Topics in the Theory of Quadratic Residues

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Preface

Although number theory as a coherent mathematical subject started with Fermat’s work in the 1630’s, modern number theory, i.e., the systematic and mathematically rigorous development of the subject from fundamental properties of the integers, began in 1801 with the appearance of Gauss’ landmark treatise *Disquisitiones Arithmeticae* [17]. A major part of the *Disquisitiones* deals with quadratic residues and non-residues: if $p$ is an odd prime, an integer $z$ is a quadratic residue (respectively, quadratic non-residue) of $p$ if there is (respectively, is not) an integer $x$ such that $x^2 \equiv z \mod p$. As we shall see, quadratic residues arise naturally as soon as one wants to solve the general quadratic congruence $ax^2 + bx + c \equiv 0 \mod m$, $a \not\equiv 0 \mod m$, and this, in fact, motivated much of the interest which Gauss himself had in them. Beginning with Gauss’ fundamental contributions, the study of quadratic residues and non-residues has subsequently led directly to many of the key ideas and techniques that are used everywhere in number theory today, and the primary goal of these lecture notes is to use this study as a window through which to view the development of some of those ideas and techniques. In pursuit of that goal, we will employ methods from elementary, analytic, and combinatorial number theory, as well as methods from the theory of algebraic numbers.

In order to follow these lectures most profitably, the reader should have some familiarity with the basic results of elementary number theory. An excellent source for this material (and much more) is the text [24] of Kenneth Ireland and Michael Rosen, *A Classical Introduction to Modern Number Theory*. A feature of this text that is of particular relevance to what we discuss is Ireland and Rosen’s treatment of quadratic and higher-power residues, which is noteworthy for its elegance and completeness, as well as for its historical perspicacity. We will in fact make use of some of their work in Chapters 3 and 7 infra.

Although not absolutely necessary, some knowledge of algebraic number theory will also be helpful for reading these notes. We will provide complete proofs of some facts about algebraic numbers and we will quote other facts without proof. Our reference for proof of the latter results is the classical treatise of Erich Hecke [22], *Vorlesungen über die Theorie der Algebraischen Zahlen*, in the very readable English translation by G. Brauer and J. Goldman. About Hecke’s text André Weil ([41], foreword) had this to say: “To improve upon Hecke, in a treatment along classical lines of the theory of algebraic numbers, would be a futile and impossible task.” We concur enthusiastically with Weil’s assessment and highly recommend Hecke’s book to all those who are interested in number theory.

We next offer a brief overview of what is to follow. The notes are arranged in a series of nine chapters. Chapter 1, an introduction to the subsequent chapters, provides some
motivation for the study of quadratic residues and non-residues by consideration of what needs to be done when one wishes to solve the general quadratic congruence mentioned above. We also record some basic results from elementary number theory that will be used frequently in the sequel. Chapter 2 provides some useful facts about quadratic residues and non-residues upon which the rest of the chapters are based. Here we also describe a procedure which provides a strategy for solving what we call the Basic Problem: if \( d \) is an integer, find all primes \( p \) such that \( d \) is a quadratic residue of \( p \). The Law of Quadratic Reciprocity is the subject of Chapter 3. We present two proofs of this fundamentally important result, both due to Gauss, and use it to implement the strategy discussed in Chapter 2 for finding all primes which have a given integer as a quadratic residue. Chapter 4 discusses some interesting and important applications of quadratic reciprocity, having to do with the structure of the finite subsets \( S \) of the positive integers possessing at least one of the following two properties: for infinitely many primes \( p \), \( S \) is a set of quadratic residues of \( p \), or for infinitely many primes \( p \), \( S \) is a set of quadratic non-residues of \( p \). Here the fundamental contributions of Dirichlet to the theory of quadratic residues enters our story and begins a major theme that will play throughout the rest of our work. The use of transcendental methods in the theory of quadratic residues, begun in Chapter 4, continues in Chapter 5 with the study of the zeta function of an algebraic number field and its application to the solution of some of the problems taken up in Chapter 4. Chapter 6 gives elementary proofs of some of the results in Chapter 5 which obviate the use made there of the zeta function. The question of how quadratic residues and non-residues of a prime \( p \) are distributed among the integers \( 1, 2, \ldots, p - 1 \) is considered in Chapter 7, and there we highlight additional results and methods due to Dirichlet which employ the basic theory of \( L \)-functions attached to Dirichlet characters determined by certain moduli. In Chapter 8 the occurrence of quadratic residues and non-residues as arbitrarily long arithmetic progressions is studied. A key issue that arises in this problem is the estimation of certain character sums over the field of \( p \) elements, \( p \) a prime, and we address this by using some ideas of Harold Davenport [4], some work of André Weil [40], and some techniques in combinatorial number theory developed in recent work of the author [45]. Our discussion concludes with Chapter 9, where the central limit theorem from the theory of probability and a theorem of Davenport and Paul Erdős [6] are used to provide evidence for the contention that as the prime \( p \) tends to infinity, quadratic residues of \( p \) are distributed randomly in the set \( \{1, 2, \ldots, p - 1\} \).

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CHAPTER 1

Introduction: solving the general quadratic congruence modulo a prime

One of the central problems of number theory, both ancient and modern, is finding solutions (in the integers) of polynomial equations with integer coefficients in one or more variables. In order to motivate our study, consider the equation

\[ ax \equiv b \mod m, \]

a linear equation in the unknown integer \( x \). Elementary number theory provides an algorithm for determining exactly when this equation has a solution, and for finding all such solutions, which essentially involves nothing more sophisticated than the Euclidean algorithm (see Proposition 1.4 below and the comments after it).

When we consider what happens for the general quadratic congruence

\[ ax^2 + bx + c \equiv 0 \mod m, \ a \not\equiv 0 \mod m, \]

things get more complicated. In order to see what the issues are, note first that

\[ (2ax + b)^2 \equiv b^2 - 4ac \mod 4am \]

iff \[ 4a^2x^2 + 4abx + 4ac \equiv 0 \mod 4am \]

iff \[ 4a(ax^2 + bx + c) \equiv 0 \mod 4am \]

iff \[ ax^2 + bx + c \equiv 0 \mod m. \]

Hence (1) has a solution iff

\[ 2ax \equiv s - b \mod 4am, \]

where \( s \) is a solution of

\[ s^2 \equiv b^2 - 4ac \mod 4am. \]

Now (2) has a solution iff \( s - b \) is divisible by \( 2a \), the greatest common divisor of \( 2a \) and \( 4am \), and so it follows that (1) has a solution iff (3) has a solution \( s \) such that \( s - b \) is divisible by
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We have hence reduced the solution of (1) to finding solutions $s$ of (3) which satisfy an appropriate divisibility condition.

Our attention is therefore focused on the following problem: if $n$ and $z$ are integers with $n \geq 2$, find all solutions $x$ of the congruence

$$x^2 \equiv z \mod n.$$  \hspace{1cm} (4)

Let

$$n = \prod_{i=1}^{k} p_i^{\alpha_i}$$

be the prime factorization of $n$, and let $\Sigma_i$ denote the set of all solutions of the congruence

$$x^2 \equiv z \mod p_i^{\alpha_i}, \ i = 1, \ldots, k.$$  \hspace{1cm} (5)

Let $s = (s_1, \ldots, s_k) \in \Sigma_1 \times \cdots \times \Sigma_k$, and let $\sigma(s)$ denote the simultaneous solution, unique mod $n$, of the system of congruences

$$x \equiv s_i \mod p_i^{\alpha_i}, \ i = 1, \ldots, k,$$

obtained via the Chinese remainder theorem (Theorem 1.3 below). It is then not difficult to show that the set of all solutions of (4) is given precisely by the set

$$\{\sigma(s) : s \in \Sigma_1 \times \cdots \times \Sigma_k\}.$$  \hspace{1cm} (6)

Consequently (4), and hence also (1), can be solved if we can solve the congruence

$$x^2 \equiv z \mod p^{\alpha},$$

where $p$ is a fixed prime and $\alpha$ is a fixed positive integer.

In articles 103 and 104 of Disquisitiones Arithmeticae [17], Gauss gave a series of beautiful formulae which completely solve (5) for all primes $p$ and exponents $\alpha$. In order to describe them, let $\sigma \in \{0, 1, \ldots, p^{\alpha} - 1\}$ denote a solution of (5).

I. Suppose first that $z$ is not divisible by $p$. If $p = 2$ and $\alpha = 1$ then $\sigma = 1$. If $p$ is odd or $p = 2 = \alpha$ then $\sigma$ has exactly two values $\pm \sigma_0$. If $p = 2$ and $\alpha > 2$ then $\sigma$ has exactly four values $\pm \sigma_0$ and $\pm \sigma_0 + 2^{\alpha-1}$.

II. Suppose next that $z$ is divisible by $p$ but not by $p^\alpha$. If (5) has a solution it can be shown that the multiplicity of $p$ as a factor of $z$ must be even, say $2\mu$, and so let $z = z_1 p^{2\mu}$. Then $\sigma$ is given by the formula

$$\sigma' p^\mu + ip^{\alpha-\mu}, \ i \in \{0, 1, \ldots, p^\mu - 1\},$$

where $\sigma'$ varies over all solutions, determined according to I, of the congruence

$$x^2 \equiv z_1 \mod p^{\alpha-2\mu}.$$
III. Finally if $z$ is divisible by $p^\alpha$, and if we set $\alpha = 2k$ or $\alpha = 2k - 1$, depending on whether $\alpha$ is even or odd, then $\sigma$ is given by the formula

$$ip^k, \ i \in \{0, \ldots, p^{\alpha-k} - 1\}.$$

We will focus on the most important special case of (5), namely when $p$ is odd and $\alpha = 1$, i.e., the congruence

(6) $$x^2 \equiv z \mod p$$

(note that when $p$ is odd, the solutions of (5) in cases I and II are determined by the solutions of (6) for certain values of $z$). The first thing to do here is to observe that the ring determined by the congruence classes of integers mod $p$ is a field, and so (6) has at most two solutions. We have that $x \equiv 0 \mod p$ is the unique solution of (6) iff $z$ is divisible by $p$, and if $s_0 \not\equiv 0 \mod p$ is a solution of (6) then so is $-s_0$, and $s_0 \not\equiv -s_0 \mod p$ because $p$ is an odd prime. These facts are motivation for the following definition:

**Definition.** If $p$ is an odd prime and $z$ is an integer not divisible by $p$, then $z$ is a quadratic residue (respectively, quadratic non-residue) of $p$ if there is (respectively, is not) an integer $x$ such that $x^2 \equiv z \mod p$.

As a consequence of our previous discussion and Gauss’ solution of (5), solutions of (1) will exist only if (among other things) for each (odd) prime factor $p$ of $4am$, the discriminant $b^2 - 4ac$ of $ax^2 + bx + c$ is either divisible by $p$ or is a quadratic residue of $p$. This remark becomes even more emphatic if the modulus $m$ in (1) is a single odd prime $p$. In that case,

$$(2ax + b)^2 \equiv b^2 - 4ac \mod p \text{ iff } ax^2 + bx + c \equiv 0 \mod p,$$

from whence the next proposition follows immediately:

**Proposition 1.1.** Let $p$ be an odd prime. The congruence

(7) $$ax^2 + bx + c \equiv 0 \mod p, \ a \not\equiv 0 \mod p,$$

has a solution iff

(8) $$x^2 \equiv b^2 - 4ac \mod p$$

has a solution, i.e., iff either $b^2 - 4ac$ is divisible by $p$ or $b^2 - 4ac$ is a quadratic residue of $p$. Moreover, if $(2a)^{-1}$ is the multiplicative inverse of $2a \mod p$ (which exists because $p$ does not divide $2a$; see Proposition 1.2 below) then the solutions of (7) are given precisely by the formula

$$x \equiv (\pm s - b)(2a)^{-1} \mod p,$$
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where $\pm s$ are the solutions of (8).

We take it as self-evident that the solution of the general quadratic congruence (1) is one of the most fundamental and most important problems in the theory of Diophantine equations in two variables. By virtue of Proposition 1.1 and the discussion which precedes it, quadratic residues and non-residues play a pivotal role in the determination of the solutions of (1). We hope that the reader will now agree: the study of quadratic residues and non-residues is important and interesting!

We now fix some notation and terminology that will be used throughout the sequel. The letter $p$ will always denote a generic odd prime, the letter $q$, unless otherwise specified, will denote a generic prime (either even or odd), $P$ is the set of all primes, $\mathbb{Z}$ is the set of all integers, and $\mathbb{Q}$ is the set of all rationals. If $m, n \in \mathbb{Z}$ with $m \leq n$ then $[m, n]$ is the set of all integers at least $m$ and no more than $n$, listed in increasing order, $[m, \infty)$ is the set of all integers exceeding $m - 1$, also listed in increasing order, and $\gcd(m, n)$ is the greatest common divisor of $m$ and $n$. If $n \in [2, \infty)$ then $U(n)$ will denote the set \{m $\in [1, n - 1] : \gcd(m, n) = 1$\}. If $z$ is an integer then $\pi(z)$ will denote the set of all prime factors of $z$. If $A$ is a set then $|A|$ will denote the cardinality of $A$, $2^A$ is the set of all subsets of $A$, and $\emptyset$ denotes the empty set. Finally, we will refer to a quadratic residue or quadratic non-residue as simply a residue or non-residue; all other residues of a modulus $m \in [2, \infty)$ will always be called ordinary residues. In particular, the minimal non-negative ordinary residues modulo $m$ are the elements of the set $[0, m - 1]$.

We recall some facts from elementary number theory that will be useful in what follows. For more information about them consult any standard text on elementary number theory, e.g., Ireland and Rosen [24] or K. Rosen [34].

If $m$ is a positive integer and $a \in \mathbb{Z}$, recall that an inverse of $a$ modulo $m$ is an integer $\alpha$ such that $a \alpha \equiv 1 \mod m$.

**Proposition 1.2.** If $m$ is a positive integer and $a \in \mathbb{Z}$ then $a$ has an inverse modulo $m$ iff $\gcd(a, m) = 1$. Moreover, the inverse is relatively prime to $m$ and is unique modulo $m$.

**Theorem 1.3.** (Chinese remainder theorem). If $m_1, \ldots, m_r$ are pairwise relatively prime positive integers and $(a_1, \ldots, a_r)$ is an $r$-tuple of integers then the system of congruences

\[
x \equiv a_i \mod m_i, \quad i = 1, \ldots, r,
\]

has a simultaneous solution that is unique modulo $\prod_{i=1}^{r} m_i$. Moreover, if

\[
M_k = \prod_{i \neq k} m_i,
\]
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and if $y_k$ is the inverse of $M_k \mod m_k$ (which exits because $\gcd(m_k, M_k) = 1$) then the solution is given by

$$x \equiv \sum_{k=1}^{r} a_k M_k y_k \mod \prod_{i=1}^{r} m_i.$$ 

Recall that a linear Diophantine equation is an equation of the form

$$ax + by = c,$$

where $a, b,$ and $c$ are given integers and $x$ and $y$ are integer-valued unknowns.

**Proposition 1.4.** Let $a, b,$ and $c$ be integers and let $\gcd(a, b) = d$. The Diophantine equation $ax + by = c$ has a solution iff $d$ divides $c$. If $d$ divides $c$ then there are infinitely many solutions $(x, y)$, and if $(x_0, y_0)$ is a particular solution then all solutions are given by

$$x = x_0 + \left(\frac{b}{d}\right)n, \quad y = y_0 - \left(\frac{a}{d}\right)n, \quad n \in \mathbb{Z}.$$ 

Given the Diophantine equation $ax + by = c$ with $c$ divisible by $d = \gcd(a, b)$, the Euclidean algorithm can be used to easily find a particular solution $(x_0, y_0)$. Simply let $k = c/d$ and use the Euclidean algorithm to find integers $m$ and $n$ such that $d = am + bn$; then $(x_0, y_0) = (km, kn)$ is a particular solution, and all solutions can then be found by using Proposition 1.4. The simple first-degree congruence $ax \equiv b \mod m$ can thus be easily solved upon the observation that this congruence has a solution $x$ iff the Diophantine equation $ax + my = b$ has the solution $(x, y)$ for some $y \in \mathbb{Z}$. 
CHAPTER 2

Basic Facts

Proposition 2.1. In every complete system of ordinary residues mod $p$, there are exactly $(p - 1)/2$ quadratic residues.

Proof. It suffices to prove that in $[1, p - 1]$ there are exactly $(p - 1)/2$ quadratic residues. Note first that $1^2, 2^2, \ldots, (\frac{p-1}{2})^2$ are all incongruent mod $p$ (if $1 \leq i, j < p/2$ and $i^2 \equiv j^2$ mod $p$ then $i \equiv j$ hence $i = j$ or $i \equiv -j$, i.e., $i + j \equiv 0$. But $2 \leq i + j < p$, and so $i + j \equiv 0$ is impossible).

Let $R$ denote the set of minimal non-negative ordinary residues mod $p$ of $1^2, 2^2, \ldots, (\frac{p-1}{2})^2$. The elements of $R$ are quadratic residues of $p$ and $|R| = (p-1)/2$. Suppose that $n \in [1, p - 1]$ is a quadratic residue of $p$. Then there exists $r \in [1, p - 1]$ such that $r^2 \equiv n$. Then $(p-r)^2 \equiv r^2 \equiv n$ and $(r, p-r) \cap [1, (p-1)/2] \neq \emptyset$. Hence $n \in R$, whence $R = \{\text{the set of quadratic residues of } p \text{ inside } [1, p - 1]\}$. QED

Remark. The proof of Proposition 2.1 provides a way to easily find the residues of any prime $p$. Simply calculate the integers $1^2, 2^2, \ldots, (\frac{p-1}{2})^2$ and then reduce mod $p$. The integers that result from this computation are the residues of $p$ inside $[1, p - 1]$.

N.B. In the next proposition, all residues and non-residues are taken with respect to a fixed prime $p$.

Proposition 2.2. (i) The product of two residues is a residue.

(ii) The product of a residue and a non-residue is a non-residue.

(iii) The product of two non-residues is a residue.

Proof. (i) If $\alpha, \alpha'$ are residues then $x^2 \equiv \alpha, y^2 \equiv \alpha' \Rightarrow (xy)^2 \equiv \alpha \alpha'$ mod $p$.

(ii) Let $\alpha$ be a fixed residue. The integers $0, \alpha, \ldots, (p-1)\alpha$ are incongruent mod $p$, hence are a complete system of ordinary residues mod $p$. If $R = \{\text{set of all residues in } [1, p-1]\}$ then by Proposition 2.2(i), $\{\alpha r : r \in R\}$ is a set of residues of cardinality $(p-1)/2$, hence Proposition 2.1 $\Rightarrow$ there are no other residues among $\alpha, 2\alpha, \ldots, (p-1)\alpha$, i.e., if $\beta \in [1, p - 1] \setminus R$ then $\alpha \beta$ is a non-residue. Statement (ii) is an immediate consequence of this.

(iii) Suppose that $\beta$ is a non-residue. Then $0, \beta, 2\beta, \ldots, (p-1)\beta$ is a complete system of ordinary residues mod $p$, and by Proposition 2.2(ii) and Proposition 2.1, $\{\beta r : r \in R\}$ is a set of non-residues and there are no other non-residues among $\beta, 2\beta, \ldots, (p-1)\beta$. Hence
\( \beta' \in [1, p - 1] \setminus \mathcal{R} \Rightarrow \beta \beta' \) is a residue. Statement \((iii)\) is an immediate consequence of this. QED

**Definition.** The Legendre symbol \( \chi_p \) of \( p \) is the function \( \chi_p : \mathbb{Z} \to [-1, 1] \) defined by

\[
\chi_p(n) = \begin{cases} 
0, & \text{if } p \text{ divides } n, \\
1, & \text{if } \gcd(p, n) = 1 \text{ and } n \text{ is a residue of } p, \\
-1, & \text{if } \gcd(p, n) = 1 \text{ and } n \text{ is a non-residue of } p.
\end{cases}
\]

The next proposition asserts that \( \chi_p \) is a completely multiplicative arithmetic function of period \( p \): this fact will play a crucial role in much of our subsequent work.

**Proposition 2.3.**

1. \( \chi_p(n) = 0 \) iff \( p \) divides \( n \), and if \( m \equiv n \mod p \) then \( \chi_p(m) = \chi_p(n) \) (\( \chi_p \) is of period \( p \)).

2. For all \( m, n \in \mathbb{Z} \), \( \chi_p(mn) = \chi_p(m) \chi_p(n) \) (\( \chi_p \) is completely multiplicative).

**Proof.** (i) If \( m \equiv n \mod p \) then \( p \) divides \( m \) (respectively, \( m \) is a residue/non-residue of \( p \)) iff \( p \) divides \( n \) (respectively, \( n \) is a residue/non-residue of \( p \)). Hence \( \chi_p(m) = \chi_p(n) \).

(ii) \( \chi_p(mn) = 0 \) iff \( p \) divides \( mn \) iff \( p \) divides \( m \) or \( n \) iff \( \chi_p(m) = 0 \) or \( \chi_p(n) = 0 \) iff \( \chi_p(m) \chi_p(n) = 0 \).

\( \chi_p(mn) = 1 \) (respectively, \( \chi_p(mn) = -1 \)) iff \( \gcd(mn, p) = 1 \) and \( mn \) is a residue (respectively, \( mn \) is a non-residue) of \( p \) iff \( \gcd(m, p) = 1 = \gcd(n, p) \) and (by Proposition 2.2) \( m \) and \( n \) are either both residues or both non-residues of \( p \) (respectively, \( \{m, n\} \) contains a residue and a non-residue of \( p \)) iff \( \chi_p(m) \chi_p(n) = 1 \) (respectively, \( \chi_p(m) \chi_p(n) = -1 \)). QED

**Remark on notation.** As a consequence of Proposition 2.3, \( \chi_p \) defines a homomorphism of the group of units in the ring \( \mathbb{Z}/p\mathbb{Z} \) into the circle group, i.e., \( \chi_p \) is a character of the group of units. This is the reason why we have chosen the character-theoretic notation \( \chi_p(n) \) for the Legendre symbol, instead of the more traditional notation \( \left( \frac{n}{p} \right) \). When \( p \) is replaced by an arbitrary integer \( m \geq 2 \), we will have more to say later about characters on the group of units in the ring \( \mathbb{Z}/m\mathbb{Z} \) and their use in what we will study here.

The next result determines the quadratic character of \(-1\).

**Theorem 2.4.**

\[
\chi_p(-1) = \begin{cases} 
1, & \text{if } p \equiv 1 \mod 4, \\
-1, & \text{if } p \equiv -1 \mod 4.
\end{cases}
\]

This theorem is due to Euler [15], who proved it in 1760. It is of considerable importance in the history of number theory because in 1795, the young Gauss (at the ripe old age of
rediscovered it. Gauss was so struck by the beauty and depth of this result that, as he testifies in the preface to *Disquisitiones Arithmeticae* [17], “I concentrated on it all of my efforts in order to understand the principles on which it depended and to obtain a rigorous proof. When I succeeded in this I was so attracted by these questions that I could not let them be.” Thus began Gauss’ work in number theory that was to revolutionize the subject.

**Proof of Theorem 2.4.** The proof that we give is Euler’s own. It is based on

**Theorem 2.5.** (Euler’s criterion) If \( a \in \mathbb{Z} \) and \( \gcd(a, p) = 1 \) then

\[
\chi_p(a) \equiv a^{(p-1)/2} \mod p.
\]

If we apply Euler’s criterion with \( a = -1 \) then

\[
\chi_p(-1) \equiv (-1)^{(p-1)/2} \mod p.
\]

Hence \( \chi_p(-1) - (-1)^{(p-1)/2} \) is either 0 or \( \pm 2 \) and is divisible by \( p \), hence

\[
\chi_p(-1) = (-1)^{(p-1)/2},
\]

and so \( \chi_p(-1) = 1 \) (respectively, \( -1 \)) iff \( (p - 1)/2 \) is even (respectively, odd) iff \( p \equiv 1 \mod 4 \) (respectively, \( p \equiv -1 \mod 4 \)). This verifies Theorem 2.4.

**Proof of Theorem 2.5.** This is an interesting application of Wilson’s theorem, which asserts that

\[
(*) \quad \text{if } q \text{ is a prime then } (q - 1)! \equiv -1 \mod q,
\]

and was in fact first stated by Abu Ali al-Hasan ibn al-Haytham in 1000 AD, over 750 years before it was attributed to John Wilson, whose name it now bears. We will use Wilson’s theorem to first prove Theorem 2.5; after that we then verify Wilson’s theorem.

Suppose that \( \chi_p(a) = 1 \), and so \( x^2 \equiv a \mod p \) for some \( x \in \mathbb{Z} \). Note now that \( 1 = \gcd(a, p) \Rightarrow 1 = \gcd(x^2, p) \Rightarrow 1 = \gcd(x, p) \) (\( p \) is prime!), hence by Fermat’s little theorem,

\[
a^{(p-1)/2} \equiv (x^2)^{(p-1)/2} = x^{p-1} \equiv 1 \mod p.
\]

Suppose that \( \chi_p(a) = -1 \), i.e., \( a \) is a non-residue. For each \( i \in [1, p - 1] \), there exists \( j \in [1, p - 1] \) uniquely determined by \( i \), such that

\[
i j \equiv a \mod p
\]

\((\mathbb{Z}/p\mathbb{Z} \text{ is a field}) \text{ and } i \neq j \) because \( a \) is a non-residue. Hence we can group the integers \( 1, \ldots, p - 1 \) into \((p - 1)/2\) pairs, each pair with a product \( \equiv a \mod p \). Multiplying all of these pairs together yields

\[
(p - 1)! \equiv a^{(p-1)/2} \mod p,
\]
and so \((\ast) \Rightarrow \)

\[-1 \equiv a^{(p-1)/2} \mod p.\]

QED

Proof of Wilson’s theorem. The implication \((\ast)\) is clearly valid when \(q = 2\), so assume that \(q\) is odd. Use Proposition 1.2 to find for each integer \(a \in [1, q-1]\) an integer \(\bar{a} \in [1, p-1]\) such that \(a\bar{a} \equiv 1 \mod q\). The integers 1 and \(q - 1\) are the only integers in \([1, q-1]\) that are their own inverses \(\mod q\), hence we may group the integers from 2 through \(q - 2\) into \((q - 3)/2\) pairs with the product of each pair congruent to 1 \(\mod q\). Hence

\[2 \cdot 3 \cdots (q-3)(q-2) \equiv 1 \mod q.\]

Multiplication of both sides of this congruence by \(q - 1\) yields

\[(q - 1)! = 1 \cdot 2 \cdots (q - 1) \equiv q - 1 \equiv -1 \mod q.\]

QED

Remark. The converse of Wilson’s theorem is also valid.

From our discussion in the introduction, if \(d\) is the discriminant of \(ax^2 + bx + c\) and if neither \(a\) nor \(d\) is divisible by \(p\) then

\[ax^2 + bx + c \equiv 0 \mod p\]

has a solution iff \(d\) is a residue of \(p\). This motivates what we will call the

**Basic Problem.** If \(d \in \mathbb{Z}\), for what primes \(p\) is \(d\) a quadratic residue of \(p\)?

We now present a strategy for solving this problem which employs Proposition 2.3 as the basic tool. Things can be stated precisely and concisely if we use the following

**Notation:** if \(z \in \mathbb{Z}\), let

\[X_{\pm}(z) = \{ p : \chi_p(z) = \pm 1\},\]

\[\pi_{\text{odd}}(z) (\text{resp., } \pi_{\text{even}}(z)) = \{ q \in \pi(z) : q \text{ has odd (resp., even) multiplicity in } z \}.\]

Suppose first that \(d > 0\), with gcd\((d, p) = 1\). If \(\pi_{\text{odd}}(d) = \emptyset\) then \(d\) is a square, so \(d\) is trivially a residue of \(p\). Hence assume that \(\pi_{\text{odd}}(d) \neq \emptyset\). Proposition 2.3 \(\Rightarrow\)

\[\chi_p(d) = \prod_{q \in \pi_{\text{odd}}(d)} \chi_p(q).\]

Hence

\[(1) \quad \chi_p(d) = 1 \text{ iff } \{|q \in \pi_{\text{odd}}(d) : \chi_p(q) = -1| \text{ is even}.}\]
Let
\[ \mathcal{E} = \{ E \subseteq \pi_{\text{odd}}(d) : |E| \text{ is even} \}. \]

If \( E \in \mathcal{E} \), let \( R_E \) denote the set of all \( p \) such that
\[ \chi_p(q) = \begin{cases} \begin{array}{ll} -1, & \text{if } q \in E, \\ 1, & \text{if } q \in \pi_{\text{odd}}(d) \setminus E. \end{array} \end{cases} \]

Then (1) \( \Rightarrow \)

\[ X_+(d) = \left( \bigcup_{E \in \mathcal{E}} R_E \right) \setminus \pi_{\text{even}}(d), \]
and this union is pairwise disjoint. Moreover

\[ R_E = \left( \bigcap_{q \in E} X_-(q) \right) \cap \left( \bigcap_{q \in \pi_{\text{odd}}(d) \setminus E} X_+(q) \right). \]

Suppose next that \( d < 0 \). Then \( d = (-1)(-d) \), hence

\[ \chi_p(d) = \prod_{q \in \{-1\} \cup \pi_{\text{odd}}(d)} \chi_p(q). \]

If we let
\[ \mathcal{E}_{-1} = \{ E \subseteq \{-1\} \cup \pi_{\text{odd}}(d) : |E| \text{ is even} \}, \]
then by applying (4) and an argument similar to the one that yielded (2) and (3) for \( X_+(d), d > 0 \), we also deduce that for \( d < 0 \),

\[ X_+(d) = \left( \bigcup_{E \in \mathcal{E}_{-1}} R_E \right) \setminus \pi_{\text{even}}(d), \]
where

\[ R_E = \left( \bigcap_{q \in E} X_-(q) \right) \cap \left( \bigcap_{q \in \{-1\} \cup \pi_{\text{odd}}(d) \setminus E} X_+(q) \right), E \in \mathcal{E}_{-1}. \]

**Example:** \( d = \pm 126 = \pm 2 \cdot 3^2 \cdot 7 \).

\( \pi_{\text{odd}}(126) = \{2, 7\}, \ \pi_{\text{even}}(126) = \{3\} \).
\( \mathcal{E} = \{\emptyset, \{2, 7\}\}, \ \mathcal{E}_{-1} = \{\emptyset, \{-1, 2\}, \{-1, 7\}, \{2, 7\}\}. \)

\[ X_+(126) = (R_{\emptyset} \cup R_{\{2, 7\}}) \setminus \{3\} \]
\[ = \left( (X_+(2) \cap X_+(7)) \cup (X_-(2) \cap X_-(7)) \right) \setminus \{3\}. \]
\[ X_+(-126) = (R_{0} \cup R_{-1,2} \cup R_{-1,7} \cup R_{2,7}) \setminus \{3\} \]
\[ = \left( (X_+(-1) \cap X_+(2) \cap X_+(7)) \cup (X_+(-1) \cap X_+(2) \cap X_+(7)) \right. \]
\[ \quad \cup \left. (X_+(1) \cap X_+(2) \cap X_+(7)) \cup (X_+(-1) \cap X_+(2) \cap X_+(7)) \right) \setminus \{3\}. \]

Theorem 2.4 and formulas (2),(3),(5), and (6) hence reduce the solution of the Basic Problem to the solution of the

**Fundamental Problem.** If \( q \) is prime, calculate \( X_\pm(q) \).

**Gauss’ lemma and the solution of the Fundamental Problem for the prime 2.**

THEOREM 2.6. \( \chi_p(2) = (-1)^{(p^2-1)/8} \).

This theorem solves the Fundamental Problem for the prime 2. It is easy to see that \((p^2-1)/8\) is even (odd) iff \( p \equiv 1 \) or 7 mod 8 \((p \equiv 3 \) or 5 mod 8). Hence

\[ X_+(2) = \{p : p \equiv 1 \text{ or } 7 \text{ mod } 8\}, \]
\[ X_-(2) = \{p : p \equiv 3 \text{ or } 5 \text{ mod } 8\}. \]

The proof of Theorem 2.6 will use a basic result in the theory of quadratic residues called Gauss’ lemma (this lemma was first used by Gauss in his third proof of the Law of Quadratic Reciprocity [18], which proof we will present in Chapter 3). To state it, let \( a \in \mathbb{Z}, \gcd(a,p) = 1 \). Consider the minimal positive ordinary residues mod \( p \) of the integers \( a, \ldots, 1/2(p-1)a \). None of these ordinary residues is \( p/2 \), as \( p \) is odd, and they are all distinct as \( \gcd(a,p) = 1 \), hence let

\[ u_1, \ldots, u_s \] be those ordinary residues that are \( > p/2 \),
\[ v_1, \ldots, v_t \] be those ordinary residues that are \( < p/2 \).

N.B. \( s + t = \frac{1}{2}(p-1) \). We then have

THEOREM 2.7. *(Gauss’ lemma)*

\[ \chi_p(a) = (-1)^s. \]

Proof of Theorem 2.6. Let \( \sigma \) be the the number of minimal positive ordinary residues mod \( p \) of the integers in the set

\[ 1 \cdot 2, 2 \cdot 2, \ldots, \frac{1}{2}(p-1) \cdot 2 \]
that exceed \( p/2 \). Gauss' lemma ⇒

\[
\chi_p(2) = (-1)^\sigma.
\]

Because each integer in (7) is less than \( p \), \( \sigma = \text{the number of integers in the set (7) that exceed } p/2 \). An integer \( 2j, j \in [1, (p - 1)/2] \) does not exceed \( p/2 \) iff \( 1 \leq j \leq p/4 \), hence the number of integers in (7) that do not exceed \( p/2 \) is \([p/4]\), where \([x]\) denotes the greatest integer not exceeding \( x \). Hence

\[
\sigma = \frac{p - 1}{2} - \left[ \frac{p}{4} \right].
\]

To prove Theorem 2.6, it hence suffices to prove that

(8) \quad \text{for all odd integers } n, \frac{n - 1}{2} - \left[ \frac{n}{4} \right] \equiv \frac{n^2 - 1}{8} \mod 2.

To see this, note first that the congruence in (8) is true for a particular integer \( n \) iff it is true for \( n + 8 \), because

\[
\frac{(n + 8) - 1}{2} - \left[ \frac{n + 8}{4} \right] = \frac{n - 1}{2} + 4 - \left( \left[ \frac{n}{4} \right] + 2 \right) \equiv \frac{n - 1}{2} - \left[ \frac{n}{4} \right] \mod 2,
\]

\[
\frac{(n + 8)^2 - 1}{8} = \frac{n^2 - 1}{8} + 2n + 8 \equiv \frac{n^2 - 1}{8} \mod 2.
\]

Thus (8) holds iff it holds for \( n = \pm 1, \pm 3 \), and it is easy to check that (8) holds for these values of \( n \). QED

Proof of Theorem 2.7. Let \( u_i, v_i \) be as defined before the statement of Gauss’ lemma. We claim that

(9) \quad \{p - u_1, \ldots, p - u_s, v_1, \ldots, v_t\} = [1, \frac{1}{2}(p - 1)].

To see this, note first that if \( i \neq j \) then \( v_i \neq v_j, u_i \neq u_j \) hence \( p - u_i \neq p - u_j \). It is also true that \( p - u_i \neq v_j \) for all \( i, j \); otherwise \( p \equiv a(k + l) \mod p \), where \( 2 \leq k + l \leq \frac{p - 1}{2} + \frac{p - 1}{2} = p - 1 \), which is impossible because \( \gcd(a, p) = 1 \). Hence

(10) \quad |\{p - u_1, \ldots, p - u_s, v_1, \ldots, v_t\}| = s + t = \frac{p - 1}{2}.

But \( 0 < v_i < p/2 \) ⇒ \( 0 < v_i \leq (p - 1)/2 \) and \( p/2 < u_i < p \) ⇒ \( 0 < p - u_i \leq (p - 1)/2 \) and so

(11) \quad \{p - u_1, \ldots, p - u_s, v_1, \ldots, v_t\} \subseteq [1, \frac{1}{2}(p - 1)].

As \( |[1, \frac{1}{2}(p - 1)]| = \frac{1}{2}(p - 1) \), (9) follows from (10) and (11).

Equation (9) ⇒

\[
\prod_{i=1}^{s} (p - u_i) \prod_{i=1}^{t} v_i = \left( \frac{p - 1}{2} \right)!.
\]
Because
\[ p - u_i \equiv -u_i \mod p \]
we conclude from the preceding equation that
\[ (-1)^s \prod_{i=1}^s u_i \prod_{i=1}^t v_i \equiv \left( \frac{p - 1}{2} \right)! \mod p. \]

Because \( u_1, \ldots, u_s, v_1, \ldots, v_t \) are the least positive ordinary residues of \( a, \ldots, \frac{1}{2}(p - 1)a \), (12) \( \Rightarrow \)
\[ (-1)^s a^{(p-1)/2} \left( \frac{p - 1}{2} \right)! \equiv \left( \frac{p - 1}{2} \right)! \mod p. \]

But \( p \) and \( \left( \frac{p - 1}{2} \right)! \) are relatively prime, and so (13) \( \Rightarrow \)
\[ (-1)^s a^{(p-1)/2} \equiv 1 \mod p \]
i.e.,
\[ a^{(p-1)/2} \equiv (-1)^s \mod p. \]

By Euler’s criterion (Theorem 2.5),
\[ a^{(p-1)/2} \equiv \chi_p(a) \mod p, \]
hence
\[ \chi_p(a) \equiv (-1)^s \mod p. \]

It follows that \( \chi_p(a) - (-1)^s \) is either 0 or \( \pm 2 \) and is also divisible by \( p \) and so
\[ \chi_p(a) = (-1)^s. \]

QED

We now need to solve the Fundamental Problem for odd primes. This will be done by using what Gauss called the *theorema aureum*, the “golden theorem”, of number theory.
CHAPTER 3

Gauss’ *theorema aureum*: the Law of Quadratic Reciprocity

**Theorem 3.1. (Law of Quadratic Reciprocity (LQR))** If $p$ and $q$ are distinct odd primes then

$$\chi_p(q)\chi_q(p) = (-1)^{\frac{1}{2}(p-1)\frac{1}{2}(q-1)}.$$

*What this says.* Note first that if $n \in \mathbb{Z}$ is odd then $\frac{1}{2}(n-1)$ is even (odd) iff $n \equiv 1 \mod 4$ ($n \equiv 3 \mod 4$). Hence

$$\chi_p(q)\chi_q(p) = 1 \text{ iff } p \text{ or } q \equiv 1 \mod 4,$$

$$\chi_p(q)\chi_q(p) = -1 \text{ iff } p \equiv q \equiv 3 \mod 4,$$

i.e.,

$$\chi_p(q) = \chi_q(p) \text{ iff } p \equiv q \equiv 1 \mod 4,$$

$$\chi_p(q) = -\chi_q(p) \text{ iff } p \equiv q \equiv 3 \mod 4.$$

Thus

if $p$ or $q \equiv 1 \mod 4$ then $p$ is a residue of $q$ iff $q$ is a residue of $p$,

and

if $p \equiv q \equiv 3 \mod 4$ then $p$ is a residue of $q$ iff $q$ is a non-residue of $p$.

This is why this theorem is called the law of quadratic *reciprocity*. The classical quotient notation for the Legendre symbol makes the reciprocity typographically explicit: in that notation, the conclusion of Theorem 3.1 reads as

$$\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = (-1)^{\frac{1}{2}(p-1)\frac{1}{2}(q-1)}.$$

**Some history.** The LQR was first conjectured by Euler [14] in an equivalent form in 1744, based on extensive numerical evidence, but he could not prove it. Legendre [27] discussed it at length in 1785; in fact he discovered the Legendre symbol in a search for a way to elegantly formulate the LQR as per the statement of Theorem 3.1. Legendre outlined several ingenious strategies for proving the LQR, but as he himself admitted, he was not able to implement any of them. Because of the attention Euler and Legendre drew to it, the proof of the LQR became one of the major unsolved problems of number theory in the eighteenth century.

The first rigorous and correct proof was discovered by Gauss in 1796. He considered this result one of his greatest contributions to mathematics, returning to it again and again.
throughout his career. Gauss eventually found six different proofs of the LQR. The first proof, which involved an extremely long and complicated induction argument, was published in *Disquisitiones Arithmeticae* ([17], articles 135-145). A major goal of Gauss’ later work in number theory was to generalize quadratic reciprocity to higher powers, in particular to cubic and bi-quadratic (fourth-power) residues. He at last achieved that goal with his sixth proof [19] of the LQR, the ideas from which Gauss used to formulate a precise statement of the law of bi-quadratic reciprocity ([20], [21]).

The establishment of generalizations of quadratic reciprocity that covered arbitrary power residues, the so-called higher reciprocity laws, was a major theme of number theory in the nineteenth century and led to many of the most important advances in the subject during that time. Further generalizations to number systems extending beyond the integers, in particular and most importantly, to rings of algebraic integers in algebraic number fields (see this chapter and the first part of Chapter 5 for the relevant definitions), was a major theme of twentieth-century number theory and led to many of the most important advances during *that* time. For an especially apt example of this latter development, we direct the reader’s attention to Hecke’s penetrating analysis of quadratic reciprocity in an arbitrary algebraic number field ([22], Chapter VIII).

**Solution of the Fundamental Problem for odd primes.**

We will now use quadratic reciprocity to solve the Fundamental Problem for odd primes. Let $q$ be an odd prime, and let $r_i^+$ (respectively, $r_i^-$), $i = 1, \ldots, \frac{1}{2}(q - 1)$ denote the residues (respectively, non-residues) of $q$ in $[1, q - 1]$.

**Case 1: $q \equiv 1 \mod 4$.**  

LQR $\Rightarrow$

$$X_{\pm}(q) = \{p : \chi_p(q) = \pm 1\}$$

$$= \{p : \chi_q(p) = \pm 1\}$$

$$\frac{1}{2}(q-1) = \bigcup_{i=1}^{\frac{1}{2}(q-1)} \{p : p \equiv r_i^\pm \mod q\}.$$  

**Example:** $q = 17$.

Residues of 17: 1,2,4,8,9,13,15,16.  
Non-residues of 17: 3,5,6,7,10,11,12,14.  

$$X_+(17) = \{p : p \equiv 1, 2, 4, 8, 9, 13, 15, \text{ or } 16 \mod 17\},$$

$$X_-(17) = \{p : p \equiv 3, 5, 6, 7, 10, 11, 12, \text{ or } 14 \mod 17\}.$$  

(Recall that $p$ always denotes an *odd* prime.)
Case 2: \( q \equiv 3 \mod 4 \).

Note first (from Theorem 2.4) that

\[
X_\pm(-1) = \{ p : p \equiv \pm 1 \mod 4 \}.
\]

Hence LQR ⇒

\[
(1) \quad X_+(q) = (X_+(-1) \cap \{ p : \chi_q(p) = 1 \}) \cup (X_-(1) \cap \{ p : \chi_q(p) = -1 \}).
\]

Now for \( i = 1, \ldots, \frac{1}{2}(q-1) \), let

\[
x \equiv x_i^\pm \mod 4q, \quad 1 \leq x_i^\pm \leq 4q-1,
\]

be the simultaneous solutions of

\[
x \equiv \pm 1 \mod 4,
\]

\[
x \equiv r_i^\pm \mod q,
\]

obtained from the Chinese remainder theorem (Theorem 1.3). If we set

\[
V(q) = \{ x_i^\pm : i \in [1, (q-1)/2] \}
\]

then (1) ⇒

\[
X_+(q) = \bigcup_{n \in V(q)} \{ p : p \equiv n \mod 4q \}.
\]

In order to calculate \( X_-(q) \), recall that \( U(4q) \) denotes the set \( \{ n \in [1, 4q-1] : \gcd(n, 4q) = 1 \} \) and then observe that

\[
V(q) \subseteq U(4q),
\]

\[
\{ p : p \neq q \} = \bigcup_{n \in U(4q)} \{ p : p \equiv n \mod 4q \}.
\]

Hence

\[
X_-(q) = \{ p : p \neq q \} \setminus X_+(q)
\]

\[
= \bigcup_{n \in U(4q) \setminus V(q)} \{ p : p \equiv n \mod 4q \}.
\]

Example: \( q = 7 \).

Residues of 7: 1,2,4
Non-residues of 7: 3,5,6.

Chinese remainder theorem ⇒ simultaneous solutions of the congruence pairs

\[
p \equiv 1 \mod 4 \text{ and } p \equiv 1 \mod 7,
\]

\[
p \equiv 1 \mod 4 \text{ and } p \equiv 2 \mod 7,
\]

\[
p \equiv 1 \mod 4 \text{ and } p \equiv 4 \mod 7,
\]
are, respectively,

\[ p \equiv 1 \mod 28, \]
\[ p \equiv 9 \mod 28, \]
\[ p \equiv 25 \mod 28, \]
\[ p \equiv 3 \mod 28, \]
\[ p \equiv 19 \mod 28, \]
\[ p \equiv 27 \mod 28. \]

Hence

\[ X_+(7) = \{ p : p \equiv 1, 3, 9, 19, 25, \text{ or } 27 \mod 28 \}. \]

We have that

\[ U(28) = \{ 1, 3, 5, 9, 11, 13, 15, 17, 19, 23, 25, 27 \}, \]
\[ V(7) = \{ 1, 3, 9, 19, 25, 27 \}, \]

hence,

\[ U(28) \setminus V(7) = \{ 5, 11, 13, 15, 17, 23 \}, \]

and so

\[ X_-(7) = \{ p : p \equiv 5, 11, 13, 15, 17, \text{ or } 23 \mod 28 \}. \]

**Solution of the Basic Problem.**

If \( d \) is a fixed but arbitrary integer, we can use formula (2) or (5) of Chapter 2 in concert with the solution of the Fundamental Problem that we now have for odd primes to calculate \( X_+(d) \), thereby solving the Basic Problem. The formulae that we have derived for the calculation of \( X_+(q) \) where \( q \) is either \(-1\) or a prime show that each of these sets is equal to a union of certain equivalence classes mod 4, 8, an odd prime, or 4 times an odd prime. It follows that when we employ formula (2) or (5) of Chapter 2 to calculate \( X_+(d) \), each of the sets \( R_E \) occurring in those formulae can hence be calculated by the method of successive substitution, a generalization of the Chinese remainder theorem that can be used to solve simultaneous congruences when the moduli of the congruences are no longer pairwise relatively prime.

