Hamiltonian analysis of Poincaré gauge theory
scalar modes

Hwei-Jang Yo* and James M. Nester†

Department of Physics and Center for Complex Systems,
National Central University, Chungli 320, Taiwan, ROC

PACS: 04.20. Fy; 04.50 +h

Short title: Hamiltonian analysis of PGT scalar modes

March 24, 2022

Abstract

The Hamiltonian constraint formalism is used to obtain the first explicit complete analysis of non-trivial viable dynamic modes for the Poincaré gauge theory of gravity. Two modes with propagating spin-zero torsion are analyzed. The explicit form of the Hamiltonian is presented. All constraints are obtained and classified. The Lagrange multipliers are derived. It is shown that a massive spin-0− mode has normal dynamical propagation but the associated massless 0− is pure gauge. The spin-0+ mode investigated here is also viable in general. Both modes exhibit a simple type of “constraint bifurcation” for certain special field/parameter values.

*Electronic address: s2234003@twncu865.ncu.edu.tw
†Electronic address: nester@joule.phy.ncu.edu.tw
1 Introduction

The Poincaré gauge theory of gravity (PGT), based on a Riemann-Cartan geometry, allows for dynamic torsion in addition to curvature. Because of its gauge structure and geometric properties it was regarded as an attractive alternative to general relativity (GR). It was quickly realized that the theory had physical difficulties with generic values for its ten coupling parameters (see, e.g., references). Consequently investigators have looked for restrictions giving viable sets of PGT parameters. By utilizing certain theoretical tests (e.g., “no-ghosts” or “no-tachyons”), sets of constraints on the parameters and possible viable PGT modes were obtained (see, e.g., references). These investigations naturally used the weak-field approximation and linearization of the theory, avoiding the inherently highly nonlinear complications in the PGT.

Subsequently, investigations indicated that certain degeneracies of the Hessian matrix, thought to be necessary for a viable theory, appeared to lead to difficulties with the initial value problem. Then a shock-wave analysis concluded that these same degeneracies allowed tachyonic propagating modes. These difficulties were more recently reconsidered. It was found that, when all the constraints are taken into account, there are no problems — at the linear order.

However, the PGT Hamiltonian analysis tells a more complicated story. The virtue of the Hamiltonian analysis is that it provides a clear vision of the possible degeneracies, corresponding constraints, and true degrees of freedom of a theory. Its application to the nonlinear PGT is revealing. It shows that the nonlinear behavior of the PGT can be — and, through a phenomenon referred to as “constraint bifurcation”, very likely will be — qualitatively different from the linearized one in the number and type of constraints, so the linearized “good modes” may very well not be viable in the full nonlinear theory. Hence, in order to understand the subtle behavior of the PGT and search for modes which truly have good propagation, it is...
important that the analysis consider the full nonlinear scope of the theory.

It has long been known that non-linear couplings involving higher spins are problematical. However, spin-zero modes were expected to be problem free. For the PGT, it has subsequently been verified that two four-parameter subclasses with only spin-zero propagating modes really do have a well posed initial value problem without any tachyonic propagation.

That leads us to examine the Hamiltonian formalism for some representative dynamic spin-zero PGT modes in order to see how (and indeed whether) the modes really manage to avoid the nearly ubiquitous (and almost certainly fatal) nonlinear “constraint bifurcation”.

Such an investigation seems to be an essential prelude to a search for any higher spin nonlinear-problem-free modes in the PGT. Our results presented here gives, to our knowledge, the only complete Hamiltonian analysis of PGT modes believed to be (linearly and nonlinearly) viable.

Also, it may be worthwhile to mention, certain interesting results which were found in studies of higher-derivative gravity (see, e.g., references). These theories contain a non-ghost scalar field, in addition to the usual graviton of GR. Some pure gravity inflationary models for the Universe were proposed based on such theories, although the issue still remains controversial. This has provided additional motivation for looking for similar situations in the PGT, but it must be kept in mind that the principle and structure between the PGT and higher-derivative gravity are quite different.

The paper is organized as follows. In section 2 we review the basic elements of the PGT and introduce its Lagrangian and field equations. In section 3 we use the Dirac theory for constrained Hamiltonian systems in the excellent “if” constraint formulation developed by Blagojević and Nikolić. The primary constraints, including ten “sure” primary constraints and thirty so-called “primary if-constraints”, are found. The total Hamiltonian density, including the canonical Hamiltonian den-
sity and all possible primary constraints, is derived. In section 4 we consider two very degenerate spin-zero modes, each having only one of the six parameters of the quadratic curvature parts being non-zero. In particular, we consider a spin-$0^+$, a massive spin-$0^-$ and the associated massless spin-$0^-$ modes. The Lagrange multipliers are derived. It is shown that the massless spin-$0^-$ mode is unphysical, being pure gauge. In section 5 the degrees of freedom are counted. The multipliers are shown to be exactly the missing ‘velocities’. Their effects in the spin-zero modes are discussed. In both scalar modes we find a simple type of “constraint bifurcation” phenomenon, which is connected with changes in the nature of the constraint reduced from the Lorentz rotation parts of the canonical Hamiltonian density. The similarity between the spin-$0^+$ case and higher-derivative gravity is noted. In the final section we present our conclusions.

Throughout the paper our PGT conventions are basically the same as Hehl’s. We have made a few adjustments to accommodate the translation of the Hamiltonian “if” constraint formalism to these conventions. The latin indices are coordinate (holonomic) indices, whereas the greek indices are orthonormal frame (anholonomic) indices. The first letters of both alphabets ($a, b, c, \ldots; \alpha, \beta, \gamma, \ldots$) run over 1, 2, 3, whereas the later ones run over 0, 1, 2, 3. Furthermore, $\eta_{\mu\nu} = \text{diag}(-, +, +, +)$; $\epsilon^{\mu\nu\gamma\delta}$ is the completely antisymmetric tensor with $\epsilon^{0123} = -1$. The meaning of a bar over a greek index is adopted from Blagojević and Nickolić.

2 Poincaré gauge theory of gravitation

In the PGT there are two sets of gauge potentials, the orthonormal frame field (tetrads) $e_i^\mu$ and the metric-compatible connection $\Gamma_{i\mu}^\nu$, which are associated with the translation and the Lorentz subgroups of the Poincaré gauge group, respectively. The associated field strengths are the torsion

$$T_{ij}^\mu = 2(\partial_i e_j)^\mu + \Gamma_{[i\mu}^\nu e_{j]}^\nu),$$  \quad (2.1)
and the curvature
\[ R_{ij}{}^{\mu}{}_{\nu} = 2(\partial_{[i} \Gamma_{j]}{}^{\nu}_{\mu} + \Gamma_{[i}{}^{\nu}_{\sigma} \Gamma_{j]}{}^{\sigma}_{\mu}), \]
which satisfy the Bianchi identities
\[ \nabla_{[i} T_{jk]}{}^{\mu} \equiv R_{[ijk]}{}^{\mu}, \]
\[ \nabla_{[i} R_{jk]}{}^{\mu\nu} \equiv 0. \]

The conventional form of the action, which is invariant under the Poincaré gauge group, has the form
\[ A = \int d^4x e (L_M + L_G), \]
where \( L_M \) stands for the matter Lagrangian density (which determines the energy-momentum and spin source currents), \( L_G \) denotes the gravitational Lagrangian density, and \( e = \text{det}(e_i{}^\mu) \). In this paper we are concerned with the gravitational propagating modes, hence we omit the matter Lagrangian density, so \( L_G \) is considered as the source-free total Lagrangian. Varying with respect to the potentials then gives the (vacuum) field equations,
\[ \nabla_j P_{\mu}{}^{ij} - \epsilon_{\mu}{}^{i} = 0, \]
\[ \nabla_j P_{\mu\nu}{}^{ij} - \epsilon_{\mu\nu}{}^{i} = 0, \]
with the field momenta
\[ P_{\mu}{}^{ij} := \frac{\partial eL_G}{\partial \partial_j e_i{}^{\mu}} = 2 \frac{\partial eL_G}{\partial T_{ji}{}^{\mu}}, \]
\[ P_{\mu\nu}{}^{ij} := \frac{\partial eL_G}{\partial \partial_j \Gamma_i{}^{\mu\nu}} = 2 \frac{\partial eL_G}{\partial R_{ji}{}^{\mu\nu}}, \]
and
\[ \epsilon_{\mu}{}^{i} := e_i{}^{\mu} eL_G - T_{\mu j}{}^{\nu} P_{\nu}{}^{ji} - R_{\mu j}{}^{\nu\sigma} P_{\nu\sigma}{}^{ji}, \]
\[ \epsilon_{\mu\nu}{}^{i} := P_{[\mu\nu]}{}^{i}. \]
The Lagrangian is chosen (as usual) to be at most of quadratic order in the field strengths, then the field momenta are linear in the field strengths:

\[ P_{\mu}^{ij} = \frac{e}{l^2} \sum_{k=1}^{3} a_k (k) T_{ji}^{\mu}, \]

