PURE INFINITENESS AND IDEAL STRUCTURE OF CROSSED PRODUCTS BY ENDOMORPHISMS OF $C_0(X)$-ALGEBRAS

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Abstract. Let $A$ be a $C_0(X)$-algebra. We consider an extendible endomorphism $\alpha : A \to A$ such that $\alpha(fa) = \Phi(f)\alpha(a)$, $a \in A$, $f \in C_0(X)$ where $\Phi$ is an endomorphism of $C_0(X)$. Pictorially speaking, $\alpha$ is a mixture of a topological dynamical system $(X, \varphi)$ dual to $(C_0(X), \Phi)$ and a continuous field of homomorphisms between the fibers $A(x)$, $x \in X$, of the corresponding $C^*$-bundle. We study relative crossed products $C^*(A, \alpha, J)$ where $J$ is an ideal in $A$. These generalize most popular constructions of this sort.

We give sufficient conditions for the uniqueness property, gauge-invariance of all ideals, the ideal property and pure infiniteness of $C^*(A, \alpha, J)$. In particular, we propose a notion of paradoxicality for reversible $C^*$-dynamical systems. The results allow to constitute a large class of crossed products $C^*(A, \alpha, J)$ which undergo Kirchberg’s classification via ideal system equivariant KK-theory and whose ideal lattice is completely described in terms of $(A, \alpha)$ and $J$.

1. INTRODUCTION.

1.1. Motivation. The notion of crossed product of a $C^*$-algebra by an injective endomorphism whose range is a corner is well established and has a number of remarkable applications [Cun77], [JKO96], [Ror95], [Exel03], [HR12]. On the other hand, the general theory of crossed products by endomorphisms appears to be a bizarre research area that deals with an abundance of different, seemingly unrelated, constructions. In [Kwa’] however, we argue that there are in essence only two different branches of that theory, one initiated by Cuntz [Cun77] and the other by Exel [Exel03]. Moreover, both of these approaches can be naturally unified in the realm of crossed products by completely positive maps; they are simply two different instances of such a construction, see [Kwa’], and actually Exel’s crossed product should be viewed as a crossed product by a transfer operator (rather than endomorphism).

In the present paper we follow the path whose origin we attributed to Cuntz. It involves a construction which is a modification of what is popularized under the name of Stacey’s crossed product. More precisely, we attach to any extendible endomorphism $\alpha : A \to A$ of a $C^*$-algebra $A$ a family of relative crossed products $C^*(A, \alpha, J)$ parametrized by an ideal $J$ in $A$. If $J = A$ then $C^*(A, \alpha, J)$ coincides with Stacey’s crossed product, and if $J = 0$, then $C^*(A, \alpha, J)$ is the partial isometric crossed product introduced in [LR04]. If $J = \ker \alpha \perp$ is the annihilator of the kernel of $\alpha$ we denote $C^*(A, \alpha, J)$ by $C^*(A, \alpha)$ and call it the (unrelative) crossed product. In the unital case this crossed product was introduced in [KL13] and in the non-unital case it was studied in [Kwa]. If the underlying endomorphism is injective, this $C^*$-algebra coincides with Stacey’s crossed product. The study of $C^*$-algebras associated to more general (not necessarily injective) endomorphisms can be motivated in many ways. Here we do it with the help of the following example, which also illustrates ideas and methods exploited further in the paper.

Example 1.1 (Polar decomposition $C^*$-algebra). Let $T \in B(H)$ be a bounded operator on a Hilbert space $H$ and let $C^*(T) := C^*(\{|T|, U, 1\}) \subseteq B(H)$ be the unital $C^*$-algebra generated by
the components of the polar decomposition $T = U|T|$ of $T$. The partial isometry $U$ implements two natural mappings on $C^*(T)$:

\[ \alpha(a) := UaU^*, \quad \alpha_*(a) := U^*aU, \quad a \in C^*(T). \]

Let

\[ A := C^* \left( \bigcup_{n \in \mathbb{N}} \{ \alpha^n(|T|), \alpha^n(1) \} \right) \]

be the smallest $C^*$-algebra that contains $\{|T|, 1\}$ and is invariant under $\alpha$. Then $\alpha : A \to A$ is a completely positive map, which is multiplicative, cf. [KL13, Lemma 1.2], if and only if

\[ U^*U \in A' \]

where $A'$ is a commutant of $A$. Suppose that (1) holds. (It is automatically satisfied, for instance, when we have a commutation relation of the form $TT^* = \gamma(T^*T)$, see [LO04, Section 6].) Then the set

\[ J := \{ a \in A : U^*Ua = a \} = A \cap U^*UA \]

is an ideal in $A$. By the universal property of $C^*(A, \alpha, J)$ (see Definition 2.9 below) there is a surjective homomorphism

\[ \Phi : C^*(A, \alpha, J) \to C^*(T), \]

which is injective on the image of $A$ in $C^*(A, \alpha, J)$. One of natural conditions implying injectivity of $\Phi$ on its whole domain can be described as follows. Let

\[ B := C^* \left( \bigcup_{n \in \mathbb{N}} \alpha_n^*(A) \right) \]

be the smallest $C^*$-algebra that contains $A$ and is invariant under $\alpha_*$. Then $\alpha : B \to B$ is an endomorphism and $\alpha_* : B \to B$ is its unique regular transfer operator, cf. [Kwa, Proposition 4.3], [Kwa']. In particular, the pair $(B, \alpha)$ is an example of what we call a reversible $C^*$-dynamical system [Kwa, Definition 2.4], see also [Kwa14', Definition 2.7] and [Kwa', Definition 4.11]. One can associate to the reversible $C^*$-dynamical system $(B, \alpha)$ a natural dual topological dynamical system $(\hat{B}, \hat{\alpha})$ where $\hat{\alpha}$ is a partial homeomorphism of the spectrum $\hat{B}$ of the $C^*$-algebra $B$. It follows from [Kwa, Theorem 4.20] or [Kwa14', Theorem 2.20 (i)] that if $\hat{\alpha}$ is topologically free, then $\Phi$ is an isomorphism, that is $C^*(A, \alpha, J) \cong C^*(T)$.

The above example indicates that crossed products $C^*(A, \alpha, J)$ model a large class of important $C^*$-algebras. For instance, $C^*$-algebras of the form $C^*(T)$, satisfying (1), appear in quantum optics and nuclear physics, cf. [LO04, Section 6]. The $C^*$-algebra $C^*(T)$ carries fundamental spectral information about $T$, e.g. the spectrum $\sigma(T)$ of $T$ is calculated in the realm of $C^*(T)$. In fact we plan to use the results of the present paper in a forthcoming article to study spectra of weighted composition operators acting on vector valued $L^2$-spaces (cf. Example 6.14 below). Moreover, if the range of $T$ is closed, then $U \in C^*(T)$. Thus if the range of $T$ is closed and the $C^*$-algebra $C^*(\{T\})$ generated by $T$ is already unital, then (by passing to a subspace of $H$ if necessary) we have $C^*(T) = C^*(\{T\})$. Such singly generated $C^*$-algebras are basic objects in the $C^*$-algebraic version of Kadison problem, cf. [TW14]. In particular, Kirchberg noticed that every separable unital properly infinite algebra is of the form $C^*(\{T\})$, see [TW14, Theorem 2.3]. The main result of [TW14] states that every separable unital $\mathcal{Z}$-stable $C^*$-algebra is singly generated.

Another source of motivation for the present work comes from the classification program for non-simple $C^*$-algebras, and most notably Kirchberg’s classification of strongly purely infinite, nuclear, separable $C^*$-algebras via ideal related $KK$-theory [Kir00]. It is important that many of such algebras with ‘nice’ primitive ideal spaces can be effectively classified using filtered $K$-theory [MN12], cf. also [Bon02], [Ror97]. The results of the present paper allow one to study a large class of noncommutative $C^*$-dynamical systems $(A, \alpha)$ whose crossed products $C^*(A, \alpha, J)$...
are classifiable in the above sense, and whose ideal structure can be controlled by \((A, \alpha)\) and \(J\). In particular, only by adding a dynamical factor to a classifiable \(C^*\)-algebra one can produce a classifiable \(C^*\)-algebra with more complex ideal structure (cf. Example 6.11 below).

1.2. Methods and results. One of the main tools we use is the reversible extension method introduced in [Kwa, Definition 2.4]. Namely, starting from any \(C^*\)-dynamical system \((A, \alpha)\) and an ideal \(J\) contained in \((\ker \alpha)^\perp\) we can construct a \(C^*\)-dynamical system \((B, \beta)\) such that \(C^*(A, \alpha, J) \cong C^*(B, \beta)\) and the structure of \(C^*(B, \beta)\) is very similar to that of crossed product by a (partial) automorphism. An instance of such a procedure is the passage from the algebra \(A\) to the algebra \(B\) in Example 1.1. The idea is to adapt classical methods, and results to \((B, \beta)\) and then express them in terms of \((A, \alpha)\) and \(J\), so they can be effectively applied to \(C^*(A, \alpha, J) \cong C^*(B, \beta)\). Unfortunately, in general, notions such as topological freeness seem to be very hard to translate from the level of the reversible system \((B, \beta)\) to the language of the initial irreversible \(C^*\)-dynamical systems \((A, \alpha)\).

One of the aims of the present paper is to show that this can be successfully done for a large class of systems \((A, \alpha)\) on \(C_0(X)\)-algebras. Since \(C_0(X)\)-algebras form already a vast class, and various regularity properties of such algebras are well studied, cf. [HRW07], [BK04], our results provide handy tools to construct interesting and non-trivial examples of crossed products. A \(C^*\)-dynamical system \((A, \alpha)\) we consider can be viewed as a convenient combination of noncommutative and topological dynamics, encoded in a pair \((\varphi, \{\alpha_x\}_{x \in \Delta})\) where \(\varphi: \Delta \to X\) is a continuous proper mapping defined on an open set \(\Delta \subseteq X\), and \(\alpha_x: A(\varphi(x)) \to A(x)\), \(x \in \Delta\), is a homomorphism between the corresponding fibers of the \(C_0(X)\)-algebra \(A\), so that

\[
\alpha(a)(x) = \alpha_x(a(\varphi(x))), \quad a \in A, x \in \Delta.
\]

Actually, any \(C^*\)-dynamical system can be equipped with this additional structure, see Example 4.6. We show that the corresponding reversible \(C^*\)-dynamical system \((B, \beta)\) has its canonically associated pair \((\tilde{\varphi}, \{\beta_x\}_{x \in \tilde{\Delta}})\) where the map \(\tilde{\varphi}\) is a partial homeomorphism. Using interrelation-ship between \(\varphi, \tilde{\varphi}\) and the system \((B, \tilde{\beta})\) dual to \((B, \beta)\), we get the first of our main results – Uniqueness Theorem (Theorem 6.3). It gives conditions, phrased in terms of \((\varphi, \{\alpha_x\}_{x \in \Delta})\) and \(A\), under which the image of any faithful representation of \((A, \alpha)\) compatible with \(J\) generates an isomorphic copy of \(C^*(A, \alpha, J)\).

In a similar manner we obtain our second main result – complete description of the ideal structure of \(C^*(A, \alpha, J)\) (Theorem 6.4). We show that if \(\varphi\) is free (e.g. it has no periodic points), then we have a bijective correspondence between ideals in \(C^*(A, \alpha, J)\) and \(J\)-pairs of ideals in \((A, \alpha)\). The latter notion was introduced in [Kwa, Definition 3.10].

Our third objective is to determine pure infiniteness of \(C^*(A, \alpha, J)\). We accomplish it also passing through the system \((B, \beta)\), to which the ideas leading to the proof of [RS12, Theorem 3.3] can be adapted. In particular, we prove that if \(\varphi\) is free and the range of \(\alpha_x, x \in \Delta\), is a full subalgebra of \(A(x)\), we have the implication (see Corollary 6.8):

\[
A \text{ is purely infinite and has the ideal property} \implies \text{the same is true for } C^*(A, \alpha, J).
\]

We would like to make some comments concerning the last mentioned result.

Firstly, by [PR07, proposition 2.14], in the presence of the ideal property, the notions of pure infiniteness and strong pure infiniteness [KR02, Definition 5.1] coincide. Thus the above implication involves in fact strongly purely infinite algebras.

Secondly, in the theorems stating pure infiniteness of crossed products by group actions in [RS12], [Jeo95], [JKO96] group partial actions in [GS], or actions by an (injective) endomorphism in [OP14], [JKO96], the coefficient algebra \(A\) is assumed to have the ideal property or even to be of real rank zero. Moreover, if the underlying group is finite or \(\mathbb{Z}\) one often assumes that \(A\) is purely infinite [Jeo95], [JKO96], and sometimes the pure infiniteness of \(A\) is necessary for pure infiniteness of the crossed product, see [Jeo95].
Thirdly, as shown in [RS12] pure infiniteness of the crossed product can be inferred from paradoxicality of an action on commutative (and hence finite) algebra. There are amenable groups that admit paradoxical actions, cf. [GS, Example 5.1], but one should not expect to have such actions for supramenable groups, see [KMR13]. Nevertheless, allowing the initial algebra to be non-commutative one can obtain a non-trivial condition of this sort even for \( \mathbb{Z} \)-actions. The reason is that both dynamic and algebraic factor of \((A, \alpha)\) may contribute to proper infiniteness of positive elements in \( C^*(A, \alpha, J) \). We propose a definition of paradoxical elements for reversible \( C^* \)-dynamical systems. In particular, if \((B, \beta)\) is a reversible \( C^* \)-dynamical system associated to \((A, \alpha)\) and \( J \), then we generalize the previous implication, showing that under certain assumptions (see Theorem 6.6(i)), we have:

\[
\text{every element in } B^+ \setminus \{0\} \text{ is Cuntz equivalent to a paradoxical element } \implies C^*(A, \alpha, J) \text{ is purely infinite and has the ideal property}
\]

Moreover, if \( A \) has real rank zero the above implication holds with projections in place of positive elements. A condition of similar nature, for injective endomorphisms, was introduced in [OP14] (see Definition 5.14 below). Combining our results with arguments of [OP14] we generalize [OP14, Theorem 3.4]. Thus we obtain another criterion for pure infiniteness of the crossed product \( C^*(A, \alpha) \) (Theorem 6.6(iii)).

Fourthly, the authors of [KS14] studied strong pure infiniteness of crossed products by discrete group actions, without passing (explicitly) through pure infiniteness. They use different methods and the relationship between our results and [KS14, Theorem 6.4] seems to require further investigation which is beyond the scope of the present paper.

1.3. Organization of the content. Sections 2 and 3 are preliminary ones.

The content of Section 2 is based mainly on [Kwa]. We briefly present indispensable definitions and results concerning general \( C^* \)-dynamical systems \((A, \alpha)\) and their crossed products \( C^*(A, \alpha, J) \). In particular, we recall a description of gauge-invariant ideals in \( C^*(A, \alpha, J) \) via \( J \)-pairs of ideals in \( A \), and the construction of the natural reversible \( J \)-extension \((B, \beta)\) of \((A, \alpha)\).

In Section 3 we recall two equivalent descriptions of \( C_0(X) \)-algebras: as algebras equipped with a continuous map \( \sigma_A : \text{Prim} A \to X \) and as continuous sections in upper semicontinuous \( C^* \)-bundles. These two viewpoints have different advantages and we use them alternatively to describe ideals, multiplier algebras and quotients of \( C_0(X) \)-algebras.

In Section 4 we introduce a notion of morphism between upper semicontinuous \( C^* \)-bundles. Such morphisms induce in a natural way homomorphisms between section algebras, which have in turn nice characterisation in the language of \( C_0(X) \)-algebras. We give examples and study basic properties of these homomorphisms.

In Section 5 we deal with reversible \( C^* \)-dynamical systems \((A, \alpha)\) where \( A \) is a \( C_0(X) \)-algebra and the endomorphism \( \alpha : A \to A \) is induced by a morphism of the associated \( C^* \)-bundle. Applying results of [Kwa] we get a description of ideal lattice in \( C^*(A, \alpha) \). Developing new tools inspired by [RS12] and [ELQ02] we obtain a theorem that reduces checking of pure infiniteness of \( C^*(A, \alpha) \) to checking whether each element in \( A^+ \setminus \{0\} \) is properly infinite in \( C^*(A, \alpha) \). We identify the latter property for a class of elements which we call paradoxical for \((A, \alpha)\). We also adapt an argument from [OP14] to give another criterion for pure infiniteness of \( C^*(A, \alpha) \) in the case \( \alpha \) is injective.

Section 6 contains our main results. They concern arbitrary (not necessarily reversible) \( C^* \)-dynamical system \((A, \alpha)\) where \( A \) is a \( C_0(X) \)-algebra and the endomorphism \( \alpha \) is induced by a morphism. Firstly we describe the natural reversible \( J \)-extension \((B, \beta)\) of \((A, \alpha)\) as a system induced by a morphism \((\tilde{\varphi}, \{\beta_x\}_{x \in \tilde{X}})\) where \( \tilde{\varphi} \) is a partial homeomorphism. Then applying to \((B, \beta)\) results of Section 5 we get uniqueness theorem, ideal-lattice description and criteria for pure infiniteness of \( C^*(A, \alpha, J) \) stated in terms of the initial morphism. We conclude the paper by considering certain more specific examples of applications our main results.
1.4. **Notation and conventions.** The set of natural numbers \( \mathbb{N} \) starts from zero. All ideals in \( C^* \)-algebras are assumed to be closed and two-sided. All homomorphisms between \( C^* \)-algebras are by definition \( * \)-preserving. For actions \( \gamma : A \times B \to C \) such as multiplications, inner products, etc., we use the notation:

\[
\gamma(A, B) = \overline{\text{span}}\{\gamma(a, b) : a \in A, b \in B\}.
\]

If \( A \) is a \( C^* \)-algebra, \( 1 \) denotes the unit in the multiplier \( C^* \)-algebra \( M(A) \). The set of positive elements in \( A \) is denoted by \( A^+ \). We write \( \approx_\varepsilon \) to indicate that \( \|a - b\| < \varepsilon \), for \( a, b \in A \). We recall, see [KR00, Theorem 4.16], that a \( C^* \)-algebra \( A \) is properly infinite if and only if every \( a \in A^+ \setminus \{0\} \) is properly infinite, e.g. \( a \oplus a \not\lesssim a \), where \( \lesssim \) is Cuntz comparison of positive elements [Cun78]. By [KR00, Proposition 3.3(iv)] for any \( a \in A^+ \setminus \{0\} \) we have

\[
(2) \quad a \text{ is properly infinite } \iff \forall_{\varepsilon > 0} \exists_{x, y \in a A_a} x^*x \approx_\varepsilon a, \quad y^*y \approx_\varepsilon a, \quad x^*y \approx_\varepsilon 0.
\]

Two elements \( a, b \in A^+ \) are Cuntz equivalent if both \( a \not\lesssim b \) and \( b \not\lesssim a \) holds. A \( C^* \)-algebra \( A \) has the ideal property [PR07], [Pas00], if every ideal in \( A \) is generated (as an ideal) by its projections.

2. **Preliminaries on \( C^* \)-dynamical systems and their crossed products**

The present section serves, in essence, as a summary of main results of [Kwa]. The exception is that we present a slightly more general definition of relative crossed products (without any restrictions on the associated ideal), so that it contains Stacey’s crossed product as a particular case. Moreover, we take an advantage to explain the issue of Stacey’s crossed product in more detail in subsection 2.4.

2.1. **Reversible \( C^* \)-dynamical systems.** A \( C^* \)-dynamical system is a pair \((A, \alpha)\) where \( A \) is a \( C^* \)-algebra and \( \alpha : A \to A \) is an extendible homomorphism. The last term means that \( \alpha \) extends to a strictly continuous endomorphism \( \overline{\alpha} : M(A) \to M(A) \), and this holds exactly when for some (and hence any) approximate unit \( \{\mu_\lambda\} \) in \( A \) the net \( \{\alpha(\mu_\lambda)\} \) converges strictly in \( M(A) \).

**Definition 2.1** (Definition 2.4 [Kwa]). A \( C^* \)-dynamical system \((A, \alpha)\) is called reversible if \( \ker \alpha \) is a complemented ideal in \( A \) and \( \alpha(A) \) is a hereditary subalgebra of \( A \).

**Remark 2.2.** An extendible endomorphism \( \alpha : A \to A \) has a hereditary range if and only if it is a corner endomorphism, that is if \( \alpha(A) \) is a corner in \( A \) (we then necessarily have \( \overline{\alpha}(1)A\overline{\alpha}(1) = \alpha(A) \)). In particular, \((A, \alpha)\) is a reversible \( C^* \)-dynamical system if and only if \( \alpha \) is a corner endomorphism with complemented kernel.

