TRAVELING FRONTS FOR FISHER-KPP LATTICE EQUATIONS IN ALMOST PERIODIC MEDIA

XING LIANG, HONGZE WANG, QI ZHOU, AND TAO ZHOU

Abstract. This paper investigates the existence of almost periodic traveling fronts for Fisher-KPP lattice equations in one-dimensional almost periodic media. By the Lyapunov exponent of the linearized operator near the unstable steady state, we give sufficient condition of the existence of minimal speed of traveling fronts. Furthermore, it is showed that almost periodic traveling fronts share the same recurrence property as the structure of the media. As applications, we give some typical examples which have minimal speed, and the proof of this depends on dynamical system approach to almost periodic Schrödinger operator.

1. INTRODUCTION

1.1. Background and main results. Since the pioneer works of [7, 23, 34], the traveling fronts of reaction-diffusion equations in unbounded domain have become an important branch of the theory of equations with diffusion. Specially, traveling fronts of the Fisher-KPP type equation in continuous media

\[ u_t - (a(x)u_x)_x = c(x)u(1 - u), \quad t \in \mathbb{R}, \ x \in \mathbb{R} \]

and more general reaction-diffusion equations in heterogeneous media received intense attention in the last few decades.

As a simplest heterogeneous case, traveling fronts in spatially periodic media were considered widely. First, the definition of spatially periodic traveling waves was provided by [53, 58] independently, and then [26] proved the existence of spatially periodic traveling waves of Fisher-KPP equations in the distributional sense. Then, in the series of works [10, 11], the authors investigated traveling fronts of Fisher-KPP type equations in high-dimensional periodic media deeply. The traveling fronts of spatially periodic Fisher-KPP type equations in discrete lattice

\[ u_t(t, n) - u(t, n + 1) - u(t, n - 1) + 2u(t, n) = c(n)u(t, n)(1 - u(t, n)), \]

were also studied in [24, 39]. Besides above works, a more general framework was given by [39, 57] to study traveling fronts for Fisher-KPP type equations and more general diffusion systems.

However, few works on traveling waves of Fisher-KPP equations exist in more complicated media. Matano [41] first gave a definition of spatially almost periodic traveling waves and provided some sufficient conditions on
the existence of spatially almost periodic traveling fronts of reaction-diffusion equations with the bistable nonlinearity. In [38], Liang showed the existence and uniqueness of the spatially almost periodic traveling front of Fisher-KPP equations in one-dimensional almost periodic media with free boundary. We also notice that the propagation problems of (temporally) nonautonomous reaction-diffusion equations were studied by Shen in a series of works [18, 51, 52].

In this paper, we are concerned with the almost periodic traveling fronts of the Fisher-KPP equation (1.2). To introduce the definition of the almost periodic traveling fronts, let us first recall the definition of classical and periodic traveling fronts. In the case where the media is homogeneous, that is, $c$ is a constant sequence, the classical traveling fronts are defined by a solution $u(t, n) = U(n - wt)$ with an invariant profile $U$ and a speed $w$; in the case where the media is periodic, that is, $c$ is a periodic sequence with period $N$, the periodic traveling fronts are defined by a solution $u(t, n) = U(n - wt, n)$ with a periodic profile $U, U(\xi, n) = U(\xi, n + N)$ and an average speed $w$. Then it is natural to consider the recurrence of the profile depending on the structure of the media if one tries to generalize the definition of traveling fronts in heterogeneous media. To be exact, in the almost periodic media, the generalized traveling fronts need to inherit the almost periodicity of the media.

Before introducing the definition of generalized traveling fronts with almost periodic recurrence, we give some notions and background. A sequence $f : \mathbb{Z} \to \mathbb{R}$ is almost periodic if $\{f(\cdot + k)|k \in \mathbb{Z}\}$ has a compact closure in $l^\infty(\mathbb{Z})$. Denote by $\mathcal{H}(f)$ the hull of the almost periodic sequence $f$, i.e., $\mathcal{H}(f) = \{f(n + k)|k \in \mathbb{Z}\}$, the closure in $l^\infty(\mathbb{Z})$. We also denote $g \cdot k = g(\cdot + k), g \in \mathcal{H}(f), k \in \mathbb{Z}$, and let $l^\infty_{loc}(\mathbb{Z})$ be the set of all sequences in $\mathbb{Z}$ with pointwise topology and $C^0(\mathbb{R} \times \mathbb{Z})$ be the set of all continuous functions with locally uniform topology.

To define the almost periodic traveling fronts, we prefer to consider not only (1.2), where $c(n)$ is an almost periodic sequence and $\inf c > 0$, but also the family of equations

\begin{equation}
(1.3) \quad u_t(t, n) - u(t, n + 1) - u(t, n - 1) + 2u(t, n) = g(n)u(t, n)(1 - u(t, n)),
\end{equation}

for any $g \in \mathcal{H}(c)$. With these, we have the following precise definition:

**Definition 1.1.** Let $v(t, n; g) : \mathbb{R} \times \mathbb{Z} \times \mathcal{H}(c) \to \mathbb{R}$. We say $v(t, n; g)$ is an almost periodic traveling front (with nonzero average speed) if $v(t, n; g)$ is an entire solution of (1.3) for any $g \in \mathcal{H}(c)$, and if the following properties hold:

1. $\{v(\cdot, \cdot, g)|g \in \mathcal{H}(c)\}$ is a one-cover of $\mathcal{H}(c)$ in $C^0(\mathbb{R} \times \mathbb{Z})$, i.e. the map $g \in \mathcal{H}(c) \to v(\cdot, \cdot, g) \in C^0(\mathbb{R} \times \mathbb{Z})$ is continuous.
2. $v(t, n; g) \to 1$ as $n \to -\infty$, $v(t, n; g) \to 0$ as $n \to \infty$, locally uniformly in $t$ and uniformly in $g$. 


For any \( g \in \mathcal{H}(c) \), there exists \( t(k; g) : \mathbb{Z} \times \mathcal{H}(c) \to \mathbb{R} \) such that \( \{ t(\cdot; g) \mid g \in \mathcal{H}(c) \} \) is a one-cover of \( \mathcal{H}(c) \) in \( l^\infty_{\text{loc}}(\mathbb{Z}) \), and \( t(k; g) \) satisfies:

\[
 t(n; g \cdot k) = t(n + k; g) - t(k; g).
\]

Moreover, \( w(g) := \lim_{|n-k|\to\infty} \frac{n-k}{t(n;g)-t(k;g)} \) exists.

(4) \( v(t + t(k; g), n + k; g) = v(t, n; g \cdot k), \quad \forall \ t \in \mathbb{R}, \ n, k \in \mathbb{Z} \).

The constant \( w(g) \) is called the average wave speed of \( v(t, n; g) \).

**Remark 1.1.** The speed \( w(g) \) is nonzero. Indeed,

\[
 t(1; g \cdot k) = t(1 + k; g) - t(k; g)
\]

is bounded with respect to \( k \), since \( \{ t(k; g) \mid g \in \mathcal{H}(c) \} \) is a one-cover of \( \mathcal{H}(c) \) in \( l^\infty_{\text{loc}}(\mathbb{Z}) \). Combining this with the definition of \( w(g) \), we deduce that \( w(g) \) is nonzero.

**Remark 1.2.** Actually, Definition 1.1 implies time recurrence of the solution (c.f. Lemma 2.2): For any \( \epsilon > 0 \), there exist relative dense sets \( \{ t_k \} \subset \mathbb{R} \) and \( \{ n_k \} \subset \mathbb{Z} \) such that

\[
 \sup_{n \in \mathbb{Z}} |v(t_k, n_k + n; g) - v(0, n; g)| \leq \epsilon,
\]

as shown in Fig 1.

It needs to point out that a more general extension of traveling fronts, so-called generalized transition front, was presented by Berestycki and Hamel [8]. For (1.2), the generalized transition front is defined as below.

**Definition 1.2.** A generalized transition front of (1.2) is an entire solution \( u = u(t, n) \) for which there exists a function \( N : \mathbb{R} \to \mathbb{Z} \) such that

\[
 \lim_{n \to -\infty} u(t, n + N(t)) = 1, \quad \lim_{n \to +\infty} u(t, n + N(t)) = 0,
\]

uniformly in \( t \in \mathbb{R} \). We say \( u \) has an average speed \( w \in \mathbb{R} \), provided

\[
 \lim_{t-s \to +\infty} \frac{N(t) - N(s)}{t - s} = w.
\]

**Remark 1.3.** In the periodic case, the almost periodic traveling front in Definition 1.1 is exactly the classical periodic traveling front [24]. However, there may exist a generalized transition front (Definition 1.2) which is not a classical traveling front [8].
A generalized transition front \( u = u(t, x) \) of (1.1) can be defined in the same way by assuming \( N = N(t) : \mathbb{R} \to \mathbb{R} \). Nadin and Rossi [45] investigated the existence of the generalized transition front of (1.1) when \( a, c \) are almost periodic. Motivated by the works [11, 24, 39, 43], especially the work of Nadin and Rossi [45], we want to construct the almost periodic traveling fronts via the eigenvalue problem of the linearized operator of (1.2) near the equilibrium state \( u \equiv 0 \):

\[
(1.5) \quad (L_g u)(n) := u(n+1) + u(n-1) - 2u(n) + g(n)u(n) = Eu(n), \quad g \in \mathcal{H}(c).
\]

Here the subscript \( g \) is to emphasize the dependence of \( g \). We will always shorten the notation \( L_c = L \).

One novelty of the paper is that we will use method from dynamical systems to study the operator \( L_g \), thus to study (1.3). Note that (1.5) can be rewritten as

\[
\begin{pmatrix} u(n+1) \\ u(n) \end{pmatrix} = A(n) \begin{pmatrix} u(n) \\ u(n-1) \end{pmatrix},
\]

where \( A(n) = \begin{pmatrix} E + 2 - g(n) & -1 \\ 1 & 0 \end{pmatrix} \). Let \( A_n(g) = A(n-1) \cdots A(0) \) be the transfer matrix. Then the Lyapunov exponent of \( L_g \) at energy \( E \) is denoted by \( L(E) \) and given by

\[
(1.6) \quad L(E) := \lim_{n \to +\infty} \frac{1}{n} \int_{\mathcal{H}(c)} \ln \| A_n(g) \| d\mu \geq 0.
\]

where \( \mu \) is the Haar measure on \( \mathcal{H}(c) \). It is known that \( L(E) \) is identical for \( g \in \mathcal{H}(c) \) (see Proposition 3.4). The Lyapunov exponent characterizes the decay rate of any solutions of (1.5), and is also a fundamental topic in smooth dynamical systems.

We always use \( \Sigma(\mathcal{L}) \) to denote the spectrum of \( \mathcal{L} \), and denote \( \lambda_1 = \max \Sigma(\mathcal{L}) \). Once we have this, we can state our main results as follows:

**Theorem 1.1.** Denote

\[
\frac{E}{L(E)}, \quad \frac{E}{\lambda_1}
\]

Then for (1.3), the following statements hold:

1. If \( w^* < w \), then for any \( w \in (w^*, w) \), there exists a time-increasing almost periodic traveling front with average wave speed \( w \);
2. If \( w^* < w \), then there exists a time-increasing generalized transition front with average speed \( w^* \);
3. There is no generalized transition fronts with average speed \( w < w^* \).

Note that the sufficient condition for the existence result in Theorem 1.1(1) is fulfilled up to a constant perturbation of \( g \) (Lemma 4.1), which was first observed in [45]. Since adding a constant to the potential doesn’t affect the spectral property of \( \mathcal{L} \), in what follows we always assume \( w^* < w \).
A remarkable fact is Proposition 3.4: For any \( E > \lambda_1 \), there exists a unique positive solution \( \phi_E(n) \) of
\[
\mathcal{L}\phi_E = E\phi_E, \quad \phi_E(0) = 1, \quad \lim_{n \to \infty} \phi_E(n) = 0,
\]
and the limit \( \mu(E) := -\lim_{n \to \pm \infty} \frac{1}{n} \ln \phi_E(n) = L(E) \). Thus the minimal speed we constructed is the same as that given in [45].

We also should point out that the established almost periodic traveling fronts share the same recurrence property as the potential \( c(n) \). To make it precisely, if the frequency of the almost periodic sequence \( c(n) \) is \( \alpha \), then the frequency of \( v(\cdot, \cdot; c) \) is also \( \alpha \), as we will show in Corollary 1.1 and Corollary 1.3.

1.2. Applications. Of course, the interesting thing is to give concrete examples in which we can establish almost periodic traveling fronts for any average wave speed \( w > w^* \) (i.e., \( w = \infty \)). Based on the work of [33], Nadin and Rossi [45] showed that if \( a, c \) are finitely differentiable quasi-periodic function with Diophantine frequency \( \alpha \), and \( c \) is small enough (the smallness must depend on \( \alpha \)), then the operator \( Lu = (a(x)u_x)_x + c(x)u \) has a positive almost periodic function. Consequently (1.1) has a time-increasing generalized transition front with average speed \( w \in (w^*, \infty) \). Here we recall that \( \alpha \) is Diophantine (denoted by \( \alpha \in \text{DC}_d(\gamma, \tau) \)), if there exist \( \gamma > 0, \tau > d \) such that
\[
\inf_{j \in \mathbb{Z}} |\langle k, \alpha \rangle - j| > \frac{\gamma}{|k|^\tau}, \quad 0 \neq k \in \mathbb{Z}^d.
\]

Therefore, the natural question is that whether one can remove the arithmetic condition of \( \alpha \), or whether one can remove the smallness of \( c \). Now we answer this question as follows:

**Corollary 1.1.** Suppose that \( c(n) = V(n\alpha) \) where \( \alpha \in \mathbb{T}^d \) is rationally independent, \( V : \mathbb{T}^d \to \mathbb{R} \) is a positive real analytic function. Then the following statements hold:

1. (1.2) has a time increasing almost periodic traveling front with average wave speed \( w \in (w^*, L(\lambda_1)) \).
2. The traveling front \( u(t, n; c) \) can be rewritten as \( u(t, n; c) = U(t + T(n), n\alpha) \), where \( U \in C^0(\mathbb{R} \times \mathbb{T}^d), T \in \ell_\text{loc}^\infty(\mathbb{Z}) \).

Corollary 1.1 (1) reduces \( w = \infty \) to \( L(\lambda_1) = 0 \). For example, based on former results [2, 59], Corollary 5.5 shows that for any irrational \( \alpha \in \mathbb{R} \setminus \mathbb{Q} \), if \( V \) is analytic and close to constant, even the closeness is independent of \( \alpha \). Hence combining Proposition 3.5 and the continuity of the Lyapunov exponent \([14, 15], L(\lambda_1) = 0 \). On the other hand, Corollary 1.1 (2) states that if \( c(n) \) is quasi-periodic with frequency \( \alpha \), then the resulting traveling front is also quasi-periodic with frequency \( \alpha \).

Corollary 1.1 (1) follows from Theorem 1.1 and the continuity of the Lyapunov exponent. Thus the assumption that \( V \) is analytic is necessary, since the Lyapunov exponent might be discontinuous in smooth topology [56].
Furthermore, Corollary 5.4 also states the results for $V$ is just a finitely differentiable quasi-periodic function as in [45]. To be exact, if $\alpha \in DC_d(\gamma, \tau)$, $V \in C^s(\mathbb{T}^d, \mathbb{R})$ with $s > 6 \tau + 2$, and $V$ is small enough, then (1.2) has a time increasing almost periodic traveling front with average wave speed $w \in (w^*, \infty)$. However, it is widely believed that if the regularity is worse, then for generic $V$, the spectrum of (1.5) has no absolutely continuous component [4]. Therefore, most probably $L(\lambda_1) > 0$ by the well-known Kotani’s theory [35].

As a concrete and typical example, we will take $V(\theta) = 2\kappa \cos \theta + C$ where $C$ is a constant such that $V(\theta) > 0$, then the corresponding linearized operator can be reduced to the well-known almost Mathieu operator (AMO):

$$(L_{2\kappa \cos -2, \alpha, \theta} u)(n) = u(n + 1) + u(n - 1) + 2\kappa \cos(\theta + n\alpha)u(n).$$

The AMO was first introduced by Peierls [48], as a model for an electron on a 2D lattice, acted on by a homogeneous magnetic field [27, 50]. Now, if $V(\theta) = 2\kappa \cos \theta + C$, then we have the following:

**Corollary 1.2.** Suppose that $c(n) = 2\kappa \cos(2\pi n\alpha) + C$ where $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and $C$ is a specified constant such that $c$ has a positive infimum. Then the following statements hold:

1. If $|\kappa| \leq 1$, then (1.3) has a time-increasing quasi-periodic traveling front with average wave speed $w \in (w^*, \infty)$.
2. If $|\kappa| > 1$, then (1.3) has a time-increasing quasi-periodic traveling front with average speed $w \in (w^*, \frac{\lambda_1}{|\ln \kappa|})$.

Note that AMO plays the central role in the Thouless et al theory of the integer quantum Hall effect [54]. This model has been extensively studied not only because of its importance in physics [46], but also as a fascinating mathematical object [5, 6, 31, 32]. For us, the example is interesting since Nadin and Rossi [45] asked is it possible to construct a rigorous example where $\lim_{E \to \lambda_1} \mu(E) > 0$? Then Corollary 1.2 (2) gives an affirmative answer to their question in the discrete setting. However, it is still open whether in this case, (1.2) has a time-increasing almost periodic traveling front with average wave speed $w \in (w^*, \infty)$.

From this aspect, we should mention that compared to [45], we are mainly concerned with Fisher-KPP lattice equations, since the main results we mentioned above [2, 3, 14, 15, 31] only work in the discrete case. Whether the corresponding results are valid for the continuous case are widely open.

We also remark that we are mainly concerned with almost periodic traveling fronts of (1.2) with average wave speed $w \in (w^*, \infty)$, while Nadin and Rossi’s example [45] and Corollary 1.1 only give examples where $c$ are quasi-periodic. Therefore, it is interesting and important to give concrete examples of real almost periodic sequence $c(n)$, such that (1.2) has almost periodic traveling front with average wave speed $w \in (w^*, \infty)$. To state the result clearly, let’s show how to construct the desired almost periodic sequence.
We assume that the frequency \( \alpha = (\alpha_j)_{j \in \mathbb{N}} \) belongs to the infinite dimensional cube \( \mathcal{R}_0 := [1, 2]^\mathbb{N} \), and we endow \( \mathcal{R}_0 \) with the probability measure \( \mathcal{P} \) induced by the product measure of the infinite dimensional cube \( \mathcal{R}_0 \). We now define the set of Diophantine frequencies that was first developed by Bourgain \([13]\):

**Definition 1.3** ([13]). Given \( \gamma \in (0, 1) \), \( \tau > 1 \), we denote by \( \text{DC}_\infty(\gamma, \tau) \) the set of Diophantine frequencies \( \alpha \in \mathcal{R}_0 \) such that

\[
|\langle k, \alpha \rangle| \geq \gamma \prod_{j \in \mathbb{N}} \frac{1}{1 + |k_j|^{\tau(j)}} \quad \forall k \in \mathbb{Z}^\infty, 0 < |\sum_{j \in \mathbb{N}} |k_j| < \infty.
\]

where \( \langle j \rangle := \max\{1, |j|\} \).

As proved in \([13, 16]\), for any \( \tau > 1 \), Diophantine frequencies \( \text{DC}_\infty(\gamma, \tau) \) are typical in the set \( \mathcal{R}_0 \) in the sense that there exists a positive constant \( C(\tau) \) such that

\[
\mathcal{P}(\mathcal{R}_0 \setminus \text{DC}_\infty(\gamma, \tau)) \leq C(\tau)\gamma.
\]

Now for any \( \alpha \in \text{DC}_\infty(\gamma, \tau) \), we say that \( c(n) : \mathbb{Z} \to \mathbb{R} \) is an almost periodic sequence with frequency \( \alpha \) and analytic in the strip \( r > 0 \) if we may write it in totally convergent Fourier series

\[
(1.7) \quad c(n) = \sum_{k \in \mathbb{Z}_k^\infty} \hat{c}(k) e^{i\langle k, \alpha \rangle n} \quad \text{such that} \quad \sum_{k \in \mathbb{Z}_k^\infty} |\hat{c}(k)||e^{r|k|_1} < \infty,
\]

where \( \mathbb{Z}_k^\infty := \{k \in \mathbb{Z}^\infty : |k|_1 := \sum_{j \in \mathbb{N}} \langle j \rangle |k_j| < \infty\} \) denotes the set of infinite integer vectors with finite support. Since \( \alpha \) is rationally independent, \( \mathcal{H}(c) = T^\infty = \prod_{i \in \mathbb{N}} T^1 \) with infinite product topology. Once we have these, we can introduce our precise results as follows:

**Corollary 1.3.** Let \( \gamma \in (0, 1) \), \( \tau > 1 \), \( r > 0 \), \( \alpha \in \text{DC}_\infty(\gamma, \tau) \). Suppose that \( c(n) \) is an almost periodic sequence with frequency \( \alpha \) and analytic in the strip \( r > 0 \). Furthermore, suppose that there exists \( \epsilon = \epsilon(\gamma, \tau, r) \) such that

\[
\sum_{k \in \mathbb{Z}_k^\infty} |\hat{c}(k)||e^{r|k|_1} < \epsilon(\gamma, \tau, r).
\]

Then following statements hold:

1. (1.2) has a time increasing almost periodic traveling front with average wave speed \( w \in (w^*, \infty) \)
2. The front \( u(t, n; c) \) can be rewritten as \( u(t, n; c) = U(t + T(n), n\alpha) \), where \( U \in C^0(\mathbb{R} \times T^\infty), T \in \ell^\infty_\text{loc}(\mathbb{Z}) \).

Finally let’s outline the novelty of the proof of these applications. The proof depends crucially on dynamical approach to almost-periodic Schrödinger operator, i.e., in order to study the spectral property of Schrödinger operator \((1.5)\), one only needs to study the corresponding Schrödinger cocycle (c.f. section 2.6). For analytic quasi-periodic potentials (Corollary 1.1 and
Corollary 1.2), the result follows from the continuity of the Lyapunov exponent for analytic cocycles. For the almost-periodic case, we need to prove the existence of positive almost-periodic functions. The key observation is Lemma 5.1, which says that if the Schrödinger cocycle is reduced to a constant parabolic cocycle, and the conjugacy is close to the identity, then the corresponding Schrödinger equation has a positive almost-periodic solution. Here reducibility means the cocycle can be conjugated to a constant cocycle (c.f. section 5). From this aspect, the powerful method is KAM. For almost-periodic Hamiltonian systems, KAM was first developed by Pöschel [49]. One can consult [13, 16, 42] for more study on similar objects. However, it is well-known traditional KAM method only works for positive measure parameters, and here we need to fix the energy to be the supremum of the spectrum, thus the corresponding cocycle is fixed. To solve the difficulty, the method is to make good use of fibred rotation number which was first developed by Herman [28] for quasiperiodic cocycles (not necessarily quasiperiodic Schrödinger cocycle), and it can be extended to the almost periodic setting.

1.3. Structure of the paper. In the section 2, we introduce some preliminary knowledge which will be needed in our proof. In the section 3, we introduce and investigate some properties of the generalized principal eigenvalue and the Lyapunov exponent which turn out to be powerful techniques in constructing the almost periodic traveling front with any average wave speed $w > w^*$. Moreover, we also show the reason why the existence of positive almost periodic solution of (1.5) implies $w = \infty$, c.f. Proposition 3.5.

In the section 4, we prove Theorem 1.1 by the following steps: First we establish the almost traveling front with any average wave speed $w > w^*$ by constructing super-sub solution, and get the monotonicity of the fronts in $t$, thus proves (1) in Theorem 1.1. Next we make use of the properties of spreading speed to deduce that even the generalized transition fronts with the average speed $w < w^*$ can not exist, and then (2) is proved. Lastly, we construct a generalized transition front with the critical wave speed $w^*$ by pulling back the solution of the Cauchy problem associated with the initial datum Heaviside function, and then we finish the proof.

In the section 5, we use KAM method to get the positive quasi-periodic (almost-periodic) solution of (1.5) with the positive infimum when $c = V(\cdot + \theta)$ is very close to a constant, where $V \in C^s(T^d, \mathbb{R})(V \in C^\omega(T^\infty, \mathbb{R}))$. This will help us to prove Corollary 1.3. At last, we finish all the proofs of applications.

2. Preliminaries

2.1. Maximum principle, existence and uniqueness for the Cauchy Problem. The maximum principle on the whole space can be stated as follows:
**Proposition 2.1** (Maximum principle[18]). Let \( v \in \ell^\infty(\mathbb{Z}) \). Assume that for any bounded interval \( I = [0, t_0] \subset [0, \infty) \), \( u \) is bounded in \( I \times \mathbb{Z} \). If \( u \) satisfies

\[
\begin{align*}
  u_t(n) - u(n + 1) - u(n - 1) + (2 - v(n))u(n) &\geq 0 \quad \text{a.e. in } I \times \mathbb{Z}, \\
  u(n) &\geq 0 \quad \text{in } \{0\} \times \mathbb{Z},
\end{align*}
\]

then \( u \geq 0 \) in \( I \times \mathbb{Z} \).

