A CLASS OF RECURSIVE SETS

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In this article one builds a class of recursive sets, one establishes properties of these sets and one proposes applications. This article widens some results of [1].

1) Definitions, properties.
One calls recursive sets the sets of elements which are built in a recursive manner: let $T$ be a set of elements and $f_i$ for $i$ between 1 and $s$, of operations $n_i$, such that $f_i : T^n \rightarrow T$. Let’s build by recurrence the set $M$ included in $T$ and such that:

(Def. 1) $1^o)$ certain elements $a_1,...,a_n$ of $T$, belong to $M$.

$2^o)$ if $(\alpha_{i_1},...\alpha_{i_n})$ belong to $M$, then $f_i(\alpha_{i_1},...\alpha_{i_n})$ belong to $M$ for all $i \in \{1,2,...,s\}$.

$3^o)$ each element of $M$ is obtained by applying a number finite of times the rules $1^o$ or $2^o$.

We will prove several proprieties of these sets $M$, which will result from the manner in which they were defined. The set $M$ is the representative of a class of recursive sets because in the rules $1^o$ and $2^o$, by particularizing the elements $a_1,...,a_n$ respectively $f_1,...,f_s$ one obtains different sets.

Remark 1: To obtain an element of $M$, it is necessary to apply initially the rule 1.

(Def. 2) The elements of $M$ are called elements $M$-recursive.
(Def. 3) One calls order of an element $a$ of $M$ the smallest natural $p \geq 1$ which has the propriety that $a$ is obtained by applying $p$ times the rule $1^o$ or $2^o$.

One notes $M_p$ the set which contains all the elements of order $p$ of $M$. It is obvious that $M_1 = \{a_1,...,a_n\}$.

$$M_2 = \bigcup_{i=1}^{s} \bigcup_{(\alpha_{i_1},...\alpha_{i_n}) \in M_1}^{(a_{i_1},...a_{i_n}) \in M_1^n} f_i(\alpha_{i_1},...\alpha_{i_n}) \setminus M_1.$$ 

One withdraws $M_1$ because it is possible that $f_j(a_{j_1},...,a_{j_n}) = a_i$ which belongs to $M_1$, and thus does not belong to $M_2$.

One proves that for $k \geq 1$ one has:
\[ M_{k+1} = \bigcup_{i=1}^{s} \left( \bigcup_{(a_n_1, \ldots, a_n_k) \in \prod_{i}^{(i)}} f_i(\alpha_i_1, \ldots, \alpha_i_n) \right) \setminus \bigcup_{h=1}^{k} M_h \]

where each

\[ \prod_{i}^{(i)} = \left\{ (\alpha_i_1, \ldots, \alpha_i_n) / \alpha_i_j \in M_{q_j}, \ j \in \{1, 2, \ldots, n_i\}; \ 1 \leq q_j \leq k \right\} \]

The sets \( M_p, \ p \in \mathbb{N}^+ \), form a partition of the set \( M \).

**Theorem 1:**

\[ M = \bigcup_{p \in \mathbb{N}^+} M_p, \text{ where } \mathbb{N}^+ = \{1, 2, 3, \ldots\} \]

**Proof:**

From the rule 1° it results that \( M_1 \subseteq M \).

One supposes that this propriety is true for values which are less than \( p \). It results that \( M_p \subseteq M \), because \( M_p \) is obtained by applying the rule 2° to the elements of \( \bigcup_{i=1}^{p-1} M_i \).

Thus \( \bigcup_{p \in \mathbb{N}^+} M_p \subseteq M \). Reciprocally, one has the inclusion in the contrary sense in accordance with the rule 3°.

**Theorem 2:** The set \( M \) is the smallest set, which has the properties 1° and 2°.

**Proof:**

Let \( R \) be the smallest set having properties 1° and 2°. One will prove that this set is unique.

Let’s suppose that there exists another set \( R' \) having properties 1° and 2°, which is the smallest. Because \( R \) is the smallest set having these proprieties, and because \( R' \) has these properties also, it results that \( R \subseteq R' \); of an analogue manner, we have \( R' \subseteq R \): therefore \( R = R' \).

