Applying Second-Order Quantifier Elimination in Inspecting Gödel’s Ontological Proof

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Abstract. In recent years, Gödel’s ontological proof and variations of it were formalized and analyzed with automated tools in various ways. We supplement these analyses with a modeling in an automated environment based on first-order logic extended by predicate quantification. Formula macros are used to structure complex formulas and tasks. The analysis is presented as a generated type-set document where informal explanations are interspersed with pretty-printed formulas and outputs of reasoners for first-order theorem proving and second-order quantifier elimination. Previously unnoticed or obscured aspects and details of Gödel’s proof become apparent. Practical application possibilities of second-order quantifier elimination are shown and the encountered elimination tasks may serve as benchmarks.

1 Introduction

Kurt Gödel’s ontological proof is bequeathed in hand-written notes by himself [14] and by Dana Scott [25]. Since transcriptions of these notes were published in 1987 [26], the proof was analyzed, formalized and varied in many different logical settings. Books by John Howard Sobel [27] and Melvin Fitting [13] include comprehensive discussions. Branden Fitelson, Paul E. Oppenheimer and Edward N. Zalta [12,24] investigated various metaphysical arguments in an automated first-order setting based on Prover9 [20]. Christoph Benzmüller and Bruno Woltzenlogel Paleo [7] initiated in 2014 the investigation of Gödel’s argument with automated systems, which led to a large number of follow-up studies concerning its verification with automated tools and the human-readable yet formal representation, e.g., [8,16,6,17,5]. Here we add to these lines of work an inspection of Gödel’s proof in a logical setting that so far has not been considered for this purpose. The expectation is that further, previously unnoticed or obscured aspects and details of the proof become apparent. The used framework, PIE (“Proving, Interpolating, Eliminating”) [29,30] embeds automated reasoners, in particular for first-order theorem proving and second-order quantifier elimination, in a system for defining formula macros and rendering \LaTeX-formatted presentations of formula macro definitions and reasoner outputs. In fact, the present paper is the generated output of such a PIE document.

1 Recently discovered further sources by Gödel are presented in [15].
2 The PIE source of this paper is available at http://cs.christophwernhard.com/pie.
The target logic of the macros is second-order logic, or, more precisely classical first-order logic extended by predicate quantifiers. The macro layer and the formulas obtained as expansions can be strictly separated. In our modeling of Gödel’s proof we proceed by expressing large-scale steps (axioms, theorems) with macros whose relationships are verified by invocations of embedded reasoners. In this sense our formalization of may be considered as semi-automated.

Aside of providing further material for the study of Gödel’s proof, the work shows possibilities of applying second-order quantifier elimination in a practical system. It appears that the functionality of the macro mechanism is necessary to express nontrivial applications on the basis of first- and second-order logic. The elimination problems that suggested themselves in the course of the investigation, some of which could not be solved by the current version of PIE, may be useful as benchmarks for implemented elimination systems.\(^3\)

The rest of the paper is structured as follows: After introducing preliminaries in Sect. 2, Gödel’s proof in the version of Scott is developed in Sect. 3. An approach to obtain the weakest sufficient precondition on the accessibility relation for Gödel’s proof with second-order quantifier elimination is then discussed in Sect. 4. Section 5 concludes the paper. Appendix A discusses further aspects of Gödel’s proof, including modal collapse and monothelism, App. C supplements Sect. 4, and App. D shows the results of macro expansion for selected tasks.

A short version of the paper appeared as [31].

### 2 Preliminaries

A PIE document is a Prolog source file that contains declarative formula macro definitions and specifications of reasoner invocations, interspersed with \LaTeX comments in the manner of literate programming [18]. The PIE processor expands the formula macros, invokes the reasoners, and compiles a \LaTeX document where the formula macro definitions and the results of reasoner invocations are pretty-printed. Alternatively, the processor’s functionality is accessible from Prolog, via the interpreter and in programs. The overall processing time for the present paper, including reasoner invocations and \LaTeX processing to produce a PDF, is about 2.5 seconds.

Formula macros without parameters can play the role of formula names. Expressions with macros expand into formulas of first-order logic extended with predicate quantifiers. Hence, some means of expression that would naturally be used in a higher-order logic formalization of Gödel’s proof are not available in the expansion results. Specifically, predicates in argument position are not permitted and there is no abstraction mechanism to construct predicates from formulas. However, these higher-order features are in Gödel’s proof actually only required with respect to specific instances that can be expressed in first-order logic.

\(^3\) Another recent system for second-order quantifier elimination on the basis of first-order logic is DLS-Forgetter [2], which, like the implementation in PIE, is based on the DLS algorithm [10]. An older resolution-based system, SCAN, [22,23] can currently be invoked via a Web interface.
As embedded reasoners we used the first-order theorem provers Prover9 [20] and CMProver [9,29], the first-order model generator Mace4 [20], and an implementation of the DLS algorithm [10] for second-order quantifier elimination, which is based on Ackermann’s Lemma [1]. Reasoner outputs computed during processing of the PIE document are presented with the introductory phrases **This formula is valid;** **This formula is not valid;** and **Result of elimination:** In addition, various methods for formula simplification, clausification and un-Skolemization are applied in preprocessing, inprocessing and for output presentation.

The generated \LaTeXX presentation of formulas and macro definitions bears some footprint inherited from the Prolog syntax that is used to write formulas in PIE documents. As in Prolog, predicate and constant symbols are written in lower case. Macro parameters and bound logical variables that are to be instantiated with fresh symbols at macro expansion are printed like Prolog variables with a capitalized initial. Where-clauses in macro definitions are used to display in abstracted form auxiliary Prolog code executed at macro expansion.

We write formulas of modal predicate logic as formulas of classical first-order logic with one additional free world variable by applying the standard translation from [4, Sec. 11.4] (see also [3, Chap. XII]), which can be defined as

\[
\begin{align*}
ST(P(t_1, \ldots, t_n)) & \equiv P(v, t_1, \ldots, t_n) \\
ST(\neg F) & \equiv \neg ST(F) \\
ST(F \lor G) & \equiv ST(F) \lor ST(G) \\
ST(\exists x F) & \equiv \exists x (e(v, x) \land ST(F)) \\
ST(\diamond F) & \equiv \exists w (r(v, w) \land \bigwedge_{i \text{ s.th. } x_i \text{ free in } F} e(w, x_i) \land ST(F)\{v \mapsto w\})
\end{align*}
\]

An \(n\)-ary predicate \(P\) in the modal logic is translated into an \(n+1\)-ary predicate, where the first argument represents a world. The binary predicates \(r\) and \(e\) are used for world accessibility and membership in the domain of a world. The logic operators \(\land, \rightarrow, \leftrightarrow, \forall, \Box\) can be understood as shorthands defined in terms of the shown operators. As target logic we neither use a two-sorted logic nor encode two-sortedness explicitly with relativizer predicates. However, the translation of modal formulas yields formulas in which all quantifications are relativized by \(r\) or by \(e\), which seems to subsume the effect of such relativizer predicates. To express that free individual symbols are of sort \textit{world} we use the unary predicate \textit{world}. Macro 4, defined below, can be used as an axiom that relates \textit{world} and \(r\) as far as needed for our purposes. The standard translation realizes with respect to the represented modal logic varying domain semantics (actualist notion of quantification), expressed with the existence predicate \(e\). Constant domain semantics (possibilist notion of quantification) can be achieved with axioms that state domain increase and decrease.

