Applications of wave equations with logarithmic nonlinearity in fluid mechanics

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Abstract. We apply statistical mechanics and Madelung hydrodynamical presentation for an effective description of strongly-interacting many-body systems, such as Bose liquids or Korteweg-type fluids. The logarithmic nonlinearity is shown to appear in equations describing such fluids. The resulting equations describe the irrotational and isothermal flow of a two-phase barotropic compressible inviscid fluid with internal capillarity and surface tension. We demonstrate spontaneous symmetry breaking in this class of fluids, which leads to a number of wave-mechanical and topological effects. We show the relationship between the “logarithmic” fluids and those described by polynomially nonlinear wave equations, such as the Gross-Pitaevskii one.

1. Introduction

It has been recently proposed to use the barotropic Korteweg capillary fluids as a model of magma transport in a conduit [1, 2]. In such a class of models, often referred as diffuse interface models, the capillary interface is viewed as a diffusion transition domain of rapid smooth variation of density, while the surface tension is intrinsically incorporated [3, 4, 5, 6]. This allows to describe flows with a spontaneous nucleation, coalescence and breakdown of density inhomogeneities in the two-phase systems [7]. The Schrödinger-like equation with a logarithmic nonlinearity is shown to appear in equations describing such fluids [1, 2].

In what follows, we place statistical and thermodynamic foundations under this approach and deduce physical implications.

2. Wave equation

Suppose we have a fluid flowing in a cylindrical channel which is in thermal contact with a reservoir of infinitely large heat capacity as to maintain constant temperature; we assume also that its microscopic structure can be regarded as a many-body system of particles, atoms and molecules, which interaction energy is larger than their kinetic energy. Then the probability $P$ of a microstate is given by a Boltzmann rule, in which the kinetic energy can be neglected compared to the potential one:

$$P \propto \exp \left( -\frac{E}{T} \right) \approx \exp \left( -\frac{U}{T} \right),$$

(1)

where $T$, $E$ and $U$ are the absolute temperature, energy and potential energy of a many-body system, respectively; we work in the units where the Boltzmann constant $k_B = 1$. At
a microscopical level, this fluid is described by an extremely large set of particles’ evolution equations, therefore certain simplifications must be made.

Namely, we employ the fluid approximation used in continuum mechanics: we must make a transition to a single equation for the fluid wavefunction written in a Madelung form [8, 1, 2]

$$\Psi = \sqrt[\rho] \exp (iS),$$

where $\rho = \rho(x, t)$ is a fluid density, and $S = S(x, t)$ is a phase which is related to the fluid velocity $\mathbf{u} = \nabla S$ (we assume the irrotational flow for simplicity).

This wavefunction should not be confused with particle wavefunctions in quantum mechanics, but rather it is a complex-valued function which stores macroscopic information about a fluid, such as the density and velocity of a fluid parcel, as discussed in Ref. [8]. Nevertheless, some mathematical similarities between these categories of functions do exist and will be used in what follows.

In particular, function $\Psi$ obeys a normalization condition

$$\int_V |\Psi|^2 dV = \int_V \rho dV = M > 0,$$

where $M$ and $V$ are the total mass and volume of the fluid. This poses restrictions upon fluid wave functions which are similar to a quantum-mechanical case: the set of all normalizable fluid wave functions must constitute a Hilbert space, such as $L^2$.

As a first derivation step of an equation for $\Psi$, it is natural to relate the probability sample space of an underlying microscopic many-body system to the volume occupied by fluid, and thus relate the probability (1) to the fluid density:

$$P \sim \rho,$$

and take into account this statistical effect upon an energy of system’s collective degrees of freedom.

Therefore, one obtains

$$|\Psi|^2 \sim \exp (-U/T),$$

from which an expression follows for an operator of the potential $U$:

$$\hat{U} \sim T \ln (A|\Psi|^2) = -K(T - T_0) \ln (\rho/\rho_0),$$

where $A = 1/\rho_0$ and $K$ are some dimensionful and dimensionless proportionality constants, respectively, and $\rho_0$ and $T_0$ are critical values of, respectively, fluid density and temperature at which the statistical effect vanishes. From now on, we assume $K = 1$ for simplicity.

