Boundary and corner terms in the action for general relativity

Ian Jubb\textsuperscript{1}, Joseph Samuel\textsuperscript{2}, Rafael D Sorkin\textsuperscript{3} and Sumati Surya\textsuperscript{2}

\textsuperscript{1} Blackett Laboratory, Imperial College, London, SW7 2AZ, United Kingdom
\textsuperscript{2} Raman Research Institute, C.V. Raman Avenue, Sadashivanagar, Bangalore 560 080, India
\textsuperscript{3} Perimeter Institute, Waterloo, Canada

E-mail: sam@rri.res.in

Received 1 December 2016, revised 31 January 2017
Accepted for publication 13 February 2017
Published 28 February 2017

Abstract

We revisit the action principle for general relativity, motivated by the path integral approach to quantum gravity. We consider a spacetime region whose boundary has piecewise $C^2$ components, each of which can be spacelike, timelike or null and consider metric variations in which only the pullback of the metric to the boundary is held fixed. Allowing all such metric variations we present a unified treatment of the spacelike, timelike and null boundary components using Cartan’s tetrad formalism. Apart from its computational simplicity, this formalism gives us a simple way of identifying corner terms. We also discuss ‘creases’ which occur when the boundary is the event horizon of a black hole. Our treatment is geometric and intrinsic and we present our results both in the computationally simpler tetrad formalism as well as the more familiar metric formalism. We recover known results from a simpler and more general point of view and find some new ones.

Keywords: action principle, boundary terms, corner terms, null boundaries

1. Introduction

The Einstein–Hilbert (EH) action for general relativity depends on the metric and its first and second derivatives. Indeed, the dependence on second derivatives is forced on us by the principle of general covariance since, there is no local coordinate scalar that can be formed from the metric and its first derivatives. By an appropriate choice of coordinates, we can make the first derivatives vanish at any point so that the only candidate for the action is the cosmological constant term.
While the EH Lagrangian does depend on the second derivatives of the metric, the dependence is rather innocuous since, as it turns out, the equations of motion are second order in metric derivatives, rather than fourth order, as one might naively expect. One can remove the dependence on second derivatives by adding a total divergence to the EH Lagrangian, which integrates to a boundary term. The appropriate action for general relativity is therefore the EH action with this boundary term. This makes the action first order in the metric derivatives: the second derivative term present in the Einstein Hilbert Lagrangian is replaced by a term of the form $(\partial g)^2$. All of this has been known for a while [3, 4].

Our first reason for revisiting the action principle for General Relativity is the path integral approach to quantum gravity. In summing over histories, we would like the quantum amplitudes to have the ‘folding’ property:

$$K(X_1, X_3) = \int dX_2 K(X_1, X_2)K(X_2, X_3),$$

where $X_1$ and $X_3$ are initial and final states respectively and $X_2$ is an intermediate state which is summed over. In the metric representation $X_0, X_3$ represent the metrics on an initial and final spatial hypersurface $\Sigma_{1,3}$ and $\Sigma_2$, an intermediate spatial geometry. We would clearly like the action to be additive under a decomposition of spacetime into pieces. There is a close relation between additivity of the action and having a first order Lagrangian. This can be clearly seen in a particle mechanics analogy. Consider the amplitude for a particle to go from $x_0$ at time $t_0$ to $x_N$ at time $T = t_N$, $K(x_0, t_0; x_N, T)$. Introducing time slices at $t_k = k\epsilon = kT/N$, we have the skeletonised version of the path integral

$$K(x_0, t_0; x_N, T) = \int dx_1 dx_2 \ldots dx_{N-1} K(x_0, t_0; x_1, x_1)K(x_1, t_1; x_2, x_2)\ldots K(x_{N-1}, t_{N-1}; x_N, T),$$

If the Lagrangian is first order, i.e. if $L$ depends only on $x$ and $\dot{x}$, the additivity of the action is immediate. One writes the short time propagator replacing $\dot{x}$ in the Lagrangian by $(x_{k+1} - x_k)/\epsilon$. This results in nearest neighbour couplings on the time lattice with the sites labeled by $k$. Decomposing the lattice into two parts separated by $t_j$ gives us the folding property equation (1). However, for a second order Lagrangian $L(x, \dot{x}, \ddot{x})$, one needs three time steps in order to define $\ddot{x}$. E.g. $\ddot{x}_k = (x_{k+1} + x_{k-1} - 2x_k)/\epsilon^2$. This brings in next nearest neighbour couplings on the time lattice, which spoils the additivity of the action.

A related point stems from the tensor nature of the gravitational field, which is not captured in the simple particle analogy above. In summing over histories that go from $X_1$ to $X_3$ via $X_2$, we allow all spacetime geometries, which on pullback agree with $X_2$. No further restriction needs to be placed on the metric. In particular, the components of the metric in directions transverse to the spacelike surfaces need not be held fixed. Textbook treatments (see [5, 6] for example) however hold all components of the metric fixed on the boundary, which is a stronger requirement. In a path integral, one typically sums over all paths without requiring continuity of all components of the metric across $\Sigma_2$. All we need is that the pullback of the four-metric to $\Sigma_2$ agrees with $X_2$.

Our second reason for revisiting the action principle is to explore boundaries of different signatures. A region in spacetime may have boundaries with components which are spacelike, timelike and null. There may also be corners where components of the boundary join. We present a formalism in which all these cases are derived in a transparent manner. The role of boundaries in gravitational physics has been emphasised in recent years. Ideas relating bulk and boundary degrees of freedom have been discussed in the context of black hole entropy. One of the possible applications of our work is in black hole physics.
The need for adding a total divergence to the Einstein–Hilbert action was realised very early in the history of General Relativity [2]. The required boundary counterterm was given a geometric interpretation by York [3] and this line of thought was carried further by Gibbons and Hawking in their work on black hole thermodynamics. When the boundary has corners, there is a need for additional corner terms. These were first discussed by Sorkin and Hartle [7, 8], and subsequently by Hayward [9], Brill and Hayward [10] for timelike and spacelike boundaries. The need for a treatment of null boundaries was recognised by Parattu et al [11, 12]. There are also several contributions by Neiman [15–18] and Epp [14]. Very recently Lehner et al [1] have given a detailed account of this problem. Our work differs from all these in several respects. We postpone a discussion of the differences to the concluding section.

Our treatment uses both the tetrad formulation and the metric formulation. We present a unified approach to the different boundary signatures. Indeed, as will appear below, this simplifies the calculation considerably. In section 2 we review some of the mathematical preliminaries. In section 3 we present the tetrad formulation, which brings out the need for the corner terms and their explicit forms. In section 4, we perform the calculation in the metric formulation, which is more familiar to readers. Section 5 contains a discussion and some open questions.

2. Mathematical preliminaries

The spacetime manifold \((M, g_{ab})\) is described by a Lorentzian metric \(g_{ab}\), where \(a, b\) are spacetime indices going over \((0, 1, 2, 3)\). Our signature is \((-+++\)). We begin with the Einstein–Hilbert action

\[
I_{EH} = \frac{1}{2} \int d^4x \sqrt{-g} R
\]  

(3)

for a spacetime \((M, g_{ab})\), where \(\partial M = \bigcup_1^n \Sigma_i\) can have several piecewise \(C^2\) components \(\Sigma_i\) whose normals \(n_{ia}\) are everywhere either timelike, spacelike or null. We have chosen units in which \(\pi G\) has been set to 1.

Consider a single component of the boundary \(\Sigma \subset \partial M\). When \(\Sigma\) is non-null, the unit normal \(n_a\) satisfies \(n^a n_a = \epsilon\) where \(\epsilon \equiv \pm 1\) depending on whether \(\Sigma\) is timelike or spacelike. When \(\Sigma\) is null the normal \(n_a\) is not unique, but for each \(n_a\) there is an equivalence class of null vectors \(l^a\) which satisfy \(n_a l^a = -1\). In order to unify the treatment of the null and non-null cases, in addition to the normal \(n_a\) to \(\Sigma\) we will find it useful to define a transverse vector \(Q^a\) to \(\Sigma\) which does not lie in \(T_p \Sigma\). For non-null \(\Sigma\), \(Q^a\) is proportional to \(n_a\), i.e. the transverse and normal directions coincide up to a sign. For null \(\Sigma\) the natural choice is \(Q^a \propto l^a\). It is this identification of the transverse vector \(Q^a\) which helps unify our treatment, rather than the normal vector \(n^a\). For a smooth boundary in spacetime, for example, it is not the normal that gives a continuous or consistent definition of the outward direction, but rather the transverse vector, as shown in figure 1. The metric can be decomposed into components along \(\Sigma\) and transverse to it, so that

\[
g_{ab} = h_{ab} + \epsilon n_a p_b \quad \text{(Non-null)}
\]

\[
g_{ab} = \sigma_{ab} - l_a n_b - n_a l_b \quad \text{(Null)}
\]  

(4)

where \(h_{ab}\) is the induced metric on \(\Sigma\) and \(\sigma_{ab}\) is the induced metric on a spatial slice of of the null boundary \(\Sigma\).

