The Logarithmic Transform of a Polynomial Function Expressed in Terms of the Lerch Function

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Abstract: This is a collection of definite integrals involving the logarithmic and polynomial functions in terms of special functions and fundamental constants. All the results in this work are new.

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1. Introduction

Perusing the current literature, we noticed there is not many articles related to a Logarithmic transform applied to a polynomial function. The intention here is to assist researchers by adding another set of mathematical tools for which to further current research in the areas requiring such transforms. The Logarithmic transform proposed by Reynolds and Stauffer [1] is used in this work and applied to a form of the Euler integral of the first kind and expressed in terms of the Lerch function.

In the present work, the authors used their contour integral method and applied it to a special case of the Beta function in [2] to derive a definite integral and expressed its closed form in terms of a special function. This derived integral formula was then used to provide formal derivations in terms of special functions and fundamental constants and summarized in a Table. The Lerch function being a special function has the fundamental property of analytic continuation, which enables us to widen the range of evaluation for the parameters involved in our definite integral. The Lerch function is a special function that generalizes the Hurwitz zeta function, the polylogarithms, and so many interesting and important special functions. The definite integral derived in this manuscript is given by:

$$\int_{0}^{1} x^n (x^{-n-1} - 1)^m \log^k \left(a (x^{-n-1} - 1)\right) \, dx$$

(1)

where the parameters $k, a, n, m$ are general complex numbers with $-1 < \text{Re}(m) < 1$ and $-1 < \text{Im}(n) < 1$. The derivation of the definite integral follows the method used by us in [1] which involves Cauchy’s integral formula. The generalized Cauchy’s integral formula is given by:

$$\frac{y^k}{\Gamma(k+1)} = \frac{1}{2\pi i} \int_{C} \frac{e^{wy}}{w^{k+1}} \, dw,$$

(2)

where $C$ is in general an open contour in the complex plane where the bilinear concomitant [1] has the same value at the end points of the contour. The method in [1] involves using a form of Equation (2) then multiply both sides by a function, then take a definite integral of both sides. This yields a definite integral in terms of a contour integral. A second contour integral is derived by multiplying Equation (2) by a function and performing some substitutions so that the contour integrals are the same.
2. Definite Integral of the Contour Integral

We use the method in [1]. The variable of integration in the contour integral is \( z = w + m \). The cut and contour are in the second quadrant of the complex \( z \)-plane. The cut approaches the origin from the interior of the second quadrant and the contour goes round the origin with a zero radius and is on opposite sides of the cut. Using Equation (2) we replace \( y \) by \( \log(a(x^{-n-1} - 1)) \) then multiply by \( x^m(x^{-n-1} - 1)^m \). Next we take the finite integral over \( x \in [0, 1] \) to get:

\[
\frac{1}{\Gamma(k+1)} \int_0^1 x^n(x^{-n-1} - 1)^m \log^k(a(x^{-n-1} - 1))dx = \frac{1}{\Gamma(n+1)} \int_C \int_0^1 a^w w^{-k-1} x^n(x^{-n-1} - 1)^m dw dx
\]

From Equation (3.249.7) in [2] where \(-1 < \Re(w + m) < 1\). The logarithmic function is given for example in Section 4.1.2 in [3]. We are able to switch the order of integration over \( w + m \) and \( x \) using Fubini’s theorem since the integrand is of bounded measure over the space \( \mathbb{C} \times [0, 1] \). The steps used to derive the final line in Equation (3) are as follows. Firstly, using Equation (3.249.7) set \( v = u \) to get the definite integral in terms of the cosecant function. Next replace \( u \to 1/u \) to get the definite integral:

\[
\int_0^1 \left(1 - x^2\right)^{-u} dx = \pi u \csc(\pi u).
\]

Next setting \( u = -u \), we will iterate this integral by looking at the following forms:

\[
\int_0^1 x^k \left(1 - x^2\right)^{-u} dx = \frac{1}{2} \pi (m + w) \csc(\pi (m + w))
\]

\[
\int_0^1 x^2 \left(1 - x^2\right)^{-u} dx = \frac{1}{3} \pi (m + w) \csc(\pi (m + w))
\]

\[
\vdots
\]

\[
\int_0^1 x^n \left(x^{-n-1} - 1\right)^{-u} dx = \frac{\pi (m + w) \csc(\pi (m + w))}{n + 1}.
\]

Next we let \( z = x^{n+1} \) which implies \( dz = (n + 1)x^n \) and the integral becomes

\[
\int_0^1 \left(1/z - 1\right)^{m+w} dz
\]

which can be derived from Equation (3.249.7) in [2].

