Evolution of entanglement under an Ising–like Hamiltonian with particle losses

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We present analytical compact solution for the density matrix and all correlation functions of two collective-macroscopic spins evolving via Ising-like Hamiltonian in the presence of particle losses. The losses introduce non-local phase noise which destroys highly entangled states arising in the evolution. On the other hand, the states appearing at relatively short timescales, possessing EPR-like entanglement will survive. Applying our solutions to the recently proposed scheme to entangle two Bose-Einstein condensates, we estimate the optimal number of atoms for EPR correlations.

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I. INTRODUCTION

There exist a number of experiments in which entanglement between massive particles is generated. Many setups are on the list: ultracold gases \[8, 9, 23\], atoms trapped in the resonant cavities \[10, 25, 26\] or superconducting qubits \[29\]. The entanglement in each of these experiments stems from different physical mechanisms. Ultracold atoms entangle simply due to atom-atom collisions \[8, 9, 23, 25\]. The diluted thermal atoms were made to interact by a common mode of light in resonant cavities \[10, 25, 26\].

Although the setups were very different, in all cases the emergence of entanglement can be understood, at least qualitatively, within a common simple theoretical model, so called one-axis twisting scheme \[11\]. In this scheme the state of the system is expressed as a macroscopic collective spin \( \hat{S}_z \) of the length proportional to the number of particles \( N \). The quantum correlations result from an unitary evolution due to the Hamiltonian proportional to \( \hat{S}_z^2 \), where \( \hat{S}_z \) is the \( z \)-th component of the collective spin. One of the mentioned above experiments has been extended \[30\], and other are proposed to be extended \[13, 17, 24\], to a setup described by two effective collective spins \( S^a \) and \( S^b \), which due to nonlocal Hamiltonian \( \hat{S}_z^a \hat{S}_z^b \) would evolve into non-local entangled states. If one manage to prepare initially both collective spins to be along \( z \) axis, then due to the latter non-local Hamiltonian they would evolve first to state possessing EPR-type of entanglement and then to more exotic non-local macroscopic superpositions, as so called "Schrödinger cat in two boxes" \[3, 13, 30\].

Preparation of such state is of fundamental interest, as it shows at macroscopic level the early quantum mechanics paradoxes \[7\]. The scheme is also considered from the quantum-computation perspective \[21\].

As one would like to extend the existing experiments into this new non-local regime, it is critical to ask again for robustness of the scheme in the presence of decoherence. In this manuscript we will investigate the role of collective particle losses, process typically found in Bose-Einstein condensates. The system we discuss, with one-body losses included via standard master equation, has a favorable theoretical feature – it has a simple compact analytical solutions, for all correlation functions and for all terms of the density matrix of the system.

After presenting our model of the non-local evolution with particle losses in Sec. II we will discuss a method of generating functions with which we find the analytical solutions, Sec. III. In Sec. IV we show the evolution of the linear entropy and so called EPR condition \[4, 5, 13, 22\], known in the Quantum Information community under the name "steering condition". In Sec. V we use the quantum trajectory method to gain an insight into the lossy dynamics. It is shown that the losses result in the non-local phase noise - loosing an atom in system "a" leads through the non-local evolution to a noise in the system "b". The entanglement captured by EPR condition is less sensitive to the action of losses, what is qualitatively understood by discussing the phase noise introduced in the frame of the quantum trajectories, given in Sec. V. We focus on this type of entanglement in the last section, where we include phenomenologically two- and three-body losses to estimate the optimal conditions for an experiment in ultracold gases.

II. MODEL

We consider nonlinear evolution of two groups (denoted with indices "a" and "b") of (pseudo)spins 1/2. We will assume, that in the initial state each from \( N + M \) spins is in the state \( |0\rangle + |1\rangle \) \( \sqrt{2} \), namely

\[
|\psi(0)\rangle = \bigotimes_{i=1}^{N} \frac{|0\rangle_{a} + |1\rangle_{a}}{\sqrt{2}} \bigotimes_{i=1}^{M} \frac{|0\rangle_{b} + |1\rangle_{b}}{\sqrt{2}}.
\]

The Hamiltonian under consideration is

\[
H = \chi (S^a_z)^2 + \chi (S^b_z)^2 - \chi_{ab} S^a_z S^b_z,
\]

where

\[
\begin{align*}
\chi &= \frac{1}{2} J_{z}^2, \\
\chi_{ab} &= \frac{1}{2} J_{z}^2 - \frac{1}{2} J_{z}^2 + \frac{1}{2} I_{z}^2,
\end{align*}
\]

where \( J_{z} \) and \( I_{z} \) are the magnetic quantum numbers.
where $S^a_z = \frac{1}{2} \sum_{i=1}^{N} \sigma^a_{iz}$ is sum of $z$-Pauli matrices of all of $N$ spins in the group "a". Similar definition holds for $S^b_z$, but with $M$ spins. In addition to the unitary evolution we consider dissipation. This is an important point, as in general entanglement existing between two subsystems is quite susceptible to very small deviations from unitarity (or in other words, to a weak entanglement with an environment). Due to the resulting decoherence, the state of the system will be given by a density operator, which we assume to obey master equation

$$\frac{d\hat{\rho}}{dt} = i[\hat{\rho}, H] + L[\hat{\rho}],$$

(3)

As the source of the dissipation we will assume one body losses modeled by the following Lindblad superoperator

$$L[\hat{\rho}] = \sum_{e} \Gamma_{e} \left( \hat{a}_{e} \hat{\rho} \hat{a}_{e}^\dagger + \hat{a}_{e}^\dagger \hat{\rho} \hat{a}_{e} - \frac{1}{2} \left\{ \hat{\rho}, \hat{a}_{e} \hat{a}_{e}^\dagger + \hat{a}_{e}^\dagger \hat{a}_{e} \right\} \right),$$

(4)

where $\hat{a}_{e}$ ($\hat{b}_{e}$) is a bosonic operator which annihilates a single atom in state $\epsilon = 0, 1$ from mode "a" ("b"). This model is particularly adequate to Bose-Einstein condensates, for which the term (4) describes losses due to interaction with external particles, coming from residue air in the vacuum chamber.

