EXACTNESS OF LIMITS AND COLIMITS IN ABELIAN CATEGORIES REVISITED

ALEJANDRO ARGUDÍN-MONROY AND CARLOS E. PARRA

Abstract. Let Σ be a small category and A be a Σ-co-complete (resp. Σ-complete) abelian category. It is a well-known fact that the category Fun(Σ, A) of functors of Σ in A is an abelian category, and that the functor colim_{Σ}(-) : Fun(Σ, A) → A (resp. lim_{Σ}(-) : Fun(Σ, A) → A) is left (resp. right) adjoint to κ_{Σ} : A → Fun(Σ, A), where κ_{Σ} is the associated constant diagram functor.

In this paper we will show that the functor colim_{Σ}(-) (resp. lim_{Σ}(-)) is exact if and only if the pair of functors (colim_{Σ}(-), κ_{Σ}) (resp. (κ_{Σ}, lim_{Σ}(-))) is Ext-adjoint. As an application of our findings, we will give new proofs of known results on the exactness of limits and colimits in abelian categories.

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1. Introduction

Let A be an abelian category. Recall that A is Ab3 if arbitrary coproducts exist; it is Ab4 if it is Ab3 and every coproduct of a set of monomorphisms is a monomorphism; and it is Ab5 if it is Ab3 and every colimit of a direct system of monomorphisms is a monomorphism. The corresponding dual notions are known as Ab3*, Ab4*, and Ab5*. This classification of abelian categories was introduced by Grothendieck in [3] in order to axiomatise the constructions of the homological algebra coming from categories of modules.

Given X, Y ∈ A, consider the ‘big’ group of extensions Ext^1_A(X, Y) (see [5, Chapter VII]). To get a glimpse of the essence of these groups, recall that Ext^1_A(X, Y) is made up of the equivalence classes of the short exact sequences of the form Y ↪ E ↔ X. So that, for every exact sequence 0 ↪ Y ↪ E ↔ X in A, we will denote as τ the extension in Ext^1_A(X, Y) representing τ. Now, recall that morphisms in

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\( \mathcal{A} \) induce morphisms between extension groups via pullback and pushout. Namely, every \( f \in \text{Hom}_A(X', X) \) induces the additive maps \( \text{Ext}_A^1(f, Y) : \text{Ext}_A^1(X, Y) \to \text{Ext}_A^1(X', Y) \), \( \eta_i \mapsto \eta_i \cdot f \), and \( \text{Ext}_A^1(Y, f) : \text{Ext}_A^1(Y, X') \to \text{Ext}_A^1(Y, X) \), \( \eta_i \mapsto f \cdot \eta_i \), for every \( Y \in \mathcal{A} \).

Now, for every \( A \in \mathcal{A} \) and for every coproduct \( \coprod_{i \in X} B_i \) (resp. product \( \prod_{i \in X} B_i \)) in \( \mathcal{A} \), consider the map \( \Psi : \text{Ext}_A^1(\coprod_{i \in X} B_i, A) \to \prod_{i \in X} \text{Ext}_A^1(B_i, A) \), \( \eta_i \mapsto (\eta_i \cdot \mu_i)_{i \in X} \) (resp. \( \Phi : \text{Ext}_A^1(\prod_{i \in X} B_i, A) \to \coprod_{i \in X} \text{Ext}_A^1(B_i, A) \), \( \eta_i \mapsto (\pi_i \cdot \eta_i)_{i \in X} \)), where \( \mu_i : B_i \to \coprod_{i \in X} B_i \) is the \( i \)-th canonical inclusion (resp. \( \pi_i : \prod_{i \in X} B_i \to B_i \) the \( i \)-th projection). In \([2]\), it was shown that the \( \text{Ab4} \) (resp. \( \text{Ab4}^* \)) categories are exactly the \( \text{Ab3} \) (resp. \( \text{Ab3}^* \)) categories where the maps \( \Psi \) (resp. \( \Phi \)) are bijective. A key point of this work is that, for every set of exact sequences \( H = \left\{ \eta_i : A \xrightarrow{f_i} E_i \to B_i \right\}_{i \in X} \) (resp. \( H = \left\{ \eta_i : B_i \xrightarrow{E_i} f_i \to A \right\}_{i \in X} \)), it can be built an exact sequence

\[ \Theta(H) : A \to E \to \prod_{i \in X} A_i \]  

( resp. \( \Theta(H) : \coprod_{i \in X} B_i \to E \to A \) ),

where \( E \) is the colimit (resp. limit) of a diagram defined through the family of morphisms \( \{ f_i \}_{i \in X} \) (see \([2]\) Section 4). It turns out that, under the right conditions, the inverse map of \( \Psi \) (resp. \( \Phi \)) can be exhibited using \( \Theta \).

Let \( \Sigma \) be a small category and \( \text{Fun}(\Sigma, \mathcal{A}) \) be the category of functors \( \Sigma \to \mathcal{A} \). Recall that \( \mathcal{A} \) is \( \Sigma \)-co-complete (resp. \( \Sigma \)-complete) if every \( F \in \text{Fun}(\Sigma, \mathcal{A}) \) admits a colimit (resp. limit) in \( \mathcal{A} \). In this context it is immediate to see that one has an adjoint pair \( (\text{colim}_\Sigma(-), \kappa^\Sigma) \) (resp. \( (\kappa^\Sigma, \lim_\Sigma(-)) \)), where \( \text{colim}_\Sigma(-) : \text{Fun}(\Sigma, \mathcal{A}) \to \mathcal{A} \) (resp. \( \lim_\Sigma(-) : \text{Fun}(\Sigma, \mathcal{A}) \to \mathcal{A} \)) is the colimit (resp. limit) functor, and where \( \kappa^\Sigma : \mathcal{A} \to \text{Fun}(\Sigma, \mathcal{A}) \) is the functor that assigns to each \( A \in \mathcal{A} \) a constant functor in \( \text{Fun}(\Sigma, \mathcal{A}) \).

