MEAN LIPSCHITZ CONDITIONS ON BERGMAN SPACE

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Abstract. For $f$ analytic on the unit disc let $r_t(f)(z) = f(e^{i t}z)$ and $f_r(z) = f(rz)$, rotations and dilations respectively. We show that for $f$ in the Bergman space $A^p$ and $0 < \alpha \leq 1$ the following are equivalent.

(i) $\|r_t(f) - f\|_{A^p} = O(|t|^\alpha)$, $t \to 0$,
(ii) $\|(f')_r\|_{A^p} = O(1 - r)^{\alpha - 1}$, $r \to 1^-$,
(iii) $\|f_r - f\|_{A^p} = O((1 - r)^\alpha)$, $r \to 1^-$.

The Hardy space analogues of these conditions are known to be equivalent by results of Hardy and Littlewood and of E. Storozhenko, and in that setting they describe the mean Lipschitz spaces $\Lambda(p, \alpha)$.

On the way, we provide an elementary proof of the equivalence of (ii) and (iii) in Hardy spaces, and show that similar assertions are valid for certain weighted mean Lipschitz spaces.

1. Introduction

Let $\mathbb{D}$ denote the unit disc in the complex plane $\mathbb{C}$. For $1 \leq p \leq \infty$ and $f : \mathbb{D} \to \mathbb{C}$ analytic the integral mean $M_p(r, f)$, $0 \leq r < 1$, is defined

$$M_p(r, f) = \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(re^{i \theta})|^p \, d\theta \right)^{1/p},$$

and

$$M_\infty(r, f) = \max_{-\pi \leq \theta < \pi} |f(re^{i \theta})|.$$

$M_p(r, f)$ is an increasing function of $r$. The Hardy space $H^p$ consists of all $f \in H(\mathbb{D})$ for which

$$\|f\|_p = \sup_{r < 1} M_p(r, f) = \lim_{r \to 1^-} M_p(r, f) < \infty.$$

Each $f$ in $H^p$ has radial limits

$$f^*(\theta) = \lim_{r \to 1^-} f(re^{i \theta})$$

almost everywhere on $\theta \in [-\pi, \pi]$. The so defined boundary function $f^*$ is $p$-integrable (essentially bounded if $p = \infty$) and $\|f\|_p$ can be recovered from $f^*$ as $\|f\|_p = \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} |f^*(\theta)|^p \, d\theta \right)^{1/p}$.

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For \(1 \leq p \leq \infty\), \(H^p\) are Banach spaces. The linear map \(f \rightarrow f^*\) identifies \(H^p\) as the closed subspace of \(L^p(-\pi, \pi)\) generated by the set of exponentials \(\{e^{int}\}_{n=0}^{\infty}\) or equivalently the subspace consisting of all functions of \(L^p(-\pi, \pi)\) whose Fourier series is “of power series type”. Additional information for Hardy spaces can be found in [Du], and we follow this reference for notation and related material.

For \(f\) analytic on \(\mathbb{D}\) we write \(r_t(f)(z) = f(e^{it}z), \ t \in \mathbb{R}\), for the rotated function. If \(f\) has boundary values \(f^*\) a.e. on \([-\pi, \pi]\) we view \(f^*\) extended periodically and we write \(\tau_t(f^*)(\theta) = f^*(\theta + t), \ t \in \mathbb{R}\) for the translated \(f^*\). It is clear that in this case \(r_t(f) = \tau_t(f^*)\).

For \(f \in H^p, \ p < \infty\), by the continuity of the integral we have \(\lim_{t \to 0} \|\tau_t(f^*) - f^*\|_p = 0\). Specifying the rate of this convergence imposes restriction on \(f\).

**Definition 1.1.** For \(1 \leq p < \infty\) and \(0 < \alpha \leq 1\) the analytic mean Lipschitz space \(\Lambda(p, \alpha)\) is the collection of \(f \in H^p\) such that
\[
\|\tau_t(f^*) - f^*\|_p \leq C|t|^{\alpha}, \ -\pi \leq t < \pi,
\]
where \(C\) is a constant. The subspace \(\lambda(p, \alpha)\) consists of all \(f \in H^p\) which satisfy
\[
\|\tau_t(f^*) - f^*\|_p = o(|t|^\alpha), \ t \to 0.
\]

Note that these spaces can be defined more generally, as subspaces of \(L^p(-\pi, \pi)\), to consist of all \(L^p\) functions \(f\) that satisfy (1.1) and (1.2) (with \(f\) in place of \(f^*\)). It was in this general setting that they were first studied in the 1920’s by Hardy, Littlewood and others, in connection with convergence and summability of Fourier series, fractional integrals and fractional derivatives, see [HL1], [HL2], [HL3]. Among several other results Hardy and Littlewood proved the following theorem.

**Theorem A.** Suppose \(1 \leq p < \infty, \ 0 < \alpha \leq 1\) and \(f \in H^p\). Then the following are equivalent

(a) \(f \in \Lambda(p, \alpha)\),
(b) \(M_p(r, f') = O((1-r)^{\alpha-1}), \ r \to 1^−\).

Note that if an analytic function satisfies (b) for some \(\alpha > 0\), then it belongs in \(H^p\).

