Moring, Kristian; Schätzler, Leah

On the Hölder regularity for obstacle problems to porous medium type equations

Published in:
Journal of Evolution Equations

DOI:
10.1007/s00028-022-00840-4

Published: 01/12/2022

Document Version
Publisher's PDF, also known as Version of record

Published under the following license:
CC BY

Please cite the original version:
Moring, K., & Schätzler, L. (2022). On the Hölder regularity for obstacle problems to porous medium type equations. Journal of Evolution Equations, 22(4), 1-46. [81]. https://doi.org/10.1007/s00028-022-00840-4

This material is protected by copyright and other intellectual property rights, and duplication or sale of all or part of any of the repository collections is not permitted, except that material may be duplicated by you for your research use or educational purposes in electronic or print form. You must obtain permission for any other use. Electronic or print copies may not be offered, whether for sale or otherwise to anyone who is not an authorised user.
On the Hölder regularity for obstacle problems to porous medium type equations

KRISTIAN MORING AND LEAH SchÄTZLER

Abstract. We show that signed weak solutions to parabolic obstacle problems with porous medium-type structure are locally Hölder continuous, provided that the obstacle is Hölder continuous.

1. Introduction

Let \( \Omega_T := \Omega \times (0, T) \), where \( \Omega \subset \mathbb{R}^n \) is an open set and \( 0 < T < \infty \). In the present paper, we are concerned with the obstacle problem to partial differential equations, whose prototype is the porous medium equation (PME for short)

\[
\partial_t (|u|^{q-1}u) - \Delta u = 0 \quad \text{in} \quad \Omega_T
\]

with a parameter \( q \in (0, \infty) \). If \( 0 < q < 1 \), the equation is degenerate and if \( q > 1 \), it is singular. More generally, for \( q \in (0, \infty) \) we are concerned with partial differential equations of the type

\[
\partial_t (|u|^{q-1}u) - \text{div} A(x, t, u, \nabla u) = 0 \quad \text{in} \quad \Omega_T,
\]

(1.1)

where \( A : \Omega_T \times \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n \) is a Carathéodory function, i.e., it is measurable with respect to \( (x, t) \in \Omega_T \) for all \( (u, \xi) \in \mathbb{R} \times \mathbb{R}^n \) and continuous with respect to \( (u, \xi) \in \mathbb{R} \times \mathbb{R}^n \) for a.e. \( (x, t) \in \Omega_T \). Moreover, we assume that \( A \) satisfies the structure conditions

\[
\begin{cases}
A(x, t, u, \xi) \cdot \xi \geq C_0 |\xi|^2, \\
|A(x, t, u, \xi)| \leq C_1 |\xi|,
\end{cases}
\]

(1.2)

where \( C_0, C_1 > 0 \) are given constants. For the basic theory for the porous medium equation and its generalizations, we refer to the monographs [12,15,30–32].

We use a variational approach to define solutions to the obstacle problem to (1.1) with an obstacle function \( \psi \in C^0(\overline{\Omega_T}) \). Heuristically, multiplying (1.1) by \( \varphi(v - u) \),
where \( \varphi \) denotes a nonnegative cutoff function with \( \text{spt} \varphi \subseteq \Omega_T \) and \( v \) is a comparison map satisfying \( v \geq \psi \), and performing integration by parts, the variational inequality

\[
\int_\Omega \int_T \left[ \partial_t(|u|^{q-1}u) \varphi(v-u) + A(x,t,u,\nabla u) \cdot \nabla (\varphi(v-u)) \right] \, dx \, dt \geq 0,
\]

must hold true for a solution \( u \) that is above the given obstacle \( \psi \). Since we do not assume regularity properties for \( u \) in the time direction, the first term is defined more rigorously in the next section. Existence results for variational solutions to the obstacle problem to porous medium type equations can be found in [1,8,27,28].

However, for the singular case can be found in [15] and for the treatment of signed solutions we mention [23]. For more recent developments, we refer to [4,9,24,26]. However, for the obstacle problem the theory is not complete yet. Hölder continuity was proven for nonnegative solutions in the degenerate case were considered. The proof in the obstacle-free case. The first proof goes back to DiBenedetto and Friedman [14], where

1.1. Background

The theory of Hölder continuity for porous medium type equations is well developed rigorously in the next section. Existence results for variational solutions to the obstacle problem to porous medium type equations can be found in [1,8,27,28].

At this stage, we will state our main result. The quantitative Hölder estimate (1.3) and thus the theorem is formulated for globally bounded local weak solutions

\[
q\text{-dist}(K, \Gamma) := \inf_{(x,t) \in K, (y,s) \in \Gamma} \left\{ |x-y| + \|u\|_\infty^{1-q} |t-s|^{\frac{1}{2}} \right\}.
\]

If the obstacle function satisfies \( \psi \in C^{0;\beta,0}_{\Omega_T} \), then also the Hölder seminorm \( [\psi]_{C^{0;\beta,0}_{\Omega_T}} \) of \( \psi \) with respect to the intrinsic \( q \)-distance is finite.

**Theorem 1.1.** Let \( u \) be a bounded local weak solution to the obstacle problem to (1.1) with \( q \in (0, \infty) \) and structure conditions (1.2) and a Hölder continuous obstacle function \( \psi \in C^{0;\beta,0}_{\Omega_T} \) for some \( \beta \in (0,1) \) in the sense of Definition 2.1. Then, \( u \) is locally Hölder continuous. More precisely, there exists an exponent \( \gamma = \gamma(n, q, C_o, C_1, \beta) \in (0, \beta) \) and a constant \( c = c(n, q, C_o, C_1, \beta) \) such that the bound

\[
|u(x_1, t_1) - u(x_2, t_2)| \leq c \max \left\{ \|u\|_{[\psi]}, [\psi] \right\} \left( |x_1 - x_2| + \|u\|_\infty^{1-q} |t_1 - t_2|^{\frac{1}{2}} \right) \frac{1}{\min\{1, q - \text{dist}(K, \Gamma)\}^\beta}
\]

holds for any compact subset \( K \subseteq \Omega_T \) and \( (x_1, t_1), (x_2, t_2) \in K \).

1.1. Background

The theory of Hölder continuity for porous medium type equations is well developed in the obstacle-free case. The first proof goes back to DiBenedetto and Friedman [14], in which nonnegative solutions in the degenerate case were considered. The proof in the singular case can be found in [15] and for the treatment of signed solutions we mention [23]. For more recent developments, we refer to [4,9,24,26]. However, for the obstacle problem the theory is not complete yet. Hölder continuity was proven for quasilinear problems in [29] and for problems with quadratic growth in [11]. In the case of porous medium type equations, Hölder continuity for nonnegative solutions
to the obstacle problem has been treated in the recent papers [7] and [10]. The former
concerns the degenerate case for the PME, and the latter the singular case for more
general equations with structural conditions analogous to (1.2). However, especially
the theory for signed solutions is missing, which we are addressing in this paper.

The obstacle problem is used as a standard tool in nonlinear potential theory; see,
e.g., [18,25]. For the porous medium equation, a crucial approximation scheme in this
connection exploits the obstacle problem in [19]. As further applications, we mention
questions related to boundary regularity addressed in [3,21]. In this context, we also
point out that there is an alternative approach to the obstacle problem, in which the
solution is defined as the smallest weak supersolution lying above the given obstacle \( \psi \). This approach allows to consider fairly irregular obstacles as in [22]. For a study
of the connection between these two different notions of solutions, we refer to [2].

1.2. Strategy of the proof

In our proof, we use a similar strategy as in [7] and [10] for nonnegative obstacles,
which relies on a De Giorgi-type iteration argument. The idea is to construct a sequence
of cylinders shrinking to a common vertex. In each of these cylinders, we consider
measure theoretic alternatives, which we will call first and second alternative. Roughly
speaking, in the first alternative the solution \( u \) is bounded away from its essential
supremum or infimum in a large portion of the considered cylinder, where “large”
means that the measure of the set where \( u \) is bounded away from its supremum or
infimum is at least a certain fraction of the measure of the whole cylinder determined
by the data \( n, q, C_o, \) and \( C_1 \). In contrast, in the second alternative \( u \) is close to the same
extremum in some positive portion of the cylinder. For a precise definition, we refer
to Sect. 5. In both alternatives, the solution is bounded away from one of its extrema
in a quantifiable way pointwise almost everywhere in a smaller cylinder, where the
distance of \( u \) to its extremum depends on the chosen portion in the second alternative.
For the precise arguments, we refer to Sect. 5 in case of the first and Sect. 6 in case of
the second alternative. We then show in Sect. 7 that when passing to the subsequent
cylinder in the sequence of nested, shrinking cylinders mentioned above, the oscillation
of the solution is reduced by a fixed amount. Finally, all cases considered in Sect. 7
are combined in Sect. 8 to deduce an oscillation decay estimate, which is in turn used
to prove the quantitative Hölder estimate (1.3).

In the heart of the De Giorgi-type iteration are energy estimates for truncations of
solutions, which are stated and proved in Sect. 3. In case of the second alternative, we
also exploit the logarithmic estimates from Sect. 4. When deriving suitable energy or
logarithmic estimates for a solution to the obstacle problem, we use a comparison map
depending on the solution itself. Then, additional attention has to be paid to guarantee
that it is admissible, especially that it is sufficiently regular in time and stays above
the given obstacle.

While the energy estimate for truncations from above takes a similar form as in
the obstacle-free case, the obstacle will play a role in the estimate for truncations
from below. In the De Giorgi-type iteration argument, this will be taken into account in the upper bound for the oscillation of \( u \). Namely, such an upper bound should be sufficiently large compared to the oscillation of the obstacle.

In order to balance the inhomogeneous scaling behavior of the PME, we will work in cylinders which respect the intrinsic geometry of the equation. In particular, we will use cylinders of the form

\[
Q_{\rho, \theta \rho^2}(x_0, t_0) := B_{\rho}(x_0) \times (t_0 - \theta \rho^2, t_0),
\]

in which the scaling parameter \( \theta \) is comparable to \( |u|^{q-1} \). In contrast to the proof for the singular equations in [10], we treat both degenerate and singular cases with cylinders taking the same form (as in the obstacle-free case [23]).

We will separate different cases in the proof: when the solution \( u \) is near zero compared to its oscillation, and when \( u \) is bounded away from zero by a fraction of its oscillation. In the latter case, the equation behaves like a linear one. Additional challenges in the case of signed solutions are given by the fact that when \( u \) is near zero, the sign of \( u \) may change in the considered cylinder. Particularly, when applying the second alternative we use a technical argument, which is visible in Lemma 7.1 to be able to avoid the set where \( u \) becomes degenerate/singular. Furthermore, when dealing with the case where \( u \) is negative and bounded away from zero in Lemma 7.3, additional care is needed in the construction of cylinders in order to ensure that the supremum of the obstacle function is small enough compared to the solution \( u \).

It would be interesting to obtain regularity up to the boundary when suitable boundary values are prescribed. However, this is a topic for further research.

2. Definitions and auxiliary results

In order to give a formal definition of solutions, we consider the class of functions

\[
K_{\psi}(\Omega_T) := \left\{ v \in C^0((0, T); L^{q+1}_{\text{loc}}(\Omega)) : v \in L^2_{\text{loc}}(0, T; H^1_{\text{loc}}(\Omega)), v \geq \psi \text{ a.e. in } \Omega_T \right\}.
\]

Admissible comparison maps will be contained in the class of functions

\[
K'_{\psi}(\Omega_T) := \left\{ v \in K_{\psi}(\Omega_T) : \partial_t v \in L^{q+1}_{\text{loc}}(\Omega_T) \right\}.
\]

Definition 2.1. We say that \( u \in K_{\psi}(\Omega_T) \) is a local weak solution to the obstacle problem associated with (1.1) if and only if

\[
\langle \partial_t |u|^{q-1} u, \varphi(v-u) \rangle + \iint_{\Omega_T} A(x, t, u, \nabla u) \cdot \nabla (\varphi(v-u)) \, dx \, dt \geq 0 \tag{2.1}
\]

holds true for all comparison maps \( v \in K'_{\psi}(\Omega_T) \) and every test function \( \varphi \in C^\infty_0(\Omega_T; \mathbb{R}_{\geq 0}) \). The time term above is defined as
\[
\langle \partial_t (|u|^{q-1}u), \varphi (v-u) \rangle := \iint_{\Omega_T} \left[ \partial_t \varphi \left( \frac{q}{q+1} |u|^{q+1} - |u|^{q-1}uv \right) - \varphi |u|^{q-1}u \partial_t v \right] \, dx \, dt.
\]

The proof of the following lemma follows the lines of [7, Theorem 5.1] and [10, Theorem 4.1].

**Lemma 2.2.** Let \( 0 < q < \frac{n+2}{(n-2)_+} \) and \( u \) be a local weak solution to the obstacle problem in the sense of Definition 2.1. Then, \( u \) is locally bounded.

For \( z_0 = (x_0, t_0) \in \Omega_T \), we will work with cylinders of the form

\[
Q_{\rho, s}(z_0) := B_{\rho}(x_0) \times (t_0 - s, t_0).
\]

The lateral boundary of \( Q_{\rho, s}(z_0) \) is \( \partial B_{\rho}(x_0) \times (t_0 - s, t_0) \) and the parabolic boundary of \( Q_{\rho, s}(z_0) \) is \( B_{\rho}(x_0) \times \{ t_0 - s \} \cup \partial B_{\rho}(x_0) \times (t_0 - s, t_0) \). For \( b \in \mathbb{R} \) and \( \alpha > 0 \), we denote the signed \( \alpha \)-power of \( b \) by

\[
b^\alpha := \begin{cases} 
|b|^\alpha & \text{if } b \neq 0, \\
0 & \text{if } b = 0.
\end{cases}
\]

Furthermore, we will write \( \sup, \inf, \) and \( \text{osc} \) instead of \( \text{ess sup}, \text{ess inf}, \) and \( \text{ess osc} \) to simplify the notation.

We will exploit the following mollification in time. For \( v \in L^p(\Omega_T) \) and \( h > 0 \), define

\[
\|v\|_h(x, t) := \frac{1}{h} \int_0^t e^{\frac{x-s}{h}} v(x, s) \, ds.
\]

We collect some useful properties of the mollification in the following lemma, see [20, Lemma 2.9] and [6, Appendix B].

**Lemma 2.3.** Let \( v \) and \( \|v\|_h \) be as above and \( p \geq 1 \). Then, the following properties hold:

(i) If \( v \in L^p(\Omega_T) \), then

\[
\|v\|_h \to v \quad \text{in } L^p(\Omega_T) \text{ as } h \to 0.
\]

(ii) Let \( v \in L^p(0, T; W^{1,p}(\Omega)) \). Then,

\[
\|v\|_h \to v \quad \text{in } L^p(0, T; W^{1,p}(\Omega)) \text{ as } h \to 0.
\]

(iii) If \( v \in C^0(\overline{\Omega_T}) \) and \( \Omega \subset \mathbb{R}^n \) is a bounded set, then

\[
\|v\|_h \to v \quad \text{uniformly in } \Omega_T \text{ as } h \to 0.
\]

(iv) The weak time derivative \( \partial_t \|v\|_h \) exists in \( \Omega_T \) and is given by the formula

\[
\partial_t \|v\|_h = \frac{1}{h} (v - \|v\|_h).
\]
In this section, we recall some standard results needed in the proofs. We begin with a special case of the Sobolev inequality, cf. [13, Chapter I, Proposition 3.1].

**Lemma 2.4.** Let \( v \in L^2(0, T; W^{1,2}_0(\Omega)) \). Then, there exists \( c = c(n) > 0 \) such that

\[
\int_0^T \int_\Omega |v|^{\frac{2(n+2)}{n}} \, dx \, dt \leq c \left( \int_0^T \int_\Omega |\nabla v|^2 \, dx \, dt \right)^{\frac{2}{n}} \sup_{0 < t < T} \int_\Omega |v|^2 \, dx.
\]

We also make use of De Giorgi’s isoperimetric inequality and the so-called fast geometric convergence [13, Chapter I, Lemma 2.2], [17, Lemma 7.1], which we state next.

**Lemma 2.5.** Let \( k < l \) be real numbers and \( B_\rho(x_0) \subset \mathbb{R}^n \). Then, for any \( v \in W^{1,1}(B_\rho(x_0)) \) there exists a constant \( c = c(n) > 0 \) such that

\[
(l - k) \left| B_\rho(x_0) \cap \{ v > l \} \right| \leq c \rho^{n+1} \int_{B_\rho(x_0) \cap \{ v < k \}} |\nabla v| \, dx.
\]

**Lemma 2.6.** Suppose that \( \{ Y_i \}_{i \in \mathbb{N}_0} \) is a sequence of positive real numbers that satisfy

\[
Y_{i+1} \leq C B^i Y_i^{1+\sigma} \quad \text{for all} \; i \geq 0,
\]

with constants \( C, \sigma > 0 \) and \( B > 1 \). Then, \( Y_i \to 0 \) as \( i \to \infty \) whenever

\[
Y_0 \leq C^{-\frac{1}{\sigma}} B^{-\frac{1}{\sigma^2}}.
\]

### 3. Energy estimates

For \( w, k \in \mathbb{R} \), let us define

\[
g_\pm(w, k) := \pm q \int_k^w |s|^{q-1}(s - k)_\pm \, ds.
\]

The following estimates follow from the definition above; see, e.g., [5, Lemma 2.2].

**Lemma 3.1.** There exists a constant \( c = c(q) > 0 \) such that for all \( w, k \in \mathbb{R} \) and \( q > 0 \), the inequality

\[
\frac{1}{c} (|w| + |k|)^{q-1} (w - k)^2_\pm \leq g_\pm(w, k) \leq c (|w| + |k|)^{q-1} (w - k)^2_\pm
\]

holds true.

Next, we give energy estimates for weak solutions to the obstacle problems. Note that in the estimate involving \((u - k)_\pm\) only levels \( k \) larger than the obstacle function are admissible, whereas there is no restriction on the admissible levels in the estimate involving \((u - k)_-\).
Lemma 3.2. Let \( z_o = (x_o, t_o) \in \Omega_T \) and \( Q_{\rho,s}(z_o) \subseteq \Omega_T \). Further, for \( \psi \in C^0(\Omega_T) \) we assume that \( u \in K^\psi(\Omega_T) \) is a local weak solution to the obstacle problem to (1.1) with structure conditions (1.2) in the sense of Definition 2.1. Then, for any function \( \varphi \in C^\infty(Q_{\rho,s}(z_o); \mathbb{R}_{\geq 0}) \) vanishing on the lateral boundary of \( Q_{\rho,s}(z_o) \) there exists \( c = c(C_o, C_1) > 0 \) such that the following estimates hold.

