MILNOR CONCORDANCE INVARIANT FOR KNOTTED SURFACES AND BEYOND

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ABSTRACT. We generalize Milnor link invariants to all types of knotted surfaces in 4–space, and more generally to all codimension 2 embeddings. This is achieved by using the notion of cut-diagram, which is a higher dimensional generalization of Gauss diagrams, associated to codimension 2 embeddings in Euclidian spaces. We define a notion of group for cut-diagrams, which generalizes the fundamental group of the complement, and we extract Milnor-type invariants from the successive nilpotent quotients of this group. We show that the latter are invariant under an equivalence relation called cut-concordance, which encompasses the topological notion of concordance. We give several concrete applications of the resulting Milnor concordance invariants for knotted surfaces, comparing their relative strength with previously known concordance invariants, and providing realization results. We also obtain classification results for Spun links up to link-homotopy, as well as a criterion for a knotted surface to be ribbon. Finally, the theory of cut-diagrams is further investigated, heading towards a combinatorial approach to the study of surfaces in 4–space.

INTRODUCTION

The study of classical knots up to concordance led to many rich developments over the last decades. In higher dimensions, the situation seems in contrast to be more simple: in odd dimensions, Levine gave a classification of knotted spheres up to concordance [18], providing in particular examples of non slice $(2k+1)$–knots for all $k \geq 1$, whereas Kervaire proved that all even-dimensional knots are slice [12]. Yet, the question of the existence of a non slice $2k$–link, i.e. a smooth embedding of disjoint copies of $S^{2k}$ in $S^{2k+2}$ which does not bound disjoint copies of $B^{2k+1}$ in $B^{2k+3}$, remains wide open to this date, even in the case $k = 1$. The search for a non trivial concordance invariant of $2k$–links thus stands as a major open problem in higher-dimensional knot theory, for which very little progress was made, see [13, Problem 1.56].

In dimension 3, a number of concordance invariants for links are known, starting with the linking number. The linking number was widely generalized by Milnor in [21, 22], into a family of numerical concordance invariants now called Milnor invariants, which can be successfully used to provide obstructions for classical links to be slice. So far, all attempts to generalize Milnor invariants to $n$–links, for $n \geq 2$, yielded trivial invariants. However, relaxing the embedded condition for $n$–links, Koschorke did provide in [15] such a non trivial generalization to higher dimensional link maps, which are continuous maps from a disjoint union of spheres to a Euclidean space such that the images of the spheres are pairwise disjoint. For 2-component link maps, Kirk also developed link-homotopy invariants [14] in the late eighties, which were recently shown to be complete link-homotopy invariants [32]. We propose here another direction for generalization: relaxing the condition that $n$–links be spherical, we define Milnor invariants for all codimension 2 embeddings. We discuss in Section 4.5.3 the earlier work of Orr [24] in the same direction. Note that Koschorke’s generalization and ours coincide only at embedded spheres, where all invariants vanish. Hence these two works can be regarded as orthogonal.

In order to define our generalization of Milnor invariants, we develop a theory of cut-diagrams. An $n$–dimensional cut-diagram consists of an $(n−1)$–dimensional diagram on some $n$–manifold, endowed with some labeling. This provides a combinatorial generalization of codimension 2 embedding diagrams, that can be seen as a higher dimensional version of Gauss diagrams. We also introduce a notion of cut-concordance for $n$–dimensional cut-diagrams via $(n+1)$–dimensional cut-diagrams. This yields an equivalence relation for cut-diagrams, that encompasses the topological concordance relation, and leads to a new theory where topological concordance can be studied in a combinatorial way. As a matter of fact, our generalization of Milnor invariants are cut-concordance invariants within this cut-diagram theory. The authors believe that this is a promising approach, which can lead to further applications.

The purpose of this paper is thus twofold, and we develop below these two aspects in further details.
A generalization of Milnor concordance invariants. One purpose of this paper is to define Milnor concordance invariants for general codimension 2 embeddings, i.e. for smooth embeddings of oriented \( n \)-dimensional manifolds in the Euclidian space of dimension \( (n + 2) \), and in particular for knotted surfaces in 4-space, possibly with boundary.

Let \( S \) be a codimension 2 embedding of \( \Sigma = \Sigma_1 \cup \cdots \cup \Sigma_\ell \) in an Euclidian space. As in the original construction of Milnor for classical links, our Milnor-type invariants are extracted from the nilpotent quotients of the fundamental group \( G(S) \) of the complement of \( S \). These are the quotients \( G(S)/G(S)_q \) by the successive terms \( G(S)_q \) of the lower central series. A key result in our construction is a presentation for these nilpotent quotients (see Theorem 3.15). This presentation widely generalizes \([22\text{ Thm. 4}]\) for usual links, known as the Chen–Milnor theorem. We stress that our proof of Theorem 3.15 is combinatorial, and leads to an algorithm for computing our higher dimensional Milnor invariants, see Remark 4.25. Building on this key result, Milnor-type invariants for \( S \) are defined by considering the Magnus expansion of some element of \( G(S)/G(S)_q \) associated with a path on \( \Sigma \), that yields a power series in \( \ell \) non-commuting variables. Although the integer coefficients in this power series depend on several choices made in the construction, we show in Section 4.1 how to extract well-defined invariants from them. For each integer \( i \) and sequence of integers \( I \), all in \( \{1, \ldots, \ell\} \), these generalized Milnor invariants come in various forms:

- a Milnor map \( M^i_I \), which is a homomorphism from \( H_1(\Sigma_i; \mathbb{Z}) \) to some cyclic group;
- Milnor loop-invariants \( \nu_S(I) \), which are positive integers extracted from closed curves on \( \Sigma \);
- Milnor arc-invariants \( \nu_S(I; i, j) \), associated to a boundary component \( p_{ij} \) of \( \Sigma \), which are residue classes extracted from paths joining \( p_{ij} \) to a previously fixed boundary component of \( \Sigma \).

In Section 5.1 we prove the following.

**Theorem.** Generalized Milnor invariants are well-defined concordance invariants of codimension 2 embeddings.

Moreover, in dimensions 1 and 2, if the indexing sequence contains no repetition, the corresponding Milnor invariants are also invariant under link-homotopy, i.e. under continuous deformations leaving distinct components disjoint at all time.

Although these invariants are all trivial for spherical \( n \)-links \( (n \geq 2) \), they turn out to be rather strong invariants for knotted manifolds with at least one non-simply connected component. This is illustrated in Section 6, where a number of applications are provided in the case of knotted surfaces:

- we give general realization results for our invariants in Subsection 6.2, in fact, we can provide infinitely many examples of knotted surfaces in 4-space with non trivial Milnor concordance invariants, assuming that at least one component is non-spherical. The spherical case thus basically appears as the only remaining case where no obstruction for sliceness exists so far;
- in Subsection 6.3 we consider a family \( W_m \) \( (m \in \mathbb{N}) \) of links made of a 2–sphere and a 2–torus. We show that \( W_m \) and \( W_{m'} \) are concordant if and only if \( m = m' \) by using our Milnor invariants while, in contrast, the Sato-Levine invariant \([31]\) and Cochran’s derivation invariants \([9]\) vanish on all \( W_m \), and Saito’s invariants \([30]\) of \( W_m \) are equal for all values of \( m \in \mathbb{N} \);
- in Subsection 6.4.1 we use our Milnor invariants to classify Spun links of 3-components up to link-homotopy (Proposition 6.7), and we characterize link-homotopically trivial Spun links of any number of components (Proposition 6.9);
- we state a general link-homotopy classification for knotted punctured spheres in Subsection 6.4.2;
- in Subsection 6.5 we provide a necessary condition for a knotted surface to be concordant to a ribbon one.

We note that our generalized Milnor invariants recover Milnor’s original invariants \([21,22]\) and Habegger-Lin’s version for string links \([11]\), as well as many of the previously known extensions of Milnor invariants, see Section 4.5.3. The relation with Orr’s invariants of surface links \([24,25]\) will in particular be discussed in Section 5.2.27.

1Here, what we call Spun links are knotted tori in 4-space, obtained from a classical link in 3-space by Artin’s Spun construction, see Subsection 6.1.
Figure 1. From a knotted surface diagram to a 2–dimensional cut-diagram.
Here, and in forthcoming figures, regions are named with capital letters, and labels are given by circled nametags

Cut-diagrams for codimension 2 embeddings and beyond. The key tool for defining our generalized Milnor invariants is the notion of cut-diagram, and building a theory of cut-diagrams is in fact another purpose of this paper.

For simplicity of exposition, let us first focus on the 2–dimensional case. Recall that a knotted surface diagram is a generic immersion of a surface $\Sigma$ into 3-space, endowed with an over/under information on the double point set. Each line of double points inherits a natural orientation from that of $\Sigma$ and of the ambient space. The lower point set forms a union $C$ of oriented immersed circles and/or intervals in $\Sigma$, which is known as the lower deck in the literature (see for example [6, § 4.1]), and which splits $\Sigma$ into connected components called regions. Each arc of $C$ can be labeled by the region where the corresponding arc of higher points is contained. The resulting labeled oriented diagram $C$ on $\Sigma$ is a 2–dimensional cut-diagram over $\Sigma$ for the given knotted surface. An example is given in Figure 1 in the case $\Sigma = S^2$. Now, we can abstractly define a notion of cut-diagram over an oriented surface $\Sigma$, irrespective of a given knotted embedding of $\Sigma$, as an oriented immersed 1–dimensional manifold $C$ on $\Sigma$, such that each arc of $C$ is labeled by a region of $\Sigma\setminus C$, subject to some natural labeling condition. This labeling condition emulates the topological constraints arising from triple and/or branch points in a knotted surface diagram (see Definition 1.4).

As explained in Subsection 1.2.2, the notion of cut-diagram generalizes naturally to any dimension. There is indeed a general notion of oriented diagrams in an $n$–dimensional manifold $\Sigma$, developed by D. Roseman in [28, 29]. Such a diagram $C$ splits $\Sigma$ into several connected regions; labeling the diagram $C$ by regions so that the same labeling condition as above is satisfied, yields an $n$–dimensional cut-diagram. In dimension $n = 1$, a cut-diagram is a union of signed points $C$ on a 1–dimensional manifold $\Sigma$, such that each point in $C$ is labeled by an arc of $\Sigma\setminus C$. This actually correspond to Gauss diagrams modulo the local move which exchanges two adjacent tails on an arc.

We stress that any diagram of a codimension 2 embedding gives rise to a cut-diagram, so that our constructions and results for cut-diagrams, outlined below, have in particular the direct topological corollaries mentioned in the previous section.

Let $C$ be a cut-diagram over $\Sigma = \Sigma_1 \cup \cdots \cup \Sigma_\ell$. We define in Section 2 the group $G(C)$ of $C$ as the group abstractly generated by all regions, and with a relation for each double points set, which is a Wirtinger-type relation among the three regions involved; see Definition 2.1. In the case where $C$ arises from a codimension 2 embedding, the group $G(C)$ coincides with the fundamental group of the complement. The Chen–Milnor type presentation mentioned in the previous section is actually a corollary of Theorem 3.15 where a presentation for $G(C)/G(C)_q$ is given for any cut-diagram $C$. And, building on this result, all generalized Milnor invariants are also defined in this more general context.

In order to build a theory of cut-diagrams, we next consider certain ‘topology-inspired’ equivalence relations.

On the one hand, we define a ‘concordance’ (and ‘self-singular concordance’) theory. Two $n$–dimensional cut-diagrams are cut-concordant if they are coherently bounded by an $(n + 1)$-dimensional cut-diagram (see Subsection 5.1). This notion generalizes the classical notion of concordance, in the sense that a cut-diagram arising from a concordance between two knotted manifolds yields a cut-concordance between the
cut-diagrams associated to the boundary manifolds. We show in Corollary 5.6 that our generalized Milnor invariants are cut-concordance invariants, what implies in particular the previously stated theorem. A key technical result here is that a cut-concordance induces isomorphisms between the nilpotent quotients of the associated groups (see Corollary 5.9 for a precise statement); this recovers in a combinatorial way a theorem of Stallings [33, Thm. 5.1], see Remark 4.25.

In Section 5.2 we also introduce a notion of self-singular cut-concordance for cut-diagrams, which encompasses link-homotopy for codimension 2 embedded objects, and we show in Proposition 5.26 that Milnor invariants associated to sequences of distinct integers are self-singular cut-concordance invariants.

On the other hand, we develop an ‘isotopy’ (and ‘link-homotopy’) theory for cut-diagrams. As explained before, cut-diagrams have their origin in the notion of diagrams for codimension 2 embeddings, and there exists in dimension 1 and 2 sets of moves on diagrams, namely Reidemeister and Roseman moves, which realize isotopies of knotted manifolds. Restricting ourselves to dimensions 1 and 2, it is thus natural to expect for cut-diagrams to be endowed with some corresponding moves. We provide such an explicit set of moves in Section 7.1. Quotiented by these moves, the notion of 1–dimensional cut-diagrams actually coincides with welded knotted objects. In this sense, the resulting theory of 2–dimensional cut-diagrams can be regarded as a 2–dimensional welded theory.

In dimension 1 and 2, there also exists diagrammatical moves realizing link-homotopy of links and knotted surfaces. Based on these local moves, we finally provide sets of self-singular moves in Section 7.3 leading to some combinatorial ‘link-homotopy’ theory for 1 and 2–dimensional cut-diagrams.

Outline of the paper and conventions. We next outline the structure of the rest of this paper. In section 1 we introduce the notion of cut-diagram, first from an intuitive point of view in low dimensions, then in full generality. Section 2 introduces the notion of peripheral system for cut-diagrams. This data consists of the group of the cut-diagram, together with a collection of elements in this group called meridians and longitudes; for a cut-diagram associated with a codimension 2 embedding, these coincide with the eponymous elements in the fundamental group of the complement. Section 3 establishes the above-mentioned Chen–Milnor-like presentation for the nilpotent quotients of the group of a cut-diagram (Theorem 3.15), which is given in terms of a choice of peripheral system. Building on this presentation, we define in Section 4 our generalized Milnor invariants for cut-diagrams; some examples are provided, and the relation with previous (partial) generalizations is also clarified. The fact that these generalized Milnor invariants indeed provide well-defined concordance invariants for codimension 2 embeddings is the main result of Section 5. More generally, we introduce in Subsection 5.1 the notion of cut-concordance for cut-diagrams, and prove the invariance in this more general setting. Subsection 5.2 is likewise devoted to self-singular cut-concordance, which is closely related to link-homotopy, and non-repeated Milnor invariants are proved to be invariants for this notion. Our main applications for knotted surfaces in 4–space, summarized above, are detailed in Section 6. The final section 7 introduces sets of local moves for cut-diagrams in dimensions 1 and 2, which provides a diagrammatic tool for the study of knotted surfaces up to isotopy or link-homotopy.

We conclude by fixing a set of conventions that will be used implicitly throughout the paper.

Convention. In this paper, everything is stated in the smooth category. In the following, unless otherwise specified, by n–dimensional knotted manifold we mean the image of a (smooth) proper codimension 2 embedding \( \Sigma \hookrightarrow B^{n+2} \) in the \((n+2)\)--dimensional ball, up to isotopy fixing the boundary. We shall also use the generic term codimension 2 embeddings for knotted manifolds of arbitrary dimension. Note that 1–dimensional knotted manifolds include usual links and string-links and, more generally, all tangles. We shall also refer to 2–dimensional knotted manifolds as knotted surfaces.

We also make use of the following notation and conventions:

- given two elements \( a, b \) of some group, their commutator is defined as \([a, b] := a^{-1}b^{-1}ab\), and the conjugate of \( a \) by \( b \) is given by \( a^b := b^{-1}ab \);
- given two normal subgroups \( G_1 \) and \( G_2 \) of a group \( G \), we denote by \( G_1 \cdot G_2 \) the normal subgroup of \( G \) made of products \( g_1g_2 \) with \( g_1 \in G_1 \) and \( g_2 \in G_2 \);
- homotopies of paths should always be understood as homotopies fixing the endpoints. Moreover, when explicit from the context, we will most often suppress the brackets for denoting the element \([y]\) of the fundamental group represented by a path \( y \);
- all homology groups are taken with coefficients in \( \mathbb{Z} \), and this shall be omitted in our notation.
Figure 2. From a knot diagram to a 1–dimensional cut-diagram.

Regions are named with capital letters, and labels on cut points are given by circled nametags.

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1. Cut-diagrams

In this section we introduce cut-diagrams, which are combinatorial objects that are derived from, and generalize, diagrams of codimension 2 embeddings. Before providing a general definition, we introduce cut-diagrams when they arise from the familiar situations of classical knot and surface diagrams.

1.1. An intuitive approach. Recall that a classical knot diagram is a generic immersion $S^1 \hookrightarrow \mathbb{R}^2$, with a finite number of transverse double points. Each double point is endowed with the over/under information on the two preimages, which is graphically encoded by cutting off a neighborhood of the lowest preimage, i.e. of the preimage with lowest coordinate in the projection axis.

These lowest points, which we shall call cut points, split $S^1$ into a number of arcs, called regions. To each cut point is associated a sign—the sign of the corresponding crossing, and a region—the overpassing arc at the crossing, i.e. the region containing the preimage with highest coordinate. The (1–dimensional) cut-diagram built from our classical knot diagram is thus a copy of $S^1$, together with a collection of cut points endowed with a sign and labeled by their associated region. See Figure 2 for a couple of examples.

Remark 1.1. Cut points can be thought of as heads of arrows in a Gauss diagram, the label being the arc where the tail is attached. Note however that the relative position of tails attached to a same arc is not specified in this language. Hence 1–dimensional cut-diagrams should rather be thought of as Gauss diagrams modulo the moves which exchanges two adjacent tails on an arc. As we shall see in Section 7.1 we actually recover in dimension 1 the theory of welded knotted objects.

Now, consider a broken surface diagram for a knotted surface in 4–space. This is a generic map $\Sigma \hookrightarrow \mathbb{R}^3$ of a surface $\Sigma$, with lines of transverse double points which may meet at triple points and/or end at branch points. As in knot diagrams, the over/under information on the two preimages at each line of double points is encoded by cutting off a neighborhood of the lowest preimage (see the left-hand side of Figure 1, 3 and 4 for some examples). Each line of double points also comes with an orientation, such that the local frame formed by a positive normal vector to the overpassing region, a positive normal vector to the underpassing region, and a positive tangent vector to the line of double points, agrees with the ambient orientation of $\mathbb{R}^3$. 
The lowest preimages of double points form a union $P$ of immersed circles and/or intervals in $\Sigma$, which splits $\Sigma$ into regions. Note that the set $P$ is known as the lower deck in the literature (see for example [6 § 4.1]). Each triple point of the surface diagram provides an over/under information at the corresponding crossing of $P$, which is encoded as in usual knot diagrams by splitting $P$ into arcs, that we call cut arcs. Each cut arc inherits an orientation, which is the orientation of the corresponding line of double points, and we label it by the region containing the preimage with highest coordinate at the corresponding line of double points. The data of the oriented and labeled cut arcs in $\Sigma$ forms a (2–dimensional) cut-diagram over $\Sigma$ for the given knotted surface. This can be thought of as an enhanced version of the lower deck. An example is given in Figure 1 in the case $\Sigma = S^2$. As the figure illustrates, a cut-diagram can be thought of as a cut-and-insert model for a broken surface diagram, where $\Sigma$ is cut along the cut arcs, and the resulting pieces are assembled according to the labels and orientations.

We stress that the labeling respects some natural conditions, summarized in Definition 1.4, which are inherited from the topological nature of the surface diagram. We investigate these conditions below, by analyzing the local cut-diagrams arising from triple and branch points. A local cut-diagram associated with a triple point is illustrated in Figure 3. A crossing among cut arcs occurs in the leftmost sheet of $\Sigma$. There, the overpassing arc is labeled by $G$, whereas the left and right underpassing cut arcs, with respect to the orientation of the overpassing arc, are labeled by $E$ and $F$, respectively; these regions $E$ and $F$ are contained in the middle sheet, and are separated by a $G$–labeled arc, with $E$ on the left and $F$ on the right side of this cut arc. We formalize this as follows.