As the initial state belongs to the symmetric subspace and dissipation is expressed with collective operator only, hence the appropriate basis will be the Fock basis:

$$|n_0, n_1, m_0, m_1\rangle = \frac{1}{\sqrt{n_0! n_1! m_0! m_1!}} \sum_{l=0}^{m_1} \sum_{k=0}^{m_0} (\hat{a}_{0}^\dagger)^{n_0} (\hat{a}_{1}^\dagger)^{n_1} (\hat{b}_{0}^\dagger)^{m_0} (\hat{b}_{1}^\dagger)^{m_1} \langle \hat{a}_{k} \hat{b}_{l} | \hat{a}_{0} \hat{b}_{0} \rangle \langle \hat{a}_{1} \hat{b}_{1} | \hat{a}_{k} \hat{b}_{l} \rangle |\langle \hat{a}_{k} \hat{b}_{l} | \hat{a}_{0} \hat{b}_{0} \rangle \langle \hat{a}_{1} \hat{b}_{1} | \hat{a}_{k} \hat{b}_{l} \rangle |\langle \hat{a}_{k} \hat{b}_{l} | \hat{a}_{0} \hat{b}_{0} \rangle \langle \hat{a}_{1} \hat{b}_{1} | \hat{a}_{k} \hat{b}_{l} \rangle |\rangle,$$

(5)

where $|\emptyset\rangle$ is the vacuum.

The density matrix written in this basis reads:

$$\hat{\rho} = \sum_{n_0 n_1 m_0 m_1} \rho^{n_0 n_1 m_0 m_1}_{k_0 k_1 l_0 l_1} |n_0 n_1 m_0 m_1\rangle \langle k_0 k_1 l_0 l_1|,$$

(6)

where the ranges of the indices within the sum are for kets, "a": $n_0 = 0, \ldots, N, n_1 = 0, \ldots, N - n_0$, for kets "b" $m_0 = 0, \ldots, M, m_1 = M - m_0$ and the indices for bras are constrained analogously.

Another states which will be helpful in the next sections are the so called phase states \cite{13, 27}, here defined as:

$$|\phi_{\alpha} \phi_{\beta}\rangle \equiv \frac{1}{\sqrt{2}} \sum_{i=1}^{N} e^{-i\phi_{\alpha}/2} |0\rangle_{ia} + e^{i\phi_{\alpha}/2} |1\rangle_{ia} \times \frac{1}{\sqrt{2}} \sum_{j=1}^{M} e^{-i\phi_{\beta}/2} |0\rangle_{jb} + e^{i\phi_{\beta}/2} |1\rangle_{jb}.$$  

(7)

III. METHODS

The master equation (3) written in the Fock basis leads to a problem which complexity grows with the number of particles: a set of numerous $(N^3 M^3)$ coupled ordinary differential equations on the density matrix terms $\rho^{n_0 n_1 m_0 m_1}_{k_0 k_1 l_0 l_1}$ needs to be solved. There exists however a mathematical technique well suited for this class of problems: method of characteristic functions. It has been already applied to problems involving Bose-Einstein condensates \cite{20}, our approach is a natural extension of the previous research. We will start with a simple example explaining the basic concepts of the method, which we will later apply to our system.

A. Simple example

In this section, we will solve a simple set of differential equations:

$$\frac{dp_n(t)}{dt} = \Gamma (n + 1)p_{n+1} - np_n,$$

(8)

where $\{p_n\}$ are the unknown function of the time and $n = 0, 1, \ldots, \infty$. There are numerous approaches to this (elementary) problem. One which renders it extremely simple is consideration of the following polynomial of $x$:

$$P(x, t) = \sum_{n} x^n p_n(t).$$

We can reconstruct time evolution of this expression by multiplying each of equations (8) by $x^n$ and summing over $n$:

$$\sum_{n} x^n \frac{dp_n(t)}{dt} = \Gamma (n + 1) x^n p_{n+1}(t) - np_n(t),$$

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$$\sum_{n} x^n \frac{dp_n(t)}{dt} = \Gamma (n + 1) x^n p_{n+1}(t) - np_n(t),$$

which can be rewritten in the form

$$\frac{\partial P(x, t)}{\partial t} = \Gamma (n + 1) x^n \frac{\partial P(x, t)}{\partial x}.$$
By explicit calculation one can find that the master equation \[ \rho^{n_0n_1m_0m_1}_{k_0k_1l_0l_1} \] couples only the density matrix terms \( \rho^{n_0n_1m_0m_1}_{k_0k_1l_0l_1} \) for which the indices difference: \[
\delta_a = (n_0 + n_1) - (k_0 + k_1), \\
\delta_b = (m_0 + m_1) - (l_0 + l_1).
\]
are equal. There is a reason to that: physically, as the particles in our system are massive, there are no coherences between different total particle number states and thus, \( \delta_a + \delta_b = 0 \). On top of that, we start the evolution with a product state, \( \hat{\rho}(t = 0) = \hat{\rho}_a \otimes \hat{\rho}_b \), so such coherences are absent in both subsystems in fact, \( \delta_a = \delta_b = 0 \).

As the evolution preserves \( \delta_a, \delta_b \), we may write \[
(n_0, n_1, m_0, m_1) = (x, y + z, u, v + r), \\
(k_0, k_1, l_0, l_1) = (x + z, y, u + r, v).
\]
In this way, instead of 8 indices used to indicate matrix elements, we will use only 6 indices. The master equation couples only these matrix elements which has the same indices z and r being offsets from diagonal.

From now on, we will use this sub-basis. The derivation of the characteristic function equation is straightforward but lengthy, hence we will just write down the final results (for a more detailed version please refer to the appendix).

We introduce a family of characteristic functions
\[
\rho^{x,y+z,u,v+r}_{x,y,u,v} = \sum_{X,Y,U,V} \rho^{x,y+z,u,v+r}_{x,y,u,v} \rho^{X,Y,U,V} \frac{1}{x!y!u!v!},
\]
indexed by \( (z, r) \) - the offset from diagonal elements used in defining sum. After laborious derivation final equation for \( h^{z,r} \) is
\[
\frac{\partial h^{z,r}}{\partial t} = \left( -\beta_0^{(r,z)} + \beta_1^{(r,z)} \right) + \left( \rho^{x,y+z,u,v+r}_{x,y,u,v} \rho^{X,Y,U,V,t} - 2(G_0 + G_1)(z + r)h^{z,r} + \right.
\]
\[
\left. \beta_0^{(z,r)} \rho^{x,y+z,u,v+r}_{x,y,u,v} \rho^{X,Y,U,V,t} \right),
\]
where
\[
\beta_0^{(p,q)} = 2(i\rho^{x,y}_{x,y} + 2\rho^{x,y}_{x,y}) \Gamma_e,
\]
in which "+" sign corresponds to \( \epsilon = 0 \), and "−" to \( \epsilon = 1 \).

