Decay rate constrained stability analysis for positive systems with discrete and distributed delays

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This paper is concerned with the decay rate constrained exponential stability analysis for continuous-time positive systems with both time-varying discrete and distributed delays. A necessary and sufficient condition is first given to ensure that a positive system with distributed delay is exponentially stable and satisfies a prescribed decay rate. Furthermore, by exploiting the monotonicity of the trajectory of a constant delay system and comparing the trajectory of the time-varying delay system with that of the constant delay system, the results are extended to positive systems with both bounded time-varying discrete delays and distributed delays.

Keywords: distributed delays; exponential stability; positive systems; time-delay systems

1. Introduction

The study of positive systems has attracted tremendous attention in the recent years. The fact that many practical models involve quantities which are intrinsically nonnegative naturally gives rise to such systems whose state variables and output signal are always constrained in the first orthant whenever both the initial condition and the input signal are nonnegative. This stimulates a variety of works on the issue of dynamic systems under positivity constraint. Positive systems have applications in a wide range of disciplines involving systems biology (de Jong 2002), pharmacokinetics (Jacquez 1985) and ecology (Caswell 2001). A typical example is the mathematical modeling of compartmental networks (Haddad, Chellaboina, & Hui, 2010), which captures the exchange of nonnegative quantities of materials among compartments and the environment with conservation of mass of materials. Along with abundant practical applications, the mathematical theory of positive systems is originated from the well-known Perron–Frobenius theorem (Berman & Plemmons 1994), which arises in the analysis of nonnegative matrices. In the past decade, a wealth of literature has been devoted to the analysis and synthesis of positive systems, just to name a few, we refer the readers to Kaczorek (2002), Kaczorek (2008), Liu (2009), Haddad et al. (2010), Li, Lam, and Shu (2010), Feng, Lam, Li, and Shu (2011), Kaczorek (2011), Ait Rami and Tadeo (2007), Ait Rami and Napp (2012), Ait Rami (2011), Zhao, Zhang, Shi, and Liu (2012), and Zhao, Zhang, and Shi (2013) and the references therein.

On the other hand, as in other dynamic systems, time delay is often encountered in the analysis of positive systems. Many works have been reported on the stability analysis of positive delay systems in the literature. The stability of both linear and nonlinear positive systems with constant delays was studied in Haddad and Chellaboina (2004) and it is pointed out that asymptotic stability can be preserved regardless of the magnitude of delays. It is further shown in Liu, Yu, and Wang (2009, 2010) that both discrete- and continuous-time positive systems with bounded time-varying delays are asymptotically stable as long as the corresponding delay-free systems are asymptotically stable. Moreover, it is found that similar results also hold for positive switched systems with unbounded time-varying delays in Liu and Dang (2010). Nevertheless, exponential stability analysis of positive delay system has not received much attention until quite recently. Decay rate constrained output-feedback stabilization is solved in Feng et al. (2011) via iterative linear matrix inequality approach by a modified cone complementarity linearization method (El Ghaoui, Oustry, & Ait Rami, 1997). In Zhu, Li, and Zhang (2012) and Zhu, Meng, and Zhang (2013), it has been shown that although bounded discrete delay has no impact on the asymptotic stability of positive systems, it does affect the decay rate of the state trajectory. The positivity and exponential stability for linear and nonlinear systems with both discrete and distributed delays were studied in Ngoc (2013). Recently, the characterization of the $L_\infty$-gain of positive systems with
discrete and distributed delays was given in Shen and Lam (2013a, b).

Motivated by the above discussion, in this paper, we address the decay rate constrained stability analysis problem for positive systems with both bounded time-varying discrete delays and distributed delays. The results obtained in this paper can be regarded as extensions of the asymptotic stability analysis for positive systems with bounded time-varying discrete delays in Liu et al. (2010) and the exponential stability analysis for positive systems with both time-varying discrete and distributed delays in Ngoc (2013). It is worth mentioning that the approach employed in this paper is different from that used in either Liu et al. (2010) or Ngoc (2013). Note that the technical proof in Liu et al. (2010) is extremely complicated. In fact, utilizing the methods in this paper, one can give an alternative proof for the results in Liu et al. (2010) and Ngoc (2013). The advantage of our approach lies in two aspects. Firstly, for the constant delay case, we make use of a Lyapunov–Krasovskii functional, which can be applied to nonlinear positive system as well as switched positive systems with average dwell time switching signal like those considered in Zhao et al. (2012) and switched positive systems with average dwell time switching signal like those considered in Zhao et al. (2012).

for all initial condition \( h > 0 \) is a constant; \( A_0 s \in C([-h, 0], R_{++n}) \).