(i) For all \( k \geq \sup_{Q_{\rho,s}(z_o)} \psi \), we have

\[
\max \left\{ \sup_{t_o - s < t < t_o} \int_{B_{\rho}(x_o) \times [t]} \varphi^2 g_+ (u, k) \, dx, \frac{C_o}{2} \int_{Q_{\rho,s}(z_o)} \varphi^2 |\nabla (u - k)_+|^2 \, dx \, dt \right\} 
\leq c \int_{Q_{\rho,s}(z_o)} [(u - k)_+^2 |\nabla \varphi|^2 + g_+ (u, k)|\partial_t \varphi^2|] \, dx \, dt 
+ \int_{B_{\rho}(x_o) \times \{t_o - s\}} \varphi^2 g_+ (u, k) \, dx.
\]

(ii) For arbitrary \( k \in \mathbb{R} \), we have

\[
\max \left\{ \sup_{t_o - s < t < t_o} \int_{B_{\rho}(x_o) \times [t]} \varphi^2 g_- (u, k) \, dx, \frac{C_o}{2} \int_{Q_{\rho,s}(z_o)} \varphi^2 |\nabla (u - k)_-|^2 \, dx \, dt \right\} 
\leq c \int_{Q_{\rho,s}(z_o)} [(u - k)_-^2 |\nabla \varphi|^2 + g_- (u, k)|\partial_t \varphi^2|] \, dx \, dt 
+ \int_{B_{\rho}(x_o) \times \{t_o - s\}} \varphi^2 g_- (u, k) \, dx.
\]

Proof. In the following, we omit \( z_o \) for simplification. We first prove (i). Let \( \varphi \in C^\infty(Q_{\rho,s}; \mathbb{R}_{\geq 0}) \) vanish on the lateral boundary \( \partial B_{\rho} \times (-s, 0) \) of \( Q_{\rho,s} \). Further, we define \( \xi_\varepsilon \in W^{1,\infty}_0([-s, 0]; [0, 1]) \) by

\[
\xi_\varepsilon (t) := \begin{cases} 
0, & \text{for } -s \leq t \leq t_1 - \varepsilon, \\
1 + \frac{t - t_1}{\varepsilon}, & \text{for } t_1 - \varepsilon < t \leq t_1, \\
1, & \text{for } t_1 < t < t_2, \\
1 - \frac{t - t_2}{\varepsilon}, & \text{for } t_2 \leq t < t_2 + \varepsilon, \\
0, & \text{for } t_2 + \varepsilon \leq t \leq 0.
\end{cases}
\]

and use \( \eta := \varphi^2 (\xi_\varepsilon)_\delta \), in which \( (\xi_\varepsilon)_\delta \) is a standard mollification [17, Sect. 2.2] of \( \xi_\varepsilon \) with \( 0 < \delta < \frac{\varepsilon}{2} \), as a test function in (2.1). Moreover, we define

\[
w_h := \|u\| - (\|u\|_h - k)_+ + \|\psi - \|\psi\|_h\|_{L^\infty(Q_{\rho,s})}.
\]

Note that there hold \( w_h \in C^0((0, T); L^{q+1}_{\text{loc}}(\Omega)) \cap L^2_{\text{loc}}(0, T; H^1_{\text{loc}}(\Omega)) \) and

\[
\partial_t w_h = \begin{cases} 
\frac{1}{h} (u - \|u\|_h) \text{ in } \{\|u\|_h \leq k\}, \\
0, & \text{otherwise}
\end{cases} \in L^{q+1}_{\text{loc}}(\Omega_T)
\]
by Lemma 2.3. Further, we have that \( w_h \geq \psi \) in \( Q_{\rho, s} \), since \( \|u\|_h \geq \|\psi\|_h \) in \( \Omega_T \) and \( k \geq \sup_{Q_{\rho, s}} \psi \). Since it is sufficient that \( w_h \) satisfies the obstacle condition in \( \text{spt}(\eta) \subset Q_{\rho, s} \), \( w_h \) is an admissible comparison map in (2.1). Therefore, we obtain that

\[
\langle \partial_t u^q, \eta(w_h - u) \rangle + \int_{\Omega_T} A(x, t, u, \nabla u) \cdot \nabla (\eta(w_h - u)) \, dx \, dt \geq 0. \quad (3.1)
\]

In the following, we treat the two terms separately. First, by the formula for \( \partial_t w_h \) above and since \( (u^q - \|u\|_h^q)(u - \|u\|_h) \geq 0 \), we conclude that

\[
\int_{Q_{\rho, s}} \eta u^q \partial_t w_h \, dx \, dt = \frac{1}{h} \int_{Q_{\rho, s}} \eta (u^q - \|u\|_h^q) (u - \|u\|_h) \, dx \, dt \leq \int_{Q_{\rho, s}} \eta \|u\|_h^q \partial_t (\|u\|_h - (\|u\|_h - k)_+) \, dx \, dt.
\]

Integrating by parts leads to

\[
\int_{Q_{\rho, s}} \eta u^q \partial_t w_h \, dx \, dt \geq \int_{Q_{\rho, s}} \eta \|u\|_h^q \partial_t (\|u\|_h - (\|u\|_h - k)_+) \, dx \, dt
\]

\[
= \int_{Q_{\rho, s}} \frac{1}{q+1} \eta \partial_t |\|u\|_h|^{q+1} - \eta \|u\|_h^{q+1} \partial_t (\|u\|_h - k)_+ \, dx \, dt
\]

\[
= \int_{Q_{\rho, s}} -\frac{1}{q+1} \partial_t \eta \|u\|_h^{q+1} + \partial_t \eta \|u\|_h^{q+1} (\|u\|_h - k)_+ \, dx \, dt
\]

\[
+ \int_{Q_{\rho, s}} \eta \partial_t (\|u\|_h^{q+1} - \|u\|_h^{q+1} (\|u\|_h - k)_+) \, dx \, dt.
\]

Writing the last term on the right-hand side of the preceding inequality as

\[
\partial_t (\|u\|_h^{q+1} (\|u\|_h - k)_+) = \partial_t \left( q \int_k^{\|u\|_h} |s|^{q-1} (s - k)_+ \, ds \right)
\]

\[
= \partial_t g_+(\|u\|_h, k)
\]

yields

\[
\int_{Q_{\rho, s}} \eta u^q \partial_t w_h \, dx \, dt
\]

\[
\geq - \int_{Q_{\rho, s}} \partial_t \eta \left[ \frac{1}{q+1} |\|u\|_h|^{q+1} - |\|u\|_h|^{q+1} (\|u\|_h - k)_+ \right] \, dx \, dt
\]

\[
- \int_{Q_{\rho, s}} \partial_t (\|u\|_h, k) \, dx \, dt.
\]

Inserting this into the first term of (3.1), we see that
\[
\langle \partial_t u^q, \eta (w_h - u) \rangle = \iint_{Q_{\rho,s}} \partial_t \eta \left( \frac{q}{q+1} |u|^{q+1} - u^q w_h \right) - \eta u^q \partial_t w_h \, dx \, dt \\
\leq \iint_{Q_{\rho,s}} \partial_t \eta \left( \frac{q}{q+1} |u|^{q+1} + \frac{1}{q+1} \|u\|_h^{q+1} - u^q \|u\|_h \right) \, dx \, dt \\
+ \iint_{Q_{\rho,s}} \partial_t \eta \left( u^q - \|u\|_h^{q-1} \|u\|_h (\|u\|_h - k)_+ \right) \, dx \, dt \\
+ \iint_{Q_{\rho,s}} \partial_t \eta \| \psi - \| \psi \|_h \|_{L^\infty(Q_{\rho,s})} \, dx \, dt \\
- \iint_{Q_{\rho,s}} \partial_t \eta g_+(\|u\|_h, k) \, dx \, dt.
\]

Hence, passing to the limit \( h \downarrow 0 \) with the aid of Lemma 2.3 (i) and (iii), we obtain that

\[
\limsup_{h \downarrow 0} \langle \partial_t u^q, \eta (w_h - u) \rangle \leq \iint_{Q_{\rho,s}} \partial_t \eta g_+(u, k) \, dx \, dt. \tag{3.2}
\]

In order to treat the second term in (3.1), observe that

\[
\nabla (\eta (w_h - u)) \longrightarrow -\nabla (\eta (u - k)_+) \text{ in } L^2(Q_{\rho,s}) \text{ as } h \downarrow 0,
\]

by Lemma 2.3 (ii). Since \( A(x, t, u, \nabla u) \in L^2(Q_{\rho,s}, \mathbb{R}^n) \) by growth condition (1.2)_2, this implies

\[
\lim_{h \downarrow 0} \iint_{Q_{\rho,s}} A(x, t, u, \nabla u) \cdot \nabla (\eta (w_h - u)) \, dx \, dt \\
= - \iint_{Q_{\rho,s}} 2(\xi \varepsilon) \delta \varphi (u - k)_+ A(x, t, u, \nabla u) \cdot \nabla \varphi \, dx \, dt \\
- \iint_{Q_{\rho,s}} \eta A(x, t, u, \nabla u) \cdot \nabla (u - k)_+ \, dx \, dt. \tag{3.3}
\]

By means of (1.2)_1 and since \( \eta \geq 0 \), we observe that

\[
\eta A(x, t, u, \nabla u) \cdot \nabla (u - k)_+ = \eta A(x, t, u, \nabla(u - k)_+) \cdot \nabla(u - k)_+ \\
\geq C_\alpha |\nabla(u - k)_+|^2.
\]

Further, by (1.2)_2 and Young’s inequality with parameter \( \frac{C_\alpha}{4C_1} \) (see [16, Appendix B]), we find that

\[
|2(\xi \varepsilon) \delta \varphi (u - k)_+ A(x, t, u, \nabla u) \cdot \nabla \varphi| \\
\leq 2C_1(\xi \varepsilon) \delta |\varphi| |\nabla(u - k)_+| (u - k)_+ |\nabla \varphi| \\
\leq \frac{C_\alpha}{2} (\xi \varepsilon)^2 |\nabla(u - k)_+|^2 + \frac{2C_1^2(\xi \varepsilon)}{C_\alpha} (\xi \varepsilon)^2 |\nabla \varphi|^2 (u - k)_+^2.
\]
Inserting the preceding two inequalities into (3.3), we obtain that
\[
\lim_{h \downarrow 0} \int_{Q_{r,s}} A(x, t, u, \nabla u) \cdot \nabla (\eta(w_h - u)) \, dx \, dt \\
\leq - \frac{C_o}{2} \int_{Q_{r,s}} \eta |\nabla (u - k)_{+}|^2 \, dx \, dt \\
+ \frac{2C_1^2}{C_o} \int_{Q_{r,s}} (\xi \delta) |\nabla \varphi|^2 (u - k)^2_{+} \, dx \, dt.
\]
Together with (3.1) and (3.2), we conclude that
\[
\frac{C_o}{2} \int_{Q_{r,s}} \eta |\nabla (u - k)_{+}|^2 \, dx \, dt \\
\leq \int_{Q_{r,s}} \partial_t \eta g^+(u, k) \, dx \, dt + \frac{2C_1^2}{C_o} \int_{Q_{r,s}} (\xi \delta) |\nabla \varphi|^2 (u - k)^2_{+} \, dx \, dt.
\]
By first passing to the limit \( \delta \downarrow 0 \), and subsequently \( \varepsilon \downarrow 0 \), we get
\[
\int_{B_r \times \{t_2\}} \varphi^2 g^+(u, k) \, dx + \frac{C_2}{2} \int_{t_1}^{t_2} \int_{B_r} \varphi^2 |\nabla (u - k)_{+}|^2 \, dx \, dt \\
\leq c \int_{Q_{r,s}} |\nabla \varphi|^2 (u - k)^2_{+} + |\partial_t \varphi^2| g^+(u, k) \, dx \, dt + \int_{B_r \times \{t_1\}} \varphi^2 g^+(u, k) \, dx.
\]
Since all terms are nonnegative, we infer the desired energy estimate by first discarding
the first term on the left-hand side and passing to the limits \( t_1 \downarrow -s \), \( t_2 \uparrow 0 \) and then
discarding the second term on the left-hand side, passing to the limit \( t_1 \downarrow -s \) and taking
the supremum over all \( t_2 \in (-s, 0) \).

In order to prove (ii), we use the comparison function \( w_h := \|u\|_h + (\|u\|_h - k)_- + \|
\psi - \|\psi\|_h\|_{L^\infty(Q_{r,s})} \) with an arbitrary level \( k \in \mathbb{R} \) and proceed similarly as in (i).

**4. Logarithmic estimates**

In this section, we will state a logarithmic estimate as in [7] and [10]. Let \( 0 < \gamma < \Gamma \)
and define
\[
\phi(a) := \phi_{\Gamma, \gamma}(a) := \left[ \log \left( \frac{\Gamma}{\Gamma + \gamma - a} \right) \right]_+ \quad \text{for } a < \Gamma + \gamma.
\]
Observe that for \( a \leq \Gamma \), we have
\[
0 \leq \phi(a) \leq \log \left( \frac{\Gamma}{\gamma} \right) \quad \text{and} \quad 0 \leq \phi'(a) \leq \frac{1}{\gamma} \quad \text{when } a \neq \gamma,
\]
with \( \phi(a) = 0 \) for \( a \leq \gamma \). Further, we have \( \phi''(a) = (\phi')^2(a) \) for \( a \neq \gamma \). Note that \( \phi^2 \)
is differentiable in \([0, \Gamma]\) such that the Lipschitz continuous derivative \((\phi^2)'\) satisfies
\[
\left( \phi^2 \right)' = 2\phi \phi' \quad \text{and} \quad \left( \phi^2 \right)'' = 2(1 + \phi) (\phi')^2 \quad \text{in } [0, \Gamma] \setminus \{\gamma\}.
\]
With this information at hand, we are able to prove the following lemma.
Lemma 4.1. Let $B_{\rho_1}(x_o) \subseteq B_{\rho_2}(x_o) \subseteq \Omega$, $0 < t_1 < t_2 < T$ and $Q_2 := B_{\rho_2}(x_o) \times (t_1, t_2)$. Further, define $\Gamma := \sup_{Q_2} (u - k)_+$ and consider some parameter $\gamma \in (0, \Gamma)$. Assume that $\psi \in C^0(\Omega_T)$ and let $u \in K_\psi(\Omega_T)$ be a local weak solution to the obstacle problem (1.1) with structure conditions (1.2) according to Definition 2.1. Then, there exists $c = c(q, C_o, C_1) > 0$ such that the following estimates hold.

(i) For any $k \geq \sup_{Q_2} \psi$, we have

$$
\sup_{t \in (t_1, t_2)} \int_{B_{\rho_1}(x_o)} \int_k^u |s|^{q-1} \left( \phi^2 \right)' ((s - k)_+) \, ds \, dx
\leq \int_{B_{\rho_2}(x_o) \times [t_1]} \int_k^u |s|^{q-1} \left( \phi^2 \right)' ((s - k)_+) \, ds \, dx
+ \frac{c}{(\rho_2 - \rho_1)^2} \int_{Q_2} \phi ((u - k)_+) \, dx \, dt.
$$

(ii) For any $k \in \mathbb{R}$, we have that

$$
\sup_{t \in (t_1, t_2)} \int_{B_{\rho_1}(x_o)} \int_k^u |s|^{q-1} \left( \phi^2 \right)' ((s - k)_-) \, ds \, dx
\leq \int_{B_{\rho_2}(x_o) \times [t_1]} \int_k^u |s|^{q-1} \left( \phi^2 \right)' ((s - k)_-) \, ds \, dx
+ \frac{c}{(\rho_2 - \rho_1)^2} \int_{Q_2} \phi ((u - k)_-) \, dx \, dt.
$$

Proof. In the following, we omit $(x_o, t_o)$ for simplicity. We start with the proof of (i). Since all terms in the asserted estimate depend continuously on $k$, we may assume that $k > \sup_{Q_2} \psi$. We would like to use

$$
w_h := \|u\|_h - \lambda \left( \phi^2 \right)' ((\|u\|_h - k)_+) + \|\psi - \|\psi\|_h\|_{L^\infty(Q_2)}
$$

with

$$
0 < \lambda \leq \frac{(k - \sup_{Q_2} \psi) \left( \phi^2 \right)'}{\sup_{[0, \Gamma]} \left( \phi^2 \right)'}
$$

as comparison map in (2.1). By Lemma 2.3 and since $\left( \phi^2 \right)'$ is a Lipschitz continuous function, we have that $w_h \in C^0((0, T); L_{loc}^{q+1}(\Omega)) \cap L_{loc}^2(0, T; H_{loc}^1(\Omega))$ with $\partial_t w_h \in L_{loc}^{q+1}(\Omega_T)$. Moreover, if $\|u\|_h \leq k$, we find that

$$
w_h = \|u\|_h + \|\psi - \|\psi\|_h\|_{L^\infty(Q_2)} \geq \|\psi\|_h + \|\psi - \|\psi\|_h\|_{L^\infty(Q_2)} \geq \psi
$$

in $Q_2$, and for $\|u\|_h > k$ we have that

$$
w_h > k - \lambda \left( \phi^2 \right)' ((\|u\|_h - k)_+) \geq \sup_{Q_2} \psi
$$
by the restriction on $\lambda$. Consequently, $w_h$ is an admissible comparison map in (2.1). Thus, for any $\varphi \in C_0^\infty(Q_2; \mathbb{R}_{\geq 0})$ we obtain that

$$I_h + \Pi_h := \langle \partial_t u^q, \varphi(w_h - u) \rangle + \iint_{\Omega_T} A(x, t, u, \nabla u) \cdot \nabla (\varphi(w_h - u)) \, dx \, dt \geq 0.$$  

In the following, we estimate these terms separately. First, we calculate

$$\partial_t w_h = \partial_t [u]_h \left(1 - \lambda (\phi^2)'\left([u]_h - k\right)_+\right) = \frac{1}{h} \left(1 - \lambda (\phi^2)'\left([u]_h - k\right)_+\right). \quad (4.1)$$

Since the derivative $\partial_t [u]_h$ vanishes a.e. in the set $\{(||u||_h - k)_+ = \gamma\}$, the terms involving $(\phi^2)'$ are well defined a.e. in $Q_2$. Further, decreasing $\lambda$ if necessary, we may assume that the last factor is positive, which allows us to estimate

$$(u^q - ||u||_h^q) \partial_t w_h = \frac{1}{h} (u^q - ||u||_h^q) (1 - \lambda (\phi^2)'\left([u]_h - k\right)_+) \geq 0,$$

by recalling $(u^q - ||u||_h^q)(u - ||u||_h) \geq 0$. Together with integration by parts and the fact that

$$\partial_t [u]_h \gamma (\phi^2)'\left([u]_h - k\right)_+ = \partial_t \left(\lambda q \int_k^{||u||_h} |s|^{q-1}(\phi^2)'((s - k)_+) \, ds\right),$$

this implies that

$$\iint_{\Omega_T} \varphi u^q \partial_t w_h \, dx \, dt$$

$$\geq \iint_{\Omega_T} \varphi [u]_h^q \partial_t w_h \, dx \, dt$$

$$= \iint_{\Omega_T} \varphi \partial_t \left(\frac{1}{q+1} [u]_h^{q+1}\right) - \lambda \varphi [u]_h^q \partial_t \left((\phi^2)'\left([u]_h - k\right)_+\right) \, dx \, dt$$

$$= \iint_{\Omega_T} \partial_t \varphi \left(-\frac{1}{q+1} [u]_h^{q+1} + [u]_h^q \lambda (\phi^2)'\left([u]_h - k\right)_+\right) \, dx \, dt$$

$$- q\lambda \iint_{\Omega_T} \partial_t \varphi \int_k^{||u||_h} |s|^{q-1}(\phi^2)'((s - k)_+) \, ds \, dx \, dt.$$  

The last term is a result of integration by parts once more. Recalling the definition of $I_h$ and inserting the preceding inequality yields

$$I_h = \iint_{\Omega_T} \partial_t \varphi \left(\frac{q}{q+1}[u]^{q+1} - u^q w_h\right) - \varphi u^q \partial_t w_h \, dx \, dt$$

$$\leq \iint_{\Omega_T} \partial_t \varphi \left(\frac{q}{q+1}[u]^{q+1} + \frac{1}{q+1} [u]_h^{q+1} - u^q [u]_h\right) \, dx \, dt$$

$$+ \iint_{\Omega_T} \lambda \partial_t \varphi (u^q - [u]_h^q)(\phi^2)'\left([u]_h - k\right)_+ \, dx \, dt.$$
\[ + \int_{\Omega_T} \partial_t \varphi u^q \| \psi - \| \psi \|_{L^{\infty}(Q_2)} \, dx \, dt + q \lambda \int_{\Omega_T} \partial_t \varphi \int_k |s|^{q-1} \left( \varphi^2 \right)' ((s - k_+)) \, ds \, dx \, dt. \]

Therefore, passing to the limit \( h \downarrow 0 \), by means of Lemma 2.3 we obtain that

\[ \lim_{h \to 0} I_h \leq q \lambda \int_{\Omega_T} \partial_t \varphi \int_k |s|^{q-1} \left( \varphi^2 \right)' ((s - k_+)) \, ds \, dx \, dt. \]  

(4.2)

Next, we turn our attention to \( \Pi_h \). By Lemma 2.3 and since \((\varphi^2)'\) is Lipschitz continuous, we find that

\[ \varphi(w_h - u) \rightharpoonup - \lambda \left( \varphi^2 \right)' ((u - k_+) \varphi \text{ weakly in } L^2(t_1, t_2; W^{1,2}(B_{\rho_1})). \]

