How to Understand Fractals and Fractal Dimension of Urban Morphology

Yanguang Chen

(Department of Geography, College of Urban and Environmental Sciences, Peking University, Beijing 100871, P. R. China. E-mail: chenyg@pku.edu.cn)

Abstract: The conventional mathematical methods are based on characteristic scales, while urban form has no characteristic scale in many aspects. Urban area is a measure of scale dependence, which indicates the scale-free distribution of urban patterns. In this case, the urban description based on characteristic scales should be replaced by urban characterization based on scaling. Fractal geometry is one of powerful tools for scaling analysis of cities, thus the concept of fractal cities emerged. However, how to understand city fractals is still a pending question. By means of logic deduction and ideas from fractal theory, this paper is devoted to discussing fractals and fractal dimensions of urban landscape. The main points of this work are as follows. First, urban form can be treated as pre-fractals rather than real fractals, and fractal properties of cities are only valid within certain scaling ranges. Second, the topological dimension of city fractals based on urban area is 0, thus the minimum fractal dimension value of fractal cities is equal to or greater than 0. Third, fractal dimension of urban form is used to substitute urban area, and it is better to define city fractals in a 2-dimensional embedding space, thus the maximum fractal dimension value of urban form is 2. A conclusion can be reached that urban form can be explored as fractals within certain ranges of scales and fractal geometry can be applied to the spatial analysis of the scale-free aspects of urban morphology. Based on fractal dimension, topological dimension, and embedding space dimension, a set of fractal indexes can be constructed to characterize urban form and growth.

Key words: fractal; fractal dimension; pre-fractal; multifractals; scaling range; fractal cities

1. Introduction
Scientific research starts from description of a phenomenon, and then focuses on understanding its work principle. The simple description is based on measurements, while the complex description relies heavily on mathematical methods (Henry, 2002). In order to describe a city, we try to express it using data. As Lord Kelvin pointed out: “When you can measure what you are speaking about, and express it in numbers, you know something about it; but when you cannot measure it, when you cannot express it in numbers, your knowledge is of a meager and unsatisfactory kind.” (Cited from Taylor, 1983, page 37) Mathematical description is not independent of measurement description, as measurement can be treated as the basic link between mathematics and empirical studies (Taylor, 1983). In order to show the results from a measurement, we should to find the characteristic scale of a thing. A characteristic scale is a special 1-dimensional measure and can be termed characteristic length, which can transform a great number of numbers into a simple number. Unfortunately, in many cases, it is impossible to find a characteristic length to describe a complex system such as a city and a system of cities. If so, we should substitute scaling concept for the characteristic scale concept. Fractal geometry can be regarded as one of the best mathematical tools for scaling analysis.

What is a fractal? This is not a problem for many scientists who are familiar with fractals. A fractal is regarded as a shape that is made of parts similar to the whole in some way (Feder, 1988). Quantitatively, a fractal is defined as a set for which the Hausdorff-Besicovitch dimension is strictly greater than the topological dimension (Mandelbrot, 1983). These definitions are suitable for the classical fractals, which belong to what is called thin fractals. The general concept of fractals is well known, but how to understand fractals is still a problem for specific subjects such as urban geography. A fractal has no characteristic scale and cannot be described with traditional measures such as length, area, volume, and density. The basic parameter used for fractal description is fractal dimension. Because the length of coastline cannot be effectively measured, Mandelbrot (1967) put forward the concept of fractal dimension. Where there is an immeasurable quantity, there is symmetry (Lee, 1988). The discovery of fractals is essentially a discovery of scaling symmetry, namely, the invariance under contraction or dilation. The immeasurability of the length of coastline enlightened Mandelbrot (1989) to think about the problem of contraction-dilation symmetry (Mandelbrot and Blumen, 1989). In urban studies, it is impossible to determine the length of urban boundary and the area within the urban boundary objectively (Batty and Longley, 1994; Frankhauser, 1994). In this case, it is impossible to quantify the population size of a city. The precondition of
determining urban population size is to determine urban boundary line effectively. Population is one of the central variables in the study of spatial dynamics of city development (Dendrinos, 1992), and it represents the first dynamics of urban evolution (Arbesman, 2012). If we cannot measure urban population size, how can we describe a city? If we cannot describe a city, how can we understand the mechanisms of urban evolution? Fortunately, today, we can employ fractal dimension of urban form to replace urban area and urban population size. However, a new problem have emerged: how to define a city fractal and determine its fractal dimension? Although fractal cities have been studied for more than 30 years, some basic problems still puzzle many theoretical geographers. I have studied city fractals for 25 years, focusing on urban fractal modeling and related spatial analysis methods. During this process, I encountered many problems, and I have been thinking of these problems and the related solutions for a long time. Now, I want to express my opinions on fractals and fractal dimension of urban form based on my own long-term research experience on fractal cities. The value of an article does not rest with the correctness of its academic viewpoints, but with the enlightenment and inspiration to readers.

2. Fractal cities and city fractals

2.1 Are cities fractals

Is the coast of Britain a real fractal line? In fact, we cannot find any real fractals (based on fractal geometry) in the real world. This is like that we cannot find circles and triangles (based on Euclidean geometry) in the real world. All of the fractal images we encounter in books and articles represent pre-fractals rather than real fractals in mathematical sense. A real fractal has infinite levels, which can only be revealed in the mathematical world, but a pre-fractal is a limited hierarchy, which can be found in any textbooks on fractals. We can use the ideas from fractal geometry to research pre-fractals, including regular pre-fractals and random pre-fractals. The coast of Britain can be regarded as a pre-fractal curve instead of a real fractal line. However, we can study the coast of Britain using the ideas from fractals and fractal dimension. Similarly, cities are not true fractals, but proved to be random pre-fractals because urban form has no characteristic scales. A great number of empirical studies show that, based on certain scaling range, urban form satisfy three necessary and sufficient conditions for fractals (Table 1). Urban form follow power laws, which indicates that cities can be
treated as pre-fractals. The basic property of a random pre-fractal object is that its scaling range is limited, and its fractal dimension value is based on the scaling range (see, e.g., Addison, 1997).

