Comment on “On the chromatic number of simple triangle-free triple systems”
by
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We have found several errors in the paper [1] and the goal here is to present corrections to all of them. Equational references with square brackets [..] are with respect to the published version. Those with round brackets (...) are with respect to this comment. The notation is from [1].

There was a substantial error in the proof of [15] (in Section 11.4.1 of [1]) and a trivial error in the calculation to the proof of [8] (in Section 9 of [1]). These are corrected in Sections 1 and 2, respectively of the current note.

1 Correction to the proof of [15]

Observe that

\[ f'_u = \sum_c \sum_{v \in N(u)} p'_u(c) p'_v(c) 1_{\kappa'(uv)=c} + \sum_c \sum_{\kappa(uv)=0} p'_u(c) p'_v(c) \]

\[ \leq \sum_c \sum_{\kappa(uv)=c} p'_u(c) p'_v(c) 1_{v \in U'} \]

\[ + \sum_c \sum_{uv \in H} (p'_u(c) p'_v(c) 1_{\gamma_u(c)=1} + p'_u(c) p'_v(c) 1_{\gamma_v(c)=1}) \]

\[ := S_1 + S_2. \]

We will bound each term separately.
1.1 $S_1$

Recall that

$$S_1 = \sum_c \sum_{\kappa(uv)=c} p_u'(c)p_v'(c)1_{v\in U'}.$$  

For each color $c$, let $D_c$ be the event that $\gamma_v(c) = 1$ for at most $\Delta \hat{p}$ vertices $v \in N(u)$. Since $P[\gamma_v(c) = 1] \leq \hat{p}\theta$,

$$P[D_c] \leq \left(\frac{\Delta}{\Delta \hat{p}}\right)\left(\hat{p}\theta\right)^{\Delta \hat{p}} \leq \left(\frac{e}{\hat{p}}\right)^{\Delta \hat{p}}\left(\hat{p}\theta\right)^{\Delta \hat{p}} = (e\theta)^{\Delta \hat{p}} < e^{-\Delta^{1/2}}.$$  

Let $D$ denote the event that $D_c$ holds for all $c$. By the union bound,

$$P[D] \leq qe^{-\Delta^{1/2}}.$$  

By (1) below,

$$E[S_1] = \sum_c \sum_{\kappa(uv)=c} E[p_u'(c)p_v'(c)1_{v\in U'}] \leq \sum_c \sum_{\kappa(uv)=c} p_u(c)p_v(c)(1-\theta(1-6\epsilon)) = f_u(1-\theta(1-6\epsilon)).$$  

Therefore,

$$E[S_1|D] = \frac{E[S_1] - E[S_1|D]P[D]}{P[D]} \leq \frac{E[S_1]}{P[D]} = \frac{f_u(1-\theta(1-6\epsilon))}{(1-qe^{-\Delta^{1/2}})} \leq \frac{f_u(1-\theta(1-7\epsilon))}{(1-qe^{-\Delta^{1/2}})}.$$  

For a vertex subset $X$, let $N(X) = \{v : \exists x \in X \text{ and } w \text{ with } xvw \in H\}$. Let $T_c$ denote the set of color trials for color $c$ at all vertices in $\{u\} \cup N(u) \cup N(N(u))$. Then the trials $T_1, \ldots, T_q$ determine the variable $S_1$. Observe that $T_c$ affects every term of the form $p_u'(c)p_v'(c)1_{v\in U'}$. For $d \neq c$, $T_c$ affects $p_u'(d)p_v'(d)1_{v\in U'}$ only if $\gamma_v(c) = 1$; this is because if $\gamma_v(c) = 0$, the trials for color $c$ have no impact on whether or not $v \in U'$. Thus, given that $\gamma_v(c) = 1$ for at most $\Delta \hat{p}$ of the variables in $T_c$, changing the values in $T_c$ can change $S_1$ by at most $d(u,c)p^2 + 2(\Delta \hat{p})\hat{p}^2$.

