Thermal instability in a gravity-like scalar theory

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We study the question of stability of the ground state of a scalar theory which is a generalization of the $\phi^3$ theory and has some similarity to gravity with a cosmological constant. We show that the ground state of the theory at zero temperature becomes unstable above a certain critical temperature, which is evaluated in closed form at high temperature.

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I. INTRODUCTION

The purely attractive nature of gravity is a source of instabilities in a gravitational system. Nevertheless, in the presence of a positive cosmological constant, such a system may remain in a static and homogeneous state at zero temperature. It is interesting to inquire whether this behavior may also hold at non-zero temperature. Instabilities in quantum gravity at finite temperature have been studied from various points of view \cite{1, 2, 3, 4, 5, 6, 7, 8}. These works indicate that, due to the inherent complexity of Einstein’s theory, a complete analysis of the stability problem at finite temperature becomes quite involved. For this reason, it may be useful to study the stability issue in a simpler model, which would mimic the behavior of gravity and yet would allow for an all-order perturbative analysis of this problem. In this spirit, such a study has already been undertaken by several authors \cite{9, 10, 11, 12} within the context of the $\phi^3$ theory in six dimensions.

With a similar motivation, we consider in this note a non-polynomial generalization of the $\phi^3$ theory with derivative interactions, which has several interesting analogies to gravity with a cosmological constant. We analyze the effective potential of this model, both at zero temperature as well as at finite temperature. At zero temperature we find that this potential has a unique minimum which indicates that the ground state of the system is stable. We next carry out an analysis of the behavior of the effective potential at finite temperature, and determine the value of the critical temperature above which the system becomes unstable. The calculation is done in the tadpole approximation, which provides the leading order contributions at high temperature to all orders. In section II we study the effective potential at zero temperature and in section III we extend this analysis to the case of non-zero temperature. We conclude this note with a brief discussion in section IV.

II. THE MODEL AND THE EFFECTIVE POTENTIAL AT ZERO TEMPERATURE

Let us consider a scalar model, which is reminiscent of (scalar) gravity with a positive cosmological constant $\Lambda$, described by the Lagrangian density

$$\mathcal{L} = \sqrt{1 + \kappa \phi} \left( \frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi + \Lambda \phi \right) + J \phi, \quad (1)$$

where we can think of $\phi$ as playing the role of the metric tensor and $J$ denotes an external source. The presence of non-polynomial and derivative interaction terms in this theory captures some of the inherent characteristics of a theory of gravity. We may also think of this model as a generalization of the $\phi^3$ theory as follows. Let us expand $L$ in a power series of $\kappa \phi$ and express $\Lambda$ and $J$ in terms of a set of new parameters $g$, $m$ and source $j$ (we assume $g > 0$) as

$$\kappa = \frac{2}{3} \frac{g}{m^2}, \quad \Lambda = \frac{9m^6}{g^2}, \quad J = \frac{3m^4}{g} + j. \quad (2)$$

Then, we can write $\mathcal{L}$ in the form

$$\mathcal{L} = \sqrt{1 - \frac{2g}{3m^2} \phi} \left( \frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi + \frac{9m^6}{g^2} \right) + \left( \frac{3m^4}{g} + j \right) \phi$$

$$= \left( 1 - \frac{1}{3} \frac{g}{m^2} \phi - \frac{1}{18} \frac{g^2}{m^4} \phi^2 + \cdots \right) \frac{1}{2} \left( \partial_{\mu} \phi \right) \left( \partial^{\mu} \phi \right)$$

$$+ \frac{9m^6}{g^2} - \frac{m^2}{2} \phi^2 - \frac{g}{3!} \phi^3 - \frac{5g^2}{(3m^2)^4} \phi^4 + \cdots + j \phi. \quad (3)$$

This shows that the theory can be interpreted as a generalization of the massive $\phi^3$ theory with non-polynomial as well as derivative coupling interactions (much like in a theory of gravity) in the presence of the source $j$ for the scalar field. We also note here that the shift in the source in (2) has been defined to eliminate, when $j$ vanishes, the linear terms in $\phi$ in the Lagrangian density.

