Third order corrections to the ground state energy of the polarized diluted gas of spin 1/2 fermions

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Abstract

We present the results of the computation of the third order corrections to the ground state energy of the diluted polarized gas of nonrelativistic spin 1/2 fermions interacting through a spin-independent repulsive two-body potential. The corrections are computed within the effective field theory approach which does not require specifying the interaction potential explicitly but only to characterize it by only a few parameters - the scattering lengths $a_0, a_1, \ldots$ and effective radii $r_0, \ldots$ - measurable in low energy fermion-fermion elastic scattering. The corrections are computed semi-analytically, that is are expressed in terms of two functions of the system’s polarization. The functions are given by the integrals which can be easily evaluated using the Mathematica built-in routines for numerical integration.

Keywords: Diluted gas of interacting fermions, itinerant ferromagnetism, effective field theory, scattering length
Introduction. The classic model of the many-body quantum mechanics, the diluted gas of nonrelativistic fermions interacting through a spin independent repulsive two-body potential [1, 2] has attracted in the recent time a renewed attention due to the advent of a new generation of experiments with cold atomic gases in which the interaction strength can be tuned in a wide range by exploiting the existence and properties of the so-called Feshbach resonance [3]. The experiments have stimulated intensive numerical studies of the system [4, 5, 6] aiming at computing its properties mainly relating to the possible application of the model to the problem of the emergence of the so-called itinerant ferromagnetism in systems of interacting fermions.

On the other hand, the application to the model of the effective field theory method in the pioneering work [7] (see also [8, 9, 10, 11]) has opened new possibilities to investigate properties of the system of interacting fermions analytically. The proposed approach has in particular greatly simplified perturbative computations of the ground-state energy of the system, automatically yielding its expansion in powers of $k_F R$ where $R$ is the lengths scale characterizing the interaction potential and $k_F$ is the Fermi wave vector of the system of $N$ fermions enclosed in the volume $V$.

The simplifications offered by the effective field theory approach allowed to complete recently [12] the computation of the fourth order, $(k_F R)^4$, contribution to the ground-state energy of the system of spin $s$ fermions with equal densities of fermions of different spin projections (unpolarized) system. It also allowed to reproduce [13] semi-analytically but in the universal setting, that is without specifying the underlying interaction potential, the order $(k_F R)^2$ correction to the ground-state energy of the spin $1/2$ fermions with the arbitrary ratio of the densities of spin up and spin down fermions (arbitrarily polarized system) which in the past has been computed by Kanno [14] using the hard spheres model interaction. This result has recently been generalized to the system of spin $s$ fermions with arbitrary proportions of densities of the $g_s = 2s + 1$ possible spin projections [15].

Computations of the ground-state energy as a function of the system’s polarization $P$ directly relates to the possibility of the phase transition to the ordered state ($P \neq 0$) at zero temperature with increasing the strength of the interaction potential (reflected in the effective field theory approach by the increasing magnitude of the scattering lengths $a_0, a_1, \ldots$ and the effective radii $r_0, \ldots$) and/or of the system’s overall density characterized by its Fermi wave vector $k_F = (3\pi^2 N/V)^{1/3}$. The first order of the perturbative expansion (equivalent to the mean-field approximation) predicts that such a transition occurs for $k_F a_0 = \pi/2$ (the Stoner’s criterion [16, 1]). Numerical investigations [4] which necessarily use a concrete form of the interaction potential indicate that the transition occurs at $k_F a_0 \approx 0.85$. Inclusion of the second order contribution in the perturbative expansion of the ground state energy yields $k_F a_0 \approx 1.054$ as the critical value of the expansion parameter.

In this letter we compute the third order correction to the ground-state energy of the system of interacting spin $1/2$ fermions for an arbitrary value of the system’s polarization $P$. As in [13] we apply the effective field theory approach proposed first in [7] and regularize the divergent integrals over wave vectors with the help of the cutoff $\Lambda$. We explicitly demonstrate the cancellation of the terms diverging in the limit $\Lambda \to \infty$ after the couplings of the effective theory Lagrangian are expressed in terms of the scattering lengths computed up to the appropriate order using the same cutoff $\Lambda$. The
final result is expressed in terms of two functions of the polarization $P$ which are given by the integrals which can be computed with sufficient accuracy by the Mathematica package built-in routine for numerical evaluation of the multidimensional integrals over a prescribed domains.

