Stabilizing Mechanism for Bose-Einstein Condensation of Interacting Magnons in Ferrimagnets and Ferromagnets

Naoya Arakawa
Department of Physics, Toho University, Funabashi, Chiba, 274-8510, Japan
(Dated: November 5, 2018)

We propose a stabilizing mechanism for the Bose-Einstein condensation (BEC) of interacting magnons in ferrimagnets and ferromagnets. By studying the effects of the magnon-magnon interaction on the stability of the magnon BEC in a ferrimagnet and two ferromagnets, we show that the magnon BEC remains stable even in the presence of the magnon-magnon interaction in the ferrimagnet and ferromagnet with a sublattice structure, whereas it becomes unstable in the ferromagnet without a sublattice structure. This indicates that the existence of a sublattice structure is the key to stabilizing the BEC of interacting magnons, and the difference between the spin alignments of a ferrimagnet and a ferromagnet is irrelevant. Our result can resolve a contradiction between experiment and theory in the magnon BEC of yttrium iron garnet. Our theoretical framework may provide a starting point for understanding the physics of the magnon BEC including the interaction effects.

Bose-Einstein condensation (BEC) has been extensively studied in various fields of physics. The BEC is a macroscopic occupation of the lowest-energy state for bosons [1]. This phenomenon was theoretically predicted in a gas of noninteracting bosons [2], and then it was experimentally observed in dilute atomic gases [3–5]. This observation opened up research of the BEC in atomic physics [1]. Since the concept of the BEC is applicable to quasiparticles that obey Bose statistics, research of the BEC has been expanded, and it covers condensed-matter physics, nuclear physics, and optical physics.

There is a critical problem with the magnon BEC. The magnon BEC was experimentally observed in yttrium iron garnet (YIG), a three-dimensional ferrimagnet [6–9]. However, a theory [10] showed that if low-energy magnons of YIG are approximated by magnons of a ferromagnet without a sublattice structure, the magnon BEC is unstable due to the attractive interaction between magnons. Note first, that YIG is often treated as the Holstein-Primakoff transformation [15–17], we derive the magnon Hamiltonian by using the Holstein-Primakoff transformation [15–17]. After remarking on several properties in the BEC of noninteracting magnons, we will focus mainly on the magnon BEC of YIG. (We will focus mainly on the magnon-magnon interaction effects in the ferrimagnet.)

We use the Heisenberg Hamiltonian as a minimal model for ferrimagnets and ferromagnets. It is given by

\[ H = 2 \sum_{\langle i,j \rangle} J_{ij} S_i \cdot S_j, \]

where \( J_{ij} \) denotes the Heisenberg exchange energy between spins at nearest-neighbor sites, and \( S_i \) denotes the spin operator at site \( i \).

We consider three cases. In the first case, we put \( J_{ij} = J \), \( \langle S_i \rangle = S_A \) for \( i \in A \), and \( \langle S_i \rangle = -S_B \) for \( i \in B \), where \( A \) and \( B \) denote \( A \) and \( B \) sublattices, respectively; each sublattice consists of \( N/2 \) sites. This case corresponds to a ferrimagnet with a two-sublattice structure [Fig. 1(a)]. In the second case, we put \( J_{ij} = -J \), \( \langle S_i \rangle = S_A \) for \( i \in A \), and \( \langle S_i \rangle = S_B \) for \( i \in B \). The second and third cases correspond to ferromagnets without sublattice and with a two-sublattice structure, respectively [Figs. 1(b) and 1(c)]. As we will show below, by studying the BEC of interacting magnons in these three cases, we can clarify the stabilizing mechanism in a ferrimagnet and the key to resolving the contradiction in the magnon BEC of YIG. (We will focus mainly on the sign of the effective interaction between magnons and its effect on the stability of the magnon BEC.)

We begin with the first case of our model. We first derive the magnon Hamiltonian by using the Holstein-Primakoff transformation [15–17]. After remarking on several properties in the BEC of noninteracting magnons, we construct the effective theory for the BEC of interacting magnons. By using this theory, we study the interaction effects in the ferrimagnet.
The magnon Hamiltonian is obtained by applying the Holstein-Primakoff transformation to the spin Hamiltonian. In general, low-energy excitations in a magnet can be described well by magnons, bosonic quasiparticles [15–17]. The magnon operators and the spin operators are connected by the Holstein-Primakoff transformation [15, 17]. This transformation for the magnon Hamiltonian is expressed as follows:

