**Abstract** – The antiferromagnetic coupling and entanglement between skyrmion lattices are treated in magnetic bilayer systems. We first formulate the problem of large bilayer skyrmions using $CP^1 \otimes CP^1$ theory. We have considered bilayer skyrmions under the presence of Dzyaloshinskii-Moriya (DMI) and Zeeman interactions confined in a two-dimensional chiral magnet such as $\text{Fe}_{0.5}\text{Co}_{0.5}\text{Si}$. We parametrize bilayer skyrmions using $SU(4)$ representation, and represent each skyrmion and antiskyrmion using Schmidt decomposition. The reduced density matrices for skyrmion and antiskyrmion has been calculated. The conditions for maximal, partial entanglement and separable bilayer skyrmions are presented. Our results can be used for generating entanglement in systems with large number of spins.

**Introduction.** – Magnetic Skyrmions are microscopic topological defects in spin textures that are characterized by the charge [1],

\[ Q = \frac{1}{4\pi} \int d^2 r \ \mathbf{n} \cdot \left( \frac{\partial \mathbf{n}}{\partial x} \times \frac{\partial \mathbf{n}}{\partial y} \right) = \pm 1. \]  

(1)

In mathematical literature, $Q$ is a topologically invariant quantity known as Pontryagin number. It counts how many times $\mathbf{n}(r) = \mathbf{n}(x, y)$ wraps the unit sphere [2]. Skyrmions were first introduced by Skyrme [3] to explain hadrons in nuclei. Interestingly it has turned out to be relevant in other condensed matter systems such as chiral magnets [4]. Theoretically, magnetic skyrmions were introduced and investigated by Bogdanov and his collaborators in [5,6]. Skyrmions can be driven by charge or spin currents in confined geometries [7]. In general, skyrmions are subject to skyrmion Hall effect (SkHE) caused by Magnus force. SkHE was predicted theoretically in [8] and has been observed experimentally [9]. Magnus force is the force acting transverse to the skyrmion velocity in the medium and can be interpreted as a manifestation of the real-space Berry phase [10,11].

SkHE is a detrimental effect since the skyrmions experience it will deviate from going in straight path. As a result, moving skyrmions can be damaged or even destroyed at the edges of thin film sample. One way of suppressing SkHE is to consider two perpendicular chiral thin films strongly coupled via antiferromagnetic (AFM) exchange coupling. It is expected that when skyrmion lattice is formed at the bottom thin film, simultaneously another skyrmion lattice is created at the top thin film with opposite topological charge. In this case, the SkHE is vanished since the Magnus force acting on the top skyrmion (antiskyrmion) is equal to the Magnus force that acts on the bottom antiskyrmion (skyrmion) with opposite sign leaving us with zero net force. Analogous scheme was proposed to suppress SkHE in nanoscale Néel skyrmions by considering two perpendicular ferromagnetic films separated by an insulator with heavy metal underneath the second ferromagnetic film [12].

Quantum signatures for large skyrmions can emerge at the phase boundary between skyrmion crystal phase (SkX) and ferromagnetic phase at zero temperature like skyrmions in $\text{Fe}_{0.5}\text{Co}_{0.5}\text{Si}$. During this phase transition a quantum liquid phase is expected to emerge [13]. In this case, the classical LLG [14] and Thiele equation [15] break down due to quantum fluctuations. The full quantum theory of bilayer skyrmions is out of the scope of this work and it can be recovered under some circumstances. As an example, for sufficiently weak antiferromagnetic exchange coupling between thin films, bilayer skyrmion (antiferromagnetically coupled skyrmion-antiskyrmion pair) can be seen as two separate skyrmions and the quantum dynamics is already known for a single large skyrmion [13]. In this work, we give a detailed theory of large bilayer skyrmions (with sizes at order of 100 nm) using HDMZ (Heisenberg exchange + Dzyaloshinskii-Moriya interaction + Zeeman interaction) model. We study the problem of entanglement in large bilayer skyrmions from general perspective using our developed continuum theory of bilayer skyrmions and $SU(4)$ representation. In final section, we study the geometry of quantum states in bilayer skyrmions.
under certain ranges of temperature and external magnetic field determined by the film parameters. Skyrmions in the first thin-film are equal in size with skyrmions in the second thin-film but with opposite topological charge. For our model to hold, we assume temperatures lower than the magnon gap and skyrmion with large radius [13]. Fortunately, skyrmions in FeCoSi support these assumptions [16]. We present a detailed theory of bilayer skyrmions written with respect to \( \mathbb{CP}^1 \otimes \mathbb{CP}^1 \)-theory. The HDMZ Hamiltonian density for each chiral magnet layer is