The ‘joins’ or intersections \(\mathcal{J}_{ij} = \Sigma_i \cap \Sigma_j\) of \(\partial M\) are allowed to be discontinuous in the sense that \(n^a_i\) and \(n^a_j\) differ at \(\mathcal{J}_{ij}\). The \(\mathcal{J}_{ij}\) are of codimension two and, like the boundary
components, may also be timelike, spacelike or null. By considering the span of the normals $n_i$ and $n_j$ and looking at the range of the polynomial $f(\alpha) = (n_i + \alpha n_j)^2$ as $\alpha$ varies over the real line, one easily arrives at the following classification: the plane of the two normals in the tangent space has Lorentzian signature and the join is spacelike if (i) at least one of the normals is timelike, or (ii) both the normals are null, or (iii) one normal is spacelike and the other null, with $n_i n_j \neq 0$ or (iv) both normals are spacelike and $n_i n_j > 1$. The plane of normals has Riemannian signature and the join is timelike if (i) both normals are spacelike and $n_i n_j < 1$. Finally, the plane of normals is null and the join is null if (i) one normal is null and the other spacelike with $n_i n_j = 0$.

In section 3 we use the Cartan tetrad formalism. This has the significant advantage offered by differential forms which can be integrated over manifolds without reference to a metric or its signature. It also has the advantage of giving us a fiducial Minkowski vector space as a reference. Given a metric $g_{ab}$ on $M$ we choose an orthonormal frame such that $g_{ab} = e_a^\mu e_b^\nu \eta_{\mu\nu}$.

The tetrad $e_a^\mu$ maps a vector $X \in T_p M$ to a point in $\eta_{\mu\nu}(0)(M)$, where $\eta_{\mu\nu}(0)(M)$ is a fixed fiducial Minkowski vector space, with $a$ the spacetime index and $\mu$ the frame index ranging over 0, 1, 2, 3. The map equation (5) is invertible, since we assume that the metric is non-degenerate. The spacetime metric $g_{ab}$ is then the pullback of the fiducial metric $\eta_{\mu\nu}$ on $M_0$. Frame indices $\mu, \nu$ are raised and lowered with $\eta_{\mu\nu}$. There is an $O(1, 3)$ gauge freedom in the choice of the $e_a^\mu$. Associated with the $e_a^\mu$ are the connection 1-forms $A_{\mu\nu}^a = e_b^\mu \nabla_a e_b^\nu$ where $\nabla_a$ is the metric compatible Christoffel connection. $A$ takes values in the Lie algebra of $O(1, 3)$ and is antisymmetric in the frame indices: $A^{\mu\nu} = -A^{\nu\mu}$. $A$ is compatible with frames and satisfies Cartan’s equation

$$de^\mu + A^{\mu\nu} \wedge e^\nu = 0.$$  

Written explicitly in the spacetime indices, the field strength of $A$ is

$$F_{\mu\nu}^{ab} = \nabla_a A_{\mu\nu}^b - \nabla_b A_{\mu\nu}^a + A_{\mu\rho}^a A_{\nu\sigma}^b - A_{\mu\rho}^b A_{\nu\sigma}^a - A_{\nu\rho}^a A_{\mu\sigma}^b + A_{\nu\rho}^b A_{\mu\sigma}^a$$

Figure 1. An illustration of how the normal and transverse vectors would be orientated on a patch of $1 + 1$ Minkowski spacetime whose boundary is a circle.
which is more succinctly expressed as $F_{\mu\nu} = dA_{\mu\nu} + A_{\rho}^{\mu} \wedge A_{\nu}^{\rho}$ where the wedge product is with respect to the spacetime indices.

Using the algebraic identity

$$\eta^{abcd} \epsilon_{\mu\nu\rho\lambda} e^{\epsilon}_{a} e^{\epsilon}_{b} = -e(2!)^2 \epsilon^{\epsilon}_{a} e^{\epsilon}_{b}$$

(8)

where $e = \sqrt{-g}$ and $\eta^{abcd}$ is the Levi-Civita tensor density, the Einstein–Hilbert action takes the form

$$I_{EH} = \frac{1}{4} \int d^4 x \epsilon_{\mu\nu\rho\lambda} e^{\epsilon}_{\mu} \wedge e^{\epsilon}_{\nu} \wedge F^{\rho\lambda}.$$  

(9)

The variation of $I_{EH}$ gives us a bulk term (which yields the equations of motion) and a boundary term. The boundary term will be expressed as the variation of a boundary action $-I_B$, which gives us a counterterm to be added to the action. The total gravitational action is therefore

$$I = I_{EH} + I_B$$  

(10)

where $I_B$ in the non-null case is the Gibbons–Hawking–York (GHY) term. From its definition, the boundary term $I_B$ is only defined up to terms that have zero variation. Certain imaginary terms that have been discussed before in the literature are of this variety. We will ignore them for the most part and comment on them in the conclusion. When the boundary is only piecewise $C^2$ the boundary contribution includes ‘corner’ terms. It is the evaluation of these various boundary components in the tetrad formulation that we will now focus on.

3. The tetrad formalism

3.1. Boundary terms

Varying the action $I_{EH}$ we find

$$\delta I_{EH} = \frac{1}{4} \left( 2 \int_M \epsilon_{\mu\nu\rho\lambda} \delta e^{\epsilon}_{\mu} \wedge e^{\epsilon}_{\nu} \wedge F^{\rho\lambda} + 2 \int_M \epsilon_{\mu\nu\rho\lambda} e^{\epsilon}_{\mu} \wedge e^{\epsilon}_{\nu} \wedge D\delta A^{\rho\lambda} \right).$$

(11)

The first term gives us Einstein’s vacuum equations in the form $e^{\epsilon}_{\nu} \wedge F^{\rho\lambda} \epsilon_{\mu\nu\rho\lambda} = 0$ and the second term reduces to a boundary contribution

$$-\delta I_B = \frac{1}{2} \int_M D(\epsilon_{\mu\nu\rho\lambda} e^{\epsilon}_{\mu} \wedge e^{\epsilon}_{\nu} \wedge \delta A^{\rho\lambda}) = \frac{1}{2} \int_M \epsilon_{\mu\nu\rho\lambda} e^{\epsilon}_{\mu} \wedge e^{\epsilon}_{\nu} \wedge \delta A^{\rho\lambda}$$

(12)

In order to calculate the boundary term we require that the pullback of the metric to the boundary $M$ is unvaried. We can, in addition, demand that the pullback of $e^{\epsilon}_{\mu}$ to the boundary $M$ also has zero variation. This permits us to take $\delta$ outside the integral and express it as the variation of

$$-I_B = \frac{1}{2} \int_{\partial M} \epsilon_{\mu\nu\rho\lambda} e^{\epsilon}_{\mu} \wedge e^{\epsilon}_{\nu} \wedge A^{\rho\lambda}.$$  

(13)

Our derivation so far is independent of the type of boundary $\partial M$. We will now show that this expression is the GHY term written in a universal form, by looking at the three types of boundaries—spacelike, timelike and null. We now choose adapted tetrads so that one of the 1-form fields $e^{\epsilon}_{\mu}$ is normal to the boundary. The natural choices for the spacelike,
timelike and null normals to $\partial M_{\alpha\beta\gamma\delta} = e^\alpha_{\mu} = n_{\mu}$, respectively, where $e^\alpha_{\mu} = (e^0_{\mu} \pm e^1_{\mu})/\sqrt{2}$. In general we write $n_{\mu} = e^\alpha_{\mu}$ where $\alpha = 0, 1, \pm 1$ depending on $\partial M$. Let us also always choose $n^\mu$ to be outward directed for $\partial M$, a well-defined concept for the non-null normals. For null normals, it is the transverse null tetrad $l_a = ± e^\alpha_{\mu}$, respectively, where $e^\alpha_{\mu} = ± e^\alpha_{\mu}$. In general we write $e^\alpha_{\mu}$ where $\alpha = ± 0, 1$, depending on $\partial M$. Let us also always choose $n_{\mu}$ to be outward directed for $\partial M$, a well-defined concept for the non-null normals. For null normals, it is the transverse null tetrad $l_a = ± e^\alpha_{\mu}$ is chosen with the same time orientation as $e^\alpha_{\mu}$ so that $e^\alpha_{\mu} = -1$.

