Inexact Gauss-Newton like methods for injective-overdetermined systems of equations under a majorant condition

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Abstract In this paper, inexact Gauss-Newton like methods for solving injective-overdetermined systems of equations are studied. We use a majorant condition, defined by a function whose derivative is not necessarily convex, to extend and improve several existing results on the local convergence of the Gauss-Newton methods. In particular, this analysis guarantees the convergence of the methods for two important new cases.

Keywords Injective-overdetermined systems of equations · Inexact Gauss-Newton like methods · Majorant condition · Local convergence

1 Introduction

Let $F$ be a continuously differentiable nonlinear function from an open set $\Omega$ in a real or complex Hilbert space $X$ to another Hilbert space $Y$. Consider the system of nonlinear equations,

$$F(x) = 0.$$  

If $F'(x)$ is injective or surjective, we say Eq. 1 is an injective-overdetermined or surjective-underdetermined system of equations, respectively.

The Gauss-Newton method (see [1, 3, 8, 9, 12]) is a generalized Newton method for solving such systems. It finds least squares solutions of Eq. 1, which may or
may not be solutions of the original problem (1). These least squares solutions are stationary points of the objective function of the nonlinear least squares problem
\[ \min_{x \in \Omega} \| F(x) \|^2. \]

This paper is focused on the case where \( F'(x) \) is injective and the least squares solutions of Eq. 1 also solve (1). The latter assumption is called zero-residual case in the theory of nonlinear least squares problems. It is worth to point that, if \( F'(x) \) is surjective, such a case always holds.

As it is well known, each Gauss-Newton iteration requires the solution of a linear system involving \( F' \). But this may be computationally expensive, because in many cases even the calculation of the derivative of \( F \) is hard to perform. Thus, inexact variations of the method must be considered for ensuring a good implementation. For more details on the different inexact versions of the Gauss-Newton and Newton methods, we refer the reader to [4, 9, 11, 15, 16, 18] and the references therein.

Denote by \( A^* \) the adjoint of the operator \( A \). Formally, the inexact Gauss-Newton like methods, which we will consider, are described as follows: Given an initial point \( x_0 \in \Omega \), define
\[
x_{k+1} = x_k + S_k, \quad B(x_k)S_k = -F'(x_k)^*F(x_k) + r_k, \quad k = 0, 1, \ldots,
\]
where \( B(x_k) \) is a suitable invertible approximation of the derivative \( F'(x_k)^*F'(x_k) \), the residual vector, \( r_k \), and the preconditioning invertible matrix, \( P_k \), (considered for the first time in [16]) for the linear systems defining the step \( S_k \) satisfy
\[ \| P_k r_k \| \leq \theta_k \| P_k F'(x_k)^*F(x_k) \|, \]
for a suitable forcing number \( \theta_k \in [0, 1) \). In particular, the above process is the inexact modified Gauss-Newton method if \( B(x_k) = F'(x_0)^*F'(x_0) \). It is the Gauss-Newton like method if \( \theta_k = 0 \), it corresponds to the inexact Gauss-Newton method if \( B(k) = F'(x_k)^*F'(x_k) \), and it represents the Gauss-Newton method if \( \theta_k = 0 \) and \( B(k) = F'(x_k)^*F'(x_k) \). It is also worth to point out that if \( F'(x) \) is invertible for all \( x \in \Omega \), the inexact Gauss-Newton like methods become the inexact Newton-like methods, which in particular include the inexact modified Newton, Newton-like, inexact Newton and Newton methods. Furthermore, methods for solving nonlinear least squares problems with simple bounds based on Gauss-Newton methods can be seen in [17] and the references therein.

Results of convergence for the Gauss-Newton methods have been discussed by many authors, see for example [1–6, 8–16, 18, 19, 22–24]. Recent research attempts to alleviate the assumption of Lipschitz continuity on the operator \( F' \). For this, the main techniques used are the majorant condition, see for example [6, 8–12], and the generalized Lipschitz condition according to X.Wang, see for example [4, 5, 13, 14, 18, 22–24]. Besides improving the convergence theory, these generalized conditions permit us to unify several unrelated convergence results existing in the literature. For example in Newton methods, convergence analysis under Lipschitz, Smale and Nesterov-Nemirovskii conditions have been unified (see for example [7, 11, 18, 22–24]). In general, it is not hard to prove that the majorant and generalized Lipschitz conditions are equivalent. However, the majorant formulation provides a
clear relationship between the majorant function and the nonlinear operator under consideration, simplifying the proof of convergence substantially.

More recently, a milder majorant condition was studied in [6, 12], which does not assume that the derivative of the majorant function is convex. In addition to classics convergence results, the lack of the aforementioned hypothesis allows us to cover new important convergence results such as ones under Hölder-like and a weaker generalized Lipschitz conditions. In the last condition, the results are obtained without the hypothesis that the function defining the condition is nondecreasing and, this way, relaxing the usual generalized Lipschitz condition. We mention that under these weaker settings, the majorant and generalized Lipschitz conditions are not equivalent and the latter one can be seen as a particular case of the first one (see the discussion on pp. 1522 of [6] or remark 4 in [12]).

In this article, under a majorant condition as in [6, 12], we present a new local convergence analysis of inexact Gauss-Newton like methods for solving (1). The convergence for this family of methods was obtained in [9]. In this paper, an analogous result is obtained under weaker assumptions, i.e., we do not assume that the derivative of the majorant function is convex. We establish the well definiteness and the convergence, along with results on the convergence rates. As mentioned before, this weaker majorant condition allows us to cover two new important special cases, namely, the convergence can be ensured under Hölder-like and a weaker generalized Lipschitz conditions. In the latter case, the results in Theorem 3.3 of [4] are generalized, because we do not assume that the function defining the condition is nondecreasing. Moreover, it is important to say that the hypothesis of convex derivative of the majorant function or monotonicity of the function which defines the generalized Lipschitz condition, are needed only to obtain the quadratic convergence rate when the inexact Gauss-Newton like methods reduce to the Gauss-Newton method.

