Noisy quantum Monty Hall game

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Abstract
The influence of spontaneous emission channel and generalized Pauli channel on quantum Monty Hall Game is analysed. The scheme of Flitney and Abbott is reformulated using the formalism of density matrices. Optimal classical strategies for given quantum strategies are found. The whole presented scheme illustrates how quantum noise may change the odds of a zero-sum game.

1 Introduction

Game theory studies decision making for a given set of rules, in order to select a strategy to maximize one’s pay-off. This theory is widely used in economics, biology, sociology and sometimes in politics. Quantum game theory, the subclass of game theory that involves quantum phenomena, lies at the crossroads of physics, quantum information processing, computer and natural sciences [1, 2].

The Monty Hall game has its roots in the popular TV show Let’s Make a Deal and it was often the source of misunderstanding. There are several quantisations of this game [3, 4, 5, 6]. We follow the scheme by Flitney and Abbott and reformulate it in the language of the density matrices in order to study the influence of noise on the game behaviour.

Problem of noise in quantum games was also studied in [7, 8, 9, 10, 11, 12, 13, 14].

This paper is organised as follows. In Section 2 the classical Monty Hall problem is introduced. In Section 3 Flitney and Abbott’s scheme of quantisation of Monty Hall game is presented. In Section 4 the noise model which is applied to the game is discussed. At last in Section 5 the results and their discussion is presented.

2 Classical Monty Hall game

In the classical scheme Monty Hall game runs as follows. There are two players: Alice and Bob. Bob’s goal is to get the prize and Alice plays the role of banker.

There are three boxes of which only one contains the prize. The game consists of successive steps:
1. Alice randomly chooses one box and hides the prize in it.
2. Bob chooses one of the boxes according to his will.
3. Alice opens one of the boxes which does not contain the prize.
4. Bob now have an option to keep his choice or to switch and chose the other closed box.
5. Alice opens the box chosen by Bob.

Bob wins if the prize is in the box he have chosen. Otherwise he loses.

The game is appealing and thought-provoking because Bob’s optimal strategy differs from intuition. To achieve higher probability of winning, Bob should switch the box fourth step of the game. Explanation of this fact is very simple. There are two possible cases in step two: Bob chooses the box with the prize inside (with probability \( \frac{1}{3} \)) or Bob chooses the box without a prize inside (with probability \( \frac{2}{3} \)). Then in third step, when Alice opens one of the boxes, Bob switches. Hence in the first case he will lose, but in the second case he will always switch to the box containing the prize. Therefore switching strategy yields to expected pay-off of \( \frac{2}{3} \) and not switching strategy only to \( \frac{1}{3} \).

### 3 Quantum Monty Hall game

Flitney and Abbott presented in [4] following quantisation of this game. Alice’s and Bob’s choices are represented by qutrits and the game starts in some initial state which will be specified further. Players’ strategies are represented by operators acting on their respective qutrits. Third qutrit represents the box opened by Alice.

The state of the system may be expressed by the normalised state vector

\[
|\Psi\rangle = |a\rangle \otimes |b\rangle \otimes |o\rangle \in \mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3,
\]

where \( a \) is Alice’s choice, \( b \) is Bob’s choice and \( o \) represents the box that has been opened.

The operator for opening of the box is defined as

\[
O = \sum_{i,j,k,l=0}^2 |\epsilon_{i,l,k}| |njk\rangle \langle ljk| + \sum_{j=0}^2 |mjj\rangle \langle jll|,
\]

and the operator for switching the box as

\[
S = \sum_{i,j,k,l=0}^2 |\epsilon_{i,l,k}| |ijk\rangle \langle ljk| + \sum_{ij=0}^2 |ijj\rangle \langle iij|,
\]

where \( |\epsilon_{i,l,k}| = 1 \Leftrightarrow i \neq l \land l \neq k \land i \neq k \), otherwise \( |\epsilon_{i,l,k}| = 0 \), \( n = (i + l) \mod 3 \) and \( m = (j + l + 1) \mod 3 \).

Bob’s not switching operator \( N \) is represented by the identity matrix acting on the state of three qutrits.

Alice and Bob are restricted to unitary transformations on their qutrits. \( A, B \in SU(3) \) are players’ movements.
The unitary operator that implements this game is given by relation

$$G_s = S \cdot O \cdot (I \otimes B \otimes A),$$

if Bob chooses to switch or by relation

$$G_n = N \cdot O \cdot (I \otimes B \otimes A),$$

if Bob chooses not to switch the box.

The final state of the game is $$\rho = G \rho G^\dagger$$, where $$x \in \{s, n\}$$ indicates Bob’s strategy and $$\rho_i$$ is the initial state of the game.