Using the relation equation (8) we see that the integrand in equation (13) can be simplified. Using the notation $e^\alpha_{\mu}$, where the hat indicates that $\alpha$ is a fixed index ($± 0, 1$), for which the summation convention does not apply, we have

$$e^\alpha_{\mu} e^\beta_{\nu} A^\alpha_{\mu} = e^b_{\mu} e^b_{\nu} A^b_{\mu} = e^b_{\mu} A^b_{\mu}$$

where the sum over $\nu$ extends over all indices except $\alpha$ because of the antisymmetry of $A$ in the frame indices. Putting in the form of $A$ we have

$$-e^b_{\mu} e^b_{\nu} \nabla a^\alpha \nabla c = -e^b_{\mu} e^b_{\nu} \nabla a^\alpha \nabla c,$n_{\mu} = e^\alpha_{\mu}.$$  

which gives the universal boundary term

$$I_B = \int_{\partial M} e(g^{bc} - e^b_{\mu} e^b_{\nu}) \nabla b n_c.$$  

Note that we have made no assumption above regarding extending the normal $n_{\mu}$ off the boundary. The normal is only defined at points on the boundary and we only use its tangential derivatives.

Observe further that in the adapted tetrads for non null normals ($\alpha = 0, 1$)

$$e^b_{\mu} e^c_{\nu} = \eta_{\alpha\beta} e^b_{\mu} e^c_{\nu} = e^b_{\mu} n_c$$

and for null normals ($\alpha = +$)

$$e^b_{\mu} e^c_{\nu} = \eta_{+} e^{-b} e^c_{\nu} = -l^b n_c.$$  

Using the decomposition equation (4) we note that

$$\eta_{\alpha\beta} e^b_{\mu} e^c_{\nu} = (\eta_{\alpha\beta} e^b_{\mu} e^c_{\nu}) \nabla b n_c = h^{bc} \nabla b n_c = K$$ (Non-null)

$$\eta_{\alpha\beta} e^b_{\mu} e^c_{\nu} = (\eta_{\alpha\beta} e^b_{\mu} e^c_{\nu}) \nabla b n_c = (\Theta - \kappa)$$ (Null)

where $K = h^{bc} \nabla b n_c$ is the extrinsic curvature of $\partial M$, $\Theta = \sigma^{ab} \nabla a n_b$ the null expansion on $\partial M$, and the surface gravity $\kappa = \sigma^{ab} \nabla a n_b$ measures the failure of $n^b$ to be affinely parameterised. For $\partial M_{\alpha\beta\gamma\delta}$ this gives the expected GHY term

$$I_B = \int_{\partial M} \sqrt{\pm h} K d^3x,$$

where $x$’s are coordinates on the boundary. For $\partial M_a$ this gives

$$I_B = \int_{\partial M_a} \sqrt{\pm (\Theta - \kappa)} d\lambda d^2x,$$

where the $x$’s are now spatial coordinates on the null surface and $\lambda$ is a parameter along the null generator satisfying $n^\mu \partial_a = \frac{\partial}{\partial \lambda}$. 


The boundary term equation (13) is not gauge invariant under $O(1, 3)$ transformations (although its variation is). This is because $A$ transforms inhomogeneously by

$$A \rightarrow \Lambda^{-1} A \Lambda + \Lambda^{-1} d \Lambda$$

with the result that

$$I_B \rightarrow I_B + \frac{1}{2} \int_{\partial M} \epsilon_{\mu \nu \rho \lambda} e^\mu \wedge e^\nu \wedge g^{\rho \lambda}$$

where $g = \Lambda^{-1} d \Lambda$ is in the Lie algebra of $O(1, 3)$.

We note that in the adapted tetrads there is a residual gauge freedom in the little group $H$, which preserves the normal. The little group is given by $H = O(3)$ for timelike, $H = O(1, 2)$ for spacelike and $H = E(2)$ for null normals. It is easily checked that the adapted boundary term is invariant under gauge transformations of the little group. In fact for $\Lambda \in H$, $h = \Lambda^{-1} d \Lambda$ satisfies $h^{\alpha \lambda} = h^{\alpha \lambda} = 0$ for $\alpha$ a fixed index labelling the normal, as above. For vector fields $\ell^\mu$, tangent to the boundary, we have $e^{\alpha}(t) = n_a \ell^a = 0$ and so the change in $I_B$ under a gauge transformation

$$\Delta I_B = \frac{1}{2} \int_{\partial M} \epsilon_{\mu \nu \rho \lambda} e^\mu \wedge e^\nu \wedge h^{\rho \lambda}$$

vanishes entirely, since the four indices of $\epsilon_{\mu \nu \rho \lambda}$ must all be distinct for a nonvanishing contribution.

We introduce four discrete elements $D$ of $O(1, 3)$ corresponding to each of the connected components of the group. They are I, P, T and PT, where P and T stand for parity and time-reversal respectively. Since these are constant matrices, the connection $A$ transforms homogeneously and the boundary term equation (13) is invariant under such transformations. These discrete elements $D$ will be needed in the next section to relate frames across a join.

### 3.2. Corner terms

The fact that the boundary term equation (13) is not gauge invariant can be exploited to identify the corner terms. By adapting our frame to the normal we have been able to derive the forms equations (21) and (22) of the boundary GHY terms for all signatures of the boundary. When there is a join of two boundary components, the adapted frames will not, in general, agree at the join. In order to pass from one frame to the other we will use the following procedure. By means of a gauge transformation in the little group $H$, we will ensure that two of the frame fields from each boundary component are tangent to the join and agree with each other at the join. By use of discrete elements in $O(1, 3)$, we will ensure that the frames are related by an element in the identity component of $O(1, 3)$. With these choices, the relation between the two frames is a Lorentz transformation in the 2-dimensional plane of normals. The change in the boundary term equation (13) under this $O(1, 3)$ gauge transformation gives us the corner terms.

Let $\Sigma_i$ and $\Sigma_j$ meet along a join $J_p$. The boundary term equation (16) is valid when an adapted frame is used, but the latter changes in going from $\Sigma_i$ to $\Sigma_j$. This corresponds to effecting an $O(1, 3)$ transformation in equation (13) which relates the adapted frames $e_{(i)}^\mu$ and $e_{(j)}^\mu$ of $\Sigma_i$, $\Sigma_j$. By operating on the frames by discrete elements $D$ of $O(1, 3)$, we can arrange that the two frames are related by an element in the identity component of $O(1, 3)$. For spacelike joins, by further gauge transformations in the little group, one can arrange that $e_{(i)}^2 = e_{(j)}^2$ and $e_{(i)}^3 = e_{(j)}^3$ and that both of these are orthogonal to the timelike plane of
normals. The two frames $e_{(i)}$ and $e_{(j)}$ are therefore related by a Lorentz boost in the timelike plane of normals,

$$e_{(i)^\mu} = \Lambda_{(i)\mu} e_{(j)^\nu}. \tag{26}$$

We define the discontinuous gauge transformation $\lambda \in O(1, 1)$ to be the identity on $\Sigma_i$ and $\Lambda_{(ij)}$ on $\Sigma_j$

$$\lambda_{ij} = \exp [\eta K \Theta_{ij}^{\mu\nu}], \tag{27}$$

where $\Theta_{ij}^{\mu\nu}$ is the Heaviside function that takes values 0 on $\Sigma_i$ and 1 on $\Sigma_j$, $\eta$ is the rapidity parameter and $K$ the boost generator in the plane of normals. $g^{\mu\nu} = (\Lambda_{ij})^{-1} d\Lambda_{(ij)^{\mu\nu}} = \eta K^{\mu\nu} d\Theta_{ij}^{\mu\nu}$ is therefore proportional to a delta function that is peaked on the join $J_{ij}$ and vanishes on $\Sigma_i$ and $\Sigma_j$. The gauge transformation of the boundary term equation (24) results in the join term

$$\frac{1}{2} \int_{J_{ij}} \eta e_{\mu\nu\rho\sigma} e^{\mu\nu} \wedge e^{\rho\sigma}, \tag{28}$$

which in this case (since only $\eta^{01} = -\eta^{10}$ is non vanishing), simplifies to

$$I_{J_{ij}} = \int_{J_{ij}} e^3 \wedge e^3 \eta = \int_{J_{ij}} d\eta, \tag{29}$$

where $dA$ is the area element of the join.

