Rigidity and deformation of discrete conformal structures on polyhedral surfaces

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Abstract

Discrete conformal structure on polyhedral surfaces is a discrete analogue of the conformal structure on smooth surfaces, which includes tangential circle packing, Thurston’s circle packing, inversive distance circle packing and vertex scaling as special cases and generalizes them to a very general context. Glickenstein [40] conjectured the rigidity of discrete conformal structures on polyhedral surfaces, which includes Luo’s conjecture on the rigidity of vertex scaling [57] and Bowers-Stephenson’s conjecture on the rigidity of inversive distance circle packings [6] on polyhedral surfaces as special cases. We prove Glickenstein’s conjecture using a variational principle.

We further study the deformation of discrete conformal structures on polyhedral surfaces by combinatorial curvature flows. It is proved that the combinatorial Ricci flow for discrete conformal structures, which is a generalization of Chow-Luo’s combinatorial Ricci flow for circle packings [7] and Luo’s combinatorial Yamabe flow for vertex scaling [57], could be extended to exist for all time and the extended combinatorial Ricci flow converges exponentially fast for any initial data if the discrete conformal structure with prescribed combinatorial curvature exists. This confirms another conjecture of Glickenstein [40] on the convergence of the combinatorial Ricci flow and provides an effective algorithm for finding discrete conformal structures with prescribed combinatorial curvatures.

The relationship of discrete conformal structures on polyhedral surfaces and 3-dimensional hyperbolic geometry is also discussed. As a result, we obtain some new convexities of the co-volume functions for some generalized 3-dimensional hyperbolic tetrahedra.

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1 Introduction

Conformal structure on Riemannian manifolds is one of the most important geometric structures that has been extensively studied in differential geometry, which defines Riemannian metrics pointwisely by scalar functions defined on the manifolds. Discrete con-
formal structure on polyhedral manifolds is a discrete analogue of the conformal structure on Riemannian manifolds, which assigns the discrete metrics by scalar functions defined on the vertices. There have been many research activities on different types of discrete conformal structures on manifolds since the work of Thurston [81], including the tangential circle packing, Thurston’s circle packing, inversive distance circle packing and vertex scaling on surfaces, sphere packing and Thurston’s sphere packing on 3-dimensional manifolds and others. Most of the existing discrete conformal structures were invented and studied individually in the past. The generic notion of discrete conformal structures on polyhedral surfaces was introduced recently independently by Glickenstein [38] and Glickenstein-Thomas [41] from Riemannian geometry perspective and by Zhang-Guo-Zeng-Luo-Yau-Gu [96] from Bobenko-Pinkall-Springborn’s observation [3] on the relationships of vertex scaling on polyhedral surfaces and 3-dimensional hyperbolic geometry, which includes the discrete conformal structures from different types of circle packings and vertex scaling on polyhedral surfaces as special cases and generalizes them to a very general context.

In this paper, we study the geometry of generic discrete conformal structures on polyhedral surfaces. The global rigidity of discrete conformal structures on polyhedral surfaces is proved, which unifies and generalizes the known rigidity results for different types of circle packings and vertex scaling on polyhedral surfaces. We further use the combinatorial curvature flows, including the combinatorial Ricci flow and combinatorial Calabi flow, to study the deformation of discrete conformal structures on polyhedral surfaces. To handle the potential singularities along the combinatorial Ricci flow, we extend the combinatorial curvature by constants. It is proved that the solution of combinatorial Ricci flow could be extended to exist for all time and the extended combinatorial Ricci flow converges exponentially fast if the discrete conformal structure with prescribed combinatorial curvature exists. The longtime behavior of combinatorial Calabi flow is also discussed. The combinatorial curvature flows provide effective algorithms for finding discrete conformal structures with prescribed combinatorial curvatures. Motivated by Bobenko-Pinkall-Springborn’s observation [3] and Zhang-Guo-Zeng-Luo-Yau-Gu’s work [96], we further discuss the relationship of discrete conformal structures on polyhedral surfaces and 3-dimensional hyperbolic geometry. A natural geometric interpolation of the structure conditions on the weights used in the main results in terms of 3-dimensional hyperbolic geometry is given. As a corollary, some new convexities of co-volume functions for some generalized hyperbolic tetrahedra in $\mathbb{H}^3$ are obtained.
1.1 Polyhedral surfaces, discrete conformal structures and the main rigidity results

Polyhedral surface is a discrete analogue of Riemannian surface. Suppose \((M, T)\) is a connected closed triangulated surface with a triangulation \(T\), which is the quotient of a finite disjoint union of triangles by identifying all the edges of triangles in pair by homeomorphism. We use \(V, E, F\) to denote the set of vertices, edges and faces in the triangulation \(T\) respectively. For simplicity, we will use one index to denote a vertex (such as \(i \in V\) or \(v_i \in V\)), two indices to denote an edge (such as \(\{ij\} \in E\)) and three indices to denote a triangle (such as \(\{ijk\} \in F\)). Denote the set of positive real numbers as \(\mathbb{R}_{>0}\) and \(|V| = N\).

**Definition 1** ([62]). A polyhedral surface \((M, T, l)\) with background geometry \(G\) (\(G = \mathbb{E}^2, \mathbb{H}^2\) or \(S^2\)) is a triangulated surface \((M, T)\) with a map \(l : E \to \mathbb{R}_{>0}\) such that any face \(\{ijk\} \in F\) could be embedded in \(G\) as a nondegenerate triangle with edge lengths \(l_{ij}, l_{ik}, l_{jk}\) given by \(l\). \(l : E \to \mathbb{R}_{>0}\) is called a Euclidean (hyperbolic or spherical respectively) polyhedral metric if \(G = \mathbb{E}^2\) (\(G = \mathbb{H}^2\) or \(G = S^2\) respectively).

The nondegenerate condition for the face \(\{ijk\} \in F\) in Definition 1 is equivalent to the edge lengths \(l_{ij}, l_{ik}, l_{jk}\) satisfy the triangle inequalities \((l_{ij} + l_{ik} + l_{jk} < 2\pi\) additionally if \(G = S^2\)). Intuitively, a polyhedral surface with background geometry \(G\) (\(G = \mathbb{E}^2, \mathbb{H}^2\) or \(S^2\)) could be obtained by gluing triangles in \(G\) isometrically along the edges in pair. For polyhedral surfaces, there may exist conic singularities at the vertices, which could be described by combinatorial curvature. The combinatorial curvature is a map \(K : V \to (-\infty, 2\pi)\) that assigns the vertex \(i \in V\) \(2\pi\) less the sum of inner angles at \(i\), i.e.

\[
K_i = 2\pi - \sum_{\{ijk\} \in F} \theta_{i}^{jk},
\]

(1.1)

where \(\theta_{i}^{jk}\) is the inner angle at \(i\) in the triangle \(\{ijk\} \in F\) of the polyhedral surface \((M, T, l)\). The combinatorial curvature satisfies the following discrete Guass-Bonnet formula ([7] Proposition 3.1)

\[
\sum_{i \in V} K_i = 2\pi \chi(M) - \lambda \text{Area}(M),
\]

(1.2)

where \(\lambda = -1, 0, 1\) for \(G = \mathbb{H}^2, \mathbb{E}^2, S^2\) respectively and \(\text{Area}(M)\) denotes the area of the surface \(M\).

**Definition 2** ([41][96]). Suppose \((M, T)\) is a triangulated connected closed surface and \(\varepsilon : V \to \{-1, 0, 1\}\), \(\eta : E \to \mathbb{R}\) are two weights defined on the vertices and edges respectively with \(\eta_{ij} = \eta_{ji}\). A discrete conformal structure on the weighted triangulated surface \((M, T, \varepsilon, \eta)\) with background geometry \(G\) is a map \(f : V \to \mathbb{R}\) such that
The edge length \( l_{ij} \) for the edge \( \{ij\} \in E \) is given by
\[
l_{ij} = \sqrt{\varepsilon_i e^{2f_i} + \varepsilon_j e^{2f_j} + 2\eta_{ij} e^{f_i+f_j}} \quad (1.3)
\]
for \( G = \mathbb{E}^2 \),
\[
l_{ij} = \cosh^{-1} \left( \sqrt{(1 + \varepsilon_i e^{2f_i})(1 + \varepsilon_j e^{2f_j}) + \eta_{ij} e^{f_i+f_j}} \right) \quad (1.4)
\]
for \( G = \mathbb{H}^2 \) and
\[
l_{ij} = \cos^{-1} \left( \sqrt{(1 - \varepsilon_i e^{2f_i})(1 - \varepsilon_j e^{2f_j}) - \eta_{ij} e^{f_i+f_j}} \right) \quad (1.5)
\]
for \( G = \mathbb{S}^2 \);

The edge length function \( l : E \to \mathbb{R}_{>0} \) defined by (1.3), (1.4), (1.5) is a Euclidean, hyperbolic and spherical polyhedral metric on \( (M, \mathcal{T}) \) respectively.

The weight \( \varepsilon : V \to \{-1,0,1\} \) is called the scheme coefficient and \( \eta : E \to \mathbb{R} \) is called the discrete conformal structure coefficient.

Two discrete conformal structures defined on the same weighted triangulated surface \( (M, \mathcal{T}, \varepsilon, \eta) \) with the same background geometry \( G \) is said to be conformally equivalent.

**Remark 1.** The relationship of the discrete conformal structure in Definition 2 and the existing special types of discrete conformal structures is contained in the following table.

| Scheme                        | \( \varepsilon_i \) | \( \varepsilon_j \) | \( \eta_{ij} \) |
|-------------------------------|---------------------|---------------------|-----------------|
| Tangential circle packing     | +1                  | +1                  | +1              |
| Thurston’s circle packing     | +1                  | +1                  | \((-1,1]\)      |
| Inversive distance circle pack| +1                  | +1                  | \((-1,+\infty)\) |
| Vertex scaling                | 0                   | 0                   | \((0, +\infty)\) |
| Discrete conformal structure  | \{+1,0,-1\}         | \{+1,0,-1\}         | \((-1, +\infty)\) |

By the table, the tangential circle packing is a special case of Thurston’s circle packing and Thurston’s circle packing is a special case of inversive distance circle packing. For simplicity, we unify all these three types of circle packings as inversive distance circle packing in the following. By the table again, the discrete conformal structure in Definition 2 contains inversive distance circle packing and vertex scaling as special cases. Furthermore, the discrete conformal structure in Definition 2 contains the mixed type of discrete conformal structures with \( \varepsilon_i = 0 \) for some vertices in \( V \) and \( \varepsilon_j = 1 \) for other
vertices \( j \in V \). There have been lots of research activities on special discrete conformal structures on polyhedral manifolds. For inversive distance circle packing on surfaces, please refer to \([1, 2, 4–8, 13, 17, 19, 21, 23, 25, 28, 33, 38, 39, 41, 46, 47, 50, 54, 61, 67, 68, 74, 76, 79, 81, 85, 87, 95, 97, 98, 99]\) and others. For vertex scaling on surfaces, please refer to \([3, 12, 22, 43–45, 57, 64, 66, 73, 77, 78, 83, 84, 88, 92, 93, 99]\) and others. There are also some research activities on tangential sphere packing and Thurston’s sphere packings on 3-dimensional manifolds, please refer to \([9, 20, 26, 27, 35, 36, 38, 39, 48, 49, 72, 80, 86]\) and others. In the following, when we mention the discrete conformal structure on polyhedral surfaces, it is referred to the generic discrete conformal structure in Definition 2 unless otherwise declared.

**Remark 2.** The Definition 2 of discrete conformal structure on polyhedral surfaces was introduced independently simultaneously by Glickenstein-Thomas [41] and by Zhang-Guo-Zeng-Luo-Yau-Gu [96]. Glickenstein-Thomas’s approach for defining discrete conformal structure on polyhedral surfaces is from Riemannian geometry perspective, where they used the notion of partial edge length introduced by Glickenstein [38, 39] and required that the deformation of discrete conformal structure depends in a reasonable form of the partial edge length. The notion of partial edge length ensures the existence of some geometric structures on the Poincaré dual of the triangulation and the conditions on the deformation of discrete conformal structure is a discrete analogue of the fact that smooth conformal change depends only on the scalar function defined on the manifolds. It is shown by Glickenstein-Thomas [41] that this definition of discrete conformal structure could be classified, which has the form presented in Definition 2 with \( \varepsilon_i \) replaced by arbitrary constant \( \alpha_i \in \mathbb{R} \). As pointed out by Thomas (page 53), one can reparameterize discrete conformal structures so that \( \alpha_i \in \{-1, 0, 1\} \) while keeping the induced polyhedral metric invariant, which gives rise to Definition 2. Zhang-Guo-Zeng-Luo-Yau-Gu’s approach [96] for defining discrete conformal structure was motivated by Bobenko-Pinkall-Springborn’s observation [3] on the relationship of vertex scaling on polyhedral surfaces and 3-dimensional hyperbolic geometry. They defined the edge lengths by embedding the triangle in a generalized tetrahedron in the extended hyperbolic 3-space. In this approach, Zhang-Guo-Zeng-Luo-Yau-Gu explicitly constructed all kinds of discrete conformal structures contained in Definition 2. The advantage of this approach is that it gives explicit geometric interpretations to the discrete Ricci energies for different kinds of special discrete conformal structures on polyhedral surfaces, which will be further discussed in Section 5.

A basic problem in discrete conformal geometry is to understand the relationship between the discrete conformal structure and its curvature. We prove the following result on the global rigidity of discrete conformal structures on polyhedral surfaces, which confirms two conjectures of Glickenstein in [40].
Theorem 1.1. Suppose \((M, \mathcal{T}, \varepsilon, \eta)\) is a weighted triangulated connected closed surface with the weights \(\varepsilon : V \to \{0, 1\}\) and \(\eta : E \to \mathbb{R}\) satisfying
\[
\varepsilon_s \varepsilon_t + \eta_{st} > 0, \quad \forall \{st\} \in E
\] (1.6)
and
\[
\varepsilon_q \eta_{st} + \eta_{qs} \eta_{qt} \geq 0, \quad \{q,s,t\} = \{i,j,k\}
\] (1.7)
for any triangle \(\{ijk\} \in F\). Then

(a) A Euclidean discrete conformal structure \(f : V \to \mathbb{R}\) on \((M, \mathcal{T}, \varepsilon, \eta)\) is determined by its combinatorial curvature \(K : V \to \mathbb{R}\) up to a vector \(c(1, 1, \cdots, 1), c \in \mathbb{R}\).

(b) A hyperbolic discrete conformal structure \(f : V \to \mathbb{R}\) on \((M, \mathcal{T}, \varepsilon, \eta)\) is determined by its combinatorial curvature \(K : V \to \mathbb{R}\).

Remark 3. If \(\varepsilon_i = 1\) for all \(i \in V\), Theorem 1.1 is reduced to the global rigidity of inversive distance circle packing on surfaces obtained by Guo [46], Luo [61] and the author [85,87], which was conjectured by Bowers-Stephenson [6]. If \(\varepsilon_i = 0\) for all \(i \in V\), Theorem 1.1 is reduced to the global rigidity of vertex scaling on surfaces obtained by Bobenko-Pinkall-Springborn [8], which was conjectured by Luo [57]. Theorem 1.1 unifies these two results and further contains the case of mixed type that \(\varepsilon_i = 1\) for some vertices \(i \in V_1 \neq \emptyset\) and \(\varepsilon_j = 0\) for the other vertices \(j \in V \setminus V_1 \neq \emptyset\). The local rigidity for discrete conformal structures on polyhedral surfaces was previously obtained by Glickenstein-Thomas [41] under a very strong condition that the discrete conformal structure induces a well-centered geometric center for each triangle in the triangulation, which is not easy to check. The local rigidity for some subcases of Theorem 1.1 was also previously obtained by Guo-Luo [47] from Thurston’s viewpoint with the standard cosine law replaced by different cosine laws in hyperbolic geometry.

1.2 Combinatorial curvature flows for discrete conformal structures on polyhedral surfaces

Finding discrete conformal structures with prescribed combinatorial curvatures on polyhedral surfaces is an important problem in discrete conformal geometry, which has lots of theoretical and practical applications [7,95]. Combinatorial curvature flow is an effective approach for handling this problem, which was pioneered by Chow-Luo’s work [7] on combinatorial Ricci flow for Thurston’s circle packing on polyhedral surfaces. For discrete conformal structures on polyhedral surfaces, the combinatorial Ricci flow was introduced by Zhang-Guo-Zeng-Luo-Yau-Gu [96]. For simplicity, set
\[
u_i = f_i
\] (1.8)
for any $i \in V$ in the Euclidean background geometry and

$$u_i = \begin{cases} f_i, & \text{if } \varepsilon_i = 0, \\ \frac{1}{2} \log \left| \frac{\sqrt{1 + \varepsilon_i e^{2f_i}} - 1}{\sqrt{1 + \varepsilon_i e^{2f_i}} + 1} \right|, & \text{if } \varepsilon_i \neq 0, \end{cases}$$

(1.9)

for the hyperbolic background geometry. For simplicity, $u : V \to \mathbb{R}$ is also called a discrete conformal structure in the following.

**Definition 3** ([96]). Suppose $(M,T,\varepsilon,\eta)$ is a weighted triangulated connected closed surface with weights $\varepsilon : V \to \{-1,0,1\}$ and $\eta : E \to \mathbb{R}$. The combinatorial Ricci flow for discrete conformal structures on polyhedral surfaces is defined as

$$\frac{du_i}{dt} = -K_i$$

(1.10)

for Euclidean and hyperbolic background geometry.

The normalized combinatorial Ricci flow for the discrete conformal structures with Euclidean background geometry is

$$\frac{du_i}{dt} = K_{av} - K_i,$$

(1.11)

where $K_{av} = \frac{2\pi \chi(M)}{N}$ is the average curvature.

**Remark 4.** There have been lots of researches on the combinatorial Ricci flow on two and three dimensional manifolds. For the combinatorial Ricci flow for circle packing on polyhedral surfaces, please refer to [7, 21, 23, 25, 28, 29, 31, 33, 85] and others. For the combinatorial Yamabe flow for vertex scaling on polyhedral surfaces, please refer to [22, 43, 44, 57, 88, 93] and others. There are also some research activities for combinatorial Ricci flow and combinatorial Yamabe flow on 3-dimensional manifolds, please refer to [11, 14, 15, 20, 26, 27, 35, 36, 58, 89, 90, 94] and others.

**Remark 5.** The combinatorial Ricci flow in Definition 3 unifies the known form of combinatorial Ricci flow or combinatorial Yamabe flow for different special discrete conformal structures on polyhedral surfaces. If $\varepsilon_i = 1$ for all $i \in V$, the combinatorial Ricci flow in Definition 3 is reduced to Chow-Luo’s combinatorial Ricci flow for circle packings on polyhedral surfaces in [7, 21, 23, 25]. If $\varepsilon_i = 0$ for all $i \in V$, the combinatorial Ricci flow in Definition 3 is reduced to Luo’s combinatorial Yamabe flow for vertex scalings on polyhedral surfaces in [57]. The combinatorial Ricci flow in Definition 3 introduced by Zhang-Guo-Zeng-Luo-Yau-Gu [96], unifies Chow-Luo’s combinatorial Ricci flow and Luo’s combinatorial Yamabe flow and generalizes them to a very general context. Specially, it
includes the mixed type case that $\varepsilon_i = 1$ for some vertices $i \in V$ and $\varepsilon_j = 0$ for the other vertices $j \in V$. The combinatorial Ricci flow in Definition \ref{def:combinatorial Ricci flow} can also be defined for spherical background geometry with $u_i$ satisfying $\frac{\partial f_i}{\partial u_i} = \sqrt{1 - \varepsilon_i e^{2f_i}}$. Please refer to \cite{41,96} for more details. In this paper, we focus on the cases of Euclidean and hyperbolic background geometry.

The combinatorial Ricci flows (1.10) and (1.11) may develop singularities, which correspond to the triangles in the polyhedral surfaces degenerate along the flows. To handle the potential singularities of the combinatorial Ricci flow (1.10) and (1.11), we extend the combinatorial curvature by constants and then extend the combinatorial Ricci flow through the singularities. We have the following result on the longtime existence and convergence for the solution of extended combinatorial Ricci flow, which confirms a conjecture of Glickenstein \cite{40} on the convergence rate of combinatorial Ricci flow and provides an effective algorithm for finding discrete conformal structures with prescribed combinatorial curvatures.

**Theorem 1.2.** Suppose $(M, T, \varepsilon, \eta)$ is a weighted triangulated connected closed surface with the weights $\varepsilon : V \to \{0, 1\}$ and $\eta : E \to \mathbb{R}$ satisfying the structure conditions (1.6) and (1.7).

(a) The solution of normalized combinatorial Ricci flow (1.11) in the Euclidean background geometry and the solution of the combinatorial Ricci flow (1.10) in the hyperbolic background geometry could be extended to exist for all time for any initial discrete conformal structure on $(M, T, \varepsilon, \eta)$.

(b) The solution of the extended combinatorial Ricci flow is unique for any initial discrete conformal structure.

(c) If there exists a Euclidean discrete conformal structure with constant combinatorial curvature on $(M, T, \varepsilon, \eta)$, the solution of the extended normalized Euclidean combinatorial Ricci flow converges exponentially fast for any initial Euclidean discrete conformal structure; If there exists a hyperbolic discrete conformal structure with zero combinatorial curvature on $(M, T, \varepsilon, \eta)$, the solution of the extended hyperbolic combinatorial Ricci flow converges exponentially fast for any initial hyperbolic discrete conformal structure.

**Remark 6.** If $\varepsilon_i = 1$ for all $i \in V$, the result in Theorem 1.2 is reduced to the convergence result for Chow-Luo’s combinatorial Ricci flow for Thurston’s circle packing obtained in \cite{7,21} and for inversive distance circle packing obtained in \cite{23,25}. If $\varepsilon_i = 0$ for all $i \in V$, the result in Theorem 1.2 is reduced to the convergence result for Luo’s combinatorial Yamabe flow for vertex scaling obtained in \cite{22}. The results in Theorem 1.2 further include
the case of mixed type that \( \varepsilon_i = 1 \) for some vertices \( i \in V \) and \( \varepsilon_j = 0 \) for the other vertices \( j \in V \). The idea of extension to handle the singularities of the combinatorial Ricci flow comes from Bobenko-Pinkall-Springborn \[3\] and Luo \[61\]. It should be mentioned that there is another way in \[43,44\] to extend the combinatorial Yamabe flow for vertex scaling on polyhedral surfaces, in which the singularities are resolved by doing surgery along the flow by edge flipping when the triangulation is not Delaunay in the polyhedral metric along the flow. In this approach, the condition on the existence of discrete conformal structure with constant combinatorial curvature in Theorem \[1.2\] is removed in \[43,44\].

Combinatorial Calabi flow is another effective combinatorial curvature flow for finding discrete conformal structures with prescribed combinatorial curvatures on polyhedral surfaces, which was introduced by Ge \[17\] (see also \[18\]) for Thurston’s Euclidean circle packing. Since then, the combinatorial Calabi flow was extensively studied, see \[16,19,28,30,33,34,56,91,99\] and others. The combinatorial Calabi flow for discrete conformal structures on polyhedral surfaces in Definition 2 is defined as follows.

**Definition 4.** Suppose \((M, \mathcal{T}, \varepsilon, \eta)\) is a weighted triangulated connected closed surface with weights \( \varepsilon : V \to \{-1, 0, 1\} \) and \( \eta : E \to \mathbb{R} \). The combinatorial Calabi flow for discrete conformal structures on polyhedral surfaces is defined as

\[
\frac{du_i}{dt} = \Delta K_i
\]

for Euclidean and hyperbolic background geometry, where \( \Delta = -\frac{\partial (K_1, \cdots, K_N)}{\partial (u_1, \cdots, u_N)} \) is the combinatorial Laplace operator for the discrete conformal structure.

**Remark 7.** The combinatorial Calabi flow introduced in Definition 4 unifies the known form of combinatorial Calabi flow for different special discrete conformal structures on polyhedral surfaces. If \( \varepsilon_i = 1 \) for all the vertices \( i \in V \), the combinatorial Calabi flow \[1.12\] in Definition 4 is reduced to the combinatorial Calabi flow for circle packings on polyhedral surfaces introduced in \[17,18,30\]. If \( \varepsilon_i = 0 \) for all the vertices \( i \in V \), the combinatorial Calabi flow \[1.12\] in Definition 4 is reduced to the combinatorial Calabi flow for vertex scaling on polyhedral surfaces introduced in \[17,99\]. The combinatorial Calabi flow in Definition 4 further contains the mixed type case that \( \varepsilon_i = 1 \) for some vertices \( i \in V \) and \( \varepsilon_j = 0 \) for the other vertices \( j \in V \). Similar to the combinatorial Ricci flow, the combinatorial Calabi flow \[1.12\] could also be defined for spherical background geometry with \( u_i \) satisfying \( \frac{\partial f}{\partial u_i} = \sqrt{1 - \varepsilon_i e^{2f}} \).

The combinatorial Calabi flow is a negative gradient flow of the combinatorial Calabi energy \( C = \frac{1}{2} \sum_{i=1}^N K_i^2 \). We have the following result on the longtime behavior of combinatorial Calabi flow \[1.12\].
Theorem 1.3. Suppose \((M, T, \varepsilon, \eta)\) is a weighted triangulated connected closed surface with the weights \(\varepsilon : V \to \{0, 1\}\) and \(\eta : E \to \mathbb{R}\) satisfying the structure conditions (1.6) and (1.7).

(a) If the solution of the combinatorial Calabi flow (1.12) converges to a nondegenerate discrete conformal structure \(u^*\), then \(u^*\) has constant combinatorial curvature.

(b) If there exists a discrete conformal factor \(u^*\) with constant combinatorial curvature \(\frac{2\pi\chi(M)}{N}\) for Euclidean background geometry and 0 for hyperbolic background geometry, then there exists a real number \(\delta > 0\) such that if the initial value \(u(0)\) of the combinatorial Calabi flow (1.12) satisfies \(||u(0) - u^*|| < \delta\), the solution of the combinatorial Calabi flow (1.12) exists for all time and converges exponentially fast to \(u^*\).

Remark 8. In the case of Thurston’s circle packing, the combinatorial Calabi flow (1.12) does not develop singularities and is proved to converge exponentially fast to circle packing metrics with constant combinatorial curvature. Please refer to [17–19, 30] for details. In the case of vertex scaling, the combinatorial Calabi flow (1.12) may develop singularities. However, by doing surgery along the combinatorial Calabi flow (1.12) by edge flipping introduced in [43, 44], it is proved [99] that the combinatorial Calabi flow with surgery for vertex scaling exists for all time and converges exponentially fast for any initial discrete conformal factor. For generic initial discrete conformal structure in Definition 2, the global convergence of the combinatorial Calabi flow (1.12) is not known up to now.