‘Inverse dynamics’ in a reversible \( C^* \)-dynamical system is implemented by a uniquely determined regular transfer operator. We recall [Exel03], [Kwa12], [Kwa’] that a transfer operator for a \( C^* \)-dynamical system \((A, \alpha)\) is a positive linear map \( \alpha_* : A \to A \) such that

\[
\alpha_*(\alpha(a)b) = a\alpha_*(b), \quad \text{for all } a, b \in A.
\]

We say that a transfer operator \( \alpha_* \) for \((A, \alpha)\) is regular, if the composition \( \alpha \circ \alpha_* \) is a conditional expectation from \( A \) onto \( \alpha(A) \). Originally, in [Exel03], such operators were called non-degenerate but we changed this nomenclature in [Kwa’] for reasons expounded therein.

The following characterization of reversible \( C^* \)-dynamical systems follows from [Kwa, Proposition 2.3] or [Kwa’, Proposition 4.5, Theorem 4.12], see also [Kwa12, Theorem 1.6]. Transfer operators satisfying condition (3) below, appear in a natural way in a number of papers, see for instance [Exel03], [ABL11], [HR12], [Kwa14’].

**Proposition 2.3.** A \( C^* \)-dynamical system \((A, \alpha)\) is reversible if and only if there exists a transfer operator \( \alpha_* \) for \((A, \alpha)\) satisfying

\[
(3) \quad \alpha(\alpha_*(a)) = \overline{\alpha}(1)\alpha\overline{\alpha}(1), \quad \text{for all } a \in A.
\]
Moreover, if \((A, \alpha)\) is a reversible \(C^*\)-dynamical system, then the above transfer operator is a unique regular transfer operator for \((A, \alpha)\); it is given by the formula
\[
\alpha_s(a) = \alpha^{-1}(\sigma(1)a\sigma(1)), \quad a \in A,
\]
where \(\alpha^{-1}\) is the inverse to the isomorphism \(\alpha : (\ker \alpha)^\perp \to \alpha(A) = \sigma(1)A\sigma(1)\).

**Remark 2.4.** It was noticed in [Kwa', Proposition 3.1] that any transfer operator \(\alpha_s : A \to A\)
(for a not necessarily extendible endomorphism \(\alpha : A \to A\)) admits a strictly continuous extension
\(\overline{\sigma}_s : M(A) \to M(A)\) such that \(\overline{\sigma}_s(\alpha(a)b) = a\overline{\sigma}_s(b)\), for all \(a \in A, b \in M(A)\). In particular, a
\(C^*\)-dynamical system \((A, \alpha)\) is reversible if and only if \((M(A), \overline{\sigma})\) is reversible. If this is the case
then the unique regular transfer operator \(\overline{\sigma}_s\) for \((M(A), \overline{\sigma})\) is the strictly continuous extension
of the regular transfer operator \(\alpha_s\) for \((A, \alpha)\).

**Example 2.5.** If \(A = C_0(X)\) where \(X\) is a locally compact Hausdorff space, then every endo-
morphism \(\alpha : A \to A\) is of the form
\[
\alpha(a)(x) = \begin{cases} a(\varphi(x)), & x \in \Delta, \\ 0, & x \notin \Delta, \end{cases}
\]
where \(\varphi : \Delta \to X\) is a continuous proper mapping defined on an open subset \(\Delta \subseteq X\). We call
the pair \((X, \varphi)\) a partial dynamical system dual to \((A, \alpha)\). The endomorphism \(\alpha\) is extendible if and
only if \(\Delta\) is closed. The pair \((A, \alpha)\) is a reversible \(C^*\)-dynamical system if and only if both \(\Delta\) and \(\varphi(\Delta)\) are clopen in \(X\) and \(\varphi : \Delta \to \alpha(\Delta)\) is a homeomorphism. In this case \((A, \alpha_s)\) is a
reversible \(C^*\)-dynamical system and \((X, \varphi^{-1})\) is its dual partial dynamical system.

2.2. **Invariant Ideals and \(J\)-Pairs.** Let \((A, \alpha)\) be a fixed \(C^*\)-dynamical system. We say that
an ideal \(I\) in \(A\) is a positive invariant ideal in \((A, \alpha)\) if \(\alpha(I) \subseteq I\). Any such ideal \(I\) induces a quotient \(C^*\)-dynamical system \((A/I, \alpha_I)\) where
\[
\alpha_I(a + I) = \alpha(a) + I \quad \text{for all } a \in A.
\]
Regarding negative invariance, we consider a family of such notions parametrized by ideals in
\((\ker \alpha)^\perp\). Namely, let \(I\) and \(J\) be ideals in \(A\) where \(J \subseteq (\ker \alpha)^\perp\). We say that \(I\) is a \(J\)-negative invariant ideal in \((A, \alpha)\) if \(J \cap \alpha^{-1}(I) \subseteq I\). If \(I\) is both positive invariant and \(J\)-negative invariant we say that \(I\) is \(J\)-invariant, and if \(J = (\ker \alpha)^\perp\) we drop the prefix ‘\(J\)’.

**Lemma 2.6 (Lemma 3.1, [Kwa]).** Suppose \((A, \alpha)\) is a reversible \(C^*\)-dynamical system and let \(I\)
be an ideal in \(A\). The following conditions are equivalent:
\begin{enumerate}
  \item \(I\) is invariant in \((A, \alpha)\),
  \item \(\alpha(I) \subseteq I\) and \(\alpha_s(I) \subseteq I\),
  \item \(\alpha(I) = \sigma(1)I\sigma(1)\),
  \item \(\alpha_s(I) = \sigma(1)I\).
\end{enumerate}
In particular, if \(I\) is invariant the quotient system \((A/I, \alpha_I)\) is reversible.

Let \(I, I'\), \(J\) be ideals in \(A\) where \(J \subseteq (\ker \alpha)^\perp\). We say that \((I, I')\) is a \(J\)-pair for \((A, \alpha)\) if
\(I\) is positive invariant, \(J \subseteq I'\) and \(I' \cap \alpha^{-1}(I) = I\).

The set of \(J\)-pairs for \((A, \alpha)\) is equipped with a natural partial order induced by inclusion:
\((I_1, I'_1) \subseteq (I_2, I'_2) \iff I_1 \subseteq I_2\) and \(I'_1 \subseteq I'_2\). We have the following relationship between \(J\)-invariant ideals and \(J\)-pairs for \((A, \alpha)\).

**Proposition 2.7.** The sets of \(J\)-invariant ideals and \(J\)-pairs for \((A, \alpha)\) form lattices. The map
\((I, I') \to I\) is an order preserving surjection from the latter onto the former, which has the right inverse given by \(I \to (I, I + J)\).

**Proof.** Combine [Kwa, Remark 3.8, Theorem 3.12 and Corollary 3.15].

**Remark 2.8.** In general, unless \(J = (\ker \alpha)^\perp\) and \((\ker \alpha)^\perp\) is a complemented ideal in \(A\), the map \((I, I') \to I\) is not injective, see [Kwa, Remark 3.8 and Example 3.9].
2.3. Covariant representations and crossed products. A representation \((\pi, U)\) of a C*-dynamical system \((A, \alpha)\) on a Hilbert space \(H\) consists of a non-degenerate representation \(\pi : A \to \mathcal{B}(H)\) and an operator \(U \in \mathcal{B}(H)\) such that

\[ U\pi(a)U^* = \pi(\alpha(a)), \quad \text{for all } a \in A. \]

Then, see [Kwa, Proposition 4.3] or [KL13, Lemma 1.2], \(U\) is necessarily a partial isometry and its initial projection \(U^*U\) commutes with the elements of \(\pi(A)\). In particular,

\[ I_{(\pi,U)} := \{ a \in A : U^*U\pi(a) = \pi(a) \} \]

is an ideal in \(A\). If an ideal \(J\) in \(A\) is contained in \(I_{(\pi,U)}\) we say that the representation \((\pi, U)\) is \(J\)-covariant. If \((\pi, U)\) is \((\ker \alpha)^\perp\)-covariant, that is if

\[ a \in (\ker \alpha)^\perp \implies \pi(a) = U^*U\pi(a) \]

we say that \((\pi, U)\) is a covariant representation. Note that if \(\alpha\) is injective, then the representation \((\pi, U)\) is covariant if and only if \(U\) is an isometry. If \((A, \alpha)\) is a reversible C*-dynamical system, then \((\pi, U)\) is covariant if and only if

\[ U^*\pi(a)U = \pi(\alpha_u(a)), \quad \text{for all } a \in A, \]

see [Kwa, Proposition 4.6]. If \(J\) is an ideal in \((\ker \alpha)^\perp\), the relative crossed product that we describe below was defined in [KL13, Definition 1.12] in unital case, and in the present setting in [Kwa, Definition 4.8]. For an arbitrary ideal \(J\) in \(A\) this crossed product is a special case of the one defined in [Kwa', Definition 2.3] where \(\alpha\) is treated as a completely positive map (see [Kwa', Example 2.13]), or in [Kwa13, Definition 4.9] where \(\alpha\) is treated as a partial morphism of \(A\).

Definition 2.9. A relative crossed product associated to a C*-dynamical system \((A, \alpha)\) and an ideal \(J\) in \(A\) is a C*-algebra \(C^*(A, \alpha, J)\) together with a non-degenerate homomorphism \(i_A : A \to C^*(A, \alpha, J)\) and an operator \(u \in M(C^*(A, \alpha, J))\) such that

a) \(i_A(a) = ui_A(a)u^*\) for each \(a \in A\),

b) \(C^*(A, \alpha, J)\) is generated (as a C*-algebra) by \(i_A(A) \cup i_A(A)u\),

c) for every \(J\)-covariant representation \((\pi, U)\) of \((A, \alpha)\) there exists a non-degenerate representation \(\pi \times U\) of \(C^*(A, \alpha, J)\) with \((\pi \times U) \circ i_A = \pi\) and \((\pi \times U)(u) = U\).

Moreover, we write \(C^*(A, \alpha) := C^*(A, \alpha, (\ker \alpha)^\perp)\) and call it the (unrelative) crossed product of \(A\) by \(\alpha\).

Proposition 2.10. For any C*-dynamical system \((A, \alpha)\) and any ideal \(J\) in \(A\), the crossed product \((C^*(A, \alpha, J), i_A, u)\) exists and is unique (up to a natural isomorphism). Moreover, we have

\[ C^*(A, \alpha, J) = \overline{\text{span}} \{ u^ni_A(a)u^m : a \in A, n, m \in \mathbb{N} \}, \]

and the homomorphism \(i_A\) is injective if and only if \(J \subseteq (\ker \alpha)^\perp\). If \(J \nsubseteq (\ker \alpha)^\perp\) then

\[ R := \{ a \in A : \alpha^n(a) \in J \text{ for all } n \in \mathbb{N} \text{ and } \lim_{n \to \infty} \alpha^n(a) = 0 \} \]

is a non-trivial positive invariant ideal in \((A, \alpha)\) such that \(q_R(J) \subseteq (\ker \alpha_R)^\perp\), and we have a natural isomorphism

\[ C^*(A, \alpha, J) \cong C^*(A/R, \alpha_R, q_R(J)). \]

Proof. If \(J \subseteq (\ker \alpha)^\perp\) the assertion follows from [Kwa, Proposition 4.9]. If \(J \nsubseteq (\ker \alpha)^\perp\) one can apply [Kwa, Proposition 4.9] to \(C^*(A/R, \alpha_R, q_R(J))\) as the latter models the crossed product \(C^*(A, \alpha, J)\) by [Kwa, Remark 4.11], cf. also [Kwa13, Example 6.24], or [KL13, Section 5.3]. In particular, universality implies that we have the isomorphism \(C^*(A, \alpha, J) \cong C^*(A/R, \alpha_R, q_R(J))\) intertwining \(i_A/R \circ q_R\) and \(i_A\). \(\square\)
Convention 2.11. In this paper, with the only exception of subsection 2.4 below, we will consider exclusively crossed products $C^*(A, \alpha, J)$ for ideals $J$ contained in $(\ker \alpha)^\perp$. The above statement shows that this is a reasonable assumption. In particular, since in this case the universal homomorphism $i_A$ is injective we will often suppress it and treat $A$ as a subalgebra of $C^*(A, \alpha, J)$.

If the $C^*$-dynamical system $(A, \alpha)$ is reversible, then the crossed product $C^*(A, \alpha)$ has a very similar structure to that of classical crossed product by an automorphism.

Proposition 2.12. Suppose $(A, \alpha)$ is a reversible $C^*$-dynamical system. The crossed product $C^*(A, \alpha)$ is a universal $C^*$-algebra generated by $A \cup Au$ where $u$ is an element of $M(C^*(A, \alpha))$ such that

$$uau^* = \alpha(a), \quad u^*au = \alpha_+(a), \quad \text{for all } a \in A.$$ 

Moreover, $C^*(A, \alpha)$ is the closure of a dense $^*$-algebra consisting of elements of the form

$$a = \sum_{k=1}^n u^k a_{-k}^* + a_0 + \sum_{k=1}^n a_k u^k, \quad a_k \in A\tau^k(1), k = 0, \pm 1, \ldots, \pm n. \quad (6)$$

The coefficients $a_k \in A\tau^k(1)$ in (6) are uniquely determined by $a$, and there is a faithful conditional expectation from $C^*(A, \alpha)$ onto $A$ such that $E(a) = a_0$.

Proof. The first part of the assertion follows from [Kwa, Proposition 4.6], see also [Kwa', Theorem 4.12]. The second part is shown in [Kwa, Proposition 4.9] modulo the fact that $au^k = a\tau^k(1)u^k \in A\tau^k(1)u^k$, for any $a \in A$, $k \in \mathbb{N}$, which follows for instance from Lemma 2.13 that we prove below. Existence of the faithful conditional expectation determined by $E(a) = a_0$ where $a$ is given by (6) is obtained in a standard manner, see for instance [Kwa, Proposition 4.12]. In particular, for $k \geq 0$ we have $E(u^k a) = a_{-k}^*$ and $E(au^k) = a_k$. Hence the coefficients $a_{\pm k}$ are uniquely determined by $a$. \hfill $\square$

2.4. Relationship with Stacey's crossed product. We pause here to explain in detail the relationship between the crossed product introduced above and Stacey’s (multiplicity one) crossed product $A \times_\alpha^1 \mathbb{N}$. It seems that the extendibility assumption on $\alpha$ is inevitable in Stacey’s construction (see also the relevant discussion in [KS14, Section 6]). We recall, [Sta93, Definition 3.1], that $A \times_\alpha^1 \mathbb{N}$ is the universal $C^*$-algebra generated by the elements $i_A(a)u^n u^m$, $a \in A$, $n, m \in \mathbb{N}$ where $\iota: A \to A \times_\alpha^1 \mathbb{N}$ is a non-degenerate homomorphism, and $u$ is an isometry in $M(A \times_\alpha^1 \mathbb{N})$ such that $i_A(a) = u\iota_A(a)u^*$ for each $a \in A$. In other words, $(i_A, u)$ is an $A$-covariant representation of $(A, \alpha)$. Thus to see that $A \times_\alpha^1 \mathbb{N} = C^*(A, \alpha, A)$ it suffices to verify the equality

$$C^*(i_A(A) \cup i_A(A)u) = C^*(\bigcup_{n,m \in \mathbb{N}} i_A(A)u^n u^m). \quad (7)$$

To this end, we will use the following lemma.

Lemma 2.13. For any representation $(\pi, U)$ of a $C^*$-dynamical system $(A, \alpha)$ we have

$$U\pi(a) = \pi(\alpha(a))U, \quad \text{for } a \in A, \quad \text{and} \quad \overline{\pi(\alpha(1))} = UU^*,$$

where $\overline{\pi}$ is the unique strict extension of $\pi$ from $A$ onto $M(A)$. In particular,

$$U\pi(A) = \pi(\alpha(A))U, \quad \pi(A)U = \pi(A\alpha(A))U = \pi(A)U\pi(A). \quad (8)$$

Proof. In view of [Kwa, Proposition 4.3] we have $U^*U \in \pi(A)'$ and $U$ is a partial isometry. Therefore for any $a \in A$ we get

$$U\pi(a) = U(U^*U)\pi(a) = U\pi(a)(U^*U) = \pi(\alpha(a))U.$$ 

For any approximate unit $\mu_\lambda$ in $A$, the net $\pi(\mu_\lambda) = U\pi(\mu_\lambda)U^*$ converges strongly both to $\overline{\pi(\alpha(1))}$ and to $UU^*$. Hence $\overline{\pi(\alpha(1))} = UU^*$. \hfill $\square$
Proposition 2.14. The relative crossed product \( C^*(A, \alpha, A) \) coincides with Stacey’s multiplicity one crossed product of \( A \) by \( \alpha \). Moreover, the following conditions equivalent:

i) \( A \) embeds into Stacey’s crossed product,

ii) Stacey’s crossed product coincides with \( C^*(A, \alpha) \),

iii) \( \alpha \) is injective.

Proof. In view of Proposition 2.10 and the discussion preceding Lemma 2.13 it suffices to show (7). It is clear that \( C^*(i_A(A) \cup i_A(A)u) \subseteq C^*(\bigcup_{n,m \in \mathbb{N}} i_A(A)u^n u^m) \). The reverse inclusion holds because, using inductively (8), for any \( n, m \in \mathbb{N} \) we see that

\[
i_A(A)u^n u^m = (i_A(A)u)(i_A(A)u) ... (i_A(A)u) i_A(A)(u^i_i(A)(\alpha(A))) ... (u^i_i(A)(\alpha(A))) = n \text{ times } m \text{ times}\]

is an element of \( C^*(i_A(A) \cup i_A(A)u) \).

\[ \square \]

2.5. Reversible extensions of \( C^* \)-dynamical systems. Fix a \( C^* \)-dynamical system \( (A, \alpha) \) and an ideal \( J \) in \((\ker \alpha)^+\). We recall a canonical construction of a reversible \( C^* \)-dynamical system \( (B, \beta) \) such that \( C^*(A, \alpha, J) \) and \( C^*(B, \beta) \) are isomorphic, see [Kwa, Subsection 3.1]. The system \( (B, \beta) \) is defined as a direct limit of approximating \( C^* \)-dynamical systems \( (B_n, \beta_n), n \in \mathbb{N} \). More precisely, we denote by \( q : A \to A/J \) the quotient map and for each \( n \in \mathbb{N} \) we put

\[
A_n := \overline{\alpha^n(1)A\alpha^n(1)}.
\]

The \( C^* \)-algebra \( B_n, n \in \mathbb{N} \), is a direct sum of the form

\[
B_n = q(A_0) \oplus q(A_1) \oplus ... \oplus q(A_{n-1}) \oplus A_n,
\]

and the endomorphism \( \beta_n : B_n \to B_n \) is given by the formula

\[
\beta_n(a_0 \oplus a_1 \oplus ... \oplus a_n) = a_1 \oplus a_2 \oplus ... \oplus q(a_n) \oplus \alpha(a_n).
\]

Thus we get a sequence \( (B_n, \beta_n), n \in \mathbb{N}, \) of \( C^* \)-dynamical systems where \( (B_0, \beta_0) = (A, \alpha) \). We consider bonding homomorphisms \( \alpha_n : B_n \to B_{n+1}, n \in \mathbb{N} \), whose action is presented by the diagram

\[
\begin{array}{ccc}
B_n & = & q(A_0) \oplus ... \oplus q(A_{n-1}) \oplus A_n \\
\alpha_n & \downarrow & id \\
B_{n+1} & = & q(A_0) \oplus ... \oplus q(A_{n-1}) \oplus q(A_n) \oplus A_{n+1}
\end{array}
\]

In other words, \( \alpha_n \) is given by the formula

\[
\alpha_n(a_0 \oplus ... \oplus a_{n-1} \oplus a_n) = a_0 \oplus ... \oplus a_{n-1} \oplus q(a_n) \oplus \alpha(a_n),
\]

where \( a_k \in q(A_k), k = 0, ..., n-1, \) and \( a_n \in A_n \). Plainly, the homomorphisms \( \alpha_n \) are injective and we have

\[
\alpha_n \circ \beta_n = \beta_{n+1} \circ \alpha_n, \quad n \in \mathbb{N}.
\]

Accordingly, we get the direct sequence of \( C^* \)-dynamical systems:

\[
(\overline{B_0, \beta_0}) \overset{\alpha_0}{\longrightarrow} (B_1, \beta_1) \overset{\alpha_1}{\longrightarrow} (B_2, \beta_2) \overset{\alpha_2}{\longrightarrow} ... ...
\]

It is shown in [Kwa, Theorem 3.1] that this direct sequence has a direct limit \( (B, \beta) \) which is a reversible \( C^* \)-dynamical system. More precisely, \( B = \lim B_n, \alpha_n \) is the \( C^* \)-algebraic direct limit, and \( \beta \) is determined by the formula \( \beta(\phi_n(a)) = \phi_n(\beta_n(a)) \) where \( \phi_n : B_n \to B \) is the natural (injective) homomorphism, \( a \in B_n \) and \( n \in \mathbb{N} \). That is we have

\[
\beta(\phi_n(a_0 \oplus a_1 \oplus ... \oplus a_n)) = \phi_{n-1}(a_1 \oplus a_2 \oplus ... \oplus a_n)
\]

where \( a_k \in A_k, k = 0, ..., n \). Moreover, by [Kwa, Theorem 3.1 and Proposition 4.7], see also [Kwa, Remark 3.3] we have the following statement.
Theorem 2.15. The $C^*$-dynamical system $(B, \beta)$ described above is reversible and identifying $A$ with a subalgebra of $B$, via the embedding $\phi_0$, the endomorphism $\beta : B \to B$ is an extension of $\alpha : A \to A$ such that

$$B = \overline{\text{span}}\{\beta^n(a) : a \in A, n \in \mathbb{N}\}, \quad \text{and} \quad \beta_s(1)A \cap A = J.$$  

Moreover, we may extend the identification $A \subseteq B$ to the equality

$$C^*(A, \alpha, J) = C^*(B, \beta) \quad \text{where} \quad \beta(b) = ubu^*, \ \beta_s(b) = u^*bu, \ b \in B.$$  

In particular, the relation $\tilde{\pi}(\sum_{k=0}^\infty \beta^k(a_k)) = \sum_{k=0}^\infty U^k \pi(a_k) U^k$, $a_k \in A$, establishes a one-to-one correspondence between $J$-covariant representations $(\pi, U, H)$ of $(A, \alpha)$ and covariant representations $(\tilde{\pi}, U, H)$ of $(B, \beta)$.

