On the Cauchy Problem for Nondegenerate Parabolic Integro-Differential Equations in the Scale of Generalized Hölder Spaces

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Abstract
Parabolic integro-differential nondegenerate Cauchy problem is considered in the scale of Hölder spaces of functions whose regularity is defined by a radially O-regularly varying Lévy measure. Existence and uniqueness and the estimates of the solution are derived.

Keywords Non-local parabolic integro-differential equations · Lévy processes

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1 Introduction

In this paper we consider for $\alpha \in (0, 2)$ the parabolic Cauchy problem
\[
\partial_t u(t, x) = Lu(t, x) - \lambda u(t, x) + f(t, x) \quad \text{in } H_T = [0, T] \times \mathbb{R}^d, \tag{1.1}
\]
with $\lambda \geq 0$ and integro-differential operator
\[
L\varphi(x) = L^\nu \varphi(x) = \int_{\mathbb{R}_0^d} [\varphi(x + y) - \varphi(x) - \chi_{\alpha} (y) y \cdot \nabla \varphi(x)] \nu(dy), \varphi \in C_0^\infty (\mathbb{R}^d),
\]
where $\nu$ is a nonnegative measure on $\mathbb{R}_0^d = \mathbb{R}^d \setminus \{0\}$ such that $\int_{\mathbb{R}_0^d} |y|^2 \wedge 1 \, d\nu < \infty$, and
\[
\alpha = \inf \left\{ \sigma < 2 : \int_{|y| \leq 1} |y|^\sigma \, d\nu < \infty \right\}.
\]
Here \( \chi_\alpha (y) = 0 \) if \( \alpha \in (0,1) \), \( \chi_\alpha (y) = 1 \{ |y| \leq 1 \} \) (y) if \( \alpha = 1 \), and \( \chi_\alpha (y) = 1 \) if \( \alpha \in (1,2) \). We say \( v \in A^\alpha \) if in addition,

\[
\int_{|y| \leq 1} |y| \, dv < \infty \text{ if } \alpha \in (0,1), \int_{|y| > 1} |y| \, dv < \infty \text{ if } \alpha \in (1,2),
\]

\[
\int_{R < |y| \leq R'} y \, dv = 0 \text{ if } \alpha = 1 \text{ for all } 0 < R < R' < \infty.
\]

The Eq. 1.1 arises naturally in the study of stochastic processes with jumps. For any \( v \in A^\alpha \), there exists a Poisson random measure \( J (ds,dy) \) on \( [0,\infty) \times \mathbb{R}^d_0 \) such that

\[
\mathbb{E} [J (ds,dy)] = \nu (dy) \, ds,
\]

and a \( \nu \)-measure \( J (ds,dy) \) so that

\[
Z^\nu_t = \int_0^t \int_{\mathbb{R}^d_0} \chi_\alpha (y) \, J (ds,dy) + \int_0^t \int_{\mathbb{R}^d_0} (1 - \chi_\alpha (y)) \, y \, J (ds,dy), \quad t \geq 0,
\]

with \( \tilde{J} (ds,dy) = J (ds,dy) - \nu (dy) \, ds \). For any bounded smooth function \( f \),

\[
u (t,x) = \mathbb{E} \int_0^t e^{-\lambda (t-s)} f (s,x + Z^\nu_{t-s}) \, ds, \quad (t,x) \in H_T,
\]

is a classical solution to Eq. 1.1, see e.g. Proposition 5 below. The characteristic function of the random variable \( Z^\nu_t \) is an exponential function

\[
\mathbb{E} \left( e^{i2\pi \xi \cdot Z^\nu_t} \right) = \exp \{ \psi^\nu (\xi) \, t \}, \xi \in \mathbb{R}^d,
\]

where

\[
\psi^\nu (\xi) = \int_{\mathbb{R}^d_0} \left[ e^{i2\pi \xi \cdot y} - 1 - i2\pi \chi_\alpha (y) \, y \cdot \xi \right] \, dv (y), \xi \in \mathbb{R}^d.
\]

is the symbol of the integro-differential operator \( L^\nu \) (\( L^\nu \) is called the generator of \( Z^\nu \)). If \( \nu (dy) = m (y) \frac{dy}{|y|^\alpha}, \) then we can get classical (pointwise) solutions to Eq. 1.1 by considering it as an equation in the standard H"older-Zygmund spaces. If \( m = 1 \), the operator \( L^\nu = c_\alpha (-\Delta)^{\alpha/2} \) is fractional Laplacian, and \( Z^\nu \) is the symmetric \( \alpha \)-stable process. It turns out that going farther away from the \( \alpha \)-stable (fractional Laplacian) case requires to have a substitute for the standard H"older space that would be capable to contain the optimal regularity determined by the generator \( L^\nu \) of Levy process \( Z^\nu \). For \( v \in A^\alpha \), set

\[
\delta (r) = \delta_v (r) = v (\{ |y| > r \}), \quad r > 0,
\]

\[
w (r) = w_v (r) = \delta_v (r)^{-1}, \quad r > 0.
\]

One of our main assumptions is that \( w (r) = w_v (r) = \delta_v (r)^{-1} \) is an O-RV function (O-regular variation function) at zero (see [2] and [3]), that is

\[
r_1 (\varepsilon) = \lim_{\varepsilon \to 0} \left( \frac{\delta (\varepsilon x)}{\delta (x)} \right)^{-1} < \infty, \quad \varepsilon > 0.
\]

By Theorem 2 in [2], the following limits exist:

\[
p_1 = p_1^v = \lim_{\varepsilon \to 0} \frac{\log r_1 (\varepsilon)}{\log \varepsilon}, \quad q_1 = q_1^v = \lim_{\varepsilon \to \infty} \frac{\log r_1 (\varepsilon)}{\log \varepsilon},
\]

and \( p_1 \leq q_1 \). It can be shown (see Remark 6) that \( p_1 \leq \alpha \leq q_1 \). In this paper, we study the problem (1.1) in the scale of spaces of generalized Hölder functions whose regularity is determined by the Lévy measure \( v \). We use \( w \) to define generalized Besov norms \( |\cdot|_{\beta, \infty} \).
and generalized spaces \( \tilde{C}^{\beta}_{\infty, \infty}(\mathbb{R}^d) , \beta > 0 \) (See Section 2.2.). They are Besov spaces of generalized smoothness (see [12–14]) with admissible sequence \( w (N^{-j})^{-\beta}, j \geq 0 \), and covering sequence \( N^j, j \geq 0 \), with \( N > 1 \). In particular (see Section 2), for \( \beta \in (0, q_1^{-1}) \), the norm \(|·|_{\beta, \infty}^\beta\) for the functions on \( \mathbb{R}^d \) is equivalent to

\[
|u|_{\beta, \infty} = \sup_x |u(x)| + \sup_{x \neq y} \frac{|u(x) - u(y)|}{w(|x - y|)^{\beta}}.
\]

When \( \nu \) is “close” to an \( \alpha \)-stable measure, they reduce to the classical Besov (or equivalently, Hölder-Zygmund) spaces.

For \( \beta > 0 \), let \( \tilde{C}^{\beta}_{\infty, \infty}(HT) \) be the space of all measurable functions \( g \) on \( HT \) such that

\[
\sup_{0 \leq s \leq T} \left\| g(s, \cdot) \right\|_{\tilde{C}^{\beta}_{\infty, \infty}(\mathbb{R}^d)} < \infty.
\]

The main result of this paper is

**Theorem 1** Let \( \beta \in (0, \infty) , \lambda \geq 0 \). Let \( \nu \in A^\alpha , \) and \( w = w_{\nu} \) be an O-RV function at zero with \( p_1, q_1 \) defined in Eq. 1.3. Assume

A. \( 0 < p_1 \leq q_1 < 1 \) if \( \alpha \in (0, 1) , 0 < p_1 \leq 1 \leq q_1 < 2 \) if \( \alpha = 1 \),

B. There is a constant \( c_0 > 0 \) such that

\[
\inf \left| \xi \right|_{\mathbb{S}^1} \int_{|y| \leq 1} \left| \xi \cdot y \right|^2 \tilde{v}_R (dy) \geq c_0, R \in (0, 1),
\]

where

\[
\tilde{v}_R (dy) = w(R) \nu (Rdy) , R \in (0, 1] ;
\]

C. There is \( N_0 > 2 \) so that

\[
\int_1^\infty w(t) \frac{1}{t^{\frac{1}{N_0}}} \frac{dt}{t^{N_0}} < \infty.
\]

Then for each \( f \in \tilde{C}^{\beta+1}_{\infty, \infty}(HT) \) there is a unique solution \( u \in \tilde{C}^{1+\beta}_{\infty, \infty}(HT) \) solving (1.1). Moreover,

\[
\sup_{s \in [0, T]} |u(s, \cdot)|_{\beta, \infty} \leq C \rho_\lambda(T) \sup_{s \in [0, T]} |f(s, \cdot)|_{\beta, \infty}, \tag{1.4}
\]

\[
\sup_{s \in [0, T]} |u(s, \cdot)|_{1+\beta, \infty} \leq C [1 + \rho_\lambda(T)] \sup_{s \in [0, T]} |f(s, \cdot)|_{\beta, \infty}, \tag{1.5}
\]

and

\[
|u(t, \cdot) - u(t', \cdot)|_{\mu+\beta, \infty} \leq C \left\{ (t - t')^{1-\mu} + [1 + \rho_\lambda(T)] |t - t'| \right\} \sup_{s \in [0, T]} |f(s, \cdot)|_{\beta, \infty}
\]

for any \( \mu \in [0, 1] \) and \( t' < t \leq T \), where \( \rho_\lambda(T) \) = \( \left( \frac{1}{\lambda} \wedge T \right) \). The constant \( C \) does not depend on \( f, \lambda, T, \mu \).

More specific examples could be the following.
Example 1 According to [9], Chapter 3, 70-74, any Lévy measure \( \nu \in \mathcal{A} \) can be disintegrated as

\[
\nu (r, S_{d-1}) = -\int_{\mathbb{R}_0^d} \int_{S_{d-1}} \nu (r, S_{d-1}) \Pi (r, S_{d-1}) d\gamma (r, S_{d-1}),
\]

where \( \gamma = \delta_{r, S_{d-1}} \) and \( \Pi (r, S_{d-1}) \), \( r > 0 \), is a measurable family of measures on the unit sphere \( S_{d-1} \) with \( \| \nu (r, S_{d-1}) \| = 1 \), \( r > 0 \). If \( \delta \) is an O-RV function, \( |\{ s \in [0, 1] : r_1 (s) < 1 \}| > 0 \), \( A, C \) and

\[
\inf_{|\xi| = 1} \int_{S_{d-1}} |\xi \cdot w|^2 \Pi (r, d\omega) \geq c_0 > 0, \quad r > 0,
\]

hold, then all assumptions of Theorem 1 are satisfied (see Corollary 6).

Example 2 Consider Lévy measures in radial and angular coordinate system \( (r = |y|, w = y / |y|) \) in the form

\[
\nu (B) = \int_{\mathbb{R}_0^d} \int_{|w| = 1} 1_B (r w) a (r, w) j (r) r^d S (d\omega) dr, \quad B \in \mathcal{B} (\mathbb{R}_0^d),
\]

where \( S (d\omega) \) is a finite measure on the unit sphere.

Assume

(i) There is \( C > 1, c > 0, 0 < \delta_1 \leq \delta_2 < 1 \), such that

\[
C^{-1} \phi (r - 2) \leq j (r) r^d \leq C \phi (r - 2)
\]

and for all \( 1 < r \leq R \),

\[
c^{-1} \left( \frac{R}{r} \right)^{\delta_1} \leq \frac{\phi (R)}{\phi (r)} \leq c \left( \frac{R}{r} \right)^{\delta_2}.
\]

(ii) There is a function \( \rho_0 (w) \) defined on the unit sphere such that \( \rho_0 (w) \leq a (r, w) \leq 1, \forall r > 0 \), and for all \( |\xi| = 1 \),

\[
\int_{S_{d-1}} |\xi \cdot w|^2 \rho_0 (w) S (d\omega) \geq c_0 > 0.
\]

Under these assumptions, it can be shown that \( B \) and \( C \) hold, and \( \delta_{r, S_{d-1}} \) is an O-RV function with \( 2\delta_1 \leq p_1 \leq q_1 \leq 2\delta_2 \). Among the options for \( \phi \) could be (see [16])

(1) \( \phi (r) = \Sigma_{i=1}^n r^{\alpha_i}, \alpha_i \in (0, 1), i = 1, \ldots, n; \)

(2) \( \phi (r) = (r + r^{\alpha})^\beta, \alpha, \beta \in (0, 1); \)

(3) \( \phi (r) = r^{\alpha} (\ln (1 + r))^{\beta}, \alpha \in (0, 1), \beta \in (0, 1 - \alpha); \)

(4) \( \phi (r) = ((r + m^{1/\alpha})^{\alpha} - m, \alpha \in (0, 1), m > 0; \)

(5) \( \phi (r) = \left[ \ln (\cosh \sqrt{r}) \right]^{\alpha}, \alpha \in (0, 1). \)

Equations in classical Hölder spaces with non-local nondegenerate operators of the form

\[
\mathcal{L} u (x) = 1_{x \in (0, 2)} \int_{\mathbb{R}_0^d} \left[ u (x + y) - u (x) - 1_{x \leq 1} y \cdot \nabla u (x) \right] m (x, y) \nu (dy)
+ 1_{x \geq 2} a^{ij} (x) \delta_{ij} u (x) + 1_{x \geq 1} \tilde{b}^i (x) \delta_i u (x) + l (x) u (x), \quad x \in \mathbb{R}_d,
\]

were considered in many papers. In [1], the existence and uniqueness of a solution to a parabolic equation with \( \mathcal{L} \) in Hölder spaces was proved analytically for \( m \) Hölder continuous in \( x \) and smooth in \( y \), \( \nu (dy) = dy / |y|^{d+\alpha} \). In [17], parabolic equation with
m Hölder in x and measurable in y was studied. The elliptic problem $L u = f$ in $\mathbb{R}^d$ with $v(dy) = dy/|y|^{d+\alpha}$ was considered in [4, 10] (see references therein). In [4], with $v(dy) = dy/|y|^{d+\alpha}$, the a priori estimates were derived in Hölder classes assuming Hölder continuity of $m$ in x, except the case $\alpha = 1$. Similar results, including the case $\alpha = 1$, were proved in [10]. In [8] (see references therein), in the classical Hölder spaces the case of a nondegenerate $\nu$ was considered. Finally, in [20], for Eq. 1.1 with $x$-dependent density $m(x, y)$ at $\nu$, under slightly different assumptions than A-C, existence and uniqueness in generalized smoothness classes is derived.