One may consider Bob’s classic probabilistic strategies. Bob controls a free parameter $$\gamma \in [0, \pi/2]$$, which represents the mixing of switching and not switching strategies. Pure switching strategy is obtained for $$\gamma = \pi/2$$ and pure not switching strategy is obtained for $$\gamma = 0$$.

Bob wins if he picks the correct box, hence expectation value of his pay-off is given by the equation

$$\langle B \rangle = 2 \sum_{ij=0}^2 \text{Tr} (|ijj\rangle\langle ijj| ( \cos^2 \gamma \rho_s + \sin^2 \gamma \rho_n )).$$

(6)

Flitney and Abbott considered two initial states, one separable

$$|\Psi_1\rangle = |0\rangle \otimes \frac{1}{\sqrt{3}}(|0\rangle + |1\rangle + |2\rangle) \otimes \frac{1}{\sqrt{3}}(|0\rangle + |1\rangle + |2\rangle),$$

and one having qutrits of Alice and Bob entangled

$$|\Psi_2\rangle = |0\rangle \otimes \frac{1}{\sqrt{3}}(|00\rangle + |11\rangle + |22\rangle).$$

(8)

The goal of the this work is to analyse the influence of noise on the game outcome. Therefore we assume, that the game is not played on the pure state but on a mixed state, that underwent non-unitary evolution trough a noisy channel.

The initial state of the game is given by $$\rho_i = \Phi(|\psi_i\rangle\langle \psi_i|)$$, where $$\Phi(\cdot)$$ denotes the quantum noisy channel.

4 Noise model

In this case, quantum Monty Hall game is implemented on qutrits, i.e. three level quantum states. One can imagine that such a game could be implemented in real physical system. One of suitable systems is 3-levels quantum state implemented on some atom. We model the noise in such system by local spontaneous emission channel parametrised by single real parameter $$t \in [0, \infty)$$. This parameter can be understood as time.

4.1 Spontaneous emission channel

Following [17] we chose an atom with so called V-configuration in which the allowed spontaneous transitions are: $$|2\rangle \rightarrow |0\rangle$$ and $$|1\rangle \rightarrow |0\rangle$$. In this analysis
we assume that each atom (qutrit) decoheres independently by the spontaneous emission. This dissipative process is characterised by two Einstein coefficients $A_1$, $A_2$, describing the irreversible depopulation from exited states $|2\rangle$ and $|1\rangle$.

For simplicity the following calculations are conducted with $A_1 = A_2 = 1$.

The following set of Kraus operators represents this channel:

$$K_0 = \begin{bmatrix} 1 & 0 & 0 \\
0 & e^{-\frac{A_1}{2}} & 0 \\
0 & 0 & e^{-\frac{A_2}{2}} \end{bmatrix}, K_1 = \sqrt{1-e^{-tA_1}} \begin{bmatrix} 0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \end{bmatrix}, K_2 = \sqrt{1-e^{-tA_2}} \begin{bmatrix} 0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0 \end{bmatrix}.$$ 

Hence action of the channel on one qutrit is

$$\Phi_{SE}(\rho) = \sum_{i=0}^{2} K_i \rho K_i^\dagger. \quad (9)$$

The extended channel acting on all three qutrits is obtained by applying the following formula

$$\Phi(\rho) = \sum_{i_1,i_2,i_3=1}^{3} K_{i_1} \otimes K_{i_2} \otimes K_{i_3} \rho K_{i_1}^\dagger \otimes K_{i_2}^\dagger \otimes K_{i_3}^\dagger. \quad (10)$$

### 4.2 Generalized Pauli channel

Generalized Pauli channel is an extension of the Pauli channel to any dimension [18]. In order to apply the generalized Pauli channel on qutrits, one defines two unitary operators:

$$X = \begin{bmatrix} 0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0 \end{bmatrix}, \quad Z = \begin{bmatrix} 1 & 0 & 0 \\
0 & e^{\frac{2}{3}i\pi} & 0 \\
0 & 0 & e^{\frac{4}{3}i\pi} \end{bmatrix}. \quad (11)$$

One can create a family of generalized Pauli channels with the help of Kraus operators, in the following way:

$$K_{i,j} = \bigcup_{i,j=1}^{2} \left\{ \sqrt{P_{i,j}}X^iY^j \right\}. \quad (12)$$

By putting $P_{0,0} = 1 - \frac{8}{9}p$ and $P_{i,j} = \frac{1}{9}p$ for $(i,j) \neq (0,0)$ we obtain one parameter family of noisy channels. The parameter $p \in [0,1]$ can be understood as probability of error occurrence. Note that for $p = 1$ action of the channel transforms any state to maximally mixed state $I/3$.

The extension of this channel acting on three qutrits is defined in analogy to the formula (10).