The initial state
\[
|\psi(0)\rangle = 2^{-N/2} \sum_{n_1=1}^{N} \sqrt{n_1} \sqrt{M-n_1} |n, N-n_1, M-m_1\rangle.
\]
leads to the following initial condition for the characteristic function \( h^{z,r}(X,Y,U,V,0) \):
\[
h^{z,r}_{|t=0} = 2^{-N-M} M!N! \frac{(X + Y)^{M-r} (U + V)^{M-r}}{(M-r)!(N-r)!}.
\]
We solve the partial differential equation \[ \rho^{x,y+z,u,v+r}_{x,y,u,v} + \frac{\partial}{\partial X} \frac{\partial}{\partial Y} \frac{\partial}{\partial U} \frac{\partial}{\partial V} \]
with the initial condition \[ \rho^{x,y+z,u,v+r}_{x,y,u,v} = 2^{(x+y+z+u+v+r)} \]
using the method of characteristics. As in the simple example we find a path in the space of parameters, here \( X, Y, U, V, t \), on which the \( h^{z,r} \) changes exponentially.

As described in the next section the characteristic functions are particularly well suited to evaluate averages of the polynomial of the spin component function, making the method a powerful tool.

C. Quantum averages

In order to evaluate average of some operator, one can express the average as a function of the density matrix terms. In principle it is possible to calculate the matrix elements from the characteristic function,
\[
\rho^{x,y+z,u,v+r}_{x,y,u,v} = 2^{(x+y+z+u+v+r)} \frac{\partial}{\partial X} \frac{\partial}{\partial Y} \frac{\partial}{\partial U} \frac{\partial}{\partial V} \]
for some \( X = Y = U = V = 0 \).

The straightforward calculation is however inefficient (there are \( O(N^3M^3) \) matrix elements in general, but not all of them are needed) and often not necessary: pseudospin-related averages can be extracted directly from the characteristic function. Let us take, for instance, operator \( S_x^a = \frac{1}{2}(a_0^+ a_0 - a_0 a_0^+) \). The \( \{ a_0^+ a_0 \} \) term is equal to \( \sum_{x,y,u,v} \sqrt{X + Y + U + V} \rho^{x+y+1,u,v} \rho^{x+y+1,u,v} \), which can be easily obtained from one of the characteristic functions:
\[
\langle a_0^+ a_0 \rangle = \sum_{x,y,u,v} \rho^{x+y+1,u,v} \rho^{x+y+1,u,v} |X = Y = U = V = 1.
\]

All important correlation function we are calculated follow the same path. We give explicit form of chosen quantum averages in the Appendix.

IV. ENTANGLEMENT IN THE SYSTEM

Linear entropy

In case without losses one can quantify the entanglement using the linear entropy defined as
\[
S_{lin} = 1 - \rho^{x,y+z,u,v+r}_{x,y,u,v} = 1 - \sum_{x,y,z} \left| \rho^{x,y+z,u,v+r}_{x,y,u,v} \right|^2,
\]
where \( \sigma = \text{Tr} \{ \rho \} \) is the reduced density matrix.

We have not found a simple equation for the linear entropy as a function of \( h^{z,r} \) functions. Instead, it is possible to compute all the matrix elements of a partially traced matrix:
\[
\rho^{x,y+z,u,v+r}_{x,y,u,v} = \sqrt{X + Y + U + V} \rho^{x+y+1,u,v} \rho^{x+y+1,u,v} |X = Y = U = V = 1.
\]

With the help of these reduced matrix elements, we evaluate the linear entropy for small \( N, M \) (of order 10) using directly the definition.
We illustrate evolution of this quantity in Fig. 1. In case without losses, \( \Gamma_0 = \Gamma_1 = 0 \) this entropy, similarly to von Neuman entropy \([3, 13]\) and negativity \([24]\) has a fractal-like structure, independent on the value of \( \chi \) (see Appendix in \([13]\)). This is reminiscent of the so called devil’s staircase known from the Ising model \([1]\). Each peak indicates a superposition of a few phase states \([7]\), with the cat-like state appearing at \( \chi_{ab} = \pi \). Precisely at \( t = \pi / \chi_{ab} \), the state is a superposition of \( |00\rangle_{ph}, |0\pi\rangle_{ph}, |\pi, 0\rangle_{ph}, |\pi, \pi\rangle_{ph} \) (the eigenstates of \( S^a_1 S^b_2 \) operator) with amplitudes depending on \( \chi \). As an example, in the case of \( \chi = 0 \) the state reads

\[
|\Psi(t)\rangle = \frac{|00\rangle_{ph} + |0\pi\rangle_{ph} + |\pi 0\rangle_{ph} - |\pi\pi\rangle_{ph}}{2}.
\]

Although the linear entropy is no a direct quantifier of entanglement when the losses are present, still its dramatic change after one (on average) loss event, as shown in Fig. 1 reveals that the entangled states at long evolution times \( (\chi_{ab} t \gg 1/\sqrt{N}) \) are, as expected, vulnerable to decoherence. Hence in the next sections we focus on short evolution times.

Figure 1: Linear entropy as a function of time in case: without losses \( \Gamma_1 = \Gamma_0 = 0.0 \) (red solid line), with \( \Gamma_1 = \Gamma_0 = 0.01 \) (blue dashed line). Top: \( N = 10 \), bottom \( N = 50 \). Other parameters \( \chi_{ab} = \chi \) and \( N = M = 10 \).

\[ E_{\text{EPR}}^2 \] calculated disregarding losses is imposed as dashed lines. The thickness of line refers to \( N \): from least to most thick, \( N = 10^2, 10^3, 10^4, 5 \times 10^4 \).

Other parameters: \( \Gamma_1 = \Gamma_0 = \chi_{ab} \) and \( \chi = 0 \). The optimal angles are \( \alpha = \beta = 0 \).

A. Short times: EPR like entanglement

We employ entanglement criterion devised in \([1][17][31]\) which is constructed from simple observable quantities and it is sensitive to EPR-like correlations \([13]\).

The entanglement criterion is rooted in the following reasoning. Let us consider two observables, \( \hat{X} \) and \( \hat{P} \). Then for a single isolated system, the expression \( L = \langle (\hat{X} - A)^2 \rangle \langle (\hat{P} - B)^2 \rangle \) is minimized by a choice \( A = \langle \hat{X} \rangle \) and \( B = \langle \hat{P} \rangle \). The Heisenberg inequality tells us, that even for this optimal choice of the numbers \( A \) and \( B \), the expression \( L \) has to be larger than \( \frac{1}{4} \left( \left| \left( \hat{X}, \hat{P} \right) \right| \right)^2 \).