The method of successive substitution works as follows. We have a series of congruences of the form

\[ x \equiv a_i \mod m_i, \quad i = 1, \ldots, k, \]
where \((m_1, \ldots, m_k)\) is a given \(k\)-tuple of moduli and \((a_1, \ldots, a_k)\) is a given \(k\)-tuple of integers, which we wish to solve simultaneously. Denoting by \(\text{lcm}(a, b)\) the least common multiple of the integers \(a\) and \(b\), one starts with

**Proposition 3.2.** The congruences

\[
x \equiv a_1 \mod m_1, \ x \equiv a_2 \mod m_2
\]

have a simultaneous solution iff \(\gcd(m_1, m_2)\) divides \(a_1 - a_2\). The solution is unique modulo \(\text{lcm}(m_1, m_2)\) and is given by

\[
x \equiv a_1 + x_0 m_1 \mod \text{lcm}(m_1, m_2),
\]

where \(x_0\) is a solution of

\[
m_1 x_0 \equiv a_2 - a_1 \mod m_2.
\]

The congruences (2) are then solved by first using Proposition 3.2 to solve the first two congruences in (2), then, if necessary, pairing the solution so obtained with the third congruence in (2) and applying Proposition 3.2 to solve that congruence pair, and continuing in this manner, successively applying Proposition 3.2 to the pair of congruences consisting of the solution obtained from step \(i - 1\) and the \(i\)-th congruence in (2). This procedure confirms that (2) has a simultaneous solution iff \(\gcd(m_i, m_j)\) divides \(a_i - a_j\) for all \(i\) and \(j\), and that the solution is unique modulo the least common multiple of \(m_1, \ldots, m_k\). Proposition 3.2 is not difficult to verify, and so we will leave that to the interested reader.

Consequently, once the residues and non-residues of each integer in \(\pi_{\text{odd}}(d)\) are determined, \(X_+(d)\) can be calculated by repeated applications of the method of successive substitutions. In particular, one finds a positive integer \(m(d)\) and a subset \(V(d)\) of \(U(m(d))\) such that

\[
X_+ (d) = \left( \bigcup_{n \in V(d)} \{ p : p \equiv n \mod m(d) \} \right) \setminus \pi_{\text{even}}(d).
\]

The modulus \(m(d)\) is determined like so: if \(d > 0\) and \(\pi_{\text{odd}}(d)\) contains neither 2 nor a prime \(\equiv 3 \mod 4\), then \(m(d)\) is the product of all the elements of \(\pi_{\text{odd}}(d)\); otherwise, \(m(d)\) is 4 times this product.

The formula for \(X_-(d)\) can now be obtained from the one for \(X_+(d)\) by first observing that as a consequence of the above determination of \(m(d)\),

\[
\pi(m(d)) \cup \{2\} = \pi_{\text{odd}}(d) \cup \{2\},
\]

and so

\[
\pi(d) \cup \{2\} = \pi(m(d)) \cup \{2\} \cup \pi_{\text{even}}(d).
\]
Upon recalling that $P$ denotes the set of all primes, it follows that
\[
X_-(d) = P \setminus (X_+(d) \cup \{2\} \cup \pi(d))
= P \setminus (\pi(m(d)) \cup \{2\} \cup X_+(d) \cup \pi_{\text{even}}(d))
= [P \setminus (\pi(m(d)) \cup \{2\})] \setminus [X_+(d) \cup \pi_{\text{even}}(d)].
\]

Because
\[
P \setminus (\pi(m(d)) \cup \{2\}) = \bigcup_{n \in U(m(d))} \{p : p \equiv n \mod m(d)\},
\]
and
\[
X_+(d) \cup \pi_{\text{even}}(d) = \left( \bigcup_{n \in V(d)} \{p : p \equiv n \mod m(d)\} \right) \cup \pi_{\text{even}}(d),
\]
it hence follows that
\[
X_-(d) = \left[ \left( \bigcup_{n \in U(m(d))} \{p : p \equiv n \mod m(d)\} \right) \setminus \left( \bigcup_{n \in V(d)} \{p : p \equiv n \mod m(d)\} \right) \right] \setminus \pi_{\text{even}}(d)
= \left( \bigcup_{n \in U(m(d)) \setminus V(d)} \{p : p \equiv n \mod m(d)\} \right) \setminus \pi_{\text{even}}(d).
\]

The set $V(d)$ that appears in the formulae which calculate $X_\pm(d)$ is obtained from applications of the method of successive substitution to the calculation of each of the sets $R_E$ which appears in equation (2) or (5) of Chapter 2. A natural question which arises asks: are all of the integers in $V(d)$ and $U(m(d)) \setminus V(d)$ which arise from these calculations required for the determination of $X_\pm(d)$? The answer is yes, if for each pair of relatively prime positive integers $m$ and $n$, it is true that the set $\{z \in \mathbb{Z} : z \equiv n \mod m\}$ contains primes. Remarkably enough, $\{z \in \mathbb{Z} : z \equiv n \mod m\}$ in fact always contains infinitely many primes. This is a famous theorem of Dirichlet [9], and the connection of that theorem to the calculation of $X_\pm(d)$ was Dirichlet’s primary motivation for proving it. Much more is to come (in Chapter 4) about Dirichlet’s theorem and its use in the study of residues and non-residues.

Example: $X_\pm(126)$.

From the example on p.16,
\[
X_+(126) = \left( (X_+(2) \cap X_+(7)) \cup (X_-(-2) \cap X_-(-7)) \right) \setminus \{3\}.
\]

Calculation of $X_+(2) \cap X_+(7)$.
\[
X_+(2) = \{p : p \equiv 1 \text{ or } 7 \mod 8\},
X_+(7) = \{p : p \equiv 1, 3, 9, 19, 25, \text{ or } 27 \mod 28\}.
\]
In order to calculate $X_+(2) \cap X_+(7)$, we need to solve at most 12 (but in fact exactly six) pairs of simultaneous congruences. We do this by applying Proposition 3.2. We have that $\gcd(8, 28) = 4$, $\text{lcm}(8, 28) = 56$, and so Proposition 3.2 $\Rightarrow X_+(2) \cap X_+(7)$ consists of the union of all odd prime simultaneous solutions of the congruence pairs

\[
x \equiv 1 \mod 8, \ x \equiv 1 \mod 28, \\
x \equiv 1 \mod 8, \ x \equiv 9 \mod 28, \\
x \equiv 1 \mod 8, \ x \equiv 25 \mod 28, \\
x \equiv 7 \mod 8, \ x \equiv 3 \mod 28, \\
x \equiv 7 \mod 8, \ x \equiv 19 \mod 28, \\
x \equiv 7 \mod 8, \ x \equiv 27 \mod 28,
\]

whose odd prime solutions are, respectively,

\[
p \equiv 1 \mod 56, \\
p \equiv 9 \mod 56, \\
p \equiv 25 \mod 56, \\
p \equiv 31 \mod 56, \\
p \equiv 47 \mod 56, \\
p \equiv 55 \mod 56.
\]

\textit{Calculation of $X_-(2) \cap X_-(7)$}.

\[
X_-(2) = \{p : p \equiv 3 \text{ or } 5 \mod 8\},
\]
\[
X_-(7) = \{p : p \equiv 5, 11, 13, 15, 17, \text{ or } 23 \mod 28\}.
\]

Hence, again according to Proposition 3.2, $X_-(2) \cap X_-(7)$ consists of the union of all odd prime simultaneous solutions of the congruence pairs

\[
x \equiv 3 \mod 8, \ x \equiv 11 \mod 28, \\
x \equiv 3 \mod 8, \ x \equiv 15 \mod 28, \\
x \equiv 3 \mod 8, \ x \equiv 23 \mod 28, \\
x \equiv 5 \mod 8, \ x \equiv 5 \mod 28, \\
x \equiv 5 \mod 8, \ x \equiv 13 \mod 28, \\
x \equiv 5 \mod 8, \ x \equiv 17 \mod 28,
\]
whose odd prime solutions are, respectively,

\[ p \equiv 11 \mod 56, \]
\[ p \equiv 43 \mod 56, \]
\[ p \equiv 51 \mod 56, \]
\[ p \equiv 5 \mod 56, \]
\[ p \equiv 13 \mod 56, \]
\[ p \equiv 45 \mod 56. \]

From this calculation of \( X_+(2) \cap X_+(7) \) and \( X_-(2) \cap X_-(7) \), it hence follows that

\[ X_+(126) = \{ p : p \equiv 1, 5, 9, 11, 13, 25, 31, 43, 45, 47, 51, \text{ or } 55 \mod 56 \}. \]

In order to calculate \( X_-(126) \), we simply delete from \( U(56) \) the minimal positive ordinary residues mod 56 that determine \( X_+(126) \): the integers resulting from that are 3, 15, 17, 19, 23, 27, 29, 33, 37, 39, 41, 53. Hence

\[ X_-(126) = \{ p \neq 3 : p \equiv 3, 15, 17, 19, 23, 27, 29, 33, 37, 39, 41, \text{ or } 53 \mod 56 \}. \]

**Proof of Theorem 3.1.**

We will give two proofs of quadratic reciprocity. The first one is a simplification, due to Eisenstein, of Gauss’ third proof [18]. It is by now the standard argument and uses an ingenious application of Theorem 2.7 (Gauss’ lemma). The second proof is a version of Gauss’ sixth and final proof. It uses ingenious calculations based on some basic facts from algebraic number theory, and anticipates some important techniques that we will use later to study various properties of residues and non-residues in greater depth.

**First proof of Theorem 3.1.**

This uses

**Lemma 3.3.** If \( a \in \mathbb{Z} \) is odd and \( \gcd(a, p) = 1 \) then

\[ \chi_p(a) = (-1)^{T(a,p)}, \]

where

\[ T(a,p) = \sum_{k=1}^{\frac{1}{2}(p-1)} \left\lfloor \frac{ka}{p} \right\rfloor. \]
Assume Lemma 3.3 for the time being, with its proof to come shortly.

We begin our first proof of Theorem 3.1 by outlining the strategy of the argument. Let \( p \) and \( q \) be distinct odd primes and consider the set \( L \) of points \((x, y)\) in the plane, where \( x, y \in [1, \infty), 1 \leq x \leq \frac{1}{2}(p-1), \text{and } 1 \leq y \leq \frac{1}{2}(q-1), \) i.e., the set of lattice points inside the rectangle with corners at \((0, 0), (0, \frac{1}{2}(q-1)), (\frac{1}{2}(p-1), 0), (\frac{1}{2}(p-1), \frac{1}{2}(q-1))\).

Let \( l \) be the line with equation \( qx = py \). To prove Theorem 3.1, one shows first that
\[
(3) \quad \text{no point of } L \text{ lies on } l.
\]

Hence
\[
L = \text{set of all points of } L \text{ which lie below } l \cup \text{set of all points of } L \text{ which lie above } l = L_1 \cup L_2,
\]
consequently
\[
(4) \quad \frac{1}{2}(p-1)\frac{1}{2}(q-1) = |L| = |L_1| + |L_2|.
\]

The next step is to count the number of points in \( L_1 \) and \( L_2 \).

The result is
\[
|L_1| = T(q, p), \quad |L_2| = T(p, q),
\]
hence from (4),
\[
\frac{1}{2}(p-1)\frac{1}{2}(q-1) = T(q, p) + T(p, q).
\]

It then follows from Lemma 3.3 that
\[
(-1)^{\frac{1}{2}(p-1)\frac{1}{2}(q-1)} = (-1)^{T(q, p)}(-1)^{T(p, q)} = \chi_p(q)\chi_q(p),
\]
which is the conclusion of Theorem 3.1. Thus, we need to verify (3), implement (5), and prove Lemma 3.3.

**Verification of (3).** Suppose that \((x, y) \in L\) satisfies \( qx = py \). Then \( q \), being prime, must divide either \( p \) or \( y \). Because \( p \) is prime and \( q \neq p \), \( q \) must divide \( y \), which is not possible because \( 1 \leq y \leq \frac{1}{2}(q-1) < q \).

**Implementation of (5).**
\[
L_1 = \{(x, y) \in L : qx > py\}
= \{(x, y) \in L : 1 \leq x \leq \frac{1}{2}(p-1), 1 \leq y < \frac{qx}{p}\}
= \bigcup_{1 \leq x \leq \frac{1}{2}(p-1)} \{(x, y) : 1 \leq y \leq \left[\frac{qx}{p}\right]\},
\]
and this union is pairwise disjoint. Hence

\[ |L_1| = \sum_{x=1}^{\frac{1}{2}(p-1)} \left\lfloor \frac{qx}{p} \right\rfloor = T(q, p). \]

\[ L_2 = \{(x, y) \in L : qx < py\} \]
\[ = \{(x, y) \in L : 1 \leq y \leq \frac{1}{2}(q-1), 1 \leq x < \frac{py}{q}\} \]
\[ = \bigcup_{1 \leq y \leq \frac{1}{2}(q-1)} \{(x, y) : 1 \leq x \leq \left\lfloor \frac{py}{q} \right\rfloor\}, \]

hence

\[ |L_2| = \sum_{y=1}^{\frac{1}{2}(q-1)} \left\lfloor \frac{py}{q} \right\rfloor = T(p, q). \]

Note that this part of the proof contains no number theory but is instead a purely geometric lattice-point count. All of the number theory is concentrated in the proof of Lemma 3.3, which is still to come. Indeed, that is the main idea in Gauss’ third proof: divide the argument into two parts, a number-theoretic part (Lemma 3.3) and a geometric part (the lattice-point count). Coupling geometry to number theory is a very powerful method for proving things, which Gauss pioneered in much of his work.

**Proof of Lemma 3.3.** We set up shop in order to apply Gauss’ lemma: take the minimal positive ordinary residues mod \( p \) of the integers \( a, \ldots, \frac{1}{2}(p-1)a \), observe as before that none of these ordinary residues is \( p/2 \), as \( p \) is odd, and they are all distinct as \( \gcd(a, p) = 1 \), hence let

\[ u_1, \ldots, u_s \text{ be those ordinary residues that are } > p/2, \]
\[ v_1, \ldots, v_t \text{ be those ordinary residues that are } < p/2. \]

By the division algorithm, for each \( j \in [1, \frac{1}{2}(p-1)] \),

\[ ja = p \left\lfloor \frac{ja}{p} \right\rfloor + \text{ remainder}, \]
\[ \text{remainder} = a \ u_k \text{ or } a \ v_l. \]

Adding these equations together, we get

\[ a \sum_{j=1}^{\frac{1}{2}(p-1)} j = p \sum_{j=1}^{\frac{1}{2}(p-1)} \left\lfloor \frac{ja}{p} \right\rfloor + \sum_{j=1}^{s} u_j + \sum_{j=1}^{t} v_j. \]

Next, recall from (9) of Chapter 2 that

\[ \{p - u_1, \ldots, p - u_s, v_1, \ldots, v_t\} = [1, \frac{1}{2}(p-1)]. \]
Hence

\[ \frac{1}{2}(p-1) \sum_{j=1}^{s} j = sp - \sum_{j=1}^{s} u_j + \sum_{j=1}^{t} v_j. \]

Subtracting (7) from (6) yields

\[ (a - 1) \frac{1}{2}(p-1) \sum_{j=1}^{s} j = pT(a, p) - sp + 2 \sum_{j=1}^{s} u_j. \]

Hence

\[ p(T(a, p) - s) \text{ is even (} a \text{ is odd!).} \]

and so

\[ T(a, p) - s \text{ is even (} p \text{ is odd!),} \]

whence

\[ (-1)^{T(a, p)} = (-1)^s. \]

Gauss' lemma now implies that

\[ \chi_p(a) = (-1)^s, \]

and so

\[ \chi_p(a) = (-1)^{T(a, p)}. \]

QED

Second proof of Theorem 3.1.

Gauss' sixth proof of quadratic reciprocity [19] appeared in 1818. He mentions in the introduction to this paper that for years he had searched for a method that would generalize to the cubic and bi-quadratic case and that finally his untiring efforts were crowned with success. The purpose of publishing this sixth proof, he states, was to bring to a close this part of the higher arithmetic dealing with quadratic residues and to say, in a sense, farewell. Our second proof of LQR is a reworking of Gauss' argument from [19] using some basic facts from the theory of algebraic numbers. We start first with a rather detailed discussion of the algebraic number theory that will be required; this is the content of Proposition 3.4 through Lemma 3.9 below. This information is then used to prove the LQR, following the development given in Ireland and Rosen [24], sections 6.2 and 6.3.

Let \( \mathbb{C} \) denote the complex numbers.

Definition. A complex number field is a nonzero subfield of \( \mathbb{C} \).

N. B. Every complex number field contains the field \( \mathbb{Q} \) of rational numbers.
**Notation:** if $A$ is a ring then $A[x]$ will denote the ring of all polynomials in $x$ with coefficients in $A$.

**Definitions.** Let $F$ be a complex number field. A complex number $\theta$ is *algebraic over $F$* if there exists $f \in F[x]$ such that $f \neq 0$ and $f(\theta) = 0$. If $\theta$ is algebraic over $F$, let

$$M(\theta) = \{ p \in F[x] : f \text{ is monic and } f(\theta) = 0 \}$$

(N.B. $M(\theta) \neq \emptyset$). An element of $M(\theta)$ of smallest degree is a *minimal polynomial of $\theta$ over $F$*.

**Proposition 3.4.** The minimal polynomial of a complex number algebraic over a complex number field $F$ is unique and irreducible over $F$.

**Proof.** Let $r$ and $s$ be minimal polynomials of the number $\theta$ algebraic over $F$. Use the division algorithm in $F[x]$ to find $d, f \in F[x]$ such that

$$r = ds + f, \ f \equiv 0 \text{ or degree of } f < \text{degree of } s.$$  

Hence

$$f(\theta) = r(\theta) - d(\theta)s(\theta) = 0.$$

If $f \neq 0$ then, upon dividing $f$ by its leading coefficient, we get a monic polynomial over $F$ of lower degree than $s$ and not identically 0 which has $\theta$ as a root, which is not possible because $s$ is a minimal polynomial of $\theta$ over $F$. Hence $f \equiv 0$ and so $s$ divides $r$ over $F$. Similarly, $r$ divides $s$ over $F$. Hence $r = \alpha s$ for some $\alpha \in F$, and as $r$ and $s$ are both monic, $\alpha = 1$, and so $r = s$. This proves that the minimal polynomial is unique.

To show that the minimal polynomial $m$ is irreducible over $F$, suppose that $m = rs$, where $r$ and $s$ are non-constant elements of $F[x]$. Then the degrees of $r$ and $s$ are both less than the degree of $m$, and $\theta$ is a root of either $r$ or $s$. Hence a constant multiple of either $r$ or $s$ is a monic polynomial in $F[x]$ having $\theta$ as a root and is of degree less than the degree of $m$, contradicting the minimality of the degree of $m$. QED

**Definition.** Let $\theta$ be algebraic over $F$. The *degree of $\theta$ over $F$* is the degree of the minimal polynomial of $\theta$ over $F$.

**Lemma 3.5.** If $\theta \in \mathbb{C}$, $F$ is a complex number field, and $f \in F[x]$ is monic, irreducible over $F$, and $f(\theta) = 0$ then $f$ is the minimal polynomial of $\theta$ over $F$.

**Proof.** Let $m$ be the minimal polynomial of $\theta$ over $F$. The division algorithm in $F[x] \Rightarrow$ there exists $q, r \in F[x]$ such that

$$f =qm + r, \ r \equiv 0 \text{ or degree of } r < \text{degree of } m.$$
But
\[ r(\theta) = f(\theta) - q(\theta)m(\theta) = 0, \]
and so if \( r \neq 0 \) then we divide \( r \) by its leading coefficient to get a monic polynomial over \( F \) that is not identically 0, has \( \theta \) as a root, and is of degree less than the degree of \( m \), which is impossible by the minimality of the degree of \( m \). Hence \( r \equiv 0 \) and so \( f = qm \). But \( f \) is irreducible over \( F \), and so either \( q \) or \( m \) is constant. If \( m \) is constant then \( m \equiv 1 \) (\( m \) is monic), not possible because \( m(\theta) = 0 \). Hence \( q \) is constant, and because \( f, m \) are both monic, \( q \equiv 1 \).

Hence \( f = m \).

**QED**

**Examples.**

(1) Let \( m \in \mathbb{Z} \setminus \{1\} \) be square-free, i.e., \( m \) does not have a square \( \neq 1 \) as a factor. Then \( \sqrt{m} \) is irrational, hence \( x^2 - m \) is irreducible over \( Q \). Lemma 3.5 \( \Rightarrow \) \( x^2 - m \) is the minimal polynomial of \( \sqrt{m} \) over \( Q \) and so \( \sqrt{m} \) is algebraic over \( Q \) of degree 2.

(2) Let \( q \) be a prime and let
\[ \zeta_q = \exp \left( \frac{2\pi i q}{q} \right). \]
Then \( \zeta_q = 1, \zeta_q \neq 1 \), hence we deduce from the factorization
\[ x^q - 1 = (x - 1) \left( \sum_{k=0}^{q-1} x^k \right) \]
that \( \zeta_q \) is a root of \( \sum_{k=0}^{q-1} x^k \).

We claim that \( \sum_{k=1}^{q-1} x^k \) is irreducible over \( Q \). To see this, note first that a polynomial \( f(x) \) is irreducible iff \( f(x+1) \) is irreducible, because \( f(x+1) = g(x)h(x) \) iff \( f(x) = g(x-1)h(x-1) \). Hence
\[ \sum_{k=0}^{q-1} x^k = \frac{x^q - 1}{x - 1} \text{ is irreducible iff } \frac{(x+1)^q - 1}{x} \text{ is irreducible.} \]

The binomial theorem \( \Rightarrow \)
\[ \frac{(x+1)^q - 1}{x} = \sum_{k=1}^{q} \binom{q}{k} x^{q-k}. \]

We now recall the following fact about binomial coefficients: \( q \) a prime \( \Rightarrow \) \( q \) divides the binomial coefficient \( \binom{q}{k}, k = 1, \ldots, q - 1 \). Hence
\[ \frac{(x+1)^q - 1}{x} = x^{q-1} + q(x^{q-2} + \ldots) + q, \]
and this polynomial is irreducible over \( Q \) by way of
Lemma 3.6. (Eisenstein’s criterion) If $q$ is prime and $f(x) = \sum_{k=0}^{n} a_k x^k$ is a polynomial in $\mathbb{Z}[x]$ whose coefficients satisfy: $q$ does not divide $a_n$, $q^2$ does not divide $a_0$, and $q$ divides $a_k$, $k = 0, 1, \ldots, n - 1$, then $f(x)$ is irreducible over $\mathbb{Q}$.

Thus $\zeta_q$ has minimal polynomial $\sum_{k=0}^{q-1} x^k$ and hence is algebraic over $\mathbb{Q}$ of degree $q - 1$.

Proof of Lemma 3.6. We assert first that if a polynomial $h \in \mathbb{Z}[x]$ does not factor into a product of polynomials in $\mathbb{Z}[x]$ of degree lower than the degree of $h$ then it is irreducible over $\mathbb{Q}$. In order to see this, suppose that $h$ is not constant (otherwise the assertion is trivial) and that $h = rs$, where $r$ and $s$ are polynomials in $\mathbb{Q}[x]$, both not constant and of lower degree than $h$. By clearing denominators and factoring out the greatest common divisors of appropriate integer coefficients, we find integers $a, b, c,$ and polynomials $g, u, v$ in $\mathbb{Z}[x]$ such that $h = ag$, degree of $r =$ degree of $u$, degree of $s =$ degree of $v$,

$$abg = cuv,$$

and all of the coefficients of $g$ (respectively, $u$, $v$) are relatively prime, i.e., the greatest common divisor of all of the coefficients of $g$ (respectively, $u$, $v$) is 1.

We claim that the coefficients of the product $uv$ are also relatively prime. Assume this for now. Then $|ab| =$ the greatest common divisor of the coefficients of $abg =$ the greatest common divisor of the coefficients of $cuv = |c|$, hence $ab = \pm c$. But then $h = \pm auv$ and this is a factorization of $h$ as a product of polynomials in $\mathbb{Z}[x]$ of lower degree.

We must now verify our claim. Suppose that the coefficients of $uv$ have a common prime factor $r$. Let $\mathbb{Z}_r$ denote the field of ordinary residue classes mod $r$. If $s \in \mathbb{Z}[x]$ and if we let $\bar{s}$ denote the polynomial in $\mathbb{Z}_r[x]$ obtained from $s$ by reducing the coefficients of $s$ mod $r$, then $s \to \bar{s}$ defines a homomorphism of $\mathbb{Z}[x]$ onto $\mathbb{Z}_r[x]$. Because $r$ divides all of the coefficients of $uv$, it hence follows that

$$0 = \bar{uv} = \bar{u} \bar{v} \text{ in } \mathbb{Z}_r[x].$$

Because $\mathbb{Z}_r$ is a field, $\mathbb{Z}_r[x]$ is an integral domain, hence we conclude from this equation that either $\bar{u}$ or $\bar{v}$ is 0 in $\mathbb{Z}_r[x]$, i.e., either all of the coefficients of $u$ or of $v$ are divisible by $r$. This contradicts the fact that the coefficients of $u$ (respectively, $v$) are relatively prime. The assertion that the product of two polynomials in $\mathbb{Z}[x]$ has all of its coefficients relatively prime whenever the coefficients of each polynomial are relatively prime is often referred to as Gauss’ lemma, not to be confused, of course, with the statement in Theorem 2.7.

Next suppose that $f(x) = \sum_{k=0}^{n} a_k x^k \in \mathbb{Z}[x]$ satisfies the hypotheses of Lemma 3.6. By virtue of what we just showed, we need only prove that $f$ does not factor into polynomials
of lower degree in $\mathbb{Z}[x]$. Suppose, on the contrary, that

$$f(x) = \left( \sum_{k=0}^{s} b_k x^k \right) \left( \sum_{k=0}^{t} c_k x^k \right)$$

is a factorization of $f$ in $\mathbb{Z}[x]$ with $b_s \neq 0 \neq c_t$ and $s$ and $t$ both less than $n$. Because $a_0 \equiv 0 \mod q$, $a_0 \not\equiv 0 \mod q^2$ and $a_0 = b_0 c_0$, one element of the set $\{b_0, c_0\}$ is $\not\equiv 0 \mod q$ and the other is $\equiv 0 \mod q$. Assume that $b_0$ is the former element and $c_0$ is the latter. As $a_n \neq 0 \mod q$ and $a_n = b_s c_t$, it follows that $b_s \neq 0 \neq c_t \mod q$. Let $m$ be the smallest value of $k$ such that $c_k \neq 0 \mod q$. Then $m > 0$, hence

$$a_m = \sum_{j=0}^{m-i} b_j c_{m-j}$$

for some $i \in [0, m-1]$. Because $b_0 \neq 0 \neq c_m \mod q$ and $c_{m-1}, \ldots, c_i$ are all $\equiv 0 \mod q$, it follows that $a_m \neq 0 \mod q$, and so $m = n$. Hence $t = n$, contradicting the assumption on $t$ and $n$.

QED

The crucial fact about algebraic numbers that we will need in order to prove the LQR is that the set of all algebraic integers (see the definition after the proof of Theorem 3.7) form a subring of the field of complex numbers. The verification of that fact is the goal of the next two results.

For use in the proof of the next theorem, we recall that if $n$ is a positive integer, then the **elementary symmetric polynomials in $n$ variables** are the polynomials in the variables $x_1, \ldots, x_n$ defined by

$$\sigma_1 = \sum_{i=1}^{n} x_i,$$

$$\vdots$$

$$\sigma_i = \text{sum of all products of } i \text{ different } x_j,$$

$$\vdots$$

$$\sigma_n = \prod_{i=1}^{n} x_i.$$

The elementary symmetric polynomials have the property that if $\pi$ is a permutation of the set $[1, n]$ then $\sigma_i(x_{\pi(1)}, \ldots, x_{\pi(n)}) = \sigma_i(x_1, \ldots, x_n)$, i.e., $\sigma_i$ is unchanged by any permutation of its variables.

**Theorem 3.7.** If $F$ is a complex number field then the set of all complex numbers algebraic over $F$ is a complex number field which contains $F$. 
Proof. Let $\alpha$ and $\beta$ be algebraic over $F$. We want to show that $\alpha \pm \beta, \alpha \beta,$ and $\alpha / \beta,$ provided that $\beta \neq 0$, are all algebraic over $F$. We will do this by the explicit construction of polynomials over $F$ that have these numbers as roots.

Start with $\alpha + \beta$. Let $f$ and $g$ denote the minimal polynomials of, respectively, $\alpha$ and $\beta$, of degree $m$ and $n$, respectively. Let $\alpha_1, \ldots, \alpha_m$ and $\beta_1, \ldots, \beta_n$ denote the roots of $f$ and $g$ in $C$, with $\alpha_1 = \alpha$ and $\beta_1 = \beta$. Now consider the polynomial

$$ (8) \prod_{i=1}^{m} \prod_{j=1}^{n} (x - \alpha_i - \beta_j) = x^{mn} + \sum_{i=1}^{mn} c_i(\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n)x^{mn-i}, $$

where each coefficient $c_i$ is a polynomial in the $\alpha_i$'s and $\beta_j$'s over $F$ (in fact, over $\mathbb{Z}$). We claim that

$$ (9) c_i(\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n) \in F, \ i = 1, \ldots, mn. $$

If this is true then the polynomial (8) is in $F[x]$ and has $\alpha_1 + \beta_1 = \alpha + \beta$ as a root, whence $\alpha + \beta$ is algebraic over $F$.

In order to verify (9), we will make use of the following result from the classical theory of equations (see Weisner [42], Theorem 49.10). Let $\tau_1, \ldots, \tau_m, \sigma_1, \ldots, \sigma_n$ denote, respectively, the elementary symmetric polynomials in $m$ and $n$ variables. Suppose that the polynomial $h$ over $F$ in the variables $x_1, \ldots, x_m, y_1, \ldots, y_n$ has the property that if $\pi$ (respectively, $\nu$) is a permutation of $[1, m]$ (respectively, $[1, n]$) then

$$ h(x_1, \ldots, x_m, y_1, \ldots, y_n) = h(x_{\pi(1)}, \ldots, x_{\pi(m)}, y_{\nu(1)}, \ldots, y_{\nu(n)}), $$

i.e., $h$ remains unchanged when its variables $x_i$ and $y_j$ are permuted amongst themselves. Then there exist a polynomial $l$ over $F$ in the variables $x_1, \ldots, x_m, y_1, \ldots, y_n$ such that

$$ h(x_1, \ldots, x_m, y_1, \ldots, y_n) = l(\tau_1(x_1, \ldots, x_m), \ldots, \tau_m(x_1, \ldots, x_m), \sigma_1(y_1, \ldots, y_n), \ldots, \sigma_n(y_1, \ldots, y_n)). $$

Observe next that the left-hand side of (8) remains unchanged when the $\alpha_i$'s and the $\beta_j$'s are permuted amongst themselves (this simply rearranges the order of the factors in the product), and so the same thing is true for each coefficient $c_i$. It thus follows from our result from the theory of equations that there exists a polynomial $l_i$ over $F$ in the variables $x_1, \ldots, x_m, y_1, \ldots, y_n$ such that

$$ c_i(\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n) = l_i(\tau_1(\alpha_1, \ldots, \alpha_m), \ldots, \tau_m(\alpha_1, \ldots, \alpha_m), \sigma_1(\beta_1, \ldots, \beta_n), \ldots, \sigma_n(\beta_1, \ldots, \beta_n)). $$
If we can prove that each of the numbers at which \( l_i \) is evaluated in this equation is in \( F \) then (9) will be verified. Hence it suffices to prove that if \( \theta \) is a number algebraic over \( F \) of degree \( n \), \( \theta_1, \ldots, \theta_n \) are the roots of the minimal polynomial \( m \) of \( \theta \) over \( F \), and \( \sigma \) is an elementary symmetric polynomial in \( n \) variables, then \( \sigma(\theta_1, \ldots, \theta_n) \in F \). But this last statement follows from the fact that \( m \) is

\[
\prod_{i=1}^{n} (x - \theta_i) = x^n + \sum_{i=1}^{n} (-1)^i \sigma_i(\theta_1, \ldots, \theta_n)x^{n-i},
\]

where \( \sigma_1, \ldots, \sigma_n \) are the elementary symmetric polynomials in \( n \) variables, and all coefficients of \( m \) are in \( F \), whence all of the coefficients of the polynomial on the right-hand side of this equation are also in \( F \).

A similar argument shows that \( \alpha - \beta \) and \( \alpha \beta \) are algebraic over \( F \).

Suppose next that \( \beta \neq 0 \) is algebraic over \( F \) and let

\[
x^n + \sum_{i=0}^{n-1} a_i x^i
\]

be the minimal polynomial of \( \beta \) over \( F \). Then \( 1/\beta \) is a root of

\[
1 + \sum_{i=0}^{n-1} a_i x^{n-i} \in F[x],
\]

and so \( 1/\beta \) is algebraic over \( F \). Then \( \alpha/\beta = \alpha \cdot (1/\beta) \) is algebraic over \( F \). QED

Notation: \( A(F) \) denotes the field of all complex numbers algebraic over \( F \).

Definition. An element of \( A(Q) \) is an algebraic integer if its minimal polynomial over \( Q \) has all of its coefficients in \( Z \).

Examples (1) and (2) \( \Rightarrow \sqrt{m}, m \) a square-free integer, and \( \exp(2\pi i/q), q \) a prime, are algebraic integers.

Theorem 3.8. The set of all algebraic integers is a subring of \( A(Q) \) containing \( Z \).

Proof. Let \( \alpha \) and \( \beta \) be algebraic integers. We need to prove that \( \alpha \pm \beta \) and \( \alpha \beta \) are algebraic integers. This can be done by first observing that the result from the theory of equations that we used in the proof of Theorem 3.7 holds mutatis mutandis if the field \( F \) there is replaced by the ring \( Z \) of integers (Weisner [42], Theorem 49.9). If we then let \( \alpha_1, \ldots, \alpha_m \) and \( \beta_1, \ldots, \beta_n \) denote the roots of the minimal polynomial over \( Q \) of \( \alpha \) and \( \beta \), respectively, then the proof of Theorem 3.7, with \( F \) replaced in that proof by \( Z \), verifies that \( \alpha \pm \beta \) and \( \alpha \beta \) are roots of monic polynomials in \( Z[x] \). We now invoke the following fact: if a complex number \( \theta \) is the root of a monic polynomial in \( Z[x] \) then it is an algebraic integer.
In order to prove the last statement about $\theta$, let $f \in \mathbb{Z}[x]$ be monic with $f(\theta) = 0$. If $m$ is the minimal polynomial of $\theta$ over $\mathbb{Q}$ then we must show that $m \in \mathbb{Z}[x]$. It follows from the proof of Proposition 3.4 that there is a $q \in \mathbb{Q}[x]$ such that $f = qm$ and so we can find a rational number $a/b$ and $u, v \in \mathbb{Z}[x]$ such that $f = (a/b)uv, u$ (respectively, $v$) is a constant multiple of $m$ (respectively, $q$), and $u$ (respectively, $v$) has all of its coefficients relatively prime.

We have that

$$bf = auv.$$ 

$f$ monic and $u, v \in \mathbb{Z}[x] \Rightarrow a$ divides $b$ in $\mathbb{Z}$, say $b = ak$ for some $k \in \mathbb{Z}$. Hence

$$kf = uv.$$ 

Because $f \in \mathbb{Z}[x]$, it follows that $k$ is a common factor of all of the coefficients of $uv$. Because of the claim that we verified in the proof of Lemma 3.6, the coefficients of $uv$ are relatively prime, hence $k = \pm 1$, and so

$$f = \pm uv.$$ 

As $f$ is monic, the leading coefficient of $u$ is $\pm 1$. But $u$ is a constant multiple of $m$ and $m$ is monic, hence $m = \pm u \in \mathbb{Z}[x]$, and so $\theta$ is an algebraic integer. QED

*Notation:* $\mathcal{R}$ will denote the ring of algebraic integers.

In the second proof of LQR, we will need the following simple lemma:

**Lemma 3.9.** $\mathcal{R} \cap \mathbb{Q} = \mathbb{Z}$.

**Proof.** If $q \in \mathcal{R} \cap \mathbb{Q}$ then $x - q$ is the minimal polynomial of $q$ over $\mathbb{Q}$, hence $x - q \in \mathbb{Z}[x]$, hence $q \in \mathbb{Z}$. QED

As a warm-up for the proof of LQR, we will reprove Theorem 2.6, which asserts that

$$\chi_\mathbb{F}(2) = (-1)^\varepsilon,$$

where $\varepsilon \equiv \frac{p^2 - 1}{8} \mod 2$,

by using algebraic number theory. Let

$$\zeta = e^{\pi/4} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i.$$ 

Then

$$\zeta^{-1} = e^{-\pi/4} = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i,$$

hence

$$\tau = \zeta + \zeta^{-1} = \sqrt{2} \in \mathcal{R},$$

and so we can work in the ring $\mathcal{R}$ of algebraic integers.
If \( p \) is an odd prime then we let

\[
(p) = \text{the ideal in } \mathcal{R} \text{ generated by } p = p\mathcal{R} = \{p\alpha : \alpha \in \mathcal{R}\}.
\]

If \( \alpha, \beta \in \mathcal{R} \), then we will write

\[\alpha \equiv \beta \mod p\]

if \( \alpha - \beta \in (p) \). Euler’s criterion (Theorem 2.5) \( \Rightarrow \)

\[
\tau^{p-1} = (\tau^2)^{(p-1)/2} = 2^{(p-1)/2} \equiv \chi_p(2) \mod p,
\]

hence

(10) \[\tau^p \equiv \chi_p(2)\tau \mod p.\]

We now make use of the following lemma, which follows from the binomial theorem and the fact that \( p \) divides the binomial coefficient \( \binom{p}{k}, k = 1, \ldots, p - 1 \).

**Lemma 3.10.** If \( \alpha, \beta \in \mathcal{R} \) then

\[(\alpha + \beta)^p \equiv \alpha^p + \beta^p \mod p.\]

Hence

(11) \[\tau^p = (\zeta + \zeta^{-1})^p \equiv \zeta^p + \zeta^{-p} \mod p.\]

The next step is to calculate \( \zeta^p + \zeta^{-p} \). Begin by noting that

\[\zeta^8 = 1,\]

hence if \( p \equiv \pm 1 \mod 8 \), then

\[\zeta^p + \zeta^{-p} = \zeta + \zeta^{-1} = \tau,\]

and if \( p \equiv \pm 3 \mod 8 \), then

\[
\begin{align*}
\zeta^p + \zeta^{-p} &= \zeta^3 + \zeta^{-3} \\
&= -(\zeta^{-1} + \zeta) \\
&= -\tau,
\end{align*}
\]

where the second line follows from the first because \( \zeta^4 = -1 \Rightarrow \zeta^3 = -\zeta^{-1} \Rightarrow \zeta^{-3} = -\zeta \).

Hence

(12) \[\zeta^p + \zeta^{-p} = (-1)^\varepsilon \tau, \quad \varepsilon \equiv \frac{p^2 - 1}{8} \mod 2.\]

(10), (11), and (12) \( \Rightarrow \)

\[\chi_p(2)\tau \equiv \zeta^p + \zeta^{-p} \equiv (-1)^\varepsilon \tau \mod p.\]
Multiply this congruence by $\tau$ and use $\tau^2 = 2$ to derive

\[(13) \quad 2\chi_p(2) \equiv 2(-1)^\varepsilon \mod p.\]

Now this congruence is in $\mathcal{R}$, so there exits $\alpha \in \mathcal{R}$ such that

$$2\chi_p(2) = 2(-1)^\varepsilon + \alpha p,$$

hence

$$\alpha = \frac{2(\chi_p(2) - (-1)^\varepsilon)}{p} \in \mathcal{R} \cap Q = Z \text{ (by Lemma 3.9)}.$$

Hence (13) is in fact a congruence in $Z$, and so

$$\chi_p(2) \equiv (-1)^\varepsilon \mod p \text{ in } Z,$$

whence, as before,

$$\chi_p(2) = (-1)^\varepsilon.$$

This proof of Theorem 2.6 depends on the equation $\tau^2 = 2$. Can one get a similar equation with an odd prime $p$ replacing the 2 on the right-hand size of this equation? Yes one can, and a proof of LQR will follow in a similar way from the ring structure of $\mathcal{R}$.

In order to see how that goes, let $\zeta = e^{2\pi i/p}$ and set

$$g = \sum_{n=0}^{p-1} \chi_p(n)\zeta^n,$$

$$p^* = (-1)^{(p-1)/2} p.$$

The sum $g$ is called a Gauss sum; these sums were first used by Gauss in his famous study of cyclotomy which concluded *Disquisitiones Arithmeticae* ([17], section VII). The analogue of the equation $\tau^2 = 2$ is given by

**Theorem 3.11.** $g^2 = p^*$.

Assume this for now; we deduce LQR from it like so: let $q$ be an odd prime, $q \neq p$. Then

$$g^{q-1} = (g^2)^{(q-1)/2} = (p^*)^{(q-1)/2} \equiv \chi_q(p^*) \mod q,$$

where the last equivalence follows from Euler’s criterion. Hence

\[(14) \quad g^q \equiv \chi_q(p^*)g \mod q,\]
where this congruence is now in $\mathcal{R}$, because $g \in \mathcal{R}$. If $n \in \mathbb{Z}$ then $\chi_p(n)^q = \chi_p(n)$ because $\chi_p(n) \in [-1, 1]$ and $q$ is odd; consequently Lemma 3.10 ⇒

\begin{equation}
(15) \quad g^q = \left( \sum_n \chi_p(n) \zeta^n \right)^q = \sum_n \chi_p(n)^q \zeta^{qn} \mod q = \sum_n \chi_p(n) \zeta^{qn} \mod q,
\end{equation}

We now need

**Lemma 3.12.** If $a \in \mathbb{Z}$ then

\[ \sum_n \chi_p(n) \zeta^{an} = \chi_p(a) g. \]

The sum on the left-hand side of this equation is another Gauss sum. Lemma 3.12 records a very important relation satisfied by Gauss sums; in addition to the use that we make of it here, it will also play an important role in some calculations that are performed in Chapter 7, where we study certain distributions of residues and non-residues.

Assume Lemma 3.12 for now; this lemma and (15) ⇒

\begin{equation}
(16) \quad g^q \equiv \chi_p(q) g \mod q.
\end{equation}

Congruences (14), (16) ⇒

\[ \chi_q(p^*) g \equiv \chi_p(q) g \mod q. \]

Multiply by $g$ and use $g^2 = p^*$ to derive

\[ \chi_q(p^*) p^* \equiv \chi_p(q) p^* \mod q, \]

and then apply Lemma 3.9 and the fact that $\chi_q(p^*)$, $\chi_p(q)$ are both ±1 as before to get

\begin{equation}
(17) \quad \chi_q(p^*) = \chi_p(q).
\end{equation}

Theorem 2.4 ⇒

\[ \chi_q(-1) = (-1)^{(q-1)/2}, \]

hence (17) ⇒

\[ \chi_p(q) = \chi_q(-1)^{(p-1)/2} \chi_q(p) = (-1)^{(p-1)(q-1)/2} \chi_q(p), \]

which is the LQR.

We must now prove Theorem 3.11 and Lemma 3.12. Since Lemma 3.12 is used in the proof of Theorem 3.11, we verify Lemma 3.12 first.
Proof of Lemma 3.12. Suppose first that \( p \) divides \( a \). Then \( \zeta^{an} = 1 \), for all \( n \) and \( \chi_p(0) = 0 \) so
\[
\sum_n \chi_p(n)\zeta^{an} = \sum_{n=1}^{p-1} \chi_p(n).
\]
Half of the terms of the sum on the right-hand side are 1 and the other half are \(-1\) (Proposition 2.1), and so this sum is 0. Because \( \chi_p(a) = 0 \) (\( p \) divides \( a \)), the conclusion of Lemma 3.12 is valid.

Suppose that \( p \) does not divide \( a \). Then
\[
\chi_p(a) \sum_n \chi_p(n)\zeta^{an} = \sum_n \chi_p(an)\zeta^{an}.
\]
Observe next that whenever \( n \) runs through a complete system of ordinary residues mod \( p \), so does \( an \), and also that \( \chi_p(an) \) and \( \zeta^{an} \) depend only on the residue class mod \( p \) of \( an \). Hence the sum on the right-hand side of (18) is
\[
\sum_{n=0}^{p-1} \chi_p(n)\zeta^n = g.
\]
Hence
\[
\chi_p(a) \sum_n \chi_p(n)\zeta^{an} = g.
\]
Now multiply through by \( \chi_p(a) \) and use the fact that \( \chi_p(a)^2 = 1 \), since \( p \) does not divide \( a \).

QED

Proof of Theorem 3.11. We must prove that \( g^2 = p^* \).

Suppose that \( \gcd(a, p) = 1 \) and let
\[
g(a) = \sum_{n=0}^{p-1} \chi_p(n)\zeta^{an}.
\]
The idea of this argument is to calculate
\[
\sum_{a=0}^{p-1} g(a)g(-a)
\]
in two different ways, equate the expressions resulting from that, and see what happens.

For the first way, use Lemma 3.12 to obtain
\[
g(a)g(-a) = \chi_p(a)\chi_p(-a)g^2
\]
\[
= \chi_p(-a^2)g^2
\]
\[
= \chi_p(-1)g^2, \; a = 1, \ldots, p - 1,
\]
Hence this and the fact that \( g(0) = \sum_{0}^{p-1} \chi_p(n) = 0 \) ⇒

\[
(19) \quad \sum_{a=0}^{p-1} g(a)g(-a) = (p-1)\chi_p(-1)g^2.
\]

Now for the second way. We have that

\[
g(a)g(-a) = \sum_{1 \leq x,y \leq p-1} \chi_p(x)\chi_p(y)\zeta^{a(x-y)}.
\]

Hence

\[
(20) \quad \sum_{a=0}^{p-1} g(a)g(-a) = \sum_{1 \leq x,y \leq p-1} \chi_p(x)\chi_p(y) \sum_{a} \zeta^{a(x-y)}.
\]

The next step is to calculate

\[
\sum_{a} \zeta^{a(x-y)}
\]

for fixed \( x \) and \( y \). If \( x \neq y \) then

\[
1 \leq |x - y| \leq p - 1
\]

and so \( p \) does not divide \( x - y \), hence \( \zeta^{x-y} \neq 1 \), hence

\[
\sum_{a} \zeta^{a(x-y)} = \frac{\zeta^{(x-y)p} - 1}{\zeta^{x-y} - 1} = 0, \quad (\zeta^p = 1 !).
\]

Hence

\[
(21) \quad \sum_{a} \zeta^{a(x-y)} = \begin{cases} 
    p, & \text{if } x = y, \\
    0, & \text{if } x \neq y.
\end{cases}
\]

Equations (20) and (21) ⇒

\[
(22) \quad \sum_{a=0}^{p-1} g(a)g(-a) = (p-1)p.
\]

Equations (19) and (22) ⇒

\[
(p-1)\chi_p(-1)g^2 = (p-1)p,
\]

hence from Theorem 2.4,

\[
g^2 = \chi_p(-1)p = (-1)^{(p-1)/2}p.
\]

\[QED\]
CHAPTER 4

Applications of Quadratic Reciprocity

Obviously, if $a \in \mathbb{Z}$ is a square then $a$ is a residue of all primes. Is the converse true, i.e., if a positive integer is a residue of all primes, must it be a square? The answer is yes; in fact a slightly stronger statement is valid:

**Theorem 4.1.** A positive integer is a residue of all but finitely many primes iff it is a square.

This theorem implies that if $S$ is a nonempty finite subset of $[1, \infty)$ then $S$ is a set of residues for all but finitely many primes iff every element of $S$ is a square. What if we weaken the requirement that $S$ be a set of residues of all but finitely many primes to the requirement that $S$ be a set of residues for only *infinitely many* primes? Then the somewhat surprising answer is asserted by

**Theorem 4.2.** If $S$ is any nonempty finite subset of $[1, \infty)$ then $S$ is a set of residues of infinitely many primes.

