On the degrees of freedom of lattice electrodynamics

Bo He* and F. L. Teixeira

ElectroScience Laboratory and Department of Electrical Engineering,
The Ohio State University, 1320 Kinnear Road, Columbus, OH 43212, USA

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Abstract

Using Euler’s formula for a network of polygons for 2D case (or polyhedra for 3D case), we show that the number of dynamic degrees of freedom of the electric field equals the number of dynamic degrees of freedom of the magnetic field for electrodynamics formulated on a lattice. Instrumental to this identity is the use (at least implicitly) of a dual lattice and of a (spatial) geometric discretization scheme based on discrete differential forms. As a by-product, this analysis also unveils a physical interpretation for Euler’s formula and a geometric interpretation for the Hodge decomposition.

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*E-mail address: he.87@osu.edu
1 Introduction

There has been continual interest in formulating physical theories on a discrete lattice [1][2][3][4]. This "latticization" process not only eliminates the infinities but also provides a computational route to study the non-perturbative region, which it is often difficult to handle by purely analytical tools. In this letter, we shall discuss aspects of a lattice version of classical electrodynamics based on a geometric discretization [5][6].

There are three main approaches to discretize Maxwell equations: finite differences [7][8], finite elements [9] and finite volumes [7]. However, naive implementations of these finite methods to discretize Maxwell equations on irregular lattices are known to cause problems, such as spurious modes [10] and late-time unconditional instabilities [11], that destroy the solutions. This is often a consequence of the failure to capture some essential physics of the continuum theory.

By applying some tools of algebraic topology and a discrete analog of differential forms, classical electrodynamics can be constructed from first principles on a lattice [5]. The purpose of this paper is to show that via such geometric discretization, the equality (necessary, but often not trivially observed in common discretization schemes) between the number of dynamic degrees of freedom (DoFs) of the electric field and the number of dynamic DoFs of the magnetic field is a simple consequence of Euler’s formula for a network of polygons for 2+1 case or volume polyhedra in the 3+1 case. As a by-product, this analysis also unveils a physical interpretation for Euler’s formula and a geometric interpretation for the Hodge decomposition.

2 Lattice electromagnetic theory

For simplicity, we consider first a TE field in 2+1 dimensions. The extension to 3+1 dimensions is considered in Section 3.2.
Maxwell equations in the Fourier domain are written in terms of differential forms as [5]

\[
\begin{align*}
    dE &= i\omega B \\
    dB &= 0 \\
    dH &= -i\omega D + J \\
    dD &= Q
\end{align*}
\]  

(1)

In the 2+1 case, \(H\) is a 0-form, \(D\), \(J\) and \(E\) are 1-forms, and \(B\) and \(Q\) are 2-forms. The operator \(d\) is the exterior derivative. From the identity \(d^2 = 0\), the electric current density \(J\) and the charge density \(Q\) satisfy the continuity equation

\[
dJ = i\omega Q
\]

(2)

Constitutive equations, which include all metric information in this framework, are written in terms of Hodge star operators (that fix an isomorphism between \(p\) forms and \(2 - p\) forms in the 2+1 case) [5]

\[
D = \star_r E, \quad B = \star_h H
\]

(3)

In numerical simulations, because of limited memory, the computational domain is taken as a closed domain \(\Omega\) with boundary \(\partial\Omega\). In the lattice, the boundary \(\partial\Omega\) is approximated by a set of linked edges \(\partial\hat{\Omega}\) (see Fig.1).

The latticization corresponds to tiling \(\Omega\) with a finite number \(N_F\) of polygons \(\Xi_m, m = 1, \ldots, N_F\), of arbitrary shape (see Fig.2)

\[
\Omega \simeq \hat{\Omega} = \bigoplus_{m=1}^{N_F} \Xi_m
\]

(4)

We require the tiling to be conformal i.e., two polygons are either connected by one single edge or are not connected at all (see Fig.3). These polygons should also be oriented, forming a cell-complex (see Fig.4) [5][12]. We denote
such oriented tiling (Fig.4) the *primal* lattice. From the primal lattice, one can construct a *dual* lattice by connecting interior points of each adjacent polygon. The dual lattice inherits an orientation from the primal lattice.

Now we consider casting Maxwell equations on a lattice using the natural latticization provided by casting differential forms of various degrees $p$ in Eq. (1) as dual elements (cochains) to $p$ dimensional geometric constituents of the lattice, i.e., *cells*: nodes, edges and faces [5]. In the primal lattice, we associate the electrostatic potential $\phi$ (0-form) with primal nodes (0-cells), the electric field intensity $E$ (1-form) with primal edges (1-cells) and the magnetic flux density $B$ (2-form) with primal cells (2-cells). In the dual lattice, we associate the magnetic field intensity $H$ (0-form) with dual nodes (0-cells), the electric flux density $D$ (1-form), the electric current density $J$ (1-form) with dual edges (1-cells), and the charge density $Q$ (2-form) with dual cells (2-cells). This is illustrated in Fig.5.

