THE HOCHSCHILD COHOMOLOGY OF A POINCARÉ ALGEBRA

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ABSTRACT. In this note, we define the notion of a cactus set, and show that its geometric realization has a natural structure as an algebra over Voronov’s cactus operad, which is equivalent to the framed 2-dimensional little disks operad $D_2$. Using this, we show that for a Poincaré algebra $A$, its Hochschild cohomology is an algebra over the (chain complexes of) $D_2$.

In [3], Chas and Sullivan considered $H_*(LM)$, the integral singular homology of the free loop space on a compact smooth oriented manifold $M$, and showed that it has the structure of a Batalin-Vilkovisky algebra, i.e. an algebra over the framed 2-dimensional little disks operad $D_2$. In this note, we consider the question of what algebraic structures have the property that its cohomology has the structure of a $D_2$-algebra. The main result of the note is that if $A$ is a Poincaré algebra, then the dual of the Hochschild homology of $A$ has the natural structure of an algebra over the framed 2-dimensional little disks operad $D_2$. To prove the theorem, we make use of Voronov’s cactus operad [9], and define a structure of cactus sets, then show that the Hochschild cohomology of a Poincaré algebra has such a structure. This gives a generalization of the string topology result of Chas and Sullivan on $H_*(LM)$. Although strictly speaking, theirs is not a Poincaré algebra, but only one up to homotopy.

In the first section, we recall the cactus operad and define cactus sets, and show that the simplicial realization of a cactus set has the structure of an algebra over the cactus operad. In Section 2, we apply this notion to the case of the Hochschild cohomology of a Poincaré algebra.

1. CACTUS OBJECTS

In this section, we will define the notion of a cactus object in a symmetric monoidal category with simplicial realization, which is a cyclic object in the sense of Connes [2] with certain extra structures, and show that the simplicial realization of a cactus object has a natural action by the cactus operad defined by Voronov [9].

In [9], Voronov defined the cactus operad $\mathcal{C} = \{\mathcal{C}(n) | n \geq 1\}$ as follows. For each $n$, an element of $\mathcal{C}(n)$ is an ordered configuration of $n$ parametrized circles (the “lobes” of the cactus), with varying positive radii, such that the sum of all the...
radii is 1, and that the dual graph of the configuration is a tree. There are also the following additional data:

1. A cyclic ordering of the lobes at each intersection point of the circles.
2. A chosen distinguished point $0_i$ on the $i$-th circle of the cactus, for $i = 1, \ldots, n$.
3. A chosen distinguished point 0 for the entire configuration. If 0 is an intersection point of circles, there is also a choice of the particular circle on which it lies.

For instance, the following configuration is an element of $C(6)$. The solid dots are the distinguished points of the lobes, and the $\times$ mark is the distinguished point for the entire cactus.

The topology on $C(n)$ is obtained as a quotient of $(S^1)^n$.

The operad structure on $\{C(n)\}$ is as follows. Given such a configuration $c \in C(n)$, the choice of 0 and the cyclic ordering of the lobes at each intersection point defines a continuous map

$$f_c : S^1 \to c.$$ 

Namely, given the standard circle with radius 1, we can wrap it around the configuration $c$, starting at the distinguished point 0 of $c$, in the manner prescribed by the cyclic ordering of the components at each intersection point: namely, whenever we arrive at an intersection point, we always continue onto the circle that comes after the one through which we arrived, in the cyclic ordering (note that the circles are parametrized, so they come with orientations). Given cacti $c \in C(n)$ and $d_i \in C(k_i)$ for $i = 1, \ldots, n$, the composition $\gamma(c; d_1, \ldots, d_n) \in C(k_1 + \cdots + k_n)$ is obtained by collapsing the $i$-th lobe of $c$ to $d_i$ via this map (with its chosen distinguished point $0_i$ identified with the distinguished point for $d_i$). The distinguished point 0$_c$ for the cactus $c$ carries along on the lobe to which it belongs.

Voronov proved the following theorem.

