A control protocol of finite dimensional quantum systems

Jianju Tang and H. C. Fu

School of Physical Sciences and Technology, Shenzhen University, Shenzhen 518060, P. R. China

An exact and analytic control protocol of two types of finite dimensional quantum systems is proposed. The system can be drive to an arbitrary target state using cosine classical fields in finite cycles. The control parameters which are time periods of interaction between systems and control fields in each cycles are connected with the probability amplitudes of target states via triangular functions and can be determined analytically.

PACS numbers: 03.67.Aa, 03.65.Ud, 02.30.Yy, 03.67.Mn

I. INTRODUCTION

Quantum control is to drive a quantum system from an initial state to an arbitrary target state through its interaction with classical control fields or with a quantum accessor. It was first proposed by Huang et. al. in 1983 \cite{1} and then attracted much attention of chemists, physicists and control scientists. Various notations in classical control theory were generalized to the quantum control system, such as open and closed control, optimal control \cite{2}, controllability \cite{3,4}, feedback control \cite{5} and so on. Coherent and incoherent (indirect) control schemes are proposed. In later case the system is controlled by its interaction with a quantum accessor which is controlled by classical fields \cite{6,7}. Typically, in the approach of quantum control, one should first model the controlled system and examine its controllability which is determined by the system Hamiltonian and interaction Hamiltonian with classical fields, and then design classical fields to stream the system to the given target state, which is referred to as the control protocol and is the issue we would like to address in this paper. Some works were proposed along this line, for example, using the Cartan decomposition of Lie groups \cite{12}.

In this paper we shall develop an explicit control protocol of finite quantum system with all distinct energy gaps such as the Mores potential, and (2) all equal energy gaps except the first one. We use cosine classical field to drive the quantum system to arbitrary target states in finite cycles and the control parameters are interaction time intervals between system and control field in each cycles. Control parameters are linked with probability amplitudes of the target states through triangular functions and can be obtained analytically.

This paper is organized as follows. In Sec. II, we formulate the controlled system and control scheme and investigated the controllability. In Sec. III, we present the control protocol of system with all distinct energy gaps and in Sec. IV we consider the system with equal energy gaps except one. We conclude in Sec.V.

II. CONTROL SYSTEMS

A. Control Systems

Consider an $N$-dimensional non-degenerate quantum system with eigen energy $E_n$ and corresponding eigenstates $|n\rangle$, described by the Hamiltonian

$$H_0 = \sum_{n=1}^{N} E_n |n\rangle\langle n|.$$  \hspace{1cm} (1)

Without losing generality, we assume $H_0$ is traceless, namely $\text{tr}H_0 = 0$. In this paper, we only consider two different types of systems, the first one having all equal energy gaps except the first one, namely

$$\mu_1 \neq \mu_2 = \mu_3 = \cdots = \mu_{N-1},$$  \hspace{1cm} (2)

where $\mu_i = E_{i+1} - E_i$ is the energy gap, and another one with all distinct energy gaps

$$\mu_i \neq \mu_j, \hspace{1cm} i \neq j = 1, 2, \cdots, N-1, \hspace{1cm} (3)$$

For later convenience, we call them the system I and system II, respectively. For System I, we also define energy gaps

$$\bar{\mu}_i = E_{i+1} - E_1, \hspace{1cm} i = 1, 2, \cdots, N-1.$$  \hspace{1cm}

The purpose of this paper is to develop a control scheme to drive the systems to an arbitrary target state from an initial state, using some independent classical fields $f_m(t)$. The total Hamiltonian of the system and control fields can be generally written as

$$H = H_0 + H_I, \hspace{1cm} H_I = \sum_{m=1}^{M} f_m(t)\hat{H}_m,$$  \hspace{1cm} (4)

where $M$ is the number of independent classical fields.

For the $N$-dimensional systems considered in this paper, the total control process includes $N-1$ cycles. In the $m$-th cycle, we first apply a classical field

