COMPUTATION OF SOME LEAFWISE COHOMOLOGY RING

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Abstract. Let $G$ be the group $SL(2, \mathbb{R})$, $P \subset G$ be the parabolic subgroup of upper triangular matrices and $\Gamma \subset G$ be a cocompact lattice. A right action of $P$ on $\Gamma \backslash G$ defines an orbit foliation $\mathcal{F}_P$. We compute the leafwise cohomology ring $H^*(\mathcal{F}_P)$ by exploiting non-abelian harmonic analysis on $G$.

1. Introduction and main result

In the rigidity theory of $C^\infty$ foliations, the leafwise cohomology is a fundamental tool (see Asaoka’s survey[2]). Specifically, in many cases the 1-dimensional leafwise cohomology group plays a crucial role to analyze the properties of $C^\infty$ foliations. For example, set $G = SL(2, \mathbb{R})$. Let $P \subset G$ be the subgroup of all upper triangular matrices, $\mathfrak{p}$ be the Lie algebra of $P$ and $\Gamma \subset G$ be a cocompact lattice. Also set $M_\Gamma = \Gamma \backslash G$. Let $\mathcal{F}_P$ be the orbit foliation induced from the natural action of $P$ on $M_\Gamma$ and $(\Omega^*(\mathcal{F}_P), d_{\mathcal{F}_P})$ be the leafwise complex. Let $Z^*(\mathcal{F}_P)$ be the space of cocycles and $B^*(\mathcal{F}_P)$ be the space of coboundaries. Also $H^*(\mathcal{F}_P)$ denote its cohomology group. Then, Matsumoto-Mitsumatsu[7, Theorem 1] proved that

\begin{equation}
H^1(\mathcal{F}_P) \cong H^1_{\text{Lie}}(\mathfrak{p}) \oplus H^1_{\text{dR}}(M_\Gamma).
\end{equation}

On the other hand, not many examples of higher leafwise cohomology groups are known, except for a linear foliation on a torus[1]. Recently, Maruhashi-Tsutaya[5, Theorem 66] provided a new example. They proved that

\begin{equation}
H^2(\mathcal{F}_P) \cong H^2_{\text{dR}}(M_\Gamma).
\end{equation}

They identified the total leafwise cohomology group $H^*(\mathcal{F}_P)$ as a linear space; however the ring structure remained unknown.

In this paper, we determine the ring structure of $H^*(\mathcal{F}_P)$ by exploiting non-abelian harmonic analysis, which method is totally independent of theirs. Let $g$ be the multiplicity of the irreducible unitary
representation $U^{-1}$ whose lowest weight is 1 in $L^2(M_{\Gamma})$ with the $G$-invariant measure. The finiteness is ensured by a duality theorem\cite[Theorem 1.4.2]{3}. This theorem says that the multiplicity of $U^1$ whose highest weight is $-1$ is also $g$. Then, our theorem below holds.

**Theorem 1.** There exist 1-cocycles $x, y_1, ..., y_{2g}$ such that

$$H^*(\mathcal{F}_p) \cong \bigwedge [x, y_1, ..., y_{2g}]/\{ y_i \wedge y_j \}_{1 \leq i, j \leq 2g}. \quad (3)$$

In the proof of Theorem 1 we also prove following: let $C_{\Gamma k} = O(25^k)$ be the constant defined in \cite[28]{} for each $k \in \mathbb{N}$. Then, for each $\eta \in B^*(\mathcal{F}_p)$, there exists $\xi \in \Omega^*(\mathcal{F}_p)$ with $\eta = d_{\mathcal{F}_p} \xi$ such that

$$||\xi||_k^2 \leq C_{\Gamma k+3} ||\eta||_{k+3}^2. \quad (4)$$

for each $k \in \mathbb{N}$, where $|| \cdot ||_k$ is a $L^2$-Sobolev norm of $k$-th order. This satisfies a tame estimate introduced in \cite[Definition II.2.1.1.]{4}. Thus our method is expected to help the further study of foliations using tameness.

The following describes the flow of this paper. First, in Section 2 we provide tools for discussion. Notations of representation theory, $L^2$-Sobolev norm, and the constants $C_{\Gamma k}$ are provided here. Second, in Section 3 and 4 we compute all explicit generators of $H^*(\mathcal{F}_p)$ in terms of representation theory. We also make a explicit estimations of $L^2$-Sobolev norms. Third, we determine the ring structure by computing eigenvalues in section 5.

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2. Preliminary

We summarize computation formulae.

2.1. From representation theory. Let $\hat{G}$ be the unitary dual of $G$. It is sufficient for us to compute $d_{\mathcal{F}_p}$ on each $\pi \in \hat{G}$ by using the differential representation. Indeed, $L^2(M_{\Gamma})$ decomposes into a countable sum of irreducible unitary representations\cite[Theorem 1.2.3]{3}. Set $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R})$ and take elements

$$X_0 = \begin{pmatrix} 0 & 1 \\ -\frac{1}{2} & 0 \end{pmatrix}, X_1 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}, X_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad (5)$$
from $\mathfrak{g}$. When we regard $X_0, X_1, X_2$ as vector fields on $M_\Gamma$, let $\omega_0, \omega_1, \omega_2 \in \Omega^1(M_\Gamma)$ be the dual forms of them. We put

$$Y = -X_0 + X_2 = \begin{pmatrix} 0 & 1 \\ \frac{1}{2} & 0 \end{pmatrix}.$$  

For $\pi \in \hat{\mathcal{G}}$, let $\pi'$ be the derivative of $\pi$, set

$$H = -i\pi'(X_0),\ E = \pi'(X_1) + i\pi'(Y),\ F = -\pi'(X_1) + i\pi'(Y).$$

When we regard $\pi'$ as the representation of a complex Lie algebra $\mathfrak{sl}(2, \mathbb{C})$, $H, E$ and $F$ are given by

$$H = \pi'(0, -\frac{i}{2}, \frac{1}{2}),\ E = \pi'(\frac{i}{2}, \frac{1}{2}, \frac{1}{2}),\ F = \pi'(\frac{i}{2}, \frac{1}{2}, -\frac{1}{2}).$$

We set

$$C = \pi'(X_0)^2 - \pi'(X_1)^2 - \pi'(Y)^2$$

which is called the Casimir element. We state the structure theory of $\hat{\mathcal{G}}$. Here, $\mathbb{N}$ denotes the set of all non-negative integers and $\frac{1}{2}\mathbb{Z}$ denotes the set of all integers and half-integers.