The following Harnack inequality is a very useful technique when we study the properties of the solution of (1.2). We will present it here for the reader’s convenience.

**Proposition 2.2** (Harnack inequality[40]). Assume that \( u \) is bounded on \((0, \infty) \times \mathbb{Z}\) and solves (2.1). Then for any \((t, n) \in (0, \infty) \times \mathbb{Z}, T > 0\), there exists a positive constant \( C = C(T) \) such that

\[
u(t, n) \leq C(T)\nu(t + T, m), \quad m \in \{n \pm 1, n\}.
\]

**Remark 2.1.** The proof of the Harnack inequality could be found in [40] with the initial value \( u(0, n) \) having a finite support. However, we should notice that the argument can be applied to (2.1) similarly with minor modification.

Combining the Harnack inequality with the maximum principle, we can deduce the strong maximum principle as follows:

**Corollary 2.1** (Strong maximum principle[40]). Under the assumption of Proposition 2.1, either \( u \equiv 0 \) or \( u > 0 \) in \( I \times \mathbb{Z} \).

The comparison principle is a consequence of the strong maximum principle, and it is useful for us to construct the almost periodic traveling front. To state it, we first give the definition of super-sub solutions:

Let \( \bar{u}, \underline{u} \in C(\mathbb{R} \times \mathbb{Z}) \) be two bounded functions. We say that \( \bar{u} \) is a supersolution of (1.2) if for any given \( n \in \mathbb{Z} \), \( \bar{u} \) is absolutely continuous in \( t \) and satisfies

\[
\bar{u}_t - \bar{u}(n + 1) - \bar{u}(n - 1) + 2\bar{u}(n) - c\bar{u}(1 - \bar{u}) \geq 0 \quad \text{for a.e. } t \in (0, \infty),
\]

and \( \underline{u} \) is a subsolution if for any given \( n \in \mathbb{Z} \), \( \underline{u} \) is absolutely continuous in \( t \) and satisfies

\[
\underline{u}_t - \underline{u}(n + 1) - \underline{u}(n - 1) + 2\underline{u}(n) - c\underline{u}(1 - \underline{u}) \leq 0 \quad \text{for a.e. } t \in (0, \infty).
\]

The strong comparison principle is given by

**Proposition 2.3** (Strong comparison principle). Let \( \bar{u} \) and \( \underline{u} \) be a supersolution and a subsolution of (1.2) respectively. If \( \bar{u}(0, n) \leq \underline{u}(0, n) \) in \( \mathbb{Z} \), then \( \bar{u} < \underline{u} \) or \( \underline{u} \equiv \bar{u} \) in \((0, \infty) \times \mathbb{Z}\).

Usually, well-behaved Cauchy Problem possesses the property that it admits a unique global solution. Moreover, existence and uniqueness is vital for us when we construct the generalized transition front with minimal speed \( w^* \).
Theorem 2.1 ([47]). For any initial value $\varphi(n) \in \ell^\infty(\mathbb{Z})$, there exists a unique $u \in C^0(\mathbb{R} \times \mathbb{Z})$ with $u(t, \cdot) \in \ell^\infty(\mathbb{Z})$ for any $t \in (0, \infty)$ such that

\[
\begin{cases}
  u_t(n) - u(n+1) - u(n-1) + 2u(n) = c(n)u(1-u) & \text{in } (0, \infty) \times \mathbb{Z} \\
  u(0, n) = \varphi(n).
\end{cases}
\]

2.2. Quadratic form and Critical operator. Denote $l_c(\mathbb{Z})$ the space of real valued functions on $\mathbb{Z}$ with compact support. The associated bilinear form $l$ of $-\mathcal{L}$ is defined on $l_c(\mathbb{Z}) \times l_c(\mathbb{Z})$ as

\[
l(\varphi, \psi) := \frac{1}{2} \sum_{n \in \mathbb{Z}} \sum_{m=\pm 1} (\varphi(n) - \varphi(n+m))(\psi(n) - \psi(n+m)) - c(n)\varphi(n)\psi(n).
\]

We denote by $l(\varphi) := l(\varphi, \varphi)$ the induced quadratic form on $l_c(\mathbb{Z})$. Furthermore, we write $l \geq 0$ if $l(\varphi) \geq 0$ for all $\varphi \in l_c(\mathbb{Z})$.

Definition 2.1 ([37]). Let $l$ be a quadratic form $l$ associated with the Schrödinger operator $-\mathcal{L}$ such that $l \geq 0$. We say the form is critical if there does not exist a positive $\varpi \in l^\infty_{\text{loc}}(\mathbb{Z})$ (i.e. $\varpi \geq 0$ and $\varpi \not\equiv 0$ on $\mathbb{Z}$) such that

\[
l(\varphi) - \sum_{n \in \mathbb{Z}} \varpi(n)\varphi(n)^2 \geq 0 \quad \text{for any } \varphi \in l_c(\mathbb{Z}).
\]

The operator $\mathcal{L}$ is said to be critical, if the quadratic form $l$ associated with $-\mathcal{L}$ is critical.

The following well known formula reveals the connection between the operator $\mathcal{L}$ and the associated quadratic form $l$.

Lemma 2.1 (Green formula[37]). For all $\varphi, \psi \in l_c(\mathbb{Z})$, one has

\[
\frac{1}{2} \sum_{n \in \mathbb{Z}} \sum_{m=\pm 1} (\varphi(n) - \varphi(n+m))(\psi(n) - \psi(n+m)) - \sum_{n \in \mathbb{Z}} c(n)\varphi(n)\psi(n)
= -\sum_{n} (\mathcal{L}\varphi)(n)\psi(n) - \sum_{n} (\mathcal{L}\psi)(n)\varphi(n).
\]

That is, $l(\varphi, \psi) = \langle -\mathcal{L}\varphi, \psi \rangle$.

The critical operator has the following important property which is useful for us to reveal the connection between the Lyapunov exponent and positive almost periodic solution (c.f. Proposition 3.5):

Proposition 2.4. [37] Let $\mathcal{L}$ be a critical operator. Then there exists a unique positive function in $l^\infty_{\text{loc}}(\mathbb{Z})$ such that $\mathcal{L}u \leq 0$ (up to scalar multiplication).

2.3. Properties of the almost periodic traveling front. Almost periodic traveling front also implies the almost periodicity of the time recurrence in the sense of the following lemma.

Lemma 2.2. Assume $u(t, n; g)$ is an almost periodic traveling front of (1.3). Then for any $\epsilon > 0$, there exist relative dense sets $\{t_k\} \subset \mathbb{R}$ and $\{n_k\} \subset \mathbb{Z}$ such that $\sup_{n \in \mathbb{Z}} |u(t_k, n + n_k; g) - u(0, n; g)| \leq \epsilon$. 
Proof. From Definition 1.1 (1), (2) and (4), for any \( k > 0 \), there exists \( \{n_k\} \subset \mathbb{Z} \) such that

\[
\sup_{n \in \mathbb{Z}} |u(0, n; g) - u(0, n; g \cdot n_k)| = \sup_{n \in \mathbb{Z}} |u(0, n; g) - u(t(n_k; g), n + n_k; g)| \leq \epsilon,
\]

if \( \sup_{n} |g - g \cdot n_k| \leq \delta \). Because \( g \) is an almost periodic sequence, we can always find such a relative dense set \( \{n_k\} \) so that \( \sup_{n \in \mathbb{Z}} |g - g \cdot n_k| \leq \epsilon \). Take \( \{t_k\} \) as \( \{t(n_k; g)\} \), then it is sufficient to prove \( \{t(n_k; g)\} \) is relative dense.

It follows from the Definition 1.1 (3) that

\[
t(n_{k+1}; g) - t(n_k; g) = t(n_{k+1} - n_k; g \cdot n_k).
\]

Since \( \{n_k\} \) is a relative dense set, i.e., there exists \( L \geq 0 \) such that for any point \( n \) in \( \mathbb{Z} \), one has \([n - L, n + L] \cap \{n_k\} \neq \emptyset \). Denote \( M := \max_{-L \leq n \leq L, g \in \mathcal{H}(c)} t(n; g) \). Since \( \{t(n, g)\} \) is a one-cover of \( \mathcal{H}(c) \) in \( \ell^\infty_{\text{loc}}(\mathbb{Z}) \) and \( \mathcal{H}(c) \) is compact, \( M \) can be obtained. Hence for any \( t \in \mathbb{R} \), one has \([-M + t, M + t] \cap \{t(n_k; g)\} \neq \emptyset \), as desired. \( \square \)

The average wave speed of almost periodic traveling front relates with the recurrence of \( c \) in the following way.

**Proposition 2.5.** The average wave speed \( w(g) \) of the almost periodic traveling front is a constant function on \( \mathcal{H}(c) \).

Proof. Recall that \( w(g) = \lim_{|n| \to \infty} \frac{n - k}{t(n; g) - t(k; g)} \) exists, and it is not zero.

Then \( \frac{1}{w(g)} = \lim_{|n| \to \infty} \frac{t(n; g) - t(k; g)}{n - k} \) exists. By Definition 1.1, \( \{t(1; g \cdot k) | k \in \mathbb{Z}\} \) has a compact closure in \( \ell^\infty_{\text{loc}}(\mathbb{Z}) \). Hence it is an almost periodic sequence. Then for any \( k \in \mathbb{Z} \),

\[
1/w(g) = \lim_{|n| \to \infty} \frac{t(n + k; g) - t(k; g)}{n} = \lim_{|n| \to \infty} \sum_{i=k}^{n+k-1} \frac{t(i+1; g) - t(i; g)}{n} = \lim_{|n| \to \infty} \sum_{i=k}^{n+k-1} \frac{t(1; g \cdot i)}{n},
\]

The limit on the right-hand side exists, and it is independent of \( g \in \mathcal{H}(c) \). Then the proof is complete. \( \square \)

An observation is that an almost traveling front is also a generalized transition front.

**Proposition 2.6.** An almost periodic traveling front of (1.3) with average wave speed \( w(g) \) is a generalized transition front of (1.3) with average speed \( w(g) \).
Proof. From Definition 1.1(3), \( w(g) = \lim_{|n-k| \to \infty} \frac{n-k}{(m;g) - t(k;g)} \) exists. Without loss of generality, we will always assume that \( w(g) > 0 \). For any \( k \in \mathbb{Z} \), there exists an absolute constant \( L \) such that for any \( |n| \geq L \),

\[
\frac{w(g)}{2} \leq \frac{n}{t(k+n;g) - t(k;g)} \leq \frac{3w(g)}{2}.
\]

(2.2)

Thus

\[
\begin{cases}
  t(k+n;g) - t(k;g) \geq \frac{2n}{3w(g)} & \text{for } n > L, \\
  t(k+n;g) - t(k;g) \leq \frac{2n}{3w(g)} & \text{for } n < -L.
\end{cases}
\]

(2.3)

From (2.3), if \( n \to \pm \infty \), then \( t(n;g) \to \pm \infty \). Hence we can deduce that for any \( t \in \mathbb{R} \), \( \{n : |t - t(n;g)| \leq 1\} < \infty \). Then

\[
N(t) := \min\{k : |t - t(k;g)| = \min_{n} |t - t(n;g)|\}
\]

is well defined.

Claim: For any \( t \in \mathbb{R} \), there exists \( M > 0 \) such that

\[
|t - t(N(t);g)| \leq \sup_{k} \sup_{|j| \leq M} |t(j+k;g) - t(k;g)|.
\]

(2.4)

Proof of Claim: As we have shown in the proof of Lemma 2.2, \( \{t(n;g)\} \) is a relative dense set, i.e. there exists \( R > 0 \) such that for any \( t \in \mathbb{R} \), \( (t-R/2, t+R/2) \cap \{t(n;g)\} \neq \emptyset \). Now by (2.3) with \( k = N(t) \), for any \( |n| \geq \max\{\frac{3w(g)}{2} R, L\} \),

\[
|t(n+N(t);g) - t(N(t);g)| \geq R.
\]

Now taking \( M = \max\{\frac{3w(g)}{2} R, L\} \), we finish the proof of the claim.

Combining (2.4) with Definition 1.1 (3), there exists an absolute constant \( C \) such that

\[
|t - t(N(t);g)| \leq \sup_{k} \sup_{|j| \leq M} |t(j;g \cdot k)| \leq C,
\]

(2.5)

since \( \mathcal{H}(c) \) is compact. At last, Definition 1.1 (2) shows that \( v(t, n; g) \to 1 \) as \( n \to -\infty \), \( v(t, n; g) \to \infty \) as \( n \to \infty \) uniformly in \( g \). Thus,

\[
v(t, n+N(t);g) = v(t-t(N(t);g), n; g \cdot N(t)) \begin{cases} 
  \to 1 & \text{as } n \to -\infty, \\
  \to 0 & \text{as } n \to +\infty,
\end{cases}
\]

uniformly in \( t \). Moreover, by (2.4) and (2.2), one has

\[
|t - s| \leq |t - t(N(t);g)| + |s - t(N(s);g)| + |t(N(t);g) - t(N(s);g)| + 2C + \frac{2}{w(g)} (N(t) - N(s)).
\]
It follows that $|t - s| \to \infty$ implies $N(t) - N(s) \to \infty$. Now applying (2.4) again, we have

$$\lim_{t-s \to \infty} \frac{N(t) - N(s)}{t - s} = \lim_{t-s \to \infty} \frac{t(N(t); g) - t(N(s); g)}{t - s}$$

$$= \lim_{t-s \to \infty} \frac{N(t) - N(s)}{t(N(t); g) - t(N(s); g)} \cdot \frac{t(N(t); g) - t(N(s); g) - s + (t - s)}{t - s}$$

$$= \lim_{N(t) - N(s) \to \infty} \frac{N(t) - N(s)}{t(N(t); g) - t(N(s); g)} = w(g).$$

2.4. SL$(2, \mathbb{R})$ cocycles, Uniformly hyperbolic. Let $X$ be a compact metric space, $(X, \nu, T)$ be ergodic, and $A : X \to \text{SL}(2, \mathbb{R})$ be a continuous map. An SL$(2, \mathbb{R})$ cocycle over $(T, X)$ is an action defined on $X \times \mathbb{R}^2$ such that

$$(T, A) : (x, v) \in X \times \mathbb{R}^2 \mapsto (Tx, A(x) \cdot v) \in X \times \mathbb{R}^2.$$

For $n \in \mathbb{Z}$, $A_n$ is defined by $(T, A)^n = (T^n, A_n)$, where $A_0(x) = \text{id}$,

$$A_n(x) = \prod_{j=0}^{n-1} A(T^j x) = A(T^{n-1} x) \cdots A(Tx)A(x), \quad n \geq 1,$$

and $A_{-n}(x) = A_n(T^{-n} x)^{-1}$. The Lyapunov exponent is defined as

$$L(T, A) = \lim_{n \to \infty} \frac{1}{n} \int_X \ln ||A_n(x)|| \, d\nu.$$

We say an SL$(2, \mathbb{R})$ cocycle $(T, A)$ is uniformly hyperbolic if, for every $x \in X$, there exists a continuous splitting $\mathbb{R}^2 = E_s(x) \oplus E_u(x)$ such that for every $n \geq 0$,

$$|A_n(x)v(x)| \leq Ce^{-cn}|v(x)|, \quad v(x) \in E_s(x),$$

$$|A_{-n}(x)v(x)| \leq Ce^{-cn}|v(x)|, \quad v(x) \in E_u(x),$$

for some constants $C, c > 0$, and it holds that $A(x)E_s(x) = E_s(Tx)$ and $A(x)E_u(x) = E_u(Tx)$ for every $x \in X$. Clearly, if $(T, A)$ is uniformly hyperbolic, then $L(T, A) > 0$. If $L(T, A) > 0$ and the splitting is not continuous, then we call $(T, A)$ is non-uniformly hyperbolic.

2.5. Fibered rotation number. Assume $X$ is compact, and $(X, T)$ is uniquely ergodic with respect to its unique invariant probability measure. For this kind of dynamically defined cocycles $(T, A)$, one can define the rotation number of the cocycle. Let $S^1$ be the set of unit vectors of $\mathbb{R}^2$, consider a projective cocycle $F_{T, A}$ on $X \times S^1$:

$$(x, v) \mapsto (Tx, \frac{A(x)v}{||A(x)v||}).$$
If \( A : X \to \text{SL}(2, \mathbb{R}) \) is continuous and homotopic to the identity, then there exists a lift \( \tilde{F}_{(T,A)} \) of \( F_{(T,A)} \) to \( X \times \mathbb{R} \) such that \( \tilde{F}_{(T,A)}(x,y) = (Tx, y + \tilde{f}_{(T,A)}(x,y)) \), where \( \tilde{f}_{(T,A)} : X \times \mathbb{R} \to \mathbb{R} \) is a continuous lift that satisfies:

- \( \tilde{f}_{(T,A)}(x,y + 1) = \tilde{f}_{(T,A)}(x,y) + 1 \);
- for every \( x \in X \), \( \tilde{f}_{(T,A)}(x, \cdot) : \mathbb{R} \to \mathbb{R} \) is a strictly increasing homeomorphism;
- if \( \pi_2 \) is the projection map \( X \times \mathbb{R} \to X \times S^1 : (x,y) \mapsto (x, e^{2\pi i y}) \), then \( F_{(T,A)} \circ \pi_2 = \pi_2 \circ \tilde{F}_{(T,A)} \).

Meanwhile, define the \( n \)-th iterate as \( \tilde{F}^n_{(T,A)}(x,y) = (T^n x, y + \tilde{f}^n_{(T,A)}(x,y)) \).

Then there exists \( \rho \in \mathbb{R} \) such that \( \frac{\tilde{f}^n_{(T,A)}(x,y)}{n} \) converges uniformly to \( \rho \) in \( (x,y) \in X \times \mathbb{R} \), and it is independent on the lift of \( F_{T,A} \), up to an addition of an integer [28]. Then \( \rho \) is called fibered rotation number of \( (T,A) \), and we denote it as rot\((T,A)\).

**Remark 2.2.** In the following, we will always take \( X = \mathbb{T}^d \) where \( d \in \mathbb{N}_+ \) or \( \infty \) (we endow it with the product topology if \( d = \infty \)), and consider the quasi-periodic (or almost-periodic) cocycle \((\alpha,A)\).

The following two lemmas are are useful for us:

**Lemma 2.3** ([1]). For any \( A \in \text{SL}(2, \mathbb{R}) \), we have

\[
|\text{rot}(\alpha, Ae^F) - \text{rot}(\alpha, A)| < 2\|A\|\|F\|_{\infty}^{\frac{1}{2}}.
\]

**Lemma 2.4** ([36]). The rotation number is invariant under the conjugation map which is homotopic to the identity. More precisely, if \( A, B : \mathbb{T}^d \to \text{SL}(2, \mathbb{R}) \) is continuous and homotopic to the identity, then

\[
\text{rot}(\alpha, B(\cdot + \alpha)^{-1}A(\cdot)B(\cdot)) = \text{rot}(\alpha, A).
\]

### 2.6. Almost periodic Schrödinger operator

For the almost periodic sequence, we also have the following:

**Proposition 2.7** ([19]). A potential \( c \in l^\infty(\mathbb{Z}) \) is almost periodic if and only if it can be represented as

\[
c(n) = V_\omega(n) = f(T^n \omega),
\]

where \( \Omega \) is a compact abelian group, \( \omega \in \Omega \), \( f : \Omega \to \mathbb{R} \) is continuous, and \( T : \Omega \to \Omega \) is a minimal translation, say \( T = \cdot + \alpha \).

As result, we define the almost periodic Schrödinger operator as a self-adjoint operator on \( l^2(\mathbb{Z}) \) :

\[
(L_{f,T,\omega}u)(n) = u(n+1) + u(n-1) - 2u(n) + f(T^n \omega)u(n), \quad \forall n \in \mathbb{Z}.
\]
It’s well known that the spectrum $\Sigma(L_{f,T,\omega})$ is a compact set of $\mathbb{R}$. Moreover $\Sigma(L_{f,T,\omega})$ is independent of $\omega$, and we shorten the notation as $\Sigma(L_{f,T})$. In particular, if $f = V$, $\Omega = T^d$, $T = \cdot + \alpha$, where $\alpha, \theta \in T^d$, we denote

$$(L_{V,\alpha,\theta}u)(n) := u(n + 1) + u(n - 1) - 2u(n) + V(n\alpha + \theta)u(n).$$

We say $\theta$ is the phase, $V$ is the potential, and $\alpha$ is the frequency.

For almost-periodic Schrödinger operator, one can define the integrated density of states (shorten as IDS), denoted by $k(E)$, as follows:

$$k(E) = \lim_{L \to \infty} \frac{\# \{\text{eigenvalues (counting multiplicity) of } L \leq E \}}{L - 1},$$

where $L$ is the restriction of $L_{V,\alpha,\omega}$ to the set $I = \{1, \ldots, L - 1\}$ with boundary conditions $u(0) = \cot \theta, u(L) = 0$. The integrated density of states will be crucial for our study of the positive almost periodic solution. Furthermore, we have the following elementary fact:

**Remark 2.3.** Suppose $E$ is at the rightmost of the spectrum. Then by the well known characterization of the biggest eigenvalue $\lambda^L$ and $E$

$$\lambda^L = \sup_{u \in l^2(\mathbb{Z}), u \neq 0, \supp u \subset I} \frac{\langle Lu, u \rangle}{\langle u, u \rangle}, \quad E = \sup_{u \in l^2(\mathbb{Z}), u \neq 0} \frac{\langle Lu, u \rangle}{\langle u, u \rangle},$$

the rightmost of the spectrum $E$ is always larger than $\lambda^L$. Thus the number of eigenvalues of $L_L$ less than or equal to $E$ is always $L - 1$. By the definition of the IDS, $k(E) = 1$.

Note that a sequence $(u_n)_{n \in \mathbb{Z}}$ is a formal solution of the eigenvalue equation $L_{f,T,\omega}u = Eu$, if and only if

$$(u(n + 1) \
\quad u(n) ) = S_E(T^n \omega) (u(n) \\quad u(n - 1)), $$

where

$$S_E^f(\omega) = \begin{pmatrix} E + 2 - f(\omega) & -1 \\ 1 & 0 \end{pmatrix}.$$ 

We call $(T, S_E^f)$ an almost-periodic Schrödinger cocycle. In this paper, we will mainly consider the following two kinds of Schrödinger cocycles.

- $X_1 = T^d, T_1 \theta = \theta + \alpha, S_1^V = \begin{pmatrix} E + 2 - V(\theta) & -1 \\ 1 & 0 \end{pmatrix}$, where $d \in \mathbb{N}_+$, $\alpha \in T^d$ is rationally independent. Then $(X_1, T_1)$ is uniquely ergodic, and we call $(\alpha, S_1^V)$ a quasi-periodic Schrödinger cocycle.
- $X_2 = T^\infty$, that is endowed with the product topology, $T_2 \theta = \theta + \alpha, S_2^V = \begin{pmatrix} E + 2 - V(\theta) & -1 \\ 1 & 0 \end{pmatrix}$, where $\alpha \in DC_\infty(\gamma, \tau)$, Then $(X_2, T_2)$ is uniquely ergodic, and $(\alpha, S_2^V)$ defines an almost-periodic Schrödinger cocycle.
The study of the spectral properties of the almost-periodic Schrödinger operator is closely related to the dynamics of almost-periodic Schrödinger cocycle. For example, we will need the following two important facts:

**Theorem 2.2** ([30]). IDS of the Schrödinger operator relates with the fibered rotation number of Schrödinger operator as follows:

\[ k(E) = 1 - 2\text{rot}(\alpha, S^f_E) \quad (\text{mod } \mathbb{Z}). \]

**Theorem 2.3** ([29]). Let \( L_{f,T,\omega} \) be an almost periodic Schrödinger operator. Then we have the following:

\[ \mathbb{R} \setminus \Sigma = \{ E \in \mathbb{R} | (T_2, S^f_E) \text{ is uniformly hyperbolic} \}. \]

### 3. Properties of the Linearized Problem

#### 3.1. Generalized principal eigenvalue for more general operator.

In this section, we will define and study the properties of generalized principal eigenvalue of a more general operator since it will be needed in our proof. Remarkably, generalized principal eigenvalue theory for elliptic operator is of its own interest, and it turns out to be very useful in studying maximum principle [9].