From (4), we see that the theory is defined only for $\phi < 3m^2/2g$ and in this case, the Hamiltonian density of the theory (with $j = 0$) is readily obtained to be

$$\mathcal{H} = \frac{1}{2} \sqrt{1 - \frac{2g}{3m^2} \phi} \left( \dot{\phi}^2 + (\partial_{\mu} \phi)^2 - \frac{18m^6}{g^2} \right) - \frac{3m^4}{g} \phi. \quad (4)$$
III. STABILITY OF THE GROUND STATE AT HIGH TEMPERATURE

Let us next study the effective potential for the theory in [3] in the presence of a heat bath at temperature $T$. This effective potential can be calculated using either the real time formalism or the imaginary time formalism [18, 19, 20] and will contain thermal contributions of various kinds in addition to the zero temperature contributions in [3]. However, using a similar analysis to that presented in reference [12], it may be shown that at high temperature $T \gg m$, the leading thermal contribution to the effective potential arises only from the tadpole (one-loop) diagram. In this regime, the thermal contribution of the tadpole at one-loop can be calculated in a straightforward manner from the thermal Feynman rules following from the Lagrangian density in [3]. The momentum integration can be done using dimensional regularization and the relation (for even values of $n$) [21]

$$\int_0^{\infty} dk k^{n-3} N_B(k) = (-1)^{n/2} (2\pi)^{n-2} B_{n-2} \frac{2}{2(n-2)},$$

where $N_B(k) = \left( e^{k/T} - 1 \right)^{-1}$ is the bosonic distribution function and $B_{n-2}$ are the Bernoulli numbers. This leads in an even $n$-dimensional space-time to the leading temperature-dependent part of the tadpole

$$\Gamma^{(T)}_{\text{tad}} = g \pi^{(n-3)/2} |B_{n-2}| \frac{T^{n-2}}{\Gamma \left( \frac{n}{2} \right)} \equiv g C_n T^{n-2}.\quad (10)$$

so that the extremum of the potential is indeed a minimum. We note here that the local minimum in the $\phi^3$ theory also occurs at the origin. However, unlike the $\phi^3$ theory where the potential is unbounded from below, here we have a potential that is better behaved as shown in Fig. 1. At zero temperature, therefore, the vacuum of the theory is stable.

$$V_{\text{eff}} = -\frac{9m^6}{g^2} \sqrt{1 - \frac{2g}{3m^2} \phi - \frac{3m^4}{g} \phi}.\quad (5)$$

The unique extremum of this effective potential satisfying $dV_{eff}/d\phi = 0$, occurs at

$$\frac{1}{\sqrt{1 - \frac{2g}{3m^2} \phi}} = 1,$$

which determines the vacuum expectation value of the scalar field in this theory to correspond to

$$\langle 0 | \phi | 0 \rangle = \langle \phi \rangle = 0.\quad (6)$$

Furthermore, from Eq. (5), we note that

$$\frac{d^2 V_{eff}}{d\phi^2} |_{\phi = 0} = m^2 > 0,\quad (7)$$

so that the extremum of the potential is indeed a minimum. We note here that the local minimum in the $\phi^3$ theory also occurs at the origin. However, unlike the $\phi^3$ theory where the potential is unbounded from below, here we have a potential that is better behaved as shown in Fig. 1. At zero temperature, therefore, the vacuum of the theory is stable.
Furthermore, at the extremum we have

$$\left. \frac{d^2 V^{(T)}_{\text{eff}}}{d\phi^2} \right|_{\langle\phi\rangle_{(T)}} = m^2 \left( 1 - \frac{g^2}{3m^2} C_n T^{n-2} \right)^3.$$  \hspace{1cm} (15)

Thus, the second derivative in (15) would be positive and the extremum will indeed correspond to a minimum as long as the temperature is smaller than

$$T_{(n)}^{cr} = \left( \frac{3m^4}{g^2 C_n} \right)^{\frac{1}{n-2}}.$$  \hspace{1cm} (16)

This, therefore, defines a critical temperature for the system (in $n$ dimensions) and for temperatures below this critical temperature the ground state will be stable. We see from Fig. 2 that for large negative values of $\phi$, the temperature dependent tadpole contribution has the effect of lowering the potential such that for temperatures below the critical temperature, there is a barrier. However, for temperatures (equal to or) above the critical temperature, there is no longer a barrier and the thermal fluctuations induce a roll off to infinity as shown in Fig. 2.

