AN ALGORITHM TO COMPUTE THE DEGREE OF A DICKSON POLYNOMIAL

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ABSTRACT

In this study, we describe an algorithm that computes the degree of a Dickson Polynomial of the First Kind from its known value at a point. Our algorithm is based on a mathematical relation between Dickson Polynomials of the First Kind and Chebyshev Polynomials of the First Kind.

Keywords: Symbolic Computation, Algorithms, Dickson Polynomials, Pohlig-Hellman Algorithm

1. INTRODUCTION

Dickson Polynomials are introduced in [1] by L.E. Dickson. Let $K$ be a finite field with characteristic $\text{char}(K) = p$ and $a \in K$. Dickson Polynomials of the First Kind are polynomials in $x$ over $K$ and they are denoted by $D_n(x, a)$ where $n$ is the degree of the polynomial. They can be defined by the recurrence relation

$$
\begin{align*}
D_0(x, a) &= 2 \\
D_1(x, a) &= x \\
D_n(x, a) &= xD_{n-1}(x, a) - aD_{n-2}(x, a), \forall n \geq 2.
\end{align*}
$$

Similarly, Dickson Polynomials of the Second Kind are denoted by $E_n(x, a)$ and they can be defined by the same recurrence relation with a different initialization at the degree $n = 0$:

$$
\begin{align*}
E_0(x, a) &= 1 \\
E_1(x, a) &= x \\
E_n(x, a) &= xE_{n-1}(x, a) - aE_{n-2}(x, a), \forall n \geq 2.
\end{align*}
$$

Wang and Yucas [2] extend the Dickson Polynomials to a family depending on a new integer parameter $k \in \mathbb{Z}_{\geq 0}$ which they call Dickson Polynomials of the $(k + 1)$-th Kind. Those polynomials are denoted by $D_{n,k}(x, a)$ and can be defined similarly:

$$
\begin{align*}
D_{0,k}(x, a) &= 2 - k \\
D_{1,k}(x, a) &= x \\
D_{n,k}(x, a) &= xD_{n-1,k}(x, a) - aD_{n-2,k}(x, a), \forall n \geq 2.
\end{align*}
$$

Here the integers $k = 0$ and $k = 1$ yield Dickson Polynomials of the First Kind and the Second Kind respectively. Alternatively, Dickson Polynomials of all kinds, can be computed via the matrix formula below:

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The matrix method gives rise to an algorithm that computes all Dickson Polynomials of the \((k + 1)\)-th Kind, \(D_{n,k}(x, a)\), in \(O(\log(n))\) scalar operations.

Dickson Polynomials are examples of orthogonal polynomials and they satisfy several useful properties. The polynomials \(D_n(x, a)\) and \(E_n(x, a)\) satisfy the differential equations

\[
\begin{align*}
(x^2 - 4a)D_n''(x, a) + xD_n'(x, a) - n^2D_n(x, a) &= 0 \\
(x^2 - 4a)E_n''(x, a) + 3x E_n'(x, a) - n(n+2)E_n(x, a) &= 0
\end{align*}
\]

and, in general, the polynomials \(D_{n,k}(x, a)\) satisfy the differential equation

\[
(x^2 - 4a)D_{n,k}''(x, a) - 4nD_{n+1,k}(x, a)D_{n,k}'(x, a) + (2n+3)xD_{n,k}'(x, a) + n(n+2)D_{n,k}(x, a) = 0.
\]

Dickson Polynomials arise in various areas in mathematics, such as integro-differential-difference equations [4-6], cryptography and number theory [7,8]. Further details about Dickson Polynomials can be found at [3-8] and references within. Equation (6) can be found at [3, Proposition 5].

We address the following problem in this article:

**Problem 1.1** From given \(p = char(K), \beta \in K \backslash \{0\}, a, b \in K\) such that \(b^2 = a\) and \(\xi = D_\delta(\beta, a) \in K\) compute the degree \(\delta\) of the Dickson Polynomial of the First Kind \(D_\delta(x, a)\).

Dickson Polynomials of the First Kind are related to Chebyshev Polynomials of the First Kind.

**Theorem 1.1** If \(a \in K, b^2 = a\), then

\[
D_n(x, a) = 2b^n T_n\left(\frac{x}{2b}\right).
\]

Chebyshev Polynomials of the First Kind have the following two useful properties.

**Theorem 1.2**. Let \(m, n \in \mathbb{Z}_{\geq 0}\). Then:

1. \(T_n(T_m(x)) = T_{nm}(x) = T_m(T_n(x))\).
2. \(T_n\left(\frac{x + \frac{1}{x}}{2}\right) = \frac{x^{n+\frac{1}{2}} + x^{-n+\frac{1}{2}}}{2}\) for all \(n \geq 0\).

