ANALYSIS OF DYNAMIC SERVICE SYSTEM BETWEEN REGULAR AND RETRIAL QUEUES WITH IMPATIENT CUSTOMERS

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Abstract. In this article, we propose a dynamic operating of a single server service system between conventional and retrial queues with impatient customers. Necessary and sufficient conditions for the stability, and an explicit expression for the joint steady-state probability distribution are obtained. We have derived some interesting and important performance measures for the service system under consideration. The first-passage time problems are also investigated. Finally, we have presented extensive numerical examples to demonstrate the effects of the system parameters on the performance measures.

1. Introduction. Any practical service system in which arriving customers place their required demands upon a limited capacity resource can be regarded as a queueing system. If the arrival times of customers and quantum leap of their demands are unpredictable, then conflicts will arise for the use of the limited resource and other facilities. As a result, the arriving customers will always find the server busy and consequently queues of waiting customers will form in the system (Takagi [38] and Gross et al. [27]).

In the conventional queueing systems, it is quite often assumed that a customer who cannot receive service immediately after arrival, either joins the waiting line to receive its service leading to unbounded queue size or leaves the system forever without receiving service, known as loss system. To alleviate the situations of either loss of the customers or overloading the system in the conventional queues, a new class of queueing system known as retrial queue has been introduced as an alternative solution among others.

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Retrial queues are featured by the fact that arriving customers or requests which find the service facility busy join the retrial group (or orbit) and try again for their requests in random order and at random time intervals. Thus, retrial queues can be considered as networks which reservice the customers after blocking.

Nowadays, retrial queueing systems are being used widely and appropriately as models for many real-life situations such as telephone switching systems, internet ticketing and reservation systems, digital cellular mobile networks, local area networks under the protocols of random multiple access, automatic repeated request call centers, internet of things, optical networks, just to name a few.

Excellent overviews of the current research on retrial queues can be seen in the books of Falin and Templeton [23] and Artalejo and Gomez-Corral [10]. Moreover, comprehensive review on these topics can be found in survey papers by Falin [22], Kulkarni and Liang [33] and Choi and Chang [18] and the bibliographies of Artalejo [7, 8].

Impatience of customers is another typical phenomenon which is commonly observed in several service systems. Impatient customers may leave (reneg from) the system before receiving service if either their waiting time exceeds their patience time or owing to uncertainty of receiving service. As a result, customers' impatience or reneging causes loss in revenues and customers' goodwill to the service provider. Hence customers' impatience in queues is known to greatly impact the system performance measures. For some aspects of queues with impatient customers, see the survey paper by Wang et al. [40] and the monograph by Gross et al. [27] and the references therein.

Due to this overwhelming, it is not surprising that a large number of studies have been devoted to characterize queues with impatient customers. The literature on queueing systems with impatient customers focus especially on performance evaluation. For related research on these topics and their applications, the reader can refer to the papers by Baccelli and Hebuterne [15], Baccelli et al. [14], Boxma and de Waal [16], Movaghar [35], Altman and Yechiali [6], Economou and Kapodistria [20], Dimou et al. [19], Yue et al. [43] and the references therein for analytical performance characterizations. Other research works pertaining to the queues with impatient customers can be found in Aksin and Harker [4], Garnett et al. [26], Koole and Mandelbaum [32], Brandt and Brandt [17], Zeltyn and Mandelboum [44] and Iravani and Balcioglu [28] for performance approximations.

In all the previous works mentioned above, queues with repeated attempts and customers’ impatience have been separately investigated. However, in the modern design of service systems, the impact of both impatience of customers and retrials on the system performance and queueing dynamics are being realized. Therefore, these scenarios should be taken into account concurrently in order to obtain an accurate performance evaluation of the complex queueing networks. For such queueing network systems, the readers are referred to Altman and Borovkov [5], Mandelbaum et al. [34], Gans et al. [25], Aguir et al. [1, 2], Wuchner et al. [42], Shin and Choo [37], Artalejo and Pla [11], Jouini et al. [30] and Tuan and Kawanishi [36].

In this paper, we examine a new generic queueing system with the coexistence of both the conventional and retrial queues with impatient customers. The novelty of this system is not only incorporating the impatience behaviour of customers, but also oscillation of the system between two phases namely, conventional queue and retrial queue. As far as we are aware, a dynamic operating queueing system
considering impatient customers of the type proposed here has not been studied and is new in the literature of queueing models.

The rest of the paper is structured as follows: We describe the mathematical model and derive the necessary and sufficient condition for the system to be stable in Section 2. The steady-state joint distribution of the status of the server and the number of customers in the orbit is obtained in Section 3. In Section 4, the factorial moments for the system under investigation are derived and the first two moments are given explicitly. Section 5 presents the key performance measures of the system under study. In Section 6, we analyze the first passage time problem and calculate the mean and variance explicitly. In Section 7, numerical illustrations and sensitivity analysis of the performance measures, including the mean busy period of the system, mean response time and mean first passage time are presented. Finally, Section 8 concludes the paper.

2. Model description. We investigate a new class of single server queueing system which makes oscillation between two phases namely, conventional queueing mode and retrial queueing mode with unlimited orbit capacity in the following manner: Whenever the number of customers in the orbit is less than or equal to the threshold level $N$, after a service completion, the server immediately fetches the next customer directly from the orbit (with probability one) and starts service. This indicates that the system operates in a conventional queueing mode. As soon as the system size becomes greater than or equal to $N + 1$, i.e., once exceeds the threshold $N$, the system switches to retrial queueing mode and continues to operate under this mode until the size of the orbit comes down to threshold level $N$ at a service completion epoch. During the retrial queueing operation mode, customers from the orbit have to make retrial attempts to access service on their own. Therefore, our dynamical service system deals with a back and forth movement between conventional queue and retrial queue operation modes.

Apart from oscillation of our service system between the two modes of queueing systems described above, customers wait in the orbit for a random duration to access the service by repeated attempts, else they immediately leave the system forever by impatience while their patience time gets expired. It is further assumed that the event of customers' impatience occurs only when the size of the orbiting customers is above the level $N + 1$ and at the same time the server is in idle status during the time intervals between retrial attempts.

It should be pointed out that the scenario of occurrence of customers' impatience in our oscillating service system is opposed to studying impatience behaviour in the literature of retrial queues where the events of impatience behavior take place when the server is in busy status and independent of the threshold level which we implemented in our analysis.

In the present system, it is important to note that for a large threshold level $N$, the retrial system behaves in a manner very similar to a conventional queue, whereas for small threshold level $N$, the idle time of server between two consecutive services will be rather lengthy if the system is in the retrial operation mode. To avoid such long idle times, the server fetches the next customer directly from the orbit as long as the number of customers in the orbit is below the threshold level $N$. Moreover, when the orbit size increases, retrial intensity also correspondingly increases and thereby the customers from the orbit generate the retrial flow more frequently in order to access service. As a result, the frequent retrial attempts reduce the idle
time of the server between services and hence the customers’ waiting in the orbit becoming impatient is controlled and the loss of customers is reduced.

Our study of dynamic oscillating service system can be applied to the performance modelling and analysis of optical networks with wavelength conversion. Optical networks are the most important part of current communication systems. It is the prominent technology for transmission of enormous amounts of data/packets over long distance with low latency. Optical fiber offers several advantages for communication with respect to copper cables. In a communication network, packets must be routed from source to destination passing through a series of links and nodes through copper cables. In copper-based transmission links, data from different sources are time-multiplexed. In recent years, the “wired” part of communication systems is almost completely replaced by the optical fiber cables in order to enhance the transmission. Thus, the optical fiber network utilizes optical fiber cables as the primary communication medium for transporting the data/packets as light signals (photons) from source (optical transceiver) to destination (photo detector).

Unlike electronic data buffering, the photons (light signals) buffering in optical switches is not an easy task, as photons cannot be stopped. Whenever there is a need to buffer photons, they are made to move locally in fiber loops. These fiber loops or fiber delay links (FDLs) originate and end at the head of a switch. When an arriving photon at the switch node finds the wavelength channel occupied at a time, it cannot be immediately served. The blocked photons, however, will always be routed to use a buffer in one of the FDLs to retry to access the free wavelength channel. Meanwhile, the photons buffered in FDLs incur a small delay since their time of arrival but without loss of service. Depending on the availability, requirement, size of photons, and other factors, the length of delay caused by these FDLs may differ. Evidently, these FDLs delay the photons by a random amount of time. Once the photons traverse one of the FDLs, they become the so called retrial photons. A retrial photon will use the free wavelength channel if available at the epoch of exit from the FDLs. Otherwise, it will attempt to re-circulate through one of the FDLs. As a result, two consecutive repeated attempts are independent of each other. This feature of the FDLs can be treated as a retrial queue (see Akar and Sohraby [3]).

In such a circumstance, the photons in FDLs buffers get highly congested which causes a huge delay in accessing the wavelength channel. At the outset, some of the photons in FDLs buffers will turn out to be outdated (less significant) and subsequently the outdated photons are selectively discarded from FDLs buffers without transmission while the local congestion measure exceeds a given threshold. A prudent choice of an effective photon discarding threshold can smooth the traffic congestion and thus reduce the system latency in the optical fiber networks. This will be at the expense of discarding moderate photons resulting in a slight reduction of the quality of service, but enhancing the efficiency of the system.