**Theorem 2. (see [9, Proposition 6.13, Theorem 6.2, 6.4, 6.5])**

For $\pi \in \hat{\mathcal{G}}$, let $M \subset \frac{1}{2}\mathbb{Z}$ and $q \in \mathbb{R}$ be given in the table below. Then, there exists an orthonormal basis $\{\phi_m\}_{m \in M}$ of $\pi$ such that

$$C\phi_m = q\phi_m,$$

$$H\phi_m = m\phi_m,$$

$$E\phi_m = \sqrt{q + m(m + 1)}\phi_{m+1},$$

$$F\phi_m = \sqrt{q + m(m - 1)}\phi_{m-1}$$

for each $m \in M$.

| $\pi$         | $M$         | $q$           | conditions                  |
|--------------|-------------|---------------|-----------------------------|
| $V^{0,\frac{1}{2}+\nu}$ | $\mathbb{Z}$ | $\frac{1}{4} + \nu^2$ | $\nu \geq 0$ |
| $V^{\frac{1}{2},\frac{1}{2}+\nu}$ | $\frac{1}{2} + \mathbb{Z}$ | $\frac{1}{4} + \nu^2$ | $\nu > 0$ |
| $V_{+}^{\frac{1}{2},\frac{1}{2}}$ | $\frac{1}{2} + \mathbb{N}$ | $\frac{1}{4}$ | - |
| $V_{-}^{\frac{1}{2},\frac{1}{2}}$ | $-\frac{1}{2} - \mathbb{N}$ | $\frac{1}{4}$ | - |
| $U^n$ | $n - \mathbb{N}$ | $n(1 - n)$ | $n \in \frac{1}{2}\mathbb{Z}$ with $n \geq 1$ |
| $U^{-n}$ | $n + \mathbb{N}$ | $n(1 - n)$ | $n \in \frac{1}{2}\mathbb{Z}$ with $n \geq 1$ |
| $V^\sigma$ | $\mathbb{Z}$ | $\sigma(1 - \sigma)$ | $\frac{1}{2} < \sigma < 1$ |
| $I$ | $\{0\}$ | 0 | - |
The notations $M$ and $q$ are sometimes denoted as $M_\pi$ and $q_\pi$, respectively. By applying Theorem 2 we write down $d_{\mathcal{F}_P}: \Omega^0(\mathcal{F}_P) \to \Omega^1(\mathcal{F}_P)$. Let $h$ be a $C^\infty$ vector of some $\pi \in \hat{G}$. We have the Fourier expansion

$$h = \sum_{m \in M} h_m \phi_m.$$  \hspace{1cm} (14)

Set $f_1 \omega_1 + f_2 \omega_2 = d_{\mathcal{F}_P} h$. Then, Fourier coefficients of $f_1$ and $f_2$ are given by

$$f_1 m = \frac{h_{m-1}}{2} \alpha_{q-1} - \frac{h_{m+1}}{2} \beta_{q+1}$$ \hspace{1cm} (15)

and

$$f_2 m = -\frac{ih_{m-1}}{2} \alpha_{q-1} + imh_m - \frac{ih_{m+1}}{2} \beta_{q+1},$$ \hspace{1cm} (16)

where

$$\alpha_{q} = \sqrt{q + m(m+1)},$$ \hspace{1cm} (17)

$$\beta_{q} = \sqrt{q + m(m-1)}. \hspace{1cm} (18)$$

Next, we write down $d_{\mathcal{F}_P}: \Omega^1(\mathcal{F}_P) \to \Omega^2(\mathcal{F}_P)$. Let $f_1, f_2$ be $C^\infty$ vectors of some $\pi \in \hat{G}$. Set $g \omega_1 \wedge \omega_2 = d_{\mathcal{F}_P}(f_1 \omega_1 + f_2 \omega_2)$. Then, $g$’s Fourier coefficients are given by

$$g_m = \frac{if_{1m-1} + f_{2m-1}}{2} \alpha_{q-1} - (imf_{1m} + f_{2m}) + \frac{if_{1m+1} - f_{2m+1}}{2} \beta_{q+1}.$$ \hspace{1cm} (19)

For convenience, we replace $h$ by $-4h$, $f_1 m$ by $2f_1 m$, $f_2 m$ by $2if_2 m$, and $g$ by $ig$. Then we always assume

$$f_1 m = -h_{m-1} \alpha_{q-1} + h_{m+1} \beta_{q+1},$$ \hspace{1cm} (20)

$$f_2 m = h_{m-1} \alpha_{q-1} - 2mh_m + h_{m+1} \beta_{q+1},$$ \hspace{1cm} (21)

or

$$g_m = (f_{1m-1} + f_{2m-1}) \alpha_{q-1} - 2(mf_{1m} + f_{2m}) + (f_{1m+1} - f_{2m+1}) \beta_{q+1}.$$ \hspace{1cm} (22)
2.2. From Sobolev space theory. In this paper, we construct formal functions by using (20), (21), and (22). To ensure their smoothness, we use $L^2$-Sobolev norms. In general, let $M$ be a compact Riemannian manifold and $(\lambda_s)_{s=0}^{\infty}$ be the sequence consisting of eigenvalues of the Laplace-Beltrami operator:

\begin{equation}
0 = \lambda_0 \leq \lambda_1 \leq \ldots \leq \lambda_s \leq \ldots \to \infty.
\end{equation}

Then, for each $k \in \mathbb{N}$, the $L^2$-Sobolev norm of $k$-th order with respect to the Bessel potential is given by

\begin{equation}
||f||_k^2 = \sum_{s=0}^{\infty} (1 + \lambda_s)^k |f_s|^2,
\end{equation}

where $f \in C^\infty(M)$ and $(f_s)_{s=0}^{\infty}$ is the Fourier coefficients of $f$. (For example, see [8, Definition 4.1]).