The definition and characterization of the positivity of system (1) are presented below.

**Definition 1 (Farina & Rinaldi 2000)** System (1) is called (internally) positive if for all initial condition \( \phi(s) \geq 0 (s \in [-h, 0]) \), the state trajectory \( x(t) \geq 0 \) for all \( t \geq 0 \).

**Lemma 2 (Ngoc, 2013, Theorem II.2 )** System (1) is (internally) positive if and only if \( A = Metzler \) and \( A_0 s \) is nonnegative for all \( s \in [-h, 0] \).

The definition of exponential stability with a given decay rate is given in the following.

**Definition 2** Given \( \alpha > 0 \), system (1) is said to be exponentially stable with decay rate \( \alpha \) if there exists a constant \( M > 0 \), such that for any initial condition \( \phi(\cdot) \in C([-h, 0], R^n) \) satisfying that \( \phi(s) \geq 0 (s \in [-h, 0]) \), the solution of system (1) satisfies that \( \|x(t; \phi)\|_{\infty} \leq M\|\phi\|e^{-\alpha t} \) for all \( t \geq 0 \).

**Remark 1** It is well known that all vector norms \( \|\cdot\|_{p} (p \in [1, \infty]) \) defined on \( R^n \) are equivalent, that is, for any \( p_1 \neq p_2 \), there exists constants \( c_1, c_2 > 0 \), such that \( c_1 \|x\|_{p_1} \leq \|x\|_{p_2} \leq c_2 \|x\|_{p_2} \) for any \( x \in R^n \). Therefore, without loss of generality, one can employ \( \infty \)-norm to define the exponential stability with a prescribed decay rate, which is more convenient for the analysis of positive systems.

In the following theorem, based on the Lyapunov–Krasovskii functional as well as a simple transformation, we
give a necessary and sufficient condition under which positive system (1) is exponentially stable with a given decay rate.

**Theorem 1** Suppose that system (1) is positive. Then, for any given $\alpha > 0$, system (1) is exponentially stable with decay rate $\alpha$ if and only if Metzler matrix $\alpha I + A + \int_{-h}^0 e^{-\alpha s}A_h(s) \, ds$ is Hurwitz.

**Proof** (Sufficiency) Construct a Lyapunov–Krasovskii functional:

$$V(x_t) = \lambda^T e^{\alpha t} x(t) + \lambda^T \int_{-h}^0 A_h(s) \int_{s}^{t} e^{\alpha (t-\tau)} x(\tau) \, d\tau \, ds,$$

where $\lambda > 0$. Taking derivative of $V(x_t)$ along the trajectory of system (1) yields that

$$\dot{V}(x_t) = \lambda^T e^{\alpha t} \left( (\alpha I + A)x(t) + \int_{-h}^0 A_h(s) x(t + s) \, ds \right)$$

$$+ \lambda^T e^{\alpha t} \left( \int_{-h}^0 e^{-\alpha s} A_h(s) \, ds \, dx(t) \right)$$

$$- \int_{-h}^0 A_h(s) x(t + s) \, ds$$

$$= \lambda^T e^{\alpha t} \left( \alpha I + A + \int_{-h}^0 e^{-\alpha s} A_h(s) \, ds \right) x(t).$$

