Harmonic oscillator force between heavy quarks

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A renormalization group procedure for effective particles is applied to quantum chromodynamics of one flavor of quarks with large mass $m$ in order to calculate light-front Hamiltonians for heavy quarkonia, $H_\lambda$, using perturbative expansion in the coupling constant $\alpha_\lambda$. $\lambda$ is the renormalization group parameter with the interpretation of an inverse of the spatial size of the color charge distribution in the effective quarks and gluons. The eigenvalue equation for $H_\lambda$ couples quark-antiquark states with sectors of a larger number of constituents. The coupling to states with more than one effective gluon, and interactions in the quark-anti-quark-gluon sector, are removed at the price of introducing an ansatz for the gluon mass, $\mu$. The simplified equation is used to evaluate a new Hamiltonian of order $\alpha_\lambda$ that acts only in the effective quark-anti-quark sector and in the non-relativistic limit turns out to contain the Coulomb term with Breit-Fermi corrections and spin-independent harmonic oscillator term with frequency $\omega = ((4/3)(\alpha_\lambda/\pi))^{1/2}(\lambda/m)^3/(\pi(1152)^{1/4})$. The latter originates from the hole excavated in the overlapping quark self-interaction gluon clouds by the exchange of effective gluons between the quarks. The new term is largely independent of the details of $\mu^2$ and in principle can fit into the ball park of phenomenology. The first approximation can be improved by including more terms in $H_\lambda$ and solving the eigenvalue equations numerically.

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I. INTRODUCTION

The purpose of this article is to describe a procedure that starts from a quantum chromodynamics (QCD) with only one flavor of massive quarks and produces the Schrödinger equation for heavy quarkonia in a single formulation of the theory. Only the first approximation for the final Hamiltonian is evaluated. In this simplest version, the procedure involves a guess for the gluon mass term. But the guess appears to have little influence on the result. The procedure itself is not limited to the simplest version and the gluon mass ansatz can be tested in future in refined calculations and phenomenology. The procedure is relativistic and can be used for quarkonia in arbitrary motion, which is a pre-requisite for application in high-energy processes. Chiral symmetry is explicitly broken in the case of heavy quarks and the issue of the spontaneous breaking of the symmetry is ignored.

The approach described here stems from the similarity renormalization group procedure for Hamiltonians [1], which has been applied to QCD [2] using the light-front (LF) form of dynamics [3]. A new ingredient is the boost-invariant creation and annihilation operator calculus for effective quarks and gluons (see below). Otherwise, the LF approach is known for a long time, mainly as a candidate for connecting two qualitatively different models of hadrons: the parton model in the infinite momentum frame (IMF) [2, 3] and the constituent model in the rest frame of a hadron [4, 5]. Many contributions in that area [6, 7, 8] have followed the seminal work on exclusive processes [9, 10]. Through the boost invariance and precisely defined notion of effective constituents, the approach described here aims at providing a bridge between the two models of hadrons in a single theoretical framework. The case of a heavy quarkonium is chosen here as the simplest one to begin with and test the method.

In the LF dynamics, the evolution of states is traced along the direction of a light-like four-vector $n^\mu$, for which $n^2 = 0$. With the conventional choice of $n = (1, 0, 0, -1)$, the variable $n x = x^0 + x^3$, $n = x^+$ plays the role of time while $P^\mu = P^0 - \lambda \rho_3$ is the Hamiltonian. In order to define it for bare particles in QCD, one has to choose a gauge. No serious alternative exists to $n A = A^+ = 0$. But the equation $D_\mu F^{\mu+} = j^+$ implies a constraint that is analogous to the Gauss law and forces the Fourier components of $A^+$ to contain inverse powers of the kinematical momentum $k^+$. Since $k^+$ ranges from 0 to infinity, the inverse powers of $k^+$ produce singularities in the region around zero.

One can impose a lower bound on $k^+$, such as $k^+ > \delta^+$, to regulate the theory [11, 12]. The parameter $\delta^+$ becomes a smallest unit of momentum that any particle, physical or virtual, can carry in such discretized theories. But the fixed unit breaks the boost invariance required for connection between the IMF and the rest frame of any hadron. Namely, when some physically relevant $P^+$ is made large, the smallest allowed $x = \delta^+ / P^+$ becomes small. In the IMF $P^+ \to \infty$ and the same small-$x$ divergences re-appear despite the presence of $\delta^+$. The key singularity is related to the $dx/x$ distribution of gluons in the parton model and seems to require a dynamical mechanism to remove. One cannot just vary $\delta^+$ together with $P^+$ because boosts cannot change the cutoff in a quantum theory constructed ab initio [12].

There exists a possibility that the small-$k^+$ singularities are related to the properties of the vacuum state. The sum rules for heavy quarkonia [13] include quark and gluon condensates [14] that may participate in the dynamics in the small-$k^+$ region [2, 3, 15, 16, 17, 18, 19]. In the QCD picture with such nontrivial ground state [21, 22], and relativistic bound-state excitations of this
state in the form of $\pi$-mesons, one can hardly hope to resolve the small-$k^+$ singularity easily. The situation simplifies a lot in the case of quarks with mass $m \gg \Lambda_{QCD}$. The small-$x$ singularity and a non-perturbative binding mechanism for quarks and gluons can interplay with each other without interference from the vacuum. In the case of light quarks, a similar interplay may be at work and contribute to the saturation mechanism of partons [24, 25, 26], but that issue is not addressed here.

The gluon mass ansatz is introduced to represent effects of the non-abelian interactions. The ansatz is inserted at the level of solving the eigenvalue equation for the Hamiltonian $H_{\lambda}$, where $\lambda$ is the renormalization group (RG) parameter. The procedure of introducing the mass ansatz is similar to the one proposed in [2] and later discussed in simplified matrix models [27, 28]. New elements are the limitation of power counting to the relative-motion variables, the exact boost invariance, removal of small-$x$ divergences through the mass ansatz as a function of the relative momenta, and no need for ad hoc potentials in the first approximation. All these features will be described in detail later.

In brief, $H_{\lambda} = T_{\lambda} + V_{\lambda}$, where $T_{\lambda}$ is the kinetic energy operator made of all terms that are bilinear in the creation and annihilation operators for the effective particles, and $V_{\lambda}$ represents all other terms. The parameter $\lambda$ defines the width of momentum-space form factors in $V_{\lambda}$. For some value of $\lambda = \lambda_0 \sim 1$ GeV, one can freely add to $H_{\lambda_0}$ a term of the form $[1 - (\alpha_s/\Lambda_0)^n]\mu^2$, where $n \geq 2$. The number $2$ ensures that corrections to the first approximation occur first in the fourth-order of expansion of $H_{\lambda_0}$ in powers of $g_0 = g_{\lambda_0}$, $\alpha_0 = g_0^2/(4\pi)$. This is the lowest order at which a perturbative shift in the gluon energy in any state can influence the contribution of that state to the dynamics of any other state in perturbation theory, keeping intact the QED-like small coupling expansion scheme with a Coulomb potential. $\mu^2$ stands for the mass term that one assigns to the effective gluons of the transverse size $1/\lambda_0$. The interpretation of $1/\lambda$ as the spatial size of the color charge distribution in the corresponding effective particles is based on the feature mentioned above that $V_{\lambda}$ contains vertex form factors of width $\lambda$ in momentum space. The mass ansatz contributes to the invariant masses through $\mu^2/x$, where $x$ is the fraction of the longitudinal momentum carried by the gluon. $\alpha_s$ denotes the large, relativistic value of the coupling constant in QCD at the scale $\Lambda_0$ with a true $\Lambda_{QCD}$ in this scheme. The term with $\mu^2$ vanishes for $\alpha_0 = \alpha_s$. Nevertheless, only $\mu^2$ counts when the ratio $\alpha_0/\alpha_s$ is small. Thus, in the weak coupling expansion,

$$H_{\lambda_0} = T_{\lambda_0} + \mu^2 + [V_{\lambda_0} - (\alpha_0/\alpha_s)^n\mu^2]$$

and the term $[V_{\lambda_0} - (\alpha_0/\alpha_s)^n\mu^2]$ is treated as a source of small corrections in comparison to $\mu^2$.

The requirement of cancellation of the small-$x$ singularities in the effective dynamics imposes some perturbatively determined constraints on the otherwise non-perturbative ansatz for $\mu^2$. These constraints restrict the behavior of $\mu^2$ as a function of the gluon motion with respect to other constituents. In future, the refined versions of the same procedure may provide constraints that come closer to the actual behavior of gluons. This behavior is hoped to be uncovered in computer simulations that one may build around the first approximation. Then, the extrapolation to $\alpha_0 = \alpha_s$ can recover the original theory from a few terms in the weak-coupling expansion if $\mu^2$ approximates the behavior of effective gluons well. Initially, the gluon mass term is viewed as a function of the relative momenta and $\Lambda_{QCD}$. The latter depends on $\alpha_0$ as $\Lambda_0 \exp(-c/\alpha_0)$ with a positive constant $c$. This means that $\Lambda_{QCD}$ vanishes to all orders in the perturbative expansion, and $\mu^2$ is considered to be on the order of $1$. Before one knows more, $\mu$ can only be estimated on the basis of implications for the resulting Schrödinger equation. The size of $\mu$ can be compared with four scales: $\Lambda_{QCD}$, $\lambda_0$, $m$, and the Bohr momentum scale, $k_B = \alpha_0 m/2$, which is distinguished non-perturbatively by the Coulomb interaction. But if $\mu^2$ is right, then $[V - (\alpha_0/\alpha_s)^n\mu^2]$ must be a source of only small corrections in the whole range of couplings between 0 and $\alpha_s$. This is taken for granted in the present article. Since the first approximation turns out to be not sensitive to the details of $\mu$, new information can be obtained only in the refined calculations.

The value of the weak coupling expansion scheme for Hamiltonians is that it starts from a local theory and leads to $H_{\lambda}$ that is capable of describing physically relevant non-perturbative dynamics even if $H_{\lambda_0}$ is calculated only in low orders. This idea is known to work in the case of QED: the Coulomb potential accounts for highly non-perturbative dynamics of atoms, including the nature of chemical bond, while the Hamiltonian itself is only of the formal order of $\alpha$. Condensed matter physics illustrates this point in still wider domain.

But when one applies the weak coupling expansion idea to QCD [2], one faces the fact that the strong coupling constant rises to 10, 30, or even 100 times larger value than in the case of atoms or positronium in QED. This leads to a complex interplay between the perturbative and non-perturbative parts of the calculation, enhanced dependence of observables on the RG parameter $\lambda$, problems with obtaining the Poincare symmetry in solutions, and amplification of artifacts due to the small-$k^+$ regularization. Most of the problems seem to come from perturbation theory in the RG part of the calculation. An exact RG procedure by definition provides $H_{\lambda}$ whose structure depends on $\lambda$ but the spectra and $S$-matrix elements do not. However, when one uses expansion in powers of $\alpha_0$ and then extrapolates to $\alpha_0 = \alpha_s$, a considerable dependence of the eigensolutions on $\lambda$ can ensue because of missing many terms. This is visible in models that are asymptotically free and produce bound states [28, 29]. One has to find the right value for $\alpha_s$ at given $\lambda_0$ from fits to bound-state observables, and perform consistency checks for whole sets of different observables [30]. In QCD, such checks involve the unknown functions of
momenta in the finite parts of the ultraviolet counterterms, unknown terms depending on $\Lambda_{QCD}$, and artifacts of the regularization of small-$k^+$ divergences. So many unknowns suggest the possibility that the approach may never achieve the desired level of predictive power. But the simplicity of the harmonic potential found here in the first approximation illustrates that there is a high degree of order in the rich structure of $H_0$. It is a consequence of preserving all seven kinematical symmetries of the LF scheme in the RG procedure. These symmetries limit the large number of terms that are allowed by the LF power counting using absolute momentum variables $\mathbf{p}$, to a much smaller number of terms that depend only on the relative momenta of the constituents. The LF symmetries must also be respected by the initial regularization prescription for the required counterterms to be simple.

The renormalization group procedure for effective particles (RGEP), which is employed here, regulates the ultra-violet and small-$k^+$ divergences by vertex factors $r$ that are inserted only in the interaction terms. In the case of the small-$k^+$ singularities, these factors limit only the ratios of momenta $k^+$. The ratios are limited from below by a dimensionless parameter $\delta$. Details of the factors $r_3$ are explained later. Every creation or annihilation operator, labeled by momentum $k^+$, enters the initial interaction Hamiltonian together with a corresponding factor $r_3(x)$, where $x = k^+ / p^+$ and $p^+$ is the sum of all momenta that label all creation operators, or, equivalently, all annihilation operators in the same interaction term (see Appendix A). In QCD with $r_3(x) \sim x^\delta$, $H_0$ contains the coupling constant $g_3$, which depends on the scale $\lambda$ in the same asymptotically free way $\delta$, which characterizes the running coupling constant dependence on the renormalization scale in the Feynman diagrams $33$, $34$.

Hamiltonians $H_0$, with small $\lambda_0$ are worth studying because their eigenstates can be expanded in the effective particle basis in the Fock space and the wave functions in this expansion are expected to correspond to the constituent picture of hadrons. This is envisioned in analogy to the models based on Yukawa theory $37$. The interactions are suppressed by the form factors and cannot copiously create new constituents, even if the coupling constant $\alpha_0$ becomes large. Exotic hadron states may have their probability distributions shifted in the number of effective particles above the constituent quark model values of 2 or 3 $37$. The effective dynamics can be in agreement with requirements of special relativity even if it is limited to a small number of effective constituents, and the RGEP provides rules for constructing the representation of the Poincare group $37$. But the key feature is that the transition between the bare and effective degrees of freedom is made in one and the same formalism. There is no need to match different formulations of the theory, such as in the case of lattice theory and the continuum perturbation theory in the Minkowski space $38$, $39$, $40$, with none of the parts meant to cover the whole range of relevant scales on its own.

