Asymptotic performance of optimal state estimation in quantum two level system

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We derived an asymptotic bound the accuracy of the estimation when we use the quantum correlation in the measuring apparatus. It is also proved that this bound can be achieved in any model in the quantum two-level system. Moreover, we show that this bound of such a model cannot be attained by any quantum measurement with no quantum correlation in the measuring apparatus. That is, in such a model, the quantum correlation can improve the accuracy of the estimation in an asymptotic setting.

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I. INTRODUCTION

Estimating unknown quantum state is an important task in quantum information. In this paper, we discuss this problem by focusing on two typical quantum effects; One is the uncertainty caused by the non-commutativity. The other is the quantum correlation between particles, \textit{e.g.}, quantum interference, quantum entanglement, \textit{etc}. Indeed, the probabilistic property in quantum mechanics is caused by the first effect. Hence, it is impossible to determine the initial quantum state based only on the single measurement. Due to this property, we need some statistical processing for identifying the unknown state. Needless to say, it is appropriate for effective processing to use a measurement drawing much information. Therefore, the optimization of measuring process is important for this purpose. The second property is also crucial for this optimization. This is because it is possible to use the quantum correlation between several particles. Hence, we compare the optimal performance in presence or absence of quantum correlation between several particles in the measuring process. This paper treat this comparison in the case of two-dimensional case, \textit{i.e.}, the qubit case.

Estimating unknown quantum state is often formulated as the identification problem of the unknown state which is assumed to be included a certain parametric quantum states family. Such a problem is called quantum estimation, and has been one of the main issues of quantum statistical inference. In this case, we often adopt mean square error (MSE) as our error criterion. Hence, the optimization of measuring process is important for this purpose. The second property is also crucial for this optimization. This is because it is possible to use the quantum correlation between several particles. Hence, we compare the optimal performance in presence or absence of quantum correlation between several particles in the measuring process. This paper treat this comparison in the case of two-dimensional case, \textit{i.e.}, the qubit case.

As is discussed by Nagaoka\textsuperscript{11}, Hayashi & Matsumoto\textsuperscript{12}, Gill & Massar\textsuperscript{3}, it is sufficient to minimize the MSE at a local setting. (For detail, see section \textsuperscript{11}).

The research of quantum estimation has been initiated by Helstrom\textsuperscript{11}. He generally solved this problem in the one-parameter case at the local setting. However, the multi-parameter case is more difficult because we need to treat the trade-off among the MSEs of the respective parameters. That is, we cannot simultaneously realize the optimal estimation of the respective parameters. This difficulty is caused by the non-commutativity. First, Yuen & Lax\textsuperscript{8} and Holevo\textsuperscript{9} derived the bound of the estimation performance in the estimation of quantum Gaussian family. In order to treat this trade-off, they minimized the sum of the weighted sum of the MSEs of respective parameters. Especially, Yuen & Lax treated the equivalent sum, and Holevo did the generally weighted sum.

After this achievement, Nagaoka\textsuperscript{4}, Fujiwara & Nagaoka\textsuperscript{6}, Hayashi\textsuperscript{9}, Gill & Massar\textsuperscript{3} calculated that of the estimation in the quantum two level system. They also adopt the same criterion. Concerning the pure states case, Fujiwara & Nagaoka\textsuperscript{10}, Hayashi\textsuperscript{9}, and Matsumoto\textsuperscript{12} proceeded to more precise treatments.

However, the above papers did not treat the performance bound of estimation with quantum correlation in measuring apparatus, which is one of the most important quantum effects. In this paper, we discuss whether the quantum correlation can improve its performance. For this purpose, we calculate the CR bound, \textit{i.e.}, the optimal decreasing coefficient of the sum of MSEs with quantum correlation in measuring apparatus, in the several specific model.

First, as a preparation, we focus on quantum Gaussian family, and prove that the above quantum correlation has no advantage for estimating the unknown state at section \textsuperscript{13}. The reason is roughly given by the following two facts. One is the fact that the optimal error without quantum correlation is given by the right logarithmic derivative (RLD) Fisher Information matrix, which is one of quantum analogues of Fisher Information matrix. The second is the fact that the CR bound can be bounded by RLD Fisher Information matrix.
Next, we proceed to the quantum two-level system, which can be regarded as the quantum analogue of the binomial distribution. In this case, as is shown in section IV quantum quantum correlation can improve the performance of estimation. As the first step, we focus on the equivalent sum of the MSEs of respective parameters in the parameterization \( \frac{1}{2} \left( \begin{array}{cc} 1 + z & x + iy \\ x - iy & 1 - z \end{array} \right) \) with the parameter \( x^2 + y^2 + z^2 \leq 1 \). As is discussed in subsection IV the asymptotically optimal estimator is given as follows.

When the quantum state is parameterized in another way: \( \frac{1}{2} \left( \begin{array}{cc} 1 + r \cos \theta & r e^{i \phi} \sin \theta \\ r e^{-i \phi} \sin \theta & 1 - r \cos \theta \end{array} \right) \) with the parameter \( 0 \leq r \leq 1 \), \( 0 \leq \phi \leq 2\pi \), \( 0 \leq \theta \leq \frac{\pi}{2} \), we can divide our estimation into two parts. One is the estimation of \( r \), the other is that of the angle \( (\theta, \phi) \).

The estimation of \( r \) can be realized by performing the projection measurement corresponding to the irreducible decomposition of the tensor product representation of SU(2), which equals the special case of the measurement used in Keyl & Werner [3], Hayashi & Matsumoto [4]. Note that they derived its error with the large deviation criterion, but did not treat its MSE. After this measurement, we perform a covariant measurement for the estimation of \( (\theta, \phi) \). By calculating the asymptotic behavior of the sum of its MSEs of respective parameters, it can be checked that it attains its lower bound given by RLD Fisher information, asymptotically. That is, this estimator is shown to be the optimal with the above mentioned criterion. Finally, by comparing the optimal coefficient without quantum correlation in measuring apparatus, we check that using this quantum effect can improve the estimation error. Furthermore, we treat the CR bound with the general weight matrix by a more technical method in subsection IV. In this discussion, the key point is the fact that this model can be asymptotically approximated by quantum Gaussian model.

This paper is organized as follows. First, we discuss the lower bounds of asymptotic error in section IV which contains reviews of the previous results. In section V quantum Gaussian model is discussed. We discuss the asymptotic approximation of spin \( j \) system by the quantum Gaussian model in section VI. Using these preliminaries, we treat quantum two level system in section VII.

