Renormalization of nonequilibrium dynamics at large $N$ and finite temperature

Jürgen Baacke\textsuperscript{1}, Katrin Heitmann\textsuperscript{2}, and Carsten Pätzold\textsuperscript{3}

Institut für Physik, Universität Dortmund
D - 44221 Dortmund, Germany

Abstract

We generalize a previously proposed renormalization and computation scheme for nonequilibrium dynamics to include finite temperature and one-loop selfconsistency as arising in the large-$N$ limit. Since such a scheme amounts essentially to tadpole resummation, it also includes, at high temperature, the hard mass corrections proportional to $T^2$. We present some numerical examples at $T = 0$ and for finite temperature; the results reproduce the essential features of those of other groups. Especially we can confirm a recently discovered sum rule for the late time behaviour.

\textsuperscript{1} e-mail: baacke@physik.uni-dortmund.de
\textsuperscript{2}e-mail: heitmann@hal1.physik.uni-dortmund.de
\textsuperscript{3}e-mail: paetzold@hal1.physik.uni-dortmund.de
1 Introduction

Nonequilibrium dynamics in quantum field theory has become, during the last years, a very active field of research in particle physics [1-5], in cosmology [6-18], and in solid state physics [19]. The application of the general formalism of nonequilibrium quantum field theory [20, 21] has been limited up to now mostly to spatially homogeneous systems (see however [22]). The typical field theoretic system considered in this context consists in a classical, spatially constant field $\phi(t)$ (Higgs, inflaton, condensate) and a quantum state of fluctuations of the same or another field, chosen initially as a Bogoliubov-transformed vacuum state or a thermal state. The classical field is started with an initial value away from a local or global minimum of the classical or effective action. The time development of this coupled system is then studied including the back reaction of the quantum field in one-loop, Hartree or large-$N$ approximations. The basic equations of motions have been derived by several groups and a considerable number of numerical studies has been performed. It has been found that the system is far from being Markovian, showing a long-time memory. The typical late-time behaviour is a stationary oscillation of the field $\phi(t)$ and of the quantum fluctuations [23, 24, 3]. Contrary to naive expectations the classical field does not come to rest and the quantum fluctuations do not thermalize in the various approximations that have been studied.

The question of a concise and covariant renormalization may not be the most urgent one in some of the above-mentioned contexts. Logarithmic corrections in the fluctuation integral, typically with coefficients $\lambda/16\pi^2$, are of course small in most cases; covariance may not be important in solid state applications; and in the case of effective theories like the sigma model for disordered chiral condensates renormalization may become replaced by physical cutoffs. Also, due to parametric resonance, suppressed by the back reaction, quantum fluctuations develop most strongly in the small momentum region; so a cutoff chosen with taste will do it for practical purposes. Nevertheless, in an expanding field of research as nonequilibrium quantum field theory one should make sure that things can be done properly. In a - still unexplored - GUT phase transition the coupling will not be as small as for the inflationary models being investigated at present. Renormalization of the large-$N$ approximation certainly will be important as a basic step if one tries to include a real rescattering of quantum fluctuations; such a rescattering - not only through the back reaction with the classical field - is presumably an important ingredient for understanding thermalization. A subject that arises in close connection with renormalization are singularities in the time variable due to initial conditions; it is a rather fundamental subject for nonequilibrium dynamics. We have addressed this problem recently [25]. Finally, from a practical point of view, our renormalization procedure also implies improving considerably the convergence of momentum integrals, and the gain in computing time can be of importance when studying more complex systems.
The basic method for the perturbative expansion which we will use here has been developed in [27]. It is based essentially on a standard resolvent expansion of the propagator. Most other groups have used the eikonal expansion which seems a natural choice in the presence of an oscillating background field and has its merit in allowing the study of adiabatic properties [3]. However, renormalization becomes more cumbersome as the relation to the Feynman graph expansion becomes more remote. Furthermore, eikonal expansions cannot be generalized to coupled fields (one would need time-ordered exponentials, not suited for numerical computations).

We will present in the next section the $O(N)$ model, the nonequilibrium equations of motion for the classical field and the fluctuation modes, the energy density and the pressure in large-$N$ approximation and in unrenormalized form. In order to prepare the discussion of renormalization we give the basic equations for a perturbative expansion of the mode equations in section 3. Large-$N$ renormalization, including finite temperature corrections is derived in section 4. Section 5 is devoted to the high temperature limit of the model. Some numerical experiments are presented in section 6. A resumé and conclusions are given in section 7.

2 Formulation of the large-$N$ equations

We consider the $O(N)$ with the Lagrangian\footnote{We deviate from the usual convention of introducing the interaction term with a factor $\lambda/8N$ in order to avoid a plethora of fractions $\frac{1}{2}$ and, even worse $\frac{1}{2} + \delta \lambda$, in the subsequent formulae.}

\[
\mathcal{L} = \frac{1}{2} \partial_\mu \phi^i \partial^\mu \phi^i - \frac{1}{2} m^2 \phi^i \phi^i - \frac{\lambda}{4N} (\phi^i \phi^i)^2
\]  

(2.1)

where $\phi^i, i = 1, \ldots, N$ are $N$ real scalar fields. The nonequilibrium state of the system is characterized by a classical expectation value which we take in the direction of $\phi_N$. We split the field into its expectation value $\phi$ and the quantum fluctuations $\psi$ via

\[
\phi^i(x, t) = \delta_N^i \sqrt{N} \phi(t) + \psi^i(x, t) .
\]  

(2.2)

In the large $N$ limit one neglects, in the Lagrangian, all terms which are not of order $N$. In particular terms containing the fluctuation $\psi_N$ of the component $\phi_N$ are at most of order $\sqrt{N}$ and are dropped, therefore. This is in contrast to the Hartree approximation where the fluctuations of $\phi_N$ are included. The fluctuations of the other components are identical, their summation produces factors $N - 1 = N(1 + O(1/N))$. Identifying all the fields $\psi_1, \ldots, \psi_{N-1}$ as $\psi$ the leading order term in the Lagrangian then takes the form

\[
\mathcal{L} = N \left( \mathcal{L}_{\phi} + \mathcal{L}_{\psi} + \mathcal{L}_I \right)
\]  

(2.3)
with
\[ L_\phi = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 - \frac{\lambda}{4} \phi^4, \]  
(2.4)
\[ L_\psi = \frac{1}{2} \partial_\mu \psi \partial^\mu \psi - \frac{1}{2} m^2 \psi^2 + \frac{\lambda}{4} (\psi^2)^2, \]  
(2.5)
\[ L_I = -\frac{\lambda}{2} \psi^2 \phi^2, \]  
(2.6)
where \( \psi^2 \) is to be identified with \( \sum \psi^i \psi^i / N \).

We decompose the fluctuating field into momentum eigenfunctions via
\[ \psi(x, t) = \int \frac{d^3k}{(2\pi)^3 2\omega_{k0}} \left[ a_k U_k(t) e^{i\mathbf{kx}} + a_k^\dagger U_k^*(t) e^{-i\mathbf{kx}} \right]. \]  
(2.7)
with \( \omega_{k0} = \sqrt{m_0^2 + k^2} \). The mass \( m_0 \) will be specified below. This field decomposition defines a vacuum state as being annihiliated by the operators \( a_k \).

