Anisotropic Quantum Hall Matrix Model

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Abstract

We consider the anisotropic effect in the quantum Hall systems by applying a confining potential that is not of parabolic type. This can be done by extending Susskind–Polychronakos’s approach to involve the matrices of two coupled harmonic oscillators. Starting from its action, we employ a unitary transformation to diagonalize the model. The operators for building up the anisotropic ground state and creating the collective excitations can be constructed explicitly. Evaluating the area of the quantum Hall droplet, we obtain the corresponding filling factor which is found to depend on the anisotropy parameter and to vary with the magnetic field strength. This can be used to obtain the observed anisotropic filling factors, i.e. $\frac{9}{2}$, $\frac{11}{2}$ and others.

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1 Introduction

Recently it has been shown experimentally that the quantum Hall (QH) phenomena can also appear in anisotropic systems. Indeed, large transport anisotropies at half fillings $\nu = \frac{9}{2}, \frac{11}{2}, \cdots$ in high-quality two-dimensional electron gases (2DEG) in GaAs [1, 2] are seen. The central observation is that the resistivity becomes strongly anisotropic close to half filling of the topmost Landau level. From a theoretical point of view, it is argued to be a signature of a novel Coulomb-induced charge-density-wave ground state whose existence had been predicted by Fogler et al. [3, 4] and by Moessner and Chalker [5].

The aim of the present paper is to study the anisotropic effect in the QH systems by means of the matrix-model language. This can be done by extending Susskind–Polychronakos’s (SP) idea to coupled harmonic oscillators. In the SP approach it has been suggested to replace the classical configuration space of $N$ electrons by a space of two $N \times N$ hermitian matrices and the time component of the vector potential by a hermitian matrix. The confining potential plays an important role, since it defines the Hamiltonian of the theory. Mainly, we are interested to investigate the Laughlin liquid by considering a confining potential that is not of parabolic type.

We develop an appropriate anisotropic model that generalizes the SP approach and investigate the basic features of these QH fluids. Making use of a unitary transformation, we end up with a diagonalized system that allows us to define creation and annihilation matrix operators. Calculating the area of the QH droplet of matrix variables, we obtain a filling factor $\nu_{\text{anis}}$, which depends on an anisotropy parameter $a(B)$, thus generalizing the isotropic (Polychronakos) factor $\nu_p$. We show that $\nu_{\text{anis}}$ can be tuned to describe some special anisotropic filling factors, i.e. $\frac{9}{2}$ and $\frac{11}{2}$. We build up the ground state of our model as well as its excitations in terms of two different representations, those corresponding to variables before and after a suitable transformation.

In section 2, we define our model by considering an action, which involves a confining potential that is not of parabolic type. Rotating the system by an angle $\varphi = \frac{\pi}{2}$, we define its new action and determine the Gauss law constraint as well as the equations of motion. Their solutions will be given and these will be used to find the corresponding solutions before the transformation in section 3. We study our model quantum mechanically by deriving the Hamiltonian and constructing a set of operators that lead to its quantization in section 4. Section 5 is devoted to building up the corresponding ground state as well as its excitations and determining the filling factor. Finally we close by emphasizing that under some conditions our ground state can be visualized similarly to the Laughlin states with $\nu_p = \frac{1}{k+1}$. 

2 Coupled matrix model

We start by recalling that Susskind [6] recently proposed an infinite non-commutative Chern-Simons matrix model for describing the Laughlin liquid [7]. Subsequently, Polychronakos [8] suggested a regularized version of the Susskind model by introducing a bosonic field \( \psi \) that is a boundary term. Basically it is a finite non-commutative Chern-Simons matrix model and allows us to reproduce the basic features of the Laughlin theory, i.e. the quantization at the filling factor

\[
\nu_p = \frac{1}{k + 1}. \tag{1}
\]

The level \( k \) of the Chern-Simons term is identified with \( B \theta \) by correspondence between the gauge fields and the matrix variables at a large number \( N \) of particles. \( B \) and \( \theta \) are, respectively, magnetic field and non-commutativity parameter.