II. LOWER BOUNDS OF ESTIMATION ERROR

A. Quasi Quantum CR bound

Let \( \Theta \) be an open set in \( \mathbb{R}^d \), and let \( S = \{ \rho_\theta; \theta \in \Theta \} \) be a family of density operators on a Hilbert space \( \mathcal{H} \) smoothly parameterized by a \( d \)-dimensional parameter \( \theta = (\theta^1, \ldots, \theta^d) \) with the range \( \Theta \). Such a family is called an \( d \)-dimensional quantum statistical model. We consider the parameter estimation problem for the model \( S \), and, for simplicity, assume that any element \( \rho_\theta \) is strictly positive. The purpose of the theory is to obtain the best estimator and its accuracy. The optimization is done by the appropriate choice of the measuring apparatus and the function from data to the estimate.

Let \( \sigma(\Omega) \) be a \( \sigma \)-field in the space \( \Omega \). Whatever apparatus is used, the data \( \omega \in \Omega \) lie in a measurable subset \( B \subset \sigma(\Omega) \) of \( \Omega \) writes

\[
\Pr\{ \omega \in B | \theta \} = P_\theta^M(B) \equiv \Tr \rho_\theta M(B),
\]

when the true value of the parameter is \( \theta \). Here, \( M \), which is called positive operator-valued measure (POVM, in short), is a mapping from subsets \( B \subset \Omega \) to non-negative Hermitian operators in \( \mathcal{H} \), such that

\[
M(\emptyset) = O, \quad M(\Omega) = I
\]

\[
M(\bigcup_{j=1}^{\infty} B_j) = \sum_{j=1}^{\infty} M(B_j) \quad (B_k \cap B_j = \emptyset, k \neq j)
\]

(see p. 53 [2] and p. 50 [3]). Conversely, some apparatus corresponds to any POVM \( M \). Therefore, we refer to the measurement which is controlled by the POVM \( M \) as ‘measurement \( M \)’. Moreover, for estimating the unknown parameter \( \theta \), we need an estimating function \( \hat{\theta} \) mapping the observed data \( \omega \) to the parameter. Then, a pair \( (\hat{\theta}, M) \) is called an estimator.

In estimation theory, we often focus on the unbiasedness condition:

\[
\int_{\Omega} \hat{\theta}^i(\omega) \Tr M(d\omega) \rho_\theta = \theta^i, \quad \forall \theta \in \Theta.
\]

(1)

Differentiating this equation, we obtain

\[
\int_{\Omega} \hat{\theta}^j(\omega) \frac{\partial}{\partial \theta^k} \Tr M(d\omega) \rho_\theta = \delta^j_k \quad (j, k = 1, 2, \ldots, n),
\]

(2)

where \( \delta^j_k \) is the Kronecker’s delta. When \( (\hat{\theta}, M) \) satisfies (1) and (2) at a fixed point \( \theta \in \Theta \), we say that \( (\hat{\theta}, M) \) is locally unbiased at \( \theta \). Obviously, an estimator is unbiased if and only if it is locally unbiased at every \( \theta \in \Theta \). In this notation, we often describe the accuracy of the estimation at \( \theta \) by the MSE matrix:

\[
V_\theta^{k,j}(\hat{\theta}, M) \equiv \int_{\Omega} (\hat{\theta}^k - \theta^k)(\hat{\theta}^j - \theta^j) \Tr M(d\omega) \rho_\theta.
\]

or

\[
\Tr V_\theta(\hat{\theta}, M) G
\]

for a given weight matrix, which is a positive-definite real symmetric matrix. Indeed, in the quantum setting, there is not necessarily minimum MSE matrix, while the minimum MSE matrix exists in the classical asymptotic setting. Thus, we usually focus on \( \Tr V_\theta(\hat{\theta}, M) \) for a given weight matrix.
We define classical Fisher information matrix \( J_0^M \) by the POVM \( M \) as in classical estimation theory:
\[
J_0^M := \left[ \int_{\omega \in \Omega} \partial_i \log \frac{dP_\theta^M}{d\omega} \partial_j \log \frac{dP_\theta^M}{d\omega} \, d\omega \right],
\]
where \( \partial_i = \partial / \partial \theta^i \). Then, \( J_0^M \) is characterized, from knowledge of classical statistics, by,
\[
(J_0^M)^{-1} = \inf_{\tilde{\theta}} \{ V_0(\tilde{\theta}, M) \mid (\tilde{\theta}, M) \text{ is locally unbiased} \},
\]
and the quasi-quantum Cramér-Rao type bound (quasi-quantum CR bound) \( \hat{C}_\theta(G) \) is defined by,
\[
\hat{C}_\theta(G) \defeq \inf \{ \text{tr} \, G V_\theta(\hat{\theta}, M) \mid (\hat{\theta}, M) \text{is locally unbiased} \},
\]
and has other expressions.
\[
\hat{C}_\theta(G) = \inf \{ \text{tr} \, G V_\theta(\hat{\theta}, M) \mid (\hat{\theta}, M) \text{satisfies the condition } \theta \}
\]
\[
= \inf \{ \text{tr} \, G (J_0^M)^{-1} \mid M \text{ is a POVM on } \mathcal{H} \}. \tag{5}
\]
As is precisely mentioned latter, the bound \( \hat{C}_\theta(G) \) is uniformly attained by an adaptive measurement, asymptotically\[2,3\]. Therefore, \( \hat{C}_\theta(G) \) expresses the bound of the accuracy of the estimation without quantum correlation in measurement apparatus.

**B. Lower bounds of quasi quantum CR bound**

1. **SLD bound and RLD bound**

In this subsection, we treat lower bounds of \( \hat{C}_\theta(G) \). Most easy method for deriving lower bound is using quantum analogues Fisher Information matrix. However, there are two analogues at least, and each of them has advantages and disadvantages. Hence, we need to treat both. One analogue is symmetric logarithmic derivative (SLD) Fisher information matrix \( J_{\theta;j,k} \):
\[
J_{\theta;j,k} \defeq \langle L_{\theta;j}, L_{\theta;k} \rangle_\theta,
\]
where
\[
\partial \rho_\theta / \partial \theta^j \rho_\theta \circ L_{\theta;j},
\]
\[
(X, Y)_\theta \defeq \text{tr} \rho_\theta (X^* \circ Y) = \text{tr}(\rho_\theta \circ Y)X^*
\]
\[
X \circ Y \defeq \frac{1}{2} (XY + YX),
\]
and \( L_{\theta,j} \) is called its symmetric logarithmic derivative (SLD). Another quantum analogue is the right logarithmic derivative (RLD) Fisher information matrix \( J_{\theta;j,k} \):
\[
J_{\theta;j,k} \defeq \text{tr} \rho_\theta \tilde{L}_{\theta;k}(\tilde{L}_{\theta;j})^* = \langle \tilde{L}_{\theta;j}, \tilde{L}_{\theta;k} \rangle_\theta.
\]
where
\[
\partial \rho_\theta / \partial \theta^j = \rho_\theta \circ \tilde{L}_{\theta;j}, \quad (A, B)_\theta \defeq \text{tr} \rho_\theta BA^*.
\]
and \( \tilde{L}_{\theta,j} \) is called its right logarithmic derivative (RLD).

