SUBNORMALITY IN $PD_3$-GROUPS AND $L^2$-BETTI NUMBERS

J.A. HILLMAN

Abstract. We reconsider work of Elkalla on subnormal subgroups of 3-manifold groups, giving essentially algebraic arguments that extend to the case of $PD_3$-groups and group pairs. However, our approach relies on an $L^2$-Betti number hypothesis which has not yet been confirmed in general.

In 1983 Elkalla considered infinite subnormal subgroups $N$ of a 3-manifold group $\pi$ which satisfy the following weak finiteness condition: “$N \leq U$, where $U$ is finitely generated and has infinite index in $\pi$”. In particular, he showed that if the 3-manifold is compact and $P^2$-irreducible and $\pi$ is “$U$-residually finite” then either $N \cong \mathbb{Z}$ and is normal in $\pi$ or $\pi$ is virtually a semidirect product $K \rtimes \mathbb{Z}$ with $K$ a $PD_2$-group which is commensurable with $U$ [7]. He gave related results for such subgroups $N \leq U < \pi$, when $\pi$ is the fundamental group of an open 3-manifold. In the light of the recently proven Virtual Fibring Theorem, “most” 3-manifold groups are virtually such semidirect products, and so we might expect a simpler picture. However it is not clear that we may assume $K$ commensurable with $U$.

When $N$ is itself finitely generated (and hence finitely presentable, by the coherence of 3-manifold groups), the situation is much simpler. If a $PD_3$-group $G$ has a non-trivial $FP_2$ subnormal subgroup $N$ then either $N \cong \mathbb{Z}$ and is normal in $G$, or $G$ is virtually a semidirect product $K \rtimes \mathbb{Z}$ and $N$ is commensurable with $K$, or $G$ is polycyclic [2]. This was subsequently extended to ascendant $FP_2$ subgroups of $PD_3$-groups and open $PD_3$-groups, in [12]; in the latter case, if $G$ is not finitely generated then $N \cong \mathbb{Z}$ and is normal in $G$.

Although Elkalla does not explicitly limit his discussion to aspherical 3-manifolds, it is easily seen that the 3-manifolds satisfying his theorems are in fact aspherical, and have $\pi_1$-injective boundaries. (See Lemma [11]) Thus it is natural to seek an essentially homological argument for Elkalla’s result which applies to $PD_3$-groups and group pairs. We shall use the Algebraic Core Theorem of [14], instead of the Geometric Core Theorem of [21], to do this. However, our arguments at present rest on a conjectural result on the vanishing of the first $L^2$-Betti number of an intermediate group, and we also replace the residual finiteness condition by the requirement that the normal closure of the intermediate subgroup have infinite index.

The first section presents some notation and terminology, in particular “almost coherent”, a homological analogue of “coherent”, and “open $PD_3$-group”, an analogue of open aspherical 3-manifold. In §2 and §3 we consider briefly the results of [7] that are either purely algebraic or can be derived algebraically. We have replaced “finitely generated” by “$FP_2$”, since these conditions are equivalent for (subgroups of) 3-manifold groups. The main results of [7] are discussed in §4. In §5 we consider briefly two special cases, and in §6 we discuss other recent work.
1. SOME TERMINOLOGY

If \( G \) is a group then \( \sqrt{G} \) is the Hirsch-Plotkin radical, the maximal locally-nilpotent normal subgroup. An easy induction up a subnormal series shows that \( \sqrt{G} \) contains all nilpotent subnormal subgroups of \( G \). If \( U \leq G \) then \( \langle \langle U \rangle \rangle_G \) is the normal closure of \( U \) in \( G \). A group is almost coherent if its finitely generated subgroups are \( FP_2 \). All 3-manifold groups are (almost) coherent \([22]\).

If \( (G, \Omega) \) is a \( PD_n \)-pair of groups, we shall say that \( G \) is the ambient group, with boundary components the conjugacy classes of point stabilizers \( G_\eta \), for \( \eta \) in the \( G \)-set \( \Omega \). (We use the formulation of [5].) The pair \( (G, \Omega) \) is proper if \( \Omega \) is nonempty, and is of \( I \)-bundle type if \( |\Omega| = 2 \).

In [12] we defined the notion of open \( PD_n \)-group in order to have a convenient term for subgroups of \( PD_3 \)-groups and to take advantage of the Algebraic Core Theorem of [14]. An open \( PD_n \)-group is a countable group \( \pi \) such that \( c.d.\pi \leq n - 1 \) and every subgroup \( G \) which is an \((n - 1)\)-dimensional duality group is the ambient group of a \( PD_n \)-pair of groups. In particular, the ambient group of a proper \( PD_n \)-pair of groups is an open \( PD_n \)-group.

This notion is of most interest when \( n = 3 \). The relevant subgroups are then \( FP_2 \) and have one end. If \( G \) is an open \( PD_3 \)-group then \( c.d. G = 2 \) by definition, and so the argument of [22] applies to show that if \( G \) is a non-trivial product then \( G \cong F \times \mathbb{Z} \) with \( F \) free. (Note that finitely generated, one-ended subgroups of \( \mathbb{Q} \oplus \mathbb{Z} \) are copies of \( \mathbb{Z}^2 \), but \( c.d. \mathbb{Q} \oplus \mathbb{Z} = 3 \). This group cannot be a subgroup of a \( PD_3 \)-group.)