$$f_m(t) = \xi_m \cos(\nu_m t),$$

where $\nu_m$ is the frequency, to control the system for time period $\tau_m$, and then turn off the control field such
that the system evaluates freely for a time period \( \tau_m \), as showed in Fig. 1.
For system I, the frequency of the control field is chosen as
\[ \nu_m = \tilde{\omega}_m, \quad \tilde{\omega}_m \equiv (E_m + 1 - E_1)\hbar^{-1}. \] (5)
It is easy to find that \( \tilde{\omega}_m \neq \omega_n \) for \( m \neq n \). This means in each cycle, only transition between \( E_m \) and \( E_1 \) occurs.
For system II, the frequency \( \nu_m \) is chosen as
\[ \nu_m = \omega_m, \quad \omega_m = (E_{m+1} - E_m)\hbar^{-1}. \] (6)
This means that, in each cycle, only transition between level \( E_m \) and \( E_{m+1} \) occurs, as all \( \omega_m \) are different. So the control process includes \( N-1 \) cycles.
For both systems, there are two processes in each cycle. We first apply the control field to control the system for a time period \( \tau_m \), and then turn off the control field and allow the system to evaluate for a time period \( \tau_m' \). We will see that the first process provides the real probability amplitude of the target state and the second process provides the phases. Therefore, the control field can be rewritten as
\[ f_m(t) = \begin{cases} \xi_m \cos(\nu_m t), & t_{m-1} \leq t \leq t_{m-1} + \tau_m, \\ 0, & \text{otherwise}, \end{cases} \] (7)
where \( t_m = \sum_{k=1}^m (\tau_k + \tau_k') \). Those \( N-1 \) control fields are independent in the sense that each \( f_m(t) \neq 0 \) in different time period.
The whole control process can be equivalently regarded as control by one control field \( f(t) = \sum_{m=1}^{N-1} f_m(t) \), where \( f(t) \) is shown in Fig. 1.
For system I, in the \( m \)-th cycle, we apply the classical field
\[ f^{(m)}(t) = \xi_m \cos(\nu_m t), \quad \nu_m = \tilde{\omega}_m, \] (8)
which causes transition between energy levels \( E_1 \) and \( E_{m+1} \). So the interaction Hamiltonian between control field and system in Schrödinger picture is
\[ H^{(m)} = f^{(m)}(t)g_m (|m+1\rangle\langle m+1| + |m+1\rangle\langle m|), \] (9)
where \( g_m \) is the coupling constant and \( \Omega_m = \xi_m g_m \).

For system II, interaction Hamiltonian between the system and the control field is
\[ H^{(m)} = f^{(m)}(t)g_m (|m\rangle\langle m+1| + |m+1\rangle\langle m|), \] (10)
where \( g_m \) and \( \Omega_m = \xi_m g_m \) as in system I, but frequency \( \nu_m = \omega_m \).
Interaction between systems and controls are illustrated in Fig. 2.

B. Complete controllability

Before presentation of the control protocol, we first examine the controllability of this control scheme, namely, to examine whether the Lie algebra generated by the skew-Hermitian operators \( iH_0 \) and \( iH_m \)
\[ \mathcal{L} = \text{Gen}\{iH_0, iH_m|m = 1, 2, ..., N-1\} \] (11)
is \( \text{su}(N) \) [3]. Here \( H_m \) for system I and II are
\[ H_m = |m+1\rangle\langle m+1| + |m+1\rangle\langle m|, \] (12)
\[ H_m = |m\rangle\langle m+1| + |m+1\rangle\langle m| \] (13)
respectively. Or equivalently, \( iH_0, iH_m \) generate the Chevalley basis of \( \text{su}(N) \) [4, 5].
\[ ix_n = i(|m\rangle\langle n+1| + |n+1\rangle\langle m|), \] (14)
\[ iy_n = |n\rangle\langle n+1| - |n+1\rangle\langle n|, \]
\[ ih_n = i(|m\rangle\langle m+1| - |m+1\rangle\langle m|), \]
where \( n = 1, 2, ..., N-1 \). In fact, it is enough to prove \( ix_n \in \mathcal{L} \) (or \( iy_n \in \mathcal{L} \)), as \( iy_n = \mu_n^{-1}[iH_0, ix_n] \) and \( ih_n = -[ix_n, iy_n]/2 \).
For system I, it is obvious that
\[ ix_1 = iH_I = i(|1\rangle\langle 2| + |2\rangle\langle 1|) \in \mathcal{L}. \]  
(15)

Then
\[ iy_2 = [iH_2, iH_I] = |2\rangle\langle 3| - |3\rangle\langle 2| \in \mathcal{L}, \]
\[ ix_2 = \frac{1}{\mu_2} [iH_2, iH_I], iH_0 = i(|2\rangle\langle 3| + |3\rangle\langle 2|) \in \mathcal{L}. \]  
(16)

Recursively, we have
\[ iy_m = \mu_m^{-1} [iH_m, iH_{m-1}], iH_0 \] .

So the system I is completely controllable. For System II, \( iH_m \) itself
\[ iH_m = i(|m\rangle\langle m+1| + |m+1\rangle\langle m|), \]  
(18)

is nothing but the generator \( ix_m \). Therefore, the system II is completely controllable.

### III. CONTROL PROTOCOL OF SYSTEM I

In this section we investigate the control protocol of System I. Suppose the system is initially on the ground state \( |\psi_0\rangle = |1\rangle \). The system is driven to an arbitrary target state after \( N - 1 \) cycles. Relationship between the control parameters \( \{\tau_m, \nu_m\} \) and probability amplitude of target states is explicitly established.