**Theorem 1** Helstrom[2]Holevo[3] If a vector \( \vec{X} = [X^1, \ldots, X^d] \) of Hermite matrixes satisfies the condition:
\[
\text{tr} \, \partial \rho_\theta / \partial \theta^k X^j = \delta^j_k, \tag{6}
\]
the matrix \( Z_\theta(\vec{X}) \):
\[
Z_{\theta}^{k,j}(\vec{X}) \defeq \text{tr} \rho_\theta X^k X^j
\]
satisfies the inequalities
\[
Z_\theta(\vec{X}) \geq (J_\theta)^{-1}, \tag{7}
\]
and
\[
Z_\theta(\vec{X}) \geq (\tilde{J}_\theta)^{-1}. \tag{8}
\]
For a proof, see Appendix A\[1\]. Moreover, the following lemma is known.

**Lemma 1** Holevo[3] When we define the vector of Hermite matrixes \( X_M \):
\[
X_M^j = \int_{\mathbb{R}^d} (\hat{\theta}^j - \theta^j) M(\hat{\theta}),
\]
then
\[
V_\theta(M) \geq Z_\theta(X_M). \tag{9}
\]
For a proof, see Appendix A\[2\]. Combining Theorem 1 and Lemma 1, we obtain the following corollary.

**Corollary 1** If an estimator \( M \) is locally unbiased at \( \theta \in \theta \), the SLD Cramér-Rao inequality
\[
V_\theta(M) \geq (J_\theta)^{-1} \tag{10}
\]
and the RLD Cramér-Rao inequality
\[
V_\theta(M) \geq (\tilde{J}_\theta)^{-1} \tag{11}
\]
hold, where, for simplicity, we regard a POVM \( \hat{M} \) with the outcome in \( \mathbb{R}^d \) as an estimator in the correspondence \( M = M \circ \hat{\theta}^{-1} \).

Therefore, we can easily obtain the inequality
\[
\text{tr} \, V_\theta(M) G \geq \text{tr} (J_\theta)^{-1} G
\]
when \( M \) is locally unbiased at \( \theta \). That is, we obtain the SLD bound:
\[
C_\theta^{\hat{S}}(G) \defeq \text{tr} (J_\theta)^{-1} G \leq \hat{C}_\theta(G). \tag{12}
\]
As was shown by Helstrom,\[2\] the equality\[12\] holds for one-parameter case. However, we need the following lemma for obtaining a bound of \( \hat{C}_\theta(G) \) from the RLD Cramér-Rao inequality.
**Lemma 2** When a real symmetric matrix $V$ and Hermite matrix $W$ satisfy

\[ V \geq W, \]

then

\[ \text{tr } V \geq \text{tr } Re W + \text{tr } |Im W|, \]

where $Re W$ (Im $W$) denotes the real part of $W$ (the imaginary part of $W$), respectively.

For a proof, see Appendix A. Since the RLD Cramér-Rao inequality yields that any locally unbiased estimator $M$ satisfies

\[ \sqrt{G}V_{\theta}(M)\sqrt{G} \geq \sqrt{G}(J_{\theta})^{-1}\sqrt{G}, \]

Lemma 2 guarantees that

\[ \text{tr } V_{\theta}(M)G \geq \text{tr } \sqrt{G} Re(J_{\theta})^{-1}\sqrt{G} + \text{tr } |\sqrt{G} Im(J_{\theta})^{-1}\sqrt{G}|. \]

(13)

Thus, we obtain the RLD bound:

\[ C_{\theta}^R(G) \overset{\text{def}}{=} \text{tr } \sqrt{G} Re(J_{\theta})^{-1}\sqrt{G} + \text{tr } |\sqrt{G} Im(J_{\theta})^{-1}\sqrt{G}| \leq \hat{C}_{\theta}(G). \]

(14)

For characterizing the relation between the RLD bound $C_{\theta}^R(G)$ and the SLD bound $C_{\theta}^S(G)$, we introduce the superoperator $D_{\theta}$ as follows:

\[ \rho_{\theta} \circ D_{\theta}(X) = i[X, \rho_{\theta}]. \]

This superoperator is called $D$-operator, and has the following relation with the RLD bound.

**Theorem 2** Holevo bound When the linear space $T_{\theta}$ spanned by $L_{\theta,1}, \ldots, L_{\theta,k}$ is invariant for the action of the superoperator $D_{\theta}$, the inverse of the RLD Fisher information matrix is described as

\[ J_{\theta}^{-1} = J_{\theta}^{-1} + \frac{1}{2} J_{\theta}^{-1} D_{\theta} J_{\theta}^{-1}, \]

(15)

where the antisymmetric matrix $D_{\theta}$ is defined by

\[ D_{\theta;k,j} \overset{\text{def}}{=} \langle D_{\theta}(L_{\theta,j}), L_{\theta,k} \rangle = i \text{Tr } \rho_{\theta}[L_{\theta,k}, L_{\theta,j}]. \]

(16)

Thus, the RLD bound is calculated as

\[ C_{\theta}^R(G) = \text{tr } G J_{\theta}^{-1} + \frac{1}{2} \text{tr } |\sqrt{G} J_{\theta}^{-1} D_{\theta} J_{\theta}^{-1} \sqrt{G}|. \]

(17)

Therefore, $C_{\theta}^R(G) \geq C_{\theta}^S(G)$, i.e., the RLD bound is better than the SLD bound.

For a proof, see Appendix A. In the following, we call the model $D$-invariant, if the linear space $T_{\theta}$ is invariant for the action of the superoperator $D_{\theta}$ for any parameter $\theta$.

