Higher Auslander algebras of type $\mathbb{A}$ and the higher Waldhausen $S$-constructions

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Abstract. These notes are an expanded version of my talk at the ICRA 2018 in Prague, Czech Republic; they are based on joint work with Tobias Dyckerhoff and Tashi Walsi [DJW19b]. In them we relate Iyama’s higher Auslander algebras of type $\mathbb{A}$ to Eilenberg–Mac Lane spaces in algebraic topology and to higher-dimensional versions of the Waldhausen $S$-construction from algebraic $K$-theory.

1. Motivation: Eilenberg–Mac Lane spaces

Let $A$ be an abelian group. For each positive integer $m$ there exists a topological space $K(A, m)$ characterised, up to weak homotopy equivalence, by the existence of natural bijections\footnote{A weak homotopy equivalence is a continuous map $X \to Y$ between topological spaces which induces a bijection $\pi_0(X) \xrightarrow{\cong} \pi_0(Y)$ on connected components as well as group isomorphisms $\pi_n(X, x) \xrightarrow{\cong} \pi_n(Y, f(x))$ for all points $x \in X$ and all $n \geq 1$. Two topological spaces are weakly homotopy equivalent if they are connected by a zig-zag of weak homotopy equivalences.} by the existence of natural bijections\footnote{Here $[X, K(A, m)]$ denotes the set of homotopy classes of continuous maps $X \to K(A, m)$, while $H^m(X; A)$ denotes the $m$-th singular cohomology group of $X$ with coefficients in $A$.}

\[
[X, K(A, m)] \xrightarrow{\cong} H^m(X; A)
\]

where $X$ is a CW complex, see Corollary III.2.17 in [GJ99]. Equivalently, the topological space $K(A, m)$ is characterised by the following two properties:

- The topological space $K(A, m)$ is path connected;
- there is an isomorphism

\[
\pi_n(K(A, m)) \cong \begin{cases} A & \text{if } n = m, \\ 0 & \text{otherwise.} \end{cases}
\]

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A topological space satisfying the above properties is called an Eilenberg–Mac Lane space. The bijection (1) exhibits Eilenberg–Mac Lane spaces as fundamental objects in algebraic topology while their characterisation in terms of higher homotopy groups makes their relative simplicity manifest.

Rather surprisingly, the standard construction of the Eilenberg–Mac Lane spaces can be expressed in terms of the higher Auslander–Reiten theory of Iyama’s higher Auslander algebras of type $A$. This observation, which we make precise below, is one of the starting points of our investigations.

1.1. Interlude: the simplex category. The modern approach to homotopy theory makes extensive use of simplicial methods and the standard construction of Eilenberg–Mac Lane spaces is no exception. We recall a minimal amount of terminology from simplicial homotopy theory needed for describing this construction.

We remind the reader of the definition of the simplex category, which is commonly denoted by $\Delta$. The objects of the simplex category are the linear posets $[n] := \{0 \to 1 \to \cdots \to n\}$, where $n$ ranges over all non-negative integers, with morphisms the monotone functions $[m] \to [n]$. We denote the set of morphisms $[m] \to [n]$ in $\Delta$ by $\Delta(m, n)$. Note that the set $\Delta(m, n)$ has a natural poset structure. Namely, $\sigma \leq \tau$ if and only if $\sigma_i \leq \tau_i$ for all $i \in [m]$. The elements of $\Delta(m, n)$ are called $m$-simplices in $\Delta^n$; an $m$-simplex is non-degenerate if the underlying monotone function is injective and is degenerate otherwise. We often identify $\Delta(m, n)$ with the set of ordered tuples $\sigma \in \mathbb{Z}^{m+1}$ which satisfy the inequalities $0 \leq \sigma_0 \leq \sigma_1 \leq \cdots \leq \sigma_m \leq n$.

A simplicial object in a category $\mathcal{C}$ is a functor $X: \Delta^{\text{op}} \to \mathcal{C}$. It is elementary to verify, using a specific presentation of $\Delta$ in terms of generators and relations, that the data of a simplicial object in $\mathcal{C}$ is equivalent to that of a diagram in $\mathcal{C}$ of the form

\[
\cdots \xrightarrow{d_1} X_2 \xrightarrow{d_2} X_1 \xrightarrow{d_3} X_0
\]

where $X_n := X([n])$ is the object of $n$-simplices of $X$. The morphisms

\[
\{ d_i: X_n \to X_{n-1} \mid i \in [n] \},
\]

called face maps, and the morphisms

\[
\{ s_i: X_n \to X_{n+1} \mid i \in [n] \},
\]

3The symbol $K(A, m)$ is more commonly used to denote the weak homotopy equivalence class (also known as the homotopy type) of an Eilenberg–Mac Lane space. For our purposes it will be convenient to abuse the notation and use this symbol to denote a preferred representative of this equivalence class.

4The reader is referred to [GJ99] for an introduction to simplicial homotopy theory.

5The poset structure on the morphism sets in $\Delta$ allows us to promote the simplex category to a 2-category; this perspective, implicit in parts of these notes, is essential in our work [BJW19b].

6Depending on the target category one speaks of simplicial sets, simplicial topological spaces, simplicial abelian groups, etc.
called degeneracy maps, are subject to a number of relations commonly referred to as the simplicial identities:

\[
\begin{align*}
\text{d}_i \circ \text{d}_j &= d_{j-1} \circ \text{d}_i, & i < j, \\
\text{s}_i \circ \text{s}_j &= s_j \circ \text{s}_{i-1}, & i > j, \\
\text{d}_i \circ \text{s}_j &= \begin{cases} 
\text{s}_{j-1} \circ \text{d}_i & i < j, \\
\text{s}_j \circ \text{d}_{i-1} & i > j + 1, \\
1 & \text{otherwise.}
\end{cases}
\end{align*}
\]

Thus, a simplicial object in \( \mathcal{C} \) is a contravariant representation in the category \( \mathcal{C} \) of the infinite quiver with (inadmissible) relations corresponding to the standard presentation of \( \Delta \).

