Piercing translates and homothets of a convex body

Adrian Dumitrescu† Minghui Jiang‡

October 21, 2009

Abstract

According to a classical result of Grünbaum, the transversal number $\tau(F)$ of any family $F$ of pairwise-intersecting translates or homothets of a convex body $C$ in $\mathbb{R}^d$ is bounded by a function of $d$. Denote by $\alpha(C)$ (resp. $\beta(C)$) the supremum of the ratio of the transversal number $\tau(F)$ to the packing number $\nu(F)$ over all families $F$ of translates (resp. homothets) of a convex body $C$ in $\mathbb{R}^d$. Kim et al. recently showed that $\alpha(C)$ is bounded by a function of $d$ for any convex body $C$ in $\mathbb{R}^d$, and gave the first bounds on $\alpha(C)$ for convex bodies $C$ in $\mathbb{R}^d$ and on $\beta(C)$ for convex bodies $C$ in the plane.

Here we show that $\beta(C)$ is also bounded by a function of $d$ for any convex body $C$ in $\mathbb{R}^d$, and present new or improved bounds on both $\alpha(C)$ and $\beta(C)$ for various convex bodies $C$ in $\mathbb{R}^d$ for all dimensions $d$. Our techniques explore interesting inequalities linking the covering and packing densities of a convex body. Our methods for obtaining upper bounds are constructive and lead to efficient constant-factor approximation algorithms for finding a minimum-cardinality point set that pierces a set of translates or homothets of a convex body.

Keywords: Geometric transversals, Gallai-type problems, packing and covering, approximation algorithms.

1 Introduction

A convex body is a compact convex set in $\mathbb{R}^d$ with nonempty interior. Let $F$ be a family of convex bodies.

The packing number $\nu(F)$ is the maximum cardinality of a set of pairwise-disjoint convex bodies in $F$, and the transversal number $\tau(F)$ is the minimum cardinality of a set of points that intersects every convex body in $F$.

Let $G$ be the intersection graph of $F$ with one vertex for each convex body in $F$ and with an edge between two vertices if and only if the two corresponding convex bodies intersect. The independence number $\alpha(G)$ is the maximum cardinality of an independent set in $G$. The clique partition number $\vartheta(G)$ is the minimum number of classes in a partition of the vertices of $G$ into cliques. Since a set of pairwise-disjoint convex bodies in $F$ corresponds to an independent set in $G$, we have $\nu(F) = \alpha(G)$. Also, since any subset of convex bodies in $F$ that share a common point corresponds to a clique in $G$, we have $\tau(F) \leq \vartheta(G)$. For the special case that $F$ is a family of axis-parallel boxes in $\mathbb{R}^d$, we indeed have $\tau(F) = \vartheta(G)$ since any subset of pairwise-intersecting boxes share a common point. In general, we clearly have the inequality $\vartheta(G) \geq \alpha(G)$, thus also $\tau(F) \geq \nu(F)$. But what else can be said about the relation between $\tau(F)$ and $\nu(F)$?

---

* A preliminary version of this paper appeared in the Proceedings of the 17th Annual European Symposium on Algorithms (ESA 2009), pages 131–142.
† Department of Computer Science, University of Wisconsin–Milwaukee, WI 53201-0784, USA. Email: ad@cs.uwm.edu. Supported in part by NSF CAREER grant CCF-0444188.
‡ Department of Computer Science, Utah State University, Logan, UT 84322-4205, USA. Email: mjiang@cc.usu.edu. Supported in part by NSF grant DBI-0743670.
functions of $F$ of pairwise-intersecting translates of a convex body in the plane, there always exists a set of three points that intersects every member of the family. It is folklore that pairwise-intersecting translates of a convex body in the plane, there always exists a set of three points that intersects every member of the family. It is folklore that pairwise-intersecting translates of a convex body in the plane, there always exists a set of three points that intersects every member of the family.

Figure 1: Piercing a family $F$ of axis-parallel unit squares. Left: all squares that intersect the highest (shaded) square contain one of its two lower vertices. Right: five squares form a 5-cycle.

For example, let $F$ be any family of axis-parallel unit squares in the plane, and refer to Figure 1. One can obtain a subset of pairwise-disjoint squares by repeatedly selecting the highest square that does not intersect the previously selected squares. Then $F$ is pierced by the set of points consisting of the two lower vertices of each square in the subset. This implies that $\tau(F) \leq 2 \cdot \nu(F)$. The factor of 2 cannot be improved below $\frac{3}{2}$ since $\tau(F) = 3$ and $\nu(F) = 2$ for a family $F$ of five squares arranged into a 5-cycle [15].

For a convex body $C$ in $\mathbb{R}^d$, $d \geq 2$, define

$$\alpha(C) = \sup_{\mathcal{F}_t} \frac{\tau(\mathcal{F}_t)}{\nu(\mathcal{F}_t)} \quad \text{and} \quad \beta(C) = \sup_{\mathcal{F}_h} \frac{\tau(\mathcal{F}_h)}{\nu(\mathcal{F}_h)}$$

where $\mathcal{F}_t$ ranges over all families of translates of $C$, and $\mathcal{F}_h$ ranges over all families of (positive) homothets of $C$. Our previous discussion (Figure 1) yields the bounds $\frac{3}{2} \leq \alpha(C) \leq 2$ for any square $C$.

Define $\alpha_1(C)$ (resp. $\beta_1(C)$) as the smallest number $k$ such that for any family $F$ of pairwise-intersecting translates (resp. homothets) of a convex body $C$, there exists a set of $k$ points that intersects every member of $F$. Note that $\alpha$ and $\beta$ generalize $\alpha_1$ and $\beta_1$. For any convex body $C$, the four numbers $\alpha(C)$, $\beta(C)$, $\alpha_1(C)$, and $\beta_1(C)$ are invariant under any non-singular affine transformation of $C$, and we have the four inequalities $\alpha_1(C) \leq \alpha(C)$, $\beta_1(C) \leq \beta(C)$, $\alpha_1(C) \leq \beta_1(C)$, and $\alpha(C) \leq \beta(C)$.

Grunbaum [14] showed that, for any convex body $C$ in $\mathbb{R}^d$, both $\alpha_1(C)$ and $\beta_1(C)$ are bounded by functions of $d$. Deriving bounds on $\alpha_1(C)$ and $\beta_1(C)$ for various types of convex bodies $C$ in $\mathbb{R}^d$ is typical of classic Gallai-type problems [10, 32], and has been extensively studied. For example, a result by Karasev [18] states that $\alpha_1(C) \leq 3$ for any convex body $C$ in the plane, i.e., for any family of pairwise-intersecting translates of a convex body in the plane, there always exists a set of three points that intersects every member of the family. It is folklore that $\alpha_1(C) = \beta_1(C) = 1$ for any parallelogram $C$ (see [14] and the references therein). Also, $\alpha_1(C) = 2$ for any affinely regular hexagon $C$ [14, Example 2], $\alpha_1(C) = \beta_1(C) = 3$ for any triangle $C$ [7], $\alpha_1(C) = 3 < 4 = \beta_1(C)$ for any (circular) disk $C$ [14, 9], and $\beta_1(C) \leq 7$ for any centrally symmetric convex body $C$ in the plane [14]. Perhaps the most celebrated result on point transversals of convex sets is Alon and Kleitman’s solution to the Hadwiger-Debrunner $(p, q)$-problem [2]. We refer to the two surveys [10, pp. 142–150] and [32, pp. 77–78] for more related results.

The two numbers $\alpha_1(C)$ and $\beta_1(C)$ bound the values of $\tau(F)$ for special families $F$ of translates and homothets, respectively, of a convex body $C$ with $\nu(F) = 1$. It is thus natural to study the general case $\nu(F) \geq 1$, and to obtain estimates on $\alpha(C)$ and $\beta(C)$. Despite the many previous bounds on $\alpha_1(C)$ and $\beta_1(C)$ [10, 32], first estimates on $\alpha(C)$ and $\beta(C)$ have been only obtained recently [21]. Note that the related problem for families of $d$-intervals (which are nonconvex) has been extensively studied [31, 20, 17, 1, 24].

Kim et al. [21] showed that $\alpha(C)$ is bounded by a function of $d$ for any convex body $C$ in $\mathbb{R}^d$, and gave the first bounds on $\alpha(C)$ for convex bodies $C$ in $\mathbb{R}^d$ and on $\beta(C)$ for convex bodies $C$ in the plane. In this

1We give a simpler construction for the lower bound $\alpha_1(C) \geq 3$ for any triangle $C$ in Appendix A.
paper, we show that \( \beta(C) \) is also bounded by a function of \( d \) for any convex body \( C \) in \( \mathbb{R}^d \), and present new or improved bounds on both \( \alpha(C) \) and \( \beta(C) \) for various convex bodies \( C \) in \( \mathbb{R}^d \) for all dimensions \( d \).

Note that in the definitions of \( \alpha \) and \( \beta \), both the convexity of \( C \) and the homothety of \( \mathcal{F}_t \) and \( \mathcal{F}_h \) are necessary for the values \( \alpha(C) \) and \( \beta(C) \) to be bounded. To see the necessity of convexity, let \( C \) be the union of a vertical line segment with endpoints \((0, 0)\) and \((0, 1)\) and a horizontal line segment with endpoints \((0, 0)\) and \((1, 0)\), where the shared endpoint \((0, 0)\) is the corner, and let \( \mathcal{F} \) be a family of \( n \) translates of \( C \) with corners at \((i/n, -i/n)\), \( 1 \leq i \leq n \) \([15]\). Then at least \( \lceil n/2 \rceil \) points are required to intersect every member of \( \mathcal{F} \). To see the necessity of homothety, let \( \mathcal{F} \) be a family of \( n \) pairwise intersecting line segments (or very thin rectangles) in the plane such that no three have a common point. Then again at least \( \lceil n/2 \rceil \) points are required to intersect every member of \( \mathcal{F} \).