The organization of the paper is as follows. First, we list some notations and one basic result used in our presentation. In Section 2 we state the main result. In Section 2.1 some properties of a sequence associated to the majorant function are established and the main relationships between the majorant function and the nonlinear function $F$ are presented. In Section 2.2 our main result is proven and some applications of this result are obtained in Section 3. Some final remarks are offered in Section 4.

1.1 Notation and auxiliary results

The following notations and results are used throughout our presentation. Let $X$ and $Y$ be Hilbert spaces. The open and closed balls at $a \in X$ with radius $\delta > 0$ are denoted, respectively by

$$B(a, \delta) := \{x \in X; \|x - a\| < \delta\}, \quad B[a, \delta] := \{x \in X; \|x - a\| \leq \delta\}.$$ 

The set $\Omega \subseteq X$ is an open set, the function $F : \Omega \rightarrow Y$ is continuously differentiable, and $F'(x)$ has a closed image in $\Omega$. The condition number of a continuous linear operator $A : X \rightarrow Y$ is denoted by $\text{cond}(A)$ and it is defined as

$$\text{cond}(A) := \|A^{-1}\| \|A\|.$$
Let \( A : X \rightarrow Y \) be a continuous and injective linear operator with closed image. The Moore-Penrose inverse \( A^\dagger : Y \rightarrow X \) of \( A \) is defined by
\[
A^\dagger := (A^* A)^{-1} A^*,
\]
where \( A^* \) denotes the adjoint of the linear operator \( A \).

The next lemma is proven in [20] (see also, [21]) for an \( m \times n \) matrix with \( m \geq n \) and \( \text{rank}(A) = \text{rank}(B) = n \). This proof holds in a more general context as we will state below.

**Lemma 1** Let \( A, B : X \rightarrow Y \) be a continuous linear operator with closed images. If \( A \) is injective and \( \|A^\dagger\| \|A - B\| < 1 \), then \( B \) is injective and
\[
\|B^\dagger\| \leq \frac{\|A^\dagger\|}{1 - \|A^\dagger\| \|A - B\|}.
\]

### 2 Local analysis for inexact Gauss-Newton like methods

In this section we state and prove the main result of the paper. It consists of a local theorem for the inexact Gauss-Newton like methods for solving (1). First, some results of a certain sequence associated to the majorant function are obtained. Then, we establish the main relationships between the majorant function and the nonlinear function \( F \). Finally, we show the well definiteness and convergence of inexact Gauss-Newton like methods, along with some results on the convergence rates. The statement of the theorem is:

**Theorem 2** Let \( X \) and \( Y \) be Hilbert spaces, \( \Omega \subseteq X \) be an open set and \( F : \Omega \rightarrow Y \) be a continuously differentiable function such that \( F' \) has a closed image in \( \Omega \). Let \( x_* \in \Omega, R > 0 \) and consider
\[
\kappa := \sup \{ t \in [0, R) : B(x_*, t) \subset \Omega \}, \quad \beta := \|F'(x_*)^\dagger\|.
\]
Suppose that \( F(x_*) = 0, F'(x_*) \) is injective and there exists a continuously differentiable function \( f : [0, R) \rightarrow \mathbb{R} \) such that
\[
\beta \|F'(x) - F'(x_* + \tau(x - x_*))\| \leq f'(\|x - x_*\|) \quad \text{for all } \tau \in [0, 1], x \in B(x_*, \kappa) \text{ and }\]
\[
h_1 \quad f(0) = 0 \text{ and } f'(0) = -1;
\]
\[
h_2 \quad f' \text{ is strictly increasing.}
\]
Let constants \( \vartheta, \omega_1 \) and \( \omega_2 \) be such that
\[
0 \leq \vartheta < 1, \quad 0 \leq \omega_2 < w_1, \quad \omega_1 \vartheta + \omega_2 < 1,
\]
and define \( v, \rho \) and \( r \) as
\[
v := \sup \{ t \in [0, R) : f'(t) < 0 \},
\]
\[
\rho := \sup \{ \delta \in (0, v) : (1 + \vartheta) \omega_1 \{ f(t)/f'(t) - 1 \}/t + \omega_1 \vartheta + \omega_2 < 1, \quad t \in (0, \delta) \}, \quad r := \min \{ \kappa, \rho \}.
\]
Consider the inexact Gauss-Newton like methods, with initial point \( x_0 \in B(x_*, r) \backslash \{ x_* \} \), defined by
\[
x_{k+1} = x_k + S_k, \quad B(x_k)S_k = -F'(x_k)^*F(x_k) + r_k, \quad k = 0, 1, \ldots,
\]
where \( B(x_k) \) is an invertible approximation of \( F'(x_k)^*F'(x_k) \) satisfying the following conditions
\[
\| B(x_k)^{-1}F'(x_k)^*F'(x_k) \| \leq \omega_1, \quad \| B(x_k)^{-1}F'(x_k)^*F'(x_k) - I \| \leq \omega_2, \quad k = 0, 1, \ldots,
\]
and the residual vector, \( r_k \), the forcing term, \( \theta_k \), and the preconditioning invertible matrix, \( P_k \), are such that
\[
\| P_k r_k \| \leq \theta_k \| P_k F'(x_k)^*F(x_k) \|, \quad 0 \leq \theta_k \text{cond}(P_k F'(x_k)^*F'(x_k)) \leq \vartheta, \quad k = 0, 1, \ldots.
\]
Define the scalar sequence \( \{ t_k \} \), with initial point \( t_0 = \| x_0 - x_* \| \), by
\[
t_{k+1} = (1 + \vartheta)\omega_1 | t_k - f(t_k)/f'(t_k) | + (\omega_1 \vartheta + \omega_2) t_k, \quad k = 0, 1, \ldots \quad (7)
\]
Then, the sequences \( \{ x_k \} \) and \( \{ t_k \} \) are well defined; \( \{ t_k \} \) is strictly decreasing, contained in \((0, r)\) and it converges to 0. Furthermore, \( \{ x_k \} \) is contained in \( B(x_*, r) \), it converges to \( x_* \) and there hold:
\[
\limsup_{k\to\infty} \left[ \frac{\| x_{k+1} - x_* \|}{\| x_k - x_* \|} \right] \leq \omega_1 \vartheta + \omega_2, \quad \lim_{k\to\infty} \frac{t_{k+1}}{t_k} = \omega_1 \vartheta + \omega_2.
\]
If, additionally, given \( 0 \leq p \leq 1 \)

h3) the function \((0, \nu) \ni t \mapsto [f(t)/f'(t) - t]/t^{p+1}\) is strictly increasing,

then the sequences \( \{ x_k \} \) and \( \{ t_k \} \) satisfy
\[
\| x_{k+1} - x_* \| \leq (1 + \vartheta)\omega_1 \left( \frac{f(t_0) - t_0 f'(t_0)}{t_0^{p+1} f'(t_0)} \right) \| x_k - x_* \|^{p+1} + (\omega_1 \vartheta + \omega_2) \| x_k - x_* \|, \quad (8)
\]
for all \( k = 0, 1, \ldots \), and
\[
\| x_k - x_* \| \leq t_k, \quad k = 0, 1, \ldots \quad (9)
\]