**Computation.** Assuming that the underlying “fundamental” spin independent two-body interaction of nonrelativistic spin $s$ fermions of mass $m_f$ is consistent with the Galileo, parity and time-reversal symmetries, the most general interaction term of the effective theory Hamiltonian which captures properties of low density system of $N$ such fermions as well as characteristics of their low energy scattering reads [7]

\[
V_{\text{int}} = \frac{C_0}{2} \int d^3x \sum \langle \psi_\alpha | \psi_\beta \rangle \psi_\beta \psi_\alpha - \frac{C_2}{16} \int d^3x \sum \langle \psi_\alpha | \psi_\beta \rangle \psi_\beta \psi_\alpha + \text{H.c.} \\
- \frac{C_{2}'}{8} \int d^3x \sum (\psi_\alpha \nabla \psi_\beta) \cdot (\psi_\beta \nabla \psi_\alpha) + \frac{D_0}{2} \int d^3x \sum \psi_\alpha \psi_\beta \psi_\gamma \psi_\delta + \ldots
\]

$\psi_\alpha$ and $\psi_\alpha^\dagger$ are the usual field operators of the second quantization formalism [2]. The coupling constants $C_0$, $C_2$, ... multiplying the local operator structures of decreasing length dimensions can be determined by computing using this interaction the amplitude the elastic scattering of two fermions parametrized in the low energy limit in terms of the scattering lengths. The result of such a procedure is [7, 17, 13]

\[
C_0 = \frac{4\pi h^2}{m_f} a_0 \left(1 + \frac{2}{\pi} a_0 \Lambda + \frac{4}{\pi^2} a_0^2 \Lambda^2 + \ldots\right),
\]

\[
C_2 = \frac{2\pi h}{m_{\text{red}}} \frac{1}{2} a_0^2 r_0 + \ldots,
\]

\[
C_{2}' = \frac{2\pi h}{m_{\text{red}}} a_0^3 \ldots
\]

$\Lambda$ is the UV cutoff imposed on the wave-vectors of the loop integrals. Divergences, absent in the underlying “fundamental” theory, appear as a result of the local (i.e. singular) nature of the interaction terms of the effective interaction Hamiltonian [1].

The ground-state energy density of the system of $N$ noninteracting nonrelativistic spin $s$ fermions (enclosed in the volume $V$) is

\[
\frac{E_{\Omega_0}}{V} = \frac{1}{6\pi^2} \sum_{\alpha=1}^{2s} \frac{\hbar^2}{2m_f} \frac{3}{5} p_{F\alpha}.
\]

$p_{F\alpha} = ((1/6\pi^2))N_\alpha/V^{1/3}$ are the respective Fermi wave vectors of $N_\alpha$ fermions with the spin projection $\alpha$ in the system; $N = \sum_{\alpha=1}^{2s} N_\alpha$. Since energy of the system of spin 1/2 fermions is (in the absence of an external magnetic field) invariant with respect to the interchange $N_\uparrow \leftrightarrow N_\downarrow$ we will in the following as in [13] denote $N_\uparrow$ (and, correspondingly, $p_{F\uparrow}$) the number of spin up fermions if $N_\uparrow \geq N_\downarrow$, and will write the system’s polarization $0 \leq P \leq 1$ as

\[
P = \frac{N_\uparrow - N_\downarrow}{N_\uparrow + N_\downarrow} = 1 - r^3, \quad \text{where} \quad r \equiv \frac{p_{F\downarrow}}{p_{F\uparrow}} = \frac{N_\downarrow^{1/3}}{N_\uparrow^{1/3}}.
\]
It will be also convenient to express the results in terms of the average Fermi wavevector $k_F = ((6\pi^2/g_s)(N/V))^{1/3}$ which does change when the numbers of fermions of different spin projections are varied (keeping constant $N = N_+ + N_-$).

The first nontrivial correction to the ground-state energy has been computed long time ago by Lenz [18]. Further corrections to $E_\Omega$ are most easily computed using the general formula\textsuperscript{1}

$$\lim_{T \to \infty} \exp(-iT(E_\Omega - E_{\Omega_0})/\hbar) = \lim_{T \to \infty} \langle \Omega_0 | T \exp\left(-\frac{i}{\hbar} \int_{-T/2}^{T/2} dt \, V_{\text{int}}^f(t)\right) | \Omega_0 \rangle. \quad (5)$$

in which $V_{\text{int}}^f(t)$ is the interaction part of the theory Hamiltonian taken in the interaction picture. In application to the considered system this formula, which can be evaluated using the standard Dyson expansion, gives the corrections $(E_\Omega - E_{\Omega_0})/V$ to the ground-state energy density as a sum of the momentum space connected vacuum Feynman diagrams (called in this context also the Hugenholtz diagrams) multiplied by $i\hbar$.

