Continual Depth-limited Responses for Computing Counter-strategies in Sequential Games

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Abstract

Limited look-ahead game solving in imperfect-information games allows defeating expert humans in large Poker, Liar’s dice, or Scotland yard. The existing algorithms of this type assume all players are perfectly rational. As a result, even a very weak opponent can only tie or lose slowly against these powerful methods. Subsequently, opponent modeling and exploitation is desirable when playing weaker opponents, which is a well-explored problem in smaller games. Recent work attempted to exploit opponent models in large games using limited look-ahead solving. We show flaws of the previous work and present new algorithms with theoretical guarantees and significantly better performance. We use only a single (optimal) value function from standard limited look-ahead solving to exploit any model efficiently. We analyze existing resolving gadgets in model exploitation and show fundamental issues of the gadgets in this setting. We propose a new full gadget, which solves these issues. Finally, we experimentally evaluate our methods on three games and show that our algorithm achieves over half of the maximum possible exploitation without risking almost any loss.

1 Introduction

Researchers aim to solve bigger and bigger games. A breakthrough that allowed defeating human experts in several large imperfect-information games is limited look-ahead solving, which adapts a well-known approach from perfect-information games to games with imperfect-information [Moravčík et al. 2017, Brown and Sandholm 2019b, Schmidt et al. 2021]. Limited look-ahead solving takes advantage of decomposition. It iteratively builds the game to some depth and solves it using a value function that can be learned, for example, by neural networks.

All theoretically sound algorithms of this type assume perfect rationality of the opponent and do not allow explicit modeling of an opponent and exploitation of his mistakes. As a result, even very weak opponents exploitable by the heuristic local best response (LBR) Lisý and Bowling [2017] can tie or lose very slowly against these methods Zarick et al. [2020]. As more and more computer-generated strategies are deployed in the real world An et al. [2013] we can not avoid the interaction of AI systems with humans. Therefore, there has been a significant amount of work towards computing strategies to use against imperfect opponents such as humans Bard et al. [2013, Wu et al. 2021, Southey et al. 2012, Mealing and Shaprud 2015, Korb et al. 2012, Milec et al. 2021, Johanson and Bowling 2009].

The process of opponent modeling and exploitation consists of opponent modeling and model exploitation. Opponent modeling requires building a model from previous data or actions observed during an online play. Model exploitation is finding a good strategy against the given model and is the main focus of this paper. In smaller games, we can trivially compute a best response, or we can use methods as restricted Nash response (RNR) Johanson et al. 2008 or data biased response Johanson and Bowling 2009 if there is uncertainty in the model and we want to be safer. However, in large games, even the best response (BR) computation is a non-trivial task, and no approach could compute it in real-time play.

Very recently, model exploitation with limited look-ahead was done by Liu et al. 2022. Therefore, it works for large games with safety considerations. However, safe exploitation search (SES) only uses a small portion of information from the model and, as a result, only exploits the opponents marginally, which can already be seen from their experiments on Leduc Hold’em. Furthermore, the authors use known methods to bind the exploitability of the algorithm, which we analyze and show why the methods break when a sub-rational model is used. We provide a counterexample to the safety theorem in Liu et al. 2022 based on our analysis.

We start with the full model exploitation and propose continual depth-limited best response (CDBR). CDBR uses a value function\footnote{Value function takes the player’s reaches at the depth limit and returns the evaluation of the information sets for both players.} to compute a response to the model
quickly. In the algorithm, we use only the single (optimal) value function, which assumes both players act rationally after the depth limit and is used in standard limited look-ahead solving [Moravčík et al. 2017]. Disregarding the model in the value function will cause the CDBR not to converge exactly to the BR, but we still provide theoretical guarantees on the performance. The computation is the same as in DeepStack with one player fixed, so the scalability is the same.

As shown [Johnson et al. 2008], we can do better than a linear combination of BR and NE using RNR. Our continual depth-limited restricted Nash response (CDRNR) adapts RNR to limited look-ahead solving. Similarly to RNR, CDRNR significantly outperforms the combination. However, it comes with drawbacks in the limited look-ahead solving. Namely, we need to keep the previously solved subgames as a path to the root to ensure theoretical soundness, which linearly increases the size of the game solved each resolve, making it scalable to games with low depth like Poker or Goofspiel but very impractical in games with high depth.

Our contributions are: 1) We formulate the algorithms to find the responses against the opponent strategy and an evaluation function, resulting in the first theoretically sound robust response applicable to large games. 2) We prove the soundness of the proposed algorithms. 3) We provide an analysis of problems that arise when using opponent models in limited look-ahead solving and propose a solution we call a full gadget. 4) We provide a counterexample to the safety theorem from [Liu et al. 2022] 5) We empirically evaluate the algorithms on three different games and compare them to SES. We show that our responses exploit the opponents, and CDBR outperforms domain-specific local best response [Lisý and Bowling 2017] on poker.

2 Background

A two-player extensive-form game (EFG) consists of a set of players $N = \{\Delta, \forall, c\}$, where $c$ denotes the chance, $\Delta$ is the maximizer and $\forall$ is the minimizer, finite set $A$ of all actions available in the game, set $H = \{a_1a_2...a_n | a_i \in A, n \in N\}$ of histories in the game. We assume that $H$ forms a non-empty finite prefix tree. We use $g \in h$ to denote that $g$ extends $h$. The root of $H$ is the empty sequence $\varnothing$. The set of leaves of $H$ is denoted $Z$, and its elements $z$ are called terminal histories. The histories not in $Z$ are non-terminal histories. By $A(h) = \{a \in A | ha \in H\}$ we denote the set of actions available at $h$. $P : H \times Z \rightarrow N$ is the player function which returns who acts in a given history. Denoting $H_i = \{h | h \in H \land P(h) = i\}$, we partition the histories as $H = H_\Delta \cup H_\forall \cup H_c \cup Z$. $\sigma_c$ is the chance strategy defined on $H_c$. For each $h \in H_c$, $\sigma_c(h)$ is a fixed probability distribution over $A(h)$. Utility functions assign each player utility for each leaf node, $u_i : Z \rightarrow \mathbb{R}$. The game is of imperfect information if some actions or chance events are not fully observed by all players. The information structure is described by information sets for each player $i$, which form a partition $\mathcal{I}_i$ of $H_i$. For any information set $I_i \in \mathcal{I}_i$, any two histories $h, h' \in I_i$ are indistinguishable to player $i$. Therefore $A(h) = A(h')$ whenever $h, h' \in I_i$. For $I_i \in \mathcal{I}_i$ we denote by $A(I_i)$ the set $A(h)$ and by $P(I_i)$ the player $P(h)$ for any $h \in I_i$.

A strategy $\sigma_i \in \Sigma_i$ of player $i$ is a function that assigns a distribution over $A(I_i)$ to each $I_i \in \mathcal{I}_i$. A strategy profile $\sigma = (\sigma_\Delta, \sigma_\forall)$ consists of strategies for both players. $\pi^*(h)$ is the probability of reaching $h$ if all players play according to $\sigma$. We can decompose $\pi^*(h) = \prod_{i \in N} \pi_i^* (h)$ into each player’s contribution. Let $\pi_i^*$ be the product of all players’ contributions except that of player $i$ (including chance). For $I_i \in \mathcal{I}_i$, define $\pi_i^*(I_i) = \sum_{h \in I_i} \pi_i^*(h)$, as the probability of reaching information set $I_i$ given all players play according to $\sigma$. $\pi_i^*(I_i)$ and $\pi_j^*(I_j)$ are defined similarly. Finally, let $\pi^*(h, z) = \pi^*(h) \pi^*(z)$ if $h \in \alpha z$ and zero otherwise. $\pi^*(h, z)$ and $\pi^*_i(h, z)$ are defined similarly. Using this notation, expected payoff for player $i$ is $u_i(\sigma) = \sum_{z \in Z} u_i(z) \pi^*(z)$. A best response (BR) of player $i$ to the opponent’s strategy $\sigma_i$ is a strategy $\sigma_i^{BR} \in BR_i(\sigma_i)$, where $u_i(\sigma_i^{BR}, \sigma_i) \geq u_i(\sigma_i', \sigma_i)$ for all $\sigma_i' \in \Sigma_i$. A tuple of strategies $(\sigma_\forall^{NE}, \sigma_i^{NE})$, $\sigma_i^{NE} \in \Sigma_i$ is a Nash Equilibrium (NE) if $\sigma_i^{NE}$ is an optimal strategy of player $i$ against strategy $\sigma_i^{NE}$. Formally: $\sigma_i^{NE} \in BR(\sigma_i^{NE}) \forall i \in \{\Delta, \forall\}$.

