Well-Posedness of Hibler’s Dynamical Sea-Ice Model

Xin Liu¹,² · Marita Thomas¹ · Edriss S. Titi³,⁴,⁵

Received: 15 May 2021 / Accepted: 14 April 2022 / Published online: 20 May 2022
© The Author(s), under exclusive licence to Springer Science+Business Media, LLC, part of Springer Nature 2022

Abstract
This paper establishes the local-in-time well-posedness of solutions to an approximating system constructed by mildly regularizing the dynamical sea-ice model of W.D. Hibler, *Journal of Physical Oceanography*, 1979. Our choice of regularization has been carefully designed, prompted by physical considerations, to retain the original coupled hyperbolic-parabolic character of Hibler’s model. Various regularized versions of this model have been widely used for the numerical simulation of the circulation and thickness of the Arctic ice cover. However, due to the singularity in the ice rheology, the notion of solutions to the original model is unclear. Instead, an approximating system, which captures current numerical study, is proposed. The well-posedness theory of such a system provides a first-step groundwork in both numerical study and future analytical study.

Keywords Well-posedness · Ice rheology · Sea-ice · Hibler sea-ice model

Mathematics Subject Classification 35A01 · 35A02 · 35Q86 · 86A05

---

Xin Liu
stleonliu@gmail.com ; stleonliu@live.com

Marita Thomas
thomas@wias-berlin.de

Edriss S. Titi
Edriss.Titi@damtp.cam.ac.uk ; titi@math.tamu.edu

¹ Weierstrass-Institut für Angewandte Analysis und Stochastik, Leibniz-Institut im Forschungsverbund Berlin, Berlin, Germany
² Isaac Newton Institute for Mathematical Sciences, University of Cambridge, Cambridge CB3 0EH, UK
³ Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Cambridge CB3 0WA, UK
⁴ Department of Mathematics, Texas A&M University, College Station, TX 77840, USA
⁵ Department of Computer Science and Applied Mathematics, Weizmann Institute of Science, 76100 Rehovot, Israel
1 Introduction

1.1 The Sea-Ice Dynamic-Thermodynamic Model

Global climate changes, especially global warming, have large impact on the Arctic sea-ice, which has, in return, determining effects on not only global climate but also the local and global ecosystem, human activities etc. (see e.g., Thomas and Dieckmann 2010). If the problem is statically determinate, as pointed out in Schreyer (2001), a sea-ice dynamical model based on the viscous-plastic rheology was introduced in Hibler (1979), where the thickness of ice plays an essential role in the thermodynamics, and characterizes the strength of the ice interaction (i.e., ice rheology). The velocity of sea-ice $u$ is described by two-dimensional momentum balance equations, where the viscosity effect is characterized by a viscous-plastic rheology, and the strength of viscosity depends on the thickness of ice. The mean ice thickness $h$ and the compactness of ice $A$ are described by two continuity equations with thermodynamic source terms. That is, with a simplified ice rheology (see (1.11a), below), the above quantities are governed by the coupled system,

\begin{align}
    m(\partial_t u + u \cdot \nabla u) + \nabla p &= \text{div} \mathbb{S} + \mathcal{F}, \\
    \partial_t h + \text{div} (h u) &= S_h, \\
    \partial_t A + \text{div} (A u) &= S_A + A \text{div} u \cdot \chi_{\{A \geq 1\}},
\end{align}

with

\begin{align}
    \text{Ice mass } m &= \rho_{\text{ice}} h, \\
    \text{Pressure } p &= c_p h \exp(c_a A), \\
    \text{Viscoplastic stress } \mathbb{S} &= \frac{\nabla u + \nabla u^\top}{|\nabla u + \nabla u^\top|} + p \frac{\text{div} u}{|\text{div} u|}, \\
    \mathcal{F} &= -m \eta u^\perp + \phi_a + \phi_w, \\
    \text{Air flow stress } \phi_a &= \rho_a C_a |U_g| (U_g \cos \phi + U_g^\perp \sin \phi), \\
    \text{Water flow stress } \phi_w &= \rho_w C_w |U_w - u| [(U_w - u) \cos \theta \\
    &\quad + (U_w - u)^\perp \sin \theta], \\
    S_h &= \left[f(h/A)A + (1 - A)f(0)\right] \cdot \chi_{\{h > 0\}}, \\
    S_A &= \left((f(0))^+ / h_0\right)(1 - A) + (-A/(2h)) \cdot (S_h)^-.
\end{align}

Here $\chi_{\{h > 0\}}, \chi_{\{A \geq 1\}}$ are the characteristic functions of sets $\{h > 0\}, \{A \geq 1\}$, defined by

\begin{align}
    \chi_{\{h > 0\}} &= \begin{cases} 1 & h > 0, \\
    0 & h \leq 0 \end{cases}, \quad \chi_{\{A \geq 1\}} &= \begin{cases} 1 & A \geq 1, \\
    0 & A < 1 \end{cases}.
\end{align}
respectively. In addition, \( \mathbf{v}^\perp = (-v_2, v_1)^\top \) for any vector \( \mathbf{v} = (v_1, v_2)^\top \); \( \rho_{\text{ice}}, \rho_{a}, \rho_{w} \) represent the density of ice, air, and water, respectively; \( c_p, c_a, C_{a}, C_{w} \) are the thermodynamic constants; and \( \mathbf{U}_g, \mathbf{U}_w, \phi, \theta \) denote the velocity and stress angle of the air and the water, which, for simplicity of presentation, are assumed to be constant in this paper.

System (1.1) is used to study simulation the evolution of sea-ice in numerical study. For instance, the model successfully reproduces many of the observed features of the circulation and thickness of the Arctic ice cover in Hibler (1979). Hibler’s model of sea-ice dynamics is the foundation for further model developments, including the elastic-viscous-plastic sea-ice dynamics model as in Hunke (2001), the Maxwell elasto-brittle rheology model as in Dansereau et al. (2016), and models with leads, ridging or tensile failure as in Schreyer et al. (2006) and Wilchinsky and Feltham (2012). See Pritchard (2001) for a summary of classical models with different descriptions of ice distribution, and Bouillon et al. (2013), Hunke and Dukowicz (1997), Mehlmann and Richter (2017), Parkinson and Washington (1979), Rampal et al. (2016), Herman (2016), Tsamados et al. (2013), Coon et al. (1998), Palmer and Johnston (2001) and the references therein for further model development and computational investigation.

Despite the steady advances that have been made in the modeling and simulation of sea ice, mathematical analysis is much less developed in this field. It is the main objective of this paper to provide rigorous mathematical analysis of Hibler’s model as a first step in this direction and as a basis for further investigation of next-generation sea-ice rheologies.

In particular, the fundamental problem of well-posedness of solutions to system (1.1) is widely open, which is related to the validity of the model as pointed out by Schreyer (2001). In Gray and Killworth (1995) and Gray (1999), the authors study the loss of hyperbolicity of the linearized system of the original Hibler’s model around divergent flows, and show that the system is ill-posed. This work is further discussed in Dukowicz (1997), Lipscomb et al. (2007), Guba et al. (2013) and Sirven and Tremblay (2015) from various perspectives. These studies do not contradict the local well-posedness result established in this paper, for the following reasons:

- Instead of the original Hibler’s model, we consider a regularization of Hibler’s model, which, as shown below in the paper, preserves the parabolicity of the momentum equations in a Sobolev space with enough regularity. This is very different from the hyperbolic equations considered in the ill-posedness studies;
- Instead of linear analysis, we consider the nonlinear well-posedness theory for regularized Hibler’s model in the Hadamard sense, including the existence, the uniqueness, and the continuous dependency on initial data in the suitable Sobolev space as shown in Theorem 1.1.

We would like to point out that the main challenge in establishing the well-posedness theory is the singularity arising in the stress tensor (1.2c) when \( |\nabla \mathbf{u}| \to 0^+ \). In fact, among the numerical investigations, such singularity is usually truncated, i.e. regularized, by replacing it with its strictly positive approximation (e.g., \( \max\{|\nabla \mathbf{u}|, \Delta_{\text{min}}\} \) or \( \sqrt{|\nabla \mathbf{u}|^2 + \varepsilon^2} \)).

Notably, we would like to point out an investigation of very singular diffusion equations in Giga et al. (1998) and Giga and Giga (2010), where the authors discuss
the notion of solutions to

$$\partial_t u = \text{div} \left( \frac{\nabla u}{|\nabla u|} \right).$$

Similarly, the positive one-homogeneity of the potential related to (1.2c) calls for a subdifferential formulation of the problem, however set in the Eulerian frame. We leave such investigation to our future study.

In this paper, due to the obstacles mentioned above, we propose to study the following regularized approximating problem of (1.1): for \( \varepsilon, \omega \in (0, 1) \),

$$m(\partial_t u + u \cdot \nabla u) + \nabla p = \text{div} S_{\varepsilon} + F, \quad (1.3a)$$
$$\partial_t h + \text{div} (hu) = S_{h, \omega}, \quad (1.3b)$$
$$\partial_t A + \text{div} (Au) = S_{A, \omega} + A \text{div} u \cdot \chi_{A^\omega}, \quad (1.3c)$$

where \( m, \ p, \) and \( F \) are as in (1.2a), (1.2b), and (1.2d), respectively, and

1. \( S_{\varepsilon} = S_{\varepsilon}(p, \nabla u) := p \frac{\nabla u + \nabla u^\top}{\sqrt{|\nabla u + \nabla u^\top|^2 + \varepsilon^2}} + p \frac{\text{div} u \|_2}{\sqrt{|\text{div} u|^2 + \varepsilon^2}}, \quad (1.4a)$$
2. \( S_{h, \omega} := [f(h/(A + \omega)))A + (1 - A)f(0)] \chi_{\{h > 0\}}, \quad (1.4b)$$
3. \( S_{A, \omega} := \frac{(f(0))^+}{h_0}(1 - A) - \frac{A}{2h} \cdot \sqrt{|S_{h, \omega}|^2 + \omega^2} - S_{h, \omega}, \quad (1.4c)$$
4. \( \chi_{A^\omega} := 1 - \frac{(1 - A)^+}{(1 - A)^+ + \omega}. \quad (1.4d)$$

To be more precise, we will establish the local in time well-posedness of strong solutions to (1.3) in domain \( \Omega := T^2 \subset \mathbb{R}^2 \):

**Theorem 1.1** Consider initial data

$$\quad (u, h, A)|_{t=0} = (u_{in}, h_{in}, A_{in}) \in (H^3(\Omega))^3 \quad (1.5)$$

to system (1.3), satisfying

$$0 < h \leq h_{in} \leq \bar{h} < \infty, \quad \text{and} \quad 0 \leq A_{in} \leq 1. \quad (1.6)$$

In addition, we assume that

$$\bar{f} \leq f \leq \overline{f}, \quad |f'| + |f''| + |f'''| \leq M_f, \quad (1.7)$$

for some constants \( f, \overline{f} \in \mathbb{R}, \ M_f \in (0, \infty) \). Then there exists a unique strong solution \( (u, h, A) \) to system (1.3) in \([0, T] \times \Omega\), for some \( T \in (0, \infty) \) depending on initial
data, with
\[ u, h, A \in L^\infty(0, T; H^3(\Omega)) \cap L^2(0, T; H^4(\Omega)), \]
\[ \partial_t u, \partial_t h, \partial_t A \in L^\infty(0, T; L^2(\Omega)), \]
and
\[ \| u, h, A \|_{L^\infty(0, T; H^3(\Omega))} + \| u \|_{L^2(0, T; H^4(\Omega))} \]
\[ + \| \partial_t u, \partial_t h, \partial_t A \|_{L^\infty(0, T; L^2(\Omega))} \leq C_{\text{in}}, \] (1.9)
where \( C_{\text{in}} \in (0, \infty) \) is some positive constant depending only on initial data. Moreover, the solution is stable with respect to perturbation of initial data.

Now, let us explain our strategy. Instead of directly constructing solutions to system (1.3), we consider another regularized system, parametrized by \((\mu, \lambda, \iota, \nu) \in (0, 1)^4:\)

\[
m(\partial_t u + u \cdot \nabla u) + \nabla p = \text{div} S_{\epsilon, \mu, \lambda} - \iota \Delta^2 u + \mathcal{F},
\]
\[ \partial_t h + \text{div} (hu) = S_{h, \omega, \nu}, \] (1.10b)
\[ \partial_t A + \text{div} (Au) = S_{A, \omega, \nu} + \text{div} u \cdot \chi^\omega_A, \] (1.10c)

where \( m, p, \mathcal{F}, \) and \( \chi^\omega_A \) are as in (1.3) and (1.4), and

\[
S_{\epsilon, \mu, \lambda} := S_{\epsilon} + S_{\mu, \lambda},
\]
\[ S_{h, \omega, \nu} := \left[ f(h^+/A^+ + \omega)A + (1 - A)f(0) \right] \cdot \chi^\nu_h, \] (1.11b)
\[ S_{A, \omega, \nu} := \left( f(0) \right)^+ \frac{h^+}{h_0 + \nu} (1 - A) - \frac{A}{2h^+ + \nu} \cdot \sqrt{\left| S_{h, \omega, \nu} \right|^2 + \omega^2} - S_{h, \omega, \nu}, \] (1.11c)

with \( S_{\epsilon} \) as in (1.3) and

\[
S_{\mu, \lambda} := \mu (\nabla u + \nabla u^\top) + \lambda \text{div} u \mathbb{I}_2,
\]
\[ \chi^\nu_h := \frac{h^+}{h^+ + \nu}. \] (1.11e)

We will construct solutions to system (1.10) through a contraction mapping argument. That is, we consider a “linearization” of (1.10), and establish a contraction mapping with respect to \( L^2 \) topology with bounds in a smooth function space. Then with a uniform-in-\((\mu, \lambda, \iota, \nu)\) estimate, we will be able to pass the limit \((\mu, \lambda, \iota, \nu) \to (0^+, 0^+, 0^+, 0^+)\), and eventually construct the strong solution to (1.3). The proof of Theorem 1.1 is then finished by showing the uniqueness and continuous dependency on the initial data. We would like to mention that the key ingredient in establishing the
well-posedness of solutions involves showing the monotonicity of $S_\epsilon(\cdash)$ in $\nabla u$, which is not trivially obvious due to the fact that $S_\epsilon(\cdash)$ is nonlinear in $\nabla u$. In particular, we will require the inequality of the type
\[
(S_\epsilon(p_1, \nabla u_1) - S_\epsilon(p_2, \nabla u_2)) : (\nabla u_1 - \nabla u_2) \gtrsim |\nabla (u_1 - u_2)|^2 + \cdots.
\]
We successfully establish this inequality by writing $S_\epsilon(p_1, \nabla u_1) - S_\epsilon(p_2, \nabla u_2)$ in a symmetric form (see (4.29), below).