We apply this fact to our case. We define the Riemannian metric on $M_\Gamma$ whose orthogonal frame is $\{X_0, X_1, Y\}$. Then the Laplace-Beltrami operator is $\Delta = -X_0^2 - X_1^2 - Y^2$. (See [10, Theorem 1]. Since $G$ is connected unimodular, we have $\text{Trace}(\text{ad}(\cdot)) = 0$.) We transform with $\Delta = C - 2X_0^2$. Then we get

\begin{equation}
\Delta \phi_m = (q + 2m^2) \phi_m.
\end{equation}

Thus a $L^2$-Sobolev norm of $k$-th order is given by

\begin{equation}
||f||_k^2 = \sum_{\pi \subset L^2(M_\Gamma), \, m \in M_\pi} (1 + q + 2m^2)^k |f_{\pi m}|^2
\end{equation}

for each $k \in \mathbb{N}$ and $f \in C^\infty(M_\Gamma)$.

We estimate the $L^2$-Sobolev norms (20) in Sections 3 and 4. We provide constants for this purpose. Observe the behavior of number $q$:

**Lemma 3.** The sequence $(q_{\pi})_{\pi \subset L^2(M_\Gamma)}$ has no accumulation points, where $q_{\pi}$ is the number of $\pi$.

**Proof.** The non-existence of accumulation points is derived from the fact that eigenvalues of $\Delta$ diverge. \qed

Set

\begin{equation}
q_\Gamma = \inf \{|q_{\pi}| \mid \pi \subset L^2(M_\Gamma), \, \pi \neq I, U^{-1}, U^1\} > 0 \text{ (Lemma 3)}.
\end{equation}

If $q_\Gamma > 1$, then we replace $q_\Gamma = 1$.

For each $k \in \mathbb{N}$, set

\begin{equation}
C_{\Gamma k} = \left(\frac{3!}{q_\Gamma^2}\right)^2 2^{5k+8}.
\end{equation}
The estimation is performed using this constant $C_{\Gamma k}$ under replacing $h$, $f_1$, $f_2$, and $g$. When a special cocycle $\eta \in Z^*(\mathcal{F}_P)$ is given, we construct $\xi \in \Omega^*(\mathcal{F}_P)$ which satisfies $\eta = d_{\mathcal{F}_P} \xi$ formally and prove
\begin{equation}
||\xi||_{k}^2 \leq C_{\Gamma k+3} ||\eta||_{k+3}^2
\end{equation}
for each $k \in \mathbb{N}$.

3. COMPUTING SECOND COCYCLES

To prove the lemmata below, we solve a linear equation for all Fourier coefficients on each $\pi \in \hat{G}$. The symbol $|_{\pi}$ means “restricted to $\pi$”.

3.1. Trivial representation. We get the following lemma directly.

Lemma 4. $H^2(\mathcal{F}_P)|_I = \{0\}$.

3.2. Corresponding to the lowest weight 1. We characterize the coboundary space.

Lemma 5. The following holds.
\begin{equation}
B^2(\mathcal{F}_P)|_{U^{-1}} = \left\{ g \omega_1 \wedge \omega_2 \left| \sum_{m=1}^{\infty} \sqrt{m} g_m = 0 \right. \right\}.
\end{equation}

Specially, $Z^2(\mathcal{F}_P)|_{U^{-1}}$ is spanned by $\phi_1 \omega_1 \wedge \omega_2$ and $B^2(\mathcal{F}_P)|_{U^{-1}}$.

Proof. Proving that the right hand side contains $B^2(\mathcal{F}_P)|_{U^{-1}}$ is easy. To prove opposite, let $N$ be a positive integer. We put
\begin{equation}
f_{1m}^{(N+1)} = \frac{N + 1 - m}{\sqrt{m(N + 1)}} \quad (1 \leq m \leq N),
\end{equation}
\begin{equation}
f_{1N+1}^{(N+1)} = 0,
\end{equation}
\begin{equation}
f_1^{(N+1)} = \sum_{m=1}^{N} f_{1m}^{(N+1)} \phi_m.
\end{equation}

We have
\begin{equation}
d_{\mathcal{F}_P}(f_1^{(N+1)} \omega_1) = (-\sqrt{N + 1} \phi_1 + \phi_{N+1}) \omega_1 \wedge \omega_2.
\end{equation}
by \cite{22}. To check it, put $g^{(N+1)} \omega_1 \wedge \omega_2 = d_{F_p}(f_1^{(N+1)} \omega_1)$. For each $2 \leq m \leq N$, 

(35) \begin{align*}
g_m^{(N+1)} &= f_1^{(N+1)} \alpha_{q_m} - 2m f_1^{(N+1)} + f_1^{(N+1)} \beta_{q_{m+1}} \\
&= \frac{N+1-(m-1)}{\sqrt{(m-1)(N+1)}} \sqrt{(m-1)m-2m} \frac{N+1-m}{\sqrt{m(N+1)}} + \frac{N+1-(m+1)}{\sqrt{(m+1)(N+1)}} \sqrt{m(m+1)} \\
&= \sqrt{\frac{m}{N+1}} (N+1-(m-1) - 2(N+1-m) + N+1-(m+1)) \\
&= 0.
\end{align*}

Next, 

(36) \begin{align*}
g_1^{(N+1)} &= -2 \cdot 1 \cdot f_1^{(N+1)} + f_1^{(N+1)} \beta_{q_2} \\
&= -2 \frac{N+1-1}{\sqrt{1 \cdot (N+1)}} + \frac{N+1-2}{\sqrt{2 \cdot (N+1)}} \sqrt{1 \cdot 2} \\
&= -\sqrt{N+1}.
\end{align*}

Finally, 

(37) \begin{align*}
g_{N+1}^{(N+1)} &= f_1^{(N+1)} \alpha_N - 2(N+1)f_1^{(N+1)} \\
&= \frac{N+1-N}{\sqrt{N(N+1)}} \sqrt{N(N+1)} - 0 \\
&= 1.
\end{align*}

Thus the formula \cite{33} is valid. We put $\xi^{(N+1)} = f_1^{(N+1)} \omega_1$.

