CLASSIFICATION OF PRE-JORDAN ALGEBRAS AND ROTA-BAXTER OPERATORS ON JORDAN ALGEBRAS IN LOW DIMENSIONS

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Abstract. This paper is devoted to the classification of complex pre-Jordan algebras in the sense of isomorphisms in dimensions ≤ 3. All Rota-Baxter operators on complex Jordan algebras in dimensions ≤ 3 and the induced pre-Jordan algebras are also presented.

Key words. Jordan algebra, pre-Jordan algebra, Rota-Baxter operator, Jordan Yang-Baxter equation.

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1. Introduction. Jordan algebras are a class of nonassociative algebras introduced in the context of axiomatic quantum mechanics ([1]). Such structures were widely studied and then applied in a lot of fields in mathematics such as differential geometry ([2], [3], [4], [5], [6]), Lie theory ([7], [8]), analysis ([4], [9]) and some fields in physics like quantum mechanics and field theory ([10]).

Pre-Jordan algebras were introduced in [11] as a class of nonassociative algebras which are closely related to Jordan algebras. There are several motivations to study such structures. For example, it is the algebraic structure behind the Jordan Yang-Baxter equation which is an analogue of the classical Yang-Baxter equation in a Lie algebra ([12], [13]). And the solutions of Jordan Yang-Baxter equation have connection with the symplectic structures on pseudo-euclidean Jordan algebras ([14]). Given a pseudo-euclidean unital Jordan algebra, we can apply the Tits Kantor Koecher construction (TKK construction) to obtain a quadratic Lie algebra ([8], [15], [16]), which is particularly important in conformal field theory because it admits a Sugawara construction ([17]). They can be regarded as the “opposite” of pre-Lie algebras ([18]) which is analogous to the Jordan algebras as the “opposite” of Lie algebras and therefore there is a close relationship with dendriform algebras ([19]) which is the “opposite” of the relationship between pre-Lie algebras and dendriform algebras (see the commutative diagram (4.15) in [11]). The operad of pre-Jordan algebras is the bisuccessor of the operad of Jordan algebras ([20]), which coincides with the fact that a Rota-Baxter operator (of weight 0) on a Jordan algebra induces a pre-Jordan algebra ([11]).

Unfortunately, we have known little on pre-Jordan algebras since we have had few explicit examples. Even except for associative algebras which automatically satisfy the pre-Jordan axioms, we have not known an explicit example of pre-Jordan algebras which are not associative. Simultaneously, the classification in the sense of isomorphism is always one of the most important problems in studying an algebraic system. Hence it is necessary and important to classify pre-Jordan algebras in low dimensions, which is the first motivation and aim of this paper. A direct consequence is that not all of 2-dimensional pre-Jordan algebras are associative.

The equations involving structural constants of both Jordan and pre-Jordan algebras are “cubic”. In general, it is difficult to give all solutions of these equations as well as give the corresponding classification in the sense of isomorphism, even more for pre-Jordan algebras since they involve two identities. An important observation is that the two identities defining pre-Jordan algebras hold if and only if one of them holds and the anticommutators are Jordan algebras (see Corollary 2.6). So for a fixed Jordan algebra, it is enough to solve the equations involving this identity in order to give all compatible pre-Jordan algebras on this Jordan algebra. Therefore there is a “strategy” to classify pre-Jordan algebras based on the classification...
of Jordan algebras of the same dimensions, which is available in low dimensions.

Besides, there is a closely related topic, namely Rota-Baxter operators on Jordan algebras. Rota-Baxter operators (on associative algebras) were introduced by G. Baxter ([21]) in 1960 and then they play important roles in various areas of mathematics and mathematical physics ([22], [23], [24]). The Rota-Baxter operators on Jordan algebras also play the similar roles and hence it is also necessary and important to give Rota-Baxter operators on Jordan algebras explicitly. Moreover, since a Rota-Baxter operator (of weight 0) on a Jordan algebra induces a (not necessarily compatible) pre-Jordan algebra, it is natural to ask whether all pre-Jordan algebras are induced by Rota-Baxter operators on Jordan algebras and if the answer is no, which kinds of pre-Jordan algebras can be obtained this way.

In this paper, we commence to classify complex pre-Jordan algebras and give Rota-Baxter operators (of weight zero) on complex Jordan algebras in dimensions \( \leq 3 \). For the former, we use the aforementioned strategy based on the known classification of complex Jordan algebras up to dimension 3 ([25], [26]) and for the latter, we find all Rota-Baxter operators on these classified Jordan algebras and the induced pre-Jordan algebras. Comparing these two sets of pre-Jordan algebras, we find that the pre-Jordan algebras from the latter only “occupy” a small part of the former (see Theorem 5.11 and Theorem 5.32) and thus we answer the above problem. Both these results on pre-Jordan algebras and Rota-Baxter operators on Jordan algebras can be regarded as a guide for a further development.

The paper is organized as follows. In Section 2, we recall some necessary notions and basic results on pre-Jordan algebras and then introduce the strategy to classify compatible pre-Jordan algebras on Jordan algebras. In Section 3, the classification of complex pre-Jordan algebras in dimensions 1 and 2 is given from the Jordan algebras we answer the above problem. Both these results on pre-Jordan algebras and Rota-Baxter operators on Jordan algebras can be regarded as a guide for a further development.

The paper is organized as follows. In Section 2, we recall some necessary notions and basic results on pre-Jordan algebras and then introduce the strategy to classify compatible pre-Jordan algebras on Jordan algebras. In Section 3, the classification of complex pre-Jordan algebras in dimensions 1 and 2 is given through the strategy in the previous section. In Section 4, the classification of complex pre-Jordan algebras in dimension 3 is given. In Section 5, we give all Rota-Baxter operators (of weight 0) on the Jordan algebras in dimensions \( \leq 3 \) and the induced pre-Jordan algebras.

Throughout the paper, all vector spaces and algebras are over the complex field \( \mathbb{C} \). We use the following notations.

Let \( A \) be an \( N \)-dimensional vector space with a bilinear multiplication denoted by \( (x, y) \mapsto x \circ y \). Let \( \{e_1, \cdots, e_N\} \) be a basis of \( A \). Set

\[
e_i \circ e_j = \sum_{k=1}^{N} a_{ij}^k e_k, \quad i, j = 1, \cdots, N.
\]

Define the \textbf{formal characteristic matrix} associated to \((A, \circ)\) under the basis \( \{e_i\} \) as

\[
\mathfrak{M}(A) = \begin{pmatrix}
e_1 \circ e_1 & \cdots & e_1 \circ e_N \\
\cdots & \cdots & \cdots \\
e_N \circ e_1 & \cdots & e_N \circ e_N
\end{pmatrix} = \begin{pmatrix}
\sum_{k=1}^{N} a_{11}^k e_k & \cdots & \sum_{k=1}^{N} a_{1N}^k e_k \\
\cdots & \cdots & \cdots \\
\sum_{k=1}^{N} a_{N1}^k e_k & \cdots & \sum_{k=1}^{N} a_{NN}^k e_k
\end{pmatrix}.
\]

Obviously \((A, \circ)\) is exactly presented by \( \mathfrak{M}(A) \). In this paper, we use \( \mathfrak{M}(A) \) to denote \((A, \circ)\). The constants \( a_{ij}^k \) are called the \textbf{structural constants} of \((A, \circ)\) under the basis \( \{e_i\} \).

2. \textbf{A strategy on the classification of pre-Jordan algebras.}

\textbf{Definition 2.1.} A vector space \( J \) with a bilinear multiplication \( \circ : J \times J \to J \) denoted by \((x, y) \mapsto x \circ y \) is called a \textbf{Jordan algebra} if the following identities hold:

\[(x \circ y) \circ z = (x \circ z) \circ y,\]

\[(x \circ y) \circ (y \circ x) = (y \circ x) \circ (x \circ y),\]

for all \( x, y \in J \). A \textbf{homomorphism} from a Jordan algebra \((J_1, \circ_1)\) to another Jordan algebra \((J_2, \circ_2)\) is a linear map \( \phi : J_1 \to J_2 \) satisfying

\[
\phi(x \circ_1 y) = \phi(x) \circ_2 \phi(y), \quad \forall x, y \in J_1.
\]
A bijective homomorphism between two Jordan algebras is called an **isomorphism**. An isomorphism from a Jordan algebra \((J, \circ)\) to itself is called an **automorphism**. The group of all automorphisms of a Jordan algebra \((J, \circ)\) is denoted by Aut\((J, \circ)\).

**Remark 2.2.** When the characteristic of the base field is neither 2 nor 3, it was pointed out in [27] that for a commutative bilinear multiplication \(\circ : J \times J \to J\), Eq. (2.4) is equivalent to the following identity

\[
(x \circ y, u, z) + (y \circ z, u, x) + (z \circ x, u, y) = 0, \quad \forall x, y, z, u \in J,
\]

where \((x, y, z)_\circ = (x \circ y) \circ z - x \circ (y \circ z)\) is the **associator** of \((J, \circ)\).

**Definition 2.3.** ([11]) A vector space \(A\) with a bilinear multiplication \(\cdot : A \times A \to A\) denoted by \((x, y) \mapsto x \cdot y\) is called a **pre-Jordan algebra** if the following identities hold:

\[
\begin{align*}
(x \circ y) \cdot (z \cdot u) + (y \circ z) \cdot (x \cdot u) + (z \circ x) \cdot (y \cdot u) & = x \cdot (y \cdot (z \cdot u)) + z \cdot (y \cdot (x \cdot u)) + ((x \circ z) \circ y) \cdot u \\
\quad & = z \cdot ((x \circ y) \cdot u) + x \cdot ((y \circ z) \cdot u) + y \cdot ((z \circ x) \cdot u), \quad \forall x, y, z, u \in A,
\end{align*}
\]

where \(x \circ y := x \cdot y + y \cdot x\). A **homomorphism** from a pre-Jordan algebra \((A_1, \cdot_1)\) to another pre-Jordan algebra \((A_2, \cdot_2)\) is a linear map \(\phi : A_1 \to A_2\) satisfying

\[
\phi(x \cdot_1 y) = \phi(x) \cdot_2 \phi(y), \quad \forall x, y \in A_1.
\]

A bijective homomorphism between two pre-Jordan algebras is called an **isomorphism**.

**Example 2.4.** According to [11], any associative algebra is a pre-Jordan algebra. And obviously, the direct sum of two pre-Jordan algebras is a pre-Jordan algebra.

Let \(A\) be a vector space with a bilinear multiplication \(\cdot : A \times A \to A\). Let \(\circ : A \times A \to A\) be the anticommutator defined by Eq. (2.12). For any \(x, y, z, u \in J\), set

\[
\begin{align*}
F(x, y, z, u) & = (x \circ y) \cdot (z \cdot u) + (y \circ z) \cdot (x \cdot u) + (z \circ x) \cdot (y \cdot u), \\
G(x, y, z, u) & = x \cdot (y \cdot (z \cdot u)) + z \cdot (y \cdot (x \cdot u)) + ((x \circ z) \circ y) \cdot u, \\
H(x, y, z, u) & = z \cdot ((x \circ y) \cdot u) + x \cdot ((y \circ z) \cdot u) + y \cdot ((z \circ x) \cdot u), \\
P(x, y, z, u) & = F(x, y, z, u) - H(x, y, z, u), \\
Q(x, y, z, u) & = F(x, y, z, u) - G(x, y, z, u).
\end{align*}
\]

**Lemma 2.5.** The notations are as above.

(a) The following identities hold:

\[
Q(x, y, z, u) = Q(z, y, x, u),
\]

\[
(x \circ y, u, z)_\circ + (y \circ z, u, x)_\circ + (z \circ x, u, y)_\circ = -P(x, y, z, u) - Q(x, u, y, z) - Q(y, u, z, x) - Q(z, u, x, y),
\]

for all \(x, y, z, u \in A\).

(b) \((A, \cdot)\) is a pre-Jordan algebra if and only if

\[
P(x, y, z, u) = Q(x, y, z, u) = 0, \quad \forall x, y, z, u \in A.
\]

**Proof.** (a). Eq. (2.9) follows from the definition of \(Q(x, y, z, u)\) immediately. Let \(x, y, z, u \in A\). Then
we have

\[(x \circ y, u, z)_0 + (y \circ z, u, x)_0 + (z \circ x, u, y)_0\]

\[= ((x \circ y) \circ u) \circ z - (x \circ y) \circ (u \circ z) + ((y \circ z) \circ u) \circ x\]

\[= ((x \circ y) \circ u) \circ z \circ z + z \circ ((y \circ z) \circ u) + z \circ (y \cdot x)\]

\[= (x \circ y) \circ (u \cdot z) - (x \circ y) \circ (z \cdot u) - (u \circ z) \cdot (y \cdot x)\]

\[+ ((y \circ z) \circ u) \cdot x + x \cdot ((y \circ z) \cdot u) + x \cdot (u \cdot (y \cdot z)) + x \cdot (u \cdot (z \cdot y))\]

\[= (x \circ y) \cdot (u \cdot z) - (x \circ y) \cdot (z \cdot u) - (u \circ z) \cdot (y \cdot z) - (u \circ x) \cdot (z \cdot y)\]

\[+ ((z \circ x) \circ u) \cdot y + y \cdot ((z \circ x) \cdot u) + y \cdot (u \cdot (z \cdot x)) + y \cdot (u \cdot (z \cdot y))\]

\[= -F(x, y, z, u) - F(x, u, y, z) - F(z, u, x, y) - F(y, u, z, x)\]

\[+ H(x, y, z, u) + G(x, u, y, z) + G(z, u, x, y) + G(y, u, z, x)\]

\[= -P(x, y, z, u) - Q(x, u, y, z) - Q(y, u, z, x) - Q(z, u, x, y).\]

Hence Eq. (2.10) holds.