As the effective theory interaction \textsuperscript{11} consists of an in principle infinite number of operator structures, diagrams which should be taken into account to obtain the order $(k_F R)^\nu$ contribution to $(E_\Omega - E_{\Omega_0})/V$ are selected by the power counting rules \textsuperscript{7, 19}

$$\nu = 5 - \sum_i V_i \Delta_i = 2 + 3L + \sum_i V_i (d_i - 2), \quad (6)$$

in which $V_i$ is the number of the vertices of type $i$ with $d_i$ derivatives and $n_i$ lines attached to the vertex; $L$ is the number of closed loops and $\Delta_i = 5 - d_i - \frac{3}{2}n_i$ characterizes the dimension interaction vertices; $\Delta_{C_0} = -1$, $\Delta_{C_1, C_2} = -3$, etc. Dimensional analysis shows that the magnitude of the coupling $C_i$ multiplying the vertex of type $i$ is $(4\pi \hbar^2/m_f) R^{-\Delta_i}$, where $R$ is the characteristic length scale of the underlying interaction potential (which, if in the assumed absence of any resonant or anomalous behaviour, implies that all $a_i \sim r_i \sim R$).

The power counting rules \textsuperscript{6} tell that the to the order $k_F^5 (k_F R)^2$ correction to $(E_\Omega - E_{\Omega_0})/V$ contribute only diagrams with two $C_0$ vertices and three loops. There is only one nonvanishing such diagram, which has been first evaluated in \textsuperscript{7} for the case of unpolarized system of spin $s$ fermions and shown to straightforwardly reproduce the well-known classic result obtained first in \textsuperscript{20} with the help of rather cumbersome methods (and since then reproduced using a variety of different approaches). In \textsuperscript{13} the corresponding three loop diagram has been evaluated semi-analytically for the case of the polarized system of spin $1/2$ fermions and the result found to numerically coincide with the analytic formula obtained by Kanno \textsuperscript{14} within the hard-spheres model of the two-body interaction (extension of the result of \textsuperscript{13} to the arbitrarily polarized system of spin $s$ fermions has been presented very recently in \textsuperscript{15}).

The order $k_F^5 (k_F R)^3$ correction is given by the Hugenholtz diagrams with either three $C_0$ vertices and four loops or by two-loop diagrams with a single $C_2$ or $C_2'$ vertex. There are only two nonvanishing diagrams of the first kind \textsuperscript{7} shown in Figure\textsuperscript{14}. Both these diagrams come with the combinatoric factor of 2 (when the interaction term of spin $1/2$ fermions is written as $C_0 \int \psi_+^\dagger \psi_+ \psi_-^\dagger \psi_-$) and both are given by an integral of a\textsuperscript{\textdagger}

\textsuperscript{1}The symbol $T$ of the chronological ordering should not be confused with $T$ denoting time.
Figure 1: The two nonvanishing order $C_0^3$ effective theory connected vacuum diagrams (the “particle-particle” diagram and the “particle-hole” diagram) contributing to the correction $E^{(3)}_{\Omega}$ to the ground-state energy of the system of spin $s$ fermions with equal densities of different spin projections and their counterparts in the case of the system of spin $1/2$ fermions and $N_\uparrow \neq N_\downarrow$. In this second case solid and dashed lines represent propagators of fermions with opposite spin projections.

product of three identical blocks which consist of two terms each; of the arising $2^3 = 8$ terms two vanish as a result of the integration while the remaining six give rise to only two different terms; the factor $2 \cdot 3$ cancels the factor $1/3!$ arising from the expansion of the exponent.

After the standard steps (for more details see [13]) the contribution of the “particle-particle” diagram to the energy density can be written in the form

$$\frac{E_{\Omega}^{(3)p-p}}{V} = \frac{128m^2}{(2\pi)^8\hbar^2} C_0^3 \left[ G^{(1)}(p_{F-},p_{F+}) + G^{(2)}(p_{F-},p_{F+}) \right],$$

(7)

with

$$G^{(1)}(p_{F-},p_{F+}) = \int_0^{s_{\text{max}}} ds \ s^2 \frac{1}{4\pi} \int d^3t \ \theta(p_{F-} - |t + s|)\theta(p_{F+} - |t - s|) \ (g(t,s))^2,$$

(8)

$$G^{(2)}(p_{F-},p_{F+}) = \int_0^{s_{\text{max}}} ds \ s^2 \frac{1}{4\pi} \int d^3t \ \theta(|t + s| - p_{F-})\theta(|t - s| - p_{F+}) \ (h(t,s))^2.$$

(9)

where the functions $g(t,s) \equiv g(|t|,|s|)$ and $h(t,s) \equiv h(|t|,|s|)$ are given below. The analogous contribution of the “particle-particle” diagram to the energy density of the unpolarized system of spin $s$ fermions is obtained by multiplying $(7)$ by the spin factor $\frac{1}{2}g_s(g_s - 1)$ and setting $p_{F-} = p_{F+} = k_F$.

