Translating the Concept of Five-Dimensional Tangent Vectors in Space-Time to Internal Gauge Symmetries in Elementary Particle Physics

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Abstract. This paper discusses the analogs of space-time five-dimensional tangent vectors for the case of internal symmetry vector spaces associated with gauge groups in elementary particle physics and their possible application in the Standard Model of quarks and leptons.

1. Introduction

Vector-like objects not directly associated with the space-time manifold are widely used in today’s theory of elementary particle. In their properties, such objects resemble ordinary tangent vectors associated with some differentiable, usually complex, manifold endowed with appropriate metric. Since, historically, such nonspacetime vectors have been first introduced to reflect certain symmetries—exact or approximate—observed in the world of elementary particles, physicists usually speak of representations with respect to a symmetry group rather than of nonspacetime vectors of a certain kind; of conjugation and contraction rather than of an antilinear/linear inner (scalar) product; of “gauging” the corresponding symmetry rather than of introducing a new kind of geometry represented by nontrivial rules of parallel transport for the considered type of nonspacetime vectors; and of the corresponding gauge fields rather than of connection coefficients.

Mathematically, both approaches are equivalent. However, when regarded as paradigms within which one searches for new physics beyond the Standard Model, they may offer one different possibilities. One particular advantage of the “geometric” approach—the one where one talks about nonspacetime vectors and their parallel transport—is that within it one may pose the following question: If the nonspacetime analogs of ordinary tangent vectors find such useful application in the theory of elementary particle, could it be that the analogs of five-dimensional tangent vectors in space-time can also be of use?

Let me recall that the so-called five-dimensional tangent vectors (see ref.[1] or ref.[2] for a simpler presentation) are a special kind of geometric objects that can be defined in space-time. Like ordinary tangent vectors, these five-dimensional vectors and the tensors constructed out of them can be used for describing certain local quantities and in this capacity find direct application in physics. For example, such familiar physical quantities as the stress-energy and angular momentum tensors prove to be parts of a single five-tensor. Similar objects can be defined in any other differentiable manifold. In the latter case they will be called \((n+1)\)-vectors, where \(n\) is the dimension of the manifold. At each point of the latter, such vectors
form a vector space whose dimension is one unit greater than that of the manifold and which is isomorphic to the direct product of the vector space of ordinary tangent vectors and of an abstract one-dimensional vector space. In other words, algebraically, each \((n+1)\)-vector is equivalent to a pair consisting of an ordinary tangent vector and a scalar. What makes the latter pair an \((n+1)\)-vector is the special way in which its components changes under parallel transport. In the particular case of the space-time manifold, the rules of parallel transport for five-vectors are such that the four-vector in the pair representing the five-vector is transported according to ordinary rules of parallel transport for four-vectors, while the scalar in the pair acquires an additive which at each infinitesimal step of the transport is proportional to the scalar product of the four-vector in the pair with the infinitesimal four-vector representing this step. In the case of \((n+1)\)-vectors of any other type the situation is similar: The rules of parallel transport for \((n+1)\)-vectors are such that the ordinary \((n\text{-dimensional})\) tangent vector in the pair representing it is transported according to the rules of parallel transport fixed by the corresponding ordinary gauge fields, while the scalar in the pair acquires an additive which at each infinitesimal step of the transport is a linear function of the vector components. In more detail, \((n+1)\)-vectors are examined in the last section of ref.[3].

2. The necessary technicalities

It is easier to understand the situation if one writes out the formulae that give the components of the covariant derivate of an \((n+1)\)-vector. Let us introduce the following notations. The components of ordinary \(n\)-dimensional nonspacetime vectors in some appropriate basis will be denoted as \(U^i, V^i, \ldots\), where the index \(i\), as all lower-case Roman indices, is assumed to run 1 through \(n\). The components of the covariant derivative of the vector field \(U^i\) are given by the usual formula:

\[
U^i;\mu = \partial_\mu U^i + C^i_{j\mu} U^j,
\]

where \(C^i_{j\mu}\) are ordinary gauge fields. The components of \((n+1)\)-vectors in a corresponding basis will be denoted as \(u^\Theta, v^\Theta, \ldots\), where the index \(\Theta\), as all upper-case Greek indices, is assumed to run 1 through \((n+1)\). The components of the covariant derivative of the vector field \(u^\Theta\) are then given by the following formulæ:

\[
u^i;\mu = \partial_\mu u^i + C^i_{j\mu} u^j
\]

and

\[
u^{(n+1)};\mu = \partial_\mu u^{(n+1)} + C^{(n+1)}_{j\mu} u^j + C^{(n+1)}_{(n+1)\mu} u^{(n+1)}
\]

where \(i, j = 1, \ldots, n\). As one can see, the expression for the first \(n\) components of the covariant derivative of an \((n+1)\)-vector field is exactly the same as it is for the field whose values are ordinary \((n\text{-dimensional})\) vectors. However, the \((n+1)\)-st component of such a covariant derivative has two kinds of terms involving connection coefficients: (i) a term proportional to \(u^{(n+1)}\), which reflects the change in the absolute value and/or phase of the \((n+1)\)-st component of the field not related to its first \(n\) components; and (ii) terms proportional to the components \(u^i\), which reflect the “admixing” of these latter components to \(u^{(n+1)}\).