One more comment is required concerning the small-$x$ divergences in the eigenvalue equation for $H_0$. In the initial studies that used $\delta^+$ to limit bare particles’ momenta and employed coupling coherence to derive certain $H_0$, $41$, $42$ $43$, $44$, $45$, $46$, one could keep only a quark-anti-quark sector in the corresponding eigenvalue problem and the resulting equation was finite in the limit $\delta^+ \rightarrow 0$. Similarly, no small-$k^+$ divergences were encountered in the case of gluonium approximated by states of only two gluons $47$. In contrast, the present RGEP approach requires inclusion of states that contain an additional effective gluon which is needed to cancel small-$x$ divergences. For example, if one keeps only a pair of the effective quark and anti-quark, the leading non-relativistic (NR) terms are free from the small-$x$ singularities $31$, but relativistic corrections are singular $47$, $48$. When the additional gluon is included, the condition of cancelation of the small-$x$ divergences becomes a guide in understanding the gluon dynamics. The rules of including the gluons must be brought under quantitative control and the well-known case of heavy quarkonia provides a laboratory for testing the approach based on the gluon mass ansatz. The tests require a first approximation to begin with and a candidate is identified in the next sections.

This paper is organized as follows. Section II describes the initial Hamiltonian of LF QCD with one heavy flavor and the procedure for deriving an effective $H_0$. Section III discusses the eigenvalue equation for a quarkonium, and introduces the ansatz for the gluon mass term. Small-$x$ effects in the dynamics are described in Section IV. The resulting potential in the Schrödinger equation for a $QQ$ bound state is described in Section V. Section VI provides a brief summary and outlook. Appendices contain key details required for completeness.

II. HAMILTONIANS

The regularized canonical Hamiltonian of LF QCD with one heavy flavor of quarks, $H$, is given in Appendix A. It includes ultraviolet counterterms. This section describes the main features of $H$ and the RGEP derivation of the effective Hamiltonian $H_0$ with a finite width $\lambda$. $H_0$ is independent of the ultraviolet regularization factors $r_{\Lambda}$ in $H$ when $\Lambda \rightarrow \infty$. The small-$x$ regularization factors $r_{\Delta}$, which are also present in $H$, and their role in $H_\Lambda$, will be discussed later.

The initial Hamiltonian has the structure

$$H = H_{\psi^2} + H_{A^2} + H_{\psi A^2} + H_{(\psi \psi)^2} + X$$

(2)

where the term $H_{\psi^2}$ denotes the kinetic energy operator for quarks, $H_{A^2}$ the kinetic energy operator for gluons, $H_{\psi A^2}$ is the interaction term that couples quarks to quarks, $H_{(\psi \psi)^2}$ is the instantaneous interaction between quarks, and $X$ denotes all other terms including the counterterms.
In terms of the creation operators for bare particles, \( b^\dagger \) for quarks, \( d^\dagger \) for anti-quarks, and the corresponding annihilation operators, the kinetic energy terms are of the form,

\[
H_{\psi^2} = \sum_{\sigma c} \int [k] \frac{k_{\perp}^2 + m^2}{k^+} \left[ b^\dagger_{\kappa \sigma c} b_{\kappa \sigma c} + d^\dagger_{\kappa \sigma c} d_{\kappa \sigma c} \right],
\]

and

\[
H_{A^2} = \sum_{\sigma c} \int [k] \frac{k_{\perp}^2}{k^+} \left[ a^\dagger_{\kappa \sigma c} a_{\kappa \sigma c} \right],
\]

where \( k \) denotes the three kinematical momentum components, \( k^+ \), \( k_{\perp}^2 = (k_1^2, k_2^2) \). The symbol in a bracket, such as \([k]\), refers to the integration measure,

\[
[k] = \frac{dk^+ d^2k_{\perp}}{16\pi^3 k^+}.
\]

The subscript \( c \) stands for color and \( \sigma \) for spin. The mass \( m \) is assumed to be very large in comparison to \( \Lambda_{QCD} \).

The quark-gluon coupling terms in \( H_{\psi A^0} \) that preserve the number of quarks and anti-quarks, have the form

\[
Y = g \sum_{123} \int [123] \tilde{r}_{3,1} \left[ j_{23} b^\dagger_{2} b^\dagger_{1} b_{3} - \bar{\jmath}_{23} d^\dagger_{2} d^\dagger_{1} d_{3} + h.c. \right].
\]

The regularization factor \( \tilde{r}_{3,1} \) is singled out to indicate its presence. The coefficients \( j_{23} \) and \( \bar{\jmath}_{23} \), are functions of the quark and gluon colors, spins, and momenta, with all details provided in the Appendix A. These coefficients contain the three-momentum conservation \( \delta \)-function factors, denoted by \( \delta \), color factors \( t_{123}^\lambda \), and products of spinors, \( j_{23}^\mu = \tilde{u}_2 \gamma^\mu u_3 \) and \( \bar{\jmath}_{23}^\mu = \bar{v}_3 \gamma^\mu v_2 \). The latter are contracted with polarization vectors for gluons, so that

\[
j_{23} = \delta t_{23}^\lambda g_{\mu \nu} j_{23}^{\mu \nu} v^*_{1},
\]

and

\[
\bar{\jmath}_{23} = \delta t_{32}^\lambda g_{\mu \nu} j_{32}^{\mu \nu} v^*_{1}.
\]

The instantaneous term \( H_{(\psi\psi)^2} \) contains

\[
Z = -g^2 \sum_{1234} \int [1234] \tilde{r}_{1234} \tilde{r}_{34} \left[ j_{12}^{\perp +} j_{34}^{\perp +}/(k_1^+ - k_2^+) \right] \times [\tilde{r}_{1, 2\tilde{r}_{4,3}} + \tilde{r}_{21, \tilde{r}_{3,4}}] b_{1}^\dagger b_{4}^\dagger d_{3} d_{2}.
\]

The current factors \( j \) and the gluon polarization vectors grow with the relative transverse momenta of the interacting particles, \( \kappa_{\perp} \). These can increase to infinity and the regularization factors \( r_\lambda \) are introduced to limit the range to a finite one. In addition, there are small-\( x \) divergences due to the inverse powers of \( x \), especially in the gluon polarization vectors that contain terms proportional to \( \kappa_{\perp}/x \). The small-\( x \) singularities are regulated by factors \( r_g \).

The RGEP procedure generates ultraviolet counter-terms contained in the operator \( X \) in Eq. (1) and renders the effective particle Hamiltonian \( H_\lambda \) which is independent of \( r_\lambda \). The procedure is defined order by order in the formal expansion in powers of the bare coupling constant \( g \). This expansion is eventually re-written in terms of the effective coupling constant, \( \hat{g} \), which replaces \( g \) in \( H_\lambda \) and depends on the ratio \( \lambda/\Lambda_{QCD} \). The procedure is designed so that energy differences in denominators of the perturbative evaluation of \( H_\lambda \) are limited from below by \( \lambda \). Therefore, no infrared divergences are encountered in the evaluation of \( H_\lambda \). Also, no perturbative intrusion into the binding mechanism is generated when \( \lambda \) is kept above the scale of typical relative momenta of the bound-state constituents. These features qualify the RGEP as a candidate for providing an answer to the well-known question of how it is possible that a simple two-body Schrödinger equation may represent a solution to a theory as complex as QCD.

A very brief recapitulation of the RGEP is provided here for completeness. The derivation of \( H_\lambda \) begins with a unitary change of the degrees of freedom from the bare quarks and gluons in Eq. (2) to the effective ones. Let \( q \) commonly denote the bare operators \( b^\dagger, d^\dagger, \) and \( a^\dagger \), and their Hermitian conjugates. The operators \( q \) are transformed by a unitary operator \( U_\lambda \) into operators \( q_\lambda \) that create or annihilate effective particles with identical quantum numbers.

\[
q_\lambda = U_\lambda q U_\lambda^\dagger.
\]

The bare point-like particles in \( H \) of Eq. (2) correspond to \( \lambda \) equal infinity. One rewrites the Hamiltonian \( H \) in terms of \( q_\lambda \) and obtains

\[
H = H_\lambda(q_\lambda).
\]

Using \( U_\lambda \), one has

\[
\mathcal{H}_\lambda \equiv H_\lambda(q) = U_\lambda^\dagger H U_\lambda.
\]

Thus, \( \mathcal{H}_\lambda \) has the same coefficient functions in front of products of \( q_\lambda \) as the effective \( H \) has in front of the unitarily equivalent products of \( q_\lambda \). Differentiating \( \mathcal{H}_\lambda \) with respect to \( \lambda \), one obtains

\[
\mathcal{H}_\lambda' = -[\mathcal{T}_\lambda, \mathcal{H}_\lambda],
\]

where \( \mathcal{T}_\lambda = U_\lambda U_\lambda^\dagger \). \( \mathcal{T}_\lambda \) is constructed using the notion of vertex form factors for effective particles. For example, if an operator without a form factor has the structure

\[
\hat{O}_\lambda = \int [123] V_\lambda(1, 2, 3) q_\lambda^\dagger q_\lambda^\dagger q_{\lambda 3},
\]

then the operator with a form factor is written as \( f_\lambda \hat{O}_\lambda \) and has the structure

\[
f_\lambda \hat{O}_\lambda = \int [123] f_\lambda(M_{12}, M_{3}) V_\lambda(1, 2, 3) q_\lambda^\dagger q_{\lambda 2} q_{\lambda 3}.
\]
Different choices of the function $f_\lambda$ imply different interactions. The choice adopted in this study is
\[
\hat{f}_\lambda(M_{12}, M_3) = \exp[-(M_{12}^2 - M_3^2)/\lambda^4]].
\] (16)

For any operator $\hat{O}$ expressible as a linear combination of products of creation and annihilation operators, $f\hat{O}$ contains a form factor $f_\lambda(M_c, M_a)$ in front of every product. $M_c$ and $M_a$ stand for the total free invariant masses of the particles created (subscript $c$) and annihilated (subscript $a$) by a given product.

The effective Hamiltonian is defined to have the structure
\[
H_\lambda = f_\lambda G_\lambda,
\] (17)
where $G_\lambda$ has to be calculated for given $f_\lambda$. One uses $G_\lambda = G_\lambda(g)$, which is introduced in the same way as $H_\lambda$ in Eq. (17). $G_I$ satisfies the differential equation
\[
G_I = \int \lambda \, ds \left[ f_s G_{1s}, \{(1-f)G_{1s}\}G_0 \right].
\] (18)

The initial condition for Eq. (18) is that $G_\lambda = H$,
\[
G_\lambda = H + \int_\lambda \, ds \left[ f_s G_{1s}, \{(1-f)G_{1s}\}G_0 \right].
\] (20)

This equation shows that one can find the counterterms $X$ in $H$ that remove regularization dependence from $G_\lambda$. $H_\lambda = f_\lambda G_\lambda$ and $H_\lambda$ is obtained by replacing $g$ by $q_\lambda$.

$G_I$ is expanded into a series of terms $\tau_n \sim g^n$,
\[
G_I = \sum_{n=1}^\infty \tau_n. \quad (21)
\]

$\tau_1$ is independent of $\lambda$. Only the term $H_{\Psi\Psi}$ needs to be discussed here. According to Eq. (20), $\tau_1 = \alpha_{21} + \alpha_{12}$, where $\alpha_{21}$ denotes terms that create a gluon and $\alpha_{12}$ the terms that annihilate a gluon (the left subscript denotes the number of creation and the right subscript the number of annihilation operators). The corresponding effective Hamiltonian interaction term is obtained by multiplying the integrand in Eq. (18) by $f_\lambda$ and transforming $qs$ into $q_{1s}$.

When one neglects the terms that change the number of particles by more than one, $\tau_2 = \beta_{11} + \beta_{22}$. Equation (21) implies
\[
\tau_2 = \{f'\tau_1, f\tau_1\} = f_2[\tau_1\tau_1],
\] (22)
with $f_2 = \{f'f - f\{f'\}$). The first factor $f$ in $f_2$ refers to invariant masses in the first $\tau$ in the square bracket, and the second $f$ in $f_2$ is for the second $\tau$. The square bracket denotes all connected terms that result from contractions that replace products $q_iq_j$ by commutators $[q_i, q_j]$. The solution for $\tau_2$ is then given by
\[
\tau_{2\lambda} = F_{2\lambda}[\tau_1\tau_1] + \tau_{2\infty},
\] (23)
where $\tau_{2\infty}$ includes $H(\psi\psi^2)$ and the second-order mass counterterms from $X$ in Eq. (22). $F_{2\lambda}$ depends on incoming and outgoing momenta in the two vertices generated by the $\tau$'s. If one labels the three successive configurations of particle momenta by $a$, $b$, and $c$, in the sequence $a\tau_ab\tau_bc$, and introduces the symbol $uv = M_{uv}^2 - M_{uv}^2$, where $M_{uv}^2$ denotes the free invariant mass of a set of particles from the configuration $uv$ that are connected to the particles in the configuration $v$ by the interaction $\tau_{uv}$ in the sequence $uv\tau_uv$, the vertex form factor of Eq. (16) in the interaction $\tau_{uv}$ can be written as
\[
f_\lambda(M_{ab}, M_{bc}) = \exp[-(ab/\lambda^4)] = f_{ab}.
\] (24)

If one then denotes the parent momentum for the vertex $\tau_{uv}$ by $P_{uv}$, and writes $P_{uv}$ in place of $P_{uv}^+$, while the all minus components of momenta of the virtual quarks and gluons are given by the eigenvalues of $G_0 = H_{\psi^2} + H_{\lambda^2}$,
\[
F_{2\lambda}(a, b, c) = \frac{p_{0b}pa + p_{0c}pc}{ba^2 + bc^2} [f_{ab}f_{bc} - 1].
\] (25)

The second-order perturbation theory renders
\[
H_\lambda = T_{q\lambda} + T_{q\lambda} + f_\lambda [Y_{qq\lambda} + V_{qq\lambda} + Z_{qq\lambda}] .
\] (26)

The kinetic energy term for effective quarks is
\[
T_{q\lambda} = \sum_{\alpha\beta} \int_k \frac{k^+}{k^+} \left[ b_{\lambda k\sigma}^\dagger b_{\lambda k\sigma} + d_{\lambda k\sigma}^\dagger d_{\lambda k\sigma} \right],
\] (27)
where
\[
m_{\lambda}^2 = m_0^2 + (4/3)g^2 \int [\kappa] \frac{\kappa^2}{k^+} \sum_{12} |f_{2\kappa}^x \varepsilon_{12}^x|^2 \times \left[ F_{2\lambda}(m_0^2, M^2, m^2) - F_{2\lambda}(m^2, M^2, m^2) \right] / k^+,
\] (28)