Let $\pi(t_i) = P(T_i = t_i | D)$ for $i = 1, 2, \ldots, q$ and let

$$\rho(t_i, t_{i+1}, \ldots, t_q) = \pi(t_i)\pi(t_{i+1}) \cdots \pi(t_q) = P(T_j = t_j, j = i, i + 1, \ldots, q | D).$$
Here we use the fact that conditioning on $D$ still leaves the choices $t_1, t_2, \ldots, t_q$ for the distinct sets of colors $T_1, T_2, \ldots, T_c$ independent of each other. Thus,

$$|E[S_1|D, T_1 = t_1, \ldots, T_c = t_c] - E[S_1|D, T_1 = t_1, \ldots, T_{c-1} = t_{c-1}, T_c = t'_c]|$$

$$= \left| \sum_{t_{c+1}, \ldots, t_q} [S_1(t_1, \ldots, t_{c-1}, t_c, t_{c+1}, \ldots, t_q) - S_1(t_1, \ldots, t_{c-1}, t'_c, t_{c+1}, \ldots, t_q)] \rho(t_{c+1}, \ldots, t_q) \right|$$

$$\leq d(u, c)p^2 + 2\Delta p^3$$

$$\leq 2t_0 \theta \Delta p^3 + 2\Delta p^3$$

$$\leq 3t_0 \theta \Delta p^3.$$

Since

$$\sum_c (3t_0 \theta \Delta p^3)^2 = 9qt^2 \theta^2 \Delta^2 p^6 \leq 9t^2 \theta^2 \Delta^{2+1/2-66/24} \leq \Delta^{-5/24},$$

the Azuma-Hoeffding inequality implies

$$P[S_1 > f_u(1 - \theta(1 - 7\epsilon)) + \Delta^{-1/12}|D] \leq P[S_1 > E[S_1|D] + \Delta^{-1/12}|D]$$

$$\leq e^{-\Delta^{5/24-2/12}}$$

$$= e^{-\Delta^{1/24}}.$$

Thus

$$P[S_1 > f_u(1 - \theta(1 - 7\epsilon)) + \Delta^{-1/12}] \leq P[S_1 > f_u(1 - \theta(1 - 7\epsilon)) + \Delta^{-1/12}|D]P[D] + P[\bar{D}]$$

$$\leq e^{-\Delta^{1/24}} (1 - qe^{-\Delta^{1/2}}) + qe^{-\Delta^{1/2}}$$

$$\leq e^{-\Delta^{1/25}}.$$

1.1.1 Proof of (1)

We prove that if $\kappa(uv) = c$,

$$E[p'_u(c)p'_v(c)1_{v \in U'}] \leq p_u(c)p_v(c)(1 - \theta(1 - 6\epsilon)).$$

(1)

We first establish the following claim.

Claim. $P[v \not\in U'|c \not\in L(v)] \geq P[v \not\in U'] \geq \theta(1 - 5\epsilon)$.

Proof of claim. The vertex $v$ is colored (i.e., not in $U'$) if and only if for some color $d \not\in B(v)$, $\gamma_v(d) = 1$ and $d \not\in L(v)$. Let $R_d$ denote the event that $\gamma_v(d) = 1$ and $d \not\in L(v)$. If $c \in B(v)$, then

$$P[v \not\in U'|c \not\in L(v)] = P[v \not\in U'].$$

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Otherwise, since $\gamma_v(c) = 1$ is independent of $c \notin L(v)$ and $R_d$ is independent of $c \notin L(v)$ for $c \neq d$,

$$
P[v \notin U'|c \notin L(v)] = P[(\cup_{d \notin B(v)} R_d) \cup (R_c)|c \notin L(v)]
= P[(\cup_{d \notin B(v)} R_d) \cup (\gamma_v(c) = 1 \cap c \notin L(v))|c \notin L(v)]
= P[(\cup_{d \notin B(v)} R_d) \cup (\gamma_v(c) = 1)]
\geq P[(\cup_{d \notin B(v)} R_d) \cup R_c]
= P[v \notin U'].
$$