\[
H_i = \frac{J}{2} (\partial_\mu n^i) \cdot (\partial_\mu n^i) + D n^i \cdot (\nabla \times n^i) - B \cdot n^i, \tag{2}
\]

We adopted Einstein summation notation for repeated indices \( \partial_i n \cdot \partial_i n \equiv \Sigma_i \partial_i n \cdot \partial_i n \). Since we are interested in two-dimensional thin films \( \mu = x, y \). The index \( i = S, A \) labels the skyrmion and anti-skyrmion respectively and \( n^i = (\sin \theta^i \cos \phi^i, \sin \theta^i \sin \phi^i, \cos \theta^i)^T \) gives the transpose magnetic moment unit written in the \( O(3) \) representation with a unit modulus constraint \( |n|^2 = 1 \). The first term in the Hamiltonian represents the exchange interaction with exchange constant \( J \), the second term is the DMI term with \( D \) being the Dzyaloshinskii-Moriya (DM) vector constant. DMI term is a manifestation of chirality in the system since it has a vanishing value for centrosymmetric crystal structures. The last term is the Zeeman interaction. According to Derrick-Hobart theorem, the Hamiltonian 2 supports the emergence of large skyrmions [17, 18]. Suppose there exists a skyrmion solution \( n^i \) to the system. We compute each contribution in the energy functional as \( E^0_H, E^0_D \), and \( E^0_Z \), where \( H, D, \) and \( Z \) denote Heisenberg exchange, Dzyaloshinskii-Moriya and Zeeman terms. Now we consider the scaling \( n = n^i(\lambda x) \). Substituting this scaled solution into each term in the energy functional gives,

\[
E(\lambda) = E^0_H - \lambda^{-1} |E^0_D| + \lambda^{-2} E^0_Z. \tag{3}
\]

This has a unique minimum point which could be found by the relation \( \lambda = \frac{2E^0_D}{E^0_H} \). It is safe to choose \( \lambda = 1 \) for consistency throughout our argument. From 3, it is not difficult to observe that skyrmion is stabilized by DMI term. When \( \lambda \to \infty \), the equation 3 implies that a skyrmion shrinks to zero without the DMI term. The perpendicular magnetic anisotropy (PMA) term was ignored since such a term does not play an important role in FeCoSi [16]. The total energy is the spatial integral of \( H_i \); \( H_i = \int d^2 x H_i \). The bilayer skyrmion can be described by the following Hamiltonian \( H_{inter} = H_S + H_A + H_{int} \). The term \( H_{int} \) is assumed to contain the AFM exchange coupling between the two chiral magnets,

\[
H_{int} = -J_{int} \int d^2 x n^i = S_n n^i = A. \tag{4}
\]

AFM interaction term is responsible for the coupling between spin degrees of freedom in skyrmion and spin degrees of freedom in antiskyrmion. AFM-coupled spins are in opposite alignment with each others. We will use a purely geometric approach in our investigation of quantum entanglement. Thus, it is more convenient to work in the equivalent \( \mathbb{CP}^1 \) formulation of nonlinear sigma model NL\( \sigma \)M [19, 20]. This can be achieved using Hopf map \( n^i = (z^i) \sigma^i z^i \). This mapping connects the classical object \( n^i \) with spinor \( z^i = \begin{bmatrix} \cos \frac{\theta_i}{2} \\ \sin \frac{\theta_i}{2} e^{i\phi_i} \end{bmatrix} \). The spinor \( z^i \) can be interpreted as the coherent-state wavefunction of spin-\( \frac{1}{2} \) particles. The DMI term can be phrased in terms of the spinor \( z_i \) as follows:

