NON-TRIVIAL COMPOSITIONS OF DIFFERENTIAL OPERATIONS AND DIRECTIONAL DERIVATIVE

Ivana Jovović, Branko Malešević

Faculty of Electrical Engineering, University of Belgrade, Bulevar kralja Aleksandra 73, 11000 Belgrade, Serbia

Abstract. In this paper we present some new results for harmonic functions and we give recurrences for an enumeration of non-trivial compositions of higher order of differential operations and Gateaux directional derivative in $\mathbb{R}^n$.

Key words: compositions of differential operations, Gateaux directional derivative, differential forms, exterior derivative, Hodge star operator, enumeration of graphs and maps

1. Non-trivial compositions of differential operations and directional derivative of the space $\mathbb{R}^3$

In the three-dimensional Euclidean space $\mathbb{R}^3$ we consider following sets

$A_0 = \{ f: \mathbb{R}^3 \to \mathbb{R} \mid f \in C^\infty(\mathbb{R}^3) \}$ and $A_1 = \{ f^i: \mathbb{R}^3 \to \mathbb{R}^3 \mid f_1, f_2, f_3 \in C^\infty(\mathbb{R}^3) \}$.

Gradient, curl, divergence and Gateaux directional derivative in direction $\vec{e}$, for a unit vector $\vec{e} = (e_1, e_2, e_3) \in \mathbb{R}^3$, are defined in terms of partial derivative operators as follows

$\text{grad } f = \nabla_1 f = \frac{\partial f}{\partial x_1} \vec{i} + \frac{\partial f}{\partial x_2} \vec{j} + \frac{\partial f}{\partial x_3} \vec{k}$, $\nabla_1: A_0 \to A_1$,

$\text{curl } f = \nabla_2 \vec{f} = \left( \frac{\partial f_3}{\partial x_2} - \frac{\partial f_2}{\partial x_3} \right) \vec{i} + \left( \frac{\partial f_1}{\partial x_3} - \frac{\partial f_3}{\partial x_1} \right) \vec{j} + \left( \frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2} \right) \vec{k}$, $\nabla_2: A_1 \to A_1$,

$\text{div } f = \nabla_3 \vec{f} = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \frac{\partial f_3}{\partial x_3}$, $\nabla_3: A_1 \to A_0$,

$\text{dir } \vec{e} \cdot f = \nabla_0 f \cdot \vec{e} = \frac{\partial f}{\partial x_1} e_1 + \frac{\partial f}{\partial x_2} e_2 + \frac{\partial f}{\partial x_3} e_3$, $\nabla_0: A_0 \to A_0$.

Let $A_3 = \{ \nabla_1, \nabla_2, \nabla_3 \}$ and $B_3 = \{ \nabla_0, \nabla_1, \nabla_2, \nabla_3 \}$. The number of compositions of the $k$th order over the set $A_3$ is $f(k) = F_{k+3}$, where $F_k$ is the $k$th Fibonacci number (see [3] for more details). A composition of differential operations that is not 0 or $\vec{0}$ is called non-trivial. The number of non-trivial compositions of the $k$th order over the set $A_3$ is $g(k) = 3$, see [5]. In paper [6], it is shown that the number of compositions of the $k$th order over the set $B_3$ is $c(k) = 2^{k+1}$.

E-mails: Ivana Jovović <ivana@etf.rs>, Branko Malešević <malesevic@etf.rs>

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According to the above results, it is natural to try to calculate the number of non-trivial compositions of differential operations from the set $B_3$. Straightforward verification shows that all compositions of the second order over $B_3$ are

\[
\begin{align*}
\text{dir}_e \text{dir}_f &= \nabla_0 \circ \nabla_0 f = \nabla_1 (\nabla_1 f \cdot \vec{e}) \cdot \vec{e}, \\
\text{grad} \text{dir}_e f &= \nabla_1 \circ \nabla_0 f = \nabla_1 (\nabla_1 f \cdot \vec{e}), \\
\Delta f &= \text{div} \text{grad} f = \nabla_3 \circ \nabla_1 f, \\
\text{curl} \text{ curl} \vec{f} &= \nabla_2 \circ \nabla_2 \vec{f}, \\
\text{dir}_e \text{ div} \vec{f} &= \nabla_0 \circ \nabla_3 \vec{f} = (\nabla_1 \circ \nabla_3 \vec{f}) \cdot \vec{e}, \\
\text{grad} \text{div} \vec{f} &= \nabla_1 \circ \nabla_3 \vec{f}, \\
\text{curl} \text{ grad} f &= \nabla_2 \circ \nabla_1 f = \vec{0}, \\
\text{div} \text{curl} \vec{f} &= \nabla_3 \circ \nabla_2 \vec{f} = 0,
\end{align*}
\]