It is possible to express the rapidity that appears in the corner term using the angle between the normals. The Lorentz boost with rapidity $\eta$ can be written as

$$e_{(i)^{\mu}} = (\exp \eta) e_{(i)^{\mu}}, e_{(j)^{-\mu}} = (\exp -\eta) e_{(i)^{\mu}}$$

and the timelike and spacelike normals as $n_{(i,j)} = (e_{(i)^{\mu}} \pm e_{(j)^{-\mu}}) / \sqrt{2}$, respectively. Using the symbols $T$, $S$, $N$ to denote a timelike, spacelike or null normal, respectively, we find that if the two normals at the join are (i) TT: $n_{(i,j)} = -\cosh \eta$, (ii) TS: $n_{(i,j)} = \sinh \eta$, (iii) TN: $n_{(i,j)} = -\exp \eta / \sqrt{2}$, (iv) SS: $n_{(i,j)} = \cosh \eta$, (v) SN: $n_{(i,j)} = \exp \eta / \sqrt{2}$ and (vi) NN: $n_{(i,j)} = -\exp \eta$.

For timelike joins, the argument is very similar. We can by gauge transformations in the little group arrange that $\eta^{00} = 0$ and $\eta^{11} = 1$ and that both of these are orthogonal to the spacelike plane of normals. The two frames $e_{(i)^{\mu}}$ and $e_{(j)^{\mu}}$ are now related by a rotation in the spacelike plane of normals

$$e_{(i)^{\mu}} = \Lambda_{(i)\mu} e_{(j)^{\nu}}. \tag{30}$$

Again, define the discontinuous gauge transformation $\lambda \in O(2)$ as the identity on $\Sigma_i$ and $\Lambda_{(ij)}$ on $\Sigma_j$, so that

$$\lambda_{ij} = \exp [\eta J \Theta_{ij}^{\mu\nu}], \tag{31}$$

where $\eta$ is now the rotation angle and $J$ the rotation generator in the plane of normals. Again this gives rise to a contribution from the join

$$\frac{1}{2} \int_{J_{ij}} \eta e_{\mu\nu\rho\sigma} e^{\mu\nu} \wedge e^{\rho\sigma}, \tag{32}$$

Since the nonvanishing components of $J$ are $J^{23} = -J^{32}$, we have the form of the corner term:

$$I_{J_{ij}} = \int_{J_{ij}} e^3 \wedge e^3 \eta = \int_{J_{ij}} d\eta, \tag{33}$$
where \(dA\) is the area element of the join. Relating the inner products to the angles follows the case of spacelike joins and we do not repeat the analysis here. A salient difference is that the angles are only defined modulo \(2\pi\). This arises because the group \(SO(1, 1)\) is simply connected \((\pi(SO(1, 1)) = 0)\), while the group \(SO(2)\) is multiply connected \((\pi(SO(2)) = \mathbb{Z})\). This ambiguity does not however affect the variation.

Null joins differ in that the plane of normals and the tangent space to the join share a one dimensional, null subspace. If \(n_i\) is spacelike and \(n_j\) is null (with \(n_in_j = 0\)), \(n_j\) belongs both to the span of normals and the tangent space to the join. It is possible to adapt a null Lorentz frame to both \(\Sigma_i\) and \(\Sigma_j\) as follows: \(e_i^j = e_j^j = n_i\), \(e_i^j = e_j^j = n_j\) and \(e_i^k = e_j^k\), \(e_i^k = e_j^k\). Since \(e_i^i = e_j^i\), we have \(\Lambda_{ij}\) equal to the identity and \(\eta = 0\). The corner term therefore vanishes.

### 3.3. Creases

A physically interesting situation covered by the above analysis occurs when one of the boundaries of spacetime is the event horizon of a dynamically evolving black hole. In this case the horizon does not remain smooth when new generators enter or leave the horizon. Suppose that we are interested in the boundary of a future set. (The case of past sets is similar.) The boundary of a future set is ruled by null generators. However, when these null generators cross because of gravitational focussing effects, they leave the boundary and enter into the interior of the future set. The horizon then develops a caustic, generically a spacetime region of codimension two, where the normal to the wavefront is discontinuous. When this happens, we have a ‘crease’ which separates regions of the null surface with different normal vectors. Locally, this is no different from a null–null join discussed above. From the analysis already presented we would expect a boundary term to appear as an integral along the crease of the rapidity parameter, just as in the NN case treated above.

### 4. The metric formalism

While the tetrad formulation is calculationally simpler, it is also true that the metric formulation is more familiar to most readers. In this section, we present the metric formulation of the above calculation, which has also recently been given in [1]. To find the boundary contribution to the action we need to consider the most general class of variations \(\delta g^{ab}\) which leave the induced metric on \(\Sigma\) fixed, so that for any \(t^a, s^a \in T \Sigma\),

\[
\delta g_{ab} n^a s^b = 0. \tag{34}
\]

Using \(\delta (g_{ab} g^{ac}) = 0\) to relate the variation of the covariant and contravariant metrics we find that

\[
\delta g_{ab} = -g_{bd} g_{ac} \delta g^{cd} \Rightarrow \delta g^{ab} n_b s_b = 0. \tag{35}
\]

From the decomposition equation (4) of \(g_{ab}\) into components transverse to and along \(\Sigma\), we see that the most general variation takes the form

\[
\delta g^{ab} = 2n^a \delta Q^b, \tag{36}
\]

where we have made the identification

\[
Q^a = \begin{cases} 
\epsilon & n^a \quad \text{(Non-null)} \\
-\ell^a & \ell^a \quad \text{(Null)} 
\end{cases} \tag{37}
\]
The 4-vector $\delta Q^a$ therefore gives the full admissible $10 - 6 = 4$ parameter degrees of freedom in this class of variations.

It is useful to decompose $\delta Q^a$ into components transverse to and along $\Sigma$

$$\delta Q^a = \alpha Q^a + r^a,$$  \hspace{1cm} (38)

where $r^a \in T \Sigma$, and such that $n^a r_a = 0$ for both null and non-null cases. When $\Sigma$ is null $r^a$ can be further decomposed as

$$r^a = \beta n^a + s^a \quad \text{(Null)}$$  \hspace{1cm} (39)

where $s^a n_a = 0$.

Using the unperturbed metric to raise and lower indices, the variation of the covariant quantities is

$$\delta Q_a = \delta g_{ab} Q^b + g_{ab} \delta Q^b, \quad \delta n_a = \delta g_{ab} n^b + g_{ab} \delta n^b,$$  \hspace{1cm} (40)

which along with equations (35)–(39) simplifies to the general expression

$$\delta n_a = -\alpha n_a, \quad \delta Q_a = \beta n_a,$$  \hspace{1cm} (41) \hspace{1cm} (42)

where $\alpha, \beta$ are independent when $\Sigma$ is null and $\beta = -\varepsilon \alpha$ when $\Sigma$ is non-null. The parameter $\alpha$ can moreover be related to a variation of the volume element in both the null and non-null cases

$$\alpha = -\delta \ln(\sqrt{-g})$$  \hspace{1cm} (43)

where we have used $\delta \ln(\sqrt{-g}) = -\frac{1}{2} g_{ab} \delta g^{ab} = -\alpha g_{ab} n^b$ using equations (36) and (38).

The boundary term resulting from the variation of the Einstein–Hilbert action has the general form

$$-\delta I_B = \frac{1}{2} \int_\Sigma dV v^a n_a$$  \hspace{1cm} (44)

where $v^a = -g^{ab} C_{cb}^e + g^{bc} C_{eb}^a$ and $dV$ is the volume element on $\Sigma$ and $C_{eb}^a$ is the variation in the metric compatible connection

$$C_{eb}^a = \frac{1}{2} g^{cd} (\nabla_e \delta g_{bd} + \nabla_b \delta g_{ed} - \nabla_d \delta g_{eb}).$$  \hspace{1cm} (45)

with $\nabla_a$ the connection compatible with $g_{ab}$.

The task is then to find the boundary term $I_B$ which has to be added to the Einstein–Hilbert action. The integrand in equation (44) can be simplified to

$$v^a n_a = -n^a g^{bc} (\nabla_a \delta g_{bc} - \nabla_b \delta g_{ac}),$$  \hspace{1cm} (46)

for all types of $\Sigma$. We now examine the two separate cases.

