ON THE NONHAMILTONIAN INTERACTION OF TWO ROTATORS. II. UNRAVELLING THE ALGEBRAIC STRUCTURE

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This short note being a continuation of the first part [1] is devoted to the unravelling of the algebraic structure, which governs the quadratic nonHamiltonian interaction of two rotators described in [1]. It should be considered in the context of a general ideology of the inverse problem of representation theory [2]. The resulted objects are formalized as Lie algebras with operators (Lie algebras with additional unary operations – see [3]).

1. Lie $\xi \rho$-algebras. First, let introduce and investigate the algebraic object, which has the general algebraic meaning besides its relation to the discussed topic.

Definition 1. The Lie $\xi \rho$-algebra is a triple $(g, \xi, \rho)$, where $g$ is a Lie algebra, $\xi \in \text{Der}(g)$, $\rho$ is an operator in $g$ commuting with $\xi$ such that the following identities hold

\[
\rho[x, y] = [\rho x, y] + [x, \rho y],
\]

\[
\xi [x, y] = [\rho x, \rho y] + [\rho x, y] + [x, \rho y],
\]

\[
\xi^2[x, y] = [\rho^2 x, y] + [x, \rho^2 y] - 2[\rho x, \rho y],
\]

where

\[
[x, y] = [\rho x, y] + [x, \rho y] - \rho[x, y] - [\xi x, \xi y].
\]

Remark 1. The second identity of definition 1 may be rewritten as

\[
\xi[\xi x, \xi y] = [S x, y] + [x, S y] - S[x, y], \quad S = \xi \rho.
\]

Remark 2. The bracket $[\cdot, \cdot]_{\rho}$ coincides with the the bracket $[\cdot, \cdot]_{\Theta}$ defined as

\[
[x, y]_{\Theta} = \frac{1}{2} (\Theta x, y + [x, \Theta y] - \Theta [x, y]), \quad \Theta = 2 \rho + \xi^2.
\]

Moreover,

\[
\Theta[x, y]_{\Theta} = [\rho^2 x, y] + [x, \rho^2 y].
\]

Example 1. Let $g = \text{gl}(n)$ be the Lie algebra of all $n \times n$ matrices. Put $\xi_q x = qx - xq$, $\rho_q x = qx q (q \in \text{Mat}_n)$. The triple $(\text{gl}(n), \xi, \rho)$ is a Lie $\xi \rho$-algebra.

Note that the mapping $\rho_q$ is quite familiar in the theory of Jordan algebras and symmetric spaces [4,5].
Example 2. Let $\ast$ be an involution in $Mat_n$ and $\mathfrak{g} = \{ x \in Mat_n : x^\ast = -x \}$. Then the restriction of $\xi$ and $\varrho$ from example 1 onto $\mathfrak{g}$ supplies it by the structure of a Lie $\xi\varrho$-algebra. In particular, $\mathfrak{so}(n)$ is a Lie $\xi\varrho$-algebra.

Remark 3. $\xi$ is a derivation of the bracket $[\cdot, \cdot]_{\varrho}$, i.e.

$$\xi [x, y]_{\varrho} = [\xi x, y]_{\varrho} + [x, \xi y]_{\varrho}.$$ 

Exercise. To describe all structures of Lie $\xi\varrho$-algebras on $\mathfrak{so}(3)$ [answer: each such structure coincides with one of example 2; hint: use remark 3].

This property is equivalent to the second identity of definition 1.

Theorem 1. In any $\xi\varrho$-algebra the bracket $[\cdot, \cdot]_{\varrho}$ is a Lie bracket compatible with $[\cdot, \cdot]$.

Let us now relate the Lie $\xi\varrho$-algebras to Lie bi-$\tilde{m}$YB-algebras of the article [6].

Definition 2 [6].

A. The Lie $\tilde{m}$YB-algebra is a Lie algebra $\mathfrak{g}$ with the bracket $[\cdot, \cdot]$, supplied by an operator $R : \mathfrak{g} \mapsto \mathfrak{g}$ such that

$$R[Rx, y] + R[x, Ry] = [Rx, Ry] + R^2[x, y].$$

B. The Lie bi-$\tilde{m}$YB-algebra is a Lie algebra $\mathfrak{g}$ with bracket $[\cdot, \cdot]$ supplied by two commuting operators $R_1$ and $R_2$ such that $(\mathfrak{g}, R_1)$ and $(\mathfrak{g}, R_2)$ are the Lie $\tilde{m}$YB-algebras with identical brackets $[\cdot, \cdot]_{R_1}$ and $[\cdot, \cdot]_{R_2}$.