In [6, 7, 18, 19] (see references therein), the Hölder regularity theory was developed for fully nonlinear integro-differential equations based on an integro-differential version of the celebrated embedding type result by Krylov and Safonov for parabolic PDEs with measurable coefficients. In [18], it goes beyond the symmetric absolutely continuous case with $v(dy) = dy/|y|^{d+\alpha}$. The constant in the Hölder norm estimate of the solution depends only on the lower bound of $\alpha$. A Hölder regularity study for fully nonlinear equations with $\nu$ satisfying the assumptions A, B of Theorem 1, would be of interest.

Our paper is organized as follows. In Section 2, notation is introduced, the scale of generalized Hölder function spaces is defined, and various equivalent norms are introduced. In particular, using some probabilistic considerations, we prove the equivalence of $|u|_{\beta, \infty}$ to the norms involving fractional powers of nondegenerate $L^\nu$. The continuity of the operator is proved as well. Study of function spaces of generalized smoothness dates back to the seventies-eignties, (see [13, 14] and references therein). Later, this interest continued in connection with the construction problems of Markov processes with jumps (see [11, 12] and references therein). In Section 3, we prove the main theorem by starting with smooth input functions. Then we derive the key uniform estimates for the corresponding smooth solutions to Eq. 1.1. We handle generalized Hölder inputs by passing to the limit. Finally, Appendix contains all needed results about O-RV functions. The regular variation functions were introduced in [15] and used in tauberian theorems which were extended to O-RV functions as well (see [2, 3], and references therein). They are very convenient for the derivation of our main estimates.

2 Notation, Function Spaces and Norm Equivalence

2.1 Basic Notation

We denote $\mathbb{N} = \{0, 1, 2, 3, \ldots \}$, $\mathbb{N}_+ = \mathbb{N}\backslash\{0\}$; $H_T = [0, T] \times \mathbb{R}^d$; $S^{d-1}$ is the unit sphere in $\mathbb{R}^d$. For $B \in \mathcal{B}(\mathbb{R}^d)$, we denote by $|B|$ the Lebesgue measure of $B$.

For a function $u$ on $H_T$, we denote its partial derivatives by $\partial_t u = \partial u/\partial t$, $\partial_i u = \partial u/\partial x_i$, $\partial^2_{ij} u = \partial^2 u/\partial x_i \partial x_j$, and denote its gradient with respect to $x$ by $\nabla u = (\partial_1 u, \ldots, \partial_d u)$ and $D^{\gamma} u = \partial^{\gamma_1} u/\partial x_{\gamma_1} \ldots \partial x_{\gamma_d}$, where $\gamma = (\gamma_1, \ldots, \gamma_d) \in \mathbb{N}^d$ is a multi-index.

We use $C^\infty_b(\mathbb{R}^d)$ to denote the set of bounded infinitely differentiable functions on $\mathbb{R}^d$ whose derivative of arbitrary order is bounded, and $C^k(\mathbb{R}^d)$, $k \in \mathbb{N}$ the class of k-times continuously differentiable functions.
We denote $S(R^d)$ the Schwartz space of rapidly decreasing functions on $R^d$, and $S'(R^d)$ denotes the space of continuous functionals on $S(R^d)$, i.e. the space of tempered distributions.

We adopt the normalized definition for Fourier and its inverse transforms for functions in $S(R^d)$, i.e.,

$$F\phi(\xi) = \hat{\phi}(\xi) := \int_{R^d} e^{-i2\pi x \cdot \xi} \phi(x) dx,$$

$$F^{-1}\phi(x) = \tilde{\phi}(x) := \int_{R^d} e^{i2\pi x \cdot \xi} \phi(\xi) d\xi, \ \phi \in S(R^d).$$

Recall that Fourier transform can be extended to a bijection on $S'(R^d)$.

Throughout the sequel, $Z_\nu$ represents the Lévy process associated to the Lévy measure $\nu \in A_\alpha$, see Eq. 1.2.

For any Lévy measure $\nu \in A_\alpha$ and $R > 0$,

$$\nu_R(B) := \int_{R^d_0} 1_B(y/R) \nu(dy), B \in B(R^d_0), \ \tilde{\nu}_R(dy) := w(R) \nu_R(dy). \quad (2.1)$$

For any Lévy measure $\nu \in A_\alpha$, we denote its symmetrization

$$\nu_{sym}(dy) = \frac{1}{2} [\nu(dy) + \nu(-dy)].$$

And $A_{\alpha sym} = \{ \mu \in A_{\alpha} : \mu = \mu_{sym} \}$.

For $\nu \in A_\alpha$, $w = w_\nu$, $\beta > 0$, and $u \in C^\infty_0(R^d)$, we denote

$$|u|_0 = \sup_x |u(x)|, \ |u|_\beta = \sup_{x,h \neq 0} \frac{|u(x+h) - u(x)|}{w(|h|)^\beta},$$

and

$$|u|_\beta := |u|_0 + |u|_\beta.$$

We have specific values assigned for $c_0, c_1, c_2, N_0, N_1$, but we allow $C$ to vary from line to line. In particular, $C(\cdots)$ represents a constant depending only on quantities in the parentheses.

### 2.2 Function Spaces and Norm Equivalence

We fix a constant $N > 1$. For such an $N$, by Lemma 6.1.7 in [5] and appropriate scaling, there exists $\phi \in C^\infty_0(R^d)$ such that supp($\phi$) = $\{ \xi : \frac{1}{N} \leq |\xi| \leq N \}$, $\phi(\xi) > 0$ in the interior of its support, and

$$\sum_{j=-\infty}^{\infty} \phi(N^{-j}\xi) = 1 \text{ if } \xi \neq 0.$$

We denote throughout this paper

$$\varphi_j = F^{-1}\left[ \phi\left( N^{-j}\xi \right) \right], \quad j = 1, 2, \ldots, \xi \in R^d, \quad (2.2)$$

$$\varphi_0 = F^{-1}\left[ 1 - \sum_{j=1}^{\infty} \phi\left( N^{-j}\xi \right) \right]. \quad (2.3)$$
Apparently, $\varphi_j \in S (\mathbb{R}^d), j \in \mathbb{N}$. They are convolution functions we use to define generalized Besov spaces. Namely, for $\beta > 0$ we write $\tilde{C}^\beta_{\infty, \infty} (\mathbb{R}^d)$ as the set of functions in $S' (\mathbb{R}^d)$ for which

$$ |u|_{\beta, \infty} := \sup_j w \left( N^{-j} \right)^{-\beta} |u * \varphi_j|_0 < \infty, \quad (2.4) $$

where $w = w_\nu$ with $\nu \in \mathfrak{A}^\alpha$.

**Lemma 1** Let $\nu \in \mathfrak{A}^\alpha$, $w = w_\nu$ be an O-RV function at zero and $A$ holds for it. Let $\beta \in (0, \infty)$. If $u \in \tilde{C}^\beta_{\infty, \infty} (\mathbb{R}^d)$, then $u$ is bounded and continuous,

$$ u (x) = \sum_{j=0}^{\infty} (u * \varphi_j) (x), x \in \mathbb{R}^d, $$

where the series converges uniformly. Moreover,

$$ |u|_0 \leq C |u|_{\beta, \infty}, u \in \tilde{C}^\beta_{\infty, \infty} (\mathbb{R}^d). $$

**Proof** Note that $u * \varphi_j$ is continuous of moderate growth and $\sum_{j=0}^{\infty} u * \varphi_j = u$ in $S' (\mathbb{R}^d)$. Obviously, by Corollary 5 in Appendix,

$$ \sum_{j=0}^{\infty} |u * \varphi_j|_0 = \sum_{j=0}^{\infty} w \left( N^{-j} \right)^{\beta} w \left( N^{-j} \right)^{-\beta} |u * \varphi_j|_0 $$

$$ \leq \sup_{j \geq 0} w \left( N^{-j} \right)^{-\beta} |u * \varphi_j|_0 \sum_{j=0}^{\infty} w \left( N^{-j} \right)^{\beta} $$

$$ \leq C |u|_{\beta, \infty} \sum_{j=0}^{\infty} w \left( N^{-j} \right)^{\beta} \leq C |u|_{\beta, \infty}. $$

Now, we prove the equivalence of $|u|_\beta$ and $|u|_{\beta, \infty}$ norms.

**Proposition 1** Let $\nu \in \mathfrak{A}^\alpha$, $w = w_\nu$ be an O-RV function at zero so that $A$ and $C$ hold for it. Let $\beta \in \left( 0, q^{-1}_1 \right)$. Then the norm $|u|_\beta$ and norm $|u|_{\beta, \infty}$ are equivalent on $C^\infty_b (\mathbb{R}^d)$. Namely, there is $C > 0$ depending only on $d, \beta, N$ such that

$$ C^{-1} |u|_\beta \leq |u|_{\beta, \infty} \leq C |u|_\beta, u \in C^\infty_b (\mathbb{R}^d). $$

**Proof** Let $u \in C^\infty_b (\mathbb{R}^d)$. Then, by Lemma 8, $|u|_\beta < \infty$. If $j = 0$, then

$$ |u * \varphi_0|_0 \leq |u|_0 \int_{\mathbb{R}^d} |\varphi_0 (y)| \, dy \leq C |u|_\beta. $$
If \( j \neq 0 \), then by the construction of \( \varphi_j \), \( \int \varphi_j (y) \, dy = \hat{\varphi}_j (0) = 0 \). Therefore, denoting \( \varphi = \mathcal{F}^{-1} \phi \),

\[
\begin{align*}
|u * \varphi_j|_0 &= \left| \int_{\mathbb{R}^d} [u(y) - u(x)] \varphi_j(x - y) \, dy \right|_0 \\
&= [u]_\beta \int_{\mathbb{R}^d} w(|y - x|) |\varphi_j(N^j(x - y))| \, dy \\
&= [u]_\beta \int_{\mathbb{R}^d} w(N^{-j} |y|)^\beta |\varphi(y)| \, dy.
\end{align*}
\]

Since for \( N_0 > d + 1 \),

\[
|\varphi(y)| \leq C (1 + |y|)^{-N_0}, \ y \in \mathbb{R}^d,
\]

for some \( C > 0 \), we have

\[
\int_{\mathbb{R}^d} w(N^{-j} |y|)^\beta |\varphi(y)| \, dy \\
\leq C \int_{|y| \leq 1} w(N^{-j} |y|)^\beta \, dy + C \int_{|y| > 1} w(N^{-j} |y|)^\beta |y|^{-N_0} \, dy = A_1 + A_2.
\]

By Lemma 8

\[
N^{-j(N_0 - d)} \leq C w(N^{-j})^\beta, \ j \geq 0,
\]

and,

\[
A_1 \leq CN^{jd} \int_0^{N^{-j}} w(r)^\beta \rho^d \frac{dr}{r} \leq C w(N^{-j})^\beta,
\]

\[
A_2 = CN^{-j(N_0 - d)} \int_{N^{-j}}^\infty w(r)^\beta r^{-(N_0 - d)} \frac{dr}{r} \\
= C \left(N^{-j(N_0 - d)} \int_{N^{-j}}^1 w(r)^\beta r^{-(N_0 - d)} \frac{dr}{r} + N^{-j(N_0 - d)} \int_1^\infty w(r)^\beta r^{-(N_0 - d)} \frac{dr}{r} \right) \\
\leq C w(N^{-j})^\beta, \ j \geq 0.
\]

That is to say \( |u|_{\beta, \infty} \leq C |u|_{\beta, u} \in C^{\infty}_b (\mathbb{R}^d) \) for some constant \( C = (\beta, d) > 0 \).