1.2. Construction via the Dold–Kan correspondence. The Dold–Kan correspondence \([Dol58, Kan58]\) is an explicit adjoint equivalence of categories

\[
C : \text{Ab}_\Delta \xrightarrow{\sim} \text{Ch}_{\geq 0}(\text{Ab}) : N
\]

between the (abelian) category of simplicial abelian groups and the (abelian) category of chain complexes of abelian groups which are concentrated in non-negative homological degree.

Let \( A \) be an abelian group and \( m \) a positive integer. As usual, let us denote by \( A[m] \) the chain complex of abelian groups whose only non-zero component consists of \( A \) placed in homological degree \( m \). The Eilenberg–Mac Lane space \( K(A, m) \) is defined as

\[
K(A, m) := |N(A[m])|,
\]

that is as the geometric realisation of the underlying simplicial set of \( N(A[m]) \).

1.3. The Eilenberg–Mac Lane space \( K(A, 1) \). Let \( A \) be an abelian group. For our purposes it is instructive to give an explicit description of the simplicial abelian group \( N(A[1]) \) in the simplest case \( m = 1 \).

**Definition 1.1.** The simplicial abelian group \( N(A[1]) \) is defined as follows. For \( n \geq 0 \), let \( N(A[1])_n \) be the abelian group of upper-triangular arrays

\[
\begin{pmatrix}
  a_{00} & a_{01} & a_{02} & \cdots & a_{0,n-1} & a_{0n} \\
  a_{11} & a_{12} & \cdots & a_{1,n-1} & a_{1n} \\
  \vdots & \vdots & \ddots & \vdots & \vdots \\
  a_{n-1,n-1} & a_{n-1,n} & \cdots & \cdots & a_{nn}
\end{pmatrix}
\]
with entries in $A$ such that for each $0 \leq i \leq n$ there is an equality $a_{ii} = 0$ and for all $0 \leq i < j < k \leq n$ the Euler relation
\begin{equation}
    a_{ij} - a_{ik} + a_{jk} = 0
\end{equation}
is satisfied. The $i$-th face map
\[ d_i : N(A[1])_n \to N(A[1])_{n-1} \]
is given by deleting the $i$-th row and the $i$-th column while the $i$-th degeneracy map
\[ s_i : N(A[1])_n \to N(A[1])_{n+1} \]
is given by repeating the $i$-th row and the $i$-th column. □

Remark 1.2. An $n$-simplex of $N(A[1])$, is completely determined by the $n$-tuple $(a_{01}, a_{02}, \ldots, a_{0n})$. Indeed, the Euler relations (2) imply that for each $1 \leq j < k \leq n$ the equality
\[ a_{jk} = a_{0k} - a_{0j} \]
is satisfied. We conclude that there is an isomorphism
\[ N(A[1])_n \cong A^n. \]
Note, however, that the above identification makes the simplicial structure of $N(A[1])$, less apparent. □

Our next task is to relate the previous construction of the Eilenberg–Mac Lane space $K(A,1)$ to the Auslander–Reiten theory of the family of quivers
\[ \mathbb{A}_n := 1 \to 2 \to \cdots \to n; \quad n \geq 0. \]
Consider the Auslander–Reiten quiver of the category of finite-dimensional representations of the quiver $\mathbb{A}_n$, which we depict as follows:

\[
\begin{array}{cccccccc}
\text{M}_{00} & \to & \text{M}_{01} & \to & \text{M}_{02} & \to & \cdots & \to & \text{M}_{0,n-1} & \to & \text{M}_{0n} \\
\downarrow & & \downarrow & & \downarrow & & \cdots & & \downarrow & & \downarrow \\
\text{M}_{11} & \to & \text{M}_{12} & \to & \cdots & \to & \text{M}_{1,n-1} & \to & \text{M}_{1n} \\
& & \downarrow & & \cdots & & \cdots & & \downarrow \\
& & & & \cdots & & \cdots & & \downarrow \\
& & & & & & \downarrow & & \downarrow \\
& & & & & & & & \text{M}_{n-1,n-1} & \to & \text{M}_{n-1,n} \\
& & & & & & & & & & \downarrow \\
& & & & & & & & & & \text{M}_{nn}
\end{array}
\]

For reasons which shall become clear shortly, we have included additional vertices $M_{ii} = 0$ in the above Auslander–Reiten quiver. The Grothendieck group $K_0(\mathbb{A}_n)$ admits a presentation by generators and relations as the quotient of the free abelian group with basis
\[ \{ [M_{ij}] | 0 \leq i \leq j \leq n \} \]
modulo the subgroup generated by the relations $[M_{ii}] = 0$, $i \in [n]$, together with the Euler relations
\begin{equation}
    [M_{ij}] - [M_{ik}] + [M_{jk}] = 0
\end{equation}
\footnote{We work over some fixed but unspecified field.}
corresponding to short exact sequences\textsuperscript{12}
\[0 \to M_{ij} \to M_{ik} \to M_{jk} \to 0\]

in the abelian category of finite-dimensional representations of the quiver $\mathbb{A}_n$, where $0 \leq i < j < k \leq n$.

