Symmetry Breaking in the Schrödinger Representation for Chern-Simons Theories*

Gerald Dunne
Department of Physics
University of Connecticut
2152 Hillside Road
Storrs, CT 06269 USA

Abstract

This paper discusses the phenomenon of spontaneous symmetry breaking in the Schrödinger representation formulation of quantum field theory. The analysis is presented for three-dimensional space-time abelian gauge theories with either Maxwell, Maxwell-Chern-Simons, or pure Chern-Simons terms as the gauge field contribution to the action, each of which leads to a different form of mass generation for the gauge fields.

1 Introduction

Gauge theories in three-dimensional space-time have the interesting feature that they permit two different mechanisms for generating masses for gauge fields. As in four dimensions, the Higgs mechanism, in which a minimally coupled scalar field acquires a nonzero vacuum expectation value, yields massive gauge field excitations [1]. Independently, one may supplement the usual Maxwell (or Yang-Mills) gauge field action with a Chern-Simons term, producing a single massive propagating transverse gauge field mode, with mass determined by the coupling coefficient of the Chern-Simons term [2]. As first considered by Pisarski and Rao [3], these two mass-generation techniques may be combined in a Maxwell-Chern-Simons-Higgs theory, giving two modes with different masses. Finally, one can dispense with

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the Maxwell term altogether and consider the Higgs mechanism for gauge theories with pure Chern-Simons ‘dynamics’, as first discussed by Deser and Yang [4].

Symmetry breaking is conventionally discussed in a Lagrangian formalism and the physical degrees of freedom may be identified by a transformation to the unitary gauge [1, 5]. Perturbative computations, and discussions of renormalizability, are performed using a gauge fixing prescription, and the massive gauge field excitations are identified with poles in the gauge field propagator. The BRS approach to the quantization of gauge theories provides the most elegant framework for a systematic investigation of more fundamental canonical issues [6]. In this paper, spontaneous symmetry breaking is analyzed using the Schrödinger representation for quantum field theory [7, 8, 10]. In this approach, one is concerned with the Hamiltonian rather than the Lagrangian, and the physical excitations are identified by a resolution of the constraints in the Weyl gauge, and by a diagonalization of the quadratic part of the Hamiltonian. For the standard Higgs mechanism this is a relatively straightforward pedagogical exercise in the use of the Schrödinger representation. However, the introduction of a Chern-Simons term modifies both the canonical structure and the constraints [9, 11]. The quadratic Hamiltonian may still be diagonalized in the physical subspace, but the explicit diagonalization is remarkably complicated due to the presence of off-diagonal mixing.

As an immediate corollary of this Schrödinger representation analysis we find a simple explanation of the fact that the two different masses identified by Pisarski and Rao [3, 12] in the gauge field propagator of Maxwell-Chern-Simons-Higgs theory coincide precisely with the two different frequencies arising in the planar quantum mechanical system of charged particles in a perpendicular magnetic field and a harmonic potential well. This correspondence is a manifestation of the usual identification of Schrödinger representation quantum field theory with Schrödinger representation quantum mechanics, as has been studied previously in the context of Chern-Simons theories without symmetry breaking [13].

This paper is organized as follows. Section 2 contains the Schrödinger representation analysis of Maxwell-Chern-Simons-Higgs theory. This discussion includes the conventional Higgs mechanism as a special case in which the Chern-Simons term is removed simply by setting its coupling coefficient to zero. By diagonalizing the quadratic Hamiltonian in the physical subspace, we find the two massive scalar modes, with masses in agreement with the Pisarski-Rao result and in agreement with the quantum mechanical analogy. In Section 3 the Chern-Simons-Higgs theory is treated in detail, leading to a single massive mode. Section 4 contains some discussion and an Appendix contains a constructive method for diagonalizing, with a symplectic canonical transformation, a general quadratic Hamiltonian.
2 Maxwell - Chern - Simons - Higgs Mechanism in the Schrödinger Representation

In this section we use the Schrödinger representation to describe the mass generation for abelian gauge fields due to the combined effects of the presence of a Chern-Simons term and the Higgs mechanism. Consider the Lagrange density

\[
\mathcal{L} = -\frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu} - \frac{\kappa}{2e^2} \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho - (D_\mu \phi)^* D^\mu \phi - V
\]

where

\[
\phi \equiv \frac{1}{\sqrt{2}} (\phi_1 + i \phi_2)
\]

is a complex scalar field, the covariant derivative is \( D_\mu = \partial_\mu + i A_\mu \), \( A_\mu \) is an Abelian gauge field, and \( F_{\mu\nu} \) is the corresponding field strength. The space-time metric is chosen to be \( \text{diag}(-1,1,1) \). The totally antisymmetric \( \epsilon \) symbol is chosen with \( \epsilon^{012} = +1 \). The scalar field potential \( V \) is chosen to have the standard symmetry-breaking form

\[
V = \alpha^2 \frac{e^2}{2} \left( |\phi|^2 - a^2 \right)^2
\]

where \( a^2 \) is some arbitrary mass scale, and \( \alpha \) is a dimensionless constant which controls the ratio of the scalar mass to the gauge mass in the broken vacuum. Note that \( e^2 \) has dimensions of mass in \( 2 + 1 \)-dimensional space-time.