2. Holevo bound

Next, we proceed to the non-D-invariant case. in this case, Lemma 1 guarantees that any locally unbiased estimator $M$ satisfies

\[ \sqrt{G}V_{\theta}(M)\sqrt{G} \geq \sqrt{G}Z_{\theta}(\bar{X}_M)\sqrt{G}, \]

where

\[ Z_{\theta}^{k,j}(\bar{X}) \overset{\text{def}}{=} \text{Tr } \rho_{\theta} X^k X^j. \]

Thus, from Lemma 2 we have

\[ \text{tr } \sqrt{G}V_{\theta}(M)\sqrt{G} \geq C_{\theta}(G, \bar{X}_M) \]

\[ \overset{\text{def}}{=} \text{tr } \sqrt{G} Re Z_{\theta}(\bar{X}_M)\sqrt{G} + \text{tr } |\sqrt{G} Im Z_{\theta}(\bar{X}_M)\sqrt{G}|. \]

(18)

Since $X_M$ satisfies the condition (9), the relation (11) yields the following theorem.

**Theorem 3** Holevo bound: The inequality

\[ C_{\theta}^H(G) \overset{\text{def}}{=} \min_{X} \left\{ C_{\theta}(G, \bar{X}) \left| \text{Tr } \frac{\partial \rho_{\theta}}{\partial \theta^k} X^j = \delta^j_k \right. \right\} \leq \hat{C}_{\theta}(G) \]

holds.

Hence, the bound $C_{\theta}^H(G)$ is called the Holevo bound. When $X$ satisfies the condition (9), the relation (11) yields that

\[ \text{tr } G Re Z_{\theta}(\bar{X}) = \text{tr } GZ_{\theta}(\bar{X}) \geq \text{tr } G J_{\theta}^{-1} = C_{\theta}^S(G), \]

which implies

\[ C_{\theta}^H(G) \geq C_{\theta}^S(G). \]

Also, the relation (13) guarantees that

\[ \sqrt{G}Z_{\theta}(\bar{X})\sqrt{G} + |\sqrt{G} Im Z_{\theta}(\bar{X})\sqrt{G}| \geq \sqrt{G}Z_{\theta}(\bar{X})\sqrt{G} + \sqrt{G} Im Z_{\theta}(\bar{X})\sqrt{G} \geq \sqrt{G}J_{\theta}\sqrt{G}. \]

Similarly to (18), the relation (11) yields

\[ C_{\theta}(G, \bar{X}) \geq C_{\theta}^R(G), \]

which implies

\[ C_{\theta}^H(G) \geq C_{\theta}^R(G). \]

(19)

Moreover, the Holevo bound has another characterization.

**Lemma 3** Let $T_{\theta}$ be the linear space spanned by the orbit of $T_{\theta}$ with respect to the action of $D_{\theta}$. Then, the Holevo bound can be simplified as

\[ C_{\theta}^H(G) = \min_{\bar{X} : \bar{X} \in T_{\theta}} \left\{ C_{\theta}(G, \bar{X}) \left| \text{Tr } \frac{\partial \rho_{\theta}}{\partial \theta^k} X^j = \delta^j_k \right. \right\}. \]

(20)
Moreover, we assume that the D-invariant model containing the original model has normally orthogonal basis \( \{ L_1, \ldots, L_m \} \) concerning SLD, and the inverse of its RLD Fisher information matrix is given by \( J \) in this basis. Then, the Holevo bound has the following expression.

\[
C_H^\theta(G) = \min_{\nu \in |\nu|} \left\{ \text{tr} \left| \sqrt{G} Z_J(v) \sqrt{G} \right| \text{Re}(d_k |J|v^3) = \delta_k^j \right\}
\]

(21)

where \( Z_j^{k,j}(v) \) is defined as \( \{ |v| |J|v^3 \} \) and a vector \( d_k \) is chosen as

\[
\frac{\partial \rho}{\partial \theta^k} = \sum_j d_{k,j} \rho \circ L_j.
\]

(22)

Note that the vector \( v^3 \) is a real vector.

For a proof, see Appendix A.3.

In the D-invariant case, only the vector \( \vec{L} = [L_0^k \equiv \sum_{j=1}^d (J_j^{-1})^{k,j} L_{0,j}] \) satisfies the condition in the right hand side (R.H.S.) of (21), i.e., \( C_H^\theta(G) = C_0(G, \vec{L}). \)

Since \( \text{Tr} \rho \circ L_0^k L_0^j = \text{Tr}(\rho \circ L_0^k + \frac{i}{2} [L_0^k, \rho_0]) L_0^j \), the equation (19) guarantees

\[
Z_\theta(\vec{L}) = \vec{J}_\theta^{-1}.
\]

(23)

That is, the equation

\[
C_\theta(G, \vec{L}) = C_0^R(G)
\]

holds. Therefore, the equality of (19) holds.

Concerning the non-D-invariant model, we have the following characterization.

**Theorem 4** Let \( S_1 \) def \( \{ \rho_{\theta_1}, \ldots, \rho_{\theta_d}, 0, \ldots, 0 \} | \theta_1, \ldots, \theta_d \subset \Theta_1 \} \subset S_2 \) def \( \{ \rho_{\theta_1}, \ldots, \rho_{\theta_d} | \theta_1, \ldots, \theta_d \subset \Theta_2 \) be two models such that \( S_2 \) is D-invariant. If a vector of Hermitian matrices \( \vec{X} = [X^k] \) satisfies the condition (21) and the condition (22), then

\[
C_{\theta,1}(G, \vec{X}) = C_{\theta,2}^R(P_{\vec{X}}^k GP_{\vec{X}}^k)
\]

(24)

for any weight matrix \( G \), where the \( d_1 \times d_2 \) matrix \( P_{\vec{X}}^k \) is defined as

\[
P_{\vec{X},j}^{k} \equiv \text{Tr} \frac{\partial \rho_{\theta}}{\partial \theta^j} X_j^k,
\]

(25)

i.e., \( P_{\vec{X},j}^{k} \) is a linear map from a \( d_2 \) dimensional space to a \( d_1 \) dimensional space. Furthermore, if the bound \( C_{\theta,2}^R(P_{\vec{X}}^k GP_{\vec{X}}^k) \) is attained in the model \( S_2 \), the quantity \( C_{\theta,1}(G, \vec{X}) \) can be attained in the model \( S_2 \).

