Non-commutative SU(N) gauge theories and asymptotic freedom

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Abstract

In this paper we analyze the one-loop renormalization of the \(\theta\)-expanded SU(N) Yang-Mills theory. We show that the freedom parameter \(a\), key to renormalization, originates from higher order non-commutative gauge interaction, represented by a higher derivative term \(bh\theta^{\mu\nu}\hat{F}_{\mu\nu}\star\hat{F}_{\rho\sigma}\star\hat{F}^{\rho\sigma}\). The renormalization condition fixes the allowed values of the parameter \(a\) to one of the two solutions: \(a = 1\) or \(a = 3\), i.e. to \(b = 0\) or to \(b = 1/2\), respectively. When the higher order interaction is switched on, \((a = 3)\), pure non-commutative SU(N) gauge theory at first order in \(\theta\)-expansion becomes one-loop renormalizable for various representations of the gauge group. We also show that, in the case \(a = 3\) and the adjoint representation of the gauge fields, the non-commutative deformation parameter \(h\) has to be renormalized and it is asymptotically free.

Key words: Standard Model, Non-commutative Geometry, Renormalization, Regularization and Renormalons

1 Introduction

For some years it was believed that field theories defined on non-commutative (NC) Minkowski space were not renormalizable. Namely, if non-commutativity is canonical,

\[
[\hat{x}^\mu, \hat{x}^\nu] = ih\theta^{\mu\nu} = \text{const},
\]

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then the algebra generated by the coordinates \( \hat{x}^\mu \), i.e. non-commutative Minkowski space and the fields on it can be represented by the algebra of functions on the ordinary \( \mathbb{R}^4 \) with the Moyal-Weyl product instead of the usual multiplication

\[
\hat{\phi}(x) \star \hat{\psi}(x) = e^{\frac{\hbar \theta_{\mu\nu}}{2} \partial_\mu \hat{\phi}(y) \partial_\nu \hat{\psi}(y) \mid_{y \to x}}.
\] (1.2)

Furthermore, the integral can be defined straightforwardly and has the trace property. Thus one can formulate field theories with the action and the variational principle. However, in the quantization of these \( \theta \)-unexpanded theories (e.g., \( \phi^4 \)) one, as a rule, meets the obstruction to renormalizability: the UV/IR mixing [1–3].

In (1.1) the non-commutative deformation parameter \( h \) has dimension length\(^2\) or energy\(^{-2}\), and can also be written as \( h = 1/\Lambda_{\text{NC}}^2 \), where the \( \Lambda_{\text{NC}} \) represents the scale of non-commutativity.

Gauge theories can be extended to a non-commutative setting in different ways. In our model, the classical action is obtained via a two-step procedure. First, the action of the non-commutative Yang-Mills (NCYM) theory is equipped with a star-product carrying information about the underlying non-commutative manifold, and, second, the star-product and non-commutative fields are expanded in the non-commutativity parameter \( h \theta \) using the Seiberg-Witten (SW) map [4]. In this approach [5–8], non-commutativity is treated perturbatively. The major advantage is that models with any gauge group and any particle content can be constructed [5,9–12], so we can construct the generalization of the standard model (SM), too. The action is gauge invariant; furthermore, it has been proved that the action is anomaly free whenever its commutative counterpart is also anomaly free [13].

In this paper, which is a continuation of two recent papers, [6] and [7], we analyze the renormalizability property of Yang-Mills theory on non-commutative space, where we confine ourselves to the \( \theta \)-expanded NC SU(N) gauge theory. Commutative gauge symmetry is the underlying symmetry of the theory and is present in each order of the \( \theta \)-expansion. Non-commutative symmetry, on the other hand, exists only in the full theory, i.e. after the summation.

There are a number of versions of the non-commutative standard model (NC-SM) in the \( \theta \)-expanded approach, [9–12]. The argument of renormalizability was previously included in the construction of field theories on non-commutative Minkowski space producing not only the one-loop renormalizable model [6], but the model containing one-loop quantum corrections free of divergences [7], contrary to previous results [14–17]. This ‘good’ behavior of the \( \theta \)-expanded non-commutative SM gauge theory is our primary motivation to re-examine one-loop renormalizability aspect of the pure NC SU(N) gauge sector. We shall
perform the analysis at first order in $\theta$, and for the fundamental representations of the matter field. Phenomenological consequences of this investigation are certainly important [12,18,19].

The plan of the paper is the following. In Section 2 we briefly review the ingredients of the SW freedom and construct the higher-order Lagrangian term which renders one-loop renormalizability of the non-commutative theories at first order in $\theta$. In Section 3 the one-loop renormalizability of the pure NC SU(N) gauge theory is worked out. Section 4 is devoted to the ultraviolet asymptotic behavior of NC SU(N) gauge theory. The discussion of the results and the concluding points are given in Section 5.