In this article, motivated mainly by the sequence \( \Theta \), we will revisit the techniques used in \([2]\) to study the exactness of \( \Sigma \)-colimits (resp. \( \Sigma \)-limits) in \( \Sigma \)-co-complete (resp. \( \Sigma \)-complete) abelian categories, for all small category \( \Sigma \). Namely, we will consider the canonical map

\[ \Psi_{F,A}^\Sigma : \text{Ext}_A^1(\text{colim}_\Sigma(F), A) \to \text{Ext}_{\text{Fun}(\Sigma, \mathcal{A})}^1(F, \kappa^\Sigma(A)) \]

( resp. \( \Phi_{F,A}^\Sigma : \text{Ext}_A^1(A, \text{lim}_\Sigma(F)) \to \text{Ext}_{\text{Fun}(\Sigma, \mathcal{A})}^1(\kappa^\Sigma(A), F) \) )

for every \( A \in \mathcal{A} \) and \( F \in \text{Fun}(\Sigma, \mathcal{A}) \) (see Remark 4.1 and Corollary 4.4). Looking for sufficient conditions for \( \Psi_{F,A}^\Sigma \) (resp. \( \Phi_{F,A}^\Sigma \)) to be bijective, we will see that every exact sequence \( \eta : \kappa^\Sigma(A) \xrightarrow{f} E \to F \) (resp. \( \eta : E \xrightarrow{f} \to \kappa^\Sigma(A) \)) in \( \text{Fun}(\Sigma, \mathcal{A}) \) induces an exact sequence \( \kappa^\Sigma(A) \xrightarrow{f} E \xrightarrow{\eta} F' \) (resp. \( F' \xrightarrow{E} E_0 \xrightarrow{f} \kappa^\Sigma(A) \)) in \( \text{Fun}(\Sigma, \mathcal{A}) \), where \( \kappa^\Sigma \) is the one-point extension of \( \Sigma \) that adds a source (resp. sink) point \( * \) (see Section 3), and \( F' \) is a functor such that \( F'|_\Sigma = F \) and \( F'(*):=0 \).

Our main result will be that, in a \( \Sigma \)-co-complete (resp. \( \Sigma \)-complete) category \( \mathcal{A} \), \( \text{colim}_\Sigma(-) \) (resp. \( \text{lim}_\Sigma(-) \)) is exact if and only if \( f_\eta \) is a monomorphism (resp. epimorphism) \( \forall \eta \in \text{Ext}_{\text{Fun}(\Sigma, \mathcal{A})}^1(\kappa^\Sigma(A), F) \) (resp. \( \forall \eta \in \text{Ext}_{\text{Fun}(\Sigma, \mathcal{A})}^1(F, \kappa^\Sigma(A)) \)), which in turn will be equivalent to \( \Phi_{F,A}^\Sigma \) (resp. \( \Psi_{F,A}^\Sigma \)) being bijective \( \forall A \in \mathcal{A} \) and \( \forall F \in \text{Fun}(\Sigma, \mathcal{A}) \). Moreover, in such case, it can be seen that \( \Psi_{\Sigma,-}^\Sigma \) (resp. \( \Phi_{\Sigma,-}^\Sigma \)) is a natural isomorphism between the respective functors. This phenomenon is what
we call an Ext-adjoint pair. An application of our results consists in reproducing with new proofs the following known results in abelian categories.

1. [6, Corollary 3.2.9] Let \( \mathcal{A} \) be an Ab3 (resp. Ab3*) abelian category. Then, \( \mathcal{A} \) is Ab4 (resp. Ab4*) whenever \( \mathcal{A} \) has enough injectives (resp. projectives).

2. [2, Theorems 4.9 and 4.11] Let \( \mathcal{A} \) be an Ab3 (resp. Ab3*) abelian category. Then, \( \mathcal{A} \) is Ab4 (resp. Ab4*) if, and only if, the canonical map

\[
\Ext^1_{\mathcal{A}}(\bigoplus_{i \in I} A_i, A) \to \prod_{i \in I} \Ext^1_{\mathcal{A}}(A_i, A)
\]

(resp. \( \Ext^1_{\mathcal{B}}(B, \prod_{i \in I} B_i) \to \prod_{i \in I} \Ext^1_{\mathcal{B}}(B, B_i) \))

is always an isomorphism of ‘big’ abelian groups.

The article is organised as follows. In Section 2 we recall the preliminaries on abelian categories and (co)limits in abelian categories that we will need throughout the article. In Section 3 we show how the one-point extension of a small category \( \Sigma \) is related to the behaviour of the colimits on the category \( \Sigma \). Finally, Section 4 contains our main results.

2. Preliminaries

In this section we recall some definitions and results, and we fix some notation that we will use throughout the paper. For more precise details, the reader is referred to [6, 7, 4].

2.1. Limits and colimits. Recall that a category is small when the isomorphism classes of its objects form a set. In this sense, every set \( \Sigma \) can be viewed as a small category whose objects are the elements of \( \Sigma \) and the only morphisms are the identity morphisms. Also, if \( \Sigma \) is a directed (resp. codirected) set, then \( \Sigma \) can be viewed as a small category whose objects are the elements of \( \Sigma \) and there is a unique morphism \( \alpha \to \beta \) exactly when \( \alpha \leq \beta \) (resp. \( \beta \leq \alpha \)).

In what follows \( \Sigma \) will denote a small category. If \( \mathcal{C} \) is a category, then a functor \( \Sigma \to \mathcal{C} \) will be called a \( \Sigma \)-diagram on \( \mathcal{C} \). In this setting, the \( \Sigma \)-diagrams on \( \mathcal{C} \) together with the respective natural transformations form a category, which will be denoted by \( \text{Fun}(\Sigma, \mathcal{C}) \). In particular, if \( \mathcal{C} \) is an abelian category, then \( \text{Fun}(\Sigma, \mathcal{C}) \) is an abelian category in the obvious way (see [7, Chapter IV, Section 7]).

Observe that the assignment \( \mathcal{C} \mapsto \kappa^\Sigma_\mathcal{C}(\mathcal{C}) \) gives a functor \( \kappa^\Sigma : \mathcal{C} \to \text{Fun}(\Sigma, \mathcal{C}) \), where \( \kappa^\Sigma(\mathcal{C}) \) is the \( \Sigma \)-diagram on \( \mathcal{C} \) such that \( i \mapsto C \) and \( \lambda \mapsto 1_C \), for each object \( i \) and morphism \( \lambda \) of \( \Sigma \); and, for every morphism \( f : A \to B \) in \( \mathcal{C} \), \( \kappa^\Sigma(f) : \kappa^\Sigma(A) \to \kappa^\Sigma(B) \) is the natural transformation \( \kappa^\Sigma_f \) given by the family of morphisms \( (\kappa^\Sigma_{f,s})_{s \in \Sigma} \) with \( \kappa^\Sigma_{f,s} = f \) for all \( s \in \Sigma \). Such functor is called the constant diagram functor.