One can also study the parameterized combinatorial curvature \(R_{\alpha, i} = \frac{K_i}{e^{\alpha u_i}}\) for discrete conformal structure in Definition 2 with \(\alpha \in \mathbb{R}\) and \(u : V \to \mathbb{R}\) defined by (1.8) and (1.9), which was introduced in [28, 31–33] for circle packings and in [88, 93] for vertex scaling on polyhedral surfaces. The global rigidity of parameterized combinatorial curvature \(R_{\alpha}\) with respect to the discrete conformal structure was established in [89], where the author further introduced the combinatorial \(\alpha\)-Ricci flow and combinatorial \(\alpha\)-Calabi flow for discrete conformal structures to study the prescribed combinatorial \(\alpha\)-curvature problem.

1.3 Relationships with 3-dimensional hyperbolic geometry

Motivated by Bobenko-Pinkall-Springborn’s observation [3] on the relationship of vertex scaling on polyhedral surfaces and 3-dimensional hyperbolic geometry, Zhang-Guo-Zeng-Luo-Yau-Gu [96] constructed all the discrete conformal structures via generalized 3-dimensional hyperbolic tetrahedra. The basic idea is to construct a generalized hyperbolic tetrahedron \(T_{Oijk}\) with the vertices \(O, v_i, v_j, v_k\) ideal or hyper-ideal. \(O\) is ideal when we study the Euclidean discrete conformal structure and hyper-ideal when we study the
hyperbolic background geometry. The vertex \( v_s \in \{v_i, v_j, v_k\} \) is hyper-ideal if \( \varepsilon_s = 1 \) and ideal if \( \varepsilon_s = 0 \). In the case that \( v_s \in \{v_i, v_j, v_k\} \) is hyper-ideal, the line segment \( Ov_s \) has nonempty intersection with \( \mathbb{H}^3 \) in the Klein model. In this way, for each pair \( v_s, v_t \) of \( v_i, v_j, v_k, \) a weight \( \eta_{st} \) can be naturally assigned via the signed edge length of \( \{v_s, v_t\} \). In the Euclidean background geometry, the edge lengths of the intersection triangle \( T_{ijk} \cap H_O \) is given by (1.3), where \( H_O \) is the horosphere attached to the ideal vertex \( O \) and \( f_s \) is minus of the signed edge length \( l_{Ov_s} \) with \( s \in \{i, j, k\} \). The case for hyperbolic background geometry is similar. By truncating the generalized hyperbolic tetrahedron \( T_{Oijk} \) with horospheres or hyperbolic planes, we can attach it with a finite hyperbolic polyhedron \( P \), the volume and co-volume of which are functions of \( l_{Ov_i}, l_{Ov_j}, l_{Ov_k} \) and \( \eta_{ij}, \eta_{ik}, \eta_{jk} \). For the details of the construction of \( T_{Oijk} \) and assignments of \( \eta_{ij}, \eta_{ik}, \eta_{jk} \), please refer to Section 5.

We find that the structure conditions (1.6) and (1.7) are direct consequences of the cosine laws for generalized hyperbolic triangles. This partially answers a question of Gortler [42]. Using the geometric explanation of the discrete conformal structures in terms of 3-dimensional hyperbolic geometry, we further obtain some new convexities of co-volume functions of generalized hyperbolic tetrahedra in extended 3-dimensional hyperbolic space.

**Theorem 1.4.** Suppose \( T = \{Oijk\} \) is a generalized tetrahedron constructed above.

(a) The weights \( \eta_{ij}, \eta_{ik}, \eta_{jk} \) on the edges \( \{ij\}, \{ik\}, \{jk\} \) satisfy the structure conditions (1.6) and (1.7).

(b) The co-volume of the generalized tetrahedron \( T = \{Oijk\} \) defined by (5.2) with fixed weights \( \eta_{ij}, \eta_{ik}, \eta_{jk} \) is a convex function of the signed edge lengths \( l_{Ov_i}, l_{Ov_j}, l_{Ov_k} \).

### 1.4 Basic ideas of the proof of Theorem 1.1

The proof for the global rigidity of discrete conformal structures uses variational principles, which was introduced by Colin de Verdière [13] for tangential circle packings on triangulated surfaces. The variational principle on triangulated surfaces and triangulated 3-manifolds has been extensively studied in [3, 7, 10, 46, 49, 55, 57, 59, 62, 65, 71, 76, 85, 87, 97] and others.

The key point using variational principle to prove the global rigidity of discrete conformal structures on polyhedral surfaces is constructing a globally defined convex function with the combinatorial curvature as gradient, which could be reduced to constructing a concave function of discrete conformal structures on a triangle with inner angles as gradient. There are two approaches to construct the concave function for discrete conformal structures on a triangle. The first approach is used to study the rigidity of vertex scaling on polyhedral surfaces. This was accomplished by Bobenko-Pinkall-Springborn [3] using
the volume function for some generalized hyperbolic tetrahedra, which is concave of the
dihedral angles and could be extended to be defined globally. This approach depends
on the concavity of the volume function of generalized hyperbolic tetrahedra in terms of
dihedral angles, which was proved by Rivin [71] for ideal tetrahedra and by Leibon [55] for
generalized hyperbolic tetrahedra with one hyper-ideal vertex and three ideal vertices. For
the other cases, Ricci energy functions with inner angles as gradient were constructed using
the volume functions of generalized hyperbolic tetrahedra by Zhang-Guo-Zeng-Luo-Yau-
Gu [96]. However, the concavities of volume functions of generalized hyperbolic tetrahedra
in these cases are not fully understood. See [75] for some partial results. The second ap-
proach is used to study the rigidity of inverse distance circle packings on polyhedral
surfaces. This was accomplished by Guo [46], Luo [61] and the author [85,87] by defining
the Ricci energy function for a triangle as the integral of a closed 1-form defined by inner
angles on the admissible space of inverse distance circle packing metrics. The difficulty
for this approach is to prove that the integral is a well-defined locally concave function
and could be extended to be a globally defined concave function.

In this paper, we adopt the second approach. The idea comes from a new proof [87] of
Bowers-Stephenson’s conjecture on the global rigidity of inverse distance circle packing
on surfaces. By solving the global version of triangle inequalities for a triangle using the
geometric center introduced by Glickenstein [38,39], we obtain an explicit characterization
for the admissible space of discrete conformal structures on the triangle, which is homotopy
equivalent to $\mathbb{R}^3$ and therefore simply connected. This implies the Ricci energy function,
deefined as the integral of a closed 1-form of the inner angles on the admissible space of
discrete conformal structures for a triangle, is well-defined. By the continuity of eigenvalues
of the hessian matrix of the Ricci energy function, the concavity of the Ricci energy
function is reduced to find a discrete conformal structure with negative definite hessian
matrix. This is accomplished by introducing the parameterized admissible space of discrete
conformal structures and choosing some “good” point in the parameterized admissible
space. As the Ricci energy function constructed by the two approaches are the same up
to some constant, the concavity of the Ricci energy function for a triangle proved here
conversely implies the convexity of the co-volume function of some generalized hyperbolic
tetrahedron with some edge lengths or dihedral angles fixed. The extension of Ricci energy
function to be defined globally follows from Luo’s generalization [61] of Bobenko-Pinkall-
Springborn’s extension in [3]. The rigidity of discrete conformal structure on polyhedral
surfaces follows from the concavity of the extended Ricci energy function for a triangle.

The main results obtained in this paper could be generalized to compact triangulated
surfaces with boundary without any further difficulty by doubling the surface across the
boundary. For simplicity, we will not state the paralleling results for compact triangulated
surfaces with boundary.
1.5 The organization of the paper

In Section 2, we study the Euclidean discrete conformal structures on polyhedral surfaces and prove a generalization of Theorem 1.1 (a). In Section 3, we study the hyperbolic discrete conformal structures on polyhedral surfaces and prove a generalization of Theorem 1.1 (b). In Section 4, we study the combinatorial Ricci flow and combinatorial Calabi flow for discrete conformal structures on polyhedral surfaces and prove generalizations of Theorem 1.2 and Theorem 1.3. In Section 5, we discuss the relationship of discrete conformal structures on polyhedral surfaces and 3-dimensional hyperbolic geometry and prove Theorem 1.4. In Section 6, we discuss some open problems and conjectures.

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2 Euclidean discrete conformal structures

2.1 Admissible space of Euclidean discrete conformal structures for a triangle

In this subsection, we study the admissible space $\Omega_{ijk}^E(\eta)$ of nondegenerate Euclidean discrete conformal structures for a triangle $\sigma = \{ijk\} \in F$ with the edge lengths $l_{ij}, l_{ik}, l_{jk}$ defined by (1.3) and the weights $\varepsilon : V \to \{0, 1\}$ and $\eta : E \to \mathbb{R}$ satisfying the structure conditions (1.6) and (1.7), which is defined to be the set of Euclidean discrete conformal structures with the edge lengths $l_{ij}, l_{ik}, l_{jk}$ satisfying the triangle inequalities.

Note that (1.3) uniquely defines a positive number for $\varepsilon : V \to \{0, 1\}$ and $f : V \to \mathbb{R}$ under the structure condition (1.6). For the triangle $\{ijk\} \in F$, the positive edge lengths $l_{ij}, l_{ik}, l_{jk}$ satisfy the triangle inequalities

$$l_{ij} < l_{ik} + l_{jk}, \quad l_{ik} < l_{ij} + l_{jk}, \quad l_{jk} < l_{ij} + l_{ik}$$

if and only if

$$0 < (l_{ij} + l_{ik} + l_{jk})(l_{ij} + l_{ik} - l_{jk})(l_{ij} - l_{ik} + l_{jk})(-l_{ij} + l_{ik} + l_{jk})$$

$$= 2l_{ij}^2 l_{ik}^2 + 2l_{ij}^2 l_{jk}^2 + 2l_{ik}^2 l_{jk}^2 + 2t_{ij}^4 t_{ik}^4 - t_{ij}^4 - t_{ik}^4 - t_{jk}^4.$$ 

(2.2)

For simplicity of notations, set

$$r_i = e^{f_i}, \quad \forall i \in V.$$ 

(2.3)
Sometimes we also call \( r \in \mathbb{R}_{\geq 0}^3 \) as a Euclidean discrete conformal structure. Then the edge length \( l_{ij} \) in the Euclidean background geometry, i.e. \((1.3)\), is given by

\[
l_{ij} = \sqrt{\varepsilon_i r_i^2 + \varepsilon_j r_j^2 + 2\eta_{ij} r_i r_j},
\]

(2.4)

Submitting (2.4) into (2.2), by direct calculations, we have

\[
(l_{ij} + l_{ik} + l_{jk})(l_{ij} + l_{ik} - l_{jk})(l_{ij} - l_{ik} + l_{jk})(-l_{ij} + l_{ik} + l_{jk})
\]

\begin{align*}
= & 4r_i^2 r_j^2 r_k^2((\varepsilon_i \varepsilon_j - \eta_{ij}^2)r_k^{-2} + (\varepsilon_i \varepsilon_k - \eta_{ik}^2)r_j^{-2} + (\varepsilon_j \varepsilon_k - \eta_{jk}^2)r_i^{-2} \\
& + 2(\varepsilon_k \eta_{ij} + \eta_{ik} \eta_{jk})r_i^{-1}r_j^{-1} + 2(\varepsilon_j \eta_{ik} + \eta_{ij} \eta_{jk})r_i^{-1}r_k^{-1} + 2(\varepsilon_i \eta_{jk} + \eta_{ij} \eta_{ik})r_j^{-1}r_k^{-1}] \\
\end{align*}

Set

\[
\kappa_i = r_i^{-1}, \kappa_j = r_j^{-1}, \kappa_k = r_k^{-1},
\]

(2.5)

\[
\gamma_i = \varepsilon_i \eta_{jk} + \eta_{ij} \eta_{ik}, \gamma_j = \varepsilon_j \eta_{ik} + \eta_{ij} \eta_{jk}, \gamma_k = \varepsilon_k \eta_{ij} + \eta_{ik} \eta_{jk},
\]

(2.6)

and

\[
Q^E = (\varepsilon_i \varepsilon_k - \eta_{ij}^2)\kappa_i^2 + (\varepsilon_i \varepsilon_k - \eta_{ik}^2)\kappa_j^2 + (\varepsilon_i \varepsilon_j - \eta_{ij}^2)\kappa_k^2 + 2\kappa_i \kappa_j \gamma_k + 2\kappa_i \kappa_k \gamma_j + 2\kappa_j \kappa_k \gamma_i.
\]

(2.7)

Then

\[
\gamma_i \geq 0, \gamma_j \geq 0, \gamma_k \geq 0
\]

(2.8)

by the structure condition (1.7) and

\[
(l_{ij} + l_{ik} + l_{jk})(l_{ij} + l_{ik} - l_{jk})(l_{ij} - l_{ik} + l_{jk})(-l_{ij} + l_{ik} + l_{jk}) = 4r_i^2 r_j^2 r_k^2 Q^E.
\]

As a consequence of the arguments above, we have the following result.

**Lemma 2.1.** Suppose \((M, \mathcal{T}, \varepsilon, \eta)\) is a weighted triangulated surface with the weights \(\varepsilon : V \rightarrow \{0, 1\}\) and \(\eta : E \rightarrow \mathbb{R}\). \{ijk\} \(\in P\) is a topological triangle in the triangulation.

Then the positive edge lengths \(l_{ij}, l_{ik}, l_{jk}\) defined by (1.3) satisfy the triangle inequalities if and only if \(Q^E > 0\).

Set

\[
\begin{align*}
    h_i &= (\varepsilon_i \varepsilon_k - \eta_{ij}^2)\kappa_i + \kappa_j \gamma_k + \kappa_k \gamma_j, \\
    h_j &= (\varepsilon_i \varepsilon_k - \eta_{ik}^2)\kappa_j + \kappa_i \gamma_k + \kappa_k \gamma_i, \\
    h_k &= (\varepsilon_i \varepsilon_j - \eta_{ij}^2)\kappa_k + \kappa_i \gamma_j + \kappa_j \gamma_i,
\end{align*}
\]

(2.9)

Then by Lemma 2.1, \( r = (r_i, r_j, r_k) \in \mathbb{R}_{\geq 0}^3 \) is a degenerate Euclidean discrete conformal structure for the triangle \{ijk\} if and only if

\[
Q^E = \kappa_i h_i + \kappa_j h_j + \kappa_k h_k \leq 0,
\]

which implies that at least one of \(h_i, h_j, h_k\) is nonpositive. We further have the following result on the signs of \(h_i, h_j, h_k\).
Lemma 2.2. Suppose $(M, T, \varepsilon, \eta)$ is a weighted triangulated surface with the weights \( \varepsilon : V \to \{0, 1\} \) and \( \eta : E \to \mathbb{R} \) satisfying the structure conditions (1.6) and (1.7). \( \{ijk\} \in F \) is a topological triangle of the triangulated surface. If \( r = (r_i, r_j, r_k) \in \mathbb{R}^3_{>0} \) is a degenerate Euclidean discrete conformal structure for the triangle \( \{ijk\} \in F \), then there is no subset \( \{s, t\} \subset \{i, j, k\} \) such that \( h_s \leq 0 \) and \( h_t \leq 0 \).

Proof. Without loss of generality, assume \( h_i \leq 0, h_j \leq 0 \). Then by the definition of \( h_i, h_j \) in (2.9), we have

\[
\kappa_j \gamma_k + \kappa_k \gamma_j \leq (\eta_{jk}^2 - \varepsilon_j \varepsilon_k) \kappa_i, \quad \kappa_i \gamma_k + \kappa_k \gamma_i \leq (\eta_{ik}^2 - \varepsilon_i \varepsilon_k) \kappa_j,
\]

which implies

\[
\eta_{jk} - \varepsilon_j \varepsilon_k \geq 0, \quad \eta_{ik} - \varepsilon_i \varepsilon_k \geq 0
\]

(2.10)

and

\[
\gamma_k^2 - (\eta_{jk}^2 - \varepsilon_j \varepsilon_k)(\eta_{ik}^2 - \varepsilon_i \varepsilon_k) = -\varepsilon_i \varepsilon_j \varepsilon_k + \varepsilon_k \eta_{ij}^2 + \varepsilon_j \varepsilon_k \eta_{ik}^2 + \varepsilon_i \varepsilon_k \eta_{jk}^2 + 2\varepsilon_k \eta_{ij} \eta_{ik} \eta_{jk} \leq 0
\]

(2.11)

by the structure conditions (1.6) and (1.7), where the property \( \varepsilon_k^2 = \varepsilon_k \) for the weight \( \varepsilon : V \to \{0, 1\} \) is used. Set

\[
F = -\varepsilon_i \varepsilon_j \varepsilon_k + \varepsilon_k \eta_{ij}^2 + \varepsilon_j \varepsilon_k \eta_{ik}^2 + \varepsilon_i \varepsilon_k \eta_{jk}^2 + 2\varepsilon_k \eta_{ij} \eta_{ik} \eta_{jk}.
\]

Then \( F \leq 0 \) by (2.11). Furthermore,

\[
F \geq (\varepsilon_i \varepsilon_j + \varepsilon_k \eta_{ij})^2 + (\varepsilon_i \varepsilon_j + \varepsilon_k \eta_{ik})^2 + (\varepsilon_i \varepsilon_j + \varepsilon_k \eta_{jk})^2 \geq 0
\]

(2.12)

by (2.10), \( \varepsilon_i, \varepsilon_j, \varepsilon_k \in \{0, 1\} \) and the structure conditions (1.6) and (1.7). Therefore, \( F = 0 \).

In the case of \( \varepsilon_k = 0 \), by the structure condition (1.6), we have

\[
\varepsilon_i \varepsilon_j + \eta_{ik} > 0, \eta_{ik} > 0, \eta_{jk} > 0.
\]

(2.13)

By \( h_i \leq 0, h_j \leq 0 \) and \( \varepsilon_k = 0 \), we have \( \eta_{ik} \eta_{jk} \kappa_j + \gamma_j \kappa_k \leq \eta_{jk}^2 \kappa_i \) and \( \eta_{ik} \eta_{jk} \kappa_i + \gamma_i \kappa_k \leq \eta_{ik}^2 \kappa_j \).

This implies

\[
\gamma_i = \varepsilon_i \eta_{jk} + \eta_{ij} \eta_{jk} = 0, \quad \gamma_j = \varepsilon_j \eta_{ik} + \eta_{ij} \eta_{jk} = 0,
\]

(2.14)

which implies \( \varepsilon_i \varepsilon_j - \eta_{ij}^2 = 0 \) by (2.13). Note that \( \varepsilon_i \varepsilon_j - \eta_{ij}^2 = (\varepsilon_i \varepsilon_j - \eta_{ij})(\varepsilon_i \varepsilon_j + \eta_{ij}) \).

Combining with (2.13) again, we have \( \varepsilon_i \varepsilon_j - \eta_{ij} = 0 \). By (2.13) once again, we have \( \varepsilon_i \varepsilon_j > 0 \), which implies \( \varepsilon_i = \varepsilon_j = \eta_{ij} = 1 \). However, in this case, (2.14) implies \( \eta_{ik} + \eta_{jk} = 0 \), which contradicts (2.13).
In the case of \( \varepsilon_k = 1 \), by \( F = 0 \) and (2.12), we have

\[
\varepsilon_i \eta_{jk} - \varepsilon_j \eta_{ik} = \eta_{ij} + \eta_{ik} \eta_{jk} = (\eta_{ik} - \varepsilon_i)(\eta_{jk} - \varepsilon_j) = \varepsilon_i(\eta_{jk} - \varepsilon_j) = \varepsilon_j(\eta_{ik} - \varepsilon_i) = 0,
\]

which implies \( \eta_{kk} = \varepsilon_i \) or \( \eta_{jk} = \varepsilon_j \). By \( \varepsilon_k = 1 \) and the structure condition (1.6), we have

\[
\varepsilon_i \varepsilon_j + \eta_{ij} > 0, \varepsilon_i + \eta_{ik} > 0, \varepsilon_j + \eta_{jk} > 0.
\]

(2.16)

If \( \eta_{ik} = \varepsilon_i \), (2.16) implies \( \varepsilon_i = \eta_{kk} = 1 \), which further implies \( \eta_{jk} = \varepsilon_j \) by (2.15). By (2.16) again, we have \( \varepsilon_j = \eta_{jk} = 1 \), which implies \( \eta_{ij} + \varepsilon_i \varepsilon_j = \eta_{ij} + \eta_{ik} \eta_{jk} = 0 \) by (2.15). This contradicts (2.16). The same arguments also apply to the case \( \eta_{jk} = \varepsilon_j \).

In summary, there exists no subset \( \{s, t\} \subset \{i, j, k\} \) such that \( h_s \leq 0 \) and \( h_t \leq 0 \). \( \square \)

As a direct corollary of Lemma 2.2, we have the following stronger result on the signs on \( h_i, h_j, h_k \) for degenerate Euclidean discrete conformal structures on the triangle \( \{ijk\} \in F \).

**Corollary 2.3.** Suppose \( (M, T, \varepsilon, \eta) \) is a weighted triangulated surface with the weights \( \varepsilon : V \to \{0, 1\} \) and \( \eta : E \to \mathbb{R} \) satisfying the structure conditions (1.6) and (1.7). \( \{ijk\} \in F \) is a topological triangle in the triangulated surface. If \( r = (r_i, r_j, r_k) \in \mathbb{R}^3_{\geq 0} \) is a degenerate Euclidean discrete conformal structure for the triangle \( \{ijk\} \in F \), then one of \( h_i, h_j, h_k \) is negative and the others are positive.

**Proof.** As \( r = (r_i, r_j, r_k) \in \mathbb{R}^3_{\geq 0} \) is a degenerate Euclidean discrete conformal structure for the triangle \( \{ijk\} \in F \), we have \( Q^E = \kappa_i h_i + \kappa_j h_j + \kappa_k h_k \leq 0 \) by Lemma 2.1 which implies at least one of \( h_i, h_j, h_k \) is nonpositive. Without loss of generality, assume \( h_i \leq 0 \). Then Lemma 2.2 implies that \( h_j > 0, h_k > 0 \). If \( h_i = 0 \), combining with \( h_j > 0, h_k > 0 \), we have \( Q^E = \kappa_i h_i + \kappa_j h_j + \kappa_k h_k > 0 \), which contradicts \( Q^E = \kappa_i h_i + \kappa_j h_j + \kappa_k h_k \leq 0 \). Therefore, \( h_i < 0, h_j > 0, h_k > 0 \). \( \square \)

**Remark 9.** Corollary 2.3 has an interesting geometric explanation as follows. For a nondegenerate Euclidean discrete conformal structure for the triangle \( \{ijk\} \in F \), there exists a geometric center \( C_{ijk} \) for the triangle \( \{ijk\} \) (39, Proposition 4), which has the same power distance to the vertices \( \{i, j, k\} \). Here the power distance of a point \( p \) to the vertex \( i \) is defined to be \( \pi_p(i) = d^2(i, p) - \varepsilon_i r_i^2 \), where \( d(i, p) \) is the Euclidean distance between \( p \) and the vertex \( i \). Denote \( h_{jk,i} \) as the signed distance of the geometric center \( C_{ijk} \) to the edge \( \{jk\} \), which is defined to be positive if the geometric center \( C_{ijk} \) is on the same side of the line determined by \( \{jk\} \) as the triangle \( \{ijk\} \) and negative otherwise (or zero if the geometric center \( C_{ijk} \) is on the line). Projections of the geometric center \( C_{ijk} \) to the edges \( \{ij\}, \{ik\}, \{jk\} \) give rise to the geometric centers of these edges, which are denoted by \( C_{ij}, C_{ik}, C_{jk} \) respectively. The signed distance \( d_{ij} \) of \( C_{ij} \) to the vertex \( i \) is defined to be positive if \( C_{ij} \) is on the same side as \( j \) along the line determined by \( \{ij\} \).
and negative otherwise (or zero if $C_{ij}$ is the same as $i$). $d_{ij}$ is defined similarly. Note that $d_{ij} + d_{ji} = l_{ij}$ and $d_{ij} \neq d_{ji}$ in general. For Euclidean discrete conformal structures, we have the following formulas \[38,39,41\]

$$h_{jk,i} = \frac{d_{ji} - d_{jk} \cos \theta_j}{\sin \theta_j}, \quad d_{ij} = \frac{\eta_i r_i^2 + \eta_j r_j^2}{l_{ij}},$$

where $\theta_j$ is the inner angle at the vertex $j$ of the Euclidean triangle $\{ijk\}$. By direct calculations,

$$h_{jk,i} = \frac{\eta_i^2 r_i^2 + \eta_j^2 r_j^2}{A l_{jk}}, \quad \text{(2.17)}$$

where $A = l_{ij} l_{ik} \sin \theta_i$. Corollary \ref{corollary} implies that the geometric center $C_{ijk}$ for a nondegenerate Euclidean discrete conformal structure does not lie in some region in the plane determined by the triangle as it tends to be degenerate. Note that $h_i, h_j, h_k$ is defined for all $(r_i, r_j, r_k) \in \mathbb{R}_3^0$, while $h_{ij,k}, h_{ik,j}, h_{jk,i}$ are defined only for nondegenerate discrete conformal structures.

Now we can solve the admissible space of Euclidean discrete conformal structures for a triangle $\{ijk\} \in F$, which shows that the admissible space of Euclidean discrete conformal structures for the triangle $\{ijk\} \in F$ with the weights satisfying the structure conditions \[1.6\] and \[1.7\] is simply connected. First, we have the following result.

**Lemma 2.4.** Suppose $(M, T, \varepsilon, \eta)$ is a weighted triangulated surface with the weights $\varepsilon: V \to \{0, 1\}$ and $\eta: E \to \mathbb{R}$ satisfying the structure conditions \[1.6\] and \[1.7\]. $\{ijk\} \in F$ is a topological triangle in the triangulated surface. If we further have $\varepsilon_j \varepsilon_k - \eta_{jk}^2 \geq 0$, $\varepsilon_i \varepsilon_k - \eta_{ik}^2 \geq 0$ and $\varepsilon_i \varepsilon_j - \eta_{ij}^2 \geq 0$, then the admissible space $\Omega^E_{ijk}(\eta)$ of Euclidean discrete conformal structure in the parameter $r \in \mathbb{R}_3^0$ and then simply connected.