**Definitions.** For a convex body \( C \) in \( \mathbb{R}^d \), denote by \(|C|\) the Lebesgue measure of \( C \), i.e., the area in the plane, or the volume in \( d \)-space for \( d \geq 3 \). For a family \( \mathcal{F} \) of convex bodies in \( \mathbb{R}^d \), denote by \(|\mathcal{F}|\) the Lebesgue measure of the union of the convex bodies in \( \mathcal{F} \), i.e., \(|\bigcup_{C \in \mathcal{F}} C|\).

For two convex bodies \( A \) and \( B \) in \( \mathbb{R}^d \), denote by \( A + B = \{a + b \mid a \in A, b \in B\} \) the Minkowski sum of \( A \) and \( B \). For a convex body \( C \) in \( \mathbb{R}^d \), denote by \( \lambda C = \{\lambda c \mid c \in C\} \) the scaled copy of \( C \) by a factor of \( \lambda \in \mathbb{R} \), denote by \( -C = \{-c \mid c \in C\} \) the reflexion of \( C \) about the origin, and denote by \( C + a = \{c + a \mid c \in C\} \) the translate of \( C \) by the vector from the origin to \( a \). Write \( C - C \) for \( C + (-C) \).

For two parallelepipeds \( P \) and \( Q \) in \( \mathbb{R}^d \) that are parallel to each other (but are not necessarily axis-parallel or orthogonal), denote by \( \lambda_i(P, Q), 1 \leq i \leq d \), the length ratios of the edges of \( Q \) to the corresponding parallel edges of \( P \). Then, for a convex body \( C \) in \( \mathbb{R}^d \), define

\[
\gamma(C) = \min_{P \subseteq Q} \left( \prod_{i=1}^{d-1} \left[ \lambda_i(P, Q) \right] \right),
\]

where \( P \) and \( Q \) range over all pairs of parallelepipeds in \( \mathbb{R}^d \) that are parallel to each other, such that \( P \subseteq C \subseteq Q \). Note that in this case \( \lambda_i(P, Q) \geq 1 \) for \( 1 \leq i \leq d \).

We review some standard definitions of packing and covering densities in the following; see [15], Chapter 1. A family \( \mathcal{F} \) of convex bodies is a packing in a domain \( Y \subseteq \mathbb{R}^d \) if \( \bigcup_{C \in \mathcal{F}} C \subseteq Y \) and the convex bodies in \( \mathcal{F} \) are pairwise-disjoint; \( \mathcal{F} \) is a covering of \( Y \) if \( Y \subseteq \bigcup_{C \in \mathcal{F}} C \). The density of a family \( \mathcal{F} \) relative to a bounded domain \( Y \) is \( \rho(\mathcal{F}, Y) = (\sum_{C \in \mathcal{F}} |C|)/|Y| \). If \( Y = \mathbb{R}^d \) is the whole space, then the upper density and the lower density of \( \mathcal{F} \) are, respectively,

\[
\overline{\rho}(\mathcal{F}, \mathbb{R}^d) = \limsup_{r \to \infty} \rho(\mathcal{F}, B^d(r)) \quad \text{and} \quad \underline{\rho}(\mathcal{F}, \mathbb{R}^d) = \liminf_{r \to \infty} \rho(\mathcal{F}, B^d(r)),
\]

where \( B^d(r) \) denote a ball of radius \( r \) centered at the origin (since we are taking the limit as \( r \to \infty \), a hypercube of side length \( r \) can be used instead of a ball of radius \( r \)). For a convex body \( C \) in \( \mathbb{R}^d \), define the packing density of \( C \) as

\[
\delta(C) = \sup_{\mathcal{F} \text{ packing}} \overline{\rho}(\mathcal{F}, \mathbb{R}^d),
\]

where \( \mathcal{F} \) ranges over all packings in \( \mathbb{R}^d \) with congruent copies of \( C \), and define the covering density of \( C \) as

\[
\theta(C) = \inf_{\mathcal{F} \text{ covering}} \underline{\rho}(\mathcal{F}, \mathbb{R}^d),
\]

where \( \mathcal{F} \) ranges over all coverings of \( \mathbb{R}^d \) with congruent copies of \( C \). If the members of \( \mathcal{F} \) are restricted to translates of \( C \), then we have the translative packing and covering densities \( \delta_T(C) \) and \( \theta_T(C) \). If the members of \( \mathcal{F} \) are further restricted to translates of \( C \) by vectors of a lattice, then we have the lattice
packing and covering densities $\delta_L(C)$ and $\theta_L(C)$. Note that the four densities $\theta_T(C)$, $\theta_L(C)$, $\delta_T(C)$, and $\delta_L(C)$ are invariant under any non-singular affine transformation of $C$. For any convex body $C$ in $\mathbb{R}^d$, we have the inequalities $\delta_L(C) \leq \delta_T(C) \leq \delta(C) \leq 1 \leq \theta(C) \leq \theta_T(C) \leq \theta_L(C)$.

For two convex bodies $A$ and $B$ in $\mathbb{R}^d$, denote by $\kappa(A,B)$ the smallest number $\kappa$ such that $A$ can be covered by $\kappa$ translates of $B$.

**Main results.** Kim et al. [21] recently proved that, for any family $F$ of translates of a convex body in $\mathbb{R}^d$, $\tau(F) \leq 2^{d-1}d! \cdot \nu(F)$, in particular $\tau(F) \leq 108 \cdot \nu(F)$ when $d = 3$, and moreover $\tau(F) \leq 8 \cdot \nu(F) - 5$ when $d = 2$. We improve these bounds for all dimensions $d$ in the following theorem:

**Theorem 1.** For any family $F$ of translates of a convex body $C$ in $\mathbb{R}^d$,

$$
\tau(F) \leq \gamma(C) \cdot \nu(F), \quad \text{where} \quad \gamma(C) \leq d(d + 1)^{d-1}.
$$

In particular, $\tau(F) \leq 48 \cdot \nu(F)$ when $d = 3$, and $\tau(F) \leq 6 \cdot \nu(F)$ when $d = 2$.

For any parallelepiped $C$ in $\mathbb{R}^d$, we can choose two parallelepipeds $P$ and $Q$ such that $P = Q = C$ hence $P \subseteq C \subseteq Q$. Then $\lambda_i(P,Q) = 1$ for $1 \leq i \leq d$, and $\gamma(C) = 2^{d-1}$. This implies the following corollary:

**Corollary 1.** For any family $F$ of translates of a parallelepiped in $\mathbb{R}^d$, $\tau(F) \leq 2^{d-1} \cdot \nu(F)$.

In contrast, for a family $F$ of (not necessarily congruent or similar) axis-parallel parallelepipeds in $\mathbb{R}^d$, the current best upper bound [11] (see also [19, 20, 26]) is

$$
\tau(F) \leq \nu(F) \log^{d-1} \nu(F)(\log \nu(F) - 1/2) + d.
$$

Kim et al. [21] also proved that, for any family $F$ of translates of a centrally symmetric convex body in the plane, $\tau(F) \leq 6 \cdot \nu(F) - 3$. The following theorem gives a general bound for any centrally symmetric convex body in $\mathbb{R}^d$ and an improved bound (if $\nu(F) \geq 5$) for any centrally symmetric convex body in the plane:

**Theorem 2.** For any family $F$ of translates of a centrally symmetric convex body $S$ in $\mathbb{R}^d$,

$$
\tau(F) \leq 2^d \cdot \frac{\theta_L(S)}{\delta_L(S)} \cdot \nu(F).
$$

Moreover, $\tau(F) \leq 24 \cdot \nu(F)$ when $d = 3$, and $\tau(F) \leq \frac{16}{3} \cdot \nu(F)$ when $d = 2$.

For special types of convex bodies in the plane, the following theorem gives sharper bounds than the bounds implied by Theorem 1 and Theorem 2. Also, as we will show later, inequality (3) below may give a better asymptotic bound than (1) and (2) for high dimensions.

**Theorem 3.** Let $F$ be a family of translates of a convex body $C$ in $\mathbb{R}^d$. Then

$$
\tau(F) \leq \min_L \kappa((C - C) \cap L, C) \cdot \nu(F),
$$

where $L$ ranges over all closed half spaces bounded by hyperplanes through the center of $C - C$. Moreover, $\tau(F) \leq 4 \cdot \nu(F) - 1$ if $C$ is a centrally symmetric convex body in the plane. Also,

(i) If $C$ is a square, then $\tau(F) \leq 2 \cdot \nu(F) - 1$,

(ii) If $C$ is a triangle, then $\tau(F) \leq 5 \cdot \nu(F) - 2$. 

4
Having presented our bounds for families of translates, we now turn to families of homothets. Kim et al. [21] proved that, for any family $F$ of homothets of a convex body $C$ in the plane, $\tau(F) \leq 16 \cdot \nu(F)$ and, if $C$ is centrally symmetric, $\tau(F) \leq 9 \cdot \nu(F)$. The following theorem gives a general bound for any convex body in $\mathbb{R}^d$, an improved bound for any centrally symmetric convex body in the plane, and additional bounds for special types of convex bodies in the plane:

**Theorem 4.** Let $F$ be a family of homothets of a convex body $C$ in $\mathbb{R}^d$. Then

$$\tau(F) \leq \kappa(C - C, C) \cdot \nu(F).$$

In particular, $\tau(F) \leq 7 \cdot \nu(F)$ if $C$ is a centrally symmetric convex body in the plane. Moreover,

(i) If $C$ is a square, then $\tau(F) \leq 4 \cdot \nu(F) - 3$,

(ii) If $C$ is a triangle, then $\tau(F) \leq 12 \cdot \nu(F) - 9$,

(iii) If $C$ is a disk, then $\tau(F) \leq 7 \cdot \nu(F) - 3$.