For \(f\) analytic on \(\mathbb{D}\) let
\[
f_r(z) = f(rz), \ 0 \leq r < 1,
\]
be the dilations of \( f \). Each \( f_r \) is analytic on a disc of radius \( 1/r > 1 \) and if \( f \in H^p \) then
\[
\| f_r - f \|_p \to 0, \quad \text{as } r \to 1^-,
\]
([Du, Theorem 2.6]). In classical terminology this states that the Fourier series of \( f^* \) (the power series of \( f \) on the boundary) is Abel summable in the \( L^p \) norm.

It seems to be not so well known that for \( f \in H^p \), membership of \( f \) in \( \Lambda(p, \alpha) \) is equivalent to a condition on the rate at which \( \| f_r - f \|_p \to 0 \).

**Theorem B ([ES]).** Let \( 1 \leq p < \infty \), \( 0 < \alpha \leq 1 \) and \( f \in H^p \). Then the following are equivalent
\[
\begin{align*}
(a) & \quad f \in \Lambda(p, \alpha), \\
(c) & \quad \| f_r - f \|_p = \text{O}((1-r)^\alpha), \quad r \to 1^-.
\end{align*}
\]

This theorem may be deduced from the work [ES] of E. A. Storozhenko. In that article the author studies the classical \( L^p \) modulus of continuity of order 1 and of higher orders for Hardy functions, and introduces a modified modulus of continuity which at order 1 coincides with the classical one. To indicate how Theorem B follows we briefly summarize the relevant information. Recall that the \( L^p \) modulus of continuity \( \omega_p(\delta, f) \) (of order 1) of a function \( f \in L^p[-\pi, \pi] \) extended periodically, is
\[
\omega_p(\delta, f) = \sup_{|t| \leq \delta} \| \tau_t(f) - f \|_p.
\]

It is easy to see that \( \Lambda(p, \alpha) \) can equivalently be defined as the space of all \( f \in H^p \) for which \( \omega_p(\delta, f^*) \leq C \delta^\alpha \) for all small positive \( \delta \). From [ES, Theorem 1 and Theorem 6] it follows that if \( f \in H^p \) then there are constants \( C_1(p), C_2(p) \) such that
\[
C_1 \omega_p(1-r, f^*) \leq \| f_r - f \|_p \leq C_2 \omega_p(1-r, f^*), \quad r \to 1^-,
\]
and from this Theorem B follows. Storozhenko attributes special cases of the result to previous work of A. I. Buadze and of R. M. Trigub, see [ES] for details. Other special cases, explicit or implicit, can be found in [Wa], [HS], [JP], [Pa3].

Note that \( M_p(r, f) = \| f_r \|_p \), so condition (b) of Theorem A can be restated as
\[
\| (f')_r \|_p = \text{O} ((1-r)^{\alpha-1}),
\]
and the theorem of Hardy and Littlewood may be interpreted as saying that smoothness of the boundary function \( f^* \) can be detected from inside the disc in terms of the dilations of \( f' \). Theorem B says that in effect that dilations of \( f \) itself can be used for detecting smoothness of \( f^* \). To stress this point of view but also to have handy the statement we prove below, we formulate
Theorem C. Let $1 \leq p < \infty$, $0 < \alpha \leq 1$ and $f \in H^p$. Then the following are equivalent:

(b) $M_p(r, f') = O((1 - r)^{\alpha - 1})$, \quad $r \to 1^-$

(c) $\|f_r - f\|_p = O((1 - r)^{\alpha})$, \quad $r \to 1^-$. 

The proofs of Theorem B or of its special cases in the above mentioned articles are technically complicated and use rather advanced techniques. We are going to present an elementary proof of Theorem C below. We then study the analogous equivalence of conditions (a), (b) and (c) on weighted mean Lipschitz spaces, on Bergman spaces and on the Dirichlet space.

2. An elementary proof of Theorem C

In what follows the letters $C, C', \ldots$ will denote constants, whose value may change at each step.

To prove that $(b) \Rightarrow (c)$ we will use the identity,

\[
(2.1) \quad f(z) - f_r(z) = \int_r^1 z f'(sz) \, ds,
\]

which is valid for all $f$ analytic on $D$, all $z \in D$ and $r < 1$.

We take $p$-integral means on the circle of radius $u \in (0, 1)$ to obtain,

\[
M_p(u, f_r - f) = \left( \frac{1}{2\pi} \int_0^{2\pi} \left| \int_r^1 u e^{i\theta} f'(sue^{i\theta}) \, ds \right|^p \, d\theta \right)^{1/p} 
\leq \int_r^1 \left( \frac{1}{2\pi} \int_0^{2\pi} |ue^{i\theta} f'(sue^{i\theta})|^p \, d\theta \right)^{1/p} \, ds 
= u \int_r^1 M_p(su, f') \, ds.
\]

If in addition $f \in H^p$ take the supremum on $u < 1$,

\[
(2.2) \quad \|f_r - f\|_p \leq \int_r^1 M_p(s, f') \, ds,
\]

and if $f$ satisfies (b) then,

\[
\|f_r - f\|_p \leq C \int_r^1 (1 - s)^{\alpha - 1} \, ds = \frac{C}{\alpha} (1 - r)^{\alpha},
\]

as desired.

For the converse suppose, $f$ is analytic on $D$, $0 < r < 1$ and $z \in D$, then

\[
z \int_r^1 (f'(sz) - f'(z)) \, ds = \int_r^1 \frac{d}{ds} (f(sz)) \, ds - zf'(z)(1 - r) 
= f(z) - f(rz) - zf'(z)(1 - r),
\]
therefore

\[(2.3) \quad (1 - r)f'(z) = \frac{f(z) - f_r(z)}{z} + \int_r^1 (f'(z) - f'(sz)) \, ds,\]

and note that \(\frac{\frac{f(z)}{z} - f_r(z)}{z}\) has removable singularity at 0.

