Mixing strategies in the matrix game

Victor N Assaul¹ and Vladimir K Lashchenov²

¹ Applied Mathematics Department, State University of Aerospace Instrumentation, B. Morskaya st. 67, St. Petersburg 190121, Russia
² Applied Mathematics Department, Admiral S.O. Makarov State University of the Sea and River Fleet, Dvinskaya St.5/7, St. Petersburg, 198035, Russia

E-mail: vicvic21@yandex.ru, vlasch_18@mail.ru

Abstract. The article considers the sequential strategies mixing in the matrix game as an algorithm for finding the optimal player’s strategy. Strategies are mixed in such a way as to obtain a row of the payment matrix with constant elements. As a result, such a row is deleted as dominated, or determines the value of a game. The games with two moves of the second player, which are solved without the use of the simplex method, are analyzed at each stage.

Let us consider a matrix game \( n \times n \) with payment matrix \( A \):

\[
A = \begin{pmatrix}
a_{11} & a_{12} & \ldots & a_{1n} \\
a_{21} & a_{22} & \ldots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
\end{pmatrix}
\]

Optimal strategies for players are usually sought by the Danzig method as solutions of a pair of dual problems [1, 2]. In this case, the simplex method procedure is used [3, 4].

Each of the players, having \( n \) pure strategies, can mix them, that is, build their convex combinations:

\[
\alpha_i a_i = \sum_{k=1}^{k} \alpha_i a_i \quad \text{or} \quad \beta_i b_i = \sum_{l=1}^{l} \beta_i b_i ,
\]

where \( \alpha_i, \beta_i \in [0, 1] \), \( \sum_{i=1}^{i} \alpha_i = \sum_{i=1}^{i} \beta_i = 1 \), \( \alpha_i \) — rows, \( \beta_i \) — payment matrix columns, \( k = 2 \ldots n; l = 2 \ldots n \).

Thus, the number of possible moves for each player increases unlimitedly. Of course, players are interested in generating new rows and columns, not worse than those available. This means that if, for example, a new row dominates one of the \( n \) source ones, then the latter can be deleted from the payment matrix (like all additional ones). In this case, the number of net strategies, together with the dimension of the payment matrix, will decrease by one. This is also true for the moves of the second player.

However, such a “joint” dominance does not exhaust all possible cases of dominance if we expand this concept to delete inactive strategies in the optimal solution.
In the case considered above, a convex combination of several rows dominated one row. Let us now consider the case when a row dominates a convex combination of several rows (in the case of columns, a convex combination of several columns dominates one of the columns).

In each of these cases, it can be argued that one of the elements of the convex combination can be deleted. This is a case of "weak" dominance.

To prove the statement, we consider the payoff functional of the first player, that is, the mathematical expectation of his winnings. For the sake of brevity, we restrict ourselves to case $n = 3$. Let the payment matrix have the form:

$$
A = \begin{pmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{pmatrix}
$$

Let $X = (x_1, x_2, 1 - x_1 - x_2)$ and $Y = (y_1, y_2, 1 - y_1 - y_2)$ be the probabilities with which the first and the second players make their moves. Then the mathematical expectation of a win for the first player is written in the form of a bilinear form:

$$
M(X, Y) = A_{11}x_1y_1 + A_{12}x_1y_2 + A_{13}x_1 + A_{21}x_2y_1 + A_{22}x_2y_2 + A_{23}x_2 + A_{31}y_1 + A_{32}y_2 + A_{33},
$$

(2)

where the coefficients of the bilinear form are found from the following formulas:

$$
A_j = a_j + a_{33} - a_{3j} - a_{j3}, A_{3j} = a_{3j} - a_{33}, A_{33} = a_{33}, i, j = 1, 2
$$

(3)

and can be reduced to matrix:

$$
\tilde{A} = \begin{pmatrix}
A_{11} & A_{12} & A_{13} \\
A_{21} & A_{22} & A_{23} \\
A_{31} & A_{32} & A_{33}
\end{pmatrix}
$$

(4)

We rewrite (2) in form:

$$
M(X, Y) = y_1(A_{11}x_1 + A_{12}x_2 + A_{13}) + y_2(A_{21}x_1 + A_{22}x_2 + A_{23}) + A_{33} - A_{3j} - A_{3j} + A_{33},
$$

Since $y_1$ and $y_2$ are non-negative, the first player who wants to maximize $M(x, y)$, must choose $x_1$ and $x_2$ so that the expressions in parentheses should be non-negative and sum $A_{33} - A_{3j} - A_{3j} + A_{33}$ as large as possible. Now we rewrite (2) in the form:

$$
M(X, Y) = x_1(A_{11}y_1 + A_{12}y_2 + A_{13}) + x_2(A_{21}y_1 + A_{22}y_2 + A_{23}) + A_{3j}y_1 + A_{3j}y_2 + A_{33}.
$$