and that only the last two are trivial. This fact leads us to use the following procedure for determining the number of non-trivial composition over the set $B_3$. Let us define a binary relation $\sigma$ on the set $B_3$ as follows: $\nabla_j \circ \nabla_i$ iff the composition $\nabla_j \circ \nabla_i$ is non-trivial. Relation $\sigma$ induces Cayley table

| $\sigma$ | $\nabla_0$ | $\nabla_1$ | $\nabla_2$ | $\nabla_3$ |
|-----------|----------|----------|----------|----------|
| $\nabla_0$ | 1        | 1        | 0        | 0        |
| $\nabla_1$ | 0        | 0        | 0        | 1        |
| $\nabla_2$ | 0        | 0        | 1        | 0        |
| $\nabla_3$ | 1        | 1        | 0        | 0        |

For convenience, we extend set $B_3$ with nowhere-defined function $\nabla_{-1}$, whose domain and range are empty set, and establish $\nabla_{-1} \circ \nabla_i$ for $i = 0, 1, 2, 3$. Thus, graph $\Gamma$ the relation $\sigma$ is rooted tree with additional root $\nabla_{-1}$

\[
\begin{align*}
g^0(0) &= 1, \\
g^0(1) &= 4, \\
g^0(2) &= 6, \\
g^0(3) &= 9.
\end{align*}
\]

Fig. 1

Here we would like to point out that the child of $\nabla_i$ is $\nabla_j$ if composition $\nabla_j \circ \nabla_i$ is non-trivial. For any non-trivial composition $\nabla_{i_k} \circ \ldots \circ \nabla_{i_1}$ there is a unique path in the tree (Fig. 1), such that the level of vertex $\nabla_{i_j}$ is $j$, $1 \leq j \leq k$. Let $g^k(i)$ be the number of non-trivial compositions of the $k^{th}$ order of functions from $B_3$ and let $g^k_{i_j}(k)$ be the number of non-trivial compositions of the $k^{th}$ order starting with $\nabla_{i_j}$. Then we have $g^k(i) = g^k_{i_1}(k) + g^k_{i_2}(k) + \ldots + g^k_{i_k}(k)$. According to the graph $\Gamma$ obtain the following equalities $g^0_{i_1}(k) = g^0_{i_1}(k-1) + g^0_{i_2}(k-1)$, $g^1_{i_1}(k) = g^1_{i_1}(k-1)$, $g^2_{i_1}(k) = g^2_{i_1}(k-1)$, $g^3_{i_1}(k) = g^3_{i_1}(k-1) + g^3_{i_2}(k-1)$. Since the only
child of $\nabla_2$ is $\nabla_2$, we can deduce $g_2^i(k) = g_1^i(k-1) = \cdots = g_1^i(1) = 1$. Putting things together we obtain the recurrence for $g_i^j(k)$:

$$
g_i^j(k) = g_0^i(k) + g_1^i(k) + g_2^i(k) + g_3^i(k)$$

$$= (g_0^i(k-1) + g_1^i(k-1)) + g_2^i(k-1) + g_3^i(k-1)$$

$$= g_i^j(k-1) + g_0^i(k-1) + g_1^i(k-1)$$

$$= g_i^j(k-1) + (g_0^i(k-2) + g_1^i(k-2)) + g_2^i(k-2) + g_3^i(k-2) - g_2^i(k-2)$$

$$= g_i^j(k-1) + g_0^i(k-2) - 1.$$

Substituting $t(k) = g_i^j(k) - 1$ into previous formula we obtain recurrence $t(k) = t(k-1) + t(k-2)$. On the base of the initial conditions $g_i^j(1) = 4$ and $g_i^j(2) = 6$, i.e. $t(1) = 3$ and $t(2) = 5$, we conclude that $g_i^j(k) = F_{k+3} + 1$.