The equations of motion for the field \( \phi(t) \) and of the fluctuations \( U_k(t) \) have been derived in this formalism by various authors [8, 26, 2]. In addition to the large-\( N \) Lagrangian, Eqs. (2.3, 2.4), one uses, on averaging over the quantum fluctuations, rules like
\[ (\psi^2)^2 \Rightarrow \langle \psi^2 \rangle^2, \]  
(2.8)
\[ \frac{\partial (\psi^2)^2}{\partial \psi} \Rightarrow 4\psi \langle \psi^2 \rangle, \]  
(2.9)
\[ \frac{\partial^2 (\psi^2)^2}{\partial \psi^2} \Rightarrow 4 \langle \psi^2 \rangle \]  
(2.10)
which follow at large \( N \) from the identification \( \psi^2 \simeq \sum \psi^i \psi^i / N \).

We include in the following the counter terms that we will need later in order to write the renormalized equations. So the equation of motion for the field \( \phi \) becomes
\[ \ddot{\phi}(t) + (m^2 + \delta m^2) \phi(t) + (\lambda + \delta \lambda) \phi(t) \left[ \phi^2(t) + \mathcal{F}(t, T) \right] = 0. \]  
(2.11)
Here \( \mathcal{F}(t, T) \) is the divergent fluctuation integral; it is given by the average of the fluctuation fields defined by the initial density matrix. For a thermal initial state of quanta with energy \( \omega_{k0} = \sqrt{k^2 + m_0^2} \) it is given by
\[ \mathcal{F}(t, T) = \langle \psi^2(x, t) \rangle = \int \frac{d^3k}{(2\pi)^3 2\omega_{k0}} \coth \frac{\beta \omega_{k0}}{2} |U_k(t)|^2. \]  
(2.12)

The mode functions satisfy the equation:
\[ \left[ \frac{d^2}{dt^2} + \omega_k^2(t) \right] U_k(t) = 0, \]  
(2.13)
and the initial conditions
\[ U_k(0) = 1 ; \ U_k(0) = -i\omega_{k0} . \] (2.14)

The time dependent frequency \( \omega_k(t) \) is given by
\[ \omega_k^2 = k^2 + \mathcal{M}^2(t) \] (2.15)
with the time dependent mass
\[ \mathcal{M}^2(t) = m^2 + (\lambda + \delta\lambda) \left[ \phi^2(t) + \mathcal{F}(t) \right] . \] (2.16)

As in our previous work we rewrite the mode equation in the form
\[ \left( \frac{d^2}{dt^2} + \omega_{k0}^2 \right) U_k(t) = -V(t)U_k(t) , \] (2.17)
whereby we have defined the time-dependent potential \( V(t) = \mathcal{M}^2(t) - \mathcal{M}^2(0) \);
we further identify \( m_0 = \mathcal{M}(0) \) as the “initial mass”. The classical equation of
motion also can be rewritten as
\[ \ddot{\phi}(t) + \mathcal{M}^2(t)\phi(t) = 0 , \] (2.18)
which is of the same form as Eq. (2.13) with \( k = 0 \), the classical field also is
referred to as “zero mode” in [1]. The average of energy with respect to the initial
density matrix is given by
\[ E = \frac{1}{2}\phi^2(t) + \frac{1}{2}(m^2 + \delta m^2)\phi^2(t) + \frac{\lambda + \delta\lambda}{4} \phi^4(t) + \delta\Lambda \]
\[ + \int \frac{d^3k}{(2\pi)^32\omega_{k0}} \coth \frac{\beta\omega_{k0}}{2} \left\{ \frac{1}{2}|\dot{U}_k(t)|^2 + \frac{1}{2}\omega_k^2(t)|U_k(t)|^2 \right\} \]
\[ - \frac{\lambda + \delta\lambda}{4} \mathcal{F}^2(t, T) . \] (2.19)

It is easy to check, using the equations of motion (2.18) and (2.13), that the energy
is conserved. The energy density is the 00 component of the energy momentum
tensor. The average of the energy momentum tensor for our system is diagonal,
its space-space components define the pressure which is given by
\[ p = \phi^2(t) - E + A \frac{d^2}{dt^2} \left[ \phi^2(t) + \mathcal{F}(t, T) \right] \]
\[ + \int \frac{d^3k}{(2\pi)^32\omega_{k0}} \coth \frac{\beta\omega_{k0}}{2} \left( \omega_k^2 + \frac{k^2}{3} \right)|U_k(t)|^2 \ldots \] (2.20)

The term proportional to \( A \) is the space-space component of the “improvement”
term \( A(g_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu)\phi^2 \) for the energy momentum tensor as introduced by [28].
It serves as a renormalization counter term, here.

\[ ^5 \text{Note that twice the last term, with positive sign, is included in the fluctuation energy, since} \]
\[ \omega_k^2(t) \text{ contains } \mathcal{F}(t, T). \]
3 Perturbative expansion

In order to prepare the renormalized version of the equations we introduce a suitable expansion of the mode functions. We have used this method exhaustively in our previous publications for the inflaton field coupled to itself [27] and to gauge bosons [3] in Minkowski-space and for the inflaton field coupled to itself in a conformally flat FRW-universe [18]. All these calculations have been done for $T = 0$. The renormalization procedure does not change for $T \neq 0$. Therefore we give here only a brief review of the perturbative expansion. For details the reader is referred to our previous work.

The mode functions can be written as

$$
\left[ \frac{d^2}{dt^2} + \omega_{k0}^2 \right] U_k(t) = -\mathcal{V}(t)U_k(t),
$$

(3.1)

with

$$
\mathcal{V}(t) = \mathcal{M}^2(t) - \mathcal{M}^2(0),
$$

(3.2)

$$
\omega_{k0} = \left[ \vec{k}^2 + \mathcal{M}^2(0) \right]^{1/2}
$$

(3.3)

(for the definition of $\mathcal{M}^2(t)$ see Eq.(2.16)). The mode functions satisfy the equivalent integral equation

$$
U_k(t) = e^{-i\omega_{k0}t} + \int_0^{\infty} dt' \Delta_{k,\text{ret}}(t - t')\mathcal{V}(t')U_k(t') .
$$

(3.4)

with

$$
\Delta_{k,\text{ret}}(t - t') = -\frac{1}{\omega_{k0}}\theta(t - t')\sin(\omega_{k0}(t - t')) .
$$

(3.5)

For $U_k(t)$ we choose the following ansatz

$$
U_k(t) = e^{-i\omega_{k0}t}(1 + f_k(t)) ,
$$

(3.6)

to separate $U_k(t)$ into the trivial part corresponding to the case $\mathcal{V}(t) = 0$ and a function $f_k(t)$ which represents the reaction to the potential. $f_k(t)$ satisfies the differential equation

$$
\ddot{f}_k(t) - 2i\omega_{k0}\dot{f}_k(t) = -\mathcal{V}(t)(1 + f_k(t)) ,
$$

(3.7)

with the initial conditions $f_k(0) = \dot{f}_k(0) = 0$ or the equivalent integral equation

$$
f_k(t) = \int_0^t dt' \Delta_{k,\text{ret}}(t - t')\mathcal{V}(t')[1 + f_k(t')]e^{i\omega_{k0}(t-t')} .
$$

(3.8)
We expand now \( f_k(t) \) with respect to orders in \( \mathcal{V}(t) \) by writing
\[
\begin{align*}
  f_k(t) &= f_k^{(1)}(t) + f_k^{(2)}(t) + f_k^{(3)}(t) + \cdots \\
  &= f_k^{(1)}(t) + f_k^{(2)}(t) ,
\end{align*}
\]
where \( f_k^{(n)}(t) \) is of \( n \)th order in \( \mathcal{V}(t) \) and \( f_k^{(n)}(t) \) is the sum over all orders beginning with the \( n \)th one:
\[
  f_k^{(n)}(t) = \sum_{l=n}^{\infty} f_k^{(n)}(t) .
\]