In what follows, we investigate the anisotropic effect in the QH systems by building up the ground state and determining its filling factor. This can be done by generalizing the SP action to the new action

\[
S = \frac{B}{2} \int dt \, \text{Tr} \sum_{a,b=1}^{2} \left\{ \epsilon^{ab} \left( \dot{X}_a + i [A_0, X_a] \right) X_b + 2 \theta A_0 - \omega X_a^2 \right\} + \mu X_1 X_2 + \psi^\dagger \left( i \dot{\psi} - A_0 \psi \right) \tag{2}
\]

where \( A_0, X_1 \) and \( X_2 \) are classical hermitian-matrix-valued variables, \( \epsilon^{ab} \) is the fully antisymmetric tensor; \( \text{Tr} \) and \( [\cdot, \cdot] \) denote operations in matrix space. \( \mu \) is playing the role of a coupling parameter between two sectors parameterized by \( X_1 \) and \( X_2 \). We note that (2) is an extension of two coupled harmonic oscillators into matrix-model language. It is clear that by switching off \( \mu \), we recover the SP model. This suggests that we are going to study a QH system of particles confined in an anisotropic potential

\[
V (X_1, X_2) = \omega (X_1^2 + X_2^2) + \mu X_1 X_2 \tag{3}
\]

instead of a parabolic confinement with single frequency \( \omega \).

As far as the action \( S \) is concerned, the full symmetry is the gauge group \( U(N) \). The matrix-model variables transform under this invariance as

\[
X_a \to UX_a U^{-1}, \quad \psi \to U \psi. \tag{4}
\]

The gauge field \( A_0 \) ensures the gauge invariance of the action, its equation of motion being the Gauss law constraint [8]

\[
G \equiv \left[ X_1, X_2 \right] - i \theta \left( 1 - \frac{1}{B \theta} \psi \psi^\dagger \right) = 0 \tag{5}
\]
representing the non-commutative aspects of the theory. The equations of motion for the variables $X_1$ and $X_2$ are different from the isotropic model. We find

$$\dot{X}_1 - \omega X_2 + \frac{\mu}{B} X_1 = 0, \quad \dot{X}_2 + \omega X_1 - \frac{\mu}{B} X_2 = 0$$

(6)

where the main difference with respect to the parabolic case is the presence of the third term in both equations. These will be solved in the next section after introducing a unitary transformation. Essentially, their solutions will determine the nature of the classical motion in the system.

3 Rotation into the principal-axis system

As (2) involves an interacting term, for a straightforward investigation of the basic features of the system we use an appropriate transformation. This can be done by defining new variables

$$Y_a = N_{ab} X_b$$

(7)

where the matrix

$$(N_{ab}) = \begin{pmatrix} \cos \frac{\varphi}{2} & -\sin \frac{\varphi}{2} \\ \sin \frac{\varphi}{2} & \cos \frac{\varphi}{2} \end{pmatrix}$$

(8)

is the unitary rotation of the elliptic system (3) into its principal-axis set with the mixing angle \(\varphi\). Clearly, \(\varphi\) should satisfy the constraint

$$\varphi = \pi(n + 1/2)$$

(9)

with \(n\) an integer. Without loss of generality we fix \(n = 0\). In this case, \(S\) transforms as

$$S' = \frac{B}{2} \int dt \ \text{Tr} \ \sum_{a,b=1}^2 \left\{ \epsilon^{ab} \left( \dot{Y}_a + i [A_0, Y_a] \right) Y_b + 2\theta A_0 - \omega_a Y_a^2 \right\} + \psi \left( i \dot{\psi} - A_0 \psi \right)$$

(10)

where the frequencies \(\omega_1\) and \(\omega_2\) are

$$\omega_1 = \omega + \frac{\mu}{B}, \quad \omega_2 = \omega - \frac{\mu}{B}.$$

(11)

Obviously \(S'\) is also invariant under the gauge group \(U(N)\) and the matrix-model variables transform as (4). The equations of motion of different matrix-model variables can be derived in the usual way. It is easily seen that for \(A_0\), we get a Gauss law constraint analogous to (5)

$$G' \equiv [Y_1, Y_2] - i \theta \left( 1 - \frac{1}{B\theta} \psi \psi^\dagger \right) = 0$$

(12)
where its trace part is given by
\[ \psi^\dagger \psi = NB\theta. \] (13)