2 NC SU(N) gauge sector effective action

According to [5], the NC parameter $\hat{\Lambda}$, the NC vector potential $\hat{V}_\mu$ and the corresponding NC field strength $\hat{F}_{\mu\nu}$ take their values in the enveloping algebra of the Lie algebra of the gauge group. As in ordinary theory, in the non-commutative case, symmetry is localized by the NC vector potential $\hat{V}_\mu$ and the NC field strength $\hat{F}_{\mu\nu}$

\[ \hat{F}_{\mu\nu} = \partial_\mu \hat{V}_\nu - \partial_\nu \hat{V}_\mu - i (\hat{V}_\mu \star \hat{V}_\nu - \hat{V}_\nu \star \hat{V}_\mu). \] (2.1)

We start by solving the gauge field transformation closure condition [5] order by order in the parameter $\hbar$. The solutions up to the first order for the vector field and the field strength read

\[ \hat{V}_\mu(x) = V_\mu(x) - \frac{1}{4} \hbar \theta^{\rho\sigma} \{ V_\sigma(x), \partial_\rho V_\mu(x) + F_{\rho\mu}(x) \} + \ldots \] (2.2)

\[ \hat{F}_{\mu\nu}(x) = F_{\mu\nu} + \frac{1}{4} \hbar \theta^{\rho\sigma} \left( 2 \{ F_{\mu\sigma}, F_{\nu\rho} \} - \{ V_\sigma, (\partial_\rho + D_\rho) F_{\mu\nu} \} \right) + \ldots \] (2.3)

The non-Abelian field strength and the covariant derivative are defined in the usual way $F_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu - i[V_\mu, V_\nu]$ and $D_\mu = \partial_\mu - i[V_\mu, \cdot]$. The relations (2.2-2.3) between non-commutative and commutative gauge symmetries are known as the Seiberg-Witten maps, [4]. For zero non-commutativity, $\hat{V}_\mu$ and $\hat{F}_{\mu\nu}$ reduce to the usual vector potential $V_\mu$ and the field strength $F_{\mu\nu}$. The SW map, which fulfills a number of requirements (hermiticity, non-uniqueness etc), leading to a physically acceptable theory, was discussed extensively in [20,21].
Clearly, the solution (2.2) is not unique. Non-uniqueness is given by the transformation

\[ \hat{V}_\mu \rightarrow \hat{V}_\mu + X_\mu, \quad \hat{F}_{\mu\nu} \rightarrow \hat{F}_{\mu\nu} + D_\mu X_\nu - D_\nu X_\mu, \]  

(2.4)

which one understands as freedom to define the physical fields \( V_\mu \) and the field strengths \( F_{\mu\nu} \).\(^1\)

The usual, or minimal NC SU(N) gauge theory action

\[ S_g = -\frac{1}{2} \text{Tr} \int d^4x \hat{F}_{\mu\nu} \ast \hat{F}^{\mu\nu}, \]  

(2.5)

expanded in the deformation parameter \( h \) using SW map (2.3) reads

\[ S_g^1 = \text{Tr} \int d^4x \left[ -\frac{1}{2} F_{\mu\nu} F^{\mu\nu} + h \theta^{\mu\nu} \left( \frac{1}{4} F_{\mu\rho\sigma} F_{\nu\rho\sigma} - F_{\mu\rho\sigma} F_{\nu\rho\sigma} \right) F^{\rho\sigma} \right]. \]  

(2.6)

One can however generalize this action introducing the terms of higher order in NC field strengths. Due to the Lorentz structure there are only two third-order interaction terms, producing the generalized NC gauge action

\[ S_*= \text{Tr} \int d^4x \left[ -\frac{1}{2} \hat{F}_{\mu\nu} \ast \hat{F}^{\mu\nu} + h \theta^{\mu\nu} \left( b \hat{F}_{\mu\nu} \ast \hat{F}_{\rho\sigma} + c \hat{F}_{\mu\rho} \ast \hat{F}_{\nu\sigma} \right) \ast \hat{F}^{\rho\sigma} \right], \]  

(2.7)

with still unspecified constants \( b \) and \( c \). We will not consider here the terms of fourth and higher order in NC field strengths. Let us stress once again that the action \( S_* \) is invariant under the NC gauge transformation. The constants \( b \) and \( c \) are going to be restricted further.

Substituting the SW map for NC field strength (2.3) into the action (2.7), we obtain

\[ S_*^1 = \text{Tr} \int d^4x \left[ -\frac{1}{2} F_{\mu\nu} F^{\mu\nu} + h \theta^{\mu\nu} \left( \frac{1}{4} F_{\mu\rho\sigma} F_{\nu\rho\sigma} - F_{\mu\rho\sigma} F_{\nu\rho\sigma} \right) F^{\rho\sigma} \right. \]

\[ \left. + h \theta^{\mu\nu} \left( b F_{\mu\nu} F_{\rho\sigma} + c F_{\mu\rho} F_{\nu\sigma} \right) F^{\rho\sigma} \right], \]

\(^1\) The transformation (2.4) introduced in [7] in fact does not produce new terms in the gauge field lagrangian as we claimed, [22]. However, this does not spoil the main result (3.21) of Ref. [7], that is the renormalizability and finiteness of the 1-loop divergent terms in the nmNCSM gauge sector for the choice of the parameter \( a = 3 \). In order to obtain terms necessary for renormalization, we have to introduce higher-order NC gauge interactions as we will see later on.
\[ S = \text{Tr} \int d^4x \left[ -\frac{1}{2} F_{\mu\nu} F^{\mu\nu} + \frac{1}{4} h \theta^{\mu\nu} \left( \left( \frac{1}{4} + b \right) F_{\mu\nu} F_{\rho\sigma} + (c - 1) F_{\mu\rho} F_{\nu\sigma} \right) F^{\rho\sigma} \right]. \]  

(2.8)

The above-given action is invariant under the classical gauge transformation. In order to obtain new contributions from \( S_1^\star \), the constant \( c \) has to respect requirement \( c \neq 1 \). In the expression (2.8) we are free to choose \( c = 0 \), and then, via simple redefinition,

\[ 1 + 4b = a, \]  

(2.9)

we arrive at the desired action \( S \) to first order in non-commutative deformation parameter \( h \):

\[ S = \text{Tr} \int d^4x \left[ -\frac{1}{2} F_{\mu\nu} F^{\mu\nu} + \frac{1}{4} h \theta^{\mu\nu} \left( \frac{a}{4} F_{\mu\nu} F_{\rho\sigma} - F_{\mu\rho} F_{\nu\sigma} \right) F^{\rho\sigma} \right]. \]  

(2.10)

Here \( a \) is an arbitrary real parameter, i.e. the freedom parameter to be determined, as before [7], from the renormalizability requirement \(^2\).