Together with the structure conditions (1.2), this implies that

\[ \lim_{h \to 0} \Pi_h = - \lambda \int_{\Omega_T} A(x, t, u, \nabla u) \cdot \nabla \left( \varphi \left( \varphi^2 \right)' ((u - k_+) \right) \, dx \, dt \]

\[ = - \lambda \int_{\Omega_T} \varphi \left( \varphi^2 \right)'' ((u - k_+) A(x, t, u, \nabla u) \cdot \nabla(u - k_+) \, dx \, dt \]

\[ - \lambda \int_{\Omega_T} \varphi \left( \varphi^2 \right)' ((u - k_+) A(x, t, u, \nabla u) \cdot \nabla \varphi \, dx \, dt \]

\[ \leq - \lambda C_o \int_{\Omega_T} \varphi \left( \varphi^2 \right)'' ((u - k_+) \nabla(u - k_+)^2 \, dx \, dt \]

\[ + \lambda C_1 \int_{\Omega_T} \varphi \left( \varphi^2 \right)' ((u - k_+) \nabla(u - k_+ || \nabla \varphi || \, dx \, dt. \]

Here, the term involving \((\varphi^2)''\) is well defined a.e. in \( \Omega_T \), since \( \nabla(u - k_+) = 0 \) a.e. in \{ (u - k_) = \gamma \}. At this point, we choose \( \varphi(x, t) = \xi_{\varepsilon}(t) \eta(x) \), where \( \xi_{\varepsilon} \) is defined as in the proof of Lemma 3.2 and \( \eta \in C^1_0(B_{\rho_2}, \mathbb{R}_{\geq 0}) \) is a cutoff function with \( \eta = 1 \) in \( B_{\rho_1} \) and \(|\nabla \eta| \leq \frac{2}{\rho_2 - \rho_1} \). Applying Young’s inequality with parameter \( \frac{C_o}{2C_1} \) (see [16, Appendix B]) yields

\[ \lim_{h \to 0} \Pi_h \leq \lambda C_o \int_{\Omega_T} \xi_{\varepsilon} \eta^2 \left( 2 \varphi (\varphi')^2 - (\varphi^2)'' \right) ((u - k_+) \nabla(u - k_+)^2 \, dx \, dt \]

\[ + \frac{2 \lambda C_1}{C_o} \int_{\Omega_T} \xi_{\varepsilon} |\nabla \eta|^2 \varphi ((u - k_+) \, dx \, dt \]

\[ \leq \frac{2 \lambda C_1}{C_o} \int_{\Omega_T} \xi_{\varepsilon} |\nabla \eta|^2 \varphi ((u - k_+) \, dx \, dt. \]

In the last line, we used that \( 2 \varphi (\varphi')^2 - (\varphi^2)'' = - 2 (\varphi')^2 \leq 0 \). Together with (4.1), we obtain that
\[-q \int \int_{\Omega_T} \xi_{\eta}^2 \int_k^u |s|^{q-1} (\phi^2) (u - k_+) \, ds \, dt \leq \frac{2C_1}{C_0} \int \int_{\Omega_T} \xi_{\eta} |\nabla \eta|^2 \phi (u - k_+) \, dx \, dt.\]

Passing to the limit \( \varepsilon \downarrow 0 \), we conclude that

\[ q \int_{B_{\rho_1} \times \{t\}} \int_k^u |s|^{q-1} (\phi^2) (u - k_+) \, ds \, dx \leq q \int_{B_{\rho_2} \times \{t_1\}} \int_k^u |s|^{q-1} (\phi^2) (u - k_+) \, ds \, dx \]

\[ + \frac{8C_1}{C_0^2 (\rho_2 - \rho_1)^2} \int \int_{Q_2} \phi (u - k_+) \, dx \, dt, \]

for any \( t \in (t_1, t_2) \), which proves (i).

For the case (ii), we start with the comparison function

\[ w_h := \|u\|_h + (\phi^2)' (\|u\|_h - k_-) + \|\psi - \|\psi\|_h\| L^\infty(Q_2) \]

\[ \geq \psi + (\phi^2)' (u - k_-) \]

\[ \geq \psi, \]

since \( \phi, \phi' \geq 0 \) and proceed similarly as in the case (i).

\[ \square \]

5. First alternative

In the following, we use parameters \( \mu^-, \mu^+ \in \mathbb{R} \) and \( \omega > 0 \) satisfying

\[ \mu^- \leq \inf_{Q_{\rho,\theta\rho^2(z_0)}} u, \quad \mu^+ \geq \sup_{Q_{\rho,\theta\rho^2(z_0)}} u \quad \text{and} \quad \omega = \mu^+ - \mu^-, \quad (5.1) \]

where slightly different factors \( \theta \approx |u|^{q-1} \) will be considered. We will distinguish between the case where \( u \) is close to the value zero in \( Q_{\rho,\theta\rho^2(z_0)} \),

\[ \mu^+ \geq -\frac{1}{4} \omega \quad \text{and} \quad \mu^- \leq \frac{1}{4} \omega, \quad (5.2) \]

the case where \( u \) is positive and bounded away from zero in \( Q_{\rho,\theta\rho^2(z_0)} \),

\[ \mu^- > \frac{1}{4} \omega, \quad (5.3) \]

and the case where \( u \) is negative and bounded away from zero in \( Q_{\rho,\theta\rho^2(z_0)} \),

\[ \mu^+ < -\frac{1}{4} \omega. \quad (5.4) \]

While the degeneracy/singularity of \((1.1)\) has to be taken into account in the first case, \((1.1)\) does not become degenerate/singular in \( Q_{\rho,\theta\rho^2(z_0)} \) in the latter two cases.
By using (5.1)\textsubscript{3}, we can write (5.2) equivalently as
\[
\mu^+ \geq \frac{1}{5} \mu^- \quad \text{and} \quad \mu^+ \geq 5 \mu^-,
\] (5.5)
which corresponds to the region $I = I_1 \cup I_2$ in Fig. 1. We split $I$ into subcases $I_1$ and $I_2$ by observing that by (5.1)\textsubscript{3} one of the cases
\[
\mu^- \leq -\frac{1}{2} \omega \quad \text{or} \quad \mu^+ \geq \frac{1}{2} \omega,
\]
must hold, which is equivalent to
\[
\mu^+ \leq |\mu^-| \quad \text{or} \quad \mu^+ \geq |\mu^-|.
\] (5.6)
Here, (5.5) and (5.6)\textsubscript{1} correspond to region $I_1$, while (5.5) and (5.6)\textsubscript{2} correspond to $I_2$ in Fig. 1. Furthermore, condition (5.3) is equivalent to
\[
\mu^+ < 5 \mu^-,
\]
which corresponds to region II in Fig. 1 and (5.4) is equivalent to
\[
\mu^+ < \frac{1}{5} \mu^-,
\]
corresponding to region III in Fig. 1.

In each of the cases (5.2) – (5.4), for some constant $\nu \in (0, 1)$ we will be concerned with the measure theoretic alternatives
\[
\begin{align*}
\left\{ \left\{ u \geq \mu^+ - \frac{1}{2} \omega \right\} \cap Q_{\rho, \theta \rho^2}(z_o) \right\} \leq \nu |Q_{\rho, \theta \rho^2}(z_o)|, \\
\left\{ \left\{ u \geq \mu^+ - \frac{1}{2} \omega \right\} \cap Q_{\rho, \theta \rho^2}(z_o) \right\} > \nu |Q_{\rho, \theta \rho^2}(z_o)|,
\end{align*}
\] (5.7)

Figure 1. Relevant ranges for the parameters $\mu^-$ and $\mu^+$
or with the measure theoretic alternatives

\[
\begin{align*}
\big|\{u \leq \mu^- + \frac{1}{2} \omega\} \cap Q_{\rho, \theta \rho^2}(z_o)\big| &\leq v |Q_{\rho, \theta \rho^2}(z_o)|, \\
\big|\{u \leq \mu^- + \frac{1}{2} \omega\} \cap Q_{\rho, \theta \rho^2}(z_o)\big| &> v |Q_{\rho, \theta \rho^2}(z_o)|.
\end{align*}
\]

(5.8)

In the so-called first alternative (5.7)\(_1\)/\((5.8)\)_1, the solution \(u\) is bounded away from its essential supremum or infimum on a large portion of the considered cylinder, whereas it is close to the essential supremum or infimum on a large part of the cylinder in the second alternative (5.7)\(_2\)/\((5.8)\)_2. In both situations, our goal is to show that \(u\) is bounded away from one of the extreme values \(\mu^+/\mu^-\) pointwise a.e. in a suitable sub-cylinder of \(Q_{\rho, \theta \rho^2}(z_o)\) together with a quantitative bound. The necessary tools for the first alternatives \((5.7)\)_1/\((5.8)\)_1 will be discussed in the present section, while we will be concerned with the second alternatives \((5.7)\)_2/\((5.8)\)_2 in Sect. 6. More precisely,

- for case I\(1\) we use the alternatives (5.7) together with Lemmas \(5.3\) and \(6.3\);
- for case I\(2\) we use the alternatives (5.8) together with Lemmas \(5.4\) and \(6.4\);
- for case II we use the alternatives (5.8) together with Lemmas \(5.2\) and \(6.4\);
- for case III we use the alternatives (5.7) together with Lemmas \(5.1\) and \(6.3\).

We start with the non-degenerate/non-singular cases (5.3) and (5.4), since the corresponding proofs are slightly easier as in the case (5.2).

5.1. De Giorgi-type lemmas in the non-degenerate/non-singular case

In this section, we state De Giorgi-type lemmas for the case where \(u\) is bounded away from its supremum and the case where \(u\) is bounded away from its infimum. However, we only prove the former one, since the proof of the latter is analogous. For the first lemma, we suppose that (5.4) holds true. Further, in the first lemma we use the scaling

\[
|\mu^-| \leq \theta \frac{1}{\pi} \leq 5|\mu^-|.
\]

(5.9)

**Lemma 5.1.** Assume that (5.4) and (5.11) hold and let \(u\) be a locally bounded, local weak solution to the obstacle problem and \(Q_{\rho, \theta \rho^2}(z_o) \subseteq \Omega_T\). Furthermore, suppose that \(\frac{1}{2} (\mu^+ + \mu^-) \geq \sup_{Q_{\rho, \theta \rho^2}(z_o)} \psi\). Then, there exists a constant \(v = v(n, q, C_o, C_1) \in (0, 1)\) such that if

\[
\big|\{u \geq \mu^+ - \frac{1}{2} \omega\} \cap Q_{\rho, \theta \rho^2}(z_o)\big| \leq v \big|Q_{\rho, \theta \rho^2}(z_o)\big|
\]

then

\[
u \leq \mu^+ - \frac{1}{4} \omega \quad a.e. \text{ in } Q_{\rho, \theta \rho^2}(z_o).
\]

**Proof.** For simplicity, we omit the reference point \(z_o\) in the notation. Observe that

\[
g_+(u, k) \approx (|u| + |k|)^{q-1} (u - k)^2 \quad \text{for any } k \in \mathbb{R}
\]
up to a constant depending only on \( q \) by Lemma 3.1. From the energy estimate, Lemma 3.2 (i), we obtain
\[
\sup_{-\theta\rho^2 < t < 0} \int_{B_\rho} \varphi^2 (|u| + |k|)^{q-1} (u - k)^2 \, dx + \int_{Q_{\rho, \theta\rho^2}} \varphi^2 |\nabla (u - k)|^2 \, dx \, dt \\
\leq c \int_{Q_{\rho, \theta\rho^2}} (u - k)^2 |\nabla \varphi|^2 \, dx \, dt + c \int_{Q_{\rho, \theta\rho^2}} (|u| + |k|)^{q-1} (u - k)^2 |\partial_t \varphi|^2 \, dx \, dt
\]
for \( k \geq \sup_{Q_{\rho, \theta\rho^2}} \psi \) and \( \varphi \in C^\infty(Q_{\rho, \theta\rho^2}; \mathbb{R}_{\geq 0}) \) vanishing on the parabolic boundary of \( Q_{\rho, \theta\rho^2} \). For \( j \in \mathbb{N}_0 \), we choose
\[
k_j = \mu^+ - \frac{\omega}{4} - \frac{\omega}{2j+2}, \quad \rho_j = \frac{\rho}{2} + \frac{\rho_j}{2j+1}, \quad B_j = B_{\rho_j}, \quad Q_j = Q_{\rho_j, \theta\rho_j^2}.
\]
Note that \( k_j \geq k_0 = \frac{1}{2}(\mu^+ + \mu^-) \geq \sup_{Q_j} \psi \geq \sup_{Q_j} \psi \) for any \( j \in \mathbb{N}_0 \), since \( Q_{\rho, \theta\rho^2} = Q_0 \supset Q_1 \supset \ldots \) and by the assumption on \( k_0 \). Furthermore, we use a smooth cutoff function \( 0 \leq \varphi \leq 1 \) vanishing on the parabolic boundary of \( Q_j \) and equal to 1 in \( Q_{j+1} \) such that
\[
|\nabla \varphi| \leq \frac{c 2^j}{\rho} \quad \text{and} \quad |\partial_t \varphi| \leq \frac{c 2^j}{\theta \rho^2}.
\]
Observe that (5.4) in particular implies that \( k_j < \mu^+ < 0 \) for any \( j \in \mathbb{N}_0 \). Thus, we know that \( 2|\mu^+| \leq 2|u| \leq |u| + |k_j| \leq 2|k_j| \) in \( A_j := \{u > k_j\} \cap Q_j \), which gives us that
\[
\min \left\{ |\mu^+|^{q-1}, |k_j|^{q-1} \right\} \sup_{-\theta\rho^2 < t < 0} \int_{B_j} \varphi^2 (u - k)^2 \, dx + \int_{Q_j} \varphi^2 |\nabla (u - k)|^2 \, dx \, dt \\
\leq \sup_{-\theta\rho^2 < t < 0} \int_{B_j} \varphi^2 (|u| + |k_j|)^{q-1} (u - k)^2 \, dx + \int_{Q_j} \varphi^2 |\nabla (u - k)|^2 \, dx \, dt \\
\leq c \frac{2^j}{\rho^2} \int_{Q_j} (u - k)^2 \, dx + c \frac{2^j}{\theta \rho^2} \int_{Q_j} (|u| + |k_j|)^{q-1} (u - k)^2 \, dx \, dt \\
\leq c \frac{2^j \omega^2}{\rho^2} \left( 1 + \max \left\{ |\mu^+|^{q-1}, |k_j|^{q-1} \right\} \frac{\theta^{-1}}{} \right) |A_j|.
\]
Observe that the bounds \( \frac{1}{2} |\mu^-| < |\mu^+| < |\mu^-| \) and \( |\mu^+| < |k_j| < |\mu^-| \) hold true by (5.4) and the definition of \( k_j \). Taking also (5.11) into account, we estimate the preceding inequality further. In particular, we conclude that
\[
\sup_{-\theta\rho_j^2 < t < 0} \int_{B_j} \varphi^2 (u - k_j)^2 \, dx \leq c \frac{2^j \omega^2}{\rho^2} |A_j|.
\]
(5.10)
Next, note that \( u - k_j \geq k_{j+1} - k_j = 2^{-(j+3)} \omega \) if \( u \geq k_{j+1} \). By Hölder’s inequality and Lemma 2.4, we find that
\[
\frac{\omega}{2j+3}|A_{j+1}| \leq \iint_{A_{j+1}} (u - k_j) \, dx \, dt \leq \iint_{Q_j} (u - k_j) + \varphi \, dx \, dt
\]
\[
\leq \left( \iint_{Q_j} [(u - k_j) + \varphi]^2 \frac{(2j+2)}{n} \, dx \, dt \right)^{\frac{n}{2(n+2)}} |A_j|^{1-\frac{n}{2(n+2)}}
\]
\[
\leq c \left( \iint_{Q_j} \left| \nabla [(u - k_j) + \varphi] \right|^2 \, dx \, dt \right)^{\frac{n}{2(n+2)}}
\times \left( \sup_{0 < t < 0} \int_{B_j} \varphi^2 (u - k_j)^2 \, dx \right)^{\frac{1}{n+2}} |A_j|^{1-\frac{n}{2(n+2)}}
\]
\[
\leq c \left( \frac{2^{2j} \omega^2}{\rho^2} |A_j| \right)^{\frac{n}{2(n+2)}} \left( \frac{2^{2j} \omega^2}{\theta \rho^2} |A_j| \right)^{\frac{1}{n+2}} |A_j|^{1-\frac{n}{2(n+2)}}
\]
\[
\leq c 2^j \omega \theta^{-\frac{1}{n+2}} \rho^{-1} |A_j|^{1+\frac{1}{n+2}}.
\]

In the penultimate line, we used the energy estimate and (5.10). This implies that
\[
|A_{j+1}| \leq c 2^j \theta^{-\frac{1}{n+2}} \rho^{-1} |A_j|^{1+\frac{1}{n+2}}.
\]

Dividing the preceding inequality by \(|Q_{j+1}|\) and denoting \(Y_j = |A_j|/|Q_j|\), we have that
\[
Y_{j+1} \leq c \left( \frac{2^{2j} \omega^2}{\theta \rho^2} |Q_j| \right)^{\frac{1}{n+2}} Y_j^{1+\frac{1}{n+2}} \leq c 2^j Y_j^{1+\frac{1}{n+2}}.
\]

Thus, we are able to conclude the proof by using Lemma 2.6. \(\square\)

Next, we suppose that (5.3) holds true. Further, we use the scaling
\[
\frac{1}{2} \mu^+ \leq \theta^{-\frac{1}{q-1}} \leq 5 \mu^+.
\]

**Lemma 5.2.** Consider \(Q_{\rho,\theta \rho^2} \Subset \Omega_T\), assume that (5.3) and (5.11) hold and let \(u\) be a locally bounded, local weak solution to the obstacle problem. Then, there exists a constant \(v = v(n, q, C_o, C_1) \in (0, 1)\) such that if
\[
\left\{ u \leq \mu^- + \frac{1}{2} \omega \right\} \cap Q_{\rho,\theta \rho^2}(z_0) \leq v \left| Q_{\rho,\theta \rho^2}(z_0) \right|
\]
then
\[
u \geq \mu^- + \frac{1}{4} \omega \quad \text{a.e. in } Q_{\frac{5}{4} \theta}(z_0).
\]

5.2. De Giorgi-type lemmas in the degenerate/singular case

Next, we will state and prove a De Giorgi-type lemma in the case where \(|\mu^+| \leq \frac{5}{4} \omega\) and \(u\) is bounded away from its supremum in a significant portion of the considered intrinsic cylinder. Note that the condition \(|\mu^+| \leq \frac{5}{4} \omega\) is in particular implied by (5.2).
Lemma 5.3. Let $u$ be a locally bounded, local weak solution to the obstacle problem and $Q_{\rho, \theta \rho^2}(z_0) \subset \Omega_T$, where $\theta = \omega^{q-1}$. Furthermore, assume that $|\mu^+| \leq \frac{5}{4} \omega$ and that $\frac{1}{2}(\mu^+ + \mu^-) \geq \sup_{Q_{\rho, \theta \rho^2}(z_0)} \psi$. Then, there exists a constant $v = v(n, q, C_0, C_1) \in (0, 1)$, such that if

$$\left| \{ u \geq \mu^+ - \frac{1}{2} \omega \} \cap Q_{\rho, \theta \rho^2}(z_0) \right| \leq v \left| Q_{\rho, \theta \rho^2}(z_0) \right|,$$

then

$$u \leq \mu^+ - \frac{1}{4} \omega \quad \text{a.e. in } Q_{\frac{\rho}{2}, \theta \left( \frac{\rho}{2} \right)^2}(z_0).$$

Proof. We omit the fixed reference point $z_0$ for simplicity. By Lemma 3.1, there holds

$$g_+(u, k) \leq c (|u| + |k|)^{q-1} (u - k)_+^2 \leq c (|u| + |k|)^q (u - k)_+$$

for any $k \in \mathbb{R}$. Further, for $\tilde{k} > k$ we have $(u - k)_+ \geq (u - \tilde{k})_+$. From the energy estimate, Lemma 3.2 (i), we obtain

$$\sup_{-\theta \rho^2 < t < 0} \int_{B_{\rho}} \varphi^2 (|u| + |k|)^{q-1} (u - \tilde{k})_+^2 \, dx + \int_{Q_{\rho, \theta \rho^2}} \varphi^2 |\nabla (u - \tilde{k})_+|^2 \, dx \, dt$$

$$\leq c \int_{Q_{\rho, \theta \rho^2}} (u - k)_+^2 |\nabla \varphi|^2 \, dx \, dt$$

$$+ c \int_{Q_{\rho, \theta \rho^2}} (|u| + |k|)^q (u - k)_+ |\partial_t \varphi|^2 \, dx \, dt$$

for any $\tilde{k} > k \geq \sup_{Q_{\rho, \theta \rho^2}} \psi$ and $\varphi \in C^\infty(Q_{\rho, \theta \rho^2}; \mathbb{R}_{\geq 0})$ vanishing on the parabolic boundary of $Q_{\rho, \theta \rho^2}$. For $j \in \mathbb{N}_0$, we define $k_j, \rho_j, B_j$, and $Q_j$ as in Lemma 5.1, and furthermore

$$\tilde{k}_j = \frac{k_j + k_{j+1}}{2}, \quad \tilde{\rho}_j = \frac{\rho_j + \rho_{j+1}}{2}, \quad \tilde{B}_j = B_{\tilde{\rho}_j}, \quad \tilde{Q}_j = Q_{\tilde{\rho}_j, \theta \tilde{\rho}_j^2}.$$

Let $0 \leq \varphi \leq 1$ such that $\varphi$ vanishes on the parabolic boundary of $Q_j$, equals to 1 in $\tilde{Q}_j$ and satisfies

$$|\nabla \varphi| \leq c \frac{2^j}{\rho} \quad \text{and} \quad |\partial_t \varphi| \leq c \frac{2^j}{\theta \rho^2}.$$

Moreover, we set $A_j = \{ u > k_j \} \cap Q_j$.

