Generalized Midpoint Fractional Integral Inequalities via $h$-Convexity

Kahkashan Mahreen$^a$, Hüseyin Budak$^b$

$^a$Department of Mathematics, COMSATS University Islamabad, Attock Campus, Pakistan
$^b$Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce, Turkey

Abstract. In this research, generalizations of midpoint type inequalities are established. $h$-convexity is used as a tool. These inequalities are for differentiable functions which involve Riemann-Liouville fractional integrals. Also, some consequences of these established inequalities are obtained.

1. Introduction

Fractional calculus plays an important role in many fields like engineering, economics, physics, and many disciplines of mathematics. For more information about fraction calculus please refer to ([11], [16], [19], [20], [24]). Similarly, it is well known that the convexity of a function plays a vital role in the field of inequalities. Here, first we define a generalized convexity namely $h$-convexity.

Definition 1.1. [33] Let $I$, $J$ be intervals in $\mathbb{R}$, $(0, 1) \subseteq J$ and let $h : J \rightarrow \mathbb{R}$ be a non-negative function, $h \neq 0$. A non-negative function $f : I \rightarrow \mathbb{R}$ is called $h$-convex if for all $x, y \in I$, $\alpha \in (0, 1)$, we have

$$f(\alpha x + (1 - \alpha)y) \leq h(\alpha)f(x) + h(1 - \alpha)f(y).$$

Next, the following inequality is known as Hermite-Hadamard inequality for convex functions: If $f : I \rightarrow \mathbb{R}$ is a convex function on the interval $I$ of real numbers and $a, b \in I$ with $a < b$, then

$$f\left(\frac{a + b}{2}\right) \leq \frac{1}{b - a} \int_a^b f(x)dx \leq \frac{f(a) + f(b)}{2}. \quad (1)$$

Both inequalities hold in the reversed direction if $f$ is concave.

In [18], U. S. Kırmacı give the following identity and using this identity, obtain some bounds for the left hand side of the inequality (1)
Lemma 1.2. Let \( f : I' \to \mathbb{R} \) be differentiable function on \( I' \), \( a, b \in I' \) (\( I' \) is interior of \( I \)) with \( a < b \). If \( f' \in L[a, b] \), then we have

\[
\frac{1}{b-a} \int_a^b f(t)dt - f\left(\frac{a+b}{2}\right) = (b-a) \left[ \int_0^1 t f'(ta + (1-t)b)dt + \int_0^1 (1-t) f'(ta + (1-t)b)dt \right].
\]

(2)

One can see \([1], [3], [5], [8], [9], [23], [25], [26], [31]\) to study the new bound for left-hand side and right-hand side of the inequality (1). Here we give the well-known Riemann-Liouville fractional integral operators which will be helpful to obtain our main results.

Definition 1.3. Let \( f \in L_1[a, b] \). The Riemann-Liouville integrals \( J_a^\alpha f \) and \( J_b^\alpha f \) of order \( \alpha > 0 \) with \( a \geq 0 \) are defined by

\[
J_a^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \quad x > a
\]

and

\[
J_b^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, \quad x < b
\]

respectively. Here, \( \Gamma(\alpha) \) is the gamma function and \( J_a^0 f(x) = J_b^0 f(x) = f(x) \).

In the following, Hermite-Hadamard type inequalities for Riemann-Liouville fractional integrals are obtained in \([28] \) and \([27]\).

Theorem 1.4. Let \( f : [a, b] \to \mathbb{R} \) be a positive function with \( 0 \leq a < b \) and \( f \in L_1[a, b] \). If \( f \) is a convex function on \( [a, b] \), then the following inequalities for fractional integrals hold:

\[
f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha + 1)}{2(b-a)^\alpha} \left[ J_a^\alpha f(b) + J_b^\alpha f(a) \right] \leq \frac{f(a) + f(b)}{2}
\]

(3)

with \( \alpha > 0 \).

Theorem 1.5. Let \( f : [a, b] \to \mathbb{R} \) be a positive function with \( a < b \) and \( f \in L_1[a, b] \). If \( f \) is a convex function on \( [a, b] \), then the following inequalities for fractional integrals hold:

\[
f\left(\frac{a+b}{2}\right) \leq \frac{2^{\alpha-1}\Gamma(\alpha + 1)}{(b-a)^\alpha} \left[ J_a^\alpha f(b) + J_b^\alpha f(a) \right] \leq \frac{f(a) + f(b)}{2}
\]

(4)

We will use the following lemmas to find our results.

Lemma 1.6. \([4]\) Let \( f : [a, b] \to \mathbb{R} \) be a differentiable mapping on \( (a, b) \) with \( a < b \). If \( f' \in L[a, b] \), then for all \( x \in [a, b] \) the following equality for fractional integrals holds:

\[
\frac{\Gamma(\alpha + 1)}{b-a} \left[ (x-a)^{1-\alpha} J_a^\alpha f(a + b - x) + (b-x)^{1-\alpha} J_b^\alpha f(a + b - x) \right] - f(a + b - x)
\]