**Proof.** By Lemma \ref{lemma} we just need to prove that for any $r = (r_i, r_j, r_k) \in \mathbb{R}_3^0$, we have $Q^E > 0$. If $\varepsilon_j \varepsilon_k - \eta_{jk}^2 \geq 0$, $\varepsilon_i \varepsilon_k - \eta_{ik}^2 \geq 0$ and $\varepsilon_i \varepsilon_j - \eta_{ij}^2 \geq 0$, then we have $Q^E \geq 0$ by the definition \[2.7\] of $Q^E$ and the structure condition \[1.7\]. If $Q^E = 0$ in this case, then $\eta_{jk}^2 = \varepsilon_j \varepsilon_k, \eta_{ik}^2 = \varepsilon_i \varepsilon_k, \eta_{ij}^2 = \varepsilon_i \varepsilon_j$. By the structure condition \[1.6\], we have $\eta_{jk} = \varepsilon_j \varepsilon_k, \eta_{ik} = \varepsilon_i \varepsilon_k, \eta_{ij} = \varepsilon_i \varepsilon_j$. Combining with the structure condition \[1.6\] again, we have $\varepsilon_i = \varepsilon_j = \varepsilon_k = \eta_{ij} = \eta_{jk} = 1$, which implies $Q^E = 4 \kappa_i \kappa_j + 4 \kappa_i \kappa_k + 4 \kappa_j \kappa_k > 0$ for any $r = (r_i, r_j, r_k) \in \mathbb{R}_3^0$. Therefore, the admissible space $\Omega^E_{ijk}(\eta) = \mathbb{R}_3^0$, which is simply connected.

By Lemma \ref{lemma}, we just need to study the case that at least one of $\varepsilon_j \varepsilon_k - \eta_{jk}^2, \varepsilon_i \varepsilon_k - \eta_{ik}^2, \varepsilon_i \varepsilon_j - \eta_{ij}^2$ is negative. Suppose $r = (r_i, r_j, r_k) \in \mathbb{R}_3^0$ is a degenerate Euclidean discrete conformal structure for this case. Then $Q^E \leq 0$ by Lemma \ref{lemma} By Proposition \ref{proposition} one of $h_i, h_j, h_k$ is negative and the other two are positive. Without loss of generality, assume
$h_i < 0, h_j > 0, h_k > 0$. By the definition (2.9) of $h_i$ and the structure condition (1.7), we have $(\gamma_i^2 - \varepsilon_i \varepsilon_k) \kappa_i > \gamma_k \kappa_j + \gamma_j \kappa_k \geq 0$, which implies $\gamma_i^2 - \varepsilon_i \varepsilon_k > 0$.

Taking $Q^E$ as a quadratic function of $\kappa_i, \kappa_j, \kappa_k$. Then $Q^E \leq 0$ is equivalent to

$$A_i \kappa_i^2 + B_i \kappa_i + C_i \geq 0,$$  \hspace{1cm} (2.18)

where

$$A_i = \eta_{jk}^2 - \varepsilon_j \varepsilon_k > 0,$$

$$B_i = -2(\gamma_k \kappa_j + \gamma_j \kappa_k) \leq 0,$$

$$C_i = (\eta_{ik}^2 - \varepsilon_i \varepsilon_k) \kappa_j^2 + (\eta_{ij}^2 - \varepsilon_i \varepsilon_j) \kappa_k^2 - 2\kappa_j \kappa_k \gamma_i.$$

**Lemma 2.5.** Suppose $(M, T, \varepsilon, \eta)$ is a weighted triangulated surface with the weights $\varepsilon : V \to \{0, 1\}$ and $\eta : E \to \mathbb{R}$ satisfying the structure conditions (1.6) and (1.7). $(ijk) \in F$ is a topological triangle in the triangulated surface. If $A_i = \eta_{jk}^2 - \varepsilon_j \varepsilon_k > 0$, then the discriminant $\Delta_i = B_i^2 - 4A_iC_i$ for (2.18) is nonnegative.

**Proof.** By direct calculations, we have

$$\frac{1}{4} \Delta_i = [(\varepsilon_k \eta_{ij} + \eta_{ik} \eta_{jk})^2 - (\eta_{jk}^2 - \varepsilon_j \varepsilon_k)(\eta_{ik}^2 - \varepsilon_i \varepsilon_k)]\kappa_j^2$$

$$+ [(\varepsilon_j \eta_{ik} + \eta_{ij} \eta_{jk})^2 - (\eta_{ij}^2 - \varepsilon_i \varepsilon_j)(\eta_{ik}^2 - \varepsilon_i \varepsilon_k)]\kappa_k^2 + 2(\gamma_j \gamma_k + A_i \gamma_i) \kappa_j \kappa_k.$$  \hspace{1cm} (2.20)

Note that the derivative of $(\varepsilon_k \eta_{ij} + \eta_{ik} \eta_{jk})^2 - (\eta_{jk}^2 - \varepsilon_j \varepsilon_k)(\eta_{ik}^2 - \varepsilon_i \varepsilon_k)$ with respect to $\eta_{ij}$ is $2\varepsilon_k \eta_{ij} + 2\varepsilon_k \eta_{ik} \eta_{jk} = 2\varepsilon_k \gamma_j \geq 0$ by the structure condition (1.7), which implies

$$(\varepsilon_k \eta_{ij} + \eta_{ik} \eta_{jk})^2 - (\eta_{jk}^2 - \varepsilon_j \varepsilon_k)(\eta_{ik}^2 - \varepsilon_i \varepsilon_k) \geq \varepsilon_k (\varepsilon_i \eta_{jk} - \varepsilon_j \eta_{ik})^2 \geq 0.$$ \hspace{1cm} (2.21)

Similarly, we have

$$(\varepsilon_j \eta_{ik} + \eta_{ij} \eta_{jk})^2 - (\eta_{ij}^2 - \varepsilon_i \varepsilon_j)(\eta_{ik}^2 - \varepsilon_i \varepsilon_k) \geq 0.$$ \hspace{1cm} (2.22)

Combining (2.20), (2.21), (2.22), $A_i = \eta_{jk}^2 - \varepsilon_j \varepsilon_k > 0$ and the structure condition (1.7), we have $\Delta_i \geq 0$. \hfill \Box

**Remark 10.** One can also take $Q^E$ as a quadratic function of $\kappa_j$ or $\kappa_k$ and define $\Delta_j, \Delta_k$ similarly. By symmetry, under the same conditions as that in Lemma 2.5 if $\eta_{ik}^2 - \varepsilon_i \varepsilon_k > 0$, then $\Delta_j \geq 0$ and if $\eta_{ij}^2 - \varepsilon_i \varepsilon_j > 0$, then $\Delta_k \geq 0$.

**Theorem 2.6.** Suppose $(M, T, \varepsilon, \eta)$ is a weighted triangulated closed connected surface with the weights $\varepsilon : V \to \{0, 1\}$ and $\eta : E \to \mathbb{R}$ satisfying the structure conditions (1.6) and (1.7). $(ijk) \in F$ is a topological triangle in the triangulated surface. Then the admissible space $\Omega_{ijk}^E(\eta)$ of nondegenerate Euclidean discrete conformal structures for the triangle
\{ijk\} \in F is nonempty and simply connected with analytical boundary components. Furthermore, the admissible space \(\Omega^E_{ijk}(\eta)\) in \(r\) could be written as
\[
\Omega^E_{ijk}(\eta) = \mathbb{R}^3_{>0} \setminus \cup_{\alpha \in \Lambda} V_\alpha,
\]
where \(\Lambda = \{q \in \{i, j, k\}| A_q = \eta^2_{st} - \varepsilon s \varepsilon t > 0, \{q, s, t\} = \{i, j, k\}\}, \cup_{\alpha \in \Lambda} V_\alpha\) is a disjoint union of \(V_\alpha\) and \(V_\alpha\) is a closed region in \(\mathbb{R}^3_{>0}\) bounded by an analytical function defined on \(\mathbb{R}^3_{>0}\).

**Proof.** We solve the admissible space of nondegenerate Euclidean discrete conformal structures for the triangle \(\{ijk\} \in F\) by giving a precise description of the space of degenerate Euclidean discrete conformal structures.

Suppose \((r_i, r_j, r_k) \in \mathbb{R}^3_{>0}\) is a degenerate Euclidean discrete conformal structure for the triangle \(\{ijk\} \in F\). Then we have \(Q^E = \kappa_i h_i + \kappa_j h_j + \kappa_k h_k \leq 0\) by Lemma 2.1. By Corollary 2.3, one of \(h_i, h_j, h_k\) is negative and the other two are positive. Without loss of generality, assume \(h_i < 0\) and \(h_j, h_k > 0\), which implies \(A_i = \eta^2_{ijk} - \varepsilon_j \varepsilon_k > 0\) by (2.9) and the structure condition (1.7). Taking \(Q^E \leq 0\) as a quadratic inequality of \(\kappa_i\). Then the solution of \(Q^E \leq 0\), i.e. \(A_i \kappa_i^2 + B_i \kappa_i + C_i \geq 0\), is
\[
\kappa_i \geq \frac{-B_i + \sqrt{\Delta_i}}{2A_i} \text{ or } \kappa_i \leq \frac{-B_i - \sqrt{\Delta_i}}{2A_i}
\]
by Lemma 2.5. Note that
\[
2A_i \kappa_i + B_i = 2(\eta^2_{ijk} - \varepsilon_j \varepsilon_k) \kappa_i - 2(\gamma_k \kappa_j + \gamma_j \kappa_k) = -2h_i,
\]
we have \(\kappa_i > \frac{-B_i}{2A_i} \geq 0\) by \(h_i < 0\) and \(A_i = \eta^2_{ijk} - \varepsilon_j \varepsilon_k > 0\), which implies the solution \((r_i, r_j, r_k) \in \mathbb{R}^3_{>0}\) of \(Q^E \leq 0\) with \(h_i < 0, h_j > 0, h_k > 0\) should be \(\kappa_i \geq \frac{-B_i + \sqrt{\Delta_i}}{2A_i}\).
Therefore, \(\mathbb{R}^3_{>0} \setminus \Omega^E_{ijk}(\eta) \subseteq \cup_{\alpha \in \Lambda} V_\alpha\), where \(\Lambda = \{q \in \{i, j, k\}| A_q = \eta^2_{st} - \varepsilon s \varepsilon t > 0, \{q, s, t\} = \{i, j, k\}\}, V_i = \{(r_i, r_j, r_k) \in \mathbb{R}^3_{>0}| \kappa_i \geq \frac{-B_i + \sqrt{\Delta_i}}{2A_i}\} = \{(r_i, r_j, r_k) \in \mathbb{R}^3_{>0}| r_i \geq \frac{2A_i}{-B_i + \sqrt{\Delta_i}}\} (2.24)
and \(V_j, V_k\) are defined similarly.

Conversely, suppose \((r_i, r_j, r_k) \in \cup_{\alpha \in \Lambda} V_\alpha \subseteq \mathbb{R}^3_{>0}\). Without loss of generality, assume \((r_i, r_j, r_k) \in V_i\) and \(A_i = \eta^2_{ijk} - \varepsilon_j \varepsilon_k > 0\). Then \(\kappa_i \geq \frac{-B_i + \sqrt{\Delta_i}}{2A_i}\), which is equivalent to \(2A_i \kappa_i + B_i \geq \sqrt{\Delta_i}\) by \(A_i > 0\). Taking the square of both sides gives \(A_i \kappa_i^2 + B_i \kappa_i + C_i \geq 0\), which is equivalent to \(Q^E \leq 0\). Therefore, \(\cup_{\alpha \in \Lambda} V_\alpha \subseteq \mathbb{R}^3_{>0} \setminus \Omega^E_{ijk}(\eta)\).

Suppose there exists some \((r_i, r_j, r_k) \in \mathbb{R}^3_{>0}\) with \((r_i, r_j, r_k) \in V_\alpha \cap V_\beta\) for some \(\alpha, \beta \in \{i, j, k\}\). Without loss of generality, assume \((r_i, r_j, r_k) \in V_i \cap V_j\). Then \(A_i > 0, A_j > 0, \)
which implies $\Delta_i \geq 0, \Delta_j \geq 0$ by Lemma 2.5 and Remark 10. By $(r_i, r_j, r_k) \in V_i$, we have $\kappa_i \geq \frac{-B_i + \sqrt{\Delta_i}}{2A_i}$, which implies $h_i = -\frac{1}{2}(2A_i \kappa_i + B_i) \leq -\frac{2}{2} \sqrt{\Delta_i} \leq 0$ by $A_i > 0$ and (2.23). Corollary 2.3 further implies $h_i < 0, h_j > 0, h_k > 0$. The same arguments applies to $(r_i, r_j, r_k) \in V_j$, which shows that $h_j < 0, h_i > 0, h_k > 0$. This is a contradiction. Therefore, $V_\alpha \cap V_\beta = \emptyset$ for $\forall \alpha, \beta \in \Lambda$. This implies $\Omega^E_{ijk}(\eta) = \mathbb{R}^3 \setminus \cup_{\alpha \in \Lambda} V_\alpha$, which is homotopy equivalent to $\mathbb{R}^3_{>0}$ and then simply connected. 

\textbf{Remark 11.} By the proof of Theorem 2.6 if $V_i$ defined by (2.24) is nonempty, then $h_i < 0, h_j > 0, h_k > 0$ for $(r_i, r_j, r_k) \in V_i$.

We further introduce the following parameterized admissible space of nondegenerate Euclidean discrete conformal structures for the triangle $\{ijk\} \in F$

$$\Omega^E_{ijk} = \{(r_i, r_j, r_k, \eta_{ij}, \eta_{ik}, \eta_{jk}) \in \mathbb{R}^3 \times \mathbb{R}^3 | \eta \text{ satisfies (1.6), (1.7) and } (r_i, r_j, r_k) \in \Omega^E_{ijk}(\eta)\}.$$ 

The parameterized admissible space $\Omega^E_{ijk}$ contains some points with good properties.

\textbf{Lemma 2.7.} The point $(r_i, r_j, r_k, \eta_{ij}, \eta_{ik}, \eta_{jk}) = (1, 1, 1, 1, 1, 1)$ is contained in $\Omega^E_{ijk}$. Furthermore, $h_i > 0, h_j > 0, h_k > 0$ at this point.

\textbf{Proof.} It is straightforward to check that $(\eta_{ij}, \eta_{ik}, \eta_{jk}) = (1, 1, 1)$ satisfies the structure conditions (1.6) and (1.7). By the definition (2.9) of $h_i, h_j, h_k$, we have

$$h_i = (\varepsilon_j \varepsilon_k - \eta^2_{ijk}) \kappa_i + \kappa_j \gamma_k + \kappa_k \gamma_j = \varepsilon_j \varepsilon_k + \varepsilon_j + \varepsilon_k + 1 > 0,$$

$$h_j = (\varepsilon_i \varepsilon_k - \eta^2_{ijk}) \kappa_j + \kappa_i \gamma_k + \kappa_k \gamma_i = \varepsilon_i \varepsilon_k + \varepsilon_i + \varepsilon_k + 1 > 0,$$

$$h_k = (\varepsilon_i \varepsilon_j - \eta^2_{ijk}) \kappa_k + \kappa_i \gamma_j + \kappa_j \gamma_i = \varepsilon_i \varepsilon_j + \varepsilon_i + \varepsilon_j + 1 > 0$$

at $(1, 1, 1, 1, 1, 1)$, which implies $Q^E = \kappa_i h_i + \kappa_j h_j + \kappa_k h_k > 0$. Therefore, $(1, 1, 1, 1, 1, 1) \in \Omega^E_{ijk}$ by Lemma 2.1.

Theorem 2.6 have the following corollary on the parameterized admissible space $\Omega^E_{ijk}$.

\textbf{Corollary 2.8.} Suppose $(M, T, \varepsilon, \eta)$ is a weighted triangulated surface with the weights $\varepsilon : V \rightarrow \{0, 1\}$ and $\eta : E \rightarrow \mathbb{R}$ satisfying the structure conditions (1.6) and (1.7). $\{ijk\} \in F$ is a topological triangle in the triangulated surface. Then the parameterized admissible space $\Omega^E_{ijk}$ is connected.

\textbf{Proof.} Set

$$\Gamma = \{(\eta_{ij}, \eta_{ik}, \eta_{jk}) \in \mathbb{R}^3 | (\eta_{ij}, \eta_{ik}, \eta_{jk}) \text{ satisfies (1.6), (1.7)}\}$$

as the space of parameters. We shall prove that $\Gamma$ is connected, from which the connectivity of $\Omega^E_{ijk}$ follows by Theorem 2.6 and the continuity of $Q$ as a function of $(r_i, r_j, r_k, \eta_{ij}, \eta_{ik}, \eta_{jk}) \in \mathbb{R}^3_{>0} \times \mathbb{R}^3$. 

21
It is obviously that $\mathbb{R}^3_{>0} \subset \Gamma$, which is connected. We will show that any point in $\Gamma$ could be connected to $\mathbb{R}^3_{>0}$ by paths in $\Gamma$. As the boundary of $\mathbb{R}^3_{>0}$ is connected to $\mathbb{R}^3_{>0}$, we just need to consider the case that some component of $(\eta_{ij}, \eta_{ik}, \eta_{jk}) \in \Gamma$ is negative. Without loss of generality, assume $\eta_{ij} < 0$, then $\varepsilon_i = \varepsilon_j = 1$ by the structure condition $\eta_{ij} + \varepsilon_i \varepsilon_j > 0$.

In the case of $\varepsilon_k = 0$, the structure conditions (1.6), (1.7) are equivalent to

$$1 + \eta_{ij} > 0, \eta_{ik} > 0, \eta_{jk} > 0$$

and

$$\eta_{jk} + \eta_{ij} \eta_{ik} \geq 0, \eta_{ik} + \eta_{ij} \eta_{jk} \geq 0, \eta_{ik} \eta_{jk} \geq 0.$$  

(2.26)

If $(\eta_{ij}, \eta_{ik}, \eta_{jk}) \in \Gamma$ and $\eta_{ij} < 0$, it is straightforward to check that $(t\eta_{ij}, \eta_{ik}, \eta_{jk})$ satisfies (2.25) and (2.26) for any $t \in [0, 1]$. This implies $(t\eta_{ij}, \eta_{ik}, \eta_{jk}) \in \Gamma$, $\forall t \in [0, 1]$, which is a path connecting $(\eta_{ij}, \eta_{ik}, \eta_{jk})$ and $\mathbb{R}^3_{>0}$. Therefore, $\Gamma$ is connected.

In the case of $\varepsilon_k = 1$, the structure conditions (1.6), (1.7) are equivalent to

$$1 + \eta_{ij} > 0, 1 + \eta_{ik} > 0, 1 + \eta_{jk} > 0$$

and

$$\eta_{jk} + \eta_{ij} \eta_{ik} \geq 0, \eta_{ik} + \eta_{ij} \eta_{jk} \geq 0, \eta_{ij} + \eta_{ik} \eta_{jk} \geq 0.$$  

(2.28)

In this case, the connectivity of $\Gamma$ has been proved in [87]. For completeness, we present the proof here. By the structure conditions (2.27) and (2.28), we have $\eta_{ij} + \eta_{ik} \geq 0$, $\eta_{ij} + \eta_{jk} \geq 0$, $\eta_{ik} + \eta_{jk} \geq 0$, which implies at most one of $\eta_{ij}, \eta_{ik}, \eta_{jk}$ is negative. By the assumption that $\eta_{ij} < 0$, we have $\eta_{ik} \geq 0, \eta_{jk} \geq 0$. It is straightforward to check that $(t\eta_{ij}, \eta_{ik}, \eta_{jk})$ satisfies (2.27) and (2.28) for any $t \in [0, 1]$. This implies $(t\eta_{ij}, \eta_{ik}, \eta_{jk}) \in \Gamma$, $\forall t \in [0, 1]$, which is a path connecting $(\eta_{ij}, \eta_{ik}, \eta_{jk})$ and $\mathbb{R}^3_{>0}$. Therefore, $\Gamma$ is connected.

2.2 Negative semi-definiteness of the Jacobian matrix in the Euclidean case

Suppose $\{ijk\}$ is a nondegenerate Euclidean triangle with edge lengths given by (1.3) and $\theta_i, \theta_j, \theta_k$ are the inner angles at the vertices $i, j, k$ in the triangle respectively. Set $u_i = f_i = \ln r_i$.

Lemma 2.9 ([38, 96]). Suppose $(M, T, \varepsilon, \eta)$ is a weighted triangulated surface with the weights $\varepsilon : V \to \{0, 1\}$ and $\eta : E \to \mathbb{R}$. Suppose $(r_i, r_j, r_k) \in \mathbb{R}^3_{>0}$ is a nondegenerate Euclidean discrete conformal structure for the topological triangle $\{ijk\} \in F$. Then

$$\frac{\partial \theta_i}{\partial u_j} = \frac{\partial \theta_j}{\partial u_i} = \frac{\partial \theta_j}{\partial u_i} = \frac{1}{A_{ij}^2} t_{r}^2 r_i^2 r_j^2 r_k^2 \left[ (\varepsilon_i \varepsilon_j - \eta_{ij}) \kappa_k^2 + \gamma_i \kappa_j \kappa_k + \gamma_j \kappa_i \kappa_k \right] = \frac{r_i^2 r_j^2 r_k^2}{A_{ij}^2} h_k$$

(2.29)
where $A = l_{ij}l_{ik} \sin \theta_i$.

**Proof.** By the chain rules, we have

$$\frac{\partial \theta_i}{\partial u_j} = \frac{\partial \theta_i}{\partial l_{jk}} \frac{\partial l_{jk}}{\partial u_j} + \frac{\partial \theta_i}{\partial l_{ik}} \frac{\partial l_{ik}}{\partial u_j} + \frac{\partial \theta_i}{\partial l_{ij}} \frac{\partial l_{ij}}{\partial u_j}.$$  (2.31)

By the derivative cosine law ([7], Lemma A1), we have

$$\frac{\partial \theta_i}{\partial l_{jk}} = \frac{l_{jk}}{l_{ij}} l_{ij} \cos \theta_k \frac{\partial \theta_k}{\partial l_{jk}}, \quad \frac{\partial \theta_i}{\partial l_{ik}} = -\frac{l_{jk}}{l_{ij}} l_{ij} \cos \theta_j \frac{\partial \theta_j}{\partial l_{ik}}, \quad \frac{\partial \theta_i}{\partial l_{ij}} = -\frac{l_{jk}}{l_{ij}} l_{ij} \cos \theta_i.$$  (2.32)

By the definition (2.4) of $l_{ij}, l_{ik}, l_{jk}$ in $r_i, r_j, r_k$, we have

$$\frac{\partial l_{jk}}{\partial u_j} = \frac{\varepsilon_i r_j}{l_{jk}} + \frac{\eta_j r_j r_k}{l_{ij}}, \quad \frac{\partial l_{ik}}{\partial u_j} = 0, \quad \frac{\partial l_{ij}}{\partial u_j} = \frac{\varepsilon_j r_j + \eta_j r_i r_j}{l_{ij}}.$$  (2.33)

Submitting (2.32) and (2.33) into (2.31), we have

$$\frac{\partial \theta_i}{\partial u_j} = \frac{l_{jk}}{l_{ij}} l_{ij} \frac{\varepsilon_j r_j + \eta_j r_j r_k}{l_{ij}} + \frac{l_{jk}}{l_{ij}} l_{ij} \frac{\varepsilon_j r_j + \eta_j r_i r_j}{l_{ij}}$$

$$= \frac{1}{Al_{ij}} \left[2(\varepsilon_j r_j + \eta_j r_j r_k)(l_{ij}^2 + (l_{ik}^2 - l_{jk}^2)(\varepsilon_j r_j + \eta_j r_i r_j)) \right]$$

$$= \frac{r_i^2 r_j^2 r_k^2}{Al_{ij}^2} \left[(\varepsilon_i r_j - \eta_i r_k)\kappa_k^2 + \gamma_i \kappa_j \kappa_k + \gamma_j \kappa_i \kappa_k \right]$$

$$= \frac{r_i^2 r_j^2 r_k^2}{Al_{ij}^2} \kappa_k h_k,$$  (2.34)

where the cosine law is used in the second line and the definition (2.4) of edge lengths is used in the third line. As the last line of (2.34) is symmetric in $i$ and $j$, we have $\frac{\partial \theta_i}{\partial u_j} = \frac{\partial \theta_j}{\partial u_i}$. Similarly, we have $\frac{\partial \theta_i}{\partial u_k} = \frac{\partial \theta_k}{\partial u_i}, \frac{\partial \theta_i}{\partial u_i} = -\frac{\partial \theta_i}{\partial u_j} - \frac{\partial \theta_k}{\partial u_k}$ follows from $\theta_i + \theta_j + \theta_k = \pi$, $\frac{\partial \theta_i}{\partial u_j} = \frac{\partial \theta_i}{\partial u_i}$ and $\frac{\partial \theta_i}{\partial u_k} = \frac{\partial \theta_i}{\partial u_i}$. □

**Remark 12.** The symmetry $\frac{\partial \theta_i}{\partial u_j} = \frac{\partial \theta_j}{\partial u_i}$ for discrete conformal structures in Lemma 2.9 was previously obtained by Glickenstein [38] and Zhang-Guo-Zeng-Luo-Yau-Gu [96]. Here we give a proof by direct calculations for completeness. Combining (2.17) and Lemma 2.9, we have [38][96]

$$\frac{\partial \theta_i}{\partial u_j} = \frac{h_{ij,k}}{l_{ij}},$$  (2.35)

which provides a nice geometric explanation for the derivative $\frac{\partial \theta_i}{\partial u_j}$. 23
Remark 13. By (2.29), (2.30) and Remark 11, if \((\eta_{ij}, \eta_{ik}, \eta_{jk})\) defined on the triangle \(\{ijk\}\) satisfies the structure conditions (1.6) and (1.7) and \((r_i, r_j, r_k) \in \Omega_{ijk}^E(\eta)\) tends to a point \((\tau_i, \tau_j, \tau_k) \in \partial V_i\) with \(V_i \neq \emptyset\), we have \(\frac{\partial \theta_i}{\partial u_j} \to +\infty, \frac{\partial \theta_j}{\partial u_k} \to +\infty\) and \(\frac{\partial \theta_k}{\partial u_i} \to -\infty\).

Lemma 2.9 shows that the Jacobian matrix

\[
\Lambda_{ijk}^E := \frac{\partial(\theta_i, \theta_j, \theta_k)}{\partial(u_i, u_j, u_k)} = \begin{pmatrix}
\frac{\partial \theta_i}{\partial u_j} & \frac{\partial \theta_i}{\partial u_k} & \frac{\partial \theta_i}{\partial u_k} \\
\frac{\partial \theta_j}{\partial u_i} & \frac{\partial \theta_j}{\partial u_k} & \frac{\partial \theta_j}{\partial u_k} \\
\frac{\partial \theta_k}{\partial u_i} & \frac{\partial \theta_k}{\partial u_j} & \frac{\partial \theta_k}{\partial u_k}
\end{pmatrix}
\]

is symmetric with \(\{t(1,1,1)^T|t \in \mathbb{R}\}\) contained in its kernel. We further have the following result on the rank of the Jacobian matrix \(\Lambda_{ijk}^E\).

