A CHARACTERIZATION OF SECANT VARIETIES OF SEVERI VARIETIES AMONG CUBIC HYPERSURFACES

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Abstract. It is shown that an irreducible cubic hypersurface with nonzero Hessian and smooth singular locus is the secant variety of a Severi variety if and only if its Lie algebra of infinitesimal linear automorphisms admits a nonzero prolongation.

1. Introduction

Let $V^m \subset \mathbb{P}^N$ be an $m$-dimensional irreducible nondegenerate smooth complex projective variety. The secant variety $SV$ is the closure of the union of lines in $\mathbb{P}^N$ joining two distinct points of $V$. It is easy to see that $V$ can be isomorphically projected to a lower-dimensional projective space if and only if $SV \neq \mathbb{P}^N$, which can only occur if $m$ is not too big. As conjectured by Hartshorne and proved by Zak ([Z1]), $SV = \mathbb{P}^N$ provided that $m > \frac{2N-4}{3}$. We call $V$ a Severi variety if $SV \neq \mathbb{P}^N$ and $m = \frac{2N-4}{3}$.

As proved by Zak ([Z1]), there exist exactly four Severi varieties:

$$v_2(\mathbb{P}^2) \subset \mathbb{P}^5, \quad \mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8, \quad G(1,5) \subset \mathbb{P}^{14}, \quad \mathbb{O} \mathbb{P}^2 \subset \mathbb{P}^{26},$$

viz. the Veronese surface ($m = 2$), the Segre variety ($m = 4$), the Grassmann variety of lines in $\mathbb{P}^5$ ($m = 8$), and the Cayley plane corresponding to the closed orbit of the minimal representation of the algebraic group $E_6$ ($m = 16$). The vector spaces corresponding to the ambient spaces $\mathbb{P}^N$ of Severi varieties can be identified with the spaces of Hermitian $3 \times 3$ matrices with coefficients in composition algebras, and under this identification the affine cones corresponding to Severi varieties are the loci of matrices whose rank does not exceed one. The secant varieties of Severi varieties are irreducible cubic hypersurfaces defined by vanishing of the determinant of the corresponding $3 \times 3$ matrix, and the projective duals of these cubics are naturally isomorphic to the corresponding Severi varieties.

Severi varieties form the third row of the so-called Freudenthal magic square, and their rich projective geometry was thoroughly studied (see e.g. [Z1], [LM] and [IM]). They are also related to homogeneous Fano contact manifolds since the latter can be recovered from Severi varieties. Thus a better understanding of Severi varieties can shed some light on the long-standing conjecture of LeBrun and Salamon predicting that all Fano contact manifolds are homogeneous. This motivates the problem of characterizing the secant varieties of Severi varieties among all cubic hypersurfaces. Following [H], we solve this problem in terms of prolongations of infinitesimal automorphisms.

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Let $W$ be a complex vector space. A prolongation of a Lie subalgebra $g \subset \mathfrak{g}l W$ is an element $A \in \text{Hom} \left( \text{Sym}^2 W, W \right)$ such that $A(w, \cdot) \in g$ for any $w \in W$. The vector space of all prolongations of $g$ is denoted by $g^1$. Let $T \subset P W$ be a smooth projective variety, let $\hat{T} \subset W$ be the corresponding affine cone, and let $\text{aut} \hat{T} \subset \mathfrak{g}l W$ be the Lie algebra of infinitesimal linear automorphisms of $\hat{T}$. We are interested in the vector space $\text{aut}^1 \hat{T}$ of all prolongations of $\text{aut} \hat{T}$.

Let $Y \subset P W$ be an irreducible cubic hypersurface defined by a cubic form $F \in \text{Sym}^3 W^*$, and let $\hat{Y}$ be its affine cone. Both $\text{aut} \hat{Y}$ and $\text{aut}^1 \hat{Y}$ can be computed effectively in terms of $F$ (cf. Section 3). In the case when $Y = SV$ for a Severi variety $V$ one has $\text{aut}^1 \hat{Y} \neq 0$ (cf. Corollary 3.11). For various reasons it makes sense to focus the study of prolongations on the case when the polar map defined by the partial derivatives of $F$ is surjective or, equivalently, the Hessian determinant of $F$ is not (identically) equal to zero. In this case we say that $Y$ is not polar defective (cf. Definition 2.2 in the next section).

J.-M. Hwang posed the following question (Question 1.3 in [H]).

**Question.** Let $Y$ be an irreducible cubic hypersurface. Is it true that if $\text{aut}^1 \hat{Y} \neq 0$ and $Y$ is not polar defective, then $Y$ is the secant variety of a Severi variety?

It turns out that in general the answer to this question is negative; cf. e.g. Example 3.13(ii). In the present paper we give a positive answer to Hwang’s question under the additional assumption that the (reduced) singular locus $Y' \subset Y$ is smooth.

**Main Theorem.** Let $Y \subset P W$ be an irreducible cubic hypersurface. Assume that

a) $Y$ is not polar defective;

b) $Y'$ is smooth;

c) $\text{aut}^1 \hat{Y} \neq 0$.

Then $Y$ is the secant variety of a Severi variety.

It should be mentioned that in [H] J.-M. Hwang proved a weaker version of this result in which assumption (c) is replaced by

$c') \exists a \neq \frac{1}{4} \ni \Xi^a_Y \neq 0$ for some $a \neq \frac{1}{4}$,

where $a \in \mathbb{C}$ is a complex number and $\Xi^a_Y \subset \text{aut}^1 \hat{Y}$ is a certain linear subspace with a rather intricate definition (cf. Theorem 1.6 in [H]), thus giving a partial answer to Question 1.5 in [H] which is a weaker form of the above Question.

As suggested in [H], the proof of the Main Theorem splits into two parts: the first one is to show that $Y = SY'_0$ for an irreducible component $Y'_0 \subset Y'$ and the second one is to go through the classification of smooth nondegenerate projective varieties with nonzero prolongation given in [FH1] and [FH2]. In this paper we mainly contribute to the first part of this strategy by exploring the relationship between dual and polar defectivity (the latter is equivalent to the classical notion of vanishing Hessian).

It is easy to see that any irreducible hypersurface with vanishing Hessian is dual defective, but the converse is not true, as is shown by the secant varieties of Severi varieties. In Theorem 2.9 we show that if $Y$ is an irreducible dual defective cubic with smooth $Y'$ such that $SY' \subset Y$, then the defining equation $F$ of $Y$ has vanishing hessian. At the second step of the proof of the Main Theorem we anyhow need to assume that $Y'$ is smooth, and so the smoothness assumption in Theorem 2.9 is not...
restrictive. In the meantime the third named author proved that Theorem 2.9 is true even without this assumption.

At the end of the paper we give examples showing that none of the conditions a)–c) of the Main Theorem can be lifted.