In the order of appearance, $m_{\lambda}^2$ is the quark mass squared that should be present in $T_{q\lambda}$ in order to fit data for quarkonia, $m_0^2 = m^2 + o(g^2)$. Factor 4/3 comes from color, $(N_c^2 - 1)/(2N_c)$. The integration measure is
\[
[x\kappa] = dx \frac{x^2}{\kappa^2} \frac{1}{[16\pi^3]x(1-x)}
\] (29)
where $x = k^+_1/k^+_3$ is the fraction of the quark momentum $k_3$ carried by the virtual gluon, and $\kappa^+ = k^+_1 - xk^+_3$ is the relative transverse momentum of the gluon with respect to the quark $2$. The effective mass does not depend on the particle motion. This is a unique property of the RGEP in LF dynamics. The small-$x$ regularization factor is
\[
\tilde{\nu}_3(x) = \frac{x^2}{\kappa^2} \theta(x),
\] (30)
where $\theta(x)$ is the Heaviside step function.

\begin{flalign}
\text{(31)}
\end{flalign}
The gluon kinetic energy term reads
\[ \mathcal{M}^2 = (m^2 + \kappa^{-2})/(1 - x) + \kappa^{-2}/x , \] (32)
and
\[ \mathcal{F}_{2\lambda}(m^2, \mathcal{M}^2, m^2)/k_5^+ = [f_{5\lambda}(\mathcal{M}^2, m^2) - 1] / (\mathcal{M}^2 - m^2) . \] (33)
The gluon kinetic energy term reads
\[ T_{g\lambda} = \sum_{\sigma c} \int [k] \frac{k_{\perp}^2 + \mu_\lambda^2}{k_{\perp}^2} \delta_{\lambda c} a_\lambda^\dagger a_{\lambda\sigma} . \] (34)
The explicit form of \( \mu_\lambda^2 \) is not needed here.
The next term in Eq. \( \text{(29)} \) is \( Y_\lambda = f_\lambda Y_{g\lambda} \),
\[ Y_\lambda = g \sum_{123} \int [123] r_{5\lambda} x_{(1/3)} r_{5\lambda} x_{(2/3)} f_\lambda (\mathcal{M}_{12}^2, m^2) \times \left[ j_{23} b_{\lambda 3} a_{\lambda 1} a_{\lambda 3} + j_{23} d_{\lambda 2} a_{\lambda 1} a_{\lambda 3} + h.c. \right] . \] (35)
The effective potential term, \( V_\lambda = f_\lambda V_{g\lambda} \), originates from the exchange of bare gluons with jumps in the invariant mass of intermediate states above \( \lambda \).
\[ V_\lambda = -g^2 \sum_{1234} \int [1234] \tilde{\delta}_{12} a_{34} V_\lambda (13, 24) b_{1}^\dagger b_{4}^\dagger d_{3} d_{4} , \] (36)
where
\[ V_\lambda (13, 24) = \frac{d_{\mu\nu}(k_5)}{k_5^2} j_{12} j_{43} f_\lambda (\mathcal{M}_{12}^2, \mathcal{M}_{24}^2) \times \left[ \theta(z) r_{5\lambda} x_{(5/4)} r_{5\lambda} x_{(5/4)} f_{2\lambda} (1, 253, 4) + \theta(-z) r_{5\lambda} x_{(5/3)} r_{5\lambda} x_{(5/2)} f_{2\lambda} (3, 154, 2) \right] . \] (37)

FIG. 1: Momentum labels in the interaction term mediated by the exchange of one high-energy gluon. The same labeling is used in the exchange of low-energy gluon in the next section and appendices.

The sum over polarizations of the intermediate gluon reads
\[ d_{\mu\nu}(k_5) = \frac{n^\mu k_5^\nu + k_5^\mu n^\nu}{k_5^2} , \] (38)
where the gluon momentum is
\[ k_{5^+} = \varepsilon(z) \left( k_{1^+} - k_{2^+} \right) , \] (39)
and \( \varepsilon(z) = \theta(z) - \theta(-z) \),
\[ z = (k_{1^+} - k_{2^+})/(k_{1^+} + k_{2^+}) , \] (40)
while \( x_5 = |z| = k_5^+/x_{(1^+)} + k_5^+/x_{(2^+)} \), and
\[ k_5^+ = k_5^+ / k_{5^+} . \] (41)
The last term in Eq. \( \text{(29)} \) is the instantaneous interaction between effective quarks, \( Z_\lambda = f_\lambda Z_{g\lambda} \),
\[ Z_\lambda = -g^2 \sum_{1234} \int [1234] \tilde{\delta}_{12} a_{34} V_\lambda (13, 24) b_{1}^\dagger b_{4}^\dagger d_{3} d_{4} , \] (42)
where
\[ Z_\lambda (13, 24) = \frac{1}{k_5^2} j_{12} j_{43} f_\lambda (\mathcal{M}_{12}^2, \mathcal{M}_{24}^2) \times \left[ \theta(z) r_{5\lambda} x_{(5/4)} r_{5\lambda} x_{(5/4)} + \theta(-z) r_{5\lambda} x_{(5/3)} r_{5\lambda} x_{(5/2)} \right] . \] (43)

FIG. 2: Momentum labels in the instantaneous gluon interaction term.

### III. EIGENVALUE EQUATION

Once \( \lambda \) is lowered in perturbation theory to some value \( \lambda_0 \) just above the scale of binding mechanism, the resulting \( H_{\lambda_0} \) can produce the mass and wave function of a bound state of interest in a numerical diagonalization. The basis states can be limited to only those that have free invariant masses within a range of size about \( \lambda \) around the eigenvalue. This has been verified numerically in a matrix model with asymptotic freedom and bound states. In that calculation, \( H_\lambda \) was derived using perturbation theory up to 6th order. A quite small set of effective basis states with energies between 4 MeV and 4 GeV was sufficient to reach accuracy close to 1% in the computation of the bound-state energy on the order of 1 GeV. In great contrast, the initial Hamiltonian of the model coupled all states in the entire range between 0.5 KeV and 65 TeV. Preliminary estimates performed in Yukawa-like theories also indicate that the form factors \( f_\lambda \) suppress large momentum changes so strongly that the effective dynamics derived in low-order perturbation theory receives only small corrections from higher order terms, even when the coupling constant is made comparable to 1. In the case of heavy quarkonia the same strategy should work even more accurately than in the Yukawa theory because \( \alpha_s \) may be very small in comparison to 1. But the weak coupling expansion for \( H_{\lambda_0} \) produces new interaction terms already in order \( \alpha_0 \). These are derived here. An ansatz for the gluon mass
allows us then to finesse the structure of the first approximation for the resulting \(QQ\)-potential.

The eigenvalue problem for \(H_\lambda\) reads

\[
H_\lambda | \Psi \rangle = E | \Psi \rangle ,
\]

where \( | \Psi \rangle \) denotes an eigenstate of the operators \(P^+\) and \(P^-\) with their eigenvalues denoted by \(P^+\) and \(P^-\) (an example of the RGEP construction of the Poincare algebra at scale \(\lambda\) in quantum field theory is given in [35]). The eigenvalue \(E\) has the form

\[
E = (M^2 + P_{\perp}^2)/P^+ .
\]

The center-of-mass motion is separated from the binding mechanism, which is a unique LF-dynamics feature preserved by the RGEP, and \(P^+\) and \(P^-\) drop out of the eigenvalue equation. The state \( | \Psi \rangle \) is written in the effective particle basis as

\[
| \Psi \rangle = |Q \bar{Q} \lambda \rangle + |Q \bar{Q} g \lambda \rangle + \ldots .
\]

The dots denote components with more than three effective particles. Such expansion does not apply in the case of the bare particles because those interact locally and the interactions disperse probability density to high momentum regions and multi-particle sectors [35]. The wave function \(\psi_{13}(\kappa_{13}, x_1)\) of the effective valence component \(|Q \bar{Q} \lambda\rangle\), is introduced by the formula

\[
|Q \bar{Q} \lambda\rangle = \int [13^\lambda] P^+ \delta \psi_{13}(\kappa_{13}, x_1) b_{13}^\lambda d_{13} | 0 \rangle ,
\]

where the quark and anti-quark quantum numbers are labeled with 1 and 3, respectively. \(\psi_{13}(\kappa_{13}, x_1)\) must have dimension of 1/\(k_{13}^\lambda\) for the canonical normalization condition to give \(\langle P^+ | P^+ \delta (P - P')\rangle\) (quantum numbers of the states \( | P \rangle \) and \( | P' \rangle \) must be the same). The relative transverse momentum of two particles, 1 and 3, is always defined as

\[
\kappa_{13} = (k_1^+ k_3^- - k_1^- k_3^+)/ (k_1^+ + k_3^+) ,
\]

and \(x_1 = k_1^+/P^+ = 1 - x_3\). The wave function depends on \(\lambda\) and quickly vanishes for \(\kappa_{13} > \lambda\). The normalization condition gives,

\[
\langle Q \bar{Q} \lambda | Q \bar{Q} \lambda\rangle = N_{QQ}(\lambda) P^+ \delta (P - P') | P = P'\rangle ,
\]

where

\[
N_{QQ}(\lambda) = \sum_{13} \int [x_1 \kappa_{13}] |\psi_{13}(\kappa_{13}, x_1)|^2 .
\]

The probability of finding other components than the \(|Q \bar{Q} \lambda\rangle\) is given by \(1 - N_{QQ}(\lambda)\). The value of \(N_{QQ}(\infty)\) is not known but it may be close to 0. On the other hand, one expects \(N_{QQ}(\lambda)\) to be close to 1 when \(m \gg \Lambda_{QCD}\) and

\[
m \gg \lambda \gg \Lambda_{QCD} .
\]

When the wave function is negligible for relative momenta much larger than such \(\lambda\), the NR approximation must be accurate in description of the relative motion of quarks. In addition, when \(\alpha_\lambda \ll 1\), the Coulomb binding mechanism is expected to work and the dominant region of momenta should lie around the Bohr momentum scale \(k_B = \alpha_\lambda m\), provided that \(\lambda \gg k_B\). At the same time, all the fermion spin and relativistic correction factors cannot become large (or diverging [51]) in the NR expansion because of the presence of \(f_\lambda\) [35]. But the dynamics of the dominant \(|Q \bar{Q} \lambda\rangle\) component receives significant contributions from the \(|Q \bar{Q} g \lambda\rangle\) component in the small-\(x_5\) region, since the coupling to the gluon sector grows like \(\kappa_{13}^\lambda/\Lambda\) when \(x_5 \rightarrow 0\). The gluon component may have a negligible contribution to the norm but has to be accounted for when \(x_5\) is small.

If one neglected sectors with gluons entirely, the eigenvalue Eq. (44) would read

\[
[T_{q\lambda} + f_\lambda (V_{q\lambda} + Z_{q\lambda})] |Q \bar{Q} \lambda\rangle = E |Q \bar{Q} \lambda\rangle .
\]

This equation is mentioned here because an analogous one was considered before [43, 44, 12] in a scheme using coupling coherence and the absolute lower bound on gluon momenta, \(k_5^+ > \delta^+\). The equation found in [43] had a finite limit when \(\delta^+ \rightarrow 0\). The resulting dynamics contained a logarithmically rising potential and reproduced some of the characteristic features of the charmonium and bottomonium spectra. This was a considerable success in view of how crude were the approximations and the fact that the potential derived in [43] and employed in [44, 45] behaved differently in the transverse and longitudinal directions. But that strategy could not work in the RGEP approach.

Three reasons can be given now for why the pure \(|Q \bar{Q} \lambda\rangle\) approximation is not allowed in solving the eigenvalue problem for \(H_\lambda\). Two of them are related to the fact that the coupling between the sectors \(|Q \bar{Q} \lambda\rangle\) and \(|Q \bar{Q} g \lambda\rangle\) is proportional to the first power of the coupling constant \(g\), being mediated by the term \(Y_\lambda\) of Eq. (45).

The first argument is non-perturbative. It is based on the result that the matrix models with asymptotic freedom and bound states lead to a successful approximation to the eigenvalue equation with \(H_\lambda\) of second order in \(g\lambda\) if and only if all important matrix elements in the properly chosen energy window are accounted for. These certainly include matrix elements on the order of \(g^2\). The second argument is perturbative, and concerns the evaluation of the effective Hamiltonian that acts in the sector \(|Q \bar{Q} \lambda\rangle\) alone. When the states \(|Q \bar{Q} g \lambda\rangle\) are lifted in energy by an amount of order 1, quantum transitions in the quark-anti-quark sector that proceed through the states \(|Q \bar{Q} g \lambda\rangle\) are formally of order \(g^2\) and must be included when one computes the quark-anti-quark dynamics in a series of powers of \(g\) up to terms of the explicit order of \(\alpha_\lambda\). The third argument is based on the fact that Eq. (52) has a finite limit when \(\delta \rightarrow 0\) only in the leading NR approximation [51]. Relativistic corrections contain singularities [47, 48] and the additional
The gluon sector has to be taken into account to remove them. The eigenvalue Eq. 44 implies that the \(|Q_λQ_λgλ⟩\) component satisfies equation

\[ [T_{qλ} + T_{gλ} + V_{qgλ} - E]\,|Q_λQ_λgλ⟩ = -Y_λ|Q_λQ_λ⟩ . \tag{53} \]

\(V_{qgλ}\) denotes interactions with sectors with more than one gluon and/or additional quark-anti-quark pairs, and the non-abelian gluon-quark and quark-anti-quark potentials of order \(α_λ\). The interactions cause a shift in the gluon energy and make the eigenvalue equation differ from a similar one for positronium. The idea of seeking an ansatz for the shift and building a corresponding first approximation in the quark-anti-quark sector may seem completely new, but it patterns QED with the exception that there one knows from the outset that the leading approximation to a Hydrogen atom or positronium is given by a two-body Schrödinger equation with a Coulomb potential \([50, 51, 52]\). The NR lattice approach to heavy quarkonia \([53, 54]\) also starts from a two-body picture. The key argument is not theoretical but comes from the phenomenology of hadrons. Theoretically, an ansatz for the energy shift in the sector \(|Q_λQ_λgλ⟩\) is an attempt to harness the giant eigenvalue problem for \(H_λ\) by turning it into the eigenvalue problem for \(H_λ\) and identifying the corrected Coulomb picture that may apply as a first approximation in QCD.

A practical way to increase the invariant mass of the three-body sector and preserve the kinematical symmetries of LF dynamics is to add a mass \(μ^2\) to \(T_{gλ}\), using the rules outlined in Section I, see Eq. 45. Since the divergence in the small-\(x\) region disappears in the case of positronium when one adds a sector with a massless photon, a gluon mass that approaches zero when \(x_5 \to 0\) can remove the small-\(x_5\) divergence in the quarkonium case. The rotational symmetry condition on \(μ^2\) can be imposed by demanding that the resulting potential in the quark-anti-quark sector is a rotationally symmetric function in the center-of-mass variables (see below).