Theorem 4.2 gives rise to the following natural and interesting question: if $S$ is a nonempty, finite subset of $[1, \infty)$, how large is the necessarily infinite set of primes

$$\{p : \chi_p \equiv 1 \text{ on } S\}?$$

(The meaning of the symbol $\equiv$ used here is as an identity of functions, *not* as a modular congruence; in subsequent uses of this symbol, its meaning will be clear from the context.)

To formulate this question precisely, we need a good way to measure the size of an infinite set of primes. This is provided by the concept of the asymptotic density of a set. If $\Sigma$ is a set of primes and $P$ denotes the set of all primes then the *asymptotic density of $\Sigma$ in $P$* is

$$\lim_{x \to +\infty} \left| \frac{\{p \in \Sigma : p \leq x\}}{\{p \in P : p \leq x\}} \right|,$$

provided that this limit exists. Roughly speaking, the density of $\Sigma$ is the “proportion” of the set $P$ that is occupied by $\Sigma$. We can in fact be a bit more precise: if $a(x)$ and $b(x)$ denote positive real-valued functions defined on $(0, +\infty)$, we say that $a(x)$ is *asymptotic to* $b(x)$ as
$x \to +\infty$, denoted by $a(x) \sim b(x)$, if

$$
\lim_{x \to +\infty} \frac{a(x)}{b(x)} = 1.
$$

The Prime Number Theorem (LeVeque, [28], chapter 7; Montgomery and Vaughn, [29], chapter 6) asserts that as $x \to +\infty$,

$$
|\{q \in \mathbb{P} : q \leq x\}| \sim \frac{x}{\log x},
$$

consequently, if $d$ is the density of $\Sigma$ then as $x \to +\infty$,

$$
|\{q \in \Sigma : q \leq x\}| \sim d \frac{x}{\log x}.
$$

We now state a theorem which provides a way to calculate the density of the set $\{p : \chi_p \equiv 1 \mbox{ on } S\}$. This will be given by a formula which depends on a certain combinatorial parameter that is determined by the prime factors of the elements of $S$. In order to formulate this result, let $F$ denote the Galois field $GF(2)$ of 2 elements, which can be concretely realized as the field $\mathbb{Z}/2\mathbb{Z}$ of ordinary residue classes mod 2. Let $A \subseteq [1, \infty)$. If $n = |A|$, then we let $F^n$ denote the vector space over $F$ of dimension $n$, arrange the elements $a_1 < \cdots < a_n$ of $A$ in increasing order, and then define the map $v : 2^A \to F^n$ like so: if $B \subseteq A$ then

$$
\text{the } i\text{-th coordinate of } v(B) = \begin{cases} 1, & \text{if } a_i \in B, \\ 0, & \text{if } a_i \notin B. \end{cases}
$$

If we recall that $\pi_{\text{odd}}(z)$ denotes the set of all prime factors of odd multiplicity of the integer $z$ then we can now state and prove the following theorem:

**Theorem 4.3.** If $S$ is a nonempty, finite subset of $[1, \infty)$, 

$$
S = \{\pi_{\text{odd}}(z) : z \in S\},
$$

$$
A = \bigcup_{X \in S} X,
$$

$$
n = |A|,
$$

and

$$
d = \text{the dimension of the linear span of } v(S) \text{ in } F^n,
$$

then the density of $\{p : \chi_p \equiv 1 \mbox{ on } S\}$ is $2^{-d}$.

Theorem 4.3 reduces the calculation of the density of $\{p : \chi_p \equiv 1 \mbox{ on } S\}$ to prime factorization of the integers in $S$ and linear algebra over $F$. If we enumerate the nonempty
elements of $S$ as $S_1, \ldots, S_m$ (if $S$ has no such elements then $S$ consists entirely of squares, hence the density is clearly 1) then $d$ is just the rank over $F$ of the $m \times n$ matrix

$$
\begin{pmatrix}
v(S_1)(1) & \cdots & v(S_1)(n) \\
\vdots & \ddots & \vdots \\
v(S_m)(1) & \cdots & v(S_m)(n)
\end{pmatrix},
$$

where $v(S_i)(j)$ is the $j$-th coordinate of $v(S_i)$. Because there are only two elementary row (column) operations over $F$, namely row (column) interchange and addition of a row (column) to another row (column), the rank of this matrix is easily calculated by Gauss-Jordan elimination. However, this procedure requires that we first find the prime factors of odd multiplicity of each element of $S$, and that, in general, is not so easy!

We proceed to prove Theorems 4.1, 4.2, and 4.3, and we will see that the LQR is an essential ingredient of the arguments.

Theorems 4.1 and 4.2 are simple consequences of

**Lemma 4.4. (Basic Lemma)** If $\Pi = \{p_1, \ldots, p_k\}$ is a nonempty finite set of primes and if $\varepsilon : \Pi \to \{-1, 1\}$ is a fixed function then there exits infinitely many primes $p$ such that

$$
\chi_p(p_i) = \varepsilon(p_i), \ i \in [1, k].
$$

N.B. This lemma asserts that if all of the integers in the set $S$ of Theorem 4.2 are prime, then the conclusion of that theorem can be strengthened considerably.

Assume Lemma 4.4 for now.

*Proof of Theorem 4.1.* Suppose that $n \in [1, \infty)$ is not a square. Then $\pi_{\text{odd}}(n) \neq \emptyset$ and

(1) \hspace{1cm} \chi_p(n) = \prod_{q \in \pi_{\text{odd}}(n)} \chi_p(q), \text{ for all } p \notin \pi(n).

Now take any fixed $q_0 \in \pi_{\text{odd}}(n)$ and define $\varepsilon : \pi_{\text{odd}}(n) \to \{-1, 1\}$ by

$$
\varepsilon(q) = \begin{cases}
-1, & \text{if } q = q_0, \\
1, & \text{if } q \neq q_0.
\end{cases}
$$

Lemma 4.4 $\Rightarrow$ there exits infinitely many primes $p$ such that

$$
\chi_p(q) = \varepsilon(q), \text{ for all } q \in \pi_{\text{odd}}(n),
$$

and so the product in (1), and hence $\chi_p(n)$, is $-1$ for all such $p \notin \pi(n)$. \hspace{1cm} \text{QED}

*Proof of Theorem 4.2.* Let

$$
X = \bigcup_{z \in S} \pi_{\text{odd}}(z).
$$
We may assume that $X \neq \emptyset$; otherwise all elements of $S$ are squares and Theorem 4.2 is trivially true in that case. Then Lemma 4.4 ⇒ there exists infinitely many primes $p$ such that

$$\chi_p(q) = 1, \text{ for all } q \in X,$$

hence for all such $p$ which are not factors of an element of $S$,

$$\chi_p(z) = \prod_{q \in \pi_{\text{odd}}(z)} \chi_p(q) = 1, \text{ for all } z \in S.$$  

QED

**Proof of Lemma 4.4.** It follows from our solution of the Fundamental Problem for all primes (Theorem 2.6 and the calculation in Chapter 3 of $X_{\pm}(q)$, $q$ an odd prime) that Lemma 4.4 is valid when $\Pi$ is a singleton, so assume that $k \geq 2$. We will make use of arithmetic progressions in this argument, and so if $a, b \in [1, \infty)$, let

$$AP(a, b) = \{a + nb : n \in [0, \infty)\}$$

denote the arithmetic progression with initial term $a$ and common difference $b$. We will find the primes that will verify the conclusion of Lemma 4.4 by looking inside certain arithmetic progressions, hence we will need the following theorem, one of the basic results in the theory of prime numbers:

**Theorem 4.5.** *(Dirichlet’s theorem on primes in arithmetic progression).* If $\{a, b\} \subseteq [1, \infty)$ and $\gcd(a, b) = 1$ then $AP(a, b)$ contains infinitely many primes.

The key ideas in Dirichlet’s proof of Theorem 4.5 will be discussed in due course. For now, assume that the elements of the set $\Pi$ in the hypothesis of Lemma 4.4 are ordered as $p_1 < \cdots < p_k$ and fix $\varepsilon : \Pi \to \{-1, 1\}$. We need to verify the conclusion of Lemma 4.4 for this $\varepsilon$. Suppose first that $p_1 = 2$ and $\varepsilon(2) = 1$. If $i \in [2, k]$ and $\varepsilon(p_i) = 1$, let $k_i = 1$, and if $\varepsilon(p_i) = -1$, let $k_i$ be an odd non-residue of $p_i$ such that $\gcd(p_i, k_i) = 1$ (if $\varepsilon(p_i) = -1$ then such a $k_i$ can always be chosen: simply pick any non-residue $x$ of $p_i$ in $[1, p_i - 1]$; if $x$ is odd, set $k_i = x$, and if $x$ is even, set $k_i = x + p_i$).

Now, suppose that $i \in [2, k]$ , $p \equiv 1 \mod 8$, and $p \in AP(k_i, 2p_i)$, say $p = k_i + 2p_i n$, for some $n \in [1, \infty)$. Then LQR ⇒

$$\chi_p(p_i) = \chi_{p_i}(p) = \chi_{p_i}(k_i + 2p_i n) = \chi_{p_i}(k_i).$$

It follows from Theorem 2.6 and the choice of $k_i$ that

$$\chi_p(2) = 1 \text{ and } \chi_p(p_i) = \varepsilon(p_i).$$
(2) if $p \equiv 1 \mod 8$ and $p \in \bigcap_{i=2}^{k} AP(k_i, 2p_i)$, then $\chi_p(p_i) = \varepsilon(p_i)$, for all $i \in [1, k]$.

We prove next that there are infinitely many primes $\equiv 1 \mod 8$ inside $\bigcap_{i=2}^{k} AP(k_i, 2p_i)$. To see this, we first use the fact that each $k_i$ is odd and an inductive construction obtained from solving an appropriate sequence of linear Diophantine equations (Proposition 1.4) to obtain an integer $m$ such that

$$AP(k_2 + 2m, 8p_2 \cdots p_k) \subseteq AP(1, 8) \cap \left( \bigcap_{i=2}^{k} AP(k_i, 2p_i) \right).$$

We then claim that $\gcd(k_2 + 2m, 8p_2 \cdots p_k) = 1$. If this is true then Theorem 4.5 $\Rightarrow AP(k_2 + 2m, 8p_2 \cdots p_k)$ contains infinitely many primes $p$, hence for any such $p$, (2) and (3) $\Rightarrow$

$$\chi_p(p_i) = \varepsilon(p_i), \ i \in [1, k],$$

the conclusion of Lemma 4.4. To verify the claim, assume by way of contradiction that $q$ is a common prime factor of $k_2 + 2m$ and $8p_2 \cdots p_k$. Then $q \neq 2$ because $k_2$ is odd, hence there is a $j \in [2, k]$ such that $q = p_j$. But (3) $\Rightarrow$ there exists $n \in [0, \infty)$ such that

$$k_2 + 2m + 8p_2 \cdots p_k = k_j + 2np_j,$$

and so $p_j$ divides $k_j$, contrary to the choice of $k_j$.

If $p_1 = 2$ and $\varepsilon(2) = -1$, a similar argument shows that $\bigcap_{i=2}^{k} AP(k_i, 2p_i)$ contains infinitely many primes $p \equiv 5 \mod 8$, hence (4) is true for all such $p$. If $p_1 \neq 2$, simply adjoin 2 to $\Pi$ and repeat this argument. QED

**Intermezzo: Dirichlet’s theorem on primes in arithmetic progression.**

Because they will play such an important role in our story, we will now discuss the key ingredients of Dirichlet’s proof of Theorem 4.5. Dirichlet [9] proved this in 1837, and it would be hard to overemphasize the importance of this theorem and the methods Dirichlet developed to prove it. As we shall see, he used *analysis*, specifically the theory of analytic functions of a complex variable, and in subsequent work [10] also the theory of Fourier series, to discover properties of the primes (for the reader who may benefit from it, we briefly discuss analytic functions, Fourier series, and some of their basic properties in Chapter 7). His use of continuous methods to prove deep results about discrete sets like the prime numbers was not only a revolutionary insight, but also caused a sensation in the nineteenth century mathematical community. Dirichlet’s results founded the subject of *analytic number theory*, which has become one of the most important areas and a major industry in number theory.
today. Later (in Chapters 5 and 7) we will also see how Dirichlet used analytic methods to study important properties of residues and non-residues.

In 1737, Euler proved that the series \( \sum_{q \in \mathbb{P}} \frac{1}{q} \) diverges and hence deduced Euclid’s theorem that there are infinitely many primes. Taking his cue from this result, Dirichlet sought to prove that

\[
\sum_{p \equiv a \pmod{b}} \frac{1}{p} \quad \text{diverges, where } a \text{ and } b \text{ are given positive relatively prime integers. To do this, he studied the behavior as } s \to 1^+ \text{ of the function of } s \text{ defined by}
\]

\[
\sum_{p \equiv a \pmod{b}} \frac{1}{p^s}.
\]

This function is difficult to get a handle on; it would be easier if we could replace it by a sum indexed over all of the primes, so consider

\[
\sum_p \delta(p)p^{-s}, \text{ where } \delta(p) = \begin{cases} 
1, & \text{if } p \equiv a \pmod{b}, \\
0, & \text{otherwise.}
\end{cases}
\]

Dirichlet’s profound insight was to replace \( \delta(p) \) by certain functions which approximate the behavior of \( \delta \) closely enough, but which are much better behaved relative to primes in the ordinary residue classes mod \( b \). We now define these functions.

Begin by recalling that if \( A \) is a commutative ring with identity 1 then a unit \( u \) of \( A \) is an element of \( A \) that has a multiplicative inverse in \( A \), i.e., there exists \( v \in A \) such that \( uv = 1 \). The set of all units of \( A \) forms a group under the multiplication of \( A \), called the group of units of \( A \). Consider now the ring \( \mathbb{Z}/b\mathbb{Z} \) of ordinary residue classes of \( \mathbb{Z} \) mod \( b \).

Proposition 1.2 \( \Rightarrow \) the group of units of \( \mathbb{Z}/b\mathbb{Z} \) consists of all ordinary residue classes that are determined by the integers that are relatively prime to \( b \). If we hence identify \( \mathbb{Z}/b\mathbb{Z} \) in the usual way with the set of ordinary non-negative minimal residues \([0, b - 1]\) on which is defined the addition and multiplication induced by addition and multiplication of ordinary residue classes, it follows that

\[
U(b) = \{ n \in [1, b - 1] : \gcd(n, b) = 1 \}
\]

is the group of units of \( \mathbb{Z}/b\mathbb{Z} \), and we set

\[
\varphi(b) = |U(b)|;
\]

\( \varphi \) is called Euler’s totient function.

Let \( T \) denote the circle group of all complex numbers of modulus 1, with the group operation defined by ordinary multiplication of complex numbers. A homomorphism of \( U(b) \)
into $T$ is called a Dirichlet character modulo $b$. We denote by $\chi_0$ the principal character modulo $b$, i.e., the character which sends every element of $U(b)$ to $1 \in T$. If $\chi$ is a Dirichlet character modulo $b$, we extend it to all integers $z$ by setting $\chi(z) = \chi(n)$ if there exists $n \in U(b)$ such that $z \equiv n \mod b$, and setting $\chi(z) = 0$, otherwise. It is then easy to verify Proposition 4.6.

A Dirichlet character $\chi$ modulo $b$ is

(i) of period $b$, i.e., $\chi(n) = 0$ iff $\gcd(n, b) > 1$ and $\chi(m) = \chi(n)$ whenever $m \equiv n \mod b$, and is

(ii) completely multiplicative, i.e., $\chi(mn) = \chi(m)\chi(n)$ for all $m, n \in \mathbb{Z}$.

We say that a Dirichlet character is real if it is real-valued, i.e., its range is either the set $\{0, 1\}$ or $[-1, 1]$. In particular the Legendre symbol $\chi_p$ is a real Dirichlet character mod $p$.

For each modulus $b$, the structure theory of finite Abelian groups can be used to explicitly construct all Dirichlet characters mod $b$; we will not do this, and instead refer the interested reader to Hecke [22], section 10 or Davenport [5], pp. 27-30. In particular there are exactly $\varphi(b)$ Dirichlet characters mod $b$.

The connection between Dirichlet characters and primes in arithmetic progression can now be made. If $\gcd(a, b) = 1$ then Dirichlet showed that

$$\frac{1}{\varphi(b)} \sum_{\chi} \chi(a)\chi(p) = \begin{cases} 1, & \text{if } p \equiv a \mod b, \\ 0, & \text{otherwise}, \end{cases}$$

where the sum is taken over all Dirichlet characters $\chi$ mod $b$. These are the so-called orthogonality relations for the Dirichlet characters. This equation says that the characteristic function $\delta(p)$ of the primes in an ordinary equivalence class mod $b$ can be written as a linear combination of Dirichlet characters. Hence

$$\sum_{p \equiv a \mod b} \frac{1}{p^s} = \sum_{p} \delta(p)p^{-s} = \sum_{p} \left( \frac{1}{\varphi(b)} \sum_{\chi} \chi(a)\chi(p) \right) p^{-s} = \frac{1}{\varphi(b)} \sum_{p} p^{-s} + \frac{1}{\varphi(b)} \sum_{\chi \neq \chi_0} \chi(a) \left( \sum_{p} \chi(p)p^{-s} \right).$$

After observing that

$$\lim_{s \to 1^+} \sum_{p} p^{-s} = +\infty,$$

Dirichlet deduced from the above equations the following lemma:

**Lemma 4.7.** $\lim_{s \to 1^+} \sum_{p \equiv a \mod b} p^{-s} = +\infty$ if for each non-principal Dirichlet character $\chi$ mod $b$, $\sum_{p} \chi(p)p^{-s}$ is bounded as $s \to 1^+$. 

Hence Theorem 4.5 will follow if one can prove that

\[(5) \quad \text{for all non-principal Dirichlet characters } \chi \mod b, \sum_p \chi(p)p^{-s} \text{ is bounded as } s \to 1^+.\]

Let \( \chi \) be a given Dirichlet character. In order to verify (5), Dirichlet introduced his next deep insight into the problem by considering the function

\[ L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}, \quad s \in \mathbb{C}, \]

which has come to be known as the \textit{Dirichlet L-function of} \( \chi \). He proved that \( L(s, \chi) \) is analytic in the half-plane \( \text{Re } s > 1 \), is analytic in \( \text{Re } s > 0 \) whenever \( \chi \) is non-principal, and satisfies the infinite-product formula

\[ L(s, \chi) = \prod_{q \in \mathbb{P}} \frac{1}{1 - \chi(q)q^{-s}}, \quad \text{Re } s > 1, \]

the \textit{Euler-Dirichlet product formula} (we will verify all of these facts about \( L \)-functions in Chapter 7). One can take the complex logarithm of both sides of this formula to deduce that

\[ \log L(s, \chi) = \sum_{n=2}^{\infty} \frac{\chi(n)\Lambda(n)}{\log n} n^{-s}, \quad \text{Re } s > 1, \]

where

\[ \Lambda(n) = \begin{cases} \log q, & \text{if } n \text{ is a power of } q, q \in \mathbb{P}, \\ 0, & \text{otherwise}. \end{cases} \]

Using algebraic properties of the character \( \chi \) and the function \( \Lambda \), Dirichlet proved that (5) is true if

\[(6) \quad \log L(s, \chi) \text{ is bounded as } s \to 1^+ \text{ whenever } \chi \text{ is non-principal.}\]

Because \( L(s, \chi) \) is continuous on \( \text{Re } s \geq 1 \), it follows that

\[ \lim_{s \to 1^+} \log L(s, \chi) = \log L(1, \chi), \]

hence (6) will hold if

\[ L(1, \chi) \neq 0 \text{ whenever } \chi \text{ is non-principal.} \]

We have at last come to the heart of the matter, namely

**Lemma 4.8.** If \( \chi \) is a non-principal Dirichlet character then \( L(1, \chi) \neq 0 \).
4. APPLICATIONS OF QUADRATIC RECIPROCITY

If $\chi$ is not real, Lemma 4.8 is fairly easy to prove, but when $\chi$ is real, this task is much more difficult to do. Dirichlet deduced Lemma 4.8 for real characters in a rather round-about way by using the classification of binary quadratic forms which Gauss developed in *Disquisitiones Arithmeticae* ([17], section V). Dirichlet established a remarkable formula which calculates $L(1, \chi)$ as the product of a certain parameter and the number of certain equivalence classes of quadratic forms; because this parameter and the number of equivalence classes are clearly positive, $L(1, \chi)$ must be nonzero. At the conclusion of Chapter 7, we will give an elegant proof of Lemma 4.8 for real characters due to de la Vallée Poussin [32].

Finally, we note that if $\chi_0$ is the principal character mod $b$ then it is a consequence of the Euler-Dirichlet product formula that

$$L(s, \chi_0) = \zeta(s) \prod_{q | b} (1 - q^{-s}),$$

where

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

is the Riemann zeta function.

At this first appearance in our story of $\zeta(s)$, probably the single most important function in analytic number theory, we cannot resist briefly discussing the

*Riemann Hypothesis*: all zeros of $\zeta(s)$ in the strip $0 < \text{Re} \ s < 1$ have real part $\frac{1}{2}$.

*Generalized Riemann Hypothesis (GRH)*: if $\chi$ is a Dirichlet character then all zeros of $L(s, \chi)$ in the strip $0 < \text{Re} \ s \leq 1$ have real part $\frac{1}{2}$.

Riemann [33] first stated the Riemann Hypothesis (in an equivalent form) in a paper that he published in 1859, in which he derived an explicit formula for the number of primes not exceeding a given real number. By general agreement, verification of the Riemann Hypothesis is the most important unsolved problem in mathematics. One of the most immediate consequences of the truth of the Riemann Hypothesis, and arguably the most significant, is the essentially optimal error estimate for the asymptotic approximation of the cardinality of the set $\{q \in P : q \leq x\}$ given in the Prime Number Theorem. This estimate asserts that there is an absolute, positive constant $C$ such that for all $x$ sufficiently large,

$$\left| \left\{ q \in P : q \leq x \right\} - \int_{2}^{x} \frac{1}{\log t} \, dt \right| \leq \frac{C}{\sqrt{x}}.$$
The integral \( \int_{2}^{x} \frac{1}{\log t} \, dt \) appearing in this inequality, the logarithmic integral of \( x \), is generally a better asymptotic approximation to the cardinality of \( \{ q \in P : q \leq x \} \) than the quotient \( x/\log x \). Hilbert emphasized the importance of the Riemann Hypothesis in Problem 8 on his famous list of 23 open problems that he presented in 1900 in his address to the second International Congress of Mathematicians. In 2000, the Clay Mathematics Institute (CMI) published a series of seven open problems in mathematics that are considered to be of exceptional importance and have long resisted solution. In order to encourage work on these problems, which have come to be known as the Clay Millennium Prize Problems, for each problem CMI will award to the first person(s) to solve it $1,000,000 (US). The proof or disproof of the Riemann Hypothesis is the second Millennium Prize Problem (as currently listed on the CMI web site).

We turn now to the

**Proof of Theorem 4.3.** We first establish a strengthened version of Theorem 4.3 in a special case, and then use it (and another lemma) to prove Theorem 4.3 in general.

**Lemma 4.9.** (Filaseta and Richman [16], Theorem 2) If \( \Pi \) is a nonempty set of primes and \( \varepsilon : \Pi \to \{-1, 1\} \) is a given function then the density of the set \( \{ p : \chi_p \equiv \varepsilon \text{ on } \Pi \} \) is \( 2^{-|\Pi|} \).

**Proof.** Let

\[
X = \{ p : \chi_p \equiv \varepsilon \text{ on } \Pi \},
\]

\[
K = \text{product of the elements of } \Pi.
\]

If \( n \in \mathbb{Z} \) then we let \([n]\) denote the ordinary residue class mod \( 4K \) which contains \( n \). The proof of Lemma 4.9 can now be outlined in a series of three steps.

**Step 1.** Use the LQR to show that

\[
X = \bigcup_{n \in U(4K): X \cap [n] \neq \emptyset} \{ p : p \in [n] \}.
\]

**Step 2.** Here we will make use of the Prime Number Theorem for primes in arithmetic progressions, to wit, if \( a \in \mathbb{Z} \), \( b \in [1, \infty) \), and \( \gcd(a, b) = 1 \) then as \( x \to +\infty \),

\[
|\{ p \in AP(a, b) : p \leq x \}| \sim \frac{1}{\varphi(b)} \frac{x}{\log x}.
\]

For a proof of this important theorem, see either LeVeque [28], section 7.4, or Montgomery and Vaughn, [29], section 11.3. In our situation it asserts that if \( n \in U(4K) \) then as \( x \to +\infty \),

\[
|\{ p \in [n] : p \leq x \}| \sim \frac{1}{\varphi(4K)} \frac{x}{\log x}.
\]
From this it follows that

\[ d_n = \frac{1}{\varphi(4K)}, \text{ for all } n \in U(4K). \]

Because the decomposition of \( X \) in Step 1 is pairwise disjoint, (7) \( \Rightarrow \)

\[ \text{density of } X = \sum_{n \in U(4K) : X \cap [n] \neq \emptyset} d_n = \frac{|\{n \in U(4K) : X \cap [n] \neq \emptyset\}|}{\varphi(4K)}. \]

**Step 3.** Use the group structure of \( U(4K) \) and the LQR to prove that

\[ |\{n \in U(4K) : X \cap [n] \neq \emptyset\}| = \frac{\varphi(4K)}{2^{|\Pi|}}. \]

From (8) and (9) it follows that the density of \( X \) is \( 2^{-|\Pi|} \), as desired, hence we need only implement Steps 1 and 3 in order to finish the proof.

**Implementation of Step 1.** We claim that

\[ \text{if } p, p' \text{ are odd primes and } p \equiv p' \mod 4K \text{ then } \chi_p \equiv \chi_{p'} \text{ on } \Pi. \]

Because \( X \) is disjoint from \( \{2\} \cup \Pi \) and

\[ P \setminus (\{2\} \cup \Pi) = \bigcup_{n \in U(4K)} \{p : p \in [n]\}, \]

the decomposition of \( X \) as asserted in Step 1 follows immediately from (10).

We verify (10) by using the LQR. Assume that \( p \equiv p' \mod 4K \) and let \( q \in \Pi \). Suppose first that \( p \) or \( q \) is \( \equiv 1 \mod 4 \). Then \( p' \) or \( q \) is \( \equiv 1 \mod 4 \), and so LQR \( \Rightarrow \)

\[ \chi_p(q) = \chi_q(p) = \chi_q(p' + 4kK) \text{ for some } k \in \mathbb{Z} = \chi_q(p'), \text{ since } q \text{ divides } 4kK = \chi_{p'}(q). \]

Suppose next that \( p \equiv 3 \equiv q \mod 4 \). Then \( p' \equiv 3 \mod 4 \) hence LQR \( \Rightarrow \)

\[ \chi_p(q) = -\chi_q(p) = -\chi_q(p') = -(-\chi_{p'}(q)) = \chi_{p'}(q). \]

**Implementation of Step 3.** Define the equivalence relation \( \sim \) on the set of residue classes \( \{[n] : n \in U(4K)\} \) like so:

\[ [n] \sim [n'] \text{ if for all odd primes } p \in [n], q \in [n'], \chi_p \equiv \chi_q \text{ on } \Pi. \]

We first count the number of equivalence classes of \( \sim \). (10) \( \Rightarrow \) the sets

\[ \{q \in \Pi : \chi_p(q) = 1\} \]
are the same for all \( p \in [n] \), and so we let \( I(n) \) denote this subset of \( \Pi \). Now if \( n \in U(4K) \) and \( p \in [n] \) then (11) \( \Rightarrow p \not\in \Pi \). Hence for all \( p \in [n] \), \( \chi_p \) takes only the values \( \pm 1 \) on \( \Pi \). It follows that 
\[
[n] \sim [n'] \text{ iff } I(n) = I(n').
\]
On the other hand, Lemma 4.4 \( \Rightarrow \) if \( S \subseteq \Pi \) then there exits infinitely many primes \( p \) such that 
\[
S = \{ q \in \Pi : \chi_p(q) = 1 \},
\]
and so we use (11) to find \( n_0 \in U(4K) \) such that \([n_0]\) contains at least one of these primes \( p \), hence 
\[
S = I(n_0).
\]

We conclude that
\[
(12) \quad \text{the number of equivalence classes of } \sim \text{ is } 2^{\Pi_\Pi}.
\]

Let \( E_n \) denote the equivalence class of \( \sim \) which contains \([n]\). We claim that 
\[
(13) \quad \text{multiplication by } n \text{ maps } E_1 \text{ bijectively onto } E_n.
\]
If this is true then \(|E_n|\) is constant as a function of \( n \in U(4K) \), hence (12) \( \Rightarrow \)
\[
(14) \quad \varphi(4K) = 2^{\Pi_\Pi}|E_n|, \text{ for all } n \in U(4K).
\]
If we now choose \( p \in X \) then there is \( n_0 \in U(4K) \) such that \( p \in [n_0] \), hence (10) \( \Rightarrow \)
\[
E_{n_0} = \{ [n] : X \cap [n] \neq \emptyset \},
\]
and so (14) \( \Rightarrow \)
\[
\varphi(4K) = 2^{\Pi_\Pi}|\{ n \in U(4K) : X \cap [n] \neq \emptyset \}|,
\]
which is (9).

It remains only to verify (13). Because \( U(4K) \) is a group under the multiplication induced by multiplication of ordinary residue classes mod \( 4K \), it is clear that multiplication by \( n \) on \( E_1 \) is injective, so we need only prove that \( nE_1 = E_n \).

\( nE_1 \subseteq E_n \).

Let \([n']\) \( \in E_1 \). We must prove: \([nn']\) \( \in E_n \), i.e., \([nn']\) \( \sim [n] \), i.e.,
\[
(15) \quad \text{if } p \in [nn'], q \in [n] \text{ are odd primes then } \chi_p \equiv \chi_q \text{ on } \Pi.
\]

In order to verify (15), let \( p \in [nn'], q \in [n], p' \in [n'], q' \in [1] \) be odd primes. Because \([n'] \sim [1] \),
\[
(16) \quad \chi_{p'} \equiv \chi_{q'} \text{ on } \Pi.
\]
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The choice of \( p, q, p', q' \) implies

\[ pq' \equiv p'q \mod 4K. \]

This congruence and the LQR when used in an argument similar to the one that was used to prove (10) \( \Rightarrow \)

(17)\[ \chi_p \chi_{q'} \equiv \chi_{p'} \chi_q \text{ on } \Pi. \]

Because \( \chi_{q'} \) and \( \chi_{p'} \) are both nonzero on \( \Pi \), we can use (16) to cancel \( \chi_{q'} \) and \( \chi_{p'} \) from each side of (17) to obtain

\[ \chi_p \equiv \chi_q \text{ on } \Pi. \]

\( E_n \subseteq nE_1. \)

Let \([n'] \in E_n\). The group structure of \( U(4K) \) \( \Rightarrow \) there exits \( n_0 \in U(4K) \) such that

(18)\[ [nn_0] = [n'], \]

so we need only show that \([n_0] \in E_1\), i.e.,

(19)\[ \chi_p \equiv \chi_q \text{ on } \Pi, \text{ for all odd primes } p \in [n_0], q \in [1]. \]

Toward that end, choose odd primes \( p' \in [n], q' \in [n'] \). Because \([n] \sim [n']\),

(20)\[ \chi_{p'} \equiv \chi_{q'} \text{ on } \Pi, \]

and (18) \( \Rightarrow \) for all \( p \in [n_0], q \in [1], \)

\[ pp' \equiv qq' \mod 4K. \]

(19) is now a consequence of this congruence, (20), and our previous reasoning. \( \text{ QED} \)

We will prove Theorem 4.3 by combining Lemma 4.9 with the next lemma, a simple result in enumerative combinatorics.

**Lemma 4.10.** If \( A \) is a nonempty finite subset of \([1, \infty), n = |A|, S \subseteq 2^A, F = \text{the Galois field of order } 2, \)

\[ v : 2^A \to F^n \text{ is the map defined on p. } 44, \text{ and} \]

\[ d = \text{the dimension of the linear span of } v(S) \text{ in } F^n, \]

then the cardinality of the set

\[ \mathcal{N} = \{ N \subseteq A : |N \cap S| \text{ is even, for all } S \in \mathcal{S} \} \]

is \( 2^{n-d} \).
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Proof. Without loss of generality take $A = [1, n]$. Observe first that if $N, T \subseteq A$, then 

$$|N \cap T| \text{ is even iff } \sum_{i=1}^{n} v(N)(i)v(T)(i) = 0 \text{ in } F.$$ 

Hence there is a bijection of the set of all solutions in $F^n$ of the system of linear equations

(*) $$\sum_{i=1}^{n} v(S)(i)x_i = 0, S \in \mathcal{S},$$

onto $\mathcal{N}$ given by

$$(x_1, \ldots, x_n) \rightarrow \{i : x_i = 1\}.$$ 

If $m = |\mathcal{S}|$ and $\sigma : F^n \rightarrow F^m$ is the linear transformation whose representing matrix is the coefficient matrix of the system (*) then

the set of all solutions of (*) in $F^n = \text{the kernel of } \sigma.$

But $d$ is the rank of $\sigma$ and so the kernel of $\sigma$ has dimension $n - d$. Hence

$$|\mathcal{N}| = |\text{the set of all solutions of (*) in } F^n| = |\text{kernel of } \sigma| = 2^{n-d}.$$ 

QED

We proceed to prove Theorem 4.3. Let $S, \mathcal{S}, A, n,$ and $d$ be as in the hypothesis of that theorem, let

$$X = \{p : \chi_p \equiv 1 \text{ on } S\},$$

$$\mathcal{N} = \{N \subseteq A : |N \cap S| \text{ is even, for all } S \in \mathcal{S}\},$$

and for each prime $p$, let

$$N(p) = \{q \in A : \chi_p(q) = -1\}.$$ 

Then since $X$ is disjoint from $A$,

$$p \in X \text{ iff } 1 = \chi_p(z) = \prod_{q \in \pi_{\text{odd}}(z)} \chi_p(q), \text{ for all } z \in S,$$

$$\text{iff } |N(p) \cap \pi_{\text{odd}}(z)| \text{ is even, for all } z \in S,$$

$$\text{iff } N(p) \in \mathcal{N}.$$ 

Hence

$$X = \bigcup_{N \in \mathcal{N}} \{p : N(p) = N\}$$

and this union is pairwise disjoint. Hence

$$\text{density of } X = \sum_{N \in \mathcal{N}} \text{(density of } \{p : N(p) = N\}).$$
Lemma 4.9 ⇒

density of \( \{ p : N(p) = N \} = 2^{-n} \) for all \( N \in \mathcal{N} \),

and so

\[
\text{density of } X = 2^{-n} |\mathcal{N}|
\]
\[
= 2^{-n} (2^n - d), \text{ by Lemma 4.10}
\]
\[
= 2^{-d}.
\]

QED

The next question which naturally arises asks: what about a version of Theorem 4.2 for quadratic non-residues, i.e., for what finite, nonempty subsets \( S \) of \([1, \infty)\) is it true that \( S \) is a set of non-residues of infinitely many primes? In contrast to what occurs for residues, this can fail to be true for certain subsets \( S \) of \([1, \infty)\), and there is a simple obstruction that prevents it from being true. Suppose that there is a subset \( T \) of \( S \) such that \( |T| \) is odd and \( \prod_{i \in T} i \) is a square, and suppose that \( S \) is a set of non-residues of infinitely many primes. We can then choose \( p > \) all the prime factors of the elements of \( T \) such that \( \chi_p(z) = -1 \), for all \( z \in T \). Hence

\[
-1 = (-1)^{|T|} = \prod_{i \in T} \chi_p(i) = \chi_p \left( \prod_{i \in T} i \right) = 1,
\]
a clear contradiction! It follows that the presence of such subsets \( T \) of \( S \) prevents \( S \) from being a set of non-residues of infinitely many primes. The next theorem asserts that those subsets are the only obstructions to \( S \) having this property.

**Theorem 4.11.** If \( S \) is a finite, nonempty subset of \([1, \infty)\) then \( S \) is a set of non-residues of infinitely many primes iff for all subsets \( T \) of \( S \) of odd cardinality, \( \prod_{i \in T} i \) is not a square.

This theorem lies considerably deeper than Theorem 4.2; in order to prove it, we will once again delve into the theory of algebraic numbers.
CHAPTER 5

The Zeta Function of an Algebraic Number Field and Some Applications

The proof of Theorem 4.11 that we will discuss in this chapter uses ideas that are closely related to the ones that Dirichlet used in his proof of Theorem 4.5, together with some technical improvements due to Hilbert [23], section 80. The key tool that we need is an analytic function attached to certain complex number fields, called the zeta function of the field. The definition of this function requires a significant amount of mathematical technology from the theory of algebraic numbers, and so we begin with a discussion of that technology.

Let $F$ be a complex number field. With respect to its addition and multiplication, $F$ is a vector space over $Q$, and we say that $F$ has degree $n$ (over $Q$) if $n$ is the dimension of $F$ over $Q$.

Definition. $F$ is an algebraic number field if the degree of $F$ is finite.

We let $F$ denote an algebraic number field of degree $n$ that will remain fixed in the discussion until indicated otherwise. Because the non-negative integral powers of a nonzero element of $F$ cannot form a set that is linearly independent over $Q$, every element of $F$ is algebraic over $Q$. The zeta function of $F$ is defined by using the ideal structure in the ring $R = \mathcal{R} \cap F$ of all algebraic integers contained in $F$, hence we need to discuss that first.

Recall that if $A$ is a commutative ring with identity then an ideal of $A$ is a subring $I$ of $A$ such that $ab \in I$ whenever $a \in A$ and $b \in I$. An ideal $I$ of $A$ is prime if $\{0\} \neq I \neq A$ and if $a, b$ are elements of $A$ such that $ab \in I$ then $a \in I$ or $b \in I$. An ideal $M$ of $A$ is maximal if $\{0\} \neq M \neq A$ and whenever $I$ is an ideal of $A$ such that $M \subseteq I$ then $M = I$ or $I = A$. A basic fact in the theory of commutative rings with identity asserts that all maximal ideals in such rings are prime ideals, with the converse false in general. However, in the ring $R$ of algebraic integers in $F$ this converse is true:

**Proposition 5.1.** An ideal of $R = \mathcal{R} \cap F$ is prime iff it is maximal.

Another remarkable fact about the ideals of $R$ is recorded in

**Proposition 5.2.** If $I$ is a non-zero ideal of $R = \mathcal{R} \cap F$ then the cardinality of the quotient ring $R/I$ is finite.
Propositions 5.1 and 5.2 indicate that the ideals of $R$ are exceptionally “large” subsets of $R$.

Proof of Propositions 5.1 and 5.2. These arguments depend on the existence of an integral basis of $R$. A subset $\{\alpha_1, \ldots, \alpha_k\}$ of $R$ is an integral basis of $R$ if for each $\alpha \in R$, there exists a $k$-tuple $(z_1, \ldots, z_k)$ of integers, uniquely determined by $\alpha$, such that

$$\alpha = \sum_{i=1}^{k} z_i \alpha_i.$$ 

It is an immediate consequence of the definition that an integral basis $\{\alpha_1, \ldots, \alpha_k\}$ is linearly independent over $Z$, i.e., if $(z_1, \ldots, z_k)$ is a $k$-tuple of integers such that $\sum_{i=1}^{k} z_i \alpha_i = 0$ then $z_i = 0$ for $i = 1, \ldots, k$. $R$ always has an integral basis (the interested reader may consult Hecke [22], section 22, Theorem 64, for a proof of this), and it is not difficult to prove that every integral basis of $R$ is a basis of $F$ as a vector space over $Q$; consequently, all integral bases of $R$ contain exactly $n$ elements.

Now for the proof of Proposition 5.1. Let $I$ be a prime ideal of $R$: we need to prove that $I$ is a maximal ideal, i.e., we take an ideal $J$ of $R$ which properly contains $I$ and show that $J = R$.

Toward that end, let $\{\alpha_1, \ldots, \alpha_n\}$ be an integral basis of $R$, and let $0 \neq \beta \in I$. If

$$x^m + \sum_{i=0}^{m-1} z_i x^i$$

is the minimal polynomial of $\beta$ over $Q$ then $z_0 \neq 0$ (otherwise, $\beta$ is the root of a nonzero polynomial over $Q$ of degree less than $m$) and

$$z_0 = -\beta^m - \sum_{i=1}^{m-1} z_i \beta^i \in I,$$

hence $\pm z_0 \in I$, and so $I$ contains a positive integer $a$. We claim that each element of $R$ can be expressed in the form

$$a \gamma + \sum_{i=1}^{n} r_i \alpha_i,$$

where $\gamma \in R$, $r_i \in [0, a - 1]$, $i = 1, \ldots, n$.

Assume this for now, and let $\alpha \in J \setminus I$. Then for each $k \in [1, \infty)$,

$$\alpha^k = a \gamma_k + \sum_{i=1}^{n} r_i \alpha_i, \quad \gamma_k \in R, \ r_i \in [0, a - 1], \ i = 1, \ldots, n,$$
hence the sequence \((\alpha^k - a\gamma_k : k \in [1, \infty))\) has only finitely many values; consequently there exist positive integers \(l < k\) such that
\[ \alpha^l - a\gamma_l = \alpha^k - a\gamma_k. \]
Hence
\[ \alpha^l(\alpha^{k-l} - 1) = \alpha^k - \alpha^l = a(\gamma_k - \gamma_l) \in I \ (a \in I). \]
Because \(I\) is prime, either \(\alpha^l \in I\) or \(\alpha^{k-l} - 1 \in I\). However, \(\alpha^l \not\in I\) because \(\alpha \not\in I\) and \(I\) is prime. Hence
\[ \alpha^{k-l} - 1 \in I \subseteq J. \]
But \(k - l > 0\) and \(\alpha \in J\) (by the choice of \(\alpha\)), and so \(-1 \in J\). As \(J\) is an ideal, this implies that \(J = R\).

Our claim must now be verified. Let \(\alpha \in R\), and find \(z_i \in Z\) such that
\[ \alpha = \sum_{i=1}^{n} z_i \alpha_i. \]
The division algorithm in \(Z \Rightarrow\) there exist \(m_i \in Z, r_i \in [1, a - 1], i = 1, \ldots, n\), such that
\[ z_i = m_i a + r_i, \ i = 1, \ldots, n. \]
Thus
\[ \alpha = a \sum_i m_i \alpha_i + \sum_i r_i \alpha_i = a\gamma + \sum_i r_i \alpha_i, \]
with \(\gamma \in R\). This completes our proof of Proposition 5.1.

We verify Proposition 5.2 next. Let \(L \neq \{0\}\) be an ideal of \(R\). We wish to show that \(|R/L|\) is finite. A propos of that, choose \(a \in L \cap Z\) with \(a > 0\) (that such an \(a\) exists follows from the previous proof of Proposition 5.1). Then \(aR \subseteq L\), hence there is a surjection of \(R/aR\) onto \(R/L\), whence it suffices to show that \(|R/aR|\) is finite.

We will in fact prove that \(|R/aR| = a^n\). Consider for this the set
\[ S = \left\{ \sum_i z_i \alpha_i : z_i \in [0, a - 1] \right\}. \]
We show that \(S\) is a set of coset representatives of \(R/aR\); if this is true then clearly \(|R/aR| = |S| = a^n\). Thus, let \(\alpha = \sum_i z_i \alpha_i \in R\). Then there exist \(m_i \in Z, r_i \in [1, a - 1], i = 1, \ldots, n\), such that \(z_i = m_i a + r_i, \ i = 1, \ldots, n\). Hence
\[ \alpha - \sum_i r_i \alpha_i = \left( \sum_i m_i \right) a \in aR \text{ and } \sum_i r_i \alpha_i \in S, \]
and so each coset of \(R/aR\) contains an element of \(S\).

Let \(\sum_i a_i \alpha_i, \sum_i a'_i \alpha_i\) be elements of \(S\) is the same coset. Then
\[ \sum_i (a_i - a'_i) \alpha_i = a\alpha, \text{ for some } \alpha \in R. \]
Hence there exists \( m_i \in \mathbb{Z} \) such that

\[
\sum_i (a_i - a'_i)\alpha_i = \sum_i m_i a\alpha_i,
\]

and so the linear independence (over \( \mathbb{Z} \)) of \( \{\alpha_1, \ldots, \alpha_n\} \) implies

\[
a_i - a'_i = m_i a, \quad i = 1, \ldots, n
\]
i.e., \( a \) divides \( a_i - a'_i \) in \( \mathbb{Z} \). Because \( |a_i - a'_i| < a \) for all \( i \), it follows that \( a_i - a'_i = 0 \) for all \( i \).

Hence each coset of \( R/aR \) contains exactly one element of \( S \). QED

By far the most important feature of the structure of proper, nonzero ideals of \( R \) is the fact that they can be factored in a unique way as the product of prime ideals. We now explain precisely what this means.

**Definition.** Let \( A \) be a commutative ring with identity, \( I, J \) (not necessarily distinct) ideals of \( A \). The (ideal) product \( IJ \) of \( I \) and \( J \) is the ideal of \( A \) generated by the set of products

\[
\{xy : (x, y) \in I \times J\},
\]
i.e., \( IJ \) is the smallest ideal of \( A \), relative to subset inclusion, which contains this set of products.

One can easily show that \( IJ \) consists precisely of all sums of the form \( \sum_i x_i y_i \), where \( x_i \in I \) and \( y_i \in J \), for all \( i \). It is also easy to show that the ideal product is commutative and associative. We then have

**Theorem 5.3.** *(Fundamental Theorem of Ideal Theory)* Every nonzero, proper ideal \( I \) of \( R \) is a product of prime ideals and this factorization is unique up to the order of the factors. Moreover, the set of prime ideal factors of \( I \) is precisely the set of prime ideals of \( R \) which contain \( I \), i.e., the set of prime ideals of \( R \) containing \( I \) is nonempty and finite, and if \( \{P_1, \ldots, P_k\} \) is this set then there exist a \( k \)-tuple \( (m_1, \ldots, m_k) \) of positive integers, uniquely determined by \( I \), such that \( I = P_1^{m_1} \cdots P_k^{m_k} \).