2.1. Underlying process and system parameters. Consider a dynamic operating (oscillating) of a single server queue described in the previous subsection. Primary customers arrive according to a Poisson process with rate \( \lambda \). The service times for all customers are independent exponential random variables with mean \( \frac{1}{\nu} \). The time-intervals describing the repeated attempts are assumed to be independent and exponentially distributed with linear intensity \( \mu_j = \alpha + j\mu \), where \( j \geq N+1 \), is the number of customers waiting in the orbit. Note that in the case of \( \alpha = 0 \)
and \( \mu > 0 \), the retrial intensity leads to the classical retrial discipline (see Falin [22]), whereas for \( \mu = 0 \) and \( \alpha > 0 \), the system turns out with the constant retrial discipline (see Fayolle [24]). During the inter-retrial times, the customers waiting in the orbit become impatient. Specifically, while the number of customers in the orbit is at least \( N + 2 \) and the server is in idle status during time intervals between retrial attempts, each waiting customer in the orbit activates an “impatience timer”, \( T \), which is exponentially distributed with mean \( \frac{1}{\mu} \). If a customer’s retrial attempt has not been completed successfully before the customer’s impatience timer expires, the customer abandons the system without getting service and never returns. To preserve mathematical tractability and provide exact analytical analysis of the system, the input stream of primary arrivals, service times, inter-arrival times, and impatient times are assumed to be mutually independent.

In the next subsection, we establish the necessary and sufficient conditions for the system to be stable.

2.2. Ergodicity condition. Let \( X_Q(t) \) be the number of customers in the retrial group / orbit at time \( t \) and \( C(t) \) represent the server status at time \( t \) as defined below:

\[
C(t) = \begin{cases} 
0, & \text{if the server is idle,} \\
1, & \text{if the server is busy.}
\end{cases}
\]

The state of the system at time \( t \) can be described by the bivariate process \( \{X(t), t \geq 0\} = \{(C(t), X_Q(t)), t \geq 0\} \) taking values on the state space \( S = \{0,1\} \times \mathbb{Z}_+ \) where \( \mathbb{Z}_+ = \{0,1,2,\cdots\} \). Indeed, the bivariate process \( \{X(t), t \geq 0\} \) is an irreducible and aperiodic continuous time Markov chain (CTMC). In what follows, \( \pi(i,j,t) = P(C(t) = i, X_Q(t) = j), i \in \{0,1\}, j \in \mathbb{Z}_+, \) is the joint probability distribution of the status of the server and the number of customers in the orbit at time \( t \). By our assumption whenever orbit size is \( \leq N \), at a service completion epoch, clearly, \( \pi(0,j,t) = 0, \) for \( j = 1,2,\cdots,N \), since on service completion the server immediately picks up a customer from the orbit with probability one and the time for this procedure is negligible. The corresponding infinitesimal generator matrix \( Q = (q_{i,j}(n,m)) \) of the CTMC \( \{X(t), t \geq 0\} \) is defined as follows:

\[
q_{0,j}(n,m) = \begin{cases} 
-\lambda - \alpha(1 - \delta_{0j}), & \text{if } (n,m) = (0,j), \ j = 0, \ j \geq N + 1, \\
\lambda + j\mu + j\xi(1 - \delta_{N+1j}), & \text{if } (n,m) = (1,j), \ j = 0, \ j \geq N + 1, \\
\alpha + j\mu, & \text{if } (n,m) = (1,j-1), \ j \geq N + 1, \\
j\xi, & \text{if } (n,m) = (0,j-1), \ j \geq N + 2, \\
0, & \text{otherwise},
\end{cases}
\]

(2.1)

\[
q_{1,j}(n,m) = \begin{cases} 
-(\lambda + \nu), & \text{if } (n,m) = (1,j), \ j \geq 0, \\
\lambda, & \text{if } (n,m) = (1,j+1), \ j \geq 0, \\
\nu, & \text{if } (n,m) = (1,j-1), \ 1 \leq j \leq N, \\
\nu, & \text{if } (n,m) = (0,j), \ j = 0, \ j \geq N + 1, \\
0, & \text{otherwise},
\end{cases}
\]

where \( \delta_{ab} \) denotes the Kronecker’s delta function, i.e., \( \delta_{ab} = 1 \) if \( a = b \) and \( \delta_{ab} = 0 \) if \( a \neq b \).
In addition, it is assumed that the process is standard in the sense that
$$\lim_{t \to 0} P\{X(t) = (n, m) | X(0) = (i, j)\} = \delta_{(i,j)(n,m)} $$
$$\forall (i,j), (n,m) \in S.$$

These transition rates among states are illustrated in Figure 1.

**Figure 1.** Dynamic oscillating queue with threshold and impatience

Observe that the matrix $Q = (q(i,j)(n,m))$ is irreducible and conservative and hence it is the so-called $Q$–matrix of the process $\{X(t), t \geq 0\}$. We assume that the $Q$–matrix uniquely determines the process $\{X(t), t \geq 0\}$, so that the process $\{X(t), t \geq 0\}$ is regular. Probabilistically the regularity condition says that the process $\{X(t), t \geq 0\}$ takes only a finite number of jumps in finite time or, equivalently, almost all sample functions of the process are step functions. Asmussen [13] has stated that an irreducible regular Markov jump process is ergodic if and only if there exists a probability solution $\Pi$ to $\Pi Q = 0$ where $Q$ is the infinitesimal generator matrix of the process $\{X(t), t \geq 0\}$. In that case $\Pi$ is the stationary distribution.

We now define the limiting joint probabilities of the CTMC $\{X(t), t \geq 0\} = \{(C(t), X_Q(t)), t \geq 0\}$ as
$$\pi(i,j) = \lim_{t \to \infty} P\{C(t) = i, X_Q(t) = j\}, i \in \{0,1\}, j \in \mathbb{Z}_+,$$
which are positive if and only if the CTMC $\{X(t), t \geq 0\}$ is ergodic. To this end, we establish the necessary and sufficient conditions for the ergodicity of the CTMC $\{X(t), t \geq 0\}$ in the following theorem.

**Theorem 2.1.** Let $\{X(t), t \geq 0\}$ be the irreducible regular CTMC with infinitesimal generator matrix, $Q = (q(i,j)(n,m))$, of the system given in (2.1). Then, the CTMC $\{X(t), t \geq 0\}$ is ergodic if and only if one of the following conditions is fulfilled:

1. If $\alpha > 0$ and $\mu = 0$, then $\nu > 0$,
2. If $\alpha > 0$ and $\mu > 0$, then $\frac{\lambda}{\nu} < 1 + \frac{\xi}{\mu}$.

**Proof.** To analyze the ergodicity of $\{X(t), t \geq 0\}$, we start with the system of equations $\Pi Q = 0$, where
$$\Pi = (\pi(0), \pi(1), \pi(2), \cdots)$$
in which $\pi(j) = (\pi(0,j), \pi(1,j)), j \geq 0$, is a probability row vector with $\pi(0,j) = 0, j = 1, 2, \cdots, N.$
If we write explicitly the equation $\overline{\Pi Q} = \overline{0}$, we can obtain the following set of equilibrium equations:

$$
\begin{align*}
\lambda \pi(0, 0) &= \nu \pi(1, 0), \\
(\lambda + \alpha + (N + 1)\mu) \pi(0, N + 1) &= \nu \pi(1, N + 1) + (N + 2) \xi \pi(0, N + 2), \\
(\lambda + \alpha + j\mu + j\xi) \pi(0, j) &= \nu \pi(1, j) + (j + 1) \xi \pi(0, j + 1), \\
&\quad j \geq N + 2, \\
(\lambda + \nu) \pi(1, 0) &= \lambda \pi(0, 0) + \nu \pi(1, 1), \\
(\lambda + \nu) \pi(1, j) &= \lambda \pi(1, j - 1) + \nu \pi(1, j + 1), \\
&\quad 1 \leq j \leq N - 1, \\
(\lambda + \nu) \pi(1, N) &= \lambda \pi(1, N - 1) + (\alpha + (N + 1)\mu) \pi(0, N + 1), \\
(\lambda + \nu) \pi(1, j) &= \lambda \pi(1, j - 1) + \lambda \pi(0, j) + (\alpha + (j + 1)\mu) \pi(0, j + 1), \quad j \geq N + 1,
\end{align*}
$$

and the normalizing condition

$$
\pi(0, 0) + \sum_{j=N+1}^{\infty} \pi(0, j) + \sum_{j=0}^{\infty} \pi(1, j) = 1. \quad (2.10)
$$

Elimination of the probabilities $\pi(1, j)$, $j \geq 0$, from (2.4), (2.5) and (2.9) yields

$$
\begin{align*}
\{\lambda[\lambda + \alpha + j(\mu + \xi)] + [\alpha\nu + j(\lambda + \nu)(\mu + \xi)]\} \pi(0, j) &= \{\lambda[\lambda + \alpha + (j - 1)(\mu + \xi)]\} \pi(0, j - 1) \\
&+ \{[\alpha\nu + (j + 1)(\lambda + \nu)(\mu + \xi)]\} \pi(0, j + 1).
\end{align*}
$$

which can be rewritten as

$$
(\alpha_j + \beta_j) \pi(0, j) = \alpha_{j-1} \pi(0, j - 1) + \beta_{j+1} \pi(0, j + 1), \quad j \geq N + 2, \quad (2.11)
$$

where $\alpha_j = \lambda[\lambda + \alpha + j(\mu + \xi)]$ and $\beta_j = [\alpha\nu + j(\lambda + \nu)(\mu + \xi)]$ for $j \geq N + 2$. Relation (2.11) can be thought of as equations for the stationary distribution of a birth and death process with birth parameter $\{\alpha_j\}$ and death parameter $\{\beta_j\}$ for $j \geq N + 2$. Thus, up to a constant multiplier, $\pi(0, j)$ coincides with the ergodic distribution of this birth and death process.