\[ P_{\mu\nu}^{ij} = -\frac{a_0 e}{l^2} e^i_{[\mu} e^j_{\nu]} + \frac{e}{\kappa} \sum_{k=1}^{6} b_k (k) R_{ji}^{\mu\nu}, \]

the three \((k) T_{ji}^{\mu}\) and the six \((k) R_{ji}^{\mu\nu}\) are the algebraically irreducible parts of the torsion and the curvature, respectively. The reciprocal frames \(e^i_{\mu}\) and \(e^i_{\mu}\) satisfy \(e^i_{\mu} e^j_{\nu} = \delta_{ij}\) and \(e^i_{\mu} e^j_{\mu} = \delta_{ij}\); the coordinate metric is defined by \(g_{ij} = e^i_{\mu} e^j_{\nu} \eta_{\mu\nu}\). The \(a_k\) and \(b_k\) are free coupling parameters. Due to the Bach-Lanczos identity only five of the six \(b_k\)'s are independent. \(a_0\) is the coupling parameter of the scalar curvature \(R := R_{\mu\nu} e^{\mu\nu}\). For the Hamiltonian formulation we associate the canonical momenta with certain components of the covariant field momenta:

\[ \pi^i_{\mu} \equiv P_{\mu}^{i0}, \]

\[ \pi_{\mu\nu}^{i} \equiv P_{\mu\nu}^{i0}. \]

3 Primary constraints and total Hamiltonian

In this section we present the primary constraints and the total Hamiltonian density of the PGT in terms of the decomposition of the canonical variables and fields. First of all, one obtains the “sure” primary constraints

\[ \pi^0_{\mu} \approx 0, \]

\[ \pi^0_{\mu\nu} \approx 0. \]

These constraints just reflect the fact that the torsion and the curvature are defined as the antisymmetric derivatives of \(e_i^{\mu}\) and \(\Gamma_i^{\mu\nu}\); they do not involve the “velocities” \(\dot{e}_0^{\mu}\) and \(\dot{\Gamma}_0^{\mu\nu}\). One will obtain further primary constraints if the Lagrangian density is singular with respect to the remaining “velocities”, \(\dot{e}_a^{\mu}\), and \(\dot{\Gamma}_a^{\mu\nu}\). (Such constraints,
so-called “primary if-constraints”, result from certain vanishing coupling parameter combinations.) The total Hamiltonian density is of the form

\[ H_{\text{tot}} = H_{\text{can}} + u_0^\mu \pi^0_{\mu} + \frac{1}{2} u_0^{\mu\nu} \pi^0_{\mu\nu} + u^A \phi_A, \]  

(3.3)

where the \( \phi_A \) are the primary “if” constraints and the \( u \)’s denote the associated Lagrange multipliers; \( H_{\text{can}} \) stands for the canonical Hamiltonian density which will be specified below.

Before we proceed to obtain the explicit form of the canonical Hamiltonian density and \( \phi_A \), it is convenient to define the decomposition of related variables and functions. We essentially follow the techniques developed by Blagojević and Nikolić. Let us note that the components of the unit normal \( n \) to the \( x^0 \) =constant hypersurface, with respect to the orthonormal frame, are given by

\[ n_\mu := \frac{-e^0_\mu}{\sqrt{-g^{00}}}. \]  

(3.4)

A vector, e.g., \( V_\mu \), can be decomposed naturally into the orthogonal and parallel components with respect to the orthonormal frame indices:

\[ V_\mu = -V_\perp n_\mu + V_\parallel, \]  

\[ V_\perp \equiv V_\mu n^\mu, \]  

\[ V_\parallel \equiv V_\nu (\delta_\mu^\nu + n_\mu n^\nu). \]  

(3.5, 3.6, 3.7)

One can easily extend the decomposition to any tensors with orthonormal frame indices. The lapse and shift functions can be written as

\[ N \equiv \frac{1}{\sqrt{-g^{00}}} = -n_\mu e^0_\mu, \]  

(3.8)

\[ N^a \equiv -\frac{g^{0a}}{g^{00}} = e^0_\mu e^a_\mu, \]  

(3.9)

and \( e = NJ \), where \( J \) is the determinant of the 3-metric. Defining the convenient “parallel” canonical momenta

\[ \pi^{\parallel}_\mu \equiv e_\sigma^a \pi^a_\mu, \quad \pi^{\parallel}_{\mu\nu} \equiv e_\sigma^a \pi^a_{\mu\nu}, \]  

(3.10)
which satisfy \( \pi^\mu n_\sigma = 0, \pi^\mu_\nu n_\sigma = 0 \), the canonical Hamiltonian density,

\[
H_{\text{can}} = \pi^a_\mu \dot{e}^a_\mu + \frac{1}{2} \pi^a_\mu_\nu \dot{\Gamma}^a_\mu_\nu - eL,
\]

(3.11)

can be rewritten in the so-called Dirac-ADM form \( ^{27,28,29} \),

\[
H_{\text{can}} = N H_\perp + N^a H_a + \frac{1}{2} \Gamma^a_0_\mu_\nu H_{\mu\nu} + \partial_a D^a,
\]

(3.12)

which is linear in \( N \) and \( N^a \). The other quantities are given by

\[
H_\perp = \pi^\mu_\perp R_\perp^\mu - JL - n^\mu \nabla_a \pi^a_\mu,
\]

(3.13)

\[
H_a = \pi^b_\mu T_{ab}^\mu + \frac{1}{2} \pi^b_\mu_\nu R_{ab}^\mu_\nu - e_a^\mu \nabla_b \pi^b_\mu,
\]

(3.14)

\[
H_{\mu\nu} = \pi^a_\mu e_{a\nu} - \pi^a_\nu e_{a\mu} - \nabla_a \pi^a_\mu_\nu,
\]

(3.15)

\[
D^a = \pi^a_\mu e_0^\mu + \frac{1}{2} \pi^a_\mu_\nu \Gamma^a_0_\mu_\nu.
\]

(3.16)

Only the super-Hamiltonian \( H_\perp \) is involved in dynamical evolution, the super-momenta \( H_a \) and Lorentz rotation parts \( H_{\mu\nu} \) are kinematical generators, consequently we concentrate on \( H_\perp \) when consistency conditions are calculated. By utilizing the forms of the total and canonical Hamiltonian density \( (3.3) \) and \( (3.12) \) in the consistency conditions for the primary constraints \( (3.1) \) and \( (3.2) \) we obtain the secondary constraints (SC)

\[
H_\perp \approx 0, \quad H_a \approx 0, \quad H_{\mu\nu} \approx 0.
\]

(3.17)

Since it is easy to check that the primary constraints \( (3.1) \) and \( (3.2) \) are first-class, i.e., \( \pi^0_\mu \) and \( \pi^0_\mu_\nu \) are unphysical variables, by using the Hamilton equation of motion we can infer that the multipliers \( u_0^\mu \) and \( u_0^\mu_\nu \) are indeed equal to \( \dot{e}_0^\mu \) and \( \dot{\Gamma}_0^\mu_\nu \). They are dynamically undetermined pure gauge multipliers.

It is necessary to understand the relation between the canonical momenta and the velocities well before one can make \( H_\perp \) more apparent. Let us consider the torsion momenta \( \pi_\perp_\nu \) at first. \( \pi_\perp_\nu \) can be decomposed into four irreducible parts as
follows:

$$
\pi_{\mu\nu} = -n_\nu \pi_{\mu\perp} + \pi_{\mu\perp}
$$

$$
= -n_\nu \pi_{\mu\perp} + \hat{\pi}_{\mu\perp} + \tilde{\pi}_{\mu\perp} + \frac{1}{3} \eta_{\mu\perp\nu},
$$

(3.18)

where $\hat{\pi}_{\mu\perp}$, $\pi$ and $\tilde{\pi}_{\mu\perp}$ are the antisymmetric part, trace part and symmetric-traceless part of $\pi_{\mu\perp}$, respectively. Manipulating the definition of the torsion momenta (2.14), one finds the following relations between the different parts of the canonical momenta and the corresponding parts of the velocities $T_{\perp\mu\nu}$:

$$
\phi_{\mu\perp} \equiv \frac{\pi_{\mu\perp}}{J} = -\frac{1}{3l^2} (a_1 - a_2) \tilde{T}_{\mu\perp} = -\frac{1}{3l^2} (2a_1 + a_2) T_{\perp\mu\perp},
$$

(3.19a)

$$
\hat{\phi}_{\mu\perp} \equiv \frac{\hat{\pi}_{\mu\perp}}{J} = -\frac{1}{3l^2} (a_1 - a_3) T_{\perp\mu\perp} = -\frac{1}{3l^2} (a_1 + 2a_3) T_{\perp\mu\perp},
$$

(3.19b)

$$
\tilde{\phi}_{\mu\perp} \equiv \frac{\tilde{\pi}_{\mu\perp}}{J} = -\frac{a_1}{l^2} T_{\perp\mu\perp},
$$

(3.19c)

$$
\phi \equiv \frac{\pi}{J} = -\frac{a_2}{l^2} T_{\perp\mu\perp},
$$

(3.19d)

where $\tilde{T}_{\mu\perp} \equiv T_{\mu\perp\nu}$, and a tensor with two indices contained in the bracket $\langle \rangle$ denotes that the tensor is symmetric-traceless with respect to the two indices. If the parameters take on the critical values: $2a_1 + a_2 = 0$, $a_1 + 2a_3 = 0$, $a_1 = 0$, and/or $a_2 = 0$, one obtains the following "primary if-constraints" (PIC): $\phi_{\mu\perp} \approx 0$, $\hat{\phi}_{\mu\perp} \approx 0$, $\tilde{\phi}_{\mu\perp} \approx 0$, and/or $\phi \approx 0$, respectively.