Let \( F \in \text{Fun}(\Sigma, \mathcal{C}) \), \( C, L \in \mathcal{C} \), and \( \rho : F \to \kappa^\Sigma(C) \), \( \varrho : \kappa^\Sigma(L) \to F \) be natural transformations. Then, we will say that the pair \( (C, \rho) \) (resp. \( (L, \varrho) \)) is a \( \Sigma \)-colimit (resp. \( \Sigma \)-limit) of \( F \) when the following condition holds: for each natural transformation \( \tau : F \to \kappa^\Sigma(D) \) (resp. \( \tau : \kappa^\Sigma(D) \to F \)), where \( D \) is an object in \( \mathcal{C} \), there is a unique morphism \( f : C \to D \) (resp. \( f : D \to L \)) such that the following diagram commutes.
If \((C, \rho)\) (resp. \((L, \ell))\) is a \(\Sigma\)-colimit (resp. \(\Sigma\)-limit) of \(F\), we use the notation \(C = \text{colim}_\Sigma(F)\) (resp. \(L = \text{lim}_\Sigma(F)\)). Now, if each \(\Sigma\)-diagram on \(C\) has a \(\Sigma\)-colimit (resp. \(\Sigma\)-limit), we say that \(C\) admits \(\Sigma\)-colimits (resp. \(\Sigma\)-limits) or that it is \(\Sigma\)-co-complete (resp. \(\Sigma\)-complete). In this case, the assignment \(F \mapsto \text{colim}_\Sigma(F)\) (resp. \(F \mapsto \text{lim}_\Sigma(F)\)) gives rise to a functor \(\text{colim}_\Sigma : \text{Fun}(\Sigma, C) \to C\) (resp. \(\text{lim}_\Sigma : \text{Fun}(\Sigma, C) \to C\)) which is the left (resp. right) adjoint to the associated constant diagram functor. The category \(C\) is called co-complete (resp. complete) when it admits \(\Sigma\)-colimits (resp. \(\Sigma\)-limits), for every small category \(\Sigma\). In this context, for every functor \(F : \Sigma \to C\), we denote by \(C_F := \text{colim}_\Sigma(F)\) (resp. \(L_F := \text{lim}_\Sigma(F)\)) and by \(\rho^F : F \to \kappa^\Sigma(C_F)\) (resp. \(\ell^F : \kappa^\Sigma(L_F) \to F\)) the associated natural transformation.

Lastly, if \(C\) admits \(\Sigma\)-colimits (resp. \(\Sigma\)-limits), observe that for each \(A \in \mathcal{A}\) there is a unique morphism \(\nabla^A : C_{\kappa^\Sigma(A)} \to A\) (resp. \(\Delta^A : A \to L_{\kappa^\Sigma(A)}\)) such that \(\kappa^\Sigma_{\nabla^A} \circ \rho^\Sigma(A) = 1_{\kappa^\Sigma(A)}\) (resp. \(\rho^\Sigma(A) \circ \kappa^\Sigma_{\Delta^A} = 1_{\kappa^\Sigma(A)}\)). We refer to such morphism as the co-diagonal morphism (resp. diagonal morphism).

### 2.2. Abelian categories

In what follows \(\mathcal{A}\) will denote an abelian category. Recall the following hierarchy among abelian categories (introduced by Grothendieck in \[3\]): we say that \(\mathcal{A}\) is

- \(\textbf{Ab3}\) if all set-indexed coproducts exist in \(\mathcal{A}\) (equivalently, if it is co-complete);
- \(\textbf{Ab4}\) if it is Ab3 and the functors \(\text{colim}_\Sigma\) are exact, for each set \(\Sigma\) viewed as a small category;
- \(\textbf{Ab5}\) if it is Ab3 and the functors \(\text{colim}_\Sigma\) are exact, for each directed set \(\Sigma\) viewed as a small category.

We will denote by \(\textbf{Abn}\) to the dual definition of \(\textbf{Abn}\) for each \(n \in \{3, 4, 5\}\).

### 3. A ONE-POINT EXTENSION OF A SMALL CATEGORY

In this section, \(\Sigma\) will denote a small category and \(\mathcal{A}\) will denote a \(\Sigma\)-co-complete abelian category. Now, from \(\Sigma\) we can construct a new category \(\Sigma_*\) which essentially consists of adding a source point to \(\Sigma\). That is, \(\Sigma_*\) is the category consisting of the objects of \(\Sigma\) plus a new object *, and the morphisms of \(\Sigma_*\) are the morphisms of \(\Sigma\) plus a single morphism from * to any other point of \(\Sigma_*\). More precisely, we define the category \(\Sigma_*\) as follows:

- \(\text{Ob}(\Sigma_*) := \text{Ob}(\Sigma) \cup \{\ast\}\);
- Let \((\alpha_i)_{i \in \text{Ob}(\Sigma)}\) be an \(\text{Ob}(\Sigma)\)-indexed family of symbols, where \(\alpha_i = \alpha_j\) with \(i, j \in \text{Ob}(\Sigma)\), when \(i\) is isomorphic to \(j\) in \(\Sigma\). In this setting, we put \(\Sigma_*(\ast, i) := \emptyset\), \(\Sigma_*(\ast, i) := \{\alpha_i\}\), \(\Sigma_*(\ast, \ast) := \{1_\ast\}\) and \(\Sigma_*(i, j) = \Sigma(i, j)\) for every \(i\) and \(j\) objects in \(\Sigma\).
3.1. A one-point extension of a \(\Sigma\)-diagram. Let \(\eta : \kappa^\Sigma(A) \xrightarrow{\phi} F \xrightarrow{\psi} G\) be an exact sequence in \(\text{Fun}(\Sigma, A)\). Define the functor \(F_\eta : \Sigma \to A\) as follows:

- For each \(i \in \text{Ob}(\Sigma)\) we assign the object \(F(i)\), and to the object \(*\) we assign the object \(A\).
- For each morphism \(\lambda\) in \(\Sigma\) we assign the morphism \(F(\lambda)\) and, for each \(\alpha_i\) we assign the morphism \(\phi_i\). Lastly, for the morphism \(1_A\) we assign the morphism \(1_A\).

It can be shown that there is an exact sequence \(\eta_* : \kappa^\Sigma_*((\Sigma)) \xrightarrow{\phi_*} F_* \xrightarrow{\psi_*} G_*\) in \(\text{Fun}(\Sigma_*, A)\), where \(G_*\) is a functor that extends \(G\) such that \(G_*(*) = 0\). In order to find the colimit of such exact sequence, we proceed as follows.

