ON THE NUMBER OF DISTINCT PRIME FACTORS OF A SUM OF SUPERPOWERS

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Abstract. Given $k, \ell \in \mathbb{N}^+$, let $x_{i,j}$ be, for $1 \leq i \leq k$ and $0 \leq j \leq \ell$, some fixed integers. Then, define $s_n := \sum_{i=1}^{k} \prod_{j=0}^{\ell} x_{i,j}^{n^j}$ for every $n \in \mathbb{N}^+$.

We prove that there exist infinitely many $n$ for which the number of distinct prime factors of $s_n$ is greater than the super-logarithm of $n$ to base $C$, for some real constant $C > 1$, if and only if there do not exist nonzero integers $a_0, b_0, \ldots, a_\ell, b_\ell$ such that $s_{2n} = \prod_{i=0}^{\ell} a_i^{(2n)^i}$ and $s_{2n-1} = \prod_{i=0}^{\ell} b_i^{(2n-1)^i}$ for all $n$. (In fact, we prove a slightly more general result in the same spirit, where the numbers $x_{i,j}$ are rational.)

In particular, for $c_1, x_1, \ldots, c_k, x_k \in \mathbb{N}^+$ the number of distinct prime factors of the sum $c_1 x_1^n + \cdots + c_k x_k^n$ is bounded, as $n$ ranges over $\mathbb{N}^+$, if and only if $x_1 = \cdots = x_k$.

1. Introduction

Given $k, \ell \in \mathbb{N}^+$, let $x_{i,j}$ be, for $1 \leq i \leq k$ and $0 \leq j \leq \ell$, some fixed rationals. Then, consider the $\mathbb{Q}$-valued sequence $(s_n)_{n \geq 1}$ obtained by taking

$$s_n := \sum_{i=1}^{k} \prod_{j=0}^{\ell} x_{i,j}^{n^j}$$

for every $n \in \mathbb{N}^+$ (see Section 2 for notation used, but not defined, in this introduction); we refer to $s_n$ a sum of superpowers of degree $\ell$, for it is more general, and has typically (i.e., except for trivial cases) a much faster growth rate, than a sum of powers. Note that $(s_n)_{n \geq 1}$ includes as a special case any integer sequence $(t_n)_{n \geq 1}$ of general term

$$\sum_{i=1}^{k} \prod_{j=1}^{\ell_i} y_{i,j}^{f_{i,j}(n)}$$

where, for each $i = 1, \ldots, k$, we let $\ell_i \in \mathbb{N}^+$ and $y_{i,1}, \ldots, y_{i,\ell_i} \in \mathbb{Q} \setminus \{0\}$, while $f_{i,1}, \ldots, f_{i,\ell_i}$ are polynomials in one variable with integral coefficients. Conversely, sequences of the form (1) can be viewed as sequences of the form (2), the latter being prototypical of scenarios where polynomials are replaced with more general functions $\mathbb{N}^+ \to \mathbb{Z}$ (see also Section 4).

Now, let $\omega(x)$ denote, for each $x \in \mathbb{Z} \setminus \{0\}$, the number of distinct prime divisors of $x$, and define $\omega(0) := \infty$. Then, for $x \in \mathbb{Z}$ and $y \in \mathbb{N}^+$ we let $\omega(xy^{-1}) := \omega(x\delta^{-1}) + \omega(y\delta^{-1})$, where $\delta$ is the greatest common divisor of $x$ and $y$.

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In addition, given $n \geq 2$ and $C > 1$, we write $\text{slog}_C(n)$ for the super-logarithm of $n$ to base $C$, that is the largest integer $l \geq 0$ for which $C \uparrow l \leq n$, where $C \uparrow 0 := 1$ and $C \uparrow l := C^{\uparrow(l-1)}$ for $l \geq 1$, cf. [5] for the notation; note that $\text{slog}_C(n) \to \infty$ as $n \to \infty$.

The main goal of the present paper is to provide necessary and sufficient conditions for the boundedness of the sequence $(\omega(s_n))_{n \geq 1}$. More precisely, we have:

**Main Theorem.** There exists a base $C > 1$ such that $\omega(s_n) > \text{slog}_C(n)$ for infinitely many $n$, hence $(\omega(s_n))_{n \geq 1}$ is an unbounded sequence, if and only if there do not exist nonzero rational numbers $a_0, b_0, \ldots, a_\ell, b_\ell$ such that $s_{2n} = \prod_{j=0}^{\ell} a_j^{(2n)^j}$ and $s_{2n-1} = \prod_{j=0}^{\ell} b_j^{(2n-1)^j}$ for all $n$.

The theorem will be proved in Section 3 (working with rationals may look pointless at first, but will eventually result into a simplification of the arguments), along with the following:

**Corollary.** Let $c_1, \ldots, c_k \in \mathbb{Q}^+$ and $x_1, \ldots, x_k \in \mathbb{Q} \setminus \{0\}$. Then, $(\omega(c_1 x_1^n + \cdots + c_k x_k^n))_{n \geq 1}$ is a bounded sequence only if $|x_1| = \cdots = |x_k|$, and this condition is also sufficient provided that $\sum_{i=1}^{k} \varepsilon_i c_i \neq 0$, where, for each $i \in [1, k]$, $\varepsilon_i := x_i \cdot |x_i|^{-1}$ is the sign of $x_i$.