#### A. Interaction Hamiltonian

Changing to the interaction picture, we obtain
\[ H_I^{(m)} = U_0^+(t) \Omega_m \cos(\nu_m t) \]
\[ \times (|1\rangle\langle m+1| + |m+1\rangle\langle 1|) U_0(t) \]
\[ = \frac{\Omega_m}{2} \left( e^{i(\nu_m - \tilde{\omega}_m)t} |1\rangle\langle m+1| + e^{i(\nu_m + \tilde{\omega}_m)t} |m+1\rangle\langle 1| \right), \]  
(19)

where \( U_0(t) = e^{-iH_0t/\hbar} \), and we have used \( \cos(\nu_m t) = (e^{\nu_m t} + e^{-\nu_m t})/2 \). As we require that the control field is resonant with the levels \( E_1 \) and \( E_m \), namely, \( \nu_m = \tilde{\omega}_m \), we can neglect the high-oscillating terms \( e^{\pm i(\tilde{\omega}_m + \nu_m)t} \) under the rotating wave approximation \cite{13}. We finally obtain
\[ H_I^{(m)} = \frac{\Omega_m}{2} \left( |1\rangle\langle m+1| + |m+1\rangle\langle 1| \right), \]  
(20)

which does not depend on time \( t \) explicitly. So the time evolution operator in interaction picture can be written as
\[ U_I^{(m)}(t) = e^{-iH_I^{(m)}t/\hbar}. \]  
(21)

From the fact
\[ \left( H_I^{(m)} \right)^{2n} = \left[ \frac{\Omega_m}{2} \right]^{2n} \left( |1\rangle\langle 1| + |m+1\rangle\langle m+1| \right), \]
\[ n > 0, \]
\[ \left( H_I^{(m)} \right)^{2n+1} = \left[ \frac{\Omega_m}{2} \right]^{2n+1} \left( |1\rangle\langle m+1| + h.c. \right), \]
\[ n \geq 0, \]
we obtain the time evolution operator as
\[ U_I^{(m)}(t) = I + (\cos(\Omega_m t) - 1) (|1\rangle\langle 1| + |m+1\rangle\langle m+1|) - i\sin(\Omega_m t) (|1\rangle\langle m+1| + h.c.) \cos(\Omega_m t) \]  
(22)

where \( \Omega_m' = \Omega_m/2\hbar \).

#### B. Cycle 1

Suppose that the system is initially on the state \( |1\rangle \). When interacting with control field for time period \( \tau_1 \), the system is on the state
\[ |\psi_1\rangle_I = U_I(\tau_1) |1\rangle_I \]
\[ = \cos(\Omega_1' \tau_1) |1\rangle_I - i\sin(\Omega_1' \tau_1) |2\rangle . \]  
(23)

Changing back to the Schrödinger picture, we have
\[ |\psi_1\rangle_S = e^{-iE_1 \tau_1/\hbar} \cos(\Omega_1' \tau_1) |1\rangle - i e^{-iE_2 \tau_1/\hbar} \sin(\Omega_1' \tau_1) |2\rangle . \]  
(24)

We then turn off the external field and allow the system to evolve for time period \( \tau_1' \). We get
\[ |\psi_1\rangle_S = e^{-i\hat{H}_0 \tau_1'/\hbar} |\psi_1\rangle_S = a_1^{(1)} |1\rangle + a_2^{(1)} |2\rangle , \]  
(25)

with
\[ a_1^{(1)} = e^{-iE_1 (\tau_1 + \tau_1')/h} \cos(\Omega_1' \tau_1), \]
\[ a_2^{(1)} = e^{-iE_2 (\tau_1 + \tau_1')/h} i \sin(\Omega_1' \tau_1). \]  
(26)

#### C. Cycle 2

For the cycle 2, the initial state is the the final state of cycle 1, namely, the state \( \langle 2 \rangle \). We first apply the control field \( f_2(t) = \xi_2 \cos(\nu_2 t) \) for time period \( \tau_2 \). Using \( \langle 28 \rangle \) for \( m = 2 \), we obtain the state in the interaction picture
\[ |\psi_2\rangle_I = U_I(\tau_2) |\psi_1\rangle_I \]
\[ = \cos(\Omega_2' \tau_2) a_1^{(1)} |1\rangle + a_2^{(1)} |2\rangle - i \sin(\Omega_2' \tau_2) a_1^{(1)} |3\rangle . \]  
(27)

