Hadronic decays of the spin-singlet heavy quarkonium under the principle of maximum conformality

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The principle of maximum conformality (PMC) provides a way to eliminate the conventional renormalization scale ambiguity in a systematic way. By applying PMC, all non-conformal terms in perturbative series are summed into the running coupling, and one obtains a unique, scale-fixed prediction at any finite order. In the paper, we make a detailed PMC analysis for the spin-singlet heavy quarkoniums decay (into light hadrons) at the next-to-leading order. After the PMC scale setting, the decay widths for all those cases are almost independent of the initial renormalization scales. The PMC scales for $\eta_c$ and $h_c$ decays are below 1 GeV, in order to achieve a confidential pQCD estimation, we adopt several low-energy running coupling models to do the estimation. By taking the MPT model, we obtain: $\Gamma(\eta_c \to LH) = 25.09^{+5.52}_{-4.28}$ MeV, $\Gamma(\eta_b \to LH) = 14.34^{+0.92}_{-0.84}$ MeV, $\Gamma(h_c \to LH) = 0.54^{+0.04}_{-0.06}$ MeV and $\Gamma(h_b \to LH) = 39.89^{+0.28}_{-0.46}$ KeV, where the errors are calculated by taking $m_c \in [1.40\text{GeV}, 1.60\text{GeV}]$ and $m_b \in [4.50\text{GeV}, 4.70\text{GeV}]$. These decay widths agree with the principle of minimum sensitivity estimations, in which the decays of $\eta_c, b$ are also consistent with the measured ones.

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Heavy quarkonium plays an important role in understanding the QCD factorization theory. Its inclusive annihilation provides one of the most interesting topics in heavy quarkonium physics. The annihilation of the spin-singlet $S$- and $P$-wave heavy quarkonium states to light hadrons (LHs) have been studied at the order of $O(\alpha_s v^2)$ within the framework of nonrelativistic QCD (NRQCD)[3]. However, those estimations still suffer from large scale uncertainties. To improve the accuracy of pQCD estimations, we adopt the principle of maximum conformality (PMC)[4–10] to set the renormalization scale.

To apply PMC, one can first finish the renormalization procedure by using an arbitrary initial scale $\mu^\text{init}_R$, and set the optimal (PMC) scales by absorbing all non-conformal terms into the running coupling via a step-by-step way. The decay widths for $|H[n]\rangle \to LH$ up to $O(\alpha_s v^2)$ can be written as:

$$\Gamma(H[n]) = C_0([n]) \alpha_s^2(\mu^\text{init}_R) \left[ 1 + \frac{\alpha_s(\mu^\text{init}_R)}{\pi} C_1([n]) \right],$$

where $H$ stands for charmonium or bottomonium, and $[n]$ stands for the color-singlet state $^1S^0_{[1]}$ or $^1P^0_{[1]}$ respectively. The leading-order (LO) coefficients

$$C_0(^1S^0_{[1]}) = \frac{4\pi}{9m_Q^2} \left[ \langle O(^1S^0_{[1]}) | s_0 - \frac{4\langle P(^1S^0_{[1]}), s_0 \rangle}{3m_Q^2} \right],$$

$$C_0(^1P^0_{[1]}) = \frac{5\pi}{6m_Q^2} \left[ \langle O(^1S^0_{[1]}), s_0 \rangle + \frac{4\langle P(^1S^0_{[1]}), s_0 \rangle}{3m_Q^2} \right],$$

where $\langle O \rangle$ and $\langle P \rangle$ are long-distance matrix elements (LDMEs), the superscript $[1]$ or $[8]$ stands for color-singlet or color-octet state, respectively. The NLO coefficient $C_1([n])$ can be divided into $\beta$-dependent non-conformal and $\beta$-independent conformal parts,

$$C_1([n]) = C_1^{(\beta)}([n]) \beta_0 + C_1^{(con)}([n]),$$

where $\beta_0 = 11 - 2n_f/3$ and the latter $\beta_1 = 51 - 19n_f/3$. For the conventional scale setting, the scale is fixed throughout the analysis once it has been set to an initial value, i.e. one usually takes $\mu_R \equiv \mu^\text{init}_R = 2m_Q$ as its central value for the present heavy quarkonium decay. Applying PMC scale setting, the $\beta_0$-dependent terms can be absorbed into PMC scale and we obtain