(b). It follows from Definition 2.3. □

**Corollary 2.6.** Let \( A \) be a vector space with a bilinear multiplication \( \cdot : A \times A \rightarrow A \). Let \( \circ : A \times A \rightarrow A \) be the anticommutator defined by Eq. (2.12). If \( Q(x, y, z, u) = 0 \) for all \( x, y, z, u \in A \), then \((A, \cdot)\) is a pre-Jordan algebra if and only if \((A, \circ)\) is a Jordan algebra.

**Proof.** If \((A, \cdot)\) is a pre-Jordan algebra, then \((A, \circ)\) is a Jordan algebra by Lemma 2.5 and Remark 2.2. Conversely, if \((A, \circ)\) is a Jordan algebra, then by Remark 2.2 and Eq. (2.10), we have

\[ P(x, y, z, u) + Q(y, z, u, x) + Q(z, u, x, y) + Q(u, x, y, z) = 0, \ \forall x, y, z, u \in A. \]

By assumption, we show that \( P(x, y, z, u) = 0 \) for all \( x, y, z, u \in A \). Therefore by Lemma 2.5 (b), \((A, \cdot)\) is a pre-Jordan algebra.

**Corollary 2.7.** ([11]) Let \((A, \cdot)\) be a pre-Jordan algebra. Then the anticommutator given by

\[ (2.12) \]

\[ x \circ y = x \cdot y + y \cdot x, \ \forall x, y \in A, \]

defines a Jordan algebra \((J(A), \circ)\), which is called the associated Jordan algebra of \((A, \cdot)\) and \((A, \cdot)\) is also called a compatible pre-Jordan algebra structure on the Jordan algebra \((J(A), \circ)\).

With the notations as above, let \( \{e_1, \cdots, e_N\} \) be a basis of \( A \) and \( \{e^1, \cdots, e^N\} \) be the dual basis. Set

\[ (2.13) \]

\[ Q_{klnm}^i = \langle e^i, Q(e_k, e_l, e_m, e_n) \rangle, \ i, k, l, m, n = 1, \cdots, N, \]

where \( \langle -, - \rangle \) is the usual pairing between \( A \) and its dual space \( A^* \). Obviously, Eq. (2.11) holds if and only if \( P_{klnm} = Q_{klnm}^i = 0 \) for all \( i, k, l, m, n \).

**Corollary 2.8.** Let \((J, \circ)\) be a Jordan algebra of dimension \( N \). The compatible pre-Jordan algebra structures on \( J \) are 1-1 corresponding to the common roots of a set of polynomials. More precisely, the set contains at most \( N^4(N^4-1) \) polynomials of degree at most 3 with \( N^4(N^4-1)/2 \) indeterminates.

**Proof.** Let \( \{e_1, \cdots, e_N\} \) be a basis of \( J \). Set

\[ e_i \circ e_j = \sum_{k=1}^N c_{ij}^k e_k, \ i, j = 1, \cdots, N. \]

That is, \( c_{ij}^k \) are the structural constants of the Jordan algebra \((J, \circ)\). Let \((J, \cdot)\) be a compatible pre-Jordan
algebra on \((J, \circ)\). Then by Eq. \((2.12)\), we can assume that

\[
(2.14) \quad e_i \cdot e_j = \begin{cases} 
\sum_{k=1}^{N} \left( \frac{c_{ji}^k}{2} + x_{ji}^k \right) e_k, & \text{if } i < j \\
\sum_{k=1}^{N} c_{ji}^k e_k, & \text{if } i = j \\
\sum_{k=1}^{N} \left( \frac{c_{ji}^k}{2} - x_{ji}^k \right) e_k, & \text{if } i > j,
\end{cases}
\]

where \(x_{ji}^k\) are indeterminates, \(1 \leq i < j \leq N, 1 \leq k \leq n\). Note that the number of these indeterminates is \(N^2(N-1)/2\). Furthermore, by Corollary 2.6, \((J, \cdot)\) is a pre-Jordan algebra if and only if

\[
Q_{klmn}^i = 0, \forall i, k, l, m, n = 1, \ldots, N.
\]

If we set \(x_{ij}^k = -x_{ji}^k\) for \(i > j\) and \(x_{ij}^k = 0\), we can write the equations as:

\[
Q_{klmn} = \sum_{r=1}^{N} \sum_{s=1}^{N} \left\{ \left( \frac{c_{mn}^r}{2} + x_{mn}^r \right) c_{kl}^r + \left( \frac{c_{kn}^r}{2} + x_{kn}^r \right) c_{lm}^r + \left( \frac{c_{ln}^r}{2} + x_{ln}^r \right) c_{mk}^r \right\} \left( \frac{c_{ij}^r}{2} + x_{ij}^r \right)
\]

\[
+ \left( \frac{c_{mn}^r}{2} + x_{mn}^r \right) \left( \frac{c_{jk}^r}{2} + x_{jk}^r \right) + \left( \frac{c_{kn}^r}{2} + x_{kn}^r \right) \left( \frac{c_{ln}^r}{2} + x_{ln}^r \right) + \left( \frac{c_{ln}^r}{2} + x_{ln}^r \right) \left( \frac{c_{ij}^r}{2} + x_{ij}^r \right)
\]

\[
+ c_{km} c_{jl} \left( \frac{c_{in}^r}{2} + x_{in}^r \right) \}
\]

\[
= 0.
\]

By Eq. \((2.9)\), we can assume that \(k \leq m\). Thus the number of these equations with indeterminates \(x_{ij}^k\) is at most \(N^4(N+1)/2\). Therefore the conclusion follows.

**Lemma 2.9.** Let \((J_1, \circ_1)\) and \((J_2, \circ_2)\) be two Jordan algebras and \(\phi: (J_1, \circ_1) \to (J_2, \circ_2)\) be an isomorphism. If \((J_1, \cdot_1)\) is a compatible pre-Jordan algebra structure on \((J_1, \circ_1)\), then the bilinear multiplication \(\cdot_2: J_2 \times J_2 \to J_2\) defined by

\[
(2.15) \quad x \cdot_2 y := \phi(\phi^{-1}(x) \cdot_1 \phi^{-1}(y)), \quad \forall x, y \in J_2,
\]

gives a compatible pre-Jordan algebra structure on \((J_2, \circ_2)\) which is isomorphic to \((J_1, \cdot_1)\).

**Proof.** It is straightforward to show that \((J_2, \cdot_2)\) is a pre-Jordan algebra which is isomorphic to \((J_1, \cdot_1)\). Moreover, for any \(x, y \in J_2\), we have

\[
x \cdot_2 y + y \cdot_2 x = \phi(\phi^{-1}(x) \cdot_1 \phi^{-1}(y)) + \phi(\phi^{-1}(y) \cdot_1 \phi^{-1}(x)) = \phi(\phi^{-1}(x) \circ_1 \phi^{-1}(y)) = x \circ_2 y.
\]

Therefore \((J_2, \cdot_2)\) is a compatible pre-Jordan algebra structure on \((J_2, \circ_2)\).

**Corollary 2.10.** Let \((J, \circ)\) be a Jordan algebra. Let \((J, \cdot_1)\) and \((J, \cdot_2)\) be two compatible pre-Jordan algebra structures on \((J, \circ)\). Then \((J, \cdot_1)\) and \((J, \cdot_2)\) are isomorphic if and only if \(\phi \in Aut(J, \circ)\) and \(\phi\) satisfies Eq. \((2.15)\). Equivalently, \(Aut(J, \circ)\) acts on the set of all compatible pre-Jordan algebra structures on \((J, \circ)\) through Eq. \((2.15)\), and the isomorphism classes of these pre-Jordan algebras are exactly the orbits of this action.

**Proof.** “⇒”: If \(\phi: J \to J\) is an isomorphism of pre-Jordan algebras from \((J, \cdot_1)\) to \((J, \cdot_2)\), then it is straightforward to show that \(\phi\) is an isomorphism from \((J, \circ)\) to itself, that is, \(\phi \in Aut(J, \circ)\). Moreover, \(\phi\) satisfies Eq. \((2.15)\) by Definition 2.3.

“⇐”: It follows from Lemma 2.9 by letting \((J_1, \circ_1) = (J_2, \circ_2) = (J, \circ)\).

Therefore, we give our strategy on classifying finite-dimensional complex pre-Jordan algebras in the sense of isomorphism as follows. We divide it into several steps.
Step 1 We classify finite-dimensional complex Jordan algebras in the sense of isomorphism.

Step 2 For a fixed Jordan algebra \((J, \circ)\) in Step 1, that is, with a fixed basis and fixed structural constants \(c_{ij}\), we give all solutions of the set of equations \(\{Q_{klmn} = 0\}\) with the indeterminates \(x_{ij}\). Thus we get all compatible pre-Jordan algebras on \((J, \circ)\).

Step 3 For the fixed Jordan algebra \((J, \circ)\) in Step 2, we first give \(\text{Aut}(J, \circ)\) and then determine the orbits by its action on the set of compatible pre-Jordan algebra structures (in fact, the solutions of the set of equations \(\{Q_{klmn} = 0\}\)) in Step 2 through Eq. (2.15).

In the following sections, we will illustrate that such a strategy is available, in particular in low dimensions. Note that the classification of complex Jordan algebras in dimensions \(\leq 3\) is known, that is, Step 1 has been already finished in our cases.

3. Classification of complex pre-Jordan algebras in dimensions \(\leq 2\). In this section, we classify all complex pre-Jordan algebras in the sense of isomorphisms in dimensions up to 2.

Proposition 3.1. Let \(\{e\}\) be a basis of a 1-dimensional vector space \(A\). Then there are exactly 2 non-isomorphic pre-Jordan algebras \((A, \cdot_1)\) and \((A, \cdot_2)\) on \(A\) given by \(e \cdot e = e\) and \(e \cdot e = 0\) respectively. Both of them are associative.

Proof. In fact, we have \(e \cdot e = \lambda e\), where \(\lambda \in \mathbb{C}\). If \(\lambda \neq 0\), then by a linear transformation given by \(e \mapsto \lambda e\), we get \(e' \cdot e' = e'\). It is obvious that both of them are associative and hence they are pre-Jordan algebras.

Lemma 3.2. ([25]) Every 2-dimensional Jordan algebra is isomorphic to one of the following (mutually non-isomorphic) Jordan algebras given by their formal characteristic matrices respectively.

\[
J_1 : \begin{pmatrix} 2e_1 & 0 \\ 0 & 2e_2 \end{pmatrix}, \quad J_2 : \begin{pmatrix} 2e_1 & 2e_2 \\ 2e_2 & 0 \end{pmatrix}, \quad J_3 : \begin{pmatrix} 2e_1 & 0 \\ 0 & 0 \end{pmatrix},
\]

\[
J_4 : \begin{pmatrix} 2e_1 & e_2 \\ e_2 & 0 \end{pmatrix}, \quad J_5 : \begin{pmatrix} 2e_2 & 0 \\ 0 & 0 \end{pmatrix}, \quad J_6 : \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.
\]

In the following content, we use the notations given in the proof of Corollary 2.8 (see Eq. (2.14)). As an illustration, we give an explicit proof for the classification of the compatible pre-Jordan algebras on the Jordan algebra \((J_1, \circ)\), whereas the proofs for the other cases are shortened by giving the “essentially available” equations only.

Proposition 3.3. There are exactly two non-isomorphic compatible pre-Jordan algebra structures on the Jordan algebra \((J_1, \circ)\) given by the following formal characteristic matrices respectively.

\[
J_{1,1} : \begin{pmatrix} e_1 & 0 \\ 0 & e_2 \end{pmatrix}, \quad J_{1,2} : \begin{pmatrix} e_1 & e_2 \\ -e_2 & e_2 \end{pmatrix}.
\]

Proof. The set of equations \(\{Q^i_{jklm} = 0\}\) is written as follows.

\[
Q^1_{1111} = Q^2_{1111} = 0, \quad Q^1_{1112} = -2x_{12}^1x_{12}^2(x_{12}^2 - 2) = 0,
\]

\[
Q^2_{1112} = -2x_{12}^1x_{12}^2(x_{12}^2 - 2)(x_{12}^2 - 1) = 0, \quad Q^1_{1121} = x_{12}^1x_{12}^2(x_{12}^2 - 1) = 0,
\]

\[
Q^2_{1121} = x_{12}^2(x_{12}^2 - 1)^2 = 0, \quad Q^1_{1122} = x_{12}^1x_{12}^2 + x_{12}^1 - x_{12}^2 + 1 = 0,
\]

\[
Q^2_{1122} = x_{12}^2(x_{12}^1 + x_{12}^1 - 2x_{12}^2 + 2) = 0, \quad Q^1_{1211} = Q^2_{1211} = 0,
\]

\[
Q^2_{1212} = 2x_{12}^1(x_{12}^1x_{12}^2 + x_{12}^1 - x_{12}^2 + 1) = 0, \quad Q^2_{1212} = 2x_{12}^1(x_{12}^1x_{12}^2 - x_{12}^2 + 1) = 0,
\]

\[
Q^2_{1211} = -x_{12}^1x_{12}^2 + 2x_{12}^2 - x_{12}^2 + 2 = 0, \quad Q^2_{1221} = -x_{12}^1x_{12}^2 + x_{12}^2 - x_{12}^2 + 1) = 0,
\]

\[
Q^2_{1222} = -x_{12}^1(x_{12}^2 + 1) = 0, \quad Q^2_{1222} = -x_{12}^1x_{12}^2(x_{12}^1 + 1) = 0,
\]

\[
Q^2_{1221} = -2x_{12}^1x_{12}^2 + x_{12}^1 + 1), \quad Q^2_{2121} = -2x_{12}^1(x_{12}^1x_{12}^2 + x_{12}^1 - x_{12}^2 + 1) = 0,
\]

\[
Q^2_{1322} = Q^2_{2122} = 0, \quad Q^2_{2121} = 2x_{12}^1(x_{12}^2 + 1)(x_{12}^1 + 2),
\]

\[
Q^2_{2221} = 2x_{12}^1x_{12}^1(x_{12}^1 + 2), \quad Q^2_{2222} = Q^2_{2222} = 0.
\]
In fact, the above equations hold if and only if the following ("essentially available") equations hold:

\[ Q_{1112}^1 = -2x_{12}^1 x_{12}^2 (x_{12}^2 - 2) = 0, \quad Q_{1121}^2 = x_{12}^2 (x_{12}^2 - 1)^2 = 0, \quad Q_{1222}^1 = -x_{12}^1 (x_{12}^1 + 1)^2 = 0. \]

It is straightforward to get all solutions as follows.