In a two-player zero-sum game, the exploitability of a strategy is the expected utility a fully rational opponent can achieve above the value of the game. Formally, exploitability $\mathcal{E}(\sigma_i)$ of strategy $\sigma_i \in \Sigma_i$ is $\mathcal{E}(\sigma_i) = u_i(\sigma_i, \sigma_i) - \min_{\sigma_i^{NE}} u_i(\sigma_i^{NE}, \sigma_i) \in BR_i(\sigma_i)$. We define gain of a strategy $\sigma_i$ against strategy $\sigma_i^{NE}$ is defined as $\mathcal{G}(\sigma_i, \sigma_i^{NE}) = u_i(\sigma_i, \sigma_i) - u_i(\sigma_i^{NE})$.

Depth-limited Solving We denote $H_i(h)$ the sequence of player $i$’s information sets and actions on the path to a history $h$. Two histories $h, h'$ where player $i$ does not act are in the same augmented information set $I_i$ if $H_i(h) = H_i(h')$. We partition the game histories into public states $PS_i \subset H$, which are closed under the membership within the augmented information sets of all players. Trunk is a set of histories $T \subset H$, closed under prefixes and public states. Subgame $S \subset H$ is a forest of trees with all the roots starting in one public state. It is closed under public states and the trees can end in terminal public states or often end after number of moves or rounds in the game. Range of a player $i$ is a probability distribution over its information sets in some public state $PS_i$, given we reached the $PS_i$, Value function is a function that takes the public state and both players’ ranges as input and outputs values for each information set in the public state for both players. We assume using an approximation of an optimal value function, which is a value function returning the values of using some NE after the depth-limit. Subgame partitioning $P$ is a partitioning that splits the game into trunk and subgames into multiple different levels based on some depth-limit or other factors (domain knowledge).
Subgames rooted in public states

Figure 1: Illustration of the depth-limited solving.

Subgame partitioning can be naturally created using the formalism of factored-observation stochastic games [Korcárik et al. 2019]. We denote by $u_i(\sigma)^T_T$ the utility for player $i$ if we use strategy profile $\sigma$ in trunk $T$ and compute values at the depth-limit using value function $V$. When resolving a subgame with just the ranges, there are no guarantees on the resulting exploitability of the strategy in the full game, and the exploitability can rise significantly [Burch et al. 2014]. To address the issue, gadgets are used to limit the increase in exploitability since they allow the opponent to deviate from its strategy in an already solved game.

3 Fully Exploiting the Opponent

Fully exploiting opponent models in small games boils down to computing a best response. This is infeasible in large games due to memory limitations. One pass BR in a depth-limited setting would not converge because, unlike terminal utilities, the value function at the depth-limit changes when our strategy changes, and we need to run CFR to recover the best response. In this section, we propose an algorithm for continual depth-limited best response (CDBR), which generalizes a best response to be used with a value function for depth-limited solving.

Continual Depth-limited Best Response

Given any extensive-form game $G$ with perfect recall, opponent’s fixed strategy $\sigma^F_G$ and some subgame partitioning $\mathcal{P}$, we define continual depth-limited best response (CDBR) recursively from the top, see Figure 1. First, we have trunk $T_1 = T$ and value function $V$. CDBR in the trunk $T_1$ for player $\Delta$ with value function $V$ is defined as $\mathcal{B}(\sigma^F_G)^{T_1}_V = \arg\max_{\sigma^\Delta} u_\Delta(\sigma^\Delta, \sigma^F_G)^{T_1}_V$. In other words, we maximize the utility over the strategy in the trunk, where we return values from the value function after the depth-limit. In each step afterward, we create a new subgame $S_i$ and create new trunk by joining the old one with the subgame, creating $T_i = T_{i-1} \cup S_i$. We fix the strategy of player $\Delta$ in the $T_{i-1}$ and maximize over strategy in the subgame. $\mathcal{B}(\sigma^F_G)^{T_i}_V = \arg\max_{\sigma^\Delta} u_\Delta(\sigma^\Delta \cup \sigma^{T_{i-1}}, \sigma^F_G)^{T_i}_V$.

We continue like that for each step, and we always create a new trunk $T_i$ using the strategy from step $T_{i-1}$ until we reach the end of the game. We denote the full CDBR strategy created by joining strategies from all possible branches $\mathcal{B}(\sigma^F_G)^{T_i}_V$.

Computing CDBR

The algorithm’s input is the game’s definition and the opponent’s strategy. We have a fixed solving depth, such that the created subgames can fit in the memory, and we can solve them in a reasonable time. First, we get the trunk from the subgame partitioning. Then we use depth-limited CFR [Tammelin et al. 2015; Brown and Sandholm 2019a; Farina et al. 2020] to compute the best response in the trunk given the value function. The value function takes ranges as input. Therefore it changes each CFR iteration. Lemma 1 shows that it converges to the best response in the trunk even if we only keep strategy from the best iteration without the need to compute the average strategy as in regular use of CFR.

After we solve the trunk, we move into the subgame, and we use value resolving without any gadgets because the opponent can not change his strategy in the trunk. We set the initial reaches in the subgame using player strategies and chance. Then we solve the resulting subgame using depth-limited CFR.

Lemma 1. Let $G$ be a zero-sum imperfect-information extensive-form game. Let $\sigma^*_G$ be the fixed opponent’s strategy, and let $T$ be some trunk of the game. If we perform CFR iterations in the trunk for player $\Delta$, then for the strategy $\hat{\sigma}_\Delta$ from the iteration with highest expected utility $\max_{\sigma^\Delta} E_{\sigma^\Delta} u_\Delta(\sigma^\Delta, \sigma^F_G)^T_T - u_\Delta(\hat{\sigma}_\Delta, \sigma^F_G)^T_T \leq \Delta \sqrt{\frac{2}{T}} |\mathcal{I}_{TR}| + TN_{S \in S} \delta$ where $\Delta$ is variance in leaf utility, $A$ is an upper bound on the number of actions, $|\mathcal{I}_{TR}|$ is a number of information sets in the trunk, $N_S$ is a number of information sets at the root of any subgame, and value function error is at most $\epsilon_S$.

4 Safe Model Exploitation

While CDBR maximizes the exploitation of the fixed opponent model, it allows a player to be exploited. In a setting where we gradually build a model during play, we need to limit our exploitability in the initial game rounds when the model is still very inaccurate.

Combination of CDBR and Nash Equilibrium

The combination of CDBR and Nash equilibrium (CDBR-NE) is the first approach to limit exploitability. We can simultaneously compute both strategies using depth-limited solving and do a linear combination in every decision node. Let $p$ be the parameter of the linear combination and let $\sigma^F_G$ be the opponent model. The Gain and Exploitability are limited accordingly

$$\sigma^L_C = p \sigma^N_E + (1-p)\mathcal{B}(\sigma^F_G)^P_V$$

$$\mathcal{E}(\sigma^L_C) = p \mathcal{E}(\sigma^N_E) + (1-p)\mathcal{E}(\mathcal{B}(\sigma^F_G)^P_V)$$

$$\mathcal{G}(\sigma^L_C, \sigma^F_G) = p \mathcal{G}(\sigma^L_C, \sigma^F_G) + (1-p)\mathcal{G}(\mathcal{B}(\sigma^F_G)^P_V, \sigma^F_G)$$

Desired exploitability or gain may be achieved by tuning the parameter $p$ while being only two times slower than the CDBR since we need to find the Nash Equilibrium.
Continual Depth-limited RNR

CDBR-NE is safe, but [2008] shows we can get a much better trade-off between gain and exploitability using RNR as it recovers the optimal Pareto set of ε-safe best responses. McCracken and Bowling [2004]. It also gives us better control of safety as it links the allowed exploitability to the achieved gain. We combine depth-limited solving with RNR to create CDRNR.