**Definition 1.2.** An ordered pair of regions $(A, B)$ is called $C$–adjacent, for some region $C$, if there exists a path $\gamma : [0, 1] \to \Sigma$ such that $\gamma([0, 1/2]) \subset A$, $\gamma([1/2, 1]) \subset B$, and $\gamma(t)$ is a regular point of a cut arc labeled by $C$, such that the tangent vector $\gamma'(t)$ is a positive normal vector for this cut arc. We define similarly the notion of $C$–adjacency for two cut arcs which are separated by a double point of a $C$–labeled cut arc.

In the local example of Figure 3 the pair of regions $(E, F)$ is $G$–adjacent (middle sheet), and we observe that likewise, the $E$–labeled and $F$–labeled cut arcs are $G$–adjacent (leftmost sheet). Figure 4 for its part, illustrates the local cut-diagram associated to a branch point. This yields a local cut-diagram with a boundary points in the interior of the surface $\Sigma$.

**Definition 1.3.** An internal endpoint of a cut-diagram over $\Sigma$ is a point of a boundary component of some cut arc, which is in the interior of $\Sigma$. A cut arc containing an internal endpoint is called terminal.

In the local example of Figure 4 the terminal cut arc is labeled by $A$, the region it is adjacent to.

We now specify the labeling condition that we shall use throughout the paper for cut-diagrams.
Definition 1.4. A labeling of the cut arcs by regions is \textit{admissible} if it satisfies the following labeling conditions:

1) any pair of $A$–adjacent cut arcs, for some region $A$, is labeled by two $A$–adjacent regions;
2) a terminal cut arc containing an internal endpoint adjacent to some region $A$, is labeled by $A$.

These two conditions are locally represented in Figure 5.

1.2. Abstract cut-diagrams. The above definition of cut-diagrams can be generalized to an abstract notion.

1.2.1. The 2–dimensional case. Let us begin with the 2–dimensional case, which will be the main focus of this paper. So let $\Sigma$ be some oriented surface, possibly with boundary. Consider a, non necessarily properly, immersed oriented 1–manifold in $\Sigma$, together with an over/under decoration at each transverse double point, as in usual knot diagrams. This immersed 1–manifold splits $\Sigma$ into pieces called \textit{regions}, and the over/under decoration at each crossing splits the immersed manifold into \textit{cut arcs}, with the induced orientation. Label each cut arc by some region, in such a way that the labeling conditions of Definition 1.4 are satisfied. The result, regarded up to ambient isotopy of $\Sigma$, is called a 2–dimensional \textit{cut-diagram} over $\Sigma$.

Note in particular that the second labeling condition imposes that all internal endpoints of a given cut arc must be adjacent to the same region. In forthcoming figures, internal endpoints satisfying the second labeling condition will be depicted with a black bullet •.

The left-hand side of Figure 6 provides an example where the first labeling condition is violated. There, the two regions $A$ and $B$ are $B$–adjacent, and are not $R$–adjacent for any other region $R$; therefore the crossing of cut arcs at the center of the figure, where an $A$–labeled and a $B$–labeled cut arcs are $C$–adjacent, does not respect the first rule.

The right-hand side of Figure 6 shows a situation where the second labeling condition cannot be satisfied. Indeed, the left internal vertex sits in region $A$, which imposes that the horizontal cut arc must be labeled by $A$; but this violates the second labeling condition because of the right internal vertex, which lies in region $B$.

As discussed in the preceding section, there is a canonical way to associate a cut-diagram to a given broken diagram of a knotted surface, and the labeling conditions are automatically satisfied in this setting. A number of examples of admissible labelings obtained in this way can be found in Section 6.

1.2.2. The general case. We now give a general definition for cut-diagrams in any dimension. The reader who is mainly interested with knotted surfaces and their invariants may safely skip this section.

Let $\Sigma$ be an $n$–dimensional oriented compact manifold, possibly with boundary. The following generalized notion of diagrams for $n$–dimensional knotted manifolds originates from the work of D. Roseman in...
A diagram on $\Sigma$ is the image of a map $\varphi : P \mapsto \Sigma$, where $P$ is an oriented $(n - 1)$-dimensional manifold, which is in generic position in the sense of $[26] \S 6.1$, equipped with a globally coherent over/under data on double points. Roughly speaking, $\varphi$ is an immersion, except at an $(n - 2)$-dimensional set of branch points. If $\partial \Sigma \neq \emptyset$, we moreover require that $\varphi(P)$ meets transversally $\partial \Sigma$, so that $\varphi^{-1}(\partial \Sigma)$ induces a $(n - 2)$-dimensional diagram on $\partial \Sigma$.

For any such diagram, the connected components of $\Sigma \setminus P$ are called regions, and the connected components of $\{\text{highest preimages of } \varphi \} \subset P$ are called cut domains. We shall keep calling them cut points and cut arcs when dealing specifically with the dimensions one and two, respectively.

**Definition 1.5.** An $n$-dimensional cut-diagram over $\Sigma$ is the data $C$ of a diagram on $\Sigma$, endowed with an admissible labeling, which associates a region to each cut domain, so that the labeling conditions of Definition 1.4 are satisfied up to ambient isotopy of $\Sigma$.

Note that the notions of $A$-adjacency and internal endpoint, of Definitions 1.2 and 1.3, apply in any dimension, so that the labeling conditions make sense. Observe also that for $n = 1$, these conditions are vacuous.

**Notation 1.6.** If $C$ is a cut-diagram with underlying diagram $\varphi : P \mapsto \Sigma$, then by abuse of notation, we shall also denote $\varphi(P)$ by $C$. In particular, we shall refer to regular and double points of $C$ as elements $x \in \Sigma$ such that $\varphi^{-1}(x)$ has 1 or 2 element(s), respectively.

As seen in the previous subsections, a 2-dimensional cut-diagram can be naturally associated to any diagram of a knotted surface in 4-space, although not any cut-diagram arises in this way. Given a diagram of a knotted $n$-dimensional manifold, the procedure generalizes naturally to produce an $n$-dimensional cut-diagram. This leads to the following notion.

**Definition 1.7.** We say that a cut-diagram is topological if it comes from the diagram of a codimension 2 embedding.

### 2. Peripheral systems of cut-diagrams

Cut-diagrams contain all the relevant information to define ‘fundamental’ groups and their distinguished elements, meridians and longitudes. The purpose of this section is to introduce these notions in the general context of cut-diagrams. For that, let $\Sigma = \bigcup_{i=1}^{\ell} \Sigma_i$ be an $n$-dimensional compact manifold with connected components $\Sigma_1, \ldots, \Sigma_{\ell}$, and let $C$ be a cut-diagram over $\Sigma$.

#### 2.1. The group of a cut-diagram

**Definition 2.1.** The group of $C$ is the group $G(C)$ abstractly generated by its regions, and satisfying the Wirtinger relation $B^{-1}A^C$ for every pair $(A, B)$ of $C$-adjacent regions. When seen as generators of $G(C)$, regions will be called meridians of $C$, and more precisely $i$-th meridian when they belong to $\Sigma_i$.

**Notation 2.2.** We denote by $\tilde{F}$ the free group generated by all meridians of $C$, and by $W$ the normal subgroup of $\tilde{F}$ generated by all Wirtinger relations, so that we have $G(C) = \tilde{F}/W$.

**Remark 2.3.** Broken surface diagrams of knotted surfaces give rise to Wirtinger presentations of the fundamental group of their exterior, see e.g. [5]. It is straightforwardly verified, using this Wirtinger presentation, that the group of a topological 2-dimensional cut-diagram, as defined in the previous section, agrees with the fundamental group of the exterior of the underlying knotted surface. In fact, it is not difficult to check that this is actually true for topological cut-diagrams of any dimension. Moreover, in that case, the above notion of meridian likewise agrees with the classical topological notion of meridian.

**Remark 2.4.** In the above definition of $G(C)$, regions and meridians are in one-to-one correspondence. However it shall sometimes be convenient to have multiple meridians for a given region; this will be the case in the proof of Theorem 5.5. More precisely, we can in general associate a finite number of meridians...
for some integer \( k_R \geq 1 \), to each region \( R \). This yields a group with all these meridians as generators, and with relations given by:

- \((R(j))^{-1}R(j)\) for any region \( R \) and \( 1 \leq j < j' \leq k_R \);
- \((B^{(1)})^{-1}(A^{(1)})\) for every pair \((A, B)\) of \( C\)-adjacent regions.

It is straightforwardly checked that such a presentation defines the same group \( G(C) \), Definition 2.1 corresponding to the case where \( k_R = 1 \) for every region \( R \). We shall refer to duplicated presentations when at least one \( k_R \) is greater than 1. We stress that, unless specified otherwise, the rest of the text will not be dealing with duplicated presentations, except for Remarks 2.9, 3.5, 3.14 and 3.18 which will track down the technical aspects of such duplicated presentations. These remarks are mainly meant to be used in the proof of Theorem 5.5 and can safely be skipped on first reading.

In the sequel, we will be mainly interested in the following quotients.

**Definition 2.5.** The lower central series \((G_q)_{q \in \mathbb{N}}\) of a group \( G \), is the descending series of subgroup defined inductively by \( G_1 := G \) and \( G_{q+1} := [G, G_q] \). For \( q \in \mathbb{N} \), the \( q \)-th nilpotent quotient of \( G \) is its quotient \( N_q G := G / G_q \) by the \( q \)-th term of its lower central series.

For any \( q \in \mathbb{N} \), the \( q \)-th nilpotent group of \( C \) is the nilpotent quotient \( N_q G(C) \) of \( G(C) \), and we have

\[
N_q G(C) \cong \tilde{F} / \tilde{F}_q \cdot W.
\]

**2.2. Generic path and associated words.** We say that an oriented path \( \gamma : [0, 1] \to \Sigma \) is generic if \( \gamma(0), \gamma(1) \notin C \) and \( \gamma([0, 1]) \) meets \( C \) only transversally, in a finite number of regular points of \( C \). For example, an oriented arc on a broken surface diagram, that avoids all branch and triple points and meets lines of double points transversally, produces a generic arc on the associated cut-diagram.

The next lemma follows from classical Thom’s transversality theory.

**Lemma 2.6.**

- Any path on \( \Sigma \) with endpoints in \( \Sigma \setminus C \) is homotopic to a generic path.
- If \( \gamma_1 \) and \( \gamma_2 \) are two homotopic generic paths, then they are related by a finite sequence of homotopies within generic paths, and the following local moves which are supported by a 2-disk in \( \Sigma \), illustrated below in each case, that intersects \( C \) transversally:

  \[ H_1 : \text{insertion/deletion of a loop going around a regular point of } C \]

  \[ H_2 : \text{insertion/deletion of a loop going around a point of } \partial C \setminus \partial \Sigma \]

  \[ H_3 : \text{insertion/deletion of a loop going around a double point of } C \]

**Remark 2.7.** Recall that all homotopies of path are assumed to fix endpoints. Notice also that, in the statement of Lemma 2.6, homotopies within generic paths include in particular the following move, which may occur in dimension 2:

Indeed, the definition of generic path allows for a path to meet transversally the cut-diagram \( C \) several times at a same point; hence it allows the seemingly singular situation shown above the double-headed arrow.
Remark 2.8. If $\gamma$ is a generic path joining two regions $A$ and $B$, then the Wirtinger relations imply that $B = A^x = A^{x'}$ in $G(C)$. This elementary observation has two important consequences:

(1) any two meridians of a same connected component of $\Sigma$ are conjugate, so that $G(C)$ is normally generated by any choice of one meridian on each connected component of $\Sigma$;

(2) if $\gamma$ is a generic loop based in a region $R$, then $[R, \tilde{w}_\gamma] = [R, w_\gamma] = 1$ in $G(C)$.

In view of Remark 2.8 (1), any conjugate of a meridian of $G(C)$ will also be called meridian by abuse of notation—see the last paragraph of Section 7.1 for a full legitimization of this terminology.

Remark 2.9. In the case of a duplicated presentation for $G(C)$, discussed in Remark 2.4, a given region no longer corresponds to a unique meridian in the group. The word $\tilde{w}_\gamma$ (and, likewise, $w_\gamma$), associated to a generic path $\gamma$, is then constructed as above, except that the letters are chosen in the larger alphabet $\{ (R^j)^{\pm 1} \mid R \text{ region of } C, 1 \leq j \leq k_B \}$ of all meridians; each letter, associated to the intersection of $\gamma$ with a cut domain labeled by some region $R$, is then chosen among the meridians $(R^j)^{\pm 1}$. There may be hence several words associated to this path $\gamma$ and we assume that $\tilde{w}_\gamma$ (and, likewise, $w_\gamma$) is just one of them, freely chosen. It is clear, from the relations in the group $G(C)$ with duplicated meridians, that any two such choices of word will describe the same group element.

2.3. Loop-longitudes. We will now define a notion of longitudes, associated to homotopy classes of paths on $\Sigma$. For that, we will need the following.

Lemma 2.10. If $\gamma$ and $\gamma'$ are be two homotopic generic paths in $\Sigma$, then $w_\gamma = w_{\gamma'}$ in $G(C)$.

Proof. Since they are homotopic, $\gamma$ and $\gamma'$ differ by a finite sequence of the moves $H_1$, $H_2$ and $H_3$, described in Lemma 2.6, it is sufficient to deal with each of them separately.

The case of move $H_3$ is clear.

In the case of move $H_2$, $\gamma$ splits into $\gamma_1, \gamma_2$, where $\gamma_1$ starts at the same region $B$ as $\gamma$, while $\gamma_2$ starts at the region $A$ supporting the local move:

![Diagram]

Figure 7. An example of word associated to a path

Given a generic path $\gamma$ on $\Sigma$, we associate unique words $\tilde{w}_\gamma$ and $w_\gamma$ of the regions of $C$ as follows. For the $k$-th intersection point between $\gamma$ and $C$ met when running $\gamma$ according to its orientation, denote by $A_k$ the label of the cut domain met at this point, and by $e_k = \pm 1$ the local sign of this intersection point. This local sign is $1$ if the local orientation given by the orientation of $\gamma$ and $C$ agrees with the ambient orientation of $\Sigma$, and is $-1$ otherwise. Then the word $\tilde{w}_\gamma$ associated to $\gamma$ is given by

$$\tilde{w}_\gamma := A_1^{e_1} \cdots A_j^{e_j} \cdots A_k^{e_k}$$

whereas the normalized word $w_\gamma$ is given by

$$w_\gamma := A_1^{\gamma_1} \cdots A_j^{\gamma_j} \cdots A_k^{\gamma_k},$$

where $A$ is the region where $\gamma$ starts, and $|\gamma|$ is the sum of the exponents $e_j$ in $\tilde{w}_\gamma$ such that $A_i$ is in the same connected component as $\gamma$; see Figure 7 for an example.

Hence any generic path on $\Sigma$ gives rise in this way to elements $\tilde{w}_\gamma$ and $w_\gamma$ in the group $G(C)$, simply by regarding each region as the associated meridian in the group.

Note that homotopies through generic paths clearly preserve the associated words.
Then we have \( w_γ = B^s \tilde{w}_γ \), while \( w_γ' = B^{-s} \tilde{w}_γ A^s \tilde{w}_γ \) for some \( s \in \mathbb{Z} \) and some \( \varepsilon = \pm 1 \). But, since \( γ_1 \) connects \( B \) to \( A \), we have \( A = B^{h_1} \) so that, in \( G(C) \):

\[
w_γ = B^{-s} \tilde{w}_γ (B^s)^{w_2} = w_γ.
\]

In the case of move \( H_1 \), we may assume, up to some moves \( H_1 \), that \( γ \) splits as follows into \( γ_1.γ_2 \):

![Diagram](image)

so that \( w_γ = D^i \tilde{w}_{γ_1} \tilde{w}_{γ_2} \) and \( w_γ' = D^i \tilde{w}_{γ_1} (A^s)^{C^s} B^{-s} \tilde{w}_{γ_2} \) for some \( s \in \mathbb{Z} \) and some \( \varepsilon = \pm 1 \), where \( A, B \) and \( C \) are the labels as shown above, and \( D \) is the starting region of \( γ \). But, by the first labeling condition, the pair \((A, B)\) is \( C \)-adjacent, hence we have \( A^C B^{-1} = 1 \) in \( G(C) \).

Now, for each \( i \in \{1, \ldots, ℓ\} \), pick an \( i \)-th meridian \( R_i \) and a basepoint \( p_i \in \Sigma_i \) in the interior of the corresponding region. This provides in particular a preferred set of normal generators for \( G(C) \). Then, thanks to Lemma \( 2.10 \) we can define the \( i \)-th loop-longitude map

\[
\lambda_i : \pi_i(Σ_i, p_i) \to G(C),
\]

\[
[γ] \mapsto w_γ
\]

where \([γ]\) is an element of \( \pi_i(Σ_i, p_i) \) represented by a generic loop \( γ \) based at \( p_i \). Observe that \( \lambda_i \) is actually a group homomorphism. Indeed, for any two loops \( γ_1, γ_2 \) based at \( p_i \), it follows from Remark \( 2.8 \)(2) that \( w_{γ_1 γ_2} = R_i^{γ_1} R_i^{γ_2} \tilde{w}_{γ_1} \tilde{w}_{γ_2} = R_i^{γ_1} \tilde{w}_{γ_1} R_i^{γ_2} \tilde{w}_{γ_2} = w_{γ_1} w_{γ_2} \).

**Definition 2.11.** The elements in \( \text{Im}(\lambda_i) \) are called \( i \)-th (preferred) loop-longitudes for \( C \). They are well-defined up to the choice of \( p_i \), that is up to a simultaneous conjugation of \( R_i \), and all \( i \)-th longitudes by some element \( w \in G(C) \).

In general, \( C \) has an infinite number of loop-longitudes, and it will sometimes be useful to consider only a finite generating set of these loops.

**Definition 2.12.** Suppose that we are given, for each component \( Σ_i \) of \( Σ \), a collection of loops \([γ_{ij}]\) based at \( p_i \), which represent a generating set for \( \pi_i(Σ_i, p_i) \). We call system of loop-longitudes for \( Σ \) the collection of associated preferred \( i \)-th loop-longitudes, each denoted by \( w_{ij} := w_{γ_{ij}} \in G(C) \).

Observe that any system \([w_{ij}]\) of loop-longitudes determines the whole loop-longitude maps \( \lambda_i \).

### 2.4. Arc-longitudes

By Section \( 2.2 \) the homotopy class of any path on \( Σ \) leads to a normalized word, which represents a well-defined element of \( G(C) \) by Lemma \( 2.10 \). It is hence possible to consider a wider range of longitudes. As a matter of fact, if \( ∂Σ \neq ∅ \), we shall also consider ‘boundary-to-boundary’ longitudes. So, for each \( i \in \{1, \ldots, ℓ\} \) such that \( ∂Σ_i \neq ∅ \), we fix boundary basepoints \( p_{ij}, j \in \{0, \ldots, |∂Σ_i| − 1\} \) on each boundary component of \( Σ_i \). All these boundary basepoints are chosen disjoint from \( C \). This provides an ordering of the components of \( ∂Σ_i \), and the one containing \( p_0 \) should be regarded as a ‘marked’ boundary component of \( Σ_i \).

We set \( \mathcal{A}(Σ_i, [p_{ij}]) \) to be the set of homotopy classes of paths from \( p_0 \) to some \( p_{ij} \) for \( j > 0 \), up to homotopy, and we define the \( i \)-th arc-longitude maps

\[
\lambda_i^0 : \mathcal{A}(Σ_i, [p_{ij}]) \to G(C),
\]

\[
[γ] \mapsto w_γ
\]

Here, \([γ]\) is the element of \( \mathcal{A}(Σ_i, [p_{ij}]) \) represented by a generic path \( γ \) from \( p_0 \) to some \( p_{ij} \). Elements of \( \text{Im}(\lambda_i^0) \) are called \( i \)-th (preferred) arc-longitudes.

We stress that the well-definedness of arc-longitudes requires the choice of the basepoints \([p_{ij}]\), and that these are kept fixed at all time.

---

5 As noted at the end of the introduction, we will usually omit the brackets in our notation in the rest of the paper.
Remark 2.13. If $\Sigma_i$ is not simply connected, for each $j > 0$, $\mathcal{A}(\Sigma_i, \{p_{i,j}\})$ contains several elements ending at $p_{i,j}$, possibly infinitely many. However, fixing a generic path $\gamma_i$ from $p_1$ to $p_0$, there is a simply transitive left action of $\pi_1(\Sigma_i, p_1)$ on $\mathcal{A}(\Sigma_i, \{p_{i,j}\})$, defined as $\tau \cdot \gamma := \tau \gamma_i \cdot \gamma$. A system of arc-longitudes, made of one arc-longitude per $j > 0$, is hence sufficient to recover all elements of $\mathcal{A}(\Sigma_i, \{p_{i,j}\})$.