The above theorem extends and improves previous results of the Gauss-Newton methods for solving injective-overdetermined systems of equations, in particular those in [4, 6, 9, 12].

Remark 1 In particular, we obtain, from Theorem 2, the convergence for inexact modified Gauss-Newton method if \( B(x_k) = F'(x_0)^*F'(x_0) \), Gauss-Newton like method if \( \vartheta = 0 \), inexact Gauss-Newton method if \( \omega_1 = 1 \) and \( \omega_2 = 0 \), and Gauss-Newton method if \( \vartheta = 0 \), \( \omega_1 = 1 \) and \( \omega_2 = 0 \). In this latter case, it is possible to prove that \( r \) is the optimal convergence radius, see Theorem 2 in [12].

Remark 2 If \( F'(x_*) \) is invertible, we obtain, from Theorem 2, the local convergence of the inexact Newton-Like methods for solving nonlinear equations, which in particular imply the convergence for the inexact modified Newton, Newton-like, inexact Newton and Newton methods. In the last case, Theorem 2 is similar to Theorem 2 in [6].
Remark 3 In particular, if \( f \)' is convex, we can prove that \( h3 \) holds for \( p = 1 \) and, therefore, in this case, we are led to the result proven in Theorem 7 of [9] with \( c = 0 \). Thus, the additional assumption that the majorant function, \( f \), has convex derivative, is sufficient to obtain the inequalities in Eq. 8 and Eq. 9, which for the Gauss-Newton method imply quadratic convergence rate. More details on the convergence of the Gauss-Newton and Newton methods under a weaker majorant condition can be found in [6, 12].

From now on, we assume that all the assumptions of Theorem 2 hold, with the exception of \( h3 \), which will be considered to hold only when explicitly stated.

2.1 Preliminary results

In this section, we will prove all the statements in Theorem 2 regarding the sequence \( \{t_k\} \) associated to the majorant function. The main relationships between the majorant function and the nonlinear operator will be also established.

2.1.1 The scalar sequence

In this part, we will check the statements in Theorem 2 involving \( \{t_k\} \). We begin by proving that \( \kappa \) and \( \nu \) are positive.

**Proposition 3** The constants \( \kappa \) and \( \nu \) defined in Eqs. 2 and 5, respectively, are positive and \( t - f(t)/f'(t) < 0 \), for all \( t \in (0, \nu) \).

**Proof** Since \( \Omega \) is open and \( x_s \in \Omega \), we can immediately conclude that \( \kappa > 0 \). As \( f' \) is continuous in 0 with \( f'(0) = -1 \), there exists \( \delta > 0 \) such that \( f'(t) < 0 \) for all \( t \in (0, \delta) \). Hence, \( \nu > 0 \).

It remains to show that \( t - f(t)/f'(t) < 0 \), for all \( t \in (0, \nu) \). Since \( f' \) is strictly increasing, \( f \) is strictly convex. So, \( 0 = f'(0) < f(t) - tf'(t) \), for \( t \in (0, R) \). If \( t \in (0, \nu) \) then \( f'(t) < 0 \), which, combined with the last inequality, yields the desired inequality. \( \square \)

According to \( h2 \) and (5), we have \( f'(t) < 0 \) for all \( t \in [0, \nu) \). Therefore, the Newton iteration map for \( f \) is well defined in \( [0, \nu) \). Let us denote it by \( n_f \):

\[
n_f: [0, \nu) \rightarrow (-\infty, 0]
\]

\[
t \mapsto t - f(t)/f'(t).
\]

(10)

**Proposition 4** Given \( n_f \) in Eq. 10, then \( \lim_{t \rightarrow 0} |n_f(t)|/t = 0 \). As a consequence, the constant \( \rho \) defined in Eq. 6 is positive and

\[
0 < \omega_1(1 + \vartheta)|n_f(t)| + (\omega_1\vartheta + \omega_2)t < t, \quad t \in (0, \rho),
\]

(11)

where \( \vartheta, \omega_1, \omega_2 \) are defined in Eq. 4.
Proof Using the definition (10), Proposition 3, $f(0) = 0$ and some algebraic manipulations give

$$\frac{|n_f(t)|}{t} = \frac{|f(t)/f'(t) - t|}{t} = \frac{1}{f'(t)} \frac{f(t) - f(0)}{t - 0} - 1, \quad t \in (0, \nu). \quad (12)$$

As $f'(0) = -1 \neq 0$, the first statement follows by taking limit in Eq. 12, as $t$ goes to 0.

Now, from Eq. 4, we have $[1 - (\omega_1 \vartheta + \omega_2)]/[\omega_1 (1 + \vartheta)] > 0$. Hence, using the first statement of the proposition and (12), we conclude that there exists a $\delta > 0$ such that

$$0 < |f(t)/(tf'(t)) - 1| < [1 - (\omega_1 \vartheta + \omega_2)]/[\omega_1 (1 + \vartheta)], \quad t \in (0, \delta),$$

or, equivalently,

$$0 < \omega_1 (1 + \vartheta)|f(t)/f'(t) - t|/t + \omega_1 \vartheta + \omega_2 < 1, \quad t \in (0, \delta).$$

Therefore, $\rho$ is positive.

To prove Eq. 11, it is enough to use the definition of $\rho$ in Eqs. 6 and 12.