Definition

Given an extensive-form game \( G \), restricted Nash response can be computed by solving a modified game \[ G \]. We add an initial chance node with two outcomes that player \( \triangle \) does not observe. We copy the whole game \( G \) after both outcomes of the chance node.

In one tree, the opponent plays the fixed strategy, and we denote it \( G^F \). The other tree is the same as the original game; we mark it \( G' \). Player \( \triangle \) does not observe the initial chance node. His infosets span over both \( G' \) and \( G^F \).

We denote the full modified game \( G^M \). Parameter \( p \) is the initial chance node probability of picking \( G^F \). Given the opponent’s fixed strategy \( \sigma^F \) and some subgame partitioning \( P \) of \( G^M \), we define continual depth-limited restricted Nash response (CDNR) recursively from the top. First, we have trunk \( T_i^M \) using \( P \) and value function \( V \). CDRNR for player \( \triangle \) in the trunk \( T_i^M \) using value function \( V \) is \( R(\sigma^F_{\triangle}, p)_{TV}^{T_i^M} = \max_{\sigma^P_{\triangle}} u_{\triangle}(\sigma_{\triangle}, BR(\sigma_{\triangle}))_{TV}^{T_i^M} \). And then, in every following step, we create the new subgame \( S^M \) and enlarge the trunk to incorporate this subgame, creating trunk \( T_i = T_{i-1} \cup S_i^M \).

Next, we fix strategy \( \sigma_{\triangle}^{T_{i-1}} \) of player \( \triangle \) in the previous trunk \( T_{i-1} \), so it can not be changed anymore, and the CDRNR is \( R(\sigma^F_{\triangle}, p)_{TV}^{T_i} = \max_{\sigma^P_{\triangle}} u_{\triangle}(\sigma_{\triangle}, BR(\sigma_{\triangle}))_{TV}^{T_i} \). where \( \sigma_{\triangle} \) is a combination of the strategy we optimize over and the fixed strategy from the previous step, formally \( \sigma_{\triangle} = \sigma_{\triangle}^S \cup \sigma_{\triangle}^{T_{i-1}} \).

To summarize, we optimize only over the strategy in the subgame used in the current step, while the strategy in the previous parts of the game is fixed for player \( \triangle \). While the strategy of the opponent is either fixed or simulated by the gadget. We denote the full CDRNR strategy \( R(\sigma^F, p)_V \).

Computing CDRNR

The input for computing CDRNR is a definition of the game, the opponent strategy, the parameter \( p \), and the value function. In the theoretical definition, the value function differs from the previous one as it spans over the modified public state. However, since the reaches of \( p_{\triangle} \) are the same for \( G^F \) and \( G^F_{\triangle} \) the resulting strategy in the value function would be the same. Hence, in practice, we can again use the same value function as in standard depth-limited solving. We only need to combine the reaches of \( p_{\triangle} \) as an input and then re-weight the values for the \( p_{\triangle} \) in \( G^F \). In the algorithm, we can create a modified game \( G^M \) explicitly, or we can use the original game tree and store everything in it because the trees are identical. The trunk is solved using depth-limited CFR with the value function, where both players act, but the opponent changes his strategy only in \( G' \). We play following the strategy in the trunk, and when we need to play outside of the trunk, we need to create a new depth-limited game.

We create a new depth-limited game, but unlike CDBR, we cannot exchange the trunk for the chance node because player \( \triangledown \) can deviate in the \( G' \) part. We use a gadget for that, and we discuss it in detail in Section 5. Then we resolve the subgame by CFR with value function, and we play according to the resolved strategy in the subgame. When we leave the subgame, we construct a new subgame and continue until the game ends.

Algorithm 1 Computing CDRNR (CDR)

Require: game \( G \), model strategy \( \sigma^F \), value function \( V \) create (virtually) modified game \( G^M \) (only CDRNR) create subgame partitioning \( P \) from \( G^M \) \( (G) \)

\( \sigma^B \) = empty strategy ready to be filled \( I = \) initial information set in which we act \( S = \) current constructed subgame

while \( I \) not null do

if \( I \) not in \( S \) then

\( S = \) construct new \( S \) from \( P \) using previous \( S \)

(\( \sigma^B \) does not delete trunk)

\( \sigma^B = \) solution of \( S \) using CFR+ with \( V \)
else

pick action \( A \) according to \( \sigma^F \) in \( I \)

get new \( I \) using \( A \) (or null if the game ends)
end if
end while

5 Gadgets and Model Exploitation

When we exploit an opponent model, we need to worsen the strategy in terms of exploitability. If we want a safe response, we need to limit how much the strategy worsens. Gadgets are used to ensure exploitability does not increase Moravcik et al. [2016]; Burch et al. [2014]; Brown and Sandholm [2017], and all the common gadgets work in scenarios where we do not expect our strategy to worsen. However, we need to worsen our strategy to exploit the opponent. For example, in RNR, we try to gain as much as possible in \( G^F \). As soon as the strategy gets worse and the exploitability increases, the common gadgets fail to quantify this increase, which is crucial in applications doing a delicate trade-off. The requirement for the gadget which would work in CDRNR is in Definition 1.

Definition 1. For each information set \( I \in I_{\triangle} \) we need the value of its part in \( G' \), formally
\[ \sum_{h \in \mathcal{H}, h' \in \mathcal{H}} \pi^R(z) u_{\Delta}(z), \] to be the same as the value we would get if we let player \( \nabla \) play BR in full \( G' \).

In the following examples, we show the requirement is not satisfied for common resolving gadgets. We tried to construct a gadget that would satisfy the condition, but in the end, we kept the previously resolved parts of the game \( G' \), which satisfies the condition, and still only increased the size of the solved part linearly. Constant-size gadget fulfilling the Definition 1 is an open problem.

**Restricted Nash Response with Gadget**

We show that commonly used resolving gadgets are either overestimating or underestimating the values from Definition 1 on an example game in Figure 2. In the game, we first randomly pick a red or green coin. Player \( \nabla \) observes this and decides to place the coin heads up (RH, GH) or tails up (RT, GT). Player \( \Delta \) can not observe anything and, in the end, chooses whether he wants to play the game (P) or quit (Q).

In equilibrium, player \( \Delta \) plays action Q, and player \( \nabla \) can mix actions up to the point where the utility for \( P \) is at most 0. This gives the value of the game 0, and counterfactual values in all inner nodes are also 0. Assuming the modified RNR game \( G_M \) with an opponent model playing \( GT \), that makes it worth for player \( \Delta \) to play \( P \) in the game, \( \nabla \) will play (RH, GH) in \( G' \) with utility -3 for player \( \Delta \) in \( G' \). We will use gadgets to resolve the game from the player \( \Delta \) information set.

**Resolving Gadget**

Resolving gadget on the game in Figure 2 has all utilities after terminate actions 0. When we resolve the gadget, the utility is 0. However, when player \( \Delta \) deviates to action \( P \), player \( \nabla \) plays follow action in all but the rightmost node, and the utility of player \( \Delta \) will be -3.5. Therefore, the common resolving gadget may overestimate the real exploitability of the strategy in the subgame. Overestimating may lead to not exploiting as much as we can and makes it impossible to prove Theorem 3 about the minimal gain of our algorithm. It might seem that normalization of the chance node could solve the problem, but it would only halve the value to -1.75, which is still incorrect.

**Full Gadget**

The only construction full-filing the requirements we found is to keep all the previously explored parts of the game in a path to the root and use a value function when we leave. From the construction, using the optimal value function, it exactly simulates the best response, which exactly measures the exploitability.

