RESEARCH ARTICLES

$p$-Adic Fractal Strings of Arbitrary Rational Dimensions and Cantor Strings

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Abstract—The local theory of complex dimensions for real and $p$-adic fractal strings describes oscillations that are intrinsic to the geometry, dynamics and spectrum of archimedean and nonarchimedean fractal strings. We aim to develop a global theory of complex dimensions for adelic fractal strings in order to reveal the oscillatory nature of adelic fractal strings and to understand the Riemann hypothesis in terms of the vibrations and resonances of fractal strings. We present a simple and natural construction of self-similar $p$-adic fractal strings of any rational fractal (i.e., Minkowski) dimension in the closed unit interval $[0,1]$. Moreover, as a first step towards a global theory of complex dimensions for adelic fractal strings, we construct an adelic Cantor string in the set of finite adèles $A_0$ as an infinite Cartesian product of every $p$-adic Cantor string, as well as an adelic Cantor-Smith string in the ring of adèles $A$ as a Cartesian product of the general Cantor string and the adelic Cantor string.

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This paper is dedicated to our advisors with gratitude and admiration.

1. INTRODUCTION

The theory of complex dimensions is a theory of oscillations that are intrinsic to fractal geometries. Complex dimensions provide a natural tool to reveal the oscillatory nature of fractality. Geometrically, a fractal is like a musical instrument tuned to play certain notes with frequencies essentially equal to the imaginary parts of the underlying complex dimensions and with amplitudes essentially equal to the real parts of the underlying complex dimensions. Physically, we can imagine a geometric wave propagating through the 'space of scales' that lies beneath the shimmering surface of a fractal with the aforementioned frequencies and amplitudes. Fractality is therefore defined as the presence of nonreal complex dimensions in the theory of complex dimensions for fractal strings and compact subsets of Euclidean spaces [19, 20, 30–32, 34].

Following the examples of the $a$-string and of the ordinary Cantor string given by the first author in the early 1990s [15–17, 29], the notion of a fractal string was conceived and defined by the first author and Carl Pomerance in their investigation and resolution of the one-dimensional Weyl-Berry conjecture for fractal drums and its connection with the Riemann zeta function in 1993, [29]. The Riemann hypothesis for the Riemann zeta function turned out to be equivalent to the solvability of the

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inverse spectral problems for fractal strings, as was established by the first author and Helmut Maier in 1995, [26]. The idea of complex dimensions started to emerge out of the work in [15–17, 29] and was used heuristically in [26] in the authors’ spectral reformulation of the Riemann hypothesis for the Riemann zeta function $\zeta(s)$ by means of a pair of complex conjugate numbers $\omega = \sigma + it$ and $\bar{\omega} = \sigma - it$ that lie symmetrically above and below the real fractal dimension $D = \sigma$ of a fractal string $\mathcal{L}$, where the geometric oscillations of order $\sigma$ disappear in the spectrum of $\mathcal{L}$ if $\zeta(\omega) = 0$. The precise notion of complex dimensions $\omega \in \mathbb{C}$, defined as the visible poles $\omega$ of the geometric zeta function $\zeta_{\mathcal{L}}(s)$ associated with a real fractal string $\mathcal{L}$, was crystallized and rigorously developed by the first and third authors in [30] and then significantly extended in [31, 32], as well as in many other works.

A higher-dimensional theory of complex dimensions for compact subsets of Euclidean spaces was initiated by the first author and Erin Pearse in a direct calculation of a tube formula for the Koch snowflake curve in 2006, [27], then pursued by the same authors and Steffen Winter in 2011, [28], and finally fully developed in the 2017 research monograph [34] by the first author, Goran Radunović, and Darko Žubrinić after the introduction of the distance zeta function by the first author in 2009 that was inspired, in part, by a result of Reese Harvey and John Polking in their study of the removable singularities of solutions of certain linear partial differential equations [9].

Motivated by the $p$-adic string theory in $p$-adic mathematical physics [43] and the vision of a global theory of complex dimensions for adèlic fractal strings, the geometric theory of complex dimensions for fractal strings was extended to the nonarchimedean field of $p$-adic numbers $\mathbb{Q}_p$ by the authors of the articles [21–25]. Similar to the theory of complex dimensions for real fractal strings, the theory of complex dimensions for $p$-adic fractal strings reveals the oscillatory nature intrinsic to the geometry of $p$-adic fractal strings via the explicit tube formulas for the volume of the inner $\varepsilon$-neighborhoods of $p$-adic fractal strings [23–25].

A basic example of a fractal is the Cantor set, which is nothing with respect to the one-dimensional Lebesgue measure on the real line and everything with respect to the zero-dimensional measure that counts points in a set since it contains as many points as the whole real line, yet itself contains no interval of nonzero length. The simplest construction of the Cantor set is by successively removing all the open middle thirds of the closed unit interval $[0, 1]$. The Cantor set is an uncountably infinite set of points in $[0, 1]$ that has zero length. Irrational numbers have the same property, but the Cantor set is closed, so it is not even dense in any interval, whereas the irrational numbers are dense in every interval.

In order to illuminate the difficulties of integrating discontinuous functions, Henry John Stephen Smith constructed a general Cantor set in 1875 by successively removing all $m - 1$ segments from $m \geq 3$ equal parts of the interval $[0, 1]$, where $m$ is an integer; “and exempt the last [open] segment from any subsequent division” in every iteration [39]. Georg Ferdinand Ludwig Philipp Cantor independently introduced the Cantor ternary set in 1883 as the set of real numbers of the form

$$x = \frac{c_1}{3} + \cdots + \frac{c_v}{3^v} + \cdots,$$

where the digit $c_v$ is 0 or 2 for each integer $v \geq 1$, as an example of a perfect set that is not everywhere dense; see [2, 6]. Through careful considerations of the Cantor set, Georg Cantor created set theory and helped lay the foundations of point-set topology.

Aus dem Paradies, das Cantor uns geschaffen hat, soll uns niemand vertreiben können [1].

David Hilbert (1862–1943).

In closing the introduction, we give an overview of the remainder of this article. In the preliminary §2, we describe the finite and infinite valuations on the field of rational numbers $\mathbb{Q}$ and their correspondence with the absolute values on $\mathbb{Q}$. We also describe the nonarchimedean field of $p$-adic numbers $\mathbb{Q}_p$ and the ring of adèles $\mathbb{A}$. In §3, for each prime number $p$, we construct a self-similar $p$-adic fractal string $\mathcal{L}_p$ of dimension $D$ and with oscillatory period $p = \frac{2\pi}{m \log p}$, for any rational number $D = \frac{k}{m} \in [0, 1]$. The construction of the $p$-adic fractal string of dimension $D = 1/2$ is especially interesting since it involves the ‘diagonal of pairs of digits’. In §4, for each prime number $p > 2$, we construct a $p$-adic Cantor string

\footnote{From his paradise Cantor with us unfolded, we hold our breath in awe, knowing we shall not be expelled.}
of dimension \( D = \frac{\log(1 + p)}{\log p} \) and with oscillatory period \( p = \frac{2\pi}{\log p} \). We also construct an adèlic Cantor-Smith string in \( \mathbb{A} \) as an infinite product of the general Cantor string and every \( p \)-adic Cantor string. In §5, we construct an adèlic Euler-Riemann string \( \mathcal{E}_A \) in \( \mathbb{A} \) with the Riemann zeta function as a possible geometric zeta function of \( \mathcal{E}_A \). In §6, we envision that a natural global theory of complex dimensions for adèlic fractal strings would provide a unified framework for understanding the vibrations and resonances in the geometry and the spectrum of adèlic fractal strings.

From the outset, we should caution the reader that for now, we do not have a rigorous definition (and let alone, theory) of an adèlic fractal string, especially one that would be compatible with the notion of a (convergent) geometric zeta function. The investigation of this fascinating question is left for future work by the authors and also possibly, by the interested readers.