Using Eq. 10, it is easy to see that the sequence $\{t_k\}$ given in Eq. 7 is equivalently defined as

$$t_0 = \|x_0 - x_*\|, \quad t_{k+1} = \omega_1 (1 + \vartheta)|n_f(t_k)| + (\omega_1 \vartheta + \omega_2)t_k, \quad k = 0, 1, \ldots. \quad (13)$$

Corollary 5 Let $\vartheta$, $\omega_1$, $\omega_2$ and $\rho$ be the constants given in Eqs. 4 and 6. Then the sequence $\{t_k\}$ defined in Eq. 7 is well defined, strictly decreasing and contained in $(0, \rho)$. Moreover, $\{t_k\}$ converges to 0 with linear rate, i.e., $\lim_{k \to \infty} t_{k+1}/t_k = \omega_1 \vartheta + \omega_2$.

Proof Since $0 < t_0 = \|x_0 - x_*\| < r \leq \rho$ (see theorem 2), using Proposition 4 and Eq. 13 it is simple to conclude that $\{t_k\}$ is well defined, strictly decreasing and contained in $(0, \rho)$. So, the first statement of the corollary holds.

Due to $\{t_k\} \subset (0, \rho)$ is strictly decreasing, it converges. So, $\lim_{k \to \infty} t_k = t_*$ with $0 \leq t_* < \rho$, which, together with Eq. 13, implies $0 \leq t_* = \omega_1 (1 + \vartheta)|n_f(t_*)| + (\omega_1 \vartheta + \omega_2)t_*$. But, if $t_* \neq 0$, Proposition 4 implies $\omega_1 (1 + \vartheta)|n_f(t_*)| + (\omega_1 \vartheta + \omega_2)t_* < t_*$, hence $t_* = 0$. Therefore, $t_k \to 0$.

Now, using $\lim_{k \to \infty} t_k = 0$, the definition of $\{t_k\}$ in Eq. 13 and the first statement in Proposition 4, we obtain that $\lim_{k \to \infty} t_{k+1}/t_k = \lim_{k \to \infty} \omega_1 (1 + \vartheta)|n_f(t_k)|/t_k + \omega_1 \vartheta + \omega_2 = \omega_1 \vartheta + \omega_2$. Hence, the linear rate is proved.

Proposition 6 If $h3$ holds, the function $(0, \nu) \ni t \mapsto |n_f(t)|/t^{p+1}$ is strictly increasing where $v$ is defined in Eq. 5.

Proof As $t - f(t)/f'(t) < 0$ for all $t \in (0, \nu)$ (see Proposition 3), the result is an immediate consequence of $h3$ and the definition in Eq. 10.
2.1.2 Relationship of the majorant function with the nonlinear function

In this part we present the main relationships between the majorant function, $f$, and the nonlinear function, $F$.

**Lemma 7** Let $\kappa$, $\beta$, $\nu$ and $r$ be the constants given in Eqs. 2, 5 and 6. If $\|x - x_*\| < \min\{\kappa, \nu\}$, then $F'(x)^*F'(x)$ is invertible and

$$\left\| F'(x)^\dagger \right\| \leq \beta / f'(\|x - x_*\|).$$

In particular, $F'(x)^*F'(x)$ is invertible in $B(x_*, r)$.

**Proof** As $\|x - x_*\| < \min\{\nu, \kappa\}$, we have $f'(\|x - x_*\|) < 0$. Hence, using the definition of $\beta$ in Eq. 2, the inequality Eq. 3 and h1, we have

$$\left\| F'(x_*)^\dagger \left\| F'(x) - F'(x_*) \right\| = \beta \| F'(x) - F'(x_*) \| \leq f'(\|x - x_*\|) - f'(0) < 1. \quad (14)$$

Since $F'(x_*)$ is injective, Eq. 14 implies, in view of Lemma 1, that $F'(x)$ is injective. So, $F'(x)^*F'(x)$ is invertible and, by the definition of $r$ in Eq. 6, we obtain that $F'(x)^*F'(x)$ is invertible for all $x \in B(x_*, r)$. Moreover, from Lemma 1 we also have

$$\left\| F'(x)^\dagger \right\| \leq \beta / \left[ 1 - \beta \| F'(x) - F'(x_*) \| \right] \leq \beta / \left[ 1 - (f'(\|x - x_*\|) - f'(0)) \right],$$

where $f'(0) = -1$ and $f'<0$ in $[0, \nu)$ are used for obtaining the last equality.

Now, it is convenient to study the linearization error of $F$ at point in $\Omega$. For this we define

$$E_F(x, y) := F(y) - \left[ F(x) + F'(x)(y - x) \right], \quad y, x \in \Omega. \quad (15)$$

We will bound this error by the error in the linearization of the majorant function $f$

$$e_f(t, u) := f(u) - \left[ f(t) + f'(t)(u - t) \right], \quad t, u \in [0, R). \quad (16)$$

**Lemma 8** If $\|x - x_*\| < \kappa$, then

$$\beta \| E_F(x, x_*) \| \leq e_f(\|x - x_*\|, 0),$$

where $\kappa$, $\beta$, $E_F$ and $e_f$ are defined in Eqs. 2, 15 and 16.

**Proof** Since $B(x_*, \kappa)$ is convex, we obtain that $x_* + \tau(x - x_*) \in B(x_*, \kappa)$, for $0 \leq \tau \leq 1$. Thus, as $F$ is continuously differentiable in $\Omega$, the definition in Eq. 15 and some simple manipulations yield

$$\beta \| E_F(x, x_*) \| \leq \int_0^1 \beta \left\| [F'(x) - F'(x_*) + \tau(x - x_*)]\right\| \cdot \|x_* - x\| \ d\tau.$$

From the last inequality and assumption Eq. 3, we obtain

$$\beta \| E_F(x, x_*) \| \leq \int_0^1 \left[ f'(\|x - x_*\|) - f'(\tau \|x - x_*\|) \right] \cdot \|x - x_*\| \ d\tau.$$
Evaluating the above integral and using the definition of $e_f$ in Eq. 16, the statement follows.