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2. Dual defective cubic hypersurfaces

Let $Y \subset \mathbb{P}^N$ be an irreducible complex projective hypersurface defined by a homogeneous polynomial $F$ of degree $d > 1$. Let $\phi : \mathbb{P}^N \to \mathbb{P}^{N^*}$ be the polar map given by the polar linear system $\mathcal{Q}$ whose members are cut out by the partial derivatives of $F$, let $X^n = Y^* \subset \mathbb{P}^{N^*}$ be the dual variety, and let $\gamma = \phi|_Y$ be the Gauss map (we refer to [1] for basic facts on dual varieties). Both $\phi$ and $\gamma$ are defined outside of the singular subset $Y' = (\text{Sing} Y)_{\text{red}}$ which is also the base locus of $\mathcal{Q}$.

Definition 2.1. The integer $\text{def } Y = \text{codim } X - 1 = N - n - 1$ is called the dual defect of $Y$. The hypersurface $Y$ is called dual defective if $\text{def } Y > 0$, i.e. $X$ fails to be a hypersurface.

It is well known that, for a general point $x \in X$, the fiber $\gamma^{-1}(x) = \perp T_{X,x} \subset Y$ is a linear subspace of dimension $\text{def } Y$ (in particular, if $\text{def } Y = 0$, then the map $\gamma : Y \to X$ is birational).

Definition 2.2. An irreducible hypersurface $Y$ is called polar defective if it satisfies one of the following equivalent conditions:

a) $\phi(\mathbb{P}^N) = Z \subset \mathbb{P}^{N^*}$;

b) $F$ has vanishing Hessian, i.e. $\det H \equiv 0$, where $H$ is the Hesse matrix formed by the second order partial derivatives of $F$.

The number $\text{pdef } Y = \text{codim } Z = N - r$, where $r = \text{dim } Z$, is called the polar defect of $Y$, so that $Y$ is polar defective if and only if $\text{pdef } Y > 0$. It is easy to see that $\text{dim } Z = \text{rk } H - 1$ and $\text{pdef } Y = \text{codim } Z = \text{cork } H$, where $\text{cork } H = N + 1 - \text{rk } H$ is the corank of $H$.

From the proof of [2, Proposition 4.9 (ii)] it follows that, for a general point $z \in Z$, the fiber $\mathcal{F}_z = \phi^{-1}(z) \subset \mathbb{P}^N$ is a union of finitely many linear subspaces of dimension $\text{pdef } Y$ passing through the linear subspace $\perp T_{Z,z} \subset Z^*$ of dimension $\text{pdef } Y - 1$. Furthermore, $Z^* \subset Y'$ and $\mathcal{F}_z \cap Y = \mathcal{F}_z \cap Y' = \mathcal{F}_z \cap Z^* = \perp T_{Z,z}$.

The simplest example of polar defective hypersurface is given by cones (in which case both $Z$ and $X$ are degenerate varieties), but there exist many more interesting examples (cf. e.g. Example 3.13 (iii)).

Proposition 2.3.

(i) $Z \supset X$, i.e. $n + 1 \leq r \leq N$;

(ii) Any polar defective hypersurface is dual defective. More precisely, $\text{def } Y \geq \text{pdef } Y$ and the inequality is strict if and only if $r > n + 1$.

Proof. (i) It is clear that $X = \gamma(Y) = \phi(Y) \subset \phi(\mathbb{P}^N) = Z$. Thus we only need to show that $Z \neq X$. Suppose to the contrary that $Z = X$, and let $x \in X$ be a general point. Then $\gamma^{-1}(x) = \perp T_{X,x}$ is a linear subspace of dimension $N - n - 1$ contained
in the \((N - n)\)-dimensional fiber \(F_x = \phi^{-1}(x) \subset \mathbb{P}^N\). Furthermore, since \(\gamma = \phi_\gamma\), \(F_x \cdot Y = \gamma^{-1}(x) = \mathbb{P}^{N-n-1}\), which is only possible if \(F_x\) is a linear subspace and \(Y\) is a hyperplane, contrary to the hypothesis that \(d > 1\).

(ii) is an immediate consequence of (i).

Remark 2.4. The converse of Proposition 2.3 (ii) is false. For example, let \(V_i^{\ast N} \subset \mathbb{P}^{N_i}, 1 \leq i \leq 4, \ N_i = 2^i, V_i = \mathbb{P}^{N_i} \) be the \(i\)-th Severi variety, and let \(Y_i = SV_i = \mathbb{P}^{N_i}\) be its secant variety. Then \(Y_i\) is a cubic hypersurface singular along \(Y_i' = V_i, X_i = \gamma_i(Y_i) \subset \mathbb{P}^{N_i}\) is also the \(i\)-th Severi variety, and \(\phi_i : \mathbb{P}^{N_i} \to \mathbb{P}^{N_i}\) is the birational Cremona transformation contracting \(Y_i\) to \(X_i\), blowing up \(V_i\) to the cubic \(SX_i\), and defining an isomorphism between the complements of the cubic hypersurfaces in \(\mathbb{P}^N\) and \(\mathbb{P}^{N_i}\) (cf. [Z1, Chap. IV]). In particular, \(def Y_i = \frac{N}{2} + 1\), but \(\text{pdef} Y_i = 0\) and \(Y_i\) is not polar defective.

Proposition 2.5. Let \(L \subset \mathbb{P}^N\) be a general hyperplane, and let \(Y_L = L \cap Y\) be the corresponding hyperplane section of \(Y\). Then either both \(Y\) and \(Y_L\) are not polar defective or \(\text{pdef} Y = \text{pdef} Y_L + 1\). In particular, if \(Y_L\) is polar defective, then so is \(Y\).

Proof. Let \(\phi : \mathbb{P}^N \to \mathbb{P}^{N\ast}\) and \(\phi_L : L \to L\ast\) be the polar maps corresponding to \(Y\) and \(Y_L\) respectively, let \(Z \subset \mathbb{P}^{N\ast}\) and \(Z_L \subset L\ast\) be their respective images, and let \(\pi : \mathbb{P}^{N\ast} \to L\ast\) be the projection with center at the point \(L\) corresponding to the hyperplane \(L\). Then

\[
\dim \phi(L) = \begin{cases} 
\dim Z = N - \text{pdef} Y & \text{if } \text{pdef} Y > 0, \\
\dim Z - 1 = N - 1 & \text{if } \text{pdef} Y = 0.
\end{cases} \tag{2.1}
\]

We use the following

Lemma 2.6. Let \(L \subset \mathbb{P}^N\) be a general hyperplane, and let \(L\) be the corresponding point in \(\mathbb{P}^{N\ast}\). Then \(L \not\in \phi(L)\).