By Lemma 1, $\alpha I + A + \int_{-h}^0 e^{-\alpha s} A_h(s) \, ds$ is a Hurwitz matrix implies that there exists a column vector $\lambda > 0$, such that $\lambda^T (\alpha I + A + \int_{-h}^0 e^{-\alpha s} A_h(s) \, ds) < 0$. Therefore, $\dot{V}(x_t) \leq 0$ for all $t \geq 0$, which implies that $V(x_t) \leq V(x_0) = \lambda^T x(0) + \lambda^T \int_{-h}^0 A_h(s) \int_{s}^{0} e^{\alpha (t-\tau)} x(\tau) \, d\tau \, ds$ for all $t \geq 0$. It is obvious that there exists $M > 0$ dependent on $h$ and $\max_{s \in [-h,0]} A_d(s)$, such that $V(x_t) \leq \|x\|_\infty M \|\phi\|$ for all $t \geq 0$. Also note that $V(x_t) \geq \lambda^T e^{\alpha t} x(t)$ and thus for all $t \geq 0$, we have

$$\min_{i=1,2,...,n} \|x_i\|_\infty \leq \min_{i=1,2,...,n} \|\lambda_i\|_\infty \|x(t)\|_\infty \leq \lambda^T e^{\alpha t} x(t) \leq M \|x\|_\infty \|\phi\|.$$

Then, it follows that $\|x(t)\|_\infty \leq (M \|\lambda\|_\infty / \min_{i=1,2,...,n} \|\lambda_i\|_\infty) e^{\alpha t} \|x(t)\|_\infty$, which reveals that system (1) is exponentially stable with decay rate $\alpha$. This completes the sufficiency part.

(Necessity) Suppose that there exists a constant $M > 0$, such that for any initial condition $\phi(s) \geq 0$ ($s \in [-h,0]$), the state trajectory of system (1) satisfies that $\|x(t; \phi)\|_\infty \leq M e^{-\alpha t} \|\phi\|$. For any constant $\epsilon$ satisfying that $0 < \epsilon < \alpha$, we define $\beta \triangleq \alpha - \epsilon > 0$ and $y(t) \triangleq e^{\beta t} x(t)$, then $y(t)$ is the solution of the following system:

$$\dot{y}(t) = \beta e^{\beta t} x(t) + e^{\beta t} \dot{x}(t)$$

$$= (\beta I + A) y(t) + \int_{-h}^0 e^{-\beta s} A_h(s) y(t + s) \, ds.$$

Note that $\|y(t)\|_\infty = e^{\beta t} \|x(t)\|_\infty \leq M e^{\beta t - \alpha t} \|\phi\| = M e^{-\epsilon t} \|\phi\|$. Therefore, system (2) is exponentially stable with decay rate $\epsilon$. By (Ngoc, 2013, Theorem III.1), we have $\beta I + A + \int_{-h}^0 e^{-\beta s} A_h(s) \, ds$ is a Hurwitz matrix. It is well known that the spectrum of a matrix is continuously varying with respect to the variations of its entries and so is the spectral abscissa. For sufficiently small $\epsilon$, we can deduce that $\alpha I + A + \int_{-h}^0 e^{-\alpha s} A_h(s) \, ds$ is a Hurwitz matrix, which completes the necessity part.

**Remark 2** Unlike (Ngoc 2013), Lyapunov method is utilized here for exponential stability analysis, which can easily take into account decay rate of the state trajectory. It would also be useful in the stability analysis of switched positive systems such as those considered in Zhao et al. (2012) and Zhao et al. (2013). In addition, this approach has potential applications in the analysis of nonlinear positive systems when the nonlinearity fulfills a Lipschitz condition.

### 4. Extension to positive systems with time-varying discrete and distributed delays

In this section, we aim to extend the results in the last section to positive systems with both time-varying discrete and distributed delays. Let us consider the following continuous-time linear system with both time-varying discrete and distributed delays:

$$\dot{x}(t) = A x(t) + A_d x(t - \tau(t)) + \int_{-h(t)}^0 A_h(s) x(t + s) \, ds,$$

where $x(t) \in \mathbb{R}^n$ stands for the state vector; $\tau(t) \leq \tau$ and $h(t) \leq h$ where $\tau, h > 0$ are constants; $\phi(\cdot) \in C([-\max[h,\tau],0], \mathbb{R}^n)$ is the initial condition; $A_h(s) \in C([-h,0], \mathbb{R}^{n \times n})$.

The positivity of system (3) is characterized in the following lemma.