The situation changes if nonclassical correlations i.e., entanglement between system "a" and another system "b" are allowed. As in the seminal paper of Einstein Podolsky and Rosen \([7]\), there exist states in which after every measurement performed in "b" we know precisely the current value of either position or momentum in "a". It is then possible to construct a better estimates for \( \hat{X} \) and \( \hat{P} \), based on observations made in the second subsystem. Mathematically, one can violate the inequality

\[
\langle (\hat{X}^a - \hat{X}^a_{\text{inf}})^2 \rangle \langle (\hat{P}^a - \hat{P}^a_{\text{inf}})^2 \rangle \geq \frac{1}{4} \left| \left( \hat{X}^a, \hat{P}^a \right) \right|^2
\]

where the new estimators \( \hat{X}^a_{\text{inf}}, \hat{P}^a_{\text{inf}} \) are functions of observables in the second subsystem (in our case linear dependence is enough to prove the existence of entanglement):

\[
\hat{X}^a_{\text{inf}} = q^a + r^a \hat{X}^b,
\]
\[
\hat{P}^a_{\text{inf}} = p^a + s^a \hat{P}^b.
\]
We can define the parameter
\[
E_{EPR}^2 = \frac{4\left(\langle \hat{X}^a - \hat{X}^a_{inf} \rangle^2 \right)}{\left\langle \left[ \hat{X}^a, \hat{P}^a \right] \right\rangle^2}, \tag{17}
\]
which, if less than 1, serves as an indicator of EPR entanglement. The optimal parameters \(q^a, r^a, p^a, s^a\), minimizing the \(E_{EPR}^2\) are defined by
\[
q^a = \frac{1}{\text{covar}(\hat{X}^a, \hat{X}^b)} \langle \hat{X}^b \rangle, \quad r^a = \frac{1}{\text{covar}(\hat{X}^a, \hat{X}^b)} \langle \hat{X}^a \hat{X}^b \rangle,
\]
\[
p^a = \frac{1}{\text{covar}(\hat{P}^a, \hat{P}^b)} \langle \hat{P}^b \rangle, \quad s^a = \frac{1}{\text{covar}(\hat{P}^a, \hat{P}^b)} \langle \hat{P}^a \hat{P}^b \rangle,
\]
where
\[
\text{covar}(\hat{A}, \hat{B}) = \langle \hat{A}\hat{B} \rangle - \langle \hat{A} \rangle \langle \hat{B} \rangle.
\]
In our case the simplest observables are pseudospins
\[
\hat{X}^a = S^a_\alpha, \quad \hat{P}^a = S^a_\alpha + \pi/2,
\]
\[
\hat{X}^b = S^b_\beta, \quad \hat{P}^b = S^b_\beta + \pi/2, \tag{18}
\]
were we used the notation
\[
S^i_\alpha = \cos \alpha S^i_x + \sin \alpha S^i_y, \tag{19}
\]
\[
S^{\alpha+\pi/2}_\alpha = \sin \alpha S^i_x - \cos \alpha S^i_y. \tag{20}
\]
To find the minimal value of \(E_{EPR}^2\) one should plug the form (18) into (17) and then minimize with respect to the free parameters \(\alpha\) and \(\beta\), used in the definition (18).

With the method of generating function we have obtained exact analytical formulas for all average fluctuations and covariances of the spin components, also in the case with particle losses. In Fig. 2 we show the resulting \(E_{EPR}^2\) as a function of time for different number of atoms. As we show, even for substantial loss (\(\Gamma\)'s comparable with other parameters, which translates to constant number of particles lost by the time of \(E_{EPR}^2\) minimum) the entanglement is preserved. This is due to timescales at which this entanglement appears, as intuitively explained at the end of the next section.

V. QUANTUM TRAJECTORIES

One can get a better understanding on what is happening during the evolution using quantum trajectories method, sometimes interpreted as a way to generate a collection of single experimental realizations.

The basic notions in quantum trajectories method are the effective Hamiltonian \(H_{\text{eff}}\) and nonunitary transition operators \(\{A_i\}\) (often called quantum jump operators), an instantaneous quantum state of a single realization (a vector in pertinent Hilbert space) and its trajectory over time. Basic evolution is simple: for a state \(|\psi(t)\rangle\), state in the next moment of time \(|\psi(t+\Delta t)\rangle\) can be calculated in a probabilistic manner:

- a transition associated with operator \(A_i\) will happen with probability \(p_i = \Delta t \langle \psi(t)|A_i^\dagger A_i|\psi(t)\rangle\), after which the state reads
  \[
  |\psi(t+\Delta t)\rangle := \frac{A_i|\psi(t)\rangle}{\langle \psi(t)|A_i^\dagger A_i|\psi(t)\rangle}.
  \]
- Hamiltonian-like evolution will happen with probability \(1 - \sum p_i\), after which the state reads
  \[
  |\psi(t+\Delta t)\rangle := \frac{\exp(-\Delta t (\sum A_i^\dagger A_i/2 + iH)) |\psi(t)\rangle}{\exp(-\Delta t (\sum A_i^\dagger A_i/2 + iH))}, \tag{21}
  \]
  where
  \[
  H_{\text{eff}} = \sum A_i^\dagger A_i/2i + H \tag{22}
  \]
is the effective Hamiltonian of the system. Note that this object is not Hermitian unless \(\sum A_i^\dagger A_i = 0\).

Please note that even if we are sure that transition did not occur (the case when Eq. (21) is applied), it has an effect on the system: effectively, the contribution to the state vector from the basis vectors undergoing transition is reduced. In experiment, this corresponds to post-selection of runs in which transition did not happen. Naturally, statistics of the final states is modified as compared to case without losses and this is precisely the effect seen here.

The connection with usual treatment using density matrices is straightforward: density matrix of the system is recovered after averaging the state projectors, \(|\psi(t)\rangle\langle\psi(t)|\) over multiple realizations. This approach could in principle be used to calculate the density matrix in an efficient manner (the number of differential equations scales linearly with the dimensionality of the Hilbert space, which corresponds to linear speedup compared to number of density matrix elements), but its power lies elsewhere: within the quantum trajectories framework, it is possible to calculate what is the state of the system if we know, for instance, that one particular transition \(A_i\) occurred.

In this paper, we will employ the quantum trajectories method to show the "microscopic" mechanisms, caused by losses and leading to decoherence as it was done in case of a single bimodal BEC. The jump operators associated with the master equation (3) are: \(A_0^{(1)} = \sqrt{\Gamma_0}a_0\) and \(A_1^{(1)} = \sqrt{\Gamma_1}a_1\) for the subsystem "a", and \(A_0^{(2)} = \sqrt{\Gamma_0}b_0\) and \(A_3^{(2)} = \sqrt{\Gamma_1}b_1\) for "b". It is enough to show the effect of losses in the subspace with \(N+M-1\) atoms, i.e. after loss of one particle. The (unnormalized) single trajectory at time \(t\) but with particle lost at time \(t_1 < t\) due to the
\( i \)-th jump operator reads

\[
|\psi(t|i, t_1)\rangle = e^{-iH - \frac{\Gamma}{2} \sum_j (A_j^{(1)})^\dagger A_j^{(1)}} (t - t_1) A_i^{(1)}
\]

\[
\rho'(t) = \int_0^t |\psi(t|i, t_1)\rangle \langle \psi(t|i, t_1)| dt_1,
\]

as the length of unnormalized \( |\psi(t|i, t_1)\rangle \) squared is proportional to the probability of quantum jump occurring at \( t_1 \).