In either case,

$$
P[v \notin U'|c \notin L(v)] \geq P[v \notin U']
= P[\cup_{d \notin B(v)} R_d]
\geq \sum_{d \notin B(v)} P[R_d] - \sum_{d,d' \notin B(v)} P[R_d]P[R_{d'}]
= \sum_{d \notin B(v)} \theta p_v(d)q_v(d) - \sum_{d,d' \notin B(v)} \theta^2 p_v(d)p_v(d')q_v(d)q_v(d')
\geq \theta \sum_{d \in C} p_v(d)q_v(d) - \theta \sum_{d \in B(v)} p_v(d)q_v(d) - \theta^2 \sum_{d,d' \notin B(v)} p_v(d)p_v(d')
\geq \theta \sum_{d \in C} p_v(d)q_v(d) - \theta |B(v)|\hat{p} - \theta^2 \sum_{d,d' \notin B(v)} p_v(d)p_v(d').
$$

Using the inequality $\prod_x (1 - x) \geq 1 - \sum_x x$ (for $x \in [0, 1]$), we obtain

$$
q_v(d) = \prod_{uvw \in H} (1 - \theta^2 p_u(d)p_w(d)) \prod_{uv \in G \kappa(uv) = d} (1 - \theta p_u(d))
\geq 1 - \sum_{uvw \in H} \theta^2 p_u(d)p_w(d) - \sum_{uv \in G \kappa(uv) = d} \theta p_u(d)
= 1 - \theta \sum_{uvw \in H} p_u(d)p_w(d) - \theta \sum_{uv \in G \kappa(uv) = d} p_u(d)
= 1 - \theta^2 \nu_v(d) - \theta f_v(d).
$$

Since $\sum_{d \in C} p_v(c) = 1 + o(1)$,

$$
\theta^2 \sum_{d,d' \notin B(v)} p_v(d)p_v(d') = \frac{1}{2} \theta^2 \sum_{d \in C} \sum_{d' \notin C:d' \neq d} p_v(d)p_v(d') \leq \frac{1}{2} \theta^2 \left( \sum_{d \in C} p_v(d) \right)^2 \leq \theta^2.
$$

By [22], $|B(v)| < \epsilon/\hat{p}$. By [8], $f_v < 3 \omega$, so $\theta f_v < 3 \epsilon$. By [7] and [10] (see [19]), $e_v \leq \omega + \Delta^{-1/10}$, so $\theta^2 e_v < \epsilon/3$. Using these three inequalities and $\sum_{d \in C} p_v(c) \geq (1 - \epsilon/3)$,
we finally obtain
\[
\Pr[v \notin U'] \geq \theta \sum_d p_v(d)(1 - \theta^2 e_v(d) - \theta f_v(d)) - \theta |B(v)| \hat{p} - \theta^2
\]
\[
\geq \theta \sum_d p_v(d) - \theta^3 \sum_d p_v(d)e_v(d) - \theta^2 \sum_d p_v(d)f_v(d) - \theta \epsilon - \theta^2
\]
\[
= \theta \sum_d p_v(d) - \theta^3 e_v - \theta^2 f_v - \theta \epsilon - \theta^2
\]
\[
\geq \theta(1 - \epsilon/3) - \theta \epsilon/3 - 3\theta \epsilon - \theta \epsilon - \theta \epsilon/3
\]
\[
= \theta(1 - 5\epsilon).
\]

We now bound \(\mathbb{E}[p'_u(c)p'_v(c)1_{v \in U'}]\). First assume that \(p'_u(c)\) and \(p'_v(c)\) are determined by Case A (see [3]). Since \(\kappa(uv) = c\), the edge containing \(u\) and \(v\) no longer exists in the hypergraph. By triangle-freeness, there are no vertices \(w\) which share an edge with both \(u\) and \(v\). Therefore the events \(c \notin L(u)\) and \(c \notin L(v)\) are independent. Also, if \(c \notin L(u)\), then \(\gamma_w(c) = 0\) for all \(w \in N_G(u)\), so in particular, \(\gamma_v(c) = 0\). Consequently,
\[
\Pr[\bar{R}_c | c \notin L(u) \cup L(v)] = \Pr[\bar{R}_c | c \notin L(u)] = \Pr[\gamma_v(c) = 0 \cup c \in L(v) | c \notin L(u)] = 1.
\]