By formulating the preceding energy estimate for these quantities and using the assumption $|\mu^+| \leq \frac{5}{4} \omega$, we conclude that

$$4\omega \geq 2|\mu^+| + \frac{1}{2} \omega + 2^{-(j+1)} \omega \geq |u| + |k_j| \geq u - k_j \geq \tilde{k}_j - k_j = 2^{-(j+4)} \omega$$

on the set where $u \geq \tilde{k}_j$. By recalling that $\theta = \omega^{q-1}$, we find that
In particular, the estimate above gives us that
\[ \min \{4^{q-1}, 2^{-(q-1)(j+4)}\} \omega^{q-1} \sup_{-\theta \rho^2 < t < 0} \int_{\tilde{B}_j} (u - \tilde{k}_j)_+^2 \, dx + \int_{\tilde{Q}_j} |\nabla (u - \tilde{k}_j)_+| \, dx \]
\[ \leq \sup_{-\theta \rho^2 < t < 0} \int_{\tilde{B}_j} \varphi^2 (|u| + |k|)^{q-1} (u - \tilde{k}_j)_+^2 \, dx + \int_{\tilde{Q}_j} \varphi^2 \nabla (u - \tilde{k}_j)_+ \, dx \]
\[ \leq c \frac{2^{2j}}{\rho^2} \int_{\tilde{Q}_j} (u - k_j)_+^2 \, dx + c \frac{2^{2j}}{\theta \rho^2} \int_{\tilde{Q}_j} (|u| + |k_j|)^q (u - k_j)_+ \, dx \]
\[ \leq c \frac{2^{2j}}{\rho^2} \omega^2 |A_j| . \]

In particular, the estimate above gives us that
\[ \sup_{-\theta \rho^2 < t < 0} \int_{\tilde{B}_j} (u - \tilde{k}_j)_+^2 \, dx \leq c \frac{2^{2+(q-1)+j}}{\rho^2} \omega^{3-q} |A_j| \tag{5.12} \]
holds true. By introducing a smooth cutoff function \(0 \leq \phi \leq 1\), such that \(\phi\) equals 1 in \(Q_{j+1}\) and vanishes on the lateral boundary of \(\tilde{Q}_j\) with \(|\nabla \phi| \leq c \rho^{-1}\), Hölder’s inequality and Lemma 2.4 imply
\[ \frac{\omega}{2^{j+4}} |A_{j+1}| \leq \int_{\tilde{Q}_j} (u - \tilde{k}_j)_+ \phi \, dx \, dt \]
\[ \leq \left( \int_{\tilde{Q}_j} \left[ (u - \tilde{k}_j)_+ + \phi \right]^2 \, dx \, dt \right)^{\frac{n}{2(n+2)}} |A_j|^{1-\frac{n}{2(n+2)}} \]
\[ \leq c \left( \int_{\tilde{Q}_j} \left[ \nabla \left[ (u - \tilde{k}_j)_+ + \phi \right] \right]^2 \, dx \right)^{\frac{1}{n+2}} \cdot |A_j|^{1-\frac{n}{2(n+2)}} \]
\[ \leq c \left( \frac{2^{2j}}{\rho^2} \omega^2 |A_j| \right)^{\frac{n}{2(n+2)}} \left( \frac{2^{2+(q-1)+j}}{\rho^2} \omega^{3-q} |A_j| \right)^{\frac{1}{n+2}} \cdot |A_j|^{1-\frac{n}{2(n+2)}} \]
\[ \leq c \left( \frac{2^{2j}}{\rho^2} \omega^2 |A_j| \right)^{\frac{n}{2(n+2)}} \left( \frac{2^{2+(q-1)+j}}{\rho^2} \omega^{3-q} |A_j| \right)^{\frac{1}{n+2}} \cdot |A_j|^{1-\frac{n}{2(n+2)}} \]
\[ \leq c 2^{\left(1+\frac{(q-1)+1}{n+2}\right)j} \rho^{-1} \omega^{1+\frac{1-q}{n+2}} |A_j|^{1+\frac{1}{n+2}} . \]

In the penultimate line, we used the energy estimate and (5.12). This implies that
\[ |A_{j+1}| \leq c 2^{\left(2+\frac{(q-1)+1}{n+2}\right)j} \rho^{-1} \omega^{1+\frac{1-q}{n+2}} |A_j|^{1+\frac{1}{n+2}} . \]

By dividing this by \(|Q_{j+1}|\) and denoting \(Y_j = |A_j|/|Q_j|\), we have
\[ Y_{j+1} \leq c 2^{\left(2+\frac{(q-1)+1}{n+2}\right)j} \rho^{-1} \omega^{1+\frac{1-q}{n+2}} \frac{|Q_j|^{1+\frac{1}{n+2}}}{|Q_{j+1}|} Y_j^{1+\frac{1}{n+2}} \]
\[ \leq c 2^{\left(2+\frac{(q-1)+1}{n+2}\right)j} Y_j^{1+\frac{1}{n+2}} . \]

Then, if \(v \leq c^{-(n+2)} B^{-(n+2)^2}\), where \(B = 2^{2+\frac{(q-1)+1}{n+2}}\) we may use Lemma 2.6 to conclude the proof. \(\square\)
We state the De Giorgi-type lemma for the case where $|\mu^-| \leq \frac{5}{4} \omega$ (which holds in particular if (5.2) is satisfied) and $u$ is away from its infimum without proof. However, it can be proven analogous to Lemma 5.3 by defining $k_j = \mu^- + \frac{\omega}{4} + \frac{\omega}{2^{j+7}}$ and exploiting the energy estimate in Lemma 3.2 (ii). Observe that in this case any level is admissible in the energy estimate.

**Lemma 5.4.** Let $u$ be a locally bounded, local weak solution to the obstacle problem and $Q_{\rho, \theta \rho^2}(z_o) \subset \Omega_T$ with $\theta = \omega^{q-1}$ and assume that $|\mu^-| \leq \frac{5}{4} \omega$. Then, there exists a constant $\nu = \nu(n, q, C_o, C_1) \in (0, 1)$ such that if $|\{u \leq \mu^- + \frac{1}{2} \omega\} \cap Q_{\rho, \theta \rho^2}(z_o)| \leq \nu |Q_{\rho, \theta \rho^2}(z_o)|$, then

$$u \geq \mu^- + \frac{1}{4} \omega \text{ a.e. in } Q_{\rho, \theta \rho^2}(z_o).$$

6. Second alternative

6.1. Second alternative near infimum

Suppose that $\mu^+, \mu^-$, and $\omega$ are given by (5.1). In this section, we are concerned with a subcase of (5.2) and the case (5.4). More precisely, we assume that either

$$-\frac{1}{4} \omega \leq \mu^+ \leq \frac{1}{2} \omega \text{ and } \theta = \omega^{q-1} \tag{6.1}$$

holds true (which corresponds to region I in Fig. 1), or

$$\mu^+ < -\frac{1}{2} \omega \text{ and } |\mu^-| \leq \frac{1}{\theta^{\frac{1}{q-1}}} \leq \frac{1}{5} \mu^- \tag{6.2}$$

(which corresponds to region III in 1). Since $\mu^+ = \mu^- + \omega$, (6.10) is equivalent to

$$-\frac{5}{4} \omega \leq \mu^- \leq -\frac{1}{2} \omega.$$

Further, observe that (6.11) implies

$$0 > -\frac{1}{2} \omega > \mu^+ \geq \mu^- > 5 \mu^+.$$

First, we prove an auxiliary lemma.

**Lemma 6.1.** Let $\eta \in (0, \frac{1}{8}]$ and $k = \mu^- + \eta \omega$. Then, there exists $c = c(q) > 0$ such that in case (6.10) there holds

$$\frac{1}{c} |\mu^-|^{q-1} \leq \theta \leq c |\mu^-|^{q-1} \text{ and } \frac{1}{c} |k|^{q-1} \leq \theta \leq c |k|^{q-1},$$

and in the case (6.11) there holds

$$\frac{1}{c} |k|^{q-1} \leq \theta \leq c |k|^{q-1}.$$

Furthermore, we have

$$\frac{|\mu^-|}{|k|} \leq (1 - 4\eta)^{-1}.$$
Proof. First consider the case (6.10). The first estimate is a direct consequence of the definition. For the second inequality, we may estimate
\[
\frac{5}{4} \omega \geq |k| = -\mu^- - \eta \omega \geq \frac{1}{2} \omega - \frac{1}{8} \omega = \frac{3}{8} \omega,
\]
which implies the claim. In case (6.11), we have
\[
|\mu^-| \geq |k| = -\mu^- - \eta \omega \geq |\mu^-| - \frac{1}{8} \omega \geq -\frac{1}{2} |\mu^-|,
\]
which concludes the proof of the inequalities for $\theta$. For the last inequality, consider first (6.10) to obtain
\[
|\mu^-| = -\mu^- - \eta \omega \geq -\mu^- + 2 \eta \mu^- = |\mu^-| (1 - 2 \eta),
\]
which implies the desired inequality. In case (6.11), we may estimate
\[
|\mu^-| = -\mu^- - \eta \omega \geq -\mu^- + 4 \eta \mu^- = |\mu^-| (1 - 4 \eta),
\]
which concludes the proof. □

Lemma 6.2. Let $Q_{\rho, \rho^2}(z_o) \subseteq \Omega_T$ be a parabolic cylinder and $v \in (0, 1)$ and $\eta \in (0, \frac{1}{8}]$. Assume that (6.10) or (6.11) holds and that $u$ is a locally bounded, local weak solution to the obstacle problem. Then, there exists $v_1 = v_1(n, q, C_0, C_1, v) \in (0, 1)$ such that if
\[
\left\{ u < \mu^- + \eta \omega \right\} \cap Q_{\rho, \frac{1}{2} \nu \theta \rho^2}(z_o) < v_1 \left| Q_{\rho, \frac{1}{2} \nu \theta \rho^2}(z_o) \right|,
\]
then
\[
u \geq \mu^- + \frac{1}{2} \eta \omega \quad \text{a.e. in } Q_{\rho, \frac{1}{2} \nu \theta \rho^2}(z_o).
\]
Proof. We omit $z_o$ for simplicity and start the proof by defining
\[
k_j = \mu^- + \left( \frac{n}{2} + \frac{n}{2 + \rho} \right) \omega, \quad \rho_j = \frac{\rho}{2 + \rho} + \frac{\rho}{2 + \rho}, \quad B_j = B_{\rho_j}, \quad Q_j := Q_{\rho_j, \frac{1}{2} \nu \theta \rho^2}.
\]
Observe that by (6.10) and (6.11) it follows that $k_j < 0$ for all $j \in \mathbb{N}_0$. Let $0 \leq \varphi \leq 1$ be a cutoff function that equals 1 in $Q_{j+1}$ and vanishes on the parabolic boundary of $Q_j$ such that
\[
|\nabla \varphi| \leq c \frac{2^j}{\rho} \quad \text{and} \quad |\partial_t \varphi| \leq c \frac{2^j}{\nu \theta \rho^2}.
\]
From the fact that $\mu^- \leq u < k_j < 0$ in the set $A_j := \{ u < k_j \} \cap Q_j$, Lemma 3.1 and the energy estimate in Lemma 3.2 (ii), we then obtain
\[
\min \left\{ |\mu^-|^{q-1}, |k_j|^{q-1} \right\} \sup_{-\theta \rho^2 < t < 0} \int_{B_j} \varphi^2 (u - k_j)^2 \, dx + \int_{Q_j} \varphi^2 |\nabla (u - k_j)|^2 \, dx \, dt \
\leq \sup_{-\theta \rho^2 < t < 0} \int_{B_j} \varphi^2 (|u| + |k_j|)^{q-1} (u - k_j)^2 \, dx + \int_{Q_j} \varphi^2 |\nabla (u - k_j)|^2 \, dx \, dt
\]
with a constant $c = c(n, q, C_0, C_1, \nu)$, where we used Lemma 6.1 in the last line. With these estimates at hand and Lemma 6.1, we infer in particular

$$
sup_{-\theta \rho^2 < t < 0} \int_{B_j} \varphi^2(u - k_j)^2 \, dx \leq c \frac{2^{2j}(\eta \omega)^2}{\theta \rho^2} |A_j|.
$$

Since $k_j - u \geq k_j - k_{j+1} = 2^{-(j+2)}\eta \omega$ in the set $\{u \leq k_{j+1}\}$, by Hölder’s inequality and Lemma 2.4 we obtain that

$$
\frac{\eta \omega}{2^{j+2}} |A_{j+1}| \leq \int_{Q_j} (u - k_j)^{-\varphi} \, dx dt
$$

$$
\leq c \left( \int_{Q_j} [(u - k_j)^{-\varphi}]^{\frac{2(n+2)}{n}} \, dx dt \right)^{\frac{n}{2(n+2)}} \frac{2j}{|A_j|}\frac{n}{2(n+2)}
$$

$$
\leq \left( \int_{Q_j} |\nabla [(u - k_j)^{-\varphi}]|^2 \, dx dt \right)^{\frac{1}{\pi^2}}
$$

$$
\left( \sup_{-\theta \rho^2 < t < 0} \int_{B_j} \varphi^2(u - k_j)^2 \, dx \right)^{\frac{1}{\pi^2}} |A_j|^{1-\frac{n}{2(n+2)}}
$$

$$
= c \frac{2^{j} \eta \omega}{\theta \rho^2} |A_j|^{1+\frac{1}{\pi^2}}.
$$

Dividing by $|Q_{j+1}|$ and denoting $Y_j = |A_j|/|Q_j|$, we conclude that

$$
Y_{j+1} \leq c 2^{2j} Y_j^{1+\frac{1}{\pi^2}}
$$

for a constant $c = c(n, q, C_0, C_1, \nu)$. Setting $v_1 \leq c^{-(n+2)}4^{-(n+2)}$, we conclude the proof by using Lemma 2.6. □

At this stage, we state the main result in this section, which allows us to deal with arbitrary $v$ in the assumed measure estimate. In contrast, in the preceding lemma $v_1$ is a fixed constant depending only on the data.
Lemma 6.3. Let $Q_{2\rho,\theta(2\rho)^2}(z_o) \subseteq \Omega_T$ be a parabolic cylinder. Assume that (6.10) or (6.11) holds and that $u$ is a locally bounded, local weak solution to the obstacle problem. Then, for any $v \in (0, 1)$ there exists a constant $a = a(n, q, C_o, C_1, v) \in (0, \frac{1}{64}]$ such that if
\[
\|\{u \geq \mu^- + \frac{1}{2}w\} \cap Q_{\rho, \theta(2\rho)^2}(z_o)\| > v \|Q_{\rho, \theta(2\rho)^2}(z_o)\|
\]
than
\[u \geq \mu^- + a\omega \quad a.e. \text{ in } Q_{\frac{\rho}{2}, \frac{1}{2}v\theta(\frac{\rho}{2})^2}(z_o).
\]

Proof. In the following, we omit $z_o$ for simplicity. Observe that from the assumption it follows that
\[
\|\{u < \mu^- + \frac{1}{2}w\} \cap Q_{\rho, \theta(2\rho)^2}\| < (1 - v) \|Q_{\rho, \theta(2\rho)^2}\|
\]
which further implies
\[
\|\{u(\cdot, t_1) < \mu^- + \frac{1}{2}w\} \cap B_{\rho}\| \leq \frac{1 - v}{1 - \frac{1}{2}v} \|B_{\rho}\|
\]
for some $t_1 \in [-\theta(2\rho)^2, -\frac{1}{2}v\theta(2\rho)^2]$. If this did not hold, we would have
\[
\|\{u < \mu^- + \frac{1}{2}w\} \cap Q_{\rho, \theta(2\rho)^2}\| \geq \int_{-\theta(2\rho)^2}^{-\frac{1}{2}v\theta(2\rho)^2} \|\{u(\cdot, t) < \mu^- + \frac{1}{2}w\} \cap B_{\rho}\| \, dt
\]
\[
\geq (1 - \frac{1}{2}v)\theta(2\rho)^2 \frac{1 - v}{1 - \frac{1}{2}v} \|B_{\rho}\|
\]
\[
= (1 - v) \|Q_{\rho, \theta(2\rho)^2}\|
\]
which contradicts (6.3). We divide the rest of the proof in three steps.

In Step 1, we show that the measure information in (6.4) can be propagated to the whole interval $(t_1, 0)$ by using logarithmic estimate, Lemma 4.1 (ii). More precisely, we show that there exists $s_o \in \mathbb{N} \geq 5$ such that for all $s \geq s_o$
\[
\|\{u(\cdot, t) < \mu^- + \frac{1}{2}w\} \cap Q_{\rho, \frac{1}{2}v\theta(\frac{\rho}{2})^2}\| < v_1 \|Q_{\rho, \frac{1}{2}v\theta(\frac{\rho}{2})^2}\|
\]
holds true.

In Step 2, we show that the measure estimate holds in a parabolic cylinder. More precisely, for parameter $v_1 \in (0, 1)$ from Lemma 6.2, we show that there exists $s_1 \in \mathbb{N} \geq 2$ such that
\[
\|\{u < \mu^- + \frac{1}{2}w\} \cap Q_{\rho, \frac{1}{2}v\theta(\frac{\rho}{2})^2}\| < v_1 \|Q_{\rho, \frac{1}{2}v\theta(\frac{\rho}{2})^2}\|
\]
holds true.

In Step 3, we use (6.6) together with Lemma 6.2 to conclude the result.