It is immediate from the above presentation of $K_0(\mathbb{A}_n)$ that there are isomorphisms
\begin{equation}
N(A[1])_n \cong \text{Hom}_\mathbb{Z}(K_0(\mathbb{A}_n), A) \cong \text{Hom}_\mathbb{Z}(\mathbb{Z}^n, A) \cong A^n,
\end{equation}
where $n \geq 1$. We leave it to the reader to verify that the Grothendieck groups $K_0(\mathbb{A}_n)$ assemble into a co-simplicial abelian group, that is into a functor
\[K_0(\mathbb{A}_\bullet) : \Delta \to \text{Ab},\]
where $K_0(\mathbb{A}_0) := 0$. Moreover, the isomorphisms \textsuperscript{14} assemble into an isomorphism of simplicial abelian groups
\begin{equation}
N(A[1])_\bullet \cong \text{Hom}_\mathbb{Z}(K_0(\mathbb{A}_\bullet), A)
\end{equation}
where $\text{Hom}_\mathbb{Z}(K_0(\mathbb{A}_\bullet), A)$ denotes the composite
\[
\Delta^{\text{op}} \xrightarrow{K_0(\mathbb{A}_\bullet)^{\text{op}}} \text{Ab}^{\text{op}} \xrightarrow{\text{Hom}_\mathbb{Z}(\text{op}, \text{Ab})} \text{Ab},
\]
and $K_0(\mathbb{A}_\bullet)^{\text{op}}$ indicates the passage to the opposite functor.

\textbf{Remark 1.3.} It is elementary to verify that the Euler relations \textsuperscript{15} are generated by the \textit{Auslander–Reiten relations}
\[\left[M_{ij}\right] - \left[M_{i+1,j}\right] \cong \left[M_{i,j+1}\right] + \left[M_{i+1,j+1}\right] = 0\]
corresponding to the almost-split sequences\textsuperscript{13}
\[0 \to M_{ij} \to M_{i+1,j} \oplus M_{i,j+1} \to M_{i+1,j+1} \to 0,
\]
where $0 \leq i < j < n$, see also \textsuperscript{Aus84}. In particular, the isomorphism \textsuperscript{16} gives a precise relationship between the simplicial abelian group $N(A[1])_\bullet$ and the Auslander–Reiten theory of the linearly oriented quivers of Dynkin type $\mathbb{A}$. \hfill $\square$

\textsuperscript{12}Equivalently, the Euler relations are induced by biCartesian squares
\[
\begin{array}{ccc}
M_{ij} & \to & M_{ik} \\
\downarrow & & \downarrow \\
M_{ii} & \to & M_{jk}
\end{array}
\]
where $0 \leq i < j < k \leq n$.

\textsuperscript{13}Equivalently, the Auslander–Reiten relations are induced by meshes
\[
\begin{array}{ccc}
M_{ij} & \to & M_{i,j+1} \\
\downarrow & & \downarrow \\
M_{i+1,j} & \to & M_{i+1,j+1}
\end{array}
\]
in the Auslander–Reiten quiver, where $0 \leq i < j < n$. 

\textsuperscript{14}Equivalently, the Auslander–Reiten relations are induced by meshes
\[
\begin{array}{ccc}
M_{ij} & \to & M_{i,j+1} \\
\downarrow & & \downarrow \\
M_{i+1,j} & \to & M_{i+1,j+1}
\end{array}
\]
in the Auslander–Reiten quiver, where $0 \leq i < j < n$. 

\textsuperscript{15}Equivalently, the Auslander–Reiten relations are induced by meshes
\[
\begin{array}{ccc}
M_{ij} & \to & M_{i,j+1} \\
\downarrow & & \downarrow \\
M_{i+1,j} & \to & M_{i+1,j+1}
\end{array}
\]
in the Auslander–Reiten quiver, where $0 \leq i < j < n$. 

\textsuperscript{16}Equivalently, the Auslander–Reiten relations are induced by meshes
\[
\begin{array}{ccc}
M_{ij} & \to & M_{i,j+1} \\
\downarrow & & \downarrow \\
M_{i+1,j} & \to & M_{i+1,j+1}
\end{array}
\]
in the Auslander–Reiten quiver, where $0 \leq i < j < n$. 

\textsuperscript{17}Equivalently, the Auslander–Reiten relations are induced by meshes
\[
\begin{array}{ccc}
M_{ij} & \to & M_{i,j+1} \\
\downarrow & & \downarrow \\
M_{i+1,j} & \to & M_{i+1,j+1}
\end{array}
\]
in the Auslander–Reiten quiver, where $0 \leq i < j < n$. 

\textsuperscript{18}Equivalently, the Auslander–Reiten relations are induced by meshes
\[
\begin{array}{ccc}
M_{ij} & \to & M_{i,j+1} \\
\downarrow & & \downarrow \\
M_{i+1,j} & \to & M_{i+1,j+1}
\end{array}
\]
in the Auslander–Reiten quiver, where $0 \leq i < j < n$.
1.4. The Eilenberg–Mac Lane space $K(A, m)$. Let $A$ be an abelian group and $m$ a positive integer. The previous discussion extends to the Eilenberg–Mac Lane space $K(A, m)$ provided that we replace the linearly oriented quivers of type $A$ by Iyama’s $m$-dimensional Auslander algebras $A_{m}(m)$ of type $A$, see [Iya11] for the definition. Indeed, the Grothendieck groups of these algebras assemble into a co-simplicial abelian group

$$K_0(A_{m}(m)) : \Delta \to \text{Ab},$$

where $K_0(A_{m}(m)) := 0$ for $n < m$. In analogy with the case $m = 1$, there is a canonical isomorphism of simplicial abelian groups

$$N(A[m])_{\bullet} \cong \text{Hom}_{\mathbb{Z}}(K_0(A_{m}(m)_{n-m+1}), A),$$

see Theorem 1.20 in [DJW19b] for a detailed description of the above isomorphism.

