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Time-frequency analysis on groups

Abstract:  Phase-space analysis or time-frequency analysis can be thought as Fourier analysis simultaneously both in time and in frequency, originating from signal processing and quantum mechanics. On groups having unitary Fourier transform, we introduce and study a natural family of time-frequency transforms, and investigate the related pseudo-differential operators.

1 Introduction

Time-frequency analysis is a subfield of Fourier analysis. It studies “time” dependent signals (functions or distributions), presenting them simultaneously both in “time” and in “frequency”, and consequently manipulating them as sharp as possible. Traditionally, “time” and “frequency” refer to real variables, where the Fourier integral transform is the essential tool. In this text, we establish time-frequency analysis on those locally compact groups that allow a unitary Fourier transform.

Time-frequency transforms in Cohen’s class present signals as joint time-frequency distributions, which are linked to the pseudo-differential operators for manipulating signals. The time-frequency concepts apply to the phase-space analysis, e.g. for position-momentum presentations of wavefunctions in quantum mechanics. One of Cohen’s original motivating examples in [5] was the deduction of the Born–Jordan phase-space transform, stemming from the Born–Jordan quantization of Heisenberg’s matrix mechanics [18, 1, 2]. Time-frequency analysis has been studied for p-adic numbers [17], on more general locally compact commutative groups [23], and on certain classes of locally compact groups [24]. However, our treatise is not reduced to those works.

For a compact group, the time-frequency plane is the Cartesian product of the group and its unitary dual. Time-frequency transforms will be “time-frequency invariant” sesquilinear mappings on pairs of test functions (trigonometric polynomials, or Schwartz–Bruhat functions), with values in the corresponding space of matrix-valued test functions on the time-frequency plane. In the non-commutative setting, the “frequency modulations” require careful rethinking. Euclidean time-frequency analysis is usually built around the symmetric Wigner transform, corresponding to the Weyl pseudo-differential quantization. However, groups often lack suitable scalings, so we build our time-frequency analysis around the always existing Rihaczek or Kohn–Nirenberg transform: this could have been the starting point for the Euclidean theory. A time-frequency transform dictates a pseudo-differential quantization, and we shall study this connection. On compact Lie groups, the Kohn–Nirenberg quantization has been treated e.g. in [29, 26, 27, 28, 11]. The compact group results are finally generalized to those locally compact groups that allow a unitary Fourier transform.
2 On Euclidean time-frequency analysis

To motivate our definitions for time-frequency analysis on compact groups \(G\), let us briefly explain how analogous concepts can be presented on Euclidean spaces \(\mathbb{R}^n\), avoiding technicalities. The general background is presented in the monographs [6] and [15]. To underline the similarities, we use quite similar notions both on \(G\) and on \(\mathbb{R}^n\). Signals are nice-enough functions \(u : \mathbb{R}^n \to \mathbb{C}\). We call variables \(x,y \in \mathbb{R}^n\) time-like (or position-like) and variables \(\xi, \eta \in \mathbb{R}^n \cong \mathbb{R}^n\) frequency-like (or momentum-like). The starting point is formula

\[
  u(x) = \int \int e^{i2\pi(x-y) \cdot \eta} u(y) \, dy \, d\eta
\]

for the Schwartz test functions \(u \in \mathcal{S}(\mathbb{R}^n)\). Define the Fourier transform \(\hat{u}\) by

\[
  \hat{u}(\eta) := \int e^{-i2\pi y \cdot \eta} u(y) \, dy.
\]

From the Schwartz test function space \(\mathcal{S}(\mathbb{R}^n)\), the Fourier transform extends to a unitary operator \(F : L^2(\mathbb{R}^n) \to L^2(\widehat{\mathbb{R}^n})\): in other words, \(F = (u \mapsto \hat{u})\) is a linear bijection satisfying

\[
  \langle u, v \rangle = \langle \hat{u}, \hat{v} \rangle,
\]

where Hilbert space \(L^2(\mathbb{R}^n)\) has the inner product defined by

\[
  \langle u, v \rangle = \int u(x) v(x)^\ast \, dx,
\]

where \(\lambda^\ast\) is the complex conjugate of \(\lambda \in \mathbb{C}\). Signal \(u\) has the norm \(\|u\| = \langle u, u \rangle^{1/2}\) and the energy \(\|u\|^2 = \langle u, u \rangle\). The symplectic Fourier transform is then \(F = \mathcal{F}^{-1} \otimes \mathcal{F}\), taking functions on the time-frequency plane (or phase-space) \(\mathbb{R}^n \times \mathbb{R}^n\) to functions on the ambiguity plane \(\widehat{\mathbb{R}^n} \times \mathbb{R}^n\). A Cohen class time-frequency transform \(D\) of signals \(u, v\) is \(D(u,v) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}\),

\[
  D(u,v)(x,\eta) = \int e^{-i2\pi y \cdot \eta} e^{i2\pi x \cdot \xi} \phi(\xi,y) \, FW(u,v)(\xi,y) \, d\xi \, dy,
\]

where \(\phi : \widehat{\mathbb{R}^n} \times \mathbb{R}^n \to \mathbb{C}\) is the ambiguity kernel, \(W(u,v) : \mathbb{R}^n \times \widehat{\mathbb{R}^n} \to \mathbb{C}\) is the Wigner transform,

\[
  W(u,v)(x,\eta) := \int e^{-i2\pi y \cdot \eta} u(x+y/2) v(x-y/2)^\ast \, dy,
\]

and \(FW(u,v) : \widehat{\mathbb{R}^n} \times \mathbb{R}^n \to \mathbb{C}\) is the ambiguity transform,

\[
  FW(u,v)(\xi,y) = \int e^{-i2\pi x \cdot \xi} u(x-y/2) v(x-y/2)^\ast \, dx.
\]
As pointed out by Gröchenig in [15], in the literature there is no precise definition of a Cohen class transform $D$. Informally, such $D$ is obtained by smoothing the Wigner transform by some tempered distribution as the convolution kernel. In light of the time-frequency results in the sequel, we would suggest that the ambiguity kernel $\phi$ should be a smooth function with polynomially bounded derivatives: in other words, then we would have a Schwartz multiplier

$$(h \mapsto F^{-1}(\phi Fh)) : \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n).$$

Indeed, the literature examples of ambiguity kernels $\phi$ seem to be smooth with polynomially bounded derivatives. Moreover, those examples in the literature are typically bounded with $|\phi(\xi, y)| \leq 1$, which yields the $L^2$-boundedness

$$\|D(u, v)\|_{L^2} \leq \|u\|_{L^2} \|v\|_{L^2}.$$ 

Hence $(u, v) \mapsto D(u, v)$ is sesquilinear: $u \mapsto D(u, v)$ is linear, and $v \mapsto D(u, v)$ conjugate-linear. The idea is that the time-frequency distribution $D[u] := D(u, u)$ would be a quasi-energy density for signal $u$ (or a quasi-probability density for wavefunction $u$). If $v(x) = e^{+i2\pi x \cdot \xi} u(x - y)$ then

$$D[v](x, \eta) = D[u](x - y, \eta - \xi), \quad \tau \in \mathbb{R},$$

reflecting the idea that $v$ is “shifted in time-frequency by $(y, \xi)$”.

For example, if the ambiguity kernel $\phi$ in (6) is given by $\phi(\xi, y) := e^{i2\pi(\xi y)}$ for $\tau \in \mathbb{R}$, this defines the Rihaczek-$\tau$-transform $D = R_{\tau}$, where

$$R_{\tau}(u, v)(x, \eta) = \int e^{-i2\pi y \cdot \eta} u(x + (\tau + 1/2)y) v(x + (\tau - 1/2)y)^* \, dy.$$  \hfill (10)

Sometimes $W_{\tau} := R_{\tau + 1/2}$ is called the Wigner-$\tau$ or Shubin-$\tau$ transform. Transforms $R_{\tau}$ and $R_{-\tau}$ are conjugates to each other in the sense that $R_{\tau}(u, v)(x, \eta)^* = R_{-\tau}(v, u)(x, \eta)$. Especially, $R_0 = W$, the Wigner transform. The Kohn–Nirenberg transform (or the Rihaczek transform) is $R := R_{-1/2}$, which will be the starting point for time-frequency analysis on groups. The anti-Kohn–Nirenberg transform refers to $R_{+1/2}$. Here $D = R$ with $\phi(\xi, y) = e^{-i\pi \xi \cdot y}$, giving

$$R(u, v)(x, \eta) = u(x) e^{-i2\pi x \cdot \eta} \hat{v}(\eta)^*.$$  \hfill (11)

It is easy to check that

$$\|R_{\tau}(u, v)\|_{L^2(\mathbb{R}^n \times \mathbb{R}^n)} = \|u\|_{L^2(\mathbb{R}^n)} \|v\|_{L^2(\mathbb{R}^n)}.$$ 

Hence the Born–Jordan transform $Q$ defined by the integral average

$$Q(u, v) = \int_0^1 W_{\tau}(u, v) \, d\tau = \int_{-1/2}^{1/2} R_{\tau}(u, v) \, d\tau$$  \hfill (12)

satisfies

$$\|Q(u, v)\|_{L^2(\mathbb{R}^n \times \mathbb{R}^n)} \leq \|u\|_{L^2(\mathbb{R}^n)} \|v\|_{L^2(\mathbb{R}^n)}.$$
The ambiguity kernel of \( D = Q \) satisfies 
\[
\phi(\xi, y) = \int_0^1 e^{2\pi i \xi \cdot y (\tau - 1)} d\tau = \text{sinc}(\xi \cdot y),
\]
where \( \text{sinc}(t) = \sin(\pi t)/(\pi t) \) for \( t \neq 0 \).

Let \( \phi \) be the ambiguity kernel of time-frequency transform \( D \). Property \( \phi(0, 0) = 1 \) corresponds to the normalization
\[
\int \int D(u, v)(x, \eta) \, dx \, d\eta = \langle u, v \rangle.
\] (13)

Properties \( \phi(\xi, 0) = 1 \) and \( \phi(0, y) = 1 \) correspond respectively to the margins
\[
\int D(u, v)(x, \eta) \, d\eta = u(x) \, v(x)^*, \quad \int D(u, v)(x, \eta) \, dx = \hat{u}(\eta) \, \hat{v}(\eta)^*.
\] (14)

Property \( |\phi(\xi, y)| = 1 \) corresponds to so-called Moyal identity
\[
\langle D(u, v), D(f, g) \rangle = \langle u, f \rangle \langle v, g \rangle^*.
\] (15)

In applied sciences and engineering, perhaps the most common time-frequency transforms date back to Gabor’s work [14]: these transforms \( D \) are of the form
\[
D(u, v)(x, \eta) := \mathcal{G}_w u(x, \eta) \mathcal{G}_w v(x, \eta)^*,
\] (16)
where the \( w \)-windowed short-time Fourier transform (STFT) is defined by
\[
\mathcal{G}_w u(x, \eta) := \int e^{-i2\pi \eta \cdot y} u(y) \, w(y - x)^* \, dy,
\] (17)
where \( \phi = \mathcal{F}W[w]^* \). Then the normalization \([13]\) means \( \|w\|^2 = \langle w, w \rangle = 1 \), and then \( D[u] = D(u, u) \) is called the \( w \)-spectrogram of \( u \).

Once choosing a time-frequency transform \( D \), it defines the \( D \)-quantization \( a \mapsto a^D \) by the \( L^2 \)-duality
\[
\langle u, a^D v \rangle = \langle D(u, v), a \rangle.
\] (18)

Here the weight function \( a : \mathbb{R}^n \times \hat{\mathbb{R}}^n \to \mathbb{C} \) is called a symbol of pseudo-differential operator \( a^D = (v \mapsto a^D v) \). Conversely, time-frequency transform \( D \) can be recovered from the quantization map \( a \mapsto a^D \), whose properties reflect the properties of \( D \). Wigner-\( \tau \)-transform \( W_{\tau} = R_{\tau-1/2} \) corresponds to so-called Weyl-\( \tau \)-quantization \( a \mapsto a^{W_{\tau}} \),
\[
a^{W_{\tau}} v(x) = \int e^{i2\pi (x - \eta) \cdot \eta} a(x + \tau(y - \eta), \eta) \, v(y) \, dy \, d\eta,
\] (19)

Especially, the Wigner transform \( W = W_{1/2} = R_0 \) corresponds to the Weyl quantization \( a \mapsto a^W \). The Rihaczek (or Kohn–Nirenberg) transform \( R = W_0 = R_{-1/2} \) corresponds to the Kohn–Nirenberg quantization \( a \mapsto a^R \),
\[
a^R v(x) = \int \int e^{i2\pi (x - \eta) \cdot \eta} a(x, \eta) \, v(y) \, dy \, d\eta = \int e^{i2\pi x \cdot \eta} a(x, \eta) \, \hat{v}(\eta) \, d\eta.
\] (20)
The Born–Jordan quantization $a \mapsto a^Q = \int_0^1 a^W \cdot d\tau = \int_{-1/2}^{1/2} a^R \cdot d\tau$ satisfies

$$a^Q v(x) = \int \int e^{i2\pi(x-y) \cdot \eta} \int_0^1 a(x + \tau(y - x), \eta) \cdot d\tau v(y) \cdot dy \cdot d\eta.$$  \hspace{1cm} (21)

Weyl introduced his quantization in 1927 in [32], and Wigner his distribution in 1932 in [33] for quantum mechanics. The Wigner distribution was independently discovered in [31], with applications to signal processing. The Born–Jordan quantization was implicit in [1] for polynomial symbols, but in the modern sense the Born–Jordan distribution was deduced by Cohen in [5]. The Kohn–Nirenberg quantization arose from the studies [22, 21] by Hörmander, Kohn and Nirenberg.

### 3 Euclidean revision

On a compact group $G$, we cannot expect to find a reasonable analogy to Wigner transform $W$, which is the central object in the Euclidean case presented above. This is simply because analogies to the Euclidean scaling $(y \mapsto y/2) : \mathbb{R}^n \to \mathbb{R}^n$ are missing on a typical compact group $G$. This problem does not disappear by a naive doubling change of variable in the integral formula: see Example 10.7. Of course, on the odd-order cyclic group $\mathbb{Z}/N\mathbb{Z}$, such scalings $y \mapsto y/2$ exist in modular arithmetic.

On the other hand, there is no necessity to start with the symmetric Wigner transform in the Euclidean case, either. Instead, we could have built the Cohen class theory around the non-symmetric Kohn–Nirenberg quantization, and this approach will work on compact groups, too.

There is another illuminating point of view: Due to the time-frequency shift-invariance, time-frequency transform $D$ is already encoded in data

$$D(u, v)(0, 0) = \langle D(u, v), \delta \rangle = \langle u, \delta^D v \rangle,$$  \hspace{1cm} (22)

where $\delta = \delta_{(0,0)}$ is the Dirac delta distribution at the time-frequency origin $(0, 0) \in \mathbb{R}^n \times \mathbb{R}^n$. Despite such a highly singular symbol $\delta$, pseudo-differential operator $\delta^D$ is typically rather well-behaving. We call $\delta^D$ the original localization operator, as $\delta^D v$ tries to be the “localization of $v$ to the time-frequency origin”, which strictly speaking cannot be achieved in view of the Heisenberg uncertainty principle. If

$$\delta^D v(z) = \int K_{\delta^D} (z, y) \cdot v(y) \cdot dy,$$  \hspace{1cm} (23)

i.e. if $K_{\delta^D}$ is the Schwartz distribution kernel of $\delta^D$, then

$$D(u, v)(x, \eta) = \iiint u(x + z) \cdot e^{-i2\pi z \cdot \eta} \cdot K_{\delta^D} (z, y) \cdot e^{+i2\pi y \cdot \eta} \cdot v(x + y) \cdot * \cdot d\tau \cdot d\eta \cdot d\eta.$$  \hspace{1cm} (24)

This formula suggests a natural variant for compact groups $G$, where time shifts do not pose problems, whereas frequency modulations are elusive.
4 Fourier analysis on compact groups

Let $e$ denote the neutral element of group $G$. A topological group $G$ is a group and a Hausdorff space, where the group operation ($(x, y) \mapsto xy$) : $G \times G \to G$ and the inversion $(x \mapsto x^{-1}) : G \to G$ are continuous.