We show that other gadgets can overestimate or underestimate exploitability, which could simply shift the distribution of the parameter \( p \), and we could still compute the same solutions. However, in Figure 3, we show the results of a game created to break the other gadgets. The game has five actions for the exploiter, and each one is crucial in reconstructing the full RNR set. The full gadget is able to recover all five actions using different \( p \), but other gadgets can only compute two actions regardless of the choice of \( p \).

**Complexity of CDRNR**

We can use other gadgets in CDRNR to obtain fast algorithms but without any theoretical guarantees. The soundness of the algorithm relies on using the full gadget, which requires solving increasingly larger parts of the game as the depth increases. This increase in size is linear with the resolving steps, so the full algorithm complexity is quadratic in the depth of the game in comparison with standard resolving or CDBR. It makes the algorithm applicable to games like Poker or Goofspiel but infeasible for deep games like Stratego.

**Soundness of CDRNR**

For the following theorems, we denote \( S \) as the set consisting of the trunk and all the subgames explored when computing the response, and \( S' \) is the same but without the last subgame. \( S^R \) denotes a border of the subgame. We also denote \( S^D \), the set of all the states where we leave the trunk not going in the currently resolving subgame.

**Theorem 3** (Gain of CDRNR). Let \( G \) be any zero-sum extensive-form game and let \( \sigma^E_V \) be any fixed opponent’s strategy in \( G \). Then we set \( G_M \) as restricted Nash response modification of \( G \) using \( \sigma^E_V \). Let \( \mathcal{P} \) be any subgame partitioning of the game \( G_M \) and using some \( p \in (0,1) \), let \( \sigma^R_S \) be a CDRNR given approximation \( \widehat{V} \) of the optimal value function \( V \) with error at most \( \epsilon_V \) and opponent strategy \( \sigma^R_S \) approximated in each step with regret at most \( \epsilon_R \), formally \( \sigma^R_S = R(\sigma^R_S, p)_{\widehat{V}} \). Let \( \sigma^{NE} \) be any Nash equilibrium in \( G \). Then \[ u_{\Delta}(\sigma^R_S, \sigma^E_V) + \sum_{S \in S^R} |S| \epsilon_R + \sum_{S \in S^D} |S| \epsilon_V \geq u_{\Delta}(\sigma^{NE}) \] and \[ |S| \epsilon_R + \sum_{S \in S^D} |S| \epsilon_V \geq u_{\Delta}(\sigma^{NE}) \]

The previous theorem states that our approaches will receive at least the value of the game when responding to the model.
Theorem 4 (Safety of CDRNR). Let \( G \) be any zero-sum extensive-form game and let \( \sigma^G_R \) be any fixed opponent's strategy in \( G \). Then we set \( G^M \) as restricted Nash response modification of \( G \) using \( \sigma^G_R \). Let \( \mathcal{P} \) be any subgame partitioning of the game \( G^M \) and using some \( p \in (0, 1) \), let \( \sigma^G_R \) be a CDRNR given approximation \( \tilde{V} \) of the optimal value function \( V \) with error at most \( \epsilon_V \), partitioning \( \mathcal{P} \) and opponent strategy \( \sigma^G_R \), which is approximated in each step with regret at most \( \epsilon_R \), formally \( \sigma^G_R = \sigma^G_R(\sigma_V^F, p)\tilde{V} \). Then exploitability has a bound \( \mathcal{E}(\sigma^G_R) \leq \mathcal{G}(\sigma^G_R, \sigma^G_V) = \frac{1}{\epsilon_R} \sum_{S \in \mathcal{S}_f} |I_S| (1-p) \epsilon_V + |S| \epsilon_R + \sum_{S \in \mathcal{S}_f} |I_S| \epsilon_V, \mathcal{E} \) and \( \mathcal{G} \) are defined in Section 2.

The last theorem is more complex, and it bounds the exploitability by the gain of the strategy against the model. With \( p = 0 \), it is reduced to the continual resolving, and with \( p = 1 \) to CDBR with unbounded exploitability. The theorem shows the parameter \( p \) directly links allowed exploitability to the gain we receive. The same works in RNR without the resolving and value errors, and as far as we know, the authors do not explicitly mention it.

6 Experiments

We compared CDBR and local best response (LBR) Lisy and Bowling [2017]. We empirically show the performance of CDRNR, and we explore the trade-off between exploitability and gain in CDRNR. The appendix contains hardware setup, domain description, algorithm details, and experiments on more domains. We use two types of opponent strategies, strategies generated by a low amount of CFR iterations and random strategies with different seeds.

Exploitation of Robust Responses

We report both gain and exploitability for CDRNR on Leduc Hold’em. Results in Figure 4 show that the proven bound on exploitability works in practice, and we see that the bound is very loose in practice. For example, with \( p = 0.5 \), the bound on the exploitability is the gain itself, but the algorithm rarely reaches even a tenth of the gain in exploitability. This shows that the CDRNR is similar to the restricted Nash response because, with a well-set \( p \), it can significantly exploit the opponent without significantly raising its exploitability. The only exception is the CDRNR with a value function, which shows the
added constants from the value function’s imprecision. When the opponent is close to optimal, we see that the exploitability can rise above the gain.

In most cases, the gain and exploitability of CDRNR are lower than that of RNR. Gain must be lower because of the different value function, but the exploitability can be higher, as seen with \( p = 0.1 \) against low iteration strategies, due to the depth-limited nature. We provide results on other domains and for more parameter values \( p \) in the appendix.

We also compare the algorithm against the best possible Nash equilibrium. It is computed by a linear program and serves as the theoretical limit of maximal gain, which does not allow exploitability. It would be impossible to compute for larger games. We can see we can gain twice as much, with exploitability still being almost zero.

We report the result using CFR with 500 iterations as a VF. As expected, the imperfect value function slightly decreases the performance but overall is comparable to the algorithms using the optimal value function.

The last comparison is with SES, which performs poorly, and its gain is only slightly above the best Nash equilibrium. Conversely, it is almost not exploitable. Our results are consistent with results in the paper [Liu et al., 2022] and are a direct consequence of using the information from the opponent model only to set the reaches to the subgame. Only reaches are not enough to do meaningful exploitation, and SES produces strategies which are very close to Nash equilibrium.

### Larger Game

We did an experiment on imperfect information Goofspiel 6, with 2 million nodes and 83 thousand infosets per player, to confirm the performance of the CDRNR scales above small games. Results in the following table clearly show the CDRNR performs well even for much larger games. Bold numbers denote CDRNR with the number marking the \( p \) used in the computation.

| \( p \) | 0 | 0.2 | 0.4 | 0.6 | 0.8 | 1 | BR |
|------|---|-----|-----|-----|-----|---|-----|
| G    | 0.32 | 0.23 | 0.51 | 1.64 | 2.4 | 4.5 | 5 | 5.56 |
| E    | 0 | 0 | 0.07 | 0.33 | 1.09 | 5.05 | 15 | 15 |

**Local Best Response vs. CDBR**

We compare LBR and CDBR in Leduc. We also compare CDBR with just the BR in II Goofspiel 5, but without LBR, which is poker specific. CDBR outperforms LBR against the majority of the strategies using one step, and as steps increase, it outperforms LBR against every strategy. With higher steps, it is able to achieve the same value as BR in the full game in both Leduc Hold’em and II Goofspiel 5. The complete results are in the appendix.

### Related Work

This section describes the related work focusing more on distinguishing our novel contributions.

**Restricted Nash response (RNR)** [Johanson et al., 2008] is an opponent-exploiting scheme. It solves the entire game and allows changing the trade-off between exploitability and gain. Essentially, it always produces a safe best response [McCracken and Bowling, 2004].

However, it is impossible to compute RNR in huge games, and we fused the RNR approach with depth-limited solving creating a novel algorithm we call CDRNR. CDRNR is the first theoretically sound robust response calculation that can be done in huge games, enabling opponent exploiting approaches in such games.