(5)

\[
= \frac{(x-a)^2}{b-a} \int_0^{1-t^\alpha} f'(tb + (1-t)(a+b-x))dt + \frac{(b-x)^2}{b-a} \int_0^{1-t^\alpha} f'(ta + (1-t)(a+b-x))dt.
\]
Lemma 1.7. [6] Let \( f : [a, b] \rightarrow \mathbb{R} \) be a differentiable mapping on \((a, b)\) with \(a < b\). If \( f' \in L^1[a, b] \), then we have the following equality for fractional integrals

\[
\frac{\Gamma(\alpha + 1)}{b - a} \left( (b - x)^{1-\alpha} f_{[a+b-x]}^\alpha f(a) + (x - a)^{1-\alpha} f_{[a+b-x]}^\alpha f(b) \right) - f(a + b - x)
= \frac{(x - a)^2}{b - a} \int_0^1 t^\alpha f' (t (a + b - x) + (1 - t) b) \, dt - \frac{(b - x)^2}{b - a} \int_0^1 t^\alpha f' (t (a + b - x) + (1 - t) a) \, dt,
\]

for all \( x \in [a, b] \).

Many authors generalized Hermite-Hadamard inequality for many fractional and conformable integral operators. One can see ([2], [7], [10], [12]-[15], [17], [21], [22], [29], [30], [32]-[36]) for more information. In the upcoming section, we established some new generalized midpoint type inequalities for Riemann-Liouville fractional integrals by the mean of \( h \)-convexity. Some important consequences are also given in the upcoming section.

2. Main Results

In this Section, by help of Lemma 1.6 and Lemma 1.7, we establish some generalized midpoint type inequalities for \( h \)-convex functions.

Theorem 2.1. \( f : [a, b] \rightarrow \mathbb{R} \) be a differentiable mapping on \((a, b)\) with \(0 < a < b\). If \(|f'|^q\) is \( h \)-convex on \([a, b]\) for some fixed \( q > 1 \), then for all \( x \in [a, b] \) the following fractional integral inequality holds:

\[
\frac{\Gamma(\alpha + 1)}{b - a} \left| (x - a)^{1-\alpha} f_{[a+b-x]}^\alpha f(a + b - x) + (b - x)^{1-\alpha} f_{[a+b-x]}^\alpha f(a + b - x) \right| - f(a + b - x) \leq \frac{1}{b - a} \left( \frac{\alpha p}{\alpha p + 1} \right)^{\frac{1}{2}} \left[ (x - a)^2 \left| f' (b) \right|^p + |f' (a + b - x)|^p \right]^{\frac{1}{2}} + (b - x)^2 \left[ |f' (a)|^p + |f' (a + b - x)|^p \right]^{\frac{1}{2}} \left( \int_0^1 h(t) \, dt \right)^{\frac{1}{2}},
\]

where \( \frac{1}{p} + \frac{1}{q} = 1 \).

Proof. By the Lemma 1.6, we have

\[
\frac{\Gamma(\alpha + 1)}{b - a} \left| (x - a)^{1-\alpha} f_{[a+b-x]}^\alpha f(a + b - x) + (b - x)^{1-\alpha} f_{[a+b-x]}^\alpha f(a + b - x) \right| - f(a + b - x) \leq \frac{(x - a)^2}{b - a} \int_0^1 [1 - t^\alpha] \left| f' (t (a + b - x) + (1 - t) a) \right| \, dt + \frac{(b - x)^2}{b - a} \int_0^1 [t^\alpha - 1] \left| f' (t a + (1 - t) (a + b - x)) \right| \, dt.
\]
Using the Hölder’s inequality and \( h \)-convexity of \( |f'|^q \), we obtain
\[
\int_0^1 [1 - t^p] |f'((tb + (1 - t)(a + b - x))] dt
\]
\[
\leq \left( \int_0^1 (1 - t^p) dt \right)^\frac{1}{p} \left( \int_0^1 |f'((tb + (1 - t)(a + b - x))]|^q dt \right)^\frac{1}{q}
\]
\[
\leq \left( \int_0^1 (1 - t^p) dt \right)^\frac{1}{2} \left( \int_0^1 |f'((tb + (a + b - x))]|^q dt \right)^\frac{1}{q} \left( \int_0^1 h(t)dt \right)^\frac{1}{2}
\]
\[
= \left( \frac{ap}{ap + 1} \right)^\frac{1}{2} \left( |f'(a)| + |f'(a + b - x)]\right) \left( \int_0^1 h(t)dt \right)^\frac{1}{2}.
\]

Here we use
\[(X - Y)^p \leq X^q - Y^q,
\]
for any \( X > Y \geq 0 \) and \( q \geq 1 \). Similarly, we have
\[
\int_0^1 [p^q - 1] |f'(ta + (1 - t)(a + b - x))] dt
\]
\[
\leq \left( \frac{ap}{ap + 1} \right)^\frac{1}{2} \left( |f'(a)| + |f'(a + b - x)]\right) \left( \int_0^1 h(t)dt \right)^\frac{1}{2}.
\]

Combining (8), (9) and (10), inequality (7) is obtained. \( \square \)

**Corollary 2.2.** Under assumption of Theorem 2.1 with \( x = \frac{a + b}{2} \), the following inequality hold:
\[
\left| \frac{2^{n-1} \Gamma(a + 1)}{(b-a)^n} \int_0^1 f_{n-1}(a + b - x) \right| \left( \frac{a + b}{2} \right) + \int_0^1 f_{n-1}(a + b - x) \right] - \frac{a + b}{2} \right)
\]
\[
\leq \frac{b - a}{4} \left( \frac{ap}{ap + 1} \right)^\frac{1}{2} \left[ \left( |f'(a)| + |f'(b)h(\frac{1}{2})|^\frac{1}{2} \right) \left( \int_0^1 h(t)dt \right)^\frac{1}{2} \right] + \left( |f'(b)| + |f'(a)h(\frac{1}{2})|^\frac{1}{2} \right) \left( \int_0^1 h(t)dt \right)^\frac{1}{2}.
\]

