HYERS–ULAM STABILITY OF HIGHER-ORDER CAUCHY-EULER DYNAMIC EQUATIONS ON TIME SCALES

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ABSTRACT. We establish the stability of higher-order linear nonhomogeneous Cauchy-Euler dynamic equations on time scales in the sense of Hyers and Ulam. That is, if an approximate solution of a higher-order Cauchy-Euler equation exists, then there exists an exact solution to that dynamic equation that is close to the approximate one. Some examples illustrate the applicability of the main results.

1. INTRODUCTION

In 1940, Ulam [26] posed the following problem concerning the stability of functional equations: give conditions in order for a linear mapping near an approximately linear mapping to exist. The problem for the case of approximately additive mappings was solved by Hyers [11] who proved that the Cauchy equation is stable in Banach spaces, and the result of Hyers was generalized by Rassias [23]. Alsina and Ger [1] were the first authors who investigated the Hyers-Ulam stability of a differential equation.

Since then there has been a significant amount of interest in Hyers-Ulam stability, especially in relation to ordinary differential equations, for example see [8, 9, 12, 13, 14, 15, 19, 20, 22, 25]. Also of interest are many of the articles in a special issue guest edited by Rassias [24], dealing with Ulam, Hyers-Ulam, and Hyers-Ulam-Rassias stability in various contexts. Also see Li and Shen [17, 18], Wang, Zhou, and Sun [27], and Popa et al [21, 6]. András and Mészáros [3] recently used an operator approach to show the stability of linear dynamic equations on time scales with constant coefficients, as well as for certain integral equations. Anderson et al [2, Corollary 2.6] proved the following concerning second-order non-homogeneous Cauchy-Euler equations on time scales:

Theorem 1.1 (Cauchy-Euler Equation). Let \( \lambda_1, \lambda_2 \in \mathbb{R} \) (or \( \lambda_2 = \overline{\lambda_1} \), the complex conjugate) be such that

\[
t + \lambda_k \mu(t) \neq 0, \quad k = 1, 2
\]

for all \( t \in [a, \sigma(b)]_T \), where \( a \in T \) satisfies \( a > 0 \). Then the Cauchy-Euler equation

\[
x^\Delta \Delta(t) + \frac{1 - \lambda_1 - \lambda_2}{\sigma(t)} x^\Delta(t) + \frac{\lambda_1 \lambda_2}{t \sigma(t)} x(t) = f(t), \quad t \in [a, b]_T
\]

(1.1)
has Hyers-Ulam stability on \([a,b]_T\). To wit, if there exists \(y \in C_{\mathbb{rd}}^\Delta[a,b]_T\) that satisfies
\[
\left| y^\Delta(t) + \frac{1 - \lambda_1 - \lambda_2}{\sigma(t)} y^\Delta(t) + \frac{\lambda_1 \lambda_2}{t\sigma(t)} y(t) - f(t) \right| \leq \varepsilon
\]
for \(t \in [a,b]_T\), then there exists a solution \(u \in C_{\mathbb{rd}}^\Delta[a,b]_T\) of (1.1) given by
\[
u(t) = e^{\sum_{k=1}^n \lambda_k t} \cdot \int_{\tau_2}^t e^{\sum_{k=1}^n \lambda_k (s)} w(s) \Delta s, \quad \text{any} \quad \tau_2 \in [a,\sigma^2(b)]_T,
\]
where for any \(\tau_1 \in [a,\sigma(b)]_T\) the function \(w\) is given by
\[
w(s) = e^{\sum_{k=1}^{n-1} \lambda_k (s)} \left[ y^\Delta(\tau_1) - \frac{\lambda_1}{\tau_1} y(\tau_1) \right] + \int_{\tau_1}^s e^{\sum_{k=1}^{n-1} \lambda_k (s)} f(\zeta) \Delta \zeta,
\]
such that \(|y - u| \leq K\varepsilon\) on \([a,\sigma^2(b)]_T\) for some constant \(K > 0\).

The motivation for this work is to extend Theorem 1.1 to the general \(n\)th-order Cauchy-Euler dynamic equation; we will show the stability in the sense of Hyers and Ulam of the equation
\[
\sum_{k=0}^n \alpha_k M_k y(t) = f(t), \quad M_0 y(t) := y(t), \quad M_{k+1} y(t) := \varphi(t) (M_k y)^\Delta(t), \quad k = 0, 1, \cdots, n - 1.
\]
This is essentially \([5, (2.14)]\) if \(\varphi(t) = t\) and \(f(t) = 0\). Throughout this work we assume the reader has a working knowledge of time scales as can be found in Bohner and Peterson \([3, 5]\), originally introduced by Hilger \([10]\).