Local best response [Lisý and Bowling, 2017] is an evaluation tool for poker. It uses given abstraction in its action space. It picks the best action in each decision set, looking at the fold probability for the opponent in the next decision node and then assuming the game is called until the end. Our second algorithm CDBR is a generalization of the LBR because we can use it on any game solvable by depth-limited solving. We have explicitly defined value function, which we can exchange for different heuristics.

Approximate best response (ABR) [Timbers et al., 2020] is also a generalization of the LBR and showed promising results in evaluating strategies. However, our approach focuses on model exploitation, which requires crucial differences, such as quick re-computation against unseen models. ABR needs to independently learn the response for every combination of opponent and game, making it unusable in the opponent modeling scenario. Our algorithms learn a single domain-specific value function and can subsequently compute strategies against any opponent in the run-time. Furthermore, ABR and even CDBR are extremely brittle, making it a bad choice if we are unsure about the opponent, which we often are in a game against an unknown opponent. On the other hand, CDRNR tackles exactly this issue and provides powerful exploitation with very limited exploitability.

Safe exploitation search (SES) [Liu et al., 2022] is a similar method to the one we propose. However, there are two significant differences. First, the method uses a max-margin gadget without the analysis we did. Hence, the main theoretical result on safety is not correct, and we provide a counterexample. Second, the method does not fix the opponent’s strategy at all and only uses opponent reaches when resolving the subgame. As a result, SES exploitation is very limited, and as we show in experiments, it is often worse than using the best Nash equilibrium. On top of that, in some games, it fundamentally cannot exploit the opponent, notably any perfect information game, even with simultaneous moves.

### Conclusion

Opponent modeling and exploitation is an essential topic in computational game theory, with recent work creating an algorithm to exploit models in depth-limited solving. We provide a counterexample to the safety theorem in the previous work and explain the problem arising from the inability of gadgets to measure exploitability. We propose a full gadget that solves the issue and propose an algorithm to quickly compute depth-limited best response
and depth-limited restricted Nash response once we have a value function, creating the first theoretically sound robust response applicable to large games. Finally, we empirically evaluate the algorithms on three games. We show CDBR outperforms LBR in poker, and CDRNR outperforms SES in any game and can achieve over half of the possible gain without almost any exploitability.

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References
Bo An, Fernando Ordóñez, Milind Tambe, Eric Shieh, Rong Yang, Craig Baldwin, Joseph DiRenzo III, Kathryn Moretti, Ben Maule, and Garrett Meyer. A deployed quantal response-based patrol planning system for the US coast guard. Interfaces, 43(5):400–420, 2013.

Noam Brown and Tuomas Sandholm. Safe and nested endgame solving for imperfect-information games. In Workshops at the thirty-first AAAI conference on artificial intelligence, 2017.

Noam Brown and Tuomas Sandholm. Solving imperfect-information games via discounted regret minimization. In Proceedings of the AAAI Conference on Artificial Intelligence, volume 33, pages 1829–1836, 2019.

Noam Brown and Tuomas Sandholm. Superhuman ai for multiplayer poker. Science, 365(6456):885–890, 2019.

Neil Burch, Michael Johanson, and Michael Bowling. Solving imperfect-information games using decomposition. In Twenty-eighth AAAI conference on artificial intelligence, 2014.

Gabriele Farina, Christian Kroer, and Tuomas Sandholm. Faster game solving via predictive blackwell approachability: Connecting regret matching and mirror descent. arXiv preprint arXiv:2007.14358, 2020.

Michael Johanson and Michael Bowling. Data biased robust counter strategies. In Artificial Intelligence and Statistics, pages 264–271, 2009.

Michael Johanson, Martin Zinkevich, and Michael Bowling. Computing robust counter-strategies. In Advances in neural information processing systems, pages 721–728, 2008.

Kevin B Korb, Ann Nicholson, and Nathalie Jitnarah. Bayesian poker. arXiv preprint arXiv:1301.6711, 2013.

Vojtěch Kořátký, Dominik Seitz, Viliam Lisý, Jan Rudolf, Shuo Sun, and Karel Ha. Value functions for depth-limited solving in imperfect-information games. arXiv preprint arXiv:1906.06112, 2019.

Viliam Lisý and Michael Bowling. Equilibrium approximation quality of current no-limit poker bots. In Workshops at the Thirty-First AAAI Conference on Artificial Intelligence, 2017.

Mingyang Liu, Chengjie Wu, Qihan Liu, Yansen Jing, Jun Yang, Pingzhong Tang, and Chongjie Zhang. Safe opponent-exploitation subgame refinement. In Advances in Neural Information Processing Systems, 2022.

Edward Lockhart, Marc Lanctot, Julien Pérola, Jean-Baptiste Lespiau, Dustin Morrill, Finbarr Timbers, and Karl Tuyls. Computing approximate equilibria in sequential adversarial games by exploitability descent. In Proceedings of the Twenty-Eighth International Joint Conference on Artificial Intelligence, IJCAI-19, pages 464–470, 2019.

Peter McCracken and Michael Bowling. Safe strategies for agent modelling in games. In AAAI Fall Symposium on Artificial Multi-agent Learning, pages 103–110, 2004.

Richard Mealing and Jonathan L Shapiro. Opponent modeling by expectation-maximization and sequence prediction in simplified poker. IEEE Transactions on Computational Intelligence and AI in Games, 9(1):11–24, 2015.

David Miletic, Jakub Černý, Viliam Lisý, and Bo An. Complexity and algorithms for exploiting quantal opponents in large two-player games. In Proceedings of the AAAI Conference on Artificial Intelligence, volume 35, pages 5575–5583, 2021.

Matej Moravčík, Martin Schmid, Karel Ha, Milan Hladík, and Stephen Gaukrodger. Refining subgames in large imperfect information games. In Proceedings of the AAAI Conference on Artificial Intelligence, volume 30, 2016.

Matej Moravčík, Martin Schmid, Neil Burch, Viliam Lisý, Dustin Morrill, Nolan Bard, Trevor Davis, Kevin Waugh, Michael Johanson, and Michael Bowling. Deep-stack: Expert-level artificial intelligence in heads-up no-limit poker. Science, 356(6337):508–513, 2017.

Martin Schmid, Matej Moravčík, Neil Burch, Rudolf Kadlec, Josh Davidson, Kevin Waugh, Nolan Bard, Finbarr Timbers, Marc Lanctot, Zach Holland, et al. Player of games. arXiv preprint arXiv:2112.03178, 2021.

Finnegan Southey, Michael P Bowling, Bryce Larson, Carmelo Piccione, Neil Burch, Darse Billings, and Chris Rayner. Bayes’ bluff: Opponent modelling in poker. arXiv preprint arXiv:1207.1411, 2012.

Oskari Tamminen, Neil Burch, Michael Johanson, and Michael Bowling. Solving heads-up limit Texas Hold’em.
In Twenty-fourth international joint conference on artificial intelligence, 2015.

Finbarr Timbers, Edward Lockhart, Marc Lanctot, Martin Schmid, Julian Schrittwieser, Thomas Hubert, and Michael Bowling. Approximate exploitability: Learning a best response in large games. arXiv preprint arXiv:2004.09677, 2020.

Zhe Wu, Kai Li, Enmin Zhao, Hang Xu, Meng Zhang, Haobo Fu, Bo An, and Junliang Xing. L2e: Learning to exploit your opponent. arXiv preprint arXiv:2102.09381, 2021.

Ryan Zarick, Bryan Pellegrino, Noam Brown, and Caleb Banister. Unlocking the potential of deep counterfactual value networks. arXiv preprint arXiv:2007.10442, 2020.
A SES Theorem Counterexample

Theorem 4.1 from Liu et al. [2022] claims that if \( \sigma'_\alpha \) is a refined strategy after search and \( \sigma^\ast \) is a NE which is the same as \( \sigma'_\alpha \) for player \( \Delta \) outside of the resolved subgame, then \( \mathcal{E}(\sigma'_\alpha) \leq \mathcal{E}(\sigma^\ast_\alpha) + \frac{\Delta}{\left(1 - (2\Delta + 1)\right)} \). \( \tau \) is the difference between the model and the real opponent strategy. We will assume those are the same. Therefore \( \tau = 0 \). \( \alpha \) corresponds to the parameter \( p \) in our case, and we set \( \alpha = 0.5 \). \( \Delta \) is the increase of the margin (max-margin) of the resolving strategy. So with the \( \alpha \) fixed and \( \tau = 0 \) we can simplify it for our example to \( \mathcal{E}(\sigma'_\alpha) \leq \mathcal{E}(\sigma^\ast_\alpha) + 4\Delta \).