By the right exactness of the functor \(\text{colim}_A : \text{Fun}(\Sigma, A) \to A\) (see [2], Chapter V, Section 1), there is an exact sequence in \(A\):

\[
\text{colim}_A(\phi) \rightarrow C_F \rightarrow \text{colim}_A(\psi) \rightarrow C_G \rightarrow 0.
\]

Consider the following commutative diagram in \(A\) with exact rows, where the left square is the pushout of \(\text{colim}_A(\phi)\) and the co-diagonal morphism.

\[
\begin{array}{ccc}
C_{\kappa^\Sigma(A)} & \xrightarrow{\text{colim}_A(\phi)} & C_F & \xrightarrow{\text{colim}_A(\psi)} & C_G & \rightarrow & 0 \\
\downarrow \nabla^A & & \downarrow \mu_\eta & & \downarrow \xi_\eta & & \downarrow 0 \\
A & \xrightarrow{f_\eta} & Z_\eta & \xrightarrow{g_\eta} & C_G & \rightarrow & 0
\end{array}
\]

Lastly, consider the morphism \(\xi_\eta : F_\eta \rightarrow \kappa^\Sigma_*(Z_\eta)\) in \(\text{Fun}(\Sigma_*, A)\) given by \(\xi_\eta := \mu_\eta \circ \rho^\Sigma_i\) for each \(i \in \text{Ob}(\Sigma)\) and \(\xi_* := f_\eta\). Notice that \(\xi_\eta\) is well-defined since \(\mu_\eta \circ \rho^\Sigma_i \circ F_\eta(\lambda) = \mu_\eta \circ \rho^\Sigma_i \circ F(\lambda) = \mu_\eta \circ \rho^\Sigma_i\) for each \(\lambda : i \rightarrow j\) morphism in \(\Sigma\), and the following diagram in \(A\) is commutative for all \(i \in \text{Ob}(\Sigma)\):

\[
\begin{array}{ccc}
F_\eta(*) = A & \xrightarrow{\xi_\eta = f_\eta} & (\kappa^\Sigma(Z_\eta))(*) = Z_\eta \\
F_\eta(\alpha_i) = \phi_i & \downarrow & (\kappa^\Sigma(Z_\eta))((\alpha_i) = 1Z_\eta) \\
F_\eta(i) = F(i) & \xrightarrow{\xi_\eta} & \kappa^\Sigma(Z_\eta)(i) = Z_\eta \\
& \mu_\eta & \rho^\Sigma_i
\end{array}
\]

Indeed, such claim follows from the following equalities of morphisms.

\[
\mu_\eta \circ (\rho^\Sigma_i \circ \phi_i) = \mu_\eta \circ (\text{colim}_A(\phi) \circ \rho^\Sigma_i(A)) \quad \text{(by definition of \(\text{colim}_A(\phi)\))}
\]

\[
= (f_\eta \circ \nabla^A) \circ \rho^\Sigma_i(A) \quad \text{(By (1.1))}
\]

\[
= f_\eta \circ 1_A = f_\eta = 1Z_\eta \circ f_\eta
\]

The following result shows that the morphism \(f_\eta\) coincide with the left morphism in the sequence \(\Theta(\eta)\) described in the introduction (see [2]) when \(\Sigma\) is a set.
**Lemma 3.1.** Let $\eta : \kappa^\Sigma(A) \overset{\phi}{\to} F \overset{\psi}{\to} G$ be an exact sequence in $\text{Fun}(\Sigma, A)$, where $\Sigma$ is a small category and $A$ is an object in $A$, and let $Z_\eta$ be as in the diagram (1.1). Then, the pair $(Z_\eta, \zeta^\eta)$ is the $\Sigma_*$-colimit of $F_\eta$. In particular, $Z_\eta = \text{colim}_{\Sigma_*}(F_\eta).

**Proof.** Let $C$ be an object in $A$ and let $\varphi : F_\eta \to \kappa^\Sigma_*(C)$ be a natural transformation in $\text{Fun}(\Sigma_*, A)$. By definition, we have that $\varphi_j \circ F(\lambda) = \varphi_j \circ F_\eta(\lambda) = \varphi_i$, for every morphism $\lambda : i \to j$ in $\Sigma$, and the following commutative diagram in $A$, for all $i \in I$:

\[
\begin{array}{ccc}
F_\eta(*) = A & \xrightarrow{\varphi_*} & (\kappa^\Sigma(C))(*) = C \\
F_\eta(\alpha_i) = \phi_i & \downarrow & (\kappa^\Sigma(C))(\alpha_i) = 1_C \\
F_\eta(i) = F(i) & \xrightarrow{\varphi_i} & (\kappa^\Sigma(C))(i) = C
\end{array}
\]

In particular, we get that the family of morphisms $(\varphi_i)_{i \in \text{Ob}(\Sigma)} := \varphi|_{\text{Ob}(\Sigma)}$ is a natural transformation in $\text{Fun}(\Sigma, A)$. Now, from the universal property of colimits, we obtain a unique morphism $\pi : CF \to C$ in $A$ such that the equality $\kappa^\Sigma_\pi \circ \rho^F = \varphi|_{\text{Ob}(\Sigma)}$ holds in $\text{Fun}(\Sigma, A)$. In particular, we get

\[
(\varphi_* \circ \nabla^A) \circ \rho^\Sigma_\phi = \varphi_* \circ 1_A = \varphi_i \circ \phi_i = (\pi \circ \rho^F_i) \circ \phi_i = (\pi \circ \text{colim}_\Sigma(\phi)) \circ \rho^\Sigma_\phi
\]

for all $i \in \text{Ob}(\Sigma)$. Using once again the universal property of colimits we deduce that $\varphi_* \circ \nabla^A = \pi \circ \text{colim}_\Sigma(\phi)$. Now, from the universal property of pushouts, we know that there is a unique morphism $\sigma : Z_\eta \to C$ such that $\sigma \circ f_\eta = \varphi_*$ and $\sigma \circ \mu_\eta = \pi$. And hence, it follows that the natural transformation $\kappa^\Sigma_\sigma$ satisfy the following equation in the category $\text{Fun}(\Sigma_*, A)$:

\[
\kappa^\Sigma_\sigma \circ \xi^\eta = \varphi.
\]

It remains to prove the uniqueness of $\sigma$. Let $\sigma' : Z_\eta \to C$ be a morphism in $A$ such that $\kappa^\Sigma_{\sigma'} \circ \xi^\eta = \varphi$. Then, $\varphi_i = (\kappa^\Sigma_{\sigma'})_\ast \circ \xi^\eta = \sigma' \circ f_\eta$ and $\varphi_i = (\kappa^\Sigma_{\sigma'})_\ast \circ \xi^\eta = (\sigma' \circ \mu_\eta) \circ \rho^F_i$, for all $i \in \text{Ob}(\Sigma)$. Thus, $\kappa^\Sigma_{\sigma' \circ \mu_\eta} \circ \rho^F = \varphi|_{\text{Ob}(\Sigma)}$. And hence, from the universal property of colimits, $\sigma' \circ \mu_\eta = \pi$. Finally, by the universal property of pushouts, we have $\sigma = \sigma'$ as desired. $\square$

4. **Main results**

In this section we will use the previous results to obtain a characterisation for the exactness of the functors $\text{colim}_\Sigma(-) : \text{Fun}(\Sigma, A) \to A$ and $\text{lim}_\Sigma(-) : \text{Fun}(\Sigma, B) \to B$, when $A$ and $B$ are $\Sigma$-co-complete and $\Sigma$-complete abelian categories, respectively, for all small category $\Sigma$.