Then let $\eta = g \omega_1 \wedge \omega_2$ be an element from the right hand side. Put 

(38) \[ \xi = \sum_{N=1}^{\infty} g_{N+1} \xi^{(N+1)} \]

We obtain $\eta = d\xi$ formally. This is determined to be smooth 2-coboundary after the Sobolev estimation (Lemma 6). \hfill \Box

**Lemma 6.** Let $\eta$ be an element from the right hand side in Lemma 5 and put $\xi$ as (38). Then, for each $k \in \mathbb{N}$, we have 

(39) \[ ||\xi||_k^2 \leq ||\eta||_{k+3}^2. \]
Proof. Fix $k \in \mathbb{N}$. It is enough to prove
\begin{equation}
\sum_{m=1}^{\infty} (1 + 2m^2)^k \left| \sum_{N=m}^{\infty} g_{N+1} \frac{N + 1 - m}{\sqrt{m}(N + 1)} \right|^2 \leq \sum_{N=1}^{\infty} (1 + 2(N + 1)^2)^{k+3} |g_{N+1}|^2.
\end{equation}
First, we estimate each term in the left hand side. Put
\begin{equation}
g'_{N+1} = (1 + 2(N + 1)^2)g_{N+1}.
\end{equation}
Then
\begin{equation}
\sum_{N=m}^{\infty} \frac{g_{N+1}}{\sqrt{m(N + 1)}} \frac{N + 1 - m}{\sqrt{m}(N + 1)} \leq \left( \sum_{N=m}^{\infty} \frac{1}{(1 + 2(N + 1)^2)^2 m(N + 1)} \right)^{\frac{1}{2}} \sum_{N=m}^{\infty} |g'_{N+1}|^2 \quad \text{(Cauchy-Schwartz)}
\end{equation}
\begin{equation}
\leq \left( \sum_{N=m}^{\infty} \frac{1}{1 + 2(N + 1)^2} \right)^{\frac{1}{2}} \sum_{N=m}^{\infty} |g'_{N+1}|^2
\end{equation}
\begin{equation}
\leq 1 \cdot \sum_{N=m}^{\infty} (1 + 2(N + 1)^2)^2 |g_{N+1}|^2.
\end{equation}
Next, we estimate the whole term in the left hand side.
\begin{equation}
\sum_{m=1}^{\infty} (1 + 2m^2)^k \left| \sum_{N=m}^{\infty} g_{N+1} \frac{N + 1 - m}{\sqrt{m}(N + 1)} \right|^2
\end{equation}
\begin{equation}
\leq \sum_{m=1}^{\infty} (1 + 2m^2)^k \sum_{N=m}^{\infty} (1 + 2(N + 1)^2)^2 |g_{N+1}|^2
\end{equation}
\begin{equation}
= \sum_{N=1}^{\infty} (1 + 2(N + 1)^2)^2 \left( \sum_{m=1}^{N} (1 + 2m^2)^k \right) |g_{N+1}|^2
\end{equation}
\begin{equation}
\leq \sum_{N=1}^{\infty} (1 + 2(N + 1)^2)^2 N(1 + 2N^2)^k |g_{N+1}|^2
\end{equation}
\begin{equation}
\leq \sum_{N=1}^{\infty} (1 + 2(N + 1)^2)^{k+3} |g_{N+1}|^2.
\end{equation}
3.3. **Corresponding to the highest weight -1.** Similar argument also holds in this case.

**Lemma 7.** The following holds.

\[
B^2(F_P)|_{U^1} = \left\{ g_1 \wedge g_2 \mid \sum_{m=-\infty}^{-1} (-1)^{-m} \sqrt{-m} g_m = 0 \right\}.
\]

Specially, \(Z^2(F_P)|_{U^1}\) is spanned by \(\phi_{-1} \omega_1 \wedge \omega_2\) and \(B^2(F_P)|_{U^1}\).

**Proof.** Proving that the right hand side contains \(B^2(F_P)|_{U^1}\) is easy. To prove opposite, let \(N\) be a positive integer. We put

\[
f^{-(N+1)}_{1m} = \frac{1}{\sqrt{-m(N+1)}} \left( \begin{array}{c} N + 1 + m \\ -N \leq m \leq -1 \end{array} \right),
\]

\[
(46) \quad f^{-(N+1)}_{1-(N+1)} = 0,
\]

\[
(47) \quad f^{-(N+1)}_{1} = \sum_{m=-N}^{-1} (-1)^{N-m} f^{-(N+1)}_{1m} \phi_m.
\]

Then, we have

\[
d_{F_p}(f^{-(N+1)}_{1} \omega_1) = (\phi^{-(N+1)}_{-1} \omega_1 + (-1)^{N+1} \sqrt{N+1} \phi_{-1} \omega_1) \omega_2
\]

by (22). To check it, put \(g^{-(N+1)} \omega_1 \wedge \omega_2 = d_{F_p}(f^{-(N+1)}_{1} \omega_1)\). For each \(-N \leq m \leq -2,\)

\[
(49) \quad (-1)^{N-(m-1)} g^{-(N+1)}_m
\]

\[
= f^{-(N+1)}_{1m-1} \alpha_{qm-1} + 2m f^{-(N+1)}_{1m} + f^{-(N+1)}_{1m+1} \beta_{qm+1}
\]

\[
= \frac{N + 1 + (m-1)}{\sqrt{|m-1|(N+1)}} \sqrt{(m-1)m} + \frac{2m(N+1+m)}{\sqrt{|m|(N+1)}} + \frac{N + 1 + (m+1)}{\sqrt{|m+1|(N+1)}} \sqrt{m(m+1)}
\]

\[
= \sqrt{\frac{|m|}{N+1}} \left( N + 1 + (m-1) - 2(N+1+m) + N + 1 + (m+1) \right)
\]

\[
= 0.
\]
Next,
\[
\begin{align*}
g_{-N+1}^{-N} &= -2(-N+1)(-N)^{N+1}f_{1-N+1}^{N+1} + (-1)^{N+N}f_{1-N+1}^{N+1}/\beta_q-N \\
&= 0 + \frac{N+1-N}{\sqrt{N \cdot (N+1)}}(-N-1)(-N) \\
&= 1.
\end{align*}
\]

Finally,
\[
\begin{align*}
g_{-1}^{-N+1} &= (-1)^{N+2}f_{-2}^{N+1}\alpha_{-2} - 2(-1)(-1)^{N+1}f_{-1}^{N+1} \\
&= (-1)^{N+1} \left( -\frac{N+1-2}{\sqrt{2 \cdot (N+1)}} \sqrt{2 \cdot 1 + 2 \frac{N+1-1}{1 \cdot (N+1)}} \right) \\
&= (-1)^{N+1}\sqrt{N+1}.
\end{align*}
\]