It is evident that \( M' \subseteq R \). One supposes that \( M_i \subseteq R \) for \( 1 \leq i < p \). Then (rule 3°), and taking in consideration the fact that each element of \( M_p \) is obtained by applying rule 2° to certain elements of \( M_i \), \( 1 \leq i < p \), it results that \( M_p \subseteq R \). Therefore \( \bigcup_{p \in \mathbb{N}^+} M_p \subseteq R \), thus \( M \subseteq R \). And because \( R \) is unique, \( M = R \).

**Remark 2.** The theorem 2 replaces the rule 3° of the recursive definition of the set \( M \) by: “ \( M \) is the smallest set that satisfies proprieties 1° and 2°°.”

**Theorem 3:** \( M \) is the intersection of all the sets of \( T \) which satisfy conditions 1° and 2°.

**Proof:**
Let $T_{12}$ be the family of all sets of $T$ satisfying the conditions 1° and 2°. We note $I = \bigcap_{A \in T_{12}} A$.

$I$ has the properties 1° and 2° because:

1) For all $i \in \{1,2,...,n\}$, $a_i \in I$, because $a_i \in A$ for all $A$ of $T_{12}$.
2) If $\alpha_1,\ldots,\alpha_n \in I$, it results that $\alpha_1,\ldots,\alpha_n$ belong to $A$ that is $A$ of $T_{12}$.

Therefore, $\forall i \in \{1,2,...,s\}$, $f_i(\alpha_1,\ldots,\alpha_n) \in A$ which is $A$ of $T_{12}$, therefore $f_i(\alpha_1,\ldots,\alpha_n) \in I$ for all $i$ from $\{1,2,...,s\}$.

From theorem 2 it results that $M \subseteq I$.

Because $M$ satisfies the conditions 1° and 2°, it results that $M \in T_{12}$, from which $I \subseteq M$. Therefore $M = I$.

(Def. 4) A set $A \subseteq I$ is called closed for the operation $f_{i_0}$ if and only if for all $\alpha_{i_0,1},\ldots,\alpha_{i_0,n}$ of $A$, one has $f_{i_0}(\alpha_{i_0,1},\ldots,\alpha_{i_0,n})$ belong to $A$.

(Def. 5) A set $A \subseteq T$ is called $M$-recursively closed if and only if:

1) $\{a_1,...,a_n\} \subseteq A$.
2) $A$ is closed in respect to operations $f_1,...,f_s$.

With these definitions, the precedent theorems become:

**Theorem 2'**: The set $M$ is the smallest $M$-recursively closed set.

**Theorem 3'**: $M$ is the intersection of all $M$-recursively closed sets.

(Def. 6) The system of elements $\langle \alpha_1,\ldots,\alpha_m \rangle$, $m \geq 1$ and $\alpha_i \in T$ for $i \in \{1,2,...,m\}$, constitute a $M$-recursive description for the element $\alpha$, if $\alpha_m = \alpha$ and that each $\alpha_i$ ($i \in \{1,2,...,m\}$) satisfies at least one of the proprieties:

1) $\alpha_i \in \{a_1,...,a_n\}$.
2) $\alpha_i$ is obtained starting with the elements which precede it in the system by applying the functions $f_j$, $1 \leq j \leq s$ defined by property 2° of (Def. 1).

(Def. 7) The number $m$ of this system is called the length of the $M$-recursive description for the element $\alpha$.

**Remark 3**: If the element $\alpha$ admits a $M$-recursive description, then it admits an infinity of such descriptions.

Indeed, if $\langle \alpha_1,\ldots,\alpha_m \rangle$ is a $M$-recursive description of $\alpha$ then $\langle \alpha_1,\ldots,\alpha_h,\alpha_1,\ldots,\alpha_m \rangle$ is also a $M$-recursive description for $\alpha$, $h$ being able to take all values from $\mathbb{N}$.
Theorem 4: The set $M$ is identical with the set of all elements of $T$ which admit a $M$-recursive description.