As technical basis for Gödel’s proof we use the presentation of Scott’s version [25] in [7, Fig. 1], shown here as Fig. 1. The identifiers \(A1-A5, T1-T3, D1-D3\) and \(C\) of the involved axioms, theorems, definitions and corollary follow [7]. In addition, Lemma \(L\) is taken from [6, Fig. 1], where it appears as \(L2\).
Either a property or its negation is positive, but not both
∀P(\text{Pos}(\neg P) \leftrightarrow \neg \text{Pos}(P))

A property necessarily implied by a positive property is positive
∀P \forall Q ((\text{Pos}(P) \land \Box \forall x (P(x) \rightarrow Q(x))) \rightarrow \text{Pos}(Q))

Positive properties are possibly exemplified
∀P \Box \exists x P(x)

A God-like being possesses all positive properties
G(x) \leftrightarrow \forall P (\text{Pos}(P) \rightarrow P(x))

The property of being God-like is positive
\text{Pos}(G)

Possibly, God exists
\Box \exists x G(x)

God-like is an essence of any God-like being
∀x (G(x) \rightarrow \text{Ess}(G, x))

Necessary existence of an individual is the necessary exemplification of all its essences
\Box \exists x \forall P (\text{Ess}(P, x) \rightarrow \Box \exists y P(y))

Necessary existence is a positive property
\text{Pos}(\Box \exists x G(x))

If a God-like being exists, then necessarily a God-like being exists
\exists x G(x) \rightarrow \Box \exists x G(x)

Necessarily, God exists
\Box \exists x G(x)

Fig. 1. Scott’s version of Gödel’s ontological argument [25], adapted from [7,6].

3 Rendering Gödel’s Ontological Proof

3.1 Positiveness – Proving Theorem T1

The first two axioms in Gödel’s proof, A1 and A2, are about positiveness of properties. Theorem T1 follows from them. The following macros render the left-to-right direction of A1 and A2, respectively.

Macro 1 $ax^\top_1(V, P)$ is defined as
world(V) \rightarrow (\text{pos}(V, N') \rightarrow \neg \text{pos}(V, P')),
where
\[ N' := \neg P, \]
\[ P' := \hat{P}. \]

Is positive is represented here by the binary predicate \text{pos}, which has a world and an individual representing a predicate as argument. $P'$ and $N'$ are individual constants that represent a supplied predicate $P$ and its complement $\lambda v x. \neg P(v, x)$, respectively. The where clause specifies that at macro expansion they are replaced by individual constants $\hat{P}$ and $\neg \hat{P}$, available for each predicate symbol $P$. 
Throughout this analysis, we expose the current world as macro parameter $V$, which facilitates identifying proofs steps where an axiom is not just applied with respect to the initially given current world but to some other reachable world.

**Macro 2** $ax_2(V, P, Q)$ is defined as

$$\text{world}(V) \rightarrow (\text{pos}(V, P') \land \forall W (r(V, W) \rightarrow (\forall X (e(W, X) \rightarrow (P(W, X) \rightarrow Q(W, X)))) \rightarrow \text{pos}(V, Q'))),$$

where $P' := \dot{P}$, $Q' := \dot{Q}$.

As an insight-providing intermediate step for proving $T_1$ we can now derive the following lemma using just a single instance of each of $ax_2^1$ and $ax_2$, where verum $(\lambda x. x = x)$ and falsum $(\lambda x. x \neq x)$ are represented as binary predicates $\top$ and $\bot$, whose first argument is a world.

**Macro 3** $\text{lemma}_1(V)$ is defined as

$$\text{world}(V) \rightarrow \neg \text{pos}(V, \bot).$$

**Macro 4** $r\_\text{world}$ is defined as $\forall v \forall w (r(v, w) \rightarrow \text{world}(w))$.

**Macro 5** $\text{topbot}\_\text{def}$ is defined as

$$\forall v \forall x (\text{world}(v) \rightarrow (\top(v, x) \leftrightarrow e(v, x))) \land \forall v \forall x (\text{world}(v) \rightarrow (\bot(v, x) \leftrightarrow \neg e(v, x))).$$

To express the precondition for $\text{lemma}_1$ we need some auxiliary macros concerning $\top$ and $\bot$. The following expresses equivalence of $\top$ and $\lambda vx. \neg \bot(v, x)$.

**Macro 6** $\text{topbot}\_\text{equiv}$ is defined as $\forall v \forall x (\text{world}(v) \rightarrow (\top(v, x) \leftrightarrow \neg \bot(v, x))).$

This formula is valid: $\text{pre}\_\text{lemma}_1(v) \rightarrow \text{lemma}_1(v)$.

$T_1$ can be rendered by the following macro with a predicate parameter.
Macro 9 \( \text{thm}_1(V, P) \) is defined as

\[
\begin{align*}
\text{world}(V) & \quad \rightarrow \\
(\text{pos}(V, P')) & \quad \rightarrow \\
\exists W (r(V, W) \land \exists X (e(W, X) \land P(W, X)))
\end{align*}
\]

where \( P' := \dot{P} \).

Macro 10 \( \text{pre}_\text{thm}_1(V, P) \) is defined as \( \text{lemma}_1(V) \land ax_2(V, P, \bot) \).

This formula is valid: \( \text{pre}_\text{thm}_1(v, p) \rightarrow \text{thm}_1(v, p) \).

Instances of \( \text{thm}_1(V, P) \) can be proven for arbitrary worlds \( V \) and predicates \( P \), from the respective instance of the precondition \( \text{pre}_\text{thm}_1(V, P) \). A further instance of \( ax_2 \) – beyond that used to prove \( \text{lemma}_1 \) – is required there, with respect to \( \bot \) and the given predicate \( P \).

3.2 Possibly, God Exists – Proving Corollary C

Axiom A3 and T1 instantiated by God-like together imply corollary C. This is rendered as follows, where God-like is represented by \( g \).

Macro 11 \( ax_3(V) \) is defined as \( \text{world}(V) \rightarrow \text{pos}(V, \dot{g}) \).

Macro 12 \( \text{coro}(V) \) is defined as \( \text{world}(V) \rightarrow \exists W (r(V, W) \land \exists X (e(W, X) \land g(W, X))) \).

Macro 13 \( \text{pre}_\text{coro}(V) \) is defined as \( \text{thm}_1(V, g) \land ax_3(V) \).

This formula is valid: \( \text{pre}_\text{coro}(v) \rightarrow \text{coro}(v) \).

Notice that, differently from the proofs reported in [7, Fig. 2], C, represented here by coro, can be proven independently from the definition of God-like. D1, which is represented here by the Macros \( \text{def}_1^{\rightarrow} \) and \( \text{def}_1^{\rightarrow \neg} \) defined below.

3.3 Essence – Proving Theorem T2

With macros \( \text{def}_1^{\rightarrow} \) and \( \text{def}_1^{\rightarrow \neg} \), defined now, we represent the left-to-right direction of D1. Actually, only this direction of D1 is required for the proving the further theorems.

Macro 14 \( \text{def}_1^{\rightarrow}(V, X, P) \) is defined as \( g(V, X) \rightarrow (\text{pos}(V, P') \rightarrow P(V, X)) \),

where \( P' := \dot{P} \).

Macro 15 \( \text{def}_1^{\rightarrow \neg}(V, X, P) \) is defined as \( g(V, X) \rightarrow (\text{pos}(V, P') \rightarrow \neg P(V, X)) \),

where \( P' := \neg P \).