Results in the spirit of the above theorem have been proved by various authors in the special case of $\mathbb{Z}$-valued sequences raising from the solution of non-degenerate linear homogeneous recurrence equations with (constant) integer coefficients of order $\geq 2$, namely in relation to a sequence $(u_n)_{n \geq 1}$ of general term

$$u_n = \alpha_1^n f_1(n) + \cdots + \alpha_h^n f_h(n),$$

(3)

where $\alpha_1, \ldots, \alpha_h$ are the nonzero (and pairwise distinct) roots of the characteristic polynomial of the recurrence under consideration and $f_1, \ldots, f_h$ nonzero polynomials in one variable with coefficients from the smallest field extension of the rational field containing $\alpha_1, \ldots, \alpha_h$, see [10, Theorem C.1]. In this respect, we recall that a recurrence is said to be non-degenerate if its characteristic polynomial has at least two distinct nonzero complex roots (so that $h \geq 2$ in the above) and the ratio of any two distinct characteristic roots is not a root of unity.

More specifically, it was proved by van der Poorten and Schlickewei [14] and, independently, by Evertse [4, Corollary 3], using Schlickewei’s $p$-adic analogue of Schmidt’s subspace theorem [8], that the greatest prime factor of $u_n$ tends to $\infty$ as $n \to \infty$. In a similar note, effective lower bounds on the greatest prime divisor and on the greatest square-free factor of a sequence of type (3) were obtained under mild assumptions by Shparlinski [11] and Stewart [12, 13], based on variants of Baker’s theorem on linear forms in the logarithms of algebraic numbers [2].

On the other hand, Luca has shown in [6] that if $(v_n)_{n \geq 1}$ is a sequence of rational numbers satisfying a recurrence of the form

$$g_0(n)v_{n+2} + g_1(n)v_{n+1} + g_2(n)v_n = 0 \quad \text{for } n \in \mathbb{N}^+,$$

where $g_0$, $g_1$ and $g_2$ are univariate polynomials over the rational field and not all zero, and $(v_n)_{n \geq n_0}$ is not binary recurrent (viz., a solution of a linear homogeneous second-order recurrence equation with integer coefficients) for some $n_0 \in \mathbb{N}^+$, then there exists a real constant $c > 0$ such
that the product of the numerators and denominators (in the reduced fraction) of the nonzero rational terms of the finite sequence \((v_n)_{1 \leq n \leq n}\) has at least \(c \log(n)\) prime factors as \(n \to \infty\).

With this said, it is perhaps worth mentioning that the present manuscript has been originally motivated by a (so-far fruitless) attempt by the authors to obtain a “suitable generalization” of the following classical result due to Zsigmondy \cite{15} to sums of powers:

**Zsigmondy’s theorem.** Let \(a, b, n \in \mathbb{N}^+\) be such that \(a > b\), \(n \geq 2\), and neither \((a, b, n) = (2, 1, 6)\), nor \(a + b\) is a power of 2 and \(n = 2\). Then, there exists a prime \(p\) such that \(p \mid a^n - b^n\), but \(p \nmid a^m - b^m\) for \(m < n\).

Continuing with the notation and the hypotheses as in the above statement, we have, in fact, that \(\omega(a^n - b^n) \geq \sigma_0(n) - 2\) for all \(n\), with \(\sigma_0(n)\) the number of (positive integer) divisors of \(n\).

To see why, let \(d > 1\) be a divisor of \(n\). By Zsigmondy’s theorem, there then exists a prime \(p\) such that \(p \mid a^d - b^d\), but \(p \nmid a^i - b^i\) for \(1 \leq i < d\), unless we have either \((a, b, d) = (2, 1, d)\), or \(a + b\) is a power of 2 and \(d = 2\). This yields the above inequality, because \(a^d - b^d \mid a^n - b^n\) and \((a, b) = (2, 1)\) only if \(a + b\) is not a power of two. On the other hand, it is known, e.g., from \cite{9} that \(\frac{1}{n} \sum_{i=1}^{n} \sigma_0(i)\) is asymptotic to \(\log(n)\) as \(n \to \infty\).

It follows that there exist a constant \(c \in \mathbb{R}^+\) and infinitely many \(n\) for which \(\omega(a^n - b^n) > c \log(n)\), which can be viewed as an analogue of our main theorem (though much stronger than the latter in the special case of the sequences to which Zsigmondy’s theorem applies) and served as a starting point for our investigations.

2. Notation and conventions

Through the paper, \(\mathbb{R}, \mathbb{Q}, \mathbb{Z},\) and \(\mathbb{N}\) are, respectively, the sets of reals, rationals, integers, and nonnegative integers. Each of these sets is endowed with its usual addition, multiplication, and (total) order \(\leq\), and we assume they have been constructed in a way that \(\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}\).

Unless noted otherwise, the letters \(h, i, j, l\) and \(k\), with or without subscripts, will stand for nonnegative integers, the letters \(m\) and \(n\) for positive integers, the letters \(p\) and \(q\) for (positive rational) primes, and the letters \(A, B\) and \(C\) for real numbers.