The curvature momenta $\pi_{\sigma\mu\nu}$ can be decomposed into six irreducible parts:

$$
\pi_{\sigma\mu\nu} = \pi_{\sigma\mu\nu} + 2\pi_{\perp[\sigma n_l]},
$$

(3.20)

and

$$
\pi_{\mu\perp} = \hat{\pi}_{\mu\perp} + \tilde{\pi}_{\mu\perp} + \frac{1}{3} \eta_{\mu\perp\nu},
$$

(3.21)

$$
\pi_{\sigma\mu\nu} = -\frac{1}{6} \epsilon_{\sigma\mu\nu\perp} \pi_{\perp} + \hat{\pi}_{\perp[\sigma n_l]} + \frac{4}{3} \eta_{\sigma\mu\nu},
$$

(3.22)

where the notation is as follows: For a spatial tensor $X_{\mu\nu\sigma\perp} = X_{\sigma\mu\nu\perp}$, $^{p}X \equiv \epsilon_{\sigma\mu\nu\perp} X_{\sigma\mu\nu\perp}$, $^{\perp}X_{\perp} \equiv X_{\mu\perp\sigma\nu}$, and $^{*}X_{\mu\perp\sigma\nu} \equiv X_{(\sigma\nu\perp\mu)} - \frac{1}{2} X_{\mu\perp\sigma\nu} + \frac{1}{2} \eta_{\mu\perp[\sigma} X_{\nu]}$ are the pseudoscalar part,
vector part, and traceless tensor part of \( X_{\mu\nu\rho} \), respectively. Identifying the irreducible parts of the curvature momenta (2.13), one finds

\[
\begin{align*}
\rho_{\phi} &\equiv \frac{\rho_{\Pi}}{J} + \frac{1}{\kappa}(b_2 - b_3)p_{R_{\perp 0}} = - \frac{1}{\kappa}(b_2 + b_3)p_{R_{\perp 0}}, \\
\tilde{\phi}_{\Pi} &\equiv \frac{\tilde{\rho}_{\Pi}}{J} - \frac{1}{\kappa}(b_4 - b_5)R_{\perp 0} = \frac{1}{\kappa}(b_4 + b_5)R_{\perp 0}, \\
s_{\phi_{\mu\nu\rho}} &\equiv \frac{s_{\rho_{\Pi\mu\nu\rho}}}{J} - \frac{1}{\kappa}(b_1 - b_2)s_{R_{\parallel \mu\nu\sigma\pi\rho}} = - \frac{1}{\kappa}(b_1 + b_2)s_{R_{\parallel \mu\nu\sigma\pi\rho}}, \\
\phi_{\perp} &\equiv \frac{\phi_{\perp}}{J} + \frac{3a_0}{l^2} + \frac{1}{2\kappa}(b_4 - b_6)R = - \frac{1}{\kappa}(b_4 + b_6)R_{\parallel}, \\
\tilde{\phi}_{\mu\nu\rho} &\equiv \frac{\tilde{\phi}_{\mu\nu\rho}}{J} + \frac{1}{\kappa}(b_1 - b_5)R_{\parallel \mu\nu\rho} = - \frac{1}{\kappa}(b_1 + b_5)R_{\parallel \mu\nu\rho}, \\
\tilde{\phi}_{\mu\nu\rho} &\equiv \frac{\tilde{\phi}_{\mu\nu\rho}}{J} + \frac{1}{\kappa}(b_1 - b_1)R_{\parallel \mu\nu\rho} = - \frac{1}{\kappa}(b_1 + b_1)R_{\parallel \mu\nu\rho},
\end{align*}
\]

where \( p_{R_{\perp 0}} := \epsilon_{\mu\nu\sigma\pi\rho}R_{\parallel \mu\nu\sigma\pi\rho}, p_{R_{\perp 0}} := \epsilon_{\mu\nu\sigma\pi\rho}R_{\parallel \mu\nu\sigma\pi\rho}, R_{\parallel} := R_{\parallel \mu\nu\sigma\pi\rho}, R_{\parallel} := R_{\parallel \mu\nu\sigma\pi\rho} \), and \( R := R_{\parallel \mu\nu\sigma\pi\rho} \). By a similar argument as used above, for various degenerate parameter combinations one can obtain any of the six expressions of (3.23a-f) as PIC’s. The relations between the critical parameter combinations and the constraints are summarized in table 1.

In order to treat all such possibilities in a concise way, the singular function

\[
\frac{\lambda(x)}{x} = \begin{cases} 
1/x, & x \neq 0, \\
0, & x = 0,
\end{cases}
\]

\[
\text{(3.24)}
\]

was introduced. The PIC’s in the total Hamiltonian density (3.3) can be given in the form

\[
u^A \phi_A = (u \cdot \phi)^T + (u \cdot \phi)^R,
\]

\[
\text{(3.25)}
\]

where

\[
(u \cdot \phi)^T \equiv [1 - \lambda(2a_1 + a_2)]u^{\perp \phi_{\perp}} + [1 - \lambda(a_1)]u^{\perp \phi_{\perp}} + [1 - \lambda(a_1 + 2a_3)]u^{\perp \phi_{\perp}} + \frac{1}{3}[1 - \lambda(a_2)]u\phi
\]

\[
\text{(3.25a)}
\]

and

\[
(u \cdot \phi)^R \equiv \frac{1}{6}[1 - \lambda(b_2 + b_3)]p_{R_{\perp 0}}^R + \frac{4}{3}[1 - \lambda(b_1 + b_2)]u^{\perp \phi_{\perp}}
\]

\[
\text{(3.25b)}
\]
\[+[1 - \lambda(b_4 + b_5)]u^\mu \phi_{\mu \nabla} + 2[1 - \lambda(b_2 + b_5)]u^\mu \phi_{\mu \nabla} + 2(1 - \lambda(b_1 + b_4))u^\mu \phi_{\mu \nabla} + \frac{2}{3}(1 - \lambda(b_1 + b_6))u^\mu \phi_{\mu \nabla}. \quad (3.25b)\]

The super-Hamiltonian \( H_\perp \) in \( H_{\text{can}} \) (3.12) then turns out to be of the form

\[
H_\perp = H_\perp^T + H_\perp^R, \tag{3.26}
\]

with

\[
H_\perp^T = -\frac{1}{2} Jl^2 \left[ \frac{3\lambda(2a_1 + a_2)}{2a_1 + a_2} \phi_{\mu \nabla} \phi_{\mu \nabla} + \frac{3\lambda(a_1 + 2a_3)}{a_1 + 2a_3} \phi_{\mu \nabla} \phi_{\mu \nabla} \right.
\]

\[
+ \frac{\lambda(a_1)}{a_1} \phi_{\mu \nabla} \phi_{\mu \nabla} + \frac{\lambda(a_2)}{3a_2} \phi^2 - J \phi^2_{\mu \nabla} - \frac{n^\mu}{\nabla a^\alpha}, \tag{3.26a}
\]

\[
H_\perp^R = -J\kappa \left[ \frac{\lambda(b_1 + b_2)}{24(b_1 + b_2)} \phi^2 + \frac{\lambda(b_1 + b_5)}{4(b_1 + b_5)} \phi_{\mu \nabla} \phi_{\mu \nabla} \right.
\]

\[
+ \frac{\lambda(b_1 + b_2)}{3(b_1 + b_2)} \phi_{\mu \nabla} \phi_{\mu \nabla} + \frac{\lambda(b_2 + b_5)}{2(b_2 + b_5)} \phi_{\mu \nabla} \phi_{\mu \nabla} \right.
\]

\[
+ \frac{\lambda(b_1 + b_4)}{2(b_1 + b_4)} \phi_{\mu \nabla} \phi_{\mu \nabla} + \frac{\lambda(b_4 + b_6)}{6(b_4 + b_6)} \phi_{\mu \nabla} \phi_{\mu \nabla} \right] - J \phi_{\mu \nabla} \phi_{\mu \nabla}, \tag{3.26b}
\]

where

\[
L^T = \frac{1}{12l^2} \left[ (2a_1 + a_3)T_{\mu \nu \rho \sigma}T_{\mu \nu \rho \sigma} \right.
\]

\[
+ 2(a_1 - a_3)T_{\mu \nu \rho \sigma}T_{\mu \nu \rho \sigma} - 2(a_1 - a_2)T_{\mu \nabla \phi_{\mu \nabla}} \right], \tag{3.26c}
\]

\[
L^R = -c_0 R_{\mu \nu \rho \sigma}R_{\mu \nu \rho \sigma} - c_1 R_{\mu \nu \rho \sigma}R_{\mu \nu \rho \sigma} \]

\[
- c_2 R_{\mu \nu \rho \sigma}R_{\mu \nu \rho \sigma} - c_3 (R_{\mu \nabla \phi_{\mu \nabla}} + R_{\mu \nabla \phi_{\mu \nabla}}) \]

\[
- c_4 R_{\mu \nabla \phi_{\mu \nabla}} - c_5 R^2 - \frac{a_0}{2l^2} R + \Lambda, \tag{3.26d}
\]