The function \( f_k^{(1)}(t) \) is identical to the function \( f_k(t) \) itself which is obtained by solving (3.7). The function \( f_k^{(2)}(t) \) can be computed by using the differential equation, via
\[
  \ddot{f}_k^{(2)}(t) - 2i\omega_{k0}\dot{f}_k^{(2)}(t) = -\mathcal{V}(t)f_k^{(1)}(t) ,
\]
or by iteration via
\[
  f_k^{(2)}(t) = \int_0^t dt' \Delta_{k,\text{ret}}(t - t')\mathcal{V}(t')f_k^{(1)}(t')e^{i\omega_{k0}(t-t')} .
\]

This iteration has the advantage for the numerical computation that it avoids computing \( f_k^{(2)} \) via the small difference \( f_k^{(1)} - f_k^{(1)} \). However, the integral equations are used as well in order to derive the asymptotic behaviour as \( \omega_{k0} \to \infty \) and to separate divergent and finite contributions. The leading orders of \( f_k(t) \) are discussed in detail in [18, 27, 3] at full length and we do not want to repeat it here. In this work we are more interested in the effects of the finite contributions at finite temperature and in the self consistent solving of the large-\( N \) limit.

## 4 Renormalization

We use the expansion and the definition introduced in the previous section in order to in order to single out the divergent terms from the fluctuation integral; we have
\[
\begin{align*}
  \mathcal{M}^2(t) &= m^2 + \delta m^2 + (\lambda + \delta \lambda) \left\{ \phi^2(t) + I_{-1}(m_0, T) \\
  &\quad - I_{-3}(m_0, T) \left[ \mathcal{M}^2(t) - \mathcal{M}^2(0) \right] + \mathcal{F}_{\text{fin}}(t, T) \right\} ,
\end{align*}
\]
where the finite part of \( \mathcal{F}(t, T) \) can be written as
\[
\begin{align*}
  \mathcal{F}_{\text{fin}}(t, T) &= \int \frac{d^3k}{(2\pi)^32\omega_{k0}} \frac{1}{2\omega_{k0}^2} \int_0^t dt' \cos(2\omega_{k0}(t - t'))\dot{\mathcal{V}}(t')\coth\frac{\beta\omega_{k0}}{2} \\
  &\quad + \int \frac{d^3k}{(2\pi)^32\omega_{k0}} \left\{ 2\text{Re}f_k^{(2)}(t) + |f_k^{(1)}(t)|^2 \right\} \coth\frac{\beta\omega_{k0}}{2} ,
\end{align*}
\]

\[6\]
and where the divergent integrals are defined as

\[ I_{-1}(m_0, T) = \int \frac{d^3k}{(2\pi)^3 2\omega_{k0}} \left( 1 + \frac{2}{e^{\beta \omega_0} - 1} \right) = I_{-1}(m_0) + \Sigma_{-1}(m_0, T) \] (4.3)

\[ I_{-3}(m_0, T) = \int \frac{d^3k}{(2\pi)^3 4\omega_{k0}^3} \left( 1 + \frac{2}{e^{\beta \omega_0} - 1} \right) = I_{-3}(m_0) + \Sigma_{-3}(m_0, T) . \] (4.4)

The integrals \( I_{-k}(m_0) \) are those which occur in the renormalization at \( T = 0 \). Their dimensionally regularized form will be given below. The additional temperature dependent terms \( \Sigma_{-k}(m_0, T) \) are finite. They are defined as

\[ \Sigma_{-1}(m_0, T) = \int \frac{d^3k}{(2\pi)^3 \omega_{k0}} \frac{1}{(e^{\beta \omega_0} - 1)} \] (4.5)

\[ \Sigma_{-3}(m_0, T) = \int \frac{d^3k}{(2\pi)^3 2\omega_{k0}^3} \frac{1}{(e^{\beta \omega_0} - 1)} . \] (4.6)

We derive some useful explicit expressions for these integrals in Appendix A.

It is convenient to include these finite terms into the definition of \( F_{\text{fin}}(t, T) \). Then the time dependent mass takes the form

\[ M^2(t) = m^2 + \delta m^2 + (\lambda + \delta \lambda) \left[ \phi^2(t) + I_{-1}(m_0) - I_{-3}(m_0) V(t) + \tilde{F}_{\text{fin}}(t, T) \right] , \]

with \( \tilde{F}_{\text{fin}}(t, T) = \Sigma_{-1}(m_0, T) - V(t) \Sigma_{-3}(m_0, T) + F_{\text{fin}}(t, T) . \) (4.7)

The time dependent mass \( M^2(t) \) contains both the renormalization constants \( \delta m \) and \( \delta \lambda \). Furthermore, its definition by this equation is implicit, \( M^2(t) \) appears also on the right hand side of Eq. (4.7) in \( V(t) \).

We now set out to fix the renormalization counter terms in such a way that the relation between the time-dependent mass and \( \phi(t) \) becomes finite. An additional constraint derives from the requirement that the renormalization counter terms should not depend on the initial condition but only on the parameters appearing in the Lagrangian, i.e., \( \lambda \) and \( m \). For the simpler case of the one-loop equations this has been achieved \[27\].

We first determine \( \delta \lambda \) by considering the difference

\[ V(t) = M^2(t) - M^2(0) \]

\[ = (\lambda + \delta \lambda) \left\{ \phi^2(t) - \phi^2(0) - I_{-3}(m_0)V(t) + \tilde{F}_{\text{fin}}(t, T) - \tilde{F}_{\text{fin}}(0, T) \right\} \] (4.9)

or

\[ V(t) \left[ 1 + (\lambda + \delta \lambda)I_{-3}(m) \right] = (\lambda + \delta \lambda) \left\{ \phi^2(t) - \phi^2(0) - [I_{-3}(m_0) - I_{-3}(m)] V(t) + \tilde{F}_{\text{fin}}(t, T) - \tilde{F}_{\text{fin}}(0, T) \right\} . \] (4.10)
We now require
\[
\frac{\lambda + \delta \lambda}{1 + (\lambda + \delta \lambda)I_{-3}(m)} = \lambda .
\]  
(4.11)

Solving with respect to $\delta \lambda$ we find
\[
\delta \lambda = \frac{\lambda^2 I_{-3}(m)}{1 - \lambda I_{-3}(m)}.
\]  
(4.12)

Inserting this relation into (4.10) we have
\[
V(t) = \lambda \left\{ \phi^2(t) - \phi^2(0) - [I_{-3}(m_0) - I_{-3}(m)] V(t) + \tilde{F}_{\text{fin}}(t, T) - \tilde{F}_{\text{fin}}(0, T) \right\}
\]  
(4.13)
or
\[
V(t) = \frac{\lambda}{1 + \lambda [I_{-3}(m_0) - I_{-3}(m)]} \left[ \phi^2(t) - \phi^2(0) + \tilde{F}_{\text{fin}}(t, T) - \tilde{F}_{\text{fin}}(0, T) \right].
\]  
(4.14)