\[
n^i \cdot (\nabla \times n^i) = \sin \theta^i \cos \theta^i (\cos \phi^i \partial_x \phi^i + \sin \phi^i \partial_y \phi^i) \\
+ (\sin \phi^i \partial_x \theta^i - \cos \phi^i \partial_y \theta^i - \sin^2 \theta^i \partial_z \phi^i) \\
= -2n^i \cdot a^i - i(z^i)(\sigma^i \cdot \nabla) z^i + i(\nabla(z^i)) \cdot \sigma^i z^i. \tag{5}
\]

The equation 2 can be re-expressed in term of the spinor \( z^i \) as

\[
H_i = \frac{J}{2} (\partial_\mu n^i) \cdot (\partial_\mu n^i) + D n^i \cdot (\nabla \times n^i) - B \cdot n^i \tag{6}
\]

\[
= 2J (\partial_\mu(z^i) + ia^i \mu a^i - i\kappa (z^i) \sigma^i) \\
(\partial_\mu z^i - ia^i \mu z^i + i\kappa a^i \sigma^i z^i) - B (z^i) \cdot \sigma^i z^i \\
= 2J(D^0 z^i) D^0 z^i - B (z^i) \cdot \sigma^i z^i,
\]

where \( D^0 = \partial_\mu - ia^i \mu + i\kappa a^i \sigma^i \) denotes the covariant derivative for thin film \( i, \kappa = \frac{D}{2r} \), and \( a^i \mu = -i(\partial_\mu \partial_\nu) \cdot z^i \) is the emergent gauge field. Each magnetic layer carries a \( \mathbb{CP}^1 \) field which is responsible for the magnetization. The \( \mathbb{CP}^1 \)-field is a two-component normalized vector with complex entries such that each field is being represented using \( SU(2) \) representation. The inclusion of DMI term in the effective Hamiltonian 6 is done simply by adding a non-Abelian gauge field proportional to Pauli matrices \( \sigma^i \). The emergent gauge field \( a^i \mu \) is usually called the real-space Berry connection. It is synthesized adiabatically varying the spin texture sufficiently slow in time. The real-space Berry phase connection can give rise to the skyrmion Hall effect. Unlike the momentum-space Berry connection which gives rise to the anomalous Hall effect [11]. Although the non-Abelian gauge field is non-dynamic (constant), it has an associated flux with it. The covariant derivative commutator gives the field tensor,

\[
F^i_{\mu \nu} = i[D^i_{\mu}, D^i_{\nu}] = f^i_{\mu \nu} + 2a^2 \epsilon_{\mu \nu \lambda} \sigma^i_\lambda. \tag{7}
\]

where the abelian part of the flux is \( f^i_{\mu \nu} = \partial_\mu a^i_\nu - \partial_\nu a^i_\mu \).

The two-dimensional emergent vector potential for a single magnetic skyrmion is

\[
a^i = -i(z^i) \nabla_2 z^i = \frac{\dot{\phi}^i}{2r} (1 - \cos \theta^i(r)) = \frac{\dot{\phi}^i}{2} \sin^2 \frac{\theta^i}{2}, \tag{8}
\]

Since \( \dot{\phi}^i = (\sin \phi^i, \cos \phi^i, 0) \), the gauge field \( a^i \) turns to be a two-dimensional object. The magnetic flux originating from this gauge field is

\[
\nabla_2 \cdot a^i = \frac{1}{2r} \sin \theta^i(r) \theta'^i(r), \tag{9}
\]

where \( \nabla_2 \equiv (\partial_x, \partial_y, 0) \).