The function $g(t,s)$ is the one which appeared in [13] in evaluating the order $C_0^3$ contribution to the energy density; it can be written in the form

$$g(t,s) = -\Lambda + g_{\text{fin}}(t,s) + \frac{t^2}{\Lambda} + O(1/\Lambda^2),$$

(10)

in which $\Lambda$ is the UV cutoff imposed on the divergent integral over the wave vectors. The finite part $g_{\text{fin}}(t,s)$ is for $0 < s \leq \frac{1}{2}(p_{F+} - p_{F-})$ given by

$$g(t,s) = \frac{1}{2} p_{F+} + \frac{1}{4} \ln \frac{(p_{F+} - t)^2 - s^2}{(p_{F+} + t)^2 - s^2} + \frac{p_{F+}^2 - s^2 - t^2}{8s} \ln \frac{(p_{F+} + s)^2 - t^2}{(p_{F+} - s)^2 - t^2},$$

(11)
and for $\frac{1}{2}(pF_+ - pF_-) < s \leq s_{\text{max}}$ by

$$g(t, s) = \frac{1}{4}(pF_+ + pF_- + 2s) + \frac{t}{4} \ln \frac{pF_+ + s - t}{pF_+ + s + t} + \frac{t}{4} \ln \frac{pF_- + s - t}{pF_- + s + t} + \frac{pF_+ - t^2 - s^2}{8s} \ln \frac{(pF_+ + s)^2 - t^2}{u_0^2 - t^2} + \frac{pF_- - t^2 - s^2}{8s} \ln \frac{(pF_- + s)^2 - t^2}{u_0^2 - t^2}, \quad (12)$$

where

$$u_0^2 = \frac{1}{2}(pF_+ + pF_- - s^2). \quad (13)$$

Compared to the form of $g(t, s)$ given in [13] we have retained in (11) the term proportional to $1/\Lambda$ to show explicitly that the finite contributions of such terms (which are absent in the dimensional regularization used in [17]) cancel out. $h(t, s)$ is a new function given by the finite integral

$$h(t, s) = \frac{1}{4\pi} \int d^3u_0 \frac{\theta(pF_- - |u + s|)\theta(pF_+ - |u - s|)}{t^2 - u_0^2 - i0}. \quad (14)$$

Its analytic form

$$h(t, s) = \frac{1}{2}pF_- - t \ln \frac{t - (pF_- - s)}{t + (pF_- - s)} - \frac{t}{4} \ln \frac{t - (pF_- + s)}{t + (pF_- + s)} + \frac{t^2 - (pF_- - s)^2}{8s} \ln \frac{t^2 - (pF_- + s)^2}{t^2 - (pF_- - s)^2}, \quad (15)$$

for $s < s_0$ and

$$h(t, s) = \frac{1}{2}(2s - pF_- - pF_+) - \frac{t}{4} \ln \frac{t - (pF_- - s)}{t + (pF_- - s)} - \frac{t}{4} \ln \frac{t - (pF_- + s)}{t + (pF_- + s)} - \frac{1}{8s} \left[ (pF_+ - s)^2 + (pF_- - s)^2 - 2u_0^2 \right] - \frac{t^2 - pF_+^2 + s^2}{8s} \ln \frac{t^2 - (pF_- - s)^2}{t^2 - u_0^2} - \frac{t^2 - pF_-^2 + s^2}{8s} \ln \frac{t^2 - (pF_- + s)^2}{t^2 - u_0^2}, \quad (16)$$

for $s_0 \leq s \leq s_{\text{max}}$, can be obtained by the same technique, introduced in [17], which served to obtain the function $g(t, s)$. Both these functions vanish for $s > s_{\text{max}} = \frac{1}{2}(pF_+ + pF_-)$, therefore the integrals over $s = |s|$ in (8) and (9) are finite. Similarly manifestly finite is the integral over $t = |t|$ in (8) while the analogous integral in (9) is finite owing to the fact that $h(t, s) \sim 1/t^2$ as $t \to \infty$.

The contribution to the energy density of the “particle-hole” diagram of Figure 1 can be written in the form

$$\frac{E_{\Omega}^{(3)p-h}}{V} = -\frac{32m^2 C_0^3}{(2\pi)^3 \hbar^4} \left[ K^{(1)}(pF_-, pF_+) + K^{(2)}(pF_-, pF_+) \right], \quad (17)$$

(the corresponding contribution of the left “particle-hole” diagram of Figure 1 to the energy density of the unpolarized system of spin $s$ fermions is obtained by multiplying
Hence the factors ± \( \) namely over the interior of the ball of radius \( f \) the function \( p \) radius \( f \) one defining \( u \) Both integrals defining the functions \( f \) and \( f \) where \( p \) else\( \text{where} [21] \). In this way we have arrived at symbolic integrations have been then simplified manually by exploiting the relations \( s \) regime of large \( K \) and the functions \( 1 \) \( (17) \) by the spin factor \( \frac{1}{2} g_s (g_s - 1) (3 - g_s) \) and setting \( p_F - = p_F + = k_F \). The functions \( K^{(1)}(p_F-, p_F+) \) and \( K^{(2)}(p_F-, p_F+) \) are given by