In the particular case of complex vectors for which the inner product is Hermitian and is positively definite, formula (1) can be presented in the following more familiar form:

\[
u^i;\mu = \partial_\mu u^i + (i/2) g (t_\alpha)_{ij} C^\alpha_{\mu} u^j + ig [2n(n+1)]^{-1/2} C^\alpha_{\mu} u^i,
\]

where the index \(\alpha\) runs 1 through \(n^2-1\); the matrices \((t_\alpha)_{ij}\) are the usual (Hermitian) generators for the fundamental representation of \(SU(n)\), normalized by the condition \(\text{Tr}(t_\alpha t_\beta) = 2\delta_{\alpha\beta}\); the fields \(C^\alpha_{\mu}\) and \(C^\alpha_{\mu}\) are real; and \(g\) is a dimensionless constant, which together with the factors
1/2 and \([2n(n+1)]^{-1/2}\) is introduced for convenience. In a similar manner, formula (2) can be cast into the following form:

\[ u^{(n+1)}_{\mu} = \partial_{\mu} u^{(n+1)} - ig [n/2(n+1)]^{1/2} C_{\mu}^a u^{(n+1)} + gX_{j\mu} u^j, \]

where \(X_{j\mu} \equiv g^{-1}C^{(n+1)}_{j\mu}\). The expression for the components of the covariant derivative of a field \(v_i\) whose values are linear forms (covariant vectors) associated with the considered type of \((n+1)\)-vectors, can be written down as follows:

\[ v_{i;\mu} = \partial_{\mu} v_i - (i/2) g v_j(t^a)_{ij} C_{\mu}^a - ig [2n(n+1)]^{-1/2} v_i C_{\mu}^a - g v_{(n+1)} X_{i\mu} \]

and

\[ v_{(n+1);\mu} = \partial_{\mu} v_{(n+1)} + ig [n/2(n+1)]^{1/2} v_{(n+1)} C_{\mu}^a. \]

One should note that the interaction with the fields \(X_{i\mu}\) is not \(C\)-invariant, which means that in this case the charge asymmetry is implemented directly in the nonspacetime degrees of freedom of the fields. A more detailed discussion of this issue can be found in ref.[3].

### 3. A simplified model

Let me now show how the concept of \((n+1)\)-vectors can be applied in the framework of the Standard Model to one generation of leptons. Let us take the first generation, where one has a doublet of left-handed particles \((e_L, \nu_L)\), where \(e\) is the electron and \(\nu\) is the electronic neutrino. Following our recipe for constructing \((n+1)\)-vectors, one should suppose that these fields are the first two components of a \((2+1)\)-vector or of a two-plus-one-vector 1-form. Furthermore, one should suppose that there exists a third left-handed field, say, \(\chi_L\), that makes up the third component of this \((2+1)\)-vector or this two-plus-one-vector 1-form.

One now has to decide whether this triplet is a \((2+1)\)-vector or a two-plus-one-vector 1-form. In the case of ordinary nonspacetime vectors, a similar question was rather trivial, for one could always choose the generators of the gauge group in such a way that the considered \(n\)-plet would transform according to the fundamental representation. If need be, one could always redefine the symmetry group so that this \(n\)-plet would transform according to its anti-fundamental representation. In the case of \((n+1)\)-vectors the choice between a fundamental and an anti-fundamental representation—alias, between a vector and a 1-form—should be made based on the kind of interaction one wishes to have between the fields involved. Indeed, if the triplet \((e_L, \nu_L, \chi_L)\) were components of a \((2+1)\)-vector, then according to equation (4), there should exist an interaction, mediated by the gauge fields \(X_{j\mu}\), by means of which the doublet \((e_L, \nu_L)\) would decay into \(\chi_L\). At the same time, since the triplet of anti-particles \((\bar{\nu}_L, \bar{\mu}_L, \bar{\chi}_L)\) in this case form a two-plus-one-vector 1-form, by virtue of equation (5), the same interaction should result in that the SU(2) singlet \(\bar{\chi}_L\) would decay into the doublet \((\bar{\nu}_L, \bar{\mu}_L)\). As a result, one would be left with the anti-particles \(\bar{\nu}_L\) and \(\bar{\mu}_L\) and particle \(\chi_L\), which is not what one observes in reality.

The alternative is that the triplet \((e_L, \nu_L, \chi_L)\) be a two-plus-one-vector 1-form. In this case, in accordance with equation (5), the particle \(\chi_L\) would decay into the doublet \((e_L, \nu_L)\), while the pair of anti-particles \((\bar{\nu}_L, \bar{\mu}_L)\) would decay into the SU(2) singlet \(\bar{\chi}_L\). In the end, one will be left with the particles \(e_L\) and \(\nu_L\) and anti-particle \(\chi_L\), the latter being a right-handed particle. Considering what one observes in reality, within this simplified model, it is natural to suppose that \(\chi_L\) is actually the right-handed electron, \(e_R\). One therefore comes to the conclusion that the SU(2) doublet \((e_L, \nu_L)\) and the anti-particle \(\bar{\nu}_R\), which is an SU(2) singlet, make up a two-plus-one-vector 1-form.

We thus see that the additional interaction that should exist if \(e_L, \nu_L,\) and \(\bar{\nu}_R\) indeed make up a two-plus-one-vector 1-form, may have very well resulted in the observed particle-antiparticle asymmetry in the Universe. Of course, to demonstrate that this is indeed so, one should construct a more complicated model that would include quarks and other generations.
4. Conclusion
The purpose of this short paper was to demonstrate how the concept of \((n+1)\)-vectors can be applied in the framework of the Standard Model to one generation of leptons, and to describe one interesting phenomenon that should occur if the assumptions made are true. The considered case should be regarded only as a toy model that just illustrates the idea, but cannot pretend to be a real model. The inclusion of quarks and of other generations into the scheme will be discussed in a subsequent paper.

References
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