**Lemma 3** (Ngoc 2013) For all delays $\tau(t)$ and $h(t)$ satisfying that $\tau(t) \leq \tau$ and $h(t) \leq h$, system (3) is positive if and only if $A$ is Metzler, $A_d$ is nonnegative and $A_h(s)$ is nonnegative for all $s \in [-h,0]$.

In the following, we always assume that $A$ is Metzler, $A_d$ is nonnegative and $A_h(s)$ is nonnegative for all $s \in [-h,0]$. In order to analyze the state trajectory of system (3), we first study the following positive systems with constant discrete...
delays and distributed delays over a fixed interval:

\[ \dot{x}(t) = Ax(t) + A_x x(t - \tau) + \int_{-h}^{0} A_s(s)x(t + s) \, ds. \]  

(4)

The stability analysis of system (4) can be performed in a manner similar to that of system (1).

**Theorem 2** Given \( \alpha > 0 \), positive system (4) is exponentially stable with decay rate \( \alpha \) if and only if \( aI + A + A_xe^{\alpha \tau} + \int_{-h}^{0} e^{-\alpha \tau}A_s(s) \, ds \) is a Hurwitz matrix.

**Proof** The sufficiency can be proved by constructing the following Lyapunov–Krasovskii functional:

\[ V(x_t) = \lambda^T e^{\alpha \tau} x(t) + \lambda^T A_x \int_{t-\tau}^{t} e^{\alpha(\theta + \tau)} x(\theta) \, d\theta + \lambda^T \int_{-h}^{0} A_s(s) \int_{t-\tau}^{t} e^{\alpha(\theta + \tau)} x(\theta) \, d\theta \, ds. \]

The necessity can be proved by following a line similar to the proof of Theorem 1 and hence the detailed proof is omitted here.

Now we are in the position to investigate the decay rate constrained exponential stability of system (3). Before moving on, the following lemma is needed for further development. The result directly follows from the linearity and the positivity of system (3).

**Lemma 4** Suppose that \( x(t; \phi_1) \) and \( x(t; \phi_2) \) are state trajectories of system (3) with initial condition \( \phi_1 \) and \( \phi_2 \), respectively. Then, \( \phi_1(s) \leq \phi_2(s) \) for \( s \in [-\max[h, \tau], 0] \) implies that \( x(t; \phi_1) \leq x(t; \phi_2) \) for \( t \geq 0 \).

The following two lemmas aim to give a monotonic property of system (4) with a particular constant initial condition.

**Lemma 5** Suppose that \( \lambda > 0 \) satisfies that \( (A + A_x + \int_{-h}^{0} A_s(s) \, ds) \lambda < 0 \). Then, the state trajectory of system (4) with initial condition \( \phi(s) \equiv \lambda \) (\( s \in [-\max[h, \tau], 0] \)) satisfies that \( x(t) \leq \lambda \) for all \( t \geq 0 \).

**Proof** Let \( e(t) \equiv x(t) - x(t) \), then \( e(t) \) satisfies that

\[ \ddot{e}(t) = A_0 e(t) + A_1 e(t - \tau) + \int_{-h}^{0} A_s(s)e(t + s) \, ds \]

\[ - (A + A_x + \int_{-h}^{0} A_s(s) \, ds) \lambda. \]

(5)

Noting that \( e(s) = 0 \) for \( s \in [-\max[h, \tau], 0] \) and system (5) is positive, it follows that \( e(t) \geq 0 \) for all \( t \geq 0 \) by regarding \( -(A + A_x + \int_{-h}^{0} A_s(s) \, ds) \lambda \) as a nonnegative input. This implies that \( x(t) \leq \lambda \) for all \( t \geq 0 \), which completes the proof.

**Lemma 6** Suppose that \( \lambda > 0 \) satisfies that \( (A + A_x + \int_{-h}^{0} A_s(s) \, ds) \lambda < 0 \). Then, the state trajectory of system (4) with initial condition \( \phi(s) \equiv \lambda \) (\( s \in [-\max[h, \tau], 0] \)) is monotonically non-increasing, that is, \( x(t_1) \geq x(t_2) \) for any \( t_2 > t_1 \geq 0 \).