Taking integral means of both sides on the circle \(|z| = u\) and using Minkowski’s inequality we have

\[(1 - r)M_p(u, f') \leq \left( \frac{1}{2\pi} \int_0^{2\pi} \left\{ \frac{f(ue^{i\theta}) - f_r(ue^{i\theta})}{u^p} \right\}^p d\theta \right)^{1/p} + \left( \frac{1}{2\pi} \int_0^{2\pi} \left\{ \int_r^1 (f'(ue^{i\theta}) - f'(sue^{i\theta})) \, ds \right\}^p d\theta \right)^{1/p} \leq \frac{1}{u} M_p(u, f_r - f) + \int_r^1 \left( \frac{1}{2\pi} \int_0^{2\pi} \left\{ f'(ue^{i\theta}) - f'(sue^{i\theta}) \right\}^p d\theta \right)^{1/p} \, ds.

Thus

\[(2.4) \quad (1 - r)M_p(u, f') \leq \frac{1}{u} M_p(u, f_r - f) + \int_r^1 M_p(u, \Phi'_s) \, ds\]

where \(\Phi'_s(z) = f(z) - \frac{1}{s} f_s(z)\) for \(r \leq s < 1\).

We estimate the two terms in the right hand side of (2.4). The quantity \(\frac{1}{u} M_p(u, f_r - f)\) is increasing in \(u\) since it is the integral mean of an analytic function. Assuming that \(f \in H^p\) we have for each \(0 < u < 1\),

\[(2.5) \quad \frac{1}{u} M_p(u, f_r - f) \leq \sup_{u < 1} \frac{1}{u} M_p(u, f_r - f) = \|f_r - f\|_p.

Next if \(f \in H^p\) then \(\Phi'_s \in H^p\) and

\[\|\Phi'_s\|_p = \|f - \frac{1}{s} f_s\|_p \leq \|f - f_s\| + \|f_s - \frac{1}{s} f_s\|_p = \|f_s - f\|_p + \frac{1 - s}{s} \|f_s\|_p \leq \frac{1}{r} (1 - s) \|f\|_p + \|f_s - f\|_p\]

for \(r \leq s < 1\). Further we use the well known estimate

\[M_p(u, F') \leq C \|F\|_p \frac{1}{1 - u}, \quad 0 \leq u < 1,\]

for functions \(F \in H^p\), with \(C\) independent of \(F\). For a proof of this one can use Cauchy’s integral formula for \(F'(z)\) as in the proof of the first part of [Du Theorem 5.5].
Applying this inequality to $\Phi[s]$ together with the inequality for $\|\Phi[s]\|_p$ we have

$$(1 - u)M_p(u, \Phi'[s]) \leq \|\Phi[s]\|_p \leq \frac{C}{r}(1 - s)\|f\|_p + C\|f_s - f\|_p,$$

valid for $0 \leq u < 1$ and $r \leq s < 1$. In particular letting $u = r$ and integrating we obtain

$$\int_r^1 M_p(r, \Phi'[s]) ds \leq \frac{C\|f\|_p}{2r}(1 - r) + \frac{C}{1 - r} \int_r^1 \|f_s - f\|_p ds.$$

Using this and letting $u = r$ in (2.4), together with (2.5) we find (2.6)

$$(1 - r)M_p(r, f') \leq \|f_r - f\|_p + \frac{C\|f\|_p^2}{2r}(1 - r) + \frac{C}{1 - r} \int_r^1 \|f_s - f\|_p ds.$$

Thus if $f$ satisfies $\|f_r - f\|_p \leq C'(1 - r)^\alpha$ then we have

$$(1 - r)M_p(r, f') \leq C'(1 - r)^\alpha + \frac{C\|f\|_p}{2r}(1 - r) + \frac{CC'}{\alpha + 1}(1 - r)^\alpha$$

therefore

$$M_p(r, f') = O((1 - r)^{\alpha - 1}), \quad r \to 1^-,$$

valid for all $0 < \alpha \leq 1$. This finishes the proof of Theorem C.

It is well known that the “little oh” analogue of the theorem of Hardy and Littlewood is valid for membership in the spaces $\lambda(p, \alpha)$. That is, $f \in \lambda(p, \alpha)$ if and only if $M_p(r, f') = o((1 - r)^{\alpha - 1})$, as $r \to 1^-$. As expected the “little oh” analogue of Theorem C also holds. The proof follows easily from the inequalities (2.2) and (2.6). We omit the details.

**Corollary 2.1.** Let $1 \leq p < \infty$, $0 < \alpha \leq 1$ and $f \in H^p$. Then the following are equivalent

1. $M_p(r, f') = o((1 - r)^{\alpha - 1})$, $r \to 1^-$,
2. $\|f_r - f\|_p = o((1 - r)^\alpha)$, $r \to 1^-$. 

**3. Weighted mean Lipschitz Spaces**

We show that the analogue of Theorem C remains valid in weighted Lipschitz spaces which are defined as follows.

Let $\omega : [0, 1) \to [0, \infty)$ be a continuous and nondecreasing function with $\omega(0) = 0$. The weighted mean Lipschitz space $\Lambda(p, \omega)$ consists of all $f \in H^p$ such that

$$(3.1) \quad \|\tau_t(f^*) - f^*\|_p = O(\omega(t)), \quad t \to 0.$$
If $\omega(t) = t^{\alpha}$ then we recover $\Lambda(p, \alpha)$. For general weights $\omega$ these spaces have been studied by various authors, see for example [BS], [Gi], [GiG] and the references therein.

