Central configurations, Morse and fixed point indices

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Abstract

We compute the fixed point index of non-degenerate central configurations for the $n$-body problem in the euclidean space of dimension $d$, relating it to the Morse index of the gravitational potential function $\bar{U}$ induced on the manifold of all maximal $O(d)$-orbits. In order to do so, we analyze the geometry of maximal orbit type manifolds, and compute Morse indices with respect to the mass-metric bilinear form on configuration spaces.

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1 Introduction: central configurations as critical points

Let $E = \mathbb{R}^d$ be the $d$-dimensional euclidean space, for $d \geq 1$. Fix an integer $n \geq 2$. The configuration space of $n$ (colored) points in $E$ is the set of all $n$-tuples of distinct points in $E$, and denoted by $\mathbb{F}_n(E)$:

$$\mathbb{F}_n(E) = \{ \mathbf{q} \in E^n : i \neq j \implies q_i \neq q_j \} = E^n \setminus \Delta,$$

where if $\mathbf{q} \in E^n$, its $n$ components are denoted by $\mathbf{q}_j$, $j = 1, \ldots, n$; points in $\mathbb{F}_n(E)$ are termed configurations of $n$ points in $E$; its complement in $E^n$ is the set of collisions

$$\Delta = \{ \mathbf{q} \in E^n : \exists (i, j), i \neq j : q_i = q_j \} = \bigcup_{1 \leq i < j \leq n} \{ \mathbf{q} \in E^n : q_i = q_j \}.$$
For \( j = 1, \ldots, n \) let \( m_j > 0 \) be fixed parameters (that can be interpreted as the mass of the \( j \)-th particle in \( E \)), under the normalization condition
\[
\sum_{j=1}^{n} m_j = 1 .
\]

If \( v, w \) are vectors in (the tangent space of) \( E^n \), then let
\[
\langle v, w \rangle_M = \sum_{j=1}^{n} m_j v_j \cdot w_j
\]
denote the mass scalar product of \( v \) and \( w \), where \( v_j \cdot w_j \) is the standard euclidean scalar product (in \( E \)) of the \( j \)-th components of \( v \) and \( w \). The unit sphere in \( \mathbb{F}_n(E) \) is termed the inertia ellipsoid and denoted by \( S_n(E) = \left\{ q \in \mathbb{F}_n(E) : \|q\|^2_M = 1 \right\} \).

It is equal to the unit sphere/ellipsoid in \( E^n \), with collisions removed, \( S_n(E) = S_n(E) \setminus \Delta \). The unit sphere/ellipsoid in \( E^n \) is denoted by \( S_n(E) = \left\{ q \in E^n : \|q\|^2_M = 1 \right\} \). To simplify notation, if possible we will use the short forms \( S \) and \( S \) instead of \( S_n(E) \) and \( S_n(E) \).

The potential function \( U : \mathbb{F}_n(E) \to \mathbb{R} \) is simply defined as
\[
\sum_{1 \leq i < j \leq n} \frac{m_i m_j}{|q_i - q_j|^\alpha},
\]
given a fixed parameter \( \alpha > 0 \). For \( \alpha = 1 \), \( U \) is the gravitational potential. It is invariant under the full group of isometries of \( E \), acting diagonally on \( \mathbb{F}_n(E) \).

Let \( D = \nabla \) denote the covariant derivative (which is the Levi-Civita connection with respect to the mass-metric) in \( \mathbb{F}_n(E) \), which is again the standard derivative. If \( F : \mathbb{F}_n(E) \to E \) is a smooth function, then \( DF = dF \) is the differential of \( F \), which is a section of the cotangent bundle \( T^*\mathbb{F}_n(E) \) defined as \( DF[v] = D_v F \) for each vector field \( v \) on \( \mathbb{F}_n(E) \). If \( v \) and \( w \) are two vector fields on \( \mathbb{F}_n(E) \), then \( D_v w \) is the (Euclidean and covariant) derivative of \( w \) in the direction of \( v \).