Let \( \tilde{\phi}, \tilde{\phi}_0 \in C^{\infty}_0 (\mathbb{R}^d) \), be such that 0 \( \not\in \text{supp} (\tilde{\phi}) \), \( \tilde{\phi} \phi = \phi, \tilde{\phi}_0 \phi_0 = \phi_0 \), where \( \phi_0 = \mathcal{F} \phi_0 \), and \( \phi, \varphi_0 \) are the functions introduced in Eqs. 2.3, 2.2. Let

\[
\tilde{\varphi} = \mathcal{F}^{-1} \tilde{\phi}, \ \tilde{\varphi}_j = \mathcal{F}^{-1} \tilde{\phi} \left( N^{-j} \right), \ j \geq 1, \tag{2.5}
\]

\[
\tilde{\varphi}_0 = \mathcal{F}^{-1} \tilde{\phi}_0. \tag{2.6}
\]

Hence

\[
\varphi_j = \varphi_j * \tilde{\varphi}_j, \ j \geq 0,
\]

where in particular,

\[
\tilde{\varphi}_j(x) = N^{jd} \tilde{\phi} \left( N^j x \right), \ j \geq 1, \ x \in \mathbb{R}^d.
\]
Obviously,
\[ |u * \varphi_0 (x + y) - u * \varphi_0 (x)| \leq \int_{\mathbb{R}^d} |\tilde{\varphi}_0 (x + y - z) - \tilde{\varphi}_0 (x - z)| |u * \varphi_0 (z)| \, dz \]
and
\[ |u * \varphi_j (x + y) - u * \varphi_j (x)| \leq N_j^d \int_{\mathbb{R}^d} |\tilde{\varphi} (N_j^j (x + y - z) - \tilde{\varphi} (N_j^j (x - z))| |u * \varphi_j (z)| \, dz \]

By Lemma 1, for \( x, y \in \mathbb{R}^d \),
\[ |u (x + y) - u (x)| \leq \sum_{j=0}^{\infty} |u * \varphi_j (x + y) - u * \varphi_j (x)| \leq C \sum_{j=0}^{\infty} \left( N_j^j |y| \wedge 1 \right) |u * \varphi_j|_0. \]

Let \( \beta q_1 < 1, k \in \mathbb{N} \). For \( |y| \in (N^{-k-1}, N^{-k}] \),
\[ |u (x + y) - u (x)| \leq C |u|_{\beta, \infty} \sup_{|y| \leq N^{-k}} \sum_{j=0}^{\infty} \left( N_j^j |y| \wedge 1 \right) w \left( N^{-j} \right)^{\beta} \]
\[ \leq C |u|_{\beta, \infty} \left[ \sum_{j=0}^{k} N_j^{j-k} w \left( N^{-j} \right)^{\beta} + \sum_{j=k+1}^{\infty} w \left( N^{-j} \right)^{\beta} \right]. \]

Then, by Lemma 8,
\[ N^{-k} \sum_{j=0}^{k} N_j^j w \left( N^{-j} \right)^{\beta} \leq CN^{-k} \int_{0}^{k+1} N^x w \left( N^{-x} \right)^{\beta} \, dx \]
\[ \leq CN^{-k} \int_{N^{-k-1}}^{1} x^{-1} w (x)^{\beta} \frac{dx}{x} \leq C w (|y|)^{\beta}. \]
Again, by Lemma 8,
\[ \sum_{j=k+1}^{\infty} w \left( N^{-j} \right)^{\beta} \leq C \int_{k+1}^{\infty} w \left( N^{-x} \right)^{\beta} \, dx \leq C \int_{0}^{N^{-k-1}} w (x)^{\beta} \frac{dx}{x} \leq C w (|y|)^{\beta}. \]

The statement is proved. \( \square \)

### 2.2.1 Equivalent Norms on \( C^\infty_b (\mathbb{R}^d) \)

Now we will introduce some other norms on \( C^\infty_b (\mathbb{R}^d) \) involving the powers of the operators \(-L^v, I - L^v:\)
\[ |u|_{v, \kappa, \beta} = |u|_{\kappa, \beta} = |u|_0 + \left| (-L^v)^{\kappa} u \right|_{\beta, \infty}, u \in C^\infty_b (\mathbb{R}^d), \]
\[ ||u||_{v, \kappa, \beta} = ||u||_{\kappa, \beta} = \left| (I - L^v)^{\kappa} u \right|_{\beta, \infty}, u \in C^\infty_b (\mathbb{R}^d). \]
with $\kappa, \beta > 0$, and $\nu$ satisfying $A$ and $B$. In addition, we assume that $\nu \in \mathfrak{A}_{\text{symb}} = \{\mu \in \mathfrak{A} : \mu = \mu_{\text{symb}}\}$ if $\kappa$ is fractional. First, we define those powers and corresponding norms on $C_{b}^{\infty}(\mathbb{R}^{d})$. Then we study their relations and extend them to $\tilde{C}_{\infty}^{\infty}(\mathbb{R}^{d})$.

For $\nu \in \mathfrak{A}_{\text{symb}}, \kappa \in (0, 1), a \geq 0$, and $f \in \mathcal{S}(\mathbb{R}^{d})$, we see easily that
\[
(a - \psi^{\nu}(\xi))^{\kappa} \hat{f}(\xi) = c_{\kappa} \int_{0}^{\infty} t^{-\kappa} \left[ e^{-at} \exp(\psi^{\nu}(\xi) t) - 1 \right] \frac{dt}{t} \hat{f}(\xi), \xi \in \mathbb{R}^{d},
\]
and define
\[
(aI - L^{\nu})^{\kappa} f(x) = \mathcal{F}^{-1} \left[ (a - \psi^{\nu})^{\kappa} \hat{f} \right](x) = c_{\kappa} \mathcal{E} \int_{0}^{\infty} t^{-\kappa} \left[ e^{-at} f(x + Z_{t}^{\nu}) - f(x) \right] \frac{dt}{t}, x \in \mathbb{R}^{d}, \tag{2.7}
\]
where
\[
c_{\kappa} = \left( \int_{0}^{\infty} (e^{-t} - 1) t^{-\kappa} \frac{dt}{t} \right)^{-1}.
\]
We denote, with $a = 0$, $f \in \mathcal{S}(\mathbb{R}^{d}), \kappa \in (0, 1)$,
\[
(-L^{\nu})^{\kappa} f := \mathcal{F}^{-1} \left[ (-\psi^{\nu})^{\kappa} \hat{f} \right].
\]

For $f \in C_{b}^{\infty}(\mathbb{R}^{d}), \kappa \in (0, 1), a \geq 0$, we define
\[
(aI - L^{\nu})^{\kappa} f(x) = c_{\kappa} \mathcal{E} \int_{0}^{\infty} t^{-\kappa} \left[ e^{-at} f(x + Z_{t}^{\nu}) - f(x) \right] \frac{dt}{t}, x \in \mathbb{R}^{d}.
\]

For $\kappa = 1$, $(aI - L^{\nu}) f = (aI - L^{\nu}) f = af - L^{\nu} f, f \in C_{b}^{\infty}(\mathbb{R}^{d})$. Note that for $\kappa \in (0, 1), a \geq 0$,
\[
(aI - L^{\nu})^{\kappa} f(x) = c_{\kappa} \mathcal{E} \int_{1}^{\infty} t^{-\kappa} \left[ e^{-at} f(x + Z_{t}^{\nu}) - f(x) \right] \frac{dt}{t} + c_{\kappa} \mathcal{E} \int_{0}^{1} t^{-\kappa} \int_{0}^{t} e^{-as} (-a + L^{\nu}) f(x + Z_{s}^{\nu}) ds \frac{dt}{t}, x \in \mathbb{R}^{d}. \tag{2.8}
\]

For $a > 0, f \in C_{b}^{\infty}(\mathbb{R}^{d})$, set
\[
(aI - L^{\nu})^{-\kappa} f(x) = c'_{\kappa} \int_{0}^{\infty} t^{\kappa} e^{-at} E f(x + Z_{t}^{\nu}) \frac{dt}{t}, x \in \mathbb{R}^{d},
\]
where
\[
c'_{\kappa} = \left( \int_{0}^{\infty} t^{\kappa} e^{-t} \frac{dt}{t} \right)^{-1},
\]
and $\nu \in \mathfrak{A}_{\text{symb}}, \kappa > 0$, or $\nu \in \mathfrak{A}, \kappa \in \mathbb{N} = \{1, 2, \ldots \}$.

Note that for $g \in \mathcal{S}(\mathbb{R}^{d})$,
\[
\mathcal{F} \left[ (aI - L^{\nu})^{-\kappa} g \right] = (a - \psi^{\nu})^{-\kappa} \hat{g}, a > 0, \kappa > 0,
\]
\[
\mathcal{F} \left[ (aI - L^{\nu})^{\kappa} g \right] = (a - \psi^{\nu})^{\kappa} \hat{g}, a \geq 0, \kappa \in (0, 1].
\]
We use the formulas above to define $(a - L^{\nu})^{\kappa} f, a > 0, \kappa = 1, 0, -1, \ldots, \nu \in \mathfrak{A}$. 

\[\copyright\ Springer\]
Remark 1 Assume $\kappa \in (0, 1], a \geq 0$ or $\kappa \in (-\infty, 0), a > 0$. It is easy to see that

a) for any $f \in C^\infty_b (\mathbb{R}^d)$, we have $(aI - L^v)^\kappa f \in C^\infty_b (\mathbb{R}^d)$ and for any multiindex $\gamma$, $D^\gamma (aI - L^v)^\kappa f = (aI - L^v)^\kappa D^\gamma f, v \in \mathcal{A}_\text{sym}^\kappa$. The same holds for $v \in \mathcal{A}^\kappa$ and $\kappa = 1, 0, -1, \ldots$

b) for any $f \in C^\infty_b (\mathbb{R}^d)$ such that for any multiindex $\gamma$, $D^\gamma f \in L^1 (\mathbb{R}^d) \cap L^2 (\mathbb{R}^d)$, we have

$$
\mathcal{F} \left[(aI - L^v)^{-\kappa} f\right] = (a - \psi^v)^{-\kappa} \hat{f}, a > 0, \kappa > 0,
$$

$$
\mathcal{F} \left[(aI - L^v)^{\kappa} f\right] = (a - \psi^v)^{\kappa} \hat{f}, a \geq 0, \kappa \in (0, 1],
$$

for $v \in \mathcal{A}_\text{sym}^\kappa$. The same holds for $v \in \mathcal{A}^\kappa$ and $\kappa = 1, 0, -1, \ldots$

The following obvious claim holds.

Lemma 2 Let $v \in \mathcal{A}_\text{sym}^\kappa$. Assume $\kappa \in (0, 1], a \geq 0$ or $\kappa \in (-\infty, 0), a > 0$. Let $f, f_n \in C^\infty_b (\mathbb{R}^d)$ be so that for any multiindex $\gamma$, $D^\gamma f_n \rightarrow D^\gamma f$ as $n \rightarrow \infty$ uniformly on compact subsets of $\mathbb{R}^d$ and

$$
\sup_{x,n} \left| D^\gamma f_n (x) \right| < \infty.
$$

Then for any multiindex $\gamma$,

$$
D^\gamma (aI - L^v)^\kappa f_n = (aI - L^v)^\kappa D^\gamma f_n \rightarrow D^\gamma (aI - L^v)^\kappa f = (aI - L^v)^\kappa D^\gamma f
$$

uniformly on compact subsets of $\mathbb{R}^d$, and

$$
\sup_{x,n} \left| (aI - L^v)^\kappa D^\gamma f_n (x) \right| < \infty.
$$

The same holds for $v \in \mathcal{A}^\kappa$ and $\kappa = 1, 0, -1, \ldots$

Remark 2 Given $f \in C^\infty_b (\mathbb{R}^d)$ there is a sequence $f_n \in C^\infty_b (\mathbb{R}^d)$ so that for any multiindex $\gamma$, $D^\gamma f_n \rightarrow D^\gamma f$ as $n \rightarrow \infty$ uniformly on compact subsets of $\mathbb{R}^d$ and

$$
\sup_{x,n} \left| D^\gamma f_n (x) \right| < \infty.
$$

Indeed, choose $g \in C^\infty_b (\mathbb{R}^d), 0 \leq g \leq 1, g (x) = 1$ if $|x| \leq 1$, and $g (x) = 0$ if $|x| > 2$. Given $f \in C^\infty_b (\mathbb{R}^d)$, take

$$
f_n (x) = f (x) g (x/n), x \in \mathbb{R}^d, n \geq 1.
$$

Lemma 3 Let $v \in \mathcal{A}_\text{sym}^\kappa$. Assume $a > 0, \kappa \in (0, 1]$. Then $(aI - L^v)^\kappa : C^\infty_b (\mathbb{R}^d) \rightarrow C^\infty_b (\mathbb{R}^d)$ is bijective whose inverse is $(aI - L^v)^{-\kappa}$:

$$(aI - L^v)^\kappa (aI - L^v)^{-\kappa} f (x) = (aI - L^v)^{-\kappa} (aI - L^v)^\kappa f (x) = f (x), x \in \mathbb{R}^d,$$

for any $f \in C^\infty_b (\mathbb{R}^d)$.

Proof It is an easy consequence of Lemma 2 and Remarks 1 and 2. 

For an integer $k \in \mathbb{N}$, we define for $v \in \mathcal{A}^\kappa$,

$$(aI - L^v)^k = (aI - L^v) \ldots (aI - L^v).$$
For a non integer $\kappa > 0$, $\kappa = [\kappa] + s$ with $s \in (0, 1)$ and $v \in \mathbb{Q}_\alpha$, we set
\[
(aI - L^\nu)^\kappa f = (aI - L^\nu)^{[\kappa]} (aI - L^\nu)^s f = (aI - L^\nu)^s (aI - L^\nu)^{[\kappa]} f, f \in C_b^\infty (\mathbb{R}^d).
\]

**Remark 3** Let $v \in \mathbb{Q}_\alpha$, and $f \in C_b^\infty (\mathbb{R}^d)$ be such that for any multiindex $\gamma$, $D^\gamma f \in L^1 (\mathbb{R}^d) \cap L^2 (\mathbb{R}^d)$. Then
\[
\mathcal{F}[(aI - L^\nu)^{-\kappa} f] = (a - \psi^\nu)^{-\kappa} \hat{f}, a > 0, s > 0, \\
\mathcal{F}[(aI - L^\nu)^{s} f] = (a - \psi^\nu)^{s} \hat{f}, a \geq 0, s > 0.
\]
The same holds with $v \in \mathbb{Q}_\alpha$ if $s \in \mathbb{N}$.

**Lemma 4** Assume $a > 0$. Then
(i) for any $\kappa, s \geq 0$, we have $(-L^\nu)^\kappa (-L^\nu)^s = (-L^\nu)^{\kappa + s}$; for any $\kappa, s \in \mathbb{R}$,
\[
(aI - L^\nu)^\kappa (aI - L^\nu)^s = (aI - L^\nu)^{\kappa + s}, \\
(aI - L^\nu)^{-\kappa} (aI - L^\nu)^{-s} = (aI - L^\nu)^{(\kappa + s)}.
\]
for $v \in \mathbb{Q}_\alpha$.

The same holds with $v \in \mathbb{Q}_\alpha$ if $\kappa, s \in \mathbb{N}$.

(ii) for any $\kappa > 0$, the mapping $(aI - L^\nu)^\kappa : C_b^\infty (\mathbb{R}^d) \to C_b^\infty (\mathbb{R}^d)$ is bijective whose inverse is $(aI - L^\nu)^{-\kappa}$:
\[
(aI - L^\nu)^\kappa (aI - L^\nu)^{-\kappa} f (x) = (aI - L^\nu)^{-\kappa} (aI - L^\nu)^\kappa f (x) = f (x), x \in \mathbb{R}^d.
\]
for any $f \in C_b^\infty (\mathbb{R}^d)$.

The same holds with $v \in \mathbb{Q}_\alpha$ if $\kappa \in \mathbb{N}$.