Changing back to Schrödinger picture, and after free evolution for time period \( \tau_2' \), the state is
\[ |\psi_2\rangle_S = U_0(\tau_2') |\psi_2\rangle_S = U_0(\tau_2') U_0(\tau_2) |\psi_2\rangle_I \]
\[ = a_1^{(2)} |1\rangle + a_2^{(2)} |2\rangle + a_3^{(2)} |3\rangle , \]  
(28)
where

\[ a_{1}^{(2)} = e^{-iE_{1}(\tau_{2}+\tau'_{2})/\hbar} \cos(\Omega'_{2}\tau_{2}) a_{1}^{(1)}, \]

\[ a_{2}^{(2)} = e^{-iE_{2}(\tau_{2}+\tau'_{2})/\hbar} a_{2}^{(1)}, \]

\[ a_{3}^{(2)} = -e^{-iE_{3}(\tau_{2}+\tau'_{2})/\hbar} i \sin(\Omega'_{2}\tau_{2}) a_{1}^{(1)}. \]  \hspace{1cm} (29)

D. From \((m-1)\)-th to \(m\)-th cycle

To obtain the explicit expression of the target state, we need the recursion relation between coefficients of the \(m\)-th cycle and \((m-1)\)-th cycle. Suppose that after the \((m-1)\)-th cycle, we obtain the state

\[ |\psi_{m-1}\rangle_{S} = \sum_{k=1}^{m} a_{k}^{(m-1)} |k\rangle. \]  \hspace{1cm} (30)

Interacting with control field for time period \(\tau_{m}\), we have the state in Schrödinger picture

\[ |\psi_{m}\rangle_{S} = e^{-iE_{1}\tau_{m}/\hbar} \cos(\Omega'_{m}\tau_{m}) a_{1}^{(m-1)} |1\rangle \]

\[ + \sum_{k=2}^{m} a_{k}^{(m-1)} e^{-iE_{k}\tau_{m}/\hbar} |k\rangle \]

\[ - e^{-iE_{m+1}\tau_{m}/\hbar} i \sin(\Omega'_{m}\tau_{m}) a_{1}^{(m-1)} |m+1\rangle. \]

After free evolution for time period \(\tau'_{m}\), the final state of the \(m\)-th cycle is

\[ |\psi_{m}'\rangle_{S} = e^{-iH\tau'_{m}/\hbar} |\psi_{m}\rangle_{S} = \sum_{k=1}^{m+1} a_{k}^{(m)} |k\rangle, \]  \hspace{1cm} (31)

where the coefficients are

\[ a_{1}^{(m)} = e^{-iE_{1}\tau_{m}/\hbar} \cos(\Omega'_{m}\tau_{m}) a_{1}^{(m-1)}, \]

\[ a_{k}^{(m)} = e^{-iE_{k}\tau_{m}/\hbar} a_{k}^{(m-1)}, \quad 2 \leq k \leq m, \]

\[ a_{m+1}^{(m)} = -e^{-iE_{m+1}\tau_{m}/\hbar} i \sin(\Omega'_{m}\tau_{m}) a_{1}^{(m-1)}, \]

and we use the notation

\[ T_{m} \equiv \tau_{m} + \tau'_{m} \]  \hspace{1cm} (35)

hereafter. Eqs. (32)-(33) establishes the relationship between the probability amplitudes of the \((m-1)\)-th cycle and the \(m\)-th cycle.

E. Target state

For 2 and 3 dimensional system, the target state has been found in subsection B and C. So we suppose that \(N \geq 4\) in the rest of this section. From (29) and (26), we can easily find that

\[ a_{1}^{(m)} = \exp \left[ -iE_{1} \sum_{i=1}^{m} T_{i} \right] \prod_{i=1}^{m} \cos(\Omega'_{i}\tau_{i}). \]  \hspace{1cm} (36)

As \(a_{m+1}^{(m)}\) depends on \(a_{1}^{(m-1)}\) only, we can easily find that

\[ a_{m+1}^{(m)} = e^{-\frac{i}{\hbar} \left[ E_{m+1} T_{m} + E_{1} \sum_{i=1}^{m-1} T_{i} \right]} \]

\[ \times i \sin(\Omega'_{m+1}\tau_{m+1}) \prod_{i=1}^{m-1} \cos(\Omega'_{i}\tau_{i}). \]  \hspace{1cm} (37)

For other coefficients, using (29), we obtain the explicit probability amplitude

\[ a_{2}^{(m)} = -e^{-\frac{i}{\hbar} \left[ \sum_{i=1}^{m} T_{i} \right]} i \sin(\Omega'_{1}\tau_{1}), \]

\[ a_{3}^{(m)} = -e^{-\frac{i}{\hbar} \left[ E_{3} \sum_{i=1}^{m-2} T_{i} + E_{1} T_{1} \right]} i \sin(\Omega'_{m+1}\tau_{m+1}) \prod_{i=1}^{m-1} \cos(\Omega'_{i}\tau_{i}). \]  \hspace{1cm} (38)