We now use the well-known criteria (Karlin and McGregor [31]) that the birth and death type equation (2.11) has the stationary distribution $\{\pi(0, j), j \geq N + 2\}$ having finite mass if and only if the following holds:

$$
S_1 = \sum_{j=N+2}^{\infty} \prod_{k=N+1}^{j-1} \frac{\alpha_k}{\beta_{k+1}} < \infty \quad \text{and} \quad S_2 = \sum_{j=N+2}^{\infty} \left( \alpha_j \prod_{k=N+1}^{j-1} \frac{\alpha_k}{\beta_{k+1}} \right)^{-1} = \infty. \quad (2.12)
$$

It is not hard to prove that for any $\alpha > 0$ and $\mu = 0$, an application of the ratio test yields the series $S_2$ diverges, whereas $S_1$ converges if and only if the service rate $\nu > 0$. Next, in the case of $\alpha \geq 0$ and $\mu > 0$, again the usage of the ratio test reveals that the series $S_2$ diverges, whereas $S_1$ converges if and only if $\frac{\lambda}{\mu} < 1 + \frac{\xi}{\mu}$.
Remark 1. The regularity of the process \( \{ X(t), t \geq 0 \} \) is equivalently defined as follows. An irreducible recurrent birth and death process is regular if and only if the series \( S_2 \) diverges (see Whitt [41]).

3. Analysis of steady-state distribution. In this section, we obtain the joint steady-state distribution of the status of the server and the number of customers in the orbit for the ergodic CTMC \( \{ X(t), t \geq 0 \} \). Besides, the joint steady-state distribution can be expressed in terms of generalized hypergeometric functions.

Theorem 3.1. If \( \mu > 0 \), \( \alpha \geq 0 \) and CTMC \( \{ X(t), t \geq 0 \} \) is regular and ergodic under the stability condition (2.2), then the joint steady-state distribution of the status of the server and the number of customers in the orbit for the system under study is given by:

\[
\pi_{(0,0)} = \begin{cases} \\
\frac{\lambda \rho^{N+1}}{\alpha + (N + 1)\mu} + \frac{(1 - \rho^{N+2})}{(1 - \rho)} \\
\begin{aligned}
&+ \frac{\lambda^2 \rho^{N+1}[\lambda + \alpha + (N + 1)\mu]}{[\alpha + (N + 1)\mu][\alpha + (N + 2)((\lambda + \nu)\xi + \mu\nu)]} \\
&\times F(1, A + N + 2; B + N + 3; D) \\
&+ \frac{\rho^N[\lambda + \alpha + (N + 1)\mu][C + N + 2]}{[\alpha + (N + 1)\mu][B + N + 2]} \\
&\times F(1, A + N + 2, C + N + 3; B + N + 3, C + N + 2; D) \\
&\left. \right\}^{-1}, \\
& \text{if } \rho \neq 1,
\end{aligned}
\end{cases}
\]

\[
\begin{aligned}
\frac{\lambda}{\alpha + (N + 1)\mu} + (N + 2) &+ \frac{\lambda [\lambda + \alpha + (N + 1)\mu]}{[\alpha + (N + 1)\mu][\alpha + (N + 2)(\mu + 2\xi)]} \\
&\times F(1, A + N + 2; E + N + 3; G) \\
&+ \frac{[\lambda + \alpha + (N + 1)\mu][C + N + 2]}{[\alpha + (N + 1)\mu][E + N + 2]} \\
&\times F(1, A + N + 2, C + N + 3; E + N + 3, C + N + 2; G) \\
&\left. \right\}^{-1}, \quad \text{if } \rho = 1.
\end{aligned}
\]
\[ \pi_{(0,j)} = \begin{cases} \frac{\lambda \rho^{N+1}}{\alpha + (N + 1) \mu} \pi(0,0), & \text{if } j = N + 1, \rho \neq 1, \\ \frac{\lambda}{\alpha + (N + 1) \mu} \pi(0,0), & \text{if } j = N + 1, \rho = 1, \\ \frac{\lambda^2 \rho^{N+1}[\lambda + \alpha + (N + 1) \mu]}{[\alpha + (N + 1) \mu][\alpha + (N + 2)(\lambda + \nu) + \mu \nu]} \pi(0,0), & \text{if } j = N + 2, \rho \neq 1, \\ \frac{\lambda [\lambda + \alpha + (N + 1) \mu]}{[\alpha + (N + 1) \mu][\alpha + (N + 2)(\mu + 2 \xi)]} \pi(0,0), & \text{if } j = N + 2, \rho = 1, \end{cases} \]

\[ \pi_{(1,j)} = \begin{cases} \rho^{N+1} \pi(0,0), & \text{if } 0 \leq j \leq N, \rho \neq 1, \\ \pi(0,0), & \text{if } 0 \leq j \leq N, \rho = 1, \\ \frac{\lambda \rho^{N+1}[\lambda + \alpha + (N + 1) \mu][\alpha + (N + 2)(\mu + \xi)]}{[\alpha + (N + 1) \mu][\alpha + (N + 2)(\lambda + \nu) + \mu \nu]} \pi(0,0), & \text{if } j = N + 1, \rho \neq 1, \\ \frac{\lambda [\lambda + \alpha + (N + 1) \mu][\alpha + (N + 2)(\mu + \xi)]}{[\alpha + (N + 1) \mu][\alpha + (N + 2)(\mu + 2 \xi)]} \pi(0,0), & \text{if } j = N + 1, \rho = 1, \end{cases} \]

where \( \rho = \frac{\lambda}{\nu}, A = \frac{\lambda + \alpha}{\mu + \xi}, B = \frac{\alpha \nu}{(\lambda + \nu) \xi + \mu \nu}, C = \frac{\alpha}{\mu + \xi}, D = \frac{\lambda(\mu + \xi)}{(\lambda + \nu) \xi + \mu \nu}, \)

\[ E = \frac{\alpha}{\mu + 2 \xi}, G = \frac{\mu + \xi}{\mu + 2 \xi}, \]

and \( F \) represents the generalized hypergeometric series (Erdelyi [21]) defined as

\[ F(\alpha_1, \alpha_2, \cdots, \alpha_p; \beta_1, \beta_2, \cdots, \beta_q; z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n(\alpha_2)_n \cdots (\alpha_p)_n}{(\beta_1)_n(\beta_2)_n \cdots (\beta_q)_n} \frac{z^n}{n!}, \quad |z| < 1, \]
in which no denominator parameter \( \beta_j \) is allowed to be 0 or a negative integer and \( (n) \) is the Pochhammer symbol defined by
\[
(x)_n = \begin{cases} 
1 & \text{for } n = 0, \\
x(x+1)(x+2)\cdots(x+n-1) & \text{for } n \geq 1.
\end{cases}
\]

Proof. To determine the limiting joint probabilities \( \pi(0, 0), \pi(0, j), j \geq N+1 \) and \( \pi(1, j), j \geq 0 \), we follow the procedure given in Falin and Templeton [23]. From (2.3), we get directly,
\[
\pi(1, 0) = \rho \pi(0, 0),
\]
and using this in (2.6), we have
\[
\pi(1, 1) = \rho^2 \pi(0, 0).
\]
With the help of the above relations, the probabilities \( \pi(1, j), j = 2, 3, 4, \ldots, N \), can be recursively computed from (2.7) as given by (3.5).
Substituting (3.5) in (2.8), after some simplifications, yielding (3.2). Now the equation (2.4) gives
\[
\pi(1, N+1) = \frac{1}{\nu} \left[ \lambda + \alpha + (N+1)\mu \right] \pi(0, N+1) - (N+1)\xi \pi(0, N+2) \right].
\]
Using the above relation in (2.9), after a little algebra, we have arrived at the result (3.3). Next, solving for \( \pi(1, j) \) from (2.5) and inserting it into (2.9), after a little algebra, we obtain
\[
[\alpha\nu + (j + 1)\left(\lambda + \nu\xi + \nu\mu\right)] \pi(0, j+1) - \lambda[\lambda + \alpha + j (\mu + \xi)] \pi(0, j) \\
= [\alpha\nu + (j + 2)\left(\lambda + \nu\xi + \nu\mu\right)] \pi(0, j+2) \\
- \lambda[\lambda + \alpha + (j + 1) (\mu + \xi)] \pi(0, j+1), \quad j \geq N+2.
\]
The above can be expressed as
\[
x_{j+1} \pi(0, j+1) - y_j \pi(0, j) = x_{j+2} \pi(0, j+2) - y_{j+1} \pi(0, j+1), \quad j \geq N+2, \quad (3.8)
\]
where
\[
x_j = [\alpha\nu + j(\lambda + \nu\xi + \nu\mu)] \quad \text{and} \quad y_j = \lambda[\lambda + \alpha + j (\mu + \xi)].
\]
Thus, from (3.8), we infer that
\[
x_{j+1} \pi(0, j+1) - y_j \pi(0, j) = K, \quad j \geq N+2, \quad (3.9)
\]
for some arbitrary constant ‘K’. The unknown constant ‘K’ can be found from (2.4), (2.5), (2.7) - (2.9) and (3.5) as follows: Inserting (3.5) into (2.8), we can get, after simplification, the probability \( \pi(0, N+1) \) as in (3.2).