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The local spin orientation \((\theta^i, \phi^i)\) is related to the local coordinate system of a single skyrmion \((r, \varphi)\) such that \(\theta_i = \theta_i(r)\) and \(\phi_i = \phi_i - \frac{\pi}{2}\). For seek of simplicity, we assume \(B = B \hat{z} > 0\). The geometric considerations of skyrmions impose the following boundary conditions on \(\theta_i\): (a) \(\theta_i(\infty) = 0\) and (b) \(\theta_i(0) = \pi\). The total energy of a single skyrmion (antiskyrmion) reads [20]

\[
E_{SK} = 4\pi J \int_0^\infty r dr \left[ \frac{1}{2} \frac{d\theta^S}{dr} + \kappa r \right]^2 \quad (10)
\]

\[-\kappa^2 + \frac{\kappa}{r} \sin \theta^i \cos \theta^i + \frac{1}{4r^2} \sin^2 \theta^i \gamma (\cos \theta^i - 1) \],

where \(\gamma = \frac{\beta}{4\pi Q}\). The total energy of large bilayer skyrmion is

\[
E_{tot} = E_{Sk}(\theta_S) + E_{Sk}(\theta_A) + E_{int}(\theta_S, \theta_A), \quad (11)
\]

In \(\mathbb{CP}^1\)-formulation, the AFM interaction term takes the form

\[
E_{int} = -2\pi J_{int} \int_0^\infty \frac{rdr}{\cos \theta_S \cos \theta_A}. \quad (12)
\]

For sufficient AFM interaction, we have the case where each spin in the first film is coupled with another opposite spin in the second film. This allows us to write \(\theta_A = \pi - \theta_S\) and express the total energy functional \((11)\) in term of a single angle \(\theta_S\) or \(\theta_A\). The total energy functional \((11)\) simplifies for fixed values of DM interaction constants \(D\), exchange couplings \(J\) and magnetic fields \(B\) in both skyrmion and its AFM coupled antiskyrmion. It takes the following simple form

\[
E_{tot} = 4\pi J \int_0^\infty r dr \left[ \frac{1}{2} \frac{d\theta^S}{dr} + 2\gamma + \frac{1}{2r^2} \sin^2 \theta^S \right] + 2\pi J_{int} \int_0^\infty r dr \cos^2 \theta^S. \quad (13)
\]

Realistically, in order for \((13)\) to make sense is to introduce a hard cutoff \(r_{SK}\) such that \(\theta_{S,A}(r) = 0\) for \(r \geq r_{SK}\). Physically, \(r_{SK}\) is a half-skyrmion distance in skyrmion phase crystal or the size of skyrmion.

It is not difficult to show that our system has a vanishing skyrmion Hall effect. For each chiral magnet film, Thiele equation can be written as [12, 15]

\[
F = G \times (v_s - \hat{R}) + \Gamma_{ij} (\beta v_s - \alpha \hat{R}), \quad (14)
\]

where \(v_s\) denotes the velocity of spin polarized current, \(\alpha\) is the Gilbert damping term, \(\beta\) is the non-adiabatic damping constant, \(\hat{R}\) represents the center of mass coordinates, \(G\) and \(\Gamma_{ij}\) are the Gyromagnetic vector and dissipative tensor respectively given by

\[
G_i = \varepsilon_{ijk} \int d^2r (n_i \partial_j n_j), \quad (15)
\]

\[
\Gamma_{ij} = \int d^2r \partial_i n_j \partial_j n_i.
\]

We introduced the non-adiabatic spin transfer torque with parameter \(\beta\) in \((14)\) to account for small dissipative forces that break the conservation of spin in the spin-transfer process. Since we have considered an external magnetic field parallel to the \(z\)-direction and an in-plane spin polarized current, by symmetry considerations, dissipation tensor has the following simple form

\[
\Gamma = \Gamma \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (16)
\]

and the Gyromagnetic vector takes the form

\[
G = 4\pi Q \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (17)
\]