Lemma 2.10. Suppose \((M, \mathcal{T}, \varepsilon, \eta)\) is a weighted triangulated surface with the weights \(\varepsilon : V \to \{0, 1\}\) and \(\eta : E \to \mathbb{R}\) satisfying the structure conditions (1.6) and (1.7). \(\{ijk\} \in F\) is a topological triangle in the triangulated surface. Then the rank of the Jacobian matrix \(\Lambda_{ijk}^E = \frac{\partial(\theta_i, \theta_j, \theta_k)}{\partial(u_i, u_j, u_k)}\) is 2 for any nondegenerate Euclidean discrete conformal structure on the triangle \(\{ijk\}\).

Proof. By the chain rules, we have

\[
\frac{\partial(\theta_i, \theta_j, \theta_k)}{\partial(u_i, u_j, u_k)} = \frac{\partial(\theta_i, \theta_j, \theta_k)}{\partial(l_{jk}, l_{ik}, l_{ij})} \cdot \frac{\partial(l_{jk}, l_{ik}, l_{ij})}{\partial(u_i, u_j, u_k)}.
\]

By the derivative cosine law (7, Lemma A1), we have

\[
\frac{\partial(\theta_i, \theta_j, \theta_k)}{\partial(l_{jk}, l_{ik}, l_{ij})} = \frac{1}{A} \begin{pmatrix}
l_{jk} & l_{ik} & l_{ij} \\
l_{ik} & l_{ij} & l_{ij} \\
l_{ij} & l_{ij} & l_{ij}
\end{pmatrix} \begin{pmatrix}
1 & -\cos \theta_k & -\cos \theta_j \\
-\cos \theta_k & 1 & -\cos \theta_i \\
-\cos \theta_j & -\cos \theta_i & 1
\end{pmatrix},
\]

which has rank 2 and kernel \(\{t(l_{jk}, l_{ik}, l_{ij})|t \in \mathbb{R}\}\) for \((l_{jk}, l_{ik}, l_{ij})\) satisfying the triangle inequalities.

Note that \(d_{ij} = \frac{\partial l_{ij}}{\partial u_i} = \frac{\varepsilon_r r_j + \eta_j r_j}{l_{ij}}\). By direct calculations,

\[
\frac{\partial(l_{jk}, l_{ik}, l_{ij})}{\partial(u_i, u_j, u_k)} = \begin{pmatrix}
0 & d_{jk} & d_{kj} \\
0 & d_{ik} & d_{ki} \\
0 & d_{ij} & d_{ji}
\end{pmatrix} = \begin{pmatrix}
l_{jk}^{-1} & l_{ik}^{-1} & l_{ij}^{-1} \\
l_{ik}^{-1} & l_{ij}^{-1} & l_{ij}^{-1}
\end{pmatrix} \cdot \begin{pmatrix}
0 & \varepsilon_r r_j + \eta_j r_k & \varepsilon_k r_k + \eta_j r_j \\
\varepsilon_r r_i + \eta_i r_k & 0 & \varepsilon_k r_k + \eta_i r_i \\
\varepsilon_r r_i + \eta_i r_j & \varepsilon_j r_j + \eta_j r_i & 0
\end{pmatrix} \begin{pmatrix}r_i \\
r_j \\
r_k\end{pmatrix}.
\]
This implies
\[
\det \frac{\partial (l_{jk}, l_{ik}, l_{ij})}{\partial (u_i, u_j, u_k)} = \frac{r_i r_j r_k}{l_{ij} l_{ik} l_{jk}} [2(\varepsilon_i \varepsilon_j \varepsilon_k + \eta_{ij} \eta_{ik} \eta_{jk}) r_i r_j r_k + r_i \gamma_i (\varepsilon_j r_j^2 + \varepsilon_k r_k^2) \\
+ r_j \gamma_j (\varepsilon_i r_i^2 + \varepsilon_k r_k^2) + r_k \gamma_k (\varepsilon_i r_i^2 + \varepsilon_j r_j^2)] \\
\geq 2r_i^2 r_j^2 r_k^2 \\
\frac{l_{ij} l_{ik} l_{jk}}{l_{ij} l_{ik} l_{jk}} (\varepsilon_i \varepsilon_j \varepsilon_k + \eta_{ij} \eta_{ik} \eta_{jk} + \gamma_i \varepsilon_j \varepsilon_k + \gamma_j \varepsilon_i \varepsilon_k + \gamma_k \varepsilon_i \varepsilon_j) \\
= 2r_i^2 r_j^2 r_k^2 \\
\frac{l_{ij} l_{ik} l_{jk}}{l_{ij} l_{ik} l_{jk}} (\varepsilon_i \varepsilon_j + \eta_{ij}) (\varepsilon_i \varepsilon_k + \eta_{ik}) (\varepsilon_j \varepsilon_k + \eta_{jk}) \\
> 0,
\]
where the structure condition (1.7) is used in the second line and the structure condition (1.6) is used in the last line. (2.37) implies that \( \frac{\partial (l_{jk}, l_{ik}, l_{ij})}{\partial (u_i, u_j, u_k)} \) is nonsingular.

Combining (2.36), (2.37) and the fact that the rank of \( \frac{\partial (\theta_i, \theta_j, \theta_k)}{\partial (l_{jk}, l_{ik}, l_{ij})} \) is 2 for \( l_{jk}, l_{ik}, l_{ij} \) satisfying the triangle inequalities, we have the rank of the Jacobian matrix \( \Lambda^E_{ijk} = \frac{\partial (\theta_i, \theta_j, \theta_k)}{\partial (u_i, u_j, u_k)} \) is 2 for any nondegenerate Euclidean discrete conformal structure on the triangle \{ijk\}. □

As a direct consequence of Lemma 2.10, we have the following negative semi-definiteness for the Jacobian matrix \( \Lambda^E_{ijk} \).

**Theorem 2.11.** Suppose \((M, T, \varepsilon, \eta)\) is a weighted triangulated surface with the weights \(\varepsilon : V \to \{0, 1\}\) and \(\eta : E \to \mathbb{R}\) satisfying the structure conditions (1.6) and (1.7). \{ijk\} \(\in F\) is a topological triangle in the triangulated surface. Then the Jacobian matrix \( \Lambda^E_{ijk} = \frac{\partial (\theta_i, \theta_j, \theta_k)}{\partial (u_i, u_j, u_k)} \) is negative semi-definite with rank 2 and kernel \( \{t(1,1,1)^T | t \in \mathbb{R}\} \) for any nondegenerate Euclidean discrete conformal structure on the triangle \{ijk\}.

**Proof.** By Lemma 2.10, the Jacobian matrix \( \Lambda^E_{ijk} \) has two nonzero eigenvalues and one zero eigenvalue. By the continuity of eigenvalues of \( \Lambda^E_{ijk} = \frac{\partial (\theta_i, \theta_j, \theta_k)}{\partial (u_i, u_j, u_k)} \) as a function of \((r_i, r_j, r_k, \eta_{ij}, \eta_{ik}, \eta_{jk}) \in \Omega^E_{ijk}\) and the connectivity of parameterized admissible space \( \Omega^E_{ijk}\) in Corollary 2.8, to prove \( \frac{\partial (\theta_i, \theta_j, \theta_k)}{\partial (u_i, u_j, u_k)} \) is negative semi-definite, we just need to prove \( \frac{\partial (\theta_i, \theta_j, \theta_k)}{\partial (u_i, u_j, u_k)} \) is negative semi-definite with rank 2 at some point in \( \Omega^E_{ijk}\). By Lemma 2.7, \( h_i > 0, h_j > 0, h_k > 0 \) at \((r_i, r_j, r_k, \eta_{ij}, \eta_{ik}, \eta_{jk}) = (1,1,1,1,1,1) \in \Omega^E_{ijk}\), which implies the \( \frac{\partial \theta_i}{\partial u_j}, \frac{\partial \theta_j}{\partial u_k}, \frac{\partial \theta_k}{\partial u_i} \) are positive by (2.17) and (2.35). Then by the following well-known result from linear algebra, \(-\Lambda^E_{ijk}\) is positive semi-definite with rank 2 and null space \( \{t(1,1,1)^T | t \in \mathbb{R}\} \) at \((r_i, r_j, r_k, \eta_{ij}, \eta_{ik}, \eta_{jk}) = (1,1,1,1,1,1) \in \Omega^E_{ijk}\).

**Lemma 2.12.** Suppose \( A = [a_{ij}]_{n \times n} \) is a symmetric matrix.
(a) If \( a_{ii} > \sum_{j \neq i} |a_{ij}| \) for all indices \( i \), then \( A \) is positive definite.

(b) If \( a_{ii} > 0 \) and \( a_{ij} < 0 \) for all \( i \neq j \) so that \( \sum_{i=1}^{n} a_{ij} = 0 \) for all \( j \), then \( A \) is positive semi-definite so that its kernel is 1-dimensional.

One can refer to \([7]\) for a proof of Lemma 2.12. Therefore, \( \Lambda_{ijk}^E \) is negative semi-definite with rank 2 and kernel \( \{t(1, 1, 1)^T | t \in \mathbb{R} \} \) at \((1, 1, 1, 1, 1) \in \Omega_{ijk}^E \).

As a direct corollary of Theorem 2.11 we have the following result on the Jacobian matrix \( \Lambda^E = \frac{\partial(K_1, \ldots, K_N)}{\partial(u_1, \ldots, u_N)} \).

**Corollary 2.13.** Suppose \((M, T, \varepsilon, \eta)\) is a weighted triangulated surface with the weights \( \varepsilon : V \to \{0, 1\} \) and \( \eta : E \to \mathbb{R} \) satisfying the structure conditions \((1.6)\) and \((1.7)\). Then the Jacobian matrix \( \Lambda^E = \frac{\partial(K_1, \ldots, K_N)}{\partial(u_1, \ldots, u_N)} \) is symmetric and positive semi-definite with rank \( N-1 \) and kernel \( \{t1 \in \mathbb{R}^N | t \in \mathbb{R} \} \) for all nondegenerate Euclidean discrete conformal structures on \((M, T, \varepsilon, \eta)\).

**Proof.** This follows from Theorem 2.11 and the fact that \( \Lambda^E = -\sum_{\{ijk\} \in F} \Lambda_{ijk}^E \), where \( \Lambda_{ijk}^E \) is extended by zeros to an \( N \times N \) matrix so that \( \Lambda_{ijk}^E \) acts on a vector \( v_1, \ldots, v_N \) only on the coordinates corresponding to vertices \( v_i, v_j \) and \( v_k \) in the triangle \( \{ijk\} \).

**Remark 14.** Under an additional condition that the signed distance of geometric center to the edges are all positive for any triangle \( \{ijk\} \in F \), Glickenstein-Thomas \([41]\) obtained the positive semi-definiteness of the Jacobian matrix \( \Lambda^E = \frac{\partial(K_1, \ldots, K_N)}{\partial(u_1, \ldots, u_N)} \). Corollary 2.13 generalizes Glickenstein-Thomas’s result in that it allows that some of the signed distance to be negative. For example, in the case that \( \varepsilon \equiv 1 \) and \( \eta \equiv 2 \) which satisfies the structures conditions \((1.6)\) and \((1.7)\), if \( r : \mathbb{V} \to (0, +\infty) \) is a map with \( r \equiv 1 \) except \( r_i = 1/5 \) for some vertex \( i \in \mathbb{V} \), then \( r \) is a nondegenerate Euclidean discrete conformal structure on \((M, T, \varepsilon, \eta)\) with \( \Lambda_{ijk}^E \) positive semi-definite by Corollary 2.13. However, in this case we have \( h_i < 0, h_j > 0, h_k > 0 \) for a triangle \( \{ijk\} \) at \( t \), which implies \( h_{jk,i} < 0, h_{jk,j} > 0, h_{jk,k} > 0 \) by \((2.17)\).

### 2.3 Rigidity of Euclidean discrete conformal structures

Theorem 2.6 and Lemma 2.9 imply the following function

\[
\mathcal{E}_{ijk}(u_i, u_j, u_k) = \int_{(u_i, u_j, u_k)} \theta_i du_i + \theta_j du_j + \theta_k du_k \quad (2.38)
\]

is a well-defined smooth function on \( \Omega_{ijk}^E(\eta) \) with \( \nabla_u \mathcal{E}_{ijk} = \theta_i \) and \( \mathcal{E}_{ijk}(u_i + t, u_j + t, u_k + t) = \mathcal{E}_{ijk}(u_i, u_j, u_k) + t\pi \), which is called the Ricci energy function for the triangle \( \{ijk\} \).

By Theorem 2.11 \( \mathcal{E}_{ijk}(u_i, u_j, u_k) \) is a locally concave function defined on \( \Omega_{ijk}^E(\eta) \). Set

\[
\mathcal{E}(u_1, \ldots, u_N) = 2\pi \sum_{i \in V} u_i - \sum_{\{ijk\} \in F} \mathcal{E}_{ijk}(u_i, u_j, u_k) \quad (2.39)
\]
to be the Ricci energy function defined on the admissible space $\Omega^E$ of nondegenerate Euclidean discrete conformal structures for $(M, \mathcal{T}, \varepsilon, \eta)$. Then $\mathcal{E}$ is a locally convex function defined on $\Omega^E$ with $\mathcal{E}(u_1 + t, \cdots, u_N + t) = \mathcal{E}(u_1, \cdots, u_N) + 2t\pi \chi(M)$ and $\nabla_u \mathcal{E} = K_i$ by Corollary 2.14 from which the local rigidity of Euclidean discrete conformal structures follows by the following well-known result from analysis.

**Lemma 2.14.** If $W : \Omega \to \mathbb{R}$ is a $C^2$-smooth strictly convex function defined on a convex domain $\Omega \subseteq \mathbb{R}^n$, then its gradient $\nabla W : \Omega \to \mathbb{R}^n$ is injective.

To prove the global rigidity of Euclidean discrete conformal structures, we need to extend the inner angles in a triangle $\{ijk\} \in F$ defined for nondegenerate Euclidean discrete conformal structures to be a globally defined function for $(r_i, r_j, r_k) \in \mathbb{R}^3_{>0}$.

**Lemma 2.15.** Suppose $(M, \mathcal{T}, \varepsilon, \eta)$ is a weighted triangulated surface with the weights $\varepsilon : V \to \{0, 1\}$ and $\eta : E \to \mathbb{R}$ satisfying the structure conditions (1.6) and (1.7). $\{ijk\} \in F$ is a topological triangle in the triangulated surface. Then the inner angles $\theta_i, \theta_j, \theta_k$ of the triangle $\{ijk\}$ defined for nondegenerate Euclidean discrete conformal structures could be extended by constants to be continuous functions $\tilde{\theta}_i, \tilde{\theta}_j, \tilde{\theta}_k$ defined for $(r_i, r_j, r_k) \in \mathbb{R}^3_{>0}$.

**Proof.** By Theorem 2.6, $\Omega_{ijk}^E(\eta) = \mathbb{R}^3_{>0} \setminus \bigcup_{\alpha \in \Lambda} V_\alpha$, where $\Lambda = \{q \in \{i, j, k\} \mid A_q = \eta_{st} - \varepsilon_{se} > 0, \{q, s, t\} = \{i, j, k\}\}$ and $V_\alpha$ is a closed region in $\mathbb{R}^3_{>0}$ bounded by the analytical function in (2.24) defined on $\mathbb{R}^2_{>0}$. If $\Lambda = \emptyset$, then $\Omega_{ijk}^E(\eta) = \mathbb{R}^3_{>0}$ and $\theta_i, \theta_j, \theta_k$ is defined for all $(r_i, r_j, r_k) \in \mathbb{R}^3_{>0}$.

Without loss of generality, suppose $\Lambda \neq \emptyset$ and $V_i$ is a connected component of $\mathbb{R}^3_{>0} \setminus \Omega_{ijk}^E(\eta)$. If $(r_i, r_j, r_k) \in \Omega_{ijk}^E(\eta)$ tends to a point $(\bar{r}_i, \bar{r}_j, \bar{r}_k)$ in the boundary $\partial V_i$ of $V_i$ in $\mathbb{R}^3_{>0}$, we have

$$4l_{ij}^2l_{ik}^2 \sin^2 \theta_i = (l_{ij} + l_{ik} + l_{jk})(l_{ij} + l_{ik} - l_{jk})(l_{ij} - l_{ik} + l_{jk})(-l_{ij} + l_{ik} + l_{jk}) \to 0.$$ 

Note that for any $r_i, r_j > 0, \varepsilon_i, r_i^2 + \varepsilon_j, r_j^2 + 2\eta_{ij}r_i r_j \geq 2(\varepsilon_i \varepsilon_j + \eta_{ij})r_i r_j > 0$ by the structure condition (1.6), we have $l_{ij}, l_{ik}$ tend to positive constants and $\sin \theta_i$ tends to zero as $(r_i, r_j, r_k) \to (\bar{r}_i, \bar{r}_j, \bar{r}_k)$, which implies $\theta_i$ tends to 0 or $\pi$. Similarly, we have $\theta_j, \theta_k$ tends to 0 or $\pi$.

Note that at $(\bar{r}_i, \bar{r}_j, \bar{r}_k) \in \partial V_i$, we have $h_i \leq 0$ by (2.24) and (2.23). Corollary 2.3 further implies that $h_i < 0, h_j > 0$ and $h_k > 0$ at $(\bar{r}_i, \bar{r}_j, \bar{r}_k) \in \partial V_i$. By the continuity of $h_i, h_j, h_k$ of $(r_i, r_j, r_k) \in \mathbb{R}^3_{>0}$, there exists some neighborhood $U$ of $(\bar{r}_i, \bar{r}_j, \bar{r}_k)$ in $\mathbb{R}^3_{>0}$ such that $h_i < 0, h_j > 0, h_k > 0$ for $(r_i, r_j, r_k) \in \Omega_{ijk}^E(\eta) \cap U$. Combining $h_k > 0, (2.35)$ and (2.17), we have

$$\frac{\partial \theta_i}{\partial u_j} = \frac{r_i^2r_j^2r_k^2}{Al_{ij}} \kappa_k h_k > 0.$$
for \((r_i, r_j, r_k) \in \Omega^E_{ijk}(\eta) \cap U\). Similarly, we have \(\frac{\partial \theta_i}{\partial u_k} > 0\) for \((r_i, r_j, r_k) \in \Omega^E_{ijk}(\eta) \cap U\). By Lemma \ref{lem:convex_function}, we have \(\frac{\partial \theta_i}{\partial u_k} = -\frac{\partial \theta_j}{\partial u_j} - \frac{\partial \theta_k}{\partial u_k} < 0\) for \((r_i, r_j, r_k) \in \Omega^E_{ijk}(\eta) \cap U\). By the form of \(V_i\), i.e.,

\[
V_i = \{(r_i, r_j, r_k) \in \mathbb{R}^3_+ | r_i \geq -B_i + \sqrt{\Delta_i}\} = \{(r_i, r_j, r_k) \in \mathbb{R}^3_+ | r_i \leq \frac{2A_i}{-B_i + \sqrt{\Delta_i}}\},
\]

we have \(\theta_i \to \pi\) as \((r_i, r_j, r_k) \to (\overline{r}_i, \overline{r}_j, \overline{r}_k)\), which implies \(\theta_j \to 0\), \(\theta_k \to 0\) as \((r_i, r_j, r_k) \to (\overline{r}_i, \overline{r}_j, \overline{r}_k)\) by \(\theta_i + \theta_j + \theta_k = \pi\). The same arguments apply to the other components of \(\mathbb{R}^3_+ \setminus \Omega^E_{ijk}(\eta)\). Then we can extend the inner angle functions \(\theta_i, \theta_j, \theta_k\) defined on \(\Omega^E_{ijk}(\eta)\) to be continuous functions defined on \(\mathbb{R}^3_+\) by setting

\[
\tilde{\theta}_i(r_i, r_j, r_k) = \begin{cases} 
\theta_i, & \text{if } (r_i, r_j, r_k) \in \Omega^E_{ijk}(\eta); \\
\pi, & \text{if } (r_i, r_j, r_k) \in V_i; \\
0, & \text{otherwise.}
\end{cases} \tag{2.40}
\]

By Lemma \ref{lem:extension} we can further extend the combinatorial curvature function \(K\) defined on \(\Omega^E\) to be defined for \(r \in \mathbb{R}^N_+\) by

\[
\tilde{K}_i = 2\pi - \sum_{\{ijk\} \in F} \tilde{\theta}_i,
\]

where \(\tilde{\theta}_i\) is the extension of \(\theta_i\) defined in Lemma \ref{lem:extension} by (2.40). The extended combinatorial curvature \(\tilde{K}_i\) still satisfies the discrete Gauss-Bonnet formula \(\sum_{i=1}^N \tilde{K}_i = 2\pi \chi(M)\). The vector \(r \in \mathbb{R}^N_+\) is called a generalized Euclidean discrete conformal structure for the weighted triangulated surface \((M, \mathcal{T}, \varepsilon, \eta)\). Sometimes, we also call a vector \(u = (u_1, \cdots, u_N) = (\ln r_1, \cdots, \ln r_N) \in \mathbb{R}^N\) as a generalized Euclidean discrete conformal structure for the weighted triangulated surface \((M, \mathcal{T}, \varepsilon, \eta)\).

Recall the following definition of closed continuous 1-form and extension of locally convex function of Luo \cite{luo2001generalized}, which is a generalization of Bobenko-Pinkall-Spingborn’s extension introduced in \cite{bobenko1996discrete}.

**Definition 5** (\cite{luo2001generalized}, Definition 2.3). A differential 1-form \(w = \sum_{i=1}^n a_i(x)dx^i\) in an open set \(U \subset \mathbb{R}^n\) is said to be continuous if each \(a_i(x)\) is continuous on \(U\). A continuous differential 1-form \(w\) is called closed if \(\int_{\partial \tau} w = 0\) for each triangle \(\tau \subset U\).

**Theorem 2.16** (\cite{luo2001generalized}, Corollary 2.6). Suppose \(X \subset \mathbb{R}^n\) is an open convex set and \(A \subset X\) is an open subset of \(X\) bounded by a real analytic codimension-1 submanifold in \(X\). If \(w = \sum_{i=1}^n a_i(x)dx_i\) is a continuous closed 1-form on \(A\) so that \(F(x) = \int_a^x w\) is locally convex on \(A\) and each \(a_i\) can be extended continuous to \(X\) by constant functions to a function \(\tilde{a}_i\) on \(X\), then \(\tilde{F}(x) = \int_a^x \sum_{i=1}^n \tilde{a}_i(x)dx_i\) is a \(C^1\)-smooth convex function on \(X\) extending \(F\).
By Lemma 2.15 and Theorem 2.16, the locally concave function $E_{ijk}$ defined by (2.38) for nondegenerate Euclidean discrete conformal structures for a triangle $\{ijk\} \in F$ could be extended to be a $C^1$ smooth concave function

$$\tilde{E}_{ijk}(u_i, u_j, u_k) = \int_{(\pi_i, \pi_j, \pi_k)} \tilde{\theta}_i du_i + \tilde{\theta}_j du_j + \tilde{\theta}_k du_k$$

(2.42)

defined on $\mathbb{R}^3$ with $\nabla_{u_i} \tilde{E}_{ijk} = \tilde{\theta}_i$. As a result, the locally convex function $E$ defined for nondegenerate Euclidean discrete conformal structures on a weighted triangulated surface could be extended to be a $C^1$ smooth convex function

$$\tilde{E}(u_1, \cdots, u_N) = 2\pi \sum_{i \in V} u_i - \sum_{\{ijk\} \in F} \tilde{E}_{ijk}(u_i, u_j, u_k)$$

(2.43)

defined on $\mathbb{R}^N$ with $\nabla_{u_i} \tilde{E} = \tilde{K}_i = 2\pi - \sum \tilde{\theta}_i$.

Using the extended Ricci energy function $\tilde{E}$, we can prove the following rigidity for generalized Euclidean discrete conformal structures on polyhedral surfaces, which is a generalization of Theorem 1.1 (a).

**Theorem 2.17.** Suppose $(M, T, \varepsilon, \eta)$ is a weighted triangulated surface with the weights $\varepsilon : V \to \{0, 1\}$ and $\eta : E \to \mathbb{R}$ satisfying the structure conditions (1.6) and (1.7). If there exists a nondegenerate Euclidean discrete conformal structure $r_A \in \Omega^E$ and a generalized Euclidean discrete conformal structure $r_B \in \mathbb{R}^N > 0$ such that $K(r_A) = \tilde{K}(r_B)$. Then $r_A = cr_B$ for some positive constant $c \in \mathbb{R}$.

**Proof.** Set

$$f(t) = \tilde{E}((1 - t)u_A + tu_B) = 2\pi \sum_{i=1}^{N} [(1 - t)u_{A,i} + tu_{B,i}] + \sum_{\{ijk\} \in F} f_{ijk}(t),$$

where $f_{ijk}(t) = -\tilde{E}_{ijk}((1 - t)u_A + tu_B)$. Then $f(t)$ is a $C^1$ smooth convex function for $t \in [0, 1]$ with $f'(0) = \tilde{E}'(1)$, which implies that $f'(t) = f'(0)$ for any $t \in [0, 1]$. Note that the admissible space $\Omega^E$ of nondegenerate Euclidean discrete conformal structures is an open subset of $\mathbb{R}^N$, there exists $\epsilon > 0$ such that $(1 - t)u_A + tu_B$ is nondegenerate for $t \in [0, \epsilon]$. Note that $f(t)$ is smooth for $t \in [0, \epsilon]$, we have

$$f''(t) = (u_B - u_A)\Lambda^E(u_B - u_A)^T = 0, \forall t \in [0, \epsilon],$$

which implies $u_B - u_A = \lambda(1, \cdots, 1)$ for some constant $\lambda \in \mathbb{R}$ by Corollary 2.13. Therefore, $r_B = cr_A$ with $c = e^\lambda$. \qed
3 Hyperbolic discrete conformal structures

3.1 Admissible space of hyperbolic discrete conformal structures for a triangle

In this subsection, we investigate the admissible space $\Omega_{ijk}^H(\eta)$ of hyperbolic discrete conformal structure for a topological triangle $\{ijk\} \in F$ with the weights $\varepsilon : V \to \{0,1\}$ and $\eta : E \to \mathbb{R}$ satisfying the structure conditions (1.6) and (1.7). To simplify the notations, set

$$S_i = e^{f_i}, C_i = \sqrt{1 + \varepsilon_i e^{2f_i}}, \kappa_i = \frac{C_i}{S_i}. \quad (3.1)$$

Then

$$C_i^2 - \varepsilon_i S_i^2 = 1 \quad (3.2)$$

and the edge length $l_{ij}$ is determined by

$$\cosh l_{ij} = C_i C_j + \eta_{ij} S_i S_j. \quad (3.3)$$

Note that (1.4) defines a positive number by the structure condition (1.6) and the inequality $(1 + a^2)(1 + b^2) \geq (1 + ab)^2$. Paralleling to Lemma 2.1 for the Euclidean background geometry, we have the following result on the triangle inequalities in the hyperbolic background geometry.