Step 1. Let us define the level $k$ by
\[k := \mu^- + \delta \omega,
\]
where \( \delta \in (0, \frac{1}{8}] \). This implies

\[ u < 0 \quad \text{in} \quad \{u < k\} \]

by the bound (6.10) or (6.11) depending on the case. Let us choose \( s_o \in \mathbb{N}_{\geq 5} \) large enough such that

\[ s_o > 1 - \frac{\log \delta}{\log 2} \geq 4, \]

which implies that

\[ 2^{1-s} < \delta \leq \frac{1}{8} \]

for every \( s \geq s_o \). First, let us suppose that

\[
\inf_{B_{\rho} \times (t_1, 0)} u > \mu^- + \frac{\delta}{2} \omega. \tag{6.7}
\]

Then, we have that

\[
\left| \{u(\cdot, t) \leq \mu^- + \frac{\delta}{2} \omega \} \cap B_{\rho} \right| = 0
\]

for all \( t \in (t_1, 0) \), which implies (6.5) since \( \frac{\delta}{2} > \frac{1}{2f} \) for \( s \geq s_o \). If (6.7) does not hold, then we have

\[
\inf_{B_{\rho} \times (t_1, 0)} u \leq \mu^- + \frac{\delta}{2} \omega \tag{6.8}
\]

and let us define

\[
H := \sup_{B_{\rho} \times (t_1, 0)} (u - k)_{-}.
\]

Now, it follows that \( \frac{\delta}{2} \omega \leq H \leq \delta \omega \), which implies

\[
\frac{1}{2} \omega \leq H \leq \frac{1}{8} \omega \quad \text{for any} \quad s \geq s_o. \tag{6.9}
\]

Further, we define the function

\[
\phi(v) := \left[ \log \left( \frac{H}{H + \frac{1}{2} \omega - v} \right) \right]_{+}
\]

for \( v < H + \frac{1}{2} \omega \). Now, we rewrite the integrals in Lemma 4.1 (ii) as

\[
\int_{k}^{u} |s|^{q-1} (\phi^2)' ((s - k)_{-}) \, ds = \int_{0}^{(u-k)_{-}} |k - \tau|^{q-1} (\phi^2)' (\tau) \, d\tau
\]

and take into account that \(|k| \leq |k - \tau| \leq |u|\). Thus, we deduce that

\[
\min \left\{ |\mu^-|^{q-1}, |k|^{q-1} \right\} I(t) := \min \left\{ |\mu^-|^{q-1}, |k|^{q-1} \right\} \int_{B_{\rho} \times \{t\}} \phi^2((u - k)_{-}) \, dx
\]

\[
\leq \max \left\{ |\mu^-|^{q-1}, |k|^{q-1} \right\} \int_{B_{\rho} \times \{t_1\}} \phi^2((u - k)_{-}) \, dx
\]

\[
+ \frac{c}{(1 - \sigma)^2 \rho^2} \int_{B_{\rho} \times (t_1, 0)} \phi ((u - k)_{-}) \, dx \, dr
\]
for any $t \in (t_1, 0)$ and $\sigma \in (0, 1)$. Since $\phi$ is increasing and by (6.9), we also find that
\[
\phi((u - k)_-) \leq \phi(H) = \log\left(\frac{2^sH}{\omega}\right) \leq \log\left(2^{s-3}\right),
\]
which together with Lemma 6.1 implies
\[
I(t) \leq (1 - 4\delta)^{-|q-1|} \left(\log\left(2^{s-3}\right)\right)^2 |\{u(\cdot, t_1) < k\} \cap B_\rho| \\
+ \frac{c \log(2^s-3)}{\theta(1 - \sigma)^2 \rho^2} |B_\rho \times (t_1, 0)| \\
\leq \left(1 - 4\delta\right)^{-|q-1|} \left(\log\left(2^{s-3}\right)\right)^2 \left(\frac{1 - \nu}{1 - \frac{1}{2}v} + \frac{c \log(2^s-3)}{(1 - \sigma)^2}\right) |B_\rho|
\]
for any $t \in (t_1, 0)$, where we used also the fact that $t_1 \geq -\theta \rho^2$. On the left-hand side, let us consider the set $B_{\sigma \rho} \cap \{u(\cdot, t) \leq \mu^- + \frac{1}{2^\sigma} \omega\}$ for $t \in (t_1, 0)$, where
\[
(u - k)_- \geq \mu^- + \delta \omega - \mu^- - \frac{1}{2^\sigma} \omega = (\delta - \frac{1}{2^\sigma}) \omega > \frac{1}{2^\sigma} \omega.
\]
Since the function $\phi((u - k)_-)$ is decreasing in $H$ and $H \leq \delta \omega$, this implies
\[
\phi((u - k)_-) \geq \left[\log\left(\frac{\delta \omega}{\delta \omega + \frac{\delta \omega}{\delta - \frac{1}{2^\sigma} \omega}}\right)\right] = \log\left(2^{s-1}\right).
\]
Therefore, we find that
\[
I(t) \geq \left(\log(2^{s-1} \delta)\right)^2 |\{u(\cdot, t) \leq \mu^- + \frac{1}{2^\sigma} \omega\} \cap B_{\sigma \rho}|.
\]
By combining the preceding estimates, we obtain that
\[
|\{u(\cdot, t) \leq \mu^- + \frac{1}{2^\sigma} \omega\} \cap B_{\sigma \rho}| \\
\leq \frac{1}{\left(\log(2^{s-1} \delta)\right)^2} \left(1 - 4\delta\right)^{-|q-1|} \left(\log\left(2^{s-3}\right)\right)^2 \left(\frac{1 - \nu}{1 - \frac{1}{2}v} + \frac{c \log(2^s-3)}{(1 - \sigma)^2}\right) |B_\rho|.
\]
Using $|B_\rho \setminus B_{\sigma \rho}| \leq n(1 - \sigma)|B_\rho|$, this yields
\[
|\{u(\cdot, t) \leq \mu^- + \frac{1}{2^\sigma} \omega\} \cap B_\rho| \\
\leq \left(1 - 4\delta\right)^{-|q-1|} \left(\log\left(\frac{2^{s-3}}{\log(2^{s-1} \delta)}\right)\right)^2 \left(\frac{1 - \nu}{1 - \frac{1}{2}v} + \frac{c \log(2^s-3)}{(1 - \sigma)^2 \log(2^{s-1} \delta)}\right) |B_\rho| + n(1 - \sigma) |B_\rho|.
\]
Let us fix
\[
\sigma := 1 - \frac{\nu^2}{8n} \in (0, 1),
\]
and $\delta$ such that

$$4\delta = \min \left\{ \frac{1}{2}, 1 - \left( \frac{1 - \nu^2}{1 - \frac{1}{2} \nu^2} \right)^{\frac{1}{|q-1|}} \right\}.$$ 

This implies that

$$(1 - 4\delta)^{-|q-1|} \leq \frac{1 - \frac{1}{4} \nu^2}{1 - \nu^2}$$

and we obtain that

$$\left| \left\{ u(\cdot, t) \leq \mu^- + \frac{1}{2} \omega \right\} \cap B_\rho \right| \leq \left[ \left( \frac{\log(2^{s_0-3})}{\log(2^{s_0-1}\delta)} \right)^2 \frac{1 - \frac{1}{2} \nu^2}{(1 + \nu)(1 - \frac{1}{2} \nu)} + c \frac{n^2 \log(2^{s_0-3})}{v^4 \left( \log(2^{s_0-1}\delta) \right)^2} + \frac{\nu^2}{8} \right] |B_\rho|$$

for all $t \in (t_1, 0)$. Let $s_o \in \mathbb{N}_{\geq 5}$ (depending on $n, q, C_o, C_1$ and $\nu$) be so large that

$$\left( \frac{\log(2^{s_o-3})}{\log(2^{s_o-1}\delta)} \right)^2 \leq (1 + \nu)(1 - \frac{1}{2} \nu)$$

and

$$c \frac{n^2 \log(2^{s_o-3})}{\left( \log(2^{s_o-1}\delta) \right)^2} \leq \frac{\nu^6}{8}.$$ 

Now, we conclude that

$$\left| \left\{ u(\cdot, t) \leq \mu^- + \frac{1}{2} \omega \right\} \cap B_\rho \right| \leq \left( 1 - \frac{1}{4} \nu^2 \right) \left| B_\rho \right|,$$

holds for all $t \in (t_1, 0)$ and $s \geq s_o$ or equivalently that (6.5) holds.

**Step 2.** To this end, we define cylinders $Q_2 = B_\rho \times (-\frac{1}{2} v \rho^2, 0]$ and $Q_1 = B_\rho \times (-v \rho^2, 0]$, which implies that $Q_2 \subset Q_1 \subset Q_{2, \rho, \theta(2\rho)^2}$. Further, we consider levels

$$k_j := \mu^- + \frac{1}{2} \omega$$

and set

$$A_j := \{ u < k_j \} \cap Q_2$$

for $j \in \mathbb{N}_{\geq s_o}$. By De Giorgi’s isoperimetric inequality from Lemma 2.5, we have that

$$(k_j - k_{j+1}) \left| \left\{ u(\cdot, t) < k_{j+1} \right\} \cap B_\rho \right| \leq c(n) \rho^{n+1} \left| \left\{ u(\cdot, t) > k_j \right\} \cap B_\rho \right| \int_{B_\rho \cap \{ k_{j+1} < u(\cdot, t) < k_j \}} |\nabla u| \, dx$$

$$\leq c(n) \rho \int_{B_\rho \cap \{ k_{j+1} < u(\cdot, t) < k_j \}} |\nabla u| \, dx.$$

Integrating over $(-\frac{1}{2} v \rho^2, 0)$, we find that
\begin{align*}
(k_j - k_{j+1})|A_{j+1}| \leq \frac{c(n)\rho}{v^2} \int_{A_j \setminus A_{j+1}} |\nabla u| \, dx \, dt \\
\leq \frac{c(n)\rho}{v^2} |A_j \setminus A_{j+1}| \frac{1}{2} \left( \int_{A_j \setminus A_{j+1}} |\nabla u|^2 \, dx \right)^{\frac{1}{2}} \\
\leq \frac{c(n)\rho}{v^2} |A_j \setminus A_{j+1}| \frac{1}{2} \left( \int_{Q_2} |\nabla (u - k_j)|^2 \, dx \, dt \right)^{\frac{1}{2}}.
\end{align*}

Applying Lemma 3.2 (ii), we estimate the integral on the right-hand side by
\begin{align*}
\int_{Q_2} |\nabla (u - k_j)|^2 \, dx \, dt &\leq c \left( \frac{1}{\rho^2} + \frac{\max \{|k_j|^{q-1}, |\mu^{-}|^{q-1}\}}{v \theta \rho^2} \right) \int_{Q_1} (u - k_j)^2 \, dx \, dt \\
&\leq \frac{c}{v \rho^2} \left( \frac{\omega}{2} \right)^2 |Q_1|,
\end{align*}
cf. Lemma 6.2. In the last line, we used Lemma 6.1. Combining the two estimates above and using $k_j - k_{j+1} = 2^{-j+1}\omega$, we infer
\[
A_{j+1}|^2 \leq \frac{c}{v^5} |A_j \setminus A_{j+1}| |Q_1|.
\]

At this stage, we sum over $j = s_0, \ldots, s_0 + s_1 - 1$ for some $s_1 \in \mathbb{N}_{\geq 2}$, which gives us
\[
s_1|A_{s_0+s_1}|^2 \leq \frac{c}{v^5} |Q_1|^2 \leq \frac{c}{v^5} |Q_2|^2.
\]
Choosing $s_1$ large enough, the estimate (6.6) holds true.

**Step 3.** Now an application of Lemma 6.2 with $\eta = 2^{-s_0-s_1}$ yields
\[
u \geq \mu^{-} + \frac{1}{2^{s_0+s_1+1}} \omega \quad \text{a.e. in } Q_{\frac{\omega}{2}, \frac{1}{2} v \theta (\frac{\omega}{2})^2}.
\]

By denoting $a = a(n, q, \omega, C_1, v) = \frac{1}{2^{s_0+s_1+1}}$, the proof is completed. \hfill \Box

6.2. Second alternative near supremum

Here, we consider a subcase of (5.2) and the case (5.3). More precisely, we assume that either
\[
-\frac{1}{2} \omega \leq \mu^{-} \leq \frac{1}{4} \omega \quad \text{and} \quad \theta = \omega^{q-1} \tag{6.10}
\]
holds true (which corresponds to region II in Fig. 1), or
\[
\mu^{-} > \frac{1}{4} \omega \quad \text{and} \quad \frac{1}{2} \mu^{-} \leq \theta \frac{1}{q-1} \leq 5 \mu^{-} \tag{6.11}
\]
(which corresponds to region II in Fig. 1). Note that (6.10)$_1$ is equivalent to
\[
\frac{1}{2} \omega \leq \mu^{+} \leq \frac{5}{2} \omega.
\]
Lemma 6.4. Let $Q_{2, \theta}(2\rho)^2(z_o) \subseteq \Omega_T$ be a parabolic cylinder such that
\[
\sup_{Q_{\rho, \theta}(z_o)} \psi \leq \frac{1}{2}(\mu^+ + \mu^-).
\]
Assume that hypothesis (6.10) or (6.11) holds and that $u$ is a locally bounded, local weak solution to the obstacle problem. Then, for any $v \in (0, 1)$ there exists a constant $a = a(n, q, C_o, C_1, v) \in (0, \frac{1}{64}]$ such that if
\[
\left| \{ u \leq \mu^- + \frac{1}{2} \omega \} \cap Q_{\rho, \theta}(z_o) \right| > v \left| Q_{\rho, \theta}(z_o) \right|,
\]
then
\[
u 
 u \leq \mu^+ - a \omega \quad \text{a.e. in } Q_{\frac{\rho}{2}, \frac{1}{2} \theta}(z_o).
\]

7. Reduction in oscillation

Throughout the rest of the paper, we denote the minimum of the parameters $\nu$ from Lemmas 5.1 to 5.4 by $\nu_o$. Further, we let $\alpha$ be the minimum of the respective parameters in Lemmas 6.3 and 6.4 corresponding to the parameter $\nu_o$ chosen above, and define $\delta = 1 - \alpha \in [\frac{3}{4}, 1)$. This implies that these parameters coincide in the following lemmas, which allows us to use them subsequently in Sect. 8.

Moreover, throughout this section, we consider parabolic cylinders of the form $Q_o := Q_{\rho_o, \theta}(z_o)$ and quantities
\[
\mu_o^+ \geq \sup_{Q_o} u, \quad \mu_o^- \leq \inf_{Q_o} u \quad \text{and} \quad \omega_o = \mu_o^+ - \mu_o^-.
\] (7.1)

Further, we assume that
\[
\sup_{Q_o} \psi \leq \frac{1}{2}(\mu_o^+ + \mu_o^-) \quad \text{and} \quad \text{osc}_{Q_o} \psi \leq \frac{1}{2} \omega_o.
\] (7.2)

We treat the following cases corresponding to (5.2)–(5.4) with $\mu^\pm, \omega$ replaced by $\mu_o^\pm, \omega_o$ separately: Either we assume that
\[
\mu_o^+ \geq -\frac{1}{4} \omega_o \quad \text{and} \quad \mu_o^- \leq \frac{1}{4} \omega_o \quad \text{and} \quad \theta = \omega_o^{q-1},
\] (7.3)
or
\[
\mu_o^- > \frac{1}{4} \omega_o \quad \text{and} \quad \theta = (\mu_o^+)^{q-1},
\] (7.4)
or
\[
\mu_o^+ < -\frac{1}{4} \omega_o \quad \text{and} \quad \theta = |\mu_o^-|^{q-1}.
\] (7.5)

First, we are concerned with the case where $u$ is near zero.
Lemma 7.1. Assume that the hypotheses (7.1), (7.2), and (7.3) are satisfied. Define 
\[ \omega_1 := \max \left\{ \delta \omega_o, 2 \text{osc}_{Q_o} \psi \right\} \]
and 
\[ Q_1 := Q_{\rho_1, \theta_1 \rho_1^2 (z_o)} \quad \text{with} \quad \theta_1 = \omega_1^{q-1}, \ \rho_1 = \lambda \rho_o, \ \lambda := \sqrt{\frac{v_o \delta (1-q)_+}{8}}. \]

Then, we have that 
\[ \text{osc}_{Q_1} u \leq \omega_1 \quad \text{and} \quad Q_1 \subset Q_o. \]

Proof. Observe that (7.3) implies that \( |\mu_o^\pm| \leq \frac{5}{4} \omega_o \). Furthermore, we have 
\[ \mu_o^+ \geq \frac{1}{2} \omega_o \quad \text{or} \quad \mu_o^- \leq -\frac{1}{2} \omega_o, \quad (7.6) \]
since \( \omega_o = \mu_o^+ - \mu_o^- \). Suppose first that (7.6)\(_1\) holds true. Then, we have \( \frac{1}{2} \omega_o \leq \mu_o^+ \leq \frac{5}{4} \omega_o \). In this case, we distinguish between the alternatives 
\[ \left\{ \begin{array}{ll} |u \leq \mu_o^- + \frac{1}{2} \omega_o| \cap Q_o \leq v_o |Q|, \\
|u \leq \mu_o^- + \frac{1}{2} \omega_o| \cap Q_o > v_o |Q|. \end{array} \right. \quad (7.7) \]

When (7.8)\(_1\) holds true, we apply Lemma 5.4. Since \( |\mu_o^-| \leq \frac{5}{4} \omega_o \), this yields 
\[ \inf_{Q_{\frac{v_o}{2}, \theta (\frac{v_o}{2})}^2 (z_o)} u \geq \mu_o^- + \frac{1}{4} \omega_o. \]

On the other hand, if (7.8)\(_2\) holds true, we may apply Lemma 6.4 and obtain 
\[ \sup_{Q_{\frac{v_o}{2}, \theta (\frac{v_o}{2})}^2 (z_o)} u \leq \mu_o^+ - a \omega_o. \]

Clearly \( \lambda \leq \frac{1}{2} \) and in the case \( 0 < q < 1 \) we can estimate 
\[ \theta_1 \rho_1^2 = \omega_1^{q-1} \cdot \frac{1}{2} v_o \delta^{1-q} \left( \frac{\rho_o}{2} \right)^2 \leq \frac{1}{2} v_o \theta \left( \frac{\rho_o}{2} \right)^2 \]
by \( \omega_1 \geq \delta \omega_o \). If \( q > 1 \) and \( \omega_1 = \delta \omega_o \) the same inequality holds true. If \( q > 1 \) and \( \omega_1 = 2 \text{osc}_{Q_o} \psi \) holds, we use that \( 2 \text{osc}_{Q_o} \psi \leq \omega_o \) by assumption. Hence, 
\[ Q_1 \subset Q_{\frac{v_o}{2}, \frac{1}{2} v_o \theta (\frac{v_o}{2})^2} \subset Q_{\frac{v_o}{2}, \theta (\frac{v_o}{2})^2} \]
follows in any case. Therefore, we have that either 
\[ \text{osc}_{Q_1} u \leq \text{osc}_{Q_{\frac{v_o}{2}, \theta (\frac{v_o}{2})^2}} u \leq \mu_o^+ - (\mu_o^- + \frac{1}{4} \omega_o) = \frac{3}{4} \omega_o \leq \delta \omega_o \leq \omega_1, \]
or 
\[ \text{osc}_{Q_1} u \leq \text{osc}_{Q_{\frac{v_o}{2}, \frac{1}{2} v_o \theta (\frac{v_o}{2})^2}} u \leq \mu_o^+ - a \omega_o - \mu_o^- = \delta \omega_o \leq \omega_1. \]
This completes the proof in case (7.6)$_1$. Now, suppose that (7.6)$_2$ holds true. Then, we have that $-\frac{5}{4}\omega_o \leq \mu^-_o \leq -\frac{1}{2}\omega_o$ and distinguish between the alternatives

$$\left\{ \begin{array}{l}
\{ u \geq \mu^+_o - \frac{1}{2}\omega_o \} \cap Q_o \leq v_o | Q | , \\
\{ u \geq \mu^+_o - \frac{1}{2}\omega_o \} \cap Q_o > v_o | Q | .
\end{array} \right.$$  (7.8)

When (7.8) holds true, we may apply Lemma 5.3, since $|\mu^+_o| \leq \frac{5}{4}\omega_o$. This implies

$$\sup_{Q_{\mu_o, \delta}} \sup_{(\frac{\mu_o}{2})^2(z_o)} u \leq \mu^+_o - \frac{1}{4}\omega_o.$$ 

On the other hand if (7.8)$_2$ holds true, we apply Lemma 6.3 to obtain

$$\inf_{Q_{\mu_o, \delta}} \inf_{(\frac{\mu_o}{2})^2(z_o)} u \geq \mu^-_o + a\omega_o.$$ 

By using similar estimates as above, we conclude that

$$\text{osc}_{Q_i} u \leq \delta\omega_o \leq \omega_1,$$

which finishes the proof. \hfill \Box

Up next, we will prove a similar result in the case where $u$ is bounded away from zero and positive.