For \(a, b \in \mathbb{R} \cup \{\infty\}\) we write \([a, b]\) for the closed interval \(\{x \in \mathbb{R} \cup \{\infty\} : a \leq x \leq b\}\), \([a, b[\) for the open interval \(\{a, b\} \setminus \{a, b\}\), and \([[a, b]\) for \([a, b]\cap \mathbb{Z}\) (we use \(\infty\) in place of \(+\infty\); moreover, for a set \(X \subseteq \mathbb{R}\) we let \(X^+ := X \cap ]0, \infty[\).

We refer to \cite{1} and \cite{3}, respectively, for basic aspects of number theory and real analysis (including notation and terms not defined here). In particular, the only topology considered on \(\mathbb{R}\) will be the usual topology, and we will use without explicit mention some of the most basic properties of the upper limit of a real sequence, see especially \cite[Theorem 18.3]{3}.

We denote by \(|\cdot|\) the usual absolute value on \(\mathbb{R}\) and by \(\log\) the natural logarithm, and assume the convention that an empty sum (of real numbers) is 0 and an empty product is equal to 1.

3. Proofs

**Proof of Main Theorem.** The “only if” part is straightforward, and the claim is trivial if at least one of the sequences \((s_{2n})_{n \geq 1}\) and \((s_{2n-1})_{n \geq 1}\) is eventually zero.
Therefore, we can just focus on the two cases below, in each of which we have to prove that there exists $B > 1$ such that $\omega(s_n) > \text{slog}_B(n)$ for infinitely many $n$.

**Case (i):** There do not exist $a_0, \ldots, a_\ell \in \mathbb{Q}$ such that $s_{2n} = \prod_{j=0}^\ell r_j^{2n}$ for all $n$, so we have in particular that $k \geq 2$ and $s_n \neq 0$ for infinitely many $n$. Notice also that $|x_{i,j}| \neq 1$ for some $(i,j) \in [1,k] \times [1,\ell]$, as otherwise $s_{2n} = \sum_{i=1}^k x_{i,0}$.

Without loss of generality, we assume that $x_{i,j} \neq 0$ for all $(i,j) \in [1,k] \times [0,\ell]$, and actually that $x_{i,j} > 0$ for $j \neq 0$, since $\prod_{j=0}^\ell x_{i,j} = x_{i,0} \cdot \prod_{j=1}^\ell |x_{i,j}|^{2n}$.

Accordingly, we may also assume, as we do, that $(x_{i,1}, \ldots, x_{i,\ell}) < (x_{j,1}, \ldots, x_{j,\ell})$ for $1 \leq i < j \leq k$, where $<$ denotes the binary relation on $\mathbb{R}^\ell$ defined by taking $(u_1, \ldots, u_\ell) < (v_1, \ldots, v_\ell)$ if and only if $|u_i| < |v_i|$ for some $i \in [1,\ell]$ and $|u_j| = |v_j|$ for $i < j \leq \ell$: This is because the $\ell$-tuples $(x_{i,1}, \ldots, x_{i,\ell})$ cannot be equal to each other for all $i \in [1,k]$, and on the other hand, if two of these tuples are equal then we can add up some terms in (1) so as to obtain a sum of superpowers of degree $\ell$, but with fewer summands.

Now, for each $(i,j) \in [1,k] \times [0,\ell]$ let $\alpha_{i,j}, \beta_{i,j} \in \mathbb{Z}$ be such that $\alpha_{i,j} > 0$ and $x_{i,j} = \alpha_{i,j}^{-1} \beta_{i,j}$, and consequently set $\tilde{x}_{i,j} := \alpha_{i,j} x_{i,j}$, where $\alpha_j := \alpha_{1,j} \cdots \alpha_{k,j}$; note that $\tilde{x}_{i,j}$ is a nonzero integer, and $\tilde{x}_{i,j} > 0$ for $j \neq 0$. Then, define $u_n := \sum_{i=1}^k \prod_{j=0}^\ell \tilde{x}_{i,j}^{\alpha_{i,j}}$ and $v_n := \prod_{j=0}^\ell \alpha_{i,j}^{\beta_{i,j}}$.

It is clear that $(u_n)_{n \geq 1}$ and $(v_n)_{n \geq 1}$ are integer sequences and $(\tilde{x}_{i,1}, \ldots, \tilde{x}_{i,\ell}) < (\tilde{x}_{j,1}, \ldots, \tilde{x}_{j,\ell})$ for $1 \leq i < j \leq k$. Moreover, $s_n = u_n v_n^{-1}$ and, hence, $\omega(s_n) \geq \omega(u_n) - \omega(v_n) = \omega(u_n) - \omega(v_1)$ for all $n$, which shows it is sufficient to prove the existence of a base $B > 1$ such that $\omega(s_{2n}) > \text{slog}_B(2n)$ for infinitely many $n$, and ultimately implies, along with the rest, that we can further assume (again, as we do) that $x_{i,j}$ is an integer for all $(i,j) \in [1,k] \times [0,\ell]$.