Thus the formula (50) is valid. This fact proves the result as in Lemma 5. The Sobolev estimation is the same as Lemma 6:
\[
||\xi||_k^2 \leq ||\eta||_{k+3}^2.
\]

\[\square\]

3.4. The other cases. Set \(\pi \neq I, U^{-1}, U^1\) and fix \(m_\pi \in \mathbb{M}\). To begin with, we observe a behaver of \(d_{F_P,\pi}\). Put \(f_1 = f_2 = f_1, f_2 = 0\) for any \(m \in \mathbb{M}\) which satisfies \(m \equiv m_\pi\) (mod 4). We consider the linear map
\[
(f_{1+m_1}, f_2, f_{1+m_2}, f_2) \mapsto (g_m, g_{m+1}, g_{m+2}, g_{m+3})
\]
for all \(m \equiv m_\pi\) by (22). The coefficient matrix is a block diagonal matrix whose block is a 4 \(\times\) 4 matrix. Each block is represented as
\[
\begin{align*}
\left( \begin{array}{cccc}
ge_m & g_{m+1} & g_{m+2} & g_{m+3} \\
\end{array} \right) &= \left( \begin{array}{cccc}
\beta_{q_{m_1}} & -\beta_{q_{m+1}} & 0 & 0 \\
-2(m+1) & -2 & \beta_{q_{m+2}} & -\beta_{q_{m+2}} \\
\alpha_{q_{m_1}} & \alpha_{q_{m+1}} & -2(m+2) & -2 \\
0 & 0 & \alpha_{q_{m+2}} & \alpha_{q_{m+2}} \\
\end{array} \right) \left( \begin{array}{c}
f_1 \\
f_2 \\
f_1 \\
f_2 \\
\end{array} \right).
\end{align*}
\]
We denote this 4 \(\times\) 4 matrix as \(A_m\). Its determinant \(\gamma_{q_m} = \det A_m\) is
\[
\gamma_{q_m} = -4q\sqrt{m^2 + m + q\sqrt{m^2 + 5m + 6}}.
\]
This value is always non-vanishing when \(m, m+3 \in \mathbb{M}\) by the lemma below.
Lemma 8. The following holds:

\[ |\gamma_{qm}| \geq 4 \min\{q^2, 1\}. \]  

Especially, we obtain \( |\gamma_{qm}| \geq 4q^2 \).

Proof. When \( \pi \neq I, U^{-n}, U^n \), since \( M \subset \frac{1}{2}\mathbb{Z} \), we have

\[
|\gamma_{qm}| = 4q\sqrt{(m + \frac{1}{2})^2 - \frac{1}{4} + q}\sqrt{(m + \frac{5}{2})^2 - \frac{1}{4} + q} \\
\geq 4q^2.
\]

When \( \pi = U^{-n} \) \((n \geq \frac{3}{2})\), \( |\gamma_{qm}| \) takes the minimum at \( m = n \):

\[
|\gamma_{qm}| \geq 4q\sqrt{n^2 + n + n(1 - n)\sqrt{n^2 + 5n + 6 + n(1 - n)}} \\
= 4q\sqrt{2n\sqrt{6(n + 1)}} \\
\geq 4 \cdot \frac{3}{2} \cdot \frac{1}{2} \sqrt{3}\sqrt{15} \\
\geq 4 \cdot 1.
\]

When \( \pi = U^n \) \((n \geq \frac{3}{2})\), \( |\gamma_{qm}| \) takes the minimum at \( m + 3 = -n \):

\[
|\gamma_{qm}| \geq 4q\sqrt{(n + 3)^2 - (n + 3) + n(1 - n)\sqrt{(n + 3)^2 - 5(n + 3) + 6 + n(1 - n)}} \\
= 4q\sqrt{6(n + 1)\sqrt{2n}} \\
\geq 4 \cdot 1.
\]

Therefore, we can determine the values \( f_{1m+1}, f_{2m+1}, f_{1m+2}, f_{2m+2} \) which satisfy (54).

Then take any \( \eta = g\omega_1 \wedge \omega_2 \in Z^2(F_P)|_\pi \). Put \( f_1m = f_2m = f_{1m+3} = f_{2m+3} = 0 \) for any \( m \in M \) which satisfies \( m \equiv m_\pi \pmod{4} \). We determine \( f_{1m+1}, f_{2m+1}, f_{1m+2}, f_{2m+2} \) in (54) and set

\[
\xi = f_1\omega_1 + f_2\omega_2.
\]

We get \( \eta = d_{F_P}\xi \) formally. We last the Sobolev estimations (Lemma 9 and 11).

Lemma 9. When \( \pi \neq I, U^{-n}, U^n \), we have

\[
||\xi||_k^2 \leq \left( \frac{3!}{q_T^2} \right)^2 2^{5(k+3)+8} ||\eta||_{k+3}^2
\]

for each \( k \in \mathbb{N} \), where \( q_T \) is defined in (27).
Proof. Fix \( k \in \mathbb{N} \) and \( m \equiv m_\pi \). Each entry of \( A_{q,m} \) is bounded from \( 2\sqrt{q+m^2} \) or \( 2\sqrt{q+(m+3)^2} \). Once we assume the latter. Here, a cofactor of \( A_{q,m} \) is degree 3 polynomial of entries of \( A_{q,m} \). Thus any entry of the cofactor matrix of \( A_{q,m} \) is bounded from \( 3!2^3(q+(m+3)^2)^{\frac{3}{2}} \).

Then, for each \( l = 1, 2 \) and \( m' = m + 1, m + 2 \),

\[
|f_{lm'}| \leq \frac{1}{|\gamma_{q,m}|} 3!2^3(q+(m+3)^2)^{\frac{3}{2}} \left( \sum_{m''=m}^{m+3} |g_{m''}| \right)
\]
\[
\leq \frac{3!}{4q_\pi^2} 2^3(q+(m+3)^2)^{\frac{3}{2}} \left( 4 \sqrt{\sum_{m''=m}^{m+3} |g_{m''}|^2} \right)
\]
\[
\leq \frac{3!}{q_\pi^2} 2^3(1+q+2(m+3)^2)^{\frac{3}{2}} \left( \sum_{m''=m}^{m+3} |g_{m''}|^2 \right)
\]

from Lemma \( \S \).