**Theorem 1.1** ([9]). *The cactus operad is naturally homotopy equivalent to $D_2$, the framed 2-dimensional little disks operad.*

**Remark:** Note that as defined, the cactus operad is unbased: there is no $C(0)$. Hence, a $C$-algebra is a structure that is non-unital. However, from the point of view of $D_2$, we can think of $D_2(0)$ as consisting of a single element, which is a solid little disk with no framed little disks inside of it. Composition with this element fills in one framed little disk. (This is also the point of view from conformal field theory, where the 0-th space of the operad “caps off” one inbound boundary component of the surface.) From this, one could say that the $C(0)$ should also consist of one configuration, which is a single point. Composition with this element contracts one
lobe of the cactus to a single point (erasing the distinguished point of that lobe). However, we shall not make use of this in this note.

Another ingredient which we will need is the notion of a cyclic set [2]. Namely, the cyclic category $\Lambda$ has objects all sets $[n] = \{0, \ldots, n\}$, same as the simplicial category $\Delta$, and the morphisms are generated by the usual face maps $d_i$ and degeneracy maps $s_i$ of the simplicial category with the usual relations, as well as one extra degeneracy map

$$s_{n+1} : [n+1] \to [n]$$

for each $n \geq 1$, which has the relation

$$(d_0s_{n+1})^{n+1} = Id : [n] \to [n].$$

In particular, in the cyclic category the number of degeneracy maps $[n+1] \to [n]$ and the number of face maps $[n] \to [n+1]$ are the same. A cyclic set is a functor $S_\bullet : \Lambda^{op} \to Sets$. In other words, a cyclic set is a simplicial set $S_\bullet$ with an extra degeneracy $S_n \to S_{n+1}$ between the $n$-th and the 0-th simplicial coordinates, for each simplicial degree $n$. In particular, if $S_\bullet$ is a cyclic set, then its simplicial realization $|S_\bullet|$ has naturally an action of $S^1$ (see [7]).

One important property of the cyclic category is that that $\Lambda \simeq \Lambda^{op}$ by reversing the faces and degeneracies (see [2] [5]). (This equivalence is not canonical, since one can compose it with any automorphism of $\Lambda$ obtained by rotation.) Hence, we also have

$$\Lambda^{op}Sets \simeq (\Lambda^{op}Sets)^{op}$$

as categories, and the dual of a cyclic set is again a cyclic set.

The notion of a cactus object can be thought of as a generalization of a cyclic object, but with the cactus configurations of $C$ taking the place of the circle. For a cactus $c \in C(n)$, we will need to consider the intersection points of circles in $c$. Given an intersection point $x$ of $c$, let the multiplicity of $x$ denote the number of circles that $x$ lies on.

We will need to consider only the combinatorial part of the structure of the cactus operad, and define the following notion of a “spiny cactus”. For each cactus $c \in C(n)$, we saw above that there is a well-defined map $f_c : S^1 \to c$. Given $c \in C(n)$ and positive numbers $j_1, \ldots, j_n$, we define the set

$$c_{(j_1-1, \ldots, j_n-1)}$$

to consist of all configurations $(c, X)$, where $X$ is a set of chosen points on $c$, such that the points 0, 0, for $i = 1, \ldots, n$, as well as all the intersection points are in $X$, and there are exact $j_i$ points on the $i$-th circle. An intersection point of multiplicity $n$ is considered to be a point on every circle which contains the point, hence, it is counted $n$ times. Note that the set of points on each lobe of the cactus comes with a cyclic ordering, starting with 0. With the provision that an intersection point appears $n$ times, there is also a cyclic ordering on the set $X$ of points on the entire cactus (having $\Sigma j_i$ points in all), arising from the cyclic ordering of the preimages of the points on $S^1$ and starting from the distinguished point 0. Let $C(n)_{(j_1, \ldots, j_n)}$ be the disjoint union of $c_{(j_1, \ldots, j_n)}$ for all $c \in C(n)$. We make the following identifications on $C(n)_{(j_1, \ldots, j_n)}$. For two configurations $(c_1, X_1) \in c_{(j_1, \ldots, j_n)}$ and $(c_2, X_2) \in c_{(j_1, \ldots, j_n)}$, we identify $(c_1, X_1)$ and $(c_2, X_2)$ if $f_{c_2}$ can be obtained from
$f_{c_1}$ by a continuous reparametrization of $S^1$, which takes the points of $X_1$ to the points of $X_2$, matching the distinguished points and intersection points exactly. This is clearly an equivalence relation, and we denote by

$$\overline{c_{(j_1,\ldots,j_n)}}$$

the equivalence class of $c_{(j_1,\ldots,j_n)}$. For instance, a representative of such a “spiny cactus” configuration $(c, X)$, where $c \in C(6)$ is the cactus pictured above, looks as follows:

\begin{center}
\begin{tikzpicture}
  \node[circle,draw] (1) at (0,0) {$1$};
  \node[circle,draw] (2) at (1,0) {$2$};
  \node[circle,draw] (3) at (2,0) {$3$};
  \node[circle,draw] (4) at (3,0) {$4$};
  \node[circle,draw] (5) at (4,0) {$5$};
  \node[circle,draw] (6) at (5,0) {$6$};