### 5 Results

Flitney and Abbott considered two combinations of Bob’s quantum strategies. One given by matrices:

$$M_1 = \begin{bmatrix} 0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 0 & 0 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0 \end{bmatrix}.$$
Figure 1: Graphical representation of the course of the game.

Together with Alice’s strategy \(I\) and the initial state \(|\Psi_1\rangle\) (eq. 7), the second with Bob’s strategy \(I\) and the initial state \(|\Psi_2\rangle\) (eq. 8) and Alice’s strategy

\[
H = \begin{pmatrix}
    \frac{1}{\sqrt{2}} & \frac{1}{2} + \frac{3-i\sqrt{7}}{4\sqrt{2}} & \frac{1}{2} + \frac{1+i\sqrt{7}}{4\sqrt{2}} \\
    -\frac{1}{2} & \frac{1}{8}(-3 + i\sqrt{7}) & \frac{1}{8}(5 + i\sqrt{7}) \\
    \frac{1}{\sqrt{2}} & \frac{1}{8}(3 + i\sqrt{7}) & \frac{1}{8}(5 + i\sqrt{7})
\end{pmatrix}.
\]

It should be noted that strategies \(M_1\) and \(M_2\) correspond to a shuffling of Bob’s choices amongst the three boxes.

We extend this analysis by sending initial pure states through quantum noisy channel. Bob’s expected pay-off \(\langle B \rangle\) as the function of noise parameters \(t\) or \(p\) and switching parameter \(\gamma\) are shown in Fig. 2 and Fig. 3. We have investigated the above combinations of strategies in presence of quantum noise modelled by spontaneous emission channel and generalized Pauli channel.

**Spontaneous emission channel** Spontaneous emission channel defined by equations 9 and 10 models the behaviour of three-level decaying atom. Hence this model represents a very likely physical scenario. Note that as \(t \to \infty\) this channel transforms any state into ground state \(|000\rangle\). Below we analyse in details four cases.

**Case 1** In the case of the separable state \(|\Psi_1\rangle\) and trivial quantum strategies: \(A = B = I\) Bob’s expected pay-off is given by the formula

\[
\langle B \rangle = \frac{1}{6} e^{-2t} \left( 3e^{2t} + (-4 + 8e^t - 3e^{2t}) \cos(2\gamma) \right).
\]

This case is presented in Fig. 2(a) One can see that in this case Bob should switch the box if the noise parameter \(t < \ln(2)\). In the limit \(t \to \infty\), \(\langle B \rangle \to \sin^2(\gamma)\) so Bob’s maximal pay-off is 1 for \(t = \infty\) and \(\gamma = \pi/2\). His minimal pay-off is 0 for \(t = \infty\) and \(\gamma = 0\).

**Case 2** In the case of the separable state \(|\Psi_1\rangle\) and following quantum strategies: \(A = I, B = M_1\) or \(M_2\), Bob’s expected pay-off is given by the formula

\[
\langle B \rangle = \frac{1}{6} e^{-2t} \left( 3e^{2t} + (2 - 4e^t + 3e^{2t}) \cos(2\gamma) \right).
\]
This case is presented in Fig. 2(b). One can see that in this case Bob should not switch the box. In the limit $t \to \infty$, $\langle B \rangle \to \cos^2(\gamma)$ so Bob’s maximal pay-off is 1 for $t = \infty$ and $\gamma = 0$. His minimal pay-off is 0 for $t = \infty$ and $\gamma = \pi/2$.

**Case 3** In the case of the entangled state $|\Psi_2\rangle$, and trivial quantum strategies $A = B = I$, Bob’s pay-off is given by the formula

$$\langle B \rangle = \frac{1}{6} e^{-2t} \left( 3e^{2t} + (-8 + 8e^t - 3e^{2t}) \cos(2\gamma) \right).$$

This case is presented in Fig. 2(c). One can see that in this case Bob should always switch the box. In the limit $t \to \infty$, $\langle B \rangle \to \sin^2(\gamma)$ so Bob’s maximal pay-off is 1 for $t = \infty$ and $\gamma = \pi/2$. His minimal pay-off is 0 for $t = \infty$ and $\gamma = 0$.

**Case 4** In the case of the entangled state $|\Psi_2\rangle$, and following quantum strategies $A = H$, $B = I$, Bob’s expected pay-off is given by the formula

$$\langle B \rangle = \frac{1}{6} e^{-2t} \left( 3e^{2t} + 2 (-1 + e^t) \cos(2\gamma) \right).$$

This case is presented in Fig. 2(d). One can see that in this case Bob should not switch the box. In the limit $t \to \infty$, $\langle B \rangle \to 1/2$. Bob’s maximal pay-off is $7/12$ for $t = \ln(2)$ and $\gamma = 0$. His minimal pay-off is $5/12$ for $t = \ln(2)$ and $\gamma = \pi/2$.