An extension of the theorem of Hardy and Littlewood was proved for these generalized Lipschitz spaces for an appropriate class of weights $\omega$. A weight $\omega$ is called a Dini weight if $\omega(t)/t$ is integrable on $[0, 1)$ and there is a constant $C$ such that the following condition is satisfied

\begin{equation}
\int_0^t \frac{\omega(s)}{s} \, ds \leq C\omega(t), \quad 0 < t < 1.
\end{equation}

A weight $\omega$ is an admissible weight if it is a Dini weight and satisfies in addition the condition

\begin{equation}
\int_1^t \frac{\omega(s)}{s^2} \, ds \leq C\frac{\omega(t)}{t}, \quad 0 < t < 1,
\end{equation}

for a constant $C$. Note that if $\omega$ satisfies this last condition then there is a constant $C > 0$ such that

\begin{equation}
\frac{\omega(t)}{t} \geq C, \quad 0 < t < 1.
\end{equation}

For admissible weights O. Blasco and G. S. de Souza proved in [BS, Theorem 2.1] the analogue of the theorem of Hardy and Littlewood.

**Theorem D** ([BS]) Suppose $1 \leq p < \infty$, $\omega$ is an admissible weight and $f$ is analytic on $D$. Then the following are equivalent

(a) $f \in \Lambda(p, \omega)$,
(b) $M_p(r, f') = O\left(\frac{\omega(1-r)}{1-r}\right)$, \quad $r \to 1^-$.

We show that the analogue of Theorem C is valid in $\Lambda(p, \omega)$.

**Theorem 3.1.** Suppose $1 \leq p < \infty$, $\omega$ is a Dini weight that satisfies (3.4) and $f \in H^p$. Then the following are equivalent

(b) $M_p(r, f') = O\left(\frac{\omega(1-r)}{1-r}\right)$, \quad $r \to 1^-$,
(c) $\|f_r - f\|_p = O(\omega(1-r))$, \quad $r \to 1^-$.

In particular the conditions (a), (b) and (c) as above are equivalent for admissible weights.

**Proof.** Suppose (b) holds for $f \in H^p$. Using (2.2) we have

\[
\|f_r - f\|_p \leq C \int_r^1 \frac{\omega(1-s)}{1-s} \, ds
\]

\[
= C \int_0^{1-r} \frac{\omega(u)}{u} \, du
\]

\[
\leq CC'(\omega(1-r))
\]
as desired, where in the last step we have used the Dini property of \( \omega \).

Suppose now (c) holds for \( f \in H^p \). Then the last term in the right hand side of (2.6) becomes

\[
\frac{C'}{1-r} \int_r^1 \frac{\omega(1-s)}{1-s} ds 
\leq C' \int_0^{1-r} \frac{\omega(u)}{u} du 
\leq C'' \omega(1-r).
\]

From this and (2.6) we have

\[
(1-r)M_p(r, f) \leq C \omega(1-r) + C'(1-r) + C'' \omega(1-r)
\]

so that

\[
M_p(r, f) \leq \frac{C \omega(1-r)}{1-r} + C', \quad \text{as } r \to 1^-.
\]

Taking into account (3.4) we find

\[
M_p(r, f) \leq C' \omega(1-r) \frac{1-r}{1-r} \text{ as } r \to 1^- \text{ and this finishes the proof.}
\]

4. Bergman spaces and the Dirichlet space.

**Bergman spaces.** Let \( dm(z) = \frac{1}{\pi} r^2 d\theta dr \) be the normalized Lebesgue area measure on \( \mathbb{D} \). Recall that for \( 1 \leq p < \infty \) the Bergman space \( A^p \) consists of the analytic functions \( f : \mathbb{D} \to \mathbb{C} \) such that

\[
\|f\|_{A^p}^p = \int_{\mathbb{D}} |f(z)|^p dm(z) = 2 \int_0^1 M_p^p(u, f) u du < \infty.
\]

For \( 1 \leq p < \infty \) \( A^p \) are Banach spaces and \( H^p \subset A^p \), but in contrast to Hardy spaces \( A^p \) contains functions which do not have boundary radial limits. Observe however that if \( g \) belongs to a Hardy space \( H^p \) with boundary function \( g^* \) then

\[
\|r_t(g^*) - g^*\|_p = \|r_t(g) - g\|_p.
\]

We may therefore use the quantity \( \|r_t(f) - f\|_{A^p} \) as a substitute for measuring "smoothness" of functions in \( A^p \).

