COMBINATORIAL HOWE DUALITY OF SYMPLECTIC TYPE

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Abstract. We give a new combinatorial interpretation of Howe dual pairs of the form $(\mathfrak{g}, \text{Sp}_{2\ell})$, where $\mathfrak{g}$ is a Lie (super)algebra of classical type. This is done by establishing a symplectic analogue of the RSK algorithm associated to this pair, in a uniform way which does not depend on $\mathfrak{g}$. We introduce an analogue of jeu de taquin sliding for spinor model of irreducible characters of a Lie superalgebra $\mathfrak{g}$ to define a $P$-tableau and show that the associated $Q$-tableau is given by a symplectic tableau due to King.

1. Introduction

The Robinson-Schensted-Knuth (simply RSK) algorithm or correspondence is one of the most fundamental results in the theory of symmetric functions with various applications and generalizations. From a viewpoint of representation theory, it can be regarded as a combinatorial aspect of Howe duality [9, 10], which is a more general principle with important applications in many areas of mathematics. Indeed, the dual pair $(\mathfrak{g}, \text{GL}_n \otimes \mathfrak{gl}_n)$, which acts on the symmetric or exterior algebra generated by $C^m \otimes C^n$ as mutual centralizers, yields the Cauchy identity or its dual as its associated character. It has been generalized to a pair $(\mathfrak{g}, \text{GL}_n)$, where $\mathfrak{g}$ is a Lie (super)algebra of type $A$ and an irreducible unitarizable highest weight $\mathfrak{g}$-module (not necessarily integrable) appears in the duality (see [3, 6, 13]). A uniform description of the RSK correspondence for this dual pair $(\mathfrak{g}, \text{GL}_n)$ together with a combinatorial model of the associated irreducible $\mathfrak{g}$-modules is studied in [19].

The purpose of this paper is to establish a symplectic analogue of the RSK algorithm associated to various Howe dual pairs, which include the symplectic group $\text{Sp}_{2\ell}$. The duality associated to a pair $(\mathfrak{g}, \text{Sp}_{2\ell})$, which we are interested in this paper, can be described as follows (see [9, 10, 11, 25, 38] and references therein). Let $\mathcal{A}$ be a $\mathbb{Z}_2$-graded linearly ordered set and let $\mathcal{E}_\mathcal{A}$ be the super exterior algebra generated by the superspace with a linear basis indexed by $\mathcal{A}$. Then $\mathcal{F}_\mathcal{A} = \mathcal{E}_\mathcal{A}^* \otimes \mathcal{E}_\mathcal{A}$ is a semisimple module over a classical Lie (super)algebra $\mathfrak{g}_\mathcal{A}$, the type of which depends on $\mathcal{A}$, and the $\ell$-fold tensor power $\mathcal{F}_\mathcal{A}^\otimes \ell$ $(\ell \geq 1)$ is a $(\mathfrak{g}_\mathcal{A}, \text{Sp}_{2\ell})$-module with the following multiplicity-free decomposition:

$$
\mathcal{F}_\mathcal{A}^\otimes \ell \cong \bigoplus_{(\lambda, \ell) \in \mathcal{P}(\text{Sp}_\mathcal{A})} V_{\mathfrak{g}_\mathcal{A}}(\lambda, \ell) \otimes V_{\text{Sp}_{2\ell}}(\lambda),
$$

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where the direct sum is over a set $\mathcal{P}(\text{Sp})_A$ of pairs $(\lambda, \ell) \in \mathcal{P} \times \mathbb{Z}_+$ with $\ell(\lambda) \leq \ell$. Here $V_{\text{Sp}_{2\ell}}(\lambda)$ is the irreducible $\text{Sp}_{2\ell}$-module corresponding to $\lambda$, and $V_{q_A}(\lambda, \ell)$ is the irreducible highest weight $g_A$-module corresponding to $V_{\text{Sp}_{2\ell}}(\lambda)$ appearing in $\mathcal{F}_A^\ell$ (see Remark 3.4).

In $[21]$ and $[22]$, the second author introduced a combinatorial object called a spinor model of type $BCD$, which gives the character of $V_{q_A}(\lambda, \ell)$ in (1.1) in case of type $C$. As a set, the spinor model $T_A(\lambda, \ell)$ consists of sequences of usual semistandard tableaux of two-columned shapes with letters in $A$, where two adjacent tableaux satisfy certain configuration. It can be viewed as a super analogue of Kashiwara-Nakashima (simply KN) tableaux of type $BCD$ $[15]$, and has interesting applications including branching multiplicities for classical groups $[12]$, crystal bases of quantum superalgebras of orthosymplectic type $[21]$ $[22]$, and generalized exponents $[12]$ $[29]$.

It is natural to ask whether we have an analogue of RSK algorithm for (1.1) in terms of spinor model, and the main result in this paper is to construct an explicit bijection

$$ (1.2) \quad \mathcal{F}_A^\ell \rightarrow \bigcup_{\lambda \in \mathcal{P}(\text{Sp})_A} T_A(\lambda, \ell) \times K(\lambda, \ell). $$

Here $\mathcal{F}_A^\ell$ is the set of $2\ell$-tuple of $A$-semistandard tableaux of single-columned shapes with letters in $A$, and $K(\lambda, \ell)$ is the set of symplectic tableaux of shape $\lambda$ due to King $[17]$, which gives the character of $V_{\text{Sp}_{2\ell}}(\lambda)$. Hence the bijection (1.2) yields the Cauchy type identity which follows from the decomposition (1.1) for arbitrary $A$.

As a special case of our bijection (1.2) by putting $A$ to be a finite set of $n$ elements with degree 0, we recover the following well-known identity $[16]$

$$ (1.3) \quad \prod_{i=1}^{n} \prod_{j=1}^{\ell} (x_i + x_i^{-1} + z_j + z_j^{-1}) = \sum_{\lambda \in (n^\ell)} sp_{\rho^\ell(\lambda, \ell)}(x)s_{\lambda}(z), $$

where $\rho^\ell(\lambda, \ell)$ is the transpose of the rectangular complement of $\lambda$ in $(n^\ell)$, and $s_{\lambda}(z)$ is the character of $V_{\text{Sp}_{2\ell}}(\lambda)$ in $z_1^\pm, \ldots, z_n^\pm$ corresponding to $\lambda$.

On the other hand, by putting $A$ to be a finite set of $n$ elements with degree 1 and using the stability of $T_A(\lambda, \ell)$ for $\ell > n$ $[24]$, we also recover another well-known classical identity due to Littlewood $[32]$ and Weyl $[39]$, 

$$ (1.4) \quad \prod_{i=1}^{n} \prod_{j=1}^{\ell} (1 - x_i z_j)(1 - x_i z_j^{-1}) = \sum_{\ell(\lambda) \leq n} sp_{\lambda}(z)s_{\lambda}(x) \prod_{1 \leq i < j \leq n} (1 - x_i x_j)^{-1}, $$

where $s_{\lambda}(x)$ is the Schur polynomial in $x_1, \ldots, x_n$, and $\ell(\lambda)$ is the length of $\lambda$.

We remark that our bijection reduced to these cases is completely different from the ones in $[37]$ and $[36]$ for (1.3) and (1.4), respectively, where the insertion algorithm in terms of the King tableaux is used.

To construct a bijection (1.2), we first introduce an analogue of jeu de taquin sliding for spinor model of a skew shape, which plays a crucial role in this paper. If $A$ is a finite set with degree 0, then a spinor model of a skew shape is naturally in one-to-one correspondence with a set of symplectic tableaux of a skew shape in the sense of $[15]$, and our sliding coincides with the symplectic sliding due to Sheats $[35]$ (see also $[27]$). A key observation to generalize
the symplectic sliding algorithm to a spinor model with respect to arbitrary \( A \) is that in terms of spinor model, the Sheats’ algorithm can be described as a sequence of Kashiwara operators with respect to an \( \mathfrak{sl}_2 \)-crystal structure on \( P_A^\ell \) defined by the jeu de taquin sliding of type \( A \). This implies that the sliding does not depend on the choice of \( A \), and enables us to define a symplectic analogue of RSK algorithm in a uniform way. In this sense, the jeu de taquin sliding for spinor model introduced here is similar to the case of type \( A \), where the related combinatorics depends not essentially on the choice of the set of letters \( A \) (cf. [1, 34]).

Then we show that for \( T \in P_A^\ell \), there exists a unique tableau \( P(T) \in T_A(\lambda, \ell) \) which can be obtained by applying our jeu de taquin sliding. We also define the recording tableau corresponding to \( P_q(T) \), which is a sequence of oscillating tableaux, and then show that it corresponds bijectively to a tableau \( Q(T) \) in \( K(\lambda, \ell) \) thanks to a recent result by Lee [31].

It would be interesting to find other applications of our RSK algorithm and new interpretation of symplectic sliding in terms of type \( A \) crystals. We first expect an orthogonal analogue of RSK correspondence associated to Howe dual pairs \( (\mathfrak{g}, O_n) \) by using the spinor model of type \( B \) and \( D \) [21, 22]. The case of type \( D \) is more interesting, where a jeu de taquin sliding for \( KN \) tableaux in this case is not known yet [28]. Also we may adopt the spinor model to realize the crystals of Kirillov-Reshetikhin crystals of classical affine type corresponding to fundamental weights to describe a combinatorial \( R \) matrix and energy function on their tensor product, which is also closely related with Kazhdan-Lusztig polynomials (cf. [30]).

The paper is organized as follows: after a brief review on necessary materials in Section 2 we recall the definition of spinor model of type \( C \) in Section 3. In Section 4 we review the symplectic sliding on \( KN \) tableaux of a skew shape introduced in [35] and further developed in [27] by using crystals. In Section 5 we define an analogue of the jeu de taquin sliding for spinor model and then the \( P \)-tableau \( P(T) \) for \( T \in P_A^\ell \). In Section 6 we introduce the recording tableau or \( Q \)-tableaux for \( P_q(T) \), say \( Q(T) \), which is a sequence of oscillating tableau, and show that it naturally corresponds to a King tableau \( Q(T) \). In Section 7 we show that the map \( T \mapsto (P(T), Q(T)) \) gives a bijection in (1.2).

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2. Preliminaries

2.1. Notations. Let \( \mathbb{Z}_+ \) denote the set of non-negative integers. Let \( \mathcal{P} \) be the set of partitions or Young diagrams \( \lambda = (\lambda_1, \lambda_2, \ldots) \). We denote by \( \lambda' = (\lambda'_1, \lambda'_2, \ldots) \) the conjugate of \( \lambda \), and by \( \lambda^n \) the skew Young diagram obtained by \( 180^\circ \)-rotation of \( \lambda \). For \( n \geq 1 \), let \( \mathcal{P}_n = \{ \lambda \in \mathcal{P} \mid \ell(\lambda) \leq n \} \), where \( \ell(\lambda) \) is the length of \( \lambda \).

Let \( \mathcal{A} \) be a linearly ordered set with a \( \mathbb{Z}_2 \)-grading \( \mathcal{A} = A_0 \sqcup A_1 \). For \( n \geq 1 \), we let

\[
[n] = \{1 < 2 < \cdots < n\}, \quad [\bar{n}] = \{\bar{n} < n - 1 < \cdots < 1\},
\]

\[
I_n = [n] \cup [\bar{n}] = \{1 < \cdots < n < \bar{n} < \cdots < 1\},
\]
where we assume that all the entries of these sets are assumed to be of degree 0, and

$$[n]' = \{1' < 2' < \cdots < n'\},$$

where we assume that all the entries are assumed to be of degree 1. For positive integers $m$ and $n$, let

$$I_{m|n} = \{1 < 2 < \cdots < m < 1' < 2' < \cdots < n'\}$$

with $(I_{m|n})_0 = [m]$ and $(I_{m|n})_1 = [n]'$.

For a skew Young diagram $\lambda/\mu$, let $\text{SST}_A(\lambda/\mu)$ be the set of $A$-semistandard (or semistandard) tableaux of shape $\lambda/\mu$, that is, tableaux with entries in $A$ such that (1) the entries in each row (resp. column) are weakly increasing from left to right (resp. from top to bottom), (2) the entries in $A_0$ (resp. $A_1$) are strictly increasing in each column (resp. row).

For $T \in \text{SST}_A(\lambda/\mu)$, let $w(T)$ be a word obtained by reading the entries of $T$ column by column from right to left, and from top to bottom in each column (cf. [7]). For a semistandard tableau $S$, we define $S \rightarrow (w(S) \rightarrow T)$.

For a skew Young diagram with two columns.

$$\lambda(2, 1, 3) = \newcommand{\Cell}{\cell} \begin{array}{ccc} \Cell & \Cell & \Cell \\ \Cell & \Cell & \Cell \\ \Cell & \Cell & \Cell \end{array}$$

For $T \in \text{SST}_A(\lambda(a, b, c))$, let $T^\ell$ and $T^r$ denote the left and right columns of $T$, respectively.

If necessary, we assume that a tableau is placed on the plane $\mathbb{P}_L$ with a horizontal line $L$. Let $U_1, \ldots, U_r$ be column tableaux (that is, tableaux of single-columned shapes), which are $A$-semistandard. Let $[U_1, \ldots, U_r]$ denote the tableau in $\mathbb{P}_L$ (not necessarily of partition shape), where the $i$-th column from the left is $U_i$ and its bottom edge lies on $L$. Similarly, let $[U_1, \ldots, U_r]$ denote the tableau, where the $i$-th column from the left is $U_i$ and its top edge lies on $L$.

For $(u_1, \ldots, u_r) \in \mathbb{Z}_r^+$, let

$$[U_1, \ldots, U_r]_{(u_1, \ldots, u_r)}, \quad [U_1, \ldots, U_r]^{(u_1, \ldots, u_r)}$$

be the tableaux obtained from $[U_1, \ldots, U_r]$ and $[U_1, \ldots, U_r]$ by sliding each $U_i$ by $u_i$ positions up and down, respectively.
Example 2.1.

\[
\begin{array}{cccc}
1 & 1 & 2 & 2' \\
2 & 3 & 3' & \\
1' & 2' & 3' & \\
\end{array}
\]

\[L \vdash \{U_1, U_2, U_3, U_4\}_{(0, 1, 1, 3)} \in \text{SST}_{4\mathfrak{a}}((4, 4, 3, 1)/(2))\]

\[
\begin{array}{cccc}
3 & \text{ } & \text{ } & \\
3 & 5 & 3 & \\
5 & 4 & \text{ } & \\
3 & 2 & \text{ } & \\
1 & \text{ } & \text{ } & \\
\end{array}
\]

\[L \vdash \{U_1, U_2, U_3\}_{(2, 1, 0)} \in \text{SST}_{3\mathfrak{a}}((3, 3, 2, 1)/(2, 1))\]

2.2. Crystal and Schützenberger’s jeu de taquin. Let us briefly recall the notion of crystals (see [8, 14] for more details). Let \( g \) be a symmetrizable Kac-Moody algebra associated to a generalized Cartan matrix \( A = (a_{ij})_{i,j \in I} \) indexed by \( I \). A \( g \)-crystal is a set \( B \) together with the maps \( \varepsilon_i, \varphi_i : B \to \mathbb{Z} \cup \{ -\infty \} \) and \( \bar{e}_i, \bar{f}_i : B \to B \cup \{ 0 \} \) for \( i \in I \) satisfying certain axioms, where \( 0 \) is a formal symbol and \( P \) is the weight lattice of \( g \). For a dominant integral weight \( \Lambda \), we denote by \( B(\Lambda) \) the crystal associated to an irreducible highest weight \( g \)-module with highest weight \( \Lambda \). A crystal \( B \) is called regular if it is isomorphic to a disjoint union of \( B(\Lambda) \)'s. In this case, we have \( \varepsilon_i(b) = \max \{ k \mid \bar{e}_i^k b \neq 0 \} \) and \( \varphi_i(b) = \max \{ k \mid \bar{f}_i^k b \neq 0 \} \) for \( b \in B \) and \( i \in I \). For example, if \( A = [r] \) with \( r \geq 2 \), then \( \text{SST}_{1[r]}(\lambda) \) is a connected regular \( \mathfrak{gl}_r \)-crystal for \( \lambda \in \mathcal{P}_r \) [15].

Let \( A \) be a \( \mathbb{Z}_2 \)-graded linearly ordered set. The Schützenberger’s jeu de taquin is also available for \( A \)-semistandard tableaux (cf. [11, 34]), where we apply sliding process provided that the resulting tableau remains semistandard. For example, when \( A = \mathbb{Z}_{2|2} \), we have

\[
\begin{array}{cccc}
1 & 2 & \text{ } & \\
1 & \text{ } & \text{ } & \\
\end{array}
\rightarrow
\begin{array}{cccc}
1 & 2 & \text{ } & \\
1 & \text{ } & \text{ } & \\
\end{array}
\]

We use this algorithm in terms of crystal operator \( \mathcal{E} \) and \( \mathcal{F} \) for \( \mathfrak{sl}_2 \) (cf. [26]), which plays an important role in this paper.