Table 1 Three necessary and sufficient conditions for fractals

| Conditions        | Formula                                                                 | Note                                                                 |
|-------------------|-------------------------------------------------------------------------|----------------------------------------------------------------------|
| Scaling law       | \( \mathcal{T}(x) = f(\lambda x) = \lambda^b f(x) \)                  | The relation between scale and the corresponding measures follow power laws |
| Fractal dimension | \( d_T < D < d_E \)                                                      | The fractal dimension \( D \) is greater than the topological dimension \( d_T \) and less than the Euclidean dimension of the embedding space \( d_E \). |
| Entropy conservation | \( \sum_i P_i^q r_i^{(1-q)D_q} = 1 \)                                  | Then Renyi entropy values of different fractal units (fractal subsets) are equal to one another. |

Note: \( T \)—scaling transform; \( x \)—scale variable; \( f(x) \)—a function of \( x \); \( \lambda \)—scale factor; \( b \)—scaling exponent; \( D \)—fractal dimension; \( d_T \)—topological dimension; \( d_E \)—Euclidean dimension of embedding space; \( q \)—order of moment; \( P_i, r_i \)—growth probability of the \( i \)th fractal set and its linear scale; \( D_q \)—generalized correlation dimension.

2.2 Fractal geometry: an approach to scale-free analysis

Fractal geometry is a powerful tool for scaling analysis of scale-free phenomena such as urban form. Scaling suggests that there is no characteristic scale in a thing. Cities, in many aspects, have no characteristic scale and cannot be effectively modeled by the conventional mathematical methods. In contrast, urban phenomena can be well characterized by fractal parameters. Natural and social phenomena can be roughly divided into two categories: one is the phenomena with characteristic scales, and the other is the phenomena without characteristic scales. The former can be termed **scaleful phenomena**, and the later can be termed **scale-free phenomena** (Table 2). For the scaleful phenomena, we can find definite length, area, volume, density, eigenvalue, mean value, standard deviation, and so on. If the spatial distribution of this kind of phenomena is converted into a probability distribution, it has clear and stable probability structure and thus can be described with Gaussian function, exponential function, logarithmic function, lognormal function, Weibull function, etc. The conventional higher mathematics can be used as an effective tool for modeling and analyzing such phenomena. On the contrary, for the scale-free phenomena, we cannot find effective length, area, volume, density, eigenvalue, mean value, standard deviation, and so forth. If the spatial distribution of this sort of phenomena is transformed into a probability distribution, it can be characterized with power functions, Cobb-Douglas function (production function), or some type...
of functions including hidden scaling. The probability structure of the scale-free distributions is not certain. Traditional advanced mathematics cannot effectively characterize such phenomena. In recent years, a number of theoretical tools for scale-free analysis are emerging, including fractal geometry, wavelet analysis, allometric theory, and complex network theory. Among various “new” tools, fractal geometry represents an excellent method for scale-free modelling and scaling analysis.

Table 2 Two types of natural and social phenomena: scaleful and scale-free phenomena

| Type                      | Probability distribution                      | Characteristics                                                                 | Example                                                                 | Mathematical tools                                                                 |
|---------------------------|----------------------------------------------|--------------------------------------------------------------------------------|-------------------------------------------------------------------------|-----------------------------------------------------------------------------------|
| Scaleful phenomena        | Normal, exponential, logarithmic, lognormal, | We can find definite length, area, volume, density, eigenvalue, mean value,     | Urban population density distribution, which follows exponential law     | Traditional higher mathematics includes calculus, linear algebra, probability theory and statistics. |
| (with characteristic scales) | Weibull, etc.                                | standard deviation, and so on                                                  |                                                                         |                                                                                   |
| Scale-free phenomena      | Power law, various hidden scaling             | We cannot find effective length, area, volume, density, eigenvalue, mean value, | Urban traffic network density distribution, which follows power law      | Fractal geometry, complex network theory, allometry theory, scaling theory         |
| (without characteristic scale) | distributions                                | standard deviation, and so on                                                  |                                                                         |                                                                                   |

A city is a complex system with multifaceted characteristics. In some aspects, a city has characteristic scales, e.g., urban population density distribution, which follows negative exponential law and can be described with Clark’s model (Clark, 1951). In another aspects, a city has no characteristic scale, e.g., urban traffic network density distribution, which follows inverse power law and can be characterized with Smeed’s model (Smeed, 1963). Where land use is concerned, urban form follow power law distribution and can be treated as random pre-fractal patterns (e.g., Batty and Longley, 1994; Frankhauser, 1994). In this sense, we cannot find effective characteristic scales for urban morphology. As a result, the traditional methods of quantitative analysis and mathematical modeling are often invalid for the research on urban form and growth. As a substitute, fractal geometry is one of feasible mathematical tools for the spatial analysis of cities.
2.3 How to define city fractals?

The angle of view for fractal studies of cities depend on the definition of embedding space. A city fractal can be defined in a 2-dimensional embedding space, and also it can be defined in a 3-dimensional embedding space (Thomas et al., 2012). Majority of fractal cities are defined in a 2-dimension embedding space based on digital maps or remote sensing images (e.g. Batty and Longley, 1994; Benguigui et al., 2000; Frankhauser, 1994). However, some scholars studies fractal cities through 3-dimensional embedding space (e.g., Qin et al., 2015). Many scholars pay attention to urban fractals defined in the 3-dimensional embedding space (Thomas et al., 2012). In fact, a fractal based on the 3-dimensional embedding space can be explored through the 2-dimensional embedding space. In the simplest case, the relationship between the fractal dimension based on 2-dimensional embedding space, \( D^{(2)} \), and the fractal dimension based on 3-dimensional embedding space, \( D^{(3)} \), is as follows, \( D^{(3)} = 1 + D^{(2)} \) (Vicsek, 1989).

Generally speaking, we define the city fractals in a 2-dimensional embedding space. The main reasons are as follows.

First, fractal dimension is used to replace the 2-dimensional urban area rather than the 3-dimensional urban volume. In order to study a city, we must describe a city; in order to describe a city, we must know its basic measures such as population size, urban area, and economic output. Unfortunately, urban form has no characteristic scales due to its fractal properties, and thus urban boundary cannot be objectively determined. Urban area cannot be objectively calculated because the measurement results depend on scales. This the well-known scale-dependence property of urban form, the cause lies in scale-free distribution of urban land use. In this case, fractal dimension of urban form can be employed to replace urban area to reflect the extent of space filling. The fractal dimension as a degree of urban space filling is exactly a substitute of urban area. Urban area is a scale-dependent measure, while fractal dimension is scaleful parameter. In this sense, fractal dimension is more effective than urban area to reflect urban spatial development. By the way, some scholars prefer to define a city fractal in a 3-dimensional space, this means that they try to calculate a fractal dimension based on 3-dimensional embedding space to replace urban volume.