The above description of $N(A[m])_{\bullet}$ utilises the presentation of the Grothendieck group $K_0(A_{m}(m)_{n-m+1})$ in terms of the basis

$$\{ [M_{\sigma}] | \sigma \in \Delta(m, n) \}$$

labelled by the indecomposable summands of the unique basic $m$-cluster-tilting $A_{n-m+1}$-module. In analogy with the case $m = 1$, one imposes relations $[M_{\sigma}] = 0$ for all degenerate $m$-simplices in $\Delta_{n}$ as well as higher Euler relations

$$\sum_{i=0}^{m}(-1)^i[M_{d_i(\sigma)}] = 0,$$

where $\sigma$ is a non-degenerate $(m+1)$-simplex in $\Delta_{n}$ and $d_i(\sigma) = (\sigma_0, \sigma_1, \cdots, \sigma_{i-1}, \tilde{\sigma}_i, \sigma_{i+1}, \cdots, \sigma_m)$. The higher Euler relations are induced by exact sequences

$$0 \to M_{d_m(\sigma)} \to M_{d_{m-1}(\sigma)} \to \cdots \to M_{d_i(\sigma)} \to M_{d_0(\sigma)} \to 0$$

in the abelian category of finite-dimensional representations of the higher Auslander algebra $A_{n-m+1}$. Finally, we note that the higher Euler relations are generated by the higher Auslander–Reiten relations

$$\sum_{v}(-1)^{|v|}[M_{\sigma+v}] = 0,$$

where $\sigma$ is a non-degenerate $m$-simplex in $\Delta_{n}$ with $\sigma_m < n$, the tuple $v$ ranges over all vertices of the $(m+1)$-cube

$$\{ 0,1 \} \times \cdots \times \{ 0,1 \},$$

and $|v| := v_0 + v_1 + \cdots + v_m$. Again in analogy with the case $m = 1$, the relations (7) are induced by the $m$-almost-split sequences

$$0 \to M_{\sigma} \to \bigoplus_{|v|=1} M_{\sigma+v} \to \cdots \to \bigoplus_{|v|=m} M_{\sigma+v} \to M_{\sigma+(1,\ldots,1)} \to 0$$

in the unique $m$-cluster-tilting subcategory associated with $A_{n-m+1}$, see [Iya07], [Iya11] and [OT12] for definitions and further details.

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14Implicit in this discussion is the Grothendieck group of an $m$-cluster-tilting subcategory, which can be defined using the ideas in [BT13].
1.5. On the purpose of these notes. In the remainder of these notes is to describe a categorified version of the above discussion where the abelian group $A$ is replaced by a refined version of a triangulated category, namely a stable $\infty$-category in the sense of Lurie [Lur17]. As such, the simplicial objects we shall consider can be thought of as ‘categorified Eilenberg–Mac Lane spaces’ as originally advocated by Dyckerhoff in [Dyc17]. The results presented in these notes can also be seen as contributions to the abstract representation theory program of Groth and Šťovíček [GS18], see also Ladicani’s Ph.D. Thesis [Lad08]. Indeed, the results presented here are for the most part higher-dimensional versions of results in [GS16a].

2. Rudiments of the theory of $\infty$-categories

The language of $\infty$-categories, developed by Joyal in [Joy02] as well as in unpublished work and by Lurie in [Lur09, Lur17], affords an adequate framework for our results. It is however impractical to include here a formal introduction to the theory of $\infty$-categories. Instead we aim to provide the reader with a minimal amount of intuition regarding the theory which we hope will be sufficient to follow our exposition in the subsequent section.

2.1. What are $\infty$-categories? An $\infty$-category is a mathematical structure\(^{17}\) which implements the idea of a higher-dimensional category with morphisms of every positive degree endowed with a coherently-associative composition law and such that the morphisms of degree greater than 1 are invertible.

In other words the theory of $\infty$-categories is a model for the theory of $(\infty,1)$-categories. In particular, for every pair of objects $x$ and $y$ in an $\infty$-category $\mathcal{C}$ there is a ‘space’ $\text{Map}_\mathcal{C}(x,y)$ of maps $x \to y$. Within this paradigm ordinary categories are those for which the non-empty spaces of maps are homotopy equivalent to discrete spaces.\(^{18}\) After suitable translation, further examples of $\infty$-categories are provided by differential graded categories and, more generally, $A_{\infty}$-categories.\(^{19}\)

An important feature of the theory of $\infty$-categories is that it allows us to formalise the notion of a universal property which only holds ‘up to coherent homotopy’. For example, an object $x$ of an $\infty$-category $\mathcal{C}$ is initial if for each object $y$ of $\mathcal{C}$ the space $\text{Map}_\mathcal{C}(x,y)$ is contractible (whence in particular non-empty).

Just as in ordinary category theory, universal properties are also captured by appropriate notions of (homotopy) limit and (homotopy) colimit. For example,

\(^{15}\)Triangulated categories are inadequate for this purpose as the cone of a morphisms lacks the necessary functoriality.

\(^{16}\)The standard reference for the theory of $\infty$-categories is Lurie’s book [Lur09]. Alternatives include Groth’s lecture notes [Gro10] and Cisinski’s book [Cis19].

\(^{17}\)More precisely, an $\infty$-category is a particular kind of simplicial set called a ‘weak Kan complex’.

\(^{18}\)See Propositions 2.3.4.5 and 2.3.4.18 in [Lur09].

\(^{19}\)See Section 1.3 in [Lur17] for the case of differential graded categories and [Fao17a, Fao17b] for the case of $A_{\infty}$-categories.

