Excited-state thermodynamics

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In the last several years, the Casimir energy for a variety of 1+1-dimensional integrable models has been determined from the exact S-matrix. It is shown here how to modify the boundary conditions to project out the lowest-energy state, which enables one to find excited-state energies. This is done by calculating thermodynamic expectation values of operators which generate discrete symmetries. This is demonstrated with a number of perturbed conformal field theories, including the Ising model, the three-state Potts model, $\mathbb{Z}_n$ parafermions, Toda minimal S-matrices, and massless Goldstinos.

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1. Introduction

A wealth of physical information about a relativistic quantum particle theory is contained in the knowledge of the asymptotic particle states and their scattering matrix. This includes all thermodynamic quantities of an infinite-volume gas of these particles, as long as the gas is dilute enough so that a particle description is valid. There is a closed-form expression for the virial coefficients (terms in the expansion of the free energy in the fugacity) depending only on the S-matrix\cite{1}. Doing the calculation in this manner is usually unwieldy and unnecessary, because one can generally calculate thermodynamic quantities directly from perturbative field theory. However, the situation is markedly different in 1+1-dimensional integrable theories. Here the S-matrix has a number of properties which simplify matters a great deal: it is completely elastic (momenta are conserved individually in a collision), the $n$-body S-matrix factorizes into a product of two-body ones, and one often can derive or conjecture it exactly\cite{2}. It is in principle possible to calculate these quantities using perturbed conformal field theory, but using the S-matrix is often more tractable, and if the S-matrix is exact, the results are non-perturbative.

The techniques for calculating the Casimir energy (the lowest eigenvalue of the one-dimensional quantum Hamiltonian) were first used in the non-relativistic case in \cite{3}. Recently, there has been a great deal of interest in this field, because the study of perturbed conformal field theory has enabled conjectures for a large number of exact S-matrices\cite{4}.\cite{5} The calculation of the Casimir energy in such a case goes by the name of the Thermodynamic Bethe Ansatz (TBA for short) and was first done for the Yang-Lee model and the three-state Potts model in ref.\cite{6}. This paper will discuss modifications of these techniques recently proposed in \cite{7}–\cite{9}, which enable the calculation of energies of excited states.

The result of the TBA is a set of coupled integral equations, one for each particle in the spectrum. These equations give the exact Casimir energy as a function of the volume of space. Although one can solve these equations only numerically, it is possible to extract either the infrared or ultraviolet asymptotics. The ultraviolet limit is where the theory becomes a conformal field theory, and thus the results can be compared with results at the conformal point. The results of this paper will be compared to conformal field theories with a variety of boundary conditions\cite{10}–\cite{11}. The TBA equations have been studied in a large variety of exactly solvable theories, including the XXZ spin chain in a magnetic field\cite{12} and its generalizations\cite{13}, the supersymmetric sine-Gordon model\cite{14},

\begin{footnote}{For a review of much of this activity, see \cite{5}.}\end{footnote}
all the simply-laced affine Toda theories\[15\][16], a number of non-unitary models\[6\][17],
the perturbed minimal models\[18\]–[21], and the perturbed N=1 and N=2 supersymmetric
minimal models\[22\]. Some of these results may even be experimentally verifiable\[23\].

The TBA allows the calculation of thermodynamic quantities for a one-dimensional
system, where the spatial dimension is a circle of circumference $L$, with $L \rightarrow \infty$. Any
thermodynamic expectation value of a $d$-dimensional quantum system at temperature $T$
can be alternatively viewed as an expectation value in a $(d + 1)$-dimensional Euclidean
quantum field theory, where the additional dimension is a circle of circumference $R = 1/T$. This is therefore equivalent to a two-dimensional theory with spacetime a torus of
periodicity $(R, L)$. We can view either the $R$-direction or the $L$-direction as space, with
the other direction then taking the role of Euclidean time. Thus the partition function
$Z(R, L)$ is equal to both

\[ Z(R, L) = \text{tr} \left[ e^{-RH_L} \right], \tag{1.1} \]

and

\[ Z(R, L) = \text{tr} \left[ e^{-LH_R} \right], \tag{1.2} \]

where $H_C$ is the Hamiltonian for the system with space a circle of circumference $C$. This
equivalence has proven very useful in conformal field theory defined on a torus, where
it goes by the name of modular invariance, and puts severe constraints on the operators
allowed in a theory\[24\] and on their fusion rules\[25\]. In the $L \rightarrow \infty$ limit, the only state
which contributes to the partition function is the ground state. Thus

\[ Z(R, L \rightarrow \infty) \rightarrow e^{-E(R)L}, \tag{1.3} \]

where $E(R)$ is the ground-state energy (lowest eigenvalue of $H_R$) of the theory on a circle
of circumference $R$, and does not depend on $L$. In any thermodynamic system the free
energy is $F = -\ln Z/R$, so (1.3) yields

\[ E(R) = \frac{R}{L} F(R, L \rightarrow \infty). \tag{1.4} \]

Therefore, the Casimir energy can be determined from the TBA.

An important check on these results can obtained by studying the limit where the
particle masses go to zero, so the system approaches a conformal field theory. A general
result of conformal field theory [20] predicts

\[ E(R) = \frac{2\pi}{R} \left( h + \widetilde{h} - \frac{c}{12} \right), \tag{1.5} \]
where $h$ and $\bar{h}$ are respectively the left and right conformal dimensions of the operator which creates the lowest energy state when acting on the vacuum. For a unitary theory with periodic boundary conditions (anti-periodic for any fermions present), this operator is the identity, so that $h = \bar{h} = 0$. This relation gives a very useful way of checking that the conjectured S-matrices used in the TBA calculation are consistent—(1.5) relates a TBA result in the $m \to 0$ limit to results obtainable from conformal field theory. This check has been successfully done in many cases [15]–[22], thus giving even stronger support for the conjectured S-matrices of these models. In addition, many of these papers numerically calculate corrections to (1.5), and compare them with results obtained with perturbation theory around the conformal point.

All of this work determined $E(R)$ for the lowest-energy state of the theory. However, the TBA formalism is not limited to this type of calculation. In several recent papers the TBA equations for a number of models were modified in order to describe the behavior of excited-state energies [7]–[9]. To check these TBA equations, the values of $h$ in (1.3) were calculated, and were found to match those of particular operators in the conformal theory. The equations were solved numerically in some cases, and the results agreed with the appropriate perturbed conformal field theory.

In this paper we demonstrate two methods of deriving excited-state energies. These methods are general and allow the determination of excited-state energies in any model with a discrete symmetry. This relates the results of [8] and [9] for massive theories to conformal results [10], and gives a simple physical interpretation of their modified TBA equations.

Both methods involve the calculation of the thermodynamic expectation value of the operators which implement this discrete symmetry, and are based on the way partition functions are calculated in conformal field theories with boundary conditions [10]–[11]. The unnormalized expectation value of an operator $A$ is

$$< A > = \text{tr} \left[ A e^{-H_{LR}} \right].$$

Thus $Z(R, L) = < 1 >$. Just as $< 1 >$ can be rewritten as (1.2) by interchanging space and Euclidean time, we rewrite

$$< A > = \text{tr} \left[ e^{-\tilde{H}_A L} \right],$$

where $\tilde{H}_A$ is the Hamiltonian for the same model, but with new boundary conditions depending on the operator $A$. If $A = 1$, then the boundary conditions are periodic for
bosons and antiperiodic for fermions, as is standard in thermodynamics. Another simple example is the case of a free scalar field $\phi$. When $A$ is the operator which sends $\phi \to -\phi$, $\tilde{H}_A$ is the Hamiltonian for the model with antiperiodic boundary conditions.