This is a finite relation for the potential $V(t)$ since the difference $[I_{-3}(m_0) - I_{-3}(m)]$ is finite. Using dimensional regularization
\[
I_{-3}(m_0) = \left\{ \int \frac{d^3k}{(2\pi)^3} \frac{1}{4\omega_k^2} \right\}_{\text{reg}} = \frac{1}{16\pi^2} \left\{ \frac{2}{\epsilon} + \ln \frac{4\pi \mu^2}{m_0^2} - \gamma \right\},
\]  
(4.15)
and therefore
\[
I_{-3}(m_0) - I_{-3}(m) = \frac{1}{16\pi^2} \ln \left( \frac{m_0^2}{m_2} \right).
\]  
(4.16)

We now go back to equation (4.1) which we take at the initial time $t = 0$:
\[
m_0^2 \equiv M^2(0) = m^2 + \delta m^2 + (\lambda + \delta \lambda) \left[ \phi^2(0) + I_{-1}(m_0) + \tilde{F}_{\text{fin}}(0, T) \right].
\]  
(4.17)

This is an implicit relation between $m_0$ and $\phi(0)$ which, however, contains still the infinite quantities $\delta \lambda, \delta m$ and $I_{-1}(m_0)$. In order to proceed we note the following explicit relation between $I_{-1}$ and $I_{-3}$ which follows from the dimensionally regularized expressions for these quantities
\[
\left\{ \int \frac{d^3k}{(2\pi)^32\omega_k} \right\}_{\text{reg}} = -\frac{m_0^2}{16\pi^2} \left\{ \frac{2}{\epsilon} + \ln \frac{4\pi \mu^2}{m_0^2} - \gamma + 1 \right\}
\]  
\[
= -m_0^2 I_{-3}(m_0) - m_0^2 \frac{m_0^2}{16\pi^2}.
\]  
(4.18)

Therefore we can rewrite (4.17) as
\[
m_0^2 = m^2 + \delta m^2 + (\lambda + \delta \lambda) \left[ \phi^2(0) - m_0^2 I_{-3}(m_0) - m_0^2 \frac{m_0^2}{16\pi^2} + \tilde{F}_{\text{fin}}(0, T) \right].
\]  
(4.19)
or
\[ m_0^2 [1 + (\lambda + \delta \lambda) I_{-3}(m)] = m^2 + \delta m^2 \]
\[ + (\lambda + \delta \lambda) \left[ \phi^2(0) - m_0^2 (I_{-3}(m_0) - I_{-3}(m)) - \frac{m_0^2}{16\pi^2} + \tilde{F}_{\text{fin}}(0, T) \right]. \]  \hspace{1cm} (4.20)

We now require the factors \( \lambda + \delta \lambda \) and \( [1 + (\lambda + \delta \lambda) I_{-3}] \) to cancel on account of (4.11), so as to obtain a relation between finite quantities. This is obviously the case if
\[ m^2 + \delta m^2 = m^2 [1 + (\lambda + \delta \lambda) I_{-3}(m)] + (\lambda + \delta \lambda) \rho \]
\hspace{1cm} (4.21)
since then
\[ m_0^2 - m^2 = \lambda \left\{ \phi^2(0) - m_0^2 [I_{-3}(m_0) - I_{-3}(m)] - \frac{m_0^2}{16\pi^2} + \rho + \tilde{F}_{\text{fin}}(0, T) \right\}. \]  \hspace{1cm} (4.22)

(4.21) fixes \( \delta m^2 \) as
\[ \delta m^2 = (\lambda + \delta \lambda) \left[ m^2 I_{-3}(m) + \rho \right] = \lambda \frac{m^2 I_{-3}(m) + \rho}{1 - \lambda I_{-3}(m)}. \]  \hspace{1cm} (4.23)
Setting \( \rho = 0 \) we have
\[ \delta m^2 = \frac{\lambda m^2 I_{-3}(m)}{1 - \lambda I_{-3}(m)}, \]
and
\[ m_0^2 - m^2 = \lambda \left\{ \phi^2(0) - m_0^2 [I_{-3}(m_0) - I_{-3}(m)] - \frac{m_0^2}{16\pi^2} + \tilde{F}_{\text{fin}}(0, T) \right\}. \]  \hspace{1cm} (4.25)
This corresponds to the \( \overline{MS} \) subtraction. Another natural choice is \( \rho = m^2/16\pi^2 \); then
\[ \delta m^2 = \lambda \frac{m^2 I_{-3}(m) + m^2/16\pi^2}{1 - \lambda I_{-3}(m)} = - \lambda \frac{I_{-1}(m)}{1 - \lambda I_{-3}(m)}, \]  \hspace{1cm} (4.26)
and
\[ m_0^2 - m^2 = \lambda \left\{ \phi^2(0) - m_0^2 [I_{-3}(m_0) - I_{-3}(m)] - \frac{m_0^2 - m^2}{16\pi^2} + \tilde{F}_{\text{fin}}(0, T) \right\}. \]  \hspace{1cm} (4.27)
This choice is analogous to the one in [27] for the one-loop equations and will be used in the following.

The “gap equation” (4.22) and the renormalized definition of the potential (4.14) constitute, along with the equations of motion the basic renormalized equations for the self consistent large-\( N \) dynamics.

The gap equation has to be solved at \( t = 0 \) and determines the relation between \( m_0 \) and \( \phi(0) \). For later times we have
\[ \mathcal{M}^2(t) = m_0^2 + \mathcal{V}(t) \]
\[ = m_0^2 + C \lambda \left[ \phi^2(t) - \phi^2(0) + \tilde{F}_{\text{fin}}(t, T) - \tilde{F}_{\text{fin}}(0, T) \right], \]  \hspace{1cm} (4.28)
with
\[ C = \left( 1 + \frac{\lambda}{16\pi^2 \ln \frac{m^2}{m_0^2}} \right)^{-1}. \tag{4.29} \]

Since the gap equation can be cast into different forms we can obtain several equivalent forms of this equation. Solving the gap equation for \( \phi^2(0) \) we find
\[ -\lambda C \phi^2(0) = -m_0^2 + C \left[ m^2 - \lambda \left( \frac{m_0^2}{16\pi^2} - \rho - \tilde{F}_{\text{fin}}(0, T) \right) \right], \tag{4.30} \]
so that
\[ M^2(t) = C \left[ m^2 + \lambda \left( \phi^2(t) - \frac{m_0^2}{16\pi^2} + \rho + \tilde{F}_{\text{fin}}(t, T) \right) \right]. \tag{4.31} \]

Having obtained a finite relation between \( \phi(t) \) and \( M(t) \) the equations of motion for the classical field \( \phi(t) \) and for the modes \( U_k(t) \) are well-defined and finite.

Here we have chosen to include the corrections of leading order, proportional to \( \Sigma_{-1}(m_0, T) \), into the finite part of the fluctuation integral. These terms are important at high temperature; they appear in the gap equation via \( \tilde{F}_{\text{fin}}(0, T) = \Sigma_{-1}(m_0, T) \approx T^2/12 \). Omitting some terms of order \( \lambda/16\pi^2 \) the gap equation \( \text{(4.22)} \) becomes
\[ m_0^2 \approx m^2 + \lambda \phi(0)^2 + \frac{\lambda}{12} T^2. \tag{4.32} \]

Therefore, at high temperature the mass circulating in the loop is dominated by the “hard” \( \lambda T^2 \) term.