Now solve for \( \pi(1, N+2) \) from (2.5) and substitute it into (2.9), after some algebraic simplification, we have
\[
[\alpha\nu + (N+3)\left(\lambda + \nu\xi + \nu\mu\right)] \pi(0, N+3) - \lambda[\lambda + \alpha + (N+2) (\mu + \xi)] \pi(0, N+2) = 0
\]
which can be rewritten as
\[
x_{N+3} \pi(0, N+3) - y_{N+2} \pi(0, N+2) = 0. \quad (3.10)
\]
By setting \( j = N+2 \) in (3.9), we obtain
\[
x_{N+3} \pi(0, N+3) - y_{N+2} \pi(0, N+2) = K. \quad (3.11)
\]
Now comparing (3.10) and (3.11), it is concluded that \( K = 0 \) for \( j \geq N+2 \). For \( K = 0 \), iterating (3.9) with respect to \( j \) and plugging (3.3) in the resulting expression, one can obtain the expressions for \( \pi(0, j), j \geq N+3 \) as given in (3.4).
So far, we have determined the joint probabilities \( \pi(0, j) \), \( j \geq N + 1 \). Next, we focus our attention in deriving the remaining joint probabilities \( \pi(1, j) \), \( j \geq N + 1 \). To this end, we use (3.2) and (3.3) in (2.4) and rearrange terms leading to the desired results for (3.6). Next, we rewrite (2.5) as

\[
\pi(1, j) = \frac{1}{\nu} [\lambda + \alpha + j(\mu + \xi)] \pi(0, j) - (j + 1)\xi \pi(0, j + 1), \quad j \geq N + 2.
\]

Substitution of (3.3) and (3.4) in the above expression gives the probability \( \pi(1, N + 2) \) as in (3.7). As before, using (3.4) recursively in (2.5), we obtain the expressions for \( \pi(1, j) \), \( j \geq N + 3 \) as given by (3.7).

Observe that all the joint probabilities \( \pi(0, j) \), \( j \geq N + 1 \), and \( \pi(1, j) \), \( j \geq 0 \), are expressed in terms of \( \pi(0, 0) \). Finally, we can get the unknown joint probability \( \pi(0, 0) \) by exploiting the normalization condition

\[
\pi(0, 0) + \sum_{j=N+1}^{\infty} \pi(0, j) + \sum_{j=0}^{\infty} \pi(1, j) = 1,
\]

implying

\[
\pi(0, 0) = \begin{cases} \\
\frac{\lambda \rho^{N+1}}{\alpha + (N + 1)\mu} + \frac{(1 - \rho^{N+2})}{(1 - \rho)} \\
+ \frac{\lambda^2 \rho^{N+1}[\lambda + \alpha + (N + 1)\mu]}{[\alpha + (N + 1)\mu][\alpha \nu + (N + 2)((\lambda + \nu)\xi + \mu\nu)]} \\
+ \sum_{j=N+3}^{\infty} \frac{\lambda^{j-N} \rho^{N+1}[\lambda + \alpha + (N + 1)\mu]}{[\alpha + (N + 1)\mu][\alpha \nu + (N + 2)((\lambda + \nu)\xi + \mu\nu)]} \\
\times \prod_{k=N+3}^{j} \frac{[\lambda + \alpha + (k - 1)(\mu + \xi)]}{[\alpha \nu + k((\lambda + \nu)\xi + \mu\nu)]} \\
+ \frac{\lambda \rho^{N+1}[\lambda + \alpha + (N + 1)\mu]}{[\alpha + (N + 1)\mu][\alpha \nu + (N + 2)((\lambda + \nu)\xi + \mu\nu)]} \\
+ \sum_{j=N+2}^{\infty} \frac{\lambda^{j-N} \rho^{N+1}[\lambda + \alpha + (N + 1)\mu]}{[\alpha + (N + 1)\mu][\alpha \nu + (j + 1)((\alpha + \nu)\xi + \mu\nu)]} \\
\times \prod_{k=N+2}^{j} \frac{[\lambda + \alpha + k(\mu + \xi)]}{[\alpha \nu + k((\lambda + \nu)\xi + \mu\nu)]}^{-1}, & \text{if } \rho \neq 1,
\end{cases}
\]
\[ \pi(0,0) = \begin{cases} 
\frac{\lambda}{\alpha + (N + 1)\mu} + (N + 2) + \frac{\lambda}{\alpha + (N + 1)\mu} \left[ \frac{\lambda + \alpha + (N + 1)\mu}{\alpha + (N + 1)\mu}\right] \\
\lambda^{j-N-1} \left[ \frac{\lambda + \alpha + (N + 1)\mu}{\alpha + (N + 1)\mu}\right] \\
\times \prod_{k=N+3}^{j} \left[ \frac{\lambda + \alpha + (k-1)(\mu+\xi)}{\lambda\alpha + k(\mu+2\xi)} \right] \\
\lambda^{j-N-1} \left[ \frac{\lambda + \alpha + (N + 1)\mu}{\alpha + (N + 1)\mu}\right] \\
\times \prod_{k=N+3}^{j} \left[ \frac{\lambda + \alpha + k(\mu+\xi)}{\lambda\alpha + k(\mu+2\xi)} \right]^{-1}, & \text{if } \rho \neq 1 \\
1 + \frac{\lambda}{\alpha + (\mu + 2\xi)} F(1, A + 1; E + 2; G) + \frac{\lambda + \alpha + (\mu + \xi)}{\alpha + (\mu + 2\xi)} \\
\times F(1, A + 1; C + 2; E + 2, C + 1; G) \right]^{-1}, & \text{if } \rho = 1,
\end{cases} \]

Utilizing the hypergeometric series, the above series expressions for \( \pi(0,0) \) will turn out to be (3.1). Hence, the joint steady-state probability distribution of the status of the server and the number of customers in the orbit can be expressed in terms of generalized hypergeometric functions.

**Remark 2.** It is observed that the generalized hypergeometric series (3.1) is convergent if and only if the CTMC \( \{X(t), t \geq 0\} \) is ergodic.

**Remark 3.** If \( \mu = 0, \alpha > 0 \) and \( \nu > 0 \), then the joint steady-state probability distribution of the status of the server and the number of customers in the orbit for the constant retrial discipline can be deduced from Theorem 2.1 as a special case.

To conclude this section, we summarize in the following Corollary, some explicit results for the predefined value \( N = 0 \).

**Corollary 1.** If the CTMC \( \{X(t), t \geq 0\} \) is ergodic, then in the special case \( N = 0, \mu > 0 \) and \( \alpha \geq 0 \), the joint steady-state probability distribution of the status of the server and the number of customers in the orbit is given by

\[
\pi(0,0) = \left\{ 
\begin{array}{ll}
1 + \frac{\lambda^2}{\alpha \nu + ((\lambda + \nu)\xi + \mu\nu)} F(1, A + 1; B + 2; D) \\
+ \frac{\lambda}{\alpha \nu + ((\lambda + \nu)\xi + \mu\nu)} \left[ \frac{\lambda + \alpha + (\mu + \xi)}{\alpha \nu + ((\lambda + \nu)\xi + \mu\nu)} \right] \\
\times F(1, A + 1, C + 2; B + 2, C + 1; D) \right]^{-1}, & \text{if } \rho \neq 1 \\
1 + \frac{\lambda}{\alpha + (\mu + 2\xi)} F(1, A + 1; E + 2; G) + \frac{\lambda + \alpha + (\mu + \xi)}{\alpha + (\mu + 2\xi)} \\
\times F(1, A + 1; C + 2; E + 2, C + 1; G) \right]^{-1}, & \text{if } \rho = 1,
\end{array} \right.
\]

(3.13)
From (3.1)-(3.7), we deduce that

$$ P_0(z) = \pi(0,0) + \sum_{j=N+1}^{\infty} \pi(0,j) z^j \quad \text{and} \quad P_1(z) = \sum_{j=0}^{\infty} \pi(1,j) z^j, \quad |z| \leq 1. $$

From (3.1)-(3.7), we deduce that

$$ P_0(z) = \left[ 1 + \frac{\lambda z \rho^{N+1}}{\alpha + (N+1)\mu} + \frac{\lambda \nu (z \rho)^{N+2}[\lambda + \alpha + (N+1)\mu]}{(\alpha + (N+1)\mu)[\alpha \nu + (N+2)((\lambda + \nu)\xi + \mu \nu)]} \right] \pi(0,0), \quad \text{if } \rho \neq 1, $$