We would like to make some remarks before going into details of the proof. Our ice rheology (1.4a) is a simplified version of the one from Hibler (1979). For some technical reasons, we are not sure whether Theorem 1.1 will apply to the original ice rheology from Hibler (1979). We have not successfully established a proper uniform-in-$\epsilon$ estimates of the solutions to (1.3). Therefore, we have not yet been able to establish a proper notion of solutions to the original system (1.1). However, our approximation (1.3) agrees with the most common numerical approaches to (1.1), which, as we explain before, is restricted to a truncated ice rheology. Thus, in this sense, our analytical results provide a solid ground for current numerical schemes of (1.1). Another issue is that we only consider the case when $h_{in} \geq \bar{h} > 0$, i.e., there is no absence of ice in the domain of study. To carry out the limit $\bar{h} \to 0^+$, more comprehensive a priori estimates are required. We leave this to future study.

Recently we have learnt an independent study by Brandt et al. (2021) about similar model, where the domain boundary and boundary conditions are taken into account instead of periodic domain. It is worth pointing out that in our regularized system (1.3) the governing equations for the evolution of $h$ and $A$ remains hyperbolic, and therefore system (1.3) is a mixed type system, while the regularized system in Brandt et al. (2021) is parabolic in all its components. In particular, due to the hyperbolicity, system (1.3) is expected to have a completely different long-time dynamics than those investigated in Brandt et al. (2021). Moreover, the additional dissipation introduced in Brandt et al. (2021) allows the authors to show global existence for small initial data.

This paper is organized as follows. In the next subsection, we will summarize some notations used in this paper. In Sect. 2, we will detail the approximation scheme to (1.10). In Sect. 3, we establish the well-posedness of solutions to (1.10) via a contraction mapping argument. Finally in Sect. 4, we establish the uniform-in-$(\mu, \lambda, \iota, \nu)$ estimates, and pass to the limit $(\mu, \lambda, \iota, \nu) \to (0^+, 0^+, 0^+, 0^+)$ to show the existence of solutions to (1.3). The well-posedness of solutions is then established in Sect. 4.3

### 1.2 Notations

We use $L^p(\cdash)$ and $H^s(\cdash)$ to denote the standard Lebesgue and Sobolev spaces, respectively. For any functional space $\mathcal{X}$ and functions $\psi, \phi, \ldots$, we denote by
\[
\|\psi, \phi, \ldots\|_{\mathcal{X}} := \|\psi\|_{\mathcal{X}} + \|\phi\|_{\mathcal{X}} + \cdots.
\]
In addition,

$$\psi^+ := \begin{cases} \psi & \text{if } \psi \geq 0, \\ 0 & \text{if } \psi < 0 \end{cases}, \quad \psi^- = \psi^+ - \psi.$$ 

Let $\partial \in \{\partial_x, \partial_y\}$. For any multi-index $(\alpha_1, \alpha_2) \in (\mathbb{Z}^+)^2$, denote by $\partial^{\alpha} := \partial_1^{\alpha_1} \partial_2^{\alpha_2}$ with $\alpha = \alpha_1 + \alpha_2$. Throughout this paper, we use the notation $X \lesssim Y$ to represent $X \leq CY$ for some generic constant $C \in (0, \infty)$, which may be different from line to line. We use $C_{a,b,...}$ to emphasize the dependency on the quantities $a, b, \ldots$. In addition, by $\mathcal{H}(\ldots)$, it represents a generic bounded function of the arguments.

2 An Approximation Scheme to Solve (1.10)

2.1 A “Linearization” of (1.10)

Given $u^o$, assumed to be smooth enough, we consider first the following coupled hyperbolic system

\begin{align}
\partial_t h_m + \text{div} (h_m u^o) &= S_{h_m, o, v}, \\
\partial_t A_m + \text{div} (A_m u^o) &= S_{A_m, o, v} + \text{div} u^o \cdot \chi_{A_m}^{o},
\end{align}

(2.1) where $S_{h_m, o, v}, S_{A_m, o, v}, \chi_{h_m}^{o},$ and $\chi_{A_m}^{o}$ are defined as in (1.11b), (1.11c), (1.11e), and (1.4d), with $h$ and $A$ replaced by $h_m$ and $A_m$, respectively. Here we use the subscript $m$ (short for ‘mapping’) and the superscript $o$ (short for ‘origin’) to label outputs and inputs in our contraction mapping.

We claim that, at least locally in time, there exists a unique solution $(h_m, A_m)$ to (2.1a) and (2.1b) with proper initial data, for smooth enough $u^o$. $(h_m, A_m)$ can be arbitrarily regular, provided that $u^o$ and initial data are regular enough. We leave the investigation of the regularity of $(h_m, A_m)$ in the subsequent sections.

We remark that such claims follow from the standard well-posedness theory of hyperbolic equations (see, e.g., Majda 1984). Hence the proof is omitted.

Let $(h_m, A_m)$ be the solution to (2.1a) and (2.1b) as above, and consider the following equation:

$$\rho_{\text{ice}} h_m \partial_t u_m + \iota \Delta^2 u_m = -\rho_{\text{ice}} h_m u^o \cdot \nabla u^o - \nabla p_m + \text{div} S_{E, \mu, \lambda, m} + \mathcal{F}_m,$$

(2.1c)

where $p_m, S_{E, \mu, \lambda, m},$ and $\mathcal{F}_m$ are defined as in (1.2b), (1.11a), and (1.2d), with $h$, $A$, and $u$ replaced by $h_m$, $A_m$, and $u^o$, respectively.

To solve the linear equation (2.1c) by, e.g., a Galerkin method, one will need to deal with the possible degeneracy of $h_m$. For this, we subsequently show that for $u^o$ smooth enough, with appropriate initial data, $h_m$ and $A_m$ satisfy certain non-degeneracy property.
2.2 Non-negativity and Uniform Bound of $A_m$: $0 \leq A_m \leq 1$

In this subsection, we show that $0 \leq A_m \leq 1$ for a smooth enough $u^\omega$. In fact, we only require that

$$\text{div } u^\omega \in L^1(0, T; L^\infty(\Omega)),$$

(2.2)

for some $T > 0$.

Non-negativity of $A_m$:

Taking the $L^2$-inner product of (2.1b) with $(-A_m^-)$ leads to, after applying integration by parts in the resultant

$$\frac{1}{2} \frac{d}{dt} \| A_m^- \|^2_{L^2(\Omega)} = \int \left( \frac{1}{2} - \frac{(1 - A_m^+)^+}{(1 - A_m)^+ + \omega} \right) \text{div } u^\omega \| A_m^- \|^2 dx$$

$$+ \int S_{A_m,\omega,\nu} (-A_m^-) dx \lesssim \| \text{div } u^\omega \|_{L^\infty(\Omega)} \| A_m^- \|^2_{L^2(\Omega)}. \quad (2.3)$$

Therefore, applying Grönwall’s inequality to (2.3) yields

$$\| A_m^- \|^2_{L^2(\Omega)} \leq e^{C \int_0^t \| \text{div } u^\omega(s) \|_{L^\infty(\Omega)} ds} \| A_m^- \|^2_{L^2(\Omega)} = 0,$$

which implies

$$A_m \geq 0.$$

Non-negativity of $1 - A_m$:

Consider the following equation for $1 - A_m$, derived from (2.1b):

$$\partial_t (1 - A_m) = -S_{A_m,\omega,\nu} - u^\omega \cdot \nabla (1 - A_m) + A_m \text{div } u^\omega \frac{(1 - A_m)^+}{(1 - A_m)^+ + \omega}.$$

(2.4)

As before, taking the $L^2$-inner product of (2.4) with $[-(1 - A_m)^-]$, after applying integration by parts, leads to

$$\frac{1}{2} \frac{d}{dt} \|(1 - A_m)^-\|^2_{L^2(\Omega)} = \int \left( \frac{1}{2} \text{div } u^\omega \|(1 - A_m)^-\|^2 + S_{A_m,\omega,\mu} (1 - A_m^-) \right) dx$$

$$- \int A_m \text{div } u^\omega \frac{(1 - A_m)^+}{(1 - A_m)^+ + \omega} (1 - A_m^-) dx,$$
which yields
\[
\frac{d}{dt} \| (1 - A_m)^{-} \|_{L^2(\Omega)}^2 \lesssim \| \text{div} \ u^o \|_{L^\infty(\Omega)} \| (1 - A_m)^{-} \|_{L^2(\Omega)}^2.
\] (2.5)

Then as before, after applying Grönwall’s inequality to (2.5), one can conclude
\[
A_m \leq 1.
\]

### 2.3 Non-Negativity, Lower and Upper Bounds of \( h_m \)

Let \( h, \bar{h} \in [0, \infty) \) be the lower and upper bounds of \( h \in \), respectively, i.e., \(-0 \leq h \leq h_{\in\min} \leq \bar{h} < \infty \) (see (1.6)). In this section, we will show that
\[
\frac{1}{4}h \leq h_m \leq 4\bar{h}
\]
locally in time. Again we assume that \( u^o \) has the regularity (2.2).

**Non-negativity of \( h_m \):**

After applying the \( L^2 \)-inner product of (2.1a) with \((-h_m^-)\) and applying integration by parts in the resultant, one has
\[
\frac{1}{2} \frac{d}{dt} \| h_m^- \|_{L^2(\Omega)}^2 = -\frac{1}{2} \int |h_m^-|^2 \text{div} \ u^o \, dx \lesssim \| \text{div} \ u^o \|_{L^\infty(\Omega)} \| h_m^- \|_{L^2(\Omega)}^2,
\] (2.6)
since the term \( S_{h_m,\omega,v} (h_m^-) \) vanishes. Therefore, applying Grönwall’s inequality to (2.6), as before in (2.3), eventually implies
\[
h_m \geq 0.
\]

**Lower and upper bounds of \( h_m \):**

Since \( A_m \in [0, 1] \), one has \(|S_{h_m,\omega,v}| \leq 3(|f| + |f|)\). Then following the characteristic method, since \( h_m \geq 0 \), one has
\[
\partial_t (e^{-\int_0^t \| \text{div} \ u^o \|_{L^\infty(\Omega)} \, ds} h_m) + \text{u}^o \cdot \nabla (e^{-\int_0^t \| \text{div} \ u^o \|_{L^\infty(\Omega)} \, ds} h_m) \leq 3(|f| + |f|) e^{-\int_0^t \| \text{div} \ u^o \|_{L^\infty(\Omega)} \, ds}.
\]

Thus, integrating in the above inequation along the characteristic path given by \( u^o \) yields
\[
h_m (x, t) \leq (\bar{h} + 3(|f| + |f|) t) \times e^{\int_0^t \| \text{div} \ u^o(s) \|_{L^\infty(\Omega)} \, ds}.
\] (2.7)

Similarly, one can show that
\[
h_m (x, t) \geq (h - 3(|f| + |f|) t) \times e^{-\int_0^t \| \text{div} \ u^o(s) \|_{L^\infty(\Omega)} \, ds}.
\] (2.8)
Then it immediately follows that \( \frac{1}{4} h \leq h_m \leq 4 \overline{h} \) provided that the following conditions are satisfied:

\[
0 < t \leq \begin{cases} 
\frac{h}{6(|\mathcal{J}| + |\mathcal{F}|)} & \text{if } h > 0, \\
\frac{\overline{h}}{3(|\mathcal{J}| + |\mathcal{F}|)} & \text{if } h = 0,
\end{cases}
\]

and \( e^{\int_0^t \| \text{div} u^o(s) \|_{L^\infty(\Omega)} \, ds} \leq e^{t^{1/2} \left( \int_0^t \| u^o(s) \|_{H^3(\Omega)}^2 \, ds \right)^{1/2}} \leq 2. \) (2.9)

### 2.4 Non-vanishing Total Ice Mass

Due to the fact that \( |S_{h_n,\omega,v}| \leq 3(|\mathcal{J}| + |\mathcal{F}|) \), one can show immediately after integrating (2.1a), that

\[
\frac{d}{dt} \int h_m \, dx \leq 3(|\mathcal{J}| + |\mathcal{F}|)|\Omega|.
\]

Therefore,

\[
\frac{1}{2} \int h_{in} \, dx \leq \int h_m \, dx \leq 2 \int h_{in} \, dx,
\]

provided

\[
t \leq \frac{\int h_{in} \, dx}{6(|\mathcal{J}| + |\mathcal{F}|)|\Omega|}.
\]

### 2.5 Well-Posedness of (2.1c) with Strictly Positive Ice Mass

Consider \( h_{in} \geq h > 0 \). Then we have shown in Sect. 2.3 that \( h_m \geq h/4 > 0 \) locally in time. Then during this local time, (2.1c) is a non-degenerate biharmonic evolutionary equation. Then following the standard Galerkin method, one can establish the well-posedness of strong solutions to (2.1c), provided that \( u^o \) is sufficiently smooth. We omit the details here and refer interesting readers to Evans (2010, section 7).