For \( T \in \text{SST}_A(\lambda(a, b, c)) \) and \( 0 \leq k \leq \min\{a, b\} \), consider a tableau \( T' \) of shape \( \lambda(a - k, b - k, c + k) \) by sliding down \( T^k \) by \( k \) positions. Let \( r_T \) be the maximal \( k \) such that \( T' \) is semistandard.

For \( T \in \text{SST}_A(\lambda(a, b, c)) \) with \( r_T = 0 \), we define

1. \( \mathcal{E}T \) to be the tableau in \( \text{SST}_A(\lambda(a - 1, b + 1, c)) \) obtained from \( T \) by applying jeu de taquin sliding to the position below the bottom of \( T^k \) when \( a > 0 \),
2. \( \mathcal{F}T \) to be the tableau in \( \text{SST}_A(\lambda(a + 1, b - 1, c)) \) obtained from \( T \) by applying jeu de taquin sliding to the position above the top of \( T^k \) when \( b > 0 \).

Here we assume that \( \mathcal{E}T = 0 \) and \( \mathcal{F}T = 0 \) when \( a = 0 \) and \( b = 0 \), respectively, where \( 0 \) is a formal symbol.
Remark 2.2. (1) When we apply jeu de taquin sliding to a corner of \( T \), we have \( \tau_T > 0 \) if and only if a vertical move occurs by [21] Lemma 6.2. Thus, we have \( \tau_{\mathcal{E} T} = 0 \) and \( \tau_{\mathcal{F} T} = 0 \) whenever \( \mathcal{E} T \) and \( \mathcal{F} T \) are defined.

(2) Let \( \varepsilon(T) = \max\{k \mid \mathcal{E}^k T \neq 0\} \), \( \varphi(T) = \max\{k \mid \mathcal{F}^k T \neq 0\} \), and let \( \text{wt}(T) = \varphi(T) - \varepsilon(T) \). Then \( B = \{\mathcal{E}^k T \mid 0 \leq k \leq a\} \cup \{\mathcal{F}^l T \mid 0 \leq l \leq b\} \) forms a regular \( \mathfrak{sl}_2 \)-crystal with respect to \( \mathcal{E} \) and \( \mathcal{F} \), where we identify the weight lattice of \( \mathfrak{sl}_2 \) with \( \mathbb{Z} \). Indeed, if we let \( P = \mathbb{Z} \epsilon_1 \oplus \mathbb{Z} \epsilon_2 \) be the weight lattice of \( \mathfrak{sl}_2 \) with \( \epsilon_1 - \epsilon_2 \) the simple root and define \( \text{wt}(T) = m_1 \epsilon_1 + m_2 \epsilon_2 \), where \( m_1 \) (resp. \( m_2 \)) is the number of boxes of \( T^k \) (resp. \( T^l \)), then we may regard \( B \) as a \( \mathfrak{sl}_2 \)-crystal.

Example 2.3. Suppose that \( \mathcal{A} = \mathbb{I}_{4|3} \).

\[
\begin{array}{c|c|c|c}
2 & 2' & \varepsilon & 2 \\
1 & 2' & \mathcal{F} & 2 \\
1' & 2' & \varepsilon & 2 \\
1' & 3 & \mathcal{F} & 2 \\
\end{array}
\]

Next, for \( (U, V) \in \text{SST}_{\mathcal{A}}((1^u)) \times \text{SST}_{\mathcal{A}}((1^v)) \) \((u, v \in \mathbb{Z}_+))\), we define

\[
(2.1) \quad \mathcal{X}(U, V) = \begin{cases} \langle XT \rangle^k, \langle XT \rangle^l & \text{if } XT \neq 0, \\ 0 & \text{if } XT = 0, \end{cases} \quad \langle \mathcal{X}, \mathcal{E}, \mathcal{F} \rangle,
\]

where \( T \) is the unique tableau in \( \text{SST}_{\mathcal{A}}(\lambda(u - k, v - k, k)) \) for some \( 0 \leq k \leq \min\{u, v\} \) such that \( \langle T^k, T^l \rangle = (U, V) \) and \( \tau_T = 0 \).

2.3. Crystal and RSK correspondence. Let \( \mathcal{A} \) be a \( \mathbb{Z}_2 \)-graded linearly ordered set. For \( r \geq 2 \), let

\[
(2.2) \quad \mathbf{E}_{\mathcal{A}}^r = \bigcup_{(u_1, \ldots, u_r) \in \mathbb{Z}_+^r} \text{SST}_{\mathcal{A}}((1^{u_1})) \times \cdots \times \text{SST}_{\mathcal{A}}((1^{u_r})).
\]

For \( (U_r, \ldots, U_1) \in \mathbf{E}_{\mathcal{A}}^r \) and \( 1 \leq i \leq r - 1 \), we define

\[
(2.3) \quad \mathcal{X}_i(U_r, \ldots, U_1) = \begin{cases} (U_r, \ldots, \mathcal{X}(U_{i+1}, U_i), \ldots, U_1) & \text{if } \mathcal{X}(U_{i+1}, U_i) \neq 0, \\ 0 & \text{if } \mathcal{X}(U_{i+1}, U_i) = 0, \end{cases}
\]

where \( \mathcal{X} \) is defined in (2.1).

Let \( P = \bigoplus_{i=1}^r \mathbb{Z} \epsilon_i \) be the weight lattice of \( \mathfrak{gl}_r \), where \( \{ \alpha_i = \epsilon_i - \epsilon_{i+1} \mid i = 1, \ldots, r - 1 \} \) is the set of simple roots. Given \( T = (U_r, \ldots, U_1) \in \mathbf{E}_{\mathcal{A}}^r \), let \( \text{wt}(T) = \sum_{i=1}^r m_i \epsilon_i \), where \( m_i \) is the number of boxes of \( U_i \).

Lemma 2.4. \( \mathbf{E}_{\mathcal{A}}^r \) is a regular \( \mathfrak{gl}_r \)-crystal with respect to \( \text{wt}, \mathcal{E}_i \) and \( \mathcal{F}_i \) for \( 1 \leq i \leq r - 1 \).

Proof. It can be proved by similar arguments as in [20, 26]. Let \( \mathbf{M}_{\mathcal{A} \times [r]} \) be the set of matrices \( \mathbf{m} = (m_{ab}) \) with non-negative integral entries \((a \in \mathcal{A}, b \in [r])\) satisfying (1) \( m_{ab} \in \{0, 1\} \) if \( a \in \mathcal{A}_0 \), (2) \( \sum_{a,b} m_{ab} < \infty \). There is a natural bijection from \( \mathbf{E}_{\mathcal{A}}^r \) to \( \mathbf{M}_{\mathcal{A} \times [r]} \), where \( (U_r, \ldots, U_1) \in \mathbf{E}_{\mathcal{A}}^r \) is sent to \( \mathbf{m} = (m_{ab}) \) such that \( m_{ab} \) is the number of occurrences of \( a \) in \( U_b \).
Let \( m = (m_{ab}) \in M_{A \times \{r\}} \) be given. For \( a \in A \), we may identify the \( a \)-th row of \( m \) with a unique tableau \( T^{(a)} \) in \( SST_{\{r\}}(u) \) (resp. \( SST_{\{r\}}(1^a) \)) if \( a \in A_0 \) (resp. \( a \in A_1 \)), where \( u = \sum_b m_{ab} \) and \( m_{ab} \) is the number of occurrences of \( b \) in \( T^{(a)} \). We may define a regular \( gl_r \)-crystal structure on \( M_{A \times \{r\}} \) by regarding \( m \) as \( \bigotimes_{a \in A} T^{(a)} \). Then we can check that the associated operators \( \vec{e}_i \) and \( \vec{f}_i \) for \( 1 \leq i \leq r - 1 \) coincide with \( \vec{e}_i \) and \( \vec{f}_i \), and the \( gl_r \)-weight is equal to \( \omega \). Hence \( M_{A \times \{r\}} \) is a regular \( gl_r \)-crystal since it is a disjoint union of tensor products of regular \( gl_r \)-crystals \( SST_{\{r\}}((u)) \) and \( SST_{\{r\}}((1^a)) \).

Let us recall the RSK correspondence, which explains the decomposition of the \( gl_r \)-crystal \( E'_A \) into its connected components. Let \( U = (U_r, \ldots, U_1) \in E'_A \) given. Let \( P(U) = (U_r \rightarrow (\cdots \rightarrow (U_2 \rightarrow U_1) \cdots)) \) and \( Q(U) \) be the corresponding recording tableau, that is, if \( \text{sh}(P(U)) = \lambda \) (the shape of \( P(U) \)) and \( P_i = (U_i \rightarrow (\cdots (U_2 \rightarrow U_1) \cdots)) \) for \( 1 \leq i \leq r \), then \( Q(U) \) is the unique tableau in \( SST_{\{r\}}(\lambda') \) such that its subtableau \( \text{sh}(P_i')/\text{sh}(P_{i-1})' \) is a horizontal strip filled with \( i \), where we assume that \( P_0 \) is the empty tableau. Then we have a bijection

\[
(2.4) \quad E'_A \xrightarrow{\kappa_A} \bigsqcup_{\lambda' \in \mathcal{P}_r} SST_A(\lambda) \times SST_{\{r\}}(\lambda').
\]

Lemma 2.5. The bijection \( \kappa_A \) is an isomorphism of \( gl_r \)-crystals, where the right-hand side is a regular \( gl_r \)-crystal with respect to the second component.

Proof. It follows from the argument in the proof of Lemma 2.4 and the symmetry of RSK correspondence.

\[\square\]

Remark 2.6. When \( A = I_m[n] \), it is shown in [18] that the bijection (2.4) is an isomorphism of \( (gl_{m|n}, gl_r) \)-bicrystals, where \( gl_{m|n} \) is a general linear Lie superalgebra associated to a superspace \( \mathbb{C}^{m|n} \). This explains the \( (gl_{m|n}, gl_r) \)-duality [2] [9] in terms of crystals (see [20] for the case of arbitrary \( A \)).

3. Spinor model of symplectic type

3.1. Spinor model of type \( C \). Let us recall our main combinatorial object, which is introduced in [21] [22]. Let

\[ \mathcal{P}(Sp) = \{ (\lambda, \ell) \mid \ell \geq 1, \lambda \in \mathcal{P}_\ell \} \]

Hereafter \( A \) denotes a \( \mathbb{Z}_2 \)-graded linearly ordered set unless otherwise specified. For \( a \in \mathbb{Z}_+ \), let

\[ T_A(a) = \bigsqcup_{c \in \mathbb{Z}_+} SST_A(\lambda(a,0,c)). \]

Note that, for \( a \neq 0 \), \( T_A(a) \) is non-empty if and only if \( SST_A((1^a)) \neq \emptyset \). For \( T \in T_A(a) \), we have \( E^a T \in SST_A(\lambda(0,a,c)) \) and define

\[ L^T = (E^a T)^L, \quad R^T = (E^a T)^R. \]

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Example 3.1. Suppose $\mathcal{A} = \mathbb{I}_{43}$ and $T \in \text{SST}_A(\lambda(2,0,2))$ as below.

\[
T = \begin{array}{ccc}
2 & 2' & x \\
\uparrow & \uparrow & \uparrow \\
1' & 2' & \\
\downarrow & \downarrow & 3'
\end{array} \xrightarrow{\mathcal{E}} \begin{array}{ccc}
2 & 3' & 1' \\
\uparrow & \uparrow & \uparrow \\
1' & 2' & \\
\downarrow & \downarrow & \downarrow
\end{array} \xrightarrow{\mathcal{E}} \begin{array}{ccc}
2 & 2' & 1' \\
\uparrow & \uparrow & \uparrow \\
1' & 2' & 3' \\
\downarrow & \downarrow & \downarrow
\end{array}.
\]

Let $\lambda = (2,2,0,2)$ and $\ell = 2$. Then, for any partition $\mu$, we have

\[
\lambda \rightarrow \lambda' = (1,2,1,3).
\]

Definition 3.2.

(1) For $a_1, a_2 \in \mathbb{Z}_+$ with $a_2 \leq a_1$ and $(T_2, T_1) \in \mathcal{T}_A(a_2) \times \mathcal{T}_A(a_1)$, we define

$T_2 < T_1$ if $[r, T_2, T_1]^{\mathbb{R}}$ and $[T_2, T_1]^{\mathbb{L}}$ are $\mathcal{A}$-semistandard.

(2) For $(\lambda, \ell) \in \mathcal{P}(\text{Sp})$, we define

$\mathcal{T}_A(\lambda, \ell) = \{ T = (T_\ell, \ldots, T_1) \mid T_\ell < \cdots < T_1 \} \subset \mathcal{T}_A(\lambda_\ell) \times \cdots \times \mathcal{T}_A(\lambda_1)$.

Example 3.3. Let $\mathcal{A} = \mathbb{I}_{43}$, and take $S \in \mathcal{T}_A(1)$ and $T \in \mathcal{T}_A(2)$ as follows.

\[
\begin{array}{ccc}
2 & 2 & 3 & 4 \\
1' & 1' & \\
\end{array} \quad \begin{array}{ccc}
1 & 1 & 1 \\
2 & 1' & 2 \\
\end{array} \quad \begin{array}{ccc}
1 & 1 & 1 \\
1' & 3 & 2' \\
\end{array} \quad \begin{array}{ccc}
1 & 1 & 1 \\
1' & 3 & 2' \\
\end{array}.
\]

Then $S < T$ since $[r, S, T]^{\mathbb{L}}$ and $[S, T]^{\mathbb{L}, \mathbb{R}}$ form semistandard tableaux.

\[
\begin{array}{ccc}
\begin{array}{ccc}
1 & 2 & 3 \\
4 & 1' & 2' \\
\end{array} \quad \begin{array}{ccc}
1 & 2' & 3 \\
4 & 1' & 2' \\
\end{array} \quad \begin{array}{ccc}
1 & 2' & 3 \\
4 & 1' & 2' \\
\end{array}.
\end{array}
\]

Put

$\mathcal{P}(\text{Sp})_A = \{ (\lambda, \ell) \in \mathcal{P}(\text{Sp}) \mid \mathcal{T}_A(\lambda, \ell) \neq \emptyset \}$.

Then, for any partition $\lambda \neq \emptyset$, we have $\text{SST}_A(\lambda') \neq \emptyset$ if and only if $(\lambda, \ell) \in \mathcal{P}(\text{Sp})_A$.

Let $x_\mathcal{A} = \{ x_a \mid a \in \mathcal{A} \}$ be the set of formal commuting variables indexed by $\mathcal{A}$. Let $\mu \in \mathcal{P}$ be given such that $\text{SST}_A(\mu) \neq \emptyset$. For $T \in \text{SST}_A(\mu)$, let $x_\mathcal{A}^T = \prod_{a \in \mathcal{A}} x_a^{m_a}$, where $m_a$ is the number of occurrences of $a$ in $T$. Note that $s_\mu(x_\mathcal{A}) = \sum_{T \in \text{SST}_A(\mu)} x_\mathcal{A}^T$ is a super-analogue of Schur function corresponding to $\mu$.

Let $t$ be a variable commuting with $x_a (a \in \mathcal{A})$. For $(\lambda, \ell) \in \mathcal{P}(\text{Sp})_A$, we define the character of $\mathcal{T}_A(\lambda, \ell)$ to be

\[
S_{(\lambda, \ell)}(x_\mathcal{A}) = t^\ell \sum_{(T_1, \ldots, T_\ell) \in \mathcal{T}_A(\lambda, \ell)} x_\mathcal{A}^{T_1} \cdots x_\mathcal{A}^{T_\ell}.
\]
Remark 3.4. The character of $T_A(\lambda, \ell)$ has an important application in representation theory. Indeed, it is motivated by the $\langle g_A, Sp_{2\ell} \rangle$-duality \[11\] for a Lie (super)algebra $g_A$.

Let us first recall the decomposition \[1.1\] for various choices of $A$. If $A = [m]$, then we have $(sp_{2m}, Sp_{2\ell})$-duality, where $V_{sp_{2m}}(\lambda, \ell)$ is a finite-dimensional irreducible $sp_{2m}$-module. If $A = [n]'$, then we have $(so_{2n}, Sp_{2\ell})$-duality, where $V_{so_{2n}}(\lambda, \ell)$ is an infinite-dimensional irreducible $so_{2n}$-module. See \[9, 10, 11\] for these dualities. In general, when $A = \mathbb{I}_{m|n}$, we have $(sp_{2m|2n}, Sp_{2\ell})$-duality \[4\], which includes both of the above cases with $n = 0$ and $m = 0$, respectively. Here $sp_{2m|2n}$ is the orthosymplectic Lie superalgebra whose Dynkin diagram is given by

![Dynkin Diagram](image)

The dualities when $A$ is an infinite $\mathbb{Z}_2$-graded set can be found in \[25, 35\].