**Corollary 2.3.** By taking \( h(t) = t^q \) in (7), the following inequality holds for \( s \)-convexity:
\[
\Gamma(a + 1) \left( \frac{b-a}{b-a} \right) \left[ (x - a)^{1-s} f_{n-1}(a + b - x) + (b - x)^{1-s} f_{n-1}(a + b - x) \right] - \frac{a + b}{2} \right)
\]
\[
\leq \frac{1}{b-a} \left( \frac{ap}{ap + 1} \right)^\frac{1}{2} \left[ (x - a)^{2-s} \left( \frac{|f'(b)| + |f'(a + b - x)]|^{s+1}}{s+1} \right)^\frac{1}{2} \right] + (b - x)^{2-s} \left( \frac{|f'(a)| + |f'(a + b - x)]|^{s+1}}{s+1} \right)^\frac{1}{2}.
\]
Remark 2.4. (i) If we take \( h(t) = t \) in Theorem 2.1, then Theorem 2.1 reduces to [4, Theorem 3].
(ii) If we take \( h(t) = t \) in Corollary 2.3, then Corollary 2.3 reduces to [4, Corollary 1].

Theorem 2.5. \( f : [a, b] \to \mathbb{R} \) be a differentiable mapping on \((a, b)\) with \( 0 \leq a < b \). If \( |f'|^q \) is h-convex on \([a, b]\) for some fixed \( q \geq 1 \), then for all \( x \in [a, b] \) the following fractional integral inequality holds:

\[
\begin{align*}
\left| \frac{\Gamma(a+1)}{b-a} \left[ (x-a)^{1-a} f_a^{\alpha} + (b-x)^{1-a} f_b^{\alpha} \right] - f(a+b-x) \right| \\
\leq \frac{1}{b-a} \left( \frac{\alpha}{\alpha + 1} \right)^{1-\frac{1}{q}} \left[ (x-a)^{2} \left( \frac{I_1}{b-a} \right) \left( f''(b)^{\alpha} + f''(a+b-x)^{\alpha} \right)^{\frac{1}{1-\frac{1}{q}}} \right] \\
+ \frac{(b-x)^2}{b-a} \left( I_1 f''(a)^{\alpha} + I_2 f''(a+b-x)^{\alpha} \right) \left[ f''(1-t^a)^{\alpha} \right] ,
\end{align*}
\]

where \( I_1 = \int_0^1 (1-t^a) h(t) dt \) and \( I_2 = \int_0^1 (1-t^a) h(1-t) dt \).

Proof. By the Lemma 1.6 and the power mean inequality, we have

\[
\begin{align*}
\left| \frac{\Gamma(a+1)}{b-a} \left[ (x-a)^{1-a} f_a^{\alpha} + (b-x)^{1-a} f_b^{\alpha} \right] - f(a+b-x) \right| \\
\leq \frac{(x-a)^2}{b-a} \int_0^1 \left| f''(tb + (1-t)(a+b-x)) \right| dt \\
+ \frac{(b-x)^2}{b-a} \int_0^1 \left| f''(ta + (1-t)(a+b-x)) \right| dt \\
\leq \frac{(x-a)^2}{b-a} \left( \int_0^1 \left| 1-t^a \right| dt \right)^{1-\frac{1}{q}} \left( \int_0^1 \left| f''(tb + (1-t)(a+b-x)) \right|^{\alpha} dt \right)^{\frac{1}{1-\frac{1}{q}}} \\
+ \frac{(b-x)^2}{b-a} \left( \int_0^1 \left| t^a - 1 \right| dt \right)^{1-\frac{1}{q}} \left( \int_0^1 \left| f''(ta + (1-t)(a+b-x)) \right|^{\alpha} dt \right)^{\frac{1}{1-\frac{1}{q}}} .
\end{align*}
\]

Using the h-convexity of \( |f'|^q \), we obtain

\[
\begin{align*}
\int_0^1 \left| f''(tb + (1-t)(a+b-x)) \right|^{\alpha} dt \\
\leq \int_0^1 (1-t^a) \left[ h(t) \left| f'(b)^{\alpha} + h(1-t) \left| f'(a+b-x)^{\alpha} \right| \right] dt \\
= \left| f'(b)^{\alpha} \right| \int_0^1 (1-t^a) h(t) dt + \left| f'(a+b-x)^{\alpha} \right| \int_0^1 (1-t^a) h(1-t) dt ,
\end{align*}
\]
and similarly, we have
\[
\int_0^1 |t^a - 1| |f'(t(a + (1 - t)(a + b - x))| dt \\
\leq \int_0^1 (1 - t^a) \left[ h(t) |f'(a)| + h(1 - t) |f'(a + b - x)| \right] dt \\
= |f'(a)| \int_0^1 (1 - t^a) h(t) dt + |f'(a + b - x)| \int_0^1 (1 - t^a) h(1 - t) dt,
\]
which completes the proof. \(\square\)