We will need in the following the fluctuation integral \( F(t, T) \) which is and will remain divergent. We need an expression in which these divergencies appear in explicit form. We use
\[ F(t, T) = I_{-1}(m_0) - I_{-3}(m_0) \left[ M^2(t) - M^2(0) \right] + \tilde{F}_{\text{fin}}(t, T) \tag{4.33} \]
and insert the expression for \( M^2(t) \) we have just derived. Using the gap equation and some reshuffling of terms we obtain
\[
F(t, T) = -\frac{m_0^2}{16\pi^2} - C \lambda I_{-3}(m_0) \phi^2(t) - m^2 C I_{-3}(m_0)
+ C \frac{\lambda}{16\pi^2} I_{-3}(m_0)(m_0^2 - m^2) + C(1 - \lambda I_{-3}(m)) \tilde{F}_{\text{fin}}(t, T).
\tag{4.34}
\]

5 Renormalization of energy and pressure

The expressions for the energy density and for the pressure have been given in section 2. Apart from the renormalization counter terms which we have already fixed in renormalizing the equation of motion, two new counter terms appear, the “cosmological constant” term \( \delta\Lambda \) in the energy density and the “improvement
term” $Ad^2(\phi^2 + \langle \psi^2 \rangle)/dt^2$ in the pressure. These terms must suffice for rendering the expressions for energy density and pressure finite.

We start with the expression (2.19) for the energy which we rewrite as

$$E = \frac{1}{2} \dot{\phi}^2(t) + \frac{1}{2} m^2 \phi^2(t) + \frac{\lambda}{4} \phi^4(t) + \mathcal{E}_{\text{fl}}(t, T) - \frac{\lambda + \delta \lambda}{4} \mathcal{F}^2(t, T) + \frac{1}{2} \delta m^2 \phi^2(t) + \frac{\delta \lambda}{4} \phi^4(t) + \delta \Lambda. \quad (5.1)$$

with

$$\mathcal{E}_{\text{fl}}(t, T) = \int \frac{d^3 k}{(2\pi)^3 2\omega_k} \coth \frac{\beta \omega_k}{2} \left\{ \frac{1}{2} |\dot{U}_k(t)|^2 + \frac{1}{2} \omega_k^2(t) |U_k(t)|^2 \right\}. \quad (5.2)$$

In the latter expression we split the Bose factor as before

$$\coth \frac{\beta \omega_k}{2} = 1 + \frac{2}{e^{\beta \omega_k} - 1}. \quad (5.3)$$

The integrations involving the second term are finite, we define

$$\Delta \mathcal{E}_{\text{fl}}(t, T) = \int \frac{d^3 k}{(2\pi)^3 2\omega_k} \frac{2}{e^{\beta \omega_k} - 1} \left\{ \frac{1}{2} |\dot{U}_k(t)|^2 + \frac{1}{2} \omega_k^2(t) |U_k(t)|^2 \right\}. \quad (5.4)$$

Those involving the first term have been discussed in [27]. Following this discussion we can decompose the integral via

$$\mathcal{E}_{\text{fl}}(t, 0) = I_1(m_0) + \frac{1}{2} \mathcal{V}(t) I_{-1}(m_0) - \frac{1}{4} \mathcal{V}^2(t) I_{-3}(m_0) + \mathcal{E}_{\text{fl, fin}}(t, 0) \quad (5.5)$$

with

$$\mathcal{E}_{\text{fl, fin}}(t, 0) = \frac{1}{2} \int \frac{d^3 k}{(2\pi)^3 2\omega_k} \left\{ \frac{1}{2} |\overline{f_k}^{-1}|^2 + \frac{\mathcal{V}(t)}{2} \left[ 2 \text{Re} f_k^{-1} + |f_k^{-1}|^2 \right] + \frac{\mathcal{V}^2(t)}{8 \omega_k^2} \right\}. \quad (5.6)$$

We denote the sum of both finite contributions as $\mathcal{E}_{\text{fl, fin}}(t, T)$. The expression for the energy now takes the form

$$E = \frac{1}{2} \dot{\phi}^2 + \frac{1}{2} m^2 \phi^2 + \frac{\lambda}{4} \phi^4 + \mathcal{E}_{\text{fl, fin}}(t, T) + I_1(m_0) + \frac{1}{2} \mathcal{V}(t) I_{-1}(m_0) - \frac{1}{4} \mathcal{V}^2(t) I_{-3}(m_0) - \frac{\lambda + \delta \lambda}{4} \mathcal{F}^2(t, T) + \frac{1}{2} \delta m^2 \phi^2 + \frac{\delta \lambda}{4} \phi^4 + \delta \Lambda. \quad (5.7)$$

In addition to the divergent integrals $I_n(m_0)$ and the counter terms further divergencies are contained in the fluctuation integral $\mathcal{F}(t, T)$; these are given explicitly.

\footnote{We have overlooked in [27] that in the expression for the fluctuation energy given there two terms cancel on account of a Wronskian identity, given in [18], Eq. (73). They did so, of course, in the numerical calculations.}
in Eq. (4.34). The analysis of Eq. (5.7), after inserting the explicit expressions for $V(t)$ and $F(t, T)$, becomes rather cumbersome. To give an outline of the typical algebraic manipulations we consider explicitly the coefficients of $\phi^4$. Collecting everything except the bare $\lambda \phi^4$ term we find that $\phi^4(t)$ is multiplied by a sum of divergent terms

$$\frac{\delta \lambda}{4} - \frac{\lambda + \delta \lambda}{4} C^2 \lambda^2 x_0^2 - \frac{1}{4} C^2 x_0 \lambda^2.$$  

(5.8)

We use the abbreviations $x_0 = I_3(m_0)$ and $x = I_3(m)$, so that

$$C = \frac{1}{1 - \lambda(x_0 - x)}$$  

(5.9)

and (see Eq. (1.12))

$$\delta \lambda = \frac{\lambda^2}{1 - \lambda x}.$$  

(5.10)

One finds that all divergent quantities combine into the finite expression

$$-\frac{1}{4} \lambda^2 C(x - x_0) = \frac{\lambda^2}{64 \pi^2} C \ln \left( \frac{m^2}{m_0^2} \right) \equiv \frac{\Delta \lambda}{4},$$  

(5.11)

so the correction to the $\phi^4$ term in the energy becomes

$$\frac{\Delta \lambda}{4} \phi^4.$$  

(5.12)

Collecting similarly all terms proportional to $\phi^2$ one finds that the correction to the mass term becomes finite as well, explicitly

$$\frac{1}{2} \Delta m^2 \phi^2 = \frac{\lambda^2}{32 \pi^2} C \left[ m^2 - m_0^2 - m^2 \ln \left( \frac{m^2}{m_0^2} \right) \right] \phi^2.$$  

(5.13)

There are further time-dependent terms proportional to $\tilde{F}_{\text{fin}}^2(t, T)$ and $\tilde{F}_{\text{fin}}(t, T)$ and constant terms. The divergent parts of the latter ones can be absorbed into $\delta \Lambda$, the coefficient of the term linear in $\tilde{F}_{\text{fin}}$ vanishes and the quadratic one has a finite coefficient. The counter term $\delta \Lambda$ can be chosen independent of $m_0$:

$$\delta \Lambda = \frac{m^4}{4(1 - \lambda x)} \left( x + \frac{1}{8 \pi^2} - \frac{\lambda}{256 \pi^4} \right),$$  

(5.14)

there remains a finite constant

$$\Delta \Lambda = \frac{1}{4} C \left[ (x_0 - x) m^4 + \frac{1}{8 \pi^2} (m^2 - m_0^2) + \frac{1}{32 \pi^2} m_0^4 + \frac{\lambda}{256 \pi^4} (m_0^2 - m^2)^2 \right].$$  