Define the Gauss-Newton step for the functions $F$ by the following equality:

$$S_F(x) := -F'(x)^\dagger F(x).$$

(17)

Lemma 9 If $\|x - x_*\| < \min(\kappa, \nu)$, then

$$\|S_F(x)\| \leq |n_f(\|x - x_*\|)| + \|x - x_*\|,$$

where $\kappa$, $\nu$, $n_f$ and $S_F$ are defined in Eqs. 2, 5, 10 and 17, respectively.

Proof Using Eq. 17, $F(x_*) = 0$ and some algebraic manipulations, it follows from Eq. 15 that

$$\|S_F(x)\| = \|F'(x)^\dagger (F(x_*) - [F(x) + F'(x)(x_* - x)]) + (x_* - x)\|
\leq \|F'(x)^\dagger\|\|E_F(x, x_*)\| + \|x - x_*\|.$$

So, the last inequality, together with the Lemmas 7 and 8, gives

$$\|S_F(x)\| \leq e_f(\|x - x_*\|, 0)/|f'(\|x - x_*\|)| + \|x - x_*\|.$$

Since $f' < 0$ in $[0, \nu)$ and $\|x - x_*\| < \nu$, we obtain from the last inequality, Eq. 16 and $h1$, that

$$\|S_F(x)\| \leq f(\|x - x_*\|)/f'(\|x - x_*\|),$$

which, combined with Eq. 10 and Proposition 3, implies the desired inequality. □

Lemma 10 Let $X$, $Y$, $\Omega$, $F$, $x_*$, $R$, $\beta$ and $\kappa$ as defined in Theorem 2. Suppose that $F(x_*) = 0$, $F'(x_*)$ is injective and there exists a continuously differentiable function $f : [0, R) \to \mathbb{R}$ satisfying (3), $h1$ and $h2$. Let $\theta$, $\omega_1$, $\omega_2$, $\nu$, $\rho$ and $r$ as defined in Theorem 2. Assume that $x \in B(x_*, r) \setminus \{x_*\}$, i.e., $0 < \|x - x_*\| < r$. Define

$$x_+ = x + S, \quad B(x)S = -F'(x)^*F(x) + r,$$

(18)

where $B(x)$ is an invertible approximation of $F'(x)^*F'(x)$ satisfying the following conditions

$$\|B(x)^{-1}F'(x)^*F'(x)\| \leq \omega_1, \quad \|B(x)^{-1}F'(x)^*F'(x) - I\| \leq \omega_2,$$

(19)

and the residual, $r$, the forcing term, $\theta$, and the preconditioning invertible matrix, $P$, are such that

$$\theta \text{cond}(PF'(x)^*F'(x)) \leq \theta, \quad \|Pr\| \leq \theta \|PF'(x)^*F(x)\|.$$

(20)

Then $x_+$ is well defined and it holds

$$\|x_+ - x_*\| \leq \omega_1(1 + \theta)|n_f(\|x - x_*\|)| + (\omega_1 \theta + \omega_2)\|x - x_*\|.$$

(21)

As a consequence,

$$\|x_+ - x_*\| < \|x - x_*\|.$$

Proof First note that, as $\|x - x_*\| < r$, it follows from Lemma 7 that $F'(x)^*F'(x)$ is invertible. Now, let $B(x)$ be an invertible approximation of it satisfying (19). Thus,
Proposition 11

Let $F(x_*) = 0$, some simple algebraic manipulations and Eq. 18 yield

$$x_+ - x_* = x - x_* - B(x)^{-1} F'(x)^* (F(x) - F(x_*) + B(x)^{-1} r.$$  

Using $F'(x)^* F'(x) F'(x)^\dagger = F'(x)^*$ and some algebraic manipulations, the above equation gives

$$x_+ - x_* = B(x)^{-1} F'(x)^* F'(x) F'(x)^\dagger (F(x_*) - [F(x) + F'(x)(x_* - x)]) + B(x)^{-1} r + B(x)^{-1} (F'(x)^* F'(x) - B(x))(x - x_*).$$

The last equation, together with Eqs. 15 and 19, implies that

$$\|x_+ - x_*\| \leq \omega_1 \|F'(x)^\dagger\| E_F(x, x_*) + \|B(x)^{-1} r\| + \omega_2 \|x - x_*\|.$$ 

On the other hand, using Eqs. 17, 19 and 20 we obtain, by simple manipulations, that

$$\|B(x)^{-1} r\| \leq \theta \|B(x)^{-1} P^{-1}\| P r$$

$$\leq \theta \|B(x)^{-1} F'(x)^* F'(x)\| (P F'(x)^* F'(x))^{-1} \|P F'(x)^* F'(x)\| \|F'(x)^\dagger F(x)\|$$

$$\leq \omega_1 \theta \|S_F(x)\|.$$ 

Hence, it follows from the two last equations that

$$\|x_+ - x_*\| \leq \omega_1 \|F'(x)^\dagger\| E_F(x, x_*) + \omega_1 \theta \|S_F(x)\| + \omega_2 \|x - x_*\|.$$ 

Combining the last equation with the Lemmas 7, 8 and 9, we obtain that

$$\|x_+ - x_*\| \leq \omega_1 \varepsilon_f(\|x - x_*\|, 0) / \|f'(\|x - x_*\|) + \omega_1 \theta |n_f(\|x - x_*\|)| + (\omega_1 \vartheta + \omega_2) \|x - x_*\|.$$ 

Now, taking into account that $f(0) = 0$, the definitions of $e_f$ and $n_f$ imply that

$$e_f(\|x - x_*\|, 0) / \|f'(\|x - x_*\|)\| = |n_f(\|x - x_*\|)|.$$ 

So, the inequality in Eq. 21 follows by combining the above two inequalities.

Take $x \in B(x_*, r)$. Since $0 < \|x - x_*\| < r \leq \rho$, the inequalities in Eqs. 11 and 21 imply that

$$\|x_+ - x_*\| \leq \omega_1 (1 + \vartheta) |n_f(\|x - x_*\|)| + (\omega_1 \vartheta + \omega_2) \|x - x_*\| < \|x - x_*\|,$$

which proves the last statement of the lemma.

\[ \square \]

2.2 Inexact Gauss-Newton like sequence

In this section, we will prove the statements in Theorem 2 involving the inexact Gauss-Newton like sequence $\{x_k\}$.