2. PRELIMINARIES

By Ostrowski’s theorem, the global field of rational numbers \( \mathbb{Q} \) has, up to equivalence, one archimedean valuation \( v_\infty(x) = \log |x|_\infty \) and infinitely many nonarchimedean valuations \( v_p(x) = -\text{ord}_p(x)\log p \), one for each prime number \( p \in \mathbb{P} \), where \( \text{ord}_p(x) \) is the power of \( p \) in the prime factorization of a nonzero rational number \( x \in \mathbb{Q}^* \).\(^2\) The valuation of a number \( x \) is positive when \( x \) is far away from \( 0 \) in the \( v \)-adic topology or more concretely, if the denominator of \( x \) is divisible by \( p \). For every number \( x \in \mathbb{Q}^* \), we have the important Artin-Whaples sum formula:

\[
\sum_{p \leq \infty} v_p(x) = 0.
\]

There is a one-to-one correspondence between valuations and absolute values on the field of rational numbers \( \mathbb{Q} \). Each valuation \( v \) on \( \mathbb{Q} \) is assigned to a unique absolute value \(| \cdot |_v \) by defining \(| x |_v = e^{v(x)} \), for any nonzero number \( x \in \mathbb{Q}^* \), and \(|0|_v = 0 \). Conversely, every absolute value \(| \cdot | \) on \( \mathbb{Q} \) is assigned to a unique valuation \( v_{|\cdot|} \) by defining \( v_{|\cdot|}(x) = \log |x| \), for all \( x \in \mathbb{Q} \). Hence, the Artin-Whaples sum formula becomes the Artin-Whaples product formula via this correspondence:

\[
\prod_{p \leq \infty} |x|_p = 1.
\]

A physical version of the idèlic product formula was discovered by Peter Freund and Edward Witten in 1987 [7]: the ordinary Veneziano string amplitude for the scattering of four open bosonic strings in their tachyon states is the inverse product of all its \( p \)-adic Veneziano string amplitudes;

\[
\prod_{p \leq \infty} \int_{\mathbb{Q}_p} |x|^0_p \cdot |1 - x|^0_p \, dx = 1.
\]

“This factorization is equivalent to the functional equation for the Riemann zeta function” [7].

The topological completion of \( \mathbb{Q} \) with respect to the metric topology induced by the infinite place of \( \mathbb{Q} \) is the local archimedean field of real numbers \( \mathbb{R} = \mathbb{Q}_\infty \). Similarly, the topological completion of \( \mathbb{Q} \) with respect to the ultrametric topology induced by a finite place is the local nonarchimedean field of \( p \)-adic numbers \( \mathbb{Q}_p \).

The (closed) unit ball in the nonarchimedean field of \( p \)-adic numbers \( \mathbb{Q}_p \) is the ring of \( p \)-adic integers \( \mathbb{Z}_p \); it is closed under multiplication and addition. Moreover, every \( p \)-adic number \( x \in \mathbb{Q}_p \) has a unique \( p \)-adic expansion as

\[
x = a_k p^k + \cdots + a_0 + a_1 p + a_2 p^2 + \cdots,
\]

for some integer \( k \in \mathbb{Z} \) such that \( a_k \neq 0 \) and where all the digits \( a_j \in \{0, 1, \ldots, p - 1\} \). In particular, every \( p \)-adic integer \( y \in \mathbb{Z}_p \) has a unique \( p \)-adic expansion as

\[
y = a_0 + a_1 p + a_2 p^2 + \cdots.
\]

\(^2\)Here, \(|x|_\infty \) denotes the usual absolute value of \( x \in \mathbb{Q}^* \).
Here, we allow \( a_0 = 0 \). In particular, \( y \) is a unit if and only if \( a_0 \neq 0 \).

Recall that according to Ostrowski’s celebrated theorem [37, (4), p. 277], the equivalence classes of (nontrivial) absolute values (also called places)\(^3\) of \( \mathbb{Q} \) are in one-to-one correspondence with the (nonisometric) metric completions of \( \mathbb{Q} \) (viewed as a field) and, more specifically, with either one of the fields of \( p \)-adic numbers \( \mathbb{Q}_p \) (corresponding to the nonarchimedean places of \( \mathbb{Q} \)), where \( p \) is an arbitrary prime number, or else with the field of real numbers \( \mathbb{Q}_\infty := \mathbb{R} \), corresponding to the unique archimedean place of \( \mathbb{Q} \) and to the ‘infinite prime’ \( p = \infty \).

The set of \( \)finite adèles\( \mathbb{A}_0 \) is the set of sequences of \( p \)-adic numbers \( x_p \) indexed by all of the prime numbers and such that \( x_p \in \mathbb{Z}_p \), for almost all primes (i.e., for all but finitely many primes):

\[
\mathbb{A}_0 = \{(x_2, x_3, x_5, \ldots) \mid x_p \in \mathbb{Q}_p \text{ for all } p \text{ and } x_p \in \mathbb{Z}_p \text{ for almost all } p\}.
\]

The set of adèles \( \mathbb{A} \) also has an archimedean component,

\[
\mathbb{A} = \mathbb{R} \times \mathbb{A}_0.
\]

Topologically, the set of finite adèles \( \mathbb{A}_0 \) is zero-dimensional because it is totally disconnected, but combinatorially, it is one-dimensional, just like \( \mathbb{Z}_p \); see Remark 3.6. On the other hand, the set of adèles \( \mathbb{A} \) is one-dimensional, both topologically and combinatorially. Therefore, every adelic fractal string based on either \( \mathbb{A}_0 \) or \( \mathbb{A} \) has fractal (i.e., Minkowski) dimension at most 1. This Minkowski–Bouligand dimension gives combinatorial information about the number of balls required to cover the set. Here, balls are translates of the basic neighborhoods of zero,

\[
\prod_{p \in \mathbb{P}} p^{r_p} \mathbb{Z}_p,
\]

where the integral exponent \( r_p \neq 0 \) for only finitely many prime numbers \( p \). These are the usual basic open sets that one uses to define the Lebesgue integral and the Fourier transform in number theory. For example, see Tate’s thesis [40].

### 2.1. Complex Dimensions of Fractal Strings

We give here a very brief overview of the classical theory of fractal strings, and point out the connection with the present paper. The reader is referred to [30–32] for more information.

A \( \)fractal string\( \mathcal{L} \) is a bounded open subset of the real line. The lengths of its connected component are denoted by \( l_1, l_2, l_3, \ldots \), and its \( \)geometric zeta function\( \) is

\[
\zeta_{\mathcal{L}}(s) = \sum_{j=1}^{\infty} l_j^s.
\]

Thus, for example, the total length of \( \mathcal{L} \) is given by \( \zeta_{\mathcal{L}}(1) \). The \( \)complex dimensions\( \) of \( \mathcal{L} \) is the set of poles of an analytic continuation of the geometric zeta function, counted with multiplicity. In particular, the series \( (2.1) \) for \( \zeta_{\mathcal{L}} \) has an abscissa of convergence \( D \) so that this series is convergent for \( \text{Re } s > D \). This value coincides with the Minkowski dimension of the boundary of \( \mathcal{L} \) and is called the \( \)dimension\( \) of \( \mathcal{L} \).

The significance of the complex dimensions is that several geometric quantities, including the counting function of the reciprocal lengths, the frequency counting function, and the volume of the tubular inner neighborhood of the boundary, are all expressed by explicit formulas that involve the complex dimensions and the residues at each complex dimension of \( \zeta_{\mathcal{L}} \). We discuss the analogue for \( p \)-adic strings in Section 4.4.1.

In this paper, we consider the set \( \mathbb{Z}_p \) as the \( p \)-adic (metric) analogue of the unit interval. With the usual \( p \)-adic metric, it has unit volume. The analogue of the unit interval in the adèles is an infinite product \( [0, 1] \times \prod_{p<\infty} \mathbb{Z}_p \). In the finite adèles, we take the infinite product \( \prod_{p<\infty} \mathbb{Z}_p \) without the archimedean component.\(^4\)

\(^3\)Two absolute values on \( \mathbb{Q} \) are said to be equivalent if one is a positive power of the other.

\(^4\)From a technical point of view, both infinite products are really ‘restricted products’.
3. $p$-ADIC FRACTAL STRINGS WITH RATIONAL DIMENSIONS BETWEEN 0 AND 1

In looking for a geometric way to create an adelic fractal string and a global theory of complex dimensions, we discovered a simple and natural construction of $p$-adic fractal strings of any rational dimension in the critical interval $[0, 1]$. The simplest example is of dimension $D = \frac{1}{2}$, which is particularly interesting since it involves the diagonal of pairs of digits.