Time-frequency analysis on non-compact locally compact groups is treated in Section 15. Let $C(G)$ be the vector space of continuous functions $u : G \to \mathbb{C}$, endowed with the norm $\|u\|_{C(G)} = \max\{|u(x)| : x \in G\}$. Especially, the unit constant function $1 = (x \mapsto 1) : G \to \mathbb{C}$ belongs to $C(G)$. Let

$$\int u(x) \, dx = \int_{G} u(x) \, dx \in \mathbb{C}$$

(25)

be the Haar integral of $u \in C(G)$: the corresponding Haar measure is the unique translation-invariant Borel probability measure on $G$. We obtain the space $L^2(G)$ of square-integrable functions or signals by completing $C(G)$ with respect to the norm $\|u\| := \langle u, u \rangle^{1/2}$ given by the the inner product $(u, v) \mapsto \langle u, v \rangle$,

$$\langle u, v \rangle := \int u(x) \, v(x)^* \, dx.$$  

(26)

Here $\|u\|^2 = \langle u, u \rangle$ is the energy of the signal.

A unitary representation of compact group $G$ on Hilbert space $\mathcal{H}_\eta$ is a strongly continuous group homomorphism $\eta : G \to \mathcal{U}(\mathcal{H}_\eta)$ to the group $\mathcal{U}(\mathcal{H}_\eta)$ of unitary operators on $\mathcal{H}_\eta$. Hence $\eta(xy) = \eta(x) \eta(y)$, $\eta(x^{-1}) = \eta(x)^{-1} = \eta(x)^*$, $\eta(e) = I$ (the identity operator on $\mathcal{H}_\eta$). The Fourier coefficient of $u \in L^2(G)$ at $\eta$ is the bounded linear operator $\tilde{u}(\eta) = \mathcal{F}u(\eta) : \mathcal{H}_\eta \to \mathcal{H}_\eta$ defined by

$$\tilde{u}(\eta) = \mathcal{F}u(\eta) := \int u(x) \eta(x)^* \, dx.$$  

(27)

The left regular representation of $G$ is $\pi_L : G \to \mathcal{U}(L^2(G))$ defined by

$$\pi_L(y)u(x) := u(y^{-1}x)$$  

(28)

for almost all $x \in G$. The left regular representation $\pi_L$ can be thought to embed the group $G$ into the “rotations” acting on Hilbert space $\mathcal{H} = L^2(G)$: thus we can study the group by tools of functional analysis. Unitary representations $\xi, \eta$ of $G$ are equivalent if there is a unitary isomorphism $U : \mathcal{H}_\xi \to \mathcal{H}_\eta$ such that

$$U \xi(x) = \eta(x)U$$
for all \( x \in G \). The corresponding equivalence class is then denoted by \([\xi] = [\eta]\). Unitary representation \( \eta \) is called irreducible if for operators \( \eta(x) \) there are no non-trivial simultaneous invariant subspaces of \( \mathcal{H}_\eta \). Let

\[
\varepsilon = (x \mapsto 1) : G \to \mathbb{U}(\mathbb{C})
\]

denote the trivial irreducible unitary representation, corresponding to “zero frequency”, a unit signal with no oscillations. We distinguish the trivial unitary representation \( \varepsilon \) from the unit constant function \( 1 = (x \mapsto 1) : G \to \mathbb{C} \), even though they are effectively the same. This convention will clarify the treatise.

The unitary dual \( \hat{G} \) of \( G \) consists of equivalence classes \([\eta]\) of irreducible unitary representations of \( G \). To make notation lighter, instead of \([\eta]\) \( \in \hat{G} \) we simply write \( \eta \in \hat{G} \). Due to the compactness of \( G \), for each \( \eta \in \hat{G} \), Hilbert space \( \mathcal{H}_\eta \) is finite-dimensional. Hence in the sequel we assume that \( \eta(x) \in \mathbb{C}^{d_\eta \times d_\eta} \) is a unitary matrix of dimension \( d_\eta \in \mathbb{Z}^+ \): there is such a choice in that equivalence class \( \eta \in \hat{G} \). The corresponding Fourier coefficient \( \hat{\eta}(\eta) \) is a matrix, belonging to \( \mathbb{C}^{d_\eta \times d_\eta} \). Function \( u \in L^2(G) \) is called a trigonometric polynomial if it has only finitely many non-zero Fourier coefficients: in this sense, trigonometric polynomials are band-limited signals. Equivalently, \( u \in L^2(G) \) is a trigonometric polynomial if and only if the span of \( \{ \pi_L(y)u : y \in G \} \) is a finite-dimensional vector space. The space of trigonometric polynomials is denoted by \( \mathcal{T}(G) \).

By the Peter–Weyl theorem, the left regular representation can be decomposed to a direct sum of irreducible unitary representations

\[
\pi_L = \bigoplus_{\eta \in \hat{G}} d_\eta \eta, \tag{29}
\]

corresponding to the Fourier decomposition of signals \( u \): in the sense of \( L^2(G) \), there is the Fourier inverse formula (Fourier series)

\[
u(x) = \sum_{\eta \in \hat{G}} d_\eta \operatorname{tr}(\eta(x) \hat{\eta}(\eta)), \tag{30}\]

where \( \operatorname{tr} \) is the usual matrix trace. Here \( \{ \sqrt{d_\eta} \eta_{jk} : \eta \in \hat{G}, 1 \leq j, k \leq d_\eta \} \) is an orthonormal basis for the Hilbert space \( L^2(G) \).

Remember that \( \operatorname{tr}(AB) = \operatorname{tr}(BA) \), but often \( \operatorname{tr}(ABC) \neq \operatorname{tr}(CBA) \). In the sequel, for matrix-valued functions \( \hat{a} \) on \( \hat{G} \), we write “non-commutative integrals”

\[
\int \hat{a}(\eta) \, d\eta := \int_{\hat{G}} \operatorname{tr}(\hat{a}(\eta)) \, d\mu_{\hat{G}}(\eta) = \sum_{\eta \in \hat{G}} d_\eta \operatorname{tr}(\hat{a}(\eta)) \tag{31}
\]

Here \( \mu_{\hat{G}} \) is the Plancherel measure. We obtain

\[
u(x) = \int \eta(x) \hat{\eta}(\eta) \, d\eta = \int \hat{\eta}(\eta) \eta(x) \, d\eta = \iint u(y) \eta(y^{-1}x) \, dy \, d\eta.
\]
Defining \( \|u\| := \langle \hat{u}, \hat{u} \rangle^{1/2} \), where
\[
\langle \hat{u}, \hat{v} \rangle := \int \hat{u}(\eta) \hat{v}(\eta)^* \, d\eta,
\]  
we obtain the Plancherel (or Parseval) identity
\[
\langle u, v \rangle = \langle \hat{u}, \hat{v} \rangle.
\]  
Especially, \( \|u\|^2 = \|\hat{u}\|^2 \) is the conservation of energy. Consequently, the Fourier transform \( \mathcal{F} = (u \mapsto \hat{u}) \) is a Hilbert space isomorphism \( \mathcal{F} : L^2(G) \to L^2(\hat{G}) \). Furthermore, Fourier transform can also be viewed as linear isomorphisms
\[
F = (u \mapsto \hat{u}) : \mathcal{S}(G) \to \mathcal{S}(\hat{G}),
\]  
where \( \mathcal{S}'(G) \) is the space of trigonometric distributions or formal trigonometric expansions \( f \). Here \( \mathcal{S}'(\hat{G}) \) consists of all functions \( \hat{f} \) on \( \hat{G} \) such that \( \hat{f}(\eta) \in \mathbb{C}^{d_n \times d_n} \) for each \( \eta \in \hat{G} \). Elements \( \hat{u} \in \mathcal{S}(\hat{G}) \subset \mathcal{S}'(\hat{G}) \) are those which have only finitely many non-zero Fourier coefficients.

On compact group \( G \), the algebra of test function can be enlarged from trigonometric \( \mathcal{S}(G) \) to the Schwartz space (or Schwartz–Bruhat space) \( \mathcal{S}(G) \), introduced by Bruhat in [3]. Let \( \mathcal{J} \) be the family of the closed normal subgroups \( K \) of \( G \) such that \( G/K \) is isomorphic to a Lie group: for short, \( G/K \) is a Lie group. Endow \( \mathcal{J} \) with the inverse inclusion order. For \( K \in \mathcal{J} \), we identify \( u \in C^\infty(G/K) \) with \( u \circ \pi_K : G \to \mathbb{C} \), where \( \pi_K = (x \mapsto xK) : G \to G/K \) is the quotient map. Hence \( C^\infty(G/K) \subset C(G) \). The reflexive space of Schwartz test functions is the inductive limit
\[
\mathcal{S}(G) := \lim_{\rightarrow} C^\infty(G/K)
\]  
of the direct system \( \{(C^\infty(G/K))_{K \in \mathcal{J}}, (f_{KL})_{K,L \in \mathcal{J}, K \subset L} \} \), where functions \( f_{KL} : C^\infty(G/K) \to C^\infty(G/L) \) are defined by \( f_{KL}(u)(xL) := u(xK) \). The strong dual of the Schwartz space \( \mathcal{S}(G) \) is the Schwartz distribution space \( \mathcal{S}'(G) \), and they are complete nuclear barreled spaces.

Function spaces are treated as subsets of distribution spaces, and we have
\[
\mathcal{S}(G) \subset \mathcal{S}(G) \subset C(G) \subset L^\infty(G) \subset L^2(G) \subset L^1(G) \subset \mathcal{S}'(G) \subset \mathcal{S}'(G).
\]  
The Fourier transform can also be viewed as linear isomorphisms
\[
\mathcal{F} = (u \mapsto \hat{u}) : \mathcal{S}(G) \to \mathcal{S}(\hat{G}),
\]  
where \( \mathcal{S}(\hat{G}) \subset L^2(\hat{G}) \) and \( \mathcal{S}'(\hat{G}) \subset \mathcal{S}'(\hat{G}) \).

There is a positive central trigonometric approximate identity, i.e. a net of central positive trigonometric polynomials \( h_\alpha \) of unit \( L^1 \)-norm such that
\[
\lim_{\alpha} \|u - h_\alpha * u\|_{L^1(G)} = 0
\]  
\[ (32) \]

\[ (33) \]

\[ (34) \]

\[ (35) \]

\[ (36) \]

\[ (37) \]

\[ (38) \]
for every \( u \in L^1(G) \), see [20] (Theorem 28.53). Let us present a brief related construction: Let \( \alpha = (U, m) \), where \( m \in \mathbb{Z}^+ \) and \( U \) is a symmetric neighborhood of \( e \in G \) meaning \( U = U_e U_e \) for a neighborhood \( U_e = U_e^{-1} \) of \( e \in G \). Choose central \( f = f_U \in C(G) \) such that \( \| f \|_{L^2} = 1 \), and \( f(x) = 0 \) whenever \( x \notin U \). Approximate \( f \) by central \( g = g_{(U, m)} \in \mathcal{F}(G) \) such that \( \| f - g \|_{C(G)} < 1/(m\| f \|_{C(G)}) \). Define central \( h = h_{(U, m)} \in \mathcal{F}(G) \) by \( h := |g|^2/\|g\|^2 \). The index pairs \( \alpha = (U, m) \) and \( \beta = (V, n) \) have the partial order

\[
\alpha \leq \beta \iff V \subset U \text{ and } m \leq n.
\]

The functions \( h_\alpha \) form a positive central trigonometric approximate identity.

Convolution \( u \ast v \) of signals \( u, v \) is the signal defined by

\[
u \ast v(x) := \int u(xy^{-1}) v(y) \, dy. \quad (39)\]

Then \( \hat{u} \ast \hat{v} = \hat{v} \hat{u} \), that is \( \hat{u} \ast v(\eta) = \hat{v}(\eta) \hat{u}(\eta) \), as

\[
\int \int \eta(x)^* u(xy^{-1}) v(y) \, dy \, dx = \int \eta(y)^* v(y) \int \eta(xy^{-1})^* u(xy^{-1}) \, dx \, dy.
\]

The unitary dual \( \hat{G} \) does not have a group structure when \( G \) is non-commutative. Nevertheless, we define a formal convolution by

\[
\hat{u} \ast \hat{v} := \mathcal{F} \left( (\mathcal{F}^{-1} \hat{u}) \mathcal{F}^{-1} \hat{v} \right). \quad (40)
\]

Here we have commutativity \( \hat{v} \ast \hat{u} = \hat{u} \ast \hat{v} \) also on non-commutative groups \( G \), since multiplication of scalar-valued functions is commutative.

Matrix \( M = [M_{jk}] \in \mathbb{C}^{d \times d} \) is positive semi-definite (or positive, for short) if

\[
0 \leq \langle M z, z \rangle := \sum_{k=1}^d (M z)_k z_k = \sum_{j,k=1}^d \overline{z}_j M_{jk} z_k.
\]

The Fourier series (or “non-commutative integral”) over \( \hat{G} \) behaves much like the Haar integral over \( G \). For instance,

\[
\int \hat{u}(\eta) \, d\eta = u(e), \quad \int u(x) \, dx = \hat{u}(\varepsilon).
\]

If \( \hat{u} \geq 0 \) in the sense that \( \hat{u}(\eta) \geq 0 \) for all \( \eta \in \hat{G} \) then \( u(e) = \int \hat{u}(\eta) \, d\eta \geq 0 \).

**Example 4.1** For \( 1 \leq p < \infty \) the Schatten-\( p \)-norm of a matrix \( M \in \mathbb{C}^{d \times d} \) is

\[
\| M \|_{S^p} := (\text{tr}(|M|^p))^{1/p},
\]

where \( |M| := (M M^*)^{1/2} \). The operator norm \( \| M \|_{op} \) or the Schatten-\( \infty \)-norm of \( M \)

\[
\| M \|_{op} = \| M \|_{S^\infty} = \lim_{p \to \infty} \| M \|_{S^p},
\]

\[
\| M \|_{L^p} := (\text{tr}(|M|^p))^{1/p},
\]

where \( |M| := (M M^*)^{1/2} \). The operator norm \( \| M \|_{op} \) or the Schatten-\( \infty \)-norm of \( M \)

\[
\| M \|_{op} = \| M \|_{S^\infty} = \lim_{p \to \infty} \| M \|_{S^p},
\]
or alternatively \( \|M\|_{op} = \sup \{ \|Mz\|_{C^d} : \|z\|_{C^d} \leq 1 \} \), where \( \|z\|_{C^d}^2 = \sum_{k=1}^{d} |z_k|^2 \).