Now we consider the operator

\[ M_{a,b,c} \phi(n) := a(n)\phi(n + 1) + b(n)\phi(n - 1) + c(n)\phi(n), \]

with \( a, b, c \) being almost periodic sequences and \( \inf a > 0, \inf b > 0 \). In particular, \( M_{1,1,-2} = L_g \) defined in (1.5).

For any (maybe unbounded) interval \( I \subset \mathbb{Z} \), we define the generalized principal eigenvalue for \( M_{a,b,c} \) as

\[ \lambda_1(M_{a,b,c}, I) = \inf \{ \lambda \in \mathbb{R} : \exists \phi > 0 \text{ in } \mathbb{Z}, M_{a,b,c,I} \phi \leq \lambda \phi \text{ in } I \}, \]

where \( M_{a,b,c,I} \) is the restriction of \( M_{a,b,c} \) to the set \( I \subset \mathbb{Z} \). If \( I \) is bounded, it is exactly the classical principal eigenvalue (the largest eigenvalue). In the case \( I = \mathbb{Z} \), \( a = b \equiv 1, c \) is replaced by \( c - 2 \), we will show below that it coincides with \( \lambda_1 = \max \Sigma(L) \). From (3.1), \( \lambda_1(M_{a,b,c}, I) \) is nondecreasing with respect to the inclusion of intervals \( I \).

**Proposition 3.1.** There holds

\[ \lambda_1(M_{a,b,c}, (-N, N)) \nearrow \lambda_1(M_{a,b,c}, \mathbb{Z}) \text{ as } N \to \infty. \]

**Remark 3.1.** The proof of this Proposition is similar to the proof in [9, Proposition 2.3] and [12, Proposition 4.2] which works for continuous elliptic operator.

**Proposition 3.2.** Let \( b(n) = a(n - 1) \), \( I \) be an interval in \( \mathbb{Z} \). Then

\[ \lambda_1(M_{a,b,c}, I) = \sup_{v \in l^2(\mathbb{Z}), v \not\equiv 0} \frac{\langle M_{a,b,c}v, v \rangle}{\langle v, v \rangle}, \]

where \( \langle \cdot, \cdot \rangle \) denotes the inner product on \( l^2(\mathbb{Z}) \). Moreover, \( \lambda_1(L, \mathbb{Z}) = \lambda_1 = \max \Sigma(L) \).
Proposition 3.3. \(a, b, c\) the hull of triple \((a, b, c)\) by \(\mathcal{H}(a, b, c)\):

\[
\begin{align*}
(a^*, b^*, c^*) & : a \cdot n_i \to a^*, b \cdot n_i \to b^*, c \cdot n_i \to c^* \text{ for some } \{n_i\}_{i \in \mathbb{Z}}.
\end{align*}
\]

Proof. For simplicity, we shorten the notation \(\mathcal{M}_{a,b,c;N} := \mathcal{M}_{a,b,c;[−N,N]}\), \(\lambda_1^N := \lambda_1(\mathcal{M}_{a,b,c;[−N,N]}\).

We only prove the case \(I = \mathbb{Z}\). Since \(b(n) = a(n - 1)\), it is clear that \(\mathcal{M}_{a,b,c;N}\) is a bounded self-adjoint operator. Then for any \(N \in \mathbb{N}_+\),

\[
(3.4) \quad \lambda_1^N = \sup_{v \in \mathcal{L}(\mathbb{Z}), v \neq 0, \operatorname{supp}v \subset [−N,N]} \frac{\langle \mathcal{M}_{a,b,c;N}v, v \rangle}{\langle v, v \rangle} = \| \mathcal{M}_{a,b,c;N} \|,
\]

here \(\| \cdot \|\) denotes the norm in the Banach space of linear bounded operators from \(L^2(\mathbb{Z})\) to \(L^2(\mathbb{Z})\).

Since \(a, b, c\) is almost periodic, hence bounded. By direct calculation,

\[
\lim_{N \to \infty} ||\mathcal{M}_{a,b,c;N}|| = \lim_{N \to \infty} ||\mathcal{M}_{a,b,c} - \mathcal{M}_{a,b,c;N}|| = 0.
\]

From (3.4) and Proposition 3.1, we have

\[
\lambda_1(\mathcal{M}_{a,b,c;N}) = \inf_{l \to \infty} \left( \mathcal{M}_{a,b,c;N} \right) = \inf_{\lambda \to \infty} \frac{\langle \mathcal{M}_{a,b,c}v, v \rangle}{\langle v, v \rangle}.
\]

In particular, \(\lambda_1(\mathcal{M}_{a,b,c;N})\) lies in the rightmost of the spectrum. □

From Proposition 3.2, we can deduce that \(\lambda_1(\mathcal{L}, \mathbb{Z}) \geq \inf c\). Indeed, taking the test function \(v_{2N}(n) = \begin{cases} 1 & |n| \leq 2N \\ 0 & \text{else} \end{cases}\) in (3.3), we deduce the generalized principal eigenvalue \(\lambda_1(\mathcal{L}, [−N,N]) \geq \inf c\). By Proposition 3.1, we have

\[
\lambda_1(\mathcal{L}, \mathbb{Z}) \geq \inf c, \text{ as desired.}
\]

Other generalizations of principal eigenvalue will be listed below, and they are all indispensable to our proof. Define the following quantities:

\[
(3.5) \quad \Delta_1(\mathcal{M}_{a,b,c}) := \sup \{ \lambda : \exists \phi \in \mathcal{S}, \mathcal{M}_{a,b,c}\phi \geq \lambda \phi \in \mathbb{Z} \};
\]

\[
(3.6) \quad \lambda_1'(\mathcal{M}_{a,b,c}) := \sup \{ \lambda : \exists \phi > 0, \phi \in L^\infty(\mathbb{Z}), \mathcal{M}_{a,b,c}\phi \geq \lambda \phi \in \mathbb{Z} \};
\]

\[
(3.7) \quad \mu_1(\mathcal{M}_{a,b,c}) := \inf \{ \mu : \exists \phi \in L^\infty(\mathbb{Z}), \inf_{n \in \mathbb{Z}} \phi > 0, \mathcal{M}_{a,b,c}\phi \leq \mu \phi \in \mathbb{Z} \};
\]

where \(\mathcal{S} = \{ \phi > 0 : \lim_{|n| \to +\infty} \frac{\log |\phi(n)|}{n} = 0, \{ \frac{\phi(n+1)}{\phi(n)} \}_{n \in \mathbb{Z}} \in L^\infty(\mathbb{Z}) \}\). We denote the hull of triple \((a, b, c)\) by \(\mathcal{H}(a, b, c)\):

\[
\{ (a^*, b^*, c^*) : a \cdot n_i \to a^*, b \cdot n_i \to b^*, c \cdot n_i \to c^* \text{ for some } \{n_i\}_{i \in \mathbb{Z}} \}.
\]
Finally, letting $a(n) = b(n+1)$, then they all coincide with $\lambda_1(M_{a,b,c})$ and $\lambda'_1(M_{a,b,c})$.

(2) $\lambda_1(M_{a^*,b^*,c^*})$, $\lambda'_1(M_{a^*,b^*,c^*})$, $\bar{\lambda}_1(M_{a^*,b^*,c^*})$, $\bar{\lambda}'_1(M_{a^*,b^*,c^*})$ are constant functions with respect to $(a,b,c)$ on $\mathcal{H}(a,b,c)$. If $a(n) = b(n+1)$, then $\lambda_1(M_{a,b,c})$, $\lambda'_1(M_{a,b,c})$ are also constant functions.

In particular, $\lambda_1(\mathcal{L}_g, \mathbb{Z})$ is a constant function with respect to $g$ on $\mathcal{H}(c)$.

Proof. (1) It is straightforward to check by (3.5), (3.6) and (3.7) that

\begin{equation}
\mu_1(M_{a,b,c}) \leq \lambda'_1(M_{a,b,c}), \quad \lambda_1(M_{a,b,c}) \leq \bar{\lambda}_1(M_{a,b,c})
\end{equation}

and

\begin{equation}
\mu'_1(M_{a,b,c}) \leq \lambda'_1(M_{a,b,c}), \quad \bar{\lambda}_1(M_{a,b,c}) \leq \bar{\lambda}_1(M_{a,b,c})
\end{equation}

Moreover, by the standard argument in [9, Theorem 1.7], we can prove that $\lambda'_1(M_{a,b,c}) \leq \lambda_1(M_{a,b,c})$ if $a(n) = b(n+1)$. From [40, Proposition 2.1], we also have $\lambda'_1(M_{a,b,c}) = \bar{\lambda}_1(M_{a,b,c})$.

For any $\epsilon > 0$, [40, Lemma 5.2] guarantees that the existence of an almost periodic function $u_\epsilon$ which satisfies

$$M_{a,b,c} e^{u_\epsilon} = \epsilon u_\epsilon e^{u_\epsilon}, \quad -\frac{\|e\|_{l^\infty(\mathbb{Z})}}{\epsilon} < u_\epsilon < \frac{2 + \|e\|_{l^\infty(\mathbb{Z})}}{\epsilon}.$$ 

Now we choose $e^{u_\epsilon} \in l^\infty(\mathbb{Z})$ with $\inf_{n \in \mathbb{Z}} e^{u_\epsilon} > 0$ as a test function in (3.7). Meanwhile, one has $\epsilon u_\epsilon \to \lambda$ uniformly by [40, Lemma 5.2]. Thus for any $\delta > 0$, there exists $\epsilon_\delta$ such that $|\epsilon\delta u_\epsilon - \lambda| < \delta$, and

$$(\lambda - \delta) e^{u_{\epsilon_\delta}} \leq M_{a,b,c} e^{u_{\epsilon_\delta}} = \epsilon_\delta u_{\epsilon_\delta} e^{u_{\epsilon_\delta}} \leq (\lambda + \delta) e^{u_{\epsilon_\delta}}.$$ 

Then letting $\delta \to 0$, $\mu_1(M_{a,b,c}) \geq \lambda \geq \bar{\lambda}_1(M_{a,b,c})$ follows by the definition. Thus by (3.9), we obtain $\lambda = \mu_1(M_{a,b,c}) = \lambda'_1(M_{a,b,c}) = \bar{\lambda}_1(M_{a,b,c}) = \bar{\lambda}_1(M_{a,b,c})$. Consequently, the desired result is obtained by (3.8).

(2) Analogous to the above argument, for any sequences $\{n_i\}_{i \in \mathbb{Z}}$ such that $a \cdot n_i \to a^*, b \cdot n_i \to b^*, c \cdot n_i \to c^*$, we have

$$M_{a_{n_i}, b_{n_i}, c_{n_i}} e^{u_{n_i} \cdot n_i} = \epsilon u_\epsilon \cdot n_i e^{u_{n_i} \cdot n_i}.$$ 

Notice when $\epsilon \to 0$, $\epsilon u_\epsilon \to \lambda$ uniformly. Hence for any $\delta > 0$, there exists $\epsilon_\delta$ such that $|\epsilon_\delta u_\epsilon \cdot n - \lambda| \leq \delta$ holds for any $n$. Thereby

$$(\lambda + \delta) e^{u_{\epsilon_\delta} \cdot n_i} \geq M_{a_{n_i}, b_{n_i}, c_{n_i}} e^{u_{\epsilon_\delta} \cdot n_i} = \epsilon_\delta u_{\epsilon_\delta} \cdot n_i e^{u_{\epsilon_\delta} \cdot n_i} \geq (\lambda - \delta) e^{u_{\epsilon_\delta} \cdot n_i}.$$ 

Note that $u_\epsilon$ is almost periodic for any $\epsilon > 0$. Passing along a subsequence $i_k \to \infty$, for any $\delta > 0$, it follows from (1) that

$$\lambda - \delta \leq \mu_1(M_{a^*, b^*, c^*}) = \lambda'_1(M_{a^*, b^*, c^*}) = \bar{\lambda}_1(M_{a^*, b^*, c^*}) = \bar{\lambda}_1(M_{a^*, b^*, c^*}) \leq \lambda + \delta.$$ 

Finally, letting $\delta \to 0$,

$$\lambda = \mu_1(M_{a^*, b^*, c^*}) = \lambda'_1(M_{a^*, b^*, c^*}) = \bar{\lambda}_1(M_{a^*, b^*, c^*}) = \bar{\lambda}_1(M_{a^*, b^*, c^*}).$$
If \( a(n) = b(n + 1) \), apply the above argument to \( \lambda_1(M_{a,b,c}) \), \( \lambda_1(M_{a,b,c}) \), then we are done. \( \square \)

3.2. The Lyapunov exponent of the linearized operator. The Lyapunov exponent is crucial to determine the average wave speed of the almost periodic traveling front. We will discuss its properties here and illustrate the connection with the existence of positive almost periodic solution. Recall

\[
(L_g u)(n) = u(n + 1) + u(n - 1) - 2u(n) + g(n)u(n),
\]

and \( L = L_c \). Note that Proposition 3.3 tells us \( \lambda_1(L_g, \mathbb{Z}) = \lambda_1(L, \mathbb{Z}) \), for any \( g \in \mathcal{H}(c) \). As a consequence of Lemma 3.2, \( \lambda_1 = \max \Sigma(L) = \lambda_1(L, \mathbb{Z}) \). Hence we do not distinguish them with a slight abuse of notation in the forthcoming paragraphs. First we state the following technical lemma which will be frequently used, and it is an immediate consequence of \([40, \text{Lemma 6.2}]\).

**Lemma 3.1.** Let \( E > \lambda_1 \), \( n_0 \in \mathbb{Z} \), and \( \phi \) defined in \([n_0, +\infty) \cap \mathbb{Z} \) satisfy

\[
L\phi \geq E\phi \text{ in } (n_0, +\infty) \cap \mathbb{Z}, \quad \lim_{n \to +\infty} \phi(n) = 0.
\]

Then there are positive constants \( C, \delta \) only depending on \( E, \lambda_1 \) and \( \|c(n)\|_{l^\infty} \) such that

\[
\phi(n) \leq C \max\{\phi(n_0), 0\} e^{-\delta(n-n_0)}.
\]

Moreover, \( \lim_{E \to \lambda_1} \delta = 0 \) and \( \lim_{E \to +\infty} \delta = +\infty \).

Now we study the Lyapunov exponent \( L(E) \) of \( L \) defined in (1.6) and our observation is stated as follows:

**Proposition 3.4.** For all \( E > \lambda_1 \), there exists a unique positive solution \( \phi_E(\cdot; g) \) of

\[
(L_g \phi) = E\phi \text{ in } \mathbb{Z}, \quad \phi(0) = 1, \quad \lim_{n \to \infty} \phi(n) = 0.
\]

and the limit

\[
L(E) = -\lim_{n \to \pm\infty} \frac{1}{n} \ln \phi_E(n; g) > 0, \quad \text{for any } g \in \mathcal{H}(c),
\]

the convergence is uniform in \( g \in \mathcal{H}(c) \).

First we need the following lemma:

**Lemma 3.2 ([55]).** Let \( T : X \to X \) be a uniquely ergodic homeomorphism of the compact metric space \( X \) and \( (T, A) \) be an \( \text{SL}(2, \mathbb{R}) \) cocycle over the probability space \((X, \mu)\). If \((T, A)\) is uniformly hyperbolic, then \( \frac{1}{n} \log \|A_n(x)\| \) converges uniformly to a constant.

**Proof of Proposition 3.4.** The existence and uniqueness of the positive solution follows from [40, Lemma 6.3].
Now let \( X = \mathcal{H}(c), T : \mathcal{H}(c) \to \mathcal{H}(c) \) with \( Tg = g \cdot 1 \) for any \( g \in \mathcal{H}(c) \), \( S^E_g = \begin{pmatrix} E + 2 - g(0) & -1 \\ 1 & 0 \end{pmatrix} \) \( \in \mathcal{C}^0(\mathcal{H}(c), \text{SL}(2, \mathbb{R})) \). Then \((T, S^E_g)\) is a cocycle defined on \((X, \mu)\), where \( \mu \) is the Haar measure on \( \mathcal{H}(c) \). Note that \( T \) is uniquely ergodic on \( \mathcal{H}(c) \) because \( \mathcal{H}(c) \) is a compact Abelian group and \( T \) is minimal.

Since \( E > \lambda_1 \), by Theorem 2.3, \((T, S^E_g)\) is uniformly hyperbolic. Then as a consequence of Lemma 3.2, the limit \( L^E := \lim_{n \to \infty} \frac{1}{n} \ln \|S^E_n(g)\| \) exists, and it is independent of \( g \in \mathcal{H}(c) \). Moreover, the convergence is uniform in \( g \in \mathcal{H}(c) \). Thus we can deduce that for any \( g \in \mathcal{H}(c) \),

\[
L^E = \lim_{n \to \infty} \frac{1}{n} \ln \|S^E_n(g)\| = \lim_{n \to \infty} \frac{1}{n} \int_{\mathcal{H}(c)} \ln \|S^E_n(g)\|d\mu = L(T_2, A) = L(E)
\]

where the last equality follows from the direct examination of the definition.

Note that in the proof of Lemma 3.2, we can deduce that the limit \( \lim_{n \to \infty} \frac{1}{n} \ln |S^E_n(g) \cdot v| \) exists for any \( v \in \mathbb{R}^2 \backslash \{0\} \). Now it follows from (2.6) that uniform hyperbolicity implies that for any \( g \in \mathcal{H}(c) \), and every \( n \geq 0 \), only \( v(g) \in E_s(g) \subset \mathbb{R}^2 \) satisfies \( \lim_{n \to \infty} \frac{1}{n} \ln |S^E_n(g) v(g)| < 0 \). Otherwise, all \( v \in E_s(g) \) violates (2.6).

Denote \( v^g_E = (\phi_E(1; g), \phi_E(0; g)) \) for \( g \in \mathcal{H}(c) \). By Lemma 3.1 and (3.10), one has

\[
\lim_{n \to \infty} \frac{1}{n} \ln |S^E_n(g) \cdot v^g_E| = \lim_{n \to \infty} \frac{\ln(\phi^2_E(n; g) + \phi^2_E(n-1; g))}{2n} < 0.
\]

Hence \( v^g_E \in E_s(g) \), and \( \lim_{n \to \infty} \frac{1}{n} \ln |S^E_n(g) \cdot v^g_E| = -L(E) \) since \( S^E_n(g) \) takes value in \( \text{SL}(2, \mathbb{R}) \).

Meanwhile, since \( S^E_{-n}(g)v^g_E \in E_s(T^{-n} g) \) in (2.6), one has

\[
|v^g_E| = |S^E_{-n}(T^{-n} g) S^E_{-n}(g)v^g_E| \leq C e^{-nL(E)} |S^E_{-n}v^g_E|.
\]

Then we can deduce that

\[
\lim_{n \to \infty} \frac{1}{n} |S^E_{-n}(g)v^g_E| = \lim_{n \to \infty} \frac{\ln(\phi^2_E(-n) + \phi^2_E(-n+1))}{n} = L(E).
\]

From equation \( (Lg \phi_E(\cdot; g)(n) = E \phi_E(n; g) \), we have

\[
\frac{\phi_E(n+1; g)}{\phi_E(n; g)} + \frac{\phi_E(n-1; g)}{\phi_E(n; g)} = 2 + g(n) \leq M,
\]

where \( M \) depends on \( ||g||_{l^\infty} \). It follows that

\[
\frac{1}{M} \phi_E(n; g) \leq \phi_E(n+1; g) \leq M \phi_E(n; g).
\]

Inserting this inequality into (3.11) and (3.12), \( L(E) = \lim_{n \to \pm \infty} \frac{\ln \phi_E(n; g)}{n} \) follows directly. Then the proof is complete.

\[ \square \]
The concavity and monotonicity of the Lyapunov exponent $L(E)$ will be needed in our construction of the almost periodic traveling front, and it is given by:

**Lemma 3.3.** The function $E \mapsto L(E)$ defined on $(\lambda_1, +\infty)$ is concave, nondecreasing and there exists $C > 0$ such that, for $E > \lambda_1$,

$$\delta \leq L(E) \leq C \sqrt{E},$$

where $\delta$ was given in Lemma 3.1 and $L(E) = \lim_{E \searrow \lambda_1} L(E)$.

**Proof.** $L(E)$ is nondecreasing and concave follows from [45, Lemma 2.5].

For $E_1 < E_2$, let $\phi_{E_1}$ and $\phi_{E_2}$ be obtained in Proposition 3.4. It is straightforward to check that $\phi_{E_2}$ is a subsolution of the equation satisfied by $\phi_{E_1}$ in $[0, \infty)$. Applying Lemma 3.1 to $\phi_{E_2} - \phi_{E_1}$, monotonicity follows directly. Also Lemma 3.1 shows that,

$$L(E) = -\lim_{n \to +\infty} \frac{1}{n} \ln \phi_E(n; g) \geq -\lim_{n \to +\infty} \frac{C}{n} + \delta = \delta.$$

Let $h_E(n; g) = e^{-n\sqrt{E-\inf c}} - \phi_E(n; g)$ with $\phi_E(\cdot; g)$ satisfying

$$L_g \phi = E\phi \text{ in } \mathbb{Z}, \; \phi_E(0) = 1, \; \lim_{n \to \infty} \phi(n) = 0.$$

We can check $h_E$ satisfies $L_g h_E \geq E h_E$ and $\lim_{n \to \infty} h_E(n; g) = 0$. By Lemma 3.1, we have $\phi_E(n; g) \geq e^{-n\sqrt{E-\inf c}}$, whence $L(E) \leq \sqrt{E - \inf c} \leq C \sqrt{E}$, as desired. \qed

The existence of positive almost periodic solution always implies the Lyapunov exponent $L(E)$ will decay to 0 as $E \searrow \lambda_1$, and it is crucial for us to determine in which case we can establish the almost periodic traveling front with average wave speed $w \in (w^*, \infty)$ (c.f. Corollaries 1.2 and 1.3). First we need a preliminary lemma about critical operator.

**Lemma 3.4.** Suppose that $\mathcal{L}$ admits a positive bounded solution $\varphi$ of $\mathcal{L}\varphi = E\varphi$. Then the associated eigenvalue is the generalized principal eigenvalue $\lambda_1$ and there hold:

1. $\mathcal{L} - \lambda_1$ is critical;
2. if $\inf \varphi > 0$, then $\mathcal{L}_g - \lambda_1$ is critical for any $g \in \mathcal{H}(c)$;
3. $\inf \varphi > 0$ if and only if $\varphi$ is almost periodic.

**Proof.** Taking $\varphi$ as the test function in (3.6), it follows from Proposition 3.3 the associated eigenvalue is exactly the generalized principal eigenvalue $\lambda_1$. (1). Assume by contradiction that $\mathcal{L} - \lambda_1$ is not critical. Denote the quadratic form associated with $\lambda_1 - \mathcal{L}$ by $h$. Then by Lemma 2.1, there exists a positive function $\varpi \in l^\infty_{loc}(\mathbb{Z})$ such that $h(u) = \langle (\lambda_1 - \mathcal{L})u, u \rangle \geq \langle \varpi u, u \rangle$ for any $u \in l_c$. That is, $\mathcal{L} - \lambda_1 \pm \varpi$ are bounded above by 0 (the supreme of the spectrum of $\mathcal{L} - \lambda_1$, see Proposition 3.2, i.e. the ground state energy of $\mathcal{L} - \lambda_1$). It follows from [20, Theorem 4] that $\varpi \equiv 0$, this is impossible since $\varpi$ positive. Hence $\mathcal{L} - \lambda_1$ is critical.
The proofs of (2) and (3) are similar to that of [45, Propostion 1.7]. □

Although the positive solution of \( L\phi = E\phi \) may not be almost periodic, almost periodicity can still be revealed in the following way:

**Lemma 3.5.** For all \( E > \lambda_1 \), the function \( \frac{\phi_E(n+1; g)}{\phi_E(n; g)} \) is almost periodic.

**Proof.** The method is similar to the proof of [45, Lemma 2.4], and Lemma 3.1 is needed in the proof. □

Once we have this, the following result follows:

**Proposition 3.5.** Assume that \( L \) admits a positive almost periodic solution \( \phi \) of \( L\phi = \lambda_1\phi \). Then we have \( \underline{L} := \lim_{E \to \lambda_1} L(E) = 0 \).

**Proof.** Denote \( \phi_E(\cdot; \cdot) := \phi_E(\cdot; c) \) which can be obtained in Proposition 3.4. Consider the analogue \( \tilde{\phi}_E \) of \( \phi_E \) but with the initial condition as follows:

\[
\tilde{L}\tilde{\phi}_E = E\tilde{\phi}_E \quad \text{in} \ Z, \quad \tilde{\phi}_E(0) = 1, \quad \lim_{n \to -\infty} \tilde{\phi}_E(n) = 0.
\]

The function \( \tilde{\phi}_E(\cdot) \) shares the same properties with \( \phi_E \). Particularly, the limit \( \underline{L}(E) := \lim_{n \to \pm \infty} \frac{1}{2} \ln \tilde{\phi}_E(n) \) exists and it is positive. Let \( \varphi_E := \sqrt{\tilde{\phi}_E\tilde{\phi}_E} \).