Definition 2.16 (Definition 3.2 in [Kwa]). Suppose $(A, \alpha)$ is a $C^*$-dynamical system and $J$ is an ideal in $(\ker \alpha)^\perp$. We call the $C^*$-dynamical system $(B, \beta)$ constructed above the natural reversible $J$-extension of $(A, \alpha)$.

Let $(B, \beta)$ be a natural reversible $J$-extension of $(A, \alpha)$ and suppose that $A = C_0(X)$ is commutative. Then $B$ is also commutative and thus we may identify it with $C_0(\widetilde{X})$ where $\widetilde{X}$ is a locally compact Hausdorff space. With this identification, $\beta$ and $\beta_s$ are given by the formulas

$$\beta(b)(\overline{x}) = \begin{cases} b(\overline{\varphi(x)}), & \overline{x} \in \Delta, \\ 0 & \overline{x} \notin \Delta, \end{cases} \quad \beta_s(b)(\overline{x}) = \begin{cases} b(\overline{\varphi^{-1}(x)}), & \overline{x} \in \overline{\varphi(\Delta)}, \\ 0 & \overline{x} \notin \overline{\varphi(\Delta)}, \end{cases}$$

where $\overline{\varphi} : \overline{\Delta} \to \overline{\varphi(\Delta)}$ is a homeomorphism between clopen subsets of $\widetilde{X}$. The pair $(\widetilde{X}, \overline{\varphi})$ is uniquely determined by $(X, \varphi)$ and the closed set

$$(9) \quad Y = \{x : a(x) = 0 \text{ for all } a \in J\},$$

which necessarily contains $X \setminus \varphi(\Delta)$. Actually, using the above construction of $(B, \beta)$ one can deduce the following description of $(\widetilde{X}, \overline{\varphi})$, see [Kwa, Proposition 4.29], cf. also [Kwa12', Theorem 3.5], [KL08, Theorem 3.5].

Proposition 2.17. Up to conjugacy with a homeomorphism, the above partial dynamical system $(\widetilde{X}, \overline{\varphi})$ can be described as follows:

$$\widetilde{X} = \bigcup_{N=0}^\infty X_N \cup X_\infty$$

where

$$X_N = \{(x_0, x_1, ..., x_N, 0, ...) : x_n \in \Delta, \varphi(x_n) = x_{n-1}, \ n = 1, ..., N, \ x_N \in Y\},$$

$$X_\infty = \{(x_0, x_1, ...) : x_n \in \Delta, \varphi(x_n) = x_{n-1}, \ n \geq 1\}.$$  

The topology on $\widetilde{X}$ is the product one inherited from $\prod_{n \in \mathbb{N}} (X \cup \{0\})$ where $\{0\}$ is a clopen singleton and $Y$ is given by (9). The homeomorphism $\overline{\varphi} : \overline{\Delta} \to \overline{\varphi(\Delta)}$ is given by the formula

$$\overline{\varphi}(x_0, x_1, ...) = (\varphi(x_0), x_0, x_1, ...), \quad \overline{\Delta} = \{(x_0, x_1, ...) \in \widetilde{X} : x_0 \in \Delta\}.$$  

Definition 2.18 (Definition 3.5, [Kwa12']). Dynamical system $(\widetilde{X}, \overline{\varphi})$ described in the assertion of Proposition 2.17 is called the natural reversible $Y$-extension of $(X, \varphi)$. 
2.6. Ideal structure of crossed products. Let $J \subseteq (\ker\alpha)^\perp$ be a fixed ideal. Standard argument shows that relations $\gamma_z(a) = a$, $\gamma_z(u) = zu$, $a \in A$, $z \in \mathbb{T}$ determine a point-wise continuous group homomorphism $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\} \ni z \mapsto \gamma_z \in \text{Aut}(C^*(A, \alpha, J))$. We call this homomorphism gauge action. Ideals in $C^*(A, \alpha, J)$ that are invariant under this action are called gauge-invariant. Combining [Kwa, Theorems 3.12, 4.15 and Proposition 4.13] we get

**Proposition 2.19.** Let $(A, \alpha)$ be a $C^*$-dynamical system, $J$ an ideal in $(\ker\alpha)^\perp$, and $(B, \beta)$ the natural reversible $J$-extension of $(A, \alpha)$. The relations

$$I = A \cap \tilde{I}, \quad I' = \{a \in A : (1 - \beta_+(1))a \in \tilde{I}\}, \quad \tilde{I} = B \cap I$$

establish lattice isomorphisms between the following classes of objects:

i) $J$-pairs $(I, I')$ for $(A, \alpha)$,

ii) invariant ideals $\tilde{I}$ in $(B, \beta)$,

iii) gauge-invariant ideals $\tilde{I}$ in $C^*(A, \alpha, J) = C^*(B, \beta)$.

Moreover, for objects satisfying (10) we have natural isomorphisms

$$C^*(A, \alpha, J)/I \cong C^*(B/\tilde{I}, \beta_{\tilde{I}}) \cong C^*(A/I, \alpha_I, q_I(I')),$$

and $(B/\tilde{I}, \beta_{\tilde{I}})$ can be identified with the natural reversible $q_I(I')$-extension of the subsystem $(A/I, \alpha_I)$.

In the favorable situation when all ideals in $C^*(A, \alpha, J)$ are gauge-invariant the above statement yields a complete description of ideal structure of $C^*(A, \alpha, J)$. We recall certain known conditions implying this. The relevant statements, see [Kwa], are deduced from the general results of [Kwa14], cf. also [Kwa14', Subsection 2.3].

**Definition 2.20.** Let $\varphi$ be a partial homeomorphism of a topological (not necessarily Hausdorff) space $X$ with domain being an open set $\Delta \subseteq X$. We say that $\varphi$ is topologically free if the set of its periodic points of any given period $n > 0$ has empty interior. A set $V \subseteq X$ is invariant if $\varphi(V \cap \Delta) = V \cap \varphi(\Delta)$. We say that $\varphi$ is (essentially) free, if it is topologically free when restricted to any closed invariant set.

**Definition 2.21.** Let $(B, \beta)$ be a reversible $C^*$-dynamical system and $\beta_+$ its unique regular transfer operator. Since $\beta_+(B) = (\ker\beta)^\perp$ is an ideal in $B$ and $\beta(B) = \overline{\beta(1)B\beta(1)}$ is a hereditary subalgebra of $B$ we have the natural identifications:

$$\overline{\beta_+(B)} = \{\pi \in \hat{B} : \pi(\beta_+(B)) \neq 0\}, \quad \overline{\beta(B)} = \{\pi \in \hat{B} : \pi(\beta(B)) \neq 0\}.$$

Thus we treat $\overline{\beta(B)}$ and $\overline{\beta_+(B)}$ as open subsets of $\hat{B}$. With these identifications the homeomorphism $\hat{\beta} : \overline{\beta(B)} \to \overline{\beta_+(B)}$ dual to the isomorphism $\beta : \beta_+(B) \to \beta(B)$ becomes a partial homeomorphism of the spectrum of $\hat{B}$, cf. [Kwa]. We refer to $\hat{\beta}$ as to the partial homeomorphism dual to $(B, \beta)$.

**Theorem 2.22.** Let $(A, \alpha)$ be a $C^*$-dynamical system, $J$ an ideal in $(\ker\alpha)^\perp$, and $(B, \beta)$ the natural reversible $J$-extension of $(A, \alpha)$.

(i) If $\hat{\beta}$ is topologically free, then every representation $(\pi, U)$ of $(A, \alpha)$ such that $J = \{a \in A : U^*U\pi(a) = \pi(a)\}$ integrates to the faithful representation of $C^*(A, \alpha, J) = C^*(B, \beta)$.

(ii) If $\hat{\beta}$ is free, then all ideals in $C^*(A, \alpha, J) = C^*(B, \beta)$ are gauge-invariant; hence they are in one-to-one correspondence with $J$-pairs in $(A, \alpha)$, see Proposition 2.19.

**Proof.** Part (i) follows from Theorem 4.20 and the last part of Proposition 4.7 in [Kwa]. Part (ii) is contained in [Kwa, Corollary 4.24].

Obviously, in order to use Theorem 2.22 in practice one has to determine topological freeness and freeness of $\hat{\beta}$ in terms of $(A, \alpha)$ and $J$. This can be readily achieved if $A$ is commutative.
Definition 2.23. Let \( \varphi \) be a partial mapping of a locally compact Hausdorff space \( X \) defined on an open set \( \Delta \subseteq X \). We say that a periodic orbit \( \mathcal{O} = \{x, \varphi(x), \ldots, \varphi^{n−1}(x)\} \) of a periodic point \( x = \varphi^n(x) \) has an entrance \( y \in \Delta \) if \( y \notin \mathcal{O} \) and \( \varphi(y) \in \mathcal{O} \). We say \( \varphi \) is topologically free outside a set \( Y \subseteq X \) if the set of periodic points whose orbits do not intersect \( Y \) and have no entrances have empty interior.

Lemma 2.24. Let \((\bar{X}, \bar{\varphi})\) be the \( Y \)-extension of a partial dynamical system \((X, \varphi)\) where \( Y \) is a closed set containing \( X \setminus \varphi(\Delta) \), see Definition 2.18. Then

(i) \( \bar{\varphi} \) is topologically free if and only if \( \varphi \) is topologically free outside \( Y \),

(ii) \( \bar{\varphi} \) is free if and only if \( \varphi \) is free (has no periodic points).

Proof. Item (i) is [Kwa, Lemma 4.32] and item (ii) is straightforward. \( \square \)

One of the aims of the present paper is to obtain effective conditions implying the properties of crossed products described in Theorem 2.22 for a class of \( C^* \)-dynamical systems on \( C_0(X) \)-algebras.

3. Preliminaries on \( C_0(X) \)-algebras and \( C^* \)-bundles

In this section we gather certain facts concerning \( C_0(X) \)-algebras. Since we find it beneficial to use two equivalent pictures of such objects: as \( C^* \)-algebras with a \( C_0(X) \)-module structure and as \( C^* \)-algebras of sections of \( C^* \)-bundles, we recall and implement both of these viewpoints. As a general reference we recommend [Wil07, Section C].

3.1. \( C^* \)-bundles and \( C^* \)-algebras. Let \( X \) be a locally compact Hausdorff space. An upper semicontinuous \( C^* \)-bundle over \( X \) is a topological space \( A = \bigsqcup_{x \in X} A(x) \) such that the natural surjection \( p : A \to X \) is open continuous, each fibre \( A(x) \) is a \( C^* \)-algebra, the mapping \( A \ni a \to \|a\| \in \mathbb{R} \) is upper semicontinuous, and the \( * \)-algebraic operations in each of the fibers are continuous in \( A \), for details see [Wil07, Definition C.16]. If additionally, the mapping \( A \ni a \to \|a\| \in \mathbb{R} \) is continuous, \( A \) is called a continuous \( C^* \)-bundle over \( X \). For each \( x \in X \), we denote by \( 0_x \) the zero element in the fiber \( C^* \)-algebra \( A(x) \), and by \( 1_x \) the unit in the multiplier algebra \( M(A(x)) \) of \( A(x) \).

We denote by \( \Gamma(A) := \{a \in C(X,A) : p(a(x)) = id\} \) the set of continuous sections of the upper semicontinuous \( C^* \)-bundle \( A \). It is a \( * \)-algebra with respect to natural pointwise operations. Moreover, the set of continuous sections that vanish at infinity

\[
\Gamma_0(A) := \{a \in \Gamma(A) : \forall \varepsilon > 0 \quad \{x \in X : \|a(x)\| \geq \varepsilon\} \text{ is compact}\}
\]

is a \( C^* \)-algebra with the norm \( \|a\| := \sup_{x \in X} \|a(x)\| \). We call \( \Gamma_0(A) \) the section \( C^* \)-algebra of \( A \). The section algebra \( \Gamma_0(A) \) determines the topology of the \( C^* \)-bundle \( A \). In pasticular, we have the following lemma (see, for instance, the proof of [Wil07, Theorem C.25]).

Lemma 3.1. A net \( \{b_i\} \) converges to \( b \) in the \( C^* \)-bundle \( A \) if and only if \( p(b_i) \to p(b) \) and for each \( \varepsilon > 0 \) there is \( a \in \Gamma_0(A) \) such that \( \|a(p(b_i)) − b\| < \varepsilon \) and we eventually have \( \|a(p(b_i)) − b_i\| < \varepsilon \).

The algebra \( \Gamma_0(A) \) is naturally equipped with the structure of \( C_0(X) \)-algebra given by

\[
(f \cdot a)(x) := f(x)a(x) \quad \text{for } f \in C_0(X), \ a \in \Gamma_0(A).
\]

3.2. \( C_0(X) \)-algebras. A \( C_0(X) \)-algebra is usually defined as a \( C^* \)-algebra \( A \) endowed with a nondegenerate homomorphism \( \mu_A \) from \( C_0(X) \) into the center \( Z(M(A)) \) of the multiplier algebra \( M(A) \) of \( A \). When \( X \) is compact \( A \) is also called a \( C(X) \)-algebra. The \( C_0(X) \)-algebra \( A \) is viewed as a \( C_0(X) \)-module where

\[
f \cdot a := \mu_A(f)a, \quad f \in C_0(X), \ a \in A.
\]

Accordingly, the structure map \( \mu_A : C_0(X) \to Z(M(A)) \) is often suppressed. Using the Dauns-Hofmann isomorphism we may identify \( Z(M(A)) \) with \( C_0(\text{Prim} A) \), and then \( \mu_A \) becomes the
operator of composition with a continuous map \( \sigma_A : \text{Prim} A \to X \). This map, called the base map, is determined by the equivalence:

\[
C_0(X \setminus \{x\}) \cdot A \subseteq P \iff \sigma_A(P) = x, \quad P \in \text{Prim}(A).
\]

Let us fix a \( C_0(X) \)-algebra \( A \) and consider a bundle \( \mathcal{A} := \bigsqcup_{x \in X} A(x) \) where

\[
A(x) := A / \left( \bigcap_{P \in \sigma_A^{-1}(x)} P \right) = A / \left( C_0(X \setminus \{x\}) \cdot A \right), \quad x \in X.
\]

It can be shown that there is a unique topology on \( \mathcal{A} := \bigsqcup_{x \in X} A(x) \) such that \( \mathcal{A} \) becomes an upper semicontinuous \( C^* \)-bundle and the \( C_0(X) \)-algebra \( A \) can be identified with \( \Gamma_0(A) \) by writing \( a(x) \) for the image of \( a \in A \) in the quotient algebra \( A(x) \).

Moreover, \( \mathcal{A} \) is a continuous \( C^* \)-bundle if and only if \( \sigma_A : \text{Prim} A \to X \) is an open map. In the latter case \( A \) is called a continuous \( C_0(X) \)-algebra. Concluding we have the following statement, see [Wil07, Theorem C.26], which in essence is due to Lee, see [Lee76, Theorem 4].

**Theorem 3.2.** A \( C^* \)-algebra \( A \) is a \( C_0(X) \)-algebra if and only if \( A \cong \Gamma_0(A) \) where \( A \) is an upper semicontinuous \( C^* \)-bundle. Moreover, \( A \) is a continuous \( C_0(X) \)-algebra if and only if \( A \) is a continuous \( C^* \)-bundle.

**Convention 3.3.** In the sequel we will freely pass (often without a warning) between the above equivalent descriptions. Thus for any \( C_0(X) \)-algebra \( A \) we will write \( A = \Gamma_0(A) \) where \( A \) is the associated \( C^* \)-bundle.

**Remark 3.4.** Let \( A \) be a \( C_0(X) \)-algebra. It follows from (11) that \( \sigma_A(\text{Prim}(A)) = \{ x \in X : \|1(x)\| \geq 1/2 \} \) is compact, because \( 1 \in \Gamma_0(A) \), and clearly the (global) unit 1 is a local unit for any point in \( X \). Conversely, suppose that \( \sigma_A(\text{Prim}(A)) \) is compact and \( A \) has local units. Consider the function \( 1 : X \to A = \bigsqcup_{x \in X} A(x) \) where for each \( x \in A \) we let \( 1(x) := 1_x \) to be the unit in \( A(x) \). Using Lemma 3.1 and local units one readily sees that \( 1 \) is a continuous section of \( A \). For any \( \varepsilon \leq 1 \) the set \( \{ x \in X : \|1(x)\| \geq \varepsilon \} \) is compact. Thus \( 1 \in \Gamma_0(A) = A \).

3.3. Multiplier algebra of a \( C_0(X) \)-algebra. We say that a \( C_0(X) \)-algebra \( A \) has local units if all fibers \( A(x) \), \( x \in X \), are unital, and for any \( x \in X \) there is \( a \in A \) such that \( a(y) = 1_y \) is the unit in \( A(y) \) for all \( y \) in a neighborhood of \( x \).

**Lemma 3.5.** A \( C_0(X) \)-algebra \( A \) is unital if and only if \( A \) has local units and the range of \( \sigma_A \) is compact.

**Proof.** If \( 1 \) is the unit in \( A \) then \( \sigma_A(\text{Prim}(A)) = \{ x \in X : \|1(x)\| \geq 1/2 \} \) is compact, because \( 1 \in \Gamma_0(A) \), and clearly the (global) unit 1 is a local unit for any point in \( X \). Conversely, suppose that \( \sigma_A(\text{Prim}(A)) \) is compact and \( A \) has local units. Consider the function \( 1 : X \to A = \bigsqcup_{x \in X} A(x) \) where for each \( x \in A \) we let \( 1(x) := 1_x \) to be the unit in \( A(x) \). Using Lemma 3.1 and local units one readily sees that \( 1 \) is a continuous section of \( A \). For any \( \varepsilon \leq 1 \) the set \( \{ x \in X : \|1(x)\| \geq \varepsilon \} \) is compact. Thus \( 1 \in \Gamma_0(A) = A \).

We have the following natural description of the multiplier algebra \( M(A) \) of a \( C_0(X) \)-algebra \( A \) as sections of the set \( M(A) := \bigsqcup_{x \in X} M(A(x)) \), see [Wil07, Lemma C.11]. We emphasize however, that in general (even when \( X \) is compact) \( M(A) \) can not be equipped with a topology making it into an upper semicontinuous \( C^* \)-bundle such that \( M(A) \subseteq \Gamma(M(A)) \), see [Wil07, Example C.13].