As in the periodic case, in the $L \to \infty$ limit

$$E_A(R) = \frac{R}{L} \ln <A>, \quad (1.8)$$

where $E_A(R)$ is the lowest eigenvalue of $\tilde{H}_A$. In the $m \to 0$ limit, (1.3) is recovered as long as the energy-momentum tensor is invariant under the action of $A$. However, $h$ is not necessarily zero, even in a unitary theory. This is because the boundary conditions can project the identity state out of the spectrum. In conformal field theory, this has been demonstrated explicitly for the Ising model and the three-state Potts model with a variety of boundary conditions\[10\].

There are two ways of calculating $< A >$ when $L \to \infty$, thus determining $E_A(R)$. The first method utilizes an imaginary chemical potential and is used when $A$ is diagonal on the space of particles. The second method works when $A$ interchanges particles, and involves truncating the Hilbert space. These methods can be combined for a general $A$.

Section 2 is a warm-up for the rest of the paper—both methods of calculating excited-state energies are applied to the Ising model, where the TBA is not needed. In section 3, the TBA in the presence of a general chemical potential is reviewed, and it is shown how this can be used to calculate expectation values of diagonal operators. Section 4 demonstrates within the TBA how one truncates the Hilbert space to obtain expectation values of non-diagonal operators. Section 5 applies these methods to models described by simply-laced Toda minimal S-matrices, including the three-state Potts model. Section 6 discusses two more models, the massless perturbation of the tricritical Ising model and a model which displays many of the characteristics of the minimal models. Section 7 contains conclusions and outlines possible future work.

2. The Ising Model

The field theory describing the $T \neq T_c$ Ising model in two spacetime dimensions consists of a single free Majorana fermion of mass $m$.\[2\] We consider an ensemble of $N$

\[2\] Another field theory describing the Ising model consists of a single massive boson with S-matrix $S = -1$. Which of the two is applicable depends on the boundary conditions, since they have different finite-volume thermodynamics\[13\]. However, the ground-state energy is the same in both theories, so it does not matter which theory we use here.
particles, and the rapdity $\theta_i$ of the $i$th particle is defined by

$$p_i = m \sinh \theta_i$$
$$E_i = m \cosh \theta_i.$$  \hfill (2.1)

Space is taken to be periodic with period $L$, so each momentum is quantized:

$$p_i = \frac{2n_i \pi}{L}. \hfill (2.2)$$

When the particle number $N$ is large, the partition function at temperature $T = 1/R$ and chemical potential $\mu$ is defined as

$$<1>_{\lambda} = Z_{\lambda}(R, L) = \text{tr} \left[ \lambda^N e^{-R H_L} \right], \hfill (2.3)$$

where the fugacity $\lambda \equiv e^{\mu R}$. This is the standard one-dimensional free-fermion partition function, which is

$$Z_{\lambda}(R, L) = \prod_{n=-\infty}^{+\infty} (1 + \lambda e^{-RE_n}), \hfill (2.4)$$

where

$$E_n = \left( \frac{(2\pi n)^2 + m^2}{L} \right)^{1/2}. \hfill (2.5)$$

Thus

$$F_{\lambda} \equiv -T \ln Z_{\lambda} = -T \sum_{n=-\infty}^{\infty} \ln(1 + \lambda e^{-RE_n}). \hfill (2.6)$$

In the limit $L \to \infty$, we can replace the sum over $n$ with an integral, and using

$$dn = \frac{L}{2\pi} dp = \frac{L}{2\pi} m \cosh \theta d\theta$$

yields

$$F_{\lambda} = -\frac{mL}{2\pi R} \int \cosh \theta \ln(1 + \lambda e^{-\epsilon(\theta)}) d\theta, \hfill (2.7)$$

where $\epsilon(\theta) \equiv mR \cosh \theta$ is the one-particle energy.

By definition, the eigenvalues of diagonal operators vary only with particle number. Looking at (2.3), one sees that to calculate the expectation value of such operators, one needs only to modify the chemical potential appropriately. Thus

$$<(-1)^N> = <1>_{\lambda=-1}, \hfill (2.8)$$
This corresponds to a chemical potential of $i\pi T$. Even though this is imaginary, (2.7) is perfectly well-defined by analytic continuation in $\lambda$. In fact, $F_{-1}$ is minus the free energy of a free boson. The argument of (2.7) has a singularity at $\lambda = -1$ and $\theta = 0$, but the integral is well-defined.

All of this analysis has been done treating the $L$-direction as space. When $R$ is the space dimension, we recover the picture of (1.7). When $A = 1$, $\tilde{H}_A$ is the Hamiltonian for free fermions with antiperiodic boundary conditions. When $A = (-1)^N$, the boundary conditions in the $R$-direction are flipped: $\tilde{H}_A$ is the Hamiltonian for free fermions with periodic boundary conditions. In the language of Ising spins, these correspond to periodic and antiperiodic boundary conditions on the spins, respectively. Thus with the equivalence (1.8), evaluating $< (-1)^N >$ gives the ground-state energy for the Ising model with antiperiodic boundary conditions.

In the limit $m \to 0$, this model approaches the Ising conformal field theory. To evaluate (2.7) in this limit involves a few tricks [13] which are discussed in section 3. Using the definition (1.8) of the ground-state energy $E_A(R)$, the result is

$$E_1(R) = -\frac{1}{2} \left( \frac{\pi}{6R} \right)$$
$$E_{(-1)^N}(R) = \left( \frac{\pi}{6R} \right)$$

(2.9)

The second equation should be clear even without the tricks since it is minus the energy of a free boson, which has $c = 1$. Equation (1.7) yields the scaling dimensions of the operator which creates these two states. $E_1(R)$ yields $c = 1/2$, because $h = \tilde{h} = 0$. $E_{(-1)^N}(R)$ shows that the operator with conformal dimensions $(1/16, 1/16)$ creates the lowest-energy state in the Ising model with antiperiodic boundary conditions. This operator is the spin field $\sigma$. This result, of course, can be derived at the conformal point [10].

We turn to operators which interchange particles. Since there is only one particle in the Ising model, this is not relevant here. However, in a system consisting of two decoupled Ising models, there is an operator which exchanges a fermion of the first system $\psi_I(\theta)$ with one from the second system $\psi_{II}(\theta)$. This operator, which we denote by $K$, commutes with the Hamiltonian, so we can compute $< K >$. The Hamiltonian is diagonal on the space of states in theories with diagonal S-matrices, and we take the states to be orthonormal. Therefore, a state must be invariant under $K$ to give a non-zero contribution to the trace in $< K >$. We define the truncated Hilbert space $\mathcal{H}_K$ as the space of such states; i.e.,
states where for every $\psi_I$ with a rapidity $\theta$, there is also a $\psi_{II}$ with the same rapidity $\theta$. Then

$$< K > = \text{tr}_{\mathcal{H}_K} \left[ \lambda_I^{N/2} \lambda_{II}^{N/2} e^{-RH_L} \right] = \sum_{\mathcal{H}_K} \lambda_I^{N/2} \lambda_{II}^{N/2} e^{-E_K},$$

(2.10)

where

$$E_K = m \sum_{i=1}^{N} \cosh(\theta_i) = 2m \sum_{i=1}^{N/2} \cosh(\theta_i).$$

The configuration sum is that of a single Ising model with $N/2$ particles, and the energy is that of a single Ising model of mass $2m$. Thus

$$< K > = \prod_{n} (1 + \lambda_I \lambda_{II} e^{-2E_n}),$$

(2.11)

where $E_n$ is defined in (2.3). In the $L \to \infty$ limit,

$$\ln < K > = -\frac{mL}{2\pi R} \int \cosh \theta \ln (1 + \lambda_I \lambda_{II} e^{-2c(\theta)}) d\theta,$$

(2.12)

where $c(\theta) = mR \cosh(\theta)$ is still the one-particle energy. Notice that in (2.12) there is no extra factor of 2 in front of the integral.