3. The Lerch Function

We use (9.550) and (9.556) in [2] where \( \Phi(z, s, v) \) is the Lerch function which is a generalization of the Hurwitz zeta \( \zeta(s, v) \) and Polylogarithm functions \( \text{Li}_n(z) \). The Lerch function has a series representation given by:

\[
\Phi(z, s, v) = \sum_{n=0}^{\infty} (v + n)^{-s} z^n
\]

where \( |z| < 1, v \neq 0, -1, \ldots \) and is continued analytically by its integral representation given by:

\[
\Phi(z, s, v) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1} e^{-vt}}{1 - re^{-t}} dt = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1} e^{-(v-1)t}}{e^t - z} dt,
\]

where \( \Re(v) > 0 \), and either \( |z| \leq 1, z \neq 1, \Re(s) > 0 \), or \( z = 1, \Re(s) > 1 \).
4. Infinite Sum of the Contour Integral

4.1. Derivation of the First Contour

In this section we will again use Cauchy’s integral Formula (2) and take the infinite sum to derive equivalent sum representations for the contour integrals. We proceed using Equation (2) and replace $y$ by $\log(a) + i\pi(2y + 1)$ and multiply both sides by $-\frac{2i\pi \text{csc}(2y+1)}{n+1}$ and take the infinite sum over $y \in [0, \infty)$ simplifying in terms of the Lerch function to get:

$$
\int_{-\infty}^{\infty} \frac{\log gamma \log \text{Li}damental constants namely, Polylogarithm

5. Definite Integral in Terms of the Lerch Function

Theorem 1. For $k, a \in \mathbb{C}$, $-1 < \text{Re}(m) < 1$, $-1 < \text{Im}(n) < 1$,

$$
\int_0^1 x^n (x^{-n-1} - 1)^m \log^k (a (x^{-n-1} - 1)) \, dx = - \frac{2(2\pi i)^k \text{e}^{2\pi i m} \Phi \left( \frac{2\pi i}{n+1}, 1 - k, \frac{-\log(a)}{2\pi i} \right)}{n+1}.
$$

Proof. Since the right-hand side of Equation (3) is equal to the sum of the right-hand sides of Equations (11) and (12) we can equate the left-hand sides and simplify the Gamma function to get the stated result. □

6. Main Results

In the following, we derive definite integrals in terms of special functions and fundamental constants namely, Polylogarithm $Li_k(z)$, Hurwitz zeta $\zeta(k, a)$, Polygamma $\psi(x)$, log-gamma log $\Gamma$, Catalan’s constant $G$, Aprey’s constant $\zeta(3)$, $\pi$, and Glaisher’s constant $A$.

Proposition 1.

$$
\int_0^1 \left( -1 + x^{-3-i} \right)^{\frac{1}{2} + i} x^{n+i} \log^{1+i} \left( \left( 1 + i \right) \left( -1 + x^{-3-i} \right) \right) \, dx
$$

$$
= \left( \frac{57}{1625} + \frac{21 i}{1625} \right) \left( -1 \right)^{\frac{1}{2} + i} + i^{2+i} \pi \left( \frac{1}{2} + i \right) \pi \Phi \left( \left( -1 \right)^{\frac{1}{2} + 2i}, -i, \frac{5}{8} - \frac{i \log(2)}{4 \pi} \right)
$$

Proof. Use Equation (13) and set $k = 1 + i, a = 1 + i, m = \frac{1}{2} + i, n = 2 + \frac{1}{2}$ and simplify. □
Proposition 2. For $-1 < \text{Re}(m) < 1$ and $-1 < \text{Im}(n) < 1$,
\[
\int_0^1 x^n (x^{-n-1} - 1)^m \, dx = \frac{\pi m \csc(\pi m)}{n+1}.
\] (15)

Proof. Use Equation (13) and set $k = 0$ and simplify using entry (2) in the Table below (64:12:7) in [4]. Note this integral for $n$, a general complex number is not defined in Wolfram Mathematica. This is a simple example of the Beta function, see Equation (3.249.7) in [2].