The corresponding momenta for the fields are given by

\[
\Pi \equiv \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = (D_0 \phi)^*
\]

\[
\Pi^* \equiv \frac{\partial \mathcal{L}}{\partial \dot{\phi}^*} = D_0 \phi
\]

\[
\pi_i \equiv \frac{\partial \mathcal{L}}{\partial A_i} = \frac{1}{e^2} F_{0i} - \frac{\kappa}{2e^2} \epsilon^{ij} A_j
\]

By a Legendre transformation we find that the Hamiltonian density is

\[
\mathcal{H} \equiv \Pi \dot{\phi} + \Pi^* \dot{\phi}^* + \pi_i \dot{A}_i - \mathcal{L}
\]

\[
= \frac{1}{2} \Pi_1^2 + \frac{1}{2} \Pi_2^2 + \frac{e^2}{2} \left( \pi_i + \frac{\kappa}{2e^2} \epsilon^{ij} A_j \right)^2 + (D_i \phi)^* D_i \phi + \frac{1}{4e^2} F_{ij} F^{ij} + V
\]

\[+ A_0 \left( -\partial_i \pi_i + \frac{\kappa}{2e^2} \epsilon^{ij} \partial_i A_j - (\phi_1 \Pi_2 - \phi_2 \Pi_1) \right)\]
The coefficient of $A_0$ in this expression for $\mathcal{H}$ is the Gauss law generator which generates fixed-time gauge transformations.

In the spontaneously broken symmetry vacuum the (real) scalar field $\phi_1$ acquires a vacuum expectation value

$$\phi_1 \rightarrow \phi_1 + \sqrt{2}a$$  \hspace{1cm} (6)

To identify the massive degrees of freedom it is sufficient to consider the quadratic part of the Hamiltonian density:

$$H_{\text{quad}} = \frac{1}{2} \Pi_1^2 + \frac{1}{2} \Pi_2^2 + \frac{e^2}{2} \left( \pi_i + \frac{\kappa}{2e^2} e^{ij} A_j \right)^2 + \frac{1}{2} \left( \partial_i \phi_1 \right) \left( \partial_i \phi_1 \right) + \frac{1}{2} \left( \partial_i \phi_2 \right) \left( \partial_i \phi_2 \right)$$

$$+ \frac{1}{4e^2} F_{ij} F^{ij} + e^2 a^2 \phi_1^2 + a^2 A_i A_i + \sqrt{2} a A_i \partial_i \phi_2$$

$$+ A_0 \left( - \partial_i \pi_i + \frac{\kappa}{2e^2} e^{ij} \partial_j A_j - \sqrt{2} a \Pi_2 \right)$$  \hspace{1cm} (7)

From this quadratic part of the Hamiltonian it is clear that the $\phi_1$ field (the “Higgs” field) separates as a real scalar field of mass $m_1 = \alpha \sqrt{2} a e$. We shall henceforth drop the $\phi_1$ field from our discussion as it is not relevant to the identification of the remaining massive modes.

It is now convenient to rewrite the Abelian gauge field $A_i$ using a Hodge decomposition into its longitudinal and transverse parts. That is, we write

$$A_i = \partial_i \lambda - e^{ij} \partial_j B$$  \hspace{1cm} (8)

where $\nabla^2$ is the Laplacian and $B$ is the ‘magnetic field’ $B = e^{ij} \partial_i A_j$. Then the gauge field momenta, $\pi_i$, become

$$\pi_i = - \partial_i \Pi_\lambda + e^{ij} \partial_j \Pi_B$$  \hspace{1cm} (9)

In terms of the new fields, $\phi_2$, $\lambda$ and $B$, the quadratic Hamiltonian density becomes

$$H_{\text{quad}} = \frac{1}{2} \Pi_1^2 - \frac{e^2}{2} \Pi_\lambda \nabla^2 \Pi_\lambda - \frac{e^2}{2} \Pi_B \nabla^2 \Pi_B - \frac{\kappa^2}{2} \nabla^2 \lambda \Pi_B + \frac{\kappa}{2 \sqrt{2}} \Pi_B \Pi_\lambda$$

$$- \frac{1}{2} \left( \phi_2 + \sqrt{2} a \lambda \right) \nabla^2 \left( \phi_2 + \sqrt{2} a \lambda \right) - \frac{\kappa^2}{8e^2} \lambda \nabla^2 \lambda + B \left( \frac{1}{2e^2} - \frac{a^2}{\nabla^2} - \frac{\kappa^2}{8e^2 \nabla^2} \right) B$$

$$+ A_0 \left( \Pi_\lambda - \sqrt{2} a \Pi_2 + \frac{\kappa}{2e^2} B \right)$$  \hspace{1cm} (10)