Therefore, by the independence of colors,
\[
\Pr[v \in U' | c \notin L(u) \cup L(v)] = \Pr[\cap_{d \notin B(v)} \bar{R}_d | c \notin L(u) \cup L(v)]
\]
\[
= \Pr[\cap_{d \notin B(v) \cup \{c\}} \bar{R}_d | \Pr[\bar{R}_c | c \notin L(u) \cup L(v)]
\]
\[
= \Pr[\cap_{d \notin B(v) \cup \{c\}} \bar{R}_d]
\]
\[
= \Pr[\cap_{d \notin B(v) \cup \{c\}} \bar{R}_d] \Pr[\bar{R}_c]/\Pr[\bar{R}_c]
\]
\[
= \Pr[\cap_{d \notin B(v)} \bar{R}_d]/\Pr[\bar{R}_c]
\]
\[
\leq \Pr[v \in U']/(1 - \theta \hat{p})
\]
\[
\leq \Pr[v \in U']/(1 + 2\theta \hat{p}).
\]

Note that this also implies \(\Pr[v \in U' | c \notin L(u)] \leq \Pr[v \in U'](1 + 2\theta \hat{p})\). If \(c \in L(v) \cup L(u)\),
then \( p'_u(c)p'_v(c) = 0 \), so by the claim,

\[
\mathbf{E}[p'_u(c)p'_v(c)1_{v \in U'}] = \mathbf{E}[p'_u(c)p'_v(c)|v \in U']\mathbf{P}[v \in U']
\]

\[
\leq \frac{p_u(c)p_v(c)}{q_u(c)q_v(c)}\mathbf{P}[c \notin L(u) \cup L(v)|v \in U']\mathbf{P}[v \in U']
\]

\[
= \frac{p_u(c)p_v(c)}{q_u(c)q_v(c)}\mathbf{P}[v \in U'|c \notin L(u) \cup L(v)]\mathbf{P}[c \notin L(u) \cup L(v)]
\]

\[
= \frac{p_u(c)p_v(c)}{q_u(c)q_v(c)}\mathbf{P}[v \in U'|c \notin L(u)]\mathbf{P}[c \notin L(u)]
\]

\[
= p_u(c)p_v(c)\mathbf{P}[v \in U'|c \notin L(u)]
\]

\[
\leq p_u(c)p_v(c)(1 - \theta(1 - 6\epsilon)).
\]

Suppose \( p'_u(c) \) is determined by Case A, and \( p'_v(c) \) is determined by Case B. Recall that the previous case showed that \( \mathbf{P}[v \in U'|c \notin L(u)] \leq \mathbf{P}[v \in U'](1 + 2\theta\hat{p}) \). If \( c \in L(u) \), then \( p'_u(c)p'_v(c) = 0 \), so

\[
\mathbf{E}[p'_u(c)p'_v(c)1_{v \in U'}] = p_v(c)\mathbf{E}[p'_u(c)|v \in U']\mathbf{P}[v \in U']
\]

\[
\leq p_v(c)\frac{p_u(c)}{q_u(c)}\mathbf{P}[c \notin L(u)|v \in U']\mathbf{P}[v \in U']
\]

\[
= p_v(c)\frac{p_u(c)}{q_u(c)}\mathbf{P}[v \in U'|c \notin L(u)]\mathbf{P}[c \notin L(u)]
\]

\[
= p_u(c)p_v(c)\mathbf{P}[v \in U'|c \notin L(u)]
\]

\[
\leq p_u(c)p_v(c)(1 - \theta(1 - 6\epsilon)).
\]

Suppose \( p'_u(c) \) is determined by Case B, and \( p'_v(c) \) is determined by Case A. If \( c \in L(v) \), then \( p'_u(c)p'_v(c) = 0 \), so by the claim,

\[
\mathbf{E}[p'_u(c)p'_v(c)1_{v \in U'}] = p_u(c)\mathbf{E}[p'_v(c)|v \in U']\mathbf{P}[v \in U']
\]

\[
\leq p_u(c)\frac{p_v(c)}{q_v(c)}\mathbf{P}[c \notin L(v)|v \in U']\mathbf{P}[v \in U']
\]

\[
= p_u(c)\frac{p_v(c)}{q_v(c)}\mathbf{P}[v \in U'|c \notin L(v)]\mathbf{P}[c \notin L(v)]
\]

\[
= p_u(c)p_v(c)\mathbf{P}[v \in U'|c \notin L(v)]
\]

\[
\leq p_u(c)p_v(c)(1 - \theta(1 - 6\epsilon)).
\]