Therefore they correspond to the following pre-Jordan algebras given by the following formal characteristic matrices respectively.

\[
J_{1,1} : \begin{pmatrix} e_1 & 0 \\ 0 & e_2 \end{pmatrix}, \quad J_{1,2} : \begin{pmatrix} e_1 & e_2 \\ -e_2 & e_2 \end{pmatrix}, \quad J_{1,3} : \begin{pmatrix} e_1 & -e_1 \\ e_1 & e_2 \end{pmatrix}.
\]

Note that \((J_{1,2}, \cdot)\) is isomorphic to \((J_{1,3}, \cdot)\) by a linear transformation given by \(e_1 \mapsto e_2, e_2 \mapsto e_1\). Moreover, \((J_{1,1}, \cdot)\) is not isomorphic to \((J_{1,2}, \cdot)\) since the former is commutative and associative, whereas the latter is neither commutative nor associative.

**Proposition 3.4.** There are exactly two non-isomorphic compatible pre-Jordan algebra structures on the Jordan algebra \((J_2, \circ)\) given by the following formal characteristic matrices respectively.

\[
J_{2,1} : \begin{pmatrix} e_1 & e_2 \\ e_2 & 0 \end{pmatrix}, \quad J_{2,2} : \begin{pmatrix} e_1 & 2e_2 \\ 0 & 0 \end{pmatrix}.
\]

**Proof.** The "essentially available" equations in the set of equations \(\{Q_{jklm}^i = 0\}\) are

\[
Q_{1222}^1 = -(x_{12}^1)^3 = 0, \quad Q_{1121}^2 = (x_{12}^2)^2 (x_{12}^2 - 1) = 0.
\]

All the solutions of these equations are

\[
(1) \quad x_{12}^1 = 0, x_{12}^2 = 0; \quad (2) \quad x_{12}^1 = 0, x_{12}^2 = 1.
\]

They correspond to the pre-Jordan algebras \((J_{2,1}, \cdot)\) and \((J_{2,2}, \cdot)\) respectively. Moreover, \((J_{2,1}, \cdot)\) is not isomorphic to \((J_{2,2}, \cdot)\) since the former is commutative and associative, whereas the latter is neither commutative nor associative.

**Proposition 3.5.** There are exactly two non-isomorphic compatible pre-Jordan algebra structures on the Jordan algebra \((J_3, \circ)\) given by the following formal characteristic matrices respectively.

\[
J_{3,1} : \begin{pmatrix} e_1 & 0 \\ 0 & 0 \end{pmatrix}, \quad J_{3,2} : \begin{pmatrix} e_1 & e_2 \\ -e_2 & 0 \end{pmatrix}.
\]

**Proof.** The "essentially available" equations in the set of equations \(\{Q_{jklm}^i = 0\}\) are

\[
Q_{1222}^1 = -(x_{12}^1)^3 = 0, \quad Q_{1121}^2 = (x_{12}^2)^2 (x_{12}^2 - 1)^2 = 0.
\]

All the solutions of these equations are

\[
(1) \quad x_{12}^1 = 0, x_{12}^2 = 0; \quad (2) \quad x_{12}^1 = 0, x_{12}^2 = 1.
\]

They correspond to the pre-Jordan algebras \((J_{3,1}, \cdot)\) and \((J_{3,2}, \cdot)\) respectively. Moreover, \((J_{3,1}, \cdot)\) is not isomorphic to \((J_{3,2}, \cdot)\) since the former is commutative and associative, whereas the latter is neither commutative nor associative.

**Proposition 3.6.** There are exactly three non-isomorphic compatible pre-Jordan algebra structures on the Jordan algebra \((J_4, \circ)\) given by the following formal characteristic matrices respectively.

\[
J_{4,1} : \begin{pmatrix} e_1 & 0 \\ e_2 & 0 \end{pmatrix}, \quad J_{4,2} : \begin{pmatrix} e_1 & e_2 \\ 0 & 0 \end{pmatrix}, \quad J_{4,3} : \begin{pmatrix} e_1 & 2e_2 \\ -e_2 & 0 \end{pmatrix}.
\]
Proof. The “essentially available” equations in the set of equations \( \{ Q_{ijklm} = 0 \} \) are
\[
Q_{1222}^1 = -(x_{12}^1)^3 = 0, \quad 8Q_{1121}^3 = (2x_{12}^2 - 3)(2x_{12}^2 - 1)(2x_{12}^2 + 1) = 0.
\]
All the solutions of these equations are
\[
(1) \ x_{12}^1 = 0, x_{12}^2 = \frac{1}{2}; \quad (2) \ x_{12}^1 = 0, x_{12}^2 = \frac{1}{2}; \quad (3) \ x_{12}^1 = 0, x_{12}^2 = \frac{3}{2}.
\]
They correspond to the pre-Jordan algebras \((J_4, \cdot), (J_4, \cdot)\) and \((J_4, \cdot)\) respectively. It is straightforward
to know that both \((J_4, \cdot)\) and \((J_4, \cdot)\) are associative whereas \((J_4, \cdot)\) is not associative since in \((J_4, \cdot)\),
we have
\[
2e_2 = (e_1 \cdot e_1) \cdot e_2 \neq e_1 \cdot (e_1 \cdot e_2) = 4e_2.
\]
Moreover, \((J_4, \cdot)\) is not isomorphic to \((J_4, \cdot)\) since \(e_1\) is a right identity in the former whereas there is
not a right identity in the latter since \(e_2 \cdot x = 0, \forall x \in J_{4,2} \).

**Proposition 3.7.** There is exactly one compatible pre-Jordan algebra structure on the Jordan algebra \((J_5, \circ)\) given by the following formal characteristic matrix.
\[
J_{5,1} : \begin{pmatrix} e_2 & 0 \\ 0 & 0 \end{pmatrix}.
\]

Proof. The “essentially available” equations in the set of equations \( \{ Q_{ijklm} = 0 \} \) are
\[
Q_{1211}^1 = 2(x_{12}^1)^2 = 0, \quad Q_{1111}^2 = -2(x_{12}^1 + (x_{12}^2)^2) = 0.
\]
There is only one solution
\[
a = b = 0.
\]
It corresponds to the pre-Jordan algebra \((J_{5,1}, \cdot)\).

**Proposition 3.8.** There is exactly one compatible pre-Jordan algebra structure on the Jordan algebra \((J_6, \circ)\) given by the following formal characteristic matrix.
\[
J_{6,1} : \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.
\]

Proof. The “essentially available” equations in the set of equations \( \{ Q_{ijklm} = 0 \} \) are
\[
Q_{1222}^1 = -(x_{12}^1)^3 = 0, \quad Q_{1121}^2 = (x_{12}^2)^3 = 0.
\]
There is only one solution
\[
x_{12}^1 = x_{12}^2 = 0.
\]
It corresponds to the pre-Jordan algebra \((J_{6,1}, \cdot)\).

Summarize the above study in this section, we have the following conclusion.

**Theorem 3.9.** Every 2-dimensional pre-Jordan algebra is isomorphic to one of the following (mutually
non-isomorphic) pre-Jordan algebras given by their formal characteristic matrices respectively.
\[
J_{1,1} : \begin{pmatrix} e_1 & 0 \\ 0 & e_2 \end{pmatrix}, \quad J_{1,2} : \begin{pmatrix} e_1 & e_2 \\ -e_2 & e_2 \end{pmatrix}, \quad J_{2,1} : \begin{pmatrix} e_1 & e_2 \\ e_2 & 0 \end{pmatrix}, \quad J_{2,2} : \begin{pmatrix} e_1 & 2e_2 \\ 0 & 0 \end{pmatrix},
\]
\[
J_{3,1} : \begin{pmatrix} e_1 & 0 \\ 0 & 0 \end{pmatrix}, \quad J_{3,2} : \begin{pmatrix} e_1 & e_2 \\ -e_2 & 0 \end{pmatrix}, \quad J_{4,1} : \begin{pmatrix} e_1 & 0 \\ e_2 & 0 \end{pmatrix}, \quad J_{4,2} : \begin{pmatrix} e_1 & e_2 \\ 0 & 0 \end{pmatrix},
\]
\[
J_{4,3} : \begin{pmatrix} e_1 & 2e_2 \\ -e_2 & 0 \end{pmatrix}, \quad J_{5,1} : \begin{pmatrix} e_2 & 0 \\ 0 & 0 \end{pmatrix}, \quad J_{6,1} : \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.
\]
Moreover, \((J_{1,1}, \cdot), (J_{2,1}, \cdot), (J_{3,1}, \cdot), (J_{5,1}, \cdot)\) and \((J_{6,1}, \cdot)\) are commutative and associative, and \((J_{4,1}, \cdot)\) and
\((J_{4,2}, \cdot)\) are associative.
4. Classification of 3-dimensional complex pre-Jordan algebras. In this section, we classify 3-dimensional complex pre-Jordan algebras in the sense of isomorphism based on the classification of 3-dimensional complex Jordan algebras given in [25].

According to [25], there are exactly 20 classes of 3-dimensional Jordan algebras:

(1) $J_i, 1 \leq i \leq 8$ are the Jordan algebras which are not associative;
(2) $A_i, 1 \leq i \leq 4$ are the associative unitary Jordan algebras;
(3) $A_i, 1 \leq i \leq 8$ are the associative but not unitary Jordan algebras.

As an illustration, we give an explicit proof for the classification of the compatible pre-Jordan algebras on the Jordan algebra $(J_1, \circ)$, whereas the proofs for the other cases are omitted.

**Proposition 4.1.** (case $J_1$) The formal characteristic matrix of the Jordan algebra $(J_1, \circ)$ is given by

$$
\begin{pmatrix}
2e_1 & 2e_2 & 2e_3 \\
2e_2 & 0 & 2e_1 \\
2e_3 & 2e_1 & 0
\end{pmatrix}.
$$

Any compatible pre-Jordan algebra structure on $(J_1, \circ)$ is isomorphic to one of the following (mutually non-isomorphic) pre-Jordan algebras given by their formal characteristic matrices respectively.

$$
\tilde{J}_{1,1}(\alpha) : \begin{pmatrix}
e_1 & e_1 + (1 + \lambda)e_2 + \frac{\lambda}{2}x & 2\alpha e_1 + 2\lambda e_2 + (2 - \lambda)e_3 \\
-e_1 + (1 - \lambda)e_2 - \frac{\lambda}{2}x & 0 & 2\lambda e_1 - 2\alpha e_2 + e_3 \\
-2\alpha e_1 - 2\lambda e_2 + \lambda e_3 & (2 - 2\lambda)e_1 + 2\alpha e_2 - e_3 & 0
\end{pmatrix},
$$

where $\alpha \in \mathbb{C}$ and $\lambda$ is a nonzero root of $x^2 - x + \alpha = 0$, and different choices of $\lambda$ give isomorphic pre-Jordan algebras,

$$
\tilde{J}_{1,2} : \begin{pmatrix}
e_1 & 2e_2 & e_3 \\
0 & 0 & 2e_1 \\
e_3 & 0 & 0
\end{pmatrix}, \quad \tilde{J}_{1,3} : \begin{pmatrix}
e_1 & 2e_2 & e_2 + e_3 \\
0 & 0 & 2e_1 \\
-e_2 + e_3 & 0 & 0
\end{pmatrix}.
$$

**Proof.** We determine the structural constants. By Corollary 2.8,

$$
Q_{1121}^3 + Q_{1122}^3 + Q_{1112}^3 = -2x_{12}^3(x_{12} + x_{13}^3 - 1) = 0,
Q_{1311}^3 + Q_{1312}^3 + Q_{1113}^3 = -2x_{13}^3(x_{12} + x_{13}^3 - 1) = 0,
2Q_{1121}^3 + Q_{1122}^3 = 2x_{12}^3(x_{12} + x_{13}^3 - 3) = 0,
Q_{1321}^3 + Q_{1323}^3 + Q_{1123}^3 = 2x_{13}^3(x_{12}^3 - x_{13}) = 0,
Q_{1331}^3 + Q_{1332}^3 + Q_{1113}^3 = 2x_{13}^3(x_{13}^3 + x_{12}^3) = 0,
2Q_{1331}^3 + Q_{1133}^3 = 2x_{13}^2(x_{13} + x_{12}^3) = 0.
$$

If at least one of \(x_{23}^3 = x_{12}^3\)
\(x_{12}^3 + x_{13}^3 = 1\) fails to hold, then
\(x_{12}^3 = x_{13}^3 = 0\). Thus,
\(Q_{2221}^1 = 2(x_{12}^3)^3 = 0\), \(Q_{4331}^1 = 2(x_{13}^3)^3 = 0\), \(Q_{2232}^3 = (x_{23}^3)^3 = 0\), \(Q_{2333}^2 = -(x_{23}^3)^3 = 0\).