Given a prime number $p$ in the infinite set of primes $\mathcal{P} = \{2, 3, 5, 7, 11, \ldots\}$, let us consider the ring of $p$-adic integers $\mathbb{Z}_p$, taken two digits at a time:

$$\mathbb{Z}_p = \{(a_0 + b_0p) + (a_1 + b_1p)p^2 + (a_2 + b_2p)p^4 + \cdots \mid a_j, b_j \in \{0, 1, 2, \ldots, p-1\}\}.$$

Let $S$ be a subset of all such two-digit numbers in base $p$, containing exactly $p$ numbers. For example, we can take $S$ to be the ‘diagonal’,

$$S = \{0, 1 + p, 2 + 2p, 3 + 3p, \ldots, p - 1 + (p - 1)p\}.$$

Then $\mathbb{Z}_p$ contains $p^2 - p$ copies of $p^2\mathbb{Z}_p$, namely,

$$a + bp + p^2\mathbb{Z}_p, \text{ for each } a + bp \in \{0, 1, 2, \ldots, p^2 - 1\} - S.$$

These are the first $p^2 - p$ substrings of a $p$-adic fractal string of length $p^{-2}$. By continuing this process with the remaining $p$ subintervals $a + bp + p^2\mathbb{Z}_p$ for each $a + bp \in S$, we obtain a self-similar $p$-adic fractal string $\mathcal{L}_p$ inside $\mathbb{Z}_p$ that is fixed by the iterated function system of similarity contraction mappings from $\mathbb{Z}_p$ into itself:

$$f_{a+bp}(x) = a + bp + p^2x, \text{ for each } a + bp \notin S.$$ 

The lengths of the string $\mathcal{L}_p$ are $l_n = p^{-2n}$, with multiplicity $\mu_n = (p^2 - p)p^{n-1}$ for each positive integer $n$. Therefore, the geometric zeta function $\zeta_{\mathcal{L}_p}(s)$ is the meromorphic continuation to all of $\mathbb{C}$ of the following convergent geometric series of $\mathbb{L}_p$:

$$\sum_{n=1}^{\infty} \mu_n \cdot l_n^s = \sum_{n=1}^{\infty} \frac{(p^2 - p)p^{n-1}}{p^{2ns}} = \frac{p^2 - p}{p^{2s} - p} \text{ for } \Re(s) > \frac{1}{2};$$

that is to say,

$$\zeta_{\mathcal{L}_p}(s) = \frac{p - 1}{p^{2(s - \frac{1}{2})} - 1}, \text{ for all } s \in \mathbb{C}. $$

Since the complex dimensions of a fractal string $\mathcal{L}_p$ are defined as the poles of the geometric zeta function $\zeta_{\mathcal{L}_p}(s)$, the set of complex dimension of $\mathcal{L}_p$ is

$$\mathcal{D}_{\mathcal{L}_p} = \{D + inp \mid n \in \mathbb{Z}\},$$

where $D = \frac{1}{2}$ is the Minkowski dimension (also sometimes called the Minkowski–Bouligand dimension) of $\mathcal{L}_p$ and $p = \frac{\pi}{\log p}$ is its oscillatory period [25]. The residue of $\zeta_{\mathcal{L}_p}(s)$ at the midfractal dimension $D = \frac{1}{2}$ is

$$\text{res} (\zeta_{\mathcal{L}_p}(s); \frac{1}{2}) = \frac{p - 1}{2\log p},$$

and this is also the residue of $\zeta_{\mathcal{L}_p}(s)$ at every other complex dimension $\omega \in \mathcal{D}_{\mathcal{L}_p}$, by the periodicity of $\zeta_{\mathcal{L}_p}(s)$.

We thus have established the following result.

**Theorem 3.1.** For each $p \in \mathcal{P}$, there is a self-similar $p$-adic fractal string $\mathcal{L}_p$ of Minkowski dimension $D = \frac{1}{2}$ and with oscillatory period $p = \frac{\pi}{\log p}$.

**Corollary 3.2.** For each positive integer $v$ and any given prime number $p \in \mathcal{P}$, there is a $p$-adic fractal string $\mathcal{L}_p$ of Minkowski dimension $D = \frac{1}{2} \log_p v$. 
**Proof.** For $S$ of cardinality $v$, the geometric zeta function of the corresponding $p$-adic fractal string $\mathcal{L}_p$ is given by

$$
\zeta_{\mathcal{L}_p}(s) = \sum_{n=1}^{\infty} \frac{(p^2 - v)v^{n-1}}{p^{2ns}} = \frac{p^2 - v}{p^{2s} - v}, \quad \text{for } \Re(s) > \frac{\log v}{\log p^2}.
$$

Therefore, $\zeta_{\mathcal{L}_p}(s)$ admits a meromorphic continuation to all of $\mathbb{C}$, given by the same expression as above; hence,

$$
\mathcal{D}_{\mathcal{L}_p} = \left\{ \frac{1}{2} \log_p v + \frac{\pi i n}{\log p} \mid n \in \mathbb{Z} \right\}
$$

is the set of complex dimensions of $\mathcal{L}_p$, as desired. This completes the proof of the corollary. \hfill \Box

We can now generalize Theorem 3.1 in order to obtain the following theorem, which is the main result of this paper.

**Theorem 3.3.** For each $p \in \mathcal{P}$, and given any integers $k$ and $m$ satisfying $m \geq 1$ and $0 \leq k \leq m$, there is a self-similar $p$-adic fractal string $\mathcal{L}_p$ of rational Minkowski dimension $D = \frac{k}{m} \in [0, 1]$ and with oscillatory period $p = \frac{2\pi}{m \log p}$.

Furthermore, the geometric zeta function $\zeta_{\mathcal{L}_p}$ of $\mathcal{L}_p$ admits a meromorphic continuation to all of $\mathbb{C}$ given by (3.1) below and its residue at $D = \frac{k}{m}$ is given by (3.2).

**Proof.** For nonnegative integers $k \leq m$, with $m \neq 0$, an analogous construction to the one given in Theorem 3.1, but now with a subset $S$ of all $m$-tuples of digits with cardinality $p^k$, creates a $p$-adic fractal string $\mathcal{L}_p$ of dimension $D = \frac{k}{m}$ and with oscillatory period $p = \frac{2\pi}{m \log p}$. More specifically, $\mathcal{L}_p$ is a self-similar $p$-adic fractal string in $\mathbb{Z}_p$ that is fixed by the iterated function system of similarity contraction mappings from $\mathbb{Z}_p$ into itself:

$$
f_{a_M}(x) = a_0 + \cdots + a_{m-1}p^{m-1} + p^m x, \quad \text{for each } a_M := a_0 + \cdots + a_{m-1}p^{m-1} \notin S.
$$

Thus, the self-similar string $\mathcal{L}_p$ has $(p^m - p^k)p^k$ substrings of length $p^{-(n+1)m}$, for $n = 0, 1, 2, \ldots$. Therefore, its geometric zeta function is given by

$$
\zeta_{\mathcal{L}_p}(s) = \frac{p^m - p^k}{p^{ms} - p^k} = \frac{p^m(1-D) - 1}{p^{m(s-D)} - 1}, \quad \text{for all } s \in \mathbb{C}. \quad (3.1)
$$

Hence, the set of complex dimensions of $\mathcal{L}_p$ is given by $\mathcal{D}_{\mathcal{L}_p} = \{ D + ip \mid n \in \mathbb{Z} \}$, where $D = \frac{k}{m}$ is the Minkowski dimension of $\mathcal{L}_p$ and $p = \frac{2\pi}{m \log p}$ is its oscillatory period. The residue of $\zeta_{\mathcal{L}_p}(s)$ at the rational dimension $D = \frac{k}{m}$ is

$$
\text{res}(\zeta_{\mathcal{L}_p}(s); \frac{k}{m}) = \frac{p^{m-k} - 1}{\log p^m} = \frac{p^{m(1-D)} - 1}{\log p^m}, \quad (3.2)
$$

and this is also the residue of $\zeta_{\mathcal{L}_p}(s)$ at every other complex dimension $\omega \in \mathcal{D}_{\mathcal{L}_p}$ due to the periodicity of $\zeta_{\mathcal{L}_p}$. \hfill \Box

From this theorem and its proof, we immediately deduce the following corollary.

**Corollary 3.4.** For each rational number $D \in [0, 1]$ and any given prime number $p \in \mathcal{P}$, there is an explicitly constructed $p$-adic self-similar fractal string $\mathcal{L}_p$ of Minkowski dimension $D$. 