**Corollary 2.6.** Under assumption of Theorem 2.5 with \(x = \frac{a+b}{2}\), the following inequality holds:
\[
\left| 2^{a-1} \Gamma(a + 1) \left[ f_a^b, f(a + b) + f_a^b, f(a + b - x) \right] - f(a + b - x) \right| \\
\leq \frac{b-a}{4} \left( \frac{a}{a+1} \right)^{1-\frac{1}{2}} \\
\times \left[ \left( I_1 + h \left( \frac{1}{2} \right) I_2 \right) |f'(b)| + h \left( \frac{1}{2} \right) I_2 |f'(a)| \right]^{\frac{1}{2}} \\
+ \left( I_1 + h \left( \frac{1}{2} \right) I_2 \right) |f'(a)| + h \left( \frac{1}{2} \right) I_2 |f'(b)| \right]^{\frac{1}{2}}.
\]

**Corollary 2.7.** By taking \(h(t) = t^s\) in (12), the following inequality holds for \(s\)-convexity:
\[
\left| \frac{1}{b-a} \left[ (x - a)^{1-a} f_a^b, f(a + b - x) + (b - x)^{1-a} f_a^b, f(a + b - x) \right] - f(a + b - x) \right| \\
\leq \frac{b-a}{4} \left( \frac{a}{a+1} \right)^{1-\frac{1}{2}} \left[ \left( \frac{x-a}{b-a} \right)^2 \left( f'(a + b - x) \left[ \frac{1}{s+1} - B(a+1,s+1) \right] + \frac{a|f'(b)|}{(s+1)(s+\alpha+1)} \right) \right]^{\frac{1}{2}} \\
+ \left( \frac{b-x}{b-a} \right)^2 \left( f'(a + b - x) \left[ \frac{1}{s+1} - B(a+1,s+1) \right] + \frac{a|f'(a)|}{(s+1)(s+\alpha+1)} \right) \right],
\]
where \(B(x, y)\) is Euler’s Beta function.

**Remark 2.8.** (i) If we take \(h(t) = t\) in Theorem 2.5, then Theorem 2.5 reduces to [4, Theorem 4].
(ii) If we take \(h(t) = t\) in Corollary 2.6, then Corollary 2.6 reduces to [4, Corollary 2].

**Theorem 2.9.** Let \(f : [a, b] \to \mathbb{R}\) be a differentiable mapping on \((a, b)\) with \(0 \leq a < b\) and \(f' \in L^1[a, b]\). If \(f'\) is \(h\)-convex on \([a, b]\), then for all \(x \in [a, b]\) the following fractional integrals inequality holds:
\[
\left| \frac{1}{b-a} \left[ (b-x)^{1-a} f_a^b, f(a) + (x-a)^{1-a} f_a^b, f(b) \right] - f(a + b - x) \right| \\
= \left( \frac{x-a}{b-a} \right)^2 \left[ I_3 |f'(a + b - x)| + I_4 |f'(b)| \right] \left( \frac{b-x}{b-a} \right)^2 \left[ I_3 |f'(a + b - x)| + I_4 |f'(a)| \right],
\]
where \(I_3 = \int_0^1 t^a h(t) dt\) and \(I_4 = \int_0^1 t^a h(1-t) dt\).
Proof. Taking the modulus in Lemma 1.7, we have
\[
\left| \frac{\Gamma(x+1)}{b-a} \left( (b-x)^{x-a} f(a) + (x-a)^{x-a} f(b) \right) - f(a+b-x) \right|
\]
\[
\leq \frac{(x-a)^2}{b-a} \int_0^1 t^\alpha \left| f'(t (a+b-x) + (1-t) b) \right| dt + \frac{(b-x)^2}{b-a} \int_0^1 t^\alpha \left| f'(t (a+b-x) + (1-t) a) \right| dt.
\]
Using h-convexity of \( f' \), we get
\[
\left| \frac{\Gamma(x+1)}{b-a} \left( (b-x)^{x-a} f(a) + (x-a)^{x-a} f(b) \right) - f(a+b-x) \right|
\]
\[
\leq \frac{(x-a)^2}{b-a} \int_0^1 \left[ t^\alpha h(t) \left| f'(a+b-x) \right| + t^\alpha h(1-t) \left| f'(b) \right| \right] dt
\]
\[
+ \frac{(b-x)^2}{b-a} \int_0^1 \left[ t^\alpha h(t) \left| f'(a+b-x) \right| + t^\alpha h(1-t) \left| f'(a) \right| \right] dt
\]
\[
= \frac{(x-a)^2}{b-a} \left[ f'(a+b-x) \right] \int_0^1 t^\alpha h(t) dt + \left| f'(b) \right| \int_0^1 t^\alpha h(1-t) dt
\]
\[
+ \frac{(b-x)^2}{b-a} \left[ f'(a+b-x) \right] \int_0^1 t^\alpha h(t) dt + \left| f'(a) \right| \int_0^1 t^\alpha h(1-t) dt,
\]
which completes the proof.