**Proposition 3.6.** Suppose \( A \) is a \( C_0(X) \)-algebra. The multiplier algebra \( M(A) \) can be naturally identified with the set of all functions \( m \) on \( X \) such that \( m(x) \in M(A(x)) \), for all \( x \in X \), and the functions \( x \mapsto m(x)a(x) \), \( x \mapsto a(x)m(x) \) are in \( A = \Gamma_0(A) \) for any \( a \in A \). Then the \( C^* \)-algebraic structure of \( M(A) \) is given by the pointwise operations and the supremum norm \( \|m\| = \sup_{x \in X} \|m(x)\| \).
3.4. Ideals and quotients of a $C_0(X)$-algebra. Fix a $C_0(X)$-algebra $A$ and let $J$ be an ideal in $A$. Assuming the standard identifications $\text{Prim} J = \{P \in \text{Prim} A : J \not\subseteq P\}$ and $\text{Prim}(A/J) = \{P \in \text{Prim} A : J \subseteq P\}$, we see that both $J$ and $A/J$ are $C_0(X)$-algebras with base maps $\sigma_A : \text{Prim}(J) \to X$ and $\sigma_A : \text{Prim}(A/J) \to X$ respectively. Moreover, we have natural isomorphisms $(A/J)(x) \cong A(x)/J(x)$ where $J(x) = \{a(x) : a \in J \subseteq A\}$, $x \in X$.

Suppose that $A$ is a continuous $C_0(X)$-algebra. Then the ideal $J$ is naturally a continuous $C_0(Y)$-algebra for any locally compact set $Y$ containing the open set $\sigma_A(\text{Prim}(J))$, because a restriction of an open map to an open set is open (independently of the codomain). The situation is quite different when dealing with a restriction to a closed set, and thus the case of the quotient restriction of an open map to an open set is open (independently of the codomain). The situation translates this to the language of $C^*$-bundles we get

**Lemma 3.7.** Suppose $J$ is an ideal in a $C^*$-algebra $A = \Gamma_0(A)$ of continuous sections of an upper semicontinuous $C^*$-bundle $\mathcal{A} = \coprod_{x \in X} A(x)$. The ideal $J$ and the quotient algebra $A/J$ can be naturally treated as algebras of continuous sections of $\mathcal{J} = \coprod_{x \in X} J(x)$ and $A/J = \coprod_{x \in X} A(x)/J(x)$ (equipped with unique topologies), respectively. If additionally $A$ is a continuous $C^*$-bundle, then $\mathcal{J}$ is a continuous $C^*$-bundle over the open set

$$\{x \in X : J(x) \neq \{0\}\} = \sigma_A(\text{Prim}(J)), \quad \text{(12)}$$

and the $C^*$-bundle $A/J$ is continuous over the locally compact subset

$$Y := \{x \in X : J(x) \neq A(x)\} = \sigma_A(\text{Prim}(A/J)) \quad \text{(13)}$$

whenever $J$ is complemented or $\sigma_A$ is injective.

**Proof.** In view of the above discussion we only need to show formulas (12) and (13). The equivalences

$$J(x) \neq \{0\} \iff J \not\subseteq \bigcap_{P \in \sigma_A^{-1}(x)} P \iff \exists P \in \sigma_A^{-1}(x) J \not\subseteq P \iff x \in \sigma_A(\text{Prim}(J))$$

prove (12). To see (13) notice that using (11) we get

$$J(x) \neq A(x) \iff \left( J + \bigcap_{P \in \sigma_A^{-1}(x)} P \right) \neq A \iff \exists P_0 \in \text{Prim}(A) \left( J + \bigcap_{P \in \sigma_A^{-1}(x)} P \right) \subseteq P_0 \iff \exists P_0 \in \text{Prim}(A) J \subseteq P_0 \text{ and } \bigcap_{P \in \sigma_A^{-1}(x)} P \subseteq P_0 \iff x \in \sigma_A(\text{Prim}(A/J)).$$

\[ \square \]

**Example 3.8.** Let $J$ be an ideal in the $C_0(X)$-algebra $A$. The annihilator $J^\perp = \{a \in A : aJ = 0\}$ of $J$ is also a $C_0(X)$-algebra with the base map $\sigma_A : \text{Prim}(J^\perp) \to X$. Moreover, since $J^\perp$ is the biggest ideal in $A$ with the property that $J \cap J^\perp = \{0\}$ it follows that $\text{Prim}(J^\perp) = \text{Int}(\text{Prim}(A/J))$. In particular, if $A$ is a unital continuous $C_0(X)$-algebra then $J^\perp$ is a continuous $C(Y)$-algebra where $Y$ is given by (13). In terms of $C^*$-bundles, $J^\perp$ can be viewed as the algebra of continuous sections of the $C^*$-bundle $\mathcal{J}^\perp := \coprod_{x \in X} J(x)^\perp$ where $J(x)^\perp$ is contained in the annihilator of $J(x)$ in $A(x)$. In particular, $J(x)^\perp = \{0\}$ for all $x \notin Y$. 


4. Homomorphisms of $C_0(X)$-algebras and morphisms of $C^*$-bundles

In this section we introduce morphisms of upper semi continuous $C^*$-bundles which induce certain homomorphisms of $C_0(X)$-algebras. We show that the arising category of $C_0(X)$-algebras has quotients and direct limits. Also we describe reversible $C^*$-dynamical systems induced by morphisms.

4.1. Morphism of $C^*$-bundles. Let $\mathcal{A} = \bigsqcup_{x \in X} A(x)$ and $\mathcal{B} = \bigsqcup_{y \in Y} B(y)$ be upper semicontinuous $C^*$-bundles. We wish to view morphism between $C^*$-bundles as a common generalization of proper mappings and $C^*$-homomorphisms. Mimicking the definition of morphisms of vector bundles, one can imagine such a morphism as a pair of continuous mappings $\alpha : \mathcal{B} \to \mathcal{A}$ and $\varphi : X \to Y$ such that the following diagram

\[
\begin{array}{ccc}
\mathcal{B} & \xrightarrow{\alpha} & \mathcal{A} \\
\downarrow{\varphi} & & \downarrow{p} \\
Y & \xrightarrow{\varphi} & X
\end{array}
\]

commutes and for each $x \in X$, $\alpha : B(\varphi(x)) \to A(x)$ is a homomorphism. Since some of these homomorphisms might be zero we will allow $\varphi$ to be defined on an open subset $\Delta$ of $X$. We recall that $0_x$ stands for the zero in $A(x)$.

**Definition 4.1.** A morphism (of upper semicontinuous $C^*$-bundles) from $\mathcal{B}$ to $\mathcal{A}$ is a pair $(\varphi, \{\alpha_x\}_{x \in \Delta})$ consisting of

1) a continuous proper map $\varphi : \Delta \to Y$ defined on an open set $\Delta \subseteq X$, and
2) a continuous bundle of homomorphisms $\{\alpha_x\}_{x \in \Delta}$ between the corresponding fibers, vanishing at the boundary of $\Delta$. Namely, we require that
   a) for each $x \in \Delta$, $\alpha_x : B(\varphi(x)) \to A(x)$ is a homomorphism;
   b) if $\{x_i\}_{i \in \Lambda} \subseteq \Delta$ and $\{b_i\}_{i \in \Lambda} \subseteq \mathcal{B}$ are nets such that $x_i \to x$, $b_i \to b$ and $p(b_i) = \varphi(x_i)$, for $i \in \Lambda$, then $\alpha_{x_i}(b_i) \to \alpha_x(b)$ if $x \in \Delta$ and $\alpha_{x_i}(b_i) \to 0_x$ if $x \notin \Delta$.

The above definition is born to work well with section algebras.

**Proposition 4.2.** Let $\varphi : \Delta \to Y$ be a proper continuous mapping where $\Delta \subseteq X$ is an open set. For each $x \in \Delta$ let $\alpha_x : B(\varphi(x)) \to A(x)$ be a homomorphism. The pair $(\varphi, \{\alpha_x\}_{x \in \Delta})$ is a morphism from $\mathcal{B}$ to $\mathcal{A}$ if and only if the formula

\[
(14) \quad \alpha(b)(x) = \begin{cases} 
\alpha_x(b(\varphi(x)), & x \in \Delta, \\
0_x, & x \notin \Delta,
\end{cases} \quad b \in \Gamma_0(\mathcal{B}), \ x \in X,
\]

yields a well defined homomorphism $\alpha : \Gamma_0(\mathcal{B}) \to \Gamma_0(\mathcal{A})$ between the section $C^*$-algebras.

**Proof.** Suppose $(\varphi, \{\alpha_x\}_{x \in \Delta})$ is a morphism. To see that (14) defines a homomorphism it suffices to show that for any $b \in \Gamma_0(\mathcal{B})$ the mapping

\[
X \ni x \mapsto \alpha(b)(x) \in A(x) \subseteq \mathcal{A}
\]

is in $\Gamma_0(\mathcal{A})$. To see it is continuous consider a net $\{x_i\}$ in $\Delta$ such that $x_i \to x \in X$, and note that applying condition 2b) from Definition 4.1 to $b_i := b(\varphi(x_i))$ we get $\alpha(b)(x_i) \to \alpha(b)(x)$. Hence $\alpha(b) \in \Gamma(A)$ and in particular $X \ni x \mapsto ||\alpha(b)(x)|| \in \mathbb{C}$ is upper semicontinuous. In particular, for any $\varepsilon > 0$ the set $\{x \in X : ||\alpha(b)(x)|| \geq \varepsilon\}$ is closed, and actually it is compact, since

\[
\{x \in X : ||\alpha(b)(x)|| \geq \varepsilon\} = \{x \in \Delta : ||\alpha_x(b(\varphi(x)))|| \geq \varepsilon\} \subseteq \{x \in \Delta : ||b(\varphi(x))|| \geq \varepsilon\}
\]

and the latter set is compact as $\varphi$ is proper and $b$ vanishes at infinity. Thus $\alpha(b) \in \Gamma_0(\mathcal{A})$.

Conversely, assume that $\alpha : \Gamma_0(\mathcal{B}) \to \Gamma_0(\mathcal{A})$ is a homomorphism satisfying (14). We need to show condition 2a) in Definition 4.1. Let $\{x_i\}_{i \in \Lambda} \subseteq \Delta$ and $\{b_i\}_{i \in \Lambda} \subseteq \mathcal{B}$ be nets such that $x_i \to x$, $b_i \to b$ and $p(b_i) = \varphi(x_i)$. Take arbitrary $\varepsilon > 0$. By Lemma 3.1 there is $a \in \Gamma_0(\mathcal{A})$ such that
\[\|a(p(b)) - b\| < \varepsilon\] and we eventually have \[\|a(\varphi(x_i)) - b_i\| < \varepsilon.\] This implies that we eventually have
\[\|a(\varphi(x_i)) - b_i\| < \varepsilon.\]
If \(x \in \Delta\), then \[\|a(\varphi(x)) - \alpha_\varphi(x)(b)\| < \varepsilon\] and since \(\alpha(a) \in \Gamma_0(\mathcal{A})\) we have \(\alpha_\varphi(x)(b_i) \to \alpha_x(b)\) by Lemma 3.1. If \(x \notin \Delta\), then \[\|a(\varphi(x))\| = 0\] and hence \[\|a(\varphi(x_i))\| \to 0\] which implies that eventually \[\|\alpha_\varphi(x_i)(b_i)\| < \varepsilon.\] Accordingly, \(\alpha_\varphi(x_i)(b_i) \to 0.\)

**Definition 4.3.** To indicate that a homomorphism \(\alpha : \Gamma_0(\mathcal{B}) \to \Gamma_0(\mathcal{A})\) is given by (14) for a certain morphism \((\varphi, \{\alpha_\varphi\}_{x \in \Delta})\) of upper semicontinuous \(C^*\)-bundles we will say that \(\alpha\) induced by a morphism.

**Remark 4.4.** Let \(A = \Gamma_0(\mathcal{A})\) and \(B = \Gamma_0(\mathcal{B})\). Note that for an induced homomorphism \(\alpha : B \to A\) the underlying mapping \(\varphi : \Delta \to Y\) is uniquely determined by \(\alpha\) on the set \(\{x \in X : \alpha(B)(x) \neq 0\} \subseteq \Delta\), which coincides with \(\Delta\) when all endomorphisms \(\alpha_x, x \in \Delta\), are nonzero. Moreover, the set \(\{x \in X : \alpha(B)(x) \neq 0\} = \bigcup_{b \in B} \{x \in X : \|\alpha(b)(x)\| > 0\}\) is open when \(A\) is a continuous \(C_0(X)\)-algebra. Thus in the latter case we can always assume that \(\alpha_x\) is nonzero for each \(x \in \Delta\).

4.2. **Homomorphisms of \(C_0(X)\)-algebras.** Throughout \(A\) stands for a \(C_0(X)\)-algebra and \(B\) for a \(C_0(Y)\)-algebra. We have the following characterizations of homomorphism induced by morphisms phrased in terms of \(C_0(X)\)-algebras.

**Proposition 4.5.** For any homomorphism \(\alpha : B \to A\) the following conditions are equivalent
1. \(\alpha\) is induced by a morphism from \(B = \bigsqcup_{y \in Y} B(y)\) to \(A = \bigsqcup_{x \in X} A(x)\),
2. there is a homomorphism \(\Phi : C_0(Y) \to C_0(X)\) such that \(\alpha(f \cdot b) = \Phi(f) \cdot \alpha(b), f \in C_0(Y), b \in B\).

If additionally \(B\) is unital and \(A\) is a continuous \(C_0(X)\)-algebra then the above conditions are equivalent to the following one
3. \(\alpha\) maps \(C_0(Y)\) ‘almost into’ \(C_0(X)\), that is \(\alpha(C_0(Y) \cdot 1) \subseteq C_0(X) \cdot \alpha(1)\).

**Proof.** Implication (i) \(\Rightarrow\) (ii) is obtained by putting \(\Phi(a) := a \circ \varphi\) for \(a \in C_0(Y)\). To show (ii) \(\Rightarrow\) (i) assume that condition (ii) holds. Then there is a continuous proper mapping \(\varphi : \Delta \to Y\) defined on an open set \(\Delta \subseteq X\) such that
\[\Phi(b)(x) = \begin{cases} b(\varphi(x)), & x \in \Delta, \\ 0, & x \notin \Delta, \end{cases} \quad b \in C_0(Y)\]

(this is a general form of homomorphism from \(C_0(Y)\) to \(C_0(X)\)). Let \(x \in \Delta\). We define a homomorphism \(\alpha_x : B(\varphi(x)) \to A(x)\) as follows. For any \(b_0 \in B(\varphi(x))\) there is \(b \in B\) such that \(b(\varphi(x)) = b_0\), and we claim that the element
\[\alpha_x(b_0) := \alpha(b)(x)\]
is well defined (does not depend on the choice of \(b\)). Indeed, let \(\tilde{b}, b \in B\) be such that \(\tilde{b}(\varphi(x)) = b(\varphi(x)) = b_0\). Then \(b(\varphi(x)) - \tilde{b}(\varphi(x)) = 0.\) Upper semicontinuity of the \(C^*\)-bundle \(B = \bigsqcup_{y \in Y} B(y)\) imply that for every \(\varepsilon > 0\) there is an open neighbourhood \(U\) of \(\varphi(x)\) such that \(\|b(y) - \tilde{b}(y)\| < \varepsilon, \) for all \(y \in U\).

Let us choose a function \(h \in C_0(Y)\) such that \(h(\varphi(x)) = 1, 0 \leq h \leq 1\) and \(h(y) = 0\) outside \(U\). We get
\[\|\alpha(b)(x) - \alpha(\tilde{b})(x)\| = \|(\Phi(h)\alpha(b) - \Phi(h)\alpha(\tilde{b}))(x)\| = \|\alpha(hb - h\tilde{b})(x)\|\]
\[\leq \|\alpha(hb - h\tilde{b})\| \leq \|hb - h\tilde{b}\| \leq \varepsilon.\]
This proves our claim. Now it is straightforward to see that (16) gives the desired homomorphism $\alpha_x : B(\varphi(x)) \to A(x)$. Moreover, for the above defined pair $(\varphi, \{\alpha_x\}_{x \in \Delta})$ the formula (14) holds. Hence in view of Proposition 4.2, $\alpha$ is induced by a morphism.

Let us now assume that $B$ is a unital and $A$ is a continuous $C_0(X)$-algebra.

Implication (ii) $\implies$ (iii) is obvious. To prove (iii) $\implies$ (ii) suppose that $\alpha(C_0(Y) \cdot 1) \subseteq C_0(X) \cdot \alpha(1)$. Then for every $f \in C_0(Y)$ there exists $g \in C_0(X)$ such that

$$\alpha(f \cdot 1)(x) = g(x)\alpha(1)(x), \quad x \in X.$$  

Clearly, the function $g$ is uniquely determined by $f$ on the set $\Delta := \{x \in X : \alpha(B)(x) \neq 0\} = \{x \in X : \alpha(1)(x) \neq 0\}$. Since the mapping $X \ni x \to \|\alpha(1)(x)\| \in \{0, 1\}$ is continuous and vanishing at infinity, $\Delta$ is open and compact. Now it is straightforward to see that the formula $\Phi(f) = g|_\Delta$ defines a homomorphism $\Phi : C_0(Y) \to C(\Delta) \subseteq C_0(X)$ satisfying condition (ii).

Using the above characterizations one can give a number of examples of homomorphisms induced by morphisms.

**Example 4.6.** Every $C_0(X)$-linear homomorphism between $C_0(X)$-algebras is induced by a morphism. In particular, any $C^*$-algebras $A$, $B$ may considered $\mathbb{C}$-algebras ($C_0(X)$-algebras where $X$ is a singleton), and then any homomorphism $\alpha : A \to B$ is induced by a morphism.

**Example 4.7** ($C^*$-algebras with Hausdorff primitive ideal space). If $A$ is a $C^*$-algebra and its primitive ideal space $X := \text{Prim}(A)$ is Hausdorff, then using Dauns-Hofmann isomorphism we may naturally treat $A$ as a continuous $C_0(X)$-algebra where the structure map $\sigma_A$ is identity, cf. [BK04, 2.2.2]. In particular, we then have $A(x) = A/x$ for $x \in X = \text{Prim}(A)$, and if $A$ is unital, an endomorphism $\alpha : A \to A$ is induced by a morphism if and only if $\alpha(Z(A)) \subseteq Z(A)\alpha(1)$.

**Example 4.8** (quotient maps). Suppose $q_I : A \to A/I$ is a quotient map and $A$ is a $C_0(X)$-algebra. Treating $A/I$ as a $C_0(Y)$-algebra with $Y = \sigma_A(\text{Prim}(A/I))$, cf. Lemma 3.7, we have

$$q_I(f \cdot a) = f|_Y \cdot q_I(a), \quad f \in C_0(X), \ a \in A.$$  

Hence condition (ii) in Proposition 4.5 is satisfied. In particular, $q_I$ is induced by the morphism $(id, \{q_{I,x}\}_{x \in Y})$ where $q_{I,x} : A(x) \to A(x)/I(x), x \in Y$, are the quotient maps.

**Example 4.9** (quotient $C^*$-dynamical systems). Suppose $(A, \alpha)$ is a $C^*$-dynamical system where $A$ is a $C_0(X)$-algebra and $\alpha$ is induced by a morphism $(\varphi, \{\alpha_x\}_{x \in \Delta})$. Let $I$ be a positive invariant ideal in $(A, \alpha)$ and put $Y = \sigma_A(\text{Prim}(A/I))$. As in Example 4.8 we treat $A/I$ as a $C_0(Y)$-algebra. We additionally assume that $Y$ is positive invariant in the sense that $\varphi(Y \cap \Delta) \subseteq Y$ (in Lemma 4.10 below we give sufficient conditions for this to hold). We claim that the quotient endomorphism $\alpha_I : A/I \to A/I$ is induced by the morphism $(\varphi|_{Y \cap \Delta}, \{\alpha_{I,x}\}_{x \in Y \cap \Delta})$ where

$$\alpha_{I,x}(a + I(\varphi(x))) := \alpha_x(a) + I(x), \quad a \in A(\varphi(x)), \ x \in Y \cap \Delta.$$  

Indeed, it is clear, that $\alpha_{I,x}$ is a well defined homomorphism from $A(\varphi(x))/I(\varphi(x))$ to $A(x)/I(x)$. Moreover, since $Y$ is $\varphi$-invariant the homomorphism $\Phi : C_0(X) \to C_0(Y)$ given by $\varphi : \Delta \to X$, see formula (15), factors thorough to the homomorphism $\Phi_Y : C_0(Y) \to C_0(Y)$ given by the restriction of $\varphi$ to $Y \cap \Delta$. For any $f \in C_0(X)$ and $a \in A$, we have

$$\alpha_I(f|_Y \cdot q_I(a)) = \alpha_I(q_I(f \cdot a)) = q_I(\alpha(f \cdot a)) = q_I(\Phi(f) \cdot \alpha(a)) = \Phi(f)|_Y \cdot q_I(\alpha(a)) = \Phi_Y(f|_Y \cdot q_I(\alpha(a))).$$  

Hence by Proposition 4.5, $\alpha_I$ is induced by a morphism, and clearly this morphism is as described above.

