SEMI-PROJECTIVE REPRESENTATIONS AND TWISTED SCHUR MULTIPLIERS

MASSIMILIANO ALESSANDRO, CHRISTIAN GLEISSNER AND JULIA KOTONSKI

Abstract. We study semi-projective representations, i.e., homomorphisms of finite groups to the group of semi-projective transformations of finite dimensional $K$-vector spaces. We extend Schur’s concept of projective representation groups to the semi-projective case under the assumption that $K$ is algebraically closed. In order to stress the relevance of the theory, we discuss two important applications, where semi-projective representations occur naturally. The first one reviews Isaacs’ treatment of the extension problem of invariant characters (over arbitrary fields) defined on normal subgroups. The second one is our original algebro-geometric motivation and deals with the problem to find linear parts of homeomorphisms and biholomorphisms of complex torus quotients.

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1. Introduction

In [Sch04], Schur developed the theory of projective representations, which are homomorphisms from a group $G$ to the group of projective transformations $\text{PGL}(V)$. Here, $G$ is a finite group and $V$ a non-trivial finite dimensional $K$-vector space. It is clear that every ordinary representation induces a projective representation. However, the converse is in general not true, more precisely the obstructions to lift are the elements of the second cohomology group $H^2(G, K^*)$, where $K^*$ is considered as a trivial $G$-module. In order to study projective representations via ordinary representations in the case $K = \mathbb{C}$, Schur showed the existence of a representation group $\Gamma$, which is a particular kind of central extension of $G$ having the property that all projective representations of $G$ lift to ordinary representations of $\Gamma$.

In our recent work (see [DG22] and [GK22]), we constructed certain quotients of complex tori by holomorphic actions of finite groups and investigated their homeomorphism and biholomorphism classes. Under mild assumptions on the fixed loci of the actions, Bieberbach’s theorems about crystallographic groups (see [Cha86, I]) imply that homeomorphisms and biholomorphisms of such quotients are induced by affine transformations. When determining the linear parts of these transformations, we came across an object similar to a projective

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representation, namely a homomorphism from a finite group to $\text{PGL}(n, \mathbb{C}) \rtimes \text{Gal}(\mathbb{C}/\mathbb{R})$. Moreover, we had to determine a particular kind of lift of this map to $\text{GL}(n, \mathbb{C}) \rtimes \text{Gal}(\mathbb{C}/\mathbb{R})$.

This example served for us as a motivation to extend Schur’s theory to semi-projective representations, i.e., homomorphisms from a finite group $G$ to the group of semi-projective transformations $\text{PGL}(V)$. Here, $\text{PGL}(V)$ is defined as the quotient of the group of semi-linearities

$$\text{PGL}(V) \simeq \text{GL}(V) \rtimes \text{Aut}(K)$$

modulo the action of the multiplicative group $K^*$. A semi-projective representation yields an action of $G$ on $K$ by automorphisms. In this way, $K^*$ becomes a $G$-module and we can consider the second cohomology group $H^2(G, K^*)$ with respect to this action. In analogy to the projective case, this group plays an important role as it is the obstruction space of the lifting problem of semi-projective representations to semi-linear representations, i.e., homomorphisms from $G$ to $\text{PGL}(V)$.

As our main result, we show that if $K$ is algebraically closed, then for any given action $\varphi$ of a finite group $G$, there exists a finite $\varphi$-twisted representation group $\Gamma$, which has the property that any semi-projective representation inducing the action $\varphi$ admits a semi-linear lift to $\Gamma$. Despite the fact that $\Gamma$ is in general not unique, it has minimal order among all groups enjoying the lifting property. This allows us to study semi-projective representations of $G$ via semi-linear representations of $\Gamma$. We also give a cohomological characterization of a group $\Gamma$ to be a $\varphi$-twisted representation group, which reduces to the classical description of a representation group in the case where the action $\varphi$ is trivial. In general, it seems to be difficult to determine explicitly a $\varphi$-twisted representation group, even in the projective case, i.e., where $\varphi$ is trivial. Indeed, there is a vast amount of literature dedicated to this problem for specific classes of groups, e.g., [Sch11], [Kar85, Section 3.7] or the more recent article [HS21]. We approach this problem in the semi-projective case via an algorithm for the case $K = \mathbb{C}$ under the assumption that $\varphi$ takes values in $\text{Gal}(\mathbb{C}/\mathbb{R})$. Our algorithm produces all $\varphi$-twisted representation groups of a given finite group $G$ and a given action $\varphi$.

Apart from the algebro-geometric application to torus quotients, there are other situations where semi-projective representations arise naturally, for example in Clifford theory: in [Isa81], Isaacs developed the concept of crossed-projective representations, which is analogous to our notion of semi-projective representations, in order to study the problem of extending $G$-invariant irreducible $L$-representations defined on a normal subgroup $N \trianglelefteq G$ to the ambient group $G$ for arbitrary fields $L$. In the section dedicated to applications and examples, we briefly review Isaacs’ work and rephrase it in our language.

We will now explain how the paper is organized: in Section 2, we introduce semi-projective representations and discuss their interplay with group cohomology. We phrase the lifting problem and give a cohomological criterion for a semi-projective representation of a finite group $G$ to lift to a semi-linear representation of an extension of $G$ by a finite abelian group $A$. In Section 3, we construct for any given finite group $G$, together with an action $\varphi$ on an algebraically closed field $K$, a $\varphi$-twisted representation group. For this purpose, we adapt Isaacs construction of a representation group in the projective case [Isa81, 11] to our setup. Then, we give a cohomological characterization of a $\varphi$-twisted representation group and show that it coincides with the classical notion in case that $\varphi$ is the trivial action. The last part of the paper, Section 4, is devoted to examples and applications. Besides basic examples of semi-projective representations and twisted representation groups, we develop an algorithm which allows us to determine all $\varphi$-twisted representation groups of a given finite group $G$ under the assumption that $K = \mathbb{C}$ and $\varphi$ maps to $\text{Gal}(\mathbb{C}/\mathbb{R})$. Running a MAGMA implementation, we determine the $\varphi$-twisted representation groups of $D_4$ for all possible actions $\varphi$. Finally, we explain the relations of semi-projective representations to the homeomorphism and biholomorphism problem of torus quotients and the extendability of $L$-representations, as already mentioned above.
2. Preliminaries

In this section, we introduce semi-linear and semi-projective representations and discuss some of their basic properties. Throughout this paper $V$ is a non-trivial finite-dimensional $K$-vector space and $G$ a finite group.

**Definition 2.1.** A bijective map $f: V \rightarrow V$ is called a semi-linear transformation if there exists an automorphism $\varphi_f \in \text{Aut}(K)$ such that for all $v, w \in V$ and all $\lambda \in K$, it holds:

\[
f(v + w) = f(v) + f(w) \quad \text{and} \quad f(\lambda v) = \varphi_f(\lambda)f(v).
\]

The set of all semi-linear transformations of $V$ forms a group, which is denoted by $\Gamma \text{L}(V)$.

In the following remark we collect some basic properties describing the structure of $\Gamma \text{L}(V)$.

**Remark 2.2.**

1. The group $\Gamma \text{L}(V)$ contains $\text{GL}(V)$ as a normal subgroup and sits inside the following short exact sequence:

\[
1 \rightarrow \text{GL}(V) \rightarrow \Gamma \text{L}(V) \rightarrow \text{Aut}(K) \rightarrow 1.
\]

This sequence splits, i.e., $\Gamma \text{L}(V) \cong \text{GL}(V) \rtimes \text{Aut}(K)$.

2. Let $v_1, \ldots, v_n$ be a basis of $V$. Then, we can associate to every $f \in \Gamma \text{L}(V)$ an invertible matrix $A_f := (a_{ij})_{ij}$ by

\[
f(v_j) = \sum_{i=1}^n a_{ij} v_i.
\]

This procedure establishes an isomorphism between $\Gamma \text{L}(V)$ and the semidirect product $\text{GL}(n, K) \rtimes \text{Aut}(K)$ with group structure

\[
(A, \varphi) \cdot (B, \psi) := (A \varphi(B), \varphi \circ \psi).
\]

Here, $\varphi(B)$ is the matrix obtained by applying the automorphism $\varphi$ to all of the entries of $B$.

In analogy to the group of *projective transformations* $\text{PGL}(V)$, the group of *semi-projective transformations* $\text{P}\Gamma \text{L}(V)$ is defined as the quotient of $\Gamma \text{L}(V)$ modulo the equivalence relation

\[
f \sim g \quad \text{if and only if there exists } \lambda \in K^*, \text{ such that } f = \lambda g.
\]

By construction, we have a short exact sequence

\[
1 \rightarrow K^* \rightarrow \Gamma \text{L}(V) \rightarrow \text{P} \Gamma \text{L}(V) \rightarrow 1.
\]

**Remark 2.3.** The structure of $\text{P} \Gamma \text{L}(V)$ is similar to the one of $\Gamma \text{L}(V)$, namely:

1. The group $\text{PGL}(V)$ is a normal subgroup of $\text{P} \Gamma \text{L}(V)$, and there is a split exact sequence

\[
1 \rightarrow \text{PGL}(V) \rightarrow \text{P} \Gamma \text{L}(V) \rightarrow \text{Aut}(K) \rightarrow 1.
\]

Note that the map $\text{P} \Gamma \text{L}(V) \rightarrow \text{Aut}(K)$ is well-defined because all representatives of a given class in $\text{P} \Gamma \text{L}(V)$ have the same automorphism.