2.5. Peripheral systems. In the sequel, we shall simply use the terminology longitude when referring to either a preferred loop- or arc-longitude. We shall denote by

$$\lambda_i : \pi_1(\Sigma_i, p_i) \sqcup \mathcal{A}(\Sigma_i, \{p_{i,j}\}) \rightarrow G(C)$$

the $i$-th longitude maps, simply defined by combining $\lambda'_i$ and $\lambda''_i$.

All the notions defined so far can now be gathered into the following.

Definition 2.14. A peripheral system of $C$ is the data $(G(C); \{R_i\}, \{\lambda_i\})$ of the group of $C$ together with the choice, for each component $\Sigma_i$, of a meridian $R_i$ and its longitude map $\lambda_i$.

Two peripheral systems $(G; \{R_i\}, \{\lambda_i\})$ and $(G'; \{R'_i\}, \{\lambda'_i\})$ of cut-diagrams over $\Sigma$ are equivalent if there exist an isomorphism $\varphi : G \rightarrow G'$ and elements $w_1, \ldots, w_\ell \in G'$ such that, for every $i \in \{1, \ldots, \ell\}$, we have $R'_i = \varphi(R_i)^{w_i}$ and

$$\lambda'_i(\gamma) = \begin{cases} \varphi(\lambda_i(\gamma))^{w_i} & \text{for every loop } \gamma \in \pi_1(\Sigma_i; p_i), \\ \varphi(\lambda_i(\gamma)) & \text{for every arc } \gamma \in \mathcal{A}(\Sigma_i, \{p_{i,j}\}). \end{cases}$$

We stress that the above notion of equivalence for peripheral systems involves only the loop-longitudes, and that arc-longitudes are not required to be conjugated. The endpoints $p_{i,j}$ of arc-longitudes are indeed fixed throughout.

For every $q \geq 1$, the $q$-th nilpotent peripheral system of $C$ is the data, associated to a peripheral system $(G(C); \{R_i\}, \{\lambda_i\})$, of the $q$-th nilpotent quotient $N_q G(C)$, together with the image of each $R_i$ in $N_q G(C)$ and the composite of each longitude map $\lambda_i$ with the projection from $G(C)$ to $N_q G(C)$. Definition 2.14 naturally induces a notion of equivalence for $q$-th nilpotent peripheral systems.

Remark 2.15. As far as the authors know, a general notion of (preferred) longitude for knotted surfaces has not been clearly defined in the literature. It is however natural to define those as the image, in the fundamental group of the surface complement, of either a cycle or a (canonically closed) boundary-to-boundary path on the surface, pushed out of the surface in such a way that it has homological intersection zero with the surface. In this setting, it is clear that the preferred longitudes of a topological 2–dimensional cut-diagram agrees with the preferred longitudes of the underlying knotted surface. This remains true in higher dimensions.

3. Chen homomorphisms and a Chen–Milnor theorem for cut-diagrams

The purpose of this section is to give an analogue, for cut-diagrams, of the Chen–Milnor presentation for the nilpotent quotients of fundamental group of link complements. In the context of links, Milnor established in [22], building on earlier works of Chen [7], such a presentation, with one generator per component, and meridian/loop-longitude commutation relations.

In what follows, $C$ denotes an $n$–dimensional cut-diagram over $\Sigma = \sqcup_{i=1}^\ell \Sigma_i$. We also denote by $C_i = C \cap \Sigma_i$ the $i$-th component of $C$.

3.1. Road networks and the Chen homomorphisms. A central tool in establishing the Chen–Milnor presentation for links is an inductively defined sequence of maps $\eta_i$. These are not only a central theoretical tool, but they also provides an effective algorithm to compute the image of preferred longitudes in nilpotent quotients, by expressing recursively each $i$-th meridian as a conjugate of a chosen meridian $R_i$. The major difference with the link case is that, in the context of cut-diagrams, there is no canonical way to express these conjugations; as a matter of fact, we shall fix paths joining any region to the basepoint of its connected component.

Fix a basepoint $p_i$ on each connected component $\Sigma_i$, away from $C$. Regions of $C$ living in $\Sigma_i$ will be denoted by $R_{i,j}$, with $R_i := R_{i,0}$ the region containing $p_i$. We also denote by $F = \langle R_i \rangle$ and $\hat{F} = \langle R_{i,j} \rangle$ the free group generated by the $R_i$ and the $R_{i,j}$, respectively.
Definition 3.1. A road network \( \alpha \) for \( C \) is the choice, on each component \( \Sigma_i \) of \( \Sigma \), of a collection of oriented generic paths \( \alpha_{ij} \), called roads, running from \( p_i \) to a point in each region \( R_{ij} \). We shall sometimes stress the choice of basepoints by referring to \( \alpha \) as a road network based at \( \{ p_i \} \).

Following Section 2.2 a word \( v_{ij} := \tilde{w}_{\alpha_{ij}} \in F \) can be associated to each road \( \alpha_{ij} \). Notice that, from the Wirtinger presentation of \( G(C) \), the relation \( R_{ij} = R_{ij}^{(v)} \) holds in \( G(C) \) (see Remark 2.8(1)).

Definition 3.2. We define Chen homomorphisms \( \eta_q : \tilde{F} \longrightarrow F \) by setting, for every \( i, j \) and \( q \geq 1 \):

\[
\eta_q^i(R_{ij}) := R_i, \\
\eta_q^{i+1}(R_{ij}) := R_i \quad \text{and} \quad \eta_q^{i+1}(R_{ij}) := R_{ij}^{(v)}. 
\]

As the notation suggests, this sequence of homomorphisms \textit{a priori} depends on the choice of road network \( \alpha \). However, for simplicity we will often drop the \( \alpha \) from the notation.

Remark 3.3. Since \( F \) is a subgroup of \( \tilde{F} \), one can compose \( \eta_q^i \) with itself, and it follows from its definition that \( \eta_q^i \circ \eta_q^i = \eta_q^i \). This simple fact will be important later, when dealing with the dependence on the choice of road network.

Remark 3.4. We use the terminology Chen homomorphisms for these \( \eta_q \) maps. In the study of links, they indeed first appeared, implicitly, in the work of Chen [7]. Specifically, in the notation of [7, §4], we have \( \eta_q = \varphi \circ \psi^{\pm} \). The inductive definition for \( \eta_q \) appeared in [22].

Remark 3.5. In the case of a duplicated presentation for \( G(C) \), \( F \) is still the free group generated by a single meridian \( R_i \) for each component \( \Sigma_i \), but \( \tilde{F} \) is now defined as the free group generated by \textit{all meridians} \( R_{ij}^{(v)} \), rather than all regions. A road network \( \alpha \) is then a choice, for each component \( \Sigma_i \), of a collection of oriented generic paths from the basepoint \( p_i \) to regions \( R_{ij}^{(v)} \), one for each meridian associated to \( R_{ij} \). This means in particular that distinct roads can be chosen for distinct meridians associated to the same region; hence several roads may run from \( p_i \) to this region. To each such road \( \bar{r} \), one associates a choice of word \( \tilde{w}_r \in \tilde{F} \) as explained in Remark 2.9 and the definition of the Chen homomorphisms proceeds just as in Definition 3.2.

Note that, if the meridian \( R_i \) is duplicated, then only one copy \( R_{ij}^{(v)} \) identifies as the ‘base’ meridian associated to \( p_i \), while other copies are treated as any other meridians, i.e. are connected to \( p_i \) by some roads and give rise to conjugations in the definition of the Chen homomorphisms. We note that Lemmas 3.7 and 3.8 below clearly hold in the duplicated setting.

We now derive some elementary properties of the Chen homomorphisms \( \eta_q \), and for that, we will often make use of the following elementary fact.

Claim 3.6. Let \( G \) be a group, \( N \) a normal subgroup of \( G \), and \( q \geq 1 \). For any \( x, y, a \in G \), if \( x \equiv y \mod G_{q}N \), then \( a^x \equiv a^y \mod G_{q+1}N \).

Proof. It follows from the equivalence \( a^x = y^{-1}[xy^{-1}, a^{-1}]ay \equiv a^y \mod G_{q+1} \).

Lemma 3.7. For all \( q \geq 1 \), and all \( w \in \tilde{F} \), \( \eta_q(w) \equiv \eta_{q+1}(w) \mod F_q \).

Proof. The claim is obvious for \( q = 1 \), since \( F_1 = F \). Assume now that the claim holds for some \( q \geq 1 \). Then it suffices to consider the case \( w = R_{ij} \) for any \( i, j \). By induction hypothesis, \( \eta_q(v_{ij}) \equiv \eta_{q+1}(v_{ij}) \mod F_q \) so that, by Claim 3.6, \( \eta_{q+1}(R_{ij}) = R_{ij}^{(v)} \equiv R_{ij}^{(v)} \equiv \eta_{q+1}(R_{ij}) \mod F_{q+1} \).

Lemma 3.8. For any generator \( R_{ij} \) of \( \tilde{F} \), \( \eta_q(R_{ij}) \equiv R_{ij} \mod \tilde{F}_q \).

Proof. We proceed by induction on \( q \). The case \( q = 1 \) is trivial. Assume now that \( \eta_q(v_{ij}) \equiv v_{ij} \mod F_q \).

Then, by Claim 3.6 and the definition of \( \eta_{q+1} \), we have that \( \eta_{q+1}(R_{ij}) = R_{ij}^{(v)} \equiv R_{ij} \mod \tilde{F}_{q+1} \).

An immediate consequence of Lemma 3.8 is the following, showing that at the \( q \)-th nilpotent level, the choice of road network has no effect on the \( \eta_q \) map.

Corollary 3.9. Let \( \alpha, \alpha' \) be two road networks for \( C \), possibly with different basepoints. Then, for any \( w \in \tilde{F} \), \( \eta_q^i(w) \equiv \eta_q^{i'}(w) \mod \tilde{F}_q \), and hence \( \eta_q^i(w) = \eta_q^{i'}(w) \) in \( N_q G(C) \).

However we shall see later, in Lemma 4.18, that the choice of road network has more subtle consequences at the level of word representatives for elements of \( N_q G(C) \).
3.2. Nilpotent quotients. We can now give a first presentation for the nilpotent quotients of \( G(C) \), with as many generators as components in \( C \), and only nilpotent and meridian-loop-longitude commutation relations. The main result of this section, the Chen–Milnor presentation of Theorem \ref{thm:chen_milnor} will later significantly reduce the number of relations.

Before stating our first presentation, we need to define walls of \( C \), which are connected components of regular points of \( C \). They actually correspond to regular ‘walls’ between adjacent regions. Note that each cut domain is a union of walls (and some higher singularity loci), so that walls inherit a labeling by regions. Consider now some wall \( a \) of \( C \), say on the \( i \)-th component and labeled by some region \( R \). Recall from Section \ref{sec:loops} that the Wirtinger relation \( R_{ik} = R_{ik}^q \) holds in \( G(C) \), where \( (R_{ij}, R_{ik}) \) is the pair of \( R \)-adjacent regions at \( a \). Consider both roads \( \alpha_{ij}, \alpha_{ik} \in \alpha \) running from \( p_i \) to \( R_{ij} \) and \( R_{ik} \), respectively, and denote as above by \( v_{ij} \) and \( v_{ik} \) the associated words in \( \bar{F} \). We define the word \( w_a \in \bar{F} \) by

\[
\gamma_a := v_{ij} R_{ik}^{-1},
\]

which is actually the (not necessarily normalized) word associated to the \( p_i \)-based loop defined by connecting \( \alpha_{ij} \) to \( \alpha_{ik}^{-1} \) with a segment crossing \( a \) transversally.

**Notation 3.10.** For all \( q \geq 1 \), we denote by \( C_{(q)} \) the normal closure of all relations \([R_i, \eta_q(w_a)] \) in \( F \), for all walls \( a \) in \( C \), with \( w_a \) defined in Equation \ref{eq:gamma_a} above.

**Lemma 3.11.** For all \( q \geq 1 \), we have an isomorphism

\[
N_q G(C) \cong \bar{F}/F_{q^c} C_{(q)}.
\]

In other words, \( N_q G(C) \) has the following presentation

\[
\left\langle R_1, \ldots, R_l \mid F_q : [R_i, \eta_q(w_a)] \text{ for all } i \text{ and all walls } a \text{ on } C_i \right\rangle.
\]

**Proof.** The homomorphism \( \eta_q \) naturally descends to a map \( N_q F \to \bar{F}/F_{q^c} C_{(q)} \). Now consider, with the above notation, the Wirtinger relator \( R_{ik} R_{ij}^q \) at some wall \( a \) on \( C \). We have

\[
\eta_q \left( R_{ik} R_{ij}^q \right) = \left( R_{ik} R_{ij}^{-1} \right)^{-1} \eta_q \left( R_{ik} R_{ij}^{q_1} \right) \eta_q \left( R_{ij} \right) = \left( R_{ik} R_{ij}^{q_1} \right)^{-1} \eta_q \left( R_{ij} \right) \eta_q \left( R_{ik} \right) = \left[ R_i \eta_q(w_{a}) \right] \eta^q \mod F_q,
\]

where the equivalence comes from Lemma \ref{lem:Wirtinger} and Claim \ref{claim:hom}. This shows that \( \eta_q \) induces a well-defined epimorphism from \( N_q G(C) \) to \( \bar{F}/F_{q^c} C_{(q)} \). It is easily checked, using Lemma \ref{lem:hom} and the above equivalence, that the natural inclusion map \( F \to \bar{F} \) yields an inverse for this latter map, which is thus an isomorphism. \( \square \)

We note that we have the following seemingly weaker form of Lemma \ref{lem:iso} with an infinite number of commutator relations. This version shall however be useful on several occasions.

**Proposition 3.12.** For all \( q \geq 1 \), the group \( N_q G(C) \) has the following presentation

\[
\left\langle R_1, \ldots, R_l \mid F_q : [R_i, \eta_q(w_{\gamma})] \text{ for all } i \text{ and all } p_i \text{-based generic loop } \gamma \text{ on } \Sigma_i \right\rangle.
\]

**Remark 3.13.** Notice that we may equally well use relations \([R_i, \eta_q(\tilde{\gamma}_{\gamma})] \), with the words \( \tilde{\gamma}_{\gamma} \) in the above presentation. Indeed we have, for some \( s \in \mathbb{Z} \):

\[
[R_i, \eta_q(\tilde{\gamma}_{\gamma})] = [R_i, \eta_q(R_i^{s} \tilde{\gamma}_{\gamma})] = [R_i, R_i^{s} \eta_q(\tilde{\gamma}_{\gamma})] = [R_i, \eta_q(\tilde{\gamma}_{\gamma})].
\]

**Proof.** Using Remark \ref{rem:relation} we have that relations in Lemma \ref{lem:iso} are special cases of the relations in Proposition \ref{prop:iso}. It is hence sufficient to show that, for any \( i \)-th preferred loop-longitude \( w_{\gamma} \), the relation \([R_i, \eta_q(\tilde{w}_{\gamma})] \) can be deduced from the relations in \( \{ [R_i, \eta_q(w_{\gamma})] \mid a \text{ a wall of } C \} \). The proof is essentially given in the figure below, as follows. Let \( \gamma \) be a \( p_i \)-based generic loop on \( \Sigma_i \). The associated word \( w_{\gamma} \) is of the form

\[
w_{\gamma} := R_i^{s_1} R_i^{s_2} R_i^{s_3} \cdots R_i^{s_n},
\]
for some \( m, s \geq 0 \) and \( a_k \in \{ \pm 1 \} \), where the \( R_{a_k} \in \tilde{F} \) are the labels of the successive walls \( a_k \) met transversally when running along \( \gamma \); see below on the left.

Then, as illustrated in the figure above, a homotopy of \( \gamma \), guided by the road network \( \alpha \) and involving only moves \( H_1 \) of Lemma 2.6 gives directly the decomposition

\[
w_\gamma = R_i^1 w_{i_1}^1 w_{i_2}^2 \cdots w_{i_n}^n,
\]

where the words \( w_{i_n} \) are defined in Equation (3.1). The commutation of \( R_i \) with \( w_\gamma \) then follows easily from the commutation of \( R_i \) with each \( w_{i_n} \).

\[ \square \]

Remark 3.14. The statement of Proposition 3.12 can be adapted to the case of a duplicated presentations for \( G(C) \). But in this case, in the presentation for \( N_q G(C) \), the relations \( [R_i, \eta_q(w_\gamma)] \) have to be considered for all \( i \), all \( p_i \)-based generic loop \( \gamma \) on \( \Sigma \), and all representative words \( w_\gamma \in \tilde{F} \). Recall indeed from Remark 2.9 that, in the case of a duplicated presentation, the definition of the word \( w_\gamma \) associated with a path \( \gamma \) involves a choice: here we consider one relation for each possible choice. The proof then follows strictly the same lines, with only the following two adjustments:

- the definition of \( w_{i_n} \), given in Equation (3.1), has to be considered in several versions, one for each meridian associated to the three involved regions \( R, R_i \), and \( R_k \);
- the relations \( (R_i^j)^{-1} R_i^j \) in \( G(C) \), corresponding to multiple meridians associated to a region \( R \) of \( \Sigma \), induce extra relations \( [R_i, \eta_q(w_{\gamma\gamma'})] \) in the statement of Lemma 3.11 where \( \gamma \) and \( \gamma' \) are the roads associated to \( R_i^j \) and \( R_i^k \), respectively. But all these relations are directly of the type meridian/loop-longitude commutations of Proposition 3.12 hence this latter statement remains true in the duplicated settings.

3.3. Chen–Milnor type presentations. For each component \( \Sigma_i \) of \( \Sigma \), pick a system of loop-longitudes \( L_i(\Sigma) := \{ w_{ij} \} \) associated with a collection of loops \( \{ \gamma_{ij} \} \) based at \( p_i \) (see Definition 3.12). The following is the first main result of this paper.

**Theorem 3.15.** If \( \mathcal{C} \) is a cut-diagram over \( \Sigma \), then for each \( q \in \mathbb{N} \) we have the following presentation for the \( q \)-th nilpotent quotient \( N_q G(C) \) of \( G(C) \):

\[
\left\langle R_1, \ldots, R_l \mid F_q; [R_i, \eta_q(w_{ij})] \text{ for all } i \text{ and all } w_{ij} \in L_i(\Sigma) \right\rangle.
\]

**Proof.** The proof builds on the presentation given in Proposition 3.12 the idea is to show that the infinite family of commutation relations in Proposition 3.12 involving all generic loops on \( \Sigma \), can be realized by just the commutation relations involving the finite collection loops in \( \{ \gamma_{ij} \} \). In fact, we shall prove more precisely that \( F_q \cdot L(\mathcal{Q}) = F_q \cdot V(\mathcal{Q}) \), where

- \( L(\mathcal{Q}) \) is the normal closure of \( \{ [R_i, \eta_q(w_{ij})] \mid \gamma \text{ any } p_i \text{-based generic loop on } \Sigma_i \} \) in \( F \),
- \( V(\mathcal{Q}) \) is the normal closure of \( \{ [R_i, \eta_q(w_{ij})] \mid w_{ij} \in L_i(\Sigma) \} \) in \( F \).

The inclusion \( F_q \cdot V(\mathcal{Q}) \subset F_q \cdot L(\mathcal{Q}) \) is clear. To prove the converse inclusion, it suffices to show that the map \( F_q / F_q \cdot V(\mathcal{Q}) \to F_q / F_q \cdot L(\mathcal{Q}) \), induced by identity, is injective. This is a consequence of Proposition 3.16 below. Indeed, since any loop on \( \Sigma_i \) is homotopic to a product of elements in \( \{ \gamma_{ij} \} \), this proposition in particular implies that \( [R_i, \eta_q(w_{ij})] \equiv 1 \mod F_q \cdot V(\mathcal{Q}) \) for any \( p_i \)-based loop \( \gamma \) on \( \Sigma_i \), and the result follows.

**Proposition 3.16.** Let \( \mathcal{C} \) be a cut-diagram over \( \Sigma \). For any two homotopic arcs \( \gamma, \gamma' \) on \( \Sigma \), we have

\[ \eta_q(w_\gamma) \equiv \eta_q(w_{\gamma'}) \mod F_q \cdot V(\mathcal{Q}). \]
We stress that the paths $\gamma, \gamma'$ in the statement may have arbitrary endpoints, and are in particular not required to start at the region $R$.