And \(x_{23}^3 + x_{13}^3 = 1 = \frac{1}{5}(3Q_{2233}^1 + 3Q_{2332}^1 - 2Q_{3132}^3 - 2Q_{2123}^2) = 0\). These lead to a contradiction, so
\(x_{23}^3 = x_{12}^3\),
\(x_{12}^3 + x_{13}^3 = 1\) always holds. What’s more, \(x_{23}^3 - 2x_{12}^3 + 1 = \frac{1}{2}(Q_{1123}^1 - Q_{1132}^1) = 0\).

We have reduced 4 indeterminates, and the remaining unknown elements follow these “essentially available”
equations:
\[
\begin{align*}
x_{12}^4(1 - x_{12}^2) &= x_{12}^3x_{13}^1 \iff Q_{1211}^1 + Q_{1121}^1 + Q_{1112}^1 = 0, \\
x_{12}^1x_{13}^2 &= x_{12}^3x_{13}^1 \iff Q_{1311}^1 + Q_{1331}^1 + Q_{1113}^1 = 0, \\
x_{12}^1x_{13}^2 &= x_{12}^3x_{13}^1 \iff Q_{1211}^1 + Q_{1212}^1 + Q_{1112}^1 = 0, \\
(x_{12}^1)^2 &= 2x_{12}^2x_{13}^3 \iff Q_{2322}^1 + Q_{2223}^1 + Q_{1112}^1 = 0, \\
(x_{13}^1)^2 &= 2x_{12}^2x_{13}^3 \iff Q_{3233}^1 + Q_{3333}^1 + Q_{1313}^1 = 0, \\
x_{12}^1x_{13}^2 &= 2x_{12}^1x_{13}^3 \iff Q_{2233}^1 + Q_{1112}^1 = 0.
\end{align*}
\]

It has been shown that the set of solutions mentioned in Corollary 2.8 1-1 corresponds to the subset of \(\mathbb{C}^5\) cut by the equations above. Their coordinates are \((x_{12}, x_{12}, x_{13}, x_{13}, x_{12})\). By Corollary 2.10, the next step is to investigate how \(\text{Aut}(\mathfrak{d}_1, \phi)\) acts on this set. It is straightforward to show that \(\text{Aut}(\mathfrak{d}_1, \phi) = H \cup \tau H\), where \(H = \{\text{diag}(1, \mu, \mu^{-1}) | \mu \neq 0\}\) and \(\tau = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\).

And by simple calculation,
\[
\begin{align*}
\text{diag}(1, \mu, \mu^{-1}) \begin{pmatrix} x_{12}^1, x_{12}^2, x_{13}^3, x_{13}^1, x_{13}^3 \end{pmatrix} &= \begin{pmatrix} x_{12}^1 \mu^{-1}, x_{12}^2, x_{13}^3 \mu^{-2}, x_{13}^1 \mu, x_{13}^2 \mu^2 \end{pmatrix}, \\
\tau \begin{pmatrix} x_{12}^1, x_{12}^2, x_{13}^3, x_{13}^1, x_{13}^3 \end{pmatrix} &= \begin{pmatrix} x_{12}^3, 1 - x_{12}^1, x_{12}^1, x_{12}^2, x_{12}^2 \end{pmatrix}.
\end{align*}
\]

Notice that \(\alpha = x_{12}^7(1 - x_{12}^2) = x_{12}^2x_{13}^2\) is invariant under \(\text{Aut}(\mathfrak{d}_1, \phi)\). And then we introduce a lemma.

Suppose \(G\) is a group acting on a set \(S\) by \(\phi\) and \(G'\) is a normal subgroup of \(G\). Note the set of all orbits by \(S/G\). \(G\) could act on \(S/G'\) by \(\phi': \phi'(g)(x) = [\phi(g)x]\) (note the orbit of \(x\) in \(S/G'\) by \([x]\)). Hence \(\phi'(g) = \text{id}\) if and only if \(g \in G'\). That is, the action of \(G/G'\) on \(S/G'\) by \(g \in G'\) is well defined. And \(\pi(g)[x] = [y]\) if and only if there is \(h \in G'\), such that \(\pi(gh)[x] = y\), which tells the following lemma.

**Lemma 4.2.** Suppose \(G\) is a group acting on a set \(S\), and \(G' \triangleleft G\) is a normal subgroup. Note the orbits of \(G\) acting on \(S\) by \(S/G'\). Then \(G/G'\) acts on \(S/G'\), and \(S/G \simeq (S/G')/(G/G')\).

When \(\alpha \neq 0, x_{12}^1x_{13}^1 = 2\alpha \neq 0\). Apply the lemma above on \(S = \{\text{tuples with } \alpha \neq 0\}\), \(G = \text{Aut}(\mathfrak{d}_1, \phi)\) and \(G' = H = \ker[\text{det: } G \to C]\). For each orbit in \(S/H\), we fix a representing element with \(x_{12}^1 = 1\), and in this way we can reduce the problem to determining the orbit of the induced action of \{1, \tau\} on the tuples with the first term 1. Consider an arbitrary tuple \((1, x_{12}^7, (2x_{12}^2)^{-1}, 2\alpha, 2x_{12}^7\alpha)\). Apply \(\tau\) on this tuple and it becomes \((1, 1 - x_{12}^2, (2 - 2x_{12}^2)^{-1}, 2\alpha, 2(1 - x_{12}^2)\alpha)\). Noticing the fact that \(x_{12}^7\) and \(1 - x_{12}^2\) are the roots of \(x^2 - x + \alpha = 0\), the discussion above ensures that a different choice of the nonzero root \(x_{12}^7\) of this equation does not make any difference on the isomorphism class. This proves the part \(\mathfrak{d}_{1, 1}(\alpha \neq 0)\).

When \(\alpha = 0 = x_{12}^2(1 - x_{12}^2), x_{12}^2 = 0\) or 1. Because \(\tau\) sends the tuples with \(x_{12}^2 = 0\) to the ones with \(x_{12}^2 = 1\), every orbit admits at least one element with \(x_{12}^2 = 1\). It is obvious that \(\{\text{tuples with } \alpha = 0\}\)/\(G \simeq \{\text{tuples with } x_{12}^2 = 1\}/H\). When \(x_{12}^2 = 1\), the equations are reduced to
\[
x_{13}^1 = 0, \quad (x_{12}^1)^2 = 2x_{12}^3, \quad x_{12}^1x_{13}^2 = 0.
\]

If \(x_{12}^1 \neq 0\), it can be fixed to 1, and this is the case \(\mathfrak{d}_{1, 1}(\alpha = 0)\).

If \(x_{12}^1 = 0\), then \(x_{13}^1 = 0\) or 1, generating \(\mathfrak{d}_{1, 2}\) and \(\mathfrak{d}_{1, 3}\) respectively.

**Proposition 4.3.** (case \(\mathfrak{d}_2\)) The formal characteristic matrix of the Jordan algebra \((\mathfrak{d}_2, \phi)\) is given by
\[
\begin{pmatrix}
2e_1 & 0 & e_3 \\
0 & 2e_2 & e_3 \\
e_3 & e_3 & 0
\end{pmatrix}.
\]

Any compatible pre-Jordan algebra structure on \((\mathfrak{d}_2, \phi)\) is isomorphic to one of the following (mutually
non-isomorphic) pre-Jordan algebras given by their formal characteristic matrices respectively.

\[
\begin{align*}
J_{2.1} : & \begin{pmatrix} e_1 & -e_1 & e_3 \\ e_1 & e_2 & 0 \\ 0 & e_3 & 0 \end{pmatrix},
J_{2.2} : & \begin{pmatrix} e_1 & 0 & e_3 \\ 0 & e_2 & e_3 \\ 0 & 0 & 0 \end{pmatrix},
J_{2.3} : & \begin{pmatrix} e_1 & 0 & e_3 \\ 0 & e_2 & e_1 + e_3 \\ 0 & 0 & -e_1 \end{pmatrix},
J_{2.4} : & \begin{pmatrix} e_1 & 0 & 0 \\ 0 & e_2 & e_3 \\ e_3 & 0 & 0 \end{pmatrix},
J_{2.5} : & \begin{pmatrix} e_1 & -e_1 & 0 \\ e_1 & e_2 & 2e_3 \\ e_3 & -e_3 & 0 \end{pmatrix},
J_{2.6} : & \begin{pmatrix} e_1 & -e_1 & e_3 \\ e_1 & e_2 & e_3 \\ 0 & 0 & 0 \end{pmatrix},
J_{2.7} : & \begin{pmatrix} e_1 & -e_1 & 2e_3 \\ e_1 & e_2 & 0 \\ -e_3 & e_3 & 0 \end{pmatrix}.
\end{align*}
\]

**Proposition 4.4.** (case \(J_3\)) The formal characteristic matrix of the Jordan algebra \((J_3, \circ)\) is given by

\[
\begin{pmatrix} 2e_1 & 0 & 0 \\ 0 & 2e_2 & e_3 \\ 0 & e_3 & 0 \end{pmatrix}.
\]

Any compatible pre-Jordan algebra structure on \((J_3, \circ)\) is isomorphic to one of the following (mutually non-isomorphic) pre-Jordan algebras given by their formal characteristic matrices respectively.

\[
\begin{align*}
J_{3.1} : & \begin{pmatrix} e_1 & -e_1 & 0 \\ e_1 & e_2 & 0 \\ 0 & e_3 & 0 \end{pmatrix},
J_{3.2} : & \begin{pmatrix} e_1 & -e_1 & 0 \\ e_1 & e_2 & e_3 \\ 0 & 0 & 0 \end{pmatrix},
J_{3.3} : & \begin{pmatrix} e_1 & -e_1 & 0 \\ e_1 & e_2 & 2e_3 \\ 0 & 0 & -e_3 \end{pmatrix},
J_{3.4} : & \begin{pmatrix} e_1 & 0 & 0 \\ 0 & e_2 & 0 \\ 0 & e_3 & 0 \end{pmatrix},
J_{3.5} : & \begin{pmatrix} e_1 & 0 & 0 \\ 0 & e_2 & 2e_3 \\ 0 & -e_3 & 0 \end{pmatrix},
J_{3.6} : & \begin{pmatrix} e_1 & 0 & 0 \\ 0 & e_2 & 0 \\ -e_3 & e_3 & 0 \end{pmatrix},
J_{3.7} : & \begin{pmatrix} e_1 & 0 & e_3 \\ -e_2 & e_3 & 0 \\ -e_3 & 0 & 0 \end{pmatrix},
J_{3.8} : & \begin{pmatrix} e_1 & 0 & e_3 \\ 0 & e_2 & e_3 \\ -e_3 & 0 & 0 \end{pmatrix},
J_{3.9} : & \begin{pmatrix} e_1 & 0 & e_3 \\ -e_2 & e_2 & 0 \\ 0 & 0 & 0 \end{pmatrix},
J_{3.10} : & \begin{pmatrix} e_1 & 0 & e_3 \\ 0 & e_2 & e_3 \\ -e_2 - e_3 & 0 & 0 \end{pmatrix},
J_{3.11} : & \begin{pmatrix} e_1 & -e_1 & e_3 \\ e_1 & e_2 & 0 \\ -e_3 & e_3 & 0 \end{pmatrix},
J_{3.12} : & \begin{pmatrix} e_1 & -e_1 & e_3 \\ e_1 & e_2 & -e_1 - e_2 \\ -e_1 - e_2 - e_3 & e_1 + e_2 + e_3 & 0 \end{pmatrix},
J_{3.13} : & \begin{pmatrix} e_1 & 0 & 0 \\ 0 & e_2 & e_3 \\ 0 & 0 & 0 \end{pmatrix}.
\end{align*}
\]

**Proposition 4.5.** (case \(J_4\)) The formal characteristic matrix of the Jordan algebra \((J_4, \circ)\) is given by

\[
\begin{pmatrix} 2e_1 & e_2 & 2e_3 \\ e_2 & 0 & 0 \\ 2e_3 & 0 & 0 \end{pmatrix}.
\]

Any compatible pre-Jordan algebra structure on \((J_4, \circ)\) is isomorphic to one of the following (mutually non-isomorphic) pre-Jordan algebras given by their formal characteristic matrices respectively.
isomorphic) pre-Jordan algebras given by their formal characteristic matrices respectively.