\[
S_\pm^z = S_A - a_\pm a_\mp, \quad S_\pm^y = \sqrt{2S_Aa_\pm^\dagger} \sqrt{1 - \frac{a_\pm a_\mp}{2S_A}}, \quad (2)
\]

\[
S_\pm^x = -S_B + b_\pm^\dagger b_\mp, \quad S_\pm^y = \sqrt{2S_Bb_\pm^\dagger} \sqrt{1 - \frac{b_\pm b_\mp}{2S_B}}, \quad (3)
\]

where \(i \in A, \quad j \in B, \quad S_\pm^z = S_\pm^y - iS_\pm^x = (S_i^z)^\dagger, \) and \(S_\pm^x = S_\pm^x + iS_\pm^y = (S_j^x)^\dagger;\) \(a_i^\dagger \) and \(a_i^\dagger \) are the operators of magnons for the \(A\) sublattice, and \(b_j^\dagger \) and \(b_j^\dagger \) are those for the \(B\) sublattice. A substitution of Eqs. (2) and (3) into Eq. (1) gives the magnon Hamiltonian.

In the magnon Hamiltonian, we consider the kinetic energy terms and the dominant terms of the magnon-magnon interaction. This is because our aim is to clarify how the magnon-magnon interaction affects the magnon BEC, which is stabilized by the kinetic energy terms. Since the kinetic energy terms come from the quadratic terms of magnon operators and the dominant terms of the interaction come from part of the quartic terms [16–17], our magnon Hamiltonian is given by \(H_{\text{mag}} = H_{\text{non}} + H_{\text{int}}\) [24], where

\[
H_{\text{non}} = 2 \sum_q J(0)(S_Ba_q^\dagger a_q + S_Ab_q^\dagger b_q) + 2 \sum_q J(q)\sqrt{S_A S_B}(a_q b_q + a_q^\dagger b_q^\dagger), \quad (4)
\]

and

\[
H_{\text{int}} = -\frac{2}{N} \sum_{q, q'} [J(0)a_q^\dagger a_{q'} b_{q'}^\dagger b_q + J(q - q')a_q^\dagger a_{q'} b_{q'}^\dagger b_q + \frac{J(q)}{\sqrt{S_A S_B}}(S_A a_q b_{q'}^\dagger b_q + S_B b_q a_{q'}^\dagger a_q)] + (\text{H.c.}), \quad (5)
\]

We have used \(a_i^\dagger = \sqrt{\frac{2}{N}} \sum_q e^{iq \cdot i} a_i, \quad b_j^\dagger = \sqrt{\frac{2}{N}} \sum_q e^{iq \cdot j} b_j\), and \(J(q) = \sum_\delta J\epsilon^q \delta \) with \(\delta\) a vector to nearest neighbors.

Before formulating the effective theory for the BEC of interacting magnons, we remark on several properties in the BEC of noninteracting magnons in our magniferm. To see the properties, we diagonalize \(H_{\text{non}}\) by using

\[
\begin{pmatrix}
\alpha_q^\dagger \\
\beta_q^\dagger
\end{pmatrix} = \begin{pmatrix}
\epsilon_q & -s_q \\
-s_q & \epsilon_q
\end{pmatrix} \begin{pmatrix}
\alpha_q \\
\beta_q
\end{pmatrix}, \quad (6)
\]

where \(\epsilon_q \equiv \cosh \theta_q\) and \(s_q \equiv \sinh \theta_q\) satisfy tanh \(2\theta_q = \frac{2\sqrt{S_A S_B J(q)}}{(S_A + S_B)J(0)}\). After some algebra, we obtain

\[
H_{\text{non}} = \sum_q \epsilon_q \alpha_q^\dagger \alpha_q + \sum_q \epsilon_q \beta_q^\dagger \beta_q, \quad (7)
\]

where \(\epsilon_q(\alpha) = (S_B - S_A)J(0) + \Delta\epsilon(q)\) and \(\epsilon_q(\beta) = (S_A - S_B)J(0) + \Delta\epsilon(q)\) with \(\Delta\epsilon(q) = \sqrt{(S_A + S_B)^2 J(0)^2 - 4S_A S_B J(q)^2}\). In Eq. (7), we have neglected the constant terms. Hereafter, we assume \(S_A > S_B\): this does not lose generality. For \(S_A > S_B\), \(\epsilon_q(0) = 0\) is the lowest energy. Thus many magnons occupy the \(q = 0\) state of the \(\alpha\) band in the BEC of noninteracting magnons in the fermigram for \(S_A > S_B\). In addition, the low-energy excitations from the condensed state are described by the \(\alpha\)-band magnons near \(q = 0\).