C. A Lie bi-$\tilde{m}$YB-algebra $(\mathfrak{g}, R_1, R_2)$ is called even-tempered if the identities

$$[R_1 x, R_2 y] + [R_2 x, R_1 y] - R_1 R_2 [x, y] = [R_1^2 x, y] + [x, R_1^2 y] - R_1^2 [x, y],$$

$$[R_1 x, R_2 y] + [R_2 x, R_1 y] - R_1 R_2 [x, y] = [R_2^2 x, y] + [x, R_2^2 y] - R_2^2 [x, y]$$

hold.

Examples of Lie $\tilde{m}$YB-algebras and bi-$\tilde{m}$YB-algebras were considered in [6].

Proposition 1. Any even-tempered Lie bi-$\tilde{m}$YB-algebra $(\mathfrak{g}, R_1, R_2)$ is a Lie $\xi\varrho$-algebra with $\xi = R_1 - R_2$, $\varrho = R_1 R_2$.

The Lie $\xi\varrho$-algebra of example 1 can be constructed from a Lie bi-$\tilde{m}$YB-algebras whereas one of example 2 can not.

2. $\xi\varrho$-structures on Lie algebras and quadratic $I$-pairs. In the definition of Lie $\xi\varrho$-algebra it is natural to consider $\xi$ as an inner derivation of $\mathfrak{g}$. In this case the defining identities of the Lie $\xi\varrho$-algebra have the form

$$\varrho [x, y]_{\varrho} = [\varrho x, y], \quad [q [x, y]_{\varrho}] = [\varrho x [q, y]] + [[q, x] \varrho y],$$

$$[q [q [x, y]_{\varrho}]] = [\varrho^2 x, y] + [x, \varrho^2 y] - 2[\varrho x, y]$$

where

$$[x, y]_{\varrho} = [\varrho x, y] + [x, \varrho y] - \varrho [x, y] - [[q, x] [q, y]]$$

and $\xi = \text{ad} q$, $q \in \mathfrak{g}$.

The following definition is self-explained.
Definition 3. The \(\xi_\mathfrak{g}\)-structure on the Lie algebra \(\mathfrak{g}\) is the set of operators \(\varrho_q\) in \(\mathfrak{g}\) (\(q \in \mathfrak{g}\)) such that (1) \(\varrho_q\) depends quadratically on \(q\) (i.e. \(\varrho_{\lambda q} = \lambda^2 \varrho_q\) and \(\varrho_{q_1+q_2} + \varrho_{q_1-q_2} = 2\varrho_{q_1} + 2\varrho_{q_2}\)), (2) the triple \((\mathfrak{g}, \text{ad} \varrho, \varrho_q)\) is a Lie \(\xi_\mathfrak{g}\) algebra for all \(q \in \mathfrak{g}\).

Below we shall consider \(\mathfrak{g}\)-equivariant \(\xi_\mathfrak{g}\)-structures on Lie algebras \(\mathfrak{g}\).

Remark 4. The structures of Lie \(\xi_\mathfrak{g}\)-algebras on \(\mathfrak{gl}(n)\) and \(\mathfrak{so}(n)\) from examples 1,2 form \(\xi_\mathfrak{g}\)-structures on them.

Definition 4. A pair \((V_1, V_2)\) of linear spaces will be called I-pair iff there are defined (nonlinear) mappings

\[
h_1 : V_2 \mapsto \text{Hom}(\Lambda^2(V_1), V_1) \\
h_2 : V_1 \mapsto \text{Hom}(\Lambda^2(V_2), V_2)
\]

such that \(\forall A \in V_2\) its image \(h_1(A) \in \text{Hom}(\Lambda^2(V_1), V_1)\) is a Lie bracket in \(V_1\) and \(\forall X \in V_1\) its image \(h_2(X) \in \text{Hom}(\Lambda^2(V_2), V_2)\) is a Lie bracket in \(V_2\). The Lie bracket in \(V_2\) corresponded to \(A\) will be denoted by \([\cdot, \cdot]_A\), whereas the Lie bracket in \(V_1\) corresponded to \(X\) will be denoted by \([\cdot, \cdot]_X\).