Let \( f \) be analytic on \( \mathbb{D} \), \( 1 \leq p < \infty \) and \( 0 \leq r < 1 \). Then the dilations \( f_r(z) = f(rz) \) belong to \( A^p \), and we define the quantity

\[
A_p(r, f) := \|f_r\|_{A^p} = \left( \int_{\mathbb{D}} |f(rz)|^p dm(z) \right)^{1/p}.
\]
Note that $A_p(r, f)$ is an area integral mean, in much the same way as $M_p(r, f) = \|f_r\|_{H^p}$ is arc-length integral mean

\[ A_p(r, f) = \left( \frac{2}{r^2} \int_0^r M_p(u, f) u \, du \right)^{1/p} = \left( \frac{1}{m(rD)} \int_{rD} |f(z)|^p \, dm(z) \right)^{1/p}. \]

Further, since $M_p(r, f)$ is increasing in $r$ it follows that $A_p(r, f)$ is also an increasing function of $r$, and if $f \in A^p$ we have

\[ A_p(r, f) = \|f_r\|_{A^p} \to \|f\|_{A^p}, \quad r \to 1^-, \]

while if $f$ is not in $A^p$ then $A_p(r, f)$ increases to infinity. An application of the Lebesgue dominated convergence theorem shows that for $f \in A^p$,

\[ \|f_r - f\|_{A^p} \to 0, \quad r \to 1^- . \]

Such area integral means were studied in the more general context of volume integral means by J. Xiao and K. Zhu in [XZ], where the authors prove basic properties of such averages for weighted volume measures $dv_\alpha(z) = (1 - |z|^2)^\alpha dv(z)$ on the unit ball of $\mathbb{C}^n$.

We show now that functions $f \in A^p$ for which $\|r_t(f) - f\|_{A^p} = O(|t|^\alpha)$ as $t \to 0$ can be characterized by the growth of $A_p(r, f')$ as well as by the limiting behavior of $\|f_r - f\|_{A^p}$.

**Theorem 4.1.** Let $1 \leq p < \infty$, $0 < \alpha \leq 1$ and $f \in A^p$. Then the following are equivalent

(a) $\|r_t(f) - f\|_{A^p} = O(|t|^\alpha)$, $t \to 0$,
(b) $A_p(r, f') = O((1 - r)^{\alpha - 1})$, $r \to 1^-$,
(c) $\|f_r - f\|_{A^p} = O((1 - r)^\alpha)$, $r \to 1^-$. 

**Proof.** We first show that (a) and (b) are equivalent. To show (b) implies (a) we follow the argument in [Du2, Theorem 5.4] adapted for area integrals. Note that if (b) holds for an analytic function $f$ then $f \in A^p$. Indeed assuming without loss of generality that $f(0) = 0$ we have

\[ f(z) = \int_0^z f'(\zeta) \, d\zeta = \int_0^1 f'(tz) \, z \, dt, \]
and an application of Minkowski’s inequality gives
\[
\left( \int_{D} |f(z)|^p \, dm(z) \right)^{1/p} = \left( \int_{D} \int_{0}^{1} \left| f'(tz) \right|^p \, dt \, dm(z) \right)^{1/p}
\]
\[
\leq \int_{0}^{1} \left( \int_{D} \left| f'(tz) \right|^p \, dm(z) \right)^{1/p} \, dt
\]
\[
= \int_{0}^{1} A_p(t, f') \, dt
\]
\[
\leq C \int_{0}^{1} (1 - t)^{\alpha - 1} \, dt
\]
\[
= C/\alpha.
\]

Suppose \( f \) is analytic on \( D \) and satisfies (b). Let \( z \in D, 0 < \delta < 1 \) and \( t > 0 \), then
\[
f(e^{it}z) - f(z) = \int_{z}^{\delta z} f'(\zeta) \, d\zeta + \int_{\delta z}^{e^{it}z} f'(\zeta) \, d\zeta + \int_{e^{it}z}^{t} f'(\zeta) \, d\zeta
\]
\[
= -\int_{\delta}^{1} f'(sz) \, ds + \int_{0}^{t} f'(\delta e^{is}z)\delta ze^{is}i \, ds + \int_{\delta}^{1} f'(se^{it}z)e^{it}z \, ds
\]
\[
= f_1(z) + f_2(z) + f_3(z).
\]
Each function \( f_i(z) \) is analytic on \( D \). We estimate the Bergman norm of each by applying Minkowski’s inequality and by using the fact that compositions of Bergman functions with rotations leave their norm invariant. For \( f_1 \) we have
\[
\|f_1\|_{A^p} = \left( \int_{D} \left| \int_{\delta}^{1} f'(sz) \, ds \right|^p \, dm(z) \right)^{1/p}
\]
\[
\leq \int_{\delta}^{1} \left( \int_{D} \left| f'(sz) \right|^p \, dm(z) \right)^{1/p} \, ds
\]
\[
\leq \int_{\delta}^{1} A_p(s, f') \, ds,
\]
and similarly we find
\[
\|f_3\|_{A^p} \leq \int_{\delta}^{1} A_p(s, f') \, ds,
\]
\[
\|f_2\|_{A^p} \leq \int_{0}^{t} A_p(\delta, f') \, ds = tA_p(\delta, f').
\]
Thus
\[
\|r_t(f) - f\|_{A^p} \leq \|f_1\|_{A^p} + \|f_2\|_{A^p} + \|f_3\|_{A^p} \leq 2 \int_{\delta}^{1} A_p(s, f') \, ds + tA_p(\delta, f'),
\]
and since by hypothesis $A_p(s, f') \leq C(1 - s)^{\alpha - 1}$, we find

$$
\|r_t(f) - f\|_A^p \leq \frac{2C}{\alpha}(1 - \delta)^\alpha + Ct(1 - \delta)^{\alpha - 1}.
$$

This holds with $\delta$ and $t$ independent of each other. Thus given $t$ with $0 < t < 1$ choose $\delta = 1 - t$. The result is

$$
\|r_t(f) - f\|_A^p \leq C\left(\frac{2}{\alpha} + 1\right)t^\alpha,
$$

with the constant independent of $t > 0$.