**Lemma 3.1.** Suppose $(M, T, \varepsilon, \eta)$ is a weighted triangulated surface with the weights $\varepsilon : V \to \{0,1\}$ and $\eta : E \to \mathbb{R}$. $\{ijk\} \in F$ is a topological triangle in the triangulation. Then the positive edge lengths $l_{ij}, l_{ik}, l_{jk}$ defined by (1.4) satisfy the triangle inequalities if and only if $Q^H > 0$, where

$$Q^H = (\varepsilon_i \varepsilon_k - \eta_{ik}^2)\kappa_i^2 + (\varepsilon_i \varepsilon_k - \eta_{ik}^2)\kappa_j^2 + (\varepsilon_i \varepsilon_k - \eta_{ik}^2)\kappa_k^2 + 2\gamma_i \kappa_i \kappa_k + 2\gamma_j \kappa_j \kappa_k$$

$$+ 2\gamma_k \kappa_i \kappa_j + \varepsilon_i \eta_{jk}^2 + \varepsilon_k \eta_{ij}^2 + \varepsilon_k \eta_{ij}^2 + 2\eta_{ij} \eta_{ik} \eta_{jk} - \varepsilon_i \varepsilon_j \varepsilon_k. \quad (3.4)$$

**Proof.** The positive edge lengths $l_{ij}, l_{ik}, l_{jk}$ defined by (1.4) for the topological triangle $\{ijk\}$ satisfy the triangle inequalities if and only if

$$0 < 4 \sinh \left( \frac{l_{ij} + l_{ik} + l_{jk}}{2} \right) \sinh \left( \frac{l_{ij} + l_{ik} - l_{jk}}{2} \right) \sinh \left( \frac{l_{ij} - l_{ik} + l_{jk}}{2} \right) \sinh \left( \frac{-l_{ij} + l_{ik} + l_{jk}}{2} \right)$$

$$= (\cosh(l_{ik} + l_{jk}) - \cosh l_{ij})(\cosh l_{ij} - \cosh(l_{jk} - l_{ik}))$$

$$= 1 + 2 \cosh l_{ij} \cosh l_{ik} \cosh l_{jk} - \cosh^2 l_{ij} - \cosh^2 l_{ik} - \cosh^2 l_{jk}. \quad (3.5)$$
Lemma 3.2. Suppose \( (\varepsilon, \eta) \) is a weighted triangulated surface with the weights \( \varepsilon : V \to \{0, 1\} \) and \( \eta : E \to \mathbb{R} \) satisfying the structure conditions (1.6) and (1.7). \( \{ijk\} \in F \) is a topological triangle in the triangulated surface. If we further have \( \varepsilon_i \varepsilon_j - \eta_{ij}^2 \geq 0 \) and \( \varepsilon_i \varepsilon_j - \eta_{ij}^2 \geq 0 \), then the admissible space \( \Omega^H_{ijk}(\eta) \) of hyperbolic discrete conformal structures \( (f_i, f_j, f_k) \) for the triangle \( \{ijk\} \) is \( \mathbb{R}^3 \) and then simply connected.

Proof. By Lemma 3.2 and the structure conditions (1.6) and (1.7), we have \( Q^H \geq 0 \) under the condition \( \varepsilon_i \varepsilon_j - \eta_{ij}^2 \geq 0 \) and \( \varepsilon_i \varepsilon_j - \eta_{ij}^2 \geq 0 \). If \( Q^H = 0 \), then we have \( G = 0 \) and

\[
(\varepsilon_i \varepsilon_j - \eta_{ij}^2)\kappa_i^2 + (\varepsilon_i \varepsilon_j - \eta_{ij}^2)\kappa_j^2 + (\varepsilon_i \varepsilon_j - \eta_{ij}^2)\kappa_k^2 + 2\gamma_i \kappa_j \kappa_k + 2\gamma_j \kappa_i \kappa_k + 2\gamma_k \kappa_i \kappa_j = 0,
\]

where (3.2) is used in the second and third equality. This completes the proof. \( \square \)

Comparing Lemma 2.1 with Lemma 3.1, we find that \( Q^H \) defined for hyperbolic background geometry in Lemma 3.1 formally has one term more than \( Q^E \) defined for Euclidean background geometry in Lemma 2.1. Set

\[
G = 2\eta_{ij} \eta_{jk} \eta_{ik} + \varepsilon_i \varepsilon_j \eta_{jk}^2 + \varepsilon_j \eta_{ik}^2 + \varepsilon_k \eta_{ij}^2 - \varepsilon_i \varepsilon_j \varepsilon_k
\]

(3.7) to be the term.

Lemma 3.2. Under the structure conditions (1.6) and (1.7), \( G \geq 0 \).

Proof. Taking \( G \) as a function of \( \eta_{ij}, \eta_{ik}, \eta_{jk} \), then \( \frac{\partial G}{\partial \eta_{ij}} = 2\gamma_k \geq 0 \) by the structure condition (1.7), which implies

\[
G(\eta_{ij}, \eta_{ik}, \eta_{jk}) \geq G(-\varepsilon_i \varepsilon_j, \eta_{ik}, \eta_{jk}) = (\varepsilon_i \eta_{jk} - \varepsilon_j \eta_{ik})^2 \geq 0
\]

by the structure condition (1.6). \( \square \)

As a direct corollary of Lemma 3.1 and Lemma 3.2, we have the following result.

Lemma 3.3. Suppose \( (M, T, \varepsilon, \eta) \) is a weighted triangulated surface with the weights \( \varepsilon : V \to \{0, 1\} \) and \( \eta : E \to \mathbb{R} \) satisfying the structure conditions (1.6) and (1.7). \( \{ijk\} \in F \) is a topological triangle in the triangulated surface. If we further have \( \varepsilon_i \varepsilon_j - \eta_{ij}^2 \geq 0 \) and \( \varepsilon_i \varepsilon_j - \eta_{ij}^2 \geq 0 \), then the admissible space \( \Omega^H_{ijk}(\eta) \) of hyperbolic discrete conformal structures \( (f_i, f_j, f_k) \) for the triangle \( \{ijk\} \) is \( \mathbb{R}^3 \) and then simply connected.

Proof. By Lemma 3.2 and the structure conditions (1.6) and (1.7), we have \( Q^H \geq 0 \) under the condition \( \varepsilon_i \varepsilon_j - \eta_{ij}^2 \geq 0 \) and \( \varepsilon_i \varepsilon_j - \eta_{ij}^2 \geq 0 \). If \( Q^H = 0 \), then we have \( G = 0 \) and

\[
(\varepsilon_i \varepsilon_j - \eta_{ij}^2)\kappa_i^2 + (\varepsilon_i \varepsilon_j - \eta_{ij}^2)\kappa_j^2 + (\varepsilon_i \varepsilon_j - \eta_{ij}^2)\kappa_k^2 + 2\gamma_i \kappa_j \kappa_k + 2\gamma_j \kappa_i \kappa_k + 2\gamma_k \kappa_i \kappa_j = 0,
\]
which formally equals \( Q^E \). Then the proof for Lemma 2.4 shows that this is impossible. Therefore, \( Q^H > 0 \) and the admissible space \( \Omega_{ijk}^H(\eta) \) of hyperbolic discrete conformal structures for the triangle \( \{ijk\} \) is \( \mathbb{R}^3 \).

By Lemma 3.3, we just need to study the admissible space \( \Omega_{ijk}^H(H) \) in the case that one of \( \varepsilon_j\varepsilon_k - \eta_{jk}^2, \varepsilon_i\varepsilon_k - \eta_{ik}^2, \varepsilon_i\varepsilon_j - \eta_{ij}^2 \) is negative. Following the case for Euclidean background geometry, we will give a precise description of the admissible space \( \Omega_{ijk}^H(\eta) \) by solving the space of degenerate hyperbolic discrete conformal structures for the triangle \( \{ijk\} \).

Set \( h_i, h_j, h_k \) as that in (2.9), then
\[
Q^H = \kappa_i h_i + \kappa_j h_j + \kappa_k h_k + G. \tag{3.8}
\]

Parallelling to the Euclidean case, we have the following result on the signs of \( h_i, h_j, h_k \) for degenerate hyperbolic discrete conformal structures for a triangle \( \{ijk\} \).

**Lemma 3.4.** Suppose \((M, T, \varepsilon, \eta)\) is a weighted triangulated surface with the weights \( \varepsilon : V \to \{0, 1\} \) and \( \eta : E \to \mathbb{R} \) satisfying the structure conditions (1.6) and (1.7). \( \{ijk\} \in F \) is a topological triangle in the triangulated surface. If \((f_i, f_j, f_k)\) is a degenerate hyperbolic discrete conformal structure for the triangle \( \{ijk\} \), then one of \( h_i, h_j, h_k \) is negative and the others are positive.

**Proof.** By Lemma 3.1, if \((f_i, f_j, f_k)\) is a degenerate hyperbolic discrete conformal structure for the triangle \( \{ijk\} \), then \( Q^H = \kappa_i h_i + \kappa_j h_j + \kappa_k h_k + G \leq 0 \), which implies \( Q^E = \kappa_i h_i + \kappa_j h_j + \kappa_k h_k \leq 0 \) by Lemma 3.2 and at least one of \( h_i, h_j, h_k \) is nonpositive. Following the arguments for Lemma 2.2, there is no subset \( \{s, t\} \subseteq \{i, j, k\} \) such that \( h_s \leq 0 \) and \( h_t \leq 0 \). Following the arguments for Corollary 2.3, one of \( h_i, h_j, h_k \) is negative and the others are positive. As the proof is parallelling to that for Lemma 2.2 and Corollary 2.3 in the Euclidean background geometry, we omit the details here. □

Suppose \((f_i, f_j, f_k) \in \mathbb{R}^3 \) is a degenerate hyperbolic discrete conformal structure for the triangle \( \{ijk\} \in F \), then one of \( h_i, h_j, h_k \) is negative by Lemma 3.4. Without loss of generality, assume \( h_i < 0 \), then \( (\eta_{jk}^2 - \varepsilon_j\varepsilon_k)\kappa_i > \gamma_j\kappa_j + \gamma_k\kappa_j \geq 0 \) by the structure condition (1.7). As \((f_i, f_j, f_k)\) is a degenerate hyperbolic discrete conformal structure, we have \( Q^H \leq 0 \) by Lemma 3.1 which is equivalent to
\[
A_i\kappa_i^2 + B_i\kappa_i + C_i \geq 0, \tag{3.9}
\]
where
\[
A_i = \eta_{jk}^2 - \varepsilon_j\varepsilon_k > 0,
B_i = -2\gamma_j\kappa_k - 2\gamma_k\kappa_j \leq 0, 
C_i = (\eta_{jk}^2 - \varepsilon_j\varepsilon_k)\kappa_i^2 + (\eta_{ij}^2 - \varepsilon_i\varepsilon_j)\kappa_k^2 - 2\gamma_i\kappa_k\kappa_j - G.
\]}

Parallelling to the Euclidean case, we have the following result for the discriminant of (3.9) in the hyperbolic background geometry.
Lemma 3.5. Suppose \((M, \mathcal{T}, \varepsilon, \eta)\) is a weighted triangulated surface with the weights \(\varepsilon : V \to \{0, 1\}\) and \(\eta : E \to \mathbb{R}\) satisfying the structure conditions (1.6) and (1.7). \(\{ijk\} \in F\) is a topological triangle in the triangulated surface. If \(A_i = \eta_{jk}^2 - \varepsilon_j \varepsilon_k > 0\), then the discriminant \(\Delta_i = B_i^2 - 4A_iC_i\) for (3.9) is nonnegative, where \(A_i, B_i, C_i\) are defined by (3.10).

Proof. By Lemma 3.2, we have \(G \geq 0\). Then the proof is reduced to the case in Lemma 2.5, which has been completed. 

Remark 15. One can also take \(QH\) as a quadratic function of \(\kappa_j\) or \(\kappa_k\) and define \(\Delta_j, \Delta_k\) similarly. By symmetry, under the same conditions as that in Lemma 3.5, if \(A_j = \eta_{ik}^2 - \varepsilon_i \varepsilon_k > 0\), then \(\Delta_j \geq 0\) and if \(A_k = \eta_{ij}^2 - \varepsilon_i \varepsilon_j > 0\), then \(\Delta_k \geq 0\).

Parallelling to Theorem 2.6, we have the following characterization for the admissible space \(\Omega_{ijk}(\eta)\) of hyperbolic discrete conformal structures for the triangle \(\{ijk\} \in F\).

Theorem 3.6. Suppose \((M, \mathcal{T}, \varepsilon, \eta)\) is a weighted triangulated surface with the weights \(\varepsilon : V \to \{0, 1\}\) and \(\eta : E \to \mathbb{R}\) satisfying the structure conditions (1.6) and (1.7). \(\{ijk\} \in F\) is a topological triangle in the triangulated surface. Then the admissible space \(\Omega_{ijk}(\eta) \subseteq \mathbb{R}^3\) of hyperbolic discrete conformal structures \(f : V \to \mathbb{R}\) for the triangle \(\{ijk\} \in F\) is nonempty and simply connected with analytical boundary components. Furthermore, \(\Omega_{ijk}(\eta) = \mathbb{R}^3 \setminus \bigcup_{\alpha \in \Lambda} V_{\alpha}\), where \(\Lambda = \{q \in \{i, j, k\} | A_q = \eta_{st}^2 - \varepsilon_s \varepsilon_t > 0, \{q, s, t\} = \{i, j, k\}\}\), \(\bigcup_{\alpha \in \Lambda} V_{\alpha}\) is a disjoint union of \(V_{\alpha}\) with

\[
V_i = \left\{ (f_i, f_j, f_k) \in \mathbb{R}^3 | f_i \geq \frac{-B_i + \sqrt{\Delta_i}}{2A_i} \right\}
\]

\[
= \left\{ (f_i, f_j, f_k) \in \mathbb{R}^3 | f_i \leq -\frac{1}{2} \ln\left(\frac{-B_i + \sqrt{\Delta_i}}{2A_i} - \varepsilon_i\right) \right\}
\]  \hspace{1cm} (3.11)

being a closed region in \(\mathbb{R}^3\) bounded by an analytical function defined on \(\mathbb{R}^2\) and \(V_j, V_k\) defined similarly.

The proof of Theorem 3.6 is parallelling to that of Theorem 2.6, we omit the details here.

Remark 16. If \((f_i, f_j, f_k) \in V_i\) is a degenerate hyperbolic discrete conformal structure for the triangle \(\{ijk\}\), then \(h_i \leq 0\) by \(-2h_i = 2A_i \kappa_i + B_i\) and (3.11), which further implies \(h_i < 0, h_j > 0, h_k > 0\) by Lemma 3.4.
Parallelling to the Euclidean case, we can introduce the following parameterized hyperbolic admissible space

\[ \Omega_{ijk}^H = \{(f_i, f_j, f_k, \eta_{ij}, \eta_{ik}, \eta_{jk}) \in \mathbb{R}^6 | \eta \text{ satisfies (1.6), (1.7)} \text{ and } (f_i, f_j, f_k) \in \Omega_{ijk}^H(\eta)\}. \]

As a direct corollary of Theorem 3.6, we have the following result for the parameterized hyperbolic admissible space \( \Omega_{ijk}^H \) parallelling to Corollary 2.8 for the parameterized Euclidean admissible space \( \Omega_{ijk}^E \).

**Corollary 3.7.** Suppose \((M, T, \varepsilon, \eta)\) is a weighted triangulated surface with the weights \(\varepsilon : V \to \{0, 1\}\) and \(\eta : E \to \mathbb{R}\) satisfying the structure conditions (1.6) and (1.7). \(\{ijk\} \in F\) is a topological triangle in the triangulated surface. Then the parameterized hyperbolic admissible space \(\Omega_{ijk}^H\) is connected.

The proof for Corollary 3.7 is the same as that for Corollary 2.8 so we omit the details of the proof here. Parallelling to the Euclidean case, the parameterized admissible space \(\Omega_{ijk}^H\) contains some points with good properties.

**Lemma 3.8.** The point \(p = (f_i, f_j, f_k, \eta_{ij}, \eta_{ik}, \eta_{jk}) = (0, 0, 1, 1, 1, 1)\) is contained in \(\Omega_{ijk}^H\). Furthermore, \(h_i(p) > 0, h_j(p) > 0, h_k(p) > 0\).

**Proof.** It is straightforward to check that \((\eta_{ij}, \eta_{ik}, \eta_{jk}) = (1, 1, 1)\) satisfies the structure conditions (1.6) and (1.7). By direct calculations, we have

\[
h_i(p) = (\varepsilon_j \varepsilon_k - 1) \sqrt{1 + \varepsilon_i} + (1 + \varepsilon_j) \sqrt{1 + \varepsilon_k} + (1 + \varepsilon_k) \sqrt{1 + \varepsilon_j}
\geq \sqrt{1 + \varepsilon_j} + \sqrt{1 + \varepsilon_k} - \sqrt{1 + \varepsilon_i}
\geq 2 - \sqrt{2}
> 0
\]

by \(\varepsilon_i, \varepsilon_j, \varepsilon_k \in \{0, 1\}\). Similarly, we have \(h_j(p) > 0, h_k(p) > 0\). Then

\[ Q^H = \kappa_i h_i + \kappa_j h_j + \kappa_k h_k + G \geq \kappa_i h_i + \kappa_j h_j + \kappa_k h_k > 0 \]

at \(p\) by Lemma 3.2, which implies \(p \in \Omega_{ijk}^H\) by Lemma 3.1.

### 3.2 Negative definiteness of the Jacobian matrix in the hyperbolic case

Suppose \(\{ijk\}\) is a nondegenerate hyperbolic triangle with edge lengths given by (1.4) and \(\theta_i, \theta_j, \theta_k\) are the inner angles at the vertices \(i, j, k\) in the triangle respectively. Set \(u_i\) to be the following function of \(f_i\)

\[
u_i = \begin{cases} f_i, & \varepsilon_i = 0; \\ \frac{1}{2} \ln \left( \frac{\sqrt{1 + e^{2f_i}} - 1}{\sqrt{1 + e^{2f_i}} + 1} \right), & \varepsilon_i = 1. \end{cases}
\]
Then
\[ \frac{\partial f_i}{\partial u_i} = \sqrt{1 + \varepsilon_i e^{2T_i}} = C_i. \] (3.13)

**Lemma 3.9** ([38, 96]). Suppose \((M, \mathcal{T}, \varepsilon, \eta)\) is a weighted triangulated surface with the weights \(\varepsilon : V \to \{0, 1\}\) and \(\eta : E \to \mathbb{R}\). Suppose \((f_i, f_j, f_k) \in \mathbb{R}^3\) is a nondegenerate hyperbolic discrete conformal structure for the topological triangle \(\{ijk\} \in \mathcal{F}\). Then
\[ \frac{\partial \theta_i}{\partial u_j} = \frac{\partial \theta_j}{\partial u_i} = \frac{S_i^2 S_j^2 S_k}{A \sinh^2 l_{ij}} \left[ (\varepsilon_i \varepsilon_j - \eta_{ij}^2) \kappa_k + \gamma_i \kappa_j + \gamma_j \kappa_i \right] = \frac{S_i^2 S_j^2 S_k}{A \sinh^2 l_{ij}} h_k, \] (3.14)

where \(A = \sinh l_{ij} \sinh l_{ik} \sin \theta_i\) and \(u_i\) is defined by (3.12).

**Proof.** By the chain rules,
\[ \frac{\partial \theta_i}{\partial u_j} = \frac{\partial \theta_i}{\partial l_{jk}} \frac{\partial l_{jk}}{\partial u_j} + \frac{\partial \theta_i}{\partial l_{ik}} \frac{\partial l_{ik}}{\partial u_j} + \frac{\partial \theta_i}{\partial l_{ij}} \frac{\partial l_{ij}}{\partial u_j}. \] (3.15)

The derivative cosine law ([7], Lemma A1) for hyperbolic triangles gives
\[ \frac{\partial \theta_i}{\partial l_{jk}} = \sinh l_{jk} A, \quad \frac{\partial \theta_i}{\partial l_{ij}} = -\sinh l_{jk} \cos \theta_j A, \] (3.16)

where \(A = \sinh l_i \sinh l_j \sin \theta_k\). By (1.4) and (3.13), we have
\[ \frac{\partial l_{jk}}{\partial u_j} = -\frac{1}{\sinh l_{jk}} (\varepsilon_j S_j^2 C_k + \eta_{jk} S_j S_k C_j), \quad \frac{\partial l_{ik}}{\partial u_j} = 0, \quad \frac{\partial l_{ij}}{\partial u_j} = \frac{1}{\sinh l_{ij}} (\varepsilon_j S_j^2 C_i + \eta_{ij} S_i S_j C_j). \] (3.17)

Submitting (3.16) and (3.17) into (3.15), by direct calculations, we have
\[ \frac{\partial \theta_i}{\partial u_j} = \frac{1}{A} (\varepsilon_j S_j^2 C_k + \eta_{jk} S_j S_k C_j) + \frac{-\sinh l_{jk} \cos \theta_j}{A} \frac{1}{\sinh l_{ij}} (\varepsilon_j S_j^2 C_i + \eta_{ij} S_i S_j C_j)
= \frac{1}{A \sinh^2 l_{ij}} (\cosh^2 l_{ij} - 1)(\varepsilon_j S_j^2 C_k + \eta_{jk} S_j S_k C_j)
+ (\cosh l_{ik} - \cosh l_{ij} \cosh l_{jk}) (\varepsilon_j S_j^2 C_i + \eta_{ij} S_i S_j C_j)
\]
\[ = \frac{S_i^2 S_j^2 S_k}{A \sinh^2 l_{ij}} \left[ (\varepsilon_i \varepsilon_j - \eta_{ij}^2) \kappa_k + \gamma_i \kappa_j + \gamma_j \kappa_i \right] = \frac{S_i^2 S_j^2 S_k}{A \sinh^2 l_{ij}} h_k, \]
where the hyperbolic cosine law is used in the second equality and the definition (1.4) for hyperbolic length is used in the third equality. Note that (3.18) is symmetric in the indices \(i\) and \(j\), this implies \(\frac{\partial \theta_i}{\partial u_j} = \frac{\partial \theta_j}{\partial u_i} \). \(\square\)
Remark 17. The result in Lemma 3.9 was previously obtained by Glickenstein [38] and Zhang-Guo-Zeng-Luo-Yau-Gu [96]. Here we give a proof by direct calculations for completeness. By (3.14) and Remark 16, if $(\eta_{ij}, \eta_{ik}, \eta_{jk})$ defined on the triangle $\{ijk\}$ satisfies the structure conditions (1.6) and (1.7) and $(f_i, f_j, f_k) \in \Omega_{ijk}^H(\eta)$ tends to a point $(\bar{f}_i, \bar{f}_j, \bar{f}_k) \in \partial V_i$ with $V_i \neq \emptyset$, then $\frac{\partial \theta_i}{\partial u_i} \to +\infty$, $\frac{\partial \theta_k}{\partial u_k} \to +\infty$. Recall the following formula obtained by Glickenstein-Thomas (\cite{41}, Proposition 9)

$$\frac{\partial A_{ijk}}{\partial u_i} = \frac{\partial \theta_j}{\partial u_i} (\cosh l_{ij} - 1) + \frac{\partial \theta_k}{\partial u_i} (\cosh l_{ik} - 1)$$

(3.19)

for the area $A_{ijk}$ of the hyperbolic triangle $\{ijk\}$, we have

$$- \frac{\partial \theta_i}{\partial u_i} = \frac{\partial A_{ijk}}{\partial u_i} + \frac{\partial \theta_j}{\partial u_i} + \frac{\partial \theta_k}{\partial u_i} \cosh l_{ij} + \frac{\partial \theta_k}{\partial u_i} \cosh l_{ik}$$

(3.20)

by the area formula for the hyperbolic triangle $\{ijk\}$. (3.20) implies $\frac{\partial \theta_i}{\partial u_i} \to -\infty$ as $(f_i, f_j, f_k) \to (\bar{f}_i, \bar{f}_j, \bar{f}_k) \in \partial V_i$.

Lemma 3.9 shows that the Jacobian matrix

$$\Lambda_{ijk}^H := \frac{\partial (\theta_i, \theta_j, \theta_k)}{\partial (u_i, u_j, u_k)} = \begin{pmatrix} \frac{\partial \theta_i}{\partial u_i} & \frac{\partial \theta_i}{\partial u_j} & \frac{\partial \theta_i}{\partial u_k} \\ \frac{\partial \theta_j}{\partial u_i} & \frac{\partial \theta_j}{\partial u_j} & \frac{\partial \theta_j}{\partial u_k} \\ \frac{\partial \theta_k}{\partial u_i} & \frac{\partial \theta_k}{\partial u_j} & \frac{\partial \theta_k}{\partial u_k} \end{pmatrix}$$

is symmetric. We further have the following result on the rank of the Jacobian matrix $\Lambda_{ijk}^H$.

Lemma 3.10. Suppose $(M, \mathcal{T}, \varepsilon, \eta)$ is a weighted triangulated surface with the weights $\varepsilon : V \to \{0, 1\}$ and $\eta : E \to \mathbb{R}$ satisfying the structure conditions (1.6) and (1.7). $\{ijk\} \in F$ is a topological triangle in the triangulated surface. Then the rank of the Jacobian matrix $\Lambda_{ijk}^H = \frac{\partial (\theta_i, \theta_j, \theta_k)}{\partial (u_i, u_j, u_k)}$ is 3 for all nondegenerate hyperbolic discrete conformal structures on the triangle $\{ijk\}$.