For any parallelepiped $C$ in $\mathbb{R}^d$, $C - C$ is a translate of $2C$ and can be covered by $2^d$ translates of $C$, thus $\kappa(C - C, C) \leq 2^d$. This implies the following corollary:

**Corollary 2.** For any family $F$ of homothets of a parallelepiped in $\mathbb{R}^d$, $\tau(F) \leq 2^d \cdot \nu(F)$.

Both Theorem 3 and Theorem 4 are obtained by a simple greedy method, used also previously by Kim et al. [21]. Although we have improved their bounds using new techniques in Theorem 1 and Theorem 2 we show that a refined analysis of the simple greedy method yields even better asymptotic bounds for high dimensions in Theorem 3 and Theorem 4. We will use the following lemma by Chakerian and Stein [7] in our analysis:

**Lemma 1 (Chakerian and Stein [7]).** For every convex body $C$ in $\mathbb{R}^d$ there exist two parallelepipeds $P$ and $Q$ such that $P \subseteq C \subseteq Q$, where $P$ and $Q$ are homothetic with ratio at most $d$.

For any convex body $C$ in $\mathbb{R}^d$, let $P$ and $Q$ be the two parallelepipeds in Lemma 1. Since $C - C \subseteq Q - Q$ and $P \subseteq C$, it follows that $\kappa(C - C, C) \leq \kappa(Q - Q, P) = \kappa(2Q, P) \leq (2d)^d$; see also [21] Lemma 4. The classic survey by Danzer, Grünbaum, and Klee [10, pp. 146–147] lists several other upper bounds due to Rogers and Danzer: (i) $\kappa(C - C, C) \leq \frac{2d}{d+1} 3^{d+1} \theta_T(C)$ for any convex body $C$ in $\mathbb{R}^d$, (ii) $\kappa(C - C, C) \leq 5^d$ and $\kappa(C - C, C) \leq 3^d \theta_T(C)$ for any centrally symmetric convex body $C$ in $\mathbb{R}^d$. Note that $\theta_T(C) \leq d \ln d + d \ln \ln d + 5d = O(d \log d)$ for any convex body $C$ in $\mathbb{R}^d$, according to a result of Rogers [28]. The following lemma summarizes the upper bounds on $\kappa(C - C, C)$:

**Lemma 2.** For any convex body $C$ in $\mathbb{R}^d$, $\kappa(C - C, C) \leq \min\{(2d)^d, \frac{2d}{d+1} 3^{d+1} \theta_T(C)\} = O(6^d \log d)$. Moreover, if $C$ is centrally symmetric, then $\kappa(C - C, C) \leq \min\{5^d, 3^d \theta_T(C)\} = O(3^d d \log d)$.

From Lemma 2 and Theorem 3 it follows that $\beta(C)$ is bounded by a function of $d$, namely by $O(6^d \log d)$, for any convex body $C$ in $\mathbb{R}^d$. Since $\min_L \kappa((C - C) \cap L, C) \leq \kappa(C - C, C)$, Lemma 2 also provides upper bounds on $\min_L \kappa((C - C) \cap L, C)$ in Theorem 3. As a result, (3) implies an upper bound $\tau(F) \leq O(6^d \log d) \cdot \nu(F)$ for any family $F$ of translates of a convex body in $\mathbb{R}^d$, which is better than the upper bound $\tau(F) \leq d(d + 1)^{d-1} \cdot \nu(F)$ in (1) when $d$ is sufficiently large. Also, (3) implies an upper bound $\tau(F) \leq 3^d \theta_T(S) \cdot \nu(F)$ for any family $F$ of translates of a centrally symmetric convex body $S$ in $\mathbb{R}^d$. Schmidt [29] showed that, for any centrally symmetric convex body $S$, $\delta_L(S) = \Omega(d/2^d)$; see also [5].
Theorem 6. For any triangle \( \alpha \) of translates of a centrally symmetric convex body in the plane. This upper bound, if true, is best possible because there exists a family \( \tau(\mathcal{F}) \in \mathbb{R}^d \) such that \( \tau(\mathcal{F}) \) for any family \( \mathcal{F} \) of congruent disks (i.e., translates of a disk) such that \( \tau(\mathcal{F}) \) for any family \( \mathcal{F} \) of pair-wise-intersecting translates of a convex body in the plane. Also, for any family \( \mathcal{F} \) of congruent disks such that \( \nu(\mathcal{F}) = 2 \), Kim et al. \[21\] confirmed that \( \tau(\mathcal{F}) \leq 2 \cdot \nu(\mathcal{F}) = 6 \). Our Corollary 1 confirms that \( \tau(\mathcal{F}) \leq 2 \cdot \nu(\mathcal{F}) \) for any family \( \mathcal{F} \) of translates of a parallelogram. The following theorem confirms the upper bound \( \tau(\mathcal{F}) \leq 3 \cdot \nu(\mathcal{F}) \) for another special case:

**Theorem 6.** For any family \( \mathcal{F} \) of translates of a centrally symmetric convex hexagon, \( \tau(\mathcal{F}) \leq 3 \cdot \nu(\mathcal{F}) \). Moreover, if \( \nu(\mathcal{F}) = 1 \), then \( \tau(\mathcal{F}) \leq 2 \).

A hexagon \( p_1p_2p_3p_4p_5p_6 \) is affinely regular if and only if (i) it is centrally symmetric and convex, and (ii) \( \frac{p_2p_1 + p_2p_3}{p_2p_4} = \frac{p_3p_4}{p_3p_5} \). Note that a centrally symmetric convex hexagon is not necessarily affinely regular. Grünbaum \[14\] showed that \( \alpha_1(C) = 2 \) for any affinely regular hexagon \( C \). Theorem 6 implies a stronger and more general result that \( 2 = \alpha_1(C) \leq \alpha(C) \leq 3 \) for any centrally symmetric convex hexagon \( C \). Theorem 3(i),(ii), and (iii) imply that \( \alpha(C) \leq 2 \) for any square \( C \), \( \alpha(C) \leq 5 \) for any triangle \( C \), and \( \alpha(C) \leq 4 \) for any disk \( C \). Theorem 4(i),(ii), and (iii) imply that \( \beta(C) \leq 4 \) for any square \( C \), \( \beta(C) \leq 12 \) for any triangle \( C \), and \( \beta(C) \leq 7 \) for any disk \( C \). We also have the lower bounds \( \beta(C) \geq \alpha(C) \geq \frac{3}{2} \).

### Table 1: Upper bounds on \( \alpha(C) \) and \( \beta(C) \) for a convex body \( C \) in \( \mathbb{R}^d \)

| Convex body \( C \) in \( \mathbb{R}^d \) | \( \alpha(C) \) upper | \( \beta(C) \) upper |
|------------------------------------------|------------------------|------------------------|
| arbitrary \( d = 2 \) | 6 | 16 |
| centr. symm. \( d = 2 \) | 4 | 7 |
| arbitrary \( d = 3 \) | 48 | \[11\] |
| centr. symm. \( d = 3 \) | 24 | \[11\] |
| arbitrary \( d > 3 \) | \( \min\{d(d+1)^{d-1}, \frac{2d}{d+1}3^{d+1}\theta_T(C)\} \) | \( \{\alpha(C)\} \) |
| centr. symm. \( d > 3 \) | \( \min\{d(d+1)^{d-1}, \frac{2d}{d+1}3^{d+1}\theta_T(C)\} \) | \( \{\alpha(C)\} \) |
| parallelepiped \( d \geq 2 \) | \( 2d \) | \( \{\beta(C)\} \) |

†By Theorem 4 and Lemma 2 for \( d = 3 \), \( (2d)^d = 216 \) and \( 5^d = 125 \).
for any square $C$, $\beta(C) \geq \alpha(C) \geq \alpha_1(C) = 3$ for any triangle $C$, $\alpha(C) \geq \alpha_1(C) = 3$ and $\beta(C) \geq \beta_1(C) = 4$ for any disk $C$ [14]. Table 2 summarizes the current best bounds on $\alpha(C)$ and $\beta(C)$ for some special convex bodies $C$ in the plane.

| Special convex body $C$ in the plane | $\alpha(C)$ lower | $\alpha(C)$ upper | $\beta(C)$ lower | $\beta(C)$ upper |
|--------------------------------------|------------------|------------------|------------------|------------------|
| centrally symmetric convex hexagon   | 2 [14]           | 3 [16]           | 2 [14]           | 7 [14]           |
| square                               | 3 [15]           | 2 [15]           | 3 [15]           | 4 [14]           |
| triangle                             | 3 [7]            | 5 [7]            | 3 [7]            | 12 [7]           |
| disk                                 | 3 [14]           | 4 [15]           | 4 [14]           | 7 [14]           |

Table 2: Lower and upper bounds on $\alpha(C)$ and $\beta(C)$ for some special convex bodies $C$ in the plane.

2 Upper bound for translates of an arbitrary convex body in $\mathbb{R}^d$

In this section we prove Theorem 1. Let $\mathcal{F}$ be a family of translates of a convex body $C$ in $\mathbb{R}^d$. Let $P$ and $Q$ be any two parallelepipeds in $\mathbb{R}^d$ that are parallel to each other, such that $\mathcal{F} \subseteq C \subseteq \mathcal{Q}$. Since the two values $\tau(\mathcal{F})$ and $\nu(\mathcal{F})$ are invariant under any non-singular affine transformation of $C$, we can assume that $P$ and $Q$ are axis-parallel and have edge lengths 1 and $e_i$, respectively, along the axis $x_i$, $1 \leq i \leq d$.

We first show that $\tau(T) \leq \lfloor e_d \rfloor \cdot \nu(T)$ for any family $T$ of $C$-translates whose corresponding $P$-translates intersect a common line $\ell$ parallel to the axis $x_d$. Define the $x_d$-coordinate of a $C$-translate as the smallest $x_d$-coordinate of a point in the corresponding $P$-translate. Set $T_1 = T$, let $C_1$ be the $C$-translate in $T_1$ with the smallest $x_d$-coordinate, and let $S_1$ be the subfamily of $C$-translates in $T_1$ that intersect $C_1$ ($S_1$ includes $C_1$ itself). Then, for increasing values of $i$, while $T_i = T \setminus \bigcup_{j=1}^{i-1} S_j$ is not empty, let $C_i$ be the $C$-translate in $T_i$ with the smallest $x_d$-coordinate, and let $S_i$ be the subfamily of $C$-translates in $T_i$ that intersect $C_i$. The iterative process ends with a partition $T = \bigcup_{i=1}^{m} S_i$, where $m \leq \nu(T)$.