Proof: Let $D$ be the set of all elements, which admit a $M$-recursive description. We will prove by recurrence that $M_p \subseteq D$ for all $p$ of $\mathbb{N}^+$. For $p = 1$ we have: $M_1 = \{a_1,\ldots,a_n\}$, and the $a_j$, $1 \leq j \leq n$, having as $M$-recursive description: $<a_j>$. Thus $M_1 \subseteq D$. Let’s suppose that the property is true for the values smaller than $p$. $M_p$ is obtained by applying the rule 2o to the elements of $\bigcup_{i=1}^{p-1} M_i$; $\alpha \in M_p$ implies that $\alpha \in f_j(\alpha_i,\ldots,\alpha_n)$ and $\alpha_i \in M_{h_i}$ for $h_j < p$ and $1 \leq j \leq n_i$. But $a_j$, $1 \leq j \leq n_i$, admits $M$-recursive descriptions according to the hypothesis of recurrence, let’s have $\langle\beta_{j_1},\ldots,\beta_{j_n}\rangle$. Then $\langle\beta_{j_1},\ldots,\beta_{j_n},\beta_{2},\ldots,\beta_{2x_2},\ldots,\beta_{n_1},\ldots,\beta_{n_{x_n}},\alpha\rangle$ constitute a $M$-recursive description for the element $\alpha$. Therefore if $\alpha$ belongs to $D$, then $M_p \subseteq D$ which is $M = \bigcup_{p \in \mathbb{N}} M_p \subseteq D$.

Reciprocally, let $x$ belong to $D$. It admits a $M$-recursive description $\langle b_1,\ldots,b_t \rangle$ with $b_1 = x$. It results by recurrence by the length of the $M$-recursive description of the element $x$, that $x \in M$. For $t = 1$ we have $\langle b_1 \rangle$, $b_1 = x$ and $b_1 \in \{a_1,\ldots,a_n\} \subseteq M$. One supposes that all elements $y$ of $D$ which admit a $M$-recursive description of a length inferior to $t$ belong to $M$. Let $x \in D$ be described by a system of length $t : \langle b_1,\ldots,b_t \rangle$, $b_t = x$. Then $x \in \{a_1,\ldots,a_n\} \subseteq M$, where $x$ is obtained by applying the rule 2o to the elements which precede it in the system: $b_1,\ldots,b_{t-1}$. But these elements admit the $M$-recursive descriptions of length which is smaller that $t : \langle b_1, b_2,\ldots,b_{t-1} \rangle$. According to the hypothesis of the recurrence, $b_1,\ldots,b_{t-1}$ belong to $M$. Therefore $b_t$ belongs also to $M$. It results that $M = D$.

Theorem 5: Let $b_1,\ldots,b_q$ be elements of $T$, which are obtained from the elements $a_1,\ldots,a_n$ by applying a finite number of times the operations $f_1, f_2,\ldots, f_s$. Then $M$ can be defined recursively in the following mode:

1) Certain elements $a_1,\ldots,a_n,b_1,\ldots,b_q$ of $T$ belong to $M$.

2) $M$ is closed for the applications $f_i$, with $i \in \{1,2,\ldots,s\}$.

3) Each element of $M$ is obtained by applying a finite number of times the rules (1) or (2) which precede.

Proof: evident. Because $b_1,\ldots,b_q$ belong to $T$, and are obtained starting with the elements $a_1,\ldots,a_n$ of $M$ by applying a finite number of times the operations $f_i$, it results that $b_1,\ldots,b_q$ belong to $M$.  

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Theorem 6: Let's have \( g_j \), \( 1 \leq j \leq r \), of the operations \( n_j \), where \( g_j : T^n \rightarrow T \) such that \( M \) to be closed in rapport to these operations. Then \( M \) can be recursively defined in the following manner:
1) Certain elements \( a_1,...,a_n \) de \( T \) belong to \( M \).
2) \( M \) is closed for the operations \( f_i \), \( i \in \{1,2,...,s\} \) and \( g_j \), \( j \in \{1,2,...,r\} \).
3) Each element of \( M \) is obtained by applying a finite number of times the precedent rules.
Proof is simple: Because \( M \) is closed for the operations \( g_j \) (with \( j \in \{1,2,...,r\} \)), one has, that for any \( \alpha_{j_1},...\alpha_{j_n} \) from \( M \), \( g_j(\alpha_{j_1},...\alpha_{j_n}) \in M \) for all \( j \in \{1,2,...,r\} \).