It is shown in \[21\] that the character of $T_A(\lambda, \ell)$ is equal to the character of $V_{q_A}(\lambda, \ell)$ when $A = \mathbb{I}_{m|n}$. Indeed, this will also follow from comparing the character identities of \[1.1\] and \[1.2\] for any $(g_A, Sp_{2\ell})$-duality (see Theorem \[7.10\]). In this sense, we call $T_A(\lambda, \ell)$ a spinor model of irreducible symplectic characters.

Remark 3.5. The definition of $T_A(\lambda, \ell)$ in Definition \[3.2\] is the same as in \[21\]. We remark that there is another definition of $T_A(\lambda, \ell)$ in \[21\], which is slightly different from Definition \[3.2\] but which has the same character.

3.2. Schur expansion. Let $(\lambda, \ell) \in \mathcal{P}(Sp)_A$ be given. Consider an embedding of sets given by

$$
\begin{align*}
T_A(\lambda, \ell) & \quad \overset{\iota}{\longrightarrow} \quad E^{2\ell}_A \\
T = (T_1, \ldots, T_1) & \quad \overset{(T_1^r, \ldots, T_1^r)}{\longrightarrow} (T_1^r, T_1^r, \ldots, T_1^r, T_1^r)
\end{align*}
$$

and identify $T_A(\lambda, \ell)$ with its image under $\iota$. By composing with $\kappa_A$ in \[24\], we have an embedding

$$
\begin{align*}
T_A(\lambda, \ell) & \quad \overset{\Phi_A}{\longrightarrow} \quad \bigsqcup_{\mu} SST_A(\mu) \times SST_{2\ell}(\mu') \\
T & \quad \longrightarrow \quad (P(T), Q(T))
\end{align*}
$$

where the union is over $\mu \in \mathcal{P}$ with $\ell(\mu') \leq 2\ell$.

Let us describe the image of $\Phi_A$ explicitly. Since $SST_{2\ell}(\mu')$ is a regular $gl_{2\ell}$-crystal, there exists an action of the Weyl group of $gl_{2\ell}$ on $SST_{2\ell}(\mu')$, which is isomorphic to $S_{2\ell}$ generated by the simple reflection $r_i$ for $1 \leq i \leq 2\ell - 1$. For $Q \in SST_{2\ell}(\mu')$, let us define the weight of $Q$ to be the sequence $(m_1, \ldots, m_{2\ell})$, where $m_i$ is the number of occurrences of $i$ in $Q$, and the $i$-signature of $Q$ to be the sequence $(\varepsilon_i(Q), \varphi_i(Q))$ (1 $\leq i \leq 2\ell - 1$).

For $\mu \in \mathcal{P}$ with $\ell(\mu') \leq 2\ell$, let $K_{\mu(\lambda, \ell)}$ be the set of $Q \in SST_{2\ell}(\mu')$ such that its weight $(m_1, \ldots, m_{2\ell})$ satisfies the following conditions:

1. $m_{2k} - m_{2k-1} = \lambda_k$ for $1 \leq k \leq \ell$,
2. $m_{2k} \geq m_{2k+2}$ for $1 \leq k \leq \ell - 1$, 

The diagram above illustrates the Schur expansion of $T_A(\lambda, \ell)$.
(3) the \((2k - 1)\)-signature of \(Q\) is \((\lambda_k, 0)\) for \(1 \leq k \leq \ell\),

(4) the \((2k)\)-signature of \(r_{2k+1}Q\) is \((0, m_{2k} - m_{2k+2})\) for \(1 \leq k \leq \ell - 1\),

(5) the \((2k)\)-signature of \(r_{2k-1}Q\) is \((\lambda_k - \lambda_{k+1} - p, m_{2k} - m_{2k+2} - p)\) with some \(p \geq 0\) for \(1 \leq k \leq \ell - 1\).

Then we have

**Theorem 3.6.** [21, Theorem 6.12] For \((\lambda, \ell) \in \mathcal{P}(\text{Sp})_A\), \(\Phi_A\) induces a bijection

\[
\Phi_A : \mathcal{T}_A(\lambda, \ell) \rightarrow \bigcup_\mu \mathcal{SST}_A(\mu) \times K_{\mu(\lambda, \ell)},
\]

where the union is over \(\mu \in \mathcal{P}\) with \(\ell(\mu') \leq 2\ell\).

**Corollary 3.7.** For \((\lambda, \ell) \in \mathcal{P}(\text{Sp})_A\), we have

\[
S_{(\lambda, \ell)}(x_A) = \ell' \sum的感觉到，
\]

where \(c_{\mu(\lambda, \ell)} = |K_{\mu(\lambda, \ell)}|\).

**Example 3.8.** Suppose that \(A = \mathbb{I}_{4|3}\) and \((\lambda, \ell) = ((3, 2, 1), 3)\).

\[
\Phi_A(T) = \begin{pmatrix}
1 & 2 & 1 & 1 \\
2 & 3 & 2 & 1 \\
3 & 4 & 1 & 2 \\
4 & 5 & 6 & 1
\end{pmatrix},
\]

\[
T \in \mathcal{T}_A(\lambda, \ell)
\]

4. Kashiwara-Nakashima Tableaux and Symplectic Jeu de Taquin

4.1. KN Tableaux of type \(C_n\).

Let us review the notion of Kashiwara-Nakashima tableaux (KN tableaux for short) of type \(C_n\) [15].

Let \(P = \bigoplus_{i=1}^n \mathbb{Z}e_i\), where \(\{e_i\mid 1 \leq i \leq n\}\) is an orthonormal basis with respect to a symmetric bilinear form \((, )\). Suppose that \(g = sp_{2n}\) of type \(C_n\) with Dynkin diagram

\[
\begin{array}{c}
\circ \quad \cdot \quad \cdots \quad \circ \quad \circ \quad \circ \quad \circ \quad \circ \\
\alpha_1 \quad \alpha_2 \quad \cdots \quad \alpha_{n-2} \quad \alpha_{n-1} \quad \alpha_n
\end{array}
\]

where \(\alpha_i = e_i - e_{i+1}\) for \(1 \leq i \leq n - 1\), and \(\alpha_n = 2e_n\). The set of dominant integral weights is given by \(P^+ = \{\omega_\lambda\mid \lambda \in \mathcal{P}_n\}\), where \(\omega_\lambda = \lambda_1 e_1 + \cdots + \lambda_n e_n\).

**Definition 4.1.** For \(\lambda \in \mathcal{P}_n\), let \(KN_\lambda\) be the set of \(T \in \mathcal{SST}_{\lambda_n}(\lambda)\) satisfying

(1) if \(T(i_1, j) = a\) and \(T(i_2, j) = b\) for some \(a\) and \(1 < i_1 < i_2 \leq \lambda_j\), then we have \(i_1 + (\lambda_j - i_2 + 1) = a\),

(2) if either \(T(p, j) = a\), \(T(q, j) = b\), \(T(r, j) = b\), \(T(s, j + 1) = a\) or \(T(p, j) = a\), \(T(q, j + 1) = b\), \(T(r, j + 1) = b\), \(T(s, j + 1) = a\) for some \(1 \leq a \leq b \leq n\), and \(p \leq q < r \leq s\), then we have \((q - p) + (s - r) < b - a\),
where $T(i,j)$ denotes the entry of $T$ in the $i$th row from the bottom and the $j$th column from the right. We call $\text{KN}_\lambda$ the set of $KN$ tableaux of type $C_n$ of shape $\lambda$.

The set $\text{KN}_{(1)}$ has an $\text{sp}_{2n}$-crystal structure such that $\text{KN}_{(1)} \cong B(\epsilon_1)$, where

\[
\begin{array}{cccccccc}
1 & 1 & 2 & 2 & \cdots & n-1 & n & n-1 & 2 & 2 & 1 & 1 \\
\end{array}
\]

with $\text{wt}(1) = \epsilon_i$ and $\text{wt}(?) = -\epsilon_i$ for $1 \leq i \leq n$. Here $b \rightarrow b'$ means $\bar{f}_ib = b'$. For $\lambda \in \mathcal{P}_n$ and $1 \leq i \leq n$, we define $\bar{e}_i$ and $\bar{f}_i$ on $\text{KN}_\lambda$ under the identification of $T \in \text{KN}_\lambda$ with $w_1 \otimes \cdots \otimes w_r \in (\text{KN}_{(1)})^{\otimes r}$ when $w(T) = w_1 \cdots w_r$. Then $\text{KN}_\lambda$ is an $\text{sp}_{2n}$-crystal with respect to $\bar{e}_i$ and $\bar{f}_i$ for $1 \leq i \leq n$, and $\text{KN}_\lambda \cong B(\omega_\lambda)$.

4.2. Bijection between $KN$ tableaux and spinor model. Suppose that $A = [\pi]$. Then we have $\mathcal{P}(\text{Sp})_n := \mathcal{P}(\text{Sp})_{[\pi]} = \{ (\lambda, \ell) \in \mathcal{P}(\text{Sp}) \mid \lambda, \ell \leq n \}$. For simplicity, we put

\[
\begin{align*}
T_n(a) &= T_{[\pi]}(a) \quad (0 \leq a \leq n), \\
T_n(\lambda, \ell) &= T_{[\pi]}(\lambda, \ell) \quad ((\lambda, \ell) \in \mathcal{P}(\text{Sp})_n).
\end{align*}
\]

For $(\lambda, \ell) \in \mathcal{P}(\text{Sp})_n$, put

\[
\rho_n(\lambda, \ell) = (n - \lambda_\ell, n - \lambda_{\ell-1}, \ldots, n - \lambda_1)',
\]

which is the conjugate of the rectangular complement of $\lambda$ in $(n^\ell)$.

\[
\begin{array}{c}
\hline
& \ell \\
\hline
\end{array}
\]

\[
\begin{array}{c}
\hline
n \\
\hline
\end{array}
\]

\[
\begin{array}{c}
\hline
\hline
\rho_n(\lambda, \ell) \\
\hline
\end{array}
\]

For $U \in \text{SST}_{[n]}((1^m))$, let $U^c$ be the tableau in $\text{SST}_{[n]}((1^{n-m}))$ such that $k$ appears in $U^c$ if and only if $\overline{k}$ does not appear in $U$ for each $k \in [n]$. For $T \in T_n(a)$, we define the following:

\begin{itemize}
  \item $T^{\text{ad}}$: the tableau obtained by putting $^LT$ below $(^RT)^c$,
  \item $T^{\text{ad}*}$: the tableau obtained by putting $^RT$ below $(^LT)^c$.
\end{itemize}

Then the map $T \mapsto T^{\text{ad}}$ is a bijection from $T_n(a)$ to $\text{KN}_{(1^{n-a})}$ [23, Lemma 3.11].
Example 4.2. Suppose that $n = 5$ and $T \in T_5(1)$ is given as follows. Then

\[
\begin{array}{c|c|c}
T_L & T & T_R \\
4 & 2 & 3 \\
3 & 1 & 1 \\
1 & & \\
\hline
\end{array}
\]

\[
\begin{array}{c|c}
\bar{(R)} & \bar{T} \\
3 & 5 \\
4 & 5 \\
1 & 1 \\
\hline
\end{array}
\quad
\begin{array}{c|c}
T_{ad} & \bar{T} \\
3 & 5 \\
5 & 3 \\
& 1 \\
\hline
\end{array}
\]

Let $C \in SST_5((1^m))$ be given with $m \geq 1$. We call $C$ admissible (or an admissible column) if $C \in \mathcal{KN}_{(1^m)}$, equivalently $C = T_{ad}$ for some $T \in T_n(n-m)$, and coadmissible (or a coadmissible column) if $C = T_{ad}^*$ for some $T \in T_n(n-m)$. We remark that the definition of an coadmissible column is equivalent to the one in [27].

Let $C \in SST_5((1^m))$ be an admissible column with $C = T_{ad}$ for $T \in T_n(n-m)$. Then we define the following:

- $lC$: the tableau obtained by putting $T_L$ below $T^c$,
- $rC$: the tableau obtained by putting $T_R$ below $T^c$,
- $C^* = T_{ad}^*$.

For $V \in SST_5((1^m))$, let $V_+$ and $V_-$ be the subtableau of $V$ consisting of its positive (unbarred) and negative (barred) letters, respectively.

Lemma 4.3. Under the above hypothesis, we have

\[
\begin{align*}
(lC)_+ &= (T^c)_+ = (T_{ad})^*_+, & (rC)_+ &= (\bar{T}^c)_+ = (T_{ad})^*_+, \\
(lC)_- &= T_L = (T_{ad})^*_-, & (rC)_- &= T_R = (T_{ad})^*_-. \\
\end{align*}
\]

Remark 4.4. The notion of admissible and coadmissible columns has been introduced in [5, 35]. For an admissible column $C$, one can check that $lC$ and $rC$ in this paper are equal to those in [27, Definition 2.2.1].

Let $C_i \in SST_5((1^{m_i}))$ $(i = 1, 2)$ be an admissible column with $m_2 \geq m_1$ and $C_i = T_{ad}^i$ for $T_i \in T_n(n-m_i)$. Following [27], define

\[
C_2 \prec C_1 \quad \text{if} \quad [rC_2, lC_1] \text{ is } \mathfrak{3}_n\text{-semi-standard.}
\]

Lemma 4.5. Under the above hypothesis, we have $C_2 \prec C_1$ if and only if $T_2 \prec T_1$.

Proof. It can be checked in a straightforward manner by using Lemma 4.3 that Definition [52][1] is equivalent to (4.1).

The following is another characterization of KN tableaux [35, Theorem A.4].
Proposition 4.6. Let $\mu \in \mathcal{P}_n$ be given. Let $m_1, \ldots, m_r$ be positive integers such that $\mu' = (m_r, \ldots, m_1)$. For $(C_r, \ldots, C_1) \in \text{SST}_{2n}((1^{m_r})) \times \cdots \times \text{SST}_{2n}((1^{m_1}))$, we have

$$[C_r, \ldots, C_1] \in \text{KN}_\mu \quad \text{if and only if} \quad C_{i+1} < C_i \text{ for } 1 \leq i \leq r - 1.$$ 

Proposition 4.7. For $(\lambda, \ell) \in \mathcal{P}(\text{Sp})_n$, we have a bijection

$$T_n(\lambda, \ell) \longrightarrow \text{KN}_{\rho_n(\lambda, \ell)}.$$ 

Proof. Let $T = (T_1, \ldots, T_1) \in T_n(\lambda, \ell)$ be given. Put $C_i = T^\text{ad}_i$ for $1 \leq i \leq \ell$. Then

$$T^\text{ad} = [C_\ell, \ldots, C_1]$$

is a tableau of shape $\rho_n(\lambda, \ell)$. By Proposition 4.6, $T^\text{ad} \in \text{KN}_{\rho_n(\lambda, \ell)}$ if and only if $C_{i+1} < C_i$ for $1 \leq i \leq \ell - 1$. Hence by Lemma 4.3, we have $T \in T_n(\lambda, \ell)$ if and only if $T^\text{ad} \in \text{KN}_{\rho_n(\lambda, \ell)}$.

The proof completes. \hfill \Box

Remark 4.8. The bijectivity of the map in Proposition 4.7 is also proved in [23, Theorem 3.14] by constructing an isomorphism of $\mathfrak{sp}_{2n}$-crystals between them. But we do not use the crystal structures on $T_n(\lambda, \ell)$ and $\text{KN}_{\rho_n(\lambda, \ell)}$ in this paper.

4.3. Admissibility and $\mathfrak{sl}_2$-crystal. In this subsection, we describe the relation between semistandard column tableaux with letters in $\mathcal{I}_n$ and admissible column tableaux using the crystal operators $\mathcal{E}$ and $\mathcal{F}$ [24].

Let

$$F_n = \bigcup_{0 \leq m \leq 2n} \text{SST}_{2n}((1^m)),$$

where we assume that $\text{SST}_{2n}((1^m))$ is the set of the empty tableau when $m = 0$.

Let $C \in \text{SST}_{2n}((1^m))$ given. Recall that $(C_+)^c$ is the tableau in $SST_{[m]}((1^m))$ such that $\tilde{k}$ appears in $(C_+)^c$ if and only if $k$ does not appear in $C_+$ for each $k \in [n]$, where $C_+ \in SST_{[m]}((1^m))$.

We define $\mathcal{E}C$ to be the unique tableau $C' \in SST_{2n}((1^{m-2}))$ such that

$$(C'_-, (C'_+)^c) = \mathcal{E}(C_-, (C_+)^c),$$

when $\mathcal{E}(C_-, (C_+)^c) \neq 0$, and $\mathcal{E}C = 0$ otherwise (see [24]). We define $\mathcal{F}C$ in a similar way.