  \draw[->] (1) to (2);
  \draw[->] (2) to (3);
  \draw[->] (3) to (4);
  \draw[->] (4) to (5);
  \draw[->] (5) to (6);
  \draw[->] (6) to (1);
\end{tikzpicture}
\end{center}

Here, the empty circles are the marked points other than the distinguished points of the cactus and of the individual lobes.

In essence, this identification removes the geometric information contained in $C(n)$, and retains only the combinatorial information. In particular, two configurations $(c_1, X_1)$ and $(c_2, X_2)$ are identified if one can be obtained from the other by changing the radii of the lobes, by moving one lobe of the cactus (or any of the chosen points) along the circumference of another lobe. However, one is not allowed to move a lobe or any of the distinguished points past each other, or past any other point of $X$.

For each $1 \leq k \leq n$, and $0 \leq i \leq j_k - 1$, we have a $(k, i)$-th cyclic degeneracy

$$\overline{c_{(j_1-1,\ldots,j_k-1,\ldots,j_n-1)}} \to \overline{c_{(j_1-1,\ldots,j_k,\ldots,j_n-1)}}$$

which is obtained by inserting a new point in the $i$-th position between two adjacent points on the $k$-th circle. The $(k, i)$-th cyclic face map

$$\overline{c_{(j_1-1,\ldots,j_k-1,\ldots,j_n-1)}} \to \overline{c'_{(j_1-1,\ldots,j_k-2,\ldots,j_n-1)}}$$

is obtained by pinching together two adjacent points at the $i$-th and $(i + 1)$-st positions on the $k$-th circle of the cactus. Note that a cyclic degeneracy always take a spiny cactus based on $c$ to a spiny cactus based on the same cactus $c$. However, a cyclic face degeneracy map may actually change the cactus configuration itself, if both points being pinched together are “special points” of the cactus, i.e. distinguished points or intersection points of the lobes. For instance, given the first “spiny” cactus configuration pictured below, if we pinch the intersection point of lobes 1 and 2 together with the intersection point of lobes 1 and 3, we get the second configuration, which is a different cactus.
The composition of cacti also translates to this model. Given cacti \( c \in C(n) \) and \( d_i \in C(m_i) \) for \( 1 \leq i \leq n \), consider the sets \( \tau_{(j_1-1,\ldots,j_n-1)} \) and
\[
\overline{d}_1(r_1,1-1,\ldots,r_{1,m_1}-1), \ldots, \overline{d}_n(r_n,1-1,\ldots,r_{n,m_n}-1).
\]
We require that for each \( 1 \leq i \leq n \),
\[
\sum_{l=1}^{m_i} r_{i,m_l} = j_i.
\]
Then we have a well-defined composition
\[
\tau_{(j_1-1,\ldots,j_n-1)} \times \overline{d}_1(r_1,1-1,\ldots,r_{1,m_1}-1) \times \cdots \times \overline{d}_n(r_n,1-1,\ldots,r_{n,m_n}-1) \rightarrow \gamma(c; d_1, \ldots, d_n)_{(s_1-1,\ldots,s_{\Sigma m_n}-1)}
\]
where \( \gamma(c; d_1, \ldots, d_n) \) is the composition of the cacti in \( C \), and \( s_1, \ldots, s_{\Sigma m_n} \) is a permutation of \( r_{1,1}, \ldots, r_{n,m_n} \), obtained by the ordering of lobes on the new cactus. In this sense, \( \tau_{(\ast,\ldots,\ast)} \) give a partial “cyclic model” of the cactus \( c \).

Let \( (S, \otimes) \) be a symmetric monoidal category, which also has simplicial realization from \( \Delta^{op}S \) to \( S \). We have the following definition of a cactus object in \( S \).