For the variables \( Y \) we obtain
\[ \dot{Y}_1 + \frac{\omega_2}{B} Y_2 = 0, \quad \dot{Y}_2 - \frac{\omega_1}{B} Y_1 = 0, \] (14)
similar to those derived by Polychronakos [8], but involving different frequencies \( \omega_1 \) and \( \omega_2 \). Their solutions can be written as
\[ Y_1 = \frac{1}{2a} \left( C e^{-i\Omega t} + C^\dagger e^{i\Omega t} \right), \quad Y_2 = \frac{i}{2a} \left( C e^{-i\Omega t} - C^\dagger e^{i\Omega t} \right) \] (15)
where the frequency is \( \Omega = \left( \frac{\omega_1 \omega_2}{\omega_1 + \omega_2} \right)^{\frac{1}{2}} \) and \( a(B) > 0 \) is fixed to be
\[ a(B) = \left( \frac{B\omega + \mu}{B\omega - \mu} \right)^{\frac{1}{2}}. \] (16)
We call it \textit{anisotropy} parameter and note that \( B\omega > \mu \) must be chosen. In Figure 1, we show how \( a(B) \) depends on \( B \) for some choices of \( \omega \) and \( \mu \). The \( N \times N \) matrices \( C, C^\dagger \) appearing in the above equations are defined by
\[ C = a Y_1 + \frac{i}{a} Y_2, \quad C^\dagger = a Y_1 - \frac{i}{a} Y_2. \] (17)
The equation of motion for the bosonic field, \( \dot{\psi} = 0 \), can be solved to get [8]
\[ \psi = \sqrt{NB\theta} |v\rangle \] (18)
where \( |v\rangle \) is a constant vector of unit length. This can be used to show that \( C \) and \( C^\dagger \) satisfy the constraint
\[ [C, C^\dagger] = 2\theta \left( 1 - N |v\rangle \langle v| \right). \] (19)
The matrices \( C, C^\dagger \) determine the classical solution for the action [10] which is given as [8]
\[ (Y_1)_{mn} = y_n \delta_{mn}, \quad (Y_2)_{mn} = z_n \delta_{mn} - \frac{i\theta}{y_m - y_n} (1 - \delta_{mn}), \] (20)
solving the Gauss law constraint [11].

Now it is easily seen that the equations of motion [6] can be solved once we express, at \( \varphi = \frac{\pi}{2}, \) \( X_a \) in terms of the \( Y_a \)
\[ X_1 = \frac{1}{\sqrt{2}} (Y_1 + Y_2), \quad X_2 = \frac{1}{\sqrt{2}} (Y_2 - Y_1). \] (21)
Thus we end up with the classical solutions of two coupled harmonic oscillators.
Figure 1: Variation of $a(B)$ in terms of magnetic field $B$. Dashed line: $\omega = \frac{\mu}{B}$, solid line: $\omega = \mu$ and strong line: $\omega = \frac{\mu}{5}$.

4 Quantizing the theory

Upon quantization the matrix elements of $X_a, Y_a$ and the components of the field $\psi$ become operators. As usual, the quantum Hamiltonian $H$ can be derived from the relation

$$H = X \frac{\partial L}{\partial \dot{X}} - L \quad (22)$$

where $\frac{\partial L}{\partial \dot{X}}$ defines the conjugate momentum and $L$ is the Lagrangian corresponding to action $(10)$. We find that $H$ can be written as a sum of a free and an interacting part

$$H = \text{Tr} \left[ \frac{B\omega}{2} (X_1^2 + X_2^2) - \mu X_1 X_2 \right] \quad (23)$$

which is nothing but the non-parabolic confining potential $V$. This means that the kinetic energy is negligible compared to $V$. With $(23)$, we actually have two possibilities to get the Hamiltonian in terms of the matrices $Y$, either by transforming $H$ via $(7)$ to obtain

$$H' = \omega_2 \text{Tr} \left( a^4 Y_1^2 + Y_2^2 \right) \quad (24)$$

or by starting straightforwardly from $(22)$, using $S'$ $(10)$ to end up with $(22)$. Let us remark that $(24)$ clearly shows the anisotropy in $H'$, i.e. the motion in the two coordinate directions $Y_{1,2}$ has different frequencies. Fixing $\omega_1 = \omega_2$, we recover the Polychronakos Hamiltonian.

The form of $H'$ is similar to the harmonic oscillator and can easily be diagonalized. We define two $N \times N$ matrices of creation and annihilation operators

$$C_{nm} = \sqrt{\frac{B}{2}} \left[ a(Y_1)_{nm} + i \frac{i}{a}(Y_2)_{nm} \right], \quad \text{and} \quad C^\dagger_{nm} = \sqrt{\frac{B}{2}} \left[ a(Y_1)_{nm} - i \frac{i}{a}(Y_2)_{nm} \right]. \quad (25)$$
Their commutator can be evaluated by calculating from (10) that the operators $Y$ satisfy the commutation relation
\[
[(Y_1)_{nm}, (Y_2)_{n'm'}] = \frac{i}{B} \delta_{nm} \delta_{n'm'}.
\] (26)

This implies
\[
[C_{nm}, C_{n'm'}^\dagger] = \delta_{nm} \delta_{n'm'}
\] (27)

where $n, m = 1, \ldots, N$. Other commutators vanish. After some algebra, we find that $H'$ can be expressed in terms of $C$ and $C^\dagger$ as
\[
H' = \Omega' \left( \hat{N} + \frac{N^2}{2} \right)
\] (28)

where $\Omega' = 4\Omega/B$ and the total number operator
\[
\hat{N} = \sum_{n,m=1}^{N} C_{nm}^\dagger C_{nm}
\] (29)

is counting the $N$ particles forming the system under consideration.