2. After choosing a *projective frame*, we can identify $\text{P} \Gamma \text{L}(V)$ with the semidirect product

\[
\text{PGL}(n, K) \rtimes \text{Aut}(K).
\]

3. If $\dim(V) \geq 3$, then the *fundamental theorem of projective geometry* characterizes the *semi-projective transformations* as the bijective self maps of the projective space $\mathbb{P}(V)$ mapping collinear points to collinear points (see [Sam88, Theorem 7]).

We can now introduce our main objects:
Definition 2.4. Let $G$ be a finite group.

1. A semi-linear representation is a homomorphism $F : G \to \Gamma L(V)$.
2. A semi-projective representation is a homomorphism $f : G \to P\Gamma L(V)$.

Remark 2.5 (The lifting problem). Note that every semi-linear representation $F : G \to \Gamma L(V)$ induces a semi-projective representation $f : G \to P\Gamma L(V)$ by composition with the quotient map:

$$
\begin{array}{ccc}
\Gamma L(V) & \longrightarrow & P\Gamma L(V) \\
F \downarrow & & \downarrow f \\
G & \longrightarrow &
\end{array}
$$

However, it is not true that every semi-projective representations can be obtained in this way. The obstruction to the existence of a lift to $\Gamma L(V)$, or more generally, the interplay between semi-linear and semi-projective representations can be described by using group cohomology in analogy to the classical theory of projective representations.

Remark 2.6. Given a semi-linear or semi-projective representation of $G$, we obtain an action

$$
\varphi : G \to \text{Aut}(K), \quad g \mapsto \varphi_g,
$$

by composition with the projection from $\Gamma L(V)$ or $P\Gamma L(V)$ to $\text{Aut}(K)$, respectively. Via this action, the abelian group $K^*$ obtains the structure of a $G$-module. In particular, we can define cocycles $Z^i(G, K^*)$, coboundaries $B^i(G, K^*)$ and the cohomology groups

$$
H^i(G, K^*) = Z^i(G, K^*)/B^i(G, K^*).
$$

For details on group cohomology, we refer the reader to the textbook [Bro94]. The basic observation is that we can associate to every semi-projective representation a well-defined class in the second cohomology group:

Proposition 2.7. Let $f : G \to P\Gamma L(V)$ be a semi-projective representation and $f_g$ be a representative of the class $f(g)$ for each $g \in G$. Then, there exists a map

$$
\alpha : G \times G \to K^* \quad \text{such that} \quad f_{gh} = \alpha(g, h) \cdot (f_g \circ f_h)
$$

for all $g, h \in G$. The map $\alpha$ is a 2-cocycle, i.e.,

$$
\varphi_g(\alpha(h, k)) \cdot \alpha(gh, k)^{-1} \cdot \alpha(g, hk) \cdot \alpha(g, h)^{-1} = 1.
$$

The cohomology class $[\alpha] \in H^2(G, K^*)$ is independent of the chosen representatives $f_g$.

Proof. Since $f$ is a homomorphism, it holds $[f_{gh}] = [f_g] \circ [f_h]$, which implies that $f_{gh}$ and $f_g \circ f_h$ differ by an element $\alpha(g, h) \in K^*$. To show that $\alpha$ is a cocycle, we use the associativity of the multiplication in $G$ to compute $f_{ghk}$ in two different ways. On the one hand, we have

$$
f_{ghk} = \alpha(g, hk) \cdot (f_g \circ f_{hk}) = \alpha(g, hk) \cdot (f_g \circ \alpha(h, k) \cdot (f_h \circ f_k))
$$

$$
= \alpha(g, hk) \cdot \varphi_g(\alpha(h, k)) \cdot (f_g \circ f_h \circ f_k).
$$

On the other hand,

$$
f_{gh}k = \alpha(gh, k) \cdot (f_{gh} \circ f_k) = \alpha(gh, k) \cdot \alpha(g, h) \cdot (f_g \circ f_h \circ f_k).
$$

Comparing the two expressions yields

$$
\alpha(g, hk) \cdot \varphi_g(\alpha(h, k)) = \alpha(gh, k) \cdot \alpha(g, h).
$$
Let $f'_g$ be another representative for $f(g)$, then there exists $\tau(g) \in K^*$ such that $f_g = \tau(g)f'_g$. Let $\alpha'$ be the 2-cocycle associated to the collection of the $f'_g$, i.e.,

$$f'_{gh} = \alpha'(g,h) \cdot (f'_g \circ f'_h) \quad \text{for all} \quad g, h \in G.$$ 

A computation as above shows that

$$\alpha'(g,h) = \varphi_g(\tau(h)) \cdot \tau(gh)^{-1} \cdot \tau(g) \cdot \alpha(g,h).$$ 

Thus, $\alpha$ and $\alpha'$ differ by the 2-coboundary $\partial \tau(g,h) = \varphi_g(\tau(h)) \cdot \tau(gh)^{-1} \cdot \tau(g)$. □

**Remark 2.8.** Let $f : G \to \text{PGL}(V)$ be a semi-projective representation.

1. If we choose id$_V$ as a representative for $f(1)$, then the 2-cocycle $\alpha$ is normalized, i.e.,

$$\alpha(1,g) = \alpha(g,1) = 1.$$ 

2. If $f$ is induced by a semi-linear representation $F$, then the attached cohomology class is trivial. Conversely, assume that $\alpha$ is a coboundary, that is

$$\alpha(g,h) = \varphi_g(\tau(h)) \cdot \tau(gh)^{-1} \cdot \tau(g) \quad \text{for some} \quad \tau : G \to K^*.$$ 

Then, the map

$$F : G \to \text{GL}(V), \quad g \mapsto F_g := \tau(g)f_g$$

is a semi-linear representation inducing $f$. Indeed, $F$ is a homomorphism, as the following computation shows:

$$F_g \circ F_h = (\tau(g) \cdot f_g) \circ (\tau(h) \cdot f_h) = \tau(g) \cdot \varphi_g(\tau(h)) \cdot (f_g \circ f_h)$$ 

$$= \tau(gh) \cdot \alpha(g,h) \cdot (f_g \circ f_h)$$ 

$$= \tau(gh) \cdot f_{gh} = F_{gh}.$$ 

In the theory of projective representations, the action $\varphi : G \to \text{Aut}(K)$ is trivial and $H^2(G, K^*)$ is called the Schur multiplier. In the semi-projective setting $\varphi$ is in general non-trivial. This motivates the next definition.

**Definition 2.9.** Let $\varphi : G \to \text{Aut}(K)$ be an action and consider the induced $G$-module structure on $K^*$. Then, we call $H^2(G, K^*)$ the $\varphi$-twisted Schur multiplier of $G$.

Up to now, we assigned to every semi-projective representation an element in $H^2(G, K^*)$. The next remark shows that all cohomology classes arise in this way.

**Remark 2.10.** Let $\varphi : G \to \text{Aut}(K)$ be an action of a finite group $G$ of order $n$ on the field $K$ and $\alpha \in Z^2(G, K^*)$ be a 2-cocycle. In analogy to the regular representation, we consider the vector space $V$ with basis $\{e_h \mid h \in G\}$ and define for every $g \in G$ an element $R_g \in \text{GL}(V)$ via

$$R_g(e_h) := \alpha(g,h)^{-1}e_{gh}.$$ 

Then, the map

$$f : G \to \text{PGL}(V) \rtimes \text{Aut}(K), \quad g \mapsto ([R_g], \varphi_g),$$

is a semi-projective representation with assigned cohomology class $[\alpha] \in H^2(G, K^*)$.

**Remark 2.10** and **Remark 2.10** show that if $H^2(G, K^*) \neq 0$, there are semi-projective representations without a semi-linear lift. In the projective case, this problem was first noticed and investigated by Schur [Sch04]. In order to study projective representations by means of ordinary linear representations, he constructed a so-called representation group $\Gamma$ of $G$; in modern terminology, a stem extension

$$1 \longrightarrow A \longrightarrow \Gamma \longrightarrow G \longrightarrow 1,$$

with $A \cong H^2(G, K^*)$. 
Such an extension has the property that for every projective representation \( f : G \to \text{PGL}(V) \) there exists an ordinary linear representation \( F : \Gamma \to \text{GL}(V) \) fitting inside the following diagram
\[
\begin{array}{cccccc}
1 & \longrightarrow & A & \longrightarrow & \Gamma & \longrightarrow & G & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
1 & \longrightarrow & K^* & \longrightarrow & \text{GL}(V) & \longrightarrow & \text{PGL}(V) & \longrightarrow & 1
\end{array}
\]
Recall that \( \text{stem} \) means that \( A \) is central and contained in the commutator group \([\Gamma, \Gamma]\).