From the expression (2.9) it is clear that the \( a \) dependence of the gauge action \( S \), (2.10), crucial to obtain renormalizability/finiteness of the nmNCSM, [7], is arising from the inclusion of the higher order gauge interaction term \( \theta^{\mu\nu} F_{\mu\nu} \ast \hat{F}_{\rho\sigma} \ast \hat{F}^{\rho\sigma} \) into the action (2.7), and from the implementation of SW map (2.3).

Finally, it is important to notice that the \( h \)-linear terms in (2.8) and (2.10) depend on the representation of gauge fields: they are proportional to the trace of the product of three group generators. Thus the non-commutative correction to the gauge field action depends on the representations of fields in a given theory.

To find the classical action, we follow [6]. For non-Abelian vector fields, from (2.10) we obtain

\[ S_{\text{NCYM}} = \int d^4x \left[ -\frac{1}{4} F^{a}_{\mu\nu} F^{a\mu\nu} + \frac{1}{4} h \theta^{\mu\nu} d^{abc} \left( \frac{a}{4} F^{a}_{\mu\nu} F^{b}_{\rho\sigma} - F^{a}_{\mu\rho} F^{b}_{\nu\sigma} \right) F^{c\rho\sigma} \right], \]  

(2.11)

where \( d^{abc} \) are totally symmetric coefficients of the SU(N) group. This action, for \( a = 1 \), corresponds to the classical action \( S_{\text{mNCSM}} \), i.e. to Eq. (24) con-

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\(^2\) It was found in [6] that for \( a = 1 \) the theory is renormalizable, while in [7] renormalizability/finiteness of the model required \( a = 3 \).
constructed in [10]. Here, for the general SU(N) gauge group, \( a, b, c = 1, \ldots, N^2 - 1 \) are the group indices. Finally, note that in [7] the action of the type (2.11) was absent owing to the special choice of the representation SU(3)\(_C\) group where the coefficient \( \kappa_5^{abc} \) was zero.

3 One-loop renormalization

To perform the one-loop renormalization of the NC SU(N) gauge part action (2.10), we apply, as before [6,7], the background-field method [23,24]. As we have already explained the details of the method in [17], here we only discuss the points needed for this computation. The main contribution to the functional integral is given by the Gaussian integral. However, technically, this is achieved by splitting the vector potential into the classical-background and the quantum-fluctuation parts, that is, \( \phi_V \rightarrow \phi_V + \Phi_V \), and by computing the terms quadratic in the quantum fields. In this way we determine the second functional derivative of the classical action, which is possible since our interactions (2.10) and/or (2.11) are of the polynomial type. The quantization is performed by the functional integration over the quantum vector field \( \Phi_V \) in the saddle-point approximation around the classical (background) configuration \( \phi_V \).

First, an advantage of the background-field method is that it guarantees covariance, as in doing the path integral the local symmetry of the quantum field \( \Phi_V \) is fixed, while the gauge symmetry of the background field \( \phi_V \) is manifestly preserved.

Since we are dealing with gauge symmetry, our Lagrangian (2.11) is singular owing to its invariance under the gauge group. Therefore, a proper quantization of (2.11) requires the presence of the gauge fixing term \( S_{gf}[\phi] \) in the one-loop effective action

\[
\Gamma[\phi] = S_{cl}[\phi] + S_{gf}[\phi] + \Gamma^{(1)}[\phi], \quad S_{gf}[\phi] = -\frac{1}{2} \int d^4x(D_\mu \Phi_V^\mu)^2, \tag{3.1}
\]

producing the standard result of the commutative part of our action (2.10). In \( S_{gf} \) from (3.1) we have chosen the Feynman gauge ‘\( \alpha = 1 \)’.

The one-loop effective part \( \Gamma^{(1)}[\phi] \) is given by

\[
\Gamma^{(1)}[\phi] = \frac{i}{2} \log \det S^{(2)}[\phi] = \frac{i}{2} \text{Tr} \log S^{(2)}[\phi]. \tag{3.2}
\]
In (3.2), the $S^{(2)}[\phi]$ is the second functional derivative of the classical action

$$S^{(2)}[\phi] = \frac{\delta^2 S_{cl}}{\delta \phi_{V_1} \delta \phi_{V_2}}.$$  \hspace{1cm} (3.3)

The structure of $S^{(2)}[\phi]$ is

$$S^{(2)} = \Box + N_1 + N_2 + T_2 + T_3 + T_4,$$  \hspace{1cm} (3.4)

where $N_1, N_2$ are commutative vertices, while $T_2, T_3, T_4$ are non-commutative ones. The indices denote the number of classical fields. The one-loop effective action computed by using the background-field method is

$$\Gamma_{\theta,2}^{(1)} = \frac{i}{2} \text{Tr} \log \left( I + \Box^{-1}(N_1 + N_2 + T_2 + T_3 + T_4) \right)$$

$$= \frac{i}{2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \text{Tr} \left( \Box^{-1}N_1 + \Box^{-1}N_2 + \Box^{-1}T_2 + \Box^{-1}T_3 + \Box^{-1}T_4 \right)^n.$$  \hspace{1cm} (3.5)

For dimensional reasons, the divergences in $\hbar\theta$-linear order are all of the forms $\hbar\theta FV^4$, $\hbar\theta F^2V^2$ and $\hbar\theta F^3$. Since the $\hbar\theta$-3-, $\hbar\theta$-4-, $\hbar\theta$-5- and $\hbar\theta$-6-vertices obtain divergent contributions, from the sum (3.5) we need to extract and compute only terms that contain up to three external field strengths.