Proof. By the chain rules, we have

$$\frac{\partial (\theta_i, \theta_j, \theta_k)}{\partial (u_i, u_j, u_k)} = \frac{\partial (\theta_i, \theta_j, \theta_k)}{\partial (l_{jk}, l_{ik}, l_{ij})} \cdot \frac{\partial (l_{jk}, l_{ik}, l_{ij})}{\partial (u_i, u_j, u_k)}.$$  

(3.21)

The derivative cosine law (\cite{7}, Lemma A1) gives

$$\frac{\partial (\theta_i, \theta_j, \theta_k)}{\partial (l_{jk}, l_{ik}, l_{ij})} = \frac{1}{A} \begin{pmatrix} \sinh l_{jk} \\ \sinh l_{ik} \\ \sinh l_{ij} \end{pmatrix} \begin{pmatrix} 1 & -\cos \theta_k & -\cos \theta_j \\ -\cos \theta_k & 1 & -\cos \theta_i \\ -\cos \theta_j & -\cos \theta_i & 1 \end{pmatrix},$$

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which implies
\[
\det \left( \frac{\partial(\theta_i, \theta_j, \theta_k)}{\partial(l_{jk}, l_{ik}, l_{ij})} \right) \\
= \frac{\sinh l_{ik} \sinh l_{ij} \sinh l_{jk}}{A^3} \det \left( \begin{array}{ccc} 1 & -\cos \theta_k & -\cos \theta_j \\ -\cos \theta_k & 1 & -\cos \theta_i \\ -\cos \theta_j & -\cos \theta_i & 1 \end{array} \right) \\
= -\frac{\sinh l_{ij} \sinh l_{ik} \sinh l_{jk}}{A^3} \left( -1 + \cos^2 \theta_i + \cos^2 \theta_j + \cos^2 \theta_k + 2 \cos \theta_i \cos \theta_j \cos \theta_k \right) \\
= -4 \sinh l_{ij} \sinh l_{ik} \sinh l_{jk} \cdot \cos \frac{\theta_i + \theta_j + \theta_k}{2} \cos \frac{\theta_i + \theta_j - \theta_k}{2} \cos \frac{\theta_i - \theta_j + \theta_k}{2} \cos \frac{\theta_i - \theta_j - \theta_k}{2}.
\]

By the area formula for hyperbolic triangles, we have \(\theta_i + \theta_j + \theta_k \in (0, \pi)\), which implies \(\frac{\theta_i + \theta_j + \theta_k}{2}, \frac{\theta_i + \theta_j - \theta_k}{2}, \frac{\theta_i - \theta_j + \theta_k}{2}, \frac{\theta_i - \theta_j - \theta_k}{2} \in (-\frac{\pi}{2}, \frac{\pi}{2})\). Then \(\det \left( \frac{\partial(\theta_i, \theta_j, \theta_k)}{\partial(l_{jk}, l_{ik}, l_{ij})} \right) < 0\).

By (1.4) and (3.13), we have
\[
\frac{\partial(l_{jk}, l_{ik}, l_{ij})}{\partial(u_i, u_j, u_k)} = \\
\left( \begin{array}{ccc} \frac{1}{\sinh l_{jk}} & 0 & \frac{1}{\sinh l_{ik}} \\ \frac{1}{\sinh l_{ik}} & 0 & \frac{1}{\sinh l_{ij}} \\ \frac{1}{\sinh l_{ij}} & 0 & \frac{1}{\sinh l_{jk}} \end{array} \right) \\
\cdot \left( \begin{array}{ccc} \varepsilon_i S_j^2 C_k + \eta_{jk} S_j S_k C_j & \varepsilon_i S_j^2 C_j + \eta_{jk} S_j S_k C_k & \varepsilon_i S_j^2 C_j + \eta_{jk} S_j S_k C_k \\ \varepsilon_i S_j^2 C_j + \eta_{jk} S_j S_k C_k & \varepsilon_i S_j^2 C_k + \eta_{jk} S_j S_k C_k & \varepsilon_i S_j^2 C_k + \eta_{jk} S_j S_k C_k \\ \varepsilon_i S_j^2 C_k + \eta_{jk} S_j S_k C_k & \varepsilon_i S_j^2 C_k + \eta_{jk} S_j S_k C_k & \varepsilon_i S_j^2 C_k + \eta_{jk} S_j S_k C_k \end{array} \right).
\]

This implies
\[
\det \left( \frac{\partial(l_{jk}, l_{ik}, l_{ij})}{\partial(u_i, u_j, u_k)} \right) \\
= \frac{\sinh l_{ij} \sinh l_{ik} \sinh l_{jk}}{S_i S_j S_k} \cdot \left[ 2(\varepsilon_i \varepsilon_j \varepsilon_k + \eta_{ij} \eta_{ik} \eta_{jk}) S_i S_j S_k C_i C_j C_k + \gamma_i S_j S_k C_i C_j C_k + \gamma_j S_i S_k C_i C_j C_k + \gamma_k S_i S_j C_i C_j C_k \left( \varepsilon_i S_j^2 C_j + \varepsilon_j S_j^2 C_j + \varepsilon_k S_j^2 C_j + \varepsilon_j S_j^2 C_j \right) \right] \\
\geq 2 \frac{S_i^2 S_j^2 S_k^2 C_i C_j C_k}{\sinh l_{ij} \sinh l_{ik} \sinh l_{jk}} \left[ (\varepsilon_i \varepsilon_j + \eta_{ij})(\varepsilon_i \varepsilon_k + \eta_{ik})(\varepsilon_j \varepsilon_k + \eta_{jk}) \right] \\
= 2 \frac{S_i^2 S_j^2 S_k^2 C_i C_j C_k}{\sinh l_{ij} \sinh l_{ik} \sinh l_{jk}} \left( \varepsilon_i \varepsilon_j + \eta_{ij} \right) \left( \varepsilon_i \varepsilon_k + \eta_{ik} \right) \left( \varepsilon_j \varepsilon_k + \eta_{jk} \right) \\
> 0,
\]

(3.22)
where the structure condition (1.7) is used in the third line and the structure condition (1.6) is used in the last line. Therefore, det \( \frac{\partial(\theta_i, \theta_j, \theta_k)}{\partial(u_i, u_j, u_k)} \) < 0 by (3.21), which implies the rank of the Jacobian matrix \( \Lambda^H_{ij} = \frac{\partial(\theta_i, \theta_j, \theta_k)}{\partial(u_i, u_j, u_k)} \) is 3.

As a consequence of Lemma 3.9 and Lemma 3.10, we have the following result on the negative definiteness of the Jacobian matrix \( \Lambda^H_{ij} = \frac{\partial(\theta_i, \theta_j, \theta_k)}{\partial(u_i, u_j, u_k)} \).

**Theorem 3.11.** Suppose \((M, T, \varepsilon, \eta)\) is a weighted triangulated surface with the weights \(\varepsilon : V \to \{0, 1\}\) and \(\eta : E \to \mathbb{R}\) satisfying the structure conditions (1.6) and (1.7). \(\{ijk\} \in F\) is a topological triangle in the triangulated surface. Then the Jacobian matrix \( \Lambda^H_{ijk} = \frac{\partial(\theta_i, \theta_j, \theta_k)}{\partial(u_i, u_j, u_k)}\) is symmetric and negative definite for all nondegenerate hyperbolic discrete conformal structures on triangle \(\{ijk\}\).

**Proof.** By Lemma 3.10, all the three eigenvalues of the Jacobian matrix \( \Lambda^H_{ijk} \) are nonzero. Taking the Jacobian matrix \( \Lambda^H_{ijk} \) as a matrix-valued function of \((f_i, f_j, f_k, \eta_{ij}, \eta_{ik}, \eta_{jk})\) in the parameterized admissible space \(\Omega^H_{ijk}\). By the continuity of eigenvalues of \( \Lambda^H_{ijk} \) and the connectivity of the parameterized admissible space \(\Omega^H_{ijk}\) in Corollary 3.7 to prove the negative definiteness of \( \Lambda^H_{ijk} = \frac{\partial(\theta_i, \theta_j, \theta_k)}{\partial(u_i, u_j, u_k)} \), we just need to find a point \( p \in \Omega^H_{ijk} \) such that the eigenvalues of \( \Lambda^H_{ijk} \) at \( p \) are negative. Taking \( p = (f_i, f_j, f_k, \eta_{ij}, \eta_{ik}, \eta_{jk}) = (0, 0, 0, 1, 1, 1) \), then \( p \in \Omega^H_{ijk} \) and \( h_i(p) > 0, h_j(p) > 0, h_k(p) > 0 \) by Lemma 3.8, which implies that \( \frac{\partial \theta_i}{\partial u_j} = \frac{\partial \theta_j}{\partial u_i} > 0 \) and \( \frac{\partial \theta_k}{\partial u_i} = \frac{\partial \theta_i}{\partial u_k} > 0 \) by Lemma 3.9. By (3.20), we have \( -\frac{\partial \theta_i}{\partial u_i} > \frac{\partial \theta_j}{\partial u_i} + \frac{\partial \theta_k}{\partial u_i} \) at \( p \). Therefore, \( \Lambda^H_{ijk} \) is negative definite and has three negative eigenvalues at \( p \) by Lemma 2.12.

As a direct corollary of Theorem 3.11, we have the following result on the Jacobian matrix \( \Lambda^H = \frac{\partial(K_1, \ldots, K_N)}{\partial(u_1, \ldots, u_N)} \) for hyperbolic discrete conformal structures.

**Corollary 3.12.** Suppose \((M, T, \varepsilon, \eta)\) is a weighted triangulated surface with the weights \(\varepsilon : V \to \{0, 1\}\) and \(\eta : E \to \mathbb{R}\) satisfying the structure conditions (1.6) and (1.7). Then the Jacobian matrix \( \Lambda^H = \frac{\partial(K_1, \ldots, K_N)}{\partial(u_1, \ldots, u_N)} \) for hyperbolic discrete conformal structures is symmetric and positive definite for all nondegenerate hyperbolic discrete conformal structures on \((M, T, \varepsilon, \eta)\).

The proof for Corollary 3.12 is the same as that for Corollary 2.13, so we omit the details of the proof here.
3.3 Rigidity of hyperbolic discrete conformal structures

Theorem 3.6 and Lemma 3.9 imply the following Ricci energy function for the triangle \{ijk\}

$$E_{ijk}(u_i, u_j, u_k) = \int_{(\pi_i, \pi_j, \pi_k)} \theta_i du_i + \theta_j du_j + \theta_k du_k$$ (3.23)

is a well-defined smooth function on \(\Omega_{ijk}^H(\eta)\) with \(\nabla u_i E_{ijk} = \theta_i\). By Theorem 3.11, \(E_{ijk}(u_i, u_j, u_k)\) is a locally strictly concave function defined on \(\Omega_{ijk}^H(\eta)\) under the structure conditions (1.6) and (1.7). Set

$$E(u_1, \cdots, u_N) = 2\pi \sum_{i \in V} u_i - \sum_{\{ijk\} \in F} E_{ijk}(u_i, u_j, u_k)$$ (3.24)

to be the Ricci energy function defined on the admissible space \(\Omega^H\) of nondegenerate hyperbolic discrete conformal structures for \((M, T, \epsilon, \eta)\). Then \(E\) is a locally strictly convex function defined on \(\Omega^H\) with \(\nabla u_i E = K_i\) by Corollary 3.12, from which the local rigidity of hyperbolic discrete conformal structures follows by Lemma 2.14.

To prove the global rigidity of hyperbolic discrete conformal structures, we need to extend the inner angles in a hyperbolic triangle \{ijk\} defined for nondegenerate hyperbolic discrete conformal structures to be a globally defined function for \((f_i, f_j, f_k) \in \mathbb{R}^3\). Parallelling to Lemma 2.15, we have the following extension for inner angles of hyperbolic triangles.

**Lemma 3.13.** Suppose \((M, T, \epsilon, \eta)\) is a weighted triangulated surface with the weights \(\epsilon : V \rightarrow \{0, 1\}\) and \(\eta : E \rightarrow \mathbb{R}\) satisfying the structure conditions (1.6) and (1.7). \(\{ijk\} \in F\) is a topological triangle in the triangulated surface. Then the inner angles \(\theta_i, \theta_j, \theta_k\) of the triangle \{ijk\} defined for nondegenerate hyperbolic discrete conformal structures could be extended by constants to be continuous function \(\tilde{\theta}_i, \tilde{\theta}_j, \tilde{\theta}_k\) defined for \((f_i, f_j, f_k) \in \mathbb{R}^3\).

**Proof.** By Theorem 3.6, \(\Omega^H_{ijk}(\eta) = \mathbb{R}^3 \setminus \bigcup_{\alpha \in \Lambda} V_{\alpha}\), where \(\Lambda = \{q \in \{i, j, k\} | A_q = \eta_{st}^2 - \epsilon_s \epsilon_t > 0, \{q, s, t\} = \{i, j, k\}\}\) and \(V_{\alpha}\) is a closed region in \(\mathbb{R}^3\) bounded by the analytical function in (3.11) defined on \(\mathbb{R}^2\). If \(\Lambda = \emptyset\), then \(\Omega^H_{ijk}(\eta) = \mathbb{R}^3\) and \(\theta_i, \theta_j, \theta_k\) is defined for all \((f_i, f_j, f_k) \in \mathbb{R}^3\).

Without loss of generality, suppose \(\Lambda \neq \emptyset\) and \(V_i\) is a connected component of \(\mathbb{R}^3 \setminus \Omega^H_{ijk}(\eta)\). Suppose \((f_i, f_j, f_k) \in \Omega^H_{ijk}(\eta)\) tends to a point \((\bar{f}_i, \bar{f}_j, \bar{f}_k) \in \partial V_i \in \mathbb{R}^3\). By direct
By the form of nondegenerate hyperbolic triangle \( \{ijk\} \), which implies
\[
\tan^2 \frac{A_{ijk}}{4} = \frac{\sinh \frac{l_{ij} + l_{ik} + l_{jk}}{2} \sinh \frac{l_{ij} + l_{ik} - l_{jk}}{2} \sinh \frac{l_{ij} - l_{ik} + l_{jk}}{2} \sinh \frac{-l_{ij} + l_{ik} + l_{jk}}{2}}{16 \cosh^2 \frac{l_{ij} + l_{ik} + l_{jk}}{4} \cosh^2 \frac{l_{ij} + l_{ik} - l_{jk}}{4} \cosh^2 \frac{l_{ij} - l_{ik} + l_{jk}}{4} \cosh^2 \frac{-l_{ij} + l_{ik} + l_{jk}}{4}}
\]
\[
= \frac{S_i^2 S_j^2 S_k^H}{64 \cosh^2 \frac{l_{ij} + l_{ik} + l_{jk}}{4} \cosh^2 \frac{l_{ij} + l_{ik} - l_{jk}}{4} \cosh^2 \frac{l_{ij} - l_{ik} + l_{jk}}{4} \cosh^2 \frac{-l_{ij} + l_{ik} + l_{jk}}{4}}
\]
(3.26)

by (3.6). (3.26) implies \( A_{ijk} \rightarrow 0 \) as \( (f_i, f_j, f_k) \rightarrow (f_i^*, f_j^*, f_k^*) \in \partial V_i \), which further implies \( \theta_i \rightarrow \pi \) by \( A_{ijk} = \pi - \theta_i - \theta_j - \theta_k \) and \( \theta_j, \theta_k \rightarrow 0 \) as \( (f_i, f_j, f_k) \rightarrow (f_i^*, f_j^*, f_k^*) \in \partial V_i \). Similar arguments apply for the other connected components of \( \mathbb{R}^3 \setminus \Omega_{ijk}^H(\eta) \).

We can extend \( \theta_i \) defined on \( \Omega_{ijk}^H(\eta) \) by constant to be a continuous function \( \tilde{\theta}_i \) defined on \( \mathbb{R}^3 \) by setting
\[
\tilde{\theta}_i(f_i, f_j, f_k) = \begin{cases} 
\theta_i, & \text{if } (f_i, f_j, f_k) \in \Omega_{ijk}^H(\eta); \\
\pi, & \text{if } (f_i, f_j, f_k) \in V_i; \\
0, & \text{otherwise}.
\end{cases}
\]
(3.27)

\( \theta_j \) and \( \theta_k \) could be extended similarly.
Remark 18. One can also use (3.20) to prove $\theta_i \to \pi$ as $(f_i, f_j, f_k) \to (T_i, T_j, T_k) \in \partial V_i$.

By Lemma 3.13 we can further extend the combinatorial curvature function $K$ defined on $\Omega^H$ to be defined for all $f \in \mathbb{R}^N$ by setting $\bar{K}_i = 2\pi - \sum_{\{ijk\} \in F} \bar{\theta}_i$, where $\bar{\theta}_i$ is the extension of $\theta_i$ defined in Lemma 3.13 by (3.27). The vector $f \in \mathbb{R}^N$ is called a generalized hyperbolic discrete conformal structure for the weighted triangulated surface $(M, T, \varepsilon, \eta)$.

Taking $\bar{\theta}_i, \bar{\theta}_j, \bar{\theta}_k$ as functions of $(u_i, u_j, u_k)$. Then the extensions $\bar{\theta}_i, \bar{\theta}_j, \bar{\theta}_k$ of $\theta_i, \theta_j, \theta_k$ are continuous functions of $(u_i, u_j, u_k) \in V_i \times V_j \times V_k$, where $V_q = \mathbb{R}$ if $\varepsilon_q = 0$ and $V_q = \mathbb{R}_{<0} = (-\infty, 0)$ if $\varepsilon_q = 1$ for $q \in \{i, j, k\}$. Combining this with Theorem 2.16, the locally concave function $\bar{E}_{ijk}$ defined by (3.23) for nondegenerated hyperbolic discrete conformal structures for a triangle $\{ijk\} \in F$ could be extended to be a $C^1$ smooth concave function

$$\bar{E}_{ijk}(u_i, u_j, u_k) = \int_{(\bar{\pi}_i, \bar{\pi}_j, \bar{\pi}_k)} \bar{\theta}_i du_i + \bar{\theta}_j du_j + \bar{\theta}_k du_k$$  \hspace{1cm} (3.28)

defined on $V_i \times V_j \times V_k$ with $\nabla_u \bar{E}_{ijk} = \bar{\theta}_i$. As a result, the locally convex function $\bar{E}$ defined by (3.24) for nondegenerate hyperbolic discrete conformal structures on a weighted triangulated surface could be extended to be a $C^1$ smooth convex function

$$\bar{E}(u_1, \ldots, u_N) = 2\pi \sum_{i \in V} u_i - \sum_{\{ijk\} \in F} \bar{E}_{ijk}(u_i, u_j, u_k)$$  \hspace{1cm} (3.29)

defined on $\mathbb{R}^{N_1} \times \mathbb{R}^{N_2}_{<0}$ with $\nabla_u \bar{E} = \bar{K}_i = 2\pi - \sum \bar{\theta}_i$, where $N_1$ is the number of vertices $v_i$ in $V$ with $\varepsilon_i = 0$ and $N_2 = N - N_1$. Sometimes we also call a vector $u \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}_{<0}$ as a generalized hyperbolic discrete conformal structure for the weighted triangulated surface $(M, T, \varepsilon, \eta)$.

Parallelling to Theorem 2.17 for generalized Euclidean discrete conformal structures, we have the following result on the rigidity of generalized hyperbolic discrete conformal structures, which is a generalization of Theorem 1.1 (b).

Theorem 3.14. Suppose $(M, T, \varepsilon, \eta)$ is a weighted triangulated surface with the weights $\varepsilon : V \to \{0, 1\}$ and $\eta : E \to \mathbb{R}$ satisfying the structure conditions (1.6) and (1.7). If there exists a nondegenerate hyperbolic discrete conformal structure $f_A \in \Omega^H$ and a generalized hyperbolic discrete conformal structure $f_B \in \mathbb{R}^N$ such that $K(f_A) = \bar{K}(f_B)$. Then $f_A = f_B$.

Proof. Set

$$f(t) = \bar{E}((1 - t)u_A + tu_B) = 2\pi \sum_{i=1}^N [(1 - t)u_{A,i} + tu_{B,i}] + \sum_{\{ijk\} \in F} f_{ijk}(t),$$

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where \( f_{ijk}(t) = -\tilde{E}_{ijk}((1-t)u_A + tu_B) \). Then \( f(t) \) is a \( C^1 \) smooth convex function for \( t \in [0,1] \) with \( f'(0) = f'(1) \), which implies that \( f'(t) = f'(0) \) for all \( t \in [0,1] \). Note that the admissible space \( \Omega^H \) of nondegenerate hyperbolic discrete conformal structures is an open subset of \( \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}_{<0} \), there exists \( \epsilon > 0 \) such that \((1-t)u_A + tu_B\) corresponds to nondegenerate hyperbolic discrete conformal structure for \( t \in [0,\epsilon] \). Note that \( f(t) \) is smooth for \( t \in [0,\epsilon] \), we have
\[
 f''(t) = (u_B - u_A) \Lambda^H (u_B - u_A)^T = 0, \quad \forall \ t \in [0,\epsilon],
\]
which implies \( u_A = u_B \) by Corollary 3.12. Therefore, \( f_A = f_B \) because the transformation \( u = u(f) \) defined by (3.12) is a diffeomorphism. \( \square \)

4 Deformation of discrete conformal structures

For further applications, we study the following modified combinatorial Ricci flow
\[
\frac{du_i}{dt} = K_i - K_i \tag{4.1}
\]
and modified combinatorial Calabi flow
\[
\frac{du_i}{dt} = \Delta(K - K)_i, \tag{4.2}
\]
where \( K : V \rightarrow (-\infty, 2\pi) \) is a function defined on the vertices with \( \sum_{i=1}^{N} K_i = 2\pi \chi(M) \) for Euclidean background geometry and \( \sum_{i=1}^{N} K_i > 2\pi \chi(M) \) for hyperbolic background geometry. The modified combinatorial Ricci flow (4.1) and modified combinatorial Calabi flow (4.2) are generalizations of the normalized combinatorial Ricci flow (1.11) and the combinatorial Calabi flow (1.12) respectively and could be used to study the prescribed combinatorial curvature problem of discrete conformal structures on polyhedral surfaces.

Lemma 4.1. Suppose \((M, \mathcal{T}, \varepsilon, \eta)\) is a weighted triangulated connected closed surface with the weights \( \varepsilon : V \rightarrow \{0,1\} \) and \( \eta : E \rightarrow \mathbb{R} \) satisfying the structure conditions (1.6) and (1.7). The modified combinatorial Ricci flow (4.1) and modified combinatorial Calabi flow (4.2) for discrete conformal structures on \((M, \mathcal{T}, \varepsilon, \eta)\) are negative gradient flows.

Proof. Set \( \mathcal{H}(u) = \mathcal{E}(u_1, \ldots, u_N) - \sum_{i=1}^{N} K_i u_i \), where \( \mathcal{E}(u_1, \ldots, u_N) \) is defined by (2.39) in the Euclidean background geometry and by (3.24) in the hyperbolic background geometry. Then \( \nabla_{u_i} \mathcal{H} = K_i - K_i \), which implies the modified combinatorial Ricci flow (4.1) is a negative gradient flow of \( \mathcal{H}(u) \).

Set \( \mathcal{C}(u) = \frac{1}{2} ||(K - K)||^2 = \frac{1}{2} \sum_{i=1}^{N} (K_i - K_i)^2 \). By direct calculations, we have \( \nabla_{u_i} \mathcal{C} = -\Delta(K - K)_i \), which implies the modified combinatorial Calabi flow (4.2) is a negative gradient flow of \( \mathcal{C}(u) \). \( \square \)
As the modified combinatorial Ricci flow \((4.1)\) and modified combinatorial Calabi flow \((4.2)\) are ODE systems, the short time existence of the solutions are ensured by the standard ODE theory. We further have the following result on the longtime existence and convergence for the solutions of modified combinatorial Ricci flow \((4.1)\) and modified combinatorial Calabi flow \((4.2)\) for initial data with small energy, which is a slight generalization of Theorem 1.3.

**Theorem 4.2.** Suppose \((M, T, \varepsilon, \eta)\) is a weighted triangulated connected closed surface with the weights \(\varepsilon : V \to \{0, 1\}\) and \(\eta : E \to \mathbb{R}\) satisfying the structure conditions \((1.6)\) and \((1.7)\).

(a) If the solution of the modified combinatorial Ricci flow \((4.1)\) or modified combinatorial Calabi flow \((4.2)\) converges to a nondegenerate discrete conformal structure \(\bar{u}\), then the combinatorial curvature for the polyhedral metric determined by the discrete conformal structure \(\bar{u}\) is \(K\).

(b) If there exists a nondegenerate discrete conformal structure \(\bar{u}\) with combinatorial curvature \(K\), then there exists a real number \(\delta > 0\) such that if the initial value \(\bar{u}(0)\) of modified combinatorial Ricci flow \((4.1)\) (modified combinatorial Calabi flow \((4.2)\) respectively) satisfies \(|\bar{u}(0) - \bar{u}| < \delta\), the solution of modified combinatorial Ricci flow \((4.1)\) (modified combinatorial Calabi flow \((4.2)\) respectively) exists for all time and converges exponentially fast to \(\bar{u}\).

**Proof.** The proof for part (a) is direct. For part (b), we only prove the Euclidean case and the proof for the hyperbolic case is almost the same.

For the modified combinatorial Ricci flow \((4.1)\), by direct calculations, we have

\[
\frac{d}{dt} \left( \sum_{i=1}^{N} u_i \right) = \sum_{i=1}^{N} (K_i - K_i) = 2\pi \chi(M) - 2\pi \chi(M) = 0,
\]  

(4.3)

where the discrete Gauss-Bonnet formula \((1.2)\) and the assumption \(\sum_{i=1}^{N} K_i = 2\pi \chi(M)\) are used in the second equality. This implies \(\sum_{i=1}^{N} u_i\) is invariant along the modified combinatorial Ricci flow \((4.1)\). Without loss of generality, assume \(\sum_{i=1}^{N} u_i(0) = 0\). Set \(\Sigma = \{u \in \mathbb{R}^{N} | \sum_{i=1}^{N} u_i = 0\}\). Then the solution \(u(t)\) of the modified combinatorial Ricci flow \((4.1)\) stays in the hyperplane \(\Sigma\) by \((4.3)\). Set \(\Gamma(u) = \bar{K} - K\) for the modified combinatorial Ricci flow \((4.1)\). Then \(\bar{u}\) is an equilibrium point of the system \((4.1)\) and \(D\Gamma(\bar{u}) = -\frac{\partial(K_1, \ldots, K_N)}{\partial(u_1, \ldots, u_N)}\) is negative semi-definite with null space \(\{t \mathbf{1} \in \mathbb{R}^{N} | t \in \mathbb{R}\}\) by Corollary 2.13. Note that the solution \(u(t)\) of modified combinatorial Ricci flow \((4.1)\) stays in the hyperplane \(\Sigma\), the normal vector of which generates the null space \(\{t \mathbf{1} \in \mathbb{R}^{N} | t \in \mathbb{R}\}\) of \(D\Gamma(\bar{u})\). This implies \(\bar{u}\) is a local attractor of the system \((4.1)\). Then the longtime
existence and exponential convergence of the solution of (4.1) follows from the Lyapunov Stability Theorem (69, Chapter 5).