Denote by $c_i$ the $x_d$-coordinate of $C_i$. Then each $C$-translate in the subfamily $S_i$, which is contained in a $Q$-translate of edge length $e_d$ along the axis $x_d$, has an $x_d$-coordinate of at least $c_i$ and at most $e_i + e_d$, and the corresponding $P$-translate, whose edge length along the axis $x_d$ is 1, contains at least one of the $[e_d]$ points on $\ell$ with $x_d$-coordinates $c_i + 1, \ldots, c_i + [e_d]$. These $[e_d]$ points form a piercing set for $S_i$, hence $\tau(S_i) \leq [e_d]$. It follows that

$$\tau(T) \leq \sum_{i=1}^{m} \tau(S_i) \leq [e_d] \cdot m \leq [e_d] \cdot \nu(T).$$

(5)

For $(a_1, \ldots, a_{d-1}) \in \mathbb{R}^{d-1}$, denote by $\ell(a_1, \ldots, a_{d-1})$ the following line in $\mathbb{R}^d$ that is parallel to the axis $x_d$:

$$\{ (x_1, \ldots, x_d) \mid (x_1, \ldots, x_{d-1}) = (a_1, \ldots, a_{d-1}) \}.$$

Now consider the following (infinite) set $\mathcal{L}$ of parallel lines:

$$\{ \ell(j_1 + b_1, \ldots, j_{d-1} + b_{d-1}) \mid (j_1, \ldots, j_{d-1}) \in \mathbb{Z}^{d-1} \},$$

where $(b_1, \ldots, b_{d-1}) \in \mathbb{R}^{d-1}$ is chosen such that no line in $\mathcal{L}$ is tangent to the $P$-translate of any $C$-translate in $\mathcal{F}$. Recall that $P$ and $Q$ are axis-parallel and have edge lengths 1 and $e_i$, respectively, along the axis $x_i$, $1 \leq i \leq d$. So we have the following two properties:

1. For any $C$-translate in $\mathcal{F}$, the corresponding $P$-translate intersects exactly one line in $\mathcal{L}$.

2. For any two $C$-translates in $\mathcal{F}$, if the two corresponding $P$-translates intersect two different lines in $\mathcal{L}$ of distance at least $e_i + 1$ along some axis $x_i$, $1 \leq i \leq d - 1$, then the two $C$-translates are disjoint.
Partition $\mathcal{F}$ into subfamilies $\mathcal{F}(j_1, \ldots, j_{d-1})$ of $C$-translates whose corresponding $P$-translates intersect a common line $l(j_1+b_1, \ldots, j_{d-1}+b_{d-1})$. Let $\mathcal{F}'(k_1, \ldots, k_{d-1})$ be the union of the families $\mathcal{F}(j_1, \ldots, j_{d-1})$ such that $j_i \equiv \lfloor e_i + 1 \rfloor k_i$ for $1 \leq i \leq d-1$. It follows from (5) that the transversal number of each subfamily $\mathcal{F}'(k_1, \ldots, k_{d-1})$ is at most $\lfloor e_d \rfloor$ times its packing number. Therefore we have

$$\tau(\mathcal{F}) \leq \sum_{(k_1, \ldots, k_{d-1})} \tau(\mathcal{F}'(k_1, \ldots, k_{d-1})) \leq \lfloor e_d \rfloor \sum_{(k_1, \ldots, k_{d-1})} \nu(\mathcal{F}'(k_1, \ldots, k_{d-1})) \leq \left( \lfloor e_d \rfloor \prod_{i=1}^{d-1} \lfloor e_i + 1 \rfloor \right) \cdot \nu(\mathcal{F}). \quad (6)$$

Since (6) holds for any pair of parallelepipeds $P$ and $Q$ in $\mathbb{R}^d$ that are parallel to each other and satisfy $P \subseteq C \subseteq Q$, it follows by the definition of $\gamma(C)$ that $\tau(\mathcal{F}) \leq \gamma(C) \cdot \nu(\mathcal{F})$. By Lemma 1 there indeed exist two such parallelepipeds $P$ and $Q$ with length ratios $\lambda_i(P, Q) = d$ for $1 \leq i \leq d$. It then follows that $\gamma(C) \leq d(d+1)^{d-1}$ for any convex body $C$ in $\mathbb{R}^d$. This completes the proof of Theorem [1]

3 Upper bound for translates of a centrally symmetric convex body in $\mathbb{R}^d$

In this section we prove Theorem [2]. Recall that $|C|$ is the Lebesgue measure of a convex body $C$ in $\mathbb{R}^d$, and that $|\mathcal{F}|$ is the Lebesgue measure of the union of a family $\mathcal{F}$ of convex bodies in $\mathbb{R}^d$. To establish the desired bound on $\tau(\mathcal{F})$ in terms of $\nu(\mathcal{F})$ for any family $\mathcal{F}$ of translates of a centrally symmetric convex body $S$ in $\mathbb{R}^d$, we link both $\tau(\mathcal{F})$ and $\nu(\mathcal{F})$ to the ratio $|\mathcal{F}|/|S|$. We first prove a lemma that links the transversal number $\tau(\mathcal{F})$ to the ratio $|\mathcal{F}|/|S|$ via the lattice covering density of $S$:

**Lemma 3.** Let $\mathcal{F}$ be a family of translates of a centrally symmetric convex body $S$ in $\mathbb{R}^d$. If there is a lattice covering of $\mathbb{R}^d$ with translates of $S$ whose covering density is $\theta$, $\theta \geq 1$, then $\tau(\mathcal{F}) \leq \theta \cdot |\mathcal{F}|/|S|$.

**Proof.** Denote by $S_p$ a translate of the convex body $S$ centered at a point $p$. Since $S$ is centrally symmetric, for any two points $p$ and $q$, $p$ intersects $S_q$ if and only if $q$ intersects $S_p$. Given a lattice covering of $\mathbb{R}^d$ with translates of $S$, every point $p \in \mathbb{R}^d$ is contained in some translate $S_q$ in the lattice covering, hence every translate $S_p$ contains some lattice point $q$.

Let $\Lambda$ be a lattice such that the corresponding lattice covering with translates of $S$ has a covering density of $\theta$. Divide the union of the convex bodies in $\mathcal{F}$ into pieces by the cells of the lattice $\Lambda$, then translate all cells (and the pieces) to a particular cell, say $\sigma$. By the pigeonhole principle, there exists a point in $\sigma$, say $p$, that is covered at most $|\mathcal{F}|/|\sigma|$ times by the overlapping pieces of the union. Let $k$ be the number of times that $p$ is covered by the pieces. Now fix $\mathcal{F}$ but translate the lattice $\Lambda$ to $\Lambda'$ until $p$ becomes a lattice point of $\Lambda'$. Then exactly $k$ lattice points of $\Lambda'$ are covered by the $S$-translates in $\mathcal{F}$. Since every $S$-translate in $\mathcal{F}$ contains some lattice point of $\Lambda'$, we have obtained a transversal of $\mathcal{F}$ consisting of $k \leq |\mathcal{F}|/|\sigma|$ lattice points of $\Lambda'$. Note that $\theta = |S|/|\sigma|$, and the proof is complete. \[\Box\]

The following lemma is a dual of the previous lemma, and links the packing number $\nu(\mathcal{F})$ to the ratio $|\mathcal{F}|/|S|$ via the lattice packing density of $S$:

**Lemma 4.** Let $\mathcal{F}$ be a family of translates of a centrally symmetric convex body $S$ in $\mathbb{R}^d$. If there is a lattice packing in $\mathbb{R}^d$ with translates of $S$ whose packing density is $\delta$, $\delta \leq 1$, then $\nu(\mathcal{F}) \geq \frac{1}{2\delta} \cdot |\mathcal{F}|/|S|$.

**Proof.** Let $S'$ be a homothet of $S$ scaled up by a factor of 2. Since $S$ is centrally symmetric, an $S$-translate is contained by an $S'$-translate if and only if the $S$-translate contains the center of the $S'$-translate. Given

\[\begin{array}{c}
\text{Proof.} \\
\text{Lemma 3 is also implied by [4] Theorem 5.}
\end{array}\]
a lattice packing in $\mathbb{R}^d$ with translates of $S'$, two $S'$-translates centered at two different lattice points are disjoint, hence two $S$-translates containing two different lattice points are disjoint.

Let $\Lambda$ be a lattice such that the corresponding lattice packing with translates of $S'$ has a packing density of $\delta$ (such a lattice exists because $S'$ is homothetic to $S$). Divide the union of the convex bodies in $\mathcal{F}$ into pieces by the cells of the lattice $\Lambda$, then translate all cells (and the pieces) to a particular cell, say $\sigma$. By the pigeonhole principle, there exists a point in $\sigma$, say $p$, that is covered at least $\lceil |\mathcal{F}|/|\sigma| \rceil$ times by the overlapping pieces of the union. Let $k$ be the number of times that $p$ is covered by the pieces. Now fix $\mathcal{F}$ but translate the lattice $\Lambda$ to $\Lambda'$ until $p$ becomes a lattice point of $\Lambda'$. Then exactly $k$ lattice points of $\Lambda'$ are covered by the $S$-translates in $\mathcal{F}$. Choose $k$ translates in $\mathcal{F}$, each containing a distinct lattice point of $\Lambda'$. Since any two $S$-translates containing two different lattice points of $\Lambda'$ are disjoint, we have obtained a subset of $k \geq \lceil |\mathcal{F}|/|\sigma| \rceil$ pairwise-disjoint $S$-translates in $\mathcal{F}$. Note that $\delta = |S'|/|\sigma| = 2^d |S|/|\sigma|$, and the proof is complete.