Any compatible pre-Jordan algebra structure on \( J \) is given by

\[
J_{4,1}(\alpha) : 
\begin{pmatrix}
\alpha + 1 & 0 & 0 \\
0 & \alpha(\alpha - 1) & 0 \\
0 & 2\alpha & \alpha \\
\end{pmatrix},
\]

\[
J_{4,2} : 
\begin{pmatrix}
e_1 & 0 & e_3 \\
e_2 & 0 & 0 \\
e_3 & 0 & 0
\end{pmatrix},
\]

\[
J_{4,3} : 
\begin{pmatrix}
e_1 & 0 & 2e_3 \\
e_2 & 0 & 0 \\
e_3 & 0 & 0
\end{pmatrix},
\]

\[
J_{4,4} : 
\begin{pmatrix}
e_1 & e_3 & e_3 \\
e_2 & -e_3 & 0 \\
e_3 & 0 & 0
\end{pmatrix},
\]

\[
J_{4,5} : 
\begin{pmatrix}
e_1 & e_3 & 2e_3 \\
e_2 & 0 & 0 \\
e_3 & 0 & 0
\end{pmatrix},
\]

\[
J_{4,6} : 
\begin{pmatrix}
e_1 & e_2 & e_3 \\
e_2 & 0 & 0 \\
e_3 & 0 & 0
\end{pmatrix},
\]

\[
J_{4,7} : 
\begin{pmatrix}
e_1 & 2e_2 & e_3 \\
e_2 & -e_2 & 0 \\
e_3 & 0 & 0
\end{pmatrix},
\]

\[
J_{4,8} : 
\begin{pmatrix}
e_1 & e_2 & 2e_3 \\
e_2 & 0 & 0 \\
e_3 & 0 & 0
\end{pmatrix},
\]

\[
J_{4,9} : 
\begin{pmatrix}
e_1 & e_2 & e_3 \\
e_2 & -e_2 & 0 \\
e_3 & 0 & 0
\end{pmatrix},
\]

\[
J_{4,10} : 
\begin{pmatrix}
e_1 & 2e_2 & e_3 \\
e_2 & -e_2 & 0 \\
e_3 & 0 & 0
\end{pmatrix},
\]

\[
J_{4,11} : 
\begin{pmatrix}
e_1 & e_2 & 0 \\
e_2 & 0 & 0 \\
e_3 & 0 & 0
\end{pmatrix},
\]

\[
J_{4,12} : 
\begin{pmatrix}
e_1 & e_2 & 2e_3 \\
e_2 & -e_2 & 0 \\
e_3 & 0 & 0
\end{pmatrix},
\]

\[
J_{4,13} : 
\begin{pmatrix}
e_1 & e_2 & e_1 - e_2 + 2e_3 \\
e_2 & e_1 & 0 \\
e_3 & 0 & 0
\end{pmatrix}.
\]

**Proposition 4.6. (case \( J_5 \))** The formal characteristic matrix of the Jordan algebra \((J_5, \circ)\) is given by

\[
\begin{pmatrix}
2e_1 & e_2 & e_3 \\
e_2 & 0 & 0 \\
e_3 & 0 & 0
\end{pmatrix}.
\]

Any compatible pre-Jordan algebra structure on \((J_5, \circ)\) is isomorphic to one of the following (mutually non-isomorphic) pre-Jordan algebras given by their formal characteristic matrices respectively.

\[
J_{5,1} : 
\begin{pmatrix}
e_1 & 0 & 0 \\
e_2 & 0 & 0 \\
e_3 & 0 & 0
\end{pmatrix},
\]

\[
J_{5,2} : 
\begin{pmatrix}
e_1 & 2e_2 & 2e_3 \\
e_2 & -e_2 & 0 \\
e_3 & 0 & 0
\end{pmatrix},
\]

\[
J_{5,3} : 
\begin{pmatrix}
e_1 & e_2 & e_3 \\
e_2 & 0 & 0 \\
e_3 & 0 & 0
\end{pmatrix},
\]

\[
J_{5,4} : 
\begin{pmatrix}
e_1 & e_2 & 0 \\
e_2 & 0 & 0 \\
e_3 & 0 & 0
\end{pmatrix},
\]

\[
J_{5,5} : 
\begin{pmatrix}
e_1 & e_2 & e_3 \\
e_2 & 0 & 0 \\
e_3 & 0 & 0
\end{pmatrix},
\]

\[
J_{5,6} : 
\begin{pmatrix}
e_1 & 2e_2 & 0 \\
e_2 & -e_2 & 0 \\
e_3 & 0 & 0
\end{pmatrix}.
\]

**Proposition 4.7. (case \( J_6 \))** The formal characteristic matrix of the Jordan algebra \((J_6, \circ)\) is given by

\[
\begin{pmatrix}
2e_1 & e_2 & 0 \\
e_2 & 2e_3 & 0 \\
e_3 & 0 & 0
\end{pmatrix}.
\]

Any compatible pre-Jordan algebra structure on \((J_6, \circ)\) is isomorphic to one of the following (mutually non-isomorphic) pre-Jordan algebras given by their formal characteristic matrices respectively.

\[
J_{6,1} : 
\begin{pmatrix}
e_1 & 2e_2 & e_3 \\
e_2 & e_3 & 0 \\
e_3 & 0 & 0
\end{pmatrix},
\]

\[
J_{6,2} : 
\begin{pmatrix}
e_1 & 0 & e_3 \\
e_2 & e_3 & 0 \\
e_3 & 0 & 0
\end{pmatrix},
\]

\[
J_{6,3} : 
\begin{pmatrix}
e_1 & e_2 & 0 \\
e_2 & 0 & e_3 \\
e_3 & 0 & 0
\end{pmatrix},
\]

\[
J_{6,4} : 
\begin{pmatrix}
e_1 & e_2 & 0 \\
e_2 & 0 & e_3 \\
e_1 & 0 & 0
\end{pmatrix}.
\]
Proposition 4.8. (case $J_7$) The formal characteristic matrix of the Jordan algebra $(J_7, \circ)$ is given by

$$
\begin{pmatrix}
2e_1 & e_2 & 2e_3 \\
e_2 & 2e_3 & 0 \\
2e_3 & 0 & 0
\end{pmatrix}.
$$

Any compatible pre-Jordan algebra structure on $(J_7, \circ)$ is isomorphic to one of the following (mutually non-isomorphic) pre-Jordan algebras given by their formal characteristic matrices respectively.

$J_{7,1} : \begin{pmatrix} e_1 & 0 & e_3 \\ e_2 & e_3 & 0 \\ e_3 & 0 & 0 \end{pmatrix}$, $J_{7,2} : \begin{pmatrix} e_1 & e_2 & 2e_3 \\ 0 & e_3 & e_1 \\ 0 & -e_1 & 0 \end{pmatrix}$, $J_{7,3} : \begin{pmatrix} e_1 & e_2 & 2e_3 \\ 0 & e_3 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $J_{7,4} : \begin{pmatrix} e_1 & 2e_2 & e_3 \\ -e_2 & e_3 & 0 \\ e_3 & 0 & 0 \end{pmatrix}$.

Proposition 4.9. (case $J_8$) The formal characteristic matrix of the Jordan algebra $(J_8, \circ)$ is given by

$$
\begin{pmatrix}
0 & 0 & 0 \\
0 & 2e_2 & e_3 \\
0 & e_3 & 0
\end{pmatrix}.
$$

Any compatible pre-Jordan algebra structure on $(J_8, \circ)$ is isomorphic to one of the following (mutually non-isomorphic) pre-Jordan algebras given by their formal characteristic matrices respectively.

$J_{8,1}(\alpha) : \begin{pmatrix} 0 & \alpha e_1 + e_3 & 0 \\ -\alpha e_1 - e_3 & e_2 & 0 \\ 0 & 0 & (\alpha + 1)(\alpha e_1 + e_3) \end{pmatrix}$, $J_{8,2} : \begin{pmatrix} 0 & -e_1 & 0 \\ e_1 & e_2 & 2e_3 \\ 0 & -e_3 & 0 \end{pmatrix}$.

$J_{8,3} : \begin{pmatrix} 0 & 0 & 0 \\ 0 & e_2 & 2e_3 \\ 0 & -e_3 & 0 \end{pmatrix}$, $J_{8,4} : \begin{pmatrix} 0 & -e_1 & 0 \\ e_1 & e_2 & e_1 + 2e_3 \\ 0 & -e_1 - e_3 & 0 \end{pmatrix}$, $J_{8,5} : \begin{pmatrix} 0 & 0 & 0 \\ 0 & e_2 & e_1 + 2e_3 \\ 0 & -e_1 - e_3 & 0 \end{pmatrix}$, $J_{8,6} : \begin{pmatrix} 0 & -e_1 & 0 \\ e_1 & e_2 & e_3 \\ 0 & 0 & 0 \end{pmatrix}$, $J_{8,7} : \begin{pmatrix} 0 & 0 & 0 \\ 0 & e_2 & 0 \\ 0 & e_3 & 0 \end{pmatrix}$, $J_{8,8} : \begin{pmatrix} 0 & -e_1 & 0 \\ e_1 & e_2 & 0 \\ 0 & e_3 & 0 \end{pmatrix}$, $J_{8,9} : \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & e_2 & 0 \end{pmatrix}$, $J_{8,10} : \begin{pmatrix} 0 & -e_1 & 0 \\ e_2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $J_{8,11} : \begin{pmatrix} 0 & -e_1 & 0 \\ e_1 & e_2 & e_1 \\ 0 & -e_1 + e_3 & 0 \end{pmatrix}$, $J_{8,12} : \begin{pmatrix} 0 & 0 & 0 \\ 0 & e_2 & 0 \\ -e_2 & 0 & 0 \end{pmatrix}$.

Proposition 4.10. (case $A_{11}$) The formal characteristic matrix of the Jordan algebra $(A_{11}, \circ)$ is given by

$$
\begin{pmatrix}
2e_1 & 0 & 0 \\
0 & 2e_2 & 0 \\
0 & 0 & 2e_3
\end{pmatrix}.
$$

Any compatible pre-Jordan algebra structure on $(A_{11}, \circ)$ is isomorphic to one of the following (mutually non-isomorphic) pre-Jordan algebras given by their formal characteristic matrices respectively.

$A_{11,1} : \begin{pmatrix} e_1 & 0 & 0 \\ 0 & e_2 & 0 \\ 0 & 0 & e_3 \end{pmatrix}$, $A_{11,2} : \begin{pmatrix} e_1 & -e_1 & 0 \\ e_1 & e_2 & 0 \\ 0 & 0 & e_3 \end{pmatrix}$, $A_{11,3} : \begin{pmatrix} e_1 & e_2 & e_3 \\ -e_2 & e_3 & 0 \\ -e_3 & 0 & e_3 \end{pmatrix}$, $A_{11,4} : \begin{pmatrix} e_1 & e_2 + e_3 & 0 \\ -e_2 - e_3 & e_2 & e_3 \\ -e_3 & e_3 & 0 \end{pmatrix}$.

Proposition 4.11. (case $A_{12}$) The formal characteristic matrix of the Jordan algebra $(A_{12}, \circ)$ is given by

$$
\begin{pmatrix}
2e_1 & 0 & 0 \\
0 & 2e_2 & 2e_3 \\
0 & 2e_3 & 0
\end{pmatrix}.
$$
Any compatible pre-Jordan algebra structure on \((A_{12}, \circ)\) is isomorphic to one of the following (mutually non-isomorphic) pre-Jordan algebras given by their formal characteristic matrices respectively.

\[
\begin{align*}
A_{12,1} : \begin{pmatrix} e_1 & -e_1 & 0 \\ e_1 & e_2 & e_3 \\ 0 & e_3 & 0 \end{pmatrix},
A_{12,2} : \begin{pmatrix} e_1 & -e_1 & 0 \\ e_1 & e_2 & 2e_3 \\ 0 & 0 & 0 \end{pmatrix},
A_{12,3} : \begin{pmatrix} e_1 & -e_1 & e_3 \\ e_1 & e_2 & e_3 \\ -e_3 & e_3 & 0 \end{pmatrix},
A_{12,4} : \begin{pmatrix} e_1 & 0 & 0 \\ 0 & e_2 & e_3 \\ 0 & e_3 & 0 \end{pmatrix},
A_{12,5} : \begin{pmatrix} e_1 & 0 & 0 \\ 0 & e_2 & 2e_3 \\ 0 & 0 & 0 \end{pmatrix},
A_{12,6} : \begin{pmatrix} e_1 & e_2 & 0 \\ -e_2 & e_2 & 2e_3 \\ 0 & 0 & 0 \end{pmatrix},
A_{12,7} : \begin{pmatrix} e_1 & e_2 & e_3 \\ -e_2 & e_2 & e_3 \\ -e_3 & e_3 & 0 \end{pmatrix}.
\end{align*}
\]

**Proposition 4.12.** (case \(A_{11}\)) The formal characteristic matrix of the Jordan algebra \((A_{13}, \circ)\) is given by

\[
\begin{pmatrix} 2e_1 & 2e_2 & 2e_3 \\ 2e_2 & 2e_3 & 0 \\ 2e_3 & 0 & 0 \end{pmatrix}.
\]

Any compatible pre-Jordan algebra structure on \((A_{13}, \circ)\) is isomorphic to one of the following (mutually non-isomorphic) pre-Jordan algebras given by their formal characteristic matrices respectively.

\[
\begin{align*}
A_{13,1} : \begin{pmatrix} e_1 & e_2 & e_3 \\ e_2 & e_3 & 0 \\ e_3 & 0 & 0 \end{pmatrix},
A_{13,2} : \begin{pmatrix} e_1 & 2e_2 & 2e_3 \\ e_2 & e_3 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\end{align*}
\]

**Proposition 4.13.** (case \(A_{14}\)) The formal characteristic matrix of the Jordan algebra \((A_{14}, \circ)\) is given by

\[
\begin{pmatrix} 2e_1 & 2e_2 & 2e_3 \\ 2e_2 & 0 & 0 \\ 2e_3 & 0 & 0 \end{pmatrix}.
\]

Any compatible pre-Jordan algebra structure on \((A_{14}, \circ)\) is isomorphic to one of the following (mutually non-isomorphic) pre-Jordan algebras given by their formal characteristic matrices respectively.

\[
\begin{align*}
A_{14,1} : \begin{pmatrix} e_1 & e_2 & 2e_3 \\ e_2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},
A_{14,2} : \begin{pmatrix} e_1 & e_2 & e_3 \\ e_2 & 0 & 0 \\ e_3 & 0 & 0 \end{pmatrix},
A_{14,3} : \begin{pmatrix} e_1 & 2e_2 & 2e_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\end{align*}
\]

**Proposition 4.14.** (case \(A_1\)) The formal characteristic matrix of the Jordan algebra \((A_1, \circ)\) is given by

\[
\begin{pmatrix} 2e_1 & 0 & 0 \\ 0 & 2e_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\]

Any compatible pre-Jordan algebra structure on \((A_1, \circ)\) is isomorphic to one of the following (mutually non-isomorphic) pre-Jordan algebras given by their formal characteristic matrices respectively.