\( \Lambda \) is the cosmological constant. The relations between the constants \( c_i \)'s and the \( b_i \)'s are given by

\[
c_0 = -\frac{1}{24\kappa}(2b_1 + 3b_2 + b_3), \tag{3.27a}
\]

\[
c_1 = -\frac{1}{6\kappa}(b_1 - b_3), \tag{3.27b}
\]

\[
c_2 = -\frac{1}{24\kappa}(2b_1 - 3b_2 + b_3), \tag{3.27c}
\]
\[ c_3 = \frac{1}{4\kappa} (b_1 + b_2 - b_4 - b_5), \]  \hspace{1cm} (3.27d)  
\[ c_4 = \frac{1}{4\kappa} (b_1 - b_2 - b_4 + b_5), \]  \hspace{1cm} (3.27e)  
\[ c_5 = -\frac{1}{24\kappa} (2b_1 - 3b_4 + b_6). \]  \hspace{1cm} (3.27f)  

We will now apply this wonderful general “if-constraint” Hamiltonian formulation to certain spin-zero modes of PGT with specific parameter combinations.

4 Spin-zero modes

The PGT propagating modes with a single spin-zero propagating mode are supposed to have ghost-free and tachyon-free characters \[2, 3, 4, 5, 16\]. There are two spin-zero modes, i.e., \(0^+\) and \(0^-\) according to table 1. These spin-zero modes were analyzed by a covariant Lagrangian technique and were shown to have a well posed initial value problem with no tachyonic propagation characteristics \[16\]. In order to better understand them, and the non-linear PGT problems, it is worthwhile to examine their full nonlinear behavior under the Hamiltonian analysis and how (and indeed whether) they really avoid the “constraint bifurcation” problems (which could be expected to arise from the many nonlinear constraints).

The restrictions \(a_1 + 2a_3 = 0\) and \(2a_1 + a_2 = 0\) have been regarded as “viable” conditions for the PGT theories, we will likewise (and in the interests of simplicity) also assume these restrictions in both of our propagating spin-zero parameter choices.

4.1 Spin-\(0^+\) mode

According to Table 1. \(\pi_\perp\) corresponds to spin-\(0^+\). We make the specific parameter choices:

\[ 2a_1 + a_2 = 0, \quad a_1 + 2a_3 = 0, \]
\[ b_1 = b_2 = b_3 = b_4 = b_5 = 0, \]  \hspace{1cm} (4.1)
In fact it is not necessary to take \( b_1, \ldots, b_5 \) to vanish. The choice can be relaxed by requiring only \( b_1 = -b_2 = b_3 = -b_4 = b_5 \neq 0 \) (refer to (3.23a-f)). However, we believe that it’s more critical to understand the character of the pure spin-0\(^+\) mode. This simple choice will greatly simplify calculations, while more general choices are expected to have the same qualitative behavior. After all the 4-covariant analysis\(^{14}\), has argued that there is a 4-parameter class of Lagrangians with a dynamic spin-0\(^+\), but each has qualitatively the same dynamic behavior. Hence, we expect that the Hamiltonian analysis of these more general choices, aside from being calculationally much more complicated and less transparent, would actually show no new interesting dynamic features, so we leave them for future work.

The corresponding super-Hamiltonian is:

\[
\mathcal{H}^+ = \mathcal{H}^T + \mathcal{H}^{R+},
\]

and

\[
\mathcal{H}^T = -\frac{l^2}{2Ja_1} \tilde{\pi}^{\mu} \tilde{\pi}^{\mu} + \frac{l^2}{12Ja_1} \pi^2 - n^\mu \nabla_c \pi^c - JL^T,
\]

\[
\mathcal{H}^{R+} = \frac{J\kappa}{6b_6} \left( \frac{\pi_\perp}{J} + 3a_0 l^2 - \frac{b_6 (R)}{2\kappa} R + \frac{Jb_6 R}{24\kappa} + \frac{a_0}{2l^2} J R - J L, \right)
\]

\[
\mathcal{T}^T = \frac{a_1}{8l^2} T^{\mu\sigma\nu} T^{\mu\sigma\nu} + \frac{a_1}{4l^2} T^{\mu\nu} T^{\mu\nu} - \frac{a_1}{2l^2} T^{\mu} T^{\mu}.
\]

Due to the parameter choice (4.1), the PIC’s are as follows:

\[
\phi_\perp \equiv \frac{\pi_\perp}{J} - \frac{a_1}{l^2} T_\perp \approx 0,
\]

\[
\phi^{\mu\nu} \equiv \frac{\pi^{\mu\nu}}{J} - \frac{a_1}{2l^2} T^{\mu\nu}_\perp \approx 0,
\]

\[
\varphi \equiv \frac{n_\pi}{J} \approx 0,
\]

\[
\phi_\perp \equiv \frac{\pi_\perp}{J} \approx 0,
\]

\[
\phi_\parallel \equiv \frac{\pi_\parallel}{J} \approx 0.
\]
According to Dirac-Bergmann algorithm it’s necessary to identify the class of these constraints. The non-vanishing Poisson brackets (PB) for the PIC’s are the following:

\[
\{ \hat{\phi}_{\mu \nu \perp}, \hat{\phi}_{\tau \sigma} \} \approx \frac{\delta_{xx'}}{3J} \varpi \eta_{\mu \nu \rho \tau \sigma}, \\
\{ \hat{\phi}_{\mu}, \hat{\phi}_{\nu \perp} \} \approx -\frac{2\delta_{xx'}}{3J} \varpi \eta_{\mu \nu}.
\]

where \( \varpi := \frac{\pi_{\perp}}{J} + \frac{3a_1}{l^2} \). Thus \( \hat{\phi}_{\perp}, \hat{\phi}_{\mu}, \hat{\phi}_{\tau} \) and \( \hat{\phi}_{\mu \nu \perp} \) are second-class — as long as \( \varpi \neq 0 \). The constraints \( \hat{\phi}, \hat{\phi}_{\mu \nu} \) and \( \hat{\phi}_{\mu \nu \perp} \) commute with the other primary constraints; according to the general PGT “if” constraint analysis, the associated SC constraints can be derived from them by calculating their time derivatives. They are

\[
\hat{\phi} \approx -\frac{1}{3} N \varpi T \approx 0, \\
\hat{\phi}_{\mu \nu \perp} \approx -\frac{l^2}{3a_1} N \varpi \hat{\pi}_{\mu \nu} \approx 0, \\
\hat{\phi}_{\mu \nu} \approx -\frac{1}{3} N \varpi T \approx 0.
\]

For the time being we put aside the highly degenerate case \( \varpi \approx 0 \). Then, assuming that \( \varpi \) does not vanish (except perhaps on a set of measure zero), these SC’s can be simplified to

\[
\hat{\phi} \equiv \hat{\phi} T \approx 0, \\
\hat{\phi}_{\mu \nu \perp} \equiv \hat{\phi}_{\mu \nu \perp} \approx 0, \\
\hat{\phi}_{\mu \nu} \equiv \hat{\phi}_{\mu \nu} T \approx 0.
\]
By identifying and eliminating all primary and secondary constraints inside, the supermomenta and Lorentz rotation parts lead to the distinct constraints:

\[ \mathcal{H}^+_a \approx 0 \Rightarrow -\pi J R_{a\perp} - \partial_a \frac{\pi}{J} + \frac{3a_1}{l_2} T^b n_\mu \nabla_b e_a^\mu \approx 0, \]

\[ \mathcal{H}^+_{\mu\nu} \approx 0 \Rightarrow \varepsilon_{\mu\nu} T_{\perp} \approx 0, \]

\[ \varepsilon_{\mu\nu} T_{\perp} + e^a_{\pi} \partial_a \varepsilon_{\mathcal{A}} \approx 0. \]

All of the “if” constraints (including the PIC’s and the SC’s) are eventually second-class by calculating their PB’s with the others. The details are shown in the appendix. The consistency conditions have to be obeyed, i.e., we must produce the time derivatives of second-class constraints and force the results to vanish weakly.