If we want to generalize Schur’s construction to the semi-projective case, we have to deal with more general finite extensions. Let us recall some facts about group extensions.

**Remark 2.11.** Let \( 1 \to A \to \Gamma \to G \to 1 \) be an extension of \( G \) by a finite abelian group \( A \) and \( s : G \to \Gamma \) a set-theoretic section.

1. There is an action of \( G \) on \( A \) defined by \( g * a := s(g) \cdot a \cdot s(g)^{-1} \). Since \( A \) is abelian, the action is independent of the choice of the section.
2. In general, \( s \) is not a homomorphism, but we may write
   \[
s(gh) = \beta(g, h)s(g)s(h)
   \]
   for some \( \beta(g, h) \in A \).
   In this way, we obtain a map \( \beta : G \times G \to A \), which is a 2-cocycle since it fulfills
   \[
   (g * \beta(h, k)) \cdot \beta(gh, k)^{-1} \cdot \beta(g, hk) \cdot \beta(g, h)^{-1} = 1.
   \]
   A different choice of a section \( s' : G \to \Gamma \) yields a cohomologous cocycle \( \beta' \). Hence, we can associate to the given extension a unique cohomology class in \( H^2(G, A) \).
3. Assume that we have an action \( \varphi : G \to \text{Aut}(K) \) on the field \( K \). Then, by composition with the projection \( \Gamma \to G \), we also obtain an action of \( \Gamma \) on \( K \) with kernel containing \( A \). In this situation, the inflation-restriction exact sequence of Hochschild and Serre \([\text{HS}53\text{, Theorem 2, p. 129}]\) reads:
   \[
   1 \longrightarrow H^1(G, K^*) \xrightarrow{\text{inf}} H^1(\Gamma, K^*) \xrightarrow{\text{res}} \text{Hom}_G(A, K^*) \xrightarrow{\text{tra}} H^2(G, K^*) \xrightarrow{\text{inf}} H^2(\Gamma, K^*)
   \]
   Here, \( \text{inf} \) and \( \text{res} \) are induced by inflation and restriction of cocycles and the transgression map \( \text{tra} \) is defined as
   \[
   \text{tra} : \text{Hom}_G(A, K^*) \to H^2(G, K^*), \quad \lambda \mapsto [\lambda \circ \beta].
   \]
   Clearly, this map depends only on the cohomology class of \( \beta \).

By using the terminology of the previous remark, we get a far-reaching generalization of Remark 2.8 (2); see \([\text{Isa}94\text{, Theorem 11.13}]\) for the corresponding statement in the projective setting.

**Theorem 2.12.** Let \( 1 \to A \to \Gamma \to G \to 1 \) be an extension of \( G \) by a finite abelian group \( A \) with associated cohomology class \( [\beta] \in H^2(G, A) \). A semi-projective representation \( f : G \to \text{PGL}(V) \) with class \( [\alpha] \in H^2(G, K^*) \) is induced by a semi-linear representation
\[
F : \Gamma \to \Gamma L(V), \quad \gamma \mapsto F_\gamma
\]
if and only if \( [\alpha] \) belongs to the image of the transgression map.

**Proof.** Assume that \( f \) is induced by a semi-linear representation \( F \). By assumption, there exists a function \( \lambda : \Gamma \to K^* \) such that \( F_\gamma = \lambda(\gamma)f_\pi(\gamma) \) for all \( \gamma \in \Gamma \). Since we assume that \( f_1 = \text{id} \), it follows that
\[
F_a = \lambda(a)f_\pi(a) = \lambda(a)\text{id} \quad \text{for all} \quad a \in A.
\]
As a result, $\lambda$ restricted to $A$ is a homomorphism. We claim that $\lambda \in \text{Hom}_G(A, K^*)$, i.e.,

$$\lambda(g \ast a) = \varphi_g(\lambda(a))$$

for all $g \in G$ and $a \in A$. Indeed, we get

$$\varphi_g(\lambda(a)) \text{id} = F_s(g) \circ (\lambda(a) \text{id}) \circ F_s(g)^{-1} = F_s(g) \circ F_a \circ F_s(g)^{-1} = F_{s(g)a}(g)^{-1} = \lambda(g \ast a) \text{id}.$$ 

By using the definition of $\beta$, we compute

$$F_{s(gh)} = F_{\beta(g,h)s(g,h)} = F_{\beta(g,h)} \circ F_s(g) \circ F_s(h)$$

$$= \lambda(\beta(g,h)) \cdot (\lambda(s(g))f_g) \circ (\lambda(s(h))f_h)$$

$$= \lambda(\beta(g,h)) \cdot (\lambda(s(g)) \cdot \varphi_g(\lambda(s(h))) \cdot (f_g \circ f_h)).$$

On the other hand,

$$F_{s(gh)} = \lambda(s(gh))f_{gh} = \lambda(s(gh)) \cdot (\alpha(g,h) \cdot (f_g \circ f_h)).$$

Comparing the results, we obtain

$$\alpha(g,h) = \lambda(\beta(g,h)) \cdot \partial(\lambda \circ s)(g,h),$$

which means that

$$[\lambda \circ \beta] = [\alpha] \in H^2(G, K^*).$$

Conversely, assume there is a function $\tau : G \to K^*$ and $\lambda \in \text{Hom}_G(A, K^*)$ such that

$$\alpha(g,h) = \lambda(\beta(g,h)) \cdot \varphi_g(\tau(h)) \cdot \tau(g)^{-1} \cdot \tau(g).$$

We define the following map

$$F : \Gamma \to \Gamma L(V), \quad a \cdot s(\lambda) \mapsto \lambda(a) \tau(g)f_g.$$ 

As in Remark 2.8, one can show that $F$ is a homomorphism inducing $f$. \hfill $\square$

A natural question arises:

**Question 2.13.** Is it possible to find for every finite group $G$ together with a fixed action $\varphi : G \to \text{Aut}(K)$ an extension

$$1 \to A \to \Gamma \to G \to 1$$

with $A$ finite and abelian

such that every semi-projective representation $f : G \to \Gamma L(V)$ with action $\varphi$ is induced by a semi-linear representation $F : \Gamma \to \Gamma L(V)$?

By virtue of Remark 2.10 and Theorem 2.12 answering this question amounts to constructing an extension with surjective transgression map

$$\text{tra} : \text{Hom}_G(A, K^*) \to H^2(G, K^*), \quad \lambda \mapsto [\lambda \circ \beta].$$

Clearly, this may only be possible if the twisted Schur multiplier is finite. In case such an extension $\Gamma$ exists, its order is bounded from below:

$$|G| \cdot |H^2(G, K^*)| \leq |G| \cdot |\text{Hom}(A, K^*)| \leq |G| \cdot |A| = |\Gamma|.$$ 

**Remark 2.14.** Note that $H^2(G, K^*)$ is in general not finite. As an example, consider $K = \mathbb{Q}(i)$ and $G = \text{Gal}(K/Q)$ acting naturally on $K$. Then, the cohomology group

$$H^2(G, K^*) \simeq \mathbb{Q}^*/\text{N}_{K/Q}(K^*)$$

is infinite. Indeed, an application of the sum of two squares theorem shows that all primes $p$ with $p \equiv 3 \mod 4$ yield non-trivial distinct elements. Nevertheless, in many important situations $H^2(G, K^*)$ is finite: e.g., if $K$ is
a finite Galois extension of $Q_p$ and $G = \text{Gal}(K/Q_p)$ is acting naturally (cf. [Neu13 II, Lemma 5.1]), or, as we shall see in the next section, if $K$ is algebraically closed and $\varphi: G \to \text{Aut}(K)$ is an arbitrary action.

3. Twisted Representation Groups: the Algebraically Closed Case

Throughout this section, $K$ is an algebraically closed field and $G$ a finite group together with a given action $\varphi: G \to \text{Aut}(K)$.

We want to provide an answer to Question 2.13 under the above assumptions. Indeed, we will construct an extension

$$1 \to A \to \Gamma \to G \to 1$$

such that the transgression map

$$\text{tra}: \text{Hom}_G(A, K^*) \to H^2(G, K^*)$$

is an isomorphism and $\Gamma$ has minimal order, namely

$$|\Gamma| = |G| \cdot |H^2(G, K^*)|.$$

Remark 3.1. Note that, under our assumptions, we mainly deal with a case similar to $K = \mathbb{C}$, where $\varphi$ acts just by the identity and/or complex conjugation. Indeed, $H := \varphi(G)$ is a finite group and $F := K^H \subset K$ is a Galois extension with Galois group $H$. Since we assume $K$ to be algebraically closed, the Artin-Schreier Theorem [AS27] implies that if $H$ is non-trivial, then it is isomorphic to $\mathbb{Z}_2$, $K = F(i)$ with $i^2 = -1$ and $\text{char}(K) = 0$. In particular, if $\text{char}(K) \neq 0$, then the action is necessarily trivial and we are in the projective setting.