Putting it all together, it follows that $x_{i,j} \geq 2$ for some $(i,j) \in [1,k] \times [1,\ell]$, and in addition there exists $N \in \mathbb{N}^+$ such that

$$\sum_{i=1}^k \prod_{j=0}^\ell x_{i,j}^{\alpha_{i,j}} \neq 0 \quad \text{for all } n \geq N \text{ and } \emptyset \neq I \subseteq [1,k].$$

We claim that it is enough, as well, to assume (once more, as we do) that $\delta_0 = \cdots = \delta_\ell = 1$, where for each $j \in [0,\ell]$ we let $\delta_j$ be the greatest common divisor of $x_{1,j}, \ldots, x_{k,j}$.

In fact, define, for $1 \leq i \leq k$ and $0 \leq j \leq \ell$, $\delta_{i,j} := \delta_{i,j}^{-1} x_{i,j}$, and let $(w_n)_{n \geq 1}$ and $(s_n)_{n \geq 1}$ be the integer sequences of general term $\prod_{j=0}^\ell s_{i,j}^{\delta_{i,j}}$ and $\sum_{i=1}^k \prod_{j=0}^\ell s_{i,j}^{\delta_{i,j}}$, respectively. Then $s_{2n} = w_{2n} s_{2n}$, and hence $\omega(s_{2n}) \geq \omega(s_{2n})$, for every $n \in \mathbb{N}^+$. On the other hand, there cannot exist $\delta_0, \ldots, \delta_\ell \in \mathbb{Z}$ such that $s_{2n} = \prod_{j=0}^\ell (\alpha_{j}(\tilde{a}_j))^{2n}$ for all $n$, as otherwise $s_{2n} = \prod_{j=0}^\ell (\delta_{j}(\tilde{a}_j))^{2n}$ (which is absurd). This leads to the claim.

With the above in mind, let $\mathcal{P}$ be the set of all (positive) prime divisors of $\mathfrak{z} := \prod_{i=1}^k \prod_{j=1}^\ell x_{i,j}$, and note that $\mathcal{P}$ is finite and nonempty, as the previous considerations yield $|\mathfrak{z}| \geq 2$.

Next, let $v_p(x)$ denote, for a prime $p$ and a nonzero integer $x$, the exponent of the largest power of $p$ dividing $x$. Then, for every $n \in \mathbb{N}^+$ we can write

$$s_{2n} = \sum_{i=1}^k \left( x_{i,0} \prod_{p \in \mathcal{P}} p^{v_p(x_{i,0})}(2n) \right).$$

(5)
where for each $i = 1, \ldots, k$ and $p \in \mathcal{P}$ we let $e_p^{(i)}$ be the function $N^+ \to N: n \mapsto \sum_{j=0}^\ell n^j v_p(x_{i,j})$.

Since we assumed $\delta_0 = \cdots = \delta_\ell = 1$, it is easily seen that for every $p \in \mathcal{P}$ there are $i, j \in [1, k]$ for which $e_p^{(i)} \neq e_p^{(j)}$, and there exist $i_p \in [1, k]$ and $n_p \geq N$ such that $e_p^{(i_p)}(n) < e_p^{(j)}(n)$ for all $n \geq n_p$ and $i \in [1, k]$ for which $e_p^{(i)} \neq e_p^{(i_p)}$. Let $n_P := \max_{p \in \mathcal{P}} n_p$ (recall that $\mathcal{P}$ is a nonempty finite set), and for each $p \in \mathcal{P}$ and $i \in [1, k]$ define $\Delta e_p^{(i)} := e_p^{(i)} - e_p^{(i_p)}$. Then set

$$
\pi_n := \prod_{p \in \mathcal{P}} p^{e_p^{(i_p)}(n)} \quad \text{and} \quad \sigma_n := \sum_{i=1}^k \left( x_{i,0} \prod_{p \in \mathcal{P}} p^{\Delta e_p^{(i)}(n)} \right). \tag{6}
$$

We observe that $|s_{2n}| = \pi_{2n} \cdot |\sigma_{2n}|$; furthermore, if $n \geq n_P$ then $\sigma_n$ is an integer, and actually a nonzero integer by (4). Thus, $\omega(s_{2n}) \geq \omega(\sigma_{2n})$ for $n \geq n_P$, and it will suffice to show that for some $B > 1$ there exist infinitely many $n$ such that $\omega(\sigma_{2n}) > \slog_B(2n)$.

On the other hand, having assumed that $(x_{1,1}, \ldots, x_{i,\ell}) < \cdots < (x_{1,k}, \ldots, x_{k,\ell})$ and $x_{i,j} > 0$ for all $(i, j) \in [1, k] \times [1, \ell]$ yields, together with (5), that

$$
\lim_{n \to \infty} \prod_{p \in \mathcal{P}} p^{e_p^{(k)}(n)} - e_p^{(i)}(n) = \lim_{n \to \infty} \prod_{j=1}^\ell \left( \frac{x_{k,j}}{x_{i,j}} \right)^{n^j} = \infty \quad \text{for each } i \in [1, k-1], \tag{7}
$$

and consequently

$$
|s_n| \sim |x_{k,0}| \cdot \prod_{j=1}^\ell x_{k,j}^j = |x_{k,0}| \cdot \prod_{p \in \mathcal{P}} p^{e_p^{(k)}(n)} \quad \text{as } n \to \infty \tag{8}
$$

and

$$
|\sigma_{2n}| = \frac{|s_{2n}|}{\pi_{2n}} \sim |x_{k,0}| \cdot \prod_{p \in \mathcal{P}} p^{\Delta e_p^{(k)}(2n)} \quad \text{as } n \to \infty. \tag{9}
$$

We want to show that the sequence $(\sigma_{2n})_{n \geq 1}$ is eventually (strictly) increasing in absolute value.