**Lemma 7.2.** Assume that (7.1), (7.2), and (7.4) hold true. For the sequence of cylinders

$$Q_i := Q_{\rho_i, \theta \rho_i^2(z_o)} , \quad \text{with} \quad \rho_i = \lambda^i \rho_o, \quad \lambda := \sqrt{\frac{v_o}{8}},$$

we define

$$\omega_i := \max \left\{ \delta\omega_{i-1}, 2 \text{osc}_{Q_{i-1}} \psi \right\} \quad \text{for } i \in \mathbb{N}_0.$$ 

Then, for any $i \in \mathbb{N}_0$ there holds

$$\text{osc}_{Q_i} u \leq \omega_i.$$ 

**Proof.** First, observe that

$$Q_{i+1} \subset Q_{\frac{\rho_i}{2}, \frac{1}{2} v_o \theta (\frac{\rho_i}{2})^2(z_o)} \subset Q_{\frac{\mu^-_o}{2}, \theta (\frac{\mu^-_o}{2})^2(z_o)} \subset Q_i$$

for any $i \in \mathbb{N}_0$. Define

$$\mu^-_i := \inf_{Q_i} u, \quad \mu^+_i = \mu^-_i + \omega_i$$

for every $i \in \mathbb{N}$. Now by the assumptions, we already have that $\omega_1 \leq \omega_o$ holds true. By induction it follows directly that

$$\frac{1}{4}\omega_{i+1} \leq \frac{1}{4}\omega_i < \mu^-_i \leq \mu^-_{i+1}$$
for all \( i \in \mathbb{N}_0 \). Since (7.4) is equivalent to \( \mu_o^+ < 5 \mu_o^- \), we have that \( \theta \frac{1}{\sigma} = \mu_o^+ < 5 \mu_o^- \leq 5 \mu_i^- \leq 5 \mu_i^+ \). Further, we know that \( \mu_i^+ = \mu_i^- + \omega_i \leq \sup_{Q_o} u + \omega_o \leq 2 \mu_o^+ = 2 \theta \frac{1}{\sigma} \). Therefore, we find that
\[
\frac{1}{2} \mu_i^+ \leq \theta \frac{1}{\sigma} \leq 5 \mu_i^+
\]
for any \( i \in \mathbb{N}_0 \). Moreover, we have that
\[
\sup_{Q_i} \psi = \inf_{Q_i} \psi + \text{osc}_{Q_i} \psi \leq \inf_{Q_i} u + \text{osc}_{Q_i-1} \psi \leq \mu_i^- + \frac{1}{2} \omega_i = \frac{1}{2} (\mu_i^+ + \mu_i^-)
\]
for every \( i \in \mathbb{N}_0 \). Assume that \( \text{osc}_{Q_i} u \leq \omega_i \) for some \( i \in \mathbb{N} \). For \( i = 0 \), this clearly holds by (7.1). Then, in particular we have that \( \mu_i^+ = \mu_i^- + \omega_i \geq \inf_{Q_i} u + \text{osc}_{Q_i} u = \sup_{Q_i} u \). Now, we distinguish between the alternatives
\[
\begin{cases}
\{ u \leq \mu_i^- + \frac{1}{2} \omega_i \} \cap Q_i \leq v_o |Q_i|, \\
\{ u \leq \mu_i^- + \frac{1}{2} \omega_i \} \cap Q_i \leq v_o |Q_i|.
\end{cases}
\]
When the first alternative holds true, by Lemma 5.2 we obtain that
\[
\inf_{Q_{\frac{\rho_i}{2}, \theta \rho_i}^2} u \geq \mu_i^- + \frac{1}{4} \omega_i.
\]
On the other hand if the second alternative holds true, Lemma 6.4 implies that
\[
\sup_{Q_{\frac{\rho_i}{2}, \frac{1}{2} v_o \theta (\frac{\rho_i}{2})}^2} u \leq \mu_i^+ - a \omega_i
\]
for some \( a = a(n, q, C_o, C_1) \in (0, \frac{1}{64}] \). Recalling that \( \delta = 1 - a \), we see that in both cases
\[
\text{osc}_{Q_{i+1}} u \leq \delta \omega_i \leq \omega_{i+1},
\]
which completes the proof. \( \square \)

Finally we state and prove a similar lemma in the case where \( u \) is bounded away from zero and negative. Observe that the problem is not symmetric in the case where the solution \( u \) is above and the case where it is below zero, since the obstacle is restricting the behavior of \( u \) only from below. Thus, in Lemma 7.3 we cannot proceed analogously to Lemma 7.2 by defining \( \mu_i^+ = \sup_{Q_i} u \) and \( \mu_i^- = \mu_i^+ - \omega_i \), since the condition \( \sup_{Q_{\rho_i, \theta \rho_i}} \psi \leq \frac{1}{2} \left( \mu_i^+ + \mu_i^- \right) \) needed for the application of Lemma 5.1 could be violated. Hence, we use a different approach.

\textbf{Lemma 7.3.} Assume that the hypotheses (7.1), (7.2), and (7.5) hold. For the sequence of cylinders
\[
Q_i := Q_{\rho_i, \theta \rho_i^2}(z_o), \quad \text{with} \quad \rho_i = \lambda^i \rho_o, \quad \lambda := \sqrt{\frac{v_o}{8}},
\]
we define
\[ \omega_i := \max\left\{ \delta \omega_{i-1}, 2 \text{osc}_{Q_{i-1}} \psi \right\} \quad \text{for } i \in \mathbb{N}_0. \]

Then, for any \( i \in \mathbb{N}_0 \) there holds
\[ \text{osc}_{Q_i} u \leq \omega_i. \]

**Proof.** First, observe that
\[ Q_{i+1} \subset Q_{\frac{1}{2}, \frac{1}{2} v_0 \theta (\frac{1}{2})^2 (z_0)} \subset Q_{\frac{1}{2}, \theta (\frac{1}{2})^2 (z_0)} \subset Q_i \]
for any \( i \in \mathbb{N}_0. \) Define
\[ \mu_i^- := \inf_{Q_i} u, \quad \mu_i^+ = \min\{ \mu_i^-, \omega_i, \mu_{i+1}^+ \} \]
for every \( i \in \mathbb{N}. \) Now by the assumptions, we already have that \( \omega_1 \leq \omega_o \) holds true and by induction it directly follows that
\[ \omega_{i+1} \leq \omega_i \quad \text{and} \quad \mu_{i+1}^+ \leq \mu_i^+ < -\frac{1}{4} \omega_i \leq -\frac{1}{4} \omega_{i+1} \]
for all \( i \in \mathbb{N}_0, \) where we have used that \( \{ \mu_i^+ \}_{i \in \mathbb{N}_0} \) is a nonincreasing sequence by definition. Since (7.5) is equivalent to \( \mu_o^- > 5 \mu_o^+ \) and \( \{ \mu_i^+ \}_{i \in \mathbb{N}_0} \) is nonincreasing, we have that \( \theta \frac{1}{q-1} = -\mu_o^- < -5 \mu_o^+ \leq -5 \mu_i^- \) and that \( \theta \frac{1}{q-1} = -\mu_o^- \geq -\mu_i^- ; \) i.e., we find that
\[ |\mu_i^-| \leq \theta \frac{1}{q-1} \leq 5 |\mu_i^-| \]
for any \( i \in \mathbb{N}_0. \) Up next, we show that the condition \( \sup_{Q_i} \psi \leq \frac{1}{2} (\mu_i^+ + \mu_i^-) \) holds true for all \( i \in \mathbb{N}_0. \) For \( i = 0 \) this is part of hypothesis (7.2). Suppose that this holds for some \( i \in \mathbb{N}. \) On the one hand if \( \mu_{i+1}^+ = \mu_{i+1}^- + \omega_{i+1}, \) we have that
\[ \sup_{Q_{i+1}} \psi = \inf_{Q_{i+1}} \psi + \text{osc}_{Q_{i+1}} \psi \leq \inf_{Q_{i+1}} u + \text{osc}_{Q_i} \psi \leq \mu_{i+1}^- + \frac{1}{2} \omega_{i+1} = \frac{1}{2} (\mu_{i+1}^+ + \mu_{i+1}^-). \]

On the other hand if \( \mu_{i+1}^+ = \mu_i^+ \), by the induction assumption and the property that \( \mu_i^- \leq \mu_{i+1}^- \) we may estimate
\[ \sup_{Q_{i+1}} \psi \leq \sup_{Q_i} \psi \leq \frac{1}{2} (\mu_i^+ + \mu_i^-) \leq \frac{1}{2} (\mu_{i+1}^+ + \mu_{i+1}^-). \]

Next, we want to show that for every \( i \in \mathbb{N}, \) there holds
\[ \sup_{Q_{i-1}} u \leq \mu_{i-1}^+ \quad \text{and} \quad \text{osc}_{Q_i} u \leq \omega_i. \quad (7.9) \]
For $i = 1$, (7.9) clearly holds by assumption. Now, we consider the alternatives

$$
\begin{cases}
\{u \geq \mu_i^+ - \frac{1}{2} \omega_i\} \cap Q_i \leq \nu_o |Q_o|,
\{u \geq \mu_i^+ - \frac{1}{2} \omega_i\} \cap Q_i > \nu_o |Q_o|.
\end{cases}
$$

If the first alternative holds true, we may apply Lemma 5.1. On the other hand, if the second alternative holds true, we use Lemma 6.3. In both cases, we find that

$$\text{osc}_{Q_i} u \leq \delta \omega_o \leq \omega_1,$$

where the last inequality holds by definition of $\omega_1$, and $\delta = 1 - a$ with the constant $a$ from Lemma 6.3. This takes care of the case $i = 1$. Now let us suppose that (7.9) holds for some $i \in \mathbb{N}$. It follows that either we have

$$\mu_{i+1} = \mu_i + \omega_i \geq \inf_{Q_i} u + \text{osc}_{Q_i} u = \sup_{Q_i} u$$

with assumption (7.9)$_2$ or that

$$\mu_{i+1} = \mu_{i-1} \geq \sup_{Q_{i-1}} u \geq \sup_{Q_i} u$$

with assumption (7.9)$_1$. The two inequalities above already prove (7.9)$_1$. Let us define

$$\tilde{\omega}_i = \mu_i^+ - \mu_i^- \leq \omega_i.$$ 

Now, we will use the alternatives

$$
\begin{cases}
\{u \geq \mu_i^+ - \frac{1}{2} \tilde{\omega}_i\} \cap Q_i \leq \nu_o |Q_i|,
\{u \geq \mu_i^+ - \frac{1}{2} \tilde{\omega}_i\} \cap Q_i > \nu_o |Q_i|.
\end{cases}
$$

(7.10)

In the first case, we apply Lemma 5.1 with $\tilde{\omega}_i$ in place of $\omega_i$. This implies that

$$\sup_{Q_{\rho_i, \theta}(\frac{\rho_i}{2})^2} u \leq \mu_i^+ - \frac{1}{4} \tilde{\omega}_i.$$ 

Since $Q_{i+1} \subset Q_{\rho_i, \theta}(\frac{\rho_i}{2})^2 \subset Q_i$, we have

$$\text{osc}_{Q_{i+1}} u \leq \sup_{Q_{\rho_i, \theta}(\frac{\rho_i}{2})^2} u - \inf_{Q_{\rho_i, \theta}(\frac{\rho_i}{2})^2} u \leq \mu_i^+ - \mu_i^- - \frac{1}{4} \tilde{\omega}_i \leq \frac{3}{4} \omega_i \leq \omega_{i+1},$$

(7.11)

since $\tilde{\omega}_i \leq \omega_i$ and by definition of $\omega_{i+1}$. If (7.10)$_2$ holds true, we use Lemma 6.3, which gives us

$$\inf_{Q_{\rho_i, \theta}(\frac{\rho_i}{2})^2} u \geq \mu_i^- + a \tilde{\omega}_i$$

for some $a = a(n, q, C_o, C_1) \in (0, \frac{1}{64}]$. Recalling that $\delta = 1 - a$, similarly to (7.11) we obtain

$$\text{osc}_{Q_{i+1}} u \leq \mu_i^+ - \mu_i^- - a \tilde{\omega}_i \leq \delta \omega_i \leq \delta \omega_{i+1} \leq \omega_{i+1}.$$ 

(7.12)

Now (7.11) and (7.12) imply (7.9)$_2$, which completes the proof. \qed
8. Proof of Theorem 1.1

In the following, we assume that $u$ is globally bounded for ease of notation. However, the argument holds for a locally bounded weak solution $u$ by restricting to a compact subset of $\Omega_T$. Thus, we can assume that

\[
\text{osc}_{\Omega_T} u \leq 1, \quad \left| \sup_{\Omega_T} u \right| \leq \frac{1}{2} \quad \text{and} \quad \left| \inf_{\Omega_T} u \right| \leq \frac{1}{2}
\]

by using the rescaling argument from Lemma A.1 with $M = 2\|u\|_{\infty}$.

Assume that $\psi \in C^{0;\beta,\frac{\beta}{2}}(\Omega_T)$ for the exponent $\beta \in (0, 1)$, i.e.,

\[
\left[ \psi \right]_{C^{0;\beta,\frac{\beta}{2}}} := \sup_{(x,t),(y,s) \in \Omega_T} \frac{|\psi(x,t) - \psi(y,s)|}{|x-y|^\beta + |t-s|^{\frac{\beta}{2}}} < \infty.
\]

Let $\varepsilon = \frac{2\beta(1-q)^+}{2+\beta(1-q)^+} \in [0, \frac{2}{3})$ and $\gamma_\varepsilon = \frac{2\beta}{2+\beta(1-q)^+} = (1 - \frac{\varepsilon}{2})\beta \in (0, \beta]$. Observe that $\varepsilon = 0$ and $\gamma_\varepsilon = \beta$ in the singular case $q > 1$. Further, consider an arbitrary point $z_\varepsilon = (x_o, t_o) \in \Omega_T$ and let $R \in (0, 1)$ be so small that $Q_{R,R^{2-\varepsilon}}(z_o) \subseteq \Omega_T$. In the following, we omit $z_\varepsilon$ from our notation for simplicity. Next, we consider the function

\[
\Psi(\rho) := \max \left\{ \rho^{\gamma_o}, 2 \text{osc}_{Q_{\rho,\rho^{2-\varepsilon}}} \psi \right\},
\]

which is continuous and increasing. By the assumption $\psi \in C^{0;\beta,\frac{\beta}{2}}(\Omega_T)$, the choice of $\varepsilon$ and the fact that $\rho \in (0, 1)$ we obtain that

\[
\text{osc}_{Q_{\rho,\rho^{2-\varepsilon}}} \psi \leq \left[ \psi \right]_{C^{0;\beta,\frac{\beta}{2}}} \max \left\{ \rho^\beta, \rho^{(2-\varepsilon)\frac{\beta}{2}} \right\} = \left[ \psi \right]_{C^{0;\beta,\frac{\beta}{2}}} \rho^{\gamma_\varepsilon}.
\]

Thus, we conclude that $u$ is Hölder continuous at $(x_o, t_o)$ in the case that the bound

\[
\text{osc}_{Q_{\rho,\rho^{2-\varepsilon}}} u \leq \Psi(\rho) \quad \forall \rho \in (0, R]
\]

holds. In order to prove the Hölder continuity of $u$ in the opposite case, observe that if the preceding inequality is false, then there exists $\rho_0 \in (0, R]$ such that

\[
\Psi(\rho_0) < \text{osc}_{Q_{\rho_o,\rho_o^{2-\varepsilon}}} u \quad \text{and} \quad \text{osc}_{Q_{\rho,\rho^{2-\varepsilon}}} u \leq 2R^{-\gamma_o}\Psi(\rho) \quad \forall \rho \in [\rho_0, R],
\]

since the map $\rho \mapsto \text{osc}_{Q_{\rho,\rho^{2-\varepsilon}}} u$ is increasing, the map $\rho \mapsto \Psi(\rho)$ is continuous and increasing and $\text{osc}_{Q_{R,R^{2-\varepsilon}}} u \leq 1 \leq R^{-\gamma_o}\Psi(\rho)$. For this choice of $\rho_0$, we define

\[
\rho_o^+ := \sup_{Q_{\rho_o,\rho_o^{2-\varepsilon}}} u, \quad \rho_o^- := \inf_{Q_{\rho_o,\rho_o^{2-\varepsilon}}} u, \quad \omega_o := \rho_o^+ - \rho_o^-.
\]

In the case $0 < q < 1$ we define $\theta_o := \omega_o^{q-1}$. Further, we compute that

\[
\theta_o = \left( \text{osc}_{Q_{\rho_o,\rho_o^{2-\varepsilon}}} u \right)^{q-1} < \Psi(\rho_o)^{q-1} \leq \rho_o^{-\varepsilon}
\]
by definition of $\rho_o$. In the singular case $q > 1$, we define

$$
\theta_o := \begin{cases} 
\omega_o^{q-1} & \text{if } \mu_o^+ \geq -\frac{1}{4}\omega_o \text{ and } \mu_o^- \leq \frac{1}{4}\omega_o, \\
(\mu_o^+)^{q-1} & \text{if } \mu_o^- > \frac{1}{4}\omega_o, \\
|\mu_o^-|^{q-1} & \text{if } \mu_o^+ < -\frac{1}{4}\omega_o.
\end{cases}
$$

By taking into account (8.1), we conclude that in any case $\theta_o \leq 1$ when $q > 1$. Hence, we have the set inclusion

$$
Q_o := Q_{\rho_o, \theta_o \rho_o^2} \subset Q_{\rho_o, \rho_o^2 - \epsilon}.
$$

Since $u \geq \psi$ a.e. in $\Omega_T$ and by the choice of $\rho_o$, we deduce that

$$
\sup_{Q_o} \psi \leq \sup_{Q_{\rho_o, \rho_o^2 - \epsilon}} \psi = \inf_{Q_{\rho_o, \rho_o^2 - \epsilon}} \psi + \text{osc}_{Q_{\rho_o, \rho_o^2 - \epsilon}} u < \inf_{Q_{\rho_o, \rho_o^2 - \epsilon}} u + \frac{1}{2} \text{osc}_{Q_{\rho_o, \rho_o^2 - \epsilon}} u
$$

$$
= \mu_o^- + \frac{1}{2}\omega_o = \frac{1}{2}(\mu_o^+ + \mu_o^-).
$$

(8.3)

By the definition of $\Psi$ and (8.2) we infer

$$
\text{osc}_{Q_o} \psi \leq \text{osc}_{Q_{\rho_o, \rho_o^2 - \epsilon}} \psi \leq \frac{1}{2}\Psi(\rho_o) < \frac{1}{2}\omega_o = \frac{1}{2}(\mu_o^+ - \mu_o^-).
$$

(8.4)

In the following, the strategy is to construct a sequence of shrinking cylinders $\{Q_i\}_{i \in \mathbb{N}_0}$ with common vertex $z_o$, such that the oscillation of $u$ in these cylinders can be reduced in a quantitative way when passing from $Q_i$ to the subsequent cylinder $Q_{i+1}$. Exactly one of the three possible cases will hold in each cylinder of the sequence: $u$ is near zero, $u$ is above and away from zero or $u$ is below and away from zero. If $u$ is near zero in $Q_i$, any of the three cases may hold in $Q_{i+1}$. However, if either of the cases where $u$ is bounded away from zero holds in $Q_i$, the same case will also hold in $Q_{i+1}$ and in every subsequent cylinder. Up next, $\mu_i^+$ will roughly denote an upper bound for $u$, $\mu_i^-$ a lower bound for $u$ and $\omega_i$ an upper bound for the oscillation of $u$ in the cylinder $Q_i$. Precise definitions are given in the following subsections. Now, we proceed as follows: Suppose that $i_o \in \mathbb{N}_0 \cup \{\infty\}$ is the largest index for which we have

$$
\mu_i^+ \geq -\frac{1}{4}\omega_i \text{ and } \mu_i^- \leq \frac{1}{4}\omega_i
$$