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$p$-ADIC NUMBERS, ULTRAMETRIC ANALYSIS AND APPLICATIONS  Vol. 13 No. 3 2021
Remark 3.5. In the archimedean world, a real fractal string is defined as a bounded open subset of the real line. Such a set has countably many connected components, and the lengths of the string are recovered as the lengths of the components. In the nonarchimedean world, this definition fails. The $p$-adic fractal string constructed above consists of infinitely many disjoint scaled copies of $\mathbb{Z}_p$ which cover a subset of $\mathbb{Z}_p$. Because of the totally disconnected nature of $\mathbb{Z}_p$, there are no boundary points where different lengths are separated from each other, and it becomes impossible to reconstruct the different components from their union. Instead, we simply list the sequence of lengths for a $p$-adic fractal string. However, see also [22, 23] where $p$-adic fractal strings, viewed as bounded open subsets of $\mathbb{Q}_p$, are also defined in terms of the lengths of the ‘$p$-adic convex components’.

Remark 3.6. Topologically, the ring of $p$-adic integers $\mathbb{Z}_p$ is zero-dimensional because it is totally disconnected. Metrically, on the other hand, it is one-dimensional, so that every $p$-adic fractal string $\mathcal{L}_p$ contained in $\mathbb{Z}_p$ has dimension at most 1. The Minkowski dimension of a fractal set is not a topological invariant, but instead, it gives combinatorial and metric information, namely, the number of balls of a given size required to cover the set. Likewise, the complex dimensions of a fractal string contain information about how much this number oscillates.

3.1. Adèlic Fractal String with Global Complex Dimension 1/2

According to Theorem 3.1, for each prime number $p \in \mathcal{P}$, there is a self-similar $p$-adic fractal string $\mathcal{L}_p$ of dimension $D = \frac{1}{2}$. Therefore, the infinite Cartesian product

$$\prod_{p \in \mathcal{P}} \mathcal{L}_p$$

is a self-similar adèlic fractal string $\mathcal{L}_{\mathbb{A}}$ in the set of finite adèles $\mathbb{A}_0$ because it is fixed by the (countably infinite) iterated function system

$$\Phi = \{ \phi_p \}_{p \in \mathcal{P}},$$

where $\phi_p(x) = f_{a + bp}(x) = a + bp + p^2x$, for each $a + bp \notin S$.

Since, for each $p \in \mathcal{P}$, the geometric zeta function of $\mathcal{L}_p$ is

$$\zeta_{\mathcal{L}_p}(s) = \frac{p - 1}{p^{2(s - \frac{1}{2})} - 1}, \text{ for all } s \in \mathbb{C}.$$ 

However, the formal infinite product of meromorphic functions

$$\zeta_{\mathcal{L}_{\mathbb{A}}}(s) = \prod_{p \in \mathcal{P}} \zeta_{\mathcal{L}_p}(s) = \prod_{p \in \mathcal{P}} \frac{p - 1}{p^{2(s - \frac{1}{2})} - 1}$$

converges only at $s = 1$, with value given by $\zeta_{\mathcal{L}_{\mathbb{A}}}(1) = 1$.

Remark 3.7. One of our long-term goals in this research program is to construct an adèlic fractal string whose geometric zeta function is the Riemann zeta function. We formulate here the following questions, which will be addressed in a later work, but for which we do not yet have definite answers. Is it possible to alter the above construction of $\mathcal{L}_{\mathbb{A}}$ so that its geometric zeta function converges for other values of $s$? What is a proper definition of the geometric zeta function for $\mathcal{L}_{\mathbb{A}}$? What is the dimension of $\mathcal{L}_{\mathbb{A}}$ and does $\mathcal{L}_{\mathbb{A}}$ have complex dimensions? Moreover, in order to study the connection with the Riemann hypothesis for the Riemann zeta function $\zeta(s)$, we ask: What is the spectrum of the vibrating adèlic fractal string $\mathcal{L}_{\mathbb{A}}$?

In addressing these questions, it may be useful to establish contact with the notion of a (vibrating) fractal membrane introduced by the first author in [19] (and viewed as a noncommutative space in the sense of Connes, [3]), along with the associated zeta function, viewed as the spectral partition function of the fractal membrane and with an Euler product and a Dirichlet series that both converge for $\text{Re}(s)$ greater than the dimension of the membrane.
4. $p$-ADIC CANTOR STRINGS

We describe a simple construction of an infinite family of $p$-adic Cantor strings $C_{S_p}$ in the nonarchimedean ring of $p$-adic integers $\mathbb{Z}_p$ and, simultaneously, of their archimedean counterparts in the real unit interval $[0, 1]$, the base-$p$ Cantor string $C_{S_p}^*$. The Minkowski dimensions of the nonarchimedean and archimedean Cantor strings vary from 0 to 1 as the prime number $p$ becomes larger. Directly above and below the dimension $D = D_p$ lie infinitely many complex dimensions, symmetrically located and periodically distributed along a discrete vertical line

$$l_D = \{\omega \in \mathbb{D}_{C_{S_p}} \mid \Re(\omega) = D\} = \mathbb{D}_{C_{S_p}}.$$  

As the prime number $p$ increases, the distribution of complex dimensions on each vertical line $l_D$ is discrete near the dimension $D = 0$, but becomes denser as the dimension tends to 1.

Lemma 4.1. For $p = 2$, there is a 2-adic fractal string $C_{S_2}$ of Minkowski dimension $D = 0$ and with oscillatory period $p = \frac{2\pi}{\log 2}$.

Proof. A simple way to create a 2-adic fractal string in the ring of 2-adic integers is to decompose $\mathbb{Z}_2$ into a disjoint union of two scaled copies of $\mathbb{Z}_2$ itself,

$$\mathbb{Z}_2 = 2\mathbb{Z}_2 \cup 1 + 2\mathbb{Z}_2$$

and then keeping the subinterval $1 + 2\mathbb{Z}_2$ as the first substring of the 2-adic fractal string $C_{S_2}$. We iterate the above process with the remaining subinterval $2\mathbb{Z}_2$, and then pursue the same construction, ad infinitum, in order to create the 2-adic Cantor string

$$C_{S_2} = 1 + 2\mathbb{Z}_2 \cup 2 + 4\mathbb{Z}_2 \cup 4 + 8\mathbb{Z}_2 \cup 8 + 16\mathbb{Z}_2 \cup \cdots.$$  

Hence, the geometric zeta function $\zeta_{C_{S_2}}$ of $C_{S_2}$ equals the meromorphic continuation of the following convergent geometric series to the entire complex plane $\mathbb{C}$:

$$\frac{1}{2^s} + \frac{1}{4^s} + \frac{1}{8^s} + \frac{1}{16^s} \cdots = \frac{1}{2^s - 1}, \quad \text{for } \Re(s) > 0;$$

that is to say,

$$\zeta_{C_{S_2}}(s) = \frac{1}{2^s - 1}, \quad \text{for all } s \in \mathbb{C}.$$  

Therefore, the set of complex dimensions for $C_{S_2}$ is obtained by solving the equation $2^s - 1 = 0$. The residue of $\zeta_{C_{S_2}}(s)$ at the least fractal dimension $D = 0$ is given by

$$\text{res}(\zeta_{C_{S_2}}; 0) = \frac{1}{\log 2},$$

and this is also the residue of $\zeta_{C_{S_2}}$ at every other complex dimension $\omega \in \mathbb{D}_{C_{S_2}}$ by the periodicity of $\zeta_{C_{S_2}}$. \hfill $\Box$

Remark 4.2. The 2-adic Cantor string $C_{S_2}$ is the topological complement of the 2-adic Cantor set $C_2$ in the ring of 2-adic integers $\mathbb{Z}_2$, see §4.4.2. The 2-adic Cantor set $C_2$ is the unique nonempty compact subset of $\mathbb{Z}_2$ that is invariant under the transformation of the iterated function system $\Phi = \{\phi\}: C_2 = \phi(C_2)$, where $\phi(x) = 2x$ is a similarity contraction mapping of $\mathbb{Z}_2$ into itself. So, technically, $C_2$ is not self-similar because the iterated function system $\Phi = \{\phi\}$ has only one map; see, e.g., [5].

Theorem 4.3. For each prime number $p > 2$, there is a self-similar $p$-adic fractal string $C_{S_p}$ of Minkowski dimension $D = \frac{\log(1-p)}{\log p}$ and with oscillatory period $p = \frac{2\pi}{\log p}$.