Here, we denote the linear space spanned by elements \( v_1, \ldots, v_l \) by \( < v_1, \ldots, v_l > \). For a proof, see Appendix A.4. Thus, if the RLD bound can be attained for any weight matrix in a larger D-invariant model, the Holevo bound can be attained for any weight matrix.

3. **Optimal MSE matrix and Optimal Fisher information matrix**

Next, we characterize POVMs attaining the Holevo bound. First, we focus on the inequality (18) for a strictly positive matrix \( G \), if and only if

\[
V_\theta(M) = \text{Re} \left[ Z_{\theta}(\vec{X}_M) + \sqrt{G} \text{Im} Z_{\theta}(\vec{X}_M) \sqrt{G} \right]^{-1} \sqrt{G},
\]

(26)

the equality of (18) holds. Thus, the Holevo bound \( C_\theta(G) \) is attained for a strictly positive matrix \( G \), if and only if

\[
V_\theta(M) = \text{Re} \left[ Z_{\theta}(\vec{X}_G) + \sqrt{G} \text{Im} Z_{\theta}(\vec{X}_G) \sqrt{G} \right]^{-1} \sqrt{G},
\]

(27)

where \( \vec{X}_G \) is a vector of Hermitian matrix satisfying \( C_{\theta}(G) = C_0(G, \vec{X}_G) \). Therefore, the equation (19) guarantees that if and only if the Fisher information matrix \( \vec{J}_\theta^{-1} \) of POVM \( M \) equals

\[
\sqrt{G} \left( \sqrt{G} \text{Re} Z_{\theta} \vec{X}_G \sqrt{G} + | \sqrt{G} \text{Im} Z_{\theta} \vec{X}_G \sqrt{G} | \right)^{-1} \sqrt{G},
\]

for any weight matrix \( G \).

C. **Quantum CR bound**

Next, we discuss the asymptotic estimation error of an estimator based on collective measurement on \( n \)-fold tensor product system \( H^{\otimes n} \) def \( H \otimes \cdots \otimes H \). In this case, we treat the estimation problem of the \( n \)-fold tensor product family \( S^{\otimes n} \) def \( \{ \rho_{\theta}^{\otimes n} \equiv \rho_\theta \otimes \cdots \otimes \rho_\theta | \theta \in \Theta \) \). Then, we discuss the limiting behavior of \( \text{tr} G\rho(M^n) \), where \( M^n \) is an estimator of the family of \( S^{\otimes n} \), and \( V_\theta(M^n) \) is its MSE matrix. In the asymptotic setting, we focus on the asymptotically unbiased condition (28) and (29) instead of the locally unbiased condition.

\[
E_{n,\theta}^j = E_{\theta}^j(M^n) \equiv \int_{\Theta} \frac{\partial}{\partial \theta} \text{Tr} M^n(d\theta) \rho_{\theta}^{\otimes n} \to \theta^j
\]

(28)

\[
A_{n,\theta,k}^j = A_{\theta,k}^j(M^n) \equiv \frac{\partial}{\partial \theta^k} E_{\theta}^j(M^n) \to \delta_k^j
\]

(29)
as \( n \to \infty \). Thus, we define the quantum Cramér-Rao type bound (quantum CR bound) \( C_{\theta}(G) \) as

\[
C_{\theta}(G) \overset{\text{def}}{=} \min_{\{M^n\}_{n=1}^{\infty}} \left\{ \lim_{n \to \infty} n \text{tr} V_{\theta}(M^n)G \right\} \{M^n\} \text{ is asymptotically unbiased} \right\}.
\]

(30)

As is independently shown by Hayashi & Matsumoto and Gill & Massar, if the state family satisfies some regularity conditions, e.g., continuity, boundedness, etc., the following two-stage adaptive estimator \( M^n \) attains the bound \( C_{\theta}(G) \). First, we choose a POVM \( M \) such that the Fisher information matrix \( J_{\theta}^M \) is strictly positive for any \( \theta \in \Theta \), and perform it on \( \sqrt{n} \) systems. Then, we obtain the MLE \( \hat{\theta} \) for the family of probability distributions \( \{P^M_\theta|\theta \in \Theta\} \) based on \( \sqrt{n} \) outcomes \( \omega_1, \ldots, \omega_\sqrt{n} \). Next, we choose the measurement \( M_{\hat{\theta}} \), which attains the quasi-quantum Cramér-Rao bound \( \tilde{C}_\theta(G) \), and perform it on the remaining \( n - \sqrt{n} \) systems. This estimator attains that \( \tilde{C}_\theta(G) \), i.e., \( \text{tr} V_{\theta}(M^n)G \geq \frac{1}{n} \tilde{C}_\theta(G) \). Also, it satisfies the conditions (A) and (B). Therefore, we obtain

\[
\tilde{C}_\theta(G) \geq C_{\theta}(G).
\]

Moreover, by applying the above statement to the family \( S^{\otimes n} \), we obtain

\[
n\tilde{C}_\theta^n(G) \geq C_{\theta}(G),
\]

where \( \tilde{C}_\theta^n(G) \) denotes the quasi-quantum Cramér-Rao bound of the family \( S^{\otimes n} \).

In the \( n \)-fold tensor product family \( S^{\otimes n} \), the SLD \( L_{\theta,n;j} \) and the RLD \( \tilde{L}_{\theta,n;j} \) are given as

\[
L_{\theta,n;j} = \sqrt{n}L_{\theta;j}, \quad \tilde{L}_{\theta,n;j} = \sqrt{n}L_{\theta;j},
\]

where

\[
X(n) \overset{\text{def}}{=} \frac{1}{\sqrt{n}} \sum_{j=1}^{n} X(n,j)
\]

\[
X(n,j) \overset{\text{def}}{=} I \otimes \cdots \otimes I \otimes X \otimes I \otimes \cdots \otimes I.
\]

Therefore, the SLD Fisher matrix of \( S^{\otimes n} \) is calculated as

\[
\text{Tr} \rho^{\otimes n}(L_{\theta,n;k} \otimes L_{\theta,n;j}) = \text{Tr} \rho^{\otimes n} \left( \sum_{l=1}^{n} L_{\theta,k;l}^{(n)} \otimes \sum_{l' = 1}^{n} L_{\theta,j;l'}^{(n)} \right)
\]

\[
= \sum_{l=1}^{n} \sum_{l' = 1}^{n} \text{Tr} \rho^{\otimes n}(L_{\theta,k;l}^{(n)} \otimes L_{\theta,j;l'}^{(n)})
\]

\[
= \sum_{l=1}^{n} \text{Tr} \rho^{\otimes n}(L_{\theta,k;l} \otimes L_{\theta,j;l})^{(n)}
\]

\[
= \sum_{l=1}^{n} \sum_{l' = 1}^{n} \text{Tr} \rho^{\otimes n}(L_{\theta,k;l} \otimes I \otimes L_{\theta,j;l} \otimes I \otimes \cdots \otimes I)
\]

\[
= \sum_{l=1}^{n} J_{\theta,k;l} = n J_{\theta,k}.
\]

Similarly, the RLD Fisher matrix of \( S^{\otimes n} \) equals the \( n \) times of \( J_{\theta} \). As is shown in Appendix A, a similar relation with respect to the Holevo bound holds as follows.