The exterior derivative $d$ can be discretized via its adjoint operator, the boundary operator, $\partial$, by applying the generalized Stoke’s theorem on each $p$-cell of the cell-complex

$$\langle \gamma^{p+1}, d\alpha^p \rangle = \langle \partial \gamma^{p+1}, \alpha^p \rangle ,$$

where $\gamma^{p+1}$ is a $p+1$ dimensional cell and $\alpha^p \in F^p(\Omega)$, with $F^p(\Omega)$ being the space of differential forms (cochains in the discrete setting) of degree $p$ on the domain $\Omega$. We denote an ordered sequence of the above pairing of cochains with each of the cells by block letters $E, B, H, D, J, Q$ in what follows. These are the DoFs of the lattice theory. In terms of these DoFs,

1Different geometrical constructions can be used to define the dual lattice. We will not dwell here into such discussion since our conclusions depend only on topological properties (connectivity) of the lattice.
the lattice analog of Maxwell’s equations is written as [5] [13]

\[ CE = i\omega B \]  \hspace{2cm} (6)
\[ SB = 0 \]  \hspace{2cm} (7)
\[ \tilde{C}H = -i\omega \tilde{D} + \mathcal{J} \]  \hspace{2cm} (8)
\[ \tilde{S}\tilde{D} = Q \]  \hspace{2cm} (9)

In the above, \( C, \tilde{C}, S, \tilde{S} \) are the \textit{incidence matrices} [5], obtained by applying the boundary operator \( \partial \) on each \( p \)-cell (discrete version of the exterior derivative \( d \)). Because the exterior derivative \( d \) is a purely topological operator (metric-free), the incidence matrices represent pure combinatorial relations, whose entries assume only \( \{ -1, 0, 1 \} \) values.

The lattice version of the Hodge isomorphism can be, in general, written as follows

\[ D = [\star_e] E, \quad B = [\star_\mu] H \]  \hspace{2cm} (10)

where both \([\star_e]\) and \([\star_\mu]\) are square invertible matrices. We will not discuss here how to construct the Hodge matrices \([\star_e]\) and \([\star_\mu]\). That the metric dependent Hodge matrices are the square invertible matrices is the only assumption needed here.

In many problems of interest, one is usually interested only in the dynamic behavior of the field, which is determined by the dynamic \textit{DoFs} of lattice theory. Here we show that the number of dynamic \textit{DoFs} of the electric field, \( \text{DoF}^d (E) \), equals to the number of dynamic \textit{DoFs} of the magnetic flux, \( \text{DoF}^d (B) \), in the discretization above. Furthermore, from the isomorphisms between \( E \) and \( D \), and between \( H \) and \( B \) (from the Hodge maps), this also implies

\[ \text{DoF}^d (E) = \text{DoF}^d (B) = \text{DoF}^d (D) = \text{DoF}^d (H) \]  \hspace{2cm} (11)

where the superscript \( d \) stands for dynamic. Moreover, for \( 2+1 \) \( TE \) field, this number equals to the total number of polygons used for tiling the domain \( \Omega \).
minus one \((N_F - 1)\).

3 Hodge decomposition

The Hodge decomposition can be written in general as

\[
F^p (\Omega) = dF^{p-1} (\Omega) \oplus \delta F^{p+1} (\Omega) \oplus \chi^p (\Omega) \quad (12)
\]

where \(\chi^p (\Omega)\) is the finite dimensional space of harmonic forms, and \(\delta\) is the codifferential operator, Hilbert adjoint of \(d\) \[14\]. Applying (12) to the electric field intensity \(E\), we obtain

\[
E = d\phi + \delta A + \chi \quad (13)
\]

where \(\phi\) is a 0-form and \(A\) is a 2-form. In Eq. (13) \(d\phi\) represents the static field and \(\delta A\) represents the dynamic field, and \(\chi\) represents the harmonic field component.

3.1 2+1 theory in a contractible domain

If domain \(\Omega\) is contractible, \(\chi\) is identically zero and the Hodge decomposition can be simplified to

\[
E = d\phi + \delta A \quad (14)
\]

In the present lattice model, the number of DoFs for the zero eigenspace \((\omega = 0)\) equals the number of internal nodes of the primal lattice \[10\] \[15\] \[16\]. This is because the DoFs of the potential \(\phi\), which is a 0-form, is associated to nodes.