The following macro \( \text{val}_\text{ess} \) renders the definiens of the essence of relationship between a predicate and an individual in D2. It is originally a formula with predicate quantification, but without application of a predicate to a predicate. The macro \( \text{val}_\text{ess} \) exposes the universally quantified predicate as parameter \( Q \), permitting to use it instantiated with some specific predicate.
Macro 16 \( \text{val}_\text{ess}(V, P, X, Q) \) is defined as
\[
P(V, X) \land (Q(V, X) \land \forall W (r(V, W) \rightarrow \forall Y (e(W, Y) \rightarrow (P(W, Y) \rightarrow Q(W, Y)))))
\]
The universally quantified version of \( \text{val}_\text{ess} \) can be be expressed by prefixing a predicate quantifier upon \( Q \). Eliminating this second-order quantifier shows another view on essence.

Input: \( \forall q \text{val}_\text{ess}(v, p, x, q) \).

Result of elimination:
\[
p(v, x) \land (r(v, w) \rightarrow (p(w, y) \land e(w, y) \rightarrow (p(w, y) \rightarrow w = v \land y = x))) \leftrightarrow \text{last_result}.
\]

We convert the elimination result “manually” to a more clear form and prove equivalence by referencing to the “last result” via a macro.

Macro 17 \( \text{last_result} \) is defined as \( F \), where
\[
\text{last_ppl_result}(F).
\]

This formula is valid:
\[
p(v, x) \land \forall w (r(v, w) \rightarrow (\forall y (e(w, y) \rightarrow (p(w, y) \rightarrow w = v \land y = x)))) \leftrightarrow \text{last_result}.
\]

The following definition now renders \( \text{D2} \) as definition of the predicate \( \text{ess} \) in terms of \( \text{val}_\text{ess} \).

Macro 18 \( \text{def}_2(V, P) \) is defined as
\[
\forall X (\text{ess}(V, P', X) \leftrightarrow \forall Q \text{val}_\text{ess}(V, P, X, Q)),
\]
where
\[
P' := \hat{P}.
\]

In App. A.1 it is shown that two observations about essence mentioned as NOTE in Scott’s version \([25]\) of Gödel’s proof can be derived in this modeling.

The following two macros render the right-to-left direction of axioms \( \text{A1} \) and \( \text{A4} \). The original axioms involve a universally quantified predicate that appears only in argument role. In the macros, it is represented by the parameter \( P \).

Macro 19 \( \text{ax}_1^+(V, P) \) is defined as
\[
\forall X (\neg \text{pos}(V, P') \rightarrow \text{pos}(V, N'))
\]
where
\[
N' := \neg P,
P' := \hat{P}.
\]

Macro 20 \( \text{ax}_4(V, P) \) is defined as
\[
\forall W (r(V, W) \rightarrow \text{pos}(W, P'))
\]
where
\[
P' := \hat{P}.
\]

Theorem \( \text{T2} \) is rendered by the following macro with \( \text{ess} \) unfolded, which permits expansion into a universal second-order formula without occurrence of a predicate in argument position.
Macro 21 \( \text{proto\_thm}_2(V, X) \) is defined as
\[
\text{world}(V) \rightarrow (\text{e}(V, X) \rightarrow (\text{g}(V, X) \rightarrow \forall Q \text{val\_ess}(V, g, X, Q))).
\]

Macro 22 \( \text{pre\_proto\_thm}_2(V, X, Q) \) is defined as
\[
\begin{align*}
\text{ax}_1^{-1}(V, Q) & \land \\
\forall W (r(V, W) \rightarrow \forall X (\text{e}(W, X) \rightarrow \text{def}_1^{-1}(W, X, Q))) & \land \\
\text{def}_1^{-1}(V, X, Q) & \land \\
\text{ax}_4(V, Q).
\end{align*}
\]

This formula is valid: \( \forall q \exists \hat{q} \exists \neg q \text{pre\_proto\_thm}_2(v, x, q) \rightarrow \text{proto\_thm}_2(v, x) \).

In this implication on the left side the constants \( \hat{q} \) and \( \neg \hat{q} \), which represent predicates \( q \) and \( \lambda x.\neg q(v, x) \) in argument positions, are existentially quantified.

3.4 Necessarily, God Exists – Proving Theorem T3

The definiens of necessary existence, which is defined in Definition D3, is rendered here by the following macro \( \text{val\_ne} \), expressed in terms of \( \text{val\_ess} \), the representation of the definiens of essence, to avoid the occurrence of a predicate representative in argument position.

Macro 23 \( \text{val\_ne}(V, X) \) is defined as
\[
\forall P (\forall Q \text{val\_ess}(V, P, X, Q) \rightarrow \\
\forall W (r(V, W) \rightarrow \exists Y (\text{e}(W, Y) \land P(W, Y)))).
\]

Eliminating the quantified predicates shows another view on necessary existence.

Input: \( \text{val\_ne}(v, x) \).

Result of elimination:
\[
\forall y (r(v, y) \rightarrow y = v) \land \forall y (r(v, y) \rightarrow e(y, x)).
\]

The elimination result can be brought into a more clear form.

This formula is valid: \( \forall w (r(v, w) \rightarrow w = v \land e(w, x)) \leftrightarrow \text{last\_result} \).

In analogy to the definition of the predicate \( \text{ess} \) in Macro 18 we define the predicate \( \text{ne} \) in terms of \( \text{val\_ne} \).

Macro 24 \( \text{def}_3(v, X) \) is defined as
\[
\text{world}(V) \rightarrow (\text{e}(V, X) \rightarrow (\text{ne}(V, X) \leftrightarrow \text{val\_ne}(V, X))).
\]

The following formula renders a fragment of the definition of necessary existence on a “shallow” level, that is, in terms of just the predicates \( \text{ess} \) and \( \text{ne} \), without referring to their definiens \( \text{val\_ess} \) and \( \text{val\_ne} \).

Macro 25 \( \text{def}_3^{-1}(V, X, P) \) is defined as
\[
\begin{align*}
\text{world}(V) & \rightarrow \\
(\text{e}(V, X)) & \rightarrow \\
(\text{ne}(V, X)) & \rightarrow \\
(\text{ess}(V, P', X)) & \rightarrow \\
(\forall W (r(V, W) \rightarrow \exists Y (\text{e}(W, Y) \land P(W, Y))))).
\end{align*}
\]

where \( P' \equiv \hat{P} \).
Correctness of \( \text{def}_{3}^{*} \) can be established by showing that it follows from the definitions of \( \text{ess} \) and \( \text{ne} \).

**This formula is valid:** \( \text{def}_{2}(v, p) \land \text{def}_{3}(v, x) \rightarrow \text{def}_{3}^{*}(v, x, p) \).

The following macro renders \( T2 \), in contrast to Macro 21 now expressed in terms of the predicate \( \text{ess} \) instead of its definiens \( \text{val}_{\text{ess}} \).

**Macro 26** \( \text{thm}_{2}(V, X) \) is defined as

\[
\text{world}(V) \rightarrow (\text{e}(V, X) \land (\text{g}(V, X) \rightarrow \text{ess}(V, g, X))).
\]

Axiom A5 \( (\text{pos}(\text{ne})) \) is represented as follows.

**Macro 27** \( \text{ax}_{5}(V) \) is defined as

\[
\text{world}(V) \rightarrow \text{pos}(V, \dot{\text{ne}}).
\]

Scott’s version [25] shows theorem \( T3 \) via the lemma \( L \), rendered as follows.

**Macro 28** \( \text{lemma}_{2}(V) \) is defined as

\[
\exists X (\text{e}(V, X) \land \text{g}(V, X)) \rightarrow \forall W (\text{r}(V, W) \rightarrow \exists Y (\text{e}(W, Y) \land \text{g}(W, Y))).
\]

**Macro 29** \( \text{pre lemma}_{2}(V, X) \) is defined as

\[
\text{ax}_{5}(V) \land \text{def}_{3}^{*}(V, X, \text{ne}) \land \text{def}_{3}^{*}(V, X, g) \land \text{thm}_{2}(V, X).
\]

**This formula is valid:** \( \forall v (\forall x \text{pre lemma}_{2}(v, x) \rightarrow \text{lemma}_{2}(v)) \).

The following formula states theorem \( T3 \), the overall result to show.