\[
\begin{align*}
A_{1,1} : \begin{pmatrix} e_1 & 0 & 0 \\ 0 & e_2 & 0 \\ 0 & 0 & 0 \end{pmatrix},
A_{1,2} : \begin{pmatrix} e_1 & 0 & e_3 \\ 0 & e_2 & 0 \\ e_3 & 0 & 0 \end{pmatrix},
A_{1,3} : \begin{pmatrix} e_1 & e_2 & 0 \\ -e_2 & e_2 & 0 \\ 0 & 0 & 0 \end{pmatrix},
A_{1,4} : \begin{pmatrix} e_1 & e_2 & e_3 \\ -e_2 & e_2 & 0 \\ -e_3 & 0 & 0 \end{pmatrix}.
\end{align*}
\]

**Proposition 4.15.** (case \(A_2\)) The formal characteristic matrix of the Jordan algebra \((A_2, \circ)\) is given by

\[
\begin{pmatrix} 0 & 0 & 0 \\ 0 & 2e_2 & 2e_3 \\ 0 & 2e_3 & 0 \end{pmatrix}.
\]
Any compatible pre-Jordan algebra structure on \((A_2, \circ)\) is isomorphic to one of the following (mutually non-isomorphic) pre-Jordan algebras given by their formal characteristic matrices respectively.

\[
A_{2,1} : \begin{pmatrix} 0 & -e_1 & 0 \\
e_1 & e_2 & e_3 \\
0 & e_3 & 0 \end{pmatrix}, \quad A_{2,2} : \begin{pmatrix} 0 & 0 & 0 \\
e_2 & 2e_3 & 0 \\
0 & 0 & 0 \end{pmatrix}, \quad A_{2,3} : \begin{pmatrix} 0 & -e_1 & 0 \\
e_1 & e_2 & 2e_3 \\
0 & 0 & 0 \end{pmatrix},
\]

\[
A_{2,4} : \begin{pmatrix} 0 & -e_1 & 0 \\
e_1 & e_2 & e_1 + 2e_3 \\
0 & -e_1 & 0 \end{pmatrix}, \quad A_{2,5} : \begin{pmatrix} 0 & 0 & 0 \\
e_2 & e_3 & 0 \\
0 & e_3 & 0 \end{pmatrix}, \quad A_{2,6} : \begin{pmatrix} 0 & e_3 & 0 \\
e_3 & e_2 & e_3 \\
0 & e_3 & 0 \end{pmatrix}.
\]

**Proposition 4.16.** (case \(A_3\)) The formal characteristic matrix of the Jordan algebra \((A_3, \circ)\) is given by

\[
\begin{pmatrix} 2e_1 & 0 & 0 \\
0 & 2e_3 & 0 \\
0 & 0 & 0 \end{pmatrix}.
\]

Any compatible pre-Jordan algebra structure on \((A_3, \circ)\) is isomorphic to one of the following (mutually non-isomorphic) pre-Jordan algebras given by their formal characteristic matrices respectively.

\[
A_{3,1} : \begin{pmatrix} e_1 & 0 & 0 \\
0 & e_3 & 0 \\
0 & 0 & 0 \end{pmatrix}, \quad A_{3,2} : \begin{pmatrix} e_1 & e_2 & e_3 \\
e_2 & e_3 & 0 \\
e_3 & 0 & 0 \end{pmatrix}.
\]

**Proposition 4.17.** (case \(A_4\)) The formal characteristic matrix of the Jordan algebra \((A_4, \circ)\) is given by

\[
\begin{pmatrix} 2e_2 & 2e_3 & 0 \\
2e_3 & 0 & 0 \\
0 & 0 & 0 \end{pmatrix}.
\]

Any compatible pre-Jordan algebra structure on \((A_4, \circ)\) is isomorphic to one of the following (mutually non-isomorphic) pre-Jordan algebras given by their formal characteristic matrices respectively.

\[
A_{4,1}(\alpha) : \begin{pmatrix} e_2 & (1 + \alpha)e_3 & 0 \\
(1 - \alpha)e_3 & 0 & 0 \\
0 & 0 & 0 \end{pmatrix}.
\]

**Proposition 4.18.** (case \(A_5\)) The formal characteristic matrix of the Jordan algebra \((A_5, \circ)\) is given by

\[
\begin{pmatrix} 2e_1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \end{pmatrix}.
\]

Any compatible pre-Jordan algebra structure on \((A_5, \circ)\) is isomorphic to one of the following (mutually non-isomorphic) pre-Jordan algebras given by their formal characteristic matrices respectively.

\[
A_{5,1} : \begin{pmatrix} e_1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \end{pmatrix}, \quad A_{5,2} : \begin{pmatrix} e_1 & e_2 & e_3 \\
e_2 & 0 & 0 \\
e_3 & 0 & 0 \end{pmatrix}, \quad A_{5,3} : \begin{pmatrix} e_1 & e_2 & 0 \\
e_2 & 0 & 0 \\
0 & 0 & 0 \end{pmatrix}.
\]

**Proposition 4.19.** (case \(A_6\)) The formal characteristic matrix of the Jordan algebra \((A_6, \circ)\) is given by

\[
\begin{pmatrix} 0 & 2e_3 & 0 \\
2e_3 & 0 & 0 \\
0 & 0 & 0 \end{pmatrix}.
\]

Any compatible pre-Jordan algebra structure on \((A_6, \circ)\) is isomorphic to one of the following (mutually non-isomorphic) pre-Jordan algebras given by their formal characteristic matrices respectively.

\[
A_{6,1}(\alpha) : \begin{pmatrix} 0 & (1 + \sqrt{\alpha})e_3 & 0 \\
(1 - \sqrt{\alpha})e_3 & 0 & 0 \\
0 & 0 & 0 \end{pmatrix}, \quad A_{6,2} : \begin{pmatrix} 0 & 2e_3 & 0 \\
0 & 0 & e_1 \\
0 & -e_1 & 0 \end{pmatrix}.
\]
Proposition 4.20. (case $A_7$) The formal characteristic matrix of the Jordan algebra $(A_7, \circ)$ is given by
\[
\begin{pmatrix}
0 & 0 & 0 \\
0 & 2e_3 & 0 \\
0 & 0 & 0
\end{pmatrix}.
\]
Any compatible pre-Jordan algebra structure on $(A_7, \circ)$ is isomorphic to one of the following (mutually non-isomorphic) pre-Jordan algebras given by their formal characteristic matrices respectively.

\[
A_{7,1} : \begin{pmatrix} 0 & 0 & 0 \\ 0 & e_3 & e_1 \\ 0 & -e_1 & 0 \end{pmatrix}, \quad A_{7,2} : \begin{pmatrix} 0 & 0 & 0 \\ 0 & e_3 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_{7,3} : \begin{pmatrix} 0 & e_3 & 0 \\ -e_3 & e_3 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\]

Proposition 4.21. (case $A_8$) The formal characteristic matrix of the Jordan algebra $(A_8, \circ)$ is given by
\[
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}.
\]
Any compatible pre-Jordan algebra structure on $(A_8, \circ)$ is isomorphic to one of the following (mutually non-isomorphic) pre-Jordan algebras given by their formal characteristic matrices respectively.

\[
A_{8,1} : \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_{8,2} : \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & e_1 \\ 0 & -e_1 & 0 \end{pmatrix}.
\]

Theorem 4.22. Every 3-dimensional pre-Jordan algebra is isomorphic to one of the pre-Jordan algebras listed in this section.

Proof. It follows by Lemma 2.9. \qed

Corollary 4.23. Adopting the notations of 3-dimensional complex associative algebras in [28], we have the following correspondences for 3-dimensional associative pre-Jordan algebras:

\[
\begin{align*}
J_{2,4} & \simeq U_1^3, \quad J_{3,4} \simeq W_4^3, \quad J_{3,13} \simeq W_7^3, \quad J_{4,2} \simeq W_9^3, \quad J_{4,6} \simeq W_{10}^3, \quad J_{5,1} \simeq C_4^3, \quad J_{5,3} \simeq C_5^3, \quad J_{5,4} \simeq C_2^3, \quad J_{8,7} \simeq W_6^3, \\
J_{8,9} & \simeq W_5^3, \quad A_{1,1} \simeq U_2^3, \quad A_{1,4} \simeq U_4^3, \quad A_{2,3} \simeq U_3^3, \quad A_{4,1} \simeq U_5^3, \quad A_{4,2} \simeq U_3^3, \quad A_{1,1} \simeq S_2^3, \quad A_{2,5} \simeq S_4^3, \quad A_{3,1} \simeq S_2^3.
\end{align*}
\]

$A_{4,1}(\alpha = 0) \simeq S_1^3, \quad A_{5,1} \simeq W_4^3, \quad A_{5,1}(\alpha = 1) \simeq W_3^2, \quad A_{5,1}(\alpha = \frac{k^2}{k^2-4}) \simeq W_3^3(k), \quad A_{7,2} \simeq W_1^3, \quad A_{7,3} \simeq W_3^3(2), \quad A_{8,1} \simeq C_0^3, \quad A_{8,2} \simeq C_1^3.$

5. Rota-Baxter operators on Jordan algebras in dimensions $\leq 3$ and the induced pre-Jordan algebras. In this section, we give all Rota-Baxter operators on complex Jordan algebras in dimensions $\leq 3$ and the induced pre-Jordan algebras. If not assigned concretely, the parameters can be any complex numbers.

Definition 5.1. ([11]) A Rota-Baxter operator (of weight 0) on a Jordan algebra $(J, \circ)$ is a linear transformation $R : J \to J$ satisfying the following identity:

\[
R(x) \circ R(y) = R(R(x) \circ y + x \circ R(y)), \quad \forall x, y \in J.
\]

Proposition 5.2. ([11]) Let $(J, \circ)$ be a Jordan algebra and $R : J \to J$ be a Rota-Baxter operator. Then the bilinear multiplication $\cdot : J \times J \to J$ given by

\[
x \cdot y := R(x) \circ y, \quad \forall x, y \in J,
\]

defines a pre-Jordan algebra.

Proposition 5.3. Keep the notations as in Proposition 3.1. The Rota-Baxter operators on the Jordan algebras $(A, \circ_1)$ given by $e \circ e = e$ are zero and the Rota-Baxter operators on the Jordan algebra $(A, \circ_2)$ given by $e \circ e = 0$ are all elements of $\text{End}(\mathbb{C})$. Both of them induce the pre-Jordan algebra $(\mathbb{C}, \cdot)$ given by $e \cdot e = 0$. 

Proof. Note the first conclusion can be obtained from the fact that all Rota-Baxter operators on a unital finite dimensional complex associative algebra are nilpotent ([29]).

Let \((J, \circ)\) be a Jordan algebra with a basis \(\{e_1, \cdots, e_N\}\). Let \(R : J \to J\) be a linear transformation on \(J\). Set

\[
S(x, y) = R(x) \circ R(y) - R(R(x) \circ y + R(y) \circ x), \quad \forall x, y \in J,
\]

(5.18)

\[
e_i \circ e_j = \sum_{k=1}^n c_{ij}^k e_k, \quad R(e_i) = \sum_{j=1}^N r_{ji} e_j.
\]

(5.19)

Hence we have

\[
S(e_j, e_k) = \sum_{i=1}^N S_{jk}^i e_i,
\]

(5.20)

where

\[
S_{jk}^i = \sum_{n=1}^N \sum_{m=1}^N \left( r_{nj} r_{mk} c_{im}^n - r_{nj} r_{im} c_{nk}^m - r_{nk} r_{im} c_{nj}^m \right).
\]

(5.21)

The following conclusion follows immediately.

**Proposition 5.4.** Keep the notations as above and let \(R = \begin{pmatrix} r_{11} & \cdots & r_{1n} \\ \cdots & \cdots & \cdots \\ r_{n1} & \cdots & r_{nn} \end{pmatrix}\). Then it is a Rota-Baxter operator on the Jordan algebra \((J, \circ)\) if and only if the entries \(r_{ij}\) satisfy the set of equations \(\{S_{ij}^k = 0\}\).

**Proposition 5.5.** Any Rota-Baxter operator on the Jordan algebra \((J_1, \circ)\) is zero, which induces the pre-Jordan algebra isomorphic to \(J_{6,1}\) through Eq. (5.17).

**Proof.** There are still the following “essentially available” equations in the set of equations \(\{S_{ij}^k = 0\}\):

\[
S_{11}^1 = -2r_{11}^2 = 0, \quad S_{22}^1 = 2r_{12}^2 - 4r_{12}r_{22} = 0, \quad S_{11}^2 = 2r_{21}^2 - 4r_{11}r_{21} = 0, \quad S_{22}^2 = -2r_{22}^2 = 0.
\]

There is only one solution:

\[
r_{11} = r_{12} = r_{21} = r_{22} = 0.
\]

Thus the conclusion holds.

**Proposition 5.6.** Any Rota-Baxter operator on the Jordan algebra \((J_2, \circ)\) is of the form \(\begin{pmatrix} 0 & 0 \\ r_{21} & 0 \end{pmatrix}\).

The induced pre-Jordan algebra through Eq. (5.17) is isomorphic to \(J_{6,1}\) when \(r_{21} = 0\) or \(J_{5,1}\) otherwise.

**Proof.** The “essentially available” equations in the set of equations \(\{S_{ij}^k = 0\}\) are:

\[
S_{11}^1 = -2r_{11}^2 - 4r_{12}r_{21} = 0, \quad S_{22}^1 = -2r_{12}^2 = 0, \quad S_{21}^2 = -2r_{22}^2 = 0.
\]

The solutions are

\[
r_{11} = r_{12} = r_{22} = 0, \quad r_{21} \in \mathbb{C}.
\]

The induced pre-Jordan algebras through Eq. (5.17) are given by their formal characteristic matrices \(\begin{pmatrix} r_{21}e_2 & 0 \\ 0 & 0 \end{pmatrix}\). When \(r_{21} = 0\), it is isomorphic to the pre-Jordan algebra \(J_{6,1}\). When \(r_{21} \neq 0\), it is isomorphic to \(J_{5,1}\).

\[\square\]
PROPOSITION 5.7. Any Rota-Baxter operator on the Jordan algebra \((J_3, \circ)\) is of the form \(
abla = \begin{pmatrix} 0 & 0 \\ r_{21} & 0 \\ r_{22} \end{pmatrix}\).