5 Determining the ground states

Via the unitary transformation (8), we can also express $H$ in terms of the operators $C$, $C^\dagger$ and build up the anisotropic ground state of our model as well as its excitations and determine the corresponding filling factor. First one has to construct a physical state $|\Phi\rangle$ obeying the Gauss law constraint (5). To proceed, we determine $|\Phi'\rangle$ corresponding to $H'$ and go on to get $|\Phi\rangle$.

The ground state satisfying $G' |\Phi\rangle = 0$ can be built as
\[
|\Phi\rangle = \left[ \epsilon^{j_1 \cdots j_N} \psi_{j_1}^\dagger (\psi^\dagger C)^{j_2} \cdots (\psi^\dagger C^{N-1})_{j_N} \right]^k |0\rangle
\] (30)

where $\epsilon^{j_1 \cdots j_N}$ are elements of the fully antisymmetric tensor. Its energy spectrum is
\[
E_k = \Omega' \left[ \frac{k}{2} N(N - 1) + \frac{N^2}{2} \right].
\] (31)

Taking $\mu = 0$, we recover the Hellerman–Van Raamsdonk ground state [9] constructed for the isotropic SP model.

Having the ground state $|\Phi'\rangle$, it is natural to ask about its anisotropic fraction $\nu_{\text{anis}}$. To answer this question, we first calculate the area $A'$ of the QH droplet of the matrices $Y$. This can be defined as
\[
A' = \frac{\pi}{N} \text{Tr} \left( Y_1^2 + Y_2^2 \right).
\] (32)
Mapping $A'$ in terms of the creation and annihilation matrix operators and evaluating the trace with respect to \(30\), we find in the large-$N$ limit

$$\nu_{\text{anis}} = \frac{\nu_p}{2} \left( a + \frac{1}{a} \right).$$  \hspace{1cm} (33)

This relation is shown for some specific values of $\omega$ and $\mu$ in Figure 2. In the limit $a = 1$, $\nu_{\text{anis}}$ coincides with $\nu_p$.

For a state that in the isotropic system would correspond to the filling factor

$$\nu_p(B) = 2\pi \rho_0 / B$$  \hspace{1cm} (34)

where $\rho_0$ is the density, an anisotropic confining potential [12] will yield an apparent filling factor $\nu_{\text{anis}}$ in dependence on $a(B)$, i.e. on $B$ according to (33). More precisely, the filling factor $\frac{9}{2}$ can be obtained from (33) by considering the following configuration

$$\nu = 1, \quad a = \frac{9}{2} \pm \frac{\sqrt{77}}{2},$$  \hspace{1cm} (35)

while for the factor $\frac{11}{2}$, one can choose

$$\nu = 1, \quad a = \frac{11}{2} \pm \frac{\sqrt{117}}{2}.$$  \hspace{1cm} (36)

More generally, the anisotropic Hall conductivity is [411]

$$\sigma_H = \frac{e^2}{h} \left( N + \frac{1}{2} \right) = \frac{e^2}{h} \nu_{\text{anis}}$$ \hspace{1cm} (37)

where $N$ is the total number of particles. Comparing (33) and (37), we find

$$a = \frac{1}{\nu_p} \left( N + \frac{1}{2} \right) \pm \frac{1}{\nu_p} \sqrt{\left( N + \frac{1}{2} \right)^2 - \nu_p^2}.$$ \hspace{1cm} (38)