Lemma 4.9. Under the above hypothesis,

1. $F_n$ is a regular $\mathfrak{sl}_2$-crystal with respect to $\mathcal{E}$ and $\mathcal{F}$;
2. $C \in F_n$ is admissible if and only if $\mathcal{E}C = 0$,
3. we have a bijection

$$F_n \longrightarrow \bigcup_{0 \leq \alpha \leq n} \text{KN}_{(1^{m-\alpha})} \times \mathbb{Z}/(\alpha + 1)\mathbb{Z},$$

$$C \longrightarrow (T, \varepsilon(C)).$$
where \( \varepsilon(C) = \max \{ k \mid \mathcal{E}^k C \neq 0 \} \), \( T = \mathcal{E}^{\max} C = \mathcal{E}^{\varepsilon(C)} C \), and \( \mathbb{Z}/(a + 1)\mathbb{Z} \) is understood as the set \( \{0, 1, \ldots, a\} \).

**Proof.** (1) It is clear that \( F_n \) is a regular \( \mathfrak{sl}_2 \)-crystal. (2) Let \( C \in \text{SST}_{2,n}((1^n)) \) be given with \( \varepsilon = \varepsilon(C) \). We may identify \( (C_-, (C_+)^c) \) as a skew tableau \( T \) of shape \( \lambda(\varepsilon, b, c) \) for some \( b, c \in \mathbb{Z}_+ \) with \( \varepsilon_T = 0 \). Let \( \overline{T} \) be a letter in \( C \) or \( T \). Then \( \overline{T} \) can be moved to the right in \( T \) by jeu de taquin or \( \mathcal{E} \) if and only if \( z \) is the smallest letter in \( C \) which does not satisfy Definition \( 4.1(1) \). This implies that \( C \) is admissible if and only if \( \varepsilon = 0 \).

(3) Let \( C \) be given. By (2), \( \mathcal{E}^{\max} C \) is admissible. If we let \( C' = \mathcal{E}^\varepsilon C \), then \( (C'_-, (C'_+)^c) \) is a skew tableau of shape \( \lambda(0, a, c) \) for some \( a, c \in \mathbb{Z}_+ \), and hence \( C' \in \text{KN}_{(1_n\cdots\_1)} \) by Proposition 4.7. Since \( F_n \) is a regular \( \mathfrak{sl}_2 \)-crystal, we have \( \varepsilon(C) \leq a \) and \( \varphi(C) + \varepsilon(C) = a \). The map sending \( C \) to \( (\mathcal{E}^{\max} C, \varepsilon(C)) \) is reversible, and hence it gives the bijection.

**Example 4.10.** For \( C \in F_5 \) below, we have

\[
C = \begin{array}{cccc}
1 & & & \\
2 & & & \\
3 & & & \\
4 & & & \\
5 & & & \\
\end{array}, \quad (C_-, (C_+)^c) = \begin{array}{cccc}
5 & 3 & & \\
4 & & & \\
\end{array}, \quad \varepsilon(C_-, (C_+)^c) = \begin{array}{cccc}
5 & 4 & & \\
\end{array}.
\]

Hence the image of \( C \) under \( 4.4 \) is

\[
(T, \varepsilon(C)) = \begin{pmatrix}
1 & \\
2 & \\
3 & \\
5 & \\
\end{pmatrix}, 1 \).
\]

### 4.4. Symplectic insertion

Let us review the insertion algorithm for KN tableaux of type \( C \) in \cite{[27]}. Let \( P(C_n) \) be the quotient of the free monoid generated by \( J_n \) subject to the relations:

1. \( yzx = yxz \) for \( x \leq y \leq z \) with \( z \neq \bar{x} \),
2. \( xzy = zxy \) for \( x < y \leq z \) with \( z \neq \bar{x} \),
3. \( g(x - 1)(x - 1) = yx \bar{x} \) for \( 1 < x \leq n \) and \( x \leq y \leq \bar{x} \),
4. \( x\bar{y}y = (x - 1)(x - 1)y \) for \( 1 < x \leq n \) and \( x \leq y \leq \bar{x} \),
5. if \( w = w(C) \) for a non-admissible column \( C \) such that every proper subword is an admissible column word (the word of an admissible column), and \( z \) is the smallest letter in \( [n] \) such that the pair \((z, \bar{z})\) occurs in \( w \) with \( N(z) > z \), where \( N(z) \) is the number of letters \( x \) in \( C \) such that \( x \leq z \) or \( x \geq \bar{z} \), then

\[
w = \tilde{w},
\]

where \( \tilde{w} \) is the column word obtained by removing the pair \((z, \bar{z})\) in \( w \).

Denote by \( W_{2,n} \) the set of words with letters in \( J_n \). For \( w, w' \in W_{2,n} \), write \( w \equiv w' \) if \( w = w' \) in \( P(C_n) \).
Proposition 4.11 \((\text{[27]}))\). For \(w \in \mathcal{W}_{3_n}\), we have a unique \(\text{KN}\) tableau \(T\) such that \(w \equiv w(T)\), which we denote by \(P(w)\).

For \(w \in \mathcal{W}_{3_n}\), we may obtain \(P(w)\) by insertion algorithm. For \(x \in \mathcal{J}_n\) and \(T \in \text{KN}_\lambda\), let us first define \(x \rightarrow T\), the insertion of \(x\) into \(T\) as follows. Assume that \(T = [C_r, \ldots, C_1]\).

*Case 1.* Suppose that \(w(C_r)x\) is the reading word of an admissible column \(C'_r\). Then

\[
(x \rightarrow T) := [C'_r, C_{r-1}, \ldots, C_1].
\]

*Case 2.* Suppose that \(w(C_r)x\) is the word of a non-admissible column and then there exists a letter \(x'\) such that \(w(C_r)x \equiv x'w(C'_r)\) for some admissible column \(C'_r\). Then we consider the insertion of \(x'\) into \(T' = [C_{r-1}, \ldots, C_1]\). If it belongs to *Case 1*, then we have \((x' \rightarrow T')\) and let

\[
(x \rightarrow T) = [C'_r, (x' \rightarrow T')].
\]

Otherwise repeat the above step until we get to *Case 1*.

*Case 3.* Suppose that \(w(C_r)x\) is the word of a non-admissible column whose proper subwords are admissible and suppose \(w(C_r)x = y_1 \cdots y_s\). Then

\[
(x \rightarrow T) := (y_s \rightarrow (y_{s-1} \rightarrow (\cdots \rightarrow (y_1 \rightarrow T')\cdots))),
\]

where \(T' = [C_{r-1}, \ldots, C_1]\).

We remark that *Case 3* does not occur during the insertion on the right-hand side of \((\text{[415]}))\) and \((\text{[416]}))\).

Now, let \(w = x_1 \cdots x_r \in \mathcal{W}_{3_n}\) be given. Define

\[
P(w) = (x_r \rightarrow (x_{r-1} \rightarrow (\cdots \rightarrow (x_2 \rightarrow [x_1] \cdots))),
\]

and define \(Q(w)\) to be the sequence \((Q_1, \ldots, Q_r)\) of partitions, where \(Q_i\) is the shape of \(P(x_1 \cdots x_i)\) for \(1 \leq i \leq r\). Note that \(Q(w)\) is an \(n\)-oscillating tableau of shape \(\mu = \text{sh}(P(w))\), that is, a sequence of partitions in \(\mathcal{P}_n\) such that \(Q_r = \mu\) and each pair \((Q_i, Q_{i+1})\) differs by one box for \(1 \leq i \leq r - 1\), i.e., \(Q_{i+1}/Q_i = \square\) or \(Q_i/Q_{i+1} = \square\).

Proposition 4.12 \((\text{[27]}))\). We have a bijection

\[
\begin{array}{ccc}
\mathcal{W}_{3_n} & \longrightarrow & \bigcup_{\mu \in \mathcal{P}_n} \text{KN}_\mu \times \text{OT}_n(\mu), \\
\downarrow & & \downarrow \\
\downarrow & & (P(w), Q(w))
\end{array}
\]

where \(\text{OT}_n(\mu)\) is the set of \(n\)-oscillating tableaux of shape \(\mu\). Moreover, the map is an isomorphism of \(\mathfrak{sp}_{2n}\)-crystals, where the operators \(\hat{e}_i\) and \(\hat{f}_i\) act on the first component on the right-hand side.

Example 4.13. For \(w = 1542543142 \in \mathcal{W}_{3_5}\), we have

\[
P(w) = \begin{pmatrix}
1 & 1 & 5 \\
5 & 5 & 4 \\
5 & 3 \\
4
\end{pmatrix}.
\]
4.5. Symplectic jeu de taquin. Let us briefly recall the algorithm of symplectic jeu de taquin for KN tableaux of type C introduced in [35]. Our review is based on [27].

Let \( T \) be a tableau of skew shape with letters in \( J_n \). We call \( T \) admissible if its columns are admissible columns, and its splitting form \( \text{spl}(T) \) is \( J_n \)-semistandard, where \( \text{spl}(T) \) is the tableau obtained by replacing each column \( T_i \) with \( \text{spl}(T_i) \). Let us denote by \( \text{KN}_{\zeta/\eta} \) the set of admissible tableaux of shape \( \zeta/\eta \).

Let us say that \( T \) is punctured if some boxes in \( T \) are removed. We put \( \bullet \) in each position of removed boxes and regard them as entries in \( T \) to keep track of the empty box in the process of sliding. We also say that a punctured tableau \( T \) is admissible if its columns are admissible and its splitting form is \( J_n \)-semistandard when we ignore the punctures \( \bullet \). Here the punctures \( \bullet \) are also duplicated in the splitting form.

Now, the symplectic jeu de taquin can be described in the following steps.

Step 1. Suppose that \( T \) is a punctured admissible tableau of skew shape with two columns, say \( T = \left[ C_2, C_1 \right]^{(c_2,c_1)} \). Suppose that \( \bullet \) is in the left column and \( \text{spl}(T) \) is as follows:

\[
\begin{align*}
\text{Example 4.14.}
\end{align*}
\]

\[
\begin{align*}
T_1 = & \begin{array}{c}
\bullet \ 3 \\
3 \ 5 \\
5 \ 3 \\
4 \\
2
\end{array} \quad \text{spl}(T_1) = & \begin{array}{c}
\bullet \bullet \ 2 \ 3 \\
1 \ 3 \ 5 \ 5 \\
5 \ 5 \ 3 \ 3 \\
4 \\
2
\end{array} \quad T'_1 = & \begin{array}{c}
2 \bullet \ 5 \\
3 \ 5 \\
5 \ 2 \\
4 \\
2
\end{array}
\end{align*}
\]
Step 2. Let $T$ be an admissible tableau of skew shape, say $T = [C_r, \ldots, C_1]^{(c_r, \ldots, c_1)}$.

Let $c$ be an inner corner at the $i$-th column (from the right). Let $C_i^*$ denote the column $C_i$ with \[ placed at the top.

We apply Step 1 to $[C_i^*, C_{i-1}]^{(c_i-1, c_{i-1})}$ to have $[C_i', C_{i-1}']^{(c_i', c_{i-1}')}$ where \[ has moved to $C_{i-1}'$. Now, we repeat Step 1 until \[ is placed at the bottom of the $j$-th column to have

$$T' = [\ldots, C_{i+1}^j, C_i^j, \ldots, C_j', C_{j-1}, \ldots]^{(c_i', \ldots, c_j')},$$

for some $j \leq i$, $C_i', \ldots, C_j'$ and $c_i', \ldots, c_j'$, where we ignore \[ in the $j$-th column. It is shown in [35] that if $T'$ is not admissible, then $C_i'$ is the only column which is not admissible. Let $C_i''$ be the unique admissible column such that $w(C_i'') \equiv w(C_i')$ with respect to $\mathcal{T}_n$. Put

$$T'' = [\ldots, C_{i+1}^j, C_i'', \ldots, C_j', C_{j-1}, \ldots]^{(c_i'', \ldots, c_j')}.$$ 

In this case, we have $c_i' = c_i - 1$. Now we define

$$(4.8) \quad \jdt_{KN}(T, c) = \begin{cases} T' & \text{if } T' \text{ is admissible,} \\ T'' & \text{if } T' \text{ has a non-admissible column.} \end{cases}$$

If $\sh(T) = \zeta/\eta$, then

$$\sh(\jdt_{KN}(T, c)) = \begin{cases} \alpha/\eta & \text{for some } \alpha \leq \zeta \text{ if } T' \text{ is non-admissible,} \\ \alpha/\beta & \text{for some } \alpha \leq \zeta, \beta \leq \eta \text{ if } T' \text{ is admissible.} \end{cases}$$

Moreover, by [27] Theorem 6.3.8, we have

$$w(\jdt_{KN}(T, c)) \equiv w(T).$$

Hence by applying the $\jdt_{KN}(\cdot, c)$ successively to the inner corners, we obtain a unique KN tableau $P(w(T))$ by induction on $|\zeta| = \sum_i \zeta_i$.

5. Sliding algorithm for spinor model

5.1. Spinor model of a skew shape.

**Definition 5.1.** Let $\mathbf{T} = (T_1, \ldots, T_\ell) \in T_A(a_\ell) \times \cdots \times T_A(a_1)$ be given for some $a_1, \ldots, a_\ell \in \mathbb{Z}_+$. Let $\lambda/\mu$ be a skew diagram with $\lambda, \mu \in \mathcal{P}_\ell$. We say that
(1) \( T \) is of shape \( \lambda/\mu \) if 
\[
a_i = \lambda_i - \mu_i, \quad \text{ht}(T_{i+1}^L) + \mu_{i+1} \leq \text{ht}(T_i^L) + \mu_i \quad (1 \leq i \leq \ell),
\]
where we put \( \text{ht}(C) = c \) for a tableau \( C \) of shape \((1^c)\).

(2) \( T \) is \( \mathcal{A} \)-admissible of shape \( \lambda/\mu \) if \( T \) is of shape \( \lambda/\mu \) and for \( 1 \leq i \leq \ell - 1 \)
\[
[T^L_{i+1}, T^L_i]_{(\mu_{i+1}, \mu_i)} \quad \left( \begin{array}{c}
T^L_{\ell+1}, T^L_1 \end{array} \right)_{(\lambda_{i+1}, \lambda_i)} \text{ are } \mathcal{A} \text{-semistandard.}
\]

We denote by \( \mathbf{T}_A(\lambda/\mu, \ell) \) the set of \( \mathcal{A} \)-admissible tableaux of shape \( \lambda/\mu \).

When \( T \) is of shape \( \lambda/\mu \), let us often identify \( T \) with a tableau in \( \mathbb{P}_L \) given by
\[
[T]_{(\mu_1, \ldots, \mu_1)} = [T_1, \ldots, T_1]_{(\mu_1, \ldots, \mu_1)} := [T_1^L, T_1^R, \ldots, T_1^L, T_1^R]_{(\mu_1, \ldots, \mu_1, \lambda_1)}.
\]
Note that \( T \) does not correspond to a KN tableau of skew shape unless \( \mathcal{A} = \{\pi\} \). So we may
not apply the algorithm in Section 4.3 to have \( T' \in \mathbf{T}_A(\nu, \ell) \) for some \( (\nu, \ell) \in \mathcal{P}(\mathcal{S}_0) \). To
overcome this problem, we introduce the notion of \( n \)-conjugate of \( T \).

**Definition 5.2.** Let \( T = (T_1, \ldots, T_1) \in \mathbf{T}_A(a_1) \times \cdots \times \mathbf{T}_A(a_1) \) be given. Let
\( (P(T), Q(T)) = \kappa_A(T) = \kappa_A(T_1^L, T_1^R, \ldots, T_1^L, T_1^R) \) with \( \nu = \text{sh}(P(T)) \),
where \( \kappa_A \) is in (2.3).

For \( n \geq \ell(\nu) \), we define the \( n \)-conjugate of \( T \) to be the unique \( \overline{T} = (\overline{T}_1, \ldots, \overline{T}_1) \in \mathbf{T}_{\{\nu\}}(a_1) \times \cdots \times \mathbf{T}_{\{\mu\}}(a_1) \) such that
\[
\kappa_{\{\nu\}}(\overline{T}) = (P(\overline{T}), Q(\overline{T})) = (H_\nu, Q(T)),
\]
where \( H_\nu \in \text{SST}_{\{\nu\}}(\nu) \) is the highest weight element, that is, the \( i \)-th row from the top is filled
with \( n - i + 1 \) for \( 1 \leq i \leq n \).

Note that if we replace \( n \) with \( m > n \), then the corresponding \( \overline{T} \) is given by replacing \( \nu \) with \( \nu + m - n \) for all \( \alpha \).