4.1. $\Sigma$ non-null

Using $g^{ab} = h^{ab} + \varepsilon n^a n^b$ reduces equation (46) to

$$v^a n_a = -n^a h^{bc} (\nabla_a \delta g_{bc} - \nabla_b \delta g_{ac}).$$  \hspace{1cm} (47)

Comparing with the variation of the extrinsic curvature $K$ of $\Sigma$ we see that
\[-2\delta K = 2h^{ab}C_{ab}\kappa - 2h^{ab}\nabla_{a}\delta n_{b} = -n^{a}h^{bc}\nabla_{c}\delta g_{ab} + 2n^{a}h^{bc}\nabla_{c}\delta g_{bc} + 2\alpha K.\] (48)

The first terms in equations (47) and (48) are the same. In [5, 6] the second term in equation (47) and the remaining terms in equation (48) are put to zero but this unnecessarily restricts the allowed variations. Allowing the full 4-parameter variation the second term in equation (47) reduces to
\[n^{a}h^{bc}\nabla_{c}\delta g_{ac} = -2\alpha K - h^{ab}\nabla_{a}n_{b},\] (49)
so that
\[v^{a}n_{a} = -2\delta K + h^{ab}\nabla_{a}n_{b}.\] (50)

Thus, in agreement with the standard results in [5, 6]
\[-\delta I_{h} + \delta I_{K} = \frac{1}{2} \int_{\Sigma} d^{3}x \sqrt{\epsilon} \, D_{a}t^{a},\] (51)
where \(D_{a}\) is the connection compatible with \(h_{ab}\) and
\[I_{K} = \int_{\Sigma} \sqrt{\epsilon} \, K.\] (52)

If \(J_{i} \subset \Sigma\) are either spacelike or timelike ‘corners’ of \(\Sigma\) with normals \(m_{(i)}^{a} \in TS_{\Sigma}\), the variation equation (51) reduces to
\[\frac{1}{2} \sum_{i} \int_{J_{i}} d^{3}x \sqrt{\epsilon} \, q_{ab} \, t^{a}m_{(i)ab} \equiv \sum_{i} \delta I_{J_{i}}\] (53)
where \(q_{ab}\) is the induced metric on \(J_{i}\) and \(\epsilon' = \pm 1\) depending on whether \(J_{i}\) is spacelike or timelike. If \(J_{i}\) is null, then
\[\delta I_{J_{i}} = \frac{1}{2} \int_{J_{i}} d\lambda \sqrt{\epsilon} \, q_{ab} \, t^{a}j_{b}\] (54)
where \(\sqrt{\epsilon} q_{ab}\) is the volume element on the 1 dimensional spatial section of \(J\) and \(j^{a}\) its null normal. As we will see in the next few sections, such corner terms will not contribute. Thus, the boundary term to be added to the action is
\[I_{b} = I_{K} - \sum_{i} I_{J_{i}}\] (55)
where \(I_{J_{i}}\) are the yet to be determined corner terms.

4.2. \(\Sigma\) null

Since the null geodesics generated by \(n^{a}\) are hypersurface orthogonal, we can suppose that they satisfy the condition \(\nabla_{a}n_{a} = 0\). Combining this with the variation \(\delta g_{ab} = 2n_{a}g_{b\kappa} - \delta^{\kappa}\) allows us to simplify equation (46) to
\[v^{a}n_{a} = n^{a}\nabla_{a}\alpha - \alpha\Theta + 2\alpha\kappa,\] (56)
where \(\Theta\) and \(\kappa\) are the null expansion and surface gravity of \(\Sigma\) respectively.

The natural analog of the GHY term is
\[ I_0 = \int \Sigma d^2x \lambda \sqrt{\sigma} \Theta \]  

(57)

and it is therefore natural to first compare this variation with equation (56). Since \( n^a = (\partial / \partial \lambda)^a \) remains invariant under this class of variations the affine parameter \( \lambda \) is unchanged, so that \( \delta I_0 \) again only involves the integrand \( \Theta \). While \( \delta \sigma_{ab} = 0 \),

\[ \delta \sigma^{ab} = \delta g^{ac} g^{bd} \sigma_{cd} + g^{ac} \delta g^{bd} \sigma_{cd} = 2n^{(a} s^{b)} \]

(58)

where we have used equations (38) and (39) so that

\[ 2 \delta \Theta = 4 n^{(a} s^{b)} \nabla_a n_b - 2 \sigma^{ab} C_{ab} \kappa_c + 2 \sigma^{ab} \nabla_a \kappa_b \]

\[ = -4 \kappa s^b n_b - 2 \alpha \Theta + 2 \alpha \Theta = 0. \]

(59)

Given the form of equation (56) it is therefore clear that an additional boundary piece is required. Instead, consider (see \[11, 12\])

\[ I_\kappa = \int \Sigma d^2x \lambda \sqrt{\sigma} \kappa, \]

(60)

whose variation again only involves the integrand \( \kappa \),

\[ 2 \delta \kappa = 2 (\delta n^{(a} s^{b)} \nabla_a n_b - l^{(a} n^{b)} C_{ab} \kappa_c + l^{(a} n^{b)} \nabla_b \kappa_a) \]

\[ = 2 \alpha \kappa + 2 n^{(a} s^{b)} \nabla_b \alpha. \]

(61)

Thus

\[ -\delta I_B = \delta I_\kappa - \frac{1}{2} \int \Sigma d^2x \lambda \sqrt{\sigma} (n^a \nabla_a \alpha + \alpha \Theta) \]

\[ = \delta I_\kappa - \frac{1}{2} \int \Sigma d^2x \lambda \frac{d(\sqrt{\sigma} \alpha)}{d\lambda} \]

\[ = \delta I_\kappa + \delta I_f(\lambda) - \delta I_f(\lambda_f) \]

(62)

where we have defined

\[ \delta I_f(\lambda) \equiv \frac{1}{2} \int_J d^2x \sqrt{\sigma(\lambda)} \alpha(\lambda) \]

(63)

and have used the expression \( \Theta = \frac{1}{\sqrt{\sigma}} \frac{d\sqrt{\sigma}}{d\lambda} \). Here, \( \lambda_{i,f} \) are the initial and final values of \( \lambda \) at the spacelike boundaries, \( J_i, J_f \) of \( \Sigma \). As one can see, it is only such spacelike corner terms that contribute for \( \Sigma \) null; there is no contribution from a null corner. The boundary term to be added to the action is therefore

\[ I_B = -I_\kappa + I_f(\lambda_f) - I_f(\lambda_i), \]

(64)

where \( I_f(\lambda) \) is a yet to be determined null corner term contribution. At this point \( I_0 \) can also be included, though its variation vanishes. This brings the boundary term into the same form as that obtained in the tetrad formulation.

Before moving on to a calculation of the corner terms it is worthwhile saying a little about the question of uniqueness of the transverse vector \( Q^a \). In the non-null case, it is easy to find a unique transverse vector. For any timelike or spacelike vector \( r^a \in T_p M \) we associate a unique transverse subspace \( K \subset T_p M \) such that \( r, v = 0 \), \( \forall v \in R \). \( r^a \) is then transverse to \( \Sigma \) and if it is normalised to \( \pm 1 \) it is the unique unit normal \( n^a \). In the null case, the situation is a little more complicated since \( l^a n_a = -1 \) does not give a unique \( l^a \) associated to every \( n^a \). We can
however enforce uniqueness as follows. If \( m_1^a, m_2^a \) are spacelike unit vectors in \( T_p \Sigma \) such that 
\[ n \cdot m_1 = 0, \]
let \( M_1, M_2 \) be their associated transverse subspaces, respectively. Then \( l^a \in M_1 \cap N_l \) is the unique transverse null vector satisfying \( l^a n_a = -1 \).

4.3. Corner terms for null–null boundary

The intersection \( \mathcal{J} \) of two null hypersurfaces \( \Sigma_{1,2} \) can be either spacelike or null. Examples of these are shown in figure 2. The achronality of a null hypersurface precludes the intersection from being timelike. When \( \mathcal{J} \) is null, as we have seen there is no corner contribution for null \( \Sigma \). Indeed, in any case, such a null intersection is not a join as per our definition in section 2. We therefore need only consider the case when \( \mathcal{J} \) is spacelike. For clarity in this section we will resort to using \( k^a \) to depict the null normal and leave \( n^a \) to denote the non-null normal.