(4.1)
\[ P_0(z) = \left[ 1 + \frac{\lambda z^{N+1}}{(\alpha + (N+1)\mu)} + \frac{\lambda z^{N+2}[\lambda + \alpha + (N+1)\mu]}{(\alpha + (N+1)\mu)[\alpha + (N+2)(\mu + 2\xi)]} \right] \times F(1, A + N + 2; E + N + 3; Gz) \pi(0,0), \quad \text{if } \rho = 1, \]

and

\[
\begin{align*}
P_1(z) = \begin{cases} 
\frac{\rho(1 - (z\rho)^{N+1})}{(1 - z\rho)} + (z\rho)^{N+1}\frac{[\lambda + \alpha + (N+1)\mu][C + N + 2]}{[\alpha + (N+1)\mu][B + N + 2]} & D \\
\frac{1 - z^{N+1}}{1 - z} + z^{N+1}\frac{[\lambda + \alpha + (N+1)\mu][C + N + 2]}{[\alpha + (N+1)\mu][E + N + 2]} & G \\
	imes F(1, A + N + 2, C + N + 3; B + N + 3, C + N + 2; Dz) \pi(0,0), & \text{if } \rho \neq 1, \\
\end{cases} \\
\frac{1}{\rho(1 - (z\rho)^{N+1})} + (z\rho)^{N+1}\frac{[\lambda + \alpha + (N+1)\mu][C + N + 2]}{[\alpha + (N+1)\mu][B + N + 2]} & D \\
\frac{1 - z^{N+1}}{1 - z} + z^{N+1}\frac{[\lambda + \alpha + (N+1)\mu][C + N + 2]}{[\alpha + (N+1)\mu][E + N + 2]} & G \\
	imes F(1, A + N + 2, C + N + 3; B + N + 3, C + N + 2; Dz) \pi(0,0), & \text{if } \rho = 1, \\
\end{cases}
\]

where \( \pi(0,0) \) is given by (3.1). Our next aim is to compute explicit expressions for the partial factorial moments for the dynamic service system under the ergodicity condition (2.2).

**Theorem 4.1.** Let \( M_k^i \), for \( i \in \{0,1\} \) and \( k \geq 1 \), denote the partial \( k \)th factorial moments of the number of customers in the orbit given that the status of the server is \( i \), defined as

\[
M_k^i = \sum_{j=k}^{\infty} j(j-1) \cdots (j-k+1) \pi(i,j).
\]

Then, the partial \( k \)th factorial moments for \( k \geq 1 \) are:

For \( \rho \neq 1 \),

\[
M_k^0 = \begin{cases} 
\frac{(N+1)!\lambda \rho^{N+1}}{(N-k+1)![\alpha + (N+1)\mu]} + \frac{(N+2)!}{(N-k+2)!} \\
\times \frac{\lambda^2 \rho^{N+1}[\lambda + \alpha + (N+1)\mu]}{[\alpha + (N+1)\mu][\alpha \nu + \alpha \nu + (N+2)(\lambda + \nu)\xi + \mu \nu]} \\
\times F(1, N + 3, A + N + 2; N - k + 3, B + N + 3; D) \pi(0,0), & \text{if } k \leq N + 1, \\
\end{cases}
\]

\[
\begin{align*}
&\left[ k!^2 \mu^2 \rho^{N+1}[\lambda + \alpha + (N+1)\mu] \\
&\times \frac{(A + N + 2)_{k-(N+2)}}{(B + N + 3)_{k-(N+2)}} \frac{(D)^{k-(N+2)}}{[\alpha + (N+1)\mu][\alpha \nu + (N+2)((\lambda + \nu)\xi + \mu \nu)]} \\
&\times F(k+1, A + k; B + k + 1; D) \pi(0,0), & \text{if } k \geq N + 2,
\end{align*}
\]

\[
\]

\[
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\[
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\[
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\[
\]
and for $\rho = 1$, 

$$
M_k^0 = \begin{cases}
\dfrac{\lambda(N + 1)!}{(N - k + 1)!}\dfrac{[\alpha + (N + 1)\mu]}{[\alpha + (N + 2)\mu + 2\xi]} \\
(\mu + \nu)\lambda^{N + 1}\dfrac{[\lambda + \alpha + (N + 1)\mu]}{[\alpha + (N + 1)\mu + (N + 2)\mu + 2\xi]} \\
\times F(1, N + 3, A + N + 2; N - k + 3, E + N + 3; G) \pi(0, 0), & \text{if } k \leq N + 1,
\end{cases}
$$

For $\rho \neq 1$, 

$$
M_k^0 = \begin{cases}
\sum_{m=k}^N \dfrac{m!}{(m - k)!}\rho^{m+1} \\
\dfrac{(N + 1)!}{(N - k + 1)!}\dfrac{[\alpha + (N + 1)\mu]}{[\alpha + (N + 2)\mu + 2\xi]} \\
\times F(1, N + 2, A + N + 2, C + N + 3; N - k + 2, B + N + 3, C + N + 2; D) \pi(0, 0), & \text{if } k \leq N,
\end{cases}
$$

and for $\rho = 1$, 

$$
M_k^1 = \begin{cases}
\sum_{m=k}^N \dfrac{m!}{(m - k)!} \\
\dfrac{(N + 1)!}{(N - k + 1)!}\dfrac{[\alpha + (N + 1)\mu]}{[\alpha + (N + 2)\mu + 2\xi]} \\
\times F(1, N + 2, A + N + 2, C + N + 3; N - k + 2, E + N + 3, C + N + 2; G) \pi(0, 0), & \text{if } k \leq N,
\end{cases}
$$
Moreover, the $k^{th}$ factorial moments, $M_k$, of the number of customers in the orbit are given by

$$M_k = M_k^0 + M_k^1, \quad k = 1, 2, 3, \ldots.$$  

The proof follows by taking successive derivatives of (4.1) and (4.2) for $k$-times with respect to $z$ at $z = 1$ after tedious algebraic calculations. In particular, the first and second partial factorial moments are derived as:

$$M_1^0 = P_0'(1) = \begin{cases} \frac{(N + 1)\lambda}{\alpha + (N + 1)\mu} + \frac{(N + 2)\lambda \rho^{\nu + 1}(\lambda + \alpha + (N + 1)\mu)}{\alpha + (N + 1)\mu[\alpha + (N + 2)(\mu + 2\xi)]} \times F(1, N + 3, A + N + 2; B + N + 3, N + 2; D) \pi(0, 0), & \text{if } \rho \neq 1, \\ \frac{(N + 1)\lambda}{\alpha + (N + 1)\mu} + \frac{(N + 2)\lambda \rho^{\nu + 1}(\lambda + \alpha + (N + 1)\mu)}{\alpha + (N + 1)\mu[\alpha + (N + 2)(\mu + 2\xi)]} \times F(1, N + 3, A + N + 2; E + N + 3, N + 2; G) \pi(0, 0), & \text{if } \rho = 1, \end{cases} \quad (4.5)$$

$$M_1^1 = P_1'(1) = \begin{cases} \frac{\rho \rho^2}{(1 - \rho)^2}[1 - N(1 - \rho)\rho^{N + 1} - \rho^N] \\ + \frac{(N + 1)\rho^{N + 1}(\lambda + \alpha + (N + 1)\mu)[C + N + 2]}{\alpha + (N + 1)\mu[B + N + 2]} D \\ \times F(1, N + 2, A + N + 2; C + N + 3; N + 1, B + N + 3, C + N + 2; D) \pi(0, 0), & \text{if } \rho \neq 1, \\ \frac{N(N + 1)}{2} + \frac{(N + 1)\lambda \rho^{\nu + 1}(\lambda + \alpha + (N + 1)\mu)}{\alpha + (N + 1)\mu[E + N + 2]} G \\ \times F(1, N + 2, A + N + 2; C + N + 3; N + 1, E + N + 3, C + N + 2; G) \pi(0, 0), & \text{if } \rho = 1. \end{cases} \quad (4.6)$$

$$M_2^0 = P_0''(1) = \begin{cases} \frac{N(N + 1)\lambda \rho^{\nu + 1}}{\alpha + (N + 1)\mu} \\ + \frac{(N + 1)(N + 2)\lambda^2 \rho^{\nu + 1}(\lambda + \alpha + (N + 1)\mu)}{\alpha + (N + 1)\mu[\alpha + (N + 2)(\mu + 2\xi)]} \times F(1, N + 3, A + N + 2; N + 1, B + N + 3; D) \pi(0, 0), & \text{if } \rho \neq 1, \\ \frac{N(N + 1)\lambda}{\alpha + (N + 1)\mu} + \frac{(N + 1)(N + 2)\lambda \rho^{\nu + 1}(\lambda + \alpha + (N + 1)\mu)}{\alpha + (N + 1)\mu[\alpha + (N + 2)(\mu + 2\xi)]} \\ \times F(1, N + 3, A + N + 2; N + 1, E + N + 3; G) \pi(0, 0), & \text{if } \rho = 1, \end{cases} \quad (4.7)$$
and

\[
M_2^1 = P_1''(1) = \begin{cases} 
\sum_{m=1}^{N-1} m(m+1)\rho^{m+2} + \frac{N(N+1)\rho^{N+1}[\lambda + \alpha + (N+1)\mu][C + N + 2]}{[\alpha + (N+1)\mu][B + N + 2]} D \\
\times F(1, N + 2, A + N + 2, C + N + 3; N, B + N + 3, C + N + 2; D) \pi(0, 0), & \text{if } \rho \neq 1, \\
\sum_{m=1}^{N-1} m(m+1) + \frac{N(N+1)\lambda + \alpha + (N+1)\mu}[C + N + 2]}{[\alpha + (N+1)\mu][E + N + 2]} G \\
\times F(1, N + 2, A + N + 2, C + N + 3; N, E + N + 3, C + N + 2; G) \pi(0, 0), & \text{if } \rho = 1,
\end{cases}
\]

where \(\pi(0, 0)\) is defined in (3.1).