To derive coefficient \(a_{k}^{(m)}\), \(3 \leq k \leq m\), we use

\[ a_{m}^{(m-1)} = e^{-iE_{m}\tau_{m}/\hbar} a_{m-1}^{(m-1)} = \exp \left[ -\frac{i}{\hbar} \left[ E_{m} \sum_{i=m-1}^{m} T_{i} \right] \right] a_{m-1}^{(m-1)} \]

\[ = -e^{-\frac{i}{\hbar} \left[ E_{m} \sum_{i=m-1}^{m} T_{i} + E_{1} \sum_{i=1}^{m-1} T_{i} \right]} \prod_{i=1}^{m-2} \cos(\Omega'_{i}\tau_{i}). \]  \hspace{1cm} (39)

According to (39) and (40), we can obtain \(a_{k}^{(m)}\)

\[ a_{k}^{(m)} = e^{-iE_{k}\tau_{m}/\hbar} a_{k}^{(m-1)} \]

\[ = e^{-iE_{k}\tau_{m}/\hbar} e^{-iE_{m} T_{m-1}/\hbar} a_{k}^{(m-2)} \]

\[ = \cdots \]

\[ = e^{-iE_{k}(T_{m} + T_{m-1} + \cdots + T_{k+1}) T_{k+1}} a_{k}^{(k)} \]

\[ = -e^{-\frac{i}{\hbar} \left[ E_{m} \sum_{i=k+1}^{m} T_{i} + E_{1} \sum_{i=1}^{k-2} T_{i} \right]} \prod_{i=1}^{k-2} \cos(\Omega'_{i}\tau_{i}), \]

\(3 \leq k \leq m\),  \hspace{1cm} (41)

where we have used the result of \(a_{k}^{(k-1)}\) given by (37) with \(m\) replaced by \(k\). Notice that when \(k = m + 1\),  \hspace{1cm} (41)
recovers the $a^{(m)}_{m+1}$ given in (37). Therefore (11) is also valid for $k = m + 1$, and all the probability amplitudes after $m$-th cycle are given by (35), (38) and (41) with $3 \leq k \leq m + 1$.

For the system we considered here with dimension $N$, we need $N - 1$ cycles to arrive at arbitrary target states. Letting $m = N - 1$, we obtain the probability amplitude of the target state

$$a^{(N-1)}_1 = \exp\left[-\frac{iE_1}{\hbar} \sum_{i=1}^{N-1} T_i \right] \prod_{i=1}^{N-1} \cos(\Omega_i' \tau_i),$$

$$a^{(N-1)}_2 = -\exp\left[-\frac{iE_2}{\hbar} \sum_{i=1}^{N-1} T_i \right] i \sin(\Omega_1' \tau_1),$$

$$a^{(N-1)}_k = -\exp\left\{ \frac{i}{\hbar} \left[ E_k \sum_{i=k-1}^{N-1} T_i + E_1 \sum_{i=1}^{k-2} T_i \right] \right\} \times i \sin(\Omega_{k-1}' \tau_{k-1}) \prod_{i=1}^{k-2} \cos(\Omega_i' \tau_i),$$

$$3 \leq k \leq N.$$

(42)

F. Control parameters

For a control problem, the target state, or in other words, the amplitude $a^{n-1}_n$ of the target state, is given. What we need to do is to determine the control parameters $\{\tau_i, \tau'_i : i = 1, 2, ..., N - 1\}$ from the probability amplitude of the target state. For convenience, we write the target state as

$$|\psi\rangle = \sum_{n=1}^{N} \gamma_n C_n |n\rangle$$

where $C_n$s are the real part of the amplitude

$$C_1 = \prod_{i=1}^{N-1} \cos(\Omega_i' \tau_i),$$

$$C_2 = \sin(\Omega_1' \tau_1),$$

$$C_n = \sin(\Omega_{n-1}' \tau_{n-1}) \prod_{i=1}^{n-2} \cos(\Omega_i' \tau_i), \quad 3 \leq n \leq N,$$

and $\gamma_n$s are phases

$$\gamma_1 = \exp\left[ -\frac{iE_1}{\hbar} \sum_{i=1}^{N-1} T_i \right],$$

$$\gamma_2 = -i \exp\left[ -\frac{iE_2}{\hbar} \sum_{i=1}^{N-1} T_i \right],$$

$$\gamma_n = -i \exp\left\{ \frac{i}{\hbar} \left[ E_n \sum_{i=n-1}^{N-1} T_i + E_1 \sum_{i=1}^{n-2} T_i \right] \right\},$$

$$3 \leq n \leq N.$$ (49)

For a given target state, namely, $C_n$ and $\gamma_n$ are given, we can calculate control parameters $\{\tau_n, \tau'_n | n = 1, 2, ..., N - 1\}$. From (45) and $C_2$, we can determine $\tau_1$. Then form (46) with $n = 3$, we can obtain $\tau_2$ from $C_3$. Repeating this process, we can obtain all parameters $\tau_n, n = 1, 2, ..., N - 1$ from (46).