As before, we interchange space and Euclidean time, and interpret $E_K$ as the ground-state energies for models with different boundary conditions. In the language of Ising spins, $\tilde{H}_K$ for $\lambda_I = \lambda_{II} = 1$ is the Hamiltonian for two Ising models, coupled only by the boundary condition $\sigma_I(0, y) = \sigma_{II}(R, y)$ and $\sigma_I(R, y) = \sigma_{II}(0, y)$, where $\sigma_I(x, y)$ and $\sigma_{II}(x, y)$ are the Ising spins at site $(x, y)$. When $\lambda_I = -1$ and $\lambda_{II} = 1$, this amounts to calculating $< K(-1)^{N/2} >$. (Remember, in the truncated Hilbert space $N/2$ is an integer.) This corresponds to the boundary condition $\sigma_I(0, y) = -\sigma_{II}(R, y)$ and $\sigma_I(R, y) = \sigma_{II}(0, y)$.

In the conformal limit $m \to 0$, (2.12) is independent of $m$. Comparing this with (2.7), we see that

$$\lim_{m \to 0} \left( \ln < K > \right) = \frac{1}{2} \lim_{m \to 0} F_{\lambda}.$$  

(2.13)

Thus we can read off the answers from (2.9), and when $\lambda = 1$,

$$E_K(R) = -\frac{1}{4} \left( \frac{\pi}{6R} \right).$$

(2.14)

Since we have two Ising models, $c = 1$, and $h + \overline{h} = (1 - 1/4)/12$. Thus the field which creates this state has conformal dimensions $(1/32, 1/32)$. When $\lambda = -1$,

$$E_K(R) = +\frac{1}{2} \left( \frac{\pi}{6R} \right).$$

(2.15)

Here $h + \overline{h} = (1 + 1/2)/12$, and the conformal dimensions are $(1/16, 1/16)$. Both of these results are in agreement with those found using modular transformations at the conformal point[27].
3. The TBA with a chemical potential

In this section we briefly derive the TBA equations in the presence of a chemical potential, following the treatment of refs. [6][16].

The starting point of the TBA is the exact two-body elastic S-matrix. For simplicity, in this paper we will treat only diagonal S-matrices, although the TBA can also be extended to models with non-diagonal S-matrices, where one must introduce massless pseudo-particles into the TBA system of equations [19]. The S-matrix element for the process $a(\theta_1)b(\theta_2) \rightarrow b(\theta_2)a(\theta_1)$ is denoted by $S_{ab}(\theta)$, where $\theta \equiv \theta_1 - \theta_2$, the difference of the particles’ rapidities. An S-matrix element in one spatial dimension is defined so that when two particles $a(\theta_1)$ and $b(\theta_2)$ are exchanged, the wavefunction is multiplied by $S_{ab}(\theta)$.

For the moment, we specialize to a theory with only one kind of particle, so that there is only one S-matrix element $S(\theta)$. The TBA applies to systems with a large number of particles $N$, where the space direction is periodic with period $L$. The periodicity leads to a quantization condition on each momentum $p_i = m \sinh(\theta_i)$:

$$e^{i p_i L} \prod_{j \neq i} S(\theta_i - \theta_j) = 1. \quad (3.1)$$

Defining $\rho_r(\theta)$ to be the rapidity density (i.e., $\rho_r(\theta)\Delta \theta$ is the number of particles with rapidity between $\theta$ and $\theta + \Delta \theta$), the logarithm of (3.1) is

$$mL \sinh(\theta_i) - i \int d\theta' \rho_r(\theta') \ln S(\theta_i - \theta') = 2\pi n_i, \quad (3.2)$$

where an integer $n_i$ is associated with each particle. All integrals in this paper run from $-\infty$ to $+\infty$ unless otherwise labeled. This equation is the generalization to an interacting model of the one-particle relation $p_i = 2\pi n_i/L$. Here there is a set of coupled equations for the momenta. We introduce a level density $\rho(\theta)$, so that $\rho(\theta)\Delta \theta$ is the number of values of $\theta_i$ which solve (3.2) in the range $\theta + \Delta \theta$. Taking the large $L$ limit, (3.2) becomes

$$\rho(\theta) = mL \cosh(\theta) + \int d\theta' \rho_r(\theta') \phi(\theta - \theta'), \quad (3.3)$$

where

$$\phi(\theta) \equiv -i \frac{\partial \ln S(\theta)}{\partial \theta}.$$
To determine the free energy, we use thermodynamics, as opposed to the statistical mechanics used in section 2. In a system with chemical potential \( \mu \) \( (\lambda \equiv e^{\mu/T}) \) at temperature \( T \), we use \( F_\lambda = E - TS - \mu N \) and minimize \( F_\lambda \) using (3.3) as a constraint. Here the energy \( E \) is
\[
E = \int \rho_r(\theta) m \cosh \theta d\theta,
\] (3.4)
the entropy \( S \) is
\[
S = \int d\theta [\rho \ln \rho - \rho_r \ln \rho_r - (\rho - \rho_r) \ln(\rho - \rho_r)],
\] (3.5)
and the particle number \( N \) is
\[
N = \int \rho_r(\theta) d\theta.
\] (3.6)
In (3.5), we have assumed that all particles are of “fermionic” type, with at most one particle of a given species with a given momentum \( \theta_i \). This is true for all known one-dimensional theories, save a free boson.\(^3\) Defining the pseudoenergy \( \epsilon(\theta) \) by
\[
\rho_r = \rho \frac{\lambda e^{-\epsilon}}{1 + \lambda e^{-\epsilon}},
\] (3.7)
the minimization of \( F_\lambda \) yields an integral equation for \( \epsilon(\theta) \):
\[
\epsilon(\theta) = mR \cosh \theta - \frac{1}{2\pi} \int d\theta' \phi(\theta - \theta') \ln(1 + \lambda e^{-\epsilon(\theta')}).
\] (3.8)
In terms of \( \epsilon \), the free energy is
\[
F_\lambda = -\frac{mL}{2\pi R} \int d\theta \cosh \theta \ln(1 + \lambda e^{-\epsilon(\theta)}).
\] (3.9)
In the case where there is more than one species of particle, one introduces particle and level densities for each species \( a \), and finds
\[
\rho^a(\theta) = m_a L \cosh(\theta) + \sum_b \int d\theta' \rho_b^a(\theta') \phi_{ab}(\theta - \theta'),
\] (3.10)
where
\[
\phi_{ab}(\theta) \equiv -i \frac{\partial \ln S_{ab}(\theta)}{\partial \theta}.
\] (3.11)
\(^3\) See [6] or [15] for a detailed explanation of this point. It is easy to treat the case where particles are “bosonic”—just replace \( \rho \) in (3.3) with \( (\rho + \rho_r) \).
\(^4\) This definition includes the chemical potential, as opposed to [16].
The pseudoenergies are defined as in (3.7), and minimizing the free energy yields the integral equation

$$\epsilon_a(\theta) = m_a R \cosh \theta - \sum_b \frac{1}{2\pi} \int d\theta' \phi_{ab}(\theta - \theta') \ln(1 + \lambda_b e^{-\epsilon_b(\theta)}).$$  \hfill (3.12)