Corollary 2.10. Under assumptions of Theorem 2.9 with \( x = \frac{a+b}{2} \), the following inequality holds:
\[
\left| \frac{\Gamma(x+1)}{(b-a)^x} \left[ \left| \frac{f(a)}{(a+b)} \right| + \left| \frac{f(b)}{(a+b)} \right| \right] - \frac{f(a+b)}{2} \right|
\]
\[
\leq \frac{(b-a)}{4} \left[ 2I_4 \left| f'(a+b) \right| + I_4 \left| f'(b) \right| + I_4 \left| f'(a) \right| \right].
\]

Corollary 2.11. By taking \( h(t) = t^s \) in (15), the following inequality holds for s-convexity:
\[
\left| \frac{\Gamma(x+1)}{b-a} \left( (b-x)^{x-a} f(a) + (x-a)^{x-a} f(b) \right) - f(a+b-x) \right|
\]
\[
= \frac{(x-a)^2}{b-a} \left( \frac{\left| f'(a+b-x) \right|}{a+s+1} + \left| f'(b) \right| B(a+1, s+1) \right)
\]
\[
+ \frac{(b-x)^2}{b-a} \left( \frac{\left| f'(a+b-x) \right|}{a+s+1} + \left| f'(a) \right| B(a+1, s+1) \right),
\]
where \( B(x, y) \) is Euler’s Beta function.

Remark 2.12. (i) If we take \( h(t) = t \) in Theorem 2.9, then Theorem 2.9 reduces to [6, Theorem 2.2].
(ii) If we take \( h(t) = t \) in Corollary 2.10, then Corollary 2.10 reduces to [27, Theorem 5].
(iii) If we take \( h(t) = t \) and \( s = 1 \) in Theorem 2.9, then Theorem 2.9 reduces to [6, Corollary 2.4].
Theorem 2.13. Let \( f : [a, b] \to \mathbb{R} \) be a differentiable mapping on \((a, b)\) with \(0 \leq a < b\) and \(f' \in L^1 [a, b]\). If \(|f'|^q\), \(q > 1\), is \(h\)-convex on \([a, b]\), then for all \(x \in [a, b]\) the following fractional integral inequality holds:

\[
\left| \frac{\Gamma(a+1)}{b-a} \left[ (b-x)^{1-a} \int_{a+b-x}^a f(a) + (x-a)^{1-a} \int_{a+x}^b f(b) \right] - f(a+b-x) \right| 
\]

\[
\leq \frac{1}{b-a} \left( \frac{1}{ap+1} \right) \left[ (x-a)^2 \left[ |f'(a+b-x)|^q + |f'(b)|^q \right] \right]^{\frac{1}{q}} + \frac{(b-x)^2 \left[ |f'(a+b-x)|^q + |f'(a)|^q \right]^{\frac{1}{q}}}{\left( \frac{1}{a+b-x} \right)} \left( \int_0^1 |h(t)| dt \right)^{\frac{1}{q}},
\]

where \(\frac{1}{q} + \frac{1}{q'} = 1\).

Proof. By the Lemma 1.7, we have

\[
\left| \frac{\Gamma(a+1)}{b-a} \left[ (b-x)^{1-a} \int_{a+b-x}^a f(a) + (x-a)^{1-a} \int_{a+x}^b f(b) \right] - f(a+b-x) \right| 
\]

\[
\leq \frac{(x-a)^2}{b-a} \int_0^1 t^q |f'(a+b-x) + (1-t)b| dt + \frac{(b-x)^2}{b-a} \int_0^1 t^q |f'(a+b-x) + (1-t)a| dt.
\]

Using the Hölder’s inequality and \(h\)-convexity of \(|f'|^q\), we obtain

\[
\int_0^1 t^q |f'(a+b-x) + (1-t)b| dt \leq \left( \int_0^1 |t^p| dt \right)^{\frac{1}{q}} \left( \int_0^1 |f'(a+b-x) + (1-t)b|^{q'} dt \right)^{\frac{1}{q'}}
\]

\[
\leq \left( \int_0^1 |t^p| dt \right)^{\frac{1}{q}} \left( \int_0^1 |h(t)| t^{q'} (a+b-x)^q + (1-t) |f'(b)|^{q'} dt \right)^{\frac{1}{q'}}
\]

\[
= \left( \frac{1}{ap+1} \right)^{\frac{1}{q}} \left( |f'(a+b-x)|^q + |f'(b)|^q \right)^{\frac{1}{q'}} \left( \int_0^1 h(t) dt \right)^{\frac{1}{q'}}.
\]

Similarly, we have

\[
\int_0^1 t^q |f'(a+b-x) + (1-t)a| dt \leq \left( \frac{1}{ap+1} \right)^{\frac{1}{q}} \left( |f'(a+b-x)|^q + |f'(a)|^q \right)^{\frac{1}{q'}} \left( \int_0^1 h(t) dt \right)^{\frac{1}{q'}}.
\]

Substituting the inequalities (19) and (20) in (18), the required result is obtained. \(\Box\)
Corollary 2.14. Under assumption of Theorem 2.13 with \( x = \frac{a+b}{2} \), the following inequality holds:

\[
\left| \frac{2^{-1} \Gamma (a+1)}{(b-a)^{\alpha}} \left[ f_{a^+} (a) + f_{b^-} (b) - f (a+b) \right] \right| + \left| \frac{2^{-1} \Gamma (a+1)}{(b-a)^{\alpha}} \left[ f_{a^+} (a) + f_{b^-} (b) - f (a+b) \right] \right|
\]

\[
\leq b-a \left( 1 + h \left( \frac{1}{2} \right) \right) \left( 1 + h \left( \frac{1}{2} \right) \right)^{\frac{1}{2}} + \left( |f'(a)|^p + |f'(b)|^p \right)^{\frac{1}{2}} \left( \int_0^1 h(t) dt \right)^{\frac{1}{2}}.
\]