**Lemma 4.10.** Suppose that $\alpha : A \to A$ is an endomorphism induced by a morphism $(\varphi, \{\alpha_x\}_{x \in \Delta})$ such that for each $x \in \Delta$ the range of $\alpha_x$ is a full subalgebra of $A(x)$. Then for any positive invariant ideal $I$ in $(A, \alpha)$ the set $Y = \sigma_A(\text{Prim}(A/I))$ is positive invariant, i.e. $\varphi(Y \cap \Delta) \subseteq Y$. 

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Proof. Put $Y_0 = \sigma_A(\text{Prim}(A/I))$ and recall that $x \in Y_0$ if and only if $I(x) \neq A(x)$, see (13). We now let $x \in \Delta \cap Y_0$ and claim that $\varphi(x) \in Y_0$. Indeed, assume on the contrary that $I(\varphi(x)) = A(\varphi(x))$. Then by positive invariance of $I$ we have $\alpha_x(A(\varphi(x))) = \alpha_x(I(\varphi(x)) \subseteq I(x)$. Hence by fullness of $\alpha_x(A(\varphi(x)))$ in $A(x)$ we get

$$I(x) = A(x)I(x)A(x) \supseteq A(x)\alpha_x(A(\varphi(x)))A(x) = A(x),$$

which contradicts the fact that $x \in Y_0$, cf. (13). Accordingly, $\varphi(Y_0 \cap \Delta) \subseteq Y_0$ and by continuity of $\varphi$ we get $\varphi(Y \cap \Delta) \subseteq Y$. \hfill \Box

4.3. Extendible morphisms and reversible $C^*$-dynamical systems. In the foregoing statement we use the description of multiplier algebras given in Proposition 3.6.

**Lemma 4.11.** Suppose $A$ is a $C_0(X)$-algebra, $B$ is a $C_0(Y)$-algebra, and $\alpha : B \to A$ is an extendible homomorphism induced by a morphism $(\varphi, \{\alpha_x\}_{x \in \Delta})$. Each $\alpha_x : B(\varphi(x)) \to A(x)$, $x \in \Delta$, is extendible and $\overline{\alpha} : M(B) \to M(A)$ is given by

$$\overline{\alpha}(m)(x) = \begin{cases} \overline{\alpha_x}(m(\varphi(x))), & x \in \Delta, \\ 0_x, & x \notin \Delta, \end{cases} m \in M(B), \ x \in X.$$

If additionally $A$ has local units and the homomorphisms $\alpha_x$, $x \in \Delta$, are non-zero, then $\Delta$ is clopen.

**Proof.** Let $\{\mu_\lambda\}$ be an approximate unit in $B$ and let $x \in \Delta$. It is immediate that $\{\mu_\lambda(\varphi(x))\}$ is an approximate unit in $B(\varphi(x))$ and $\alpha_x(\mu_\lambda(\varphi(x))) = \alpha(\mu_\lambda)(x)$ converges strictly in $A(x)$. Hence the homomorphisms $\alpha_x$, $x \in \Delta$, are extendible. Recall that $\overline{\alpha}$ is determined by the formula $\overline{\alpha}(m)a = \lim_\lambda \alpha(m\mu_\lambda)a$, where $a \in A$, $m \in M(B)$. It follows that for any $x \in \Delta$ we have

$$\overline{\alpha}(m)a)(x) = \lim_\lambda \alpha(m\mu_\lambda)(x)a(x) = \lim_\lambda \alpha_x((m\mu_\lambda)(\varphi(x)))a(x)$$

$$= \lim_\lambda \alpha_x(m(\varphi(x))\mu_\lambda(\varphi(x)))a(x) = \overline{\alpha_x}(m(\varphi(x)))a(x).$$

Thus we get (17).

Now suppose that $A$ has local units and all $\alpha_x$, $x \in \Delta$, are non-zero. Assume that there is a point $x_0$ in the boundary of the set $\Delta$. Choose $a \in A$ such that $a(\varphi(x)) = 1_x$ is the unit in $A(x)$ for every $x$ in a neighbourhood $U$ of $x_0$. Then the compact set $\{x \in X : \|\alpha(1)a(x)\| \geq 1/2\}$ contains $\Delta \cap U$ but does not contain $x_0$. This leads to a contradiction, since $x_0$ is in the closure of $\Delta \cap U$. \hfill \Box

Suppose now that $(A, \alpha)$ is a reversible $C^*$-dynamical system where $A$ is a $C_0(X)$-algebra and $\alpha$ is induced by a morphism $(\varphi, \{\alpha_x\}_{x \in \Delta})$. Then the range $\alpha(A) = \overline{\alpha}(1)A\overline{\alpha}(1)$ is a corner in $A$ determined by the multiplier

$$\overline{\alpha}(1)(x) = \begin{cases} \overline{\alpha_x}(1_{\varphi(x)}), & x \in \Delta, \\ 0_x, & x \notin \Delta, \end{cases} \ x \in X.$$

In general all we can say about the kernel of $\alpha$ and its annihilator is that

$$\ker \alpha = \{a \in A : a(y) \in \bigcap_{x \in \varphi^{-1}(y)} \ker \alpha_x \text{ for all } y \in \varphi(\Delta)\}$$

and $(\ker \alpha)^\perp$ is contained in

$$\{a \in A : a|_{X \setminus \varphi(\Delta)} = 0 \text{ and } a(y) \in \bigcap_{x \in \varphi^{-1}(y)} (\ker \alpha_x)^\perp \text{ for } y \in \varphi(\Delta)\}.$$  

Nevertheless, we have the following statement.

**Proposition 4.12.** Suppose that $(A, \alpha)$ is a reversible $C^*$-dynamical system where $A$ is a $C_0(X)$-algebra and $\alpha$ is induced by a morphism $(\varphi, \{\alpha_x\}_{x \in \Delta})$. 

(i) If all of the endomorphisms $\alpha_x$, $x \in \Delta$, are injective then $\varphi$ is injective.

(ii) If $\varphi$ is injective then the unique regular transfer operator for $(A, \alpha)$ is determined by the formula

$$\alpha_x(a)(x) = \begin{cases} 
\alpha_x(a(\varphi^{-1}(x))), & x \in \varphi(\Delta), \\
0, & x \notin \varphi(\Delta),
\end{cases} \quad a \in A, \ x \in X,$$

where for each $x \in \varphi(\Delta)$, $\alpha_{x, \varphi} : A(\varphi^{-1}(x)) \to A(x)$ is a completely positive generalized inverse to $\alpha_{x^{-1}} : A(x) \to A(\varphi^{-1}(x))$ (the homomorphisms $\alpha_{x^{-1}}$ has complemented kernel and corner range). The mappings $\alpha_{x, \varphi}$ have strictly continuous extensions $\overline{\alpha}_{x, \varphi}$ and the strictly continuous extension $\overline{\alpha}_{x}$ of $\alpha_{x}$ is given by the formula

$$\overline{\alpha}_{x}(a)(x) = \begin{cases} 
\overline{\alpha}_{x}(a(\varphi^{-1}(x))), & x \in \varphi(\Delta), \\
0, & x \notin \varphi(\Delta),
\end{cases} \quad a \in M(A), \ x \in X,$$

where we use the description of multipliers given in Proposition 3.6.

Proof. (i). Injectivity of $\alpha_x$'s imply that $\ker \alpha = \{a \in A : a(x) = 0 \text{ for all } x \in \varphi(\Delta)\}$ and therefore $(\ker \alpha)^\perp \subseteq \{a \in A : a(x) = 0 \text{ for all } x \notin \varphi(\Delta)\}$. Let $x, y \in \Delta$ be two different points. Take any $b \in \ker(\alpha)$ such that $b(x) \neq 0$ and any $h \in C_0(X)$ such that $h(x) = 1$ and $h(y) = 0$. Since $\alpha(A) = \overline{\alpha}(1) \ker \overline{\alpha}(1)$ we see that $c := bh$ is in $\ker(\alpha)$. Obviously, $c(x) \neq 0$ and $c(y) = 0$. Thus for any $a \in \ker(\alpha)$ we have $\alpha_x(a(\varphi(x))) \neq 0$ and $\alpha_y(a(\varphi(y))) = 0$. Thus by injectivity of $\alpha_x$ and $\alpha_y$ we get $\varphi(x) \neq \varphi(y)$. Hence $\varphi$ is injective.

(ii). Fix $x \in \Delta$ and $b_0 \in \ker(\overline{\alpha}_{x}(1))A(x)\overline{\alpha}_{x}(1)$, Then there is $b \in \alpha_{x}(A) = \overline{\alpha}_{x}(1)A\overline{\alpha}_{x}(1)$ such that $b(x) = b_0$. Hence there is a unique $a \in (\ker \alpha)^\perp$ such that $\alpha(a) = b$. Accordingly, $b_0 = b(x) = \alpha_x(a(\varphi(x)))$ where $a(\varphi(x)) \in (\ker \alpha_x)^\perp$ (here we use injectivity of $\varphi$ and that $(\ker \alpha_x)^\perp$ is contained in the set (18)). It follows that the range of $\alpha_x : A(\varphi(x)) \to A(x)$ is the corner $\overline{\alpha}_{x}(1)A(x)\overline{\alpha}_{x}(1)$ and $\alpha_x : (\ker \alpha_x)^\perp \to \alpha_x(A(\varphi(x)))$ is an isomorphism. The latter fact implies that the kernel $\ker \alpha_x$ is complemented in $A(\varphi(x))$. We may define the map

$$\alpha_{x, \varphi}(a) := \alpha_x^{-1}(\overline{\alpha}_{x}(1)\alpha(\varphi(x)))^{-1} \overline{\alpha}_{x}(1)\varphi(x), \quad a \in A(x),$$

where $\alpha_x^{-1}$ is the inverse to the isomorphism $\alpha_x : (\ker \alpha_x)^\perp \to \overline{\alpha}_{x}(1)A(x)\overline{\alpha}_{x}(1)$. Clearly, the maps have $\alpha_{x, \varphi}$ strictly continuous extensions which are given by (20) with $\alpha_x^{-1}$ replaced by the inverse to the strictly continuous isomorphism $\overline{\alpha}_{x} : M((\ker \alpha_x)^\perp) \to \overline{\alpha}_{x}(1)A(x)\overline{\alpha}_{x}(1)$, cf. Lemma 4.11. Now one readily sees that the homomorphisms $\alpha_{x, \varphi}$ fulfill the requirements of the assertion. \hfill $\square$

Injectivity of the map $\varphi$ in the above proposition is essential.

Example 4.13. Consider a reversible $C^*$-dynamical system $(A, \alpha)$ where $A = C^3$ and $\alpha(a) = (a_1, a_2, a_3)$ for $a = (a_1, a_2, a_3) \in A$. Then the regular transfer operator $\alpha(a) = (0, a_1, a_3)$ for $(A, \alpha)$ is actually an endomorphism. Treating $A$ as a $C(\{1, 2\})$-algebra where $a(1) = a_1 \in C$ and $a(2) = (a_2, a_3) \in C^2$, for $a \in A$, the endomorphism $\alpha$ is induced by the morphism $\varphi(\{a_1, a_2\})$ where

$$\varphi(1) = \varphi(2) = 2, \quad \alpha_1(a_2, a_3) = a_2, \quad \alpha_2(a_2, a_3) = (0, a_3).$$

But $\alpha_*$ is not induced by a morphism because the fiber $\alpha_*(a)(2) = (a_1, a_3)$ of $\alpha_*(a)$ depends on two fibers of $a$.

4.4. Direct limits. Let us consider a direct sequence $B_0 \xrightarrow{\alpha_0} B_1 \xrightarrow{\alpha_1} \ldots$, where for each $n \in \mathbb{N}$, $B_n$ is a $C_0(X_n)$-algebra and $\alpha_n : B_n \to B_{n+1}$ is a homomorphism induced by a morphism $(\varphi_n, \{x_n\}_{x_n \in X_{n+1}})$ (we may assume that the sets $X_n$ are disjoint, so a homomorphism $\alpha_n$ is uniquely determined by the point $x \in X_{n+1}$). We show that the $C^*$-algebraic direct limit $B := \lim \{B_n, \alpha_n\}$ is naturally a $C_0(\overline{\Delta})$-algebra over the the topological inverse limit space $\overline{\Delta} := \lim \{X_{n+1}, \varphi_n\}$.
To this end, we attach to each \( \bar{x} = (x_0, x_1, \ldots) \in \bar{X} \) the direct limit \( C^* \)-algebra

\[
B(\bar{x}) := \lim \{B_n(x_n), \alpha_{x_{n+1}}\}
\]

of the direct sequence \( B_0(x_0) \xrightarrow{\alpha_{x_0}} B_1(x_1) \xrightarrow{\alpha_{x_2}} B_2(x_2) \xrightarrow{\alpha_{x_3}} \ldots \). We let \( \phi_{\bar{x}, n} : B_n(x_n) \to B(\bar{x}) \) and \( \phi_n : B_n \to B \) be the natural homomorphisms:

\[
\phi_n(b_n) = \left[0, \ldots, 0, b_n, \alpha_{n+1}(b_n), \ldots\right],
\]

\[
\phi_{\bar{x}, n}(b_n) = \left[0, \ldots, 0, b_n(x_n), \alpha_{x_{n+1}}(b_n(x_n)), \ldots\right]
\]

where \( b_n \in B_n \) and \( \bar{x} \in \bar{X} \). The following statement can be viewed as a generalization of [HRW07, Proposition 1.7] to the non-unital case. In contrast to [HRW07] we prove it using the \( C^* \)-bundle approach.

**Proposition 4.14.** Retain the above notation and suppose additionally that the mappings \( \varphi_n \) are surjective, \( n \in \mathbb{N} \). There is a unique topology on \( B = \bigsqcup_{\bar{x} \in \bar{X}} B(\bar{x}) \) making it an upper semicontinuous \( C^* \)-bundle over \( \bar{X} \) such that

\[
\text{(21)} \quad B \ni \phi_n(b_n) \mapsto \phi_n(b_n)(\bar{x}) := \phi_{\bar{x}, n}(b_n(x_n)), \quad \bar{x} \in \bar{X}, b_n \in B_n,
\]

establishes the natural isomorphism from \( B = \lim \{B_n, \alpha_n\} \) onto \( \Gamma_0(B) \).

If additionally, all the \( C^* \)-algebras \( B_n \) are continuous \( C_0(X_n) \)-algebras and all the endomorphisms \( \alpha_x, x \in X_{n+1}, n \in \mathbb{N} \), are injective, then the \( C^* \)-bundle \( B = \bigsqcup_{\bar{x} \in \bar{X}} B(\bar{x}) \) is continuous.

**Proof.** For \( \bar{x} \in \bar{X} \) and \( m > n \), we put

\[
\alpha_{\bar{x}, [n, m]} := \alpha_m \circ \cdots \circ \alpha_{x_{n+2}} \circ \alpha_{x_{n+1}} \quad \text{and} \quad \alpha_{[n, m]} := \alpha_{m-1} \circ \cdots \circ \alpha_{n+1} \circ \alpha_n.
\]

These are the bonding homomorphisms from \( B(x_n) \) to \( B(x_m) \) and from \( B_n \) to \( B_m \), respectively. Let \( \bar{x} \in \bar{X}, b_n \in B_n \). To check that the map (21) is well defined assume that \( \phi_n(b_n) = 0 \). Then for any \( \varepsilon > 0 \) and sufficiently large \( m \) we have \( \|\alpha_{[n, m]}(b_n)\| < \varepsilon \), and all the more \( \|\alpha_{\bar{x}, [n, m]}(b_n(x_n))\| = \|\alpha_{[n, m]}(b_n(x_m))\| < \varepsilon \). This implies that \( \phi_{\bar{x}, n}(b_n(x_n)) = 0 \). Hence (21) is well defined and it is straightforward to see that it yields a surjective homomorphism from \( B \) onto \( B(\bar{x}) \).

We show that for a fixed \( \phi_n(b_n) \in B \), the mapping

\[
\bar{X} \ni \bar{x} \mapsto \|\phi_n(b_n)(\bar{x})\| \in \mathbb{C}
\]

is upper semicontinuous. Suppose that \( \bar{x} \in \bar{X} \) is such that \( \|\phi_n(b_n)(\bar{x})\| < K \). Then there is \( m > n \) such that \( \|\alpha_{\bar{x}, [n, m]}(b_n(x_n))\| < K \). Since \( \alpha_{[n, m]}(b_n(x_m)) = \alpha_{\bar{x}, [n, m]}(b_n(x_n)) \) and \( X_m \ni x \mapsto \|\alpha_{[n, m]}(b_n(x))\| \) is upper semicontinuous, there is an open neighborhood \( U \) of \( x_m \) such that \( \|\alpha_{[n, m]}(b_n(x))\| < K \) for all \( x \in U \). It follows that the set

\[
\bar{U} := \{\bar{y} = (y_0, y_1, \ldots) \in \bar{X} : y_m \in U\}
\]

is an open neighborhood of \( \bar{x} \) such that for \( \bar{y} \in \bar{U} \) we have

\[
\|\phi_n(b_n)(y)\| \leq \|\alpha_{\bar{y}, [n, m]}(b_n(y_n))\| < K.
\]

This proves the upper semicontinuity of (22).

We wish to show that (22) vanishes at infinity. Let \( \varepsilon > 0 \). By upper semicontinuity of (22) the set \( \{\bar{x} \in \bar{X} : \|\phi_n(b_n)(\bar{x})\| \geq \varepsilon\} \) is closed, and clearly, it is a subset of \( \{\bar{x} \in \bar{X} : \|b_n(x_n)\| \geq \varepsilon\} \). However, the latter set is compact because the map \( \bar{X} \ni \bar{x} \mapsto x_n \in X_n \) is proper and \( X_n \ni x \mapsto \|b_n(x)\| \) is vanishing at the infinity. Hence \( \{\bar{x} \in \bar{X} : \|\phi_n(b_n)(\bar{x})\| \geq \varepsilon\} \) is compact as well.

Now, by Fell’s theorem, see [Wil07, Theorem C.25], there is a unique topology on \( B \) such that (21) defines a surjective homomorphism from \( B \) onto \( \Gamma_0(B) \). We still need to show that this homomorphism is injective.
To this end, assume that $\phi_n(b_n)$ is non-zero. Then there exists $\varepsilon > 0$ such that $\|\alpha_{[m,n]}(b_n)\| > \varepsilon$ for all $m > n$. Thus, for each $m > n$, the set

$$D_m := \{x \in X_m : \|\alpha_{[m,n]}(b_n)(x)\| \geq \varepsilon\}$$

is nonempty, and it is compact because $X_m \ni x \to \|\alpha_{[m,n]}(b_n)(x)\|$ vanishes at infinity. Note that $\varphi_m(D_{m+1}) \subseteq D_m$. Thus the sets

$$\tilde{D}_m := \{\tilde{x} \in \tilde{X} : x_m \in D_m\}$$

form a decreasing sequence of compact nonempty sets (nonemptiness follows from surjectivity of the mappings $\varphi_m$). Hence there is $\tilde{x}_0 \in \bigcap_{m>n} \tilde{D}_m$ and plainly

$$\|\phi_n(b_n)(\tilde{x}_0)\| \geq \varepsilon > 0.$$  

This finishes the proof of the first part of the assertion.

Assume now that for each $n \in \mathbb{N}$, $B_n$ is a continuous $C_0(X_n)$-algebra and all of the endomorphisms $\alpha_x$, $x \in X_n$, are injective. Then $\|\phi_n(b_n)(\tilde{x})\| = \|b_n(x_n)\|$ for all $\tilde{x} \in \tilde{X}$, $b_n \in B_n$, $n \in \mathbb{N}$. Hence mapping (22), as a composition of two continuous mappings $\tilde{X} \ni \tilde{x} \to x_n \in X_n$ and $X_n \ni x_n \to \|b_n(x_n)\|$, is continuous.