### 3 Well-Posedness of Solutions to (1.10) with \( h > 0 \) and \( \iota > 0 \) fixed

In this section, we aim at showing that the map defined by

\[
\mathcal{M} : u^o \mapsto u_m,
\]

(3.1)
where $u_m$ is the unique solution to (2.1c) with $h_m$ and $A_m$ being solutions to (2.1a) and (2.1b), respectively, is bounded in $\mathcal{X}_{T^*}$ and contracting with contraction constant $1/2$ in $L^\infty(0, T^*; L^2(\Omega)) \cap L^2(0, T^*; H^3(\Omega))$, where

$$\mathcal{X}_{T^*}:=\{u|u \in L^\infty(0, T^*; H^2(\Omega)) \cap L^2(0, T^*; H^3(\Omega)), \partial_t u, \nabla^4 u \in L^2(0, T^*; L^2(\Omega))\}.$$ (3.2)

for some $T^* \in (0, \infty)$ to be determined. Throughout this section, unless stated otherwise, the initial data for $u_m, u^o, h, A$ are assumed to be $u_{in}, u_{in}, h_{in}, A_{in}$, given in Theorem 1.1, respectively.

Consequently, one can apply the Banach fixed-point theorem, i.e., the contraction mapping theorem, to show the existence of solutions to system (1.10).

Let $c_{in} \in (0, \infty)$ be the bound of the initial data defined by

$$\|\nabla h_{in}, \nabla A_{in}\|_{L^4(\Omega)} + \|u_{in}\|_{H^2(\Omega)} \leq c_{in}$$ (3.3)

### 3.1 Uniform Bounds

Let $u^o \in \mathcal{X}_{T^*}$ satisfy

$$\sup_{0 \leq s \leq t} \|u^o(s)\|^2_{H^2(\Omega)} + \int_0^t \left(\|\partial_t u^o(s)\|^2_{L^2(\Omega)} + \|u^o(s)\|^2_{H^3(\Omega)}\right) ds \leq c_o,$$ (3.4)

with $t \in [0, T^*]$, for some $c_o \in (0, \infty)$ to be determined later.

**Estimates for $h_m$ and $A_m$**

Aside from the point-wise estimates deduced in Sects. 2.2 and 2.3, we shall need a uniform $H^1$-estimate for $A_m$ and $h_m$.

We record the equation after applying $\partial \in \{\partial_x, \partial_y\}$ to (2.1a), as follows:

$$\partial_t \partial h_m + u^o \cdot \nabla \partial h_m + \partial u^o \cdot \nabla h_m + \partial h_m \text{div} u^o + h_m \text{div} \partial u^o = \partial S_{h_m, o, v}.$$ (3.5)

Then taking the $L^2$-inner product of (3.5) with $4|\partial h_m|^2 \partial h_m$ leads to, after applying integration by parts,

$$\frac{d}{dt} \|\partial h_m\|^4_{L^4(\Omega)} = -3 \int \text{div} u^o |\partial h_m|^4 dx$$

$$- 4 \int (\partial u^o \cdot \nabla h_m + h_m \text{div} \partial u^o) |\partial h_m|^2 \partial h_m dx$$

$$+ 4 \int \partial S_{h_m, o, v} |\partial h_m|^2 \partial h_m dx$$

$$\lesssim \|\nabla u^o\|_{L^\infty(\Omega)} \|\nabla h_m\|^4_{L^4(\Omega)} + \|h_m\|_{L^\infty(\Omega)} \|\nabla^2 u^o\|_{L^4(\Omega)} \|\nabla h_m\|^2_{L^4(\Omega)} + \int |\partial S_{h_m, o, v}| |\partial h_m|^2 \partial h_m dx.$$ (3.6)
Meanwhile, simple calculation shows that
\[
\left| \partial S_{h,m,\omega,v} \right| \lesssim \left( \frac{1}{\omega} + \frac{1}{v^{1/2}} \right) \left| \partial h_m \right| + \left( 1 + \frac{|h_m|}{\omega^2} \right) \left| \partial A_m \right| ,
\]
where we have used (1.7). Consequently, one concludes from (3.6) that
\[
\frac{d}{dt} \left\| \nabla h_m \right\|_{L^4(\Omega)}^4 \lesssim \left( \left\| \nabla u^0 \right\|_{L^\infty(\Omega)} + \frac{1}{\omega} + \frac{1}{v} \right) \left\| \nabla h_m \right\|_{L^4(\Omega)}^4
+ \left( 1 + \frac{|h_m|}{\omega^2} \right) \left\| \nabla h_m \right\|_{L^4(\Omega)}^3
+ \left\| h_m \right\|_{L^\infty(\Omega)} \left\| \nabla^2 u^0 \right\|_{L^4(\Omega)} \left\| \nabla h_m \right\|_{L^4(\Omega)}^3 , \tag{3.7}
\]

The estimate for \( \nabla A_m \) is obtained from (2.1b) in a similar fashion, we record it here:
\[
\frac{d}{dt} \left\| \nabla A_m \right\|_{L^4(\Omega)}^4 \lesssim \left( \left\| \nabla u^0 \right\|_{L^\infty(\Omega)} + \frac{\left\| \nabla u^0 \right\|_{L^\infty(\Omega)}}{\omega} \right) \left\| \nabla A_m \right\|_{L^4(\Omega)}^4
+ \left( 1 + \frac{|h_m|}{\omega^2} \right) \left\| \nabla A_m \right\|_{L^4(\Omega)}^3
+ \left( \frac{1}{\sqrt{v}} + \frac{1}{\omega^2} \right) \left\| \nabla h_m \right\|_{L^4(\Omega)} \left\| \nabla A_m \right\|_{L^4(\Omega)}^3 + \left\| \nabla^2 u^0 \right\|_{L^4(\Omega)} \left\| \nabla A_m \right\|_{L^4(\Omega)}^3 , \tag{3.8}
\]
where we have used the fact that \( A_m \in [0, 1] \).

After combining (3.7) and (3.8) and applying Grönwall’s inequality, one can derive that
\[
\sup_{0 \leq s \leq t} \left\| \nabla h_m(s), \nabla A_m(s) \right\|_{L^4(\Omega)}^4 \leq e^{H_{h,A,1}(t)} \left( \left\| \nabla h_{in}, \nabla A_{in} \right\|_{L^4(\Omega)}^4 + G_{h,A,1}(t) \right) , \tag{3.9}
\]
where
\[
H_{h,A,1}(t) := C_{\omega,v} \int_0^t \left( 1 + \left\| \nabla u^0(s) \right\|_{L^\infty(\Omega)} + \left\| h_m(s) \right\|_{L^\infty(\Omega)} \right)
+ \left\| \nabla^2 u^0(s) \right\|_{L^4(\Omega)} + \left\| h_m(s) \right\|_{L^\infty(\Omega)} \left\| \nabla^2 u^0(s) \right\|_{L^4(\Omega)} \right) ds , \tag{3.10}
\]
\[
G_{h,A,1}(t) := \int_0^t \left( 1 + \left\| h_m(s) \right\|_{L^\infty(\Omega)} + \left\| \nabla^2 u^0(s) \right\|_{L^4(\Omega)} \right) ds . \tag{3.11}
\]

On the other hand, in direct consequence of equations (2.1a) and (2.1b), one has
\[
\left\| \partial_t h_m, \partial_t A_m \right\|_{L^4(\Omega)} \leq C \left( 1 + 1/v + \left\| \nabla u^0 \right\|_{L^4(\Omega)} \right)
+ \left\| h_m \right\|_{L^\infty(\Omega)} \left\| \nabla u^0 \right\|_{L^4(\Omega)} + \left\| u^0 \right\|_{L^\infty(\Omega)} \left\| \nabla h_m, \nabla A_m \right\|_{L^4(\Omega)} , \tag{3.12}
\]
\hspace{1cm} Springer
where we have used the fact that $0 \leq A_m \leq 1$ and (1.7).

Estimates for $u_m$

Taking the $L^2$-inner product of (2.1c) with $2u_m + 2\partial_t u_m - 2\Delta u_m$ leads to, after applying integration by parts,

$$
\frac{d}{dt} \rho_{\text{ice}} h_m^{1/2} u_m^{1/2} \nabla^2 u_m + \rho_{\text{ice}} h_m^{1/2} \nabla^2 u_m^2 + 2\|\rho_{\text{ice}} h_m^{1/2} \partial_t u_m + \nabla h_m \cdot \nabla u_m^2 + \|L^2(\Omega) \|_2 \leq \|

= \int \rho_{\text{ice}} \partial_t h_m \|u_m\|^2 dx + 2 \int \rho_{\text{ice}} (\nabla h_m \cdot \nabla) u_m \cdot \partial_t u_m dx

- \int \rho_{\text{ice}} \partial_t h_m \|\nabla u_m\|^2 dx - 2 \int \rho_{\text{ice}} h_m (u^o \cdot \nabla) u^o \cdot (u_m + \partial_t u_m - \Delta u_m) dx

- 2 \int \nabla p_m \cdot (u_m + \partial_t u_m - \Delta u_m) dx + 2 \int \mathcal{F}_m \cdot (u_m + \partial_t u_m - \Delta u_m) dx

+ 2 \int \text{div} S_{\varepsilon, \mu, \lambda, m} \cdot (u_m + \partial_t u_m - \Delta u_m) dx.

(3.13)

We obtain the following estimates for the $R_j$ terms by applying Hölder’s inequality and the Sobolev embedding inequality:

$$
R_1 \lesssim \|\partial_t h_m\|_{L^2(\Omega)} \|u_m\|^2_{L^4(\Omega)},

R_2 \lesssim \|\partial_t u_m\|_{L^2(\Omega)} \|\nabla u_m\|_{L^4(\Omega)} \|\nabla h_m\|_{L^4(\Omega)},

R_3 \lesssim \|\partial_t h_m\|_{L^2(\Omega)} \|\nabla u_m\|^2_{L^4(\Omega)},

R_4 \lesssim \|h_m\|_{L^\infty(\Omega)} \|u^o\|_{L^4(\Omega)} \|\nabla u^o\|_{L^4(\Omega)} \|\partial_t u_m, u_m, \nabla^2 u_m\|_{L^2(\Omega)},

R_5 \lesssim \|\nabla p_m\|_{L^2(\Omega)} \|\partial_t u_m, u_m, \nabla^2 u_m\|_{L^2(\Omega)},

R_6 \lesssim \left(1 + \|u^o\|_{L^2(\Omega)} + \|h_m\|_{L^\infty(\Omega)} \|u^o\|_{L^2(\Omega)}\right) \|\partial_t u_m, u_m, \nabla^2 u_m\|_{L^2(\Omega)},

R_7 \lesssim \left(\frac{1}{\varepsilon} \|p_m\|_{L^2(\Omega)} \|\nabla^2 u^o\|_{L^2(\Omega)} + (\mu + \lambda) \|\nabla^2 u^o\|_{L^2(\Omega)} + \|\nabla p_m\|_{L^2(\Omega)} \right)

\times \|\partial_t u_m, u_m, \nabla^2 u_m\|_{L^2(\Omega)}.

To deduce the above estimates, consider $\varepsilon > 0$ and let $t$ satisfy (2.9) and (2.11). Therefore, the estimates in Sect. 2.3 guarantee that $0 < 1/4\varepsilon \leq h_m \leq 4\varepsilon < \infty$. Consequently, (3.13) yields, after applying the Sobolev embedding inequality and
Hölder’s inequality,

\[
\frac{d}{dt} \left\| \rho^{1/2}_{\text{ice}} h^{1/2}_m \mathbf{u}_m \right\|_{L^2(\Omega)}^2 + 2 \left\| \rho^{1/2}_{\text{ice}} h^{1/2}_m \partial_t \mathbf{u}_m, t^{1/2} \nabla^2 \mathbf{u}_m \right\|_{L^2(\Omega)}^2 \\
\leq C_{\varepsilon, \mu, \lambda, h} \left( \left\| \partial_t h_m \right\|_{L^4(\Omega)} + \left\| \nabla h_m, \nabla A_m \right\|_{L^4(\Omega)}^2 \right) \\
\times \left( \left\| \rho^{1/2}_{\text{ice}} h^{1/2}_m \mathbf{u}_m, \rho^{1/2}_{\text{ice}} h^{1/2}_m \nabla \mathbf{u}_m, t^{1/2} \nabla^3 \mathbf{u}_m \right\|_{L^2(\Omega)}^2 + 1 \right).
\]  

(3.14)

Furthermore, consider \( t \) small enough such that

\[
H_{h, A, t}(t) + G_{h, A, t}(t) \leq C_{\omega, \nu, \bar{h}, \varepsilon, \mu, \lambda, \omega, \nu, \varepsilon} \left( t^{1/2} + \left( \int_0^t \left\| \nabla \mathbf{u}^o(s) \right\|_{H^2(\Omega)}^2 ds \right)^{1/2} \right) \\
\leq C_{\omega, \nu, \bar{h}} t^{1/2} \left( t^{1/2} + \varepsilon_o^{1/2} \right) \leq 1,
\]

where we have applied Hölder’s inequality. Then (3.9) and (3.12) imply that, after applying the Sobolev embedding inequality,

\[
\left\| \nabla h_m, \nabla A_m, \partial_t h_m, \partial_t A_m \right\|_{L^4(\Omega)} \leq C_{\omega, \nu, \bar{h}, \varepsilon} \left( 1 + \varepsilon_o^{1/2} \right).
\]