We now construct the effective theory for the BEC of interacting magnons. To construct it as simple as possible, we utilize the properties in the BEC of noninteracting magnons. As described above, in the fermigram for \(S_A > S_B\) the condensed state is the \(q = 0\) state of the \(\alpha\) band and the low-energy noncondensed states are the small-\(q\) states of the \(\alpha\) band. Thus we can reduce \(H_{\text{mag}}\) to an effective Hamiltonian \(H_{\text{eff}}\), which consists of the kinetic energy term of the \(\alpha\) band and the intraband terms of the magnon-magnon interaction for the \(\alpha\) band; \(H_{\text{eff}}\) is given by \(H_{\text{eff}} = H_0 + H'\), where \(H_0\) is the first term of Eq. (7), and \(H'\) is obtained by substituting Eq. (6) into Eq. (5) and retaining the intraband terms. This \(H_{\text{eff}}\) is sufficient for studying properties of the BEC of interacting magnons at temperatures lower than a Curie temperature, because the dominant excitations come from the first, second, and third cases, respectively. The direction and length of an arrow represent the direction and size of an ordered spin. The ordered spins are ferrimagnetic in panel (a) and ferromagnetic in panels (b) and (c); sublattice degrees of freedom are present in panels (a) and (c) and absent in panel (b).
the small-$q$ magnons in the $a$ band and the interband terms may be negligible in comparison with the intra-band terms. Then we can further simplify $H'$. Since its main effects can be taken into account in the mean-field approximation, the leading term of $H'$ is given by \cite{21}
\begin{equation}
H' = -\frac{4}{N} \sum_{q, q'} \Gamma_{aa}(q, q') n_{q', a} a_{q}^\dagger a_{q'},
\end{equation}
where $\Gamma_{aa}(q, q') = J(0) \sqrt{q_{S} S_{q}} c_{q} s_{q} c_{q'} s_{q'} - 2J(q - q') c_{q} s_{q} c_{q'} s_{q'} + \left(\frac{J(q)}{\sqrt{S_{q} S_{q}'} c_{q} s_{q} S_{q} S_{q}'} c_{q'} s_{q'} - \frac{J(q)}{\sqrt{S_{q} S_{q}'} c_{q} s_{q} S_{q} S_{q}'} c_{q'} s_{q'}\right) + J(q - q') c_{q} s_{q} c_{q'} s_{q'} - 2J(q - q') c_{q} s_{q} c_{q'} s_{q'} + (\text{H.c.}).$

By applying the mean-field approximation to $H_{\text{int}}$, the leading term of the magnon-magnon interaction is reduced to $H' = -\frac{4}{N} \sum_{q, q'} \Gamma_{aa}(q, q') n_{q, a} c_{q} c_{q'}$ and $n_{q, a} = \langle a_{q}^\dagger a_{q} \rangle$. By combining Eq. (5) with $H_0 = \sum_{q} \epsilon_{a}(q) a_{q}^\dagger a_{q}$, we obtain
\begin{equation}
H_{\text{eff}} = \sum_{q} \epsilon_{a}(q) a_{q}^\dagger a_{q},
\end{equation}
with $\epsilon_{a}(q) = \epsilon_{a}(q) - \frac{\hbar}{2} \sum_{q} \Gamma_{aa}(q, q') n_{q, a}$.

By using the theory described by $H_{\text{eff}}$, we study the interaction effects on the stability of the magnon BEC. Since the magnon energy should be nonnegative, the magnon BEC remains stable even for interacting magnons as long as $\epsilon_{a}(0)$ remains the lowest energy. This is realized if $H'$ is the repulsive interaction. If $H'$ is the attractive interaction, the magnon BEC becomes unstable. Thus we need to analyze the sign of $\Gamma_{aa}(q, q')$ in Eq. (3). Since the dominant low-energy excitations are described by the $a$-band magnons near $q = 0$, we estimate $\Gamma_{aa}(q, q')$ in Eq. (3) in the long-wavelength limits $|q|, |q'| \rightarrow 0$. For a concrete simple example we perform this estimation in a three-dimensional case on the cubic lattice. By expressing $J(q)$ in a Taylor series around $q = 0$ and retaining the leading correction, we get $J(q) \approx J(0) [1 - \frac{q^{2}}{6}]$. Then, by using this expression and performing some calculations \cite{21}, we obtain the expression of $\Gamma_{aa}(q, q')$ including the leading correction in the long-wavelength limits. The derived expression is
\begin{equation}
\Gamma_{aa}(q, q') \approx -\frac{2}{9} J(0) q^{2} q'^{2} \frac{(S_{a} S_{b})^{2}}{(S_{a} - S_{b})^{3}},
\end{equation}
The combination of Eqs. (10) and (8) shows that the leading term of the magnon-magnon interaction is repulsive. Thus the magnon BEC remains stable in the ferromagnet even with the magnon-magnon interaction.