\[
K^{(1)}(p_F-, p_F+) = \int_0^\infty ds s^2 \frac{1}{4\pi} \int d^3t \theta(|t + s| - p_F-) \theta(p_F+ - |t - s|) \left( f_1(t \cdot s, s) \right)^2, \tag{18}
\]

\[
K^{(2)}(p_F-, p_F+) = \int_0^\infty ds s^2 \frac{1}{4\pi} \int d^3t \theta(p_F- - |t + s|) \theta(|t - s| - p_F+) \left( f_2(t \cdot s, s) \right)^2, \tag{19}
\]

and the functions \( f_1(t \cdot s, s) \) and \( f_2(t \cdot s, s) \) are given by the integrals

\[
f_1(t \cdot s, s) = \frac{1}{4\pi} \int d^3u \frac{\theta(p_F- - |u + s|) \theta(|u - s| + p_F+)}{(u - t) \cdot s + i0}, \tag{20}
\]

and

\[
f_2(t \cdot s, s) = \frac{1}{4\pi} \int d^3u \frac{\theta(|u + s| - p_F-) \theta(p_F+ + |u - s|)}{(u - t) \cdot s - i0}. \tag{21}
\]

Both integrals defining the functions \( f_1 \) and \( f_2 \) are over manifestly finite domains: the one defining \( f_1 \) is over the interior of the ball of radius \( p_F- \) and exterior of the sphere of radius \( p_F+ \) and the one defining \( f_2 \) - the other way around. In \( K_1(p_F-, p_F+) \) \( [18] \) the function \( f_1 \) is then integrated (over \( d^3t \)) again over a manifestly finite domain - namely over the interior of the ball of radius \( p_F+ \) and the exterior of the sphere of radius \( p_F- \) while the function \( f_2 \) is in \( K_2(p_F-, p_F+) \) \( [19] \) integrated over the interior of the ball of \( p_F- \) and the exterior of the sphere of radius \( p_F+ \). The straightforward analysis shows that the poles at \( t \cdot s = u \cdot s \) are never within the integration domains. Hence the factors ±i0 are irrelevant. It is also clear that \( K_2(k_F, k_F) = K_1(k_F, k_F) \).

The most difficult part of the computation is obtaining analytical expressions for the functions \( f_1 \) and \( f_2 \). The formulae for \( f_1 \) (for \( f_2 \)) have been obtained by shifting the center of the \( u \)-space in the regime of small \( s \) to the center of the \( p_F+ \)-sphere (of the \( p_F- \)-sphere) and to the center of the \( p_F- \)-sphere (of the \( p_F+ \)-sphere) in the regime of large \( s \), introducing then the polar coordinated and taking the resulting integrals analytically with the help of the Mathematica routines; the results of the symbolic integrations have been then simplified manually by exploiting the relations which follow from the definitions of the integration domains (details will be published elsewhere \[21\]). In this way we have arrived at

\[
f_1(t \cdot s, s) = \frac{1}{2s} \times \begin{cases} f_1^{(a)}(t \eta - s, s) & f_1^{(b)}(t \eta + s, s) \\ f_1^{(c)}(t \eta + s, s) & f_1^{(d)}(t \eta - s, s) \end{cases}, \quad f_2(t \cdot s, s) = \frac{1}{2s} \times \begin{cases} f_2^{(a)}(t \eta + s, s) & f_2^{(b)}(t \eta - s, s) \\ f_2^{(c)}(t \eta - s, s) & f_2^{(d)}(t \eta + s, s) \end{cases}, \tag{22}
\]

where

\[
f_1^{(a)}(t, s) = -2s^2 + \frac{t}{2} (p_F+ - p_F- - 2s) - s p_F- s p_F+ \xi_0
\]

\[
= \frac{p_F^2}{2} + \ln \left( \frac{t - p_F+ \xi_0}{t + p_F+} \right) + \frac{t^2}{2} \ln \left( \frac{t + 2s + p_F-}{t + p_F+} \right)
\]

\[
+ \frac{1}{4} \left( p_F^2 - 4s^2 - 4st \right) \left( -2 \ln \left( \frac{p_F+ + 4s \xi_0}{p_F- - 2s} \right) \right)
\]

