EVERYWHERE DIFFERENTIABILITY OF ABSOLUTE MINIMIZERS FOR LOCALLY STRONGLY CONVEX AND CONCAVE HAMILTONIAN \( H(p) \in C^0(\mathbb{R}^n) \) WITH \( n \geq 3 \)

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Abstract. Suppose that \( n \geq 3 \) and \( H(p) \in C^0(\mathbb{R}^n) \) is a locally strongly convex and concave Hamiltonian. We obtain the everywhere differentiability of all absolute minimizers for \( H \) in any domain of \( \mathbb{R}^n \).

1. Introduction

Let \( n \geq 2 \) and suppose that \( H \in C^0(\mathbb{R}^n) \) is convex and coercive (i.e., \( \liminf_{p \to \infty} H(p) = \infty \)). Aronsson 1960’s initiated the study of minimization problems for the \( L^\infty \)-functional

\[
\mathcal{F}_H(u, \Omega) = \text{esssup}_{x \in \Omega} H(Du(x))
\]

for any domain \( \Omega \subset \mathbb{R}^n \) and function \( u \in W^{1,\infty}_{\text{loc}}(\Omega) \); see \([2, 3, 4, 5]\). Given any domain \( \Omega \subset \mathbb{R}^n \), by Aronsson a function \( u \in W^{1,\infty}_{\text{loc}}(\Omega) \) is called an absolute minimizer for \( H \) in \( \Omega \) (write \( u \in \text{AM}_H(\Omega) \) for simplicity) if

\[
\mathcal{F}_H(u, V) \leq \mathcal{F}_H(v, V) \text{ whenever } V \subset \Omega, \quad v \in W^{1,\infty}_{\text{loc}}(V) \cap C(V) \text{ and } u = v \text{ on } \partial V.
\]

It turns out that the absolute minimizer is the correct notion of minimizers for such \( L^\infty \)-functionals. The existence of absolute minimizers for given continuous boundary in bounded domains was proved by Aronsson \([4]\) for \( \frac{1}{2}|p|^2 \) and Barron-Jensen-Wang \([9]\) for general \( H(p) \in C^0(\mathbb{R}^n) \); while their uniqueness was built up by Jensen \([26]\) for \( \frac{1}{2}|p|^2 \) (see also \([1, 8, 13]\)), and by Jensen-Wang-Yu \([27]\) and Armstrong-Crandal-Julin-Smart \([7]\) for \( H(p) \in C^2(\mathbb{R}^n) \) and \( H(p) \in C^0(\mathbb{R}^n) \), respectively, with \( H^{-1}(\min H) \) having empty interior.

Moreover, if \( H \in C^1(\mathbb{R}^n) \) is convex and coercive, absolute minimizers coincide with viscosity solutions to the Aronsson equation (a highly degenerate nonlinear elliptic equation)

\[
(1.1) \quad \mathcal{A}_H(u) := \sum_{i,j=1}^n H_{p_i} (Du) H_{p_j} (Du) u_{x_i} u_{x_j} = 0 \quad \text{in } \Omega,
\]

see Jensen \([20]\) for \( H(p) = \frac{1}{2}|p|^2 \), and Crandall-Wang-Yu \([15]\) and Yu \([33]\) (and also \([7, 9, 10, 23, 13]\)) in general. Here \( H_{p_i} = \frac{\partial H}{\partial p_i} \) for \( H \in C^1(\mathbb{R}^n) \), \( u_{x_i} = \frac{\partial u}{\partial x_i} \) for \( u \in C^1(\mathbb{R}^n) \), and \( u_{x_i x_j} = \frac{\partial^2 u}{\partial x_i \partial x_j} \) for \( u \in C^2(\mathbb{R}^n) \). For the theory of viscosity solution see \([14]\). In the special case \( H(p) = \frac{1}{2}|p|^2 \), the Aronsson equation \((1.1)\) is the \( \infty \)-Laplace equation

\[
(1.2) \quad \Delta_{\infty} u := \sum_{i,j=1}^n u_{x_i} u_{x_j} u_{x_i x_j} = 0 \quad \text{in } \Omega
\]

and its viscosity solutions are called as \( \infty \)-harmonic functions. If \( H \in C^0(\mathbb{R}^n) \) but \( \not\in C^1(\mathbb{R}^n) \), we refer to \([13, 7]\) for further discussions and related problems on the Euler–Lagrange equation for absolute minimizers.

The regularity of absolute minimizer is then the main issue in this field.

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By Aronsson [6], \( \infty \)-harmonic functions are not necessarily \( C^2 \)-regular; indeed \( \infty \)-harmonic functions \( x_1^{4/3} - x_2^{1/3} \) in whole \( \mathbb{R}^n \) is not \( C^2 \)-regular. Such a function also leads to a well-known conjecture on the \( C^{1,1/3}, \) and \( W^{2, t}_{\text{loc}} \)-regularity with \( 1 \leq t < 3/2 \) of \( \infty \)-harmonic functions. A seminar step towards this is made by Crandall-Evans [11], who obtained their linear approximation property. They [12] also proved that all bounded \( \infty \)-harmonic functions in whole \( \mathbb{R}^n \) with \( n \geq 2 \) must be constant functions.

Next, when \( n = 2 \), Savin [30] established their interior \( C^1 \)-regularity and then deduced the corresponding Liouville theorem, that is, all \( \infty \)-harmonic functions in whole plane with a linear growth at \( \infty \) (that is, \( |u(x)| \leq C(1 + |x|) \) for all \( x \in \mathbb{R}^2 \)) must be linear functions. Later, the interior \( C^{1, \alpha} \)-regularity for some \( 0 < \alpha < 1/3 \) was proved by Evans-Savin [17] and the boundary \( C^1 \)-regularity by Wang-Yu [32]. Recently, Koch-Zhang-Zhou [28] proved that \( |Du|^\alpha \in W^{1,2}_{\text{loc}} \) for all \( \alpha > 0 \) and all \( \infty \)-harmonic functions \( u \) in planar domains, which is sharp as \( \alpha \to 0 \); also that the distributional determinant \( -\det D^2 u \) is a nonnegative Radon measure.

Moreover, when \( n \geq 3 \), Evans-Smart [18, 19] obtained their everywhere differentiability; Miao-Wang-Zhou [29] and Hong-Zhao [25] independently observed an asymptotic Liouville property, that is, if \( u \) is a \( \infty \)-harmonic function in whole \( \mathbb{R}^n \) with a linear growth at \( \infty \), then \( \lim_{R \to \infty} \frac{1}{2R^2} u(Rx) = e \cdot x \) locally uniformly for some vector \( e \) with \( |e| = \|Du\|_{L^\infty(\mathbb{R}^n)} \). But \( C^4, C^{1, \alpha} \)-regularity and the corresponding Liouville theorem of \( \infty \)-harmonic functions are completely open.

On the other hand, if \( H \in C^2(\mathbb{R}^n) \) is locally strongly convex, Wang-Yu [31] obtained the linear approximation property of absolute minimizer, and when \( n = 2 \), the \( C^1 \)-regularity and hence the corresponding Liouville theorem. In this paper, we say that \( H \in C^0(\mathbb{R}^n) \) is locally strongly convex (resp. concave) if for any convex subset \( U \) of \( \mathbb{R}^n \), there exists \( \lambda > 0 \) depending on \( U \) (resp. \( \Lambda > 0 \)) such that

\[
H(p) - \frac{\lambda}{2} |p|^2 \quad \text{(resp. } \frac{\Lambda |p|^2}{2} - H(p) \text{)} \quad \text{is convex in } U.
\]

Note that \( H \in C^2(\mathbb{R}^n) \) implies that \( H \) is always locally strongly concave. In particular, the \( l_\alpha \)-norm for \( 2 < \alpha < \infty \) provides a class of typical example of locally strongly convex and concave but non-Hilbertian Hamiltonians.

Recently, under the assumptions that \( H \in C^0(\mathbb{R}^n) \) is convex and coercive, it was shown by Fa-Wang-Zhou [20] that \( H \) is not a constant in any line segment if and only if all absolute minimizers for \( H \) have the linear approximation property; moreover, when \( n = 2 \), if and only if all absolute minimizers for \( H \) are \( C^1 \)-regular, and also if and only if the corresponding Liouville theorem holds. In [21], we proved that if \( H \in C^2(\mathbb{R}^2) \) is locally strongly convex and concave, then \( H(Du)^\alpha \in W^{1,2}_{\text{loc}} \) for all \( \alpha > \frac{1}{2} - \tau_H \) for all absolute minimizers \( u \) in planar domains, where \( 0 < \tau_H \leq \frac{1}{2} \) and \( \tau_H = 1/2 \) when \( H \in \mathcal{C}^2(\mathbb{R}^2) \); and also that the distributional determinant \( -\det D^2 u \) is a nonnegative Radon measure. But, when \( n \geq 3 \), the everywhere differentiability, \( C^1, C^{1, \alpha} \)-regularity and the Liouville theorem is not clear.

If \( n \geq 3 \) and \( H \in C^0(\mathbb{R}^n) \) is locally strongly convex and concave, this paper aims to prove the following everywhere differentiability (Theorem 1.1 below) and asymptotic Liouville property (Theorem 1.2 below) of absolute minimizers.

**Theorem 1.1.** Suppose that \( n \geq 3 \) and \( H \in C^0(\mathbb{R}^n) \) is locally strongly convex and concave. Let \( \Omega \subset \mathbb{R}^n \) be any domain. If \( u \in AM_H(\Omega) \), then \( u \) is differentiable everywhere in \( \Omega \).

**Theorem 1.2.** Suppose that \( n \geq 3 \) and \( H \in C^0(\mathbb{R}^n) \) is locally strongly convex/concave. If \( u \in AM_H(\mathbb{R}^n) \) with a linear growth at \( \infty \), then there exists a unique vector \( e \) such that

\[
H(e) = \|H(Du)\|_{L^\infty(\mathbb{R}^n)} \quad \text{and} \quad \lim_{R \to \infty} \frac{1}{R} u(Rx) = e \cdot x \quad \text{locally uniformly in } \mathbb{R}^n.
\]

When \( n \geq 3 \), it is unclear to us whether the assumption for \( H \) in Theorems 1.1, 1.2 can be relaxed to the weaker (and also necessary in some sense) assumption that \( H \in C^0(\mathbb{R}^n) \) is convex and coercive.
and is not a constant in any line segment. By [20], if $H \in C^0(\mathbb{R}^n)$ is convex and coercive, and is constant in some line-segment, then both of Theorems 1.1, 1.2 are not necessarily true.

In particular, it would be interesting to prove the everywhere differentiability of absolute minimizer for $l_\alpha$-norm with $1 < \alpha < 2$. Recall that if $2 < \alpha < \infty$, then $l_\alpha$-norm belongs to $C^2(\mathbb{R}^n)$ and is convex, and hence both of the conclusions of Theorem 1.1, 1.2 holds. If $\alpha = 1$ or $\infty$, the $l_\alpha$-norm will be constant in some line-segment.

By Remark 1.3 below, we only need to prove Theorems 1.1, 1.2 when $H \in C^0(\mathbb{R}^n)$ satisfies (H1) $H$ is strongly convex and concave in $\mathbb{R}^n$, that is, there exist $0 < \lambda \leq \Lambda < \infty$ such that both of $H(p) - \frac{1}{\lambda}|p|^2$ and $\frac{1}{\Lambda}|p|^2 - H(p)$ are convex in $\mathbb{R}^n$.

(H2) $H(0) = \min_{p \in \mathbb{R}^n} H(p) = 0.$

Remark 1.3. Suppose that $H \in C^0(\mathbb{R}^n)$ is locally strongly convex and concave.

(i) If $u \in AM_H(\Omega)$ for some domain $\Omega \subset \mathbb{R}^n$, letting $U \subset \Omega$ be arbitrary subdomain, we have $k = \|Du\|_{L^\infty(U)} < \infty$. Next, by [21] Lemma A.8, there exists a $\bar{H}$ which is strongly convex/concave in $\mathbb{R}^n$ and $\bar{H} = H$ in $B(0, k + 1)$. Thus $u \in AM_{\bar{H}}(U)$. The strongly convexity of $\bar{H}$ implies that there exists a $p_0 \in \mathbb{R}^n$ such that $\min_{p \in \mathbb{R}^n} \bar{H}(p) = H(p_0)$. Set $\bar{H}(p) = H(p + p_0) - H(p_0)$ for $p \in \mathbb{R}^n$. Then $\bar{H}$ satisfies (H1)&(H2). Write $\bar{u}(x) = u(x) - p_0 \cdot x$ for all $x \in U$. We have $\bar{u} \in AM_{\bar{H}}(U)$. Since $u$ and $\bar{u}$ have the same regularity in $U$, we only need to prove the everywhere differentiability of $\bar{u}$ in $U$.