Note that \(F\) in equation \((14)\) vanishes since HDMZ action is translationally invariant \(r \rightarrow r + \delta r\) provided that Zeeman field is uniform. Thus, we obtain the following coupled equations

\[
\begin{pmatrix} \alpha\Gamma & -4\pi Q \\ 4\pi Q & \alpha\Gamma \end{pmatrix} \begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} = \begin{pmatrix} \beta\Gamma & -4\pi Q \\ 4\pi Q & \beta\Gamma \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}, \quad (18)
\]

This system is non-singular and always has a solution of the form

\[
\begin{pmatrix} X \\ Y \end{pmatrix} = \frac{\beta}{\alpha} v_s + \frac{\alpha - \beta}{\alpha^3 (4\pi Q^2) + \alpha} (v_s + \frac{\alpha\Gamma}{4\pi Q} \hat{z} \times v_s).
\]

From the last result, we observe that skyrmion (antiskyrmion) velocity \(v_{SK}\) is a combination of drag velocity \(v_s\) and Magnus term proportional to the topological charge \(Q\). Since we dealt with two thin films of chiral magnets where skyrmions in one layer has opposite charge with the second i.e. \(Q_S = -Q_A\). This implies the vanishing of Magnus term contribution for the whole system.

**SU(4) Parametrization of Bilayer Skyrmion.** At the perfect coupling between skyrmion-antiskyrmion pairs such that each spin in skyrmion is AFM-coupled to a spin in anti-skyrmion, the system skyrmion-antiskyrmion pair can be described by a four-component wavefunction. Thus we can represent the spin degrees of freedom in bilayer skyrmions using \(SU(4)\) symmetry. The \(SU(4)\) skyrmions were studied before in multicomponent quantum Hall system [21] and graphene [22]. It was found that skyrmions in these systems are stabilized mainly by the competition between Zeeman and Coulomb interactions, unlike skyrmions in chiral magnets. However, both skyrmions share the same topological properties in common regardless of the system details. Since we have AFM coupled skyrmion-antiskyrmion pairs, our system resembles the spin-pseudospin skyrmions in term of parametrization despite the fact that we now have two skyrmions instead of one. For large bilayer skyrmions, we consider the properties of \(SU(2) \otimes SU(2)\) skyrmion-antiskyrmion pairs under the presence of DM and Zeeman interactions (HDMZ model). We do this from a perspective of entanglement between the spin degrees of freedom in skyrmion and its AFM coupled anti-skyrmion. Because of Zeeman interaction term, the full \(SU(4)\) symmetry
breaks down to $U(1) \otimes U(1)$ symmetry where each symmetry group corresponds to a rotation of spin in the skyrmion or antiskyrmion along the applied magnetic field direction (in our case, the $z$-direction). Thus, we have the symmetry breaking sequence $SU(4) \rightarrow SU(2) \otimes SU(2) \rightarrow U(1) \otimes U(1)$. Interestingly, the DMI term written in term of spinors $z^i$ preserves the full $SU(2) \otimes SU(2)$ symmetry. This is due to the embedding of DMI term in the covariant derivative that acts on the spinor $z^i$ as a nondiagonal term.

We parametrize the $SU(4)$ bilayer skyrmion using a Schmidt decomposition [23]. According to Schmidt decomposition, every pure state in the Hilbert space $\mathcal{H}_{12} = \mathcal{H}_1 \otimes \mathcal{H}_2$ can be written in the form

$$|\psi\rangle = \sum_{i=0}^{N-1} \lambda_i |e_i\rangle \otimes |f_i\rangle,$$

(20)

Where $\{|e_i\rangle\}_{i=0}^{N-1}$ is an orthonormal basis for $\mathcal{H}_1$, $\{|f_i\rangle\}_{i=0}^{N-1}$ is an orthonormal basis for $\mathcal{H}_2$, $N \leq \min\{N_1, N_2\}$, and $\lambda_i$ are non-negative real numbers such that $\sum_{i=0}^{N-1} \lambda_i^2 = 1$. Thus, we can express the wavefunction as

$$|\Psi(r)\rangle = \cos \frac{\alpha}{2} |\varphi^S\rangle \otimes |\varphi^A\rangle + \sin \frac{\alpha}{2} e^{i\beta} |\chi^S\rangle \otimes |\chi^A\rangle$$