We then learn the Lagrange multipliers. The procedure can be understood from the following expression:

\[ \dot{\phi}_B \equiv \int \{ \phi_B, H'_{\text{tot}} \} \]

\[ = \int \{ \phi_B, \mathcal{H}'_{\text{can}} \} + u^A \{ \phi_B, \phi'_A \} \approx 0. \]  

(4.21)

If \( \phi_B \) is second-class, then it is possible to solve for some \( u^A \). In \( H_{\text{tot}} (3.3) \) only the super-Hamiltonian \( H_{\perp} \) (namely \( H^+_{\perp} \) here) involving time evolution is important.

By proceeding with the step described by (4.21), it is straightforward to get the Lagrange multipliers:

\[ u_{\mathcal{P}} = -\frac{1}{2} N J \vec{T}', \quad \dot{u}_{\mathcal{P}} = 0, \]

\[ \dot{u}_{\mathcal{P}} = \frac{1}{2} N J R_{\mathcal{P}}, \quad \dot{u}_{\mathcal{P}} = -\frac{1}{2} N J R_{\mathcal{P}} \]

(4.22a)

\[ n_u = \frac{1}{N} J^{p} R_{a\perp}, \]

(4.22b)

(4.22c)

\[ s_{u_{\mathcal{P}}} = \frac{1}{2} N J s R_{\mathcal{P}} \]

(4.22d)

\[ \dot{u}_{\mathcal{P}} = \frac{1}{2} N J \left[ \frac{\pi}{\pi_{\perp}} n_{\mathcal{P}} e^{a_{\pi}} \nabla_a n_\sigma - \frac{3a_1}{l_2} \eta_{\mathcal{P}} e^{a_{\pi}} \nabla_a \vec{T}_\pi - \vec{R}_{\mathcal{P}} \right]. \]

(4.22e)
Now the canonical Hamilton equations of motion can be derived directly from the completed total Hamiltonian density,

\[ \dot{q}_A = \int \{q_A, H'_{\text{tot}}\}, \quad (4.23) \]

\[ \dot{\pi}_A = \int \{\pi_A, H'_{\text{tot}}\}, \quad (4.24) \]

where \(q_A\) represents the collection of canonical variable \(e_i^{\mu}\) and \(\Gamma^i_{\mu \nu}\), \(\pi_A\) represents the collection of the conjugate momenta \(\pi^i_{\mu}\) and \(\pi^i_{\mu \nu}\).

### 4.2 Massive spin-0\(^{-}\) mode

From table 1, \(p_\pi\) corresponds to the spin-0\(^{-}\) dynamic mode. We consider the simple parameter choice:

\[ 2a_1 + a_2 = 0, \quad a_1 + 2a_3 = 0, \]

\[ b_1 = b_2 = b_4 = b_5 = b_6 = 0, \quad (4.25) \]

\[ a_0 \neq 0, \quad b_3 \neq 0. \]

The corresponding super-Hamiltonian is:

\[ H_{\perp}^- = H_{\perp}^T + H_{\perp}^{R-}, \quad (4.26) \]

and

\[ H_{\perp}^{R-} = -\frac{J_\kappa}{24b_3} \left( \frac{p_\pi}{J} - \frac{b_3 p R_{\perp}}{\kappa} \right)^2 - \frac{Jb_3 p R_{\perp} p R_{\perp}^\perp}{24\kappa} + \frac{a_0}{2l^2} JR - J\Lambda. \quad (4.26a) \]

In this case the PIC’s are:

\[ \phi_{\vec{\pi}_\perp} \equiv \frac{\vec{\pi}_{\perp}}{J} - \frac{a_1}{l^2} T_{\vec{\pi}} \approx 0, \quad (4.27) \]

\[ \hat{\phi}_{\pi_{\mu \nu}} \equiv \hat{\pi}_{\pi_{\mu \nu}} - \frac{a_1}{2l^2} T_{\pi_{\mu \nu}} \approx 0, \quad (4.28) \]
The non-zero PB’s for the PIC’s are the following:

\[ \{ \hat{\phi}^\perp, \phi'^\perp \} \approx -\frac{\delta_{xx'} p_{\pi}}{6 J} \epsilon_{\mu \nu \sigma \perp}, \]  
(4.34)

\[ \{ \hat{\phi}^\perp, \phi'^{\mu \nu} \} \approx -\frac{\delta_{xx'} m_{\pi \mu \nu \rho}}{J}, \]  
(4.35)

\[ \{ \hat{\phi}, \phi'^{\perp} \} \approx \frac{2 \delta_{xx'} p_{\pi}}{J} m_{\mu \nu \rho}, \]  
(4.36)

\[ \{ \hat{\phi}, \phi'^{\mu \nu} \} \approx \frac{\delta_{xx'} p_{\pi}}{6 J} \epsilon_{\mu \nu \sigma \perp}. \]  
(4.37)

where \( m = \frac{a_0 - a_1}{l^2} \). We first consider the generic “massive” case: \( m \neq 0 \). Since the determinant of the PB’s matrix is \( J^{-12}[(\rho_{\pi}/6J)^2 + m^2]^6 > 0 \), the constraints \( \phi^\perp, \hat{\phi}^\perp, \hat{\phi}^\perp, \) and \( \hat{\phi}^\perp \) are second-class.

Similarly we have the time derivatives of \( \phi^\perp, \hat{\phi}^\perp, \) and \( \phi^{\mu \nu} \).

\[ \dot{\phi}^\perp \approx \frac{1}{12} p_{T \mu} \frac{p_{\pi}}{J} + m \frac{l^2 \pi}{a_1 J} \approx 0, \]  
(4.38)

\[ \dot{\pi}_{\pi} \approx \frac{1}{9} \frac{p_{\pi}}{J} \epsilon_{(\mu, \nu)} T_{\pi \mu \nu} + m \frac{l^2 \pi_{\mu \nu}}{a_1 J} \approx 0, \]  
(4.39)

\[ \dot{\phi}^{\mu \nu} \approx \frac{1}{6 a_1 J^2} \epsilon_{\mu \nu \rho \sigma} \pi_{\rho \sigma} + m T_{\mu \nu \rho \sigma} \approx 0. \]  
(4.40)

Generically these can be replaced by the three simpler SC constraints:

\[ \chi^\perp \equiv \frac{1}{12} p_{T \mu} \frac{p_{\pi}}{J} + m \frac{l^2 \pi}{a_1 J} \approx 0, \]  
(4.41)

\[ \tilde{\chi} \equiv \frac{\pi_{\mu \nu}}{J} \approx 0, \]  
(4.42)

\[ \tilde{s} \chi \equiv \pi_{\mu \nu} \approx 0. \]  
(4.43)
The supermomenta and Lorentz rotation parts are:
\[ \mathcal{H}_\perp^a \approx 0 \Rightarrow \quad \frac{a_1}{4l^2} \vec{T}^\sigma T_{\perp} + \frac{a_1}{12l^2} \eta^\sigma \epsilon_{\mu \nu} T^\perp \]
\[ - \frac{1}{12} \frac{p_{\pi}}{J} R^\perp - \frac{a_0}{l^2} \nu_{\perp} - \frac{a_1}{l^2} T^a \delta_{\lambda}^a = 0, \]
\[ \mathcal{H}_\mu^\perp \approx 0 \Rightarrow \quad \frac{1}{12} \frac{p_{\pi}}{J} \epsilon_{\mu \nu} T^\perp + m T^\perp = 0, \]
\[ \frac{1}{6} \epsilon_{\mu \nu} \epsilon^\alpha \partial_\alpha + \frac{1}{J} T^\perp = m T^\perp \approx 0, \]

where \( \eta R^\perp = e^{\nu \sigma} R_{\mu \nu \sigma} \). Utilizing again the standard Dirac-Bergmann algorithm the corresponding Lagrange multipliers are found to be

\[ u^\perp = \quad \frac{1}{2} \nu R^\perp, \quad \hat{u}^\perp = NJT^\perp, \]
\[ \bar{u} = \quad \frac{1}{2} \nu R^\perp, \quad \hat{u}^\perp = -\frac{1}{2} NJR^\perp, \]
\[ s^\perp = \frac{1}{4} \nu \left[ \nu T^\perp T^\perp + 3 \frac{\nu^2}{4} \nu T^\perp T^\perp - \frac{1}{16} T^2 + \frac{a_0}{2a_1} \pi - \frac{2l^2}{a_1} \right] \]
\[ \frac{1}{4} \nu \left[ \nu T^\perp T^\perp + 3 \frac{\nu^2}{4} \nu T^\perp T^\perp - \frac{1}{16} T^2 + \frac{a_0}{2a_1} \pi - \frac{2l^2}{a_1} \right], \]
\[ \hat{s}^\perp = \quad \frac{1}{2} \nu \left[ \nu T^\perp T^\perp + 3 \frac{\nu^2}{4} \nu T^\perp T^\perp - \frac{1}{16} T^2 + \frac{a_0}{2a_1} \pi - \frac{2l^2}{a_1} \right], \]
\[ \hat{u}^\perp = \quad \frac{1}{2} \nu \left[ \nu T^\perp T^\perp + 3 \frac{\nu^2}{4} \nu T^\perp T^\perp - \frac{1}{16} T^2 + \frac{a_0}{2a_1} \pi - \frac{2l^2}{a_1} \right]. \]

\[ \hat{u}^\perp = \quad \frac{1}{2} \nu \left[ \nu T^\perp T^\perp + 3 \frac{\nu^2}{4} \nu T^\perp T^\perp - \frac{1}{16} T^2 + \frac{a_0}{2a_1} \pi - \frac{2l^2}{a_1} \right], \]