In the counterexample, we use a game that starts with a chance node with six actions with uniform probability. Then player \( \Delta \) has an infoset spanning all the actions with a choice to terminate the game and receive 0 or continue. If the game continues, the player \( \triangledown \) can pick 2 or 0 in one branch or -1 or 0 in the remaining five branches and can distinguish the branches.

As a first step, we create an RN modification of the game with player \( \triangledown \) strategy in the fixed part is to pick 2 in the first branch and 0 in all the others. We solve the blueprint to depth 2, which gives us no strategy but returns counterfactual values for the infoset nodes. The counterfactual values are all 0. We can see the RN modification in Figure 5 on the left.

In the second step, we start resolving in the infoset of player \( \Delta \). We construct the SES gadget and fix the opponent to make a mistake (this would correspond to the algorithm exp-strat from Liu et al. [2022] and not SES directly. However, it is much more apparent why the problems arise, and the game is smaller. We can easily construct a bigger game that will be equivalent to this one using only bad opponent reaches). We can give the opponent a choice to play the game as it is or go to the game where he is fixed, as shown in the example. In the blueprint solution, he will always choose the normal game. When doing the resolving part, starting after the choice, we can have a strategy where he made a mistake and went to the fixed game, which will exactly correspond to the gadget version (except that there would also be the other parts with reach 0). We can see the resolving game in Figure 5 on the right.

In the third step, we solve the game and obtain strategy for player \( \Delta \) using action 2 with probability 1. Action 2 is better for player \( \Delta \) because he receives 2/12 from the first branch, and in part, with the gadget, the player \( \triangledown \) can only pick one of the branches punishing him for -1/12 giving expected utility of 1/12 compared to 0 of action 1. Player \( \triangledown \) will pick 0 in the first branch and -1 in all the other branches, and any mix of actions 2 - 6 in the gadget part.

The problem is that the resulting strategy has exploitability 5/6, which is caused by the fact that in the original game, we reach all the branches through the chance node, while the max-margin gadget further limits the reach so that the range sums to 1. Finally, we look at the bound. The bound is connected to the margin-increase \( \Delta \). In the game, the margin increase from the change was 1/6, granting us an exploitability bound of 5/6. However, as we explained earlier, the exploitability of the resulting strategy is 5/6. Hence, we come to a contradiction.

B Local Best Response vs. CDBR

We compare LBR and CDBR in Leduc Hold’em. We also compare CDBR with just the BR in imperfect information Goofspiel, but without LBR, which is poker specific. We show that CDBR and LBR are very similar with smaller steps, and as we increase the depth-limit of CDBR, it starts outperforming LBR. The behavior differs in every strategy because LBR assumes the player continues the game by only calling till the end, while CDBR uses the perfectly rational extension. Furthermore, it is possible to exchange the value function of CDBR, and both concepts would be very similar. However, we would lose the guarantee that CDBR will never perform worse than the value of the game.

Looking at the results in Figure 6, we can see that both concepts are good at approximating the best response, with CDBR actually being better against both strategies. LBR looks at one following action, so in terms of comparability, it is best compared to the CDBR1. Next, we observe a lack of monotonicity in step increase, which is linked to the counterexample in Figure 9. When we increase the depth-limit, the algorithm can exploit some early mistake that causes it to miss a sub-tree where the opponent makes a much bigger mistake in the future. We can clearly see the difference between the algorithm with guarantees and LBR without them. Against strategy from 34 CFR iterations, LBR can no longer achieve positive gain and only worsens with more iterations. In contrast, CDBR can always achieve at least zero gain (assuming we have an optimal value function).

C Counterexample Gadget Game

Examples in Figure 4 are the games used to generate the Figure 3. The plot in the figure combines two games that have pure actions with the same gain and exploitability. The full gadget reconstructs the Pareto set using all the actions in both games. Other gadgets fail in one of the presented games in Figure 7. In Figure 8, we show the expected utility of all the actions in the CDBNR version of the game, showing that for the resolving gadget and max-margin gadget, two actions dominate all the others, and we can not select any of the remaining actions.

D Experiment Details

Experimental Setup

For all experiments, we use Python 3.8 and C++17. We solved linear programs using Gurobi 9.0.3, and experiments were done on an Intel i7 1.8GHz CPU with 8GB RAM. We used Leduc Hold’em, imperfect information Goofspiel 5, and Liar’s dice for the smaller detailed experiments. We used Goofspiel 6 for the large experiment,
Figure 5: RNR version of the game used in the counterexample (left) and game with the max-margin gadget, which is used in SES. (right)
and we only ran it against the strategy generated by the CFR with three iterations. We used the torch library for the neural network experiment. For most of the experiments, we wanted to solve the concepts perfectly with perfect value function, so we used LP and fixed the parts of the game that needed to be fixed. For the neural network experiment with CDBR, we used CFR+ to solve the subgame and the neural network as a value function. For the value function experiment in CDRNR, we used CFR+ with 1000 iterations to solve the game and CFR+ with 500 iterations as a value function in the subgames.

**Domain Definition**

*Leduc Hold’em* is a medium-sized poker game. Both players give one chip at the beginning of the match and receive one card from a deck with six cards of 2 suits and three ranks. Then players switch and can call or bet. After a bet, the opponent can also fold, which finishes the game, and he forfeits all the staked money. After both players call or after at most two bets public card is revealed, and another betting round begins. In the first round, the bet size is two, and in the second, it is 4. If the game ends without anyone, folding cards are compared, and the player with pair always wins, and if there is no pair, the player with the higher card wins. If both have the same card, the money is split. *Goofspiel* is a bidding card game where players are trying to obtain the most points. Cards are shuffled and set face-down. Both players have $K$ cards with values from 1 to $K$. These cards may be used as a bid. After bidding with that card, the player cannot play it again. Each turn, the top point card is revealed, and players simultaneously play a bid card; the point card is given to the highest bidder or discarded if the bids are equal. In this implementation, we use a fixed deck with $K = 5$ and $K = 6$. *Liar’s Dice* is a game where players have some number of dice and secretly roll. The first player bids rolled numbers, and the other player can either bid more or disbelieve the first player. When bidding ends with disbelief action, both players show dice. If the bid is right, the caller loses one die, and if the bidder is wrong, the bidder loses one die. Then the game continues, but for our computation, we use a version that ends with the loss of a die, and we use only a single die with four sides for each player.

**E Proofs**

**Lemma 1.** Let $G$ be zero-sum imperfect-information extensive-form game with perfect recall. Let $\sigma^F_\nu$ be fixed opponent’s strategy, let $T$ be some trunk of the game. If we perform CFR iterations in the trunk for player $\Delta$ then for the best iterate $\hat{\sigma}_\Delta \max_{\sigma^*_\Delta, \Sigma_\Delta} u_\Delta(\sigma^*_\Delta, \sigma^F_\nu)_T \leq \Delta \sqrt{\frac{1}{2}|T_{TR}| + N_S|S|}$ where $\Delta$ is variance in leaf utility, $A$ is an upper bound on the number of actions, $|T_{TR}|$ is number of information sets in the trunk, $N_S$ is the number of information sets at the root of any subgame and value function error is at most $\epsilon_S$.