In topological terms, the corresponding covering spaces \( X_G = K(G, 1) \) of \( K(\pi, 1) \) must satisfy Poincaré-Lefshetz duality relative to some “boundary” \( \partial X_G \). (Note, however, that we are not assuming that \( G \) is finitely presentable, and so \( (X_G, \partial X_G) \) may not be a \( PD_n \)-pair in the usual sense.)

2. AUXILIARY RESULTS AND HYPOTHESES

The main result of the first section of [7] is Theorem 1.5 of loc. cit., which is in fact a consequence of the following result of Burns.

**Theorem.** (Burns [3, Theorem 1.1]) Let \( G = A \ast_B \), where \( A \) and \( B \) are non-trivial. If \( N < H \) are subgroups of \( G \) with \( N \) subnormal in \( G \), \( N \neq 1 \) and \( H \) finitely generated then \( [G : H] \) is finite.

Burns’ argument extends to certain generalized free products with amalgamation \( A \ast_C B \), where \( N \not\leq C \), in particular to the case when \( A \) and \( B \) are free of rank \( > 1 \) and \( C \) is a maximal infinite cyclic subgroup in each of \( A \) and \( B \) [3, Theorem 2.3].

We may use Burns’ Theorem to show that our focus on \( PD_3 \)-pairs of groups is adequate to recover Elkalla’s result for 3-manifolds.

**Lemma 1.** Let \( M \) be a compact orientable 3-manifold which has no \( S^2 \) boundary components. Suppose that \( \pi = \pi_1(M) \) has non-trivial subgroups \( N \leq U \) such that \( N \) is subnormal in \( \pi \), \( U \) is finitely generated and \( [\pi : U] = \infty \). Then \( M \) is aspherical and the boundary components are \( \pi_1 \)-injective.

**Proof.** The group \( \pi \) is infinite and is indecomposable, by Burns’ Theorem and is not infinite cyclic since it has a non-trivial subgroup of infinite index, and \( N \) is also infinite, since \( M \) is orientable. The result is then a standard consequence of the Loop and Sphere Theorems. \( \square \)
Hence $M$ is determined by its peripheral system $(\pi, \Omega)$, where $\Omega = \pi_0(\partial\tilde{M})$, and this is a $PD_3$-pair of groups.

The next result extends the corresponding result for normal subgroups [21]. (Neither this lemma nor the version from [21] is in the reference [5].)

**Lemma.** [7, Lemma 2.4] If $G$ is a graph of groups and $N < G_e$ is a subgroup of a vertex group which is subnormal in $\pi G$ then $N$ is conjugate to a subgroup of some edge group $G_e$, where $e$ has $v$ as one vertex.

Ar present, we appear to need one or the other of several hypotheses on the ambient groups of the $PD_3$-pairs of groups that we shall consider.

**Hypothesis A.** If $(G, \Omega)$ is a $PD_3$-pair of groups and $G_\omega$ is the stabilizer of a point $\omega \in \Omega$ then $G_\omega \cap gG_\omega g^{-1}$ is finitely generated, for all $g \in G$.

If $(G, \Omega)$ is the peripheral system of an aspherical 3-manifold pair $(M, \partial M)$ then Hypothesis A holds. For if $F$ is a component of $\partial M$ such that $G_\omega = \pi_1(F)$ and $M_F$ is the covering space with fundamental group $F$ then the fundamental groups of the components of the preimage of $F$ in $M_F$ are finitely generated [13, proposition 1.3]. These groups are the intersections of $\pi_1(F)$ with its conjugates in $\pi_1(M)$.

The second hypothesis relates to a different aspect of group theory.

**Hypothesis B.** If a finitely generated group $G$ has infinite subgroups $N \leq U$ such that $N$ is subnormal in $G$, $U$ is finitely generated and $[G : U]$ is infinite, then $\beta^{(2)}_1(G) = 0$.

Hypothesis B holds for 3-manifold groups $G$ which satisfy the conclusions of Elkalla’s main theorems. It also holds if $N$ is normal or 2-step subnormal, i.e., if $N < N_1 < G$ with each subgroup normal in the next. This is evidence that the hypothesis might be appropriate. (See Theorems 7 and 8 below.) It holds if $N$ is finitely generated [17, Chapter 7] so we may assume also that $[U : N] = \infty$.

**Lemma 2.** Let $(G, \Omega)$ be a $PD_3$-pair of groups. If $\beta^{(2)}_1(G) = 0$ then $\chi(G) = 0$ and so $\chi(G_\eta) = 0$ for all $\eta \in \Omega$.

**Proof.** If $\Omega = \emptyset$ then $G$ is a $PD_3$-group, and so $\chi(G) = 0$, by Poincaré duality.

If $\Omega$ is non-empty then $c.d.G = 2$, and so the Strong Bass Conjecture holds [6]. Hence the $L^2$-Euler characteristic formula applies, and so $\chi(G) = \beta^{(2)}_2(G) \geq 0$. But $\chi(G) = \frac{1}{2}\sum \chi(G_\eta) \leq 0$, where the summation is over the orbits of $G$ in $\Omega$, by Poincaré-Lefshetz duality with coefficients $F_2$. Hence $\chi(G) = 0$, and so $\chi(G_\eta) = 0$ for all $\eta \in \Omega$.