(5.15)
So the expression for the energy can really be rendered finite with counter terms independent of the initial condition. Explicitly we find

\[ E = \frac{1}{2} \dot{\phi}^2 + \frac{1}{2} (m^2 + \Delta m^2) \phi^2 + \frac{\lambda + \Delta \lambda}{4} \phi^4 \]

\[ + \mathcal{E}_{\text{fin}}(t, T) - \frac{\lambda}{4} C \mathcal{F}_{\text{fin}}^2(t, T) + \Delta \Lambda. \]  

We finally have to give a finite expression for the pressure, using our last free counter term. We write the pressure in the form

\[ p = \dot{\phi}^2(t) - \mathcal{E} + p_{\text{fl}}(t, T) + A \frac{d^2}{dt^2} \left[ \phi^2(t) + \mathcal{F}(t, T) \right]. \]

Here we have anticipated a special form of the counter term, indeed for the expression in brackets one can choose a priori an arbitrary Lorentz scalar, the additional piece of the energy momentum tensor being trivially conserved on account of its tensor structure \( \partial_\mu \partial_\nu - g_{\mu\nu} \partial^2 \). Of course it has to be suited for the renormalization procedure. The fluctuation part of the pressure consists again of three parts, a divergent one, a finite one independent of the temperature and a finite integral involving the thermal distribution function \( \frac{1}{\exp(\omega k_0/T) - 1} \).

The analysis for \( T = 0 \) has been performed in \[27\]. Following the discussion there we can write \( p_{\text{fl}} \) as

\[ p_{\text{fl}}(t, T) = p_{\text{fl, fin}}(t, 0) + \Delta p_{\text{fl}}(t, T) - \frac{m_0^4}{96 \pi^2} - \frac{m_0^2}{48 \pi^2} \mathcal{V}(t) - \frac{1}{6} (I_3(m_0) + \frac{1}{48 \pi^2}) \ddot{\mathcal{V}}(t). \]

\[ \Delta p_{\text{fl}}(t, T) \] is given by

\[ \Delta p_{\text{fl}}(t, T) = \int \frac{d^3 k}{(2\pi)^3 2\omega_{k_0}} \frac{2}{e^{\beta \omega_{k_0}} - 1} \left( \omega_{k_0}^2 + \frac{k^2}{3} \right) |U_k(t)|^2, \]

the \( T = 0 \) finite part by

\[ p_{\text{fl, fin}}(t, 0) = \int \frac{d^3 k}{(2\pi)^3 2\omega_{k_0}} \left\{ \left( \frac{1}{6} \omega_{k_0}^2 - \frac{m_0^2}{24 \omega_{k_0}^2} \right) f_k(t) \right\} \]

\[ + \left( \frac{1}{12 \omega_{k_0}^2} + \frac{m_0^2}{24 \omega_{k_0}^2} \right) \cos(2\omega_{k_0}t) \ddot{\mathcal{V}}(0) \]

\[ + |\mathcal{F}_{k}^{(1)}(t)|^2 - 2 \text{Re} \left[ i \omega_{k_0} \mathcal{F}_{k}^{(1)}(t) + i \omega_{k_0} \mathcal{F}_{k}^{(1)*}(t) \right] \right\}. \]

We call the sum of both finite fluctuation integrals \( p_{\text{fl, fin}}(t, T) \). Now we have to consider the divergent terms. We observe that \( \ddot{\mathcal{V}}(t) \) is given by

\[ \ddot{\mathcal{V}}(t) = \lambda C \frac{d^2}{dt^2} \left[ \phi^2(t) + \mathcal{F}_{\text{fin}}(t, T) \right]. \]
On the other hand, using Eq. (4.34) we have
\[
\frac{d^2}{dt^2} F(t, T) = \frac{d^2}{dt^2} \left[ -\lambda C x_0 \phi^2(t) + C (1 - \lambda x) \tilde{F}_{\text{fin}}(t, T) \right]
\] (5.23)
and therefore
\[
A \frac{d^2}{dt^2} (\phi^2(t) + F(t, T)) = A \frac{d^2}{dt^2} C (1 - \lambda x) \left[ \phi^2(t) + \tilde{F}_{\text{fin}}(t, T) \right]
\] (5.24)
As apparent from Eq. (5.22) this matches in form with the divergent term
\[ I_{-3}(m_0) \ddot{V}(t)/6. \]
Insisting again in choosing the counter term independent of the initial condition we fix
\[
A = \frac{\lambda x}{6(1 - \lambda x)} = \frac{\lambda I_{-3}(m)}{6(1 - \lambda I_{-3}(m))}
\] (5.25)
and retain a finite term
\[
- \frac{1}{96\pi^2} \left[ \ln \left( \frac{m^2}{m_0^2} \right) + 2 \right] \ddot{V}(t). \] (5.26)
The final result for the pressure reads
\[
p = \dot{\phi}^2(t) - E + p_{\text{fl,fin}}(t, T) - \frac{m_0^4}{96\pi^2} - \frac{m_0^2}{48\pi^2} V(t) - \frac{1}{96\pi^2} \left[ \ln \left( \frac{m^2}{m_0^2} \right) + 2 \right] \ddot{V}(t). \] (5.27)
A further quantity of interest is the particle number density. It does not need to be renormalized after subtraction of the initial particle number density and is given by
\[
n(t) - n(0) = \int \frac{d^3k}{(2\pi)^3} \coth \frac{\beta \omega_{k0}}{2} \left\{ \frac{1}{4} \left[ |U_k(t)|^2 + \frac{1}{\omega_{k0}^2} |\dot{U}_k(t)|^2 \right] - \frac{1}{2} \right\}
\]
\[
= \int \frac{d^3k}{(2\pi)^3} \coth \frac{\beta \omega_{k0}}{2} \frac{\left| \tilde{f}^{(1)}(t) \right|^2}{4\omega_{k0}^2}. \] (5.28)

6 **Numerical results**

We have implemented numerically the renormalized formalism derived in the previous sections. The results show essentially the same features as those found by other groups [30, 3].

We have chosen several parameter sets which are displayed in Table 1. All parameter sets start with an initial value of the mass \( m_0 = 3 \). The gap equation is then solved for \( \phi(0) \). We have chosen two parameter sets at \( T = 0 \), one with \( \lambda = 1 \) and one with \( \lambda = 5 \). Another two parameter sets have finite temperature, \( T = 3 \).
and $T = 10$, respectively, and $\lambda = 1$. Parameter sets with smaller temperatures showed very little deviations from the $T = 0$ case and are not presented.

For the first parameter set the numerical results are displayed in Figs. 1 a-e. The classical field $\phi(t)$ is seen to oscillate with an amplitude decreasing slowly to an asymptotic value. The potential $V(t)$, the difference between $M^2(t)$ and $m_0^2 = M^2(0)$ reaches an asymptotic average of $-4.0$ with small oscillations. Classical and fluctuation energy are shown in Fig. 1c. We denote as fluctuation energy the quantity $\mathcal{E}_{\text{fl,fin}}(t, T) = \mathcal{E}_{\text{fl,fin}}(t, 0) + \Delta \mathcal{E}_{\text{fl,fin}}(t, T)$, see Eqs. (5.6, 5.4). The remaining parts of the total energy (5.16) are considered as the classical energy. The separation is somewhat arbitrary as finite parts of the leading order fluctuation energy are contained in $\Delta \lambda, \Delta m^2$ and $\Delta \Lambda$. So the fact that the “classical energy” becomes even negative is deceptive, the classical amplitude $\phi(t)$ has not decreased to zero. It has decreased roughly by a factor of 3, implying a decrease of energy by a factor 9. Nevertheless the production of fluctuation energy is important, as also seen from the particle number displayed in Fig. 1e. The energy is seen to be conserved, apart from some numerical noise in the early stage of evolution, due to badly convergent integrals. The pressure is seen to approach an asymptotic value of $\simeq 5$, somewhat smaller than the ultrarelativistic limit of $E/3 = 6.7$.