Now we show the identity (11). Recall the Euler’s formula for a general
network of polygons without holes (Fig. 2)

\[ N_V - N_E = 1 - N_F \]  \hspace{1cm} (15)

Here, \( N_V \) is the number of vertices (nodes), \( N_E \) the number of edges, and \( N_F \) the number of faces (cells). For any \( \partial \Omega \), it is easy to verify that

\[ N_V^b - N_E^b = 0 \]  \hspace{1cm} (16)

where \( N_V^b \) is the number of vertices on the boundary and \( N_E^b \) the number of edges on the boundary (the superscript \( b \) stands for boundary). Note that cochains on \( \partial \tilde{\Omega} \) are not associated to DoFs, since they are fixed from the boundary conditions. Using the Hodge decomposition (13), the number of dynamic (\( \omega \neq 0 \)) DoFs of the electric field, corresponding to \( \delta A \), is given by

\[
\text{DoF}^d (E) = \ N_E^{in} - N_V^{in} \\
= (N_E - N_E^b) - (N_V - N_V^b) \\
= N_E - N_V
\]  \hspace{1cm} (17)

where the superscript \( \text{in} \) stands for internal. Since \( E \) is given along the boundary, then, for \( \omega \neq 0 \), \( \int_{\tilde{\Omega}} B \) is fixed by

\[ i\omega \int_{\tilde{\Omega}} B = \int_{\partial \tilde{\Omega}} E \]  \hspace{1cm} (18)

This corresponds to one constraint on \( B \). Subtracting one degree of freedom from the constraint (18), the number dynamic DoFs of the magnetic flux \( B \) is

\[
\text{DoF}^d (B) = N_F - 1
\]  \hspace{1cm} (19)
From Euler’s formula (15), we then have the identity

$$DoF^d(E) = DoF^d(B)$$  \hspace{1cm} (20)

Furthermore, thanks to the Hodge isomorphism, the identity (11) follows directly.

### 3.2 3+1 theory in a contractible domain

The source free Maxwell equations in 3+1 dimensions read as

$$dE = i\omega B,$$  \hspace{1cm} (21)

$$dB = 0,$$  \hspace{1cm} (22)

$$dH = -i\omega D,$$  \hspace{1cm} (23)

$$dD = 0$$  \hspace{1cm} (24)

where now $H$ and $E$ are 1-forms, and $D$ and $B$ are 2-forms. The spatial domain $\Omega$ is again (approximately) tiled by a set of polyhedra $\widehat{\Omega}$ and the boundary $\partial\Omega$ is by a polyhedron $\partial\widehat{\Omega}$. Using Euler’s formula for $\widehat{\Omega}$, we have

$$N_V - N_E = 1 - N_F + N_P$$ \hspace{1cm} (25)

and Euler’s formula for the boundary polyhedron $\partial\widehat{\Omega}$

$$N_V^b - N_E^b = 2 - N_F^b$$ \hspace{1cm} (26)

where $N_P$ is now the number of polyhedra. Combining Eq. (25) and (26), we obtain

$$(N_E - N_E^b) - (N_V - N_V^b) = (N_F - N_F^b) - (N_P - 1)$$ \hspace{1cm} (27)
Using the Hodge decomposition (14), the number of dynamic DoFs of the electric field (corresponding to $\delta A$) is

$$\text{DoF}^d(E) = N_E^{in} - N_V^{in} = (N_E - N_E^b) - (N_V - N_V^b)$$  \hspace{1cm} (28)$$

Each polyhedron produces one constraint for the magnetic flux $B$ from Eq. (22). Furthermore, this set of constraints span the condition at the boundary $\partial \hat{\Omega}$. The total number of the constrains for $B$ is therefore $(N_P - 1)$. Consequently, the number of DoFs for the magnetic flux $B$ is

$$\text{DoF}^d(B) = N_F^{in} - (N_P - 1) = (N_F - N_F^b) - (N_P - 1)$$  \hspace{1cm} (29)$$

Identity (11) then follows from Eq. (27), (28) and (29).