Although we use only this consequence of Hypothesis B, it seems to us that $L^2$-methods might prove the best route to achieving this conclusion, and it is for this reason that we highlight Hypothesis B.

**Theorem.** (Gildenhuys-Strebel [8, Theorem 3.3]) Let $G = \cup_{n \in \mathbb{N}} G_n$ be a strictly increasing union of $FP_2$, one-ended groups. If $[G_{n+1} : G_n] < \infty$ for all $n$ then $G$ is not finitely generated and $c.d.G = 3$.

3. Finitely generated subnormal subgroups

Most of the results of §2 of [7] and many of those of §3 of [7] are either pure algebra or can be subsumed into the results of [12]. (We shall in fact refer to the book [11].) The exceptions are Lemma 2.5, which appears to need either Hypothesis
A or Hypothesis B, and Theorem 2.6 and Theorem 3.7. We consider these in §4, and state the $PD_3$-pair analogues of the other results here.

The first lemma corresponds to Lemma 2.1 of [7].

**Lemma 3.** Let $1 \neq N < G_1 \leq G_2$, where $N$ is a normal $FP_2$ subgroup of $G_1$, $G_1$ is not finitely generated and is normal in $G_2$, and $G_2$ is a $PD_3$-group or an open $PD_3$-group. Then $N \cong \mathbb{Z}$ and $N$ is normal in $G_2$.

**Proof.** Since $G_1$ is itself an open $PD_3$-group, $N \cong \mathbb{Z}$ [11, Theorem 9.11]. Hence $N \leq \sqrt{G_2}$. If $G_2$ is also an open $PD_3$-group then $N$ is normal in $G_2$, by the same theorem. If $G_2$ is a $PD_3$-group then either $\sqrt{G} \cong \mathbb{Z}$ or $G_2$ is polycyclic [11, Theorem 8.10]. Since $G_1$ is not finitely generated, $G_2$ is not polycyclic. Hence $N$ is normal in $G_2$, in either case. \[\square\]

The next result includes Proposition 2.2 of [7].

**Lemma.** [11, Theorem 9.11] Let $G$ be an open $PD_3$-group which is not finitely generated. If $N$ is a non-trivial subnormal subgroup of $G$ then $N \cong \mathbb{Z}$ and $N$ is normal in $G$. \[\square\]

The next result includes Corollary 2.3 and Theorem 3.1 of [7], and again can be deduced from Theorem 12 (and Corollary 13) of [12]. Note that finitely generated free groups (including $\mathbb{Z}$) are surface groups, in the terminology of [7].

**Theorem.** [11, Theorem 9.12] Let $(G, \Omega)$ be a $PD_3$-pair of groups and $N < G$ a non-trivial subnormal $FP_2$ subgroup of infinite index in $G$. Then $N$ is a surface group, and

1. if $N \cong \mathbb{Z}$ then either $N$ is normal in $G$ or $G$ is polycyclic;
2. if $N \not\cong \mathbb{Z}$ then $[G : N_G(N)]$ is finite and $G$ is virtually a semidirect product $F \rtimes \mathbb{Z}$, where $N$ is commensurable with $F$. If $\Omega = \emptyset$ then $F$ is a $PD_2$-group; otherwise $F$ is free of finite rank. \[\square\]

The formulation of Theorem 3.1 of [7] is in terms of 3-manifolds, and allows for the possibility of $RP^2 \times S^1$. This is easily to deal with.

Lemma 2.4 of [7] is stated in §2 above, while Lemma 3.2, Theorem 3.3 and Lemma 3.6 of [7] can be deduced from known results for $PD_3$-groups, and Lemma 3.5 is standard.

Lemma 3.4 does not involve finiteness assumptions, but is essentially algebraic. The proof in [7] uses the fact that 3-manifold groups have restricted torsion and is otherwise straightforward, and holds for $G$ any torsion-free group. It is claimed in the proof that $[N_r : S]$ finite, and this is used in the proof of [7, Theorem 3.7]. This claim is not clear to us, but we believe that it can be avoided in the application. In any case, we shall not make use of this Lemma 3.4 below.

**4. ELKALLA’S MAIN RESULTS**

In the main results of [7] the finiteness conditions on the subnormal subgroup $N$ are replaced by a hypothesis on an intermediate subgroup. The prototype for such results is the following:

**Theorem.** (Greenberg [9, Theorem 4]) Let $S$ be a $PD_2$-group with non-trivial subgroups $N \leq U$ such that $N$ is subnormal in $S$ and $U$ is finitely generated. If $S$ is hyperbolic (i.e., if $\chi(S) < 0$) then $[S : U] < \infty$. \[\square\]
This also follows from [3] Theorem 2.3], for if \( \chi(S) < 0 \) then \( S \cong A \ast \mathbb{Z} B \) with \( A \) and \( B \) free of rank \( > 1 \), while \( N \) cannot be cyclic. (It would also follow from Hypothesis B, since \( \chi(S) = -\beta_1^2(S) \), by the \( L^2 \) Euler characteristic formula and Poincaré duality.)

The statement of Lemma 2.5 of [7] is (after minor reformulation):

**Lemma.** (Elkalla [7, Lemma 2.5]) Let \( M \) be a compact 3-manifold, let \( F \) be an incompressible boundary component, and let \( N < \phi = \pi_1(F) \) be a non-trivial subnormal subgroup of \( \pi = \pi_1(M) \). Then either \( [\pi : \phi] < \infty \) or \( \chi(F) = 0 \) and \( N \cong \mathbb{Z} \).