**Proof** The statement is an easy consequence of Lemma 2 and Remarks 1, 2, and 3. □

**Lemma 5** Let $v \in \mathbb{Q}_\alpha$, satisfy $A$.

(i) Let $a \geq 0$, $\kappa > 0$, $m = [\kappa] + 1$. For any $f \in C_b^\infty (\mathbb{R}^d),$
\[
\sup_{R \in (0, 1]} \left| (a - L^\nu)^\kappa D^\gamma f (x) \right| \leq C (1 + a)^\kappa \max_{|\mu| \leq |\gamma| + 2m} |D^\mu f|_0 < \infty.
\]
If, in addition, for any multiindex $\gamma$, \( \int_{\mathbb{R}^d} |D^\gamma f (x)| dx < \infty \), then
\[
\sup_{R \in (0, 1]} \int_{\mathbb{R}^d} \left| (a - L^\nu)^\kappa D^\gamma f (x) \right| dx \leq C (1 + a)^\kappa \max_{|\mu| \leq |\gamma| + 2m} \int_{\mathbb{R}^d} |D^\mu f (x)| dx.
\]
The same holds with $v \in \mathbb{Q}_\alpha$ satisfying $A$ if $\kappa \in \mathbb{N}$.

(ii) Let $a > 0$, $\kappa > 0$. For any $f \in C_b^\infty (\mathbb{R}^d),$
\[
\sup_{R \in (0, 1]} \left| (a - L^\nu)^{-\kappa} D^\gamma f (x) \right| \leq C a^{-\kappa} \max_{|\mu| \leq |\gamma|} |D^\mu f|_0 < \infty.
\]
If in addition, for any multiindex $\gamma$, $\int_{\mathbb{R}^d} |D^\gamma f(x)| \, dx < \infty$, then

$$
\sup_{t \in (0, 1]} \int_{\mathbb{R}^d} \left| (a - L_{\tilde{\nu} R})^{-\kappa} D^\gamma f(x) \right| \, dx \leq C a^{-\kappa} \max_{|\mu| \leq |\gamma|} \int_{\mathbb{R}^d} |D^\mu f(x)| \, dx.
$$

The same holds with $\nu \in \mathcal{A}^\alpha$ satisfying $A$ if $\kappa \in \mathbb{N}$.

**Proof** (i) Let $\kappa \in (0, 1]$. Then

$$
\left( a - L_{\tilde{\nu} R} \right)^\kappa f(x) = c_\kappa \int_0^\infty \int_0^t e^{-at} t^{-\kappa} E \left[ \left| D^\gamma f(x + Z_t^R) - f(x) \right| \, dt \right] \, \frac{dt}{t} = c_\kappa \int_1^\infty \ldots
$$

$$
+ c_\kappa \int_0^1 \int_0^t e^{-as} E \left[ (L_{\tilde{\nu} R} - a) f(x + Z_s^R) \right] \, ds \, \frac{dt}{t}, \ x \in \mathbb{R}^d.
$$

By Lemma 9, we have

$$
\sup_{t \in (0, 1]} \int_{\mathbb{R}^d} \left| E g \left( x + Z_t^R \right) \right| \, dx < \infty \text{ if } \alpha \in (0, 1),
$$

$$
\sup_{t \in (0, 1]} \int_{\mathbb{R}^d} \left| \left| y \right|^2 \wedge 1 \right| \tilde{\nu} R(dy) < \infty \text{ if } \alpha = 1,
$$

$$
\sup_{t \in (0, 1]} \int_{\mathbb{R}^d} \left| \left| y \right|^2 \wedge \left| y \right| \right| \tilde{\nu} R(dy) < \infty \text{ if } \alpha \in (1, 2),
$$

and both inequalities easily follow. Applying them repeatedly we obtain the claim for an arbitrary $\kappa > 0$.

(ii) Indeed, for any $\kappa > 0$, $a > 0$, and any multiindex $\gamma$,

$$
D^\gamma \left( a - L_{\tilde{\nu} R} \right)^{-\kappa} f(x) = c_\kappa \int_0^\infty e^{-at} t^{-\kappa} E D^\gamma f \left( x + Z_t^R \right) \, \frac{dt}{t}, \ x \in \mathbb{R}^d,
$$

and the claim obviously follows.

□

**Lemma 6** Let $\nu \in \mathcal{A}^\alpha$ satisfy $A$ and $B$. Let $g \in \mathcal{S}(\mathbb{R}^d)$ be such that $\hat{g} \in C_0^\infty(\mathbb{R}^d)$, $0 \notin \text{supp}(\hat{g})$. Then there are constants $C, c$ so that

$$
\sup_{t \in (0, 1]} \int_{\mathbb{R}^d} \left| E g \left( x + Z_t^R \right) \right| \, dx \leq Ce^{-ct}, \ t > 0.
$$

**Proof** Let $F(t, x) = E g \left( x + Z_t^R \right)$, $x \in \mathbb{R}^d$, $t > 0$. We choose $\epsilon > 0$ so that $\text{supp}(\hat{g}) \subseteq \{ \xi : |\xi| \leq \epsilon^{-1} \}$. Let $\tilde{v}_{R, \epsilon}(dy) = \chi_{|\xi| \leq \epsilon} \tilde{v} R(dy)$, $R \in [0, 1]$. Then for $\xi \in \text{supp}(\hat{g})$ and $|y| \leq \epsilon$,

$$
1 - \cos (\xi \cdot y) \geq \frac{1}{\pi} |\xi \cdot y|^2 = \frac{|\xi|^2}{\pi} |\tilde{\xi} \cdot y|^2
$$
with \( \hat{\xi} = \xi / |\xi| \). Therefore there is \( c_0 > 0 \) so that for any \( \xi \in \text{supp} (\hat{\tilde{g}}) \) and \( R \in (0, 1) \),

\[
\Re \psi \tilde{\nu}_{R,t}(\xi) = \int_{|y| \leq \varepsilon} [1 - \cos (\xi \cdot y)] \tilde{\nu}_R(dy) \\
\geq \frac{|\xi|^2}{\pi} \int_{|y| \leq \varepsilon} |\hat{\xi} \cdot y|^2 \tilde{\nu}_R(dy) \geq c_0 |\xi|^2.
\]

Then

\[
\hat{F}(t, \xi) = \exp \left\{ \psi \tilde{\nu}_{R,t}(\xi) t \right\} \hat{\tilde{g}}(\xi) = \exp \left\{ \psi \tilde{\nu}_{R,t}(\xi) t \right\} \exp \{ \psi(\xi) t \} \hat{\tilde{g}}(\xi), \xi \in \mathbb{R}^d,
\]

where \( \exp \{ \psi(\xi) t \} \) is a characteristic function of a probability distribution \( P_{R,t}(dy) \) on \( \mathbb{R}^d \). Hence

\[
F(t,x) = \int_{\mathbb{R}^d} H(t,x - y) P_{R,t}(dy), x \in \mathbb{R}^d,
\]

with

\[
H(t,x) = \mathcal{F}^{-1} \left[ \exp \left\{ \psi \tilde{\nu}_{R,t} \right\} \hat{\tilde{g}} \right] = \mathbb{E}g(x + Z_{t}^{\tilde{\nu}_{R,t}}), x \in \mathbb{R}^d.
\]

Since

\[
\int_{\mathbb{R}^d} |F(t,x)| \, dx \leq \int_{\mathbb{R}^d} |H(t,x)| \, dx,
\]

it is enough to prove that

\[
\int_{\mathbb{R}^d} |H(t,x)| \, dx \leq C e^{-at}, t > 0. \tag{2.10}
\]

Now, Eq. 2.9 implies that for any multiindex \( |\gamma| \leq n = [\frac{d}{2}] + 3, \)

\[
\int_{\mathbb{R}^d} |x^\gamma H(t,x)|^2 \, dx \leq C \int_{\mathbb{R}^d} D^\gamma \left[ \hat{\tilde{g}}(\xi) \exp \left\{ \psi \tilde{\nu}_{R,t}(\xi) t \right\} \right] |^2 \, d\xi
\leq C_1 e^{-ct}, t > 0.
\]

Hence, denoting \( d_0 = [\frac{d}{2}] + 1, \)

\[
\int_{\mathbb{R}^d} |H(t,x)| \, dx = \int_{\mathbb{R}^d} \left( 1 + |x|^2 \right)^{-d_0} |H(t,x)| \left( 1 + |x|^2 \right)^{d_0} \, dx
\leq C \int_{\mathbb{R}^d} |H(t,x)|^2 \left( 1 + |x|^2 \right)^{2d_0} \, dx \leq C_1 e^{-ct}, t > 0.
\]

Thus (2.10) follows, and

\[
\int_{\mathbb{R}^d} \left| \mathbb{E}g(x + Z_{t}^{\tilde{\nu}_{R,t}}) \right| \, dx \leq C_1 e^{-ct}, t > 0. \tag{2.11}
\]

\( \square \)

**Corollary 1** Let \( \nu \in \mathcal{M}_{\text{sym}}^\alpha \) satisfy A and B. Let \( g \in \mathcal{S}(\mathbb{R}^d) \) be such that \( \hat{g} \in C_0^\infty(\mathbb{R}^d), 0 \notin \text{supp}(\hat{g}) \). Then for \( a \geq 0, \kappa > 0, R \in (0, 1), \)

\[
(a - L_{R}^\nu)^{-\kappa} g(x) = -\mathcal{F}^{-1} \left[ (a - \psi \tilde{\nu}_R)^{-\kappa} \hat{g} \right](x)
= c_{\kappa} \int_0^\infty e^{-at} t^\kappa \mathbb{E}g(x + Z_{t}^{\tilde{\nu}_{R,t}}) \frac{dt}{t}, x \in \mathbb{R}^d,
\]
is $C_\infty^b$ -function, and for every multiindex $\gamma$, we have $D^\gamma (a - L\tilde{\nu}_R)^{-\kappa} g = (a - L\tilde{\nu}_R)^{-\kappa} D^\gamma g$, $R > 0$, and

$$\sup_{R \in (0, 1], a \geq 0} \int \mathbb{R}^d \left| D^\gamma (a - L\tilde{\nu}_R)^{-\kappa} g (x) \right|^p dx < \infty, \ p \geq 1.$$

The same holds with $v \in \mathcal{Q}^d$ satisfying $A$ and $B$ if $\kappa \in \mathbb{N}$.

**Proof** Take $\eta \in C_\infty^0 (\mathbb{R}^d)$ so that $\hat{\eta} \hat{g} = \hat{\hat{g}}, \ 0 \notin \text{supp} (\eta)$, and let $\tilde{\eta} = \mathcal{F}^{-1} \eta$. Let

$$F_R (t, x) = E g \left( x + Z^{\tilde{\nu}_R}_t \right), \ t > 0, x \in \mathbb{R}^d.$$

Then

$$\hat{F}_R (t, \xi) = \exp \left\{ \psi^{\tilde{\nu}_R} (\xi) t \right\} \eta (\xi) \hat{g} (\xi), \ \xi \in \mathbb{R}^d,$$

and

$$F_R (t, x) = \int \mathbb{R}^d H_R (t, x - y) g (y) dy = \int \mathbb{R}^d g (x - y) H_R (t, y) dy, \ x \in \mathbb{R}^d,$$

with $H_R (t, x) = \mathcal{F}^{-1} \left[ \exp \left\{ \psi^{\tilde{\nu}_R} (\xi) \right\} \eta \right] (\xi) = E \tilde{\eta} (x + Z^{\tilde{\nu}_R}_t), \ t > 0, x \in \mathbb{R}^d$.

By Lemma 6,

$$\sup_{R \in (0, 1]} \int \mathbb{R}^d |H_R (t, y)| dy \leq C e^{-ct}, \ t > 0.$$

Hence $F_R (t, \cdot) \in C_\infty^b (\mathbb{R}^d), \ t > 0$, and for each multiindex $\gamma$ and $p \geq 1$,

$$\sup_R \left( \int \mathbb{R}^d \left| D^\gamma F_R (t, x) \right|^p dx \right)^{1/p} \leq C \left( \int \mathbb{R}^d \left| D^\gamma g (x) \right|^p dx \right)^{1/p} e^{-ct}, \ t > 0.$$

**Corollary 2** Let $v \in \mathcal{Q}^d$ satisfy $A$ and $B$. Let $g \in S (\mathbb{R}^d)$ be such that $\hat{\hat{g}} \in C_\infty^0 (\mathbb{R}^d), \ 0 \notin \text{supp} (\hat{\hat{g}})$. Then there are constants $C, c > 0$ so that

$$\sup_{R \in (0, 1]} \int \mathbb{R}^d \left| E L\tilde{\nu}_R g (x + Z^{\tilde{\nu}_R}_t) \right| dx \leq C e^{-ct}, \ t > 0.$$

**Proof** Let $h \in C_\infty^0 (\mathbb{R}^d), \ 0 \leq h \leq 1$, and $h (\xi) = 1$ if $\xi \in \text{supp} (g), \ h (\xi) = 0$ in a neighborhood of zero. Let

$$G_R (t, x) = E L\tilde{\nu}_R g \left( x + Z^{\tilde{\nu}_R}_t \right), \ x \in \mathbb{R}^d.$$

Then

$$\hat{G}_R (t, \xi) = \exp \left\{ \psi^{\tilde{\nu}_R} (\xi) t \right\} \hat{\hat{g}} (\xi)$$

$$= \exp \left\{ \psi^{\tilde{\nu}_R} (\xi) t \right\} h (\xi) \psi^{\tilde{\nu}_R} (\xi) \hat{\hat{g}} (\xi), \ \xi \in \mathbb{R}^d.$$

Hence

$$G_R (t, x) = \int \mathbb{R}^d H_R (t, x - y) B_R (y) dy, \ x \in \mathbb{R}^d,$$

where

$$B_R (x) = L\tilde{\nu}_R g (x), \ H_R (t, x) = Eh \left( x + Z^{\tilde{\nu}_R}_t \right), \ x \in \mathbb{R}^d.$$
Thus, by Lemma 6,
\[
\sup_{R \in (0, 1]} \int_{\mathbb{R}^d} |G_R (t, x)| \, dx \leq \sup_{R \in (0, 1]} \int_{\mathbb{R}^d} |H_R (t, x)| \, dx \sup_{R \in (0, 1]} \int_{\mathbb{R}^d} |B_R (x)| \, dx
\leq C e^{-ct}, \quad t > 0.
\]

Lemma 7 Let \( v \in \mathfrak{A}_{\text{sym}}^{|\nu} \), satisfy A and B. Then

(i) for each \( \beta, \kappa > 0 \), there is \( C > 0 \) so that
\[
|(-L^v)^\kappa u|_{\beta, \infty} \leq C |u|_{\beta + \kappa, \infty}, \quad u \in C^\infty_b \left( \mathbb{R}^d \right),
\]
\[
|(I - L^v)^\kappa u|_{\beta, \infty} \leq C |u|_{\beta + \kappa, \infty}, \quad u \in C^\infty_b \left( \mathbb{R}^d \right),
\]

(ii) for each \( 0 < \kappa < \beta \), there is \( C > 0 \) so that
\[
|u|_{\beta, \infty} \leq C \left[ |(-L^v)^\kappa u|_{\beta - \kappa, \infty} + |u|_0 \right], \quad u \in C^\infty_b \left( \mathbb{R}^d \right),
\]
\[
|u|_{\beta, \infty} \leq C \left[ |(I - L^v)^\kappa u|_{\beta - \kappa, \infty} \right], \quad u \in C^\infty_b \left( \mathbb{R}^d \right).
\]
The same holds with \( v \in \mathfrak{A}^{|\nu} \) satisfying A and B if \( \kappa \in \mathbb{N} \).