$$\Gamma(H[n]) = C_0([n]) \alpha_s^2(\mu^\text{PMC}_R) \left[ 1 + \frac{\alpha_s(\mu^\text{PMC}_R)}{\pi} C_1^{(con)}([n]) \right],$$

where the PMC scale $\mu^\text{PMC}_R = \mu^\text{init}_R \exp \left[ -C_1^{(\beta)}([n]) \right]$. The non-conformal $C_1^{(\beta)}([^1S^0])$ and $C_1^{(\beta)}([^1P^1])$, and the conformal $C_1^{(con)}([^1S^0])$ and $C_1^{(con)}([^1P^1])$ at the initial scale $\mu^\text{init}_R$ can be derived from Refs.[1,2], i.e.

$$C_1^{(\beta)}([^1S^0]) = -\frac{a}{a + b} \frac{1}{72} \left[ 36 \ln \left( \frac{4m_Q^2}{(\mu^\text{init}_R)^2} \right) - 96 \right]$$

$$-\frac{b}{a} \frac{1}{144} \left[ 72 \ln \left( \frac{4m_Q^2}{(\mu^\text{init}_R)^2} \right) - 246 \right]$$

$$C_1^{(con)}([^1S^0]) = -\frac{a}{a + b} \frac{1}{72} \left[ 93 \pi^2 - 852 \right]$$

$$-\frac{b}{a} \frac{1}{144} \times \left[ 192 \ln \left( \frac{m_P^2}{4m_Q^2} \right) + 237 \pi^2 - 2258 \right],$$

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\[ C_1^{(\text{con})} ([^1 P_1]) = - \frac{A}{A + B} \frac{1}{72} (129\pi^2 - 1392) \]
\[ - \frac{B}{A + B} \frac{1}{288} \left[ 168 \ln \left( \frac{\mu_F^2}{4m_Q^2} \right) + 735\pi^2 - 6892 \right] \]
\[ + \frac{C}{A + B} \left[ 7\pi^2 - 112 - 24 \ln \left( \frac{\mu_F^2}{4m_Q^2} \right) \right] \]
\[ + \frac{D}{A + B} \left[ 1740 \ln \left( \frac{\mu_F^2}{4m_Q^2} \right) - 555\pi^2 + 9236 \right], \]
\[ C_1^{(\beta)} ([^1 P_1]) = - \frac{A}{A + B} \frac{1}{72} \left[ 36 \ln \left( \frac{\mu^2_R}{\mu^2_{\text{init}}} \right)^2 \right] - 96 \]
\[ - \frac{B}{A + B} \frac{1}{288} \left[ 144 \ln \left( \frac{4m_Q^2}{\mu^2_{\text{init}}} \right) - 492 \right], \]
where \( \mu_F \) is the factorization scale and the coefficients
\[ a = \frac{4\pi}{9m_Q^2} (\mathcal{O}(1^1 S_0^{[1]})), s_0, b = -\frac{16\pi}{27m_Q^2} (\mathcal{P}(1^1 S_0^{[1]})), s_0, \]
\[ A = \frac{5\pi}{6m_Q^2} (\mathcal{O}(1^1 S_0^{[8]})), s, B = -\frac{10\pi}{9m_Q^4} (\mathcal{P}(1^1 S_0^{[8]})), s, \]
\[ C = \frac{5\pi}{486m_Q^2} (\mathcal{O}(1^1 P_1^{[1]})), p_1, D = \frac{\pi}{3645m_Q^2} (\mathcal{P}(1^1 P_1^{[1]})), p_1. \]

After PMC scale setting, the PMC scale may close to or even smaller than \( \Lambda_{\text{QCD}} \) in certain processes, which could lead to Landau pole problem for the running coupling. A small scale is reasonable since at higher orders more gluons are involved and all of them can share the typical momentum flow of the process and result in smaller renormalization scales. Such a small scale can explain the discrepancies between the conventional QCD predictions with the experimental data, e.g. it can shrink the gap between the pQCD estimation and experimental measurement for the top pair forward and backward asymmetry at the TEVATRON to be within \( 1\sigma \).

Moreover, the commensurate scale relations among different renormalization schemes can smear such problem to a certain degree, since those relations between observables can be tested at quite low momentum transfers.