Proof. If \(\phi\) fails to be dominant, we can just take any \(L\) for which \(L \not\in Z\). Suppose now that \(\phi\) is dominant. Let \(\Gamma \subset \mathbb{P}^N \times \mathbb{P}^{N\ast}\) denote the closure of the graph of \(\phi\), and let \(\Gamma_\phi : \mathbb{P}^N \to \mathbb{P}^{N\ast}\) and \(\Gamma_\phi^\ast : \mathbb{P}^{N\ast} \to \mathbb{P}^N\) be the natural projections. Let \(L\) be a hyperplane for which \(L \not\in D \cup X\), where \(D = q(p^{-1}(Y')) \subset \mathbb{P}^{N\ast}\). Then \(L \not\in \phi(L)\) since otherwise \(L = \phi(v)\) for a point \(v \in L \setminus Y\) and from the Euler formula it follows that \(v \in Y \cap L\) and \(L = \phi(v) \in \gamma(Y) = X\), a contradiction.

We return to the proof of Proposition 2.5. Since \(Z_L = \phi_L(L) = \pi(\phi(L))\) and, by Lemma 2.6, \(\phi(L)\) is not a cone with vertex \(L\), from (2.1) it follows that

\[
\dim Z_L = \dim \phi(L) = \begin{cases} 
\dim Z & \text{if } \text{pdef} Y > 0, \\
\dim Z - 1 & \text{if } \text{pdef} Y = 0.
\end{cases}
\]

and so

\[
\text{pdef} Y_L = N - \dim Z_L - 1 = \begin{cases} 
N - \dim Z - 1 = \text{pdef} Y - 1 & \text{if } \text{pdef} Y > 0, \\
N - (\dim Z - 1) = \text{pdef} Y & \text{if } \text{pdef} Y = 0.
\end{cases}
\]
From now on we restrict our attention to the case when $Y$ is an irreducible dual defective cubic hypersurface.

**Proposition 2.7.** Let $Y \subset \mathbb{P}^n$ be a cubic hypersurface. Then $SY' \subset Y$.

*Proof.* If a line joining two distinct points of $Y'$ were not contained in $Y$, it would meet $Y$ with multiplicity at least four while $\deg Y = 3$. □

Consider the conormal variety $\mathcal{P} \subset X \times Y$, $\mathcal{P} = \{(x, y) | x \in \text{Sing } X, \gamma x + y \in T_{X,x}\}$, where $\text{Sing } X$ is the open subset of nonsingular points of $X$ and $T_{X,x}$ is the embedded tangent space to $X$ at $x$, and let $p : \mathcal{P} \to X$ and $\pi : \mathcal{P} \to Y$ denote the projections. Then $\pi$ is birational, $\gamma = p \circ \pi^{-1}$, $p|_{\text{Sm } X}$ is a $(\mathbb{P}^n-n-1)$-bundle and, for $x \in \text{Sm } X$, $\pi(\mathcal{P}_x) = \gamma^{-1}(x)$. For a general $x \in X$, the intersection $\mathcal{P}_x' = \mathcal{P}_x \cap Y'$ contains all the points $y$ for which the hyperplane section $\gamma x \cap X$ fails to have a nondegenerate quadratic singularity at $x$. The locus $\mathcal{P}_x'$ of such points in $\mathcal{P}_x$ is defined by vanishing of the determinant of a nondegenerate quadratic form, so $\mathcal{P}_x \cap Y'$ is a hypersurface, and since $Y'$ is defined by quadratic equations, $\deg \mathcal{P}_x' \leq 2$. The locus of $\mathcal{P}_x'$ in $\mathcal{P}$ will be denoted by $\mathcal{P}'$.

**Lemma 2.8.** Suppose that $SY' \subset Y$, and let $x \in X$ be a general point. Then $\mathcal{P}_x'$ is a hyperplane in $\mathcal{P}_x$.

*Proof.* In fact, if, for a general $x \in X$, $\mathcal{P}_x'$ were a quadric, then one would have $S\mathcal{P}_x' = \mathcal{P}_x$, hence $SY' = Y$. □

It is clear that $\mathcal{P}'$ is a rational section of the morphism $\mathcal{P} \to X$. Denote by $\pi'$ the restriction of $\pi$ on $\mathcal{P}'$ and by $Y''$ the image of $\mathcal{P}'$ in $Y$. Then $Y'' = \pi(\mathcal{P}') \subset Y'$ is an irreducible subvariety. For a general point $y \in Y''$, put $X_y = p((\pi')^{-1}(y))$.

By Proposition 2.3, polar defectivity implies dual defectivity, and Remark 2.4 shows that the converse is not true even for cubics. However, in the examples in Remark 2.4 one has $SY'_i = Y_i$. This is not accidental: it turns out that a dual defective cubic hypersurface is polar defective provided that $SY' \subset Y$. The main goal of this section is to prove this under the additional assumption that $Y'$ is smooth (cf. however Remark 2.15).

**Theorem 2.9.** Let $Y \subset \mathbb{P}^n$ be an irreducible dual defective cubic hypersurface. Suppose that $Y'$ is smooth and $SY' \subset Y$. Then $Y$ is polar defective.

*Proof of Theorem 2.9.* We split the proof into a series of lemmas.

**Lemma 2.10.** It suffices to prove Theorem 2.9 in the case when $Y$ is not a cone and $N = n + 2$ (i.e. $\text{def } Y = 1$).

*Proof.* Since cones are polar defective, we can assume that $Y$ is not a cone. Let $L \subset \mathbb{P}^N$, $\dim L = n + 2$ be a general linear subspace, let $Y_L = L \cap Y$, and put $X_L = Y_L^*$. From Theorem 1.21 in [1] or Proposition 2.4 in [2] it follows that $X_L$ is obtained from $X$ by projecting from the (general) linear subspace $L \subset \mathbb{P}^{N+1}$, hence $\dim X_L = \dim X = n$ and $\text{def } Y = \text{def } Y_L + 1$. The hypotheses that $Y'$ is smooth and $SY' \subset Y$ are clearly stable with respect to passing to a general linear section. Therefore Lemma 2.10 follows from Proposition 2.5 □

From now on we assume that $N = n + 2$ and $Y$ is not a cone. We denote by $\langle A \rangle$ the linear span of a subset $A \subset \mathbb{P}^N$. 
Let \( x \in X \) be a general point, and let \( l = l_x = \pi(P_x) = \gamma^{-1}(x) \). By Lemma 2.8 \( l \cap Y' = l \cap Y'' = y \) is a single point. Varying \( x \in X \), we see that the lines \( l_x \) sweep out a dense subset in \( Y \) while by our hypothesis \( Y' \) is smooth and the embedded tangent spaces \( T_{Y',y} \) are contained in the subvariety \( SY' \subseteq Y \); hence we may assume that \( l \not\in T_{Y',y} \). The line \( l \) is blown down by the map \( \phi \) defined by the polar linear system \( \mathcal{Q} \), hence all quadrics from \( \mathcal{Q} \) meet \( l \) only at \( y \).