In this line of thought, we conclude that in the interests of the second player, the expressions in parentheses should be non-negative, and the amount should be as small as possible. These conditions can be written as a pair of dual tasks:

$$
A_{3j} - A_{3j} \rightarrow \max \quad A_{3j}y_1 + A_{32}y_2 + A_{33} \rightarrow \min
$$

$$
-A_{11}x_1 - A_{21}x_2 \leq A_{3j} \quad y_1 \geq 0
$$
Suppose that the elements of the third row of payment matrix A are equal to the linear combination of the first two rows:

\[ a_{3i} = \alpha a_{1i} + (1 - \alpha) a_{2i}, \quad i = 1, 2, 3. \]  

(6)

Obviously, the third line can be deleted because it is dominated by a convex combination of the first two rows. We show that one of the first two rows can also be deleted.

Taking into account formulas (3) for matrix A elements, we obtain:

\begin{align*}
A_{11} &= (1 - \alpha)(a_{11} - a_{13} - a_{21} + a_{23}); \\
A_{12} &= (1 - \alpha)(a_{12} - a_{13} - a_{22} + a_{23}); \\
A_{13} &= (1 - \alpha)(a_{13} - a_{23}); \\
A_{21} &= -\alpha(a_{11} - a_{13} - a_{21} + a_{23}); \\
A_{22} &= -\alpha(a_{12} - a_{13} - a_{22} + a_{23}); \\
A_{23} &= -\alpha(a_{13} - a_{23}); \\
A_{31} &= \alpha(a_{11} - a_{13}) + (1 - \alpha)(a_{21} - a_{23}); \\
A_{32} &= \alpha(a_{12} - a_{13}) + (1 - \alpha)(a_{22} - a_{23}); \\
A_{33} &= \alpha a_{13} + (1 - \alpha)a_{23};
\end{align*}

It is easy to see that the first two rows of matrix \( \tilde{A} \) are proportional, therefore, taking into account the fact that the coefficient of proportionality is negative, the third and the fourth inequalities in dual problem (5) become opposite, that is, they can only be fulfilled as equalities. This means that one of them can simply be removed as redundant. Then the corresponding variable of the original problem (x1 or x2) will be equal to zero, and the remaining one will be positive, since the corresponding inequality of the dual problem is fulfilled as equality.

If condition (6) is replaced by the inequality:

\[ a_{3i} \geq \alpha a_{1i} + (1 - \alpha) a_{2i}, \quad i = 1, 2, 3, \]

the third row of the payment matrix will increase, and this cannot lead to an increase in the probabilities with which the other two rows remain unchanged, and one of them will still have a zero component in the optimal plan.

Let us explain the above with an example. Consider the payment matrix:

\[
A = \begin{pmatrix}
214 \\
470 \\
543
\end{pmatrix}.
\]

It is easy to verify that there is no explicit dominance of rows and columns, and there is no joint dominance. We will check the weak dominance starting with the columns. The first and the third columns cannot dominate the second, since their first elements are larger than the first element of the second column, and the second elements are smaller than the second element of the second column. The same can be said about the first and the second columns, if we compare their first and second elements with the corresponding elements of the third column. Finally, a weak dominance of the second and the third columns over the first is impossible, since the first element in the third row is the largest. At the same time, the third row at \( \frac{1}{2} \leq \alpha \leq \frac{3}{4} \) dominates weakly the first two, and one of these rows can be deleted. It is easy to establish by direct verification that this is the second line. Then we come to the payment matrix:
in which the first column dominates the second, so it can be deleted. For the remaining matrix:

\[ A_2 = \begin{pmatrix} 14 \\ 43 \end{pmatrix} \]

it’s easy to find the optimal strategy and the value of a game. Taking into account deleted rows and columns we get: \( P^*(0, \frac{1}{4}, 0, \frac{3}{4}), Q^*(0, \frac{1}{4}, \frac{3}{4}), \nu = \frac{13}{4} \).

Thus, using three forms of dominance, you can get rid of all inactive strategies of the first and second players. True, with an increase in the size of the payment matrix, this process becomes too burdensome.