(8.5)

for all $i \in \{0, 1, \ldots, i_o - 1\}$. This means that up to the index $i_o - 1$, we apply the reduction in oscillation for the case where $u$ is near zero (see Sect. 8.1). For every $i \geq i_o$, it then follows that either

$$
\mu_i^- > \frac{1}{4}\omega_i \text{ or } \mu_i^+ < -\frac{1}{4}\omega_i
$$

(8.6)

holds true and we use the results on a reduction in oscillation for either one of the cases where $u$ is away from zero (see Sects. 8.2–8.3). Observe that it is also possible that $i_o = 0$, which means that the case where $u$ is near zero never occurs, or that $i_o = \infty$ and the iteration is carried out for $u$ near zero completely.
8.1. Reduction in oscillation near zero

Suppose that $i_o > 0$, otherwise we can skip this part and move directly to either Sect. 8.2 or 8.3 depending on which case in (8.6) holds true. For $i \in \{1, 2, \ldots, i_o\}$, define

\[
\begin{aligned}
\rho_i := \lambda \rho_{i-1}, & \quad \omega_i := \max\{\delta \omega_{i-1}, 2 \osc Q_{i-1} \psi\}, \quad \theta_i := \omega_i^{q-1}, \\
\lambda := \sqrt{\frac{1}{8} v_o \delta (1-q)^+} & \quad Q_i := Q_{\rho_i \theta_i \rho_i^2}, \\
\mu^-_i := \inf Q_i u & \quad \text{and} \quad \mu^+_i := \mu^-_i + \omega_i.
\end{aligned}
\]

This implies

\[
\sup Q_i \psi = \inf Q_i \psi + \osc Q_i \psi \leq \inf Q_i u + \osc Q_{i-1} \psi \leq \mu^-_i + \frac{1}{2} \omega_i = \frac{1}{2} \left( \mu^-_i + \mu^+_i \right),
\]

and

\[
\osc Q_i \psi \leq \osc Q_{i-1} \psi \leq \frac{1}{2} \omega_i = \frac{1}{2} \left( \mu^+_i - \mu^-_i \right)
\]

for every $i \in \{1, 2, \ldots, i_o\}$. Now, we claim that

\[
Q_{i_o} \subset Q_{i_o-1} \subset \cdots \subset Q_0 \quad \text{and} \quad \osc Q_i u \leq \omega_i \quad \text{for all } i \in \{0, 1, \ldots, i_o\}.
\]

(8.7)

Clearly, this holds true by definitions when $i = 0$. Suppose that the statement holds true for some $i < i_o$. Then, we have that

\[
\mu^+_i = \mu^-_i + \omega_i \geq \mu^-_i + \osc Q_i u = \sup Q_i u,
\]

by assumption. Now, we are in a point of using Lemma 7.1, which implies that

\[
Q_{i+1} \subset Q_i \quad \text{and} \quad \osc Q_{i+1} u \leq \omega_{i+1},
\]

which proves (8.7).

8.2. Reduction in oscillation above and away from zero

Suppose that $i_o \in \mathbb{N}_0$ is the first index for which there holds that

\[
\mu^-_{i_o} > \frac{1}{4} \omega_{i_o}.
\]

(8.8)

Now, we define $\theta_* = \left( \mu^+_{i_o} \right)^{q-1}$. In the case $0 < q < 1$ it directly follows that $\theta_* \leq \left( \omega_{i_o} \right)^{q-1} = \theta_{i_o}$. If $q > 1$ and $i_o = 0$ we have $\theta_* = \theta_0$. If $i_o \geq 1$ we deduce that

\[
\frac{1}{4} \omega_{i_o} < \mu^-_{i_o} = \inf Q_{i_o} u \leq \sup Q_{i_o-1} u \leq \mu^+_{i_o-1} \leq \frac{5}{4} \omega_{i_o-1} \leq \frac{5}{43} \omega_{i_o},
\]

which proves (8.8).
by using (8.5) for the index $i_o - 1$. Since (8.8) is equivalent to $\mu_{i_o}^+ < 5\mu_{i_o}^-$, this implies that

$$\mu_{i_o}^+ < 5\mu_{i_o}^- \leq \frac{25}{48} \omega_{i_o}.$$  

Now, we can deduce the bound $\theta_* < \left(\frac{25}{48} \omega_{i_o}\right)^{q-1} = \left(\frac{25}{48}\right)^{q-1} \theta_{i_o}$ when $q > 1$, which is taken into account in the following construction. For the cylinders $i > i_o$ let

$$Q_i := Q_{\hat{\rho}_i, \theta_* \hat{\rho}_i^2} \quad \text{with} \quad \hat{\rho}_i = \hat{\lambda} i^{1-i_o} \left(\frac{48}{25}\right)^{(q-1)/2} \rho_{i_o}, \quad \hat{\lambda} := \sqrt{\nu_o / 8},$$

and let $Q_{i_o}^* = Q_{\hat{\rho}_{i_o}, \theta_* \hat{\rho}_{i_o}^2} \subset Q_{i_o}$, where $Q_{i_o}$ is the cylinder obtained in the last section after application of the last iteration step or it is $Q_{i_o} = Q_0$ if $i_o = 0$. Now, we clearly have that $Q_{i_o} \supset Q_{i_o}^* \supset Q_{i_o+1} \supset \ldots$ and

$$\inf_{Q_{i_o}^*} u \geq \inf_{Q_{i_o}} u = \mu_{i_o}^- \quad \text{and} \quad \sup_{Q_{i_o}^*} u \leq \sup_{Q_{i_o}} u \leq \mu_{i_o}^+.$$  

Further, we find that

$$\sup_{Q_{i_o}^*} \psi \leq \sup_{Q_{i_o}} \psi \leq \frac{1}{2} \left(\mu_{i_o}^+ + \mu_{i_o}^-\right).$$

where the last inequality follows from Sect. 8.1 if $i_o > 0$ and from (8.3) if $i_o = 0$. Finally, we obtain that

$$\mu_{i_o}^+ - \mu_{i_o}^- = \omega_{i_o} \geq 2 \text{osc}_{Q_{i_o-1}} \psi \geq 2 \text{osc}_{Q_{i_o}} \psi \geq 2 \text{osc}_{Q_{i_o}^*} \psi,$$

which follows from Sect. 8.1 if $i_o > 0$, and from (8.4) if $i_o = 0$. Now, we are in the position to use Lemma 7.2, which implies

$$\text{osc}_{Q_i} u \leq \omega_i \quad \text{for all} \ i > i_o.$$  

8.3. Reduction in oscillation below and away from zero

Suppose that $i_o \in \mathbb{N}_0$ is the first index for which there holds that

$$\mu_{i_o}^+ < -\frac{1}{4} \omega_{i_o}. \quad (8.9)$$

Now, we define $\theta_* = \left(\mu_{i_o}^-\right)^{q-1}$. If $0 < q < 1$, it follows that $\theta_* \leq \left(\omega_{i_o}\right)^{q-1} = \theta_{i_o}$. In the case $q > 1$, $\theta_* = \theta_{i_o}$ if $i_o = 0$, and if $i_o > 0$ we deduce that

$$-\frac{1}{4} \omega_{i_o} > \mu_{i_o}^+ \geq \sup_{Q_{i_o}} u \geq \inf_{Q_{i_o-1}} u = \mu_{i_{o-1}}^- \geq -\frac{5}{4} \omega_{i_{o-1}} \geq -\frac{5}{48} \omega_{i_o}$$

by using the condition (8.5) for the index $i_o - 1$ in the penultimate inequality. Since (8.9) is equivalent to $5\mu_{i_o}^+ < \mu_{i_o}^-$, this implies

$$-\mu_{i_o}^- < -5\mu_{i_o}^+ \leq \frac{25}{48} \omega_{i_o}.$$
For the cylinders $i > i_o$ we define

$$Q_i := Q_{\hat{\rho}_i, \theta_i \hat{\rho}_i^2}, \quad \text{with} \quad \hat{\rho}_i = \hat{\lambda}^{i-i_o} \left(\frac{4\delta}{25}\right)\frac{(q-1)^+}{2} \rho_{i_o}, \quad \hat{\lambda} := \frac{\sqrt{v_o}}{8},$$

and let $Q_{i_o}^* = Q_{\hat{\rho}_{i_o}, \theta_{i_o} \hat{\rho}_{i_o}^2} \subset Q_{i_o}$, where $Q_{i_o}$ is the cylinder obtained in Sect. 8.1 after the last iteration step. Analogously to the Sect. 8.2, we are now in the position to use Lemma 7.3, which implies

$$\text{osc}_Q u \leq \omega_i \quad \text{for all} \quad i > i_o.$$

### 8.4. Proof of the oscillation decay estimate

We define

$$r_i := \left\{ \begin{array}{ll}
\left(\frac{\delta_i}{\omega_o}\right)_{q-1}^{\frac{q-1}{2}} \hat{\rho}_i & \text{if} \quad q > 1, \\
\min\left\{ 1, \left(\frac{\delta_i}{\omega_o}\right)_{q-1}^{\frac{q-1}{2}} \right\} \hat{\rho}_i & \text{if} \quad 0 < q < 1,
\end{array} \right.$$

where $\hat{\rho}_i := \rho_i$ for $i < i_o$. We claim that

$$Q_{r_i} := Q_{r_i, r_i^2} \subset Q_i$$

for any $i \in \mathbb{N}_0$.

Recall that either $\theta_* = \left(\frac{\mu_{i_o}^+}{\omega_o}\right)^{q-1}$ if (8.8) holds, or $\theta_* = \left|\frac{\mu_{i_o}}{\omega_o}\right|^{q-1}$ if (8.9) holds.

Indeed, $r_i \leq \hat{\rho}_i$ and if $q > 1$ we know that $\theta_i = \omega_i^{q-1} \geq \left(\delta_i \omega_o\right)^{q-1}$ for $i \leq i_o$ and $\theta_* \geq \left(\frac{1}{4}\omega_o\right)^{q-1} \geq \left(\frac{\delta_i}{\omega_o}\right)^{q-1}$ if $i_o > 0$. If $q > 1$ and $i_o = 0$, we have $\theta_* \geq \left(\frac{\delta_i}{\omega_o}\right)^{q-1}$. Moreover, if $0 < q < 1$ we have that $\theta_i = \omega_i^{q-1} \geq \omega_o^{q-1}$ for $i \leq i_o$ and $\theta_* \geq \left(\frac{25}{4}\omega_o\right)^{q-1} \geq \left(\frac{\delta_i}{\omega_o}\right)^{q-1}$ by definition of $i_o$ if $i_o > 0$. In case $0 < q < 1$ and $i_o = 0$, we can use (8.1) such that $\theta_* \geq 1$. Therefore, we find that

$$\text{osc}_{Q_{r_i}} u \leq \text{osc}_Q u \leq \omega_i \leq \delta_i \omega_o + 2 \sum_{j=0}^{i-1} \delta^j \text{osc}_{Q_{i-1-j}} \psi. \quad \text{(8.10)}$$

When $i - 1 - j \leq i_o$, by the fact that $\rho_{i-1-j} = \lambda^{i-1-j} \rho_o$ and the definition of $\lambda$ we estimate

$$\text{osc}_{Q_{i-1-j}} \psi \leq c[\psi]_{C^{0,\beta/2}} \left( \rho_{i-1-j}^\beta + \left( \theta_{i-1-j} \rho_{i-1-j}^2 \right)^\beta \right)$$

$$= c[\psi]_{C^{0,\beta/2}} \left( 1 + \omega_{i-1-j} \right) \rho_{i-1-j}^\beta$$

$$\leq c[\psi]_{C^{0,\beta/2}} \left( 1 + \delta^\beta \frac{\rho(i-1-j)(1-q)^+}{\omega_o^2} \right) \rho_{i-1-j}^\beta$$
\[ \leq c[\psi] C_{0, \beta, \frac{\beta}{2}} \left( 1 + \frac{\beta(q-1)}{2} \right) \delta^{-(i_1-j) \frac{\beta(1-q)+}{2}} \rho_0^{\beta} \]

Then let \( i_1 - j > i_o \). Recall that by construction we have \( \theta_* \leq \theta_i \leq \left( \delta^{q_0} \omega_o \right)^{q-1} \) if \( 0 < q < 1 \), and in case \( q > 1 \) we can simply use the rescaling to estimate \( \theta_* \leq 1 \).

By the definitions of \( \lambda \) and \( \hat{\lambda} \) we obtain in a similar way as above that

\[ \text{osc } Q_{i_1-j} \psi \leq c[\psi] C_{0, \beta, \frac{\beta}{2}} \left( \rho_{i_1-j}^{\beta} + (\theta_* \hat{\rho}_{i_1-j}^{2})^{\frac{\beta}{2}} \right) \leq c[\psi] C_{0, \beta, \frac{\beta}{2}} \left( 1 + \theta_*^{\frac{\beta}{2}} \right) \rho_{i_1-j}^{\beta} \]

\[ \leq c[\psi] C_{0, \beta, \frac{\beta}{2}} \left( 1 + \delta^{-\frac{i_o(1-q)+}{2}} \omega_o^{\frac{\beta(q-1)}{2}} \right) \rho_{i_1-j}^{\beta} \]

\[ \leq c[\psi] C_{0, \beta, \frac{\beta}{2}} \left( 1 + \omega_o^{\frac{\beta(q-1)}{2}} \right) \delta^{-\frac{i_o(1-q)+}{2}} \rho_{i_1-j}^{\beta} \]

\[ \leq c[\psi] C_{0, \beta, \frac{\beta}{2}} \left( 1 + \omega_o^{\frac{\beta(q-1)}{2}} \right) \left( \frac{\delta}{\omega_o} \right)^{\frac{\beta(i_1-j)}{2}} \rho_0^{\beta}. \]

Using the estimates above in (8.10) gives us

\[ \text{osc } Q_{i} u \leq \delta^i \omega_o + c[\psi] C_{0, \beta, \frac{\beta}{2}} \left( 1 + \omega_o^{\frac{\beta(q-1)}{2}} \right) \rho_0^{\beta} \sum_{j=0}^{i-1} \delta^j \left( \frac{\delta}{\omega_o} \right)^{\frac{\beta(i-j)}{2}}. \]

Setting

\[ \tau := \max \left\{ \delta, \left( \frac{\delta}{\omega_o} \right)^{\frac{\beta}{2}} \right\}, \]

we conclude from the preceding inequality that

\[ \text{osc } Q_{i} u \leq \delta^i \omega_o + c[\psi] C_{0, \beta, \frac{\beta}{2}} \left( 1 + \omega_o^{\frac{\beta(q-1)}{2}} \right) \rho_0^{\beta}. \]

By the fact that \( i \tau^{i-1} \leq c(\tau) \sqrt{i} \), we infer

\[ \text{osc } Q_{i} u \leq \delta^i \omega_o + c[\psi] C_{0, \beta, \frac{\beta}{2}} \sqrt{i} \left( 1 + \omega_o^{\frac{\beta(q-1)}{2}} \right) \rho_0^{\beta} \]

for a constant \( c = c(n, q, C_o, C_1, \beta) \). Let us define \( \eta = \left( \delta \right)^{(q-1)^+} \lambda \) and

\[ \gamma_1 := \min \left\{ \log \frac{\tau}{2 \log \eta}, \frac{2 \beta}{2 \log \eta} \right\} \]

depending on \( n, q, C_o, C_1 \) and \( \beta \) and observe that \( \gamma_1 \leq \frac{\log \tau}{2 \log \eta} \leq \frac{\log \delta}{2 \log \eta} \). Now, we have

\[ \text{osc } Q_{i} u \leq \eta^\gamma i \omega_o + c[\psi] C_{0, \beta, \frac{\beta}{2}} \eta^\gamma i \left( 1 + \omega_o^{\frac{\beta(q-1)}{2}} \right) \rho_0^{\beta}. \]
Observe that (8.2) implies
\[ \rho_0^{-\gamma_o} \leq \omega_o \leq c \max \{1, [\psi]_C^{0, \beta, \frac{\eta}{2}}\} R^{-\gamma_o} \rho_0^{\gamma_o}. \] (8.11)

Let \( q > 1 \). Recall that in this case \( \gamma_o = \beta \) and by using \( \omega_o \leq 1 \) we obtain
\[ \text{osc}_{Q_{\tilde{r}_i}} u \leq c \max \{1, [\psi]_C^{0, \beta, \frac{\eta}{2}}\} \eta^{\gamma_i} R^{-\beta} \rho_0^\beta. \]

Furthermore, we have that
\[ r_i = \left( \frac{\delta^i}{4} \omega_o \right) \frac{q-1}{2} \tilde{\rho}_i \geq \left( \frac{\delta^{i+1}}{25} \omega_o \right) \frac{q-1}{2} \lambda^i \rho_o \geq \delta \frac{(q-1)(1-i)}{2} \eta^i \omega_o \frac{q-1}{2} \rho_o \geq \eta^i \omega_o \frac{q-1}{2} \rho_o \]
for \( i \in \mathbb{N} \) by definitions of \( r_i, \tilde{\rho}_i \) and \( \eta \). Now, since \( \varrho_o \in (0, 1) \) and \( \beta - \gamma_1 - \gamma_1 \beta \frac{q-1}{2} \geq 0 \) by definition of \( \gamma_1 \), by using the estimate above and (8.11) we have
\[ \text{osc}_{Q_{\tilde{r}_i}} u \leq c \max \{1, [\psi]_C^{0, \beta, \frac{\eta}{2}}\} \left( \frac{r_i}{\varrho_o} \omega_o \frac{1-q}{2} \right)^\gamma R^{-\beta} \rho_0^\beta \]
\[ \leq c \max \{1, [\psi]_C^{0, \beta, \frac{\eta}{2}}\} R^{-\beta} \rho_0^{1-q} \rho_o^{\gamma_1} \rho_o^{\gamma_1} R^{-\gamma_1 \beta} \rho_o^{\gamma_1} \leq c \max \{1, [\psi]_C^{0, \beta, \frac{\eta}{2}}\} R^{-\beta} \rho_0^{\gamma_1}, \]
for \( c = c(n, q, C_o, C_1, \beta) \).