For the present \( \eta_c \) and \( h_c \) decay, their PMC scales are smaller than 1 GeV. At the low energy region, the natural extension of the running coupling is somewhat questionable. To achieve a more accurate pQCD estimation, we adopt several low-energy models suggested in the literature to do our discussion, i.e.

- The APT model, which is based on the analytic perturbative theory and takes the form
  \[ \alpha_{\text{APT}}(\mu_R^2) = \frac{4\pi}{\beta_0} \left( \ln \frac{1}{x} + \frac{1}{1 - x} \right), \tag{3} \]
  where \( x = \frac{\mu_R^2}{\Lambda_{\text{QCD}}^2} \).

- The WEB model, which is suggested by Webber to suppress the power correction of APT model and takes the form
  \[ \alpha_{\text{WEB}}(\mu_R^2) = \frac{4\pi}{\beta_0} \left[ \ln \frac{x + b}{(1 - x)(1 + b)} \left( \frac{1 + c}{x + c} \right)^p \right], \tag{4} \]
  where \( b = 1/4 \) and \( p = c = 4 \).

- The MPT model, which is based on the ‘massive’ analytic pQCD theory. It suggests to use the effective glueball mass as the infrared regulator. The main idea of MPT is to change the logarithm \( \ln \mu_R^2/\Lambda_{\text{QCD}}^2 \) by \( \ln(\xi + \mu_R^2/\Lambda_{\text{QCD}}^2) \), in which the parameter \( \xi \) corresponds to the “effective gluonic mass” \( m_g = \sqrt{\xi}\Lambda_{\text{QCD}} \). It takes the following form
  \[ \alpha_{\text{MPT}}(\mu_R^2) = \frac{\alpha_{\text{crit}}}{1 + \alpha_{\text{crit}} \frac{\beta_0}{\beta_1} \ln(1 + x/\xi) + \alpha_{\text{crit}} \frac{\beta_0}{\beta_1} \ln \left[ 1 + \alpha_{\text{crit}} \frac{\beta_0}{\beta_1} \ln(1 + x/\xi) \right]. \tag{5} \]
  with \( \alpha_{\text{crit}} = 0.61 \) and \( \xi = 10 \). It is noted that the moment of the spin-dependent structure function calculated within MPT model is consistent with the experiment data down to a few hundred MeV.

- The BPT model, which takes the following form
  \[ \alpha_{\text{BPT}}(\mu_R^2) = \frac{4\pi}{\beta_0 t_B} \left( 1 - \frac{2\beta_1}{\beta_2} \ln t_B \right), \tag{6} \]
  where \( t_B = \ln \left( \frac{\mu_R^2 + m_B^2}{\Lambda_{\text{QCD}}^2} \right) \) and \( m_B = 1 \text{ GeV} \).

- The CON model, which takes the following form
  \[ \alpha_{\text{CON}}(\mu_R^2) = \frac{4\pi/\beta_0}{\ln \left[ x + 4M_g^2(\mu_R^2)/\Lambda_{\text{QCD}}^2 \right]}, \tag{7} \]
  where \( M_g^2(\mu_R^2) \) stands for the running gluon mass, determined by the gluon mass \( m_g = 0.34 \):
  \[ M_g(\mu_R^2) = m_g^2 \left[ \ln \left[ x + 4m_g^2(\mu_R^2)/\Lambda_{\text{QCD}}^2 \right] \right]^{1/2}. \tag{8} \]

- The GI model, which takes the following form
  \[ \alpha_{\text{GI}}(\mu_R^2) = \sum_{k=1}^{3} \alpha_k \exp \left[ -\mu_R^2/4\gamma_k^2 \right], \tag{9} \]
where $\alpha_1 = 0.25$, $\alpha_2 = 0.15$, $\alpha_3 = 0.2$, $\gamma_1^2 = 1/4$, $\gamma_2^2 = 5/2$ and $\gamma_3^2 = 250$.

Fig. 1. Various effective running strong coupling models versus the scale $\mu_R$. $\alpha_{QCD}$ stands for the conventional behavior for the strong running coupling.