Here \( \|M\|_{S_i} = \text{tr}(|M|) \) is the trace class norm, and \( \|M\|_{HS} = \|M\|_{S_2} \) is the Hilbert-Schmidt norm. The Lebesgue spaces \( L^p(\widehat{G}) \) have the norms given by

\[
\|\widehat{u}\|_{L^p(\widehat{G})} := \left( \int |\widehat{u}(\eta)|^p \, d\eta \right)^{1/p},
\]

\[
\|\widehat{u}\|_{L^{\infty}(\widehat{G})} := \sup_{\eta \in \widehat{G}} \|\widehat{u}(\eta)\|_{op}.
\]

If \( 1 \leq p, q \leq \infty \) such that \( 1/p + 1/q = 1 \), then

\[
|u \ast v(\epsilon)| = \left| \int \widehat{\nu}(\eta) \widehat{u}(\eta) \, d\eta \right| \leq \int |\widehat{\nu}(\eta)\widehat{u}(\eta)| \, d\eta \leq \|\widehat{u}\|_{L^p(\widehat{G})} \|\widehat{\nu}\|_{L^q(\widehat{G})}.
\]

**Dirac and Kronecker deltas.** In distributional sense, the Fourier inverse

\[
u(x) = \int \eta(x) \widehat{u}(\eta) \, d\eta
\]

for the Dirac delta distribution \( \delta_\varepsilon \in C'(G) \) at \( \varepsilon \in G \). Also, for \( \eta \in \widehat{G} \),

\[
\int \eta(x) \ast dx = \text{I}(\eta) = \delta_\varepsilon(\eta) \, I \in \mathbb{C}^{d_\eta \times d_\eta},
\]

where the Kronecker delta \( \delta_\varepsilon \) at \( \varepsilon \in \widehat{G} \) satisfies \( \delta_\varepsilon(\eta) := \begin{cases} 1 & \text{if } \varepsilon = \eta \in \widehat{G}, \\ 0 & \text{if } \varepsilon \neq \eta \in \widehat{G}. \end{cases} \)

**Example 4.2** The compact commutative Lie groups \( G \) are easy to list up to an isomorphism: such a \( G \) can be a product of a discrete cyclic group \( \mathbb{Z}/N\mathbb{Z} \) and a flat torus \( \mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n \) for some \( N, n \in \mathbb{N} = \{0, 1, 2, 3, \cdots \} \). Let us review the notion above in the familiar case of the torus group \( G = \mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n \). The Haar measure on \( G \) is given by the usual Lebesgue measure, and for functions \( u \in L^2(G) \) the traditional Fourier coefficient transform \( \hat{u} : \mathbb{Z}^n \rightarrow \mathbb{C} \) is defined by

\[
\hat{u}(\eta) := \int_{\mathbb{T}^n} e^{-i2\pi \eta \cdot y} u(y) \, dy.
\]

The inverse Fourier transform is given by the \( L^2 \)-converging Fourier series

\[
u(x) = \sum_{\eta \in \mathbb{Z}^n} e^{i2\pi \eta \cdot x} \hat{u}(\eta).
\]

Here the irreducible unitary representations are one-dimensional

\[ x \mapsto e^{i2\pi x \cdot \eta}, \]

and we may obviously identify \( \widehat{G} \) with \( \mathbb{Z}^n \), which is a non-compact discrete commutative group. The convolutions are now given by

\[
u \ast v(x) = \int_{\mathbb{T}^n} v(x - y) \nu(y) \, dy, \quad \hat{u} \ast \hat{v}(\xi) = \sum_{\eta \in \mathbb{Z}^n} \hat{u}(\xi - \eta) \hat{v}(\eta).
\]
5 Hopf algebras of functions and distributions

Test function space $\mathcal{T}(G)$ of trigonometric polynomials and $\mathcal{S}(G)$ of Schwartz functions can be endowed with Hopf algebra structures. Notice that

$$\mathcal{T}(G \times G) \cong \mathcal{T}(G) \otimes \mathcal{T}(G),$$

$$\mathcal{S}(G \times G) \cong \mathcal{S}(G) \hat{\otimes} \mathcal{S}(G),$$

where $\otimes$ denotes the algebraic tensor product, and $\hat{\otimes}$ the projective tensor product. The commutative unital $C^*$-algebra $C(G)$ of continuous functions has involution $\iota : C(G) \to C(G)$ given by $\iota(u) := u(x)^*$. Let us define mappings

$$m_0 : C(G \times G) \to C(G), \quad m_0w(x) := w(x, x), \quad (51)$$

$$\eta_0 : C \to C(G), \quad \eta_0(\lambda) := \lambda \mathbf{1}, \quad (44)$$

$$\Delta_0 : C(G) \to C(G \times G), \quad \Delta_0u(x, y) := u(xy), \quad (45)$$

$$\varepsilon_0 : C(G) \to C, \quad \varepsilon_0u(x) := u(e), \quad (46)$$

$$S_0 : C(G) \to C(G), \quad S_0u(x) := u(x^{-1}). \quad (47)$$

When restricting these mappings respectively to trigonometric polynomials to and Schwartz test functions, $\mathcal{T}(G)$ and $\mathcal{S}(G)$ can be regarded as Hopf algebras. By dualizing the structure of $\mathcal{T}(G)$, we obtain mappings

$$\left( m_1, \eta_1, \Delta_1, \varepsilon_1, S_1 \right)$$

$$:= \left( \Delta_0^*, \varepsilon_0^*, m_0^*, \eta_0^*, S_0^* \right)$$

where for $f, g \in \mathcal{T}'(G)$ we have

$$m_1(f \otimes g) = f * g, \quad \mathcal{F}m_1(f \otimes g)(\xi) = \hat{f}(\xi) \hat{g}(\xi), \quad (48)$$

$$\eta_1(\lambda) = \lambda \delta_e, \quad \mathcal{F}(\eta_1(\lambda))(\xi) = \lambda I \in \mathbb{C}^d \times d', \quad (49)$$

$$\Delta_1f(x, y) = f(x) \delta_x(y), \quad \Delta_1^*(\xi, \eta) = \hat{f}(\xi \otimes \eta), \quad (50)$$

$$\varepsilon_1(f) = \int f(x) dx, \quad \varepsilon_1(f) = \hat{f}(\varepsilon), \quad (51)$$

$$S_1f(x) = f(x^{-1}), \quad \overline{S_1f(\eta)} = \overline{\hat{f}(\eta^*)}^T. \quad (52)$$

Here $M^T$ is the transpose of matrix $M$, and $\eta^* \in \hat{G}$ is the contragredient representation of $\eta \in G$, defined by $\eta^*(x) := \eta(x^{-1})^T$.

6 Symplectic Fourier transform

We call $G \times \hat{G}$ the time-frequency plane (or the position-momentum space, or the phase-space), where time-frequency points $(x, \eta) \in G \times \hat{G}$ comprise of time $x \in G$ and of frequency $\eta \in \hat{G}$. We shall deal with Hilbert space $L^2(G \times \hat{G})$, where the inner product is given by

$$\langle b, a \rangle = \iiint b(x, \eta) a(x, \eta)^* d\eta dx. \quad (53)$$
Here the matrix elements of $x \mapsto a(x, \eta) \in \mathbb{C}^{d_n \times d_n}$ belong to $L^2(G)$ for all $\eta \in \hat{G}$. The ambiguity plane

$$\hat{G} \times G = \{ (\xi, y) : \xi \in \hat{G}, y \in G \}$$

(54)
is the Fourier dual to the time-frequency plane $G \times \hat{G}$ by the symplectic Fourier transform $F$, which is the linear isomorphism

$$F = (\mathcal{F} \otimes I)(I \otimes \mathcal{F}^{-1}) : L^2(G \times \hat{G}) \rightarrow L^2(\hat{G} \times G).$$

(55)

Thus if $a \in L^2(G \times \hat{G})$ then $Fa \in L^2(\hat{G} \times G)$,

$$Fa(\xi, y) = \int \xi(x)^* \int \eta(y) a(x, \eta) \, d\eta \, dx.$$  

(56)

As in traditional signal processing, here we may call $y \in G$ the time-delay or lag variable, and $\xi \in \hat{G}$ the frequency-delay or Doppler variable. The inverse symplectic Fourier transform is then given by

$$a(x, \eta) = \int \eta(y)^* \int \xi(x) Fa(\xi, y) \, d\xi \, dy.$$  

(57)

Then

$$\langle Fa, Fb \rangle = \langle a, b \rangle, \quad \| Fa \|^2 = \langle Fa, Fa \rangle = \langle a, a \rangle = \| a \|^2.$$  

Matrix-valued functions on $G \times \hat{G}$ and $\hat{G} \times G$ can be multiplied “pointwise”:

$$(ab)(x, \eta) := a(x, \eta)b(x, \eta), \quad ((Fa)Fb)(\xi, y) := Fa(\xi, y)Fb(\xi, y).$$

Then the convolution $a \ast b$ of $a, b$ on $G \times \hat{G}$ is defined by

$$a \ast b := F^{-1}((Fb)Fa).$$

(58)

For example, $a \ast I = \lambda I$, where

$$\lambda = Fa(\varepsilon, e) = \iint a(x, \eta) \, d\eta \, dx \in \mathbb{C}.$$  

(59)

We shall also need spaces of matrix-valued test functions and distributions. Especially, we have linear isomorphisms

$$(I \otimes \mathcal{F}) : \mathcal{S}(G \times G) \rightarrow \mathcal{S}(G \times \hat{G}),$$

(60)

$$F : \mathcal{S}(G \times \hat{G}) \rightarrow \mathcal{S}(\hat{G} \times G),$$

(61)

where we have the projective tensor product isomorphisms

$$\mathcal{S}(G \times G) \cong \mathcal{S}(G) \otimes \mathcal{S}(G),$$

$$\mathcal{S}(G \times \hat{G}) \cong \mathcal{S}(G) \otimes \mathcal{S}(\hat{G}),$$

$$\mathcal{S}(\hat{G} \times G) \cong \mathcal{S}(\hat{G}) \otimes \mathcal{S}(G).$$

Then $\mathcal{S}'(\ldots)$ will denote the respective distribution space corresponding to the test function space $\mathcal{S}(\ldots)$. 

12
7 Kohn–Nirenberg quantization

The Kohn–Nirenberg quantization of pseudo-differential operators serves as the starting point to acquire all the different time-frequency transforms. The idea of the Kohn–Nirenberg pseudo-differential operators on compact Lie groups was introduced by Taylor in [29], and further investigated e.g. in [26, 27, 28, 11].

**Definition 7.1** The Kohn–Nirenberg symbol \( a \in \mathcal{S}'(G \times \hat{G}) \) of linear mapping \( B : \mathcal{S}(G) \rightarrow \mathcal{S}'(G) \) is defined by

\[
a(x, \eta) = \eta(x)^* B \eta(x),
\]

where matrix elements of \( B\eta \) belong to \( \mathcal{S}'(G) \). Then

\[
B u(x) = \int \eta(x) a(x, \eta) \hat{\eta}(\eta) \, d\eta = \int a(x, \eta) \hat{\eta}(\eta) \eta(x) \, d\eta,
\]

and we call \( a^R := B \) the Kohn–Nirenberg pseudo-differential operator with symbol \( a \). The invertible mapping \( a \mapsto a^R \) is called the Kohn–Nirenberg quantization. For \( u, v \in \mathcal{S}(G) \), we define the corresponding Kohn–Nirenberg (or the Rihaczek) time-frequency transform \( R(u, v) \in \mathcal{S}(G \times G) \) by

\[
\langle u, a^R v \rangle = \langle R(u, v), a \rangle
\]

for all symbols \( a \in \mathcal{S}'(G \times \hat{G}) \). Then the Kohn–Nirenberg ambiguity transform is \( FR(u, v) = (\mathcal{F} \otimes I)(I \otimes \mathcal{F}^{-1}) R(u, v) \in \mathcal{S}(\hat{G} \times G) \).

**Remark 7.2** Combining (63) and (64), we obtain

\[
R(u, v)(x, \eta) = u(x) \eta(x)^* \hat{v}(\eta)^* \in \mathcal{B}(\mathcal{H}_u).
\]

Especially, \( R(u, v)(e, \varepsilon) = u(e) \hat{v}(\varepsilon)^* \in \mathbb{C} \). Notice that the same definition extends directly to distributions \( u, v \in \mathcal{S}'(G) \), so that \( R(u, v) \in \mathcal{S}'(G \times G) \), and then \( FR(u, v) \in \mathcal{S}(\hat{G} \times G) \). Moreover,

\[
FR(u, v)(\xi, y) = \int \xi(x)^* \int \eta(y) R(u, v)(x, \eta) \, d\eta \, dx
\]

\[
= \int \xi(x)^* u(x) \nu(xy^{-1})^* \, dx \in \mathcal{B}(\mathcal{H}_\xi).
\]

Especially, \( FR(u, v)(\varepsilon, e) = \langle u, v \rangle = \langle \hat{u}, \hat{v} \rangle \). Notice that \( FR(u, v)(\xi, y) = \widehat{f_y}(\xi) \), where \( f_y(x) = u(x) \nu(xy^{-1})^* \). The Cauchy–Schwarz inequality yields

\[
\int |f_y(x)| \, dx \leq \|u\| \|v\|.
\]

**Remark 7.3** On a compact group \( G \), the Kohn–Nirenberg transform \( R \) maps \( \mathcal{F}(G) \times \mathcal{F}(G) \) to \( \mathcal{F}(G \times G) \). Why? Let \( u, v \in \mathcal{F}(G) \). Since

\[
u(x) = (I \otimes \xi) \Delta u(x, y), \quad \nu(xy^{-1})^* = (I \otimes S) \Delta \nu(x, y),
\]

this shows both \((x, y) \mapsto u(x)\) and \((x, y) \mapsto \nu(xy^{-1})^*\) belong to \( \mathcal{F}(G \times G) \). Hence also \((x, y) \mapsto u(x) \nu(xy^{-1})^*\) belongs to \( \mathcal{F}(G \times G) \). With a similar reasoning, we see that the Kohn–Nirenberg transform maps \( \mathcal{F}(G) \times \mathcal{F}(G) \) to \( \mathcal{F}(G \times G) \).
8 Time-frequency transforms and quantizations

Time-frequency transform \((u, v) \mapsto D(u, v)\) will be “time-frequency invariant”, taking test functions of time to matrix-valued test functions of time-frequency. More precisely:

**Definition 8.1** Time-frequency transform \(D : \mathcal{S}(G) \times \mathcal{S}(G) \to \mathcal{S}(G \times \hat{G})\) is a mapping of the form
\[
D(u, v) := F^{-1}(\phi_D FR(u, v)),
\]
where \(\phi_D \in \mathcal{S}'(\hat{G} \times G)\) is the ambiguity kernel (or Doppler-lag kernel). Time-frequency transform is called band-limited if it maps \(\mathcal{T}(G) \times \mathcal{T}(G)\) to \(\mathcal{T}(G \times \hat{G})\).

**Remark 8.2** Trigonometric function \(u \in \mathcal{T}(G)\) is band-limited in the sense that it has only finitely many non-zero Fourier coefficients. For the Kohn–Nirenberg transform \(R\), notice that \(\phi_R(\xi, y) = I\) for all \((\xi, y) \in \hat{G} \times G\). Thus we may have \(\phi_D \notin \mathcal{T}(G \times \hat{G})\). Nevertheless,
\[
FD(u, 1)(\xi, y) = \phi_D(\xi, y) \hat{u}(\xi)
\]
for all \(u \in \mathcal{T}(G)\). Hence the matrix elements of \(y \mapsto \phi_D(\xi, y)\) belong to \(\mathcal{T}(G \times \hat{G})\) for each \(\xi \in \hat{G}\). Band-limitedness of \(D\) is equal to that these matrix elements would be trigonometric polynomials: for instance, the Kohn–Nirenberg transform \(R\) is band-limited. Time-frequency transform can also be expressed by
\[
D(u, v)(x, \eta) = \int \eta(y)^* \int \xi(x) \phi_D(\xi, y) FR(u, v)(\xi, y) d\xi d\eta = R(u, v) * \psi_D(x, \eta),
\]
where \(\psi_D = F^{-1}(\phi_D)\) is the time-frequency kernel of \(D\), corresponding to the ambiguity kernel \(\phi_D = F(\psi_D)\). Sometimes we need the time-lag kernel \(\varphi_D = (I \otimes \mathcal{F})\psi_D = (\mathcal{F}^{-1} \otimes I)\phi_D\). Notice that the kernels
\[
\psi_D(x, \eta), \quad \varphi_D(x, y), \quad \phi_D(\xi, y)
\]
contain the same information, with different variables \(x, y \in G\) and \(\xi, \eta \in \hat{G}\). With the approach above, we have avoided finding “frequency modulations” on non-commutative groups; the commutative case works still fine, and yet we obtain many essential features also in the non-commutative setting.