As the conventions and the notation are the same as in [6], we only encounter and discuss the intermediate and the final results.

Using the previously introduced notation, the vertices read

$$(N_1)^{a\alpha\beta} = -2i(V^\mu)^{a\mu}g^{\alpha\beta}\partial_\mu,$$  \hspace{1cm} (3.6)

$$(N_2)^{a\alpha\beta} = -2f^{abc}F^{a\alpha\beta} - (V^\mu V_\mu)^{a\mu}g^{\alpha\beta},$$  \hspace{1cm} (3.7)

which are the same as in the commutative case. Non-commutative vertices are

$$(T_2)^{a\alpha\beta} = \frac{h}{8} d^{abc} \left\{ \left[ (a\theta^{r\sigma}F_{r\rho}^{\alpha\nu}g^{\beta\mu} - 2(a - 1)\theta^{a\mu}F^{\alpha\beta\nu} + 4\theta^{a\rho}F^{\alpha\beta\rho}g^{\mu\nu} \right. \right.$$ 

$$+ 4\theta^{\alpha\rho}F^{a\rho\nu}g^{\beta\mu}) - (\beta \leftrightarrow \nu) \left[ (\alpha \leftrightarrow \beta) \right] \partial_\nu \right\},$$

$$(T_3)^{a\alpha\beta} = \frac{i h}{4} \left\{ d^{abcd} \left[ -2a\theta^{\alpha\mu}(V_\nu)^{abc}F^{d\beta\nu} - 2a\theta^{\beta\nu}(V_\nu)^{abc}F^{d\alpha\mu} - a\theta^{\rho\sigma}(V_\mu)^{abc}F_{\rho\sigma}^{d\alpha\beta} + a\theta^{\rho\sigma}(V_\alpha)^{abc}F_{\rho\sigma}^{d\beta\mu} - \right. \right.$$ 

$$\left. - 2\theta^{a\mu}(V_\nu)^{bc}F^{d\alpha\mu}g^{\beta\nu} + 2\theta^{\beta\rho}(V_\mu)^{bc}F_{\rho\sigma}^{d\beta\nu} \right\}.$$
resulting contributions are given by

\[ + 2\theta^{\mu \nu}(V^\nu)_{bc}F_{\nu \rho}^d - 2\theta^{\mu}_\rho(V^\alpha)_{bc}F^{d \beta \rho} - 2\theta^{\beta}_\rho(V^\alpha)_{bc}F^{d \mu \rho} \]

\[ + 2\theta^{\beta}_\rho(V^\mu)_{bc}F^{d \alpha \rho} + 2\theta^{\nu}_\rho(V^\nu)_{bc}F^{d \mu \rho} \]

\[ + 2\theta^{\alpha \beta}(V^\nu)_{bc}F^{d \mu \nu} + 2\theta^{\alpha \nu}(V^\nu)_{bc}F^{d \beta \nu} + 2\theta^{\beta \nu}(V^\nu)_{bc}F^{d \alpha \nu} \]

\[-2\theta^{\nu}_\rho(V^\nu)_{bc}F^{d \alpha \rho}g^{\beta \mu} + 2\theta^{\mu \nu}(V^\nu)_{bc}F^{d \alpha \nu} \]

\[- [a \leftrightarrow b, \alpha \leftrightarrow \beta] \partial_{\mu} \right] (3.9) \]

\[(T_4)_{abcd} = \frac{h}{8} g^{cde} \left[ \left( -4a^\alpha a^\beta (V^\nu)_{ac}F^{e \mu} - a^\alpha (V^\nu)_{ac}F^{e \mu}g^{\alpha \beta} \right) + 4\theta^{\alpha \beta} (V^\nu)_{ac}F^{e \mu} + 4\theta^{\alpha \beta} (V^\nu)_{ac}F^{e \mu}g^{\alpha \beta} \right] \]

\[ + 2\theta^{\alpha \beta}(V^\nu)_{ac}F^{e \mu} + 4\theta^{\alpha \beta}(V^\nu)_{ac}F^{e \mu}g^{\alpha \beta} \]

\[ + 2\theta^{\alpha \beta}(V^\nu)_{ac}F^{e \mu} + 4\theta^{\alpha \beta}(V^\nu)_{ac}F^{e \mu}g^{\alpha \beta} \]