Notice that the fields $\phi_2$ (the “Goldstone field”) and $\lambda$ (the longitudinal part of the gauge field) are naturally related in this Hamiltonian. It is clearly convenient to define new fields

$$\chi = \phi_2 + \sqrt{2} a \lambda$$
\[ \rho = \phi_2 - \sqrt{2}a\lambda \] (11)

The main advantage of such a transformation is that the Gauss law constraint now takes the very simple form

\[ \Pi_\rho = \frac{\kappa}{4\sqrt{2}ae^2} B \] (12)

In terms of the fields \( B, \chi \) and \( \rho \), the Hamiltonian density becomes

\[
\mathcal{H}_{\text{quad}} = \frac{1}{2} \Pi_\chi \left( 1 - \frac{2e^2a^2}{\nabla^2} \right) \Pi_\chi + \frac{1}{2} \Pi_\rho \left( 1 - \frac{2e^2a^2}{\nabla^2} \right) \Pi_\rho - \frac{e^2}{2} \Pi_B \nabla^2 \Pi_B
+ \Pi_\chi \left( 1 + \frac{2e^2a^2}{\nabla^2} \right) \Pi_\rho + \frac{\kappa}{4\sqrt{2}a} \nabla^2 \rho \Pi_B - \frac{\kappa}{4\sqrt{2}a} \nabla^2 \chi \Pi_B + \frac{a\kappa}{\sqrt{2}\nabla^2} B \Pi_\chi - \frac{a\kappa}{\sqrt{2}\nabla^2} B \Pi_\rho
- \frac{1}{2} \left( 1 + \frac{\kappa^2}{32e^2a^2} \right) \chi \nabla^2 \chi - \frac{\kappa^2}{64e^2a^2} \rho \nabla^2 \rho + B \left( \frac{1}{2e^2} - \frac{a^2}{\nabla^2} - \frac{\kappa^2}{8e^2\nabla^2} \right) B + \frac{\kappa^2}{32e^2a^2} \chi \nabla^2 \rho
+ A_0 \left( -2\sqrt{2}a\Pi_\rho + \frac{\kappa}{2e^2} B \right) \] (13)

This form of the Hamiltonian density looks very complicated. However, the Hamiltonian simplifies considerably when we now solve the Gauss law constraint (12), as can be done explicitly by expressing a physical wavefunctional \( \Psi[B, \chi, \rho] \) as

\[
\Psi[B, \chi, \rho] = \exp \left( \frac{i\kappa}{4\sqrt{2}ae^2} \int B \rho \right) \psi[B, \chi] \] (14)

where the wavefunctional \( \psi \) is independent of the field \( \rho \). This implies that \( \Psi \) automatically satisfies the Gauss law constraint, and we can derive a new effective Hamiltonian density for the wavefunctional \( \psi \) by acting on \( \Psi \) with \( \mathcal{H}_{\text{quad}} \).

\[
\mathcal{H}_{\text{quad}} = \frac{1}{2} \Pi_\chi \left( 1 - \frac{2e^2a^2}{\nabla^2} \right) \Pi_\chi - \frac{e^2}{2} \Pi_B \nabla^2 \Pi_B + \frac{a\kappa}{2\sqrt{2}} \left( \frac{3}{\nabla^2} + \frac{1}{2e^2a^2} \right) B \Pi_\chi - \frac{\kappa}{4\sqrt{2}a} \nabla^2 \chi \Pi_B
+ B \left( \frac{1}{2e^2} + \frac{\kappa^2}{64a^2e^4} - \frac{9\kappa^2}{32e^2\nabla^2} - \frac{a^2}{\nabla^2} \right) B - \frac{1}{2} \left( 1 + \frac{\kappa^2}{32a^2e^2} \right) \chi \nabla^2 \chi \] (15)

Notice that all dependence on the field \( \rho \) has disappeared from the Hamiltonian density. This serves as a nontrivial check that the Gauss law has been solved correctly. This is also the Schrödinger representation version of the usual statement that the longitudinal component \( \lambda \) of the gauge field “eats” the Goldstone boson \( \phi_2 \) - they combine into the fields \( \rho \) and \( \chi \) as in (11) and one of these fields, \( \rho \), disappears when Gauss’ law is solved.

The final result is that we are left with a quadratic Hamiltonian density involving just the fields \( B \) and \( \chi \), which we can therefore view as (in a sense to be made precise below) the “physical” fields.
To truly identify the two independent physical fields, we first normalize the fields so that the kinetic parts of the Hamiltonian density take the form \( \frac{1}{2} \) (momentum)\(^2\). Define the rescaled fields

\[
B \equiv \sqrt{-e^2 \nabla^2} \tilde{B} \quad \chi \equiv \sqrt{1 - \frac{2e^2a^2}{\nabla^2}} \tilde{\chi}
\]