Theorem 5.3, one of the most important theorems in algebraic number theory, was proved by R. Dedekind in 1871, and appeared as Supplement X in his famous series of addenda to Dirichlet’s landmark text *Vorlesungen über Zahlentheorie* [11]. For its proof we refer to Dedekind [7], section 25, Theorem 4 and Hecke [22], section 25, Theorem 72.

Proposition 5.2 \( \Rightarrow \) if \( I \neq \{0\} \) is an ideal of \( R \) then \( |R/I| \) is finite. We set

\[
N(I) = |R/I|,
\]
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and call this the *norm of I*. The norm function $N$ on nonzero ideals is multiplicative with respect to the ideal product, i.e., we have

**Proposition 5.4.** If $I, J$ are (not necessarily distinct) nonzero ideals of $R$ then

$$N(IJ) = N(I)N(J).$$

**Proof.** Hecke [22], section 27, Theorem 79. QED

Now, let

$I = \text{the set of all nonzero ideals of } R.$

If $n \in [1, \infty)$, let

$$Z(n) = |\{I \in I : N(I) \leq n\}|.$$

**Proposition 5.5.** $Z(n) < +\infty$, for all $n \in [1, \infty)$.

Perhaps the most elegant way to verify Proposition 5.5 is to make use of the ideal class group of $R$. In order to define this group, we first declare that the ideals $I$ and $J$ of $R$ are *equivalent* if there exist nonzero elements $\alpha$ and $\beta$ of $R$ such that $\alpha I = \beta J$. This defines an equivalence relation on the set of all ideals of $R$, and we refer to the corresponding equivalence classes as the *ideal classes* of $R$. If we let $[I]$ denote the ideal class which contains the ideal $I$ then we can define a multiplication on the set of ideal classes by declaring that the product of $[I]$ and $[J]$ is $IJ$. It can be shown that when endowed with this product (which is well-defined), the ideal classes of $R$ form an Abelian group, called the *ideal-class group of $R$* (Hecke [22], section 33). It is easy to see that the set of all principal ideals of $R$, i.e., the set of all ideals of the form $\alpha R$, $\alpha \in R$, is an ideal class of $R$, called the *principal class*, and one can prove that the principal class is the identity element of the ideal-class group. It is one of the fundamental theorems of algebraic number theory that the ideal-class group is always *finite* (see Hecke [22], section 33, Theorem 96), and the order of the ideal-class group of $R$ is called the *class number of $R$*. We can now turn to the

**Proof of Proposition 5.5.** Let $C$ be an ideal class of $R$ and for each $n \in [1, \infty)$, let $Z_C(n)$ denote the set

$$\{I \in C \cap I : N(I) \leq n\}.$$

We claim that $|Z_C(n)|$ is finite. In order to verify this, let $J$ be a fixed nonzero ideal in $C^{-1}$ (the inverse of $C$ in the ideal-class group), and let $0 \neq \alpha \in J$. Then there is a unique ideal $I$ such that $\alpha R = IJ$, and since $[I] = C[IJ] = C[\alpha R] = C$, it follows that $I \in C \cap I$. Moreover, the map $\alpha R \to I$ is a bijection of the set of all nonzero principal ideals contained in $J$ onto $C \cap I$. Proposition 5.4 $\Rightarrow$

$$N(\alpha R) = N(I)N(J),$$
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\[ N(I) \leq n \text{ iff } N(\alpha R) \leq nN(J). \]

Hence there is a bijection of \( Z_C(n) \) onto the set
\[ \mathcal{J} = \{ \{0\} \neq \alpha R \subseteq J : N(\alpha R) \leq nN(J) \}, \]
and so it suffices to show that \( \mathcal{J} \) is a finite set.

That \( |\mathcal{J}| \) is finite will follow if we prove that there is only a finite number of principal ideals of \( R \) whose norms do not exceed a fixed constant. Suppose that this latter statement is false, i.e., there are infinitely many elements \( \alpha_1, \alpha_2, \ldots \) of \( R \) such that the principal ideals \( \alpha_i R, i = 1, 2, \ldots \) are distinct and \( (N(\alpha_1 R), N(\alpha_2 R), \ldots) \) is a bounded sequence. As all of the numbers \( N(\alpha_i R) \) are positive integers, we may suppose with no loss of generality that \( N(\alpha_i R) \) all have the same value \( z \).

We now wish to locate \( z \) in each ideal \( \alpha_i R \). Toward that end, use the primitive element theorem (Hecke [22], section 19, Theorem 52) to find \( \theta \in F \), of degree \( n \) over \( Q \), such that for each element \( \nu \) of \( F \), there is a unique polynomial \( f \in \mathbb{Q}[x] \) such that \( \nu = f(\theta) \) and the degree of \( f \) does not exceed \( n-1 \). For each \( i \), we hence find \( f_i \in \mathbb{Q}[x] \) of degree no larger than \( n-1 \) and for which \( \alpha_i = f_i(\theta) \). If \( \theta_1, \ldots, \theta_n \), with \( \theta_1 = \theta \), are the roots of the minimal polynomial of \( \theta \) over \( Q \), then one can show that
\[
N(\alpha_i R) = \left| \prod_{k=1}^{n} f_i(\theta_k) \right|
\]
(Hecke [22], section 27, Theorem 76). Moreover, the degree \( d_i \) of \( \alpha_i \) over \( Q \) divides \( n \) in \( \mathbb{Z} \), and if \( \alpha_i^{(1)}, \ldots, \alpha_i^{(d_i)} \), with \( \alpha_i^{(1)} = \alpha_i \), denote the roots of the minimal polynomial of \( \alpha_i \) over \( Q \), then the numbers on the list \( f_i(\theta_k), k = 1, \ldots, n \), are obtained by repeating each \( \alpha_i^{(j)} n/d_i \) times (Hecke [22], section 19, Theorem 54). If \( c_0 \) denotes the constant term of the minimal polynomial of \( \alpha_i \) over \( Q \), it follows that
\[
\prod_{k=1}^{n} f_i(\theta_k) = \left( \prod_{k=1}^{d_i} \alpha_i^{(k)} \right)^{n/d_i} = ((-1)^{d_i} c_0)^{n/d_i} \in \mathbb{Z}.
\]
Because \( f_i(\theta_k) \) is an algebraic integer for all \( i \) and \( k \), it hence follows that
\[
\frac{z}{\alpha_i} = \pm \prod_{k=2}^{n} f_i(\theta_k) \in \mathcal{R} \cap F = R,
\]
whence \( z \in \alpha_i R \), for all \( i \).

If we now let \( \{\beta_1, \ldots, \beta_n\} \) be an integral basis of \( R \) then the claim in the proof of Proposition 5.1 shows that for each \( i \) there exists \( \gamma_i \in R \) and \( z_{ij} \in [0, z-1], j = 1, \ldots, n, \)
such that
\[ \alpha_i = z\gamma_i + \sum_{j=1}^{n} z_{ij} \beta_j. \]
Because \( z \in \alpha_i R \), it follows that
\[ \alpha_i R = zR + \left( \sum_{j=1}^{n} z_{ij} \beta_j \right) R, \text{ for all } i. \]
However, the sum \( \sum_{j=1}^{n} z_{ij} \beta_j \) can have only finitely many values; we conclude that the ideals \( \alpha_i R, i = 1, 2, \ldots \) cannot all be distinct, contrary to their choice.

We now have what we need to easily prove that \( Z(n) \) is finite. Let \( C_1, \ldots, C_h \) denote the distinct ideal classes of \( R \). The set of all the ideals of \( R \) is the (pairwise disjoint) union of the \( C_i \)'s hence \( \{ I \in \mathcal{I} : N(I) \leq n \} \) is the union of \( \mathcal{J}_{C_1}(n), \ldots, \mathcal{J}_{C_h}(n) \). Because each set \( \mathcal{J}_{C_i}(n) \) is finite, so therefore is \( \{|I \in \mathcal{I} : N(I) \leq n|\} = Z(n) \). QED

In particular, Proposition 5.5 \( \Rightarrow \mathcal{I} \) is countable, and so if \( s \in \mathbb{C} \) then the formal series
\[ (*) \sum_{I \in \mathcal{I}} \frac{1}{N(I)^s} \]
is defined, relative to some fixed enumeration of \( \mathcal{I} \). As we will see, the zeta function of \( F \) will be defined by this series. However, in order to do that precisely and rigorously, a careful examination of the convergence of this series must be done first. That is what we will do next.

If we let
\[ L(n) = |\{ I \in \mathcal{I} : N(I) = n \}|, \quad n \in [1, \infty), \]
then by formal rearrangement of its terms, we can write the series (*) as
\[ (**) \sum_{n=1}^{\infty} \frac{L(n)}{n^s}. \]
The series (**) is a Dirichlet series, i.e., a series of the form
\[ \sum_{n=1}^{\infty} \frac{a_n}{n^s}, \]
where \( (a_n) \) is a given sequence of complex numbers. The \( L \)-function of a Dirichlet character is another important example of a Dirichlet series.

We will determine the convergence of the series (*) by studying the convergence of the Dirichlet series (**)'. This will be done by way of the following proposition, which describes how a Dirichlet series converges.
Proposition 5.6. Let \((a_n)\) be sequence of complex numbers, let

\[ S(n) = \sum_{k=1}^{n} a_k, \]

and suppose that there exists \(\sigma \geq 0, C > 0\) such that

\[ \left| \frac{S(n)}{n^\sigma} \right| \leq C, \text{ for all } n \text{ sufficiently large}. \]

Then the Dirichlet series

\[ \sum_{n=1}^{\infty} \frac{a_n}{n^s} \]

converges in the half-plane \(\Re s > \sigma\) and uniformly in each closed and bounded subset of this half-plane. Moreover, if

\[ \lim_{n \to \infty} \frac{S(n)}{n} = d \]

then

\[ \lim_{s \to 1^+} (s-1) \sum_{n=1}^{\infty} \frac{a_n}{n^s} = d. \]

Proof (according to Hecke [22], section 42, Lemmas (a), (b), (c)). Let \(m\) and \(h\) be integers, with \(m > 0\) and \(h \geq 0\), and let \(K \subseteq \{ s : \Re s > \sigma \}\) be a compact (closed and bounded) set. Then

\[
\sum_{n=m}^{m+h} \frac{a_n}{n^s} = \sum_{n=m}^{m+h} \frac{S(n) - S(n-1)}{n^s} = \frac{S(m+h)}{(m+h)^s} - \frac{S(m-1)}{m^s} + \sum_{n=m}^{m+h-1} S(n) \left( \frac{1}{n^s} - \frac{1}{(n+1)^s} \right)
\]

\[
= \frac{S(m+h)}{(m+h)^s} - \frac{S(m-1)}{m^s} + s \sum_{n=m}^{m+h-1} S(n) \int_{n}^{n+1} \frac{dx}{x^{s+1}}.
\]

If we now use the stipulated bound on the quotients \(S(n)/n^\sigma\), it follows that

\[
\left| \sum_{n=m}^{m+h} \frac{a_n}{n^s} \right| \leq \frac{2C}{m \Re s - \sigma} + C|s| \int_{m}^{\infty} \frac{dx}{x^{\Re s - \sigma+1}}
\]

\[
= \frac{2C}{m \Re s - \sigma} + C|s| \frac{1}{\Re s - \sigma} \frac{1}{m \Re s - \sigma}.
\]

Because \(K\) is a compact subset of \(\Re s > \sigma\), it is bounded and lies at a positive distance \(\delta\) from \(\Re s = \sigma\), i.e., there is a positive constant \(C'\) such that

\[ \Re s - \sigma \geq \delta \text{ and } |s| \leq C', \text{ for all } s \in K. \]
Hence there is a positive constant $C''$, independent of $m$ and $h$, such that

$$\left| \sum_{n=m}^{m+h} \frac{a_n}{n^s} \right| \leq C'' \left( 1 + \frac{1}{\delta} \right) \frac{1}{m^\delta}, \text{ for all } s \in K.$$ 

As $m$ and $h$ are chosen arbitrarily and $\delta$ depends on neither $m$ nor $h$, this estimate implies that the Dirichlet series converges uniformly on $K$, and as $K$ is also chosen arbitrarily, it follows that the series converges to a function continuous in $\text{Re } s > \sigma$.

We now assume that

$$\lim_{n \to \infty} \frac{S(n)}{n} = d;$$

we wish to verify that

$$\lim_{s \to 1^+} (s-1) \sum_{n=1}^{\infty} \frac{a_n}{n^s} = d.$$ 

From what we have just shown, it follows that the Dirichlet series now converges for $s > 1$. Let

$$S(n) = dn + \varepsilon_n n, \text{ where } \lim_{n \to \infty} \varepsilon_n = 0,$$

$$\varphi(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}, s > 1.$$ 

Then for $s > 1$, we have that

$$|\varphi(s) - d\zeta(s)| = s \sum_{n=1}^{\infty} n \varepsilon_n \int_{n}^{n+1} \frac{dx}{x^{s+1}}$$

$$< s \sum_{n=1}^{\infty} |\varepsilon_n| \int_{n}^{n+1} \frac{dx}{x^s}.$$ 

Let $\varepsilon > 0$, and choose an integer $N$ and a positive constant $A$ such that $|\varepsilon_n| < \varepsilon$, for all $n \geq N$, and $|\varepsilon_n| \leq A$, for all $n$. Then

$$|(s-1)\varphi(s) - d(s-1)\zeta(s)| < As(s-1) \sum_{n=1}^{N-1} \int_{n}^{n+1} \frac{dx}{x} + \varepsilon s(s-1) + \sum_{n=N}^{\infty} \int_{n}^{n+1} \frac{dx}{x^s}$$

$$= As(s-1) \log N + \varepsilon s(s-1) \int_{N}^{\infty} \frac{dx}{x^s}.$$ 

Because the last expression has limit $\varepsilon$ as $s \to 1$, it follows that

$$\lim_{s \to 1^+} ((s-1)\varphi(s) - d(s-1)\zeta(s)) = 0.$$ 

We now claim that

$$\lim_{s \to 1^+} (s-1)\zeta(s) = 1;$$
if this is so, then

$$\lim_{s \to 1^+} (s - 1)\varphi(s) = d,$$

as desired. This claim can be verified upon noting that

$$\int_n^{n+1} \frac{dx}{x^s} < \frac{1}{n^s} < \int_{n-1}^{n} \frac{dx}{x^s},$$

for all \( n \in [2, \infty) \) and for all \( s > 1 \).

Hence

$$\frac{1}{s - 1} = \int_1^{\infty} \frac{dx}{x^s} < \sum_1^{\infty} \frac{1}{n^s} = \zeta(s) < 1 + \int_1^{\infty} \frac{dx}{x^s} = \frac{s}{s - 1},$$

and so

$$1 < (s - 1)\zeta(s) < s, \text{ for all } s > 1,$$

from which the claim follows immediately. QED

Because each function \( a_n/n^s \) is an entire function of \( s \), a Dirichlet series which satisfies the hypotheses of Proposition 5.6 is a series of functions each term of which is analytic in \( \text{Re } s > \sigma \) and which also converges uniformly on every compact subset of \( \text{Re } s > \sigma \). Hence the sum of the series is analytic in \( \text{Re } s > \sigma \).

We wish to apply Proposition 5.6 to the series (**), and so we must study the behavior of the sequence

$$Z(n) = \sum_{k=1}^{n} L(k).$$

The required behavior of this sequence is given by the following theorem, another very important result of Dedekind: for a proof, consult Hecke [22], section 42, Theorem 122.

**Theorem 5.7.** *(Dedekind’s Ideal Distribution Theorem).* The limit

$$\lim_{n \to \infty} \frac{Z(n)}{n} = \lambda$$

exists, is positive, and its value is given by the formula

$$\lambda = \frac{2^{r+1} \pi^{e} \rho}{w \sqrt{|d|} h},$$
where
\[d = \text{discriminant of } F,\]
\[e = \frac{1}{2} \text{(number of complex embeddings of } F \text{ over } Q),\]
\[h = \text{class number of } R,\]
\[r = \text{unital rank of } R,\]
\[\rho = \text{regulator of } F,\]
\[w = \text{order of the group of roots of unity in } R.\]

Thus the number of nonzero ideals of \(R\) whose norms do not exceed \(n\) is asymptotic to \(\lambda n\) as \(n \to +\infty\).

Although we will make no further use of them, readers who are interested in the definition of the discriminant of \(F\) and the regulator of \(F\), should see, respectively, the definition on p. 73 and the definition on p. 116 of Hecke [22]. The parameter \(e\) in the statement of Theorem 5.7 is equal to the parameter \(r_2\) defined on p. 109 of [22] and the unital rank of \(R\) is the parameter \(r_1 + r_2 - 1\) defined on p. 109 of [22]. The integers \(d, e, h, r, w,\) and the real number \(\rho\) are fundamental parameters associated with \(F\) which govern many aspects of the arithmetic and algebraic structure of \(F\) and \(R\); Theorem 5.7 is a remarkable example of how these parameters work in concert to do that.

Theorem 5.7 \Rightarrow the hypotheses of Proposition 5.6 are satisfied for \(a_n = L(n)\) with \(\sigma = 1,\) hence the series (**) converges to a function analytic in \(\Re s > 1.\)

We now let \(s > 1.\) Because \(L(n) \geq 0\) for all \(n,\) the convergence of (**) is absolute for \(s > 1,\) hence we can rearrange the terms of (**) in any order without changing its value. It follows that the value of the series

\[\sum_{I \in \mathcal{I}} \frac{1}{N(I)^s}\]

for \(s > 1\) is finite, is independent of the enumeration of \(\mathcal{I}\) used to define the series, and is given by the value of the Dirichlet series (**).

**Definition.** The (Dedekind-Dirichlet) zeta function of \(F\) is the function \(\zeta_F(s)\) defined for \(s > 1\) by

\[\zeta_F(s) = \sum_{I \in \mathcal{I}} \frac{1}{N(I)^s}.\]

**Remark.** One can show without difficulty that if \(\sum_n a_n/n^s\) is a Dirichlet series which satisfies the hypotheses of Proposition 5.6 then \(\sum_n a_n/n^s\) converges absolutely in \(\Re s > 1 + \sigma.\) If we apply this fact to the series (**), it follows that (**) converges absolutely in \(\Re s > 1 + \sigma.\)
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$s > 2$. Hence the value of the series

$$\sum_{I \in \mathcal{I}} \frac{1}{N(I)^s}$$

for $\text{Re } s > 2$ is finite, is independent of the enumeration of $\mathcal{I}$ used to define the series, and is given by the value of the series (**). Although we will make no use of this fact, it follows that the zeta function of $F$ can be defined by the series (**) not only for $s > 1$, but also for $\text{Re } s > 1$, and when so defined, is analytic in that half-plane.

For future reference, we observe that Proposition 5.6 and Theorem 5.7 $\Rightarrow$

**Lemma 5.8.** If $\zeta_F(s)$ is the zeta function of $F$ then

$$\lim_{s \to 1^+} (s - 1)\zeta_F(s) = \lambda > 0.$$  

If $F = Q$ then $R = \mathcal{R} \cap Q = \mathbb{Z}$, hence the nonzero ideals of $R$ in this case are the principal ideals $n\mathbb{Z}, n \in [1, \infty)$. Then

$$N(n\mathbb{Z}) = |\mathbb{Z}/n\mathbb{Z}| = n,$$

and so

$$\{I \in \mathcal{I} : |N(I)| = n\} = \{n\mathbb{Z}\}.$$

Hence

$$\zeta_Q(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},$$

the Riemann zeta function.

The next theorem gives a product formula for $\zeta_F(s)$ that is reminiscent of the product formula for the Dirichlet $L$-function of a Dirichlet character that we pointed out in Chapter 4. It is a very useful tool for analyzing certain features of the behavior of $\zeta_F(s)$ and will play a key role in our proof of Theorem 4.11.

**Theorem 5.9. (Euler-Dedekind product formula for $\zeta_F$)** Let $\mathcal{Q}$ denote the set of all prime ideals of $R$. Then

$$\zeta_F(s) = \prod_{I \in \mathcal{Q}} \frac{1}{1 - N(I)^{-s}}, \ s > 1.$$  

**Proof.** Note that because a prime ideal $I$ of $R$ is proper, $N(I) > 1$, and so each term of this product is defined for $s > 1$. In order to prove the theorem we will need some standard facts about the convergence of infinite products.
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Definitions. Let \((a_n)\) be a sequence of complex numbers such that \(a_n \neq -1\), for all \(n\). The infinite product

\[
\prod_{1}^{\infty} (1 + a_n)
\]

converges if

\[
\lim_{n \to \infty} \prod_{1}^{n} (1 + a_k)
\]

exists and is finite, and it converges absolutely if

\[
\prod_{1}^{\infty} (1 + |a_n|)
\]

converges.

**Proposition 5.10.** (i) \(\prod_{n}(1 + a_n)\) converges absolutely iff the series \(\sum_{n} |a_n|\) converges.

(ii) The limit of an absolutely convergent infinite product is not changed by any rearrangement of the factors.

**Proof.** See Nevanlinna and Paatero [30], Sections 13.1, 13.2. QED

Returning to the proof of Theorem 5.9, we next consider the product on the right-hand side of (1). Because \(N(I) \geq 2\) for all \(I \in \mathbb{Q}\) it follows that for \(s > 1\),

\[
0 < \frac{1}{1 - N(I)^{-s}} - 1 = \frac{N(I)^{-s}}{1 - N(I)^{-s}} \leq 2N(I)^{-s},
\]

hence

\[
\sum_{I \in \mathbb{Q}} \left( \frac{1}{1 - N(I)^{-s}} - 1 \right) \leq 2 \sum_{I \in \mathbb{Q}} N(I)^{-s} < +\infty
\]

and so by Proposition 5.10, the product on the right-hand side of (1) converges absolutely for \(s > 1\) and its value is independent of the order of the factors.

The next step is to prove that this product converges to \(\zeta_F(s)\) for \(s > 1\). Let

\[
\Pi(x) = \prod_{I \in \mathbb{Q}; N(I) \leq x} \frac{1}{1 - N(I)^{-s}};
\]

this product is finite by Proposition 5.5 and

\[
\lim_{x \to +\infty} \Pi(x) = \prod_{I \in \mathbb{Q}} \frac{1}{1 - N(I)^{-s}}.
\]

We have that

\[
\frac{1}{1 - N(I)^{-s}} = \sum_{n=0}^{\infty} \frac{1}{N(I)^{ns}},
\]
hence \( \Pi(x) \) is a finite product of absolutely convergent series, which we can hence multiply together and rearrange terms in any order without altering the sum. Proposition 5.4 \( \Rightarrow \) each term of this sum is either 1 or of the form

\[ N(I_1^{\alpha_1} \cdots I_r^{\alpha_r})^{-s}, \]

where \((\alpha_1, \ldots, \alpha_r)\) is an \( r \)-tuple of positive integers, \( I_i \) is a prime ideal for which \( N(I_i) \leq x, i = 1, \ldots, r \), and all products of powers of prime ideals \( I \) with \( N(I) \leq x \) of this form occur exactly once. Hence

\[ \Pi(x) = 1 + \sum \frac{1}{N(I)^s}, \]

where the sum here is taken over all ideals \( I \) of \( R \) such that all prime ideal factors of \( I \) have norm no greater than \( x \). Now Theorem 5.3 \( \Rightarrow \) all nonzero ideals of \( R \) have a unique prime ideal factorization, hence

\[ \zeta_F(s) - \Pi(x) = \sum \frac{1}{N(I)^s}, \]

where the sum here is taken over all ideals \( I \neq \{0\} \) of \( R \) such that at least one prime ideal factor of \( I \) has norm greater than \( x \). Hence this sum does not exceed

\[ \sum_{n > x} \frac{L(n)}{n^s}, \]

hence

\[ \lim_{x \to +\infty} (\zeta_F(s) - \Pi(x)) = \lim_{x \to +\infty} \sum_{n > x} \frac{L(n)}{n^s} = 0. \]

QED

If \( F = Q \) then the prime ideals of \( R = Z \) are the principal ideals generated by the rational primes \( q \in Z \), and so Theorem 5.9 \( \Rightarrow \)

\[ \zeta(s) = \prod_q \frac{1}{1 - q^{-s}}, \quad s > 1, \]

the Euler-product expansion of Riemann's zeta.

We are now going to use Theorem 5.9 to obtain a factorization of \( \zeta_F \) over rational primes that is the analog of the product expansion (2) of the Riemann zeta function. In order to derive it, we need some more information about the structure of prime ideals of \( R \).

**Proposition 5.11.** (i) If \( I \in Q \) then there exists a rational prime \( q \in Z \) such that \( I \cap Z = qZ \). In particular \( q \) is the unique rational prime contained in \( I \).

(ii) If \( I \in Q \) and \( q \) is the rational prime in \( I \) then \( R/I \) is a finite field of characteristic \( q \), hence there exists a unique positive integer \( d \) such that \( N(I) = q^d \).
Proof. (i) The proof of Proposition 5.1 ⇒ $I \cap \mathbb{Z} \neq \{0\}$ and $I \cap \mathbb{Z} \neq \mathbb{Z}$ because 1 ∉ $I$. Hence $I \cap \mathbb{Z}$ is a prime ideal of $\mathbb{Z}$, and is hence generated in $\mathbb{Z}$ by a unique prime number $q$.

(ii) $I$ is a maximal ideal of $R$ (Proposition 5.1): a standard result in elementary ring theory asserts that if $M$ is a maximal ideal in a commutative ring $A$ with identity then the quotient ring $A/M$ is a field, hence $R/I$ is a field, and is finite by Proposition 5.2.

To see that $R/I$ has characteristic $q$, note first that $I \cap \mathbb{Z} = q\mathbb{Z}$, and so there is a natural isomorphism of the field $\mathbb{Z}/q\mathbb{Z}$ into $R/I$ such that the identity in $\mathbb{Z}/q\mathbb{Z}$ is mapped onto the identity of $R/I$. Because $\mathbb{Z}/q\mathbb{Z}$ has characteristic $q$, it follows that if $\bar{1}$ is the identity in $R/I$ then $q\bar{1} = 0$ in $R/I$, and $q$ is the least positive integer $n$ such that $n\bar{1} = 0$ in $R/I$. Hence $R/I$ has characteristic $q$.

Remark. It is a consequence of Theorem 5.3 and Proposition 5.11 that $R$ contains infinitely many prime ideals.

Definition. If $I \in \mathcal{Q}$ then the integer $d$ from Proposition 5.11(ii) is called the degree of $I$, denoted $\text{deg } I$.

If $n \in \mathbb{Z}$ then the ideal $nR$ is contained in a prime ideal of $R$ (Theorem 5.3) and so Proposition 5.11(i) ⇒ $\mathcal{Q}$ can be expressed as the pairwise disjoint union

$$\bigcup_{q \text{ a rational prime}} \{I \in \mathcal{Q} : q \in I\};$$

hence Theorem 5.9, Proposition 5.11(ii) ⇒ we can factor $\zeta_F$ as

$$(3) \quad \zeta_F(s) = \prod_{q \text{ a rational prime}} \left( \prod_{I \in \mathcal{Q}, q \in I} \frac{1}{1 - q^{-(\text{deg } I)s}} \right), \quad s > 1.$$}

The ideal $qR$ of $R$ is contained in only finitely many prime ideals (because of Theorem 5.3) and so each product inside the parentheses in (3) is finite; these finite products are called the elementary factors of $\zeta_F$.

**The zeta function of a quadratic field.**

Let $d \neq 1$ be a square-free integer. Then $\sqrt{d}$ is an algebraic integer with minimal polynomial $x^2 - d$ over $Q$. It is not difficult to show that the complex number field generated by $\sqrt{d}$ over $Q$, i.e., the smallest subfield of the complex numbers containing $\sqrt{d}$ and $Q$, the so-called quadratic field determined by $d$, is

$$Q(\sqrt{d}) = \{u + v\sqrt{d} : (u, v) \in Q \times Q\}.$$}

With more effort, one can also show that

$$\mathcal{R} \cap Q(\sqrt{d}) = \{m + n\omega : (m, n) \in \mathbb{Z} \times \mathbb{Z}\},$$
where
\[ \omega = \begin{cases} \sqrt{d}, & \text{if } d \equiv 2 \text{ or } 3 \mod 4, \\ 1 + \sqrt{d} \over 2, & \text{if } d \equiv 1 \mod 4 \end{cases} \]
(Hecke [22], pp. 95, 96).

Let \( F = \mathbb{Q}(\sqrt{d}), R = \mathcal{R} \cap F \). We want to calculate the Euler-product expansion of \( \zeta_F \) by means of (3). This requires the determination of the prime-ideal factorization of each ideal \( qR \) of \( R \), \( q \) a rational prime, and the calculation of the degree of each factor. This is done in

**Proposition 5.12.** *(Decomposition law in \( \mathbb{Q}(\sqrt{d}) \)) Let \( p \) be an odd prime.

(i) If \( \chi_p(d) = 1 \) then \( pR \) factors into the product of two distinct prime ideals, each of degree 1.

(ii) If \( \chi_p(d) = 0 \) then \( pR \) is the square of a prime ideal \( I \), and \( \deg I = 1 \).

(iii) If \( \chi_p(d) = -1 \) then \( pR \) is prime in \( R \), of degree 2.

If \( d \equiv 1 \mod 8 \) then

(iv) \( 2R \) factors into the product of two distinct prime ideals, each of degree 1.

If \( d \equiv 2 \) or \( 3 \mod 4 \) then

(v) \( 2R \) is the square of a prime ideal \( I \), and \( \deg I = 1 \).

If \( d \equiv 5 \mod 8 \) then

(vi) \( 2R \) is prime in \( R \) of degree 2.

**Proof.** Hecke [22], section 29, Theorem 90. QED

Proposition 5.12 \( \Rightarrow \) if \( p \) is an odd prime in \( \mathbb{Z} \) then the corresponding elementary factor of \( \zeta_F \) is

\[ \frac{1}{(1 - p^{-s})^2}, \text{ if } \chi_p(d) = 1, \]
\[ \frac{1}{1 - p^{-s}}, \text{ if } \chi_p(d) = 0, \]
\[ \frac{1}{1 - p^{-2s}}, \text{ if } \chi_p(d) = -1, \]

and the elementary factor corresponding to 2 is

\[ \frac{1}{(1 - 2^{-s})^2}, \text{ if } d \equiv 1 \mod 8, \]
\[ \frac{1}{1 - 2^{-s}}, \text{ if } d \equiv 2 \text{ or } 3 \mod 4, \]
\[ \frac{1}{1 - 2^{-2s}}, \text{ if } d \equiv 5 \mod 8. \]
Observe next that each of the elementary factors corresponding to $p$ can be expressed as
\[ \frac{1}{1 - p^{-s}} \frac{1}{1 - \chi_p(d)p^{-s}}. \]

Hence (2) and (3) \(\Rightarrow\)

(4) \[ \zeta_{Q(\sqrt{d})}(s) = \theta(s) \zeta(s) \prod_p \frac{1}{1 - \chi_p(d)p^{-s}}, \quad s > 1, \]
where
\[ \theta(s) = \begin{cases} 
\frac{1}{1 - 2^{-s}}, & \text{if } d \equiv 1 \mod 8, \\
1, & \text{if } d \equiv 2 \text{ or } 3 \mod 4, \\
\frac{1}{1 + 2^{-s}}, & \text{if } d \equiv 5 \mod 8. 
\end{cases} \]

We will use this factorization of $\zeta_{Q(\sqrt{d})}(s)$ to prove, in due course, the following lemma, the crucial fact that we will need to prove Theorem 4.11.

**Lemma 5.13.** If $a \in \mathbb{Z}$ is not a square then
\[ \sum_p \chi_p(a)p^{-s} \]
remains bounded as $s \to 1^+$.

Note that Lemma 5.13 is very similar in form and spirit to the hypothesis of Lemma 4.7, which was a key step in Dirichlet’s proof of Theorem 4.5. We will eventually see that this is no accident!

**Proving Theorem 4.11 and related results.**

We now have assembled all of the ingredients necessary for a proof of Theorem 4.11. Let $S$ be a nonempty finite subset of \([1, \infty)\) and suppose that for each subset $T$ of $S$ such that $|T|$ is odd,
\[ \prod_{i \in T} i \text{ is not a square}. \]

Let
\[ X = \{ p : \chi_p \equiv -1 \text{ on } S \}. \]
We must prove: $|X| = +\infty$.

Consider the sum
(5) \[ \Sigma(s) = \sum_{(p)} \left( \prod_{i \in S} (1 - \chi_p(i)) \right) \cdot \frac{1}{p^s}, \quad s > 1, \]
where \((p)\) means that the summation is over all primes \(p\) such that \(p\) divides no element of \(S\). Then

\[
\Sigma(s) = 2^{|S|} \sum_{p \in X} \frac{1}{p^s}, \quad s > 1,
\]

hence \(|X| = +\infty\) will follow if we can show that

\[
\lim_{s \to 1^+} \Sigma(s) = +\infty.
\]

In order to get (6), we first calculate that

\[
\prod_{i \in S} (1 - \chi_p(i)) = 1 + \sum_{\emptyset \neq T \subseteq S} (-1)^{|T|} \chi_p \left( \prod_{i \in T} i \right),
\]

plug this into (5) and interchange the order of summation to obtain

\[
\Sigma(s) = \sum_{(p)} \frac{1}{p^s} + \sum_{\emptyset \neq T \subseteq S} (-1)^{|T|} \left( \sum_{(p)} \chi_p \left( \prod_{i \in T} i \right) \cdot \frac{1}{p^s} \right).
\]

Now divide \(\{T : \emptyset \neq T \subseteq S\}\) into \(U \cup V \cup W\), where

\[
U = \left\{ \emptyset \neq T \subseteq S : |T| \text{ is even and } \prod_{i \in T} i \text{ is a square} \right\},
\]

\[
V = \left\{ \emptyset \neq T \subseteq S : |T| \text{ is even and } \prod_{i \in T} i \text{ is not a square} \right\},
\]

\[
W = \{T \subseteq S : |T| \text{ is odd}\}.
\]

Then

\[
\Sigma(s) = (1 + |U|) \sum_{(p)} \frac{1}{p^s}
\]

\[
+ \sum_{T \in V} \left( \sum_{(p)} \chi_p \left( \prod_{i \in T} i \right) \cdot \frac{1}{p^s} \right)
\]

\[
- \sum_{T \in W} \left( \sum_{(p)} \chi_p \left( \prod_{i \in T} i \right) \cdot \frac{1}{p^s} \right)
\]

\[
= \Sigma_1(s) + \Sigma_2(s) - \Sigma_3(s).
\]

Because the range of the summation here is over all but finitely many primes, Lemma 5.13, the definition of \(V\) and the hypothesis on \(S \Rightarrow \Sigma_2(s)\) and \(\Sigma_3(s)\) remain bounded as \(s \to 1^+\), and so (6) will follow once we prove Lemma 5.13 and verify that

\[
\lim_{s \to 1^+} \sum_{(p)} \frac{1}{p^s} = +\infty.
\]
We check (7) first. Because the summation range in (7) is over all but finitely many primes, we need only show that

\[
\lim_{s \to 1^+} \sum_p \frac{1}{p^s} = +\infty.
\]

To see (8), recall from the proof of Proposition 5.6 that

\[
\lim_{s \to 1^+} (s - 1)\zeta(s) = 1,
\]

hence

\[
\lim_{s \to 1^+} \log \zeta(s) = \lim_{s \to 1^+} \log \frac{1}{s - 1} + \lim_{s \to 1^+} \log (s - 1)\zeta(s) = +\infty.
\]

Now let \( s > 1 \). The mean value theorem \( \Rightarrow \)

\[
|\log(1 + x)| \leq 2|x| \text{ for } |x| \leq \frac{1}{2},
\]

and so

\[
|\log(1 - q^{-s})| \leq 2q^{-s}, \text{ for all } q \in P.
\]

Because \( \sum_q q^{-s} < \sum_{n=1}^{\infty} n^{-s} < \infty \) it follows that the series

\[
\sum_q \log(1 - q^{-s})
\]

is absolutely convergent. Hence

\[
\log \zeta(s) = \log \left( \prod_q \frac{1}{1 - q^{-s}} \right) \text{ (from (2))}
\]

\[
= - \sum_q \log(1 - q^{-s})
\]

\[
= \sum_q \frac{1}{q^s} + \sum_q \left( - \log(1 - q^{-s}) - \frac{1}{q^s} \right)
\]

\[
= \sum_q \frac{1}{q^s} + \sum_q \left( \sum_{n \geq 2} \frac{1}{nq^{ns}} \right),
\]
5. THE ZETA FUNCTION OF AN ALGEBRAIC NUMBER FIELD AND SOME APPLICATIONS

where we use the series expansion $\log(1 - x) = -\sum_1^\infty x^n/n, |x| < 1$, to obtain the last equation. Then

$$0 < \sum_{n\geq 2} \frac{1}{nq^{ns}} = \frac{1}{q^{2s}} \left( \sum_{n=0}^\infty \frac{1}{(n+2)q^{ns}} \right)$$

$$\leq \frac{1}{q^{2s}} \sum_{n=0}^\infty q^{-ns}$$

$$= \frac{1}{q^{2s}} \frac{1}{1 - q^{-s}}$$

$$< \frac{2}{q^2}, \text{ for all } q \geq 2 \text{ and for all } s \geq 1.$$

and so

$$0 < \sum_q \left( \sum_{n\geq 2} \frac{1}{nq^{ns}} \right) < 2 \sum_q \frac{1}{q^2} < +\infty \text{ for all } s \geq 1.$$

It follows that

$$\sum_q \frac{1}{q^s} = \log \zeta(s) + H(s), \ H(s) \text{ bounded on } s > 1,$$

hence this equation and (9) $\Rightarrow$ (8).

It remains only to prove Lemma 5.13. Let $d \neq 1$ be a square-free integer. Then the factorization (4) of $\zeta_F, F = Q(\sqrt{d}) \Rightarrow$

$$\zeta_F(s) = \theta(s)\zeta(s)L(s), \text{ where } L(s) = \prod_p \frac{1}{1 - \chi_p(d)p^{-s}}.$$

Lemma 5.8 $\Rightarrow$

$$\lim_{s \to 1^+} (s - 1)\zeta_F(s) = \lambda > 0,$$

hence

$$\lim_{s \to 1^+} L(s) = \lim_{s \to 1^+} \frac{1}{\theta(s)} \frac{(s - 1)\zeta_F(s)}{(s - 1)\zeta(s)}$$

$$= \frac{\lambda}{\theta(1)} > 0,$$

hence

(10) $\lim_{s \to 1^+} \log L(s)$ is finite.
Now let $s > 1$. Then

$$\log L(s) = -\sum_p \log(1 - \chi_p(d)p^{-s})$$

$$= \sum_p \sum_{n=1}^{\infty} \frac{\chi_p(d)^n}{np^{ns}}$$

$$= \sum_p \chi_p(d)p^{-s} + \sum_p \sum_{n=2}^{\infty} \frac{\chi_p(d)^n}{np^{ns}}.$$ 

Because

$$\left| \sum_p \sum_{n=2}^{\infty} \frac{\chi_p(d)^n}{np^{ns}} \right| \leq \sum_p \sum_{n \geq 2} \frac{1}{np^{ns}},$$

the second term on the right-hand side of the last equation in (11) can be estimated as before to verify that it is bounded on $s > 1$. Hence (10) and (11) imply

$$\sum_p \chi_p(d)p^{-s} \text{ is bounded as } s \to 1^+.$$ 

The integer $d$ here can be any integer $\neq 1$ that is square-free, but every integer is the product of a square and a square-free integer, hence (12) remains valid if $d$ is replaced by any integer which is not a square. QED

The technique used in the proof of Theorem 4.11 can also be used to obtain an interesting generalization of Basic Lemma 4.4 which answers the following question: if $S$ is a nonempty, finite subset of $[1, \infty)$ and $\varepsilon : S \to \{-1, 1\}$ is a given function, when does there exist infinitely many primes $p$ such that $\chi_p \equiv \varepsilon$ on $S$? There is a natural obstruction to $S$ having this property very similar to the obstruction that prevents the conclusion of Theorem 4.11 from being true for $S$. Suppose that there exits a subset $T \neq \emptyset$ of $S$ such that $\prod_{i \in T} i$ is a square. If we choose $i_0 \in T$ and define

$$\varepsilon(i) = \begin{cases} 
-1, & \text{if } i = i_0, \\
1, & \text{if } i \in S \setminus \{i_0\},
\end{cases}$$

then $\chi_p \not\equiv \varepsilon$ on $S$ for all sufficiently large $p$: otherwise there exits a $p$ exceeding all prime factors of the elements of $T$ such that

$$-1 = \prod_{i \in T} \varepsilon(i) = \chi_p \left( \prod_{i \in T} i \right) = 1.$$ 

By tweaking the proof of Theorem 4.11, we will show that this is the only obstruction to $S$ having this property.
Theorem 5.14. Let $S$ be a nonempty finite subset of $[1, \infty)$. The following statements are equivalent:

(i) The product of all the elements in each nonempty subset of $S$ is not a square;
(ii) If $\varepsilon : S \to \{-1, 1\}$ is a fixed but arbitrary function, then there exist infinitely many primes $p$ such that $\chi_p \equiv \varepsilon$ on $S$.

Proof. Consider the sum

$$\Sigma_\varepsilon(s) = \sum_{(p)} \left( \prod_{i \in S} \left( 1 + \varepsilon(i) \chi_p(i) \right) \right) \cdot \frac{1}{p^s}, \ s > 1.$$ 

If

$$X_\varepsilon = \{ p : \chi_p \equiv \varepsilon \text{ on } S \}$$

then

$$\Sigma_\varepsilon(s) = 2^{|S|} \sum_{p \in X_\varepsilon} \frac{1}{p^s}.$$ 

Also,

$$\Sigma_\varepsilon(s) = \sum_{(p)} \frac{1}{p^s} + \sum_{\emptyset \neq T \subseteq S} \prod_{i \in T} \varepsilon(i) \left( \sum_{(p)} \chi_p \left( \prod_{i \in T} i \right) \cdot \frac{1}{p^s} \right).$$

Lemma 5.13 and the hypotheses on $S \Rightarrow$ the second term on the right-hand side of this equation is bounded as $s \to 1^+$ hence (7) $\Rightarrow$

$$\lim_{s \to 1^+} \Sigma_\varepsilon(s) = +\infty,$$

and so $X_\varepsilon$ is infinite. QED

Definition. Any set $S$ satisfying statement (ii) of Theorem 5.14 will be said to support all patterns.

Remark. The proof of Theorems 4.11 and 5.14 follows exactly the same strategy as Dirichlet’s proof of Theorem 4.5. One wants to show that a set $X$ of primes with a certain property is infinite. Hence take $s > 1$, attach a weight of $p^{-s}$ to each prime $p$ in $X$ and then attempt to prove that the weighted sum

$$\sum_{p \in X} \frac{1}{p^s}$$

of the elements of $X$ is unbounded as $s \to 1^+$. In order to achieve this (using ingenious methods!), one writes this weighted sum as $\sum_p 1/p^s$ plus a term that is bounded as $s \to 1^+$. The similarity of all of these arguments is no accident; Theorem 5.14 is in fact also due to Dirichlet, and appeared in his great memoir [10], *Recherches sur diverses applications de*
l’analyse infinitésimal à la théorie des nombres, of 1839-40, which together with [9] founded modern analytic number theory. The proof of Theorem 5.14 given here is a variation on Dirichlet’s original argument due to Hilbert [23], section 80, Theorem 111.

A straightforward modification of the proof of Theorem 4.3 can now be used to establish

**Theorem 5.15.** If $S$ is a nonempty, finite subset of $[1, \infty)$ such that for all subsets $T$ of $S$ of odd cardinality, $\prod_{i \in T} i$ is not a square, $S$ and $v : 2^S \to F^n$ are defined by $S$ as in the statement of Theorem 4.3, and $d$ is the dimension of the linear span of $v(S)$ in $F^n$, then the density of the set \{\(p : \chi_p \equiv -1 \text{ on } S\} \text{ is } 2^{-d}.

A straightforward modification of the proof of Lemma 4.9 can also be used to establish

**Theorem 5.16.** (Filaseta and Richman, [16], Theorem 2) If $S$ is a nonempty, finite subset of $[1, \infty)$ such that the product of all the elements in each nonempty subset of $S$ is not a square and $\varepsilon : S \to \{-1, 1\}$ is a fixed but arbitrary function, then the density of the set \{\(p : \chi_p \equiv \varepsilon \text{ on } S\} \text{ is } 2^{-|S|}.
CHAPTER 6

Elementary Proofs

Although Dirichlet’s work on prime numbers and quadratic residues created a sensation among his contemporary mathematical colleagues, it also received significant criticism. This criticism did not dispute the validity of his results, which were, of course, rigorously and correctly arrived at, but instead focused on the suitability of his methods. It was widely thought that methods used in number theory which adhered to the “true spirit” of the subject, and were thus best used in its development, should involve only ideas and techniques that deal with or stem directly from the fundamental structure of the integers, avoiding in particular methods from areas like analysis that were “foreign to” or “violated” that philosophy. This viewpoint in fact was championed originally by none other than Leonard Euler, and was continued by Lagrange, Legendre, and Gauss in much of their fundamental contributions to number theory. Because of the profound influence of these great mathematicians (and others!), the subject of elementary number theory, i.e., the practice of number theory using methods which have their basis in the algebra and/or the geometry of the integers, and which, in particular, avoid the use of any of the infinite processes coming from analysis, has attained major importance. Indeed, one of the most spectacular results of twentieth-century mathematics was the discovery by Selberg [36], [37] and Erdős [13] in 1949 of the long-sought elementary proofs of the Prime Number Theorem, Dirichlet’s theorem on primes in arithmetic progression, and the Prime Number Theorem for primes in arithmetic progression.

In the spirit of the philosophy espoused in the previous paragraph, we will now give elementary proofs of Theorems 4.11 and 5.14 which use only Lemma 4.4 and linear algebra over GF(2), thereby avoiding the use of zeta functions over quadratic fields. Taking into account the fact that Dirichlet’s theorem and the Prime Number Theorem for primes in arithmetic progression also have elementary proofs, the proofs that we have given of Theorems 4.1, 4.2, 4.3, 5.15, and 5.16 are already elementary.