**Proof.** Note that the case $q = 1$ of the claim is trivial, and assume inductively that Proposition 3.15 holds for some $q \geq 1$. We first observe the inclusion

$$F_{q+1} \cdot V(q) \subset F_{q+1} \cdot V(q+1), \text{ for all } q \geq 1.$$  

This is true, since by Lemma 3.7 and Claim 3.6, we have $[a_i, \eta_q(w_{ij})] \equiv [a_i, \eta_{q+1}(w_{ij})] \bmod F_{q+1}$ for any $i, j$.

Now, two homotopic arcs $\gamma, \gamma'$ differ by a sequence of the three local moves $H_1$, $H_2$ and $H_3$ of Lemma 2.6.

So it suffices to check the claim when $\gamma$ and $\gamma'$ differ by one of these three moves. In the rest of this proof, for a given region $R$, we shall denote by $v_R$ the word associated with the road, from the road network $\alpha$, running to $R$.

- If $\gamma$ and $\gamma'$ differ by move $H_1$, then $w_\gamma = w_{\gamma'}$ in $\mathcal{F}$ so the desired equivalence holds trivially.

- We now consider the case where $\gamma$ and $\gamma'$ differ by move $H_2$, with $\gamma$ corresponding to the left-hand side of the move. We then have

$$w_\gamma = B^i \tilde{w}_y \tilde{w}_z \text{ and } w_{\gamma'} = B^{i+e} \tilde{w}_y A^e \tilde{w}_z,$$

where $\gamma_1, \gamma_2$ are paths such that $\gamma = \gamma_1 \cdot \gamma_2$, $s \in \mathbb{Z}$, $e = \pm 1$, and $B$ is the region of $\Sigma_i$ where $\gamma_1$ starts:

![Diagram](https://via.placeholder.com/150)

By definition of the Chen homomorphism, we have

$$\eta_q(w_\gamma) = \eta_q(v_{B}^{-1} R_y \omega^{i+e} (R_y)^{-1}) \eta_q(v_B) \eta_q(\tilde{w}_y) \eta_q(\tilde{w}_z).$$

Observe that $\eta_q(\tilde{w}_y) \equiv \eta_q(v_{B}^{-1} v_B) \equiv \eta_q(v_{B}) \equiv \eta_q(\tilde{w}_y) \bmod F_q \cdot V(q)$.

This implies, by Claim 3.6 that

$$\eta_{q+1}(w_{\gamma'}) \equiv \eta_{q+1}(v_{B}^{-1} R_y \omega^{i+e} (R_y)^{-1}) \eta_{q+1}(v_B) \eta_{q+1}(\tilde{w}_y) \eta_{q+1}(\tilde{w}_z) \equiv \eta_{q+1}(w_\gamma) \bmod F_{q+1} \cdot V(q),$$

where the last equivalence simply uses the fact that $[R_y, \eta_q(\tilde{w}_y)] = [R_y, \eta_q(\tilde{w}_y)] \equiv 1 \bmod V(q)$. The conclusion then follows from (3.2).

- Suppose now that $\gamma$ and $\gamma'$ differ by move $H_3$, with $\gamma$ corresponding to the left-hand side of the move. Then, up to some moves $H_1$, we may assume that the situation is as shown below, so that there exists two arcs $\gamma_1, \gamma_2$ on $\Sigma_i$ such that

$$w_\gamma = D^i \tilde{w}_y \tilde{w}_z \text{ and } w_{\gamma'} = D^i \tilde{w}_y (A^e)^{-1} B^{-e} \tilde{w}_z,$$

for some $s \in \mathbb{Z}$ and some $e \in \{\pm 1\}$, and where $R_{ik}$ is the starting region of $\gamma_1$ and $(A, B)$ is the pair of $C$-adjacent cut domains on $\Sigma_j$ involved in the move.

![Diagram](https://via.placeholder.com/150)

One can then easily check, using the definition of the Chen homomorphisms, that we have

$$\eta_{q+1}(w_{\gamma'}) = \eta_{q+1}(D^i \tilde{w}_y) [X, R_y^{-1}] \eta_{q+1}(v_B) \equiv X \equiv \eta_q(v_A C v_B^{-1}) \bmod F_q.$$  

where we set $X := \eta_q(v_A) \eta_{q+1}(C) \eta_q(v_B^{-1})$. By Lemma 3.7, we have $X \equiv \eta_q(v_A C v_B^{-1}) \bmod F_q$. And the word $v_A C v_B^{-1}$ is associated to a $p_j$-based loop on $\Sigma_j$, which is thus homotopic to a product $\xi$ of
elements in $\{\gamma^a_{pk}\}$. So the induction hypothesis gives that $X \equiv \eta_q(\tilde{w}_G) \mod F_q \cdot V(q)$. Claim 3.6 then gives

$$[X, R^\xi_j] = (R^\xi_j)^X R^{-\xi}_j \equiv (R^\xi_j)^{\eta_q(\tilde{w}_G)} R^{-\xi}_j \equiv 1 \mod F_{q+1} \cdot V(q).$$

And by (3.2), this implies that

$$\eta_{q+1}(w_v) \equiv \eta_{q+1}(D^v w) \eta_{q+1}(\tilde{w}_G) \equiv \eta_{q+1}(w_v) \mod F_{q+1} \cdot V(q+1).$$

This concludes the proof.

**Remark 3.17.** Theorem 3.15 does not only provide a presentation for $N_qG(C)$ derived from a system of loop-longitudes, it also gives, via the Chen homomorphisms, an algorithm to compute representative words for any element in $N_qG(C)$, and in particular for the longitudes in these quotient.

**Remark 3.18.** The statement of Theorem 3.15 still holds in the context of duplicated presentation for $G(C)$. Following Remark 3.14, Proposition 3.12 can indeed be used in the same way. The proof of Proposition 3.16 requires however one extra argument: in that case, two representative words for a given longitude differ from a finite sequence of four moves, the three moves $H_1$, $H_2$ and $H_3$ of Lemma 2.6, and a fourth move which replaces a letter $R^j$ by another meridian $R^{j'}$ associated to the same region $R$ of $\Sigma_k$, for some $k$. It is hence necessary to deal, in the induction process, with this last move. So let $w = w_1(R^j)^{k_1}$ and $w' = w_1(R^{j'})^{k_2}$ be two words differing by this move, and denote by $v$ and $v'$ the two words associated to the roads joining the basepoint $p_k$ to $R^j$ and $R^{j'}$, respectively. Since $v'v^{-1}$ corresponds to a $p_k$–based loop, by induction hypothesis, we have $\eta_q(v'v^{-1}) \equiv \eta_q(\tilde{w}_G) \mod F_q \cdot V(q)$ for some product $\eta_q$ of elements in $\{\gamma^a_{pk}\}$; it follows from Claim 3.6 that

$$[\eta_q(v'v^{-1}), R^\xi_j] \equiv [\eta_q(\tilde{w}_G), R^\xi_j] \equiv 1 \mod F_{q+1} \cdot V(q).$$

Then observe that

$$\eta_{q+1}(w_v) \equiv \eta_{q+1}(w_1)(R^\xi_j)^{\eta_q(v)} \eta_{q+1}(w_2) = [\eta_q(v'v^{-1}), R^\xi_j]^{\eta_q(v) \eta_{q+1}(v^{-1})} \eta_{q+1}(w) \equiv \eta_{q+1}(w) \mod F_{q+1} \cdot V(q+1),$$

so that $\eta_{q+1}(w_v) \equiv \eta_{q+1}(w) \mod F_{q+1} \cdot V(q+1)$ by (3.2). This concludes the proof.

A major advantage of duplicated presentations is that it allows, in the algorithm leading to the explicit relations in Theorem 3.15, to use distinct roads for each occurrence of a same region in the longitudes $\{w_j\}$. This fact will play a key role to prove, in Theorem 5.5, the invariance of the nilpotent peripheral system under concordance.

4. **Milnor invariants for cut-diagrams**

The purpose of this section is to define Milnor invariants for any cut-diagram $C$ on $\Sigma$. In the classical setting of links in the 3–sphere, Milnor invariants are extracted from the coefficients in the Magnus expansion of the fundamental group of the link exterior. Now, as discussed in Sections 2.3 and 2.4, the notion of (preferred) longitudes for cut-diagrams comes in two flavors: loop-longitudes and arc-longitudes. As a matter of fact, although coming from a same global construction that associates a collection of ‘Milnor numbers’ to any generic path on $\Sigma$ (Section 4.1), Milnor invariants defined from loop and arc-longitudes have rather different behaviors, and we shall discuss them separately (Sections 4.2 and 4.3).

Let us recall some notation used in the previous sections: $\Sigma = \bigcup_{i=1}^{l} \Sigma_i$ is an oriented $n$–dimensional manifold, and $C$ is a cut-diagram over $\Sigma$. For all $i$, regions of $\Sigma_i$ are denoted by $[R_i]$. We also fix, for each $i \in \{1, \ldots, l\}$ such that $\partial \Sigma_i \neq \emptyset$, a point $p_0$ on each boundary component of $\Sigma_i$ ($k \in \{0, \ldots, |\partial \Sigma_i| - 1\}$), and consider the component containing $p_0$ as a marked boundary component.

4.1. **Milnor numbers associated with a longitude.** Fix an integer $q \geq 1$. Although we are going to define here invariants for the cut-diagram $C$, we emphasize that our construction actually will only depends on its $q$–th nilpotent peripheral system, as defined in Section 2.5. When referring to longitudes, we will often implicitly mean their images in the $q$–th nilpotent quotient $N_qG(C)$.

Let us choose, for each $i$, an $i$–th meridian $R_i$, and denote by $F := \langle R_i \rangle$ the free group generated by these meridians. We also pick a basepoint $p_0$ on the region $R_i$. By Theorem 3.15, the above choice of an $i$–th meridian $R_i$ for each component of $C$ provides a generating set for $N_qG(C)$.
We also fix a system of loop-longitudes $L_i(\Sigma)$ for $\Sigma$. Recall that, for each $i$, the system $L_i(\Sigma)$ is given by the $i$-th loop-longitudes associated with a collection of loops on $\Sigma$ representing a generating set for $\pi_1(\Sigma, p_i)$.

Now, let $\gamma$ be a generic path in $\pi_1(\Sigma, p_i) \cup \mathcal{A}(\Sigma, \{p_j\})$, for some $i \in \{1, \ldots, \ell\}$. Let $\omega_\gamma$ be a representative word in $[R_1^{\pm 1}]$ for the associated $i$-th longitude in $N_G(C)$. This word is not to be confused with the normalized word $w_\gamma$, defined in Section 2.2 which is a word in the larger alphabet $\{R_i^{\pm 1}\}$. In what follows, and abusing terminologies, we shall often blur the distinction between the path $\gamma$ and the associated $i$-th longitude, seen either in $G(C)$ or in $N_G(C)$.

**Definition 4.1.** The Magnus expansion of an element $w \in F$ is the formal power series $E(w) \in \mathbb{Z}[\langle\ell\rangle]$ in $\ell$ noncommuting variables $X_1, \ldots, X_\ell$, obtained by substituting each $R_i$ by $1 + X_i$, and each $R_i^{-1}$ by $\sum_{k \geq 0} (-1)^k X_i^k$. For any sequence $I$ of at most $q$ integers in $\{1, \ldots, \ell\}$, we denote by $\mu_C(I; w)$ the coefficient of the monomial $X_{j_1} \cdots X_{j_\ell}$ in $E(w)$, meaning that

$$E(w) = 1 + \sum_{j_1 \cdots j_\ell} \mu_C(j_1 \cdots j_\ell; w) X_{j_1} \cdots X_{j_\ell},$$

where the sum runs over all sequences of (possibly repeated) integers in $\{1, \ldots, \ell\}$.

In particular, we can obtain in this way a collection of integers $\mu_C(I; \omega_\gamma)$, one for each sequence $I$ of integers in $\{1, \ldots, \ell\}$, from the representative word $\omega_\gamma$. Of course, the coefficients $\mu_C(I; \omega_\gamma)$ are not invariants for the $i$-th longitude $\gamma$, as they depend on the choice of the representative word $\omega_\gamma$. But Theorem 3.15 tells us precisely how two different representatives may differ. Indeed we have a presentation

$$N_G(C) \cong \left\langle R_i \mid F_q ; [R_i, \omega_j] \text{ for all } i, j \right\rangle,$$

where, for each $i$, the $\omega_j$’s are given word representatives in $\{R_i^{\pm 1}\}$ for the elements of the system of $i$-th loop-longitudes $L_i(\Sigma)$. Note that we do not need here the explicit formula for $\omega_i$, given in Theorem 3.15 in terms of the Chen homomorphisms.

In order to extract invariants from the numbers $\mu_C(I; \omega_\gamma)$, we introduce the following indeterminacies.

**Definition 4.2.** For any sequence $Ii$ of integers in $\{1, \ldots, \ell\}$, we define

- $m_c(I) := \gcd\{\mu_C(I; \omega_j) \text{ for all } j\}$;
- $\Delta_c(I)$ is the greatest common divisor of all $m_c(J)$, where $J$ is any sequence obtained from $I$ by deleting at least one index, and possibly permuting the resulting sequence cyclically.

We emphasize the fact that the definition of $\Delta_c(I)$ only involves loop-longitudes on $\Sigma_i$ for all $i = 1, \ldots, \ell$.

**Proposition 4.3.** For any sequence $I$ of at most $q - 1$ integers in $\{1, \ldots, \ell\}$ and any $\gamma \in \pi_1(\Sigma, p_i) \cup \mathcal{A}(\Sigma, \{p_j\})$ for some $i \in \{1, \ldots, \ell\}$, the residue class

$$\overline{n}_c(I; \gamma) := \mu_C(I; \omega_\gamma) \mod \Delta_c(I)$$

does not depend on the representative word $\omega_\gamma$.

This allows for the following definition.

**Definition 4.4.** The classes $\overline{n}_c(I; \gamma)$ are called Milnor numbers associated with $\gamma$.

We stress that, for Proposition 4.3, a choice of one meridian per component has been made, and $\overline{n}_c(I; \gamma)$ and $\Delta_c(I)$ may depend on this choice; this will be further discussed in the next subsection. The indeterminacy $\Delta_c(I)$ also depends on the choice of system of loop-longitude, which is addressed in Lemma 4.7.

Observe also that Milnor numbers are well-defined regardless of the initially fixed value for $q$. Indeed, the lower central series being decreasing, any representative word for an element of $G(C)$ in $N_G(C)$ is also a representative word for its image in $N_G(C)$, for any $q' < q$. In particular, by taking $q$ to be sufficiently large, Milnor numbers are defined for sequences of arbitrary length.

**Proof of Proposition 4.3.** We begin with a definition that already appears in the work of Milnor in the classical link case, and which will also be a key tool in our more general context. For every $i \in \{1, \ldots, \ell\}$, we denote by $D_i$ the set of all power series $\sum \nu(i_1 \cdots i_\ell) X_i \cdots X_i$ in $\mathbb{Z}[\langle\ell\rangle]$ whose coefficients satisfy $\nu(i_1 \cdots i_\ell) \equiv 0 \mod \Delta_C(i_1 \cdots i_\ell)$ for all $i_1 \cdots i_k$ with $k < q$. 


Note that, by definition, the Magnus expansions of two words representing an \( i \)-th longitude yield the same residue classes up to degree \( q \), if their difference is in \( D_1 \). Moreover, following verbatim Milnor’s argument in the link case, the following statements are straightforwardly checked.

**Claim 4.5.**

(i). The set \( D_1 \) is a two-sided ideal of \( \mathbb{Z}[\ell] \); see [22] (16).

(ii). For any \( j \in \{1, \ldots, l \} \) and any representative word \( \omega_j \in F \) for an \( i \)-th loop-longitude, both \( X_j(E(\omega_j) - 1) \) and \( (E(\omega_j) - 1)X_j \) are elements of \( D_1 \); see [22] (17).

(iii). Let \( \mu_c(j_1 \cdots j_k; \omega_j; X_j \cdots X_k) \) be any term in \( E(\omega_j) \), for some word \( \omega_j \in F \) representing an \( i \)-th loop-longitude, and let \( \mu_c(j_1 \cdots j_k; \omega_j) \) be a monomial obtained from \( X_j \cdots X_k \) by inserting at least one copy of \( X_j \) anywhere, then \( \mu_c(j_1 \cdots j_k; \omega_j)(X_j \cdots X_k) \) is in \( D_1 \); see [22] (18).

(iv). For any \( j \) and \( k \), both \( X_j(E(\omega_j) - 1) \) and \( (E(\omega_j) - 1)X_j \) are in \( D_1 \); see [22] (19).

(v). For any element \( w \in F_q \), \( E(w) \) is equivalent to \( 1 \) modulo \( D_1 \); see [22] (20).

Now, by (4.1), two word representatives of an element in \( D_1 G(C) \) differ by a finite sequence of insertions/deletions of

1. \( R_j^{-1}R_j^1 \) for some \( j \);
2. \( q \)-iterated commutators;
3. commutators \( [R_i, \omega_j] \) for some \( i \) and \( j \).

It is hence sufficient to check that the insertion of any such word modifies the Magnus expansion by an element of \( D_1 \). Case (1) is just clear. Case (2) is a direct consequence of (v) and (i). And case (3) can be shown, using (iv) and (i), with the exact same argument as in [22] pp. 294.

Furthermore, the techniques of the proof above also apply to the following.

**Lemma 4.6.** Let \( I \) be a sequence of integers in \( \{1, \ldots, l \} \), and let \( \omega_i \in F \) be a representative word of an \( i \)-th loop-longitude. Let \( w_1, w_2 \in F \). Then

\[
\mu_c(I; w_1, \omega_i w_2) \equiv \mu_c(I; \omega_i) + \mu_c(I; w_1, w_2) \mod D_1(I).
\]

**Proof.** The Magnus expansion being a group morphism, we have that

\[
E(w_1, \omega_i w_2) = E(w_1, w_2) + \left( 1 + (E(w_1) - 1) \right) \left( 1 + (E(w_2) - 1) \right).
\]

Using (ii) and (i) of Claim 4.5, this implies that \( E(w_1, \omega_i w_2) - E(\omega_i) - E(w_1, w_2) + 1 \) is in \( D_1 \), which gives the conclusion.

As a first application to this lemma, we have the following, which shall be of importance in the next subsection.

**Lemma 4.7.** For any sequence \( I \) of integers in \( \{1, \cdots, l \} \), we have that both \( \{\Delta_c(I), m_c(I)\} \) and the indeterminacy \( \Delta_c(I) \) are independent of the systems of loop-longitudes \( L(\Sigma) \), and of the choice of representative words \( \omega_j \).

**Proof.** If for some \( i, j \), we denote by \( \omega''_j \) another choice of representative word for the element of \( L(\Sigma) \) represented by \( \omega_j \), then by Proposition 4.3, we have \( \mu_c(I; \omega''_j) \equiv \mu_c(I; \omega_j) \mod \Delta_c(I) \). This shows that

\[
\gcd \{\Delta_c(I), m_c(I)\} = \gcd \{\Delta_c(I), \mu_c(I; \omega''_j) \mid \omega''_j \text{ any representative for the elements of } L(\Sigma) \}.
\]

Then, by definition of a loop-system, any \( i \)-th loop-longitude decomposes as a product of elements of \( L(\Sigma) \). By Lemma 4.6, we therefore have

\[
\gcd \{\Delta_c(I), m_c(I)\} = \gcd \{\Delta_c(I), \mu_c(I; \omega_i) \mid \omega_i \text{ any representative of any } i \text{-th loop-longitude} \},
\]

and the result follows for \( \gcd \{\Delta_c(I), m_c(I)\} \). The statement about \( \Delta_c(I) \) is obtained by induction on the length of \( I \), since \( \Delta_c(I) \) is the greatest common divisor of all \( m_c(J) \) and \( \Delta_c(J) \), when \( J \) runs over all sequences obtained from \( I \) by deleting one index and applying possibly a cyclic permutation. □
At this point, each generic path, and in particular each longitude of a cut-diagram, yields Milnor numbers. We now need to analyse how these numbers provide actual invariants of the cut-diagrams. On one hand, the initial choice of \( i \)-th meridians \( R_i \) must be discussed. On the other hand, comparing longitudes for two different cut-diagrams is in general not possible, as there is no natural identification between the elements of the fundamental groups of the underlying manifolds, and we must further refine the construction to get intrinsic invariants of our cut-diagram. As a matter of fact, the situation is quite different, depending on whether we start with loop-longitudes or arc-longitudes. The main point is that loop-longitudes are based at our chosen basepoint \( p_i \) in \( R_i \), so that invariance of the associated Milnor numbers must address the question of basepoint change, while arc-longitudes come with a natural choice of basepoint. We will hence discuss these two cases separately, in the next two subsections.