We define the scaled free energy density \(f_\lambda(mR)\) as

$$f_\lambda(mR) \equiv \frac{R^2}{L} F_\lambda,$$

and it is given by

$$f_\lambda(mR) = \sum_a f^a_\lambda(mR),$$

$$f^a_\lambda(mR) \equiv -\frac{mR}{2\pi} \int d\theta \cosh \theta \ln(1 + \lambda_a e^{-\epsilon_a(\theta)}).$$  \hfill (3.13)

The conformal limit of (3.13) is the limit \(mR \to 0\). To evaluate this, we define \(\epsilon^{kink}(\theta) \equiv \epsilon(\theta - x)\), where \(x = \ln(mR/2)\). When \(x \to -\infty\), this yields equations which do not depend on \(mR\):

$$\epsilon^{kink}_a(\theta) = \epsilon^\theta - \sum_b \frac{1}{2\pi} \int d\theta' \phi_{ab}(\theta - \theta') \ln(1 + \lambda_b e^{-\epsilon^{kink}_b(\theta)}).$$  \hfill (3.14)

and

$$f^a(0) = \frac{1}{\pi} \int e^\theta \ln(1 + \lambda_a e^{-\epsilon^{kink}_a(\theta)}).$$  \hfill (3.15)

There are a few tricks which enable us to evaluate this without knowing the full function \(\epsilon(\theta)\). We take the derivative of (3.14) and substitute for \(e^\theta\) in (3.15). The \(\phi_{ab}\) can be removed from the resulting expression by using (3.14) again. Every term depends on \(\theta\) only through \(\epsilon^{kink}\), and also has a \(\partial \epsilon^{kink}/\partial \theta\) in it. Since the shifted \(\epsilon(\theta)\) is always decreasing, the integral over \(\theta\) can be replaced by an integral over \(\epsilon^{kink}\), yielding

$$2f_\lambda(0) = -\frac{1}{\pi} \sum_a \int_{C_a} d\epsilon \left[ \frac{\epsilon_a e^{-\epsilon}}{1 + \lambda_a e^{-\epsilon}} + \ln(1 + \lambda_a e^{-\epsilon}) \right],$$  \hfill (3.16)

where the contour \(C_a\) runs in the complex-\(\epsilon\) plane from \(\epsilon^{kink}_a(-\infty)\) to \(\epsilon^{kink}_a(\infty)\). In theories without massless particles, \(\epsilon^{kink}(\infty) = \infty\). To determine \(\epsilon^{kink}(-\infty)\), we use the property of (3.14) that \(\epsilon^{kink}(\theta)\) is flat from \(\theta = -\infty\) to around \(\theta = 0\), and that \(\phi_{ab}\) falls off
exponentially. This enables us to pull \( \ln(1 + \lambda_b e^{-\epsilon_b kink(\infty)}) \) out of the integral in (3.12), which yields a system of equations for \( x_a \equiv \exp(\epsilon_a kink(-\infty)) = \exp(\epsilon_a(0)) \):

\[
x_a = \prod_b (1 + \frac{\lambda_b}{x_b}) N_{ab},
\]

(3.17)

where

\[
N_{ab} \equiv -\frac{1}{2\pi} \int \phi_{ab}(\theta)d\theta.
\]

(3.18)

When \( \lambda_a = 1 \), the contour \( C_a \) lies on the real axis, but in the cases with imaginary chemical potential we will discuss later, it may not. Of course, one can deform \( C_a \), paying attention to the logarithmic cuts in (3.16) starting at \( \mu_a - \epsilon_a = i(2n + 1)\pi \).

We write (3.16) in the form

\[
f_{\lambda}(0) = -\frac{1}{\pi} \sum_a \mathcal{L}_{\lambda_a}(x_a),
\]

(3.19)

When \( \lambda_a \) are either 1 or \(-1\), and all \( x_a \geq 1 \) for \( a \) where \( \lambda_a = -1 \), then the contours \( C_a \) run along the real axis and do not go through the logarithmic singularity. In this case, a change of variables in the integral (3.16) yields

\[
\mathcal{L}_1(x) = L(1/(1 + x))
\]

\[
\mathcal{L}_{-1}(x) = -L(1/x),
\]

(3.20)

where

\[
L(x) = -\frac{1}{2} \int_0^x dy \left[ \frac{\ln y}{(1-y)} + \frac{\ln(1-y)}{y} \right].
\]

There are many known identities involving the Rogers dilogarithm function \( L(x) \), and they can often be used to evaluate (3.19) exactly [13][28].

Since this derivation allows an arbitrary chemical potential for each particle, the expectation value of any symmetry operator diagonal on the space of particles is obtained by setting \( \lambda_a \) to be the value of \( A \) on the particle \( a \). A variety of such examples will be presented in sections 5 and 6.

4. Non-diagonal operators

In this section, we show how to calculate the expectation value of a symmetry operator \( K \) which is not diagonal on the space of particles. This involves truncating the Hilbert
space, just like in the (Ising)\(^2\) model in the second half of section 2. The procedure for calculating
\[
<K> \equiv \text{tr} \left[ Ke^{-RHL} \right]
\] (4.1)
basically follows that of section 3: we find the appropriate quantization of momenta in periodic boundary conditions, and use this as a constraint while minimizing the free energy. This results in an integral equation for the pseudoenergy \(\epsilon(\theta)\) and an expression for the free energy in terms of \(\epsilon(\theta)\). If there are no particles neutral under \(K\), the asymptotics of these equations are obtainable, allowing the calculation of the conformal dimensions of the operator which creates the excited state. In others, it seems that one must resort to a numerical calculation.

The relation which determines the level densities \(\rho^a(\theta)\) remains (3.10), but we must modify the free energy. The Hamiltonian is diagonal in theories with diagonal S-matrices, so only states invariant under \(K\) contribute to the sum in (4.1). Thus, as before, \(<K>\) is the sum of \(e^{-RHL}\) over states on which \(K = 1\). This restriction changes the entropy. The species of particles form multiplets under \(K\), and all species of particles in a multiplet must have the same level density and particle density (this is true even in the unrestricted case). The crucial effect of the restriction is that the entropy from only one species of each multiplet contributes to the free energy. This is because in order to specify a given state with \(K = 1\), one needs only to specify the set of momenta for one species in a given multiplet; the other sets must be identical by \(K\) invariance. Minimizing the free energy, one finds the coupled integral equations are modified to
\[
\epsilon_a(\theta) = n_a m_a R \cosh \theta - \sum_b \frac{1}{2\pi} \int d\theta' \phi_{ab}(\theta - \theta') \ln(1 + \lambda_b e^{-\epsilon_b(\theta')}),
\] (4.2)
where \(n_a\) is the number of species in the multiplet \(a\). The scaled free energy, defined as \(f_K(mR) \equiv \frac{R}{L} \ln <K>_{\lambda} = RE_K(R)\), is
\[
f_K(mR) = \sum_a \frac{1}{n_a} f^a_{\lambda},
\] (4.3)
\[
f^a_{\lambda} \equiv -\frac{mR}{2\pi} \int d\theta \cosh \theta \ln(1 + \lambda_a e^{-\epsilon_a(\theta)}).
\]}

\(^5\) We do not treat operators which take a particle into a superposition of particle states.
In the limit \( mR \to 0 \), doing the same substitutions as in section 3, one finds that (specializing to the case where all \( n_a \) are 1 or 2)

\[
f_K(0) = -c_\lambda \left( \frac{\pi}{6} \right) - \frac{3}{2} \sum_A f_A.
\]

(4.4)

where the sum over \( A \) is only over particles with \( n_A = 2 \), and \( c_\lambda \) is the result obtained without truncating the Hilbert space; e.g., \( c_1 \) is the central charge derived by the methods of section 3 with \( \lambda = 1 \). Unfortunately, I can find a closed-form expression for \( f_K(0) \) like (3.16) only in the case where all the \( n_a \) are the same. In the case where all \( n_a = 2 \), (4.4) yields

\[
f_K(0) = -\frac{c_\lambda}{4} \left( \frac{\pi}{6} \right).
\]

(4.5)

When \( \lambda = 1 \), (1.3) gives \( h + \bar{h} = (3/4c)/12 \), and the operator which creates this excited state has conformal dimensions \( (c/32, c/32) \). This is quite an odd result—it applies to any theory with a conjugation symmetry that leaves no particles neutral. This includes the situation where two copies of a model are coupled by boundary conditions, as in the (Ising)\(^2\) model treated in section 2.