The first step towards our goal is to prove the finiteness of the twisted Schur multiplier, or more generally, of all higher cohomology groups $H^i(G, K^*)$. In order to show this, we adapt the proof of the finiteness of the Schur multiplier given in [Isa94].

Lemma 3.2. [Isa94, Lemma 11.14] Let $A$ be an abelian group (not necessarily finite) and $Q \leq A$ with $Q$ divisible, i.e., for all positive integers $n$, the maps

$$Q \to Q, \quad \alpha \mapsto \alpha^n$$

are surjective. Assume $|A : Q| < \infty$. Then, $Q$ is complemented in $A$.

Lemma 3.3. Under our assumptions, the groups $B^i(G, K^*)$ are divisible.

Proof. Let $n$ be a positive integer and $\beta \in B^i(G, K^*)$ a coboundary. Then, there is a function $\tau: G^{i-1} \to K^*$ such that $\beta = \partial \tau$, where

$$\partial \tau(g_1, \ldots, g_i) := \varphi_{g_1}(\tau(g_2, \ldots, g_i)) \cdot \left( \prod_{j=2}^{i} \tau(g_1, \ldots, g_{j-2}, g_{j-1}g_j, g_{j+1}, \ldots, g_i)^{(-1)^{j-1}} \right) \cdot \tau(g_1, \ldots, g_{i-1})^{(-1)^i}.$$

Since we assume $K$ to be algebraically closed, for all $(g_1, \ldots, g_{i-1}) \in G^{i-1}$, there is an element $\nu(g_1, \ldots, g_{i-1}) \in K^*$ such that $\nu(g_1, \ldots, g_{i-1})^n = \tau(g_1, \ldots, g_{i-1})$. As $\varphi_{g_1}$ is a field automorphism, it holds

$$\beta(g_1, \ldots, g_i) = \partial \tau(g_1, \ldots, g_i) = \partial \nu^n(g_1, \ldots, g_i) = (\partial \nu(g_1, \ldots, g_i))^n.$$

Now, we are ready to prove the finiteness of the higher cohomology groups $H^i(G, K^*)$.

Proposition 3.4. For each $i \geq 1$, the cohomology groups $H^i(G, K^*)$ are finite with exponent dividing the order of $G$. Moreover, $B^i(G, K^*)$ has a complement in $Z^i(G, K^*)$. 
Proof. It is well known that $\alpha^{[G]} \in B^i(G, K^*)$ for every cocycle $\alpha \in Z^i(G, K^*)$, see [Bro94] III, Corollary 10.2. In other words, the exponent of $H^i(G, K^*)$ divides the order of $G$. Take a cocycle $\alpha \in Z^i(G, K^*)$ and consider the group $A := \langle B^i(G, K^*), \alpha \rangle$. By construction, $A/B^i(G, K^*) = \langle [\alpha] \rangle$, which implies that the order of the quotient divides the order of $G$. Since $B^i(G, K^*)$ is divisible, it is complemented in $A$ thanks to Lemma 3.2. Thus, there exists a subgroup $W \leq A$ such that

$$W \cap B^i(G, K^*) = \{1\} \quad \text{and} \quad WB^i(G, K^*) = A.$$ 

Note that, for all $\gamma \in W$, it holds

$$\gamma^{[G]} \in W \cap B^i(G, K^*) = \{1\}.$$ 

This shows that $W$ is contained in the group

$$U := \{\eta \in Z^i(G, K^*) \mid \eta^{[G]} = 1\}.$$ 

In particular,

$$\alpha \in A = WB^i(G, K^*) \leq UB^i(G, K^*).$$ 

Since $\alpha \in Z^i(G, K^*)$ is arbitrary, the above relation implies

$$Z^i(G, K^*) = UB^i(G, K^*).$$ 

The group $U$ is finite because it consists of functions $G^i \to K^*$ with image contained in the group of $|G|$-th roots of unity. It follows that

$$|H^i(G, K^*)| = |Z^i(G, K^*)/B^i(G, K^*)| \leq |U| < \infty,$$

and Lemma 3.2 implies that $B^i(G, K^*)$ has a complement in $Z^i(G, K^*)$. □

The main result of this section is the following.

Theorem 3.5. Let $G$ be a finite group and $K$ an algebraically closed field. Let $\varphi : G \to \text{Aut}(K)$ be a fixed action. Then, there exists an extension

$$1 \to A \to \Gamma \to G \to 1$$

of $G$ with $A$ finite and abelian such that the transgression map

$$\text{tra} : \text{Hom}_G(A, K^*) \to H^2(G, K^*)$$

is an isomorphism.

Proof. Take a complement $M$ of $B^2(G, K^*)$ in $Z^2(G, K^*)$. Such a group $M$ exists and is finite thanks to Proposition 3.3. Consider $A := \text{Hom}(M, K^*)$ and define via $\varphi$ an action on it:

$$(g * a)(m) := \varphi_g(a(m)).$$

We define a map $\beta : G \times G \to A$ by

$$\beta(g, h)(m) := m(g, h) \quad \text{for} \quad m \in M.$$ 

A straightforward computation shows that $\beta$ is a 2-cocycle:

$$\partial \beta(g, h, k)(m) = \left( g * \beta(h, k) \cdot \beta(gh, k)^{-1} \cdot \beta(g, hk) \cdot \beta(g, h)^{-1} \right)(m)$$

$$= \varphi_g(\beta(h, k)(m)) \cdot \beta(gh, k)^{-1}(m) \cdot \beta(g, hk)(m) \cdot \beta(g, h)^{-1}(m)$$

$$= \varphi_g(m(h, k)) \cdot m(gh, k)^{-1} \cdot m(g, hk) \cdot m(g, h)^{-1}$$

$$= \partial m(g, h, k) = 1.$$
Despite the fact that the cocycle $\beta$ is in general not normalized, we can consider a normalized cocycle $\beta'$ in its cohomology class. Then, it is clear from literature (see [Bro94, IV]) that $\beta'$ defines an extension

$$1 \to A \to \Gamma \to G \to 1,$$

where $\Gamma := A \times G$ with product structure

$$(a, g) \cdot (b, h) := (a(g \ast b)\beta'(g, h), gh).$$

We point out that the conjugation action of $G$ on $A$ is given by $g \ast a$.

Now, we claim that the transgression map

$$\text{tra}: \text{Hom}_G(A, K^*) \to H^2(G, K^*), \quad \lambda \mapsto [\lambda \circ \beta] = [\lambda \circ \beta'],$$

is surjective. Any class in $H^2(G, K^*)$ is represented by a (unique) element $m_0 \in M \leq Z^2(G, K^*)$. Consider the evaluation homomorphism at $m_0$, that is

$$\lambda: A \to K^*, \quad a \mapsto a(m_0).$$

Note that $\lambda$ is $G$-equivariant, in fact

$$\lambda(g \ast a) = (g \ast a)(m_0) = \varphi_g(a(m_0)) = \varphi_g(\lambda(a)) \quad \text{for all} \quad g \in G.$$

Furthermore, we have

$$(\lambda \circ \beta)(g, h) = \lambda(\beta(g, h)) = \beta(g, h)(m_0) = m_0(g, h).$$

This shows that $\text{tra}(\lambda) = [m_0]$ and thus the desired surjectivity. Finally, the injectivity follows from

$$|M| = |H^2(G, K^*)| \leq |\text{Hom}_G(A, K^*)| \leq |\text{Hom}(A, K^*)| \leq |A| \leq |M|. \quad \square$$

Remark 3.6.

(1) From the above chain of inequalities, it follows that

(a) all characters of $A$ are $G$-equivariant, namely $\text{Hom}_G(A, K^*) = \text{Hom}(A, K^*)$,

(b) $A \simeq \text{Hom}(A, K^*)$,

(c) $A \simeq H^2(G, K^*)$,

(d) the group $\Gamma$ has minimal order $|\Gamma| = |G| \cdot |H^2(G, K^*)|$,

(e) $H^1(G, K^*) \simeq H^1(\Gamma, K^*)$ by the inflation-restriction sequence

$$0 \to H^1(G, K^*) \to H^1(\Gamma, K^*) \to \text{Hom}_G(A, K^*) \simeq H^2(G, K^*).$$

(2) If $\text{char}(K) \neq 0$, the action $\varphi$ is trivial, see Remark 3.6. Moreover, property (1b) amounts to saying that $\text{char}(K) \nmid |A|$. Thus, as a byproduct, we found a general property of the Schur multiplier, namely

$$\text{char}(K) \nmid |H^2(G, K^*)|,$$

whenever $G$ is a finite group and $K$ is algebraically closed.

Remark 3.6 motivates the following definition:

**Definition 3.7.** Let $\varphi: G \to \text{Aut}(K)$ be an action of a finite group $G$ on an algebraically closed field $K$. A group $\Gamma$ is called a $\varphi$-twisted representation group of $G$ if there exists an extension

$$1 \to A \to \Gamma \to G \to 1$$

with $A$ finite and abelian

such that the following conditions hold:

(1) $\text{char}(K) \nmid |A|$,

(2) $\text{Hom}_G(A, K^*) = \text{Hom}(A, K^*)$. 
(3) the transgression map

\[ \text{tra: } \text{Hom}_G(A, K^*) \to H^2(G, K^*) \]

is an isomorphism.