Injectivity of the endomorphisms $\alpha_x$, $x \in X_{n+1}$, in the second part of the above assertion is essential.

**Example 4.15.** Consider the stationary inductive limit given by the continuous $C_0(\mathbb{N})$-algebra $A := C_0(\mathbb{N}, \mathbb{C}^2)$ and the endomorphism $\alpha : A \to A$ induced by a morphism $(\varphi, \{\alpha_n\}_{n \in \mathbb{N}})$ where

$$\phi(0) = 0, \quad \alpha_0 = \text{id}, \quad \text{and} \quad \phi(n) = n - 1, \quad \alpha_n(a, b) = (a, 0), \quad \text{for } n > 0.$$  

The resulting direct limit $B = \lim \{A, \alpha\}$ can be viewed as a $C_0(\{-\infty\} \cup \mathbb{Z})$-algebra with the obvious topology on $\{-\infty\} \cup \mathbb{Z}$, and fibers $B_{-\infty} = \mathbb{C}^2$ and $B_0 = \mathbb{C}$, $n \in \mathbb{Z}$. The image of the constant function $\mathbb{N} \ni n \to (a, b) \in \mathbb{C}^2$ (treated as an element of $A$) in the algebra $B$ corresponds to the section $f$ with $f(-\infty) = (a, b)$ and $f(n) = a$ for $n \in \mathbb{Z}$. If $|a| < |b|$, the function $\{-\infty\} \cup \mathbb{Z} \ni x \to \|f(x)\|$ is not lower semicontinuous at $-\infty$.

5. Crossed Products of Reversible $C^*$-Dynamical System on $C_0(X)$-Algebras

Throughout this subsection we consider a reversible $C^*$-dynamical system $(A, \alpha)$ induced by a morphism $(\varphi, \{\alpha_x\}_{x \in \Delta})$ with injective $\varphi$ (we can always set up this structure, cf. Example 4.6). Thus the unique regular transfer operator $\alpha_\ast$ for $(A, \alpha)$ is of the form described in Proposition 4.12.

5.1. Description of Ideal Structure. The partial homeomorphism $\hat{\alpha} : \alpha(A) \to \alpha_\ast(A)$ dual to the endomorphism $\alpha : A \to A$, cf. Definition 2.21, factors through to the partial homeomorphism of the primitive ideal space $\text{Prim}(A)$. We denote the latter mapping by the same symbol $\hat{\alpha} : \text{Prim}(\alpha(A)) \to \text{Prim}(\alpha_\ast(A))$. Thus we have $\hat{\alpha}(\ker \pi) = \ker \hat{\alpha}(\pi)$ for $\pi \in \alpha(A)$. With the identifications $\text{Prim}(\alpha(A)) = \{P \in \text{Prim}(A) : \alpha(A) \not\subseteq P\}$ and $\text{Prim}(\alpha_\ast(A)) = \{P \in \text{Prim}(A) : \alpha_\ast(A) \not\subseteq P\}$ we have

$$\hat{\alpha}(P) = \alpha^{-1}(P) = \alpha_\ast(P) + (1 - \varpi_\ast(1))A, \quad P \in \text{Prim}(\alpha(A)).$$

The partial dynamical system $(\text{Prim}(A), \hat{\alpha})$ is an extension of $(X, \varphi)$:

**Lemma 5.1.** With the above notation the following diagram

$$\begin{array}{ccc}
\text{Prim}(\alpha(A)) & \xrightarrow{\hat{\alpha}} & \text{Prim}(\alpha_\ast(A)) \\
\sigma_A \downarrow & & \sigma_A \\
\Delta & \xrightarrow{\varphi} & \varphi(\Delta)
\end{array}$$
commutes. In particular, if \( \varphi \) is free then \( \hat{\alpha} \) is free, and if \( A \) is a continuous \( C_0(X) \)-algebra and \( \varphi \) is topologically free, then \( \hat{\alpha} \) is also topologically free.

**Proof.** Using, among other things, equality (12) we see that

\[
\sigma_A(\text{Prim}(\alpha(A))) = \sigma_A(\text{Prim}(\alpha \alpha(A)A)) = \{ x \in (\alpha \alpha(A)A)(x) \neq 0 \} = \{ x \in X : \alpha(A)(x) \neq 0 \} \subseteq \Delta,
\]

\[
\sigma_A(\text{Prim}(\alpha_s(A))) = \{ x \in \alpha_s(A)(x) \neq 0 \} \subseteq \varphi(\Delta).
\]

Now let \( P \in \text{Prim}(\alpha(A)) \). Then \( x := \sigma_A(P) \) is in \( \Delta \). By (11), \( C_0(\Delta \setminus \{ x \}) \cdot A \subseteq P \). Applying to this inclusion \( \alpha_s \) we get \( C_0\left( \varphi(\Delta) \setminus \{ \varphi(x) \} \right) \cdot \alpha_s(A) \subseteq \alpha_s(P) \). Since \( A = \alpha_s(A) + (1 - \pi_s(1))A \)
we conclude that

\[
C_0\left( \varphi(\Delta) \setminus \{ \varphi(x) \} \right) \cdot A \subseteq \alpha_s(P) + (1 - \pi_s(1))A
\]

which in view of (23) and (11) means that \( \sigma_A(\hat{\alpha}(P)) = \varphi(x) \).

**Proposition 5.2.** Suppose that the map \( \varphi \) is free (has no periodic points). Then all the ideals in \( C^*(A, \alpha) \) are gauge-invariant. In particular, we have a one-to-one correspondence between the ideals \( I \) in \( C^*(A, \alpha) \) and invariant ideals \( I \) in \( (A, \alpha) \) where \( I = \mathcal{I} \cap A \). Moreover, we have

\[
C^*(A, \alpha)/I \cong C^*(A/I, \alpha_I) \quad \text{where} \quad \mathcal{I} \subseteq C^*(A, \alpha) \text{ and } I = A \cap \mathcal{I}.
\]

If \( A \) has the ideal property then \( C^*(A, \alpha) \) has the ideal property.

**Proof.** By Lemma 5.1 the mapping \( \hat{\alpha} \) is free on the level of \( \text{Prim}(A) \) and hence all the more on the level of \( \hat{A} \). Hence it is enough to apply Theorem 2.22(ii), see also Proposition 2.19, or [Kwa, Corollary 4.24]. Now assume that \( A \) has the ideal property. Since any ideal \( \mathcal{I} \subseteq C^*(A, \alpha) \) is generated by \( I = A \cap \mathcal{I} \), denoting by \( P(\mathcal{I}) \) and \( P(I) \) respectively the sets of projections in \( \mathcal{I} \) and \( I \), we see that the ideal generated by \( P(\mathcal{I}) \) in \( C^*(A, \alpha) \) contains

\[
C^*(A, \alpha)P(\mathcal{I})C^*(A, \alpha) = C^*(A, \alpha)AP(\mathcal{I})AC^*(A, \alpha) = C^*(A, \alpha)IC^*(A, \alpha) = \mathcal{I}.
\]

Hence \( C^*(A, \alpha)P(\mathcal{I})C^*(A, \alpha) = \mathcal{I} \) which shows the ideal property for \( C^*(A, \alpha) \).

5.2. **Pure infiniteness criterion.** Now, we take up our route to conditions ensuring pure infiniteness of \( C^*(A, \alpha) \). A technical device introduced in the following lemma will allow us to adapt the arguments in [ELQ02] to our setting. Recall that any \( C^*-\)algebra \( B \) is a \( M(B) \)-bimodule where \((m \cdot b) := mb \) and \((b \cdot m) := (mb)^* \), for \( m \in M(B), b \in B \).

**Lemma 5.3.** The action of \( h \in C_0(X) \) on \( A \) as a multiplier of \( A \) extends to the action on \( C^*(A, \alpha) \) as a multiplier of \( C^*(A, \alpha) \) which is uniquely determined by the formulas

\[
h \cdot (a^n) := (h \cdot a) u^n, \quad (aa^n(b)u^n) \cdot h := au^n(b \cdot h) = aa^n(b \cdot h)u^n,
\]

where \( a, b \in A, n \in \mathbb{N} \).

**Proof.** Recall that \( A \) is a non-degenerate subalgebra of \( C^*(A, \alpha) \). In other words multiplication from the left defines a non-degenerate homomorphism from \( A \) into \( M(C^*(A, \alpha)) \). This homomorphisms extends uniquely to the homomorphism from \( M(A) \) into \( M(C^*(A, \alpha)) \). Composing the latter with \( \mu_A : C_0(X) \to Z(M(A)) \) we get a multiplier action of \( C_0(X) \) on \( C^*(A, \alpha) \) that clearly satisfies (24). In view of the second part of Proposition 2.12 formulas (24) determine this action uniquely.

In the following statement we use the \( C_0(X) \)-bimodule structure on \( C^*(A, \alpha) \) described in the previous lemma (to increase readability we will suppress the symbol \( \cdot \)). We also use the conditional expectation \( E : C^*(A, \alpha) \to A \) described in Proposition 2.12.

**Lemma 5.4** (cf. Proposition 2.4 in [ELQ02]). Suppose that either \( \varphi \) is topologically free and \( A \) is a continuous \( C_0(X) \)-algebra, or that \( \varphi \) is free. Then for every \( a \in C^*(A, \alpha) \) and every \( \varepsilon > 0 \) there is \( h \in C_0(X) \) such that
that

\[ x \in V \] such that \( x_0 \) is not a fixed point for \( \varphi^k \) for all \( k = 1, \ldots, n \). Indeed, if \( \varphi \) is free existence of such \( x_0 \) is obvious. Otherwise, \( A \) is a continuous \( C_0(X) \)-algebra, hence \( V \) is open, and the existence of \( x_0 \) is guaranteed by topological freeness of \( \varphi \). Applying [ELQ02, Lemma 2.3] (whose proof works in our case; it suffices to replace the partial crossed product convolution formula with (24)) we see that for each \( k = \pm 1, \ldots, \pm n \) there exists \( h_k \in C_0(X) \) such that

\[
h_k(x_0) = 1, \quad \|h_k(a_k u^{|k|})h_k\| \leq \frac{\varepsilon}{2^n}, \quad \text{and } 0 \leq h_k \leq 1.
\]

Let \( h := \prod_{k=\pm 1, \ldots, \pm n} h_k \). Then (iii) is immediate, and (i) holds because \( \|ha_0h\| \geq \|a_0(x_0)\| > \|a_0\| - \varepsilon \). For (ii), we have

\[
\|ha_0h - hah\| \leq \sum_{k=\pm 1, \ldots, \pm n} \|h(a_k u^{|k|})h\| \leq \sum_{k=\pm 1, \ldots, \pm n} \|h_k(a_k u^{|k|})h_k\| < \varepsilon.
\]

\[ \square \]

Applying Lemma 5.4 and a standard reasoning (see for instance the proof of [ELQ02, Theorem 2.6]) we get the following corollary. This statement also follows from Theorem 2.22 combined with Lemma 5.1.

**Corollary 5.5.** Suppose that either \( A \) is a continuous \( C_0(X) \)-algebra and \( \varphi \) is topologically free, or that \( \varphi \) is free. A representation of the crossed product \( C^*(A, \alpha) \) is faithful if and only if it is faithful on \( A \).

In the case of real rank zero algebras we have the following lemma, which might be known to experts.

**Lemma 5.6.** Let \( A \subseteq B \) be \( C^* \)-algebras and let \( A \) be of real rank zero. The following conditions are equivalent

(i) Every non-zero positive element in \( A \) is properly infinite in \( B \).

(ii) Every non-zero projection in \( A \) is properly infinite in \( B \).

In particular, \( A \) is purely infinite if and only if every non-zero projection in \( A \) is properly infinite in \( A \).

**Proof.** Implication (i)\( \Rightarrow \) (ii) is trivial. Assume that (ii) holds and let \( a \in A \) be a non-zero positive element. By [BP91, Theorem 2.6] there is an approximate unit \( \{p_\lambda : \lambda \in \Lambda\} \) in \( a^\prime A a^\prime \) consisting of projections. Thus, by [KR00, Proposition 2.7(i)], \( p_\lambda \precsim a \) for all \( \lambda \), in \( A \) and all the more in \( B \). Applying [KR00, Lemma 3.17(ii)] we see that \( \{p_\lambda : \lambda \in \Lambda\} \subseteq J(a) := \{x \in B : a \oplus |x| \precsim a\} \). Thus \( B\{p_\lambda : \lambda \in \Lambda\}B \subseteq J(a) \) because \( J(a) \) is an ideal, see [KR00, Lemma 3.12(i)]. On the other hand \( J(a) \subseteq BaB \) by [KR00, Lemma 3.12(ii)] and since we clearly have \( BaB \subseteq B\{p_\lambda : \lambda \in \Lambda\}B \) it follows that \( J(a) = BaB \). Hence [KR00, Lemma 3.12(iv)] tells us that \( a \) is properly infinite in \( B \). \[ \square \]

**Remark 5.7.** The equivalence of (i) and (ii) above answers the question posed in the proof of [GS, Theorem 4.4]: it shows that [GS, Theorem 4.4] can be deduced from [GS, Theorem 4.2].

We are ready to prove the main result of this subsection. If \( I \) is an invariant ideal in \( (A, \alpha) \) we denote by \( E_I : C^*(A/I, \alpha_I) \to A/I \) the canonical conditional expectation.
Theorem 5.8. Suppose $(A, \alpha)$ is a reversible $C^*$-dynamical system such that all ideals in $C^*(A, \alpha)$ are gauge-invariant and for any invariant ideal $I$ in $(A, \alpha)$ and every non-zero positive element $b \in C^*(A/I, \alpha_I)$ with $\|b\| = 1$ there is $h \in M(A/I)$ such that

\[
\|h\| = 1, \quad \|hbh - hE_I(b)h\| < 1/4, \quad \|hE_I(b)h\| > 3/4.
\]

These assumptions are satisfied, for instance, when $\varphi$ is free and for each $x \in \Delta$ the range of $\alpha_x$ is full in $A(x)$.

If $A$ has the ideal property, then $C^*(A, \alpha)$ has the ideal property and the following statements are equivalent:

(i) Every non-zero positive element in $A$ is properly infinite in $C^*(A, \alpha)$.

(ii) $C^*(A, \alpha)$ is purely infinite.

(iii) Every non-zero hereditary $C^*$-subalgebra in any quotient $C^*(A, \alpha)$ contains an infinite projection.

If $A$ is of real rank zero, then each of the above conditions is equivalent to

(i’) Every non-zero projection in $A$ is properly infinite in $C^*(A, \alpha)$.

In particular, if $A$ is purely infinite and has the ideal property, the same holds for $C^*(A, \alpha)$.

Proof. If $\varphi$ is free and for each $x \in \Delta$ the range of $\alpha_x$ is full in $A(x)$, the assumptions are satisfied by Lemma 5.4 which thanks to Lemma 4.10 may be applied to quotient systems, see Example 4.9.

Suppose that $A$ has the ideal property. In the same way as in the proof of Proposition 5.2 we get that $C^*(A, \alpha)$ has the ideal property. In the presence of the ideal property the equivalence (ii)$\iff$(iii) holds for any $C^*$-algebra, see [PR07, Propositions 2.11]. If $A$ is if real rank zero the equivalence (i)$\iff$(i’) is ensured by Lemma 5.6. Implication (ii)$\iff$(i) is a trivial consequence of the definition of pure infiniteness we adopted, see [KR00, Theorem 4.16]. Thus it suffices to show the implication (i)$\Rightarrow$(iii) and the argument goes along the same lines as the proof of [RS12, Theorem 3.3] or [GS, Theorem 4.2]. Namely, let us fix an ideal $I$ in $C^*(A, \alpha)$ and a non-zero hereditary $C^*$-subalgebra $B$ in the quotient $C^*(A, \alpha)/I$. We need to show that $B$ contains an infinite projection.

By Proposition 5.2, we have $C^*(A, \alpha)/I \cong C^*(A/I, \alpha_I)$ where $I := A \cap I$ is an invariant ideal in $(A, \alpha)$. Fix a non-zero positive element $b$ in $B$. We may assume that $\|b\| = 1$. Choose $h \in M(A/I)$ satisfying (25). Putting $a := (hE(b)h - 1/2)_+$ one concludes, exactly as in the proof of [RS12, Lemma 3.2], that $a$ is non-zero and $a \lesssim b$ relative to $C^*(A/I, \alpha_I)$. Note that $a$ is properly infinite in $C^*(A/I, \alpha_I)$ as a non-zero homomorphic image of a properly infinite positive element in $C^*(A, \alpha)$, by the assumption in (i). Since $A$ has the ideal property we can find a projection $q \in A$ that belongs to the ideal in $A$ generated by the preimage of $a$ in $A$ but not to $I$. Then $q + I$ belongs to the ideal in $A/I$ generated by $a$, whence $q + I \lesssim a \lesssim b$, by [KR00, Proposition 3.5(ii)]. From the comment after [KR00, Proposition 2.6] we can find $z \in C^*(A/I, \alpha_I)$ such that $q + I = z^*bz$. With $v := b^{1/2}z^*z^{1/2}$ it follows that $v^*v = g + I$, whence $p := vv^* = b^{1/2}z^*z^{1/2}b^{1/2}$ is a projection in $B$, which is equivalent to $q$. By the assumption in (i), $q$ and hence also $p$ is properly infinite. \[\square\]

Remark 5.9. We recall, see [PR07, Propositions 2.11, 2.14], that in the presence of the ideal property the pure infiniteness of a $C^*$-algebra is equivalent to strong pure infiniteness, weak pure infiniteness, and many other notions of infiniteness appearing in the literature. Thus the list of equivalent conditions in Theorem 5.8 can be considerably extended.

5.3. Paradoxicality for $C^*$-dynamical systems. In view of [RS12], [GS], [KMR13] it is natural to call the $C^*$-dynamical system $(A, \alpha)$ paradoxical or purely infinite if the condition (i) in Theorem 5.8 is satisfied. In view of [KMR13] one can not expect to have paradoxical $\mathbb{Z}$-actions on abelian algebras. Nevertheless, allowing the algebra $A$ to be non-commutative makes it possible to obtain a non-trivial criterion for singly generated actions.
Definition 5.10. We will say that a non-zero positive element \( a \) in \( A \) is **paradoxical** for the reversible \( C^* \)-dynamical system \((A, \alpha)\) if for any \( \varepsilon > 0 \) there are numbers \( t_1, \ldots, t_{n+m} \in \mathbb{N} \), and elements \( a_{\pm k} \in aAa^{\pm t_k}(a) \), \( k = 1, 2, \ldots, (n + m) \) such that

\[
(26) \quad a \approx_{\varepsilon} \sum_{k=1}^{n} \alpha^{t_k}(a^*_{-k}a_{-k}) + \alpha^{t_k}(a^*_k a_k) \approx_{\varepsilon} \sum_{k=n+1}^{n+m} \alpha^{t_k}(a^*_{-k}a_{-k}) + \alpha^{t_k}(a^*_k a_k),
\]

\[
(27) \quad \|a^*_{-k}a_k\| < \frac{\varepsilon}{n \cdot m} \quad \text{for all } k \neq l, \|a^*_{\pm k}a_{\pm l}\| < \frac{\varepsilon}{n \cdot m} \quad \text{if } l \neq k.
\]

**Proposition 5.11.** Every paradoxical element \( a \) in \( A \) is properly infinite in \( C^*(A, \alpha) \).

**Proof.** Let \( \varepsilon > 0 \) and choose numbers \( t_{\pm k} \in \mathbb{N} \) and elements \( a_{\pm k} \in A \) as in Definition 5.10. Let us consider the elements

\[
x := \sum_{k=1}^{n} a_{-k}u^{*t_k} + a_k u^{t_k}, \quad y := \sum_{k=n+1}^{n+m} a_{-k}u^{*t_k} + a_k u^{t_k}.
\]

Using the commutation relations \( a_{-k}u^{*t_k} = u^{t_k} a^*(a_{-k}) \) and \( \alpha^{t_k}(a)u^{t_k} = u^{t_k} a \), and our choice of elements \( a_{\pm k} \in aAa^{\pm t_k}(a) \) we get \( x, y \in aC^*(A, \alpha)a \). Similarly using (26) and (27) simple calculations show that \( x^*x \approx_{4\varepsilon} a \), \( y^*y \approx_{4\varepsilon} a \) and \( x^*y \approx_{4\varepsilon} 0 \). Hence \( a \) is properly infinite in \( C^*(A, \alpha) \), cf. (2).