Consider the following property of the payment matrix. If the payment matrix contains a constant row (column), then this row (column) is not mixed with other rows (columns). In other words, such a row (column) is either deleted as dominated (dominant) or determines the value of a game. Consider the 3x3 payment matrix and assume that the third row in it is constant:

\[ A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ c & c & c \end{pmatrix} \]    \hspace{1cm}  (7)

The matrix of a bilinear form corresponding to the payoff function of the first player then takes the form:

\[ \tilde{A} = \begin{pmatrix} a_{11} - a_{13} & a_{13} - a_{11} & -c \\ a_{21} - a_{23} & a_{23} - a_{21} & -c \\ 0 & 0 & c \end{pmatrix} \]

Consider the conditions of the first problem from a pair of dual problems (5):

\[ (a_{13} - c)x_1 + (a_{23} - c)x_2 \rightarrow \max \]
\[ (a_{13} - a_{11})x_1 + (a_{23} - a_{21})x_2 \leq 0 \]
\[ (a_{13} - a_{12})x_1 + (a_{23} - a_{22})x_2 \leq 0 \]
\[ x_1 \geq 0 \]
\[ x_2 \geq 0 \]

The tolerance region of variables in problem (8) is limited by axes \( x_1 \) and \( x_2 \) in the first quarter and by two straight lines passing through the origin. If we take into account the additional restriction \( x_1 + x_2 \leq 1 \), it becomes clear that this region is either a triangle with a vertex at the origin and with sides described by equations

\[ (a_{13} - a_{11})x_1 + (a_{23} - a_{21})x_2 = 0 \]
\[ (a_{13} - a_{12})x_1 + (a_{23} - a_{22})x_2 = 0 \]

or the origin point. In the first case the objective function reaches a maximum at one of the vertices of the specified triangle, that is, at one of the points on the line segment \( x_1 + x_2 = 1 \) (then \( x_1 = 1 - x_2 \) or at the origin (then \( x_1 = x_2 = 0 \)). In the second case also \( x_1 = x_2 = 0 \). In the first
of the described options, the third row of the payment matrix is deleted, in the other two first rows are deleted, that is, the value of a game is equal to $c$. It proves the formulated property.

Based on the above, each player should strive to mix their pure strategies in such a way as to get a constant row or column. Consider an algorithm for sequential strategies mixing aimed at obtaining a constant row in the payment matrix. For mixing, we select the elements of two columns of the payment matrix. Since the choice of columns is the prerogative of the second player, we choose those columns that give the lowest value of a game. From the elements of this column, we select 2x2 submatrices that allow strategies mixing, that is, for which there is a convex combination giving a row with constant elements. There should be $n-1$ such submatrices, where $n$ is the number of rows in the payment matrix. Then we get a matrix with $n-1$ row and two identical columns.

Repeating the specified algorithm for the resulting matrix, we arrive at a new payment matrix with an even smaller number of rows and with three constant columns. At the end of the procedure, you get one constant row that determines value of a game. At each step, dominant columns and dominated rows should be deleted if any. We illustrate the proposed algorithm with examples.

Let the payment matrix $A$ be:

$$A = \begin{pmatrix} 261 \\ 524 \\ 315 \end{pmatrix}$$

Consider three submatrixes of matrix $A$:

$$A_1 = \begin{pmatrix} 26 \\ 52 \\ 31 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 21 \\ 54 \\ 35 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 61 \\ 24 \\ 15 \end{pmatrix}$$

In the first of them, the second row dominates the third and therefore the third row is deleted. The value of a game of the remaining square matrix is:

$$v_1 = \frac{2 \cdot 2 - 5 \cdot 6}{(2 + 2) - (5 + 6)} = \frac{26}{7}.$$ 

In the second submatrix, the first row is deleted, and the value of a game is $v_2 = \frac{13}{3}$. In matrix $A_3$ at $\frac{1}{5} \leq \alpha \leq \frac{1}{4}$ the combination of the first and third rows $\alpha \cdot a_1 + (1-\alpha) \cdot a_3$ dominates the second row, which is deleted, in this case $v_3 = \frac{29}{9}$. We choose the third submatrix and compose three second-order matrixes from it:

$$\begin{pmatrix} 61 \\ 24 \end{pmatrix}, \begin{pmatrix} 61 \\ 15 \end{pmatrix}, \begin{pmatrix} 24 \\ 15 \end{pmatrix}.$$

In the third of these matrixes, the rows do not mix, because segments $[1,2]$ and $[4,5]$ do not intersect, and the first two admit mixing, leading to the constant row: $\frac{2}{7}a_1 + \frac{5}{7}a_2$, for the first and $\frac{4}{9}a_1 + \frac{5}{9}a_3$ for the third matrixes. We construct the corresponding row combinations for the original matrix $A$:

$$T = \frac{2}{7}a_1 + \frac{5}{7}a_2 = \left(\frac{29}{7}, \frac{22}{7}, \frac{22}{7}\right) = \left(\frac{261}{63}, \frac{198}{63}, \frac{198}{63}\right)$$
which are reduced to the common denominator for convenience of calculations.