(3.16)

Consequently, (3.14) yields the following estimate:

\[
\sup_{0 \leq s \leq t} \left\| \mathbf{u}_m(s) \right\|_{H^2(\Omega)}^2 + \int_0^t \left( \left\| \partial_t \mathbf{u}_m(s) \right\|_{L^2(\Omega)}^2 + \left\| \mathbf{u}_m(s) \right\|_{H^3(\Omega)}^2 \right) ds \\
\leq C_{\varepsilon, \mu, \lambda, \omega, v, h, \bar{h}, \varepsilon} \left[ \left( \frac{C_{\varepsilon, \mu, \lambda, \omega, v, h, \bar{h}, \varepsilon}}{C_{\varepsilon, \mu, \lambda, \omega, v, h, \bar{h}, \varepsilon}^2 \left( 1 + \varepsilon_o \right) t} - 1 \right)^2 \left( 1 + \varepsilon_o \right) + 1 \right] \\
\leq C_{\varepsilon, \mu, \lambda, \omega, v, h, \bar{h}, \varepsilon} \left[ 2 \left( \frac{2 C_{\varepsilon, \mu, \lambda, \omega, v, h, \bar{h}, \varepsilon} - 1}{C_{\varepsilon, \mu, \lambda, \omega, v, h, \bar{h}, \varepsilon}^2 \left( 1 + \varepsilon_o \right)} \right)^2 + 1 \right],
\]

(3.17)

provided that \( t \) is small enough and where we have made the choice

\[
\varepsilon_o := C_{\varepsilon, \mu, \lambda, \omega, v, h, \bar{h}, \varepsilon} \left[ 2 \left( \frac{2 C_{\varepsilon, \mu, \lambda, \omega, v, h, \bar{h}, \varepsilon} - 1}{C_{\varepsilon, \mu, \lambda, \omega, v, h, \bar{h}, \varepsilon}^2 \left( 1 + \varepsilon_o \right)} \right)^2 + 1 \right].
\]

(3.18)

where the right-hand side is as in (3.17). Then (2.9), (2.11), (3.15), and (3.17) imply that, there exists \( T^* \in (0, \infty) \) such that

\[
\sup_{0 \leq s \leq t} \left\| \mathbf{u}_m(s) \right\|_{H^2(\Omega)}^2 + \int_0^t \left( \left\| \partial_t \mathbf{u}_m(s) \right\|_{L^2(\Omega)}^2 + \left\| \mathbf{u}_m(s) \right\|_{H^3(\Omega)}^2 \right) ds \leq \varepsilon_o,
\]

(3.19a)

and

\[
\frac{1}{4} h \leq h_m \leq 4 \bar{h}, \quad \frac{1}{2} \int h_{in} dx \leq \int h_m dx \leq 2 \int h_{in} dx,
\]

(3.19b)
for $t \in [0, T^*]$. In addition, using equation (1.10), it is easy to obtain

$$\int_0^t \|\Delta^2 u_m(s)\|_{L^2(\Omega)}^2 \, ds \leq C_{\overline{r}, \mu, \lambda, \epsilon} c_o. \tag{3.20}$$

Therefore, $\mathcal{M}$, defined in (3.1), maps $\mathcal{X}_{T^*}$ into itself for such choices of $T^*$ and $c_o$.

We remark here that, $c_0 \to \infty$ as $\iota \to 0^+$, i.e., the estimates we obtain here depend on $\iota > 0$. We will remove the dependency of $\iota$ in Sect. 4.

### 3.2 Contraction Mapping and Well-Posedness

For $j = 1, 2$, consider $u^j \in \mathcal{X}_{T^*}$ satisfying (3.4), and let $h_{m,j}, A_{m,j},$ and $u_{m,j} = \mathcal{M}(u^j)$, be the solutions to (2.1a), (2.1b), and (2.1c), respectively, with $u^o$ replaced by $u^j$ and with the same initial data. Then we have the estimates of $h_{m,j}, A_{m,j},$ and $u_{m,j}$ as in Sects. 2.2 and 2.3, as well as (3.16) and (3.19a).

In the following, let $\sigma \in (0, 1)$ be a constant to be determined later. Denote by

$$\delta h_m := h_{m,1} - h_{m,2}, \quad \delta A_m := A_{m,1} - A_{m,2},$$
$$\delta u_m := u_{m,1} - u_{m,2}, \quad \delta u^o := u^o_1 - u^o_2. \tag{3.21}$$

The notations

$$\delta p_m, \delta S_{\epsilon, \mu, \lambda, m}, \delta F_m, \delta S_{h_{m,\omega, v}}, \delta S_{A_{m,\omega, v}}, \delta \chi_{A_m}^o,$$

have similar meanings. Then $\delta h_m, \delta A_m, \delta u_m$ satisfy

$$\partial_t \delta h_m + \text{div} (\delta h_m \mathbf{u}^o_1) + \text{div} (h_{m,2} \delta \mathbf{u}^o) = \delta S_{h_{m,\omega, v}}, \tag{3.22a}$$
$$\partial_t \delta A_m + \text{div} (\delta A_m \mathbf{u}^o_1) + \text{div} (A_{m,2} \delta \mathbf{u}^o) = \delta S_{A_{m,\omega, v}},$$
$$+ \delta A_m \text{div} \mathbf{u}^o_1 \cdot \mathbf{X}_{A_m,1} + A_{m,2} \text{div} \delta \mathbf{u}^o \cdot \mathbf{X}_{A_m,1} + A_{m,2} \text{div} \mathbf{u}^o_2 \cdot \delta \mathbf{X}_{A_m}^o. \tag{3.22b}$$
$$\rho_{\text{ice}} h_{m,1} \partial_t \delta \mathbf{u}_m + \rho_{\text{ice}} \delta h_m \partial_t \mathbf{u}_{m,2} + \iota \Delta^2 \delta \mathbf{u}_m = -\rho_{\text{ice}} h_{m,1} \mathbf{u}^o_1 \cdot \nabla \delta \mathbf{u}^o$$
$$- \rho_{\text{ice}} h_{m,1} \delta \mathbf{u}^o \cdot \nabla \mathbf{u}^o_2 - \rho_{\text{ice}} \delta h_m \mathbf{u}^o_2 \cdot \nabla \mathbf{u}^o_2 - \nabla \delta p_m + \text{div} \delta S_{\epsilon, \mu, \lambda, m} + \delta F_m. \tag{3.22c}$$

After taking the $L^2$-inner product of (3.22a) and (3.22b) with $4|\delta h_m|^2 \delta h_m$ and $4|\delta A_m|^2 \delta A_m$, respectively, and applying integration by parts in the resultant, one has
\[
\begin{align*}
\frac{d}{dt} \| \delta h_m, \delta A_m \|^4_{L^4(\Omega)} &= -3 \int_{\Omega} \text{div } \mathbf{u}_1^\rho (|\delta h_m|^4 + |\delta A_m|^4) \, dx \\
R_8 &= -4 \int (\delta \mathbf{u}^\rho \cdot \nabla h_{m,2}) |\delta h_m|^2 \delta h_m + \delta \mathbf{u}^\rho \cdot \nabla A_{m,2} |\delta A_m|^2 \delta A_m \, dx \\
R_9 &= -4 \int (h_{m,2} \text{div } \delta \mathbf{u}^\rho |\delta h_m|^2 \delta h_m + A_{m,2} \text{div } \delta \mathbf{u}^\rho |\delta A_m|^2 \delta A_m) \, dx \\
R_{10} &= +4 \int \text{div } \mathbf{u}_1^\rho |\delta A_m|^4 \chi_{A_{m,1}} \, dx + 4 \int A_{m,2} \text{div } \delta \mathbf{u}^\rho |\delta A_m|^2 \delta A_m \chi_{A_{m,1}} \, dx \\
R_{11} &= +4 \int A_{m,2} \text{div } \mathbf{u}_2^\rho |\delta A_m|^2 \delta A_m \delta \chi_{A_{m}} \, dx + 4 \int \delta S_{h_{m,\omega,v}} |\delta h_m|^2 \delta h_m \, dx \\
R_{12} &= +4 \int \delta S_{h_{m,\omega,v}} |\delta A_m|^2 \delta A_m \, dx \\
R_{13} &= +4 \int \delta h_{m,\omega,v} |\delta A_m|^2 \delta A_m \, dx \\
R_{14} &= +4 \int \delta \mathbf{u}_1^\rho |\delta A_m|^4 \chi_{A_{m,1}} \, dx + 4 \int A_{m,2} \text{div } \delta \mathbf{u}^\rho |\delta A_m|^2 \delta A_m \chi_{A_{m,1}} \, dx \\
R_{15} &= +4 \int A_{m,2} \text{div } \mathbf{u}_2^\rho |\delta A_m|^2 \delta A_m \delta \chi_{A_{m}} \, dx + 4 \int \delta S_{h_{m,\omega,v}} |\delta h_m|^2 \delta h_m \, dx \\
R_{16} &= +4 \int \delta S_{h_{m,\omega,v}} |\delta A_m|^2 \delta A_m \, dx \\
R_{17} &= +4 \int \delta h_{m,\omega,v} |\delta A_m|^2 \delta A_m \, dx.
\end{align*}
\]

In the following, we sketch the estimates of the \( R_j \) terms by applying Hölder’s inequality and the Sobolev embedding inequality:

\[
R_8 + R_{11} + R_{13} \lesssim \left( \| \text{div } \mathbf{u}_1^\rho \|_{L^\infty(\Omega)} + \left( \frac{1}{\omega} + \frac{1}{\omega^2} \right) \| \text{div } \mathbf{u}_2^\rho \|_{L^\infty(\Omega)} \right) \\
\times \| \delta h_m, \delta A_m \|^4_{L^4(\Omega)}, \\
R_9 \lesssim \| \delta \mathbf{u}^\rho \|_{L^\infty(\Omega)} \| \nabla h_{m,2}, \nabla A_{m,2} \|_{L^4(\Omega)} \| \delta h_m, \delta A_m \|^3_{L^4(\Omega)}, \\
R_{10} + R_{12} \lesssim (\tilde{h} + 1) \| \text{div } \delta \mathbf{u}^\rho \|_{L^4(\Omega)} \| \delta h_m, \delta A_m \|^3_{L^4(\Omega)}, \\
R_{14} + R_{15} \lesssim C_{\tilde{h},\omega,v} \| \delta h_m, \delta A_m \|^4_{L^4(\Omega)},
\]

where we have used the identity

\[
\delta \left( \frac{g}{g + \epsilon} \right) = \frac{\delta g}{g_1 + \epsilon} - \frac{g_2 \delta g}{(g_1 + \epsilon)(g_2 + \epsilon)}
\]

for \( g = (1 - A_m)^+ = 1 - A_m \) in the estimate of \( \delta \chi_{A_m}^\omega \) in \( R_{13} \). In view of (3.23) and (3.24), one has

\[
\frac{d}{dt} \| \delta h_m, \delta A_m \|^2_{L^4(\Omega)} \leq C_{\epsilon,\omega,v} \| \delta h_m, \delta A_m \|^2_{L^4(\Omega)} + \sigma \| \delta \mathbf{u}^\rho \|^2_{H^2(\Omega)},
\]
where we have used (3.4) and (3.9). Consequently, applying Grönwall’s inequality to (3.25) yields

\[
\sup_{0 \leq s \leq t} \| \delta h_m(s), \delta A_m(s) \|^2_{L^4(\Omega)} \leq \sigma \left( \int_0^t \| \delta u^o(s) \|^2_{H^2(\Omega)} \, ds \right) e^{C_{\sigma, \varepsilon, \omega, \rho, \tau_0, \varepsilon_m} \left( t + t^{1/2} \right)},
\]

(3.26)

where we have also employed Young’s inequality.