**Lemma 4** Let \( C_{\theta}^H^n(G) \) be the Holevo bound of \( S^{\otimes n} \), then

\[
C_{\theta}^H^n(G) = \frac{1}{n} C_{\theta}^H(G).
\]

(31)

Thus, we can evaluate \( C_{\theta}(G) \) as follows. It proof will be given in Appendix A.

**Theorem 5** The quantum CR bound is evaluated as

\[
C_{\theta}(G) \geq C_{\theta}^H(G).
\]

(32)

Therefore, if there exists estimators \( M^n \) for \( n \)-fold tensor product family \( S^{\otimes n} \) such that

\[
n \text{tr} G V_{\theta}(M^n) \rightarrow C_{\theta}^H(G),
\]

then the relation

\[
C_{\theta}(G) = C_{\theta}^H(G) = \lim_{n \to \infty} n C_{\theta}^H(G)
\]

(33)

holds. Furthermore, if the relation (A) holds in a D-invariant model, any submodel of it satisfies the relation (B).

**D. General error function**

In the above discussion, we focus only on the trace of the product of the MSE matrix and a weight matrix. However, in general, we need to take the error function \( g(\theta, \hat{\theta}) \) other than the above into consideration. In this case, similarly to (30), we can define the asymptotic minimum error \( C_{\theta}(g) \) as

\[
C_{\theta}(g) \overset{\text{def}}{=} \min_{\{M^n\}_{n=1}^{\infty}} \left\{ \lim_{n \to \infty} n R_{\theta}^g(M^n) \right\} \{M^n\} \text{ is asymptotically unbiased} \right\},
\]

where

\[
R_{\theta}^g(M^n) \overset{\text{def}}{=} \int_{\mathbb{R}^d} g(\theta, \hat{\theta}) \text{Tr} M^n(d\hat{\theta}) \rho_\theta^{\otimes n}.
\]

We assume that when \( \hat{\theta} \) is close to \( \theta \), the error function \( g \) can be approximated by the symmetric matrix \( G^g \) as follows:

\[
g(\hat{\theta}, \theta) \approx \sum_{k,l} G^g_{k,l}(\hat{\theta}^k - \theta^k)(\hat{\theta}^l - \theta^l).
\]

Similarly to subsection II.C if we choose suitable adaptive estimator \( M^n \), the relation \( R_{\theta}^g(M^n) \leq \frac{n}{n} \tilde{C}_\theta(G^g) \) holds. Thus, \( C_{\theta}(g) \leq \tilde{C}_\theta(G^g) \). Also, we obtain \( C_{\theta}(g) \leq n \tilde{C}_\theta(G^g) \).
Conversely, for a fixed \( \theta_0 \), we choose local chart \( \phi(\theta) \) at a neighborhood \( U_{\theta_0} \) of \( \theta_0 \) such that

\[
g(\theta_0, \theta) = \sum_{k,l} C_{k,l}^2 (\phi^k(\theta) - \phi^k(\theta_0))(\phi^l(\theta) - \phi^l(\theta_0)),
\]

for \( \forall \theta \in U_{\theta_0} \). By applying the above discussions to the family \( \{\rho_{\theta} | \theta \in U_{\theta_0}\} \), we obtain

\[
C_\theta(g) \geq C_\theta^H(G^g).
\]

### III. QUANTUM GAUSSIAN STATES FAMILY

Next, we review the estimation of expected parameter of the quantum Gaussian state. In this case, Yuen & Lax derived quasi CR bound for the specific weight matrix and Holevo did it for arbitrary weight matrix. This model is essential for the asymptotic analysis of quantum two-level system. In the boson system, the coherent state with complex amplitude \( \alpha \) is described by the coherent vector \( |\alpha\rangle \equiv e^{-1/2} \sum_n \frac{\alpha_n}{\sqrt{n!}} |n\rangle \), where \( |n\rangle \) is the \( n \)-th number vector. The quantum Gaussian state is given as

\[
\rho_{\theta,NN} \equiv \frac{1}{N+1} \int_C |\alpha\rangle\langle\alpha| e^{-i\alpha^2/2} \, d\alpha.
\]

In particular, the relations

\[
\rho_{0,NN} = \frac{1}{N+1} \sum_{n=0}^{\infty} \left( \frac{N}{N+1} \right)^n |n\rangle\langle n|,
\]

\[
\rho_{\theta,NN} = W_{\theta_1,\theta_2} \rho_{0,NN} W_{\theta_1,\theta_2}^*,
\]

hold, where \( \theta = \frac{1}{\sqrt{2}}(\theta^1 + \theta^2 i) \) and \( W_{\theta_1,\theta_2} \equiv e^{(i\theta_1 P + i\theta_2 Q)} \).

For the estimation of the family \( \mathcal{S}_{NN} \equiv \{\rho_{\theta,NN} | \theta \in \frac{1}{\sqrt{2}}(\theta^1 + \theta^2 i)\} \), the following estimator is optimal. Let \( G \) be the weight matrix, then the matrix \( \hat{G} = \sqrt{\det G} G^{-1} \) has the determinant 1. We choose the squeezed state \( |\phi_G\rangle \langle \phi_G| \) such that

\[
\begin{pmatrix}
\langle \phi_G | Q | \phi_G \rangle \\
\langle \phi_G | P | \phi_G \rangle
\end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\]

\[
\begin{pmatrix}
\langle \phi_G | Q^2 | \phi_G \rangle \\
\langle \phi_G | Q \circ P | \phi_G \rangle \\
\langle \phi_G | P^2 | \phi_G \rangle
\end{pmatrix} = \frac{\hat{G}}{2},
\]

then the relation

\[
|\langle \phi_G, \frac{1}{\sqrt{2}}(\theta^1 + \theta^2 i) | \rangle|^2 = \exp(- \sum_{k,j} \theta^k ((\hat{G} + I)^{-1})_{k,j} \theta^j)
\]