**Macro 30** \( \text{thm}_{3}(V) \) is defined as

\[
\text{world}(V) \rightarrow \forall W (\text{r}(V, W) \rightarrow \exists Y (\text{e}(W, Y) \land \text{g}(W, Y))).
\]

**Macro 31** \( \text{pre thm}_{3}(V) \) is defined as \( \text{r}_{\text{world}}^{1} \land \forall v \text{lemma}_{2}(v) \land \text{coro}(V) \).

**Macro 32** \( \text{euclidean} \) is defined as \( \forall x \forall y \forall z (\text{r}(x, y) \land \text{r}(x, z) \rightarrow \text{r}(z, y)) \).

**Macro 33** \( \text{symmetric} \) is defined as \( \forall x \forall y (\text{r}(x, y) \rightarrow \text{r}(y, x)) \).

**This formula is valid:** \( \text{symmetric} \lor \text{euclidean} \rightarrow (\text{pre thm}_{3}(v) \rightarrow \text{thm}_{3}(v)) \).

As observed in [7], \( T3 \) can not be just proven in the modal logic \( S5 \), but also in \( KB \), whose accessibility relation is just constrained to be symmetric. We have shown here in a single statement that the proof is possible for a Euclidean as well as a symmetric accessibility relation by presupposing the disjunction of both properties. Precondition \( \text{pre thm}_{3} \) includes \( \text{coro} \) instantiated with just the current world and \( \text{lemma}_{2} \) with a universal quantifier upon the world parameter. In fact, as shown now, using \( \text{lemma}_{2} \) there just instantiated with the current world would not be sufficient to derive \( \text{thm}_{3} \).

**This formula is not valid:** \( \text{symmetric} \lor \text{euclidean} \rightarrow (\text{r}_{\text{world}}^{1} \land \text{lemma}_{2}(v) \land \text{coro}(v) \rightarrow \text{thm}_{3}(v)) \).

In App. A further aspects of Gödel’s proof are modeled, in particular modal collapse and monotheism.
4 On Weakening the Frame Condition for Theorem T3

In the proof of thm\textsubscript{3} from pre\textsubscript{thm}\textsubscript{3} we used the additional frame condition euclidean \lor symmetric. The observation that the weaker KB instead of S5 suffices to prove T3 was an important finding of [7]. Hence, the question arises whether the precondition on the accessibility relation can be weakened further.

In general, the weakest sufficient condition [19,11,28] of a formula G on a set Q of predicates within a formula F can be expressed as the second-order formula \[ \forall p\_1 \ldots \forall p\_n (F \rightarrow G), \] where \( p\_1, \ldots, p\_n \) are all predicates that occur free in \( F \rightarrow G \) and are not members of Q. This formula denotes the weakest (with respect to entailment) formula \( H \) in which only predicates in Q occur free such that \( H \rightarrow (F \rightarrow G) \) is valid. Second-order quantifier elimination can be applied to this formula to “compute” a weakest sufficient condition, that is, converting it to a first-order formula, which, of course, is inherently not possible in all cases.

For T3, the weakest sufficient condition on the accessibility relation \( r \) and the domain membership relation and \( e \) is the second-order formula

\[ \forall g \forall v (pre\_thm\_3(v) \rightarrow thm\_3(v)). \]

Unfortunately, elimination of the second-order quantifier upon \( g \) fails for this formula (at least with the current version of PIE). But elimination succeeds for a simplified variant of the problem, which considers just propositional modal logic and combines two instances of Lemma lemma\textsubscript{2} with an unfolding of C. The way in which this simplification was obtained is outlined in App. C.

Macro 34 lemma\textsubscript{2-simp}(V) is defined as

\[ g(V) \rightarrow \forall W (r(V,W) \rightarrow g(W)). \]

Macro 35 pre\_thm\_3\_simp\_inst(V) is defined as

\[ \forall W (r(V,W) \land g(W) \land lemma\_2\_simp(W)). \]

Macro 36 thm\textsubscript{3-simp}(V) is defined as

\[ \forall W (r(V,W) \rightarrow g(W)). \]

This formula is valid: euclidean \lor symmetric \rightarrow (pre\_thm\_3\_simp\_inst(v) \rightarrow thm\_3\_simp(v)).

Input: \[ \forall g \forall v (pre\_thm\_3\_simp\_inst(v) \rightarrow thm\_3\_simp(v)). \]

Result of elimination:

\[ \forall x\forall y\forall z (r(x,y) \land r(x,z) \rightarrow r(y,x) \lor r(y,z) \land x = y \lor y = z). \]

We write the resulting first-order formula in a slightly different form, give it a name, verify equivalence to the original form and show some of its properties.

Macro 37 frame\_cond\_simp is defined as

\[ \forall x\forall y\forall z (r(x,y) \land r(x,z) \land y \neq z \rightarrow r(y,x) \lor r(y,z)). \]

This formula is valid: frame\_cond\_simp \leftrightarrow last\_result.
Macro 38  reflexive is defined as $\forall x r(x, x)$.

This formula is valid: reflexive $\rightarrow$ (symmetric $\lor$ euclidean $\leftrightarrow$ frame$_{\_}$cond$_{\_}$simp).
This formula is valid: symmetric $\lor$ euclidean $\rightarrow$ frame$_{\_}$cond$_{\_}$simp.
This formula is not valid: frame$_{\_}$cond$_{\_}$simp $\rightarrow$ symmetric $\lor$ euclidean.

Thus, the obtained frame condition frame$_{\_}$cond$_{\_}$simp is under the assumption of reflexivity equivalent to symmetric $\lor$ euclidean, and without that assumption strictly weaker. The following statement shows that this weaker frame condition also works for our original problem, proving T3.

This formula is valid: frame$_{\_}$cond$_{\_}$simp $\rightarrow$ (pre$_{\_}$thm$_3(v) \rightarrow$ thm$_3(v))$.

Hence, via the detour through elimination applied to a simplified problem, we have found a strictly weaker frame condition for T3 than symmetric $\lor$ euclidean, but, since elimination has just been performed on the second-order formula representing the simplified problem, we do not know whether it is the weakest one.

5  Conclusion

We reconstructed Gödel’s ontological proof in an environment that integrates automated first-order theorem proving, second-order quantifier elimination, a formula macro mechanism and \LaTeX-based formula pretty-printing, supplementing a number of previous works that render Gödel’s proof in other automated theorem proving environments. Particular observations of interest for the study of Gödel’s proof that became apparent through our modeling include the following:

1. The presentation of the derivation of theorem T1 exhibits the few actually used instantiations of axioms A1 and A2. The derivation is via a lemma, which might be useful as internal interface in the proof because it can be justified in alternate ways.
2. Corollary C can be shown independently from the actual definition of God-like (D1) just on the basis of the assumption that T1 applies to God-like.\footnote{This is also apparent in [5, Fig. 4, line 20].}
3. In the whole proof, definition D1 is only used in the left-to-right direction.\footnote{This applies if A3 is given as in Scott’s version, but not if it is derived from further general properties of positive, as in Gödel’s original version and in [5, Fig. 4, line 19].}
4. Second-order quantifier elimination yields first-order representations of essence (definition D2) and necessary existence (definition D3).
5. Lemma L can be derived independently from the definiens of essence. Here the predicate ess appears in the respective expanded formula passed to the reasoner, but not its definiens.
6. For the derivation of theorem T3 an accessibility relationship is sufficient that, unless reflexivity is assumed, is strictly weaker than the disjunction of the Euclidean property and symmetry.