**Proof.** Using Theorem 2 from [Burch et al. 2014](#).

![Figure 6: Gain comparison of best response (BR), local best response (LBR - only poker), and continual depth-limited best response (CDBR) in Leduc Hold’em (top), Imperfect information Goofspiel (middle) and Small Liar’s dice (bottom) against strategies from CFR using a small number of iterations (left) and random strategies (right). The a stands for the average of the other values in the plot. The number after CDBR stands for the number of actions CDBR was allowed to look in the future, and CDBRNN is one step CDBR with a neural network as a value function.](#)
Figure 7: Example games to show the inability of common gadgets to reconstruct the whole Pareto set in the CDRNR setting. Left: Game 1 to break the max-margin gadget. Right: Game 2 to break the resolving gadget.
we know that regret for player $\Delta$ is bounded by
\[ R^\Delta = \frac{1}{T} \max_{\sigma^*_{S\Delta}} \sum_{i=1}^{T} (u_\Delta(\sigma^*_{S\Delta}) - u_\Delta(\sigma^*_{S\Delta}))^+ \leq \Delta \sqrt{AT_I T_R} + T N_{S\Delta}. \]
Then we can directly map the regret to a different regret, that uses time-independent loss function $l(\sigma^*_{S\Delta}) = -u_\Delta(\sigma^*_{S\Delta})^+$. We can then use Lemma 2 from [Lockhart et al. 2019] and get $l(\sigma^*_{S\Delta}) - \min_{\sigma^*_{S\Delta}} l(\sigma^*_{S\Delta}) \leq \frac{R^\Delta}{T}$. Substituting $l$ and $R^\Delta$ back we get

\[ \max_{\sigma^*_{S\Delta}} u_\Delta(\sigma^*_{S\Delta}, \sigma^F_{V}) - u_\Delta(\sigma^*_{S\Delta}, \sigma^F_{V})_V \leq \Delta \sqrt{\frac{A}{T}} T_I T_R + T N_{S\Delta}. \]

\[ \square \]

**Theorem 2.** Let $G$ be zero-sum extensive-form game with perfect recall. Let $\sigma^F_{V}$ be fixed opponent's strategy, let $P$ be any subgame partitioning of the game and let $\sigma^B_{S\Delta}$ be a CDBR given approximation $V$ of the optimal value function $V$ with error at most $e_V$, partitioning $P$ and opponent strategy $\sigma^F_{V}$ approximated in each step with regret at most $e_R$, formally $\sigma^B_{S\Delta} = \sigma^F_{V}(\sigma^B_{S\Delta})^+$. Let $\sigma^{NE}$ be any Nash equilibrium. Then $u_\Delta(\sigma^B_{S\Delta}, \sigma^F_{V}) + |S|e_R + \sum_{S\Delta \in S} |I_{S\Delta}| e_V \geq u_\Delta(\sigma^{NE}).$

**Proof.** Using subgame partitioning $P$, let $T_i$ be the trunk of the game from the properties of a NE $u_\Delta(\sigma^{NE}) \leq u_\Delta(\sigma^{NE}, \sigma^F_{V})_V \leq u_\Delta(\sigma^{NE}, \sigma^F_{V})_V$. To compute CDBR we are maximizing in the trunk and using error in the value function and non-zero regret of the computed strategy $u_\Delta(\sigma^B_{S\Delta}, \sigma^F_{V})_V + |I_{T_i}| e_V + e_R \geq u_\Delta(\sigma^{NE}, \sigma^F_{V})_V$. We continue using induction over steps with induction assumption that in step $i$, $u_\Delta(\sigma^B_{S\Delta}, \sigma^F_{V})_V + e_i \geq u_\Delta(\sigma^{NE})$. We already know it holds for $T_1$. Now we assume we have trunk $T_{i-1}$ for which the induction step holds and trunk $T_i$ which is $T_{i-1}$ joined with new subgame $S_{i-1}$. Our algorithm recovers approximate equilibrium in $S_{i-1}$ using $V$ at the boundary $S^B_{i-1}$, which means $u_\Delta(\sigma^B_{S\Delta}, \sigma^F_{V})_V + |I_{S^B_{i-1}}| e_V + e_R \geq u_\Delta(\sigma^{NE_{S^B_{i-1}}} \cup \sigma^{B_{T_{i-1}}}, \sigma^F_{V})_V$. If we use equilibrium for the opponent in the subgame $S_{i-1}$ we can replace equilibrium in the subgame by the value function and we have $u_\Delta(\sigma^B_{S\Delta}, \sigma^F_{V})_V + |I_{S^B_{i-1}}| e_V + e_R \geq u_\Delta(\sigma^{B_{T_{i-1}}}, \sigma^F_{V})_V$. We use the errors through the subgames will give the desired result $u_\Delta(\sigma^B_{S\Delta}, \sigma^F_{V}) + |S|e_R + \sum_{S\Delta \in S} |I_{S\Delta}| e_V \geq u_\Delta(\sigma^{NE})$. We omit last subgame from the accumulated value function error because the last step does not use value function.

\[ \square \]

**Theorem 3.** Let $G$ be any zero-sum extensive-form game with perfect recall and let $\sigma^F_{V}$ be any fixed opponent's strategy in G. Then we set $G_M$ as restricted Nash response modification of G using $\sigma^F_{V}$. Let $P$ be any subgame partitioning of the game $G_M$ and using some $p \in (0, 1)$, let $\sigma^S_{G_M}$ be a CDRN given approximation $\bar{V}$ of the optimal value function $V$ with error at most $e_V$ and opponent strategy $\sigma^F_{V}$ approximated in each step with regret at most $e_R$, formally $\sigma^S_{G} = \sigma^F_{V}(\sigma^S_{G})^+$. Let $\sigma^{NE}$ be any Nash equilibrium in $G$. Then $u_\Delta(\sigma^S_{G}, \sigma^F_{V}) + \sum_{S\Delta \in S} |I_{S\Delta}| e_V + |S|e_R + \sum_{S\Delta \in S} |I_{S\Delta}| e_V \geq u_\Delta(\sigma^{NE})$.

**Proof.** Let $T^M_i$ be a trunk of a modified game $G^M$ using partitioning $P$. We use $u^G(\sigma)$ as utility in $G$. Utility of player $\Delta$ for playing Nash equilibrium of the G in trunk $T_i$ will be higher or the same as game value of $G$, formally $u^G(\sigma^{NE}) \leq u^G(\sigma^{NE}, \sigma^F_{V})_V + e_i \geq u^G(\sigma^{NE})$ and we already showed it holds for $T_1$. Now we assume we have trunk $T_{i-1}$ for which the induction step holds and trunk $T_i$ which is $T_{i-1}$ joined with new subgame $S_{i-1}$. Our algorithm recovers approximate equilibrium in $S^M_{i-1}$ and we want similar equation as for the CDBR. Part of the game tree $G^F$ has the errors bounded as in CDBR but because we use gadget in $G$ we need to also consider error in actions ending with value function player $\nabla$ can play in the top with error bounded by $e_V$. We have $|I_{S\Delta}|$ of actions leading out of the tree so the error increase in the $G'$ going to the next subgame is at most $|I_{S\Delta}|e_V + |I_{S^B_{i-1}}|e_V + |I_{S^B_{i-1}}|e_R \geq u^G(\sigma^{NE}, \sigma^F_{V})_V + |I_{S^B_{i-1}}|e_V \geq u^G(\sigma^{NE}, \sigma^F_{V})_V + |I_{S^B_{i-1}}|e_R \geq u^G(\sigma^{NE}, \sigma^F_{V})_V + |I_{S^B_{i-1}}|e_R \geq u^G(\sigma^{NE}, \sigma^F_{V})_V$. Then $\sigma^S_{G}$ is a combination of the strategy we approximated in the subgame and the fixed strategy from previous step, formally $\sigma^S_{G} = \sigma^{NE_{S^B_{i-1}}} \cup \sigma^{B_{T_{i-1}}}, \sigma^F_{V})_V$. Joining it with the induction assumption we have $u^G_M(\sigma^{S_{G}}, \sigma^F_{V})_V + (1-p)|I_{S\Delta}|e_V + |I_{S^B_{i-1}}|e_V + |I_{S^B_{i-1}}|e_R \geq u^G_M(\sigma^{NE}, \sigma^F_{V})_V + \sum_{S\Delta \in S} |I_{S\Delta}| e_V + |S|e_R + \sum_{S\Delta \in S} |I_{S\Delta}| e_V$. However, we still need to show it works for $u^G_M(\sigma^{S_{G}}, \sigma^F_{V})$. We can do it by replacing strategy of player $\nabla$ in the $G'$ by $\sigma^F_{V}$ which will effectively transform $G^M$ game back to $G$ with player $\nabla$ playing $\sigma^F_{V}$. Since we did this transformation by changing the strategy that was a best response the utility can only increase and $u^G_M(\sigma^{S_{G}}, \sigma^F_{V}) \geq u^G_M(\sigma^{NE}, \sigma^F_{V})$. This concludes the proof.