**Remark 8.3** If \(u, v \in \mathcal{T}(G)\) then \(\phi_D FR(u, v) \in \mathcal{T}'(\hat{G} \times G)\), so that we can define
\[
D(u, v) := F^{-1}(\phi_D FR(u, v)) \in \mathcal{T}'(G \times \hat{G}).
\]

**Definition 8.4** Let \(D\) be a time-frequency transform. The corresponding \(D\)-quantization \(a \mapsto a^D\) satisfies
\[
\langle u, a^D v \rangle = \langle D(u, v), a \rangle.
\]
Linear operators \(a^D = (v \mapsto a^D v)\) are called \(D\)-pseudo-differential operators.
In the sequel, we investigate how the properties of different kernels affect the properties of the time-frequency transform \( D \) and the \( D \)-quantization \( a \mapsto a^D \). Due to \([74]\), \( a^D v \in \mathcal{S}'(G) \) if \( v \in \mathcal{S}(G) \) and \( a \in \mathcal{S}'(G \times \widehat{G}) \). Moreover, if \( v \in \mathcal{S}'(G) \) and \( a \in \mathcal{S}(G \times \widehat{G}) \), then \( a^D v \in \mathcal{S}(G) \). Thereby we have

\[
D^D : \mathcal{S}'(G) \to \mathcal{S}'(G) \quad \text{if} \quad a \in \mathcal{S}'(G \times \widehat{G}),
\]

\[
D^D : \mathcal{S}'(G) \to \mathcal{S}(G) \quad \text{if} \quad a \in \mathcal{S}(G \times \widehat{G}).
\]

Different quantizations can be linked to the Kohn–Nirenberg case:

**Lemma 8.5** Let \( D \) be a time-frequency transform, and let \( a \in \mathcal{S}'(G \times \widehat{G}) \). Then \( a^D = b^R \), where \( Fb(\xi, y) = \phi_D(\xi, y)^* Fa(\xi, y) \).

**Proof.** Noticing that

\[
\langle D(u, v), a \rangle = \langle FD(u, v), Fa \rangle = \langle FR(u, v), Fb \rangle = \langle R(u, v), b \rangle,
\]

we obtain \( \langle u, a^D v \rangle = \langle u, b^R v \rangle \). QED

**Definition 8.6** For a time-frequency transform \( (u, v) \mapsto D(u, v) \), we call

\[
D[u] := D(u, u)
\]

the *time-frequency distribution* of signal \( u \). Notice that \( D[\lambda u] = |\lambda|^2 D[u] \) for all \( \lambda \in \mathbb{C} \), so define the equivalence class \([u]\) of *indistinguishable signals* by

\[
[u] := \{ \lambda u : \lambda \in \mathbb{C}, \; |\lambda| = 1 \}.
\]

Value \( D[u](x, \eta) \in \mathcal{B}(\mathcal{H}_0) \) presents an idealized operator-valued energy density at time-frequency \( (x, \eta) \in G \times \widehat{G} \) for a scalar-valued signal \( u : G \to \mathbb{C} \). With the complex scalars, numeric data families

\[
\left( \langle u, a^D u \rangle \right)_{u \in \mathcal{S}(G)} \quad \text{and} \quad \left( \langle u, a^D v \rangle \right)_{u, v \in \mathcal{S}(G)}
\]

mediate the same information. Thereby the *invertibility* of time-frequency transform \( D \) refers to the invertibility of the mapping \([u] \mapsto D[u]\). This amounts to the properties of ambiguity kernel \( \phi_D \). Invertibility is not merely “being bijective”, it deals also with the numerical stability (cf. the inverse problem for the traditional heat equation). For invertibility, we need \( \phi_D(\xi, y) \) to be invertible for almost every \((\xi, y) \in \widehat{G} \times G\), and numerically that \( \phi_D \) grows or decays at infinity at most polynomially. The Kohn–Nirenberg transform is invertible, since

\[
\int \eta(y) R[u](x, \eta) \, d\eta = u(x) u(xy)^*.
\]

**Example 8.7** An analogue of Wigner-\(\tau\)-pseudo-differential operators on certain families of locally compact groups was introduced and studied in \([24]\). On a compact group, this Wigner-\(\tau\)-quantization would formally correspond to our time-frequency transform \( D \), which has the ambiguity kernel of the form

\[
\phi_D(\xi, y) = \xi(\tau(y)),
\]

where \( \tau : G \to G \) is a suitable function.
Boundedness in energy. What if also $\phi_D \in L^\infty(\hat{G} \times G)$? In other words, $\phi_D$ would be bounded in the sense that $\|\phi_D\|_{L^\infty} < \infty$ for

$$\|\phi_D\|_{L^\infty} = \sup_{(\xi,y) \in \hat{G} \times G} \|\phi_D(\xi,y)\|_{op} ,$$

where $\|M\|_{op}$ is the spectral norm of operator $M$. We obtain the following boundedness result on $L^2$-spaces, where norms $\|f\|$ are the appropriate $L^2$-norms:

Theorem 8.8 Let $\phi_D \in L^\infty(\hat{G} \times G)$ for a time-frequency transform $D$. Then

$$\|D(u,v)\| \leq \|\phi_D\|_{L^\infty} \|u\| \|v\| ,$$

$$\|a^D v\| \leq \|\phi_D\|_{L^\infty} \|a\| \|v\| ,$$

for all $u, v \in L^2(G)$ and $a \in L^2(G \times \hat{G})$. For the Kohn–Nirenberg transform, $\|R(u,v)\| = \|u\| \|v\|$, and $\|a^R v\| \leq \|a\| \|v\|$.

Proof. In the special case of the Kohn–Nirenberg transform, $\|\phi_R\|_{L^\infty} = 1$ as $\phi_R(\xi,y) = I$ for all $(\xi,y) \in \hat{G} \times G$. Moreover,

$$\|R(u,v)\|^2 = \langle R(u,v), R(u,v) \rangle = \int \int |u(x)|^2 d\xi d\eta \int \hat{\phi}(\xi,\eta) \hat{\phi}(\xi,\eta) d\xi d\eta$$

$$= \int \|u\|^2 \|v\|^2 .$$

The $L^2$-norm is preserved in the symplectic Fourier transform:

$$\|D(u,v)\| = \|FD(u,v)\| = \|\phi_D FR(u,v)\| .$$

Let $\|M\|_{HS} = (\text{tr}(MM^*))^{1/2}$ denote the Hilbert–Schmidt norm. Recall that $\|MN\|_{HS} \leq \|M\|_{op} \|N\|_{HS}$. Thereby

$$\|\phi_D FR(u,v)\|^2 = \int \int \|\phi_D(\xi,y) FR(u,v)(\xi,y)\|^2_{HS} d\xi d\eta \leq \int \int \|\phi_D(\xi,y)\|^2_{op} \|FR(u,v)(\xi,y)\|^2_{HS} d\xi d\eta \leq \|\phi_D\|_{L^\infty}^2 \|FR(u,v)\|^2 .$$

Inequality (81) follows from this, because $\|R(u,v)\| = \|u\| \|v\|$. Hence by the Cauchy–Schwarz inequality we obtain

$$|\langle u, a^D v \rangle| = |\langle D(u,v), a \rangle| \leq \|D(u,v)\| \|a\| \leq \|\phi_D\|_{L^\infty} \|u\| \|v\| \|a\| ,$$

completing the proof. QED
Let us emphasize the invariance under the time translations, in the sense that \( D[v](x, \eta) = D[u](y x, \eta) \) if \( v(x) = u(y x) \). The frequency modulations are more elusive in the non-commutative case, but nevertheless, the message of the next result is that the information can be “shifted” to the specific point \((e, \varepsilon)\) in the time-frequency plane:

**Theorem 8.9** Time-frequency transform \( D \) can be recovered from the evaluation mapping \((u \mapsto D[u](e, \varepsilon)) : \mathscr{S}(G) \to \mathbb{C} \).

**Proof.** For \( u, v \in \mathscr{S}(G) \) we have

\[
D(u, v)(x, \eta) = \int \eta(y)^* \int \xi(x) \phi_D(\xi, y) \int \xi(t)^* u(t) v(ty^{-1})^* \, dt \, d\xi \, dy.
\]

Especially,

\[
\langle u, \delta^D u \rangle = D[u](e, \varepsilon) = \int u(x) \left( \int \varphi_D(x^{-1}, y^{-1} x)^* u(y) \, dy \right)^* \, dx,
\]

where \( \delta = \delta_{(e, \varepsilon)} \) is the Dirac–Kronecker delta distribution at \((e, \varepsilon) \in G \times \hat{G} \). Hence from knowing all \( D[u](e, \varepsilon) \) we obtain \( \varphi_D \) and thereby \( D \).

**QED**

**Remark 8.10** Notice that in the statement of the previous Theorem on a compact group \( G \), we could replace the test function space \( \mathscr{S}(G) \) by \( \mathcal{T}(G) \).

**Original localizations.** Let us call \((e, \varepsilon) \in G \times \hat{G}\) the **origin of the time-frequency plane** \( G \times \hat{G} \). As seen in the proof of Theorem 8.9 above, the original localization pseudo-differential operator \( \delta^D = (\delta_{(e, \varepsilon)})^D : \mathscr{S}(G) \to \mathscr{S}^*(G) \) encodes all the information about the time-frequency transform \( D \). The original localization \( \delta^D \) is bounded on \( L^2(G) \) if and only if

\[
|D(u, v)(e, \varepsilon)| = |\langle u, \delta^D v \rangle| \leq c \|u\| \|v\| \tag{84}
\]

for all \( u, v \in \mathscr{S}(G) \), where \( c < \infty \) is a constant. Original localizations provide an alternative way to understand time-frequency transforms. Notice that if \( K_{\delta^D} \) is the Schwartz integral kernel of the original localization \( \delta^D \), then

\[
D(u, v)(x, \eta) = \int \int u(x z) \eta(z)^* K_{\delta^D}(z, y)^* \eta(y) v(xy)^* \, dz \, dy,
\]

in analogy to the Euclidean case \[24\]. Here \( K_{\delta^D}(z, y)^* = \varphi_D(z^{-1}, y^{-1} z) \), that is \( \varphi_D(x, y) = K_{\delta^D}(x^{-1}, x^{-1} y^{-1})^* \). Hence

\[
\phi_D(\xi, y) = \int K_{\delta^D}(x, xy^{-1})^* \xi(x) \, dx.
\]
Moreover, if we define amplitude $a_D$ by

$$a_D(x,y,\eta) := \int a(t,\eta) K_{\delta_D}(t^{-1}x, t^{-1}y) \, dt$$

then $a_D^D v(x) = \int K_{a_D^D}(x,y) v(y) \, dy$ for the Schwartz kernel $K_{a_D^D}$:

$$K_{a_D^D}(x,y) = \int \eta(y^{-1}x) a_D(x, \eta) \, d\eta.$$ (88)

### Example 8.11

Since $R(u,v)(e,\varepsilon) = u(e) \hat v(\varepsilon)^*$, the Kohn–Nirenberg original localization is given by

$$\delta^R v(x) = \hat v(\varepsilon) \delta_e(x) = \int v(y) \, dy \, \delta_e(x).$$ (89)

Here $\delta^R : \mathcal{S}(G) \to \mathcal{S}'(G)$ is unbounded on $L^2(G)$ unless $G$ is finite. Amplitude $a_R$ of $a_R^R$ satisfies $a_R(x,y,\eta) = a(x,\eta)$. The so-called anti-Kohn–Nirenberg transform $R^*$ satisfies $R^*(u,v)(e,\varepsilon) = \hat u(\varepsilon) v(e)^*$. Its original localization satisfies $\delta(R^*) v(x) = v(e)$, and its amplitudes are given by $a_{R^*}(x,y,\eta) = a(y,\eta)$.

### 9 Uncertainty and original localizations

In this section we discuss original localizations related to the Heisenberg uncertainty principle in quantum mechanics. Our quantum states $u$ are unit vectors in the Hilbert space $\mathcal{H} = L^2(G)$, identifying states $u,v$ whenever $[u] = [v]$. Bounded observables are self-adjoint operators $A : \mathcal{H} \to \mathcal{H}$, and

$$\begin{cases}
\mu = \mu_A^u := \langle Au, u \rangle, \\
\sigma = \sigma_A^u := \|Au - \mu u\|
\end{cases}$$

are the expectation and the deviation of measurement $A$ in state $u$, respectively. For instance, let $A = \sum_{\alpha \in J} \alpha P_\alpha$ where $P_\alpha$ is an orthogonal projection, with distinct measured values $\alpha \in J \subset \mathbb{R}$. Then the interpretation is the following: in initial state $u$, our measurement gives value $\alpha \in J$ with probability $\|P_\alpha u\|^2$, and then $u$ collapses to state $P_\alpha u/\|P_\alpha u\|$. Let $A, B$ be bounded observables. The uncertainty observable of the pair $(A,B)$ is

$$-i\hbar^{-1}[A,B] = -i\hbar^{-1}(AB - BA),$$

where we normalize the Dirac–Planck constant so that $\hbar := (2\pi)^{-1}$. Applying Cauchy–Schwarz inequality, we obtain the Heisenberg uncertainty inequality

$$|\mu_{-i\hbar^{-1}[A,B]}^u| \leq 2\hbar^{-1} \sigma_A^u \sigma_B^u.$$ (92)

Suppose above $A$ would be a “position operator” and $B$ a “momentum operator”: $Au = f u$ and $Bu = \hat u \hat g$ (that is $Bu = g * u$), initially with $f,g \in \mathcal{S}(G)$
(later considering \( f, g \in \mathcal{S}(G) \)), where for self-adjointness we should have real-valued “coordinate function” \( f \), and \( g(z)^* = g(z^{-1}) \). If \( A, B \) are able to distinguish \((e, \varepsilon) \in G \times \hat{G} \) in a reasonable fashion, then a good candidate for an original localization \( \delta^D \) would be given by
\[
\delta^D := -\text{i}2\pi [A, B].
\]

Then
\[
D(u, v)(e, \varepsilon) = \int\int\text{i}2\pi (f(x) - f(y)) g(xy^{-1})^* u(x) v(y) dy dx.
\]

We shall return to this uncertainty commutator approach when dealing with cyclic groups in Section 16. If here \( f = \delta_e \in \mathcal{S}(G) \) and \( g = 1 \) then
\[
D(u, v) = \text{i}2\pi (R(u, v) - R^*(u, v)),
\]
where the conjugate transforms \( R^*(u, v) := R(v, u) \) will be studied in Section 10.

10 Symmetry

**Definition 10.1** Let \( D : \mathcal{S}(G) \times \mathcal{S}(G) \rightarrow \mathcal{S}(G \times \hat{G}) \) be a time-frequency transform. We define its conjugate
\[
D^* : \mathcal{S}(G) \times \mathcal{S}(G) \rightarrow \mathcal{S}(G \times \hat{G})
\]
by \( D^*(u, v) := D(v, u)^* \), more precisely
\[
D^*(u, v)(x, \eta) := D(v, u)(x, \eta)^*
\]
for all \( u, v \in \mathcal{S}(G) \) and \((x, \eta) \in G \times \hat{G} \). We call time-frequency transform \( D \) symmetric if \( D^* = D \). The \( D \)-quantization is symmetric if for all \( u, v \in \mathcal{S}(G) \)
\[
\langle a^D u, v \rangle = \langle u, a^D v \rangle
\]
whenever \( a \in \mathcal{S}(G \times \hat{G}) \) satisfies \( a(x, \eta)^* = a(x, \eta) \) for all \((x, \eta) \in G \times \hat{G} \).