**Lemma 1.** There exists $p \in \mathcal{P}$ such that $\Delta e_p^{(k)}(n) \to \infty$ as $n \to \infty$.

**Proof of Lemma 1.** Suppose the contrary is true. Then, for each $p \in \mathcal{P}$ we must have $e_p^{(k)} = e_p^{(i_p)}$, since $\Delta e_p^{(k)}(n)$ can be understood as a homogeneous polynomial with integral coefficients in the variable $n$ and $\Delta e_p^{(k)}(n) = e_p^{(k)}(n) - e_p^{(i_p)}(n) \geq 0$ for $n \geq n_P$. Therefore, we get by (7) that

$$
\prod_{p \in \mathcal{P}} p^{e_p^{(k)}(n)} \leq \prod_{p \in \mathcal{P}} p^{e_p^{(i_p)}(n)} \leq \prod_{p \in \mathcal{P}} p^{e_p^{(k)}(n)} = \prod_{p \in \mathcal{P}} p^{e_p^{(i_p)}(n)}
$$

for all $n \geq n_P$ and $i \in [1, k]$. But this is absurd, as it implies that $e_p^{(i)} = e_p^{(i_p)}$ for all $p \in \mathcal{P}$ and $i \in [1, k]$, and hence, in view of (5), $s_{2n} = (x_{1,0} + \cdots + x_{k,0}) \cdot \prod_{p \in \mathcal{P}} p^{e_p^{(i_p)}(2n)}$ for all $n$. •

With this in hand, let $A > |x_{k,0}| \cdot \prod_{j=1}^\ell x_{j,k}$. Then, using that $\Delta e_p^{(k)}$ is eventually nondecreasing for every $p \in \mathcal{P}$ (recall that $\Delta e_p^{(k)}$ is a polynomial function and $\Delta e_p^{(k)}(n) \geq 0$ for large $n$), we get from (8) and Lemma 1 that there exists $n_0 \geq n_P$ such that, for every $n \geq n_0$,

$$
\sigma_{2n}^2 \leq s_{2n}^2 < A(2n)^\ell \quad \text{and} \quad |\sigma_{2n+2}| > |\sigma_{2n}|. \tag{10}
$$
Now, denote by $Q_\kappa$ the set of all prime divisors of $\sigma_\kappa$ and let $Q_\kappa^* := Q_\kappa \setminus P$; we note that $Q_\kappa$ is finite for $n \geq n_0$ (recall that $\sigma_\kappa \neq 0$ for $n \geq n_P$). Next, let

$$\lambda := \max_{p \in P} v_p(\sigma_{2n_0}) + \max_{p \in P} \Delta e_p^{(i)}(2n_0),$$

and then

$$\alpha := k \cdot \max_{1 \leq i \leq k} |x_{i,0}| \cdot \prod_{p \in P} p^\lambda \quad \text{and} \quad \beta := \prod_{p \in P} p^{\lambda-1}(p-1).$$

Lastly, suppose for a fixed $\kappa \in \mathbb{N}$ that we have determined even integers $r_0, \ldots, r_\kappa \in \mathbb{N}^+$ such that $n_0 \leq r_0 \leq \cdots \leq r_\kappa$, and define $\beta_\kappa := \beta \cdot \prod_{p \in Q_\kappa} p^{v_p(\sigma_{r_\kappa})}(p-1)$.

By taking $r_0 := 2n_0$ and $r_{\kappa+1} := 2\beta_\kappa + r_\kappa$, the recursive construction we have just described then yields a (strictly) increasing sequence $(r_\kappa)_{\kappa \geq 0}$ of even integers $\geq n_0$ with the property that, however we choose a prime $p \in P$ and an index $i \in [1, k]$, it holds

$$\Delta e_p^{(i)}(r_{\kappa+1}) \equiv \Delta e_p^{(i)}(r_\kappa) \mod q^{\alpha-1}(q-1) \quad \text{for all } q \in P$$

and

$$\Delta e_p^{(i)}(r_{\kappa+1}) \equiv \Delta e_p^{(i)}(2n_0) \mod q^{\alpha-1}(q-1) \quad \text{for all } q \in Q_\kappa^*,$$

where we use that $r_{\kappa+1} \equiv r_\kappa \mod m$ whenever $m | \beta_\kappa$ and, as was already mentioned, $\Delta e_p^{(i)}$ is essentially an integral polynomial. In particular, (11) and a routine induction imply that

$$\Delta e_p^{(i)}(r_\kappa) \equiv \Delta e_p^{(i)}(2n_0) \mod q^{\alpha-1}(q-1) \quad \text{for all } p, q \in P, \ i \in [1, k] \text{ and } \kappa \in \mathbb{N}. \quad (13)$$

Also, we get from (10) and $r_\kappa \geq 2n_0$ that there exists $B \geq A$ such that, for all $\kappa$,

$$r_{\kappa+1} \leq r_\kappa + 2\beta \cdot \prod_{p \in Q_\kappa} p^{v_p(\sigma_{r_\kappa})}(p-1) \leq r_\kappa + 2\beta \sigma_{r_\kappa}^2 < r_\kappa + 2\beta A^{\ell} < B^{r^\ell}. \quad (14)$$

Based on these premises, we now prove a few more lemmas; to ease notation, we will denote by $I_p$, for each $p \in P$, the set of all indices $i \in [1, k]$ such that $e_p^{(i)} \neq e_p^{(i_p)}$, and let $I_p^* := [1, k] \setminus I_p$.