Then by the direct computation that

Thus,

\[
(\underline{L} - E)\varphi_E = -\frac{1}{2} \varphi_E q_E
\]

with \( q_E = \frac{1}{2} \left( \frac{\phi_E(n+1)}{\phi_E(n)} - \frac{\phi_E(n)}{\phi_E(n+1)} \right)^2 - \frac{1}{2} \left( \frac{\phi_E(n-1)}{\phi_E(n)} - \frac{\phi_E(n)}{\phi_E(n-1)} \right)^2.
\]

We claim that \( q_E \) converges uniformly to 0 in \( R \) as \( E \to \lambda_1 \). Otherwise, there exist \( \epsilon > 0 \) and two sequences \( \{E_i\}_i \) and \( \{n_i\}_i \) such that \( E_i \to \lambda_1 \) and \( |q_E(n_i)| \geq \epsilon \) for all \( i \in N \). According to Lemma 3.5, \( \phi_{E_i(n_i)} \) is almost periodic, hence \( \{q_{E_i}\} \) is uniformly bounded. Passing along a subsequence \( n_{i_k} \) in (3.14), \( \varphi_{E_i(n_{i_k})} \) converges pointwise to a positive solution \( \varphi^* \) of

\[
(\underline{L}_{g_*} - \lambda_1)\varphi^* = -\frac{1}{2} \varphi^* q
\]

where \( g_* = \lim_{k \to \infty} c \cdot n_{i_k} \) and \( q \) is the limit of \( \{q_{E_{i_k}(n+n_{i_k})}\}_k \). Since \( |q(0)| \geq \epsilon \), \( \underline{L}_{g_*} \) admits a supersolution which is not a solution. And from the assumption, Lemma 3.4 (2),(3) and Proposition 2.4, \( \underline{L}_{g_*} \) admits a unique positive supersolution. However, as will see, \( \varphi \) and \( \tilde{\phi} \) are both positive supersolutions. This is impossible! Therefore we have \( q_E \to 0 \) uniformly as \( E \to \lambda_1 \). Finally we have

\[
\underline{L}(E) + L(E) = \lim_{n \to \infty} \frac{1}{n} \sum_{0}^{n} \left( -\ln \frac{\phi_E(n+1)}{\phi_E(n)} + \ln \frac{\tilde{\phi}_E(n+1)}{\tilde{\phi}_E(n)} \right)
\]
tends to 0 as $E \to \lambda_1$. Then they both tend to 0 as $E \to \lambda_1$ due to the positivity of $L(E)$ and $\tilde{L}(E)$.

\section{Construction of the fronts}

\subsection{Construction of almost periodic fronts}

In this section, we start to prove Theorem 1.1. First we will consider (1) in Theorem 1.1 in this subsection. The basic idea is to apply the super-sub solution method.

From now on, we set
\[
\sigma_E(n; g) := -\ln \frac{\phi_E(n + 1; g)}{\phi_E(n; g)} \quad \text{for any } n \in \mathbb{Z}, \ g \in \mathcal{H}(c),
\]
where $\phi_E(n; g)$ is given by Proposition 3.4. Notice that $\sigma_E(n; g)$ is almost periodic by Lemma 3.5.

Before constructing the almost periodic traveling front, we should first notice the following facts:

\subsection*{Lemma 4.1}

Let $w^*, w$ be defined in Theorem 1.1. Then

1. $w^* < w$ provided $g$ in the equation (1.3) replaced by $g + c_0$ with $c_0$ large enough.

2. $w^*$ is a minimum provided $w^* < w$. Moreover, for all $w \in (w^*, w)$, there exists $E > \lambda_1$ such that $w = \frac{E}{L(E)}$ and $w > \frac{E'}{L(E')}$ for $E' - E > 0$ small enough.

\subsection*{Proof}

The proofs can be referred to [45, Lemmas 1.6 and 3.2].

Now afterwards, we take $w \in (w^*, w)$, and let $E > \lambda_1$ be as in Lemma 4.1. For a given almost periodic sequence $\sigma$, we define the operator

\begin{equation}
\mathcal{L}_g^\sigma \phi := e^{\sum_0^{n-1} \sigma \mathcal{L}_g(e^{\sum_0^{n-1} \sigma \phi})} = e^{-\sigma(n)\phi(n + 1) + e^{\sigma(n-1)}\phi(n - 1) + g(n)\phi(n)}.
\end{equation}

\subsection*{Proposition 4.1}

For any $g \in \mathcal{H}(c)$, there exist absolute constants $\delta > 0$, $\epsilon \in (0, 1)$ and a function $\theta \in l^\infty(\mathbb{Z})$ satisfying

\[
\inf_{\mathbb{Z}} \theta > 0, \quad -\mathcal{L}_g^{(1+\epsilon)\sigma_E(\cdot; g)}\theta \geq (\delta - (1 + \epsilon)E)\theta \quad \text{in } \mathbb{Z}.
\]

\subsection*{Proof}

As we choose $E$ in the Lemma 4.1, there exists $\epsilon \in (0, 1)$ such that

\[
\frac{E}{L(E)} > \frac{(1 + \epsilon)E}{L((1 + \epsilon)E)}.
\]

Now we define
\[
F(\kappa) := \frac{1}{L(E)} - \frac{1 + \epsilon}{L((1 + \epsilon)E)}.
\]

Then $F(E\epsilon) = \frac{1}{L(E)} - \frac{1 + \epsilon}{L((1 + \epsilon)E)} > 0$ and $F(0) = -\frac{\epsilon}{L(E)} < 0$. As $L(E)$ is concave in $(\lambda_1, \infty)$, it is continuous. Hence $F$ is continuous. Then there exists $\kappa \in (0, \epsilon E)$ so that $F(\kappa) = 0$. 

Consider now the function
\[ \zeta(n; g) := \frac{\phi_{E+n}(n; g)}{\phi_E(n; g)} \] for all \( n \in \mathbb{Z} \).

First, since \( e^{-(1+\epsilon)\sum_{0}^{n-1} \sigma_E(n; g)} = \phi_E^{1+\epsilon}(n; g) \), the positive function \( \zeta \) satisfies
\[ \mathcal{L}^{(1+\epsilon)\sigma_E(n; g)}_g \zeta = e^{(1+\epsilon)\sum_{0}^{n-1} \sigma_E(n; g)} \mathcal{L}_g \left( e^{-(1+\epsilon)\sum_{0}^{n-1} \sigma_E(n; g)} \zeta \right) = \frac{\mathcal{L}_g(\zeta \phi_E^{1+\epsilon}(n; g))}{\phi_E^{1+\epsilon}(n; g)} = (E + \kappa) \zeta. \]

Then it follows that
\[ -(\mathcal{L}^{(1+\epsilon)\sigma_E(n; g)}_g - (1 + \epsilon)E) \zeta = (\epsilon E - \kappa) \zeta \] in \( \mathbb{Z} \).

Moreover,
\[ \lim_{n \to \pm \infty} \frac{1}{n} \ln \zeta(n; g) = \lim_{n \to \pm \infty} \frac{1}{n} \ln \phi_{E+n}(n; g) - \lim_{n \to \pm \infty} \frac{1 + \epsilon}{n} \ln \phi_E(n; g) \]
\[ = -L(E + \kappa) + (1 + \epsilon)L(E) = 0, \]
where the last equality follows from the definition of \( \kappa \).

Next, it follows from Lemma 3.5 that \( \zeta^{(1+\epsilon)\sigma_E(n; g)} \) is in \( l^\infty(\mathbb{Z}) \), and the coefficients of \( \mathcal{M}_{a,b,c} = \mathcal{L}^{(1+\epsilon)\sigma_E(n; g)}_g - (1 + \epsilon)E \) are almost periodic. Applying Proposition 3.3 with \( \mathcal{M}_{a,b,c} \), then \( \lambda := \tilde{\lambda}_1(\mathcal{M}_{a,b,c}) = \lambda_1(\mathcal{M}_{a,b,c}) = \kappa - \epsilon E. \) Let \( \delta' = \epsilon E - \kappa > 0 \). Thus for any \( 0 < \delta < \delta' \), there exists a function \( \theta := e^{u_{\epsilon'}(\cdot; g)} \) with \( |\epsilon' u_{\epsilon'} - \lambda| < \delta' - \delta \) which is defined in the proof of Proposition 3.3 satisfies
\[ \mathcal{L}^{(1+\epsilon)\sigma_E(n; g)}_g \theta = (1 + \epsilon)E \theta = \epsilon' u_{\epsilon'} \theta \leq (\lambda + \delta' - \delta) \theta \leq -\delta \theta \]
in \( \mathbb{Z} \).

Moreover \( 0 < \inf \theta \leq \sup \theta < +\infty \) since \( u_{\epsilon'}(\cdot; g) \) is bounded. \( \square \)

It is clear that the choice of \( \theta \) is not unique. Let us now define \( \{\theta(\cdot; g)\} \) in the following certain way. Note that the function \( \theta = e^{u_{\epsilon'}} \) with \( |\epsilon' u_{\epsilon'} - \lambda| < \delta' - \delta \) in Proposition 4.1 is the unique almost periodic solution (see [40, Lemma 5.1]) of
\[ (L^{(1+\epsilon)\sigma_E(n; g)} - (1 + \epsilon)E) \theta = \epsilon' u_{\epsilon'} \theta \] in \( \mathbb{Z} \).

Indeed, almost periodicity follows from the inequality
\[ \|u_{\epsilon'} n_i - u_{\epsilon'} n_j\|_{\infty} \leq \frac{C}{\epsilon} \max \{ \|a n_i - a n_j\|_{1\infty}, \|b n_i - b n_j\|_{1\infty}, \|c n_i - c n_j\|_{1\infty} \}, \]
for any sequence \( \{n_i\}_i, \{n_j\}_j \), where \( a = e^{-(1+\epsilon)\sigma_E(n; g)}, b = e^{(1+\epsilon)\sigma_E(n-1; g)} \).

The details can be found in the proof of [40, Lemma 5.2]. Since \( \epsilon' u_{\epsilon'}(n) \to \lambda \) uniformly in \( n \) which follows from [40, Lemma 5.2], one can choose a fixed \( \epsilon_0 > 0 \) sufficiently small such that \( |\epsilon_0 u_{\epsilon_0} - \lambda| < \delta' - \delta \), and it is independent of \( g \in \mathcal{H}(c) \). In this way, we always denote \( \theta(n; g) = e^{u_{\epsilon_0}(n; g)} \).
Define for all \((t, n) \in \mathbb{R} \times \mathbb{Z}, g \in \mathcal{H}(c)\):

\[
\overline{u}(t, n; g) := \min \{ 1, \phi_E(n; g) e^{\epsilon t} \},
\]

\[
\underline{u}(t, n; g) := \max \{ 0, \phi_E(n; g) e^{\epsilon t} - A\theta(n; g) \phi_E^{1+\epsilon}(n; g) e^{(1+\epsilon)Et} \},
\]

where \(\epsilon\) and \(\theta(\cdot; g)\) are given by Propositions 4.1 and \(A\) is a positive constant that is to be specified. Notice that \(\epsilon\) is independent of \(g\).

Denote \(S_g = \{ \bar{u} \text{ is an entire solution of (1.3)} \} | u(t, n; g) \leq \bar{u}(t, n) \leq \overline{u}(t, n; g) \text{ in } \mathbb{R} \times \mathbb{Z} \} \).

**Proposition 4.2.** There exists a solution \(u\) of (1.3) satisfying \(u \in S_g\). Moreover, \(u = u(t, n; g)\) is increasing in \(t\).

**Proof.** By the calculation, \(\phi_E(n; g) e^{\epsilon t}\) is a supersolution on the \(\mathbb{R} \times \mathbb{Z}\) of (1.2). Then \(\overline{u}\) is also a supersolution of the equation (1.2). Take \((t, n) \in \mathbb{R} \times \mathbb{Z}\) so that \(u(t, n; g) > 0\) and set \(\zeta := \phi_E(n; g) e^{\epsilon t}\). Then we have:

\[
\underline{u}(t, n; g) - \underline{u}(t, n + 1; g) - \underline{u}(t, n - 1; g) + 2\underline{u}(t, n; g) - g(n)\underline{u}(t, n; g) \\
= - (1 + \epsilon)AE\theta(\cdot; g) \phi_E^{1+\epsilon}(\cdot; g) e^{(1+\epsilon)Et} + AE(1+\epsilon)Et \phi_E^{1+\epsilon}(\cdot; g) L_g^{(1+\epsilon)}(\sigma_E(g)\theta(\cdot; g) \\
= A\zeta^{1+\epsilon}[L_g^{(1+\epsilon)} g(\cdot; g) \theta(\cdot; g) - (1 + \epsilon) Et \theta(\cdot; g)] \\
\leq - A\delta \theta \zeta^{1+\epsilon}.
\]

Therefore, as 0 obviously solves (1.2), for \(u\) to be a subsolution it is sufficient to choose \(A\) so large that, for all \((t, n)\) such that \(u(t, n; g) > 0\), one has

\[A\delta \theta (\cdot; g) \zeta^{1+\epsilon} \geq g\zeta^2.\]

Observe that \(\underline{u}(t, n; g) > 0\) if and only if \(A\theta(\cdot; g) \zeta^\epsilon(t, n; g) < 1\), that is, \(\zeta^{\epsilon-1}(t, n) > (A\theta(n; g))^{\frac{1}{\epsilon}-1}\). Therefore, we can choose \(A\) so that \(A \geq \frac{\sup g}{\inf \theta(\cdot; g)}\).

The above argument also shows that \(\underline{u} < (A\theta(\cdot; g))^{-\frac{1}{\epsilon}}\). \(A\) can be chosen such that \(\underline{u} < 1\), whence \(\underline{u} \leq \overline{u}\).

Define the sequence \(\{u_i\}\) as follows: \(u_i\) is the solution of (1.3) for \(t > -i\) with initial condition \(u_i(-i, n; g) = \overline{u}(-i, n; g)\). By the comparison principle Proposition 2.3, \(u_i\) satisfies

\[\forall t > -i, n \in \mathbb{Z}, \underline{u}(t, n; g) \leq u_i(t, n; g) \leq \overline{u}(t, n; g).\]

Thus, for \(i, j \in \mathbb{N}\) with \(j < i\) and for any \(0 < h < 1\), using the monotonicity of \(\overline{u}\), we will get

\[u_j(-j, n; g) = \overline{u}(-j, n; g) \geq \overline{u}(-j - h, n; g) \geq u_i(-j - h, n; g).\]

Note that \(u_i(\cdot - h, \cdot; g)\) is also a solution of (1.3). The comparison principle Proposition 2.3 gives us

\[\forall j < i, 0 < h < 1, t > -j, n \in \mathbb{Z}, g \in \mathcal{H}(c) u_j(t, n; g) \geq u_i(t - h, n; g).\]

Now by the arguments before, we can prove \(\{u_i\}_i\) converges locally uniformly to a global function \(\underline{u} \leq u \leq \overline{u}\) of (1.3). Then passing to the limit as \(i, j \to \infty\), \(u(t, n; g) \geq u(t - h, n; g)\) for all \((t, n) \in \mathbb{R} \times \mathbb{Z}\) and \(0 < h < 1\), \(g \in \mathcal{H}(c)\). This means that \(u\) is nondecreasing in \(t\). If the monotonicity were
not strict, then the parabolic maximum principle Corollary 2.1 would imply that \( u \) is constant in time, which contradicts \( u \leq \bar{u} \leq \bar{u} \). Then we finish the proof. \( \square \)

One should notice the following fact:

**Proposition 4.3.** For any \( \tilde{u} \in S_g \), either \( \tilde{u}(t, n) < u(t, n; g) \), or \( \tilde{u}(t, n) \equiv u(t, n; g) \) in \( \mathbb{R} \times \mathbb{Z} \).

**Proof.** Since \( \tilde{u} \in S_g \), then, as constructed in Proposition 4.2,

\[
\forall j \in \mathbb{Z}_+, \quad \tilde{u}(-j, n) \leq \bar{u}(-j, n; g) = u_j(-j, n; g).
\]

Therefore, the comparison principle Proposition 2.3 gives \( \tilde{u}(t, n) \leq u_j(t, n; g) \) for \( t > -j \). Taking \( j \to \infty \), we have

\[
\tilde{u}(t, n) \leq u(t, n; g), \quad \forall (t, n) \in \mathbb{R} \times \mathbb{Z}.
\]

Let \( u = \tilde{u}(t, n), \bar{u} = u(t, n; g) \). By Proposition 2.3, one has either \( \tilde{u}(t, n) < u(t, n; g) \), or \( \tilde{u}(t, n) \equiv u(t, n; g) \) in \( \mathbb{R} \times \mathbb{Z} \). Thus the proof is complete. \( \square \)

In fact, \( u(t, n; g) \) is an almost periodic traveling front of (1.3) as we will show afterwards. To prove it, we shall notice some facts which will be represented below.

**Proposition 4.4.** Let \( E > \lambda_1 \). For any \( g \in \mathcal{H}(c) \), the following properties hold:

1. \( \phi_E(n; g \cdot k) = \frac{\phi_E(n+k; g)}{\phi_E(k; g)} \).
2. \( \{ \phi_E(n; g) | g \in \mathcal{H}(c) \} \) is a one-cover of \( \mathcal{H}(c) \) in \( l^\infty(\mathbb{Z}) \).
3. \( \{ \frac{\phi_E(n+1; g)}{\phi_E(n; g)} | g \in \mathcal{H}(c) \} \) is a one-cover of \( \mathcal{H}(c) \) in \( l^\infty(\mathbb{Z}) \).
4. \( \theta(n+k; g) = \theta(n; g \cdot k) \).
5. \( \{ \theta(n; g) | g \in \mathcal{H}(c) \} \) is a one-cover of \( \mathcal{H}(c) \) in \( l^\infty(\mathbb{Z}) \).

Before proving this proposition, we give an observation about one-cover.

**Remark 4.1.** We claim that \( \{ \Phi_g | g \in \mathcal{H}(c) \} \) is said to be a one-cover of \( \mathcal{H}(c) \) in some metric space \( X \) if and only if

\[
\Phi_{g_{n_k}} \to \Phi_{g^*} \quad \text{in } X \quad \text{provided that } g \cdot n_k \to g^*.
\]

In fact, (4.4) holds straightforward if \( \{ \Phi_g | g \in \mathcal{H}(c) \} \) is a one-cover. Assume that \( g_k \to g^* \) and \( g \) is the metric of \( X \). Then for any \( k \in \mathbb{Z} \), there exists \( \{n_k\} \) such that \( \varrho(\Phi_{g_{n_k}}, \Phi_{g_k}) + \|g \cdot n_k - g_k\|_{\ell^\infty} < 1/k \). Therefore, \( g \cdot n_k \to g^* \) since \( g_k \to g^* \), and thus \( \Phi_{g_{n_k}} \to \Phi_{g^*} \) by (4.4). Hence \( \Phi \) is continuous since

\[
\varrho(\Phi_{g^*}, \Phi_{g_k}) \leq \varrho(\Phi_{g_{n_k}}, \Phi_{g_k}) + \varrho(\Phi_{g_{n_k}}, \Phi_{g^*}) \to 0 \quad \text{as } k \to \infty.
\]

**Proof of Proposition 4.4.** (1) follows from the uniqueness of \( \phi_E(n; g) \) which follows from Proposition 3.4.

(2) By Remark 4.1, it is sufficient to prove \( \Phi_{g_{n_k}} \to \Phi_{g^*} \) provided that \( g \cdot n_k \to g^* \). For \( g \cdot n_k \to g^* \), \( \phi_E(n; g \cdot n_k) \) converges locally uniformly, up to some
subsequence, to some function \( \tilde{\phi}_E(n) \). Moreover, we have \( \tilde{\phi}_E(n) \leq Ce^{-\delta n} \) for \( n > 0 \) since \( \phi_E(n; g \cdot n_k) \leq Ce^{-\delta n} \) from Lemma 3.1. Therefore \( \tilde{\phi} \) satisfies

\[
L_{g} \phi = E \phi \text{ in } \mathbb{Z}, \quad \phi(0) = 1, \quad \lim_{n \to \infty} \phi(n) = 0,
\]

which yields that \( \tilde{\phi} = \phi_E(n; g^*) \) by uniqueness. That is to say, all the convergence subsequences converge to the same limit. Hence \( \phi_E(n; g \cdot n_k) \to \phi_E(n; g^*) \) locally uniformly in \( n \in \mathbb{Z} \).

(3). We want to prove that \( \frac{\phi_E(1; g \cdot (m_k + n_k))}{\phi_E(1; g^*)} \to \frac{\phi_E(1; g^*)}{\phi_E(1; g^*)} \) in \( l^\infty(\mathbb{Z}) \) if \( g \cdot n_k \to g^* \).

If not, then there exists a sequence \( \{m_k\}_k \) such that, up to extraction,

\[
\lim_{k \to \infty} \left| \frac{\phi_E(m_k + 1; g \cdot n_k)}{\phi_E(m_k; g \cdot n_k)} - \frac{\phi_E(m_k + 1; g^*)}{\phi_E(m_k; g^*)} \right| > 0,
\]

i.e., by (1),

\[
(5.5) \quad \lim_{k \to \infty} \left| \frac{\phi_E(1; g \cdot (m_k + n_k))}{\phi_E(1; g^*)} - \frac{\phi_E(1; g^*)}{\phi_E(1; g^*)} \right| > 0.
\]

Note that \( \lim_{k \to \infty} g \cdot (m_k + n_k) = \lim_{k \to \infty} g \cdot n_k = \lim_{k \to \infty} g^* \cdot m_k \) since \( g \) is almost periodic. Then using (2), we have

\[
\lim_{k \to \infty} \left| \frac{\phi_E(1; g \cdot (m_k + n_k))}{\phi_E(1; g^*)} - \frac{\phi_E(1; g^*)}{\phi_E(1; g^*)} \right| = 0
\]

as \( k \to \infty \), which contradicts (5.5).

(4). \( \theta(n + k; g) = \theta(n; g \cdot k) \) follows from the uniqueness.

(5). By the similar argument in (2), \( \{\theta(n; g)|g \in \mathcal{H}(c)\} \) is a one-cover of \( \mathcal{H}(c) \) in \( l^\infty(\mathbb{Z}) \).

With this at hand, we can prove that \( u \) satisfies (4) of Definition 1.1.

**Lemma 4.2.** For any \( g \in \mathcal{H}(c) \), we have

\[ u(t + t(k; g), n + k; g) = u(t, n; g \cdot k) \forall t \in \mathbb{R}, \; n, k \in \mathbb{Z}, \]

where \( t(k; g) = -\frac{1}{E} \ln \phi_E(k; g) \).

**Proof.** From the definitions of \( u(t, n; g) \) and \( u(t, n; g) \), together with Proposition 4.4 (1), we can verify that

\[
\overline{u}(t + t(k; g), n + k; g) = \overline{u}(t, n; g \cdot k),
\]

\[
u(t + t(k; g), n + k; g) = u(t, n; g \cdot k).
\]

Combining these two equations, we can check that

\[
u(t + t(k; g), n + k; g) \in \mathcal{S}_g, \quad \nu(t - t(k; g), n - k; g \cdot k) \in \mathcal{S}_g.
\]

Therefore, \( u(t + t(k; g), n + k; g) \leq u(t, n; g \cdot k) \) and \( u(t - t(k; g), n - k; g \cdot k) \leq u(t, n; g) \) by Proposition 4.3, which gives \( u(t + t(k; g), n + k; g) = u(t, n; g \cdot k) \) for any \( (t, n) \in \mathbb{R} \times \mathbb{Z} \).

To prove \( u \) satisfying (2) of Definition 1.1, we will consider \( u(t, 0; g \cdot n) \).