We begin with Theorem 5.14: let $S$ be a nonempty finite subset of $[1, \infty)$ such that

\[(1) \quad \text{for all } \emptyset \neq T \subseteq S, \prod_{i \in T} i \text{ is not a square.}\]
Recall that the square-free part $\sigma(z)$ of $z \in [1, \infty)$ is
\[
\sigma(z) = \prod_{q \in \pi_{\text{odd}}(z)} q,
\]
and observe that if $\emptyset \neq T \subseteq [1, \infty)$ is finite then
\[
\prod_{i \in T} i \text{ is not a square iff } \prod_{i \in T} \sigma(i) \text{ is not a square.}
\]
(There is an integer $n$ such that $\prod_{i \in T} i = \prod_{i \in T} \sigma(i) \times n^2$, so the multiplicity $m$ of a prime factor $q$ of $\prod_{i \in T} i$ in $\prod_{i \in T} i$ is congruent mod 2 to the multiplicity $m'$ of $q$ in $\prod_{i \in T} \sigma(i)$ hence $m$ is odd iff $m'$ is odd.) Also
\[
\chi_p(z) = \chi_p(\sigma(z)), \quad \text{for all } p \notin \pi(z).
\]
Hence, upon replacing $S$ by the set formed from the integers $\sigma(z)$ for $z \in S$, we may suppose with no loss of generality that all elements of $S$ are square-free. Hence
\[
z = \prod_{q \in \pi(z)} q, \quad z \in S,
\]
$\pi(z) \neq \emptyset$, for all $z \in S(1 \notin S)$, and if $\{w, z\} \subseteq S$ then $\pi(w) \neq \pi(z)$.

We look next for a purely combinatorial condition on the sets $\pi(z), z \in S$, that is equivalent to condition (1). The following notation will be helpful with regard to that: if $T \subseteq S$, let
\[
\Pi(T) = \bigcup_{i \in T} \pi(i),
\]
\[
S(T) = \{\pi(i) : i \in T\},
\]
\[
p(T) = \prod_{i \in T} i,
\]
and let
\[
\Pi = \bigcup_{i \in S} \pi(i),
\]
\[
S = \{\pi(i) : i \in S\}.
\]
Now
\[
\Pi(T) = \quad \text{the set of all prime factors of } p(T)
\]
and
\[
\text{the multiplicity in } p(T) \text{ of } q \in \Pi(T) = |\{X \in S(T) : q \in X\}|.
\]
Hence
\[
(2) \quad p(T) \text{ is not a square iff } \{q \in \Pi(T) : |\{X \in S(T) : q \in X\}| \text{ is odd } \} \neq \emptyset.
\]
Condition (2) can be elegantly expressed by using the symmetric difference operation on sets. Recall that if $A$ and $B$ are sets then the symmetric difference $A \triangle B$ of $A$ and $B$ is the set $(A \setminus B) \cup (B \setminus A)$. The symmetric difference operation is commutative and associative, hence if $A_1, \ldots, A_k$ are distinct sets then the repeated symmetric difference
\[ \triangle_i A_i = A_1 \triangle \cdots \triangle A_k \]
is unambiguously defined. In fact, one can show that
\[ (3) \quad \triangle_i A_i = \left\{ a \in \bigcup_i A_i : \left| \left\{ A_j : a \in A_j \right\} \right| \text{ is odd} \right\}. \]

(2) and (3) \Rightarrow
\[ p(T) \text{ is not a square iff } \triangle_{i \in T} \pi(i) \neq \emptyset. \]

Hence

condition (1) holds iff for all nonempty subsets $T$ of $S$, $\triangle_{i \in T} \pi(i) \neq \emptyset$.

As the map $i \to \pi(i)$ is a bijection of $S$ onto $S$, it follows that

(4) \quad condition (1) holds iff for all nonempty subsets $T$ of $S$, $\triangle_{T \subseteq T} T \neq \emptyset$.

Recall now from Chapter 5 that $S$ is said to support all patterns if for each function $\varepsilon : S \to \{-1, 1\}$, the set $\{p : \chi_p \equiv \varepsilon \text{ on } S\}$ is infinite. Consequently from (4), in order to prove Theorem 5.14, we must show that

(5) \quad if $\triangle_{T \subseteq T} T \neq \emptyset$ for all $\emptyset \neq T \subseteq S$ then $S$ supports all patterns.

Hence we next look for a combinatorial condition on $S$ which guarantees that $S$ supports all patterns. This is provided by

**Lemma 6.1.** Suppose that $S$ satisfies the following condition:

(6) \quad for each nonempty subset $T$ of $S$, there exists a subset $N$ of $\Pi$ such that

$T \in S \Rightarrow |N \cap T|$ is odd iff $T \in T$.

Then $S$ supports all patterns.

**Proof.** Let $\varepsilon$ be a function of $S$ into $\{-1, 1\}$. We must prove: $\{p : \chi_p \equiv \varepsilon \text{ on } S\}$ is infinite.

The map $\pi(i) \to \varepsilon(i), i \in S$ defines a function $\varepsilon'$ of $S$ into $\{-1, 1\}$. Let

$T = (\varepsilon')^{-1}(-1)$. 
If $T = \emptyset$ then $\varepsilon \equiv 1$, hence apply Theorem 4.2. Suppose that $T \neq \emptyset$, and then find $N \subseteq \Pi$ such that $N$ satisfies the conclusion of (6) for this $T$. Basic Lemma 4.4 ⇒ there are infinitely many primes $p$ for which

$$\{q \in \Pi : \chi_p(q) = -1\} = N.$$ 

Let $p$ be any one of these primes which divides no element of $S$.

We claim that $\chi_p \equiv \varepsilon$ on $S$. To verify this, note first that (7) ⇒

$$\chi_p(i) = (-1)^{|N \cap \pi(i)|}, \text{ for all } i \in S.$$ 

Hence

$$i \in S \cap \chi_p^{-1}(-1) \text{ iff } |N \cap \pi(i)| \text{ is odd.}$$ 

Since the conclusion of (6) holds for $N$ and $T$, it follows that

$$|N \cap \pi(i)| \text{ is odd iff } \pi(i) \in T, \text{ for all } i \in S.$$ 

The definition of $\varepsilon' \Rightarrow$

$$\pi(i) \in T \text{ iff } i \in \varepsilon^{-1}(-1),$$ 

Hence

$$S \cap \chi_p^{-1}(-1) = \varepsilon^{-1}(-1),$$ 

and so $\chi_p \equiv \varepsilon$ on $S$. QED

Remark. The converse of Lemma 6.1 is valid.

In order to verify statement (5), and hence prove Theorem 5.14, it suffices by virtue of Lemma 6.1 to prove that if

$$\Delta_{T \in \mathcal{T}} T \neq \emptyset \text{ for all } \emptyset \neq T \subseteq S$$ 

then

$$\text{for each } \emptyset \neq T \subseteq S, \text{ there exits } N \subseteq \Pi \text{ such that } T \in S \Rightarrow |N \cap T| \text{ is odd iff } T \in \mathcal{T}.$$ 

We have now completely removed residues and non-residues from the scene and have reduced everything to proving the following purely combinatorial statement about finite sets:

if $A$ is a nonempty finite set, $\emptyset \neq S \subseteq 2^A \setminus \{\emptyset\}$, and $S$ satisfies (8), then,

with $\Pi$ replaced by $A$, $S$ satisfies (9).

This can be done via linear algebra over $F = GF(2)$. We may suppose with no loss of generality that $A = [1, n]$ for some $n \in [1, \infty)$. Let

$$v : 2^A \rightarrow F^n$$
be the map defined on p. 44. If \( \mathcal{S} = \{S_1, \ldots, S_m\} \), note that if \( \emptyset \neq \mathcal{T} \subseteq \mathcal{S} \) then there is a bijection of the set of solutions over \( F \) of the \( n \times m \) system of linear equations

\[
\sum_i v(T)(i)x_i = 1, \ T \in \mathcal{T},
\]

\[
\sum_i v(S)(i)x_i = 0, \ S \in \mathcal{S} \setminus \mathcal{T},
\]

onto the set

\[
\{N \subseteq [1, n] : N \text{ satisfies the conclusion of (9) (with } \Pi \text{ replaced by } A \text{) for } \mathcal{T}\}
\]
given by

\[(x_1, \ldots, x_n) \to \{i : x_i = 1\}.
\]

Hence (9) holds with \( \Pi \) replaced by \( A \) iff the linear transformation of \( F^n \to F^m \) with matrix

\[
B = \begin{pmatrix}
v(S_1)(1) & \cdots & v(S_1)(n) \\
\vdots & \vdots & \vdots \\
v(S_m)(1) & \cdots & v(S_m)(n)
\end{pmatrix}
\]

is surjective, i.e., \( B \) has rank \( m \), i.e., the row vectors of \( B \) are linearly independent over \( F \).

We now show that

(10) the row vectors of \( B \) are linearly independent over \( F \) iff \( \mathcal{S} \) satisfies (8);

this will prove Theorem 5.14 using only Lemma 4.4 and linear algebra over \( F \)!

If \( w = (w_1, \ldots, w_n) \in F^n \), recall that the support \( \text{supp}(w) \) of \( w \) is the set

\[\text{supp}(w) = \{i : w_i = 1\}.
\]

It is easy to see that if \( \emptyset \neq U \subseteq F \) then

\[\text{supp}\left( \sum_{w \in U} w \right) = \Delta_{w \in U} \text{supp}(w),\]

and so

(11) \( \sum_{w \in U} w \neq 0 \) iff \( \Delta_{w \in U} \text{supp}(w) \neq \emptyset \).

Observe now that

(12) \( U \) is linearly independent over \( F \) iff \( 0 \neq \sum_{w \in S} w \), for all \( \emptyset \neq S \subseteq U \).

(10) is now a consequence of (11), (12), and the fact that

\[\text{supp}(v(T)) = T, \text{ for all } T \in \mathcal{S}.
\]
Now for Theorem 4.11. Let $S$ be a nonempty finite subset of $[1, \infty)$ such that

(13) $p(T)$ is not a square for all $T \subseteq S$ with $|T|$ odd.

If we replace $S$ by the set $S'$ of integers formed by the square-free parts of the elements of $S$ then (13) is true with $S$ replaced by $S'$ hence we may again suppose with no loss of generality that all integers in $S$ are square-free.

The argument now proceeds along the same line of reasoning that we used to prove Theorem 5.14. It follows as before that, with $S = \{\pi(i) : i \in S\}$,

(14) condition (13) holds iff $\triangle_{T \in \mathcal{T}} T \neq \emptyset$ for all $\mathcal{T} \subseteq S$ with $|\mathcal{T}|$ odd.

We then look for a combinatorial condition on $S$ which implies that the set of primes

$$\{p : \chi_p \equiv -1 \text{ on } S\}$$

is infinite, in analogy with Lemma 6.1. Such a condition is provided by

**Lemma 6.2.** If there exits a subset $N$ of $\Pi = \bigcup_{i \in S} \pi(i)$ such that

$$|N \cap \pi(i)| \text{ is odd for all } i \in S,$$

then

$$\{p : \chi_p \equiv -1 \text{ on } S\}$$

is infinite.

**Proof.** Let $N$ be a subset of $\Pi$ which satisfies the hypothesis of Lemma 6.2. As before, use Lemma 4.4 to find infinitely many primes $p$ such that

$$\{q \in \Pi : \chi_p(q) = -1\} = N;$$

then for all such $p$ which divides no element of $S$,

$$\chi_p(i) = (-1)^{|N \cap \pi(i)|} = -1, \text{ for all } i \in S.$$

QED

The final step is to prove that if $A$ is a nonempty finite set, $\emptyset \neq S \subseteq 2^A \setminus \{\emptyset\}$, and

(15) $\triangle_{T \in \mathcal{T}} T \neq \emptyset$ for all $\mathcal{T} \subseteq S$ with $|\mathcal{T}|$ odd,

then there is a subset $N$ of $A$ such that

$$|N \cap S| \text{ is odd, for all } S \in \mathcal{S},$$

which can be done again by linear algebra over $F$. 

QED
We may take $A = [1, n]$, list the elements of $S$ as $S = \{S_1, \ldots, S_m\}$ and then observe that there is a bijection of the set of solutions in $F^n$ of the system of equations
\[ \sum_i v(S_j)(i)x_i = 1, \ j = 1, \ldots, m, \]
on the set
\[ \{N \subseteq [1, n] : |N \cap S| \text{ is odd, for all } S \in S\}. \]
This system has a solution iff the matrices
\[
B = \begin{pmatrix}
    v(S_1)(1) & \ldots & v(S_1)(n) \\
    \vdots & & \vdots \\
    v(S_m)(1) & \ldots & v(S_m)(n)
\end{pmatrix}
\]
and
\[
B' = \begin{pmatrix}
    v(S_1)(1) & \ldots & v(S_1)(n) & 1 \\
    \vdots & & \vdots & \vdots \\
    v(S_m)(1) & \ldots & v(S_m)(n) & 1
\end{pmatrix}
\]
have the same rank (over $F$), hence we must verify that if (15) holds then $B$ and $B'$ have the same rank.

Assuming that (15) is valid, we let $v_1, \ldots, v_m, v'_1, \ldots, v'_m$ denote the row vectors of $B$ and $B'$, respectively. We will use (15) to prove that
\begin{equation}
(16) \quad \text{for all } \emptyset \neq T \subseteq [1, m], \sum_{i \in T} v_i = 0 \iff \sum_{i \in T} v'_i = 0.
\end{equation}

(16) $\Rightarrow$ if $\mathcal{L}$ (respectively, $\mathcal{L}'$) is the set of all sets of linearly independent rows of $B$ (respectively, $B'$) then the map $v_i \rightarrow v'_i$ induces a bijection $\Lambda$ of $\mathcal{L}$ onto $\mathcal{L}'$ such that
\[ |\Lambda(L)| = |L|, \text{ for all } L \in \mathcal{L}. \]
Hence
\[ \text{rank of } B = \max_{L \in \mathcal{L}} |L| = \max_{L \in \mathcal{L}'} |L| = \text{rank of } B'. \]

In order to verify (16), note first that if $\emptyset \neq T \subseteq [1, m]$ then
\begin{equation}
(17) \quad \text{i-th coordinate of } \sum_{j \in T} v_j = \text{i-th coordinate of } \sum_{j \in T} v'_j, \ i = 1, \ldots, n,
\end{equation}
hence $\sum_{j \in T} v'_j = 0 \Rightarrow \sum_{j \in T} v_j = 0$. If $\sum_{j \in T} v_j = 0$ then (15) $\Rightarrow |T|$ is even, and so
\[ (n + 1)-\text{st coordinate of } \sum_{j \in T} v'_j \text{ is } |T| \cdot 1 = 0, \]
hence (17) $\Rightarrow \sum_{j \in T} v'_j = 0$. QED
We close this chapter by discussing what happens if instead of subsets of \([1, \infty)\) we allow nonempty, finite subsets of \(Z \setminus \{0\}\) in the hypotheses of all of the theorems in Chapters 4 and 5. Theorem 4.1 remains valid if the positive integer in its hypothesis is replaced by a non-zero integer, and Theorems 4.2, 4.11, 5.14, and 5.16 remain valid with no change in their statements if the set \(S\) in the hypotheses there is replaced by an arbitrary nonempty, finite subset of \(Z \setminus \{0\}\). In this more general situation, the integer \(-1\) behaves like an additional prime, and once that is taken into account, all of our arguments, both elementary and non-elementary, can be modified without too much additional effort to verify these more general results. If the subset of \([1, \infty)\) in the hypotheses of Theorems 4.3 and 5.15 is replaced by a nonempty, finite subset \(S\) of \(Z \setminus \{0\}\) and if the dimension \(d\) is determined by \(S\) as in the statements of those theorems, then the density of the sets in their conclusions is now either \(2^{-d}\) or \(2^{-(1+d)}\), with the latter value occurring if either \(-1 \in S\) or the sets \(\pi_{\text{odd}}(z), z \in S\), possess a certain combinatorial structure. However, the proof of this version of Theorems 4.3 and 5.15 proceeds along the same lines as the arguments that we have given, with only a few additional technical adjustments (see Wright [44], section 3 for the details).
CHAPTER 7

Dirichlet $L$-functions and the Distribution of Quadratic Residues

In this chapter we will prove

**Theorem 7.1.** (i) If $p \equiv 3 \mod 4$ then
\[
\sum_{0<n<p/2} \chi_p(n) > 0.
\]

(ii) If $p \equiv 1 \mod 4$ then
\[
\sum_{0<n<p/4} \chi_p(n) > 0.
\]

(iii) If $p > 3$ then
\[
\sum_{0<n<p/3} \chi_p(n) > 0.
\]

Dirichlet [10] proved this in 1839, and this theorem yields interesting and important information about how residues and non-residues of $p$ are distributed throughout $[1, p-1]$. In order to see how that goes, we consider an interval $I$ of the real line of finite length and, following Berdnt [1], define the **quadratic excess of $I$** to be the sum
\[
q(I) = \sum_{n \in I} \chi_p(n).
\]

If $q(I) > 0$ (respectively, $q(I) < 0$) then the number of residues (respectively, non-residues) of $p$ inside $I$ exceeds the number of non-residues (respectively, residues) of $p$ there, and if $q(I) = 0$ then the number of residues and non-residues are the same. Hence Theorem 7.1 $\Rightarrow$ if $p \equiv 3 \mod 4$ (respectively, $p \equiv 1 \mod 4$, $p > 3$ ) then the number of residues inside the interval $(0, p/2)$ (respectively, $(0, p/4), (0, p/3)$) exceeds the number of non-residues there.

By taking Proposition 2.1 and Theorem 2.4 into account, we can say more. If $\{X_1, \ldots, X_k\}$ is a set of pairwise disjoint intervals of finite length such that $[1, p-1] = Z \cap (\bigcup_i X_i)$ then Proposition 2.1 $\Rightarrow$

\[
(1) \quad \sum_i q(X_i) = 0.
\]

Now, let
\[
I_1 = (0,p/3), \quad I_2 = (p/3,2p/3), \quad I_3 = (2p/3, p),
\]
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\[ J_1 = (0, p/4), \ J_2 = (p/4, p/2), \ J_3 = (p/2, 3p/4), \ J_4 = (3p/4, p). \]

Assume first that $p \equiv 3 \mod 4$. Theorem 2.4 $\Rightarrow \chi_p(-1) = -1$, hence

\[
q(I_1) = \sum_{0 < n < p/3} \chi_p(n)
\]
\[
= - \sum_{0 < n < p/3} \chi_p(-n)
\]
\[
= - \sum_{0 < n < p/3} \chi_p(p - n)
\]
\[
= - \sum_{2p/3 < n < p} \chi_p(n)
\]
\[
= -q(I_3),
\]

hence by (1) and Theorem 7.1 (iii),

\[ q(I_2) = 0 \text{ and } q(I_3) < 0. \]

It follows that $(p/3, 2p/3)$ contains the same number of residues as non-residues of $p$ and the number of non-residues in $(2p/3, p)$ exceeds the number of residues there.

Assume next that $p \equiv 1 \mod 4$. Theorem 2.4 $\Rightarrow \chi_p(-1) = 1$ hence the minus signs in (2) can be dropped to conclude that

\[ q(I_1) = q(I_3) \]

and so by (1) and Theorem 7.1(iii),

\[ q(I_3) > 0 \text{ and } q(I_2) = -q(I_1) - q(I_3) < 0. \]

It follows that the number of non-residues (respectively, residues) of $p$ in $(p/3, 2p/3)$ (respectively, $(2p/3, p)$) exceeds the number of residues (respectively, non-residues) there.

Similar arguments show that if $p \equiv 1 \mod 4$ then

\[ q(J_1) = q(J_4), \ q(J_2) = q(J_3), \ q(J_1) = -q(J_3), \]

hence Theorem 7.1(ii) $\Rightarrow$ the number of residues (respectively, non-residues) of $p$ in each of the intervals $(0, p/4)$ and $(3p/4, p)$ (respectively, $(p/4, p/2)$ and $(p/2, 3p/4)$) exceeds the number of non-residues (respectively, residues) there.

The proof of Theorem 7.1 depends on formulas for the quadratic excesses there given in terms of certain Dirichlet $L$-functions. Recall from Chapter 4 that if $\chi$ is a Dirichlet character then the $L$-function of $\chi$ is defined by the Dirichlet series

\[ L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}, \ s \in \mathbb{C}. \]
The facts about these \( L \)-functions that we will need are recorded in

**Lemma 7.2.** Let \( \chi \) be a Dirichlet character mod \( m \).

(i) If \( \chi \) is non-principal then \( L(s, \chi) \) is analytic in the half-plane \( \Re s > 0 \).

(ii) \( L(s, \chi) \) has the absolutely convergent Euler-product expansion given by

\[
L(s, \chi) = \prod_q \frac{1}{1 - \chi(q)q^{-s}}, \quad \Re s > 1,
\]

where the product is taken over all prime numbers \( q \).

(iii) If \( \chi \) is real-valued and non-principal then \( L(1, \chi) > 0 \).

**Proof.** (i) This will follow immediately from Proposition 5.6 after we prove that the sums

\[
\sum_{k=1}^{n} \chi(k)
\]

are uniformly bounded as a function of \( n \). To see this, we claim first that

\[
\sum \chi(k) = 0, \quad \text{whenever this sum is taken over any complete system of ordinary residues mod } m.
\]

Assuming this is true, we take \( n \in [1, \infty) \), write \( n = r + lm, \ 0 \leq r < m \), and then calculate that

\[
\sum_{k=1}^{n} \chi(k) = \sum_{k=1}^{lm-1} \chi(k) + \sum_{k=0}^{r} \chi(k + lm)
\]

\[
= \sum_{k=0}^{r} \chi(k + lm), \text{ by (4)}
\]

\[
= \sum_{k=0}^{r} \chi(k),
\]

hence

\[
\left| \sum_{k=1}^{n} \chi(k) \right| \leq \sum_{k=0}^{r} |\chi(k)| \leq m - 1.
\]

In order to verify (4) use the fact that \( \chi \) is periodic of period \( m \) (Proposition 4.6) and the fact that \( k \) in (4) runs through a complete set of ordinary residues mod \( m \) to write

\[
\sum_{k} \chi(k) = \sum_{k \in U(m)} \chi(k),
\]

so we need only show that this latter sum is 0.
Because \( \chi \) is non-principal, there is a \( k_0 \in U(m) \) such that \( \chi(k_0) \neq 1 \). The map \( k \to kk_0 \)

is a bijection of \( U(m) \) onto \( U(m) \), hence

\[
\sum_{k \in U(m)} \chi(k) = \sum_{k \in U(m)} \chi(kk_0) = \chi(k_0) \sum_{k \in U(m)} \chi(k),
\]

hence

\[
(1 - \chi(k_0)) \sum_{k \in U(m)} \chi(k) = 0.
\]

As \( 1 - \chi(k_0) \neq 0 \), it follows that

\[
\sum_{k \in U(m)} \chi(k) = 0.
\]

(ii) This product formula can be derived by appropriate modifications of our proof of Theorem 5.9, which verified the product formula for the zeta function of an algebraic number field. Note first that

\[
\left| \frac{1}{1 - \chi(q)q^{-s}} - 1 \right| = \left| \frac{\chi(q)q^{-s}}{1 - \chi(q)q^{-s}} \right| \leq \frac{q^{-\Re s}}{1 - q^{-\Re s}} \leq 2q^{-\Re s}, \text{ for all } q \geq 2, \Re s > 1,
\]

consequently Proposition 5.10 \( \Rightarrow \) the product in (ii) is absolutely convergent for \( \Re s > 1 \).

The proof of Theorem 5.9 can now be easily modified by replacing the set of prime ideals of \( R \), the set of nonzero ideals of \( R \), Proposition 5.4 and Theorem 5.3 in that proof by, respectively, the set \( P \) of all primes, the set \([1, \infty)\), the complete multiplicativity of \( \chi \), and the fundamental theorem of arithmetic to obtain

\[
\sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_q \frac{1}{1 - \chi(q)q^{-s}}, \text{ for } \Re s > 1.
\]

(iii) If \( \chi \) is real then every value of \( \chi \) is 0 or \( \pm 1 \), hence each factor in the Euler product expansion of \( L(s, \chi) \) is positive for \( s > 1 \). Consequently \( L(s, \chi) \) is not less than 0, and so by the continuity of \( L(s, \chi) \) on \( s > 0 \) it follows that

\[
L(1, \chi) = \lim_{s \to 1^+} L(s, \chi) \geq 0.
\]

But Dirichlet’s fundamental Lemma 4.8 \( \Rightarrow \) \( L(1, \chi) \neq 0 \), hence \( L(1, \chi) > 0 \). QED

In light of Lemma 7.2(iii), Theorem 7.1(i) will follow immediately from

**Theorem 7.3.** If \( p \equiv 3 \mod 4 \) then

\[
q(0, p/2) = \frac{\sqrt{p}}{\pi} (2 - \chi_p(2)) L(1, \chi_p).
\]
In order to state the $L$-function formulae that will produce Theorem 7.1(ii) and (iii), we will need to make use of the fact that if $\chi_m$ and $\chi_n$ are Dirichlet characters of modulus $m$ and $n$, and if $\gcd(m, n) = 1$, then the point-wise product $\chi_m\chi_n$ is a Dirichlet character of modulus $mn$. This follows from the fact that if $\gcd(m, n) = 1$ then the Chinese remainder theorem $\Rightarrow U(mn)$ is isomorphic to the direct product $U(m) \times U(n)$, and so the point-wise product $\chi_m\chi_n$ clearly defines a homomorphism of $U(mn)$ into the circle group.

Our proof of Theorem 7.1 (ii) will make use of the character $\chi_{4p}$ of modulus $4p$ given by point-wise multiplication of $\chi_p$ and the character $\chi_4$ of modulus 4 defined by

$$
\chi_4(n) = \begin{cases} (-1)^{(n-1)/2}, & n \text{ odd}, \\ 0, & n \text{ even}. \end{cases}
$$

Also, if $p > 3$ then we let $\chi_{3p}$ denote the point-wise product of $\chi_3$ and $\chi_p$.

Again, because of Lemma 7.2(iii) , Theorem 7.1(ii) and (iii) will follow, respectively, from

**Theorem 7.4.** If $p \equiv 1 \mod{4}$ then

$$q(0, p/4) = \frac{\sqrt{p}}{\pi} L(1, \chi_{4p}).$$

**Theorem 7.5.** Let $p > 3$.

(i) If $p \equiv 1 \mod{4}$ then

$$q(0, p/3) = \frac{\sqrt{3p}}{2\pi} L(1, \chi_{3p}).$$

(ii) If $p \equiv 3 \mod{4}$ then

$$q(0, p/3) = \frac{\sqrt{p}}{2\pi} (3 - \chi_p(3)) L(1, \chi_p).$$

For point of emphasis, in order to prove Theorem 7.1, it now suffices to prove Theorems 7.3, 7.4, and 7.5.

In addition to $L$-functions, our derivation of the formulae in Theorems 7.3, 7.4, and 7.5 will also employ some very useful properties of Gauss sums. Recall from the second proof of quadratic reciprocity in Chapter 3 the Gauss sums

$$G(n, p) = \sum_{j=0}^{p-1} \chi_p(j) \exp \left( \frac{2\pi i nj}{p} \right).$$

In that proof (Lemma 3.12 and Theorem 3.11), we showed that

$$G(n, p) = \chi_p(n) G(1, p)$$

(5)
and that
\[ G(1, p)^2 = \begin{cases} 
p, & \text{if } p \equiv 1 \text{ mod } 4, \\
-p, & \text{if } p \equiv 3 \text{ mod } 4. 
\end{cases} \]

Determining the sign of \( G(1, p) \) from this equation turns out to be a very difficult problem, and was solved by Gauss in 1805 after four long years of intense effort on his part. The plus sign is the correct one in both cases; we will present a very nice proof of this fact due to L. Kronecker, according to the account of it given in Ireland and Rosen [24], section 6.4.

**Theorem 7.6.**

\[ G(1, p) = \begin{cases} 
\sqrt{p}, & \text{if } p \equiv 1 \text{ mod } 4, \\
\imath \sqrt{p}, & \text{if } p \equiv 3 \text{ mod } 4. 
\end{cases} \]

*Proof.* Let \( \zeta = \exp(2\pi i/p) \). The argument proceeds through a series of claims and their verifications.

**Claim 1.**

\[ (-1)^{(p-1)/2} p = \prod_{k=1}^{(p-1)/2} (\zeta^{2k-1} - \zeta^{-2k+1})^2. \]

**Claim 2.**

\[ \prod_{k=1}^{(p-1)/2} (\zeta^{2k-1} - \zeta^{-2k+1}) = \begin{cases} 
\sqrt{p}, & \text{if } p \equiv 1 \text{ mod } 4, \\
\imath \sqrt{p}, & \text{if } p \equiv 3 \text{ mod } 4. 
\end{cases} \]

Once that Claim 1 is verified, we deduce from Theorem 3.11 that

\[ G(1, p) = \varepsilon \prod_{k=1}^{(p-1)/2} (\zeta^{2k-1} - \zeta^{-2k+1}). \]

where \( \varepsilon = \pm 1 \). The conclusion of Theorem 7.6 will then be at hand once we verify Claim 2 and prove that \( \varepsilon = 1 \). Hence we make

**Claim 3.** \( \varepsilon = 1 \).

To verify Claim 1, start with the factorization

\[ x^p - 1 = (x - 1) \prod_{j=1}^{p-1} (x - \zeta^j). \]

Divide this equation by \( x - 1 \) and set \( x = 1 \) to derive that

\[ p = \prod_r (1 - \zeta^r), \]
where this product is taken over any complete system of ordinary residues mod $p$. It is easy to see that the integers $\pm (4k - 2), k = 1, \ldots, (p - 1)/2$, is such a system of residues, and so

$$
p = \prod_{1}^{(p-1)/2} (1 - \zeta^{4k-2}) \prod_{1}^{(p-1)/2} (1 - \zeta^{-(4k-2)})
$$

$$
= \prod_{1}^{(p-1)/2} (\zeta^{-(2k-1)} - \zeta^{2k-1}) \prod_{1}^{(p-1)/2} (\zeta^{2k-1} - \zeta^{-(2k-1)})
$$

$$
= (-1)^{(p-1)/2} \prod_{1}^{(p-1)/2} (\zeta^{2k-1} - \zeta^{-2k+1})^2.
$$

Now for Claim 2. Claim 1 $\Rightarrow$

$$
\left( \prod_{1}^{(p-1)/2} (\zeta^{2k-1} - \zeta^{-2k+1}) \right)^2 = (-1)^{(p-1)/2} p,
$$

hence Claim 2 will follow from this equation once the sign of the product in Claim 2 is determined. That product is

$$
i^{(p-1)/2} \prod_{1}^{(p-1)/2} 2 \sin \frac{(4k-2)\pi}{p}.
$$

Observe now that for $k \in [1, (p - 1)/2]$,

$$
\sin \frac{(4k-2)\pi}{p} < 0 \ \text{iff} \ \frac{p + 2}{4} < k \leq \frac{p - 1}{2},
$$

hence this product has precisely $(p - 1)/2 - \lceil (p + 2)/4 \rceil$ negative factors, and so the number of negative factors is either $(p - 1)/4$ or $(p - 3)/4$ if, respectively, $p \equiv 1$ or 3 mod 4. It is now easy to see from this that the product in Claim 2 is a positive number if $p \equiv 1$ mod 4 or is $i \times (a$ positive number) if $p \equiv 3$ mod 4.

In order to verify Claim 3, consider the polynomial

$$
f(x) = \sum_{j=1}^{p-1} \chi_p(j)x^j - \varepsilon \prod_{k=1}^{(p-1)/2} (x^{2k-1} - x^{p-2k+1}).
$$

Then

$$
f(\zeta) = G(1, p) - \varepsilon \prod_{1}^{(p-1)/2} (\zeta^{2k-1} - \zeta^{-2k+1}) = 0
$$

and

$$
f(1) = \sum_{j=1}^{p-1} \chi_p(j) = 0.
$$
Now the minimal polynomial of $\zeta$ over $\mathbb{Q}$ is $\sum_{k=0}^{p-1} x^k$, and so we conclude from the proof of Proposition 3.4 that $\sum_{k=0}^{p-1} x^k$ divides $f(x)$ in $\mathbb{Q}[x]$. As $x - 1$ and $\sum_{k=0}^{p-1} x^k$ have no common roots (over the complex numbers), they are relatively prime in $\mathbb{Q}[x]$. Because $x - 1$ also divides $f(x)$ in $\mathbb{Q}[x]$, it follows that $x^p - 1 = (x - 1)(\sum_{k=0}^{p-1} x^k)$ must also divide $f(x)$ in $\mathbb{Q}[x]$. Hence there exists $h \in \mathbb{Q}[x]$ such that $f(x) = (x^p - 1)h(x)$. Now replace $x$ by $e^z$ to obtain the equation

$$
\sum_{j=1}^{p-1} \chi_p(j) e^{jz} - \varepsilon \prod_{k=1}^{(p-1)/2} \left(e^{(2k-1)z} - e^{z(p-2k+1)}\right) = (e^{pz} - 1)h(e^z).
$$

Insert the power series expansion of $e^z$ into this equation and then deduce that the coefficient of $z^{(p-1)/2}$ on the left-hand side of the equation is

$$
\frac{1}{((p-1)/2)!} \sum_{j=1}^{p-1} \chi_p(j) j^{(p-1)/2} - \varepsilon \prod_{k=1}^{(p-1)/2} (4k - p - 2),
$$

while the coefficient of $z^{(p-1)/2}$ on the right-hand side is of the form $pA/B$, where $A$ and $B$ are integers and $\gcd(B, p) = 1$. Now equate coefficients, multiply through by $B((p-1)/2)!$ and reduce mod $p$ to derive

$$
\sum_{j=1}^{p-1} \chi_p(j) j^{(p-1)/2} \equiv \varepsilon \left(\frac{p-1}{2}\right)! \prod_{k=1}^{(p-1)/2} (4k - 2) \\
\equiv \varepsilon \prod_{k=1}^{(p-1)/2} 2k \prod_{k=1}^{(p-1)/2} (2k - 1) \\
\equiv \varepsilon (p-1)! \\
\equiv -\varepsilon \mod p,
$$

where the last congruence follows from Wilson’s theorem. But then by Euler’s criterion (Theorem 2.5),

$$
j^{(p-1)/2} \equiv \chi_p(j) \mod p,
$$

hence

$$
p - 1 = \sum_{j=1}^{p-1} \chi_p(j)^2 \equiv -\varepsilon \mod p,
$$

and so

$$
\varepsilon \equiv 1 \mod p.
$$

Because $\varepsilon = \pm 1$, it follows that $\varepsilon = 1$. QED
Proof of Theorem 7.3. Here \( p \equiv 3 \mod 4 \). We will present a proof due to Bruce Berdnt [1], which uses an elegant application of contour integration from complex analysis. We begin by discussing the requisite facts from that subject.

Let \( \emptyset \neq U \subset \mathbb{C} \) be an open set. A function \( f : U \to \mathbb{C} \) is **analytic in** \( U \) if for each \( z \in U \),

\[
\lim_{w \to z} \frac{f(w) - f(z)}{w - z} = f'(z)
\]

exists and is finite, i.e., \( f \) has a complex derivative at each point of \( U \). A complex-valued function with domain \( \mathbb{C} \) is said to be **entire** if it is analytic in \( \mathbb{C} \). We will use the following fundamental theorem about analytic functions in our proof of Lemma 4.8 for real Dirichlet characters:

**Theorem 7.7.** (Taylor-series expansion of analytic functions) If \( f \) is analytic in \( U \) then the \( n \)-th order derivative \( f^{(n)}(z) \) exists and is finite for all \( z \in U \) and for all \( n \in [1, \infty) \). Moreover, if \( a \in U \) and \( r > 0 \) is the distance of \( a \) to the boundary of \( U \) then

\[
f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (z - a)^n , \quad |z - a| < r.
\]

Theorem 7.7 highlights the remarkable regularity which all analytic functions possess: not only is an analytic function always infinitely differentiable, but it even has a convergent Taylor-series expansion in a neighborhood of each point in its domain. This is far from true for real-valued differentiable functions.

Now let \( I \) denote the closed unit interval on the real line, and let \( \gamma : I \to U \) be a **contour in** \( U \), i.e., a continuous, piecewise-smooth function defined on \( I \) with range in \( U \). Let \( \{\gamma\} \) denote the range of \( \gamma \). If \( g : \{\gamma\} \to \mathbb{C} \) is a function continuous on \( \{\gamma\} \), \( u = \text{Re}(g) \), and \( v = \text{Im}(g) \), then the **contour integral of** \( g \) **along** \( \gamma \), denoted by

\[
\int_{\gamma} g(z) \, dz,
\]

is defined by

\[
\oint_{\gamma} (u \, dx - v \, dy) + i \oint_{\gamma} (v \, dx + u \, dy),
\]

where, from multi-variable calculus, \( \oint_{\gamma} \) denotes standard line integration in the plane along \( \gamma \) of real-valued functions continuous on \( \{\gamma\} \). Since it would take us too far afield to give a detailed account of the properties of this integral, we instead refer to J.B. Conway [2], section IV.1 for that. We will need only the basic estimate

\[
(6) \quad \left| \int_{\gamma} g(z) \, dz \right| \leq \left( \max \{|g(z)| : z \in \{\gamma\}\} \right) \text{(length of } \gamma \text{)}.
\]
A contour $\gamma$ is **closed** if $\gamma(0) = \gamma(1)$. The next theorem is one of the most important and most useful in all of complex analysis.

**Theorem 7.8.** (Cauchy’s integral theorem) If $f$ is analytic in $U$ and $\gamma$ is a closed contour in $U$ which does not wind around any point in $\mathbb{C} \setminus U$ then

$$\int_{\gamma} f(z) \, dz = 0.$$ 

The next theorem provides a very useful formula for computing certain contour integrals of functions which are analytic outside of a finite set of points. In order to state it, some terminology needs to be defined, and so we will do that first.

A closed contour $\gamma$ is a **Jordan contour** if $\gamma$ is an injective function on the interval $(0, 1)$. If $\gamma$ is a Jordan contour then $\gamma$ divides $\mathbb{C}$ into a pairwise disjoint union

$$V \cup \{\gamma\} \cup W,$$

where $V$ and $W$ are open sets and

the boundary of $V = \{\gamma\} = \text{the boundary of } W$.

Suppose that as $t$ increases from 0 to 1, $\gamma(t)$ traverses $\{\gamma\}$ in the counterclockwise direction: we then say that $\gamma$ is **positively oriented**. If $\gamma$ is positively oriented then as $t$ increases from 0 to 1, $\gamma(t)$ winds around either all of the points of $V$ or all of the points of $W$ exactly once. The set for which this occurs, either all of the points of $V$ or $W$, is called the **interior of $\gamma$**. The set $\mathbb{C} \setminus (\{\gamma\} \cup (\text{interior of } \gamma))$ is the **exterior of $\gamma$**. It can be shown that the interior of $\gamma$ is a bounded set and the exterior of $\gamma$ is unbounded. All of the facts in this paragraph are the contents of the Jordan Curve Theorem: for a proof, consult Dugundji [12], section XVII.5.

A function $f$ has an **isolated singularity** at a point $a$ if there is an $r > 0$ such that $f$ is analytic in $0 < |z - a| < r$, but $f'(a)$ does not exist. An isolated singularity of $f$ at $a$ is a **pole of order** $m \in [1, \infty)$ if there exists $\delta > 0$ and a function $g$ analytic in $|z - a| < \delta$ such that $g(a) \neq 0$ and

$$f(z) = \frac{g(z)}{(z - a)^m}, \quad 0 < |z - a| < \delta.$$ 

The **residue** of $f$ at this pole, denoted $\text{Res}(f, a)$, is the number

$$\frac{g^{(m-1)}(a)}{(m - 1)!}.$$ 

If the order of the pole at $a$ is 1 then it is called a **simple pole**, and its residue there is

$$g(a) = \lim_{z \to a} (z - a) f(z).$$
We can now state the result on the calculation of contour integrals that we need.

**Theorem 7.9.** *(The residue theorem)* Let \( U \) be an open subset of \( \mathbb{C} \), \( f \) a function analytic in \( U \) except for poles located in \( U \). If \( \gamma \) is a positively oriented Jordan contour in \( U \) which does not wind around a point in \( \mathbb{C} \setminus U \) and which does not pass through any of the poles of \( f \), and if \( a_1, \ldots, a_n \) are the poles of \( f \) that are in the interior of \( \gamma \), then

\[
\frac{1}{2\pi i} \int_{\gamma} f(z)dz = \sum_{k=1}^{n} \text{Res}(f, a_k).
\]

For proof of Theorems 7.7, 7.8, and 7.9, consult, respectively, Conway [2], sections IV.2, IV.5, and V.2.

We will apply Theorems 7.8 and 7.9 in the following situation. Let \( U \) be an open set, \( h \) and \( g \) functions analytic in \( U \), and suppose that \( a \in U \) is a zero of \( g \), i.e., \( g(a) = 0 \). Moreover suppose that \( a \) is a simple zero, i.e., \( g'(a) \neq 0 \). Then \( h/g \) has a simple pole at \( a \) iff \( h(a) \neq 0 \), and if \( h(a) \neq 0 \) then L'Hospital's rule \( \Rightarrow \)

\[
\text{Res}(h/g, a) = \lim_{z \to a} \frac{(z-a)h(z)}{g(z)} = \frac{h(a)}{g'(a)}.
\]

Hence Theorems 7.8 and 7.9 \( \Rightarrow \)

**Lemma 7.10.** Let \( U \) be an open subset of \( \mathbb{C} \), let \( h \) and \( g \) be analytic in \( U \), and suppose \( g \) has only simple zeros in \( U \). If \( \gamma \) is a positively oriented Jordan contour in \( U \) which does not wind around a point in \( \mathbb{C} \setminus U \) and does not pass through any of the zeros of \( g \), and \( a_1, \ldots, a_n \) are the zeros of \( g \) in the interior of \( \gamma \), then

\[
\frac{1}{2\pi i} \int_{\gamma} \frac{h(z)}{g(z)}dz = \sum_{k=1}^{n} \frac{h(a_k)}{g'(a_k)}.
\]

Now let

\[ F(z) = \sum_{0<j<p/2} \chi_p(j) \cos \left( \left( 1 - \frac{4j}{p} \right) \pi z \right), \]

\[ f(z) = \frac{\pi F(z)}{z \cos(\pi z)}. \]

We will prove Theorem 7.3 by integrating \( f(z) \) around rectangles and then applying Lemma 7.10.

Note first that the numerator and denominator of \( f \) are entire functions, then that the zeros of the denominator of \( f \) occur at \( z = 0, z_n = (2n - 1)/2, n \in \mathbb{Z} \), and that they are all simple. In order to apply Lemma 7.10 to \( f \), we therefore need to calculate

\[
\frac{d}{dz} \left( \frac{\pi F(z)}{z \cos(\pi z)} \right) \text{ at } z = 0, \quad z_n, \quad n \in \mathbb{Z}.
\]
At $z = 0$ this is
\begin{equation}
\pi F(0) = \pi \sum_{0 < j < p/2} \chi_p(j) = \pi q(0, p/2),
\end{equation}
and at $z = z_n$, it is
\begin{equation}
(-1)^n \frac{F(z_n)}{z_n}.
\end{equation}

We claim that
\begin{equation}
(-1)^n \frac{F(z_n)}{z_n} = -\frac{\sqrt{p}}{2n - 1} \chi_p(2n - 1), \ n \in \mathbb{Z}.
\end{equation}

In order to check this, we will first use the identity
\begin{equation}
\cos z = \frac{e^{iz} + e^{-iz}}{2}
\end{equation}
to calculate $F(z_n)$ as a Gauss sum. Toward that end, let $\alpha_j = 1 - (4j/p)$; then
\begin{align*}
\exp \left( i \frac{2n - 1}{2} \alpha_j \pi \right) &= \exp \left( i \frac{2n - 1}{2} \pi \right) \exp \left( -i \frac{2\pi j (2n - 1)}{p} \right) \\
&= (-1)^{n+1} i \exp \left( -i \frac{2\pi j (2n - 1)}{p} \right),
\end{align*}
and similarly
\begin{align*}
\exp \left( -i \frac{2n - 1}{2} \alpha_j \pi \right) &= (-1)^n i \exp \left( i \frac{2\pi j (2n - 1)}{p} \right),
\end{align*}

Hence (9) $\Rightarrow$
\begin{align*}
F(z_n) &= \frac{(-1)^{n+1} i}{2} \sum_{0 < j < p/2} \chi_p(j) \exp \left( -\frac{2\pi ij (2n - 1)}{p} \right) \\
&\quad + \frac{(-1)^n i}{2} \sum_{0 < j < p/2} \chi_p(j) \exp \left( \frac{2\pi ij (2n - 1)}{p} \right).
\end{align*}

Observe now that the exponential factors here are periodic of period $p$ in the variable $j$ and, as $p \equiv 3 \mod 4$, $\chi_p(-1) = -1$. We can hence shift the summation in the first term on the right-hand side of this equation to express that term as
\begin{align*}
\frac{(-1)^n i}{2} \sum_{p/2 < j < p} \chi_p(j) \exp \left( \frac{2\pi ij (2n - 1)}{p} \right),
\end{align*}

and then simplify:
\begin{align*}
F(z_n) &= \frac{(-1)^n i}{2} \sum_{0 < j < p} \chi_p(j) \exp \left( \frac{2\pi ij (2n - 1)}{p} \right) \\
&= \frac{(-1)^n i}{2} G(2n - 1, p).
\end{align*}
Hence (10), (5), and Theorem 7.6 \Rightarrow
\[ (-1)^n \frac{F(z_n)}{z_n} = \frac{i}{2z_n} G(2n - 1, p) \]
\[ = \frac{i}{2n - 1} \chi_p(2n - 1) G(1, p) \]
\[ = -\frac{\sqrt{p}}{2n - 1} \chi_p(2n - 1). \]

This verifies (8).

Now for the contour around which we will integrate \( f \). Let \( \gamma_N \) denote the positively oriented rectangle centered at 0, with horizontal side length \( 4pN \) and vertical side length \( 2\sqrt{N} \), where \( N \) is a fixed positive integer. \( \gamma_N \) is clearly a Jordan contour, and the zeros of \( z \cos \pi z \) inside \( \gamma_N \) are 0 and \( z_n, n \in [-pN + 1, pN] \). Hence (7), (8), and Lemma 7.10 \Rightarrow

\[ \frac{1}{2\pi i} \int_{\gamma_N} f(z) dz = \pi q(0, p/2) - \sqrt{p} \sum_{n=-pN+1}^{pN} \frac{\chi_p(2n - 1)}{2n - 1}. \]

Because \( \chi_p(-1) = -1 \),
\[ \frac{\chi_p(k)}{k} = \frac{\chi_p(-k)}{-k}, \text{ for all } k \in Z \setminus \{0\}, \]
hence
\[ \sum_{n=-pN+1}^{pN} \frac{\chi_p(2n - 1)}{2n - 1} = 2 \sum_{n=1}^{pN} \frac{\chi_p(2n - 1)}{2n - 1}. \]

We claim that
\[ \lim_{N \to \infty} \frac{1}{2\pi i} \int_{\gamma_N} f(z) \, dz = 0. \]

Assuming this for a moment, we deduce from (11), (12) and (13) that

\[ q(0, p/2) = \frac{2\sqrt{p}}{\pi} \lim_{N \to \infty} \sum_{n=1}^{pN} \frac{\chi_p(2n - 1)}{2n - 1}. \]

In order to evaluate the limit on the right-hand side of (14), note that for each integer \( M > 1 \),
\[ \frac{\chi_p(2)}{2} \sum_{k=1}^{M-1} \frac{\chi_p(k)}{k} = \sum_{k=1}^{M-1} \frac{\chi_p(2k)}{2k}, \]
hence
\[ \sum_{k=1}^{2M-1} \frac{\chi_p(k)}{k} = \frac{\chi_p(2)}{2} \sum_{k=1}^{M-1} \frac{\chi_p(k)}{k} = \sum_{k=1}^{M} \frac{\chi_p(2n - 1)}{2n - 1}. \]
Letting $M \to \infty$ in this equation, we obtain
\[
\lim_{M \to \infty} \sum_{1}^{M} \frac{\chi_p(2n-1)}{2n-1} = \left(1 - \frac{\chi_p(2)}{2}\right) \sum_{1}^{\infty} \frac{\chi_p(k)}{k} = \left(1 - \frac{\chi_p(2)}{2}\right) L(1, \chi_p).
\]
Hence (14) $\Rightarrow$
\[
q(0, p/2) = \frac{\sqrt{p}}{\pi} (2 - \chi_p(2)) L(1, \chi_p),
\]
the conclusion of Theorem 7.3.