The above result differs from the stability of the magnon BEC in the ferromagnet without a sublattice structure. This can be seen by applying a similar theory to the second case of our model and comparing the result with the above result. The Holstein-Primakoff transformation in the ferromagnet without a sublattice structure is expressed as $S_{i}^{a} = S - c_{i}^{a} c_{i}$, $S_{i}^{-} = c_{i}^{a} \sqrt{2S - c_{i}^{a} c_{i}}$, and $S_{i}^{+} = (S_{i}^{-})^{\dagger}$ for all $i$; $c_{i}$ and $c_{i}^{a}$ are the magnon operators. By using this transformation and the Fourier transformations of the magnon operators, such as $c_{i} = \frac{1}{\sqrt{N}} \sum_{q} e^{i q i} c_{q}$, we obtain the magnon Hamiltonian $H_{\text{mag}} = H_{\text{non}} + H_{\text{int}}$.

where $H_{\text{non}} = \sum_{q} \epsilon(q) c_{q} c_{q}$ with $\epsilon(q) = 2S[J(0) - J(q)]$ and $H_{\text{int}} = -\frac{\hbar}{N} \sum_{q, q'} J(0) c_{q} c_{q'} c_{q} c_{q'} + J(q - q') c_{q} c_{q'} c_{q} c_{q'} - 2J(q - q') c_{q} c_{q'} c_{q} c_{q'} + (\text{H.c.}).$.

Then, by applying the mean-field approximation to $H_{\text{int}}$, the leading term of the magnon-magnon interaction is reduced to $H' = -\frac{2}{N} \sum_{q, q'} \Gamma(q, q') n_{q} c_{q} c_{q'}$, where $(\Gamma(q, q') = J(0) + J(q - q') - J(q) - J(q')$ and $n_{q} = \langle c_{q} c_{q'} \rangle$. Since $\Gamma(q, q') \geq 0$, the magnon-magnon interaction becomes attractive. Thus the BEC of interacting magnons becomes unstable in the ferromagnet without a sublattice structure.

In order to understand the key to causing the above difference, we study the stability of the BEC of interacting magnons in the third case of our model. As we can see from Fig. (4), the difference between the third and first cases is about the spin alignment, and the difference between the third and second cases is about the sublattice structure. Thus, by comparing the result in the third case with the result in the first or second case, we can deduce which of the two, the differences in the spin alignment and in the sublattice structure, causes the difference in the stability of the BEC of interacting magnons.