4.2. Milnor invariants for loop-longitudes. We first focus on the case where \( \gamma \) is (an element of \( \pi_1(\Sigma_i; p_i) \) representing) an \( i \)-th loop-longitude.

As mentioned above, we need to analyse the effect of replacing a basepoint by a conjugate. Such a basepoint change, say on the \( j \)-th component, transforms a nilpotent peripheral system into an equivalent one; the \( j \)-th meridian and loop-longitudes are then accordingly conjugated, as in Definition 2.14.

By Lemma 4.6, the residue class \( \overline{\mu}(I; \gamma) \) remains unchanged when replacing \( \gamma \) by a conjugate. Actually, Lemma 4.6 implies that, for fixed \( i \)-th meridians \( R_i \), Milnor numbers for loop-longitudes only depend on their homology class. Hence we can set the following.

Notation 4.8. For \( \lambda \in H_1(\Sigma_i) \) and any sequence \( I \) of integers in \( \{1, \ldots, \ell\} \), we set \( \overline{\mu}(I; \lambda) := \overline{\mu}(I; \gamma) \), where \( \gamma \) is a generic loop representing \( \lambda \).

In fact, the following will be proved at the end of this subsection.

Proposition 4.9. Let \( I \) be any sequence of integers in \( \{1, \ldots, \ell\} \), and let \( \lambda \in H_1(\Sigma_i) \). Then \( \Delta_C(I) \) and \( \overline{\mu}(I; \lambda) \) do not depend on the choice of meridians \( (R_i) \).

This leads to the following.

Definition 4.10. For any sequence \( I \) of integers in \( \{1, \ldots, \ell\} \), we define the Milnor map
\[
M^C_I : H_1(\Sigma_i) \to \mathbb{Z}/\Delta(I)[\mathbb{Z},
\]
by sending any \( \lambda \in H_1(\Sigma_i) \) to the residue class \( \overline{\mu}(I; \lambda) \).

Remark 4.11. In the familiar case of classical links, \( H_1(\Sigma_i) \cong \mathbb{Z} \) is generated by a preferred \( i \)-th longitude, for each \( i \), and we recover Milnor’s link invariants as the image of the corresponding Milnor maps; see Section 4.5.

Assuming Proposition 4.9, we obtain the following.

Theorem 4.12. The Milnor maps are well-defined invariants of \( C \).

Furthermore, we can define numerical Milnor invariants of the cut-diagram \( C \) from loop-longitudes; they essentially record the images of Milnor maps, but will also appear in the next section as indeterminacies for Milnor invariants defined from arc-longitudes.

Definition 4.13. For any sequence \( I \) of integers in \( \{1, \ldots, \ell\} \), we set
\[
\nu_C(I) := \gcd\{\Delta_C(I), m_C(I)\}
\]
and call it Milnor loop-invariant of \( C \) associated with the sequence \( I \) and the \( i \)-th component.

Here we use the usual convention that \( \gcd(0, 0) = 0 \). By Lemma 4.7 and still assuming Proposition 4.9, we have the following.

Theorem 4.14. The Milnor loop-invariants are well-defined invariants of \( C \).

Remark 4.15. As in the classical settings, Milnor invariants are ordered by their length, which is the number of indices in the indexing sequence. For a cut-diagram \( C \), suppose that there is a largest \( k > 0 \) such that \( m_C(I) = 0 \) for any sequence \( J \) of less than \( k \) indices (we set \( m_C(I) = 0 \) for any \( i \), as a convention). Then for any sequence \( I \) of length \( k + 1 \) we have \( \Delta_C(I) = 0 \), and the first non-vanishing Milnor (loop-)invariants of \( C \) are the nontrivial invariants \( \nu_C(I) \), which are simply given by \( \nu_C(I) = m_C(I) \in \mathbb{Z} \).
Proof of Lemma 4.18.

By definition of the homomorphism $\bar{\eta}$, this new alphabet of the form (4.1) for a given word in the alphabet $\mathbb{N}$, homomorphisms turn out to provide the desired dictionary between the two presentations of $\mathbb{N}_qG(C)$. This shows that the word $\bar{\eta}'(w)$ is a representative word of $\mathbb{N}_qG(C)$. In order to state this key lemma, we need to introduce some further notation.

Notation 4.17. Given an element $x$ in the free group generated by a set $\{e_1, \ldots, e_l\}$, we will use the notation $x(e_1, \ldots, e_l) \in \bar{F}$ for the element in a group $\bar{F}$ obtained by substituting each $e_i$ by $f_i \in \bar{G}$.

Now, let $\alpha$ be a word in $C$ based at $\{p_i\}$, and let $\alpha'$ be another choice of road network, based at $\{p_i'\}$. Consider $\alpha'$ the road $a_1'$ running from $p_1'$ to the basepoint $p_1$ of $a_1$, and set $v_1 := w_{a_1}$, the associated (unnormalized) word. The maps $\bar{\eta}'$ take values in the free group $\mathbb{F}_q = (\mathbb{R}, \mathbb{W})$, while the maps $\bar{\eta}$ take values in the free group $F' = (\mathbb{R}', \mathbb{W})$, and both can be seen as taking values in the free group $\bar{F} = (\mathbb{R}, \mathbb{W})$.

Recall that $W$ is the normal subgroup of $\bar{F}$ generated by all Wirtinger relations, see Notation 2.2. Then for any $w \in \bar{F}$, one can see $\bar{\eta}'(w)$ as an element of $F'/F_q \cdot \bar{\eta}'(W)$, while $\bar{\eta}(w)$ is seen as an element of $F'/F_q \cdot \bar{\eta}(W)$.

**Lemma 4.18.** In the above notation, for any $w \in \langle R_i \rangle$, we have

$$\bar{\eta}'(w) = \bar{\eta}(w) \left[ R_i \mapsto R_i' \right] \text{ in } F'/F_q \cdot \bar{\eta}'(W).$$

**Remark 4.19.** This lemma indeed provides the desired dictionary between our two presentations, as follows. Let $w$ be a word of $\langle R_i \rangle$, representing some given element in $\mathbb{N}_qG(C)$, and which happens to only involve the letters $R_i'$. Any element of $F$ can be seen as sitting in $\langle R_i \rangle$ in this way. Then we have by definition of the Chen homomorphisms that $w = \bar{\eta}(w)$. But we also have that $\bar{\eta}'(w)$ is a representative word of $\left\{ R_i' \right\}$. Lemma 4.18 tells us that this representative is obtained by replacing each $R_i$ by $R_i' \bar{\eta}(v_i)$ in the word $w$.

**Proof of Lemma 4.18.** By definition of the homomorphism $\bar{\eta}'$, and regarding $\bar{\eta}'(w)$ as an element of $\bar{F}$, we have

$$\bar{\eta}'(w) = w[R_i \mapsto \bar{\eta}'(R_i)] \mod \bar{F}_q \cdot W.$$

Now, a direct consequence of Lemma 3.18 is that, for any $i, j$, we have

$$\bar{\eta}'(\bar{\eta}'(R_{ij})) = \bar{\eta}'(R_{ij}) \equiv R_{ij} \mod \bar{F}_q \cdot W.$$

There is hence some element $g \in \bar{F}_q \cdot W$ such that

$$w[R_i \mapsto \bar{\eta}'(R_{ij})] = w[R_i \mapsto \bar{\eta}'(\bar{\eta}'(R_{ij}))] \bar{F}_q \cdot W,$$

and applying $\bar{\eta}'$ to this equality gives, according to Remark 3.3

$$w[R_i \mapsto \bar{\eta}'(R_{ij})] = w[R_i \mapsto \bar{\eta}'(\bar{\eta}'(R_{ij}))] \bar{\eta}'(g).$$

This shows that the word $\bar{\eta}'(w) = w[R_i \mapsto \bar{\eta}'(R_{ij})]$ in $F'/F_q \cdot \bar{\eta}'(W)$ is obtained from $\bar{\eta}(w) = w[R_i \mapsto \bar{\eta}(R_{ij})]$ in $F'/F_q \cdot \bar{\eta}(W)$ by substituting each $R_i \in F$ by $\bar{\eta}(R_i)$. Now, by the inductive definition of $\bar{\eta}'$, we have that $\bar{\eta}'(R_{ij}) = R_{ij}' \bar{\eta}(v_i)$, and Lemma 3.7 tells us that $\bar{\eta}'(v_i) \equiv \bar{\eta}(v_i) \mod F_q^{i+1}$. The conclusion then follows from Claim 3.6.

We can finally prove Proposition 4.9.
Proof of Proposition 4.9 The proof is by induction on the length of $I$, the initial case being trivial, and the statement on the indeterminacy $\Delta_C(I)$ being an immediate consequence of the induction hypothesis.

Proving the statement on $\overline{\mu}_C(I; \lambda)$ involves again the ideal $D_I$ introduced in the proof of Proposition 4.3. Pick two $i$-th loop-longitudes $\gamma$ and $\gamma'$, respectively based at $p_i$ and $p'_i$ and both representing $\lambda \in H_1(\Sigma_i)$. Starting with a representative word $\omega \in F$ for $\gamma$, Lemma 4.18 tells us how a representative word $\omega'$ for $\gamma'$ is obtained from $\omega$ by replacing each $R_j$ by a conjugate of itself. It then follows from Claim 4.5 (iii), in a strictly similar way as with $[22, (13)]$ in the link case, that $E(\omega) - E(\omega')$ falls in $D_I$, hence the result. \hfill $\Box$

A key technical point of the above proof is Claim 4.5 (iii), which is only valid for loop-longitude; this is closely related to the fact that the indeterminacies $\Delta(I)$ are defined in terms of loop-longitude only. This technical point explains that Proposition 4.9 is only valid for Milnor invariants extracted from loop-longitude. As a matter of fact, this independence under the choice of meridians does not hold when dealing with arc-longitudes, as we shall now explain.

4.3. Milnor invariants from arc-longitudes. We thus now focus on the case where $\gamma$ is an $i$-th arc-longitude. Dealing with an $i$-th arc-longitude obviously assumes that $\Sigma_i$ is not closed. It is hence rather natural to assume, at a first stage, that each component of $\Sigma$ has nonempty boundary. Note that this is in particular the case for string links $[11]$ and concordances in 4–space $[19]$, which we aim at generalizing here, see Section 4.3. The general case, where $\Sigma$ is allowed to contain closed components, shall be shortly addressed in Remark 4.21.

Recall that we chose basepoints $\{p_i\}$ on each boundary component, among which the point $p_0$ indicates a marked boundary component of $\Sigma_i$. This piece of data comes with the cut-diagram $C$, and is fixed, once and for all. This provides a canonical choice of $i$-th meridian, materialized by the basepoint $p_0$, and hence well-defined Milnor numbers associated to each arc-longitude. In other words, as a corollary of Proposition 4.3 we obtain the following.

Theorem 4.20. If $\Sigma$ has no closed component, Milnor numbers associated to arc-longitudes are well-defined invariants of $C$.

Remark 4.21. In the general case, where some component $\Sigma_j$ of $\Sigma$ may have empty boundary, the above result remains true, provided that we fix a choice of $j$-th meridian for $\Sigma_j$. Hence, in general, Milnor numbers associated to arc-longitudes are well-defined invariants of $C$, enhanced with a basepoint on each closed connected component of $\Sigma$ (in addition to the already fixed boundary basepoints).

In general, there are however several, possibly infinitely many, different arc-longitudes running from $p_0$ to some other boundary basepoint $p_{ij}$. But using the action of loop-longitudes noted in Remark 2.13 all arc-longitudes can be recovered from a single one. More precisely, for any two arcs $\gamma, \gamma'$ running from $p_0$ to $p_{ij}$, there exists by Remark 2.13 some $\tau \in \pi_1(\Sigma_i, p_0)$ such that $\gamma' = \tau \gamma$. By Lemma 4.6, we then have for any sequence $I$ of integers in $\{1, \ldots, \ell\}$,

$$\mu_C(I; \gamma') \equiv \mu_C(I; \gamma) + \mu_C(I; \tau) \mod \Delta_C(I).$$

Since $\tau$ decomposes as a product of elements in a system of longitudes (and their inverses), we therefore observe that

$$\mu_C(I; \gamma') \equiv \mu_C(I; \gamma) \mod v_C(I).$$

where $v_C(I) = \gcd \{\Delta_C(I), m_C(I)\}$ was introduced in Definition 4.13.

We can thus define intrinsic Milnor invariants from arc-longitudes, as follows.

Definition 4.22. For any sequence $I$ of integers in $\{1, \ldots, \ell\}$, we set

$$v_0^C(I; ij) := \mu_C(I; \gamma_{ij}) \mod v_C(I),$$

where $\gamma_{ij}$ is any path from $p_0$ to $p_{ij}$. We call it Milnor arc-invariant of $C$ associated with the sequence $I$ and the $j$-th boundary component of $\Sigma$.

Remark 4.23. As in Remark 4.16 we have that Milnor arc-invariants are well-defined invariants for nilpotent peripheral systems up to equivalence, since our notion of equivalence fixes the boundary, see Definition 2.14.

\footnote{Observe that, since $\Sigma$ has nonempty boundary, we picked here $p_i = p_0$ as canonical basepoint for $\Sigma_i$ for defining this action.}
4.4. First examples. We now give a couple of basic computational examples for our invariants. More examples can be found in Section 6 where a number of applications are also provided.

We first consider Milnor loop-invariants. Consider the diagram of the 3–component surface link $B$ shown in Figure 8 along with the associated cut-diagram $C_B$. A system of loop-longitudes for the third component is given by the loops $(\alpha_3, \beta_3)$ represented in dashed lines in the figure. On one hand, since $\beta_3$ avoids all cut arcs, we clearly have $\mu_{C_B}(I; \beta_3) = 0$ for all sequences $I$. On the other hand, using regions $E$ and $F$ of the first and second components as preferred meridians for these components, we directly have that $w_{\alpha_3} = EFE^{-1}F^{-1} \in G(C_B)$. Hence we have $\nu_{C_B}(I) = 0$ for any length 2 sequence $I$, and $\nu_{C_B}(123) = \nu_{C_B}(213) = 1$ and $\nu_{C_B}(I) = 0$ for $I \in \{312, 132, 231, 321\}$.

We now turn to Milnor arc-invariants. Consider the diagram of the 3–component knotted surface $M$ shown on the left-hand side of Figure 9 and the cut-diagram $C_M$ for $M$ shown on the right-hand side. For each component of $C_M$, we choose the preferred boundary component to be the bottom one, so that preferred meridians are given by regions $A, D$ and $E$ for the three components.

It is easily seen that $\nu_{C_M}(I) = 0$ for any sequence $I$, since each component admits a system of longitude given by a single loop running parallel to the lower boundary component; Milnor arc-invariants of $B$ are thus well-defined integers, given as follows. A choice of arc-longitude is specified by an arc running from top to bottom on each component. Such a choice $\gamma_i$ ($i = 1, 2, 3$) is represented with dashed lines in the figure. We clearly have $w_{\gamma_1} = DD^{-1} = 1$ and $w_{\gamma_2} = 1$ in $G(C_M)$. Moreover, we have $w_{\gamma_3} = B^{-1}C = DA^{-1}D^{-1}A$ in $G(C_M)$. As a result, we obtain that $\mu_{C_M}(12; \gamma_3) = -\mu_{C_M}(21; \gamma_3) = 1$ and $\mu_{C_M}(I; \gamma_i) = 0$ for $i = 1, 2$ and any sequence $I$.

We shall provide in Section 6 a number of applications and further computational examples involving the various invariants defined so far.
We stress that, at this point, the above computations are only for the given cut-diagrams representing the surface links $B$ and $M$. As mentioned in the introduction, these are in fact topological invariants of the represented surface links, i.e. do not depend on the chosen cut-diagram: this will be formally stated in Corollary 5.7.

4.5. Relation to previous works.

**Definition 4.24.** We call Milnor invariants of $C$ the collection of invariants defined in Sections 4.2 and 4.3, namely Milnor maps $M^i_C$, Milnor loop-invariants $\nu_C(Ii)$ and Milnor arc-invariants $\nu^\partial_C(I;ij)$, for all indices $i, j$ and all sequences of indices $I$.

This general definition recovers and generalizes many of the previously known versions of Milnor invariants, as follows.

4.5.1. *The classical dimension.* In the case of 1–dimensional cut-diagrams, the situation is quite special since $\Sigma$ always comes with a single preferred longitude $l_i$. Our Milnor invariants then boil down to a unique numerical invariant associated to each sequence $I$, which correspond to the classical Milnor invariants.

For links, $l_i$ is a loop-longitude. As discussed in Remark 4.11, we have for any cut-diagram $C_L$ representing a link $L$, and any sequence $I$:

$$\overline{\mu}_L(Ii) = \overline{\mu}^l_{C_L}(Ii) \in \mathbb{Z}/\Delta_{C_L}(Ii)\mathbb{Z},$$

where $\overline{\mu}_L(Ii)$ is the classical Milnor invariant defined in [21, 22]. Note also that, on the other hand, we have $m_{C_L}(I) = |\mu_L(I)|$ and $\Delta_{C_L}(I) = \Delta_L(I)$, so that

$$m_{C_L}(I) \equiv |\mu_L(I)| \mod \Delta_L(I).$$

For string-links, $l_i$ is an arc-longitude and we recover Habegger–Lin’s Milnor invariants for string-links [11]. More precisely, for any cut-diagram $C_L$ representing a string link $L$, since each component is homologically trivial we have $\nu_C(Ii) = 0$ for any sequence $I$, and

$$\mu_L(I) = \nu^\partial_C(I;ii) \in \mathbb{Z}.$$

As we shall explain in Section 7.1 the more general case of 1–dimensional cut-diagrams relates to the theory of welded knotted objects. Then, the case where $\Sigma$ is a union of $\ell$ circles recovers the welded extension of Milnor’s link invariant of [8, 23], while the case of $\ell$ intervals recovers Milnor invariants of welded string links defined in [21], see also [16].

4.5.2. *Milnor invariants of concordances.* In the case of 2–dimensional cut-diagrams over $\ell$ annuli, Milnor arc-invariants of Definition 4.22 are precisely Milnor invariants for concordances, i.e. for embedded annuli cobounded by two links, as defined in [19, 4]. Note that in this setting, loop-longitudes are given by parallel copies of the boundary components, so that, in the general case where the latter form a nontrivial link, both $\Delta_{C}(I)$ and $\nu_C(I)$ involve Milnor invariants of the link at the boundary; see [19] for a complete description (note that in [19], the indeterminacy $\nu_C(I)$ of Definition 4.22 is denoted by $\overline{\Lambda}(I)$).

**Remark 4.25.** A key ingredient in [19, 4] for defining Milnor invariants of concordances, i.e. knotted annuli in 4–space, is a result of Stallings [33, Thm. 5.1] that provides the Chen-Milnor presentation for the nilpotent quotients of the fundamental group of the complement. This was already the case when defining Milnor invariants of string links [11]. In this paper, rather than using similar techniques for general knotted surfaces in 4–space, we made use of the notion of cut-diagram to derive Theorem 3.15. Indeed, this combinatorial method not only provides the desired presentation, but also an algorithm for expressing longitudes as words in the meridians. In other words, this approach also gives an algorithm for computing higher dimensional Milnor invariants. As a matter of fact, rather than using Stallings theorem as an external tool, we instead give, as a byproduct, a combinatorial proof for this result in our generalized setting, see Corollary 5.9.
4.5.3. *Orr invariants.* Orr introduced in [24, 25] (based) concordance invariants $\theta^q_\ell$ for (based) codimension 2 embeddings. The construction can be roughly summarized as follows.