Proof Let \( u \in C^\infty_b \left( \mathbb{R}^d \right), \tilde{\phi}, \tilde{\varphi}_0 \in C^\infty_0 \left( \mathbb{R}^d \right) \), be such that \( \tilde{\phi} \tilde{\varphi} = \phi, \tilde{\varphi}_0 \tilde{\varphi}_0 = \phi_0 \), where \( \phi_0 = \mathcal{F}^{-1} \varphi_0 \), and \( \tilde{\phi}, \varphi_0 \) are the functions in the definition of the spaces.

(i) Let \( r \in [0, 1] \). Then
\[
(r - L^v)^\kappa u \ast \varphi_j = \mathcal{F}^{-1} \left[ (r - \psi^v)^\kappa \tilde{\phi} \left( N^{-j} \cdot \right) \tilde{\varphi} \left( N^{-j} \cdot \right) \right]
\]
\[
= \int_{\mathbb{R}^d} H^j_r (x - y) u \ast \varphi_j (y) \, dy, \quad x \in \mathbb{R}^d, \quad j \geq 1,
\]
\[
(r - L^v)^\kappa u \ast \varphi_0 = \mathcal{F}^{-1} \left[ (r - \psi^v)^\kappa \tilde{\varphi}_0 \tilde{\varphi}_0 \right] = \int_{\mathbb{R}^d} H^0_r (x - y) u \ast \varphi_0 (y) \, dy, \quad x \in \mathbb{R}^d
\]
where \( H^j_r = \mathcal{F}^{-1} \left[ (r - \psi^v)^\kappa \tilde{\phi} \left( N^{-j} \cdot \right) \right], \quad j \geq 1, \quad H^0_r = \mathcal{F}^{-1} \left[ (r - \psi^v)^\kappa \tilde{\varphi}_0 \right] \) Let
\[
G_j = w \left( N^{-j} \right)^{-\kappa} \mathcal{F}^{-1} \left[ \left( r w \left( N^{-j} \cdot \right) - \psi \tilde{\varphi}_{N^{-j}} \right)^\kappa \tilde{\phi} \left( N^{-j} \cdot \right) \right], \quad j \geq 1.
\]
Since
\[
(r - \psi^v (\xi))^{\kappa} \tilde{\phi} \left( N^{-j} \xi \right) = w \left( N^{-j} \right)^{-\kappa} \left[ r w \left( N^{-j} \cdot \right) - \psi \tilde{\varphi}_{N^{-j}} \left( N^{-j} \xi \right) \right]^{\kappa} \tilde{\phi} \left( N^{-j} \xi \right), \xi \in \mathbb{R}^d,
\]
it follows by Lemma 5 that
\[
\int_{\mathbb{R}^d} \left| H^j_r (x) \right| \, dx = \int_{\mathbb{R}^d} \left| G_j (x) \right| \, dx
\]
\[
= w \left( N^{-j} \right)^{-\kappa} \int_{\mathbb{R}^d} \left| \left( r w \left( N^{-j} \cdot \right) - L^\kappa \tilde{\varphi} \right) \left( N^{-j} \xi \right) \right| \, dx
\]
\[
\leq C w \left( N^{-j} \right)^{-\kappa}, \quad j \geq 0.
\]
(ii) Let $0 < \kappa < \beta, r \in [0, 1]$. Then for $j \geq 1$,

$$u * \varphi_j = (r - L^\nu)^\kappa (r - L^\nu)^{-\kappa} u * \varphi_j$$

$$= \mathcal{F}^{-1} \left[ (r - \psi^\nu)^{-\kappa} \tilde{\phi} \left( N^{-j} \right) (r - \psi^\nu)^\kappa \hat{u} \phi \left( N^{-j} \right) \right]$$

$$= \int_{\mathbb{R}^d} H^j_r (x - y) (r - L^\nu)^\kappa u * \varphi_j (y) dy, x \in \mathbb{R}^d, j \geq 1,$$

where

$$H^j_r = \mathcal{F}^{-1} \left[ (r - \psi^\nu)^{-\kappa} \tilde{\phi} \left( N^{-j} \right) \right], j \geq 1, r \geq 0.$$

Let

$$G_j = w \left( N^{-j} \right)^\kappa \mathcal{F}^{-1} \left[ (rw - \psi^\nu N^{-j})^{-\kappa} \tilde{\phi} \left( N^{-j} \right) \right], j \geq 1.$$

It follows by Corollary 1 that there is $C$ independent of $r \geq 0, j \geq 1$, so that

$$\int_{\mathbb{R}^d} \left| H^j_r (x) \right| dx = \int_{\mathbb{R}^d} \left| G_j (x) \right| dx = w \left( N^{-j} \right)^\kappa \int_{\mathbb{R}^d} \left| (rw - \psi^\nu N^{-j})^{-\kappa} \tilde{\phi} (x) \right| dx$$

$$\leq C w \left( N^{-j} \right)^\kappa.$$

On the other hand,

$$|u * \varphi_0|_0 \leq C |u|_0.$$

The statement follows.

The same holds with $\nu \in \mathcal{A}^\alpha_{sym}$ satisfying $A$ and $B$ if $\kappa \in \mathbb{N}$.

For $\beta > 0, \kappa > 0$, we define the following norms:

$$|u| \nu, \kappa, \beta = |u|_0 + \left| (r - L^\nu)^\kappa u \right|_{\beta, \infty}, u \in C^\infty_b (\mathbb{R}^d),$$

$$\|u\| \nu, \kappa, \beta = \left| (I - L^\nu)^\kappa u \right|_{\beta, \infty}, u \in C^\infty_b (\mathbb{R}^d),$$

with $\nu$ satisfying $A$ and $B$. An immediate consequence of Lemma 7 is the following norm equivalence.

**Corollary 3** Let $\nu \in \mathcal{A}^\alpha_{sym}$ be a Lévy measure satisfying $A$ and $B$, $\beta > 0, \kappa > 0$. Then norms $|u| \nu, \kappa, \beta$, $\|u\| \nu, \kappa, \beta$ and $|u| \beta + \kappa, \infty$ are equivalent on $C^\infty_b (\mathbb{R}^d)$.

The same holds with $\nu \in \mathcal{A}^\alpha$ satisfying $A$ and $B$ if $\kappa \in \mathbb{N}$.

**Proof** Let $\beta, \kappa > 0$. By Lemma 7,

$$\left| (I - L^\nu)^\kappa u \right|_{\beta, \infty} \leq C |u|_{\beta + \kappa, \infty} \leq C \left[ \left| (I - L^\nu)^\kappa u \right|_{\beta, \infty} + |u|_0 \right]$$

$$= C |u| \nu, \beta, \kappa, u \in C^\infty_b (\mathbb{R}^d).$$

On the other hand, by Lemmas 1 and 7,

$$|u|_0 \leq C |u|_{\beta, \infty} \leq C |u|_{\beta + \kappa, \infty} \leq C \left| (I - L^\nu)^\kappa u \right|_{\beta, \infty},$$

$$\left| (L^\nu)^\kappa u \right|_{\beta, \infty} \leq C |u|_{\beta + \kappa, \infty} \leq C \left| (I - L^\nu)^\kappa u \right|_{\beta, \infty}, u \in C^\infty_b (\mathbb{R}^d).$$
Corollary 4 Let \( \nu \in \mathfrak{N}_{\text{sym}} \) and \( \pi \in \mathfrak{N} \) be a Lévy measure satisfying \( A \) and \( B \) such that \( w_\pi \sim w_\nu \). Then for any \( \kappa \in \mathbb{N}, \beta > 0 \), there are constants \( c, C > 0 \) so that

\[
\left| (L^\pi)^\kappa u \right|_{\beta, \infty} \leq C_1 |u|_{\nu; \kappa, \beta} \leq C_2 \left[ |(L^\pi)^\kappa u|_{\beta, \infty} + |u|_0 \right], u \in C^\infty_b \left( \mathbb{R}^d \right).
\]

Proof Indeed, by Corollary 3,

\[
|L^\pi \kappa u|_{\beta, \infty} \leq C |u|_{\kappa + \beta, \infty} \leq C |u|_{\nu; \kappa, \beta}
\]

\[
\leq C |u|_{\kappa + \beta, \infty} \leq C_2 \left[ |(L^\pi)^\kappa u|_{\beta, \infty} + |u|_0 \right], u \in C^\infty_b \left( \mathbb{R}^d \right).
\]

2.2.2 Extension of Norm Equivalence to \( \tilde{C}^{\beta}_{\infty\infty} \left( \mathbb{R}^d \right) \)

We extend the definition of \( (a - L^\nu)^\kappa \) and the norm equivalence (see Corollary 3 above) from \( C^\infty_b \left( \mathbb{R}^d \right) \) to \( \tilde{C}^{\beta}_{\infty\infty} \left( \mathbb{R}^d \right) \). We start with the following observation.

Remark 4 Let \( 0 < \beta' < \beta \). Then for each \( \epsilon > 0 \) there is a constant \( C_\epsilon > 0 \) so that

\[
|u|_{\beta', \infty} \leq \epsilon |u|_{\beta, \infty} + C_\epsilon |u|_0, u \in \tilde{C}^{\beta}_{\infty\infty} \left( \mathbb{R}^d \right).
\]

Indeed, For each \( \epsilon > 0 \) there is \( K > 1 \) so that \( w \left( N^{-j} \right)^{\beta - \beta'} \leq \epsilon \) if \( j \geq K \). Hence

\[
w \left( N^{-j} \right)^{\beta'} |u \ast \varphi_j|_0 = w \left( N^{-j} \right)^{\beta - \beta'} w \left( N^{-j} \right)^{-\beta} |u \ast \varphi_j|_0 \]

\[
\leq \epsilon |u|_{\beta, \infty} + \max_{k < K} w \left( N^{-k} \right)^{-\beta'} |u \ast \varphi_k|_0 \]

\[
\leq \epsilon |u|_{\beta, \infty} + C_\epsilon |u|_0.
\]

Proposition 2 Let \( \beta \in (0, \infty), u \in \tilde{C}^{\beta}_{\infty\infty} \left( \mathbb{R}^d \right), \) and

\[
u_n = \sum_{j=0}^{n} u \ast \varphi_j, n \geq 1.
\]

Then \( u_n \in C^\infty_b \left( \mathbb{R}^d \right) \),

\[
|u|_{\beta, \infty} \leq \liminf_n |u_n|_{\beta, \infty}, |u_n|_{\beta, \infty} \leq C |u|_{\beta, \infty}
\]

for some \( C > 0 \) that only depends on \( d, N \). Moreover, for any \( 0 < \beta' < \beta \),

\[
|u_n - u|_{\beta', \infty} \leq C \left( N^{-n} \right)^{(\beta - \beta')} |u|_{\beta, \infty}, n > 1.
\]

Proof Set \( u_n = \sum_{j=0}^{n} u \ast \varphi_j, n \geq 1 \). Obviously, \( u_n \in C^\infty_b \left( \mathbb{R}^d \right) \), \( n \geq 1 \), and by Lemma 1, \( u = \sum_{j=0}^{\infty} u \ast \varphi_j \) is a bounded continuous function. Since

\[
\varphi_k = \sum_{l=-1}^{1} \varphi_{k+l} \varphi_k, k \geq 1, \varphi_0 = (\varphi_0 + \varphi_1) \varphi_0,
\]

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we have for \( n > 1 \),
\[
(u - u_n) * \varphi_k = 0, \quad k < n,
\]
\[
(u - u_n) * \varphi_k = (u * \varphi_{k-1} + u * \varphi_k + u * \varphi_{k+1}) * \varphi_k, \quad k > n + 1,
\]
\[
(u - u_n) * \varphi_n = (u * \varphi_{n+1}) * \varphi_n,
\]
\[
(u - u_n) * \varphi_{n+1} = (u * \varphi_{n+1} + u * \varphi_{n+2}) * \varphi_{n+1}.
\]

Hence there is a constant \( C \) so that for all \( n > 1 \)
\[
| (u - u_n) * \varphi_j |_0 \leq C |u * \varphi_j |_0, \quad j \geq n
\]
\[
|u_n * \varphi_j |_0 \leq C |u * \varphi_j |_0, \quad j \geq 0, \quad n > 1,
\]
and
\[
\sup_{j < n} w \left( N^{-j} \right)^{-\beta} |u * \varphi_j |_0 = \sup_{j < n} w \left( N^{-j} \right)^{-\beta} |u_n * \varphi_j |_0 \leq |u_n |_{\beta, \infty}.
\]

Thus
\[
|u - u_n|_{\beta, \infty} = \sup_j w \left( N^{-j} \right)^{-\beta} |(u - u_n) * \varphi_j |_0
\]
\[
\leq C \sup_{j \geq n} w \left( N^{-j} \right)^{-\beta} |u * \varphi_j |_0 \leq C |u |_{\beta, \infty},
\]
and for any \( 0 < \beta' < \beta \),
\[
|u - u_n|_{\beta', \infty} = \sup_{j \geq n} w \left( N^{-j} \right)^{-\beta'} |(u - u_n) * \varphi_j |_0
\]
\[
= \sup_{j \geq n} w \left( N^{-j} \right)^{\beta - \beta'} w \left( N^{-j} \right)^{-\beta} |(u - u_n) * \varphi_j |_0
\]
\[
\leq w \left( N^{-n} \right)^{\beta - \beta'} |u - u_n|_{\beta, \infty} \leq C w \left( N^{-n} \right)^{\beta - \beta'} |u |_{\beta, \infty}.
\]

The statement follows. \( \square \)

Using the approximating sequence introduced in Proposition 2, we can extend \((-L)^{\kappa} u, (I - L)^{\kappa} u, 0 < \kappa < \beta\), to all \( u \in \tilde{C}^{\beta}_{\infty, \infty} (\mathbb{R}^d) \), \( \beta > 0 \).