**Proposition 11** Let $\vartheta, \omega_1, \omega_2, r, \{x_k\}$ and $\{t_k\}$ as defined in Theorem 2. Then $\{x_k\}$ is well defined, contained in $B(x_*, r)$ and it converges to $x_*$ with linear rate, i.e.,

$$\limsup_{k \to \infty} \left( \|x_{k+1} - x_*\| / \|x_k - x_*\| \right) \leq \omega_1 \vartheta + \omega_2. \quad (22)$$

If $h3$ holds, the sequences $\{x_k\}$ and $\{t_k\}$ satisfy

$$\|x_{k+1} - x_*\| \leq (1 + \vartheta) \omega_1 \left( \frac{f(t_0) - t_0 f'(t_0)}{t_0^{p+1} f'(t_0)} \right) \|x_k - x_*\|^{p+1} + (\omega_1 \vartheta + \omega_2) \|x_k - x_*\|, \quad (23)$$
for all \( k = 0, 1, \ldots, \) and

\[
\|x_k - x_*\| \leq t_k, \quad k = 0, 1, \ldots
\]  

(24)

**Proof** Since \( x_0 \in B(x_*, r)/\{x_*\} \), i.e., \( 0 < \|x_0 - x_*\| < r \), a combination of Lemma 7, the last inequality in Lemma 10 and an induction argument, we conclude that \( \{x_k\} \) is well defined and it remains in \( B(x_*, r) \).

We will now prove that \( \{x_k\} \) converges to \( x_* \). Since \( \|x_k - x_*\| < r \leq \rho \), for \( k = 0, 1, \ldots \), we obtain from Lemma 10 with \( x_* = x_{k+1}, x = x_k, r = r_k \), \( B(x) = B(x_k), P = P_k \) and \( \theta = \theta_k \), and Proposition 4, that

\[
0 \leq \|x_{k+1} - x_*\| \leq \omega_1(1 + \theta)|n_f(\|x_k - x_*\|)| + (\omega_1 \vartheta + \omega_2)\|x_k - x_*\| < \|x_k - x_*\|, \quad k = 0, 1, \ldots
\]

(25)

So, \( \{\|x_k - x_*\|\} \) is a bounded and strictly decreasing sequence. Therefore \( \{\|x_k - x_*\|\} \) converges. Let \( \ell_* = \lim_{k \to \infty} \|x_k - x_*\| \). Since \( \{\|x_k - x_*\|\} \) remains in \( (0, \rho) \) and is strictly decreasing, we have \( 0 \leq \ell_* < \rho \). Thus, taking the limit in Eq. 25 with \( r \) converging to 0 and using the continuity of \( n_f \) in \( [0, \rho] \), we obtain that \( 0 \leq \ell_* = \omega_1(1 + \theta)|n_f(\ell_*)| + (\omega_1 \vartheta + \omega_2)\ell_* \). But, if \( \ell_* \neq 0 \), Proposition 4 implies \( \omega_1(1 + \theta)|n_f(\ell_*)| + (\omega_1 \vartheta + \omega_2)\ell_* < \ell_* \), hence \( \ell_* = 0 \). Therefore, the convergence \( x_k \to x_* \) is proved.

In order to establish inequality Eq. 22, note that inequality Eq. 25 implies

\[
\left[\|x_{k+1} - x_*\|/\|x_k - x_*\|\right] \leq \omega_1(1 + \vartheta)[|n_f(\|x_k - x_*\|)|/\|x_k - x_*\|]+\omega_1 \vartheta + \omega_2, \quad k = 0, 1, \ldots
\]

Hence, as \( \{\|x_{k+1} - x_*\|/\|x_k - x_*\|\} \) is bounded and \( \lim_{k \to \infty} \|x_k - x_*\| = 0 \), the desired inequality follows from the first statement in Proposition 4.

Now, inequality Eq. 25 also implies

\[
\|x_{k+1} - x_*\| \leq \omega_1(1 + \vartheta)|n_f(\|x_k - x_*\|)|/\|x_k - x_*\|^{p+1} + (\omega_1 \vartheta + \omega_2)\|x_k - x_*\|, \quad k = 0, 1, \ldots
\]

(26)

Thus, using that \( \{\|x_k - x_*\|\} \) is strictly decreasing, Proposition 6 and the definition of \( n_f \) in Eq. 10, the last inequality gives Eq. 23.

To end the proof, we will show inequality Eq. 24 by induction. As \( m_0 = \|x_0 - x_*\| \), it is immediate for \( k = 0 \). Assume that \( \|x_k - x_*\| \leq t_k \). Hence, Eq. 26, Proposition 6 and the definition of \( t_{k+1} \) in Eq. 13 imply

\[
\|x_{k+1} - x_*\| \leq \omega_1(1 + \vartheta)|n_f(t_k)| + (\omega_1 \vartheta + \omega_2)t_k = t_{k+1},
\]

Therefore, inequality Eq. 24 holds.

Proof of Theorem 2 follows from Corollary 5 and Proposition 11.

### 3 Special cases

In this section, we present some special cases of Theorem 2.
3.1 Convergence results under Hölder-like and Smale conditions

In this section, we present a local convergence theorem for the inexact Gauss-Newton like methods under a Hölder-like condition, see \cite{6, 12, 13}. We also provide a Smale’s theorem on the inexact Gauss-Newton like methods for analytical functions, cf. \cite{19}.