(ii) If $u \in AM_{\bar{H}}(\mathbb{R}^n)$ has a linear growth at $\infty$, then by [20] we have $k := \|Du\|_{L^\infty(\mathbb{R}^n)} < \infty$. Let $\bar{u}$ and $\bar{H}$ as above. Then $u$ is linear if and only if $\bar{u}$ is linear. So we only need to prove $\bar{u}$ is linear.

Unless otherwise specifying, we always assume that $H \in C^0(\mathbb{R}^n)$ satisfies (H1)&(H2) below. Note that the geometric&variational approach used in dimension 2 (see Savin [30] and also [20, 31]) is not enough to prove Theorems 1.1, 1.2 since it includes a key planar topological argument. Moreover, since $H \in C^0(\mathbb{R}^n)$ does not have Hilbert structure necessarily, it is not clear whether one can prove Theorem 1.1 by using the idea of Evans-Smart [18]—a PDE approach based on maximal principle (see also Remark 2.6 (i)). But, in Section 2, we are able to prove Theorems 1.1, 1.2 by borrowing some idea of Evans-Smart [19]—a PDE approach based on an adjoint argument, and using the following crucial ingredients:

(a) the linear approximation property of any given absolute minimizer $u$ for $H$ as obtained in Fa-Wang-Zhou [20] and Wang-Yu [31] (see Lemmas 2.1&2.5).

(b) a stability result in [21] (see Lemma 2.2) which allows to approximate $u$ via absolute minimizers $u^\gamma$ of a Hamiltonian $H^\gamma$, where $H^\gamma$ is a smooth approximation of $H$ and satisfies (H1)&(H2) with the same constants $\lambda, \Lambda$.

(c) a uniform approximation to $u^\gamma$ via smooth functions $u^{\gamma, \epsilon}$ (see Theorem 2.3), which is an appropriate modification of Evans’ approximation via $e^{\frac{\epsilon}{2}H^\gamma}$-harmonic functions in [10]. The point is that none of $k \geq 3$ -order derivatives of $H^\gamma$ is involved in the linearization of the equation (2.2) for $u^{\gamma, \epsilon}$.

(d) an integral flatness estimate for $u^{\gamma, \epsilon}$ (see Theorem 2.4).

Theorem 2.3 will be proved in Section 3. The novelty in the proof of Theorem 2.3 is that we use viscosity solutions to certain Hamilton-Jacobi equation as barrier functions to get a boundary regularity of $u^{\gamma, \epsilon}$ and then conclude the uniform approximation of $u^{\gamma, \epsilon}$ to $u^\gamma$. The reason to use $u^{\gamma, \epsilon}$ instead of $e^{\frac{\epsilon}{2}H^\gamma}$-harmonic functions is that the linearization of $e^{\frac{\epsilon}{2}H^\gamma}$-harmonic equation contains 3-order derivatives of $H^\gamma$; see Remark 2.3 (i) for details.

Theorem 2.4 will be proved in Section 5. To this end, we generalize in Section 4 the adjoint arguments of [19] to Hamiltonian $H^\gamma$ and $u^{\gamma, \epsilon}$. Since none of $k \geq 3$ -order derivatives of $H^\gamma$ is involved in the equation for $u^{\gamma, \epsilon}$, all key estimates in Theorem 2.3 and Section 4 rely only on $\lambda$ and $\Lambda$. This is indeed important to get Theorem 2.4 Moreover, since $H \in C^0(\mathbb{R}^n)$ does not have Hilbert
structure in general, some new ideas are needed to get Theorem 2.4 in Section 5; in particular, the test function used in the proof of flatness estimates in [19] is not enough to us, as an another novelty we find a suitable test function and build up some related estimates.

2. Proofs of Theorems 1.1-1.2

Considering Remark 1.3, we always assume that $H \in C^0(\mathbb{R}^n)$ satisfies (H1)&(H2). To prove Theorem 1.1, let $\Omega$ be any domain of $\mathbb{R}^n$, and $u \in AM_H(\Omega)$. We recall the following linear approximation property of $u$ as established by [20].

**Lemma 2.1.** For any $x \in \Omega$ and any sequence $\{r_j\}_{j \in \mathbb{N}}$ which converges to 0, there exist a subsequence $\{r_{j_k}\}_{k \in \mathbb{N}}$ and a vector $e_{\{r_{j_k}\}_{k \in \mathbb{N}}}$ such that

$$\lim_{k \to \infty} \sup_{y \in B(0,1)} \left| \frac{u(x + r_{j_k}y) - u(x)}{r_{j_k}} - e_{\{r_{j_k}\}_{k \in \mathbb{N}}} \cdot y \right| = 0$$

and

$$H(e_{\{r_{j_k}\}_{k \in \mathbb{N}}}) = \lim_{r \to 0} \|H(Du)\|_{L^\infty(B(x,r))}.$$  

For each $x \in \Omega$, denote by $\mathcal{D}u(x)$ the collection of all possible vector $e_{\{r_{j_k}\}_{k \in \mathbb{N}}}$ as above. Observe that $u$ is differentiable at $x$ if and only if $\mathcal{D}u(x)$ is a singleton; in this case $\mathcal{D}u(x) = \{Du(x)\}$.

To see that $\mathcal{D}u(x)$ is a singleton, we need the following approximation to $u$ given in [21]. Precisely, let $\{H^\gamma\}_{\gamma \in (0,1]}$ be a standard smooth approximation to $H$ as below. For each $\gamma \in (0,1]$, let $\tilde{H}^\gamma = \eta_\gamma \ast H$, where $\eta_\gamma$ is standard smooth mollifier. Since $H^\gamma$ is strictly convex there exists a unique point $p^\gamma \in \mathbb{R}^2$ such that $\tilde{H}^\gamma(p^\gamma) = \min_{p \in \mathbb{R}^2} \tilde{H}^\gamma(p)$. Set

$$(2.1) \quad H^\gamma(p) = \tilde{H}^\gamma(p + p^\gamma) - \tilde{H}^\gamma(p^\gamma) \quad \forall p \in \mathbb{R}^n.$$  

Obviously, $H^\gamma$ satisfies (H2); by [21] Appendix A, $\{H^\gamma\}_{\gamma \in (0,1]}$ satisfies (H1) with the same $\lambda$ and $\Lambda$, and $H^\gamma \to H$ locally uniformly as $\gamma \to 0$. For each $\gamma \in (0,1]$ and $U \subseteq \Omega$, let

$$u^\gamma \in AM_{H^\gamma}(U) \cap C^0(\overline{U})$$

with $u^\gamma = u$ on $\partial U$.

We then have the following result; see [21] for $n = 2$ and note that the proofs in [21] also works for $n \geq 3$.

**Lemma 2.2.** We have

$$\|H^\gamma(Du^\gamma)\|_{L^\infty(U)} \leq C\Lambda\|u^\gamma\|_{C^{0,1}(\partial U)} \quad \forall \gamma \in (0,1],$$

and $u^\gamma \to u$ in $C^0(U)$ as $\gamma \to 0$.

Next, for any $\gamma \in (0,1]$, to approximate $u^\gamma$ in a smooth way we consider the following Dirichlet problem:

$$(2.2) \quad \mathcal{A}_{H^\gamma}(v) + \epsilon \Delta v = 0 \quad \text{in} \ U; \ v = u^\gamma \text{ on } \partial U.$$  

The following result is proved in Section 3.

**Theorem 2.3.** For each $\epsilon, \gamma \in (0,1]$, there exists a unique solution $u^{\gamma,\epsilon} \in C^\infty(U) \cap C^0(\overline{U})$ to (2.2). Moreover, the following hold.

(i) We have

$$\|u^{\gamma,\epsilon}\|_{C^0(\overline{U})} \leq \|u^\gamma\|_{C^0(\partial U)} \quad \forall \epsilon \in (0,1].$$

(ii) We have

$$\|Du^{\gamma,\epsilon}\|_{C^0(\overline{U})} \leq C_0(\lambda, \Lambda, \text{dist}(V, \partial U), \|u^\gamma\|_{C^0(\partial U)}) \quad \text{for any } V \subseteq U \text{ and } \epsilon \in (0,1],$$

where the constant $C_0(\lambda, \Lambda, \text{dist}(V, \partial U), \|u^\gamma\|_{C^0(\partial U)})$ depends only on $\lambda$, $\Lambda$, $\text{dist}(V, \partial U)$ and $\|u^\gamma\|_{C^0(\partial U)}$. 

(iii) There exist $\epsilon_*>0$ and $C_*>0$ depending on $H^\gamma$ and $\|u^\gamma\|_{C^{0,1}(\overline{U})}$ such that for any $0<\epsilon<\epsilon_*$, we have

$$|u^{\gamma,\epsilon}(x) - u^\gamma(x_0)| \leq C_*|x-x_0| \quad \forall x \in U, \ x_0 \in \partial U.$$  

(iv) We have $u^{\gamma,\epsilon} \to u^\gamma$ in $C^0(\overline{U})$ as $\epsilon \to 0$.

The existence and uniqueness of $u^{\gamma,\epsilon}$, and also Theorem 2.3 (i) follow from the classical elliptic theory; Theorem 2.3 (iv) from Theorem 2.3 (ii) and (iii). Theorem 2.3 (ii) follows from the approach by [13] based on the maximal principle and the linearized operator arising from (2.2):

$$(2.3) \quad -H_{p_j}(Du^{\gamma,\epsilon})H_{p_j}(Du^{\gamma,\epsilon})v_{x_j,x_j} - 2H_{p_j,p_l}(Du^{\gamma,\epsilon})u^{\gamma,\epsilon}_{x_j,x_l}H_{p_j}(Du^{\gamma,\epsilon})v_{x_l} - \epsilon \Delta v.$$  

Since none of $k \geq 3$ order derivatives of $H$ is involved in (2.3), we will conclude that the constant $C_0$ in Theorem 2.3 (ii) depends at most on $\lambda, \Lambda$ dist$(V, \partial U)$ and $\|u^\gamma\|_{C^0(\partial U)}$. To get Theorem 2.3 (iii), we need new ideas. Indeed, unlike the case $H(p) = \frac{1}{2}|p|^2$, where we use $|x|^{\gamma}$ as a barrier function to conclude Theorem 2.3 (iii) from the comparison principle, the novelty here is that due to we take viscosity solutions $L^b_{\sigma}$ of certain Hamilton-Jacobi equation as barrier functions; see Lemmas 3.1-3.2.

In Section 5, we establish the following flatness estimate of $u^{\gamma,\epsilon}$, which is crucial to show that $\mathcal{D}u(x)$ is a singleton. Denote by $e_n$ the vector $(0, \cdots, 0, 1)$.

**Theorem 2.4.** Suppose that $U = B(0,3)$ and for some $\gamma, \epsilon \in (0,1]$, $u^{\gamma,\epsilon}$ satisfies

$$(2.4) \quad \max_{B(0,3)} |u^{\gamma,\epsilon} - x_n| \leq \tau$$  

for some $0<\tau<1$ and

$$(2.5) \quad H^\gamma(Du^{\gamma,\epsilon}(x_0)) \geq H^\gamma(e_n) - \delta$$  

for some $0<\delta<H(e_n)/2$ and $x_0 \in B(0,1)$. Then

$$(2.6) \quad |Du^{\gamma,\epsilon}(x_0) - e_n|^2 \leq C_1(\lambda, \Lambda) \left[ \tau + \frac{\tau}{\epsilon} \right],$$  

where $\mu = \frac{1}{10n}$. Above $C_1(\lambda, \Lambda)$ is a constant depending only on $\lambda$ and $\Lambda$.

The proof of Theorem 2.4 relies on a generalization of the adjoint method of Evans-Smart [19] to the equation (2.2) as developed in Section 4. Moreover, since $H$ does not have Hilbert structure necessarily, we can not follow the argument of Evans-Smart to get Theorem 2.4, where they take $u^{\gamma,\epsilon}_n - 1$ as a test function. The novelty here is to take $|Du^{\gamma,\epsilon} - e_n|^2$ as a test function. With aid of the estimates in Section 4, by using the strongly convexity/concavity of $H$ and some careful analysis, we are able to prove Theorem 2.4. Again, since none of $k \geq 3$ order derivatives of $H$ are involved in the linearized operator and hence in the whole procedure, we conclude that all constants in estimates in Section 4 and hence $C_1$ in Theorem 2.4 depend on at most $\lambda$ and $\Lambda$.

With the aid of Theorem 2.4, Theorem 2.3 and Lemma 2.11 by some necessary modifications of the arguments of [13] we are able to prove that for any $x \in \Omega$, $\mathcal{D}u(x)$ is singleton, and hence that $u$ is differentiable everywhere in $\Omega$; for reader’s convenience we give the details.

**Proof of Theorem 2.4** By Remark 1.3 we assume $H \in C^0(\mathbb{R}^n)$ satisfies (H1)&(H2). Let $\Omega$ be any domain of $\mathbb{R}^n$, and $u \in AM_H(\Omega)$. It suffices to prove that $\mathcal{D}u(x)$ is singleton. We prove this by contradiction. Assume that $\mathcal{D}u(x_0)$ contains at least two vectors $a \neq b$ with $H(a) = H(b)$ for some $x_0 \in \Omega$. Note that $a, b \neq 0$. We may assume that $x_0 = 0 \in \Omega$, $u(0) = 0$, $a = e_n$ without loss of generality. Set $\theta = |b - e_n| > 0$. We obtain a contradiction by the following 4 steps.