\[
+ \ln \left( \frac{t^2 - (p_F^2 - 4s^2 - 4st) \xi_0^2}{(t^2 - p_F^2 + 4s^2 + 4st) \xi_0^2} \right) \right) \right).
\]
\[ f_1^{(b)}(t, s) = \frac{t}{2} p_{F-}(1 + 1/\xi_0^t) + 2s^2 + \frac{1}{2} (t - 2s)p_{F+} - st + \frac{t}{2\xi_0^t} p_{F-} - sp_{F-} \xi_0^t \]
\[ + \frac{t^2}{2} \ln \left( 1 + \frac{p_{F-}}{t} \right) + \frac{p_{F-}}{2} \ln \left( \frac{t - \xi_0^t p_{F-}}{t + p_{F-}} \right) - \frac{t^2}{2} \ln \left( 1 + \frac{p_{F+} - 2s}{t} \right) \]
\[ + \frac{1}{4} (p_{F+}^2 - 4s^2 + 4st) \left\{ -2 \ln \left( \frac{2s + p_{F+}}{4s \xi_0^t - p_{F-}} \right) \right\} \]
\[ + \ln \left[ \frac{p_{F+}^2 - 4s^2 + 4st - t^2}{t^2 - (p_{F+}^2 - 4s^2 + 4st) \xi_0^t} \right] \left[ \frac{t p_{F+} + p_{F+}^2 - 4s^2 + 2st}{(p_{F+}^2 - 4s^2 + 4st) \xi_0^t - t p_{F-}} \right] , \]
\[ f_1^{(c)}(t, s) = -tp_{F-} + \frac{1}{2} (p_{F-}^2 - t^2) \ln \left( \frac{t - p_{F-}}{t + p_{F-}} \right) . \]

and
\[ f_2^{(a)}(t, s) = \frac{t}{2} p_{F+} + 2s^2 + \frac{1}{2} (2s - t)p_{F+} - st - \xi_0^t s p_{F-} - \]
\[ + \frac{1}{2} p_{F-}^2 \ln \left( \frac{t - \xi_0^t p_{F-}}{t + p_{F-}} \right) - \frac{1}{2} t^2 \ln \left( \frac{t - 2s - p_{F+}}{t + p_{F-}} \right) \]
\[ + \frac{1}{4} (p_{F+}^2 - 4s^2 + 4st) \left\{ 2 \ln \left( \frac{2s + p_{F+}}{p_{F-}} \right) \right\} \]
\[ + \ln \left[ \frac{t^2 - p_{F+}^2 + 4s^2 - 4st}{t^2 - (p_{F+}^2 - 4s^2 + 4st) \xi_0^t} \right] \left[ \frac{t p_{F+} + p_{F+}^2 - 4s^2 + 2st}{(p_{F+}^2 - 4s^2 + 4st) \xi_0^t - t p_{F-}} \right] , \]
\[ f_2^{(b)}(t, s) = -2s^2 - ts + sp_{F-} + \frac{t}{2} p_{F-} - \frac{t}{2\xi_0^t} p_{F+} - \xi_0^t s p_{F+} \]
\[ - \frac{t^2}{2\xi_0^t} \ln \left( 1 - \frac{\xi_0^t p_{F+}}{t} \right) + \frac{t^2}{2} \ln \left( 1 + \frac{2s - p_{F-}}{t} \right) \]
\[ + \frac{1}{4} (p_{F-}^2 - 4s^2 - 4st) \left\{ -2 \ln \left( \frac{2s - p_{F-}}{p_{F+}} \right) \right\} \]
\[ + \ln \left[ \frac{t^2 - (p_{F+}^2 - 4s^2 - 4st) \xi_0^t}{t^2 - (p_{F+}^2 - 4s^2 + 4st) \xi_0^t} \right] \left[ \frac{t p_{F+} + (p_{F+}^2 - 4s^2) \xi_0^t}{(p_{F+}^2 - 4s^2 + 4st) \xi_0^t - t p_{F-}} \right] \]
\[ - \frac{t}{2} p_{F+} (1 - 1/\xi_0^t) - \frac{1}{2} p_{F+}^2 \ln \left( \frac{\xi_0^t p_{F-} - t}{p_{F-} - t} \right) - \frac{t^2}{2} \ln \left( 1 - \frac{p_{F+}}{t} \right) + \frac{t^2}{2\xi_0^t} \ln \left( 1 - \frac{\xi_0^t p_{F+}}{t} \right) , \]
\[ f_2^{(c)}(t, s) = -tp_{F+} + \frac{1}{2} (p_{F+}^2 - t^2) \ln \left( \frac{t - p_{F+}}{t + p_{F+}} \right) . \]

In these formulae
\[ \xi_0 = \frac{p_{F-}^2 - p_{F+}^2 - 4s^2}{4s p_{F+}} , \quad \xi_0^t = \frac{p_{F-}^2 + p_{F+}^2 + 4s^2}{4s p_{F-}} . \tag{23} \]

Once the functions \( f_1(t, s) \) and \( f_2(t, s) \) are given in their analytic forms, the functions \( K^{(1)}(p_{F-}, p_{F+}) \) and \( K^{(2)}(p_{F-}, p_{F+}) \) can be evaluated using the Mathematica package built-in instruction for numerical integration over a specified domain.