Let $L$ be an embedding of an $\ell$–component $n$–manifold $\Sigma$ into $S^{n+2}$, and let $G(L)$ be the fundamental group of its complement. Suppose that the nilpotent quotients satisfy $G(L)/G(L)_q \cong F/F_q$, where $F$ is the free group of rank $\ell$ (see below regarding this assumption). The natural homomorphisms

$$F \rightarrow F/F_q \quad \text{and} \quad G(L) \rightarrow G(L)/G(L)_q \cong F/F_q,$$

induce continuous maps

$$f^\ell_q : K(F, 1) \rightarrow K\left(\frac{F}{F_q}, 1\right) \quad \text{and} \quad \phi^\ell_q : S^{n+2} \setminus L \rightarrow K\left(\frac{F}{F_q}, 1\right)$$

respectively, where $K(G, 1)$ denotes an Eilenberg-MacLane space for a group $G$. The map $\phi^\ell_q$ extends to a map $\overline{\phi}^\ell_q$ from $S^{n+2}$ to the mapping cone $\text{Cone}(f^\ell_q)$ of the map $f^\ell_q$. The fundamental class $\theta^\ell_q \in \pi_{n+2}(\text{Cone}(f^\ell_q))$ of $\overline{\phi}^\ell_q$ is an invariant of $L$ for each $q \geq 1$, and Orr shows that these $\theta^\ell_q$ actually are (based) concordance invariants.

As noted above, the definition of Orr’s invariants requires that $G(L)/G(L)_q \cong F/F_q$. Observe that this condition is fulfilled if $H_1(\Sigma) = 0$, which is the (stronger) assumption taken in [24] for the definition. We note that $G(L)/G(L)_q \cong F/F_q$ if and only if all Milnor invariants of $L$ (in the sense of the present paper) with length $< q$ vanish. Hence Orr invariants actually are invariants for codimension 2 embeddings with vanishing higher-dimensional Milnor invariants.

Moreover, as discussed in Remark 5.27, we expect that Orr’s construction can be adapted to create new link-homotopy invariants of knotted surfaces.

5. **Cut-concordance and self-singular cut-concordance**

For this section, we fix an $n$–dimensional manifold $\Sigma$.

5.1. **Cut-concordance.** We consider the situation where two cut-diagrams over $\Sigma$ cobound a cut-diagram over $\Sigma \times [0, 1]$. This yields general notion of concordance for cut-diagrams.

5.1.1. **Definition and first properties.**

**Definition 5.1.** Two cut-diagrams $C_0$ and $C_1$ over $\Sigma$ are *cut-concordant* if there is a cut-diagram $C$ over $\Sigma \times [0, 1]$ which intersects $\Sigma \times [0]$ and $\Sigma \times [1]$ as $C_0$ and $C_1$, respectively. This means that, for $\varepsilon \in [0, 1]$:

- there is an orientation preserving diffeomorphism $\psi_\varepsilon : \Sigma \rightarrow \Sigma \times [\varepsilon]$ sending $C_\varepsilon$ to $C \cap (\Sigma \times [\varepsilon])$,

where $C \times (\Sigma \cap [\varepsilon])$ is given the inward induced orientation if $\varepsilon = 0$ and the outward one if $\varepsilon = 1$;

- the following diagram commutes:

$$\begin{array}{ccc}
\{\text{cut-domains of } C_\varepsilon\} & \xrightarrow{\text{labeling}} & \{\text{regions of } C_\varepsilon\} \\
\psi^{\varepsilon}_x & \Downarrow & \psi^{\varepsilon}_y \\
\{\text{cut-domains of } C\} & \xrightarrow{\text{labeling}} & \{\text{regions of } C\}
\end{array}$$

where $\psi^{\varepsilon}_x$ (resp. $\psi^{\varepsilon}_y$) sends a cut-domain $d_\varepsilon$ (resp. a region $r_\varepsilon$) of $C_\varepsilon$ to the unique cut-domain $d$ (resp. region $r$) of $C$ such that $\psi^{\varepsilon}_x(d_\varepsilon) \subset d$ (resp. $\psi^{\varepsilon}_y(r_\varepsilon) \subset r$).

Moreover, if $\partial \Sigma \neq \emptyset$, we also require that $C \cap (\partial \Sigma \times [0, 1]) \cong (C_0 \cap \partial \Sigma) \times [0, 1]$. We also say that $C$ is a cut-concordance between $C_0$ and $C_1$.

Cut-concordance defines a natural equivalence relation on cut-diagrams, thus providing a coherent theory for studying these objects. Moreover, this notion generalizes the topological notion of concordance in the following sense.

**Proposition 5.2.** Two topological cut-diagrams, associated to two concordant codimension 2 embeddings, are cut-concordant.
Proof. Let $\mathcal{C}$ (resp. $\mathcal{C}'$) be a cut-diagram associated to a diagram $D$ (resp. $D'$), which is the projection of a fixed embedding $L$ (resp. $L'$) in $\mathbb{R}^{n+2}$. If there exists a concordance from $L$ to $L'$, then Roseman’s theory of projection for codimension 2 embeddings [28]-[29] ensures the existence of an equivalent concordance which projects to a diagram, such that the restriction to the boundary coincides with $D$ and $D'$. The associated cut-diagram is the desired cut-concordance between $\mathcal{C}$ and $\mathcal{C}'$. □

In particular, since an isotopy is a special instance of concordance, we have the following.

Corollary 5.3. Two topological cut-diagrams associated to a same codimension 2 embedding, are cut-concordant.

Moreover, in any fixed dimension, two 1-component cut-diagrams are always cut-concordant. Indeed, we have the following.

Lemma 5.4. Let $\Sigma$ be an oriented $n$–dimensional manifold. Any 1-component cut-diagram over $\Sigma$ is cut-concordant to the empty cut-diagram over $\Sigma$.

Proof. Let $\mathcal{C}$ be a 1-component cut-diagram over $\Sigma$. Then $\mathcal{C} \times [0, 1/2] \subset \Sigma \times [0, 1]$ simply defines a cut concordance from $\mathcal{C}$ to the trivial diagram, where $\mathcal{C} \times [1/2] \subset \Sigma \times [0, 1]$ is a union of internal points. □

5.1.2. Concordance invariance results. It is well-known that equivalent classes of nilpotent peripheral systems and Milnor invariants of link in $S^3$ are concordance invariants. This remains true for cut-diagrams. Indeed, we prove below the following.

Theorem 5.5. Equivalence classes of nilpotent peripheral systems for cut-diagrams are invariant under cut-concordance.

By Remarks [4.16] and [4.23] Milnor invariants indexed by sequences $I$ of at most $q \geq 1$ indices only depend on the $q$–th nilpotent peripheral system. As a consequence, we obtain the following.

Corollary 5.6. Milnor invariants are invariant under cut-concordance.

Corollaries [5.3] and [5.6] then provide our main topological invariance result.

Corollary 5.7. Milnor invariants are well-defined concordance invariants for codimension 2 embeddings.

This allows for the following definition.

Definition 5.8. For any knotted manifold $S$ we define Milnor invariants—which are, for any sequence $I$ of indices in $[1, \ldots, \ell]$, Milnor maps $M^I_S$, Milnor loop-invariants $\nu_S(I)$, and Milnor arc-invariants $\nu_S^\alpha(I; ij)$—as the corresponding invariants for any cut-diagram associated to $S$.

Recall that, in dimension 3, Milnor invariants are not only invariant under ambient isotopy; they are also invariant under ‘isotopy’ in the sense of [21]. In particular, Milnor invariants are trivial for knots. We observe that, by Lemma 5.4, the same holds in any dimension.

Let us now prove the main invariance result.

Proof of Theorem 5.5. Let $\mathcal{C}$ be a cut-concordance between two cut-diagrams $\mathcal{C}_0$ and $\mathcal{C}_1$ over a manifold $\Sigma$. We fix $\varepsilon \in [0, 1]$ and $q \in \mathbb{N}$, and call $\varepsilon$–regions the regions of $\mathcal{C}$ intersecting $\Sigma \times \{\varepsilon\}$.

Let’s consider first the simpler case where each $\varepsilon$–region has a connected intersection with $\Sigma \times \{\varepsilon\}$. Then there is a one-to-one correspondence between the regions of $\mathcal{C}_0$ and the $\varepsilon$–regions of $\mathcal{C}$. Since $\mathcal{C}$ is a diagram on the product $\Sigma \times [0, 1]$, we can freely choose basepoints $\{p^\varepsilon_f\}$ and defining-cycles for the loop-longitudes of $\mathcal{C}$ on the boundary $\Sigma \times \{\varepsilon\}$; in particular, a system of loop-longitudes $L(\Sigma \times [0, 1])$ can be associated with a collection of loops in $\Sigma \times \{\varepsilon\}$. Moreover, when choosing a road network $a_\varepsilon$ for $\mathcal{C}$, we can freely pick the roads to the $\varepsilon$–regions as living on $\Sigma \times \{\varepsilon\}$. With such choices, it is immediate that applying Theorem 3.15 to $\mathcal{C}$ and $\mathcal{C}_\varepsilon$, will provide the exact same presentation for $N_{q} G(\mathcal{C})$ and $N_{q} G(\mathcal{C}_\varepsilon)$.

Now, if there are $\varepsilon$–regions whose intersections with $\Sigma \times \{\varepsilon\}$ are non connected, then regions of $\mathcal{C}_\varepsilon$ and $\varepsilon$–regions of $\mathcal{C}$ are no longer in one-to-one correspondence. In that case, we duplicate generators, as explained in Remark 2.4 so that we get an injection $\iota$ of the meridians of $\mathcal{C}_\varepsilon$ into the meridians of $\mathcal{C}$. Then for every region/meridian $R$ of $\mathcal{C}_\varepsilon$, the road to $\iota(R)$ can be chosen so that it stays within $\Sigma \times \{\varepsilon\}$. Moreover, when associating words to loop-longitudes and roads on $\Sigma \times \{\varepsilon\}$ as explained in Remark 2.9, we use $\iota$ to choose
letters. This being done, the duplicated version of Theorem 3.15 given in Remark 3.18 provides, as above, the exact same presentation for \(N_qG(C)\) and \(N_qG(C_2)\).

This shows that the nilpotent quotients \(N_qG(C_0)\) and \(N_qG(C_1)\) are isomorphic, being both isomorphic to \(N_qG(C)\). More precisely, \(N_qG(C_0)\) and \(N_qG(C_1)\) can be given presentations which coincide with the presentations of \(G(C)\) associated to the above road networks \(\alpha_0\) and \(\alpha_1\), respectively. We can then use Lemma 4.18 to get an explicit isomorphism \(\psi_{0\to1}\) between these two presentations. In particular, \(\psi_{0\to1}\) sends the \(i\)-th meridian of \(C_0\) to the conjugate of the \(i\)-th meridian of \(C_1\) by \(\eta_1^q(v_i)\), where \(v_i\) is the word associated to the path \(\gamma_i\) of \(\alpha_1\) joining the \(i\)-th meridian of \(C_1\subset C\) to the \(i\)-th meridian of \(C_0\subset C\). Moreover, conjugation by \(\gamma_i\) provides a bijection between \(p_i^1\)-based cycles in \(\Sigma \times \{1\}\) and \(p_i^0\)-based cycles in \(\Sigma \times \{0\}\), up to homotopy. Hence, using Proposition 3.16 \(\psi_{0\to1}\) hence induces a bijection between \(i\)-th longitudes of \(C_0\) and those of \(C_1\), realized as the conjugation by \(\eta_1^q(v_i)\).

We thus proved that the nilpotent peripheral systems for \(C_0\) and \(C_1\) are equivalent. \(\square\)

As announced in Remark 4.25 a noteworthy consequence of the former half of this proof is a combinatorial Stallings Theorem, as follows.

**Corollary 5.9.** Given a cut-concordance \(C\) between two cut-diagrams \(C_0\) and \(C_1\), the inclusion maps \(\psi_0\) and \(\psi_1\) of Definition 5.1 induce isomorphisms for all \(q \geq 1\):

\[
N_qG(C_0) \xrightarrow{\sim} N_qG(C) \xleftarrow{\sim} N_qG(C_1). 
\]

This provides a combinatorial proof of Stallings Theorem [33, Thm. 5.1] in the case of a topological concordance between two codimension 2 embeddings, and a generalization to cut-concordances.

### 5.2. Self-singular cut-concordance

We now introduce the notion of self-singular concordance for cut-diagrams, in connection with the topological notion of link-homotopy. In this subsection, we shall essentially revisit many of the previous notions and results in this extended context, focusing on the points that need to be adapted.

#### 5.2.1. Self-singular cut-diagrams and cut-concordance

Cut-diagrams were, for a large part, motivated by the study of knotted manifolds and their Milnor invariants. Until now, we considered only the regular case of embedded surfaces, but Milnor’s work was first introduced in the context of link-homotopy of links [21]. Recall that a *link-homotopy* is a continuous deformation during which distinct components remain disjoint, but each component may intersect itself. Hence this study involves self-singularities, which are self-intersections within a same connected component.

Let us focus for a moment on the 2-dimensional case. Singular knotted surfaces are immersed surfaces with only a finite number of singularities which are transverse double points. We define therefore *self-singular knotted surfaces* as singular knotted surfaces for which the two preimages of each double point belong to the same connected component of the associated abstract surface; these are also called *link-maps* in the literature, see for instance [31, Sec. 3.4].

Broken surface diagrams were generalized to this (self-)singular context in [4]; there, a singularity corresponds to the local model given on the left-hand side of Figure 10.

The routine explained in Section 1.1 that associates a cut-diagram to a broken surface diagram, produces then a cut-diagram which may violate the second labeling condition. However, as Figure 10 illustrates, any terminal cut arc of a cut-diagram associated to a self-singular knotted surface, is always labeled by a region.
that belongs to the same connected component. This leads to the following general definition, for \( \Sigma \) an \( n \)-dimensional oriented compact manifold, possibly with boundary.

**Definition 5.10.** A self-singular cut-diagram over \( \Sigma \) is the data \( C \) of a diagram on \( \Sigma \), endowed with an \( s \)-admissible labeling. Here, a labeling of oriented cut domains by regions, is called \( s \)-admissible if it satisfies the first labeling condition of Definition 1.4 and the following modified second condition:

\( \gamma \)

1. a terminal cut domain is labeled by a region which belong to its connected component.

\( \gamma \)

2. a terminal cut domain may not be labeled by the region it is adjacent to, but by any region which belongs to its connected component.

\( \gamma \)

Obviously, the labeling condition 2) of Definition 1.4 is a special case of condition 2) above, so that cut-diagrams are special cases of self-singular cut-diagrams. Internal endpoints satisfying 2) but not 2) will be depicted with a white circle \( \circ \) on cut-diagrams.

We can now define a ‘self-singular’ notion of concordance for cut-diagrams.

**Definition 5.11.** Two (self-singular) cut-diagrams \( C_0 \) and \( C_1 \) over \( \Sigma \), are self-singular concordant if there is a self-singular cut-diagram \( C \) over \( \Sigma \times [0, 1] \) which intersects, in the sense of Definition 5.1, \( \Sigma \times [0] \) and \( \Sigma \times [1] \) as \( C_0 \) and \( C_1 \), respectively. We call \( C \) a self-singular cut-concordance between \( C_0 \) and \( C_1 \).

As emphasized above, the notion of self-singular cut-concordance encompasses the notion of cut-concordance, but it also generalizes the topological notion of ‘self-singular concordance’. Two links are called self-singular concordant if they cobound disjointly immersed annuli in 4–space. Considering a self-singular cut-diagram representing such self-immersed annuli, as explained above, we easily have the following.

**Proposition 5.12.** Two 1–dimensional (topological) cut-diagrams, representing self-singular concordant links, are self-singular cut-concordant.

**Remark 5.13.** Note that this statement is likely to hold in any dimension, the only missing ingredient being a generalization, to the singular case, of Roseman’s result on the existence of diagrams in higher dimensions.

**5.2.2. Reduced peripheral system.** Let \( C \) be a self-singular cut-diagram over \( \Sigma = \bigcup_{\nu} \Sigma_{\nu} \). Since the second labeling condition plays no role in its definition, a group \( G(C) \) can be defined for \( C \) in the same way as in the regular case of Definition 2.1 using a Wirtinger presentation. However, in this self-singular case, longitudes are no longer well-defined elements in \( G(C) \). Consequently, we will consider the following quotient, first introduced by Milnor [21] in the study of links up to link-homotopy.

**Definition 5.14.** The reduced quotient of \( G(C) \) is defined as

\[
\mathrm{RG}(C) := \frac{G(C)}{\langle R, R^g \rangle} \text{ for all meridians } R \text{ and all element } g \in G(C).
\]

**Remark 5.15.** The reduced quotient \( \mathrm{RG}(C) \) is equivalently defined as the quotient of \( G(C) \) by the normal subgroup generated by commutators \( \langle R_i, R_j \rangle \) where \( \{R_1, \ldots, R_\ell\} \) is any given set of one meridian per component.

**Lemma 5.16.** If \( \gamma \) and \( \gamma' \) are two homotopic generic paths in \( \Sigma_{\nu} \), for some \( \nu \in \{1, \ldots, \ell\} \), then \( w_\gamma = w_{\gamma'} \) in the quotient \( \mathrm{RG}(C)/N_{\nu} \) of \( \mathrm{RG}(C) \) by the normal subgroup \( N_{\nu} \) generated by \( i \)-th meridians.

Note that \( N_{\nu} \) can equivalently be defined as the normal subgroup of \( \mathrm{RG}(C) \) generated by any \( i \)-th meridian.

**Proof.** The proof follows the exact same lines as the proof of Lemma 2.10. The only difference concerns move \( H_2 \), for which the terminal cut domain may not be labeled by the region it is adjacent to, but by any conjugate of it. But we directly get \( w_{\gamma'} = w_\gamma \) in \( \mathrm{RG}(C)/N_{\nu} \).

For any \( \nu \in \{1, \ldots, \ell\} \), we have by Lemma 5.16 a well-defined map

\[
\lambda_{i, \nu} : \pi_1(\Sigma_{\nu}, p_i) \sqcup \mathcal{A}(\Sigma_{\nu}, \{p_{i, j}\}) \rightarrow \mathrm{RG}(C)/N_{\nu},
\]

defined by composing the \( i \)-th longitude map \( \lambda_i \) with the projection map from \( G(C) \) to \( \mathrm{RG}(C)/N_{\nu} \). Elements in the image of \( \lambda_{i, \nu} \) are called reduced \( i \)-th longitudes. Note that these are cosets of the form \( \lambda N_{\nu} \), where \( \lambda \) is a word representing an \( i \)-th longitude.

**Definition 5.17.** The data \( \{\mathrm{RG}(C); \{R_i\}, \{\lambda_{i, \nu}\}\} \) is called a reduced peripheral system of \( C \). Definition 2.14 naturally induces a notion of equivalence for reduced peripheral systems.
Now, for each component \( \Sigma_i \) of \( \Sigma \), let \( R_i \) be a choice of meridian, specified by a basepoint \( p_i \in \Sigma_i \setminus C \). As noted in Remark 3.12, \( G(C) \) is normally generated by these meridians. It follows, by a result of Habegger and Lin [11] Lem. 1.3], that its reduced quotient is nilpotent of degree at most \( \ell \), so that \( RG(C) \cong N_q RG(C) \) for any integer \( q \geq \ell \).

As in Section 3.3, pick for each component \( \Sigma_i \) of \( \Sigma \), a system of loop-longitudes \( L_i(\Sigma) = \{ w_{ij} \} \) in \( G(C) \), and recall that \( F := \langle R_i \rangle \) and \( F := \langle R_{ij} \rangle \). We have the following Chen–Milnor type presentation for \( RG(C) \), similar to Theorem 3.15.