\(^{20}\)In $\infty$-category theory it is customary to drop the qualifier ‘homotopy’ from the terminology as only homotopy-invariant notions make sense in this context.
given two morphisms $x \to y'$ and $x \to y''$ in an $\infty$-category $\mathcal{C}$, we may define the (homotopy) pushout of the diagram\(^{21}\)

\[
\begin{array}{ccc}
  x & \longrightarrow & y' \\
  \downarrow & & \downarrow \\
  y'' & \longrightarrow & \\
\end{array}
\]

by means of a suitable universal property, that is as a (homotopy) colimit diagram of the form\(^{22}\)

\[
\begin{array}{ccc}
  x & \longrightarrow & y' \\
  \downarrow & & \downarrow \\
  y'' & \longrightarrow & z \\
\end{array}
\]

The space of all possible (homotopy) (co)limit diagrams of a fixed shape is either empty (if no (co)limit of the diagram exists in $\mathcal{C}$) or contractible\(^{23}\).

### 2.2. Stable $\infty$-categories

In the sequel we are mostly concerned with the following class of $\infty$-categories, which the reader might want to think of as refined versions of triangulated categories.

**Definition 2.1.** An $\infty$-category $\mathcal{A}$ is stable\(^{24}\) if it has the following properties:

1. The $\infty$-category $\mathcal{A}$ is pointed, that is $\mathcal{A}$ has a zero object.
2. For every morphism $f: x \to y$ in $\mathcal{A}$ there exist (coherently commutative) squares in $\mathcal{A}$ of the form

\[
\begin{array}{ccc}
  x & \longrightarrow & y \\
  \downarrow & & \downarrow \\
  0 & \longrightarrow & z \\
\end{array}
\]

and

\[
\begin{array}{ccc}
  w & \longrightarrow & x \\
  \downarrow & & \downarrow \\
  0 & \longrightarrow & y \\
\end{array}
\]

which are a (homotopy) pushout square and a (homotopy) pullback square, respectively. The objects $z$ and $w$ are called the cofibre of $f$ and the fibre of $f$, respectively\(^{25}\).

3. A diagram in $\mathcal{A}$ of the form

\[
\begin{array}{ccc}
  x & \longrightarrow & y \\
  \downarrow & & \downarrow \\
  0 & \longrightarrow & z \\
\end{array}
\]

\(^{21}\)Strictly speaking, specifying a coherently commutative diagram in an $\infty$-category involves an infinite amount of data. For expository purposes we nonetheless display such diagrams as we would display commutative diagrams in ordinary categories.

\(^{22}\)We decorate the square to indicate that it is a (homotopy) pushout square.

\(^{23}\)This means that, for the purposes of $\infty$-category theory, (co)limit cones of a fixed diagram in an $\infty$-category are unique in the appropriate sense. See Proposition 1.2.12.9 and Definition 1.2.13.4 in \cite{Lur09} for details.

\(^{24}\)Although we have not provided formal definitions of any of the notions involved, the property of being stable is sufficiently intuitive for it to be worth to be included in these notes. A detailed treatment of the theory of stable $\infty$-categories can be found in Chapter 1 in \cite{Lur17}.

\(^{25}\)The cofibre and the fibre of a morphism in a stable $\infty$-category are the $\infty$-categorical versions of the cone and the co-cone of a morphism in a triangulated category. Following the established convention, we adopt the ‘topological’ terminology which further reminds us of the homotopical nature of these concepts.
is a (homotopy) pushout square if and only if it is a (homotopy) pullback square. A diagram as above which satisfies these additional properties is called a fibre-cofibre sequence.

Remark 2.2. Stable $\infty$-categories are defined in terms of properties and not in terms of additional structure, the latter being the case for triangulated categories. Roughly speaking, this is the reason why stable $\infty$-categories enjoy better formal properties than triangulated categories.

Every $\infty$-category $\mathcal{C}$ has an associated homotopy category $\text{Ho}(\mathcal{C})$ which is in fact an ordinary category. The following result relates stable $\infty$-categories to triangulated categories, see Theorem 1.1.2.14 in [Lur17].

**Proposition 2.3.** Let $A$ be a stable $\infty$-category. Then, the homotopy category $\text{Ho}(A)$ is (canonically) a triangulated category.

Remark 2.4. The suspension $\Sigma(x)$ of an object $x$ of a stable $\infty$-category $A$ is characterised by the existence of a fibre-cofibre sequence

$$
\begin{array}{ccc}
x & \longrightarrow & 0 \\
\downarrow & & \downarrow \\
0 & \longrightarrow & \Sigma(x)
\end{array}
$$

Similarly, exact triangles in the homotopy category $\text{Ho}(A)$ are induced by diagrams in $A$ of the form

$$
\begin{array}{ccc}
x & \longrightarrow & y & \longrightarrow & 0 \\
\downarrow & \downarrow & \downarrow & & \downarrow \\
0 & \longrightarrow & z & \longrightarrow & \Sigma(x)
\end{array}
$$

in which each square is a fibre-cofibre sequence.

The good formal behaviour of stable $\infty$-categories is illustrated by the following statement, see Proposition 1.1.3.1 in [Lur17].

**Proposition 2.5.** Let $A$ be a stable $\infty$-category and $K$ a small $\infty$-category. Then, the $\infty$-category $\text{Fun}(K,A)$ of functors $K \rightarrow A$ is also stable.

### 3. The higher Waldhausen $S$-constructions

In this section we establish a link between the higher dimensional Auslander algebras of type $A$ and the higher Waldhausen $S$-constructions introduced by Dyckerhoff [Dyc17] and Poguntke [Pog17], and by Hesselholt and Madsen [HM15].