Here \(\text{Hom}(H_1, H_2)\) denotes the space of all linear operators from \(H_1\) to \(H_2\), \(\Lambda^2(H)\) is a skew square of the linear space \(H\), so \(\text{Hom}(\Lambda^2(H), H)\) is the space of all skew–symmetric bilinear binary operations in \(H\). The Lie brackets in \(H\) form a submanifold \(\text{Lie}(H)\) of the space \(\text{Hom}(\Lambda^2(H), H)\).

Theorem 2. Any \(\xi_\mathfrak{g}\)-structure on the Lie algebra \(\mathfrak{g}\) induces a structure of the quadratic I-pair on \((\mathfrak{g}, \mathfrak{g})\) with

\[
[x, y]_q = [x, y] + [x, y]_{\varrho_q}.
\]

Remark 5. The I-pair of [1] is just the I-pair constructed from the \(\xi_\mathfrak{g}\)-structure on \(\mathfrak{so}(3)\) from remark 4.

3. Lie \(\xi_\mathfrak{g}\)-algebras, Lie \(R_\mathfrak{g}\)-algebras and Lie \(\Theta_\mathfrak{g}\)-algebras. First, let us describe a relation between Lie \(\xi_\mathfrak{g}\)-algebras and Lie \(R_\mathfrak{g}\)-algebras of the article [7].

Definition 5A [7]. The Lie \(R_\mathfrak{g}\)-algebra is a triple \((\mathfrak{g}, R, \varrho)\), where \(\mathfrak{g}\) is the Lie algebra with the bracket \([\cdot, \cdot]\) and \(R, \varrho\) are two operators in it such that the following two identities

\[
\varrho[x, y]_R = [\varrho x, \varrho y], \\
R[x, y]_R + \varrho[x, y]_R = [Rx, Ry] + [\varrho x, \varrho y].
\]

holds for all \(x, y\) and \(R\) from \(\mathfrak{g}\). Here

\[
[x, y]_R = [Rx, y] + [x, Ry] - R[x, y], \\
[x, y]_\varrho = \varrho[x, y] + [\varrho x, y] - \varrho[x, y] + [Rx, Ry] - R[x, y]_R.
\]

A Lie \(R_\mathfrak{g}\)-algebra \((\mathfrak{g}, R, \varrho)\) is called regular if the identity

\[
R[x, y]_R = 2([\varrho x, y] + [x, \varrho y])
\]

holds.

Lie \(R_\mathfrak{g}\)-algebras are deeply related to certain quadratic bunches of Lie algebras [7]. In particular, the bracket \([\cdot, \cdot]_R\) is a Lie one (i.e. obeys the Jacobi identity).
**Proposition 2.** Let \((\mathfrak{g}, R, \varrho)\) be a regular Lie \(R\varrho\)-algebra with commuting \(R\) and \(\varrho\). Let us define the sequence of operators \(R_n\) as

\[
R_{n+1} = RR_n - \varrho R_{n-1}, \quad R_0 = 1, \quad R_1 = R.
\]

The triples \((\mathfrak{g}, R_n, \varrho^n)\) are regular Lie \(R\varrho\)-algebras.

**Proposition 3.** Let \((\mathfrak{g}, \xi, \varrho)\) be a Lie \(\xi\varrho\)-algebra. Let us define the sequences of operators \(\xi_n\) and \(R_n\) as

\[
\begin{align*}
R_{n+1} &= \Theta R_n - \varrho^2 R_{n-1}, \quad R_0 = 1, \quad R_1 = \Theta; \\
\xi_{n+1} &= \Theta \xi_n + \varrho^2 \xi_{n-1}, \quad \xi_0 = \xi, \quad \xi_1 = (\Theta + \varrho)\xi,
\end{align*}
\]

where \(\Theta = \varrho + \xi^2\). The triples \((\mathfrak{g}, R_n, \varrho^{2n})\) are regular Lie \(R\varrho\)-algebras, whereas the triples \((\mathfrak{g}, \xi_n, \varrho^{2n+1})\) are Lie \(\xi\varrho\)-algebras.