**Proposition 3** Let \( \nu \) be a Lévy measure satisfying A and B, \( \beta > 0 \) and \( u \in \tilde{C}^{\beta}_{\infty, \infty} (\mathbb{R}^d) \). Let \( u_n \in C^\infty_b (\mathbb{R}^d) \) be an approximating sequence of \( u \) in Proposition 2. Then for each \( \kappa \in (0, \beta) \) there are bounded continuous functions, denoted \((I - L)^{\kappa} u, (-L)^{\kappa} u, 0 < \kappa < \beta\), and a constant \( C \) (independent of \( u \) and \( n \)) so that for any \( 0 < \beta' < \beta - \kappa \),
\[
\left| (I - L)^{\kappa} u_n - (I - L)^{\kappa} u \right|_{\beta', \infty} + \left| (I - L)^{\kappa} u - (I - L)^{\kappa} u_n \right|_{\beta', \infty} \leq C w \left( N^{-n} \right)^{\beta - \kappa - \beta'} |u |_{\beta, \infty} \to 0 \text{ as } n \to \infty.
\]

Moreover, for each \( \kappa \in (0, \beta) \) there is \( C > 0 \) independent of \( u \) in \( \tilde{C}^{\beta}_{\infty, \infty} (\mathbb{R}^d) \) so that
\[
\left| (I - L)^{\kappa} u \right|_{\beta - \kappa, \infty} \leq C |u |_{\beta, \infty}, \quad \left| (I - L)^{\kappa} u \right|_{\beta - \kappa, \infty} \leq C |u |_{\beta, \infty},
\]
and
\[
|u |_{\beta, \infty} \leq C \left[ \left| (I - L)^{\kappa} u \right|_{\beta - \kappa, \infty} + |u |_0 \right], \quad |u |_{\beta, \infty} \leq C \left| (I - L)^{\kappa} u \right|_{\beta - \kappa, \infty}.
\]
Proof Let \( u \in \tilde{C}^{\beta,\infty}(\mathbb{R}^d) \). By Proposition 2, there is a a sequence \( u_n \in C^\infty_b(\mathbb{R}^d) \) such that
\[
|u|_{\beta,\infty} \leq \liminf_n |u_n|_{\beta,\infty}, \quad |u_n|_{\beta,\infty} \leq C |u|_{\beta,\infty},
\]
and for each \( 0 < \kappa < \beta \), there is \( C \) so that (see Lemma 7 as well)
\[
\left| (-L^\nu)^\kappa u_n - (-L^\nu)^\kappa u_m \right|_{\beta' \infty} + \left| (I - L^\nu)^\kappa u_n - (I - L^\nu)^\kappa u_m \right|_{\beta' \infty} \\
\leq C \left| u_n - u_m \right|_{\beta'} \leq C \left[ w \left( N^{-n}(\beta - \kappa') + w \left( N^{-m}(\beta - \kappa') \right) \right) |u|_{\beta,\infty},
\]
\( \beta' \in (0, \beta - \kappa) \). Hence there are bounded continuous functions, denoted \((-L^\nu)^\kappa u, (I - L^\nu)^\kappa u\), so that
\[
\left| (-L^\nu)^\kappa u_n - (-L^\nu)^\kappa u \right|_{\beta' \infty} + \left| (I - L^\nu)^\kappa u_n - (I - L^\nu)^\kappa u \right|_{\beta' \infty} \\
\leq C w \left( N^{-n}(\beta - \kappa') \right) |u|_{\beta,\infty} \rightarrow 0 \text{ as } n \rightarrow \infty.
\]
Now, for each \( m > 1 \), and \( a = 0,1 \),
\[
\sup_{j \leq m} w \left( N^{-j} \right)^{\beta - \kappa} \left| (a - L^\nu)^\kappa u \ast \varphi_j \right|_0 = \\
= \lim_{n \rightarrow \infty} \sup_{j \leq m} w \left( N^{-j} \right)^{\beta - \kappa} |(a - L^\nu)^\kappa u_n \ast \varphi_j|_0 \\
\leq \sup_n \left| (a - L^\nu)^\kappa u_n \right|_{\beta - \kappa} \leq \sup_n |u_n|_{\beta,\infty} \leq C |u|_{\beta,\infty}.
\]
Hence \((a - L^\nu)^\kappa u \in \tilde{C}^{\beta - \kappa,\infty}(\mathbb{R}^d), a = 0,1 \), and Eq. 2.12 holds.

Now for every \( j \geq 0 \), we have
\[
\left[ (a - L^\nu)^\kappa u \ast \varphi_j \right] = \lim_n \left[ (a - L^\nu)^\kappa u_n \ast \varphi_j \right] = (a - L^\nu)^\kappa \left[ u \ast \varphi_j \right]
\]
uniformly. By the definition of the approximation sequence (see proof of Proposition 2),
\[
\left| (a - L^\nu)^\kappa \left[ u_n \ast \varphi_j \right] \right|_0 \leq C \left| (a - L^\nu)^\kappa \left[ u \ast \varphi_j \right] \right|_0, j \geq 0.
\]
Hence
\[
|u|_{\beta,\infty} \leq \liminf_n |u_n|_{\beta,\infty} \leq C \liminf_n \left| (I - L^\nu)^\kappa u_n \right|_{\beta - \kappa,\infty} \\
\leq C \left| (I - L^\nu)^\kappa u \right|_{\beta - \kappa,\infty},
\]
and similarly,
\[
|u|_{\beta,\infty} \leq \liminf_n |u_n|_{\beta,\infty} \leq C \liminf_n \left[ \left| (-L^\nu)^\kappa u_n \right|_{\beta - \kappa,\infty} + |u_n|_0 \right] \\
\leq C \left[ \left| (-L^\nu)^\kappa u \right|_{\beta - \kappa,\infty} + |u|_0 \right].
\]
The statement is proved. \( \square \)

**Proposition 4** Let \( \nu \in \mathcal{Q}^a_{sym} \) be a Lévy measure satisfying \( A \) and \( B \), \( \beta > 0, \kappa > 0 \). Then norms \( |u|_{v,\kappa,\beta}, \|u\|_{v,\kappa,\beta} \) and \( |u|_{\beta + \kappa,\infty} \) are equivalent on \( \tilde{C}^{\beta + \kappa,\infty}(\mathbb{R}^d) \).
Proof. We show the equivalence by repeating proof of Corollary 3 where the equivalence of the same norms on $C^\infty_b (\mathbb{R}^d)$ was derived. Only instead of Lemma 7 we use Proposition 3.

Remark 5. Let $\nu \in \mathcal{A}_\text{sym}^{\alpha}$ be a Lévy measure satisfying $A$ and $C$, $\beta \in \left(0, q^{-1}_1\right)$. The claim of Proposition 1 about the equivalence of the norms $|u|_\beta, \infty$ and $|u|_\beta$ can be easily extended from $C^\infty_b (\mathbb{R}^d)$ to $u \in \tilde{C}^\beta_{\infty, \infty} (\mathbb{R}^d)$.

3 Proof of Main Theorem

We assume in this section that $A$, $B$ and $C$ hold. First we solve the equation with smooth input functions.

Proposition 5. Let $\nu \in \mathcal{A}_\text{sym}^\alpha$, $\beta \in \left(0, 1\right)$, $\lambda \geq 0$. Assume that $f(t, x) \in C^\infty_b (HT)$. Then there is a unique solution $u \in C^\infty_b (HT)$ to

$$
\partial_t u(t, x) = L^\nu u(t, x) - \lambda u(t, x) + f(t, x),
$$

$$
u(t, x) = 0, \quad (t, x) \in [0, T] \times \mathbb{R}^d.
$$

Proof: Existence. Denote

$$
e^{-\lambda(t-s)} f \left( s, x + Z^\nu_t - Z^\nu_s \right) - f(s, x)
$$

$$= -\lambda \int^t_s F(r, Z^\nu_r) dr + \int^t_s \int_{\mathbb{R}^d} \chi_\alpha(y) y \cdot \nabla F r, Z^\nu_r \tilde{J} (dr, dy)
$$

$$+ \int^t_s \int_{\mathbb{R}^d} \left[ F(r, Z^\nu_r - y) - F(r, Z^\nu_r) - \chi_\alpha(y) y \cdot \nabla F (r, Z^\nu_r) \right] J(dr, dy).$$

Take expectation for both sides and use the stochastic Fubini theorem,

$$
e^{-\lambda(t-s)} \mathbb{E} f \left( s, x + Z^\nu_t - Z^\nu_s \right) - f(s, x)
$$

$$= -\lambda \int^t_s e^{-\lambda(r-s)} \mathbb{E} f \left( s, x + Z^\nu_r - Z^\nu_s \right) dr + \int^t_s L^\nu e^{-\lambda(t-s)} \mathbb{E} f \left( s, x + Z^\nu_r - Z^\nu_s \right) dr.$$

Integrate both sides over $[0, t]$ with respect to $s$ and obtain

$$
\int^t_0 e^{-\lambda(t-s)} \mathbb{E} f \left( s, x + Z^\nu_t - Z^\nu_s \right) ds - \int^t_0 f(s, x) ds
$$

$$= -\lambda \int^t_0 \int^r_0 e^{-\lambda(r-s)} \mathbb{E} f \left( s, x + Z^\nu_r - Z^\nu_s \right) ds dr
$$

$$+ \int^t_0 \int^r_0 L^\nu e^{-\lambda(t-s)} \mathbb{E} f \left( s, x + Z^\nu_r - Z^\nu_s \right) ds dr,$$

which shows $u(t, x) = \int^t_0 e^{-\lambda(t-s)} \mathbb{E} f \left( s, x + Z^\nu_t - Z^\nu_s \right) ds$ solves (3.1) in the integral sense. Obviously, as a result of the dominated convergence theorem and Fubini’s theorem, $u \in C^\infty_b (HT)$. And by the equation, $u$ is continuously differentiable in $t$. 

\[ \text{Springer} \]
UNIQUENESS Suppose there are two solutions $u_1, u_2$ solving the equation, then $u := u_1 - u_2$ solves
\[
\begin{align*}
\partial_t u (t, x) &= L^\nu u (t, x) - \lambda u (t, x), \\
u (0, x) &= 0.
\end{align*}
\] (3.2)

Fix any $t \in [0, T]$. Apply the Itô formula to $v (t-s, Z_s^\nu) := e^{-\lambda s} u (t-s, x+Z_s^\nu)$, $0 \leq s \leq t$, over $[0, t]$ and take expectation for both sides of the resulting identity, then
\[
u (t,x) = -\mathbb{E} \int_0^t e^{-\lambda s} \left[ (-\partial_t u - \lambda u + L^\nu u) (t-s, x+Z_s^\nu) \right] ds = 0.
\]

3.1 Hölder Estimates of the Smooth Solution

First we derive the estimates of the solution corresponding to a smooth input function.

Proposition 6 Let $\nu \in \mathbb{Q}^\alpha$, $\beta > 0$ and $A - C$ hold. Let $u \in C^\infty_b (H_T)$ be the unique solution $u$ to Eq. 3.1 with $f \in C^\infty_b (H_T)$. Then
\[
\sup_{s \in [0, T]} |u (s, \cdot)|_{\beta, \infty} \leq C_\rho_\lambda (T) \sup_{s \in [0, T]} |f (s, \cdot)|_{\beta, \infty},
\] (3.3)
\[
\sup_{s \in [0, T]} |u (s, \cdot)|_{1+\beta, \infty} \leq C [1 + \rho_\lambda (T)] \sup_{s \in [0, T]} |f (s, \cdot)|_{\beta, \infty}
\] (3.4)

and for any $\mu \in [0, 1]$, $t' < t \leq T$,
\[
|u (t, \cdot) - u (t', \cdot)|_{\mu+\beta, \infty} \leq C \left\{ (t-t')^{1-\mu} + [1 + \rho_\lambda (T)] |t - t'| \right\} \sup_{s \in [0, T]} |f (s, \cdot)|_{\beta, \infty},
\] (3.5)

where $\rho_\lambda (T) = \frac{1}{\lambda} \wedge T$, $C = C (p_1, q_1, \beta, c_0, d)$. The constant $C$ does not depend on $\lambda, f, T, \mu$.