If non-experts in automated reasoning are addressed, the syntactical presentation of Gödel’s argument is of particular importance [8]. We approached this
requirement by means of formula macro definitions with the representation of input formulas by Prolog terms and \LaTeX{} pretty-printing for output formulas.

Most automated formalizations of metaphysical arguments, e.g., [12,24,7,8,5], seem closely tied to a particular system or combination of systems. Of course, processing a PIE document similarly depends on the PIE system with specific embedded reasoners. However, a system-independent view on the formalization is at least obtainable: The underlying target logic of the macro expansion is just the well-known classical first-order logic extended with predicate quantification. Reasoning tasks are only performed on the expanded formulas. The PIE system can output these explicitly (see, e.g., App. D), providing a low-level, but system-independent logical representation of the complete formalization. As a further beneficial aspect, such an explicit low-level formalization may prevent the unnoticed interaction with features of involved special logics.

A limitation of our approach might be that there is no automated support for the passage from the low to the high level, i.e., folding into formula macros. If trust in proofs is an issue, steps in the overall workflow for which no proof representations are produced may be objectionable. This concerns macro expansion, formula normalization (see, however, [21]), pre- and postprocessing of formulas, and in particular second-order quantifier elimination, for which the creation of proofs seems an unexplored terrain. A practical makeshift is comparison with the few other elimination systems [2, Sect. 4].

In principle it should be possible to integrate second-order quantifier elimination as used here also into automated reasoning environments based on other paradigms, in particular the heterogeneous environments that involve forms of higher-order reasoning and are applied in [7,8,6,5].

Concerning second-order quantifier elimination, an issue that might be worth further investigation is the generalization of the method applied here ad-hoc to weaken the precondition on the accessibility relation: We started from an elimination problem that could not be solved (at least with the current implementation of PIE), moved to a simpler problem and then verified that the solution of the simpler problem is also applicable to the original problem, where it does not represent the originally desired unique weakest sufficient condition, but nevertheless a condition that is weaker than the condition known before.

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Appendices

A Further Aspects

A.1 Notes from Scott’s Version Concerning Essence

The observations $\text{Ess}(P, x) \land \text{Ess}(Q, x) \rightarrow \Box P = Q$ and $\text{Ess}(P, x) \rightarrow \forall y (P(y) \rightarrow y = x)$ are stated as NOTE in Scott’s version [25] of Gödel’s proof. We express them here with the predicate version $\text{ess}$ of essence to facilitate their use as axioms in other statements.

Macro A-1 $\text{note}_1(V, P, Q)$ is defined as

$$\text{world}(V) \rightarrow (\exists X (\text{ess}(V, P', X) \land \text{ess}(V, Q', X))) \rightarrow \forall W (r(V, W) \rightarrow \forall Y (e(W, Y) \rightarrow (P(W, Y) \leftrightarrow Q(W, Y))))),$$

where $P' := \dot{P}$, $Q' := \dot{Q}$.

This formula is valid: $\text{def}_2(v, p_1) \land \text{def}_2(v, p_2) \rightarrow \text{note}_1(v, p_1, p_2)$.

Macro A-2 $\text{note}_2(V, P, X)$ is defined as

$$\text{world}(V) \rightarrow (\exists X (\text{ess}(V, P', X))) \rightarrow \forall W (r(V, W) \rightarrow \forall Y (e(W, Y) \rightarrow (P(W, Y) \rightarrow Y = X))))),$$

where $P' := \dot{P}$.

This formula is valid: $\text{def}_2(v, p) \rightarrow \text{note}_2(v, p, x)$.

A.2 Modal Collapse

A well-known objection to Gödel’s theory is that it implies modal collapse [26]. To show this we need to relate $\text{world}$ and $r$ with $r_{\text{world}}$, which strengthens $r_{\text{world}}$, defined as Macro 4.

Macro A-3 $r_{\text{world}}$ is defined as

$$\forall v \forall w (r(v, w) \rightarrow \text{world}(v) \land \text{world}(w)).$$

Macro A-4 $\text{collapse}$ is defined as

$$\forall x \forall y (r(x, y) \rightarrow y = x).$$

In our representation, modal collapse can be derived from the following preconditions, selected after Fitting’s reconstruction [13, Chapter 11, Section 8] of Sobel’s proof [27,26].
Macro A-5 \( \text{pre\_collapse} \) is defined as
\[ \forall x \forall v \text{thm}_2(v, x) \land \forall x \forall v \text{thm}_3(v) \land \forall v \text{def}_2(v, g) \land \text{r\_world} \land \text{reflexive}. \]

This formula is valid: \( \text{pre\_collapse} \rightarrow \text{collapse} \).

In presence of \( \text{collapse} \), the choice between frame conditions symmetric and euclidean (or the modal logics \( \text{KB} \) and \( \text{S5} \)) becomes immaterial, as both properties are implied by \( \text{collapse} \). Also A-4 is in presence of \( \text{collapse} \) redundant.

This formula is valid: \( \text{collapse} \rightarrow \text{symmetric} \land \text{euclidean} \).

This formula is valid: \( \text{collapse} \rightarrow \text{ax}_4(v, p) \).

A.3 Monotheism

In Fitting’s system the proposition \( \exists x \forall y (g(y) \leftrightarrow y = x) \) can be derived [13, Section 7.1]. This can be proven in our representation from \( \text{thm}_2, \text{note}_2 \) and \( \text{thm}_3 \) under the additional assumption of reflexivity of the accessibility relation. Without that assumption, it can be shown that \( \Box \exists x \exists y (G(y) \leftrightarrow y = x) \):

Macro A-6 \( \text{pre\_monotheism} \) is defined as
\[ \forall x \forall v \text{thm}_2(v, x) \land \forall x \forall v \text{note}_2(v, g, x) \land \forall x \forall v \text{thm}_3(v) \land \text{r\_world}. \]

Macro A-7 \( \text{monotheism} \) is defined as
\[ \forall v \exists x (e(v, x) \land \forall y (e(v, y) \rightarrow (g(v, y) \leftrightarrow y = x))). \]

This formula is valid: \( \text{pre\_monotheism} \land \text{reflexive} \rightarrow \text{monotheism} \).

Macro A-8 \( \text{nec\_monotheism} \) is defined as
\[ \forall v \forall w (r(v, w) \rightarrow \exists x (e(w, x) \land \forall w_1 (r(w, w_1) \rightarrow \forall y (e(w_1, y) \rightarrow (g(w_1, y) \leftrightarrow y = x))))). \]

This formula is valid: \( \text{pre\_monotheism} \rightarrow \text{nec\_monotheism} \).

B Alternate Weaker Preconditions for \( \text{pre\_lemma}_1 \)

The precondition \( \text{pre\_lemma}_1 \) used in Sect. 3.1 to derive \( \text{lemma}_1 \) includes \( \text{topbot\_def} \land \text{topbot\_equiv\_equal} \).