2. Non-trivial compositions of differential operations and directional derivative of the space $\mathbb{R}^n$

We start this section by recalling some definitions of the theory of differential forms. Let $\mathbb{R}^n$ denote the $n$-dimensional Euclidean space and consider set of smooth functions $A_0 = \{ f : \mathbb{R}^n \to \mathbb{R} | f \in C^\infty(\mathbb{R}^n) \}$. The set of all differential $k$-forms on $\mathbb{R}^n$ is a free $A_0$-module of rank $\binom{n}{k}$ with the standard basis $\{dx_{i_1} \cdots dx_{i_k} | 1 \leq i_1 < \ldots < i_k \leq n \}$, denoted $\Omega^k(\mathbb{R}^n)$. Differential $k$-form $\omega$ can be written uniquely as $\omega = \sum_{I \in \mathcal{I}(k,n)} \omega_I dx_I$, where $\omega_I \in A_0$ and $\mathcal{I}(k,n)$ is the set of multi-indices $I = (i_1, \ldots, i_k)$, $1 \leq i_1 < \ldots < i_k \leq n$. The complement of multi-index $I$ is multi-index $J = (j_1, \ldots, j_{n-k}) \in \mathcal{I}(n-k, n)$, $1 \leq j_1 < \ldots < j_{n-k} \leq n$, where components $j_p$ are elements of the set $\{1, \ldots, n\} \backslash \{i_1, \ldots, i_k\}$. We have $dx_I dx_J = \sigma(I) dx_J \cdots dx_{i_k}$, where $\sigma(I)$ is a signature of the permutation $(i_1, \ldots, i_k, j_1, \ldots, j_{n-k})$. Note that $\sigma(J) = (-1)^{k(n-k)} \sigma(I)$. With the notions mentioned above we define $\star_k(\omega) = \omega = \sum_{I \in \mathcal{I}(k,n)} \omega_I dx_I$. Map $\star_k : \Omega^k(\mathbb{R}^n) \to \Omega^{n-k}(\mathbb{R}^n)$ defined by $\star_k(\omega) = \sum_{I \in \mathcal{I}(k,n)} \omega_I dx_I$ is Hodge star operator and it provides natural isomorphism between $\Omega^k(\mathbb{R}^n)$ and $\Omega^{n-k}(\mathbb{R}^n)$. The Hodge star operator twice applied to a differential $k$-form yields $\star_{n-k}(\star_k(\omega)) = (-1)^{k(n-k)} \omega$. So for the inverse of the operator $\star_k$ holds $\star_k^{-1}(\psi) = (-1)^{k(n-k)} \star_{n-k}(\psi)$, where $\psi \in \Omega^{n-k}(\mathbb{R}^n)$.

A differential 0-form is a function $f(x_1, x_2, \ldots, x_n) \in A_0$. We define $df$ to be the differential 1-form $df = \sum_{I=1}^n \frac{\partial f }{\partial x_i} dx_i$. Given a differential $k$-form $\sum_{I \in \mathcal{I}(k,n)} \omega_I dx_I$, the exterior derivative $d_k \omega$ is differential $(k+1)$-form $d_k \omega = \sum_{I \in \mathcal{I}(k,n)} d \omega_I dx_I$.

The exterior derivative $d_k$ is a linear map from $k$-forms to $(k+1)$-forms which obeys Leibnitz rule: If $\omega$ is a $k$-form and $\psi$ is a $l$-form, then $d_k(\omega \psi) = d_k \omega \psi + (-1)^k \omega d_k \psi$. The exterior derivative has a property that $d_{k+1}(d_k \omega) = 0$ for any differential $k$-form $\omega$.

For $m = \lfloor n/2 \rfloor$ and $k = 0, 1, \ldots, m$ let us consider the following sets of functions

$$A_k = \{ f : \mathbb{R}^n \to \mathbb{R}^{\binom{n}{k}} | f_1, \ldots, f_{\binom{n}{k}} \in C^\infty(\mathbb{R}^n) \}.$$
Let $p_k : \Omega^k(\mathbb{R}^n) \to A_k$ be presentation of differential forms in coordinate notation. Let us define functions $\varphi_i$ ($0 \leq i \leq m$) and $\varphi_{n-j}$ ($0 \leq j < n-m$) as follows

$$
\varphi_i = p_i : \Omega^i(\mathbb{R}^n) \to A_i
$$

and

$$
\varphi_{n-j} = p_{j} \ast_{j}^{-1} : \Omega^{n-j}(\mathbb{R}^n) \to A_j.
$$

Then, according to [7], the combination of Hodge star operator and the exterior derivative generates one choice of differential operations $\nabla_k = \varphi_k d_{k-1} \varphi_{k-1}$, $1 \leq k \leq n$, in $n$-dimensional space $\mathbb{R}^n$.

List of differential operations in $\mathbb{R}^n$

Formulae for the number of compositions of differential operations from the set $A_n$ and corresponding recurrences are given by Malešević in [6, 7], see also appropriate integer sequences in [14] and [15]. The following theorem provides a natural characterization of the number of non-trivial compositions of differential operations from the set $A_n$. For the proof we refer reader to [6].