**Proposition 3.8.** If \( \varphi: G \to \text{Aut}(K) \) is the trivial action, then an extension as in Definition 3.7 is a stem extension.

**Proof.** Since \( \varphi \) is trivial, the restriction-inflation sequence reads

\[ 1 \to \text{Hom}(G, K^*) \to \text{Hom}(\Gamma, K^*) \to \text{Hom}_G(A, K^*) \to H^2(G, K^*). \]

As the transgression map is an isomorphism, the restriction \( \text{Hom}(\Gamma, K^*) \to \text{Hom}_G(A, K^*) \) has to be trivial, which implies \( A \leq [\Gamma, \Gamma] \). Suppose it does not, then the map from \( A \) to the abelianization \( \Gamma^{ab} \) is non-trivial. We write \( \Gamma^{ab} \simeq \mathbb{Z}_{d_1} \times \ldots \times \mathbb{Z}_{d_m} \) and, w.l.o.g., we can assume that the induced map \( A \to \mathbb{Z}_{d_1} \) is not the zero-map. If \( p := \text{char}(K) \mid d_1 \), then we write \( d_1 = p^k l_1 \) with \( p \nmid l_1 \neq 1 \) and obtain a non-trivial map \( A \to \mathbb{Z}_{d_1} \to \mathbb{Z}_{l_1} \), since \( p \nmid |A| \). Replacing \( d_1 \) by \( l_1 \), if necessary, we may assume that there exists a primitive \( d_1 \)-th root of unity. This yields a character \( \lambda \in \text{Hom}(\Gamma, K^*) \) such that the restriction \( \lambda_A: A \to K^* \) is non-trivial. Thus, we get a contradiction.

Assume now that \( A \) is not contained in the center of \( \Gamma \). Then, there exist \( a \in A \) and \( \gamma \in \Gamma \) such that

\[ \gamma a \gamma^{-1} a^{-1} \neq 1. \]

Since \( \text{char}(K) \nmid |A| \), a similar argument as before shows that there exists a character \( \lambda \in \text{Hom}(A, K^*) \) such that \( \lambda(\gamma a \gamma^{-1} a^{-1}) \neq 1 \). As \( \varphi \) is the trivial action, this means that \( \lambda \notin \text{Hom}_G(A, K^*) \), which contradicts the assumption \( \text{Hom}_G(A, K^*) = \text{Hom}(A, K^*) \).

The next proposition shows that Definition 3.7 is well-posed.

**Proposition 3.9.** In the projective case, i.e., when the \( G \)-action on \( K \) is trivial, Definition 3.7 reduces exactly to the classical notion of a representation group (cf. [Isa91, Corollary 11.20]), i.e.,

1. the extension \( 1 \to A \to \Gamma \to G \to 1 \) is stem,
2. \( |A| = |H^2(G, K^*)| \).

**Proof.** If the extension fulfills the conditions of Definition 3.7, then Proposition 3.8 implies that it is stem. Since \( \text{char}(K) \nmid |A| \), we have that \( A \simeq \text{Hom}(A, K^*) \) and then (2) follows from the fact that the transgression map is an isomorphism.

Conversely, suppose we have a stem extension

\[ 1 \to A \to \Gamma \to G \to 1 \]

such that \( |A| = |H^2(G, K^*)| \). First of all, Remark 3.6 (2) implies that \( \text{char}(K) \nmid |A| \). Since the extension is stem, \( A \leq Z(\Gamma) \) and therefore the action of \( G \) on \( A \) is trivial implying \( \text{Hom}_G(A, K^*) = \text{Hom}(A, K^*) \). Furthermore, the inflation-restriction sequence yields that, for a stem extension, the transgression map is injective because \( A \leq [\Gamma, \Gamma] \). Since

\[ |\text{Hom}(A, K^*)| = |A| = |H^2(G, K^*)|, \]

we conclude that the transgression map is also surjective and hence an isomorphism.

**Remark 3.10.** We want to point out that only the order of a \( \varphi \)-twisted representation group \( \Gamma \) is unique, whereas the group itself is in general not (see examples in Section 3), even in the projective case. Here, it is known that the group \( \Gamma \) is unique up to isomorphism if \( |G^{ab}| \) and \( |H^2(G, K^*)| \) are coprime [BT82, p. 92]. Note that the
latter condition is fulfilled if for instance $G$ is perfect. However, there are groups with a unique representation group, even though $|G^{ab}|$ and $|H^2(G, K^*)|$ are not coprime. An example is the metacyclic group

$$G := \langle a, b \mid a^8 = b^4 = 1, bab^{-1} = a^5 \rangle,$$

which has abelianization $\mathbb{Z}_2^2$, Schur multiplier $\mathbb{Z}_2$ and

$$\Gamma := \langle a, b \mid a^{16} = b^4 = 1, bab^{-1} = a^5 \rangle$$

as the unique representation group.

Now, we want to give a numerical criterion to decide whether a given extension is a $\varphi$-twisted representation group or not.

**Proposition 3.11.** Let $\varphi : G \to \text{Aut}(K)$ be a non-trivial action of a finite group $G$ on an algebraically closed field $K$. Let

$$1 \longrightarrow A \longrightarrow \Gamma \longrightarrow G \longrightarrow 1$$

be an extension by a finite abelian group $A$. Then, $\Gamma$ is a $\varphi$-twisted representation group if and only if the following conditions are satisfied:

1. $|A| = |H^2(G, K^*)|$,  
2. $|\text{Hom}_G(A, K^*)| = |\text{Hom}(A, K^*)|$ and  
3. $|H^1(G, K^*)| = |H^1(\Gamma, K^*)|$.

**Proof.** Clearly, every $\varphi$-twisted representation group fulfills the three conditions. Conversely, if they hold, then the inflation-restriction sequence together with (3) implies that the transgression map is injective. Condition (1), together with Remark 3.6(2), implies $\text{char}(K) \nmid |A|$. Therefore, by using condition (2), we have

$$|\text{Hom}_G(A, K^*)| = |\text{Hom}(A, K^*)| = |A|.$$  

Thus, the transgression map is also surjective and hence an isomorphism. \qed

### 4. Examples and Applications

In this section, we present basic examples of semi-projective representations. Furthermore, we develop an algorithm to compute all $\varphi$-twisted representation groups for a given finite group $G$ and a given action $\varphi$ under the assumption $K = \mathbb{C}$ and that $\varphi$ maps to $\text{Gal}(\mathbb{C}/\mathbb{R})$. In the end, as we have announced in the introduction, we discuss two more involved situations, where semi-projective representations arise naturally. The first one deals with a purely representation theoretic question from Clifford theory, namely the extendability of $G$-invariant irreducible $L$-representations defined on a normal subgroup $N \trianglelefteq G$ to the ambient group $G$, where $L$ is an arbitrary field. Isaacs investigated this problem in [Isa81] by using the concept of crossed-projective representations, which is analogous to our notion of a semi-projective representation. The second one is the original geometric motivation of the authors. It deals with the problem to find linear parts of homeomorphisms and biholomorphisms of complex torus quotients, cf. [DG22], [GK22] and [HL21]. We show that this problem reduces, in some occasions, to a lifting problem of a certain semi-projective representation.

#### 4.1. Basic Examples of Semi-Projective Representations and Twisted Representation Groups.

**Example 4.1.** Consider $K = \mathbb{C}$ as a $G = \mathbb{Z}_2$-module, where $1 \in \mathbb{Z}_2$ acts via complex conjugation $\text{conj}(z) = \overline{z}$. In this example, a twisted representation group $\Gamma$ is of order 4 because

$$H^2(\mathbb{Z}_2, \mathbb{C}^*) \simeq (\mathbb{C}^*)^{\mathbb{Z}_2}/N_{\mathbb{C}/\mathbb{R}}(\mathbb{C}^*) = \mathbb{Z}_2.$$
It is easy to see that $\Gamma$ must be isomorphic to $\mathbb{Z}_4$. Indeed, since the transgression map is required to be an isomorphism, the extension

$$0 \rightarrow \mathbb{Z}_2 \rightarrow \Gamma \rightarrow \mathbb{Z}_2 \rightarrow 0$$

has to be non-split, which implies $\Gamma \simeq \mathbb{Z}_4$. Consider the semi-projective representation

$$f: \mathbb{Z}_2 \rightarrow \text{PGL}(2, \mathbb{C}) \ltimes \mathbb{Z}_2, \quad 1 \mapsto \left( \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right), \text{conj}.\)$$

Its cohomology class in $H^2(\mathbb{Z}_2, \mathbb{C}^*)$ is represented by the normalised 2-cocycle $\alpha$ with $\alpha(1,1) = -1$, see Remark 2.10. It has no lift to a semi-linear representation of $\mathbb{Z}_2$. A semi-linear lift to $\Gamma$ is given by

$$F: \mathbb{Z}_4 \rightarrow \text{GL}(2, \mathbb{C}) \ltimes \mathbb{Z}_2, \quad 1 \mapsto \left( \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right), \text{conj}.\)$$

In the following, we explain how to use a computer algebra system, such as MAGMA [BCP97], to produce all twisted representation groups of a given finite group $G$ in the case $K = \mathbb{C}$. We assume that $\varphi: G \rightarrow \text{Aut}(\mathbb{C})$ takes values in $\text{Gal}(\mathbb{C}/\mathbb{R}) \simeq [\text{id}, \text{conj}]$, cf. Remark 3.1.