By Lemma 3 and Lemma 4 we have, for any family $\mathcal{F}$ of translates of a centrally symmetric convex body in $\mathbb{R}^d$, $$\tau(\mathcal{F}) \leq \theta_L(S) \cdot \frac{|\mathcal{F}|}{|S|} = 2^d \cdot \frac{\theta_L(S)}{\delta_L(S)} \cdot \frac{\delta_L(S)}{2^d} \cdot \frac{|\mathcal{F}|}{|S|} \leq 2^d \cdot \frac{\theta_L(S)}{\delta_L(S)} \cdot \nu(\mathcal{F}).$$

Smith [30] proved that, for any centrally symmetric convex body $S$ in 3-space, $\theta_L(S) \leq 3 \cdot \delta_L(S)$. This immediately implies that, for any family $\mathcal{F}$ of translates of a centrally symmetric convex body $S$ in 3-space, $\tau(\mathcal{F}) \leq 2^d \cdot 3 \cdot \nu(\mathcal{F}) = 24 \cdot \nu(\mathcal{F})$. A similar inequality for the planar case was proved by Kuperberg [22]: for any (not necessarily centrally symmetric) convex body $C$ in the plane, $\theta(C) \leq \frac{4}{3} \cdot \delta(C)$. However, this result is not about lattice covering and packing, so we cannot use it to obtain the bound in Theorem 2 for the planar case. Instead, we prove the following “sandwich” lemma:

**Lemma 5.** Let $\mathcal{F}$ be a family of translates of a (not necessarily centrally symmetric) convex body $C$ in $\mathbb{R}^d$. Let $A$ and $B$ be two centrally symmetric convex bodies in $\mathbb{R}^d$ such that $A \subseteq C \subseteq B$. Then $$\tau(\mathcal{F}) \leq 2^d \cdot \frac{|B|}{|A|} \cdot \frac{\theta_L(A)}{\delta_L(B)} \cdot \nu(\mathcal{F}).$$

**Proof.** Since $A \subseteq C$, it follows by Lemma 3 that $$\tau(\mathcal{F}) \leq \theta_L(A) \cdot \frac{|\mathcal{F}|}{|A|}.$$ Since $C \subseteq B$, it follows by Lemma 4 that $$\nu(\mathcal{F}) \geq \frac{\delta_L(B)}{2^d} \cdot \frac{|\mathcal{F}|}{|B|}.$$ Putting these together yields $$\tau(\mathcal{F}) \leq \theta_L(A) \cdot \frac{|\mathcal{F}|}{|A|} = 2^d \cdot \frac{|B|}{|A|} \cdot \frac{\theta_L(A)}{\delta_L(B)} \cdot \frac{\delta_L(B)}{2^d} \cdot \frac{|\mathcal{F}|}{|B|} \leq 2^d \cdot \frac{|B|}{|A|} \cdot \frac{\theta_L(A)}{\delta_L(B)} \cdot \nu(\mathcal{F}).$$

We also need the following lemma which is now folklore [5, Theorem 2.5 and Theorem 2.8]:

**Lemma 6.** For any centrally symmetric convex body $S$ in the plane, there are two centrally symmetric convex hexagons $H$ and $H'$ such that $H \subseteq S \subseteq H'$ and $|H|/|H'| \geq 3/4$.

Note that $\theta_L(H) = \delta_L(H) = 1$ for a centrally symmetric convex hexagon $H$. Set $A = H$, $B = H'$, and $C = S$ in the previous two lemmas, and we have, for any family $\mathcal{F}$ of translates of a centrally symmetric convex body in the plane, $$\tau(\mathcal{F}) \leq 2^2 \cdot \frac{4}{3} \cdot \frac{1}{3} \cdot \nu(\mathcal{F}) = \frac{16}{3} \cdot \nu(\mathcal{F}).$$

This completes the proof of Theorem 2.
4 Upper bound by greedy decomposition and lower bound by packing and covering

In this section we prove Theorems 3, 4, and 5.

Proof of Theorem 3. Let $\mathcal{F}$ be a family of translates of a convex body $C$ in $\mathbb{R}^d$. Without loss of generality, assume that $\kappa((C - C) \cap L, C)$ is minimized when $L = \{(x_1, \ldots, x_d) | x_d \geq 0\}$. Perform a greedy decomposition as follows. For $i = 1, 2, \ldots$, while $T_i = \mathcal{F} \setminus \bigcup_{j=1}^{i-1} S_j$ is not empty, let $C_i$ be the translate of $C$ in $T_i$ that contains a point of the largest $x_d$-coordinate, and let $S_i$ be the subfamily of translates in $T_i$ that intersect $C_i$ ($S_i$ includes $C_i$ itself). The iterative process ends with a partition $\mathcal{F} = \bigcup_{i=1}^{m} S_i$, where $m \leq \nu(\mathcal{F})$. We next show that $\tau(S_i) \leq \kappa((C - C) \cap L, C)$.

Choose any point in $C$ as a reference point. We have the following lemma:

Lemma 7. Let $A$ and $B$ be two translates of $C$ with reference points $a$ and $b$, respectively. Then,

(i) $A$ contains $b$ if and only if $-(B - b) + b$ contains $a$,

(ii) If $A$ intersects $B$, then $a$ is contained in a translate of $C - C$ centered at $b$.

Proof. (i) $b \in A \iff b - a \in A - a = B - b \iff a - b \in -(B - b) \iff a \in -(B - b) + b$. (ii) Let $c \in A \cap B$. Then $c \in A \Rightarrow c - a \in A - a \Rightarrow a - c \in -(A - a)$, and $c \in B \Rightarrow c - b \in B - b = A - a$. It follows that $a - b = (a - c) + (c - b) \in -(A - a) + (A - a) = C - C$.

By Lemma 7 (ii), the reference point of each translate of $C$ in $S_i$ is contained in a translate of $C - C$ centered at the reference point of $C_i$. Since the translate of $C - C$ is covered by $\kappa(C - C, -C)$ translates of $-C$, it follows by Lemma 7 (i) that each translate of $C$ in $S_i$ contains one of the $\kappa(C - C, -C)$ corresponding reference points. Therefore,

$$\tau(S_i) \leq \kappa(C - C, -C) = \kappa(C - C, C).$$

The stronger bound $\tau(S_i) \leq \kappa((C - C) \cap L, C)$ follows by our choice of $C_i$. We have

$$\tau(\mathcal{F}) \leq \sum_{i=1}^{m} \tau(S_i) \leq \kappa((C - C) \cap L, C) \cdot m \leq \kappa((C - C) \cap L, C) \cdot \nu(\mathcal{F}).$$

In the special case that $C$ is a centrally symmetric convex body in the plane, $C - C$ is a translate of $2C$. Assume without loss of generality that $C$ is centered at the origin. Then $C - C = 2C$. We have the following lemma on covering $2C$ with translates of $C$; this lemma is implicit in a result by Grünbaum [14, Theorem 4], we nevertheless present our own simple proof here for completeness:

Lemma 8. Let $C$ be a centrally symmetric convex body in the plane. Then $2C$ can be covered by seven translates of $C$, including one translate concentric with $2C$ and six others centered at the six vertices, respectively, of an affinely regular hexagon $H_C$ concentric with $2C$.

Proof. Refer to Figure 2. Let the center $o$ of $C$ be the origin. Let $p_2$ and $p_5$ be the intersections of the boundary of $2C$ and an arbitrary line $\ell$ through the origin. Choose two points $p_1$ and $p_6$ on the boundary of $2C$ on one side of the line $\ell$, and choose two points $p_3$ and $p_4$ on the other side, such that $\overrightarrow{p_1p_6} = \overrightarrow{p_3p_4} = \frac{1}{2} \overrightarrow{p_2p_5}$. Then $p_1p_2p_3p_4p_5p_6$ is an affinely regular hexagon. Let $2H$ be this hexagon inscribed in $2C$. Consider the (shaded) hexagon $H'$ that is a translate of $H$ with two opposite vertices $p_1$ and $p_6$. Let $q_1$ and $q_6$ be the midpoints of $p_1o$ and $p_6o$, respectively. Then $q_1$ and $q_6$ are also vertices of $H'$. The two
Figure 2: Covering $2C$ with seven translates of $C$. $2H = p_1p_2p_3p_4p_5p_6$ is an affinely regular hexagon inscribed in $2C$; $o$ is the center of $2C$; $o'$ is the intersection of the two lines extending $p_2p_1$ and $p_5p_6$; $q_1$ and $q_6$ are the midpoints of $p_1o$ and $p_6o$, respectively.

hexagons $2H$ and $H'$ are homothetic with ratio 2 and with homothety center at the intersection $o'$ of the two lines extending $p_2p_1$ and $p_5p_6$. Let $C'$ be a translate of $C$ such that $H'$ is inscribed in $C'$. Then $C'$ covers the part of $2C$ between the two rays $op_1$ and $op_6$. It follows that $2C$ is covered by seven translates of $C$, one centered at the origin, and six others centered at the midpoints of the six sides of $2H$, respectively. The six midpoints are clearly the vertices of another (smaller) affinely regular hexagon concentric with $2C$. Let $H_C$ be this hexagon, and the proof is complete.

Choose the halfplane $L$ through the center of $2C$ and any two opposite vertices of the hexagon $2H = p_1p_2p_3p_4p_5p_6$ in Lemma 8. Then $\kappa((C - C) \cap L, C) \leq 4$. It follows that $\tau(F) \leq 4 \cdot \nu(F)$ for any family $F$ of translates of a centrally symmetric convex body in the plane.

Figure 3: Piercing a subfamily $S_i$ of translates that intersect the highest translate $C_i$ (dark-shaded). (a) The centers of the squares are contained in the light-shaded rectangle; the squares can be pierced by two points. (b) The lower-left vertices of the triangles are contained in the light-shaded trapezoid; the triangles can be pierced by five points. (c) The centers of the disks are contained in the light-shaded half-disk; the disks can be pierced by four points.