If both \( p'_u(c) \) and \( p'_v(c) \) are determined by Case B, then \( p'_u(c) \) and \( p'_v(c) \) are independent
of each other and of \( v \in U' \). Hence
\[
E[p'_u(c)p'_v(c)1_{v \in U'}] = E[p'_u(c)] E[p'_v(c)] E[1_{v \in U'}] = p_u(c)p_v(c) \Pr[v \in U'] \\
\leq p_u(c)p_v(c)(1 - \theta(1 - 6\epsilon)).
\]

### 1.2 \( S_2 \)

By (2) below,
\[
E[S_2] = \sum_c \sum_{uvw} (E[p'_u(c)p'_v(c)1_{\gamma_w(c) = 1}] + E[p'_u(c)p'_w(c)1_{\gamma_v(c) = 1}]) \\
= \sum_c \sum_{uvw} E[p'_u(c)p'_v(c)|\gamma_w(c) = 1] \Pr[\gamma_w(c) = 1] \\
+ \sum_c \sum_{uvw} E[p'_u(c)p'_w(c)|\gamma_v(c) = 1] \Pr[\gamma_v(c) = 1] \\
\leq \sum_c \sum_{uvw} (p_u(c)p_v(c)\Pr[\gamma_w(c) = 1] + p_u(c)p_w(c)\Pr[\gamma_v(c) = 1]) \\
= \sum_c \sum_{uvw} (p_u(c)p_v(c)\theta p_w(c) + p_u(c)p_w(c)\theta p_v(c)) \\
= 2\theta e_u.
\]

Let
\[
S_{2,c} = \sum_{uvw} (p'_u(c)p'_v(c)1_{\gamma_w(c) = 1} + p'_u(c)p'_w(c)1_{\gamma_v(c) = 1}),
\]
and
\[
\hat{S}_2 = \sum_c \min\{S_{2,c}, 2\Delta\hat{p}^3\}.
\]

Then \( \hat{S}_2 \) is the sum of \( q \) independent random variables, each bounded by \( 2\Delta\hat{p}^3 \). By [23],
\[
\Pr[\hat{S}_2 \geq E[\hat{S}_2] + \Delta^{-1/10}] \leq e^{-\frac{\Delta^{-1/5}}{4q\Delta\hat{p}^3}} \leq e^{-\Delta^{-1/5} - 1/2 - 2 + 66/24} = e^{-\Delta^{1/20}/4}.
\]

Observe that if \( S_2 \neq \hat{S}_2 \), then \( S_{2,c} > 2\Delta\hat{p}^3 \) for some color \( c \). This would imply that \( \gamma_w(c) = 1 \) for at least \( \Delta\hat{p} \) neighbors \( w \) of \( u \). Therefore,
\[
\Pr[S_2 \neq S_{2,c}] \leq q \left( \frac{2\Delta}{\Delta\hat{p}} \right)^{\Delta\hat{p}} \leq q \left( \frac{2\epsilon}{\hat{p}} \right)^{\Delta\hat{p}} = q(2\epsilon\theta)^{13/24}.
\]
Since $E[S_2] > E[\hat{S}_2]$, this implies
\[
\Pr[S_2 > E[S_2] + \Delta^{-1/10}] \leq \Pr[S_2 > E[\hat{S}_2] + \Delta^{-1/10}] \\
\leq \Pr[S_2 \neq \hat{S}_2] + \Pr[\hat{S}_2 > E[\hat{S}_2] + \Delta^{-1/10}] \\
\leq q(2e^\theta)^{\Delta^{13/24}} + e^{-\Delta^{1/20}/4} \\
\leq e^{-\Delta^{1/21}}.
\]

Therefore, with probability at least $1 - e^{-\Delta^{1/21}} - e^{-\Delta^{1/25}}$
\[
f_u' \leq f_u(1 - \theta(1 - 7\epsilon)) + \Delta^{-1/12} + 2\theta e_u + \Delta^{-1/10} \\
\leq f_u(1 - \theta(1 - 7\epsilon)) + 2\theta e_u + \Delta^{-1/21},
\]
which is [15].