Second, the general principle of model building is based on reduction of dimension. The effective skill of scientific quantitative analysis is to reduce dimension instead of to increase
dimension. The basic relation between spatial dimension $n$ and the degree of analytical complexity $C$ can be expressed as $C=n(n-1)/2$. The well-known Clark’s law of urban population density distribution in a 2-dimensional space is actually based on a 1-dimensional space, but this model reflect the geographical information in a 3-dimensional space (Clark, 1951). In other word, the population distribution in the 3-dimensional space is projected to the 2-dimensional space by population density, and then the mathematical expression is established on the base of the 1-dimensional space with the help of statistical averaging. The same is the case with Smeed’s model on urban traffic density distribution (Batty and Longley, 1994; Smeed, 1963). If we study a city fractal through a 3-dimensional embedding space, the amount of work and difficulty of fractal dimension calculation is considerably increased, and the accuracy of fractal parameter estimation is reduced, but the increment of gained geographic information is very limited. In short, it is hard to promote the analytical effect of fractal cities significantly by substituting the 2-dimensional embedding space with the 3-dimensional embedding space.

**Third, the allometric scaling relation between population and land use suggests that urban form should be defined in a 2-dimensional space.** The allometric scaling exponent $b$ is the ratio of the fractal dimension of urban form $D_f$ to the dimension of urban population $D_p$, that is, $b=D_f/D_p$. Empirical studies show that the $b$ values are close to 0.85. If $D_f\geq2$, then we have $D_p>2/0.85=2.35$. Based on Clark’s law and scaling analysis, urban population distribution proved to be a 2-dimension phenomena ($D_p=2$) (Chen and Feng, 2012). If the urban form is defined in a 3-dimensional embedding space, the fractal dimension $D_f$ values will come between 2 and 3, and the allometric scaling exponent $b$ values will be greater than 1. However, the observational values of allometric scaling exponent $b$ values range from $2/3$ to 1 in the most cases, that is, $2/3<b<1$ (Chen, 2010; Lee, 1989; Louf and Barthelemy, 2014a). This suggests that the dimension of urban form, $D_f$, comes between 1 and 2. In fact, in urban studies, fractal dimension is a concept of comparability. The fractal dimension value depends on the definition of embedding space.

If a city fractal is defined in a 2-dimensional embedding space, the fractal form includes two aspects: urban area and urban boundary. The above discussion is actually based on urban area, but urban boundary can be treated fractal lines (Batty and Longley, 1988; Batty and Longley, 1994; Benguigui et al, 2006; Chen, 2011; Longley and Batty, 1989a; Longley and Batty, 1989b; De Keersmaecker et al, 2003). The closed urban boundary curve is termed *urban envelope*, in which
we can determine a Euclidean urban area (Batty and Longley, 1994; Longley et al, 1991). The length of urban boundary and the Euclidean area within the urban envelope follow the geometric measure relation as follows

\[ A = aL^{2/D_b}, \]  

where \( A \) refers to the Euclidean area of a city, \( L \) denotes the length of urban perimeter, \( a \) is the proportionality coefficient, and \( D_b \) is the fractal dimension of urban boundary, which can be termed boundary dimension (Chen, 2011). In fractal, equation (1) can be generalized to the more general expression as below (Benguigui et al, 2006; Chen, 2013):

\[ A = aL^{D_f/D_b}, \]  

where \( A \) refers to the Euclidean area of a city, and \( D_f \) is the fractal dimension of “urban area”. Equation (2) is in fact an allometric scaling relation of urban shape (Chen, 2013). The topological dimension of urban boundary is \( d_T^b=1 \), so the boundary dimension is greater than 1. The fractal parameter value comes between 1 and 2, that is, \( 1 < D_b < 2 \). Now, a question appears. What determines the lower limit of fractal dimension of urban morphology, urban area or urban boundary? The answer is clear. If we study urban form and try to substitute urban area with form dimension, it is the topological dimension of urban area that determines the least value of the fractal dimension; on the other, if we research urban boundary and attempt to replace urban perimeter length with boundary dimension, it is the topological dimension of urban boundary that determine the minimum value of the fractal dimension. In the most cases, we study urban area (fractal subsets) rather than urban boundary (fractal lines).

2.4 The lower and upper limits of fractal dimension

Fractal dimension values have strict lower limit and upper limit. This is beyond doubt. However, what are the lower limit and upper limit of the fractal dimension of urban from? This is still a pending question. Empirically, if a city fractal is defined in a 2-dimensional embedding space, the fractal dimension value come between 0 and 2 (Chen, 2012; Chen, 2018; Shen, 2002; Thomas et al, 2007). In theory, the lower and upper limits of fractal dimension of urban form rely on the topological dimension and embedding dimension. In many cases, the box-counting method is employed to estimate the fractal dimension values of urban form. The lower limit of the fractal
dimension $D_{\text{min}}$ depends on the topological dimension of urban form $d_T$, while the upper limit $D_{\text{max}}$ depends on the Euclidean dimension of the embedding space $d_E$. As indicated above, the embedding space can be defined as a 2-dimensional space, thus the Euclidean dimension of $d_E=2$, so we have $D_{\text{max}} \leq d_E=2$. As for the topological dimension of urban form, $d_T$, in theory, it should be $d_T=0$. Therefore, we have $D_{\text{min}} \geq d_T=0$.

How to determine the topological dimension of urban form? As we know, the Lebesgue measures of real fractals are zero (Mandelbrot, 1982). This suggests that, if we treated urban form as a fractal, the urban area of land use should be treated as zero. Please note that this is based on theoretical understanding, which is different from reality. How to understand the assumption that the area of urban fractal is 0? This means that an urban fractal can be reduced to either a point set or a space-filling curve under the limit conditions. For a point set, the topological dimension is $d_T=0$; while for a space-filling curve, the topological dimension is $d_T=1$. In fact, using ArcGIS technique, we can reduce a city fractal to a point data rather than a space-filling curve. This indicates that the topological dimension of city fractals is $d_T=0$ instead of $d_T=1$. According to Shen (2002), the box dimension values of Baltimore come between 0.6641 and 1.7211 from 1792 to 1992 year. The time span is about 200 years.