We now need only to verify (13). This requires appropriate estimates of $f$ along the sides of $\gamma_N$. Consider first the function
\[
g(z) = \frac{\cos(\alpha \pi z)}{\cos(\pi z)}, \quad \alpha = 1 - \frac{4j}{p},
\]
coming from a term of $F(z)/\cos \pi z$. Using (9), we calculate that for $z = x + iy$,
\[
|g(z)|^2 = h(z) e^{2\pi(\alpha-1)|y|}, \quad \text{where}
\]
\[h(z) = \frac{e^{-4\pi \alpha|y|} + 2e^{-2\pi(\alpha-1)|y|} \cos 2x + 1}{e^{-4\pi|y|} + 2e^{-2\pi|y|} \cos 2x + 1}.
\]
We have
\[
\alpha - 1 \leq -4/p, \quad \text{for all } \alpha,
\]
\[
h(z) < 4/(1/2) = 8, \quad \text{for all } |y| \geq 1,
\]
and so
\[
|g(z)| < 2\sqrt{2} e^{-(4\pi/p)|y|}, \quad \text{for all } |y| \geq 1.
\]
Hence
\[
|F(z)| < p\sqrt{2} e^{-(4\pi/p)|y|}, \quad \text{for all } |y| \geq 1.
\]
From (15) it follows that
\[
|f(z)| < \frac{p\sqrt{2} e^{-(4\pi/p)|y|}}{\sqrt{N}}, \quad \text{for all } z \text{ on the horizontal sides } H_N \text{ of } \gamma_N.
\]
By (15), $F(z)/\cos(\pi z)$ is bounded on the vertical line $\Re z = 2p$. But $F(z)/\cos(\pi z)$ is periodic of period $2p$, hence there is a constant $B$, independent of $N$, such that
\[
\left|\frac{F(z)}{\cos(\pi z)}\right| \leq B, \quad \text{for all } z \text{ on the vertical sides } V_N \text{ of } \gamma_N.
\]
Hence
\[
|f(z)| \leq \frac{B}{2pN}, \quad \text{for all } z \text{ on the vertical sides } V_N \text{ of } \gamma_N.
\]
The estimates (6), (16), and (17) ⇒

\[
\left| \int_{fN} f(z) \, dz \right| \leq \left| \int_{H_N} f(z) \, dz \right| + \left| \int_{V_N} f(z) \, dz \right| \leq \frac{p\sqrt{2} \, e^{-(4\pi/p)\sqrt{N}}}{\sqrt{N}} \cdot 8pN + \frac{B}{2pN} \cdot 4\sqrt{N} \to 0, \text{ as } N \to \infty.
\]

QED

Proof of Theorem 7.4. Here \( p \equiv 1 \mod 4 \). The proof we give is based on the convergence of Fourier series and is very much in the same spirit as Dirichlet’s original argument. We therefore preface the proof proper with a brief discussion of Fourier series and their convergence.

If \( f \) is a real-valued function defined and integrable over \(-\pi \leq x \leq \pi\), then the Fourier series \( S(f, x) \) of \( f \) is the series defined by

\[
\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx),
\]

where

\[
a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \, dx,
\]

\[
a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx,
\]

\[
b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx, \quad n = 1, 2, \ldots;
\]

\( a_n \) and \( b_n \) are called, respectively, the Fourier cosine and sine coefficients of \( f \).

Recall that a real-valued function \( f \) defined on a closed and bounded interval \( J = \{ x : c \leq x \leq d \} \) of the real line is piecewise differentiable on \( J \) if there is a finite partition of \( \{ x : c \leq x < d \} \) into subintervals such that for each subinterval \( a \leq x < b \), there exists a function \( g \) differentiable on \( a \leq x \leq b \) such that \( f \equiv g \) on \( a < x < b \). A function \( f \) that is piecewise differentiable on \( J \) is clearly piecewise continuous there, hence if \( c < x < d \) then the one-sided limits

\[
f_{\pm}(x) = \lim_{t \to x^\pm} f(t), \quad \lim_{t \to c^+} f(t), \quad \text{and} \quad \lim_{t \to d^-} f(t)
\]

exist and are finite. It follows that if \( f \) is defined on the entire real line, is periodic of period \( 2\pi \), and is piecewise differentiable on \(-\pi \leq x \leq \pi\) then both one-sided limits of \( f \) at any real number exist and are finite, and so the functions \( f_{\pm}(x) = \lim_{t \to x^\pm} f(t) \) are both defined and real-valued on the entire real line.
We will use the following basic theorem on the convergence of Fourier series, a variant of which was first proved by Dirichlet [8] in 1829.

**Theorem 7.11.** If \( f \) is defined on \((-\infty, +\infty)\), is periodic of period \( 2\pi \), and is piecewise differentiable on \(-\pi \leq x \leq \pi\), then the Fourier series \( S(f, x) \) of \( f \) converges to

\[
\frac{f_+(x) + f_-(x)}{2}, \quad -\infty < x < +\infty.
\]

In particular, if \( f \) is continuous at \( x \) then \( S(f, x) \) converges to \( f(x) \).

**Proof.** Let

\[
S_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx),
\]

denote the \( n \)-th partial sum of the Fourier series of \( f \). The key idea of this argument, due to Dirichlet, and used more or less in all convergence proofs of Fourier series, is to first express \( S_n(x) \) in an integral form that is more amenable to an analysis of the convergence involved. Using the definition of the Fourier cosine and sine coefficients of \( f \), we thus calculate that

\[
S_n(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \left( \frac{1}{2} + \sum_{k=1}^{n} \cos kx \cos kt + \sin kx \sin kt \right) dt
\]

\[
= \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \left( \frac{1}{2} + \sum_{k=1}^{n} \cos k(x - t) \right) dt.
\]

Using the trigonometric identity

\[
\frac{1}{2} + \sum_{k=1}^{n} \cos k\theta = \frac{\sin \left( n + \frac{1}{2} \right) \theta}{2 \sin \left( \frac{\theta}{2} \right)},
\]

it follows that

\[
S_n(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) D_n(x - t) dt,
\]

where

\[
D_n(\theta) = \frac{\sin \left( n + \frac{1}{2} \right) \theta}{2 \sin \left( \frac{\theta}{2} \right)}
\]

is the **Dirichlet kernel** of \( S_n(x) \) (at \( \theta = k\pi, k \) an even integer, we define \( D_n(\theta) \) to be \( n + \frac{1}{2} \), so as to make \( D_n \) a function continuous on \(-\infty < \theta < +\infty\)). Using the facts that \( f \) and \( D_n \) are of period \( 2\pi \) and \( D_n \) is an even function, we can rewrite the integral formula for \( S_n \) as

\[
S_n(x) = \frac{1}{\pi} \int_{0}^{\pi} (f(x + t) + f(x - t)) D_n(t) dt , \quad -\infty < x < +\infty.
\]

If we now let \( f \equiv 1 \) in this equation and check that for this \( f \), \( S_n \equiv 1 \), we find that

\[
1 = \frac{2}{\pi} \int_{0}^{\pi} D_n(t) dt.
\]
After multiplying this equation by \( \frac{1}{2}(f_+(x) + f_-(x)) \) and then subtracting the equation resulting from that from the last integral formula for \( S_n \), it follows that

\[
S_n(x) - \frac{f_+(x) + f_-(x)}{2} = \frac{1}{\pi} \int_0^\pi \left( f(x + t) - f_+(x) + f(x - t) - f_-(x) \right) D_n(t) dt.
\]

Now let

\[
\Xi(t) = \frac{f(x + t) - f_+(x) + f(x - t) - f_-(x)}{2 \sin \left( \frac{t}{2} \right)}, \quad 0 < t \leq \pi.
\]

With an eye toward defining \( \Xi \) at \( t = 0 \) so as to make \( \Xi \) right-continuous there, we study the behavior of \( \Xi(t) \) as \( t \to 0^+ \). To that end, first rewrite \( \Xi(t) \) as

\[
\Xi(t) = \left( \frac{f(x + t) - f_+(x)}{t} + \frac{f(x - t) - f_-(x)}{t} \right), \quad \frac{t}{2 \sin \left( \frac{t}{2} \right)}, \quad 0 < t \leq \pi.
\]

Because \( f \) is periodic of period \( 2\pi \) and \( f \) is piecewise differentiable on \( -\pi \leq \xi \leq \pi \), there exists subintervals \( a \leq \xi < b, b \leq \xi < c \) of the real line and functions \( g \) and \( h \) differentiable on \( a \leq \xi \leq b \) and \( b \leq \xi \leq c \), respectively, such that \( b < x < c \) and \( f(\xi) \) equals, respectively, \( g(\xi) \) or \( h(\xi) \) whenever \( a < \xi < b \) or \( b < \xi < c \). A moment’s reflection now confirms that

\[
\lim_{t \to 0^+} \frac{f(x + t) - f_+(x)}{t} = h'(x),
\]

\[
\lim_{t \to 0^+} \frac{f(x - t) - f_-(x)}{t} = \begin{cases} -h'(x) & \text{if } x > b, \\ -g'(b) & \text{if } x = b, \end{cases}
\]

and so we conclude that \( \lim_{t \to 0^+} \Xi(t) \) exists and is finite. If we take \( \Xi(0) \) to be this finite limit, then \( \Xi \) is defined and piecewise continuous on \( 0 \leq t \leq \pi \).

It follows that the functions

\[
\Xi(t) \sin \left( n + \frac{1}{2} \right)t, \quad 0 \leq t \leq \pi,
\]

and

\[
\left( f(x + t) - f_+(x) + f(x - t) - f_-(x) \right) D_n(t), \quad 0 \leq t \leq \pi,
\]

are both piecewise continuous on \( 0 \leq t \leq \pi \) and agree on \( 0 < t \leq \pi \). The latter function can hence be replaced by the former function in the integrand of the integral on the right-hand side of the above formula for \( S_n - \frac{f_+ + f_-}{2} \) to obtain the equation

\[
S_n(x) - \frac{f_+(x) + f_-(x)}{2} = \frac{1}{\pi} \int_0^\pi \Xi(t) \sin \left( n + \frac{1}{2} \right)t dt.
\]

The conclusion of Theorem 7.11 will now follow if we prove that

\[
\lim_{n \to +\infty} \frac{1}{\pi} \int_0^\pi \Xi(t) \sin \left( n + \frac{1}{2} \right)t dt = 0.
\]
In order to do that, use the formula for the sine of a sum to write
\[
\int_0^\pi \Xi(t) \sin \left( n + \frac{1}{2} \right) t \ dt = \int_{-\pi}^\pi \alpha(t) \sin nt \ dt + \int_{-\pi}^\pi \beta(t) \cos nt \ dt,
\]
where
\[
\alpha(t) = \begin{cases} 
0, & \text{if } -\pi \leq t < 0, \\
\Xi(t) \cos \left( \frac{t}{2} \right), & \text{if } 0 \leq t \leq \pi,
\end{cases}
\]
\[
\beta(t) = \begin{cases} 
0, & \text{if } -\pi \leq t < 0, \\
\Xi(t) \sin \left( \frac{t}{2} \right), & \text{if } 0 \leq t \leq \pi.
\end{cases}
\]
Because \( \alpha \) and \( \beta \) are functions piecewise continuous on \( -\pi \leq t \leq \pi \), our proof will be done upon verifying that if a function \( \psi \) is piecewise continuous on \( -\pi \leq t \leq \pi \) and if \( a_n \) and \( b_n \) are the Fourier cosine and sine coefficients of \( \psi \) then
\[
\lim_{n \to \infty} a_n = 0 = \lim_{n \to \infty} b_n
\]
(This very important fact is known as the Riemann-Lebesgue lemma). In order to see that, note that the set of functions \( \left\{ \frac{1}{\sqrt{\pi}} \cos nt : n \in [0, \infty) \right\} \cup \left\{ \frac{1}{\sqrt{\pi}} \sin nt : n \in [1, \infty) \right\} \) is orthonormal with respect to the inner product defined by integration over the interval \( -\pi \leq t \leq \pi \), hence a straightforward calculation using this fact shows that if \( \sigma_n \) denotes the \( n \)-th partial sum of the Fourier series of \( \psi \) then
\[
0 \leq \frac{1}{\pi} \int_{-\pi}^{\pi} (\psi - \sigma_n)^2 \ dx = \frac{1}{\pi} \int_{-\pi}^{\pi} \psi^2 \ dx - \left( \frac{a_0^2}{2} + \sum_{k=1}^{n} (a_k^2 + b_k^2) \right),
\]
and so
\[
\frac{a_0^2}{2} + \sum_{k=1}^{n} (a_k^2 + b_k^2) \leq \frac{1}{\pi} \int_{-\pi}^{\pi} \psi^2 \ dx < +\infty, \text{ for all } n \in [1, \infty)
\]
(this is Bessel’s inequality). Hence the series
\[
\frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2)
\]
converges, and so \( a_n \) and \( b_n \) both tend to 0 as \( n \to +\infty \). QED

Remark. Another very useful class of real-valued functions for which the conclusion of Theorem 7.11 is also valid is the functions \( f \) that are defined on the whole real line, periodic of period \( 2\pi \), and are of bounded variation on \( -\pi \leq x \leq \pi \). This means that the supremum of the sums
\[
\sum_{i=1}^{m} |f(x_i) - f(x_{i-1})|
\]
as \(\{ -\pi = x_0 < x_1 < \cdots < x_m = \pi \} \) varies over all divisions of the interval \(-\pi \leq x \leq \pi\) by a finite number of points \(x_0, \ldots, x_m\) is finite. Elementary real analysis \(\Rightarrow\) if \(f\) is of bounded variation on \(-\pi \leq x \leq \pi\) then \(f\) is the difference of two functions both of which are non-decreasing on \(-\pi \leq x \leq \pi\), and so if \(f\) is also defined on the entire real line and is periodic of period \(2\pi\) then the one-sided limits \(f_{\pm}(x)\) exist and are finite for all \(x\). That Theorem 7.11 is valid for all functions of bounded variation on \(-\pi \leq x \leq \pi\) is in fact what Dirichlet proved in his landmark paper \([8]\). This version of Theorem 7.11 also works in our proof of Theorems 7.4 and 7.5 \(\textit{infra}\); we have proved Theorem 7.11 for piecewise differentiable functions because the argument which covers that situation is a bit more elementary than the one which suffices for functions of bounded variation. For a proof of the latter theorem, the interested reader should consult Zygmund \([47]\), Theorem II.8.1. However, note well: a function that is piecewise differentiable need not be of bounded variation and a function of bounded variation is not necessarily piecewise differentiable.

Now for the proof of Theorem 7.4. Let \(f\) be the function defined on \((-\infty, +\infty)\) which is

\[
1, \text{ for } 0 < x < \pi/2, 3\pi/2 < x \leq 2\pi, \\
0, \text{ for } x = \pi/2, 3\pi/2, \\
-1, \text{ for } \pi/2 < x < 3\pi/2,
\]
on \(0 \leq x \leq 2\pi\), and is periodic of period \(2\pi\). Clearly \(f\) is piecewise differentiable on \(-\pi \leq x \leq \pi\), hence calculation of the Fourier series of \(f\) and Theorem 7.11 \(\Rightarrow\)

\[
(18) \quad f(x) = -4\pi \sum_{n=1}^{\infty} \frac{(-1)^n}{2n-1} \cos(2n-1)x, -\infty < x < +\infty.
\]

Next, let \(\chi = \chi_{4p} = \chi_4\chi_p\). Multiply the equation of Gauss sums

\[
G(2n-1, \chi_p) = \chi_p(2n-1)G(1, p),
\]
from (5), by

\[
\frac{(-1)^n}{2n-1}
\]
to obtain

\[
(19) \quad \frac{(-1)^n\chi_p(2n-1)}{2n-1}G(1, p) = \sum_{j=1}^{p-1} \chi_p(j) \frac{(-1)^n}{2n-1} \exp\left(\frac{2\pi i(2n-1)j}{p}\right).
\]

By virtue of Theorem 7.6,

\[
G(1, p) = \sqrt{p},
\]
and so, upon taking the real part of (19), we arrive at

\[ \sqrt{p} \frac{(-1)^n \chi_p(2n - 1)}{2n - 1} = \sum_{j=1}^{p-1} \chi_p(j) \frac{(-1)^n}{2n - 1} \cos \left( \frac{(2n - 1) \cdot 2\pi j}{p} \right). \]  

The definition of \( \chi_4 \Rightarrow \)

\[ \chi(k) = 0, \ k \text{ even}, \]

\[ \chi(2n - 1) = (-1)^{n+1} \chi_p(2n - 1), \]

hence

\[ \sum_{n=1}^{\infty} \frac{(-1)^n \chi_p(2n - 1)}{2n - 1} = - \sum_{k=1}^{\infty} \frac{\chi(k)}{k} = -L(1, \chi). \]

On the other hand, we have from (18) that

\[ -\frac{\pi}{4} f \left( \frac{2\pi j}{p} \right) = \sum_{n=1}^{\infty} \frac{(-1)^n}{2n - 1} \cos \left( \frac{(2n - 1) \cdot 2\pi j}{p} \right), \ j = 1, \ldots, p - 1. \]

Consequently, we can sum (20) from \( n = 1 \) to \( \infty \), interchange the order of summation on the right-hand side of the equation that results from that, and then use (21) and (22) to deduce that

\[ \sqrt{p} \ L(1, \chi) = \frac{\pi}{4} \sum_{j=1}^{p-1} f \left( \frac{2\pi j}{p} \right) \chi_p(j). \]

The final step is to evaluate the right-hand side of (23). Note that

\[ 0 < j < \frac{p}{4} \text{ iff } 0 < \frac{2\pi j}{p} < \frac{\pi}{2}, \]

\[ \frac{p}{4} < j < \frac{p}{2} \text{ iff } \frac{\pi}{2} < \frac{2\pi j}{p} < \pi, \]

\[ \frac{p}{2} < j < \frac{3p}{4} \text{ iff } \pi < \frac{2\pi j}{p} < \frac{3\pi}{2}, \]

\[ \frac{3p}{4} < j < p \text{ iff } \frac{3\pi}{2} < \frac{2\pi j}{p} < 2\pi. \]

Hence, according to the definition of \( f \),

right-hand side of (23) = \( \frac{\pi}{4} \left( q(0, p/4) - q(p/4, p/2) - q(p/2, 3p/4) + q(3p/4, p) \right). \)

But by way of (3),

\[ q(0, p/4) = q(3p/4, p), \]

\[ q(p/4, p/2) = -q(0, p/4), \]

\[ q(p/2, 3p/4) = -q(0, p/4), \]
and so
right-hand side of (23) = \pi q(0, p/4),
whence
\[ q(0, p/4) = \frac{\sqrt{p}}{\pi} L(1, \chi). \]

**QED**

**Proof of Theorem 7.5.**

(i) We have here that \( p \equiv 1 \mod 4 \), and we will use Fourier series once more. Let \( f \) be the function that is

- 1, for \( 0 \leq x < 2\pi/3 \), \( 4\pi/3 < x \leq 2\pi \),
- \( 1/2 \), for \( x = 2\pi/3, 4\pi/3 \),
- 0, for \( 2\pi/3 < x < 4\pi/3 \),

on \( 0 \leq x \leq 2\pi \) and is periodic of period \( 2\pi \). Calculation of the Fourier series of \( f \) and Theorem 7.11 \( \Rightarrow \)

\[ f(x) = \frac{2}{3} + \frac{\sqrt{3}}{\pi} \sum_{n=1}^{\infty} \frac{a_n}{n} \cos nx, \quad -\infty < x < +\infty, \]

where

\[ a_n = \begin{cases} 
0, & \text{if } 3 \text{ divides } n, \\
1, & \text{if } n \equiv 1 \mod 3, \\
-1, & \text{if } n \equiv 2 \mod 3.
\end{cases} \]

Observe now that

\[ a_n = \chi_3(n), \quad \text{for all } n, \]

and so

\[ f(x) = \frac{2}{3} + \frac{\sqrt{3}}{\pi} \sum_{n=1}^{\infty} \frac{\chi_3(n) \cos nx}{n}, \quad -\infty < x < +\infty. \]

(24)

Now multiply both sides of

\[ G(n, \chi_p) = \chi_p(n)G(1, p) \]

by

\[ \frac{\sqrt{3}}{\pi n} \chi_3(n), \]

equate real parts in the equation which results, and then use Theorem 7.6, (24), and summation of the resulting terms from \( n = 1 \) to \( \infty \) as was done in the proof of Theorem 7.4 to obtain

\[ \frac{\sqrt{3}p}{\pi} L(1, \chi_3p) = \frac{\sqrt{3}}{\pi} G(1, p) \sum_{n=1}^{\infty} \frac{\chi_3(n)\chi_p(n)}{n} = \sum_{j=1}^{p-1} \left( f \left( \frac{2\pi j}{p} \right) - \frac{2}{3} \right) \chi_p(j). \]
Because $\sum_{j=1}^{p-1} \chi_p(j) = 0$, the sum on the right is

$$
\sum_{j=1}^{p-1} f \left( \frac{2\pi j}{p} \right) \chi_p(j) = \sum_{0<j<p/3} \chi_p(j) + \sum_{2p/3<j<p} \chi_p(j), \text{ by definition of } f,
$$

$$
= 2 \sum_{0<j<p/3} \chi_p(j), \text{ because } \chi_p(-1) = 1,
$$

$$
= 2q(0, p/3).
$$

Hence

$$
q(0, p/3) = \frac{\sqrt{3p}}{2\pi} L(1, \chi_{3p}).
$$

(ii) This follows by either contour integration or the method of Fourier series along the same lines of argument that we have used before: for the details, see Berndt [1], section 4.

Remark. Berndt’s paper is well worth studying; in it, he establishes many other results on positivity and negativity of the quadratic excess over various intervals: for example if $p \equiv 11, 19 \mod 40$ then $q(0, p/10) > 0$ and if $p \equiv 5 \mod 24$ then $q(3p/8, 5p/12) < 0$. He also gives a very interesting discussion of the history of this problem with numerous pertinent references to the literature.

Because of the crucial role that it has played in the work done in this chapter, we will now prove Lemma 4.8 for real, non-principal Dirichlet characters $\chi$, i.e., if $\chi(Z) = [-1, 1]$ then $L(1, \chi) \neq 0$. The proof that we will present is due to de la Vallée Poussin [32] and is one of the most elegant arguments available for this. Following Davenport [5], pp. 32-34, we start by recalling some well-known facts about analytic continuation of Riemann’s zeta.

Following long tradition in these matters, we let $s = \sigma + it$ denote a complex variable.

Proposition 5.6 $\Rightarrow$ $\zeta(s)$ is analytic in $\sigma > 1$; we want to show that $\zeta$ can be extended to a function analytic in $\sigma > 0$ except for a simple pole at $s = 1$. Let $\sigma > 1$ and then write

$$
\zeta(s) = \sum_{n=1}^{\infty} n^{-s} = \sum_{n=1}^{\infty} n(n^{-s} - (n + 1)^{-s})
$$

$$
= s \sum_{n=1}^{\infty} n \int_{n}^{n+1} x^{-(s+1)} dx
$$

$$
= s \int_{1}^{\infty} [x] x^{-(s+1)} dx,
$$

QED
where \([x]\) denotes the greatest integer which does not exceed \(x\). Now let \([x] = x - (x)\), so that \((x)\) denotes the fractional part of \(x\). This gives

\[
(25) \quad \zeta(s) = \frac{s}{s-1} - s \int_1^\infty (x)x^{-(s+1)}dx, \quad \sigma > 1.
\]

The integral on the right is absolutely convergent for \(\sigma > 0\), uniformly convergent for \(\sigma \geq u > 0\), and all Riemann sums of the integrand are entire functions of \(s\), hence this integral defines a function analytic on \(\sigma > 0\). Consequently the right-hand side of (25) extends \(\zeta(s)\) to a function analytic in \(\sigma > 0\) except for a simple pole at \(s = 1\). It hence follows that

\[
(26) \quad \lim_{s \to 1^+} \zeta(s) = +\infty.
\]

By taking \(\chi\) in Lemma 7.2(ii) to be the principal character, we deduce next that \(\zeta\) has the Euler-product expansion

\[
\zeta(s) = \prod_q (1 - q^{-s})^{-1}, \quad \sigma > 1,
\]

and we have from the estimate in the proof of (8) in Chapter 5 that the series

\[
\sum_q \log(1 + q^{-\sigma})
\]

is absolutely convergent for \(\sigma > 1\). Hence

\[
|\zeta(s)| \geq \prod_q (1 + q^{-\sigma})^{-1} = \exp \left( -\sum_q \log(1 + q^{-\sigma}) \right) > 0, \quad \sigma > 1,
\]

and so \(\zeta(s)\) never vanishes in \(\sigma > 1\).

Now let \(\chi\) be a real, non-principal Dirichlet character and suppose by way of contradiction that \(L(1, \chi) = 0\). Because \(L(s, \chi)\) is analytic for \(\sigma > 0\) (Lemma 7.2(i)) and \(\zeta\) has only a simple pole at \(s = 1\) for \(\sigma > 0\), it follows that

\[
L(s, \chi)\zeta(s) \text{ is analytic in } \sigma > 0.
\]

Because \(\zeta(2s) \neq 0\) in \(\sigma > 1/2\), the function

\[
\psi(s) = \frac{L(s, \chi)\zeta(s)}{\zeta(2s)}
\]

is analytic in \(\sigma > 1/2\). (26) \(\Rightarrow\) \(\lim_{s \to 1^+} \zeta(2s) = +\infty\), hence

\[
(27) \quad \lim_{s \to \frac{1}{2}^+} \psi(s) = 0.
\]
For $\sigma > 1$, $\psi$ has the Euler-product expansion

$$
\psi(s) = \prod_q \frac{(1 - \chi(q)q^{-s})^{-1}(1 - q^{-s})^{-1}}{(1 - q^{-2s})^{-1}}.
$$

Let $m = \text{the modulus of } \chi$. $\chi(q) = 0$ iff $q$ divides $m$, and the factor of the Euler product corresponding to such $q$ is

$$
1 + q^{-s}.
$$

If $\chi(q) = -1$ then the factor corresponding to $q$ is

$$
\frac{(1 + q^{-s})^{-1}(1 - q^{-s})^{-1}}{(1 - q^{-2s})^{-1}} = 1.
$$

Hence

$$
\psi(s) = \prod_{q|m}(1 + q^{-s}) = \prod_{q: \chi(q)=1} \frac{1 + q^{-s}}{1 - q^{-s}}, \sigma > 1.
$$

(28)

(We note incidentally that $X = \{q : \chi(q) = 1\}$ is infinite; otherwise

$$
\psi(s) = \prod_{q|m}(1 + q^{-s}) \prod_{q \in X} \frac{1 + q^{-s}}{1 - q^{-s}}
$$

is a finite product, hence $\lim_{s \to \frac{1}{2}^+} \psi(s) > 0$, contrary to (27)).

Next let

$$
\phi(s) = \frac{\psi(s)}{\prod_{q|m}(1 + q^{-s})}.
$$

As the denominator here is nonzero in $\sigma > 0$, $\phi(s)$ is analytic in $\sigma > 1/2$, and (27) $\Rightarrow$

$$
(29) \quad \lim_{s \to \frac{1}{2}^+} \phi(s) = 0.
$$

We will now show that the product expansion (28) of $\phi$ $\Rightarrow$ $\phi(s) > 1$ for $\frac{1}{2} < s < 2$; this contradicts (29) and so Lemma 4.8 follows for real non-principal characters.

Because

$$
\frac{1 + q^{-s}}{1 - q^{-s}} = 1 + 2 \sum_{n=1}^{\infty} q^{-ns}, \sigma > 1,
$$

we can use (28) to express $\phi(s)$ as a Dirichlet series

$$
\phi(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}, \sigma > 1,
$$

where the coefficients $a_n$ are calculated like so: $a_1 = 1$, and if $n \geq 2$ then

$$
a_n = \begin{cases} 
2^{\pi(n)}, & \text{if } \pi(n) \subseteq \{q : \chi(q) = 1\}, \\
0, & \text{otherwise.}
\end{cases}
$$
In particular, $a_n \geq 0$, for all $n$.

Because $\psi$ is analytic in $\sigma > \frac{1}{2}$, Theorem 7.7 $\Rightarrow \psi$ has a Taylor series expansion centered at 2 with radius of convergence at least $\frac{3}{2}$, i.e.,

$$
\phi(s) = \sum_{m=0}^{\infty} \frac{\phi^{(m)}(2)}{m!} (s - 2)^m, \ |s - 2| < \frac{3}{2}.
$$

We can calculate $\phi^{(m)}(2)$ by term-by-term differentiation of the Dirichlet series: this series is locally uniformly convergent in $\sigma > 1$ and so we can apply the theorem which asserts that a series of functions analytic in an open set $U$ and locally uniformly convergent there has a sum that is analytic in $U$ and the derivative can be calculated by term-by-term differentiation of the series. The result is

$$
\phi^{(m)}(2) = (-1)^m \sum_{n=1}^{\infty} \frac{a_n (\log n)^m}{n^2} = (-1)^m b_m, b_m \geq 0.
$$

Hence

$$
\phi(s) = \sum_{m=0}^{\infty} \frac{b_m}{m!} (2 - s)^m, \ |s - 2| < \frac{3}{2}.
$$

If $\frac{1}{2} < s < 2$ then all terms of this series are non-negative, hence $\phi(s) \geq \phi(2) > 1$ for $\frac{1}{2} < s < 2$.

**Remark.** Because the statements in Theorem 7.1 are so important in the theory of quadratic residues, elementary proofs of them would be of great interest. However, despite numerous efforts by many people during the intervening 174 years, those proofs continue to remain elusive.
CHAPTER 8

Quadratic Residues and Non-residues in Arithmetic Progression

The following question began to attract interest in the early 1900’s: if \( s \) is a fixed positive integer and \( p \) is sufficiently large, does there exist an \( n \in [1, \infty) \) such that \( \{n, n+1, \ldots, n+s-1\} \) is a set of residues (respectively, non-residues) of \( p \) inside \([1, p-1]\), i.e., for all sufficiently large primes \( p \), does \([1, p-1]\) contain arbitrarily long sets of consecutive residues, (respectively, non-residues) of \( p \)? For \( s = 2, 3, 4, \) and \( 5 \), various authors showed that the answer is yes; in fact it was shown that if \( R_s(p) \) (respectively, \( N_s(p) \)) denotes the number of sets of \( s \) consecutive residues (respectively, non-residues) of \( p \) inside \([1, p-1]\) then as \( p \to +\infty \),

\[
R_s(p) \sim 2^{-s}p \sim N_s(p), \text{ for } s = 2, 3, 4, \text{ and } 5.
\]

This shows in particular that for \( s = 2, 3, 4, \) and \( 5 \), not only are \( R_s(p) \) and \( N_s(p) \) both positive, but as \( p \to +\infty \), they both tend to \( +\infty \). Based on this evidence and extensive numerical calculations, the speculation was that (1) in fact is valid without any restriction on \( s \), and in 1939, Harold Davenport [4] proved that this is indeed the case.

Davenport established the validity of (1) in general by yet another application of the Dirichlet-Hilbert trick that was used in the proof of Theorems 4.11 and 5.14. Let \( \mathbb{Z}_p \) denote the field \( \mathbb{Z}/p\mathbb{Z} \) of \( p \) elements. Then \( U(p) \) can be viewed as the group of nonzero elements of \( \mathbb{Z}_p \), and if \( \varepsilon \in \{-1, 1\} \) then the sum

\[
2^{-s} \sum_{x=1}^{p-s} \prod_{i=0}^{s-1} (1 + \varepsilon \chi_p(x+i))
\]

is \( R_s(p) \) (respectively, \( N_s(p) \)) when \( \varepsilon = 1 \)(respectively, \( \varepsilon = -1 \)). A la Dirichlet-Hilbert, Davenport rewrote this sum as

\[
2^{-s}(p-s) + 2^{-s} \sum_{\emptyset \neq T \subseteq [0, s-1]} \varepsilon^{|T|} \left( \sum_{x=1}^{p-s} \chi_p \left( \prod_{i \in T} (x+i) \right) \right),
\]

and then proceeded to estimate the size of the second term of this sum. This term is a sum of terms of the form

\[
\pm \sum_{x=1}^{p-s} \chi_p(f(x)),
\]

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where \( f \) is a monic polynomial of degree at most \( s \) over \( \mathbb{Z}_p \) with distinct roots in \( \mathbb{Z}_p \). Using results from the theory of certain \( L \)-functions due to Hasse, Davenport found absolute constants \( C > 0 \) and \( 0 < \sigma < 1 \) such that

\[
\left| \sum_{x=1}^{p-s} \chi_p(f(x)) \right| \leq Csp^{-\sigma}, \quad \text{for all } p \text{ large enough.}
\]

This estimate, the heart of Davenport’s argument, implies that the modulus of the second term in (2) does not exceed \( Csp^{\sigma} \), and so

\[
|R_s(p) - 2^{-s}(p - s)| \leq Csp^{\sigma}, \quad \text{for all } p \text{ large enough.}
\]

Hence

\[
\left| \frac{R_s(p)}{2^{-s}p} - 1 \right| \leq \frac{s}{p} + Cs2^sp^{\sigma - 1} \rightarrow 0 \text{ as } p \rightarrow +\infty.
\]

The same argument also works for \( N_s(p) \).

It transpires that Davenport’s technique is quite flexible and can be used to investigate the occurrence of residues and non-residues with specific arithmetical properties. We are going to use it to detect arbitrarily long arithmetic progressions of residues and non-residues of a prime.

Our point of departure from Davenport’s work is to notice that the sequence \( \{x, x + 1, \ldots, x + s - 1\} \) of \( s \) consecutive positive integers is an instance of the sequence \( \{x, x + b, \ldots, x + b(s - 1)\} \), an arithmetic progression of length \( s \) and common difference \( b \), with \( b = 1 \). Thus, if \( (b, s) \in [1, \infty) \times [1, \infty) \), and we set

\[
AP(b; s) = \left\{ \{n + ib : i \in [0, s - 1]\} : n \in [1, \infty) \right\},
\]

the family of all arithmetic progressions of length \( s \) and common difference \( b \), it is natural to inquire about the asymptotics as \( p \rightarrow +\infty \) of the number of elements of \( AP(b; s) \) that are sets of quadratic residues (respectively, non-residues) of \( p \) that occur inside \( [1, p - 1] \). We also consider the following related question: if \( a \in [0, \infty) \), set

\[
AP(a, b; s) = \left\{ \{a + b(n + i) : i \in [0, s - 1]\} : n \in [1, \infty) \right\},
\]

the family of all arithmetic progressions of length \( s \) taken from a fixed arithmetic progression

\[
AP(a, b) = \{a + bn : n \in [1, \infty)\}.
\]

We then ask for the asymptotics of the number of elements of \( AP(a, b; s) \) that are sets of quadratic residues (respectively, non-residues) of \( p \) that occur inside \( [1, p - 1] \). Solutions
of these problems will provide interesting insights into how often quadratic residues and non-residues appear as arbitrarily long arithmetic progressions.

We will in fact consider the following generalization of these questions. For each \( m \in [1, \infty) \), let
\[
a = (a_1, \ldots, a_m) \quad \text{and} \quad b = (b_1, \ldots, b_m)
\]
be \( m \)-tuples of nonnegative integers such that \( (a_i, b_i) \neq (a_j, b_j) \), for all \( i 
eq j \). When the \( b_i \)'s are distinct and positive, we set
\[
AP(b; s) = \left\{ \bigcup_{j=1}^{m} \{ n + ib_j : i \in [0, s-1] \} : n \in [1, \infty) \right\},
\]
and when the \( b_i \)'s are all positive, we set
\[
AP(a, b; s) = \left\{ \bigcup_{j=1}^{m} \{ a_j + b_j(n + i) : i \in [0, s-1] \} : n \in [1, \infty) \right\}.
\]
If \( m = 1 \) then we recover our original sets \( AP(b; s) \) and \( AP(a, b; s) \). We now pose

**Problem 1** (respectively, **Problem 2**): determine the asymptotics as \( p \to +\infty \) of the number of elements of \( AP(b; s) \) (respectively, \( AP(a, b; s) \)) that are sets of quadratic residues of \( p \) inside \([1, p-1]\).

We also pose as **Problem 3** and **Problem 4** the problems which result when the phrase “quadratic residues” in the statements of Problems 1 and 2 is replaced by the phrase “quadratic non-residues”.

**Weil sums and their estimation.**

In order to solve Problems 1-4, we will require estimates of sums of the form
\[(*) \sum_{x=1}^{N} \chi_p(f(x)), \]
where \( f \) is a polynomial in \( Z_p[x] \) and \( N \) is a fixed integer in \([1, p-1]\).

Suppose first that \( N = p - 1 \). In this case there is an elegant way to calculate the sum \((*)\) in terms of the number of rational points on an algebraic curve over \( Z_p \).

If \( F \) is a field, \( \overline{F} \) is an algebraic closure of \( F \), and \( g(x, y) \) is a polynomial in two variables with coefficients in \( F \), then the set of points
\[
C = \{(x, y) \in \overline{F} \times \overline{F} : g(x, y) = 0\}
\]
is an algebraic curve over \( F \). A point \((x, y) \in C \) is a rational point of \( C \) over \( F \) if \((x, y) \in F \times F \). If \( F \) is finite then the set of rational points on an algebraic curve over \( F \) is evidently finite, and so the determination of the cardinality of the set of rational points is an interesting and very important problem in combinatorial number theory. In 1948, A. Weil’s great treatise
[40] on the geometry of algebraic curves over finite fields was published, which contained, among many other results of fundamental importance, an upper estimate of the number of rational points in terms of $\sqrt{|F|}$ and certain geometric parameters associated with an algebraic curve. The Weil bound has turned out to be very important for various problems in number theory; in particular, we will now show how it can be employed to obtain good estimates of the sums $(\ast)$ when $N = p - 1$.

Let $f \in \mathbb{Z}_p[x]$ and consider the algebraic curve $C$ over $\mathbb{Z}_p$ defined by the polynomial $y^2 - f(x)$.

We will calculate the so-called complete Weil sum

$$\sum_{x=0}^{p-1} \chi_p(f(x))$$

in terms of the number of rational points of $C$ over $\mathbb{Z}_p$.

Let $\mathcal{R}(p)$ denote the set of rational points of $C$, i.e.,

$$\mathcal{R}(p) = \{(x, y) \in \mathbb{Z}_p \times \mathbb{Z}_p : y^2 = f(x)\},$$

and let

$$S_0 = \{x \in \mathbb{Z}_p : f(x) = 0\},$$

$$S_+ = \{x \in \mathbb{Z}_p \setminus S_0 : \chi_p(f(x)) = 1\},$$

$$S_- = \{x \in \mathbb{Z}_p \setminus S_0 : \chi_p(f(x)) = -1\}.$$

If $x \in S_+$ then there are exactly two solutions $\pm y_0 \neq 0$ of $y^2 = f(x)$ in $\mathbb{Z}_p$, hence $(x, \pm y_0) \in \mathcal{R}(p)$. Conversely, if $(x, y) \in \mathcal{R}(p)$ and $y \neq 0$ then $0 \neq y^2 = f(x)$, hence $x \in S_+$ and $y = \pm y_0$.

We conclude that

$$|\mathcal{R}(p)| = |S_0| + 2|S_+|.$$ (2)

Because $\mathbb{Z}_p$ is the pairwise disjoint union of $S_0, S_+$, and $S_-$,

$$|S_0| + |S_+| + |S_-| = p.$$ (3)

Observe now that

$$\sum_{x=0}^{p-1} \chi_p(f(x)) = |S_+| - |S_-|.$$ (4)
Equations (2), (3), (4) ⇒

$$|\mathcal{R}(p)| = |S_0| + |S_+| + |S_-| + \sum_{x=0}^{p-1} \chi_p(f(x))$$

$$= p + \sum_{x=0}^{p-1} \chi_p(f(x)),$$

i.e.,

$$\sum_{x=0}^{p-1} \chi_p(f(x)) = |\mathcal{R}(p)| - p.$$

We are ready to apply Weil’s estimate of $|\mathcal{R}(p)|$. In this case, Weil ([40], Corollaire IV.3) proved that if $y^2 - f(x)$ is non-singular over $\mathbb{Z}_p$, which means essentially that $f$ is monic of degree at least 1 and there does not exist a polynomial $g \in \mathbb{Z}_p[x]$ such that $f = g^2$, then

$$|\mathcal{R}(p)| = 1 + p - r(p),$$

where $1 \leq r(p) < d\sqrt{p}$, $d = \text{degree of } f$

(for an elementary proof, see Schmidt [35], Theorem 2.2C). If $f \in \mathbb{Z}_p[x]$ is monic with distinct roots in $\mathbb{Z}_p$ then $f$ cannot be the square of a polynomial over $\mathbb{Z}_p$, and so $y^2 - f(x)$ is non-singular over $\mathbb{Z}_p$. Hence (5) and (6) ⇒

**Theorem 8.1.** (complete Weil-sum estimate) If $f \in \mathbb{Z}_p[x]$ is monic of degree $d \geq 1$ and $f$ has distinct roots in $\mathbb{Z}_p$ then

$$\left| \sum_{x=0}^{p-1} \chi_p(f(x)) \right| < d\sqrt{p}.$$

The work of Weil in [40] is another seminal development in modern number theory. There Weil used methods from algebraic geometry to study number-theoretic properties of curves, thereby founding the modern subject of arithmetic algebraic geometry. This not only introduced important new techniques in both number theory and geometry, but it also led to the formulation of innovative strategies for attacking a wide variety of problems which until then had been intractable. Certainly one of the most striking examples of that is the proof of Fermat’s Last Theorem by Andrew Wiles [43] in 1995 (with an able assist from Richard Taylor [39]), which employed arithmetic algebraic geometry as one of its crucial tools.

We now turn to the problem of estimating the sums (*) when $N < p - 1$. An incomplete Weil sum is a sum of the form

$$\sum_{x=M}^{N} \chi_p(f(x)).$$
where \( f \in \mathbb{Z}_p[x] \), and either \( 0 \leq M \leq N < p - 1 \) or \( 0 < M \leq N \leq p - 1 \). Our solution of Problems 2 and 4 will require an estimate of incomplete Weil sums similar to the estimate of complete Weil sums provided by Theorem 8.1, and also independent of the parameters \( M \) and \( N \). When \( f(x) = x \), Polya proved in 1918 that

\[
\left| \sum_{x=M}^{N} \chi_p(x) \right| \leq \sqrt{p} \log p,
\]

and Vinogradov in the same year showed that if \( \chi \) is a non-principal Dirichlet character mod \( m \) then

\[
\left| \sum_{x=M}^{N} \chi(x) \right| \leq 6 \sqrt{m} \log m.
\]

Assuming the Generalized Riemann Hypothesis, in 1977 Montgomery and Vaughn improved this to

\[
\left| \sum_{x=M}^{N} \chi(x) \right| \leq C \sqrt{m} \log \log m.
\]

By an earlier result of Paley (which holds without assuming GRH), this estimate, except for the choice of the constant \( C \), is best possible. It follows that an estimate of (***) that is independent of \( M \) and \( N \) will most likely behave more or less like (an absolute constant) \( \times \sqrt{p} \log p \). In fact, we will prove

**Theorem 8.2. (incomplete Weil-sum estimate)** There exists \( p_0 > 0 \) such that the following statement is true: if \( p \geq p_0 \), if \( f \in \mathbb{Z}_p[x] \) is monic of degree \( d \geq 1 \) with distinct roots in \( \mathbb{Z}_p \), and \( N \in [0, p - 1] \), then

\[
\left| \sum_{x=0}^{N} \chi_p(f(x)) \right| \leq d(1 + \log p) \sqrt{p}.
\]

Our proof of Theorem 8.2 will make use of certain homomorphisms of the additive group of \( \mathbb{Z}_p \) into the circle group, defined like so. Let

\[
e_p(\theta) = \exp\left(\frac{2\pi i \theta}{p}\right).
\]

If \( n \in \mathbb{Z} \) then we set

\[
\psi(m) = e_p(mn), \ m \in \mathbb{Z}.
\]

Because \( \psi(m) = \psi(m') \) whenever \( m \equiv m' \mod p \), \( \psi \) defines a homomorphism of the additive group of \( \mathbb{Z}_p \) into the circle group, hence \( \psi \) is called an *additive character* mod \( p \).
Now for each \( n \in \mathbb{Z}, \zeta = e_p(n) \) is a \( p \)-th root of unity, i.e., \( \zeta^p = 1 \), and from the factorization
\[
(1 - \zeta) \left( \sum_{k=0}^{p-1} \zeta^k \right) = 1 - \zeta^p = 0
\]
we see that
\[
\sum_{k=0}^{p-1} \zeta^k = 0
\]
unless \( \zeta = 1 \). Applying this with \( \zeta = e_p(n - a) \), we obtain
\[
\frac{1}{p} \sum_{x=0}^{p-1} e_p(-ax)e_p(nx) = \begin{cases} 
1, & \text{if } n \equiv a \pmod{p}, \\
0, & \text{otherwise},
\end{cases}
\]
the so-called orthogonality relations of the additive characters. These relations are quite similar to the orthogonality relations satisfied by Dirichlet characters, the latter of which Dirichlet used to prove Lemma 4.7, on his way to the proof of Theorem 4.5.

Proof of Theorem 8.2. Let \( f \in \mathbb{Z}_p[x] \) be monic of degree \( d \geq 1 \), with distinct roots in \( \mathbb{Z}_p \), let \( N \in [1, p - 1] \) and set
\[
S(N) = \sum_{x=1}^{N} \chi_p(f(x)).
\]
The strategy of this argument is to use the orthogonality relations of the additive characters to express \( S(N) \) as a sum of terms \( \lambda(x)S(x), x = 0, 1, \ldots, p - 1 \), where \( \lambda(x) \) is a sum of additive characters and \( S(x) \) is a sum that is a “twisted” or “hybrid” version of a complete Weil sum. Appropriate estimates of these terms are then made to obtain the conclusion of Theorem 8.2.