\[ \square \]

**Theorem 4.** Let $G$ be any zero-sum extensive-form game with perfect recall and let $\sigma^F_{V}$ be any fixed opponent’s strategy in G. Then we set $G_M$ as restricted Nash response modification of G using $\sigma^F_{V}$. Let $P$ be any subgame
partitioning of the game $G^M$ and using some $p \in (0,1)$, let $\sigma^p_\mathcal{E}$ be a CDRNR given approximation $V$ of the optimal value function $V$ with error at most $\varepsilon V$, partitioning $\mathcal{P}$ and opponent strategy $\sigma^p_\mathcal{F}$, which is approximated in each step with regret at most $\varepsilon R$, formally $\sigma^p_\mathcal{E} = \sigma^p_\mathcal{E}(\sigma^p_\mathcal{F}, p) V$. Then exploitability has a bound $\mathcal{E}(\sigma^p_\mathcal{E}) \leq \mathcal{G}(\sigma^p_\mathcal{E}, \sigma^p_\mathcal{F}) + \frac{1}{p} \sum_{S \in \mathcal{S}^M} |I_{SO}|(1-p)\varepsilon_V + |S|\varepsilon_R + \sum_{S \in \mathcal{S}^M} |I_{SO}| \varepsilon_V$, $\mathcal{E}$ and $\mathcal{G}$ are defined in Section 4.

Proof. We will examine the exploitability increase in each step. First, we define gain in a single step as $\mathcal{G}(\sigma^p, \sigma^p_\mathcal{F})^V = u_\Delta(\sigma^p, \sigma^p_\mathcal{F})^V - u_\Delta(\sigma, \sigma^p_\mathcal{F})^V$ for $i > 0$ and $\mathcal{G}(\sigma, \sigma^p_\mathcal{F})^{V_0} = u_\Delta(\sigma, \sigma^p_\mathcal{F})^{V_0} - u_\Delta(\sigma^p_\mathcal{F})^{V_0}$. This is consistent with full definition of gain because sum of gains over all steps will results in $u_\Delta(\sigma^p_\mathcal{F})^{V_0} = u_\Delta(\sigma, \sigma^p_\mathcal{F})^{V_0}$. We define exploitability in a single step similarly as $\mathcal{E}(\sigma^p_\mathcal{F})^V = u_\mathcal{V}(\sigma^p_\mathcal{F}, BR(\sigma^p_\mathcal{F}))^V - u_\mathcal{V}(\sigma^p_\mathcal{F})^V$ for $i > 0$ and $\mathcal{E}(\sigma^p_\mathcal{F})^{V_0} = u_\mathcal{V}(\sigma^p_\mathcal{F}, BR(\sigma^p_\mathcal{F}))^{V_0} - u_\mathcal{V}(\sigma^p_\mathcal{F})^{V_0}$. In each step we approximate the strategy in the modified game, having full utility in step written as $\mathcal{G}(\sigma^p_\mathcal{E}, \sigma^p_\mathcal{F})^V = p - \mathcal{E}(\sigma^p_\mathcal{F})^V(1-p)$. If we had exact equilibrium in the subgame this would always be at least 0. However, we have $V$ instead of $V$, values at the top of the created strategy are not exact and the exploited strategy has regret $\varepsilon R$. As in the previous proof the error is bounded by $I_{SO}(1-p)\varepsilon_V + |I_{SM}| \varepsilon_R$ and we can write $\mathcal{G}(\sigma^p_\mathcal{E}, \sigma^p_\mathcal{F})^M = p - \mathcal{E}(\sigma^p_\mathcal{E})^M(1-p) + |I_{SO}|(1-p)\varepsilon_V + |I_{SM}| \varepsilon_R \geq 0$. We reorganize the equation to get $\mathcal{G}(\sigma^p_\mathcal{E}, \sigma^p_\mathcal{F})^M \geq \mathcal{E}(\sigma^p_\mathcal{E})^M(1-p)\varepsilon_V + |I_{SM}| \varepsilon_R \geq \mathcal{E}(\sigma^p_\mathcal{E})^M$ and summing over all the steps gives us $\mathcal{G}(\sigma^p_\mathcal{E}, \sigma^p_\mathcal{F})^V + \sum_{S \in \mathcal{S}^M} |I_{SO}|(1-p)\varepsilon_V + |S|\varepsilon_R + \sum_{S \in \mathcal{S}^M} |I_{SO}| \varepsilon_V \geq \mathcal{E}(\sigma^p_\mathcal{E}) \square$

### F CDBR Against Nash Strategy
**Observation 5.** An example in Figure 9 shows that CDBR can perform worse than a Nash equilibrium against the fixed opponent because of the perfect opponent assumption after the depth-limit. An example is a game of matching pennies with a twist. Player $\sigma$ can choose in the case of the tails whether he wants to give the opponent 10 instead of only 1. A rational player will never do it, and the equilibrium is a uniform strategy as in normal matching pennies.

Now we have an opponent model that plays $h$ with probability $\frac{2}{3}$ and always plays $x$. The best response to the model will always play $T$ and get payoff $\frac{10}{3}$. Nash equilibrium strategy will get payoff 2, and CDBR with depth-limit 2 will cut the game before the $x/y$ choice. Assuming the opponent plays perfectly after the depth-limit and chooses $y$, $\Delta$ will always play $H$. Playing $H$ will result in receiving payoff $\frac{2}{3}$, which is higher than the value of the game $\frac{1}{2}$ but lower than what Nash equilibrium can get against the model.

### G Additional Empirical Results
**CDRN**R We show more results for Goofspiel, Leduc Hold’em, and Liar’s dice with different values of $p$. SX is CDRNR with step size denoted by $X$. We also evaluate SES and only use the highest step value of 5. Next, we show the same setup as in the main text with exactly the same partitioning as they used in SES, and we include more values of $p$.

**Repeated RPS** In Figure 14 we show the strategy sets recovered for all possible $p$ against one strategy in two round biased RPS where after the round information is revealed. As we explained before, we can see that SES cannot gain anything in a game where only information imperfections are simultaneous moves. Exp-strat can exploit only the second round, and it gains half of the maximum, while the other algorithms can gain the maximum and are more or less successful in achieving the best trade-off. The full gadget is the best, followed by the other gadgets without theoretical guarantees, and then by a combination of Nash and CDBR.
Figure 8: Value of each action based on $p$ in the games shown in Figure 7. max-margin gadget in Game 1 (top), full gadget on both Game 1 and 2 (middle) and resolving gadget on Game 2 (bottom).

Figure 9: Example of game where step best response is worse than NE against fixed strategy $\sigma(h) = \frac{3}{5}, \sigma(x) = 1$.

Figure 10: Additional results for CDRNR showing the performance of CDRNR with varying step-size. Generated on Goofspiel 5.
Figure 11: Additional results for CDRNR showing the performance of CDRNR with varying step-size. Generated on Leduc Hold'em.

Figure 12: Additional results for CDRNR with different values of $p$. Generated on Leduc Hold'em split only by the round.
Figure 13: Additional result on Liar’s dice. For every $p$ it exactly mimics the RNR so we only show one value.

Figure 14: Results showing gain and exploitability trade-off in two round biased RPS. Max-margin and resolving gadget overlaps.