Let \( \nabla^S \) denote the covariant derivative (Levi-Civita connection) on \( S \), induced by the mass-metric of \( \mathbb{F}_n(E) \) restricted to \( S \), i.e. the restriction to \( S \) of the Riemannian structure of \( \mathbb{F}_n(E) \). If \( v \) and \( w \) are two vector fields defined in a neighborhood of \( S \), then the covariant derivative \( \nabla^S_v w \) is equal, at \( x \in S \), to the orthogonal projection of \( D_v w \), projected orthogonally to the tangent space \( T_x S \) (cf. proposition 3.1 at page 11 of [7], or proposition 2).
1.2 at page 371 of [8]). The same holds with $S \subset S$ instead of $S$. If $\Pi$ denote the projection $T\mathbb{F}_n(E) \mapsto T\mathbb{S}$, then $\nabla^S_v w = \Pi D_v w$.

If $F: \mathbb{F}_n(E) \to \mathbb{R}$ is a smooth function, and $f = F|_S$ is its restriction to $S$, then $\nabla^S f = df$ is the restriction of $dF$ to the tangent bundle $T\mathbb{S}$. Let $\text{grad}(f) = df^\sharp$ and $\text{grad}(F) = dF^\sharp$ denote the gradients of $f$ and $F$ respectively (i.e., the images of the differentials under the musical isomorphisms induced by the mass-metric). For each $x \in S$, $df^\sharp(x) \in T_xS$ and $dF^\sharp(x) \in T_x\mathbb{F}_n(E)$ satisfy the equations

$$\langle df^\sharp, v \rangle_M = df[v] = \langle dF^\sharp, v \rangle_M = dF[v]$$

for any $v \in T_xS$, and hence $\text{grad}(f) = df^\sharp$ is the projection of $\text{grad}(F) = dF^\sharp$ on the tangent space $T_xS$. A critical point of $f$ is a point $x \in S$ such that $df = 0 \iff \text{grad}(f) = 0$, which is equivalent to say that $\text{grad}(F)$ is orthogonal to $T_xS$.

The Hessian of the function $f$, at a critical point $x$ of $f$ in $S$, is (cf. page 343 of [8]) equal to the bilinear form $\text{Hess}(f)[v, w]$, defined on the tangent space $T_xS$ as

$$\text{Hess}(f)[v, w](x) = \nabla^S_v \nabla^S_w f - \nabla^S_{\nabla^S_v w} f)(x) = (\nabla^S_v \nabla^S_w f)(x)$$

where $v$ and $w$ are two vector fields defined in a neighborhood of $x$.

The Hessian of $F$ is simply the symmetric matrix of all the second derivatives $D^2F$:

$$\text{Hess}(F)[v, w](x) = (D_v D_w F)(x) = D^2F(x)[v, w]$$

$$= \sum_{j=1}^n \sum_{\beta=1}^d \sum_{\gamma=1}^d \frac{\partial^2 F}{\partial q_{i\beta} \partial q_{j\gamma}} v_{i\beta} w_{j\gamma}$$

where $q_{i\beta}, v_{i\beta}$ and $w_{j\gamma}$ are the $d$ cartesian components in $E$ ($\mathbb{R}^d$ as the tangent space of $E$) of $q_i$, $v_i$ and $w_j$ respectively.

Using the mass-metric, if $\mathcal{N}$ denotes the unit vector field normal to $T_xS$ in $T_x\mathbb{F}_n(E)$, the projection of $\nabla^S_v u$ of any vector field $u$ on $T_xS$ is

$$\nabla^S_v u = D_v u - \langle D_v u, \mathcal{N} \rangle_M \mathcal{N}$$

and

$$df^\sharp = dF^\sharp - \langle dF^\sharp, \mathcal{N} \rangle_M \mathcal{N}.$$
Because of the product rule for each function $\varphi$ and each vector field $u$
\[ \nabla^S_v (\varphi u) = \varphi \nabla^S_v u + (d\varphi[v])u \]
which implies that
\[ \langle \nabla^S_v (\langle dF^z, N \rangle_M N), w \rangle_M = \langle dF^z, N \rangle_M \langle \nabla^S_v N, w \rangle_M \]