**Lemma 2.11.** Suppose that \( \langle Y'' \rangle \subset Y' \). Then \( Y \) is polar defective.

**Proof.** Put \( P_x = (l_x, Y'') \supset \langle Y'' \rangle \). Since \( l_x \) meets \( Y'' \) at a single point \( y \), \( \langle Y'' \rangle \) is a hyperplane in \( P_x \), and so by our hypothesis \( \langle Y'' \rangle \) is a fixed component of the restriction \( \mathcal{Q}_{P_x} \) of the polar system of quadrics on \( P_x \) and \( \phi_{P_x} \) is a linear projection. Since \( \phi(l_x) = x \), we conclude that the restriction of the polar map \( \phi \) on \( P_x \) coincides with the projection from the point \( y \) and \( \dim \phi(P_x) = \dim \langle Y'' \rangle = \dim P_x - 1 \).

Denote by \( \mathcal{P} \) the closure of the locus of \( P_x \). Since \( P_x \supset l_x \) and the locus of \( l_x \) is dense in \( Y \), we have \( \mathcal{P} \supset Y \). On the other hand, \( P_x \not\subset Y \) for a general \( x \in X \) since otherwise \( Y \) would be a cone with vertex \( Y'' \). Hence \( \mathcal{P} = \mathbb{P}^N \), and so \( \dim \phi(\mathbb{P}^N) = N - 1 \) and \( \phi(\mathbb{P}^N) \) is a hypersurface in \( \mathbb{P}^{N*} \) passing through \( X \).

From now on we assume that \( \langle Y'' \rangle \not\subset Y' \).

We skip the proof of the following elementary lemma on linear systems of conics in a plane.

**Lemma 2.12.** Let \( P = \mathbb{P}^2 \) be a plane, let \( l, l' \subset P \) be two distinct lines, and let \( y = l \cap l' \). Let \( \mathcal{Q}_P \) be a linear system of conics in \( P \) meeting both \( l \) and \( l' \) only at \( y \), and let \( \phi_P \) be the rational map defined by \( \mathcal{Q}_P \). Then all members of \( \mathcal{Q}_P \) are unions of pairs of lines passing through \( y \), \( \phi_P(P) = \mathbb{P}^1 \), and there are two possibilities:

a) \( y \) is the only base point of \( \mathcal{Q}_P \);

b) \( \mathcal{Q}_P \) has a fixed component \( \ell = \ell_P \), where \( \ell \) is a line passing through \( y \) and distinct from \( l \) and \( l' \).

More precisely, in case a) \( \mathcal{Q}_P \) is composite with a pencil of lines, i.e. \( \phi_P \) is a composition of the projection \( P \dashrightarrow \mathbb{P}^1 \) with center at \( y \) with the double covering \( \mathbb{P}^1 \rightarrow \mathbb{P}^1 \) defined by a pencil in the linear system \( |O_{\mathbb{P}^1}(2)| \), and in case b) \( \phi_P : P \dashrightarrow \mathbb{P}^1 \) is the projection with center at \( y \).

We will apply this lemma in the case when, as above, \( l = l_x = \pi(P_x) = \gamma^{-1}(x) \), where \( x \in X \) is a general point, \( l \cap Y' = l \cap Y'' = y \), \( l \not\in T_{Y',y} \). Let \( l' \ni y \) be a general tangent line to \( Y' \) at the point \( y \). Since \( Y' \) is smooth, from Proposition 2.7 it follows that \( l' \subset Y' \). Furthermore, we may assume that \( l' \not\subset Y'' \) since otherwise, for a general point \( y \in Y'' \), \( T_{Y',y} \subset Y' \), whence the component of \( Y' \) containing \( Y'' \) coincides with the linear subspace \( T_{Y',y} \) and \( (Y'') \subset Y' \) contrary to our assumption. Hence \( l' \) is blown down by \( \phi \) and all quadrics from \( \mathcal{Q} \) meet \( l' \) only at \( y \). Let \( P = \langle l, l' \rangle \) be the plane spanned by the lines \( l \) and \( l' \).

**Lemma 2.13.** \( P, l, l' \) and \( \mathcal{Q}_P = \mathcal{Q}_{l,P} \) satisfy the hypotheses of Lemma 2.12 and condition a) of that lemma.

**Proof.** The first claim is clear since by our assumption \( l' \not\subset Y' \).

Suppose to the contrary that condition b) holds. Then \( \ell \not= l' \) is also a tangent line to \( Y' \) at \( y \), and so \( l_x \subset P = \langle l, l' \rangle = \langle \ell, l' \rangle \subset T_{Y',y} \), contrary to our choice of \( x \) (here we used the hypothesis that \( Y' \) is nonsingular).
Lemma 2.14. Suppose that \( Y'' \not\subset Y' \). Then \( Y \) is polar defective.

Proof. Consider the family of planes \( P = \langle l, l' \rangle \) satisfying, by the preceding lemma, condition a) of Lemma 2.12. By Lemma 2.12, \( \phi(P) = \mathbb{P}^1 \) and the general fiber of \( \phi_P \) is a union of two lines.

Let \( P \) denote the closure of the locus of planes \( P \). Since \( P \supset l = l_x \) and the locus of \( l_x \) is dense in \( Y \), we have \( P \supset Y \). On the other hand, a general plane \( P \) from our family is not contained in \( Y \) since a general fiber of \( \phi_P \) consists of two distinct lines while a general fiber of \( \phi_Y = \gamma \) is a single line. Hence \( P = \mathbb{P}^N \), \( \dim \phi(\mathbb{P}^N) = N - 1 \) and \( \phi(\mathbb{P}^N) \) is a hypersurface in \( \mathbb{P}^N \) passing through \( X \). \( \square \)

This completes the proof of Theorem 2.9. \( \square \)

Remark 2.15. The assumption that \( Y' \) is smooth in the statement of Theorem 2.9 is not necessary. The third named author proved that the theorem holds even without this assumption. However, our Main Theorem 3.12 is false without the smoothness hypothesis (condition b) in its statement) as is shown by Example 3.13(ii). Thus this assumption is not restrictive for the purposes of the present paper and we do not give a proof of Theorem 2.9 in full generality.

In Section 3 we will use Theorem 2.9 via its

Corollary 2.16. Let \( Y \subset \mathbb{P}^N \) be an irreducible cubic hypersurface. Assume that

a) \( Y \) is dual defective, but not polar defective;

b) the singular locus \( Y' \) of \( Y \) is smooth.

Then there exists an irreducible component \( Y'_0 \subset Y' \) such that \( Y = SY'_0 \).