5. Examples from Toda minimal S-matrices

5.1. The Three-State Potts Model

The three-state Potts model at its conformal point has central charge \( c = 4/5 \). Perturbation by the thermal operator leaves the theory integrable, and its exact S-matrix was conjectured in [29]. The particle spectrum consists of a particle (labeled 1) and its antiparticle \( \bar{1} \). These particles are in the two-dimensional representation of the model’s \( S_3 \) symmetry. This symmetry is generated by the operators \( A \), which is diagonal and multiplies particle 1 \((1)\) by \( e^{2\pi i/3} \) \( (e^{-2\pi i/3}) \), and \( K \), which exchanges the two particles.

Calculating \( \langle K \rangle \) requires merely reading off the answer from section 4. The conformal dimensions of the operator creating the associated excited state are found from equation (1.3), and in this case are \( (1/40, 1/40) \). In the language of the lattice Potts model (where a spin \( \sigma \) takes the values 1, \( e^{2\pi i/3} \) or \( e^{-2\pi i/3} \)), this corresponds to finding the lowest eigenvalue of the Hamiltonian with “twisted” boundary conditions, where \( \sigma(0, y) = \sigma^*(R, y) \). This is in agreement with the conformal result [10]. We note that even though the \( (1/40, 1/40) \) operator is not part of the operator content of the three-state Potts model on the torus, it is present with these boundary conditions. This is analogous to the
situation in the Ising model, where the free fermion is not part of the toroidal operator content, but does belong in the case of antiperiodic boundary conditions [10].

Calculating $< A >$ requires the introduction of a chemical potential, as in section 3. In this case, we must have a complex fugacity: $\lambda_1 = e^{2\pi i/3}$ and $\lambda_\tau = e^{-2\pi i/3}$. We define the TBA equations here by analytic continuation from $\lambda_1 = \lambda_\tau = 1$, whose TBA system was derived in [6]. In this analytic continuation, we keep $\lambda_1 = (\lambda_\tau)^*$, enabling us to require that $e^\lambda = (e^{\lambda})^*$. This specifies which branch of the logarithm is taken in (3.12) and (3.13). The free energy (3.13) remains real because $f_\lambda = (f_{\lambda'}^*)$. These resulting TBA equations are those discussed in [8].

To evaluate $f$ in the conformal limit, we must find $x_a$ by solving (3.17). In this case, $N_{11} = N_{22} = 1/3$, and $N_{12} = N_{21} = 2/3$. This equation has more than one solution, but the correct one is the continuation of the $\lambda = 1$ result of $x_1 = x_\tau = (\sqrt{5} + 1)/2$. It is

$$\frac{x_1}{e^{\frac{i\pi}{3}}} = \frac{x_\tau}{e^{-\frac{i\pi}{3}}} = \frac{1}{2}(1 - \sqrt{5}),$$

which with our analytic continuation yields

$$x_1 = \frac{1}{2}(\sqrt{5} - 1)e^{-i\frac{\pi}{3}}$$

$$x_\tau = \frac{1}{2}(\sqrt{5} - 1)e^{+i\frac{\pi}{3}}.$$ (5.1)

Thus the contour $C_1$ in (3.18) starts at $\ln x_1 = \ln[(\sqrt{5} - 1)/2] - i\pi/3$ and goes off to positive infinity, asymptotically approaching the real $\epsilon$-axis from below. It goes over the logarithmic singularity at $\epsilon = i\pi/3$. The contour $C_\tau$ is of course the complex conjugate of $C_1$. The actual shape of these contours is determined by the solutions $\epsilon_a(\theta)$, but we do not need this information to evaluate (3.16). We can deform the contour $C_1$ so that $Im(\epsilon)$ is always $-i\pi/3$. This goes right through the singularity of the integrand, but the integral is well-defined. Shifting $\epsilon \to \epsilon + i\pi/3$ in (3.19) yields

$$2f_A(0) = -\frac{1}{\pi} \int_{\ln |x_1|}^{\infty} d\epsilon \left[ \frac{-\epsilon e^{-\epsilon}}{1 - e^{-\epsilon}} + \ln(1 - e^{-\epsilon}) + \left( \frac{i\pi}{3} \right) \frac{e^{-\epsilon}}{1 - e^{-\epsilon}} \right] + \text{complex conjugate.}$$ (5.2)

The last term in the square brackets is a result of the shift, and can be evaluated explicitly, giving

$$\left( -\frac{1}{\pi} \right) \frac{i\pi}{3} \ln(1 - e^{-\epsilon}) \bigg|_{\ln |x_1|}^{\infty} + \text{complex conjugate.}$$
This is not zero because the argument of the logarithm at \( \epsilon = \ln |x_1| \) is negative, and the way we have analytically continued means that the logarithm from \( C_1 \) gives \(-i\pi\), whereas

the logarithm from \( C_T \) gives \(+i\pi\). Thus the total contribution of this term is \(-2\pi/3\). To evaluate the other two terms, we split the contour into two parts, one running from \( \ln |x_1| \) to zero, and the other from zero to infinity. Then the change of variables \( x = 1 - e^{\epsilon} \) in the first part and \( x = e^{-\epsilon} \) in the second yields

\[
f_A(0) = -\frac{1}{\pi} \left[ -2L \left( \frac{1}{2} (3 - \sqrt{5}) \right) - 2L(1) + \frac{\pi^2}{3} \right]. \tag{5.3}
\]

Since \( L(1) = \pi^2/6 \) and \( L(\frac{1}{2} (3 - \sqrt{5})) = \pi^2/15 \), this yields

\[
E_A = +\frac{4}{5} \left( \frac{\pi}{6R} \right). \tag{5.4}
\]

The operator which creates this state has dimensions \((1/15, 1/15)\), as was derived in \([8]\). In the lattice Potts model, this corresponds to “cyclic” boundary conditions, where \( \sigma(0, y) = \exp(2\pi i/3)\sigma(R, y) \). This result has been derived at the conformal point \([10]\).