26 Fibre-cofibre sequences in stable $\infty$-categories are the $\infty$-categorical versions of exact triangles in triangulated categories.

27 Proposition 2.3 should be compared with Happel’s theorem which states that the stable category of a Frobenius exact category is a triangulated category [Hap88].

28 The decoration indicates that the square is both a (homotopy) pushout and a (homotopy) pullback. In fact, in a stable $\infty$-category these two notions coincide, see Proposition 1.1.3.4 in [Lur17].

29 Proposition 2.5 should be compared with the elementary fact that the category of functors from a small category into an abelian category is again abelian. Indeed, (homotopy) limits and (homotopy) colimits in functor $\infty$-categories are computed point-wise, see Corollary 5.1.2.3 in [Lur09].
in the case \( m = 2 \). We begin by reminding the reader of what is known in the classical situation, that is in the case \( m = 1 \).

### 3.1. The Waldhausen \( S \)-construction.

The following construction is due to Waldhausen [Wal85]. It is the main ingredient in the definition of the algebraic \( K \)-theory space of a stable \( \infty \)-category, see for example [BGT13].

**Definition 3.1.** Let \( \mathcal{C} \) be a stable \( \infty \)-category and \( n \geq 0 \). We let \( S(\mathcal{C})_n \) be the (stable) \( \infty \)-category of diagrams of the form

\[
\begin{array}{c}
X_{00} \to X_{01} \to X_{02} \to \cdots \to X_{0,n-1} \to X_{0n} \\
\downarrow \downarrow \downarrow \cdots \downarrow \\
X_{11} \to X_{12} \to \cdots \to X_{1,n-1} \to X_{1n} \\
\downarrow \cdots \downarrow \\
\cdots \cdots \\
X_{n-1,n-1} \to X_{n-1,n} \\
\downarrow \\
X_{nn}
\end{array}
\]

which satisfy the following two conditions:

- For each \( i \in [n] \) the object \( X_{ii} \) is a zero object of \( \mathcal{C} \);
- for each \( 0 \leq i < j < k \leq n \) the square

\[
\begin{array}{c}
X_{ij} \to X_{ik} \\
\downarrow \downarrow \\
X_{ii} \to X_{jk}
\end{array}
\]

is a (homotopy) pushout diagram and a (homotopy) pullback diagram.

Size issues aside, the \( \infty \)-categories \( S(\mathcal{C})_n \) assemble into a simplicial object \( S(\mathcal{C})_\bullet \), called the **Waldhausen \( S \)-construction of** \( \mathcal{C} \), which takes values in the \( \infty \)-category of stable \( \infty \)-categories and exact functors between them.

The following elementary observation can be viewed as a categorification of the isomorphism (4). Proofs can be found in Lemma 7.3 in [BGT13] and Lemma 1.2.2.4 in [Lur17]. A version in the related framework of stable derivators is proven in Theorem 4.6 in [GS16a] by means of a version of the knitting algorithm.

**Proposition 3.2 (Waldhausen).** Let \( \mathcal{C} \) be a stable \( \infty \)-category and \( n \geq 1 \). The restriction functor

\[
S(\mathcal{C})_n \longrightarrow \text{Fun}(01 \to 02 \to \cdots, 0n, \mathcal{C}),
\]

which sends an object \( X \) of \( S(\mathcal{C})_n \) to the diagram

\[
X_{01} \to X_{02} \to \cdots \to X_{0,n-1} \to X_{0n},
\]

is an equivalence of (stable) \( \infty \)-categories.
The following theorem extends the foregoing proposition to arbitrary orientations of the Dynkin diagram \( A_n \) and can be proven using combinatorial versions of the classical reflection functors. A proof, carried out in the related framework of stable derivators, can be found in [GŠ16a].

**Theorem 3.3 (Groth–Šťovíček).** Let \( \mathcal{C} \) be a stable \( \infty \)-category and \( n \geq 1 \). Let \( S \) be a slice in the poset \( \Delta(1, n) \). The restriction functor

\[
S(\mathcal{C})_\bullet \to \text{Fun}(S, \mathcal{C})
\]

is an equivalence of (stable) \( \infty \)-categories. □

Our aim in this section is to provide higher-dimensional versions of Proposition 3.2 and Theorem 3.3 in terms of certain higher-dimensional versions of the Waldhausen \( S \)-construction.

### 3.2. The higher Waldhausen \( S \)-constructions

Before proceeding we introduce further terminology. Let \( I = [1] \) be the poset \( \{0 \to 1\} \) and \( m \) a non-negative integer. An \( m \)-cube in an \( \infty \)-category \( \mathcal{C} \) is a functor \( X : I^m \to \mathcal{C} \). The isomorphism

\[
I^{m+1} \cong I \times I^m
\]

together with the adjunction

\[
\text{Fun}(I \times I^m, \mathcal{C}) \cong \text{Fun}(I, \text{Fun}(I^m, \mathcal{C}))
\]

allow us to view an \( (m + 1) \)-cube in \( \mathcal{C} \) as morphism in the \( \infty \)-category \( \text{Fun}(I^m, \mathcal{C}) \) of \( m \)-cubes in \( \mathcal{C} \). In the case of a stable \( \infty \)-category, this identification allows for an inductive treatment of hyper-cubes as illustrated by the following definition.