Finally if $t < 0$ then $\|r_t(f) - f\|_A^p = \|f - r_{-t}(f)\|_A^p$ with $-t > 0$ and the assertion follows. This completes the proof of the direction (b) $\Rightarrow$ (a).

We now show that (a) implies (b). Suppose $f \in A^p$ satisfies (a). Let $0 < u < 1$ and use the Cauchy formula to write $f'(uz)$ as line integral over the contour $\gamma(t) = ze^{it}, -\pi \leq t \leq \pi$,

$$
f'(uz) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta) - f(z)}{(\zeta - uz)^2} d\zeta = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{f(ze^{it}) - f(z)}{(ze^{it} - uz)^2} ze^{it} dt
$$

$$
= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{f(ze^{it}) - f(z)}{ze^{it} - u} \frac{e^{it}}{(e^{it} - u)^2} dt.
$$

From Minkowski’s inequality we have

$$
A_p(u, f') = \left(\int_{\mathbb{D}} |f'(uz)|^p \ dm(z)\right)^{1/p}
$$

$$
= \left(\int_{\mathbb{D}} \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{f(ze^{it}) - f(z)}{z} \frac{e^{it}}{(e^{it} - u)^2} dt \right|^p \ dm(z)\right)^{1/p}
$$

$$
\leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\int_{\mathbb{D}} \left| \frac{f(ze^{it}) - f(z)}{z} \right|^p \ dm(z)\right)^{1/p} \frac{1}{|e^{it} - u|^2} dt
$$

$$
= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left\| \frac{f(ze^{it}) - f(z)}{z} \right\|_{A^p} \frac{1}{|e^{it} - u|^2} dt.
$$

At this point we need the fact that if $g \in A^p$ and $g(0) = 0$ then there is a constant $C = C_p$ such that

$$
\left\| \frac{g(z)}{z} \right\|_{A^p} \leq C_p \|g\|_{A^p},
$$
see [Zhu, Lemma 4.26]. Using this and the hypothesis (a) we have

\[
A_p(u, f') \leq \frac{C_p}{2\pi} \int_{-\pi}^{\pi} \|f(ze^{it}) - f(z)\|_{Ap} \frac{1}{|e^{it} - u|^2} dt
\]

\[
\leq \frac{C_p C}{2\pi} \int_{-\pi}^{\pi} \frac{|t|^\alpha}{|e^{it} - u|^2} dt
\]

\[
= \frac{C_p C}{\pi} \int_{0}^{\pi} \frac{t^\alpha}{1 - 2u \cos(t) + u^2} dt.
\]

Now use the standard inequality

\[
1 - 2u \cos(t) + u^2 = (1 - u)^2 + 4u \sin^2\left(\frac{t}{2}\right) \geq (1 - u)^2 + \frac{4u t^2}{\pi^2}
\]

which is valid for \(0 \leq t \leq \pi\). Making the change of variable \(s = \frac{2\sqrt{u}}{(1-u)\pi} t\) and assuming \(u \geq 1/4\) we obtain

\[
\int_{0}^{\pi} \frac{t^\alpha}{1 - 2u \cos(t) + u^2} dt \leq \int_{0}^{\pi} \frac{t^\alpha}{(1 - u)^2 + \frac{4ut^2}{\pi^2}} dt \leq C_\alpha (1 - u)^{\alpha-1},
\]

where \(C_\alpha = \pi^{\alpha+1} \int_{0}^{\infty} \frac{s^\alpha}{1+s^2} ds\) is a constant, finite for \(0 < \alpha < 1\), which does not depend on \(u \in (1/4, 1)\). Thus if \(0 < \alpha < 1\) then \(A_p(u, f') = O((1 - u)^{\alpha-1})\) as \(u \to 1^-\), the desired conclusion.

It remains to treat the case \(\alpha = 1\). Here we have to prove that if \(f \in Ap\) and \(\|r_t(f) - f\|_{Ap} \leq Ct\) as \(t \to 0\) then \(f' \in Ap\). To do this let \(z \in \mathbb{D}\) and observe that

\[
\lim_{t \to 0} \frac{f(e^{it}z) - f(z)}{t} = z \lim_{t \to 0} \frac{f(e^{it}z) - f(z)}{e^{it}z - z} \frac{e^{it} - 1}{t} = iz f'(z),
\]

i.e. \(\frac{f(e^{it}z) - f(z)}{t}\) converges pointwise to \(iz f'(z)\) on \(\mathbb{D}\) as \(t \to 0\). By the assumption we also have \(\|r_t(f) - f\|_{Ap} \leq C\). Thus by Fatou’s Lemma,

\[
\int_{\mathbb{D}} |zf'(z)|^p dm(z) = \int_{\mathbb{D}} \liminf_{t \to 0} \left| \frac{f(e^{it}z) - f(z)}{t} \right|^p dm(z)
\]

\[
\leq \liminf_{t \to 0} \int_{\mathbb{D}} \left| \frac{f(e^{it}z) - f(z)}{t} \right|^p dm(z)
\]

\[
\leq C^p,
\]

so that \(zf'(z)\) and therefore \(f'(z)\) are in \(Ap\).