Corollary 2.15. By taking \( h(t) = t^s \) in (17), the following result holds for \( s \)-convexity:

\[
\left| \frac{\Gamma (a+1)}{b-a} \left( b-a \right)^{s-1} \left[ x-a \right]^{1-\alpha} f_{a^+} (a) + \left( x-a \right)^{1-\alpha} f_{b^-} (b) - f (a+b-x) \right|
\]

\[
\leq b-a \left( \frac{1}{\alpha + 1} \right)^{1-\frac{s}{2}} \left( 1 + h \left( \frac{1}{2} \right) \right) \left( 1 + h \left( \frac{1}{2} \right) \right)^{\frac{1}{2}} + \left( |f'(a)|^p + |f'(b)|^p \right)^{\frac{1}{2}} \left( \int_0^1 h(t) dt \right)^{\frac{1}{2}}.
\]

Remark 2.16. (i) If we take \( h(t) = t \) in Theorem 2.13, then Theorem 2.13 reduces to [6, Theorem 2.5].

(ii) If we take \( h(t) = 1 \) in Corollary 2.14, then Corollary 2.14 reduces to [6, Corollary 2.6].

(iii) If we take \( h(t) = t \) and \( \alpha = 1 \) in Theorem 2.13, then Theorem 2.13 reduces to [25, Theorem 4].

Theorem 2.17. Let \( f : [a, b] \to \mathbb{R} \) be differentiable mapping on \( (a, b) \) with \( 0 \leq a < b \) and \( f' \in L^1 [a, b] \). If \( |f'(a)|^q + |f'(b)|^q \geq 1 \), is \( h \)-convex on \( [a, b] \), then for all \( x \in [a, b] \) the following fractional integral inequality holds:

\[
\left| \frac{\Gamma (a+1)}{b-a} \left( b-a \right)^{s-1} \left[ x-a \right]^{1-\alpha} f_{a^+} (a) + \left( x-a \right)^{1-\alpha} f_{b^-} (b) - f (a+b-x) \right|
\]

\[
\leq b-a \left( \frac{1}{\alpha + 1} \right)^{1-\frac{s}{2}} \left( 1 + h \left( \frac{1}{2} \right) \right) \left( 1 + h \left( \frac{1}{2} \right) \right)^{\frac{1}{2}} + \left( |f'(a)|^p + |f'(b)|^p \right)^{\frac{1}{2}} \left( \int_0^1 h(t) dt \right)^{\frac{1}{2}}.
\]

Proof. By the Lemma 1.7 and the power mean inequality, we have

\[
\left| \frac{\Gamma (a+1)}{b-a} \left( b-a \right)^{s-1} \left[ x-a \right]^{1-\alpha} f_{a^+} (a) + \left( x-a \right)^{1-\alpha} f_{b^-} (b) - f (a+b-x) \right|
\]

\[
\leq \frac{(x-a)^2}{b-a} \int_0^1 \left( t^s \right)^{\frac{1}{2}} \left( t^s \right)^{\frac{1}{2}} dt + \left( b-x \right)^2 \left( t^s \right)^{\frac{1}{2}} \left( t^s \right)^{\frac{1}{2}} dt
\]

\[
\leq b-a \left( \frac{1}{\alpha + 1} \right)^{1-\frac{s}{2}} \left( 1 + h \left( \frac{1}{2} \right) \right) \left( 1 + h \left( \frac{1}{2} \right) \right)^{\frac{1}{2}} + \left( |f'(a)|^p + |f'(b)|^p \right)^{\frac{1}{2}} \left( \int_0^1 h(t) dt \right)^{\frac{1}{2}}.
\]
Using the $h$-convexity of $|f''|$ , we obtain
\[
\int_0^1 t^\alpha \left| f' \left( t (a+b-x) + (1-t) b \right) \right|^\eta dt \\
\leq \int_0^1 t^\alpha \left[ h(t) \left| f' \left( a+b-x \right) \right|^\eta + h(1-t) \left| f'' \left( b \right) \right|^\eta \right] dt \\
= \left| f' \left( a+b-x \right) \right|^\eta \int_0^1 t^\alpha h(t) dt + \left| f'' \left( a \right) \right|^\eta \int_0^1 t^\alpha h(1-t) dt.
\]
Similarly, we have
\[
\int_0^1 t^\alpha \left| f' \left( t (a+b-x) + (1-t) a \right) \right|^\eta dt \\
\leq \left| f' \left( a+b-x \right) \right|^\eta \int_0^1 t^\alpha h(t) dt + \left| f'' \left( a \right) \right|^\eta \int_0^1 t^\alpha h(1-t) dt.
\]
This completes the proof. □