since $N$ is orthogonal to $w$. The same argument can be applied to show that for any vector field $u$ (not necessarily tangent to $S$)
\[ \langle \nabla^S_v u, w \rangle_M = \langle D_v u, w \rangle_M, \]
and therefore that, evaluated at the critical point $x$,
\[ \text{Hess}(f)[v, w] = D^2 F[v, w] - \langle dF^z, q \rangle_M \langle D_v q, w \rangle_M \]

The inertia ellipsoid $S$ is defined by the equation $\|q\|^2_M = 1$, or equivalently $h(q) = \frac{1}{2}$ where $h(q) = \frac{1}{2} \|q\|^2_M$. The normal unit vector $N$ is equal to $dh^z = q$, and thus
\[ \text{Hess}(f)[v, w] = D^2 F[v, w] - \langle dF^z, q \rangle_M \langle D_v q, w \rangle_M \]

If $F = U$, then $U$ is homogeneous of degree $-\alpha$, and therefore $\langle dU^z, q \rangle_M = dU(q)[q] = -\alpha U(q)$. The following equation follows, at any critical point $x$ of the restriction of $U$ to $S$.
\[ (1.1) \quad \text{Hess}(U|_S)[v, w] = D^2 U(x)[v, w] + \alpha U(x) \langle v, w \rangle_M \]

A central configuration is a configuration $q \in F_n(E)$ with the property that there exists a multiplier $\lambda \in \mathbb{R}$ such that
\[ (1.2) \quad dU^z(q) = \lambda q, \]
where $dU^z$ is the gradient in $E^n$ of the potential function $U$, with respect to the mass-metric. Equation (1.2) implies that $\lambda = -\alpha \frac{U(q)}{\|q\|^2_M}$ (for more on central configurations see e.g. [17] (§369–§382bis at pp. 284–306), [13], [10], [12], [14], [1], [6], [2], [11], [5]). An equivalent definition for a normalized (i.e. $q \in S$) central configuration is the following:
(1.3) \( q \in S_n(E) \) is a central configuration if and only if it is a critical point for the restriction \( U|_S \) of the potential function to \( S = S_n(E) \).

Let \( c: E^n \to E^n \) be the isometry defined as \( c(q) = q' \), with

\[
q'_j = q_j - 2q_0
\]

for each \( j = 1, \ldots, n \), and with \( q_0 = \sum_{j=1}^n m_jq_j \). It is an isometry, since

\[
\|q'_j\|^2_M = \sum_{j=1}^n m_j|q_j - 2q_0|^2 = \sum_{j=1}^n m_j(|q_j|^2 + 4|q_0|^2 - 4q_j \cdot q_0) = \sum_{j=1}^n m_j|q_j|^2 + 4(\sum_{j=1}^n m_j)|q_0|^2 - 4|q_0|^2 = \|q\|^2_M.
\]

It is the orthogonal reflection around the space of all configurations with center of mass \( q_0 \) equal to zero: \( cq = q \iff q_0 = 0 \). It is easy to see that if \( q \) is a central configuration then \( cq = q \), and hence \( q \) has center of mass \( q_0 \) in 0. Let \( Y \) be defined as \( Y = \{ q \in E^n : q_0 = 0 \} \), and \( S^c = S \cap Y, S^c_c = S \cap Y \). In other words, elements of \( S^c \) are normalized configurations with center of mass in 0. Since the potential function is invariant up to translations, \( U(cq) = U(q) \), and any critical point of the restriction \( U|_{S^c} \) is a critical point of \( U|_S \) (for example, by Palais Principle of Symmetric Criticality [13]). Thus it is equivalent to define central configurations as critical points of \( U|_{S^c} \) or as critical points of \( U|_S \).