**Definition 1.2.** A cactus object \( S_\bullet \) in \( S \) is a cyclic object in \( S \), with additionally a structure map for each \( (c, X) \in \tau_{j_1,\ldots,j_n} \)
\[
\mu_{c,X} : S_{j_1-1} \otimes S_{j_2-1} \otimes \cdots \otimes S_{j_n-1} \rightarrow S_{j_1+\cdots+j_n-1}.
\]
These maps are compatible with the face and degeneracy maps of the cyclic set in the manner prescribed by the cactus \( c \). Namely, there is a cyclic ordering of the segments of \( X \) over the entire cactus, and the total simplicial degree is exactly \( j_1+\cdots+j_n-1 \). For \( 1 \leq k \leq n \) and \( 0 \leq i \leq j_k-1 \), let \( f(k,i) \) be the position in this overall cyclic ordering corresponding to the \( i \)-th segment of the \( k \)-th lobe. For the \( (k,i) \)-th degeneracy, we require the following diagram to commute:
\[
\begin{array}{ccc}
S_{j_1-1} \otimes \cdots \otimes S_{j_{k-1}} \otimes \cdots \otimes S_{j_n-1} & \xrightarrow{\sigma_{i}} & S_{j_1+\cdots+j_n-1} \\
\downarrow{\sigma_{f(k,i)}} & & \downarrow{\sigma_{f(k,i)}} \\
S_{j_1-1} \otimes \cdots \otimes S_{j_k} \otimes \cdots \otimes S_{j_{n+1-1}} & \xrightarrow{\mu_{c,X}} & S_{j_1+\cdots+j_n-1}
\end{array}
\]
where $\sigma_i$ is the $i$-th degeneracy. Similarly, for the $(k,i)$-th face map for $(c,X)$, the following diagram commutes:

$$
\begin{array}{c}
S_{j_1-1} \otimes \cdots \otimes S_{j_k-1} \otimes \cdots \otimes S_{j_n-1} \\
\downarrow \delta_i \\
S_{j_1-1} \otimes \cdots \otimes S_{j_k-1} \otimes \cdots \otimes S_{j_n-1}
\end{array}
\rightsquigarrow
\begin{array}{c}
S_{j_1+\cdots+j_n-1} \\
\delta_{f(k,i)}
\end{array}
$$

where $\delta_i$ denotes the $i$-th face map.

Further, the maps $\mu_{c,X}$ are compatible with the composition of cacti in the following manner. Given $(c,X)$ as above, for each $k$ between 1 and $n$, suppose we also have $(d,k,Y_k) \in \mathcal{T}_k(r_{k,1}^{-1},\ldots,r_{k,m_k}^{-1})$, such that $\Sigma r_{k,i} = j_k$. Then we require the following diagram to commute:

$$
\begin{array}{c}
(S_{r_{1,1}^{-1}} \otimes \cdots \otimes S_{r_{1,m_1}^{-1}}) \otimes \cdots \otimes (S_{r_{n,1}^{-1}} \otimes \cdots \otimes S_{r_{n,m_n}^{-1}}) \\
\downarrow \\
S_{j_1-1} \otimes \cdots \otimes S_{j_n-1}
\end{array}
\rightsquigarrow
\begin{array}{c}
S_{\Sigma r_{jk}^{-1}}
\end{array}
$$

For simplicity, we state and prove the following proposition in the context of sets. For the category of chain complexes, which will be the case relevant in the next section, the argument of the proof goes through by taking the chain complexes of the standard simplices (and the cactus operad).

**Proposition 1.3.** If $S_\bullet$ is a cactus set, then the simplicial realization of $S_\bullet$ has naturally an action of the cactus operad $\mathcal{C}$.

**Proof.** For a cactus $c \in \mathcal{C}(n)$, we will define a map

$$m_c : \Delta^{j_1-1} \times \cdots \times \Delta^{j_n-1} \to \Delta^{j_1+\cdots+j_n-1+m}$$