The following is a weaker formula that is also sufficient for deriving \( \text{lemma}_1 \):

Macro B-1 \( \text{topbot\_alt}_1 \) is defined as
\[ \forall v (\text{world}(v) \rightarrow \exists x (e(v, x) \rightarrow \top(v, x))) \land \forall v (\text{world}(v) \rightarrow (\text{pos}(v, \bot) \rightarrow \text{pos}(v, \neg \top))). \]

This formula is valid: \( \text{topbot\_def} \land \text{topbot\_equiv\_equal} \rightarrow \text{topbot\_alt}_1 \).
Macro B-2  \textit{pre\_lemma\_1\_drop\_topbot}(V) is defined as \( F \),
where
\[
F \text{ is like } \text{pre\_lemma\_1}(V) \text{ except }
\top \text{ instead of } \text{topbot\_def}
\]
\[
\top \text{ instead of } \text{topbot\_equiv\_equal}.
\]
This formula is valid: \( \text{topbot\_alt\_1} \land \text{pre\_lemma\_1\_drop\_topbot}(v) \rightarrow \text{lemma\_1}(v) \).

A third possibility to derive \text{pre\_lemma\_1} is with the formula \text{topbot\_alt\_2} defined below, which is like \text{topbot\_alt\_1} except that \( \top \) in the first conjunct is replaced by \( \neg \bot \):

Macro B-3  \textit{topbot\_alt\_2} is defined as \( F \),
where
\[
F \text{ is like } \text{topbot\_alt\_1} \text{ except }
\neg \bot(V,X) \text{ instead of } \top(V,X).
\]
This formula is valid: \( \text{topbot\_def} \land \text{topbot\_equiv\_equal} \rightarrow \text{topbot\_alt\_2} \).
This formula is valid: \( \text{topbot\_alt\_2} \land \text{pre\_lemma\_1\_drop\_topbot}(v) \rightarrow \text{lemma\_1}(v) \).

C The Way to the Simplified Elimination Task of Sect. 4

In Sect. 4 a simplified version of the second-order formula
\[
\forall g \forall v (\text{pre\_thm\_3}(v) \rightarrow \text{thm\_3}(v))
\]
was used to compute a weakest sufficient condition by second-order quantifier elimination. The simplification was in two respects: an adaption to propositional modal logic and the explicit involvement of only two instances of \text{lemma\_2} with an unfolding of \text{coro}. The adaption to propositional modal logic alone did not lead to success of elimination (with the current version of PIE). The instantiation and unfolding was suggested by the manual inspection of a clausal tableau proof by \text{CMProver} in the propositional modal logic setting.

Macro C-1  \textit{coro\_simp}(V) is defined as
\[
\exists W (r(V,W) \land g(W)).
\]

Macro C-2  \textit{pre\_thm\_3\_simp}(V) is defined as
\[
\forall v \text{lemma\_2\_simp}(v) \land \text{coro\_simp}(V).
\]
This formula is valid: \( \text{symmetric} \rightarrow (\text{pre\_thm\_3\_simp}(v) \rightarrow \text{thm\_3\_simp}(v)) \).

The tableau is shown in Fig. 2. Actually only two instances of \( \forall v \text{lemma\_2\_simp}(v) \) are used in the proof. Formula \text{pre\_thm\_3\_simp\_inst} (Macro 35) is a version of \text{pre\_thm\_3\_simp} with the required two instances, the second one inserted into an unfolding of \text{coro\_simp}.
Fig. 2. Clausal tableau proof of $\text{symmetric} \rightarrow (\text{pre}_\text{thm}_3\text{_simp}(v) \rightarrow \text{thm}_3\text{_simp}(v))$. The two instances of $\forall v \text{lemma}_2\text{_simp}(v)$ appear in the clausal tableau as the two ternary clauses.
D Expansions

This appendix shows for selected proving and elimination tasks the first- or
second-order formulas submitted after full macro expansion to reasoners.

D.1 Formulas whose Validity was Proven by Prover9 Embedded in PIE

Before these expanded formulas are passed by PIE to a first-order system, pre-
processing is applied, which may involve the rearrangement of universal second-
order variables into a prefix such that the validity of the resulting universal
second-order formula equals first-order validity after stripping off the prefix.

Original Formula: \( pre\_lemma_1(v) \rightarrow lemma_1(v) \).

Expanded Formula:
\[
\begin{align*}
\forall x \forall y \ (r(x, y) & \rightarrow world(y)) \\
\forall x \forall y \ (world(x) & \rightarrow (\top(x, y) \leftrightarrow e(x, y))) \\
\forall x \forall y \ (world(x) & \rightarrow (\bot(x, y) \leftrightarrow \neg e(x, y))) \\
(\forall x \forall y \ (world(x) & \rightarrow (\top(x, y) \leftrightarrow \neg \bot(x, y)))) & \rightarrow \\
\bot = \neg \top & \\
\forall x \ (r(v, x) & \rightarrow \forall y \ (e(x, y) \rightarrow (\bot(x, y) \rightarrow \top(x, y)))) & \rightarrow \\
\exists x \ (r(v, x) & \rightarrow \exists y \ (e(x, y) \land g(x, y)))) & \\
\forall x \forall y \ (r(x, y) & \rightarrow \forall y \ (e(x, y) \land g(x, y)))) & \\
\end{align*}
\]

Original Formula: \( pre\_thm_1(v, g) \rightarrow thm_1(v, g) \).

Expanded Formula:
\[
\begin{align*}
\forall x \forall y \ (r(x, y) & \rightarrow world(y)) \\
\forall x \forall y \ (world(x) & \rightarrow (\top(x, y) \leftrightarrow e(x, y))) \\
\forall x \forall y \ (world(x) & \rightarrow (\bot(x, y) \leftrightarrow \neg e(x, y))) \\
(\forall x \forall y \ (world(x) & \rightarrow (\top(x, y) \leftrightarrow \neg \bot(x, y)))) & \rightarrow \\
\bot = \neg \top & \\
\forall x \ (r(v, x) & \rightarrow \forall y \ (e(x, y) \rightarrow (\bot(x, y) \rightarrow \top(x, y)))) & \rightarrow \\
\exists x \ (r(v, x) & \rightarrow \exists y \ (e(x, y) \land g(x, y)))) & \\
\forall x \forall y \ (r(x, y) & \rightarrow \forall y \ (e(x, y) \land g(x, y)))) & \\
\end{align*}
\]

Original Formula: \( pre\_coro(v) \rightarrow coro(v) \).

Expanded Formula:
\[
\begin{align*}
\forall x \forall y \ (r(x, y) & \rightarrow world(y)) \\
\forall x \forall y \ (world(x) & \rightarrow (\top(x, y) \leftrightarrow e(x, y))) \\
\forall x \forall y \ (world(x) & \rightarrow (\bot(x, y) \leftrightarrow \neg e(x, y))) \\
(\forall x \forall y \ (world(x) & \rightarrow (\top(x, y) \leftrightarrow \neg \bot(x, y)))) & \rightarrow \\
\bot = \neg \top & \\
\forall x \ (r(v, x) & \rightarrow \forall y \ (e(x, y) \rightarrow (\bot(x, y) \rightarrow \top(x, y)))) & \rightarrow \\
\exists x \ (r(v, x) & \rightarrow \exists y \ (e(x, y) \land g(x, y)))) & \\
\forall x \forall y \ (r(x, y) & \rightarrow \forall y \ (e(x, y) \land g(x, y)))) & \\
\end{align*}
\]
Original Formula:

\[ pre_{\text{proto_thm}_2(v, x, q)} \rightarrow \text{proto_thm}_2(v, x). \]