**Theorem 5.18.** For every \( q \geq \ell \), the reduced quotient of \( G(C) \) has the following presentation:

\[
RG(C) \cong \left\langle R_1, \ldots, R_\ell \left| \begin{array}{c}
[R_i, R_j^q] \text{ for all } i, \text{ and all } g \in F \\
[R_i, \eta(w_{ij})] \text{ for all } i \text{ and all } w_{ij} \in L(\Sigma)
\end{array} \right. \right\rangle.
\]

**Proof.** This presentation for \( RG(C) \) is obtained in a completely similar way as Theorem 3.15 is shown from Proposition 3.12. Hence we just sketch the argument below, stressing only the new ingredients. It is indeed easily checked that the Chen homomorphisms \( \eta \) descend to well-defined homomorphisms \( RF \rightarrow RF \). The arguments proving Lemma 3.11 and Proposition 3.12 then apply verbatim to show that, for any \( q \geq 1 \), the \( q \)-th nilpotent quotient of \( RG(C) \) has the following presentation:

\[
(5.1) \quad N_q RG(C) \cong \left\langle R_1, \ldots, R_\ell \left| \begin{array}{c}
F_q \\
[R_i, R_j^q] \text{ for all } i, \text{ and all } g \in F \\
[R_i, \eta(w_{ij})] \text{ for all } i \text{ and all } w_{ij} \in L(\Sigma)
\end{array} \right. \right\rangle.
\]

From this point, we can use an induction on \( q \) to rework the presentation so that the third type of relations are substituted by commutation relations of the form \( [R_i, \eta(w_{ij})] \), as in the proof of Theorem 3.15. The only new ingredient here is that move \( H_2 \) may now involve a terminal cut domain which is not labeled by the region supporting the move, but just by any region in the same connected component. Suppose hence that two arcs \( \gamma, \gamma' \) on \( \Sigma \) differ by such a move \( H_2 \), with \( \gamma \) corresponding to the left-hand side of the move. By the labeling condition 2'), the cut domain involved in this move is labeled by some region \( A \) of \( \Sigma \). We then have

\[
w_{\gamma} = B^s \tilde{w}_{\gamma_1} \tilde{w}_{\gamma_2} \quad \text{and} \quad w_{\gamma'} = B^s \tilde{w}_{\gamma_1} A^s \tilde{w}_{\gamma_2},
\]

where \( \gamma_1, \gamma_2 \) are paths such that \( \gamma = \gamma_1 \gamma_2, s \in \mathbb{Z}, e = \pm 1, B \) is the region of \( \Sigma \), and \( \gamma_1 \) starts. Denoting by \( v_B \), for any region \( R \), the word associated to the road, in the road network \( \alpha \), running to the region \( R \), we have, by the definition of the Chen homomorphisms, that

\[
(5.2) \quad \eta_{q+1}(w_{\gamma'}) = (R_\ell^e)^{\eta_{q}(v_B)} (R_i^{\eta_{q}(v_A)})^{\eta_{q}(v_B)} \eta_{q+1}(w_{\gamma}).
\]

Since \( \eta_{q+1}(w_{\gamma'}) \) and \( \eta_{q+1}(w_{\gamma}) \) differ by a product of conjugates of \( R_i \), we have that \( [R_i, \eta_{q+1}(w_{\gamma})] \) and \( [R_i, \eta_{q+1}(w_{\gamma})] \) are equivalent, modulo the normal closure of the relations \( [R_i, R_j^q] \) for all \( i \) and all \( g \in F \).

Now, by Habegger–Lin’s result recalled above, we have that \( RG(C) \) is a nilpotent group of order at most \( \ell \). Picking any \( q \geq \ell \), it follows that (5.1) provides a presentation for \( RG(C) \cong N_q RG(C) \), and also that the relations involving \( q \)-th iterated commutators are trivially satisfied. Removing the latter, we obtain hence the presentation of the statement. \( \Box \)

A consequence of this proof, and more specifically of Equation (5.2) and Proposition 3.16, is the following.

**Proposition 5.19.** Let \( C \) be a self-singular cut-diagram over \( \Sigma \). For any two homotopic arcs \( \gamma, \gamma' \) on \( \Sigma \), there exists a product \( N \) of conjugates of \( R_i \) such that

\[
\eta_q(w_{\gamma}) \equiv N\eta_q(w_{\gamma'}) \mod F_q V(q).
\]

The duplication process for the group of a cut-diagram, introduced in Remark 2.4, more generally applies for self-singular cut-diagrams. Then, replacing Theorem 3.15 with Theorem 5.18, Proposition 3.16 with Proposition 5.19, and using this duplication process, the same lines of proof as for Theorem 5.5 lead to the following invariance result.

**Theorem 5.20.** Equivalence classes of reduced peripheral systems for self-singular cut-diagrams are self-singular cut-concordance invariants.
Moreover, just like Corollary 5.9 follows as an immediate byproduct of the proof of Theorem 5.5 we obtain in this way the following reduced version of the combinatorial Stallings theorem.

**Corollary 5.21.** Given a self-singular cut-concordance \( C \) between two cut-diagrams \( C_0 \) and \( C_1 \), the boundary inclusion maps (in the sense of Definition 5.4) induce isomorphisms:
\[
RG(C_0) \xrightarrow{\cong} RG(C) \xleftarrow{\cong} RG(C_1).
\]

5.2.3. Non-repeated Milnor invariants. Milnor invariants first appeared in [21] as a tool for studying links up to link-homotopy. As a matter of fact, at first, Milnor only considered invariants indexed by a sequence of pairwise distinct indices, and proved that they are indeed invariant under link-homotopy. Actually, they are more generally invariant under self-singular concordance. We now investigate a similar phenomenon in the setting of cut-diagrams.

**Definition 5.22.** We say that a Milnor number \( \bar{\mu}_C(I; \gamma) \) for a cut-diagram \( C \), a sequence \( I \) of indices, and a path \( \gamma \) on the \( i \)-th component of \( \Sigma \), is non-repeated if all the indices in \( I_i \) are pairwise distinct.

Accordingly, as in Definition 4.24, we shall call non-repeated Milnor invariants of a cut-diagram \( C \), all Milnor maps \( M^I \), Milnor loop-invariants \( \nu_C(I;i) \) and Milnor arc-invariants \( \nu_C^0(I;i,j) \), for indices \( i, j \) and sequences \( I \) such that all indices in \( I_i \) are pairwise distinct.

We have the following, which underlies a ‘reduced’ version of Proposition 4.3.

**Proposition 5.23.** Let \( C \) be a cut-diagram over \( \Sigma = \bigsqcup_{i=1}^l \Sigma_i \), and \( i \in \{1, \ldots, l\} \). Let \( I \) be a sequence of pairwise distinct integers in \( \{1, \ldots, l\} \setminus \{i\} \), and \( \gamma \) be some path in \( \pi_i(\Sigma_i) \sqcup \mathcal{A}(\Sigma_i, \{p_{ij}\}) \). Then
\[
\bar{\mu}_C(I; \gamma) \equiv \mu_C(I; \omega) \mod \Delta_C(I),
\]
where \( \omega \) is any word in the alphabet \( \{R_{j}^0\} \) such that \( \omega \cdot N_{0,i} \) is the reduced \( i \)-th longitude associated to \( \gamma \).

**Proof.** By Theorem 5.18 and the definition of reduced longitudes, two choices of representative words \( \omega \) and \( \omega' \) are related by a finite sequence of additions/deletions of

1. \( R_j^{\pm 1}R_{j'}^{\pm 1} \) for some \( j \) and \( j' \);
2. commutators \( [R_j, \eta_k(w_{ij})] \) for some \( j \) and \( k \);
3. commutators \( [R_j, R_j'] \) for some \( j \) and \( g \in F \);
4. conjugates \( R_g \) for some \( g \in F \).

It has already been checked in the proof of Proposition 4.3 that the residue class of \( \mu_C(I; \omega_j) \) is invariant under the operations (1) and (2) (there, we denoted \( \eta_k(w_{ij}) \) by \( \omega_{ij} \)). It is hence sufficient to check it for operations (3) and (4). Let us first consider operation (3). Suppose that the words \( \omega \) and \( \omega' \) differ by a factor \( [R_j, R_j'] \), for some \( j \) and \( g \in F \). A standard argument on the Magnus expansion shows that \( E([R_j, R_j'])^{-1} \) is a sum of monomials, each containing two occurrences of the variable \( X_j \). This readily implies that \( \bar{\mu}_C(I; \omega) \equiv \mu_C(I; \omega') \mod \Delta_C(I) \) for any sequence \( I \) of pairwise distinct integers. Concerning operation (4), since the sequence \( I \) does not contain \( i \), it is easily seen that \( \mu_C(I; \omega) = \mu_C(\{I; I_i; \rho_{R_i}^g(\omega)\}) \), where the operator \( \rho_{R_i}^g \) removes all occurrences of \( R_i \). If \( \omega' \) is obtained from \( \omega \) by inserting a \( R_i^0 \) factor, then \( \rho_{R_i}^g(\omega') \) is obtained from \( \rho_{R_i}^g(\omega) \) by inserting \( \rho_{R_i}^g(\omega') \), and the result then follows from case (1). \( \square \)

By Proposition 5.23 non-repeated Milnor numbers can be extracted from the reduced peripheral system. By Theorem 5.20 we hence obtain the following.

**Proposition 5.24.** Non-repeated Milnor invariants for cut-diagrams are invariant under self-singular cut-concordance.

**Remark 5.25.** Note that Propositions 5.23 and 5.24 are stated for cut-diagrams that are not self-singular. However, following the same lines as in Section 5.2.3, non-repeated Milnor invariants can actually be generalized to well-defined invariants for all self-singular cut-diagrams, and these two propositions remain true in this more general setting.

Let us return to topology. As hinted at the beginning of Subsection 5.2.3, self-singular cut-concordance can be thought of as a combinatorial generalization of link-homotopy.
In dimension 1, the trace of a link-homotopy gives a self-singular concordance, hence a self-singular cut-concordance between the associated diagrams. Proposition 5.24 thus recovers, in a combinatorial way, the link-homotopy invariance for non-repeated Milnor invariants of classical links, known since the seminal works of Milnor [21].

In higher dimensions, it seems less obvious that the trace of a link-homotopy leads in general to a self-singular cut-concordance. However, in the 2-dimensional case, we have a general diagrammatic theory for link-homotopy in terms of local moves [4]. This is recalled in Subsection 7.3 where the following will be proved.

**Proposition 5.26.** Non-repeated Milnor invariants for knotted surfaces are invariant under link-homotopy.

**Remark 5.27.** Let $L$ be an embedding of an $\ell$–component $n$–manifold $\Sigma$ into $S^{n+2}$. One can adapt the definition of Orr’s invariant [23] reviewed in Section 4.5.3 but replacing the nilpotent quotient $F/\mathcal{F}_q$ with the reduced free group $RF$. Under the assumption that $RG(L) \cong RF$, we obtain in this way ‘new’ invariants $\theta'_R = \{\theta'_R\} \in \pi_{n+2}(\text{Cone}(f'_R))$ of $L$, where

$$j'_R : K(F, 1) \longrightarrow K(RF, 1),$$

and $\overline{\theta'_R} : S^{n+2} \longrightarrow \text{Cone}(f'_R)$ is the extension of $\phi'_R : S^{n+2} \setminus L \longrightarrow K(RF, 1)$.

Although this will not be further developed in the present paper, we expect that these invariants $\theta'_R$ are (based) link-homotopy invariants; note that one possible ingredient for proving such a result is the reduced Stallings theorem stated in Corollary 5.21. Since the above assumption that $RG(L) \cong RF$ is equivalent to saying that all non-repeated Milnor invariants of $L$ vanish, we could obtain in this way ‘new’ link-homotopy invariants for codimension 2 embeddings with vanishing non-repeated Milnor invariants. Note however that in the classical dimension $n = 1$, all links with vanishing non-repeated Milnor invariants are link-homotopically trivial [22], so that this construction is only relevant for higher dimensions. In this respect, it is natural to raise the question whether all surface surface links with vanishing non-repeated Milnor invariants are link-homotopically trivial.

6. Examples and applications

In this section, we gather several topological applications of Milnor invariants for cut-diagrams. These examples involve almost exclusively knotted surfaces. Recall indeed that Milnor invariants for cut-diagrams induce Milnor invariants for knotted manifolds, see Definition 5.8.

6.1. Milnor invariants of Spun links. Spun links refer to a classical construction due to Artin [11], which produces knotted surfaces from classical tangles as follows. Consider in $\mathbb{R}^4$ the upper 3–dimensional space $\mathbb{R}^3 = \{(x, y, z, 0) \mid x, y \in \mathbb{R}, z \geq 0\}$; the $(x, y)$–plane $P_{xy} = \{(x, y, 0, 0) \mid x, y \in \mathbb{R}\}$ sits as the boundary of $\mathbb{R}^3$. Given a 1–dimensional compact manifold $X$ properly embedded in $\mathbb{R}^3$, the **Spun of $X$** is obtained by spinning $X$ around $P_{xy}$ inside $\mathbb{R}^4 > \mathbb{R}^3$:

\[(6.1) \quad \text{Spun}(X) := \{ (x, y, z \cos \theta, z \sin \theta) \mid (x, y, z, 0) \in X, \theta \in [0, 2\pi] \}. \]

Note that the Spuns of a tangle and of its mirror image are isotopic.

Observe that a closed component of $X$ yields an toroidal component of Spun($X$), while a knotted arc in $X$ produces a spherical component. A diagram of $X$ naturally gives a broken surface diagram of Spun($X$), where each crossing yields a circle of double points without triple or branch point, hence a (topological) 2–dimensional cut-diagram.

Our Milnor invariants are well-behaved under the Spun construction. In particular, we have the following.

**Lemma 6.1.** Let $X$ be a tangle in $\mathbb{R}^3$ as above.

If the $i$-th component of $X$ is a knot, then for any sequence $I$ we have

$$m_{\text{Spun}X}(I) \equiv \nu_X(I) \mod \Delta_X(I).$$

If the $i$-th component of $X$ is an arc, then for any sequence $J$ we have $m_{\text{Spun}X}(J_i) = \nu_{\text{Spun}X}(J_i) = 0$.

**Remark 6.2.** When the $i$-th component of $X$ is a knot, we have in particular that the first non-vanishing Milnor loop-invariants of $\text{Spun}(X)$ are given by the first non-vanishing Milnor invariants of $X$ by $\nu_{\text{Spun}X}(I_i) = \nu_X(I_i)$.
Proposition. Suppose that the \( i \)-th component of \( X \) is a knot, so that the \( i \)-th component of \( \text{Spun}(X) \) is a torus. Let \( C_X \) be a cut-diagram for \( X \), whose \( i \)-th component is denoted by \( \sigma_i \). A cut-diagram \( C_S \) for \( \text{Spun}(X) \) is obtained from \( C_X \) for \( X \) by setting \( C_S := C_X \times S^1 \); we denote by \( \Sigma \) the \( i \)-th component of \( C_S \). A system of \( i \)-th loop-longitudes is provided by a choice \((\alpha_i, \beta_i)\) of two loops in \( \Sigma \), given by:

\[
\alpha_i = \sigma_i \times \{*\} \quad \text{and} \quad \beta_i = \{*\} \times S^1
\]

for some point \(*\) of \( \sigma_i \) which is not a cut point. By construction, the loop \( \beta_i \) is disjoint from all cut arcs in \( \Sigma \), hence we have that \( \mu_{\text{Spun}(X)}(I; \beta_i) = 0 \) for any sequence \( I \), and \( \mu_{\text{Spun}(X)}(I) = \mu_{\text{Spun}(X)}(I; \alpha_i) \). Observe that there are one-to-one correspondences between the cut arcs (resp. regions) in \( C_S \), and cut points (resp. regions) in \( C_X \), and that the cut arcs of \( C_S \) contain no crossing. (See Figure 11 for an example.) It follows not only that the groups \( G(C_S) \) and \( G(C_X) \) have same presentation, which is a well-known fact on Spun links, but also that the \( i \)-th longitude \( \lambda_i(\alpha_i) \in G(C_S) \) coincides, up to sign, with the \( i \)-th longitude in \( G(C_X) \) given by \( \sigma_i \). This shows that \( \mu_{\text{Spun}(X)}(I; \alpha_i) = \mu_{C_S}(I; \alpha_i) = |\mu_X(I)| \), and the result follows.

The second part of the statement is clear: if the \( i \)-th component of \( X \) is an arc with boundary in \( P_{ij} \), then the \( i \)-th component of \( \text{Spun}(X) \) is a sphere, hence there is no nontrivial loop-longitude in \( \Sigma \).

Remark 6.3. The Spun construction can be generalized to any dimension in a straightforward way. Hence one can iterate this construction, starting with a tangle in \( \mathbb{R}^3 \), to build a codimension 2 knotted submanifold. The argument of the above proof apply at each step and we can obtain a similar result as Lemma 6.1 in any dimension.

6.2. Realization results. A natural question is to ask whether, for any given sequence \( I \) of integers, there exists a knotted surface whose Milnor invariant (or Milnor map) indexed by \( I \) takes the value 1.

The examples given in Section 4–4 provide such realization results for length 3 Milnor loop/arc-invariants, and can be straightforwardly generalized to (non-repeating) sequences of any length. In particular, the first example involving the knotted surface \( B \) of Figure 8 can be generalized straightforwardly to realize any Milnor loop-invariant, by a knotted surface containing a single non-spherical component.

Alternatively, we can give a general realization result using the Spun construction. Consider the \((n + 1)\)-component link \( M_{n+1} \) shown below, known as Milnor’s link.

\[
\begin{array}{c}
\begin{array}{c}
\includegraphics[width=0.5\textwidth]{spun.png}
\end{array}
\end{array}
\]

Milnor observed that all Milnor invariants of length \( \leq n \) vanish for \( M_{n+1} \), and that for any permutation \( \sigma \) in \( S_{n-1} \), we have

\[
\mu_{M_{n+1}}(\sigma(1) \cdots \sigma(n-1)n n + 1) = \begin{cases} 1 & \text{if } \sigma = \text{Id}, \\ 0 & \text{otherwise.} \end{cases}
\]

Using Lemma 6.1 and Remark 6.2, we directly obtain a similar realization result for the first non-vanishing Milnor loop invariants \( v(1 \cdots n + 1) \) of knotted surfaces, by considering \( \text{Spun}(M_{n+1}) \). More generally, by iterating the Spun construction and using Remark 6.3, we obtain in this way a similar result in any dimension.

6.3. A classification result up to concordance. We now compare the relative strength of our Milnor invariants of knotted surfaces with previous invariants defined in the literature.

Definition 6.4. For every \( m \in \mathbb{N} \), we define \( W_m \) as the spun surface obtained by spinning the tangle \( X_m \) described in Figure 11.

Remark 6.5. The family \( W_m \) can be extended to negative values of \( m \), but this amounts to spinning the mirror image of the tangle \( X_m \). Hence, \( W_m \) is isotopic to \( W_m \).

Many of the previously known concordance invariants of knotted surfaces cannot detect this family of Spun links. Indeed, for all \( m \in \mathbb{N} \), we have that

- the Sato–Levine invariant [31] vanishes on \( W_m \),
- Cochran’s derivation invariants [9] all vanishes on \( W_m \),
- Saito’s invariants [30] of \( W_m \) are equal for all values of \( m \in \mathbb{N} \),
• since $W_m$ is link-homotopic to a union of trivially embedded torus and sphere for all $m \in \mathbb{N}$, all link-homotopy invariants vanish on $W_m$.

In contrast, using Milnor loop-invariants we show the following.

**Proposition 6.6.** For any $m_1, m_2 \in \mathbb{N}$, $W_{m_1}$ and $W_{m_2}$ are concordant if and only if $m_1 = m_2$.

**Proof.** This is an immediate consequence of the fact that, for all $m \in \mathbb{N}$,

$$\nu_{W_m}(2121) = m.$$  

This computation can be performed directly on a cut-diagram, as given on the right-hand side of Figure [11] for $m = 3$; although it is technically slightly more involved, this computation follows the exact same lines as the one for $B$ in Section 5.4.

Alternatively, since $W_m$ is a Spun link, one can also use Lemma 6.1 for this computation. Closing the second component of $X_m$ yields a 2–component link which is the Whitehead $m$–double of the negative Hopf link, in the sense of [20]. Using [20] Thm. 1.1], we obtain that $\mu_{\Sigma_{s}}(11) = 0$ for any sequence $I$ of length $\leq 3$, and that $\mu_{\Sigma_{s}}(2211) = -m$. By Lemma 6.1 we deduce that $\nu_{W_m}(2211) = m_{W_m}(2211) = m$, as desired.  

**6.4. Link-homotopy classification results.** The next two applications provide link-homotopy classification results for Spun links and for knotted punctured spheres.

**6.4.1. Spun links up to link-homotopy.** Milnor showed in [21] that links of 3 components are classified up to link-homotopy by their non-repeated Milnor invariants of length $\leq 3$. Using the good behavior of our Milnor invariants under the Spun construction, we show the following.

**Proposition 6.7.** Let $L$ and $L'$ be 3–component links. Then Spun($L$) and Spun($L'$) are link-homotopic if and only if $L$ is link-homotopic to either $L'$ or its mirror image.