To complete the proof of Theorem 3, we apply the greedy decomposition algorithm to some simple types of convex bodies in the plane: squares, triangles, and disks. We use some known bounds on $\tau(F)$ for families $F$ with small $\nu(F)$, for example, $\alpha_1(C)$ for $\nu(F) = 1$, to obtain slightly better upper bounds for these special cases. We refer to Figure 3, where the $x_1$ and $x_2$ axes are the $x$ and $y$ axes.

First let $C$ be a square, and refer to Figure 3(a). Corollary 1 implies that $\tau(F) \leq 2 \cdot \nu(F)$ for any family $F$ of translates of $C$. We obtain a slightly better bound by a tighter analysis of the greedy decomposition
algorithm. Assume that \( C \) is axis-parallel and has side length 1. Choose the center of \( C \) as the reference point. Then the centers of the squares in \( \mathcal{S}_i \) are contained in the light-shaded rectangle of width 2 and height 1, which is covered by two unit squares centered at the two lower vertices of \( C_i \). Each square in \( \mathcal{S}_i \) contains one of the two lower vertices of \( C_i \), thus \( \tau(\mathcal{S}_i) \leq 2 \). Consider two cases:

1. \( m \leq \nu(F) - 1 \). Then
   \[
   \tau(F) \leq \sum_{i=1}^{m} \tau(\mathcal{S}_i) \leq 2 \cdot (\nu(F) - 1) = 2 \cdot \nu(F) - 2.
   \]
2. \( m = \nu(F) \). Then \( \nu(S_m) = 1 \). It follows that \( \tau(S_m) \leq \alpha_1(C) = 1 \) [14]. Then
   \[
   \tau(F) \leq \sum_{i=1}^{m} \tau(\mathcal{S}_i) \leq 2 \cdot (\nu(F) - 1) + 1 = 2 \cdot \nu(F) - 1.
   \]

Next let \( C \) be a triangle, and refer to Figure 3(b). Assume that \( C \) has a horizontal lower side. Choose the lower-left vertex of \( C \) as the reference point. The lower-left vertices of the triangles in \( \mathcal{S}_i \) are contained in the light-shaded trapezoid, which can be covered by five translates of \(-C\). Hence each triangle in \( \mathcal{S}_i \) contains one of the upper-right vertices of these five translates, thus \( \tau(\mathcal{S}_i) \leq 5 \). The proof can be finished in the same way as for squares by considering the two cases \( m \leq \nu(F) - 1 \) and \( m = \nu(F) \), and using the fact that \( \alpha_1(C) = 3 \) for any triangle \( C \) [7].

Finally let \( C \) be a disk, and refer to Figure 3(c). Assume that \( C \) has radius 1. Choose the center of \( C \) as the reference point. Then the centers of the disks in \( \mathcal{S}_i \) are contained in the light-shaded half-disk of radius 2. It is well known (see [13]) that a disk of radius 2 can be covered by seven disks of radius 1, with one disk in the middle and six others around in a hexagonal formation. Therefore the half-disk of radius 2 can be covered by four disks of radius 1. The center of each disk in \( \mathcal{S}_i \) is contained by one of the four disks; by symmetry, each disk in \( \mathcal{S}_i \) contains the center of one of the four disks, thus \( \tau(\mathcal{S}_i) \leq 4 \). Again, the proof can be finished by considering the two cases \( m \leq \nu(F) - 1 \) and \( m = \nu(F) \) as done for squares and triangles, and using the fact that \( \alpha_1(C) = 3 \) for any disk \( C \) [14]. Indeed the same argument shows that \( \tau(F) \leq 4 \cdot (\nu(F) - 1) + 3 = 4 \cdot \nu(F) - 1 \) for any centrally symmetric convex body \( C \) in the plane since \( \alpha_1(C) \leq 3 \) also holds [13]. This completes the proof of Theorem 3.

**Proof of Theorem 4** Let \( F \) be a family of homothets of a convex body \( C \) in \( \mathbb{R}^d \). We again use greedy decomposition. The only difference in the algorithm is that \( C_i \) is now chosen as the smallest homothet of \( C \) in \( T_i \). By our choice of \( C_i \), each homothet in \( \mathcal{S}_i \) contains a translate of \( C_i \) that intersects \( C_i \). Hence the bound \( \tau(\mathcal{S}_i) \leq \kappa(C - C, C) \) follows in a similar way as the derivation of (7).

Let now \( C \) be a centrally symmetric convex body in the plane. By Lemma 8 we have \( \kappa(C - C, C) \leq 7 \). Then \( \tau(\mathcal{S}_i) \leq \kappa(C - C, C) \leq 7 \), from which it follows that \( \tau(F) \leq 7 \cdot \nu(F) \) for any centrally symmetric convex body \( C \) in the plane.

The analysis for special types of convex bodies \( C \) in the plane (squares, triangles, and disks) is also similar to the corresponding analysis in the proof of Theorem 3. We obtain the bound \( \tau(\mathcal{S}_i) \leq \kappa(C - C, C) \) and show that \( \kappa(C - C, C) \leq 4 \) for any square \( C \), \( \kappa(C - C, C) \leq 12 \) for any triangle \( C \), and \( \kappa(C - C, C) \leq 7 \) for any disk \( C \), then use \( \beta_1(C) \) instead of \( \alpha_1(C) \) to bound \( \tau(S_m) \) in case 2. As discussed in the introduction, it is known that \( \beta_1(C) = 1 \) for any square \( C \) [14], \( \beta_1(C) = 3 \) for any triangle \( C \) [7], and \( \beta_1(C) = 4 \) for any disk \( C \) [14, 9]. This completes the proof of Theorem 4.
Proof of Theorem 5. Let $C$ be a convex body in $\mathbb{R}^d$ and $n$ be a positive integer. We will show that $\beta(C) \geq \alpha(C) \geq \theta_T(C)/\delta_T(C)$ by constructing a family $\mathcal{F}_n$ of $n^{2d}$ translates of $C$, such that

$$\lim_{n \to \infty} \frac{\tau(\mathcal{F}_n)}{\nu(\mathcal{F}_n)} \geq \frac{\theta_T(C)}{\delta_T(C)}. \quad (8)$$

By Lemma 1, there exist two homothetic parallelepipeds $P$ and $Q$ with ratio $d$ such that $P \subseteq C \subseteq Q$. Without loss of generality (via an affine transformation), we can assume that $P$ and $Q$ are axis-parallel hypercubes of side lengths 1 and $d$, respectively, and that $P$ is centered at the origin. Now choose the origin as the reference point of $C$. Let $\mathcal{F}_n = \{ C + \tau \mid \tau \in T_n \}$ be a family of translates of $C$ corresponding to a set of $n^{2d}$ regularly placed reference points

$$T_n = \{(t_1/n, \ldots, t_d/n) \mid (t_1, \ldots, t_d) \in \mathbb{Z}^d, 1 \leq t_1, \ldots, t_d \leq n^2\}.$$ 

Denote by $H(\ell)$ any axis-parallel hypercube of side length $\ell$.

We first obtain an upper bound on $\nu(\mathcal{F}_n)$. For each $C + \tau \in \mathcal{F}_n$, we have $C + \tau \subseteq C + T_n \subseteq Q + T_n$. Note that $Q + T_n$ is an axis-parallel hypercube of side length exactly $n - \frac{1}{n} + d$. Denote by $\delta_T(X,Y)$ the supremum of the packing density of a domain $Y \subseteq \mathbb{R}^d$ by translates of $X$. By a volume argument, we have

$$\nu(\mathcal{F}_n) \leq \frac{\delta_T(C, Q + T_n) \cdot |Q + T_n|}{|C|} = \frac{\delta_T(C, H(n - \frac{1}{n} + d)) \cdot (n - \frac{1}{n} + d)^d}{|C|}. \quad (9)$$

We next obtain a lower bound on $\tau(\mathcal{F}_n)$. By Lemma 7(i), piercing the family $\mathcal{F}_n$ of translates of $C$ is equivalent to covering the corresponding set $T_n$ of reference points by translates of $-C$. Let $S_n$ be any set of points such that $T_n \subseteq -C + S_n$, that is, $T_n$ is covered by the set $\{-C + s \mid s \in S_n\}$ of translates of $-C$. We also have $\frac{1}{n}P \subseteq -\frac{1}{n}C$ since $P \subseteq C$. It follows that

$$-\frac{1}{n}P + T_n \subseteq -\frac{1}{n}C + (-C + S_n) = -(1 + \frac{1}{n})C + S_n.$$ 

Thus $\tau(\mathcal{F}_n)$ is at least the minimum number of translates of $-(1 + \frac{1}{n})C$ that cover $-\frac{1}{n}P + T_n$. Note that $-\frac{1}{n}P + T_n$ is an axis-parallel hypercube of side length exactly $n$. Denote by $\theta_T(X,Y)$ the infimum of the covering density of a domain $Y \subseteq \mathbb{R}^d$ by translates of $X$. Again by a volume argument, we have

$$\tau(\mathcal{F}_n) \geq \frac{\theta_T(\frac{1}{n}P + T_n, -(1 + \frac{1}{n})C) \cdot |-(1 + \frac{1}{n})C|}{\theta_T((1 + \frac{1}{n})C, H(n)) \cdot n^d} = \frac{\theta_T((1 + \frac{1}{n})C, H(n)) \cdot n^d}{(1 + \frac{1}{n})^d \cdot |C|}. \quad (10)$$

From the two inequalities (9) and (10), it follows that

$$\frac{\tau(\mathcal{F}_n)}{\nu(\mathcal{F}_n)} \geq \frac{\theta_T((1 + \frac{1}{n})C, H(n))}{\delta_T(C, H(n - \frac{1}{n} + d)) \cdot (1 + \frac{1}{n})^d(1 - \frac{1}{n^2} + \frac{d}{n})^d}. \quad (11)$$

Taking the limit as $n \to \infty$, we have $\theta_T((1 + \frac{1}{n})C, H(n)) \to \theta_T(C)$, $\delta_T(C, H(n - \frac{1}{n} + d)) \to \delta_T(C)$, and $(1 + \frac{1}{n})^d(1 - \frac{1}{n^2} + \frac{d}{n})^d \to 1$. This yields (8) as desired.