Proposition 3.
\[
\int_0^1 x^n (x^{-n-1} - 1)^m \log(x^{-n-1} - 1) \, dx = \frac{\pi(1 - \pi m \cot(\pi m)) \csc(\pi m)}{n+1}.
\] (16)

Proof. Use Equation (13) and set $k = 1$ and simplify using entry (1) in the Table below (64:12:7) in [4].

Proposition 4.
\[
\int_0^1 \frac{1 - 2x}{\sqrt{(1-x)x \log(\frac{1}{x} - 1)}} \, dx = \frac{4G}{\pi}. \] (17)

Proof. Use Equation (13) and form a second equation by replacing $m \to -m$ and take their difference. The using the resulting equation set $k = -1, a = 1, m = 1/2, n = 0$, and simplify using entry (4) in Table below (64:12:7), Equations (64:4:1) and (64:7:1) in [4] and Equation (2) in [5].

Proposition 5.
\[
\int_0^1 x^n (x^{-n-1} - 1)^m \log^k(1 - x^{-n-1}) \, dx
= -\frac{1}{\pi^k(2\pi)^i\kappa(2n-2m)}(kLi_{-k}(e^{2im\pi}) + 2i\pi mL_{-k}(e^{2im\pi})).
\] (18)

Proof. Use Equation (13) and set $a = 1$ and simplify using Equation (64:12:2) in [4].

Proposition 6.
\[
\int_0^1 \frac{\log(1 - \frac{1}{x})}{\sqrt{x^2 - 1}} \, dx = \frac{\sqrt{2}}{\pi} \left(-\sqrt{2}i\zeta\left(\frac{1}{2}\right) + 2i\sqrt{2}\zeta\left(\frac{3}{2}\right) + 4\sqrt{2}\zeta\left(\frac{3}{2}\right)ight). \] (19)

Proof. Use Equation (13) and set $a = -1, n = -1/2, k = 1/2, m = -1/2$, and simplify using entry (4) in the Table below (64:12:7) and entry (2) in the Table below (64:7) in [4].

Proposition 7. For $-1 < \text{Re}(m) < 1, -1 < \text{Re}(p) < 1$,
\[
\int_0^1 \left( -\frac{x^p \sqrt{x^{-p-1} - 1}}{\log(x^{-p-1} - 1)} + \frac{x^n \sqrt{x^{-n-1} - 1}}{\log(x^{-n-1} - 1)} \right) \, dx = \frac{8G - i\pi^2}{4\pi(n+1)(p+1)}(n-p).
\] (20)

Proof. Use Equation (13) and form a second equation by replacing $n \to p$ and take their difference. Next, using the resulting equation set $k = -1, a = 1, m = 1/2$, and simplify using entry (4) in Table below (64:12:7), Equations (64:4:1) and (64:7:1) in [4], and Equation (2) in [5].
Proposition 8.
\[
\int_0^1 x^n \sqrt{x^{-n-1}} \log \left( \log \left( x^{-n-1} - 1 \right) \right) dx = \frac{n}{4 \pi n + 7} \left( -4 \log \Gamma \left( -\frac{3}{4} \right) + 4 \log \Gamma \left( -\frac{1}{4} \right) - 2i + i\pi + 2 \log \left( \frac{4\pi}{\pi} \right) \right).
\] (21)

Proof. Use Equation (13) and set \( m = 1/2, \ a = 1 \) and simplify using entry (4) in Table below (64:12:7) in [4]. Next we take the first partial derivative with respect to \( k \) and apply L'Hopital's rule as \( k \to 0 \) and simplify using Equation (64:10:2) in [4].