where $m$ is 1+ the total of the multiplicities of the intersection points of $c$. Namely, for each $1 \leq k \leq n$, let $r_k$ be the radius of the $k$-th lobe of the cactus. Given a point with barycentric coordinates $(s_{k,1},\ldots,s_{k,j_k}) \in \Delta^{j_k-1}$, with $s_{k,1} + \cdots + s_{k,j_k} = 1$, by scaling $r_k$ so that the circumference of the $k$-th lobe is 1, starting from the distinguished point of the $k$-th lobe, $(s_{k,1},\ldots,s_{k,j_k})$ determine $j_k$ points on the $k$-th lobe, which divide the circle into $j_k$ subintervals having lengths $s_{k,1},\ldots,s_{k,j_k}$. For each circle, we need to make sure that any intersection points are among the dividing points; we ensure this by using degeneracy maps $\Delta^{j_k} \to \Delta^{j_k+1}$ to add the intersection points on each circle. (Note that hence, each intersection point of $c$ is added as many times as its multiplicity. If on a particular circle, it happens to be a dividing point from the original barycentric coordinates, it is still added, and we will get a point in a higher simplex some of whose barycentric coordinates are 0.) Similarly, we also add the distinguished point 0 of the cactus, on the lobe on which it lies. Starting at the distinguished point 0 for the entire cactus, and using the well-defined flow (which has degree 1 on each circle) on the cactus, we get a well-defined subdivision of a single circle with circumference 1 into $j_1+\cdots+j_k-1$ subintervals. The length of these, in cyclic order, are $t_1,\ldots,t_{j_1+\cdots+j_n}$, with $t_1+\cdots+t_{j_1+\cdots+j_n} = 1$. Hence,

$$(t_1,\ldots,t_{j_1+\cdots+j_n})$$
is a point in $\Delta^{j_1 + \cdots + j_n - 1}$. Note that this map uses the actual cactus $c \in C(n)$, including the radii of its circles and the exact locations of its distinguished points, instead of just its class in $C(n)$. It is straightforward to see that this map is associative, from the associativity of the composition of cacti in $C$, and is compatible with the cyclic set structure of rotation.

Now for a cactus set $S_\bullet$, for each configuration $c \in C(n)$, and $j_1, \ldots, j_n$, we have

$$S_{j_1 - 1} \times \cdots \times S_{j_n - 1} \to S_{j_1 + \cdots + j_n + m - 1}$$

and

$$\Delta^{j_1 - 1} \times \cdots \times \Delta^{j_n - 1} \to \Delta^{j_1 + \cdots + j_n + m - 1}.$$ 

Here, $m$ is 1 + the total of the multiplicities of the intersection points of $c$, and the map on $S_\bullet$ is obtained from the structure map of a cactus set by inserting degeneracies at the appropriate positions.

Now recall that the simplicial realization of $S_\bullet$ is given by the coequalizer

$$\bigvee_{l=m} S_l \times \Delta^n \rightrightarrows \bigvee_{l=1} S_l \times \Delta^n \to |S_\bullet|.$$ 

For $c \in C(n)$, we have a map

$$(\bigvee_{l=1} S_l \times \Delta^l)^n \to \bigvee_{l=1} S_l \times \Delta^l.$$ 

Namely, consider the source as the sum over all $l_1, \ldots, l_n$ of

$$(S_{l_1 - 1} \times \cdots \times S_{l_n - 1}) \times (\Delta^{l_1 - 1} \times \cdots \times \Delta^{l_n - 1}).$$ 

As above, an element of $\Delta^{l_1 - 1} \times \cdots \times \Delta^{l_n - 1}$ determines a set $X$ of distinguished points on $c$. We have the map $m_c$, which takes it to $\Delta^{l_1 + \cdots + l_n - 1 + m}$. Further, using the spiny cactus $(c, X)$, we get a map from

$$S_{l_1 - 1} \times \cdots \times S_{l_n - 1} \to S_{l_1 + \cdots + l_n - 1 + m}$$ 

(after inserting degeneracies at the appropriate places of each $S_{l_i - 1}$, corresponding to the positions of the intersection points on each lobe of the cactus and to the distinguished point for the entire cactus). As both maps respect degeneracies and face maps, it is straightforward to check that this induces a well-defined map

$$c : |S_\bullet|^n \to |S_\bullet|,$$

and to check that it gives an operad action on $|S_\bullet|$. \hfill $\square$

2. The Hochschild cohomology of a Poincaré algebra

Recall the following definition of Poincaré algebras, which is a Frobenius algebra that is also commutative.