In practice, the lower and upper limits of fractal dimension of urban form depend on the methods of defining study area. There are two approaches to obtaining the time series of the fractal dimension values of urban growth and form (Chen, 2012). One is based on constant study area (Batty and Longley, 1994; Shen, 2002), and the other, based on variable study area (Benguigui et al., 2000; Feng and Chen, 2010). Each approach has its advantages and disadvantages (Table 3). If we define a study area with fixed size for different years, the largest box can be determined by the urban boundary of the recent year. Then, the largest box can be applied to digital maps of the city in previous years (Figure 1(a)). Using the same set of boxes, we estimate the fractal dimension values of urban form in different years. Based on this approach, the fractal dimension values of a city’s form in different years are more comparable. If the sample path is very long, the original urban form can be treated as a point. As a result, the fractal values may come between 0 and 2 (Chen, 2012; Thomas et al., 2007). In contrast, if we define a variable study area, the size of the largest box is determined by the urban boundary in a given year. Thus, the largest boxes are different for different years (Figure 1(b)). Based on this approach, the comparability of fractal dimension values of urban
form in different years is reduced. But these fractal dimension values can better reflect the degree of urban space filling. As a result, the fractal values may come between 1 and 2 (Chen, 2012).

Table 3 Two approaches to defining the study area for fractal dimension estimation of urban form

| Approach            | Property       | Merit                                           | Demerit                                         | Dimension range |
|---------------------|----------------|-------------------------------------------------|-------------------------------------------------|-----------------|
| Constant study area | Fixed size     | The comparability of fractal parameters of different year is strong | Fractal dimension values cannot well reflect the space filling extent. | Come between 0 and 2 |
| Variable study area | Unfixed size   | Fractal dimension values can better reflect the space filling extent. | The comparability of fractal parameters of different year is weak. | Come between 1 and 2 |

Figure 1 The sketch maps for two types of approaches to defining study areas for fractal dimension estimation of urban form (by Chen, 2012)

Note: The square frames surrounding the growing fractals represent the study area of fractal dimension measurements. Figure 1(a) shows a fixed study area, and Figure 1(b) displays a variable study area, the size of which depends on the extent of fractal city cluster.

3. Fractal modeling of urban form

3.1 Two research directions of fractal cities

A complete scientific research process comprises two elements. One is to describe a system, and
the other is to understand the mechanism of the system’s work. In short, scientific studies should proceed first by describing how things work and later by understanding why (Gordon, 2005). Accordingly, scientific method contains two elements: description and understanding. Concretely speaking, as stated by Henry (2002, p14): “The two main elements of this scientific method are the use of mathematics and measurement to give precise determinations of how the world and its parts work, and the use of observation, experience, and where necessary, artificially constructed experiments, to gain understanding of nature.” A comparison between the two elements of scientific process can be drawn as follows (Table 4). The most important method of scientific description is to establish mathematical models.

| Table 4 A complete scientific research process consists of two elements |
|-----------------------------|------------|-----------------|----------------|----------------|-----------------|
| Element             | Level  | Method                          | Purpose                          | Result                            | Finding                               |
| Description         | Macro level | Mathematics, measurement, and computation | Data, numbers | Show characters of a system’s behavior | How a system works |
| Understanding       | Micro level | Observation, experience, experiments, and simulation | Insight, sharpen questions | Reveal dynamical mechanism | Why the system works in this way |

Fractal theory comprise two related parts: one is the scaling theory of complex systems, and the other is the mathematical method known as fractal geometry. As a complex system theory, it can be employed to understand complexity of cities; as a geometry, it can be used to describe cities from the angle of view of scaling analysis. In fact, a mathematical theory plays two roles in any scientific research (Table 5). One is make models and develop theory (mathematical modeling), and the other is process experimental and observational data (statistical analysis). In urban studies, fractal geometry can serve two functions. One is to establish models for cities and systems of cities, and the other is to make empirical analysis of cities using observational data. Many scholars utilize fractal geometry to process the observational data of urban geography, but I emphasize the basic function: mathematical modeling. No matter what types of studies are made, there is no contradiction between models and observed data. All models relies heavily on observational data.
Table 5 Two functions of fractal geometry in urban studies

| Function   | Use                                      | Purpose            | Approach                                      |
|------------|------------------------------------------|--------------------|-----------------------------------------------|
| Theoretical| Present postulates and make models       | Develop urban theory | Build mathematical models based on fractals or fractal dimension |
| Empirical  | Process experiment and observational data | Solve practical problems in reality | Rely heavily on fractal dimension |

In fact, one of the main tasks in scientific research is to make models. As Neumann (1961) said: “The sciences do not try to explain, they hardly even try to interpret, they mainly make models.” I agree with Hamming (1962), who said: “The purpose of modelling is insight, not numbers.” Karlin (1983) has similar viewpoint: “The purpose of models is not to fit the data, but to sharpen the questions.” However, the confidence level of a model depends heavily on the relationship between mathematical expression and observed data. In order to verify a mathematical model, we must fit it to observational data and illustrate the statistical relationships and analytical effect. I am very much in favor of his viewpoint of Louf and Barthelemy (2014b), who said: “The success of natural sciences lies in their great emphasis on the role of quantifiable data and their interplay with models. Data and models are both necessary for the progress of our understanding: data generate stylized facts and put constraints on models. Models on the other hand are essential to comprehend the processes at play and how the system works. If either is missing, our understanding and explanation of a phenomenon are questionable. This issue is very general, and affects all scientific domains, including the study of cities.” The basic functions of mathematical models are explanation and prediction. As Fotheringham and O’Kelly (1989) pointed out, “All mathematical modelling can have two major, sometimes contradictory, aims: explanation and prediction.” Not only that, as Kac (1969) observed, “The main role of models is not so much to explain or predict—although ultimately these are the main functions of science—as to polarize thinking and to pose sharp questions.” The chief uses of fractal models lie in explanation and prediction. Let’s take the logistic model of fractal dimension curves as an example. The model can be used to explain the speed change characteristics of urban growth (Chen, 2018). It can tell us when the growth rate of a city will peak. It can also tell us the maximum space-filling index of a city’s land use. What is more, the model can sharpen questions for us. For example, the similarity and difference between the model of fractal dimension
curves of Chinese cities such as Beijing and that of the cities in western countries such as London, Baltimore and Tel Aviv give rise to new thinking about the spatial dynamics of urban evolution.