4.3 Massless spin-0⁻ mode

In the spin-0⁻ mode there is a special “massless” subcase given by

\[ m = 0, \quad \text{i.e.,} \quad a_1 = a_0. \]
We take the other parameters in the massless case to be the same as those of the
massive spin-0\(^{-}\) mode. Therefore we will only mention the parts that differs from
those in the massive spin-0\(^{-}\) case. The supermomenta and Lorentz rotation parts
can be further simplified
\[
\mathcal{H}_a \approx 0 \Rightarrow -\frac{1}{12} J^\mu p_\pi \approx -\frac{a_0}{l^2} R_{\perp} - \frac{a_0}{l^2} \nabla a_\mu + \frac{1}{3} e_a \pi \partial_\alpha \pi \approx 0, \tag{4.48}
\]
\[
\mathcal{H}_{\mu \nu} \approx 0 \Rightarrow \frac{-p_\pi}{J} T_{\mu \nu} \approx 0, \tag{4.49}
\]
\[
\partial_\alpha \frac{-p_\pi}{J} + \frac{p_\pi}{J} T_a \approx 0. \tag{4.50}
\]

The non-zero PB’s for the PIC’s that remain, eqs (4.34, 4.37), are \{\phi_{\mu \pi \perp}, \phi_{\perp \pi \perp}\}, \{\phi_{\perp \pi}, \phi'_{\perp \pi \perp}\}. Whether the PB’s matrix is singular is determined by \(p_\pi/J\). Let us first
suppose the \(p_\pi\) does not vanish (except perhaps on a set of measure zero). Then the
class of \(\phi_{\perp \pi}, \phi'_{\perp \pi \perp}, \phi_{\perp \pi \perp}\) and \(\phi'_{\mu \pi \perp}\) won’t change. The SC’s are
\[
\chi_{\perp} \equiv \frac{\nu T}{J} \approx 0 \tag{4.51}
\]
\[
\tilde{\chi}_{\mu \pi \perp} \equiv \frac{\nu T}{J} \approx 0, \tag{4.52}
\]
\[
\check{\chi}_{\mu \pi \perp} \equiv \frac{\nu T}{J} \approx 0. \tag{4.53}
\]

The Lagrange multipliers are
\[
u \pi \perp = -\frac{1}{2} N J T_{\perp}, \quad \hat{u}^{\mu \perp} = 0, \tag{4.54a}
\]
\[
u \pi = \frac{1}{2} N J R_{\perp}^{\perp}, \quad \hat{u}^{\perp} = -\frac{1}{2} N J R_{\perp}^{\perp}, \tag{4.54b}
\]
\[
u \pi_{\mu \perp} = \frac{1}{2} N J \check{R}_{\mu \pi \perp}, \tag{4.54c}
\]
\[
u \pi_{\mu \perp} = \frac{1}{2} N J R_{\mu \pi \perp} \tag{4.54d}
\]

We find that \(\chi_{\perp}\) commutes with \(\phi_{\perp}\), so its consistency condition leads to, instead
of determining \(u^\perp\), a tertiary constraint (a quite unusual occurrence):
\[
\zeta_{\perp} \equiv \frac{\nu \pi}{J} - \frac{b_3}{2\kappa} p R_{\sigma \perp} \approx 0. \tag{4.55}
\]
However, $\zeta_\perp$ still commutes with $\phi_\perp$ thus $u_\perp$ remains undetermined. Then the consistency condition of $\zeta_\perp$, i.e., $\dot{\zeta}_\perp \approx 0$, and the substitution of the known multipliers in (4.54a-d) lead, in turn to the constraint

$$\xi_\perp \equiv \rho R_\perp \approx 0.$$  

(4.56)

(Because of (4.49) the constraint $\xi_\perp \approx 0$ can be derived directly from the Bianchi identity $\nabla_{[\mu} T_{\nu\rho]}^{\mu} = R_{(abc)}^{\mu}$. ) Then there is no need to go further, since the constraint $\zeta_\perp$, along with $\xi_\perp$, contradicts our assumption that $p_\pi$ does not vanish. Consequently, for the massless spin-0\(^{-}\) case, $p_\pi$ must definitely vanish.

Knowing now that $p_\pi$ definitely vanishes, so then does the rhs of (4.38, 4.39, 4.40), which tells us that the massless spin-0\(^{-}\) doesn’t have any propagating torsion modes and is, consequently, just equivalent to GR.

## 5 Discussion

In the PGT, there are forty dynamic variables coming from sixteen tetrad components and twenty-four connection components. The number of total variables counts eighty because the same number of canonical momenta accompany the variables. Nonetheless, constraints eliminate many unphysical variables. The super-Hamiltonian $\mathcal{H}_\perp$, super-momenta $\mathcal{H}_a$, and the Lorentz generators $\mathcal{H}_{\mu\nu}$ are (generally) ten first-class constraints. There are also ten “sure” first-class primary constraints. This total of twenty first-class constraints (because of their gauge nature) offset forty variables—in general.

In the dynamic spin-0\(^{+}\) and massive spin-0\(^{-}\) modes that we have considered here, we found that the twenty-three PIC’s and eleven SC’s are second-class. They thus eliminate thirty-four unphysical variables. Therefore, there are only $80 - 40 - 34 = 6$ true physical variables. It simply means that the number of degrees of freedom is reduced to three: the scalar/pseudoscalar mode and the two helicity states of the usual massless graviton.
However, for the massless spin-0$^-$ mode, we have shown that the pseudoscalar canonical momentum must vanish weakly in order to make the theory self-consistent. In that case we have the usual 4 + 4 “sure” first class constraints associated with the translational gauge freedom and the 6 “sure” first class Lorentz gauge freedom constraints (3.2). All of the “if” constraints (4.27-4.33) then turn out to be first class. Their consistency conditions, as well as the consistency condition for (3.2), degenerate to the single first class condition $p_\pi \approx 0$. Thus we have a total of $4 + 4 + 6 + 23 + 1 = 38$ first class constraints leading to $80 - 2 \times 38 = 4$ physical initial values for just the usual 2 degrees of freedom for the graviton. In this special case the propagating torsion modes are unphysical, being pure gauge.

One can determine the positivity of the non-zero coupling parameters from the Hamiltonian density (4.2a, 4.2b, 4.26a). Since the kinetic energy density must be positive definite, we have

$$a_1 > 0, \quad \frac{b_3}{\kappa} < 0, \quad \frac{b_6}{\kappa} > 0.$$  \hspace{1cm} (5.1)

The results are consistent with restrictions that earlier researchers have proposed, e.g., references 2, 3, 4, 5.

It is no wonder that the Lagrange multipliers are identical to the missing velocities. Let us examine these in the spin-0$^+$ mode. The time derivatives of canonical variables are given in (4.23). After reducing the results we truly find

$$T_{\perp \mu} \equiv u_{\perp \mu}, \quad T_{\perp [\mu \nu]} \equiv \hat{u}_{\perp [\mu \nu]},$$

$$R_{\perp \mu} \equiv \hat{u}_{\perp \mu}, \quad R_{\perp [\mu \nu]} \equiv \hat{u}_{\perp [\mu \nu]}$$ \hspace{1cm} (5.2a)

$$sR_{\perp [\mu \nu \sigma]} \equiv \hat{s}u_{\perp [\mu \nu \sigma]}, \quad R_{\perp [\mu \nu \sigma]} \equiv \hat{u}_{\perp [\mu \nu \sigma]}$$ \hspace{1cm} (5.2b)

$$pR_{\perp \mu \nu} \equiv \hat{p}u_{\perp \mu \nu}.$$  \hspace{1cm} (5.2c)

The situation of the massive spin-0$^-$ mode is much the same. On the other hand, the expressions for the multipliers, the $u$’s, show that each missing velocity is determined by non-velocity terms, hence they do not have a physically independent dynamical
status. This fact reflects the details of the Lagrangian field equations. Due to the specific parameter combinations (4.1) and (4.25), the coefficients attached to the time derivatives of those missing velocities vanish and the related field equations become constraints.