We may assume that \( M \) is orientable, the components of \( \partial M \) are aspherical and \([\pi : \phi] = \infty\). It then follows from Lemma 11 that \( M \) is aspherical and has \( \pi_1 \)-injective boundary, and so the peripheral system of \((M, \partial M)\) is a \( PD_3 \)-pair of groups which satisfies Hypothesis A [13, proposition 1.3].

**Lemma 4.** Let \( (G, \Omega) \) be a \( PD_3 \)-pair of groups which is not of \( I \)-bundle type, and such that Hypothesis A holds. Suppose that \( G \) has a non-trivial subnormal subgroup \( N \) which is a subgroup of \( G_\omega \), for some \( \omega \in \Omega \). Then \( \chi(G_\omega) = 0 \), \( N \cong \mathbb{Z} \) and \( N \) is normal in \( G \).

**Proof.** Let \( R = \mathbb{F}_2[G_\omega \backslash G] \) be the right \( \mathbb{Z}[G] \)-module with \( \mathbb{Z} \)-basis the right cosets of \( G_\omega \) in \( G \), and let \( \mathcal{R} \cong \mathbb{F}_2[G/G_\omega] \) be the conjugate left module. The long exact sequence for the pair with coefficients \( R \) gives an exact sequence

\[
H_3(G, \Omega; R) \to H_2(\Omega; R) \to H_2(G; R) = H_2(G_\omega, \mathbb{F}_2).
\]

Since \( \Omega \) is non-empty, \( c.d. G = 2 \), and since \( (G, \Omega) \) is not of \( I \)-bundle type, \( [G : G_\omega] = \infty \) and \( \text{Comm}_G(G_\omega) = G_\omega \) [16, Lemma 2.2]. Hence \( H_3(G, \Omega; R) \cong H^0(G; \mathbb{R}) = 0 \), by Poincaré-Lefshetz duality, and so \( H_2(\Omega; R) \cong \mathbb{F}_2 \). This module is the sum \( \bigoplus_\eta H^2(G_\eta \cap G_\omega; \mathbb{F}_2) \), indexed by the orbits of \( G_\omega \) acting on \( \Omega \). The summand corresponding to \( \omega \) is clearly non-zero. Hence if \( \eta \neq \omega \) then \( H^2(G_\eta \cap G_\omega; \mathbb{F}_2) = 0 \), and so \([G_\omega : G_\eta \cap G_\omega] = \infty\).

Let \( \bar{N} = N_0 \cdots N_k = G \) be a subnormal series, and let \( r = \min\{i | N_i \not\subseteq G_\omega \} \).

If \( g \in N_r \backslash G_\omega \) then \( g \) normalizes \( N_{r-1} \), and so \( N \leq U = G_\omega \cap gG_\omega g^{-1} \). This subgroup \( U \) has infinite index in \( G_\omega \), since \( \text{Comm}_G(G_\omega) = G_\omega \). Since \( N \leq U \) and \( U \) is finitely generated, by Hypothesis A, \( \chi(G_\omega) = 0 \), by Greenberg’s Theorem. Hence \( G_\omega \cong \mathbb{Z}^2 \) or \( \pi_1(Kb) \), and so \( N \) is \( FP_2 \). Since \( (G, \Omega) \) is not of \( I \)-bundle type, \( G \) is not polycyclic. It then follows from [11] Theorem 9.12] that \( N \cong \mathbb{Z} \) and is normal in \( G \).

If some \([G : G_\omega]\) is finite then \( G \) is a \( PD_3 \)-group and \((G, \Omega)\) is of \( I \)-bundle type. In this case \( G \) has many non-trivial subnormal subgroups of infinite index, all free (and of infinite rank, if \( \chi(G) < 0 \)).

**Lemma 5.** Let \((G, \Omega)\) and \((G_1, \Omega_1)\) be orientable \( PD_3 \)-pairs of groups, with \( G \) a proper subgroup of \( G_1 \). Assume that \( \chi(G) = 0 \) and \((G, \Omega)\) is not of \( I \)-bundle type. If \( G_1 \) has no non-trivial abelian normal subgroup, then \( G_1 \) splits over a virtually abelian subgroup, and \( G \) stabilizes a vertex of the corresponding \( G_1 \)-tree.

**Proof.** Since \( G \) is not an \( I \)-bundle the boundary components of \( G \) are distinct. If each represents some boundary component of \( G_1 \) then the embedding \( j : G \to G_1 \) extends to an embedding \( Dj \) of the doubles \( DG \to DG_1 \). But the image of \( Dj \) has
infinite index, contradicting the fact that \( DG \) and \( DG_1 \) are \( PD_3 \)-groups. Hence \( G \) has a boundary component \( T \) which is not conjugate into a boundary component of \( G_1 \). Since \( \chi(G) \) is half the sum of the Euler characteristics of its boundary components, \( \chi(T) = 0 \). Since \( G_1 \) has the maximal condition on centralizers \( [4] \) (see also \([12]\)) and has no non-trivial abelian normal subgroup, it splits over a subgroup commensurable with a conjugate of \( T \) \([15\) Theorem A1]. Since \( G \) does not split over \( T \) it must fix a vertex of the \( G_1 \)-tree corresponding to the splitting. \( \square \)

The next lemma is a variation on Lemma \([4]\)

**Lemma 6.** Let \((W, \Omega)\) be a proper \( PD_3 \)-pair of groups such that \( W \) has non-trivial subgroups \( N \leq V \), where \( N \) is subnormal in \( W \), \( V \) is \( FP_2 \), has one end and \( \chi(V) = 0 \), and \(|W : V|\) is infinite. Then \( \chi(W) = 0, N \cong \mathbb{Z} \) and \( N \) is normal in \( W \).