**Example 5.3.** Suppose \( \mathcal{A} = I_{4|3} \) and take \( T = (T_3, T_2, T_1) \in \mathbf{T}_A(1) \times \mathbf{T}_A(1) \times \mathbf{T}_A(2) \) below.
Then we have
\[
T = \begin{pmatrix}
2 & 1 & 1 & 1 & 2' & 2'
3 & 2 & 2 & 3 & 1' & 1'
1'
\end{pmatrix}, \quad \kappa_A(T) = \begin{pmatrix}
1 & 1 & 1 & 2 & 2' & 2'
2 & 2 & 3 & 2' & 3'
3 & 4 & 1' & 1'
4 & 4 & 5 & 5
6 & 6
\end{pmatrix}, \quad (H_\nu, Q(T)) = \begin{pmatrix}
1 & 1 & 2 & 2 & 2
2 & 2 & 3 & 3 & 5
3 & 3 & 4 & 1
4 & 4 & 5 & 2
5 & 5
\end{pmatrix}.
\]

Hence the 5-conjugate \( \overline{T} \), the inverse image of \( (H_\nu, Q(T)) \) under \( \kappa_{\{\nu\}} \) with \( \nu = (5, 5, 4, 3, 1) \), is
\[
\overline{T} = \begin{pmatrix}
5 & 5 & 5 & 4 & 4
4 & 4 & 4 & 4
3 & 3 & 3 & 3
2 & 2 & 2 & 2
\end{pmatrix}.
\]

The following two lemmas play an important role in this paper.

**Lemma 5.4.** Under the above hypothesis, \( T \) is \( \mathcal{A} \)-admissible of shape \( \lambda/\mu \) if and only if its \( n \)-conjugate \( \overline{T} \) is \( \{\nu\} \)-admissible of shape \( \lambda/\mu \).
Proof. It follows directly from [21, Lemma 6.2] and \( Q(T) = Q(T) \).

Lemma 5.5. Under the above hypothesis, \( T \) is \([\pi]\)-admissible of shape \( \lambda/\mu \) if and only if

\[
\left[ T^{\text{ad}}_{\ell}, \ldots, T^{\text{ad}}_{1} \right]_{\rho_{\nu_{1}}(\mu, \ell)}
\]

is admissible of shape \( \rho_{n+\mu_{1}}(\lambda, \ell)/\rho_{\mu_{1}}(\mu, \ell) \).

Proof. Suppose that \( T \) is \([\pi]\)-admissible of shape \( \lambda/\mu \). Let \( m \) be a sufficiently large positive integer. Let \( X = \{ u_{m} < \cdots < u_{1} < 1 < \cdots < n \} \) be a linearly ordered set of degree 0, where \( u_{1}, \ldots, u_{m} \) are formal symbols, and let \( \Xi = \{ \pi < \cdots < \overline{\pi} < \pi_{1} < \cdots < \pi_{m} \} \) of degree 0 similarly. Since \( T \) is \([\pi]\)-admissible of shape \( \lambda/\mu \), there exists \( S = (S_{\ell}, \ldots, S_{1}) \in T_{\Xi}(\lambda, \ell) \), where \( T \) can be obtained from \( S \) by ignoring the letters in \( \{ \pi_{1}, \ldots, \pi_{m} \} \).

Let \( J_{m,n} = \{ u_{m} < \cdots < u_{1} < 1 < \cdots < n < \pi < \cdots < \pi_{1} < \cdots < \pi_{m} \} \). We may define \( S^{\text{ad}} = (S_{\ell}^{\text{ad}}, \ldots, S_{1}^{\text{ad}}) \) as in Proposition 4.7, which is a KN tableau of shape \( \rho_{m+n}(\lambda, \ell) \) with letters in \( J_{m,n} \). Then we can check that \( [T_{\ell}^{\text{ad}}, \ldots, T_{1}^{\text{ad}}]^{\rho_{\nu_{1}}(\mu, \ell)} \) can be obtained from \( [S_{\ell}^{\text{ad}}, \ldots, S_{1}^{\text{ad}}] \) by ignoring the letters in \( \{ u_{m}, \ldots, u_{1} \} \). This implies that \( [T_{\ell}^{\text{ad}}, \ldots, T_{1}^{\text{ad}}]^{\rho_{\nu_{1}}(\mu, \ell)} \) is \( J_{n}\)-admissible of shape \( \rho_{n+\mu_{1}}(\lambda, \ell)/\rho_{\mu_{1}}(\mu, \ell) \). The proof of the converse is similar.

The following is an analogue of Proposition 4.7 for skew shapes.

Corollary 5.6. We have a bijection

\[
T_{n}(\lambda/\mu, \ell) \rightarrow \text{KN}_{\rho_{n+\nu_{1}}(\lambda, \ell)/\rho_{\nu_{1}}(\mu, \ell)}.
\]

\[
T = [T_{\ell}, \ldots, T_{1}]_{(\mu_{\ell}, \ldots, \mu_{1})} \rightarrow T^{\text{ad}} := [T_{\ell}^{\text{ad}}, \ldots, T_{1}^{\text{ad}}]^{\rho_{\nu_{1}}(\mu, \ell)}
\]
Example 5.7. For $\mathbf{T} = [T_3, T_2, T_1]_{(0,1,2)} \in T_5(\lambda/\mu, 3)$ with $\lambda = (4, 2, 1)$ and $\mu = (2, 1, 0)$ given below, we have $\mathbf{T}^\text{ad} \in \mathbf{K}(6,5,3)/(2,1,0)$ as follows.

\[
\begin{array}{c|c|c|c|c|c}
& 5 & 5 & 4 & 2 & 1 \\
1 & 2 & 3 & 4 & 5 & 1 \\
\end{array}
\quad
\begin{array}{c|c|c|c|c|c}
& 5 & 5 & 4 & 2 & 1 \\
1 & 2 & 3 & 4 & 5 & 1 \\
\end{array}
\]

$\mathbf{T} = [T_3, T_2, T_1]_{(0,1,2)}$ $\mathbf{T}^\text{ad} = [T_3^\text{ad}, T_2^\text{ad}, T_1^\text{ad}]_{(2,1,0)}$

5.2. Jeu de taquin for spinor model of a skew shape. Now, let us introduce an analogue of jeu de taquin for $\mathbf{T} \in T_\mathcal{A}(\lambda/\mu, \ell)$.

5.2.1. We first consider the case when $\ell = 2$. Suppose that $\mathbf{T} = (T_2, T_1) \in T_\mathcal{A}(a_2) \times T_\mathcal{A}(a_1)$ is given for some $a_1, a_2 \in \mathbb{Z}_+$. Let

\[
d(T_1, T_2) = \min \left\{ d \left| d \in \mathbb{Z}_+, \ [T_2, T_1]_{(0,d)} \text{ is } \mathcal{A}\text{-admissible (of a skew shape)} \right. \right\}.
\]

Note that we have $T_2 < T_1$ if and only if $d(T_1, T_2) = 0$. Let us assume that $\mathbf{T} = (T_2^\mathcal{A}, T_2^\mathcal{L}, T_1^\mathcal{L}, T_1^\mathcal{R}) \in \mathcal{E}_\mathcal{A}^\mathcal{A}$. Recall that

\[
\mathcal{E}_3^{a_2} \mathbf{T} = (T_2^\mathcal{L}, T_2^\mathcal{R}, T_1^\mathcal{L}, T_1^\mathcal{R}), \quad \mathcal{E}_1^{a_1} \mathbf{T} = (T_2^\mathcal{L}, T_2^\mathcal{R}, T_1^\mathcal{L}, T_1^\mathcal{R}).
\]

Suppose that $d = d(T_1, T_2) > 0$. Let $\mathbf{T}'$ be given by applying a sequence of jeu de taquin’s as follows:

**Case 1.** Suppose that $[T_2, T_1^\mathcal{L}]_{(0,d-1)}$ is not $\mathcal{A}$-semistandard. Then we put

\[
\mathbf{T}' = (U_4, U_3, U_2, U_1) = \begin{cases} 
T_3^{a_2-1} \mathcal{E}_2 \mathcal{E}_3^{a_2} \mathbf{T} & \text{if } \mathcal{E}_3(\mathcal{E}_2 \mathcal{E}_3^{a_2} \mathbf{T}) = 0, \\
T_3^{a_2} \mathcal{E}_2 \mathcal{E}_3^{a_2} \mathbf{T} & \text{if } \mathcal{E}_3(\mathcal{E}_2 \mathcal{E}_3^{a_2} \mathbf{T}) = 1.
\end{cases}
\]

**Case 2.** Suppose that $[T_2, T_1^\mathcal{L}]_{(0,d-1)}$ is $\mathcal{A}$-semistandard, but $[T_2, T_1^\mathcal{L}]_{(0,d-1)}$ is not. Then we put

\[
\mathbf{T}' = (U_4, U_3, U_2, U_1) = T_3^{a_1+1} \mathcal{E}_2 \mathcal{E}_3^{a_1} \mathbf{T}.
\]

**Lemma 5.8.** Under the above hypothesis, $\mathbf{T}'$ is well-defined.

**Proof.** First, consider **Case 1.** Since $[T_2, T_1^\mathcal{L}]_{(0,d-1)}$ is not $\mathcal{A}$-semistandard, we have $\mathcal{E}_2(\mathcal{E}_3^{a_2} \mathbf{T}) > 0$ and hence $\mathcal{E}_2 \mathcal{E}_3^{a_2} \mathbf{T} \neq 0$. Let

\[
(V_4, V_3, V_2, V_1) = \mathcal{E}_2 \mathcal{E}_3^{a_2} \mathbf{T}.
\]

Since $\mathcal{E}_3[\mathcal{T}_2^\mathcal{L}, T_2^\mathcal{R}, T_1^\mathcal{L}, T_1^\mathcal{R}]_{(0,d,d+\alpha)} = \mathcal{E}_3(V_4, V_3, V_2, V_1)_{(0,d,d+\alpha)}$, it is not difficult to see that $\mathcal{E}_3(\mathcal{E}_2 \mathcal{E}_3^{a_2} \mathbf{T}) = 0, 1$. Hence $\mathbf{T}'$ is well-defined. Similarly, we can check the well-definedness of $\mathbf{T}'$ in **Case 2.** \(\square\)
Example 5.9. Suppose that \( A = \mathbb{I}_{4|3} \).

1. The following is an example of Case 1 with \( \varepsilon_3(\mathcal{E}_3\mathcal{E}_3^2 \mathbf{T}) = 0 \).

\[
\begin{array}{cccc}
1 & 1 & 2 & 2' \\
2 & 3 & 1' & 2' \\
3' & 2' & 1' & 2' \\
\end{array} \quad \xrightarrow{\mathcal{E}_3} \quad \begin{array}{cccc}
1 & 3 & 1' & 2' \\
1' & 2' & 3' & 2' \\
1' & 2' & 3' & 2' \\
\end{array}
\]

\[ \mathbf{T} = [T_2, T_1]_{(0,1)} \]

2. The following is an example of Case 2.

\[
\begin{array}{cccc}
2 & 4 & 1 & 1 \\
3 & 2' & 1' & 2' \\
\end{array} \quad \xrightarrow{\mathcal{F}_2} \quad \begin{array}{cccc}
2 & 2 & 1' & 1' \\
3 & 2' & 4 & 3 \\
1' & 2' & 3' & 2' \\
\end{array}
\]

\[ \mathbf{T} = [T_2, T_1]_{(0,2)} \]

3. The following is an example of Case 1 with \( \varepsilon_3(\mathcal{E}_3^2 \mathcal{E}_3 \mathbf{T}) = 1 \).

\[
\begin{array}{cccc}
2 & 2 & 3 & 2' \\
3 & 1 & 1 & 2 \\
1' & 2' & 3' & 2' \\
\end{array} \quad \xrightarrow{\mathcal{E}_2} \quad \begin{array}{cccc}
2 & 2 & 3 & 2' \\
3 & 1 & 1 & 2 \\
1' & 2' & 3' & 2' \\
\end{array} \quad \xrightarrow{\mathcal{F}_1} \quad \begin{array}{cccc}
2 & 2 & 3 & 2' \\
3 & 1 & 1 & 2 \\
1' & 2' & 3' & 2' \\
\end{array}
\]

\[ \mathbf{T} = [T_2, T_1]_{(0,1)} \]
Proposition 5.10. Under the above hypothesis, there exists a unique pair $T' = (T'_2, T'_1)$ such that

$$T' = (T'_2, T'_1) \in T_A(a_2 + 2\epsilon - 1) \times T_A(a_1 + 1),$$

$$d(T'_1, T'_2) \leq d(T_1, T_2) - 1,$$

where $\epsilon = \varepsilon_3(\mathcal{E}_2^a T)$ in Case 1, and $\epsilon = 0$ otherwise.

**Proof.** Suppose first that $A = \overline{A}$. Let $d = d(T_1, T_2)$. Since $[T_2, T_1]_{(0,d)}$ is $[\overline{A}]$-admissible, we have $[T_2^{ad}, T_3^{ad}]^{(d,0)}$ is admissible by Lemma 5.5.

Let $T = [C_2, C_1]^{(c_2, c_1)} = [T_2^{ad}, T_3^{ad}]^{(d,0)}$. Let $c$ be the inner corner of $T$. We claim that the algorithm in Section 4.5 to have $T' = \text{adj}_K^d(T, c)$ corresponds to either (5.3) or (5.4).

First, we put $\bullet$ at the top of $C_2$ and apply Step 1.1 as far as possible to have (4.7). Since $d = d(T_1, T_2) > 0$, we should apply Step 1.2 to (4.7) with $d' > b$.

Suppose that $b \in [\overline{A}]$. Then it is straightforward to see from Lemma 4.3 and 5.2 that applying Step 1.2-(b) and sliding $\bullet$ to the bottom of the column corresponds to (5.3). This implies that $(U_4, U_3) = ((T_2)^{1_L}, (T_2)^{1_R})$ for some $T_2 \in T_n(a_2 - 1)$ and $(U_2, U_1) = ((T'_1)^{1_L}, (T'_1)^{1_R})$ for some $T'_1 \in T_n(a_1 + 1)$. Furthermore, since $T' = [C'_2, C'_1]^{(c_2-1, c_1)}$ it follows from Lemma 5.5 that $[T'_2, T'_1]_{(0,d-1)}$ is $A$-admissible and

$$\text{(5.5)}$$

$$(T')^{ad} = T' = [C'_2, C'_1]^{(c_2-1, c_1)},$$

where $(T')^{ad}$ is given in Corollary 5.6. This implies that $d(T'_1, T'_2) \leq d - 1$.

Next, suppose that $b \in [n]$. It is easy to see that exchanging $\bullet$ with $\circ$ in Step 1.2-(a) to have $C'_1$ corresponds to $\mathcal{E}_2^{a_2} T$. If $C'_2$ is admissible, which is equivalent to $\varepsilon_3(\mathcal{E}_2^{a_2} T) = 0$, then the process to have $C'_2$ corresponds to applying $T_3^{a_2-1}$ to $\mathcal{E}_2^{a_2} T$. As in the previous case, we have $(U_4, U_3) = ((T_2)^{1_L}, (T_2)^{1_R})$ and $(U_2, U_1) = ((T'_1)^{1_L}, (T'_1)^{1_R})$ for some $T_2 \in T_n(a_2 - 1)$ and $T'_1 \in T_n(a_1 + 1)$, and $[T'_2, T'_1]_{(0,d-1)}$ is $A$-admissible with (5.3), which implies $d(T'_1, T'_2) \leq d(T_1, T_2) - 1$. On the other hand, if $C'_2$ is not admissible, which is equivalent to $\varepsilon_3(\mathcal{E}_2^{a_2} T) = 1$, then it is not difficult to see that the process to have $C'_2$ corresponds to applying $T_3^{a_2}$ to $\mathcal{E}_2^{a_2} T$. Hence, we have $(U_4, U_3) = ((T_2)^{1_L}, (T_2)^{1_R})$ for some $T_2 \in T_n(a_2 + 1)$ and $(U_2, U_1) = ((T'_1)^{1_L}, (T'_1)^{1_R})$ for some $T'_1 \in T_n(a_1 + 1)$. In this case, we have $T' = [C'_2, C'_1]^{(c_2-1, c_1)}$, and hence $[T'_2, T'_1]_{(0,d)}$ is $A$-admissible.

Moreover, it can be shown that $[C'_2, C'_1]^{(c_2-1, c_1)}$ is also admissible, hence $[T'_2, T'_1]_{(0,d-1)}$ is $A$-admissible with (5.3), which implies $d(T'_1, T'_2) \leq d(T_1, T_2) - 1$.