8
Evaluation of the contribution to the energy density of the interactions proportional to the couplings $C_2$ and $C'_2$ is straightforward (no complicated integrals are involved). The result is

$$\frac{E_{\Omega}^{(C_2)}}{V} = \frac{C_2}{240\pi^2} p_{F-}^3 p_{F+}^3 (p_{F-}^2 + p_{F+}^2),$$

$$\frac{E_{\Omega}^{(C'_2)}}{V} = \frac{C'_2}{120\pi^2} \left[ p_{F-}^8 + p_{F+}^8 + \frac{1}{2} p_{F+}^3 p_{F-}^3 (p_{F+}^2 + p_{F-}^2) \right].$$

(24)

These formulae agree for $p_{F-} = p_{F+} = k_F$ with the ones for $g_s = 2$ obtained in [7].

Combining (7) with (17) and (24), adding the result (3) (for $\alpha = +, -$), the known order $k_F^5 (k_F R)^3$ contribution $C_0 p_{F-}^2 p_{F+}^2 / 36\pi^4$, the contribution of order $k_F^6 (k_F R)^3$ obtained in [13] and finally expressing the couplings $C_0, C_2$ and $C'_2$ in terms of the $s$ and $p$-wave scattering lengths $a_0, a_1$ and the $s$-wave effective radius $r_0$ using (2) one easily finds (using the results of [13]) that up to the order $k_F^6 (k_F R)^3$ all terms diverging with $\Lambda \to \infty$ cancel out. One observes that the finite contribution arising in (25) from the term proportional to $1/\Lambda$ after it is multiplied by the term $\propto \Lambda$ present in $C_0^2$ cancels against the finite term $-2t^2$ arising from squaring the function (10) in the contribution of the “particle-particle” diagram. Such terms must cancel because they would be absent had one used Dimensional Regularization instead of the cutoff $\Lambda$. Defining then

$$G(p_{F-}, p_{F+}) = G_{\text{fin}}^{(1)}(p_{F-}, p_{F+}) + G^{(2)}(p_{F-}, p_{F+}),$$

$$K(p_{F-}, p_{F+}) = K^{(1)}(p_{F-}, p_{F+}) + K^{(2)}(p_{F-}, p_{F+}),$$

(26)

one arrives at the final formula

$$\frac{E_{\Omega}}{V} = \frac{1}{6\pi^2} \frac{\hbar^2}{2m_f} \left\{ \frac{3}{5} p_{F-}^5 + p_{F+}^5 \right\} + \frac{4}{3\pi} p_{F-}^3 p_{F+}^3 a_0 + \frac{192}{\pi^2} a_0^2 J(p_{F-}, p_{F+})$$

(25)
The function $J(p_{F-}, p_{F+})$ is defined in [13]. It is also easy to see that $J(p_{F-}, p_{F+}) = p_{F+}^{-}J(r, 1)$, $G(p_{F-}, p_{F+}) = p_{F+}^{-}G(r, 1)$ and $K(p_{F-}, p_{F+}) = p_{F+}^{-}K(r, 1)$. The plot of the function $J(r, 1)$ has been given in [13]. The functions $(192/\pi^3)G(r, 1)$ and $(48/\pi^3)K(r, 1)$ are shown here in Figures 2 and 3, respectively.

In the limit $p_{F-} = p_{F+} = k_F$ the result (27) should coincide with

$$
\frac{E_\Omega}{V} = \frac{1}{6\pi^2} \frac{\hbar^2}{2m_f} \left\{ g_s \left( \frac{3}{5} k_F^5 + g_s(g_s - 1) \frac{2}{3\pi} k_F^8 a_0 + g_s(g_s - 1) \frac{4(11 - 2\ln 2)}{35\pi^2} k_F a_0^2 \right) + \frac{1}{10\pi} k_F^8 a_0^2 r_0 + g_s(g_s + 1) \frac{1}{5\pi} k_F a_0^3 \right\},
$$

for $g_s = 2$ given in [7] and [12] with $N_1 = 0.07550 \pm 0.00003$ and $N_2 = 0.05741 \pm 0.00002$ in [7], and $N_1 = 0.0755732$ and $N_2 = 0.0573879$ in [12]. Numerical evaluation of the functions $(192/\pi^3)G(1, 1)$ and $(48/\pi^3)K(1, 1)$ - the endpoints in Figures 2 and 3 respectively - gives $N_1 = 0.0755617$ and $N_2 = 0.057387$ in good agreement with the numbers obtained in [7] and [12]. (In [13] it has been found that $J(1, 1) = 0.0114449$ which with high accuracy equals $(11 - 2\ln 2)/840$).