Let the normals to the two null boundary components \( \Sigma_{1,2} \) be \( k^a_{1,2} \). In order to fix the relative signs of the corner terms equation (63) it is important to define first what is meant by an outward pointing null normal. We will define this using the transverse vector \( l_a \) rather than the normal \( k^a \). For a join arising in the causal diamond shown in figure 2 the join is outward convex in the following sense. The outward pointing transverse vector \( l_1^a \) to \( \Sigma_1 \) in \( T_\mathcal{J} \) is along the positive \( \partial \partial u_a \) direction. Hence \( k_1^a \) is along the positive \( \partial \partial v_a \) direction. On the other hand, \( l_2^a \) for \( \Sigma_2 \) is in the negative \( \partial \partial u_a \) direction, which makes \( k_2^a \) lie in the negative \( \partial \partial v_a \) direction. Thus, in this case the parameters \( \lambda_{1,2} \) on \( \Sigma_{1,2} \) both take their initial values on \( \mathcal{J} \). This is an outward convex join. Conversely at an outward concave join \( \lambda_{1,2} \) both take their final values on \( \mathcal{J} \).

Thus, from equations (63) and (64) the total contribution to \( \mathcal{J} \) is

\[
\delta I_{\mathcal{J}} = \pm \frac{1}{2} \int_{\mathcal{J}} d^2 x \sqrt{\sigma} (\alpha_1 + \alpha_2) \tag{65}
\]
depending on whether the join is concave or convex. Here \( \alpha_{1,2} \) come from the variations of \( l^a_1 \) and \( l^a_2 \) on \( \mathcal{J} \). The variations of the metric on \( \Sigma_1 \) and \( \Sigma_2 \) are, respectively

\[
\delta g_{1,2}^{ab} = k_{1,2}^a (\alpha_{1,2} k_{1,2}^b + \beta_{1,2} k_{1,2}^b + s_{1,2}) \tag{66}
\]

which at \( \mathcal{J} \) must match up, i.e.

\[
\delta g_{1}^{ab} \mid_{\mathcal{J}} = \delta g_{2}^{ab} \mid_{\mathcal{J}} \tag{67}
\]

Of the 4 null vectors \( k_{1,2}^a, l_{1,2}^a \), we pick two linearly independent ones to be the normals \( k_{1,2}^a \), so that \( l_{1,2}^a = u_{1,2} k_{1,2}^a + v_{1,2} k_{1,2}^a \). Using \( h_1 h_1 = h_2 h_2 = 0, h_1 k_{1,2} = -1, v_1 = u_2 = \frac{-1}{k_{1,2}} \) and
\( u_1 = v_2 = 0 \), which means that \( n_{12}^a = -\frac{1}{\delta_1 k_2} k_1^2 \). Denoting the equality on \( J \) by \( \equiv \), equation (67) then implies that
\[
\begin{align*}
\alpha_1 &\equiv \alpha_2 \\
\beta_1 &\equiv \beta_2 \\
s_1^a &\equiv s_2^a \equiv 0
\end{align*}
\] (68)
where we have used the linear independence of \( k_1^a \) and \( k_2^a \) and the directions tangent to \( J \). Noting that \( \delta(k_1,k_2) = k_1 \delta k_2 = -\alpha_2(k_1,k_2) \) allows us to express \( \alpha_1 \) as the variation \( \alpha_1 = -\delta(\ln(|k_1,k_2|)) \), so that the corner term can be written as
\[
I_J = \mp \int_{J \Sigma} d^2x \sqrt{\sigma} \ln(|k_1,k_2|),
\] (69)
with the sign depending on whether \( J \) is concave or convex outward.

4.4. Corner terms for null-spacelike or null-timelike boundary

The null-spacelike join can only be spacelike, while the null-timelike join can be either spacelike or null. We will first consider the case when \( J \) is spacelike. If \( \Sigma_1 \) is null and \( \Sigma_2 \) non-null, the corner contributions to the spacelike join \( J \) come from equations (53) and (63) so that
\[
\delta I_J = \pm \frac{1}{2} \int_J d^2x \sqrt{\sigma} \alpha_1 - \frac{1}{2} \int_J d^2x \sqrt{\sigma} t^a m_a
\] (70)
where \( m^a \) is normal to \( J \) in \( \Sigma_2 \) and the \( \pm \) sign in front of the first term is positive or negative if it is an initial or final boundary, respectively, with respect to the outward directed normal to \( \Sigma_2 \). Again, we will see that \( J \) can be thought of as concave or convex outward, and this determines the sign of the first term, but also of the second term.

The variations of the metric on \( \Sigma_1 \) and \( \Sigma_2 \) are
\[
\begin{align*}
\delta g_{1}^{ab} &= k^a(\alpha_1 b^b + \beta b^b) + s_1^b \\
\delta g_{2}^{ab} &= n^a(\alpha_2 b^b + t^b).
\end{align*}
\] (71)
Decomposing \( t^a \in T \Sigma_2 t^a = r^a + s_2^a \), where \( s_2^a \in T J \) and \( r^a \) is transverse to \( T J \), and using the \( k^a, l^a \) basis we express \( n^a \) as \( n^a = u_1 k^a + v_1 l^a \), \( r^a = u_2 k^a + v_2 l^a \), where the normalisation \( n^a m_a = \epsilon \Rightarrow v_1 = \epsilon \frac{1}{2u_0} \). Using the matching condition equation (67) and the fact that \( v_1 \) can be arbitrary, we find that
\[
\begin{align*}
\alpha_2 &\equiv -\frac{v_2}{v_1} \equiv -2u_1 v_2 \\
\alpha_1 &\equiv v_1(\alpha_2 u_1 + u_2) \\
\beta_1 &\equiv u_1(\alpha_2 u_1 + u_2) \equiv 2u_1^2 \alpha_1 \\
s_1^a &\equiv s_2^a \equiv 0.
\end{align*}
\] (72)
Since the normal to a spacelike \( J \) in \( T \Sigma_2 \) is spacelike when \( \Sigma_2 \) is spacelike and timelike when \( \Sigma_2 \) is timelike, \( m^a m_a = -\epsilon \). Combining this with \( m^a m_a = 0 \), we can express \( m^a = \epsilon |u_1| k^a + \frac{1}{2 |m|} l^a \). Here we use the fact that since \( m^a \) is outward directed with respect to \( \Sigma_2 \), it is \( \epsilon \) times the sense of the outward directed \( k^a \) as shown in figures 3 and 4.
Thus, \[ \alpha = \mp \frac{\alpha_1}{u_1} \] where \( |u_1| = \pm 1 \) depending on the orientation of \( \Sigma_2 \) with respect to \( \Sigma_1 \). Specifically, \( u_1 = n.l \), the inner product of the transverse vectors (which determine the ‘outward’ directions) of \( \Sigma_1 \) and \( \Sigma_2 \). When \( u_1 < 0 \), \( J \) is an initial boundary with respect to the affine parameter \( \lambda \) on \( \Sigma_1 \), and when \( u_1 > 0 \), \( J \) is a final boundary. Thus, \( \int_{\delta J} \sigma \alpha = \pm \frac{\alpha_1}{u_1} \int_{\Sigma_1} d^2x \sqrt{\sigma} \) \( \alpha_1 \). (74)

Again, using \( \delta(n,k) = -\alpha_1(n,k) \Rightarrow \alpha_1 = -\delta(\ln(|n,k|)) \) we find that the corner term is

\[ I_J = \mp \int_{\delta J} d^2x \sqrt{\sigma} \ln(|n,k|). \]

Finally, let us consider the case when \( \Sigma_2 \) is timelike and \( J \) is null. An example of this is shown in figure 5.