3.2 Two approaches to modeling cities

As indicated above, one of important tasks of fractal urban studies it to make models. As Longley (1999, Page 605) pointed out, “In the most general terms, a ‘model’ can be defined as a ‘simplification of reality’, nothing more, nothing less.” In scientific research, mathematical models can be classified into two categories: mechanistic models and parametric models (Su, 1988). Accordingly, there exist two approaches to establishing mathematical models: analytical method and experimental method (Zhao and Zhan, 1991) (Table 6). The so-called analytical method is the approach to deriving a mathematical model with the help of the existing scientific theories and laws, and in light of the relationship and evolution of the various components of the studied system. The process is as follows: establish a functional equation based on one or more postulates, and then find the general solution to the functional equation. The solution to the equation is exactly the theoretical model (mechanistic or structural model) we need. The experimental method is to select a most possible model in a set of hypothetical or imaginary models so that the model can be well fitted to the observational or experimental data. What is more, the model will not give rise to logical contradiction and difficulty in interpretation. Thus we have an empirical model (parametric or functional model). In geography, the traditional gravity model is an empirical model, which is obtained by analogy with Newton's law of universal gravitation. In contrast, the spatial interaction model of Wilson (1968) is a theoretical model. The model is derived by constructing the postulates and solving the maximum entropy equation of traffic flows. The two types of models are not opposed, but can be transformed into each other. An effective theoretical model must be an empirical model, which must be well fitted to observation data. On the other hand, an empirical model will become a theoretical model by mathematical demonstration. A typical example is Clark's urban population density model (Clark, 1991). The model was originally presented as an empirical model based on observation data (Batty and Longley, 1994). However, it has become a theoretical model because it can be derived from the postulates of spatial entropy maximization of urban population distribution (Chen, 2008). In an article, limited to the conditions at the time, we may fulfil some aspect of the research work, not necessarily complete all the research process.
### Table 6 Two types of models and methods of model building

| Model type                  | Property               | Building method     | Principle                      | Example                        |
|-----------------------------|------------------------|---------------------|--------------------------------|--------------------------------|
| Mechanistic model (structural model) | Theoretical model    | Analytical method    | Postulates and demonstration | Wilson’s spatial interaction model |
| Parametric model (functional model) | Empirical model      | Experimental method | Data and fitting               | Traditional gravity model       |

### 3.4 Fractal models and parameters of cities

We have at least three approach to develop mathematical models of urban form by using ideas from fractal theory. The first is to make new models, the second is to improve the old models, and the third is to borrow models from other disciplines (Table 7). A typical example is the models of fractal dimension curve of urban form, different approaches result in different models, and different models are suitable for different situations (Chen, 2012; Chen, 2018). It is necessary to briefly comment on the third way. In scientific research, a mathematical model can be transplanted from a field and applied to another field. The logistic function was originally proposed by Verhulst in 1838 to prediction population growth (Banks, 1994). Today, the well-known logistic function has been employed to predict many growing phenomena in many different fields, including urbanization level and fractal dimension growth (Chen, 2018). Similarly, Boltzmann equation can also be generalized to other fields and to model urban growth (Benguigui et al, 2001; Chen, 2012). The allometric growth equation of urban geography came from biology (Naroll and Bertalanffy, 1956; Chen, 2011). The gravity model of geography resulted from Newton's law of universal gravitation by analogy, and the spatial autocorrelation models of geography come from mathematical biology. These examples are too numerous to enumerate. The uniqueness of different fields is always determined by the physical meaning of model parameters rather than by the expression of mathematical models. The mathematical expression of the model is often general, but the parameters are for special purposes. The same mathematical model can be applied to many different fields, but different fields have different parameter meanings.

### Table 7 Three approaches to develop models for fractal dimension curves of urban form
The notion of maximum and minimum of fractal dimension discussed above is important for making models of the fractal dimension curves of urban form. The fractal dimension curve results from the time series of urban growth. In theory, we can calculate the fractal dimension values of a city’s form in different times. This values compose a sample path of fractal dimension, and further form a curve of fractal dimension change of urban morphology. A sample path can be regarded as a subset of a time series (Diebold, 2007). Due to the lower and upper limits of urban fractal dimension, a fractal dimension curve takes on squashing effect and can be described with one of sigmoid functions such as logistic function and Boltzmann’s equation (Chen, 2012; Chen, 2014; Chen, 2018). On the other hand, how to determine fractal parameter values depends on specific research objectives and data conditions. This is a complex problem and needs to be judged on the basis of long-term research experience. Even for theoretical research, if the sample path of fractal dimension is short, we can take $D_{\text{min}}=1$ and adopt the quadratic Boltzmann equation. For example, in one of studies made by Chen (2018), the time span is about 25 years (1984-2008). All the fractal dimension values are greater than 1. On the other hand, even for application research, if the sample path of fractal dimension is very long, we can take $D_{\text{min}}=0$ and adopt the quadratic logistic function. For instance, for the study of Shen (2002), the time span is about 200 years (1792-1992). One of the fractal dimension values for early years is less than 1. The situations can be classified into four groups and tabulated as below (Table 8). In the revised manuscript, we make two models to predict Beijing’s urban growth. One is the quadratic logistic model based on $D_{\text{min}}=0$, and the other is the
quadratic Boltzmann model based on $D_{\text{min}}=1$. The effect of and conclusions from the two models are similar to one another.

Table 8 Four cases for the lower limit of fractal dimension curves of urban form

|                | Fixed study area                               | Variable study area                                           |
|----------------|-----------------------------------------------|--------------------------------------------------------------|
| **In theory**  | $D_{\text{min}}=0$, logistic function         | $D_{\text{min}}=0$, long sample path, logistic function;     |
|                |                                               | $D_{\text{min}}=1$, usual cases, Boltzmann equation          |
| **In practice**| $D_{\text{min}}=1$, short sample path, Boltzmann equation; $D_{\text{min}}=0$, usual cases, logistic function | $D_{\text{min}}=1$, Boltzmann equation                      |

4. Questions and discussion

4.1 Problems of fractal dimension values

The concept of fractal dimension proceeded from Hausdorff’s fractional dimension. Today, there are various definitions for fractal dimension, and the common fractal dimensions in urban studies is box dimension and similarity dimension. The box dimension is mainly suitable for the spatial structure of cities and systems of cities, while the similarity dimension is chiefly applied to urban hierarchies, including hierarchies of cities and hierarchies of urban internal elements such as land use patches. Generally speaking, fractal dimension values come between the topological dimension and the Euclidean dimension of embedding space. For a regular fractal, if fractal copies/units have no overlapping, its Hausdorff dimension will equal similarity dimension. Empirically, both Hausdorff dimension and similarity dimension can be represented with box dimension. All these dimension values are less than the Euclidean dimension of the embedding space and greater than the topological dimension of fractal objects. However, if fractal copies have overlapped parts, the similarity dimension will exceed the dimension of embedding space in value. Thus, similarity dimension will not equal Hausdorff dimension or box dimension. In contrast, the box dimension will never exceed the embedding dimension.