In terms of these rescaled fields, the Hamiltonian density is

\[
\mathcal{H}_{\text{quad}} = \frac{1}{2} \Pi^2 \chi + \frac{1}{2} \Pi^2_B + \frac{\kappa}{4\sqrt{2ae}} \sqrt{2e^2a^2 - \nabla^2} \tilde{\chi} \Pi_B = -\frac{\kappa}{4\sqrt{2ae}} \frac{6a^2e^2 + \nabla^2}{\sqrt{2e^2a^2 - \nabla^2}} \tilde{B} \Pi \tilde{\chi} \\
+ \frac{1}{2} \left( 1 + \frac{\kappa^2}{32a^2e^2} \right) \tilde{\chi} \left( 2a^2e^2 - \nabla^2 \right) \tilde{\chi} \\
+ \frac{1}{2} \tilde{B} \left( 2a^2e^2 \left( 1 + \frac{9\kappa^2}{32a^2e^2} \right) - \left( 1 + \frac{\kappa^2}{32a^2e^2} \right) \nabla^2 \right) \tilde{B}
\]

Notice that in the absence of a Chern-Simons term (i.e. set \( \kappa = 0 \)), this Hamiltonian simplifies dramatically to

\[
\mathcal{H}_{\text{quad}} |_{\kappa=0} = \frac{1}{2} \Pi^2 \chi + \frac{1}{2} \Pi^2_B + \frac{1}{2} \tilde{\chi} \left( 2a^2e^2 - \nabla^2 \right) \tilde{\chi} + \frac{1}{2} \tilde{B} \left( 2a^2e^2 - \nabla^2 \right) \tilde{B}
\]

from which we clearly see that the fields \( \tilde{B} \) and \( \tilde{\chi} \) represent scalar degrees of freedom with equal masses \( \sqrt{2ae} \), to be compared with the Higgs field \( \phi_1 \) mass, \( \alpha \sqrt{2ae} \). The special choice of \( \alpha = 1 \) in the potential (3) makes the gauge and Higgs masses degenerate and corresponds to the self-dual point of the Abelian Higgs model [15].

When \( \kappa \neq 0 \), there are cross-terms in the Hamiltonian (17) which mix the fields and momenta, thereby complicating the direct identification of the masses of the elementary fields. The general technique for diagonalizing such a Hamiltonian density is described in detail in the Appendix. One can consider the quadratic quantum mechanical Hamiltonian of the form

\[
H = \frac{1}{2} p_1^2 + \frac{1}{2} p_2^2 + b_1 q_1 p_2 + b_2 q_2 p_1 + \frac{1}{2} c_1^2 q_1^2 + \frac{1}{2} c_2^2 q_2^2 \\
\equiv \frac{1}{2} \xi^T h \xi
\]

where in the second line we have written \( \xi \) as a phase-space vector \( \xi = (p_1, p_2, q_1, q_2) \), and \( h \) is the \( 4 \times 4 \) real symmetric matrix

\[
h = \begin{pmatrix}
1 & 0 & 0 & b_2 \\
0 & 1 & b_1 & 0 \\
0 & b_1 & c_1 & 0 \\
b_2 & 0 & 0 & c_2^2
\end{pmatrix}
\]
In order to identify the normal mode frequencies of such a Hamiltonian we need to be able to diagonalize $h$ with a symplectic (rather than an orthogonal) matrix.

Since the Hamiltonian equations of motion for a quadratic Hamiltonian such as (19) are linear and have the form

$$\dot{\xi} = -\mathcal{E} h \xi$$  \hspace{1cm} (21)

where the $4 \times 4$ matrix $\mathcal{E}$ is

$$\mathcal{E} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$  \hspace{1cm} (22)

the eigenmodes of the quadratic Hamiltonian (19) are given by the eigenvalues of the matrix $i\mathcal{E} h$. A simple calculation gives the squares of the eigenvalues of $i\mathcal{E} h$ as

$$\omega_+^2 = \frac{1}{2} (c_1^2 + c_2^2 - 2b_1b_2) \pm \frac{1}{2} \sqrt{(c_1^2 + c_2^2 - 2b_1b_2)^2 - 4 (c_1^2 - b_1^2)(c_2^2 - b_2^2)}$$  \hspace{1cm} (23)

We can now apply this result directly to the hamiltonian density (17) with phase-space fields $\xi = (\Pi_\tilde{\chi}, \Pi_\tilde{B}, \tilde{\chi}, \tilde{B})$. Then the normal modes of the hamiltonian (17) are given by Equation (23) as

$$\omega_+^2 = -\nabla^2 + 2a^2 e^2 + \kappa^2 \pm \frac{\kappa^2}{2} \sqrt{\kappa^2 + 8a^2 e^2}$$  \hspace{1cm} (24)

It is worth commenting on the remarkable simplicity of this result, especially considering the very complicated form of the Hamiltonian (17). Notice that each $\omega_\pm^2$ is positive, and that each has the form $-\nabla^2 + (\text{mass})^2$. This clearly identifies the Hamiltonian as one describing two scalar fields, with masses (squared) given by

$$m_\pm^2 = 2a^2 e^2 + \frac{\kappa^2}{2} \pm \frac{\kappa^2}{2} \sqrt{\kappa^2 + 8a^2 e^2}$$  \hspace{1cm} (25)

which agrees with the masses found by Pisarski and Rao [3] in their analysis of the covariant gauge propagator. This result can alternatively be expressed as

$$m_\pm = \frac{\kappa}{2} \left( \sqrt{1 + 8a^2 e^2} \kappa^2 \pm 1 \right).$$  \hspace{1cm} (26)