Furthermore, let us assume our flow to be Hamiltonian, and consider an operator form for the energy conservation, which can be written as the following identity:

$$\hat{H} \Psi = (\hat{T} + \hat{U}) \Psi,$$

where $\hat{H} = i\eta \partial_t$ is a Liouville-type generator of time translations, $\eta$ being a positive scaling constant, $\hat{T} \propto \frac{1}{2} \mathbf{p} \cdot \mathbf{p}$ is a kinetic energy operator, $\mathbf{p} \propto -i \nabla$ is a generator of spatial translations, and $\hat{U}$ is a potential energy operator derived above. Altogether, this identity can be recast in the form of a logarithmic Schrödinger-like equation:

$$i\partial_t \Psi = \left[ -\frac{D}{2} \nabla^2 - b \ln \left( |\Psi|^2 / \rho_0 \right) \right] \Psi,$$
where \( b = (T - T_0) / \eta \) and \( \mathcal{D} \) are positive constants. This equation must be supplemented with the normalization condition, as discussed above.

Substituting Eq. (2) into Eq. (8), one can recover the hydrodynamic laws for mass and momentum conservation for a two-phase compressible inviscid fluid with internal capillarity which flow is irrotational and isothermal:

\[
\begin{align*}
\partial_t \rho + \nabla \cdot (\rho \mathbf{u}) &= 0, \\
\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} - \frac{1}{\rho} \nabla \cdot T &= 0,
\end{align*}
\]

with the stress tensor \( T \) of the Korteweg form with capillarity effects [4], which is used to model fluid mixtures with phase changes and diffuse interfaces [6, 7]:

\[
T = -\frac{\mathcal{D}}{4\rho} \nabla p \otimes \nabla p - \tilde{p} \mathbf{I},
\]

where \( \mathbf{I} \) is the identity matrix,

\[
\tilde{p} = p(\rho) - \frac{1}{4} \mathcal{D} \nabla^2 \rho = -b \rho - \frac{1}{4} \mathcal{D} \nabla^2 \rho
\]

is a capillary pressure, and

\[
p(\rho) = -b \rho
\]

is a barotropic equation of state for the fluid pressure \( p \) [1, 2]. The stress tensor is also related to the chemical potential by the formula:

\[
\nabla \mu = -\frac{1}{\rho} \nabla \cdot T = -\nabla \left( \frac{\mathcal{D}}{2} \frac{\sqrt{p}}{\sqrt{\rho}} + b \ln \left( \rho / \rho_0 \right) \right),
\]

where a scale factor \( \eta \) is incorporated into \( \mu \) for brevity, which means that the chemical potential can be derived from Eq. (8) for stationary functions \( \Psi(x, t) = \exp(-i\mu t)\Psi(x) \). Thus, Eq. (8) is a concise form of writing two hydrodynamic equations, which makes it useful for studies of the Korteweg-type fluids, especially considering a large information accumulated about properties of wave equations with logarithmic nonlinearity in different branches of physics [9, 10, 11, 12, 13, 14, 15], including a theory of logarithmic Bose-Einstein condensates and superfluids [16, 17, 18, 19, 20].

Besides, by using a standard averaging procedure with respect to the inner product in the Hilbert space of fluid wavefunctions, one can show that an averaged form of Eq. (8) can be written as a formula for a wave-mechanical internal energy of the fluid:

\[
\mathcal{U} = \langle \hat{\mathbf{H}} \rangle = \mathcal{F} + TS,
\]

where \( \mathcal{F} = \langle \hat{T} \rangle \) is a wave-mechanical free energy, and

\[
S = -\int_V \rho \ln(\rho / \rho_0) \, dV;
\]

is an entropy function; according to aforesaid

\[
S \sim -\int_V P \ln P \, dV.
\]