(34)

holds. The POVM

\[
M_{\hat{G}}(d\theta^1 d\theta^2) \equiv W_{\theta_1,\theta_2} |\phi_G\rangle \langle \phi_G| W_{\theta_1,\theta_2}^* \frac{d\theta^1 d\theta^2}{2\pi}
\]

satisfies the unbiased condition

\[
E_\theta(M_{\hat{G}}) = \theta^i.
\]

Moreover, \( \mathcal{S} \) guarantees that \( \text{Tr} \rho_{\theta,NN} M_{\hat{G}}(d\theta^1 d\theta^2) \) is the normal distribution with covariance matrix \( (N + \frac{1}{2}) I + \frac{\hat{G}}{2} \). Therefore, its error can calculated as follows.

\[
\begin{align*}
\text{tr} G \nu_{\theta,NN} & = \text{tr} G \left((N + \frac{1}{2}) I + \frac{\hat{G}}{2}\right) \\
& = (N + \frac{1}{2}) \text{tr} G + \frac{1}{2} \text{tr} G \sqrt{\det \hat{G}} G^{-1} \\
& = (N + \frac{1}{2}) \text{tr} G + \sqrt{\det G}. \quad (35)
\end{align*}
\]

For its details, the following theorem holds.

**Theorem 6 Holevo** The POVM \( M_{\hat{G}} \) satisfies

\[
\left( \begin{pmatrix} \int (\hat{\theta})^2 M_{\hat{G}}(d\hat{\theta}) \\
\int \hat{\theta}^2 M_{\hat{G}}(d\hat{\theta}) \\
\int (\hat{\theta}^2)^2 M_{\hat{G}}(d\hat{\theta}) \end{pmatrix} \right) = \left( \begin{pmatrix} Q^2 & Q \circ P \\
Q \circ P & P^2 \end{pmatrix} + \frac{\sqrt{\det G}}{2} G^{-1} \otimes I \right).
\]

(36)

It is proved in Appendix. Its optimality is showed as follows. The derivatives can be calculated as

\[
\frac{\partial \rho_{\theta,NN}}{\partial \theta^1} = -i[P, \rho_{\theta,NN}] = \frac{1}{N + \frac{1}{2}} (Q - \theta^1) \circ \rho_{\theta,NN},
\]

\[
\frac{\partial \rho_{\theta,NN}}{\partial \theta^2} = i[Q, \rho_{\theta,NN}] = \frac{1}{N + \frac{1}{2}} (P - \theta^2) \circ \rho_{\theta,NN}.
\]

Therefore, we can calculate as

\[
L_{0,1} = \frac{1}{N + \frac{1}{2}} (Q - \theta^1), \quad L_{0,2} = \frac{1}{N + \frac{1}{2}} (P - \theta^2)
\]

\[
J_{\theta} = \left( \begin{pmatrix} N + \frac{1}{2} \end{pmatrix}^{-1} \frac{1}{0} \begin{pmatrix} 0 \end{pmatrix},
\right)
\]

\[
J_{\theta}^{-1} D_{\theta} J_{\theta}^{-1} = \left( \begin{pmatrix} 0 \end{pmatrix}, \begin{pmatrix} -1 \end{pmatrix} \begin{pmatrix} 0 \end{pmatrix}\right),
\]

where we use the relation (10). Thus, since

\[
\text{tr} \left[ \frac{1}{2} \sqrt{G} \begin{pmatrix} 0 & -i \\
i & 0 \end{pmatrix} \sqrt{G}\right] = \sqrt{\det G},
\]

the RLD Fisher information matrix is

\[
J_{\hat{\theta}}^{-1} = \begin{pmatrix} N + \frac{1}{2} \\
i/2 \end{pmatrix} \begin{pmatrix} i/2 \end{pmatrix}.
\]

Thus, the RLD bound is calculated as

\[
C_{\hat{\theta}}^R(G) = (N + \frac{1}{2}) \text{tr} G + \sqrt{\det G}.
\]
which equals the right hand side of (33). Thus, from (14), we obtain the optimality of $M_G$, i.e., Yuen, Lax, and Holevo’s result:

$$\hat{C}_\theta(G) = (N + 1/2) \text{tr} G + \sqrt{\det G}.$$  

Furthermore, for the $n$-fold tensor product model $S_n^\otimes n$, we can define a suitable estimator as follows. First, we perform the measurement $M_G$ on the individual system, and obtain $n$ data $(\theta_1^n, \theta_2^n, \ldots, \theta_n^n, \theta_n^n)$. We decide the estimate as $\hat{\theta}^n \defeq \frac{1}{n} \sum_{i=1}^n \theta_i^n$. In this case, the MSE matrix equals $\frac{1}{n}((N + \frac{1}{2})I + \frac{1}{2})$. Therefore, Theorem [1] guarantees

$$C_\theta(G) = \hat{C}_\theta(G) = (N + 1/2) \text{tr} G + \sqrt{\det G},$$

which implies that there is no advantage for using the quantum correlation in the measurement apparatus in the estimation of the expected parameter of quantum Gaussian family.

IV. ASYMPTOTIC BEHAVIOR OF SPIN $j$ SYSTEM

In this section, we discuss how the spin $j$ system asymptotically approaches to the quantum Gaussian state as $j$ goes to infinity. Accardi and Bach [13, 16] focused on the limiting behaviour of the $n$-tensor product space of spin 1/2, but we focus on that of spin $j$ system. Let $J_{j,1}, J_{j,2}, J_{j,3}$ be the standard generators of the spin $j$ representation of Lie algebra $\mathfrak{su}(2)$. That is, the representation space $\mathcal{H}_j$ is spanned by $|j, m \rangle$, $m = j, j - 1, \ldots, -j + 1, -j$, satisfying

$$J_{j,3}|j, m \rangle = m|j, m \rangle.$$  

The matrices $J_{j,\pm} \defeq J_{j,1} \pm iJ_{j,2}$ are represented as

$$J_{j,+}|j, m \rangle = \sqrt{(j - m)(j + m + 1)}|j, m + 1 \rangle$$

$$J_{j,-}|j, m \rangle = \sqrt{(j - m + 1)(j + m)}|j, m - 1 \rangle.$$  

For any complex $z = x + iy$, $|z| < 1$, we define the special unitary matrix

$$U_z \defeq \left( \begin{array}{cc} \sqrt{1 - |z|^2} & -z^* \\ z & \sqrt{1 - |z|^2} \end{array} \right),$$

and denote its representation on $\mathcal{H}_j$ by $U_{j,z}$. The spin coherent vector $|j, z \rangle \defeq U_{j,z}|j, j \rangle$ satisfies

$$\langle j, m|j, z \rangle = \sqrt{\binom{2j}{j + m} \alpha^{j-m}(1 - |\alpha|^2)^{j-m}}.$$  