The trajectory \( |\psi(t|i, t_1)\rangle \) can be written as

\[
|\psi(t|i, t_1)\rangle \propto e^{\pm i t_1 \left( \frac{\chi_{ab}}{2} S_a^b - 3 \chi S_a^a \right)} (A_i^{(1)} |\psi(t)\rangle),
\]

where the "+" sign corresponds to \( i = 0 \) and "-" to \( i = 1 \).

In Eq. \( 25 \), the proportionality factor does not depend on \( t_1 \) and \( |\psi(t)\rangle \) is the "original" state vector, as if the evolution remained unperturbed by quantum jumps. The density matrix in subspace with one lost particle can be obtained according to Eq. \( 24 \).

The prefactor \( e^{\pm i t_1 \left( \frac{\chi_{ab}}{2} S_a^b - 3 \chi S_a^a \right)} \) corresponds to nonlocal, \( t_1 \)-dependent rotation of the state vector, which after integration destroys the state coherence. This is probably the major mechanism underlying the decoherence for short times, already known under the name "phase noise". In every quantum trajectory the evolving state is "entangled" in a similar degree. The only effect is the averaging over the random time \( t_1 \) at which the atom has been lost. This is the averaging over the random time \( t_1 \) which results in the deterioration of the entanglement. This effect can be seen by calculating the cut of the Husimi function \( Q \):

\[
Q(\phi_a, \phi_b) = \langle \phi_a, \phi_b | \rho' | \phi_a, \phi_b \rangle,
\]

with respect to the phase states \( |\phi_a, \phi_b \rangle \), i.e. equatorial product states, eigenvectors of \( S_a^\alpha S_b^\beta \):

\[
|\phi_a, \phi_b \rangle = e^{i \phi_a S_a^a} e^{i \phi_b S_b^b} \sum \sqrt{\binom{N}{n} \binom{M}{m}} \langle n, N-n, m, M-m |.\]

In Fig. 3(a) we present the cut of the Husimi function at time \( \chi_{ab} t = \pi \) in the case without lost atom, where the state is the so called Schrödinger cat in two boxes, given already in Eq. \( 15 \). In the panel (b) of Fig. 3 we visualize with the Husimi function the state \( 24 \), restricted to the subspace with \( N + M - 1 \) atoms. The state is smeared due to the phase noise described above.

Figure 3: Husimi Q functions for \( N = 10, \chi_{ab} = \chi \), \( \Gamma = 0.01 \chi \), in the case where particles are lost only in the mode \( |0\rangle \) of condensate "a". The Husimi functions are computed in \( \chi_{ab} t = \pi \).

Panels Fig. 3(c) and (d) show the Husimi functions of single trajectories \( |\psi(t = \frac{\pi}{\chi_{ab}} |0, t_1)\rangle \) for two chosen times \( t_1 \).

Due to the phase noise the Schrödinger cats disappear upon single atom loss - the time of loss event is of the order of \( t_1 \sim \pi/\chi_{ab} \) hence the resulting phase-noise is of the order of \( \delta \phi \sim \chi_{ab} t_1 \sim \pi \). On the other hand, the EPR entanglement is relatively robust simply because it appears early enough, at time-scales \( \tau_{ent} \approx 1/\chi_{ab} N^2/3 \). The average number of lost atoms during this time is around \( N_{loss} \approx \Gamma \tau_{ent} \), hence the number of lost atom scales like \( N_{loss} \sim \Gamma N^{1/3}/\chi_{ab} \). Although the number of lost atoms grows when the total number of atoms is increased, but the total phase noise decreases: \( \Delta \phi \sim N_{loss} |\chi_{ab} \tau_{ent}| \sim (\Gamma/\chi_{ab}) N^{-1/3} \).

VI. CONNECTION WITH BOSE–EINSTEIN CONDENSATES

A. Hamiltonian

One of the scheme proposed as physical implementation of the Hamiltonian \( 1 \) is based on the clouds of atoms cooled down for the Bose-Einstein condensated to appear \( 12 \) \( 13 \). The Bose-Einstein condensates are well
Figure 4: The scheme showing the central stage of the protocol to create the EPR-entangled states based on four Bose-Einstein condensates trapped in state-dependent potentials. As sketched in the plot, we assume that all BECs are in the Thomas-Fermi regime, hence they have parabolic shapes. The condensates "a0" and "b1" have lower densities than condensates "a1" and "b0" due to repulsive interactions between different species, i.e. collisions [0]-[1]. After the interaction time, the traps holding BECs should be moved to reach two bimodal condensates, "a" and "b". Only then the measurements should be taken to prove the EPR entanglement.

isolated quantum systems with practically all parameters tunable. Hence they are good candidates to test foundation of quantum mechanics and to implement the quantum information proposals. Here we will briefly sketch the main results regarding coherent evolution.

The scheme [13] is based on two Bose-Einstein condensates, "a" and "b", each consisting of two-level atoms with internal states denoted with |0⟩ and |1⟩. Each of the four components can be independently manipulated using appropriate state-dependent potentials [23]. In the central stage of the scheme, atoms in the states |0⟩ from condensate "a" overlap with atoms in the state |1⟩ from condensate "b" as shown in Fig. 4. To model mathematically the Hamiltonian of such four gaseous clouds, we will benefit from the simple form of the wavefunctions of BECs. Namely, the many body wave-function ψ_N(r_1,r_2,...,r_N) of a condensate to a good approximation is just equal to a single-body wave-function (orital) ψ^{GPE}(r) occupied by all atoms, namely ψ_N(r_1,r_2,...,r_N) = ∏_{i=1}^{N} ψ^{GPE}(r_i). In case of the four BECs, the system has to be described with four orbitals ψ^{GPE}_σ, where σ = a, b and ε = 0, 1. In the stationary situation the orbitals can be calculated with the help of the four coupled Gross-Pitaevskii equations:

\[ \mu_{εσ}ψ_{εσ}(r) = \left( \hat{T} + V_{εσ} + \sum_{ε'σ'} g_{ε'σ'}N_{ε'σ'}|ψ_{ε'σ'}|^2 \right) ψ_{εσ}(r), \]  

(26)

where \( \hat{T} = -\frac{h^2}{2m}\Delta \) is the kinetic energy operator, \( V_{εσ} = \frac{1}{2}m\omega^2(z^2 + y^2 + (z - z_{σ})^2) \) is the potential trapping the BEC "εσ" centered at \( z_{σ} \) and \( g_{εσ} \) is the coupling constant for two-body collisions of atoms in internal states |ε⟩ |ε′⟩. The energy of such four BECs is approximately

\[ E_{GPE} = \sum_{εσ} N_{εσ} \int ψ_{εσ}(r) \left( \hat{T} + V_{εσ} \right) ψ_{εσ}(r) dr + \frac{1}{2} \sum_{εσε'σ'} N_{εσ}N_{ε'σ'}|ψ_{ε'σ'}(r)|^2 |ψ_{εσ}(r)|^2. \]  