For the modified combinatorial Calabi flow (4.2), we have

$$\frac{d}{dt}(\sum_{i=1}^{N} u_i) = \sum_{i=1}^{N} \Delta(K - K)_i = \sum_{j=1}^{N} \sum_{i=1}^{N} \Lambda_{ij}^E(K - K)_j = 0$$

by the kernel of $\Lambda^E$ is $\{t(1, \cdots, 1) \in \mathbb{R}^N | t \in \mathbb{R}\}$ in Corollary 2.13, which implies $\sum_{i=1}^{N} u_i$ is invariant along the flow (4.2). Set $\Gamma(u) = \Delta(K - \bar{K})$, Then $\bar{u}$ is an equilibrium point of the system (4.2) and $D\Gamma(\bar{u}) = -\left(\frac{\partial(K_1, \cdots, K_N)}{\partial(u_1, \cdots, u_N)}\right)^2$ is negative semi-definite with null space $\{t(1, \cdots, 1) \in \mathbb{R}^N | t \in \mathbb{R}\}$ by Corollary 2.13. The rest of the proof is the same as that for the modified combinatorial Ricci flow (4.1), we omit the details here.

□

For general initial value, the modified combinatorial Ricci flow (4.1) and the modified combinatorial Calabi flow (4.2) may develop singularities, which correspond to the triangles in the triangulation degenerate or the discrete conformal structure $f$ tends to infinity along the combinatorial curvature flows. For the combinatorial Ricci flow, we can extend it through the singularities to ensure the longtime existence and convergence for general initial value.

**Definition 6.** Suppose $(M, \mathcal{T}, \varepsilon, \eta)$ is a weighted triangulated connected closed surface with the weights $\varepsilon : V \to \{0, 1\}$ and $\eta : E \to \mathbb{R}$ satisfying the structure conditions (1.6) and (1.7). The extended modified combinatorial Ricci flow is defined to be

$$\frac{du_i}{dt} = \bar{K}_i - \bar{K}_i,$$

where $\bar{K}_i = 2\pi - \sum_{\{ijk\} \in E} \bar{\theta}_i$ is an extension of the combinatorial curvature $K_i$ with $\bar{\theta}_i$ given by (2.40) in the Euclidean background geometry and by (3.27) in the hyperbolic background geometry.

Note that the extended combinatorial curvature $\bar{K}$ is only a continuous function of the generalized discrete conformal structures and does not have continuous derivatives. Remark 13 and Remark 17 further imply that $\bar{K}$ is not Lipschitz. For such ODE systems, there may exist more than one solution by the standard ODE theory. However, we can prove the following uniqueness for the solution of extended modified combinatorial Ricci flow (4.4) with any generalized discrete conformal structures as initial value, which is a generalization of Theorem 1.2 (b).

**Theorem 4.3.** Suppose $(M, \mathcal{T}, \varepsilon, \eta)$ is a weighted triangulated connected closed surface with the weights $\varepsilon : V \to \{0, 1\}$ and $\eta : E \to \mathbb{R}$ satisfying the structure conditions (1.6) and (1.7). The solution of extended combinatorial Ricci flow (4.4) is unique for any initial generalized discrete conformal structure $f : V \to \mathbb{R}^N$ on $(M, \mathcal{T}, \varepsilon, \eta)$. 

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Proof. We take Ge-Hua’s trick in \[20\] to prove the uniqueness of the solution of (4.4). As the proofs for the Euclidean and hyperbolic background geometry are all the same, we will not mention the background geometry explicitly in the following of the proof.

Note that the extended combinatorial Ricci energy function \( \tilde{\mathcal{E}} \) for \((M, \mathcal{T}, \varepsilon, \eta)\), defined by (2.43) for the Euclidean background geometry and by (3.29) for the hyperbolic background geometry, is a \( C^1 \) smooth convex function with \( \nabla_u \tilde{\mathcal{E}} = \tilde{K} \). By mollifying \( \tilde{\mathcal{E}} \) using the standard mollifier \( \varphi_\varepsilon(u) = \frac{1}{\varepsilon^N} \varphi(\frac{u}{\varepsilon}) \) with

\[
\varphi(u) = \begin{cases} 
C e^{\frac{1}{1-|u|^2}}, & |u| < 1, \\
0, & |u| \geq 1,
\end{cases}
\]

where the positive constant \( C \) is chosen to make \( \int_{\mathbb{R}N} \varphi(u) du = 1 \), we have \( \tilde{\mathcal{E}}_\varepsilon = \varphi_\varepsilon * \tilde{\mathcal{E}} \) is a smooth convex function and \( \tilde{\mathcal{E}}_\varepsilon \to \tilde{\mathcal{E}} \) in \( C^1_{\text{loc}} \) as \( \varepsilon \to 0 \). Suppose \( u_A \) and \( u_B \) are two different discrete conformal structures and set \( f(t) = \tilde{\mathcal{E}}_\varepsilon(tu_A + (1-t)u_B) \), then \( f(t) \) is a smooth convex function of \( t \in [0,1] \) with \( f'(t) = \nabla \tilde{\mathcal{E}}_\varepsilon(tu_A + (1-t)u_B) \cdot (u_A - u_B) \) and \( f''(t) \geq 0 \). Therefore,

\[
(\nabla \tilde{\mathcal{E}}_\varepsilon(u_A) - \nabla \tilde{\mathcal{E}}_\varepsilon(u_B)) \cdot (u_A - u_B) = f'(1) - f'(0) = f''(\xi) \geq 0 \tag{4.5}
\]

for some \( \xi \in (0,1) \). Note that \( \tilde{\mathcal{E}}_\varepsilon \to \tilde{\mathcal{E}} \) in \( C^1_{\text{loc}} \) as \( \varepsilon \to 0 \). Letting \( \varepsilon \to 0 \), (4.5) gives

\[
(\tilde{\mathcal{K}}(u_A) - \tilde{\mathcal{K}}(u_B)) \cdot (u_A - u_B) \geq 0. \tag{4.6}
\]

Suppose \( u_A(t) \) and \( u_B(t) \) are two solutions of the extended combinatorial Ricci flow (4.4) with \( u_A(0) = u_B(0) \). Set \( f(t) = ||u_A(t) - u_B(t)||^2 \). Then \( f(0) = 0 \), \( f(t) \geq 0 \) and

\[
\frac{df(t)}{dt} = 2 \left( \frac{du_A(t)}{dt} - \frac{du_B(t)}{dt} \right) \cdot (u_A(t) - u_B(t)) = -(\tilde{\mathcal{K}}(u_A) - \tilde{\mathcal{K}}(u_B)) \cdot (u_A - u_B) \leq 0
\]

by (4.6), which implies \( f(t) \equiv 0 \). Therefore, \( u_A(t) = u_B(t) \). \qed

For the longtime existence and convergence of the extended combinatorial Ricci flow (4.4), we have the following result in the Euclidean background geometry, which is a generalization of Theorem 1.2 (a) (c) in the Euclidean background geometry.

Theorem 4.4. Suppose \((M, \mathcal{T}, \varepsilon, \eta)\) is a weighted triangulated connected closed surface with the weights \( \varepsilon : V \to \{0,1\} \) and \( \eta : E \to \mathbb{R} \) satisfying the structure conditions (1.6) and (1.7). The solution of extended combinatorial Ricci flow (4.4) in the Euclidean background geometry exists for all time for any initial generalized discrete conformal structure \( u : V \to \mathbb{R}^N \) on \((M, \mathcal{T}, \varepsilon, \eta)\). Furthermore, if there exists a nondegenerate Euclidean discrete conformal structure \( \overline{u} \in \Omega^E \) with combinatorial curvature \( \overline{\mathcal{K}} \), then the solution of the extended combinatorial Ricci flow (4.4) in the Euclidean background geometry converges exponentially fast to \( \overline{u} \) for any initial generalized Euclidean discrete conformal structure \( u(0) \in \mathbb{R}^N \) with \( \sum_{i=1}^N u(0) = \sum_{i=1}^N \overline{u}_i \).
Proof. Suppose $u(t)$ is a solution of the extended Euclidean combinatorial Ricci flow (4.4) with initial generalized Euclidean discrete conformal structure $u(0) \in \mathbb{R}^N$, then $|\frac{du_i}{dt}| = |\tilde{K}_i - K_i| \leq |\tilde{K}_i| + (d_i + 2)\pi$, where $d_i$ is the number of vertices adjacent to the vertex $i \in V$. This implies $|u_i(t)| \leq |u_i(0)| + |\tilde{K}_i| + (d_i + 2)\pi |t < +\infty$ for $t \in (0, +\infty)$. Therefore, the solution of the extended Euclidean combinatorial Ricci flow (4.4) exists for all time.

Note that the extended inner angles $\tilde{\theta}_i, \tilde{\theta}_j, \tilde{\theta}_j$ for a triangle $\{ijk\}$ in Lemma 2.15 satisfy $\tilde{\theta}_i + \tilde{\theta}_j + \tilde{\theta}_j = \pi$. This implies the extended combinatorial curvature $\tilde{K}$ satisfies the discrete Gauss-Bonnet formula $\sum_{i=1}^N \tilde{K}_i = 2\pi \chi(M)$, which further implies $\frac{d(\sum_{i=1}^N u_i)}{dt} = \sum_{i=1}^N (\tilde{K}_i - K_i) = 0$ along the extended combinatorial Ricci flow (4.4) in the Euclidean background geometry. Therefore, $\sum_{i=1}^N u_i$ is invariant along the extended Euclidean combinatorial Ricci flow (4.4). Without loss of generality, assume $\sum_{i=1}^N u_i(0) = 0$, then the solution $u(t)$ of the extended Euclidean combinatorial Ricci flow (4.4) stays in the hyperplane $\Sigma := \{u \in \mathbb{R}^N | \sum_{i=1}^N u_i = 0\}$.

Set $\tilde{\mathcal{H}}(u) = \tilde{E}(u) - \sum_{i=1}^N \tilde{K}_i du_i$, where $\tilde{E}(u)$ is the extended Ricci energy function defined by (4.23). Then $\tilde{\mathcal{H}}(u)$ is a $C^1$ smooth convex function defined on $\mathbb{R}^N$ with $\tilde{\mathcal{H}}(u) \geq \tilde{\mathcal{H}}(\tilde{u}) = 0$ and $\nabla \tilde{\mathcal{H}}(\tilde{u}) = K(\tilde{u}) = 0$ by the assumption $K(\tilde{u}) = K$. This further implies $\lim_{u \in \Sigma, u \to \infty} \tilde{\mathcal{H}}(u) = +\infty$ by Corollary 2.13 and the following property of convex functions, a proof of which could be found in [32] (Lemma 4.6).

Lemma 4.5. Suppose $f(x)$ is a $C^1$ smooth convex function on $\mathbb{R}^n$ with $\nabla f(x_0) = 0$ for some $x_0 \in \mathbb{R}^n$, $f(x)$ is $C^2$ smooth and strictly convex in a neighborhood of $x_0$, then $\lim_{x \to \infty} f(x) = +\infty$.

By direct calculations, we have
\begin{align}
\frac{d}{dt} \tilde{\mathcal{H}}(u(t)) = \nabla_u \tilde{\mathcal{H}} \cdot \frac{du}{dt} = -\sum_{i=1}^N (\tilde{K}_i - \tilde{K}_i)^2 \leq 0,
\end{align}
which implies $0 = \tilde{\mathcal{H}}(\tilde{u}) \leq \tilde{\mathcal{H}}(u(t)) \leq \tilde{\mathcal{H}}(u(0))$ along the extended Euclidean combinatorial Ricci flow (4.4). This further implies the solution $u(t)$ of the extended Euclidean combinatorial Ricci flow (4.4) stays in a compact subset $U$ of $\Sigma$ by $\lim_{u \in \Sigma, u \to \infty} \tilde{\mathcal{H}}(u) = +\infty$. Therefore, $\tilde{\mathcal{H}}(u(t))$ is bounded along the flow (4.4) and the limit $\lim_{t \to +\infty} \tilde{\mathcal{H}}(u(t))$ exists by (4.7). Taking $t_n = n$, then there exists $\xi_n \in (n, n + 1)$ such that
\begin{align}
\tilde{\mathcal{H}}(u(t_{n+1})) - \tilde{\mathcal{H}}(u(t_n)) = -\sum_{i=1}^N \left(\tilde{K}_i(u(\xi_n)) - \tilde{K}_i\right)^2 \to 0, \text{ as } n \to \infty.
\end{align}
Note that $u(\xi_n) \in U \subset \subset \Sigma$, there exists a subsequence of $\xi_n$, still denoted by $\xi_n$ for simplicity, such that $u(\xi_n) \to u^*$ for some $u^* \in U \subset \subset \Sigma$. Then $\tilde{K}(u^*) = \tilde{K} = K(\tilde{u})$ by
the continuity of $\bar{K}$ and (4.8). Therefore, $u^* = \bar{u}$ by Theorem 2.17 and there is a sequence $\xi_n \in (0, +\infty)$ such that $u(\xi_n) \to \bar{u}$ as $n \to \infty$.

Set $\Gamma(u) = \bar{K} - \bar{K}$ for the extended Euclidean combinatorial Ricci flow (4.4). Then $\bar{u}$ is an equilibrium point of the system (4.4) and $D\Gamma|_{\bar{u}} = -\frac{\partial(K_1, \ldots, K_N)}{\partial(u_1, \ldots, u_N)}|_{\bar{u}}$ is negative definite with null space $\{t \mathbf{1} \in \mathbb{R}^N|t \in \mathbb{R}\}$ generated by the normal vector of $\Sigma$ by Corollary 2.13. Note that the solution $u(t)$ of the extended Euclidean combinatorial Ricci flow (4.4) stays in $\Sigma$. This implies $\bar{u}$ is a local attractor of the extended Euclidean combinatorial Ricci flow (4.4). Then the exponential convergence of the solution $u(t)$ to $\bar{u}$ follows from the Lyapunov stability theorem (69, Chapter 5).

For the hyperbolic version of Theorem 4.4, we need some more arguments.

**Lemma 4.6.** Suppose $i, j$ are two adjacent vertices in $V$ and the weight $\eta_{ij}$ satisfies the structure condition (1.6). If the edge length $l_{ij}$ is defined by (1.4), $\epsilon_i = 1$ and $\epsilon_j \epsilon \in \{0, 1\}$, then there exist two positive constants $\lambda = \lambda(\epsilon_j, \eta_{ij})$ and $\mu = \mu(\eta_{ij})$ such that

$$\lambda(C_iC_j + S_iS_j) \leq \cosh l_{ij} \leq \mu(C_iC_j + S_iS_j),$$

where $C_i, C_j, S_i, S_j$ are defined by (3.1).

**Proof.** By (1.4), the edge length $l_{ij}$ satisfies $\cosh l_{ij} = C_iC_j + \eta_{ij}S_iS_j \leq (1 + |\eta_{ij}|)(C_iC_j + S_iS_j).$ Therefore, we can take $\mu = 1 + |\eta_{ij}|$.

If $\epsilon_j = 1$, then $\eta_{ij} > -1$ by the structure condition (1.6). In this case, $C_i = \sqrt{1 + e^{2\nu} > e^\nu = S_i$ and similarly $C_j > S_j$. If $\eta_{ij} > 0$, then $\cosh l_{ij} \geq \min\{1, \eta_{ij}\}(C_iC_j + S_iS_j)$, where $\min\{1, \eta_{ij}\} > 0$. If $-1 < \eta_{ij} \leq 0$, by $C_i > S_i, C_j > S_j$, we have $\cosh l_{ij} \geq (1 + \eta_{ij})C_iC_j > \frac{1}{2}(1 + \eta_{ij})(C_iC_j + S_iS_j)$, where $\frac{1}{2}(1 + \eta_{ij}) > 0$.

If $\epsilon_j = 0$, then $\eta_{ij} > 0$ by the structure condition (1.6). In this case, $\cosh l_{ij} \geq \min\{1, \eta_{ij}\}(C_iC_j + S_iS_j)$ with $\min\{1, \eta_{ij}\} > 0$. $\square$

**Lemma 4.7.** Suppose $(M, T, \epsilon, \eta)$ is a weighted triangulated connected closed surface with the weights $\epsilon : V \to \{0, 1\}$ and $\eta : E \to \mathbb{R}$ satisfying the structure conditions (1.6) and (1.7). $i \in V$ is a vertex with $\epsilon_i = 1$. Then for any $\epsilon > 0$, there exists a positive number $L = L(\epsilon, \eta, \epsilon)$ such that if $f_i > L$, the inner angle $\theta_i$ at the vertex $i$ of the nondegenerate hyperbolic triangle $\{ijk\} \in F$ with edge lengths defined by (1.4) is smaller than $\epsilon$.

**Proof.** By the hyperbolic cosine law, we have

$$\cos \theta_i = \frac{\cosh l_{ij} \cosh l_{ik} - \cosh l_{jk}}{\sinh l_{ij} \sinh l_{ik}} = \frac{\cosh(l_{ij} + l_{ik}) + \cosh(l_{ij} - l_{ik}) - 2 \cosh l_{jk}}{\cosh(l_{ij} + l_{ik}) - \cosh(l_{ij} - l_{ik})} = \frac{1 + \nu - 2\omega}{1 - \nu},$$

(4.9)
where \( \nu = \frac{\cosh(l_{ij} - l_{ik})}{\cosh(l_{ij} + l_{ik})} \) and \( \omega = \frac{\cosh l_{jk}}{\cosh(l_{ij} + l_{ik})} \). By Lemma 4.6, we have

\[
0 < \nu < \frac{\cosh(l_{ik})}{\cosh(l_{ij} + l_{ik})} < \frac{1}{\cosh l_{ij}} < \frac{1}{\lambda C_i} < \frac{1}{\lambda S_i} \quad (4.10)
\]

and

\[
0 < \omega < \frac{\mu(C_j C_k + S_j S_k)}{\lambda^2 (C_i C_j + S_i S_j) (C_i C_k + S_i S_k)} < \frac{\mu(C_j C_k + S_j S_k)}{\lambda^2 (C_i^2 C_j C_k + S_i^2 S_j S_k)} < \frac{\mu}{\lambda^2 S_i^2} \quad (4.11)
\]

where \( C_i > S_i \) is used for \( \epsilon_i = 1 \). Note that \( S_i = e^{l_i} \), (4.10) and (4.11) imply \( \nu, \omega \to 0 \) uniformly as \( f_i \to +\infty \). By (4.9), \( \theta_i \) tends to 0 uniformly as \( f_i \to +\infty \). Therefore, for any \( \epsilon > 0 \), there exists \( L > 0 \) such that if \( f_i > L \), then \( \theta_i < \epsilon \).

**Remark 19.** Suppose \( \{ijk\} \) is a topological triangle in \( F \) with \( \epsilon_i = 1 \), the weights \( \epsilon, \eta \) satisfying the structure condition (1.6) and edge lengths defined by (1.4). By (4.11) in the proof of Lemma 4.7, there exists a positive constant \( L = L(\epsilon, \eta) \) such that if \( f_i > L \), then \( l_{jk} < l_{ij} + l_{ik} \).

As a corollary of Lemma 4.7, we have the following estimation of the extended inner angle \( \tilde{\theta} \).

**Corollary 4.8.** Suppose \((M, T, \epsilon, \eta)\) is a weighted triangulated connected closed surface with the weights \( \epsilon : V \to \{0, 1\} \) and \( \eta : E \to \mathbb{R} \) satisfying the structure conditions (1.6) and (1.7). \( \{ijk\} \) is a topological triangle in \( F \) with \( \epsilon_i = 1 \). Then for any \( \epsilon > 0 \), there exists a positive number \( L = L(\epsilon, \eta, \epsilon) \) such that if \( f_i > L \), the extended inner angle \( \tilde{\theta}_i \) defined by (3.27) at the vertex \( i \) in the generalized hyperbolic triangle \( \{ijk\} \in F \) with edge lengths defined by (1.4) is smaller than \( \epsilon \).

**Proof.** By Remark 19, there exists a constant \( L_1 = L_1(\epsilon, \eta) > 0 \), if \( f_i > L_1 \), then \( l_{jk} < l_{ij} + l_{ik} \). If the generalized hyperbolic triangle does not degenerate, then by Lemma 4.7 for \( \epsilon > 0 \), there exists a constant \( L_2 = L_2(\epsilon, \eta, \epsilon) > 0 \) such that if \( f_i > L_2 \), then \( \theta_i = \tilde{\theta}_i < \epsilon \). If the generalized hyperbolic triangle \( \{ijk\} \) degenerates, we claim that \( \theta_i = 0 \). Then the result in the corollary follows by taking \( L = \max\{L_1, L_2\} \).

Now we prove the claim. By (3.22) in the proof of Lemma 3.10 the following map

\[
F : \mathbb{R}^3 \to \mathbb{R}_{>0}^3
\]

\[
(f_i, f_j, f_k) \mapsto (l_{jk}, l_{ik}, l_{ij})
\]

is injective. Furthermore, \( F \) is proper by Lemma 4.6. Therefore, \( F : \mathbb{R}^3 \to F(\mathbb{R}^3) \) is a diffeomorphism by invariance of domain. Set \( \mathcal{L} = \{(l_{jk}, l_{ik}, l_{ij}) \in \mathbb{R}_{>0}^3 | l_{st} > l_{qs} + l_{qt}, \{q, s, t\} = \{i, j, k\}\} \). Then \( \Omega_{ijk}^H(\eta) = F^{-1}(F(\mathbb{R}^3) \cap \mathcal{L}) \) with the boundary of \( \Omega_{ijk}^H(\eta) \)
in $\mathbb{R}^3$ mapped homeomorphic to the boundary of $\mathcal{F}(\mathbb{R}^3) \cap \mathcal{L}$ in $\mathcal{F}(\mathbb{R}^3)$. If $f_i > L_1$ and the generalized hyperbolic triangle $\{ijk\}$ degenerates, then $(l_{jk}, l_{ik}, l_{ij})$ is in the region $W_j := \{(l_{jk}, l_{ik}, l_{ij}) \in \mathbb{R}_{>0}^3 | l_{ik} \geq l_{ij} + l_{jk}\}$ or $W_k := \{(l_{jk}, l_{ik}, l_{ij}) \in \mathbb{R}_{>0}^3 | l_{ij} \geq l_{ik} + l_{jk}\}$, the boundary of which are $\partial W_j = \{(l_{jk}, l_{ik}, l_{ij}) \in \mathbb{R}_{>0}^3 | l_{ik} = l_{ij} + l_{jk}\}$ and $\partial W_k = \{(l_{jk}, l_{ik}, l_{ij}) \in \mathbb{R}_{>0}^3 | l_{ij} = l_{ik} + l_{jk}\}$ respectively. Note that $\theta_i$ is a continuous function with $\tilde{\theta}_i = 0$ on $\partial W_j$ and $\partial W_k$. Therefore, $\tilde{\theta}_i = 0$ for degenerate hyperbolic discrete conformal structures $(f_i, f_j, f_k) \in \mathbb{R}^3$ with $f_i > L_1$.

Now we can prove the hyperbolic version of Theorem 4.4, which generalizes Theorem 1.2 (a) (c) in the hyperbolic background geometry.

**Theorem 4.9.** Suppose $(M, \mathcal{T}, \varepsilon, \eta)$ is a weighted triangulated connected closed surface with the weights $\varepsilon : V \to \{0, 1\}$ and $\eta : E \to \mathbb{R}$ satisfying the structure conditions (1.6) and (1.7). Then the solution of extended combinatorial Ricci flow (4.4) in the hyperbolic background geometry exists for all time for any initial generalized hyperbolic discrete conformal structure $u$ on $(M, \mathcal{T}, \varepsilon, \eta)$. Furthermore, if there exists a nondegenerate hyperbolic discrete conformal structure $\overline{u}$ with combinatorial curvature $\overline{K}$, then the solution of the extended combinatorial Ricci flow (4.4) in the hyperbolic background geometry converges exponentially fast to $\overline{u}$ for any initial generalized hyperbolic discrete conformal structure $u(0)$.

**Proof.** By (3.12), $u_i \in \mathbb{R}$ for vertex $i$ with $\varepsilon_i = 0$ and $u_i \in \mathbb{R}_{<0} = (-\infty, 0)$ for vertex $i$ with $\varepsilon_i = 1$. Therefore, $u = (u_1, \ldots, u_N) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$, where $N_1$ is the number of vertices in $V$ with $\varepsilon = 0$ and $N_2 = N - N_1$. If $u(t)$ is a solution of the extended hyperbolic combinatorial Ricci flow (4.4), then $|u_i(t)| \leq |u_i(0)| + |\overline{K}_i| + (d_i + 2)\pi |t| < +\infty$ for $t \in (0, +\infty)$, where $d_i$ is the number of vertices adjacent to the vertex $i \in V$. This implies $u_i(t)$ is bounded for the vertex $i \in V$ with $\varepsilon_i = 0$ and bounded from below for $i \in V$ with $\varepsilon_i = 1$ in finite time. We claim that $u_i(t)$ is uniformly bounded from above in $(-\infty, 0)$ for $i \in V$ with $\varepsilon_i = 1$. Then the longtime existence for the solution of the extended hyperbolic combinatorial Ricci flow (4.4) follows.

We use Ge-Xu’s trick in [31] to prove the claim. Suppose there exists some $i \in V$ such that $\lim_{t \uparrow T} u_i(t) = 0$ for $T \in (0, +\infty)$, which corresponds to $\lim_{t \uparrow T} f_i(t) = +\infty$ by (3.12). By Corollary 4.8, for $\varepsilon = \frac{1}{d_i}(2\pi - \overline{K}_i) > 0$, where $d_i$ is the degree of the vertex $v_i$, there exists a constant $c < 0$ such that if $u_i > c$, then $\tilde{\theta}_i < \varepsilon$ and then $\overline{K}_i > \overline{K}_i$. Choose a time $t_0 \in (0, T)$ such that $u_i(t_0) > c$, the existence of which is ensured by $\lim_{t \uparrow T} u_i(t) = 0$. Set $a = \inf\{t < t_0 | u_i(s) > c, \forall s \in [t, t_0]\}$, then $u_i(a) = c$. Note that for $t \in (a, t_0]$, $u_i'(t) = \overline{K}_i - \overline{K}_i < 0$ along the flow (4.4), we have $u_i(t_0) < u(a) = c$, which contradicts the assumption that $u_i(t_0) > c$. The arguments here further imply that $u_i(t)$ is uniformly bounded from above in $(-\infty, 0)$ for all $i \in V$ with $\varepsilon_i = 1$.