Let’s examine two kinds of fractal dimension of the fractals with overlapped parts. The interior boundary line of the Sierpinski gasket is a typical fractal line with overlapped parts (Figure 2). The initiator is a straight line segment with length of unit (Figure 3(a)), the generator is a curve consisting
of 5 straight line segments with length of 1/2 unit (Figure 3(b)). From step 3 on, fractal copies begin to overlap one another, and the overlapped parts are marked with red circles (Figure 3(c), Figure 3(d)).

![Diagram](image)

**Figure 2** The interior boundary line of the Sierpinski gasket (The first four steps)

![Diagram](image)

**Figure 3** A special fractal line with overlapped parts (The first four steps)

The similarity dimension and box dimension can be calculated by the ideas from fractal dimension. In the $m$th step, the length (linear size) of line segments can be expressed as

$$s_m = \left(\frac{1}{2}\right)^{m-1},$$

where $m=1,2,3,\ldots$ denotes the ordinal numeration of steps. The number of line segments in each step can be counted by two different ways. One is to repeat counting the overlapped parts, and the other is to count the overlapped parts only one times. For example, for the curve of step 3 (Figure 2(c), Figure 3(c)), the number of line segments is $N_3=5^2=25$ according to the first counting way, and $N_3=3\times5+2^2=19$ according to the second counting way. According to the first way with repeated counting, the line segment number in the $m$th step is

$$N_m = 5^{m-1}. \quad (4)$$

Thus the *similarity dimension* is
\[ D_s = \frac{\ln(N_{m+1} / N_m)}{\ln(s_{m+1} / s_m)} = \frac{\ln 5}{\ln 2} = 2.322 > d = 2. \]  \hspace{1cm} (5)

According to the second way without repeated counting, the line segment number of step \( m \) is

\[ N_m = 3N_{m-1} + 2^{m-1}, \]  \hspace{1cm} (6)

where \( N_0 = 0 \) for \( m = 1 \). By recurrence, we have

\[ N_m = \sum_{j=0}^{m-1} \left( \frac{3^{m-1-j}}{2^j} \right) = 3^{m-1} \sum_{j=0}^{m-1} \left[ \frac{2}{3} \right]^j, \]  \hspace{1cm} (7)

where \( j = 1, 2, \ldots, m-1 \). Under the condition of limit, the result is

\[ N_m = \lim_{m \to \infty} \left[ 3^{m-1} \sum_{j=0}^{m-1} \left( \frac{2}{3} \right)^j \right] = 3^{m-1} \frac{1}{1 - 2/3} = 3^m. \]  \hspace{1cm} (8)

This suggests that when \( m \) becomes large enough, \( N_m \) will approaches \( 3^m \). So the box dimension is

\[ D_b = -\frac{\ln N_m}{\ln s_m} = \frac{m \ln 3}{(m - 1) \ln 2} \xrightarrow{m \to \infty} \frac{\ln 3}{\ln 2} \approx 1.585 < d = 2. \]  \hspace{1cm} (9)

For this special regular fractal, box dimension equals Hausdorff dimension in theory. Therefore, for the regular monofractals with overlapped units, we have the following relation: Hausdorff dimension = box dimension < embedding space dimension < similarity dimension. However, for the regular monofractals without overlapped units, the dimension relation is as follows: Hausdorff dimension = box dimension = similarity dimension < embedding space dimension.

### 4.2 New measurements based on fractal dimension

Fractal dimension is a measure for scale-free phenomena, which have no characteristic scales and cannot be effectively described by traditional mathematical methods. Where cities is concerned, the meanings and uses of fractal dimension of urban form rest with at least three aspects: degree of space filling, degree of spatial uniformity, degree of spatial complexity. As a space-filling index, fractal dimension can be used to reflect the replacement process of urban and rural space in theory. Unfortunately, it is both impossible and unnecessary to distinguish between urban area and rural area strictly. When we define a study area for a fractal cities, it comprises urban buildings, rural buildings, and other types of land. Various types of land form a hierarchy with cascade structure of land use based on different levels of scales (Kaye, 1989). In the urban regions, there are rural buildings, and in the rural regions, there are urban buildings. If we examine a city’s form from
various spatial scales, we can find interlaced distributions of urban and rural land and buildings. The hierarchy with cascade structure of urban and rural landscapes should be described with multifractals (Chen, 2016). To solve the problem, we can use the concepts space-filling extent, $U(t)$, and space-saving extent, $V(t)$, to replace urban land use and rural land use (Chen, 2012).

Fractal dimension can be treated as a basic measure of urban growth, and this measure is used to replace urban area. As indicated once and again above, due to scale-dependence of urban spatial measurements, urban area cannot be objectively determined, while fractal dimension is a scale-free parameter, which can be employed to substitute urban area to reflect space filling and land use extent. Based on fractal dimension of urban form, a set of urban measurements or indexes can be defined to describe city development. The measurements are tabulated as follows (Table 9). (1) Fractal dimension range, the difference between the upper limit and lower limit of fractal dimension values, $D_{\text{max}}-D_{\text{min}}$. (2) Space-filling degree, the difference between the fractal dimension value at time $t$ and the lower limit of fractal dimension value, $D(t)-D_{\text{min}}$. (3) Space-saving degree, or space-remaining degree, the difference between the upper limit of fractal dimension value and the fractal dimension value at time $t$, $D_{\text{max}}-D(t)$. (4) Space-filling ratio, the ratio of space-filling degree to fractal dimension range, $(D(t)-D_{\text{min}})/(D_{\text{max}}-D_{\text{min}})$. (5) Space-saving ratio, or space-remaining ratio, the ratio of space-saving degree to fractal dimension range, $(D_{\text{max}}-D(t))/(D_{\text{max}}-D_{\text{min}})$. (6) Fractal dimension odd, the ratio of space-saving degree to the fractal dimension value at time $t$, $(D_{\text{max}}-D(t))/D(t)$. The basic relationships between these indexes are as below: (a) The space-filling degree plus space-saving degree equals fractal dimension range; (b) The space-filling ratio plus space-saving ratio equals 1; (c) If $D_{\text{min}}=0$, then the space-saving degree divided by space-filling degree equals fractal dimension odd. In fact, the space-filling ratio is a normalized fractal dimension, and the normalized fractal dimension proved to equal the normalized spatial entropy of urban form (Chen, 2012; Chen, 2018). The spatial entropy reflects the land-use extent of an urban region, namely, the degrees of space-filling and spatial uniformity.