The divergent parts are calculated in the momentum representation via dimensional regularization, by picking relevant terms out of the expansion (3.5). The resulting contributions are given by

\[ D_1^{\text{div}} = \frac{i}{2} \text{Tr} \left( \left( \square^{-1} N_1 \right)^2 \left( \square^{-1} T_4 \right) \right)^{\text{div}} \]

\[ = \frac{h}{(4\pi)^2} \int d^4x \left[ \frac{\alpha - 3}{4} (\theta^{\alpha \mu}F^a_{\alpha \nu} + \theta^{\alpha \nu}F^{a \alpha \mu})(V^\mu V^\nu V^\beta V^\nu)_{bc} \right] \]

\[ + \frac{3\alpha - 4}{4} \theta^{\alpha \beta} F^a_{\alpha \beta}(V^\mu V^\nu V^\nu V^\nu)_{bc} \right], \]

\[ D_2^{\text{div}} = -\frac{i}{2} \text{Tr} \left( \left( \square^{-1} N_1 \right)^3 \left( \square^{-1} T_3 \right) \right)^{\text{div}} \]

\[ = \frac{h}{(4\pi)^2} \int d^4x \left[ \frac{7 - 3\alpha}{6} \theta^{\alpha \beta} F^a_{\alpha \beta}(V^\mu V^\nu V^\nu V^\nu + V^\nu V^\nu V^\mu V^\nu \right] \]

\[ + V^\mu V^\nu V^\nu V^\nu)_{bc} + \frac{3 - 2\alpha}{6} (\theta^{\alpha \mu}F^a_{\alpha \nu} \]

\[ + \theta^{\alpha \nu}F^{a \alpha \mu})(V^\mu V^\nu V^\mu V^\nu + V^\nu V^\nu V^\nu V^\nu \right] \]

\[ + V^\mu V^\nu V^\nu V^\nu)_{bc} \right], \]

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\[ D_3^{\text{div}} = \frac{i}{2} \text{Tr} \left( \left( \Box^{-1} N_1 \right)^4 \left( \Box^{-1} T_2 \right) \right)^{\text{div}} \]

\[ = \frac{h}{(4\pi)^2 \epsilon} d^{abc} \int d^4x \left[ \frac{7a - 11}{12} \theta^{\alpha\beta} F^a_{\alpha\beta} (V_\mu V^\mu V_\nu + V_\nu V_\mu V^\mu) + V_\mu V_\nu V^\mu V^\nu \right. \]

\[ + \left. \frac{a - 3}{12} (\theta^{\alpha\mu} F^a_{\alpha\nu}) + \theta_{\alpha\nu} F^{a\alpha\mu} (2V_\rho V_\mu V_\nu + 2V_\rho V_\nu V_\mu) \right] \]

\[ + V_\rho V_\mu V_\nu V_\rho + V_\mu V_\rho V_\nu V_\nu \right] \] (3.13)

\[ D_4^{\text{div}} = -\frac{i}{2} \text{Tr} \left( \left( \Box^{-1} N_2 \right) \left( \Box^{-1} T_4 \right) \right)^{\text{div}} \]

\[ = \frac{h}{(4\pi)^2 \epsilon} d^{abc} \int d^4x \left[ \frac{4 - 3a}{4} \theta^{\alpha\beta} F^a_{\alpha\beta} (V_\mu V_\nu V^\mu V^\nu)^{bc} \right. \]

\[ + \left. \frac{2 - a}{2} (\theta^{\mu\nu} F^a_{\mu\nu} + \theta_{\alpha\nu} F^{a\alpha\mu})(V_\mu V^\rho V_\nu)^{bc} \right] \]

\[ + 2(a + 1) i\theta_{\alpha\nu} F^a_{\beta\mu} (V^\mu F^{a\beta\nu})^{bc} \]

\[ + \frac{2}{2} i\theta^{\alpha\beta} F^a_{\mu\nu} (V^\nu F^{a\alpha\beta})^{bc} + 2i\theta^{\alpha\beta} F^a_{\beta\mu} (V^\nu F^{a\alpha\nu})^{bc} \]

\[ + 2\theta^{\alpha\beta} F^a_{\mu\nu} (V^\nu F^{a\alpha\beta})^{bc} + \left. 4i\theta^{\alpha\beta} F^a_{\mu\nu} (V^\nu F^{a\alpha\beta})^{bc} \right] \] (3.14)

\[ D_5^{\text{div}} = \frac{i}{2} \text{Tr} \left( \left( \Box^{-1} N_2 \right)^2 \left( \Box^{-1} T_4 \right) \right)^{\text{div}} \]

\[ = \frac{h}{(4\pi)^2 \epsilon} d^{abc} \int d^4x \left[ \frac{a - 3}{2} (\theta^{\mu\nu} F^a_{\mu\nu} + \theta_{\alpha\nu} F^{a\alpha\mu})(F_{\mu\nu} F^{\mu\nu})^{bc} \right. \]

\[ + \frac{3a - 4}{4} \theta^{\alpha\beta} F^a_{\alpha\beta} (F_{\mu\nu} F^{\mu\nu})^{bc} \]

\[ + \frac{4a - 7}{4} \theta^{\alpha\beta} F^a_{\alpha\beta} (V^\nu V_\mu V_\mu V_\nu)^{bc} \] \] (3.15)

\[ D_6^{\text{div}} = \frac{i}{2} \text{Tr} \left( \left( \Box^{-1} N_1 \right) \left( \Box^{-1} N_2 \right) \left( \Box^{-1} T_3 \right) \right)^{\text{div}} \]

\[ + \left( \Box^{-1} N_2 \right) \left( \Box^{-1} N_1 \right) \left( \Box^{-1} T_3 \right)^{\text{div}} \]