**Lemma 2.** $Q_\kappa \subseteq Q_{r_{\kappa+1}}$ for every $\kappa$.

**Proof of Lemma 2.** Pick $\kappa \in \mathbb{N}$ and $q \in Q_{r_\kappa}$. If $i \in I_p$, then $\Delta e_p^{(i)}(n) = 0$ for all $n$, and hence $p^{\Delta e_p^{(i)}(n)} = 1$. If, on the other hand, $i \in I_p^*$, then $\Delta e_p^{(i)}(n) > 0$ for $n \geq n_P$, with the result that $p^{\Delta e_p^{(i)}(n)} \equiv 0 \mod q$ if $q = p$, and $p^{\Delta e_p^{(i)}(n)} \equiv p^m \mod q$ if $p \neq q$ and $\Delta e_p^{(i)}(n) \equiv m \mod (q-1)$, in the light of Fermat’s little theorem.

So putting it all together, we get from (11), (12) and $r_{\kappa+1} > r_\kappa \geq 2n_0 > n_P$ that

$$p^{\Delta e_p^{(i)}(r_{\kappa+1})} \equiv p^{\Delta e_p^{(i)}(r_\kappa)} \mod q \quad \text{for all } p \in P \text{ and } i \in [1, k],$$

which in turn implies that

$$\sigma_{r_{\kappa+1}} = \sum_{i=1}^k \left( x_{i,0} \prod_{p \in P} p^{\Delta e_p^{(i)}(r_{\kappa+1})} \right) \equiv \sum_{i=1}^k \left( x_{i,0} \prod_{p \in P} p^{\Delta e_p^{(i)}(r_\kappa)} \right) \equiv \sigma_{r_\kappa} \equiv 0 \mod q.$$ 

This concludes the proof, by the arbitrariness of $\kappa \in \mathbb{N}$ and $q \in Q_{r_\kappa}$.

**Lemma 3.** Let $q \in P$ and $\kappa \in \mathbb{N}$. Then $v_q(\sigma_{r_\kappa}) \leq \alpha - 1$.\hfill \blacksquare
Proof of Lemma 3. The claim is straightforward if \( \kappa = 0 \), since \( r_0 = 2n_0 \) and \( v_q(\sigma_{2n_0}) \leq \lambda < \alpha \). So assume for the rest of the proof that \( \kappa \geq 1 \). Then, we have from (6) that

\[
\sigma_n = \sum_{i \in \mathcal{I}_q} \left( x_{i,0} \prod_{p \in \mathcal{P}} p^{\Delta e^{(i)}_p(n)} \right) + \sum_{i \in \mathcal{I}_q^s} \left( x_{i,0} \prod_{q \neq p \in \mathcal{P}} p^{\Delta e^{(i)}_p(n)} \right) \quad \text{for all } n.
\]

(15)

If \( i \in \mathcal{I}_q \), \( n > 2n_0 \) and \( n \equiv 2n_0 \mod \beta \) then \( q^\alpha \) divides \( \prod_{p \in \mathcal{P}} p^{\Delta e^{(i)}_p(n)} \), because \( n \mid \Delta c^{(i)}_p(n) \) and \( \Delta c^{(i)}_p(n) \neq 0 \), hence \( \alpha < \beta < \beta + 2n_0 \leq n \leq \Delta c^{(i)}_p(n) \).

On the other hand, it is seen by induction that \( r_\kappa = 2n_0 \mod \beta \) (recall that \( r_\kappa = r_{\kappa-1} \mod \beta \)). Thus, we get from the above, equations (15) and (13), [1, Theorem 2.5(a)], and Euler’s totient theorem that

\[
\sigma_{r_\kappa} \equiv \sum_{i \in \mathcal{I}_q} \left( x_{i,0} \prod_{q \neq p \in \mathcal{P}} p^{\Delta e^{(i)}_p(2n_0)} \right) \equiv \sum_{i \in \mathcal{I}_q} \left( x_{i,0} \prod_{q \neq p \in \mathcal{P}} p^{\Delta e^{(i)}_p(2n_0)} \right) \mod q^\alpha.
\]

(16)

But \( \emptyset \neq \mathcal{I}_q^s \subseteq [1, k] \), so it follows from (4) that

\[
0 < \frac{1}{2n_0} \sum_{i \in \mathcal{I}_q^s} \prod_{j=0}^{\ell} x_{i,j}^{(2n_0)} = \sum_{i \in \mathcal{I}_q} \left( x_{i,0} \prod_{q \neq p \in \mathcal{P}} p^{\Delta e^{(i)}_p(2n_0)} \right) \leq \max_{1 \leq i \leq k} |x_{i,0}| \cdot \prod_{p \in \mathcal{P}} p^\lambda = \alpha < q^\alpha,
\]

which, together with (16), yields that \( v_q(\sigma_{r_\kappa}) < \alpha \).