**Theorem 4.1.** Let $\Sigma$ be a small category and let $A$ be a $\Sigma$-co-complete abelian category. Then, the following conditions are equivalent:

1. The functor $\text{colim}_\Sigma : \text{Fun}(\Sigma, A) \to A$ is exact;
2. $f_\eta$ is a monomorphism, for every exact sequence $\eta : \kappa^\Sigma(A) \overset{\phi}{\to} F \overset{\psi}{\to} G$ in $\text{Fun}(\Sigma, A)$, where $A$ is an object in $A$. 
Proof. (a) ⇒ (b). It is clear by properties of the pushouts (see (1.1)).

(b) ⇒ (a). Let $H \xrightarrow{\zeta} N \xrightarrow{\chi} G$ be an exact sequence in $\text{Fun}(\Sigma, \mathcal{A})$. Consider the following commutative diagram in $\text{Fun}(\Sigma, \mathcal{A})$ with exact rows, where the left square is the pushout of $\zeta$ and $\psi^H$:

\[
\begin{array}{ccccccccc}
0 & \to & H & \xrightarrow{\zeta} & N & \xrightarrow{\chi} & G & \to & 0 \\
\rho^H & | & \downarrow & | & \downarrow & | & \downarrow & | & \\
0 & \to & \kappa^\Sigma(C_H) & \xrightarrow{\phi} & F & \xrightarrow{\psi} & G & \to & 0
\end{array}
\]

From the universal property of pushouts, we obtain a unique natural transformation $\nu : F \to \kappa^\Sigma(C_N)$ such that $\nu \circ \omega = \rho^N$ and $\nu \circ \phi = \kappa^\Sigma_{\text{colim}\Sigma}(\zeta)$ since $\rho^N \circ \zeta = \kappa^\Sigma_{\text{colim}\Sigma}(\zeta) \circ \rho^H$. Note that there is a unique morphism $\gamma : C_F \to C_N$ such that $\kappa^\Sigma_\gamma \circ \rho^F = \nu$ by the universal property of colimits.

Now, set $\eta := \kappa^\Sigma(C_H) \xrightarrow{\phi} F \xrightarrow{\psi} G$ and, for each $i \in \text{Ob}(\Sigma)$, consider the following commutative diagram in $\mathcal{A}$ with exact rows, where the upper and lower left squares are pushouts (here $f_\eta$ is a monomorphism by the assumption (b)):

\[
\begin{array}{ccccccccc}
0 & \to & H(i) & \xrightarrow{\zeta_i} & N(i) & \xrightarrow{\chi_i} & G(i) & \to & 0 \\
\rho^H_i & | & \downarrow & | & \downarrow & | & \downarrow & | & \\
0 & \to & C_H & \xrightarrow{\phi_i} & F(i) & \xrightarrow{\psi_i} & G(i) & \to & 0 \\
\rho^F_i & | & \downarrow & | & \downarrow & | & \downarrow & | & \\
C_{\kappa^\Sigma(C_H)} & \xrightarrow{\text{colim}\Sigma(\phi)} & C_F & \xrightarrow{\text{colim}\Sigma(\psi)} & C_G & \to & 0 \\
\n\end{array}
\]

In particular, we have the following equalities:

\[
(\gamma \circ \text{colim}\Sigma(\phi)) \circ \rho^\Sigma_i(C_H) = \gamma \circ (\rho^F_i \circ \phi_i) = v_i \circ \phi_i = \text{colim}\Sigma(\zeta) = \text{colim}\Sigma(\zeta) \circ 1_{C_H} = (\text{colim}\Sigma(\zeta) \circ \nabla^H) \circ \rho^\Sigma_i(C_H)
\]

for all $i \in \text{Ob}(\Sigma)$. So that $\kappa^\Sigma_{\text{colim}\Sigma(\phi)} \circ \rho^\Sigma(C_H) = \kappa^\Sigma_{\text{colim}\Sigma(\zeta) \circ \nabla^H \circ \rho^\Sigma(C_H)}$ and hence $\text{colim}\Sigma(\zeta) \circ \nabla^H = \gamma \circ \text{colim}\Sigma(\phi)$. Therefore, there is a unique morphism $\theta : Z_\eta \to C_N$ such that $\theta \circ \mu_\eta = \gamma$ and $\theta \circ f_\eta = \text{colim}\Sigma(\zeta)$. We claim that the pair $(Z_\eta, \kappa^\Sigma_{\mu_\eta} \circ (\rho^F \circ \omega) : N \xrightarrow{\kappa^\Sigma(Z_\eta)} \text{colim}\Sigma)$ is a colimit of $N$. Indeed, by the universal property of colimits, there is a unique morphism $m : C_N \to Z_\eta$ in $\mathcal{A}$ such that $\kappa^\Sigma_m \circ \rho^N = \kappa^\Sigma_{\mu_\eta} \circ (\rho^F \circ \omega)$. Therefore, the claim holds once we show that $m$ is an
isomorphism in \(A\). For this, observe that, on the one hand:

\[
(\theta \circ m) \circ \rho^N_i = \theta \circ (\mu_\eta \circ (\rho^F_i \circ \omega_i)) = \gamma \circ (\rho^F_i \circ \omega_i) = \upsilon \circ \omega_i = \rho^N_i \circ 1_{C_N} \circ \rho^N_i \quad \text{for all } i \in Ob(\Sigma).
\]

And hence, \(\theta \circ m = 1_{C_N} \) by the universal property of colimits.