Proof Since $f \in C^\infty_b (H_T)$, by Lemma 1,
\[
f (t, x) = (f (t, \cdot) \ast \varphi_0 (\cdot))(x) + \sum_{j=1}^{\infty} (f (t, \cdot) \ast \varphi_j (\cdot))(x)
\]
\[
= f_0 (t, x) + \sum_{j=1}^{\infty} f_j (t, x), (t,x) \in H_T.
\]

Accordingly, for $j \geq 0$,
\[
u_j (t, x) = u (t, x) \ast \varphi_j (x) = \int_0^t e^{-\lambda (t-s)} \mathbb{E} f_j (s, x+Z_{t-s}^\nu) ds, (t,x) \in H_T,
\]
is the solution to Eq. 3.1 with input $f_j = f \ast \varphi_j$. In terms of Fourier transform,
\[
\hat{u}_j(t, \xi) = \int_0^t \exp \{-(\lambda - \psi_\nu^{\ast}(\xi))(t-s)\} \hat{f}(s, \xi) \phi(N^{-j} \xi) \, ds
\]
\[
= \int_0^t e^{-\lambda(t-s)} \exp \{\psi N^{-j} \left(N^{-j} \xi\right) w(N^{-j})^{-1}(t-s)\} \phi(N^{-j} \xi) \hat{f}_j(s, \xi) \, ds, \quad j \geq 1.
\]

Denote $w_j = w(N^{-j})^{-1}$. Then for $j \geq 0$,
\[
u_j(t, x) = \int_0^t e^{-\lambda(t-s)} \int_{\mathbb{R}^d} H_j(t-s, x-y) f_j(s, y) \, dy \, ds, \quad t \in [0, T], \ x \in \mathbb{R}^d,
\]
with
\[
H_j(t, x) = N^{jd} E_{\tilde{\varphi}} \left(N^j x + Z_{\tilde{\varphi}_j}(t)\right), \ (t, x) \in H_T, \ j \geq 1,
\]
\[
H_0(t, x) = E_{\tilde{\varphi}_0}(x + Z^\nu t), \ (t, x) \in H_T.
\]

Hence
\[
\int_{\mathbb{R}^d} |H_j(t, x)| \, dx = \int_{\mathbb{R}^d} |G_j(t, x)| \, dx, \ i > 0, \ j \geq 0, \quad (3.6)
\]
with $G^0 = H^0$ and
\[
G_j(t, x) = E_{\tilde{\varphi}} \left(x + Z_{\tilde{\varphi}_j}(t)\right), \ (t, x) \in \mathbb{R}^d, \ j \geq 1.
\]
First we estimate the solution itself. For $j \geq 1$, by Lemma 6,
\[
|u_j(t, \cdot)|_0 \leq |f_j|_0 \int_0^t e^{-\lambda(t-s)} \int_{\mathbb{R}^d} |G_j(t-s, x)| \, dx \, ds
\]
\[
\leq |f_j|_0 \int_0^t e^{-\lambda(t-s)} e^{cw_j(t-s)} \, ds \leq C w_j^{-1} |f_j|_0.
\]
Directly,
\[
|u_0(t, \cdot)|_0 \leq |f_0|_0 \int_0^t e^{-\lambda(t-s)} \, ds \leq \left(\frac{1}{\lambda} + T\right) |f_0|_0.
\]
Hence
\[
|u|_{1+\beta, \infty} \leq C \left[1 + \left(\frac{1}{\lambda} + T\right)\right] |f|_{\beta, \infty}.
\]
Now we estimate time differences. For fixed $0 < t' < t \leq T, \ j \geq 0$,
\[
u_j(t, x) - \nu_j(t', x)
\]
\[
= \int_{t'}^t e^{-\lambda(t-s)} \int_{\mathbb{R}^d} H_j(t-s, x-y) f_j(s, y) \, dy \, ds
\]
\[
+ \left(e^{-\lambda(t-t')} - 1\right) \int_0^{t'} e^{-\lambda(t'-s)} \int_{\mathbb{R}^d} H_j(t-s, x-y) f_j(s, y) \, dy \, ds
\]
\[
+ \int_0^{t'} e^{-\lambda(t'-s)} \int_{\mathbb{R}^d} \left[H_j(t-s, x-y) - H_j(t'-s, x-y)\right] f_j(s, y) \, dy \, ds
\]
\[
= A_1^j(x) + A_2^j(x) + A_3^j(x), \ x \in \mathbb{R}^d.
\]
First, by Lemma 6, for \( j \geq 1 \),
\[
\left| A_j^1 \right|_0 \leq \int_{t'}^t e^{-\lambda(t-s)} \int_{\mathbb{R}^d} |G^j(t-s,y)| dy ds \left| f_j \right|_0 \\
\leq C \int_{t'}^t e^{-\lambda(t-s)} e^{-cw_j(t-s)} ds \left| f_j \right|_0 \leq C \int_{t'}^t e^{-cw_j(t-s)} ds \left| f_j \right|_0 \\
\leq Cw_j^{-1} \left[ 1 - e^{-cw_j(t-t')} \right] \left| f_j \right|_0.
\]
And
\[
\left| A_j^0 \right| \leq C \left| f_0 \right| \int_{t'}^t e^{-\lambda(t-s)} ds \leq C \left| f_j \right|_0 |t - t'|.
\]
By Eq. 3.6 and Lemma 6, for \( j \geq 1 \),
\[
\left| A_j^2 \right|_0 \leq \left( 1 - e^{-\lambda(t-t')} \right) \int_0^{t'} e^{-\lambda(t'-s)} \int_{\mathbb{R}^d} |G^j(t-s,y)| dy ds \left| f_j \right|_0 \\
\leq C \left( 1 - e^{-\lambda(t-t')} \right) \int_0^{t'} e^{-\lambda(t'-s)} e^{-cw_j(t-s)} ds \left| f_j \right|_0.
\]
Thus for \( j \geq 1 \),
\[
\left| A_j^2 \right|_0 \leq C \left( 1 - e^{-\lambda(t-t')} \right) \int_0^{t'} e^{-\lambda(t'-s)} ds \left| f_j \right|_0 \leq C \left| f_j \right|_0 |t - t'|.
\]
in the mean time,
\[
\left| A_j^2 \right|_0 \leq C \left| f_j \right|_0 \int_0^{t'} e^{-cw_j(t'-s)} ds \leq C \left| f_j \right|_0 w_j^{-1}.
\]
For \( j = 0 \),
\[
\left| A_j^0 \right|_0 \leq C \left( 1 - e^{-\lambda(t-t')} \right) \int_0^{t'} e^{-\lambda(t'-s)} ds \left| f_j \right|_0 \\
\leq C \left| t - t' \right| \lambda \int_0^{t'} e^{-\lambda(t'-s)} ds \left| f_j \right|_0 \leq C \left| f_j \right|_0 |t - t'|.
\]
At last, for \( j \geq 1 \),
\[
\left| A_j^3 \right|_0 \leq \left| f_j \right| \int_0^{t'} \int_{\mathbb{R}^d} \left| G^j(t-s,y) - G^j(t'-s,y) \right| dy ds.
\]
Note for \( s \leq t' \),
\[
G^j(t-s,y) - G^j(t'-s,y) = E \left[ \tilde{\phi} \left( y + Z_{\tilde{\nu}^{-j}_{w_j(t-s)}} \right) - \tilde{\phi} \left( y + Z_\nu^{\tilde{\nu}^{-j}_{w_j(t'-s)}} \right) \right] \\
= E \int_{w_j(t'-s)}^{t'-s} L^{\tilde{\nu}^{-j}_{w_j(t'-s)}} \tilde{\phi} \left( y + Z_\nu^{\tilde{\nu}^{-j}_{w_j(t'-s)}} \right) dr,
\]
and by Corollary 2,
\[
\int |G^j(t - s, y) - G^j(t' - s, y)| dy
\leq C \int_{w_j(t' - s)}^{w_j(t - s)} e^{-cr} dr \leq Ce^{-cw_j(t' - t)}. 
\]
Thus for \( j \geq 1 \),
\[
|A^j_3|_0 \leq C |f_j|_0 \left[ 1 - e^{-cw_j(t' - t)} \right] \int_0^{t'} e^{-cw_j(t' - s)} ds
\leq C w_j^{-1} |f_j|_0 \left[ 1 - e^{-cw_j(t' - t)} \right] \left[ 1 - e^{-cw_j(t')} \right]
\leq C |f_j|_0 w_j^{-1} \left( 1 - e^{-cw_j(t')} \right).
\]
In addition,
\[
|A^0_3|_0 \leq C |f_0|_0 \int_0^{t'} e^{-\lambda(t' - s)} ds \left| t - t' \right|
\leq C \left( \frac{1}{\lambda} \wedge T \right) |f_0|_0 \left| t - t' \right|.
\]
Summarizing,
\[
|u_0(t, \cdot) - u_0(t', \cdot)|_0 \leq C \left[ 1 + \left( \frac{1}{\lambda} \wedge T \right) \right] |f_0|_0 \left| t - t' \right|,
\]
and
\[
|u_j(t, \cdot) - u_j(t', \cdot)|_0
\leq C |f_j|_0 \left[ \left( |t - t'| \wedge w_j^{-1} \right) + w_j^{-1} \left( 1 - e^{-cw_j(t' - t)} \right) \right]
\leq C |f_j|_0 w_j^{-1} \left[ \left( |t - t'| \wedge w_j \right) \wedge 1 + \left( 1 - e^{-cw_j(t' - t)} \right) \right],
\]
which leads to
\[
|u_j(t, \cdot) - u_j(t', \cdot)|_0 \leq C w_j^{-\mu} \left( t - t' \right)^{1-\mu}, \quad \mu \in [0, 1], \; j \geq 1.
\]
Thus
\[
|u(t, \cdot) - u(t', \cdot)|_{\mu+\beta, \infty}
\leq C \sup_{s \in [0, T]} |f(s, \cdot)|_{\beta, \infty} \left\{ (t - t')^{1-\mu} + \left[ 1 + \left( \frac{1}{\lambda} \wedge T \right) \right] \left| t - t' \right| \right\}
\]
for any \( \mu \in [0, 1] \). The statement is proved.

3.2 General Hölder Inputs

Existence and Estimates  Given \( f \in \tilde{C}^\beta_{\infty, \infty}(HT) \), set
\[
f_n(t, \cdot) = \sum_{j=0}^n f(t, \cdot) * \varphi_j, \; t \in [0, T].
\]
By Proposition 2, $f_n \in C_b^\infty (H_T)$,

$$\sup_{s \in [0,T]} |f_n(s, \cdot)|_{\beta,\infty} \leq C \sup_{s \in [0,T]} |f(s, \cdot)|_{\beta,\infty},$$

$$\sup_{s \in [0,T]} |f(s, \cdot)|_{\beta,\infty} \leq \liminf_n \sup_{s \in [0,T]} |f_n(s, \cdot)|_{\beta,\infty},$$

and for any $0 < \beta' < \beta$,

$$|f_n - f|_0 \leq C \sup_{s \in [0,T]} |f_n(s, \cdot) - f(s, \cdot)|_{\beta',\infty} \to 0 \text{ as } n \to \infty.$$

According to Propositions 5 and 6, for each $f_n \in C_b^\infty (\mathbb{R}^d)$, there is a corresponding solution $u_n \in C_b^\infty (H_T)$:

$$u_n(t, x) = \int_0^t \left[ L^v u_n(r, x) - \lambda u_n(r, x) + f_n(r, x) \right] dr, \quad (t, x) \in [0, T] \times \mathbb{R}^d. \quad (3.9)$$

By Proposition 6,

$$\sup_{s \in [0,T]} \left| L^v u_m(s, \cdot) - L^v u_n(s, \cdot) \right|_{\beta',\infty} \leq C \sup_{s \in [0,T]} \left| L^v u_m(s, \cdot) - L^v u_n(s, \cdot) \right|_{\beta',\infty} \leq C \sup_{s \in [0,T]} \left| u_m(s, \cdot) - u_n(s, \cdot) \right|_{1+\beta',\infty}$$

$$\leq C \sup_{s \in [0,T]} \left| f_m(s, \cdot) - f_n(s, \cdot) \right|_{\beta',\infty} \to 0, \text{ as } m, n \to \infty$$

for all $\beta' \in (0, \beta)$, which by Lemma 1 implies that

$$|u_n - u_m|_0 + \left| L^v u_m - L^v u_n \right|_0 \to 0 \text{ as } m, n \to \infty.$$

So, there is $u \in \tilde{C}_{\infty,\infty}^{1+\beta}(H_T)$ for any $\beta' \in (0, \beta)$ such that

$$\sup_{s \in [0,T]} \left| u_n(s, \cdot) - u(s, \cdot) \right|_{1+\beta',\infty} \to 0 \text{ as } n \to \infty.$$

Passing to the limit in Eq. 3.9 we see that Eq. 3.9 holds for $u$. Let $\beta' \in (0, \beta)$ and $\beta - \beta' < q_1^{-1}$. Then

$$\sup_{s \in [0,T]} \left| (-L^v)^{1+\beta'} u_n(s, \cdot) \right|_{\beta-\beta',\infty} \leq C \sup_{s \in [0,T]} \left| u_n(s, \cdot) \right|_{1+\beta,\infty} \leq C \sup_{s \in [0,T]} \left| f_n(s, \cdot) \right|_{\beta,\infty} \leq C \sup_{s \in [0,T]} \left| f(s, \cdot) \right|_{\beta,\infty}$$

implies that

$$\left| (-L^v)^{1+\beta'} u_n(t, x) - (-L^v)^{1+\beta'} u_n(t, y) \right| \leq C \left| f \right|_{\beta,\infty} w(|x - y|)^{\beta-\beta'}, \quad x, y \in \mathbb{R}^d.$$

and passing to the limit we see that

$$\left| (-L^v)^{1+\beta'} u(t, x) - (-L^v)^{1+\beta'} u(t, y) \right| \leq C \left| f \right|_{\beta,\infty} w(|x - y|)^{\beta-\beta'}, \quad x, y \in \mathbb{R}^d.$$

Hence $(-L^v)^{1+\beta'} u \in \tilde{C}_{\infty,\infty}^{1+\beta'}(\mathbb{R}^d)$, i.e., $u \in \tilde{C}_{\infty,\infty}^{1+\beta}(H_T)$ and

$$\sup_{s \in [0,T]} \left| u(s, \cdot) \right|_{1+\beta,\infty} \leq C \sup_{s \in [0,T]} \left| f(s, \cdot) \right|_{\beta,\infty}.$$

The convergence of $u_n$ to $u$ implies easily other estimates.
Uniqueness Suppose there are two solutions $u_1, u_2 \in \mathcal{C}_1^1 + \beta \infty (H_T)$ to Eq. 1.1, then $u := u_1 - u_2$ solves

$$u(t, x) = \int_0^t [L^\nu u(r, x) - \lambda u(r, x)] dr, \quad (t, x) \in [0, T] \times \mathbb{R}^d.$$  \hspace{2cm} (3.10)

Let $g \in C_0^\infty (\mathbb{R}^d)$, $0 \leq g \leq 1$, $\int g dx = 1$. For $\varepsilon > 0$, set

$$u_\varepsilon(t, x) = \int_{\mathbb{R}^d} u(t, y) g_\varepsilon(x - y) dy = \int_{\mathbb{R}^d} u(t, x - y) g_\varepsilon(y) dy, \quad (t, x) \in H_T,$$

with $g_\varepsilon(x) = \varepsilon^{-d} g(x/\varepsilon), x \in \mathbb{R}^d$. Then $u_\varepsilon \in \mathcal{C}_1^\infty (H_T)$ solves (3.10). Hence $u_\varepsilon = 0$ for all $\varepsilon > 0$. Thus $u = 0$, the solution is unique.