Lemma 4. Let \( \kappa \in \mathbb{N}^+ \) and \( q \in \mathcal{Q}_{r_\kappa} \). Then \( v_q(\sigma_{r_\kappa}) = v_q(\sigma_{r_{\kappa+1}}) \).

Proof of Lemma 4. If \( q \notin \mathcal{P} \), then we have from (6) and (12), [1, Theorem 2.5(a)], and Euler’s totient theorem that \( \sigma_{r_{\kappa+1}} \equiv \sigma_{r_\kappa} \mod q^\nu(\sigma_{r_{\kappa+1}})^{+1} \), and we are done.

If, on the other hand, \( q \in \mathcal{P} \), then we get from Lemma 3 that \( v_q(\sigma_{r_\kappa}) \leq \alpha - 1 \), which, along with (16), entails that \( \sigma_{r_{\kappa+1}} \equiv \sigma_{r_\kappa} \mod q^\nu(\sigma_{r_{\kappa+1}})^{+1} \), and consequently \( v_q(\sigma_{r_\kappa}) = v_q(\sigma_{r_1}) \).


At long last, we are almost there. In fact, since the sequence \( \sigma_{r_\kappa} \) is (strictly) increasing in modulus (as was mentioned before) and \( r_\kappa \) is even and \( \geq 2n_0 \) for all \( \kappa \), it follows from Lemmas 2-4 that \( \emptyset \neq \mathcal{Q}_{r_\kappa} \subseteq \mathcal{Q}_{r_{\kappa+1}} \), and hence \( \omega(\sigma_{r_\kappa}) < \omega(\sigma_{r_{\kappa+1}}) \), for all \( \kappa \in \mathbb{N}^+ \). By a routine induction, this in turn implies that \( \omega(\sigma_{r_\kappa}) \geq \kappa \) for all \( \kappa \).

On the other hand, if we let \( C := \max(B^\ell, r_\ell) \), then we get from (14) and another induction that \( r_\kappa < C \parallel \kappa \) for all \( \kappa \in \mathbb{N}^+ \), which, together with the above considerations, ultimately leads to \( \slog_C(r_\kappa) < \kappa \leq \omega(\sigma_{r_\kappa}) \), and hence to the desired conclusion.

Case (ii): There do not exist \( b_0, \ldots, b_\ell \in \mathcal{Q} \) such that \( s_{2n-1} = \prod_{j=0}^{\ell} b_j^{(2n-1)^j} \) for all \( n \). Then, we are reduced to the previous case by taking

\[
y_{i,j} := \prod_{h=j}^{\ell} x_{i,h}^{(-1)^{h-j}(k)} \quad \text{for } 1 \leq i \leq k \text{ and } 0 \leq j \leq \ell,
\]
and noticing that for every \( n \in \mathbb{N}^+ \) we have \( s_{2n-1} = t_{2n} \), where \((t_n)_{n \geq 1}\) is the integer sequence of general term \( \sum_{i=1}^{k} \prod_{j=0}^{\ell} y_{i,j}^{n} \) (we omit further details).

**Proof of Corollary.** Suppose to a contradiction that there are \( c_1, \ldots, c_k \in \mathbb{Q}^+ \) and \( x_1, \ldots, x_k \in \mathbb{Q} \setminus \{0\} \) such that \( |x_i| \neq |x_j| \) for some \( i, j \in [1, k] \) and \((\omega(u_n))_{n \geq 1}\) is bounded, where \( u_n := \sum_{i=1}^{k} c_i x_i^n \) for all \( n \), and let \( k \) the minimal positive integer for which this is pretended to be true.

Then \( k \geq 2 \), and there is no loss of generality in assuming, as we do, that \( |x_1| \leq \cdots \leq |x_k| \neq |x_1| \). Furthermore, we get from the main theorem of this paper that there must exist \( c, x \in \mathbb{Q}^+ \) such that \( u_{2n} = c x^{2n} \). So now, we have two cases, each of which will lead to a contradiction (the rest is trivial and we may omit details):

**Case (i):** \( x \leq |x_k| \). We have \( cy^{2n} = \sum_{i=1}^{k} c_i y_i^{2n} \) for all \( n \), where \( y_i := |x_i| \cdot |x_k|^{-1} \) for \( 1 \leq i \leq k \) and \( y := x \cdot |x_k|^{-1} \). Let \( h \) be the maximal index in \([2, k]\) such that \( y_{h-1} < y_k \), which exists because \( y_1 < y_k \). Since \( 0 < y \leq 1 \) and \( 0 < y_i < 1 \) for \( 1 \leq i < h \), we find that

\[
c \cdot \lim_{n \to \infty} y^{2n} = c_h + \cdots + c_k,
\]

which is possible only if \( y = 1 \), as \( c_h, \ldots, c_k > 0 \). But then \( c = c_1 + \cdots + c_k \), and consequently \( \sum_{i=1}^{h-1} c_i y_i^{2n} = 0 \) for all \( n \), which is absurd, because \( h \geq 2 \) and \( c_1, \ldots, c_{h-1} > 0 \).