(21)

$$= \begin{pmatrix}
\cos \frac{\alpha}{2} 
& \cos \theta^S e^{i\phi^S} + \sin \theta^S e^{i(\beta - \phi^S - \phi^A)} \\
\cos \frac{\alpha}{2} \sin \theta^S e^{i\phi^S} - \sin \theta^S \cos \frac{\alpha}{2} e^{i(\beta - \phi^S - \phi^A)} \\
\cos \frac{\alpha}{2} \sin \theta^S e^{i\phi^A} - \sin \theta^S \cos \frac{\alpha}{2} e^{i(\beta - \phi^A - \phi^S)} \\
-\sin \frac{\alpha}{2} e^{-i\beta} \\
\cos \frac{\alpha}{2}
\end{pmatrix},$$

where $\alpha \in [0, \pi]$ and $\beta \in [0, 2\pi]$ are functions of $r$, and the local two-component spinors $|\varphi^S\rangle$, $|\chi^S\rangle$, $|\varphi^A\rangle$ and $|\chi^A\rangle$ are constructed as follows

$$|\varphi^S\rangle = \left( \cos \frac{\theta^S}{2}, \sin \frac{\theta^S}{2} e^{i\phi^S} \right),$$

and

$$|\chi^S\rangle = \left( \cos \frac{\theta^S}{2}, \sin \frac{\theta^S}{2} e^{-i\phi^S} \right),$$

where $\theta^S \in [0, \pi]$ and $\phi^S \in [0, 2\pi]$ are the usual polar angles defining the vector $n^i$. We can read off directly the reduced density matrices using the Schmidt decomposition. The reduced density matrices for spins in skyrmion and antiskyrmion are

$$\rho_S = Tr_A(|\Psi(r)\rangle \langle \Psi(r)|) = \cos^2 \frac{\alpha}{2} |\varphi^S\rangle \langle \varphi^S| + \sin^2 \frac{\alpha}{2} | \chi^S\rangle \langle \chi^S|,$$

$$\rho_A = Tr_S(|\Psi(r)\rangle \langle \Psi(r)|) = \cos^2 \frac{\alpha}{2} |\varphi^A\rangle \langle \varphi^A| + \sin^2 \frac{\alpha}{2} | \chi^A\rangle \langle \chi^A|.$$

(22)

(23)

It is convenient to express the wavefunction 21 as $|\Psi(r)\rangle$ such that entanglement measure can be written gentry as [23]

$$\mathcal{E} = 4 |z_1 z_4 - z_2 z_3|^2.$$
theory for describing large bilayer skyrmions in section. The
distinguishability of any two arbitrary states in large skyrmion
or antiskyrmion is supposed to be difficult to observe. The in-
finitesimal form of the Fubini-Study distance approaches the
Fubini-Study metric tensor

$$
\text{ds}^2 = \frac{Z\bar{Z}dZ.d\bar{Z} - Z.d\bar{Z}d\bar{Z}}{(Z\bar{Z})(\bar{Z}\bar{Z})},
$$

(26)

Here $Z\bar{Z} = Z\bar{Z}$. From Fubini-Study metric, time-energy
uncertainty relation can be derived directly for each single spin
[24]. As a spin-coherent state goes through a closed loop, it
will gain the phase $\gamma = \oint \psi|\frac{\delta}{\delta\psi}|\psi\rangle$. It was found
that this phase is equal to the Riemannian curvature $K = \frac{1}{\rho^2}$
of the phase space of spin-coherent state up to a constant. When
$S = 1/2$ (like large 2D skyrmions), the curvature is equal to
its maximum value $K = 1$ [26].