Claim 10. For each \( s', s'' \in \{0, 1, 2, 3\} \), we have

\[
1 + q + 2(m + s')^2 \leq 2^5(1 + q + 2(m + s'')^2).
\]

We continue by assuming this claim.

\[
(1 + q + 2m'^2)^k |f_{tm'}|^2
\]
\[
\leq \left( \frac{3!}{q_\pi^2} \right)^2 2^6(1 + q + 2m'^2)^k(1 + q + 2(m + 3)^2)^{\frac{3}{2}} \sum_{m''=m}^{m+3} |g_{m''}|^2
\]
\[
\leq \left( \frac{3!}{q_\pi^2} \right)^2 2^{5(k+3)+6} \sum_{m''=m}^{m+3} (1 + q + 2m'^2)^{k+3} |g_{m''}|^2 \quad \text{(Claim \[\S\])}.
\]

Add together for \( l = 1, 2 \) and \( m' = m + 1, m + 2 \), and get

\[
\sum_{l=1,2, m'=m+1, m+2} (1 + q + 2m'^2)^k |f_{lm'}|^2
\]
\[
\leq \left( \frac{3!}{q_\pi^2} \right)^2 2^{5(k+3)+8} \sum_{m''=m}^{m+3} (1 + q + 2m''^2)^{k+3} |g_{m''}|^2
\]

Finally, the desired inequality is obtained by adding up for \( m \equiv m_\pi \). \( \square \)

Proof of Claim \[\S\]. We find a constant \( c > 1 \) which satisfies

\[
1 + q + 2(m + s)^2 \leq c(1 + q + 2(m + s'')^2).
\]
By transposition, we obtain

$$2c(m + s'')^2 - 2(m + s')^2 + (c - 1)(1 + q) \geq 0.$$  

(68)

Since $\pi \neq I, U^{-n}, U^n$, $q$ is positive. Then it is enough to satisfy

$$2c(m + s'')^2 - 2(m + s')^2 + c - 1 \geq 0.$$  

(69)

In the left hand side, we obtain

$$2c(m + s'')^2 - 2(m + s')^2 = 2(c - 1) \left( m + \frac{cs'' - s}{c - 1} \right)^2 - 2c \frac{(s'' - s)^2}{c - 1} \geq - \frac{18c}{c - 1}.$$  

Then it is enough to satisfy

$$- \frac{18c}{c - 1} + c - 1 \geq 0$$  

or

$$-18c + (c - 1)^2 \geq 0.$$  

(72)

Roughly, this is valid for $c \geq 20$. Thus we can set $c = 2^5$. $\square$

**Lemma 11.** When $\pi = U^{-n}, U^n (n \geq \frac{3}{2})$, we have

$$||\xi||_k^2 \leq \left( \frac{3!}{q^2} \right)^2 25(k+3) + 8 ||\eta||_{k+3}$$  

(73)

for each $k \in \mathbb{N}$.

**Proof.** We prove the case $\pi = U^{-n}$. (The proof for case $\pi = U^n$ is similar.) Fix $k \in \mathbb{N}$ and $m \equiv m_\pi$. Each entry of $A_{qm}$ is bounded from $2(m + 3)$. Then any entry of the cofactor matrix of $A_{qm}$ is bounded from

$$3!2^3(m + 3)^3.$$  

(74)

Then, for each $l = 1, 2$ and $m' = m + 1, m + 2,$

$$|f_{lm'}| \leq \frac{1}{|\gamma_{qm}|} 3!2^3(m + 3)^3 \left( \sum_{m''=m}^{m+3} |g_{m''}| \right)$$  

(75)

$$= \frac{3!}{q^2} 2^3(m + 3)^3 \sqrt{\sum_{m''=m}^{m+3} |g_{m''}|^2}$$

from Lemma. ☐
Claim 12. The following holds.

\[ m + 3 \leq \sqrt{1 + n(1 - n) + 2(m + 3)^2}. \]  

We continue by assuming this claim.

\[ |f_{lm'}| \leq \frac{3!}{q_{E}} 2^3 (1 + n(1 - n) + 2(m + 3)^2)^{\frac{3}{2}} \sum_{m''=m}^{m+3} |g_{m''}|^2 \quad (\text{Claim 12}). \]

Claim 13. For each \( s', s'' \in \{0, 1, 2, 3\} \), we have

\[ 1 + n(1 - n) + 2(m + s')^2 \leq 2^5 (1 + n(1 - n) + 2(m + s'')^2). \]

Assume this claim. The rest is the same as after Claim [10] of the proof of Lemma [9]. \qed

Proof of Claim 12. Transforming the desired inequality, we obtain

\[ 1 + n(1 - n) + (m + 3)^2 \geq 0. \]

The left hand side takes the minimum if \( m = n \). Then

\[ 1 + n(1 - n) + (n + 3)^2 = 7n + 10. \]

This is positive. \qed

Proof of Claim 13. We find a constant \( c > 13 \) which satisfies

\[ 1 + n(1 - n) + 2(m + s')^2 \leq c(1 + n(1 - n) + 2(m + s'')^2). \]

By transposition, we obtain

\[ 2c(m + s'')^2 - 2(m + s')^2 + (c - 1)(1 + n(1 - n)) \geq 0. \]

Since \( c > 3 \) and \( m \geq n \geq \frac{3}{2} \), we have

\[ 2c(m + s'')^2 - 2(m + s')^2 \geq 2cm^2 - 2(m + 3)^2 \]

\[ = 2(c - 1) \left( m - \frac{3}{c - 1} \right)^2 - \frac{18c}{c - 1} \]

\[ \geq 2cn^2 - 2(n + 3)^2. \]

Then it is enough to satisfy

\[ 2cn^2 - 2(n + 3)^2 + (c - 1)(1 + n(1 - n)) \geq 0. \]
Since $c > 13$ and $n \geq \frac{3}{2}$, the left hand side is estimated by the below:

\begin{align*}
2cn^2 - 2(n + 3)^2 + (c - 1)(1 + n(1 - n)) \\
= (c - 1)n^2 + (c - 13)n + (c - 19) \\
\geq (c - 1)\left(\frac{3}{2}\right)^2 + (c - 13)\frac{3}{2} + (c - 19) \\
= \frac{19}{4}c - \frac{163}{4}.
\end{align*}

This is positive under $c > 13$. Thus we can set $c = 2^5$ roughly. \qed

Then the following holds.