Recall that Proposition 3.11 provides necessary and sufficient numerical conditions for an extension $\Gamma$ of $G$ by a finite abelian group $A$ to be a $\varphi$-twisted representation group. The results from the previous section say that $A$ must be isomorphic to $H^2(G, \mathbb{C}^*)$. Furthermore, condition (3) of the proposition requires $H^1(\Gamma, \mathbb{C}^*)$ and $H^1(\Gamma, \mathbb{C}^*)$ to be of the same size. In order to check this, we determine the above cohomology groups. Since we want to use a computer, it is necessary to replace the module $\mathbb{C}^*$ by a discrete module. Identifying complex conjugation with multiplication by $-1$, the homomorphism $\varphi$ induces an action of $G$ on $\mathbb{Z}$ that is also denoted by $\varphi$. In this way, we can consider $\varphi$ as a complex character of $G$ of degree 1 with values in $\{\pm 1\}$. Furthermore, the exponential sequence

$$0 \rightarrow \mathbb{Z} \rightarrow \mathbb{C} \overset{2\pi i}{\rightarrow} \mathbb{C}^* \rightarrow 1$$

becomes a sequence of $G$-modules. Since the cohomology groups $H^n(G, \mathbb{C})$ vanish for $n \geq 1$, see [Bro94, III, Corollary 10.2], the corresponding long exact sequence induces isomorphisms

$$H^n(G, \mathbb{C}^*) \simeq H^{n+1}(G, \mathbb{Z}) \quad \text{for all} \quad n \geq 1.$$

Similarly, we have these isomorphisms for the cohomology groups of $\Gamma$. In order to check the second condition of the proposition, we make use of the identity $\text{Hom}(A, \mathbb{C}^*) = \text{Irr}(A)$, which holds since $A$ is abelian.

These considerations lead to Algorithm 1. It takes as inputs a finite group $G$ and an action $\varphi$, which is given as a character with values in $\{\pm 1\}$, and it returns all $\varphi$-twisted representation groups of $G$.

The reader can find a MAGMA implementation on the webpage

[http://www.staff.uni-bayreuth.de/~bt300503/publi.html](http://www.staff.uni-bayreuth.de/~bt300503/publi.html)

**Example 4.2.** Running our code, we compute the $\varphi$-twisted representation groups of the dihedral group

$$D_4 = \langle s, t \mid s^2 = t^4 = 1, \; stst^{-1} = t^3 \rangle$$

for all possible actions $\varphi: D_4 \rightarrow \text{Aut}(\mathbb{C})$ given as characters with values in $\{\pm 1\}$:

| $\varphi(s)$ | $\varphi(t)$ | $A = H^2(D_4, \mathbb{C}^*)$ | $\varphi$-twisted representation groups |
|-------------|-------------|----------------|--------------------------------|
| 1           | 1           | $\mathbb{Z}_2$ | $\langle 16, 7 \rangle$, $\langle 16, 8 \rangle$, $\langle 16, 9 \rangle$ |
| -1          | -1          | $\mathbb{Z}_2 \times \mathbb{Z}_2$ | $\langle 32, 14 \rangle$, $\langle 32, 13 \rangle$ |
| 1           | -1          | $\mathbb{Z}_2 \times \mathbb{Z}_2$ | $\langle 32, 9 \rangle$, $\langle 32, 10 \rangle$, $\langle 32, 14 \rangle$, $\langle 32, 13 \rangle$ |
| -1          | 1           | $\mathbb{Z}_2 \times \mathbb{Z}_2$ | $\langle 32, 2 \rangle$, $\langle 32, 10 \rangle$, $\langle 32, 13 \rangle$ |

Here, the symbol $\langle n, d \rangle$ denotes the $d$-th group of order $n$ in MAGMA’s Database of Small Groups.
Algorithm 1 ϕ-twisted representation groups

function TwistedRepresentationGroups\((G, ϕ)\)

input: Finite group \(G\), \(ϕ \in \text{Irr}(G)\) of degree one with values in \(\{±1\}\)

output: List of all \(ϕ\)-twisted representation groups of \(G\)

\(A \leftarrow H^3(G, \mathbb{Z})\)

\((Γ_1, \ldots, Γ_k) \leftarrow \text{extensions of } G \text{ by } A\)

\(L \leftarrow \text{empty list}\)

for \(j = 1, \ldots, k\) do

\(\text{test} \leftarrow \text{true}\)

for \(χ \in \text{Irr}(A)\) do

if \(χ\) is not \(G\)-invariant then

\(\text{test} \leftarrow \text{false}\)

end if

end for

if \(\text{test} = \text{true} \text{ and } \#H^2(G, \mathbb{Z}) = \#H^2(Γ_j, \mathbb{Z})\) then

\(L \leftarrow \text{append}(L, Γ_j)\)

\(\triangleright \text{add } Γ_j \text{ to the list } L\)

end if

end for

return \(L\)

4.2. Extendability of \(L\)-Representations.
Let \(L\) be a field and \(χ \in \text{Irr}_L(N)\) an irreducible character defined on a normal subgroup \(N \trianglelefteq G\). Assume that \(χ\) is \(G\)-invariant, i.e.,

\(χ(gng^{-1}) = χ(n) \quad \text{for all } g \in G, n \in N.\)

Then, we can ask the question if \(χ\) can be extended to an irreducible character of the ambient group \(G\). Clearly, the \(G\)-invariance is a necessary condition for the extendibility, but in general not sufficient. In the following, we will describe how this problem relates to the theory of semi-projective representations.

Remark 4.3. Let \(K\) be an algebraically closed field containing \(L\). Then, the character \(χ\) splits as follows

\(χ = m(η_1 + \ldots + η_r), \quad \text{where } η_i \in \text{Irr}_K(N).\)

The irreducible characters \(η_1, \ldots, η_r\) form a single orbit under the action of \(\text{Gal}(K/L)\). The common multiplicity \(m\) of the constituents \(η_i\) is called the Schur index of \(χ\).

Let us call \(η := η_1\) and \(F\) the subfield of \(K\) generated by \(L\) and the values of \(η\). The extension \(L \subset F\) is Galois of degree \(r\) with abelian Galois group. By [Isa81, Lemma 2.1], the \(\text{Gal}(F/L)\)-orbit of \(η\) consists of all constituents \(η_i\) of \(χ\). We now make the crucial assumption that \(m = 1\), which by [Isa94, Theorem 9.21]) is automatically fulfilled in the case \(\text{char}(L) \neq 0\). Under this assumption, the character \(η\) is afforded by an irreducible \(F\)-representation \(ρ: N \to \text{GL}(n, F)\), cf. [Isa94, Corollary 10.2]. By the \(G\)-invariance of \(χ\), there exists for all \(g \in G\) an element \(ϕ_g \in \text{Gal}(F/L)\) such that \(η^g = ϕ_g \circ η\), where \(η^g(n) := η(gng^{-1})\). Clearly, \(ϕ_g\) is unique and \(ϕ_n = \text{id}\) for all \(n \in N\). Thus, we obtain an action \(ϕ: G \to \text{Gal}(F/L)\) which factors through the quotient map \(π: G \to G/N\). Since the \(F\)-representations \(ρ^g\) and \(ϕ_g(ρ)\) are irreducible and their characters \(η^g\) and \(ϕ_g \circ η\) agree, they are equivalent according to [Isa94, Corollary 9.22]. Thus, for all \(g \in G\), there exists a matrix \(A_g \in \text{GL}(n, F)\) such that

\[(*) \quad A_g \cdot ϕ_g(ρ) \cdot A_g^{-1} = ρ^g.\]
In order to show that they are equal, it suffices to prove that the following two matrices

\[ A \]

This is an immediate consequence of Schur's lemma and the identity

\[ \rho \]

which we leave to the reader.

**Proposition 4.4.** The map \( f : G \to \text{PGL}(n, F) \times \text{Gal}(F/L) \) from above is a semi-projective representation.