2. HYERS-ULAM STABILITY FOR HIGHER-ORDER CAUCHY-EULER DYNAMIC EQUATIONS

In this section we establish the Hyers-Ulam stability of the higher-order non-homogeneous Cauchy-Euler dynamic equation on time scales of the form
\[
\sum_{k=0}^n \alpha_k M_k y(t) = f(t), \quad M_0 y(t) := y(t), \quad M_{k+1} y(t) := \varphi(t) (M_k y)^\Delta(t), \quad k = 0, 1, \cdots, n - 1 \quad (2.1)
\]
for given constants \(\alpha_k \in \mathbb{R}\) with \(\alpha_n \equiv 1\), and for functions \(\varphi, f \in C_{\mathbb{rd}}[a,b]_T\), using the following definition.

**Definition 2.1** (Hyers-Ulam stability). Let \(\varphi, f \in C_{\mathbb{rd}}[a,b]_T\) and \(n \in \mathbb{N}\). If whenever \(M_k x \in C_{\mathbb{rd}}^\Delta[a,b]_T\) satisfies
\[
\left| \sum_{k=0}^n \alpha_k M_k x(t) - f(t) \right| \leq \varepsilon, \quad t \in [a,b]_T
\]
there exists a solution \(u\) of (2.1) with \(M_k u \in C_{\mathbb{rd}}^\Delta[a,b]_T\) for \(k = 0, 1, \cdots, n - 1\) such that \(|x - u| \leq K\varepsilon\) on \([a,\sigma^n(b)]_T\) for some constant \(K > 0\), then (2.1) has Hyers-Ulam stability \([a,b]_T\).
Lemma 2.3. Before proving the Hyers-Ulam stability of (2.1) we will need the following lemma, which allows us to factor (2.1) using the elementary symmetric polynomials \([7]\) in the \(n\) symbols \(\rho_1, \cdots, \rho_n\) given by

\[
\begin{align*}
    s_1^n &= s_1(\rho_1, \cdots, \rho_n) = \sum_i \rho_i \\
    s_2^n &= s_2(\rho_1, \cdots, \rho_n) = \sum_{i<j} \rho_i \rho_j \\
    s_3^n &= s_3(\rho_1, \cdots, \rho_n) = \sum_{i<j<k} \rho_i \rho_j \rho_k \\
    s_4^n &= s_4(\rho_1, \cdots, \rho_n) = \sum_{i<j<k<\ell} \rho_i \rho_j \rho_k \rho_\ell \\
    &\vdots \\
    s_t^n &= s_t(\rho_1, \cdots, \rho_n) = \sum_{i_1<i_2<\cdots<i_t} \rho_{i_1} \rho_{i_2} \cdots \rho_{i_t} \\
    &\vdots \\
    s_n^n &= s_n(\rho_1, \cdots, \rho_n) = \rho_1 \rho_2 \rho_3 \cdots \rho_n.
\end{align*}
\]

In general, we let \(s^n_j\) represent the \(j\)th elementary symmetric polynomial on \(j\) symbols. Then, given the \(\alpha_k\) in (2.2), introduce the characteristic values \(\lambda_k \in \mathbb{C}\) via the elementary symmetric polynomial \(s^n_k\) on the \(n\) symbols \(-\lambda_1, \cdots, -\lambda_n\), where

\[
\alpha_k = s^n_{n-k} = s_{n-k}(-\lambda_1, \cdots, -\lambda_n) = \sum_{i_1<i_2<\cdots<i_{n-k}} (-1)^{n-k} \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_{n-k}}, \quad \alpha_n = s_0 = 1. \tag{2.2}
\]

Lemma 2.3 (Factorization). Given \(y, \varphi \in \mathcal{C}_T[a, b]\) and \(\alpha_k \in \mathbb{R}\) with \(\alpha_n \equiv 1\), let \(M_k y \in \mathcal{C}_T^\Delta[a, b]\), where \(M_0 y(t) := y(t)\) and \(M_{k+1} y(t) := \varphi(t) (M_k y)^\Delta(t)\) for \(k = 0, 1, \cdots, n-1\). Then we have the factorization

\[
\sum_{k=0}^n \alpha_k M_k y(t) = \prod_{k=1}^n (\varphi D - \lambda_k I) y(t), \quad n \in \mathbb{N}, \tag{2.3}
\]

where the differential operator \(D\) is defined via \(Dx = x^\Delta\) for \(x \in \mathcal{C}_T^\Delta[a, b]\), and \(I\) is the identity operator.