**Theorem 10.2** Mapping \( D^* \) defined in (94), (95) is a time-frequency transform. Moreover, the following conditions are equivalent:
(a) For all \( u \in \mathcal{S}(G) \) we have \( D[u](e, \varepsilon) \in \mathbb{R} \).
(b) Time-frequency transform \( D \) is symmetric.
(c) The \( D \)-quantization is symmetric.
(d) The time-lag kernel \( \varphi_D = (I \otimes \mathcal{F}^{-1}) \psi_D = (\mathcal{F}^{-1} \otimes I) \phi_D \) satisfies
\[
\varphi_D(x, y)^* = \varphi_D(xy, y^{-1}).
\]

**Remark 10.3** The superficial non-symmetry in the appearance of (96) is just due to the fact that the Kohn–Nirenberg transform itself is not symmetric. Moreover, in the statement of the previous Theorem on a compact group \( G \), we can replace the test function space \( \mathcal{S}(G) \) by \( \mathcal{F}(G) \).
Proof. On the one hand,

\[
D(u, v)(x, \eta) = \int \eta(y)^* \int \xi(x) \phi_D(\xi, y) FR(u, v)(\xi, y) \, d\xi \, dy
\]

\[
= \int \eta(y)^* \int \xi(x) \phi_D(\xi, y) \int \xi(z)^* u(z) v(zy^{-1})^* \, dz \, d\xi \, dy
\]

\[
= \int \eta(y)^* \int \varphi_D(z^{-1}x, y) u(z) v(zy^{-1})^* \, dz \, dy.
\]

On the other hand,

\[
D(v, u)(x, \eta)^* = \int \eta(y) \int \varphi_D(z^{-1}x, y)^* v(z)^* u(zy^{-1}) \, dz \, dy
\]

\[
= \int \eta(y)^* \int \varphi_D(z^{-1}x, y^{-1})^* u(z) v(z)^* \, dz \, dy
\]

\[
= \int \eta(y)^* \int \varphi_D(yz^{-1}x, y^{-1})^* u(z) v(zy^{-1})^* \, dz \, dy,
\]

showing that

\[
\varphi_D^*(x, y) = \varphi_D(yx, y^{-1})^*,
\]

and leading to the equivalence of conditions (b) and (d). In the special case of \((x, \eta) = (e, \varepsilon)\) and \(v = u\), this gives also the equivalence of (d) and (a). Moreover, if \(D\) is symmetric and \(a^* = a\), then

\[
\langle a^D u, u \rangle = \langle u, a^D u \rangle^* = \langle D[u], a \rangle^* = \langle D[u]^*, a^* \rangle = \langle D[u], a \rangle = \langle u, a^D u \rangle,
\]

so that \(a \mapsto a^D\) is also symmetric. Thus (b) implies (c). Now suppose \(a \mapsto a^D\) is symmetric. Let \((h_\alpha)_\alpha\) be a bounded left approximate identity with \(0 \leq h_\alpha \in \mathcal{S}(G)\). Define \(a_\alpha \in \mathcal{S}(G \times \hat{G})\) by \(a_\alpha(x, \eta) := h_\alpha(x) \delta_\varepsilon(\eta) I\). Then \(a_\alpha(x, \eta)^* = a_\alpha(x, \eta)\), and

\[
D[u](e, \varepsilon) = \langle D[u], \delta_{(e, \varepsilon)} \rangle = \lim_\alpha \langle D(u, u), a_\alpha \rangle = \lim_\alpha \langle u, (a_\alpha)^D u \rangle,
\]

which is real-valued due to the symmetry of the quantization. Hence condition (c) implies (a). QED

Remark 10.4 Clearly, \((D^*)^* = D\). Notice also that

\[
(a^D)^* = (a^*)^{(D^*)},
\]

and especially \((\delta^D)^* = \delta^{(D^*)}\). This follows from

\[
\langle u, (a^D)^* v \rangle = \langle v, a^D u \rangle^* = \langle D(v, u)^*, a^* \rangle = \langle D^*(u, v), a^* \rangle = \langle u, (a^*)^{(D^*)} v \rangle.
\]

Example 10.5 The conjugate \(R^*\) of the Kohn–Nirenberg transform \(R\) satisfies

\[
R^*(u, v)(x, \eta) = R(v, u)(x, \eta)^* = \hat{u}(\eta) \eta(x) v(x)^*.
\]
The corresponding pseudo-differential quantization satisfies
\[
\langle u, a^{(R^*)} v \rangle = \langle R^*(u, v), a \rangle = \int \int \int \hat{u}(\eta) \eta(x) v(x)^* a(x, \eta)^* \, d\eta \, dx = \int u(y) \left( \int \int a(x, \eta) v(x) \eta(x^{-1} y) \, d\eta \, dx \right)^* dy,
\]
leading to
\[
a^{(R^*)} v(x) = \int \eta(y^{-1} x) a(y, \eta) v(y) \, d\eta \, dy.
\] (100)
Mapping \( a \to a^{(R^*)} \) is called the anti-Kohn–Nirenberg quantization. It is easy to find that \( \phi_{R^*}(\xi, y) = \xi(y) \).

**Example 10.6** Let \( D \) be a time-frequency transform. Then
\[
D = \frac{D + D^*}{2} + \frac{D - D^*}{2i},
\]
where the symmetric time-frequency transforms \( (D + D^*)/2 \) and \(-i(D - D^*)/2\) could be called the respective real and imaginary parts of \( D \).

**Example 10.7** For the moment, let us try to introduce Wigner distribution on compact groups \( G \). The Euclidean space Wigner transform \( W \) satisfies
\[
W(u, v)(x, \eta) = \int_{\mathbb{R}^n} e^{-i2\pi y \cdot \eta} u(x + y/2) v(x - y/2)^* \, dy = 2^n \int_{\mathbb{R}^n} e^{-i2\pi z \cdot \eta} u(x + z) v(x - z)^* \, dz.
\]
It would be tempting to define the “Wigner transform \( W \)” of \( u, v \in \mathcal{S}(G) \) by
\[
W(u, v)(x, \eta) := \int \eta(z)^* u(xz) v(xz^{-1})^* \, dz \quad (101)
\]
possibly up to a constant multiple, depending on \( G \). The problem here is that \( (xz^{-1})^{-1}(xz) = z^2 \) is not the lag \( z \) in time. Transform \( W \) would also be formally symmetric, as \( W(v, u)(x, \eta)^* = W(u, v)(x, \eta) \). Nevertheless,
\[
FW(u, 1)(\xi, y) = \int \xi(x)^* \int \eta(y) \int \eta(z)^* u(xz) \, dz \, d\eta \, dx = \int \xi(x)^* u(xy) \, dx = \xi(y) \tilde{u}(\xi).
\]
So for such \( W \) to be a time-frequency transform, we would have \( \phi_W(\xi, y) = \xi(y) \), meaning that \( W = R^* \), the anti-Kohn–Nirenberg transform: this is possible only when \( G = \{e\} \) is the trivial group of one element. Hence, \( W \) defined in
formula (101) is a dead-end in time-frequency analysis, and it does not make sense to talk about a corresponding Weyl-like pseudo-differential quantization: especially, \((u,v) \mapsto W(u,v)\) would not be modulation-invariant for commutative \(G \neq \{e\}\). However, consider such a compact group \(G\), where \((y \mapsto y^2) : G \to G\) is a bijection: its inverse \((y \mapsto y^{1/2}) : G \to G\) is a homeomorphism of the compact Hausdorff space \(G\). Then

\[
W(u,v)(x,\eta) := \int \eta(y)^* u(xy^{1/2}) v(xy^{-1/2})^* \, dy
\]

defines the natural Wigner transform on \(G\), where \(y^{-1/2} = (y^{1/2})^{-1}\), and \(\phi_W(\xi, y) = \xi(y^{1/2})\). Especially, it is possible to define the Wigner time-frequency transform on finite cyclic groups of odd order, or on \(p\)-adic groups for primes \(p \neq 2\). Related questions on commutative locally compact groups have been treated in [23].

### 11 Normalization, and time-frequency margins

**Definition 11.1** We call time-frequency transform \(D\) normalized if

\[
\iint D(u,v)(x,\eta) \, d\eta \, dx = \langle u,v \rangle = \langle \hat{u}, \hat{v} \rangle
\]

for all \(u,v \in \mathcal{S}(G)\). Especially for \(v = u\) formula (103) yields the energy \(\|u\|^2 = \|\hat{u}\|^2\). We say that the \(D\)-quantization has correct traces if

\[
\text{tr}(a^D) = \iint a(t,\eta) \, d\eta \, dt
\]

for all \(a \in \mathcal{S}(G \times \hat{G})\).

**Theorem 11.2** The following conditions are equivalent:

(a) \(\iint D[1](x,\eta) \, d\eta \, dx = 1\).

(b) Time-frequency transform \(D\) is normalized.

(c) The \(D\)-quantization has correct traces.

(d) The ambiguity kernel satisfies \(\phi_D(\varepsilon,e) = 1 \in \mathbb{C}\).

*Especially, the Kohn–Nirenberg transform \(R\) is normalized.*

**Remark 11.3** Condition (a) in the previous Theorem is relevant only for compact groups \(G\).
Proof. Conditions (a), (b) and (d) are equivalent, because
\[ \iint D(u,v)(x,\eta)\,d\eta\,dx = FD(u,v)(\varepsilon,e) = \phi_D(\varepsilon,e)FR(u,v)(\varepsilon,e) = \phi_D(\varepsilon,e)(u,v). \]

Let \( a \in S(G \times \hat{G}) \). By Lemma 8.5, we see that \( aD = bR \), where \( b := F^{-1}(\phi_D^*Fa) \in S(G \times \hat{G}) \). Hence
\[ \iint b(x,\eta)\,d\eta\,dx = Fb(\varepsilon,e) = \phi_D(\varepsilon,e)^*Fa(\varepsilon,e) = \phi_D(\varepsilon,e)^*\iint a(x,\eta)\,d\eta\,dx. \]

Moreover, \( b(x,\eta) = \eta(x)^*(bR\eta)(x) \), so that
\[ \text{tr}(aD) = \text{tr}(bR) = \sum_{\eta \in \hat{G}} d_{\eta} \sum_{j,k=1}^{d_{\eta}} \langle bR\eta_{jk},\eta_{jk} \rangle = \iint b(x,\eta)\,d\eta\,dx. \]

Thus conditions (c) and (d) are equivalent. QED

Remark 11.4 Let us find how the Schwartz kernel \( K \in S(G \times G) \) of \( aD \) is related to the symbol \( a \in S(G \times \hat{G}) \) in the previous proof:
\[ \langle u,aDv \rangle = \langle D(u,v),a \rangle = \iint D(u,v)(t,\eta)\,a(t,\eta)^*\,d\eta\,dt \]
\[ = \iint \int \eta(y)^* \int \xi(t)\phi_D(\xi,y) \int \xi(x)^*u(x)v(xy^{-1})^*\,dx\,dy\,a(t,\eta)^*\,d\eta\,dt \]
\[ = \int \int \int \eta(y)(a(t,\eta)\int v(xy^{-1})\xi(x)\phi_D(\xi,y)^*\xi(t)^*\,d\xi\,dy\,d\eta\,dt)^*\,dx. \]

Hence we obtain
\[ K(x,z) = \iint \eta(z^{-1}x)(a(t,\eta)\int \xi(t^{-1}x)\phi_D(\xi,z^{-1}x)^*\,d\xi\,d\eta\,dt. \]

Here naturally \( \text{tr}(aD) = \int K(x,x)\,dx. \)

Definition 11.5 We say that time-frequency transform \( D \) has the correct time margins if
\[ \int D(u,v)(x,\eta)\,d\eta = u(x)v(x)^* \quad (105) \]
for all \( u,v \in S(G) \) and \( x \in G \). We say that \( D \)-quantization is correct in time if
\[ aDv(x) = f(x)v(x) \quad (106) \]
for all \( v \in S(G) \) and for all symbols \( a \) of the time-like form \( a(x,\eta) = f(x)I \), where \( f \in S(G) \).
Theorem 11.6 The following conditions are equivalent:
(a) $D[\delta_e] = \delta_e \otimes I$. In other words, $D[\delta_e](x,\eta) = \delta_e(x) I$.
(b) Time-frequency transform $D$ has the correct time margins.
(c) The $D$-quantization is correct in time.
(d) The ambiguity kernel satisfies $\phi_D(\xi, e) = I$ for all $\xi \in \hat{G}$.

Proof. For any time-frequency transform $D$ we have

$$ F(D[\delta_e])(\xi, y) = \phi_D(\xi, y) \int \xi(z) \delta_e(z) (zy^{-1})^* dz = \phi_D(\xi, e) \delta_e(y). $$

On the other hand, if $D[\delta_e](x,\eta) = \delta_e(x) I$ then

$$ F(D[\delta_e])(\xi, y) = \int \xi(x)^* \eta(y) \delta_e(x) d\eta dx = \delta_e(y) I. $$

Thus conditions (a) and (d) are equivalent. By the Fourier inverse formula,

$$ \int D(u,v)(x,\eta) d\eta = \int \int \xi(x)^* \eta(y) \delta_e(x) d\eta dx = \phi_D(\xi, e) \delta_e(y). $$

so that conditions (b) and (d) are equivalent. Now assume condition (b), and let $a(x,\eta) = f(x)$. Then

$$ \langle u, a D v \rangle = \langle D(u,v), a \rangle = \int \int D(u,v)(x,\eta) a(x,\eta)^* d\eta dx $$

$$ = \int \int D(u,v)(x,\eta) f(x)^* dx $$

$$ = \int u(x) v(x)^* f(x)^* dx, $$

so $a^D v(x) = f(x) v(x)$. That is, condition (b) implies (c). Finally, assume condition (c). Let $(h_\alpha)_\alpha$ be a bounded left approximate identity with $0 \leq h_\alpha \in \mathcal{S}'(G)$. By translation, it is enough to check the time margins at $x = e$:

$$ \int D(u,v)(e,\eta) d\eta = \int \int D(u,v)(t,\eta) \delta_e(t) d\eta dt $$

$$ = \lim_\alpha \langle D(u,v), h_\alpha \otimes I \rangle $$

$$ = \lim_\alpha \langle u, (h_\alpha \otimes I) D v \rangle $$

$$ = \lim_\alpha \langle u, h_\alpha v \rangle $$

$$ = u(e) v(e)^*. $$

24
This proves condition (b) of the correct margins in time. \textbf{QED}

**Definition 11.7** We say that time-frequency transform $D$ has the \textit{correct frequency margins} if
\begin{equation}
\int D(u, v)(x, \eta) \, dx = \hat{u}(\eta) \hat{v}(\eta)^* \tag{107}
\end{equation}
for all $u, v \in \mathcal{S}(G)$ and $\eta \in \hat{G}$. As a special case $v = u$ of (107), matrix $\hat{u}(\eta) \hat{u}(\eta)^*$ is the “energy density” of $u$ at frequency $\eta \in \hat{G}$. We say that $D$-quantization is \textit{correct in frequency} if
\begin{equation}
b^D v(x) = v * g(x), \quad \text{i.e.} \quad \widehat{b^D v}(\eta) = \hat{g}(\eta) \hat{v}(\eta), \tag{108}
\end{equation}
for all $v \in \mathcal{S}(G)$ and for all symbols $b$ of the frequency-like form $b(x, \eta) = \hat{g}(\eta)$, where $g \in \mathcal{S}(G)$.

**Theorem 11.8** The following conditions are equivalent:

(a) $D[1] = 1 \otimes \delta_\varepsilon I$. In other words, $D[1](x, \eta) = \delta_\varepsilon(\eta) I$.

(b) Time-frequency transform $D$ has the correct frequency margins.

(c) The $D$-quantization is correct in frequency.

(d) The ambiguity kernel satisfies $\phi_D(\varepsilon, y) = 1 \in \mathbb{C}$ for all $y \in G$.