**Lemma 4.3.** The function \( u(t, 0; g \cdot n) \) satisfies

\[
(5.6) \quad \lim_{t \to -\infty} u(t, 0; g \cdot n) = 0, \quad \lim_{t \to +\infty} u(t, 0; g \cdot n) = 1, \quad \text{uniformly in } n \in \mathbb{Z}.
\]
Proof. For any \((t, n) \in \mathbb{R} \times \mathbb{Z}\), we have 
\[
    u(t, 0; g \cdot n) \leq \overline{u}(t, 0; g \cdot n) \leq \phi_E(0; g \cdot n)e^{Et} = e^{Et},
\]
Meanwhile, 
\[
    u(t, 0; g \cdot n) \geq u(t, 0; g \cdot n) \geq e^{Et} \phi_E(0; g \cdot n) - A\theta(0; g \cdot n)\phi_E^{1+\epsilon}(0; g \cdot n)e^{(1+\epsilon)Et} \\
    \geq e^{Et} - A(\sup_n \theta(n; g))e^{(1+\epsilon)Et},
\]
where the last inequality follows from \(\theta(n + k; g) = \theta(n; g \cdot k)\). Therefore, 
\[
    \forall (t, n) \in \mathbb{R} \times \mathbb{Z}, \quad e^{Et}(1 - Me^{Et}) \leq u(t, 0; g \cdot n) \leq e^{Et},
\]
for some positive constant \(M\). From the second inequality, we deduce that 
\[
    \lim_{t \to -\infty} u(t, 0; g \cdot n) = 0 \quad \text{uniformly in } n \in \mathbb{Z}.
\]
From the first inequality of (4.7), we can see that \(\inf_{n \in \mathbb{Z}} u(t, 0; g \cdot n) > 0\) for \(t < 0\) small enough. Therefore, it follows from the monotonicity of \(u\) in \(t\) that 
\[
    \forall t \in \mathbb{R}, \quad \inf_{[t, +\infty) \times \mathbb{Z}} u(t, 0; g \cdot n) > 0,
\]
and the quantity \(\vartheta := \lim_{t \to +\infty} \inf_{n \in \mathbb{Z}} u(t, 0; g \cdot n) > 0\) is well defined. To conclude the proof we only need to prove \(\vartheta = 1\). Let \(\{n_k\}_{k \in \mathbb{Z}_+} \subset \mathbb{Z}\) be such that 
\[
    u(k, 0; g \cdot n_k) \to \vartheta \quad \text{as } k \to \infty.
\]
Consider the family of functions \(\{p^k\}_{k \in \mathbb{Z}_+}\) 
\[
    p^k(t, n) := u(t + k, n; g \cdot n_k) = u(t + k - t(n; g \cdot n_k), 0; g \cdot (n + n_k)),
\]
where the last equality follows from Lemma 4.2. Then we have 
\[
    p^k(0, 0) = u(k, 0; g \cdot n_k) \to \vartheta \quad \text{as } k \to \infty,
\]
and, for any \((t, n) \in \mathbb{R} \times \mathbb{Z}\), 
\[
    \liminf_{k \to \infty} p^k(t, n) = \liminf_{k \to \infty} u(t + k - t(n; g \cdot n_k), 0; g \cdot (n + n_k)) \geq \vartheta
\]
by (4.8). Moreover, the sequence \(\{p^k\}_k\) converges, up to sequences, to a function \(p\) satisfying 
\[
    p(t, n) - p(t, n + 1) - p(t, n - 1) + 2p(t, n) = g^*(n)p(t, n)(1 - p(t, n)) \text{ in } \mathbb{R} \times \mathbb{Z},
\]
where \(g^* \in \mathcal{H}(c)\). Note that \(p\) reaches its minimum \(\vartheta\) at \((0, 0)\). Then we have 
\[
    g^*(0)\vartheta(1 - \vartheta) \leq 0.
\]
Therefore, \(\vartheta = 1\) since \(g^*(0) \geq \inf_c c > 0\) and \(0 < \vartheta \leq 1\). 

Now we need the uniform convergence in \(\mathbb{R} \times \mathbb{Z}\) to explain why \(u(t, n; g)\) is an almost periodic traveling front. Before that, the following lemma is needed. 

Lemma 4.4. Let two bounded uniformly continuous functions \(u^1\) and \(u^2\) be a subsolution and a supersolution of (1.3), respectively, i.e., 
\[
    u^1(t, n) - u^1(t, n + 1) - u^1(t, n - 1) + 2u^1(t, n) \leq g(n)u^1(t, n)(1 - u^1(t, n)), \quad t \in \mathbb{R},
\]
\[
    u^2(t, n) - u^2(t, n + 1) - u^2(t, n - 1) + 2u^2(t, n) \geq g(n)u^2(t, n)(1 - u^2(t, n)), \quad t \in \mathbb{R},
\]
and satisfy \(0 \leq u^1 \leq u^2\) in \(\mathbb{Z} \times \mathbb{R}\).
If \( \inf_{n \in \mathbb{Z}} (u^2 - u^1)(t_0 + t(n; g), n) = 0 \) for some \( t_0 \in \mathbb{R} \), then
\[
\forall t < t_0, \quad \inf_{n \in \mathbb{Z}} (u^2 - u^1)(t + t(n; g), n) = 0.
\]

**Proof.** Let \( \{n_k\} \subset \mathbb{Z} \) be such that
\[
u^2(t_0 + t(n_k; g), n_k) - u^1(t_0 + t(n_k; g), n_k) \to 0 \text{ as } k \to \infty.
\]

Consider
\[
u_k(t, n) := u^i(t + t(n_k; g), n + n_k), \quad i = 1, 2.
\]

Then the nonnegative function \( w_k := u^2 - u^1 \) satisfies
\[
\lim_{k \to \infty} w_k(t_0, 0) = 0
\]
and
\[
\frac{d}{dt} w_k(t, n) - D w_k(t, n) \geq (g \cdot n_k)(n)(1 - u_k^2(t, n) - u_k^1(t, n)) w_k(t, n),
\]
where \( D w_k(t, n) = w_k(t, n + 1) - w_k(t, n - 1) + 2 w_k(t, n) \). Take \( M \) large enough such that
\[
M + \sup_{n \in \mathbb{Z}} (g \cdot n_k)(1 - u_k^2 - u_k^1) > 1.
\]

Then the nonnegative function \( v_k(t, n) := e^M w_k(t, n) \) satisfies
\[
\frac{d}{dt} v_k(t, n) - v_k(t, n + 1) - v_k(t, n - 1) + 2 v_k(t, n) \geq v_k(t, n).
\]

Now from Proposition 2.2 we have
\[
v_k(t, n) \leq C(t_0 + t) v_k(t_0, n), \quad C(t_0 + t) \text{ is a constant which is independent of } k.
\]

Therefore, \( w_k(t, n) \leq e^{M(t + t)} C(t_0 - t) w_k(t_0, 0) \), and this yields that
\[
0 \leq \lim_{k \to \infty} w_k(t, 0) \leq \lim_{k \to \infty} e^{M(t + t)} C(t_0 - t) w_k(t_0, 0) = 0.
\]

Therefore,
\[
0 \leq \inf_{n \in \mathbb{Z}} (u^2 - u^1)(t + t(n; g), n) \leq \inf_{k \in \mathbb{Z}} (u^2 - u^1)(t + t(n_k; g), n_k)
\]
\[
= \inf_{k \in \mathbb{Z}} w_k(t, 0) \leq \lim_{k \to \infty} w_k(t, 0) = 0.
\]

Thus the proof is complete. \( \square \)

Using Lemma 4.4, we have the following result about uniform convergence.

**Theorem 4.1.** Assume that \( g^* = \lim_{k \to \infty} g \cdot n_k \) for some sequence \( \{n_k\} \subset \mathbb{Z}_+ \).
Then \( u(t + t(n; g \cdot n_k), n; g \cdot n_k) \to u(t + t(n; g^*), n; g^*) \) uniformly in \( \mathbb{R} \times \mathbb{Z} \).

**Proof.** First, considering a sequence \( \{n_k\} \subset \mathbb{Z} \) where \( c(\cdot + n_k) \) converges uniformly in \( \mathbb{R} \), we prove that \( u(t + t(n; g \cdot n_k), n; g \cdot n_k) \) converges uniformly in \( \mathbb{R} \times \mathbb{Z} \). Assume by contradiction that it is false. Then there exist \( \{t_k\} \subset \mathbb{R} \), \( \{m_k\} \subset \mathbb{Z} \), and two subsequences \( \{n_k\} \subset \mathbb{Z} \) such that
\[
\liminf_{k \to \infty} (u(t_k + t(m_k; g \cdot n_k^1), m_k; g \cdot n_k^1) - u(t_k + t(m_k; g \cdot n_k^2), m_k; g \cdot n_k^2)) > 0,
\]
i.e., \( \liminf_{k \to \infty} (u(t_k, 0; g \cdot (m_k + n_k^1)) - u(t_k, 0; g \cdot (m_k + n_k^2))) > 0 \). By Lemma 4.3, \( \{t_k\} \) is bounded. Let \( \zeta \) be a limit point of \( \{t_k\} \). Then by \( 0 \leq u_t \leq \)
4 + \|c\|_{L^\infty(Z)} which follows from (1.3), \( u \) is uniform continuous in \( t \). Thus,

\[
\liminf_{k \to \infty} (u(\zeta, 0; g \cdot (m_k + n_k^1)) - u(\zeta, 0; g \cdot (m_k + n_k^2))) > 0,
\]

Consider for \( i = 1, 2 \),

\[
p^i_k(t, n) := u(t, n; g \cdot (m_k + n_k^i)) = u(t - t(n; g \cdot (m_k + n_k^i)), 0; g \cdot (n + m_k + n_k^i)),
\]

where the second equality follows from Lemma 4.2. Then, up to subsequences, \( p^i_k(t, n) \) and \( g \cdot (m_k + n_k^i)(i = 1, 2) \) converge locally uniformly to \( p^i \) and \( g^{**} \), respectively, as \( k \to \infty \). Moreover \( p^1(\zeta, 0) - p^2(\zeta, 0) > 0 \), and

\[
p^i(t, n) = p^i(t, n+1)+p^i(t, n-1) - 2p^i(t, n)+g^{**}(n)p^i(t, n)(1-p^i(t, n)), \quad i = 1, 2.
\]

Recall \( t(n; g) = -\frac{1}{2} \ln \phi_E(n; g) \). Thanks to (2) of Proposition 4.4 and the fact that \( u \) is continuous in \( t \), we have for \( i = 1, 2 \),

\[
\lim_{k \to \infty} u(t, 0; g \cdot (n + m_k + n_k^i)) = \lim_{k \to \infty} p^i_k(t + t(n; g \cdot (m_k + n_k^i)), n) = p^i(t + t(n; g^{**}), n)
\]

Combining this with (4.6) and (4.7), we have \( p^1(t + t(n; g^{**}), n)/p^2(t + t(n; g^{**}), n) \to 1 \) as \( t \to \pm \infty \) uniformly in \( n \in \mathbb{Z} \). Note that by (4.8),

\[
\kappa^* := \sup_{\mathbb{R} \times \mathbb{Z}} \frac{p^1(t + t(n; g^{**}), n)}{p^2(t + t(n; g^{**}), n)}
\]

is finite and \( \kappa^* > 1 \) since \( p^1(\zeta, 0) - p^2(\zeta, 0) > 0 \). Moreover, by the uniform continuity of \( p^1 \) and \( p^2 \), we can find some finite \( \ell \) such that

\[
\sup_{n \in \mathbb{Z}} \frac{p^1(\ell + t(n; g^{**}), n)}{p^2(\ell + t(n; g^{**}), n)} = \sup_{\mathbb{R} \times \mathbb{Z}} \frac{p^1(t + t(n; g^{**}), n)}{p^2(t + t(n; g^{**}), n)} = \kappa^*.
\]

By the direct computation, we can show that \( \kappa^* p^2 \) is a supersolution of (1.3) with \( g \) replaced by \( g^{**} \) since \( \kappa^* > 1 \). Now we can apply Lemma 4.4 to deduce that

\[
\forall t < \ell, \inf_{n \in \mathbb{Z}} (\kappa^* p^2(t + t(n; g^{**}), n) - p^1(t + t(n; g^{**}), n)) = 0
\]

which contradicts (4.7) when \( t \) is sufficiently large. Hence \( u(t + t(n; g \cdot n_k), n; g \cdot n_k) \) converges uniformly in \( \mathbb{R} \times \mathbb{Z} \).

Let us now show that \( \lim_{k \to \infty} u(t+t(n; g \cdot n_k), n; g \cdot n_k) = u(t+t(n; g^*), n; g^*) \).

Denote \( v(t + t(n; g^*), n) := \lim_{k \to \infty} u(t + t(n; g \cdot n_k), n; g \cdot n_k) \). Using (2) of Proposition 4.4 again, we have

\[
\lim_{k \to \infty} u(t, n; g \cdot n_k) = \lim_{k \to \infty} u(t - t(n; g \cdot n_k) + t(n; g \cdot n_k), n; g \cdot n_k) = v(t, n).
\]

Hence \( v \in S_{g^*} \), and thus \( v(t, n) \leq u(t, n; g^*) \) by Proposition 4.3. We claim that \( v(t, n) \equiv u(t, n; g^*) \). Otherwise, \( v(t, n) < u(t, n; g^*) \). Note that by Lemma 4.2,

\[
v(t + t(n; g^*), n) = \lim_{k \to \infty} u(t + t(n; g \cdot n_k), n; g \cdot n_k) = \lim_{k \to \infty} u(t, 0; g \cdot (n + n_k)).
\]
Then, by the similar reasons as before, we deduce that
\[
\kappa' := \sup_{\mathbb{R} \times \mathbb{Z}} \frac{u(t + t(n; g^*), n)}{v(t + t(n; g^*), n)}
\]
is finite, \(\kappa' > 1\) since \(v(t, n) < u(t, n; g^*)\), and \(\kappa' v\) is a supersolution of (1.3) with \(g\) replaced by \(g^*\). Applying Lemma 4.4 as before, we obtain a contradiction. Hence \(v(t, n) \equiv u(t, n; g^*)\), that is to say,
\[
u(t + t(n; g \cdot n_k), n; g \cdot n_k) \to u(t + t(n; g^*), n; g^*) \quad \text{as} \quad k \to \infty
\]
universally in \((t, n) \in \mathbb{R} \times \mathbb{Z}\). \(\square\)

We are now in the position to prove the existence of almost periodic traveling front.

**Theorem 4.2.** \(u(t, n; g)\) is an almost periodic traveling front with the average wave speed \(w \in (w^*, \bar{w})\).

**Proof.** We prove that \(u(t, n; g)\) satisfies (1) of Definition 1.1 first. Consider \(g \cdot n_k \to g^*\), and note that \(\{t(n; g) | g \in H(c)\} = \{-\frac{1}{2} \ln \phi_E(n; g) | g \in H(c)\}\) is a one-cover of \(H(c)\) in \(L^\infty_{loc}(\mathbb{Z})\) by Proposition 4.4 (2). Then combining Theorem 4.1, we have
\[
u(t, n; g \cdot n_k) = u(t - t(n; g \cdot n_k) + t(n; g \cdot n_k), n; g \cdot n_k)
\to u(t - t(n; g^*) + t(n; g^*), n; g^*)
= u(t, n; g^*)
\]
locally uniformly in \((t, n) \in \mathbb{R} \times \mathbb{Z}\). Hence \(\{u(t, n; g) | g \in H(c)\}\) is a one-cover of \(C(\mathbb{R} \times \mathbb{Z})\).

Note that \(u(t, n; g) = u(t - t(n; g), 0; g \cdot n)\) by Lemma 4.2. From the choice of \(t(n, g)\), we have \(t(n; g) \to \pm \infty\) as \(n \to \pm \infty\). Then by Lemma 4.3, we deduce that \(u(t, n; g)\) satisfies (2) of Definition 1.1.

By Proposition 3.4, one has
\[
w(g) = \lim_{|n| \to \infty} \frac{\frac{n}{t(n + k; g) - t(k; g)}}{L(E)} = \frac{\frac{nE}{\ln \phi_E(n; g \cdot k)}}{E} \in (w^*, \bar{w}).
\]
Now combining this with Proposition 4.4, (3) of Definition 1.1 is proved. Finally, Lemma 4.2 gives rise to (4). Hence we finish the proof. \(\square\)

Now we ends the proof of (1) in Theorem 1.1.

4.2. **Non-existence of Fronts with Speed Less than** \(w^*\). Next we turn to prove (3) in Theorem 1.1, i.e., there is even no generalized transition front with average speed \(w < w^*\). Compared to the previous section, we only consider the generalized transition front of (1.2). However, the similar arguments can be applied to (1.3) with minor modification. From now on, for the sake of simplicity, we denote \(\mathcal{D}\phi(t, n) = \phi(t, n + 1) + \phi(t, n - 1) - \)
2\phi(t, n), t, n \in \mathbb{R} \times \mathbb{Z} and u(t, n; s, u_0), t \geq s, n \in \mathbb{Z} a solution of (1.2) with initial value \( u_0 \) starting at time \( s \).

Now we begin with a lemma which will provide a lower bound of the average speed of generalized transition front.

**Lemma 4.5.** Let \( u(t, n; 0, u^{(0)}) \) be a solution of (1.2) with its initial value \( u(0, n; 0, u^{(0)}) = u^{(0)} \), where

\[
u^{(0)}(n) = 0 \text{ if } n > 0, \text{ and } \inf_{n \leq 0} u^{(0)}(n) = \alpha \in (0, 1]. \]

Then \( \lim_{t \to \infty} \inf_{n \leq wt} u(t, n; 0, u^{(0)}) = 1 \), \( \forall 0 \leq w < w^* \).

To prove this lemma, we need

**Proposition 4.5.** Let \( g \in \mathcal{H}(c) \) and \( u^g(t, n; 0, u_0) \) be the solution of

\[
u_t = D u(t, n) = g(n)u(t, n)(1 - u(t, n)) \text{ in } \mathbb{R} \times \mathbb{Z}, \]

\[
u(0, n) = u_0(n) \text{ with } u_0(0) = \alpha \in (0, 1] \text{ and } u_0(n) = 0, \text{ if } n \neq 0. \]

Then for any \( 0 \leq w < w^* \), we have \( \lim_{t \to \infty} \inf_{0 \leq n \leq wt} u^g(t, n; 0, u_0) = 1 \) exists uniformly in \( g \).

**Proof.** Step 1: Show that \( \lim_{t \to \infty} \inf_{0 \leq n \leq wt} u^g(t, n; 0, u_0) = 1 \) for any \( 0 \leq w < w^* \).

Denote \( \mathcal{L}_{g,p} \phi := e^p \mathcal{L}_g(e^{-p} \phi) \), where \( \mathcal{L}_g \) is the linearized operator given by \( (\mathcal{L}_g \phi)(n) = \phi(n + 1) + \phi(n - 1) - 2\phi(n) + g(n)\phi(n) \). Then as proved in [40, Theorem 2.1], we have

\[
\lim_{t \to \infty} \inf_{0 \leq n \leq wt} u^g(t, n; 0, u_0) = 1 \text{ for any } 0 \leq w < \inf_{p > 0} \frac{\lambda_1(\mathcal{L}_{g,p})}{p},
\]

where \( \lambda_1(\mathcal{L}_{g,p}) \) is defined as (3.5) with \( \mathcal{M}_{a,b,c} \) replaced by \( \mathcal{L}_{g,p} \). Proposition 3.3 yields that \( \lambda_1(\mathcal{L}_{g,p}) = \lambda_1(\mathcal{L}_{c,p}) \) for any \( g \in \mathcal{H}(c) \), where \( \lambda_1(\mathcal{L}_{g,p}) \) is defined as (3.6), and then we denote it by \( k(p) \). It still needs to show that

\[
\inf_{p > 0} \frac{k(p)}{p} = w^* = \inf_{E > \lambda_1} \frac{E}{L(E)} \text{ with } L(E) \text{ given by (1.6)}.
\]

In fact, by similar arguments to the proof of [40, Theorem 6.1], we can deduce that the map \( L : (\lambda_1, +\infty) \to (L, +\infty), \) where \( L = \lim_{E \to \lambda_1} L(E) \geq 0 \), admits an inverse, which is exactly \( k : (L, +\infty) \to (\lambda_1, +\infty) \). If \( L > 0 \), then \( k(p) \equiv k(0) = \lambda_1 \) for any \( p \in [0, L] \). Hence

\[
w^* = \inf_{E > \lambda_1} \frac{E}{L(E)} = \inf_{p > L} \frac{k(p)}{p} = \inf_{p > 0} \frac{k(p)}{p}.
\]

Step 2: Construct a appropriate subsolution to obtain the uniform convergence.

Suppose by contradiction that there exist \( \delta \in (0, 1) \) and a sequence \( \{g_l(t_l, n_l) \subset \mathcal{H}(c) \times \mathbb{R}_+ \times \mathbb{Z} \text{ with } 0 \leq n_l \leq wt_l \text{ and } t_l \to +\infty \text{ such that} \)

\[
u^{g_l}(t_l, n_l; 0, u_0) \leq 1 - \delta.
\]
Without loss of generality, we assume that \( g_l \to g^* \) in \( l^\infty(Z) \). Let \( g^*(n) = (1 - s)g^*(n) + s \inf_{n \in \mathbb{Z}} c(n)/2 \) and \( u^g(t, n; 0, u_0) \) be the solution of
\[
\begin{aligned}
u_t(t, n) - Du(t, n) = g^*(n)u(t, n)(1 - u(t, n)) \quad \text{in} \quad \mathbb{R} \times \mathbb{Z},
\phantom{u(0, \cdot) = u_0.}
u(0, \cdot) = u_0.
\end{aligned}
\]
(4.11)

Then we choose \( N \) large enough such that for \( l \geq N \) and \( s \in (0, 1) \), \( \|g_l - g^*\|_{l^\infty} \leq \inf_{n \in \mathbb{Z}} c(n)/2 \). Hence for \( l \geq N \),
\[
g_l(n) \geq g^*(n) - \|g_l - g^*\|_{l^\infty} \geq g^*(n) - s \inf_{n \in \mathbb{Z}} c(n)/2 \geq g^*(n)
\]
(4.12)
since \( \inf_{n \in \mathbb{Z}} g^*(n) = \inf_{n \in \mathbb{Z}} c(n) \). From this, for \( l \geq N \), we have
\[
u_l^{g^*}(t, n; 0, u_0) - Du^{g^*}(t, n; 0, u_0) = g^*(n)u^{g^*}(t, n; 0, u_0)(1 - u^{g^*}(t, n; 0, u_0))
\]
\[
\leq g_l(n)u^{g^*}(t, n; 0, u_0)(1 - u^{g^*}(t, n; 0, u_0)).
\]

That is to say, \( u^{g^*}(t, n; 0, u_0) \) is a subsolution of (4.10) with \( g \) replaced by \( g_l \) for any \( l \geq N \).

Step 3: End the proof.

As proved in [40, Theorem 2.1], we have
\[
\lim_{t \to +\infty} \inf_{0 \leq n \leq wt} u^{g^*}(t, n; 0, u_0) = 1 \quad \text{for any} \ 0 \leq w < \inf_{p > 0} \frac{\lambda_1(L_{g^*, p})}{p}
\]
[40, Proposition 3.1] also tells us that
\[
\lim_{s \to 0} \inf_{p > 0} \frac{\lambda_1(L_{g^*, p})}{p} = \inf_{p > 0} \frac{\lambda_1(L_{g, p})}{p} = \inf_{E > \lambda_1} \frac{E}{L(E)} = w^*.
\]

Hence for any \( 0 \leq w < w^* \), we can take \( s > 0 \) small such that \( 0 \leq w < \inf_{p > 0} \frac{\lambda_1(L_{g^*, p})}{p} \). Moreover, we have \( \lim_{t \to +\infty} \inf_{0 \leq n \leq wt} u^{g^*}(t, n; 0, u_0) = 1 \). On the other hand, since \( u^{g^*}(t, n; 0, u_0) \) is a subsolution, it follows from Proposition 2.3 that we have \( u^{g^*}(t, n; 0, u_0) \leq u^{g_l}(t, n; 0, u_0) \) for \( l \) large enough. Therefore,
\[
\lim_{t \to +\infty} \inf_{0 \leq n \leq wt} u^{g^*}(t, n; 0, u_0) \leq \limsup_{t \to +\infty} u^{g^*}(t, n; 0, u_0)
\]
\[
\leq \limsup_{t \to +\infty} u^{g_l}(t, n; 0, u_0) \leq 1 - \delta.
\]
That’s impossible. Hence \( \lim_{t \to +\infty} \inf_{0 \leq n \leq wt} u^g(t, n; 0, u_0) = 1 \) exists uniformly in \( g \in \mathcal{H}(c) \).

\[\square\]

Proof of Lemma 4.5. Consider the solution \( u_k(t, n) \) of
\[
\begin{aligned}
u_t(t, n) - Du(t, n) = c(n + k)u(t, n)(1 - u(t, n)), \; (t, n) \in \mathbb{R} \times \mathbb{Z},
u(0, \cdot) = \alpha, \; \text{and} \; u(0, n) = 0 \quad \text{if} \; n \neq 0.
\end{aligned}
\]
Then for any $0 \leq w < w^*$, we have $\lim_{t \to \infty} \inf_{0 \leq n \leq w} u_k(t, n) = 1$ exists uniformly in $k$ by Proposition 4.5. Therefore, the solution $u(t, n; 0, u_k^{(0)})$ of
\[
\begin{cases}
  u_t(t, n) - D u(t, n) = c(n) u(t, n)(1 - u(t, n)), & (t, n) \in \mathbb{R} \times \mathbb{Z}, \\
  u(0, n) = u_k^{(0)}(n), & n \in \mathbb{Z},
\end{cases}
\]
where $u_k^{(0)}(-k) = \alpha$ and $u(0, n) = 0$ if $n \neq -k$, satisfies
\[
\lim_{t \to \infty} \inf_{k \leq n \leq w t - k} u(t, n; 0, u_k^{(0)}) = 1 \text{ for any } w < w^* \text{ uniformly in } k \in \mathbb{N},
\]
since $u(t, n; 0, u_k^{(0)}) = u_k(t, n - k)$ by Theorem 2.1. Then it follows from Proposition 2.3 that
\[
\lim_{t \to \infty} \inf_{n \leq w t} u(t, n; 0, u_k^{(0)}) = 1, \forall 0 \leq w < w^*.
\]

Using Lemma 4.5, we finish the proof of (3) of Theorem 1.1.