Given a gluon mass ansatz, the whole eigenvalue problem for \(H_λ\) is limited to only two coupled equations,

\[
(T_q + T_g)|QQg⟩ + Y|QQg⟩ = E|QQg⟩ ,
Y|QQg⟩ + [T_q + f(V_{qg} + Z_{qg})]|QQ⟩ = E|QQ⟩
\]

(54)

\(V_{qg}\) differs from \(T_{gλ}\) of Eq. 45 by replacement of the perturbative \(μ_λ^2\) by the ansatz \(μ^2\). It is understood that \(μ^2\) may depend on \(λ\). All terms of order \(g^2\) in the three-body sector are ignored because they do not contribute to the dynamics of the \(|QQ⟩\) component in second-order perturbation theory (see below). This dynamics is described by the Hamiltonian \(H_{QQ}\) that acts only in the quark-anti-quark sector. One should keep in \(H_{QQ}\) terms of formal orders 1, \(g\), and \(g^2\), when the effective Hamiltonian \(H_λ\) is calculated up to the terms of order \(g^2\), while \(μ^2 \sim 1\). The bare \(g\) is understood to go over in higher order calculations to a suitably defined \(\bar{g}_λ\) \([53, 54]\). The perturbative expansion is applied only in the evaluation of \(H_{QQ}\). Solution for the bound-state spectrum of \(H_{QQ}\) is not perturbative.

To evaluate \(H_{QQ}\) as a power series in \(g\), one can introduce an operator \(R\) that expresses the 3-body component through the 2-body one,

\[
|QQg⟩ = R|QQ⟩ . \tag{55} \]

Since \(Y\) is of order \(g\), \(R\) is expected to be at least of order \(g\). If \(P\) denotes the projector on the \(|QQ⟩\) sector, one has \(R = (1 - P)R = RP\) and \(PR = R(1 - P) = 0\). The effective Hamiltonian in the \(|QQ⟩\) sector is then given by the formula \([55, 57]\) (see also \([35]\) concerning the context of RGEP)

\[
H_{QQ} = \frac{1}{\sqrt{P + R^1}}(P + R^1)\,H_λ\,P + R \frac{1}{\sqrt{P + R^1}} . \tag{56} \]

In the first order in \(g\),

\[
RT_q - (T_q + T_g)R = Y . \tag{57} \]

Consequently, the second-order expression for the matrix elements of \(H_{QQ}\) between different states \(i\) and \(j\) in the \(|QQ⟩\) sector, is

\[
\langle i|H_{QQ}|j⟩ = \langle i|T_q + f(V_{qg} + Z_{qg})|j⟩ + \frac{1}{2} |Y\left(\frac{1}{E_j - T_q - T_g} + \frac{1}{E_i - T_q - T_g}\right)|Y|j⟩ . \tag{58} \]

The effective eigenvalue equation for heavy quarkonia, \(H_{QQ}|QQ⟩ = E|QQ⟩\), takes the form

\[
\left[(\kappa_{13}^2 + m_λ^2)\frac{m_λ^2}{x_1} + m_λ^2(3)\frac{3}{x_3} - M^2\right] \psi(\kappa_{13}^2, x_1) - \frac{4}{3} g^2 v_\lambda(13, 24) v_\lambda(13, 24) ψ(\kappa_{24}^2, x_2) = 0 , \tag{59} \]

where

\[
v_\lambda(13, 24) = V_\lambda(13, 24) + Z_\lambda(13, 24) + \frac{1}{2x_5} d_{\mu\nu}(k_5) j_{12}^\mu j_{23}^\nu w_\lambda(13, 24) . \tag{60} \]
and

\begin{equation}
\omega_\lambda(13, 24) = \left\{ \frac{\theta(z) \tilde{r}_\delta(x_{5/4}) \tilde{r}_\delta(x_{5/4}) f_{\lambda}(m^2, \mathcal{M}^2_{53}) f_\lambda(\mathcal{M}^2_{3}, m^2)}{(\kappa^2_{5} + m^2)/x_1 - (\kappa^2_{5} + m^2)/x_2} + \frac{\theta(-z) \tilde{r}_\delta(x_{5/3}) \tilde{r}_\delta(x_{5/3}) f_{\lambda}(m^2, \mathcal{M}^2_{51}) f_\lambda(\mathcal{M}^2_{1}, m^2)}{(\kappa^2_{5} + m^2)/x_1 - (\kappa^2_{5} - \kappa^2_{3} + m^2)/x_3} + \frac{\theta(z) \tilde{r}_\delta(x_{5/3}) \tilde{r}_\delta(x_{5/3}) f_{\lambda}(m^2, \mathcal{M}^2_{51}) f_\lambda(\mathcal{M}^2_{1}, m^2)}{(\kappa^2_{5} + m^2)/x_1 - (\kappa^2_{5} - \kappa^2_{3} + m^2)/x_3} \right\} .
\end{equation}

The terms with \( \theta(z) \) describe the emission of the gluon by the quark and absorption by the anti-quark, while the terms with \( \theta(-z) \) describe the gluon emission by the anti-quark and absorption by the quark, see Fig. A. The first square bracket corresponds to the first term in the large round bracket in Eq. (58), and the second bracket corresponds to the second term. The mass terms, \( m^2_j \), originate from the emission and re-absorption of the effective gluon by the same quark, in which case both terms in the bracket of Eq. (58) are equal. The mass terms read

\begin{equation}
m^2_1(1) = (4/3) g^2 \int [\kappa^2 \tilde{r}_\delta^2(x) f_\delta^2(m^2, \mathcal{M}^2) \frac{j^\nu j^{\mu*} d_{\mu\nu}(k)/x_1}{(\kappa^2_{13} + m^2)/x_1 - (\kappa^2_{13} + \mathcal{M}^2_{1})/x_1},
\end{equation}

where

\begin{equation}
\mathcal{M}^2_{1} = [\kappa^2_{1} + \mu^2(1', 5', 3)]/x + (\kappa^2_{1} + m^2)/(1 - x),
\end{equation}

and,

\begin{equation}
m^2_3(3) = (4/3) g^2 \int [\kappa^2 \tilde{r}_\delta^2(x) f_\delta^2(m^2, \mathcal{M}^2) \frac{j^\nu j^{\mu*} d_{\mu\nu}(k)/x_3}{(\kappa^2_{13} + m^2)/x_3 - (\kappa^2_{13} + \mathcal{M}^2_{3})/x_3},
\end{equation}

where

\begin{equation}
\mathcal{M}^2_{3} = [\kappa^2_{1} + \mu^2(1', 5', 3')]/x + (\kappa^2_{1} + m^2)/(1 - x).
\end{equation}

\( \mathcal{M} \) is given by Eq. (22). The subscript '1' denotes the intermediate quark and 5' denotes the intermediate gluon in the self-interaction of the effective quark 1, and, similarly, 3' and 5' denote the intermediate anti-quark and gluon in the self-interaction of the anti-quark 3.

The gluon four-momentum \( k_5 \) in the sum over polarizations, i.e., in \( d_{\mu\nu}(k_5) \) in Eq. (58), can be written as

\begin{equation}
k_5^\alpha = \varepsilon(z) q^\alpha_{ij} + n^\alpha \left( k_5^- - \varepsilon(z) q_{ij}^- \right)/2,
\end{equation}

where \( i j \) refers to quarks 1 and 2, or anti-quarks 3 and 4, \( q_{ij} = k_i - k_j \).

The quark momentum four-vectors are on the mass shell. Since the gluon connects two vertices, one momentum \( k_5 \) in \( d_{\mu\nu}(k_5) \) is contracted with the current carried by the quark, and the other with the current of the anti-quark. In the self-interactions, both momenta are contracted with the same current. The momentum \( k_5 \) contracted with current \( j^\alpha_5 \) can be expressed through \( q^\alpha_5 \). But the current conservation implies that the terms proportional to \( q_{ij} \) give zero. Therefore, one can replace Eq. (58) in the gluon exchange terms by

\begin{equation}
d_{\mu\nu}(k_5) = -g_{\mu\nu} + n_{\mu} n_{\nu} \times \left[ k_5^- + \varepsilon(z) (k_5^- - k_3^- + k_4^-)/2 \right]/k_5^+.
\end{equation}

In the quark self-interaction one has

\begin{equation}
d_{\mu\nu}(k_5) = -g_{\mu\nu} + n_{\mu} n_{\nu} \frac{k_5^- + k_3^- - k_4^-}{k_5^+},
\end{equation}

with an analogous result for the anti-quark.

The terms with the metric \( g_{\mu\nu} \) are regular in the small-\( x_5 \) region, while the terms with \( n_{\mu} n_{\nu} \) are singular. The metric terms lead in the well-known way to the Breit-Fermi spin-dependent terms with a Coulomb potential. A discussion of the Breit-Fermi terms and gluons in the context of QCD can be found in [12] and references therein. The singular terms with \( n_{\mu} n_{\nu} \) are independent of the quark spin. It is shown below that the latter generate the harmonic force between quarks when combined with the fermions’ self-interactions, which are also independent of the spin. Thus, the harmonic force appears without Breit-Fermi terms. This result sets the
In both cases, and the last factor of \( w \), where one had to guess whether a confining potential appeared with or without Breit-Fermi terms. The spin-independent harmonic force is akin in this respect to the lattice picture and the original charmonium model based on the Coulomb force \[34, 35, 36, 37\].

In the explicit discussion of singular small-\( x \) features of Eq. (13) in the next section, all the \( g_{\mu\nu} \) terms are omitted. The reader should keep their presence in mind until they are re-inserted in Section IV. The symbols of mass, wave function, and potential are provided with a tilde as a reminder about the need to include the \( g_{\mu\nu} \) terms. Also, expressions for the quark masses are simplified by considering from now on only \( \lambda = \lambda_0 \). The subscript 0 indicates that \( \lambda = \lambda_0 \). The ansatz for \( \mu^2 \) is understood to correspond to \( \lambda_0 \).

With the \( g_{\mu\nu} \) terms hidden and \( \lambda = \lambda_0 \), the eigenvalue equation reads

\[
\left( \frac{\kappa_{13}^2 + m_0^2}{x_1 x_3} + \frac{\hat{m}_2^2}{x_1} + \frac{\hat{m}_3^2}{x_3} - M^2 \right) \hat{\psi}(\kappa_{13}, x_1) - \frac{4 g^2}{316 \pi} \int \frac{dx_2 dx_4}{x_2 x_4} \frac{2 \hat{f}_{12}^3}{x_3^2} \frac{1}{x_5^2} \hat{\psi}_{0}(13, 24) \hat{\psi}(\kappa_{24}, x_2) = 0,
\]

where

\[
\hat{\psi}_{0}(13, 24) = f_{13, 24} [k_0^- + \varepsilon(z)(k_1^- - k_2^- + k_3^- - k_4^-)/2] \left[ \theta(z) \hat{r}_{5/1} \hat{r}_{5/4} \mathcal{F}_{1.253.4} + \theta(-z) \hat{r}_{5/3} \hat{r}_{5/2} \mathcal{F}_{3.154.2} \right] + f_{13, 24} [\theta(z) \hat{r}_{5/1} \hat{r}_{5/4} + \theta(-z) \hat{r}_{5/3} \hat{r}_{5/2}] + \frac{1}{2} [k_1^2 + \varepsilon(z)(k_1^- - k_2^- + k_3^- - k_4^-)/2] \psi_{0},
\]

and the last factor of \( \psi_{0} \equiv P^+ w_{\lambda_0}(13, 24) \) is abbreviated to

\[
\psi_{0} = \frac{\theta(z) \hat{r}_{5/1} \hat{r}_{5/4}}{k_1^- - k_5^-(2, 5, 3) - k_2^-} + \frac{\theta(-z) \hat{r}_{5/3} \hat{r}_{5/2}}{k_3^- - k_5^-(1, 5, 4) - k_1^-} + \frac{\theta(z) \hat{r}_{5/1} \hat{r}_{5/4}}{k_1^- - k_5^-(2, 5, 3) - k_3^-} + \frac{\theta(-z) \hat{r}_{5/3} \hat{r}_{5/2}}{k_2^- - k_5^-(1, 5, 4) - k_1^-}.
\]

The compact notation includes

\[
f_{i, j} \equiv f_{\lambda_0} (M_1^2, M_2^2), \quad \hat{r}_{5/4} \equiv \hat{r}_{5}(x_5/i), \quad \mathcal{F}_{i, k, j} \equiv \mathcal{F}_{2\lambda_0}(i, k, j), \quad \hat{k}_0^-(i, j, k) = [k_5^-(i, j, k) + \mu^2(i, i, j, k)/k_5^-(i, j, k)], \quad \kappa_5^+(i, j, k) = \varepsilon(z)(\kappa_{13}^-(i, j, k) - \kappa_{24}^-(i, j, k)).
\]

The mass terms with the \( g_{\mu\nu} \) terms suppressed are

\[
\hat{m}_i^2 = \frac{4 g^2}{3} \int [xk] \hat{r}_3^2(x) f_{i, k}^2 \frac{|j^+|^2}{k_0^+} \frac{1}{x^2} \frac{x(M^2 - m^2)}{m^2 - M_i^2},
\]

for \( i = 1, 3 \). \( M_1 \) is given by Eq. (86) and \( M_3 \) by Eq. (85). In both cases, \( M_2 \) is given by Eq. (82), and the factor \( (j^+/k_0^+)^2 = 4(1 - x) \).

IV. SMALL-\( x \) BEHAVIOR

All small-\( x \) singularities of the eigenvalue Eq. (54) are contained in Eqs. (50), (71), and (76). We first discuss the exchange terms, then the mass terms, and finally the net effect of the interplay between these terms.

The analysis hinges on the properties of the energy of motion of a gluon with respect to the parent quark, \( p_i^+ k_5^- = x_i k_5^-(2)/x_5 \), \( p_i \) is the momentum of the parent quark \( i \). The momenta \( k_5^- \) ranges under the integrals from 0 to \( \infty \), while \( x_5 \) can reach 0 (in the mass terms, the integrals are over \( \kappa^- \) and \( x \)). Appendices B and C provide definitions of all variables used in the description of the integrands. The key difficulty is that the ratio of two variables of different kinds, \( \kappa^- \) and \( x \), varies quickly with a change of any one of them. This complexity is related to the power counting rules for the Hamiltonian densities on the LF [2]. But the analysis described here concerns only the relative motion of the effective particles and it is simplified by taking advantage of the NR limit after the finiteness of the small-\( x \) dynamics with the gluon mass ansatz is established.