**Lemma 14.** $H^2(\mathcal{F}_P)\mid_\pi = \{0\}$.

### 3.5. The whole sum

For any $\eta \in B^2(\mathcal{F}_P)$ and $\pi \subset L^2(M_1)$, let $\eta_\pi$ be the $\pi$-component. Then we can get smooth cochain $\xi_\pi \in \Omega^1(\mathcal{F}_P)\mid_\pi$ such that

\begin{equation}
||\xi_\pi||^2_k \leq C_{\Gamma k+3}||\eta_\pi||^2_{k+3}
\end{equation}

for each $k \in \mathbb{N}$. Thus we get $\eta = d_{\mathcal{F}_P} \xi$ and

\begin{equation}
||\xi||^2_k \leq C_{\Gamma k+3}||\eta||^2_{k+3},
\end{equation}

where $\xi = \sum_\pi \xi_\pi$.

We summarize the discussion so far. We put

\begin{equation}
x = \omega_1,
\end{equation}

\begin{equation}
y_j = \phi_1(\omega_1 - \omega_2) \text{ in } j\text{-th } U^{-1},
\end{equation}

\begin{equation}
y_{g+j} = \phi_{-1}(\omega_1 + \omega_2) \text{ in } j\text{-th } U^1.
\end{equation}

These are 1-cocycles. Then, we got the following.

**Proposition 15.** The set $\{x \wedge y_1, \ldots, x \wedge y_{2g}\}$ is basis for $H^2(\mathcal{F}_P)$, where the number $g$ is the multiplicity of $U^{-1}$ and $U^1$.

**Remark 16.** This recovers the result [2] by Maruhashi-Tsutaya.

### 4. Computing first cocycles

We also solve a linear equation.

#### 4.1. Trivial representation

We get the following lemma directly.

**Lemma 17.** $H^1(\mathcal{F}_P)\mid_I = C\omega_1$. 
4.2. **Corresponding to the lowest weight 1.** We also characterize the coboundary space. Before that, we prove that the special 1-cocycles are trivial.

**Lemma 18.** If \( f_1 \omega_1 \in Z^1(\mathcal{F}_P)|_{U^{-1}} \), then \( f_1 = 0 \).

Proof. Put \( g \omega_1 \wedge \omega_2 = d_{\mathcal{F}_P}(f_1 \omega_1) \) and \( f_{10} = 0 \). Using (22), for any positive integer \( N \geq 3 \), we compute

\[
\sum_{m=1}^{N-1} \sqrt{m} g_m
\]

\[
= \sum_{m=1}^{N-2} \sqrt{m} \left( f_{1m-1} \sqrt{m(m-1)} - 2mf_1 + f_{1m+1} \sqrt{m(m+1)} \right)
\]

\[
= \sum_{m=0}^{N-2} \sqrt{m+1} f_{1m} \sqrt{m(m+1)} + \sum_{m=1}^{N-1} \sqrt{m} (-2mf_1) + \sum_{m=2}^{N} \sqrt{m-1} f_{1m} \sqrt{(m-1)m}
\]

\[
= \sum_{m=0}^{N-2} (m+1) \sqrt{m} f_{1m} + \sum_{m=1}^{N-1} (-2m) \sqrt{m} f_{1m} + \sum_{m=2}^{N} (m-1) \sqrt{m} f_{1m}
\]

\[
= 0 + \sum_{m=N-1}^{N-1} (-2m) \sqrt{m} f_{1m} + \sum_{m=N-1}^{N} (m-1) \sqrt{m} f_{1m}
\]

\[
= -N \sqrt{N-1} f_{1N-1} + (N-1) \sqrt{N} f_{1N}.
\]

Since \( g = 0 \), it means

\[
f_{1N} = \sqrt{\frac{N}{N-1}} f_{1N-1}.
\]

Thus we obtain

\[
f_{1N} = \sqrt{N} f_{11}.
\]

Then \( f_1 = 0 \) because \( \sum_{N=1}^{\infty} |f_{1N}|^2 < \infty \). □

**Lemma 19.** The following holds.

\[
B^1(\mathcal{F}_P)|_{U^{-1}} = \left\{ f_1 \omega_1 + f_2 \omega_2 \in Z^1(\mathcal{F}_P)|_{U^{-1}} \mid \sum_{m=1}^{\infty} \sqrt{m} f_{2m} = 0 \right\}.
\]

Specially, \( Z^1(\mathcal{F}_P)|_{U^{-1}} \) is spanned by \( \phi_1(\omega_1 - \omega_2) \) and \( B^1(\mathcal{F}_P)|_{U^{-1}} \).
Proof. Proving that the right hand side contains $B^1(\mathcal{F}_P)|_{U^{-1}}$ is easy. To prove opposite, let $N$ be a positive integer. We put
\begin{equation}
 h_m = \frac{N + 1 - m}{\sqrt{m(N + 1)}} \quad (1 \leq m \leq N),
\end{equation}
\begin{equation}
 h = \sum_{m=1}^{N} h_m \phi_m.
\end{equation}
Then, we have
\begin{equation}
 d_{\mathcal{F}_P} h = (\text{some function}) \omega_1 + (-\sqrt{N+1} \phi_1 + \phi_{N+1}) \omega_2
\end{equation}
by (21). This formula and Lemma 18 prove the result. \qed

As in Section 3.2, the following Sobolev estimation holds:
\begin{equation}
 |||\xi||^2_k \leq |||\eta||^2_{k+3},
\end{equation}
where $\eta = f_1 \omega_1 + f_2 \omega_2$ is an element of the right hand side in Lemma 19 and $\xi = h$ is a 0-cochain constructed as Lemma 5.

4.3. Corresponding to the highest weight -1. Similar argument also holds in this case.

Lemma 20. The following holds.
\begin{equation}
 B^1(\mathcal{F}_P)|_{U^1} = \left\{ f_1 \omega_1 + f_2 \omega_2 \in Z^1(\mathcal{F}_P)|_{U^1} \left| \sum_{m=-\infty}^{-1} (-1)^{-m} \sqrt{-m} f_{2m} = 0 \right. \right\}.
\end{equation}
Specially, $Z^1(\mathcal{F}_P)|_{U^1}$ is spanned by $\phi_{-1}(\omega_1 + \omega_2)$ and $B^1(\mathcal{F}_P)|_{U^1}$.