Proof. From Theorem 2.9 it follows that \( Y = SY' \) and, for a general point \( x \in X \), \( \deg \mathcal{P}_x = 2 \). Suppose that the secant variety of an arbitrary component of \( Y' \) is a proper subvariety of \( Y \). Then there exist two distinct irreducible components \( Y_1, Y_2 \subset Y' \) such that \( Y = S(Y_1, Y_2) \), where \( S(Y_1, Y_2) \) is the join of \( Y_1 \) and \( Y_2 \), i.e. the closure of the subvariety in \( \mathbb{P}^N \) swept out by the lines \( \langle y_1, y_2 \rangle \), where \( y_1 \) (resp. \( y_2 \)) runs through the subset of general points in \( Y_1 \) (resp. \( Y_2 \)). Since, by our hypotheses, \( Y' \) is smooth, \( Y_1 \cap Y_2 = \emptyset \). Corollary 2.16 now follows from

Lemma 2.17. Let \( Y_1, Y_2 \subset \mathbb{P}^N \) be irreducible subvarieties such that their join \( Y = S(Y_1, Y_2) \) is a nonconic cubic hypersurface. Then \( Y_1 \cap Y_2 \neq \emptyset \).

Proof. Let \( z_0 : \cdots : z_{2N+1} \) be homogeneous coordinates in \( \mathbb{P}^{2N+1} \). Consider two copies \( \Lambda_1 \) and \( \Lambda_2 \) of \( \mathbb{P}^N \) embedded as disjoint linear subspaces in \( \mathbb{P}^{2N+1} \) as follows:

\[
\Lambda_1 = \{ z_{N+1} = \cdots = z_{2N+1} = 0 \}, \quad \Lambda_2 = \{ z_0 = \cdots = z_N = 0 \}.
\]

Let \( \tilde{Y}_1 \subset \Lambda_1 \) and \( \tilde{Y}_2 \subset \Lambda_2 \) be the corresponding embeddings of \( Y_1 \) and \( Y_2 \), and let \( \tilde{Y} = S(\tilde{Y}_1, \tilde{Y}_2) \) be their join. It is easy to see that \( \deg \tilde{Y} = \deg \tilde{Y}_1 \cdot \deg \tilde{Y}_2 \).

Let \( \tilde{Y} \subset \Lambda \) and \( \tilde{Y} \not\subset \Lambda \) be the corresponding embeddings of \( Y_1 \) and \( Y_2 \), and let \( \pi_\Lambda : \mathbb{P}^{2N+1} \to \mathbb{P}^N \) denote the projection with center at \( \Lambda \). Then \( Y = \pi_\Lambda(\tilde{Y}) \) and \( Y_1 \cap Y_2 \neq \emptyset \) if and only if \( \Lambda \cap \tilde{Y} \neq \emptyset \). Thus if \( Y_1 \cap Y_2 = \emptyset \), then \( 3 = \deg \tilde{Y} = \deg \tilde{Y} = \deg Y_1 \cdot \deg Y_2 \) and either \( Y_1 \) or \( Y_2 \) is linear, contrary to the hypothesis that \( Y \) is not a cone. \( \square \)

Corollary 2.16 is proved. \( \square \)
3. Cubic hypersurfaces admitting nontrivial prolongations

To study group actions on projective algebraic varieties it is often more convenient to consider the corresponding affine cones. We start with interpreting in the affine language some of the notions introduced in the preceding section.

Let $\mathbb{P}^N = \mathbb{P} W$, where $W$ is an $(N+1)$-dimensional complex vector space, and let $F \in \text{Sym}^2 W^*$ be an irreducible cubic form. For $w \in W$, we denote by $F_{ww} \in W^*$ (resp. $F_w \in \text{Sym}^2 W^*$) the linear function (resp. quadratic form) on $W$ defined by $u \mapsto F(w, w, u)$ (resp. $(u, v) \mapsto F(w, u, v)$). The affine polar map $\hat{\phi} : W \to W^*$ is given by $w \mapsto F_{ww}$ and the Hessian by $F_w$. Let $\hat{Y} = \{w \in W \mid F(w, w, w) = 0\} \subset W$ be the affine cubic hypersurface defined by $F$. The indeterminancy locus of $\hat{\phi}$ coincides with the singular locus $\text{Sing} \hat{Y} = \{w \in W \mid F_{ww} = 0\}$. For any $y \in \text{Sm} \hat{Y}$ we have

$$T_{\hat{Y}, y} = \text{Ker} F_{yy} = \{v \in W \mid F(y, y, v) = 0\}. \quad (3.1)$$

The affine Gauss map $\hat{\gamma} : \hat{Y} \to W^*$, $\hat{\gamma} = \hat{\phi}^* |_{\hat{Y}}$ maps a smooth point $y \in \hat{Y}$ to the hyperplane $T_{\hat{Y}, y}$. The closure of $\hat{\gamma}(\hat{Y})$ in $W^*$ is called the affine dual variety of $\hat{Y}$ and is denoted by $\hat{X}$.

Denote by $\text{aut} \hat{Y} \subset \mathfrak{gl} W$ the Lie algebra of infinitesimal linear automorphisms of $\hat{Y}$:

$$\text{aut} \hat{Y} = \{g \in \mathfrak{gl} W \mid g(y) \in T_{\hat{Y}, y} \quad \forall y \in \text{Sm} \hat{Y}\}. \quad \text{By (3.1), this can be rewritten as follows:}$$

$$\text{aut} \hat{Y} = \{g \in \mathfrak{gl} W \mid F(g(y), y, y) = 0 \quad \forall y \in \text{Sm} \hat{Y}\}. \quad \text{A prolongation of aut \hat{Y} is an element } A \in \text{Hom (Sym}^2 W, W) \text{ such that } A(w, \cdot) \in \text{aut} \hat{Y} \text{ for all } w \in W. \text{ The vector space of all prolongations of aut \hat{Y} will be denoted by aut}^1 \hat{Y}. \text{ In other words, an element } A \in \text{Hom (Sym}^2 W, W) \text{ is contained in aut}^1 \hat{Y} \text{ if and only if}$$

$$F(A(y, w), y, y) = 0 \quad \forall y \in \text{Sm} \hat{Y}, \forall w \in W. \quad (3.2)$$

**Proposition 3.1.** Let $Y \subset \mathbb{P} W$ be an irreducible cubic hypersurface. If $\text{aut}^1 \hat{Y} \neq 0$, then $Y$ is dual defective.

*Proof.* The differential of $\hat{\phi}$ at a point $w \in W$ is given by the Hessian of $F$ or, equivalently, by the form $F_w$. Since, for $y \in \text{Sm} \hat{Y}$, the affine Gauss fiber $\hat{\gamma}^{-1}(\hat{\gamma}(y))$ coincides with the linear subspace $\langle y, \text{Ker} d_y \hat{\phi} \rangle$, to prove the proposition it suffices to show that, for a general point $y \in \hat{Y}$, the form $F_y$ is degenerate, i.e. there exists a nonzero vector $v \in W$ such that $F(y, v, w) = 0$ for a general point $w \in W$.