**Generalized Pauli channel** This family of bi-stochastic channels is interesting from quantum information point of view. It applies random unitary rotations on the quantum state.

**Case 5** In the case of the separable state $|\Psi_1\rangle$ and quantum strategies: $A = I = I$ or $M_1$ or $M_2$, Bob’s expected pay-off is given by the formula

$$\langle B \rangle = \frac{1}{6} (1 - p) \cos(2\gamma) + 3 - p.$$  

This case is presented in Fig. 3(a). One can see that in this case Bob should switch the box if $p < 1$. Bob’s maximal pay-off is $2/3$ for $p = 0$ and $\gamma = 0$. His minimal pay-off is $1/3$ for $p = 1$ and any $\gamma$.

**Case 6** In the case of the entangled state $|\Psi_2\rangle$ and quantum strategies: $A = B = I$, Bob’s expected pay-off is given by the formula

$$\langle B \rangle = \frac{1}{6} \left( 2p^3 - 4p^2 + (2p^3 - 8p^2 + 9p - 3) \cos(2\gamma) + p + 3 \right)$$

This case is presented in Fig. 3(b). One can see that Bob should switch the box if $p > 3/2 - \sqrt{3}/2$. Bob’s maximal pay-off is 1 for $p = 0$ and $\gamma = \pi/2$. His minimal pay-off is $1/3$ for $p = 1$ and any $\gamma$. 
Case 7 In the case of the entangled state $|\Psi_2\rangle$ and quantum strategies: $A = H$, $B = I$, Bob's expected pay-off is given by the formula
\[
\langle B \rangle = \frac{1}{12} \left( p^3 + (p^2 - 4p + 3) p \cos(2\gamma) - 2p^2 - p + 6 \right)
\]
(19)
This case is presented in Fig. 3(c). One can see that Bob should switch the box if $p < 1$. Bob's maximal pay-off is $1/27 \left( 9 + 2\sqrt{6} \right)$, for $p = 1 - \sqrt{2}/3$ and $\gamma = 0$. His minimal pay-off is $1/3$ for $p = 1$ and any $\gamma$.

Conclusions We have studied two noise models, that are likely to occur in the physical implementation of the quantum version of the Monty Hall game.

The calculation have shown that existence of noise heavily influences the expectation value of the quantum Monty Hall game. We can observe different behaviour of the game for different strategies, initial states and quantum channels.

In the case of spontaneous emission channel the initial state of the game approaches the ground state $|000\rangle$ as noise parameter goes to infinity. Therefore for large amounts of noise the game is played (quantum strategies are applied) on the state $|000\rangle$ rather than on states $|\Psi_1\rangle$ or $|\Psi_2\rangle$. When we compare the asymptotic behaviour of the pay-off functions in the cases 1 and 3 we see that they converge to the same limit $\sin^2 \gamma$. The difference between those cases is that in case 1 Bob should switch the gate only if noise parameter is larger than $\ln(2)$.

It should be noted that for $t = \infty$ Bob knows exactly what is the initial state of the game — where the prize is hidden. Therefore given Alice will not act (her strategy is to apply an identity), Bob can always win and the game becomes unfair. This can be observed in cases 1, 2 and 3.

Generalized Pauli channel transforms any input state towards the maximally mixed state. Therefore for maximal value of noise parameter any correlations are lost. It can be easily seen that if initial state of this game is maximally mixed then, Bob's pay-off is always equal to $1/3$ and is independent of Alice's and Bob's strategies. The noise can influence the outcomes of the game in a non-trivial way. In example in case 6 Bob should change his strategy, to switching the box, when noise parameter is larger than $3/2 - \sqrt{3}/2$.

Obtained results show, that if Bob knows the properties of the noise in the system implementing quantum Monty Hall game he can use this knowledge to change his strategy in order to maximize his pay-off. In some cases the noise can be more influential than strategies and therefore can impede successful implementation of quantum games. More careful studies, that take into account imperfections of the physical device realizing this game, would be needed if a proposition of concrete physical implementation of this game will appear.

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Figure 2: The behaviour on quantum Monty Hall game under the influence of spontaneous emission channel. Bob’s expected pay-off $\langle B \rangle$ is plotted as the function of noise parameter $t$ and switching parameter $\gamma$. The colours vary from light (maximal possible pay-off) to dark (minimal pay-off).
Figure 3: The behaviour on quantum Monty Hall game under the influence of generalized Pauli channel. Bob’s expected pay-off $\langle S_B \rangle$ is plotted as the function of noise parameter $p$ and switching parameter $\gamma$. The colours vary from light (maximal possible pay-off) to dark (minimal pay-off).
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