Any arbitrary state vector for bipartite composite system reads

$$
|\Psi\rangle = \frac{1}{\sqrt{N}} \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} C_{ij} |i\rangle \otimes |j\rangle,
$$

(27)

where $C_{ij}$ is $N \times N$ matrix with complex entries. The density
matrix for the composite system is given by $\rho_{ij,kl} = \frac{1}{N} C_{ij} C^*_{kl}$. Since the system is in pure state, its density matrix
has rank one. Now suppose we perform experiment in one of
the two chiral magnets, the reduced density matrix for this sub-
system is the partially traced density matrix $\rho_A = Tr_B \rho :=
\sum_{\gamma} Tr_{\text{H}_{\gamma}} \rho$ which equals to $\rho^A_{ik} = \sum_{j=0}^{N-1} \rho_{ij,jk}$. The rank of this
subsystem density matrix may be greater than one. The global
state of the chiral magnets system may be written as a product
state spanned in the total Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$

$$
|\Psi\rangle = |A\rangle \otimes |B\rangle = \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} (a_i |i\rangle) \otimes (b_j |j\rangle),
$$

(28)

So the matrix $C_{ij} = a_i b_j$ is the dyadic product of two vectors
$a$ and $b$. It is not difficult to notice that such global state
of this kind is disentangled or separable since the partially
traced matrix and the matrix $C_{ij}$ have rank one and the
subsystems are in pure states of their own. On other hand, the
maximally entangled state can be identified using the condition
$\rho^A_{ik} = \frac{1}{N} I_N$ which corresponds to $\sum_{j=0}^{N-1} C_{ij} C^*_{kj} = \delta_{ik}$. It
means that we know nothing at all about the state of the sub-
systems even though the global state is precisely determined.
The maximally entangled states form an orbit of the group of
local unitary transformations. In our case, this group is
$SU(2)/\mathbb{Z}$ = $SO(3)$ which is identical to the real projective space
$\mathbb{R}P^3$. In general, the group $U(N)/\mathbb{Z}$ = $SU(N)/\mathbb{Z}$ is a Lagrangian
submanifold of $CP^{N^2-1}$. Between these two cases, the
separable and maximally entangled cases, the Von Neumann
entropy $S = -Tr(\rho_A \ln \rho_A)$ takes some intermediate value
with a possibility of partial entanglement between the spins.

Conclusion. - In this letter, the problem of antiferromag-
netically coupled skyrmions (bilayer skyrmion) has been studied
in detail using continuum theory approach. This was done
by considering two thin films formed from the same chiral mag-
net separated by an insulating spacer with antiferromagnetic
coupling between chiral films. We assumed each chiral film to
host Bloch skyrmions under certain range of temperatures and
external magnetic fields determined by the film parameters.
Skyrmions in the first thin-film are equal in size with skyrmions
in the second thin-film but with opposite topological charge.
We give a representation for the spin degrees of freedom based
on $SU(4)$ Lie algebra. Moreover, we have computed the den-
sity matrices for the spin degrees of freedom in skyrmion and
its AFM-coupled antiskyrmion using Schmidt decomposition.
Utilizing from the computed density matrices, we found the
conditions for maximal or partial entanglement and separabil-
ity within bilayer skyrmions. The full $SU(4)$ symmetry is
broken to $SU(2) \otimes SU(2)$ symmetry during the Schmidt de-
composition process while Zeeman interaction term causes the
breaking to $U(1) \otimes U(1)$ symmetry. Interestingly DMI terms
preserves the $SU(2) \otimes SU(2)$ symmetry.
The geometry of quantum states in bilayer skyrmions can be
described using complex projective space $CP^3$ endowed with
a unitary-invariant Fubini-Study metric. Geometrically, the en-
tangled states can be described naturally using $CP^3$ space. We
have two extreme cases corresponding to maximally entangled
and separable states. The space of maximally entangled states
happens to be the real projective space $RP^3$ while for separa-
ble states is simply the space $CP^1 \otimes CP^1$. The entangle-
ment in skyrmion-antiskyrmion pairs can be extended to the
whole skyrmion lattice $SkX$. We can have maximally entan-
gled, partially entangled and separable cases for each coupled
pairs. However, for uncoupled skyrmion-antiskyrmion pairs,
the formalism studied in this letter can not be accurate in term
of entanglement conditions. This is mainly because of the fact
that skyrmion-antiskyrmion pairs are treated as antiferromag-
netically coupled pairs throughout our formalism. This allowed
us to impose conditions on energy eigenstates of skyrmion and
antiskyrmion accordingly. We need to consider each skyrmion
and antiskyrmion as $XXZ$ spin chains and calculate the corre-
spanding entanglement entropy [28]. Other possible scenarios
such as having saturated ferromagnetic phase or general helical
spin phase in one layer and skyrmion lattice in the second layer
are theoretically possible. However we do not find these struc-
tures to be of great interest at least in term of having a vanishing
Magnus force for the whole system.