We now consider the special case that $C$ is the $d$-dimensional unit ball $B^d$ in $\mathbb{R}^d$. We clearly have $\theta_T(B^d) \geq 1$. Kabatjanskiï and Levenšteïn [16] showed that $\delta_T(B^d) = \delta(B^d) \leq 2^{-0.599 \pm O(1)d}$ as $d \to \infty$; see also [5] p. 50]. Therefore we have

$$\beta(B^d) \geq \alpha(B^d) \geq \frac{\theta_T(B^d)}{\delta_T(B^d)} \geq 2^{0.599 \pm O(1)d} \text{ as } d \to \infty. \quad (12)$$

This completes the proof of Theorem 5.
centers of all translates of it suffices to show that parallelogram symmetric convex hexagon that two points (the centers of translates of $H$ in an axis-parallel unit square, then we can choose supporting lines of a pair of parallel edges of $H$ are parallel to each other, with length ratios $w = \lambda_1(P, Q) \leq 2$ and $h = \lambda_2(P, Q) = 1$, such that $P \subseteq H \subseteq Q$. Let $H = p_1p_2p_3p_4p_5p_6$. Without loss of generality (via an affine transformation), the parallelogram $p_2p_3p_5p_6$ is an axis-parallel rectangle of width 1/2 and height 1. If the hexagon is contained in an axis-parallel unit square, then we can choose $P$ and $Q$ as the rectangle $p_2p_3p_5p_6$ and the square, whose length ratios are $w = 2$ and $h = 1$. Suppose otherwise. Assume that $p_1$ is higher than $p_4$. Then we choose $P$ as the parallelogram $p_1p_3p_4p_6$ and $Q$ as the (dashed) parallelogram circumscribing $H$ and parallel to $P$. Let $u$ be the intersection of $p_1p_3$ and $p_2p_5$, and let $v$ be the intersection of $p_4p_6$ and $p_2p_5$. The length ratios of $P$ and $Q$ are $w = |p_2p_5|/|uv|$ and $h = 1$, where $w$ is maximized to 2 when $p_1$ and $p_4$ are the midpoints of the two vertical sides of the unit square.

We refer to Figure 5 for the special case $\nu(F) = 1$. We will prove that $\tau(F) \leq 2$. The centrally symmetric convex hexagon $H$ is the intersection of three strips $S_1$, $S_2$, and $S_3$, each bounded by the two supporting lines of a pair of parallel edges of $H$. Without loss of generality, assume that the strip $S_1$ is horizontal. Let $A$ be the highest translate of $H$ in $F$, and let $B$ be any other translate of $H$ in $F$. Then the $y$-coordinates of the centers of $A$ and $B$ differ by at most the width of the strip $S_1$. This implies that the centers of all translates of $H$ in $F$ are contained in a translate of $S_1$. Apply the same argument to the other two strips $S_2$ and $S_3$. It follows that the centers of all translates of $H$ in $F$ are contained a hexagon $H'$ that is the intersection of three strips $S'_1$, $S'_2$, and $S'_3$, which are translates of $S_1$, $S_2$, and $S_3$, respectively. Let $H_{12}$, $H_{13}$, and $H_{23}$ be the three unique translates of $H$ contained in $S'_1 \cap S'_2$, $S'_1 \cap S'_3$, and $S'_2 \cap S'_3$, respectively. Then any two of the three translates of $H$, say $H_{12}$ and $H_{13}$, cover the hexagon $H'$. It follows by symmetry that two points (the centers of $H_{12}$ and $H_{13}$) are enough to pierce all members of $F$. This completes the proof of Theorem 6.

5 Upper bound for translates of a centrally symmetric convex hexagon

In this section we prove Theorem 6. Let $F$ be a family of translates of a centrally symmetric convex hexagon $H$ in the plane.

We refer to Figure 4 for the special case $\nu(F) \geq 1$. We will prove that $\tau(F) \leq 3 \cdot \nu(F)$. By Theorem 1 it suffices to show that $\gamma(H) \leq 3$. We will show that $\gamma(H) \leq 3$ by finding two parallelograms $P$ and $Q$ that are parallel to each other, with length ratios $w = \lambda_1(P, Q) \leq 2$ and $h = \lambda_2(P, Q) = 1$, such that $P \subseteq H \subseteq Q$. Let $H = p_1p_2p_3p_4p_5p_6$. Without loss of generality (via an affine transformation), the parallelogram $p_2p_3p_5p_6$ is an axis-parallel rectangle of width 1/2 and height 1. If the hexagon is contained in an axis-parallel unit square, then we can choose $P$ and $Q$ as the rectangle $p_2p_3p_5p_6$ and the square, whose length ratios are $w = 2$ and $h = 1$. Suppose otherwise. Assume that $p_1$ is higher than $p_4$. Then we choose $P$ as the parallelogram $p_1p_3p_4p_6$ and $Q$ as the (dashed) parallelogram circumscribing $H$ and parallel to $P$. Let $u$ be the intersection of $p_1p_3$ and $p_2p_5$, and let $v$ be the intersection of $p_4p_6$ and $p_2p_5$. The length ratios of $P$ and $Q$ are $w = |p_2p_5|/|uv|$ and $h = 1$, where $w$ is maximized to 2 when $p_1$ and $p_4$ are the midpoints of the two vertical sides of the unit square.

We refer to Figure 5 for the special case $\nu(F) = 1$. We will prove that $\tau(F) \leq 2$. The centrally symmetric convex hexagon $H$ is the intersection of three strips $S_1$, $S_2$, and $S_3$, each bounded by the two supporting lines of a pair of parallel edges of $H$. Without loss of generality, assume that the strip $S_1$ is horizontal. Let $A$ be the highest translate of $H$ in $F$, and let $B$ be any other translate of $H$ in $F$. Then the $y$-coordinates of the centers of $A$ and $B$ differ by at most the width of the strip $S_1$. This implies that the centers of all translates of $H$ in $F$ are contained in a translate of $S_1$. Apply the same argument to the other two strips $S_2$ and $S_3$. It follows that the centers of all translates of $H$ in $F$ are contained a hexagon $H'$ that is the intersection of three strips $S'_1$, $S'_2$, and $S'_3$, which are translates of $S_1$, $S_2$, and $S_3$, respectively. Let $H_{12}$, $H_{13}$, and $H_{23}$ be the three unique translates of $H$ contained in $S'_1 \cap S'_2$, $S'_1 \cap S'_3$, and $S'_2 \cap S'_3$, respectively. Then any two of the three translates of $H$, say $H_{12}$ and $H_{13}$, cover the hexagon $H'$. It follows by symmetry that two points (the centers of $H_{12}$ and $H_{13}$) are enough to pierce all members of $F$. This completes the proof of Theorem 6.

6 Conclusion

We believe that our bounds in Lemma 5 and Lemma 4 are not tight. We have the following conjectures:
Figure 5: (a) A centrally symmetric convex hexagon $H$ is the intersection of three strips $S_1$, $S_2$, and $S_3$. (b) The centers of all translates of $H$ in $\mathcal{F}$ are contained in the shaded hexagon $H'$ that is the intersection of three strips $S'_1$, $S'_2$, and $S'_3$; the shaded hexagon $H'$ is covered by any two of the three translates of $H$: $H_{12} \subseteq S'_1 \cap S'_2$, $H_{13} \subseteq S'_1 \cap S'_3$, and $H_{23} \subseteq S'_2 \cap S'_3$. $H_{12}$ is shown in bold lines.

Conjecture 1. Let $\mathcal{F}$ be a family of translates of a centrally symmetric convex body $S$ in the plane. Then $\tau(\mathcal{F}) \leq |\mathcal{F}|/|S|$.

Conjecture 2. Let $\mathcal{F}$ be a family of translates of a centrally symmetric convex body $S$ in the plane. Then $\nu(\mathcal{F}) \geq \frac{1}{4} \cdot |\mathcal{F}|/|S|$.

If both conjectures were to hold (note that they hold for the special cases when $S$ is a parallelogram or a centrally symmetric convex hexagon since $\theta_L(S) = \delta_L(S) = 1$ in such cases), then we would have an alternative proof of essentially the same bound $\tau(\mathcal{F}) \leq 4 \cdot \nu(\mathcal{F})$ as in Theorem 3 for any family $\mathcal{F}$ of translates of a centrally symmetric convex body in the plane. Conjecture 2 is related to another recent conjecture [3] in the spirit of Rado [27]:

Conjecture 3 (Bereg, Dumitrescu, and Jiang [3]). For any set $S$ of (not necessary congruent) closed disks in the plane, there exists a subset $I$ of pairwise-disjoint disks such that $|I|/|\mathcal{F}| \geq \frac{1}{4}$.

Note that a disk $D$ is centrally symmetric; for any family $\mathcal{F}$ of congruent disks (i.e., translates of a disk) in the plane, $\nu(\mathcal{F}) \geq \frac{1}{4} \cdot |\mathcal{F}|/|D|$ if and only if there exists a subset $I$ of pairwise-disjoint disks such that $|I|/|\mathcal{F}| \geq \frac{1}{4}$. 