**Proof.** Since \((W, \Omega)\) is a proper \( PD_3 \)-pair of groups, \( c.d.W = 2 \), and since \( W \) has a subgroup of infinite index and one end, \( W \) is not a \( PD_2 \)-group. Since \( V \) is \( FP_2 \) and has one end, \( c.d.V = 2 \) and \( V \) is the ambient group of a proper \( PD_3 \)-pair of groups \((V, \Upsilon) \([13]\).

If \( W \) splits over a subgroup \( T \) which is virtually \( \mathbb{Z}^2 \), then \( N \leq T \) \([7\) Lemma 2.4] (see \S 2 above). Hence \( N \) is \( FP_2 \) and virtually abelian, and so \( \sqrt{W} \neq 1 \). If \((V, \Upsilon)\) is of \( I \)-bundle type and \( W \) has no such splitting then \( W \) has a non-trivial abelian normal subgroup, by Lemma \([5]\). If \((V, \Upsilon)\) is of \( I \)-bundle type then \( V \) is virtually \( \mathbb{Z}^2 \), since \( \chi(V) = 0 \), and so \( N \) is \( FP_2 \) and virtually abelian. Hence \( 1 \neq \sqrt{N} \leq \sqrt{W} \). Thus \( \sqrt{W} \neq 1 \), in all cases.

Since \( c.d.W = 2 \) and \( W \) is not a \( PD_2 \)-group, \( \sqrt{W} \cong \mathbb{Z} \) \([1\) Corollary 8.6]. Moreover, \( W/\sqrt{W} \) is virtually free \([1\) Theorem 8.4]. Hence \( \chi(W) = 0 \). The other hypotheses imply that \(|N : N \cap \sqrt{W}|\) is finite. Hence \( N \) is virtually cyclic, and so \( N \cong \mathbb{Z} \), since it is torsion free. Hence \( N \) is normal in \( W \) \([11\) Theorem 9.12]. \( \square \)

The statement of Theorem 2.6 of \([7]\) is (after minor reformulation):

**Theorem.** (Elkalla \([7\) Theorem 2.6]) Let \( G \) be a 3-manifold group which is not finitely generated and which has non-trivial subgroups \( N \leq U \), where \( N \) is subnormal in \( G \) and \( U \) is finitely generated. Then \( N \cong \mathbb{Z} \) and \( N \) is normal in \( G \). \( \square \)

We may suppose that \( G = \pi_1(M) \), where \( M \) is a non-compact 3-manifold. Then \( N \) is infinite and \( G \) is indecomposable, by \([7\) Proposition 2.2 and Theorem 1.5] (or Burns’ Theorem). On passing to the orientable covering space, if necessary, we see that \( M \) is aspherical. Our versions of Elkalla’s Theorems 2.6 and 3.7 (Theorems \([7\) and \([8\) below) requires a condition that would follow from Lemma \([2]\) if Hypothesis B holds sufficiently widely.

**Theorem 7.** Let \( G \) be an almost coherent open \( PD_3 \)-group which is not finitely generated and which has non-trivial subgroups \( N \leq U \), where \( N \) is subnormal in \( G \) and \( U \) is finitely generated. Assume that \( \chi(V) = 0 \) for all finitely generated subgroups \( V \) of \( G \) which contain \( U \) as a subgroup of infinite index. Then \( N \cong \mathbb{Z} \) and \( N \) is normal in \( G \).

**Proof.** Since \( G \) is almost coherent, we may write \( G = \bigcup_{n \geq 0} U_n \) as a properly increasing union of \( FP_3 \) subgroups, with \( U_0 = U \). If \(|U : N|\) is finite then \( N \) is \( FP_2 \), and the result is known. Hence we may assume that \(|U : N| = \infty \). In particular, \( U \) is not infinite cyclic.
It follows from the Kurosh subgroup theorem that if \([U_{n+1} : U_n]\) is finite then \(U_{n+1}\) has strictly fewer indecomposable factors than \(U_n\), unless both groups are indecomposable \([21]\). Hence if \([U_n : U]\) is finite for all \(n\) then \(U_n\) is indecomposable, for all \(n \gg 0\). Since \(G\) is torsion-free, it follows that \(U_n\) has one end, for all \(n\), and so \(c.d.\, U_n = 2\). But then \(c.d.\, G > 2\) \([8, \text{Theorem 3.3}]\), which gives a contradiction. Thus we may as well assume that \([U_{n+1} : U_n]\) = \(\infty\), for all \(n\).