Proposition 9.
\[
\int_0^1 \sqrt{x^{-1}} \log \left( \log \left( \frac{1}{x} - 1 \right) \right) dx = \frac{1}{2\pi \pi} \left( -4 \log \Gamma \left( -\frac{3}{4} \right) + 4 \log \Gamma \left( -\frac{1}{4} \right) \right)
\] (22)

Proof. Use Equation (21) and set \( n = -1/2 \) and simplify.

Proposition 10.
\[
\int_0^1 x \left( \frac{1}{x} - 1 \right)^p \left( \frac{1}{x} - 1 \right)^m dx = \frac{1}{4\pi} \left( ie^{i\pi m} \Phi \left( e^{2i\pi n}, 2, \frac{1}{2} \right) - ie^{i\pi p} \Phi \left( e^{2i\pi m}, 2, \frac{1}{2} \right) + 4\pi m \tan^{-1} \left( e^{i\pi m} \right) - 4\pi p \tan^{-1} \left( e^{i\pi p} \right) \right).
\] (23)

Proof. Use Equation (13) and form a second equation by replacing \( m \to p \) and take their difference. Next using the resulting equation, we set \( a = 1, k = -1, n = 1 \), and simplify using entry (5) in the Table below (64:12:7) in [4].

Proposition 11.
\[
\int_0^1 x \left( \sqrt{x^{-1}} - \sqrt{x} \right) dx = \frac{\left( 2iG - 6 - \sqrt{-7} \Phi \left( -1^{2/3}, 2, \frac{1}{2} \right) + \pi^2 + 2i\pi \log(3) \right)}{24\pi}.
\] (24)

Proof. Use Equation (23) and set \( m = 1/2, p = 1/3 \) and simplify.

Proposition 12.
\[
\int_0^1 x \left( \frac{1}{x^n} - 1 \right)^p \left( \frac{1}{x^n} - 1 \right)^m dx = \frac{1}{2\pi n + 2\pi} \left( ie^{i\pi m} \Phi \left( e^{2i\pi p}, 2, \frac{1}{2} \right) - ie^{i\pi p} \Phi \left( e^{2i\pi m}, 2, \frac{1}{2} \right) + 4\pi m \tan^{-1} \left( e^{i\pi m} \right) \right)
\] (25)

Proof. Use Equation (13) and form a second equation by replacing \( m \to p \) and take their difference. Use the resulting equation and set \( a = 1, k = -1 \) and simplify.

Proposition 13.
\[
\int_0^1 \frac{2x^{n+1} - 1}{x^n \sqrt{x^{-n-1} - 1} \log(x^{-n-1} - 1)} dx = -\frac{4G}{\pi n + \pi}.
\] (26)

Proof. Use Equation (25) and set \( m = 1/2, p = -1/2 \) and simplify using entry (4) in Table below (64:12:7) in [4].
Proposition 14.

\[
\int_{0}^{1} x^{n-1} \sqrt{x^n - 1} \log^k (1 - x^{-n}) \, dx = \frac{\pi}{8k} \left( -2^k \left( \frac{2k+1}{2} \right)^k \xi(-k) \cos \left( \frac{2k+1}{2} \right) \Gamma(k+1) \right).
\]  

(27)

**Proof.** Use Equation (13) and set \( m = 1/2 \) and simplify using entry (5) in the Table below (64:12:7) in [4]. Next set \( a = -1, n \to n - 1 \) and simplify using entry (20 in the Table below (64:7) in [4]. \( \square \)

Proposition 15.

\[
\int_{0}^{1} x^{n-1} \sqrt{x^n - 1} \log(1 - x^{-n}) \, dx = \frac{(2 + i\pi)\pi}{2\pi}.
\]  

(28)

**Proof.** Use Equation (27) and apply L’Hôpital’s rule as \( k \to 1 \) and simplify. \( \square \)

Proposition 16.