Furthermore, the geometric zeta function $\zeta_{C_{S_p}}$ of $C_{S_p}$ admits a meromorphic continuation to all of $\mathbb{C}$ given by (4.1) below and its residue at $D$ is given by (4.2) below.
**Proof.** Reminiscent of Smith’s construction of the general Cantor set, we create a \( p \)-adic fractal string in the ring of \( p \)-adic integers \( \mathbb{Z}_p \) by decomposing \( \mathbb{Z}_p \) into the disjoint union

\[
p\mathbb{Z}_p \cup 1 + p\mathbb{Z}_p \cup \ldots \cup (p - 1) + p\mathbb{Z}_p
\]

and then keeping the subintervals

\[
1 + p\mathbb{Z}_p, 3 + p\mathbb{Z}_p, \ldots, (p - 2) + p\mathbb{Z}_p
\]

as the first generation of sub-intervals of the \( p \)-adic fractal string \( \mathcal{CS}_p \). By iterating this process with the remaining sub-intervals

\[
p\mathbb{Z}_p, 2 + p\mathbb{Z}_p, \ldots, (p - 1) + p\mathbb{Z}_p,
\]

we obtain the \( p \)-adic Cantor string

\[
\mathcal{CS}_p = \bigcup_{j=1}^{\frac{p-1}{2}} 2j - 1 + p\mathbb{Z}_p \cup \bigcup_{j=1}^{\frac{p-1}{2}} 2j - 1 + p^2\mathbb{Z}_p \cup \cdots.
\]

Thus, we get \( \mu_n = \frac{(p-1)(p+1)^{n-1}}{2} \) sub-intervals of length \( l_n = p^{-n} \), for each positive integer \( n \). Hence, the geometric zeta function \( \zeta_{\mathcal{CS}_p}(s) \) of the \( p \)-adic Cantor string \( \mathcal{CS}_p \) coincides with the meromorphic continuation to the whole complex plane \( \mathbb{C} \) of the following convergent geometric series:

\[
\sum_{n=1}^{\infty} \mu_n \cdot l_n^s = \frac{p-1}{p+1} \sum_{n=1}^{\infty} \left( \frac{1+p}{2p^s} \right)^n = \frac{p-1}{2p^s - p - 1}, \quad \text{for} \ \Re(s) > \frac{\log(1+p)}{\log p};
\]

that is to say,

\[
\zeta_{\mathcal{CS}_p}(s) = \frac{p-1}{2p^s - p - 1}, \quad \text{for all} \ s \in \mathbb{C}.
\]

(4.1)

Therefore, the set of complex dimensions of the \( p \)-adic Cantor string \( \mathcal{CS}_p \) is given by

\[
\mathcal{D}_{\mathcal{CS}_p} = \{ D + inp \mid n \in \mathbb{Z} \},
\]

where \( D = \frac{\log(1+p)}{\log p} \) is the Minkowski dimension of \( \mathcal{CS}_p \) and \( \mathbf{p} = \frac{2\pi}{\log p} \) is its oscillatory period. The residue of \( \zeta_{\mathcal{CS}_p} \) at the fractal dimension \( D = \frac{\log(1+p)}{\log p} \) is given by

\[
\text{res}(\zeta_{\mathcal{CS}_p}(s); D) = \frac{p-1}{(p+1)\log p}
\]

(4.2)

and this is also the residue of \( \zeta_{\mathcal{CS}_p} \) at every other complex dimension \( \omega \in \mathcal{D}_{\mathcal{CS}_p} \), by the periodicity of \( \zeta_{\mathcal{CS}_p} \); see (4.1) above.

\[\square\]

**Remark 4.4.** Since the limit of \( D = \frac{\log(1+p)}{\log p} \) is 1 and the limit of \( \mathbf{p} = \frac{2\pi}{\log p} \) is 0, as \( p \to \infty \), the periodic distribution of complex dimensions on the vertical line \( l_D \) is discrete near the dimension \( D = 0 \), but becomes denser as the dimension \( D \) tends to 1, as \( p \to \infty \).

**Remark 4.5.** We recover the 3-adic Cantor string in [21] when we choose \( p = 3 \). When \( p = 5 \), we recover the 5-adic Cantor string in [14] (building on [21]), which motivated our construction of the \( p \)-adic Cantor strings as well as their archimedean counterparts in §4.4.2.

### 4.1. Geometric Waves

Nonreal complex dimensions are the source of oscillations in the geometry of a fractal string. The real parts of the complex dimensions correspond to the amplitudes of geometric waves propagating through the ‘space of scales’ that lies beneath the surface of a fractal string and the imaginary parts of the complex dimensions correspond to the frequencies of those geometric waves. The lengths out of which a fractal string is composed of can be thought of as being the underlying scales of the system; see [20].
For each complex dimension \( \omega = D + i n p \in \mathbb{D}_{CS_p} \), the residue of \( \zeta_{CS_p}; \omega \) is given by \( \text{res}(\zeta_{CS_p}; \omega) = \frac{p - 1}{(p + 1)^2} \). Therefore, for any sufficiently small \( \varepsilon > 0 \), the volume of the inner \( \varepsilon \)-neighborhoods of the \( p \)-adic Cantor string \( CS_p \) is given by the explicit fractal tube formula

\[
V_{CS_p}(\varepsilon) = \frac{p - 1}{(p + p^2) \log p} \times \varepsilon^{1-D} \sum_{n=1}^{\infty} \frac{\cos(np \log \varepsilon) - i \sin(np \log \varepsilon)}{1 - D - i np},
\]

which follows from the exact fractal tube formula for any self-similar \( p \)-adic fractal string \( L_p \) with simple complex dimensions (see [24, 25]):

\[
V_{L_p}(\varepsilon) = \sum_{\omega \in \mathbb{D}_{L_p}} \frac{\text{res}(\zeta_{L_p}; \omega)}{p} \times \frac{\varepsilon^{1-\omega}}{1 - \omega}.
\]

Consequently, every \( p \)-adic Cantor string has logarithmic oscillations of order \( D \) in its geometry because the limit of \( \frac{V_{CS_p}(\varepsilon)}{\varepsilon^{1-D}} \) does not exist in \((0, +\infty)\), as \( \varepsilon \to 0^+ \). Therefore, \( p \)-adic Cantor strings are not Minkowski measurable and hence, their Minkowski contents do not exist. However, the average Minkowski content, defined as a Cesàro logarithmic average, exists for each \( p \)-adic Cantor string \( CS_p \) and is given by

\[
M_{av}(CS_p) = \frac{p - 1}{(p + p^2) \log(\frac{2p}{p + 1})},
\]

while \( M_{av}(L_p) \) is defined by the formula

\[
M_{av}(L_p) := \lim_{T \to \infty} \frac{1}{\log T} \int_{T^{-1}}^{1} \frac{V_{L_p}(\varepsilon) \, d\varepsilon}{\varepsilon^{1-D} - \varepsilon} = \frac{1}{p(1 - D)} \text{res}(\zeta_{L_p}; D),
\]

which is valid for any self-similar \( p \)-adic fractal string \( L_p \) (see [24]).

### 4.1.1. \( V_{L_p}(\varepsilon) \) associated with a geometric wave.

For each prime \( p \in \mathbb{P} \) and a sufficiently small \( \varepsilon > 0 \), the volume \( V_{L_p}(\varepsilon) \) of the inner \( \varepsilon \)-neighborhoods of the \( p \)-adic fractal string \( L_p \) of rational dimension \( D = \frac{k}{m} \) and with oscillatory period \( p = \frac{2\pi}{m \log p} \) (constructed in Theorem 3.3 above) is given by the following exact explicit fractal tube formula (see [24, 25]),

\[
V_{L_p}(\varepsilon) = \frac{p^{m(1-D)} - 1}{p \log p^m} \times \varepsilon^{1-D} \sum_{n=1}^{\infty} \frac{\cos(np \log \varepsilon) - i \sin(np \log \varepsilon)}{1 - D - i np},
\]

since the residue of \( \zeta_{L_p} \) at every complex dimension \( \omega = D + i np \in \mathbb{D}_{L_p} \) is given by

\[
\text{res}(\zeta_{L_p}(s); \omega) = \frac{p^{m(1-D)} - 1}{\log p^m}.
\]

Consequently, the \( p \)-adic fractal string \( L_p \) is not Minkowski measurable. However, its average Minkowski content exists and is given by

\[
M_{av}(L_p) = \frac{p^{m(1-D)} - 1}{p \log p^m(1-D)}.
\]

The notion of Minkowski content is a generalization of the notion of volume of a smooth manifold in \( N \)-dimensional Euclidean space to an arbitrary bounded subset of \( \mathbb{R}^N \) of any fractal dimension \( D \in [0, N] \). Intuitively, the \( D \)-dimensional Minkowski content of a fractal set of Minkowski dimension \( D \) is its \( D \)-dimensional fractal volume. Motivated in parts by some of the main results of [29] and [18], Alain Connes showed that the Minkowski content is a natural analogue of the volume of a compact smooth Riemannian spin manifold for a fractal set in the case of the ordinary Cantor set or string [3]. Benoit Mandelbrot proposed the Minkowski content as a measure of fractal lacunarity because the value of the Minkowski content allows one to compare the lacunarity of fractal sets of the same Minkowski dimension [35]. (See also [13, 29–32, 34].)
On the other hand, Minkowski measurability is a kind of ‘fractal regularity’ for the underlying geometry. The existence of nonreal complex dimensions of a fractal string $L$, along with the simplicity of the dimension $D$ as a pole of its geometric zeta function $\zeta_L(s)$, determines the Minkowski nonmeasurability of $L$ [30–32]; in the Euclidean case, this result has been extended to higher-dimensional fractals in [34].