**Proof.** For any time-frequency transform $D$ we have
\[F(D[1])(\xi, y) = \phi_D(\xi, y) \int \xi(z) \, dz = \phi_D(\varepsilon, y) \delta_\varepsilon(\xi).\]

On the other hand, $D[1](x, \eta) = \delta_\varepsilon(\eta) I$ gives here
\[F(D[1])(\xi, y) = \int \xi(x)^* \int \eta(y) \delta_\varepsilon(\eta) I \, d\eta \, dx = \int \xi(x)^* \, dx = \delta_\varepsilon(\xi).\]

Hence conditions (a) and (d) are equivalent. By $\mathcal{F}, \mathcal{F}^{-1}$ canceling each other, we obtain
\[
\int D(u, v)(x, \eta) \, dx = \int \int \eta(y)^* \int \xi(x) \phi_D(\xi, y) FR(u, v)(\xi, y) \, d\xi \, dy \, dx
\]
\[= \int \int \eta(y)^* \phi_D(\varepsilon, y) FR(u, v)(\varepsilon, y) \, dy
\]
\[= \int \int \eta(y)^* \phi_D(\varepsilon, y) u(z) v(zy^{-1})^* \, dy \, dz.
\]
Especially,
\[
\int D(u, \delta_\varepsilon)(x, \eta) \, dx = \int \eta(y)^* \phi_D(\varepsilon, y) u(y) \, dy
\]
which equals to $\hat{a}(\eta)$ for all $u \in \mathcal{S}(G)$ if and only if $\phi_D(\varepsilon, y) = 1$ for all $y \in G$:
in that case also

$$D(u, v)(x, \eta) \, dx = \iint \eta(z)^* u(z) \eta(zy^{-1}) v(zy^{-1})^* \, dy \, dz = \hat{u}(\eta) \hat{v}(\eta)^*.$$  

Thus conditions (b) and (d) are equivalent. Now assume condition (b), and let $b(x, \eta) = \hat{g}(\eta)$. Then

$$\langle u, b^D v \rangle = \langle D(u, v), b \rangle = \iint D(u, v)(x, \eta) b(x, \eta)^* \, d\eta \, dx$$

$$= \iint D(u, v)(x, \eta) \, dx \hat{g}(\eta)^* \, d\eta$$

$$= \int \hat{u}(\eta) \hat{v}(\eta)^* \hat{g}(\eta)^* \, d\eta$$

$$= \int \hat{u}(\eta) (\hat{g}(\eta) \hat{v}(\eta))^* \, d\eta$$

$$= \langle \hat{u}, \hat{g} \hat{v} \rangle = \langle u, v * g \rangle.$$  

Hence condition (c) follows from (b). Finally, assume condition (c). Then

$$\int D(u, v)(x, \eta) \, dx = \iint D(u, v)(x, \omega) \delta_{\eta}(\omega) \, d\omega \, dx$$

$$= \langle D(u, v), 1 \otimes \delta_\eta I \rangle$$

$$= \langle u, (1 \otimes \delta_\eta I) D v \rangle$$

$$= \langle \hat{u}, \delta_\eta \hat{v} \rangle$$

$$= \hat{u}(\eta) \hat{v}(\eta)^*,$$

so that we obtain condition (b) of the correct margins in frequency.  

QED

Example 11.9 In a sense, on a finite group $G$ of $|G|$ elements, the minimal time-frequency transform $D$ having the correct margins would satisfy

$$\phi_D(\xi, y) = \begin{cases} 1 & \text{if } \xi = \varepsilon \text{ or } y = e, \\ 0 & \text{otherwise.} \end{cases} \tag{109}$$  

Then

$$D(u, v)(x, \eta) = \hat{u}(\eta) \hat{v}(\eta)^* + \frac{1}{|G|} (u(x)^* v(x) - \langle u, v \rangle) I. \tag{110}$$  

Such $D$ could be added to other time-frequency transforms that would otherwise have zero margins: for instance, this happens when the original localization comes from a commutator of position and momentum operators, like on cyclic groups in Section [16].
12 Positivity

From the application point of view, a reasonable time-frequency transform ought to be at least normalized: this does not pose any problems. However, it turns out that the pointwise positivity is typically conflicting with the margin properties, and thus positivity may not be an utterly desirable property.

**Definition 12.1** Positivity of time-frequency transform $D$ means

$$ D[u](x, \eta) \geq 0 $$

for all $u \in \mathcal{S}(G)$ and all $(x, \eta) \in G \times \hat{G}$. Positivity of the $D$-quantization $a \mapsto a^D$ means that for all $u \in \mathcal{S}(G)$

$$ \langle u, a^D u \rangle \geq 0 $$

whenever $a \in \mathcal{S}(G \times \hat{G})$ is positive in the sense that $a(x, \eta) \geq 0$ for all $(x, \eta) \in G \times \hat{G}$.

**Example 12.2** In the trivial case of the one-element group $G = \{e\}$, defining $D(u, v)(x, \eta) := u(e) v(e)^*$ gives a positive time-frequency transform with the correct margins in time and in frequency. For time-frequency transforms, positivity is a special case of symmetry:

**Theorem 12.3** The following conditions are equivalent:

(a) For all $u \in \mathcal{S}(G)$ we have $D[u](e, \varepsilon) \geq 0$.

(b) Time-frequency transform $D$ is positive.

(c) The $D$-quantization is positive.

(d) The time-lag kernel satisfies $\varphi_D(x, y) = \int \kappa(x, z) \kappa(yx, z)^* \, dz$ for some $\kappa$.

**Proof.** Condition (b) trivially implies (a). Assume condition (a). Let $K_{\delta^D}$ denote the Schwartz kernel of the original localization $\delta^D : \mathcal{S}(G) \to \mathcal{S}'(G)$. Then for any $u \in \mathcal{S}(G)$ and $z = (z_k)_{k=1}^{d_n} \in \mathbb{C}^{d_n}$ we have

$$ \langle D[u](t, \eta) z, z \rangle = \sum_{j,k=1}^{d_n} z_j^* z_k \, D[u]_{j,k}(t, \eta) $$

$$ = \sum_{j,k=1}^{d_n} z_j^* z_k \, \iint K_{\delta^D}(x, y) \, u(tx) \, u(ty)^* \, \eta_{j,k}(yx) \, dx \, dy $$

$$ = \sum_{\ell=1}^{d_n} \iint K_{\delta^D}(x, y) \, u_{\ell}(x) \, u_{\ell}(y)^* \, dx \, dy $$

$$ = \sum_{\ell=1}^{d_n} D[u_{\ell}](e, \varepsilon) \geq 0, $$
where \( u_t(x) := \sum_{k=1}^{d_n} z_k \eta_{k\ell}(x)^* u(tx) \). Hence condition (a) implies (b). Let \( a \geq 0 \).
Then \( a^* = a = (a^{1/2})^2 \), where \( a^{1/2}(x, \eta) = a(x, \eta)^{1/2} \) is the positive square root of \( a(x, \eta) \), and

\[
\langle u, a D u \rangle = \langle D[u], a \rangle \\
= \iint D[u](x, \eta) a(x, \eta) \, d\eta \, dx \\
= \iint D[u](x, \eta) (a(x, \eta)^{1/2})^2 \, d\eta \, dx \\
= \iint a(x, \eta)^{1/2} D[u](x, \eta) a(x, \eta)^{1/2} \, d\eta \, dx \geq 0,
\]

where the last inequality follows because the “integrand” \( a^{1/2} D[u] a^{1/2} \) is positive. Notice that here both the Haar integral and the “non-commutative integral” are positive functionals. Hence condition (b) implies (c). Now suppose \( a \mapsto a^D \) is positive and \( a \in \mathcal{S}(G) \). Take \( (h_{\alpha})_\alpha \) be a bounded left approximate identity, where \( 0 \leq h_{\alpha} \in \mathcal{S}(G) \) such that \( \lim_\alpha (u, h_{\alpha}) = u(e) \). Define \( a_{\alpha} \in \mathcal{S}(G \times \hat{G}) \) by \( a_{\alpha}(x, \eta) := h_{\alpha}(x) \delta_\varepsilon(\eta)I \). Then \( a_{\alpha}(x, \eta) \geq 0 \), and

\[
D[u](e, \varepsilon) = \langle D[u], \delta_{(e, \varepsilon)} \rangle = \lim_\alpha \langle D(u, u), a_{\alpha} \rangle = \lim_\alpha \langle u, (a_{\alpha})^D u \rangle,
\]

which is non-negative due to the positivity of the quantization. Hence condition (c) implies (a). Assuming (d), from (83) we obtain

\[
D[u](e, \varepsilon) = \int u(x) \left( \int \varphi_D(x^{-1}, y^{-1}x)^* u(y) \, dy \right)^* \, dx \\
= \iint u(x) u(y)^* \varphi_D(x^{-1}, y^{-1}x) \, dy \, dx \\
= \iint \int u(x) u(y)^* \kappa(x^{-1}, z) \kappa(y^{-1}, z)^* \, dz \, dy \, dx \\
= \int \left| \int u(x) \kappa(x^{-1}, z) \, dz \right|^2 \, dx \geq 0,
\]

yielding condition (a). Finally, let \( \delta^D = A^2 \) for a positive operator \( A \). Then

\[
\varphi_D(x, y) = K_{A^2}(x^{-1}, x^{-1}y^{-1})^* \\
= \int K_A(x^{-1}, z)^* K_A(z, x^{-1}y^{-1})^* \, dz \\
= \int K_A(x^{-1}, z)^* K_A(x^{-1}y^{-1}, z) \, dz \\
= \int \kappa(x, z) \kappa(yx, z)^* \, dz,
\]

when setting \( \kappa(x, z) = K_A(x^{-1}, z)^* \). Thus condition (d) follows from (a). QED
**Spectrograms.** A simple example of positive original localization operators is an orthogonal projection \( \delta^D : L^2(G) \to L^2(G) \) onto the 1-dimensional subspace spanned by a unit-energy window \( w \in \mathcal{S}(G) \):

\[
\delta^D v := \langle v, w \rangle w.
\]  

(111)

The window here should be “focused at \((e, \varepsilon) \in G \times \hat{G}\)” in a reasonable sense: most of energy of \( w \) should be nearby \( e \in G \), and most of energy of \( \hat{w} \) should be nearby \( \varepsilon \in \hat{G} \). In any case, now \( K_{\delta^D}(x,y) = w(y)^* w(x) \), and

\[
D(u, v)(x, \eta) = \iint u(xz) \eta(z)^* K_{\delta^D}(z, y)^* \eta(y) v(xy)^* \, dy \, dz
\]

(112)

\[= \mathcal{G}_w u(x, \eta) \mathcal{G}_w v(x, \eta)^*, \]

(113)

where

\[\mathcal{G}_w u(x, \eta) := \int \eta(y)^* u(y) w(x^{-1}y)^* \, dy\]

(114)

defines the \( w \)-windowed short-time Fourier transform \( \mathcal{G}_w u \) of signal \( u \). Notice that

\[\mathcal{G}_w \delta_e(x, \eta) = w(x^{-1})^* =: \hat{w}(x), \quad (115)\]

\[\mathcal{G}_w \mathcal{1}(x, \eta) = \hat{\eta}(\eta)^* \eta(x)^*. \quad (116)\]

Clearly \( D[u](x, \eta) := D(u, u)(x, \eta) \geq 0 \), and we may call it the \( w \)-spectrogram of signal \( u \) at \((x, \eta) \in G \times \hat{G}\). Actually, such a short-time Fourier transform formula on unimodular groups was briefly mentioned in [4], as an analogue to the Euclidean case. Let us find the corresponding ambiguity kernel \( \phi_D \):

\[
FD(u, v)(\xi, y) = \iint \xi(x)^* \int \eta(y) \int t u(t) w(x^{-1}t)^* \, dt \int w(x^{-1}s) v(s)^* \eta(s) \, ds \, dy \, dx
\]

\[
= \iint \xi(x)^* \int u(t) w(x^{-1}t)^* w(x^{-1}ty^{-1}) v(ty^{-1})^* \, dt \, dx
\]

\[
= \iint \left( \int \xi(x^{-1}t) w(x^{-1}t)^* w(x^{-1}ty^{-1}) \, dx \right) \xi(t)^* u(t) v(ty^{-1})^* \, dt
\]

\[
= \iint \left( \int \xi(z) w(z)^* w(zy^{-1}) \, dz \right) \xi(t)^* u(t) v(ty^{-1})^* \, dt
\]

\[
= \left( \int \xi(z) w(z)^* w(zy^{-1}) \, dz \right)^* \int \xi(t)^* u(t) v(ty^{-1})^* \, dt
\]

\[
= \phi_D(\xi, y) FR(u, v)(\xi, y),
\]

where

\[
\phi_D(\xi, y) = \int \xi(z) w(z)^* w(zy^{-1}) \, dz = FR(w, w)(\xi, y)^*.
\]
Hence $\varphi_D(x,y) = (\mathcal{F}^{-1} \otimes I) \phi_D(x,y) = w(x^{-1})^* w(x^{-1}y^{-1}) = \tilde{w}(x) \tilde{w}(yx)^*$.

The energy normalization means then the energy normalization of the window:

$$1 = \phi_D(\varepsilon,e) = \int |w(x)|^2 \, dx = \|w\|^2.$$  

The correct margins in time would mean

$$I = \phi_D(\xi,e) = \int \xi(x) |w(x)|^2 \, dx = \hat{|w|^2}(\xi),$$

i.e. $|w|^2 = \delta_e$, the Dirac delta at $e \in G$. From another point of view, here

$$D[\delta_e](x,\eta) = |w(x^{-1})|^2 = |\tilde{w}(x)|^2, \\
D[1](x,\eta) = \overline{\tilde{w}(\eta)^* \tilde{w}(\eta)}.$$  

Consequently, it is too much to ask for the correct margins here, but the energy normalization follows just from $\|w\| = 1$.

**Remark 12.4** Let $D$ be a positive time-frequency transform satisfying the correct margins both in time (105) and in frequency (107). Suppose $\delta_D$ is bounded on $L^2(G)$. By the spectral decomposition of $\delta_D$, then $G$ must be the trivial group of just one element $e$, and $D(u,v)(x,\eta) = u(e) v(e)^*$.  

#### 13 Unitarity

**Definition 13.1** Time-frequency transform $D$ is called unitary if it satisfies the Moyal identity

$$\langle D(u,v), D(f,g) \rangle = \langle u,f \rangle \langle v,g \rangle^*$$  

for all $u,v \in \mathcal{S}(G)$ and $f,g \in \mathcal{S}'(G)$. The $D$-quantization $a \mapsto a^D$ is called unitary if

$$\langle a,b \rangle = \langle a^D,b^D \rangle$$  

for all $a, b \in \mathcal{S}(G \times \hat{G})$, where $\langle a^D,b^D \rangle = \operatorname{tr} (a^D (b^D)^*)$.  

**Theorem 13.2** The following conditions are equivalent:

(a) $\langle D(u,1), D(\delta_e,\delta_y) \rangle = u(e)$ for all $u \in \mathcal{S}(G)$ and $y \in G$.

(b) Time-frequency transform $D$ is unitary.

(c) The $D$-quantization is unitary.

(d) Ambiguity operators $\phi_D(\xi,y)$ are unitary for all $(\xi,y) \in \hat{G} \times G$.

Especially, the Kohn–Nirenberg transform is unitary.

**Remark 13.3** In condition (a) of Theorem 13.2 on non-compact $G$ we may approximate the constant $1 \notin \mathcal{S}(G)$ within $\mathcal{S}(G)$.  