2 Fixed points, \( SO(d) \)-orbits and projective configuration spaces

Following [3, 4], consider the function \( F: S_n(E) \to S_n(E) \) defined as

\[
F(q) = -\frac{dU^1(q)}{\|dU^2(q)\|_M}
\]

where \( dU^2 \) is the gradient of \( U \), with respect to the mass-metric.

First, consider the isometry \( c \) defined above in [1.4]. Since \( F(cq) = cF(q) \), \( F(S) \subset S^c \). Moreover, as the image of \( F \) is in \( S^c \), if \( F^c \) denotes the restriction \( F^c: S^c \to S^c \),

\[
(2.2) \quad \text{Fix}(F^c) = \text{Fix}(F),
\]

and the fixed point indexes are exactly the same.

Let \( O(d) \) be the special orthogonal group, acting diagonally on \( E^n \), and \( SO(d) \) the special orthogonal subgroup. The inertia ellipsoid \( S \), \( S \) and \( Y \) are \( O(d) \)-invariant in \( E^n \), and so are \( S^c \) and \( S^c_c \). Let \( \pi: S \to S/G \) denote the quotient map onto the space of \( G \)-orbits, for \( G = SO(d) \) or \( G = O(d) \).
Since $U$ is a $G$-invariant function, $F$ is a $G$-equivariant map, and hence it induces a map on the quotient spaces:

\[(2.3)\]

\[
\begin{array}{ccc}
S & \xrightarrow{F} & S \\
\pi & \downarrow & \pi \\
S/G & \xrightarrow{\pi} & S/G
\end{array}
\]

A fixed point of $F$ is a normalized configuration $q$ such that $F(q) = q$. A fixed point of $f$ is a conjugacy class $[q]$ of configurations such that $f([q]) = [q]$, i.e. it is a conjugacy class $[q]$ such that $F(q) = gq$ for some $g \in G$. It follows from Theorem (2.5) of [4] that if $G = SO(d)$, then $F(q) = gq \iff F(q) = q$, or equivalently that

\[(2.4)\]

\[G = SO(d) \implies \pi(\text{Fix}(F)) = \text{Fix}(f),\]

and hence also that $\pi(\text{Fix}(F^c)) = \text{Fix}(f^c)$.

(2.5) Remark. Elements in $S/G$ are called projective configurations: for $d = 2$ and $G = SO(2)$, $S/G$ is the $(n - 1)$-dimensional complex projective space $\mathbb{P}^{n-1}(\mathbb{C})$, and $S^c$ is a hyperplane in it, hence a $(n - 2)$-dimensional complex projective space $\mathbb{P}^{n-2}(\mathbb{C})$. For $n = 3$, it is the Riemann sphere. Projective configurations are projective classes of elements $[q_1 : q_2 : q_3]$ in $\mathbb{P}^1(\mathbb{C}) \subset \mathbb{P}^2(\mathbb{C})$ such that $m_1q_1 + m_2q_2 + m_3q_3 = 0$, $q_j \in \mathbb{C}$, and $q_1 \neq q_2$, $q_1 \neq q_3$, $q_2 \neq q_3$.

For $d = 1$, projective configurations are equivalence classes under the action of the orthogonal group $G = O(1) = \mathbb{Z}_2$.

The following Corollary of (2.4) shows that the difference is minor.

(2.6) Corollary. If $q \in S$ is a central configuration such that $F(q) = gq$, with $g \in O(d)$ (acting diagonally on $E^n$), then $g = 1$.

Proof. Let $E' = E \oplus \mathbb{R}$ be the euclidean space of dimension $d+1$, and $E \subset E'$ one of its $d$-dimensional subspaces. If $q \in S \subset \mathbb{F}_n(E)$, then $q \in S \subset \mathbb{F}_n(E) \subset \mathbb{F}_n(E')$, and there exists $g' \in SO(d+1)$ such that $g'E = E$ and the restriction of $g'$ to $E$ is equal to $g$: it follows that $F(q) = g'q$, in $\mathbb{F}_n(E')$, and therefore $g' = 1$, from which it follows that $g = 1$.

q.e.d.