Now, suppose that $A$ is arbitrary. Let $\overline{T}$ be the $n$-conjugate of $T$ for a sufficiently large $n$. Let $X$ be the composite of operators $\mathcal{E}_i$ and $\mathcal{F}_i$ in (5.3) or (5.4). Then we have by definition of $\overline{T}$ and Lemma 5.5

$$\text{(5.6)}$$

$$Q(X T) = X Q(T) = X Q(\overline{T}) = Q(X \overline{T}).$$

By the previous arguments for the case of $A = [\overline{A}]$, there exists $\overline{T} = (\overline{T}_2, \overline{T}_1)$ such that

$$X \overline{T} = \overline{T} \in T_n(a_2 + 2\epsilon - 1) \times T_n(a_1 + 1),$$
By Lemma [21, Lemma 6.2] and (5.6), we have

\[ X T = T' \in \mathcal{T}_A(a_2 + 2\varepsilon - 1) \times \mathcal{T}_A(a_1 + 1), \]

and the \( n \)-conjugate of \( T'_i \) is \( \overline{T'_i} \) for \( i = 1, 2 \). Finally, it follows from the arguments for \( A = [n] \) and Lemma 5.3 that \( d(T'_1, T'_2) \leq d(T_1, T_2) - 1 \). \( \square \)

**Definition 5.11.** For \( T = (T_2, T_1) \in \mathcal{T}_A(a_2) \times \mathcal{T}_A(a_1) \) with \( d(T_1, T_2) > 0 \), we define

\[ \text{jdt}_{\text{spin}}(T) = T', \]

where \( T' \) is given in (5.3) and (5.4).

The following corollaries follow from the proof of Proposition 5.10.

**Corollary 5.12.** Under the above hypothesis, if \( A = [n] \), then we have

\[ (\text{jdt}_{\text{spin}}(T))^\text{ad} = \text{jdt}_{K^N} (T^\text{ad}, c), \]

where \( c \) is the inner corner of \( T^\text{ad} \).

**Corollary 5.13.** Under the above hypothesis, we have

\[ \text{jdt}_{\text{spin}}(T) = \text{jdt}_{\text{spin}}(T), \]

\[ (\text{jdt}_{\text{spin}}(T))^\text{ad} = \text{jdt}_{K^N} (T^\text{ad}, c), \]

where \( \overline{\cdot} \) denotes the \( n \)-conjugate for a sufficiently large \( n \), \( (\cdot)^\text{ad} \) is given in (4.2) or (5.1), and \( c \) is the inner corner of \( T^\text{ad} \).

5.2.2. Now we consider a general case. Let \( \lambda/\mu \) be a skew diagram with \( \lambda, \mu \in \mathcal{P}_\ell \) and let \( T = (T_\ell, \ldots, T_1) \in \mathcal{T}_A(\lambda+\mu, \ell) \) be given. Let \( c \) be an inner corner of \( \lambda/\mu \) in the \( i \)-th row from the top.

Let us define an analogue of jeu de taquin sliding on \( T \) with respect to \( c \). We first take a sufficiently large \( n \) and the \( n \)-conjugate \( \overline{T} \) of \( T \). Then we consider

\[ \text{jdt}_{K^N} (\overline{T^\text{ad}}, b), \]

where \( b \) is the inner corner of \( T^\text{ad} \) in the \( (i+1) \)-th column from the right and \( (\cdot)^\text{ad} \) is as in (5.1). Recall that (5.7) is obtained by applying a sequence of \( \text{jdt}_{K^N} \) to two neighboring components in \( \overline{T^\text{ad}} \) (see Step 2 in Section 4.5). By Definition 5.11 and Corollary 5.13 there exists a composite of operators \( \mathcal{E}_i \) and \( \mathcal{F}_i \), say \( \mathcal{X} \), such that \( (\mathcal{X} \overline{T})^\text{ad} = \text{jdt}_{K^N} (\overline{T^\text{ad}}, b) \).

**Definition 5.14.** Under the above hypothesis, we define

\[ \text{jdt}_{\text{spin}}(T, c) = \mathcal{X} T. \]

Note that \( \text{jdt}_{\text{spin}}(T, c) \) is independent of the choice of \( \mathcal{X} \) since

\[ (\text{jdt}_{\text{spin}}(T, c))^\text{ad} = (\mathcal{X} T)^\text{ad} = (\mathcal{X} T)^\text{ad} = \text{jdt}_{K^N} (\overline{T^\text{ad}}, b), \]

and \( \overline{\cdot} \) and \( (\cdot)^\text{ad} \) are injective on the connected component of \( T \).
**Theorem 5.15.** Let $\lambda/\mu$ be a skew diagram with $\lambda, \mu \in \mathcal{P}_\ell$ and let $T = (T_1, \ldots, T_1) \in T_A(\lambda/\mu, \ell)$ be given. There exists a unique $P(T) \in T_A(\nu, \ell)$ for some $(\nu, \ell) \in \mathcal{P}(Sp)A$, which can be obtained from $T$ by applying $\jdt_{spin}(\cdot, c)$ finitely many times with respect to inner corners. In particular, if $A = [\pi]$, then we have

$$P(T)^{ad} = P\left( w\left( T^{ad}\right) \right).$$

**Proof.** Let us first prove the existence of $P(T)$. Let $\tilde{T}$ be the $n$-conjugate of $T$ for a sufficiently large $n$. Let $U = \tilde{T}$ and $V = \tilde{T}^{ad}$. By Section 4.5 there exists a sequence $\psi = V_0, \ldots, V_r$ such that

$$(5.9) \quad V_{i+1} = \jdt_{KN}(V_i, b_i) \quad (1 \leq i \leq r - 1),$$

for some inner corner $b_i$ in $\text{sh}(V_i)$, and $V_r \in \text{KN}_\delta$ for some $\delta \in \mathcal{P}_n$ with $\delta_1 \leq \ell$. By Corollary 5.13 there exists a sequence $U = U_0, \ldots, U_r$ such that $U_r^{ad} = V_i$ and

$$(5.10) \quad U_{i+1} = \jdt_{spin}(U_i, c_i) \quad (1 \leq i \leq r - 1)$$

for some inner corners $c_i$ in $\text{sh}(U_i)$. Again by Corollary 5.13 there exists a sequence $T = T_0, \ldots, T_r$ such that $T_i = U_i$ (the $n$-conjugate of $T_i$) and

$$(5.11) \quad T_{i+1} = \jdt_{spin}(T_i, c_i) \quad (1 \leq i \leq r - 1).$$

Since $V_r \in \text{KN}_\delta$, we conclude from Lemmas 5.4 and 5.9 that $T_r \in T_A(\nu, \ell)$ with $\rho_n(\nu, \ell) = \delta$. We put $P(T) = T_r$.

Now let us prove the uniqueness. Suppose that there exists a sequence $T = T_0, \ldots, T_s$ such that $T_{i+1} = \jdt_{spin}(T_i', c_i')$ for some $c_i'$ (1 $\leq i \leq s - 1$) and $T_s' \in T_A(\xi, \ell)$ for some $(\xi, \ell) \in \mathcal{P}(Sp)A$. We claim that $T_r = T_s'$.

If we put $T_i' = \tilde{T}_i$ and $V_i' = (U_i')^{ad}$ for 1 $\leq i \leq s$, then they also satisfy (5.9) and (5.10) by Corollary 5.13. We have $V_s' \in \text{KN}_{\rho_n(\xi, \ell)}$ by Lemmas 5.4 and 5.9. By Proposition 4.11 we have

$$V_s' = P\left( w(V_0') \right) = P\left( w(V_0) \right) = V_r,$$

and hence $\xi = \nu$. On the other hand, let $\chi$ and $\chi'$ be composites of $\mathcal{E}_i$ and $\mathcal{F}_i$ (1 $\leq i \leq 2\ell - 1$) such that

$$\chi T = T_r, \quad \chi' T = T_s',$$

which implies $\chi U = U_r$ and $\chi' U = U_s'$, respectively. Since $V_r = V_s'$, we have $\chi U = U_r = U_s' = \chi' U$, and

$$Q(\chi T) = \chi Q(T) = \chi Q(U) = Q(\chi U)$$

$$= Q(\chi' U) = \chi' Q(U) = \chi Q(T) = Q(\chi' T).$$

Recall that $Q(T) = Q(U)$ by definition of the $n$-conjugate. Since $P(\chi T) = P(T) = P(\chi' T)$, we have $\Phi_A(T_i) = \Phi_A(T_i')$ and hence $T_r = T_s'$ by Theorem 3.6.

Finally, if $A = [\pi]$, then it follows directly from Corollary 5.12 that $P(T)^{ad} = P\left( w\left( T^{ad}\right) \right)$. This completes the proof.
Example 5.16. Let $T = [T_3, T_2, T_1]_{(0,1,2)}$ be the tableau in Example 5.3. Then $P(T)$ can be obtained as follows (for detailed computation, see Example 5.9).

\[
T = [T_3, T_2, T_1]_{(0,1,2)} \quad \quad \quad T_1 = [T_3', T_2', T_1']_{(0,2,2)}
\]

\[
T = [T_3, T_2, T_1]_{(0,1,2)} \quad \quad \quad T_1 = [T_3', T_2', T_1']_{(0,2,2)}
\]

where

\[
\text{jdt}_{\text{spin}}(T, c_1) = \mathcal{E}_2 \mathcal{E}_3 T,
\]

\[
\text{jdt}_{\text{spin}}(T_1, c_2) = \mathcal{F}_3 \mathcal{F}_4 T_1,
\]

\[
\text{jdt}_{\text{spin}}(T_2, c_3) = \mathcal{E}_4 T_2.
\]

The corresponding jeu de taquin for the 5-conjugate $U$ and $V = U^{\text{ad}}$ is given as follows.
6. Recording tableaux for spinor model

6.1. Oscillating tableaux of shape \((\lambda, \ell)\). Recall that an oscillating tableau is a sequence of partitions \(Q = (Q_1, \ldots, Q_s)\) for some \(s \geq 1\) such that each pair \((Q_i, Q_{i+1})\) differs by one box for \(1 \leq i \leq s - 1\). We say that an oscillating tableau \(Q = (Q_1, \ldots, Q_s)\) is vertical if \(Q_1 \supseteq \cdots \supseteq Q_r \supseteq \cdots \supseteq Q_s\) for some \(1 \leq r \leq s\) and \(Q_r/Q_1\) and \(Q_r/Q_s\) is a skew diagram of vertical strip. We denote by \(|Q| = s\) the length of \(Q\).

Let \((\lambda, \ell) \in \mathcal{P}(Sp)\) be given. For \(n \geq \lambda_1\), we define \(O(\lambda, \ell; n)\) to be the set of a sequence of oscillating tableaux \(Q = (Q^{(1)} : \cdots : Q^{(\ell)})\) such that

1. \(Q\) is itself an oscillating tableau,
2. \(Q^{(i)} = (Q_{i,1}, \ldots, Q_{i,s_i})\) is a vertical oscillating tableau for \(1 \leq i \leq \ell\),
3. \(\ell(Q_{i,j}) \leq n\) for \(1 \leq i \leq \ell\) and \(1 \leq j \leq s_i\),
4. \(Q_{1,1} = \square\) and \(Q_{\ell,s_\ell} = \rho_n(\lambda, \ell)\).

Let us consider a stable limit of \(Q\). More precisely, let \(\sigma(Q) = (\hat{Q}^{(1)} : \cdots : \hat{Q}^{(\ell)})\), where \(\hat{Q}^{(i)}\) is a vertical oscillating tableau with \(|\hat{Q}^{(i)}| = s_i + 1\) given as follows;

\[
\hat{Q}^{(i)} = ((i) \cup Q_{i-1,s_{i-1}}, (i) \cup Q_{i,1}, \ldots, (i) \cup Q_{i,s_i}) \quad (1 \leq i \leq \ell).
\]

Here we note that \(Q_{i,k}\) is partition with \(\ell(Q^i_{i,k}) \leq i\) for \(1 \leq k \leq s_i\) since \(Q^{(i)}\) is vertical for \(1 \leq j \leq i - 1\), and denote by \((i) \cup Q_{i,k}\) the partition obtained by adding \(i\) to \(Q_{i,k}\) as its first part.

Hence \(\sigma(Q) = (\hat{Q}^{(1)} : \cdots : \hat{Q}^{(\ell)}) \in O(\lambda, \ell; n + 1)\). Indeed, \(\sigma : O(\lambda, \ell; n) \rightarrow O(\lambda, \ell; n + 1)\) is injective for \(n \geq \lambda_1\), and induces an equivalence relation on \(\bigsqcup_{n \geq \lambda_1} O(\lambda, \ell; n) \times \{n\}\), where

\[
(Q', m) \sim (Q, n) \quad \text{if and only if} \quad \sigma^{m-n}(Q) = Q',
\]
for \( Q' \in \mathbf{O}(\lambda, \ell; m) \) and \( Q \in \mathbf{O}(\lambda, \ell; n) \) with \( m \geq n \). For example, if
\[
(Q^{(1)} : Q^{(2)}) = \left( \begin{array}{ccc}
\Box & \Box & \Box \vline \Box & \Box & \Box \vline \Box & \Box & \Box \\
\Box & \Box & \Box \vline \Box & \Box & \Box \vline \Box & \Box & \Box \\
\Box & \Box & \Box \vline \Box & \Box & \Box \vline \Box & \Box & \Box \\
\Box & \Box & \Box \vline \Box & \Box & \Box \vline \Box & \Box & \Box \\
\Box & \Box & \Box \vline \Box & \Box & \Box \vline \Box & \Box & \Box \\
\Box & \Box & \Box \vline \Box & \Box & \Box \vline \Box & \Box & \Box \\
\Box & \Box & \Box \vline \Box & \Box & \Box \vline \Box & \Box & \Box \\
\Box & \Box & \Box \vline \Box & \Box & \Box \vline \Box & \Box & \Box \\
\Box & \Box & \Box \vline \Box & \Box & \Box \vline \Box & \Box & \Box \\
\Box & \Box & \Box \vline \Box & \Box & \Box \vline \Box & \Box & \Box \\
\end{array} \right),
\]
then
\[
(\hat{Q}^{(1)}, \hat{Q}^{(2)}) = \left( \begin{array}{ccc}
\Box & \Box & \Box \vline \Box & \Box & \Box \vline \Box & \Box & \Box \\
\Box & \Box & \Box \vline \Box & \Box & \Box \vline \Box & \Box & \Box \\
\Box & \Box & \Box \vline \Box & \Box & \Box \vline \Box & \Box & \Box \\
\Box & \Box & \Box \vline \Box & \Box & \Box \vline \Box & \Box & \Box \\
\Box & \Box & \Box \vline \Box & \Box & \Box \vline \Box & \Box & \Box \\
\Box & \Box & \Box \vline \Box & \Box & \Box \vline \Box & \Box & \Box \\
\Box & \Box & \Box \vline \Box & \Box & \Box \vline \Box & \Box & \Box \\
\Box & \Box & \Box \vline \Box & \Box & \Box \vline \Box & \Box & \Box \\
\Box & \Box & \Box \vline \Box & \Box & \Box \vline \Box & \Box & \Box \\
\Box & \Box & \Box \vline \Box & \Box & \Box \vline \Box & \Box & \Box \\
\end{array} \right).
\]

We define
\[
\mathbf{O}(\lambda, \ell) = \{ [Q, n] \mid Q \in \mathbf{O}(\lambda, \ell; n) \ (n \geq \lambda_1) \},
\]
where \([Q, n]\) is the equivalence class of \( Q \in \mathbf{O}(\lambda, \ell; n) \) with respect to \( \boxtimes \). We call \([Q, n] \in \mathbf{O}(\lambda, \ell)\) an oscillating tableau of shape \((\lambda, \ell)\).

**Remark 6.1.** Let \( Q = (Q^{(1)} : \cdots : Q^{(\ell)}) \in \mathbf{O}(\lambda, \ell; n) \) with \(|Q^{(i)}| = s_i\). Each \( Q^{(i)} \) can be identified with a tableau \( U^{(i)} \in SST_{s_i}(\{1^{n-i}\}) \) for \( 1 \leq i \leq \ell \), where \( a \) (resp. \( \overline{a} \)) occurs in \( U^{(i)} \) if and only if a box is added (resp. removed) in the \( a \)-th row in \( Q^{(i)} \). Hence we may view \( Q^{(i)} \in F_n \), and apply \( \varepsilon \) and \( \overline{\varepsilon} \). Indeed, \( \varepsilon Q^{(i)} \) (resp. \( \overline{\varepsilon} Q^{(i)} \)) corresponds to removing (resp. adding) two components in \( Q^{(i)} \) if it is not \( 0 \).

Next, let us define the weights of the elements in \( \mathbf{O}(\lambda, \ell) \). Let \( a = (a_1, \ldots, a_\ell) \in \mathbb{Z}_+^\ell \) be given. We assume that \( n \) is sufficiently large so that \( n - a_i > 0 \) for \( 1 \leq i \leq \ell \). Define \( \mathbf{O}(\lambda, \ell; n)_a \) to be the set of \( Q = (Q^{(1)} : \cdots : Q^{(\ell)}) \in \mathbf{O}(\lambda, \ell; n) \) such that
\[
|Q^{(i)}| = (n - a_i) + 2\varepsilon(Q^{(i)}) \quad (1 \leq i \leq \ell),
\]
where \( Q^{(i)} \) is viewed an element in the regular \( \mathfrak{sl}_2 \)-crystal \( F_n \) by Remark 6.1.