### 5.2. Z\(_{n+1}\) parafermions

The Ising model and the 3-state Potts model are the first two models in the hierarchy of \( Z_{n+1}\) parafermions\([30]\). At the conformal point these models have central charge \( c = 2n/(n + 3) \). There are “spin” fields \( \sigma_k \) \((k = 1\ldots n)\) which have conformal dimension \( h = \tilde{h} = k(n+1-k)/(2(n+1)(n+3)) \), and “C-disorder” fields \( \Phi_s \) \((s = 0, 1, 2\ldots \leq (n+1)/2)\), which have conformal dimensions \( h = \tilde{h} = [n-1+(n+1-2s)^2]/16(n+3) \). These conformal field theories remain integrable when perturbed by the primary field with dimensions \((2/(n+3), 2/(n+3))\), and the S-matrix is conjectured to be the \( A_n \) minimal Toda S-matrix\([31]\). There are \( n \) particles with mass \( m_a = \sin(\pi a/(n+1)) \), \( a = 1\ldots n \).

The value of \( N_{ab} \) for a simply-laced Toda minimal S-matrix are given by the matrix relation \( N = I(2 - I) \), where \( I \) is the incidence matrix \((\equiv 2 - C, \text{where } C \text{ is the Cartan matrix})\) of the Lie algebra corresponding to that Toda theory. It was shown in \([33]\) that the relation \((3.17)\) for simply-laced Toda theories can be transformed to the simpler form

\[
x_a = \prod_b (\lambda_a + x_a)^{1/2} I_{ab}. \tag{5.5}
\]

\(\text{For arguments as to why this is so, see } [32].\)
The symmetry of a Toda S-matrix is the symmetry of the extended Dynkin diagram. For $A_n$ this is the dihedral group $D_{n+1}$, and is generated by two operators $A$ and $K$. The diagonal $\mathbb{Z}_{n+1}$ symmetry is generated by $A$, which when acting on the particle $a$ gives the phase $\exp(2\pi ia/(n+1))$. $K$ implements a $\mathbb{Z}_2$ symmetry, which exchanges particle $a$ with the particle $n + 1 - a$. For more details on these and other Toda S-matrices, and on the TBA in these theories, see [15] and [33].

By using the method of section 3, we can determine $<A^k>$ by setting the chemical potentials $\lambda_a = \exp(i2\pi ka/(n+1))$. For the $A_n$ Toda minimal S-matrices, the solution of (5.5) is

$$\frac{x_a}{\lambda_a} = \frac{\sin \left(\frac{(k+1)a\pi}{n+3}\right) \sin \left(\frac{(k+1)(a+2)\pi}{n+3}\right)}{\sin^2 \left(\frac{(k+1)\pi}{n+3}\right)}.$$  \hspace{1cm} (5.6)

These values should be used with (3.16) to determine the ground-state energy in the conformal limit. In the Ising ($n=1$) and three-state Potts ($n=2$) models, the excited state associated with $<A^k>$ is created by $\sigma_k$. Therefore, we conjecture that this holds true for this entire hierarchy. This has been checked for $n = 4$ in [3]. In the next section, the $n = 3$ case is discussed in detail. However, I was unable to find a general way of evaluating the integral in (3.16), and evaluating them case-by-case is straightforward but tedious.

By using the method of section 4, we find the integral equations which determine $<K>$. For $n$ even, there are no particles neutral under $K$, so we can read off the answer in the conformal limit from (4.5). The operator which creates the associated excited state has dimensions $n/16(n+3)$, and is the $\Phi_{n/2}$ field. To evaluate $<K>$ in the conformal limit for $n$ odd would require solving the integral equation (4.2) numerically, and substituting the result into (4.3). However, it can be easily estimated using

$$f_K(0) \approx \sum_a \frac{1}{n_a^2} L(\frac{x_a}{1 + x_a}) \hspace{1cm} (5.7)$$

where $L$ is defined in (3.20), and the $x_a$ are those of the untruncated model. (The $x_a$ do not change because (3.12) and (4.2) are identical in the $m \to 0$ limit.) Evaluating the dilogarithms numerically, we find that

$$f_K(0) \approx -\frac{1}{2} \left(\frac{\pi}{6}\right)$$

for all odd $n$. The associated field has dimensions $(n - 1)/16(n + 3)$, and is $\Phi_{(n+1)/2}$. One should be able to verify this in the conformal limit by using the analysis of [34].
5.3. $\mathbb{Z}_4$ parafermions (in detail)

In this section we calculate the conformal limit of $< A >$ and $< A^2 >$ for $\mathbb{Z}_4$ parafermions. The S-matrix is that of the $A_3$ Toda theory, where the mass spectrum consists of three particles 1, 2 and 3, with masses $1/\sqrt{2}$, 1 and $1/\sqrt{2}$, respectively.

To calculate $< A^2 >$, we set $\lambda_2 = 1$, $\lambda_1 = \lambda_3 = -1$. The solution of (5.5) is given in (5.6), and is $x_1 = x_3 = 1$, and $x_2 = 0$. It is not the unique solution, but it is easy to verify that this in fact is the solution obtained by deforming $\lambda_1$ and $\lambda_3$ continuously from 1 to −1. Since all the $\lambda_a$ are either 1 or −1, we use (3.19) and (3.20) to obtain

$$f_{A^2}(0) = -\frac{1}{\pi} [L(1) - 2L(1)] = \left(\frac{\pi}{6}\right),$$

(5.8)

since $L(1) = \pi^2/6$. The equivalence (1.8) means that this can be interpreted as the lowest eigenvalue of the Hamiltonian with boundary conditions appropriate to $A$. (The lattice-model interpretation of these boundary conditions will be given in the next subsection.)

The conformal dimensions of the operator creating this state are given by the asymptotic formula (1.5), and are $(1/12, 1/12)$. This operator is the spin field $\sigma_2$.

The calculation of $< A >$ in the conformal limit is a bit more involved. We set $\lambda_a = (i)^a$, and the solution of (5.5) is $x_1 = 0$, $x_2 = 1$. Using (3.20), we find

$$L_{\lambda_2} = -\frac{1}{\pi} L(1) = \frac{\pi}{6}.$$

Since $x_1$ and $x_3$ are real the contours in (3.10) run along the real axis:

$$L_{\lambda_1} + L_{\lambda_3} = -\frac{1}{2\pi} \int_{-\infty}^{\infty} d\epsilon \left[ \frac{\epsilon i e^{-\epsilon}}{1 + ie^{-\epsilon}} + \ln(1 + ie^{-\epsilon}) \right] + \text{complex conjugate.}$$

(5.9)

Even though the two terms in the square brackets individually diverge when $\epsilon \to \infty$, their sum does not, and the integral is well-defined. Since the integral is over the real axis, we can add it and its complex conjugate, obtaining

$$L_{\lambda_1} + L_{\lambda_3} = -\frac{1}{2\pi} \int_{-\infty}^{\infty} d\epsilon \left[ \frac{\epsilon e^{-2\epsilon}}{1 + e^{-2\epsilon}} + \ln(1 + e^{-2\epsilon}) \right]$$

$$= -\frac{1}{2\pi} L(1),$$

(5.10)

where we do the change of variables $x = 1/(1 + \exp(2\epsilon))$ to obtain the dilogarithm in the second line. Adding all the contributions together, we obtain

$$f_A(0) = \frac{1}{2} \left(\frac{\pi}{6}\right),$$

(5.11)

and associated operator is the spin field $\sigma_1$, with conformal dimensions of $(1/16, 1/16)$. We also have $< A^3 > = < A >$, which agrees with the general parafermion symmetry $\sigma_k \leftrightarrow \sigma_{n+1-k}$.
5.4. $D_n$ Toda theories

The $D_n$ Toda S-matrices describe a hierarchy of models which in the conformal limit all have $c = 1$. They correspond to a scalar field with radius $\sqrt{n/2}$ orbifolded by its $\mathbb{Z}_2$ symmetry [35]. They are the continuum limit of the lattice Ashkin-Teller model, which consists of two Ising spins at each site with a four-spin coupling specified by the value of $n$. The $(\text{Ising})^2$ model and the $\mathbb{Z}_4$ parafermions are the first two models in this series, since $D_2 = A_1 + A_1$ and $D_3 = A_3$. The $D_n$ model has $n$ particles, with $m_0 = m_{\overline{0}} = 1$ and $m_a = 2\sin(\pi a/2(n - 1))$, where $a$ runs from 1 to $(n - 2)$. As before, the symmetry is that of the extended Dynkin diagram. For more details of these models, see [36] or [15]. The TBA was applied to these models in [15].