As the mean and variance of the number of customers in the orbit are key performance measures for the system under investigation, these quantities can be calculated immediately from the above results.

5. Performance measures. In this section, we present some key performance measures for our dynamic oscillating service system under the ergodicity condition:

1. The probability that the server is busy as

\[
P_1(1) = \begin{cases} 
\rho(1 - \rho)^{N+1} + \rho^{N+1}[\lambda + \alpha + (N+1)\mu][C + N + 2]}{[\alpha + (N+1)\mu][B + N + 2]} D \\
\times F(1, A + N + 2, C + N + 3; B + N + 3, C + N + 2; D) \pi(0, 0), & \text{if } \rho \neq 1, \\
(N + 1) + \frac{[\lambda + \alpha + (N+1)\mu][C + N + 2]}{[\alpha + (N+1)\mu][E + N + 2]} G \\
\times F(1, A + N + 2, C + N + 3; E + N + 3, C + N + 2; G) \pi(0, 0), & \text{if } \rho = 1.
\end{cases}
\]
2. The probability that the server is idle during the time intervals between retrial attempts as

\[
P_0(1) = \begin{cases} 
\left( 1 + \frac{\lambda \rho^{N+1}}{[\alpha + (N + 1)\mu]} + \frac{\lambda^2 \rho^{N+1} [\lambda + \alpha + (N + 1)\mu]}{[\alpha + (N + 1)\mu][\alpha \nu + (N + 2)(\lambda + \nu)\xi + \mu\nu]} \right) \times F(1, A + N + 2; B + N + 3; D) \pi(0, 0), & \text{if } \rho \neq 1, \\
\left( 1 + \frac{\lambda}{[\alpha + (N + 1)\mu]} + \frac{\lambda [\lambda + \alpha + (N + 1)\mu]}{[\alpha + (N + 1)\mu][\alpha + (N + 2)(\mu + 2\xi)]} \right) \times F(1, A + N + 2; E + N + 3; G) \pi(0, 0), & \text{if } \rho = 1.
\end{cases}
\]

(5.2)

3. The mean number, \(E(X_Q)\), of customers in the orbit is given by

\[E(X_Q) = M_0^0 + M_1^1.\]

(5.3)

4. The mean number, \(E(X_S)\), of customers in the system is obtained as

\[E(X_S) = M_0^0 + M_1^1 + P_1(1).\]

(5.4)

5. The regeneration cycle, \(T\), of our system is the time elapsed between two consecutive primary customer arrivals finding the system empty. Thus the mean length of the regeneration cycle is

\[
E(T) = \begin{cases} 
\frac{1}{\pi(0, 0)} \left[ \frac{(1 - \rho^{N+2})}{\lambda(1 - \rho)} + \frac{\rho^{N+1}}{[\alpha + (N + 1)\mu]} \right. \\
\left. + \frac{\lambda \rho^{N+1} [\lambda + \alpha + (N + 1)\mu]}{[\alpha + (N + 1)\mu][\alpha \nu + (N + 2)(\lambda + \nu)\xi + \mu\nu]} \right] \times F(1, A + N + 2; B + N + 3; D) \frac{\rho^{N+1} [\lambda + \alpha + (N + 1)\mu][C + N + 2]}{\lambda [\alpha + (N + 1)\mu][B + N + 2]} D \\
\times F(1, A + N + 2, C + N + 3; B + N + 3, C + N + 2; D); & \text{if } \rho \neq 1, \\
\left[ \frac{(N + 2)}{\lambda} + \frac{1}{[\alpha + (N + 1)\mu]} + \frac{[\lambda + \alpha + (N + 1)\mu]}{[\alpha + (N + 1)\mu][\alpha + (N + 2)(\mu + 2\xi)]} \right. \\
\left. \times F(1, A + N + 2; E + N + 3; G) + \frac{[\lambda + \alpha + (N + 1)\mu][C + N + 2]}{\lambda [\alpha + (N + 1)\mu][E + N + 2]} G \\
\times F(1, A + N + 2, C + N + 3; E + N + 3, C + N + 2; G) \right] & , & \text{if } \rho = 1.
\end{cases}
\]

(5.5)
6. The expected amount, $E(T_B)$, of time in a regenerative cycle during which the server being busy is derived as

$$E(T_B) = P_1(1) \ E(T) = \frac{P_1(1)}{\lambda \pi(0,0)}. \quad (5.6)$$

7. The expected amount, $E(T_I)$, of time in a regenerative cycle during which the server is idle is given by

$$E(T_I) = P_0(1) \ E(T) = \frac{P_0(1)}{\lambda \pi(0,0)}. \quad (5.7)$$

8. The total sojourn time, $W_S$, of a customer in the system is measured from the moment of its primary arrival until departure, either by completion of service or as a result of abandonment. By Little's law, the mean sojourn time $E(W_S)$ in the system is determined as

$$E(W_S) = \frac{1}{\lambda} \left[ M_1^0 + M_1^1 + P_1(1) \right] = \frac{E(X_S)}{\lambda}. \quad (5.8)$$

9. The busy period, $L$, of our dynamic service system is defined as the period that starts at the epoch when an arriving primary customer finds an empty system (i.e., the server is idle and no customer is in the orbit) and ends at the departure epoch of the customer due to either completion of service or reneging at which the system becomes empty again. The mean length, $E(L)$, of the system busy period of our service system is obtained directly by the theory of regenerative process as

$$E(L) = \frac{1}{\lambda} \left[ \frac{1}{\pi(0,0)} - 1 \right] = E(T) - \frac{1}{\lambda}$$

$$= \left\{ \begin{array}{ll}
\frac{1}{\lambda} \left[ \frac{\lambda \rho^{N+1}}{\alpha + (N+1)\mu} + \frac{\rho(1 - \rho^{N+1})}{(1 - \rho)} \right] \\
\frac{\lambda^2 \rho^{N+1}}{[\alpha + (N+1)\mu][\alpha + (N+1)\mu]} \left[ \frac{1}{\alpha + (N+1)\mu} \right] F(1, A + N + 2; B + N + 3; D) \\
\frac{\rho^{N+1}[\lambda + \alpha + (N+1)\mu][C + N + 2]}{[\alpha + (N+1)\mu][B + N + 2]} D \\
\times F(1, A + N + 2, C + N + 3; B + N + 3, C + N + 2; D) \\
\end{array} \right. \right. \quad \text{if } \rho \neq 1,$$

$$= \left\{ \begin{array}{ll}
\frac{1}{\lambda} \left[ \frac{\lambda}{\alpha + (N+1)\mu} + (N + 1) + \frac{\lambda [\lambda + \alpha + (N+1)\mu]}{[\alpha + (N+1)\mu][\alpha + (N+2)(\mu + 2\xi)]} \right] \\
\times F(1, A + N + 2; E + N + 3; G) + \frac{[\lambda + \alpha + (N+1)\mu][C + N + 2]}{[\alpha + (N+1)\mu][E + N + 2]} G \\
\times F(1, A + N + 2, C + N + 3; E + N + 3, C + N + 2; G) \\
\end{array} \right. \right. \quad \text{if } \rho = 1. \quad (5.9)
10. When the system is in state \((0, j), j \geq N+2\), the average rate of abandonment due to impatience is calculated as

\[
R_A = \sum_{j=N+2}^{\infty} j \xi \pi(0, j) = \xi [P_0^1(1) - (N+1) \pi(0, N+1)]
\]

\[
= \left\{ \begin{array}{ll}
\xi \left[ \frac{(N+2) \lambda^2 \rho^{N+1}[\lambda + \alpha + (N+1)\mu]}{[\alpha + (N+1)\mu][\alpha + \nu + (N+2)((\lambda + \nu)\xi + \mu\nu)]} \times F(1, N + 3, A + N + 2; B + N + 3, N + 2; D) \right] \pi(0, 0), & \text{if } \rho \neq 1, \\
\xi \left[ \frac{(N+2)\lambda[\lambda + \alpha + (N+1)\mu]}{[\alpha + (N+1)\mu][\alpha + \nu + (N+2)(\mu + 2\xi)]} \times F(1, N + 3, A + N + 2; E + N + 3, N + 2; G) \right] \pi(0, 0), & \text{if } \rho = 1. 
\end{array} \right.
\]

11. The proportion of customers served is computed as

\[
P_S = \frac{\nu P_1(1)}{\lambda} = \frac{P_1(1)}{\lambda}
\]

\[
= \left\{ \begin{array}{ll}
\frac{1}{\rho} \left[ \frac{\rho(1 - \rho^{N+1})}{(1 - \rho)} + \rho^{N+1} \frac{[\lambda + \alpha + (N+1)\mu][C + N + 2]}{[\alpha + (N+1)\mu][B + N + 2]} D \\
\times F(1, A + N + 2, C + N + 3; B + N + 3, C + N + 2; D) \right] \pi(0, 0), & \text{if } \rho \neq 1, \\
\frac{(N+1) + [\lambda + \alpha + (N+1)\mu][C + N + 2]}{[\alpha + (N+1)\mu][E + N + 2]} G \\
\times F(1, A + N + 2, C + N + 3; E + N + 3, C + N + 2; G) \right] \pi(0, 0), & \text{if } \rho = 1. 
\end{array} \right.
\]

6. **First passage time analysis.** The investigation of the random variables associated with first passage times is highly essential for various problems in Markovian queueing networks. In fact, it would be helpful for computing the performance measures of the state-dependent arrival and service rates in complex queueing network systems.