At this point, it may be instructive to give a diagrammatic representation of the solution which minimizes the effective potential and determines the vacuum expectation value of the field. Expanding out the effective potential in (12) we note that we can write

$$V^{(T)}_{\text{eff}} = -\frac{9m^6}{g^2} + \frac{m^2}{2} \phi^2 + \frac{g}{3!} \phi^3 + \frac{5g^2}{4! (3m^2)} \phi^4 + \cdots + \Gamma^{(T)}_{\text{tad}} \langle\phi\rangle.$$  \hspace{1cm} (17)

Furthermore, recognizing that for constant field configurations we can represent $G^{-1} = -\left(\Box + m^2\right) \rightarrow -m^2$ we can write the equation for the minimum of the effective potential also as

$$G^{-1} \phi = \Gamma^{(T)}_{\text{tad}} + \frac{g}{2!} \phi^2 + \frac{5g^2}{3! 3m^2} \phi^3 + \cdots \text{, or},$$

$$\phi = G \left( \Gamma^{(T)}_{\text{tad}} + \frac{g}{2!} \phi^2 + \frac{5g^2}{3! 3m^2} \phi^3 + \cdots \right),$$  \hspace{1cm} (18)

where $G = -1/m^2$ represents the scalar propagator evaluated at zero momentum. Iterating this equation we obtain a perturbative solution of Eq. (18) as a power series

in $\Gamma^{(T)}_{\text{tad}}$, which can be represented graphically as

Here the lines denote the zero-momentum (and zero temperature) propagator $G$, the blobs represent the temperature-dependent tadpole $\Gamma^{(T)}_{\text{tad}}$, defined in (10), and the vertices are: $\lambda_3 = g$, $\lambda_4 = 5g^2/(3m^2)$, $\lambda_5 = 35g^3/(9m^4)$, $\cdots$. It can be checked that the sum of the above series of diagrams yields identically the result in (14).

### IV. DISCUSSION

We have studied the question of stability of the ground state of a scalar theory which may be thought as a generalization of the conventional $\phi^3$ theory with nonpolynomial and derivative interactions. In this model, which
is somewhat analogous to gravity with a cosmological constant, the critical temperature above which the theory becomes unstable is given by Eq. (16), which is our main result.

Let us now evaluate this temperature in six space-time dimensions, in order to compare it with the critical temperature which occurs in the renormalizable $(\phi^3)_6$ model [12]. Using the value of $C_n$ given in Eq. (11), we find that

$$T^{cr}_{(6)} = \left( \frac{810}{\pi} \right)^{1/4} \frac{m}{\sqrt{g}}. \quad (20)$$

This critical temperature is somewhat higher than the one found in the $(\phi^3)_6$ model, which indicates that the present theory may be a bit more stable under thermal fluctuations.

In conclusion, let us point out an important difference between these two theories. The model described by the Lagrangian density [3] involves effective dimensionful coupling constants, with dimension of inverse powers of mass, and is, like gravity, non-renormalizable. It turns out that, in order to be able to neglect higher order thermal loops, it is necessary to assume the condition

$$\frac{g^{2T^{n-4}}}{m^2} = \left( \frac{g^{2T^{n-6}}}{m^2} \right) T^2 \ll 1. \quad (21)$$

This relation is somewhat similar to the one required in four dimensional thermal gravity [6]: $G_N T^2 \ll 1$, where $G_N$ is the gravitational constant. We remark that Eq. (21) is consistent with the result (16) for the critical temperature, provided the coupling constant $g$ is sufficiently small. For completeness we note here that all of our results reduce, in six space-time dimensions, to those of [12] upon proper truncation.

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