The symmetry structure is slightly different for $n$ odd or even. For all $n$, there is a $\mathbb{Z}_2$ generated by $K$, which interchanges 0 with $\overline{0}$. (For $n=4$, this becomes $S_3$, which interchanges 1, 0 and $\overline{0}$.) For $n$ even, the diagonal symmetry is $\mathbb{Z}_2 \times \mathbb{Z}_2$, and is generated by two operators $(A_1, A_2)$, which act on the particles with eigenvalues $\alpha_1$ and $\alpha_2$, where $\alpha_1(a) = \alpha_2(a) = (-1)^a$, $\alpha_1(0) = \alpha_2(\overline{0}) = 1$, and $\alpha_1(\overline{0}) = \alpha_2(0) = -1$. For $n$ odd, the diagonal symmetry is $\mathbb{Z}_4$, and is generated by $A$, with eigenvalues $\alpha(a) = (-1)^a$, $\alpha(0) = i$ and $\alpha(\overline{0}) = -i$.

To calculate $< A_1 A_2 >$ for $n$ even or $< (A)^2 >$ for $n$ odd, we set $\lambda_0 = \lambda_{\overline{0}} = -1$ and $\lambda_a = 1$. The solution of (3.17) is

$$
x_0 = x_{\overline{0}} = 1
$$

$$
x_a = \frac{\sin((a-1)\pi/n)\sin((a+1)\pi/n)}{\sin^2(\pi/a)}.
$$

(5.12)

Since all $\lambda_a$ are $\pm 1$, (3.19) and (3.20) are applicable in the conformal limit. Notice that $x_1 = 0$ and that the $x_a$ for $a \geq 2$ are those for the $A_{n-3}$ Toda theory, as displayed in (5.6) with $k = 0$. Thus

$$
f(A_1 A_2)(0) = -\frac{\pi}{6} \left(\frac{2(n - 3)}{n} + 1 - 1 - 1\right)
$$

$$
= -\frac{\pi}{6} (1 - \frac{6}{n}).
$$

(5.13)

The associated excited state is created by the field with conformal dimensions $(1/4n, 1/4n)$. This corresponds to anti-periodic boundary conditions on both Ashkin-Teller spins, and has been derived in the conformal limit[27].

18
To calculate $A_1 \to A_2$ for $n$ even, we set $\lambda_0 = 1$, $\lambda_{\overline{0}} = -1$ and $\lambda_a = (-1)^a$. The solution of (3.17) is

$$x_0 = x_{\overline{0}} = 1$$
$$x_a = 1 \quad a \text{ odd}$$
$$x_a = 0 \quad a \text{ even.}$$

Thus (3.19) and (3.20) yield

$$f_{A_1}(0) = -\frac{\pi}{6} \left(-1 + \frac{1}{2} + 1 - 1 + \ldots + 1 - 1\right)$$
$$= \frac{1}{2} \left(\frac{\pi}{6}\right).$$

The associated excited state is created by the field with dimensions $(1/16, 1/16)$ and is called the spin field. This corresponds to anti-periodic boundary conditions on one of the Ashkin-Teller spins, and has also been derived in the conformal limit [27].

Calculating $<A>$ for $n$ odd requires setting $\lambda_0 = i$, $\lambda_{\overline{0}} = -i$ and $\lambda_a = (-1)^a$. The solution of (3.17) is (5.14) in this case as well, but we cannot use (3.19) this time. However, evaluating (3.16) for the three particles $0, \overline{0}$ and 1 is identical to the calculation of $<A>$ done for the $D_3$ model ($Z_4$ parafermions) treated in the previous subsection. The rest of the terms in (3.16) can be evaluated with (3.19), so we find

$$f_A(0) = -\frac{\pi}{6} \left(-\frac{1}{2} + 1 - 1 + \ldots + 1 - 1\right)$$
$$= \frac{1}{2} \left(\frac{\pi}{6}\right),$$

and the associated field has conformal dimensions $(1/16, 1/16)$ here as well.

Evaluating $<K>$ exactly for $n > 2$ would require a numerical calculation. However, using the approximate formula (5.7) yields conformal dimensions of

$$(h, \overline{h}) \approx \left(\frac{1}{16} L(1/n), \frac{1}{16} L(1/n)\right) \approx \left(\frac{1}{16n}, \frac{1}{16n}\right),$$

which is the result obtained at the conformal point.

5.5. $E_6$, $E_7$ Toda theories

The tricritical Ising model ($c = 7/10$) perturbed by its thermal operator is believed to be described by the seven-particle $E_7$ Toda S-matrix [37]. The solution of (5.3) for $\lambda_a = 1$
is given in [15]. This model has a diagonal $Z_2$ symmetry which multiplies particles 1, 3 and 6 (in the notation of [13]) by $-1$. The solution of (5.5) for $\lambda_1 = \lambda_3 = \lambda_6 = -1$ is

$$x_4 = x_7 = 0$$
$$x_1 = x_3 = x_6 = 1$$
$$x_2 = x_5 = \frac{1 + \sqrt{5}}{2}.$$  

Thus (3.19) and (3.20) yield

$$f_A(0) = -\frac{1}{2\pi} \left( -1 + \frac{2}{5} - 1 + 1 + \frac{2}{5} - 1 + 1 \right) = \frac{1}{5} \left( \frac{\pi}{6} \right),$$  

which means that the associated operator has conformal dimensions $(3/80, 3/80)$.

The tricritical three-state Potts model ($c = 6/7$) perturbed by its thermal operator is believed to be described by the six-particle $E_6$ Toda S-matrix [38]. The solution of (5.5) for $\lambda_a = 1$ is given in [17]. This model has an $S_3$ symmetry generated by the operators $A$ and $K$. The diagonal operator $A$ multiplies particles 1 and 3 by $\exp(2\pi i/3)$, $\overline{1}$ and $\overline{3}$ by $\exp(-2\pi i/3)$, and leaves particles 2 and 4 neutral (using the labels of [15]). $K$ interchanges 1 with $\overline{1}$ and 3 with $\overline{3}$, while leaving 2 and 4 neutral. The calculation of $f_A(0)$ was outlined in [8], and the associated operator has dimensions $(1/21, 1/21)$. To calculate $f_K(0)$ would require a numerical calculation, since there are particles left neutral by this symmetry. The approximate formula (5.7) yields the conformal dimensions of $(1/56, 1/56)$ for the associated operator.