Taking the \( L^2 \)-inner product of (3.22c) with \( 2 \delta u_m \) and applying integration by parts in the resultant yields

\[
\rho_{\text{ice}} \frac{d}{dt} \| \delta u_m \|^2_{L^2(\Omega)} + 2 \| \nabla^2 \delta u_m \|^2_{L^2(\Omega)} = \rho_{\text{ice}} \int \partial_t h_{m,1} \| \delta u_m \|^2 \, dx
\]

\[
-2 \int \rho_{\text{ice}} \delta h_m \partial_t \delta u_m \cdot \delta u_m \, dx - 2 \int \rho_{\text{ice}} h_{m,1} (u^o_1 \cdot \nabla) \delta u^o \cdot \delta u_m \, dx
\]

\[
-2 \int \rho_{\text{ice}} h_{m,1} (\delta u^o \cdot \nabla) u^o_2 \cdot \delta u_m \, dx - 2 \int \rho_{\text{ice}} \delta h_m (u^o_2 \cdot \nabla) u^o_2 \cdot \delta u_m \, dx
\]

\[
+ 2 \int \delta \rho_m \text{div} \delta u_m \, dx + 2 \int \text{div} \delta S_{\varepsilon, \mu, \lambda, \varepsilon_m} \cdot \delta u_m \, dx
\]

\[
+ 2 \int \delta F_m \cdot \delta u_m \, dx.
\]

(3.27)

In the following, again, we sketch the estimates for the terms \( \mathcal{R}_j \) by applying Hölder’s inequality, the Gagliardo–Nirenberg inequality, and the Sobolev embedding inequality:

\[
\mathcal{R}_{16} \lesssim \| \partial_t h_{m,1}, \| L^4(\Omega) \| \delta u_m \|^{3/2}_{L^2(\Omega)} \| \delta u_m \|^{1/2}_{H^1(\Omega)},
\]

\[
\mathcal{R}_{17} \lesssim \| \partial_t \delta u_m, 2 \| L^2(\Omega) \| \delta h_m \| L^4(\Omega) \| \delta u_m \|^{1/2}_{L^2(\Omega)} \| \delta u_m \|^{1/2}_{H^1(\Omega)},
\]

\[
\mathcal{R}_{18} \lesssim \| h \| L^2(\Omega) \| \nabla \delta u^o \| L^2(\Omega) \| \delta u_m \| L^2(\Omega),
\]

\[
\mathcal{R}_{19} \lesssim \| h \| L^2(\Omega) \| \delta u^o \|^{1/2}_{L^2(\Omega)} \| \delta u^o \|^{1/2}_{H^1(\Omega)} \| \nabla u^o_2 \|_{L^4(\Omega)} \| \delta u_m \|_{L^2(\Omega)},
\]

\[
\mathcal{R}_{20} \lesssim \| \delta h_m \|_{L^4(\Omega)} \| u^o_2 \|_{H^2(\Omega)} \| \nabla u^o_2 \|_{L^4(\Omega)} \| \delta u_m \|_{L^2(\Omega)},
\]

\[
\mathcal{R}_{21} \lesssim (1 + \bar{h}) \| \delta h_m \|_{L^4(\Omega)} \| \delta A_m \|_{L^2(\Omega)} \| \nabla \delta u_m \|_{L^2(\Omega)},
\]

\[
\mathcal{R}_{23} \lesssim (1 + \bar{h} + \sum_{j=1}^2 \| u^o_j \|_{H^2(\Omega)})(\| \delta u^o \|_{L^2(\Omega)} + \| \delta h_m \|_{L^2(\Omega)}) \| \delta u_m \|_{L^2(\Omega)}.
\]
To estimate $\mathcal{R}_{22}$, we rewrite it as

\[
\mathcal{R}_{22} = 2 \int \text{div} \left[ \mu (\nabla \delta u^o + (\nabla \delta u^o)^\top) + \lambda \text{div} \delta u^o \right] \cdot \delta u_m \, dx \\
+ 2 \int \text{div} \left[ p_{m,1} \delta \left( \frac{\nabla \delta u^o}{\sqrt{|\nabla \delta u^o| + (\nabla \delta u^o)^\top|^2 + \varepsilon^2}} \right) \right] + p_{m,1} \delta \left( \frac{\text{div} u^o_2}{\sqrt{|\text{div} u^o_2|^2 + \varepsilon^2}} \right) \cdot \delta u_m \, dx \\
+ 2 \int \delta p_m \left[ \frac{\nabla u^o_2 + (\nabla u^o_2)^\top}{\sqrt{|\nabla u^o_2 + (\nabla u^o_2)^\top|^2 + \varepsilon^2}} + \frac{\text{div} u^o_2}{\sqrt{|\text{div} u^o_2|^2 + \varepsilon^2}} \right] : \nabla \delta u_m \, dx.
\]

Therefore, applying Hölder’s inequality and the Sobolev embedding inequality implies

\[
\mathcal{R}_{22} \lesssim C_{\epsilon, \mu, \lambda} \left( 1 + \bar{h} + \| \nabla h_{m,1} \|_{L^4(\Omega)} \right) \| \delta u^o \|_{H^2(\Omega)} \| \delta u_m \|_{L^2(\Omega)} \\
+ C_{\epsilon} \bar{h} \sum_{j=1}^2 \| \nabla^2 u^o_j \|_{L^2(\Omega)} \| \nabla \delta u^o \|_{L^4(\Omega)} \| \delta u_m \|_{L^2(\Omega)} \frac{1}{\sqrt{2}} \| \delta u_m \|_{H^1(\Omega)} \\
+ (1 + \bar{h}) \| \delta h_m \|_{L^2(\Omega)} \| \delta A_m \|_{L^2(\Omega)} \| \nabla \delta u_m \|_{L^2(\Omega)},
\]

where we have used the identity

\[
\delta \left( \frac{g}{\sqrt{|g|^2 + \varepsilon^2}} \right) = \frac{\delta g}{\sqrt{|g|^2 + \varepsilon^2}} - \frac{g \delta |g|^2}{\sqrt{|g|^2 + \varepsilon^2} \sqrt{|g|^2 + \varepsilon^2} + \sqrt{|g|^2 + \varepsilon^2} + \sqrt{|g|^2 + \varepsilon^2}}
\]

for $g = \nabla u^o + (\nabla u^o)^\top$ and $\text{div} u^o_2$, respectively.

Then, after substituting the bounds in (3.16) and (3.19a) and applying interpolation inequalities, one can obtain from (3.27) that

\[
\rho_c(\frac{d}{dt}) \| h_{m,1} \|_{L^2(\Omega)}^{1/2} \| \delta u_m \|_{L^2(\Omega)}^2 + \| \delta u_m \|_{H^2(\Omega)}^2 \leq C_{\sigma, \epsilon, \mu, \lambda, \epsilon_o, \epsilon_m} \| \delta u_m \|_{L^2(\Omega)}^2 \\
+ C_T (1 + \| \partial_t u_{m,2} \|_{L^2(\Omega)}^2) \left( \| \delta h_m \|_{L^2(\Omega)}^2 + \| \delta h_m \|_{L^2(\Omega)}^2 + \| \delta A_m \|_{L^2(\Omega)}^2 \right) \\
+ \sigma \| \delta u^o \|_{L^2(\Omega)}^2,
\]

where Young’s inequality is applied.

Thus, after substituting (3.26) into (3.28) and applying Grönwall’s inequality to the resultant, one has

\[
\sup_{0 \leq s \leq t} \| \delta u_m(s) \|_{L^2(\Omega)}^2 + \int_0^t \| \delta u_m(s) \|_{H^2(\Omega)}^2 \, ds \leq \sigma C_{\epsilon, \bar{\epsilon}, \bar{h}, \bar{h}, \epsilon_o, \epsilon_m} \\
\times \exp \left[ C_{\sigma, \epsilon, \mu, \lambda, \epsilon_o, \epsilon_m} \left( t + t^2 \right) \right] \int_0^t \| \delta u^o \|_{H^2(\Omega)}^2 \, ds.
\]
Therefore, after choosing $\sigma$ and $t$ small enough, one can conclude that

$$\sup_{0 \leq s \leq t} \| \delta u_m(s) \|_{L^2(\Omega)}^2 + \int_0^t \| \delta u_m(s) \|_{H^2(\Omega)}^2 \, ds$$

$$\leq \frac{1}{2} \left( \sup_{0 \leq s \leq t} \| \delta u^o(s) \|_{L^2(\Omega)}^2 + \int_0^t \| \delta u^o(s) \|_{H^2(\Omega)}^2 \, ds \right).$$

(3.29)

Now we update the smallness of $T^*$, so that (3.29) holds true for $t \in (0, T^*)$. Then the map $M$, defined in (3.1), is contracting with constant $1/2$. By means of Banach’s fixed point theorem, we conclude that there exists a unique solution to (1.10) in $X_T$.

What is left is to show that such solutions are stable. Namely, they continuously depend on the initial data. Let $(u_j, h_j, A_j)$ be two solutions to (1.10), associated with initial data $(u_{in,j}, h_{in,j}, A_{in,j})$, $j = 1, 2$, satisfying (3.3). Then it is easy to check that (3.26) and (3.29) still hold true with $\delta u^o, \delta u_m, \delta h_m, \delta A_m$ replaced by $\delta u := u_1 - u_2, \delta h := h_1 - h_2, \delta A := A_1 - A_2$, with additional initial data on the righthand side, i.e.,

$$\sup_{0 \leq s \leq t} \left( \| \delta h(s) \|_{L^4(\Omega)}^2 + \| \delta A(s) \|_{L^2(\Omega)}^2 + \| \delta u(s) \|_{H^2(\Omega)}^2 \right)$$

$$\leq C_{\varepsilon, \omega, h, \overline{h}, \varepsilon_0, c_{in}} \left( \| h_{in,1} - h_{in,2}, A_{in,1} - A_{in,2} \|_{L^4(\Omega)}^2 + \| u_{in,1} - u_{in,2} \|_{L^2(\Omega)}^2 \right).$$

(3.30)

Hence, we have established the local-in-time well-posedness of strong solutions to system (1.10). We would like to remind readers that the estimates obtained in this section depend on $(\mu, \lambda, \iota, \nu)$. In the next section, we aim at removing such dependency.

4 Well-Posedness of Solutions to (1.3) with $h > 0$

4.1 $(\mu, \lambda, \iota, \nu)$-Independent Estimates of Solutions to (1.10)

We shall only present the uniform-in-$(\mu, \lambda, \iota, \nu)$ a priori estimate in this subsection, based on which the standard different quotient argument can be established.

Throughout this section, we use the notation $X \lesssim Y$ to represent $X \leq CY$ for some generic constant $C \in (0, \infty)$, which may be different from line to line, and depend on $\varepsilon, \omega, h, \overline{h}$, but is independent of $(\mu, \lambda, \iota, \nu)$.

To begin with, let

$$\mathcal{E}(t) := \sup_{0 \leq s \leq t} \| u(s), h(s), A(s) \|_{H^3(\Omega)}^2 + \int_0^t \| u(s) \|_{H^4(\Omega)}^2 \, ds,$$

(4.1)
and

\[ E(t) := \sup_{0 \leq s \leq t} \| u(s), h(s), A(s) \|_{H^3(\Omega)}^2 + \int_0^t \left( \frac{\| \nabla^3 (\nabla u(s) + \nabla u^\top(s)) \|^2}{\| \nabla u(s) + \nabla u^\top(s) \|^2 + \varepsilon^2} + \frac{\| \nabla^3 \text{div}(u(s)) \|^2}{\| \text{div} u(s) \|^2 + \varepsilon^2} \right) \, dx \, ds. \] (4.2)

One can easily check that \( \mathcal{E} \) and \( \mathcal{E} \) are essentially equivalent in the sense that estimates on one imply estimates on the other. Indeed, it is trivial that \( \mathcal{E} \lesssim \mathcal{E} \). On the other hand, applying integration by parts yields that

\[ \int |\nabla^4 u|^2 \, dx = \frac{1}{2} \int |\nabla^3 (\nabla u + \nabla u^\top)|^2 \, dx - \int |\nabla^3 \text{div} u|^2 \, dx \lesssim (\varepsilon^3 + \| u \|_{H^3(\Omega)}^3) \int \left( \frac{|\nabla^3 (\nabla u + \nabla u^\top)|^2}{\| \nabla u + \nabla u^\top \|^2 + \varepsilon^2} + \frac{|\nabla^3 \text{div} u|^2}{\| \text{div} u \|^2 + \varepsilon^2} \right) \, dx. \] (4.3)

Therefore, we have

\[ E(t) \lesssim \mathcal{E}(t) \lesssim (1 + t + \mathcal{E}^2(t))E(t). \] (4.4)

**Estimates for \( h \) and \( A \)**

It is easy to check that (2.3), (2.5), (2.6), (2.7), and (2.8) also hold true with \( A_m, h_m, u^o \) replaced by \( A, h, u \), respectively. Therefore, for \( s \in (0, t) \) with \( t \) satisfying (2.9), with \( u^o \) replaced by \( u \), we have

\[ 0 \leq A \leq 1, \quad 0 < \frac{1}{4} \varepsilon \leq h \leq 4 \varepsilon. \] (4.5)

Notice that the smallness of \( t \) here is independent of \( (\mu, \lambda, \iota, \nu) \).

Next, we shall establish the regularity estimates of \( A \) and \( h \). Indeed, after applying \( \partial^3 \) to (1.10b) and (1.10c), one can obtain the following equations:

\[ \partial_t \partial^3 h + u \cdot \nabla \partial^3 h = \partial^3 S_{h, \mu, \nu} - \partial^3 (h \text{div} u) \]
\[ + (u \cdot \nabla h) - \partial^3 (u \cdot \nabla h), \] (4.6a)

\[ \partial_t \partial^3 A + u \cdot \nabla \partial^3 A = \partial^3 S_{A, \omega, \nu} + \partial^3 (A \text{div} u \cdot \chi_A^o) \]
\[ - \partial^3 (A \text{div} u) + (u \cdot \nabla \partial^3 A - \partial^3 (u \cdot \nabla A)). \] (4.6b)

Then, applying the \( L^2 \)-inner product of (4.6a) and (4.6b) with \( 2 \partial^3 h \) and \( \partial^3 A \), respectively, and integration by parts in the resultant leads to

\[ \frac{d}{dt} \| \partial^3 h \|_{L^2(\Omega)}^2 = \int \left( \text{div} u |\partial^3 h|^2 - 2 \partial^3 (h \text{div} u) \partial^3 h \right) \, dx \]
\[ + 2 \int (\mathbf{u} \cdot \nabla \partial^3 h - \partial^3 (\mathbf{u} \cdot \nabla h)) \partial^3 h \, dx + 2 \int \partial^3 S_{\mathbf{h},\mu,\nu} \partial^3 h \, dx, \quad (4.7a) \]

\[ \frac{d}{dt} \| \partial^3 A \|_{L^2(\Omega)}^2 = \int \left( \text{div} \mathbf{u}| \partial^3 A|^2 - 2 \partial^3 (A \text{div} \mathbf{u}) \partial^3 A \right) \, dx \]

\[ + 2 \int (\mathbf{u} \cdot \nabla \partial^3 A - \partial^3 (\mathbf{u} \cdot \nabla A)) \, dx + 2 \int \partial^3 S_{A,\mu,\nu} \partial^3 A \, dx \]

\[ + 2 \int \partial^3 (A \text{div} \mathbf{u} \cdot \chi_A^\omega) \partial^3 A \, dx, \quad (4.7b) \]