Approximation algorithms. A computational problem related to the results of this paper is finding a minimum-cardinality point set that pierces a given set of geometric objects. This problem is NP-hard even for the special case of axis-parallel unit squares in the plane [12], and it admits a polynomial-time approximation scheme for the general case of fat objects in $\mathbb{R}^d$ [8] (see also [6] for similar approximation schemes for several related problems). These approximation schemes have very high time complexities $n^{O(1/\epsilon^d)}$, and hence are impractical. Our methods for obtaining the upper bounds in Theorems 1, 2, 3, and 4 are constructive and lead to efficient constant-factor approximation algorithms for piercing a set of translates or homothets of a convex body. The approximation factors, which depend on the dimension $d$, are the multiplicative factors in the respective bounds on $\tau(\mathcal{F})$ in terms of $\nu(\mathcal{F})$ in the theorems, see also Table 1 and Table 2. For instance, Theorem 1 yields a factor-6 approximation algorithm for piercing translates of a convex body in the plane, and Theorem 4 yields a factor-216 approximation algorithm for piercing homothets of a convex body in 3-space.
Note. After completion of this work and shortly before journal submission, we learned that very recently, Naszódi and Taschuk [25] independently obtained some results similar in nature to our Theorems 4 and 5. There are however differences in the specific bounds:

1. They proved\[β(C) ≤ 2^d \left(\frac{2d}{d+1}\right)(d \ln d + d \ln \ln d + 5d)\]for any convex body \(C\) in \(\mathbb{R}^d\), and that \(β(C) ≤ 3^d(d \ln d + d \ln \ln d + 5d)\) for any centrally symmetric convex body \(C\) in \(\mathbb{R}^d\). Note that their upper bound for the centrally symmetric case is essentially the same as our bound \(β(C) ≤ \frac{2d}{d+1} \theta_T(C)\) by Theorem 4 and Lemma 2. Their upper bound for the general case, however, is weaker than our bound \(β(C) ≤ 2^d \left(\frac{2d}{d+1}\right)^{\frac{d}{d}+1} \theta_T(C)\), also by Theorem 4 and Lemma 2. By Stirling’s formula, \(\left(\frac{2d}{d+1}\right)^{d+1} = Θ(\frac{8d}{\sqrt{d}})\) in their bound with the factor \(2^d \left(\frac{2d}{d+1}\right)^{d+1} = Θ(\frac{8d}{\sqrt{d}})\) in our bound.

2. They also derived the following lower bound: for sufficiently large \(d\), there is a convex body \(C\) in \(\mathbb{R}^d\) such that \(α(C) ≥ \frac{1}{2} (1.058)^d\). This lower bound is analogous to our exponential lower bound in Theorem 5 if \(C\) is the unit ball \(B^d\) in \(\mathbb{R}^d\), then \(α(C) ≥ 2^d(0.599 ± o(1))^d = (1.51)^d\) as \(d → ∞\). Recall that our lower bound for the unit ball \(B^d\) follows from a general lower bound for any convex body \(C\) in \(\mathbb{R}^d\), namely, \(α(C) ≥ \frac{\theta_T(C)}{\theta_T(C)}\). A comparison shows that their lower bound is both weaker and less general than ours.

References

[1] N. Alon, Piercing \(d\)-intervals, Discrete and Computational Geometry, 19 (1998), 333–334.

[2] N. Alon and D.J. Kleitman, Piercing convex sets and the Hadwiger-Debrunner \((p, q)\)-problem, Advances in Mathematics, 96 (1992), 103–112.

[3] S. Bereg, A. Dumitrescu, and M. Jiang, Maximum area independent sets in disk intersection graphs, International Journal of Computational Geometry & Applications, to appear.

[4] S. Bereg, A. Dumitrescu, and M. Jiang, On covering problems of Rado, Algorithmica, doi:10.1007/s00453-009-9298-z, to appear.

[5] P. Braß, W. Moser, and J. Pach, Research Problems in Discrete Geometry, Springer, New York, 2005.

[6] P. Carmi, M.J. Katz, and N. Lev-Tov, Polynomial-time approximation schemes for piercing and covering with applications in wireless networks, Computational Geometry: Theory and Applications, 39 (2008), 209–218.

[7] G.D. Chakerian and S.K. Stein, Some intersection properties of convex bodies, Proceedings of the American Mathematical Society, 18 (1967), 109–112.

[8] T. Chan, Polynomial-time approximation schemes for packing and piercing fat objects, Journal of Algorithms, 46 (2003), 178–189.

[9] L. Danzer, Zur Lösung des Gallaischen Problems über Kreisscheiben in der Euklidischen Ebene, Studia Scientiarum Mathematicarum Hungarica, 21 (1986), 111–134.

\[Naszódi and Taschuk [25] used the terms \(d \log d + \log \log d + 5d\) instead of \(d \ln d + d \ln \ln d + 5d\) throughout their paper, which are clearly misprints. Recall that \(\theta_T(C) < d \ln d + d \ln \ln d + 5d\) for any convex body \(C\) in \(\mathbb{R}^d\) [28].\]
[10] L. Danzer, B. Grünbaum, and V. Klee, Helly’s theorem and its relatives, in: *Proceedings of Symposia in Pure Mathematics*, vol. 7, American Mathematical Society, 1963, pp. 101–181.

[11] D.G. Fon-Der-Flaass and A.V. Kostochka, Covering boxes by points, *Discrete Mathematics*, 120 (1993), 269–275.

[12] R.J. Fowler, M.S. Paterson, and S.L. Tanimoto, Optimal packing and covering in the plane are NP-complete, *Information Processing Letters*, 12 (1981), 133–137.

[13] E. Friedman, Circles covering circles, http://www.stetson.edu/~efriedma/circovcir/.

[14] B. Grünbaum, On intersections of similar sets, *Portugaliae Mathematica*, 18 (1959), 155–164.

[15] A. Gyárfás and J. Lehel, Covering and coloring problems for relatives of intervals, *Discrete Mathematics*, 55 (1985), 167–180.

[16] G.A. Kabatjanskiĭ and V.I. Levenšteĭn, Bounds for packing on a sphere and in space (in Russian), *Problemnye Peredači Informacii*, 14 (1978), 3–25. English translation: *Problems of Information Transmission*, 14 (1978), 1–17.

[17] T. Kaiser, Transversals of $d$-intervals, *Discrete and Computational Geometry*, 18 (1997), 195–203.

[18] R.N. Karasev, Transversals for families of translates of a two-dimensional convex compact set, *Discrete and Computational Geometry*, 24 (2000), 345–353.

[19] G. Károlyi, On point covers of parallel rectangles, *Periodica Mathematica Hungarica*, 23 (1991), 105–107.

[20] G. Károlyi and G. Tardos, On point covers of multiple intervals and axis-parallel rectangles, *Combinatorica*, 16 (1996), 213–222.

[21] S.-J. Kim, K. Nakprasit, M.J. Pelsmajer, and J. Skokan, Transversal numbers of translates of a convex body, *Discrete Mathematics*, 306 (2006), 2166–2173.

[22] W. Kuperberg, An inequality linking packing and covering densities of plane convex bodies, *Geometriae Dedicata*, 87 (1987), 59–66.

[23] J. Kynčl and M. Tancer, The maximum piercing number for some classes of convex sets with $(4, 3)$-property, *Electronic Journal of Combinatorics*, 15 (2008), #R27.

[24] J. Matoušek, Lower bounds on the transversal numbers of $d$-intervals, *Discrete and Computational Geometry*, 26 (2001), 283–287.

[25] M. Naszódi and S. Taschuk, On the transversal number and VC-dimension of families of positive homothets of a convex body, *Discrete Mathematics*, doi:10.1016/j.disc.2009.07.030, to appear.

[26] F. Nielsen, On point covers of $c$-oriented polygons, *Theoretical Computer Science*, 263 (2001), 17–29.

[27] R. Rado, Some covering theorems (I), *Proceedings of the London Mathematical Society*, 51 (1949), 241–264.

[28] C.A. Rogers, A note on coverings, *Mathematika*, 4 (1957), 1–6.

[29] W.M. Schmidt, On the Minkowski-Hlawka theorem, *Illinois Journal of Mathematics*, 7 (1963), 18–23.
A Lower bound for translates of a triangle

In this section we prove the lower bound $\alpha_1(T) \geq 3$ for any triangle $T$ by a very simple\footnote{Simpler than the previous constructions \cite{7,23} that give the same lower bound.} construction:

**Proposition 1.** For any triangle $T$, there exists a family $\mathcal{F}$ of nine translates of $T$ such that $\nu(\mathcal{F}) = 1$ and $\tau(\mathcal{F}) = 3$.

![Figure 6: Three pairwise-tangent translates $A$, $B$, and $C$ of a triangle $T$. The dashed triangle is $B_C$.](image)

**Proof.** We refer to Figure 6. Let $A$, $B$, and $C$ be three translates of $T$ that are pairwise-tangent with intersections at three vertices $a$, $b$, and $c$. We obtain six more translates of $T$ as follows. Translate a copy of $T$ for a short distance $\epsilon$ from $B$ toward $C$, and let $B_C$ be the resulting translate. Similarly obtain $A_B$, $A_C$, $B_A$, $C_A$, and $C_B$. Let $\mathcal{F}$ be the family of nine translates $A$, $B$, $C$, $A_B$, $A_C$, $B_C$, $B_A$, $C_A$, and $C_B$. It is clear that any two members of $\mathcal{F}$ intersect. We next show that three points are necessary to pierce all members of $\mathcal{F}$. Suppose for contradiction that two points are enough. Then one of the two points must be $a$, $b$, or $c$ since $A$, $B$, and $C$ are pairwise-tangent. Assume that $a$ is one of the two points. Then the other point must intersect the three translates $A$, $B_A$, and $C_A$ that do not contain the point $a$. But these three translates do not have a common point when $\epsilon$ is sufficiently small. We have reached a contradiction.

By repeating the configuration of nine translates in Proposition 1, we can obtain a family $\mathcal{F}$ of $9 \nu(\mathcal{F})$ translates of a triangle such that $\tau(\mathcal{F}) = 3 \cdot \nu(\mathcal{F})$ for any $\nu(\mathcal{F}) \geq 1$. 

---

[70] E. Smith, An improvement of an inequality linking packing and covering densities in 3-space, *Geometriae Dedicata*, **117** (2006), 11–18.

[71] G. Tardos, Transversals of 2-intervals, a topological approach, *Combinatorica*, **15** (1995), 123–134.

[72] R. Wenger, Helly-type theorems and geometric transversals, in: *Handbook of Discrete and Computational Geometry*, 2nd edition, CRC Press, 2004, pp. 73–96.