**Proof** Given any constant \( c > 0 \), define \( e(t) \equiv x(t) - x(t + c) \), then \( e(t) \) satisfies that

\[ \ddot{e}(t) = A_0 e(t) + A_1 e(t - \tau) + \int_{-h}^{0} A_s(s)e(t + s) \, ds. \]

(6)

Note that the initial condition \( e(s) = 0 \) for \( s \in [-\max[h, \tau], 0] \), if \( x(t_1) \leq x(t_2) \) for all \( t \geq 0 \), which completes the proof.

Based on the monotonicity of constant delay system (4) with constant initial condition \( \phi(s) \equiv \lambda \) (\( s \in [-\max[h, \tau], 0] \)), we compare the state trajectories of systems (3) and (4) under this initial condition.

**Lemma 7** Suppose that \( \lambda > 0 \) satisfies that \( (A + A_x + \int_{-h}^{0} A_s(s) \, ds) \lambda < 0 \) and that \( x_1(t) \) and \( x_2(t) \) are the state trajectories of systems (3) and (4) with the same initial condition \( \phi(s) \equiv \lambda \) (\( s \in [-\max[h, \tau], 0] \)), respectively. Then \( x_1(t) \leq x_2(t) \) for all \( t \geq 0 \).

**Proof** Let \( e(t) \equiv x_2(t) - x_1(t) \), then \( e(t) \) is the solution of system

\[ \dot{e}(t) = A_0 e(t) + A_1 e(t - \tau) + \int_{-h(t)}^{0} A_s(s)e(t + s) \, ds \]

\[ + A_x (x_2(t - \tau) - x_2(t - \tau(t))) \]

\[ + \int_{-h(t)}^{0} A_s(s)x_2(t + s) \, ds. \]

(7)

By Lemma 6, \( x_2(t - \tau) - x_2(t - \tau(t)) \geq 0 \) for all \( t \geq 0 \). Noting that \( e(s) = 0 \) for \( s \in [-\max[h, \tau], 0] \) and the error system (7) is positive, it follows that \( e(t) \geq 0 \) for all \( t \geq 0 \) by regarding \( A_x (x_2(t - \tau) - x_2(t - \tau(t))) + \int_{-h(t)}^{0} A_s(s)x_2(t + s) \, ds \) as a nonnegative input. Therefore, \( x_2(t) \geq x_1(t) \) for all \( t \geq 0 \), which completes the proof.

In the light of Lemmas 4 and 7, we are ready to state the main theorem of this section.

**Theorem 3** Given \( \alpha > 0 \), positive system (3) is exponentially stable with decay rate \( \alpha \) for all delays \( \tau(t) \) and \( h(t) \) satisfying that \( \tau(t) \leq \tau \) and \( h(t) \leq h \) if and only if positive system (4) is exponentially stable with decay rate \( \alpha \), or equivalently, \( \alpha I + A + A_x e^{\alpha \tau} + \int_{-h}^{0} e^{-\alpha \tau}A_x(s) \, ds \) is a Hurwitz matrix.
Proof  Necessity holds trivially. In the following we prove the sufficiency part. Suppose that positive system (4) is exponentially stable with decay rate $\alpha$ and that $x_1(t;\phi)$ and $x_2(t;\phi)$ are the state trajectories of systems (3) and (4) with the same initial condition $\phi$. Then, from (Ngoc, 2013, Theorem III.1), it can be concluded that there exists a column vector $\lambda > 0$, such that $(A + A_r + \int_{-h}^0 A_h(s) ds)\lambda < 0$ for any initial condition $\phi$, one can define $c = \|\lambda\|/\min_{j=1,2,...,n}\{\lambda_j\}$, then it is obvious that $c\lambda \geq \phi(s)$ for $s \in [-\max\{h, r\}, 0]$. For initial condition $\phi \equiv c\lambda$, by Lemma 7, we can conclude that $x_1(t; c\lambda) \leq x_2(t; c\lambda)$ for all $t \geq 0$. By Lemma 4, this implies that $x_1(t; \phi) \leq x_1(t; c\lambda) \leq x_2(t; c\lambda)$ for all $t \geq 0$. Since system (4) is exponentially stable with decay rate $\alpha$, there exists constant $M > 0$, such that $\|x_2(t; c\lambda)\| \leq Me^{-\alpha t}\|\lambda\| = (M\|\lambda\|\|/\min_{j=1,2,...,n}\{\lambda_j\})e^{-\alpha t}\|\phi\|$. This further implies that $\|x_1(t; \phi)\| \leq \|x_2(t; c\lambda)\| \leq (M\|\lambda\|\|/\min_{j=1,2,...,n}\{\lambda_j\})e^{-\alpha t}\|\phi\|$. Hence, system (3) is exponentially stable with decay rate $\alpha$, which completes the proof. \[\square\]

The above theorem can be easily extended to the multiple discrete delay case, and hence we only present the results in the following corollary while the detailed proof is omitted.