In comparison with graphene and multicomponent Hall sys-
tems, intimate relation between the entanglement conditions in
large bilayer skyrmions and $SU(4)$-skyrmions has been found.
However the system which has been investigated in this let-
ter is different from that studied in Graphene and multicomponent
quantum Hall systems. For example, they dealt in graphene
case with spin-valley pseudospin degrees of freedom in a single
skyrmion [27]. In contrast, we have considered two skyrmions
with AFM coupling between its internal spins. This is the rea-
son why we used $CP^1 \otimes CP^1$-theory instead of $CP^3$-theory.
However, the space of entangled states is $CP^3$-manifold as ex-
pected [24, 29].

As a last comment, we propose the usage of entanglement in
skyrmion-antiskyrmion lattices for probing the geometric na-
ture of quantum entanglement. This will help in turn further
understand and possibly manipulating skyrmion lattices in per-

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forming quantum technological tasks such as generating entanglement in systems with large number of spins.

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\[SU(4) \text{ Representation.} - \text{ The special unitary group } SU(N) \text{ has } (N^2 - 1) \text{ generators, where } -1 \text{ is because of the condition } \det(M) = 1 \text{ where } M \text{ is any element from } SU(N).\]

We denote the generators as \(\lambda_A, A = 1, 2, \ldots, N^2 - 1\). We choose the following normalization condition between generators \(\text{Tr}(\lambda_A \lambda_B) = 2\delta_{AB}\). Their commutator and anti-commutator are [30]

\[
\begin{align*}
[\lambda_A, \lambda_B] &= 2i f^{ABC} \lambda_C, \\
\{\lambda_A, \lambda_B\} &= \frac{4}{N} + 2d^{ABC} \lambda_C,
\end{align*}
\]

where \(f^{ABC}\) and \(d^{ABC}\) are the structure constants of \(SU(N)\). When \(\lambda_A = \sigma_A\) (Pauli matrix) we have \(f^{ABC} = \varepsilon^{ABC}\) and \(d^{ABC} = 0\) in the case of \(SU(2)\).

Since our developed model of bilayer skyrmions in chiral magnets is based on \(SU(4)\) we will give a specific attention to this group. \(SU(4)\) has 15 generators while \(SU(2) \otimes SU(2)\) has 6 generators in total. Embedding \(SU(2) \otimes SU(2)\) into \(SU(4)\) we find the matrix representation for skyrmion \(S\) and its AFM-coupled antiskyrmion \(A\)

\[
\begin{align*}
\tau^S_x &= \begin{pmatrix} \sigma_x & 0 \\ 0 & \sigma_x \end{pmatrix}, \\
\tau^S_y &= \begin{pmatrix} \sigma_y & 0 \\ 0 & \sigma_y \end{pmatrix}, \\
\tau^S_z &= \begin{pmatrix} \sigma_z & 0 \\ 0 & \sigma_z \end{pmatrix}, \\
\tau^A_x &= \begin{pmatrix} 0 & i 2 \\ i 2 & 0 \end{pmatrix}, \\
\tau^A_y &= \begin{pmatrix} 0 & -i 2 \\ i 2 & 0 \end{pmatrix}, \\
\tau^A_z &= \begin{pmatrix} i 2 & 0 \\ 0 & -i 2 \end{pmatrix}.
\end{align*}
\]

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