In the following analysis, we focus on evaluating the mean and variance of the first passage time random variable for the oscillating service system between conventional queue and retrial queue operation modes under the assumption that the CTMC \(\{X(t), t \geq 0\}\) is ergodic.

To do so, let us now define the random variable, \(\tau_j\), which is the down crossing time until first passage to state \((1, N)\) starting from state \((0, j)\). Here \(j, (j \geq N+1)\), represents the number of customers in the orbit while the system is in the retrial queueing mode. The states of the system and transition rates are shown in Figure 1. Recall that a customer leaves the orbit either by starting service due to the successful
retrial attempt or by an abandonment (customers’ impatience). Jouini and Dallery [29] have obtained closed-form expressions for the moments of order \( k, \quad (k \geq 1) \), of random variables related to first passage times of a general birth-death process. We now adopt their results in our dynamic oscillating service system context to get the first two moments of the first passage time random variable \( \tau_j \) for \( j \geq N + 1 \).

To this end, we define the potential coefficients, say \( \phi_j \), of our oscillating queueing system as follows:

For \( j \geq N + 1 \),

\[
\phi_N = 1 \quad \text{and} \quad \phi_j = \prod_{k=N+1}^{j} \frac{\alpha_{k-1}}{\beta_k},
\]

where

\[
\alpha_N = \alpha_{N+1} = \lambda, \quad \beta_{N+1} = \nu, \quad \alpha_j = \lambda[\lambda + \alpha + j(\mu + \xi)] \quad \text{and} \quad \beta_j = [\alpha \nu + j((\lambda + \nu)\xi + \mu \nu)], \quad j \geq N + 2,
\]

such that the regularity condition

\[
\sum_{j=N}^{\infty} \frac{1}{\alpha_j \phi_j} \sum_{k=N}^{j} \phi_k = \infty,
\]

is fulfilled due to the ergodicity condition (Anderson [12] and Whitt [41]). Thus, from Jouini and Dallery [29], the mean, \( E(\tau_j) \), and the second moment, \( E(\tau_j^2) \), of \( \tau_j \) can be obtained for \( j \geq N + 1 \), after laborious calculation and making use of hypergeometric series, as

\[
E(\tau_j) = \sum_{k=N+1}^{j} \frac{1}{\beta_k \phi_k} \sum_{m=k}^{\infty} \phi_m
\]

\[
\begin{cases}
\frac{1}{\nu} + \frac{\rho}{[\alpha \nu + (N+2)((\lambda + \nu)\xi + \mu \nu)]} F(1, A + N + 2; B + N + 3; D) \\
+ \sum_{k=N+2}^{j} \frac{1}{[\alpha \nu + k((\lambda + \nu)\xi + \mu \nu)]} F(1, A + k; B + k + 1; D),
\end{cases}
\]

\[
= \begin{cases}
\frac{1}{\lambda} + \frac{1}{\lambda(\alpha + (N+2)(\mu + 2 \xi))} F(1, A + N + 2; E + N + 3; G) \\
+ \sum_{k=N+2}^{j} \frac{1}{\lambda(\alpha + k(\mu + 2 \xi))} F(1, A + k; E + k + 1; G),
\end{cases}
\]

if \( \rho \neq 1 \),

\[
\text{and}
\]

\[
E(\tau_j^2) = 2 \sum_{k=N+1}^{j} \frac{1}{\beta_k \phi_k} \sum_{m=k}^{\infty} \frac{1}{\beta_m \phi_m} \left( \sum_{l=m}^{\infty} \phi_l \right)^2,
\]

if \( \rho = 1 \),

(6.1)
whence

\[
E(\tau_j^2) = \begin{cases} 
2 \left[ \left( \frac{1}{\nu} + \frac{\rho F(1, A + N + 2; B + N + 3; D)}{[\alpha
\nu + (N + 2)(\lambda + \nu)\xi + \mu\nu]} \right) \right]^2 \\
+ \rho \left( \frac{F(1, A + N + 2; B + N + 3; D)}{[\alpha
\nu + (N + 2)(\lambda + \nu)\xi + \mu\nu]} \right)^2 \\
+ \rho \sum_{l=N+3}^\infty \left( \frac{F(1, A + l; B + l + 1; D)}{[\alpha
\nu + l(\lambda + \nu)\xi + \mu\nu]} \right)^2 \times \prod_{s=N+3}^l \left( \frac{(A + s - 1)}{(B + s - 1)} \right) \\
+ \sum_{k=l+1}^j \left( \frac{F(1, A + k; B + k + 1; D)}{[\alpha
\nu + k(\lambda + \nu)\xi + \mu\nu]} \right)^2 \times \prod_{s=l+1}^k \left( \frac{(A + s - 1)}{(B + s - 1)} \right) \right], & \text{if } \rho \neq 1,
\end{cases}
\]

Herein, we used the condition \( \left( \sum_{j=N}^\infty \phi_j \right)^{-1} \left( \sum_{m=N}^\infty \beta_m \phi_m \left( \sum_{j=m+1}^\infty \phi_j \right)^2 \right) < \infty \) for the existence of \( E(\tau_j^2) \) (see Jouini and Dallery [30]).

Finally, knowing that \( \text{Var}(\tau_j) = E(\tau_j^2) - (E(\tau_j))^2 \) for \( j \geq N + 1 \).

In the following analysis, we investigate a couple of characteristics pertaining to the conventional queue before visiting state \((1, N)\).

First, we compute the hitting probabilities in our dynamic oscillating service system namely, the probabilities \( U_j \) that the system size hits state \((1, N)\) before hitting state \((0, 0)\) starting from an initial state \((1, j)\), \( j \leq N \), in the classical queueing mode. By the notion of ruin probabilities (Taylor and Karlin [39]), it can be shown that
\[ U_j = \begin{cases} 1 - \left( \frac{\nu}{\lambda} \right)^j, & \text{if } j = 1, 2, 3, \ldots, N-1, \text{ and } \rho \neq 1, \\ 1, & \text{if } j = 1, 2, 3, \ldots, N-1, \text{ and } \rho = 1, \\ j/N, & \text{if } j = 1, 2, 3, \ldots, N, \end{cases} \]  

(6.3)

with \( U_0 = 0 \) and \( U_N = 1 \).

On the other hand, we can compute the complementary probabilities \( V_j = 1 - U_j \), \( j = 0, 1, 2, \ldots, N \), that the system size hits state \((0, 0)\) before hitting state \((1, N)\) starting from an initial state \((1, j)\), \( j \leq N \), in the classical queueing mode. This result is a straightforward calculation and hence we omit it here.

Second characterisation is the expected number, \( E(X_{CQ}) \), of customers in the conventional queueing mode before it turns into retrial queueing mode, i.e., before reaching state \((1, N+1)\). This quantity is measured as

\[ E(X_{CQ}) = \sum_{j=0}^{N} (j+1)\pi(1,j) = \pi(0,0) \sum_{j=0}^{N} (j+1)\rho^{j+1}, \]

where we have used the results given in (3.5). Hence, using a little algebra and calculus, the above expression becomes,

\[ E(X_{CQ}) = \begin{cases} \frac{\rho}{(1-\rho)^2} \left( 1 - (N+2)\rho^{N+1}(1-\rho) - \rho^{N+2} \right) \pi(0,0), & \text{if } \rho \neq 1, \\ \frac{(N+1)(N+2)}{2} \pi(0,0), & \text{if } \rho = 1, \end{cases} \]

(6.4)

where \( \pi(0,0) \) is defined in (3.1).

7. **Numerical illustrations and sensitivity analysis.** In this section, we present some numerical results that illustrate graphically the qualitative behavior of the performance characteristics of the proposed system. To this purpose, we study the influence of the system parameters on the following performance measures of our dynamic oscillating queueing system:

- the joint probability \( \pi(0,0) \) that the server is in idle status and no customers in the system,
- the average busy period, \( E(L) \), of the system,
- the average total sojourn time, \( E(W_S) \), of an arbitrary customer in the system,
- the average rate, \( R_A \), of abandonment of customers due to impatience,
- the proportion, \( P_S \), of customers served successfully,
- the mean, \( E(\tau_j) \), of first-passage time to level \((1, N)\) starting from level \((0, j)\) for \( j > N \).