From the theorems 5 and 6 it results:
Theorem 7: The set \( M \) can be recursively defined in the following manner:
1) Certain elements \( a_1,...,a_n,b_1,...,b_q \) of \( T \) belong to \( M \).
2) \( M \) is closed for the operations \( f_i \), \( i \in \{1,2,...,s\} \) and for the operations \( g_j \) (\( j \in \{1,2,...,r\} \)) previously defined.
3) Each element of \( M \) is defined by applying a finite number of times the previous 2 rules.

(Def. 8) The operation \( f_i \) conserves the property \( P \) iff for any elements \( \alpha_{i_1},...\alpha_{i_n} \) having the property \( P \), \( f_i(\alpha_{i_1},...\alpha_{i_n}) \) has the property \( P \).

Theorem 8: If \( a_1,...,a_n \) have the property \( P \), and if the functions \( f_1,...,f_s \) preserve this property, then all elements of \( M \) have the property \( P \).
Proof:
\[ M = \bigcup_{p \in \mathbb{N}} M_p \]. The elements of \( M_1 \) have the property \( P \).

Let’s suppose that the elements of \( M_i \) for \( i < p \) have the property \( P \). Then the elements of \( M_p \) also have this property because \( M_p \) is obtained by applying the operations \( f_1,f_2,...,f_s \) to the elements of: \( \bigcup_{i=1} M_i \), elements which have the property \( P \). Therefore, for any \( p \) of \( \mathbb{N} \), the elements of \( M_p \) have the property \( P \).

Thus all elements of \( M \) have it.

Corollary 1: Let’s have the property \( P \): "\( x \) can be represented in the form \( F(x) \)".

If \( a_1,...,a_n \) can be represented in the form \( F(a_1),...,F(a_n) \), and if \( f_1,...,f_s \) maintains the property \( P \), then all elements \( \alpha \) of \( M \) can be represented in the form \( F(\alpha) \).

Remark. One can find more other equivalent def. of \( M \).

2) APPLICATIONS, EXAMPLES.
In applications, certain general notions like: $M$ - recursive element, $M$ - recursive description, $M$ - recursive closed set will be replaced by the attributes which characterize the set $M$. For example in the theory of recursive functions, one finds notions like: recursive primitive functions, primitive recursive description, primitively recursive closed sets. In this case “$M$” has been replaced by the attribute “primitive” which characterizes this class of functions, but it can be replaced by the attributes ”general”, ”partial”.

By particularizing the rules 1° and 2° of the def. 1, one obtains several interesting sets:

**Example 1:** (see [2], pp. 120-122, problem 7.97).

**Example 2:** The set of terms of a sequence defined by a recurring relation constitutes a recursive set.

Let’s consider the sequence: $a_{n+k} = f(a_n, a_{n+1}, \ldots, a_{n+k-1})$ for all $n$ of $\mathbb{N}^+$, with $a_i = a_0^i$, $1 \leq i \leq k$. One will recursively construct the set $A = \{a_m\}_{m \in \mathbb{N}^+}$ and one will define in the same time the position of an element in the set $A$:

1°) $a_1^0, \ldots, a_k^0$ belong to $A$, and each $a_i^0$ ($1 \leq i \leq k$) occupies the position $i$ in the set $A$;

2°) if $a_n, a_{n+1}, \ldots, a_{n+k-1}$ belong to $A$, and each $a_j$ for $n \leq j \leq n+k-1$ occupies the position $j$ in the set $A$, then $f(a_n, a_{n+1}, \ldots, a_{n+k-1})$ belongs to $A$ and occupies the position $n+k$ in the set $A$.