The singularity in the effective gluon exchange term is tempered by the product of two vertex form factors, \( \mathcal{F} \). The form factors vanish exponentially fast when \( k_5^- \to \infty \). This prevents \( x_5 \) from becoming small unless \( k_5^- \) vanishes at least as fast as \( \sqrt{x_5} \). Therefore, the measurement of integration over transverse momenta is on the order of \( x_5 \) when \( x_3 \to 0 \) and it reduces the divergence to a logarithmic one. The logarithmic divergence is taken care of using the gluon mass ansatz. The mechanism of reducing singularities to only logarithmic ones does not work in the instantaneous interaction term \( Z_\Lambda(13, 24) \) and in the terms in \( V_\Lambda(13, 24) \) that come without \( ff \) in \( \mathcal{F}_{2\Lambda} \). But all the terms without \( ff \) are inde-
pendent of $\mu^2$ and the $dx_5/x_5^2$ and $dx_5/x_5$ singularities cancel out in them perturbatively.  

In the fermion self-interactions an analogous pattern of the singularities occurs. But one has to also consider the size of $m_0^2$. The latter is determined by the size of the free ultraviolet-finite part of the quark mass counterterm in the initial Hamiltonian of Eq. (2). That size is related to an ansatz for a gluon mass term in the sectors $|Q g|$ and $|\bar{Q} g|$ in the eigenvalue equations for states with quantum numbers of a single fermion, see Appendix C.

Eventually, the gluon mass ansatz leads to the result that the single-quark eigenvalue diverges logarithmically in the limit $\delta \to 0$, while the quark self-interaction in the quarkonium dynamics becomes finite. The self-interaction and effective gluon exchange, both finite due to the chosen behavior of the gluon mass ansatz, lead together to the harmonic potential which is described in the next section.

According to the Appendix C the dominant exchange terms in Eq. (70) read

$$\bar{v}_0(13, 24) = \theta(z) \bar{v}_{+\text{low}} + \theta(-z) \bar{v}_{-\text{low}},$$

(79)

where $\bar{v}_{+\text{low}}$ is given in Eq. (B19), and $\bar{v}_{-\text{low}}$ in Eq. (B20). In the limit $x_5 \to 0$,

$$\bar{v}_{+\text{low}} = f_{1.52 f53.4} \frac{\mu^2(2, 5, 3)}{q^2 + \mu^2(2, 5, 3)},$$

(80)

and

$$\bar{v}_{-\text{low}} = f_{3.54 f51.2} \frac{\mu^2(1, 5, 4)}{q^2 + \mu^2(1, 5, 4)}.$$  

(81)

Since $q^{1,2}$ is on the order of $|z|$, one obtains the result that if $\mu^2$ vanishes faster than $q^{1,2}$, i.e., faster than $x_5$, the potential produces a finite effective in the limit of $\delta \to 0$.

In the denominators of Eqs. (79) and (80) there also appears $q^2 = (2mz)^2$, which is negligible in comparison to the leading terms on the order of $z$ but can be included here on the basis of hindsight to take advantage of the NR nature of the quarks’ motion with respect to each other. The larger is the quark mass $m$ for fixed $\lambda_0$ and the smaller is $\Lambda_{QCD}$, the more accurate the NR picture actually becomes after the small-$x$-divergences are removed. Writing $q_x = q t$, with $q = |q|, t = \cos \theta$, the singular factor $1/x^2$ equals $4m^2/q_2^2 = (4m^2/q^2)t^{-2}$. The integration measure $d^2q$ is proportional to $q^2$ and the small-$x$ singularity is actually produced by the angle integration $dt/t^2$. $\mu^2$ should vanish for $t \to 0$ in order to remove the singularity. An example of such behavior is used below to provide a constructive context for the steps that follow. The final result is not sensitive to the details of the example. Given that $\mu^2$ vanishes faster than $q^2$, one can write

$$\mu^2(i, 5, j) = \mu_0^2(i, 5, j)/q^2,$$

(82)

and determine behavior of $c(i, 5, j)$ from the condition that

$$c(i, 5, j) = \frac{\mu^2(i, 5, j)}{1 + \mu^2(i, 5, j)}$$

(83)

should vanish for $x_5 \to 0$.

The only information about the three-particle sector that is available in the relativistic construction of $\bar{q}$ and $c(i, 5, j)$ are the $\perp$ and $\parr$ components of the momenta $k_i$, $k_5$, and $k_j$. Two physical constraints are used in defining a helpful vector $\bar{q}$: the definition must respect all kinematical LF symmetries (to render a boost-invariant description of quarkonia), and it must reduce to $q_z = 2mz$ for $z \to 0$ when $|\bar{k}_{13}|/m$ and $|\bar{k}_{24}|/m$ approach 0 in the NR limit. A geometrically motivated candidate for $\bar{q}$ is provided by the difference between the square of the free invariant mass of three effective particles in the state $|QQg\rangle$ and the square of the invariant mass of the $QQ$-pair in this state. The difference reads

$$M_{\text{inv}}^2 - M_{ij}^2 = \frac{\kappa_5^2 + x_5^2 M_{ij}^2}{x_5(1 - x_5)}.$$  

(84)

Multiplication by $x_5(1 - x_5)$ produces an expression that tends in the limit of $x_5 \to 0$ to the three-momentum transfer squared that appears in the energy denominators in the small-$x$ dynamics. The components of $\bar{q}$ are therefore defined as $q_{\perp}^\perp = \kappa_5$ and

$$q_z = z M_{ij}.$$  

(85)

Further analysis of all exchange terms shows that if the ansatz mass $\mu^2$ behaves like

$$\mu^2 \sim x_5^{1+\delta}$$

(86)

(or like $q^2 x_5^{\delta}$) with $0 < \delta < 1/2$, the factors $\bar{v}_{\pm,\text{low}}$ of Eqs. (B19) and (B20) vanish in the limit $x_5 \to 0$ as $x_5^{\delta_{\text{low}}}$ independently of the terms in the energy denominators on the order of $x_5^{3/2}$ or smaller. Thus, the gluon exchange term becomes finite when

$$c(i, 5, j) = c(t)$$

(87)

and $c(t)$ is a function that behaves as

$$c(t) = c |t|^{\delta_0/2},$$

(88)

for $t \to 0$, with $c$ a constant.

With this ansatz, the quark mass terms also become finite in the limit $\delta \to 0$. Appendix C shows details of how $m_0^2$ is chosen in agreement with the physical picture explained in the Introduction and at the beginning of this section. Eq. (75) gives $m_0^2 + \bar{m}_i^2 = m^2 + \delta m_i^2$ with $i = 1, 3$ and

$$\delta m_i^2 = \frac{4\alpha_0}{3\pi^2} \int dz x^2 \kappa(x) f_{\lambda_0}(m^2, M^2) \times \frac{M^2 - m^2}{x^2} \left( \frac{1}{M_0^2 - m^2} - \frac{1}{M_t^2 - m^2} \right).$$

(89)

The function $M_0^2$ is given by Eq. (99) in terms of the gluon mass function $\mu_0^2$ that must satisfy the condition (10). The quark self-energies are positive if

$$\mu^2(i, 5, j) > \mu_0^2.$$  

(90)
A simple way to satisfy this condition is to set \( \mu_0^2 = 0 \). Then,

\[
\delta m_i^2 = \frac{4\alpha}{3\pi^2} \int dx \frac{\kappa^+}{\kappa} f_{\lambda_0}^2(m^2, M^2) \frac{1}{x^2} \times \frac{\mu^2(i', 5', j)}{\mu^2(i', 5', j) + (\kappa^+ - x^2m^2)/(1 - x)},
\]

where \( i = 1 \) and \( j = 3 \) or vice versa. The factors \( r_s \) are no longer needed.

The small-\( x \) regularization disappears from the quarkonium dynamics entirely. Finite phenomenological parameters that describe the small-\( x \) behavior of the gluon mass ansatz, such as \( \delta_\mu \) in Eq. (50), become responsible for the regularization of the exchange and self-interaction terms, preserving their distinct properties. The mass terms grow when \( \delta_\mu \) decreases, while the effective gluon exchange potential provides a negative contribution that increases in magnitude at similar rate and compensates the size of the masses at small momentum transfers. The net result is described in the next section.

\[ \text{V. THE } Q\bar{Q} \text{ SCHRODINGER EQUATION} \]

The condition (51) validates the NR and weak coupling limits after the small-\( x \) divergences are removed by the gluon mass ansatz. This section then identifies the leading structure in \( H_{Q\bar{Q}} \) in formal order of \( \alpha_0 \). The additional simplification in the case of small \( \alpha_0 \) is that the dominant interaction in Eq. (59) becomes equal to the well-known Coulomb term and one can find the leading correction analytically.

Equation (59) can be re-written using the relative three-momentum variables described in Appendix B [see Eq. (B2)]. The integration measure is

\[
dx d^3k_{13} d^3k_{24} = \frac{4d^3k_{13}}{M_{24}},
\]

and Eq. (59) takes the form

\[
\left[ 4(m^2 + 13\vec{k}_{13}^2) + \frac{\delta m_1^2}{x_1} + \frac{\delta m_3^2}{x_3} - (2m + B)^2 \right] \psi(\vec{k}_{13})
+ \int \frac{d^3k_{24}}{(2\pi)^3 \sqrt{m^2 + k_{24}^2}} U(\vec{k}_{13}, \vec{k}_{24}) \psi(\vec{k}_{24}) = 0.
\]

The mass corrections include now the \( g_{\mu\nu} \) terms that were suppressed in the previous section, \( \delta m_1^2 = \delta m_1^2 + \delta m_2^2 \), and the potential is

\[
U(\vec{k}_{13}, \vec{k}_{24}) = \frac{4}{3} f_{13,24} 4\pi \alpha
\times \left\{ \frac{4\sqrt{3}f_{13,24}x_4}{x_3^2} \left[ \theta(z) \bar{v}_{+ \text{low}} + \theta(-z) \bar{v}_{- \text{low}} \right] + v_g \right\},
\]

where \( v_g \) denotes the \( g_{\mu\nu} \) contribution in the exchange term.

Since the form factor \( f_{13,24} \) cuts off changes of the relative momenta above \( \lambda_0 \) exponentially fast, one can focus on the eigenstates with lowest \( M^2 \) and take advantage of the conditions \( |\vec{k}_{13}| \ll m \) and \( |\vec{k}_{24}| \ll m \) that are satisfied in the entire domain of physically relevant probability distribution. For such states, one can expand Eq. (B3) in powers of \( \vec{k}/m \), with the exception of the form factors that are needed for convergence. The Coulomb force defines the momentum scale of the inverse of the quarkonium Bohr radius, \( k_B = r_B^{-1} = \alpha_0 m/24 \). When \( \lambda_0 \gg k_B \), the form factor \( f_{13,24} \) does not differ from 1 in the dynamically dominant region; \( f_{13,24} \) matters only when one extends the expansion to high powers of \( \vec{k}/m \). These would lead to divergent integrals with Coulomb wave functions and require counterterms [50, 51]. The latter are not needed here and the lowest terms dominate (55). The binding energy, \( B \), is small in comparison to \( m \). Writing the quarkonium mass as \( M = 2m + B \) and neglecting \( \sim B^2/m \), one obtains

\[
\left[ \frac{\vec{k}_{13}^2}{m} - B + \frac{\delta m_1^2}{2m} + \frac{\delta m_2^2}{2m} \right] \psi(\vec{k}_{13})
+ \int \frac{d^3k_{24}}{(2\pi)^3} V_{Q\bar{Q}}(\vec{k}_{13} - \vec{k}_{24}) \psi(\vec{k}_{24}) = 0.
\]

The structure of \( V_{Q\bar{Q}} \) and the size of the mass corrections \( \delta m_1^2 \) and \( \delta m_2^2 \) need explanation.

The two vertex form factors that appear inside the exchange and mass terms in Eq. (59), have arguments given in Appendix B [see Eqs. (B8) to (B13)]. When one writes the product of the two vertex form factors in the NR limit as \( \exp(-u^2) \), \( u \) reads

\[
u = \sqrt{\frac{m}{\lambda_0}} \frac{1}{t} \frac{q}{\lambda_0}.
\]

The limit \( m/\lambda_0 \gg 1 \) enforces \( q \ll \lambda_0 \), the more so the smaller is \( t \). The Coulomb binding mechanism is intact for \( \lambda_0 \) as small as several times \( k_B \) [55], which is much smaller than \( m \) in the weak coupling limit. Thus, the momentum transfer \( q \) is much smaller than \( k_B \) in all terms that contain \( ff \). These terms become then negligible in comparison to the Coulomb term, unless they have singularly small denominator factors for small \( t \). That is the case for the mass and exchange terms when \( \delta_\mu \) becomes small. In the presence of the \( g_{\mu\nu} \) contributions that were omitted in Section IV, these terms are found as follows.