**Lemma 6.2.** Under the above hypothesis,
\begin{enumerate}
\item \( \varphi(Q^{(i)}) + \varepsilon(Q^{(i)}) = a_i \) for \( 1 \leq i \leq \ell \),
\item \( \varepsilon(Q^{(i)}) = \varepsilon(\hat{Q}^{(i)}) \) for \( 1 \leq i \leq \ell \), where \( \sigma(Q) = (\hat{Q}^{(1)} : \cdots : \hat{Q}^{(\ell)}) \),
\item \( \sigma(\mathbf{O}(\lambda, \ell; n)_a) \subset \mathbf{O}(\lambda, \ell; n + 1)_a \).
\end{enumerate}

**Proof.** Consider \( Q^{(i)} \). If \( \varepsilon Q^{(i)} = 0 \), then the tableau \( U^{(i)} \in SST_{s_i}(\{1^{n-a_i}\}) \) corresponding to \( Q^{(i)} \) is admissible by Lemma 4.9(2). So, in general, each \( Q^{(i)} \) belongs to a regular \( \mathfrak{sl}_2 \)-crystal with the highest weight \( a_i \) by \( (6.2) \), and \( \varphi(Q^{(i)}) + \varepsilon(Q^{(i)}) = a_i \) for \( 1 \leq i \leq \ell \). This proves (1).

Let \( \hat{U}^{(i)} \) correspond to \( \hat{Q}^{(i)} \). Then it is obtained from \( U^{(i)} \) by replacing \( k \) (resp. \( \overline{k} \)) with \( k + 1 \) (resp. \( k + 1 \)), and adding the box \( \overline{1} \) at the top. Then both \( (\hat{U}^{(i)})^\varepsilon \) and \( \hat{U}^{(i)} \) are obtained by replacing \( k \) with \( k + 1 \) and so we obtain (2).

Finally, we have \( |\hat{Q}^{(i)}| = |Q^{(i)}| + 1 = (n + 1 - a_i) + 2\varepsilon(Q^{(i)}) = (n + 1 - a_i) + 2\varepsilon(\hat{Q}^{(i)}) \), which implies (3). \( \square \)
Hence by Lemma 6.2, we have the following weight decomposition

\[ O(\lambda, \ell) = \bigsqcup_{a \in \mathbb{Z}_+^\ell} O(\lambda, \ell)_a, \]

where \( O(\lambda, \ell)_a \) is the set of equivalence classes \([Q, n]\) of \( Q \in O(\lambda, \ell; n)_a \). We call \([Q, n] \in O(\lambda, \ell)_a\) an oscillating tableau of shape \((\lambda, \ell)\) with weight \( a \).

**Example 6.3.** Consider an oscillating tableau \( Q = (Q^{(1)} : Q^{(2)} : Q^{(3)}) \in O(\lambda, \ell; n)\) with \( \lambda = (3, 2, 1), \ell = 3, \) and \( n = 5 \) as follows.

\[
Q = (Q^{(1)} : Q^{(2)} : Q^{(3)}) = \begin{pmatrix}
\square \quad \square \quad \square \quad \square \quad \square \\
\square \quad \square \quad \square \quad \square \quad \square \\
\square \quad \square \quad \square \quad \square \quad \square \\
\square \quad \square \quad \square \quad \square \quad \square \\
\square \quad \square \quad \square \quad \square \quad \square \\
\square \quad \square \quad \square \quad \square \quad \square \\
\end{pmatrix}
\]

If we consider each \( Q^{(i)} \) as an element in \( F_a \), then we see that \( \varepsilon(Q^{(1)}) = 2, \varepsilon(Q^{(2)}) = 0, \) and \( \varepsilon(Q^{(3)}) = 1 \) and hence the weight of \([Q, n]\) is \( a = (2, 1, 1) \).

### 6.2. Admissible oscillating tableaux.

Let \( (\lambda, \ell) \in \mathcal{P}(\mathcal{S}p) \) be given. For \( n \geq \lambda_1 \), let

\[ O_c(\lambda, \ell; n) = \big\{ Q \big| Q = (Q^{(1)} : \cdots : Q^{(\ell)}) \in O(\lambda, \ell; n), \varepsilon(Q^{(i)}) = 0 (1 \leq i \leq \ell) \big\}. \]

For \( a \in \mathbb{Z}_+^\ell \) and a sufficiently large \( n \), let \( O_c(\lambda, \ell; n)_a = O_c(\lambda, \ell; n) \cap O(\lambda, \ell; n)_a \). By Lemma 6.2, we have \( \sigma(O_c(\lambda, \ell; n)_a) \subset O_c(\lambda, \ell; n + 1)_a \). Hence we have the following weight decomposition

\[ O_c(\lambda, \ell) := \bigsqcup_{a \in \mathbb{Z}_+^\ell} O_c(\lambda, \ell)_a, \]

where \( O_c(\lambda, \ell)_a \) is the set of equivalence classes \([Q, n]\) of \( Q \in O_c(\lambda, \ell; n)_a \). We call \([Q, n] \in O_c(\lambda, \ell)\) an *admissible oscillating tableau of shape* \((\lambda, \ell)\).

**Proposition 6.4.** For \((\lambda, \ell) \in \mathcal{P}(\mathcal{S}p)\) and \( a = (a_1, \ldots, a_\ell) \in \mathbb{Z}_+^\ell \), we have a bijection

\[
O(\lambda, \ell)_a \longrightarrow O_c(\lambda, \ell)_a \times \mathbb{Z}/(a + 1)\mathbb{Z},
\]

\[
[Q, n] \longrightarrow ([Q, n], \varepsilon(Q))
\]

where \( \mathbb{Z}/(a + 1)\mathbb{Z} = \mathbb{Z}/(a_1 + 1)\mathbb{Z} \times \cdots \times \mathbb{Z}/(a_\ell + 1)\mathbb{Z} \), and

\[ Q_c = (E^{\max}Q^{(1)} : \cdots : E^{\max}Q^{(\ell)}), \quad \varepsilon(Q) = \left( \varepsilon(Q^{(1)}), \ldots, \varepsilon(Q^{(\ell)}) \right) \]

for \( Q = (Q^{(1)} : \cdots : Q^{(\ell)}) \in O(\lambda, \ell; n)_a \).

**Proof.** The map is a well-defined bijection by Lemmas 6.3 (3) and 6.2 (2). \( \square \)
Example 6.5. Let \([Q, 5] \in \mathbf{O}(\lambda, \ell)\) be given in Example 6.3. Then the image of \([Q, 5]\) under (6.3) is

\[
[(E_{\max}^{(1)}: E_{\max}^{(2)}: E_{\max}^{(3)}), (2, 0, 1)],
\]

where

\[
E_{\max}^{(1)} = \begin{pmatrix}
\square & \square & \square \\
\square & \square & \square \\
\square & \square & \square
\end{pmatrix},
\]

\[
E_{\max}^{(2)} = \begin{pmatrix}
\square & \square & \square & \square & \square \\
\square & \square & \square & \square & \square \\
\square & \square & \square & \square & \square \\
\square & \square & \square & \square & \square
\end{pmatrix},
\]

\[
E_{\max}^{(3)} = \begin{pmatrix}
\square & \square & \square & \square & \square \\
\square & \square & \square & \square & \square \\
\square & \square & \square & \square & \square \\
\square & \square & \square & \square & \square \\
\square & \square & \square & \square & \square
\end{pmatrix}.
\]

6.3. King tableaux. Let us recall another combinatorial model for irreducible symplectic characters. For \(\ell \geq 2\), let

\[
J_{\ell} = \{1 < T < 2 < \ldots < \ell < T\},
\]

where we assume that all the entries are of degree 0.

For \((\lambda, \ell) \in P(\text{Sp})\), let \(K(\lambda, \ell)\) be the set of \(T \in \text{SST}_{J_{\ell}}(\lambda)\) such that all entries in the \(i\)th row are larger than or equal to \(i\). It is known as the set of King tableaux of shape \(\lambda\).

For \(n \geq \lambda_1\), let \(K(\lambda, \ell; n)\) denote the set \(K(\lambda, \ell)\), where the columns of the tableaux are enumerated by \(n, n - 1, \ldots, 1\) from the left.

Let \(K \in K(\lambda, \ell; n)\) be given. We define a sequence of vertical oscillating tableaux \(Q(K; n) = (Q^{(1)}: \ldots : Q^{(\ell)})\) as follows: for \(1 \leq i \leq \ell\) and \(1 \leq j \leq n\),

1. the letter \(i\) is contained in the \(j\)th column of \(K\) if and only if there is no step in \(Q^{(i)}\) such that a box is added in the \(j\)th row,
2. the letter \(i\) is contained in the \(j\)th column of \(K\) if and only if there is a step in \(Q^{(i)}\) such that a box is deleted in the \(j\)th row.

Theorem 6.6. [31, Theorem 2.7] For \(\lambda \subseteq (n^\ell)\), we have a bijection

\[
K(\lambda, \ell; n) \rightarrow \mathbf{O}(\lambda, \ell; n)
\]

\[
K \rightarrow Q(K; n)
\]

Example 6.7. Let \(\lambda = (3, 2, 1) \subseteq (5^3)\) with \(n = 5\) and \(\ell = 3\), and

\[
K = \begin{pmatrix}
1 & 1 & 2 \\
3 & 3 & 3 \\
\end{pmatrix} \in K(\lambda, \ell; n).
\]
Corollary 6.8. For $\lambda \subseteq (n^\ell)$, we have a bijection

\[
Q(K; n) = \begin{pmatrix}
\square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square \\
\square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square \\
\square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square \\
\square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square \\
\square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square \\
\square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square \\
\square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square \\
\square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square \\
\square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square \\
\square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square \\
\square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square \\
\square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square \\
\square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square
\end{pmatrix}.
\]

Proof. It follows from the definition of $Q(K; n)$ and $\sigma$ that $\sigma(Q(K; n)) = Q(K; n + 1)$ for $K \in K(\lambda, \ell; n)$. Hence the map is a well-defined bijection.

\section{Symplectic RSK correspondence}

7.1. Pieri rule for spinor model. Let $a = (a_1, \ldots, a_\ell) \in \mathbb{Z}_+^\ell$ be given. Let $T = (T_\ell, \ldots, T_1) \in T_A(a_\ell) \times \cdots \times T_A(a_1)$ be given. We may regard $T = (T_\ell, \ldots, T_1) \in T_A(\zeta/\eta, \ell)$ for some skew diagram $\zeta/\eta$ with $\zeta, \eta \in \mathcal{P}_\ell$.

By Theorem 5.15, there exists a unique $P(T) \in T_A(\lambda, \ell)$ for some $(\lambda, \ell) \in \mathcal{P}(\text{Sp})_A$, which is obtained by applying $jdt_{\text{spin}}$ finitely many times with respect to inner corners $c$.

Let us define a recording tableau for $P(T)$. We choose first a sufficiently large $n$. Let $T = (T_\ell, \ldots, T_1)$ be the $n$-conjugate of $T$. By Corollary 5.6, $T^\text{ed} \in KN_{\alpha/\beta}$ for some skew diagram $\alpha/\beta$. Let

\[
Q(T; n) = Q\left(\bar{T}^\text{ed}\right),
\]

where the right-hand side means $Q(w)$ for $w = w(T^\text{ed})$ given in Proposition 4.12.

Lemma 7.1. Under the above hypothesis, we have

(1) $Q(T; n) \in O_\text{c}(\lambda, \ell; n)_a$.
(2) $\sigma(Q(T; n)) = Q(T; n + 1)$.

Proof. (1) Note that

\[
\bar{T}^\text{ed} = \left[\bar{T}_\ell^\text{ed}, \ldots, \bar{T}_1^\text{ed}\right]^{\beta'},
\]

where $\bar{T}_i^\text{ed} \in SST_{n}\left((1^{n-a_i})\right)$. Let $w^{(i)} = w_{i,1} \cdots w_{i,n-a_i} = w(T_i^\text{ed})$. For $1 \leq i \leq \ell$ and $1 \leq k \leq n - a_i$, put

\[
Q_{i,k} = \text{shP}\left(w^{(1)} \cdots w^{(i-1)} w_{i,1} \cdots w_{i,k}\right),
\]

where we assume that $w^{(0)}$ is the empty word. Then by Lemma 4.9 and Proposition 4.12, we have

- $Q^{(i)} = (Q_{i,1}, \ldots, Q_{i,n-a_i})$ is a vertical oscillating tableau with $\varepsilon(Q^{(i)}) = 0$,
- $Q_{i,1}, \ldots, Q_{i,n-a_i}$ are all 0.

7. Symplectic RSK correspondence

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\[
Q(T; n) = Q\left(\bar{T}^\text{ed}\right),
\]

where the right-hand side means $Q(w)$ for $w = w(T^\text{ed})$ given in Proposition 4.12.
• $Q(T; n) = Q(w^{(1)} \cdots w^{(\ell)}) = (Q^{(1)} : \cdots : Q^{(\ell)})$,

which implies that $Q(T; n) \in O_\ell(\lambda, \ell; n)_\mathfrak{a}$.

(2) It can be checked in a straightforward manner. So we leave it to the reader. □

Now, we define

\begin{equation}
(7.2) \quad Q_\mathfrak{a}(T) = [Q(T; n), n] \in O_\ell(\lambda, \ell; n)_\mathfrak{a},
\end{equation}

which is well-defined by Lemma 7.1. The following is one of the main results in this paper.

**Theorem 7.2.** For $\mathfrak{a} = (a_1, \ldots, a_\ell) \in \mathbb{Z}_+^\ell$, we have a bijection

\begin{equation}
(7.3) \quad \bigoplus_{(\lambda, \ell) \in \mathcal{P}(\text{Sp})_\mathfrak{a}} T_A(a_\ell) \times \cdots \times T_A(a_1) \longrightarrow \bigoplus_{(\lambda, \ell) \in \mathcal{P}(\text{Sp})_\mathfrak{a}} T_A(\lambda, \ell) \times O_\ell(\lambda, \ell; n)_\mathfrak{a}.
\end{equation}

\begin{equation}
(7.4) \quad T \quad \xrightarrow{\gamma'} \quad P(T) \quad \xrightarrow{\gamma} \quad U \quad \xrightarrow{a^*} \quad V \quad \xrightarrow{a^*} \quad P(V)
\end{equation}

where $\gamma$ is a sequence of $jdt_{KN}$'s \(4.3\), and $\gamma'$ is the corresponding sequence of $jdt_{\text{spin}}$'s \(5.8\). Recall $Q_\mathfrak{a}(T) = [Q(V), n]$.

Let us first prove the injectivity of the map. Let $T'$ be given such that $(P(T), Q_\mathfrak{a}(T)) = (P(T'), Q_\mathfrak{a}(T'))$. Let $U'$ be its $n$-conjugate and $V' = U'^{\text{rad}}$. By definition, this implies that $P(V) = P(V')$ and $Q(V) = Q(V')$. We claim that

\begin{equation}
(7.5) \quad \bigoplus_{(\lambda, \ell) \in \mathcal{P}(\text{Sp})_n} \mathcal{K}N_{(n-a_\ell)} \times \cdots \times \mathcal{K}N_{(n-a_1)} \longrightarrow \bigoplus_{(\lambda, \ell) \in \mathcal{P}(\text{Sp})_n} \mathcal{K}N_{\rho_n(\lambda, \ell)} \times O_\ell(\lambda, \ell; n)_\mathfrak{a}
\end{equation}

\begin{equation}
V \longmapsto \bigoplus_{(\lambda, \ell) \in \mathcal{P}(\text{Sp})_n} \mathcal{K}N_{\rho_n(\lambda, \ell)} \times O_\ell(\lambda, \ell; n)_\mathfrak{a}
\end{equation}

is a bijection. In fact, the map is a morphism of $\mathfrak{sp}_{2n}$-crystals by Proposition \(1.12\) which in particular implies that $Q(V)$ is constant on its connected component. Using the combinatorial rule of tensor product decomposition \(33\), we see that $Q(V)$ uniquely determines a highest weight element in the connected component and hence the map is injective. On the other hand, for a given pair $(H_{\rho_n(\lambda, \ell)}, Q)$ on the right-hand side of \(7.5\), one can construct directly $T$ such that $\tilde{e}_i T = 0$ for $1 \leq i \leq n$ and $Q(T) = Q$ again by \(33\). This implies the surjectivity of \(7.5\). Hence, we have $V = V'$ and $T = T'$ by \(7.4\).