Thus, \( T \) and \( S \) must be thermodynamically conjugated:

\[
T \sim b \sim \left( \frac{\partial \mathcal{U}}{\partial S} \right)_V,
\]

if one treats \( \mathcal{U} \) as a thermodynamical potential.
3. Symmetry breaking and solitary waves

Spontaneous symmetry breaking is a widely known phenomenon in physics, through which a system undergoes a transition from a more symmetric to less symmetric state. One must differentiate this from the dynamical symmetry breaking phenomenon, in which a state of a system, described as a solution of evolution equations, has a different (lesser) symmetry from what it is expected from the properties of evolution equations themselves. The latter are coming usually from Lagrangian or Hamiltonian formulations and written in a form of differential equations, with positions as variables.

Perhaps a simplest example of the spontaneous symmetry breaking phenomenon would be a long solid rod or bar made of elastic material, to which one applies a pressure force in the direction parallel to the main axis. Since a bar is rotationally invariant with respect to that axis, one could expect that it can not bend (since it cannot “decide” which direction to bend), only splat, i.e., expand in the direction perpendicular to the main axis. In reality however, most long bars are more likely to eventually bend at some point than to splat into a rotationally symmetric shape. The microscopic reasons for that are of course small anisotropies inside the molecular or crystal structure of real bars as many-body systems, but in the idealized model, i.e., perfectly radial symmetric single body, the only explanation and description one can offer is the spontaneous symmetry breaking: a bar undergoes a transition from initially rotationally symmetric shapes to one of bent shapes which break this symmetry.

The most known mechanism by which the spontaneous symmetry breaking can be introduced or described is the Higgs mechanism. Originally proposed for purposes of physics of elementary particles, where a single-body idealized description and small fluctuations thereof are not only natural but also inevitable in most cases, it can be extended to condensed matter theory, mathematical ecology, financial mathematics, and, as we shall see below, fluid mechanics. Its basic idea is the following.

Suppose that our system’s evolution equation can be somehow written in the form of Lagrange-Euler equations minimizing an action whose potential part has a shape with a few extrema. One of those extrema usually corresponds to a maximally symmetric configuration, while others do to configuration with less symmetry. A simplest example of such potential would be a Mexican-hat one where the very peak of the dome is placed in the origin (where we have a configuration which is symmetric with respect to $Z_2$ or $U(1)$ transformation of a variable), while an infinite number of possible minima at the perimeter does not obey this symmetry. Since the local minimum of the potential (hence energy) is realized on those non-symmetric solutions, it is more energetically favorable for a system to stay in a less-symmetric states even if it was originally in the symmetric state. The transition from the latter will readily occur, sooner or later, due to its instability against small fluctuations (in the elementary particles’ case, quantum).

Thus, our task now to check whether our system belongs to this class of dynamical systems, which would allow us to reliably apply the above-mentioned formalism and make certain predictions.

Using a variational approach, Eq. (8) can be derived as an Euler-Lagrange equation for the Galilean-invariant Lagrangian density:

$$\mathcal{L} = \frac{i}{2}(\Psi \partial_t \Psi^* - \Psi^* \partial_t \Psi) + \frac{D}{2} |\nabla \Psi|^2 + \mathcal{V}(|\Psi|^2),$$  \hspace{1cm} (16)$$

with the potential density given by

$$\mathcal{V}(\rho) = -b\rho \left[ \ln \left( \frac{\rho}{\rho_0} \right) - 1 \right] + V_0,$$  \hspace{1cm} (17)$$

where $V_0 = V(0)$ being an arbitrary constant; for simplicity we can assume $V_0 = 0$. For complex values of $\Psi$, this potential has an upside-down Mexican-hat shape, with local maxima
at $|\Psi_e| = \sqrt{\rho_0}$, which is placed inside a square potential well; the latter occurs due to the normalization condition which bounds $|\Psi|$ from above.