6. Examples from the minimal models

6.1. The massless perturbation of the tricritical Ising model

The conformal tricritical Ising model (the $n = 4$ minimal model) perturbed by the $\phi_{1,3}$ operator

$$S_{TCI} \rightarrow S_{TCI} + g \int d^2x \phi_{1,3}$$

is described by two different field theories, depending on the sign of $g$. For $g$ positive, one obtains a theory of massive kinks with a non-diagonal S-matrix [39]. Since the S-matrix is not diagonal, one must introduce pseudoparticles into the TBA [19]. Due to the difficulty in
undersstanding the effect of boundary conditions on the pseudoparticles, we will not discuss this case here. When \( g \) is positive, the model flows to the Ising model \((n = 3 \text{ minimal model})\). Thus there must be massless excitations, which are the only particles which will remain in the infrared limit. As shown in \([20]\), one would expect there to be a right-moving and a left-moving massless fermion, which in the infrared limit become the free Majorana fermion of the Ising model. The S-matrix for these fermions was conjectured in \([20]\), and the TBA system was derived. Since these particles are massless, the rapidity relations (2.1) do not apply but instead are replaced by

\[
P_{\text{right}} = \frac{M}{2} e^\theta, \\
P_{\text{left}} = -\frac{M}{2} e^{-\theta},
\]

where \( M \) is the scale of the theory, which depends on \( g \). The TBA equations (3.12) are changed accordingly.

We calculated \( \langle (-1)^N \rangle \) in the Ising model in section 2, and showed that it corresponded to periodic boundary conditions in the \( R \)-direction on the fermion. The same is true here. The calculation can be done by introducing fugacities \( \lambda_{\text{right}} = \lambda_{\text{left}} = -1 \). The analysis of section 3 is easily repeated to take into account the modified rapidities (6.1). One then obtains the TBA equations conjectured in [8] and [9] as describing the excited-state energy for this model. We thus have provided a physical interpretation of these equations. These papers show that the operator which creates the excited state has dimensions \((3/80,3/80)\) in the conformal limit.

6.2. The “staircase” model

A simple TBA system with many intriguing similarities to the minimal models has recently been studied [21]. The system consists of a single particle \( \Phi \) with mass \( m \) and with S-matrix

\[
S(\theta) = \frac{\sinh \theta - i \cosh \theta_0}{\sinh \theta + i \cosh \theta_0},
\]

where \( \theta_0 \) is a free parameter. Using (3.11) and (3.18), one finds \( N_{\Phi\Phi} = -1 \). Solving (3.17) yields \( x_\Phi = 0 \), which means that in the \( m \to 0 \) limit, \( c = 1 \). This S-matrix is the analytic continuation of the S-matrix for the sinh-Gordon model [10]

\[
S_{shG} = \int \left[ \frac{1}{2} (\partial_\mu \Phi)^2 - 2\mu \cosh \beta \Phi \right] d^2x.
\]
to complex coupling \( \gamma = \frac{\pi}{2} + i\theta_0 \), where

\[
\gamma = \frac{\beta^2/8}{1 + \beta^2/8\pi}.
\]

In [21] it is shown that at large \( \theta_0 \), \( f(mR) \) develops an unusual “staircase” structure. For all other known models, the structure of \( f(mR) \) (as a function of \( mR \) on a log scale) is a plateau at \( f(0) \), which eventually falls off smoothly to \( f(\infty) = 0 \). One then can read off the central charge of the conformal theory from the value of \( f(0) \). In this “staircase” model at large \( \theta_0 \), \( f(mR) \) develops plateaus in various regimes of \( mR \), making the plot of \( f(mR) \) look something like a staircase. Even more intriguing is that the values of \( f(mR) \) on the plateaus are the values which give the central charge of the minimal models, \( c = 1 - 6/n(n+1) \). Thus the picture one has is that one starts out at a \( c = 1 \) conformal theory at \( m = 0 \), and then as \( m \) is increased, the renormalization group flow takes one into the neighborhood of the fixed point corresponding to each of the minimal models. The trajectory spends some time in the neighborhood of each model, before the instability takes over, forcing it to flow to the next model. This goes on until it finally reaches the \( c = 0 \) trivial model, where all the degrees of freedom have been scaled away. The staircase structure was derived numerically and analytically. It is derived analytically by showing that the TBA system defined by (6.2) can be accurately approximated by the TBA system conjectured in [19] as describing the flow of the ground state from the \( n \)-th to the \( (n-1) \)-th minimal model [41]. For a much more detailed description of the staircase model, see the original paper [21].

Since there is no pole in the S-matrix (6.2) corresponding to \( \Phi \) as a bound state of itself, this model must have a \( \mathbb{Z}_2 \) symmetry \( \Phi \to -\Phi \). This is also apparent from the sinh-Gordon description (6.3). Thus in our usual manner we can set \( \lambda = -1 \) and calculate \( <(-1)^N> \). The solution of (3.17) is \( x_\Phi = 2 \), so

\[
f_{-1}(0) = \frac{1}{2}(\frac{\pi}{6}). \quad (6.4)
\]

Thus the associated operator in the conformal limit has dimensions \((1/16, 1/16)\). It is easy to repeat the argument of [21] to show that \( f_1(mR) \) develops plateaus, only this time the TBA system can be approximated by the TBA system studied in sections 5.1 and 5.3 of ref. [4]. This is identical to the system conjectured for the ground-state energies [19], except in this case all the \( \lambda_a = -1 \) instead of 1. The plateaus do not form a staircase here; for \( n \) odd (even) the plateaus are at values greater (less) than \( f_{-1}(0) \). The plateaus take
values of $f_{-1}$ corresponding to the conformal dimensions of the field $\phi_{n/2,n/2}$ for $n$ even and $\phi_{(n+1)/2,(n+1)/2}$ for $n$ odd. This is conjectured to describe the flow from the $n$-th to the $(n-1)$-th minimal model of an excited-state energy for $n$ even, but not $n$ odd\cite{9}. This is for several reasons. The first is that the numerical results for $n$ odd beyond lowest order do not agree with perturbation theory around the conformal point. The second is that one expects the flows in minimal models to take the $\phi_{i,i}$ operator in the $n$-th minimal model to the $\phi_{i,i}$ operator in the $(n-1)$-th minimal model\cite{12,42}; this is true here only for $n$ even.

So while our result shows that excited states in the staircase model resemble those in the minimal models in some respects, there is a crucial difference. Although the flow down the chain of operators seems very natural from the point-of-view of the staircase model, its relation to flows in the minimal models is still not completely clear.

7. Conclusions and speculations

We have seen that the TBA is useful for calculating not only the Casimir energy of the lowest-energy state in the theory, but for a number of excited states as well. This is done by choosing the boundary conditions to project out the ground state. In the limit $R \rightarrow \infty$, all of the states discussed here have $E \rightarrow 0$, and become degenerate with the ground state\cite{7}. It is not clear if it will be possible to project out all such states and obtain information on massive states, in the manner of \cite{43,44}.

The most obvious open direction is to understand how these results can be used to study excited states in theories with non-diagonal S-matrices, where one must introduce massless pseudo-particles into the TBA\cite{19}. TBA equations in some such situations have been conjectured, but it is not obvious how to interpret these equations as boundary conditions on the field theory. However, this hopefully is only an interpretational issue: all the formalism should be applicable to these cases.

There are many other operator expectation values that one can calculate. For example, it is straightforward to calculate $< (-1)^{Na} >$ for any particle $a$ in a diagonal scattering theory, since $(-1)^{Na}$ is a symmetry of the S-matrix even if it is not a symmetry of the field theory or of the fusion rules. The interpretation of this in terms of boundary conditions is not immediately obvious. However, in the minimal models on the lattice (on and off

\[7\] I thank Tim Klassen for pointing this out.

23
criticality) it is possible to define boundary conditions so that any of the primary operators creates the lowest-energy state. This seems to be true for all rational conformal field theories as well. It is thus not difficult to believe that once suitable ways are found of implementing the boundary conditions, one will be able to calculate the energy for all such excited states by calculating operator expectation values with the TBA.

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