Homological calculations on configurations spaces for the sake of central configurations have been done by Palmore [14], Pacella [12] and McCord [9]. We can arrange all the spaces inertia ellipsoids and the corresponding
projective quotients as in diagram (2.7)

\[
\begin{array}{c}
\mathbb{S}^c_n(\mathbb{R}) \xrightarrow{\iota_1} \mathbb{S}^c_n(\mathbb{R}^2) \xrightarrow{\iota_2} \mathbb{S}^c_n(\mathbb{R}^3) \xrightarrow{\iota_3} \cdots \\
\mathbb{S}^c_n(\mathbb{R})/SO(1) \xrightarrow{\gamma_1} \mathbb{S}^c_n(\mathbb{R}^2)/SO(2) \xrightarrow{\gamma_2} \mathbb{S}^c_n(\mathbb{R}^3)/SO(3) \xrightarrow{\gamma_3} \cdots
\end{array}
\]

(2.7)

For each \( d \), \( \mathbb{S}^c_n(\mathbb{R}^d) \) is a deformation retract of \( \mathbb{P}^c_n(\mathbb{R}^d) \), which in turn is a deformation retraction of \( \mathbb{F}_n(\mathbb{R}^d) \) (where \( \mathbb{F}_n(E) \) denotes the space of all configurations with center of mass in 0). The Poincaré polynomial for the cohomology of the configuration space \( \mathbb{F}_n(\mathbb{R}^d) \) is equal to

\[
P(t) = \prod_{k=1}^{n-1} (1 + kt^{d-1}),
\]

as shown e.g. in Theorem 3.2 of [16] (see also Proposition 2.11.2 of [11]).

Now, note that in the sequence of projections

\[
\mathbb{S}^c_n(\mathbb{R}^d) \to \mathbb{S}^c_n(\mathbb{R}^d)/SO(d) \to \mathbb{S}^c_n(\mathbb{R}^d)/O(d)
\]

the second map corresponds to the projection given by the action of the quotient group \( \mathbb{Z}_2 = O(d)/SO(d) \) on the quotient space \( \mathbb{S}^c_n(\mathbb{R}^d)/SO(d) \) (\( SO(d) \) is normal in \( O(d) \)). For \( d \geq 2 \), let \( h \) be the orthogonal reflection of \( \mathbb{R}^d \) around \( \mathbb{R}^{d-1} \subset \mathbb{R}^d \): its coset \( hSO(d) \) is the generator of \( O(d)/SO(d) \), and hence the image \( \text{Im}(\tilde{t}_{d-1}) \) in \( \mathbb{S}^c_n(\mathbb{R}^d)/SO(d) \) is fixed by \( O(d)/SO(d) \). Actually, it is equal to the fixed point subset of \( O(d)/SO(d) \) in \( \mathbb{S}^c_n(\mathbb{R}^d)/SO(d) \). Outside the image of \( \tilde{t}_{d-1} \), therefore the \( \mathbb{Z}_2 \) action is free: let \( \mathcal{M}_n(\mathbb{R}^d) \) denote the manifold

\[
\mathcal{M}_n(\mathbb{R}^d) = (\mathbb{S}^c_n(\mathbb{R}^d)/SO(d) \setminus \text{Im}(\tilde{t}_{d-1})) / \mathbb{Z}_2 = \mathbb{S}^c_n(\mathbb{R}^d)/O(d) \setminus \text{Im}(\tilde{t}_{d-1}),
\]

where the last equality holds since \( \tilde{t}_{d-1} \) factors through \( \mathbb{S}^c_n(\mathbb{R}^{d-1}) \).

The next proposition follows from the dimension of \( SO(d) \) and the previous remarks.