The induced pre-Jordan algebra is isomorphic to \(J_{6,1}\).

Proof. The “essentially available” equations in the set of equations \(\{S_{ij}^k = 0\}\) are:

\[
S_{11}^1 = -2r_{11}^2 = 0, \quad S_{22}^1 = r_{12}^2 = 0.
\]

The solutions are

\[
r_{11} = r_{12} = 0, \quad r_{21} \in \mathbb{C}, \quad r_{22} \in \mathbb{C}.
\]

All induced multiplications vanish.

PROPOSITION 5.8. Any Rota-Baxter operator on the Jordan algebra \((J_4, \circ)\) is of the form \(
abla = \begin{pmatrix} \lambda & -\lambda^2 \\ \mu^2 & -\lambda \mu \end{pmatrix}\), when \(\lambda \neq 0\), the induced pre-Jordan algebra is isomorphic to \(J_{2,2}\), when \(\lambda = 0, \mu \neq 0\), the induced pre-Jordan algebra is isomorphic to \(J_{5,1}\), when \(\lambda = \mu = 0\), the induced pre-Jordan algebra is isomorphic to \(J_{6,1}\).

Proof. Set \(r_{12} = -\lambda^2, r_{21} = \mu^2\), and then the “essentially available” equations in the set of equations \(\{S_{ij}^k = 0\}\) are:

\[
S_{11}^1 = -2r_{11}^2 + 2\lambda^2\mu^2 = 0, \quad S_{22}^1 = -r_{22}^2 + \lambda^2\mu^2 = 0, \quad S_{12}^1 = \lambda^2(r_{11} + r_{22}) = 0.
\]

By choosing the sign of \(\lambda\) properly, we may assume \(r_{11} = \lambda\mu\). If \(r_{22} = \lambda\mu\), then \(\lambda^3\mu = 0, r_{22} = 0 = -\lambda\mu\), so \(r_{22} = -\lambda\mu\) always holds.

We obtain the algebra with the following formal characteristic matrix: \(
\begin{pmatrix} 2\lambda\mu e_1 + \mu^2 e_2 & \lambda\mu e_2 \\ -2\lambda^2 e_1 - \lambda\mu e_2 & -\lambda^2 e_2 \end{pmatrix}.
\)

If \(B \neq 0\): Under the basis \(e'_1 = -\lambda^{-2}e_2, e'_2 = \lambda e_1 + \mu e_2\), the algebra above has the following formal characteristic matrix:

\[
\begin{pmatrix} e'_1 & 2e'_2 \\ 0 & 0 \end{pmatrix}.
\]

If \(\lambda = 0\): The formal characteristic matrix is \(
\begin{pmatrix} \mu^2 e_2 & 0 \\ 0 & 0 \end{pmatrix},
\)

and the rest of proof is the same as Theorem 5.6.

PROPOSITION 5.9. Any Rota-Baxter operator on the Jordan algebra \((J_5, \circ)\) is of the following forms:

(a) \(
\begin{pmatrix} 0 & 0 \\ r_{21} & r_{22} \end{pmatrix},
\)

the induced pre-Jordan algebra is isomorphic to \(J_{6,1}\);

(b) \(
\begin{pmatrix} 2r_{22} & 0 \\ r_{21} & r_{22} \end{pmatrix} (r_{22} \neq 0),
\)

the induced pre-Jordan algebra is isomorphic to \(J_{5,1}\).

Proof. The “essentially available” equations in the set of equations \(\{S_{ij}^k = 0\}\) are:

\[
S_{12}^1 = -2r_{12}^2, \quad S_{11}^1 = 2r_{11}(r_{11} - 2r_{22}).
\]

The solutions are

\[
r_{11} = 0 \text{ or } 2r_{22}, \quad r_{12} = 0, \quad r_{21} \in \mathbb{C}, \quad r_{22} \in \mathbb{C}.
\]

This completes the first assertion.

Pre-Jordan algebras induced by the former ones are isomorphic to \(J_{6,1}\). And the latter ones induce the formal characteristic matrices \(
\begin{pmatrix} 2r_{22}e_2 & 0 \\ 0 & 0 \end{pmatrix},
\)

which are isomorphic to \(J_{5,1}\).

PROPOSITION 5.10. Every linear transformation on \((J_6, \circ)\) is a Rota-Baxter operator. The induced pre-Jordan algebra is isomorphic to \(J_{6,1}\).

Proof. Obvious.

Summarizing the above study, we have the following conclusion:
**Theorem 5.11.** Each pre-Jordan algebra obtained from Rota-Baxter operators on 2-dimensional Jordan algebras through Eq. (5.17) is isomorphic to one of the pre-Jordan algebras $J_{2,2}$, $J_{5,1}$ and $J_{6,1}$.

Next we give all Rota-Baxter operators on 3-dimensional Jordan algebras and the induced pre-Jordan algebras through Eq. (5.17). We only list the results while omitting the proof.

**Proposition 5.12.** Any Rota-Baxter operator on the Jordan algebra $(\beta_1, \circ)$ is of the following forms:

(a) \[
\begin{pmatrix}
0 & -\frac{r_{13}}{2r_{23}} & r_{13} \\
0 & -\frac{r_{23}}{2r_{23}} & -\frac{r_{23}}{2r_{23}} \\
0 & -\frac{r_{23}}{2r_{23}} & r_{13}
\end{pmatrix}
\] \( (r_{23} \neq 0) \),

when $r_{13} = 0$, the induced pre-Jordan algebra is isomorphic to $A_{4,1}(\alpha = 1)$,

when $r_{13} \neq 0$, the induced pre-Jordan algebra is isomorphic to $A_{2,2}$;

(b) \[
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
r_{32} & 0 & 0
\end{pmatrix}
\]

when $r_{32} = 0$, the induced pre-Jordan algebra is isomorphic to $A_{8,1}$,

when $r_{32} \neq 0$, the induced pre-Jordan algebra is isomorphic to $A_{4,1}(\alpha = 1)$.

**Proposition 5.13.** Any Rota-Baxter operator on the Jordan algebra $(\beta_2, \circ)$ is of the following forms:

(a) \[
\begin{pmatrix}
0 & 0 & 0 \\
r_{31} & 0 & 0 \\
r_{32} & 0 & 0
\end{pmatrix}
\]

when $r_{31} \neq r_{32}$, the induced pre-Jordan algebra is isomorphic to $A_{6,1}(\alpha = 1)$,

when $r_{31} = r_{32} = 0$, the induced pre-Jordan algebra is isomorphic to $A_{8,1}$,

when $r_{31} = r_{32} \neq 0$, the induced pre-Jordan algebra is isomorphic to $A_{7,2}$;

(b) \[
\begin{pmatrix}
r_{31} & -r_{33} & r_{33} \\
r_{32} & -r_{33} & 0 \\
r_{33} & -r_{33} & 0
\end{pmatrix}
\]

when $r_{32} = 0$, the induced pre-Jordan algebra is isomorphic to $A_{2,2}$;

(c) \[
\begin{pmatrix}
r_{31} & r_{13} & 0 \\
r_{32} & r_{13} & 0 \\
r_{33} & r_{13} & 0
\end{pmatrix}
\]

the induced pre-Jordan algebra is isomorphic to $A_{2,2}$.

**Proposition 5.14.** Any Rota-Baxter operator on the Jordan algebra $(\beta_3, \circ)$ is of the following forms:

(a) \[
\begin{pmatrix}
0 & 0 & 0 \\
r_{31} & 0 & 0 \\
r_{32} & 0 & 0
\end{pmatrix}
\]

the induced pre-Jordan algebra is isomorphic to $A_{6,1}(\alpha = 1)$;

(b) \[
\begin{pmatrix}
0 & r_{22} & r_{23} \\
0 & -r_{23} & -r_{23} \\
0 & -r_{23} & -r_{23}
\end{pmatrix}
\]

the induced pre-Jordan algebra is isomorphic to $A_{2,2}$;

(c) \[
\begin{pmatrix}
0 & r_{12} & r_{13} \\
0 & r_{12} & r_{13} \\
r_{32} & -r_{12} & 0
\end{pmatrix}
\]

when $r_{12} \neq 0$ or $r_{12} = r_{32} = 0$, the induced pre-Jordan algebra is isomorphic to $A_{14,3}$,

when $r_{12} = 0$, the induced pre-Jordan algebra is isomorphic to $A_{5,2}$,

when $r_{12} = r_{13} = r_{32} = 0$, the induced pre-Jordan algebra is isomorphic to $A_{8,1}$.
Proposition 5.15. Any Rota-Baxter operator on the Jordan algebra \((\mathfrak{J}_4, \circ)\) is of the following forms:

(a) \[
\begin{pmatrix}
  r_{11} & r_{12} & 0 \\
  r_{21} & -r_{11} & 0 \\
  r_{31} & 0 & 0
\end{pmatrix}
\] (\(r_{12}^2 + r_{12}r_{21} = 0\)),
when \(r_{31} = 0\), the induced pre-Jordan algebra is isomorphic to \(A_{14,3}\),
when \(r_{31} \neq 0\), the induced pre-Jordan algebra is isomorphic to \(A_{13,2}\);

(b) \[
\begin{pmatrix}
  0 & 0 & 0 \\
  r_{21} & 0 & r_{23} \\
  0 & 0 & 0
\end{pmatrix}
\] (\(r_{23} \neq 0\)),
the induced pre-Jordan algebra is isomorphic to \(A_{6,1}(\alpha = 1)\);

(c) \[
\begin{pmatrix}
  r_{21} & 0 & 0 \\
  0 & 0 & 0 \\
  r_{31} & 0 & 0
\end{pmatrix}
\] (\(r_{21} \neq 0\)),
the induced pre-Jordan algebra is isomorphic to \(A_{7,2}\);

(d) \[
\begin{pmatrix}
  0 & 0 & 0 \\
  0 & 0 & 0 \\
  r_{31} & r_{32} & 0
\end{pmatrix}
\]
when \(r_{31} = r_{32} = 0\), the induced pre-Jordan algebra is isomorphic to \(A_{8,1}\),
when \(r_{32} \neq 0\), the induced pre-Jordan algebra is isomorphic to \(A_{6,1}(\alpha = 1)\),
when \(r_{32} = 0, r_{31} \neq 0\), the induced pre-Jordan algebra is isomorphic to \(A_{7,2}\);

(e) \[
\begin{pmatrix}
  r_{21} & r_{31} & r_{33} \\
  -2r_{33} & r_{23} & 0 \\
  r_{31} & -2r_{33} & r_{33}
\end{pmatrix}
\] (\(r_{23}, r_{33} \neq 0\)),
the induced pre-Jordan algebra is isomorphic to \(A_{6,1}(\alpha = 1)\).

Proposition 5.16. Any Rota-Baxter operator on the Jordan algebra \((\mathfrak{J}_5, \circ)\) is of the following forms:

(a) \[
\begin{pmatrix}
  0 & 0 & 0 \\
  0 & 0 & 0 \\
  0 & 0 & 0
\end{pmatrix}
\]
the induced pre-Jordan algebra is isomorphic to \(A_{8,1}\);

(b) \[
A \begin{pmatrix}
  0 & 1 & 0 \\
  0 & 0 & 0 \\
  0 & 0 & 0
\end{pmatrix} A^{-1}(A \in \text{Aut}(\mathfrak{J}_5, \circ)),
\]
the induced pre-Jordan algebra is isomorphic to \(\mathfrak{J}_{4,8}\);

(c) \[
A \begin{pmatrix}
  0 & 0 & 1 \\
  0 & 0 & 0 \\
  0 & 0 & 0
\end{pmatrix} A^{-1}(A \in \text{Aut}(\mathfrak{J}_5, \circ)),
\]
the induced pre-Jordan algebra is isomorphic to \(A_{6,1}(\alpha = 1)\);

(d) \[
A \begin{pmatrix}
  0 & 0 & 0 \\
  0 & 0 & 0 \\
  1 & 0 & 0
\end{pmatrix} A^{-1}(A \in \text{Aut}(\mathfrak{J}_5, \circ)),
\]
the induced pre-Jordan algebra is isomorphic to \(A_{7,2}\),
where \(\text{Aut}(\mathfrak{J}_5, \circ) = \begin{pmatrix}
  1 & 0 & 0 \\
  \lambda_1 & \lambda_2 & \lambda_3 \\
  \lambda_4 & \lambda_5 & \lambda_6
\end{pmatrix}\) (\(\lambda_2\lambda_6 - \lambda_3\lambda_5 \neq 0\)).

Proposition 5.17. Any Rota-Baxter operator on the Jordan algebra \((\mathfrak{J}_6, \circ)\) is of the following forms:

(a) \[
\begin{pmatrix}
  0 & r_{12} & 0 \\
  0 & 0 & 0 \\
  0 & 0 & r_{33}
\end{pmatrix}
\] (\(r_{12} \neq 0\),

20
the induced pre-Jordan algebra is isomorphic to $A_{2,2}$;

(b) \[
\begin{pmatrix}
0 & 0 & 0 \\
r_{31} & r_{32} & r_{33}
\end{pmatrix},
\]
the induced pre-Jordan algebra is isomorphic to $A_{8,1}$;

(c) \[
\begin{pmatrix}
r_{11} & -\frac{r_{11}^2}{r_{21}} & 0 \\
r_{21} & -r_{11} & 0 \\
r_{31} & r_{21} - \frac{2r_{11}r_{31}}{r_{21}} & -r_{11} + \frac{r_{11}^2 + r_{31}}{r_{21}}
\end{pmatrix} \quad (r_{21} \neq 0),
\]
when $r_{11} = 0$, the induced pre-Jordan algebra is isomorphic to $A_{4,1}(\alpha = 1)$, when $r_{11} \neq 0$, the induced pre-Jordan algebra is isomorphic to $A_{2,2}$.