In all our numerical experiments, we have chosen the parametric values satisfying the stability condition given in (2.2).

In Figure 2(a) – 2(e), we present some graphs to show the performance measures such as \( \pi(0,0) \), \( E(L) \), \( E(W_S) \), \( R_A \) and \( P_S \) that are affected by the intensity of impatience \( \xi \) for the set of parametric values \((\alpha, \mu, N) = (3, 4, 5)\). Besides, in each figure, three curves are drawn according to the traffic load \( \rho = 0.9, 1.0 \) and \( 1.1 \). Figure 2(a) shows that the probability \( \pi(0,0) \) increases with increasing values of \( \xi \), whereas it decreases for increasing values of \( \rho \). This is due to the fact that increase in the intensity of impatience may cause reneging of more customers from the orbit. Hence the system becomes empty with high probability. However, \( \pi(0,0) \) decreases
as the traffic load $\rho$ increases as expected. Figure 2(b) illustrates the impact of $\xi$ on the mean busy period length $E(L)$. Note that the descriptor $E(L)$ decreases significantly for increasing values of $\xi$ in contrast to the behavior of $\pi(0, 0)$, but it increases with respect to $\rho$ for a fixed value of $\xi$. We now depict the mean sojourn time, $E(W_S)$, of a customer in the system versus $\xi$ in Figure 2(c). It can be seen that the curves of $E(W_S)$ decrease rapidly in the beginning for increasing values of $\xi$ before stabilizing to their limiting values, but the curves of $E(W_S)$ increase for increasing values of traffic load $\rho$. Next, Figure 2(d) displays the trend of the mean rate, $R_A$, of abandonment of a customer from the orbit against $\xi$. It is shown that the descriptor $R_A$ increases in both $\xi$ and $\rho$. This phenomenon is explained as follows. While the traffic load increases, the service system is occupied by more number of customers. This asserts that the impatient customers start leaving the system with high rate resulting in the increase of $R_A$. Finally, we present in Figure 2(e), the effect of $\xi$ on the proportion, $P_S$, of customers served successfully. When the intensity of impatience $\xi$ increases, the measure $P_S$ decreases rapidly in the beginning following a slow decrease as opposed to the trend of the descriptor $R_A$. Evidently, Figures 2(a) – 2(e) show the impact not only of $\xi$ but also of the traffic load $\rho$.

Next, we explore the sensitivity of the retrial intensity to various system performance measures. Figures 3(a) – 3(e) reveal the effect of the retrial intensity $\mu$ on the measures $\pi(0, 0)$, $E(L)$, $E(W_S)$, $R_A$ and $P_S$ for the fixed set of parametric values $(\alpha, \xi, N) = (3, 5, 5)$. As before, in each figure, three curves are plotted corresponding to the traffic load $\rho = 0.9$, 1.0, and 1.1. The probability, $\pi(0, 0)$, of empty system is analyzed as a function of $\mu$ and the results are plotted in Figure 3(a). Note that $\pi(0, 0)$ seems to be concave functions of $\mu$ for all three values of $\rho$. However, $\pi(0, 0)$ decreases for increasing values of $\rho$ for a fixed value of $\mu$ as expected. Figure 3(b) exhibits the effect of $\mu$ on the mean, $E(L)$, of busy period of the system for three values of system load $\rho$. Comparing Figures 3(b) with Figure 3(a), we observe that the behavior of $E(L)$ is completely opposite to the trend of $\pi(0, 0)$ as the function of both $\mu$ and $\rho$ which is in accordance with intuitive expectation. In Figure 3(c), we present the trend of the mean $E(W_S)$, which looks like the convex functions of $\mu$ for all the three values of $\rho$. In addition, the same tendency can also be observed in $E(L)$. The influence of $\mu$ on the mean rate, $R_A$, of abandonment is depicted in Figure 3(d). As $\mu$ increases, the descriptor $R_A$ decreases gradually in order to attain its limiting value. Moreover, the measure $R_A$ decreases much faster in $\rho = 0.9$ when compared to $\rho = 1$ and $\rho = 1.1$. The above phenomenon implies that for relatively high values of the retrial intensity, the reneging behaviour of impatient customers from the orbit can be prevented. Thus, the mean rate of abandonment, $R_A$, of customers decreases owing to prevention of customers’ impatience, whereas it increases with the increase of the system load $\rho$ as it should be.

The impact of $\mu$ on the proportion, $P_S$, of customers served successfully is studied in Figure 3(e). For very small values of $\mu$, the descriptor $P_S$ decreases and then it starts increasing for relatively high values of $\mu$. Further, the measure $P_S$ increases significantly for increasing values of the system load $\rho$ for a fixed value of $\mu$ as is to be expected.

Our next set of numerical examples deal with the mean, $E(\tau_j)$, of first-passage time at $(1, N)$ starting from $(0, j)$ for $j \geq N + 1$. In Figures 4(a) and 4(b), we demonstrate the behaviour of $E(\tau_j)$, parameterized by three values of $\rho = 0.9, 1.0$ and 1.1. The impact of varying the impatience intensity $\xi$ on $E(\tau_j)$ is sketched in
Figure 4(a) for the set of parameters \((\alpha, \mu, N, j) = (3, 4, 5, 10)\). We have noticed that \(E(\tau_j)\) decreases rapidly in \(\rho = 1.1\) when compared to \(\rho = 0.9\) and \(\rho = 1.0\) for increasing values of \(\xi\) as expected. Figure 4(b) describes the variation of \(E(\tau_j)\) with respect to the retrial intensity \(\mu\) for \((\xi, \alpha, N, j) = (5, 3, 5, 10)\). Clearly, the measure \(E(\tau_j)\) is a decreasing function of \(\mu\), whereas it is increasing in \(\rho\).

In Figures 4(c) – 4(e), we analyze the behaviour of the measure \(E(\tau_j)\) against the system parameters \(\xi, \mu\) and \(\lambda\) for varying the state \((0, j)\) with the number of customers \(j\) in the orbit, which is above the threshold \(N\). In each figure, three curves are plotted according to \(j = 10, 15\) and 20 for the chosen parametric values \((\lambda, \nu, \alpha, \mu, N) = (18, 20, 3, 4, 5)\). The trend of the mean \(E(\tau_j)\) is displayed in Figure 4(c) versus the values of \(\xi\), indicating a sharp decrease in the beginning for small values of \(\xi\) and then decreases slowly to attain its limiting value while \(\xi\) approaches larger values. We can also see that \(E(\tau_j)\) is an increasing function of \(j\), the number of customers in the orbit, for a fixed value of \(\xi\) as is to be expected. We now discuss the behaviour of \(E(\tau_j)\) as a function of \(\mu\). Results are plotted in Figure 4(d) for \((\lambda, \nu, \xi, \alpha, N) = (18, 20, 5, 3, 5)\). One can see that \(E(\tau_j)\) is a decreasing function of \(\mu\) and the rate of decrease being more prominent for decreasing values of \(j\) as is to be.

We plot in Figure 4(e), the influence of the primary arrival rate \(\lambda\) on \(E(\tau_j)\) for fixed set of parametric value \((\xi, \nu, \alpha, \mu, N) = (5, 20, 3, 4, 5)\). From a quick look at this figure, it appears that \(E(\tau_j)\) increases gradually for small values of \(\lambda\) and then starts growing rapidly while the arrival rate \(\lambda\) approaches larger values.

It is noteworthy that the trend of \(E(\tau_j)\) versus \((\lambda, j)\) in Figure 4(e) is completely opposite to the behaviour of \(E(\tau_j)\) against \((\xi, j)\) in Figure 4(c) which is consistent with our intuition.

Lastly, Figure 4(f) describes the behaviour of the mean \(E(\tau_j)\) versus the service rate \(\nu\) for fixed set of parametric value \((\lambda, \xi, \alpha, \mu, N) = (18, 5, 3, 4, 5)\). Evidently, \(E(\tau_j)\) is a decreasing function of \(\nu\), but it is increasing with \(j\) for a fixed value of \(\nu\). Moreover, while comparing Figures 4(c) and 4(f), it is observed that the trend of \(E(\tau_j)\) versus \(\nu\) shows completely opposite to \(E(\tau_j)\) against \(\lambda\) as is to be expected.

We encountered no numerical difficulties to provide some illustrations for studying the effect of parameters on the measures of the system for the range of values of the threshold size \(N\) of interest for the system under study.

8. Conclusion. In this report, we have analyzed a single server oscillating system between conventional and retrial queueing models with reneging of customers. The steady-state condition for the system to be stable and the joint steady-state probabilities of the server’s status and the number of customers in the system have been obtained. We have subsequently derived several key performance measures of the system under study. In particular, the mean busy period of the system, mean response time and mean first passage time which could be used to investigate the sensitivity analyses for different values of traffic load of the system. The proposed system can be applied to investigate the various measures of the optical fiber networks.

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