Since the behavior of the running coupling is universal, the model parameters have been determined by each group via comparing with the known data for high energy processes. Further more, to apply those low-energy models, we also need to set the scale parameters. Using the two-loop $\alpha_s$ running with $\alpha(M_T) = 0.33$ $[18]$, we predict the scale parameters for the conventional QCD running behavior, i.e. we have $\Lambda_{QCD}^{(n_f=3)} = 0.387$ GeV, $\Lambda_{QCD}^{(n_f=4)} = 0.333$ GeV and $\Lambda_{QCD}^{(n_f=5)} = 0.231$ GeV. For APT model, we have $\Lambda_{QCD}^{(n_f=3)} = 0.254$ GeV. For QCD model, we have $\Lambda_{QCD}^{(n_f=3)} = 0.214$ GeV. For BPT model, we have $\Lambda_{QCD}^{(n_f=3)} = 0.453$ GeV. For MPT model, we have $\Lambda_{QCD}^{(n_f=3)} = 0.294$ GeV. For CON model, we have $\Lambda_{QCD}^{(n_f=3)} = 0.222$ GeV. A comparison of running coupling has been presented in Fig.1. Except for the GI model, Fig.1 shows that as required, the low-energy models mainly change the low-energy behavior and their high-energy behaviors are almost unchanged in comparison with the conventional running coupling.

For the color-singlet LDMEs, they can be related to the wavefunction at the origin for S-wave states or the first derivative of the wavefunction at the origin for P-wave states. We adopt the wavefunctions of B-T potential model$[19]$ to fix those LDMEs, i.e. $|R_{np}|^2 = 0.810$ GeV, $|R_{np}|^2 = 0.647$ GeV, $|R_{n\bar{p}}|^2 = 0.075$ GeV, $|R_{n\bar{p}}|^2 = 1.417$ GeV. For the charmion quark LDMEs involving S-wave state, we obtain $⟨\mathcal{O}(1)S_{0}^{[1]}⟩_{q_c} = \frac{N_c}{2π} |R_{n\bar{p}}|^2 = 0.387$ GeV and $⟨\mathcal{O}(1)S_{0}^{[1]}⟩_{q_c} = m_c^2 (\mu^2)_{q_c} ⟨\mathcal{O}(1)S_{0}^{[1]}⟩_{q_c} = 0.198$ GeV. For the LDMEs involving P-wave state, we have $⟨\mathcal{O}(1)P_{1}^{[1]}⟩_{h_c} = \frac{N_c}{2π} |R_{n\bar{p}}|^2 = 0.102$ GeV and $⟨\mathcal{O}(1)P_{1}^{[1]}⟩_{h_c} = m_c^2 (\mu^2)_{h_c} ⟨\mathcal{O}(1)P_{1}^{[1]}⟩_{h_c} = 0.055$ GeV. The color-octet LDMEs can be determined by the evolution equations$[2,3,20,21]$.

$$
\frac{d⟨\mathcal{O}(1)S_{0}^{[8]}⟩_{h_c}}{dμ_R^2} = \frac{-7α_s}{9π} \frac{⟨\mathcal{O}(1)P_{1}^{[1]}⟩_{h_c}}{m_c^2} + \frac{16α_s}{9π} \frac{⟨\mathcal{O}(1)P_{1}^{[1]}⟩_{h_c}}{2N_cm_c^2},
$$

$$
\frac{d⟨\mathcal{O}(1)P_{1}^{[1]}⟩_{h_c}}{dμ_R^2} = \frac{16α_s}{9π} \frac{⟨\mathcal{O}(1)P_{1}^{[1]}⟩_{h_c}}{2N_cm_c^2},
$$

Setting $μ_F = 2m_c$, we obtain $⟨\mathcal{O}(1)S_{0}^{[8]}⟩_{h_c} = 0.0045$ GeV and $⟨\mathcal{O}(1)P_{1}^{[1]}⟩_{h_c} = 0.0028$ GeV. For the case of bottomonium, we have $⟨\mathcal{O}(1)S_{0}^{[1]}⟩_{b_b} = 3.092$ GeV, $⟨\mathcal{O}(1)P_{1}^{[1]}⟩_{b_b} = 2.748$ GeV, $⟨\mathcal{O}(1)P_{1}^{[1]}⟩_{h_c} = 1.804$ GeV, $⟨\mathcal{O}(1)S_{0}^{[1]}⟩_{h_c} = 0.0074$ GeV and $⟨\mathcal{O}(1)P_{1}^{[1]}⟩_{h_c} = 0.0068$ GeV. In the calculation, we have taken $⟨v^2⟩_{h_c} ≈ ⟨v^2⟩_{q_c} = 0.223$ and $⟨v^2⟩_{b_b} ≈ ⟨v^2⟩_{q_c} = 0.042$ $[2]$. When varying the quark masses, those LDMEs shall be changed accordingly.