Let $A \in \text{aut}^1 \hat{Y}$ be a nontrivial prolongation. From (3.2) it follows that, for any $u \in W$, the cubic forms $F(A(u, w), w, w)$ and $F(w, w, w)$ define the same hypersurface $Y \subset \mathbb{P}^N$. Therefore, for all $u, w \in W$,

$$F(A(u, w), w, w) = \lambda(u) F(w, w, w), \quad (3.3)$$

where $\lambda = \lambda^A \in W^* \setminus 0$. Replacing $w$ by $tu + sw$, we get an identity of the form

$$\sum_{i=0}^{3} c_i(u, w) t^i s^{3-i} \equiv 0,$$

hence $c_i = 0$ for all $i$, $0 \leq i \leq 3$. In particular, $c_0 = 0$ is
Since substituting (3.6) in (3.4), we get equation hence it follows that assumption that Proposition 3.1 was proved as Corollary 4.5 in [H] under the additional Remark 3.2.

Let Lemma 3.3.

V a cubic hypersurface. Then \( V \subset \mathbb{P}^N \) is a cone over a smooth nondegenerate projective variety \( V \subset \mathbb{P}^N \). It is well known that all such varieties are conics over linear sections of the Segre variety \( \mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5 \) (to see this it suffices to notice that a section of \( \tilde{Y} \) by a general \( \mathbb{P}^3 \subset \mathbb{P}^N \) is a twisted cubic curve). Therefore \( Y \) is a cone over an outer projection of such a linear section. However, \( \mathbb{P}^1 \times \mathbb{P}^2 \) and its irreducible linear sections are nonsingular and their projections are cubic hypersurfaces with linear singular loci. Hence the singular locus of the cone \( \tilde{Y} \) is also linear, and so \( \tilde{V} \subset \text{Sing } \tilde{Y} \) is degenerate, contrary to our assumption.

Existence of nontrivial prolongations of \( \text{aut } \tilde{V} \) imposes strong geometric restrictions on a smooth nondegenerate projective variety \( V \subset \mathbb{P}^N \). For example, from [HM] it follows that there exist lots of \( \mathbb{C}^* \)-actions on \( V \) the closures of whose orbits are conics in \( \mathbb{P}W \). In particular, this implies that \( V \) is conic connected. Complete classification
of such varieties \( V \) is carried out in \cite{FH1} and \cite{FH2}. In the case when \( V \) is linearly normal, this classification is as follows.

**Theorem 3.4** (Theorem 7.13 in \cite{FH2}). Let \( V \subset \mathbb{P}W \) be a nondegenerate irreducible smooth linearly normal variety such that \( \text{aut}^1 V \neq 0 \). Then \( V \subset \mathbb{P}W \) is projectively equivalent to one of the following varieties.

1. A rational homogeneous variety from the following list:
   \[ v_2(\mathbb{P}^{m}), \; \mathbb{P}^a \times \mathbb{P}^b, \; G(1, m), \; Q^n, \; S_5, \; \mathbb{O}\mathbb{P}^2, \]
   viz. the second Veronese embedding of \( \mathbb{P}^m \), the Segre embedding of \( \mathbb{P}^a \times \mathbb{P}^b \),
   the Plücker embedding of the Grassmann variety of lines in \( \mathbb{P}^m \), the \( m \)-dimensional nonsingular quadric, the 10-dimensional spinor variety \( S_5 \subset \mathbb{P}^{15} \) parametrizing 4-dimensional linear subspaces from one family on the 8-dimensional quadric, and the 16-dimensional Cayley plane in \( \mathbb{P}^{26} \);

2. A smooth section of \( G(1, 4) \subset \mathbb{P}^9 \) by a linear subspace of codimension 1 or 2 in \( \mathbb{P}^9 \);

3. The section of \( S_5 \subset \mathbb{P}^{15} \) by a linear subspace \( L \) of codimension 1, 2 or 3 in the ambient \( \mathbb{P}^{15} \) which is general in the sense that \( L \cap S_5 \) is smooth and if \( \text{codim}_{\mathbb{P}^{15}} L > 1 \), \( L \) contains one of the 10-dimensional family of \( \mathbb{P}^4 \)'s lying on \( S_5 \);

4. The blowup \( \text{Bl}_{\mathbb{P}^4}(\mathbb{P}^m) \) embedded by the linear system of quadric hypersurfaces in \( \mathbb{P}^n \) passing through \( \mathbb{P}^8 \).

Now a straightforward computation of the dimension of the secant varieties of the varieties from the above list yields the following

**Corollary 3.5** (Theorem 2.1 in \cite{H}). Let \( V \subset \mathbb{P}W \) be a nondegenerate irreducible linearly normal smooth subvariety with \( \text{aut}^1 \hat{V} \neq 0 \). If \( SV \subset \mathbb{P}W \) is a hypersurface, then \( V \subset \mathbb{P}W \) is a Severi variety.

**Example 3.6.** Let us explain why the varieties listed in Theorem 3.4(1) have nonzero prolongations. Let \( V^n \subset \mathbb{P}^N \) be a variety from this list. By Theorem 3.8 in Chapter III of \cite{Z}, \( V^n = \psi(\mathbb{P}^n) \), where the rational map \( \psi : \mathbb{P}^n \to \mathbb{P}^N \) is given by the linear system of quadrics passing through a base locus \( B \subset \mathbb{P}^{n-1} \subset \mathbb{P}^n \), where \( B = \emptyset \) for \( V = v_2(\mathbb{P}^m), \; B = \mathbb{P}^{a-1} \cup \mathbb{P}^{b-1} \) for \( V = \mathbb{P}^a \times \mathbb{P}^b \) (meaning that \( B = \mathbb{P}^{a-1} \cup \mathbb{P}^{b-1} \) and \( \mathbb{P}^{a-1} \cap \mathbb{P}^{b-1} = \emptyset \)), \( B = \mathbb{P}^1 \times \mathbb{P}^{m-2} \) (Segre embedding) for \( V = G(1, m) \), \( B = \mathbb{P}^{m-2} \) for \( V = Q^n, \; B = G(1, 4) \) for \( V = S_5 \), and \( B = S_5 \) for \( V = \mathbb{O}\mathbb{P}^2 \). In coordinates, let \( \mathbb{P}^{n-1} = P(U), \; \mathbb{P}^n = P(C \oplus U) \) and \( \mathbb{P}^N = P(W) \), where \( W = C \oplus U \oplus K \), where \( U \) and \( K \) are complex vector spaces. The linear system of quadrics defining \( \psi \) contains the reducible quadrics formed by \( \mathbb{P}^{n-1} \) and an arbitrary hyperplane in \( \mathbb{P}^n \), and \( \psi(t : u) = (t^2 : tu : \sigma(u, u)) \), where \( \sigma : \text{Sym}^2 U \to K \) corresponds to the linear system of quadrics in \( \mathbb{P}^{n-1} \) passing through \( B \). The map \( \psi \) identifies the affine space \( U \) with an open subset of \( V \) and, in the coordinates \( (w_0, w_1, w_2) \) corresponding to the decomposition \( W = C \oplus U \oplus K \), \( V \) is locally defined by the equations \( w_0w_2 = \sigma(w_1, w_1) \).