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Appendix

We simply state a few results that were used in this paper. Let $\nu \in \mathcal{A}_\alpha$, and

$$\delta(r) = \delta_\nu(r) = \nu(|y| > r) > 0, r > 0,$$

$$w = w_\nu(r) = \delta^{-1}(r), r > 0, \lim_{r \to 0} w(r) = 0.$$

We assume that $w = w_\nu$ is an O-RV function at zero, i.e.,

$$r_1(\varepsilon) = \lim_{x \to 0} \frac{\delta(\varepsilon x)^{-1}}{\delta(x)^{-1}} < \infty, \varepsilon > 0.$$

By Theorem 2 in [2], the following limits exist:

$$p_1 = \lim_{\varepsilon \to 0} \frac{\log r_1(\varepsilon)}{\log \varepsilon} \leq q_1 = \lim_{\varepsilon \to \infty} \frac{\log r_1(\varepsilon)}{\log \varepsilon}. \hspace{2cm} (1)$$

Lemma 8 Assume $w = w_\nu$ is an O-RV function at zero.

a) Let $\beta > 0$ and $\tau > -\beta p_1$. There is $C > 0$ so that

$$\int_0^x t^\tau w(t)^{\beta} \frac{dt}{t} \leq C x^\tau w(x)^{\beta}, x \in (0, 1],$$

and $\lim_{x \to 0} x^\tau w(x)^{\beta} = 0$.

b) Let $\beta > 0$ and $\tau < -\beta q_1$. There is $C > 0$ so that

$$\int_x^1 t^\tau w(t)^{\beta} \frac{dt}{t} \leq C x^\tau w(x)^{\beta}, x \in (0, 1],$$

and $\lim_{x \to 0} x^\tau w(x)^{\beta} = \infty$.

c) Let $\beta < 0$ and $\tau > -\beta q_1$. There is $C > 0$ so that

$$\int_0^x t^\tau w(t)^{\beta} \frac{dt}{t} \leq C x^\tau w(x)^{\beta}, x \in (0, 1],$$

and $\lim_{x \to 0} x^\tau w(x)^{\beta} = 0$.  

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d) Let \( \beta < 0 \) and \( \tau < -\beta p_1 \). There is \( C > 0 \) so that

\[
\int_x^1 t^\tau w(t)^\beta \frac{dt}{t} = \int_1^{x^{-1}} t^{-\tau} \left( \frac{1}{t} \right)^\beta \frac{dt}{t} \leq C x^\tau w(x)^\beta, \ x \in (0, 1),
\]

and \( \lim_{x \to 0} x^\tau w(x)^\beta = \infty \).

**Proof** The claims follow easily by Theorems 3, 4 in [2]. Because of the similarities, we will prove c) only. Let \( \beta < 0 \) and \( \tau > -\beta q_1 \). Then

\[
\lim_{\varepsilon \to 0} \frac{w\left(\frac{1}{\varepsilon}\right)^\beta}{w\left(\frac{1}{\tau}\right)^\beta} = \lim_{x \to 0} \frac{w(x)^{-\beta}}{w\left(\frac{\varepsilon - 1}{x}\right)^{-\beta}} = \lim_{x \to 0} \frac{w\left(\varepsilon x\right)^{-\beta}}{w\left(x\right)^{-\beta}} = r_1(\varepsilon)\beta < \infty, \ \varepsilon > 0.
\]

Hence \( w\left(\frac{1}{\tau}\right)^\beta, \ t \geq 1, \) is an O-RV function at infinity with

\[
p = \lim_{\varepsilon \to 0} \frac{\log r_1(\varepsilon)^{-\beta}}{\log \varepsilon} = -\beta p_1 \leq -\beta q_1 = \lim_{\varepsilon \to \infty} \frac{\log r_1(\varepsilon)^{-\beta}}{\log \varepsilon} = q.
\]

Then for \( x \in (0, 1) \),

\[
\int_0^x t^\tau w(t)^\beta \frac{dt}{t} = \int_{x^{-1}}^\infty t^{-\tau} \left( \frac{1}{t} \right)^\beta \frac{dt}{t} \leq C x^\tau w(x)^\beta
\]

by Theorem 3 in [2], and \( \lim_{x \to 0} x^\tau w(x)^\beta = 0 \) according to Theorem 4 in [2].

**Corollary 5** Assume \( w = w_v \) is an O-RV function at zero and \( p_1 > 0 \). Let \( N > 1, \beta > 0 \). Then

\[
\sum_{j=0}^\infty w\left(N^{-j}\right)^\beta < \infty.
\]

**Proof** Indeed,

\[
\sum_{j=0}^\infty w\left(N^{-j}\right)^\beta \leq \int_0^\infty w\left(N^{-x}\right)^\beta dx \leq C \int_0^1 w(t)^\beta \frac{dt}{t} < \infty,
\]

because, by Lemma 8a),

\[
\int_0^x w(t)^\beta \frac{dt}{t} \leq C w(x)^\beta, \ x \in [0, 1].
\]

We will need some Lévy measure moment estimates.
Lemma 9 Let $v \in \mathbb{R}^\alpha$, and $w = w_v$ be an O-RV function at zero with $p_1, q_1$ defined in Eq. 1. Assume

$$0 < p_1 \leq q_1 < 1 \text{ if } \alpha \in (0, 1),$$
$$0 < p_1 \leq 1 \leq q_1 < 2 \text{ if } \alpha = 1,$$
$$1 < p_1 \leq q_1 < 2 \text{ if } \alpha \in (1, 2).$$

Then

(i)

$$\sup_{R \in (0, 1]} \int_{|y| \leq 1} (|y| \wedge 1) \tilde{v}_R (dy) < \infty \text{ if } \alpha \in (0, 1),$$
$$\sup_{R \in (0, 1]} \int_{|y| \leq 1} (|y|^2 \wedge 1) \tilde{v}_R (dy) < \infty \text{ if } \alpha = 1,$$
$$\sup_{R \in (0, 1]} \int_{|y| \leq 1} (|y|^2 \wedge |y|) \tilde{v}_R (dy) < \infty \text{ if } \alpha \in (1, 2).$$

(ii)

$$\inf_{R \in (0, 1]} \int_{|y| \leq 1} |y|^2 \tilde{v}_R (dy) \geq c_1,$$

for some $c_1 > 0$.

Proof (i) Let $\alpha \in (0, 1)$. Then by Lemma 8 (recall $v_R (dy) = v (Rdy), \tilde{v}_R (dy) = w (R) v_R (dy), \delta (R) = w (R)^{-1}$),

\[
\int_{|y| \leq 1} |y| v_R (dy) = R^{-1} \int_{|y| \leq R} |y| v (dy)
= R^{-1} \int_0^R [\delta (s) - \delta (R)] ds,
\]

and

\[
\int_{R_0^d} (|y| \wedge 1) \tilde{v}_R (dy) = R^{-1} w (R) \int_0^R w (s)^{-1} ds \leq C, \ R \in (0, 1].
\]

Let $\alpha = 1$. Then similarly using Lemma 8, we have

\[
\int \left( |y|^2 \wedge 1 \right) \tilde{v}_R (dy) = 2R^{-2}w (R) \int_0^R s^2 w (s)^{-1} \frac{ds}{s} \leq C, \ R \in (0, 1].
\]

Let $\alpha \in (1, 2)$. Then similarly,

\[
R^{-1} \int_{|y| > R} |y| v (dy) = R^{-1} \int_0^\infty \delta (s \vee R) ds
= \delta (R) + R^{-1} \int_R^\infty \delta (s) ds = \delta (R) + R^{-1} \int_0^\infty w (s)^{-1} ds
\]

and with $R \in (0, 1]$,

\[
R^{-2} \int_{|y| \leq R} |y|^2 v (dy) = 2R^{-2} \int_0^R s^2 [w (s)^{-1} - w (R)^{-1}] \frac{ds}{s}
= 2R^{-2} \int_0^R s^2 w (s)^{-1} \frac{ds}{s} - w (R)^{-1}.
\]
Hence, by Lemma 8,
\[
\int \left( |y|^2 \wedge |y| \right) v_R (dy) 
\leq 2R^{-2} \int_0^R s^2 w (s)^{-1} \frac{ds}{s} + R^{-1} \int_1^R w (s)^{-1} ds + R^{-1} \int_1^\infty w (s)^{-1} ds 
= 2R^{-2} \int_0^R s^2 w (s)^{-1} \frac{ds}{s} + R^{-1} \int_1^R w (s)^{-1} ds + R^{-1} \int_{|y|>1} |y| v (dy) 
\leq C w (R)^{-1}, \quad R \in (0, 1].
\]

(ii) By Eq. 2, for \( R \in (0, 1] \),
\[
\int_{|y|\leq 1} |y|^2 \tilde{v}_R (dy) = w (R) \int_{|y|\leq 1} |y|^2 v_R (dy) 
= 2R^{-2} \int_0^R s^2 \left[ \frac{w (R)}{w (Rs)} - 1 \right] ds \frac{s}{s} = 2 \int_0^1 s^2 \left[ \frac{w (R)}{w (Rs)} - 1 \right] ds.
\]
Hence, by Fatou’s lemma,
\[
\lim_{R \to 0} \int_{|y|\leq 1} |y|^2 \tilde{v}_R (dy) \geq 2 \int_0^1 s^2 \left[ \frac{1}{r_1 (s)} - 1 \right] ds = c_1 > 0
\]
if \(|\{s \in [0, 1] : r_1 (s) < 1\}| > 0\), because
\[
\lim \inf_{R \to 0} \frac{w (R)}{w (Rs)} = \frac{1}{\lim sup_{R \to 0} \frac{w (Rs)}{w (R)}} = \frac{1}{r_1 (s)}, \quad s \in (0, 1].
\]

According to [9], Chapter 3, 70-74, any Lévy measure \( v \in \mathcal{A}^\alpha \) can be disintegrated as
\[
v (\Gamma) = - \int_0^\infty \int_{S_{d-1}} \chi_\Gamma (rw) \Pi (r, dw) d\delta (r), \quad \Gamma \in \mathcal{B} \left( \mathbb{R}^d_0 \right),
\]
where \( \delta = \delta_v \), and \( \Pi (r, dw), \ r > 0 \), is a measurable family of measures on the unit sphere \( S_{d-1} \) with \( \Pi (r, S_{d-1}) = 1, \ r > 0 \). The following is a straightforward consequence of Lemma 9(ii).

**Corollary 6** Let \( v \in \mathcal{A}^\alpha \),
\[
v (\Gamma) = - \int_0^\infty \int_{S_{d-1}} \chi_\Gamma (rw) \Pi (r, dw) d\delta (r), \quad \Gamma \in \mathcal{B} \left( \mathbb{R}^d_0 \right),
\]
where \( \delta = \delta_v, \Pi (r, dw), \ r > 0 \), is a measurable family of measures on \( S_{d-1} \) with \( \Pi (r, S_{d-1}) = 1, \ r > 0 \). Assume \( w = w_v = \delta_v^{-1} \) is an O-RV function at zero satisfying assumptions of Lemma 9, and
\[
\inf_{|\xi|_1 = 1} \int_{S_{d-1}} \left| \frac{\xi}{|\xi|_1} \cdot w \right|^2 \Pi (r, dw) \geq c_0 > 0.
\]

Then assumption \( \mathcal{B} \) holds.
Proof Indeed, for $|\hat{\xi}| = 1$, $R \in (0, 1]$, with $C > 0$,

$$\int_{|y| \leq 1} |\hat{\xi} \cdot y|^2 v_R (dy) = R^{-2} \int_{|y| \leq R} |\hat{\xi} \cdot y|^2 v (dy) = -R^{-2} \int_0^R \int_{S_{d-1}} |\hat{\xi} \cdot w|^2 \Pi (r, dw) r^2 \delta (r)$$

$$\geq -c_0 R^{-2} \int_0^R r^2 \delta (r) = c_0 R^{-2} \int_{|y| \leq R} |y|^2 v (dy) = c_0 \int_{|y| \leq 1} |y|^2 v_R (dy).$$

Hence by Lemma 9(ii),

$$\inf_{R \in (0, 1]} \inf_{|\hat{\xi}| = 1} \int_{|y| \leq 1} |\hat{\xi} \cdot y|^2 \tilde{v}_R (dy) \geq c_0 \inf_{R \in (0, 1]} \int_{|y| \leq 1} |y|^2 \tilde{v}_R (dy)$$

$$\geq c_0 c_1 > 0.$$

Remark 6 Let $\alpha \in (0, 2)$, $\nu \in \mathfrak{X}^\alpha$, and $w_\nu$ be an O-RV function at zero, $p_1 > 0$. By Theorems 3 and 4 in [2], for any $\sigma \in (0, p_1)$,

$$\int_{r < |y| \leq 1} |y|^\sigma v (dy) = \sigma \int_r^1 t^\sigma w (t)^{-1} \frac{dt}{t} - \delta (1)$$

$$\geq cr^\sigma w (r)^{-1} - \delta (1) \to \infty$$

as $r \to 0$. Hence $p_1 \leq \alpha$. On the other hand for any $\sigma > q_1$, by Lemma 9,

$$\int_{0 < |y| \leq 1} |y|^\sigma v (dy) \leq \sigma \int_0^1 t^\sigma w (t)^{-1} \frac{dt}{t} < \infty,$$

and $\alpha \leq q_1$.

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