Corollary 1 Suppose that $A$ is Metzler, $A_r$, $i = 1, 2, \ldots, N$, are all nonnegative and $A_h(s)$ is nonnegative for all $s \in [-h, 0]$. Consider the following continuous-time linear delay system:

$$\dot{x}(t) = Ax(t) + \sum_{i=1}^{N} A_r x(t - \tau_i(t)) + \int_{-h(t)}^{0} A_h(s)x(t + s) ds.$$ \label{eq:8}

System (8) is exponentially stable with decay rate $\alpha > 0$ for all delays $\tau_i(t)$ and $h(t)$ satisfying that $\tau_i(t) \leq \tau_i$ ($i = 1, 2, \ldots, N$) and $h(t) \leq h$, if and only if one of the following conditions holds:

1. $\alpha I + A + \sum_{i=1}^{N} A_r e^{\alpha \tau_i} + \int_{-h}^{0} e^{-\alpha s}A_h(s) ds$ is a Hurwitz matrix.
2. There exists a column vector $\lambda > 0$, such that $(\alpha I + A + \sum_{i=1}^{N} A_r e^{\alpha \tau_i} + \int_{-h}^{0} e^{-\alpha s}A_h(s) ds)\lambda < 0$.

5. Numerical example

Let us consider positive delay system (3) with the following system matrices:

$$A = \begin{bmatrix} -5 & 1 \\ 0.8 & -6 \end{bmatrix}, \quad A_r = \begin{bmatrix} 0.3 & 0.4 \\ 0.2 & 0.5 \end{bmatrix},$$

$$A_h(s) = \begin{bmatrix} 0 & -0.1s \\ -0.1s^3 & -0.1s \end{bmatrix}, \quad s \in [-3, 0].$$

The delays are given as $\tau_i(t) = 4 + 2\sin t$ and $h(t) = 2 + \sin t$, respectively. According to Theorem 3, one can easily check through linear programming that the decay rate that can be achieved is $\alpha = 0.192$. Given initial conditions $\phi(s) = [1 - \sin(0.1r) 1 + \cos(0.1r)]^T$ ($t \in [-6, 0]$), the state trajectory of system (3) is depicted in Figure 1. One can observe that the state of system (3) satisfies that $\|x(t)\| \leq y(t) = 3e^{-0.192t}$. Therefore, it holds that $\ln(\|x(t)\|) \leq \ln(y(t)) = \ln(3e^{-0.192t})$, which can be verified from Figure 2.

6. Conclusions

In this paper, the decay rate constrained exponential stability problem of positive systems with both time-varying discrete and distributed delays has been investigated. An explicit characterization has been given to ensure that a positive system with distributed delays is exponentially stable and satisfies a prescribed decay rate. The results have been further extended to positive systems with both bounded time-varying discrete delays and distributed delays. In fact,
this paper has provided a very simple alternative proof for the results in Liu et al. (2010) and Ngoc (2013). It is worth noting that our approach only relies on the monotonicity and positivity of the system, hence it is flexible and also has potential applications in the analysis of switched positive systems with delays. Applications of the methods proposed in this paper to these topics would be our future research directions.