(27)

The equations (26) and (27) describe the BECs orbital and BECs energy in situation where the number of atoms in each from the four components is fixed, given by the integers \{N_{σ}\}. The initial state considered in this paper is a superposition of Fock states with \( N + M \) atoms differently distributed between the four modes. The full description of such system is quite involved [12]. Here we restrict the analysis to a simplified model, which stems from the formula for BECs energy (27) expanded in the Taylor series around the average values of the BECs occupation \( N_{a0} = N/2 \) and \( N_{b0} = M/2 \):

\[ \hat{H} \approx χ_σ (S_z^σ)^2 + χ_{ab} S_z^a S_z^b + \sum_σ ν_σ S_z^σ + χ_σ N_σ S_z^σ + \tilde{E}_{GPE} \]  

(28)

The parameters \( χ_σ, \tilde{χ}_σ, χ_{ab} \) are combinations of the second derivatives of the energy \( E_{GPE} \) with respect to the number of atoms:

\[ χ_σ = \frac{1}{2\hbar}(\chi_{0σ} + \chi_{1σ} - 2\chi_{01σ}) \]  

(29)

\[ \chi_{0σ} = \frac{∂^2 E_{GPE}}{∂N_0σ}; \quad \chi_{1σ} = \frac{∂^2 E_{GPE}}{∂N_1σ}; \quad \chi_{01σ} = \frac{∂^2 E_{GPE}}{∂N_0σ∂N_1σ}. \]

\[ χ_{ab} = \frac{∂^2 E_{GPE}}{∂N_{0a}∂N_{1b}}; \quad \tilde{χ}_σ = \frac{1}{2\hbar}(\chi_{1σ} - \chi_{0σ}) \]

In the symmetric situation, when \( N = M \) and the coupling constants \( g_{00} \) and \( g_{11} \) are equal (which is close to the real situation in Rubidium-87), the Hamiltonian reduces to the form presented until now in our paper, Eq. (2). Comparison between the evolution given by the Hamiltonian (28) and the more involved model in which the spatial and time dependence of the orbitals \( ψ^{GPE}_σ \) is accounted for is given in [12].

To estimate analytically the Hamiltonian parameters for a real systems we follows the papers [15, 16]. In the limit of large number of atoms, the kinetic terms in the Gross-Pitaevskii equations (26) can be neglected. This is the standard Thomas-Fermi approximation. Then the equations (26) reduce to a set of algebraic equations, which in the case of symmetric couplings \( g_{00} = g_{11} = g \) can be solved analytically (due to symmetry one has \( ψ_{00}(r) = ψ_{11}(r) \) and, up to a translation, \( ψ_{01}(r) = ψ_{10}(r) \)). Having analytical form of the orbitals \( ψ_{εσ} \), one can use the formula for GPE-energy (27) to derive the coefficients of the Hamiltonian (28). The results can be
in fact deduced from the paper \cite{16} (page 373):

\[
\chi = \frac{1}{5} \left( \frac{15a}{2l_{\text{osc}}} \right)^{2/5} \left( 1 + \left( \frac{a}{a + a_0} \right)^{3/5} \right) \omega N^{-3/5},
\]

\[
\chi_{ab} = \frac{2}{5} \left( \frac{15a_0}{2l_{\text{osc}}} \right)^{2/5} \left( \frac{a}{a + a_0} \right)^{3/5} \omega N^{-3/5},
\]

where \(a = \frac{m}{4\pi\hbar^2}g\) and \(a_0 = \frac{m}{4\pi\hbar^2}g_0\) are the scattering lengths, \(m\) is the mass of a single atom, \(\omega\) and \(l_{\text{osc}} = \hbar/(m\omega)\) are the trap frequency and oscillatory length, respectively.

### B. Many-body losses

Real life condensates do note evolve in accordance to Hamiltonian only: losses are present and often are a limiting factor in experiments. From the experimental standpoint, most important types of losses are:

1. one-body losses, arising from interaction with ambient molecules (imperfect vacuum) – the jump operators are linear in ladder operators \(\hat{a}_0, \hat{a}_1, \hat{b}_0, \hat{b}_1\), as described before

2. two-body losses, when the interaction between two atoms makes them change spin – the jump operators are quadratic. For \(^{87}\text{Rb}\) spin states \(|F = 1, m_F = -1\rangle = |0\rangle, |F = 2, m_F = 1\rangle = |1\rangle\) the possible operators are \(A_{11}^{(2)} = \sqrt{\gamma_{11}^{(2)}} \hat{a}_1 \hat{a}_1\), \(A_{01}^{(2)} = \sqrt{\gamma_{01}^{(2)}} \hat{a}_0 \hat{a}_1\) and analogously for "b" subsystem. The conservation of the magnetization during spin exchange collision prevent 0-0 losses, i.e. \(\gamma_{00}^{(2)} = 0\).

3. three-body losses, happening during inelastic three body collision during which one of the interacting atoms gain enough energy to leave the condensate and two others form a stable bimolecule. The jump operators corresponding to these losses are given by the third order polynomials of annihilation operators, for instance all losses in "a" are of the form

\[
A_{\epsilon_1 \epsilon_2 \epsilon_3}^{(3)} = \sqrt{\gamma_{\epsilon_1 \epsilon_2 \epsilon_3}} \hat{a}_{\epsilon_1} \hat{a}_{\epsilon_2} \hat{a}_{\epsilon_3},
\]

The two- and three-body losses are originated in the two- and three-body collisions, respectively. Hence they depend on the atomic densities. For instance the rates \(\gamma_{\epsilon_1 \ldots \epsilon_m}^{(m)}\) of \(m\)-body losses in the component "a" are given by

\[
\gamma_{\epsilon_1 \ldots \epsilon_m}^{(m)} = \frac{K_{\epsilon_1 \ldots \epsilon_m}^{(m)}}{m} \int |\phi_{\epsilon_1, a}|^2 \ldots |\phi_{\epsilon_m, a}|^2 d^3r,
\]

where \(K_{\epsilon_1 \ldots \epsilon_m}^{(m)}\) are atomic constants.