Directly applying Hölder’s inequality and the Sobolev embedding inequality leads to the following estimates:

\[ I_1 + I_2 + I_4 + I_5 + I_7 \lesssim \mathcal{H}(\| \mathbf{u}, h, A \|_{H^3(\Omega)}) \]

\[ + \| \mathbf{u} \|_{H^4(\Omega)} \| h, A \|_{H^3(\Omega)}^2. \quad (4.8) \]

Similarly,

\[ I_3 + I_6 \lesssim \mathcal{H}(\| h, A \|_{H^3(\Omega)}). \quad (4.9) \]

Therefore, after substituting estimates (4.8) and (4.9) into (4.7a) and (4.7b), one can derive that

\[ \frac{d}{dt} \| \partial^3 h, \partial^3 A \|_{L^2(\Omega)}^2 \lesssim \mathcal{H}(\| \mathbf{u}, h, A \|_{H^3(\Omega)}) + \| \mathbf{u} \|_{H^4(\Omega)} \| h, A \|_{H^3(\Omega)}^2. \]

Similar estimates also hold for lower order derivatives. Hence we have shown that

\[ \frac{d}{dt} \| h, A \|_{H^3(\Omega)}^2 \leq \mathcal{H}(\| \mathbf{u}, h, A \|_{H^3(\Omega)}) + C_{\omega, h, h} \| \mathbf{u} \|_{H^4(\Omega)} \| h, A \|_{H^3(\Omega)}^2, \]

for some constant \( C_{\omega, h, h} \in (0, \infty) \), independent of \( \iota \) and \( \nu \). Consequently, applying Grönwall’s inequality concludes that

\[ \sup_{0 \leq s \leq t} \| h(s), A(s) \|_{H^3(\Omega)}^2 \leq e^{C_{\omega, h, h} \int_0^t \| \mathbf{u}(s) \|_{H^4(\Omega)} \, ds} \cdot \left( \| h_{in}, A_{in} \|_{H^3(\Omega)}^2 + \int_0^t \mathcal{H}(\| \mathbf{u}(s), h(s), A(s) \|_{H^3(\Omega)}) \, ds \right). \quad (4.10) \]
Estimates for $u$

After applying $\partial^3$ to (1.10a), one can obtain the following equation:

$$
\begin{align*}
& m(\partial_t \partial^3 u + u \cdot \nabla \partial^3 u) + \nabla \partial^3 p = \text{div} \, \partial^3 S_e + \text{div} \, \partial^3 S_{\mu, \lambda} \\
& - \iota \Delta^2 \partial^3 u + \partial^3 F + [m \partial_t \partial^3 u - \partial^3 (m \partial_t u)] \\
& + [m u \cdot \nabla \partial^3 u - \partial^3 (m u \cdot \nabla u)].
\end{align*}
$$

(4.11)

Then, applying the $L^2$-inner product of (4.11) with $2\partial^3 u$ and integration by parts in the resultant leads to

$$
\begin{align*}
\frac{d}{dt} \| \rho_{\text{ice}} h^{1/2} \partial^3 u \|_{L^2(\Omega)}^2 + 2\mu \| \nabla \partial^3 u \|_{L^2(\Omega)}^2 + 2(\mu + \lambda) \| \text{div} \, \partial^3 u \|_{L^2(\Omega)}^2 \\
+ 2\iota \| \nabla^2 \partial^3 u \|_{L^2(\Omega)}^2 = \frac{1}{\mathcal{I}_8} \left[ \int [\rho_{\text{ice}} \partial_t h + \text{div} (\rho_{\text{ice}} h u)] |\partial^3 u|^2 \, dx \right] \\
- \frac{2}{\mathcal{I}_9} \int \partial^3 S_e : \nabla \partial^3 u \, dx + 2 \left[ \int [m \partial_t \partial^3 u - \partial^3 (m \partial_t u)] \cdot \partial^3 u \, dx \right] \\
+ \frac{2}{\mathcal{I}_{10}} \int [\rho_{\text{ice}} h u \cdot \nabla \partial^3 u - \partial^3 (\rho_{\text{ice}} h u \cdot \nabla u)] \cdot \partial^3 u \, dx \\
+ \frac{2}{\mathcal{I}_{11}} \int \partial^3 \text{div} \, \partial^3 u \, dx - 2 \int \partial^2 F \cdot \partial^4 u \, dx.
\end{align*}
$$

(4.12)

The estimates of $\mathcal{I}_j$, $j \in \{8, 11, 12\}$, are standard, which we will record below. Applying Hölder’s inequality and the Sobolev embedding inequality yields that

$$
\begin{align*}
\mathcal{I}_8 & \lesssim \left( \| \partial_t h \|_{L^2(\Omega)} + \| \text{div} (h u) \|_{L^2(\Omega)} \right) \| \partial^3 u \|_{L^2(\Omega)} \| \partial^3 u \|_{H^1(\Omega)} \\
& \lesssim \left( \| h \|_{L^\infty(\Omega)} + \| \nabla h \|_{L^4(\Omega)} \right) \| u \|_{H^5(\Omega)}^2 \| u \|_{H^4(\Omega)}, \\
\mathcal{I}_{11} & \lesssim \| h \|_{H^3(\Omega)} \| u \|_{H^3(\Omega)}^2 \| u \|_{H^4(\Omega)}, \\
\mathcal{I}_{12} & \lesssim (\| A \|_{H^3(\Omega)}^3 + 1) \| h \|_{H^3(\Omega)} \| u \|_{H^4(\Omega)}.
\end{align*}
$$

(4.13)

To estimate $\mathcal{I}_{13}$, notice that

$$
\| \partial^2 F \|_{L^2(\Omega)} \lesssim \| \partial^2 ([U_w - u](U_w - u)) \|_{L^2(\Omega)} + \| h \|_{H^2(\Omega)} \| u \|_{H^2(\Omega)} + \text{l.o.t},
$$

where l.o.t represents lower order terms of $u$. Direct calculation yields that

$$
\partial^2 ([U_w - u](U_w - u)) = [U_w - u] \partial^2 (U_w - u) + \frac{\partial (U_w - u)}{|U_w - u|} \partial (U_w - u).
$$
which implies
\[
\| \partial^2 (|U_w - u| (U_w - u)) \|_{L^2(\Omega)} \lesssim \| U_w - u \|^2_{H^2(\Omega)} + \| U_w - u \|^3_{H^2(\Omega)}.
\]

Therefore, we have
\[
I_{13} \lesssim \| \partial^2 F \|_{L^2(\Omega)} \| \partial^4 u \|_{L^2(\Omega)} \lesssim (\| u \|^3_{H^3(\Omega)} + \| h \|^2_{H^2(\Omega)} + 1) \| u \|_{H^4(\Omega)}.
\]  

In order to estimate $I_{10}$, we first rewrite $I_{10}$ as follows,
\[
I_{10} = -2 \int \partial^3 m \partial_t u \cdot \partial^3 u \, dx + 6 \int \partial m \partial_t \partial u \cdot \partial^4 u \, dx, \tag{4.15}
\]

where we have applied integration by parts. Next, we will use equation (1.10a) to substitute $\partial_t u$ and $\partial_t \partial u$ in (4.15). Indeed, after rearranging (1.10a), it follows
\[
\partial_t u = \frac{\text{div} S_{\epsilon, \mu, \lambda}}{m} + \frac{F}{m} - \frac{\nabla p}{m} - u \cdot \nabla u - \iota \frac{\Delta^2 u}{m},
\]
\[
\partial_t \partial u = \frac{\text{div} \partial S_{\epsilon, \mu, \lambda}}{m} - \frac{\text{div} S_{\epsilon, \mu, \lambda}}{m^2} \partial m + \frac{\partial F}{m} - \frac{F}{m^2} \partial m
\]
\[
- \frac{\nabla \partial p}{m} + \frac{\nabla p}{m^2} \partial m - \partial u \cdot \nabla u - u \cdot \nabla \partial u
\]
\[
- \iota \frac{\Delta^2 \partial u}{m} + \iota \frac{\Delta^2 u}{m^2} \partial m.
\]

Then similarly as before, directly applying Hőlder’s inequality and the Sobolev embedding inequality leads to,
\[
\| \partial_t u \|_{L^4(\Omega)} + \| \partial_t \partial u \|_{L^4(\Omega)} \lesssim \mathcal{H}(\| u \|_{H^3(\Omega)}, \| A \|_{H^2(\Omega)}, \| h \|_{H^2(\Omega)}) + \iota (1 + \| h \|_{H^2(\Omega)}) \| u \|_{H^5(\Omega)}.
\]

Therefore, one can derive that,
\[
I_{10} \lesssim \| \partial^3 m \|_{L^2(\Omega)} \| \partial_t u \|_{L^4(\Omega)} \| \partial^3 u \|_{L^4(\Omega)}
\]
\[
+ \| \partial m \|_{L^\infty(\Omega)} \| \partial_t \partial u \|_{L^2(\Omega)} \| \partial^4 u \|_{L^2(\Omega)}
\]
\[
\lesssim \mathcal{H}(\| u \|_{H^3(\Omega)}, \| A \|_{H^2(\Omega)}, \| h \|_{H^2(\Omega)}) \| u \|_{H^4(\Omega)}
\]
\[
+ \iota (\| h \|_{H^3(\Omega)} + \| h \|^2_{H^2(\Omega)}) \| u \|_{H^5(\Omega)} \| u \|_{H^4(\Omega)}.
\]  

Lastly, we will estimate $I_9$. Notice that,
\[
I_9 = - \int \partial^3 \left( p \frac{\nabla u + \nabla u^\top}{\sqrt{|\nabla u + \nabla u^\top|^2 + \epsilon^2}} \right) : \partial^3 (\nabla u + \nabla u^\top) \, dx
\]
\[-2 \int \vartheta^3 \left( p \frac{\text{div } u}{\sqrt{\text{div } u^2 + \varepsilon^2}} \right) \partial^3 \text{div } u \ dx.\]

Denote by \( D u \in \{ \nabla u + \nabla u^\top, \text{div } u \} \). In this notation, estimating \( I_9 \) amounts to determining an estimate for

\[ \int \vartheta^3 \left( p \frac{D u}{\sqrt{|D u|^2 + \varepsilon^2}} \right) \cdot \partial^3 D u \ dx. \]

Direct calculation shows that

\[
\int \vartheta^3 \left( p \frac{D u}{\sqrt{|D u|^2 + \varepsilon^2}} \right) \cdot \partial^3 D u \ dx = \int p \left( \frac{|\partial^3 D u|^2}{|D u|^2 + \varepsilon^2} - \frac{(D u \cdot \partial^3 D u)^2}{(|D u|^2 + \varepsilon^2)^{3/2}} \right) dx \\
-3 \int p \frac{(D u \cdot \partial D u)(\partial^2 D u \cdot \partial^3 D u) + (D u \cdot \partial^2 D u)(\partial D u \cdot \partial^3 D u)}{(|D u|^2 + \varepsilon^2)^{3/2}} dx_{\mathcal{L}_1} \\
-3 \int p \frac{(\partial D u \cdot \partial^2 D u)(D u \cdot \partial^3 D u)}{(|D u|^2 + \varepsilon^2)^{3/2}} dx_{\mathcal{L}_2} \\
+9 \int p \frac{(D u \cdot \partial D u)(D u \cdot \partial^2 D u)(D u \cdot \partial^3 D u)}{(|D u|^2 + \varepsilon^2)^{5/2}} dx_{\mathcal{L}_3} \\
-3 \int p \frac{|\partial D u|^2(\partial D u \cdot \partial^3 D u)}{(|D u|^2 + \varepsilon^2)^{3/2}} dx_{\mathcal{L}_4} \\
+9 \int p \frac{(D u \cdot \partial D u)^2(\partial D u \cdot \partial^3 D u) + |\partial D u|^2(D u \cdot \partial D u)(D u \cdot \partial^3 D u)}{(|D u|^2 + \varepsilon^2)^{5/2}} dx_{\mathcal{L}_5} \\
-15 \int p \frac{(D u \cdot \partial D u^3)(D u \cdot \partial^3 D u)}{(|D u|^2 + \varepsilon^2)^{7/2}} dx_{\mathcal{L}_6} \\
+3 \int \left[ \vartheta p \vartheta^2 \left( \frac{D u}{\sqrt{|D u|^2 + \varepsilon^2}} \right) \cdot \partial^3 D u + \vartheta^2 p \vartheta \left( \frac{D u}{\sqrt{|D u|^2 + \varepsilon^2}} \right) \cdot \partial^3 D u \right] dx_{\mathcal{L}_7} \\
+ \int \vartheta^3 p \frac{D u \cdot \partial^3 D u}{\sqrt{|D u|^2 + \varepsilon^2}} dx_{\mathcal{L}_8}.\]
Notice that
\[
\frac{|\partial^3 \mathbf{D}u|^2}{\sqrt{|\mathbf{D}u|^2 + \epsilon^2}} - \frac{(\mathbf{D}u \cdot \partial^3 \mathbf{D}u)^2}{(|\mathbf{D}u|^2 + \epsilon^2)^{3/2}} \geq \epsilon^2 \frac{|\partial^3 \mathbf{D}u|^2}{(|\mathbf{D}u|^2 + \epsilon^2)^{3/2}}.
\]

Therefore, applying Hölder’s inequality and the Sobolev embedding inequality implies that,
\[
|\mathcal{L}_4| + |\mathcal{L}_5| + |\mathcal{L}_6| + |\mathcal{L}_7| + |\mathcal{L}_8| \lesssim \|p\|_{H^3(\Omega)}(1 + \|\mathbf{D}u\|_{H^2(\Omega)})\|\partial^3 \mathbf{D}u\|_{L^2(\Omega)},
\]
\[
|\mathcal{L}_1| + |\mathcal{L}_2| + |\mathcal{L}_3| \lesssim \|p\|_{L^\infty(\Omega)}\|\mathbf{D}u\|_{H^2(\Omega)}^{3/2}\|\mathbf{D}u\|_{H^3(\Omega)}^{3/2}.
\]