(2.9) The subspace of all points in \( \mathbb{S}^c_n(\mathbb{R}^d)/O(d) \) with maximal orbit type is the open subspace \( \mathcal{M}_n(\mathbb{R}^d) \) defined in (2.8), and it is a manifold of dimension

\[
\dim \mathcal{M}_n(\mathbb{R}^d) = d(n - 1) - 1 - d(d - 1)/2
\]
For $d = 1$, it is the projective space $\mathbb{P}^{n-2}(\mathbb{R})$ minus collisions. For $d = 2$, it is a $(2n - 4)$ dimensional manifold (where $\mathbb{P}^{n-2}(\mathbb{C})$ minus collinear and minus collisions is its double cover).

(2.10) $\mathbb{S}_c^n(\mathbb{R}^2)/SO(2)$ has the same homotopy type of $\mathbb{F}_{n-2}(\mathbb{R}^2) \setminus \{p, q\}$, where $p, q$ are two arbitrary distinct points of $\mathbb{R}^2$.

Proof. It is Lemma 4.1 of [9]. $q.e.d.$

It follows that the Poincaré polynomial (where $\beta_j$ are Betti numbers) of $\mathbb{S}^c_n(\mathbb{R}^2)/SO(2)$ is

(2.11) $p(t) = \prod_{k=2}^{n-1} (1 + kt) = \sum_{j=0}^{n-2} \beta_j t^j$.

(see also Proposition 2.11.3 of [11]). McCord in [9] proved also that

$$\dim H^k(M_n(\mathbb{R}^2)) = \begin{cases} \sum_{j=0}^{k} \beta_j & \text{if } k \leq n - 3 \\ 0 & \text{otherwise,} \end{cases}$$

while Pacella in (2.4) of [12] computed the $SO(3)$-equivariant homology (using Borel homology) Poincaré series of $\mathbb{S}^c_n(\mathbb{R}^2) \sim \mathbb{F}_n(\mathbb{R}^3)$ as

$$p^{SO(3)}(t) = \prod_{k=2}^{n-1} (1 + kt^2) \frac{1}{1 - t^2}$$

(2.12) Remark. The projective quotient $\mathbb{S}^c_n(\mathbb{R}^2)/SO(2)$ is a manifold (it is the projective space $\mathbb{P}^{n-2}(\mathbb{C})$ with collisions removed). It contains $\mathbb{S}^c_n(\mathbb{R})/O(1)$ as a submanifold (the collinear configurations). For $d \geq 3$ the isotropy groups of the action start being non-trivial, and the filtration of subspaces of constant orbits type in $\mathbb{S}^c_n(\mathbb{R}^d)/SO(d)$ is given by the horizontal arrows $\bar{i}_j$ in diagram (2.7).

### 3 Fixed points and Morse indices

Let $q \in \mathbb{S}^c_n(\mathbb{R}^d)$ a central configuration, and hence a fixed point of the map $F$ defined above in (2.1), such that its $O(d)$-orbits lies in the maximal orbit type submanifold $M_n(\mathbb{R}^d) \subset \mathbb{S}^c(\mathbb{R}^d)/O(d)$.

(3.1) If $DF : T_q\mathbb{S} \to T_q\mathbb{S}$ denotes the differential of $F$ at the central configuration $q$, then for any $v, w \in T_q\mathbb{S}$ the following equation holds:

$$D^2U(q)[v, w] = -\alpha U(q)\langle DF[v], w \rangle_M$$
Proof. As we have seen in the introduction, \( \langle D_dU^2, w \rangle_M = D^2U[v, w] \), and if \( q \) is a normalized central configuration then by (1.2) \( dU^2(q) = \lambda q \) with \( \lambda = -\alpha \frac{U(q)}{\|q\|^2_M} = -\alpha U(q) \). It follows that \( \langle dU^2, w \rangle_M = 0 \), being \( w \) tangent to \( S \), and \( \|dU^2\|_M = -\lambda = \alpha U(q) \). Also, 

\[
\langle DF[v], w \rangle_M = \langle D_v \left( -\frac{dU^2}{\|dU^2\|_M} \right), w \rangle_M = -\langle \left( \frac{D_v dU^2}{\|dU^2\|_M} \right), w \rangle_M - \langle D_v \left( \frac{1}{\|dU^2\|_M} \right) dU^2, w \rangle_M = \langle DF[v], w \rangle_M - 0 = -\frac{1}{\alpha U(q)} D^2U(q)[v, w].
\]

q.e.d.