Then let \( 0 < q < 1 \). Observe that by using the definition of \( r_i \) and \( \omega_o \in (0, 1) \) we have \( \eta^i \leq \tilde{\rho}_i \leq c(q) \frac{r_i}{\varrho_o} \). By using this fact together with \( \omega_o^{q-1} \leq \rho_o^q \), (8.11)2, \( \varrho_o \in (0, 1) \) and \( \gamma_1 \leq \gamma_o \) we can conclude
\[ \text{osc}_{Q_{\tilde{r}_i}} u \leq \eta^{\gamma_i} \omega_o + c \max \{1, [\psi]_C^{0, \beta, \frac{\eta}{2}}\} \eta^{\gamma_i} \left( 1 + \omega_o \frac{\beta(q-1)}{2} \right)^\beta \rho_o^q \]
\[ \leq c \max \{1, [\psi]_C^{0, \beta, \frac{\eta}{2}}\} \eta^{\gamma_i} R^{-\gamma_1 \beta} \rho_o^{\gamma_1} \leq c \max \{1, [\psi]_C^{0, \beta, \frac{\eta}{2}}\} \varrho_o^{\gamma_1} R^{-\gamma_1 \beta} \rho_o^{\gamma_1} \]
\[ \leq c \max \{1, [\psi]_C^{0, \beta, \frac{\eta}{2}}\} R^{-\gamma_1 \beta} \rho_o^{\gamma_1}, \]
for a constant \( c = c(n, q, C_o, C_1, \beta) \). Thus, for all \( q > 0 \) we have
\[ \text{osc}_{Q_{\tilde{r}_i}} u \leq c \max \{1, [\psi]_C^{0, \beta, \frac{\eta}{2}}\} \left( \frac{r_i}{\varrho_o} \right)^{\gamma_1}. \] (8.12)

We conclude this section by showing that the last estimate holds for an arbitrary radius \( r \in (0, R) \). First, let us consider \( r \in (0, r_o) \). Choose \( i \in \mathbb{N}_0 \) such that \( r_{i+1} < r < r_i \). By (8.12) and the fact that \( \omega_o \leq 1 \), we find that
\[ \text{osc}_{Q_r} u \leq \text{osc}_{Q_{\tilde{r}_i}} u \leq c \max \{1, [\psi]_C^{0, \beta, \frac{\eta}{2}}\} \left( \frac{r_i}{\varrho_o} \right)^{\gamma_1} \rho_o^{\gamma_1} \]
\[ \leq c \max \{1, [\psi]_C^{0, \beta, \frac{\eta}{2}}\} \eta^{-\gamma_1} \left( \frac{\delta^i}{4} \omega_o \right)^{\gamma_1} \left( \frac{r_i}{\varrho_o} \right)^{\gamma_1} \rho_o^{\gamma_1} \]
\[ \leq c \max \{1, [\psi]_C^{0, \beta, \frac{\eta}{2}}\} \eta^{-\gamma_1} \left( \frac{\delta^{i+1}}{25} \omega_o \right)^{\gamma_1} \left( \frac{r_i}{\varrho_o} \right)^{\gamma_1} \rho_o^{\gamma_1}. \]
\[ \leq c \max\{1, [\psi]_{C^{0, \beta, \frac{q}{2}}} \} \left( \delta^{\frac{q-1}{2}} \eta \right)^{\gamma_1} R^{-\gamma_0} r_{i+1}^{\gamma_1} \]
\[ \leq c \max\{1, [\psi]_{C^{0, \beta, \frac{q}{2}}} \} \frac{r^{\gamma_1}}{R^{\gamma_0}} \]
for a constant \( c = c(n, q, C_0, C_1, \beta) \). Next, let us assume that \( r \in [r_0, \rho_0) \). By (8.11)_2 and since \( \rho_0 \leq \left( \frac{25}{4} \right)^{\frac{1}{2}} r_0 \) when \( 0 < q < 1 \) and \( r_0 \geq c(q) \rho_0^{\frac{1+\beta}{q+1}} \) implying \( r_{\gamma_1}^{\gamma_1} \geq c \rho_0^{\gamma_0} \) by \( \gamma_0 \geq \gamma_1 + \gamma_1 \beta \frac{q-1}{2} \) when \( q > 1 \), we obtain that
\[ \text{osc}_{Q_r} u \leq \text{osc}_{Q_{\rho_0, \rho_0^{\leq-\epsilon}}} u \leq 2 R^{-\gamma_0} \Psi(\rho_0) \leq c \max\{1, [\psi]_{C^{0, \beta, \frac{q}{2}}} \} \left( \frac{\rho_0}{R} \right)^{\gamma_0} \]
\[ \leq c \max\{1, [\psi]_{C^{0, \beta, \frac{q}{2}}} \} \frac{\rho_0^{\gamma_1}}{r_0^{\gamma_1}} \frac{r^{\gamma_1}}{R^{\gamma_0}} \leq c \max\{1, [\psi]_{C^{0, \beta, \frac{q}{2}}} \} \frac{r^{\gamma_1}}{R^{\gamma_0}}. \]

In the remaining case \( r \in [\rho_0, R] \), we have that
\[ \text{osc}_{Q_r} u \leq \text{osc}_{Q_{r, r^{\leq-\epsilon}}} u \leq 2 R^{-\gamma_0} \Psi(r) \leq c \max\{1, [\psi]_{C^{0, \beta, \frac{q}{2}}} \} \left( \frac{r}{R} \right)^{\gamma_0} \]
\[ \leq c \max\{1, [\psi]_{C^{0, \beta, \frac{q}{2}}} \} \frac{r^{\gamma_1}}{R^{\gamma_0}}. \]

Altogether, recalling that we have omitted \((x_0, t_0)\) in our notation, we infer
\[ \text{osc}_{Q_{r, r^{\leq-\epsilon}}} u \leq c \max\{1, [\psi]_{C^{0, \beta, \frac{q}{2}}} \} \frac{r^{\gamma_1}}{R^{\gamma_0}} \quad \text{for all } r \in (0, R). \quad (8.13) \]

8.5. Quantitative Hölder estimate

Denote the parabolic boundary of \( \Omega_T \) by \( \Gamma \) and consider \((x_1, t_1), (x_2, t_2) \in K \) for a compact subset \( K \subset \Omega_T \). Without loss of generality, assume that \( t_2 > t_1 \). Define the (non-intrinsic) parabolic distance
\[ \text{dist}(K, \Gamma) := \inf_{(x,t) \in K, (y,s) \in \Gamma} \{ |x - y| + |t - s|^{\frac{1}{2}} \}. \]
Since
\[ |u(x_1, t_1) - u(x_2, t_2)| \leq |u(x_1, t_1) - u(x_2, t_1)| + |u(x_2, t_1) - u(x_2, t_2)|, \]
we can prove quantitative Hölder estimates with respect to the space and time variables separately. We only give the proof of the latter, since the proof of the former is analogous. To this end, note that for
\[ R := \frac{1}{2} \min \left\{ 1, \text{dist}(K, \Gamma)^{\frac{q}{q-\epsilon}} \right\} \]
we have that
\[ Q_{R, R^{\leq-\epsilon}} \subseteq \Omega_T. \]
We remark that in the obstacle-free case, starting with any intrinsic cylinder $Q_{R, \omega_R^{q-1}, R^2}$ with any intrinsic cylinder as in Sects. 8.1 – 8.4 can be repeated and thus an analogous oscillation decay estimate to (8.13) is derived without assuming that $Q_{R, \omega_R^{q-1}, R^2}$ is contained in any specific non-intrinsic cylinder. In contrast, in the present situation the construction of intrinsic cylinders contained in a non-intrinsic cylinder of the form $Q_{R, R^2 - \epsilon}$ in the beginning of Sect. 8 seems to be unavoidable in the degenerate case $0 < q < 1$, since it is used to deal with the factor $\omega_\beta (q - 1) 2^o R \omega$, which is related to the oscillation of the obstacle $\psi$ in intrinsic cylinders of the form $Q_{r, \omega_r^{q-1}, r^2}$.

Now, we distinguish between the cases $t_2 - t_1 < R^2$ and $t_2 - t_1 \geq R^2$. (8.14)

If (8.14)_1 holds, choose $r \in (0, R)$ such that $t_2 - t_1 = r^2$. Applying the oscillation decay estimate (8.13) in $Q_{r, r^2}(x_2, t_2)$ and recalling the definition of $\gamma_0$ leads to

$$|u(x_2, t_1) - u(x_2, t_2)| \leq c \max \{1, \psi \}_{C^{a, \beta, \frac{\beta}{2}}} \frac{|t_2 - t_1|^\gamma_0}{R \gamma_0} \leq c \max \{1, \psi \}_{C^{a, \beta, \frac{\beta}{2}}} \frac{|t_2 - t_1|^\gamma_0}{\min \{1, \text{dist}(K, \Gamma)\}}^\beta.$$

In the case (8.14)_2, we use (8.1) and the facts that $R \leq 1$ and $\gamma_1 \leq \gamma_0$ to estimate

$$|u(x_2, t_1) - u(x_2, t_2)| \leq \frac{R^{\gamma_1}}{R \gamma_0} \leq 2 \frac{|t_2 - t_1|^\gamma_0}{\min \{1, \text{dist}(K, \Gamma)\}}^\beta.$$

Together with analogous estimates for the space variables, we arrive at

$$|u(x_1, t_1) - u(x_2, t_2)| \leq c \max \{1, \psi \}_{C^{a, \beta, \frac{\beta}{2}}} \left(\frac{|x_1 - x_2|}{|t_2 - t_1|^\frac{1}{2}}\right)^\gamma_1 \frac{\min \{1, \text{dist}(K, \Gamma)\}}{\psi}_{C^{a, \beta, \frac{\beta}{2}}, q},$$

for bounded local weak solutions $u$ of the obstacle problem satisfying (8.1). Taking the rescaling argument from Lemma A.1 with $M = 2\|u\|_{\infty}$ into account, we infer the quantitative Hölder estimate (1.3) with $\gamma = \gamma_1$ for general bounded local weak solutions $u$ in the sense of Definition 2.1. In particular, note that rescaling does not affect the Hölder exponent $\beta$ of the rescaled obstacle function $\tilde{\psi}$ and there holds $[\tilde{\psi}]_{C^{a, \beta, \frac{\beta}{2}}} = \frac{1}{M} [\psi]_{C^{a, \beta, \frac{\beta}{2}}, q}$, see Appendix A.

**Acknowledgements**

K. Moring has been supported by the Magnus Ehrnrooth Foundation and Foundation for Aalto University Science and Technology.

**Data Availability Statement** Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

**Declarations**

**Conflict of interests** Both authors have no conflicts of interests.
Appendix A: Rescaling argument

Let $M > 0$ and consider the rescaled functions

$$
\tilde{u}(x, t) := \frac{1}{M} u(x, M^{q-1}t) \quad \text{and} \quad \tilde{\psi}(x, t) := \frac{1}{M} \psi(x, M^{q-1}t),
$$

(A.1)

together with

$$
\tilde{A}(x, t, v, \xi) := \frac{1}{M} A(x, M^{q-1}t, Mv, M\xi)
$$

(A.2)

for $(x, t) \in \Omega_{\tilde{T}} := \Omega \times (0, \tilde{T}) := \Omega \times (0, M^{1-q}T)$, such that the vector field $\tilde{A}$ satisfies the same structure conditions as $A$.

**Lemma A.1.** Let $\tilde{u}$ and $\tilde{\psi}$ be defined as in (A.1) and $\tilde{A}$ as in (A.2). Then, $\tilde{u}$ is a weak solution to the obstacle problem with obstacle $\tilde{\psi}$ and

$$
\partial_t (|\tilde{u}|^{q-1} \tilde{u}) - \text{div} \tilde{A}(x, t, \tilde{u}, \nabla \tilde{u}) = 0 \quad \text{in} \ \Omega_{\tilde{T}}
$$

in the sense of Definition 2.1.

**Proof.** For $\tilde{\psi} \in C^0(\Omega_{\tilde{T}})$, we compute that

$$
[\tilde{\psi}]_{C^{0;\beta,\frac{\beta}{2}}} = \sup_{(x, \tilde{t}), (y, \tilde{s}) \in \Omega_{\tilde{T}}} \frac{|\tilde{\psi}(x, \tilde{t}) - \tilde{\psi}(y, \tilde{s})|}{|x - y|^\beta + |\tilde{t} - \tilde{s}|^{\frac{\beta}{2}}}
$$

$$
= \frac{1}{M} \sup_{(x, t), (y, s) \in \Omega_T} \frac{|\psi(x, t) - \psi(y, s)|}{|x - y|^\beta + M^{(1-q)\frac{\beta}{2}}|t - s|^{\frac{\beta}{2}}}
$$

$$
\leq \frac{1}{M} \max \{1, M^{(q-1)\frac{\beta}{2}}\} [\psi]_{C^{0;\beta,\frac{\beta}{2}}}.
$$

Hence, we have that $\tilde{\psi} \in C^{0;\beta,\frac{\beta}{2}}(\Omega_{\tilde{T}})$ and that $[\tilde{\psi}]_{C^{0;\beta,\frac{\beta}{2}}} = \frac{1}{M} [\psi]_{C^{0;\beta,\frac{\beta}{2}};q}$. Further, it is clear that there holds $\tilde{u} \geq \tilde{\psi}$ a.e. in $\Omega_{\tilde{T}}$ and we compute that

$$
\tilde{u} \in C^0((0, \tilde{T}); L^{q+1}_{\text{loc}}(\Omega)) \cap L^2_{\text{loc}}(0, \tilde{T}; H^1_{\text{loc}}(\Omega)),
$$
i.e., we find that $\tilde{u} \in K_{\psi}(\Omega_T)$. Now, we consider $\tilde{\varphi} \in C_0^\infty(\Omega_T; \mathbb{R}_{\geq 0})$ and $\tilde{v} \in K_{\psi}'(\Omega_T)$. First, observe that $\varphi(x, t) := \tilde{\varphi}(x, M^{1-q} t) \in C_0^\infty(\Omega_T; \mathbb{R}_{\geq 0})$. Furthermore, we state that $v(x, t) := M\tilde{v}(x, M^{1-q} t)$ is an admissible comparison map related to $u$ and $\psi$. To this end, check that $v \in C^0((0, T); L_{\text{loc}}^{q+1}(\Omega)) \cap L_{\text{loc}}^2(0, T; H^1_{\text{loc}}(\Omega))$ and

$$v(x, t) = M\tilde{v}(x, M^{1-q} t) \geq M\tilde{\psi}(x, M^{1-q} t) = \psi(x, t)$$

for a.e. $(x, t) \in \Omega_T$. Moreover, compute that

$$\partial_t v(x, t) = \partial_t (M\tilde{v}(x, M^{1-q} t)) = M^{2-q} \partial_t \tilde{v}(x, M^{1-q} t) \in L^{q+1}(\Omega_T).$$

Altogether, this implies that $v \in K_{\psi}'(\Omega_T)$. With these considerations at hand, a straightforward computation shows that

$$\langle \partial_t \tilde{u}^q, \tilde{\varphi} (\tilde{v} - \tilde{u}) \rangle + \iint_{\Omega_T} \tilde{A}(x, \tilde{t}, \tilde{u}, \nabla \tilde{u}) \cdot \nabla (\tilde{\varphi} (\tilde{v} - \tilde{u})) \, dx \, dt$$

$$= M^{-(q+1)} \left[ \langle \partial_t u^q, \varphi (v - u) \rangle + \iint_{\Omega_T} A(x, t, u, \nabla u) \cdot \nabla (\varphi (v - u)) \, dx \, dt \right] \geq 0,$$

since $u$ is a weak solution associated with the obstacle $\psi$ in the sense of Definition 2.1.

REFERENCES

[1] H. Alt and S. Luckhaus, Quasilinear elliptic-parabolic differential equations, Math. Z. 183 (1983), (3), 311–341.
[2] B. Avelin and T. Lukkari, A comparison principle for the porous medium equation and its consequences, Rev. Mat. Iberoam. 33 (2017), no. 2, 573–594.
[3] A. Björn, J. Björn, U. Gianazza and J. Siljander, Boundary regularity for the porous medium equation, Arch. Ration. Mech. Anal. 230 (2018), no. 2, 493–538.
[4] V. Bögelein, F. Duzaar and U. Gianazza, Continuity estimates for porous medium type equations with measure data, J. Funct. Anal. 267 (2014), 3351–3396.
[5] V. Bögelein, F. Duzaar and N. Liao, On the Hölder regularity of signed solutions to a doubly nonlinear equation, J. Funct. Anal. 281 (2021), no. 9, Paper No. 109173, 58 pp.
[6] V. Bögelein, F. Duzaar, and P. Marcellini, Parabolic systems with $p, q$-growth: a variational approach, Arch. Ration. Mech. Anal. 210 (2013), no. 1, 219–267.
[7] V. Bögelein, T. Lukkari and C. Scheven, Hölder regularity for degenerate parabolic obstacle problems, Ark. Mat. 55 (2017), no. 1, 1–39.
[8] V. Bögelein, T. Lukkari and C. Scheven, The obstacle problem for the porous medium equation, Math. Ann. 363 (2015), no. 1-2, 455–499.
[9] M. Bonforte and N. Simonov, Quantitative a priori estimates for fast diffusion equations with Caffarelli-Kohn-Nirenberg weights. Harnack inequalities and Hölder continuity, Adv. Math. 345 (2019), 1075–1161.
[10] Y. Cho and C. Scheven, Hölder regularity for singular parabolic obstacle problems of porous medium type, Int. Math. Res. Not. IMRN 2020, no. 6, 1671–1717.
[11] H. Choe, On the regularity of parabolic equations and obstacle problems with quadratic growth nonlinearities, J. Differential Equations 102 (1993), no. 1, 101–118.
[12] P. Daskalopoulos and C. E. Kenig, Degenerate diffusions: Initial value problems and local regularity theory, EMS Tracts in Mathematics, 1, European Mathematical Society (EMS), Zürich, 2007.
[13] E. DiBenedetto, *Degenerate parabolic equations*, Universitext, Springer-Verlag, New York, 1993.

[14] E. DiBenedetto and A. Friedman, *Hölder estimates for nonlinear degenerate parabolic systems*, J. Reine Angew. Math. 357 (1985), 1–22.

[15] E. DiBenedetto, U. Gianazza and V. Vespri, *Harnack’s inequality for degenerate and singular parabolic equations*, Springer Monographs in Mathematics, Springer, New York, 2012.

[16] L. C. Evans, *Partial Differential Equations*, American Mathematical Society, 2. edition, 2010.

[17] E. Giusti, *Direct methods in the calculus of variations*, World Scientific Publishing Co., 2003.

[18] J. Heinonen, T. Kilpeläinen and O. Martio, *Nonlinear potential theory of degenerate elliptic equations*, The Clarendon Press, Oxford University Press, New York (1993).

[19] J. Kinnunen and P. Lindqvist, *Definition and properties of supersolutions to the porous medium equation*, J. Reine Angew. Math. 618 (2008), 135–168.

[20] J. Kinnunen and P. Lindqvist, *Pointwise behaviour of semicontinuous supersolutions to a quasilinear parabolic equation*, Ann. Mat. Pura Appl. (4) 185 (2006), no. 3, 411–435.

[21] J. Kinnunen, P. Lindqvist and T. Lukkari, *Perron’s method for the porous medium equation*, J. Eur. Math. Soc. (JEMS) 18 (2016), no. 12, 2953–2969.

[22] R. Korte, P. Lehtelä and S. Sturm, *Lower semicontinuous obstacles for the porous medium equation*, J. Differential Equations 266 (2019), no. 4, 1851–1864.

[23] N. Liao, *A unified approach to the Hölder regularity of solutions to degenerate and singular parabolic equations*, J. Differential Equations 268 (2020), no. 10, 5704–5750.

[24] N. Liao, *Hölder regularity for porous medium systems*, Calc. Var. 60 (2021), 156.

[25] P. Lindqvist, *On the definition and properties of p-superharmonic functions*, J. Reine Angew. Math. 365 (1986), 67–79.

[26] M. Mizuno, *Hölder estimates for solutions of the Cauchy problem for the porous medium equation with external forces*, Manuscripta math. 141 (2013), 273–313.

[27] L. Schätzler, *The obstacle problem for degenerate doubly nonlinear equations of porous medium type*, Ann. Mat. Pura Appl. (4) 200 (2021), no. 2, 641–683.

[28] L. Schätzler, *The obstacle problem for singular doubly nonlinear equations of porous medium type*, Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. 31 (2020), no. 3, 503–548.

[29] M. Struwe and M. A. Vivaldi, *On the Hölder continuity of bounded weak solutions of quasi-linear parabolic inequalities*, Ann. Math. Pura Appl. (4) 139 (1985), no. 1, 175–189.

[30] J.L. Vázquez, *Smoothing and decay estimates for nonlinear diffusion equations. Equations of porous medium type*, Oxford Lecture Series in Mathematics and its Applications, 33. Oxford University Press, Oxford, 2006.

[31] J.L. Vázquez, *The porous medium equation: Mathematical theory*, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, Oxford, 2007.

[32] Z. Wu, J. Zhao, J. Yin, and H. Li, *Nonlinear diffusion equations*, World Scientific Publishing Co., Inc., River Edge, NJ, 2001, Translated from the 1996 Chinese original and revised by the authors.

Kristian Moring
Department of Mathematics and Systems Analysis
Aalto University
P. O. Box 11100 00076 Aalto
Finland
E-mail: kristian.moring@aalto.fi

Leah Schätzler
Fachbereich Mathematik
Paris-Lodron-Universität Salzburg
Hellbrunner Str. 34
5020 Salzburg
Austria
E-mail: leahanna.schaezler@plus.ac.at

Accepted: 30 August 2022