For the other parameter sets we display just the classical amplitude and the potential $V(t)$, need to verify a sum rule (see below). For the finite temperature case $T = 3$ the particle number develops almost identically to the one for parameter set 1, and reaches an asymptotic value of $\simeq 8.5$. This can be compared with the thermal particle number density $3.29$. The situation changes strongly in the high temperature situation, parameter set 4. There, the particle production, displayed in Fig. 4c, is insignificant with respect to the thermal particle number density which is $n(0) \simeq 122$. The other Figures for this parameter set show clearly that the interaction of the classical amplitude with fluctuations is suppressed.

An interesting topic which has emerged recently \cite{30} is a sum rule for the late time behaviour, which was found to be satisfied numerically with high precision. Adapted to our notation and definitions, and generalized (naively) to finite temperature, it reads

$$M^2(\infty) \simeq C \left( m^2 - \frac{\lambda}{16\pi^2} (m_0^2 - m^2) + \Sigma_{-1}(m_0, T) \right) + \lambda C \frac{\phi^2(0)}{2}. \quad (6.29)$$

This value of $M(\infty)$ is determined by the lower limit of a parametric resonance band, essentially it implies that the classical oscillation becomes stationary if its frequency, i.e. $M(t)$, settles in such a way as to avoid resonant excitation. We have verified this sum rule for our parameter sets, the left and right hand sides of the sum rule are compared in Table 1. For $T = 0$ the agreement is excellent, for $T \neq 0$, a case not considered in \cite{30}, the deviations are of the order of 10%.
7 Conclusions

We have derived in this paper the renormalized equations of motion, the energy and the pressure for the nonequilibrium evolution of a scalar $O(N)$ model. The regularization has been done in a covariant way, using dimensional regularization. As in the case of the one-loop equations studied previously [27] it was possible to fix all the renormalization counter terms independent of the initial conditions, though the divergent integrals appearing in the unrenormalized expressions do depend on the “initial mass” $m_0$ instead of the renormalized mass. As a renormalization convention we have chosen a slightly modified $\overline{MS}$ scheme, it can of course be modified to another suitable convention like renormalization at the minimum of the effective potential.

We have restricted the formalism to the case of unbroken symmetry. The special - and highly interesting - aspects of the broken symmetry case have been investigated in 3. Here it was our aim to present a general framework, which has to be adapted to specific physical models. In the case of the one-loop equations it was indeed possible to extend the formalism to nonabelian gauge theories [3].

The formalism developed here represents at the same time a rather convenient computation scheme. The CPU time requirements are of the same order as the one for the one-loop equations. Typically, the examples we have presented took 1-2 hours each on a small workstation. We have not attempted to meet the same standards in numerical precision as other groups, nevertheless our results show the same general features as those of other groups. This is presumably due to the fact that the system as such has a stable and essentially predictable late time behavior [30], indeed our results fulfil an asymptotic sum rule formulated in this Reference.

A Some thermal integrals

In this Appendix we give, without claim of originality, some explicit expressions for the thermal integrals as we have used them in the numerical computations. In deriving these relations we have relied on the integral tables of Prudnikov, Brychkov and Marichev [29].

The finite temperature part of the tadpole graph, which constitutes a correction to the mass, is given by the integral

$$\Sigma_{-1}(m_0, T) = \int \frac{d^3k}{(2\pi)^3} \omega_k \left( e^{\beta \omega_k} - 1 \right)$$  \hspace{1cm} (A.1)

$$= \frac{m_0^2}{2\pi^2} \sum_{n=1}^{\infty} \left\{ \tau_{nm}^{-2} + \sum_{j=0}^{\infty} \frac{1}{4j!(j+1)!} \left[ 2\ln \frac{\tau_{nm}}{2} - \psi(j+1) - \psi(j+2) \right] \left( \frac{\tau_{nm}^2}{4} \right)^j \right\}$$

where $\tau_{nm}$ stands for $nm_0/T$. For large values of $\tau_{nm}$ (this means for small $T$) the integrand is dominated by momenta of order $k \approx T$. Therefore one can expand
ω_{k0} w.r.t. powers of m/k; the integral is then well approximated by

\[ \Sigma_{-1}(m_0, T) \simeq \frac{m_0^2}{2\pi^2} \sum_{n=1}^{\infty} \sqrt{\frac{\pi}{2}} e^{-T_{nm}} T_{nm}^{3/2} \left\{ 1 + \frac{3}{8} T_{nm}^{-2} - \frac{15}{128} T_{nm}^{-2} + \frac{105}{1024} T_{nm}^{-3} + \mathcal{O}\left(T_{nm}^{-4}\right) \right\}. \]  

(A.2)

For \( T \gg m_0 \) we find directly from Eq. (A.1) the well-known approximation

\[ \Sigma_{-1}(m_0, T) \simeq \frac{1}{2\pi^2} \zeta(2) T^2 = \frac{T^2}{12}. \]  

(A.3)

It yields the hard thermal loop corrections to the mass.

The finite temperature part of the fish graph, which can be considered as a finite correction to the coupling constant, is given by

\[ \Sigma_{-3}(m_0, T) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_{k0}^3} \frac{1}{e^{\beta \omega_{k0}} - 1} \]  

(A.4)

\begin{align*}
\text{For small } T \text{ or large } T_{nm} \text{ we find the approximation} \\
\Sigma_{-3}(m_0, T) &\simeq -\frac{1}{4\pi^2} \sum_{n=1}^{\infty} \sum_{j=0}^{\infty} \left\{ \frac{1}{2(2j-1)j!} \left[ \ln \frac{T_{nm}}{2} - \psi(j+1) - \frac{1}{2j-1} \right] \left( \frac{T_{nm}^2}{4} \right)^j + \frac{\pi}{2} T_{nm} \right\}.
\end{align*}

(A.5)

For large temperatures this integral behaves linear in \( T \), more precisely

\[ \Sigma_{-3}(m_0, T) \simeq -\frac{1}{4\pi^2} \sum_{n=1}^{\infty} \sum_{j=0}^{\infty} \frac{\sqrt{\pi}}{2} e^{-T_{nm}} T_{nm}^{3/2} \left\{ 1 - \frac{21}{8} T_{nm}^{-1} - \frac{1185}{128} T_{nm}^{-2} - \frac{42735}{1024} T_{nm}^{-3} + \mathcal{O}\left(T_{nm}^{-4}\right) \right\}. \]  

(A.6)

The finite temperature part associated with the quartic divergence in the energy is given by the Planck formula

\[ \Sigma_{1}(m_0, T) = \int \frac{d^3k}{(2\pi)^3} e^{\beta \omega_{k0}} e^{-\beta \omega_{k0}} - 1 \]  

(A.7)

\begin{align*}
\text{For large } T_{nm} \text{ or small } T \text{ we find} \\
\Sigma_{1}(m_0, T) &\simeq \frac{m_0^4}{2\pi^2} \sum_{n=1}^{\infty} \left\{ 6 T_{nm}^{-4} \left( 1 - \frac{T_{nm}}{12} \right) \\
&+ \frac{1}{16} \sum_{j=0}^{\infty} \frac{2j+1}{j!(3)_j} \left( 2 \ln \frac{T_{nm}}{2} - \psi(3+j) - \psi(1+j) + \frac{2}{2j+1} \right) \right\}. \end{align*}