If \(n \geq 1\) then \(U_n\) is indecomposable, by Burn’s Theorem. Hence it has one end, and so is the ambient group of a proper \(PD_3\)-pair of groups \((U_n, \Omega_n)\), by the Algebraic Core Theorem of \([13]\). We may apply Lemma \([6]\) to the pair \((W, \Omega) = (U_2, \Omega_2)\) and the subgroups \(N < V = U_1\), since \(\chi(U_1) = 0\), by assumption. Hence \(N \cong \mathbb{Z}\), and so \(N\) is normal in \(G\) \([11, \text{Theorem 9.12}]\). \(\square\)

The final result of \([7]\) involves a relative notion of residual finiteness. If \(U\) is a subgroup of a finitely generated group \(G\) then \(G = \pi_1(M)\) is \(U\)-residually finite if for all \(g \in G \setminus U\) there is a subgroup \(H\) of finite index in \(G\) such that \(U \leq H\) and \(g \notin H\). We have again modified the formulation of the theorem slightly.

**Theorem.** (Elkalla \([7, \text{Theorem 3.7}]\)) Let \(M\) be a compact, connected \(\mathbb{P}^2\)-irreducible 3-manifold with aspherical boundary. If \(\pi = \pi_1(M)\) has nontrivial subgroups \(N < U\) of infinite index and such that \(N\) is normal in \(\pi\) and \(U\) is indecomposable and finitely generated, and if \(\pi\) is \(U\)-residually finite then either

1. \(N \cong \mathbb{Z}\); or
2. \(M\) has a finite covering space which fibres over \(S^1\), with fibre a compact surface \(F\) such that the image of \(\pi_1(F)\) in \(\pi\) is commensurable with \(U\).

In our adaptation, we shall assume that \([G : \langle \langle U \rangle \rangle_G]\) is infinite, rather than that \(G\) is \(U\)-residually finite. Neither hypothesis appears to imply the other. (However, if \(G\) is \(U\)-residually finite then \(G/\langle \langle U \rangle \rangle_G\) is residually finite.)

**Theorem 8.** Let \((G, \Omega)\) be a \(PD_3\)-pair of groups, where \(G\) is almost coherent and has nontrivial subgroups \(N \subseteq U\), where \(N\) is subnormal in \(G\), \(U\) is \(FP_2\) and has one end, and \([G : \langle \langle U \rangle \rangle_G] = \infty\). Assume that \(\chi(V) = 0\) for all finitely generated subgroups \(V\) of \(G\) which contain \(U\) as a subgroup of infinite index. Then either

1. \(N \cong \mathbb{Z}\) and either \(N\) is normal in \(G\) or \(G\) is polycyclic; or
2. \(\Omega = \emptyset\) and \(G\) is virtually a semidirect product \(K \rtimes \mathbb{Z}\), where \(K\) is commensurable with \(U\).

**Proof.** Let \(K = \langle \langle U \rangle \rangle_G\). Then \(c.d.\, K = 2\), since \([G : K] = \infty\) and \(U \leq K\) has one end. If \(K\) is not finitely generated then it is an open \(PD_3\)-group, and \(N \cong \mathbb{Z}\), by Theorem \([7]\) if \(K\) is finitely generated then it is \(FP\), since \(G\) is almost coherent and \(c.d.\, K = 2\). Hence \(c.d.\, G = 3\), since \(K\) is normal in \(G\) and \([G : K] = \infty\) \([11, \text{Corollary 8.6}]\). Therefore \(\Omega\) is empty and \(G\) is a \(PD_3\)-group. Moreover, \(G\) is virtually a semidirect product \(K \rtimes \mathbb{Z}\) and \(K\) is a \(PD_2\)-group \([12]\). Since \(U \leq K\) has one end, \([K : U]\) is finite, and so (2) holds. \(\square\)

**Corollary 9.** If \(N \leq N_1 \leq G\) where \(N\) is normal in \(N_1\) and \(N_1\) is normal in \(G\) then the theorem holds without the explicit assumption in the second sentence.

**Proof.** The hypothesis on \(V\) holds if \(N\) is normal or 2-step subnormal in \(G\), by Lemma 2 and \([20, \text{Corollary 1.4}]\). \(\square\)
It remains an open question whether every $PD_3$-group is the fundamental group of an aspherical 3-manifold, but this seems very plausible. Thus we might expect the following ideal theorem to hold.

**Theorem.** Let $(G,\Omega)$ be a $PD_3$-group pair, where $G$ is almost coherent and has non-trivial subgroups $N \leq U$, where $N$ is subnormal in $G$, $U$ is $FP_2$ and has one end, and $[G : U] = \infty$. Then either

1. $N \cong \mathbb{Z}$; or
2. $\Omega = \emptyset$ and $G$ is virtually a semidirect product $K \rtimes \mathbb{Z}$, where $U$ is commensurable with $K$.

In attempting to prove this, we may assume that $U = N_G(U)$, for otherwise $U$ is a $PD_2$-group and $G$ is virtually a semidirect product $K \rtimes \mathbb{Z}$, with $K$ and $U$ commensurable, by Corollary 2.16.2 of [10]. We may also assume that $N = \langle \langle z \rangle \rangle_{N_1}$ is the normal closure of a single element $z$ in the next term $N_1$ of a subnormal series. (We could then replace $N_i$ by $\langle \langle z \rangle \rangle_{N_{i+1}}$, for all $i < k$.)

5. SPECIAL CASES

It is natural to ask whether the situation of Elkalla’s results can be reduced to the special case when $N$ is normal in $G$. (We might also ask whether “subnormal” can be relaxed to “ascendant”, as in [12]. However, the arguments for the ascendant case involve the Gildenhuys-Strebel Theorem, and thus seem to depend on the base of the ascending series being $FP_2$.)