30
Proof. As \( \phi_R(\xi, y) \equiv I \), the Kohn–Nirenberg transform satisfies condition (d). Moreover, it is unitary, because

\[
\langle R(u, v), R(f, g) \rangle = \int \int u(x) \eta(x)^* \hat{v}(\eta)^* \hat{g}(\eta) f(x)^* \, d\eta \, dx
\]

\[
= \int u(x) f(x)^* \, dx \int \hat{v}(\eta)^* \hat{g}(\eta) \, d\eta
\]

\[
= \langle u, f \rangle \langle \hat{g}, \hat{v} \rangle = \langle u, f \rangle \langle v, g \rangle^*.
\]

Assume (d), i.e. the unitarity of the ambiguity operators \( \phi_D(\xi, y) \). Then

\[
\langle D(u, v), D(f, g) \rangle = \langle FD(u, v), FD(f, g) \rangle
\]

\[
= \int \int \phi_D(\xi, y) FR(u, v)(\xi, y) FR(f, g)(\xi, y)^* \phi_D(\xi, y)^* \, d\xi \, dy
\]

\[
= \int \int FR(u, v)(\xi, y) FR(f, g)(\xi, y)^* \, d\xi \, dy
\]

\[
= \langle FR(u, v), FR(f, g) \rangle
\]

\[
= \langle R(u, v), R(f, g) \rangle.
\]

Thus condition (d) implies (b), as we already know that \( R \) is unitary. Condition (b) implies condition (a), because for \((u, v, f, g) = (u, 1, \delta_e, \delta_y)\) we have

\[
\langle u, f \rangle \langle v, g \rangle^* = u(e).
\]

Now assume condition (a), and let \((u, v, f, g) = (u, 1, \delta_e, \delta_y)\), and \(M(\omega, t) := \phi_D(\omega, t)^* \phi_D(\omega, t)\). Then

\[
u(e) = \langle D(u, v), D(f, g) \rangle
\]

\[
= \langle FD(u, v), FD(f, g) \rangle
\]

\[
= \int \int M(\xi, t) \int \xi(x)^* u(x) v(xt^{-1})^* dx \left( \int \xi(z)^* f(z) g(xt^{-1})^* dz \right)^* \, d\xi \, dt
\]

\[
= \int M(\xi, y^{-1}) \hat{u}(\xi) \, d\xi.
\]

Since this holds for every \( u \in \mathcal{S}(G) \), we have \( M(\xi, y^{-1}) = I \) for every \((\xi, y) \in \hat{G} \times G\). Hence condition (d) follows from (a).

Finally, let us consider the Hilbert–Schmidt inner product of operators:

\[
\langle a^D, b^D \rangle = \int \int K_{a^D}(x, y) K_{b^D}(x, y)^* \, dx \, dy
\]

\[
= \int \int \int \xi(yt) \phi_D(\xi, t)^* Fa(\xi, t) \, d\xi \int Fb(\omega, t)^* \phi_D(\omega, t) \omega(yt)^* \, d\omega \, dx \, dy
\]

\[
= \int \int \phi_D(\xi, t)^* Fa(\xi, t) Fb(\xi, t)^* \phi_D(\xi, t) \, d\xi \, dt.
\]

It is clear that this equals to \( \langle Fa, Fb \rangle = \langle a, b \rangle \) for all \( a, b \in \mathcal{S}(G \times \hat{G}) \) if and only if condition (d) holds: thus conditions (c) and (d) are equivalent. QED
Remark 13.4 By the previous Theorem, unitary time-frequency transforms satisfy the Moyal identity (117) also for all \( u, v, f, g \in L^2(G) \). As a consequence of the unitarity of the Kohn–Nirenberg transform, the energy densities \( D[v_\alpha] \) uniformly cover the time-frequency plane \( G \times \hat{G} \) for any time-frequency transform \( D \):

**Corollary 13.5** Let \( D \) be normalized, i.e. \( \phi_D(\varepsilon, e) = 1 \). Let \( (v_\alpha)_{\alpha \in J} \) be an orthonormal basis of \( L^2(G) \). Then \( b^R = I \), where

\[
b = \sum_{\alpha \in J} D[v_\alpha].
\]

**Proof.** Notice that

\[
\langle u, v \rangle = \langle \sum_{\alpha \in J} (u, v_\alpha) v_\alpha, v \rangle = \sum_{\alpha \in J} \langle u, v_\alpha \rangle \langle v, v_\alpha \rangle^*.
\]

Thus by the previous Theorem, for the Kohn–Nirenberg transform \( R \) we have

\[
\langle u, v \rangle = \sum_{\alpha \in J} \langle R(u, v), R(v_\alpha, v_\alpha) \rangle = \langle R(u, v), \sum_{\alpha \in J} R[v_\alpha] \rangle = \langle u, a^R v \rangle,
\]

yielding \( a^R = I \) with

\[
a = \sum_{\alpha \in J} R[v_\alpha].
\]

Now

\[
\sum_{\alpha \in J} D[v_\alpha] = \sum_{\alpha \in J} R[v_\alpha] * \psi_D = I * \psi_D = \lambda I,
\]

where \( \lambda = \int \int \psi_D(x, \eta) \, d\eta \, dx = \phi_D(\varepsilon, e) = 1 \).

QED

### 14 Inner invariance

Let us study the invariance under inner automorphisms \( (x \mapsto z^{-1}xz) : G \to G \). We denote \( u_z(x) := u(z^{-1}xz) \) for \( u \in \mathcal{S}(G) \) and \( x, z \in G \).

**Definition 14.1** Time-frequency transform \( D \) is called *inner* if it satisfies

\[
D(u_z, v_z)(x, \eta) = \eta(z) D(u, v)(z^{-1}xz, \eta) \eta(z)^* \quad (120)
\]

for all \( u, v \in \mathcal{S}(G) \), \( (x, \eta) \in G \times \hat{G} \) and \( z \in G \). The \( D \)-quantization \( a \mapsto a^D \) is called *inner* if

\[
(a^D(v_z))_{z^{-1}} = a^D v \quad (121)
\]

for all \( v \in \mathcal{S}(G) \) and \( z \in G \) whenever \( a \in \mathcal{S}(G \times \hat{G}) \) satisfies \( a(z^{-1}xz, \eta) = \eta(z)^* a(x, \eta) \eta(z) \) for all \( (x, \eta) \in G \times \hat{G} \) and \( z \in G \).
Theorem 14.2 The following conditions are equivalent:

(a) \( D[u_z](e, \varepsilon) = D[u](e, \varepsilon) \) for all \( u \in \mathcal{S}(G) \) and \( z \in G \).

(b) Time-frequency transform \( D \) is inner.

(c) The \( D \)-quantization is inner.

(d) \( \phi_D(\xi, zy^{-1}) = \xi(\eta) \phi_D(\xi, y) \eta(\xi)^* \) for all \( (\xi, y) \in \hat{G} \times G \) and \( z \in G \).

Especially, the Kohn–Nirenberg transform is inner.

Proof. Condition (a) is a special case of condition (b). Condition (d) implies condition (b), because

\[
D(u_z, v_z)(x, \eta) = \int \eta(y)^* \int \xi(t) \phi_D(\xi, y) \int \xi(t)^* u_z(t) v_z(t y^{-1})^* \, dt \, d\xi \, dy
\]

\[
= \int \eta(y)^* \int \xi(t) \phi_D(\xi, y) \int \xi(t z^{-1})^* u(t) v(t z^{-1} y^{-1})^* \, dt \, d\xi \, dy
\]

\[
= \int \eta(z) \phi_D(\xi, z) \int \xi(t)^* u(t) v(t y^{-1})^* \, dt \, d\xi \, dy
\]

\[
= \eta(z) D(u, v)(z^{-1} xz, \eta) \eta(z)^*.
\]

Suppose \( a \in \mathcal{S}(G \times \hat{G}) \) is inner invariant: now assuming condition (b), we obtain condition (c), because

\[
\langle u, (a^D(v_z))_{z^{-1}} \rangle = \langle u_z, a^D(v_z) \rangle = \langle D(u_z, v_z), a \rangle \overset{(b)}{=} \langle D(u, v), a \rangle = \langle u, a^D v \rangle.
\]

Now assume condition (c). Let \( (h_\alpha)_\alpha \) be an inner invariant approximate identity in \( \mathcal{S}(G) \). Let \( a_\alpha(x, \eta) = h_\alpha(x) \delta_\eta I : H_\eta \to H_\eta \). Then for all \( u \in \mathcal{S}(G) \) and \( z \in G \) we have

\[
D[u](e, \varepsilon) = \langle u, \delta_{(e, \varepsilon)}^D \rangle = \lim_{\alpha} \langle u, a_\alpha^D u \rangle \overset{(c)}{=} \lim_{\alpha} \langle u, (a_\alpha^D(u_z))_{z^{-1}} \rangle = D[u_z](e, \varepsilon).
\]

Hence condition (c) implies condition (a). Finally, conditions (a) and (d) are equivalent, because for the kernel \( \varphi_D = (\mathcal{F}^{-1} \otimes I)\phi_D \) on one hand

\[
D[u](e, \varepsilon) = \iint \varphi_D(x^{-1}, y) u(x) u(xy^{-1})^* \, dx \, dy,
\]
and on the other hand
\[ D[u_z](e,\varepsilon) = \int\int \varphi_D(x^{-1}, y) u_z(x) u_z(xy^{-1})^* \, dx \, dy \]
\[ = \int\int \varphi_D(x, z^{-1}xz) u(z^{-1}xy^{-1}z)^* \, dx \, dy \]
\[ = \int\int \varphi_D(zx^{-1}z^{-1}, y) u(x) u(zx^{-1}y^{-1}z)^* \, dx \, dy \]
\[ = \int\int \varphi_D(zx^{-1}z^{-1}, zyz^{-1}) u(x) u(xy^{-1})^* \, dx \, dy. \]

This completes the proof. \hfill QED

15 On locally compact groups

Time-frequency analysis on compact groups was presented above so that the results turn out to have natural counterparts on those locally compact groups that allow reasonable Fourier analysis. We shall consider two families of such groups: the Abelian ones, and the type I second-countable unimodular locally groups.

15.1 Locally compact Abelian groups

For locally compact Abelian groups, time-frequency analysis has been studied e.g. in [23], and Kohn–Nirenberg pseudo-differential operators have been treated in [16]. We just have to modify the definitions a bit, and then the results would hold as such. In the commutative case, the frequency matrices would be just one-dimensional scalars, which drastically simplifies many of the proofs.

What to change? Let \( G \) be a locally compact Abelian group. Now \( \hat{G} \) is the character group of \( G \), consisting of the characters \( \eta: G \rightarrow U(1) \), i.e. continuous scalar unitary homomorphisms. By the Pontryagin–van Kampen duality theorem, \( \hat{G} \) is a locally compact Abelian group. The group operation is given by the multiplication of the characters, and the topology is the natural compact-open topology. In the non-compact case, we choose a positive regular group-invariant measure on \( G \) to be the Haar measure: this is unique up to a scalar multiple, and \( G \) has then infinite measure. After this, we choose the Haar measure on \( \hat{G} \) so that the Fourier transform and the Fourier inverse transform formulas match:

\[ \hat{u}(\eta) = \int_G u(y) \eta(y)^* \, dy, \quad u(x) = \int_{\hat{G}} \eta(x) \hat{u}(\eta) \, d\eta \quad (122) \]

for those \( u \in L^1(G) \) for which \( \hat{u} \in L^1(\hat{G}) \). Then we let the test function space to be \( \mathcal{S}(G) \), the Schwartz–Bruhat space on \( G \). The corresponding tempered distribution space is denoted by \( \mathcal{S}'(G) \).

Why we did not choose Eymard’s Fourier algebra \( A(G) \) for a space of test functions on compact groups \( G \)? Here \( u \in A(G) \) has the norm \( \|u\|_{A(G)} := \)
The Fourier algebra looks initially an inviting alternative, especially as on the compact Abelian groups it coincides with the Feichtinger algebra. The Feichtinger algebra has turned out to be a natural setting for time-frequency analysis on locally compact Abelian groups, see e.g. \cite{Feichtinger, Feichtinger2, Feichtinger3}. However, on non-commutative compact groups the Kohn–Nirenberg transform would not map $A(G) \times A(G)$ to $A(G \times \hat{G})$, and we would have the similar difficulties with the Kohn–Nirenberg quantization, which is our starting point for the time-frequency analysis on groups. The difficulties boil down to that the co-multiplication $\Delta$ does not necessarily map $A(G)$ to $A(G \times G)$, as

\[
\|u\|_{A(G)} = \sum_{\eta \in \hat{G}} d_\eta \text{tr}(\hat{u}(\eta)),
\]

\[
\|\Delta u\|_{A(G \times G)} = \sum_{\eta \in \hat{G}} d_\eta^2 \text{tr}(\hat{u}(\eta)),
\]

where dimensions $d_\eta$ may grow arbitrarily large. Of course, $d_\eta = 1$ when the group is commutative, and then $\Delta : A(G) \to A(G \times G)$ is an isometry, and the Kohn–Nirenberg transform behaves well.

All in all, on a locally compact Abelian group $G$, a time-frequency transform is a mapping

\[
D : \mathcal{S}(G) \times \mathcal{S}(G) \to \mathcal{S}(G \times \hat{G})
\]

such that

\[
FD(u, v)(\xi, y) = \phi_D(\xi, y) FR(u, v)(\xi, y),
\]

where the ambiguity kernel $\phi_D : \hat{G} \times G \to \mathbb{C}$ defines a Schwartz multiplier $h \mapsto F^{-1}(\phi_D Fh)$. Then we have the translation-modulation invariance

\[
D[M_\xi T_y u](x, \eta) = D[u](x - y, \xi^{-1} y),
\]

where $T_y u(x) := u(x - y)$ and $M_\xi u(x) := \xi(x) u(x)$.

In case of the compact group $G$, the approximate identities on $G \times \hat{G}$ could be treated merely on $G$. This is not enough on non-compact locally compact Abelian groups $G$, but the modification for $G \times \hat{G}$ is easy. Notice that in the calculations for non-compact $G$, distribution $1 \notin \mathcal{S}(G)$ occasionally has to be approximated by test functions.

15.2 Type I second-countable unimodular groups

Let $G$ be a type I second-countable unimodular locally compact group. For background information, see e.g. \cite{Feichtinger, Feichtinger2, Feichtinger3}. Unimodularity of $G$ means that the left-invariant Haar measure coincides with the right-invariant Haar measure: briefly, it is the Haar measure of $G$. Recall that a topological space is second-countable when its topology has a countable base. In our convention, topological groups are always Hausdorff spaces, and consequently second-countable locally compact groups are metrizable with a complete metric. Moreover, second-countable locally compact groups are of type I if and only if they are postliminal: this
means that for each $\eta \in \hat{G}$ the compact linear operators $M : H_\eta \to H_\eta$ belong to the closure of $\{u(\eta) : u \in L^1(G)\}$.

On such a group $G$, the Schwartz–Bruhat space $\mathcal{S}(G)$ will be the test function space, with the corresponding Schwartz–Bruhat distributions $\mathcal{S}'(G)$. The time-frequency analysis results on compact groups are carried to $G$ without major changes in formulations and proofs. The unit constant function $1 : G \to \mathbb{C}$ is a distribution which does not belong to $\mathcal{S}(G)$ on non-compact $G$, but it can be approximated by the test functions.

16 Example of finite cyclic groups

Consider time-frequency analysis on the finite cyclic group $G = \mathbb{Z}/N\mathbb{Z}$, where $\hat{G} \cong G$. First, label spaces $G, \hat{G}$ by functions $f : G \to \mathbb{R}$ and $\hat{g} : \hat{G} \to \mathbb{R}$. Define respective position and momentum operators $A, B : L^2(G) \to L^2(G)$ by

$$ Au := fu, \quad \hat{B}u := \hat{g}\hat{u}. \tag{123} $$

The uncertainty observable of measurement pair $(A, B)$ is

$$ \delta^{D_{\mathbb{Z}/N\mathbb{Z}}}(x, y) = \int K_{\mathbb{Z}/N\mathbb{Z}}(x, y) v(y) dy, \tag{125} $$

where

$$ K_{\mathbb{Z}/N\mathbb{Z}}(x, y) = i2\pi (f(y) - f(x)) g(x - y) \tag{126} $$

corresponds to the time-lag kernel $\varphi_{D_{\mathbb{Z}/N\mathbb{Z}}(x, y)} = i2\pi (f(y) - f(x)) g(x - y)^\ast$. \hspace{1cm} (127)

As $D(u, v)(0, 0) = \langle u, \delta^D v \rangle$, by the time-frequency shift-invariance

$$ |D_{\mathbb{Z}/N\mathbb{Z}}(u, v)(x, \eta)| \leq 2\pi \|AB - BA\| \|u\| \|v\| \leq 4\pi \|f\|_{L^\infty} \|\hat{g}\|_{L^\infty} \|u\| \|v\| \tag{128} $$

for all $(x, \eta) \in G \times \hat{G}$. For the ambiguity kernel $\phi_{D_{\mathbb{Z}/N\mathbb{Z}}(x, \eta)} = i2\pi \hat{f}(-\xi) \left(1 - e^{i2\pi\xi/N}\right) g(y)^\ast$.