**Definition 2.1.** An associative algebra $A$ is a Poincaré algebra if it is commutative, and has the property that there is an augmentation $A \to K$, where $K$ is the base field, such that the adjoint map

$$A \to A^\vee$$

obtained from

$$A \otimes A \to A \to K$$

is an isomorphism. (Here, $A^\vee = \text{Hom}_K(A, K)$ is the dual of $A$ in the category of $K$-modules.)
In particular, by the isomorphism between $A$ and $A^\vee$, there is a coalgebra structure on $A$ which is dual to the algebra structure on $A$:

$$A \cong A^\vee \to (A \otimes A)^\vee \simeq A^\vee \otimes A^\vee \xrightarrow{\cong} A \otimes A.$$ 

The motivating example is the cohomology of an orientable manifold $M$, which is isomorphic to its dual $H^*_s(M)$ by Poincaré duality.

We consider the dual to $C_{cyclic}(A)$, which in this case is the cyclic bar construction $B_{cyclic}(A)$ (now considered as an algebra instead of a coalgebra). Note that for commutative $A$, this gives the Hochschild homology $HH_*(A)$. The main theorem of this section is the following.

**Theorem 2.2.** Let $A$ be a Poincaré algebra. Then the dual of its Hochschild homology $HH_*(A)^\vee$ naturally has the structure of the an algebra over the chain complexes of $C$.

By (2), the algebra multiplication of $A$ is dual to the coproduct on $A$ as a coalgebra, and it is straightforward to check that the algebra unit of $A$ is dual to the coalgebra counit of $A$. Hence, the degeneracies are given by the unit $K \to A$, and the face maps are given by multiplication $A \otimes A \to A$, and $B_{cyclic}(A)$ is the usual bar construction $B(A)$ of $A$, with an extra degeneracy. In fact, we can visualize $B_{cyclic}^n(A)$ as the tensor product of $n + 1$ copies of $A$ arranged in a circle as below, with the appropriate coface and codegeneracy maps. (Note that it can also be considered as a cocyclic object, with the degeneracies being the cofaces, and the faces being the codegeneracies, since the categories of cyclic and cocyclic objects are canonically equivalent.)

$$\begin{array}{c}
A 
\otimes
A \\
\otimes
A \\
\vdots \\
\otimes
A \\
\otimes
A \\
\end{array}$$

We claim that this has the dual structure to a cactus object in the category of module over the base field $K$. (Note that here, the product $\times$ of sets is replaced by $\otimes$ of algebras). In other words, for every cactus configuration $(c, X)$, where there are $j_k$ points on the $k$-th lobe of the cactus, there is a map

$$B_{cyclic}^{\Sigma j_k - 1}(A) \to B_{cyclic}^{j_1 - 1}(A) \otimes \cdots \otimes B_{cyclic}^{j_n - 1}(A)$$

and these maps commute with the face and degeneracy maps, as well as with the composition of cacti. We will need the following lemma.

**Lemma 2.3.** The cactus operad $C$ is generated by $C(1)$ and $C(2)$.

**Proof.** For any cactus configuration $c \in C(n)$, we can obtain $c$ from $S^1$ by a succession of operations that pinches one of the lobes into two, which increases the number of lobes by one. Note that each cactus $c$ determines a parametrization on $S^1$, which gives the parametrization on its lobes. Now $C(1)$ consists of parametrized circles with one distinguished point. Hence, composition with elements of $C(1)$ allows us to reparametrize $S^1$. Thus, to obtain any $c \in C(n)$, we can begin with the
correct parametrization on $S^1$. However, each pinching operation is precisely operad composition with an element of $\mathcal{C}(2)$ in one position (and with the identity in $\mathcal{C}(1)$ in all other positions). Finally, any way of ordering the lobes can be obtained by the action of the symmetric group on $\mathcal{C}(n)$. □

Given such a cactus configuration $(c, X)$, we consider $B^{\Sigma j_k - 1}_{cyclic}(A)$. From the intersection points of $(c, X)$, we get a list of identifications in the set $\{1, \ldots, \Sigma j_k\}$. For each intersection point, we "pinch" $B^{\Sigma j_k - 1}_{cyclic}$ by tensoring together the copies of $A$ at the positions to be identified, using the product $A \otimes A \rightarrow A$. This gives a tensor product of copies of $A$, with one copy of $A$ corresponding to each point in $X$ in the configuration $c$, but with only one copy of $A$ at each intersection point. By Proposition 2.3, it suffices to consider the situation when $c$ is a cactus with two lobes. In this case, the configuration will be of the form

\[
\begin{array}{cccccc}
A & A \\
\otimes & \otimes & \otimes & \otimes \\
A & A & A \\
\vdots & \vdots & \vdots & \vdots \\
A & A \\
\end{array}
\]