Combining (3.1) with equation (1.1) the following corollary holds.

(3.2) Corollary. If \( q \) is as above, then for each \( v, w \in T_qS \)

\[
\text{Hess}(U|_S)[v, w] = \alpha U(q) \left( \langle v, w \rangle_M - \langle DF[v], w \rangle_M \right)
\]

Finally, consider again the group \( O(d) \) acting on \( S_n^c(\mathbb{R}^d) \). Let \( F \) and \( q \) be the map and the central configuration defined above. Recall that \( f: S/O(d) \to S/O(d) \) denotes the map defined on the quotient. Let \( [q] \in M_n(\mathbb{R}^d)S/O(d) \) denote the projective class (i.e. the \( O(d) \)-orbit of \( q \)) of \( q \), which is a fixed point of \( f \), and is a critical point of the map \( \bar{U}: M_n(\mathbb{R}^d) \to \mathbb{R} \) induced on \( M_n \) by \( U \), defined simply as \( \bar{U}([x]) = U(x) \) for each \( x \in S_n^c(\mathbb{R}^d) \).

(3.3) Theorem. The point \([q]\) is a non-degenerate critical point of \( \bar{U} \) if and only if it is a non-degenerate fixed point of \( f \). If \( \text{ind}([q], f) \) denotes the fixed point index of \([q]\) for \( f \), and \( \mu([q]) \) the Morse index of \([q]\), then the following equation holds:

\[
\text{ind}([q], f) = (-1)^{\mu([q])}.
\]

Proof. The point \([q]\) is a non-degenerate critical point if and only if the dimension of the kernel of the Hessian \( \text{Hess}(U|_S)(q) \) is equal to the dimension of \( SO(d) \), i.e. \( d(d - 1)/2 \). By (3.2) the kernel is equal to the eigenspace of \( DF(q) \) corresponding to the eigenvalue 1, which has dimension \( d(d - 1)/2 \) if and only if the fixed point \([q]\) is non-degenerate. Now, if this holds then the index \( \text{ind}([q], f) \) is equal to the number \((-1)^e\). where \( e \) is the number
of negative eigenvalues $1 - f'$, which is the same as the number of negative eigenvalues of $1 - F'$. Again by (3.2) and since $U > 0$, $e$ is equal to the number of negative eigenvalues of $\text{Hess}(U|_S)$, which is by definition the Morse index $\mu([q])$.

q.e.d.

(3.4) Remark. Unfortunately, a former version of this statement had a wrong formula for $\text{ind}(q)$. In fact, in (3.5) of [4] one should put $\epsilon = 0$, and not $\epsilon = d(n - 1) - 1 - d(d - 1)/2 = \dim \mathbb{M}_n(\mathbb{R}^d)$. The error occurred because I used the wrong sign of $U$ in (3.1) ($V = -U$ instead of $U$).

(3.5) Example. For $d = 1$ and any $n$, all critical points are local minima of $U$, and hence $\mu = 0$, and fixed points have index 1. The map induced on the quotient can be regularized on binary collisions (see [4,3]), hence the map on the quotient can be extended to a self-map $f: \mathbb{P}^1(\mathbb{R}) \to \mathbb{P}^1(\mathbb{R})$ with three fixed points of index 1. Therefore the Lefschetz number of $f$ is 3, and $f$ has degree $-2$.

For $d = 2$ and $n = 3$, the three Euler configurations have $\mu = 1$, while the two Lagrange points have $\mu = 1$, hence the map $f$ induced on the quotient $\mathbb{P}^1(\mathbb{C})$ (again, by regularizing the binary collisions) has Lefschetz number equal to $L(f) = 2 - 3 = -1$. Therefore the degree of $f$ is equal to $-2$.

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