Now define \( A : \text{Sym}^2 W \to W \) by
\[
A((w_0, w_1, w_2), (w'_0, w'_1, w'_2)) = \left( w_0w'_0, \frac{w_0w'_0 + w'_0w_1}{2}, \sigma(w_1, w'_1) \right).
\]
Then $A \neq 0$ and in [FH2, Proposition 7.11] it is shown that, for any $v \in \hat{V}$ and $w \in W$, one has $A(v, w) \in T_{V,w}$, i.e. $A \in \text{aut}^1 \hat{V}$.

**Remark 3.7.** In [FH1 Proposition 3.4] it is shown that for the varieties $V$ from Theorem 3.3(1) one actually has $\text{aut}^1 \hat{V} \simeq W^*$.

**Example 3.8.** Now let $V = V_{i_1} \subset \mathbb{P}^{N_1}$, $n_i = \dim V_i = 2^i$, $N_i = \frac{3n_i}{2} + 2$ be the $i$-th Severi variety, $1 \leq i \leq 4$ (cf. Remark 2.4), and let $Y = Y_i = SV_i$ be the corresponding cubic hypersurface. Being special cases of Theorem 3.3(1), Severi varieties have nonzero prolongations constructed in Example 3.6. Furthermore, the description of the base loci $B$ given in Example 3.6 shows that $\dim U_i = \dim V_i = n_i$, $Q = Q_i = \psi_i(\mathbb{P}U_i) \subset V_i$ is a smooth quadric of dimension $n_i/2 = 2^{i-1}$, $y = y_i = \psi_i(C)$ is a point in $\langle Q \rangle \setminus Q$, where $Q$ is the entry locus of $y$, i.e. the locus of points $v \in V$ for which there exists a point $v' \in V$ such that $\langle v, v' \rangle \ni y$, and the hyperplane $H = T_{V,y}$ is tangent to $V$ along $Q$ and to $Y$ along $\langle Q \rangle$ (cf. [Z1, ch. IV, Theorem 2.4]). Conversely, by [Z1, ch. III, Theorem 3.8] (or by the formula for $\psi$ given in Example 3.6), if $\pi : \mathbb{P}W \longrightarrow \mathbb{P}^n$ is the projection with center $\langle Q \rangle = \mathbb{P}^{\frac{n}{2} + 1}$, then $\pi(H) = \mathbb{P}^{n-1} \subset \mathbb{P}^n$, $\pi \circ \psi = \text{Id}$, $\pi_W : V \longrightarrow \mathbb{P}^n$ is the birational isomorphism inverse to $\psi$, and $\pi_{V,H,V} : V \setminus H \cap V \iso U = \mathbb{P}^n \setminus \mathbb{P}^{n-1}$. Furthermore, if $x = \frac{1}{2}H = \gamma(y) \in X = Y^*$, then, by duality, $\frac{1}{2}T_X(x) = \langle Q \rangle$ and $\frac{1}{2}\langle Q \rangle = T_X(x) \subset SX = V^*$. Recall that $X$ is also a Severi variety (projectively isomorphic to $V$) and the linear subspaces $T_X(x)$ sweep out $SX$. Hence, in particular, a hyperplane $L$ is tangent to $V$ if and only if it contains one of the quadratic entry loci of type $Q$.

Fixing $Q = \psi(\mathbb{P}U)$ as above, we observe that the projection $\pi$ establishes a projective equivalence between the hyperplanes in $\mathbb{P}W$ passing through $Q$ and the hyperplanes in $\mathbb{P}(\mathbb{C} \oplus U)$, and this equivalence is respected by the map $\psi : \mathbb{P}(\mathbb{C} \oplus U) \longrightarrow V$. Thus the argument in Example 3.6 shows that for any $L \supset Q$ one has $\text{aut}^1 \hat{V}_L \neq 0$, where $V_L = L \cap V$.

**Corollary 3.9.** Let $V \subset \mathbb{P}^N$ be a Severi variety, let $L \subset \mathbb{P}^N$ be a hyperplane, and put $V_L = L \cap V$. The following conditions are equivalent:

a) $V_L$ is singular (i.e. $L$ is tangent to $V$, viz. $\frac{1}{2}L \in V^*$);

b) $\text{aut}^1 \hat{V}_L \neq 0$.

**Proof.** As we saw in Example 3.8 a) $\Rightarrow$ b). On the other hand, suppose that $V_L$ is nonsingular, but $\text{aut}^1 V_L \neq 0$. Then $V_L$ satisfies the hypotheses of Theorem 3.4 (to see that $V_L$ is linearly normal without introducing new arguments it suffices to observe that $SV_L = L \cap SV$ because the entry loci $Q$ have dimension $\frac{n}{2} > 0$ and apply Lemma 3.3). However, the list of varieties in Theorem 3.4 does not contain hyperplane sections of Severi varieties.

**Proposition 3.10.** Let $V \subset \mathbb{P}^N = \mathbb{P}W$ be an irreducible smooth variety, and let $Y = SV$ be its secant variety. Suppose that $V$ is an irreducible component of $Y' = \text{Sing} Y$. Then $\text{aut} \hat{Y} = \text{aut} \hat{V}$ and $\text{aut}^1 \hat{Y} = \text{aut}^1 \hat{V}$.

**Proof.** Let $g \in \text{aut} \hat{V} \subset \mathfrak{gl} W$, and let $\xi_0 \subset \text{GL} W$ be the 1-parameter subgroup generated by $g$. Since $\hat{V}$ is invariant with respect to $\xi_0$, the same is true for $\hat{Y} = SV$, hence $v \in \text{aut} \hat{Y}$. Conversely continuous families of automorphisms of $\hat{Y}$ preserve every irreducible component of its singular locus $Y'$, which proves the first claim. The second claim follows from the first one by the definition of prolongations.
Corollary 3.11. Let \( V \subset \mathbb{P}^N \) be a Severi variety, and let \( Y = SV \) be its secant variety. Then \( \text{aut}^1 \hat{V} \neq 0 \).