**Step 1.** Fix $\tau_0 \in (0,1]$ such that

$$(2.7) \quad C_1(\lambda, \Lambda)\frac{\tau^2}{32} \leq \frac{\theta^2}{32}, \quad \forall \delta < \delta_0, \tau < \tau_0.$$
Since $e_n \in \mathcal{D}u(0)$ we can find a sequence $\{r_j\}_{j \in \mathbb{N}}$ which converges to 0 such that
\[
\max_{B(0,3)} |u_j(x) - x_n| = \max_{B(0,3r_j)} \frac{|u(x) - a \cdot x|}{r_j} \to 0 \quad \text{as} \; j \to \infty,
\]
where $u_j(x) = u(r_jx)/r_j$ for $x \in \Omega$. For each $\tau \in (0, \tau_0]$, there exists a $j_\tau$ such that if $j \geq j_\tau$,
\[
\max_{B(0,3)} |u_j(x) - x_n| < \frac{\tau}{4} \quad \forall \; j \geq j_\tau.
\]
For any $\gamma \in (0, 1)$ and $j \in \mathbb{N}$, let
\[
u_j^\gamma \in AM_{H^\gamma}(B(0,3)) \quad \text{with} \quad u_j^\gamma = u_j \text{ on } \partial B(0,3).
\]
By Lemma 2.2 for each $j \geq j_\tau$, $u_j^\gamma \to u_j$ as $\gamma \to 0$, there exists $\gamma_{j,\tau} > 0$ such that
\[
\max_{B(0,3)} |u_j^\gamma - x_n| \leq \frac{\tau}{2} \quad \forall \gamma \in (0, \gamma_{j,\tau}].
\]
By Theorem 2.3 (iv), for each $j \geq j_\tau$ and $\gamma < \gamma_{j,\tau}$, there is an $\epsilon_{j,\gamma,\tau} \in (0, 1]$ that
\[
\max_{B(0,3)} |u_j^{\gamma,\epsilon} - x_n| \leq \tau \quad \forall \epsilon < \epsilon_{j,\gamma,\tau}.
\]
**Step 2.** Since $b \in \mathcal{D}(u)(0)$ by \[20\], there exist a sequence $\{s_k\}_{k=1}^\infty$ which converge to zero such that
\[
\max_{B(0,s_k/r_j)} \frac{|u_j(x) - b \cdot x|}{s_k/r_j} = \max_{B(0,s_k)} \frac{|u(x) - b \cdot x|}{s_k} \to 0 \quad \text{as} \; k \to \infty
\]
for all $j \in \mathbb{N}$. For each $\eta \in (0, \tau)$ and $j \geq j_\tau$, there exist $k_{\eta,j} \in \mathbb{N}$ such that for all $k \geq k_{\eta,j}$, we have $s_k/r_j \leq 1$ and
\[
\max_{B(0,s_k/r_j)} \frac{|u_j - b \cdot x|}{s_k/r_j} \leq \frac{\eta}{4}.
\]
Since $u_j^\gamma \to u_j$ in $B(0,3)$, for each $k \geq k_{\eta,j}$ we can find $\gamma_{k,j,\eta} < \gamma_{j,\tau}$ such that for each $\gamma \in (0, \gamma_{k,j,\eta})$,
\[
\max_{B(0,s_k/r_j)} \frac{|u_j^{\gamma,\epsilon} - b \cdot x|}{s_k/r_j} \leq \frac{\eta}{2}.
\]
Since $u_j^{\gamma,\epsilon} \to u_j^\gamma$ in $B(0,3)$, for each $\epsilon < \epsilon_{j,\gamma,k,j,\eta}$, we further find $\epsilon_{\gamma,k,j,\eta} < \epsilon_{j,\gamma,\tau}$ such that for all $\epsilon < \epsilon_{\gamma,k,j,\eta}$,
\[
\max_{B(0,s_k/r_j)} \frac{|u_j^{\gamma,\epsilon} - b \cdot x|}{s_k/r_j} \leq \eta.
\]
**Step 3.** For each $\eta \in (0, \tau)$, there exists $\gamma_{\eta} \in (0, 1)$ such that
\[
|H^\gamma(e_n) - H^\gamma(b)| \leq \eta,
\]
where we have used $H(e_n) = H(b)$.
For each $\eta \in (0, \tau)$, $j \geq j_\tau$, $k \geq k_{\eta,j}$, $\gamma < \min\{\gamma_{k,j,\eta}, \gamma_{\eta}\}$ and $\epsilon < \epsilon_{j,\gamma,k,j,\eta}$, by Lemma \[18, 19, 2.11\] implies that there is a point $x_{\epsilon,\gamma,k,j,\eta} \in B(0,s_k/r_j)$ at which
\[
|Du_j^{\gamma,\epsilon}(x_{\epsilon,\gamma,k,j,\eta}) - b| \leq 4\eta.
\]
We further have that
\[
|H^\gamma(e_n) - H^\gamma(Du_j^{\gamma,\epsilon}(x_{\epsilon,\gamma,k,j,\eta}))| \leq C_2(\lambda, \Lambda, b)\eta.
\]
Indeed, by convexity of $H$, we have
\[
H^\gamma(b) - H^\gamma(Du_j^{\gamma,\epsilon}(x_{\epsilon,\gamma,k,j,\eta})) \geq \langle D_pH(Du_j^{\gamma,\epsilon}(x_{\epsilon,\gamma,k,j,\eta})), b - Du_j^{\gamma,\epsilon}(x_{\epsilon,\gamma,k,j,\eta}) \rangle.
\]
Since $|D_p H^\gamma(p)| \leq \Lambda |p|$ and $|Du_j^{\gamma,\epsilon}(x^0)| \leq |b| + 4$, one has
\[ H^\gamma(Du_j^{\gamma,\epsilon}(x,\gamma, k, j, \eta)) - H^\gamma(b) \leq |D_p H^\gamma(Du_j^{\gamma,\epsilon}(x,\gamma, k, j, \eta))| |Du_j^{\gamma,\epsilon}(x,\gamma, k, j, \eta) - b| \leq C(\lambda, \Lambda, b)\eta. \]
A similar estimate holds for $H^\gamma(b) - H^\gamma(Du_j^{\gamma,\epsilon}(x,\gamma, k, j, \eta))$. Thus
\[ |H^\gamma(Du_j^{\gamma,\epsilon}(x,\gamma, k, j, \eta)) - H^\gamma(b)| \leq C(\lambda, \Lambda, b)\eta. \]
This together with Remark 2.6. (i) Recall that Evans [16] suggested another approximation to prove Theorem 1.2; here we omit the details and also refer to [25, 29].

**Step 4.** Let $\delta_0 \in (0, 1]$ such that
\[ C_1(\lambda, \Lambda)\delta \leq \frac{\theta^2}{32}, \quad \forall \delta < \delta_0; \]
For each $\mu \in (0, \frac{\Lambda}{8n}], \delta \in (0, \delta_0]$, let $\epsilon_{\mu, \delta, \theta} \in (0, 1]$ such that
\[ C_1(\lambda, \Lambda)\frac{1}{\epsilon} e^{-\frac{4}{\epsilon}} \leq \frac{\theta^2}{32}, \quad \forall \epsilon \in (0, \epsilon_{\mu, \delta, \theta}]. \]
Let $\eta < \min\{\delta_0/C_2(\lambda, \Lambda, b), \theta/16\}$ and $\delta = C_2(\lambda, \Lambda, b)\eta$. For $\tau < \tau_\theta$, $j \geq j_\theta$, $k \geq k_{j, \eta}$, $\gamma < \min\{\gamma_{k, j, \eta}, \gamma_n\}$, and $\epsilon < \min\{\epsilon_{\gamma, k, j, \eta}, \epsilon_{\mu, \delta, \theta}\}$, by Theorem 2.4 (2.13) and (2.10) imply that
\[ |Du^{\gamma,\epsilon}(x,\gamma, k, j, \eta) - e_n|^2 \leq C_1(\lambda, \Lambda) \left[ \tau + C_2(\lambda, \Lambda, b)\eta + \frac{1}{\epsilon} e^{-\frac{4}{\epsilon}} \right] \leq \frac{\theta^2}{8}. \]
Thus by (2.12) one has
\[ \theta = |e_n - b| \leq 4\eta + \frac{\theta}{2} \leq \frac{3\theta}{4}, \]
which is a contradiction as desired. The proof of Theorem 1.1 is complete. \[\square\]

To prove Theorem 1.2 let $u \in AM_H(\mathbb{R}^n)$ with a linear growth at $\infty$. By [20], $\|Du\|_{L^\infty(\mathbb{R}^n)} < \infty$ and moreover $u$ has the linear approximation property at $\infty$ as below.

**Lemma 2.5.** For any sequence $\{r_j\}_{j \in \mathbb{N}}$ which converges to $\infty$, there exist a subsequence $\{r_{jk}\}_{k \in \mathbb{N}}$ and a vector $e_{\{r_{jk}\}_{k \in \mathbb{N}}}$ such that
\[ \lim_{k \to \infty} \sup_{y \in B(0,1)} \left| \frac{u(r_{jk})}{r_{jk}} - e_{\{r_{jk}\}_{k \in \mathbb{N}}} \cdot y \right| = 0 \]
and
\[ H(e_{\{r_{jk}\}_{k \in \mathbb{N}}}) = \|H(Du)\|_{L^\infty(\mathbb{R}^n)}. \]

Denote by $\mathcal{D}(\infty)$ the collection of all possible $e_{\{r_{jk}\}}$ as above. Following the proof of Theorem 1.1 line by line and letting $r_k \to \infty$ as $k \to \infty$, we are able to prove that $\mathcal{D}(\infty)$ is singleton, and hence prove Theorem 1.2 here we omit the details and also refer to [25, 29].

We end this section by the following remark.

**Remark 2.6.** (i) Recall that Evans [16] suggested another approximation to $u^\gamma$ via $e^{1/H}$-harmonic functions $\tilde{u}^{\gamma,\epsilon}$, that is, smooth solutions to
\[ \text{div} \left( e^{1/H(Du)} D_p H(Du) \right) = e^{1/H(Du)} [\partial H[v] + \epsilon \text{div} \left( D_p H^\gamma(Du) \right)] = 0 \quad \text{in} \ U; \ v = u^\gamma \quad \text{on} \ \partial U. \]
But note that the 3-order derivative of $H^\gamma$ appears in third terms of the linearized operator
\[ (2.17) \quad - H_{p_i}(D\tilde{u}^{\gamma,\epsilon}) H_{p_j}(D\tilde{u}^{\gamma,\epsilon}) v_{x_i x_j} - 2H_{p_i p_j}(D\tilde{u}^{\gamma,\epsilon}) \tilde{u}^{\gamma,\epsilon}_{x_i x_j} H_{p_j}(D\tilde{u}^{\gamma,\epsilon}) v_{x_i} - \epsilon \text{div} \left( D_{pp}^2 H^\gamma(D\tilde{u}^{\gamma,\epsilon}) Dv \right). \]
If we want to get Theorem 2.3 and 2.4 for $\tilde{u}^{\gamma,\epsilon}$ so that the constants $C_0, C_1$ are independent of 3-order derivative of $H^\gamma$ or $H$, some extra efforts are needed. To avoid such extra efforts, we prefer to consider the approximation equation (2.2).
If \( H(p) = \frac{1}{p} |p|^2 \), a flatness estimate stronger than Theorem 2.4 is also given in [18] via the maximal principle,

\[
|Du^\gamma| \leq u_{x_n}^\gamma + C\sqrt{\tau} \quad \text{in } B(0, 1) \text{ for all } \epsilon \in (0, 1].
\]

Note that in this case, \( H^\gamma = H \) and \( u^\gamma = u \), and \( u^\gamma \epsilon \) is then reduced to \( u^\epsilon \). From this Evans-Smart [18] concluded the everywhere differentiability of \( \infty \)-harmonic functions \( u \). But for \( H \in C^0(\mathbb{R}^n) \) satisfying (H1) and (H2), since \( H \) does not necessarily have a Hilbert structure, it is still unclear whether there is some estimate similar to (2.18), and also whether the approach in [18] can be used to prove Theorem 1.1.