We first decompose \( S(N) \) like so:
\[
S(N) = \sum_{k=1}^{N} \sum_{j=0}^{p-1} \delta_{jk} \chi_p(f(j)), \quad \delta_{jk} = \begin{cases} 
1, & \text{if } j = k, \\
0, & \text{if } j \neq k.
\end{cases}
\]
\[
= \sum_{k=1}^{N} \sum_{j=0}^{p-1} \chi_p(f(j)) \left( \frac{1}{p} \sum_{x=0}^{p-1} e_p(xk)e_p(-xj) \right), \quad \text{by (7)}
\]
\[
= \frac{1}{p} \sum_{x=0}^{p-1} \left( \sum_{k=1}^{N} e_p(xk) \right) \sum_{j=0}^{p-1} \chi_p(f(j)) e_p(-xj)
\]
\[
= \frac{1}{p} \sum_{x=0}^{p-1} \lambda(x)S(x),
\]
where
\[ \lambda(x) = \sum_{k=1}^{N} e_p(xk), \quad S(x) = \sum_{k=0}^{p-1} e_p(-xk)\chi_p(f(k)). \]

The next step is to estimate \(|\lambda(x)|\) and \(|S(x)|, x = 0, 1, \ldots, p - 1\). To get a useful estimate of \(|\lambda(x)|\), use the trigonometric identities
\[
\sum_{k=1}^{N} \cos k\theta = \frac{\sin \left(\left( N + \frac{1}{2}\right) \theta \right) - \sin \left(\frac{\theta}{2}\right)}{2 \sin \left(\frac{\theta}{2}\right)},
\]
\[
\sum_{k=1}^{N} \sin k\theta = \frac{\cos \left(\frac{\theta}{2}\right) - \cos \left(\left( N + \frac{1}{2}\right) \theta \right)}{2 \sin \left(\frac{\theta}{2}\right)},
\]
to calculate that
\[ |\lambda(x)| = \left| \sin \left(\frac{N\pi x}{p}\right) \right| \left| \sin \left(\frac{\pi x}{p}\right) \right|. \]

Now use the estimate
\[ \frac{2|\theta|}{\pi} \leq |\sin \theta|, \quad |\theta| \leq \frac{\pi}{2}, \]
to get
\[ (8) \quad |\lambda(x)| \leq \frac{p}{2|x|}, \quad 0 < |x| < \frac{p}{2}. \]

\(\lambda(x)\) and \(S(x)\) are periodic in \(x\) of period \(p\), hence
\[ (9) \quad S(N) = \frac{1}{p} \sum_{|x|<p/2} \lambda(x)S(x). \]

Note that \(\lambda(0) = N\), hence (8), (9) \(\Rightarrow\)
\[ \left| S(N) - \frac{N}{p} S(0) \right| \leq \frac{1}{2} \sum_{0<|x|<p/2} |x|^{-1}|S(x)|. \]

An estimate of each sum \(S(x)\) is now required. These are so-called hybrid or mixed Weil sums, and consist of terms \(e_p(-xy)\chi_p(f(y)), y = 0, 1, \ldots, p - 1\), which are the terms of the complete Weil sum \(\sum_{y=0}^{p-1} \chi_p(f(y))\) that are “twisted” by the multiplier \(e_p(-xy)\). As Perel’muter [31] proved in 1963 by means of the arithmetic algebraic geometry of Weil (see also Schmidt [35], Theorem 2.2G for an elementary proof), this twisting causes no problems, i.e., we have the estimate
\[ |S(x)| \leq d\sqrt{p}, \quad \text{for all } x \in \mathbb{Z}. \]
Hence

\[ |S(N)| \leq \frac{N}{p} |S(0)| + \frac{1}{2} \sum_{0 < |x| < p/2} |x|^{-1} |S(x)| \]

\[ \leq d \sqrt{p} \left( 1 + \sum_{1 \leq n < p/2} \frac{1}{n} \right). \]

Because

\[ \lim_{p \to +\infty} \left( \gamma + \log \frac{p}{2} - \sum_{1 \leq n < p/2} \frac{1}{n} \right) = 0, \]

where \( \gamma = 0.57721\ldots \) is Euler’s constant, we are done. QED

**Solution of Problems 1 and 3.**

We begin with some terminology and notation that will allow us to state our results precisely and concisely. Let \( W = \{z_1, \ldots, z_r\} \) be a nonempty, finite subset of \([0, \infty)\) with its elements indexed in increasing order \( z_i < z_j \) for \( i < j \). We let

\[ S(W) = \left\{ \{n + z_i : i \in [1, r]\} : n \in [1, \infty) \right\}, \]

the set of all shifts of \( W \) to the right by a positive integer. Let \( \varepsilon \) be a choice of signs for \([1, r]\), i.e., a function from \([1, r]\) into \( \{-1, 1\} \). If \( S = \{n + z_i : i \in [1, r]\} \) is an element of \( S(W) \), we will say that the pair \((S, \varepsilon)\) is a residue pattern of \( p \) if

\[ \chi_p(n + z_i) = \varepsilon(i), \text{ for all } i \in [1, r]. \]

The set \( S(W) \) has the universal pattern property if there exists \( p_0 > 0 \) such that for all \( p \geq p_0 \) and for all choices of signs \( \varepsilon \) for \([1, r]\), there is a set \( S \in S(W) \cap 2^{1,p-1} \) such that \((S, \varepsilon)\) is a residue pattern of \( p \). \( S(W) \) hence has the universal pattern property if and only if for all \( p \) sufficiently large, \( S(W) \) contains a set that exhibits any fixed but arbitrary pattern of quadratic residues and non-residues of \( p \). This property is inspired directly by Davenport’s work: using this terminology, we can state the result of [4, Corollary of Theorem 5] for quadratic residues as asserting that if \( s \in [1, \infty) \) then \( S([0, s-1]) \) has the universal pattern property, and moreover, for any choice of signs \( \varepsilon \) for \([1, s]\), the cardinality of the set

\[ \{S \in S([0, s-1]) \cap 2^{1,p-1} : (S, \varepsilon) \text{ is a residue pattern of } p\} \]

is asymptotic to \( 2^{-s} p \) as \( p \to +\infty \). Note that if \( \varepsilon \) is the choice of signs that is either identically 1 or identically −1 on \([1, s]\), then we recover the results of Davenport that were discussed at the beginning of this chapter.

Suppose now that there exists nontrivial gaps between elements of \( W \), i.e., \( z_{i+1} - z_i \geq 2 \) for at least one \( i \in [1, r-1] \). It is then natural to search for elements \( S \) of \( S(W) \) such that the quadratic residues (respectively, non-residues) of \( p \) inside \([\min S, \max S]\) consists
precisely of the elements of $S$, so that $S$ acts as the “support” of quadratic residues or non-residues of $p$ inside the minimal interval of consecutive integers containing $S$. We formalize this idea by declaring $S$ to be a residue (respectively, non-residue) support set of $p$ if $S = \{\text{the set of all residues of } p \text{ inside } [1, p - 1] \} \cap \{\min S, \max S\}$ (respectively, $S = \{\text{the set of all non-residues of } p \text{ inside } [1, p - 1] \} \cap \{\min S, \max S\}$). We then define $\mathcal{S}(W)$ to have the residue (respectively, non-residue) support property if there exist $p_0 > 0$ such that for all $p \geq p_0$, there is a set $S \in \mathcal{S}(W) \cap 2^{[1, p-1]}$ such that $S$ is a residue (respectively, non-residue) support set of $p$.

We now use Davenport’s method to establish the following proposition, which generalizes [4, Corollary of Theorem 5] for quadratic residues.

**Proposition 8.3.** If $W$ is any nonempty, finite subset of $[0, \infty)$, then $\mathcal{S}(W)$ has the universal pattern property and both the residue and non-residue support properties. Moreover, if $\varepsilon$ is a choice of signs for $[1, |W|]$,

$$c_\varepsilon(W)(p) = \left|\{S \in \mathcal{S}(W) \cap 2^{[1, p-1]} : (S, \varepsilon) \text{ is a residue pattern of } p\}\right|,$$

and

$$c_\sigma(W)(p) = \left|\{S \in \mathcal{S}(W) \cap 2^{[1, p-1]} : S \text{ is a residue (respectively, non-residue) support set of } p\}\right|,$$

then as $p \to +\infty$,

$$c_\varepsilon(W)(p) \sim 2^{-|W|} p \text{ and } c_\sigma(W)(p) \sim 2^{-(1+\max W - \min W)} p.$$

**Proof.** Suppose that the asserted asymptotics of $c_\varepsilon(W)(p)$ has been established for all nonempty, finite subsets $W$ of $[0, \infty)$. Then the asserted asymptotics for $c_\sigma(W)(p)$ can be deduced from that by means of the following trick. Let $W \subseteq [0, \infty)$ be nonempty and finite. Define the choice of signs $\varepsilon$ for $[\min W, \max W]$ to be 1 on $W$ and $-1$ on $[\min W, \max W] \setminus W$. Now for each $p$, let

$$\mathcal{S}(p) = \{S \in \mathcal{S}(W) \cap 2^{[1, p-1]} : S \text{ is a residue support set of } p\},$$

$$\mathcal{R}(p) = \{S \in \mathcal{S}([\min W, \max W]) \cap 2^{[1, p-1]} : (S, \varepsilon) \text{ is a residue pattern of } p\}.$$

If to each $E \in \mathcal{R}(p)$ (respectively, $F \in \mathcal{S}(p)$), we assign the set $f(E) = E \cap \{\text{set of all residues of } p \text{ inside } [1, p - 1] \}$ (respectively, $g(F) = [\min F, \max F]$), then $f$ (respectively, $g$) maps $\mathcal{R}(p)$ (respectively, $\mathcal{S}(p)$) injectively into $\mathcal{S}(p)$ (respectively, $\mathcal{R}(p)$). Hence $\mathcal{R}(p)$ and $\mathcal{S}(p)$ have the same cardinality. Because of our assumption concerning the asymptotics of $c_\varepsilon([\min W, \max W])(p)$, it follows that as $p \to +\infty$,

$$c_\sigma(W)(p) = |\mathcal{S}(p)| = |\mathcal{R}(p)| \sim 2^{-|\min W, \max W|} p = 2^{-(1+\max W - \min W)} p.$$
This establishes the conclusion of the proposition with regard to residue support sets, and the conclusion with regard to non-residue support sets follows by repeating the same reasoning after $\varepsilon$ is replaced by $-\varepsilon$.

If $\varepsilon$ is now an arbitrary choice of signs for $[1, |W|]$, it hence suffices to deduce the asserted asymptotics of $c_\varepsilon(W)(p)$. Letting $r(p) = p - \max W - 1$, we have for all $p$ sufficiently large that

$$c_\varepsilon(W)(p) = 2^{-|W|} \sum_{x=1}^{r(p)} \prod_{i=1}^{|W|} \left(1 + \varepsilon(i)\chi_p(x + z_i)\right).$$

This sum can hence be rewritten as

$$2^{-|W|}r(p) + 2^{-|W|} \sum_{\emptyset \neq T \subseteq [1,|W|]} \prod_{i \in T} \varepsilon(i) \left(\sum_{x=1}^{r(p)} \chi_p \left(\prod_{i \in T} (x + z_i)\right)\right).$$

The asserted asymptotics for $c_\varepsilon(W)(p)$ now follows from an application of Theorem 8.1 to the Weil sums in the second term of this expression. QED

Now, let $(k, s) \in [1, \infty) \times [1, \infty)$, $\{b_1, \ldots, b_k\} \subseteq [1, \infty)$ and $b = (b_1, \ldots, b_k)$. We will apply Proposition 8.3 to the family of sets defined by

$$AP(b; s) = \left\{\bigcup_{j=1}^k \left\{n + ib_j : i \in [0, s - 1]\right\} : n \in [1, \infty)\right\};$$

we need only to observe that

$$AP(b; s) = S\left(\bigcup_{j=1}^k \left\{ib_j : i \in [0, s - 1]\right\}\right),$$

for then the following theorem is an immediate consequence of Proposition 8.3. In particular, if the choice of signs $\varepsilon$ in the theorem is taken to be either identically 1 or identically $-1$, we obtain the solution of Problems 1 and 3.

**Theorem 8.4.** (Wright [45], Theorem 2.3) If $(k, s) \in [1, \infty) \times [1, \infty)$, $\{b_1, \ldots, b_k\} \subseteq [1, \infty)$ and $b = (b_1, \ldots, b_k)$, then $AP(b; s)$ has the universal pattern property and both the residue and non-residue support properties. Moreover, if $b = \max\{b_1, \ldots, b_k\}$,

$$\gamma = \left|\bigcup_{j=1}^k \{ib_j : i \in [0, s - 1]\}\right|,$$

$\varepsilon$ is a choice of signs for $[1, \gamma]$,

$$c_\varepsilon(p) = |\{S \in AP(b; s) \cap 2^{[1,p-1]} : (S, \varepsilon) \text{ is a residue pattern of } p\}|,$$

and

$$c_\sigma(p) = |\{S \in AP(b; s) \cap 2^{[1,p-1]} : S \text{ is a residue (respectively, non-residue) support set of } p\}|.$$
then as \( p \to +\infty \),
\[
c_{\varepsilon}(p) \sim 2^{-\gamma}p \text{ and } c_{\sigma}(p) \sim 2^{-(1+b(s-1))}p.
\]

**Solution of Problems 2 and 4.**

Let \((m, s) \in [1, \infty) \times [1, \infty)\), let \(a = (a_1, \ldots, a_m)\), (respectively, \(b = (b_1, \ldots, b_m)\)) be an \(m\)-tuple of nonnegative (respectively, positive) integers such that \((a_i, b_i) \neq (a_j, b_j)\) for \(i \neq j\), let \((a, b)\) denote the \(2m\)-tuple \((a_1, \ldots, a_m, b_1, \ldots, b_m)\) (we will call \((a, b)\) a **standard \(2m\)-tuple**), and recall that
\[
AP(a, b; s) = \{ m \cup \{a_j + b_j(n + i) : i \in [0, s - 1] \} : n \in [1, \infty) \}.
\]

Because of certain arithmetical interactions which can take place between the elements of the sets in \(AP(a, b; s)\), the asymptotic behavior as \( p \to +\infty \) of the number of elements of \(AP(a, b; s) \cap 2[1,p-1]\) which are sets of residues (respectively, non-residues) of \(p\) is somewhat more complicated than what occurs for \(AP(b; s)\) as per Theorem 8.4.

In order to explain the situation, we set
\[
q_{\varepsilon}(p) = |\{ A \in AP(a, b; s) \cap 2[1,p-1] : \chi_p(a) = \varepsilon, \text{ for all } a \in A \}|
\]
and note that the value of \(q_{\varepsilon}(p)\) for \(\varepsilon = 1\) (respectively, \(\varepsilon = -1\)) counts the number of elements of \(AP(a, b; s)\) that are sets of residues (respectively, non-residues) of \(p\) that are located inside \([1,p-1]\). As we mentioned before, it will transpire that the asymptotic behavior of \(q_{\varepsilon}(p)\) depends on certain arithmetical interactions that can take place between the elements of \(AP(a, b; s)\). In order to see how this goes, first consider the set \(B\) of distinct values of the coordinates of \(b\). If we declare the coordinate \(a_i\) of \(a\) and the coordinate \(b_i\) of \(b\) to **correspond** to each other, then for each \(b \in B\), we let \(A(b)\) denote the set of all coordinates of \(a\) whose corresponding coordinate of \(b\) is \(b\). We then relabel the elements of \(B\) as \(b_1, \ldots, b_k\), say, and for each \(i \in [1, k]\), set
\[
S_i = \bigcup_{a \in A(b_i)} \{a + b_i j : j \in [0, s - 1]\},
\]
and then let
\[
\alpha = \sum_i |S_i|, \ b = \max\{b_1, \ldots, b_k\}.
\]

Next, suppose that
\[(***)\] if \((i, j) \in [1, k] \times [1, k]\) with \(i \neq j\) and \((x, y) \in A(b_i) \times A(b_j)\), then either \(b_ib_j\) does not divide \(yb_i - xb_j\) or \(b_i b_j\) divides \(yb_i - xb_j\) with a quotient that exceeds \(s - 1\) in modulus.
Then as $p \to +\infty$, $q_\varepsilon(p)$ is asymptotic to $(b \cdot 2^\alpha)^{-1}p$. On the other hand, if the assumption $(***)$ does not hold then the asymptotic behavior of $q_\varepsilon(p)$ falls into two distinct regimes, with each regime determined in a certain manner by the integral quotients

$$
\frac{yb_i - xb_j}{b_i b_j}, \ (x, y) \in A(b_i) \times A(b_j),
$$

whose moduli do not exceed $s - 1$. More precisely, these quotients determine a positive integer $e < \alpha$ and a collection $\mathcal{S}$ of nonempty subsets of $[1, k]$ such that each element of $\mathcal{S}$ has even cardinality and for which the following two alternatives hold:

(i) if $\prod_{i \in S} b_i$ is a square for all $S \in \mathcal{S}$, then as $p \to +\infty$, $q_\varepsilon(p)$ is asymptotic to $(b \cdot 2^{\alpha-e})^{-1}p$, or

(ii) if there is an $S \in \mathcal{S}$ such that $\prod_{i \in S} b_i$ is not a square, then there exist two disjoint, infinite sets of primes $\Pi_+$ and $\Pi_-$ whose union contains all but finitely many of the primes and such that $q_\varepsilon(p) = 0$ for all $p \in \Pi_-$, while as $p \to +\infty$ inside $\Pi_+$, $q_\varepsilon(p)$ is asymptotic to $(b \cdot 2^{\alpha-e})^{-1}p$. Thus we see that when $(***)$ does not hold and $p \to +\infty$, either $q_\varepsilon(p)$ is asymptotic to $(b \cdot 2^{\alpha-e})^{-1}p$ or $q_\varepsilon(p)$ asymptotically oscillates infinitely often between 0 and $(b \cdot 2^{\alpha-e})^{-1}p$.

In light of what we have just discussed, it will come as no surprise that the solution of Problems 2 and 4 for $AP(a, b; s)$ involves a bit more effort than the solution of Problems 1 and 3 for $AP(b; s)$. In order to analyze the asymptotic behavior of $q_\varepsilon(p)$, we follow the same strategy as before: using an appropriate sum of products involving $\chi_p$, $q_\varepsilon(p)$ is expressed as a sum of a dominant term and a remainder. If the dominant term is a non-constant linear function of $p$ and the remainder term does not exceed an absolute constant $\times \sqrt{T} \log p$, then the asymptotic behavior of $q_\varepsilon(p)$ will be in hand.

We in fact will implement this strategy when the set $AP(a, b; s)$ in the definition of $q_\varepsilon(p)$ is replaced by a slightly more general set; for a precise statement of what we establish, see Theorem 8.9 below. We then deduce the solution of Problems 2 and 4 from this more general result, where, in particular, we indicate more precisely the manner in which the integral quotients (z) whose moduli do not exceed $s - 1$ determine the parameter $e$ and collection of sets $\mathcal{S}$ discussed above.

We begin the analysis of $q_\varepsilon(p)$ by taking a closer look at the structure of $AP(a, b; s)$. Let $\mathcal{J}$ denote the set of all subsets $J$ of $[1, m]$ that are of maximal cardinality with respect to the property that $b_j$ is equal to a fixed integer $b_J$ for all $j \in J$. We note that $\{J : J \in \mathcal{J}\}$ is a partition of $[1, m]$ and that $b_J \neq b_{J'}$ whenever $\{J, J'\} \subseteq \mathcal{J}$. Because $(a_i, b_i) \neq (a_j, b_j)$
whenever \( i \neq j \), it follows that if \( J \in \mathcal{J} \) then the integers \( a_j \) for \( j \in J \) are all distinct. Let
\[
S_J = \bigcup_{j \in J} \{a_j + b_j i : i \in [0, s - 1]\}, \quad J \in \mathcal{J}.
\]

Then
\[
\bigcup_{j=1}^m \{a_j + b_j (n + i) : i \in [0, s - 1]\} = \bigcup_{J \in \mathcal{J}} b_J n + S_J, \quad \text{for all } n \in [1, \infty).
\]

It follows that \( AP(a, b; s) \) is a special case of the following more general situation. Let \( k \in [1, \infty) \), let \( B = \{b_1, \ldots, b_k\} \) be a set of positive integers, and let \( S = (S_1, \ldots, S_k) \) be a \( k \)-tuple of finite, nonempty subsets of \([0, \infty)\). By way of analogy with the expression of the elements of \( AP(a, b; s) \) according to (11), we will denote by \( AP(B, S) \) the collection of sets defined by
\[
\left\{ \bigcup_{i=1}^k b_i n + S_i : n \in [1, \infty) \right\}.
\]

We are interested in the number of elements of \( AP(B, S) \) that are sets of quadratic residues or, respectively, quadratic non-residues of a prime \( p \), and so if \( \varepsilon \in \{-1, 1\} \), we let
\[
q_\varepsilon(p) = |\{A \in AP(B, S) \cap 2^{[1, p-1]} : \chi_p(a) = \varepsilon, \text{ for all } a \in A\}|,
\]
and seek an asymptotic formula for \( q_\varepsilon(p) \) as \( p \to +\infty \).

Toward that end, begin by noticing that there is a positive constant \( C \), depending only on \( B \) and \( S \), such that for all \( n \geq C \),
\[
\text{(12)} \quad \text{the sets } b_i n + S_i, i \in [1, k], \text{ are pairwise disjoint, and}
\]
\[
\text{(13)} \quad \bigcup_{i=1}^k b_i n + S_i \text{ is uniquely determined by } n.
\]

Because of (12) and (13), if
\[
\alpha = \sum_i |S_i| \text{ and } r(p) = \min_i \left[ \frac{p - 1 - \max S_i}{b_i} \right],
\]
then the sum
\[
2^{-\alpha} \sum_{x=1}^{r(p)} \prod_{i=1}^k \prod_{j \in S_i} (1 + \varepsilon \chi_p(b_i x + j))
\]
differs from \( q_\varepsilon(p) \) by at most \( O(1) \), hence, as per the strategy as outlined in the introduction, this sum can be used to determine the asymptotics of \( q_\varepsilon(p) \).
Apropos of that strategy, let
\[ T = \bigcup_{i=1}^{k} \{ (i, j) : j \in S_i \}, \]
and then rewrite the above sum as
\begin{equation}
2^{-\alpha}r(p) + 2^{-\alpha} \sum_{\emptyset \neq T \subseteq T} \varepsilon[T] \prod_{i=1}^{k} \chi_p(b_i)^{|\{(j, i) \in T\}|} \sum_{x=1}^{r(p)} \chi_p\left( \prod_{(i,j) \in T} (x + \bar{b}_i j) \right),
\end{equation}
where \( \bar{b}_i \) denotes the inverse of \( b_i \) modulo \( p \), which clearly exists for all \( p \) sufficiently large. Our intent now is to estimate the modulus of certain summands in the second term of (14) by means of Theorem 8.2.

Let \( \Sigma(p) \) denote the second term of the sum in (14). In order to carry out the intended estimate, we must first remove from \( \Sigma(p) \) the terms to which Theorem 8.2 cannot be applied. Toward that end, let
\[ E(p) = \{ \emptyset \neq T \subseteq T: \text{the distinct elements, modulo } p, \text{ in the list } \bar{b}_i j, (i, j) \in T, \text{ each occurs an even number of times} \}. \]
We then split \( \Sigma(p) \) into the sum \( \Sigma_1(p) \) of terms taken over the elements of \( E(p) \) and the sum \( \Sigma_2(p) = \Sigma(p) - \Sigma_1(p) \). The sum \( \Sigma_2(p) \) has no more than \( 2^\alpha - 1 \) terms each of the form
\[ \pm 2^{-\alpha} \sum_{x=1}^{r(p)} \chi_p\left( \prod_{(i,j) \in T} (x + \bar{b}_i j) \right), \quad \emptyset \neq T \in 2^T \setminus E(p). \]
Since \( \emptyset \neq T \notin E(p) \), the polynomial in \( x \) in this term at which \( \chi_p \) is evaluated can be reduced to a product of at least one and no more than \( \alpha \) distinct monic linear factors in \( x \) over \( \mathbb{Z}_p \), and so the sum in each of the above terms of \( \Sigma_2(p) \) is an incomplete Weil sum to which Theorem 8.2 can be applied. It therefore follows from that theorem that
\[ \Sigma_2(p) = O(\sqrt{p} \log p) \text{ as } p \to +\infty. \]
We must now estimate
\[ \Sigma_3(p) = 2^{-\alpha}r(p) + \Sigma_1(p), \]
and, as we shall see, it is precisely this term that will produce the dominant term which determines the asymptotic behavior of \( q_c(p) \).

Since each element of \( E(p) \) has even cardinality,
\[ \Sigma_1(p) = 2^{-\alpha} \sum_{T \in E(p)} \prod_{i=1}^{k} \chi_p(b_i)^{|\{(j, i) \in T\}|} \sum_{x=1}^{r(p)} \chi_p\left( \prod_{(i,j) \in T} (x + \bar{b}_i j) \right). \]
We now examine the sum over $x \in [1, r(p)]$ on the right-hand side of this equation. Because $T \in E(p)$, each term in this sum is either 0 or 1, and a term is 0 precisely when the value of $x$ in that term agrees with the minimal nonnegative residue mod $p$ of $-\bar{b}_j$, for some element $(i, j)$ of $T$. However, there are at most $\alpha/2$ of these values at which $x$ can agree for each $T \in E(p)$ and so it follows that $\Sigma_3(p)$ differs by at most $O(1)$ from

$$\Sigma_4(p) = 2^{-\alpha} r(p) \left( 1 + \sum_{T \in E(p)} \prod_{i=1}^k \chi_p(b_i) \chi_p(\{j: (i, j) \in T\}) \right).$$

Consequently,

(15) for all $p$ sufficiently large, $q_e(p) - \Sigma_4(p) = O(\sqrt{p} \log p)$,

and so it suffices to calculate $\Sigma_4(p)$ in order to determine the asymptotics of $q_e(p)$.

This calculation requires a careful study of $E(p)$. In order to pin this set down a bit more firmly, we make use of the equivalence relation $\approx$ defined on $T$ as follows: if $((i, j), (l, m)) \in T \times T$ then $(i, j) \approx (l, m)$ if $b_ij = b_l m$. For all $p$ sufficiently large, $(i, j) \approx (l, m)$ if and only if $\bar{b}_i j \equiv \bar{b}_l m$ mod $p$, and so if we let $\mathcal{E}(A)$ denote the set of all nonempty subsets of even cardinality of a finite set $A$, then

for all $p$ sufficiently large, $E(p)$ consists of all subsets $T$ of $T$ such that there exists a nonempty subset $S$ of equivalence classes of $\approx$ and elements $E_S \in \mathcal{E}(S)$ for $S \in S$ such that

(16) $T = \bigcup_{S \in S} E_S$.

In particular, it follows that for all $p$ large enough, $E(p)$ does not depend on $p$, hence from now on, we delete the “$p$” from the notation for this set.

The description of $E$ given by (16) mandates that we determine the equivalence classes of the equivalence relation $\approx$. In order to do that in a precise and concise manner, it will be convenient to use the following notation: if $b \in [1, \infty)$ and $S \subseteq [0, \infty)$, we let $b^{-1} S$ denote the set of all rational numbers of the form $z/b$, where $z$ is an element of $S$. We next let

$$K = \left\{ \emptyset \neq K \subseteq [1, k]: \bigcap_{i \in K} b_i^{-1} S_i \neq \emptyset \right\}.$$

If $K \in K$ then we set

$$T(K) = \left( \bigcap_{i \in K} b_i^{-1} S_i \right) \cap \left( \bigcap_{i \in [1, k] \setminus K} (Q \setminus b_i^{-1} S_i) \right).$$

Let

$$K_{\text{max}} = \{ K \in K : T(K) \neq \emptyset \}.$$

Using the theory of linear Diophantine equations, it is then straightforward to verify that the equivalence classes of $\approx$ consist precisely of all sets of the form

$$\{(i, tb_i) : i \in K\},$$

where $K \in \mathcal{K}_{\text{max}}$ and $t \in T(K)$.

Observe next that if the set

$$\left\{\{(i, tb_i) : i \in K\} : K \in \mathcal{K}, t \in \bigcap_{i \in K} b_i^{-1}S_i\right\}$$

is ordered by inclusion then the equivalence classes of $\approx$ are the maximal elements of this set. Hence $T(K) \cap T(K') = \emptyset$ whenever $\{K, K'\} \subseteq \mathcal{K}_{\text{max}}$. Consequently, if $(K, K') \in \mathcal{K}_{\text{max}} \times \mathcal{K}_{\text{max}}, \emptyset \neq \sigma \subseteq K, \emptyset \neq \sigma' \subseteq K', t \in T(K)$, and $t' \in T(K')$, then $\{(i, tb_i) : i \in \sigma\}$ and $\{(i, t'b_i) : i \in \sigma'\}$ are each contained in distinct equivalence classes of $\approx$ if and only if $t \neq t'$. The following lemma is now an immediate consequence of (16) and the structure just obtained for the equivalence classes of $\approx$.

**Lemma 8.5.** If $T \in \mathcal{E}$ then there exists a nonempty subset $\mathcal{S}$ of $\mathcal{K}_{\text{max}}$, a nonempty subset $\Sigma(S)$ of $\mathcal{E}(S)$ for each $S \in \mathcal{S}$ and a nonempty subset $T(\sigma, S)$ of $T(S)$ for each $\sigma \in \Sigma(S)$ and $S \in \mathcal{S}$ such that

the family of sets $\left\{T(\sigma, S) : \sigma \in \Sigma(S), S \in \mathcal{S}\right\}$ is pairwise disjoint, and

$$T = \bigcup_{S \in \mathcal{S}} \left[ \bigcup_{\sigma \in \Sigma(S)} \left( \bigcup_{t \in T(\sigma, S)} \{(i, tb_i) : i \in \sigma\} \right) \right].$$

We have now determined via Lemma 8.5 the structure of the elements of $\mathcal{E}$ precisely enough for effective use in the calculation of $\Sigma_4(p)$. However, if we already know that $q_\varepsilon(p) = 0$, the value of $\Sigma_4(p)$ is obviated in our argument. It would hence be very useful to have a way to mediate between the primes $p$ for which $q_\varepsilon(p) = 0$ and the primes $p$ for which $q_\varepsilon(p) \neq 0$. We will now define and study a gadget which does that.

Denote by $\Lambda(\mathcal{K})$ the set

$$\bigcup_{\mathcal{K} \in \mathcal{K}_{\text{max}}} \mathcal{E}(\mathcal{K}).$$

Then $\Lambda(\mathcal{K})$ is empty if and only if every element of $\mathcal{K}_{\text{max}}$ is a singleton.

Suppose that $\Lambda(\mathcal{K})$ is not empty. We will say that $p$ is an \emph{allowable prime} if no element of $B$ has $p$ as a factor. If $p$ is an allowable prime, then the $(B, S)$-\emph{signature of $p$} is defined to be the multi-set of $\pm 1$’s given by

$$\left\{\chi_p\left(\prod_{i \in I} b_i\right) : I \in \Lambda(\mathcal{K})\right\}.$$
We declare the signature of \( p \) to be \textit{positive} if all of its elements are 1, and \textit{non-positive} otherwise. Let

\[
\Pi_+(B, S) \quad \text{(respectively,} \quad \Pi_-(B, S)\text{)}
\]

denote the set of all allowable primes \( p \) such that the \((B, S)\)-signature of \( p \) is positive (respectively, non-positive).

We can now prove the following two lemmas: the first records some important information about the signature, and the second implies that we need only calculate \( \Sigma_4(p) \) for the primes \( p \) in \( \Pi_+(B, S) \).

**Lemma 8.6.** (i) The set \( \Pi_+(B, S) \) consists precisely of all allowable primes \( p \) for which each of the sets

\[
\{b_i : i \in I\}, \; I \in \Lambda(K),
\]

is either a set of residues of \( p \) or a set of non-residues of \( p \). In particular, \( \Pi_+(B, S) \) is always an infinite set.

(ii) The set \( \Pi_-(B, S) \) consists precisely of all allowable primes \( p \) for which at least one of the sets \((\sharp)\) contains a residue of \( p \) and a non-residue of \( p \), \( \Pi_-(B, S) \) is always either empty or infinite, and \( \Pi_-(B, S) \) is empty if and only if for all \( I \in \Lambda(K) \), \( \prod_{i \in I} b_i \) is a square.

**Proof.** Suppose that \( p \) is an allowable prime such that each of the sets \((\sharp)\) is either a set of residues of \( p \) or a set of non-residues of \( p \). Then

\[
\chi_p\left(\prod_{i \in I} b_i\right) = 1
\]

whenever \( I \in \Lambda(K) \) because \(|I|\) is even, i.e., \( p \in \Pi_+(B, S) \). On the other hand, let \( p \in \Pi_+(B, S) \) and let \( I = \{i_1, \ldots, i_n\} \in \Lambda(K) \). Then because \( p \in \Pi_+(B, S) \),

\[
\chi_p(b_{ij}b_{ij+1}) = 1, \; j \in [1, n - 1],
\]

and these equations imply that \( \{b_i : i \in I\} \) is either a set of residues of \( p \) or a set of non-residues of \( p \). This verifies the first statement in (i), and the second statement follows from the fact (Theorem 4.2) that there are infinitely many primes \( p \) such that \( B \) is a set of residues of \( p \).

Statement (ii) of the lemma follows from (i), the definition of \( \Pi_-(B, S) \), and the fact (Theorem 4.1) that a positive integer is a residue of all but finitely many primes if and only if it is a square.

QED

**Lemma 8.7.** If \( p \in \Pi_-(B, S) \) then \( q_e(p) = 0 \).
8. QUADRATIC RESIDUES AND NON-RESIDUES IN ARITHMETIC PROGRESSION

**Proof.** If \( p \in \Pi_-(B,S) \) then there is an \( I \in \Lambda(\mathcal{K}) \) such that

\[
\chi_p \left( \prod_{i \in I} b_i \right) = -1.
\]

Because \( I \) is nonempty and of even cardinality, there exists \( \{m,n\} \subseteq I \) such that

\[
(17) \quad \chi_p(b_mb_n) = -1.
\]

Because \( \{m,n\} \) is contained in an element of \( \mathcal{K}_{\text{max}} \), it follows that \( b_m^{-1}S_m \cap b_n^{-1}S_n \neq \emptyset \), and so we find a non-negative rational number \( r \) such that

\[
(18) \quad rb_m \in S_m \text{ and } rb_n \in S_n.
\]

By way of contradiction, suppose that \( q_\varepsilon(p) \neq 0 \). Then there exists a \( z \in [1,\infty) \) such that \( b_mz + S_m \) and \( b_nz + S_n \) are both contained in \( [1,p-1] \) and

\[
(19) \quad \chi_p(b_mz + u) = \chi_p(b_nz + v), \text{ for all } u \in S_m \text{ and for all } v \in S_n.
\]

If \( d \) is the greatest common divisor of \( b_m \) and \( b_n \) then there is a non-negative integer \( t \) such that \( r = t/d \). Hence by (18) and (19),

\[
\chi_p(b_m/d)\chi_p(dz + t) = \chi_p(b_mz + rb_m) = \chi_p(b_nz + rb_n) = \chi_p(b_n/d)\chi_p(dz + t).
\]

However, \( dz + t \in [1,p-1] \) and so \( \chi_p(dz + t) \neq 0 \). Hence

\[
\chi_p(b_m/d) = \chi_p(b_n/d),
\]

and this value of \( \chi_p \), as well as \( \chi_p(d) \), is nonzero because \( d, b_m/d, \) and \( b_n/d \) are all elements of \( [1,p-1] \). But then

\[
\chi_p(b_mb_n) = \chi_p(d^2)\chi_p(b_m/d)\chi_p(b_n/d) = 1,
\]

contrary to (17). QED

With Lemmas 8.5 and 8.7 in hand, we now calculate the sum \( \Sigma_4(p) \) that arose in (15). By virtue of Lemma 8.7, we need only calculate \( \Sigma_4(p) \) for \( p \in \Pi_+(B,S) \), hence let \( p \) be an allowable prime for which

\[
(20) \quad \chi_p \left( \prod_{i \in I} b_i \right) = 1, \text{ for all } I \in \Lambda(\mathcal{K}).
\]

We first recall that

\[
(21) \quad \Sigma_4(p) = 2^{-a_T}(p) \left( 1 + \sum_{T \in E} \prod_{i=1}^k \chi_p(b_i)^{|\{j:(i,j) \in T\}|} \right),
\]
and so we must evaluate the products over \( T \in E \) which determine the summands of the third factor on the right-hand side of (21). Toward that end, let \( T \in E \) and use Lemma 8.5 to find a nonempty subset \( S \) of \( \mathcal{K}_{\max} \), a nonempty subset \( \Sigma(S) \) of \( \mathcal{E}(S) \) for each \( S \in \mathcal{S} \) and a nonempty subset \( T(\sigma, S) \) of \( T(S) \) for each \( \sigma \in \Sigma(S) \) and \( S \in \mathcal{S} \) such that

the sets \( T(\sigma, S), \sigma \in \Sigma(S), S \in \mathcal{S}, \) are pairwise disjoint, and

\[
T = \bigcup_{S \in \mathcal{S}} \left[ \bigcup_{\sigma \in \Sigma(S)} \left( \bigcup_{t \in T(\sigma, S)} \{(n, tb_n) : n \in \sigma\} \right) \right].
\]

Then

\[
\{ j : (i, j) \in T \} = \bigcup_{S \in \mathcal{S}} \left( \bigcup_{\sigma \in \Sigma(S), i \in \sigma} \{tb_i : t \in T(\sigma, S)\} \right)
\]

and this union is pairwise disjoint. Hence

\[
|\{ j : (i, j) \in T \}| = \sum_{S \in \mathcal{S}} \sum_{\sigma \in \Sigma(S), i \in \sigma} |T(\sigma, S)|.
\]

Thus from this equation and (20) we find that

\[
\prod_{i=1}^{k} \chi_p(b_i)^{|\{ j : (i, j) \in T \}|} = \prod_{i \in \cup_{S \in \mathcal{S}} \cup_{\sigma \in \Sigma(S)} \sigma} \chi_p(b_i)^{\sum_{S \in \mathcal{S}} \sum_{\sigma \in \Sigma(S), i \in \sigma} |T(\sigma, S)|}
\]

\[
= \prod_{S \in \mathcal{S}} \left( \prod_{\sigma \in \Sigma(S)} \left( \chi_p \left( \prod_{i \in \sigma} b_i \right) \right)^{|T(\sigma, S)|} \right)
\]

\[
= 1.
\]

Hence

\[
(22) \quad \sum_{T \in E} \prod_{i=1}^{k} \chi_p(b_i)^{|\{ j : (i, j) \in T \}|} = |E|,
\]

and so we must count the elements of \( E \). In order to do that, note first that the pairwise disjoint decomposition (16) of an element \( T \) of \( E \) is uniquely determined by \( T \), and, obviously, uniquely determines \( T \). Hence if \( \mathcal{D} \) denotes the set of all equivalence classes of \( \approx \) of cardinality at least 2 then

\[
|E| = \sum_{\emptyset \neq S \subseteq \mathcal{D}} \prod_{S \in \mathcal{S}} |\mathcal{E}(S)|
\]

\[
= -1 + \prod_{D \in \mathcal{D}} (1 + |\mathcal{E}(D)|)
\]

\[
= -1 + \prod_{D \in \mathcal{D}} 2^{|D|-1}
\]

\[
= -1 + 2^{-|\mathcal{D}|} \cdot 2\sum_{D \in \mathcal{D}} |D|.
\]
However, $\mathcal{D}$ consists of all sets of the form
\[
\{(i, tb_i) : i \in K\}
\]
where $K \in \mathcal{K}_{\text{max}}, |K| \geq 2,$ and $t \in T(K)$. Hence
\[
|\mathcal{D}| = \sum_{K \in \mathcal{K}_{\text{max}}: |K| \geq 2} |T(K)|,
\]
\[
\sum_{D \in \mathcal{D}} |D| = \sum_{K \in \mathcal{K}_{\text{max}}: |K| \geq 2} |K||T(K)|,
\]
and so if we set
\[
e = \sum_{K \in \mathcal{K}_{\text{max}}} |T(K)|(|K| - 1),
\]
then
\[\tag{23} |E| = 2^e - 1.\]

Equations (21), (22), and (23) now imply

**Lemma 8.8.** If
\[
\alpha = \sum_i |S_i|, \quad e = \sum_{K \in \mathcal{K}_{\text{max}}} |T(K)|(|K| - 1), \quad \text{and} \quad r(p) = \min_i \left[ \frac{p - 1 - \max_i S_i}{b_i} \right],
\]
then
\[
\Sigma_4(p) = 2^{e-\alpha} r(p), \quad \text{for all} \quad p \in \Pi_+(B, S).
\]

All of the ingredients are now assembled for a proof of the following theorem, which determines the asymptotic behavior of $q_\varepsilon(p)$.

**Theorem 8.9.** *(Wright [45], Theorem 6.1)* Let $\varepsilon \in \{-1, 1\}, k \in [1, \infty)$, and let $B = \{b_1, \ldots, b_k\}$ be a set of positive integers and $S = (S_1, \ldots, S_k)$ a $k$-tuple of finite, nonempty subsets of $[0, \infty)$. If $\mathcal{K}_{\text{max}}$ is the set of subsets of $[1, k]$ defined by $B$ and $S$ as on p. 129, let
\[
\Lambda(\mathcal{K}) = \bigcup_{K \in \mathcal{K}_{\text{max}}} \mathcal{E}(K),
\]
\[
\alpha = \sum_i |S_i|, \quad b = \max_i \{b_i\}, \quad e = \sum_{K \in \mathcal{K}_{\text{max}}} |T(K)|(|K| - 1), \quad \text{and} \quad q_\varepsilon(p) = |\{A \in AP(B, S) \cap 2^{[1, p-1]} : \chi_p(a) = \varepsilon, \text{ for all } a \in A\}|.
\]
(i) If the sets $b_1^{-1}S_1, \ldots, b_k^{-1}S_k$ are pairwise disjoint then
\[
q_\varepsilon(p) \sim (b \cdot 2^\alpha)^{-1} p \quad \text{as} \quad p \to +\infty.
\]
(ii) If the sets $b_1^{-1}S_1, \ldots, b_k^{-1}S_k$ are not pairwise disjoint then


(a) the parameter \( e \) is positive and less than \( \alpha \);

(b) if \( \prod_{i \in I} b_i \) is a square for all \( I \in \Lambda(K) \) then
\[
q_\varepsilon(p) \sim (b \cdot 2^{\alpha-e})^{-1}p \text{ as } p \to +\infty;
\]

(c) if there exists \( I \in \Lambda(K) \) such that \( \prod_{i \in I} b_i \) is not a square then

\( \alpha \) the set \( \Pi_+(B,S) \) of primes with positive \((B,S)\)-signature and the set \( \Pi_-(B,S) \) of primes with non-positive \((B,S)\)-signature are both infinite,

(\( \beta \)) \( q_\varepsilon(p) = 0 \) for all \( p \) in \( \Pi_-(B,S) \), and

(\( \gamma \)) as \( p \to +\infty \) inside \( \Pi_+(B,S) \),
\[
q_\varepsilon(p) \sim (b \cdot 2^{\alpha-e})^{-1}p.
\]

Proof. If the sets \( b_1^{-1}S_1, \ldots, b_k^{-1}S_k \) are pairwise disjoint then every element of \( K_{\max} \) is a singleton set, hence all of the equivalence classes of the equivalence relation \( \approx \) defined above on \( \bigcup_{i=1}^k \{(i,j) : j \in S_i\} \) by the set \( B \) are singletons. It follows that the set \( E \) which is summed over in (21) is empty and so
\[
\Sigma_4(p) = 2^{-\alpha}r(p), \text{ for all } p \text{ sufficiently large.}
\]

Upon recalling that
\[
r(p) = \min_i \left[ \frac{p-1 - \max S_i}{b_i} \right],
\]
the conclusion of \((i)\) is an immediate consequence of (15) and (24).

Suppose that the sets \( b_1^{-1}S_1, \ldots, b_k^{-1}S_k \) are not pairwise disjoint. Then \( \Lambda(K) \) is not empty and so conclusion \((a)\) is an obvious consequence of the definition of \( e \). If \( \prod_{i \in I} b_i \) is a square for all \( I \in \Lambda(K) \) then it follows from its definition that \( \Pi_+(B,S) \) contains all but finitely many primes, and so \((b)\) is an immediate consequence of (15) and Lemma 8.8. On the other hand, if there exists \( I \in \Lambda(K) \) such that \( \prod_{i \in I} b_i \) is not a square then \((a)\) follows from Lemma 8.6, \((\beta)\) follows from Lemma 8.7, and \((\gamma)\) is an immediate consequence of (15) and Lemma 8.8.

QED

Theorem 8.9 shows that the elements of \( \Lambda(K) \) contribute to the formation of quadratic residues and non-residues inside \( AP(B,S) \). If no such elements exist then \( q_\varepsilon(p) \) has the expected minimal asymptotic approximation \((b \cdot 2^{\alpha})^{-1}p\) as \( p \to +\infty \). In the presence of elements of \( \Lambda(K) \), the parameter \( e \) is positive and less than \( \alpha \), the asymptotic size of \( q_\varepsilon(p) \) is increased by a factor of \( 2^e \), and whenever \( \Pi_-(B,S) \) is empty, \( q_\varepsilon(p) \) is asymptotic to \((b \cdot 2^{\alpha-e})^{-1}p\) as \( p \to +\infty \). However, the most interesting behavior occurs when \( \Pi_-(B,S) \) is not empty; in that case, as \( p \to +\infty \), \( q_\varepsilon(p) \) asymptotically oscillates infinitely often between 0 and \((b \cdot 2^{\alpha-e})^{-1}p\).
Remark. If we observe that the cardinality of the set
\[ \bigcup_{i=1}^{k} b_i^{-1}S_i \]
is equal to the number of equivalence classes of the equivalence relation \( \approx \) that was defined on the set
\[ T = \bigcup_{i=1}^{k} \{(i, j) : j \in S_i \}, \]
then it follows that
\[ | \bigcup_{i=1}^{k} b_i^{-1}S_i | = \sum_{K \in K_{\text{max}}} |T(K)|. \]
But we also have that
\[ \alpha = |T| = \sum_{K \in K_{\text{max}}} |T(K)||K|. \]
Consequently, the exponents in the power of 1/2 that occur in the asymptotic approximation to \( q_\varepsilon (p) \) in Theorem 8.9 are in fact all equal to the cardinality of \( \bigcup_{i=1}^{k} b_i^{-1}S_i \).