**Theorem 2.1.** All non-trivial compositions of differential operations from the set $A_n$ are given in the following form

$$(\forall i \circ \nabla_{n+1-i} \circ \cdots \circ \nabla_{i+1} \circ \nabla)$$

where $2i, 2(i-1) \neq n, 1 \leq i \leq n$. Term in bracket is included in if the number of differential operations is odd and left out otherwise.

**Theorem 2.2.** Let $g(k)$ be the number of non-trivial compositions of the $k$th order of differential operations from the set $A_n$. Then we have

$$
g(k) = \begin{cases} 
  n & : 2 \n; \\
  n & : 2 \div n, k = 1; \\
  n - 1 & : 2 \div n, k = 2; \\
  n - 2 & : 2 \div n, k > 2.
\end{cases}
$$
Hodge dual to the exterior derivative $d_k : \Omega^k(\mathbb{R}^n) \to \Omega^{k+1}(\mathbb{R}^n)$ is codifferential
$
\delta_{k-1}, \text{ a linear map } \delta_{k-1} : \Omega^k(\mathbb{R}^n) \to \Omega^{k-1}(\mathbb{R}^n), \text{ which is a generalization of the divergence, defined by }
$
\delta_{k-1} = (-1)^n (k-1)! \ast_{n-(k-1)} d_{n-k} \ast_k = (-1)^k \ast_{k-1} d_{n-k} \ast_k .

Note that $\nabla_{n-j} = (-1)^{j+1} p_j \delta_j p_{j+1}^1$, for $0 \leq j < n - m - 1$. The codifferential can be coupled with the exterior derivative to construct the Hodge Laplacian, also known as the Laplace-de Rham operator, $\Delta_k : \Omega^k(\mathbb{R}^n) \to \Omega^k(\mathbb{R}^n)$, a harmonic generalization of Laplace differential operator, given by $\Delta_0 = \delta_0 d_0$ and $\Delta_k = \delta_k d_k + d_{k-1} \delta_{k-1}$, for $1 \leq k \leq m$, see [9]. The operator $\Delta_0$ is actually the negative of the Laplace-Beltrami (scalar) operator. A $k$-form $\omega$ is called harmonic if $\Delta_k(\omega) = 0$. We say that $f \in A_k$ is a harmonic function if $\omega = p_k^{-1}(\tilde{f})$ is harmonic $k$-form. If $k \geq 1$ harmonic function $\tilde{f}$ is also called harmonic field.

For function $\tilde{f} \in A_k$, $1 \leq k \leq m$, according to Proposition 4.15. from [3], holds
$\Delta_k(p_k^{-1}\tilde{f}) = 0$ iff $\delta_{k-1}(p_k^{-1}\tilde{f}) = 0$ and $d_k(p_k^{-1}\tilde{f}) = 0$. In fact, we obtain the following

Lemma 2.1. Let $\tilde{f} \in A_k$, $1 \leq k \leq m$, then
$\Delta_k(p_k^{-1}\tilde{f}) = 0 \iff \nabla_{n-(k-1)}(\tilde{f}) = 0 \wedge \nabla_{k+1}(\tilde{f}) = 0 .
$

For harmonic function $f \in A_0$ we have $\Delta_0 f = \delta_0 d_0 f = 0$, hence $\nabla_n \circ \nabla_1 f = 0$ and finally $\nabla_1 \circ \nabla_n \circ \nabla_1 \circ \cdots \circ \nabla_n \circ \nabla_1 f = 0$. We can now rephrase Theorem 2.1 for harmonic functions.

Theorem 2.3. All compositions of the second and higher order in $\ast$ acting on harmonic function $f \in A_0$ are trivial. All compositions of the first and higher order in $\ast$ acting on harmonic field $\tilde{f} \in A_k$, $1 \leq k \leq m$, are trivial.

We say that $f \in A_k$, $0 \leq k \leq m$, is coordinate-harmonic function or that $f$ satisfies harmonic coordinate condition, if all its coordinates are harmonic functions. Malešević [3] showed that all compositions of the third and higher order in $\ast$ acting on coordinate-harmonic function $f$ are trivial in $\mathbb{R}^3$. Based on the previous statement for coordinate-harmonic functions in $\mathbb{R}^3$ we formulate.

Conjecture 2.1. All compositions of the third and higher order in $\ast$ acting on coordinate-harmonic function $f \in A_k$, $0 \leq k \leq m$, are trivial.