The linear canonical (symplectic) transformation used to find these normal modes of the Hamiltonian density (17) transforms the quadratic Hamiltonian density into its diagonalized oscillator form:

$$\mathcal{H}_{\text{quad}} = \frac{1}{2} \sqrt{-\nabla^2 + m_+^2} \left( a_+^\dagger a_+ + a_+ a_+^\dagger \right) + \frac{1}{2} \sqrt{-\nabla^2 + m_-^2} \left( a_-^\dagger a_- + a_- a_-^\dagger \right)$$  \hspace{1cm} (27)
where \(a_\pm\) are annihilation operators for two independent scalar fields. The explicit linear transformation relating the phase space fields \(\Pi_\tilde{\chi}, \Pi_\tilde{\beta}, \tilde{\chi}\) and \(\tilde{\beta}\) with the oscillator fields \(a_\pm\) and \(a_\pm^\dagger\) is extremely complicated and is described in the Appendix. These oscillators satisfy the canonical commutation relations

\[
[a_\pm(x), a_\pm^\dagger(y)] = \delta(x - y)
\]

\[
[a_\pm(x), a_\pm^\dagger(y)] = 0 = [a_\pm(x), a_\mp(y)]
\]

(28)

Re-introducing the Higgs field, \(\phi_1\), the ground state wavefunctional in the symmetry breaking vacuum is

\[
\Psi_0[\phi_+, \phi_-, \phi_1] = \text{det}^{1/4} \left( \frac{\omega_+\omega_-\omega_1}{\pi^3} \right) \exp \left[ -\frac{1}{2} \int (\phi_+\omega_+\phi_+ + \phi_-\omega_-\phi_- + \phi_1\omega_1\phi_1) \right]
\]

(29)

where \(\omega_\pm = \sqrt{-\nabla^2 + m_\pm^2}\), \(\omega_1 = \sqrt{-\nabla^2 + 2\alpha^2e^2a^2}\), and \(\phi_\pm\) are Schrödinger representation fields given by

\[
\phi_\pm = \frac{1}{\sqrt{2\omega_\pm}} (a_\pm + a_\pm^\dagger)
\]

(30)

The result (26) for the masses in the broken phase may also be understood in the context of Chern-Simons quantum mechanics, in which many features of an Abelian topologically massive gauge theory may be analyzed in an analogue quantum mechanical model describing the planar motion of a particle in a uniform perpendicular magnetic field [13]. The Chern-Simons coefficient \(\kappa e^2\) plays the role of the external magnetic field strength, and the Chern-Simons modifications to the equations of motion behave like the Lorentz force modifications to the free-particle equations of motion. Extending this analogy to include a Higgs-like mass term \(a^2e^2A_iA_i\) for the gauge fields is like adding an additional isotropic harmonic potential to the analogue quantum mechanical model. Indeed, the two-dimensional quantum mechanical Hamiltonian

\[
H = \frac{1}{2} \left( p_i + \frac{\kappa}{2e^2} \epsilon^{ij} q_j \right)^2 + a^2e^2q_iq_i
\]

(31)

is well known to be separable into two independent harmonic oscillator Hamiltonians of frequencies

\[
\omega_\pm = \frac{\kappa}{2} \left( \sqrt{1 + 8\frac{a^2e^2}{\kappa^2}} \pm 1 \right)
\]

(32)

which agree precisely with the masses in (26). This comparison is due to the standard reduction of Schrödinger representation quantum field theory to Schrödinger representation quantum mechanics in a derivative expansion. The field theoretical frequencies \(\omega_\pm^2 = -\nabla^2 + m_\pm^2\) reduce to \(m_\pm^2\), and the Lagrange density reduces to the corresponding quantum mechanical Lagrangian.
3 Chern-Simons-Higgs Mechanism in the Schrödinger Representation

In this section, we discuss the Schrödinger representation implementation of the Higgs mechanism in the presence of a Chern-Simons, but without a Maxwell kinetic term for the gauge field. This “pure Chern-Simons” Higgs mechanism was first discussed by Deser and Yang. It warrants separate investigation in the Schrödinger representation because, as is well known, the canonical structure is very different without a Maxwell kinetic term for the gauge field.