To break this up into $A^{\otimes j_1 - 1} \otimes \cdots \otimes A^{\otimes j_n - 1}$, we use the coproduct structure $A \rightarrow A \otimes A$ to "break apart" the circles: i.e. we break apart the copy of $A$ at each intersection point into several copies, one for each lobe of the cactus at that intersection point. Hence, the configuration will look like

\[
\begin{array}{cccccc}
A & A \\
\otimes & \otimes & \otimes & \otimes \\
A & A \otimes A & A \\
\vdots & \vdots & \vdots & \vdots \\
A & A \\
\end{array}
\]

This gives maps of the form

\[
B_{cyclic}^{j_1 + \cdots + j_n - 1}(A) \rightarrow B_{cyclic}^{j_1 - 1}(A) \otimes \cdots \otimes B_{cyclic}^{j_n - 1}(A).
\]

Note that since $A$ is associative as an algebra there is no ambiguity in the ordering when we are pinching together three or more copies of $A$ at an intersection point with three or more lobes. Further, since $A$ is also coassociative and cocommutative as a coalgebra, there is also no ambiguity when breaking apart circles at an intersection point of three or more lobes of the cactus. Also, note that in order to consider each circle of $A$'s along a lobe of the cactus as $A^{\otimes j_k - 1}$, we begin counting at the copy of $A$ at the position corresponding to the distinguished point of that lobe, and go in the direction of the parametrization of the lobe given in the cactus data.

It remains to check that this is compatible with the faces and degeneracies, as well as the composition of operads. By Proposition 2.3, it suffices to consider the case of $\mathcal{C}(2)$, with only two copies of $A$ being pinched together. It is clear that if the face or degeneracy map does not involve the $A$'s being pinched, there is no problem of compatibility with the faces or degeneracies. We need to consider the
case when the face or degeneracy involves one of the copies of $A$ being pinched. For the degeneracies, we need to compare the following maps, starting from $B^n_{cyclic}(A)$. First, we have the pinching followed by a degeneracy:

$$
\begin{array}{cccccc}
A \\
\otimes \otimes \cdots \otimes \cdots \\
\vdots \quad \mu \quad \vdots \\
\otimes \otimes \cdots \otimes \cdots \\
A \\
\cdots \otimes \cdots \cdots \cdots \otimes A \cdots \\
\vdots \quad A \otimes A \quad \eta \quad A \otimes A \\
\downarrow \quad \vdots \\
\cdots \otimes \cdots \cdots \cdots \otimes \cdots \cdots
\end{array}
$$

Recall that here, $\mu$ is the product on $A$, and $\psi$ is the coproduct. These two together form the pinching operation. The last map $\eta$ is the unit $K \to A$, which inserts the new copy of $A$ in the circle on the right.

On the other hand, taking the degeneracy first, and then pinching is

$$
\begin{array}{cccccc}
A \\
\otimes \otimes \cdots \otimes \cdots \\
\vdots \quad \mu \quad \vdots \\
\otimes \otimes \cdots \otimes \cdots \\
A \\
\cdots \otimes \cdots \cdots \cdots \otimes A \cdots \\
\vdots \quad A \cdots \quad \psi \quad A \cdots \\
\downarrow \quad \vdots \\
\cdots \otimes \cdots \cdots \cdots \otimes \cdots \cdots
\end{array}
$$

The first map $\eta$ inserts the right one of the two $A$’s on the top of the second row. The product $\mu$ coming next multiplies together the left copy of the two $A$’s on top with the copy of $A$ at the bottom of the circle, and the last map $\psi$ is the coproduct splitting the resulting single copy of $A$ into two. It is easy to see that these two compositions are equal to each other.