3. Proof of Theorem 2.3

Let \( H, H^\gamma, u, u^\gamma \) and \( u^\gamma \epsilon \) be as in Section 2. Note that \( H^\gamma \) satisfies (H1)\&(H2) with the same \( \lambda \) and \( \Lambda \). Since \( |H^\gamma_p(p)|^2 \leq \Lambda^2 |p|^2 \) implies

\[
\epsilon |\xi|^2 \leq [H^\gamma_p(p)H^\gamma_{ij}(p)] |\xi_i| |\xi_j| \leq \Lambda^2 (|p|^2 + 1)|\xi|^2 \quad \forall \xi \in \mathbb{R}^n,
\]

by a standard quasilinear elliptic theory (see [24]), there exists a unique smooth solution \( u^\gamma \epsilon \in C^\infty(U) \cap C^0(\bar{U}) \) to (2.2). Theorem 2.3 (i) follows from the known maximum principle. We also note that by a standard argument, \( u^\gamma \epsilon \to u^\gamma \) in \( C^0(\bar{U}) \) (that is Theorem 2.3 (iv)) follows from Theorem 2.3 (ii)\&(iii), and the uniqueness of \( u^\gamma \epsilon \) in [18]; here we omit the details. Below we only need to prove Theorem 2.3 (ii)\&(iii). For simplicity, we write \( u^\gamma \) as \( H \), \( u^\gamma \) as \( u \), and we write \( u^\gamma \epsilon \) as \( u^\epsilon \) by abuse of notation.

We prove Theorem 2.3 (ii) using the approach of Evans-Smart [19] here. Denote by \( L_\epsilon \) the linearized operator obtained from \( \mathcal{A}_H[u^\epsilon] + \epsilon \Delta u^\epsilon = 0 \), that is,

\[
L_\epsilon(v) := -H_{p_i}(Du^\epsilon)H_{p_j}(Du^\epsilon)v_{x_i x_j} - 2H_{p_i p_j}(Du^\epsilon)u^\epsilon_{x_i x_j}H_{p_j}(Du^\epsilon)v_{x_l} - \epsilon \Delta v
\]
for \( v \in C^\infty(U) \). Note that

\[
L_\epsilon(u^\epsilon_{x_k}) = (\mathcal{A}_H[u^\epsilon] + \epsilon \Delta u^\epsilon)_{x_k} = 0 \quad \text{in } U \quad \forall k = 1, \ldots , n.
\]

**Proof of Theorem 2.3 (ii).** We choose \( \zeta \in C^\infty_c(U) \) such that

\[
0 \leq \zeta \leq 1, \quad \zeta = 1 \quad \text{in } V, \quad |D\zeta| \leq 4 \frac{1}{\text{dist} (V, \partial U)}, \quad |D^2\zeta| \leq C_0 \frac{1}{[\text{dist} (V, \partial U)]^2}.
\]

Define an auxiliary function

\[
w = \zeta^2 |Du^\epsilon|^2 + \alpha(u^\epsilon)^2,
\]
where \( \alpha > 0 \) will be determined later. If \( w \) attains its maximum on \( \partial U \), then

\[
\max_V |Du^\epsilon|^2 \leq \sup_U w = \max_{\partial U} \alpha u^2,
\]
this implies Theorem 2.3 (ii).

Assume that \( w \) attains its maximum at some \( x_0 \in U \). Since \( Dw(x_0) = 0 \) and \( D^2 w(x_0) \) is nonpositive definite, we have \( L_\epsilon(w) \geq 0 \) at \( x_0 \). Below we estimate \( L_\epsilon(w) \) at \( x_0 \) from above. Note that

\[
L_\epsilon(w) = \zeta^2 L_\epsilon(|Du^\epsilon|^2) + |Du^\epsilon|^2 L_\epsilon(\zeta^2) + \alpha L_\epsilon((u^\epsilon)^2)
+ [-4\zeta \langle D_p H(Du^\epsilon), D\zeta \rangle \langle D^2 u^\epsilon D_p H(Du^\epsilon), D\zeta \rangle - 8\epsilon \zeta \langle D^2 u^\epsilon Du^\epsilon, D\zeta \rangle]
\]
A direct calculation gives

\[
L_\epsilon(|Du^\epsilon|^2) = -2H_{p_i}(Du^\epsilon)H_{p_j}(Du^\epsilon)[u^\epsilon_{x_k} u^\epsilon_{x_k x_i x_j} + u^\epsilon_{x_k x_i} u^\epsilon_{x_k x_j}]
- 4H_{p_i p_j}(Du^\epsilon)H_{p_j}(Du^\epsilon)u^\epsilon_{x_i x_j} u^\epsilon_{x_k x_i} u^\epsilon_{x_k x_j} - 2\epsilon [u^\epsilon_{x_k} u^\epsilon_{x_k x_i} + (u^\epsilon_{x_k x_i})^2]
= 2u^\epsilon_{x_k} L_\epsilon(u^\epsilon_{x_k}) - 2|D^2 u^\epsilon D_p H(Du^\epsilon)|^2 - 2\epsilon |D^2 u^\epsilon|^2.
\]
By $L_{\epsilon}(u_{x_0}^\epsilon) = 0$, and $D^2u^\epsilon D_pH(Du^\epsilon) = D[H(Du^\epsilon)]$ we obtain
\[
\zeta^2 L_{\epsilon}(|Du^\epsilon|^2) = -\frac{2}{2}\zeta^2|D[H(Du^\epsilon)]|^2 - 2\epsilon \zeta^2|D^2u^\epsilon|^2.
\]
Similarly using (2.2), we have
\[
L_{\epsilon}((u^\epsilon)^2) = -2H_{p_1}(Du^\epsilon)^2 + u_{x_1}^\epsilon u_{x_1}^\epsilon + u_{x_i}^\epsilon u_{x_i}^\epsilon
\[
-4u_{x_i}^\epsilon H_{p_1}(Du^\epsilon)H_{p_1}(Du^\epsilon)u_{x_1}^\epsilon u_{x_i}^\epsilon
\[
-2\epsilon u_{x_1}^\epsilon u_{x_i}^\epsilon + (u_{x_i}^\epsilon)^2
\[
= -2(D_pH(Du^\epsilon), Du^\epsilon)^2 - 2\epsilon|Du^\epsilon|^2 - 4\epsilon(D_{pp}^2H(Du^\epsilon)D[H(Du^\epsilon)], Du^\epsilon).
\]
Since (H1)\&(H2) implies
\[
\langle D_pH(p), p \rangle \geq \frac{\lambda}{2}|p|^2, \ |D_{pp}^2H(p)\rangle \leq \Lambda \langle \xi | \ \forall p, \xi \in \mathbb{R}^n,
\]
by Young’s inequality, we obtain
\[
\alpha L_{\epsilon}((u^\epsilon)^2) \leq -\alpha \lambda^2|Du^\epsilon|^4 + C(\alpha, \Lambda)|D[H(Du^\epsilon)]|^4/3 + |u^\epsilon|^4|Du^\epsilon|^4.
\]
Since
\[
L_{\epsilon}(\zeta^2) = -2H_{p_1}(Du^\epsilon)H_{p_1}(Du^\epsilon)[\zeta_{x_1}^2 + \zeta_{x_i}^2]
\[
-4\epsilon H_{p_1}(Du^\epsilon)H_{p_1}(Du^\epsilon)\zeta_{x_1}^2 \zeta_{x_i}^2 - 2\epsilon \zeta_{x_1}^2 + \zeta_{x_i}^2
\]
using (H1)\&(H2) and Young’s inequality we also obtain
\[
|Du^\epsilon|^2L_{\epsilon}(\zeta^2) \leq C(\Lambda)|Du^\epsilon|^4||D\zeta|^2 + |D^2\zeta|\zeta| + \frac{1}{4}|D[H(Du^\epsilon)]|^2\zeta^2 + C(\Lambda)||D\zeta|^2 + |D^2\zeta|\zeta|.
\]
Similarly,
\[
-4\epsilon \langle D_pH(Du^\epsilon), D\zeta \rangle \langle D^2u^\epsilon D_pH(Du^\epsilon), Du^\epsilon \rangle - 8\epsilon \zeta \langle D^2u^\epsilon D_u^\epsilon, D\zeta \rangle
\]
\[
\leq \frac{1}{4}|\zeta|^2|D[H(Du^\epsilon)]|^2 + \frac{1}{4}|\zeta|^2|D^2u^\epsilon|\zeta^2 + C(\Lambda)||D\zeta|^2 + C(\alpha)||\zeta|^2.
\]
In conclusion, we have
\[
L_{\epsilon}(w) \leq -\zeta^2|D[H(Du^\epsilon)]|^2 + C(\alpha, \Lambda)|D[H(Du^\epsilon)]|^4/3 - [\alpha \lambda^2 - C(\Lambda)(|D\zeta|^2 + |D^2\zeta|\zeta)]|Du^\epsilon|^4
\]
\[
- |u^\epsilon|^4|Du^\epsilon|^4 + C(\Lambda)||D\zeta|^2 + |D^2\zeta|\zeta| + C(\alpha)||\zeta|^2.
\]
At $x_0$, $L_{\epsilon}(w) \geq 0$ implies that
\[
\zeta^2|D[H(Du^\epsilon)]|^2 + [\alpha \lambda^2 - C(\Lambda)(|D\zeta|^2 + |D^2\zeta|\zeta)]|Du^\epsilon|^4
\]
\[
\leq C(\alpha, \Lambda)|D[H(Du^\epsilon)]|^4/3 + |u^\epsilon|^4|Du^\epsilon|^4 + C(\Lambda)||D\zeta|^2 + |D^2\zeta|\zeta|.
\]
Multiplying the above inequality with $\zeta^4$ yields
\[
|D[H(Du^\epsilon)]|^2 \zeta^6 + [\alpha \lambda^2 - C(\Lambda)(|D\zeta|^2 + |D^2\zeta|\zeta)]|Du^\epsilon|^4 \zeta^4
\]
\[
\leq C(\alpha, \Lambda)|D[H(Du^\epsilon)]|^4/3 \zeta^4 + |u^\epsilon|^4|Du^\epsilon|^4 \zeta^4 + C(\Lambda)||D\zeta|^2 + |D^2\zeta|\zeta| \zeta^4.
\]
By Young’s inequality we have
\[
C(\alpha, \Lambda)|D[H(Du^\epsilon)]|^4/3 \zeta^4 \leq \frac{1}{2}|D[H(Du^\epsilon)]|^2 \zeta^6 + C(\alpha, \Lambda),
\]
and hence
\[
[\alpha \lambda^2 - C(\Lambda)(|D\zeta|^2 + |D^2\zeta|\zeta)]|Du^\epsilon|^4 \zeta^4 \leq |u^\epsilon|^4|Du^\epsilon|^4 \zeta^4 + C(\Lambda)||D\zeta|^2 + |D^2\zeta|\zeta| \zeta^4 + C(\alpha, \Lambda).
\]
Choosing $\alpha = \alpha(\lambda, \Lambda, \|u^\epsilon\|_{C^0(U)}, \ dist (V, \partial U))$ so that
\[
\alpha \lambda^2 - C(\Lambda) \frac{C_0 + 16}{(\dist (V, \partial U))^2} \geq \|u^\epsilon\|^4_{C^0(U)} + 1,
\]
we have
\[ \zeta^4 |Du^e|^4 |_{x=x_0} \leq C(\lambda, \Lambda, \|u^e\|_{C^{0}(U)}, \text{dist} (V, \partial U)). \]

Hence,
\[ \sup_{V} |Du^e|^4 \leq \left[ \sup_{U} w \right]^2 \leq \zeta^4 |Du^e|^4 |_{x=x_0} + \alpha [u^e(0)]^2 \leq C(\lambda, \Lambda, \|u^e\|_{C^{0}(U)}, \text{dist} (V, \partial U)) \]
as desired. \hfill \Box

To prove Theorem 2.3 (iii), we need the following Lemma 3.1, which can be found in [22, Lemma 3.2 and Lemma 3.4]. For each \( t > 0, \delta > 0, \sigma > 0 \) and \( x, y \in U \), define
\[ L^\delta(x, y) := \inf \left\{ \int_0^t \left[ \sigma + L(\xi(s)) \right] e^{-\delta(t-s)} ds \bigg| t > 0, \xi \in C(0, t; x, y; U) \right\}, \]
where \( C(0, t; x, y; U) \) is the set of all rectifiable curves \( \xi : [0, t] \to U \) that joins \( x \) to \( y \), and
\[ L(q) = \sup_{p \in \mathbb{R}^n} \{ p \cdot q - H(p) \}, \quad \forall q \in \mathbb{R}^n. \]

For each \( \sigma > 0 \), we also need to the notion of generalized cones, that is
\[ \mathcal{C}_\sigma^H(x) = \max_{H(p)=\sigma} \{ p \cdot x \}, \quad \forall x \in U. \]

By the strongly convexity of \( H \), one always has that
\[ \sqrt{2\sigma/\lambda|x|} \leq \mathcal{C}_\sigma^H(x) \leq \sqrt{2\sigma/\lambda|x|} \]

**Lemma 3.1.** Assume that \( H \in C^\infty(\mathbb{R}^n) \) satisfy (H1)&(H2).

(i) For all \( \sigma > 0, \delta \geq 0 \) and \( x, y \in U \), we have
\[ \mathcal{C}_\sigma^H(y - x) \geq L^\delta(x, y) \geq 0. \]

(ii) When \( \frac{\delta}{\sigma} L^\delta(x, y) < \ln \sqrt{2} \), we also have
\[ \mathcal{C}_\sigma^H(y - x) \leq e^{\frac{2\delta}{\sigma} L^\delta(x, y)} L^\delta(x, y). \]

(iii) For any domain \( V \subseteq \mathbb{R}^n \) and \( x_0 \in \partial V \), we have \( L^\delta(x_0, \cdot) \) is a viscosity sup-solution of
\[ A_H(v) = -\frac{\delta}{2} \quad \text{in} \ V \setminus \{x_0\} \quad \text{whenever} \ 0 < \delta < \delta_{\sigma, V} = \frac{\sigma}{2 \sup \{ \mathcal{C}_\sigma^H(y - x) : x, y \in \partial V \}} \]
and \( L^\delta(\cdot, x_0) \) is a viscosity sub-solution of
\[ A_H(v) = \frac{\delta}{2} \quad \text{in} \ V \setminus \{x_0\} \quad \text{whenever} \ 0 < \delta < \delta_{\sigma, V}. \]

We also need the following comparison principle, see [33, Appendix, Theorem 2].