Therefore,
\[
\int \partial^3 \left( p \frac{\mathbf{D}u}{\sqrt{|\mathbf{D}u|^2 + \epsilon^2}} \right) : \partial^3 \mathbf{D}u \; dx \geq \epsilon^2 \int \frac{p|\partial^3 \mathbf{D}u|^2}{(|\mathbf{D}u|^2 + \epsilon^2)^{3/2}} \; dx
\]
\[
- \|p\|_{H^3(\Omega)}(1 + \|\mathbf{D}u\|_{H^2(\Omega)})\|\partial^3 \mathbf{D}u\|_{L^2(\Omega)}
\]
\[
- \|p\|_{L^\infty(\Omega)}\|\mathbf{D}u\|_{H^2(\Omega)}^{3/2}\|\mathbf{D}u\|_{H^3(\Omega)}^{3/2}.
\]

Thus, we have shown that, thanks to the fact \(p \geq c_p h/4 > 0\),
\[
\mathcal{I}_9 \leq -\frac{\epsilon^2 c_p h}{4} \int \left( \frac{|\partial^3 (\nabla \mathbf{u} + \nabla \mathbf{u}^\top)|^2}{(|\nabla \mathbf{u} + \nabla \mathbf{u}^\top|^2 + \epsilon^2)^{3/2}} + 2\frac{|\partial^3 \text{div} \mathbf{u}|^2}{(|\text{div} \mathbf{u}|^2 + \epsilon^2)^{3/2}} \right) \; dx \tag{4.17}
\]
\[
+ \mathcal{H}(\|\mathbf{u}, A, h\|_{H^3(\Omega)})\|\mathbf{u}\|_{H^4(\Omega)} + \|\mathbf{u}\|_{H^3(\Omega)}^{3/2}\|\mathbf{u}\|_{H^4(\Omega)}^{3/2}.
\]

In addition, notice that, according to (4.3),
\[
\|\mathbf{u}\|_{H^4(\Omega)} \lesssim \|\mathbf{u}\|_{H^3(\Omega)} + \|\nabla^2 \mathbf{u}\|_{L^2(\Omega)} \lesssim \|\mathbf{u}\|_{H^3(\Omega)}
\]
\[
+ \left( \epsilon^3 + \|\mathbf{u}\|_{H^3(\Omega)}^3 \right) \int \left( \frac{|\nabla^2 (\nabla \mathbf{u} + \nabla \mathbf{u}^\top)|^2}{(|\nabla \mathbf{u} + \nabla \mathbf{u}^\top|^2 + \epsilon^2)^{3/2}} + \frac{|\nabla^2 \text{div} \mathbf{u}|^2}{(|\text{div} \mathbf{u}|^2 + \epsilon^2)^{3/2}} \right) \; dx \right)^{1/2} \tag{4.18}
\]

To sum up, after substituting estimates (4.13), (4.14), (4.16), (4.17), and (4.18) into (4.12), and applying Young’s inequality, one can derive that
\[
\frac{d}{dt} \|\mathbf{u}\|_{L^2(\Omega)}^{1/2}h^{1/2}\partial^3 \mathbf{u}\|_{L^2(\Omega)}^2 + 2\|\nabla^2 \partial^3 \mathbf{u}\|_{L^2(\Omega)}^2 - 2\|\mathbf{u}\|_{L^2(\Omega)}^2
\]
\[
+ \frac{\epsilon^2 c_p h}{8} \int \left( \frac{|\partial^3 (\nabla \mathbf{u} + \nabla \mathbf{u}^\top)|^2}{(|\nabla \mathbf{u} + \nabla \mathbf{u}^\top|^2 + \epsilon^2)^{3/2}} + 2\frac{|\partial^3 \text{div} \mathbf{u}|^2}{(|\text{div} \mathbf{u}|^2 + \epsilon^2)^{3/2}} \right) \; dx
\]
\[
\leq \mathcal{H}(\|\mathbf{u}, A, h\|_{H^3(\Omega)}, t).
\]
which implies, recalling \( \iota \in (0, 1) \),

\[
\sup_{0 \leq s \leq t} \| \nabla^3 u(s) \|^2_{L^2(\Omega)} + (t - \iota^2) \int_0^t \| \nabla^5 u(s) \|^2_{L^2(\Omega)} \, ds \\
+ \int_0^t \int \left( \frac{|\partial^3 (\nabla u(s) + \nabla u^\top(s))|^2}{(|\nabla u(s) + \nabla u^\top(s)|^2 + \varepsilon^2)^{3/2}} + 2 \frac{|\partial^3 \text{div} u(s)|^2}{(|\text{div} u(s)|^2 + \varepsilon^2)^{3/2}} \right) \, dx \, ds \\
\leq C_{\varepsilon, h, h} \| \nabla^3 u_{in} \|^2_{L^2(\Omega)} + \int_0^t \mathcal{H}(\| u(s), A(s), h(s) \|_{H^3(\Omega)}, \iota) \, ds,
\]

for some constant \( C_{\varepsilon, h, h} \in (0, \infty) \), independent of \( \mu, \lambda, \iota, \text{and} \nu \).

Similar estimates also hold for lower order derivatives. Thus one can conclude that, for \( \iota \ll 1 \) small enough,

\[
\sup_{0 \leq s \leq t} \| u(s) \|^2_{H^3(\Omega)} \\
+ \int_0^t \int \left( \frac{|\partial^3 (\nabla u(s) + \nabla u^\top(s))|^2}{(|\nabla u(s) + \nabla u^\top(s)|^2 + \varepsilon^2)^{3/2}} + 2 \frac{|\partial^3 \text{div} u(s)|^2}{(|\text{div} u(s)|^2 + \varepsilon^2)^{3/2}} \right) \, dx \, ds \\
\leq C_{\varepsilon, h, h} \| u_{in} \|^2_{H^3(\Omega)} + \int_0^t \mathcal{H}(\| u(s), A(s), h(s) \|_{H^3(\Omega)}) \, ds. \tag{4.19}
\]

**Uniform Estimates**

The summation of (4.10) and (4.19) leads to

\[
\mathcal{E}(t) \leq \left( e^{C_{\varepsilon, h, h} \varepsilon^{1/2} + C_{\varepsilon, h, h}} \times \left( \| h_{in}, A_{in}, u_{in} \|^2_{H^3(\Omega)} + t \times \mathcal{H}(\mathcal{E}(t)) \right) \right) \\
\leq \left( e^{C_{\varepsilon, h, h} \varepsilon^{1/2} + \mathcal{E}(t) + \varepsilon^3(t)} + C_{\varepsilon, h, h} \right) \\
\times \left( \| h_{in}, A_{in}, u_{in} \|^2_{H^3(\Omega)} + t \times \mathcal{H}(\mathcal{E}(t)) \right),
\]

where we have applied (4.3) and Young’s inequality in the second inequality. Consequently, for \( t \) small enough, independent of \( \mu, \lambda, \iota, \nu \), one can conclude that

\[
\mathcal{E}(t) \leq C_{\varepsilon, \mu, \lambda, \iota, \nu} \times \| h_{in}, A_{in}, u_{in} \|^2_{H^3(\Omega)}, \tag{4.20}
\]

and, thanks to (4.3),

\[
\mathcal{E}(t) \leq \mathcal{E}_{in}^2, \tag{4.21}
\]
for some constant $C_{in} \in (0, \infty)$, depending only on $\varepsilon, \omega, h, \bar{h}$, and
\[
\left\| h_{in}, A_{in}, u_{in} \right\|_{H^3(\Omega)}.
\]
Thus we have established the $(\mu, \lambda, \iota, \nu)$-independent estimates. Therefore, together with the well-posedness theory in Sect. 3 and continuity arguments, the existence time of solutions to (1.10) can be extended to some $T^{**} \in (0, \infty)$, independent of $(\mu, \lambda, \iota, \nu)$, which might be larger than $T^*$.

### 4.2 Limit as $(\mu, \lambda, \iota, \nu) \to (0^+, 0^+, 0^+, 0^+)$

Denote by $(u_{\mu, \lambda, \iota, \nu}, h_{\mu, \lambda, \iota, \nu}, A_{\mu, \lambda, \iota, \nu})$, the solution constructed above to system (1.10). With (4.1), (4.21), and by comparison in system (1.10), it is easy to check that we have the following uniform-in-$(\mu, \lambda, \iota, \nu)$ estimates:
\[
\begin{align*}
\left\| u_{\mu, \lambda, \iota, \nu}, h_{\mu, \lambda, \iota, \nu}, A_{\mu, \lambda, \iota, \nu} \right\|_{L^\infty(0, T^{**}; H^3(\Omega))} + \left\| u_{\mu, \lambda, \iota, \nu}, h_{\mu, \lambda, \iota, \nu}, A_{\mu, \lambda, \iota, \nu} \right\|_{L^2(0, T^{**}; H^4(\Omega))} \\
+ \left\| \partial_t u_{\mu, \lambda, \iota, \nu}, \partial_t h_{\mu, \lambda, \iota, \nu}, \partial_t A_{\mu, \lambda, \iota, \nu} \right\|_{L^\infty(0, T^{**}; L^2(\Omega))} \leq C_{in},
\end{align*}
\]
for some constant $C_{in} \in (0, \infty)$, and $T^{**} \in (0, \infty)$, independent of $\mu, \lambda, \iota$, and $\nu$. Therefore, applying the Aubin-Lions lemma yields that there exists $(u, h, A)$ satisfying (1.8) and (1.9), such that, as $(\mu, \lambda, \iota, \nu) \to (0^+, 0^+, 0^+, 0^+)$,
\[
\begin{align*}
u_{\mu, \lambda, \iota, \nu} &\to u \quad \text{in} \quad C(0, T^{**}; H^2(\Omega)), \\
h_{\mu, \lambda, \iota, \nu} &\to h \quad \text{in} \quad C(0, T^{**}; H^2(\Omega)), \\
A_{\mu, \lambda, \iota, \nu} &\to A \quad \text{in} \quad C(0, T^{**}; H^2(\Omega)), \\
(u_{\mu, \lambda, \iota, \nu}, h_{\mu, \lambda, \iota, \nu}, A_{\mu, \lambda, \iota, \nu}) &\to (u, h, A) \quad \text{in} \quad L^\infty(0, T^{**}; H^3(\Omega)), \\
u_{\mu, \lambda, \iota, \nu} &\to u \quad \text{in} \quad L^2(0, T^{**}; H^4(\Omega)), \\
(\partial_t u_{\mu, \lambda, \iota, \nu}, \partial_t h_{\mu, \lambda, \iota, \nu}, \partial_t A_{\mu, \lambda, \iota, \nu}) &\to (\partial_t u, \partial_t h, \partial_t A) \quad \text{in} \quad L^\infty(0, T^{**}; L^2(\Omega)),
\end{align*}
\]
and it is easy to verify that $(u, h, A)$ satisfies system (1.3) in $(0, T^{**}]$.

### 4.3 Well-Posedness of Solutions for System (1.3)

To deduce the well-posedness of solutions to system (1.3), it remains to establish the uniqueness and the continuous dependency of solutions on initial data. Indeed, this can be done following similar arguments as in Sect. 3.2, which we will sketch below.

Denote by $(u_j, h_j, A_j), j = 1, 2,$ two solutions to system (1.3) with initial data $(u_{in,j}, h_{in,j}, A_{in,j})$ within $(0, T_{j}^{**}), j = 1, 2$, as constructed above, respectively. In particular, (1.8) and (1.9) hold for $(u_j, h_j, A_j), j = 1, 2$. Further, let $\delta u := u_1 - u_2, \delta h := h_1 - h_2, \delta A := A_1 - A_2$, and $T_{12}^{**} := \min\{T_{1}^{**}, T_{2}^{**}\} \in (0, \infty)$. The triple $(\delta u, \delta h, \delta A)$ satisfies the following equations:
\[
\rho_{ice} h_1 \partial_t \delta u + \rho_{ice} \delta h \partial_t u_2 = \text{div} \delta \bar{S}_e - \nabla \delta p
\]
\[-\rho_{\text{ice}} h_1 \mathbf{u}_1 \cdot \nabla \mathbf{u} - \rho_{\text{ice}} h_1 \mathbf{u}_1 \cdot \nabla \mathbf{u}_2 - \rho_{\text{ice}} h \mathbf{u}_2 \cdot \nabla \mathbf{u}_2 + \delta \mathcal{F}, \quad (4.24a)\]
\[\partial_t \delta h + \text{div} (\delta h \mathbf{u}_1) + \text{div} (h_2 \delta \mathbf{u}) = \delta S_{h, \omega}, \quad (4.24b)\]
\[\partial_t \delta A + \text{div} (\delta A \mathbf{u}_1) + \text{div} (A_2 \delta \mathbf{u}) = \delta S_{A, \omega} + \delta A \text{div} \mathbf{u}_1 \cdot \chi_{A_1}^\omega + A_2 \text{div} \mathbf{u}_2 \cdot \delta \chi_{A}^\omega. \quad (4.24c)\]