The Hamiltonian \( H_{Q\bar{Q}} \) has the structure

\[
H_{Q\bar{Q}} = m^2 + \delta m^2(ff, g + n, 0) - \delta m^2(ff, g + n, \mu)
+ f(1 - ff)|(g + n, 0) + z|
+ f(ff)|(g + n, \mu) + z|
\]

where \( g \) denotes the \( g_{\mu\nu} \) terms, \( n \) denotes the singular \( n_{\mu
u} \) terms, and \( z \) denotes the instantaneous terms. The
gluon mass ansatz in energy denominators is indicated by an extra variable in the brackets, and 0 says that the gluon mass is 0. The last two terms can be re-arranged as

\[ f(1 - ff)((g, 0) + (n, 0) + z) + f(ff)[(g, 0) + (g, \mu) - (g, 0) + (n, \mu) + z]. \]  

(98)

The contribution of \( (n, 0) + z \) in the first term vanishes in the leading NR limit, see Appendix B. Two of the terms with \( (g, 0) \) combine to \( f(g, 0) \), and reduce to the Coulomb term with the Breit-Fermi spin corrections. The remaining terms, with \( f \) in front also being equivalent to 1,

\[ f(ff)((g, \mu) - (g, 0) + (n, \mu) + z), \]  

(99)

add to the Coulomb term and produce together \( V_{QQ} \) in Eq. (101). The mass terms can be re-written, in the same fashion, as

\[ (ff)\delta m^2[(g, 0) - (g, \mu) + (n, 0) - (n, \mu)]. \]  

(100)

Expressions (99) and (100) show that the exchange potential and the mass terms have similar structures with opposite signs. A change of variables from \( x \) and \( \kappa \) to \( q_\perp = x m \) and \( q_\perp = \kappa \) in the mass terms produces integrals in which the factor \( ff \) ensures that \( q = |\vec{q}| \ll m \) and one can again use the expansion in powers of the ratio of \( q/m \). Since the integrands are symmetric functions of \( q_\perp \), one can extend the integration to negative \( q_\perp \) and divide the result by 2, which produces the same integrands as in the exchange terms. Hence,

\[ V_{QQ}(\vec{q}) = (1 + BF)V_C(\vec{q}) + W(\vec{q}), \]  

(101)

where \( BF \) denotes the Breit-Fermi spin-dependent factors,

\[ V_C(\vec{q}) = -\frac{4}{3} \frac{4\pi \alpha}{\vec{q}^2}, \]  

(102)

\[ W(\vec{q}) = \frac{4}{3} \frac{4\pi \alpha}{\vec{q}^2} \left[ \frac{1}{\vec{q}^2} - \frac{1}{q_\perp^2} \right] \frac{\mu^2}{\mu^2 + \vec{q}^2} \times \exp \left[ -2 \left( \frac{mq_\perp}{\sqrt{2} \lambda_0^2} \right)^2 \right], \]  

(103)

with \( \mu^2 = \theta(z) \mu^2(2, 5, 3) + \theta(-z) \mu^2(1, 5, 4) \), and

\[ \frac{\delta m^2}{m} = -\int \frac{d^3q}{(2\pi)^3} W(\vec{q}), \]  

(104)

with \( \mu^2 \) equal \( \mu^2(1', 5', 3) \) for \( i = 1 \) and \( \mu^2(1, 5', 3') \) for \( i = 3 \).

If the gluon mass ansatz is 0, \( W = 0 \) and the quarkonium dynamics reduces to the same as in QED with additional color charge factor 4/3. A finite gluon mass ansatz introduces new dynamics which is discussed in the remaining part of this section.

\[ W \] is large and negative when \( \delta_\mu \) is small. The exchange term tends to compensate the positive contribution of the mass terms. This can be made transparent by re-writing Eq. (95) as

\[ \left[ \frac{\vec{k}^2}{m} - B \right] \psi(\vec{k}) + \int \frac{d^3q}{(2\pi)^3} (1 + BF)V_C(\vec{q})\psi(\vec{k} - \vec{q}) + \int \frac{d^3q}{(2\pi)^3} W(\vec{q}) \left[ \psi(\vec{k} - \vec{q}) - \psi(\vec{k}) \right] = 0. \]  

(105)

There is no need to trace the small relativistic corrections before the main NR picture is identified. Only this picture is discussed below.

Since \( |\vec{q}| \) in \( W \) is constrained to values much smaller than \( k_B \), one can expand the wave function in the Coulomb region under the integral in the Taylor series and consider the lowest terms as candidates for the first approximation,

\[ \psi(\vec{k} - \vec{q}) = \psi(\vec{k}) - q_\perp \frac{\partial}{\partial k_\perp} \psi(\vec{k}) + \frac{1}{2} q_\perp q_\perp \frac{\partial^2}{\partial k_\perp \partial k_j} \psi(\vec{k}) + \ldots \]  

(106)

The terms with odd powers of \( q_\perp \) average to 0. The bilinear terms contain \( q_\perp^2 \times (1 - t^2) \times \cos^2 \phi \), or \( \sin^2 \phi \), for \( i = j = 1, 2 \), respectively, and \( t^2 \), for \( i = 3 \). The integral over \( \phi \) produces \( \pi/2 \) times a vector

\[ \vec{w}(t) = (1 - t^2, 1 - t^2, 2t^2). \]  

(107)

The variable \( q \) can be changed to \( u \) of Eq. (100), and introducing the constant

\[ b = \frac{\sqrt{2m}}{\lambda_0^2}, \]  

(108)

one obtains the vector

\[ \vec{v} = \int_{0}^{1} dt \, t(1 - t^2) \, \vec{w}(t) \, \tau(t), \]  

(109)

\[ \tau(t) = \int_{0}^{\infty} du \frac{\alpha}{(b/\mu)^2 + u^2} \, e^{-u^2}. \]  

(110)

that appears in the resulting interaction term:

\[ W_{QQ} = -\frac{4}{3} \frac{\alpha}{2\pi} b^{-3} \sum_{i=1}^{3} \tau_i \frac{\partial^2}{\partial k_i^2}. \]  

(111)

The next non-vanishing terms in the Taylor expansion contain the fourth and higher even powers of \( \vec{q} \). They are expected to be small in the momentum region dominated by the Coulomb dynamics and do not count around the bottom of the harmonic potential. The remaining question is if the harmonic approximation can be rotationally symmetric.
The interaction $W_{QQ}$ given by Eq. (11) is rotationally symmetric when all components of $\tilde{\tau}$ are equal, or

$$\int_0^1 dt \, t (1-t^2) (1-3t^2) \, \tau(t) = 0 \, .$$  

(112)

The function $\tau(t)$ depends on $\mu$ in a limited way because the integral over $u$ in Eq. (11) extends only from 0 to about 1, $b/t$ is large, and $b\mu/t \gg 1$ produces $\tau(t) = \beta$,

$$\beta = \sqrt{\pi}/4 \, .$$  

(113)

Behavior of $\tau(t)$ near $t = 0$ does not matter because of the factor $t$ in Eq. (112), and the condition (88) is of little consequence if $\mu^2$ raises quickly from 0 at $t = 0$. For any ansatz of Eq. (88) with a small $\delta_\mu$,

$$\tau(t) = \frac{c^2(t)}{1 + c^2(t)} \beta \, ,$$  

(114)

which is equivalent to the constant $\beta$ if $c(t) \gg 1$ for $t \neq 0$. The factorization feature is independent of the shape of the RGEP form factor $f$, which reflects the dependence of the effective dynamics on the RG scheme, but $\beta$ is stable for $f_{ab/\lambda}$ of similar shapes as functions of $ab/\lambda^2$, see Eq. (24).

The first approximation for heavy quarkonium dynamics in position space can be defined by the Fourier transform of the eigenvalue equation for $H_{Q\bar{Q}}$, with

$$\langle \hat{r} | \hat{p} \rangle = \exp (i \hat{k} \hat{r}) \, .$$  

(121)

This transform exists only in the relative motion variables, since the motion of the quarkonium as a whole is described in a relativistic fashion and the relation between the relative motion of quarks and the motion of the bound state as a whole does not coincide for large speeds with the one known in NR theory. The Schrödinger equation reads

$$\left[ 2m - \frac{\Delta r}{m} - \frac{4\alpha}{3} \left( \frac{1}{r} + BF \right) + \frac{k^2}{2} \right] \psi(r) = M \psi(r) \, ,$$  

(122)

where $k = m\omega^2/2$, and

$$\omega = \sqrt{\frac{4}{3} \frac{\alpha}{\pi} \lambda (\frac{\lambda}{m})^2 \sqrt{\frac{\beta}{6}} \, .}$$  

(123)

The number $\beta$ depends on the shape of the RGEP form factor $f$, which reflects the dependence of the effective dynamics on the RG scheme, but $\beta$ is stable for $f_{ab/\lambda}$ of similar shapes as functions of $ab/\lambda^2$, see Eq. (24).

VI. CONCLUSION

The Coulomb interaction between quarks in heavy quarkonia is corrected by the potential well that is excavated by the one effective gluon exchange in the overlapping self-interaction gluon clouds of the quarks. At the bottom, the well shape is a quadratic function of the distance between the quarks. The resulting harmonic oscillator force plays the role of a confining one in a limited range. At distances much larger than the Bohr radius the quadratic approximation stops working. Emission of additional gluons and pairs of light quarks will further change the rate of growth of the potential. The size of these effects should be computable in the present approach by evaluating effective Hamiltonians order by order in a weak coupling expansion and solving eigenvalue problems for them numerically.
The effective particle approach is of interest because it describes the relative motion of quarks independently of the speed of the quarkonium as a whole. This result is obtained at the price of setting up QCD in its Hamiltonian version in LF dynamics, with a host of difficulties in the renormalization program that had to be overcome. Further advances in the RGEP and methods of solving the eigenvalue equations for Hamiltonians $H_\lambda$ are expected to reflect the well known features of interactions of relativistic particles. The first approximation for $H_{QQ}$ can be expected to work well in the refined calculations because it appears to be largely independent in its structure from the details of the RGEP vertex form factors and the gluon mass ansatz. The first approximation also appears to involve the least possible degree of complexity as a basis around which a meaningful successive approximation scheme can emerge. A few percent accuracy in evaluating effective Hamiltonians is known to be achievable using essentially the same method in the case of elementary matrix models with asymptotic freedom and bound states.

Since the approach developed here is boost invariant, it can connect physical images of hadrons in different frames as soon as the hadron dynamics is understood in one of them. Although the light quarks are expected to behave differently from the heavy ones, one should note that the Schrödinger equation with $H_\lambda$ does not lead to the spread of probability towards large relative momenta and large numbers of effective particles. The spread is halted because the interaction terms in $H_\lambda$ contain form factors. These form factors are the reason for hope that one of them. Although the light quarks are expected to be expected to work well in the refined calculations because it appears to be largely independent in its structure from the details of the RGEP vertex form factors and the gluon mass ansatz. The first approximation also appears to involve the least possible degree of complexity as a basis around which a meaningful successive approximation scheme can emerge. A few percent accuracy in evaluating effective Hamiltonians is known to be achievable using essentially the same method in the case of elementary matrix models with asymptotic freedom and bound states.

Aside from QCD, the same scheme for setting up and solving quantum field theory should be tested in the case of QED. There, the effective mass ansatz for virtual photons is much more restricted and small-$x$ effects are of less significance. On the other hand, QED is not asymptotically free and its effective nature requires better understanding. The RGEP approach may help in defining QED as an effective theory. But one needs to first verify if perturbation theory with $H_\lambda$ can produce covariant $S$-matrix in QED in orders higher than second.

**APPENDIX A: REGULARIZED LF HAMILTONIAN OF QCD**

The canonical LF Hamiltonians of gauge theories, similar to the Hamiltonians in the infinite momentum frame [68, 69, 70], are well known [71, 72], and extensive literature exists on the light-like axial gauges [73, 74]. The Hamiltonian given below is further specified by inclusion of the ultraviolet and small-$x$ regularization factors that render a computable operator. This means that $H$ does not require a separate regularization prescription for evaluating loop integrals. The same regularized Hamiltonian was used in [32] but the quark terms needed here were not explicitly given there. $H$ is supplied with counterterms $H_{\Delta \delta}$. Their structure is known from considerations similar to [2]. Details can be calculated using RGEP. The initial Lagrangian is

$$L = \bar{\psi}(i\mathcal{D} - m)\psi - \frac{1}{4} F^{\mu\nu} F_{\mu\nu}^a,$$  \hspace{1cm} (A1)  

with one flavor of quarks of mass $m$,

$$F^{\mu\nu} = -i[D^\mu, D^\nu]/g,$$  \hspace{1cm} (A2)  

and

$$D_\mu = \partial_\mu + igA_\mu,$$  \hspace{1cm} (A3)  

where $A = A^a t^a$, $[t^a, t^b] = i f^{abc} t^c$, and $T_\gamma(t^a t^b) = \delta^{ab}/2$. The classical Nether generator of evolution in $x^+$ takes the form (the Gauss law constraint is formally solved in $A^+ = 0$ gauge and the counterterms are added as the last term from hindsight),

$$H = H_{\psi^2} + H_{A^2} + H_{A^3} + H_{A^4} + H_{\psi A^2}$$
$$+ H_{\psi A A^2} + H_{[\theta A A]}^2 + H_{[\theta A A](\psi \psi)} + H_{(\psi \psi)^2} + H_{\Delta \delta},$$  \hspace{1cm} (A4)  

where each term is an integral over the LF hyperplane,

$$H_i = \int dx^- d^2 x^+ \mathcal{H}_i,$$  \hspace{1cm} (A5)  

and,

$$\mathcal{H}_{\psi^2} = \frac{1}{2} \bar{\psi} \gamma^+ \partial^- \frac{1}{2} m^2 \psi,$$  \hspace{1cm} (A6)  

$$\mathcal{H}_{A^2} = -\frac{1}{2} A^++(\partial^-)^2 A^+, \hspace{1cm} (A7)$$

$$\mathcal{H}_{A^3} = g i \partial_\alpha A_\mu [A^\alpha A^\beta]^a,$$  \hspace{1cm} (A8)  

$$\mathcal{H}_{A^4} = -\frac{1}{4} g^2 [A_\alpha, A_\beta]^a [A^\alpha, A^\beta]^a,$$  \hspace{1cm} (A9)  

$$\mathcal{H}_{\psi A^2} = g \bar{\psi} A^2 \psi,$$  \hspace{1cm} (A10)  

$$\mathcal{H}_{\psi A A^2} = \frac{1}{2} g^2 \bar{\psi} \frac{\gamma^+}{(i\partial^+)^2} \psi A^2,$$  \hspace{1cm} (A11)  

$$\mathcal{H}_{[\theta A A]^2} = \frac{1}{2} g^2 [i\partial^+ A^+, A^+]^a \frac{1}{(i\partial^+)^2} [i\partial^+ A^+, A^+]^a,$$  \hspace{1cm} (A12)  

$$\mathcal{H}_{[\theta A A](\psi \psi)} = g^2 \bar{\psi} \gamma^+ t^a \psi - \frac{1}{(i\partial^+)^2} [i\partial^+ A^+, A^+]^a,$$  \hspace{1cm} (A13)  

$$\mathcal{H}_{(\psi \psi)^2} = \frac{1}{2} g^2 \bar{\psi} \gamma^+ t^a \psi - \frac{1}{(i\partial^+)^2} \bar{\psi} \gamma^+ t^a \psi.$$  \hspace{1cm} (A14)  

A quantum Hamiltonian is introduced by expanding the fields into Fourier components at $x^+ = 0$ and imposing commutation relations on the latter. They define creation and annihilation operators for bare particles.