If we look at the multipliers for the two spin-zero modes, \( u_{\mu \perp} \), \( u_{\mu \nu} \), \( \vec{u}_{\mu} \), and \( \hat{u}_{\mu \nu \perp} \) essentially have the same expressions. \( \hat{u}_{\mu \nu \perp} \) vanishes in the spin-0\(^+\) and massless spin-0\(^-\) modes because \( T_{\mu \nu \perp} \approx 0 \) there. It is clear that, since \( \phi_{\mu \perp} \), \( \hat{\phi}_{\mu \nu} \), \( \vec{\phi}_{\mu} \), and \( \hat{\phi}_{\mu \nu \perp} \) contain the variables related to spin-one modes, they should not be involved in the field equations of spin-zero modes. As to the multipliers \( \hat{u}_{\mu \nu \perp} \) and \( s_{\mu \nu \perp} \), which are related to the spin-2\(^+\) and spin-2\(^-\) modes, in both of the modes they are more involved and complicated because the usual massless spin-two graviton is mixed up here.

We note that the constraints (4.20) in the spin-0\(^+\) case deduced from the Lorentz rotation parts are intriguing. Since \( \varpi \) is generically a function of spatial coordinates, the constraint will generally prevent \( \varpi \) from vanishing unless it vanishes globally. However \( \varpi \) could vanish permanently and become a new “constraint” if its initial value is zero. (Assuming that \( \hat{T}_{\mu} \) cannot become unbounded.) From (3.23d) the relation between \( \varpi \) and the (affine not Riemannian) scalar curvature is given by

\[
\varpi \equiv \frac{b_0}{2\kappa} R - 3m,
\]

therefore \( \varpi = 0 \) indicates that the (affine) scalar curvature is equal to \( 6km/b_0 \), a constant. The constant forms a barrier for the scalar field to cross. Whether the scalar mode evolves on such an affine (anti) de-Sitter spacetime background or is frozen depends upon the initial conditions. This forms a new kind of constraint bifurcation. The usual type, which originates from the PB’s matrix of the constraints being not of constant rank, has been linked to acausal propagating modes. (Because of the mass terms, as can be seen from (4.46), the massive spin-0\(^-\) mode avoids getting involved in this new type of constraint bifurcation.)
It is straightforward to work out, from the covariant analysis of the scalar PGT modes\([14]\), the Lagrangian field equations for the spin-0\(^+\), case with \(\varpi = 0\). They just turn out to be those of GR with the addition of a corrected cosmological constant depending on \(\Lambda\) and the value of the constant (affine) scalar curvature, \(6\kappa m/b_6\). All coefficients of the torsion components in the field equations vanish. Consequently the torsion is really pure gauge in this case. A further very special highly degenerate subcase, with \(\varpi = 0 = m\), needs no additional discussion; it has just the behavior expected in the \(\varpi \rightarrow 0\) limit.

This covariant analysis is in accord with the corresponding Hamiltonian analysis which shows that the \(4 + 4\) “sure” translational gauge generators, the 6 “sure” first class rotation gauge generators and the 23 primary “if” constraints give rise to just one secondary constraint: \(\varpi \approx 0\). All the “if” constraints turn out to be first class. Thus we again have a total of 38 first class constraints and the usual 2 degrees of freedom for the graviton plus purely gauge torsion.

For solutions with special symmetries, of course there are other possibilities. In particular, consider the case of homogeneous cosmologies. If \(\varpi\) is dependent on time only, the constraint (4.20) simplifies, thereby allowing for simpler solutions for scalar fields on a Robertson-Walker-like spacetime. But here we are concerning ourselves with the dynamic structure of the general theory, not with the peculiarities of solutions with special symmetries.

We remark that the fourth-derivative gravity which people have investigated recently has similar degeneracy problem\([15]\). Careful consideration is required to understand if the relation between its degeneracy and constraint bifurcation phenomenon exits, but such a study is beyond the purpose of this paper.
6 Conclusion

In this paper the Dirac-“if” constraint Hamiltonian formalism for the PGT has been applied to the study of the full nonlinear behavior of its spin-zero modes with certain specific coupling parameter choices. It is shown that by using the Hamiltonian analysis one can clearly identify the degeneracy and the corresponding constraints, and the true dynamic degrees of freedom of the PGT as well as the positivity of its coupling parameters.

For the spin-0$^-$ mode, there is a massive, non-ghost pseudoscalar field, in addition to the usual spin-two graviton, propagating dynamically. But the corresponding massless mode has additional gauge freedom in the torsion modes, yielding a dynamics which is essentially equivalent to that of GR (with presumably undetectable purely gauge torsion). Therefore, the magnitude of the mass \( m \) will determine the viability and detectability of this mode. If \(|m| \gg 0\), we might expect to find this mode at about the Planck-scale range. On the other hand if \(|m| \to 0\), the differences between the PGT and GR are too tiny to be detected. In both of these situations we can only obtain almost the same effects as GR on the ordinary scale. There is also a scalar field propagating in the spin-0$^+$ mode case. This mode encounters a type of constraint bifurcation phenomenon, depending on whether \( \varpi \) vanishes, which divides the phase-space into two subspaces, since \( \varpi \) cannot vanish in general unless it vanishes globally because of the Lorentz rotation constraint (4.20). However, as we discussed above, the scalar momentum could be only a function of time which turns off the effect of the phenomenon. Then we have a solution which can be applied to spatially homogeneous cosmological models.

The constraint bifurcation phenomenon is generally expected to occur in the full nonlinear PGT with higher spin propagating modes. Somewhat surprisingly, we also see the “constraint bifurcation” phenomenon in these spin-zero cases, albeit in a very simple essentially “all or nothing” fashion.
This work is an important complement to the earlier work on PGT scalar modes and is a necessary preliminary to our next step: examining the spin-one and spin-two modes and comparing the nonlinear results to the linearized ones of PGT.

Acknowledgments

This work was supported by the National Science Council of the R.O.C. under grants No. NSC87-2112-M-008-007 and No. NSC88-2112-M-008-018.

Appendix: Non-zero Poisson brackets

In order to classify the SC constraints and derive the Lagrange multipliers, the values of all PB’s should be calculated. Here the results of the calculations of non-zero PB’s in the spin-zero modes of the PGT are presented.

Results in the spin-$0^+$ mode

The followings just show that all constraints in the spin-$0^+$ are second-class provided $\varpi \neq 0$.

\[
\{p_X, \hat{u}^{\mu\nu} \hat{\phi}_{\mu \nu}^{\dagger}\} \approx 2\delta_{xx'} \epsilon_{\mu\nu} \sigma^\perp e^a \nabla_a \hat{u}_{xx'}^{\perp}, \\
\{p_X, p_{\phi'}\} \approx 24 \frac{\delta_{xx'}}{J}, \\
\{\chi_{\mu \nu}^{\perp}, u^{\mu \nu} \phi_{\mu \nu}^{\dagger}\} \approx \frac{\delta_{xx'}}{J} \eta_{\mu \sigma} \epsilon_{\lambda \rho} \nabla_\lambda \hat{u}_{xx'}^{\perp}, \\
\{\chi_{\mu \nu}^{\perp}, \phi_{\mu \nu}^{\dagger}\} \approx \frac{\delta_{xx'}}{J} \eta_{\mu \sigma} \epsilon_{\lambda \rho} \nabla_\lambda \hat{u}_{xx'}^{\perp}, \\
\{\chi_{\mu \nu}^{\perp}, \phi_{\mu \nu}^{\dagger}\} \approx \frac{\delta_{xx'}}{J} \eta_{\mu \sigma} \epsilon_{\lambda \rho} \nabla_\lambda \hat{u}_{xx'}^{\perp}, \\
\{\chi_{\mu \nu}, u^{\mu \nu} \phi_{\mu \nu}^{\dagger}\} \approx \frac{\delta_{xx'}}{J} \eta_{\mu \sigma} \epsilon_{\lambda \rho} \nabla_\lambda \hat{u}_{xx'}^{\perp}, \\
\{\chi_{\mu \nu}, \phi_{\mu \nu}^{\dagger}\} \approx \frac{\delta_{xx'}}{J} \eta_{\mu \sigma} \epsilon_{\lambda \rho} \nabla_\lambda \hat{u}_{xx'}^{\perp}, \\
\{\chi_{\sigma \mu \nu}, \phi_{\sigma \mu \nu}^{\dagger}\} \approx \frac{\delta_{xx'}}{J} \eta_{\mu \sigma} \epsilon_{\lambda \rho} \nabla_\lambda \hat{u}_{xx'}^{\perp}, \\
\{\chi_{\sigma \mu \nu}, \phi_{\sigma \mu \nu}^{\dagger}\} \approx \frac{\delta_{xx'}}{J} \eta_{\mu \sigma} \epsilon_{\lambda \rho} \nabla_\lambda \hat{u}_{xx'}^{\perp}, \\
\{\chi_{\sigma \mu \nu}, \phi_{\sigma \mu \nu}^{\dagger}\} \approx \frac{\delta_{xx'}}{J} \eta_{\mu \sigma} \epsilon_{\lambda \rho} \nabla_\lambda \hat{u}_{xx'}^{\perp}, \\
\{\chi_{\sigma \mu \nu}, \phi_{\sigma \mu \nu}^{\dagger}\} \approx \frac{\delta_{xx'}}{J} \eta_{\mu \sigma} \epsilon_{\lambda \rho} \nabla_\lambda \hat{u}_{xx'}^{\perp},
\]