**Remark 6.8.** Of course the link-homotopy classification for 2–component Spun links is also implied by this statement, for example by adding a third trivial and unlinked component.

**Proof.** The ‘if’ part of the statement is clear. Indeed spinning a link-homotopy between $L$ and $L'$ provides a link-homotopy between Spun($L$) and Spun($L'$), and we already noted that the Spun of a link and of its mirror image are isotopic.

Next we suppose that $F = \text{Spun}(L)$ and $F' = \text{Spun}(L')$ are link-homotopic. Set $L = L_1 \sqcup L_2 \sqcup L_3$ and $L' = L'_1 \sqcup L'_2 \sqcup L'_3$, and set $F_i = \text{Spun}(L_i)$ and $F'_i = \text{Spun}(L'_i)$ for $i \in \{1, 2, 3\}$.

For each $i$, let $a_i, b_i \in H_i(F_i) \ (\text{resp. } a'_i, b'_i \in H_i(F'_i))$ be the cycles corresponding to $L_i \times \{\ast\}$ and $\{\ast\} \times S^1$, respectively. Suppose that $a'_i$ corresponds to $\gamma_i \in H_i(\Sigma_i)$ by link-homotopy deformation from $F'$ to $F$.

By link-homotopy invariance (Proposition 5.26), and by a similar observation as in the proof of Lemma 6.1, we have that the Milnor map of Definition 4.10 satisfies

\begin{equation}
M_i^b(\alpha+i) = aM_i^b(\alpha) \equiv a\mu_L(\alpha) \mod \Delta_L(\alpha)
\end{equation}

for any non-repeated sequence $\alpha$ and any $a, b \in \mathbb{Z}$. It follows that $\text{Im}(M_i^b) = \mu_L(\alpha)\mathbb{Z}$ whenever $\Delta_L(\alpha) = 0$.

Moreover, if $\mu_L(\alpha) \neq 0$, then $\mu_L(i; a\alpha + b\beta)$ generates $\text{Im}(M_i^b)$ if and only if $a = \pm 1$.  

![Figure 11](image-url)  

**Figure 11.** Definition of the knotted surfaces $W_m = \text{Spun}(X_m)$, and a cut-diagram for $W_3$.
Suppose that at least one linking number of $L'$, say $\mu_L(12) = \mu_L(21)$, is nonzero. Since for $i \in \{1, 2\}$, $\mu_F(I; \alpha')$ generates $\text{Im}(M^{21}_F)$, we have that $\mu_L(2; \gamma_1)$ and $\mu_L(1; \gamma_2)$ generate $\text{Im}(M^{32}_F)$ and $\text{Im}(M^{12}_F)$, respectively. Hence we have

\[ \gamma_1 = e\alpha_1 + b_1\beta_1 \in H_1(\Sigma_1) \quad \text{and} \quad \gamma_2 = \delta\alpha_2 + b_2\beta_2 \in H_1(\Sigma_2) \]

for some $b_1, b_2 \in \mathbb{Z}$ and $e, \delta \in \{\pm 1\}$. Applying (6.3) with $I \in \{21, 31, 32\}$ provides the following values for Milnor invariants of $F$:

\[ \epsilon\mu_L(21) = eM^{21}_F(\alpha_1) = M^{21}_F(\gamma_1) = \mu_L(21), \]
\[ \epsilon\mu_L(31) = eM^{31}_F(\alpha_1) = M^{31}_F(\gamma_1) = \mu_L(31), \]
\[ \delta\mu_L(12) = \delta M^{12}_F(\alpha_2) = M^{12}_F(\gamma_2) = \mu_L(12), \]
\[ \delta\mu_L(32) = \delta M^{32}_F(\alpha_2) = M^{32}_F(\gamma_2) = \mu_L(32). \]

This implies $\epsilon = \delta$, and hence $\mu_L(ij) = \epsilon\mu_L(ij)$ for any distinct $i, j \in \{1, 2, 3\}$. In particular we have $\Delta_L(2,3,1) = \Delta_L(231)$. Further applying (6.3) with $I = 231$ then gives

\[ \epsilon\mu_L(231) = eM^{231}_F(\alpha_1) = M^{231}_F(\gamma_1)\mu_L(231) \mod \Delta_L(231). \]

This, and the known relations among Milnor link invariants [22], imply that $\mu_L(I) \equiv e\mu_L(I) \mod \Delta_L(I)$ for any non-repeated sequence $I$ of length 3.

Now, if $e = 1$, then $L$ and $L'$ share all non-repeated Milnor invariants of length $\leq 3$, hence are link-homotopic by Milnor’s classification result. If $e = -1$, then the same holds for $L$ and the mirror image of $L'$. This concludes the proof in the case where at least one linking number of $L$ is nonzero.

(ii) Suppose now that all linking numbers of $L$ vanish. Then by (6.3), all Milnor invariants of length 2 vanish for $F$, hence also vanish for $L'$. Consequently, $\mu_L(123)$ and $\mu_L(123)$ are integers, both generating $\text{Im}(M^{231}_F) = \text{Im}(M^{123}_F)$. This implies that there exists some $e \in \{\pm 1\}$ such that $\mu_L(I) = e\mu_L(I)$ for any non-repeated sequence $I$ of length $\leq 3$, and the above argument still applies to conclude. \(\square\)

For Spun links with an arbitrary number of components, we also have the following consequence of Lemma 6.1.

**Proposition 6.9.** Let $L$ be an $m$–component link. The following are equivalent:

(i) $\text{Spun}(L)$ is link-homotopically trivial;

(ii) $\gamma_{\text{Spun},L}(I) = m_{\text{Spun},L}(I) = 0$ for any non-repeated sequence $I$;

(iii) $L$ is link-homotopically trivial.

**Proof.** Implication (i) $\Rightarrow$ (ii) is a consequence of Proposition 5.26. Condition (ii) implies by Lemma 6.1 that all non-repeated Milnor invariants of $L$ are zero, which implies (iii) according to Milnor [21]. Implication (iii) $\Rightarrow$ (i) is clear, as above: spinning a link-homotopy between $L$ and the trivial link $U$ provides a link-homotopy between $\text{Spun}(L)$ and $\text{Spun}(U)$. \(\square\)

6.4.2. **Link-homotopy classification of knotted punctured spheres.** We next mention another link-homotopy classification result derived from our construction, whose proof will appear in a forthcoming paper.

Let $n \geq 2$ be some integer, and $p_1, \ldots, p_n$ some positive integers. We call knotted punctured spheres any knotted surface which is a proper embedding of $\bigcup_{i=1}^n S^2_{p_i}$ into the 4–ball $B^4$, where $S^2_{p_i}$ is the $2$–sphere with $p_i$ holes, and such that the boundary is mapped to the trivial link with $\sum_i p_i$ components in $S^3 = \partial B^4$. Notice that taking $p_i = 1$ for all $i$ yields the notion of linked disks introduced by Le Dimet in [17], while taking $p_i = 2$ for all $i$ produces 2–string links, as studied in [4].

We have the following classification result, which generalizes [4, Thm. 4.8].

**Theorem 6.10.** Knotted punctured spheres are classified up to link-homotopy by Milnor invariants.

Note that knotted punctured spheres do not contain any nontrivial loop longitude, since the boundary is assumed to be a trivial link. Therefore, we actually make use here of Milnor arc-invariants.

**Remark 6.11.** This result can actually be extended to a wider notion of knotted punctured sphere, where the boundary is assumed to form a slice link. The slice condition ensures that loop longitudes are still all trivial as nilpotent group elements.
6.5. An obstruction for ribbonness. Our next application of Milnor invariants for knotted surfaces is a criterion for ribbon knotted surfaces.

Recall that a ribbon immersed 3–manifold in 4–space is an immersion such that the only singularities are 2–disks with two preimages, one embedded in the interior and the other properly embedded. A ribbon surface in 4–space is the boundary of some ribbon immersed 3–dimensional handlebodies in 4–space. We call the latter a ribbon filling of the ribbon surface; note that a given ribbon surface generally doesn’t admit a unique ribbon filling. These are the natural analogues of the notion of ribbon introduced by Fox in the classical dimension; as a matter of fact, this definition generalizes naturally to any dimension. Spun links turn out to be examples of ribbon surfaces. This is seen, in the notation of Section 6.1, by spinning not only the given tangle $X$, but also all projection rays from the tangle to the plane $P_X$, around which $X$ is spun.

We now show how Milnor maps can be used to obstruct ribbonness. Let $C$ be some cut-diagram over an $n$–dimensional manifold $\Sigma$, and let $I_i$ be some sequence of indices.

**Definition 6.12.** The free kernel of the Milnor map $M_C^H : H_1(\Sigma) \to \mathbb{Z}/\Delta_C(I_i)\mathbb{Z}$, is the maximal submodule $\text{Ker}_0(M_C^H) \subset \text{Ker}(M_C^H)$ such that $H_1(\Sigma)/\text{Ker}_0(M_C^H)$ is torsion-free.

Now, a ribbon filling $H$ of a genus $g$ component $S_i$ of a knotted surface $S$, induces a rank $g$ submodule $\text{Ker}_0(M_C^H) \subset \text{Ker}(M_C^H)$, which is clearly contained in the free kernel of any Milnor map on $H_1(\Sigma_i)$. This implies the following obstruction result for ribbonness.

**Proposition 6.13.** Let $S$ be a knotted surface which is concordant to a ribbon one. If $S_i$ is a genus $g$ component of $S$, then rank $(\cap_{i \in I} \text{Ker}_0(M_C^H)) \geq g$ for any nonempty set $I$ of sequences ending with $i$. If, moreover, all the sequences in $I$ are non-repeated, then $S$ is not even link-homotopic to a ribbon one.

Let us give a concrete application of this criterion. For every $m \in \mathbb{Z}$, we denote by $S_m$ the knotted surface shown on the left-hand side of Figure 12. It is obtained by spinning a 3-component link as illustrated in Figure 12, and while spinning, moving the third unknot $m$ times around the second one. The knotted surface $S_1$ appears in [4], where it was proved that it is not link-homotopic, hence not concordant, to any ribbon surface. In fact, the argument given in [4] shows that $S_m$ is not link-homotopic to a ribbon surface for all odd values of $m$. Using free kernels, we can actually prove the following.

**Proposition 6.14.** The knotted surface $S_m$ is link-homotopic to a ribbon surface if and only if $m = 0$.

**Proof.** Since $S_0$ is a Spun link, it is a ribbon surface. Let us now show that, if $m \neq 0$, then $S_m$ is not concordant to a ribbon surface. For the sake of simplicity, we give the proof for $m = 3$: the general case is handled in a strictly similar way. The right-hand side of Figure 12 gives a cut-diagram $C_3$ for the knotted surface $S_3$. We denote by $\alpha_3, \beta_3 \in H_1(\Sigma_3)$ the cycles shown in dashed lines on the cut-diagram of Figure 12. It is easily computed that $\text{Ker}_0(M_C^H) = \langle \beta_3 \rangle$ and $\text{Ker}_0(M_C^H) = \langle \alpha_3 \rangle$, so that $\text{rank} (\text{Ker}_0(M_C^H) \cap \text{Ker}_0(M_C^H)) = 0$. By Proposition 6.13 this shows that $S_3$ cannot be link-homotopic to a ribbon surface.

7. Local moves for cut-diagrams

In this last section, we introduce sets of local moves on 1 and 2–dimensional cut-diagrams, which emulate usual equivalence relations on codimension 2 embeddings, such as isotopy or link-homotopy.
7.1. Topological moves for cut-diagrams. As discussed in Section 1.1, there is a procedure which associates a cut-diagram to any given diagram of a codimension 2 embedding; recall that we called topological a cut-diagram arising in this way. We now introduce topological moves on cut-diagrams, which generalize the topological notion of isotopy in the sense that they relate any two cut-diagrams associated to diagrams of a same knotted manifold.

Let us first focus on the 1–dimensional case. It is straightforward to translate each of the three Reidemeister moves of knot theory into the language of 1–dimensional cut-diagrams, leading to the three 1–dimensional topological moves given in Figure 13. By construction, the assignment of a 1–dimensional cut-diagram to any given tangle yields a well-defined map

$$\Xi_1 : \{\text{tangles}\}/\text{isotopy rel. boundary} \rightarrow \{\text{1–dim. cut-diagrams}\}/\text{topological moves}.$$

Now, as noticed in Remark 1.1, 1–dimensional cut-diagrams are in bijection with Gauss diagrams up to Tail-Commute moves. Since Gauss diagrams up to Reidemeister and Tail-Commute moves exactly correspond to welded knotted objects, 1–dimensional cut-diagrams up to topological moves are just a mere reformulation of the welded theory. In particular, the map $$\Xi_1$$ is in fact known to be injective; see Remark 7.6. As a matter of fact, cut-diagrams can be seen as a generalization, in any dimension, of the notion of welded knotted objects, see [35, Question 80].

Next we turn to the 2–dimensional case. Two broken surface diagram represent isomorphic knotted surfaces if and only if they differ by a sequence of the seven Roseman moves given in [27]. These Roseman moves are straightforwardly translated into moves on cut-diagrams, and it is easily seen that they are all generated by the 2–dimensional topological moves, given in Figure 14. This leads to a well-defined map

$$\Xi_2 : \{\text{knotted surfaces}\}/\text{isotopy rel. boundary} \rightarrow \{\text{2–dim. cut-diagrams}\}/\text{top. moves}.$$

Remark 7.1. Although motivated by Roseman moves, 2–dimensional topological moves are reminiscent of Reidemeister and cobordism moves on links and knotoids. Note that further natural moves can be derived, such as the following, which is a consequence of the first and third moves in the upper part of Figure 14:

7Note that such an admissible relabeling may not exist, in which case the move is not valid; if several different relabelings are admissible, then the move exists in several valid versions.

Now, for knotted manifolds, isotopies are special cases of concordances; it is hence natural to ask whether this is still true for cut-diagrams.

**Proposition 7.2.** In dimension 1 and 2, two cut-diagrams related by a sequence of topological moves, are cut-concordant.
some of these moves merge two regions, or split a region into two: in the former case, all the cut arcs labeled by one of the two merging regions are relabeled by the new region, and in the latter case, the cut arcs labeled by the split region are relabeled in an admissible way by any of the two new regions; other moves make a region disappear: such moves are only valid if this region never occurs as the label of any cut arc.

This is proved, in a rather straightforward way, by giving explicit concordances realizing each of the topological moves. In dimension 1, for example, the three topological moves of Figure 13 are respectively realized by the three cut-concordances shown on the bottom line of Figure 15. The 2–dimensional case is left to the reader.

Remark 7.3. The three moves in Figure 15, called topological ∂–moves, correspond to the three Reidemeister moves on a link bounding a knotted surface. Actually, these move can be used to describe knotted surfaces up to ambient isotopies that are not required to fix the boundary. The map $\Xi_2$ indeed descends to a well-defined map

$$\tilde{\Xi}_2 : \{\text{knotted surfaces}\}/\text{isotopy} \rightarrow \{\text{2–dim. cut-diagrams}\}/\text{top. and \partial–moves}.$$ 

To conclude this section, we outline how a similar map $\Xi_n$ can be defined for any $n \geq 3$. Roseman moves, given in [27] for broken surface diagram, are merely a special case of [29], where Roseman defines local
moves for diagrams in any dimension. By simply translating these general Roseman moves in the language of cut-diagrams, one can thus define topological moves in any dimension. In this way, \( n \)-dimensional welded objects can be defined directly, without resorting to any ‘virtual’ theory.

### 7.2. Topological moves and peripheral system

The notion of group and peripheral elements—meridians and preferred longitudes—was defined for cut-diagrams in Section 2. This is actually a well-defined invariant of cut-diagrams up to topological moves.

**Proposition 7.4.** In dimension 1 and 2, the peripheral system is invariant under all topological moves of cut-diagrams.

**Proof.** This is proved by analyzing one by one all topological moves, noting that each is supported in a ball that we will denote by \( B \). Details are left to the reader, but the strategy in each case is to compare generators and relators within \( B \), and check that the resulting presentations are related by Tietze transformations. It follows that the groups, before and after the move, are isomorphic. It then remains to show that the preferred longitudes are preserved by this isomorphism. For that, it suffices to note that any longitude can be described by a path on \( \Sigma \) which is disjoint from \( B \); note however that this may require to first move the component basepoint outside of \( B \).

It follows that nilpotent peripheral systems are also invariant under topological moves in dimensions 1 and 2. This provides, in these dimensions, a direct proof of the following result, which can also be obtained as a consequence of Proposition 7.2 and Corollary 5.6.

**Corollary 7.5.** In dimension 1 and 2, Milnor numbers are invariant under all topological moves of cut-diagrams.

In the 2–dimensional case, it was already noted in Remarks 2.3 and 2.15 that for a topological cut-diagram, the notions of group and peripheral elements coincide with the fundamental group and peripheral elements for the exterior of the underlying knotted surfaces. In other words, Remarks 2.3 and 2.15 tell that \( \Xi_2 \) preserves the peripheral system. The same is true in dimension 1. Using the Wirtinger presentation associated with a diagram of some tangle \( L \), it is indeed straightforwardly checked that the group of \( \Xi_1(L) \) is isomorphic to the fundamental group of the exterior of \( L \), and that under this isomorphism, the combinatorial meridians and preferred longitudes of \( \Xi_1(L) \) correspond to the topological meridians and longitudes of \( L \). Combined with Corollary 7.5, this observation provides a direct proof of Corollary 5.7 in dimension 1 and 2.

**Remark 7.6.** Since, and since, in the 1–dimensional case, the peripheral system is a complete invariant, we have that \( \Xi_1 \) is an injective map. This, of course, is a mere reformulation of a well-known fact on the inclusion of classical tangles into welded tangles [10]. This naturally raises the question of the injectivity of \( \Xi_2 \), for which the above argument no longer applies since the peripheral system is not a complete invariant for knotted surfaces [34].

### 7.3. Cut-diagram moves for link-homotopy in dimensions 1 and 2

In addition to the topological moves of Section 7.1, we give here, in dimension 1 and 2, a list of local moves that generate, for topological cut-diagrams, the notion of link-homotopy.

We call **self-singular moves** the local move of Figure 16. In terms of link diagrams, the 1–dimensional move amounts to replacing a classical crossing of two strands of a same component, by a virtual one. This is the **self-virtualization** move, which is known to imply link-homotopy for classical links, see [2]. In dimension 2, Roseman moves have been extended to the case of self-singular surfaces in [3], where three self-singular Roseman moves were introduced. It is easily verified that the two 2–dimensional self-singular moves shown on the right-hand side of Figure 16 realize these self-singular Roseman moves for broken surface diagrams. As a consequence of [3], Prop. 2.4, we thus have that two cut-diagrams of link-homotopic knotted surfaces, are related by a sequence of 2–dimensional topological and self-singular moves.

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8We only mention here the special case of **finger moves**, generalized in any dimension as the insertion/deletion of a sphere (with any orientation and label) enclosing a region that does never appears as a cut domain label. Note that this moves indeed validates the terminology ‘meridian’ given to any conjugate of a region, since it allows the realization of any of them as a region.
Figure 16. Self-singular moves in dimensions 1 and 2: here, A and B are always regions of a same connected component; as in Figure[13] when going from top to bottom in the dimension 1 move, every A–label becomes either an A or an A′–label, and when going from bottom to top, all the A and A′–labels become A–labels.

Now, the reduced peripheral system is invariant, in dimensions 1 and 2, under these self-singular moves; this is a straightforward verification that we leave to the reader. Consequently, any invariant derived from the reduced peripheral system is also invariant under these moves; by Proposition 5.23 we have hence the following.

Proposition 7.7. In dimensions 1 and 2, non-repeated Milnor invariants of cut-diagrams are invariant under self-singular moves.

In view of the above discussion, this implies in particular that non-repeated Milnor invariants for knotted surfaces are invariant under link-homotopy, as stated in Proposition 5.26.

Remark 7.8. It should be noted that the self-singular equivalence induced by the above set of self-singular moves is not ‘minimal’ to generate link-homotopy for topological cut-diagrams. In dimension 1, it is indeed sufficient to consider the self-crossing change move to generate link-homotopy, but several prior works support the fact that self-virtualization is the natural extension of link-homotopy for welded objects; see [2, 4, 3]. Likewise, in dimension 2, the above moves are very likely to be stronger than necessary, but they appear to be a natural generalization of the 1–dimensional self-virtualization move.

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