If $N$ is normal in $G$ then it is contained in all of the conjugates of $U$, and so $N \leq \text{Core}_G(U) = \cap_{g \in G} gUg^{-1}$. Thus if $U$ is a finitely generated subgroup of a 3-manifold group $\pi$ and $\pi$ is $U$-residually finite then Elkalla’s results imply that either $\text{Core}_\pi(U) \cong \mathbb{Z}$ or $U$ is commensurable with a subgroup $V$ such that $[\pi : N_\pi(V)] < \infty$, since $\text{Core}_\pi(U) \leq U$ and is normal in $\pi$.

If $[U : \text{Core}_G(U)]$ is finite then $\text{Core}_G(U)$ is $FP_2$, and is normal in $G$, so we may apply [12]. (In this case we also have $\text{Comm}_G(U) = G$.) Thus we may also assume that $[U : \text{Core}_G(U)] = \infty$. If $G$ is almost coherent then we may also assume that $U = N_G(U)$. (However, $U$ is not malnormal in $G$ in the cases of interest, since $\text{Core}_G(U) \neq 1$.) We cannot expect that $\text{Core}_G(U)$ is always finitely generated. See [12] §9.6 for an example.

We may also compare the following three conditions

1. $[G : N_G(U)] < \infty$;
2. $[U : \text{Core}_G(U)] < \infty$;
3. $U$ is commensurable with $V \leq G$ such that $[G : N_G(V)] < \infty$.

It is easy to see that (1) and (2) each imply (3), since we may take $V = U$ or $\text{Core}_G(U)$, respectively, and (2) is equivalent to “$U$ is commensurable with a normal subgroup”. On the other hand, if $G = D_\infty$ and $U$ is of order 2 then (2) and (3) clearly hold, but $N_G(U) = U$, and so (1) fails. This counterexample can be lifted to the flat group $G_2 = \mathbb{Z}^2 \rtimes_{-1} \mathbb{Z}$, which is also an extension of $D_\infty$ by $\mathbb{Z}^2$, to give a counterexample which is a $PD_3$-group. Nor does (3) imply (2). For let $G = G_3$, with presentation

$$\langle t, x, y | txt^{-1} = y, tyt^{-1} = x^{-1}y^{-1}, xy = yx \rangle,$$

and let $U = \langle t^3, x \rangle$. Then $[G : N_G(U)] = 3$ and $N_G(U)/U \cong \mathbb{Z}$, but $\text{Core}_G(U) \cong \mathbb{Z}$.

Let $G$ be a $PD_3$-group or an open $PD_3$-group, and consider the hypotheses:
(1) \(C(G)\): Let \(U\) be an \(FP_2\) subgroup of \(G\) such that \([G : U] = \infty\). Then either \(\text{Core}_G(U) = 1\) or \(Z\), or \(U\) is commensurable with a subgroup \(V\) such that \([G : N_G(V)] < \infty\): and

(2) \(S(G)\): Let \(N < U\) be non-trivial subgroups of \(G\) such that \(N\) is subnormal in \(G, U\) is \(FP_2\) and \([G : U] = \infty\). Then either \(N \cong Z\) or \(U\) is commensurable with a subgroup \(V\) such that \([G : N_G(V)] < \infty\) and \(N_G(V)/V\) has two ends.

We have not assumed that \(U\) be indecomposable or that \(G\) be \(U\)-residually finite, and so these hypotheses may be too strong to hold for all 3-manifold groups. On the other hand, Elkalla does not need “\(U\)-residually finiteness” for the case when \(G\) is not finitely generated.

Clearly \(S(G) \Rightarrow C(G)\), and \(\forall G, S(G) \Rightarrow \forall G, C(G)\). Can either of these implications be reversed?

Elkalla’s results were obtained in the early 1980s, only a few years after Thurston had made the then seemingly preposterous suggestion that hyperbolic 3-manifolds might all be virtually fibred. This was confirmed recently, by Agol and others. It had made the then seemingly preposterous suggestion that hyperbolic 3-manifolds \(\pi\) group \(P D\) is a closed graph manifold. However, this is not in itself enough to establish \(G\) subnormal, \(K\). This is clear already when \(\pi \cong \mathbb{Z}^3\). If \(\chi(K) < 0\) and \(\beta_1(G) > 1\) then \(G\) admits epimorphisms to \(\mathbb{Z}\) with kernels \(PD_2\)-groups of arbitrarily high genus \(23\).

When \(\beta_1(G) = 1\) there is an essentially unique epimorphism to \(\mathbb{Z}\). In this case, if \(G\) has non-trivial subgroups \(N < U\) with \(N\) subnormal, \(U\) finitely generated and one-ended and \([G : U] = \infty\) is \(U \leq K\)? (This is so if \(U\) is is subnormal in \(G\) \(12\).