An algorithm that computes the degree of a Chebyshev Polynomial of the First Kind by using its known value at a point is given in [9]. That algorithm makes use of Theorem 1.1 and the idea lying behind of the Pohlig-Hellman Algorithm (which is also known as Silver-Pohlig-Hellman Algorithm) [10]. More details about the Pohlig-Hellman Algorithm and a survey of several discrete logarithm algorithms can be found at [11]. The algorithm in [9], at the end, computes and returns the mixed-radix form of the unknown degree of the Chebyshev Polynomial.

In this paper, we make use of Theorem 1.1, Theorem 1.2(2) and the algorithm in [9] to introduce a method which solves Problem 1.1.
2. DISCUSSION, RESULTS AND ALGORITHM

We want to solve Problem 1.1, i.e., we want to compute the degree $\delta$ from given the value $\xi = D_\delta(\beta, a) \in K$ at $x = \beta$. We assume that $\beta \in K \setminus \{0\}$, $a, b \in K$ such that $b^2 = a$ and $p = \text{char}(K)$ are known. We may assume, without loss of generality, $\beta = b \left( \omega + \frac{1}{\omega} \right)$ for some unknown $\omega \in \overline{K}$. We do not need to know $\omega \in \overline{K}$. We make use of Theorem 1.1 and Theorem 1.2(2) and proceed as follows:

$$\xi = D_\delta(\beta, a) = D_\delta b \left( b \left( \omega + \frac{1}{\omega} \right), a \right) = 2b^\delta T_\delta \left( \frac{\omega + \frac{1}{\omega}}{2} \right). \quad (8)$$

From the last equation we get

$$\zeta = \xi \left( 2b^\delta \right)^{-1} = T_\delta \left( \frac{\omega + \frac{1}{\omega}}{2} \right). \quad (9)$$

If $\zeta = \xi \left( 2b^\delta \right)^{-1}$ is known, then algorithm in [9] can compute the degree $\delta$ from given $\zeta = T_\delta(\gamma)$, where $\gamma = \left( \omega + \frac{1}{\omega} \right)/2$. Since the degree $\delta$ is unknown, here also $\zeta = \xi \left( 2b^\delta \right)^{-1}$ remains unknown. Note that, since $b \in K$ is a known value, here two cases occur:

1. If it is given that the order of $b \in K$ divides $\delta$, then $\zeta = \xi \left( 2b^\delta \right)^{-1} = \xi/2$. In this case, one can directly use the algorithm in [9] to compute $\delta$.

2. Otherwise, one can compute the order $m$ of $b \in K$ first. Then:

$$\zeta^m = \left( \xi \left( 2b^\delta \right)^{-1} \right)^m = \xi^m2^{-m} = \left( \frac{\xi}{2} \right)^m \quad (10)$$

From $\zeta^m = (\xi/2)^m$, one can compute $\xi$. Once $\xi$ is computed, one can use the algorithm in [9] and can compute $\delta$.

We summarize our algorithm as follow:

**Algorithm 2.1**

Input:
- $a, b \in K$ such that $b^2 = a$
- $p = \text{char}(K) \geq 3$
- $\beta = b \left( \omega + \frac{1}{\omega} \right) \in K \setminus \{0\}$
- $\xi = D_\delta(\beta, a) \in K$

Output:
- The order $n$ of $\omega$
- $\delta \mod n$ or $-\delta \mod n$
1. Use Theorem 1.1 and Theorem 1.2(2) to get \( \zeta = T_\delta(\gamma) \), where \( \gamma = \left( \omega + \frac{1}{\omega} \right)/2 \), from \( \xi = D_\delta(\beta, a) \).
   a. If it is given that the order of \( b \) divides \( \delta \), let \( \zeta = \xi/2 \), and proceed to Step 2.
   b. Otherwise:
      i. Compute the order \( m \) of \( b \).
      ii. Compute \( \zeta \) from \( \zeta^m = (\xi/2)^m \).

2. Use algorithm in [9] to compute \( \zeta \) from \( \zeta = T_\delta(\gamma) \) where \( \gamma = \left( \omega + \frac{1}{\omega} \right)/2 \) and return order \( n \) of \( \omega \), and, \( \delta \ mod \ n \) or \( -\delta \ mod \ n \).

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CONFLICT OF INTEREST

The author stated that there are no conflicts of interest regarding the publication of this article.

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