(A.8)

As an approximation for large \( T_{nm} \) or small \( T \) we find

\[ \Sigma_{1} \simeq \frac{m_0^4}{2\pi} \sum_{n=1}^{\infty} e^{-T_{nm}} \sqrt{\frac{\pi}{2}} T_{nm}^{3/2} \left\{ 1 + \frac{27}{8} T_{nm}^{-1} + \frac{705}{128} T_{nm}^{-2} + \frac{2625}{1024} T_{nm}^{-3} + \mathcal{O}\left(T_{nm}^{-4}\right) \right\}. \]  

(A.9)

For large temperatures one obtains

\[ \Sigma_{1}(m_0, T) \simeq \frac{\pi^2}{30} T^4. \]  

(A.9)
References

[1] D. Boyanovsky, H. J. de Vega, and R. Holman, Phys. Rev. D 51, 734 (1995).
[2] F. Cooper, Y. Kluger, E. Mottola, and J. P. Paz, Phys. Rev. D 51, 2377 (1995).
[3] F. Cooper, S. Habib, Y. Kluger, and E. Mottola, Phys. Rev. D 55, 6471 (1997).
[4] D. Boyanovsky, H. J. de Vega, R. Holman, and S. Prem Kumar, Phys. Rev. D 56, 1939 (1997); ibid. 56, 3929 (1997).
[5] J. Baacke, K. Heitmann, and C. Pätzold, Phys. Rev. D 55, 7815 (1997).
[6] A. Ringwald, Z. Phys. C 34, 481 (1987); Ann. Phys. 177, 129 (1987); A. Ringwald, Quantenfeldtheorie und frühes Universum, Inauguraldissertation, Heidelberg 1988.
[7] E. Calzetta and B. L. Hu, Phys. Rev. D 35, 495 (1987); ibid., 37, 2878 (1988).
[8] D. Boyanovsky, H. J. de Vega, and R. Holman, Phys. Rev. D 49, 2769 (1994).
[9] D. Boyanovsky, D. Cormier, H. J. de Vega, and R. Holman, Phys. Rev. D 55, 3373 (1997).
[10] D. Boyanovsky, D. Cormier, H. J. de Vega, R. Holman, A. Singh, and M. Srednicki, Phys. Rev. D 56, 1939 (1997).
[11] S. Y. Khlebnikov and I. I. Tkachev, Phys. Rev. Lett. 77, 219 (1996).
[12] S. Y. Khlebnikov and I. I. Tkachev, Phys. Lett. B 390, 80 (1997).
[13] D. T. Son, Phys. Rev. D 54, 3745 (1996).
[14] P. B. Greene, L. Kofman, A. Linde, and A. A. Starobinskii, Phys. Rev. D 56, 6175 (1997).
[15] L. Kofman, A. Linde, and A. A. Starobinskii, Phys. Rev. Lett. 76, 1011 (1996).
[16] D. I. Kaiser, Phys. Rev. D 53, 1776 (1996); ibid. 56, 706 (1997).
[17] S. A. Ramsey and B. L. Hu, Phys. Rev. D 56, 678 (1997).
[18] J. Baacke, K. Heitmann, and C. Pätzold, Phys. Rev. D 56, 6556 (1997).
[19] see, e.g., W. H. Zurek, Phys. Rept. 276, 177 (1996).
[20] J. Schwinger, J. Math. Phys. (N.Y.) 2, 407 (1961).

[21] L. V. Keldysh, Zh. Eksp. Teor. Fiz. 47, 1515 (1964); Sov. Phys. JETP 20, 1018 (1965).

[22] M. A. Lampert, J. F. Dawson, and F. Cooper, Phys. Rev. D 54, 2213 (1996).

[23] D. Boyanovsky, H. J. de Vega, R. Holman, and J. F. J. Salgado, Phys. Rev. D 54, 7570 (1996).

[24] H. J. de Vega and J. F. J. Salgado, Phys. Rev. D 56, 6524 (1997).

[25] J. Baacke, K. Heitmann, and C. Pätzold, On the choice of initial states in nonequilibrium dynamics, Dortmund preprint DO-TH 97/24, hep-th 9711144, November 1997.

[26] F. Cooper, S. Habib, Y. Kluger, E. Mottola, J. P. Paz, and P. R. Anderson, Phys. Rev. D 50, 2848 (1994) and references therein.

[27] J. Baacke, K. Heitmann, and C. Pätzold, Phys. Rev. D 55, 2320 (1997).

[28] C. G. Callan, S. Coleman and R. Jackiw, Ann. Phys. (NY) 59, 42 (1970).

[29] A. P. Prudnikov, Yu. A. Brychkov, O. I. Marichev, Integrals and Series, Vol.1, Gordon and Breach Science Publishers, 1986.

[30] D. Boyanovsky, C. Destri, H. J. de Vega, R. Holman, and J. F. J. Salgado, hep-ph 9711384.
Table Captions

**Table 1**: Parameter sets and sum rule. We display the parameters of the 4 numerical simulations. The mass unit is set by $m = 1$; $\phi(0)$ follows from the gap equation (4.22), $M^2(\infty)$ is the sum of $m_0^2$ and of $V(\infty)$ read from the corresponding Figures. R.h.s. is the right hand side of Eq.(6.29).

Figure Captions

Fig. 1a: $\phi(t)$ for parameter set 1.
Fig. 1b: The potential $V(t)$ for parameter set 1.
Fig. 1c: Classical energy (long-dashed line), fluctuation energy $\mathcal{E}_{\text{fl,fin}}(t, T)$ (short-dashed line) and total energy (solid line) for parameter set 1.
Fig. 1d: The pressure $p(t)$ for parameter set 1.
Fig. 1e: Particle number $n(t) - n(0)$ for parameter set 1.
Fig. 2a: $\phi(t)$ for parameter set 2.
Fig. 2b: The potential $V(t)$ for parameter set 2.
Fig. 3a: $\phi(t)$ for parameter set 3.
Fig. 3b: The potential $V(t)$ for parameter set 3.
Fig. 4a: $\phi(t)$ for parameter set 4.
Fig. 4b: The potential $V(t)$ for parameter set 4.
Fig. 4c: Particle number $n(t) - n(0)$ for parameter set 4.

| set # | $\lambda$ | $T$ | $m_0$ | $\phi(0)$ | $M^2(\infty)$ | r.h.s. |
|-------|-----------|-----|-------|-----------|---------------|-------|
| 1     | 1         | 0   | 3     | 2.815     | 5.0           | 4.98  |
| 2     | 5         | 0   | 3     | 1.235     | 4.95          | 4.90  |
| 3     | 1         | 3   | 3     | 2.76      | 5.15          | 5.39  |
| 4     | 1         | 10  | 3     | 1.25      | 8.24          | 9.45  |

Table 1
Figure 1e

The graph illustrates the function \( n(t) - n(0) \) over time \( t \). The graph shows a clear oscillatory behavior with increasing amplitude as \( t \) increases. The x-axis represents time \( t \), and the y-axis represents \( n(t) - n(0) \).
Figure 1a
Figure 3a

The graph shows the function $\phi(t)$ plotted against $t$ from $0$ to $100$. The y-axis represents $\phi(t)$, ranging from $-3$ to $3$, and the x-axis represents time $t$, ranging from $0$ to $100$. The function appears to oscillate with decreasing amplitude over time.
Figure 4c
Figure 4a