The surjectivity of the map follows from \(7.2\) and the bijection \(7.5\). □
Example 7.3. Let $T = (T_3, T_2, T_1)$ be given in Example 5.3. By Example 5.16 we get

$$P(T) = \begin{pmatrix}
2 & 2 \\
3 & 4 \\
1' & 2'
\end{pmatrix}, \begin{pmatrix}
1 & 1 \\
3 & 2' \\
1' & 3'
\end{pmatrix}, \begin{pmatrix}
2 & 2 \\
1' & 2'
\end{pmatrix}.$$ 

Since

$$T^{ad} = \begin{pmatrix}
1 \\
2 & 5 \\
1 & 5 & 4 \\
4 & 4 \\
4 & 3 \\
2
\end{pmatrix},$$

we have by Example 4.13

$$Q_\varphi(T) = \left(\begin{pmatrix}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{pmatrix}, \begin{pmatrix}
5
\end{pmatrix}\right).$$

7.2. RSK correspondence. The goal of this subsection is to establish an analogue of RSK correspondence for spinor model. Let

$$\mathbf{F}^\ell_{A} = \mathbf{E}^{2\ell}_{A} \quad (\ell \geq 1).$$

Lemma 7.4. We have a bijection

$$\mathbf{F}^1_{A} \longrightarrow \bigsqcup_a T_A(a) \times \mathbb{Z}/(a+1)\mathbb{Z},$$

$$T \longrightarrow (T^{max}, \varphi(T))$$

where the union is over $a \geq 0$ such that $T_A(a) \neq \emptyset$.

Proof. Let $T \in \mathbf{F}^1_{A}$ be given. Then we have $T^{max} = T^{\varphi(T)} \in SST_{A}(\lambda(a, 0, c))$ for some $a, c \in \mathbb{Z}_+$ by Lemma 2.4 where $\varphi = \varphi(T)$. It is clear that the map is injective. The connected component of $T$ is a regular $\mathfrak{sl}_2$-crystal with highest weight $a$, and the map is also surjective. $\Box$

For $a = (a_1, \ldots, a_\ell) \in \mathbb{Z}_+^\ell$, put

$$T_A(a) = T_A(a_\ell) \times \cdots \times T_A(a_1).$$

Corollary 7.5. We have a bijection

$$\mathbf{F}^\ell_{A} \longrightarrow \bigsqcup_a T_A(a) \times \mathbb{Z}/(a+1)\mathbb{Z},$$

$$T \longrightarrow (T^{max}, \varphi(T))$$

where the union is over $a \geq 0$ such that $T_A(a) \neq \emptyset$. 

Proof. Let $T \in \mathbf{F}^\ell_{A}$ be given. Then we have $T^{max} = T^{\varphi(T)} \in SST_{A}(\lambda(a, 0, c))$ for some $a, c \in \mathbb{Z}_+$ by Lemma 2.4 where $\varphi = \varphi(T)$. It is clear that the map is injective. The connected component of $T$ is a regular $\mathfrak{sl}_2$-crystal with highest weight $a$, and the map is also surjective. $\Box$
where

\[ \mathcal{T} \max (T) = (\mathcal{T} \max (U_{2\ell}, U_{2\ell-1}), \ldots, \mathcal{T} \max (U_2, U_1)), \]
\[ \varphi(T) = (\varphi(U_{2\ell}, U_{2\ell-1}), \ldots, \varphi(U_2, U_1)), \]

for \( T = (U_2, \ldots, U_1) \in T_A(a) \) and the union is over \( a \in \mathbb{Z}_+^\ell \) such that \( T_A(a) \neq \emptyset \).

Example 7.6. Suppose that \( A = \mathbb{I}_{4|3} \). If

\[
T = \begin{pmatrix}
3 & 2 & 1' & 1' & 2' & 3' \\
4 & 1 & 1 & 1 & 2' & 1' \\
2' & 3 & 1' & 2' & 1' & 3' \\
1 & 2 & 1' & 2' & 1' & 3' \\
\end{pmatrix} \in F^3_A,
\]

then we have

\[
\mathcal{T} \max (T) = \begin{pmatrix}
2 & 4 & 1 & 1 & 2' & 1' & 2' & 1' & 3' \\
3 & 2 & 1 & 1 & 2' & 1' & 2' & 1' & 3' \\
1' & 2' & 1' & 2' & 1' & 3' \\
\end{pmatrix}, \quad \varphi(T) = (1, 0, 2).
\]

Now we are ready to state our main result in this paper. Consider the composition of the following sequence of bijections.

\[
F^\ell_A \xrightarrow{(7.6)} \bigcup_{a \in \mathbb{Z}_+^\ell} T_A(a) \times \mathbb{Z}/(a + 1)\mathbb{Z}
\]
\[
\xrightarrow{(7.3)} \bigcup_{a \in \mathbb{Z}_+^\ell \times \mathbb{Z}/(a + 1)\mathbb{Z}} T_A(\lambda, \ell) \times \mathbb{O}_c(\lambda, \ell)_a \times \mathbb{Z}/(a + 1)\mathbb{Z}
\]
\[
\xrightarrow{(6.3)} \bigcup_{(\lambda, \ell) \in \mathbb{Z}/(a + 1)\mathbb{Z}} T_A(\lambda, \ell) \times \mathbb{O}(\lambda, \ell)
\]
\[
\xrightarrow{(6.3)} \bigcup_{(\lambda, \ell) \in \mathbb{Z}/(a + 1)\mathbb{Z}} T_A(\lambda, \ell) \times \mathbb{K}(\lambda, \ell)
\]

Let \( (P(T), Q(T)) \) denote the image of \( T \in F^\ell_A \) under the composition of \( (7.6), (7.3), \) and \( (6.3) \). Let \( (P(T), Q(T)) \) denote the image of \( (P(T), Q(T)) \) under \( (6.3) \). Hence we obtain the following correspondence, which is the main result in this paper.

**Theorem 7.7.** For \( \ell \geq 1 \), we have a bijection

\[
F^\ell_A \xrightarrow{} \bigcup_{(\lambda, \ell) \in \mathbb{Z}/(a + 1)\mathbb{Z}} T_A(\lambda, \ell) \times \mathbb{K}(\lambda, \ell)
\]

\[
T \xrightarrow{} (P(T), Q(T))
\]

**Example 7.8.** Let \( T \in F^3_A \) be the one given in Example 7.6. Combining Examples 7.6, 7.3, 5.16, 6.5, and 6.3, we have \( (P(T), Q(T)) \in T_A(\lambda, 3) \times \mathbb{O}(\lambda, 3) \) for \( \lambda = (3, 2, 1) \), where
The oscillating tableau $Q(T)$ corresponds to a King tableau $K$ in Example 6.7 under (6.4). Hence $Q(T) \in K(\lambda, 3)$, where

$$Q(T) = \begin{array}{cccc}
1 & 1 & 2 \\
3 & 3 & 4 \\
1' & 2' & 1' & 2' & 3'
\end{array}.$$

**Remark 7.9.** When $A = [\ell]$, the right-hand side of the bijection in Theorem 7.7 has an $(\mathfrak{sp}_{2n}, \mathfrak{sp}_{2\ell})$-bicrystal structure. On the other hand, $F^\ell_A$ is an $\mathfrak{sp}_{2n}$-crystal by (7.6), and the bijection is an isomorphism of $\mathfrak{sp}_{2n}$-crystals. However, we do not know how to define an $\mathfrak{sp}_{2\ell}$-crystal structure on $F^\ell_A$ directly so that the bijection is an isomorphism of $(\mathfrak{sp}_{2n}, \mathfrak{sp}_{2\ell})$-bicrystals.

### 7.3. Cauchy type identity

Let $z = z_\ell = \{ z_1, \ldots, z_\ell \}$ be formal commuting variables, which commute with $x = x_A = \{ x_a | a \in A \}$ (cf. Section 3.1). For $A = [n]$, write $x_n = x_{[n]} = \{ x_1, \ldots, x_n \}$.

Let $(\lambda, \ell) \in \mathcal{P}(\text{Sp})$ be given. For $K \in K(\lambda, \ell)$, let $z^K = \prod_{i \in [\ell]} z_i^{m_i - m_\ell}$, where $m_i$ (resp. $m_\ell$) is the number of occurrences of $i$ (resp. $\ell$) in $K$. Then put

$$sp_\lambda(z) = \sum_{K \in K(\lambda, \ell)} z^K.$$

It is well-known that $sp_\lambda(z)$ is the character of the irreducible highest weight module of $\text{Sp}_{2\ell}$ with highest weight corresponding to $\lambda$.

Let $U = (U_2, \ldots, U_1) \in F^\ell_A$ be given with $u_i = \text{ht}(U_i)$. Let $x^U = \prod_{i=1}^{2\ell} x^{U_i}$ and $z^U = \prod_{i \in [\ell]} z_i^{u_{2i-1}}$. Then we have

$$\text{ch} F^\ell_A : = \sum_U x^U z^U = \prod_{j=1}^{\ell} \frac{\prod_{a \in A_0} (1 + x_a z_j) (1 + x_a z_j^{-1})}{\prod_{a \in A_1} (1 - x_a z_j) (1 - x_a z_j^{-1})}.$$
Theorem 7.10. We have the following identity
\[ t^\ell \prod_{j=1}^{\ell} \frac{(1 + x a_j z_j)(1 + x z_j^{-1})}{(1 - x a_j z_j)(1 - x z_j^{-1})} = \sum_{(\lambda, \ell) \in \mathcal{P}(\mathfrak{sp})_{\lambda}} S_{(\lambda, \ell)}(x, \mathfrak{z}) s_{P(\lambda, \ell)}(\mathfrak{z}). \]

Proof. Let \( U = (U_{2\ell}, \ldots, U_1) \in F^\lambda_L \) be given. If \( U \) is mapped to \((T, K)\) by Theorem 7.7, then it suffices to show that \( x^U z^U = z^T z^K \). Since \( x^U = x^T \) is clear, it remains to show that \( z^U = z^K \).

Suppose that \( U \) is mapped to \((T, \varphi)\) by (7.6) where \( T = (T_\ell, \ldots, T_1) \in T_A(a) \) for some \( a \in \mathbb{Z}_L \) and \( \varphi = (\varphi_\ell, \ldots, \varphi_1) = (\varphi(U_{2\ell}, U_{2\ell-1}), \ldots, \varphi(U_2, U_1)) \in \mathbb{Z}/(a + 1)\mathbb{Z} \).

Choose a sufficiently large \( n \) and take the \( n \)-conjugate \( \overline{T} = (\overline{T}_\ell, \ldots, \overline{T}_1) \) of \( T \) and \( \mathfrak{ad} = (\overline{T}_\ell, \ldots, \overline{T}_1) \). By (7.3), \( T \) is mapped to \((P(T), Q_0(T))\). Here \( P(T) \in T_A(\lambda, \ell) \) for some \( (\lambda, \ell) \in \mathcal{P}(\mathfrak{sp}) \) and \( Q_0(T) = [Q(T; n), n] \in O(\lambda, \ell) \).

For \( (\lambda, \ell) \in \mathcal{P}(\mathfrak{sp}) \) and \( Q_0(T) = [Q(T; n), n] \in O(\lambda, \ell) \), define
\[ Q(T; n) = (Q^{(1)} : \cdots : Q^{(\ell)}), \]
as given in (7.1). Note that \( |Q^{(i)}| = n - a_i \) for \( 1 \leq i \leq \ell \). By (6.3), \( Q(T; n), \varphi \) is mapped to \([Q', n]\) where \( Q' = (Q^{(1)} : \cdots : Q^{(\ell)}) \in O(\lambda, \ell; n) \). Then we have \( |Q^{(i)}'| = |Q^{(i)}| + 2\varphi_i \) for \( 1 \leq i \leq \ell \).

Let \( u_j = \text{ht}(U_j) \) for \( 1 \leq j \leq 2\ell \), and \( t_i^\pm = \text{ht}(\overline{T}_i^{\pm}) \) for \( 1 \leq i \leq \ell \). By considering the \( \mathfrak{sl}_2 \)-weight of \((U_{2\ell}, U_{2\ell-1})\), we have
\[ u_{2i} - u_{2i-1} + 2\varphi = (n - t_i^+) - t_i^- = n - (t_i^+ + t_i^-) = a_i. \]

On the other hand, it is straightforward to see from the bijection in Theorem 6.6 that
\[ a_i - 2\varphi = n - |Q^{(i)}| = m_i - m_i^- \]
for \( 1 \leq i \leq \ell \). Hence \( m_i = m_i^- = u_{2i} - u_{2i-1} \). This proves \( z^U = z^K \). \( \square \)

Let us end this section with well-known identities which can be recovered from Theorem 7.10 under special choices of \( A \).

First, assume that \( A = [n] \). Let \( P = \mathbb{Z}_{\geq 0} \mathbf{e}_i \) be the weight lattice for \( \mathfrak{sp}_{2n} \) in Section 4.1 and let \( \mathbb{Z}[P] \) be its group ring with a \( \mathbb{Z} \)-basis \( \{ e^\mu \mid \mu \in P \} \). Note that \( \varpi_n = \epsilon_1 + \cdots + \epsilon_n \), the \( n \)-th fundamental weight. For \( 0 \leq a \leq n \) and \( T \in T_n(a) \), define
\[ \text{wt}(T) = \varpi_n - \sum_{i=1}^n m_i \epsilon_i, \]
where \( m_i \) is the number of occurrences of \( \mathbf{i} \) in \( T \). For \( a = (a_1, \ldots, a_\ell) \in \mathbb{Z}_+^\ell \) and \( T = (T_\ell, \ldots, T_1) \in T_n(a) \), define \( \text{wt}(T) = \sum_{i=1}^\ell \text{wt}(T_i) \). For a tableau \( K \) with letters in \( \mathcal{J}_n \), let
\[ \text{wt}(K) = \sum_{i=1}^n (m_i - m_i^-) \epsilon_i, \]
where \( m_a \) is the number of occurrences of \( a \) in \( K \) for \( a \in \mathcal{J}_n \). It is easy to check that \( \text{wt}(T) = \text{wt}(T^{\text{ad}}) \) for any \( T \in T_n(a) \), and hence (4.2) and (5.1) are weight-preserving bijections.
By identifying $x_i^\tau = e^{-i} \in \mathbb{Z}[P]$ for $i \in [n]$ and $t = e^{\infty n} = x_1 \cdots x_n$ in (3.1), we have
\[
S_{\lambda, \ell}(x_A) = \sum_{t \in T_n(\lambda, \ell)} t^i x_A^T = \sum_{t \in T_n(\lambda, \ell)} e^{\text{wt}(T)} = \sum_{t \in \mathbb{K}N_{\mu}(\lambda, \ell)} e^{\text{wt}(T)} = sp_{\mu}(\lambda, \ell)(x_n).
\]
The following identity follows immediately from Theorem 7.10 and the identity $x_i + x_i^{-1} + z_j + z_j^{-1} = x_i(1 + x_i^{-1}z_j)(1 + x_i^{-1}z_j^{-1})$.

**Corollary 7.11 (10).** For $n, \ell \geq 1$, we have
\[
\prod_{i=1}^{n} \prod_{j=1}^{\ell} (x_i + x_i^{-1} + z_j + z_j^{-1}) = \sum_{\lambda \in \mathbb{N}(n, \ell)} sp_{\mu}(\lambda, \ell)(x_n) sp_{\lambda}(z).
\]
Next, assume that $A = [n]'$. For $\ell \geq n$, there exists a bijection in [24, Theorem 6.5]
\[
T_A(\lambda, \ell) \quad \longrightarrow \quad \bigcup_{\beta: \text{even}} SST_A(\lambda') \times SST_A(\beta'),
\]
which gives the identity
\[
S_{\lambda, \ell}(x_A) = t^i s_{\lambda}(x_A) \sum_{\beta: \text{even}} s_{\beta}(x_A) = t^i s_{\lambda}(x_n) \sum_{\beta: \text{even}} s_{\beta}(x_n).
\]
Here we call a partition $\beta$ even if all of its parts are even. Also note that we have
\[
s_{\mu}(x_A) = \sum_{T \in SST_A(\mu)} x_A^T = \sum_{T \in SST_A([n])} x_A^T = s_{\mu}(x_n)
\]
for $\mu \in \mathcal{P}$ by identifying $x_i^\tau = x_i$ for $i \in [n]$. By Theorem 7.10, we also recover the well-known classical identity due to Littlewood [32] and Weyl [39].

**Corollary 7.12 (32, 39).** For $\ell \geq n \geq 1$, we have
\[
\prod_{i=1}^{n} \prod_{j=1}^{\ell} (1 - x_i z_j)^{-1}(1 - x_i z_j^{-1})^{-1} = \sum_{\ell(\lambda) \leq n} sp_{\lambda}(z)s_{\lambda}(x_n) \prod_{1 \leq i < j \leq n} (1 - x_i x_j)^{-1}
\]
\[
= \sum_{\ell(\lambda) \leq n} sp_{\lambda}(z)s_{\lambda}(x_n) \sum_{\beta: \text{even}} s_{\beta}(x_n).
\]

**Remark 7.13.** The bijection in Theorem 7.11 even when reduced to the above cases is completely different from the ones in [37] and [39] for the identities in Corollaries 7.11 and 7.12 respectively, where the insertion algorithm in terms of the King tableaux is used.

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