**Lemma 3.2.** Assume that \( H \in C^\infty(\mathbb{R}^n) \) satisfy (H1)&(H2). For any \( \sigma > 0 \) and domain \( V \subseteq \mathbb{R}^n \), assume that \( u_1 \in C^0(\partial V) \) is a viscosity sup-solution of
\[ A_H(u_1) + \epsilon \Delta u_1 = -\delta \quad \text{in} \ V \]
and \( u_2 \in C^0(\partial V) \) is a viscosity sub-solution of
\[ A_H(u_2) + \epsilon \Delta u_2 = 0 \quad \text{in} \ V. \]

If either \( u_1 \in C^{0,1}(V) \) or \( u_2 \in C^{0,1}(V) \), then
\[ \max_{V} (u_2 - u_1) \leq \max_{\partial V} (u_2 - u_1). \]

From Lemma 3.1 and 3.2 we deduce the following.
Lemma 3.3. Assume that $H \in C^\infty(\mathbb{R}^n)$ satisfy (H1) & (H2). For any domain $V \subset \mathbb{R}^n$ and $x_0 \in V$ for all $\sigma > 0$ and $0 < \delta < \delta_0$, there exist constant $\mu_1, \mu_2 > 0$ depending on $\sigma, \delta, H$ such that for all $\epsilon \in (0,1)$, $L_\sigma^\delta(x_0, \cdot)$ is a viscosity super-solution of
\[ \mathcal{A}_H(v) + \epsilon \Delta v = -\frac{\delta v}{2} + \epsilon \mu_1 \text{ in } V \setminus \{x_0\} \]
and $L_\sigma^\delta(\cdot, x_0)$ is a viscosity sub-solution of
\[ \mathcal{A}_H(v) + \epsilon \Delta v = \frac{\delta v}{2} - \epsilon \mu_2 \text{ in } V \setminus \{x_0\}. \]

Proof of Lemma 3.3. For any $\phi \in C^2(V)$ and $L_\sigma^\delta(x_0, x) - \phi(x)$ attains its locally minimum at $y \in V \setminus \{x_0\}$, it suffice to prove that
\[ \mathcal{A}_H(\phi)(y) + \epsilon \Delta \phi(y) \leq -\frac{\delta \sigma}{2} + \epsilon \mu_1. \]

Without loss of generality, we may assume that $L_\sigma^\delta(x_0, x) - \phi$ attains its a strictly minimum at $y_0 \in V \setminus \{x_0\}$. Since $L_\sigma^\delta(x_0, x)$ is semiconcave, for any $\eta > 0, r > 0$, by Lemma A.3 in [14] there exist $x^{r, \eta} \in B(y_0, \sigma)$ and $p^{r, \eta} \in B(0, \eta)$ such that $L_\sigma^\delta(x_0, x) - \phi(x) - \langle p^{r, \eta}, x \rangle$ has a local minimal at $x^{r, \eta}$ and $L_\sigma^\delta(x_0, x)$ is twice differentiable at $x^{r, \eta}$. Also, the semiconcave property of $L_\sigma^\delta(x_0, x)$ implies that there exists $\mu_1 > 0$ depending on $\sigma, \delta, H$ such that
\[ D^2 \mathcal{L}_\sigma^\delta(x_0, x) \leq \mu_1 I \]
in the sense of distributions, where $I_n$ is identity matrix. Since $L_\sigma^\delta(x_0, x)$ is twice differentiable at $x^{r, \eta}$, by Lemma 3.1 and 3.3, we have
\[ \langle D^2 \mathcal{L}_\sigma^\delta(x_0, x^{r, \eta})D_p H(D \mathcal{L}_\sigma^\delta(x_0, x^{r, \eta})), D_p H(D \mathcal{L}_\sigma^\delta(x_0, x^{r, \eta})) \rangle + \epsilon \Delta \mathcal{L}_\sigma^\delta(x_0, x^{r, \eta}) \leq -\frac{\delta \sigma}{2} + \epsilon \mu_1. \]

On the other hand, since $L_\sigma^\delta(x_0, x) - \phi(x) - \langle p^{r, \eta}, x \rangle$ has a local minimal at $x^{r, \eta}$, we have $D\mathcal{L}_\sigma^\delta(x_0, x^{r, \eta}) = D\phi(x^{r, \eta}) + p^{r, \eta}$ and $D^2 \mathcal{L}_\sigma^\delta(x_0, x^{r, \eta}) \geq D^2 \phi(x^{r, \eta})$. Thus
\[ \langle D^2 \mathcal{L}_\sigma^\delta(x_0, x^{r, \eta})D_p H(D \phi(x^{r, \eta}) + p^{r, \eta}), D_p H(D \phi(x^{r, \eta}) + p^{r, \eta}) \rangle + \epsilon \Delta \phi(x^{r, \eta}) \]
\[ \geq \langle D^2 \phi(x^{r, \eta})D_p H(D \phi(x^{r, \eta}) + p^{r, \eta}), D_p H(D \phi(x^{r, \eta}) + p^{r, \eta}) \rangle + \epsilon \Delta \phi(x^{r, \eta}). \]
Combining (3.4) and (3.5), we have
\[ \langle D^2 \mathcal{L}_\sigma^\delta(x_0, x^{r, \eta})D_p H(D \phi(x^{r, \eta}) + p^{r, \eta}), D_p H(D \phi(x^{r, \eta}) + p^{r, \eta}) \rangle + \epsilon \Delta \phi(x^{r, \eta}) \leq -\frac{\delta \sigma}{2} + \epsilon \mu_1. \]

Letting $r = \eta \to 0$ and noting $p^{r, \eta} \to 0, x^{r, \eta} \to y_0$, this leads to (3.2).

Similarly, we can prove that $-\mathcal{L}_\sigma^\delta(x, x_0)$ is viscosity sub-solution of
\[ \mathcal{A}_H(v) + \epsilon \Delta v = -\frac{\delta v}{2} - \epsilon \mu_2 \text{ in } V \setminus \{x_0\}. \]
The proof is complete. \qed

We are able to prove Theorem 2.2 (iii) as below.

Proof of Theorem 2.2 (iii). Note that $u \in C^{0,1}(\overline{U})$. Letting $\sigma > 8\Lambda \|u\|_{C^{0,1}(\overline{U})}^2$, we have
\[ |u(y) - u(x)| \leq \|u\|_{C^{0,1}(\overline{U})}|y - x| \leq \frac{1}{4} \mathcal{E}_\sigma^H(y - x), \quad \forall x, y \in U. \]
Moreover, there exist $\delta(\sigma, U) > 0$ such that for all $x, y \in U$ and $\delta < \delta(\sigma, U)$, we have
\[ \frac{\delta}{\sigma} L_\sigma^\delta(x, y) < \ln \sqrt{2} \]
and hence, by Lemma 3.1
\[ \mathcal{E}_\sigma^H(y - x) \leq e^{\frac{4\delta}{\sigma} L_\sigma^\delta(x, y)} L_\sigma^\delta(x, y) \leq 4 L_\sigma^\delta(x, y). \]
By (3.7), for all $\sigma > 8\Lambda \|u\|_{C^{0,1}(\overline{V})}^2$ and $\delta < \delta(\sigma, U)$, we have
\begin{equation}
|u(y) - u(x)| \leq L_\sigma^\delta(x, y), \quad \forall x, y \in U.
\end{equation}

Note that
\[\mathcal{A}_H(u^\epsilon) + \epsilon \Delta u^\epsilon \geq 0 \quad \text{in } U\]
in viscosity sense and by Lemma 3.3,
\[\mathcal{A}_H(L_\sigma^\delta(x, 0, x)) + \epsilon \Delta L_\sigma^\delta(x, 0, x) \leq -\frac{\delta \sigma}{2} + \epsilon n \mu_1 \quad \text{in } U\]
in viscosity sense. For all $\sigma > 8\Lambda \|u\|_{C^{0,1}(\overline{V})}^2$ and $\delta < \delta(\sigma, U)$ and if $0 < \epsilon < \frac{\delta \sigma}{2n \mu_1}$, by Lemma 3.2 we have
\begin{equation}
u^\epsilon(x) - u(x_0) \leq L_\sigma^\delta(x, 0, x_0), \quad \forall x \in U, x_0 \in \partial U.
\end{equation}
By similar argument, for all $\sigma > 8\Lambda \|u\|_{C^{0,1}(\overline{V})}^2$ and $\delta < \delta(\sigma, U)$, if $0 < \epsilon < \frac{\delta \sigma}{2n \mu_2}$, we have
\begin{equation}u^\epsilon(x) - u(x_0) \geq -L_\sigma^\delta(x, 0, x_0), \quad \forall x \in \partial U, x_0 \in \partial U.
\end{equation}
We therefore conclude that for $\sigma = 8\Lambda \|u\|_{C^{0,1}(\overline{V})}^2$ and $\delta < \delta(\sigma, U)$ if $0 < \epsilon < \min\{\frac{\delta \sigma}{2n \mu_1}, \frac{\delta \sigma}{2n \mu_2}\}$,
\[|u^\epsilon(x) - u(x_0)| \leq C_H^\delta(x, 0, x), \quad \forall x \in U, x_0 \in \partial U.
\]
Thus, there exist $\epsilon_*$ and $C$ depending on $U$, $\|u\|_{C^{0,1}(\overline{V})}$, $H$, $\delta, \sigma$ such that for all $0 < \epsilon < \epsilon_*$, we have
\[|u^\epsilon(x) - u(x_0)| \leq C|x - x_0| \quad \forall x \in U, x_0 \in \partial U.
\]
The proof of Theorem 2.2 is complete. \hfill \Box

4. A generalization of Evans-Smart' adjoint method

Let $H, H^\gamma, u, u^\gamma$ and $u^{\gamma, \epsilon}$ be as in Section 2. For convenience, we write $H^\gamma$ as $H$, and $u^\gamma$ as $u, u^{\gamma, \epsilon}$ as $u^\epsilon$ below. Let $L_\epsilon$ be the linearized operator given in (3.1), and $L_\epsilon^*$ be its dual operator, that is,
\begin{equation}
L_\epsilon^*(v) := -[H_{p_j}(Du^\epsilon)H_{p_j}(Du^\epsilon)v]_{x, x_j} + 2[H_{p_j}(Du^\epsilon)u^\epsilon_{j, x_j}H_{p_j}(Du^\epsilon)v]_{x, x_j} - \epsilon \Delta v
\end{equation}
for any $v \in C^\infty(U)$. Observe that
\[\int_{\mathbb{R}^n} L_\epsilon(v)(x)w(x) \, dx = \int_{\mathbb{R}^n} v(x)L_\epsilon^*(w)(x) \, dx \quad \forall v, w \in C^\infty_c(U).
\]
Fix a smooth domain $V \subseteq U$. For each point $x_0 \in V$, we consider the adjoint problem
\begin{equation}
L_\epsilon^*(v) = \delta_{x_0} \quad \text{in } V; v = 0 \text{ on } \partial V,
\end{equation}
where $\delta_{x_0}$ denotes the Dirac measure at $x_0$. Equivalently,
\[\int_V v(x)L_\epsilon(\phi)(x) \, dx = \phi(x_0) \quad \forall \phi \in C^\infty_c(V); v = 0 \text{ on } \partial V.
\]
Then we have the following result.

**Theorem 4.1.** For each point $x_0 \in V$, there exists a unique solution $\Theta^\epsilon \in C^\infty(\overline{V} \setminus \{x_0\})$ of the linear adjoint problem (4.2) such that $\Theta^\epsilon \geq 0$ in $V$.