**Proposition 5.18.** Any Rota-Baxter operator on the Jordan algebra $(\mathfrak{d}_7, \circ)$ is of the following forms:

(a) \[
\begin{pmatrix}
0 & 0 & 0 \\
r_{31} & r_{32} & 0
\end{pmatrix},
\]
the induced pre-Jordan algebra is isomorphic to $A_{6,1}(\alpha = 1)$;

(b) \[
\begin{pmatrix}
r_{12} & 0 \\
r_{31} & 0 & 0
\end{pmatrix},
\]
when $r_{12} = r_{31} = 0$, the induced pre-Jordan algebra is isomorphic to $A_{6,1}$, when $r_{12} = 0, r_{31} \neq 0$, the induced pre-Jordan algebra is isomorphic to $A_{7,2}$, when $r_{12} \neq 0, r_{31} = 0$, the induced pre-Jordan algebra is isomorphic to $A_{4,3}$, when $r_{12} \neq 0, r_{31} \neq 0$, the induced pre-Jordan algebra is isomorphic to $A_{1,3,2}$;

(c) \[
\begin{pmatrix}
r_{11} & -\frac{r_{11}^2}{r_{21}} & 0 \\
r_{21} & -r_{11} & 0 \\
r_{31} & r_{21} & 0
\end{pmatrix} \quad (r_{21} \neq 0),
\]
when $r_{11} = 0$, the induced pre-Jordan algebra is isomorphic to $A_{4,1}(\alpha = 0)$, when $r_{11} \neq 0, r_{21} + r_{11}r_{31} \neq 0$, the induced pre-Jordan algebra is isomorphic to $A_{1,3,2}$, when $r_{11} \neq 0, r_{21} + r_{11}r_{31} = 0$, the induced pre-Jordan algebra is isomorphic to $A_{1,4,3}$.

**Proposition 5.19.** Any Rota-Baxter operator on the Jordan algebra $(\mathfrak{d}_8, \circ)$ is of the following forms:

(a) \[
\begin{pmatrix}
r_{11} & r_{12} & 0 \\
r_{31} & 0 & 0
\end{pmatrix},
\]
when $r_{31} \neq 0$, the induced pre-Jordan algebra is isomorphic to $A_{6,1}(\alpha = 1)$, when $r_{31} = 0, r_{32} \neq 0$, the induced pre-Jordan algebra is isomorphic to $A_{7,2}$, when $r_{31} = r_{32} = 0$, the induced pre-Jordan algebra is isomorphic to $A_{6,1}$;

(b) \[
\begin{pmatrix}
r_{22} & r_{23} \\
r_{32} & r_{33}
\end{pmatrix},
\]
the induced pre-Jordan algebra is isomorphic to $A_{2,2}$;

(c) \[
\begin{pmatrix}
r_{11} & r_{12} & r_{13} \\
r_{31} & 0 & 0 \\
r_{32} & 0 & 0
\end{pmatrix} \quad (r_{13} \neq 0),
\]
the induced pre-Jordan algebra is isomorphic to $A_{8,1}$.

**Proposition 5.20.** Any Rota-Baxter operator on $A_1$, is zero and the induced pre-Jordan algebra is isomorphic to $A_{8,1}$.

**Proposition 5.21.** Any Rota-Baxter operator on the Jordan algebra $(A_{12}, \circ)$ is of the following form:
When \( r_{31} \neq 0 \), the induced pre-Jordan algebra is isomorphic to \( A_{6,1}(\alpha = 1) \); when \( r_{31} = 0, r_{32} \neq 0 \), the induced pre-Jordan algebra is isomorphic to \( A_{7,2} \); when \( r_{31} = r_{32} = 0 \), the induced pre-Jordan algebra is isomorphic to \( A_{8,1} \).

**Proposition 5.22.** Any Rota-Baxter operator on the Jordan algebra \((A_{13}, \circ)\) is of the following forms:

(a) \[
\begin{pmatrix}
0 & 0 & 0 \\
r_{32} & 0 & 0 \\
r_{31} & r_{32} & 0
\end{pmatrix},
\]

\( (r_{32} \neq 0) \),

the induced pre-Jordan algebra is isomorphic to \( A_{4,1}(\alpha = \frac{1}{3}) \);

(b) \[
\begin{pmatrix}
0 & 0 & 0 \\
r_{31} & r_{32} & 0
\end{pmatrix},
\]

when \( r_{32} \neq 0 \), the induced pre-Jordan algebra is isomorphic to \( A_{6,1}(\alpha = 1) \);

when \( r_{32} = 0, r_{31} \neq 0 \), the induced pre-Jordan algebra is isomorphic to \( A_{7,2} \);

when \( r_{31} = r_{32} = 0 \), the induced pre-Jordan algebra is isomorphic to \( A_{8,1} \).

**Proposition 5.23.** Any Rota-Baxter operator on the Jordan algebra \((A_{14}, \circ)\) is of the following forms:

(a) \[
\begin{pmatrix}
0 & 0 & 0 \\
r_{21} & 0 & 0 \\
r_{31} & 0 & 0
\end{pmatrix},
\]

when \( r_{21} = r_{31} = 0 \), the induced pre-Jordan algebra is isomorphic to \( A_{8,1} \), otherwise, the induced pre-Jordan algebra is isomorphic to \( A_{7,2} \);

(b) \[
A \begin{pmatrix}
0 & 0 & 0 \\
r_{31} & 1 & 0
\end{pmatrix} A^{-1}(A \in \text{Aut}(A_{14}, \circ)) ,
\]

the induced pre-Jordan algebra is isomorphic to \( A_{6,1}(\alpha = 1) \),

where \( \text{Aut}(A_{14}, \circ) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda_1 & \lambda_2 \\ 0 & \lambda_3 & \lambda_4 \end{pmatrix} (\lambda_1 \lambda_4 - \lambda_2 \lambda_3 \neq 0) \).

**Proposition 5.24.** Any Rota-Baxter operator on the Jordan algebra \((A_1, \circ)\) is of the following form:

\[
\begin{pmatrix}
0 & 0 & 0 \\
r_{31} & r_{32} & r_{33}
\end{pmatrix},
\]

the induced pre-Jordan algebra is isomorphic to \( A_{8,1} \).

**Proposition 5.25.** Any Rota-Baxter operator on the Jordan algebra \((A_2, \circ)\) is of the following forms:

(a) \[
\begin{pmatrix}
r_{11} & r_{12} & 0 \\
r_{31} & r_{32} & 0
\end{pmatrix},
\]

when \( r_{31} \neq 0 \), the induced pre-Jordan algebra is isomorphic to \( A_{6,1}(\alpha = 1) \);

when \( r_{31} = 0, r_{32} \neq 0 \), the induced pre-Jordan algebra is isomorphic to \( A_{7,2} \);

when \( r_{31} = r_{32} = 0 \), the induced pre-Jordan algebra is isomorphic to \( A_{8,1} \);

(b) \[
\begin{pmatrix}
0 & 0 & 0 \\
r_{11} & r_{12} & r_{13}
\end{pmatrix} (r_{13} \neq 0),
\]

the induced pre-Jordan algebra is isomorphic to \( A_{8,1} \).

**Proposition 5.26.** Any Rota-Baxter operator on the Jordan algebra \((A_3, \circ)\) is of the following forms:
(a) \[
\begin{pmatrix}
0 & 0 & 0 \\
r_{31} & 0 & 0 \\
r_{32} & r_{33} & 0
\end{pmatrix},
\]
the induced pre-Jordan algebra is isomorphic to \(A_{8,1}\);

(b) \[
\begin{pmatrix}
0 & 0 & 0 \\
r_{31} & 0 & 0 \\
r_{32} & r_{33} & 0
\end{pmatrix}
\begin{pmatrix}
r_{33} \\
0 \\
r_{33}
\end{pmatrix}
\]
\((r_{33} \neq 0)\),
the induced pre-Jordan algebra is isomorphic to \(A_{7,2}\).

**Proposition 5.27.** Any Rota-Baxter operator on the Jordan algebra \((A_4, \circ)\) is of the following forms:

(a) \[
\begin{pmatrix}
0 & 0 & 0 \\
r_{31} & 0 & 0 \\
r_{32} & r_{33} & 0
\end{pmatrix}
\]
\((r_{33} \neq 0)\),
the induced pre-Jordan algebra is isomorphic to \(A_{8,1}\);

(b) \[
\begin{pmatrix}
0 & 0 & 0 \\
r_{21} & r_{22} & 0 \\
r_{31} & r_{32} & 0
\end{pmatrix}
\]
when \(r_{22} \neq 0\), the induced pre-Jordan algebra is isomorphic to \(A_{6,1}(\alpha = 1)\),
when \(r_{22} = 0, r_{21} \neq 0\), the induced pre-Jordan algebra is isomorphic to \(A_{7,2}\),
when \(r_{21} = r_{22} = 0\), the induced pre-Jordan algebra is isomorphic to \(A_{8,1}\);

(c) \[
\begin{pmatrix}
r_{11} & 0 & 0 \\
\frac{1}{2} r_{11} & r_{11} & 0 \\
\frac{1}{3} r_{31} & \frac{1}{3} r_{32} & \frac{1}{3} r_{11}
\end{pmatrix}
\]
\((r_{11} \neq 0)\),
the induced pre-Jordan algebra is isomorphic to \(A_{4,1}(\alpha = \frac{1}{4})\).

**Proposition 5.28.** Any Rota-Baxter operator on the Jordan algebra \((A_5, \circ)\) is of the following form:

\[
\begin{pmatrix}
0 & 0 & 0 \\
r_{21} & r_{22} & r_{23} \\
r_{31} & r_{32} & r_{33}
\end{pmatrix}
\]
the induced pre-Jordan algebra is isomorphic to \(A_{8,1}\).

**Proposition 5.29.** Any Rota-Baxter operator on the Jordan algebra \((A_6, \circ)\) is of the following forms:

(a) \[
\begin{pmatrix}
r_{11} & r_{12} & 0 \\
0 & 0 & 0 \\
r_{31} & r_{32} & 0
\end{pmatrix}
\]
\((r_{11} \neq 0)\),
the induced pre-Jordan algebra is isomorphic to \(A_{6,1}(\alpha = 1)\);

(b) \[
\begin{pmatrix}
0 & 0 & 0 \\
r_{31} & 0 & 0 \\
r_{32} & r_{33} & 0
\end{pmatrix}
\]
\((r_{33} \neq 0)\),
the induced pre-Jordan algebra is isomorphic to \(A_{8,1}\);

(c) \[
\begin{pmatrix}
r_{11} & 0 & 0 \\
\frac{1}{2} r_{11} & r_{11} & 0 \\
\frac{3}{2} r_{31} & \frac{3}{2} r_{32} & \frac{3}{2} r_{11}
\end{pmatrix}
\]
\((r_{11} \neq 0, r_{33} \neq 0, r_{11} \neq r_{33})\),
the induced pre-Jordan algebra is isomorphic to \(A_{6,1}(\alpha = (1 - \frac{2 r_{33}}{r_{11}})^2)\);

(d) \[
\begin{pmatrix}
0 & r_{12} & 0 \\
r_{31} & 0 & 0 \\
r_{32} & r_{33} & 0
\end{pmatrix}
\]
when \(r_{12} \neq 0\), the induced pre-Jordan algebra is isomorphic to \(A_{7,2}\),
when \(r_{12} = 0\), the induced pre-Jordan algebra is isomorphic to \(A_{8,1}\);

(e) \[
\begin{pmatrix}
r_{11} & \frac{1}{2} r_{11} & 0 \\
r_{21} & r_{11} & 0 \\
r_{31} & r_{32} & r_{11}
\end{pmatrix}
\]
\((r_{21} \neq 0)\),
the induced pre-Jordan algebra is isomorphic to \(A_{7,2}\);
(f) \[
\begin{pmatrix}
0 & 0 & 0 \\
r_{21} & r_{22} & 0 \\
r_{31} & r_{32} & 0
\end{pmatrix} \quad \text{(} r_{22} \neq 0 \text{)},
\]
the induced pre-Jordan algebra is isomorphic to $A_{6,1}(\alpha = 1)$.

**Proposition 5.30.** Any Rota-Baxter operator on the Jordan algebra $(A_7, \circ)$ is of the following forms:

(a) \[
\begin{pmatrix}
r_{11} & r_{12} & r_{13} \\
0 & 0 & 0 \\
r_{31} & r_{32} & r_{33}
\end{pmatrix},
\]
the induced pre-Jordan algebra is isomorphic to $A_{8,1}$;

(b) \[
\begin{pmatrix}
r_{11} & r_{12} & 0 \\
0 & 2r_{33} & 0 \\
r_{31} & r_{32} & r_{33}
\end{pmatrix} \quad \text{(} r_{33} \neq 0 \text{)},
\]
the induced pre-Jordan algebra is isomorphic to $A_{7,2}$.

**Proposition 5.31.** All linear transformations of $A_8$ are Rota-Baxter operators. The induced pre-Jordan algebra is isomorphic to $A_{8,1}$.

Summarizing the above study, we have the following conclusion:

**Theorem 5.32.** Each 3-dimensional pre-Jordan algebra induced by a Rota-Baxter operator on a Jordan algebra is isomorphic to one of the following pre-Jordan algebras:

$J_{4,8}$, $A_{1,2}$, $A_{4,3}$, $A_{2,2}$, $A_{4,1}(\alpha = 0, \frac{1}{2}, 1)$, $A_{6,1}(\alpha \in \mathbb{C})$, $A_{7,2}$, $A_{8,1}$.

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