The evolution in the presence of one-body losses is given by the master equation \(\mathcal{E}\), but with \(\chi\) and \(\chi_{ab}\) being functions of the total number of atoms, see equations \(30\), \(31\). From the preceding analysis of time-scales we expect that the number of atoms lost until the \(E_{\text{EPR}}\) reaches minimum will be small. Hence we neglect the time dependence of \(\chi\) and \(\chi_{ab}\) resulting from the changes in time of the total number of atoms \(N\) and \(M\). In this model, in which \(\chi\) and \(\chi_{ab}\) are function of the total initial number of atoms, the conclusions from the previous parts do not holds.

Taking into considerations the effect of number of particles on parameters, we may plot the minimum \(E_{\text{EPR}}\) as a function of \(N\), which we take for simplicity to be equal to \(M\), see Fig. \(5\). Each point from this figure has been obtained by optimizing with respect to \(\alpha, \beta\) and time in formula \(17\). The atomic constants used in these plots are close to the parameters of Rubidium 87. In the upper panel of Fig. \(5\), corresponding to trap frequency \(\omega = 2\pi \times 200\text{Hz}\), one can see that for increasing number of atoms the effect of 1-body losses (green dashed line) is stronger and stronger. This comes from the fact that the nonlinearities \(\chi\) and \(\chi_{ab}\) tend to 0 with the number of atoms. The time to reach the minimum of \(E_{\text{EPR}}\) increases what, for fixed 1-body loss rate constant, lead to severe losses in the limit \(N \rightarrow \infty\). To counteract this effect one should increase the gas density to enhance the
nonlinearities and make the unitary evolution faster, as illustrated in the lower panel of Fig. 3, corresponding to trap frequency $\omega = 2\pi \times 1000$Hz. The side effect of making the confinement steeper are increased density-dependent losses, i.e. two and three-body losses.

We covered the existence of one-body losses in fully analytical way. However, the method of characteristic functions does not work even for two-body losses as it leads to the partial differential equations of order 2 with unknown solutions. Therefore, a procedure for heuristic incorporation of higher-order losses into one body losses was devised. The prescription is deduced from the method of quantum trajectories: we increase the rate of one-body losses such that the average number of lost atom at the interval $t, t + \Delta t$ is the same as there would be all losses present. Precisely the number of atoms lost in BEC "a" with the internal state $|0\rangle$ in the small time interval $\Delta t$ is equal to

$$N_{a0}(t+\Delta t) - N_{a0}(t) \approx \sum_{m=1}^{3} \sum_{\epsilon} n_{a0,\epsilon} \left( \langle \hat{A}_m^{(m)} \rangle \hat{A}_m^{(m)} \right) \Delta t,$$

where $\hat{A}_m^{(m)}$ are jump operators corresponding to $m$-body losses, and $n_{a0,\epsilon}$ is the number of lost atoms in the component "a0" due to the jump $\hat{A}_m^{(m)}$. We approximate the average of $m$-body operators $\left( \langle \hat{A}_m^{(m)} \rangle \hat{A}_m^{(m)} \right)$ with an average of 1-body operator $\Gamma_{a_0}^{\mathrm{eff}} \langle \hat{a}_0^\dagger \hat{a}_0 \rangle$ with the effective rate $\Gamma_{a_0}^{\mathrm{eff}}$ chosen in such a way that at least in the limit of weak losses and short evolution times, when $\langle N_{a1} \rangle \approx \langle N_{a0} \rangle \approx N/2$, the total rate of loosing particles will be similar in both models [33]. This leads to the following formula for the effective rate:

$$\Gamma_{a_0}^{\mathrm{eff}} = \gamma_c + N \left( \gamma_{11}^{(2)} + \frac{1}{2} \gamma_{01}^{(2)} \right) + \frac{3}{4} \gamma_{10}^{(3)} N^2. \quad (34)$$

As the loss rates constants $K_2$ are asymmetric [18] and, as shown schematically in Fig. 4 the densities of overlapping BECs are different from the BECs in the outer traps, then each from the four BECs would have a different rate of losses. We do not account this asymmetry. Keeping in mind that our analysis of the many-body losses is only qualitative, we will do the analysis for symmetry. Keeping in mind that our analysis of the many-body differences in both models [33]. This leads to the following formula for the effective rate:

$$\Gamma_{a_0}^{\mathrm{eff}} = \gamma_c + N \left( \gamma_{11}^{(2)} + \frac{1}{2} \gamma_{01}^{(2)} \right) + \frac{3}{4} \gamma_{10}^{(3)} N^2. \quad (34)$$

In Fig. 5 we show the results which include phenomenologically the two and three body losses according to the prescription described above. As opposed to the result without proper scaling, there exists an optimal number of particles for which the minimum $E_{EPR}^2$ is attained. It is easily explained heuristically: in the limit of large number of atoms, the density has to increase. In high densities regime however the atoms scatter very often, what leads to increased number of three-body losses. One could counteract this effect by decreasing the trap frequency, this leads to small coefficients $\chi_{ab}$ though and EPR entanglement occurs at later times, in this case the system is limited by one-body losses. The mechanism is similar to the limits on the squeezing in Bose-Einstein condensates discussed in [14].

Summarizing this part: in our heuristic analysis including 2 and 3-body losses, where we assumed stronger losses than should be in reality, still one can obtain EPR-entangled condensates consisting of thousands of atoms, as shown in Fig. 5.

VII. CONCLUSIONS

We presented the analytical solution of the master equation describing the two macroscopic spins interacting via $\hat{S}_z^{(a)} \hat{S}_z^{(b)}$ term, but undergoing the one-body particle losses. The intuitive picture is gained via the quantum trajectory method, which shows that the mechanism underlying the decoherence is the phase noise, which here is non-local. We discussed the fate of the non-local entangled states predicted in the unitary evolution. As expected, the so called "Schrödinger cat in two boxes" loose the quantum coherence once a single particle is lost. Much more robust is the entanglement captured by the EPR condition, appearing at short evolution times. We discuss possibility of producing them from the perspective of the Bose-Einstein condensates.

We hope that our work can be used in the Quantum Information community as a simple, still rich, model with the analytical solutions even in the presence of particle losses. It is known that the $\hat{S}_z^{(a)} \hat{S}_z^{(b)}$-scheme can lead to many interesting and potentially useful, entangled states [3]. The next steps in research could be finding an optimal situations for production of the EPR state in BEC taking into account all losses (we touched the problem in Sec. [14] or trying to use another entanglement witness to find another correlated states appearing in the dissipative evolution.