| Measurement (fractal index) | Definition | Special case 1 ($D_{\text{min}}=0$) | Special case 2 ($D_{\text{min}}=0$, $D_{\text{max}}=d_E$) | Meaning |
|----------------------------|------------|----------------------------------|-------------------------------------------------|---------|
| Fractal dimension range    | $D_{\text{max}}-D_{\text{min}}$ | $D_{\text{max}}$             | $d_E$                                           | Available space |
The analytical process and discussion of this paper is based on the standard definition of fractals. A fractal has three elements, i.e., form, chance, and dimension (Mandelbrot, 1977). The first definition of Mandelbrot (1982, page 15) based on dimension and chance is as follows: “A fractal is by definition a set for which the Hausdorff -Besicovitch dimension strictly exceeds the topological dimension.” The second definition based on form and chance is as below: “A fractal is a shape made of parts similar to the whole in some way.” The second definition is given by Mandelbrot but published by Feder (1988, page 11). The quantitative criterion of fractals is Hausdorff -Besicovitch dimension. Recent years, Jiang and his co-workers tried to relax the definition of fractals and give the third definition as follows: A set or pattern is fractal if the scaling of far more small things than large ones recurs multiple times (Jiang and Yin, 2014). According to the new definition, the quantitative criterion of fractals is replaced by the head/tail index (Jiang, 2013; Jiang, 2015): the ht-index of a fractal set or fractal pattern is at least three (Jiang and Yin, 2014). The new definition and criterion of fractals are very interesting and instructive. Unfortunately, I am a conservative person who respects very much the fractal definition based on strict mathematical thinking. The new fractal definition goes beyond my understanding for the time being, and this results in a shortcoming of this study.

Definitions of concepts or terms are most likely to lead to ambiguity, misunderstanding, and controversy. Therefore, scientific should sidesteps the terminological minefield so that we can move beyond the semantic debate (Gallagher and Appenzeller, 1999). On the one hand, we should leave certain room for developing and consolidating a definition as the research approach continues to
5. Conclusions

Fractal geometry provides us a new mathematical framework of describing urban morphology. To characterize urban form and explain urban growth, we need various fractal dimensions. To understand fractal dimension concept, we should know the notions of topological dimension and Euclidean dimension of embedding space in which fractal cities are defined. Fractal theory can be employed to make spatial analysis for the scale-free aspects of urban morphology. The main points of this paper can be summarized as follows.

First, fractal geometry is powerful tool of scale-free analysis, and urban morphology is typical scale-free geographical phenomenon. Therefore, fractal theory can be naturally applied to urban studies. Cities are not true fractals, but they can be treated as random pre-fractals, which bear fractal properties within certain scaling ranges. If urban form had characteristic scales, we would be able to calculate urban area and urban perimeters. Thus urban form can be described with the methods from traditional advanced mathematics. Unfortunately, urban form has no characteristic scales, it belong to scale-free distributions. A great many studies show that urban form follow power laws indicative of fractal nature. In this case, it is a advisable selection to employ fractal geometry to describe urban morphology and make scaling analysis of urban patterns and processes.
Second, the most proper dimension of embedding space for city fractals is 2 dimension rather than 3 dimension. The upper limit of fractal dimension of urban form should not exceed the embedding dimension. A city fractal can be defined in a 2-dimensional space, and it can also be defined in a 3-dimensional space. It is better to define city fractals in a 2-dimensional space. On the one hand, fractal dimension is used to replace urban area, which cannot be objectively measured due to scale-free distribution of cities. Urban area is a measure defined in 2-dimensional space. Therefore, city fractals can be defined in 2-dimensional space so that fractal dimension can be employed to well replace urban area. On the other hand, the criterion of scientific method is to reduce dimensions rather than increase dimensions. Moreover, more available datasets of cities are based on 2-dimensional space. It is simpler and more effective to analyze a city fractal through 2-dimensional space.

Third, the topological dimension of urban form is 0 dimension rather than 1 dimension. The lower limit of fractal dimension is equal to or greater than the topological dimension. In theory, urban form can be reduced to point sets, so the topological dimension of city fractals is $d_T=0$. The lower limit of fractal dimension of urban form is $D_{\text{min}}=0$. The topological dimension of urban boundary is 1, but the most important city fractals are based on urban area instead of urban boundary. In practice, the lower limit of fractal dimension of urban form can also be treated as $D_{\text{min}}=1$ especially when the sample path is short. Based on the constant study area and fixed largest box, the lower limit of fractal dimension of urban form should be taken as $D_{\text{min}}=0$. Based on the variable study area and unfixed largest box, the lower limit of fractal dimension of urban form should be taken as $D_{\text{min}}=1$. Based on the constant study area, fixed largest box and long sample path (time span is very large), the fractal dimension values of urban form is sometimes $D<1$. How to take the $D_{\text{min}}$ value, it depends on the concrete situation.

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References
Addison PS (1997). *Fractals and Chaos: An Illustrated Course*. Bristol and Philadelphia: Institute of Physics Publishing.

Arbesman S (2012). *The Half-Life of Facts: Why Everything We Know Has An Expiration Date*. New York: Penguin Group

Banks RB (1994). *Growth and Diffusion Phenomena: Mathematical Frameworks and Applications*. Berlin Heidelberg: Springer-Verlag

Batty M, Longley PA (1994). *Fractal Cities: A Geometry of Form and Function*. London: Academic Press

Batty M, Longley PA (1988). The morphology of urban land use. *Environment and Planning B: Planning and Design*, 15(4): 461-488

Benguigui L, Blumenfeld-Lieberthal E, Czamanski D (2006). The dynamics of the Tel Aviv morphology. *Environment and Planning B: Planning and Design*, 33: 269-284

Benguigui L, Czamanski D, Marinov M (2001). City growth as a leap-frogging process: an application to the Tel-Aviv metropolis. *Urban Studies*, 38(10): 1819-1839

Benguigui L, Czamanski D, Marinov M, Portugali J (2000). When and where is a city fractal? *Environment and Planning B: Planning and Design*, 27(4): 507–519

Chen YG (2008). A wave-spectrum analysis of urban population density: entropy, fractal, and spatial localization. *Discrete Dynamics in Nature and Society*, vol. 2008, Article ID 728420, 22 pages

Chen YG (2011). Derivation of the functional relations between fractal dimension and shape indices of urban form. *Computers, Environment and Urban Systems*, 35(6): 442–451

Chen YG (2012). Fractal dimension evolution and spatial replacement dynamics of urban growth. *Chaos, Solitons & Fractals*, 45 (2): 115–124

Chen YG (2013). A set of formulae on fractal dimension relations and its application to urban form. *Chaos, Solitons & Fractals*, 54(1): 150-158