**Definition 3.4.** We say that a 0-cube in a stable \( \infty \)-category \( \mathcal{A} \), which is nothing but an object of \( \mathcal{A} \), is biCartesian if it is a zero object of \( \mathcal{A} \). Inductively, we say that an \( (m + 1) \)-cube \( X \) in a stable \( \infty \)-category \( \mathcal{A} \) is biCartesian if its cofibre (taken in the stable \( \infty \)-category \( \text{Fun}(I^m, \mathcal{A}) \)) is a biCartesian \( m \)-cube in \( \mathcal{A} \). For example, a 1-cube in \( \mathcal{A} \) is biCartesian if its underlying morphism is an equivalence. □

We are now ready to state the definition of the \( m \)-dimensional Waldhausen \( S \)-construction of a stable \( \infty \)-category [HM15, Dyc17, Pog17].

**Definition 3.5.** Let \( \mathcal{A} \) be a stable \( \infty \)-category and \( m \) a non-negative integer. For \( n \geq 0 \) we denote by \( S^{(m)}(\mathcal{A})_n \) the full subcategory of \( \text{Fun}(\Delta(m, n), \mathcal{A}) \) spanned by the diagrams \( X \) satisfying the following two conditions:

- For every degenerate \( m \)-simplex \( \sigma \) in \( \Delta^n \) the object \( X_\sigma \) is a zero object of \( \mathcal{A} \);
- for each non-degenerate \( (m + 1) \)-simplex in \( \Delta^n \) consider the \( (m + 1) \)-cube

\[
q : I^{m+1} \to \Delta(m, n)
\]

given by \( q(v)_i = \sigma_{i+v} \) for each \( v \in I^{m+1} \) and \( i \in [m] \). Then, the induced \( (m + 1) \)-cube

\[
X \circ q : I^{m+1} \to \mathcal{A}
\]

---

33 As left implicit above, the Hasse quiver of \( \Delta(1, n) \) can be identified with the Auslander–Reiten quiver of the quiver \( \Lambda_n \) (with additional degenerate vertices). Slices in \( \Delta(1, n) \) can then be defined in the usual way.

34 The notion of a biCartesian hyper-cube can be defined directly as a certain (homotopy) colimit diagram, see Proposition 1.2.4.13 and Lemma 1.2.4.5 in [Lur17].

35 Equivalences are the \( \infty \)-categorical analogues of isomorphisms in ordinary category theory.
is biCartesian\[36\].

Size issues aside, the (stable) $\infty$-categories $S^{(m)}(A)_{n}$, $n \geq 0$, assemble into a simplicial object $S^{(m)}(A)$, called the $m$-dimensional Waldhausen $S$-construction of $A$, which takes values in the $\infty$-category of stable $\infty$-categories and exact functors between them.

**Remark 3.6.** The second exactness conditions appearing in the definition of the $m$-dimensional Waldhausen $S$-construction are equivalent to the condition that the $(m + 1)$-cubes of the form $X \circ q$ are biCartesian, where

$$q(v)_i = \sigma_i + v_i, \quad i \in [m]$$

and $\sigma$ ranges over all non-degenerate $m$-simplices in $\Delta^n$ with $\sigma_m < n$. For example, for $m = 2$ these are the cubes

\[
\begin{array}{ccc}
X_{ijk} & \rightarrow & X_{ij,k+1} \\
X_{i+1,j,k} & \rightarrow & X_{i+1,j,k+1} \\
X_{i+1,j+1} & \rightarrow & X_{i+1,j+1,k+1}
\end{array}
\]

where $0 \leq i < j < k < n$. These conditions should be thought of as a categorification of the higher Auslander–Reiten relations \[36\].

**Example 3.7.** Let $A$ be a stable $\infty$-category, $m = 2$, and $n = 4$. After discarding redundant information pertaining additional zero objects, an object of $S^{(2)}(A)_4$ can be identified with a diagram of the form

\[
\begin{array}{cccc}
& X_{012} & \rightarrow & X_{013} & \rightarrow & X_{014} \\
0 & \downarrow & 0 & \downarrow & 0 & \downarrow \\
& \rightarrow & X_{023} & \rightarrow & X_{024} \\
0 & \downarrow & \rightarrow & X_{123} & \rightarrow & X_{124} \\
& \downarrow & \rightarrow & X_{134} & \rightarrow & X_{234} \\
& \downarrow & \rightarrow & \rightarrow & \rightarrow
\end{array}
\]

(8)

\[36\]These exactness conditions should be thought of as categorifications of the higher-dimensional Euler relations \[6\]. For example, for $m = 2$ the cubes which are required to be biCartesian are those of the form

\[
\begin{array}{ccc}
X_{ijk} & \rightarrow & X_{ijl} \\
X_{jik} & \rightarrow & X_{jlk} \\
X_{ikl} & \rightarrow & X_{jkl}
\end{array}
\]

where $0 \leq i < j < k < l \leq n$; these cubes are $\infty$-categorical analogues of 4-term exact sequences.
in which all ‘unit cubes’ are biCartesian. A standard application of the theory of Kan extensions in ∞-categories shows that such a diagram is determined by its restriction to the (coherently commutative) diagram

\[
\begin{align*}
X_{012} &\rightarrow X_{013} &\rightarrow X_{014} \\
\downarrow & & \downarrow & \downarrow \\
0 &\rightarrow X_{023} &\rightarrow X_{024} \\
\downarrow & & \downarrow & \\
0 &\rightarrow X_{034}
\end{align*}
\]

The latter diagram can be thought of as a (coherent) representation of the Auslander algebra \( \mathcal{A}_3^{(2)} \) of the quiver \( \mathcal{A}_3 \) in the stable ∞-category \( \mathcal{A} \). It is also worth noting that diagram \( \mathcal{A}_3^{(2)} \) agrees with the ‘higher-dimensional Auslander–Reiten quiver’ of the unique cluster-tilting subcategory associated with \( \mathcal{A}_3^{(2)} \), see Section 6 in \[Iya11\] or \[OT12\]. □

The following observation relates the \( m \)-dimensional Waldhausen \( S \)-construction to the \( m \)-dimensional Auslander algebras of type \( \mathcal{A} \); it can be viewed as a higher-dimensional analogue of Proposition 3.2. It can be proven either constructively, by means of a higher-dimensional version of the knitting algorithm, or by means of the point-wise formulas for ∞-categorical Kan extensions. A proof can be found in Proposition 2.10 in \[DJW19b\].