1.2.1 Proof of (2)

We prove that
\[
E[p_u'(c)p_v'(c) | \gamma_w(c) = 1] \leq p_u(c)p_v(c).
\] (2)

We assume first that both $p_u'(c)$ and $p_v'(c)$ are determined by Case A. If $c \in L(u)$ or $c \in L(v)$, then $p_u'(c)p_v'(c) = 0$, so
\[
E[p_u'(c)p_v'(c) | \gamma_w(c) = 1] \leq \frac{p_u(c)p_v(c)}{q_u(c)q_v(c)} \Pr[c \notin L(u) \cup L(v) | \gamma_w(c) = 1].
\]

Since
\[
\Pr[c \notin L(u)] = \prod_{uxy \in H} (1 - \Pr[\gamma_x(c) = 1, \gamma_y(c) = 1]) \prod_{\kappa(ux) = c} (1 - \Pr[\gamma_x(c) = 1]) = q_u(c),
\]
we see that
\[
\Pr[c \notin L(u)|c \notin L(v), \gamma_w(c) = 1] \\
= (1 - \Pr[\gamma_v(c) = 1]) \prod_{uxy \in H - uvw} (1 - \Pr[\gamma_x(c) = 1, \gamma_y(c) = 1]) \prod_{\kappa(ux) = c} (1 - \Pr[\gamma_x(c) = 1]) \\
= \frac{1 - \Pr[\gamma_v(c) = 1]}{1 - \Pr[\gamma_v(c) = 1, \gamma_w(c) = 1]}q_u(c) \\
= \frac{1 - \theta p_u(c)}{1 - \theta^2 p_u(c)p_v(c)}q_u(c).
\]

Similarly,
\[
\Pr[c \notin L(v)|\gamma_w(c) = 1] = \frac{1 - \theta p_u(c)}{1 - \theta^2 p_u(c)p_v(c)}q_u(c).
\]
Therefore, using $\theta p_w(c) \leq 1$,

$$\begin{align*}
P[c \notin L(u) \cup L(v) | \gamma_w(c) = 1] &= P[c \notin L(u) | c \notin L(v), \gamma_w(c) = 1]P[c \notin L(v) | \gamma_w(c) = 1] \\
&= \frac{q_u(c)(1 - \theta p_v(c)) q_v(c)(1 - \theta p_u(c))}{1 - \theta^2 p_v(c)p_w(c)} \frac{1}{1 - \theta^2 p_u(c)p_w(c)} \\
&\leq q_u(c)q_v(c),
\end{align*}$$

and (2) follows.

If $p'_u(c)$ or $p'_v(c)$ is determined by Case B, then these values are independent, and (2) follows in a similar way.

## 2 Correction to the proof of property [8]

There was a trivial error in the calculation justifying [8]. We correct here for completeness. Replace the last sentence with: So, using $f_u \leq 3(1 - \theta/4)^t \omega$,

$$\begin{align*}
f'_u &\leq 3(1 - \theta(1 - 7\epsilon))(1 - \theta/4)^t \omega + 2\theta \omega (1 - \theta/3)^t + \theta \Delta^{-1/22} \\
&= 3(1 - \theta/4)^{t+1}\omega + \omega(1 - \theta/4)^t \left(-\theta (9/4 - 21\epsilon) + 2\theta \left(\frac{1 - \theta/3}{1 - \theta/4}\right)^t\right) + \theta \Delta^{-1/22} \\
&\leq 3(1 - \theta/4)^{t+1}\omega + \omega(1 - \theta/4)^t (-\theta (9/4 - 21\epsilon) + 2\theta) + \theta \Delta^{-1/22} \\
&\leq 3(1 - \theta/4)^{t+1}\omega - \omega \theta (1/4 - 21\epsilon)(\log \Delta)^{-O(1)} + \theta \Delta^{-1/22} \\
&\leq 3(1 - \theta/4)^{t+1}\omega.
\end{align*}$$

## References

[1] A.M. Frieze and D. Mubayi, On the chromatic number of simple triangle-free triple systems, Electronic Journal of Combinatorics 15 (2008), no. 1, Research Paper 121, 27 pp.