A natural choice for the position labeling function $f : G \to \mathbb{R}$ could be

$$ f(x) := x/N \quad \text{for} \quad 0 \leq x < N \tag{130} $$

(here $f(x) := x/N$ for $0 < x \leq N$ would be another good choice, but it ultimately leads to the same limit as $N \to \infty$ in the next section). Observe that for $0 < \eta < N$

$$ 0 = N^{-1} \sum_{x=0}^{N-1} ((x+1)/N - x/N) e^{-i2\pi x\eta/N} = e^{i2\pi\eta/N} \left(\hat{f}(\eta) + N^{-1}\right) - \hat{f}(\eta), \tag{129} $$

$$ 36 $
that the Kohn–Nirenberg quantization

\[ \hat{f}(\eta) = \frac{-1/N}{1 - e^{-i2\pi\eta/N}}, \tag{131} \]

so that if \( \hat{g}(\eta) = f(\eta) \) (i.e. \( g(y) = N\hat{f}(-y) \)) then

\[
\phi_{DZ/NZ}(\xi, y) = \begin{cases} 
\frac{12\pi}{N} \left( \frac{1-e^{i2\pi y/N}}{1-e^{-i2\pi y/N}} \right) & \text{if } \xi \neq 0 \text{ and } y \neq 0, \\
0 & \text{if } \xi = 0 \text{ or } y = 0.
\end{cases} \tag{132}
\]

Let us define the time-frequency transform \( Q_{Z/NZ} \) on the finite cyclic group \( G = \mathbb{Z}/N\mathbb{Z} \) by its ambiguity kernel, where

\[
\phi_{Q_{Z/NZ}}(\xi, y) = \begin{cases} 
\frac{12\pi}{N} \left( \frac{1-e^{i2\pi y/N}}{1-e^{-i2\pi y/N}} \right) & \text{if } \xi \neq 0 \text{ and } y \neq 0, \\
1 & \text{if } \xi = 0 \text{ or } y = 0.
\end{cases} \tag{133}
\]

That is, we summed (132) and (109), obtaining the correct margins.

**Theorem 16.1** Mapping \( [u] \mapsto Q_{Z/NZ}[u] \) is invertible for all \( N \in \mathbb{Z}^+ \). The corresponding \( Q_{Z/NZ} \)-quantization is invertible if and only if \( N \) is prime or \( N = 1 \).

**Proof.** Let \( D = Q_{Z/NZ} \). Case \( N = 1 \) is trivial. Assume now that \( N \) is prime. Then ambiguity kernel \( \phi_D \) has no zeros, so let \( g(\xi, y) := \phi_D(\xi, y)^{-1} \). Hence starting from \( D[u] = F^{-1}(\phi_D FR[u]) \) we find \( FR[u] = g FD[u] \), and from it we obtain \( u(x) u(x-y)^* \) for all \( x, y \in \mathbb{Z}/N\mathbb{Z} \). Thus \( [u] \mapsto D[u] \) is invertible when \( N \) is prime. What about the invertibility of the \( D \)-quantization \( a \mapsto a^D \)? Recall that the Kohn–Nirenberg quantization \( a \mapsto a^R \) is invertible: linear mapping \( A : L^2(\mathbb{Z}/N\mathbb{Z}) \to L^2(\mathbb{Z}/N\mathbb{Z}) \) is of the form \( A = a^R \), where \( a(x, y) = \eta(x)^* A \eta(x) \) for some \( \eta(x) := e^{i2\pi x/N} \). Then the \( D \)-quantization \( b \mapsto b^D \) is invertible, because

\[
\langle u, a^R v \rangle = \langle R(u, v), a \rangle = \langle FR(u, v), Fa \rangle = \langle FD(u, v), Fb \rangle = \langle D(u, v), b \rangle
\]

where \( Fb = g^* Fa \) : here \( a^R = b^D \). This concludes the case of prime \( N \).

Finally, let us consider divisible \( N \geq 4 \). Now \( \phi_D(\xi, y) = 0 \) if and only if \( \xi, y \) are zero divisors modulo \( N \). In this case, \( b^D = 0 \) if \( b \) is a symbol such that \( Fb \) is supported only on the zero divisors. Hence the \( D \)-quantization is not injective, nor surjective (due to the finite-dimensionality). However, it turns out that \( [u] \mapsto D[u] \) is still invertible. Finding \( [u] \) from \( D[u] \) is reduced to phase retrieval, as we easily get the time margins \( [u(x)]^2 = \sum_{y=1}^{N} D[u](x,y) \).

Especially, case \( u = 0 \) is trivial, so assume \( u \neq 0 \). Knowing \( D[u] \), we also find

\[
F^{-1} D[u](\xi, y) = \phi_D(\xi, y) \frac{1}{N} \sum_{z=1}^{N} e^{-i2\pi \xi/N} u(z) u(z-y)^*.
\]

37
From this, since \( \frac{1 - e^{i2\pi y/N}}{1 - e^{i2\pi \xi/N}} = \sum_{k=0}^{y-1} e^{i2\pi k\xi/N} \) for \( 0 < y < N \), we obtain numbers

\[
E(x, y) := \sum_{k=0}^{y-1} u(x + k) u(x + k - y)^*
\]

for all \( x \). We may recover only the equivalence class \([u]\) of \( u \), but suppose we know the complex phase of some \( u(z) \neq 0 \). We proceed recursively as follows:

We find numbers \( u(z+1) \) and \( u(z-1) \) from \( E(z+1, 1) \) and \( E(z, 1) \), respectively.

If we have already recovered numbers \( u(z \pm h) \) for \( 0 \leq h < j \), then we stably obtain numbers of \( u(z+j) \) and \( u(z-j) \) by finding their complex phases from \( E(z+1, j) \) and \( E(z, j) \), respectively. This completes the proof. QED

Remark 16.2 In the previous proof, the stable algorithm for \( D[u] \mapsto [u] \) can be built around any point \( z \in \mathbb{Z}/N\mathbb{Z} \) for which \( u(z) \neq 0 \). Let us also note the estimates

\[
|\phi_D(\xi, y)| \leq |\phi_D(1, y)| = \frac{2\pi}{N} \left| 1 - e^{i2\pi/N} \right|^{-1} \leq \frac{\pi}{2}
\]

for all \( N \geq 2 \) and \( \xi, y \). Without losing generality, for \( 0 < y \leq N/2 \) this follows by observing that

\[
\phi_D(\xi, y) = \frac{i2\pi}{N} \left( 1 - e^{-i2\pi y/N} \right)^{-1} \sum_{k=0}^{y-1} e^{i2\pi k\xi/N}.
\]

By the geometry of the unit circle, the optimal bounds

\[
|\phi_D(\xi, y)| \leq \frac{2\pi}{N} \left| 1 - e^{i2\pi/N} \right|^{-1}
\]

form a monotonically decreasing sequence with the limit 1 as \( N \to \infty \).

17 Limit of cyclic case: Born–Jordan

Next we study what happens to transforms \( D_{\mathbb{Z}/N\mathbb{Z}} \) when we take the limit \( N \to \infty \) interpreting either that \( \mathbb{Z}/N\mathbb{Z} \) tends to the compact circle group \( \mathbb{T} = \mathbb{R}/\mathbb{Z} \) or to the non-compact group \( \mathbb{Z} \) of integers. We also study the further limiting time-frequency transforms on the real line \( \mathbb{R} \).

Starting from natural time-frequency transforms of signals on \( \mathbb{Z}/N\mathbb{Z} \), we study the limiting cases on compact \( \mathbb{T} \) and non-compact \( \mathbb{Z} \), and their limits on \( \mathbb{R} \). At limit \( N \to \infty \) to compact group \( \mathbb{T} \), from transforms \( D_{\mathbb{Z}/N\mathbb{Z}} \) in the previous section we obtain time-frequency transform \( D_T \) with ambiguity kernel \( \phi_{D_T} : \mathbb{Z} \times \mathbb{T} \to \mathbb{C} \), where

\[
\phi_{D_T}(\xi, y) = \begin{cases} 
-\xi^{-1} \left( 1 - e^{i2\pi \xi y} \right) / \left( 1 - e^{-i2\pi y} \right) & \text{if } \xi \neq 0 \text{ and } y \neq 0, \\
1 & \text{if } \xi \neq 0 \text{ and } y = 0, \\
0 & \text{if } \xi = 0.
\end{cases}
\]
Indeed, \( y \mapsto \phi_{D_T}(\xi, y) \) is a trigonometric polynomial:

\[
\phi_{D_T}(\xi, y) = \begin{cases} 
\frac{1}{|\xi|} \sum_{k=0}^{\lfloor |\xi|/2 \rfloor} e^{-i2\pi yk} & \text{if } \xi < 0, \\
\frac{1}{\xi} \sum_{k=1}^{\lfloor |\xi|/2 \rfloor} e^{i2\pi yk} & \text{if } \xi > 0.
\end{cases}
\]

Indeed, \( (h \mapsto F^{-1}(\phi_D Fh)) : \mathcal{S}(\mathbb{T} \times \mathbb{T}) \to \mathcal{S}(\mathbb{T} \times \mathbb{Z}) \) is a Schwartz multiplier. Moreover, time-frequency transform \( D_T \) is band-limited, mapping \( \mathcal{S}(\mathbb{T}) \times \mathcal{F}(\mathbb{T}) \) to \( \mathcal{S}(\mathbb{T} \times \mathbb{Z}) \). Since \( |\phi_{D_T}(\xi, y)| \leq 1 \), by Theorem 8.8 we have the \( L^2 \)-bounds

\[
\|D_T(u, v)\| \leq \|u\| \|v\|, \quad (135)
\]

\[
\|a^{D_T}v\| \leq \|a\| \|v\|. \quad (136)
\]

Analogously, we have time-frequency transform \( D_Z \) on non-compact group \( \mathbb{Z} \), with ambiguity kernel \( \phi_{D_Z} : \mathbb{T} \times \mathbb{Z} \to \mathbb{C} \),

\[
\phi_{D_Z}(\xi, y) = \begin{cases} 
y^{-1} \left( 1 - e^{i2\pi y} \right) / \left( 1 - e^{i2\pi \xi} \right) & \text{if } \xi \neq 0 \text{ and } y \neq 0, 
1 & \text{if } \xi = 0 \text{ and } y \neq 0, 
0 & \text{if } y = 0.
\end{cases} \quad (137)
\]

Hence time-lag kernel \( \varphi_{D_Z} : \mathbb{Z} \times \mathbb{Z} \to \mathbb{C} \) is given by

\[
\varphi_{D_Z}(x, y) = \begin{cases} 
1/|y| & \text{if } -y < x \leq 0 \text{ or } 0 < x \leq -y, \\
0 & \text{otherwise}.
\end{cases} \quad (138)
\]

Here \( \varphi_{D_Z}(x, y) = K_Z(-x, -x - y)^* \) (equivalently, \( K_Z(x, y) = \varphi_{D_Z}(-x, x - y)^* \)), with

\[
\delta^{D_Z}v(x) = \sum_{y \in \mathbb{Z}} K_Z(x, y) v(y), \quad (139)
\]

with kernel \( K_Z : \mathbb{Z} \times \mathbb{Z} \to \mathbb{C} \) given by

\[
K_Z(x, y) = \begin{cases} 
1/|x - y| & \text{if } y < 0 \text{ or } x < 0 \leq y, \\
0 & \text{otherwise}.
\end{cases} \quad (140)
\]

At the continuum limit on \( \mathbb{R} \), we obtain time-frequency transform \( D_{\mathbb{R}} \), with

\[
\delta^{D_{\mathbb{R}}}v(x) = \int_{\mathbb{R}} K_{\mathbb{R}}(x, y) v(y) dy, \quad (141)
\]

where Schwartz kernel \( K_{\mathbb{R}} : \mathbb{R} \times \mathbb{R} \to \mathbb{C} \) is given by

\[
K_{\mathbb{R}}(x, y) = \begin{cases} 
1/|x - y| & \text{if } xy < 0, \\
0 & \text{otherwise}.
\end{cases} \quad (142)
\]
Hence $D_{\mathbb{R}} = Q$ is the Born–Jordan transform,

$$Q(u, v)(x, \eta) = \int_{\mathbb{R}} e^{-i2\pi y\eta} \frac{1}{y} \int_{x-y/2}^{x+y/2} u(t+y/2) v(t-y/2)^* \, dt \, dy.$$  \hfill (143)

Time-frequency transform $D_{\mathbb{Z}/\mathbb{N}\mathbb{Z}}$ has zero margins in both time and in frequency, but the margins for $Q$ are correct.

**Alternative way.** Above, we went from $\mathbb{Z}/\mathbb{N}\mathbb{Z}$ to $\mathbb{R}$ via $\mathbb{Z}$. What if our route would have been via $\mathbb{T}$ instead? The outcome must still be the Born–Jordan transform. Let us check this process: Time-frequency transform $D_{\mathbb{T}}$ on compact group $\mathbb{T}$ has time-lag kernel $\varphi_{D_{\mathbb{T}}}: \mathbb{T} \times \mathbb{T} \to \mathbb{C}$, where for $y \neq 0$ we have

$$\varphi_{D_{\mathbb{T}}}(x, y) = i2\pi \frac{w(x) - w(x+y)}{1 - e^{-i2\pi y}},$$  \hfill (144)

with the sawtooth wave $w: \mathbb{T} \to \mathbb{R}$ satisfying $w(x) = x$ for $0 < x < 1$. Now

$$\delta^{D_{\mathbb{T}}} v(x) = \int K_{\mathbb{T}}(x, y) v(y) \, dy,$$  \hfill (145)

with kernel $K_{\mathbb{T}}: \mathbb{T} \times \mathbb{T} \to \mathbb{C}$ given by $K_{\mathbb{T}}(x, y) = \varphi_{D_{\mathbb{T}}}(x, x-y)^*$,

$$K_{\mathbb{T}}(x, y) = -i2\pi \frac{1 - (x-y)}{1 - e^{i2\pi(x-y)}}$$  \hfill (146)

when $-1 < y < 0 < x < 1$ and $x-y \neq 1$: if here $x, y \to 0$, we again obtain the Born–Jordan transform $Q$ as the continuum limit. Properties of the Born–Jordan transform were studied in [30], where also closely related variants of $D_{\mathbb{T}}, D_{\mathbb{Z}}$ were introduced.

### 18 Computed pictures of discrete distributions

In the following pictures, we present three different discrete time-frequency distributions for the same signal: the periodic and non-periodic Born–Jordan distributions, and a spectrogram. The original speech signal of the author has 1000 samples, with sampling rate of 4000 Hz. The pictures were produced using Matlab. In the grey-scale time-frequency distribution pictures, higher values are darker in shade. For the spectrogram, zero value corresponds to white. For the other time-frequency images, zero value corresponds to mid-grey.
Figure 1: Speech signal “Why?”, sampling rate 4000 Hz.

Figure 2: Time-frequency distribution $Q_z[u]$ for signal $u$ (“Why?”).
Figure 3: Time-frequency distribution $Q_{z/Nz}[u]$ for the periodized signal $u$ (‘‘...Why Why Why Why...’’), zooming into a single period of 250 ms.

Figure 4: Spectrogram for periodized signal $u$ (‘‘...Why Why Why Why...’’), with a Gaussian window, zooming into a single period of 250 ms.
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