Table 1. Total decay widths for $Γ(H[n])$ under the PMC scale setting with various low-energy running coupling models, where $μ_R^{P PMC} = 2m_Q$ and $[n] = 1S_{0}^{[1]}$ or $1P_{1}^{[1]}$. The errors are calculated by taking $m_c ∈ [1.40GeV, 1.60GeV]$ and $m_b ∈ [4.50GeV, 4.70GeV]$.

| Model   | $Γ(η_c)$ (MeV) | $Γ(η_b)$ (MeV) | $Γ(η_c)$ (MeV) | $Γ(η_b)$ (MeV) |
|---------|----------------|----------------|----------------|----------------|
| QCQ      | 26.70±3.15     | 14.08±0.84     | 14.33±0.81     | 14.33±0.81     |
| WEB      | 25.97±0.35     | 14.34±0.84     | 14.34±0.84     | 14.34±0.84     |
| APT      | 26.70±3.15     | 14.34±0.84     | 14.34±0.84     | 14.34±0.84     |
| MPT      | 25.79±0.33     | 14.34±0.84     | 14.34±0.84     | 14.34±0.84     |
| CON      | 25.79±0.33     | 14.34±0.84     | 14.34±0.84     | 14.34±0.84     |
| GI       | 25.79±0.33     | 14.34±0.84     | 14.34±0.84     | 14.34±0.84     |

After applying the PMC scale setting, we present the total decay widths for the conventional and the six low-energy effective running coupling models in Table 1. As for the bottomonium case, the PMC scales are larger than 1 GeV, the results of all low-energy models agree well with each other. As for the charmonium case, the conventional running coupling could be questionable, since the PMC scales for $η_c$ and $h_c$ are less than 1 GeV, i.e. $μ_R^{P PMC} = 0.93$ GeV and $μ_R^{P PMC} = 0.98$ GeV. It is found that the total decay widths with the six low-energy effective models are consistent with each other, which are caused by the fact that those models have similar behavior around the region close to 1 GeV. In the following, we adopt
MPT model to do a detail discussion on the heavy quarkonium decay properties.

Table 2. Scale dependence of $\Gamma(H[n])$ within the MPT model under the conventional scale setting and the PMC scale setting, where three typical (initial) scales are adopted.

| $\mu_R^{init}$ | Conventional | PMC |
|----------------|--------------|-----|
| m_Q | 2m_Q | 4m_Q | m_Q | 2m_Q | 4m_Q |
| $\Gamma(\eta_c)$ (MeV) | 27.78 | 20.38 | 14.01 | 25.09 | 25.09 | 25.09 |
| $\Gamma(\eta_b)$ (MeV) | 12.91 | 10.14 | 7.94 | 14.34 | 14.34 | 14.34 |
| $\Gamma(\gamma)$ (MeV) | 0.57 | 0.41 | 0.28 | 0.54 | 0.54 | 0.54 |
| $\Gamma(h_b)$ (KeV) | 45.92 | 39.27 | 31.98 | 39.89 | 39.89 | 39.89 |

Table 3. Total decay width $\Gamma(H[n])$ within the MPT model under the conventional scale setting and the PMC scale setting, where $\mu_R^{init} = 2m_Q$.

| $\mu_R^{init}$ | Conventional | PMC |
|----------------|--------------|-----|
| m_Q | 2m_Q | 4m_Q | m_Q | 2m_Q | 4m_Q |
| $\eta_c$ (MeV) | 11.62 | 8.76 | 20.38 | 31.95 | -6.85 | 25.09 |
| $\eta_b$ (MeV) | 6.25 | 3.89 | 10.14 | 15.68 | -1.33 | 14.35 |
| $h_c$ (MeV) | 0.23 | 0.18 | 0.41 | 0.61 | -0.07 | 0.54 |
| $h_b$ (keV) | 27.97 | 11.30 | 39.27 | 70.12 | -30.23 | 39.89 |