Proof. We proceed by mathematical induction on \(n \in \mathbb{N}\), utilizing the substitution defined in (2.2). For \(n = 1\),

\[
\sum_{k=0}^n \alpha_k M_k y(t) = \alpha_0 M_0 y(t) + \alpha_1 M_1 y(t) = s_1(-\lambda_1) y(t) + 1 \cdot \varphi(t) y^\Delta(t) = (\varphi D - \lambda_1 I) y(t)
\]
and the result holds. Assume (2.3) holds for \( n \geq 1 \). Then we have \( \alpha_{n+1} \equiv 1 \) and

\[
\sum_{k=0}^{n+1} \alpha_k M_k y(t) = \alpha_0 y(t) + \sum_{k=1}^{n} \alpha_k M_k y(t) + M_{n+1} y(t)
\]

\[
= s_{n+1}^{n+1} y(t) + \sum_{k=1}^{n} s_{n+1-k}^{n+1} M_k y(t) + \varphi(t) (M_n y)(t)
\]

\[
= -\lambda_{n+1} s_n^n y(t) + \sum_{k=1}^{n} (s_{n+1-k}^n - \lambda_{n+1} s_{n-k}^n) M_k y(t) + \varphi(t) D (M_n y)(t)
\]

\[
= -\lambda_{n+1} \left[ s_n^n y(t) + \sum_{k=1}^{n} s_{n-k}^n M_k y(t) \right] + \sum_{k=1}^{n} s_{n+1-k}^{n+1} M_k y(t) + \varphi(t) D (M_n y)(t)
\]

\[
= -\lambda_{n+1} \sum_{k=0}^{n} s_{n-k}^n M_k y(t) + \varphi(t) D \left( \sum_{k=1}^{n} s_{n-k}^n M_k y(t) + M_n y \right)(t)
\]

\[
= -\lambda_{n+1} \sum_{k=0}^{n} s_{n-k}^n M_k y(t) + \varphi(t) D \sum_{k=0}^{n} s_{n-k}^n M_k y(t)
\]

\[
= (\varphi(t) D - \lambda_{n+1} I) \sum_{k=0}^{n} s_{n-k}^n M_k y(t)
\]

\[
= (\varphi(t) D - \lambda_{n+1} I) \sum_{k=0}^{n} \alpha_k M_k y(t)
\]

\[
= (\varphi(t) D - \lambda_{n+1} I) \prod_{k=1}^{n} (\varphi - \lambda_k I) y(t)
\]

and the proof is complete. \( \square \)

**Theorem 2.4 (Hyers-Ulam Stability).** Given \( y, \varphi, f \in C_{rd}[a, b]_T \) with \( |\varphi| \geq A > 0 \) for some constant \( A \), and \( \alpha_k \in \mathbb{R} \) with \( \alpha_n \equiv 1 \), consider (2.1) with \( M_k y \in C_{rd}^A[a, b]_T \) for \( k = 0, \ldots, n-1 \). Using the \( \lambda_k \) from the factorization in Lemma 2.3, assume

\[
\varphi(t) + \lambda_k \mu(t) \neq 0, \quad k = 1, 2, \ldots, n
\]

(2.4)

for all \( t \in [a, \sigma^{n-1}(b)]_T \). Then (2.1) has Hyers-Ulam stability on \([a, b]_T\).

**Proof.** Let \( \varepsilon > 0 \) be given, and suppose there is a function \( x \), with \( M_k x \in C_{rd}^A[a, b]_T \), that satisfies

\[
\left| \sum_{k=0}^{n} \alpha_k M_k x(t) - f(t) \right| \leq \varepsilon, \quad t \in [a, b]_T.
\]
We will show there exists a solution $u$ of (2.1) with $M_ku \in C_{\text{id}}^\Delta[a,b]_T$ for $k = 0, 1, \cdots, n-1$ such that $|x-u| \leq K\varepsilon$ on $[a,\sigma^n(b)]_T$ for some constant $K > 0$.