\[
\int_{0}^{1} \frac{x^n \sqrt{x^n - 1}}{\log^2 (1 - x^{-n})} \, dx = -\frac{\pi}{48(n+1)} - \frac{3i\zeta(3)}{8\pi^2(n+1)}.
\]  

(29)

**Proof.** Use Equation (13) and set \( m = 1/2 \) and simplify in terms of the Hurwitz zeta function using entry (4) in the Table below (64:12:7) in [4]. Next set \( k = -2, a = -1 \) and simplify. \( \square \)

Proposition 17.

\[
\int_{0}^{1} \frac{x^{n-1}}{\log^2 \left( \frac{1}{x} - 1 \right)} \, dx = -\frac{\pi}{8\pi} \left( 2\pi m \left( \Phi \left( e^{2im\pi}, 1, \frac{n-ia}{2\pi} \right) - \Phi \left( e^{2im\pi}, 2, \frac{n+ia}{2\pi} \right) \right) \right).
\]  

(30)

**Proof.** Use Equation (13) and replace \( a \to e^a, k \to -1 \). Next form a second equation by replacing \( a \to -a \) and take their difference and set \( n = 1 \). \( \square \)

Proposition 18.

\[
\int_{0}^{1} \frac{x^n(x^n - 1)^m \log(x^n - 1)}{\log^2(x^n - 1)} \, dx = -\frac{\pi}{2\pi} \left( 2\pi m \left( \Phi \left( e^{2im\pi}, 1, \frac{x^n + 1}{2\pi} \right) + \Phi \left( e^{2im\pi}, 2, \frac{x^n - 1}{2\pi} \right) \right) \right).
\]  

(31)

**Proof.** Use (13) set \( k = -1, a = e^{ia} \) rationalize the dominator and equate real and imaginary parts. \( \square \)

Proposition 19.

\[
\int_{0}^{1} \frac{\sqrt{\frac{1}{\pi^2} - 1} x \log \left( \frac{1}{x} - 1 \right)}{\log^2 \left( \frac{1}{x} - 1 \right) + 4\pi^2} \, dx = \frac{1 - G}{\pi}
\]  

(32)

and

\[
\int_{0}^{1} \frac{\sqrt{\frac{1}{\pi^2} - 1} x}{\log^2 \left( \frac{1}{x} - 1 \right) + 4\pi^2} \, dx = -\frac{\pi - 4}{16\pi}.
\]  

(33)
Proof. Use (31) set $a = 2\pi, m = 1/2, n = 1$, and simplify using entry (4) in the Table below (64:12:7) in [4].

\begin{equation}
\int_0^1 \frac{x^n}{a^2 + \log^2(\frac{1}{x^n} - 1)} \, dx = \frac{\psi^{(1)}(\frac{a + \pi}{2\pi})}{2\pi a}. \tag{34}
\end{equation}

Proof. Use (31) and set $m = 0$ and simplify using Equation (64:4:1) in [4].

Proposition 21.

\[ \int_0^1 \frac{x}{\log^2(\frac{1}{x^n} - 1) + \pi^2} \, dx = \frac{6\zeta(3) + \pi^2}{48\pi^4}. \tag{35} \]

Proof. Use Equation (34) take the first partial derivative with respect to $a$ and set $a = \pi, n = 1$ and simplify using Equation (44:12:5) in [4].

Proposition 22.

\[ \int_0^1 \frac{\sqrt{\frac{1}{x^n} - 1} \log(\frac{1}{x^n} - 1)}{\log(\frac{x^n}{x^n - 1})} \, dx = -\frac{\pi^2}{24} - \frac{\pi}{2} \log(2). \tag{39} \]

Proof. Use Equation (38) and set $k = -1, m = -1/2, n = 0$, and simplify using entry (4) in Table below (64:12:7) and entry (2) below Table (64:7) in [4].