As in Section 3.3, the following Sobolev estimation holds:
\begin{equation}
 |||\xi||^2_k \leq |||\eta||^2_{k+3},
\end{equation}
where $\eta = f_1 \omega_1 + f_2 \omega_2$ is an element of the right hand side in Lemma 20 and $\xi = h$ is some 0-cochain.

4.4. The other cases. Set $\pi \neq I, U^{-1}, U^1$ and fix $m_\pi \in \mathbb{M}$. We still start with proving the trivialness of the special 1-cocycles.

Lemma 21. Let $f_1 \omega_1 + f_2 \omega_2 \in Z^1(\mathcal{F}_P)|_{\pi}$. Assume $f_{1m} = f_{2m} = f_{1m+3} = f_{2m+3} = 0$ for any $m \in \mathbb{M}$ which satisfies $m \equiv m_\pi \pmod{4}$. Then, $f_1 = f_2 = 0$. 
Proof. Under the assumption, 1-cocycle conditions (22) is realized as the kernel of the linear map
\[(f_1 m_{+1}, f_2 m_{+1}, f_1 m_{+2}, f_2 m_{+2}) \mapsto (g m, g m_{+1}, g m_{+2}, g m_{+3})\]
for each \(m \equiv m_\pi\). (See (54).) Recall that the determinant \(\gamma_{q m}\) is not 0. Thus \(f_1 = f_2 = 0\). □

We consider the linear map
\[(h_m, h_{m+1}, h_{m+2}, h_{m+3}) \mapsto (f_1 m_{+1}, f_2 m_{+1}, f_1 m_{+2}, f_2 m_{+2})\]
for each \(m \equiv m_\pi + 2\) in (20) and (21). The coefficient matrix is also a block diagonal matrix whose block is a \(4 \times 4\) matrix. Each block is represented as
\[
\begin{pmatrix}
-\alpha_{q m} & 0 & \beta_{q m+2} & 0 \\
\alpha_{q m} & -2(m + 1) & \beta_{q m+2} & 0 \\
0 & -\alpha_{q m+1} & 0 & \beta_{q m+3} \\
0 & \alpha_{q m+1} & -2(m + 2) & \beta_{q m+3}
\end{pmatrix}
\begin{pmatrix}
h_m \\
h_{m+1} \\
h_{m+2} \\
h_{m+3}
\end{pmatrix}
\]
Its determinant is also \(\gamma_{q m}\) defined in (55).

Lemma 22. \(H^1(\mathcal{F}_P)|_\pi = \{0\}\).

Proof. Take any \(f_1 \omega_1 + f_2 \omega_2 \in Z^1(\mathcal{F}_P)|_\pi\). From (102), we can construct \(h\) such that \(f_1 \omega_1 + f_2 \omega_2 - d_{\mathcal{F}_P} h\) satisfies the assumption of Lemma 21. Then \(f_1 \omega_1 + f_2 \omega_2 = d_{\mathcal{F}_P} h\). □

As in Section 3.4, the following Sobolev estimation holds:
\[(104) \quad ||\xi||_k^2 \leq \left(\frac{3!}{q_P}\right)^2 25(k+3)+8 ||\eta||_{k+3}^2,\]
where \(\eta = f_1 \omega_1 + f_2 \omega_2\) is any 2-cocycle and \(\xi = h\) is some 0-cochain.

4.5. The whole sum. As in Section 3.5, for each \(\eta \in B^1(\mathcal{F}_P)\), we have \(\xi \in \Omega^0(\mathcal{F}_P)\) such that
\[(105) \quad ||\xi||_k^2 \leq C_{r, k+3} ||\eta||_{k+3}^2.\]
The following holds.

Proposition 23. The set \(\{x, y_1, \ldots, y_g\}\) is basis for \(H^1(\mathcal{F}_P)\), where the number \(g\) is the multiplicity of \(U^{-1}\) and \(U^1\).

Remark 24. This recovers the result (7) by Matsumoto-Mitsumatsu.
5. Determining the ring structure

We can prove our main theorem by combining the above preparation with the following lemma.

**Lemma 25.** Let $\phi_1, \phi'_1 \in L^2(M_\Gamma)$ be weight vectors of $U^{-1}$. Here, $\phi_1$ and $\phi'_1$ do not necessarily belong to the same irreducible component. Also, let $\phi_{-1}, \phi'_{-1} \in L^2(M_\Gamma)$ be weight vectors of $U^1$. Then,

\begin{align}
(106) & \quad (X_0^2 - X_1^2 - Y^2)(\phi_1 \phi'_1) = -2\phi_1 \phi'_1, \\
(107) & \quad (X_0^2 - X_1^2 - Y^2)(\phi_{-1} \phi'_{-1}) = -2\phi_{-1} \phi'_{-1}, \\
(108) & \quad X_0(\phi_1 \phi_{-1}) = 0.
\end{align}

Especially, $\phi_1 \phi'_1, \phi_{-1} \phi'_{-1}$ and $\phi_1 \phi_{-1}$ orthogonal to $U^{-1}$ and $U^1$.

**Proof.** Formulae are proved easily. The first two of them mean that $\phi_1 \phi'_1$ and $\phi_{-1} \phi'_{-1}$ are eigenvectors corresponding to $-2$ of the Casimir element. On the other hand, the Casimir element vanishes on $U^{-1}$ and $U^1$. Then, they orthogonal to $U^{-1}$ and $U^1$. Also $\phi_1 \phi_{-1}$ does by (108). Indeed, the set $\mathbb{M}$ of $U^{-1}$ and $U^1$ does not contain 0. □

**Proof of Theorem 1** Generators of $H^*(F_\mathbb{P})$ are given in Proposition 15 and 23. The vanishing of $y_i \wedge y_j$ in $H^2(F_\mathbb{P})$ follows from Lemma 25 for each $1 \leq i, j \leq 2g$. □

**Remark 26.** When $\Gamma$ is the fundamental group of a closed orientable hyperbolic surface, the vanishing of $y_i \wedge y_j$ is implied by the ring structure of $H^*_{dR}(M_\Gamma)$. In fact, we get the embedding $H^*_{dR}(M_\Gamma)/H^3_{dR}(M_\Gamma) \subset H^*(F_\mathbb{P})$ as rings from (1) and (2). The ring structure of $H^*_{dR}(M_\Gamma)$ is determined by the Thom-Gysin sequence and [6, Lemma 1].

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