To this end, set

\[
\begin{align*}
g_1 &= \varphi x - \lambda_1 x = (\varphi D - \lambda_1 I)x \\
g_2 &= \varphi g_1 - \lambda_2 g_1 = (\varphi D - \lambda_2 I)g_1 \\
&\quad \vdots \\
g_k &= \varphi g_{k-1} - \lambda_k g_{k-1} = (\varphi D - \lambda_k I)g_{k-1} \\
&\quad \vdots \\
g_n &= \varphi g_{n-1} - \lambda_n g_{n-1} = (\varphi D - \lambda_n I)g_{n-1}.
\end{align*}
\]

This implies by Lemma 2.3 that

\[
g_n(t) - f(t) = \sum_{k=0}^{n} \alpha_k M_k x(t) - f(t),
\]

so that

\[
|g_n(t) - f(t)| \leq \varepsilon, \quad t \in [a,b]_T.
\]

By the construction of $g_n$ we have $|\varphi g_{n-1} - \lambda_n g_{n-1} - f| \leq \varepsilon$, that is

\[
|\varphi g_{n-1} - \lambda_n g_{n-1} - f| \leq \varepsilon \leq \varepsilon = \varepsilon.
\]

By [2] Lemma 2.3 and (2.4) there exists a solution $w_1 \in C_{\text{id}}^\Delta[a,b]_T$ of

\[
w(t) = \sum_{k=0}^{n} \alpha_k M_k x(t) - f(t),
\]

$t \in [a,b]_T$, where $w_1$ is given by

\[
w_1(t) = e^{\lambda_n}(t,\tau_1)g_{n-1}(\tau_1) + \int_{\tau_1}^{t} e^{\lambda_n}(t,s)\frac{f(s)}{\varphi(s)}\Delta s, \quad \text{any} \quad \tau_1 \in [a,\sigma(b)]_T,
\]

and there exists an $L_1 > 0$ such that

\[
|g_{n-1}(t) - w_1(t)| \leq L_1\varepsilon/A, \quad t \in [a,\sigma(b)]_T.
\]

Since $g_{n-1} = \varphi g_{n-2} - \lambda_{n-1} g_{n-2}$, we have that

\[
|\varphi g_{n-2} - \lambda_{n-1} g_{n-2}(t) - w_1(t)| \leq L_1\varepsilon/A, \quad t \in [a,\sigma(b)]_T.
\]

Again we apply [2] Lemma 2.3 to see that there exists a solution $w_2 \in C_{\text{id}}^\Delta[a,\sigma(b)]_T$ of

\[
w(t) = \sum_{k=0}^{n} \alpha_k M_k x(t) - f(t),
\]

$t \in [a,b]_T$, where $w_2$ is given by

\[
w_2(t) = e^{\lambda_n}(t,\tau_1)g_{n-1}(\tau_1) + \int_{\tau_1}^{t} e^{\lambda_n}(t,s)\frac{f(s)}{\varphi(s)}\Delta s, \quad \text{any} \quad \tau_1 \in [a,\sigma(b)]_T,
\]

and there exists an $L_2 > 0$ such that

\[
|g_{n-1}(t) - w_2(t)| \leq L_2\varepsilon/A, \quad t \in [a,\sigma(b)]_T.
\]

Since $g_{n-1} = \varphi g_{n-2} - \lambda_{n-1} g_{n-2}$, we have that

\[
|\varphi g_{n-2} - \lambda_{n-1} g_{n-2}(t) - w_2(t)| \leq L_2\varepsilon/A, \quad t \in [a,\sigma(b)]_T.
\]

Again we apply [2] Lemma 2.3 to see that there exists a solution $w_3 \in C_{\text{id}}^\Delta[a,\sigma(b)]_T$ of

\[
w(t) = \sum_{k=0}^{n} \alpha_k M_k x(t) - f(t),
\]

$t \in [a,b]_T$, where $w_3$ is given by

\[
w_3(t) = e^{\lambda_n}(t,\tau_1)g_{n-1}(\tau_1) + \int_{\tau_1}^{t} e^{\lambda_n}(t,s)\frac{f(s)}{\varphi(s)}\Delta s, \quad \text{any} \quad \tau_1 \in [a,\sigma(b)]_T,
\]

and there exists an $L_3 > 0$ such that

\[
|g_{n-1}(t) - w_3(t)| \leq L_3\varepsilon/A, \quad t \in [a,\sigma(b)]_T.
\]
Thus there exists a solution \( w_2(t) \) given by

\[
w_2(t) = e^{\lambda_{n-1}(t, \tau_2) g_{n-2}(\tau_2)} + \int_{\tau_2}^t e^{\lambda_{n-1}(t, \sigma(s)) \frac{w_1(s)}{\varphi(s)}} \Delta s, \quad \forall \tau_2 \in [a, \sigma^2(b)]_T,
\]
and there exists an \( L_2 > 0 \) such that

\[
|g_{n-2}(t) - w_2(t)| \leq L_2 A^2 \varepsilon, \quad t \in [a, \sigma^2(b)]_T.
\]