**Proof.** Consider problem
\[L_\epsilon(w) = 0 \quad \text{in } V; w = 0 \quad \text{on } \partial V.
\]
By Theorem 2.3 there exists a unique solution $\omega \equiv 0$ on $\overline{V}$. So that 0 is not an eigenvalue of the operator $L_\epsilon$, and hence 0 is not an eigenvalue of $L_\epsilon^*$. Applying standard linear elliptic PDE theory,
there exists smooth Green’s function \( \Theta^\epsilon \in C^\infty(\overline{B}(0, 2 \setminus \{x_0\}) \). Next we show that \( \Theta^\epsilon \geq 0 \). For any \( f \in C^\infty(V) \) and \( \epsilon \geq 0 \) in \( V \), we introduce the solution \( \omega^\epsilon \) of the linear boundary value problem (4.3)

\[
L_\epsilon(\omega^\epsilon) = f \quad \text{in } V; \quad \omega^\epsilon = 0 \quad \text{on } \partial V.
\]

By Theorem 2.3, we know that there exists a unique solution \( 0 \leq \omega^\epsilon \in C^\infty(V) \). Multiply the equation in (4.3) by \( \Theta^\epsilon \), we have

\[
\int_V f \Theta^\epsilon \, dx = \int_V L_\epsilon(\omega^\epsilon) \Theta^\epsilon \, dx
\]

\[
= \int_V [-H_{p_i}(Du^\epsilon)H_{p_j}(Du^\epsilon)\omega^\epsilon_{x_i x_j} - 2H_{p_j}(Du^\epsilon)H_{p_ip_i}(Du^\epsilon)u^\epsilon_{x_i x_j} - \epsilon \omega^\epsilon_{x_i x_j}] \Theta^\epsilon \, dx.
\]

By integration by parts, \( \omega^\epsilon|_{\partial V} = 0 \) and \( \Theta^\epsilon|_{\partial V} = 0 \), we have

\[
\int_V -H_{p_i}(Du^\epsilon)H_{p_j}(Du^\epsilon)\omega^\epsilon_{x_i x_j} \Theta^\epsilon \, dx
\]

\[
= \int_V -H_{p_i}(Du^\epsilon)H_{p_j}(Du^\epsilon)\Theta^\epsilon \omega^\epsilon_{x_i x_j} \, dx - \int_{\partial V} H_{p_j}(Du^\epsilon)H_{p_j}(Du^\epsilon)\Theta^\epsilon \omega^\epsilon_{x_j} \cos(\vec{N}, x_i) \, ds
\]

\[
+ \int_{\partial V} (H_{p_i}(Du^\epsilon)H_{p_j}(Du^\epsilon)\Theta^\epsilon)_{x_i x_j} \omega^\epsilon \cos(\vec{N}, x_j) \, ds
\]

\[
= \int_V -H_{p_i}(Du^\epsilon)H_{p_j}(Du^\epsilon)\Theta^\epsilon \omega^\epsilon_{x_i x_j} \, dx,
\]

where \( \vec{N} \) denotes the outward pointing unit normal along \( \partial V \). By similar calculation, which lead to

\[
\int_V f \Theta^\epsilon \, dx = \int_V L^*_\epsilon(\Theta^\epsilon) \omega^\epsilon \, dx = \omega^\epsilon(x_0) \geq 0.
\]

Since for all \( f \geq 0 \) holds, that is \( \Theta^\epsilon \geq 0 \).

**Lemma 4.2.** Denote by \( \vec{N} \) denotes the outward pointing unit normal along \( \partial V \). Then

\[
\cos(\vec{N}, x_i) = -\frac{\Theta^\epsilon_{x_i}}{|D\Theta^\epsilon|} \quad \forall i = 1, \cdots, n.
\]

We have the following connection of between operator \( L_\epsilon \) and \( \Theta_\epsilon \).

**Lemma 4.3.** For any \( v \in C^\infty(\overline{V}) \), we have

\[
\int_V L_\epsilon(v) \Theta^\epsilon \, dx + \int_{\partial V} v \rho^\epsilon \, ds = v(x_0),
\]

where dentes

\[
\rho^\epsilon := \frac{\langle D_p H(Du^\epsilon), D\Theta^\epsilon \rangle^2}{|D\Theta^\epsilon|} + \epsilon |D\Theta^\epsilon|.
\]

**Proof.** By integrate by parts and \( \Theta^\epsilon|_{\partial V} = 0 \), we have

\[
\int_V L_\epsilon(v) \Theta^\epsilon \, dx
\]

\[
= \int_V [-H_{p_i}(Du^\epsilon)H_{p_j}(Du^\epsilon)v_{x_i x_j} - 2H_{p_j}(Du^\epsilon)H_{p_ip_i}(Du^\epsilon)u^\epsilon_{x_i x_j}v_{x_j} - \epsilon v_{x_i x_j}] \Theta^\epsilon \, dx
\]

\[
= \int_V [-H_{p_i}(Du^\epsilon)H_{p_j}(Du^\epsilon)\Theta^\epsilon]_{x_i x_j} + 2(H_{p_j}(Du^\epsilon)H_{p_ip_i}(Du^\epsilon)u^\epsilon_{x_i x_j} \Theta^\epsilon)_{x_i} - \epsilon (\Theta^\epsilon)_{x_i x_j} v \, dx
\]

\[
+ \int_{\partial V} v[(H_{p_i}(Du^\epsilon)H_{p_j}(Du^\epsilon)\Theta^\epsilon)_{x_i} + \epsilon (\Theta^\epsilon)_{x_j}] \cos(\vec{N}, x_j) \, ds,
\]
Proof. Let \( \phi(r) = \epsilon e^{\frac{r}{\epsilon}}(\alpha - r) \) and \( \alpha_{\epsilon} := H(Du^\epsilon(x_0)) \), where \( 0 < \mu \leq \frac{\lambda}{8\mu} \). Similarly to (4.4) we have

\[
L_\epsilon((\phi \circ H)(Du^\epsilon)) = -(\phi \circ H)_{p_kp_s}(Du^\epsilon)[H_{p_k}(Du^\epsilon)H_{p_j}(Du^\epsilon)u^\epsilon_{x_kx_j}u^\epsilon_{x_sx_i} + \epsilon u^\epsilon_{x_kx_i}u^\epsilon_{x_sx_i}] \\
\]

as desired.

We further need an exponential estimate.

Lemma 4.6. Moreover, for all \( 0 < \mu < \frac{\lambda}{8\mu} \) we have

\[
\int_{\partial V} e^{\frac{r}{\epsilon}[H(Du^\epsilon(x_0)) - H(Du^\epsilon)]} \rho^\epsilon ds \\
+ \mu \int_V e^{\frac{r}{\epsilon}[H(Du^\epsilon(x_0)) - H(Du^\epsilon)]} D^2_{pp}H(Du^\epsilon)D[H(Du^\epsilon)], D[H(Du^\epsilon)] \Theta^\epsilon dx \\
+ \epsilon \mu \int_V e^{\frac{r}{\epsilon}[H(Du^\epsilon(x_0)) - H(Du^\epsilon)]} H_{p_kp_s}(Du^\epsilon)u^\epsilon_{x_kx_i}u^\epsilon_{x_sx_i} \Theta^\epsilon dx \\
\leq 2\epsilon.
\]

Proof. Let

\[
\phi(r) = \epsilon e^{\frac{r}{\epsilon}[\alpha_{\epsilon} - r]} \quad \text{and} \quad \alpha_{\epsilon} := H(Du^\epsilon(x_0)),
\]

where \( 0 < \mu \leq \frac{\lambda}{8\mu} \). Similarly to (4.4) we have

\[
L_\epsilon((\phi \circ H)(Du^\epsilon)) = -(\phi \circ H)_{p_kp_s}(Du^\epsilon)[H_{p_k}(Du^\epsilon)H_{p_j}(Du^\epsilon)u^\epsilon_{x_kx_i}u^\epsilon_{x_sx_j} + \epsilon u^\epsilon_{x_kx_i}u^\epsilon_{x_sx_i}]
\]

as desired.

Since \( L_\epsilon(1) = 0 \) in \( V \), the following follows from Lemma 4.3 obviously.

Corollary 4.4. We have

\[
\int_{\partial V} \rho^\epsilon ds = 1.
\]

Letting \( v = H(Du^\epsilon) \) in Lemma 4.3 we also have the following.

Lemma 4.5. We have

\[
\int_V \langle D^2_{pp}H(Du^\epsilon)D[H(Du^\epsilon)], D[H(Du^\epsilon)] \rangle + \epsilon H_{p_kp_s}(Du^\epsilon)u^\epsilon_{x_kx_i}u^\epsilon_{x_sx_j} \rangle \Theta^\epsilon dx \leq \|H(Du^\epsilon)\|_{L^\infty(V)}.
\]

Proof. By Lemma 4.3

\[
-\int_V L_\epsilon(H(Du^\epsilon)) \Theta^\epsilon dx = \int_{\partial V} H(Du^\epsilon) \rho^\epsilon ds - H(Du^\epsilon(x_0)) \leq \|H(Du^\epsilon)\|_{L^\infty(V)}.
\]

Write

\[
L_\epsilon(H(Du^\epsilon)) = -H_{p_i}(Du^\epsilon)H_{p_j}(Du^\epsilon)[H_{p_kp_s}(Du^\epsilon)u^\epsilon_{x_kx_i}u^\epsilon_{x_sx_j} + H_{p_k}(Du^\epsilon)u^\epsilon_{x_kx_i} u^\epsilon_{x_sx_j}] \\
- 2H_{p_j}(Du^\epsilon)H_{p_kp_s}(Du^\epsilon)u^\epsilon_{x_kx_i}u^\epsilon_{x_sx_j} + H_{p_k}(Du^\epsilon)u^\epsilon_{x_kx_i} u^\epsilon_{x_sx_j} \\
- \epsilon[H_{p_kp_s}(Du^\epsilon)u^\epsilon_{x_kx_i} u^\epsilon_{x_sx_j} + H_{p_k}(Du^\epsilon)u^\epsilon_{x_kx_i} u^\epsilon_{x_sx_j}].
\]

Since \( L_\epsilon(u^\epsilon_{x_sx_i}) = 0 \), we have

\[
(4.4) \quad L_\epsilon(H(Du^\epsilon)) = -(H_{p_kp_s}(Du^\epsilon)H_{p_j}(Du^\epsilon)u^\epsilon_{x_kx_i}u^\epsilon_{x_sx_j} + \epsilon u^\epsilon_{x_kx_i}u^\epsilon_{x_sx_j}) \\
= -\langle D^2_{pp}H(Du^\epsilon)D[H(Du^\epsilon)], D[H(Du^\epsilon)] \rangle - \epsilon H_{p_kp_s}(Du^\epsilon)u^\epsilon_{x_kx_i}u^\epsilon_{x_sx_j}
\]

as desired.
By the Young's inequality, we have
\[ (\phi \circ H)_{pkp_\epsilon} = (\phi'' \circ H)_{pk} H_{p_\epsilon \epsilon} + \epsilon (\phi' \circ H)_{pkp_\epsilon} u_{x,s,x_i}^\epsilon. \]

Since
\[ (\phi \circ H)_{pkp_\epsilon} = (\phi'' \circ H)_{pk} H_{p_\epsilon \epsilon} + (\phi' \circ H)_{pkp_\epsilon}, \]
and \( \mathcal{A}_H[u^\epsilon] = -\epsilon \Delta u^\epsilon \), we get
\[
L_\epsilon((\phi \circ H)(Du^\epsilon)) \\
= \left| ((\phi'' \circ H)(Du^\epsilon))_{pk}(Du^\epsilon)_{p_\epsilon} \right| + (\phi' \circ H)(Du^\epsilon)_{pkp_\epsilon}(Du^\epsilon) \\
\times \left| (Du^\epsilon)_{p_\epsilon}(Du^\epsilon)_{p_\epsilon} u_{x,s,x_j}^\epsilon + \epsilon u_{x,s,x_j}^\epsilon \right| \\
= -\epsilon (\phi' \circ H)(Du^\epsilon)_{pkp_\epsilon}(Du^\epsilon)_{p_\epsilon} u_{x,s,x_i}^\epsilon - \phi''(H(Du^\epsilon))\epsilon^2(\Delta u^\epsilon)^2. \]

Note that
\[ \phi'(r) = -\mu e^{\frac{H}{\epsilon}[\alpha_r - r]} \leq 0, \quad \phi''(r) = \frac{\mu^2}{\epsilon} e^{\frac{H}{\epsilon}[\alpha_r - r]} \geq 0. \]

Since the strongly convexity of \( H \) implies
\[ |p|^2 \leq \frac{1}{\lambda} \langle D^2_H(Du^\epsilon)p, p \rangle, \]
by \( 1 - \frac{M}{\lambda} \geq 1/2 \) we have
\[
L_\epsilon((\phi \circ H)(Du^\epsilon)) \geq \frac{H}{2} e^{\frac{H}{\epsilon}[\alpha_r - H(Du^\epsilon)]} \langle D^2_H(Du^\epsilon)D[H(Du^\epsilon)], D[H(Du^\epsilon)] \rangle \\
+ \frac{H}{2} e^{\frac{H}{\epsilon}[\alpha_r - H(Du^\epsilon)]} (H(Du^\epsilon))_{p_\epsilon \epsilon} u_{x,s,x_i}^\epsilon. \]

Since Lemma \[4.3\] implies
\[ \int_{\partial V} (\phi \circ H)(Du^\epsilon) \rho^\epsilon \, ds + \int_V L_\epsilon((\phi \circ H)(Du^\epsilon)) \Theta^\epsilon \, dx = (\phi \circ H)(Du^\epsilon(x_0)) = \epsilon. \]
We obtain the desired estimate. \( \square \)

Applying Lemma \[4.7\], we will get the following upper bound.