Proposition 26.

\[ \int_0^1 \frac{1}{\sqrt{x^n - 1} \log(\frac{x^n}{x^n - 1})} \, dx = i^{k+1}(2^k - 1)k(2\pi)^k\zeta(1 - k) - (2i)^k(2^{k+1} - 1)\pi^{k+1}\zeta(-k). \tag{40} \]

Proof. Use Equation (13) and set $a = -1, m = 1/2, n = 0$, and simplify using entry (4) in the Table below (64:12:7) and entry (2) below Table (64:7) in [4].
Proposition 27.
\[
\int_0^1 \sqrt{\frac{1}{x} - 1} \log \left( \frac{x-1}{x} \right) \, dx = \pi + \frac{i\pi^2}{2}.
\] (41)

**Proof.** Use Equation (40) and set \( k = 1 \) and simplify. \( \square \)

Proposition 28.
\[
\int_0^1 \sqrt{\frac{1}{x} - 1} \, dx = \frac{1}{24} \left( \pi - 12i \log(2) \right) \] (42)

**Proof.** Use Equation (40) and apply L'Hopital's rule as \( k \to -1 \) and simplify. \( \square \)

Proposition 29.
\[
\int_0^1 \sqrt{\frac{1}{x} - 1} \log \left( \log \left( \frac{1}{x} - 1 \right) \right) \, dx = \frac{i\pi^2}{4} + \left( \pi - i \right) \log(2) \] (43)

**Proof.** Use Equation (40) to take the first partial derivative with respect to \( k \) and apply L'Hopital's rule as \( k \to 0 \) and simplify. \( \square \)

Proposition 30.
\[
\int_0^1 x^2 \left( \frac{1}{x^2} - 1 \right)^m \log^k \left( 1 - \frac{1}{x} \right) \, dx = -\frac{1}{2} (2i\pi)^k e^{-i\pi m} \left( k \text{Li}_{1-k} \left( e^{2i\pi} \right) \right) + 2i\pi e^{i\pi m} \text{Li}_{1-k} \left( e^{2i\pi} \right) \] (46)

**Proof.** Use (13) and set \( a = -1, n = 2 \), and simplify using Equation (64:12:2) in [4]. \( \square \)

Proposition 31.
\[
\int_0^1 x^m \log \left( \frac{1 - 1}{x^2} \right) \, dx = \frac{1}{9} \pi \left( 3i + 3(-1)^{5/6} + 2i \right) \] (47)

**Proof.** Use Equation (46) and set \( k = 1, m = -1/3 \), and simplify using entry (4) in the Table below (25:12:5) in [4]. \( \square \)
Proposition 34.

\[
\int_0^1 \frac{x \log \left( \frac{x^2}{1-x^2} \right)}{\sqrt{\frac{1}{x^2} - 1 \left( \log^2 \left( \frac{1}{x^2} - 1 \right) + \pi^2 \right)}} \, dx = \frac{\pi}{48}
\]  

(48)

and

\[
\int_0^1 \frac{x}{\sqrt{\frac{1}{x^2} - 1 \left( \log^2 \left( \frac{1}{x^2} - 1 \right) + \pi^2 \right)}} \, dx = \frac{\log(2)}{4\pi}.
\]  

(49)

**Proof.** Use Equation (46) and set \( k = -1, m = -1/2 \), and rationalize the denominator and equate real and imaginary parts and simplify. \( \square \)

Proposition 35.

\[
\int_0^1 \frac{\log \left( \frac{1}{\sqrt{x}} \right)}{\sqrt{\frac{1}{x^2} - 1 \sqrt{x}}} \, dx = (2i)^{k+1} \left( 1 - 2^k \right) k\pi^2 \zeta(1-k) - i^k \left( 2^{k+1} - 1 \right) (2\pi)^{k+1} \zeta(-k).
\]  

(50)

**Proof.** Use (13) and set \( a = -1, m = -1/2, n = -1/2 \), and simplify using entry (4) in the Table below (64:12:7) and entry (2) in the Table below (64:7) in [4]. \( \square \)

Proposition 36.

\[
\int_0^1 \frac{1}{\sqrt{\frac{1}{x^2} - 1 \sqrt{x}} \log \left( 1 - \frac{1}{\sqrt{x}} \right)} \, dx = -\frac{\pi}{12} - i \log(2).
\]  