The excited states for the anisotropic model can be constructed as

$$|\Phi'_\text{ex}\rangle = \prod_{n=1}^{N-1} (\text{Tr} C^\dagger_n)^{c_n} [e^{i j_1 \cdots j_N} \psi_{j_1}^\dagger \psi_{j_2}^\dagger \cdots (\psi_{j_N}^\dagger C^\dagger_{j_N})^k] |0\rangle$$ \hspace{1cm} (39)

and their energies are

$$E_k(c_n) = \Omega' \left[ \frac{k}{2} N(N-1) + \frac{N^2}{2} + \sum_{n=1}^{N} n c_n \right]$$ \hspace{1cm} (40)

where the $c_n$ are non-negative integers.
The anisotropic ground state $|\Phi\rangle$ corresponding to the matrices $X$ can be obtained by expressing the matrices $C, C^\dagger$ in terms of those corresponding to $X$. They are defined by

$$A_{nm} = \sqrt{\frac{B}{2}} (X_1 + iX_2)_{nm}, \quad A^\dagger_{nm} = \sqrt{\frac{B}{2}} (X_1 - iX_2)_{nm}.$$  \hspace{1cm} (41)

By using the unitary transformation (7), (25) can be written as

$$C = \bar{\eta} (A - iA^\dagger), \quad C^\dagger = \eta (A^\dagger + iA)$$  \hspace{1cm} (42)

where $\eta$ is given by

$$\eta = \frac{1}{2\sqrt{2}} \left\{ \left( a - \frac{1}{a} \right) - i \left( a + \frac{1}{a} \right) \right\}.$$  \hspace{1cm} (43)

Inserting (42) into (30) and normalizing, we obtain

$$|\Phi\rangle = \left\{ \epsilon^{j_1...j_N} \psi_{j_1}^{\dagger} \left[ \eta \psi_{j_1}^{\dagger} (A^\dagger + iA) \right]_{j_2} \cdots \left[ \eta^{N-1} \psi_{j_1}^{\dagger} (A^\dagger + iA)^{N-1} \right]_{j_N} \right\}^k |0\rangle.$$  \hspace{1cm} (44)

Its filling factor can be obtained from the corresponding area

$$A = \frac{\pi}{N} \text{Tr}(X_1^2 + X_2^2).$$  \hspace{1cm} (45)

Using the mapping (7), one can see that $A$ coincides with $A'$ (32), as it should so that the ground states (44) and (30) have the same fraction $\nu_{\text{anis}}$.

The excited states of $H$ can be derived from (39) in the same way as (44), giving

$$|\Phi\rangle_{\text{ex}} = \prod_{n=1}^{N-1} \left[ \text{Tr} \eta^n (A^\dagger + iA)^n \right] c_n \left\{ \epsilon^{j_1...j_N} \psi_{j_1}^{\dagger} \left[ \eta \psi_{j_1}^{\dagger} (A^\dagger + iA) \right]_{j_2} \cdots \left[ \eta^{N-1} \psi_{j_1}^{\dagger} (A^\dagger + iA)^{N-1} \right]_{j_N} \right\}^k |0\rangle.$$  \hspace{1cm} (46)

We close this section by noting that all the states obtained here coincide with those for the isotropic SP model if we set $a(B) = 1$.

6 Concluding remarks

We emphasize that the resulting filling factor for the anisotropic system can be mapped onto the isotropic one (11) after making use of a second transformation (12). This can be done by defining the new matrices

$$W_1 = aY_1, \quad W_2 = \frac{1}{a}Y_2.$$  \hspace{1cm} (47)
In terms of these, the system becomes identical to the original rotationally symmetric matrix model proposed by Polychronakos [8]. This has a circular droplet ground state with the standard filling factor. The transformation

\[ X_1, X_2 \rightarrow Y_1, Y_2 \rightarrow W_1, W_2 \]  

is area-preserving, the first is a rotation by \( \frac{\pi}{2} \) and the second is a dilation by "\( a \)" in one direction and "\( \frac{1}{a} \)" in the other. In this case, we end up with the filling factor (1).

In summary, we investigated the Laughlin liquids by considering a confining potential that is of anisotropic type. This can be done by applying matrix-model theory to two coupled harmonic oscillators and studying the Laughlin states of the FQHE. Our model is a generalization of that proposed by Susskind and Polychronakos and reproduces its basic features in the limit \( \mu = 0 \). Our most important result shows that the filling factor is anisotropy dependent. This suggests that our model may be a good starting point for the interpretation of recent experiments in anisotropic 2DEG.

A direct connection of our model to the Calogero-Sutherland (CS) models [13, 14] as in Refs. [8, 10] would be highly beneficial. An interpretation of the anisotropy of \( X_1 \) and \( X_2 \) in such a mapping as two coupled CS models or as CS models with two types of particles might be possible. While such models exist [15, 16], they are solvable only for special sets of parameters and the inclusion of a continuously varying \( \mu \) appears quite challenging.
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