**Lemma 4.7.** We have
\[ \int_V |H(Du^\epsilon)|^2 \Theta^\epsilon \, dx \leq C(\lambda, \Lambda, \eta)(1 + \|u^\epsilon\|_{L^\infty(V)}) \left[ 1 + \|H(Du^\epsilon)\|_{L^\infty(V)} \right] + C(\lambda, \Lambda)\eta^2 \int_V \Theta^\epsilon \, dx. \]

**Proof.** By \( \mathcal{A}_H(u^\epsilon) + \epsilon \Delta u^\epsilon = 0 \), a direct calculation implies that
\[ L_\epsilon(\frac{1}{2} u^\epsilon^2) = -\langle D_p H(Du^\epsilon), Du^\epsilon \rangle^2 + \epsilon |Du^\epsilon|^2 - 2u^\epsilon \langle D^2_H(Du^\epsilon)D[H(Du^\epsilon)], Du^\epsilon \rangle. \]

By the convexity of \( H \) and \( H(0) = 0 \), we have
\[ \langle D_p H(Du^\epsilon), Du^\epsilon \rangle^2 \geq |H(Du^\epsilon)|^2. \]

By the Young's inequality, we have
\[ 2u^\epsilon \langle D^2_H(Du^\epsilon)D[H(Du^\epsilon)], Du^\epsilon \rangle \leq C(\eta) |u^\epsilon|^2 \langle D^2_H(Du^\epsilon)D[H(Du^\epsilon)], D[H(Du^\epsilon)] \rangle + \eta \langle D^2_H(Du^\epsilon)Du^\epsilon, Du^\epsilon \rangle. \]

By the strongly concavity of \( H \), we have
\[ \eta \langle D^2_H(Du^\epsilon)Du^\epsilon, Du^\epsilon \rangle \leq \frac{2\Lambda}{\lambda} H(Du^\epsilon) \leq \frac{1}{2} |H(Du^\epsilon)|^2 + C(\lambda, \Lambda)\eta^2. \]

Thus
\[ \int_V |H(Du^\epsilon)|^2 \Theta^\epsilon \, dx \leq - \int_V L_\epsilon(\frac{1}{2} u^\epsilon^2) \Theta^\epsilon \, dx + \frac{1}{2} \int_V |H(Du^\epsilon)|^2 \Theta^\epsilon \, dx + C(\lambda, \Lambda)\eta \int_V \Theta^\epsilon \, dx. \]
and hence by the Young's inequality,

\[ C(\eta) \int_V |u^\epsilon|^2 \langle D_{pp}^2 H(Du^\epsilon) D[H(Du^\epsilon)] \rangle, D[H(Du^\epsilon)] \rangle \Theta^\epsilon \, dx \]

By Lemma 4.3,

\[ - \int_V L_\epsilon \left( \frac{1}{2}(u^\epsilon)^2 \right) \Theta^\epsilon \, dx \leq 2\|u^\epsilon\|_L^\infty(V), \]

and hence,

\[ \int_V \|H(Du^\epsilon)\|^2 \Theta^\epsilon \, dx \leq C(\eta, \lambda, \Lambda)[1 + \|u^\epsilon\|^2_\infty(V)] [1 + \|H(Du^\epsilon)\|_L^\infty(V)] + C(\lambda, \Lambda)\eta^2 \int_V \Theta^\epsilon \, dx. \]

Moreover, we also need an integral estimate of \( \Theta^\epsilon \).

**Lemma 4.8.** Let \( x_0 \in V \) and \( \alpha_\epsilon := H(Du^\epsilon(x_0)) > 0 \).

(i) For any \( 0 < \mu < \frac{1}{8n} \) and \( 0 < \beta < H(Du^\epsilon(x_0)) \), we have

\[ \int_{V \cap \{H \leq \beta\}} \Theta^\epsilon \, dx \leq C(\lambda, \Lambda) \frac{1}{\epsilon} e^{\mu [\beta - H(Du^\epsilon(x_0))]}. \]

(ii) If \( \lim \inf_{\epsilon \to 0} H(Du^\epsilon(x_0)) \geq \alpha > 0 \), we have

\[ \int_V \Theta^\epsilon \, dx \leq C(\lambda, \Lambda) \frac{1}{\epsilon} e^{-\frac{\alpha}{\epsilon}} + C(\lambda, \Lambda, \|u\|_L^\infty(V), \text{dist}(V, \partial U)) \frac{1}{\alpha^2}. \]

**Proof.** For each \( 0 < \mu < \frac{1}{8n} \), define

\[ \phi(r) = \epsilon e^{\frac{\mu}{\epsilon}(\alpha_\epsilon - r)} \]

and set \( v(x) = (\phi \circ H)(Du^\epsilon(x))|x|^2 \). By Lemma 4.3,

\[ \int_V L_\epsilon(v) \Theta^\epsilon \, dx = v(x_0) - \int_{\partial V} v \rho^\epsilon \, ds. \]

Then

\[ L_\epsilon(v) = (\phi \circ H)(Du^\epsilon)L_\epsilon(|x|^2) + |x|^2 L_\epsilon((\phi \circ H)(Du^\epsilon)) \]

\[ - 2\langle D_{pp} H(Du^\epsilon), D(\phi \circ H)(Du^\epsilon) \rangle \langle D_p H(Du^\epsilon), D|x|^2 \rangle - 2\epsilon \langle D(\phi \circ H)(Du^\epsilon), D|x|^2 \rangle. \]

Write

\[ K := e^{\frac{\mu}{\epsilon}[\alpha_\epsilon - H(Du^\epsilon)]}[\langle D_{pp}^2 H(Du^\epsilon) D[H(Du^\epsilon)], D[H(Du^\epsilon)] \rangle + \epsilon H_{pp} H(Du^\epsilon) u^\epsilon_{x_i x_j} u^\epsilon_{x_i x_j}]. \]

By 4.5, we have

\[ |x|^2 L_\epsilon((\phi \circ H)(Du^\epsilon)) \leq 4\mu K. \]

Note that

\[ L_\epsilon(|x|^2) = -H_{p^2}(Du^\epsilon)H_{i j}(Du^\epsilon)\langle |x|^2 \rangle_{x_i x_j} - 2H_{p^2}(Du^\epsilon)H_{x_i x_j}(Du^\epsilon) u^\epsilon_{x_i x_j} \langle |x|^2 \rangle_{x_i x_j} - \epsilon(\langle |x|^2 \rangle_{x_i x_j}) \]

and hence by the Young's inequality,

\[ \langle D_p H(Du^\epsilon), D[H(Du^\epsilon)] \rangle = \mathcal{A}_H[u^\epsilon] = -\epsilon \Delta u^\epsilon \text{ we also have} \]

\[ - 2\langle D_p H(Du^\epsilon), D(\phi \circ H)(Du^\epsilon) \rangle \langle D_p H(Du^\epsilon), D|x|^2 \rangle - 2\epsilon \langle D(\phi \circ H)(Du^\epsilon), D|x|^2 \rangle \]

\[ = -2(\phi \circ H)(Du^\epsilon)[\langle D_p H(Du^\epsilon), D[H(Du^\epsilon)] \rangle \langle D_p H(Du^\epsilon), D|x|^2 \rangle + \epsilon \langle D[H(Du^\epsilon)], D|x|^2 \rangle] \]

\[ = -2(\phi \circ H)(Du^\epsilon)[-\epsilon \Delta u^\epsilon \langle D_p H(Du^\epsilon), D|x|^2 \rangle + \epsilon \langle D[H(Du^\epsilon)], D|x|^2 \rangle), \]
by $\phi'(r) = -\mu e^{\frac{\mu}{\epsilon}(\alpha - H(Du^\epsilon))} \leq 0$ and Young’s inequality, which is bounded by

$$C(\lambda, \Lambda) \mu K + \frac{1}{8} (\phi \circ H)(Du^\epsilon) |D_pH(Du^\epsilon)|^2 + \frac{c}{8} (\phi \circ H)(Du^\epsilon).$$

Thus

$$L_\epsilon(v) \leq - (\phi \circ H)(Du^\epsilon) [D_pH(Du^\epsilon)]^2 + \epsilon n] + C(\lambda, \Lambda) \mu K.$$

Therefore, applying Lemma 4.6 we get

$$\int_V (\phi \circ H)(Du^\epsilon) [D_pH(Du^\epsilon)]^2 + \epsilon n] \Theta^\epsilon \, dx$$

$$\leq 4 \int_{\partial V} (\phi \circ H)(Du^\epsilon) \Theta^\epsilon \, ds + C(\lambda, \Lambda) \int_V \mu K \Theta^\epsilon \, dx \leq C(\lambda, \Lambda) \epsilon.$$

Thus $L_\epsilon(\nu) \leq - (\phi \circ H)(Du^\epsilon) [D_pH(Du^\epsilon)]^2 + \epsilon n] + C(\lambda, \Lambda) \mu K.$

Therefore, applying Lemma 4.6 we get

$$\int_V (\phi \circ H)(Du^\epsilon) [D_pH(Du^\epsilon)]^2 + \epsilon n] \Theta^\epsilon \, dx$$

$$\leq 4 \int_{\partial V} (\phi \circ H)(Du^\epsilon) \Theta^\epsilon \, ds + C(\lambda, \Lambda) \int_V \mu K \Theta^\epsilon \, dx \leq C(\lambda, \Lambda) \epsilon.$$

We conclude that

$$e^2 \int_V e^\frac{\mu(\alpha - H(Du^\epsilon))}{\epsilon} \Theta^\epsilon \, dx \leq C(\lambda, \Lambda) \epsilon,$$

and hence

$$\int_{V \cap \{H(Du^\epsilon) \leq \beta\}} \Theta^\epsilon \, dx \leq C(\lambda, \Lambda) \frac{1}{\epsilon} e^{-\frac{\mu(\alpha - \beta)}{\epsilon}}.$$  (4.6)

This implies that

$$\int_V \Theta^\epsilon \, dx = \int_{V \cap \{H(Du^\epsilon) \leq \frac{\beta}{2}\}} \Theta^\epsilon \, dx + \int_{V \cap \{H(Du^\epsilon) > \frac{\beta}{2}\}} \Theta^\epsilon \, dx$$

$$\leq C(\lambda, \Lambda) \frac{1}{\epsilon} e^{-\frac{\mu(\alpha_0 - \alpha)}{2\epsilon}} + \frac{4}{\alpha^2} \int_V [H(Du^\epsilon)]^2 \Theta^\epsilon \, dx.$$  

Let $C_0(\lambda, \Lambda) \eta^2 \frac{4}{\alpha^2} = \frac{1}{2}$, that is, $\eta^2 = \frac{\alpha^2}{8C_0(\lambda, \Lambda)}$. Apply Lemma 4.7 Theorem 2.3 and $\alpha_\epsilon \geq \alpha$ for all $\epsilon \in (0, 1]$, we have

$$\int_V \Theta^\epsilon \, dx \leq C(\lambda, \Lambda) \frac{1}{\epsilon} e^{-\frac{\mu(\alpha_0 - \alpha)}{2\epsilon}} + C(\lambda, \Lambda, \|u\|_{L^\infty(V)}, \text{dist}(V, \partial U)) \frac{1}{\alpha^2}.$$  

\[\square\]

5. PROOF OF THEOREM 2.4

Let $U = B(0, 3)$ and $V = B(0, 2)$ in this section. Let $H$, $H^\gamma$, $u$, $u^\gamma$ and $u^\gamma, \epsilon$ be as in Section 2. For convenience, we write $H^\gamma$ as $H$, and $u^\gamma$ as $u$, $u^\gamma, \epsilon$ as $u^\epsilon$ below.

Note that the condition (2.4) and Theorem 2.3 implies that

$$\sup_{U} |u^\epsilon| \leq 4 \quad \text{and} \quad \sup_{V} |Du^\epsilon| \leq C(\lambda, \Lambda).$$

Moreover, let $L_\epsilon$ and $\Theta^\epsilon$ is given in Theorem 4.1. The condition (2.5) implies that Lemma 4.8 (ii) holds, that is

$$\int_{\partial V} \Theta^\epsilon \, dS \leq C(\lambda, \Lambda).$$

The proof of Theorem 2.4 is then divided into 3 steps.