\[ = \frac{h}{(4\pi)^2 \epsilon} d^{abc} \int d^4x \left[ \frac{a - 3}{2} \theta^{\mu\nu} F^a_{\mu\nu} (2V_\mu V_\nu V^\mu V^\nu + V_\rho V_\nu V_\mu V^\nu) \right. \]

\[ + \left. V_\rho V_\mu V_\nu V_\rho + V_\mu V_\rho V_\nu V_\nu \right] \] (3.16)
\[
\sum_{i=1}^{7} D_i \text{div} = \frac{N}{(4\pi)^2 \epsilon} \frac{h}{d^{abc}} \int d^4 x \left( -\frac{25a - 3}{48} F_{\mu\rho}^a F_{\sigma\rho}^b F^{c\rho\sigma} + \frac{a + 21}{12} F_{\mu\rho}^a F_{\nu\sigma}^b F^{c\rho\sigma} \right), \tag{3.18}
\]

has been obtained after the dimensional regularization and summation of all the contributions. Therefore, the total divergent contribution to the effective action (2.11) is

\[
\]
\[ \Gamma_{\text{div}} = \frac{\alpha}{6} \frac{N}{(4\pi)^2 \epsilon} \int d^4 x F_{\mu \nu}^a F^{\mu \nu a} \]  

(3.19)

\[ + \frac{N}{(4\pi)^2 \epsilon} \hbar \theta^{\mu \nu} d_{abc} \int d^4 x \left( -\frac{25\alpha - 3}{48} F_{\mu \nu}^a F_{\rho \sigma}^b + \frac{a + 21}{12} F_{\mu \rho}^a F_{\nu \sigma}^b \right) F^{c \rho \sigma}. \]

In the above expression the ghost contribution to the one-loop effective action is included.

We are interested in the renormalization of the theory. Our starting Lagrangian in \( D = 4 - \epsilon \) dimensional space-time has the following form:

\[ \mathcal{L} = -\frac{1}{4} F_{\mu \nu}^a F^{a \mu \nu} + \frac{1}{4} g \mu^{\epsilon/2} h \theta^{\mu \nu} d_{abc} \left( \frac{a}{4} F_{\mu \nu}^a F_{\rho \sigma}^b - F_{\mu \rho}^a F_{\nu \sigma}^b \right) F^{c \rho \sigma}, \]

(3.20)

where \( g \) is the gauge coupling constant and \( \mu \) is the subtraction point mass parameter or the so-called renormalization point. In order to cancel divergences, counter terms should be added to the starting action, which produces the bare Lagrangian from (3.19):

\[ \mathcal{L}_0 = -\frac{1}{4} F_{\mu \nu}^a F^{a \mu \nu} - \frac{11 N g^2}{6(4\pi)^2 \epsilon} F_{\mu \nu}^a F^{a \mu \nu} \]

\[ + \frac{1}{4} g \mu^{\epsilon/2} h \theta^{\mu \nu} d_{abc} \left( \frac{a}{4} F_{\mu \nu}^a F_{\rho \sigma}^b - F_{\mu \rho}^a F_{\nu \sigma}^b \right) F^{c \rho \sigma} \]

\[ - \frac{N g^3 \mu^{\epsilon/2}}{(4\pi)^2 \epsilon} h \theta^{\mu \nu} d_{abc} \left( -\frac{25\alpha - 3}{48} F_{\mu \nu}^a F_{\rho \sigma}^b + \frac{a + 21}{12} F_{\mu \rho}^a F_{\nu \sigma}^b \right) F^{c \rho \sigma} \]

\[ = -\frac{1}{4} F_{0 \mu \nu}^a F_0^{a \mu \nu} + \frac{1}{4} g \mu^{\epsilon/2} h \theta^{\mu \nu} d_{abc} \left[ \frac{a}{4} \left( 1 + \frac{25\alpha - 3}{3a} \frac{N g^2}{(4\pi)^2 \epsilon} \right) F_{\mu \rho}^a F_{\nu \sigma}^b \right] + \left( 1 + \frac{21 + a}{3} \frac{N g^2}{(4\pi)^2 \epsilon} \right) F_{\mu \rho}^a F_{\nu \sigma}^b. \]

(3.21)

It is easy to see that in order to keep the ratio of the coefficients of two terms from (3.21) the same as in the classical action (2.11), one has to impose the condition

\[ \left( -\frac{25\alpha - 3}{48} \right) : \left( \frac{a + 21}{12} \right) = \frac{a}{4} : (-1). \]

(3.22)

Interestingly enough, this equation has two solutions, \( a = 1 \) and \( a = 3 \). In our previous paper [6] the action (2.11) was discussed and renormalizability was proved for \( a = 1 \). In this case, the divergences are canceled through redefinition of the gauge potential and the coupling constant.

The case \( a = 3 \) is different since the non-commutative deformation parameter \( h \) has to be renormalized. The bare gauge field, the coupling and the NC...
deformation parameter are defined as follows:

\[ V_0^{\mu} = V^{\mu} \sqrt{1 + \frac{22Ng^2}{3(4\pi)^2\epsilon}}, \]
\[ g_0 = \frac{g^{\mu/2}}{\sqrt{1 + \frac{22Ng^2}{3(4\pi)^2\epsilon}}}, \]
\[ h_0 = \frac{h}{1 - \frac{2Ng^2}{3(4\pi)^2\epsilon}}, \]

with an arbitrary choice for the renormalization point \( \mu \). Any change in \( \mu \) is compensated by the corresponding change in the charge \( g \), the NC deformation parameter \( h \) and for the scale of the fields. The above result means that it is not possible to renormalizes our action, for \( a = 3 \), only through the renormalization of the vector potential and the coupling constant.