$$\psi = \sum_{\alpha c} \int [k] \left[ \chi_{\alpha}^uk_{\alpha}\epsilon_{\alpha}e^{-ikx} + \chi_{\alpha}^u\kappa_{\alpha}\kappa_{\alpha}\epsilon_{\alpha}e^{ikx} \right].$$  \hspace{1cm} (A15)
The integration measure is
\[ [k] = \frac{\theta(k^+ + k^+ d^2 k^-)}{16\pi^3 k^+}. \] (A16)

\[ \{b_{k\sigma c}, b_{k'\sigma' c'}\} = \{d_{k\sigma c}, d_{k'\sigma' c'}\} = 16\pi^3 k^+ \delta_{\sigma \sigma'} \delta_{cc'} \delta^3(k - k'). \] (A17)

\[ \delta^3(k - k') = \delta(k^+ - k'^+) \delta(k^- - k'^-) \delta(k^\perp - k'^\perp). \] (A18)

The symmetries of the matrix \( B(k, m) \) are given by \( u_{\sigma} = \frac{1}{\sqrt{k + m}} \left[ \Lambda_+ k^+ + \Lambda_-(m + k^+ \alpha^\perp) \right] \), (A19)

\[ \left[ \begin{array}{c} X_\sigma \\ 0 \end{array} \right] \]

\[ \left[ \begin{array}{c} 0 \\ \xi_{\sigma} \end{array} \right] \]

where \( \xi_{\sigma} = -i \sigma_2 \chi_{\sigma} = \sigma \chi_{-\sigma} \). The gluon field at \( x^+ = 0 \) is

\[ A^\mu = \sum_{\sigma} \int [k] \left[ t^c \epsilon^\mu_{k\sigma c} a_{k\sigma c} e^{-ik\hat{x}} + t^c \epsilon^\mu_{k\sigma c} a^\dagger_{k\sigma c} e^{ik\hat{x}} \right], \] (A20)

and the commutation relations read,

\[ [k_{a\sigma c}, a^\dagger_{k'\sigma' c'}] = 16\pi^3 k^+ \delta_{\sigma \sigma'} \delta_{cc'} \delta^3(k - k'). \] (A21)

The symbols introduced in this operator occur in all other terms and require explanation for completeness, see also [32]. The conservation of momentum in the interaction vertices is enforced by the factor

\[ \delta(p^1 - p) = 16\pi^3 \delta^3 \left( \sum_{a} p_{a1} - \sum_{a} p_a \right). \] (A22)

The regularization factors are given by

\[ r_{\Delta}(p,d) = r_{\Delta}(p,d) r_{\Delta}(p,p - d), \] (A23)

where

\[ r_{\Delta}(p,d) = r_{\Delta}(\kappa_{d/p}^{-2}) r_{\delta}(x_{d/p}) \theta(x_{d/p}). \] (A24)

The gluon spin vertex factor reads

\[ Y_{123} = if \epsilon_{123} \frac{1}{x_{1/3}^2} \left[ \frac{1}{x_{1/3}^2} - \frac{1}{x_{1/3}^2} \right]. \] (A25)

The quartic gluon vertex is

\[ \epsilon \equiv \epsilon^\perp, \kappa \equiv \kappa^1_{1/3}. \] (A26)

except that the boosts are applied to the polarization vectors \( \epsilon_{\sigma}^a = (0,0,\epsilon^\perp|^a|) \) that correspond to the selected state of a gluon moving along the z-axis [12]. Terms that contain the ratio \( k^+/k^+ \), which mix the transverse and longitudinal momenta, are again only a shorthand notation for writing interactions. The independent transverse field \( A^\perp \) contains only polarization vectors \( \epsilon_{\sigma}^a \) that have dimension 1. \( A^\perp \) has dimension of \( k^+ a \), which matches the required \( 1/x_{\perp} \) on the LF when \( a \sim 1/k^+ \), as promised.
\[ H_{A^s} = \sum_{1234} \int [1234] \delta(p^i - p) \frac{g^2}{4} \left[ \Xi_{A^s} 1234 a_1^\dagger a_2^\dagger a_3 a_4 + X_{A^s} 1234 a_1^\dagger a_3 a_4 + \Xi_{A^s} 1234 a_1^\dagger a_2 a_3 a_1 \right], \]  

(A33)

where

\[ \Xi_{A^s} 1234 = \frac{2}{3} \left( \bar{\nu}_1 + \bar{\nu}_3 \right) \sum_{1234} \left( \epsilon_1^+ \epsilon_2^+ \epsilon_3^+ \epsilon_4^+ \right) f_{ac1c2} f_{ac3c4} + \bar{\nu}_1 + \bar{\nu}_3 \sum_{1234} \left( \epsilon_1^+ \epsilon_2^+ \epsilon_3^+ \epsilon_4^+ \right) f_{ac1c3} f_{ac2c4} + \bar{\nu}_1 + \bar{\nu}_3 \sum_{1234} \left( \epsilon_1^+ \epsilon_2^+ \epsilon_3^+ \epsilon_4^+ \right) f_{ac1c4} f_{ac2c3}, \]  

(A34)

The instantaneous fermion interaction reads

\[ H_{\psi \chi} = \sum_{1234} \int [1234] \delta(p^i - p) \bar{\psi}_3 \gamma_4 \left[ \bar{u}_2 \gamma_1 u_3 t_{12}^1 b_1^\dagger a_3 b_3 - \bar{v}_3 \gamma_1 v_2 t_{32}^1 d_1^\dagger a_3 d_3 + \bar{u}_1 \gamma_2 v_3 t_{12}^3 b_1^\dagger d_3 a_3 + h.c. \right], \]  

(A36)

where the spin vertex factors are

\[ \bar{u}_2 \gamma_1 u_3 = \sqrt{\bar{x}_3 / x_2} \chi_2 \left[ i(\kappa_{1/3} \times \epsilon_1^+)^3 \sigma_3 + \frac{x_2 + x_3}{x_1} \kappa_{1/3}^{+} \epsilon_1^+ - m \frac{x_1}{x_3} \sigma_1 \epsilon_1^+ \sigma_3 \right] \chi_3, \]  

(A37)

\[ \bar{v}_3 \gamma_1 v_2 = \sqrt{\bar{x}_3 / x_2} \chi_1 \left[ -i(\kappa_{1/3} \times \epsilon_1^+)^3 \sigma_3 + \frac{x_2 + x_3}{x_1} \kappa_{1/3}^{+} \epsilon_1^+ - m \frac{x_1}{x_3} \sigma_1 \epsilon_1^+ \sigma_3 \right] \chi_2, \]  

(A38)

\[ \bar{u}_1 \gamma_2 v_3 = \sqrt{\bar{x}_3 / x_1} \sqrt{\bar{x}_3 / x_2} \chi_1 \left[ -i(\kappa_{1/3} \times \epsilon_1^+)^3 + \frac{x_1 - x_2}{x_3} \kappa_{1/3}^{+} \sigma_3 - m \sigma_1 \epsilon_1^+ \right] \chi_2. \]  

(A39)

The instantaneous fermion interaction reads

\[ H_{\psi \chi} = \sum_{1234} \int [1234] \delta(p^i - p) \left( g^2 / 2 \right) 2 \sqrt{\bar{x}_1 x_4} \cdot \{ \} , \]  

(A40)

where the curly brackets \{ \} contain the operators ordered according to the rule \( b^i a^i \cdot a^i d^i \);
where

\[
\Xi_{(\partial AA)^2} = \frac{1}{6} \left[ \hat{r}_{1+2,1} \hat{r}_{4,3} \epsilon_1^* e_2^* \cdot e_3^* e_4 \frac{(x_1 - x_2)(x_3 + x_4)}{(x_1 + x_2)^2} f_{\alpha c_1 \epsilon_2} f_{\alpha c_3 c_4} \right. \\
+ \hat{r}_{1+3,1} \hat{r}_{4,2} \epsilon_1^* e_3^* \cdot e_2^* e_4 \frac{(x_1 - x_3)(x_2 + x_4)}{(x_1 + x_3)^2} f_{\alpha c_1 \epsilon_3} f_{\alpha c_2 c_4} \\
+ \left. \hat{r}_{3+2,1} \hat{r}_{4,1} \epsilon_3^* e_3^* \cdot e_1^* e_4 \frac{(x_3 - x_2)(x_1 + x_4)}{(x_3 + x_2)^2} f_{\alpha c_2 \epsilon_3} f_{\alpha c_1 c_4} \right], \tag{A43}
\]

\[
X_{(\partial AA)^2} = \frac{1}{4} \left[ \hat{r}_{1+2,1} \hat{r}_{3+4,3} \epsilon_1^* e_2^* \cdot e_3^* e_4 \frac{(x_1 - x_2)(x_3 - x_4)}{(x_1 + x_2)^2} f_{\alpha c_1 \epsilon_2} f_{\alpha c_3 c_4} \\
- [\hat{r}_{3,1} \hat{r}_{2,4} + \hat{r}_{1,3} \hat{r}_{4,2}] \epsilon_1^* e_3^* \cdot e_2^* e_4 \frac{(x_1 + x_3)(x_2 + x_4)}{(x_2 - x_4)^2} f_{\alpha c_1 c_3} f_{\alpha c_2 c_4} \\
- [\hat{r}_{3,2} \hat{r}_{1,4} + \hat{r}_{2,3} \hat{r}_{4,1}] \epsilon_1^* e_4^* \cdot e_2^* e_4 \frac{(x_2 + x_3)(x_1 + x_4)}{(x_1 - x_4)^2} f_{\alpha c_1 c_4} f_{\alpha c_2 c_3} \right]. \tag{A44}
\]

The instantaneous gluon interaction between quarks and gluons reads

\[
H_{(\partial AA)(\psi \psi)} = \sum_{1234} \int [1234] \delta(p^1 - p) \frac{g^2}{2} i f^{a_12} \epsilon^{a_34} 2 \sqrt{x_1 x_2 x_3 x_4} \{ \}, \tag{A45}
\]

where the brackets \{ \} contain,

\[
\{ \} = \frac{x_2 - x_1}{2(x_1 + x_2)} \left[ \hat{r}_{1+2,1} \hat{r}_{3+4,3} \epsilon_1^* e_2^* \cdot e_3^* e_4 \frac{(x_1 - x_2)(x_3 - x_4)}{(x_1 + x_2)^2} f_{\alpha c_1 \epsilon_2} f_{\alpha c_3 c_4} \\
- \epsilon_1^* e_2^* \frac{x_1 + x_2}{(x_1 + x_4)^2} \hat{r}_{3+4,3} \hat{r}_{2,1} \epsilon_3^* e_3^* \cdot e_2^* e_4 \frac{(x_1 + x_3)(x_2 + x_4)}{(x_2 - x_4)^2} f_{\alpha c_1 c_3} f_{\alpha c_2 c_4} \\
+ \epsilon_1^* e_2^* \frac{x_1 + x_2}{(x_1 - x_2)^2} \left[ \hat{r}_{2,1} \hat{r}_{4,3} + \hat{r}_{1,2} \hat{r}_{4,3} \right] \epsilon_1^* \epsilon_4^* - (\hat{r}_{2,1} \hat{r}_{4,3} + \hat{r}_{1,2} \hat{r}_{4,3}) \chi_3^4 b_3^a a_1^a \epsilon_2^* e_4^* + h.c. \right]. \tag{A46}
\]

Finally, the instantaneous gluon interaction between quarks reads

\[
H_{(\psi \psi)^2} = \sum_{1234} \int [1234] \delta(p^1 - p) \frac{g^2}{2} \frac{\epsilon^{a_12} \epsilon^{a_34}}{4 \sqrt{x_1 x_2 x_3 x_4}} \{ \}, \tag{A47}
\]

where the brackets \{ \} contain,

\[
\{ \} = -\frac{1}{2} \left[ \frac{\chi^1 \chi^2 \chi^3 \chi^4}{(x_1 - x_2)^2} \epsilon^{a_12} \epsilon^{a_34} \left[ \hat{r}_{1,2} \hat{r}_{4,3} + \hat{r}_{2,1} \hat{r}_{3,4} \right] - \frac{\chi^1 \chi^2 \chi^1 \chi^4}{(x_3 - x_2)^2} \epsilon^{a_12} \epsilon^{a_34} \left[ \hat{r}_{3,2} \hat{r}_{4,1} + \hat{r}_{2,3} \hat{r}_{1,4} \right] \right] b_1^a b_2^b b_4^c \\
+ \frac{1}{2} \left[ \frac{\chi^1 \chi^2 \chi^3 \chi^4}{(x_1 - x_2)^2} \epsilon^{a_12} \epsilon^{a_34} \left[ \hat{r}_{1,2} \hat{r}_{4,3} + \hat{r}_{2,1} \hat{r}_{3,4} \right] - \frac{\chi^1 \chi^2 \chi^1 \chi^4}{(x_3 - x_2)^2} \epsilon^{a_12} \epsilon^{a_34} \left[ \hat{r}_{3,2} \hat{r}_{4,1} + \hat{r}_{2,3} \hat{r}_{1,4} \right] \right] d_1^a d_2^b d_4^c \\
+ \left( \frac{\chi^1 \chi^2 \chi^3 \chi^4}{(x_1 - x_2)^2} \epsilon^{a_12} \epsilon^{a_34} \left[ \hat{r}_{1,2} \hat{r}_{4,3} + \hat{r}_{2,1} \hat{r}_{3,4} \right] - \frac{\chi^1 \chi^2 \chi^1 \chi^4}{(x_3 - x_2)^2} \epsilon^{a_12} \epsilon^{a_34} \left[ \hat{r}_{3,2} \hat{r}_{4,1} + \hat{r}_{2,3} \hat{r}_{1,4} \right] \right] b_1^a d_1^b b_2^c \right. \\
- \left( \frac{\chi^1 \chi^2 \chi^3 \chi^4}{(x_1 - x_2)^2} \epsilon^{a_12} \epsilon^{a_34} \left[ \hat{r}_{1,2} \hat{r}_{4,3} + \hat{r}_{2,1} \hat{r}_{3,4} \right] - \frac{\chi^1 \chi^2 \chi^1 \chi^4}{(x_3 - x_2)^2} \epsilon^{a_12} \epsilon^{a_34} \left[ \hat{r}_{3,2} \hat{r}_{4,1} + \hat{r}_{2,3} \hat{r}_{1,4} \right] \right] b_1^a d_1^b b_2^c \left. \right) + h.c. \right) \\
- 2 \frac{\chi^1 \chi^2 \chi^3 \chi^4}{(x_1 - x_2)^2} \epsilon^{a_12} \epsilon^{a_34} \left[ \hat{r}_{1,2} \hat{r}_{4,3} + \hat{r}_{2,1} \hat{r}_{3,4} \right] b_1^a d_1^b d_2^c + 2 \frac{\chi^1 \chi^2 \chi^3 \chi^4}{(x_1 + x_3)^2} \epsilon^{a_12} \epsilon^{a_34} \left[ \hat{r}_{1,2} \hat{r}_{4,3} + \hat{r}_{2,1} \hat{r}_{3,4} \right] b_1^a d_1^b d_2^c. \tag{A48}
\]