Mixing the first two rows of the resulting matrix, we get a matrix with a constant row 67/21:

Thus, the value of a game is 67/21, and the first player’s optimal strategy

Consider the 4x4 payment matrix and show how the proposed method is implemented in this case.

Having considered all 6 possible options for the 4x2 submatrixes of matrix A, we choose one for which the value of a game is minimal. In this case, it is submatrix A1:

Row mixing allow 4 submatrixes of matrix A1

\[
b_1 = \frac{1}{2} a_1 + \frac{1}{2} a_2 = \left(\frac{9}{2}, \frac{3}{2}, \frac{4}{2}\right),  \quad b_2 = \frac{4}{7} a_1 + \frac{3}{7} a_4 = \left(\frac{12}{7}, \frac{13}{7}, \frac{15}{7}\right),  \\
b_3 = \frac{1}{4} a_2 + \frac{3}{4} a_3 = \left(\frac{9}{4}, \frac{17}{4}, \frac{10}{4}\right),  \quad b_4 = \frac{4}{5} a_1 + \frac{1}{5} a_4 = \left(\frac{4}{5}, \frac{23}{5}, \frac{13}{5}\right).
\]

\(a_i\) - the rows of matrix A, \(i = 1 \ldots 4\), \(b_j\) - the rows of matrix B obtained by mixing strategies, \(j = 1 \ldots 4\).

\[
B = \begin{bmatrix}
630 & 210 & 280 & 280 \\
240 & 260 & 300 & 300 \\
315 & 595 & 350 & 350 \\
112 & 644 & 364 & 364
\end{bmatrix}
\]

The second row of matrix B can be deleted as dominated.

In the resulting matrix D, we consider 3x2 submatrixes.
The following submatrixes of matrix D allow mixing:

\[
D = \frac{1}{20} \begin{bmatrix}
90 & 30 & 40 & 40 \\
45 & 85 & 50 & 50 \\
16 & 92 & 52 & 52
\end{bmatrix}.
\]

The following submatrixes of matrix D allow mixing:

\[
\begin{bmatrix}
90 & 30 \\
45 & 85 \\
16 & 92
\end{bmatrix}
\]

\[
\begin{bmatrix}
90 & 40 \\
45 & 50 \\
16 & 52
\end{bmatrix}
\]

\[
\begin{bmatrix}
90 & 40 \\
45 & 50 \\
16 & 52
\end{bmatrix}
\]

\[
\begin{bmatrix}
90 & 40 \\
45 & 50 \\
16 & 52
\end{bmatrix}
\]

For further strategies mixing, we choose the second of them (D₂) since it has the lowest value of a game. Strategies mixing is possible for the following submatrixes of matrix D₂:

\[
\begin{bmatrix}
90 & 40 \\
45 & 50 \\
16 & 52
\end{bmatrix}
\]

Denote the rows of matrix D by \(d_i\). Then, as a result of mixing, we can obtain two new rows:

\[
e_1 = \frac{1}{11}d_1 + \frac{10}{11}d_2 = \frac{1}{220}(540,880,540,540), \quad e_2 = \frac{18}{43}d_1 + \frac{25}{43}d_3 = \frac{1}{860}(2020,2840,2020,2020).
\]

We shall reduce these strategies into one matrix:

\[
E = \frac{1}{9460} \begin{bmatrix}
23220 & 37840 & 23220 & 23220 \\
22220 & 31240 & 22220 & 22220
\end{bmatrix}.
\]

In matrix E, the dominant second column and the dominated second row can be deleted. As a result, we get a constant row with a value of 23220/9460 = 27/11. This is the price of the game. Optimal strategies are restored bearing in mind the history of the row mixing:

\[
P^* = \frac{1}{11}d_1 + \frac{10}{11}d_2 = \frac{1}{11}(\frac{1}{2}a_1 + \frac{1}{2}a_2) + \frac{10}{11}(\frac{3}{4}a_3) = \frac{1}{22}a_1 + \frac{6}{22}a_2 + \frac{15}{22}a_3 = (\frac{1}{22}, \frac{6}{22}, \frac{15}{22}).
\]

It is easy now to find the optimal strategies for the second player:

\[
Q^* = (\frac{2}{11}, 0, \frac{2}{11}, \frac{7}{11}).
\]

The examples given illustrate the possibilities of the method proposed in this paper for reducing the payment matrix to a constant-row matrix by strategies mixing. In this case, at each step, games with an \(n \times 2\) matrix are selected in accordance with the minimax principle: for sequential mixing, row elements appearing in two columns with a minimum game price of the second player are selected. The solution to such problems is elementary and does not require the use of the simplex method. In addition, the proposed search scheme for the optimal strategy reveals the structure of a multidimensional problem as a linear combination of two-dimensional problems solutions.

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