**Proposition 4.6.** Let $u$ be a generalized transition front of equation (1.2) and let $N$ be such that (1.4) holds. Then
\[
\lim_{t-s \to +\infty} \inf_{t-s \to +\infty} \frac{N(t) - N(s)}{t - s} \geq w^*.
\]
Particularly, there exists no generalized transition front with average speed $w < w^*$.

**Proof.** First by (1.4) and Proposition 2.3, we can check that
\[
(4.13) \quad \alpha := \inf_{t \in \mathbb{R}, n \leq 0} u(t, n + N(t)) > 0 \quad \text{and} \quad \beta := \sup_{t \in \mathbb{R}, n > 0} u(t, n + N(t)) < 1.
\]
It is clear that $\alpha < 1, \beta > 0$. Assume that, by contradiction, there exist $t_k$ and $s_k$ such that
\[
\lim_{k \to \infty} t_k - s_k = +\infty \quad \text{and} \quad \lim_{k \to \infty} \frac{N(t_k) - N(s_k)}{t_k - s_k} = w < w^*.
\]
Set $v_k(t, n) := u(t + s_k, n + N(s_k))$. It is clear that $v_k(0, n) \geq u_k^{(0)}(n)$, where $u_k^{(0)}(n) = 0$ if $n > 0$, and $u_k^{(0)}(n) = \alpha$ if $n \leq 0$. Thus by Proposition 2.3, we have $v_k(t, n) \geq u(t, n; 0, u_k^{(0)})$ for $t \geq 0$ and $n \in \mathbb{Z}$, which yields
\[
(4.14) \quad u(t_k, N(t_k) + 1) = v_k(t_k - s_k, N(t_k) - N(s_k) + 1) \geq u(t_k - s_k, N(t_k) - N(s_k) + 1; 0, u_k^{(0)}).
\]
For the left-hand side of (4.14), we have $u(t_k, N(t_k) + 1) \leq \beta < 1$ by (4.13). But from Lemma 4.5, the right-hand side of (4.14) converges to 1 as $k \to \infty$ since $\lim_{k \to \infty} \frac{N(t_k) - N(s_k) + 1}{t_k - s_k} = w < w^*$, which is a contradiction. □

In all, we have proved (3) in Theorem 1.1.
4.3. Construction of the critical fronts. At last, to verify (2) of Theorem 1.1, we only need to consider (1.3) with \( g = c \), i.e., (1.2). First we want to construct the critical front with average speed \( w^* \). By critical front we mean that

**Definition 4.1.** We say that an entire solution \( u \) of (1.2) with \( 0 < u < 1 \), is a critical traveling front (to the right) if for all \( (t_0, n_0) \in \mathbb{R} \times \mathbb{Z} \), \( v \) is an entire solution of (1.2) such that \( v(t_0, n_0) = u(t_0, n_0) \) and \( 0 < v < 1 \), then

\[
   u(t_0, n) \geq v(t_0, n) \quad \text{if} \quad n \leq n_0 \quad \text{and} \quad u(t_0, n) \leq v(t_0, n) \quad \text{if} \quad n > n_0.
\]

Before going any further, we introduce some useful lemmas.

**Lemma 4.6.** Let \( u(t, n) \) be an entire solution of (1.2) with \( 0 < u < 1 \). There exists \( \delta \in (0,1) \), such that \( |u(t, n) - u(t, n + 1)| \leq 1 - \delta \) for all \( t,n \in \mathbb{R} \times \mathbb{Z} \).

**Proof.** Suppose that the conclusion fails. Then for any \( k \in \mathbb{Z}_+ \), there exist \( t_k \) and \( n_k \) such that \( |u(t_k, n_k) - u(t_k, n_k + 1)| > 1 - 1/k \). After passing a subsequence, we have \( u(s_k, m_k) - u(s_k, m_k + 1) > 1 - \varepsilon_k \) or \( u(s_k, m_k + 1) - u(s_k, m_k) > 1 - \varepsilon_k \) for some \( s_k, m_k \), and \( \varepsilon_k \to 0 \). We only prove the former case. Note that \( 0 < u(t, n) < 1 \). Then

\[
   u(s_k, m_k) > 1 - \varepsilon_k \quad \text{and} \quad u(s_k, m_k + 1) < \varepsilon_k.
\]

Note also that \( \|u\|_{\infty(\mathbb{Z})} < 4 + \|g\|_{\infty(\mathbb{Z})} \). Then there exists \( S \) which is independent of \( k \) such that \( u(s_k - S, m_k) \geq 3/4 - \varepsilon_k > 1/2 \) for \( k \) large. On the other hand, from Proposition 2.2, there exists a constant \( C(S) \) which only depends on \( S \) such that

\[
   1/2 < u(s_k - S, m_k) \leq C(S)u(s_k, m_k + 1) \leq C(S)\varepsilon_k \to 0 \quad \text{as} \quad k \to 0.
\]

This yields a contradiction. \( \square \)

With Lemma 4.6 at hand, we have the following equivalent definition of generalized transition front.

**Lemma 4.7.** Let \( u(t, n) \) be a solution of (1.2) with \( 0 < u < 1 \) and

\[
   u(t, n) \to 0 \quad \text{as} \quad n \to \infty, \quad u(t, n) \to 1 \quad \text{as} \quad n \to -\infty \quad \text{for any} \quad t \in \mathbb{R}.
\]

Then \( u \) is a generalized transition front if and only if

\[
   \sup_{t \in \mathbb{R}} \text{diam}\{n \in \mathbb{Z} | \varepsilon \leq u(t, n) \leq 1 - \varepsilon\} < \infty \quad \text{for any} \quad \varepsilon \in (0, 1/2).
\]

**Proof.** As we see, (4.16) is satisfied if \( u \) is a generalized transition front. Next we prove that \( u \) is a generalized transition front provided (4.16) holds. Set \( N(t) := \sup\{n | u(t, n) \geq 1/4\} \).

Now we claim that \( 1/4 \leq \inf_{t \in \mathbb{R}} u(t, N(t)) \leq \sup_{t \in \mathbb{R}} u(t, N(t)) < 1 \). In fact, if \( \sup_{t \in \mathbb{R}} u(t, N(t)) = 1 \), then there exists a sequence \( \{t_k\} \subset \mathbb{Z} \) such that \( u(t_k, N(t_k)) \to 1 \). After passing to a subsequence, \( u(t + t_k, n + N(t_k)) \) converges locally uniformly to a function \( v(t, n) \). Then \( v(0, 0) = 1 \), and thus
By Corollary 2.1. On the other hand, the definition of \( N(t) \) gives \( v(0, 1) \leq 1/4 \), which is a contradiction.

For any \( \varepsilon < \varepsilon_0 := \min\{1 - \sup_{t \in \mathbb{R}} u(t, N(t)), 1/4, \delta/2\} \), where \( \delta \) was given in Lemma 4.6, we have

\[
N(t) \in \{ n \in \mathbb{Z} | \varepsilon_0 \leq u(t, n) \leq 1 - \varepsilon_0 \} \subset \{ n \in \mathbb{Z} | \varepsilon \leq u(t, n) \leq 1 - \varepsilon \}.
\]

Denote \( L_\varepsilon := \sup_{t} \text{diam}\{ n \in \mathbb{Z} | \varepsilon \leq u(t, n) \leq 1 - \varepsilon \} \).

If \( n > L_\varepsilon + 1 \) or \( n < -L_\varepsilon - 1 \). Note that \( u(t, n) \to 0 \) as \( n \to \infty \) and \( u(t, n) \to 1 \) as \( n \to -\infty \) for any \( t \in \mathbb{R} \). Combining this with Lemma 4.6, we have \( u(t, n + N(t)) \leq \varepsilon \) for \( n > L_\varepsilon + 1 \), and \( u(t, n + N(t)) \geq 1 - \varepsilon \) for \( n < -L_\varepsilon - 1 \). Therefore, (1.4) holds.

Let us now construct critical front. Fix any \( \theta \in (0, 1) \). For any \( k \in \mathbb{Z}_+ \), we define

\[
H_k(n) = \begin{cases} 
1 & \text{if } n \leq -k \\
0 & \text{if } n > -k.
\end{cases}
\]

Then by Lemma 4.5 and the continuity of \( u(t, n; 0, H_k) \) with respect to \( t \), we can define \( s_k := \min\{ s | u(s, 0; 0, H_k) = \theta \} > 0 \). In particular, \( u(s_k, 0; 0, H_k) = \theta \). Note that by Theorem 2.1, \( u(t, n; s, H_k) = u(t - s, n; 0, H_k) \) for any \( t \geq s \).

Then \( u(0, 0; -s_k, H_k) = \theta \) for any \( k \in \mathbb{Z}_+ \). The idea is to take the limit of some subsequence from \( \{ u(t, n; -s_k, H_k) \} \), and prove the resulting function is exactly the critical traveling front. Moreover, it is exactly a generalized transition front with average speed \( w^* \). Before that, an observation about some important properties of \( s_k \) is given by

**Lemma 4.8.** \( s_k \) is strictly increasing and converges to \( +\infty \).

Proof. We first show that \( s_k \) is strictly increasing. Assume by contradiction, that \( s_k \geq s_{k+1} \) for some \( k \in \mathbb{Z}_+ \). Note that

\[
u(t+s_k-s_{k+1}, n; -s_k-H_k)_{t=-s_k} = H_{k+1} \leq H_k = u(t, n; -s_k, H_k)_{t=-s_k}.
\]

Then by Proposition 2.3, we have

\[
u(t+s_k-s_{k+1}, n; -s_k, H_{k+1}) < u(t, n; -s_k, H_k) \text{ for } t > -s_k.
\]

In particular, \( \theta = u(0, 0; -s_k, H_{k+1}) < u(s_{k+1} - s_k, 0; -s_k, H_k) \). Notice that \( u(-s_k, 0; -s_k, H_k) = 0 \). Then by intermediate value theorem there exists \( -s_k < \tau < s_{k+1} - s_k \leq 0 \) such that

\[
\theta = u(\tau, 0; -s_k, H_k) = u(\tau + s_k, 0; 0, H_k).
\]

Therefore the definition of \( s_k \) gives \( s_k \leq \tau + s_k \). That is impossible since \( \tau < 0 \). Hence \( s_k < s_{k+1} \).
Then, we prove that \( \lim_{k \to \infty} s_k = +\infty \). Suppose by contradiction that after passing to a subsequence, \( \lim_{k \to \infty} s_k = s_\infty < +\infty \). Let \( \phi_{E,k} \) be a solution of
\[
\phi(n+1) + \phi(n-1) - 2\phi(n) + c(n-k)\phi(n) = E\phi(n), \quad n \in \mathbb{Z},
\]
with \( \phi(0) = 1, \lim_{n \to +\infty} \phi(n) = 0 \), where \( E > \lambda_1 \). Then \( \mathbf{u}_k(t,n+k) : = \min\{1, \phi_{E,k}(n+k)e^{Et}\} \) is a supersolution of
\[
u_t(n) - u(n+1) - u(n-1) + 2u(n) = c(n)u(n)(1-u(n)), \quad (t,n) \in \mathbb{R} \times \mathbb{Z}.
\]
Note that for any \( t \leq 0 \),
\[
\mathbf{u}_k(t,k) = \min\{1, \phi_{E,k}(k)e^{Et}\} \leq \phi_{E,k}(k) \leq Ce^{-\delta k},
\]
for some constant \( C \) only depending on \( E, \lambda_1 \) and \( \|g\|_{l∞} \), and the last inequality follows from Lemma 3.1. Then we can take \( K \) large such that \( \mathbf{u}_K(t,K) \leq \theta/2 \) for any \( t < 0 \). Moreover, combining the Proposition 3.4, there exists \( K_1 \) such that for any \( t \geq -s_\infty \) and \( n \in \mathbb{Z} \),
\[
H_{K_1}(n) \leq \mathbf{u}_K(-s_\infty, n + K) \leq \mathbf{u}_K(t, n + K).
\]
Then \( u(-s_{K_1}, n; -s_{K_1}, H_{K_1}) \leq \mathbf{u}_K(-s_\infty, n + K) \leq \mathbf{u}_K(-s_{K_1}, n + K) \). Thus for any \( t \geq -s_{K_1} \) and \( n \in \mathbb{Z} \), it follows from Proposition 2.3
\[
u_t(n; -s_{K_1}, H_{K_1}) \leq \mathbf{u}_K(t, n + K).
\]
In particular, we have \( \theta = u(0,0; -s_{K_1}, H_{K_1}) \leq \mathbf{u}_K(0, K) \leq \theta/2 \), which is a contradiction. Then we complete the proof. \( \square \)

As \( s_k \to \infty \), there exists a subsequence of \( \{u(t,n; -s_k, H_k)\} \) such that it converges to some entire solution \( u(t,n) \). Moreover, \( u(t,n) \) is "steeper" than any other entire solution in the following sense (see [21] for continuous case).

**Lemma 4.9.** Let \( u(t,n) \) be a limit of some subsequences of \( u(t,n; -s_k, H_k) \).
Assume that \( v \) is an entire solution of (1.2) with \( v(t,n) \in (0,1) \) on \( \mathbb{R} \times \mathbb{Z} \).
Then for any \( t \in \mathbb{R} \), there exists \( n_t \in \mathbb{Z} \cup \{±\infty\} \) such that
\[
u(t,n) \geq v(t,n) \text{ if } n \leq n_t, \text{ and } u(t,n) \leq v(t,n) \text{ if } n > n_t.
\]

First we need the following proposition whose proof can be found in [25, Lemma 4]:

**Proposition 4.7** ([25]). Consider the solution \( u_1(t,n) \) and \( u_2(t,n) \) of (1.2).
Denote \( w(t,n) = u_1(t,n) - u_2(t,n) \). If \( w(t_0,n_0) > 0 \) for some \( (t_0,n_0) \in \mathbb{R} \times \mathbb{Z} \),
\( w(t_0,n) > 0 \) for \( n < n_0, w(t_0,n) < 0 \) for \( n > n_0 \), then the following hold:

1. For any \( t \geq t_0 \), if \( w(t,n) > 0 \) for some \( n \in \mathbb{Z} \), then \( w(t,m) > 0 \) for any \( m < n \).
2. For any \( t \geq t_0 \), if \( w(t,n) < 0 \) for some \( n \in \mathbb{Z} \), then \( w(t,m) < 0 \) for any \( m > n \).
Proof of Lemma 4.9. First we prove that if \( u(t_0, n_0) < v(t_0, n_0) \) for some \((t_0, n_0) \in \mathbb{R} \times \mathbb{Z} \), then \( u(t_0, m) \leq v(t_0, m) \) for any \( m > n_0 \).

Suppose by contradiction that \( u(t_0, m) > v(t_0, m) \) for some \( m > n_0 \). Then we have

\[
(4.17) \quad u(t_0, n_0; -s_k, H_k) < v(t_0, n_0), \quad \text{and} \quad u(t_0, m; -s_k, H_k) > v(t_0, m)
\]

for some \( k \) large enough. It is clear to check that

\[
\begin{cases}
1 = u(-s_k, n; -s_k, H_k) > v(-s_k, n), \; n \leq -k, \\
0 = u(-s_k, n; -s_k, H_k) < v(-s_k, n), \; n > -k.
\end{cases}
\]

Applying Proposition 4.7 with \( w(t, n) := u(t, n; -s_k, H_k) - v(t, n) \), we can conclude that for \( t \geq -s_k \) and \( n \in \mathbb{Z} \) where \( w(t, n) < 0 \), one has \( w(t, m) < 0 \) for any \( m > n \). This contradicts (4.17).

Similarly, if \( u(t_0, n_0) > v(t_0, n_0) \) for some \((t_0, n_0) \in \mathbb{R} \times \mathbb{Z} \), then \( u(t_0, m) \geq v(t_0, m) \) for any \( m < n_0 \). Therefore the existence of \( n \) follows directly. \( \square \)

Now we begin to construct an entire solution by taking the limit of \( \{u(t, n; -s_k, H_k)\} \):

Lemma 4.10. The limit \( u(t, n) := \lim_{k \to \infty} u(t, n; -s_k, H_k) \) exists locally uniformly in \((t, n) \in \mathbb{R} \times \mathbb{Z} \). Moreover, \( u \) is an entire solution of (1.2) with \( u(0, 0) = 0 \).

Proof. Define \( w(t, n) := u(t, n; -s_k, H_k) - u(t, n; -s_{k+1}, H_{k+1}) \) for \( t \geq -s_k \) and \( n \in \mathbb{Z} \). Then for \( t > -s_k \), \( n \in \mathbb{Z} \), \( w \) satisfies

\[
w(t, n) - w(t, n + 1) - w(t, n - 1) + 2w(t, n) = c(n)f_k(t, n)w(t, n),
\]

where \( f_k(t, n) = 1 - u(t, n; -s_k, H_k) - u(t, n; -s_{k+1}, H_{k+1}) \). Now we prove this lemma in the following two steps.

Step 1: Show that \( w(0, n) \geq 0 \) for \( n < 0 \), and \( w(0, n) \leq 0 \) for \( n > 0 \).

Assume, by contradiction, that there is \( n_1 < 0 \) such that \( w(0, n_1) < 0 \) (the proof is similar in the case where there is \( n_1 > 0 \) such that \( w(0, n_1) > 0 \)). Note that

\[
\begin{cases}
\forall n \leq -k, w(-s_k, n) = 1 - u(-s_k, n; -s_{k+1}, H_{k+1}) > 0, \\
\forall n > -k, w(-s_k, n) = -u(-s_k, n; -s_{k+1}, H_{k+1}) < 0.
\end{cases}
\]

We can deduce from Proposition 4.7 that if \( w(t, n) < 0 \) for some \( t > -s_k \) and \( n \in \mathbb{Z} \), then \( w(t, m) < 0 \) for any \( m > n \) (similarly, if \( w(t, n) > 0 \) for some \( t > -s_k \) and \( n \in \mathbb{Z} \), then \( w(t, m) > 0 \) for \( m < n \)). Therefore, \( w(0, m) < 0 \) for \( m > n_1 \) contradicts with \( w(0, 0) = 0 \).

Step 2: Show that \( u_1(t, n) = u_2(t, n) \), where \( u_i(t, n) \) \((i = 1, 2)\) are any two limits of different subsequences of \( u(t, n; -s_k, H_k) \).

From Step 1, we have \( u(0, n; -s_k, H_k) \geq u(0, n; -s_{k+1}, H_{k+1}) \) for \( n < 0 \)

and \( u(0, n; -s_k, H_k) \leq u(0, n; -s_{k+1}, H_{k+1}) \) for \( n > 0 \). That is to say, the sequence \( \{u(0, n; -s_k, H_k)\} \) is nonincreasing if \( n < 0 \) and is nondecreasing if \( n > 0 \). Thus the limit \( u_0(n) := \lim_{k \to \infty} u(0, n; -s_k, H_k) \) is well defined for
n ∈ Z. Hence \( u_1(0, n) = u_2(0, n) \), which yields that \( u_1(t, n) = u_2(t, n) \) for \( t > 0 \) and \( n \in \mathbb{Z} \) by Proposition 2.3.

Now we prove \( u_1(t, n) = u_2(t, n) \) for all \( (t, n) \in \mathbb{R} \times \mathbb{Z} \). From Lemma 4.9, there exist \( n_i^1 \in \mathbb{Z} \cup \{ \pm \infty \} \), \((i = 1, 2)\) such that
\[
\begin{align*}
\quad u_1(t, n) &\geq u_2(t, n) \quad \text{if } n \leq n_i^1, \quad \text{and} \quad u_1(t, n) \leq u_2(t, n) \quad \text{if } n > n_i^1, \\
\quad u_2(t, n) &\geq u_1(t, n) \quad \text{if } n \leq n_i^2, \quad \text{and} \quad u_2(t, n) \leq u_1(t, n) \quad \text{if } n > n_i^2.
\end{align*}
\]

It’s clear that \( u_1(t, n) = u_2(t, n) \) if \( n_i^1 = n_i^2 \) for any \( t < 0 \). Now we assume, without loss of generality, that \( n_1^1 < n_2^2 \) for some \( \tau < 0 \). It follows directly that
\[
(4.18) \quad \begin{cases} 
\quad u_1(\tau, n) = u_2(\tau, n) \quad \text{if } n \leq n_1^1 \text{ or } n > n_2^2, \\
\quad u_1(\tau, n) \leq u_2(\tau, n) \quad \text{if } n_1^1 < n \leq n_2^2.
\end{cases}
\]

It’s still needed to prove \( u_1(\tau, n) = u_2(\tau, n) \) for \( n_1^1 < n \leq n_2^2 \). If not, then \( u_1(t, n) < u_2(t, n) \) for \( t > \tau \) and \( n \in \mathbb{Z} \) by Proposition 2.3. That is impossible since \( u_1(t, n) = u_2(t, n) \) for any \( t > 0 \) and \( n \in \mathbb{Z} \). Hence \( u_1(t, n) = u_2(t, n) \) in \( \mathbb{R} \times \mathbb{Z} \) with \( u(0, 0) = \theta \) which follows from \( u(0, 0; -s_k, H_k) = \theta \) for any \( k \in \mathbb{Z}_+ \).

Now we prove (2) of Theorem 1.1 by verifying that the above solution is exactly a time increasing generalized transition front with average speed \( w^* \).

**Theorem 4.3.** \( u(t, n) \) in Lemma 4.10 is a critical traveling front and a time increasing generalized transition front with average speed \( w^* \).

**Proof.** We prove that \( u(t, n) \) is a critical traveling front first. Assume that \( v \) is an entire solution of (1.2) such that \( v(t_0, n_0) = u(t_0, n_0) \) and \( 0 < v < 1 \). Then by Lemma 4.9, we have
\[
u(t_0, n) \geq v(t_0, n) \quad \text{if } n \leq n_0 \text{ and } u(t_0, n) \leq v(t_0, n) \quad \text{if } n > n_0.\]

That is to say, \( u(t, n) \) is a critical traveling front.

Let \( u_w(t, n) \) be the almost periodic traveling front \( u(t, n; c) \) obtained in Theorem 4.2 with average wave speed \( w > w^* \). Thus by Proposition 2.6, \( u_w(t, n) \) is also a generalized transition front with average speed \( w \). Then a similar argument to that of [44, Theorem 3.1] yields that \( \sup_{t \in \mathbb{R}} \text{diam}\{n \in \mathbb{Z} \mid \varepsilon \leq u(t, n) \leq 1 - \varepsilon \} < \infty \) for any \( \varepsilon \in (0, 1/2) \). Therefore, it follows from Lemma 4.7 that \( u(t, n) \) is a generalized transition front.

It remains to show that \( u(t, n) \) possesses an average speed \( w^* \). Set \( N(t) := \sup\{n \mid u(t, n) \geq 1/4\} \) and \( M_w(t) := \sup\{n \mid u_w(t, n) \geq 1/4\} \). Then by similar arguments to that of [44, Theorem 3.6], we can find \( L \) such that, for any \( s \in \mathbb{R} \), there exists \( \bar{s} \in \mathbb{R} \) satisfying
\[
N(s + \tau) - N(s) \leq M_w(\bar{s} + \tau) - M_w(\bar{s}) + L, \quad \forall \tau > 0.
\]

Then
\[
\lim_{\tau \to +\infty} \sup_{\bar{s}} \frac{N(s + \tau) - N(s)}{\tau} \leq \lim_{\tau \to +\infty} \sup_{\bar{s}} \frac{M_w(\bar{s} + \tau) - M_w(\bar{s}) + L}{\tau} = w,
\]
for all $w^* < w < w_1$, which gives $\limsup_{t-s \to +\infty} \frac{N(t)-N(s)}{t-s} \leq w^*$. On the other hand, since $u(t, n)$ is a generalized transition front and $N(t)$ satisfies (1.4), we have $\liminf_{t-s \to +\infty} \frac{N(t)-N(s)}{t-s} \geq w^*$ by Proposition 4.6. Hence $w^*$ is the average speed of $u$.