Theorem 12 Let $\mathbb{X}$ and $\mathbb{Y}$ be Hilbert spaces, $\Omega \subseteq \mathbb{X}$ be an open set and $F : \Omega \to \mathbb{Y}$ be a continuously differentiable function such that $F'$ has a closed image in $\Omega$. Let $x_0 \in \Omega$, $R > 0$, $\beta := \|F'(x_0)\|$ and $\kappa := \sup \{t \in [0, R) : B(x_0, t) \subset \Omega\}$. Suppose that $F(x_0) = 0$, $F'(x_0)$ is injective and there exists a constant $K > 0$ and $0 < p \leq 1$ such that
\[
\beta \|F'(x) - F'(x_0 + \tau(x - x_0))\| \leq K(1 - \tau^p)\|x - x_0\|^p, \quad x \in B(x_0, \kappa) \quad \tau \in [0, 1].
\]
Take $0 \leq \vartheta < 1$, $0 \leq \omega_2 < \omega_1$ such that $\omega_1 \vartheta + \omega_2 < 1$. Let
\[
r = \min \left\{ \kappa, \left[ \frac{(1 - \omega_1 \vartheta - \omega_2)(p + 1)}{K(1 - \omega_1 \vartheta - \omega_2 + p(1 + \omega_1 - \omega_2))} \right]^{1/p} \right\}.
\]
Consider the inexact Gauss-Newton like sequence $\{x_k\}$ as defined in Theorem 2 with initial point $x_0 \in B(x_0, r) \setminus \{x_0\}$. Define the scalar sequence $\{t_k\}$, with initial point $t_0 = \|x_0 - x_0\|$, by
\[
t_{k+1} = \frac{(1 + \vartheta)\omega_1 p Kt_k^{p+1}}{(p + 1)[1 - K t_k^p]} + (\omega_1 \vartheta + \omega_2)t_k, \quad k = 0, 1, \ldots.
\]
Then, the sequences $\{x_k\}$ and $\{t_k\}$ are well defined; $\{t_k\}$ is strictly decreasing, contained in $(0, r)$ and it converges to 0. Furthermore, $\{x_k\}$ is contained in $B(x_0, r)$, it converges to $x_0$,
\[
\|x_{k+1} - x_0\| \leq \frac{(1 + \vartheta)\omega_1 p K}{(p + 1)[1 - K \|x_0 - x_0\|^p]} \|x_k - x_0\|^{p+1} + (\omega_1 \vartheta + \omega_2) \|x_k - x_0\|, \quad k = 0, 1, \ldots,
\]
and
\[
\|x_k - x_0\| \leq t_k, \quad k = 0, 1, \ldots.
\]

Proof It is immediate to prove that $F, x_0$ and $f : [0, \kappa) \to \mathbb{R}$, defined by $f(t) = Kt^{p+1}/(p + 1) - t$, satisfy the inequality Eq. 3 and the conditions h1, h2 and h3 in Theorem 2. In this case, it is easily seen that $v$ and $\rho$ as defined in Eqs. 5 and 6, respectively, satisfy
\[
\rho = \left[ \frac{(1 - \omega_1 \vartheta - \omega_2)(p + 1)}{K(1 - \omega_1 \vartheta - \omega_2 + p(1 + \omega_1 - \omega_2))} \right]^{1/p} \leq v = [1/K]^{1/p},
\]
and, as a consequence, $r = \min(\kappa, \rho)$. Therefore, the statements of the theorem follow from Theorem 2.

Remark 4 For $p = 1$ in the previous theorem, we obtain the convergence of the inexact Gauss-Newton like methods under a Lipschitz condition, as obtained in Theorem 16 of \cite{9} with $c = 0$.

Below, we present a theorem corresponding to Theorem 2 under Smale’s condition.
**Theorem 13** Let $\mathcal{X}$ and $\mathcal{Y}$ be Hilbert spaces, $\Omega \subseteq \mathcal{X}$ be an open set and $F : \Omega \rightarrow \mathcal{Y}$ an analytic function such that $F'$ has a closed image in $\Omega$. Let $x_\ast \in \Omega$, $R > 0$, $\beta := \|F'(x_\ast)^\dagger\|$ and $\kappa := \sup \{ t \in [0, R) : B(x_\ast, t) \subseteq \Omega \}$. Suppose that $F(x_\ast) = 0$, $F'(x_\ast)$ is injective and

$$
\gamma := \sup_{n>1} \beta \frac{\|F^{(n)}(x_\ast)\|^{1/(n-1)}}{n!} < +\infty.
$$

Take $0 \leq \vartheta < 1$, $0 \leq \omega_2 < \omega_1$ such that $\omega_1 \vartheta + \omega_2 < 1$. Let $a := (1 + \vartheta)\omega_1$, $b := (1 - \omega_1 \vartheta - \omega_2)$ and

$$
r := \min \left\{ \kappa, a + 4b - \sqrt{(a + 4b)^2 - 8b^2} \right\}.
$$

Consider the inexact Gauss-Newton like sequence $\{x_k\}$ as defined in Theorem 2 with initial point $x_0 \in B(x_\ast, r) \setminus \{x_\ast\}$. Define the scalar sequence $\{t_k\}$, with initial point $t_0 = \|x_0 - x_\ast\|$, by

$$
t_{k+1} = \frac{(1 + \vartheta) \omega_1 \gamma t_k^2}{2(1 - \gamma t_k)^2 - 1} + (\omega_1 \vartheta + \omega_2) t_k, \quad k = 0, 1, \ldots.
$$

Then, the sequences $\{x_k\}$ and $\{t_k\}$ are well defined; $\{t_k\}$ is strictly decreasing, contained in $(0, r)$ and it converges to 0. Furthermore, $\{x_k\}$ is contained in $B(x_\ast, r)$, it converges to $x_\ast$,

$$
\|x_{k+1} - x_\ast\| \leq \frac{\gamma}{2(1 - \gamma \|x_0 - x_\ast\|)^2 - 1} \|x_k - x_\ast\|^2 + (\omega_1 \vartheta + \omega_2) \|x_k - x_\ast\|, \quad k = 0, 1, \ldots,
$$

and

$$
\|x_k - x_\ast\| \leq t_k, \quad k = 0, 1, \ldots.
$$

**Proof** In this case, the real function, $f : [0, 1/\gamma) \rightarrow \mathbb{R}$, defined by $f(t) = t/(1 - \gamma t) - 2t$, is a majorant function for the function $F$ on $B(x_\ast, 1/\gamma)$. Hence, as $f$ has a convex derivative, the proof follows the same pattern as outlined in Theorem 18 of [9].

### 3.2 Convergence result under a generalized Lipschitz condition

In this section, we present a local convergence theorem for the inexact Gauss-Newton like methods under a generalized Lipschitz condition according to X.Wang (see [13, 22]). It is worth to point out that the result in this section does not assume that the function defining the generalized Lipschitz condition is nondecreasing. Thus, Theorem 3.3 in [4] is generalized for the zero-residual case.