**Corollary 2.18.** Under assumption of Theorem 2.17 with $x = \frac{a+b}{2}$, the following inequality holds:
\[
\left| \frac{2^{\frac{n-1}{2}} \Gamma (\alpha + 1)}{(b-a)^{\eta}} \left[ \int_0^1 f'' \left( \frac{t}{n} \right) f'' \left( \frac{t}{n} \right) dt + \int_0^1 f'' \left( \frac{t}{n} \right) f'' \left( \frac{t}{n} \right) dt \right] \right| \\
\leq \frac{b-a}{4} \left( \frac{1}{\alpha + 1} \right)^{1-\frac{1}{2}} I_3 \left| f' \left( \frac{a+b}{2} \right) \right|^\eta + I_4 \left| f'' \left( b \right) \right|^\eta \\
+ \frac{b-a}{4} \left( \frac{1}{\alpha + 1} \right)^{1-\frac{1}{2}} I_3 \left| f' \left( \frac{a+b}{2} \right) \right|^\eta + I_4 \left| f'' \left( a \right) \right|^\eta \\
\leq \frac{b-a}{4} \left( \frac{1}{\alpha + 1} \right)^{1-\frac{1}{2}} \left[ h \left( \frac{1}{2} I_5 + I_3 \right) \right] \left| f' \left( a \right) \right|^\eta + \left( h \left( \frac{1}{2} I_5 + I_4 \right) \right) \left| f'' \left( b \right) \right|^\eta \\
+ \frac{b-a}{4} \left( \frac{1}{\alpha + 1} \right)^{1-\frac{1}{2}} \left[ h \left( \frac{1}{2} I_5 + I_3 \right) \right] \left| f' \left( a \right) \right|^\eta + \left( h \left( \frac{1}{2} I_5 + I_4 \right) \right) \left| f'' \left( b \right) \right|^\eta.
\]

**Corollary 2.19.** By putting $h(t) = t^\eta$ (22), the following inequality holds for $s$-convexity:
\[
\left| \frac{2^{\frac{n-1}{2}} \Gamma (\alpha + 1)}{(b-a)^{\eta}} \left[ \int_0^1 f'' \left( \frac{t}{n} \right) f'' \left( \frac{t}{n} \right) dt + \int_0^1 f'' \left( \frac{t}{n} \right) f'' \left( \frac{t}{n} \right) dt \right] \right| \\
\leq \frac{1}{\left( \alpha + 1 \right)^{1-\frac{1}{2}}} \left( \frac{x-a}{2} \right)^2 \left| f' \left( \frac{a+b-x}{\alpha + s + 1} \right) \right|^\eta + B \left( \alpha + 1, s + 1 \right) \left| f'' \left( b \right) \right|^\eta \\
+ \frac{b-a}{\left( \alpha + 1 \right)^{1-\frac{1}{2}}} \left( \frac{x-a}{2} \right)^2 \left| f' \left( \frac{a+b-x}{\alpha + s + 1} \right) \right|^\eta + B \left( \alpha + 1, s + 1 \right) \left| f'' \left( b \right) \right|^\eta,
\]
where $B(x, y)$ is Euler’s Beta function.
Remark 2.20. (i) If we take \( h(t) = t \) in Theorem 2.17, then Theorem 2.17 reduces to [6, Theorem 2.7].
(ii) If we take \( h(t) = t \) in Corollary 2.18, then Corollary 2.18 reduces to [6, Corollary 2.8].
(iii) If we take \( h(t) = t \) and \( \alpha = 1 \) Theorem 2.17, then Theorem 2.17 reduces to [25, Theorem 5].

3. Conclusions

The generalized midpoint inequalities and some related results have been obtained for \( h \)-convex functions. The obtained inequalities have direct consequences in midpoint type inequalities for Riemann-Liouville fractional integral operators via convex and \( s \)-convex functions.