Continuing in this manner, we see that for \( k = 1, 2, \ldots, n-1 \) there exists a solution \( w_k \in C^\Delta_{\text{id}}[a, \sigma^{k-1}(b)]_T \) of

\[
w^\Delta(t) - \frac{\lambda_{n-k+1}}{\varphi(t)} w(t) - \frac{w_{k-1}(t)}{\varphi(t)} = 0, \quad \text{or} \quad \varphi(t) w^\Delta(t) - \lambda_{n-k+1} w(t) - w_{k-1}(t) = 0,
\]
\( t \in [a, \sigma^{k-1}(b)]_T \), where \( w_k \) is given by

\[
w_k(t) = e^{\lambda_{n-k+1}(t, \tau_k) g_{n-k}(\tau_k)} + \int_{\tau_k}^t e^{\lambda_{n-k+1}(t, \sigma(s)) \frac{w_{k-1}(s)}{\varphi(s)}} \Delta s, \quad \forall \tau_k \in [a, \sigma^k(b)]_T,
\]
and there exists an \( L_k > 0 \) such that

\[
|g_{n-k}(t) - w_k(t)| \leq \prod_{j=1}^k L_j A^k \varepsilon, \quad t \in [a, \sigma^k(b)]_T.
\]

In particular, for \( k = n-1 \),

\[
|g_1(t) - w_{n-1}(t)| \leq \prod_{j=1}^{n-1} L_j A^{n-1} \varepsilon, \quad t \in [a, \sigma^{n-1}(b)]_T
\]

implies by the definition of \( g_1 \) that

\[
\left| x^\Delta(t) - \frac{\lambda_1}{\varphi(t)} x(t) - \frac{w_{n-1}(t)}{\varphi(t)} \right| \leq \prod_{j=1}^{n-1} L_j A^n \varepsilon, \quad t \in [a, \sigma^{n-1}(b)]_T.
\]

Thus there exists a solution \( w_n \in C^\Delta_{\text{id}}[a, \sigma^{n-1}(b)]_T \) of

\[
w^\Delta(t) - \frac{\lambda_1}{\varphi(t)} w(t) - \frac{w_{n-1}(t)}{\varphi(t)} = 0, \quad \text{or} \quad \varphi(t) w^\Delta(t) - \lambda_1 w(t) - w_{n-1}(t) = 0,
\]
\( t \in [a, \sigma^{n-1}(b)]_T \), where \( w_n \) is given by

\[
w_n(t) = e^{\lambda_1(t, \tau_n) x(\tau_n)} + \int_{\tau_n}^t e^{\lambda_1(t, \sigma(s)) \frac{w_{n-1}(s)}{\varphi(s)}} \Delta s, \quad \forall \tau_n \in [a, \sigma^n(b)]_T,
\]
and there exists an \( L_n > 0 \) such that

\[
|x(t) - w_n(t)| \leq K\varepsilon : = \prod_{j=1}^n L_j A^n \varepsilon, \quad t \in [a, \sigma^n(b)]_T.
\]
By construction,

\[
(\varphi D - \lambda_1 I) w_n(t) = w_{n-1}(t)
\]

\[
\prod_{k=1}^{2} (\varphi D - \lambda_k I) w_n(t) = (\varphi D - \lambda_2 I) w_{n-1}(t) = w_{n-2}(t)
\]

\[
\vdots
\]

\[
\prod_{k=1}^{n} (\varphi D - \lambda_k I) w_n(t) = (\varphi D - \lambda_n I) w_1(t) \equiv f(t)
\]

on \([a, \sigma^{n-1}(b)]_{\mathbb{T}}\), so that \(u = w_n\) is a solution of (2.1), with \(u \in C^\Delta_{\text{rd}}[a, \sigma^{n-1}(b)]_{\mathbb{T}}\) and \(|x(t) - w_n(t)| \leq K\varepsilon\) for \(t \in [a, \sigma^n(b)]_{\mathbb{T}}\) by (2.8). Moreover, using (2.7) and (2.6), we have an iterative formula for this solution \(u = w_n\) in terms of the function \(x\) given at the beginning of the proof. 

\[\square\]

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