At last, we need to prove $u(t, n)$ is a time increasing traveling front. Otherwise, there exist some $t', \tau, n'$ such that $u(t' + \tau, n') < u(t', n')$. Since $\alpha := \inf u(0, n) \in (0, 1)$, then by the similar argument in the proof of Proposition 4.6, we can deduce that $u(t, n) \geq u(t, n; 0, u^{(0)})$ with $u^{(0)}(n) = 0$ for $n > 0$ and $u^{(0)}(n) = \alpha$ for $n \leq 0$. Thus, we have $\limsup_{t \to \infty} u(t, n) = 1$.

Combining this with the intermediate value theorem, there exists $T_0 > \tau$ such that $u(t' + T_0, n') = u(t', n')$.

As $u(t, n)$ is a critical traveling front, so is $v(t, n) := u(t + T_0, n)$. Combining $u(t', n') = v(t', n')$ and the definition of critical traveling front, one has

$$u(t', n) \geq v(t', n') \text{ if } n < n' \text{ and } u(t', n) \leq v(t', n) \text{ if } n > n',$$

$$v(t', n) \geq u(t', n) \text{ if } n < n' \text{ and } v(t', n) \geq u(t', n) \text{ if } n > n'.$$

Hence $u(t', n) = v(t', n)$ for $n \in \mathbb{Z}$. Now we get the conclusion that $u(t, n) = v(t, n) = u(t + T_0, n)$ for any $(t, n) \in \mathbb{R} \times \mathbb{Z}$ by the similar arguments in Step 2 of Lemma 4.10. This means $u$ is a time periodic transition front. Then the maximum 1 of $u(t, 0)$ can be attained since $\limsup_{t \in \mathbb{R}} u(t, 0) = 1$. This contradicts Corollary 2.1. Then the proof is complete.

Summing up Theorem 4.2, Proposition 4.6 and Theorem 4.3, we have proved Theorem 1.1.

5. Positive almost periodic solution of the Discrete Schrödinger equation

5.1. Criterion of existence of positive almost periodic solution.

Theorem 1.1 is an incomplete answer since we cannot determine whether (1.2) has an almost periodic traveling front with average wave speed $w \geq w_1$, even the generalized transition front. However, in the case $w = \infty$, we can get a complete answer by Theorem 1.1. In this section, we will provide some conditions on $c$ in (1.2) to guarantee $w = \infty$. The idea is to apply Proposition 3.5 and to use KAM method for constructing the positive almost periodic solution of $(\mathcal{L}_{V, \alpha, \theta} u)(n) = u(n+1) + u(n-1) - 2u(n) + V(n\alpha + \theta)u(n) = \lambda_1 u(n)$ with $V, \alpha, \theta$ to be specified and $\lambda_1 = \max \Sigma(\mathcal{L}_{V, \alpha, \theta})$.

We will first provide a simple criterion, which says that if the Schrödinger cocycle can be reduced to constant parabolic cocycle, and the conjugacy is close to the identity, then the corresponding Schrödinger equation has a positive almost-periodic solution. Recall that an almost-periodic cocycle $(\alpha, A)$ is said to be $C^s$ reducible where $0 \leq s \leq \infty$, if there exist $B(\cdot) \in$
$C^s(\mathbb{T}^d, \text{SL}(2, \mathbb{R}))$ and a constant matrix $\tilde{A}$ such that

$$B(\theta + \alpha)^{-1}A(\theta)B(\theta) = \tilde{A}.$$  

Denote the norm in $C^s(\mathbb{T}^d, \text{SL}(2, \mathbb{R}))$ as $\|F\|_s := \sup_{|l| \leq s, \theta \in \mathbb{T}^d} \|\partial^l F(\theta)\|$. Then we have the following:

**Lemma 5.1.** Suppose that there exists $B \in C^0(\mathbb{T}^d, \text{SL}(2, \mathbb{R}))$ with $\|B - \text{id}\|_C \leq \frac{1}{100}$ such that

\begin{equation}
B^{-1}(\theta + \alpha)S_E^V(\theta)B(\theta) = \tilde{A},
\end{equation}

where $\tilde{A} \in \text{SL}(2, \mathbb{R})$ is a parabolic matrix (i.e. the trace $|\text{tr}(\tilde{A})| = 2$). Then

\begin{equation}
(L_{V,\alpha,\theta}u)(n) = u(n+1) + u(n-1) - 2u(n) + V(\theta + n\alpha)u(n) = Eu(n)
\end{equation}

has an almost-periodic positive solution.

**Proof.** Without loss of generality, we assume $\text{tr}(\tilde{A}) = 2$. Then one can find some $R_\eta := \begin{pmatrix} \cos \eta & -\sin \eta \\ \sin \eta & \cos \eta \end{pmatrix} \in \text{SO}(2, \mathbb{R})$ with $\eta \in [0, 2\pi)$ such that

\begin{equation}
R_\eta \begin{pmatrix} 1 \\ 0 \\ \eta \\ 1 \end{pmatrix} R^{-\eta} = \tilde{A}.
\end{equation}

Denote $B(\theta) := \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$ and $C(\theta) := R_\eta B(\theta) = \begin{pmatrix} C_{11}(\theta) & C_{12}(\theta) \\ C_{21}(\theta) & C_{22}(\theta) \end{pmatrix}$. It directly follows that

\begin{equation}
C^{-1}(\theta + \alpha)S_E^V(\theta)C(\theta) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\end{equation}

Now we can write (5.4) as

$$\begin{pmatrix} E + 2 - V(\theta) & -1 \\ -1 & 1 \end{pmatrix} C(\theta) = C(\theta + \alpha) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

which gives us

\begin{align*}
(E + 2 - V(\theta))C_{11}(\theta) - C_{21}(\theta) &= C_{11}(\theta + \alpha), \\
C_{11}(\theta) &= C_{21}(\theta + \alpha),
\end{align*}

that is

$$C_{11}(\theta + \alpha) + C_{11}(\theta - \alpha) - 2C_{11}(\theta) + V(\theta)C_{11}(\theta) = EC_{11}(\theta).$$

Hence $u(n) = C_{11}(n\alpha + \theta) = C_{21}((n+1)\alpha + \theta)$ is an almost periodci solution.

Now we distinguish this into the following two cases:

**Case 1:** If $\cos^2 \eta > \frac{1}{2}$, $\sin^2 \eta < \frac{1}{2}$. Moreover if $\cos \eta > 0$, then by the assumption $\|B - \text{id}\|_{C^s} \leq \frac{1}{100}$,

$$C_{11}(\theta) = \cos \eta B_{11}(\theta) - \sin \eta B_{21}(\theta) > \frac{\sqrt{2}}{4}$$

for any $\theta \in \mathbb{T}^d$. 

Moreover, we have the estimates verify that $C_{\cos \eta} < C_{\sin \eta}$. This implies $C_{21}(\eta) = \cos \eta B_{21}(\eta) + \sin \eta B_{11}(\eta) \geq \frac{\sqrt{2}}{2}$ for any $\theta \in \mathbb{T}^d$. Hence $C_{21}(\eta) = -C_{21}(\eta)$ is an positive almost periodic solution; If $\sin \eta < 0$, $-C_{21}(\eta)$ is an positive almost periodic solution of (5.2).

In all, the proof is complete. □

5.2. Analytic quasi-periodic potential. Motivated by Lemma 5.1, to prove the existence of positive almost periodic solution of (5.2), we only need to prove that the corresponding Schrödinger cocycle is reducible to a parabolic constant cocycle:

$$e^{-Y(\theta+\alpha)}S^Y_E(\theta)e^{Y(\theta)} = \tilde{A}.$$  

Moreover, the conjugation is close to constant.

First we state the reducibility result for analytic quasi-periodic potential. To prove this, we first need a non-resonance cancelation lemma. The result will be the basis of our proof, and we will also use this when we deal with analytic almost periodic potentials.

Let $B$ be a $\text{sl}(2, \mathbb{R})$ valued Banach algebra. Assume that for any given $\eta > 0$, $\alpha \in \mathbb{T}^d$ where $d \in \mathbb{N} \cup \{\infty\}$ and $A \in \text{SL}(2, \mathbb{R})$, we have a decomposition of the Banach space $B$ into non-resonant spaces and resonant spaces, i.e. $B = B^{\text{nre}}(\eta) \oplus B^{\text{re}}(\eta)$. Here $B^{\text{nre}}(\eta)$ is defined in the following way: for any $Y \in B^{\text{nre}}(\eta)$, we have

$$A^{-1}Y(\theta + \alpha)A \in B^{\text{nre}}(\eta), \quad |A^{-1}Y(\theta + \alpha)A - Y(\theta)| \geq \eta|Y(\theta)|,$$

where $|\cdot|$ is the norm of the Banach space $B$.

Once we have this, we have the following:

**Lemma 5.2.** Assume that $A \in \text{SL}(2, \mathbb{R})$, $\epsilon \leq (4\|A\|)^{-4}$ and $\eta \geq 13\|A\|^2$. For any $F \in B$ with $|F| \leq \epsilon$, there exist $Y \in B$ and $F^{\text{re}} \in B^{\text{re}}(\eta)$ such that

$$e^{-Y(\theta+\alpha)}Ae^{F(\theta)}e^{Y(\theta)} = Ae^{F^{\text{re}}(\theta)}.$$  

Moreover, we have the estimates $|Y| \leq \epsilon^2$ and $|F^{\text{re}}| \leq 2\epsilon$.

**Remark 5.1.** The proof of the lemma for $B := C^\omega_r(\mathbb{T}^d, \text{su}(1,1))$ with $d \in \mathbb{N}$ could be found in [17, Lemma 3.1], and we can easily see that the proof works for any other Banach algebra.

In our application, we will set $B := C^\omega_r(\mathbb{T}^d, \text{sl}(2, \mathbb{R}))$ where $d \in \mathbb{N} \cup \{\infty\}, r > 0$ (the definition of $C^\omega_r(\mathbb{T}^\infty, \text{sl}(2, \mathbb{R}))$ will be introduced later). Define the norm in $B$ as $|F| := \sup_{|\alpha| \leq r} |F(\alpha)|$. The non-resonant space $B^{\text{nre}}$ will take the truncating operator $T_K$ on $B$: For any $K > 0$, we define

$$T_K F(\theta) = \sum_{k \in \mathbb{Z}^d, |k| \leq K} \hat{F}(k)e^{i(k,\theta)}.
and define $\mathcal{R}_K$ as

$$\mathcal{R}_K F(\theta) = \sum_{k \in \mathbb{Z}^d, |k| \geq K} \hat{F}(k)e^{i(k,\theta)}.$$  

Obviously, $\mathcal{T}_K F + \mathcal{R}_K F = F$. Now as a direct application of Lemma 5.2, we have the following:

**Proposition 5.1.** Let $\alpha \in \text{DC}_d(\gamma, \tau)$, $r, \delta, \gamma > 0$, $\tau > d \geq 1$. Suppose that $A \in \text{SL}(2, \mathbb{R})$, $F' \in C_{\rho}^\omega(\mathbb{T}^d, \text{sl}(2, \mathbb{R}))$. For any $r' \in (0, r)$, there exists $c = c(\gamma, \tau, d)$ such that if $|\text{rot}(\alpha, A)| \leq 2\|A\|e^2$, and

$$|F|_r \leq \epsilon < \frac{c(r - r')^{6(1+\delta)\tau}}{\|A\|^6},$$

then there exist $Y, F' \in C_{\rho}^\omega(\mathbb{T}^d, \text{sl}(2, \mathbb{R}))$ and $A' \in \text{SL}(2, \mathbb{R})$ such that

$$e^{-Y(\theta + \alpha)}Ae^{F'(\theta)}e^{Y(\theta)} = A'e^{F'(\theta)}.$$

Moreover, we have the following estimates:

$$|Y|_{r'} \leq \epsilon^2, \quad |F'|_{r'} \leq 4\epsilon^2, \quad \|A - A'\| \leq 2\epsilon\|A\|.$$

**Proof.** We only need to apply the non-resonance cancelation lemma (Lemma 5.2). In this case, we will define

$$\Lambda_K = \{f \in C_{\rho}^\omega(\mathbb{T}^d, \text{sl}(2, \mathbb{R})) \mid f(\theta) = \sum_{k \in \mathbb{Z}^d, \theta \leq |k| \leq K} \hat{f}(k)e^{i(k,\theta)}\},$$

where $K = \frac{2}{r - r'}|\ln \epsilon|$, and prove that for any $Y \in \Lambda_K$, the operator

$$Y \to A^{-1}Y(\theta + \alpha)A - Y(\theta)$$

has a bounded inverse.

Thus we only need to consider the equation

$$A^{-1}Y(\theta + \alpha)A - Y(\theta) = \mathcal{T}_K G(\theta) - \hat{G}(0).$$

Without loss of generality, we assume that $A = \begin{pmatrix} e^{i\rho} & p \\ 0 & e^{-i\rho} \end{pmatrix}$, where $e^{\pm i\rho}$ are the two eigenvalues of $A$, $p \in \mathbb{R}$, and write

$$Y = \begin{pmatrix} Y_{11} & Y_{12} \\ Y_{21} & -Y_{11} \end{pmatrix}, \quad G = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & -G_{11} \end{pmatrix}.$$

Taking the Fourier transformation for the above equation and comparing the Fourier coefficients, we can get

$$\begin{cases}
\hat{Y}_{11}(k) = \frac{\hat{G}_{11}(k) + pe^{i(\rho + (k,\alpha))}\hat{Y}_{21}(k)}{e^{i(k,\alpha)} - 1}, \\
\hat{Y}_{12}(k) = \frac{\hat{G}_{12}(k) + pe^{i(k,\alpha)}\hat{Y}_{21}(k) - 2pe^{i((k,\alpha) - \rho)}\hat{Y}_{11}(k)}{e^{i(k,\alpha) - 2\rho} - 1}, \\
\hat{Y}_{21}(k) = \frac{\hat{G}_{21}(k)}{e^{i(k,\alpha) + 2\rho} - 1}.
\end{cases}$$
Note for any $k \in \mathbb{Z}^d$ with $0 < |k| \leq K$, if $\epsilon$ satisfies (5.5), we have
\[
|\langle k, \alpha \rangle| \geq \frac{\gamma}{|k|^r} \geq \frac{\gamma}{|K|^r} \geq \frac{C}{\ln \epsilon^r} \geq e^{\frac{1}{\epsilon^{2r+\delta}}},
\]
\[
|2\text{rot}(\alpha, A) - \langle k, \alpha \rangle| \geq \frac{\gamma}{|K|^r} - 4 \|A\|\epsilon^\frac{2}{3} \geq e^{\frac{1}{\epsilon^{2r+\delta}}},
\]
which implies that
\[
|Y(\theta)|_r \leq \epsilon^{-\frac{1}{2r+\delta}} |\mathcal{T}_K G(\theta)|_r,
\]
and then $\Lambda_K \subset \mathcal{B}_r^{\text{pre}}(e^{\frac{1}{2r+\delta}})$.

Since $\epsilon^{\frac{1}{2r+\delta}} \geq 13 \|A\|^2 e^{\frac{1}{2}}$, it follows from Lemma 5.2 there exist $Y, F^{\text{pre}} \in C_\epsilon^\omega(T^d, \text{sl}(2, \mathbb{R}))$ such that
\[
e^{Y(\theta+\alpha)} A e^{F(\theta)} e^{-Y(\theta)} = A e^{F^{\text{pre}}(\theta)},
\]
where $\mathcal{T}_K F^{\text{pre}} = \hat{F}^{\text{pre}}(0)$, $\mathcal{R}_K F^{\text{pre}} = \sum_{|k| > K} \hat{F}^{\text{pre}}(k) e^{i(k, \theta)}$. Moreover, we have the estimates $|Y|_r \leq \epsilon^\frac{1}{2r}, |F^{\text{pre}}|_r \leq 2\epsilon$. Consequently, for any $r' \in (0, r)$, we have
\[
|\mathcal{R}_K F^{\text{pre}}(\theta)|_{r'} = \sum_{|k| > K} \hat{F}^{\text{pre}}(k) e^{i(k, \theta)} \leq 2\epsilon e^{-K(r-r')} K^d \leq 2\epsilon^2.
\]
Furthermore, we can compute that
\[
e^{F^{\text{pre}}(0) + \mathcal{R}_K F^{\text{pre}}(\theta)} = e^{\hat{F}^{\text{pre}}(0)} (\text{id} + e^{-\hat{F}^{\text{pre}}(0)} O(\mathcal{R}_K F^{\text{pre}}(\theta))) = e^{\hat{F}^{\text{pre}}(0)} e^{F'(\theta)},
\]
with the estimate
\[
|F'(\theta)|_{r'} \leq 2|\mathcal{R}_K F^{\text{pre}}|_{r'} \leq 4\epsilon^2.
\]
Finally, if we denote $A' = A e^{\hat{F}^{\text{pre}}(0)}$, then we get
\[
\|A' - A\| \leq 2\|A\| \|\text{id} - e^{\hat{F}^{\text{pre}}(0)}\| \leq 2\|A\|\epsilon.
\]

5.3. Finitely differentiable quasi-periodic potential. Now we want to get the reducibility result for finitely differentiable case. Note that for any $f \in C^\alpha(T^d, \text{sl}(2, \mathbb{R}))$, by [60, Lemma 2.1], there exist an analytic sequence $\{f_j\}_{j \geq 1}$, $f_j \in C_\epsilon^\omega(T^d, \text{sl}(2, \mathbb{R}))$ and a universal constant $C' > 1$ such that
\[
\|f_j - f\|_s \to 0, \text{ if } j \to +\infty,
\]
\[
|f_j|_\frac{1}{2} \leq C'\|f\|_s,
\]
\[
|f_{j+1} - f_j|_\frac{1}{2+j} \leq C' \left(\frac{1}{j}\right)^s \|f\|_s.
\]

The basic idea is we approximate a finitely differentiable cocycle by an analytic cocycle. If the analytic cocycle is reducible, then the finitely differentiable cocycle is also reducible. In our case, we will set
\[
l_1 = M > \max \left\{ (8\|A\|)^2, 4\epsilon^{\frac{1}{2}} \right\}, \quad l_j = [M(1+\delta)^{j-1}], \quad j \geq 2.
\]
If we assume
\[ C'\|F\|_s \leq \frac{c}{\|A\|^6 \|M\|^{1+\frac{1}{2}}} = \frac{c}{\|A\|^6 \|M\|^{1+\frac{1}{2}}} , \]
then by (5.6) we have
\[ |F_{l_{k+1}} - F_{l_k}|_{l_{k+1}} \leq \frac{c}{\|A\|^6 \|M\|^{1+\frac{1}{2}},} \]
\[ |F_{l_k}|_{l_k} \leq \frac{c}{\|A\|^6 \|M\|^{1+\frac{1}{2}.}} \]

Consequently,
\[ \|F - F_l\|_0 \leq \sum_{m=k}^{\infty} |F_{l_m} - F_{l_{m+1}}|_{l_{m+1}} \leq \frac{c}{\|A\|^6 \|M\|^{1+\frac{1}{2}} s_{l_k+1}^{2\frac{1}{2}}}. \]

We also define
\[ \epsilon_0(r, r') = \frac{c(r - r')^{6\tau(1+\delta)}}{\|A\|^6}. \]

Then for any \( s > 6\tau(1+\delta)^3 + 1 \), where \( 0 < \delta < 1 \), we can compute that for any \( m \geq 2^{\frac{1}{2}} \),
\[ \epsilon_m := \frac{c}{\|A\|^6 m^{1+\frac{1}{2}}} \leq \epsilon_0(\frac{1}{m}, \frac{1}{m^{1+\delta}}). \]

With these parameters, we have the following:

**Corollary 5.1.** Let \( \alpha \in DC_d(\gamma, \tau), \quad \gamma > 0, \quad \tau > d, \quad A \in SL(2, \mathbb{R}), \quad F \in C^s(\mathbb{T}^d, sl(2, \mathbb{R})) \) with \( s \geq 6\tau(1+\delta)^3 + 1, \) where \( 0 < \delta < 1 \). If \( \text{rot}(\alpha, Ae^F) = 0 \), and
\[ \|F\|_s \leq \epsilon \leq \frac{c}{C''\|A\|^6 \|M\|^{1+\frac{1}{2}}}, \]
then there exist \( \tilde{A} \in SL(2, \mathbb{R}), \quad Y \in C(\mathbb{T}^d, sl(2, \mathbb{R})) \) with \( \|Y\|_0 \leq 2\epsilon^{\frac{1}{2}} \) such that
\[ e^{-Y(\theta + \alpha)} Ae^{F(\theta)} e^{Y(\theta)} = \tilde{A}. \]

**Proof.** To prove this, we only need to show inductively that there exist \( Y_{l_k}, \quad F'_{l_k} \in C^\omega_{l_{k+1}}(\mathbb{T}^d, sl(2, \mathbb{R})) \) and \( A_{l_k} \in SL(2, \mathbb{R}) \) such that
\[ e^{-Y_{l_k}(\theta + \alpha)} Ae^{F_{l_k}(\theta)} e^{Y_{l_k}(\theta)} = A_{l_k} e^{F'_{l_k}(\theta)}, \]
with the estimates
\[ |Y_{l_k}|_{l_{k+1}} \leq \sum_{i=1}^{k} \epsilon_{l_i}^{\frac{1}{2}}, \quad |F'_{l_k}|_{l_{k+1}} \leq \epsilon_{l_k}^{\frac{3}{2}}, \quad \|A_{l_k} - A\| \leq 2\|A||\epsilon_{l_k}. \]
Once this holds, \( \|Y_{k}\|_0 \leq |Y_{l}|_{\frac{1}{l_{k+1}}} \leq 2\epsilon \frac{1}{2}, \) and as a consequence of (5.12),

\[
e^{-Y_{k}(\theta+\alpha)} Ae^{F(\theta)} e^{Y_{k}(\theta)}
\]

(5.13) \( = A_{l_{k}} e^{F'_{l_{k}}(\theta)} + e^{-Y_{k}(\theta+\alpha)}(Ae^{F(\theta)} - Ae^{F'_{l_{k}}(\theta)})e^{Y_{k}(\theta)} \)

\( = A_{l_{k}} \tilde{G}_{l_{k}}. \)

By (5.10), we have

\[
\|\tilde{G}_{l_{k}} - \text{id}\|_0 \leq \|F'_{l_{k}}\|_0 + \|A_{l_{k}}^{-1}\| \|e^{-Y_{k}(\theta+\alpha)}(Ae^{F(\theta)} - Ae^{F'_{l_{k}}(\theta)})e^{Y_{k}(\theta)}\|_0 \leq 4\epsilon_{l_{k}}^{2} + 2||A||^{2} \times \frac{c}{||A||^{0} M_{\frac{3}{2}+\delta_{k}+3}} \leq \epsilon_{l_{k+1}},
\]

Taking limits of (5.13), we then have the desired results.

Now let’s finish the iteration.