In particular, the triple \((\mathfrak{g}, \Theta, \varrho^2)\) is a Lie \(R\varrho\)-algebra for any Lie \(\xi\varrho\)-algebra \((\mathfrak{g}, \xi, \varrho)\). So algebras from the least class may be characterized as “square roots” of algebras from the first one. Note once more that Lie \(R\varrho\)-algebras admit an interpretation in terms of quadratic bunches of Lie algebras [7], however, an analogous interpretation for the Lie \(\xi\varrho\)-algebras is not known.

**Definition 5B.** The Lie \(\Theta\varrho\)-algebra is a triple \((\mathfrak{g}, \Theta, \varrho)\), where \(\mathfrak{g}\) is the Lie algebra with the bracket \([\cdot, \cdot]\) and \(\Theta, \varrho\) are two commuting operators in it such that the following identities

\[
\begin{align*}
\varrho[x, y]_{\Theta} &= [\varrho x, \varrho y], \quad \Theta[x, y]_{\Theta} = [\varrho^2 x, y] + [x, \varrho^2 y], \\
[\Theta x, \varrho y] + [\varrho x, \Theta y] &= \varrho[\varrho x, y] + \varrho[x, \varrho y] + [x, \varrho y]_{\Theta} + [\varrho x, y]_{\Theta}, \\
[\varrho x, \varrho y]_{\Theta} &= \varrho([\Theta x, \Theta y] - [\varrho^2 x, y] - [x, \varrho^2 y] - \varrho^2[x, y])
\end{align*}
\]

holds for all \(x\) and \(y\) from \(\mathfrak{g}\). Here

\[
[x, y]_{\Theta} = \frac{1}{2}([\Theta x, y] + [x, \Theta y] - \Theta[x, y]).
\]

A Lie \(\Theta\varrho\)-algebra is called special if the identity

\[
([\operatorname{ad} x, \Theta] - \operatorname{ad} \Theta x) \cdot \varrho + 2\varrho \cdot \operatorname{ad}(\varrho x) = 0
\]

holds.

**Example 3.** Let \(\mathfrak{A}\) be an associative algebra, \(\mathfrak{A}_{[\cdot, \cdot]}\) be its commutator algebra, \(\Theta x = q^2 x + x q^2\), \(\varrho x = q x q\) \((q \in \mathfrak{A})\). The triple \((\mathfrak{A}_{[\cdot, \cdot]}, \Theta, \varrho)\) is a special Lie \(\Theta\varrho\)-algebra.

**Remark 5.** The bracket \([\cdot, \cdot]_{\Theta}\) obeys the Jacobi identity.

**Proposition 4.** Let \((\mathfrak{g}, \Theta, \varrho)\) be a Lie \(\Theta\varrho\)-algebra. Let us define the sequence of operators \(\Theta_n\) as

\[
\Theta_{n+1} = \Theta \Theta_n - \varrho^2 \Theta_{n-1}, \quad \Theta_0 = 1, \quad \Theta_1 = \Theta.
\]

The triples \((\mathfrak{g}, \Theta_n, \varrho^{2n})\) are Lie \(\Theta\varrho\)-algebras.

Let us now relate the Lie \(\Theta\varrho\)-algebras to Lie \(\xi\varrho\)-algebras and Lie \(R\varrho\)-algebras.

**Proposition 5.**

A. Let \((\mathfrak{g}, \xi, \varrho)\) be a Lie \(\xi\varrho\)-algebra. The triple \((\mathfrak{g}, \Theta, \varrho)\) \((\Theta = 2\varrho + \xi^2)\) is the Lie \(\Theta\varrho\)-algebra.

B. Let \((\mathfrak{g}, R, \varrho)\) be a regular Lie \(R\varrho\)-algebra with commuting \(R\) and \(\varrho\). The triple \((\mathfrak{g}, \Theta, \varrho)\) \((\Theta = R^2 - 2\varrho)\) is the Lie \(\Theta\varrho\)-algebra.

It is rather interesting to describe the classes of Lie \(\xi\varrho\)-algebras and regular \(R\varrho\)-algebras, to which the special \(\Theta\varrho\)-algebras are related.
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