If there exists a nondegenerate hyperbolic discrete conformal structure $\overline{u}$ with com-
binatorial curvature $\overline{K}$, then $\overline{u}$ is a critical point of the $C^1$ smooth convex function $\overline{H}(u) = \overline{E}(u) - \int_0^u \sum_{i=1}^N \overline{K}_i du_i$, where $\overline{E}(u)$ is the extended Ricci energy function defined by (3.29). Note that $0 = \overline{H}(\overline{u}) \leq \overline{H}(u)$ and $\nabla \overline{H}(\overline{u}) = 0$, by Lemma 4.5, we have $\lim_{u \to \infty} \overline{H}(u) = +\infty$. Further note that

$$\frac{d}{dt} \overline{H}(u(t)) = \nabla_a \overline{H} \cdot \frac{du}{dt} = - \sum_{i=1}^N (\overline{K}_i - K_i)^2 \leq 0$$

along the extended hyperbolic combinatorial Ricci flow (4.4), we have $0 \leq \overline{H}(u(t)) \leq \overline{H}(u(0))$, which implies the solution $u(t)$ of the extended hyperbolic combinatorial Ricci flow (4.4) lies in a compact subset of $\mathbb{R}^N$. Combining with the fact that $u_i(t)$ is uniformly bounded from above in $(-\infty, 0)$ for any vertex $i \in V$ with $\epsilon_i = 1$, the solution $u(t)$ of the extended hyperbolic combinatorial Ricci flow (4.4) lies in a compact subset of $\mathbb{R}^{N_1} \times \mathbb{R}_{<0}^{N_2}$. The proof in the following is the same as that for Theorem 4.4, so we omit the details here.

Remark 20. By Remark 13 and Remark 17, the extended combinatorial curvature $\overline{K}$ is not Lipschitz. As a result, the combinatorial Laplace operator $\Delta = -\frac{\partial (K_1, \ldots, K_N)}{\partial (u_1, \ldots, u_N)}$ can not be extended by extending the combinatorial curvature $K$ to be $\overline{K}$. Therefore, the combinatorial Calabi flow can not be extended in the way used for the combinatorial Ricci flow in this section. In the special case that $\epsilon_i = 0$ for all $i \in V$, i.e. the case of vertex scaling, there is another way introduced in [43, 44] to extend the combinatorial Yamabe flow, where one does surgery on the combinatorial Yamabe flow by edge flipping when the triangulation is not Delaunay in the polyhedral metric along the combinatorial Yamabe flow. The method of doing surgery by edge flipping also applies to combinatorial Calabi flow for vertex scaling of polyhedral metrics [99]. It is proved that the combinatorial Yamabe flow with surgery [43, 44] and the combinatorial Calabi flow with surgery [99] exists for all time and converges exponentially fast.

5 Relationships of discrete conformal structures on polyhedral surfaces and 3-dimensional hyperbolic geometry

5.1 Construction of discrete conformal structures via generalized hyperbolic tetrahedra

The relationship of discrete conformal structures on polyhedral surfaces and 3-dimensional hyperbolic geometry was first discovered by Bobenko-Pinkall-Springborn [3] in the case of vertex scaling, which was further studied in [95, 96]. In this section, we extend the interpolation for the Ricci energy of discrete conformal structures on a triangle in terms of
co-volume functions of some generalized tetrahedra in $\mathbb{H}^3$ to more general cases and study the convexity of the co-volume functions.

We use the Klein model for $\mathbb{H}^3$ with $S^2$ as the ideal boundary $\partial \mathbb{H}^3$. Suppose $\{ijk\}$ is a Euclidean or hyperbolic triangle generated by discrete conformal structures in Definition 2. The Ricci energy for the triangle $\{ijk\}$ is closely related to the co-volume of a generalized tetrahedron $T_{Oijk}$ in the extended hyperbolic space $\mathbb{H}^3$, whose vertices are truncated by a hyperbolic plane in $\mathbb{H}^3$ or by a horosphere in $\mathbb{H}^3$. In the following, we briefly describe the construction of $T_{Oijk}$ for $\varepsilon \in \{0,1\}$. One can also refer to [3,95,96] for more information.

The generalized tetrahedron $T_{Oijk}$ has 4 vertices $O,v_i,v_j,v_k$, which are ideal or hyper-ideal. The vertex $O$ is called the bottom vertex.

1. For the Euclidean background geometry, $O$ is ideal, i.e. $O \in \partial \mathbb{H}^3$, and the generalized hyperbolic tetrahedron $T_{Oijk}$ is truncated by a horosphere $H_O$ at $O$. Please refer to Figure 1 for a generalized hyperbolic tetrahedron $T_{Oijk}$ with $O$ ideal. The Euclidean triangle $\{ijk\}$ is the intersection of the generalized hyperbolic tetrahedron $T_{Oijk}$ with the horosphere $H_O$ at $O$. For the hyperbolic background geometry, $O$ is hyper-ideal, i.e. $O \notin \mathbb{H}^3 \cup \partial \mathbb{H}^3$, and the generalized hyperbolic tetrahedron is truncated by a hyperbolic plane $P_O$ in $\mathbb{H}^3$ dual to $O$. Please refer to Figure 2 for a generalized hyperbolic tetrahedron $T_{Oijk}$ with $O$ hyper-ideal. The hyperbolic triangle $\{ijk\}$ is the intersection of the hyperbolic plane $P_O$ with the generalized hyperbolic tetrahedron $T_{Oijk}$.

![Figure 1: Tetrahedron for PL metric](image1)

![Figure 2: Tetrahedron for PH metric](image2)

2. For $v_q \in \{v_i,v_j,v_k\}$, if the corresponding $\varepsilon_q = 1$, then the vertex $v_q$ is hyper-ideal and the generalized tetrahedron $T_{Oijk}$ is truncated by a hyperbolic plane $P_q$ in $\mathbb{H}^3$. 

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dual to \(v_q\). If \(O\) is also hyper-ideal, then \(P_O \cap P_q = \emptyset\), which is equivalent to the line segment \(Ov_q\) has nonempty intersection with \(\mathbb{H}^3\) in the Klein model. If \(\varepsilon_q = 0\), then the vertex \(v_q\) is ideal and the generalized tetrahedron \(T_{Oijk}\) is truncated by a horosphere \(H_q\) at \(v_q\). For simplicity, we choose the horosphere \(H_q\) so that it has no intersection with the hyperplane or horospheres attached to other vertices of the generalized hyperbolic tetrahedron \(T_{Oijk}\).

(3) The signed edge length of \(Ov_i, Ov_j, Ov_k\) are \(-u_i, -u_j, -u_k\) respectively.

(4) For the edge \(v_iv_j\) in the extended hyperbolic space, the weight \(\eta_{ij}\) is assigned as follows.

(a) If \(v_i, v_j\) are hyper-ideal and spans a spacelike or lightlike subspace \(P_{ij}\), then \(\eta_{ij} = \cos \beta_{ij}\), where \(\beta_{ij}\) is determined by \(-v_i \circ v_j = \|v_i\| \cdot \|v_j\| \cdot \cos \beta_{ij}\). Here we take \(v_i, v_j\) as points in the Minkowski space, \(\circ\) is the Lorentzian inner product in the Minkowski space and \(\|\cdot\|\) is the norm of a spacelike vector. In fact, in this case, the hyperbolic planes \(P_i\) and \(P_j\), dual to \(v_i\) and \(v_j\) respectively, intersect in \(\mathbb{H}^3\) and \(\beta_{ij}\) is the dihedral angle determined by \(P_i\) and \(P_j\) in the truncated tetrahedron.

(b) If \(v_i, v_j\) are hyper-ideal and spans a timelike subspace, then \(P_i\) and \(P_j\) do not intersect in \(\mathbb{H}^3\). Denote \(\lambda_{ij}\) as the hyperbolic distance of \(P_i\) and \(P_j\), then \(\eta_{ij} = \cosh \lambda_{ij} = -\frac{v_i \circ v_j}{\|v_i\| \cdot \|v_j\|}\).

(c) If \(v_i, v_j\) are ideal, we choose the horospheres \(H_i, H_j\) at \(v_i, v_j\) with \(H_i \cap H_j = \emptyset\) and set \(\lambda_{ij}\) to be the distance from \(H_i \cap \overrightarrow{v_i v_j}\) to \(H_j \cap \overrightarrow{v_i v_j}\), where \(\overrightarrow{v_i v_j}\) is the geodesic from \(v_i\) to \(v_j\). Then \(\eta_{ij} = \frac{1}{2}e^{\lambda_{ij}}\).

(d) If \(v_i\) is ideal and \(v_j\) is hyper-ideal, we choose the horosphere \(H_i\) at \(v_i\) to have no intersection with the hyperbolic plan \(P_j\) dual to \(v_j\). Set \(\lambda_{ij}\) to be the distance from \(H_i\) to \(P_j\). Then \(\eta_{ij} = \frac{1}{2}e^{\lambda_{ij}}\).

In this setting, it can be checked that the lengths for the edges in the Euclidean triangle \(H_O \cap T_{Oijk}\) and in the hyperbolic triangle \(P_O \cap T_{Oijk}\) are given by \((1.3)\) and \((1.4)\) respectively, where \(u_i = f_i\) for the Euclidean background geometry and \(u_i\) is defined by \((3.12)\) in terms of \(f_i\) for the hyperbolic background geometry.

By the hyperbolic cosine laws for generalized hyperbolic triangle \(v_iv_jv_k\) in the extended hyperbolic plane, it can be checked that

\[
\eta_{st} + \varepsilon_s \varepsilon_t > 0, \quad \varepsilon_s \eta_{tq} + \eta_{st} \eta_{sq} > 0, \quad \{s, t, q\} = \{i, j, k\},
\]

which proves Theorem 1.4 (a). We suggest the readers to refer to Appendix A of [47] for a full list of formulas of hyperbolic sine and cosine laws for generalized hyperbolic triangles used here.
In the case that $\varepsilon_i = \varepsilon_j = \varepsilon_k = 1$, this can be proved in a geometric approach. Note that $\eta_{ij} = -\frac{v_i \times v_j}{||v_i|| \cdot ||v_j||}$ in this case. Taking $\eta_{ij} + \eta_{ik} \eta_{jk} > 0$ for example. Note that

\[(v_j \otimes v_k) \circ (v_k \otimes v_i) = \left( \frac{v_j}{||v_j||} \otimes \frac{v_k}{||v_k||} \right) \circ \left( \frac{v_k}{||v_k||} \otimes \frac{v_i}{||v_i||} \right) \]

\[= \frac{v_j}{||v_j||} \circ \frac{v_k}{||v_k||} \circ \frac{v_i}{||v_i||} \]

\[= - (\eta_{ij} + \eta_{ik} \eta_{jk}), \tag{5.1} \]

where $\otimes$ is the Lorentzian cross product defined by $x \otimes y = J(x \times y)$ with $J = \text{diag}\{-1, 1, 1\}$ for $x, y \in \mathbb{R}^3$. To prove $\eta_{ij} + \eta_{ik} \eta_{jk} > 0$, we just need to prove $(v_j \otimes v_k) \circ (v_k \otimes v_i) < 0$. Please refer to [70] (Chapter 3) for more details on Lorentzian cross product and (5.1). In the following, we use $P_{st} = \text{span}(v_s, v_t)$ to denote the two dimensional plane spanned by $v_s$ and $v_t$ in the Minkowski space, where $\{s, t\} \subset \{i, j, k\}$. By symmetry, we just need to consider the following six cases.

![Figure 3: Generalized triangles](image-url)
(a) If $P_{ik}$ and $P_{jk}$ are spacelike, then $v_j \otimes v_k$, $v_k \otimes v_i$ are timelike with the same parity, which implies $(v_j \otimes v_k) \circ (v_k \otimes v_i) < 0$. Please refer to Figure 3 (a).

(b) If $P_{ik}$ and $P_{jk}$ are timelike, then $v_j \otimes v_k$, $v_k \otimes v_i$ are spacelike and $(v_j \otimes v_k) \circ (v_k \otimes v_i) = -||v_j \otimes v_k|| \cdot ||v_k \otimes v_i|| \cosh d(P_{ik}, P_{jk}) < 0$. Please refer to Figure 3 (b).

(c) If $P_{ik}$ is spacelike and $P_{jk}$ is timelike, then $v_j \otimes v_k$ is spacelike and $v_k \otimes v_i$ is timelike. Then $(v_j \otimes v_k) \circ (v_k \otimes v_i) = -||v_j \otimes v_k|| \cdot ||v_k \otimes v_i|| \sinh d < 0$, where $d$ is the distance of $\frac{v_k \otimes v_i}{||v_k \otimes v_i||}$ to $P_{jk}$. Please refer to Figure 3 (c).

(d) If $P_{ik}$ is spacelike and $P_{jk}$ is lightlike, then $v_j \otimes v_k$ is lightlike and $v_k \otimes v_i$ is timelike with the same parity as $v_j \otimes v_k$. Then $(v_j \otimes v_k) \circ (v_k \otimes v_i) < 0$. Please refer to Figure 3 (d).

(e) If $P_{ik}$ is timelike and $P_{jk}$ is lightlike, then $v_j \otimes v_k$ is lightlike, $v_k \otimes v_i$ is spacelike with the same parity as $v_j \otimes v_k$. Then $(v_j \otimes v_k) \circ (v_k \otimes v_i) < 0$. Please refer to Figure 3 (e).

(f) If $P_{ik}$ is lightlike and $P_{jk}$ is lightlike, then $v_j \otimes v_k$ is lightlike and $v_k \otimes v_i$ is lightlike with the same parity as $v_j \otimes v_k$. Furthermore, $v_j \otimes v_k$ and $v_k \otimes v_i$ are linearly independent. Then $(v_j \otimes v_k) \circ (v_k \otimes v_i) < 0$. Please refer to Figure 3 (f).

Remark 21. In the case of Thurston’s circle packings, similar explanation of the structure condition [L.7] in terms of the spherical cosine law was recently obtained by Zhou [98].

5.2 Convexities of co-volume functions of generalized hyperbolic tetrahedra

For the generalized hyperbolic tetrahedron $T_{ijjk}$ above, we have attached it with a generalized hyperbolic polyhedron $P$ in the extended hyperbolic space by truncating it by the hyperbolic plane or horosphere attached to the vertices $O, v_i, v_j, v_k$. If $P$ is a finite hyperbolic polyhedron in $H^3$, we set $P = P$. Otherwise, the generalized hyperbolic polyhedron $P$ has ideal or hyper-ideal vertices and we need to further truncate $P$ to get a finite hyperbolic polyhedron $P$. For example, in the case $\varepsilon_s = \varepsilon_t = 1$ and $P_{st}$ is lightlike for $\{s, t\} \subset \{i, j, k\}$, the generalized hyperbolic polyhedron $P$ has at least one ideal vertex $P_s \cap P_t \cap \partial H^3$ and we need further use a horosphere at $P_s \cap P_t \cap \partial H^3$ to truncate $P$ to get a finite hyperbolic polyhedron $P$ in $H^3$. Please refer to Figure 3 (d)(e)(f) for this case. Another example is the case that $\varepsilon_i = \varepsilon_j = \varepsilon_k = 1$, $P_{ij}, P_{ik}, P_{jk}$ are spacelike and the generalized hyperbolic triangle $\Delta v_i v_j v_k$ is tangential to $\partial H^3$. In this case, $P$ has an ideal vertex at $P_{ij} \cap P_{ik} \cap P_{jk} \cap \partial H^3$ and we need further use a horosphere at $P_{ij} \cap P_{ik} \cap P_{jk} \cap \partial H^3$ to truncate $P$ to obtain a finite hyperbolic polyhedron $P$ in $H^3$. A third example is the
case that $\varepsilon_i = \varepsilon_j = \varepsilon_k = 1$, $P_{ij}, P_{ik}, P_{jk}$ are spacelike, the generalized hyperbolic triangle $\triangle v_i v_j v_k$ has no intersection with $\partial \mathbb{H}^3$ and the point $P_{ij} \cap P_{ik} \cap P_{jk}$ is hyper-ideal. In this case, we further need to use a hyperbolic plane $P_{ijk}$ dual to $P_{ij} \cap P_{ik} \cap P_{jk}$ to truncate $\widetilde{P}$ to get a finite hyperbolic polyhedron $P$. One can refer to Figure 1 and Figure 2 for this case.

Denote the volume of the finite hyperbolic polyhedron $P$ by $V$. By the Schl"afli formula \cite{71}, we have
\[ dV = \frac{1}{2}(-u_i d\theta_i - u_j d\theta_j - u_k d\theta_k + \lambda_{ij} d\beta_{ij} + \lambda_{ik} d\beta_{ik} + \lambda_{jk} d\beta_{jk}). \]
If $v_q, v_s \in \{v_i, v_j, v_k\}$ are spacelike and $P_{qs}$ is non-timelike, then $\beta_{qs}$ is fixed, otherwise $\lambda_{qs}$ is fixed. Set
\[ \mu_{qs} = \begin{cases} 0, & \text{if } \varepsilon_q = \varepsilon_s = 1 \text{ and } P_{qs} \text{ is non-timelike;} \\ 1, & \text{otherwise}. \end{cases} \]
Define the co-volume by
\[ \hat{V} = 2V - u_i \theta_i - u_j \theta_j - u_k \theta_k + \mu_{ij} \lambda_{ij} \beta_{ij} + \mu_{ik} \lambda_{ik} \beta_{ik} + \mu_{jk} \lambda_{jk} \beta_{jk}. \quad (5.2) \]
Then we have
\[ d\hat{V} = -\theta_i du_i - \theta_j du_j - \theta_k du_k. \quad (5.3) \]
By Theorem 2.11 and Theorem 3.11 (5.3) implies the co-volume function $\hat{V}$ defined by (5.2) is convex in $u_i, u_j, u_k$, which implies the co-volume function $\hat{V}$ is convex in the edge lengths $l_{Ov_i} = -u_i, l_{Ov_j} = -u_j, l_{Ov_k} = -u_k$. This completes the proof of Theorem 1.4 (b).

6 Open problems

6.1 Convergence of discrete conformal structures to the Riemann mapping

Thurston conjectured that the tangential circle packing could be used to approximate the Riemann mapping, which was proved by Rodin-Sullivan \cite{74}. Thurston’s conjecture was then further studied by lots of mathematicians, see \cite{50, 52} and others. In the case of vertex scaling, the corresponding convergence to Riemann mapping was recently proved by Luo-Sun-Wu \cite{64} in the Euclidean background geometry and by Wu-Zhu \cite{84} in the hyperbolic background geometry. See also \cite{12, 63, 83} for related works. For the discrete conformal structure, which is a generalization of circle packings and vertex scaling, it is convinced that Thurston’s conjecture is still true.
6.2 Discrete uniformization theorems for discrete conformal structures

Another interesting question about discrete conformal structure on polyhedral surfaces is the existence of discrete conformal structure with prescribed combinatorial curvature. In the special case that the prescribed combinatorial curvature is 0, this corresponds to the discrete uniformization theorem. In the case of vertex scaling of polyhedral metrics, the discrete uniformization theorems for polyhedral metrics were recently established in [43, 44, 77]. Note that the case of vertex scaling corresponds to $\epsilon_i = 1$ for all vertex $i \in V$ in our case. This motivates us to study the discrete uniformization theorem for discrete conformal structures.

Suppose $(M, V)$ is a marked surface and $V$ is a nonempty finite subset of $M$. $\varepsilon : V \rightarrow \{0, 1\}$ is a weight defined on $V$. $(M, V, \varepsilon)$ is called a weighted marked surface. Motivated by Glickenstein’s work [35, 37–39], we introduce the following definition of weighted Delaunay triangulation.

**Definition 7.** Suppose $(M, V, \varepsilon)$ is a weighted marked surface with a PL metric $d$. $T$ is a geometric triangulation of $(M, V, \varepsilon)$ with every triangle $\{ijk\}$ in the triangulation having a well-defined geometric center $C_{ijk}$. Suppose $\{ij\}$ is an edge shared by two adjacent Euclidean triangles $\{ijk\}$ and $\{ijl\}$. The edge $\{ij\}$ is called weighted Delaunay if $h_{ijk} + h_{ijl} \geq 0$, where $h_{ijk}, h_{ijl}$ are the signed distance of $C_{ijk}, C_{ijl}$ to the edge $\{ij\}$ respectively. The triangulation $T$ is called weighted Delaunay in $d$ if every edge in the triangulation is weighted Delaunay.

One can also define the weighted Delaunay triangulation using the power distance in Remark 9. For a PL metric $d$ on $(M, V, \varepsilon)$, its weighted Voronoi decomposition is defined to be the connection of 2-cells $\{R(v) | v \in V\}$, where $R(v) = \{x \in M | \pi_v(x) \leq \pi_{v'}(x) \text{ for all } v' \in V\}$ is defined by the power distance. The dual cell-decomposition $\mathcal{C}(d)$ of the weighted Voronoi decomposition is called the weighted Delaunay tessellation of $(M, V, \varepsilon, d)$. A weighted Delaunay triangulation $\mathcal{T}$ of $(M, V, \varepsilon, d)$ is a geometric triangulation of the weighted Delaunay tessellation $\mathcal{C}(d)$ by further triangulating all non-triangular 2-dimensional cells without introducing extra vertices. As the power distance is a generalization of Euclidean distance, the weighted Delaunay triangulation is a generalization of the Delaunay triangulation.

Following Gu-Luo-Sun-Wu [44], we introduce the following new definition of discrete conformality, which allows the triangulation of the weighted marked surface $(M, V, \varepsilon)$ to be changed.

**Definition 8.** Two piecewise linear metrics $d, d'$ on $(M, V, \varepsilon)$ are discrete conformal if there exist sequences of PL metrics $d_1 = d, \ldots, d_m = d'$ on $(M, V, \varepsilon)$ and triangulations $\mathcal{T}_1, \ldots, \mathcal{T}_m$ of $(M, V, \varepsilon)$ satisfying
(a) (Weighted Delaunay condition) each $T_i$ is weighted Delaunay in $d_i$.

(b) (Discrete conformal condition) if $T_i = T_{i+1}$, there exists two functions $u_i, u_{i+1} : V \to \mathbb{R}$ such that if $e$ is an edge in $T_i$ with end points $v$ and $v'$, then the lengths $l_{d_i}(e)$ and $l_{d_{i+1}}(e)$ of $e$ in $d_i$ and $d_{i+1}$ are defined by (1.3) using $u_i$ and $u_{i+1}$ respectively with the same weight $\eta : E \to \mathbb{R}$.

(c) if $T_i \neq T_{i+1}$, then $(S, d_i)$ is isometric to $(S, d_{i+1})$ by an isometry homotopic to identity in $(S, V)$.

The space of PL metrics on $(M, V, \varepsilon)$ discrete conformal to $d$ is called the conformal class of $d$ and denoted by $\mathcal{D}(d)$.

Motivated by Gu-Luo-Sun-Wu’s discrete uniformization theorem for PL metrics in [44], we have the following conjecture on the discrete conformal uniformization for Euclidean discrete conformal structures on weighted marked surfaces.

**Conjecture 1.** Suppose $(M, V, \varepsilon)$ is a closed connected weighted marked surface with $\varepsilon : V \to \{0, 1\}$, $\chi(M) = 0$ and $d$ is a PL metric on $(M, V, \varepsilon)$. There exists a PL metric $d' \in \mathcal{D}(d)$, unique up to scaling and isometry homotopic to the identity on $(M, V, \varepsilon)$, such that $d'$ is discrete conformal to $d$ and the discrete curvature of $d'$ is 0.

For the hyperbolic background geometry, one can define the weighted Delaunay similarly with $h_{i,j,k} + h_{i,j,l} \geq 0$ replaced by $\tanh h_{i,j,k} + \tanh h_{i,j,l} \geq 0$ and define the discrete conformality similarly. We have the following conjecture on the discrete uniformization for hyperbolic discrete conformal structures on weighted marked surfaces.

**Conjecture 2.** Suppose $(M, V, \varepsilon)$ is a closed connected weighted marked surface with $\varepsilon : V \to \{0, 1\}$, $\chi(M) < 0$ and $d$ is a PH metric on $(M, V, \varepsilon)$. There exists a unique PH metric $d' \in \mathcal{D}(d)$ on $(M, V, \varepsilon)$ so that $d'$ is discrete conformal to $d$ and the discrete curvature of $d'$ is 0.

One can also study the prescribing combinatorial curvature problem for the discrete conformal structures on polyhedral surfaces. Results similar to the main results in [43, 44] are convinced to be true for the discrete conformal structures on polyhedral surfaces.

**6.3 Convergence of combinatorial curvature flows with surgery**

In Theorem[1, 2] we extend the combinatorial Ricci flow through the singularities of the flow to ensure the convergence of the flow under the assumption that there exists a discrete conformal structure with constant combinatorial curvature. This method can not be applied to the combinatorial Calabi flow by Remark[20]. Furthermore, we do not hope the combinatorial curvature flows develop singularities in practical applications. One way to
avoid the singularities is to do surgery along the combinatorial curvature flows before the singularities develops. Motivated by the surgery by edge flipping introduced in [43,44] for vertex scaling, we introduced the following surgery for combinatorial curvature flows of discrete conformal structures on polyhedral surfaces.

Along the Euclidean combinatorial curvature flows (Euclidean combinatorial Ricci flow or combinatorial Calabi flow) for discrete conformal structures on a weighted marked surface \((M, V, \varepsilon)\) with a triangulation \(T\), if \(T\) is weighted Delaunay in \(d(u(t))\) for \(t \in [0, T]\) and not weighted Delaunay in \(d(u(t))\) for \(t \in (T, T+\varepsilon)\), \(\varepsilon > 0\), there exists an edge \(\{ij\} \in E\) such that \(h_{ij,k} + h_{ij,l} \geq 0\) for \(t \in [0, T]\) and \(h_{ij,k} + h_{ij,l} < 0\) for \(t \in (T, T+\varepsilon)\). We replace the triangulation \(T\) by a new triangulation \(T'\) at time \(t = T\) by replacing two triangles \(\{ijk\}\) and \(\{ijl\}\) adjacent to \(\{ij\}\) by two new triangles \(\{ikl\}\) and \(\{jkl\}\). This is called a surgery by flipping on the triangulation \(T\), which is an isometry of \((M, V, \varepsilon)\) in the PL metric \(d(u(T))\). After the surgery at time \(t = T\), we run the Euclidean combinatorial curvature flow on \((M, V, \varepsilon, T')\) with initial metric coming from the Euclidean combinatorial curvature flow on \((M, V, \varepsilon, T)\) at time \(t = T\). The surgery by flipping for hyperbolic combinatorial curvature flows could defined similarly.

We have the following conjecture on the longtime existence and convergence of the combinatorial Ricci flow and combinatorial Calabi flow with surgery.

**Conjecture 3.** Suppose \((M, V, \varepsilon)\) is a closed connected weighted marked surface with \(\varepsilon: V \to \{0, 1\}\). For any initial PL or PH metric on \((M, V, \varepsilon)\), the solution of combinatorial Ricci flow and combinatorial Calabi flow with surgery exists for all time and converges exponentially fast.

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