Consider the Lagrange density

$$L = -\frac{\kappa}{2\epsilon^2} \epsilon^{\mu
u\rho} A_\mu \partial_\nu A_\rho - (D_\mu \phi)^* D^\mu \phi - V$$

where the scalar field potential is as defined in (3). The scalar field canonical momenta, $\Pi$ and $\Pi^*$, are as in (4). However, since the Lagrange density (33) is first-order in time derivatives, the gauge fields $A_1$ and $A_2$ are canonically conjugate to one another [14, 9]. One is free to choose a convenient polarization, and here I choose to identify $A_1$ as a coordinate field, with $A_2$ as the corresponding momentum field:

$$\pi_1 = -\frac{\kappa}{\epsilon^2} A_2$$

The Hamiltonian density now reads

$$\mathcal{H} = \Pi \dot{\phi} + \Pi^* \dot{\phi}^* + \pi_1 \dot{A}_1 - L$$

$$\mathcal{H} = \Pi^* \Pi + (D_i \phi)^* D_i \phi + V + A_0 \left( \frac{\kappa}{\epsilon^2} \epsilon^{ij} \partial_i A_j + i (\phi^* \Pi^* - \phi \Pi) \right)$$

where we have dropped a total time derivative term [1] which does not affect the identification of the physical modes.

In the spontaneously broken vacuum, we shift the scalar field as in (6), and (just as before) the real Higgs scalar field $\phi_1$ separates from the rest of the quadratic Hamiltonian density, identifying itself as a real scalar field of mass $\alpha \sqrt{2}a$. The remainder of the quadratic Hamiltonian density is

$$\mathcal{H}_{quad} = \frac{1}{2} \Pi_2^2 + \frac{1}{2} (\partial_i \phi_2) (\partial^i \phi_2) + a^2 A_i A_i + \sqrt{2}a A_i \partial_i \phi_2 + A_0 \left( \frac{\kappa}{\epsilon^2} \epsilon^{ij} \partial_i A_j - \sqrt{2}a \Pi_2 \right)$$

We now make the Hodge decomposition for the gauge field as in (8), but note that this is now to be viewed as a linear transformation of the phase space fields $A_1$ and $A_2$ to the longitudinal and transverse fields $\lambda$ and $B$, respectively. Thus, $B$ and $\lambda$ are now to be viewed...
as conjugate fields, and we can choose \( \lambda \) as the coordinate field, in which case we identify \( B \) as the corresponding momentum

\[
B = \frac{e^2}{\kappa} \Pi_\lambda
\]  

(37)

In terms of these fields, the quadratic Hamiltonian density becomes

\[
\mathcal{H}_{\text{quad}} = \frac{1}{2} \Pi_2^2 - \frac{a^2 e^4}{\kappa^2} \Pi_\lambda \frac{1}{\sqrt{2}} \Pi_\lambda - \frac{1}{2} \left( \phi_2 + \sqrt{2} a \lambda \right) \nabla^2 \left( \phi_2 + \sqrt{2} a \lambda \right) + A_0 \left( \Pi_\lambda - \sqrt{2} a \Pi_2 \right)
\]  

(38)

In this form of the Hamiltonian we recognize once again the field combinations \( \chi = \phi_2 + \sqrt{2} a \lambda \) and \( \rho = \phi_2 - \sqrt{2} a \lambda \), defined before in (11). With these fields, the Gauss law constraint takes the simple form

\[
\Pi_\rho = 0
\]  

(39)

Thus, when acting on physical state wavefunctionals \( \Psi[\chi, \rho] = \Phi[\chi] \) which are annihilated by \( \Pi_\rho \), the effective quadratic Hamiltonian density is

\[
\mathcal{H}_{\text{quad}} = \frac{1}{2} \Pi_\chi \left( 1 - \frac{4 a^4 e^4}{\kappa^2 \nabla^2} \right) \Pi_\chi - \frac{1}{2} \chi \nabla^2 \chi
\]  

(40)

To identify the physical mode the field \( \chi \) is rescaled as \( \chi = \sqrt{\frac{1 - 4 a^4 e^4}{\kappa^2 \nabla^2}} \tilde{\chi} \), in which case the quadratic Hamiltonian density becomes

\[
\mathcal{H}_{\text{quad}} = \frac{1}{2} \Pi_\chi^2 + \frac{1}{2} \chi \left( -\nabla^2 + \frac{4 a^4 e^4}{\kappa^2} \right) \tilde{\chi}
\]  

(41)

which clearly identifies the single scalar mode of mass

\[
m = \frac{2 a^2 e^2}{\kappa},
\]  

(42)

in agreement with the result of Deser and Yang [4].

To conclude this section, we note that it is possible to arrive at this mass from the results of the previous section by considering the limit in which the Maxwell term is removed from the Lagrangian density (11). Comparing the Lagrange densities (11) and (33), we see that formally one can arrive at the pure Chern-Simons case via the limit

\[
e^2 \rightarrow \infty
\]  

\[
\kappa \rightarrow \infty
\]  

with \( \frac{\kappa}{e^2} \) fixed

(43)
In this limit the masses \( m_{\pm} \) in (26) behave as

\[
\begin{align*}
    m_+ & \rightarrow \infty \\
    m_- & \rightarrow \frac{2a^2 e^2}{\kappa}
\end{align*}
\]

so that excitations associated with \( m_+ \) decouple, leaving the single mode of mass \( m = 2a^2 e^2 / \kappa \). This is precisely analogous to the lowest Landau level projection in the analogue quantum mechanical problem in which the external magnetic field strength goes to infinity and the resulting dynamics is governed by the frequency of the harmonic well \([16, 13]\).