**Proposition 3.8 (Dyckerhoff–J–Walde).** Let \( \mathcal{A} \) be a stable ∞-category and \( n \geq m \geq 1 \). The restriction functor

\[
S_{m}(\mathcal{A})_m \longrightarrow \text{Fun}_*(P(m,n), \mathcal{A})
\]

is an equivalence of (stable) ∞-categories, where

\[
P(m,n) := \{ \sigma \in \Delta(m,n) | \sigma_0 = 0 \}
\]

and \( \text{Fun}_*(P(m,n), \mathcal{A}) \) is the full subcategory of \( \text{Fun}(P(m,n), \mathcal{A}) \) spanned by those functors which send degenerate \( m \)-simplices in \( \Delta^n \) to zero objects in \( \mathcal{A} \). □

**Remark 3.9.** After discarding further redundant information, the poset \( P(m,n) \) appearing in \[Proposition 3.8\] can be replaced by a smaller poset which models perfectly the quiver with relations of the higher Auslander algebra \( \mathcal{A}_m^{(m)} \). This fact is perhaps more transparent from the description of Iyama’s higher Auslander algebras of type \( \mathcal{A} \) given in Section 2 in \[JK16\]. □

As explained in \[IO11\], the following theorem can be regarded as a higher-dimensional version of \[Theorem 3.3\]. A proof can be found in Theorem 2.41 in \[DJW19b\].

---

\[\text{This approach is analogous to the proof of Theorem 4.6 in} \text{ GS16a. The aforementioned knitting algorithm can be deduced, for example, from the proof of Theorem 5.27 in} \text{ IO11.}\]

\[\text{This approach is analogous to the proof of} \text{ Proposition 3.2 given for example in Lemma 1.2.2.4 in} \text{ Lur17.}\]
Theorem 3.10 (Dyckerhoff–J–Walde). Let $A$ be a stable $\infty$-category and $n \geq m \geq 1$. Let $S$ be a slice in the poset $\Delta(m, n)$. The restriction functor

$S^{(m)}(A)_n \to \text{Fun}_*(S, A)$

is an equivalence of (stable) $\infty$-categories, where $S$ denotes the convex hull of $S$ in the poset $\Delta(m, n)$. \hfill \Box

Remark 3.11. The proof of Theorem 3.10 relies on combinatorial versions of the derived equivalences induced by higher-dimensional reflection functors in the sense of [IO11]; these reflection functors rely on the operation of slice mutation also introduced in [IO11]. In more detail, if $S$ and $S'$ are slices in $\Delta(m, n)$ which are mutation of each other, then there exists a slightly larger poset $S \odot S'$ containing both $S$ and $S'$ as well as a distinguished $(m+1)$-cube. This larger poset allows us to realise the aforementioned reflection functors by means of equivalences of stable $\infty$-categories

$\text{Fun}_*(S, A) \xrightarrow{\sim} \text{Fun}^{ex}_*(S \odot S', A) \xrightarrow{\sim} \text{Fun}_*(S', A)$

induced by the restriction functors, where the objects of the stable $\infty$-category $\text{Fun}^{ex}_*(S \odot S', A)$ satisfy the additional requirement that their restriction along the distinguished $(m+1)$-cube in $S \odot S'$ is a biCartesian $(m+1)$-cube in $A$, see Figure 1. The proof of Theorem 3.10 is obtained by combining these equivalences with Proposition 3.8 and the transitivity of the mutation operation on slices proven in [IO11]. See [DJW19b] for details. \hfill \Box

3.3. Recollements. We make the following general observation, which makes the inductive nature of the $m$-dimensional Waldhausen $S$-construction of $A$ readily apparent. Indeed, Proposition 3.12 is reminiscent of Iyama’s inductive description of the higher Auslander algebras of type $A$ by means of cones of translation quivers, see Section 6 in [Iya11]. A proof can be found in Proposition 2.51 in [DJW19b].

Proposition 3.12 (Dyckerhoff–J–Walde). Let $A$ be a stable $\infty$-category and $n \geq m \geq 1$ integers. For each $i \in \{n\}$ the functor $s_i: S^{(m)}(A)_n \to S^{(m)}(A)_{n+1}$ is

The notion of a ‘slice’ is analogous to that introduced in [IO11].

Combinatorial versions of classical reflection functors are investigated in [GS16a] in the case of quivers of type $A$ and in [GS16b] for general trees in the related framework of stable derivators. A further generalisation, corresponding to the generalised reflection functors of Ladkani [Lad08], was obtained in [DJW19a].

Figure 1. An example of slice mutation.
part of a recollement of stable $\infty$-categories

$$
S^{(m)}(A)_n \leftrightarrow d_i \leftrightarrow S^{(m)}(A)_{n+1} \leftrightarrow d_{i+1} \leftrightarrow S^{(m-1)}(A)_n.
$$

In particular, the sequence of adjunctions

$$
d_0 \dashv s_0 \dashv d_1 \dashv s_1 \dashv \cdots \dashv s_n \dashv d_{n+1}
$$

is part of a ladder of recollements in the sense of [BBD82, AHKLY17]. □

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