**Proof.** It is well known that for Severi varieties \( Y' = V \). Hence the claim follows from Example 3.6 and Proposition 3.10. \( \square \)

We are now ready to prove the Main Theorem.

**Theorem 3.12.** Let \( Y \subset \mathbb{P}W \) be an irreducible cubic hypersurface. Assume that

- a) \( Y \) is not polar defective;
- b) the singular locus \( Y' \) is smooth;
- c) \( \text{aut}^1 \hat{Y} \neq 0 \).

Then \( Y \) is the secant variety of a Severi variety.

**Proof.** By Proposition 3.1, \( Y \) is dual defective. By Corollary 2.10, \( Y = SY_0' \) for some irreducible (smooth) component \( Y_0' \subset Y' \), and by Lemma 3.3, \( Y_0' \) is linearly normal. By Proposition 3.10, we have \( \text{aut}^1 \hat{Y}_0' = \text{aut}^1 \hat{Y} \neq 0 \). Thus the claim is an immediate consequence of Corollary 3.5. \( \square \)

We conclude with giving examples showing that none of the hypotheses a)-c) in the statement of Theorem 3.12 can be lifted.

**Examples 3.13.** Let \( V = V_i \subset \mathbb{P}^N \), \( i = 1, \ldots, 4 \) be a Severi variety, and let \( Y = SV \) be its secant variety. Then \( X = Y'' \simeq V \) and \( V' \simeq SX \) (projective equivalence, cf. [Z1, ch. IV, (2.5.4)])]. There are three types of hyperplanes \( L \subset \mathbb{P}^N \) with respect to \( V \) and \( Y' \) corresponding to the filtration \( X \subset SX \subset \mathbb{P}^N \) or, equivalently, to the three orbits \( O_1, O_2, O_3 \) of the action of \( \text{GL}_3 \) on \( \mathbb{P}W \) characterized by the rank of nonzero Hermitian \( 3 \times 3 \) matrices over composition algebras over \( \mathbb{C} \) making up \( W' \) (so that \( X = O_1, SX \setminus X = O_2 \) and \( \mathbb{P}W' \setminus SX = O_3 \)). Let \( L \in \mathbb{P}^N \) be a hyperplane, and put \( V_L = L \cap V, Y_L = L \cap Y \). As we observed in the proof of Corollary 3.9, \( Y_L = SV_L \). In the following examples we study the properties of the cubic hypersurfaces \( Y_L \) depending on the position of \( L \).

(i) Suppose that \( L \) is a general hyperplane, i.e. \( L \in O_3 \). Then \( \text{Sing} Y_L = Y_L'' = L \cap Y'' = L \cap V = V_L \) and from Corollary 3.9 and Proposition 3.10 it follows that \( \text{aut}^1 Y_L = 0 \). As we recalled in the proof of Lemma 2.10, \( \text{def} Y_L = \text{def} Y - 1 > 0 \) while \( \text{pdef} Y_L = 0 \) and \( Y_L'' \) is smooth; so conditions a) and b) of the Main Theorem are satisfied. This example shows that condition c) of the Main Theorem cannot be lifted. It is worthwhile to note that in this example \( Y_L'' = V_L \) is a homogeneous variety, and so \( \text{aut} Y_L \) is big; nevertheless, \( \text{aut}^1 Y_L = 0 \).

(ii) Suppose now that \( L \in O_2 \). Then \( L \) is tangent to \( V \) at a unique point \( v \in V \) and, by Example 3.8, \( L \supset Q \ni v \), where \( Q \subset V \) is a nonsingular \( \frac{N}{2} \)-dimensional quadric. Furthermore, \( \text{Sing} Y_L = Y_L'' = V_L \) and from Corollary 3.9 and Proposition 3.10 it follows that \( \text{aut}^1 Y_L \neq 0 \). It is clear that \( \text{def} Y_L > 0 \), and from the proof of Proposition 2.5 it follows that \( \text{pdef} Y_L = 0 \). Thus conditions a) and c) of the Main Theorem are satisfied. This example shows that condition b) of the Main Theorem cannot be lifted.

(iii) Suppose finally that \( L \subset \mathbb{P}W \) is most special, i.e. \( L \in O_1 \). Then \( L \) is tangent to \( V \) along a nonsingular \( \frac{N}{2} \)-dimensional quadric \( Q \) (the entry locus
of a smooth point \( y \in Y \); cf. Example 3.8], and \( L \) is tangent to \( Y \) along (i.e. at the nonsingular points of) the \( (\frac{n}{2} + 1) \)-dimensional linear subspace \( \langle Q \rangle \). From Corollary 3.9 and Proposition 3.10 it follows that \( \text{aut}^1 Y_L \neq 0 \), and so condition c) of the Main Theorem is satisfied. Furthermore, \( \text{Sing} Y_L = Y'_L \cup \langle Q \rangle \), where \( Y'_L = V_L = L \cap V \). For \( i > 1 \), \( \text{Sing} Y_L = V_L \cup \langle Q \rangle \) is reducible: for \( i = 2 \) it has three components (\( \mathbb{P}^3 \) and two Segre varieties \( \mathbb{P}^1 \times \mathbb{P}^2 \), all meeting along \( Q \)), and for \( i = 3, 4 \) there are two components \( (\mathbb{P}^{n/2+1} \text{ and } V_L \text{ meeting along } Q_L) \). Thus, for \( i > 1 \), \((Y_L)'\) is singular (and \( \text{Sing} (Y_L)' = \text{Sing} Y'_L = Q \)) and condition b) fails. For \( i = 1 \), the singular locus \((Y_L)'\) is a plane containing the reduced singular locus \( Y'_L \) which is a conic, and so condition b) of the Main Theorem is satisfied. However, it is clear that \( x = \perp L = \gamma(y) = \phi(y) \in \phi(L) \), and from the proof of Lemma 2.6 it follows that \( \text{pdef} Y = 1 \) and condition a) of the Main Theorem fails. More precisely, the dual variety \( X_L = Y'_L^* \) is the projection of the Severi variety \( X \) from the point \( x \in X \), and under this projection the point \( x \) is blown up to a linear subspace \( \Lambda = \perp \langle Q \rangle \subset X_L \) (cf. Example 3.8). From our construction and Proposition 2.5 it follows that \( Z = \phi(\langle Y_L \rangle) = S(\Lambda, X^n) \) is the cone with vertex \( \Lambda \) over the variety \( \pi_{\Lambda}(X^n) = \pi_{T_X, x}(X) \subset \mathbb{P}^{n/2+1} \) which is easily seen to be a nonsingular quadric. Summing up, the polar image of any hyperplane from the orbit \( O_1 \) is a quadratic cone with vertex \( \mathbb{P}^n \) over an \( \frac{n}{2} \)-dimensional quadric. This example (particularly for \( i = 1 \)) shows that condition a) of the Main Theorem cannot be lifted.

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