4 Ultraviolet asymptotic behavior of NC SU(N) gauge theory

In this section we investigate the high-energy behavior of our theory (2.11) by employing the renormalization group equation (RGE) and compute the relevant \( \beta \) functions. The RGE provides a framework within which we discuss the ultraviolet (UV) asymptotic behavior of renormalizable gauge field theory (GFT), i.e. the behavior of the relevant amplitudes in a physically uninteresting region, i.e. in a region for large \( g \) and/or far from the origin.

Since the gauge coupling constant \( g \) in our theory (2.11) depends on the renormalization point \( \mu \) satisfying the same beta function

\[ \beta_g = \mu \frac{\partial}{\partial \mu} g(\mu) = -\frac{11Ng^3(\mu)}{3(4\pi)^2}, \]

as for the commutative Yang-Mills theory without fermions and with gauge independence in lowest order \( (g^3) \), our theory is UV stable. This means that (2.11) belongs to the class of asymptotically approaching free-field theories, or in short ‘asymptotically free theory’. The solution to (4.1) is the very well-known result [25,26]

\[ \alpha_s(\mu) = \frac{g^2(\mu)}{4\pi} = \frac{6\pi}{11N} \frac{1}{\ln \frac{\Lambda}{\mu}}. \]

In (4.2), \( \Lambda \) is an integration constant not predicted by the theory, thus it is
a free parameter to be determined from the experiment. The QCD (physical) interpretation of $\Lambda$ is that it represents the marking of the boundary between a world of quasi-free quarks and gluons and the world of protons, pions, and so on. For typical QCD energies $\mu$ with $m_b \ll \mu \ll m_t$, where fermions are included ($N_f = 5$), the study of hadronic production in $e^+e^-$ annihilation at the $Z$ resonance has given a direct measured value $\alpha_s(m_Z) = 0.12$ corresponding to $\Lambda = \Lambda_{\text{QCD}} \simeq 250$ MeV.

The $\beta$ function for the NC deformation parameter $h$ can be easily computed from (3.25) and (4.1):

$$
\beta_h = \mu \frac{\partial}{\partial \mu} h(\mu) = -\frac{11N_f g^2(\mu)}{24\pi^2} h(\mu).
$$

(4.3)

Since both $\beta$ functions (4.1) and (4.3) are negative, both the coupling $g$ and the NC deformation parameter $h$ decrease with increasing energy and our theory is considered to be UV stable.

Solving equation (4.3) we obtain

$$
h(\mu) = \frac{h_0}{\ln \frac{\mu}{\Lambda}},
$$

(4.4)

which is the running NC deformation parameter $h$. Here $h_0$ is an additional integration constant whose physical interpretation is going to be discussed later. From the above expression we see that with the increase of energy the NC deformation parameter decreases, which might seem counterintuitive in the view of Heisenberg uncertainty relations. However, there are many arguments for the modification of uncertainty relations at high energy [27]. For example, if the commutation relation is

$$
[x, p] = i\hbar(1 + \beta p^2),
$$

(4.5)

where $\beta$ is a constant and has dimension $\text{energy}^{-2}$, then one can easily see that in the region of the large momentum $\Delta x$ grows linearly [30],

$$
\Delta x = \frac{\hbar}{2} \left( \frac{1}{\Delta p} + \beta \Delta p \right).
$$

(4.6)

From this example it follows that large energies do not necessarily correspond to small distances, that is the behavior of the running NC deformation param-

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3 The physical effects of the modifications at high energies, like (4.5), in 1 and 3 dimensions were analyzed in [28,29] and references therein.
eter (3.25) does not imply that non-commutativity vanishes at small distances. This is related to the UV/IR correspondence.

Owing to the necessity of the renormalization of the non-commutativity deformation parameter $h$, the scale of non-commutativity $\Lambda_{NC}$ becomes a function of energy $\mu$ too,

$$\Lambda_{NC}(\mu) = \Lambda_\theta \sqrt{\ln \frac{\mu}{\Lambda}}.$$  \hspace{1cm} (4.7)

Equivalently to (4.4), the scale $\Lambda_{NC}$ becomes the running scale of non-commutativity. Here $\Lambda_\theta$ is an additional integration constant, namely the dimension of energy, not predicted by the theory.

Even though that the physical interpretation of $h_0$ and/or $\Lambda_\theta$ is not quite clear, it seems that, owing to the energy dependence, they have to be proportional to the scale of non-commutativity $\Lambda_{NC}$. If one could think of $h_0$ and/or $\Lambda_\theta$ as a boundary between a world of commutative fields (particles) and non-commutative quantum fields, then, according to (4.4) and (4.7), it would be obvious to assume that in a first approximation $h_0 = 1/\Lambda_\theta^2 = 1/\Lambda_{NC}^2$. Considering typical QCD energies, the factor $\sqrt{\ln(m_Z/\Lambda_{QCD})} \simeq 2.4$, which means that at such energies the scale of non-commutativity $\Lambda_{NC}$ is effectively shifted by a factor of $\simeq 2.4$ up.

5 Discussion and conclusion

We have constructed a version of the SU(N) model on non-commutative Minkowski space at first order in the non-commutative deformation parameter $h$, which has the one-loop multiplicative renormalizable gauge sector.