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Appendix A: Characteristic function

The master equation \([3]\) written in the Fock basis, namely \(\frac{d}{dt} \langle x + z, y, u + r, v | \rho | x, y + z, u, v + r \rangle\), read

\[
\frac{d}{dt} \rho_{x+y+z, u+v+r} = \left( \chi_a 4z(y - x) + \chi_b 4r(v - u) + 2\chi_{ab} (r(y - x) + z(v - u)) \right) \rho_{x+y+z, u+v+r} - \gamma_0 (2x + 2u + z + r + \gamma_1 (2y + 2v + z + r)) \rho_{x+y+z, u+v+r} + \gamma_0 \rho_{x+y+z, u+v+r} \sqrt{(x + 1)(x + z + 1) + \rho_{x+y+z, u+v+r} \sqrt{(u + 1)(u + r + 1)} + \gamma_1 \left( \rho_{x+y+z, u+v+r} \sqrt{(y + 1)(y + z + 1) + \rho_{x+y+z, u+v+r} \sqrt{(v + 1)(v + r + 1)} \right)}.
\] (A1)

Hence for fixed \(z\) and \(r\) we obtain a set of a coupled differential equation on the density matrix terms \(\rho_{x+y+z, u+v+r}\) labeled with \(x, y, u\) and \(v\). This set can be elegantly solved with help of the characteristic function:

\[
h^{z,r}(X, Y, U, V, t) = \sum_{x,y,u,v} \sqrt{\frac{(x+z)!}{x!y!u!v!}} X^x Y^y U^u V^v \rho_{x+y+z, u+v+r}.
\]

By summing the equations for individual \(\rho_{x+y+z, u+v+r}\) with weights \(\sqrt{\frac{(x+z)!}{x!y!u!v!}}\) one obtain a closed, equation on the characteristic function \([10]\): \(A2)\)

\[
\frac{\partial h^{z,r}}{\partial t} = \left( -\beta_0^{(r,z)} X + \gamma_0 \right) \frac{\partial h^{z,r}}{\partial X} + \left( \beta_1^{(r,z)} Y + \gamma_1 \right) \frac{\partial h^{z,r}}{\partial Y} + \left( -\beta_1^{(x,z)} U + \gamma_0 \right) \frac{\partial h^{z,r}}{\partial U} + \left( \beta_0^{(x,z)} V + \gamma_1 \right) \frac{\partial h^{z,r}}{\partial V} - 2 \left( \gamma_0 + \gamma_1 \right) (z + r) h^{z,r}.
\] (A2)

This first order partial differential equation can be solved with the standard methods, as the method of characteristics. For the initial state investigated in the paper the final result is

\[
h^{z,r}(X, Y, U, V, t) = \frac{N! M! e^{-\frac{1}{2} (\Gamma_0 + \Gamma_1) (r+z) t}}{2r+z(M-r)! (N-z)!} \left( L_{z,r}(X, Y, t) \right)^{N-z} \left( L_{z,r}(U, V, t) \right)^{M-r},
\] (A3)

where

\[
L_{z,r}(X, Y, t) = \frac{1}{2} \left( \frac{\Gamma_0 + (A_{z,r} X - \Gamma_0) e^{-A_{z,r} t}}{A_{z,r}} + \frac{\Gamma_1 + (B_{z,r} Y - \Gamma_1) e^{-B_{z,r} t}}{B_{z,r}} \right),
\] (A4)

\[
A_{z,r} = \gamma_0 + i \chi + i r \chi_{ab}/2
\] (A5)

\[
B_{z,r} = \gamma_1 - i \chi - i r \chi_{ab}/2
\] (A6)

Appendix B: Quantum averages

Using the characteristic function \([A3]\), all number-of-particles-preserving correlators can be calculated (the others are 0 in our system). For example, \(\langle a_1^\dagger a_0 \rangle = h^{1.0} | x=y=v=1 \). Unfortunately, the explicit equations are complicated and would not fit on one page, so we will define the averages by intermediate functions. Therefore, we define

\[
\alpha^{(z,r)} = z \chi + r \chi_{ab}/2
\] (B1)

and subsequently

\[
f_{z,r}(t) = L_{z,r} | x=y=u=v=1 = \frac{1}{2} \left( \frac{\Gamma_0 + i \alpha^{(z,r)} e^{-\Gamma_0 t - i \alpha^{(z,r)}}}{\Gamma_0 + i \alpha^{(z,r)}} + \frac{\Gamma_1 - i \alpha^{(z,r)} e^{-\Gamma_1 t + i \alpha^{(z,r)}}}{\Gamma_1 - i \alpha^{(z,r)}} \right).
\] (B2)
The quantum averages necessary to evaluate the steering condition $E_{\text{EPR}}$ read:

$$\langle S_x^a \rangle = \Re h^{1,0} \quad \langle S_y^a \rangle = \Im h^{1,0} \quad \langle S_z^a \rangle = \frac{N}{4} \left( e^{-i\Gamma_1 t} - e^{-i\Gamma_0 t} \right)$$

$$N(t) = \frac{1}{2}N \left( e^{-\Gamma_0 t} + e^{-\Gamma_1 t} \right)$$

$$\langle (S_y^a)^2 \rangle = \frac{1}{4} \left( N(t) + \frac{N(N-1)}{2} e^{-\Gamma_0 t} \Gamma_1 (1 - \Re \left\{ F_{2,0}^{-2} f_{0,3}^M \right\}) \right)$$

$$\langle (S_z^a)^2 \rangle = \frac{N(N-1)}{16} \left( e^{-\Gamma_1 t} - e^{-\Gamma_0 t} \right)^2 + \frac{1}{4} N(t)$$

$$\langle S_y^a S_y^a \rangle = \frac{MN e^{-(\Gamma_0 + \Gamma_1) t}}{8} \Re \left\{ f_{1,1}^{N+M-2} f_{1,1}^{N-1} f_{1,1}^{M-1} \right\}$$

$$\langle S_y^a S_z^a \rangle = \frac{MN e^{-(\Gamma_0 + \Gamma_1) t/2}}{8} \Im \left\{ f_{1,0}^{N-1} f_{0,1}^{M-1} \left( e^{-(\Gamma_0 + i\chi_{ab}/2)t} - e^{-(\Gamma_1 - i\chi_{ab}/2)t} \right) \right\}$$

$$\langle S_z^a S_z^a \rangle = \frac{N(N-1)}{4} e^{-\frac{i}{2}(\Gamma_0 + \Gamma_1) t} \Im \left\{ f_{1,0}^{N-2} f_{0,1}^{M} \left( e^{-(\Gamma_1 - i\chi)t} - e^{-(\Gamma_0 + i\chi)t} \right) \right\}$$,

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[32] We will omit this part — the calculation, while simple, is lengthy and generally the easiest method is to employ Mathematica to perform this laborious task.

[33] Technically, we contract operators. For example $\langle \Gamma_{001}^{(3)} \hat{a}_0\hat{a}_0^\dagger \hat{a}_1\hat{a}_0^\dagger \hat{a}_0 \rangle \Rightarrow \Gamma_{001}^{(3)} \left( \frac{N}{2} \right)^2 \langle \hat{a}_0^\dagger \hat{a}_0 \rangle$. 