3°) each element of $B$ is obtained by applying a finite number of times the rules 1° or 2°.

**Example 3:** Let $G = \{e, a^1, a^2, \ldots, a^p\}$ be a cyclic group generated by the element $a$. Then $\left( G, \square \right)$ can be recursively defined in the following manner:

1°) $a$ belongs to $G$.

2°) if $b$ and $c$ belong to $G$ then $b\square c$ belongs to $G$.

3°) each element of $G$ is obtained by applying a finite number of times the rules 1 or 2.

**Example 4:** Each finite set $ML = \{x_1, x_2, \ldots, x_n\}$ can be recursively defined (with $ML \subseteq T$):

1°) The elements $x_1, x_2, \ldots, x_n$ of $T$ belong to $ML$.

2°) If $a$ belongs to $ML$, then $f(a)$ belongs to $ML$, where $f : T \to T$ such that $f(x) = x$;

3°) Each element of $ML$ is obtained by applying a finite number of times the rules 1° or 2°.

**Example 5:** Let $L$ be a vectorial space on the commutative corps $K$ and $\{x_1, \ldots, x_m\}$ be a base of $L$. Then $L$, can be recursively defined in the following manner:

1°) $x_1, \ldots, x_m$ belong to $L$;

2°) if $x, y$ belong to $L$ and if $a$ belongs to $K$, then $x \perp y$ $y$ belong to $L$ and $a \times x$ belongs to $L$;

3°) each element of $L$ is recursively obtained by applying a finite number of times the rules 1° or 2°.
The operators $\bot$ and $\ast$ are respectively the internal and external operators of the vectorial space $L$.

**Example 6:** Let $X$ be an $A$-module, and $M \subset X$ ($M \neq \emptyset$), with $M = \{x_i\}_{i \in I}$.

The sub-module generated by $M$ is:

$$\langle M \rangle = \{x \in X / x = a_1 x_1 + \ldots + a_n x_n, \ a_i \in A, \ x_i \in M, \ i \in \{1, \ldots, n\}\}$$

can be recursively defined in the following way:

1°) for all $i$ of $\{1, 2, \ldots, n\}$, $\{1, 2, \ldots, n\}x_i \in \langle M \rangle$;

2°) if $x$ and $y$ belong to $\langle M \rangle$ and $a$ belongs to $A$, then $x + y$ belongs to $\langle M \rangle$, and $ax$ also;

3°) each element of $\langle M \rangle$ is obtained by applying a finite number of times the rules 1° or 2°.

In accordance to the paragraph 1 of this article, $\langle M \rangle$ is the smallest sub-set of $X$ that verifies the conditions 1° and 2°, that is $\langle M \rangle$ is the smallest sub-module of $X$ that includes $M$. $\langle M \rangle$ is also the intersection of all the subsets of $X$ that verify the conditions 1° and 2°, that is $\langle M \rangle$ is the intersection of all sub-modules of $X$ that contain $M$. One also directly refines some classic results from algebra.

One can also talk about sub-groups or ideal generated by a set: one also obtains some important applications in algebra.

**Example 7:** One also obtains like an application the theory of formal languages, because, like it was mentioned, each regular language (linear at right) is a regular set and reciprocally. But a regular set on an alphabet $\Sigma = \{a_1, \ldots, a_n\}$ can be recursively defined in the following way:

1°) $\emptyset, \{e\}, \{a_1\}, \ldots, \{a_n\}$ belong to $R$.

2°) if $P$ and $Q$ belong to $R$, then $P \cup Q$, $PQ$, and $P^*$ belong to $R$, with $P \cup Q = \{x / x \in P \text{ or } x \in Q\}$; $PQ = \{xy / x \in P \text{ and } y \in Q\}$, and $P^* = \bigcup_{n=0}^{\infty} P^n$ with $P^n = \underbrace{P \cdot P \cdot \ldots \cdot P}_{n \text{ times}}$ and, by convention, $P^0 = \{e\}$.

3°) Nothing else belongs to $R$ other than those which are obtained by using 1° or 2°.

From which many properties of this class of languages with applications to the programming languages will result.

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