One approach to a coordinate investigation of Conjecture 2.1 in $\mathbb{R}^4$ can be found in Gilbert N. Lewis and Edwin B. Wilson papers [1], [2] (see also [4]). Similar problem for coordinate-harmonic functions can be considered in Discrete Exterior Calculus [10, 11] and Combinatorial Hodge Theory [12, 13].

Let $f \in A_0$ be a scalar function and $e = (e_1, \ldots, e_n) \in \mathbb{R}^n$ be a unit vector. Gateaux directional derivative in direction $e$ is defined by
$\text{dir}_e f = \nabla_0 f = \sum_{k=1}^n \frac{\partial f}{\partial x_k} e_k : A_0 \to A_0$.
Let us extend set of differential operations $\mathcal{A}_n = \{\nabla_1, \ldots, \nabla_n\}$ with directional derivative to the set $\mathcal{B}_n = \mathcal{A}_n \cup \{\nabla_0\} = \{\nabla_0, \nabla_1, \ldots, \nabla_n\}$. Recurrences for counting compositions of differential operations from the set $\mathcal{B}_n$ can be found in [8]. Corresponding integer sequences are given in [14].

The number of non-trivial compositions of differential operations from the set $\mathcal{B}_n$ is determined by the binary relation $\nu$, defined by:

$$\nabla_i \nu \nabla_j \iff (i = 0 \land j = 0) \lor (i = 0 \land j = 1) \lor (i = n \land j = 0) \lor (i + j = n + 1 \land 2i \neq n).$$

Applying Theorem 2.2 to cases $i = 2, \ldots, n - 1$ we conclude that the number of non-trivial compositions of the $k^{th}$ order starting with $\nabla_2, \ldots, \nabla_{n-1}$ can be express by formula

$$j(k) = g(k) - 2 = \begin{cases} n - 2 : 2 \n \n - 2 : 2 | n \text{, } k = 1; \\ n - 3 : 2 | n \text{, } k = 2; \\ n - 4 : 2 | n \text{, } k > 2. \end{cases}$$

Let $g^c(k)$ be the number of non-trivial the $k^{th}$ order compositions of operations from the set $\mathcal{B}_n$. Let $g_n^c(0)$, $g_n^c(1)$ and $g_n^c(2)$ be the numbers of non-trivial the $k^{th}$ order compositions starting with $\nabla_0$, $\nabla_1$ and $\nabla_{n}$, respectively. Then we have $g^c(k) = g_n^c(0) + g_n^c(1) + f(k) + g_n^c(2)$. Denote $g^c(k) = g_n^c(0) + g_n^c(1) + g_n^c(2)$. Hence, the following three recurrences are true $g_n^c(0) = g_n^c(1) = g_n^c(k-1)$, $g_n^c(2) = g_n^c(k-1) + g_n^c(2)$. Thus, the recurrence for $g^c(k)$ is of the form

$$g^c(k) = g_n^c(k) + g_n^c(k) + g_n^c(k) + (g_n^c(k-1) + g_n^c(k-1))$$

$$= g_n^c(k) + g_n^c(k-1) + g_n^c(k-1)$$

$$= g_n^c(k-1) + g_n^c(k-1) + g_n^c(k-2) + g_n^c(k-2)$$

$$= g_n^c(k-1) + g_n^c(k-2).$$

With initial conditions $g^c(1) = 3$, $g^c(2) = 5$ we deduce $g^c(k) = F_{k+3}$. Therefore, we have proved the following theorem.

**Theorem 2.4.** The number of non-trivial compositions of the $k^{th}$ order over the set $\mathcal{B}_n$ is

$$g^c(k) = F_{k+3} + j(k) = \begin{cases} F_{k+3} + n - 2 : 2 \n \n + 1 : 2 | n \text{, } k = 1; \\ n + 2 : 2 | n \text{, } k = 2; \\ F_{k+3} + n - 4 : 2 | n \text{, } k > 2. \end{cases}$$

**Corollary 2.1.** In the case $n = 3$ follows formula $g^c(k) = F_{k+3} + 1$ from the first section.
Remark 2.1. The values of function $g^c(k)$ are given in [14] as the following sequences $A_{001611}$ ($n=3$), $A_{000045}$ ($n=4$), $A_{157726}$ ($n=5$), $A_{157725}$ ($n=6$), $A_{157729}$ ($n=7$), $A_{157727}$ ($n=8$), $A_{187107}$ ($n=9$), $A_{187179}$ ($n=10$) for $k > 2$.

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