All $\tau'_i$ can be obtained from (49). From $\gamma_2$ and $\gamma_3$, we can obtain

$$\sum_{i=1}^{N-1} T_i, \quad E_3 \sum_{i=2}^{N-1} T_i + E_1 T_1,$$

(50)

from which we find $T_1$ and $\sum_{i=2}^{N-1} T_i$. From $\gamma_4$, we find

$$E_4 \sum_{i=3}^{N-1} T_i + E_1 (T_1 + T_2),$$

(51)

from which as well as $T_1$ and $\sum_{i=2}^{N-1} T_i$, we can obtain $T_2$. Repeating this process, we can obtain all $T_i$ and thus all $\tau'_i$.

IV. CONTROL PROTOCOL OF SYSTEM II

A. Time evolution operator

For system II, the interaction Hamiltonian is given in Eq. (10). This Hamiltonian is same as (9) for system I except the state $|1\rangle$ is replaced by $|n\rangle$. So we can follow exactly the same procedure as in last section, namely, changing to the interaction picture, using rotating wave approximation, and obtaining a time-independent Hamiltonian in interaction picture

$$H^{(m)}_I = \frac{\Omega_m}{2} (|m\rangle \langle m+1| + |m+1\rangle \langle m|).$$

(52)

Using

$$\left(H_I^{(m)}\right)^{2n} = \left(\frac{\Omega_m}{2}\right)^{2n} (|m\rangle \langle m| + |m+1\rangle \langle m+1|)$$

$$\left(H_I^{(m)}\right)^{2n+1} = \left(\frac{\Omega_m}{2}\right)^{2n+1} (|m\rangle \langle m| + |m+1\rangle \langle m+1|)$$

$$\left(\frac{\Omega_m}{2}\right)^{2n+1} (|m\rangle \langle m| + |m+1\rangle \langle m+1|), \quad (n \geq 0),$$

we can obtain the time evolution operator in the interaction picture

$$U_I^{(m)}(t) = I + \left[ \cos(\Omega_m t) - 1 \right] (|m\rangle \langle m| + |m+1\rangle \langle m+1|)$$

$$- i \sin(\Omega_m t) (|m\rangle \langle m| + |m+1\rangle \langle m+1|),$$

(53)

where $\Omega_m = \Omega_m/(2\hbar)$. 
B. Determine amplitude $a_m$

For this model, the cycle 1 is exactly the same as the system I. So after the cycle 1, the system is driven to the state

$$|\psi_1^l\rangle_S = a_1^{(1)} |1\rangle + a_2^{(1)} |2\rangle,$$

where

$$a_1^{(1)} = e^{-iE_1 T_1 / \hbar} \cos(\Omega_1 \tau_1),$$

$$a_2^{(1)} = -e^{-iE_2 T_1 / \hbar} i \sin(\Omega_1 \tau_1).$$

(54)

Different from system I, in cycle 2, the control field $f(t) = \xi_2 \cos(\omega_2 t)$ causes transition between |2⟩ and |3⟩. We can find the state after the cycle 2 in Schrödinger picture as

$$|\psi_2^l\rangle_S = a_1^{(2)} |1\rangle + a_2^{(2)} |2\rangle + a_3^{(2)} |3\rangle,$$

where

$$a_1^{(2)} = e^{-iE_1 T_2 / \hbar} a_1^{(1)},$$

$$a_2^{(2)} = e^{-iE_2 T_2 / \hbar} \cos(\Omega_2 \tau_2) a_2^{(1)},$$

$$a_3^{(2)} = -e^{-iE_3 T_2 / \hbar} i \sin(\Omega_2 \tau_2) a_2^{(1)}.$$  

(55)

To obtain the target state, we first find the recursion relations between the $(m-1)$-th cycle and the $m$-th cycle. To this end, we suppose that, after $m-1$ cycles, the system is on the state

$$|\psi_{m-1}^l\rangle_S = \sum_{k=1}^{m} a_k^{(m-1)} |k\rangle.$$  

(56)

Then after interactions with the control field for time period $\tau_m$, and free evolution for time period $\tau'_m$, we find the final state after cycle $m$ as

$$|\psi_m^l\rangle_S = \sum_{k=1}^{m+1} a_k^{(m)} |k\rangle,$$

with $(m \geq 2)$

$$a_k^{(m)} = e^{-iE_k T_m / \hbar} a_k^{(m-1)}, \quad 1 \leq k \leq m - 1,$$

$$a_m^{(m)} = e^{-iE_m T_m / \hbar} \cos(\Omega_m \tau_m) a_m^{(m-1)},$$

$$a_{m+1}^{(m)} = -e^{-iE_{m+1} T_m / \hbar} i \sin(\Omega_m \tau_m) a_m^{(m-1)}.$$  