We have shown in [6] that if the gauge fields are in the adjoint representation of SU(N), the action (2.10) for the freedom parameter $a = 1$ is renormalizable. Trying to extend this result to the gauge group of the standard model $U(1)_Y \otimes SU(2)_L \otimes SU(3)_C$, we have seen [7] that the action of the type (2.5) with SW map (2.3) and/or, (2.10), for $a = 1$ cannot be renormalized. However, with a suitable choice of the representations of the gauge group, the theory is renormalizable and finite for the freedom parameter $a = 3$ [7]. This naturally poses a question: is the obtained result, $a = 3$, just an outcome of a specific interaction among gauge bosons in the NCSM or is there something new?

In order to answer the above question, in this paper we reconsider the renormalizability of the ‘building blocks’ for arbitrary values of the freedom parameter $a$ i.e. of the non-commutative SU(N) gauge theories described by the
action (2.11) and for the gauge fields in the adjoint representation.

As a solution to the problem of the origin of the freedom parameter $a$, we propose the framework where the general non-commutative action was expanded in terms of NC field strengths, with the $\ast$-product defining multiplication. This way the higher order non-commutative gauge interaction was introduced and higher derivative interaction terms have been constructed:

\[
( b h \theta^{\mu \nu} \hat{F}_{\mu \nu} * \hat{F}_{\rho \sigma} * \hat{F}^{\rho \sigma} ) \text{ and } ( c h \theta^{\mu \nu} \hat{F}_{\mu \rho} * \hat{F}_{\nu \sigma} * \hat{F}^{\rho \sigma} ).
\]

Seiberg-Witten mapping (2.3) and freedom in constants $b$ and $c$, ($b \neq 0, c = 0$), lead to relation between $a$ and $b$: ($a = 1 + 4b$). Thus, in the above defined framework, the NC SU(N) gauge field theory is described by the NC action,

\[
S^a = \text{Tr} \int d^4 x \left( -\frac{1}{2} \hat{F}_{\mu \nu} * \hat{F}^{\mu \nu} + \frac{a - 1}{4} h \theta^{\mu \nu} \hat{F}_{\mu \rho} * \hat{F}^{\rho \sigma} \right), \quad (5.1)
\]

which, together with well known Seiberg-Witten mapping, (2.3), leads to the action in terms of commutative fields, (2.10), that is one-loop renormalizable only for $a = 1, 3$ and for various representations of the gauge potential, ([6–8], and this work). Those results suggest that in the further investigations of the renormalizability properties, of the $\theta$-expanded non-commutative gauge field theories involving fermion and Higgs fields, the gauge action (5.1) should be used as a starting point. Factor $a$ from (5.1) and/or (2.10) can be kept as the freedom parameter generically, i.e. during the computations.

To obtain 1-loop renormalizable NC SU(N) gauge theory for $a = 3$ we had to pay a prize by the renormalization of the non-commutative deformation parameter $h$. However, by doing that we gain the beta function $\beta_h$ which allow us to analyze the ultraviolet behavior of the parameter $h$.

The analysis of the ultraviolet asymptotic behavior of NC SU(N) gauge theory in the case $a = 3$ is in order next. The relevant beta functions, $\beta_g$ and $\beta_h$, have been computed and they are both negative, thus our theory is completely UV stable. The non-commutative deformation parameter $h$ becomes the running non-commutative deformation parameter and it is asymptotically free. However, owing to the inverse square behavior, the non-commutative scale runs according to (4.7). The function (4.7) is very smooth and mild, showing a small change of the scale of non-commutativity as the energy increases. We consider this property very welcome, because it shows a large degree of stability of our theory within a wide range of energy.

The necessity of the $h$ renormalization jeopardizes the previous hope that the NC SU(N) gauge theory might be renormalizable to all orders in $h\theta^{\mu \nu}$. This means that most probably the theory would need to be renormalized independently order by order in the non-commutative parameter $h\theta^{\mu \nu}$.
The one-loop renormalizability of the non-commutative SU(N) gauge sector is certainly a very encouraging result, both theoretically and experimentally. So far, this property has not concerned fermions: the results on the renormalizability of NC theories including the Dirac fermions are not yet positive, [15,17], i.e. till now fermions in non-commutative theory have still spoilt the UV stable behavior of (4.1) and (4.3) owing to non-renormalizability.

Non-Abelian commutative theories are completely renormalizable. However, fermions spoil the stable behavior of a gauge boson beta function, but they leave room to spare, and the theory becomes asymptotically free as long as $\frac{11}{4} C_2(G) > T(R)$ [25]. The possibility that something similar could happen in the case of the NCSM [7,9–12], at first order in the NC parameter $h\theta^{\mu\nu}$, is a matter for other, fermion involving, studies.

The present result has deep impact on $\theta$-expanded non-commutative gauge field theory and it could be an indication that the inclusion of fermions into a renormalizable theory might be possible within the framework of some higher non-commutative interaction symmetry, similar to the scheme outlined by (5.1). More careful choices of freedom parameters, of the representations of a gauge group and of the renormalization of the non-commutative parameter as well, are certainly necessary.

We hope that the answer to the obvious question, ‘why the freedom parameter $a$ is so special?’ has been well explained in this article. Clearly, it led us to discovery of the key role of the higher order non-commutative gauge interaction in one-loop renormalization of the $\theta$-expanded gauge theories, at first order in $h\theta$, and hence, to discovery that the non-commutative deformation parameter $h$ vanishes at very high energies, i.e. that $h$ is asymptotically free, opposite to the previous expectations. Those discoveries are certainly of the paramount importance for further research of the non-commutative gauge field theory properties.

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