The stability in the third case can be studied in a similar way to that in the first case. In the third case, the Holstein-Primakoff transformation of $S_{i}$ for $i \in A$ is the same as Eq. (2), whereas that of $S_{j}$ for $j \in B$ is given by $S_{j}^{z} = S_{B} - b_{j}^{\dagger} b_{j}$, $S_{j}^{+} = \sqrt{2S_{B}} b_{j}^{\dagger} \sqrt{2S_{B}} [1 - (b_{j} b_{j}^{\dagger} / 2S_{B})]$, and $S_{j}^{+} = (S_{j}^{+})^{\dagger}$; this difference arises from the different alignment of the spins belonging to the $B$ sublattice. In a similar way to the first case, we obtain the magnon Hamiltonian $H_{\text{mag}} = H_{\text{non}} + H_{\text{int}}$, where $H_{\text{non}}$ and $H_{\text{int}}$ are given by
\begin{equation}
H_{\text{non}} = 2 \sum_{q} J(0) (S_{a} b_{q}^{\dagger} a_{q} + S_{B} b_{q}^{\dagger} b_{q}),
\end{equation}
and
\begin{equation}
H_{\text{int}} = -\frac{2}{N} \sum_{q, q'} \left[J(0) a_{q} a_{q'} b_{q}^{\dagger} b_{q}^{\dagger} b_{q} + J(q - q') a_{q} a_{q'} b_{q}^{\dagger} b_{q}^{\dagger} b_{q} - \frac{J(q)}{\sqrt{S_{a} S_{B}}} \left(S_{a} b_{q}^{\dagger} b_{q}^{\dagger} a_{q} a_{q'} + S_{B} b_{q}^{\dagger} a_{q} a_{q'} b_{q}^{\dagger} b_{q}^{\dagger}\right)\right] + (\text{H.c.}),
\end{equation}
respectively, with $a_{i} = \sqrt{\frac{2}{N}} \sum_{q} e^{iqiq_i} a_{q}$ and $b_{j} = \sqrt{\frac{2}{N}} \sum_{q} e^{iqjq_j} b_{q}$. In addition, $H_{\text{non}}$ can be diagonalized by using $a_{q} = c_{q} a_{q} - s_{q} b_{q}$ and $b_{q} = -s_{q} a_{q} + c_{q} b_{q}$, where $c_{q} \equiv \cosh \theta_{q}$ and $s_{q} \equiv \sinh \theta_{q}$ satisfy $\tan h 2\theta_{q} = -\frac{2 \sqrt{S_{a} S_{B}} J(0)}{S_{a} S_{B}}$. The diagonalized $H_{\text{non}}$ is
\begin{equation}
H_{\text{non}} = \sum_{q} \epsilon_{a}(q) c_{q} c_{q} + \epsilon_{B}(q) b_{q}^{\dagger} b_{q},
\end{equation}
with $\epsilon_{a}(q)$ and $\epsilon_{B}(q)$, which are the same as those in the first case. Thus,
the ferromagnet and ferrimagnet with the two-sublattice structure have the same properties of the BEC of noninteracting magnons. Then we can construct the effective theory for the BEC of interacting magnons in the third case in a similar way. For \( S_A > S_B \), in the third case, the BEC of interacting magnons can be effectively described by \( H_{\text{eff}} = \sum_q c_q^* \sigma_A^a q \sigma_A^a q + c_q^* \sigma_A^a q \sigma_A^a q - \frac{1}{N} \sum_q \Gamma_{\alpha\alpha}(q, q') n_{q' \alpha} \), where \( \Gamma_{\alpha\alpha}(q, q') = J(0)(c_q^2 + c_q^2) + 2J(q - q')c_q c_q c_{q'} c_{q'} + \frac{j(q)}{\sqrt{S_A S_B}} c_q c_q (S_A S_B^2 + S_B^2 c_{q'}^2) + \frac{j(q)}{\sqrt{S_A S_B}} c_q c_{q'} (S_A S_B^2 + S_B^2 c_{q'}^2) \). By estimating \( \tilde{\Gamma}_{\alpha\alpha}(q, q') \) in the long-wavelength limits in a similar way, we obtain \( \tilde{\Gamma}_{\alpha\alpha}(q, q') \approx -\frac{2}{3} J(0) q^2 q'^2 (S_A S_B) \). Thus the BEC of interacting magnons is stable in the ferromagnet with the two-sublattice structure.

Combining the results in the three cases, we find that the difference between the interaction effects in the ferromagnet and in the ferromagnet without a sublattice structure arises not from the difference in the spin alignment, but from the difference in the sublattice structure. This can resolve the contradiction between experiment \cite{6-9} and theory \cite{10} because that theory uses a ferromagnet without a sublattice structure. This also suggests that the existence of a sublattice structure is necessary to reconsider some results of YIG if the results are deduced by using a ferromagnet without a sublattice structure, in particular, the results depend on the sign of the magnon-magnon interaction. Our theoretical framework can then be used to study the BEC of interacting magnons in other magnets as long as the low-energy magnons can be described by a single magnon band. For the magnets whose low-energy magnons have degeneracy, an extension of this framework enables us to study the BEC of interacting magnons. Thus our theory may provide a starting point for understanding properties of the BEC of interacting magnons in various magnets.

In summary, we have studied the stability of the BEC of interacting magnons in a ferrimagnet and ferromagnets, and we proposed the stabilizing mechanism. By adopting the Holstein-Primakoff transformation to the Heisenberg Hamiltonian, we have derived the magnon Hamiltonian, which consists of the kinetic energy terms and the dominant terms of the magnon-magnon interaction. We then construct the effective theory for the BEC of interacting magnons by utilizing the properties for noninteracting magnons and the mean-field approximation. From the analyses using this theory, we have deduced that in the ferrimagnet and ferromagnet with the sublattice structure the magnon BEC remains stable even in the presence of the magnon-magnon interaction, whereas it becomes unstable in the ferromagnet without a sublattice. This result shows that the existence of a sublattice structure is the key to stabilizing the BEC of interaction magnons, whereas the difference in the spin alignments is irrelevant. In addition, this result is consistent with the experimental results \cite{6-9} of YIG and the
theoretical result \[10\] of a ferromagnet without a sublattice structure.