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References
Ait Rami, M. (2011). Solvability of static output-feedback stabilization for LTI positive systems. Systems & Control Letters, 60(9), 704–708.
Ait Rami, M., & Napp, D. (2012). Characterization and stability of autonomous positive descriptor systems. IEEE Transactions on Automatic Control, 57(10), 2668–2673.
Ait Rami, M., & Tadeo, F. (2007). Controller synthesis for positive linear systems with bounded controls. IEEE Transactions on Circuits and Systems II: Express Briefs, 54(2), 151–155.
Berman, A., & Plemmons, R. J. (1994). Nonnegative matrices. SIAM: Philadelphia, PA.
Caswell, H. (2001). Matrix population models: Construction, analysis and interpretation. Sunderland, MA: Sinauer Associates.
El Ghaoui, L., Oustry, F., & Ait Rami, M. (1997). A cone complementarity linearization algorithm for static output-feedback and related problems. IEEE Transactions on Automatic Control, 42(8), 1171–1176.
Farina, L., & Rinaldi, S. (2000). Positive linear systems: Theory and applications. New York, NY: Wiley-Interscience.
Feng, J., Lam, J., Li, P., & Shu, Z. (2011). Decay rate constrained stabilization of positive systems using static output feedback. International Journal of Robust and Nonlinear Control, 21(1), 44–54.
Haddad, W. M., & Chellaboina, V. (2004). Stability theory for nonnegative and compartmental dynamical systems with time delay. Systems & Control Letters, 51(5), 355–361.
Haddad, W. M., Chellaboina, V. S., & Hui, Q. (2010). Nonnegative and compartmental dynamical systems. Princeton, NJ: Princeton University Press.
Jacquez, J. (1985). Compartmental analysis in biology and medicine. Ann Arbor, MI: University of Michigan Press.
de Jong, H. (2002). Modeling and simulation of genetic regulatory systems: A literature review. Journal of Computational Biology, 9(1), 67–103.
Kaczorek, T. (2002). Positive 1D and 2D systems. London: Springer-Verlag.
Kaczorek, T. (2008). Fractional positive continuous-time linear systems and their reachability. International Journal of Applied Mathematics and Computer Science, 18(2), 223–228.
Kaczorek, T. (2011). Positive linear systems consisting of $n$ subsystems with different fractional orders. IEEE Transactions on Circuits and Systems I: Regular Papers, 58(6), 1203–1210.
Li, P., Lam, J., & Shu, Z. (2010). $H_\infty$ positive filtering for positive linear discrete-time systems: An augmentation approach. IEEE Transactions on Automatic Control, 55(10), 2337–2342.
Liu, X. (2009). Constrained control of positive systems with delays. IEEE Transactions on Automatic Control, 54(7), 1596–1600.
Liu, X., & Dang, C. (2010). Stability analysis of positive switched linear systems with delays. IEEE Transactions on Automatic Control, 56(7), 1684–1690.
Liu, X., Yu, W., & Wang, L. (2009). Stability analysis of positive systems with bounded time-varying delays. IEEE Transactions on Circuits and Systems II: Express Briefs, 56(7), 600–604.
Liu, X., Yu, W., & Wang, L. (2010). Stability analysis for continuous-time positive systems with time-varying delays. IEEE Transactions on Automatic Control, 55(4), 1024–1028.
Ngoc, P. H. A. (2013). Stability of positive differential systems with delay. IEEE Transactions on Automatic Control, 58(1), 203–209.
Shen, J., & Lam, J. (2013a). $L_\infty$-gain analysis for positive systems with distributed delays. Automatica. Advance online publication. doi:10.1016/j.automatica.2013.09.037
Shen, J., & Lam, J. (2013b). On $L_\infty$ and $L_\infty$ gains for positive systems with bounded time-varying delays. International Journal of Systems Science. Advance online publication. doi:10.1080/00207721.2013.843217
Zhao, X., Zhang, L., & Shi, P. (2013). Stability of a class of switched positive linear time-delay systems. International Journal of Robust and Nonlinear Control, 23(5), 578–589.
Zhao, X., Zhang, L., Shi, P., & Liu, M. (2012). Stability of switched positive linear systems with average dwell time switching. Automatica, 48(6), 1132–1137.
Zhu, S., Li, Z., & Zhang, C. (2012). Exponential stability analysis for positive systems with delays. IET Control Theory & Applications, 6(6), 761–767.
Zhu, S., Meng, M., & Zhang, C. (2013). Exponential stability for positive systems with bounded time-varying delays and static output feedback stabilization. Journal of the Franklin Institute, 350(3), 617–636.