On the other hand, from the above diagram we have that

\[
f_\eta = f_\eta \circ 1_{C_N} = f_\eta \circ \nabla^C_H \circ \rho^F_i = \mu_\eta \circ \rho^F_i \circ \phi_i,
\]

for all \(i \in Ob(\Sigma)\). In particular, for every \(i \in Ob(\Sigma)\), we get

\[
f_\eta \circ \rho^H_i = \mu_\eta \circ \rho^F_i \circ \phi_i \circ \rho^H_i = \mu_\eta \circ \rho^F_i \circ \omega_i \circ \zeta_i = m \circ \rho^N_i \circ \zeta_i = m \circ \text{colim}_{\Sigma}(\zeta) \circ \rho^H_i.
\]

Therefore, \(f_\eta = m \circ \text{colim}_{\Sigma}(\zeta) = \mu_\eta \circ \rho^F_i \circ \phi_i\) for all \(i \in Ob(\Sigma)\). Now, for each \(i \in Ob(\Sigma)\), we get the following commutative diagram in \(A\) (recall that \(\mu_\eta \circ (\rho^F_i \circ \omega_i) = m \circ \rho^N_i\)):

\[\begin{array}{c}
C_H \\
\phi_i \\
\downarrow \omega_i \\
\downarrow \text{m} \circ \rho^N_i \\
F(i)
\end{array}\]

From the universal property of the upper pushout in the diagram, we obtain that \(\mu_\eta \circ \rho^F_i = m \circ \upsilon_i\), for all \(i \in Ob(\Sigma)\). But, each \(\upsilon_i\) coincide with the composition \(\gamma \circ \rho^F_i\), and hence \(\mu_\eta = m \circ \gamma\). Now, notice that the composition \(m \circ \theta\) and \(1_{Z_\eta}\) complete the following diagram in \(A\) and, therefore, such morphisms coincide and our claim holds (since \(m \circ \theta = 1_{Z_\eta}\) and \(\theta \circ m = 1_{C_N}\)).

\[\begin{array}{c}
C_H \\
\downarrow f_\eta \\
Z_\eta
\end{array}\]

Finally, from the equality \(\theta \circ f_\eta = \text{colim}_{\Sigma}(\zeta)\) we get that the morphism \(\text{colim}_{\Sigma}(\zeta)\) is a monomorphism as desired. \(\square\)

**Remark 4.1.** In the setting of the previous theorem, we get that for each \(F \in \text{Fun}(\Sigma, A)\) and \(A \in Ob(A)\) there is a natural map

\[
\Psi^\Sigma_{F,A} : \text{Ext}^1_A(C_F, A) \rightarrow \text{Ext}^1_{\text{Fun}(\Sigma, A)}(F, \kappa^\Sigma(A))
\]

both in \(A\) and in \(F\), defined as \(\Psi^\Sigma_{F,A}(\epsilon) := \kappa^\Sigma(\epsilon) \cdot \rho^F\), where \(\kappa^\Sigma(\epsilon) \cdot \rho^F\) denote the upper exact sequence in the following commutative diagram in \(\text{Fun}(\Sigma, A)\) with exact
Theorem 4.2. Let \( \Sigma \) be a category. Then, the following conditions are equivalent:

\[ \begin{array}{c}
0 \to \kappa^\Sigma(A) \xrightarrow{G_{\kappa^\Sigma(e)}} F \to 0
\end{array} \]

\[ \kappa^\Sigma(e) : 0 \to \kappa^\Sigma(A) \to \kappa^\Sigma(B) \to \kappa^\Sigma(C_F) \to 0 \]

Theorem 4.2. Let \( \Sigma \) be a small category and let \( A \) be a \( \Sigma \)-co-complete abelian category. Then, the following conditions are equivalent:

1. The functor \( \text{colim}_A^\Sigma : \text{Fun}(\Sigma, A) \to A \) is exact;
2. The map \( \Psi_{F,A}^\Sigma \) is bijective, for every \( F \in \text{Fun}(\Sigma, A) \) and \( A \in \text{Ob}(A) \);
3. The bifunctors \( \text{Ext}^1_A(\text{colim}_A^\Sigma(\cdot), ?) \) and \( \text{Ext}^1_{\text{Fun}(\Sigma,A)}(\cdot, \kappa^\Sigma(\cdot)) \) are naturally isomorphic in the obvious way.

Proof. (b) \( \Leftrightarrow \) (c) Is clear by definition of the maps \( \Psi_{F,A}^\Sigma \) (see Remark 4.1).

(a) \( \Rightarrow \) (b) Let \( F \) be a functor in \( \text{Fun}(\Sigma, A) \) and let \( A \) be an object in \( A \). By Lemma 4.2 our task is reduced to check that \( \Psi_{F,A}^\Sigma \) is surjective. For this, consider an arbitrary exact sequence \( \eta : \kappa^\Sigma(A) \to G \to F \) in \( \text{Fun}(\Sigma, A) \). Now, using the exactness of the functor \( \text{colim}_A^\Sigma \), we obtain the following commutative diagram in \( A \) with exact rows, where the left square is a pushout (see (1.1)).

\[
\begin{array}{c}
0 \to C_{\kappa^\Sigma(A)} \xrightarrow{\text{colim}_A^\Sigma(\phi)} C_G \xrightarrow{\text{colim}_A^\Sigma(\psi)} C_F \to 0
\end{array}
\]

Let \( \epsilon \) be the lower exact sequence in the above diagram. We claim that \( \Psi_{F,A}^\Sigma(\epsilon) = \eta \).

To show this, notice that we have the following commutative diagram in \( \text{Fun}(\Sigma, A) \):

\[
\begin{array}{c}
0 \to \kappa^\Sigma(A) \xrightarrow{\phi} G \xrightarrow{\psi} F \to 0
\end{array}
\]
Using the fact that \( \kappa^{\Sigma}_{\varphi} \circ \rho^{\Sigma}(A) = 1_{\kappa^{\Sigma}(A)} \), we obtain that the right outer rectangle in the above diagram is a pullback (see [6, dual of Lemma 5.2, p.35]). And, hence \( \Psi_{\mathcal{F}, \mathcal{A}}(\varphi) = \mathfrak{f} \).

(b) \( \Rightarrow \) (a) By Theorem 4.1.1 it is enough to check that \( f_{\eta} \) is a monomorphism, for every exact sequence \( \eta : \kappa^{\Sigma}(A) \xrightarrow{\varphi} G \xrightarrow{\psi} F \) in \( \text{Fun}(\Sigma, \mathcal{A}) \), where \( A \) is an object in \( \mathcal{A} \). Observe that, by hypothesis, there exists an exact sequence \( \epsilon : A \xrightarrow{f} B \xrightarrow{g} C_F \) in \( \mathcal{A} \) such that \( \Psi_{\mathcal{F}, \mathcal{A}}(\varphi) = \mathfrak{f} \). So that we have the following commutative diagram in \( \text{Fun}(\Sigma, \mathcal{A}) \).

\[
\begin{array}{ccc}
\Psi_{\mathcal{F}, \mathcal{A}}(\epsilon) : 0 & \xrightarrow{\phi} & \kappa^{\Sigma}(A) \\
\kappa^{\Sigma}(\epsilon) : 0 & \xrightarrow{\kappa^{\Sigma}(f)} & \kappa^{\Sigma}(B) \\
\end{array}
\]