**Proof.** We need to show that \( f \) is a homomorphism, i.e.,

\[ f(n_1 s(\gamma_1) n_2 s(\gamma_2)) = f(n_1 s(\gamma_1)) \circ f(n_2 s(\gamma_2)). \]

For this purpose, we rewrite the left-hand side as

\[
f(n_1 s(\gamma_1) n_2 s(\gamma_2)) = f(n_1 s(\gamma_1) n_2 s(\gamma_2) s(\gamma_1 \gamma_2)^{-1} s(\gamma_1 \gamma_2))
\]

\[
= ([\rho(n_1 s(\gamma_1) n_2 s(\gamma_2) s(\gamma_1 \gamma_2)^{-1}) A_{s(\gamma_1 \gamma_2)}], \varphi_{\gamma_1 \gamma_2})
\]

\[
= ([\rho(n_1 s(\gamma_1) n_2 s(\gamma_1)^{-1}) \rho(s(\gamma_1) s(\gamma_2) s(\gamma_1 \gamma_2)^{-1}) \cdot A_{s(\gamma_1 \gamma_2)}], \varphi_{\gamma_1 \gamma_2}).
\]

Similarly, the right-hand side becomes

\[
f(n_1 s(\gamma_1)) \circ f(n_2 s(\gamma_2)) = ([\rho(n_1) A_{s(\gamma_1)}], \varphi_{\gamma_1}) \circ ([\rho(n_2) A_{s(\gamma_2)}], \varphi_{\gamma_2})
\]

\[
= ([\rho(n_1) A_{s(\gamma_1)} \varphi_{\gamma_1} (\rho(n_2) A_{s(\gamma_2)})], \varphi_{\gamma_1 \gamma_2})
\]

\[
= ([\rho(n_1 s(\gamma_1) n_2 s(\gamma_1)^{-1}) \cdot A_{s(\gamma_1)} \cdot \varphi_{\gamma_1} (A_{s(\gamma_2)})], \varphi_{\gamma_1 \gamma_2}).
\]

In order to show that they are equal, it suffices to prove that the following two matrices

\[ C_{(\gamma_1, \gamma_2)} := \rho(s(\gamma_1) s(\gamma_2) s(\gamma_1 \gamma_2)^{-1}) \cdot A_{s(\gamma_1 \gamma_2)} \quad \text{and} \quad D_{(\gamma_1, \gamma_2)} := A_{s(\gamma_1)} \cdot \varphi_{\gamma_1} (A_{s(\gamma_2)}) \]

differ by a constant \( \bar{\alpha}(\gamma_1, \gamma_2) \) in \( F^* \), namely

\[ \bar{\alpha}(\gamma_1, \gamma_2) \cdot D_{(\gamma_1, \gamma_2)} = C_{(\gamma_1, \gamma_2)}. \]

This is an immediate consequence of Schur's lemma and the identity

\[
C_{(\gamma_1, \gamma_2)} \cdot \varphi_{\gamma_1 \gamma_2}(\rho) \cdot C_{(\gamma_1, \gamma_2)}^{-1} = D_{(\gamma_1, \gamma_2)} \cdot \varphi_{\gamma_1 \gamma_2}(\rho) \cdot D_{(\gamma_1, \gamma_2)}^{-1},
\]

which we leave to the reader.

**Remark 4.5.** We observe from the proof of Proposition 4.3 that the cohomology class of \( f \) is represented by a cocycle \( \alpha : G \times G \to F^* \), which is constant on \( N \). For this reason, it induces a cocycle \( \bar{\alpha} : G/N \times G/N \to F^* \) whose class in \( H^2(G/N, F^*) \) is independent of the chosen section \( s : G/N \to G \) and of the chosen \( A_s(\gamma) \), which we recall to be unique only up to a scalar in \( F^* \).

It is clear from Remark 2.8 (2) that \( f \) lifts to a semi-linear representation of \( G \) if and only if \( [\alpha] \) is trivial in \( H^2(G, F^*) \). However, this semi-linear representation might not be an extension of \( \rho \), cf. [Isa94] p. 179.

**Theorem 4.6.** [Isa94] Theorem 4.3 The representation \( \rho : N \to \text{GL}(n, F) \) extends to a semi-linear representation

\[ \hat{\rho} : G \to \text{GL}(n, F) \times \text{Gal}(F/L) \]

if and only if \( [\bar{\alpha}] \) is trivial in \( H^2(G/N, F^*) \).

**Proof.** Given a semi-linear extension

\[ \hat{\rho} : G \to \text{GL}(n, F) \times \text{Gal}(F/L), \quad g \mapsto (B_g, \varphi_g), \]

the matrices \( B_g \) fulfill the conjugation equation (2). Thus, setting \( A_g := B_g \), one can see that \( \hat{\rho} \) is a lift of the semi-projective representation \( f \) and then it is clear that \( \bar{\alpha} = 1 \) as a cocycle.
Assume now that $[\hat{\alpha}]$ is trivial, where the representative $\hat{\alpha}$ is constructed as above choosing the matrices $A_\gamma$ and the section $s: G/N \to G$ such that $s(1) = 1$ and $A_1$ is the identity matrix $E_n$. Then, there exists a function $\tau: G/N \to F^*$ such that

$$\hat{\alpha}(\gamma_1, \gamma_2) = \varphi_{\gamma_1}(\tau(\gamma_2))\tau(\gamma_1\gamma_2)^{-1}\tau(\gamma_1).$$

Define the following map:

$$\hat{\rho}: G \to \text{GL}(n, F) \rtimes \text{Gal}(F/L), \quad n \cdot s(\gamma) \mapsto (\tau(\gamma)\rho(n)A_s(\gamma), \varphi_{\gamma}).$$

Clearly, by our choice of $s$ and $A_\gamma$, the map $\hat{\rho}$ is an extension of $\rho$. Indeed, since $\hat{\alpha}(1, 1) = 1$, it follows that $\tau(1) = 1$ and we obtain

$$\hat{\rho}(n) = (\tau(1)\rho(n)A_s(1), \varphi_{1}) = (\rho(n), \text{id}).$$

It remains to show that $\hat{\rho}$ is a homomorphism. In order to have a compact notation, we use the matrices $C_{(\gamma_1, \gamma_2)}$ and $D_{(\gamma_1, \gamma_2)}$, as defined in the proof of Proposition [4.4] and compute

$$\hat{\rho}(n_1s(\gamma_1)) \circ \hat{\rho}(n_2s(\gamma_2)) = (\tau(\gamma_1)\rho(n_1)A_s(\gamma_1), \varphi_{\gamma_1}) \circ (\tau(\gamma_2)\rho(n_2)A_s(\gamma_2), \varphi_{\gamma_2})$$

$$= (\tau(\gamma_1)\varphi_{\gamma_1}(\tau(\gamma_2))\cdot \rho(n_1s(\gamma_1)n_2s(\gamma_1)^{-1})\cdot D_{(\gamma_1, \gamma_2)}, \varphi_{\gamma_1\gamma_2})$$

$$= (\tau(\gamma_1\gamma_2) \cdot \rho(n_1s(\gamma_1)n_2s(\gamma_1)^{-1}) \cdot \hat{\alpha}(\gamma_1, \gamma_2) \cdot D_{(\gamma_1, \gamma_2)}, \varphi_{\gamma_1\gamma_2})$$

$$= (\tau(\gamma_1\gamma_2) \cdot \rho(n_1s(\gamma_1)n_2s(\gamma_1)^{-1}) \cdot C_{(\gamma_1, \gamma_2)}, \varphi_{\gamma_1\gamma_2})$$

$$= \hat{\rho}(n_1s(\gamma_1)n_2s(\gamma_2)).$$

\[\square\]

Remark 4.7. The extension $\hat{\rho}$ can be considered as an ordinary representation over the field $L$. Its character $\chi_{\hat{\rho}}$ is an extension of $\chi \in \text{Irr}_L(N)$, see [Isa81, Theorem 3.1].

4.3. Homeomorphisms and Biholomorphisms of Torus Quotients.

In order to describe the representation theoretic problem, we will briefly sketch the geometric setup. For details, we refer to the articles [DC22] and [GR22].

Let $G$ be a finite group acting homomorphically and faithfully on a compact complex torus $T = \mathbb{C}^n/\Lambda$. Such an action is always affine-linear, i.e., of the form

$$\phi(g)z = \rho(g)z + t(g),$$

where the linear part $\rho: G \to \text{GL}(n, \mathbb{C})$ is a representation such that $\rho(g) \cdot \Lambda = \Lambda$ and the translation part $t: G \to T$ is a $1$-cocycle

$$\rho(g)t(h) - t(gh) + t(g) = 0.$$

Here, we view the torus $T$ as a $G$-module via $\rho$. Since a quotient of a complex torus by a finite group of translations is again a complex torus, we may assume that $\rho$ is faithful, or equivalently, $\phi$ is translation-free. Suppose that $\phi'$ is another action with the same linear part $\rho$, but a different translation part $t'$. If these actions are free, or at least free in codimension one, then Bieberbach’s theorems from crystallographic group theory (see [Cha86, II]) allow us to decide if the quotients $X$ and $X'$ of $T$ with respect to these actions are homeomorphic or not. It turns out that $X$ and $X'$ are homeomorphic if and only if there exist a matrix $C \in \text{GL}(2n, \mathbb{R})$ with $C \cdot \Lambda = \Lambda$, an automorphism $\psi$ of the group $G$ and an element $d \in T$, such that

1. $C \cdot \rho_{\psi} \cdot C^{-1} = \rho_{\psi} \circ \psi,
2. (\rho_{\psi}(g) - \text{id})d = C\ell(\psi^{-1}(g)) - t'(g)$ for all $g \in G$.