After taking the $L^2$-inner product of (4.24a), (4.24b), and (4.24c) with $2 \delta \mathbf{u}$, $2 \delta h$, and $2 \delta A$, respectively, and applying integration by parts in the resultant, one has

\[
\frac{d}{dt} \| \rho_{\text{ice}} h_1^{1/2} \delta \mathbf{u} \|_{L^2(\Omega)}^2 = 2 \int \delta S_{\varepsilon} : \nabla \delta \mathbf{u} \, dx + \int \rho_{\text{ice}} \partial_t h_1 |\delta \mathbf{u}|^2 \, dx \quad \tag{4.25} \]

\[
\frac{d}{dt} \| \delta h \|_{L^2(\Omega)}^2 = - \int \text{div} \mathbf{u}_1 |\delta h|^2 \, dx - 2 \int \text{div} (h_2 \delta \mathbf{u}) \delta h \, dx + 2 \int \delta S_{h, \omega} \delta h \, dx, \quad \tag{4.26} \]

\[
\frac{d}{dt} \| \delta A \|_{L^2(\Omega)}^2 = - \int \text{div} \mathbf{u}_1 |\delta A|^2 \, dx - 2 \int \text{div} (A_2 \delta \mathbf{u}) \delta A \, dx + 2 \int \delta S_{A, \omega} \delta A \, dx + \int \text{div} \mathbf{u}_1 \cdot \chi_{A_1}^\omega |\delta A|^2 \, dx \quad \tag{4.27} \]

Then it is straightforward to check that, thanks to the uniform bounds in (1.9),

\[
\sum_{15 \leq j \leq 28} I_j \lesssim \| \delta \mathbf{u}, \delta h, \delta A \|_{L^2(\Omega)}^2 + \| \delta \mathbf{u}, \delta h, \delta A \|_{L^2(\Omega)} \| \nabla \delta \mathbf{u} \|_{L^2(\Omega)}. \quad (4.28)\]
To estimate $\mathcal{I}_{14}$, we will have to investigate the monotonicity of $S_\varepsilon$, which is an important ingredient in our proof. Notice that

$$2\delta S_\varepsilon : \nabla \delta u = \delta \left( p \frac{\nabla u + \nabla u^\top}{\sqrt{|\nabla u + \nabla u^\top|^2 + \varepsilon^2}} \right) : \delta (\nabla u + \nabla u^\top) + 2\delta \left( p \frac{\text{div} u}{\sqrt{\text{div} u^2 + \varepsilon^2}} \right) \delta \text{div} u.$$ 

For $Du \in \{\nabla u + \nabla u^\top, \text{div} u\}$, direct calculation yields that

$$\delta \left( p \frac{Du}{\sqrt{|Du|^2 + \varepsilon^2}} \right) = \frac{1}{2} \left( \frac{p_1}{\sqrt{|Du_2|^2 + \varepsilon^2}} + \frac{p_2}{\sqrt{|Du_1|^2 + \varepsilon^2}} \right) \delta Du$$

$$- \frac{1}{2} \frac{(Du_1 + Du_2) \cdot \delta Du \times (p_1 Du_1 + p_2 Du_2)}{\sqrt{|Du_1|^2 + \varepsilon^2} \sqrt{|Du_2|^2 + \varepsilon^2} (\sqrt{|Du_1|^2 + \varepsilon^2} + \sqrt{|Du_2|^2 + \varepsilon^2})}$$

$$+ \frac{\delta p}{2} \left( \frac{Du_1}{\sqrt{|Du_1|^2 + \varepsilon^2}} + \frac{Du_2}{\sqrt{|Du_2|^2 + \varepsilon^2}} \right).$$

Therefore

$$\delta \left( p \frac{Du}{\sqrt{|Du|^2 + \varepsilon^2}} \right) \cdot \delta Du = \frac{\delta p}{2} \left( \frac{Du_1}{\sqrt{|Du_1|^2 + \varepsilon^2}} + \frac{Du_2}{\sqrt{|Du_2|^2 + \varepsilon^2}} \right) \cdot \delta Du$$

$$+ \frac{1}{2} \frac{M}{\sqrt{|Du_1|^2 + \varepsilon^2} \sqrt{|Du_2|^2 + \varepsilon^2} (\sqrt{|Du_1|^2 + \varepsilon^2} + \sqrt{|Du_2|^2 + \varepsilon^2})},$$

with

$$M := \left( p_1 \sqrt{|Du_1|^2 + \varepsilon^2} + p_2 \sqrt{|Du_2|^2 + \varepsilon^2} \right) \left( \sqrt{|Du_1|^2 + \varepsilon^2} + \sqrt{|Du_2|^2 + \varepsilon^2} \right)$$

$$\times |\delta Du|^2 - \left( (Du_1 + Du_2) \cdot \delta Du \right) \times \left( (p_1 Du_1 + p_2 Du_2) \cdot \delta Du \right)$$

$$\geq C_{h_0, c_{in}} \varepsilon |\delta Du|^2,$$

for some constant $C_{h_0, c_{in}} \in (0, \infty)$ depending on $h$ and $c_{in}$. Therefore, one can derive that

$$\mathcal{I}_{14} \lesssim -C_{h_0, c_{in}} (\|\nabla \delta u + \nabla \delta u^\top\|^2_{L^2(\Omega)} + \| \delta u \|^2_{L^2(\Omega)})$$

$$+ \| \delta h, \delta A \|^2_{L^2(\Omega)} \| \nabla \delta u \|^2_{L^2(\Omega)},$$

for some constant $C_{\varepsilon, h_0, c_{in}} \in (0, \infty)$ depending on $\varepsilon$, $h_0$, and $c_{in}$. In addition, using integration by parts, one can derive that,

$$\|\nabla \delta u\|^2_{L^2(\Omega)} \lesssim \|\nabla \delta u + \nabla \delta u^\top\|^2_{L^2(\Omega)} + \| \delta u \|^2_{L^2(\Omega)}.$$

Consequently, after substituting (4.28), (4.30), and (4.31) into (4.25), (4.26), and (4.27), summing up the results, and applying Young’s inequality, one can conclude
that
\[
\frac{d}{dt} \| \rho^{1/2}_{\text{ice}} h^{1/2}_{1} \delta \mathbf{u}, \delta h, \delta A \|_{L^2(\Omega)}^2 \leq C_{c_{\text{in}}} \| \rho^{1/2}_{\text{ice}} h^{1/2}_{1} \delta \mathbf{u}, \delta h, \delta A \|_{L^2(\Omega)}^2,
\]
which, after applying Grönwall’s inequality, yields
\[
\sup_{0 \leq s \leq T^{**}_{12}} \| \delta \mathbf{u}(s), \delta h(s), \delta A(s) \|_{L^2(\Omega)}^2 \leq C_{c_{\text{in}}} \| \delta \mathbf{u}_{\text{in}}, \delta h_{\text{in}}, \delta A_{\text{in}} \|_{L^2(\Omega)}^2,
\]
with some constant $C_{c_{\text{in}}} \in (0, \infty)$, depending on the initial data. The uniqueness and the continuous dependence on initial data of solutions to system (1.3) follow from (4.32).

Acknowledgements XL and MT gratefully acknowledge the partial funding by the Deutsche Forschungsgemeinschaft (DFG) through project AA2-9 Variational methods for viscoelastic flows and gelation within MATH+.

MT also gratefully acknowledges the partial funding by the DFG through project C09 Dynamics of rock dehydration on multiple scales (Project Number 235221301) within CRC 1114 Scaling Cascades in Complex Systems. Moreover XL and EST are thankful for the kind hospitality of Freie Universität Berlin where part of this work was done and partially supported by the Einstein Stiftung/Foundation - Berlin, through the Einstein Visiting Fellow Program. EST and XL would also like to thank the Isaac Newton Institute for Mathematical Sciences for support and hospitality during the programme TUR when part of this work was undertaken. This work was supported by EPSRC Grant Number EP/R014604/1. XL’s work was partially supported by a grant from the Simons Foundation, during his visit to the Isaac Newton Institute. The authors also thank the reviewers for the helpful comments during the submission of this work.

References

Bouillon, S., Fichefet, T., Legat, V., Madec, G.: The elastic-viscous-plastic method revisited. Ocean Model. 71, 2013

Brandt, F., Disser, K., Haller-Dintelmann, R., Hieber, M.: Rigorous analysis and dynamics of Hibler’s sea ice model. arXiv:2104.01336 (2021)

Coon, M.D., Knoke, G.S., Echert, D.C., Pritchard, R.S.: The architecture of an anisotropic elastic-plastic sea ice mechanics constitutive law. J. Geophys. Res. 103(C10), 21915–21925 (1998)

Dansereau, V., Weiss, J., Saramito, P., Lattes, P.: A Maxwell elasto-brittle rheology for sea ice modelling. Cryosphere 10, 1339–1359 (2016). https://doi.org/10.5194/tc-10-1339-2016

Dukowicz, J.K.: Comments on “Stability of the viscous-plastic sea ice rheology”. J. Phys. Oceanogr. 27, 480–481 (1997)

Evans, L.C.: Partial Differential Equations. Graduate Studies in Mathematics, vol. 19. American Mathematical Society, Providence (2010)

Giga, M.-H., Giga, Y.: Very singular diffusion equations: second and fourth order problems. Jpn. J. Ind. Appl. Math. 27(3), 323–345 (2010)

Giga, M.-H., Giga, Y., Kobayashi, R.: Very singular diffusion equations. In: Taniguchi Conference on Mathematics Nara ‘98, 10304010, pp. 93–125 (1998)

Gray, J.M.N.T.: Loss of hyperbolicity and ill-posedness of the viscous-plastic sea ice rheology in uniaxial divergent flow. J. Phys. Oceanogr. 29, 2920–2929 (1999)

Gray, J.M.N.T., Killworth, P.D.: Stability of the viscous-plastic sea ice rheology. J. Phys. Oceanogr. 25, 971–978 (1995)

Guba, O., Lorenz, J., Sulsky, D.: On well-posedness of the viscous-plastic sea ice model. J. Phys. Oceanogr. 43(10), 2185–2199 (2013). https://doi.org/10.1175/JPO-D-13-014.1

Herman, A.: Discrete-element bonded-particle sea ice model DESIgn, version 1.3a—model description and implementation. Geosci. Model Dev. 9, 1219–1241 (2016). https://doi.org/10.5194/gmd-9-1219-2016
Hibler, W.D.: A dynamic thermodynamic sea ice model. J. Phys. Oceanogr. 9(4), 815–846 (1979)
Hunke, E.C., Dukowicz, J.K.: An elastic-viscous-plastic model for sea ice dynamics. J. Phys. Oceanogr. 27(9), 1849–1867 (1997)
Hunke, E.C.: The elastic-viscous-plastic sea ice dynamics model. In: Dempsey, J.P., Shen, H.H. (eds.) IUTAM Symposium on Scaling Laws in Ice Mechanics and Ice Dynamics. Solid Mechanics and Its Applications, vol. 94, pp. 289–297. Springer, Dordrecht (2001)
Lipscomb, W.H., Hunke, E.C., Maslowski, W., Jakacki, J.: Ridging, strength, and stability in high-resolution sea ice models. J. Geophys. Res. 112, C03S91 (2007). https://doi.org/10.1029/2005JC003355
Majda, A.: Compressible Fluid Flow and Systems of Conservation Laws in Several Space Variables. Applied Mathematical Sciences, vol. 53. Springer, New York (1984)
Mehlmann, C., Richter, T.: A finite element multigrid-framework to solve the sea ice momentum equation. J. Comput. Phys. 348, 847–861 (2017)
Palmer, A., Johnston, I.: Ice velocity effects and ice force scaling. In: Dempsey, J.P., Shen, H.H. (eds.) IUTAM Symposium on Scaling Laws in Ice Mechanics and Ice Dynamics. Solid Mechanics and Its Applications, vol. 94, pp. 115–126. Springer, Dordrecht (2001)
Parkinson, C.L., Washington, W.M.: A large-scale numerical model of sea ice. J. Geophys. Res. 84(C1), 311 (1979)
Pritchard, R.S.: Sea ice dynamics models. In: Dempsey, J.P., Shen, H.H. (eds.) IUTAM Symposium on Scaling Laws in Ice Mechanics and Ice Dynamics. Solid Mechanics and Its Applications, vol. 94, pp. 265–288. Springer, Dordrecht (2001)
Rampal, P., Bouillon, S., Olason, E., Morlighem, M.: neXtSIM: a new Lagrangian sea ice model. Cryosphere 10, 1055–1073 (2016). https://doi.org/10.5194/tc-10-1055-2016
Schreyer, H.L.: Modeling failure initiation in sea ice based on loss of ellipticity. In: Dempsey, J.P., Shen, H.H. (eds.) IUTAM Symposium on Scaling Laws in Ice Mechanics and Ice Dynamics. Solid Mechanics and Its Applications, vol. 94, pp. 239–250. Springer, Dordrecht (2001)
Schreyer, H.L., Sulsky, D.L., Munday, L.B., Coon, M.D., Kwok, R.: Elastic–decohesive constitutive model for sea ice. J. Geophys. Res. 111, C11S26 (2006). https://doi.org/10.1029/2005JC003334
Sirmen, J., Tremblay, B.: Analytical study of an isotropic viscoplastic sea ice model in idealized configurations. J. Phys. Oceanogr. 45, 331–354 (2015). https://doi.org/10.1175/JPO-D-13-0109.1
Thomas, D.N., Dieckmann, G.S.: Sea Ice. Wiley, Hoboken (2010)
Tsamados, M., Feltham, D.L., Wilchinsky, A.V.: Impact of a new anisotropic rheology on simulations of Arctic sea ice. J. Geophys. Res. Oceans 118, 91–107 (2013). https://doi.org/10.1029/2012JC007990
Wilchinsky, A.V., Feltham, D.L.: Rheology of discrete failure regimes of anisotropic sea ice. J. Phys. Oceanogr. 42, 1065–1082 (2012). https://doi.org/10.1175/JPO-D-11-0178.1

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.