**Example 5.12.** Every properly infinite positive element in \( A \) is paradoxical, independently of \( \alpha \); it suffices to apply (2) with \( n = m = 1, t_1 = t_2 = 0, a_1 = x, a_2 = y, \) and \( a_i = 0 \), for \( i = -1, -2 \).

**Example 5.13.** The Cuntz algebra \( \mathcal{O}_n = C^*(S_1, \ldots, S_n) \) is naturally isomorphic to the crossed product of the \( n \)-\( \infty \)-UHF core \( \mathcal{F}^n \) by the endomorphism \( \alpha(a) := S_1 a S_1^* \), \( a \in \mathcal{F}^n \), see [Cun77]. Every non-zero projection in \( \mathcal{F}^n \) is equivalent to a paradoxical one, for the system \( (\mathcal{F}^n, \alpha) \).

Indeed, since \( \mathcal{F}^n = \bigcup_{k \in \mathbb{N}} \mathcal{F}^k \) where \( \mathcal{F}^k \approx M_{n^k} \) it suffices to show that the projection \( a := S_{1}^k S_{-1}^k \) is paradoxical, \( k \in \mathbb{N} \). To this end put \( n = m = 1, t_1 = t_2 = 1, a_1 = S_{1}^{k+1} S_{-1}^{k+1}, a_2 = S_{1}^k S_{2} S_{1}^{k+1}, \) and \( a_i = 0 \), for \( i = -1, -2 \). Using that \( \alpha^*(a) = S_1^* a S_1 \), one readily checks that conditions in Definition 5.10 are satisfied with \( \varepsilon = 0 \).

A different condition describing, what can also be viewed as, a non-commutative paradoxical action was considered in [OP14]. The authors of [OP14] introduced a property of an injective endomorphism of real rank zero algebras that, under the assumptions of Theorem 5.8, implies condition (iii) in Theorem 5.8. We can adjust their argument to obtain a criterion for pure infiniteness.

**Definition 5.14 (cf. Definition 3.2 and Theorem 3.4 in [OP14]).** Let \( \alpha \) be an extendible monomorphism of a \( C^* \)-algebra \( A \). We say that

(i) \( \alpha \) **contracts projections** if for any non-zero projecton \( p \in A \) there is \( n \in \mathbb{N} \) such that \( \alpha^n(p) \) is Murray-von Neumann equivalent to a proper subprojection of \( p \).

(ii) \( \alpha \) **residually contracts projections** if for any invariant ideal \( I \) in \((A, \alpha)\), the quotient endomorphism \( \alpha_I \) contracts projections.

**Proposition 5.15.** Retain the assumptions of Theorem 5.8. Additionally suppose that \( A \) is of real rank zero and \( \alpha \) is a monomorphism that residually contracts projections. Then \( C^*(A, \alpha) \) is purely infinite and has the ideal property.

**Proof.** The argument from the last three paragraphs of the proof of [OP14, Theorem 3.4] carries over to our situation, note that the property described in [RS12, Lemma 3.2], follows from (25), see the proof of [RS12, Lemma 3.2]. This argument shows that condition (iii) in Theorem 5.8 is satisfied. \( \square \)
6. Crossed products of irreversible $C^*$-dynamical systems on $C_0(X)$-algebras

Now we turn to study the ideal structure and pure infiniteness of crossed products $C^*(A,\alpha, J)$ where $A$ is $C_0(X)$-algebra and $\alpha$ is an arbitrary endomorphism induced by a morphism. The tactic is as follows. We first consider the natural reversible $J$-extension $(B,\beta)$ of $(A,\alpha)$. We show that it is induced by a morphism, in a natural way. Then applying results of the previous section to this morphism we deduce the main results of the paper. We conclude by discussing several examples.

6.1. Reversible extension of a $C^*$-dynamical system induced by a morphism. We fix an arbitrary $C^*$-dynamical system $(A,\alpha)$ where $A$ is a $C_0(X)$-algebra and $\alpha$ is induced by a morphism $(\varphi,\{\alpha_x\}_{x\in X})$. We let $J$ be an ideal in $(\ker \alpha)^\perp$ and denote by $(B,\beta)$ the reversible $J$-extension of $(A,\alpha)$. We put

$$Y = \sigma_A(\text{Prim}(A/J)).$$

In view of (13) and the fact that $J$ is contained in the set (18) we see that $Y$ contains $X \setminus \varphi(\Delta)$. We denote by $(\tilde{X},\tilde{\varphi})$ the reversible $Y$-extension of the partial dynamical system $(X,\varphi)$, see Definition 2.18. Our aim is to use Proposition 4.14 to describe $B$ as a $C_0(\tilde{X})$-algebra and $\beta$ as an endomorphism induced by a morphism $(\tilde{\varphi},\{\tilde{\beta}_x\}_{x\in \Delta})$ for a certain field of homomorphisms $\tilde{\beta}_x$, $x \in \Delta$.

We start by fixing indispensable notation. For $n \in \mathbb{N} \setminus \{0\}$ and $x \in X$ that belongs to the domain of $\varphi^n$ we put

$$\alpha_{(x,n)} := \alpha_x \circ \alpha_{\varphi(x)} \circ \cdots \circ \alpha_{\varphi^{n-1}(x)}, \quad p_{(x,n)} := \text{tr}_{(x,n)}(1_{\varphi^n(x)}).$$

For $n = 0$ we put $\alpha_{(x,0)} := id$ and $p_{(x,0)} = 1_x$. To each $n \in \mathbb{N}$ and $x \in X$ belonging to the domain of $\varphi^n$, we associate the following corner in $A(x)$:

$$A_n(x) := p_{(x,n)} A(x) p_{(x,n)}.$$

We construct the fibre $C^*$-algebra $B(\tilde{x})$ as follows. If $\tilde{x} = (x_0, x_1, \ldots) \in \tilde{X}_\infty$, we let

$$B(\tilde{x}) := \lim_{\to} \{A_n(x_n), \alpha_{x_{n+1}}\}$$

to be the inductive limit of the sequence $A_0(x_0) \xrightarrow{\alpha_{x_1}} A_1(x_1) \xrightarrow{\alpha_{x_2}} A_2(x_3) \xrightarrow{\alpha_{x_3}} \ldots$. If $\tilde{x} = (x_0, x_1, \ldots, x_N, 0, \ldots) \in \tilde{X}_N$, we simply put

$$B(\tilde{x}) = A_N(x_N)/J(x_N).$$

In other words, $B(\tilde{x}) = q_{x_N}(A_N(x_N))$ where $(id, \{q_x\}_{x \in \tilde{X}})$ is the morphism that induces the quotient map $q : A \to A/J$, see Example 4.8. Note that certain fibers $B(\tilde{x})$, $\tilde{x} \in \tilde{X}$, might be zero algebras.

We shall represent the dense $*$-subalgebra $\bigcup_{n \in \mathbb{N}} B_n$ of $B$ as an algebra of sections of $B = \bigcup_{\Delta \subseteq X} B_\Delta$. For every $a = (a_0 + J) \oplus \cdots \oplus (a_{n-1} + J) \oplus a_n \in B_n$ we define the section $\pi(a)$ of $B$ by the formula

$$\pi(a)(\tilde{x}) = \begin{cases} a_N(x_N) + J(x_N), & \text{if } \tilde{x} \in X_N, \ N < n, \\ a_{(x_{n-n-N},n-n)}(a_n(x_n)) + J(x_N), & \text{if } \tilde{x} \in X_N, \ N \geq n, \\ 0, \ldots, 0, a_n(x_n), a_{x_{n+1}}(a_n(x_n)), a_{x_{n+2}}(a_n(x_n)), \ldots, & \text{if } \tilde{x} \in X_\infty. \end{cases}$$

Let $\tilde{x} \in \tilde{X}$. We define an endomorphism $\tilde{\beta}_x : B(\tilde{\varphi}(\tilde{x})) \to B(\tilde{x})$ and a positive linear map $\beta_{*,\tilde{\varphi}(\tilde{x})} : B(\tilde{x}) \to B(\tilde{\varphi}(\tilde{x}))$ as follows. If $\tilde{x} \in X_\infty$ we let them to act as shifts

$$\tilde{\beta}_x[a_0, a_1, a_2, \ldots] := [a_1, a_2, \ldots],$$

$$\beta_x(a_0, a_1, a_2, \ldots) := [0, p_{(x_0,1)} a_0 p_{(x_0,1)}, p_{(x_1,2)} a_1 p_{(x_1,2)}, \ldots].$$

If $\tilde{x} \in X_N$ we simply put $\tilde{\beta}_x := id$ and $\beta_x(a) := q_{x_N}(p_{(x_N,N+1)} a p_{(x_N,N+1)})$ where we use the inclusion $B(\tilde{\varphi}(\tilde{x})) = q_{x_N}(A_{N+1}(x_N)) \subseteq B(\tilde{x}) = q_{x_N}(A_N(x_N))$. 
Now we are ready to show that reversible extensions of any $C^*$-dynamical system induced by a morphism are also induced by morphisms, in a natural way.

**Theorem 6.1.** Retain the above notation. There is a unique topology on $B = \bigsqcup_{\bar{x} \in \tilde{X}} B_{\bar{x}}$ making it into an upper semicontinuous $C^*$-bundle over $\tilde{X}$ such that $\pi$ establishes the isomorphism $B \cong \Gamma_0(B)$. Identifying $B$ with the algebra of continuous sections of $B$ we have

$$\beta(a)(\bar{x}) = \begin{cases} \beta_x(a(\varphi(\bar{x}))), & \bar{x} \in \tilde{\Delta}, \\ 0, & \bar{x} \notin \tilde{\Delta}, \end{cases}$$

$$\beta_s(a)(\bar{x}) = \begin{cases} \beta_s, x(a(\varphi^{-1}(\bar{x}))), & \bar{x} \in \varphi(\Delta), \\ 0, & \bar{x} \notin \varphi(\Delta). \end{cases}$$

Moreover, if either

(i) $A$ is a continuous $C_0(X)$-algebra, all ideals $J(x)$ are trivial (i.e. either $\{0\}$ or $A(x)$), all homomorphisms $\alpha_x$ are injective and $\sigma_A(\text{Prim}(A \setminus J))$ is clopen, or

(ii) $\sigma_A$ is a homeomorphism, i.e. $A$ is a $C^*$-algebra with Hausdorff primitive ideal space, cf. Example 4.7,

then $B = \bigsqcup_{\bar{x} \in \tilde{X}} B_{\bar{x}}$ is a continuous $C^*$-bundle.

**Proof.** Notice that $B_n, n \in \mathbb{N}$, is naturally a $C_0(X_n)$-algebra with

$$X_n := Y \sqcup Y \cap \Delta_1 \sqcup \ldots \sqcup Y \cap \Delta_n \sqcup \Delta_n.$$ 

Moreover, the bonding homomorphism $\alpha_n : B_n \to B_{n+1}$ is induced by the morphism $(\phi_n, \{\alpha_{x,n}\}_{x \in X_{n+1}})$ where $\phi_n : X_{n+1} \to X_n$ is given by the diagram

$$\begin{array}{c}
X_n = Y \sqcup Y \cap \Delta_1 \sqcup \ldots \sqcup Y \cap \Delta_n \sqcup \Delta_n \\
\varphi_n \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
X_{n+1} = Y \sqcup Y \cap \Delta_1 \sqcup \ldots \sqcup Y \cap \Delta_n \sqcup \Delta_{n+1} \quad \downarrow \varphi
\end{array}$$

For $x \in X_{n+1}$ and $a = a_0 + J \oplus \ldots \oplus a_{n-1} + J \oplus a_n \in B_n$ where $a_k \in A_k, k \leq n$, we have

$$\alpha_{x,n}(a) = \begin{cases} a_k + J(x), & x \in Y \cap \Delta_k \quad \text{a $k$-th summand of $X_{n+1}$ with $k \leq n$} \\
\alpha_x(a_n), & x \in \Delta_n \quad \text{the last summand of $X_{n+1}$}. \end{cases}$$

Since $\varphi(\Delta) \cup Y = X$, the mappings $\phi_n : X_{n+1} \to X_n$ are surjective and we can apply Proposition 4.14 to the inductive system $\{B_n, \alpha_n\}_{n \in \mathbb{N}}$. A closer inspection, see the proof of , shows that the $C^*$-bundle given by Proposition 4.14 and the one described above can be naturally identified.

Let $a = a_0 + J \oplus \ldots \oplus a_{n-1} + J \oplus a_n \in B_n$ and $\bar{x} \in \tilde{X}$. Note that $\pi(\beta_n(a)(\bar{x}))$ is equal to

$$\begin{cases} a_{N+1}(x_N) + J(x_N), & \bar{x} \notin \tilde{\Delta}, \\
\alpha(x_{n+1-n}, N+1-n)(a_n(x_{n-1})) + J(x_N), & \bar{x} \in \tilde{\Delta}, N \geq n, \\
0, \ldots, 0, \alpha_{x_n}(a_n(x_{n-1})), \alpha_{x_{n+1}}(a(x_{n-1})), \ldots, & \bar{x} \in \varphi(\Delta). \end{cases}$$

Note that for $x \notin \Delta_k$ we have $a_k(x) = 0$ because $a_k \in \overline{\alpha^{-1}(1)} A \alpha^{-1}(1)$. Thus $\bar{x} \notin \tilde{\Delta}$ implies $\pi(\beta_n(a)(\bar{x})) = 0$, because then either $x_N \in \Delta_N \setminus \Delta_{N+1}$, when $\bar{x} \in \tilde{\Delta}$, or $x_N \in \Delta_n \setminus \Delta_{n+1}$, otherwise. On the other hand one readily sees that $\pi(\beta_n(a)(\bar{x})) = \beta_x(a(\varphi(\bar{x})))$ for $\bar{x} \notin \Delta$. Hence, in view of Proposition 4.2, $\beta : B \to B$ is induced by the morphism $(\varphi, \{\beta_x\}_{x \in \tilde{\Delta}})$.

Invoking Proposition 4.12 and formula (20) one concludes that the transfer operator $\beta_s$ satisfies the formula described in the assertion with the mappings

$$\beta_{s,\tilde{\varphi}}(b) := \beta_x^{-1}(1 \tilde{\varphi}(\bar{x})) \alpha_{\tilde{\varphi}}^{-1}(1 \tilde{\varphi}(\bar{x})), \quad b \in B(\bar{x}), \bar{x} \in \tilde{\Delta},$$

where $\alpha_{\tilde{\varphi}}$ is the isomorphism that establishes the isomorphism $B \cong \Gamma_0(B)$. The second of the above equations is justified by the fact that $\beta_x$ is an isomorphism of $C(X)$-algebras, and the first is justified by the fact that $\varphi$ is a homeomorphism.
where $\beta_x^{-1}$ is the inverse to the isomorphism $\beta_x : (\ker \beta_x)^{-1} \to \overline{\beta}_x^{-1}(1, \bar{x})B(\bar{x})\overline{\beta}_x^{-1}(1, \bar{x})$. The reader will readily see that these maps coincide with the maps we have previously described. This proves the first part of the assertion.

For the second part of the assertion it suffices to apply the second part of Proposition 4.14. Indeed, if all of the ideals $J(x)$ are trivial and all of the homomorphisms $\alpha_x$ are injective, then all of the homomorphisms $\alpha_{x,n}$ are injective. \hfill \blacksquare

**Remark 6.2.** The morphism $(\bar{\varphi}, \{\beta_x\}_{x \in \Delta})$ constructed above can be viewed as a canonical one. In particular, $\bar{\varphi}$ is always a partial homeomorphism and all the homomorphisms $\beta_x$ are injective. Thus even when the initial system $(A, \alpha)$ is already reversible and $J = (\ker \alpha)^{-1}$, so that $(A, \alpha) = (B, \beta)$, the morphism $(\varphi, \{\alpha_x\}_{x \in \Delta})$ in general differs from $(\bar{\varphi}, \{\beta_x\}_{x \in \Delta})$. For instance, for the reversible dynamical system $(A, \alpha)$ described in Example 4.13 we obtain (omitting zero fibers) that $B = A = \mathbb{C}^3$ is naturally a $C_0(\{1, 2, 3\})$-algebra and $\beta = \alpha$ is induced by the morphism $(\varphi, \{\alpha_1, \alpha_3\})$ where $\varphi(1) = 2, \varphi(3) = 3, \alpha_1 = \alpha_3 = id$.

6.2. **Main results.** In this subsection we fix a $C^*$-dynamical system $(A, \alpha)$ where $A$ is a $C_0(X)$-algebra and $\alpha$ is induced by a morphism $(\varphi, \{\alpha_x\}_{x \in \Delta})$ (recall that any $C^*$-dynamical system may be trivially treated as induced by a morphism, cf. Example 4.6). We also fix an ideal $J$ in $(\ker \alpha)^{-1}$. The first two results should be compared with [Kwa, Proposition 4.33] and [Kwa, Proposition 4.36], respectively.

**Theorem 6.3** (Uniqueness property). Suppose that $\varphi$ is topologically free outside $\sigma_A(\text{Prim}(A/J))$ and either (i) or (ii) in Theorem 6.1 holds. Then every faithful representation $(\pi, U)$ of $(A, \alpha)$, such that $J = \{a \in A : U^*U\pi(a) = \pi(a)\}$, integrates to a faithful representation of $C^*(A, \alpha, J)$.

**Proof.** By Theorem 6.1 and Lemmas 2.24, 5.1 we may apply Theorem 2.22(i) to the reversible $J$-extension $(B, \beta)$ of $(A, \alpha)$. Alternatively, in view of Theorem 6.1 and Lemma 2.24, we could apply Corollary 5.5 modulo the fact that $(\pi, U)$ yields a faithful representation of $B$ if and only if $J = \{a \in A : U^*U\pi(a) = \pi(a)\}$, see the last part of [Kwa, Proposition 4.7]. \hfill \square

The following result implies the uniqueness property described above and it does not impose continuity on the $C_0(X)$-algebra $A$. Still it requires freeness of $\varphi$ which is much stronger than topological freeness.

**Theorem 6.4** (Ideal lattice description). If $\varphi$ is free, then all ideals in $C^*(A, \alpha, J)$ are gauge-invariant and, in particular, they are in one-to-one correspondence with $J$-pairs for $(A, \alpha)$.

**Proof.** Apply Theorem 6.1, Lemma 2.24(ii) and Proposition 5.2. See also Proposition 2.19. \hfill \square

To show our final main result we need a simple lemma. Recall that an element $a \in M(A)$ is full in $A$ if $A\{a\}A = A$. Clearly, if $\alpha : A \to B$ is an extendible homomorphism then $\overline{\alpha(1)}$ is full in $B$ iff $\alpha(A)$ is full in $B$.

**Lemma 6.5.** Let $\alpha : A \to B$ and $\beta : B \to C$ be extendible homomorphisms such that $\overline{\alpha(1)}$ is full in $B$ and $\overline{\beta(1)}$ is full in $C$. Then $\overline{\beta(\alpha(1))}$ is full in $C$ and (equivalently) in $\overline{\beta(1)}\overline{C\beta(1)}$.

**Proof.** $C\overline{\beta(1)}C = C\overline{\beta(B)\beta(1)}\overline{\beta(B)}C = C\overline{\beta(B\alpha(1))B}C = C\overline{\beta(B)}C = C$. \hfill \square

**Theorem 6.6** (Pure infiniteness). Suppose the reversible $J$-extension $(B, \beta)$ of $(A, \alpha)$ satisfies the assumptions of Theorem 5.8. This holds, for instance, if $\varphi$ is free and for each $x \in \Delta$ the range of $\alpha_x$ is a full subalgebra of $A(x)$.

Any of the following conditions implies $C^*(A, \alpha, J)$ is purely infinite and has the ideal property:

(i) all the corners $\overline{\alpha^N(1)}A\overline{\alpha^N(1)}$, $N \in \mathbb{N}$, have the ideal property, and every element in $B^\delta \setminus \{0\}$ is Cuntz equivalent to an element paradoxical for $(B, \beta)$,

(ii) $A$ is of real rank zero and every non-zero projection in $B$ is equivalent to a paradoxical projection for $(B, \beta)$,

(iii) $A$ is of real rank zero, $\alpha$ is injective, residually contracting projections, and $J = A$. 