Next we show that (b) and (c) are equivalent. Assume \(f\) is analytic on \(\mathbb{D}\) and satisfies (b). Fix \(r \in (0, 1)\) and take integral means on \(|z| = u \in (0, 1)\) in the equation (2.1) to obtain

\[
M_p(u, f_r - f) \leq u \int_r^1 M_p(su, f') ds \leq \int_r^1 M_p(su, f') ds.
\]
Recalling that (b) implies that $f \in A^p$ and we find
\[ \|f_r - f\|_{A^p} = \left(2 \int_0^1 M^p_p(u, f_r - f)u \; du \right)^{1/p} \]
\[ \leq \left(2 \int_0^1 \left(\int_r^1 M_p(su, f') \; ds\right)^p u \; du \right)^{1/p} \]
\[ \leq \int_r^1 \left(2 \int_0^1 M^p_p(su, f')u \; du \right)^{1/p} \; ds \]
\[ = \int_r^1 \left(\frac{2}{s^2} \int_0^s M^p_p(v, f')v \; dv \right)^{1/p} \; ds \]
\[ = \int_r^1 A_p(s, f') \; ds. \]

Since $A_p(s, f') \leq C(1 - s)^{a - 1}$ the integration gives $\|f_r - f\|_{A^p} \leq C'(1 - r)^a$ so (c) holds.

Finally we prove (c) implies (b). Observe first that if $f, g$ are analytic on $\mathbb{D}$ then since $\|(f + g)_r\|_{A^p} \leq \|f_r\|_{A^p} + \|g_r\|_{A^p}$ we will have
\[ A_p(r, f + g) \leq A_p(r, f) + A_p(r, g) \]
for each $0 \leq r < 1$.

Assume $f \in A^p$ and fix $r \in (0, 1)$. Taking area integral means in both sides of (2.3) we find that for each $0 < u < 1$
\[ (1 - r)A_p(u, f') \leq A_p(u, F_{[r]}) + A_p(u, \Psi_{[r,s]}), \]
where $F_{[r]}(z) = \frac{f(z) - f_r(z)}{z}$ and
\[ \Psi_{[r,s]}(z) = \int_r^1 (f'(z) - f'(sz)) \; ds. \]

For the term $A_p(u, F_{[r]})$ in (1.1) we obtain
\[ A_p(u, F_{[r]}) = \|(F_{[r]})_u\|_{A^p} \leq \|F_{[r]}\|_{A^p} \leq C_p \|f - f_r\|_{A^p} \]
for each $0 < u < 1$. Now we look at the the term $A_p(u, \Psi_{[r,s]})$ and use Minkowski’s inequality,
\[ A_p(u, \Psi_{[r,s]}) = \|\Psi_{[r,s]}(uz)\|_{A^p} \]
\[ = \left(\int_0^1 \int_0^{2\pi} \left|\int_r^1 (f'(vue^{i\theta}) - f'(svue^{i\theta})) \; ds\right|^p \; \frac{d\theta}{\pi} \; v \; dv \right)^{1/p} \]
\[ \leq \left(\int_r^1 \left(\int_0^1 2^{1/p} M_p(uv, \Phi_{[s]}') \; ds\right)^p v \; dv \right)^{1/p} \]
where $\Phi_s(z) = f(z) - \frac{1}{s} f_s(z)$ for $r \leq s < 1$, and further

$$\leq \int_r^1 \left( 2 \int_0^1 M_p^p(u v, \Phi_s) v \, dv \right)^{1/p} ds$$

$$= \int_r^1 \left( 2 \int_0^1 M_p^p(v, (\Phi_s')_u) v \, dv \right)^{1/p} ds$$

$$= \int_r^1 A_p(u, \Phi_s') \, ds.$$ 

At this point we need the inequality

(4.3)  \[ A_p(u, F') \leq \frac{C \| F \|_{A_p}}{1 - u}, \quad 0 \leq u < 1, \]

for $F \in A^p$, where $C$ is a constant independent of $F$. A proof of this is as follows. For $F \in A^p$ the Bergman norm of $\| F \|_{A_p}$ is equivalent to the quantity

$$|F(0)|^p + \int_D |F'(z)|^p (1 - |z|^2)^p \, dm(z),$$

see [Zhu, page 85]. Thus there is a constant $C$ such that for $0 < u < 1$,

$$\| F \|_{A_p}^p \geq C \int_D |F'(z)|^p (1 - |z|^2)^p \, dm(z)$$

$$\geq C \int_{uD} |F'(z)|^p (1 - |z|^2)^p \, dm(z)$$

$$\geq C (1 - u^2)^p \int_{uD} |F'(z)|^p \, dm(z).$$

Then for $1/2 < u < 1$,

$$A_p(u, F') = \left( \frac{1}{u^2} \int_{uD} |F'(z)|^p \, dm(z) \right)^{1/p} \leq \frac{C'}{1 - u} \| F \|_{A_p}.$$ 

On the other hand there is a constant $C''$ such that

$$|F'(z)| \leq C'' \| F \|_{A_p},$$

for $F \in A^p$, and $|z| \leq 1/2$, [Zhu, page 99, exer. 24]. Therefore $A_p(u, F') \leq C'' \| F \|_{A_p}$ when $0 \leq u \leq 1/2$. Choosing $C = \max \{ C', C'' \}$ gives (4.3).
Using (4.3) and observing that if \( f \in A^p \) then \( \Phi[s] \in A^p \) we have

\[
\frac{1}{C}(1 - u)A_p(u, \Phi'[s]) \leq \|\Phi[s]\|A^p = \| f - \frac{1}{s}f_s \|A^p \leq \| f - f_s \|A^p + \| f_s - \frac{1}{s}f_s \|A^p \leq \| f_s - f \|A^p + \frac{1}{s}\| f_s \|A^p \leq \frac{1}{r}(1 - s)\| f \|A^p + \| f_s - f \|A^p
\]

for \( r \leq s < 1 \) and \( 0 \leq u < 1 \). The proof can now be finished as in the case of Hardy spaces. Namely choose \( u = r \) and integrate to obtain

\[
\int_r^1 A_p(r, \Phi'[s]) ds \leq \frac{C\| f \|A^p}{2r}(1 - r) + \frac{C'}{1 - r} \int_r^1 \| f_s - f \|A^p ds.
\]