References

[1] M. Alomari, M. Darus, U. S. Kirmaci, Refinements of Hadamard-type inequalities for quasi-convex functions with applications to trapezoidal formula and to special means, Comput. Math. Appl., 59 (2010), 225–232.
[2] G. A. Anastassiou, General fractional Hermite-Hadamard inequalities using \( \alpha \)-convexity and \((s, m)\)-convexity, Frontiers in Time Scales and Inequalities, (2016), 237–255.
[3] A. G. Azpeitia, Convex functions and the Hadamard inequality, Rev. Colombiana Math., 28 (1994), 7–12.
[4] H. Budak, P. Agarwal, New generalized midpoint type inequalities for functional integral, Miskolc Math. Notes, 20 (2019), 81–93.
[5] J. de la Cal, J. Carcamo, L. Escauriaza, A general multidimensional Hermite-Hadamard type inequality, J. Math. Anal. Appl., 356 (2009), 659–663.
[6] H. Budak, R. Kapucu, New generalization of midpoint type inequalities for fractional integral, An. Stiint. Univ. Al. I. Cuza Iasi. Mat. (N.S.) Tomul LXVII, 2021, f. 1 https://doi.org/10.47743/anstim.2021.00009
[7] H. Chen, U. N. Katugampola, Hermite-Hadamard and Hermite-Hadamard-Fejér type inequalities for generalized fractional integrals, J. Math. Anal. Appl. 446 (2017), 1274–1291.
[8] S. S. Dragomir and C. E. M. Pearce, Selected Topics on Hermite-Hadamard Inequalities and Applications, RGMIA Monographs, Victoria University, (2000). Online:[http://rgmia.org/papers monographs/Master2.pdf].
[9] S. S. Dragomir, R. P. Agarwal, Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula, Appl. Math. lett. 11(5) (1998), 91–95.
[10] G. Farid, A. U. Rehman, M. Zahra, On Hadamard type inequalities for \( k \)-fractional integrals, Konuralp J. Math, 4(2) (2016), 79–86.
[11] R. Gorenflo, F. Mainardi, Fractional calculus: integral and differential equations of fractional order, Springer Verlag, Wien (1997), 223–276.
[12] M. Iqbal, S. Qaisar, M. Muddassar, A short note on integral inequality of type Hermite-Hadamard through convexity, J. Comput. Anal. App, 21(5) (2016), 946–953.
[13] I. İşcan and S. Wu, Hermite-Hadamard type inequalities for harmonically convex functions via fractional integrals, Appl. Math. Comput., 238 (2014), 237–244.
[14] I. İşcan, Generalization of different type integral inequalities for \( s \)-convex functions via fractional integrals, Applicable Analysis, 93(9), 1846-1862.
[15] M. Jilei, B. Samet, On Hermite-Hadamard type inequalities via fractional integrals of a function with respect to another function, J. Nonlinear Sci. Appl., 9 (2016), 1252–1260.
[16] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, Theory and Applications of Fractional Differential Equations, North-Holland Mathematics Studies, 204, Elsevier Sci. B.V., Amsterdam, (2006).
[17] M. Kirano, B. T. Torebek, Hermite-Hadamard, Hermite-Hadamard-Fejer, Dragomir-Agarwal and Pachpatte type inequalities for convex functions via fractional integrals, arXiv:1701.00092.
[18] U. S. Kirmaci, Inequalities for differentiable mappings and applications to special means of real numbers to midpoint formula, Appl. Math. Comput., 147(5) (2004), 137–146, doi: 10.1016/S0096-3003(02)00657-4.
[19] J. A. T. Machado, Discrete-time fractional-order controllers, Fract. Calc. Appl. Anal., 4(2001), 47–66.
[20] S. Miller, B. Ross, An introduction to the Fractional Calculus and Fractional Differential Equations, John Wiley Sons, USA, (1993), p.2.
[21] M. A. Noor, M. U. Awan, Some integral inequalities for two kinds of convexes via fractional integrals, TJMM, 5(2), (2013), 129–136.
[22] M. E. Özdemir, M. Avci-Ardıç, H. Kavurmacı-Onalan, Hermite-Hadamard type inequalities for \( s \)-convex and \( s \)-concave functions via fractional integrals,Turkish J. Science, 1(1) (2016), 28–40.
[23] M. E. Onder, M. Avci, H. Kavurmacı,Hermite-Hadamard-type inequalities via \((s, m)\)-convexity, Comput. Math. Appl. 61 (2011), 2614–2620.
[24] I. Podlubni, Fractional Differential Equations, Academic Press, San Diego, 1999.
[25] S. Qaisar, S. Hussain, On Hermite-Hadamard type inequalities for functions whose first derivative absolute values are convex and concave, Fasciculi Mathematici, 58 (2017), 155–166.
[26] A. Saglam, M. Z. Sarıkaya, H. Yıldırım, Some new inequalities of Hermite-Hadamard’s type, Kyungpook Math. J., 50(2010), 399–410.
[27] M. Z. Sarıkaya, H. Yıldırım, On Hermite-Hadamard type inequalities for Riemann-Liouville fractional integrals, Miskolc Mathematical Notes, 7(2) (2016), 1049–1059.
[28] M. Z. Sarikaya, E. Set, H. Yaldiz, N. Basak, Hermite-Hadamard’s inequalities for fractional integrals and related fractional inequalities, Math. Comput. Model., DOI:10.1016/j.mcm.2011.12.048, 57 (2013), 2403–2407.

[29] M. Z. Sarikaya, H. Budak, Generalized Hermite-Hadamard type integral inequalities for fractional integrals, Filomat 30(5) (2016), 1315–1326.

[30] M. Z. Sarikaya, A. Akkurt, H. Budak, M. E. Yildirim, H Yildirim, Hermite-hadamard’s inequalities for conformable fractional integrals, An International Journal of Optimization and Control: Theories and Applications (IJOCTA), 9(1), 49-59.

[31] E. Set, M. E. Ozdemir, M. Z. Sarikaya, New inequalities of Ostrowski’s type for s-convex functions in the second sense with applications, Facta Universitatis, Ser. Math. Inform. 1(27) (2012), 67–82.

[32] E. Set, M. Z. Sarikaya, M. E. Ozdemir, H. Yildirim, The Hermite-Hadamard’s inequality for some convex functions via fractional integrals and related results, Journal of Applied Mathematics, Statistics and Informatics (JAMSI), 10(2), 2014.

[33] S. Varošanec, On h-convexity, J. Math. Anal. Appl., 326 (2007), 303–311.

[34] J. Wang, X. Li, M. Fečkan, Y. Zhou, Hermite-Hadamard-type inequalities for Riemann-Liouville fractional integrals via two kinds of convexity, Appl. Anal. 92(11) (2012), 2241–2253.

[35] J. Wang, X. Li, C. Zhu, Refinements of Hermite-Hadamard type inequalities involving fractional integrals Bull. Belg. Math. Soc. Simon Stevin, 20 (2013), 655–666.

[36] Y. Zhang, J. Wang, On some new Hermite-Hadamard inequalities involving Riemann-Liouville fractional integrals, J. Inequal. Appl. 2013, 220(2013).