Chen YG (2014). Urban chaos and replacement dynamics in nature and society. *Physica A: Statistical Mechanics and its Applications*, 413: 373-384

Chen YG (2016). Defining urban and rural regions by multifractal spectrums of urbanization. *Fractals*, 2016, 24(1): 1650004

Chen YG (2018). Logistic models of fractal dimension growth of urban morphology. *Fractals*, 26(1): 1850033

Chen YG, Feng J (2012). Fractal-based exponential distribution of urban density and self-affine fractal forms of cities. *Chaos, Solitons & Fractals*, 45(11): 1404-1416

Clark C (1951). Urban population densities. *Journal of Royal Statistical Society*, 114(4): 490-496

De Keersmaecker M-L, Frankhauser P, Thomas I (2003). Using fractal dimensions for characterizing
intra-urban diversity: the example of Brussels. *Geographical Analysis*, 35(4): 310-328

Dendrinos DS (1992). *The Dynamics of Cities: Ecological Determinism, Dualism and Chaos*. London and New York: Routledge

Diebold FX (2007). *Elements of Forecasting* (4th edition). Mason, Ohio: Thomson/South-Western

Feder J (1988). *Fractals*. New York: Plenum Press

Feng J, Chen YG (2010). Spatiotemporal evolution of urban form and land use structure in Hangzhou, China: evidence from fractals. *Environment and Planning B: Planning and Design*, 37(5): 838-856

Fotheringham AS, O’Kelly ME (1989). *Spatial Interaction Models: Formulations and Applications*. Boston: Kluwer Academic Publishers, page 2.

Frankhauser P (1994). *La Fractalité des Structures Urbaines (The Fractal Aspects of Urban Structures)*. Paris: Economica

Gallagher R, Appenzeller T (1999). Beyond reductionism. *Science*, 284: 79

Gordon K (2005). The mysteries of mass. *Scientific American*, 293(1): 40-46/48

Hamming RW (1962). *Numerical Methods for Scientists and Engineers*. New York: McGraw-Haw

[Quoted in Van der Leeuw S.E. and McGlade J. (1997 eds.) *Time, Process and Structured Transformation in Archaeology*. London and New York: Routledge, page 57]

Henry J (2002). The Scientific Revolution and the Origins of Modern Science (2nd Edition). New York: Palgrave, page 14

Jiang B (2013). Head/tail breaks: A new classification scheme for data with a heavy-tailed distribution. *The Professional Geographer*, 65 (3), 482–494

Jiang B (2015). Head/tail breaks for visualization of city structure and dynamics. *Cities*, 43, 69–77

Jiang B, Yin J (2014). Ht-index for quantifying the fractal or scaling structure of geographic features. *Annals of the Association of American Geographers*, 104(3): 530-541

Kac M (1969). Some mathematical models in science. *Science*, 166: 695-699

Karlin S (1983). Eleventh RA Fischer Memorial Lecture. *Royal Society* [20 April 1983]

Kaye BH (1989). *A Random Walk Through Fractal Dimensions*. New York: VCH Publishers

Lee TD (1988). *Symmetries, Asymmetries, and the World of Particles*. Seattle and London: University of Washington Press

Lee Y (1989). An allmetric analysis of the US urban system: 1960–80. *Environment and Planning A*, 21(4): 463–476

Longley PA (1999). Computer simulation and modeling of urban structure and development. In: M. Pacione (ed). *Applied Geography: Principles and Practice*. London and New York: Routledge., pp605-619

Longley PA, Batty M (1989a). Fractal measurement and line generalization. *Computer & Geosciences*, 15(2): 167-183

24
Longley PA, Batty M (1989b). On the fractal measurement of geographical boundaries. *Geographical Analysis*, 21(1): 47-67

Longley PA, Batty M, Shepherd J (1991). The size, shape and dimension of urban settlements. *Transactions of the Institute of British Geographers (New Series)*, 16(1): 75-94

Louf R, Barthelemy M (2014a). How congestion shapes cities: from mobility patterns to scaling. *Scientific Reports*, 4, 5561

Louf R, Barthelemy M (2014b). Scaling: lost in the smog. *Environment and Planning B: Planning and Design*, 41: 767-769

Mandelbrot BB (1967). How long is the coast of Britain? Statistical self-similarity and fractional dimension. *Science*, 156: 636-638

Mandelbrot BB (1977). *Fractals: Form, Chance, and Dimension*. San Francisco: W. H. Freeman

Mandelbrot BB (1983). *The Fractal Geometry of Nature*. New York: W. H. Freeman and Company

Mandelbrot BB, Blumen A (1989). Fractal geometry: what is it, and what does it do? *Proceedings of the Royal Society of London A: Mathematical and Physical Sciences*, 423 (1864): 3-16

Naroll RS, Bertalanffy L von (1956). The principle of allometry in biology and social sciences. *General Systems Yearbook*, 1: 76-89

Neumann J von (1961). *Collected works (Vol.6)*. New York/Oxford: Pergamon Press, page492

Qin J, Fang CL, Wang Y, Li QY, Zhang YJ (2015). A three dimensional box-counting method for estimating fractal dimension of urban form. *Geographical Research*, 34 (1): 85-96 [In Chinese]

Shen G (2002). Fractal dimension and fractal growth of urbanized areas. *International Journal of Geographical Information Science*, 16(5): 419-437

Smeed RJ (1963). Road development in urban area. *Journal of the Institution of Highway Engineers*, 10(1): 5-30

Su MK (1988). *Principle and Application of System Dynamics*. Shanghai: Shanghai Jiao Tong University press6 [In Chinese]

Taylor PJ (1983). *Quantitative Methods in Geography*. Prospect Heights, Illinois: Waveland Press

Thomas I, Frankhauser P, Badariotti D (2012). Comparing the fractality of European urban neighbourhoods: do national contexts matter? *Journal of Geographical Systems*, 14(2): 189-208

Thomas I, Frankhauser P, De Keersmaecker M-L (2007). Fractal dimension versus density of built-up surfaces in the periphery of Brussels. *Papers in Regional Science*, 86(2): 287-308

Vicsek T (1989). *Fractal Growth Phenomena*. Singapore: World Scientific Publishing Co.

West D, West BJ (2013). Physiologic time: a hypothesis. *Physics of Life Reviews*, 10(2): 210–224

Wilson AG (1968). Modelling and systems analysis in urban planning. *Nature*, 220: 963-966

Zhao CY, Zhan YH (1991). *Foundation of Control Theory*. Beijing: Tsinghua University press6 [In Chinese]