4.2. $p$-Adic Cantor Sets and Their Archimedean Counterparts

Fix a prime $p$. The $p$-adic Cantor string $C^*_p$ is the set-theoretic complement of the self-similar $p$-adic Cantor set $C_p$ in the ring of $p$-adic integers $\mathbb{Z}_p$. The archimedean counterpart, in the archimedean field of real numbers $\mathbb{R}$, of the nonarchimedean $p$-adic Cantor set $C_p$ is the base-$p$ Cantor set $C^*_p$. The complement of the base-$p$ Cantor set $C^*_p$ in the real unit interval $[0,1]$ is the base-$p$ Cantor string $C^*_p$.

**Lemma 4.6.** For each prime $p > 2$, the $p$-adic Cantor set $C_p$ is a self-similar set in $\mathbb{Z}_p$; that is to say, $C_p$ is the unique nonempty invariant compact set of an iterated function system of $(p+1)/2$ similarity contraction mappings from $\mathbb{Z}_p$ into itself:

$$C_p = \Phi_p(C_p) = \bigcup_{k=1}^{p+1} \phi_k(C_p),$$

where $\Phi_p = \{\phi_1, \phi_2, \ldots, \phi_{p+1}\} = \{0 + px, 2 + px, \ldots, p - 1 + px\}$.

**Proof.** This follows from Part (i) of Theorem 4.2 in [22].

**Lemma 4.7.** For each $p > 2$, the $p$-adic Cantor set $C_p$ is a subset of the $p$-adic integers $\mathbb{Z}_p$ whose elements only contain even digits in their $p$-adic expansions:

$$C_p = \{a_0 + a_1 p + a_2 p^2 + a_3 p^3 + \cdots \in \mathbb{Z}_p | a_j \in \{0, 2, 4, \ldots, p - 1\}, \text{ for all } j \geq 0\}.$$  

**Proof.** Since we remove all the odd digits at every stage in the construction of the $p$-adic Cantor set $C_p$, the elements of $C_p$ must consist only of even digits in their $p$-adic expansions; see [14, 21].

Let $C^*_p$ be a subset of the unit interval $[0,1]$ on the real line whose elements only contain even digits in their base-$p$ expansion:

$$C^*_p := \{a_0 + a_1 p^{-1} + a_2 p^{-2} + a_3 p^{-3} + \cdots | a_j \in \{0, 2, 4, \ldots, p - 1\}, \text{ for all } j \geq 0\}.$$

Then, the base-$p$ Cantor set $C^*_p$ is homeomorphic to the $p$-adic Cantor set $C_p$ via the continuous map

$$\sum_{i=0}^{\infty} a_i \cdot p^{-i} \mapsto \sum_{i=0}^{\infty} a_i \cdot p^i$$

that sends each element of the compact set $C^*_p$ in the complete metric space $(\mathbb{R}, | \cdot |_{\infty})$ to an element of the compact set $C_p$ in the complete ultrametric space $(\mathbb{Q}_p, | \cdot |_p)$. Therefore, the base-$p$ Cantor set $C^*_p$ can be considered as a natural archimedean counterpart of the nonarchimedean $p$-adic Cantor set $C_p$.

The complement of the base-$p$ Cantor set $C^*_p$ in the real unit interval $[0,1]$ is an archimedean fractal string $C^*_p$, called the base-$p$ Cantor string. Hence, the base-$p$ Cantor string $C^*_p$ is the subset of the unit interval $[0,1]$ whose elements must contain at least one odd digit in their base-$p$ expansions. In the same way, the complement of the $p$-adic Cantor set $C^*_p$ in the ring of $p$-adic integers $\mathbb{Z}_p$ is the $p$-adic Cantor string $C^*_p$. Therefore, every element of the $p$-adic Cantor string $C^*_p$ contains at least one odd digit in its $p$-adic expansion.

The base-$p$ Cantor string $C^*_p$ and the $p$-adic Cantor string $C^*_p$ both have the same sequence of lengths with respect to each respective metric. It follows that the base-$p$ Cantor string $C^*_p$ can be considered as a natural archimedean counterpart of the nonarchimedean $p$-adic Cantor string $C^*_p$. Therefore, we consider the Cartesian product $C^*_p \times C^*_p$ as a natural symmetric fractal string in the archimedean-nonarchimedean space $\mathbb{R} \times \mathbb{Q}_p$. 

\[ p\text{-ADIC NUMBERS, ULTRAMETRIC ANALYSIS AND APPLICATIONS} \quad \text{Vol. 13 No. 3 2021} \]
4.3. Adèlic Cantor String $\mathcal{CS}_{\mathbb{A}_0}$

Let $\mathcal{P}$ denote the set of rational primes. For each $p \in \mathcal{P}$, let $\mathcal{CS}_p$ be the $p$-adic Cantor string. Then, an infinite Cartesian product of every $p$-adic Cantor string is an adèlic fractal string $\mathcal{CS}_{\mathbb{A}_0}$ in the set of finite adèles $\mathbb{A}_0$:

$$\mathcal{CS}_{\mathbb{A}_0} = \prod_{p \in \mathcal{P}} \mathcal{CS}_p.$$ 

**Theorem 4.8.** The adèlic Cantor string $\mathcal{CS}_{\mathbb{A}_0}$ is a self-similar string in $\mathbb{A}_0$, with respect to the countably infinite iterated function system $\Phi = \{\Phi_p\}_{p \in \mathcal{P}}$ of $\mathbb{A}_0$, as constructed in the proof.

**Proof.** Let $\Phi = \{\Phi_2, \Phi_3, \Phi_5, \ldots\}$ be an iterated function system in the set of finite adèles $\mathbb{A}_0$, where each $\Phi_p = \{px, 2 + px, \ldots, p - 1 + px\}$ is an iterated function system (IFS) of similarity contraction mappings in $\mathbb{Z}_p$ that acts trivially on every other components of $\mathbb{A}_0$. Moreover, let $\mathcal{C}_{\mathbb{A}_0} = \mathcal{C}_2 \times \mathcal{C}_3 \times \mathcal{C}_5 \times \ldots$ be the adèlic Cantor set in $\mathbb{A}_0$. Then, it is easy to check that $\Phi$ is a contraction with respect to the Hausdorff metric induced by the standard norm $\prod_{p = 2}^{\infty} |\cdot|_p$ on $\mathbb{A}_0$, and $\Phi(\mathcal{C}_{\mathbb{A}_0}) = \mathcal{C}_{\mathbb{A}_0}$ (which is the natural definition of a self-similar fractal) since

$$\Phi(\mathcal{C}_{\mathbb{A}_0}) = \Phi_2(\mathcal{C}_2) \times \Phi_3(\mathcal{C}_3) \times \Phi_5(\mathcal{C}_5) \times \cdots = \mathcal{C}_2 \times \mathcal{C}_3 \times \mathcal{C}_5 \times \cdots = \mathcal{C}_{\mathbb{A}_0}.$$ 

Thus, the adèlic Cantor set $\mathcal{C}_{\mathbb{A}_0}$ is self-similar, with respect to the countably infinite IFS $\Phi = \{\Phi_p\}_{p \in \mathcal{P}}$ constructed above.