4 Conclusions

This paper presents a pedagogical exercise in implementing the field theoretic Schrödinger representation for gauge theories in \( 2 + 1 \) dimensions involving mass generation effects via the Higgs mechanism and/or Chern-Simons terms. The presence of a Chern-Simons term introduces off-diagonal mixing in the quadratic Hamiltonian, a feature which complicates the diagonalization procedure in an interesting manner. The Schrödinger representation approach explains why the resulting two massive modes have masses corresponding to the two characteristic frequencies of an analogue quantum mechanical model of planar particles in a perpendicular magnetic field and a harmonic potential well. This also shows that the relationship between the pure Chern-Simons-Higgs theory, with one massive gauge mode, and the Maxwell-Chern-Simons-Higgs theory, with two massive gauge modes, is one of a truncation of the Hilbert space as one mode decouples to infinity, analogous to the projection of dynamics onto the lowest Landau level in the quantum mechanical model. Even in the quantum mechanical case, there are well-known subtleties \([16, 13]\) involved with the combination of such a truncation of the Hilbert space with the development of a perturbation expansion. In field theoretical models, such as Maxwell-Chern-Simons-Higgs theory or Chern-Simons-Higgs theory, these subtleties are compounded by regularization concerns. Nevertheless, this Schrödinger representation analysis clearly identifies this phenomenon as the physical source of various delicate issues that have been found in the analysis of Chern-Simons theories with symmetry breaking using more conventional covariant perturbation theory \([17, 18, 19]\), where it has been found that the computation of loop effects does not necessarily commute with the taking of such projections.
5 Appendix

In this appendix I present the details of the symplectic diagonalization of the Hamiltonian

\[ H = \frac{1}{2} \xi^T h \xi \]  

(45)

where \( \xi \equiv (p_1, p_2, q_1, q_2) \) and \( h \) is the real symmetric 4 \( \times \) 4 matrix

\[
h = \begin{pmatrix}
1 & 0 & 0 & b_2 \\
0 & 1 & b_1 & 0 \\
0 & b_1 & c_1^2 & 0 \\
b_2 & 0 & 0 & c_2^2 \\
\end{pmatrix}.
\]  

(46)

That is, we shall seek a real matrix \( S \) such that

\[
h = S^T \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & \omega_+^2 & 0 \\
0 & 0 & 0 & \omega_-^2 \\
\end{pmatrix} S
\]  

(47)

with \( S \) being symplectic (not orthogonal!)

\[
S \mathcal{E} S^T = \mathcal{E}
\]  

(48)

and where \( \omega_{\pm} \) are the eigenvalues of the matrix \( i \mathcal{E}h \). The condition that \( S \) be symplectic is required so that the canonical structure of the phase space variables is preserved by the linear transformation \( \xi \to \xi' = S \xi \).

In the familiar special case (31) of two-dimensional motion of a particle in a uniform magnetic field and an isotropic harmonic potential, the matrix \( h \) has the form

\[
h = \begin{pmatrix}
1 & 0 & 0 & -\frac{B}{2} \\
0 & 1 & \frac{B}{2} & 0 \\
0 & \frac{B}{2} & \Omega^2 & 0 \\
-\frac{B}{2} & 0 & 0 & \Omega^2 \\
\end{pmatrix}
\]  

(49)

where \( B \) is the magnetic field strength, \( \Omega^2 \equiv \omega^2 + \frac{B^2}{c^2} \), and \( \omega \) is the frequency of the isotropic harmonic potential. Then the eigenvalues of \( i \mathcal{E}h \) are

\[
\omega_{\pm} = \Omega \pm \frac{B}{2}
\]  

(50)
and the real symplectic matrix $S$ which diagonalizes $h$ is

$$
S = \begin{pmatrix}
\sqrt{\frac{\omega_+}{2\Omega}} & 0 & 0 & -\sqrt{\frac{1}{2\Omega} \omega_+} \\
-\sqrt{\frac{\omega_-}{2\Omega}} & 0 & 0 & -\sqrt{\frac{1}{2\Omega} \omega_-} \\
0 & 1 & \sqrt{2\Omega} \omega_+ & 0 \\
0 & 1 & \sqrt{2\Omega} \omega_- & 0
\end{pmatrix}
$$

(51)

The linear transformation $\xi \rightarrow \xi' = S\xi$ with this form of $S$ is the familiar one \[13\] which separates this model into two independent harmonic oscillators of frequency $\omega_{\pm} = \Omega \pm \frac{B}{2}$. However, in the general case, with $h$ given by (46), it is considerably more complicated to construct the symplectic matrix $S$ which diagonalizes $h$. To introduce the general method for constructing such an $S$, consider the equations of motion arising from the Hamiltonian (45):

$$
\dot{\xi} = -\mathbf{E} h \xi
$$

(52)

This shows that the normal modes of this Hamiltonian system are given by the eigenvalues of the matrix $i\mathbf{E}h$. Further, as outlined below, the symplectic matrix $S$ is constructed from the eigenvectors of $i\mathbf{E}h$.