For the face map, we need to compare the following maps, starting from $B^n_{cyclic}(A)$. To apply the face first, then pinch, we have

$$
\begin{array}{cccccc}
A \otimes A \\
\otimes \otimes \cdots \otimes \cdots \\
\vdots \quad \mu \quad \vdots \\
\otimes \otimes \cdots \otimes \cdots \\
A \\
\cdots \otimes \cdots \cdots \cdots \otimes \cdots \cdots \\
\vdots \quad A \cdots \quad \psi \quad A \cdots \\
\downarrow \quad \vdots \\
\cdots \otimes \cdots \cdots \cdots \otimes \cdots \cdots
\end{array}
$$

The first $\mu$ is a face, multiplying together the two copies of $A$ at the top of the circle. The second $\mu$ and the coproduct $\psi$ together make up the pinching operation: the product $\mu$ multiplies together the copies of $A$ at the top and the bottom of the circle, and the $\psi$ splits the resulting single copy of $A$ into two copies.
On the other hand, doing the pinching first, then the face gives

\[ A \otimes A \otimes A \otimes A \cdots \]

\[ \vdots \quad \mu \quad \vdots \quad A \quad \psi \]

\[ A \otimes A \otimes A \otimes A \cdots \]

Here, the first \( \mu \) and the \( \psi \) form the pinching operation, and the last \( \mu \) is the face map. The first \( \mu \) multiplies together the copies of \( A \) at the top left and the bottom of the circle, and the coproduct \( \psi \) splits the resulting single \( A \) (at the center of the configuration of two circles) into two copies. Finally, the last \( \mu \) multiplies together one of the two resulting \( A \)'s at the center of the configuration (the one on the right) with the copy of \( A \) immediately above it in the circle on the right. It is easy to see that by the associativity of \( A \) as an algebra, the two compositions are equal.

Thus,

\[ B_{cyclic}(A) \]

has the dual structure \([2]\) to a cactus set. To turn it into a cactus set, we take its dual

\[ B_{cyclic}(A)^\vee = \text{Hom}_K(B_{cyclic}(A), K). \]

This dual is taken termwise, so the \( n \)-th stage is \((A^\vee)^{\otimes n+1} \simeq A^{\otimes n+1}\), and the face and degeneracy maps are also dualized. As seen above, this is also the cyclic cobar construction \( C_{cyclic}(A) \) on \( A \) as a coalgebra, via the isomorphism \( A \simeq A^\vee \), with a shift by the dimension of \( A \). This is still a cyclic module (since the dual of a cyclic module is a cyclic module), and now the cactus structure maps go in the right direction:

\[ C_{cyclic,j_1-1}(A) \otimes \cdots \otimes C_{cyclic,j_n-1}(A) \to C_{cyclic,j_1+\cdots+j_n-1}(A). \]

Hence, applying the totalization functor, by the result from the previous section, we get an action of the chain complexes cactus operad \( \mathcal{C} \) on the dual of the Hochschild homology \( HH_\ast(A)^\vee \). Voronov’s theorem tells us that the cactus operad is equivalent to \( D_2 \).

To get to the Hochschild cohomology \( HC_\ast(A) \) of \( A \), recall that for a commutative algebra \( A \), \( HC_\ast(A) \) can be calculated as the cohomology of the cyclic cobar construction \( C_{cyclic}(A) \) of \( A \) ([3], Proposition 2.8). (This is the same as the usual cobar construction \( C(A) = C_K(K, A, K) \) over \( K \) of \( A \) as a coalgebra, i.e. \( C^n_{cyclic}(A) = A^{\otimes n+1} \), with coface maps given by the counit on \( A \) as a coalgebra (which is the same as the augmentation \( A \to K \)), and codegeneracy maps given by multiplying together two adjacent copies of \( A \). The only difference is that there is an extra codegeneracy between the first and last copies of \( A \) at each stage, and that there is also a rotational action on the \( n \)-th stage by \( \mathbb{Z}/(n+1) \).) Thus, for a Poincaré algebra \( A \), the dual of the Hochschild homology gives \( HC_\ast(A) \).

**Remark:** The motivating example for \( A \) is the cochain complex \( C_\ast(M) \) for \( M \) a smooth compact orientable manifold, given in Chas and Sullivan [3]. However, we
must note that this case is only an example in the philosophical sense, since $C^*(M)$ is only a Poincaré algebra up to homotopy, which we do not address in this paper. By Theorem 3 of [4],

$$HC^*(C^*(M)) \simeq C_*(LM^{\nu_M})$$

where $\nu_M$ is the stable normal bundle of $M$. This corresponds to the $\mathcal{D}_2$-algebra structure on $H_*(LM)$ given in [3], and the shift is by the dimension of the manifold $M$.

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