Proof. We first suppose that $\varphi$ is free and the range of $\alpha_x$ is a full subalgebra of $A(x)$, for $x \in \Delta$. To prove that $(B, \beta)$ satisfies the assumptions of Theorem 5.8 we show that the morphism $(\bar{\varphi}, \{\bar{\beta}_x\}_{x \in \Delta})$ associated to $(B, \beta)$ in Theorem 6.1 is such that $\bar{\varphi}$ is free and the range of $\bar{\beta}_x$ is a full subalgebra of $B(\bar{x})$, for each $\bar{x} \in \bar{\Delta}$. Freedom of $\bar{\varphi}$ is clear, cf. Lemma 2.24. To see that $\bar{\beta}_x(1_{\bar{x}(\bar{x})})$ is full in $B(\bar{x})$, for $\bar{x} \in \bar{\Delta}$, note first that Lemma 6.5 implies that for any $x_N \in X$, in the domain of $\varphi^N$, the projection 

$$p_{(x_N,N+1)} = \overline{\varphi}(x_N,N+1)(1_{\varphi(x_0)}) = (\overline{\varphi}(x_N,N) \circ \overline{\varphi}_N)(1_{\varphi(x_0)})$$

is full in 

$$A_N(x_N) = p_{(x_N,N)}A(x_N)p_{(x_N,N)} = \overline{\varphi}(x_N,N)(1_{x_0})A_N(\overline{\varphi}(x_N,N))(1_{x_0}).$$

Thus if $\bar{x} \in \bar{\Delta} \cap X_N$, then $\bar{\beta}_x(1_{\varphi(\bar{x})}) = \overline{\varphi}_N(p_{(x_N,N+1)})$ is full in $B(\bar{x}) = q_{x_N}(A_N(x_N))$. Now let $\bar{x} \in \bar{\Delta} \cap X_\infty$ and note that the direct limit $\lim_{x_N} \{A_N(x_N), \alpha_{x,N+1}\}$ sits naturally in the multiplier algebra of $B(\bar{x}) = \lim_{\Xi,x_N} \{A_N(x_N), \varphi(x_N)\}$ and is preserved by the strictly continuous extension $\bar{\beta}_x$ of $\beta_x$. Formally, $\bar{\beta}_x$ acts on $\lim_{\Xi,x_N} \{A_N(x_N), \varphi(x_N)\}$ by the same formula as $\beta_x$. Thus 

$$\bar{\beta}_x(1_{\varphi(\bar{x})}) = \bar{\beta}_x(1_{\varphi(x_0)}p(x_0,1), p(x_1,2), \ldots) = [p(x_0,1), p(x_1,2), \ldots] \in M(\bar{x}),$$

and since $p_{(x_N,N+1)}$ is full $A_N(x_N)$ for each $N \in \mathbb{N}$ it follows that $\bar{\beta}_x(1_{\varphi(\bar{x})})$ is full in the direct limit $B(\bar{x}) = \lim_{\Xi} \{A_N(x_N), \alpha_{x,N+1}\}$.

Let us now consider the general case. It is easy to see that the ideal property is preserved under taking direct sums and quotients. It is also preserved when passing to direct limits [Pas00, Proposition 2.2]. Hence if the algebras $\overline{\varphi}^N(1)A\overline{\varphi}^N(1), N \in \mathbb{N}$, have the ideal property, then $B$ has the ideal property. Thus assuming (i) $B$ has the ideal property and the assertion follows from Proposition 5.11 and Theorem 6.1. Namely, condition (i) in Theorem 6.1 is satisfied (an element which is Cuntz equivalent to a properly infinite element is properly infinite).

Now assume (ii). Since direct sums, hereditary subalgebras and direct limits of real rank zero algebras have real rank zero, see [BP91, Corollary 2.8 and Proposition 3.1], we conclude that $B$ has real rank zero. Thus similarly as above it suffices to apply Proposition 5.11 and Theorem 6.1 (this time condition (i’) plays the crucial role).

Assuming (iii), one readily checks, cf. the proof of [OP14, Theorem 3.4], that $\beta$ residually contracts projections. Hence the assertion follows from Proposition 5.15. 

\textbf{Remark 6.7.} It seems to remain an open problem whether a corner of a $C^*$-algebra with the ideal property has the ideal property, which explains our assumption in item (i).

In our applications we will focus on the following:

\textbf{Corollary 6.8.} Assume that $\varphi$ is free and for each $x \in \Delta$ the range of $\alpha_x$ is a full subalgebra of $A(x)$. If $A$ is purely infinite and has the ideal property, then $C^*(A, \alpha, J)$ is purely infinite and has the ideal property, for any ideal $J \subseteq (\ker \alpha)^\perp$.

\textbf{Proof.} If we assume (i) then the algebras $\overline{\varphi}^N(1)A\overline{\varphi}^N(1), N \in \mathbb{N}$, have the ideal property by [PR07, Proposition 2.10]. Moreover, pure infiniteness is preserved under taking direct sums, quotients [KR00, Proposition 4.3] and direct limits [KR00, Proposition 4.18]. Thus (i) implies that $B$ is purely infinite and has the ideal property. Therefore the assertion follows from Theorem 5.8. 

It was shown in [KL13, Subsection 4.4] (under the assumption that $A$ is unital, but the arguments carry out easily to our more general situation) that the crossed product $C^*(A, \alpha, J)$ is naturally isomorphic to a relative Cuntz-Pimsner algebra $O(X, J)$ associated to the $C^*$-correspondence given by 

$$X := \alpha(A)A, \quad a \cdot x := \alpha(a)x, \quad x \cdot a := xa, \quad \langle x, y \rangle_A := x^*y,$$
where \( a \in A \), \( x, y \in X \), see, for instance, [KL13] for all the relevant definitions. Thus general theorems on Cuntz-Pimsner algebras apply to \( C^*(A, \alpha, J) \) which, in particular, allows us to conclude with the following.

**Corollary 6.9.** Suppose that \( \varphi \) is free and for each \( x \in \Delta \) the range of \( \alpha_x \) is a full subalgebra of \( A(x) \). If \( A \) is purely infinite, nuclear, separable \( C^* \)-algebra with the ideal property, then \( C^*(A, \alpha, J) \) is strongly purely infinite, nuclear and separable.

If additionally \( A \) and \( J \) satisfy UCT, then \( C^*(A, \alpha, J) \) satisfies UCT.

**Proof.** If \( A \) is separable, then clearly \( C^*(A, \alpha, J) \) is separable. If \( A \) is purely infinite and has the ideal property, then in view of Theorem 6.6, see Remark 5.9, \( C^*(A, \alpha, J) \) is strongly pure infinite. For general crossed products, it follows from [Kat04, Theorem 7.2] that if \( A \) is nuclear then \( C^*(A, \alpha, J) \) is nuclear. Similarly, the argument leading to [Kat04, Proposition 8.8] (in separable case) implies that if both \( A \) and \( J \) satisfy the UCT then \( C^*(A, \alpha, J) \) satisfies the UCT. \( \square \)

6.3. **Examples.** The simplest way to mix noncommutative and topological dynamics is to use tensor product. This already allows one to produce interesting examples.

**Example 6.10** (‘Tensoring’ a \( C^* \)-dynamical system with a partial dynamical system). Let \((A_0, \alpha_0)\) be any \( C^* \)-dynamical system and let \((X, \varphi)\) be any partial dynamical system where the domain \( \Delta \) of \( \varphi \) is clopen in \( X \). Let us consider the \( C^* \)-dynamical system \((A, \alpha)\) where \( A := C_0(X) \otimes A_0 = C_0(X, A_0) \) and

\[
\alpha(a)(x) := \alpha_0(\alpha(\varphi(x))) \quad \text{for} \quad x \in \Delta, \quad \alpha(a)(x) = 0 \quad \text{for} \quad x \notin \Delta.
\]

We can apply our main results to \((A, \alpha)\) in the following cases:

1. Assume \( \alpha_0 \) is injective and \( \varphi \) is topologically free outside a clopen subset \( Y \) of \( X \) such that \( Y \cup \varphi(\Delta) = X \). Then by Theorem 6.3, see item (i) in Theorem 6.1, for any faithful non-degenerate representation \( \pi : A \to \mathcal{B}(H) \) and an operator \( U \in \mathcal{B}(H) \) such that

\[
U \pi(a) U^* = \pi(\alpha(a)), \quad a \in A,
\]

we have

\[
C^*(\pi(A) \cup \pi(A)U) \cong C^*(A, \alpha; J)
\]

where \( J := \{a \in A : a(y) = 0 \text{ for } y \in Y\} \). Note that if \( \varphi(\Delta) \) is open, we may take \( Y := X \setminus \varphi(\Delta) \) and then \( C^*(A, \alpha; J) = C^*(A, \alpha) \).

2. Suppose \( \varphi \) is free. Then for any ideal \( J \) in \((\ker \alpha)^\perp \) we have a one-to-one correspondence between ideals in \( C^*(A, \alpha; J) \) and \( J \)-pairs for \((A, \alpha)\), by Theorem 6.4.

3. Suppose that \( X \) is zero dimensional, \( \varphi \) is free and \( \alpha_0(A_0) \) is full in \( A_0 \). If \( A_0 \) is purely infinite and has the ideal property, then by [PR07, Proposition 4.6] the \( C^* \)-algebra \( A = C_0(X) \otimes A_0 \) is purely infinite and has the ideal property. In this case for any ideal \( J \) in \((\ker \alpha)^\perp \) the crossed product \( C^*(A, \alpha, J) \) is purely infinite, by Theorem 6.6 (i). If \( A_0 \) is an arbitrary \( C^* \)-algebra with real rank zero, then \( A = C_0(X) \otimes A_0 \) has real rank zero by [BP91, Corollary 3.4], and thus if \( \alpha \) residually contracts projections then \( C^*(A, \alpha, J) \) is purely infinite by Theorem 6.6 (ii).

To illustrate the importance of item (3) in the above example let us show it can be used to produce non-simple classifiable \( C^* \)-algebras from simple ones, only by adding an appropriate ‘dynamical ingredient’. Starting from an arbitrary Kirchberg algebra we will construct a classifiable \( C^* \)-algebra with a non-Hausdorff primitive ideal space with two points. Such algebras were the first to be considered in classification of non-simple infinite \( C^* \)-algebras [Ror97, Bon02].
Example 6.11. Let $A_0$ be a Kirchberg algebra and let $\alpha_0 : A_0 \to A_0$ be a non-zero endomorphism. Let $\phi_0 : C \to C$ be a minimal homeomorphism of the Cantor set $C$. We define $X := C \cup \{x_0\}$ to be a disjoint sum of $C$ and a clopen singleton $\{x_0\}$. We let $\phi : X \to X$ be any extension of $\phi_0$ such that $\phi(x_0) \in C$. Then the $C^*$-dynamical system $(A, \alpha)$ considered in Example 6.10 satisfies the assumptions of Corollary 6.9. Hence $C^*(A, \alpha)$ is strongly purely infinite, nuclear and separable.

In order to establish the ideal structure of $C^*(A, \alpha)$ let $(B, \beta)$ be the natural reversible $(\ker \alpha)^\perp$-extension system and $(\tilde{\varphi}, \{\beta_x\}_{x \in \tilde{X}})$ the associated morphism described in Theorem 6.1. Then $X_\infty \subseteq \tilde{X}$ can be identified with $C$ and $X_N = \{x_N\}$, $N \in \mathbb{N}$, are clopen singletons converging to the point $\phi(x_0) \in C = X_\infty$. The fibers $B(x_N)$ are isomorphic to $A_0$ and for $x \in C = X_\infty$, $B(x)$ is isomorphic to the stationary direct limit $\varprojlim \{A_0, \alpha_0\}$. The map $\tilde{\varphi}$ sends $x_N$ to $x_{N+1}$ and when restricted to $C = X_\infty$ coincides with $\phi_0$. Using Lemma 2.6 one sees that the only non-trivial invariant ideal in $B$ is

$$I = \{b \in B : b(x) = 0 \text{ for all } x \in C = X_\infty\}.$$

Since $I$ is not complemented in $B$ we deduce, see Proposition 5.2, that $\text{Prim}(C^*(A, \alpha))$ has two elements and is non-Hausdorff. Moreover, both $C^*(A, \alpha)$ and its the only non-trivial quotient $C^*(A, \alpha)/I \cong C^*(B/I, \beta_I/I)$ satisfy the UCT, see Corollary 6.9.

The homomorphisms between the non-commutative fibers may contribute to pure infiniteness in an essential way. The following is somewhat trivial but a significant example.

Example 6.12 (Cuntz algebras). Consider the system $(F_n, \alpha)$ introduced in Example 5.13. We showed there it satisfies condition (ii) in Theorem 6.6. Moreover, for every $b \in C^*(F_n, \alpha) \cong \mathcal{O}_n$ with $\|b\| = 1$ there is an element $Q \in F_n$, constructed in the proof of [Cun77, Proposition 1.7] such that $\|Q\| = 1$, $\|QbQ - QE(b)Q\| < 1/4$ and $\|QE(b)Q\| > 3/4$. Since $C^*(F_n, \alpha) \cong \mathcal{O}_n$ is simple and $(F_n, \alpha)$ is reversible, Theorem 6.6 implies that $C^*(F_n, \alpha) \cong \mathcal{O}_n$ is purely infinite.

Another natural class of examples arise from dynamics on algebras with Hausdorff primitive ideal space, such algebras were studied for instance in [BK04].

Example 6.13 (Systems on algebras with Hausdorff primitive ideal space). Suppose $A$ is a $C^*$-algebra with a Hausdorff primitive spectrum $X := \text{Prim}(A)$, and treat $A$ as a $C_0(X)$-algebra, cf. Example 4.7. Assume also that $(A, \alpha)$ is a $C^*$-dynamical system where $\alpha$ is induced by a morphism $(\varphi, \{\alpha_x\}_{x \in \Delta})$; if $A$ is unital it is equivalent to assuming that $\alpha(Z(A)) \subseteq Z(A)\alpha(1)$, see Proposition 4.5. Applying our main results we get:

1. If $\varphi$ is topologically free outside a set $Y$ where $Y = \overline{\sigma_A(\text{Prim}(A/J))}$ and $J$ is an ideal in $(\ker \alpha)^\perp$, then for any faithful non-degenerate representation $\pi : A \to B(H)$ and an operator $U \in B(H)$ such that

$$U\pi(a)U^* = \pi(\alpha(a)), \quad a \in A, \quad J = \{a \in A : \pi(a) \in U^*\pi(A)U\}$$

we have $C^*(\pi(A) \cup \pi(A)U) \cong C^*(A, \alpha; J)$, by Theorem 6.3.

2. If $\varphi$ is free, then for any ideal $J$ in $(\ker \alpha)^\perp$ we have a one-to-one correspondence between ideals in $C^*(A, \alpha; J)$ and $J$-pairs for $(A, \alpha)$, by Theorem 6.4.

3. Suppose $X$ is zero dimensional and each of the fibers $A(x)$, $x \in X$, contains a non-zero projection. By [Pas06, Theorem 2.5], the latter is equivalent for $A$ to have the ideal property. Thus if $A$ is purely infinite and $\varphi$ is free, then for any ideal $J$ in $(\ker \alpha)^\perp$ the crossed product $C^*(A, \alpha, J)$ is purely infinite and has the ideal property, see Theorem 6.6 (i).

Our next and last example indicates how Theorem 6.3 can be used to study spectral properties of weighted composition operators on vector-valued spaces. Let us consider the weighted shift operator $(aUf)(x) = \rho(x)^{\frac{1}{2}}a(x)f(\varphi(x))$, acting in the space $L^2_{\mu}(X; H)$ of functions on a compact
metric space $X$ with values in a separable Hilbert space $H$. Here $\varphi$ is a topologically free homeomorphism of $X$, the measure $\mu$ is quasi-invariant with respect to $\varphi$, $\rho = \frac{d\mu\varphi}{d\mu}$, and $a$ is a continuous function on $X$ with values in the algebra $\mathcal{K}(H)$ of compact operators on $H$. The authors of [LS90] computed the spectrum $\sigma(aU)$ of $aU$ in terms of exact Lyapunov exponents of the cocycle constructed from $a$ and $\varphi$, corresponding to measures ergodic for $(X, \varphi)$. The authors of [AL94], see also [CL99], noticed that the same computation can be performed in a far more general (abstract) setting. The only crucial assumptions are that $a$ is an element of a $C^*$-algebra $A \cong C(X, \mathcal{K}(H))$ and $U$ is a unitary such that $U(a)U^* = a\varphi$ is an automorphism of $A$. They call such operators abstract weighted shifts. The tools we have developed provide good foundation for a further generalization of this formalism to operators associated with endomorphisms of $C_0(X)$-algebras.

**Example 6.14** ($C^*$-algebras generated by abstract weighted shift operators). Let $X$ be a compact Hausdorff space and $\mu$ a probability borel measure on $X$ with $\text{supp}\, \mu = X$. To each continuous field $\mathcal{H} = \{(H(x))_{x \in X}, \Gamma\}$ of separable Hilbert spaces over $X$, see [Dix77, 10.1.2], we can attach the Hilbert space $L^2_\mu(\mathcal{H})$ which is a completion of the space $\Gamma$ equipped with the inner product:

$$
\langle f, g \rangle := \int_X \langle f(x), g(x) \rangle_x \, d\mu, \quad f, g \in \Gamma \subseteq \prod_{x \in X} H(x).
$$

Moreover, as it is explained in [Dix77, 10.7.1], there is a continuous $C^*$-bundle structure on $\mathcal{A}_0 := \bigsqcup_{x \in X} \mathcal{K}(H(x))$ where the $C^*$-algebra $A_0 = \Gamma(\mathcal{A}_0)$ is spanned by the elements $\Theta_{\xi, \eta}(t)$, $\xi, \eta \in \Gamma$, where $\Theta_{\xi, \eta}(t) := \Theta_{\xi(t), \eta(t)}$ is the one-dimensional operator in $H(t)$, $t \in X$. Both of the algebras $C(X)$ and $A$ are represented in a faithful and natural way on $L^2_\mu(\mathcal{H})$, cf. [Dix77, 10.1.9, 10.7.3]. Then $C(X) \subseteq Z(M(A_0)) \subseteq M(A_0) \subseteq B(L^2_\mu(\mathcal{H}))$ and we may consider the unital $C^*$-subalgebra $A := C^*(A_0 \cup C(X))$ of $B(L^2_\mu(\mathcal{H}))$ generated by $A_0$ and $C(X)$. Then $A$ is a continuous $C(X)$-algebra such that $A(x) = \mathcal{K}(H(x)) \oplus \mathbb{C}1$. Suppose now $U \in B(L^2_\mu(\mathcal{H}))$ is a partial isometry such that

$$
UAU^* \subseteq A, \quad U^*U \in A'
$$

and

$$
UC(X)U^* \subseteq C(X)U^*U
$$

Relations (28) are equivalent to stating that $\alpha(\cdot) = U(\cdot)U^*$ yields an endomorphism of $A$. While (29) says that $\alpha : A \to A$ is induced by a morphism $(\varphi, \{\alpha_x\}_{x \in \Delta})$, cf. Proposition 4.5. Assume also that $\alpha$ preserves $A_0$, which is trivially satisfied when all the spaces $H(x)$, $x \in X$, are finite dimensional, but one can also readily show it holds when all the spaces $H(x)$, $x \in X$, have dimension greater than one. Then the restriction of $\alpha$ to $A_0$ is an extendible endomorphism $\alpha : A_0 \to A_0$ induced by the morphism $(\varphi, \{\alpha_x\}_{x \in \Delta})$. We can view the operators $aU$, $a \in A_0$, as abstract weighted shift operators (what is actually important here are the relations (28) and (29), not a particular form of $U$ or representation of $A$; concrete examples of weighted composition operators obeying (28), (29) can be found, for instance, in [Kwa12], [KL08]). In particular, if $\varphi$ is topologically free, then as in Example 6.13(1) we get

$$
C^*(A_0 \cup \{aU : a \in A_0\}) \cong C^*(A_0, \alpha, J)
$$

where $J = A_0 \cap U^*A_0U$. Thus the spectrum $\sigma(aU)$ of $aU$, $a \in A_0$, can be calculated in the realm of $C^*(A_0, \alpha, J)$. For instance, existence of the gauge circle action on $C^*(A_0, \alpha, J)$ readily implies that $\sigma(aU)$ is invariant under the rotations around the origin of $\mathbb{C}$. In other words $\sigma(aU)$ is a sum of annuli, and one can expect to determine their radii, using Riesz projections and a version of Lyapunov exponents of the arising cocycles, cf. [LS90], [AL94], [CL99].
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