Then use (4.1) with \( u = r \) and (4.2) to find

\[
(1 - r)A_p(r, f') \leq C_p\| f_r - f \|A^p + \frac{C\| f \|A^p}{2r}(1 - r) + \frac{C'}{1 - r} \int_r^1 \| f_s - f \|A^p ds.
\]

Using the assumption \( \| f_r - f \|A^p \leq C'(1 - r)^\alpha \) we have

\[
(1 - r)A_p(r, f') \leq C'_p(1 - r)^\alpha + \frac{C\| f \|A^p}{2r}(1 - r) + \frac{CC'}{\alpha + 1}(1 - r)^\alpha
\]

therefore

\[
A_p(r, f') = O((1 - r)^{\alpha - 1}), \quad r \to 1^-, \quad \text{and this completes the proof.}
\]

The Dirichlet space. We next show that the analogue of theorem (4.1) is valid in the Dirichlet space \( \mathcal{D} \). This is in fact a corollary of theorem (4.1) so we will be brief. Recall that \( \mathcal{D} \) contains those analytic \( f \) such that \( f' \in A^2 \). It is a Hilbert space with the norm

\[
\| f \|_{\mathcal{D}} = (|f(0)|^2 + \| f' \|_{A^2})^{1/2}.
\]

For \( f \in \mathcal{D} \) the dilations \( f_r, r < 1 \), are in \( \mathcal{D} \). We define the quantity

\[
D(r, f) = \| f_r \|_{\mathcal{D}}, \quad 0 < r < 1.
\]

Further for \( w \in \mathbb{D} \) write \( f_w(z) = f(wz) \). Using the triangle inequality we obtain

\[
\| f_w - f \|_{\mathcal{D}} \leq |1 - w|\| f \|_{\mathcal{D}} + \| (f')_w - f' \|_{A^2},
\]

and

\[
\| (f')_w - f' \|_{A^2} \leq \frac{|1 - w|}{|w|}\| f \|_{\mathcal{D}} + \| f_w - f \|_{\mathcal{D}}.
\]
In particular letting $w = e^{it}$ we see that for $0 < \alpha \leq 1$, each of the conditions
$$\|r_t(f) - f\|_D = O(|t|^\alpha)$$
and
$$\|r_t(f') - f'\|_{A^2} = O(|t|^\alpha)$$
implies the other.

Similarly letting $w = r \in (0, 1]$ we find that the two conditions
$$\|f_r - f\|_D = O((1 - r)^\alpha)$$
and
$$\|(f')_r - f'\|_{A^2} = O((1 - r)^\alpha)$$
are equivalent.

Collecting all these observations we obtain as a corollary of Theorem (4.1)
the following analogue on $D$.

**Theorem 4.2.** Let $0 < \alpha \leq 1$ and $f \in D$. Then the following are equivalent

(a) $\|r_t(f) - f\|_D = O(|t|^\alpha)$, $t \to 0$,

(b) $D(r, f') = O((1 - r)^{\alpha - 1})$, $r \to 1^-$,

(c) $\|f_r - f\|_D = O((1 - r)^\alpha)$, $r \to 1^-$.

5. Concluding Remarks

The classical holomorphic Lipschitz spaces $\Lambda_\alpha(D)$, $0 < \alpha \leq 1$, are defined
to contain all analytic functions on $D$ such that
$$|f(z) - f(w)| \leq C|z - w|^\alpha, \quad z, w \in D.$$  

Such functions are continuous on the close disc so $\Lambda_\alpha(D)$ are contained in
the disc algebra $A$. For these spaces it was proved already by Hardy and
Littlewood that $f \in \Lambda_\alpha(D)$ if and only if
$$M_\infty(r, f') = O((1 - r)^{\alpha - 1}), \quad r \to 1^-,$$
[Dn] Theorem 5.1. If we use the identities (2.1) and (2.2) and take the $\infty$-
means $M_\infty(r, f)$ instead of the $p$-means $M_p(r, f)$ we can obtain the analogue
of Theorem C for the disc algebra, i.e. if $0 < \alpha \leq 1$ and $f$ is analytic on $D$
and continuous on $\overline{D}$ then the following conditions are equivalent

(i) $M_\infty(r, f') = O((1 - r)^{\alpha - 1}), \quad r \to 1^-$

(ii) $\|f_r - f\|_\infty = O((1 - r)^\alpha), \quad r \to 1^-$

This result is already known in a much more general setting, see for example
[Pa1] for a detailed description.

Notice that all available relevant results can be included in a single statement
as follows: Let $X$ denote any of the Banach spaces $H^p$, $A^p$, $D$ or $A$,
and $\|\|_X$ be the corresponding norm. Then if $0 < \alpha \leq 1$ and $f \in X$ the
following are equivalent

(a) $\|r_t(f) - f\|_X = O(|t|^\alpha)$, $t \to 0$,

(b) $\|(f')_r\|_X = O((1 - r)^{\alpha - 1}), \quad r \to 1^-$,

(c) $\|f_r - f\|_X = O((1 - r)^\alpha), \quad r \to 1^-.$

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