6. Work of Moon, Sahattchieve and Sánchez-Peralta

After writing up the above material (and shortly before the original submission to the archive in June 2021) we learned of the papers \(18\) and \(19\). These papers obtain versions of Elkalla’s main result with no residual-finiteness hypothesis. In \(18\) it assumed that \(G = \pi_1(M)\), where \(M\) is either a geometric 3-manifold or splits along an embedded torus into one or two geometric pieces, and that \(N\) is normal in \(G\). This is subsumed in the more recent result of \(19\), which assumes that \(M\) is closed or has toroidal boundary, the intersections of \(N\) with the fundamental groups of each of the pieces of a JSJ decomposition are not cyclic, and \(N\) has a subnormal series of length \(n\) in which \(n - 1\) terms are finitely generated. This is shown first for geometric 3-manifolds, and then extended inductively across the tori of a JSJ decomposition. (Thus in the notation of \(5\), the property \(C(\pi)\) holds for all 3-manifold groups \(\pi\).)

We shall comment on the condition that \(N\) should have a subnormal series of length \(n\) in which \(n - 1\) terms are finitely generated. If \(N\) is itself finitely generated, then we may invoke \(2\). If \(N\) is not finitely generated then either \(N\) is normal or it is a normal subgroup of a finitely generated subnormal subgroup \(N_1\). In the latter case we may again invoke \(2\), to conclude that \(N_1\) is normal in \(G\) and either \(N_1 \cong \mathbb{Z}\), (in which case \(N \cong \mathbb{Z}\) and is normal in \(G\)), or \(G/N_1\) is virtually \(\mathbb{Z}\) (and so is residually finite). Thus the major novelty in \(19\) is the case when \(N\) is normal, (and is contained in a finitely generated subgroup \(U\) of infinite index).
Since then P. Sánchez-Peralta has shown that Hypothesis B holds if $N$ is normal or 2-step subnormal in $G$ \[20\]. This establishes Corollary \[9\] but our arguments still need to assume that $[G : \langle (U) \rangle_G] = \infty$.

REFERENCES

[1] Bieri, R. Homological Dimensions of Discrete Groups, Queen Mary College Lecture Notes, London (1976).
[2] Bieri, R. and Hillman, J.A. Subnormal subgroups in 3-dimensional Poincaré duality groups, Math. Z. 206 (1991), 67–69.
[3] Burns, R.G. On the finitely generated subgroups of an amalgamated product of two groups, Trans. Amer. Math. Soc. 169 (1972), 293–306.
[4] Castel, F. Centralisateurs d’éléments dans les $PD_3$-paires, Comment. Math. Helvetici 82 (2007), 499–517.
[5] Dicks, W. and Dunwoody, M.J. Groups acting on Graphs, Cambridge University Press (1989).
[6] Eckmann, B. Cyclic homology of groups and the Bass conjecture, Comment. Math. Helv. 61 (1986), 193–202.
[7] Elkalla, H. Subnormal subgroups in 3-manifold groups, J. London Math. Soc. 30 (1984), 342–360.
[8] Gildenhuys, D. and Strebel, R. On the cohomological dimension of soluble groups, Canadian Math. Bull. 24 (1981), 385–392.
[9] Greenberg, L. Discrete groups of motions, Canadian J. Math 12 (1960), 415–426.
[10] Hillman, J.A. Four-Manifolds, Geometries and Knots, Geometry and Topology Monographs, vol. 5, Geometry and Topology Publications (2002 - revisions 2007 and 2014).
[11] Hillman, J. A. Poincaré Duality in Dimension 3, Open Book Series 3, MSP, Berkeley (2020).
[12] Hillman, J.A. Centralizers and normalizers of subgroups of $PD_3$-groups and group pairs, J. Pure Appl. Alg. 204 (2006), 244–257.
[13] Jaco, W. and Shalen, P. Periperal structure of 3-manifold groups, Invent. Math 38(1976), 55–87.
[14] Kapovich, M. and Kleiner, B. Coarse Alexander duality and duality groups, J. Diff. Geom. 69 (2005), 279–352.
[15] Kropholler, P.H. An analogue of the torus decomposition theorem for certain Poincaré duality groups, Proc. London Math. Soc. 60 (1990), 503–529.
[16] Kropholler, P.H. and Roller, M. A. Splittings of Poincaré duality groups. II, J. London Math. Soc. 38 (1988), 410–420.
[17] Lück, W. $L^2$-Invariants: Theory and Applications to Geometry and $K$-Theory, Ergebnisse der Mathematik und ihrer Grenzgebiete 3 Folge, Bd. 44, Springer-Verlag, Berlin - Heidelberg - New York (2002).
[18] Moon, M. A generalization of a theorem of Griffiths to 3-manifolds, Top. Appl. 149 92005), 17–32.
[19] Sahattchieve, J. A. A fibering theorem for 3-manifolds, J. Groups, Complexity and Cryptology 13 (2021), 2:1–2:15. (Erratum ibid, website (2023).)
[20] Sánchez-Peralta, P. On vanishing criteria of $L^2$-Betti numbers of groups, arXiv: 2307.07031 [math.GR]
[21] Scott, G.P. Normal subgroups in 3-manifold groups, J. London Math. Soc. 13 (1976), 5–12.
[22] Scott, G.P. Fundamental groups of non-compact 3-manifolds, Proc. London Math. Soc. 34 (1977), 303–326.
[23] Tollefson, J. 3-Manifolds fibering over $S^1$ with nonunique connected fiber, Proc. Amer. Math. Soc. 21 (1969), 79–80.

SCHOOL OF MATHEMATICS AND STATISTICS, UNIVERSITY OF SYDNEY, NSW 2006, AUSTRALIA

Email address: jonathanhillman47@gmail.com