3.3 2+1 theory in a non-contractible domain

Now consider a non-contractible two-dimensional domain $\Omega$ with a finite number $g$ of holes (genus). This is illustrated in Fig. 6 for $g = 1$. Along the boundary of each hole, the electric field $E$ is constrained by

$$\int E = M$$  \hspace{1cm} (30)$$

where the magnetic current density $^2 M$ (passing through the hole) is a known quantity. The equation (30) accounts for the possible existence of

\[^2\text{In physical terms, the magnetic current density } M \text{ is identified with the "displacement magnetic current density" } i\omega B, \text{ which is given for some cases. In some other cases, } M \text{ comes also from the equivalent magnetic current density by the surface equivalence theorem [17]. It should be emphasized that the equivalent magnetic current results from an impressed electric field } E, \text{ not from the movement of any "magnetic charge".}\]
the harmonic forms $\chi$ on $\Omega$. In particular, the number of holes $g$ is equal to the dimension of the space of harmonic forms $\chi$ and gives the number of independent constraint equations (30). Subtracting $g$ from Eq. (17), the number of dynamic DoFs of the electric field in this case becomes

$$
\text{DoF}^d (E) = N_{E}^{in} - N_{V}^{in} - g
= N_{E} - N_{V} - g
$$

(31)

whereas the number of DoFs of the magnetic flux $\text{DoF}^d (B)$ remains $N_F - 1$.

Since Euler’s formula for a network of polygons with $g$ holes is

$$
N_{V} - N_{E} = (1 - g) - N_{F}
$$

(32)

we have that from Eq. (19), (31) and (32), the identity (11) is again satisfied.

### 3.4 Euler’s formula and Hodge decomposition

From the above considerations, we can trace the following correspondence in the 2+1 case

$$
N_{E} = N_{V} + (N_{F} - 1) + g
$$

(33)

The number of edges $N_{E}$ corresponds to the dimension of the space of (discrete) electric field intensity $E$ (1-forms), which is the sum of the number of nodes $N_{V}$ (dimension of the space of discrete 0-forms $\phi$), the number of faces $(N_{F} - 1)$ (dimension of the space of discrete 2-form $A$) and the number of holes $g$ (dimension of the space of harmonic form $\chi$). These correspondences attach a physical meaning to Euler’s formula and a geometric interpretation to the Hodge decomposition. We note that the identity (33) can be viewed
as

\[
N_{E}^{\text{in}} = N_{V}^{\text{in}} + (N_{F} - 1) + g \quad (34)
\]

\[
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow
\]

\[
E = d\phi + \delta A + \chi
\]

since only the internal edges and nodes describe the degrees of freedom. We can simply drop the superscript \textit{in} because the identity (16). For the 3+1 case, a similar correspondence could also be drawn.

4 Concluding remarks

Based on a geometric discretization\(^3\), we have shown that Euler’s formula matches the algebraic properties of the discrete Helmholtz decomposition in an exact way. Furthermore, we have showed that the number of dynamic \textit{DoFs} for the electric field equals the number of dynamic \textit{DoFs} for the magnetic field on such lattices\(^4\).

Regarding the time discretization, we also remark that simplectic integrators, originally developed for Hamiltonian systems [22], can provide a time discretization that respects the simplectic structure [23]. However, it is not trivial to formulate the Maxwell’s equations on a lattice as the canonical equations of the Hamiltonian, because electrodynamics can be thought of as a \textit{constrained} dynamic system. A Hamiltonian requires that the canonical pair \((E, H)\) should have the same number of degrees of freedom. Identity (11) suggests that it is indeed possible to formulate electrodynamics on a lattice as the canonical equations of the Hamiltonian.

\(^3\)One key feature of this scheme is the use of a dual lattice and of a geometric discretization scheme based on differential forms, also proposed in different contexts in [18][19][20].

\(^4\)For the case of high order 1-forms [21], the \textit{DoFs} of 1-forms could associate with the faces and volumes. However, the dimension of the range space of 1-forms (e.g. \(E\)) equals the dimension of 2-forms (e.g. \(B\)) if such 2-forms have zero range space due to \(dB = 0\), so the identity (11) still holds.
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Figure captions

Fig. 1. The curved boundary $\partial \Omega$ is approximated by a set of linked edges $\partial \hat{\Omega}$.

Fig. 2. Tiling the computation region with polygons.

Fig. 3 (a) conformal tiling (cell complex); (b) non-conformal tiling.

Fig. 4. Oriented polygons.

Fig. 5. Solid lines represent the primal lattice. Primal nodes (vertices) are paired with $\phi$ (e.g., node 1), primal edges with $E$ (e.g., edge 15) and primal cells with $B$ (e.g., cell 12345). Dashed lines represent the dual lattice. Dual nodes are paired with $H$ (e.g., node 4'), dual edges with $(D, J)$ (e.g., edge 3'4') and dual cells with $Q$ (e.g., cell 1'2'3'4'5').

Fig. 6. 2+1 theory in a non-contractible domain (network of polygons with a hole, illustrated by a triangle 123).
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