(A.1)
\{\chi_{\sigma\mu\nu}, \phi'_{\tau\rho}\} \approx \frac{\delta_{xx'}}{J}[\chi(\eta_{\rho\sigma\tau\rho\tau\nu}) - \chi(\eta_{\rho\sigma\tau\rho\tau\nu})], \quad (A.8)

**Results in the spin-0\(^-\) mode**

\{\chi_{\perp}, u^{\perp\lambda} \phi'_{\mu\perp}\} \approx \frac{\delta_{xx'} v_{\mu}}{6J} \epsilon^{\mu\nu\sigma} u^{\perp\sigma} T_{\mu\nu\perp} - 2\delta_{xx'} m e^a_{\pi} \nabla_a u^{\perp\mu}, \quad (B.1)

\{\chi_{\perp}, \hat{u}^{\perp\lambda} \phi'_{\mu\perp}\} \approx \frac{\delta_{xx'} v_{\rho}}{6J} \epsilon^{\rho\sigma\pi} c^a_{\tau} \hat{u}^{\perp\pi} \nabla_a \hat{u}^{\perp\mu} + \frac{\delta_{xx'}}{J} m \hat{u}^{\perp\mu} T_{\mu\nu\perp}, \quad (B.2)

\{\chi_{\perp}, \phi'_{\perp}\} \approx \frac{-\delta_{xx'} 6a_0}{J} m, \quad (B.3)

\{\tilde{\chi}_{\rho\sigma\perp}, u^{\rho\sigma\perp} \phi'_{\tau\perp}\} \approx \frac{\delta_{xx'} a_1}{12} \chi(\eta_{\nu\sigma\rho\sigma\rho\perp}) - \chi(\eta_{\nu\sigma\rho\sigma\rho\perp})] \nabla_a n^\rho, \quad (B.4)

\{\tilde{\chi}_{\rho\sigma\perp}, \hat{u}^{\rho\sigma\perp} \phi'_{\tau\perp}\} \approx \frac{\delta_{xx'} a_0}{12} \chi(\eta_{\nu\sigma\rho\sigma\rho\perp}), \quad (B.5)

\{\tilde{\chi}_{\rho\sigma\perp}, \tilde{\phi}'_{\tau\perp}\} \approx \frac{\delta_{xx'} a_1}{12} \chi(\eta_{\nu\sigma\rho\rho\perp}) \nabla_a \hat{u}^{\rho\perp}, \quad (B.6)

\{\tilde{\chi}_{\rho\sigma\perp}, \hat{u}^{\rho\rho\perp} \phi'_{\tau\perp}\} \approx \frac{\delta_{xx'} a_0}{12} \chi(\eta_{\nu\rho\sigma\rho\perp}), \quad (B.7)

\{\hat{\chi}_{\rho\sigma\perp}, \hat{u}^{\rho\rho\perp} \phi'_{\tau\perp}\} \approx \frac{\delta_{xx'} a_1}{12} \chi(\eta_{\nu\rho\sigma\rho\perp}) \nabla_a \hat{u}^{\rho\perp}, \quad (B.8)

\{\hat{\chi}_{\rho\sigma\perp}, \tilde{\phi}'_{\tau\perp}\} \approx \frac{\delta_{xx'} a_0}{12} \chi(\eta_{\nu\rho\rho\perp}) \nabla_a \hat{u}^{\rho\perp}, \quad (B.9)

**References**

1. F.W. Hehl, in: *Proc. of 6th Course of the International School of Cosmology and Gravitation on Spin, Torsion and Supergravity*, eds. P.G. Bergmann and V. de Sabatta (Plenum, New York, 1980).

2. K. Hayashi and T. Shirafuji, *Prog. Theor. Phys.* **64**, 866, 883, 1435, 2222 (1980).

3. E. Sezgin and P. van Nieuwenhuizen, *Phys. Rev.* **D21**, 3269 (1980).

4. E. Sezgin, *Phys. Rev.* **D24**, 1677 (1980).

5. R. Battiti and M. Toller, Lett. *Nuovo Cimento* **44**, 35 (1985).

6. D.C. Chern, J.M. Nester, and H.J. Yo, *Int. J. Theor. Phys.* **A1**, 1993 (1992).
7. R. Kuhfuss and J. Nitsch, *Gen. Rel. Grav.* **18**, 1207 (1986).
8. A. Dimakis, *Ann. Inst. Henri Poincaré* **51** 371, 389, (1989).
9. J. Lemke, *Phys. Lett.* **A143**, 13 (1990).
10. R.D. Hecht, J. Lemke and R.P. Wallner, *Phys. Lett.* **A151**, 12 (1990); *Phys. Rev.* **D44**, 2442 (1991).
11. R.D. Hecht, J.M. Nester and V.V. Zhytnikov, “Does the Poincaré Gauge Theory really have problems?”, NCU, 1994 (unpublished report).
12. W.H. Cheng, D.C. Chern, and J.M. Nester, *Phys. Rev.* **D38**, 2656 (1988).
13. H. Chen, J.M. Nester, and H.J. Yo, *Acta Phys. Pol.* **B29**, 961 (1998).
14. G. Velo and D. Zwanzinger, *Phys. Rev.* **188**, 2218 (1969).
15. C. Aragone and S. Deser, *Il Nuovo Cim.* **A3**, 709 (1971).
16. R.D. Hecht, J.M. Nester, and V.V. Zhytnikov, *Phys. Lett.* **A222**, 37-42, (1996).
17. K. Stelle, *Gen. Rel. Grav.* **9**, 353-371 (1978).
18. P. Teyssandier and P. Tourrenc, *J. Math. Phys.* **24**, 2793-2799 (1983).
19. B. Whitt, *Phys. Lett.* **B145**, 176-178 (1984).
20. A. Hindawi, B.A. Ovrut, and D. Waldram, *Phys. Rev.* **D53**, 5583-5896 (1996), [hep-th/9509142](http://arxiv.org/abs/hep-th/9509142).
21. A.A. Starobinsky, *Phys. Lett.* **B91**, 99 (1980).
22. M.B. Mijić, M.S. Morris, and W.-M. Suen, *Phys. Rev.* **D34** 2934 (1986).
23. J.D. Barrow and S. Cotsakis, *Phys. Lett.* **B214**, 515 (1988).
24. J.Z. Simon, *Phys. Rev.* **D45**, 1953 (1992).
25. M. Blagojević and I.A. Nicolić, *Phys. Rev.* **D28**, 2455 (1983).
26. I.A. Nikolić, *Phys. Rev.* **D30**, 2508 (1984).
27. P.A.M. Dirac, *Can. J. Math.* **2**, 129 (1950); *Lectures on Quantum Mechanics*, Yeshiva University-Belfer Graduate School of Science (Academic, New York, 1964).
28. A. Hanson, T. Regge, and C. Teitelboim, *Constrained Hamiltonian Systems* (Academia Nazionale dei Lincei, Roma, 1976).
29. K. Sundermeyer, *Constrained Dynamics* (Springer, Berlin, 1982).
| $J^p$ | Kinetic Parameter Combinations | Constraints | Mass Parameter Combinations |
|-------|--------------------------------|-------------|----------------------------|
| 0+    | (i) $a_2$                      | $\phi, \chi$ | $a_0, 2a_0 + a_2$          |
|       | (ii) $b_4 + b_6$               | $\phi_\perp, \chi_\perp$ |                            |
| 1+    | (i) $a_1 + 2a_3$               | $\hat{\phi}_{\mu\nu}, \hat{\chi}_{\mu\nu}$ | $a_1 - a_0, \frac{a_0}{2} + a_3$ |
|       | (ii) $b_2 + b_5$               | $\hat{\phi}_{\mu\nu\perp}, \hat{\chi}_{\mu\nu\perp}$ |                            |
| 2+    | (i) $a_1$                      | $\tilde{\phi}_{\mu\nu}, \tilde{\chi}_{\mu\nu}$ | $a_0, a_1 - a_0$          |
|       | (ii) $b_1 + b_4$               | $\tilde{\phi}_{\mu\nu\perp}, \tilde{\chi}_{\mu\nu\perp}$ |                            |
| 1−    | (i) $2a_1 + a_2$               | $\phi_{\mu\perp}, \chi_{\mu\perp}$ | $a_1 - a_0, 2a_0 + a_2$    |
|       | (ii) $b_4 + b_5$               | $\tilde{\phi}_{\mu\nu}, \tilde{\chi}_{\mu\nu}$ |                            |
| 0−    | $b_2 + b_3$                    | $p\phi, p\chi$ | $a_0 + a_3$                |
| 2−    | $b_1 + b_2$                    | $s\phi_{\mu\nu\perp}, s\chi_{\mu\nu\perp}$ | $a_1 - a_2$                |

Table 1: Primary 'if'-constraints, critical parameter values and masses