(51)

**Proof.** Use Equation (50) and apply L'Hopital’s rule as \( k \to -1 \) and simplify using Equation (45:12:5) in [4]. \( \square \)

Proposition 37.

\[
\int_0^1 \frac{\log \left( \log \left( 1 - \frac{1}{\sqrt{x}} \right) \right)}{\sqrt{\frac{1}{x^2} - 1 \sqrt{x}}} \, dx = -\frac{1}{8} \pi \left( -2i\pi (-36 \log(A) + \log(128) + 3 \log(\pi)) + 12 + 3\pi^2 + 24 \log(2) \right).
\]  

(52)

**Proof.** Use Equation (50) take the first partial derivative with respect to \( k \) then set \( k = -1 \) and simplify using Equation (44:12:7) in [4]. \( \square \)

Proposition 38.

\[
\int_0^1 \frac{\log \left( \log \left( 1 - \frac{1}{\sqrt{x}} \right) \right)}{\sqrt{\frac{1}{x^2} - 1 \sqrt{x}}} \, dx = \frac{i\pi^2}{2} + (\pi + i) \log(4).
\]  

(53)

**Proof.** Use Equation (50) and apply L’Hopital’s rule as \( k \to 0 \) and simplify using Equation (45:12:5) in [4]. \( \square \)

Proposition 39.

\[
\int_0^1 \frac{\log \left( \log \left( \frac{1}{\sqrt{x}} - 1 \right) \right)}{\sqrt{\frac{1}{x^2} - 1 \sqrt{x}}} \, dx = \frac{1}{2} \pi \left( -4 \log \Gamma \left( -\frac{3}{4} \right) + 4 \log \Gamma \left( -\frac{1}{4} \right) + 2i + i\pi ight. + 2 \log \left( \frac{\sqrt{2}}{\pi} \right).
\]  

(54)
Proof. Use Equation (13) and set \( a = 1, m = -1/2, n = -1/2 \), and simplify using entry (4) in the Table below (64:12:7) in [4]. Next, take the first partial derivative with respect to \( k \) followed by applying L’Hopital’s rule as \( k \to 0 \) and simplify using Equation (64:10:2) in [4]. □

Proposition 40.

\[
\int_0^1 \log\left(\frac{\sqrt{x} - 1}{\sqrt{x} + 1}\right) \log\left(\frac{\sqrt{x} - 1}{\sqrt{x} + 1}\right) \, dx = \pi \left( -4iC + 4 \log \Gamma\left( -\frac{3}{4} \right) - 4 \log \Gamma\left( -\frac{1}{4} \right) \right) - 2 - i\pi + \log\left( \frac{81}{16} \right) - 2 \log(\pi) \right). \tag{55}
\]

Proof. Use Equation (13) and set \( a = 1, m = -1/2, n = -1/2 \), and simplify using entry (4) in the Table below (64:12:7) in [4]. Next, take the first partial derivative with respect to \( k \) and set \( k = 1 \) and simplify using Equation (64:10:2) in [4], where \( C \) is Catalan’s constant Equation (1:7:4) in [4]. □

Proposition 41.

\[
\int_0^1 \log^2\left( \frac{\sqrt{x} - 1}{\sqrt{x} + 1}\right) \log\left(\frac{\sqrt{x} - 1}{\sqrt{x} + 1}\right) \, dx = \frac{1}{2} \pi^3 \left( 64\zeta'\left( -2, \frac{3}{4} \right) - 64\zeta'\left( -2, \frac{1}{4} \right) \right) + i\pi + \log\left( \frac{16}{21} \right) + 2 \log(\pi) + 16iG. \tag{56}
\]

Proof. Use Equation (13) and set \( a = 1, m = -1/2, n = -1/2 \), and simplify using entry (4) in the Table below (64:12:7) in [4]. Next, take the first partial derivative with respect to \( k \) and set \( k = 2 \) and simplify using Equation (64:10:2) in [4], where \( G \) is Catalan’s constant Equation (1:7:4) in [4]. □