Useful color identities are: \( t^a t^b t^c = -t^a/(2N_c), t^a t^b t^c = \delta^{ab}/N_c + \delta^{bc} c^d, \) \( d^{abc} d^{abd} = [(N_c^2 - 1)/N_c] \delta^{cd} \) and \( f^{abc} t^c = i(N_c/2)t^a \).
APPENDIX B: $Q_\Lambda \bar{Q}_\Lambda$ INTERACTION

Several factors are needed to estimate the small-$x$ behavior of the potential kernel $\tilde{v}_0(13, 24)$ in Eq. (71). Momentum are labeled according to Figs. 1 and 2.

\[ f_{13,24} = \exp \left[-(\mathcal{M}_{13}^2 - \mathcal{M}_{24}^2)^2/\lambda^4 \right], \quad (B1) \]
where
\[ \mathcal{M}_{ij}^2 \equiv 4(m^2 + |\vec{k}_{ij}|^2) = \frac{\kappa_{ij}^2 + m^2}{x_i x_j}, \quad (B2) \]

with
\[ \kappa_{ij}^1 = \kappa_{ij}^x, \quad (B3) \]
\[ \kappa_{ij}^3 = (x_i - 1/2)\mathcal{M}_{ij}, \quad (B4) \]

\[ \mathcal{M}_{13}^2 - \mathcal{M}_{24}^2 = 4(\kappa_{13}^x + \kappa_{24}^x) \vec{q}, \quad (B5) \]
where
\[ \vec{q} = \vec{k}_{13} - \vec{k}_{24}, \quad (B6) \]
is the momentum transfer that goes over to the standard one in the NR limit.

\[ \mathcal{M}_{13}^2 - \mathcal{M}_{24}^2 = \frac{2\kappa_{ij}^1 q^1 - q^1 - z(1 - 2x_1 + z)\mathcal{M}_{13}^2}{(x_1 - z)(x_3 + z)}, \quad (B7) \]

In the last term in Eq. (71), the form factors $f_{1,52} f_{53,4}$ have arguments
\[ \mathcal{M}_{52}^2 - m^2 = \frac{x_1}{x_1 - z} D_1, \quad (B8) \]

where
\[ D_1 \equiv \frac{x_1}{|z|} \left( q^1 + \frac{z}{x_1} \kappa_{13}^1 \right)^2 + m^2 |z| x_1, \quad (B9) \]

and,
\[ \mathcal{M}_{53}^2 - m^2 = D_3, \quad (B10) \]

where
\[ D_3 = \frac{x_3}{|z|} \left( q^1 + \frac{z}{x_3} \kappa_{13}^1 \right)^2 + m^2 |z| x_3, \quad (B11) \]

while the form factors $f_{3,54} f_{51,2}$ have the arguments,
\[ \mathcal{M}_{54}^2 - m^2 = \frac{x_3}{x_3 + z} D_3, \quad (B12) \]

and,
\[ \mathcal{M}_{51}^2 - m^2 = D_1. \quad (B13) \]

The last term in Eq. (71) can be written with the coefficient $1 = (1 - f_{13,24}) + f_{13,24}$ but only the second term counts at small $z$ because $(1 - f_{13,24})$ is proportional to the momentum transfer squared. The factor $f_{13,24}$ becomes common to all terms in Eq. (71) and is taken out in front. The LF instantaneous term can be split into the part $ff$ that joins the low-energy exchange $1-ff$ that goes with the high-energy exchange. This way one obtains

\[ \left( \frac{\kappa_{13}^1 + m^2}{x_1 x_3} + \frac{\vec{m}_1^2}{x_1} + \frac{\vec{m}_3^2}{x_3} - M^2 \right) \psi(\kappa_{13}^x, x_1) - \frac{4\alpha}{3\pi^2} \int dx_2 d^2 \kappa_{24} \sqrt{\frac{x_1 x_3}{x_2 x_4}} \frac{f_{13,24}}{x_5^2} \tilde{v}_0(13, 24) \psi(\kappa_{24}^x, x_2) = 0, \quad (B14) \]

where
\[ \tilde{v}_0(13, 24) = \theta(z) \tilde{f}_{5/1} \tilde{f}_{5/4} (\tilde{v}_{+\, \text{high}} + \tilde{v}_{+\, \text{low}}) + \theta(-z) \tilde{f}_{5/3} \tilde{f}_{5/2} (\tilde{v}_{-\, \text{high}} + \tilde{v}_{-\, \text{low}}). \quad (B15) \]
The gluon mass ansatz contributes to the low-energy exchange terms only. In terms of the invariant masses from Eqs. (B8)-(B13),

\[ \tilde{v}_{+\, \text{high}} = \frac{f_{1,52} f_{53,4} - 1}{2} \left\{ \frac{x_1^2 + x_3^2}{x_1 x_3} \frac{(\mathcal{M}_{52}^2 - m^2)(\mathcal{M}_{53}^2 - m^2)}{(\mathcal{M}_{52}^2 - m^2)^2 + (\mathcal{M}_{53}^2 - m^2)^2} - 1 \right\}, \quad (B16) \]
\[ \tilde{v}_{-\, \text{high}} = \frac{f_{3,54} f_{51,2} - 1}{2} \left\{ \frac{x_2^2 + x_5^2}{x_2 x_5} \frac{(\mathcal{M}_{54}^2 - m^2)(\mathcal{M}_{51}^2 - m^2)}{(\mathcal{M}_{54}^2 - m^2)^2 + (\mathcal{M}_{51}^2 - m^2)^2} - 1 \right\}. \quad (B17) \]

These have the same limit when $z \to 0$ for fixed $q^1$,

\[ \lim_{z \to 0} \tilde{v}_{+\, \text{high}} = \lim_{z \to 0} \tilde{v}_{-\, \text{high}} \]
The terms on the order of \( z^2 \) and higher are finite when divided by the square of \( x_5 = |z| \). Terms linear in \( z \) produce an integral convergent in the sense of principal value \( \frac{1}{2} \). When \( q^\perp \sim \sqrt{z} \to 0 \), \( d^2k_{24} \) removes one power of \( z \) from the denominator in Eq. (20), while \( \tilde{v}_{\pm \text{high}} \) vanish for \( z \to 0 \). The contributions of \( \tilde{v}_{\pm \text{high}} \) are \( k^2/m^2 \) times smaller than the dominant terms and can be ignored in the first approximation. One can see this by integrating \( \tilde{v}_{\pm \text{high}} \) with a Coulomb wave function.

The low-energy terms read

\[
\tilde{v}_{+ \text{low}} = \frac{f_{5,52} f_{53,4}}{4} \left[ 2 - \frac{(M^2_{52} - m^2)/x_4 - \mu^2(2,5,3)/x_5}{(M^2_{52} - m^2)/x_1 + \mu^2(2,5,3)/x_5} - \frac{(M^2_{52} - m^2)/x_1 - \mu^2(2,5,3)/x_5}{(M^2_{52} - m^2)/x_4 + \mu^2(2,5,3)/x_5} \right] ,
\]

\[
\tilde{v}_{- \text{low}} = \frac{f_{5,54} f_{51,2}}{4} \left[ 2 - \frac{(M^2_{53} - m^2)/x_2 - \mu^2(1,5,4)/x_5}{(M^2_{52} - m^2)/x_3 + \mu^2(1,5,4)/x_5} - \frac{(M^2_{54} - m^2)/x_3 - \mu^2(1,5,4)/x_5}{(M^2_{51} - m^2)/x_2 + \mu^2(1,5,4)/x_5} \right] .
\]

**APPENDIX C: MASS TERMS**

The mass terms with \( i = 1,3 \) in the eigenvalue Eq. (13) are given in Eq. (21), with \( M^2_i \) given in Eq. (20), and \( M^2_3 \) in Eq. (22). \( m_3^2 \) originates from Eq. (25) with \( \lambda = \lambda_0 \). Namely, the quark mass counterterm in \( X \) of Eq. (4) adds \( \delta m^2_3 \) to the original mass parameter \( m^2 \) in Eq. (3) and the free ultraviolet-finite part of the counterterm is such that \( m_3^2 \) appears in Eq. (70). The condition on \( m_3^2 \) that the eigenstates of \( H_{\lambda_0} \) with quantum numbers of a single quark have eigenvalues growing to infinity is fulfilled below by representing gluon interactions in the case of the single quark state by a new gluon mass ansatz. The resulting value of \( m_3^2 \) enters into the quarkonium dynamics. The determination of the ultraviolet-finite part of the mass counterterm in \( X \) in Eq. (2) is thus based on the picture that comes out from simultaneous consideration of two eigenvalue equations, one for the state with quantum numbers of a single quark (or an anti-quark, the result is the same), and another one for the quarkonium. The key physical assumption made in the comparison is that the binding of effective quarks in the quarkonium state occurs at the expense of change in their individual structure. While the buildup of self-interacting clouds of gluons around single quarks leads to the infinite quark masses, in the case of a colorless pair the main parts of the gluon clouds can recombine into a colorless object that may fly out of the region of strong interaction with the quarks, leaving behind only the minimal remnants of the gluon clouds required to form the quarkonium eigenstate with a finite mass. The new finite balance is described using the gluon mass ansatz parameter \( \delta_\mu \). The finite balance can be achieved because the quark-anti-quark state looks neutral from large distances and does not continue to generate gluons over infinite distances along the LF. This scenario is partly similar to the one originally developed in the LF dynamics in [44, 45], and studied in [46, 47]. The main differences are related to the fact that the physical picture that emerges here in the finite effective theory with the gluon mass ansatz relies on the phenomenological parameter \( \delta_\mu \). A formal cutoff parameter \( \delta^+ \) of the canonical theory, the coupling coherence phenomenon that may work over many scales of an ultraviolet cutoff, and the condition of transverse locality are not employed in the new picture. Instead, the present scenario can be studied in higher orders of perturbation theory according to the known rules [30, 32] that explicitly preserve the boost invariance, cluster decomposition property, and unitary connection with the initial theory.

The eigenstate of \( H_{\lambda_0} \) with a single quark quantum numbers and momentum \( p \) with components \( p^+ \) and \( p^- \), has the eigenvalue

\[
p^- = (p^{z^2} + m^2)/p^+ ,
\]

and the decomposition in the effective particle basis,

\[
|p\rangle = |Q_{\lambda_0}\rangle + |Q_{\lambda_0}Q_{\lambda_0}\rangle + \ldots .
\]

The new gluon mass ansatz is introduced in the quark-gluon component. It is different than in the quarkonium case because the states have different quantum numbers and are made of different numbers of effective particles. Dropping the subscript \( \lambda_0 \) as in Eq. (51), the eigenvalue problem is written as

\[
(T_g + \tilde{T}_g) |Qg\rangle + Y |Q\rangle = E |Qg\rangle ,
\]

\[
Y |Qg\rangle + T_g |Q\rangle = E |Q\rangle .
\]

The new ansatz enters through the kinetic energy \( \tilde{T}_g \), which contains

\[
\tilde{\mu}_{\lambda_0}^2 = \mu_{\lambda_0}^2 + \mu_{\lambda_0}^2(x, \kappa^\perp) ,
\]

where \( x \) and \( \kappa^\perp \) refer to the relative motion of the effective gluon with respect to the quark. The operator

\[
= \frac{f_{3,54} f_{51,2} - 1}{2} \frac{x_1 - x_3}{x_1 x_3 (x_1^2 + x_3^2)} \left( q^\perp - \frac{\kappa_{13}^\perp}{2} \right) \frac{- \kappa_{13}^\perp}{q^\perp} z + o(z^2) .
\]
$R$ from Eq. 67 is now replaced by the one with $\hat{P}$ that projects on the single effective quark basis state with kinematical momentum components $p^+$ and $p^\perp$. In the perturbative expansion in $g$, only the second term on the right-hand side of Eq. 64 contributes in orders up to $g^2$. Thanks to the boost invariance, the resulting eigenvalue condition reduces to an equation for $\tilde{m}^2$, which is independent of $p$,

\begin{align}
\tilde{m}^2 &= m_0^2 - \frac{4\alpha}{3\pi^2} \int dx d^2k\perp \tilde{r}^2(x) \\
&\times f_M^2(m_0^2, M^2) \frac{1}{x^2} \frac{M^2 - m_0^2}{M_0^2 - m_0^2}, \tag{C5}
\end{align}

$M_0^2 = [\kappa^\perp + \mu_0^2(x, \kappa^\perp)]/x + (\kappa^\perp + m^2)/(1 - x)$.

(C6)

For $\tilde{m}^2$ to be positive and growing to infinity when $\delta \to 0$, one can write $m_0^2$ in the integral form,

\begin{align}
m_0^2 &= m^2 + \frac{4\alpha}{3\pi^2} \int dx d^2k\perp \tilde{r}^2(x) \\
&\times f_M^2(m_0^2, M^2) \frac{1}{x^2} \frac{M^2 - m_0^2}{M_0^2 - m_0^2}, \tag{C7}
\end{align}

with some function $M_0^2$ that satisfies the condition,

\begin{equation}
M_0^2 > M_0^2 > m^2. \tag{C8}
\end{equation}

This condition can be satisfied by writing,

\begin{equation}
M_0^2 = [\kappa^\perp + \mu_0^2(x, \kappa^\perp)]/x + (\kappa^\perp + m^2)/(1 - x), \tag{C9}
\end{equation}

and assuming that

\begin{equation}
\mu_0^2 > \mu_0^2 \geq 0. \tag{C10}
\end{equation}

As long as the difference $\mu_0^2 - \mu_0^2$ does not vanish for $x \to 0$, the single quark mass will tend to $\infty$ when $\delta \to 0$. But this may easily happen here because the larger is the gluon mass ansatz $\mu_Q^2$, the stronger the single quark mass eigenvalue diverges in the limit $\delta \to 0$, while $\mu_0^2$ remains free to vanish in the limit $x \to 0$ and lead to a finite mass contribution in the quarkonium dynamics. Using Eq. (C7) for $m_0^2$ one obtains Eq. (89).

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