(57)

(58)

(59)

From those recursion relations, and initial conditions (55) we can find all the explicit expressions of $a_k^{(m)}$. It is easy to see that

$$a_1^{(m)} = \exp \left[ -i E_1 / \hbar \sum_{i=1}^{m} T_i \right] \cos(\Omega_1 \tau_1),$$

$$a_2^{(m)} = -\exp \left[ i E_2 / \hbar \sum_{i=1}^{m} T_i \right] \cos(\Omega_2 \tau_2) i \sin(\Omega_1 \tau_1).$$  

(60)

(61)

$$a_{m+1}^{(m)} = \left[ -e^{-iE_{m+1} T_m / \hbar} i \sin(\Omega_m \tau_m) a_m^{(m-1)} \right] a_{m+1}^{(m-1)} = \cdots$$

$$= (-i)^{m-1} \exp \left[ -i \frac{m}{\hbar} \sum_{i=2}^{m} E_{i+1} T_i \right] \prod_{i=2}^{m} \sin(\Omega'_i \tau_i) a_2^{(1)}$$

$$= \exp \left[ -i \frac{m}{\hbar} \sum_{i=1}^{m} E_{i+1} T_i - i \frac{\pi}{2} m \right] \prod_{i=1}^{m} \sin(\Omega'_i \tau_i).$$  

(62)

Then using (52) and $a_m^{(m-1)}$, which is obtained from (61) by replacing $m$ by $m - 1$, we have

$$a_k^{(m)} = e^{-iE_k (\tau_m + \tau_m') / \hbar} a_k^{(m-1)} = e^{-iE_k (\tau_m + \tau_m' + \tau_{m+1}) / \hbar} a_k^{(m-2)} = \cdots$$

$$= \exp \left[ -i \frac{E_k}{\hbar} \sum_{i=k+1}^{m} T_i \right] a_k^{(k)}$$

$$= \exp \left[ -i \frac{E_k}{\hbar} \sum_{i=k+1}^{m} T_i \right] e^{-iE_k T_k \hbar} \cos(\Omega'_k \tau_k) a_k^{(k-1)}$$

$$= \exp \left[ i \frac{E_k}{\hbar} \sum_{i=k+1}^{m} T_i - i \frac{\pi}{2} \sum_{i=1}^{k} E_{i+1} T_i \right.$$  

$$- i \frac{\pi}{2} (k - 1) \left] \cos(\Omega'_k \tau_k) \prod_{i=1}^{k-1} \sin(\Omega'_i \tau_i).$$  

(63)

(58)

(56)

One can check that (65) includes the case $k = 2$ and $k = m$ as special cases.

Therefore, after $N - 1$ cycles, we arrive at the target state

$$a_1^{(N-1)} = \exp \left[ -i E_1 / \hbar \sum_{i=1}^{N-1} T_i \right] \cos(\Omega'_1 \tau_1),$$

$$a_m^{(N-1)} = \exp \left[ -i E_m / \hbar \sum_{i=m}^{N-1} T_i - i \frac{1}{\hbar} \sum_{i=1}^{m-1} E_{i+1} T_i \right.$$  

$$- i \frac{\pi}{2} (m - 1) \left] \cos(\Omega'_m \tau_m) \prod_{i=1}^{m-1} \sin(\Omega'_i \tau_i),$$

$$2 \leq m \leq N - 1,$$

$$a_N^{(N-1)} = \exp \left[ -i \frac{E_N}{\hbar} \sum_{i=1}^{N-1} E_{i+1} T_i - i \frac{\pi}{2} (N - 1) \right]$$

$$\times \prod_{i=1}^{N-1} \sin(\Omega'_i \tau_i).$$  

(64)

(65)

(66)

(67)

(68)

(69)

C. Control parameters

To determine control parameters $\tau_i, \tau_i'$, $1 \leq i \leq N - 1$, we write the target state as

$$|\psi\rangle = \sum_{n=1}^{N} a_n |n\rangle = \sum_{n=1}^{N} \gamma_n C_n |n\rangle,$$

