Open problems (for AGNES)

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Below are a few basic questions and speculations related to the moduli spaces of curves, K3 surfaces, maps, and sheaves presented in the problem session of the AGNES conference in Amherst (April 2010).

(i) On the virtual class:

Let \( X \) be a nonsingular, projective variety over \( \mathbb{C} \). Let \( \overline{M}_g(X, \beta) \) be the moduli space of stable maps and let

\[
\pi: \overline{M}_g(X, \beta) \to \overline{M}_g
\]

be the forgetful morphism, see [5] for background. The moduli space of stable maps carries a virtual class \( \overline{M}_g(X, \beta)^{\text{vir}} \) obtained from deformation theory [1, 2, 13]. Tautological classes in the Chow and cohomology rings of \( \overline{M}_g \) are defined efficiently in [7]. For a discussion of the properties (many conjectural) of tautological classes, see also [6, 21].

**Q1.** Does \( \pi_*[\overline{M}_g(X, \beta)]^{\text{vir}} \in H^*(\overline{M}_g) \) lie in the tautological ring in cohomology?

**Q2.** When does \( \pi_*[\overline{M}_g(X, \beta)]^{\text{vir}} \in A^*(\overline{M}_g) \) lie in the tautological ring in Chow?

I would guess the answer to **Q1** is yes. If \( X \) is a curve, an affirmative answer to **Q1** follows from the results of [7]. We know \( \pi_*[\overline{M}_g(X, \beta)]^{\text{vir}} \) does not
always lie in the tautological ring in Chow — counterexamples can be found when $X$ is a curve. A wild speculation, motivated by the Bloch-Beilinson conjecture, is that the answer to Q2 is yes when $X$ is defined over over $\mathbb{Q}$.

(ii) **On the Virasoro constraints:**

The spaces $\overline{M}_g(X, \beta)$ determine the Gromov–Witten invariants of $X$. These are conjectured to satisfy the Virasoro constraints [4]. Virasoro constraints are known to hold now in many, but not all, cases [8, 14]. A very interesting variety for which the Virasoro constraints are unknown is the Enriques surface.

**Q3.** Prove the Virasoro constraints in case $X$ is an Enriques surface.

A study of the Gromov-Witten theory of the Enriques surface, closely related to modular forms, has been started in [18]. The Enriques surfaces is perhaps the most basic variety where new techniques are required to establish the Virasoro constraints.

(iii) **On the moduli of sheaves:**

Let $X$ be a nonsingular, projective 3-fold. The Gromov-Witten theory of $X$, defined via $\overline{M}_g(X, \beta)$, is conjecturally [15] equivalent to the Donaldson-Thomas theory of $X$. The latter is defined via the moduli of ideal sheaves of curves in $X$ [3, 25], or more recently, in terms of the moduli spaces of stable pairs [22].

**Q4.** Prove the GW/DT correspondence for 3-folds.

The toric cases of Q4 are known [16]. Algebraic cobordism results [12] suggest the possibility of reducing to the toric case using degeneration methods.

Donaldson–Thomas invariants are defined only in dimension 3 because a virtual fundamental class for the moduli space of sheaves is required. Deformations are given by $\text{Ext}^1(E, E)$, obstructions by $\text{Ext}^2(E, E)$, and to
define the virtual fundamental class we need (roughly) the vanishing

$$\text{Ext}^i(E, E) = 0 \text{ for } i > 2.$$ 

On 3-folds, the vanishing can often be obtained using Serre duality and stability. However, there are parallel examples of enumerative computations in higher dimensions in Gromov-Witten theory [11, 23]. Moreover, many aspects of Joyce’s counting theory are valid in higher dimensions [9].

Q5. Define Donaldson–Thomas invariants in dimensions > 3.

(iv) On the moduli of K3 surfaces:

Let $M_{2n}^{K3}$ denote the moduli space of polarized $K3$ surfaces $(S, L)$ of degree $L^2 = 2n$. Little appears to be known about the cycle theory of $M_{2n}^{K3}$.

Q6. What is the analogue of the tautological ring for $M_{2n}^{K3}$?

A natural guess for Q6 is the subring generated by the classes of the Noether–Lefschetz loci. The Noether-Lefschetz loci parameterize $K3$ surfaces with higher rank Picard lattices.

Q7. Do the Noether-Lefschetz divisors span Pic($M_{2n}^{K3}$) $\otimes \mathbb{Q}$?

Let $X$ be a compact Calabi-Yau 3-fold expressed as $K3$-fibration over $\mathbb{P}^1$,

$$\pi : X \rightarrow \mathbb{P}^1.$$ 

Given an ample line bundle $L$ on $X$, the family $\pi$ determines a morphism of the base $\mathbb{P}^1$ to the moduli of polarized $K3$ surfaces. Via [19], the Gromov-Witten theory of $X$ in $\pi$-fiber classes is calculated in terms of the Noether–Lefschetz numbers of $\pi$ and the Katz-Klemm-Vafa [10] conjecture concerning $\lambda_g$ integrals in the reduced Gromov-Witten theory of a fixed $K3$ surface. The KKV conjecture is proven for all classes in genus 0 in [24] and all genera in primitive classes in [20].
Q8. Prove the Katz-Klemm-Vafa conjecture for in all genera and in all classes on $K3$ surfaces.

A solution to Q8 would provide a large class of exact formulas for higher genus Gromov-Witten invariants of compact Calabi-Yau 3-folds. Unlike the local toric cases, mathematical results for higher genus Gromov-Witten invariants have been difficult to obtain, see [26] for the genus 1 theory of the quintic 3-fold.

Q9. Find effective mathematical methods for calculating the higher genus Gromov-Witten invariants of compact Calabi-Yau 3-folds.

Effective methods for the Enriques Calabi-Yau in genus $g \leq 2$ have been found in [18]. Complete, but less effective, techniques for the quintic are explained in [17]. At present, the holomorphic anomaly equation in topological string theory is more effective than the higher genus mathematical methods.

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