Note first of all that the eigenvalues of $i\mathbf{E}h$ are real, and occur in $\pm$ pairs. This is because

$$
\det (i\mathbf{E}h - \omega I) = \det \left((i\mathbf{E}h - \omega I)^T\right)
= \det (-ih\mathbf{E} - \omega I)
= \det (i\mathbf{E}h + \omega I)
$$

(53)

Further, if $v$ is an eigenvector of $i\mathbf{E}h$ with eigenvalue $\omega$, then $v^*$ is an eigenvector with eigenvalue $-\omega^* = -\omega$. Now let $V$ be the $4 \times 4$ matrix whose columns are formed by the eigenvectors of $i\mathbf{E}h$. Then these columns may be normalized in such a way that

$$
V^\dagger \mathbf{E} V = i
\begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
$$

(54)

In particular, note that eigenvectors of $i\mathbf{E}h$ with different eigenvalues are orthogonal with respect to the inner product $w^\dagger \mathbf{E} v$. Using equation (54) we see that the equations of motion for $\xi$ may be written as

$$
V^\dagger \mathbf{E} \dot{\xi} = -i (V^\dagger h V)
\begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
V^\dagger \mathbf{E} \xi
$$
These equations of motion are now in the diagonal “oscillator” form
\[ a_\pm = i\omega_\pm a_\pm \]
\[ \dot{a}_\pm = -i\omega_\pm a_\pm \] (56)
which would follow from the Hamiltonian
\[ H = \frac{\omega_+}{2} (a_+^a a_+ + a_+ a_+^a) + \frac{\omega_-}{2} (a_-^a a_- + a_- a_-^a) \] (57)
So we see that the transformation from the initial phase space variables \( \xi \equiv (p_1, p_2, q_1, q_2) \) to the oscillators \( a_\pm \) and \( a_\pm^a \) is
\[ \begin{pmatrix} a_+^a \\ a_-^a \\ a_+ \\ a_- \end{pmatrix} = V^\dagger E \begin{pmatrix} p_1 \\ p_2 \\ q_1 \\ q_2 \end{pmatrix} \equiv V^\dagger E \xi \] (58)
We may alternatively wish to present the diagonalized Hamiltonian in terms of new canonical coordinates and momenta, \( q_\pm \) and \( p_\pm \), related to the oscillators \( a_\pm \) and \( a_\pm^a \) as
\[ q_\pm \equiv \pm \frac{1}{\sqrt{2\omega_\pm}} (a_\pm + a_\pm^a) \]
\[ p_\pm \equiv \mp i\sqrt{\frac{\omega_\pm}{2}} (a_\pm - a_\pm^a) \] (59)
These coordinates and momenta are thus related to the original ones by the net transformation
\[ \begin{pmatrix} p_+ \\ p_- \\ q_+ \\ q_- \end{pmatrix} = S \begin{pmatrix} p_1 \\ p_2 \\ q_1 \\ q_2 \end{pmatrix} \] (60)
where \( S \) is the matrix
\[ S = \begin{pmatrix} i\sqrt{\frac{\omega_+}{2}} & 0 & -i\sqrt{\frac{\omega_-}{2}} & 0 \\ 0 & -i\sqrt{\frac{\omega_-}{2}} & 0 & i\sqrt{\frac{\omega_+}{2}} \\ \frac{1}{\sqrt{2\omega_+}} & 0 & \frac{1}{\sqrt{2\omega_-}} & 0 \\ 0 & \frac{1}{\sqrt{2\omega_-}} & 0 & -\frac{1}{\sqrt{2\omega_+}} \end{pmatrix} V^\dagger E \] (61)
By construction, this matrix $S$ is a real symplectic matrix. Furthermore, by construction, it diagonalizes $h$ in the sense of (47).

Note that this result is completely general, for any Hamiltonian of the form (19). However, for such a general Hamiltonian the explicit expression for the normalized eigenvectors of $i \mathcal{E} h$ (recall that these eigenvectors form the columns of the matrix $V$) is very complicated - note that even the expression for the eigenvalues (23) is quite involved. It is, however, instructive to illustrate this procedure in the special case (49). Here the matrix $V$ of eigenvectors of $i \mathcal{E} h$ (recall that these eigenvectors form the columns of the matrix $V$) is

$$V = \frac{\sqrt{\Omega}}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ -i & i & i & -i \\ \frac{-i}{\overline{\Omega}} & \frac{i}{\overline{\Omega}} & \frac{i}{\overline{\Omega}} & \frac{-i}{\overline{\Omega}} \end{pmatrix}$$

(62)

It is straightforward to check that this matrix $V$ is correctly normalized, as in (54), and that using it in equation (61) yields the correct real symplectic diagonalizing matrix (51).

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