**First step:** First by our assumption (5.11), and then by (5.9) we have

\[|F_{l_1}|_{l_1} \leq \epsilon_{l_1} \leq \epsilon_{0}(\frac{1}{l_1}, \frac{1}{l_2}),\]

and by Lemma 2.3, we have

\[|\text{rot}(\alpha, A)| = |\text{rot}(\alpha, A_{l_1}) - \text{rot}(\alpha, A)| \leq 2\|A||\|F_{l_1}\|_{0}^{\frac{1}{2}} \leq 2\|A||\epsilon_{l_1}^{\frac{1}{2}}.\]

It follows from Proposition 5.1 that there exist \( Y_{l_{1}}, F'_{l_{1}} \in C_{\frac{2}{l_1}}^{\omega} (\mathbb{T}^{d}, \text{sl}(2, \mathbb{R})) \), \( A_{l_{1}} \in \text{SL}(2, \mathbb{R}) \) such that

\[e^{-Y_{l_{1}}(\theta+\alpha)} A e^{F_{l_{1}}(\theta)} e^{Y_{l_{1}}(\theta)} = A_{l_{1}} e^{F'_{l_{1}}(\theta)},\]

with the estimates \( |Y_{l_{1}}|_{l_{1}} \leq \epsilon_{l_1}^{\frac{1}{2}}, |F'_{l_{1}}|_{l_{1}} \leq 4\epsilon_{l_1}^{2}, \|A_{l_{1}} - A\| \leq 2\|A\|\epsilon_{l_1}.\)

**Induction step:** Now at the \((k+1)-th\) step, first notice that if we write

\[
e^{-Y_{k}(\theta+\alpha)} Ae^{F'_{k+1}(\theta)} e^{Y_{k}(\theta)}
\]

(5.14) \( = A_{l_{k}} e^{F'_{l_{k}}(\theta)} + e^{-Y_{k}(\theta+\alpha)}(Ae^{F_{k}(\theta)} - Ae^{F'_{l_{k}}(\theta)})e^{Y_{k}(\theta)} \)

\( = A_{l_{k}} \tilde{G}_{l_{k}}(\theta), \)

then we have

\[
|G_{l_{k}} - \text{id}|_{l_{k+1}} \leq |F'_{l_{k}}|_{l_{k+1}} + \|A_{l_{k}}^{-1}\| \|e^{-Y_{k}(\theta+\alpha)}(Ae^{F_{k}(\theta)} - Ae^{F'_{l_{k}}(\theta)})e^{Y_{k}(\theta)}|_{l_{k+1}} \leq 4\epsilon_{l_{k}}^{2} + 4||A||^{2} \times \frac{c}{||A||^{0} M_{\frac{3}{2}+\delta_{k}+3}} \leq \frac{1}{2} \epsilon_{l_{k+1}}.
\]

Then by implicit function theorem, there exists \( \tilde{F}_{l_{k}} \in C_{\frac{2}{l_{k+1}}}^{\omega} (\mathbb{T}^{d}, \text{sl}(2, \mathbb{R})) \) with

\[|\tilde{F}_{l_{k}}|_{l_{k+1}} \leq 2|G_{l_{k}} - \text{id}|_{l_{k+1}} \leq \epsilon_{l_{k+1}} \leq \epsilon_{0}(\frac{1}{l_{k+1}}, \frac{1}{l_{k+2}}),\]

such that \( G_{l_{k}}(\theta) = e^{\tilde{F}_{l_{k}}(\theta)}. \)
On the other hand, by Lemma 2.4, we have
\[ \text{rot}(\alpha, A_l \tilde{G}_l(\theta)) = \text{rot}(\alpha, e^{Y_l(\theta + \alpha)} A e^{F(\theta)} e^{Y_l(\theta)}) = \text{rot}(\alpha, A e^{F(\theta)}) = 0. \]
Then by Lemma 2.3 and (5.14), we have
\[ |\text{rot}(\alpha, A) - \text{rot}(\alpha, A_l \tilde{G}_l(\theta))| \leq 2\|\tilde{G}_l - \text{id}\|_{l^2} \leq 2\|A\|_{l^2}. \]

Consequently, we can apply Proposition 5.1 to the cocycle \((\alpha, A_l e^{\tilde{F}_l})\), and there exist \(\tilde{Y}_{l+1}, F'_{l+1} \in C_{\lambda, \gamma}^\omega (\mathbb{T}^d, \text{sl}(2, \mathbb{R}))\) such that
\[ e^{-\tilde{Y}_{l+1}(\theta + \alpha)} A_l e^{\tilde{F}_l(\theta)} e^{\tilde{Y}_{l+1}(\theta)} = A_{l+1} e^{F'_{l+1}(\theta)}, \]
with \(|\tilde{Y}_{l+1}|_{l^2} \leq \frac{1}{l^2}, |F'_{l+1}|_{l^2} \leq \frac{1}{l^2}, \|A_{l+1} - A_l\| \leq 2\|A_l\|_{l^2}. \)
Also note if \(B, D\) are small \(\text{sl}(2, \mathbb{R})\) matrices, then there exists \(C \in \text{sl}(2, \mathbb{R})\) such that
\[ e^{B} e^{D} = e^{B + D + C}, \]
where \(C\) is a sum of terms at least 2 orders in \(B, D\). Thus there exists \(Y_{l+1} \in C_{\lambda, \gamma}^\omega (\mathbb{T}^d, \text{sl}(2, \mathbb{R}))\) such that \(e^{Y_{l+1}} = e^{\tilde{Y}_{l+1}} e^{Y_{l+1}}\), with the estimate
\[ |Y_{l+1}|_{l^2} \leq \sum_{i=1}^{l+1} \frac{1}{l^2}. \]
Moreover, by (5.12) and (5.14), we have
\[ e^{-Y_{l+1}(\theta + \alpha)} A e^{F_{l+1}(\theta)} e^{Y_{l+1}(\theta)} = A_{l+1} e^{F'_{l+1}(\theta)}, \]
and thus we finish the iteration.

**Theorem 5.1.** Let \(\alpha \in \text{DC}_d(\gamma, \tau), \gamma > 0, \tau > d, V \in C^s(\mathbb{T}^d, \mathbb{R})\) with \(s > 6\tau + 2\). There exists \(\epsilon = \epsilon(\gamma, \tau, d, s)\) such that if \(\|V\|_s \leq \epsilon\), then
\[ (L_{V, \alpha, \theta} u)(n) = u(n + 1) + u(n - 1) - 2u(n) + V(n\alpha + \theta)u(n) = \lambda_1 u(n) \]
has a positive quasi-periodic solution.

**Proof.** Write
\[ A = \begin{pmatrix} E + 2 & -1 \\ 1 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ V & 0 \end{pmatrix}. \]
Then \(S^E_{\lambda}(\theta) = A e^{F(\theta)}\) which is close to constant. Now we consider the energy \(E\) which lies in the extreme right endpoint of the spectrum. Since the spectrum is compact, and it is included in \([-4 + \text{inf} V, \text{sup} V]\), then \(\|A\| \leq 6\).

By the assumption that \(s > 6\tau + 2\), there exists \(0 < \delta < 1\), such that \(s > 6\tau(1 + \delta)^2 + 1\). For such selected \(\delta\), we can take
\[ \epsilon \leq \frac{c}{6^3 C M^{\frac{2}{1+\delta}}}. \]
Proposition 5.2. Let integer vectors with finite support.

Then we can distinguish the following two cases:

**Case 1:** If $\hat{A}$ is hyperbolic, then $(\alpha, \hat{A})$ is uniformly hyperbolic, which implies that $(\alpha, A e^{F(\theta)})$ is uniformly hyperbolic, since uniformly hyperbolic is conjugacy invariant. However, this contradicts with Theorem 2.3.

**Case 2:** If $\hat{A}$ is parabolic, since $\|e^Y - \text{id}\|_0 \leq 2\|Y\|_0 \leq 4e^\frac{1}{2} < \frac{1}{100}$, then the result follows from Lemma 5.1. Thus we finish the proof.

\[ \square \]

5.4. Analytic Almost-periodic potential. Now we consider the reducibility results in the almost periodic case. First we define the almost periodic functions in the context of analytic functions on a thickened infinite dimensional torus $T_\infty^r$, where $T_\infty^r$ is defined as

$$\theta = (\theta_j)_{j \in \mathbb{N}}, \quad \theta_j \in \mathbb{C} : \Re(\theta_j) \in \mathbb{T}, |\Im(\theta_j)| \leq r(j).$$

**Definition 5.1.** For any $r > 0$, we define the space of analytic functions $T_\infty^r \rightarrow \mathfrak{sl}(2, \mathbb{R})$ as

$$C^\omega_r(T_\infty^r, \mathfrak{sl}(2, \mathbb{R})) := \left\{ F(\theta) = \sum_{k \in \mathbb{Z}_*^\infty} \hat{F}(k)e^{i(k, \theta)} \in \mathcal{F} : |F|_r := \sum_{k \in \mathbb{Z}_*^\infty} e^{r|k|} |\hat{F}(k)| < \infty \right\},$$

where $\mathcal{F}$ denotes the space of pointwise absolutely convergent formal Fourier series $T_\infty^r \rightarrow \mathfrak{sl}(2, \mathbb{R})$ as

$$F(\theta) = \sum_{k \in \mathbb{Z}_*^\infty} \hat{F}(k)e^{i(k, \theta)}, \quad \hat{F}(k) \in \mathfrak{sl}(2, \mathbb{R}),$$

and $\mathbb{Z}_*^\infty := \{ k \in \mathbb{Z}_\infty : |k|_1 := \sum_{j \in \mathbb{N}} |j| |k_j| < \infty \}$ denotes the set of infinite integer vectors with finite support.

Now we state the following Proposition:

**Proposition 5.2.** Let $r, \gamma > 0, \tau > 1$. Suppose that $\alpha \in \text{DC}_\infty(\gamma, \tau)$, $A \in \text{SL}(2, \mathbb{R})$, $F \in C^\omega_r(T_\infty^\gamma, \mathfrak{sl}(2, \mathbb{R}))$. For any $r' \in (0, r)$, there exists $c = c(\gamma, \tau)$ such that if $|\text{rot}(\alpha, A)| \leq 2\|A\|e^\frac{1}{2}$, and

$$|F|_r \leq \epsilon < \frac{1}{\|A\|^6},$$

then there exist $Y, F' \in C^\omega_r(T_\infty^\gamma, \mathfrak{sl}(2, \mathbb{R}))$ and $A' \in \text{SL}(2, \mathbb{R})$ such that

$$e^{-Y(\theta + \alpha)} A e^{F(\theta)} e^{Y(\theta)} = A' e^{F'(\theta)}.$$
Moreover, we have the estimates

\[ |Y|_{r'} \leq \epsilon^\frac{1}{2}, \quad |F'|_{r'} \leq \epsilon^2, \quad \|A - A'\| \leq 2\epsilon\|A\|. \]

**Proof.** Similar to Proposition 5.1, what we need is to apply the non-resonance cancelation lemma (Lemma 5.2). In this case, we will take \( B = C_r^\omega(\mathbb{T}^\infty, \mathfrak{sl}(2, \mathbb{R})) \), and define

\[ \Lambda_K = \{ f \in C_r^\omega(\mathbb{T}^\infty, \mathfrak{sl}(2, \mathbb{R})) \mid f(\theta) = \sum_{k \in \mathbb{Z}^\infty, |k|_1 \leq K} \hat{f}(k)e^{i(k, \theta)} \}, \]

where \( K = \frac{2}{r-r'}|\ln \epsilon| \). By [42, Lemma 2.5], \( C_r^\omega(\mathbb{T}^\infty, \mathfrak{sl}(2, \mathbb{R})) \) is Banach algebra. Moreover, since \( \alpha \in \text{DC}_\infty(\gamma, T) \), we have the following estimate:

**Lemma 5.3.** ([42, Lemma C.2]) Let \( \alpha \in \text{DC}_\infty(\gamma, T) \). Then there holds the estimate

\[ \sup_{k \in \mathbb{Z}^\infty, |k|_1 \leq K} \prod_{i \in \mathbb{Z}} \frac{1}{1 + |k_i|^{r'} \langle i \rangle^{r'}} \leq (1 + K)^{2rK^\frac{1}{2}}. \]

By similar calculation in Proposition 5.1, these facts imply that for any \( Y \in \Lambda_K \),

\[ |A^{-1}Y(\theta + \alpha)A - Y(\theta)|_{r'} \geq \epsilon^\frac{1}{2}|Y(\theta)|_{r'}, \]

and then \( \Lambda_K \subset B_r^{\text{free}}(\epsilon^\frac{1}{2}) \).

Since \( \epsilon^\frac{1}{2} \geq 13\|A\|^2\epsilon^\frac{1}{2} \), by Lemma 5.2, there exist \( Y, F^{\text{free}} \in C_r^\omega(\mathbb{T}^\infty, \mathfrak{sl}(2, \mathbb{R})) \), such that

\[ e^{-Y(\theta + \alpha)}Ae^{F(\theta)}e^Y(\theta) = Ae^{F^{\text{free}}(\theta)}, \]

where \( T_KF^{\text{free}} = \hat{F}^{\text{free}}(0), \ \mathcal{R}_KF^{\text{free}} = \sum_{|k|_1 > K} \hat{F}^{\text{free}}(k)e^{i(k, \theta)} \). Moreover, we have the estimates \( |Y|_{r'} \leq \epsilon^\frac{1}{2}, \ |F^{\text{free}}|_{r'} \leq 2\epsilon \). Meanwhile, by [42, Lemma 2.3], one has \( |\mathcal{R}_KF|_{r'} \leq e^{-(r-r')K}|F|_{r'} \). Then for any \( r' \in (0, r) \), we have

\[ |\mathcal{R}_KF^{\text{free}}(\theta)|_{r'} = \left| \sum_{|k|_1 > K} \hat{F}^{\text{free}}(k)e^{i(k, \theta)} \right|_{r'} \leq 2\epsilon e^{-K(r-r')} \leq 2\epsilon^3. \]

Furthermore, one can compute that

\[ e^{\hat{F}^{\text{free}}(0) + \mathcal{R}_KF^{\text{free}}(\theta)} = e^{\hat{F}^{\text{free}}(0)}(\text{id} + e^{-\hat{F}^{\text{free}}(0)}O(\mathcal{R}_KF^{\text{free}}(\theta))) = e^{\hat{F}^{\text{free}}(0)}e^{F'(\theta)}, \]
with the estimate
\[ |F'(\theta)|_{r'} \leq 2|R_K F_{re}|_{r'} \leq 2|F|_{r'} \leq e^2. \]

Finally, if we denote \( A' = Ae^{F_{re}(0)} \), then we get
\[ \|A' - A\| \leq 2\|A\|\|\text{id} - e^{F_{re}(0)}\| \leq 2\|A\|\epsilon. \]

\[ \square \]

As a consequence, we have the following:

**Corollary 5.2.** Let \( \alpha \in DC_\infty(\gamma, \tau), A \in \text{SL}(2, \mathbb{R}), F \in C_\omega^r(T^\infty, \text{sl}(2, \mathbb{R})). \) For any \( 0 < \tilde{r} < r \), there exists \( \epsilon = \epsilon(\tau, \gamma, r) > 0 \), such that if \( \text{rot}(\alpha, Ae^F) = 0 \), and
\[ (5.17) |F|_r \leq \epsilon \leq \frac{c_0^{-1} (r_1 - r)^2}{\|A\|^6}, \]
then there exist \( \tilde{A} \in \text{SL}(2, \mathbb{R}), Y \in C_\omega^r(T^\infty, \text{sl}(2, \mathbb{R})) \) with \( |Y|_{\tilde{r}} \leq 2\epsilon^2 \), such that
\[ e^{-Y_{\theta+\alpha}} A e^F(\theta) e^Y(\theta) = \tilde{A}. \]

**Proof.** We will prove this by induction. First we define the sequence
\[ r_0 = r, \quad \epsilon_0 = \epsilon, \quad r_k - r_{k+1} = \frac{r - \tilde{r}}{(k + 2)^2}, \quad \epsilon_k = \epsilon^{2k}. \]

Now assume that we are at the \((k+1)\)-step, i.e., we already construct \( Y_k, F_k \in C_\omega^r(T^\infty, \text{sl}(2, \mathbb{R})) \) such that
\[ (5.18) e^{-Y_k(\theta+\alpha)} A e^F(\theta) e^Y_k(\theta) = A_k e^{F_k(\theta)}, \]
with the estimates
\[ |Y_k|_{r_k} \leq \sum_{i=1}^{k-1} \epsilon_i^{1/2}, \quad |F_k|_{r_k} \leq \epsilon_k, \quad \|A_k - A_{k-1}\| \leq 2\|A_{k-1}\|\epsilon_{k-1}. \]

First by Lemma 2.4, we have
\[ \text{rot}(\alpha, A_k e^{F_k}) = \text{rot}(\alpha, Ae^F) = 0, \]
since the conjugacy \( e^{Y_k} \) is homotopic to the identity. Then by Lemma 2.3, we have
\[ |\text{rot}(\alpha, A_k)| \leq 2\|A_k\|\epsilon_k^{1/2}. \]

By our selection of \( \epsilon_0 \) and the sequence \( r_k \), one can check that
\[ \epsilon_k \leq \frac{c_0^{-1} (r_k - r_{k+1})^2}{\|A_k\|^6}. \]

Therefore by Proposition 5.2, there exist \( A_{k+1} \in \text{SL}(2, \mathbb{R}), Y_{k+1}, F_{k+1} \in C_\omega^r(T^\infty, \text{sl}(2, \mathbb{R})) \) such that
\[ (5.19) e^{-Y_{k+1}(\theta+\alpha)} A_k e^{F_k(\theta)} e^{Y_{k+1}(\theta)} = A_{k+1} e^{F_{k+1}(\theta)}. \]
Then the following well-known result of Bourgain-Jitomirskaya:

It follows directly that

\[ \| e^{Y_{k+1}} \| \leq 2 \| A_k \| \epsilon_k. \]

Thus there exists \( Y_{k+1} \in C^\omega_{r_{k+1}}(\mathbb{T}^\infty, \text{sl}(2, \mathbb{R})) \) such that

\[ e^{Y_{k+1}} = e^{\tilde{Y}_{k+1}} e^{Y_k}, \]

with the estimate \( |Y_{k+1}|r_{k+1} \leq \sum_{i=1}^k \epsilon_i^2 \). Moreover, by (5.18) and (5.19), we have

\[ e^{-Y_{k+1}(\theta + \alpha)} A e^{F(\theta)} e^{Y_{k+1}(\theta)} = A_{k+1} e^{F_{k+1}(\theta)}. \]

Taking limits of (5.20), we get the desired results.

**Corollary 5.3.** Let \( c(n) = V(n \alpha + \theta) \) be an almost periodic sequence with frequency \( \alpha \in DC_\infty(\gamma, \tau) \) and analytic in the strip \( r > 0 \). There exists \( \epsilon = \epsilon(\gamma, \tau) > 0 \), such that if \( |V|_r \leq \epsilon \), then

\[ (L_{V, \alpha, \theta} u)(n) = u(n + 1) + u(n - 1) - 2u(n) + V(n \alpha + \theta) u(n) = \lambda_1 u(n) \]

has a positive almost periodic solution.

**Proof.** The proof is same as Theorem 5.1 if we replace Corollary 5.1 by Corollary 5.2.

### 5.5. Proof of the applications

In the final subsection, we will give the applications of Theorem 1.1 in various settings, including the quasi-periodic case and almost-periodic case. First, we give the proof of Corollary 1.1.

**Proof of Corollary 1.1.** By Theorem 1.1, (1.2) has a time increasing almost periodic traveling front with average wave speed \( w \in (w^*, w) \), where

\[ w^* = \inf_{E > \lambda_1} \frac{E}{L(E)} \text{ and } w = \lim_{E \searrow \lambda_1} \frac{E}{L(E)}. \]

Recall that

\[ L(E) = \lim_{n \rightarrow +\infty} \frac{1}{n} \int_{H(c)} \log \| A_n(g) \| d\mu \]

with \( A_n = A(n - 1) \cdots A(0) \), where \( A(n) = \begin{pmatrix} E + 2 - g(n) & -1 \\ 1 & 0 \end{pmatrix} \). In the quasi-periodic case, it is well known that any \( g \in H(c) \) has the form

\[ g(n) = V(n \alpha + \theta), \]

for some \( \theta \in \mathbb{T}^d \), where \( V \) is a continuous function on \( \mathbb{T}^d \). Then it is straightforward to check that for any \( g(\cdot) = V(\theta + \cdot \alpha) \), one has

\[ \begin{pmatrix} E + 2 - g(\cdot) & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} E + 2 - V(\theta + \cdot \alpha) & -1 \\ 1 & 0 \end{pmatrix}. \]

It follows directly that

\[ A_n(g) = A(n - 1) \cdots A(0) = S^E_{\theta + (n - 1)\alpha} \cdots S^E_{\theta + \alpha} S^E_{V(\theta)}. \]

Then \( L(E) = L(\alpha, S^E_{V}\theta) \), and Corollary 1.1 follows from Theorem 1.1 and the following well-known result of Bourgain-Jitomirskaya:
Theorem 5.2 ([14, 15]). For any \(d \in \mathbb{N}_+\), let \(\alpha \in \mathbb{T}^d\) be rationally independent and \(V\) be an analytic function on \(\mathbb{T}^d\). Then \(L(\alpha, S_E^V)\) is a continuous function of \(E\).

Now we begin to prove Corollary 1.1 (2). Define function \(U\) on \(\mathbb{R} \times \mathbb{T}^d\) as
\[
U(t, \theta) := u(t, 0; V(\alpha + \theta)).
\]

Moreover, from Remark 4.1 and Theorem 4.1, one can prove that \(U\) is continuous on \(\mathbb{R} \times \mathbb{T}^d\). Denote \(T(n) = -t(n; c)\), one has
\[
\lambda_{l} = u(t, n; c) = u(t - t(n; c), 0; V(n + \cdot \alpha + \theta)) = U(t + T(n), n\alpha)
\]
which follows from Lemma 4.2, i.e., \(u(t, n; c) = u(t - t(n; c), 0; c \cdot n)\). Thus the proof is complete. \(\Box\)

As a direct consequence of Corollary 1.1, we give the following:

Proof of Corollary 1.2. By [3, Theorem 10] and [15, Corollary 2], it follows that for any \(E\) which belongs to the spectrum of almost Mathieu operator
\[
(\mathcal{L}_{2\kappa - 2, \alpha, \theta}u)(n) = u(n + 1) + u(n - 1) + 2\kappa \cos(\theta + n\alpha)u(n),
\]
its Lyapunov exponent satisfies
\[
L(E) = \max(0, \ln |\kappa|).
\]
Then the result follows from Corollary 1.1. \(\Box\)

Next we give some typical examples such that \(L = \lim_{E \to \lambda_1} L(E) = 0\) which implies that \(\lambda = \infty\). First we consider the quasi-periodic case, if the potential \(V\) is finitely differentiable, then we have the following:

Corollary 5.4. Let \(\alpha \in DC_d(\gamma, \tau), \gamma > 0, \tau > d, V \in C^s(\mathbb{T}^d, \mathbb{R})\) with \(s > 6\tau + 2\). There exists \(\epsilon = \epsilon(\gamma, \tau, d, s)\) such that if \(\|V\|_s \leq \epsilon\), then (1.2) with \(c(n) = V(n\alpha)\) has a time increasing almost periodic traveling front with average wave speed \(w \in (w^*, \infty)\).

Proof. By the assumption and Corollary 5.1, we know
\[
(\mathcal{L}_{V, \alpha, 0}u)(n) = u(n + 1) + u(n - 1) + 2u(n) + V(n\alpha)u(n) = \lambda_1 u(n)
\]
has a positive almost periodic solution. It follows that \(L = \lim_{E \to \lambda_1} L(E) = 0\) by Proposition 3.5. Then the result follows from Theorem 1.1. \(\Box\)

Lemma 5.1 and Proposition 3.5 shows that if the corresponding cocycle is reducible, and the conjugacy is close to constant, then \(L = \lim_{E \to \lambda_1} L(E) = 0\). However, in some cases, we can relax the condition, and the right concept is “Almost reducible”.

Definition 5.2. An analytic cocycle \((\alpha, A)\) is \(C^\omega\)-almost reducible if the closure of its analytic conjugacy class contains a constant.
Corollary 5.5. Let \( r > 0 \), \( \alpha \in \mathbb{R} \setminus \mathbb{Q} \). There exists \( \epsilon = \epsilon (r) \) such that if \( V \in C^\omega_c (\mathbb{T}, \mathbb{R}) \), and \( |V|_r \leq \epsilon \), then (1.2) with \( c(n) = V(n\alpha) \) has a time increasing almost periodic traveling front with average wave speed \( w \in (w^*, \infty) \).

Proof. By [59, Corollary 1.3] and [2, Corollary 1.2], there exists \( \epsilon = \epsilon (r) \) such that if \( |V|_r \leq \epsilon \), then one frequency analytic quasi-periodic Schrödinger cocycle \( (\alpha, S^E_V) \) is almost reducible. Clearly by its definition, any almost-reducible cocycle is not non-uniformly hyperbolic. Thus either \( L(E) = 0 \) or \( (\alpha, S^E_V) \) is uniformly hyperbolic. Then \( L(\lambda_1) = 0 \) follows from Theorem 2.3 since we only consider \( \lambda_1 \) which is the right endpoint of the spectrum. By Corollary 1.1, the result follows directly.

Finally we give the application for almost-periodic potentials:

Proof of Corollary 1.3. Choose \( \epsilon = \epsilon (\gamma, \tau, r) \) defined in Corollary 5.3 such that \( \sum_{k \in \mathbb{Z}} |\hat{c}(k)| e^{\epsilon |k|} < \epsilon (\gamma, \tau, r) \). Then by Corollary 5.3, we know

\[
(L_{V, \alpha, 0} u)(n) = u(n+1) + u(n-1) - 2u(n) + V(n\alpha)u(n) = \lambda_1 u(n)
\]

has a positive almost periodic solution. It follows from Proposition 3.5 that \( L = \lim_{E \to \lambda_1} L(E) = 0 \). Then Corollary 1.3 (1) follows from Theorem 1.1. By the similar arguments in the proof of Corollary 1.1, (2) can be obtained. \( \square \)

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School of Mathematical Sciences, University of Science and Technology of China, Hefei, Anhui 230026, China
Email address: xliang@ustc.edu.cn

Chern Institute of Mathematics and LPMC, Nankai University, Tianjin 300071, China
Email address: hongzew@mail.nankai.edu.cn

Chern Institute of Mathematics and LPMC, Nankai University, Tianjin 300071, China
Email address: qizhou@nankai.edu.cn

School of Mathematical Sciences, University of Science and Technology of China, Hefei, Anhui 230026, China
Email address: tzhou910@ustc.edu.cn