In Table 2 and Table 3 we present the decay widths with the MPT model before and after PMC scale setting. As shown by Table 2, under the conventional scale setting, the decay widths depend heavily on the choice of scale. By varying $\mu_R \equiv \mu_R^{init}$ from $m_Q$ to $4m_Q$, the decay widths for $\eta_c$, $\eta_b$, $h_c$ and $h_b$ are changed by about 50%, 39%, 51% and 30% respectively. On the other hand, after the PMC scale setting, we observe that all of the decay widths remain almost unchanged by varying $\mu_R^{init}$ from $m_Q$ to $4m_Q$, thus the scale ambiguity is eliminated even at the NLO level. There is residual scale dependence due to unknown higher-order $\{\beta_3\}$-terms, which however will be highly exponentially suppressed. After the PMC scale setting, one can absorb/resum the $\{\beta_3\}$-terms into the running coupling, and in principle, the LO decay widths shall be increased and the NLO decay widths shall be decreased. Thus, the pQCD convergence can be generally improved. As shown by Table 4, this is indeed the case for most of the decays. For convenience, we define a parameter $K$, which equals to the ratio of the NLO decay width to the LO decay width. Under the conventional scale setting, the $K$ factors for $\eta_c$, $\eta_b$ and $h_c$ and $h_b$ are 75%, 62% and 78% respectively; while they change down to 21%, 8% and 11%, respectively. The only exception is $h_b$ decay, whose $K$ factor is about 40% even after PMC scale setting, which means one needs to finish at least the NNLO calculation to achieve a better pQCD convergence for $h_b$ decay.

In the literature, another scale setting method, i.e. principle of minimum sensitivity (PMS)[22], has also been suggested. Following the standard PMS procedure, we obtain

$$\Gamma(H[n]) = C_0([n]) \alpha_s^2(\mu_{R^{(\text{PMS})}}) \left( 3 + \cot \beta_4(\mu_{R^{(\text{PMS})}}) / \pi \right) \left( 3 \left( 1 + \cot \beta_4(\mu_{R^{(\text{PMS})}}) / \pi \right) \right),$$

where $c = \beta_1 / 2 \beta_0$, and the PMS effective coupling is a solution of the following equation:

$$\rho_1([n]) = \frac{2 \pi}{\alpha_s(\mu_{R^{(\text{PMS})}})} + 2 \cot \beta_4(\mu_{R^{(\text{PMS})}}) \left( 1 + \cot \beta_4(\mu_{R^{(\text{PMS})}}) / \pi \right),$$

Here $\rho_1([n]) = \frac{\alpha_s(\mu_{R^{(\text{PMS})}})}{2 \pi} - \frac{\alpha_s(\mu_{R^{(\text{init})}})}{2 \pi} - \frac{\alpha_s(\mu_{R^{(\text{init})}})}{2 \pi}.$

Table 4. Comparison of PMC and PMS estimations for $\Gamma(H[n])$ together with the experimental measurements[24]. The MAP model is adopted and the errors are calculated by taking $m_c \in [1.40\text{GeV}, 1.60\text{GeV}]$ and $m_b \in [4.50\text{GeV}, 4.70\text{GeV}].$

| $\Gamma(\eta_c)$ (MeV) | 32.0 ± 0.9 | 25.09 ± 0.44 | 31.57 ± 0.92 |
| $\Gamma(\eta_b)$ (MeV) | 10.8 ± 0.9 | 14.34 ± 0.92 | 13.2 ± 0.41 |
| $\Gamma(h_c)$ (MeV) | 0.54 ± 0.06 | 0.66 ± 0.08 | 0.96 ± 0.06 |
| $\Gamma(h_b)$ (keV) | 39.89 ± 0.46 | 42.95 ± 1.42 |