Here, the representation $\rho_{\psi}: G \to \text{GL}(2n, \mathbb{R})$ is the decomplexification of $\rho$. If such $C$ and $d$ exist, then a homeomorphism is given by

$$\Xi: X \to X', \quad x \mapsto Cx + d.$$
The quotients $X$ and $X'$ are biholomorphic if and only if $C$ can be chosen as a $\mathbb{C}$-linear matrix, see [DG22, Remark 4.6] or [HL21, Section 3].

Note that condition (1) says that the representations $\rho_R$ and $\rho_R \circ \psi$ are equivalent. In particular,

$$\psi \in \text{Stab}(\chi_R) := \{ \psi \in \text{Aut}(G) \mid \chi_R = \chi_R \circ \psi \},$$

where $\chi_R := \text{tr}(\rho_R)$.

Condition (2) says that the cocycles $t'$ and $C \cdot (t \circ \psi^{-1})$ differ by a coboundary, i.e., they are equal in the cohomology group $H^1(G, T)$.

Concretely, if the torus $T$ and the two actions $\phi$ and $\phi'$ are explicitly given, one can easily check the second condition, for example by a computer, provided that the full list of candidates for $C$ is known.

The problem to determine the solutions $C$ of the conjugation equation in condition (1) relates to semi-projective representations, in analogy to the extension problem discussed in Subsection 4.2, where we had to solve a similar conjugation equation, see [DG22]. Note that for each $\psi \in \text{Stab}(\chi_R)$ the representations $\rho_R$ and $\rho_R \circ \psi$ are equivalent because they have the same character. Thus, there exists a matrix $C_\psi \in \text{GL}(2n, \mathbb{R})$ fulfilling condition (1).

Assume now that $\rho$ is irreducible and of complex type, i.e., the Schur index $m(\chi) = 1$, where $\chi = \text{tr}(\rho)$. Then, the matrix $C_\psi$ is unique up to an element in the endomorphism algebra $\text{End}_G(\rho_R) \simeq \mathbb{C}$. Since $\chi_R = \chi + \mathfrak{r}$, the automorphism $\psi$ either stabilizes $\chi$ or maps $\chi$ to $\mathfrak{r}$. In the first case the matrix $C_\psi$ is $\mathbb{C}$-linear, whereas in the second case $\mathbb{C}$-antilinear. This yields a semi-projective representation

$$f : \text{Stab}(\chi_R) \to \text{PGL}(n, \mathbb{C}) \rtimes \mathbb{Z}_2.$$

Since $\rho$ is faithful, the representation $f$ is also faithful. The candidates for the linear part $C$ of potential homeomorphisms are the elements in the group

$$\mathcal{N} := \{ C \in \text{GL}(n, \mathbb{C}) \rtimes \mathbb{Z}_2 \mid [C] \in \text{im}(f), \ C \cdot \Lambda = \Lambda \}.$$

By construction, the group $\mathcal{N}$ sits inside the short exact sequence

$$1 \to A \to \mathcal{N} \to S \to 1,$$

where $A := \{ \mu \in \mathbb{C}^* \mid \mu \Lambda = \Lambda \}$ and $S \leq \text{Stab}(\chi_R)$ is the subgroup of automorphisms $\psi$ such that $f(\psi)$ has a representative $C_\psi$ with $C_\psi \cdot \Lambda = \Lambda$.

**Proposition 4.8.** The group $A$ is a finite cyclic group. In particular, $\mathcal{N}$ is finite.

**Proof.** We claim that $|\mu| = 1$ for all $\mu \in A$. Suppose there exists an element $\mu \in A$ with modulus different from 1; note that we can always assume $|\mu| < 1$, otherwise we replace $\mu$ by its inverse. Let $v \in \Lambda$ be a non-zero element of minimal norm. Then, $w := \mu v \in \Lambda$ has norm strictly less then $v$, which contradicts the minimality of $v$. Thus, $|\mu| = 1$ and the map defined by multiplication with $\mu$ restricted to closed balls $\overline{B_r}$ of any radius $r$. If $r$ is chosen large enough so that $\overline{B_r}$ contains a non-zero element of $\Lambda$, then the multiplication-homomorphism

$$A \to \text{Sym}(\overline{B_r} \cap \Lambda), \quad \mu \mapsto (v \mapsto \mu v)$$

is injective. Since $A$ is discrete, the intersection $\overline{B_r} \cap \Lambda$ is finite and it follows that $A$ is a finite cyclic group. □

**Remark 4.9.** The inclusion $i : \mathcal{N} \to \text{GL}(n, \mathbb{C}) \rtimes \mathbb{Z}_2$ is by construction a semi-linear lift of the semi-projective representation $f|_S : S \to \text{PGL}(n, \mathbb{C}) \rtimes \mathbb{Z}_2$.

**Example 4.10.** We discuss the example from [DG22], the one from [DG22] is similar. Here, the dimension is three and the lattice of the torus $T = \mathbb{C}^3/\Lambda$ is one of the following

$$\Lambda_1 := \mathbb{Z}[\zeta_3]^3 + \langle (u, u, u) \rangle \quad \text{or} \quad \Lambda_2 := \Lambda_1 + \langle (u, -u, 0) \rangle,$$

where $u := \frac{1}{3}(1 + 2\zeta_3)$.
The group $G$ is the Heisenberg group of order 27. It can be presented as

$$\text{He}(3) := \langle g, h, k \mid g^3 = h^3 = k^3 = [g, k] = [h, k] = 1, \ [g, h] = k \rangle.$$ 

This group has two irreducible complex three-dimensional representations: the first one, called the Schrödinger representation, is given by

$$\rho(g) := \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \rho(h) := \begin{pmatrix} 1 & \zeta_3 & 0 \\ \zeta_3 & 1 & \zeta_3 \\ 0 & 0 & 1 \end{pmatrix}, \quad \rho(k) := \begin{pmatrix} \zeta_3 \\ \zeta_3 \\ \zeta_3 \end{pmatrix},$$

and the second one is its complex conjugate $\overline{\rho}$. Note that they both have Schur index one. Furthermore, the decomplexification $\rho_1$ of $\rho$ is the unique irreducible 6-dimensional representation of $\text{He}(3)$. Hence, $\text{Stab}(\chi_{\mathbb{R}})$ is the full automorphism group of $\text{He}(3)$.

In this example, $A = \langle \zeta_6 \rangle \cong \mathbb{Z}_6$ and, for both lattices $\Lambda_1$ and $\Lambda_2$, the group $\mathcal{N}$ contains the $\mathbb{C}$-linear maps

$$C_1 := \begin{pmatrix} \zeta_3 \\ \zeta_3^2 \\ 1 \end{pmatrix}, \quad C_2 := -u \cdot \begin{pmatrix} 1 & \zeta_3 & \zeta_3^2 \\ \zeta_3 & 1 & \zeta_3 \\ \zeta_3^2 & \zeta_3 & 1 \end{pmatrix}, \quad C_3 := u \cdot \begin{pmatrix} 1 & 1 & 1 \\ 1 & \zeta_3 & \zeta_3^2 \\ \zeta_3 & 1 & \zeta_3 \end{pmatrix}$$

and the $\mathbb{C}$-antilinear map $C_4(z_1, z_2, z_3) = (\overline{z}_1, \overline{z}_2, \overline{z}_3)$. A MAGMA computation shows that the elements $C_1, \ldots, C_4$ generate a subgroup of $\mathcal{N}$ of order 2592 $= |A| \cdot |\text{Stab}(\chi_{\mathbb{R}})|$. Hence, this subgroup is actually equal to $\mathcal{N}$ and every class in the image of

$$f: \text{Stab}(\chi_{\mathbb{R}}) \to \text{PGL}(n, \mathbb{C}) \rtimes \mathbb{Z}_2$$

is represented by an element in $\mathcal{N}$. However, even if the semi-projective representation $f$ lifts to $\mathcal{N}$, this group is not a $\varphi$-twisted representation group for the action $\varphi: \text{Stab}(\chi_{\mathbb{R}}) \to \text{Aut}(\mathbb{C})$ induced by $f$. Indeed, a MAGMA computation yields $H^1(\text{Stab}(\chi_{\mathbb{R}}), \mathbb{C}) \cong \mathbb{Z}_3$ and $H^1(\mathcal{N}, \mathbb{C}^*) \cong \mathbb{Z}_6$, which violates the third condition of Proposition 3.11.

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Massimiliano Alessandro
Università degli Studi di Genova, DIMA Dipartimento di Matematica, I-16146 Genova, Italy
Email address: alessandro@dima.unige.it, massimiliano.alessandro@uni-bayreuth.de

Christian Gleissner
University of Bayreuth, Universitätsstr. 30, D-95447 Bayreuth, Germany
Email address: christian.gleissner@uni-bayreuth.de

Julia Kotonski
University of Bayreuth, Universitätsstr. 30, D-95447 Bayreuth, Germany
Email address: julia.kotonski@uni-bayreuth.de