Now, from the universal property of colimits, there is a unique morphism \( \gamma : Z_{\eta} \rightarrow B \) such that \( \kappa^{\Sigma}_{\epsilon} \circ \rho^{\Sigma} = u \). Using the fact that \( \kappa^{\Sigma}(f) = u \circ \phi \) and \( \kappa^{\Sigma}(\text{colim}_{\Sigma}(\phi)) \circ \rho^{\Sigma}(A) = \rho^{G} \circ \phi \), we deduce that

\[
\kappa^{\Sigma}(h \circ \text{colim}_{\Sigma}(\phi)) \circ \rho^{\Sigma}(A) = \kappa^{\Sigma}_{\epsilon} \circ \kappa^{\Sigma}(\text{colim}_{\Sigma}(\phi)) \circ \rho^{\Sigma}(A) = \kappa^{\Sigma}_{\epsilon} \circ \rho^{G} \circ \phi = u \circ \phi = \kappa^{\Sigma}(f) = \kappa^{\Sigma}(f) \circ 1_{\kappa^{\Sigma}(A)} = \kappa^{\Sigma}(f) \circ \kappa^{\Sigma}_{\mathcal{A}} \circ \rho^{\Sigma}(A) = \kappa^{\Sigma}(f \circ \nabla^{A}) \circ \rho^{\Sigma}(A).
\]

Then, by the universal property of colimits, \( h \circ \text{colim}_{\Sigma}(\phi) = f \circ \nabla^{A} \). Now, from the universal property of pushouts, we get a unique morphism \( \gamma : Z_{\eta} \rightarrow B \) such that \( \gamma \circ f_{\eta} = f \) (see (1.1)). Hence, \( f_{\eta} \) is a monomorphism.

\[\square\]

**Remark 4.3.** Let \( \mathcal{X} \) and \( \mathcal{Y} \) be abelian categories, and \( F : \mathcal{X} \rightarrow \mathcal{Y} \) and \( G : \mathcal{Y} \rightarrow \mathcal{X} \) be functors such that \( (F, G) \) is an adjoint pair. We will say that \( (F, G) \) is an Ext-adjoint pair if there is a natural isomorphism \( \text{Ext}^{1}_{\mathcal{X}}(-, G(?)) \cong \text{Ext}^{1}_{\mathcal{Y}}(F(-), ?) \). Note that the statement (c) of the above theorem can be restated by saying that \( (\text{colim}_{\Sigma}, \kappa^{\Sigma}) \) is an Ext-adjoint pair. It should be noted that not every adjoint pair is Ext-adjoint. The reader can find counterexamples of this in [1] pp.29–30.

By duality we get the following straightforward corollary.

**Corollary 4.4.** Let \( \Sigma \) be a small category and let \( \mathcal{B} \) be a \( \Sigma \)-complete abelian category. Then, the following conditions are equivalent:

1. the functor \( \text{lim}^{\Sigma} : \text{Fun}(\Sigma, \mathcal{B}) \rightarrow \mathcal{B} \) is exact;
2. the maps \( \Phi_{\mathcal{B}, \mathcal{F}}^{\Sigma}(\varphi) : \text{Ext}^{1}_{\mathcal{B}}(B, L_{\mathcal{F}}) \rightarrow \text{Ext}^{1}_{\text{Fun}(\Sigma, \mathcal{B})}(\kappa^{\Sigma}(B), F) \) defined as \( \Phi_{\mathcal{B}, \mathcal{F}}^{\Sigma}(\varphi) = \varphi^{F} \circ \kappa^{\Sigma}(\epsilon) \), where \( \varphi^{F} \circ \kappa^{\Sigma}(\epsilon) \) denotes the lower exact sequence in the below commutative diagram in \( \text{Fun}(\Sigma, \mathcal{B}) \) with exact rows (here the left square is
a pushout), are natural isomorphisms both in $B$ and in $F$.

$$\begin{array}{cccccc}
\kappa^\Sigma(\epsilon) : & 0 & \longrightarrow & \kappa^\Sigma(L_F) & \longrightarrow & \kappa^\Sigma(A) & \longrightarrow & \kappa^\Sigma(B) & \longrightarrow & 0 \\
0 & \downarrow{e^F} & & \downarrow{\ast} & & \downarrow{\ast} & & \downarrow{\ast} & & 0 \\
F & \longrightarrow & G & \longrightarrow & \kappa^\Sigma(B) & \longrightarrow & 0
\end{array}$$

(3) the bifunctors $\text{Ext}_B^1(?, \lim_{\Sigma}(-))$ and $\text{Ext}_{\text{Fun}(\Sigma, B)}^1(\kappa^\Sigma(?), -)$ are naturally isomorphic in the obvious way.

Now, we give a new proof of the following result, which is a particular case of the dual of [6, Theorem 3.2.8].

**Corollary 4.5.** In the setting of the previous corollary. If $\mathcal{B}$ has enough projectives, then the functor $\lim_{\Sigma}$ is exact if and only if the functor $\kappa^\Sigma$ preserves projective objects.

**Proof.** ($\Rightarrow$) It follows by Corollary [4.4](#). ($\Leftarrow$) Let $B$ be an object in $\mathcal{B}$ and let $F$ be a functor in $\text{Fun}(\Sigma, \mathcal{B})$. Consider an exact sequence in $\mathcal{B}$ of the form:

$$\epsilon : 0 \longrightarrow A \longrightarrow P \longrightarrow B \longrightarrow 0,$$

where $P$ is a projective object of $\mathcal{B}$. Applying the functors $\Gamma := \text{Hom}_{\mathcal{B}}(-, \lim_{\Sigma}(F))$ and $\Upsilon := \text{Hom}_{\text{Fun}(\Sigma, \mathcal{B})}(-, F)$ on $\epsilon$ and $\kappa^\Sigma(\epsilon)$, respectively, we get the commutative diagram with exact rows

$$\begin{array}{cccccc}
0 & \longrightarrow & \Gamma(B) & \longrightarrow & \Gamma(P) & \longrightarrow & \Gamma(A) & \longrightarrow & \Gamma^1(B) & \longrightarrow & \Gamma^1(P) = 0 \\
0 & \downarrow{f_1} & \downarrow{f_2} & \downarrow{f_3} & \downarrow{\Phi_{B,F}^\Sigma} & & & & & \\
\Upsilon(\kappa^\Sigma(B)) & \longrightarrow & \Upsilon(\kappa^\Sigma(P)) & \longrightarrow & \Upsilon(\kappa^\Sigma(A)) & \longrightarrow & \Upsilon^1(\kappa^\Sigma(B)) & \longrightarrow & \Upsilon^1(\kappa^\Sigma(P))
\end{array}$$

where $\Gamma^1$ and $\Upsilon^1$ denote the functors $\text{Ext}_B^1(-, \lim_{\Sigma}(F))$ and $\text{Ext}_{\text{Fun}(\Sigma, \mathcal{B})}^1(-, F)$ respectively, and the morphisms $f_i$'s are the natural isomorphisms given by the adjoint pair $(\kappa^\Sigma, \text{lim}_{\Sigma})$. Using the hypothesis on $\kappa^\Sigma$, we know that $\Upsilon^1(\kappa^\Sigma(P)) = 0$ and hence $\Phi_{B,F}^\Sigma$ is an isomorphism by the Five Lemma. Thus, this implication follows by Corollary [4.4](#).