A comparison of the PMC and PMS estimations by using MPT model has been put in Table 4, in which the present experimental results on $\eta_c$ and $\eta_b$ are also presented. Due to the large errors from the bound state parameters such as the quark masses, both the PMC and PMS estimations are consistent with the experimental results within reasonable regions. As an estimation of $h_c$ decay, it has two dominant decay channels, by taking $\Gamma(h_c \to \gamma \eta_c) = 385$ KeV[23], we obtain $\Gamma^{th}(h_c) \simeq 0.19$ MeV, which is consistent with $\Gamma^{exp}(h_c) = 0.73$ ± 0.45 ± 0.28 MeV[24]. As for $h_b$, it also contains two decay channels, the decay width $\gamma \eta_b$ has been estimated by Refs.[25,26,27]. By taking $\Gamma(h_b \to \gamma \eta_b) = 37.0$ KeV[26], we obtain $\Gamma^{th}(h_b(1P) \to \eta_b(1S) \gamma) = (48.1^{+0.9}_{-0.7})$ %, which agrees $\Gamma^{exp}(h_b(1P) \to \eta_b(1S) \gamma) = (49.2 \pm 5.7^{+5.6}_{-3.3})$ %[18].

In summary, we have made a detailed discussion on the decay widths for spin-singlet heavy quarkonia under the PMC scale setting. After PMC scale setting, the renormalization scale uncertainty has been has been eliminated even at the NLO level. Thus, after applying PMC scale setting, it can eliminate an important theoretical error and increase the precision of QCD tests, which shall also increase the sensitivity of the collider experiments to new physics beyond the standard model[28]. More over, we show that the PMC estimations are also consistent with the PMS estimations for the present decay channels. The remaining uncertainties are from the bound state parameters such as the quark masses and the LDMEs. As a final remark, it is noted that the PMC scales of $\eta_c$ and $h_c$
are below 1 GeV, in such low-energy region, a proper low-energy running coupling is necessary. We have applied several low-energy running coupling models for the estimation. The results show that the decay widths under those models have similar results and are in reasonably consistent with the data.

References

[1] Li J Z, Ma Y Q and Chao K T 2011 Phys. Rev. D 83 114038.
[2] Li J Z, Ma Y Q and Chao K T 2013 Phys. Rev. D 88 034002.
[3] Bodwin G T, Braaten E and Lepage G P 1995 Phys. Rev. D 51, 1125 ; 1997 Erratum-ibid. D 55 5853.
[4] Brodsky S J and Wu X G 2012 Phys. Rev. Lett. 109 042002.
[5] Brodsky S J and Wu X G 2012 Phys. Rev. D 85 034032.
[6] Brodsky S J and Wu X G 2012 Phys. Rev. D 86 014021.
[7] Brodsky S J and Wu X G 2012 Phys. Rev. D 86 054018.
[8] Mojaza M, Brodsky S J and Wu X G 2013 Phys. Rev. Lett. 110 192001; Brodsky S J, Mojaza M and Wu X G 2013 arXiv:1304.4631.
[9] Brodsky S J and Giustino L D 2012 Phys. Rev. D 86 085026.
[10] Brodsky S J and Lu H J 1995 Phys. Rev. D 51 3652.
[11] Shirkov D V and Solovtsov I L 1997 Phys. Rev. Lett. 79 1209.
[12] Webber B R 1998 JHEP 10 012.
[13] Shirkov D V 2013 Phys. Part. Nucl. Lett. 10 186.
[14] Badalian A M and Kuzmenko D S 2002 Phys. Rev. D 65 016004.
[15] Cornwall J M 1982 Phys. Rev. D 26 1453.
[16] Godfrey S and Isgur N 1985 Phys. Rev. D 32 189.
[17] Beringer J et al [Particle Data Group] 2012 Phys. Rev. D 86 010001.
[18] Eichten E J and Quigg C 1995 Phys. Rev. D 52 1726.
[19] Gremm M and Kapustin A 1997 Phys. Lett. B 407 323.
[20] Fan Y, Li J Z, Meng C and Chao K T 2012 Phys. Rev. D 85 034032.
[21] Stevenson P M 1981 Phys. Rev. D 23 2916.
[22] Chao K T, Ding Y B and Qin D H 1993 Phys. Lett. B 301 282.
[23] Ablikim M et al [The BESIII Collaboration] 2010 Phys. Rev. Lett. 104 132002.
[24] Brambilla N et al [Quarkonium Working Group], hep-ph/0412158.
[25] Godfrey S and Rosner J L 2002 Phys. Rev. D 66 014012.
[26] Li B Q and Chao K T 2009 Commun. Theor. Phys. 52 653.
[27] Wu X G, Brodsky S J and Mojaza M 2013 Prog. Part. Nucl. Phys. 72 44.