4.4. Adèlic Cantor-Smith String $\mathcal{CS}_\mathbb{A}$

For each positive integer $m > 2$, let $\mathcal{C}_m$ be the general Cantor set as constructed by Smith in 1875 [39]. Then, the complement of $\mathcal{C}_m$ in $[0, 1]$ is an ordinary fractal string $\mathcal{CS}_m$ consisting of all the deleted segments in the construction of $\mathcal{C}_m$. Thus, $\mathcal{CS}_m$ can be represented as a sequence of lengths $l_n = m^{-n}$ with multiplicity $\mu_n = (m - 1)^{n-1}$, for each positive integer $n$. Therefore, the geometric zeta function $\zeta_{\mathcal{CS}_m}$ of $\mathcal{CS}_m$ is given by

$$\zeta_{\mathcal{CS}_m}(s) = \frac{1}{m^s + 1 - m}, \text{ for all } s \in \mathbb{C}.$$ 

When $m = 3$, $\mathcal{CS}_3$ is the ordinary Cantor string $\mathcal{CS}$, as constructed and studied by the first author and C. Pomerance in [16, 17, 29] and then significantly expounded upon (by the first and third authors) in [30–32].

The Cartesian product of $\mathcal{CS}_m$ and the adèlic Cantor string $\mathcal{CS}_{\mathbb{A}_0}$ is a self-similar adèlic string $\mathcal{CS}_\mathbb{A}$ in the ring of adèles $\mathbb{A}$. We call it the adèlic Cantor-Smith string.

The formal infinite product of meromorphic functions

$$\zeta_{\mathcal{CS}_\mathbb{A}}(s) = \zeta_{\mathcal{CS}_m}(s) \cdot \zeta_{\mathcal{CS}_{\mathbb{A}_0}}(s) = \frac{1}{m^s + 1 - m} \cdot \frac{1}{2^s - 1} \prod_{p > 2} \frac{p - 1}{2p^s - p - 1}$$

could be a geometric zeta function $\zeta_{\mathcal{CS}_\mathbb{A}}$ for the adèlic Cantor-Smith string $\mathcal{CS}_\mathbb{A}$. However, this product converges only for $s = 1$, with $\zeta_{\mathcal{CS}_\mathbb{A}}(1) = 1$. We could therefore ask the very same questions in the present context about the adèlic Cantor-Smith string as those asked in Remark 3.7 about the adèlic fractal string $\mathcal{L}_\frac{1}{2}$. 

**p-ADIC NUMBERS, ULTRAMETRIC ANALYSIS AND APPLICATIONS** Vol. 13 No. 3 2021
5. ADÉLIC EULER STRING $\varepsilon_{A_0}$

For each $p \in \mathcal{P}$, let $\varepsilon_p = \bigcup_{n=0}^{\infty} (a_n + p^n \mathbb{Z}_p)$ be the $p$-adic Euler string with geometric zeta function

$$\zeta_{\varepsilon_p}(s) = \frac{1}{1 - \frac{1}{p^s}}, \text{ for all } s \in \mathbb{C}, \quad (5.1)$$

as constructed by the authors in [25, Section 2.3] via the infinite multiplicative convolution product of (local) measures, as in [32, Chapter 4, esp. Section 4.2]. Then, the infinite Cartesian product

$$\prod_{p \in \mathcal{P}} \varepsilon_p$$

can be considered as an adèlic fractal string $\varepsilon_{A_0}$ in the set of finite adèles $\mathbb{A}_0$. We call $\varepsilon_{A_0}$ the adèlic Euler string. The adèlic Euler string $\varepsilon_{A_0} = \prod_{p} \varepsilon_p$ is self-similar since every $\varepsilon_p$ is self-similar of Minkowski dimension zero.

Let $\zeta_{\varepsilon_p}(s)$ be the geometric zeta function of the $p$-adic Euler string $\varepsilon_p$, as given by (5.1) above. Then the infinite product of complex meromorphic functions

$$\prod_{p \in \mathcal{P}} \zeta_{\varepsilon_p}(s) = \prod_{p \in \mathcal{P}} \frac{1}{1 - \frac{1}{p^s}} = \sum_{n \in \mathbb{N}} \frac{1}{n^s}$$

is the Riemann zeta function $\zeta(s)$. Therefore, we may consider the Riemann zeta function $\zeta(s)$ as the geometric zeta function $\zeta_{\varepsilon_{A_0}}(s)$ of the adèlic Euler string $\varepsilon_{A_0}$; that is to say:

$$\zeta_{\varepsilon_{A_0}}(s) = \zeta(s), \text{ for all } s \in \mathbb{C}.$$ 

With this in mind, we may also consider the only simple pole of the Riemann zeta function $\zeta(s)$ at $s = 1$ as the global Minkowski dimension of the adèlic Euler string $\varepsilon_{A_0}$.

**Remark 5.1.** As is explained in Remark 3.5, the main information is the sequence of lengths (or volumes), $1, p^{-1}, p^{-2}, \ldots$, of the Euler string. In [25, Section 2.3], the authors chose the centers $a_0 = 0$, and $a_n = p^{-1} + 1 + \cdots + p^{-n-2}$ for $n \geq 1$.

5.1. Adèlic Euler-Riemann String $\varepsilon_{A}$

Let $\varepsilon_{A_0} = \prod_p \varepsilon_p$ be the adèlic Euler string and let the local positive measure $h = \sum_{n \in \mathbb{N}} \delta_{\{n\}}$ be the harmonic string (as in [30–32]). Then the Cartesian product

$$\varepsilon_{A} = h \times \varepsilon_{A_0}$$

can be considered as an adèlic fractal string in the ring of adèles $\mathbb{A}$. Here, $h$ is a fractal string in the sense of measures, as in [32, Section 4.2, (4.19)]. We call it the adèlic Euler-Riemann string.

Let $\zeta_{\varepsilon_{A_0}}(s)$ be the geometric zeta function of the adèlic Euler string $\varepsilon_{A_0}$ and $\zeta_h(s)$ be the geometric zeta function of the harmonic string $h$. Then, their product

$$\zeta_h(s) \cdot \zeta_{\varepsilon_{A_0}}(s) = \zeta(s) \cdot \zeta(s)$$

is the square of the Riemann zeta function, which could be viewed as the geometric zeta function $\zeta_{\varepsilon_{A}}(s)$ for the adèlic Euler-Riemann string $\varepsilon_{A}$.
6. EPILOGUE

In *Number Theory as the Ultimate Physical Theory*, Igor Volovich has suggested that $p$-adic numbers can possibly be used to describe the geometry of spacetime at very high energies, and hence, very small scales, because measurements in the ‘archimedean’ geometry of spacetime at fine scales have no certainty [44]. Furthermore, Stephen Hawking and other authors have also suggested that the fine scale structure of spacetime may be fractal [8, 10, 36, 45]. Moreover, in [19], the first author has suggested that fractal strings and fractal membranes may be related to T-duality in string theory and the functional equation for the Riemann zeta function $\zeta(s)$, and that the $p$-adic and adèle analogues of these notions may be helpful for understanding the underlying noncommutative spacetimes and their moduli spaces introduced in [19].

The (yet to be rigorously developed) global theory of complex dimensions for adèle fractal strings would provide a natural and unified framework for understanding the vibrations and resonances in the geometry and the spectrum of adèle fractal strings as well as the pole and zeros of the Riemann zeta function $\zeta(s)$. The theory would also precisely describe the oscillatory nature intrinsic to the geometry of adèle fractal strings as geometric waves propagating through the space of scales that lies beneath the surface of the adèle fractal strings. Moreover, it might shed light on the Riemann hypothesis for the Riemann zeta function $\zeta(s)$ via the inverse spectral problems for adèle fractal strings [11, 17, 19, 26, 30–32, 41, 42].

We leave the possible rigorous construction and development of such a theory to future work of the authors and to the interested reader.