Theorem 8.9 will now be applied to the situation of primary interest to us here, namely to the family of sets \( AP(a, b; s) \) determined by a standard 2\( m \)-tuple \((a, b)\). In this case, the decomposition (11) of the sets in \( AP(a, b; s) \) shows that there is a set \( B = \{b_1, \ldots, b_k\} \) of positive integers (the set of distinct values of the coordinates of \( b \)), a \( k \)-tuple \((m_1, \ldots, m_k)\) of positive integers such that \( m = \sum_i m_i \), and sets
\[ A_i = \{a_{i1}, \ldots, a_{im_i}\} \]
of non-negative integers, all uniquely determined by \((a, b)\), such that if we let
\[ S_i = \bigcup_{j=1}^{m_i} \{a_{ij} + b_it : t \in [0, s - 1]\}, \quad i \in [1, k], \]
and set
\[ S = (S_1, \ldots, S_k) \]
then
\[ AP(a, b; s) = AP(B, S). \]
It follows that
\[ b_i^{-1}S_i = \bigcup_{q \in b_i^{-1}A_i} \{q + j : j \in [0, s - 1]\}, \quad i \in [1, k]. \]
These sets then determine the subsets of \([1, k]\) that constitute
\[ K = \{\emptyset \neq K \subseteq [1, k] : \bigcap_{i \in K} b_i^{-1}S_i \neq \emptyset\}. \]
and hence also the elements of $K_{\text{max}}$, according to the recipe given on p. 126. The sets in $K_{\text{max}}$, together with the parameters

$$\alpha = \sum_{i} |S_i|, \quad b = \max \{b_i\}, \quad c = \sum_{K \in K_{\text{max}}} |T(K)|(\|K\| - 1),$$

when used as specified in Theorem 8.9, then determine precisely the asymptotic behavior of the sequence $q_\varepsilon(p)$ that is defined upon replacement of $AP(B, S)$ by $AP(a, b; s)$ in the statement of Theorem 8.9, thereby solving Problems 2 and 4. In particular, the sets $b_{i}^{-1}S_1, \ldots, b_{k}^{-1}S_k$ are pairwise disjoint if and only if

(26) if $(i, j) \in [1, k] \times [1, k]$ with $i \neq j$ and $(x, y) \in A_i \times A_j$, then either $b_i b_j$ does not divide $y b_i - x b_j$ or $b_i b_j$ divides $y b_i - x b_j$ with a quotient that exceeds $s - 1$ in modulus.

Hence the conclusion of statement $(i)$ of Theorem 8.9 holds for $AP(a, b; s)$ when condition (26) is satisfied, while the conclusions of statement $(ii)$ of Theorem 8.9 hold for $AP(a, b; s)$ whenever condition (26) is not satisfied. In the section below we will show, among other things, that for each integer $m \in [2, \infty)$ and for each of the hypotheses in the statement of Theorem 8.9, there exists infinitely many standard $2m$-tuples $(a, b)$ which satisfy that hypothesis.

**An interesting class of examples.**

In order to apply Theorem 8.9 to a standard $2m$-tuple $(a, b)$, we need to calculate the parameters $\alpha$ and $c$, the set $\Lambda(K)$, and the associated signatures of the allowable primes. In general, this can be somewhat complicated, but there is a class of standard $2m$-tuples for which these computations can be carried out by means of easily applied algebraic and geometric formulae, which we will discuss next.

Let $k \in [2, \infty)$. We will say that a standard $2k$-tuple $(a, b)$ of integers is *admissible* if it satisfies the following two conditions:

(27) the coordinates of $b$ are distinct, and,

(28) $a_i b_j - a_j b_i \neq 0$ for $i \neq j$.

If $s \in [1, \infty)$ and $(a, b)$ is admissible then it follows trivially from (27) that

$$S_i = \{a_i + b_j : j \in [0, s - 1]\}, \quad i \in [1, k],$$

hence

$$|S_i| = s, \quad i \in [1, k],$$

and so the parameter $\alpha$ in the statement of Theorem 8.9 for $AP(a, b; s)$ is $ks$.
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We turn next to the calculation of the parameter $e$. Let $q_i = a_i/b_i, i \in [1, k]$; (28) ⇒ the $q_i$’s are distinct, and without loss of generality, we suppose that the coordinates of $a$ and $b$ are indexed so that $q_i < q_{i+1}$ for each $i \in [1, k-1]$. Let $\mathcal{R}$ denote the set of all subsets $R$ of $\{q_1, \ldots, q_k\}$ such that $|R| \geq 2$ and $R$ is maximal relative to the property that $w - z$ is an integer for all $(w, z) \in R \times R$. We note that $\mathcal{R}$ is just the set of all equivalence classes of cardinality at least 2 of the equivalence relation $\sim$ defined on the set $\{q_1, \ldots, q_k\}$ by declaring that $q_i \sim q_j$ if $q_i - q_j \in \mathbb{Z}$. After linearly ordering the elements of each $R \in \mathcal{R}$, we let $D(R)$ denote the $(|R| - 1)$-tuple of positive integers whose coordinates are the distances between consecutive elements of $R$. Then if $M_R(s)$ denotes the multi-set formed by the coordinates of $D(R)$ which do not exceed $s - 1$, it can be shown that

$$e = \sum_{R \in \mathcal{R}} \sum_{r \in M_R(s)} (s - r)$$

(29)

(see Wright [45], section 8). We note in particular that $e = 0$ iff the set $\{R \in \mathcal{R} : M_R(s) \neq \emptyset\}$ is empty and that this occurs iff the sets $b_i^{-1}S_i, i \in [1, k]$, are pairwise disjoint. Formula (29) shows that $e$ can be calculated solely by means of information obtained directly and straightforwardly from the set $\{q_1, \ldots, q_k\}$.

In order to calculate the signature of allowable primes, the set $\Lambda(K)$ must be computed. There is an elegant geometric formula for this computation that is based on the concept of what we will call an overlap diagram, and so those diagrams will be described first.

Let $(n, s) \in [1, \infty) \times [1, \infty)$ and let $\mathbf{g} = (g(1), \ldots, g(n))$ be an $n$-tuple of positive integers. We use $\mathbf{g}$ to construct the following array of points. In the plane, place $s$ points horizontally one unit apart, and label the $j$-th point as $(1, j-1)$ for each $j \in [1, s]$. This is row 1. Suppose that row $i$ has been defined. One unit vertically down and $g(i)$ units horizontally to the right of the first point in row $i$, place $s$ points horizontally one unit apart, and label the $j$-th point as $(i + 1, j - 1)$ for each $j \in [1, s]$. This is row $i + 1$. The array of points so formed by these $n + 1$ rows is called the overlap diagram of $\mathbf{g}$, the sequence $\mathbf{g}$ is called the gap sequence of the overlap diagram, and a nonempty set that is formed by the intersection of the diagram with a vertical line is called a column of the diagram. N.B. We do not distinguish between the different possible positions in the plane which the overlap diagram may occupy. A typical example with $n = 3, s = 8$, and gap sequence $(3, 2, 2)$ looks like

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We need to describe how and where rows overlap in an overlap diagram. Begin by first noticing that if \((g(1), \ldots, g(n))\) is the gap sequence, then row \(i\) overlaps row \(j\) for \(i < j\) if and only if
\[
\sum_{r=i}^{j-1} g(r) \leq s - 1;
\]
and in particular, row \(i\) overlaps row \(i + 1\) if and only if \(g(i) \leq s - 1\). Now let \(G\) denote the set of all subsets \(G\) of \([1, n]\) such that \(G\) is a nonempty set of consecutive integers maximal with respect to the property that \(g(i) \leq s - 1\) for all \(i \in G\). If \(G\) is empty then \(g(i) \geq s\) for all \(i \in [1, n]\), and so there is no overlap of rows in the diagram. Otherwise there exists \(m \in [1, 1 + [(n - 1)/2]]\) and strictly increasing sequences \((l_1, \ldots, l_m)\) and \((M_1, \ldots, M_m)\) of positive integers, uniquely determined by the gap sequence of the diagram, such that \(l_i \leq M_i\) for all \(i \in [1, m]\), \(1 + M_i \leq l_{i+1}\) if \(i \in [1, m - 1]\), and
\[
G = \{[l_i, M_i] : i \in [1, m]\}.
\]
In fact, \(l_{i+1} \leq 1 + M_i\) if \(i \in [1, m - 1]\), lest the maximality of the elements of \(G\) be violated. It follows that the intervals of integers \([l_i, 1 + M_i], i \in [1, m]\), are pairwise disjoint.

The set \(G\) can now be used to locate the overlap between rows in the overlap diagram like so: for \(i \in [1, m]\), let
\[
B_i = [l_i, 1 + M_i],
\]
and set
\[
B_i = \text{the set of all points in the overlap diagram whose labels are in } B_i \times [0, s - 1].
\]
We refer to \(B_i\) as the \(i\)-th block of the overlap diagram; thus the blocks of the diagram are precisely the regions in the diagram in which rows overlap.

We will now use the elements of \(R\) to construct a series of overlap diagrams. Let \(R\) be an element of \(R\) such that \(D(R)\) has at least one coordinate that does not exceed \(s - 1\). Next, consider the nonempty and pairwise disjoint family of all subsets \(V\) of \(R\) such that \(|V| \geq 2\) and \(V\) is maximal with respect to the property that the distance between consecutive elements of \(V\) does not exceed \(s - 1\). List the elements of \(V\) in increasing order and then for each \(i \in [1, |V| - 1]\) let \(q_V(i)\) denote the distance between the \(i\)-th element and the \((i + 1)\)-th element on that list. N.B. \(q_V(i) \in [1, \infty)\), for all \(i \in [1, |V| - 1]\). Finally, let \(D(V)\) denote the overlap diagram of the \(|V| - 1\)-tuple \((q_V(i) : i \in [1, |V| - 1])\). Because \(q_V(i) \leq s - 1\) for all \(i \in [1, |V| - 1]\), \(D(V)\) consists of a single block.

Using a suitable positive integer \(m\), we index all of the sets \(V\) that arise from all of the elements of \(R\) in the previous construction as \(V_1, \ldots, V_m\) and then define the \emph{quotient}
diagram of \((a, b)\) to be the \(m\)-tuple of overlap diagrams \((\mathcal{D}(V_n) : n \in [1, m])\). We will refer to the diagrams \(\mathcal{D}(V_n)\) as the blocks of the quotient diagram.

The quotient diagram \(\mathcal{D}\) of \((a, b)\) will now be used to calculate the set \(\Lambda(K)\) determined by \((a, b)\) and hence the associated signature of an allowable prime. In order to see how this goes, we will need to make use of a certain labeling of the points of \(\mathcal{D}\) which we describe next. Let \(V_1, \ldots, V_m\) be the subsets of \(\{q_1, \ldots, q_k\}\) that determine the sequence of overlap diagrams \(\mathcal{D}(V_1), \ldots, \mathcal{D}(V_m)\) which constitute \(\mathcal{D}\), and then find the subset \(J_n\) of \([1, k]\) such that \(V_n = \{q_j : j \in J_n\}\), with \(j \in J_n\) listed in increasing order (note that this ordering of \(J_n\) also linearly orders \(q_j, j \in J_n\)). The overlap diagram \(\mathcal{D}(V_n)\) consists of \(|J_n|\) rows, with each row containing \(s\) points. If \(i \in [1, |J_n|]\) is taken in increasing order then there is a unique element \(j\) of \(J_n\) such that the \(i\)-th element of \(V_n\) is \(q_j\). Proceeding from left to right in each row, we now take \(l \in [1, s]\) and label the \(l\)-th point of row \(i\) in \(\mathcal{D}(V_n)\) as \((j, l - 1)\). N.B. This labeling of the points of \(\mathcal{D}(V_n)\) does not necessarily coincide with the labeling of the points of an overlap diagram that was used before to define the blocks of the diagram.

Next let \(C\) denote a column of one of the diagrams \(\mathcal{D}(V_n)\) which constitute \(\mathcal{D}\). We identify \(C\) with the subset of \([1, k] \times [0, s - 1]\) defined by
\[
\{(i, j) \in [1, k] \times [0, s - 1] : (i, j) \text{ is the label of a point in } C\},
\]
let \(C_n\) denote the set of all subsets of \([1, k] \times [0, s - 1]\) which arise from all such identifications, and then set \(C = \bigcup_n C_n\). If \(\theta\) denotes the projection of \([1, k] \times [0, s - 1]\) onto \([1, k]\) then one can show (Wright [46], Lemma 2.5) that \(K \in K_{\text{max}}\) if and only if there exists a \(T \in C\) such that \(K = \theta(T)\), and so
\[
\Lambda(K) = \bigcup_{T \in C} \mathcal{E}(\theta(T)).
\]
When this formula for \(\Lambda(K)\) is now combined with (29), it follows that all of the data required for an application of Theorem 8.9 can be easily read off directly from the set \(\{q_1, \ldots, q_k\}\) and the quotient diagram of \((a, b)\).

At this juncture, some concrete examples which illustrate the mathematical technology that we have introduced are in order. But before we get to those, recall that if \((a, b)\) is an admissible \(2k\)-tuple, \(B\) is the set formed by the coordinates \(b_1, \ldots, b_k\) of \(b\), \(S_i = \{a_i + b_j : j \in [0, s - 1]\}\), \(a_i\) the \(i\)-th coordinate of \(a\), \(i \in [1, k]\), and \(S\) is the \(k\)-tuple of sets \((S_1, \ldots, S_k)\), then the pair \((B, S)\) determines by way of Theorem 8.9 the asymptotic behavior of \(|\{A \in AP(a, b; s) \cap 2^{[1, p - 1]} : \chi_p(a) = \varepsilon, \text{ for all } a \in A\}|\), \(\varepsilon \in \{-1, 1\}\). Hence for this pair, we use the more specific notation \(\Pi_{\pm}(a, b)\) for the sets \(\Pi_{\pm}(B, S)\) in the statement of Theorem 8.9.

Now for the examples. Let \(m \in [1, +\infty)\) and for each \(n \in [1, m]\), let \(\mathcal{D}(n)\) be a fixed but arbitrary overlap diagram with \(k_n\) rows, \(k_n \geq 2\), and gap sequence \((d(i, n) : i \in [1, k_n - 1])\),
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with no gap exceeding \( s - 1 \). Let \( k_0 = 0, k = \sum_n k_n \). We will now exhibit infinitely many admissible \( 2k \)-tuples \((a, b)\) whose quotient diagram is \( \Delta = (D(n) : n \in [1, m]) \). This is done by taking the \((k - 1)\)-tuple \((d_1, \ldots, d_{k-1})\) in the following lemma to be

\[
d_i = \begin{cases} 
  d \left( i - \sum_0^n k_j, n + 1 \right), & \text{if } n \in [0, m - 1] \text{ and } i \in \left[ 1 + \sum_0^n k_j, -1 + \sum_0^{n+1} k_j \right], \\
  s, & \text{elsewhere,}
\end{cases}
\]

and then letting \((a, b)\) be any \( 2k \)-tuple obtained from the construction in the lemma.

**Lemma 8.10.** For \( k \in [2, \infty) \), let \((d_1, \ldots, d_{k-1})\) be a \((k - 1)\)-tuple of positive integers. Define \( k \)-tuples \((a_1, \ldots, a_k)\), \((b_1, \ldots, b_k)\) of positive integers inductively as follows: let \((a_1, b_1)\) be arbitrary, and if \( i > 1 \) and \((a_i, b_i)\) has been defined, choose \( t_i \in [2, \infty) \) and set

\[
a_{i+1} = t_i(a_i + d_i b_i), \quad b_{i+1} = t_i b_i.
\]

Then

\[
a_i - a_j = \sum_{r=j}^{i-1} d_r, \quad \text{for all } i > j.
\]

**Proof.** This is a straightforward calculation using the recursive definition of the \( k \)-tuples \((a_1, \ldots, a_k)\) and \((b_1, \ldots, b_k)\). QED

We can use Lemma 8.10 to also find infinitely many admissible \( 2k \)-tuples \((a, b)\) with quotient diagram \( \Delta \) and such that the set \( \Pi_-(a, b) \) is empty. To do this, simply choose the integer \( b_1 \) and all subsequent \( t_i \)'s used in the above construction from Lemma 8.10 to be squares. This shows that there are infinitely many admissible \( 2k \)-tuples with a specified quotient diagram which satisfy the hypotheses of Theorem 8.9(ii)(b). On the other hand, if \( b_1 \) and all the subsequent \( t_i \)'s are instead chosen to be distinct primes, it follows that the \( 2k \)-tuples determined in this way all have quotient diagram \( \Delta \) and each have \( \Pi_-(a, b) \) of infinite cardinality, and so there are infinitely many admissible \( 2k \)-tuples with specified quotient diagram which satisfy the hypotheses of Theorem 8.9(ii)(c). We also note that if all of the coordinates of \((d_1, \ldots, d_{k-1})\) in Lemma 8.10 are chosen to exceed \( s - 1 \) then we obtain infinitely many admissible \( 2k \)-tuples which satisfy the hypothesis of Theorem 8.9(i).

With this cornucopia of examples in hand, for \( \varepsilon \in \{-1, 1\} \), we let \( q_\varepsilon(p) \) denote the cardinality of the set

\[
\{ A \in AP(a, b; s) \cap 2^{[1, p-1]} : \chi_p(a) = \varepsilon, \text{ for all } a \in A \},
\]

where \((a, b)\) is admissible. We will now use the quotient diagram of \((a, b)\), formulae (29), (31), and Theorem 8.9 to study how \((a, b)\) determines the asymptotic behavior of \( q_\varepsilon(p) \) in specific situations. We will illustrate how things work when \( k = 2 \) and \( 3 \), and for when “minimal” or “maximal” overlap is present in the quotient diagram of \((a, b)\).
When \( k = 2 \), there is only at most a single overlap of rows in the quotient diagram of \((a, b)\), and if, e.g., \( a_1b_2 - a_2b_1 = qb_1b_2 \) with \( 0 < q \leq s - 1 \), then the quotient diagram looks like

\[
\begin{array}{cccccccccc}
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\leftarrow & q & \rightarrow & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\end{array}
\]

where \( \alpha = 2s \) and, because of (29), \( e = s - q \). Formula (31) shows that the signature of \( p \) is \( \{\chi_p(b_1b_2)\} \), and so we conclude from Theorem 8.9 that when \( b_1b_2 \) is a square,

\[
q_\varepsilon(p) \sim (b \cdot 2^{s+q})^{-1}p, \quad \text{as } p \to +\infty,
\]

and when \( b_1b_2 \) is not a square, \( \Pi_+(a, b) \) is the set of all allowable primes \( p \) such that \( \{b_1, b_2\} \) is either a set of quadratic residues of \( p \) or a set of quadratic non-residues of \( p \), \( \Pi_-(a, b) \) is the set of all allowable primes \( p \) such that \( \{b_1, b_2\} \) contains a quadratic residue of \( p \) and a quadratic non-residue of \( p \),

\[
q_\varepsilon(p) = 0, \quad \text{for all } p \in \Pi_-(a, b),
\]

and as \( p \to +\infty \) inside \( \Pi_+(a, b) \),

\[
q_\varepsilon(p) \sim (b \cdot 2^{s+q})^{-1}p.
\]

When \( k = 3 \) there are exactly three types of overlap possible in the quotient diagram of \((a, b)\), determined, e.g., when either

(i) exactly one,

(ii) exactly two, or

(iii) exactly three

of \( b_1b_2, b_2b_3, \) and \( b_1b_3 \) divide, respectively, \( a_2b_1 - a_1b_2, a_3b_2 - a_2b_3, \) and \( a_3b_1 - a_1b_3 \) with positive quotients not exceeding \( s - 1 \).

In case (i), with \( a_2b_1 - a_1b_2 = qb_1b_2 \), say, the block in the quotient diagram of \((a, b)\) is formed by a single overlap between rows 1 and 2, and this block looks exactly like the overlap diagram that was displayed for \( k = 2 \) above. It follows that the conclusions from (29), (31), and Theorem 8.9 in case (i) read exactly like the conclusions in the \( k = 2 \) case described before, except that the exponent of the power of \( 1/2 \) in the coefficient of \( p \) in the asymptotic approximation is now \( 2s + q \) rather than \( s + q \).

In case (ii), with \( a_2b_1 - a_1b_2 = qb_1b_2 \) and \( a_3b_2 - a_2b_3 = rb_2b_3 \), say, the block in the quotient diagram is formed by an overlap between rows 1 and 2 and an overlap between rows 2 and 3, but no overlap between rows 1 and 3. Hence the diagram looks like
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\[ q \leftrightarrow r \rightarrow \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdOTS
Here \( \alpha = ks, \ e = k - 1 \), and the signature of \( p \) is \( \{ \chi_p(b_i b_{i+1}) : i \in [1, k - 1] \} \). Hence via Theorem 8.9, if \( b_i b_{i+1}, i \in [1, k - 1] \), are all squares then

\[
q_\varepsilon(p) \sim (b \cdot 2^{1+k(s-1)})^{-1}p \quad \text{as} \quad p \to +\infty,
\]

and if at least one of those products is not a square, then \( \prod_+ (a, b) \) consists of all allowable primes \( p \) such that \( \{b_1, \ldots, b_k\} \) contains a quadratic residue of \( p \) and a quadratic non-residue of \( p \),

(35) \[
q_\varepsilon(p) = 0, \quad \text{for all} \quad p \in \Pi_-(a, b), \quad \text{and}
\]

\[
q_\varepsilon(p) \sim (b \cdot 2^{1+k(s-1)})^{-1}p \quad \text{as} \quad p \to +\infty \quad \text{inside} \quad \Pi_+(a, b).
\]

Maximal overlap \( (k \geq 3) \). Here we take the quotient diagram to consist of a single block with gap sequence \( (1, 1, \ldots, 1) \) and \( k = s \), so that the overlap between each pair of rows is as large as possible: the diagrams for \( k = 3, 4, \) and 5 look like

\[
\begin{array}{cccccccc}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\end{array}
\]

We have in this case that \( \alpha = k^2, \ e = (k - 1)^2 \), and the signature of \( p \) is

\[
\left\{ \chi_p \left( \prod_{i \in I} b_i \right) : I \in \mathcal{E}([1, k]) \right\}.
\]

Hence if \( \prod_{i \in I} b_i \) is a square for all \( I \in \mathcal{E}([1, k]) \) then

\[
q_\varepsilon(p) \sim (b \cdot 2^{2k-1})^{-1}p \quad \text{as} \quad p \to +\infty,
\]

and if one of these products is not a square then \( \Pi_+(a, b) \) and \( \Pi_-(a, b) \) are determined by \( \{b_1, \ldots, b_k\} \) as before, (35) holds, and

\[
q_\varepsilon(p) \sim (b \cdot 2^{2k-1})^{-1}p \quad \text{as} \quad p \to +\infty \quad \text{inside} \quad \Pi_+(a, b).
\]
It follows from our discussion after the proof of Theorem 8.9 that an increase in the number of overlaps between rows in the quotient diagram of \((a, b)\) leads to an increase in the asymptotic number of elements of \(AP(a, b; s) \cap 2^{[1,p-1]}\) that are sets of residues or non-residues of \(p\), and these examples now verify that principle quantitatively. In order to make this explicit, note first that Lemma 8.10 can be used to generate examples in which the \((k - 1)\)-tuple \((d_1, \ldots, d_{k-1})\) varies arbitrarily, while at the same time \(b = \max\{b_1, \ldots, b_k\}\) always takes the same value. Hence we may assume in the discussion to follow that the value of \(b\) is constant in each set of examples, and so the only parameter that is relevant when comparing asymptotic approximations to \(q_\varepsilon(p)\) is the exponent of the power of \(1/2\) in the coefficient of that approximation. When \(k = 2\), there is either no overlap between rows or exactly 1 overlap; in the former case, the exponent in the power of \(1/2\) that occurs in the asymptotic approximation to \(q_\varepsilon(p)\) is \(2s\) and in the latter case this exponent is less than \(2s\). When \(k = 3\) there are 0, 1, 2, or 3 possible overlaps between rows, with the last three possibilities occurring, respectively, in cases \((i)\), \((ii)\), and \((iii)\) above. It follows that \(q < s\) in case \((i)\), \(q + r \geq s\) in case \((ii)\) and \(q + r < s\) in case \((iii)\). Hence the exponent in the power of \(1/2\) that occurs in the asymptotic approximation to \(q_\varepsilon(p)\) is \(3s\) when no overlap occurs, is greater than \(2s\) and less than \(3s\) in case \((i)\), is at least \(2s\) and less than \(3s\) in case \((ii)\), and is less than \(2s\) in case \((iii)\). If we also take \(k = s\) when there is minimal overlap in the quotient diagram and compare that to what happens when there is maximal overlap there, we see that the exponent in the power of \(1/2\) that occurs in the asymptotic approximation of \(q_\varepsilon(p)\) is quadratic in \(k\), i.e., \(k^2 - k + 1\), in the former case, but only linear in \(k\), i.e., \(2k - 1\), in the latter case.

Suppose that \((a, b)\) is a standard \(2k\)-tuple and assume that there exists an \(I \in \Lambda(K)\) such that \(\prod_{i \in I} b_i\) is not a square. Then, in accordance with Theorem 8.9, the sets \(\Pi_+(a, b)\) and \(\Pi_-(a, b)\) are both infinite, and so it is of interest to calculate their density. Because \(\Pi_+(a, b)\) and \(\Pi_-(a, b)\) are disjoint sets with only finitely many primes outside of their union, it follows that

\[
\text{the density of } \Pi_+(a, b) + \text{ the density of } \Pi_-(a, b) = 1,
\]

so it suffices to calculate only the density of \(\Pi_+(a, b)\).

In order to keep the technicalities from becoming too complicated, we will describe this calculation for the following special case: assume that \((a, b)\) is admissible, the square-free parts \(\sigma_i = \sigma(b_i)\) of the coordinates \(b_i\) of \(b\) are distinct and for each nonempty subset of \(T\) of \([1, k]\), \(\prod_{i \in T} \sigma_i\) is not a square.
This condition is satisfied, for example, if

\[ b_i \text{ is square-free for all } i \text{ and } \pi(b_i) \text{ is a proper subset of } \pi(b_{i+1}), \text{ for all } i \in [1, k - 1]. \]

Moreover for each \( k \in [2, \infty) \), Lemma 8.9 can be used to construct infinitely many admissible \( 2k \)-tuples with a specified fixed but arbitrary quotient diagram which satisfy (37).

Let \((D(V_1), \ldots, D(V_m))\) be the quotient diagram of \((a, b)\) and let \( D_i \) be the subset of \([1, k]\) such that \( V_i = \{q_j : j \in D_i\}, i \in [1, m]\); as the sets \( V_1, \ldots, V_m \) are pairwise disjoint, so also are the sets \( D_1, \ldots, D_m \).

Now, let \( C_i \) denote the set of columns of the overlap diagram \( D(V_i) \), realized as subsets of \([1, k] \times [0, s - 1]\) as per the identification given by (30), and let

\[ \Lambda_i(K) = \bigcup_{C \in C_i} \mathcal{E}(\theta(C)). \]

Then

\[ \bigcup_{I \in \Lambda_i(K)} I = \bigcup_{C \in C_i} \theta(C) = D_i, \ i \in [1, m], \]

and so it follows from the pairwise disjointness of the \( D_i \)'s, these equations, and (31) that

\[ \Lambda(K) = \bigcup_i \Lambda_i(K), \text{ and this union is pairwise disjoint.} \]

Next, for each \( I \in \Lambda(K) \) let

\[ S(I) = \{\sigma_i : i \in I\}, \]

and then set

\[ \mathcal{M}_1 = \{I \in \Lambda(K) : 1 \in S(I)\}. \]

If \( \mathcal{M}_1 \neq \emptyset \) then there is a unique element \( n_0 \) of \( \bigcup_i D_i \) such that \( \sigma_{n_0} = 1 \), hence it follows from (38) and (39) that there is a unique element \( i_0 \) of \([1, m]\) such that

\[ \mathcal{M}_1 = \{I \in \Lambda_{i_0}(K) : n_0 \in I\}. \]

It can then be shown that if

\[ \sigma = \sum_i |D_i| \quad \text{and} \]

\[ m = \text{the number of blocks in the quotient diagram of } (a, b), \]

then the density of \( \Pi_+ (a, b) \) is

\[ 2^{m-\sigma}, \text{ if } \mathcal{M}_1 = \emptyset \text{ or } \mathcal{M}_1 = \Lambda_{i_0}(K), \text{ or} \]

\[ 2^{1-\sigma} (2^m - 1), \text{ if } \emptyset \neq \mathcal{M}_1 \neq \Lambda_{i_0}(K). \]
It follows that whenever \((a, b)\) is an admissible \(2k\)-tuple for which the square-free parts of the coordinates of \(b\) are distinct and satisfy condition (36), the cardinality of \(\bigcup_i D_i\), the number of blocks \(m\) in the quotient diagram, and the set \(\mathcal{M}_1\) completely determine the density of \(\Pi_+(a, b)\) by means of formulae (40) and (41). Those formulae show that each element of \(\bigcup_i D_i\) contributes a factor of 1/2 to the density of \(\Pi_+(a, b)\) and each block of the quotient diagram of \((a, b)\) contributes essentially a factor of 2 to the density. Because \(|V_i| \geq 2\) for all \(i\), it follows that \(|D_i| \geq 2\) for all \(i\) and so \(\sigma \geq 2m\); in particular, the density of \(\Pi_+(a, b)\) is at most \(2^{-m}\) whenever \(\mathcal{M}_1 = \emptyset\) or \(\mathcal{M}_1 = \Lambda_{io}(\mathcal{K})\) and is at most \((2^m - 1)/2^{2m-1}\), otherwise. This gives an interesting number-theoretic interpretation to the number of blocks in the quotient diagram. In fact, if for each \(k \in [2, \infty)\), we let \(A_k\) denote the set of all admissible \(2k\)-tuples which satisfy condition (36), and then set \(A = \bigcup_{k \in [2, \infty)} A_k\) and take \(m \in [1, \infty)\), then Lemma 8.10 can be used to show that there exists infinitely many elements \((a, b)\) of \(A\) such that the quotient diagram of \((a, b)\) has \(m\) blocks and the density of \(\Pi_+(a, b)\) is \(2^{-m}\) (respectively, \((2^m - 1)/2^{2m-1}\)). One can also show that if \(\{l, n\} \subseteq [1, \infty)\), with \(l \geq 2n\), then there are infinitely many elements \((a, b)\) of \(A\) such that the density of \(\Pi_+(a, b)\) is \(2^{1-l}(2^n - 1)\).

For more details in this situation and for what transpires for arbitrary standard \(2m\)-tuples, we refer the interested reader to Wright [46].
CHAPTER 9

Are quadratic residues randomly distributed?

Extensive numerical calculations performed over the years indicate that, at least in certain subintervals of \([1, p-1]\), residues and non-residues of \(p\) occur in very irregular patterns. This has led to speculation about whether residues occur more or less randomly. In this section, we will provide some evidence to support the contention that residues and non-residues are indeed distributed in this manner.

The method which we will use to detect random behavior employs the central limit theorem from the mathematical theory of probability. Let \((\Omega, \mu)\) denote a probability space, i.e., a measure space \(\Omega\) equipped with a nonnegative, countably additive measure \(\mu\) such that \(\mu(\Omega) = 1\). Suppose that \(X_1, X_2, \ldots\) is a sequence of real-valued random variables defined on \(\Omega\) which are (stochastically) independent, identically distributed, and each random variable has mean 0 and variance 1. If we set

\[
S_n = \sum_{k=1}^{n} X_k, \quad n \in [1, \infty),
\]

then the central limit theorem (Chung [3], Theorem 6.4.4) asserts that for each real number \(\lambda\),

\[
\lim_{n \to +\infty} \mu\left( \left\{ \omega \in \Omega : \frac{S_n(\omega)}{\sqrt{n}} \leq \lambda \right\} \right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\lambda} e^{-t^2/2} dt,
\]

i.e., as \(n \to +\infty\), \(S_n/\sqrt{n}\) becomes normally distributed.

Now let \(p\) be a prime. We convert the set \([0, p-1]\) into a (discrete and finite) probability space by assigning probability \(1/p\) to each element of \([0, p-1]\). This induces the probability measure \(\mu_p\) on \([0, p-1]\) defined by

\[
\mu_p(S) = \frac{|S|}{p}, \quad S \subseteq [0, p-1].
\]

For each positive integer \(h < p\), consider the sums

\[
S_h(x) = \sum_{n=x+1}^{x+h} \chi_p(n), \quad x = 0, \ldots, p-1,
\]

which is just the quadratic excess of the interval \((x, x+h+1)\) that we studied in Chapter 7. The function \(S_h\) is a random variable on \(([0, p-1], \mu_p)\), and so by way of analogy with
(1), we consider the distribution function

\[ \lambda \to \mu_p\left( \left\{ x \in [0, p-1] : \frac{S_h(x)}{\sqrt{h}} \leq \lambda \right\} \right), \quad \lambda \in (-\infty, +\infty), \]

of \( S_h/\sqrt{h} \).

We next let \( h = h(p) \) be a function of \( p \) and look for conditions on the growth of \( h(p) \) which guarantee that for each real number \( \lambda \),

\[ \lim_{p \to +\infty} \frac{1}{p} \left| \left\{ x \in [0, p-1] : \frac{S_{h(p)}(x)}{\sqrt{h(p)}} \leq \lambda \right\} \right| = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\lambda} e^{-t^2/2} dt, \]

It is easy to see that a necessary condition for (4) to occur is that \( \lim_{p \to +\infty} h(p) = +\infty \). If (4) is valid then, as we see from (2) and (3), when \( p \to +\infty \) the sums \( S_{h(p)} \) satisfy a “central limit theorem” relative to the probability spaces \((\mathbb{Z}/p\mathbb{Z}, \mu_p)\). If (4) can be verified, then upon comparing it to (1), we conclude that for \( p \) sufficiently large, at least with respect to sampling using \( \chi_p \) in the intervals \([x + 1, x + h(p)]\), \( x = 0, 1, \ldots, p - 1 \), residues and non-residues of \( p \) appear to behave as if they are distributed randomly and independently!

The following theorem of Davenport and Erdős ([6], Theorem 5) provides conditions on \( h(p) \) which imply that (4) is true:

**Theorem 9.1.** If \( h : \mathbb{P} \to [1, \infty) \) is any function such that

\[ \lim_{p \to +\infty} h(p) = +\infty, \quad \lim_{p \to +\infty} \frac{h(p)}{\sqrt{p}} = 0, \text{ for all } r \in [1, \infty) \]

(e.g., \( h(p) = [\log^N p] \), where \( N \) is any fixed positive integer), then for each real number \( \lambda \),

\[ \lim_{p \to +\infty} \frac{1}{p} \left| \left\{ x \in [0, p-1] : \frac{S_{h(p)}(x)}{\sqrt{h(p)}} \leq \lambda \right\} \right| = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\lambda} e^{-t^2/2} dt. \]

The proof of this theorem relies on the following lemma: we will first state the lemma, use it to prove Theorem 9.1, and then prove the lemma.

**Lemma 9.2.** Let \( r \) be a fixed positive integer, and let \( h \) be an integer and \( p \) a prime such that \( r < h < p \). Then there exists numbers \( 0 \leq \theta \leq 1, 0 \leq \theta' \leq 1 \) such that

\[ \left| \sum_{x=0}^{p-1} S_h(x)^{2r} - (p - \theta r)(h - \theta' r) \prod_{i=1}^{r} (2i - 1) \right| \leq 2rh^{2r} \sqrt{p}, \]

\[ \left| \sum_{x=0}^{p-1} S_h(x)^{2r-1} \right| \leq 2rh^{2r} \sqrt{p}. \]
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Proof of Theorem 9.1. Let \( r \) be a fixed positive integer. Then by the hypotheses satisfied by \( h(p) \), we have that \( r < h(p) < p \) for all \( p \) sufficiently large, hence Lemma 9.2 implies that for all such \( p \),

\[
\left| \frac{1}{p} \sum_{x=0}^{p-1} (h(p)^{-1/2} S_{h(p)}(x))^{2r} - \left( 1 - \frac{\theta r}{p} \right) \left( 1 - \frac{\theta' r}{h(p)} \right) \prod_{i=1}^{r} (2i - 1) \right| \leq 2r \frac{h(p)^r}{\sqrt{p}},
\]

\[
\left| \frac{1}{p} \sum_{x=0}^{p-1} (h(p)^{-1/2} S_{h(p)}(x))^{2r-1} \right| \leq 2r \frac{h(p)^r}{\sqrt{p}}.
\]

Letting \( p \to +\infty \) in these inequalities, we deduce from the growth conditions on \( h(p) \) that if \( r \) is any positive integer and

\[
\mu_r = \begin{cases} 
\frac{r/2}{\prod_{i=1}^{r/2} (2i - 1)}, & \text{if } r \text{ is even}, \\
0, & \text{if } r \text{ is odd},
\end{cases}
\]

then

(7) \[ \lim_{p \to +\infty} \frac{1}{p} \sum_{x=0}^{p-1} (h(p)^{-1/2} S_{h(p)}(x))^r = \mu_r. \]

Now for each real number \( s \), let

\[ N_p(s) = \frac{1}{p} \left\lfloor \left\{ x \in [0, p-1] : S_{h(p)}(x) \leq s \right\} \right\rfloor. \]

The function \( N_p \) is nondecreasing in \( s \), constant except for possible discontinuities at certain integral values of \( s \), and is right-continuous at every value of \( s \). Because

\[ |S_{h(p)}(x)| \leq h(p), \text{ for all } x, \]

it follows that

\[ N_p(s) = \begin{cases} 
0, & \text{if } s < -h(p), \\
1, & \text{if } s \geq h(p).
\end{cases} \]

We also have that

(8) \[ \frac{1}{p} \sum_{x} (h(p)^{-1/2} S_{h(p)}(x))^r = \frac{1}{p} \sum_{s=-h(p)}^{h(p)} \left( \sum_{x:S_{h(p)}(x)=s} (h(p)^{-1/2} s)^r \right) \]

\[ = \frac{1}{p} \sum_{s=-h(p)}^{h(p)} (h(p)^{-1/2} s)^r |\{ x : S_{h(p)}(x) = s \}| \]

\[ = \sum_{s=-h(p)}^{h(p)} (h(p)^{-1/2} s)^r (N_p(s) - N_p(s - 1)), \]
and so if we let

\[ \Phi_p(t) = N_p(th(p)^{-1/2}), \]

then the last sum in (8) can be written as the Stieltjes integral

\[ \int_{-\infty}^{\infty} t^r d\Phi_p(t). \]

Putting

\[ \Phi(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-u^2/2} du, \]

we have

\[ \int_{-\infty}^{\infty} t^r d\Phi(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} t^r e^{-t^2/2} dt = \mu_r, \]

due to (7), (8) \Rightarrow

\[ \lim_{p \to +\infty} \int_{-\infty}^{\infty} t^r d\Phi_p(t) = \int_{-\infty}^{\infty} t^r d\Phi(t), \text{ for all } r \in [0, \infty). \]

By virtue of the definition of \( \Phi_p \), the conclusion of Theorem 9.1 can be stated as

\[ \lim_{p \to +\infty} \Phi_p(\lambda) = \Phi(\lambda), \text{ for all real numbers } \lambda. \]

We will deduce (10) from (9) by an appeal to the classical theory of moments.

Suppose by way of contradiction that (10) is false for some \( \lambda \); then there exists \( \delta > 0 \) such that

\[ |\Phi_p(\lambda) - \Phi(\lambda)| \geq \delta \text{ for infinitely many } p. \]

Using the first and second Helly selection theorems ([38], Introduction, section 3), we find a subsequence of these \( p \), say \( p' \), and a nondecreasing real-valued function \( \Phi^* \) defined on \(( -\infty, \infty )\) such that

\[ \lim_{t \to \infty} \Phi^*(t) = 0, \lim_{t \to -\infty} \Phi^*(t) = 1, \]

\[ \Phi^* \text{ is right-continuous at all points of } (-\infty, \infty), \]

\[ \lim_{p' \to +\infty} \Phi_{p'}(t) = \Phi^*(t), \text{ for all points } t \text{ at which } \Phi^* \text{ is continuous,} \]

and

\[ \lim_{p' \to +\infty} \int_{-\infty}^{\infty} t^r d\Phi_{p'}(t) = \int_{-\infty}^{\infty} t^r d\Phi^*(t), \text{ for all } r \in [0, \infty). \]

By way of (9) and (15),

\[ \int_{-\infty}^{\infty} t^r d\Phi^*(t) = \int_{-\infty}^{\infty} t^r d\Phi(t), \text{ for all } r \in [0, \infty). \]
The Weierstrass approximation theorem, which asserts that each function continuous on a closed and bounded interval of the real line is the uniform limit on that interval of a sequence of polynomials, and (16) ⇒

\[ \int_{-\infty}^{\infty} f \, d\Phi^*(t) = \int_{-\infty}^{\infty} f \, d\Phi(t), \]

for all real-valued functions \( f \) continuous on \((-\infty, \infty)\) of compact support. (12), (13), and (17) ⇒

\[ \Phi^*(t) = \Phi(t), \quad \text{for all } t \in (-\infty, \infty). \]

Hence \( \Phi^* \) is continuous everywhere in \((-\infty, \infty)\), and so by (14) and (18),

\[ \lim_{p' \to +\infty} \Phi_{p'}(\lambda) = \Phi(\lambda), \]

and this contradicts (11).

It remains to prove Lemma 9.2. The argument here makes use of another interesting application of the Weil-sum estimates available from Theorem 8.1.

Consider first the case with \( 2r \) as the exponent. We have that

\[ \sum_{x=0}^{p-1} (S_h(x))^{2r} = \sum_{(n_1, \ldots, n_{2r}) \in [1, h]^{2r}} \sum_{x=0}^{p-1} \chi_p \left( \prod_{i=1}^{2r} (x + n_i) \right). \]

In order to estimate the absolute value of this sum, we divide the elements \((n_1, \ldots, n_{2r})\) of \([1, h]^{2r}\) into two types: \((n_1, \ldots, n_{2r})\) is of type 1 if it has at most \( r \) distinct coordinates, each of which occurs an even number of times; all other elements of \([1, h]^{2r}\) are of type 2.

If \((n_1, \ldots, n_{2r})\) is of type 1 then the polynomial \( \prod_{i=1}^{2r} (x + n_i) \) is a perfect square in \((\mathbb{Z}/p\mathbb{Z})[x]\). If \( s \) is the number of distinct coordinates of \((n_1, \ldots, n_{2r})\), then \( \chi_p \left( \prod_{i=1}^{s} (x + n_i) \right) = 0 \) whenever there is a distinct coordinate \( n_j \) of \((n_1, \ldots, n_{2r})\) such that \( x \equiv -n_j \mod p \), and \( \chi_p \left( \prod_{i=1}^{s} (x + n_i) \right) = 1 \) otherwise. It follows that the value of the sum

\[ \sum_{x=0}^{p-1} \chi_p \left( \prod_{i=1}^{2r} (x + n_i) \right) \]

is at least \( p - r \), and this value is clearly at most \( p \). Hence there exists a number \( 0 \leq \theta \leq 1 \) such that the sum (19) is

\[ F(h, r)(p - \theta r), \]

where \( F(h, r) \) denotes the cardinality of the set of all elements of \([1, h]^{2r}\) of type 1.

On the other hand, if \((n_1, \ldots, n_{2r})\) is of type 2 then the polynomial \( \prod_{i=1}^{2r} (x + n_i) \) reduces modulo \( p \) to a product of at least one and at most \( 2r \) distinct linear factors over \( \mathbb{Z}/p\mathbb{Z} \), hence
Theorem 8.1 \[\Rightarrow\]
\[\left| \sum_{x=0}^{p-1} \chi_p \left( \prod_{i=1}^{2r} (x + n_i) \right) \right| \leq 2r \sqrt{p}.\]

Hence the contribution of the elements of type 2 to the sum (19) has an absolute value that does not exceed \(2rh2^r \sqrt{p}\).

An appropriate estimate of the size of \(F(h,r)\) is now required. Following Davenport and Erdős, we note first that the number of ways of choosing \(\text{exactly} \ r\) distinct integers from \([1,h]\) is \(h(h-1) \cdots (h-r+1)\), and the number of ways of arranging these as \(r\) pairs is \(\prod_{i=1}^{r} (2i-1)\). Hence

\[F(h,r) \geq h(h-1) \cdots (h-r+1) \prod_{i=1}^{r} (2i-1)\]

On the other hand, the number of ways of choosing at most \(r\) distinct elements from \([1,h]\) is at most \(h^r\), and when these have been chosen, the number of different ways of arranging them in \(2r\) places is at most \(\prod_{i=1}^{r} (2i-1)\). Hence

\[F'(r,h) \leq h^r \prod_{i=1}^{r} (2i-1)\]

Hence there is a number \(0 \leq \theta' \leq 1\) such that

\[F(r,h) = (h - \theta' r)^r \prod_{i=1}^{r} (2i-1)\]

The conclusion of Lemma 9.2 for odd exponents follows from these estimates, and when the sum has an even exponent, the desired conclusion is now obvious, because in this case there are no elements of type 1.

**Remark.** More recently, Kurlberg and Rudnick [26] and Kurlberg [25] have provided further evidence of the random behavior of quadratic residues by computing the limiting distribution of normalized consecutive spacings between representatives of the squares in \(Z/nZ\) as \(|\pi(n)| \to +\infty\). In order to describe their work there, let \(S_n \subseteq [0,n-1]\) denote the set of representatives of the squares in \(Z/nZ\), i.e., the set of quadratic residues modulo \(n\) inside \([0,n-1]\) (N.B. It is not assumed here that a quadratic residue mod \(n\) is relatively prime to \(n\)). Order the elements of \(S_n\) as \(r_1 < \cdots < r_N\) and then let \(x_i = (r_{i+1} - r_i)/s\), where \(s = (r_N - r_1)/N\) is the mean spacing; \(x_i, i = 1, \ldots, N - 1\), are the distances between
consecutive elements of $S_n$ normalized to have mean distance 1. If $t$ is any fixed positive real number then it is shown in [25] and [26] that

$$\lim_{|\pi(n)| \to +\infty} \frac{|\{x_i : x_i \leq t\}|}{|S_n| - 1} = 1 - e^{-t},$$
i.e., for all $n$ with $|\pi(n)|$ large enough, the normalized spacings between quadratic residues of $n$ follow (approximately) a Poisson distribution. Among many other things, the Poisson distribution governs the number of customers and their arrival times in queueing theory, and so the results of Kurlburg and Rudnick can be interpreted to say that if the number of prime factors of $n$ is sufficiently large then quadratic residues of $n$ appear consecutively in the set $[0, n - 1]$ in the same way as customers arriving randomly to join a queue.
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