7. Summary Table of Integrals

In this section we produce a summary Table 1 of definite integrals.

| \( f(x) \) | \( \int_0^1 f(x) \, dx \) |
| --- | --- |
| \( x^n (x^{-n-1} - 1)^m \) | \( \frac{n m \csc(m \pi)}{\pi (1 - m \csc(m \pi)) \csc(m \pi)} \) |
| \( x^n (x^{-n-1} - 1)^m \log(x^{-n-1} - 1) \) | \( -\frac{1}{2} \sqrt{-1} \sqrt{-1} \left( -4i\zeta\left( \frac{3}{2} \right) + 2i \sqrt{2} \zeta\left( \frac{1}{2} \right) + 4 \zeta\left( \frac{3}{2} \right) - 2 \sqrt{2} \zeta\left( \frac{1}{2} \right) \right) \) |
| \( \frac{\log(1 - x)}{\sqrt{2} \sqrt{1 - x} \log(1 - x)} \) | \( \frac{1}{2} \left( 8 i \pi (\pi^2 - 1) \right) \) |
| \( x^n \log\left( x^{-n-1} - 1 \right) \) | \( \frac{\log(1 - x^{-n-1})}{\sqrt{2} \sqrt{-1} \log(1 - x^{-n-1})} \) |
| \( x^n \log\left( x^{-n-1} - 1 \right) \) | \( \frac{1}{2} \left( 8 i \pi (\pi^2 - 1) \right) \) |
| \( x^n \log\left( x^{-n-1} - 1 \right) \) | \( \frac{\log(1 - x^{-n-1})}{\sqrt{2} \sqrt{-1} \log(1 - x^{-n-1})} \) |
| \( x^n \log\left( x^{-n-1} - 1 \right) \) | \( \frac{1}{2} \left( 8 i \pi (\pi^2 - 1) \right) \) |
| \( x^n \log\left( x^{-n-1} - 1 \right) \) | \( \frac{\log(1 - x^{-n-1})}{\sqrt{2} \sqrt{-1} \log(1 - x^{-n-1})} \) |
| \( x^n \log\left( x^{-n-1} - 1 \right) \) | \( \frac{1}{2} \left( 8 i \pi (\pi^2 - 1) \right) \) |
Table 1. Cont.

\| \int_0^1 f(x) \, dx \|
\| \sqrt{\frac{4}{9} - 1} \log \left(\frac{4}{9} - 1\right) \| \| 1 - \frac{G}{\pi} \|
\| \log \left(\frac{4}{9} - 1\right) + 4\pi^2 \| \| - \frac{\pi - 4}{16\pi} \|
\| \log \left(\frac{4}{9} - 1\right) + 4\pi^2 \| \| \log(2) \| \| \frac{1}{k^2} \|
\| \log \left(\frac{4}{9} - 1\right) + 4\pi^2 \| \| \frac{6\pi^3 + 4\pi^2}{48\pi^2} \|
\| \log \left(\frac{4}{9} - 1\right) + \frac{1}{k^2} \| \| \frac{1}{\sqrt{\frac{4}{9} - 1}} \|
\| \log \left(\frac{4}{9} - 1\right) + \frac{1}{k^2} \| \| (1)^{1/3} \left(48G + \pi \log(4096) - 5\pi\right) \|

8. Discussion

In this present work, we used our contour integral method to derive the logarithmic transform in terms of the Lerch function. We then used this new integral relation to derive new definite integrals in terms of the other special functions and fundamental constants. A table of definite integrals summarizing our results was produced for easy reading.

9. Conclusions

In this paper, we derived a method for expressing definite integrals in terms of special functions using our contour integration method. The contour we used was specific to solving integral representations in terms of the Lerch function. The present results in this work can be extended to fractional calculus and fractal calculus [6,7]. There is also a possible connection to the variational iteration method and the homotopy perturbation method [8]. We expect that other contours and integrals can be derived using this method for future work. The results presented were numerically verified for both real and imaginary values of the parameters in the integrals using Mathematica by Wolfram.

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