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REFERENCES

1. C. Andreae and L. Franke, *Georg Cantor: Das erste Diagonalverfahren*, Oberstufe Mathematik-Projekt Unendlichkeit, www.rudolf-web.de (2004).
2. G. F. L. P. Cantor, “Über unendliche, lineare Punktmannichfaltigkeiten, Part 5,” Math. Ann. 21, 545–591 (1883).
3. A. Connes, *Noncommutative Geometry* (Academic Press, San Diego, 1994).
4. M. du Sautoy, *The Music of the Primes* (Harper Collins, New York, 2003).
5. K. J. Falconer, *Fractal Geometry: Mathematical Foundations and Applications*, third edition (John Wiley & Sons, Chichester, 2014).
6. J. F. Fleron, “A note on the history of the Cantor set and Cantor function,” Math. Magaz. 67 (2), 136–140 (1994).
7. P. Freund and E. Witten, “Adelic string amplitudes,” Phys. Lett. B 199, 191 (1987).
8. G. W. Gibbons and S. W. Hawking, (eds.), *Euclidean Quantum Gravity* (World Scientific Publ., Singapore, 1993).
9. R. Harvey and J. Polking, “Removable singularities of solutions of linear partial differential equations,” Acta Math. 125, 39–56 (1970).
10. S. W. Hawking and W. Israel, (eds.), *General Relativity: An Einstein Centenary Survey* (Cambridge Univ. Press, Cambridge, 1979).
11. H. Herichi and M. L. Lapidus, *Quantized Number Theory, Fractal Strings and the Riemann Hypothesis: From Spectral Operators to Phase Transitions and Universality* (World Scientific Publishing, Singapore, 2021).
12. A. E. Ingham, *The Distribution of Prime Numbers* (Cambridge Univ. Press, Cambridge, 1932).
13. M. Kesseböhmer and S. Kombrink, “Fractal curvature measures and Minkowski content for self-conformal subsets of the real line,” Adv. Math. 230, 2474–2512 (2012).
14. A. Kumar, M. Rani and R. Chugh, “New 5-adic Cantor sets and fractal string,” SpringerPlus 2, 654 (2013).
15. M. L. Lapidus, “Fractal drum, inverse spectral problems for elliptic operators and a partial resolution of the Weyl–Berry conjecture,” Trans. Amer. Math. Soc. 325, 465–529 (1991).
16. M. L. Lapidus, “Spectral and Fractal Geometry: From the Weyl-Berry conjecture for the vibrations of fractal drums to the Riemann zeta-function,” in: Differential Equations and Mathematical Physics, Proc. Fourth UAB Internat. Conf., Birmingham, USA, March 1990 (Bennewitz, C., ed.), pp. 151–182 (Academic Press, New York, 1992).

17. M. L. Lapidus, “Vibrations of fractal drums, the Riemann hypothesis, waves in fractal media, and the Weyl-Berry conjecture,” in: Ordinary and Partial Differential Equations, vol. IV, Proc. 12th Internat. Conf., Dundee, Scotland, UK, June 1992 (Sleeman, B.D., Jarvis, R.J., eds.), Pitman Research Notes in Mathematics Series 289, pp. 126–209 (Longman Sci. and Tech., London, 1993).

18. M. L. Lapidus, “Analysis on fractals, Laplacian on self-similar sets, noncommutative geometry and spectral dimensions,” Topol. Meth. Nonlin. Anal. 4 (1), 137–195 (1994).

19. M. L. Lapidus, In Search of the Riemann Zeros: Strings, Fractal Membranes and Noncommutative Spacetimes (Amer. Math. Soc., Providence, R.I., 2008).

20. M. L. Lapidus, “An overview of complex fractal dimensions: From fractal strings to fractal drums, and back,” in: Horizons of Fractal Geometry and Complex Dimensions (R. G. Niemeyer, E. P. J. Pearse, J. A. Rock, T. Samuel, eds.), Contemp. Math. 731, 143–265 (Amer. Math. Soc., Providence, R.I., 2019).

21. M. L. Lapidus and H. Lü, “Nonarchimedean Cantor set and string,” J. Fixed Point Theory Appl. 3, 181–190 (2008).

22. M. L. Lapidus and H. Lü, “Self-similar $p$-adic fractal strings and their complex dimensions,” $p$-Adic Num. Ultrametr. Anal. Appl. 1, 167–180 (2009).

23. M. L. Lapidus and H. Lü, “The geometry of $p$-adic fractal strings: A comparative survey,” in: Advances in Nonarchimedean Analysis, Proc. 11th Internat. Conf. on $p$-Adic Functional Analysis (Clermont-Ferrand, France, July 2010), (J. Araujo, B. Diarra, A. Escassut, eds.), Contemp. Math. 551, 163–206 (Amer. Math. Soc., Providence, R.I., 2011).

24. M. L. Lapidus, H. Lü and M. van Frankenhuysen, “Minkowski measurability and exact fractal tube formulas for $p$-adic self-similar strings,” in: Fractal Geometry and Dynamical Systems in Pure Mathematics I: Fractals in Pure Mathematics (D. Carfi, M. L. Lapidus, E. P. J. Pearse, M. van Frankenhuysen, eds.), Contemp. Math. 600, 161–184 (Amer. Math. Soc., Providence, R.I., 2013).

25. M. L. Lapidus, H. Lü and M. van Frankenhuysen, “Minkowski dimension and explicit tube formulas for $p$-adic fractal strings,” Fractal Fract. 2, 26th paper, 1–30 (2018).

26. M. L. Lapidus and M. van Frankenhuysen, “The Riemann hypothesis and inverse spectral problems for fractal strings,” J. London Math. Soc. 52 (2), 15–34 (1995).

27. M. L. Lapidus and E. P. J. Pearse, “A tube formula for the Koch snowflake curve, with applications to complex dimensions,” J. London Math. Soc. 74 (2), 397–414 (2006).

28. M. L. Lapidus, E. P. J. Pearse and S. Winter, “Pointwise tube formulas for fractal sprays and self-similar tilings with arbitrary generators,” Adv. Math. 227, 1349–1398 (2011).

29. M. L. Lapidus and C. Pomerance, “The Riemann zeta function and the one-dimensional Weyl-Berry conjecture for fractal drums,” Proc. London Math. Soc. 66 (3) No. 1, 41–69 (1993).

30. M. L. Lapidus and M. van Frankenhuysen, Fractal Geometry and Number Theory: Complex Dimensions of Fractal Strings and Zeros of Zeta Functions (Birkhäuser, Boston, 2000).

31. M. L. Lapidus and M. van Frankenhuysen, Fractal Geometry, Complex Dimensions and Zeta Functions: Geometry and Spectra of Fractal Strings, Springer Monographs in Math. (Springer, New York, 2006).

32. M. L. Lapidus and M. van Frankenhuysen, Fractal Geometry, Complex Dimensions and Zeta Functions: Geometry and Spectra of Fractal Strings, Springer Monographs in Math. (Springer, New York, 2013).

33. M. L. Lapidus, A Panorama of Number Theory: From Euler and Riemann to Weil, and Beyond, book in preparation (for Springer/Birkhäuser, 2021).

34. M. L. Lapidus, G. Radunović and D. Žubrinić, Fractal Zeta Functions and Fractal Drums: Higher-Dimensional Theory of Complex Dimensions, Springer Monographs in Math. (Springer, New York, 2017).

35. B. B. Mandelbrot, “Measures of fractal lacunarity: Minkowski content and alternatives,” in: Fractal Geometry and Stochastics, Progress in Probability 37, (C. Bandt, S. Graf, M. Zähle, eds.) (Birkhäuser, Basel, 1995).

36. L. Nottale, Fractal Spacetime and Microphysics: Towards a Theory of Scale Relativity (World Scientific Publishing, Singapore, 1993).

37. A. Ostrowski, “Über einige Lösungen der Funktionalgleichung $\varphi(x) \cdot \varphi(y) = \varphi(xy)$,” Acta Math. (2nd ed.), 41 (1), 271–284 (1916).

38. G. F. B. Riemann, “Über die Anzahl der Primzahlen unter einer gegebenen Größe,” English translation in Riemann’s Zeta Function (H. M. Edwards, Dover edition, 2001).

39. H. J. S. Smith, “On the integration of discontinuous functions,” Proc. London Math. Soc. 6 (1), 140–153 (1875).

40. J. Tate, “Fourier analysis in number fields and Hecke’s zeta functions,” in: Algebraic Number Theory, 305–347 (Cassels, J.W.S., Fröhlich, A., eds.) (Academic Press, New York, 1967).
41. M. van Frankenhuijsen, *The Riemann Hypothesis for Function Fields*, London Math. Soc., Student Texts 80 (Cambridge Univ. Press, Cambridge, 2014).
42. M. van Frankenhuijsen, “The spectral operator and resonances,” in: Horizons of Fractal Geometry and Complex Dimensions (R. G. Niemeyer, E. P. J. Pearse, J. A. Rock, T. Samuel, eds.), Contemp. Math. 731, 115–131 2019.
43. V. S. Vladimirov, I. V. Volovich and E. I. Zelenov, *p-Adic Analysis and Mathematical Physics* (World Scientific Publishing, Singapore, 1994).
44. I. V. Volovich, “Number theory as the ultimate physical theory,” CERN-TH.4781/87, July 1987. Published in: *p-Adic Num. Ultrametr. Anal. Appl.* 2 (1), 77–87 (2010).
45. J. A. Wheeler and K. W. Ford, *Geons, Black Holes, and Quantum Foam: A Life in Physics* (Norton, W.W., New York, 1998).