Expanded Formula:

\[ \forall p \exists y \exists z \left( \left( \text{world}(v) \rightarrow \left( \neg \text{pos}(v, y) \rightarrow \text{pos}(v, z) \right) \right) \land \right. \\
\left. \forall u \left( r(v, u) \rightarrow \left( \forall w \left( e(u, w) \rightarrow \left( g(u, w) \rightarrow \left( \text{pos}(u, y) \rightarrow p(u, w) \right) \right) \right) \right) \right) \land \\
\left( g(v, x) \rightarrow \left( \text{pos}(v, z) \rightarrow \neg p(v, x) \right) \right) \land \\
\left( \text{world}(v) \rightarrow \left( \text{pos}(v, y) \rightarrow \forall u \left( r(v, u) \rightarrow \text{pos}(u, y) \right) \right) \right) \rightarrow \\
\left( \text{world}(v) \right) \rightarrow \\
\left( e(v, x) \rightarrow \right) \\
\left( g(v, x) \land \right. \\
\left. \forall p \left( g(v, x) \land \right. \\
\left. \forall y \left( r(v, y) \rightarrow \forall z \left( e(y, z) \rightarrow \left( g(y, z) \rightarrow p(y, z) \right) \right) \right) \right) \right). \]

Original Formula:

\[ \forall x \left( \forall y \text{pre_lemma}_2(x, y) \rightarrow \text{lemma}_2(x). \right) \]

Expanded Formula:

\[ \forall x \left( \forall y \left( \left( \text{world}(x) \rightarrow \text{pos}(x, \text{nie}) \right) \land \right. \\
\left. \left( g(x, y) \rightarrow \left( \text{pos}(x, \text{nie}) \rightarrow \text{ne}(x, y) \right) \right) \right) \land \\
\left( \text{world}(x) \right) \rightarrow \\
\left( e(x, y) \rightarrow \right) \\
\left( \text{ne}(x, y) \rightarrow \right) \\
\left( \text{ess}(x, \tilde{g}, y) \rightarrow \right. \\
\left. \forall z \left( r(x, z) \rightarrow \exists u \left( e(z, u) \land g(z, u) \right) \right) \right) \land \\
\left( \text{world}(x) \rightarrow \left( e(x, y) \rightarrow \left( g(x, y) \rightarrow \text{ess}(x, \tilde{g}, y) \right) \right) \right) \) \rightarrow \\
\left( \text{world}(x) \right) \rightarrow \\
\left( \exists y \left( e(x, y) \land g(x, y) \right) \rightarrow \right) \\
\left( \forall y \left( r(x, y) \rightarrow \exists z \left( e(y, z) \land g(y, z) \right) \right) \right) \land \\
\left( \text{world}(v) \rightarrow \exists x \left( r(x, v) \land \right. \\
\left. \exists y \left( e(x, y) \land g(x, y) \right) \right) \right) \rightarrow \\
\left( \text{world}(v) \rightarrow \forall x \left( r(v, x) \rightarrow \exists y \left( e(x, y) \land g(x, y) \right) \right) \right). \]

Original Formula:

\[ \text{symmetric} \lor \text{euclidean} \rightarrow \left( \text{pre_thm}_3(v) \rightarrow \text{thm}_3(v) \right). \]

Expanded Formula:

\[ \forall x \forall y \left( r(x, y) \rightarrow r(y, x) \right) \lor \\
\forall x \forall y \forall z \left( r(x, y) \land r(x, z) \rightarrow r(z, y) \right) \rightarrow \\
\left( \forall x \forall y \left( r(x, y) \rightarrow \text{world}(y) \right) \right) \land \\
\forall x \left( \text{world}(x) \rightarrow \right. \\
\left. \left( \exists y \left( e(x, y) \land g(x, y) \right) \rightarrow \right. \\
\left. \forall y \left( r(x, y) \rightarrow \exists z \left( e(y, z) \land g(y, z) \right) \right) \right) \right. \land \\
\left( \text{world}(v) \rightarrow \exists x \left( r(x, v) \land \right. \\
\left. \exists y \left( e(x, y) \land g(x, y) \right) \right) \right) \rightarrow \\
\left( \text{world}(v) \rightarrow \forall x \left( r(v, x) \rightarrow \exists y \left( e(x, y) \land g(x, y) \right) \right) \right). \]

Original Formula:

\[ \text{frame_cond_simp} \rightarrow \left( \text{pre_thm}_3(v) \rightarrow \text{thm}_3(v) \right). \]
Expanded Formula:

\[
\forall x \forall y \forall z \left( r(x, y) \land r(x, z) \land y \neq x \land y \neq z \rightarrow r(y, x) \lor r(y, z) \right) \quad \rightarrow \\
\left( \forall x \forall y \left( r(x, y) \rightarrow \text{world}(y) \right) \right) \quad \land \\
\forall x \left( \text{world}(x) \rightarrow \\
\quad \left( \exists y \left( e(x, y) \land g(x, y) \right) \rightarrow \\
\quad \quad \forall y \left( r(x, y) \rightarrow \exists z \left( e(y, z) \land g(y, z) \right) \right) \right) \right) \quad \land \\
\left( \text{world}(v) \rightarrow \exists x \left( r(v, x) \land \exists y \left( e(x, y) \land g(x, y) \right) \right) \right) \quad \rightarrow \\
\left( \text{world}(v) \rightarrow \forall x \left( r(v, x) \rightarrow \exists y \left( e(x, y) \land g(x, y) \right) \right) \right).
\]

D.2 Formulas whose Validity was Proven by Prover9 after Second-Order Quantifier Elimination with PIE

Original Formula:

\[
def_{2}(v, p) \land \exists v, x \rightarrow \exists y \left( e(x, y) \land g(x, y) \right).
\]

Expanded Formula:

\[
\left( \text{world}(v) \rightarrow \\
\quad \forall y \left( \text{ess}(v, \dot{p}, y) \land \\
\quad \forall q \left( p(v, y) \land \\
\quad \quad \forall z \left( r(v, z) \rightarrow \forall u \left( e(z, u) \rightarrow (p(z, u) \rightarrow q(z, u)) \right) \right) \right) \right) \rightarrow \\
\left( \text{world}(v) \rightarrow \\
\quad \exists v, x \left( \text{ne}(v, x) \rightarrow \\
\quad \quad \forall q \left( \forall p_{1} \left( q(v, x) \land \\
\quad \quad \quad \forall z \left( r(v, y) \rightarrow \forall z \left( e(y, z) \rightarrow (q(y, z) \rightarrow p_{1}(y, z)) \right) \right) \right) \rightarrow \\
\quad \quad \forall y \left( r(v, y) \rightarrow \exists z \left( e(y, z) \land q(y, z) \right) \right) \right) \right) \rightarrow \\
\left( \text{world}(v) \rightarrow \\
\quad \forall v, x \left( \text{ne}(v, x) \rightarrow \\
\quad \quad \forall y \left( r(v, y) \rightarrow \exists z \left( e(y, z) \land p(y, z) \right) \right) \right) \right).
\]