(70)
in which

\[ C_1 = \cos(\Omega_1 \tau_1) \]
\[ C_m = \cos(\Omega'_m \tau_m) \prod_{i=1}^{m-1} \sin(\Omega'_i \tau_i), \quad (2 \leq m \leq N - 1) , \]
\[ C_N = \prod_{i=1}^{N-1} \sin(\Omega'_i \tau_i), \quad \]

and phase \( \gamma_n \)

\[ \gamma_1 = \exp \left[ \frac{iE_1}{\hbar} \sum_{i=1}^{N-1} T_i \right] , \]
\[ \gamma_m = \exp \left[ \frac{iE_m}{\hbar} \sum_{i=m}^{N-1} T_i - i \frac{m-1}{2} \right] , \]
\[ \gamma_N = \exp \left[ -i \frac{N-1}{2} \sum_{i=1}^{N-1} E_{i+1} T_i - i \frac{\pi}{2} (N-1) \right]. \quad (72) \]

For a given target state, namely, \( C_n \) and \( \gamma_n \) are given, we can determine the control parameters \( \{\tau_n, \gamma'_n\}_{n=1}^{N-1} \). From \( C_1 \) we can determine \( \tau_1 \), and then from \( C_2 \) and the obtained \( \tau_1 \). Recursively we can obtain all \( \tau_n \).

For \( \gamma'_n \), from \( \gamma_2 \) and \( \gamma_3 \), we obtain

\[ \sum_{i=1}^{N-1} T_i, \quad E_3 \sum_{i=2}^{N-1} + E_2 T_i, \quad \] respectively. As \( E_2 \neq E_3 \), we obtain \( T_1 \) and \( \sum_{i=2}^{N-1} T_i \).

From \( \gamma_4 \), we can obtain

\[ E_4 \sum_{i=3}^{N-1} T_i + E_3 T_2 \quad (74) \]

from which we obtain \( T_2 \) as well as \( \sum_{i=3}^{N-1} T_i \). Repeating this process, we can obtain all \( T_i \) and thus \( \gamma'_i \).

V. CONCLUSION

In this paper we proposed a protocol to drive two types of finite dimensional quantum system to an arbitrary given target states. The control parameters are time periods \( \{\tau_m, \gamma'_m\}_{m=1, 2, \ldots, N-1} \) which can be explicitly determined from the probability amplitudes of the given target states. Relationship between control parameters and amplitudes is triangular functions and can be solved explicitly. The control fields in this protocol is the usual electric field described by the cosine function.

We have \( 2(N-1) \) real control parameters. In the target state there are \( N \) complex or \( 2N \) real parameters. Taking into account the normalization condition of target state, one has \( 2(N-1) \) real parameters, the same as the number of the real control parameters. From this fact we can conclude that we can drive the system to an arbitrary target state by choosing appropriate control parameters \( \{\tau_m, \gamma'_m\} \).

As further works, we would like to consider the indirect control protocol of finite quantum system by generalizing the control scheme in this paper. We also would like to consider the control protocol in the presentence of environment.

ACKNOWLEDGEMENT

This work is supported by the National Science Foundation of China under grand number 11075108.

[1] G. M. Huang, T. J. Tarn and J. W. Clark, J. Math. Phys. 24 2608 (1983)
[2] M. A. Daleh, A. M. Peirce, and H. Rabitz, Phys. Rev. A 37 4950 (1988); A. Bartana, K. Kosloff, and D. J. Tannor, Chem. Phys. 267, 95 (2001); U. Boscoain, G. Charlot, J.-P. Gauthier, S. Duerin and H.-R. Jauslin, J. Math. Phys. 43 2107 (2002).
[3] V. Ramakrishna and H. Rabitz, Phys. Rev. A 54 1715 (1996).
[4] S. G. Schirmer, H. Fu and A. I. Solomon, Phys. Rev. A 63, 063410 (2001).
[5] H. Fu, S. G. Schirmer and A. I. Solomon, Phys. A: Math. Gen. 34 (2001).
[6] G. Turinici, Mathematical Models and Methods for ab Initio Quantum Chemistry (Lecture Notes in Chemistry vol 74) ed. M. Defranceschi and C. Le Bris (Berlin: Springer, 2000); G. Turinici and H. Rabitz, Chem. Phys. 267 (2001)
[7] H. M. Wiseman and G. J. Milburn, Quantum measurement and control, Cambridge Press, 2010.
[8] H. C. Fu, H. Dong, X. F. Liu and C. P. Sun, Phys. Rev. A. 75 052317 (2007).
[9] H. C. Fu, H. Dong, X. F. Liu and C. P. Sun, J. Phys. A. 42 045303 (2009).
[10] R. Romano and D. DAlessandro, Phys. Rev. Lett. 97 080402 (2006); Phys. Rev. A 73 022323 (2006).
[11] A. Pechen and H. Rabitz, Phys. Rev. A 73 062102 (2006).
[12] D. DAlessandro and F. Albertini J. Phys. A. 40 2439 (2007).
[13] M. O. Scully, M. S. Zubairy, Quantum Optics, 2000.