The following result is a key for a direct consequence of the above corollary.

**Lemma 4.6.** Let $A$ be an abelian category. Then, the functor $\kappa^\Sigma$ preserve injective and projective objects, for every set $\Sigma$ viewed as a small category.

**Proof.** We show that the functor $\kappa^\Sigma$ preserves projective objects. The rest of the statement can be proved by similar arguments. Let $P$ be a projective object in $A$ and let $\epsilon : F \xrightarrow{\phi} G \xrightarrow{\phi} \kappa^\Sigma(P)$ be an exact sequence in $\text{Fun}(\Sigma, A)$. In such case, $\phi_i$ is a split epimorphism, for all $i \in \text{Ob}(\Sigma) = \Sigma$. Thus, for each $i \in \Sigma$, we can take a morphism $\xi_i : \kappa^\Sigma(P)(i) = P \to G(i)$ such that $\phi_i \circ \xi_i = 1_{\kappa^\Sigma(P)(i)}$. Now, using the fact that the only morphisms in $\Sigma$ are the identity morphisms, we obtain that the family of morphisms $\xi := (\xi_i)_{i \in \Sigma}$ is a natural transformation from the functor $\kappa^\Sigma(P)$ to the functor $G$. Finally, note that $\xi \circ \phi = 1_{\kappa^\Sigma(P)}$ and hence $\xi$ is the trivial extension in $\text{Ext}_{\text{Fun}(\Sigma, A)}^1(\kappa^\Sigma(P), F)$. Therefore, $\kappa^\Sigma(P)$ is a projective object in $\text{Fun}(\Sigma, A)$ as desired. $\square$
From Corollary 4.5 and its dual, together with the Lemma 4.6, we deduce the Corollary 3.2.9 in [6].

**Corollary 4.7.** Let $\mathcal{A}$ be an Ab$3$ abelian category and let $\mathcal{B}$ be an Ab$3^*$ abelian category. Then, the following assertions hold:

1. $\mathcal{A}$ is Ab$4$, whenever $\mathcal{A}$ has enough injectives;
2. $\mathcal{B}$ is Ab$4^*$, whenever $\mathcal{A}$ has enough projectives.

The following result is straightforward.

**Lemma 4.8.** Let $\mathcal{A}$ be an abelian category and let $\Sigma$ be a set. Given an exact sequence $\epsilon : H \xrightarrow{\phi} G \xrightarrow{\psi} F$ in $\text{Fun}(\Sigma, \mathcal{A})$, consider the exact sequence $\epsilon_i : H(i) \xrightarrow{\phi_i} G(i) \xrightarrow{\psi_i} F(i)$ for all $i \in \Sigma$. Then, the assignment $\epsilon \mapsto \prod_{i \in \Sigma} \epsilon_i$ give rise a family of isomorphisms

$$\Xi_{F,A}^\Sigma : \text{Ext}^1_{\text{Fun}(\Sigma, \mathcal{A})}(F, \kappa^\Sigma(A)) \rightarrow \prod_{i \in \Sigma} \text{Ext}^1_{\mathcal{A}}(F(i), A)$$

$$\Theta_{F,A}^\Sigma : \text{Ext}^1_{\text{Fun}(\Sigma, \mathcal{A})}(\kappa^\Sigma(A), F) \rightarrow \prod_{i \in \Sigma} \text{Ext}^1_{\mathcal{A}}(A, F(i))$$

for all $F$ in $\text{Fun}(\Sigma, \mathcal{A})$ and $A$ in $\text{Ob}(\mathcal{A})$.

Let $\mathcal{A}$ be an Ab$3$ abelian category and $\Sigma$ be a set viewed as a small category. In this context, $\text{colim}_\Sigma(F) = \prod_{i \in \Sigma} F(i)$ for all $F$ in $\text{Fun}(\Sigma, \mathcal{A})$. We can use this to get as a corollary [2, Theorems 4.9 and 4.11].

**Corollary 4.9.** Let $\mathcal{A}$ be an Ab$3$ abelian category and let $\mathcal{B}$ be an Ab$3^*$ abelian category. Then, the following assertions hold:

1. $\mathcal{A}$ is Ab$4$ if, and only if, the canonical map

$$\text{Ext}^1_{\mathcal{A}}(\prod_{i \in I} A_i, A) \rightarrow \prod_{i \in I} \text{Ext}^1_{\mathcal{A}}(A_i, A)$$

is an isomorphism, for all $A$ and $(A_i)_{i \in I}$ object and family of objects in $\mathcal{A}$, respectively.

2. $\mathcal{B}$ is Ab$4^*$ if, and only if, the canonical map

$$\text{Ext}^1_{\mathcal{B}}(B, \prod_{i \in I} B_i) \rightarrow \prod_{i \in I} \text{Ext}^1_{\mathcal{B}}(B, B_i)$$

is an isomorphism, for all $B$ and $(B_i)_{i \in I}$ object and family of objects in $\mathcal{B}$, respectively.

**Proof.** We prove statement (a), statement (b) follows by duality. Let $I$ be a set and let $(A_i)_{i \in I}$ be a family of objects in $\mathcal{A}$. Then, the assignment $i \mapsto A_i$ give rise to a functor $F : I \rightarrow \mathcal{A}$. Notice that for each $A$ in $\text{Ob}(\mathcal{A})$, we have that the map in (a) coincide with the compositions $\Xi_{F,A}^\Sigma \circ \Psi_{F,A}$ (see Theorem 4.2 and Lemma 4.8). On the other hand, every functor in $\text{Fun}(I, \mathcal{A})$ can be viewed as an $I$-indexed family of objects in $\mathcal{A}$. Then, the result follows by Theorem 4.2 and Lemma 4.8. \qed
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Instituto de Matemática y Estadística Rafael Laguardia, Facultad de Ingeniería, Universidad de la República, Julio Herrera y Reissig 565, Montevideo, URUGUAY
Email address: argudin@ciencias.unam.mx

Instituto de Ciencias Físicas y Matemáticas, Edificio Emilio Pugin, Campus Isla Teja, Universidad Austral de Chile, 5690000 Valdivia, CHILE
Email address: carlos.parra@uach.cl