As discussed in section 4.3 there is no contribution to a non-spatial corner from \( \Sigma_1 \), and the contribution from \( \Sigma_2 \) is given by equation (54). Moreover, the normal \( \mathbf{j}^\mu \) to \( \Sigma_1 \) coincides with that of \( \Sigma_4 \), i.e. \( \mathbf{j}_\ell = \mathbf{k}_a \). Choosing the spatial basis \( \{ \tilde{x}^a, \tilde{y}^a \} \) on \( \Sigma_4 \) such that \( \tilde{y}^a \) is in \( T \mathbf{J} \) and noting
that $n^\alpha k_\alpha = 0$, $n^a = w_1 \delta^a$, with $n.n = 1 \Rightarrow w_1^2 = 1$. If we expand $\gamma^a = u^a k^a + v^a l^a + w_1 \delta^a + z_2 \delta^a$, $n^\alpha t_\alpha = 0 \Rightarrow w_2 = 0$. The variations of the metric

$$\delta g_1^{\alpha \beta} = k^{(a} (\alpha l^b) + \beta k^b) + s_1^{b)}$$

$$\delta g_2^{\alpha \beta} = n^{(a} (\alpha m^b) + l^b).$$

(76)

Expanding $s_1^a = \gamma_1 \delta^a + \gamma_2 \delta^a$, the matching condition equation (67) implies that all the variables in the variation except for $\alpha_2$ and $\gamma_1$ which are related by $w_1 \alpha_2 = \gamma_1$, vanish on $J$. Since $t^a l_\alpha = -v_2 = 0$ there is no corner term contribution. This is consistent with the fact that the inner product of the normals $k.n = 0$ and that $\delta(n,k) = \alpha_2(k.n) = 0.$
4.5. Reparametrisation and the null boundary action

Combining all the boundary terms we find that

\[ I_B = \sum_i I_{K_i} - \sum_j I_{N_j} + \sum_k I_{\partial_k}, \]  

(77)

where \( i, j, k \) range over the number of non-null boundary components, the number of null boundary components and the number of corners, respectively. The null boundary term equation (60) is not invariant under reparametrisation. Let us consider the reparametrisation of the null vector

\[ \tilde{\kappa}^a = f(\lambda, x)k^a. \]  

(78)

where \( f \) is strictly positive and \( x \) is a local coordinate on the null generators. The surface gravity associated with \( \tilde{\kappa}^a \) then transforms as

\[ \tilde{\kappa} = f(\lambda, x)\kappa - \frac{df}{d\lambda}, \]  

(79)

so that

\[ I_\kappa = \int_{\Sigma} d^2x \sqrt{\sigma} (d\tilde{\kappa}) = \int_{\Sigma} d^2x \sqrt{\sigma} (d\lambda \kappa) - \int_{\Sigma} d^2x \sqrt{\sigma} \left( \frac{d\ln f(\lambda, x)}{d\lambda} \right) \]

\[ = I_\kappa \int_{\Sigma} d^2x \sqrt{\sigma} \ln f(\lambda, x) + \int_{\Sigma} d^2x \sqrt{\sigma} \ln f(\lambda, x) + \int_{\Sigma} d^2x d\lambda \frac{d\sqrt{\sigma}}{d\lambda} [\ln f(\lambda, x)]. \]  

(80)

The second and third terms exactly cancel the corner contribution (equations (69) and (75))

\[ \mp \int_{\partial} d^2x \sqrt{\sigma} \ln(|\tilde{k}|n) = \mp \int_{\partial} d^2x \sqrt{\sigma} \ln(|k|n) \mp \int_{\partial} d^2x \sqrt{\sigma} \ln f(\lambda, x). \]  

(81)
which is negative or positive depending on whether $\lambda|_{J}$ is an final or initial value. Here $n^a$ represents the normal to the ‘other’ surface at the join $\mathcal{J}$ which can be either null or non-null. The presence of the last term in equation (80), which can be rewritten as

$$\Delta I_B = \int_{\Sigma} d^2 x \sqrt{\sigma} \Theta(\lambda, x) d\lambda \ln f(\lambda, x)$$

(82)

shows that the boundary action is not invariant under reparametrisation. Let us now interpret this. Let us note first that under allowed variations, (those that hold the boundary geometry fixed) the variation of $\Delta I_B$ vanishes, since it depends only on the boundary geometry. Thus the variation of the boundary action is reparametrisation invariant although the action itself is not. As a general rule, it is differences in the action that are important. Presumably, we can assume this to be true in quantum gravity as well.

Recall the discussion in section 3.1, where we noted that the boundary action is not gauge invariant under general gauge transformations although its variation is. This is exactly what is happening here. The surface gravity $\kappa$ is a component of a connection and (as seen in equation (79)) transforms inhomogeneously under gauge transformations. Reparametrisation changes the ‘size’ of the null normal $k$ and is therefore not in the little group. The behaviour of the boundary action under reparametrisation is an example of the general phenomenon discussed there.

We also clarify that this lack of reparametrisation invariance of the null boundary action does not result in any arbitrariness in physical quantities. This is because what appears in physical quantities is the difference of two connections, which is a gauge covariant quantity. This point is explained further in the conclusion.

If one wishes, one could add ‘counter terms’ to the boundary action to render it reparametrisation invariant. For example

$$- \int_{\Sigma} d^2 x \sqrt{\sigma} d\lambda [\Theta(\lambda, x) \ln \frac{d\lambda}{dt}],$$

(83)

with $t$ being an arbitrary affine parameter, does the job. Another possibility [1] is

$$- \int_{\Sigma} d^2 x \sqrt{\sigma} d\lambda [\Theta \ln \Theta].$$

(84)

A third possibility is

$$-1/2 \int_{\Sigma} d^2 x \sqrt{\sigma} d\lambda [\Theta \ln s_{ab}],$$

(85)

where $s_{ab}$ is the shear tensor of the null geodesic congruence ruling the null surface. Of these, the first equation (83) depends on a choice of affine parametrisation, which brings in some arbitrariness, since the parameter $t$ can be rescaled by $t \rightarrow c(x) t$, where $c(x)$ depends on the null generator. A more serious problem is that including this counterterm spoils the additivity of the action for regions separated by a null boundary. For, the notion of an ‘affine’ parameter in general will depend on which region we use to define the affine parameter. The two counter terms will therefore differ in value and therefore spoil the additivity of the action, which was one of our prime concerns.

The second and third equations (84) and (85) do not suffer from this ambiguity. However, they too have a problem: the counterterm is not differentiable if the expansion or shear vanishes. Our view is that there no real need to add a counterterm at all since the lack of reparametrisation invariance does not manifest itself in physical quantities.
5. Conclusion

The main new advance of this paper is the realisation that the tetrad formulation of Einstein’s theory permits a unified approach to boundaries of all signatures. The calculations are considerably simplified and the use of differential forms permits us to integrate over boundary manifolds regardless of their signature. Our derivation of the corner terms too is extremely simple. Our methods are complementary to [1, 11, 12] and our perspective is somewhat different. The differential form version of the boundary term also makes it obvious that the boundary corrected action is additive. In any splitting of a spacetime into pieces, the boundary term $I_B$ equation (13) appears twice on the shared boundary with opposite orientation and so cancel out. The gauge non invariance of the boundary action does not affect us here since the difference of the two connections is a gauge covariant object. In particular, the reparametrisation non invariance of the null boundary action does not spoil the additivity of the action.

In this paper we have worked within the Dirichlet formalism for gravity in which the pull-back metric $q_{ab}$ is held fixed on the boundary during the variation. One can also conceive of ‘Neuman gravity’ in which the conjugate variable is held fixed. For example if the boundary is spacelike, the quantity $\sqrt{q}(K^{ab} - 1/2Kq^{ab})$ related to the extrinsic curvature is conjugate to the three-metric. There has been recent work [19] exploring this possibility, albeit in the Euclidean context. Such alternate formalisms are of interest since it is far from clear which ensemble would prove the most advantageous in quantisation. It is also possible that these different choices may lead to different quantum theories. For example, it is known in statistical mechanics that conjugate ensembles may not always be equivalent. Such issues are particularly acute in the case of long range forces like gravity. A classic example is the stability question of a black hole in equilibrium with thermal radiation in a box.