Thus, the maximally symmetric state $|\Psi|$ is not the most energetically favorable one here, since there could exist one or more states which located between local maxima and one of walls of the above-mentioned square well. This indicates a possibility of spontaneous symmetry breaking (or restoration), therefore, Eq. (8) should have solutions of a solitary wave type, and indeed such solutions exist.

One solitary wave solution of Eq. (8) can be found exactly. It corresponds to the ground state (i.e., the one with a lowest eigenvalue of the chemical potential $\mu$) and can be written in the form

$$\Psi_{(g)}(x, t) = \exp (-i\omega t - ik \cdot x) \sqrt{\rho_g(x)}, \quad (18)$$

where $k$ is a constant vector, $\omega = \mu/\eta$ is a wave frequency, and the density eigenfunction and frequency eigenvalue are, respectively:

$$\rho_g(x) = \tilde{\rho} \exp \left[ -\frac{(x - x_0)^2}{\ell^2} \right], \quad (19)$$

$$\omega_g = \mu_g/\eta = \frac{1}{2} D k^2 + b \left[ d - \ln \left( \frac{\tilde{\rho}}{\rho_0} \right) \right], \quad (20)$$

where $d$ is a number of spatial dimensions of the fluid, and $k = 0$ if $b \neq 0$. The propagating solutions can be generated by means of a Galilean transformation $x \rightarrow x - ut$.

The value of $\tilde{\rho}$ can be found from the normalization condition (3) and a formula for the $d$-dimensional Gaussian integrals $\int_V \exp \left(-x^T A x\right) dV = \pi^{d/2} / \sqrt{\det A}$, $A$ being a $d \times d$ positive-definite matrix. Thus

$$\tilde{\rho} = M/V, \quad V = \pi^{d/2} \ell^d, \quad \ell = \sqrt{D/(2b)}, \quad (21)$$

are the wave’s peak density value, effective volume and Gaussian width, respectively.

It should be noted that, for the same set of boundary conditions, Eq. (8) allows also other normalized (eigen)solutions which correspond to excited states in the Hilbert space of the problem, with different eigenvalues for $\mu$. This means that the Korteweg-type fluid can “choose” one of these solutions spontaneously. Though, the ground state is a preferable one as it corresponds to a minimum of the wave-mechanical energy.

Furthermore, the system’s behavior changes drastically when the nonlinear coupling $b$ changes its sign; this could happen if, in accordance with (6), it becomes proportional to $T - T_c$ where $T_c$ is some transition temperature. In that case, the potential (17) acquire a conventional Mexican hat shape, so the states $|\Psi_e|$ become global minima and thus become energetically favorable. This leads to occurrence of multiple topological sectors as well as topological soliton solutions whose stability is ensured by a conserved topological charge [21].

Thus, the overall theory predicts two types of phase transitions in these fluids, at least: the transition related to the sign flip of the nonlinear coupling, and the transitions related to spontaneous symmetry breaking for a given value (sign) of this coupling.

4. Conclusion

In this paper, we have explored a mapping between nonlinear Schrödinger-like wave equations of a logarithmic type and flow equations of the Korteweg-type materials including fluids with internal surface tension and capillary effects. This mapping has allowed us to rewrite the flow equations in the form of a single Schrödinger-like equation which can be studied by means of the conventional wave mechanics which has been hitherto popular in describing mostly microscopical systems.
In particular, studying Korteweg fluids in terms of the Schrödinger-like wave equations reveals an existence of the Hilbert space of states represented by different solutions, for the same set of boundary conditions. This means that the fluid can randomly “choose” either of these states (with a preference to the lower-energy ones, of course), or it can even be in a superposition of these states (an analogue of an entangled pure state in quantum mechanics) or their statistical ensemble (an analogue of a mixed state in quantum mechanics).

Analogies of this kind were known in studies of propagation of classical electromagnetic waves in media [22] but not in a theory of Korteweg fluids. This open further perspectives for searching phenomena in classical fluids with internal surface tension and capillary effects, which are analogous to quantum-statistical phenomena in quantum dissipative systems where mixed states are inevitable due to interaction of a (sub)system with its environment.

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