Step 1. We first show that

$$\int_{V} [H(e_n) - H(Du^\epsilon)]_+ \Theta^\epsilon \, dx \leq C(\lambda, \Lambda) |\delta + \frac{1}{\epsilon} e^{-\frac{\mu(\alpha - \beta)}}|.$$
Here and below $f_+ = \max\{f, 0\}$. Observe that
\[
\int_{\Omega} [H(e_n) - H(Du^\epsilon)]^+ \Theta^\epsilon \, dx = \int_{\Omega\cap \{H(Du^\epsilon) \leq H(e_n) - 2\delta\}} [H(e_n) - H(Du^\epsilon)] \Theta^\epsilon \, dx \\
+ \int_{\Omega\cap \{H(e_n) - 2\delta \leq H(Du^\epsilon) \leq H(e_n)\}} [H(e_n) - H(Du^\epsilon)] \Theta^\epsilon \, dx.
\]

By Lemma 1.8 (i), we have
\[
\int_{\Omega\cap \{H(Du^\epsilon) \leq H(e_n) - 2\delta\}} [H(e_n) - H(Du^\epsilon)] \Theta^\epsilon \, dx \leq H(e_n) \frac{1}{\epsilon} e^{-\frac{2\delta}{H(Du^\epsilon)(x_0)}} \leq \frac{1}{\epsilon} e^{-\frac{2\delta}{H(Du^\epsilon)}}.
\]

By Lemma 1.8 (ii), we also have
\[
\int_{\Omega\cap \{H(e_n) - 2\delta \leq H(Du^\epsilon) \leq H(e_n)\}} [H(e_n) - H(Du^\epsilon)] \Theta^\epsilon \, dx \leq 2\delta \int_{\Omega} \Theta^\epsilon \, dx \leq C(\lambda, \Lambda)\delta.
\]

**Step 2.** We show that
\[
\int_{\Omega} (D_p H(Du^\epsilon), Du^\epsilon - e_n)^2 \Theta^\epsilon \, dx \leq C(\lambda, \Lambda) \tau [1 + \frac{1}{\epsilon} e^{-\frac{2\delta}{H(Du^\epsilon)}}].
\]

Taking $v = (u^\epsilon - x_n)^2$ in Lemma 1.3, we have
\[
(u^\epsilon(x_0) - x_n^0)^2 = \int_{B(0,2)} L_\epsilon((u^\epsilon - x_n)^2) \Theta^\epsilon \, dx + \int_{\partial B(0,2)} (u^\epsilon - x_n)^2 \rho \, ds
\]
and hence, by (2.3),
\[
0 \leq \int_{B(0,2)} L_\epsilon((u^\epsilon - x_n)^2) \Theta^\epsilon \, dx + \tau^2.
\]

Since $\mathcal{H}(u^\epsilon) + \epsilon \Delta u^\epsilon = 0$, one has
\[
L_\epsilon((u^\epsilon - x_n)^2) = -H_{p_i}(Du^\epsilon)H_{p_j}(Du^\epsilon)[2(u^\epsilon_{x_i} - \delta_{nj})(u^\epsilon_{x_j} - \delta_{ni}) + 2(u^\epsilon - x_n)u^\epsilon_{x_i,x_j}] \\
- 4H_{p_i}(Du^\epsilon)H_{p_j}(Du^\epsilon)u^\epsilon_{x_i}u^\epsilon_{x_j}(u^\epsilon - x_n)(u^\epsilon_{x_i} - \delta_{ni}) - \epsilon [2(u^\epsilon_{x_i} - \delta_{nj})(u^\epsilon_{x_j} - \delta_{ni}) + 2(u^\epsilon - x_n)u^\epsilon_{x_i,x_j}] \\
= -2\langle D_p H(Du^\epsilon), Du^\epsilon - e_n \rangle^2 - 4(u^\epsilon - x_n)(D_p^2 H(Du^\epsilon)D[H(Du^\epsilon)] + Du^\epsilon - e_n - 2\epsilon |Du^\epsilon - e_n|^2,
\]
and hence, by (2.3),
\[
L_\epsilon((u^\epsilon - x_n)^2) \leq -2\langle D_p H(Du^\epsilon), Du^\epsilon - e_n \rangle^2 + 4\tau |D_p^2 H(Du^\epsilon)D[H(Du^\epsilon)], Du^\epsilon - e_n|.
\]

By Young’s inequality,
\[
|\langle D_p^2 H(Du^\epsilon)D[H(Du^\epsilon)], Du^\epsilon - e_n\rangle| \leq \frac{1}{2} \langle D_p^2 H(Du^\epsilon)D[H(Du^\epsilon)], D[H(Du^\epsilon)] \rangle + \frac{1}{2} \langle D_p^2 H(Du^\epsilon)(Du^\epsilon - e_n), (Du^\epsilon - e_n) \rangle.
\]

By the strongly concavity/convexity of $H$, we know that
\[
\langle D_p^2 H(Du^\epsilon)(Du^\epsilon - e_n), (Du^\epsilon - e_n) \rangle \leq \frac{\Lambda}{2} |Du^\epsilon - e_n|^2
\]
and
\[
\langle D_p H(Du^\epsilon), e_n - Du^\epsilon \rangle + \frac{\lambda}{2} |Du^\epsilon - e_n|^2 \leq H(e_n) - H(Du^\epsilon).
\]

Thus
\[
L_\epsilon((u^\epsilon - x_n)^2) \leq -2\langle D_p H(Du^\epsilon), Du^\epsilon - e_n \rangle^2 + 2\tau \langle D_p^2 H(Du^\epsilon)D[H(Du^\epsilon)], D[H(Du^\epsilon)] \rangle
\]
+ 2\tau \frac{\Lambda}{\lambda} [H(e_n) - H(Du^\epsilon)] - 2\tau \frac{\Lambda}{\lambda} \langle D_p H(Du^\epsilon), e_n - Du^\epsilon \rangle,

Plugging this in (5.1), one gets

$$
\int_{B(0,2)} \langle D_p H(Du^\epsilon), Du^\epsilon - e_n \rangle^2 \Theta^\epsilon \, dx
\leq \tau^2 + 2\tau \int_{B(0,2)} \langle D_{pp}^2 H(Du^\epsilon)D[H(Du^\epsilon)], D[H(Du^\epsilon)] \rangle \Theta^\epsilon \, dx
+ 2\tau \frac{\Lambda}{\lambda} \int_{B(0,2)} [H(e_n) - H(Du^\epsilon)] \Theta^\epsilon \, dx
+ 2\tau \frac{\Lambda}{\lambda} \int_{B(0,2)} |\langle D_p H(Du^\epsilon), e_n - Du^\epsilon \rangle| \Theta^\epsilon \, dx.
$$

Note that

$$
\int_{B(0,2)} \langle D_{pp}^2 H(Du^\epsilon)D[H(Du^\epsilon)], D[H(Du^\epsilon)] \rangle \Theta^\epsilon \, dx \leq C(\lambda, \Lambda).
$$

By Young’s inequality,

$$
2\tau \frac{\Lambda}{\lambda} \int_{B(0,2)} |\langle D_p H(Du^\epsilon), e_n - Du^\epsilon \rangle| \Theta^\epsilon \, dx
\leq \frac{1}{2} \int_{B(0,2)} \langle D_p H(Du^\epsilon), Du^\epsilon - e_n \rangle^2 \Theta^\epsilon \, dx + \tau^2 \frac{\Lambda}{\lambda} \int_{B(0,2)} \Theta^\epsilon \, dx
\leq \frac{1}{2} \int_{B(0,2)} \langle D_p H(Du^\epsilon), Du^\epsilon - e_n \rangle^2 \Theta^\epsilon \, dx + C(\lambda, \Lambda) \tau^2.
$$

Step 3. Set

$$
v := \zeta^2 |Du^\epsilon - e_n|^2,
$$

where \( \zeta \in C_0^\infty(B(0,2)), \ 0 \leq \zeta \leq 1 \) in \( B(0,2) \) and \( \zeta = 1 \) in \( B(0,1) \). Lemma 4.3 gives that

$$
(5.2) \quad |Du^\epsilon(x_0) - e_n|^2 = \int_{B(0,2)} L_\epsilon(\zeta^2 |Du^\epsilon - e_n|^2) \Theta^\epsilon \, dx,
$$

where we used \( \zeta|_{\partial B(0,2)} = 0 \). One has

$$
L_\epsilon(\zeta^2 |Du^\epsilon - e_n|^2) = |Du^\epsilon - e_n|^2 L_\epsilon(\zeta^2) + \zeta^2 L_\epsilon(|Du^\epsilon - e_n|) - 2H_{p_1}(Du^\epsilon)H_{p_j}(Du^\epsilon)(\zeta^2)_{x_i}(|Du^\epsilon - e_n|^2)_{x_j} - 2\epsilon(\zeta^2)_{x_i}(|Du^\epsilon - e_n|)_{x_i}.
$$

Owing to \( L_\epsilon(u_\epsilon^i) = 0 \), we further compute

$$
L_\epsilon(|Du^\epsilon - e_n|^2) = -H_{p_1}(Du^\epsilon)H_{p_j}(Du^\epsilon)[2u_{x_j} u_{x_i}^\epsilon + 2(u_{x_j} - \delta_{ns})u_{x_i}^\epsilon] - 4(u_{x_j}^\epsilon - \delta_{ns})H_{p_1}(Du^\epsilon)H_{p_jp_i}(Du^\epsilon)u_{x_j}^\epsilon u_{x_i}^\epsilon
- \epsilon[2u_{x_j}^\epsilon u_{x_i}^\epsilon + 2(u_{x_j}^\epsilon - \delta_{ns})u_{x_i}^\epsilon]
- 2|D[H(Du^\epsilon)]|^2 - 2\epsilon|D^2 u^\epsilon|^2.
$$

Note that

$$
L_\epsilon(\zeta^2) = -H_{p_1}(Du^\epsilon)H_{p_j}(Du^\epsilon)[2\zeta_{x_j} + 2\zeta_{x_i} x_j] - 4\zeta H_{p_1}(Du^\epsilon)H_{p_jp_i}(Du^\epsilon)u_{x_j}^\epsilon \zeta_{x_i}
- \epsilon[2\zeta_{x_j} + 2\zeta_{x_i} x_j]
\leq C|D_p H(Du^\epsilon)|^2 + CA \zeta |D[H(Du^\epsilon)]| + C\epsilon
$$
and hence
\[ |Du^\varepsilon - e_n|^2 L_\varepsilon(\zeta^2) \leq \frac{1}{8} \xi^2 |D[H(Du^\varepsilon)]|^2 + C(\lambda, \Lambda) |Du^\varepsilon - e_n|^2 [1 + |Du^\varepsilon|^2] \]
and
\[ -2H_{p_i}(Du^\varepsilon)H_{p_j}(Du^\varepsilon) \langle \partial_\lambda (\zeta^2) \rangle_{x_i} |Du^\varepsilon - e_n|^2 \rangle_{x_j} - 2\varepsilon (\zeta^2)_{x_i} |Du^\varepsilon - e_n|^2 \rangle_{x_i} \]
\[ \leq C |D[H(Du^\varepsilon)]| |D_p H(Du^\varepsilon)| |Du^\varepsilon - e_n| + C \zeta \varepsilon |Du^\varepsilon - e_n||D^2 u^\varepsilon| \]
\[ \leq \frac{1}{8} \xi^2 [D[H(Du^\varepsilon)]|^2 + \varepsilon |D^2 u^\varepsilon|^2 + C(\lambda, \Lambda) |Du^\varepsilon - e_n|^2 [1 + |Du^\varepsilon|^2]. \]

We conclude that
\[ L_\varepsilon(\zeta^2|Du^\varepsilon - e_n|^2) \leq C(\lambda, \Lambda) |Du^\varepsilon - e_n|^2 [1 + |Du^\varepsilon|^2]. \]

In view of (5.2), we conclude that
\[ |Du^\varepsilon(x_0)|^2 - e_n|^2 \leq C(\lambda, \Lambda) \|Du^\varepsilon\|_{L^\infty(V)} \int_{B(0,2)} |Du^\varepsilon - e_n|^2 \Theta^\varepsilon dx. \]

Since \( H \) is strongly convex,
\[ H(e_n) \geq H(Du^\varepsilon) + \langle D_p H(Du^\varepsilon), e_n - Du^\varepsilon \rangle + \frac{\lambda |Du^\varepsilon - e_n|^2}{2}. \]

This implies that
\[ \frac{\lambda}{2} \int_{B(0,2)} |Du^\varepsilon - e_n|^2 \Theta^\varepsilon dx \leq - \int_{B(0,2)} \langle D_p H(Du^\varepsilon), Du^\varepsilon - e_n \rangle \Theta^\varepsilon dx + \int_{B(0,2)} [H(e_n) - H(Du^\varepsilon)] \Theta^\varepsilon dx \]
\[ \leq C(\lambda, \Lambda) \tau + \delta + \frac{1}{\varepsilon} e^{-\Theta^\varepsilon}. \]

The proof of Theorem 2.4 is complete.

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