A notable feature of the boundary term equation (13) is that it is not gauge invariant although its variation is. One must bear in mind that the boundary action is only determined up to a functional of the boundary data that is held fixed, in our case the pullback of the metric to the boundary. One may worry that the value of the action changes under change of gauge. However, there is no cause for concern. In a path integral formulation observable quantities are related to the absolute value squared of the Feynman amplitude in equation (1). This leads to a closed time path integral of the Schwinger–Keldysh formalism. The quantity that appears in the exponent is now $\mathcal{S}(X_3, \Gamma) - \mathcal{S}(X_3, \Gamma')$, where $\Gamma'$ and $\Gamma'$ are histories going between $X_1$ and $X_3$. While the two histories share the same final geometry $X_3$, they have different values of the connection at the final point. The two boundary terms at $X_3$ then combine to give a gauge invariant answer, since the difference of two connections transforms homogenously. Another situation that arises is when one considers asymptotically flat spacetimes, takes the boundary to infinity and interprets the boundary term in terms of the total mass. In this case as is well known, we need to make a background subtraction in order to get a finite answer. Once again, this subtraction results in a gauge invariant boundary term, since the difference of two connections is a gauge covariant object. The gauge non invariance of the boundary term is precisely what we have exploited in order to identify the corner terms. This remark has a parallel in the metric formulation too. The integrand in the boundary term equation (44) is also not coordinate invariant since it depends on the affine connection. The general allowed variation of the metric equation (36) can (at points of the boundary) be interpreted as a diffeomorphism generated by the vector field $\xi = \phi Q_{\alpha}$, where $\phi$ is any function that vanishes on the boundary. Under such a diffeomorphism, the integrand in the boundary term changes by a total derivative and this permits us to identify the corner terms in the metric formulation.
In the literature, it is suggested that the corner terms [15] or their close analogs [13] may pick up imaginary contributions. (Imaginary contributions figure heavily in the Lorentzian Gauss–Bonnet theorem as well.) Using our methods, such contributions would not be detected, as they have zero variation. However, the origin of such terms can be understood when the normal changes from timelike to spacelike. We have chosen different adapted frames depending on whether the normal to the boundary is null, spacelike or timelike. This is because no Lorentz transformation can connect these different normals. However, in connecting spacelike normals to timelike normals, it is possible to use complex Lorentz transformations. If we complexify the Lorentz group to $O(2, \mathbb{C})$, the element $\Lambda = \begin{pmatrix} \cosh(\eta + i\pi/2) & \sinh(\eta + i\pi/2) \\ \sinh(\eta + i\pi/2) & \cosh(\eta + i\pi/2) \end{pmatrix}$ which has complex rapidity, $\eta + i\pi/2$ does the job of connecting spacelike and timelike normals. Thus every time the normal crosses a null direction, (crossing counted with sign), the action picks up an imaginary contribution $i\pi/2 \int dA$. This imaginary area term has been interpreted as black hole entropy by Neiman and we refer the reader to [15] for a fuller discussion. While such a term affects the value of the Action, it does not affect the variation, since the variation of the area vanishes. Note however, that no Lorentz transformation (real or complex) can relate a null normal to a spacelike or timelike one. It seems necessary to use different canonical forms for null and non-null normals.

The case of null boundaries has not received much attention till the recent works of Neimann [15–18], Parattu et al [11, 12] and Lehner et al [1]. Neimann was mainly interested in imaginary contributions to the action at the join of null boundaries. He used affine parametrisations to describe the null generators, which is unnecessarily restrictive in the present context. The treatment of Parattu et al [11] allows for arbitrary parametrisation of the null generators and correctly identifies the form of the boundary action for null surfaces. However, these authors do not consider the corner terms, which are necessary for a complete treatment of the boundary action. In a second paper [12], they attempt a unified description of both the null and non null case. Their treatment is coordinate bound and makes assumptions about the behaviour of the normal away from the boundary. Lehner et al [1] provide a metric treatment of the null boundary terms and identify the corner terms. They also have a detailed discussion of reparameterisation invariance and suggest counterterms to be added to the boundary action.

In the present work, we use the power of Cartan’s tetrad formulation and differential forms to considerably simplify the treatment. Differential forms give us a unified approach to boundaries of all signatures. We compute the corner terms quite simply using the local Lorentz invariance of the tetrad formalism. In the mathematical section we also give a classification of all possible corner signatures, including the case of null joins (see figure 5) that have not been considered in the above works. In order to reach a wider audience we also translate our results into the metric language which is more familiar to readers. We have also noted the contribution which come from ‘creases’ that appear in spacetimes with a dynamically evolving black hole exterior. Finally, we offer a perspective on reparameterisation invariance (RI) in the null case, which differs slightly from [1]. Rather than try to restore RI, we note that the lack of RI in the boundary action does not affect any physical quantity in the path integral.

We close with a remark regarding the asymptotics of gravitational fields. Let us compare the value of the action in the second order Einstein–Hilbert form and the first order form. For asymptotically flat spacetimes, the metric tends to its flat asymptotic form $g_0$ at the rate
\( (g - g_0) = O(1/r) \). As a result, the difference between the connection \( \Gamma \) of \( g \) and the flat connection \( \Gamma_0 \), \( \Delta \Gamma = \Gamma - \Gamma_0 \) goes as \( \Delta \Gamma = O(r^{-1/2}) \) and \( R = O(1/r^3) \). The Einstein Hilbert form diverges logarithmically at radial infinity \( (\int Rr^2dr \approx \int dr/r^2) \) but the first order form converges: \( \int (\Delta \Gamma)^2r^2dr \approx \int dr/r^2 \). This allows an interpretation of the 4-momentum as a well defined variation of the action, i.e. as a Noether charge. While there has been much work on null infinity [20], we are not aware of any discussion of boundary counterterms in this context, for example, in the derivation of the Bondi mass. The issue of null boundaries has been neglected until the recent interest generated by [11, 12]. There has been recent work [21, 22] reviving the topic of asymptotic null infinity [23–25] and relating it to soft theorems in particle physics. We hope that our treatment of null boundaries may help understand null asymptotics of gravitational fields.

Acknowledgment

SS was supported in part under an agreement with Theiss Research and funded by a grant from the FQXI Fund on the basis of proposal FQXi-RFP3-1346 to the Foundational Questions Institute. IJ is supported by the EPSRC. RDS’s research was supported in part by NSERC through grant RGPIN-418709-2012. This research was supported in part by Perimeter Institute for Theoretical Physics. Research at Perimeter Institute is supported by the Government of Canada through Industry Canada and by the Province of Ontario through the Ministry of Economic Development and Innovation.

We gratefully acknowledge an email correspondence with Yang Run Qiu, which led us to improve the paper.

References

[1] Lehner L, Myers R C, Poisson E and Sorkin R D 2016 Phys. Rev. D 94 084046
[2] Einstein A 1918 Physica Z 19 115
[3] York J W 1972 Role of conformal three-geometry in the dynamics of gravitation Phys. Rev. Lett. 28 1082–5
[4] Gibbons G W and Hawking S W 1977 Action integrals and partition functions in quantum gravity Phys. Rev. D 15 2752–6
[5] Wald R M 1984 General Relativity 1st edn (Chicago, IL: University of Chicago Press)
[6] Poisson E 2004 A Relativist’s Toolkit: The Mathematics of Black-Hole Mechanics (Cambridge: Cambridge University Press)
[7] Sorkin R 1975 Phys. Rev. D 12 385
Sorkin R 1981 Phys. Rev. D 23 565 (erratum)
[8] Hartle J B and Sorkin R 1981 Gen. Relativ. Gravit. 13 541
[9] Hayward G 1993 Gravitational action for space-times with nonsmooth boundaries Phys. Rev. D 47 3275–80
[10] Brill D and Hayward G 1994 Is the gravitational action additive? Phys. Rev. D 50 4914–9
[11] Parattu K, Chakraborty S, Majhi B R and Padmanabhan T 2016 A boundary term for the gravitational action with null boundaries Gen. Relativ. Gravit. 48 94
[12] Parattu K, Chakraborty S and Padmanabhan T 2016 Variational principle for gravity with null and non-null boundaries: a unified boundary counter-term Eur. Phys. J. C 76 129
[13] Louko J and Sorkin R D 1997 Class. Quantum Grav. 14 179
[14] Epp R J 1995 arXiv:gr-qc/9511060
[15] Neiman Y 2012 On-shell actions with lightlike boundary data (arXiv:1212.2922 [hep-th])
[16] Neiman Y 2013 The imaginary part of the gravity action and black hole entropy J. High Energy Phys. JHEP04(2013)071
[17] Neiman Y 2013 Action and entanglement in gravity and field theory *Phys. Rev. Lett.* **111** 261302
[18] Neiman Y 2013 Imaginary part of the gravitational action at asymptotic boundaries and horizons *Phys. Rev. D* **88** 024037
[19] Krishnan C and Raju A 2016 arXiv:1605.01603 [hep-th]
[20] Bondi H, van der Burg M G J and Metzner A W K 1962 *Proc. R. Soc. A* **269** 21
[21] Cachazo F and Strominger A 2014 arXiv:1404.4091 [hep-th]
[22] Ashtekar A 2015 *Surveys in Differential Geometry* vol 20 ed L Bieri and S-T Yau (Boston, MA: International Press) arXiv:1409.1800 [gr-qc]
[23] Ashtekar A 1981 *J. Math. Phys.* **22** 2885
[24] Ashtekar A 1981 *Phys. Rev. Lett.* **46** 573
[25] Ashtekar A and Streubel M 1981 *Proc. R. Soc. A* **376** 585