**Theorem 14** Let $\mathcal{X}$ and $\mathcal{Y}$ be Hilbert spaces, $\Omega \subseteq \mathcal{X}$ be an open set and $F : \Omega \rightarrow \mathcal{Y}$ be a continuously differentiable function such that $F'$ has a closed image in $\Omega$. Let $x_\ast \in \Omega$, $R > 0$, $\beta := \|F'(x_\ast)^\dagger\|$ and $\kappa := \sup \{ t \in [0, R) : B(x_\ast, t) \subseteq \Omega \}$. Suppose
that \( F(x_*) = 0 \), \( F'(x_*) \) is injective and there exists a positive integrable function \( L : [0, R) \to \mathbb{R} \) such that
\[
\beta \left\| F'(x) - F'(x_* + \tau(x - x_*)) \right\| \leq \int_{\tau \| x - x_* \|}^{\| x - x_* \|} L(u) du, \tag{27}
\]
for all \( \tau \in [0, 1] \), \( x \in B(x_*, \kappa) \). Let \( \tilde{\nu} := \sup \{ t \in [0, R) : \int_{0}^{t} L(u) du - 1 < 0 \} \),
\[
\tilde{\rho} := \sup \left\{ t \in (0, \delta) : \frac{(1 + \vartheta) \omega_1 \int_{0}^{t} L(u) du}{\varrho (1 - \int_{0}^{t} L(u) du)} + \omega_1 \vartheta + \omega_2 < 1, \ t \in (0, \delta) \right\}, \quad \tilde{\nu} = \min \{ \kappa, \tilde{\rho} \}.
\]
Consider the inexact Gauss-Newton like sequence \( \{x_k\} \) as defined in Theorem 2 with initial point \( x_0 \in B(x_*, r) \setminus \{x_*\} \). Define the scalar sequence \( \{t_k\} \), with initial point \( t_0 = \|x_0 - x_*\| \), by
\[
t_{k+1} = \frac{(1 + \vartheta) \omega_1 \int_{0}^{t_k} L(u) du}{1 - \int_{0}^{t_k} L(u) du} + (\omega_1 \vartheta + \omega_2) t_k, \quad k = 0, 1, \ldots.
\]
Then, the sequences \( \{x_k\} \) and \( \{t_k\} \) are well defined; \( \{t_k\} \) is strictly decreasing, contained in \( (0, r) \) and it converges to 0. Furthermore, \( \{x_k\} \) is contained in \( B(x_*, r) \), it converges to \( x_* \) and there hold:
\[
\limsup_{k \to \infty} \frac{\|x_{k+1} - x_*\|}{\|x_k - x_*\|} \leq \omega_1 \vartheta + \omega_2, \quad \lim_{k \to \infty} \frac{t_{k+1}}{t_k} = \omega_1 \vartheta + \omega_2.
\]
If, additionally, given \( 0 \leq \rho \leq 1 \)
\[ h ) \quad \text{the function } (0, \nu) \ni t \mapsto t^{1-\rho} L(t) \text{ is nondecreasing}, \]
then the sequences \( \{x_k\} \) and \( \{t_k\} \) satisfy
\[
\|x_{k+1} - x_*\| \leq (1 + \vartheta) \omega_1 \left( \frac{\int_{0}^{t_k} L(u) du}{\varrho (1 - \int_{0}^{t_k} L(u) du)} \right) \|x_k - x_*\|^{\rho+1} + (\omega_1 \vartheta + \omega_2) \|x_k - x_*\|,
\]
for all \( k = 0, 1, \ldots \), and
\[
\|x_k - x_*\| \leq t_k, \quad k = 0, 1, \ldots.
\]
Proof Let \( \tilde{f} : [0, \kappa) \to \mathbb{R} \) be a differentiable function defined by
\[
\tilde{f}(t) = \int_{0}^{t} L(u)(t - u) du - t.
\]
Note that the derivative of the function \( f \) is given by
\[
\tilde{f}'(t) = \int_{0}^{t} L(u) du - 1.
\]
Since \( L \) is integrable, \( \tilde{f}' \) is continuous (in fact \( \tilde{f}' \) is absolutely continuous). Hence, it is easy to see that Eq. 27 becomes Eq. 3 with \( f' = \tilde{f}' \). Moreover, since \( L \) is positive,
the function \( f = \tilde{f} \) satisfies the conditions \( \mathbf{h1} \) and \( \mathbf{h2} \) in Theorem 2. Direct algebraic manipulation yields

\[
\frac{1}{t^{p+1}} \left[ \frac{\tilde{f}(t)}{\tilde{f}'(t)} - t \right] = \left[ \frac{1}{t^{p+1}} \int_0^t L(u) u \, du \right] \frac{1}{|\tilde{f}'(t)|}.
\]

If assumption \( \mathbf{h} \) holds, then Lemma 2.2 of [23] implies that the first term on the right hand side of the above equation is nondecreasing in \((0, \nu)\). Now, since \( 1/|\tilde{f}'| \) is strictly increasing in \((0, \nu)\), the above equation implies that \( \mathbf{h3} \) in Theorem 2, with \( f = \tilde{f}, \nu = \tilde{\nu}, \rho = \tilde{\rho} \) and \( r = r \). Therefore, the result follows from Theorem 2 with \( f = \tilde{f}, \nu = \tilde{\nu}, \rho = \tilde{\rho} \) and \( r = r \).

**Remark 5** If \( f' \) in Theorem 2 is convex, the inequalities Eqs. 3 and 27 are equivalent. Otherwise, the equivalence does not hold. For more details see the discussion on pp. 1522 of [6] or remark 4 in [12].

### 4 Final remarks

We presented, under a milder majorant condition, a local convergence of inexact Gauss-Newton like methods for solving injective-overdetermined systems of equations. Our main theorem was proved without the additional assumption that the majorant function has convex derivative. Among other things, the lack of this assumption allowed us to obtain two new important special cases, namely, the convergence was ensured under Hölder-like and generalized Lipschitz conditions.

It would be interesting to study results of semi-local convergence for the inexact Gauss-Newton like methods under a similar majorant condition. This analysis will be carried out in the future.

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