Original Formula:

\[
def_{2}(v, p_{1}) \land \exists v, x \rightarrow \exists y \left( e(x, y) \land p(y, z) \right).
\]
Expanded Formula:

\[
\forall x (\text{ess}(v, p_1, x)) \quad \forall p (p_1(v, x) \land (p(v, x) \rightarrow \forall y (r(v, y) \rightarrow \forall z (e(y, z) \rightarrow (p_1(y, z) \rightarrow p(y, z))))))) \quad \land \\
\forall x (\text{ess}(v, p_2, x)) \quad \forall p (p_2(v, x) \land (p(v, x) \rightarrow \forall y (r(v, y) \rightarrow \forall z (e(y, z) \rightarrow (p_2(y, z) \rightarrow p(y, z))))))) \\
\forall x (\text{def}_2(v, p) \rightarrow \text{note}_2(v, p, x)) \\
\forall y (\text{ess}(v, p, y) \rightarrow \forall q (q(v, y) \land (q(v, y) \rightarrow \forall z (r(v, z) \rightarrow \forall u (e(z, u) \rightarrow (p(z, u) \rightarrow q(z, u)))))))) \\
\forall y (r(v, y) \rightarrow \forall y (e(x, y) \rightarrow (p_1(x, y) \leftrightarrow p_2(x, y)))))) \\
\forall x (\text{pre}_2(v, x) \rightarrow \text{collapse} \\
\forall y (\text{world}(x) \rightarrow (e(y, x) \rightarrow (g(y, x) \rightarrow \text{ess}(y, g, x)))) \quad \land \\
\forall x (\text{world}(x) \rightarrow \forall y (r(x, y) \rightarrow \exists z (e(y, z) \land g(y, z)))) \quad \land \\
\forall x (\text{world}(x) \rightarrow \forall y (\text{ess}(x, g, y) \leftrightarrow \forall p (g(x, y) \land (p(x, y) \rightarrow \forall z (r(x, z) \rightarrow \forall u (e(z, u) \rightarrow (g(z, u) \rightarrow p(z, u)))))))) \quad \land \\
\forall x (\text{world}(x) \land \text{world}(y)) \\
\forall x r(x, x) \\
\forall x \forall y (r(x, y) \rightarrow y = x).}

Original Formula:

\[
def_2(v, p) \rightarrow \text{note}_2(v, p, x).
\]

Expanded Formula:

\[
\forall y (\text{ess}(v, p, y) \\ \rightarrow \\
\forall q (q(v, y) \\ \land \\
(q(v, y) \\ \rightarrow \\
\forall z (r(v, z) \\ \rightarrow \\
\forall u (e(z, u) \\ \rightarrow \\
(p(z, u) \rightarrow q(z, u)))))))) \\
\rightarrow \\
\forall y (r(v, y) \\ \rightarrow \\
\forall y (e(x, y) \\ \rightarrow \\
(p_1(x, y) \leftrightarrow p_2(x, y)))))).
\]

Original Formula:

\[
\text{pre}_2(v, x) \rightarrow \text{collapse}.
\]

Expanded Formula:

\[
\forall x \forall y (\text{world}(y) \rightarrow (e(y, x) \rightarrow (g(y, x) \rightarrow \text{ess}(y, g, x)))) \\ \land \\
\forall x (\text{world}(x) \rightarrow \forall y (r(x, y) \rightarrow \exists z (e(y, z) \land g(y, z)))) \\ \land \\
\forall x (\text{world}(x) \\ \rightarrow \\
\forall y (\text{ess}(x, g, y) \\ \leftrightarrow \\
\forall p (g(x, y) \\ \land \\
(p(x, y) \\ \rightarrow \\
\forall z (r(x, z) \\ \rightarrow \\
\forall u (e(z, u) \\ \rightarrow \\
(g(z, u) \rightarrow p(z, u)))))))) \\ \land \\
\forall x (\text{world}(x) \land \text{world}(y)) \\
\forall x r(x, x) \\ \rightarrow \\
\forall x \forall y (r(x, y) \rightarrow y = x).
\]
D.3 Formulas whose Non-Validity was Proven by Mace4 Embedded in PIE

Original Formula:
\[ \text{symmetric} \lor \text{euclidean} \rightarrow (r_{\text{world}} \land \text{lemma}_2(v) \land \text{coro}(v) \rightarrow \text{thm}_3(v)). \]

Expanded Formula:
\[ \forall x \forall y (r(x, y) \rightarrow r(y, x)) \lor \forall x \forall y \forall z (r(x, y) \land r(x, z) \rightarrow r(z, y)) \rightarrow \forall x \forall y (r(x, y) \rightarrow \text{world}(y)) \land \text{world}(v) \rightarrow (\exists x (e(v, x) \land g(v, x)) \rightarrow \forall x (r(v, x) \rightarrow \exists y (e(x, y) \land g(x, y)))) \land (\text{world}(v) \rightarrow \exists x (r(v, x) \land \exists y (e(x, y) \land g(x, y)))) \rightarrow (\text{world}(v) \rightarrow \forall x (r(v, x) \rightarrow \exists y (e(x, y) \land g(x, y))))). \]

D.4 Formulas on which Second-Order Quantifier Elimination was Performed with PIE

Original Formula:
\[ \text{val}_\text{ess}(v, p, x, q). \]

Expanded Formula:
\[ \forall q (p(v, x) \land (q(v, x) \rightarrow (\forall y (r(v, y) \rightarrow \forall z (e(y, z) \rightarrow (p(y, z) \rightarrow q(y, z)))))), \land \forall y (r(v, y) \rightarrow \forall z (e(y, z) \rightarrow (p(y, z) \rightarrow q(y, z))))). \]

Original Formula:
\[ \text{val}_\text{ne}(v, x). \]

Expanded Formula:
\[ \forall p (\forall q (p(v, x) \land (q(v, x) \rightarrow (\forall y (r(v, y) \rightarrow \forall z (e(y, z) \rightarrow (p(y, z) \rightarrow q(y, z))))))) \rightarrow (\forall y (r(v, y) \rightarrow \exists x (e(x, z) \land p(y, z)))), \land (\forall p (\forall q (p(v, x) \land (q(v, x) \rightarrow (\forall y (r(v, y) \rightarrow \forall z (e(y, z) \rightarrow (p(y, z) \rightarrow q(y, z))))))) \rightarrow (\forall y (r(v, y) \rightarrow \exists x (e(x, z) \land p(y, z)))), \land (\forall y (r(v, y) \rightarrow \exists x (e(x, z) \land \exists y (e(x, y) \land g(x, y)))))) \rightarrow (\forall y (r(v, y) \rightarrow \exists x (e(x, z) \land \exists y (e(x, y) \land g(x, y))))). \]

Original Formula:
\[ \forall x (\text{pre}_\text{thm}_3\text{\_simp}_\text{inst}(x) \rightarrow \text{thm}_3\text{\_simp}(x)). \]

Expanded Formula:
\[ \forall p (\forall x ((p(x) \lor r(x, z) \rightarrow p(y))) \land (r(x, y) \land (p(y) \lor \forall z (r(y, z) \rightarrow p(z)))) \rightarrow (\forall y (r(x, y) \rightarrow (p(y)))), \land (r(x, y) \land (p(y) \lor \forall z (r(y, z) \rightarrow p(z)))) \rightarrow (\forall y (r(x, y) \rightarrow (p(y)))). \]
D.5 Formulas on which Second-Order Quantifier Elimination with PIE Failed

Original Formula:  
$$\text{pre}_3 \_\text{simp}(v) \rightarrow \text{thm}_3 \_\text{simp}(v).$$

Expanded Formula:  
$$\forall p (\forall x (p(x) \rightarrow \forall y (r(x, y) \rightarrow p(y))) \land \\
\exists x (r(v, x) \land p(x)) \rightarrow \\
\forall x (r(v, x) \rightarrow p(x))).$$

Original Formula:  
$$\forall x (\text{pre}_3 \_x) \rightarrow \text{thm}_3 \_x.$$ 

Expanded Formula:  
$$\forall p \forall x (\forall y \forall z (r(y, z) \rightarrow \text{world}(z)) \land \\
\forall y (\text{world}(y)) \rightarrow \\
(\exists z (e(y, z) \land p(y, z)) \rightarrow \\
\forall z (r(y, z) \rightarrow \exists u (e(z, u) \land p(z, u)))) \land \\
(\text{world}(x) \rightarrow \exists y (r(x, y) \land \exists z (e(y, z) \land p(y, z)))) \rightarrow \\
(\text{world}(x) \rightarrow \forall y (r(x, y) \rightarrow \exists z (e(y, z) \land p(y, z)))).$$