Expressed in terms of $k_F = (3\pi^2 N/V)^{1/3}$ and $r$ the formula (27) takes the form

$$
\frac{E_\Omega}{V} = \frac{k_F^5}{3\pi^2} \frac{\hbar^2 k_F^2}{2m_f} \left\{ \frac{4}{5} k_F a_0 + \frac{4}{9\pi} r_0 + \frac{2}{r_0} \right\} (k_F a_0) + \frac{100}{27}\left( \frac{2}{1 + r^3} \right)^{7/3} J(r, 1) (k_F a_0)^2 + \frac{80}{9\pi^3} \left( \frac{2}{1 + r^3} \right)^{8/3} [4 G(r, 1) - K(r, 1)] (k_F a_0)^3
$$

Figure 3: Plot of the function $(48/\pi^3)K(r, 1)$. 

$$
+ \frac{384}{\pi^3} a_0^3 G(p_{F-}, p_{F+}) - \frac{96}{\pi^3} a_0^3 K(p_{F-}, p_{F+})
$$

$$
+ \frac{1}{10\pi} a_0^2 r_0 p_{F-} p_{F+} (p_{F-}^2 + p_{F+}^2)
$$

$$
+ \frac{1}{5\pi} a_0^3 [2p_{F-}^8 + 2p_{F+}^8 + p_{F-}^8 p_{F+}^8 (p_{F-}^2 + p_{F+}^2)] \right\}.
$$


Figure 4: Energy density $E_{\Omega}/V$ in units $(3/5)\hbar^2k_F^2/6\pi^2m_f = (N/V)(\hbar^2k_F^2/2m_f)(3/5)$ of the gas of spin 1/2 fermions as a function of its polarization $P = (N_+ - N_-)/N$ for different values (from below) 0.2 (blue), 0.4 (yellow), 0.6 (green) of the expansion parameter $k_Fa_0$. The last curve (red) corresponding to $k_Fa_0 = 0.6$ shows the same quantity but without the order $(k_Fa_0)^3$ correction.

The third order corrections computed in this work (the last two lines in the above formula) are rather small. For $r_0 = a_1 = 0$ (i.e. without the contribution on the dimension $R^{-6}$ operators) the ratio of the order $(k_Fa_0)^3$ contribution to the first term in the curly brackets increases from 0.00003 at $k_Fa_0 = 0.1$ to 0.03 at $k_Fa_0 = 1$. This can be compared to the analogous ratio of the order $(k_Fa_0)^2$ term which at $r = 1$ increases from 0.00185 to 0.185. These ratios decrease further with decreasing $r$ (increasing polarization) and become exactly zero at $r = 0$ due to the Pauli exclusion which forbids any contribution to the ground state energy to be generated by the interaction operator proportional to $C_0$.

The plot of the system’s ground state energy density as a function of the polarization $P$ related to $r$ by $r(P) = ((1-P)/(1+P))^{1/3}$ is shown in Fig. 4 for three different values of the expansion parameter $k_Fa_0$ (keeping $r_0 = a_1 = 0$). All curves merge at $P = 1$ as a result of the Pauli exclusion principle. The curve corresponding to $k_Fa_0 = 0.6$ can be directly compared to the lowest curve shown in Fig. 3 of ref. [4] which shows a numerical estimate of the exact ground state energy obtained using the Quantum Monte Carlo method for a specific model repulsive potential. Consistently with the comparison of the ground state energies of the unpolarized system ($P = 0$ or $r = 1$) made in Fig. 2 of ref. [4], our green curve (for $k_Fa_0 = 0.6$) is systematically below its counterpart in Fig. 3 of ref. [4] but the comparison with the red curve of Figure 4 shows that the third order correction computed in this work has the tendency to reduce the difference between the perturbative and Monte Carlo estimates. In general, the comparison with the results of ref. [4] show that the perturbative expansion is reliable up to $k_Fa_0 \lesssim 0.5$. 

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Summary. In this work we have reproduced the third order formula for the ground-state energy of the unpolarized gas of spin $s$ fermions and extended it to the case of the arbitrarily polarized gas of spin $1/2$ fermions. We have checked the cancellation of all ultraviolet divergences occurring when the result is expressed in terms of the $s$-wave scattering length $a_0$ and worked out analytically the most important integrals occurring in the computation of the relevant Feynman (Hughenholtz) diagrams. This allowed to compute the remaining integrals numerically using the standard Mathematica built-in routines; the resulting final third order formula for the ground state energy of the arbitrarily polarized gas of spin $1/2$ fermions is given in terms of two new functions of the system’s polarization for which the convenient interpolating formula can be easily obtained. The numerical results suggest that for $k_F R < \sim 0.5$ the perturbation series for the ground-state energy is well convergent but is not reliable for higher values of the expansion parameter for which the system is expected to exhibit the phase transition to the ordered phase. One may hope, however, that supplemented with a reliable extrapolation procedure the perturbative series will be able to give valuable information about the nature of the phase transition. Further extension of our work to the case of arbitrary mixture of different spin projections of spin $s$ fermions (in the spirit of [15]) is straightforward.

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