**Case (ii):** \( x > |x_k| \). Then \( c = \sum_{i=1}^{k} c_i z_i^{2n} \) for all \( n \), where \( z_i := |x_i| \cdot x^{-1} \) for \( 1 \leq i \leq k \). But this is still absurd, since \( z_1, \ldots, z_k \in ]0,1[ \), and hence \( \sum_{i=1}^{k} c_i z_i^{2n} \to 0 \) as \( n \to \infty \).

4. **Closing remarks**

We list here some questions we hope to pick up in future work: Let \( \tau \) be a (strictly) increasing function from \( \mathbb{N}^+ \) into itself. What can be said about the behavior of \( \omega(s_{\tau(n)}) \) as \( n \to \infty \)? In particular, is it true that \( \limsup_{n \to \infty} \omega(s_{\tau(n)}) = \infty \)? Is it possible to obtain nontrivial bounds on the greatest prime divisors of \( s_{\tau(n)} \) as \( n \to \infty \)? And what about the asymptotic growth of the average of the function \( \mathbf{R}^+ \to \mathbf{N} : x \mapsto \# \{ n \leq x : \omega(s_{\tau(n)}) \geq h \} \) for a fixed \( h \in \mathbb{N}^+ \)? (If \( S \) is a set, we write \#\( S \) the cardinality of \( S \).)

In this paper, we have considered the case where \( \tau \) is the identity or, more in general, a polynomial function (by the considerations made in relation to equation 2 in the introduction). So it could be interesting to answer the above questions under the assumption that \( \tau \) is, e.g., a geometric progression, which however may be hard, as an affirmative answer would then imply the existence of infinitely many composite Fermat numbers (to the best of our knowledge, still a longstanding open problem).

On the other hand, the basic question addressed in the present manuscript has the following algebraic generalization (we refer to [7, Ch. 1] for background on divisibility and related topics in ring theory): Given a unique factorization domain \( \mathbb{F} = (F, +, \cdot) \), let \( \theta, j \) be, for \( 1 \leq i \leq k \) and \( 0 \leq j \leq \ell \), some fixed elements in \( F \), and for \( x \in F \) let \( \omega_F(x) \) denote the number of non-associate primes dividing \( x \), where two nonzero elements in \( F \) are non-associate (in \( F \)) if their ratio is not a unit of \( F \). What can be said about the sequence \((\theta_n)_{n \geq 1}\) of general term \( \sum_{i=1}^{k} \prod_{j=0}^{\ell} \theta_{i,j}^{n} \) if
the sequence \((\omega_\varphi(\vartheta_n))_{n \geq 1}\) is bounded? More specifically, does anything in the lines of our main theorem hold true?

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**References**

[1] T. M. Apostol, *Introduction to Analytic Number Theory*, Undergrad. Texts Math., Springer-Verlag: New York, 1976.
[2] A. Baker and G. Wüstholz, *Logarithmic forms and Diophantine geometry*, New Mathematical Monographs 9, Cambridge University Press: Cambridge, 2007.
[3] R. G. Bartle, *The Elements of Real Analysis*, John Wiley & Sons, 1976 (2nd edition).
[4] J.-H. Evertse, *On sums of \( S \)-units and linear recurrences*, Compositio Math. 53 (1984), No. 2, 225–244.
[5] D. E. Knuth, *Mathematics and Computer Science: Coping with Finiteness*, Science 194 (1976), No. 4271, 1235–1242.
[6] F. Luca, *Prime divisors of binary holonomic sequences*, Adv. Appl. Math. 40 (2008), No. 2, 168–179.
[7] R. A. Mollin, *Algebraic Number Theory*, Discrete Math. Appl., CRC: Boca Raton (FL), 2011 (2nd ed.).
[8] H. P. Schlickewei, *Linearformen mit algebraischen Koeffizienten*, Manuscripta Math. 18 (1976), 147–185.
[9] R. A. Smith and M. V. Subbarao, *The average number of divisors in an arithmetic progression*, Canad. Math. Bull. 24 (1981), No. 1, 37–41.
[10] T. N. Shorey and R. Tijdeman, *Exponential Diophantine Equations*, Cambridge Tracts in Mathematics 87, Cambridge Univ. Press: Cambridge, 1986.
[11] I. E. Shparlinski, *Prime divisors of recurrent sequences*, Izv. Vyssh. Uchebn. Zaved. Math. 215 (1980), 101–103.
[12] C. L. Stewart, *On divisors of terms of linear recurrence sequences*, J. reine angew. Math. 333 (1982), 12–31.
[13] _, “On Prime Factors of Terms of Linear Recurrence Sequences”, in J. M. Borwein, I. Shparlinski, and W. Zudilin (eds.), *Number Theory and Related Fields - In Memory of Alf van der Poorten*, Springer Proc. Math. Stat. 43, Springer: New York (2013), 341–359.
[14] A. J. van der Poorten and H. P. Schlickewei, *The growth conditions for recurrence sequences*, Macquarie Math. Reports 82-0041 (1982).
[15] K. Zsigmondy, *Zur Theorie der Potenzreste*, Monatsh. Math. 3 (1892), 265–284.

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