We also define the state $\rho_{j,p}$ as

$$\rho_{j,p} \defeq \frac{1 - p}{1 - p^2j + 1} \sum_{m=-j}^j p^{j-m}|j, m \rangle \langle j, m |.$$

Defining the isometry $W_j$ from $\mathcal{H}_j$ to $L^2(\mathbb{R})$ as

$$W_j : |j, m \rangle \rightarrow |j - m \rangle,$$

we can regard the space $\mathcal{H}_j$ as a subspace of $L^2(\mathbb{R})$.

Theorem 7 Under the above imbedding, we obtain the following limiting characterization

$$\rho_{j, p} \rightarrow \rho_{0, \frac{p}{1-p}}$$ (37)

$$|j, \frac{z}{\sqrt{2j}} \rangle \rightarrow |z|z \rangle$$ (38)

in the trace norm topology. Moreover, when $j$ goes to infinity, the limiting relations

$$\text{Tr} \rho_{j, p}(a - \frac{1}{\sqrt{2j}}J_{j,+}^*)^*(a - \frac{1}{\sqrt{2j}}J_{j,+}) \rightarrow 0$$ (39)

$$\text{Tr} \rho_{j, p}(a^* - \frac{1}{\sqrt{2j}}J_{j,-})^*(a^* - \frac{1}{\sqrt{2j}}J_{j,-}) \rightarrow 0$$ (40)

$$\text{Tr} \rho_{j, p}(Q - \frac{1}{\sqrt{j}}J_{j,3})^2 \rightarrow 0$$ (41)

$$\text{Tr} \rho_{j, p}(P - \frac{1}{\sqrt{j}}J_{j,3})^2 \rightarrow 0$$ (42)

$$\text{Tr} \rho_{j, p}Q^2 \rightarrow \text{Tr} \rho_{0, \frac{p}{1-p}}Q^2$$ (43)

$$\text{Tr} \rho_{j, p}P^2 \rightarrow \text{Tr} \rho_{0, \frac{p}{1-p}}P^2$$ (44)

$$\text{Tr} \rho_{j, p}(Q \circ P) \rightarrow \text{Tr} \rho_{0, \frac{p}{1-p}}(Q \circ P)$$ (45)

$$\text{Tr} \rho_{j, p}((Q - \frac{1}{\sqrt{j}}J_{j,3}) \circ Q) \rightarrow 0$$ (46)

$$\text{Tr} \rho_{j, p}((Q - \frac{1}{\sqrt{j}}J_{j,3}) \circ P) \rightarrow 0$$ (47)

$$\text{Tr} \rho_{j, p}((P - \frac{1}{\sqrt{j}}J_{j,3}) \circ Q) \rightarrow 0$$ (48)

$$\text{Tr} \rho_{j, p}((P - \frac{1}{\sqrt{j}}J_{j,3}) \circ P) \rightarrow 0$$ (49)

hold, where we abbreviate the isometry $W_j$.

V. ESTIMATION IN QUANTUM TWO-LEVEL SYSTEM

Next, we consider the estimation problem of $n$-fold tensor product family of the full parameter model $S_{\text{full}} \defeq \{ \rho_{\theta} \defeq \frac{1}{2} I + \sum_{i=1}^3 \theta^i \sigma_i | ||\theta|| \leq 1 \}$ on the Hilbert space $\mathbb{C}^2$, where

$$\sigma_1 = \frac{1}{2} \left( \begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array} \right), \sigma_2 = \frac{1}{2} \left( \begin{array}{cc} 0 & -i \\ i & 0 \end{array} \right), \sigma_3 = \frac{1}{2} \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right).$$

In this parameterization, the SLDs at the point $(0, 0, r)$ can be expressed as

$$L_{(0,0,r), 1} = 2\sigma_1, \quad L_{(0,0,r), 2} = 2\sigma_2$$

$$L_{(0,0,r), 3} = \left( \begin{array}{cc} \frac{1}{1-r^2} & 0 \\ 0 & \frac{1}{1-r^2} \end{array} \right) = \frac{1}{1-r^2}(2\sigma_3 - r I).$$
Then, the SLD Fisher matrix $J_{(0,0,r)}$ and RLD Fisher matrix $\tilde{J}_{(0,0,r)}$ at the point $(0,0,r)$ can be calculated as

\[
J_\theta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{1-r^2} \end{pmatrix}, \quad \tilde{J}_{\theta}^{-1} = \begin{pmatrix} 1 - ir & 0 \\ ir & 1 & 0 \\ 0 & 0 & 1 - r^2 \end{pmatrix}.
\]

(50)

We can also check that this model is D-invariant. The state $\rho_\theta$ is described in the notations in Section 17 as

\[
\rho_\theta = U_{e^{i\phi}} \sin \phi/2 \rho_{1/2,p(\|\theta\|)} U_{e^{i\phi}}^* \sin \phi/2,
\]

where $p(r) = \frac{1+r}{2}$ and $\frac{\theta^2 + \theta^2}{\|\theta\|^2} = e^{i\psi} \sin \phi$.

On the other hand, as was proved by Nagaoka[4], Hayashi[3], Gill & Massar[5], in any model of the quantum two level system, the quasi CR bound can be calculated as

\[
\hat{C}_\rho(G) = \left( \text{tr} \sqrt{\hat{J}_\theta^{1/2} G \hat{J}_\theta^{1/2}} \right)^2 = \left( \text{tr} \sqrt{\sqrt{G} \hat{J}_\theta^{-1} \sqrt{G}} \right)^2,
\]

(51)

where the second equation follows from the unitary equivalence between $AA^*$ and $A^*A$.

### A. Covariant approach

As the first step of this problem, we focus on the risk function $g$ covariant for SU(2). Then, the risk function $R(\theta, \tilde{\theta})$ can be expressed by $g(\|\hat{\theta}\|, \|\theta\|, \phi)$, where $\phi$ is the angle between $\tilde{\theta}$ and $\theta$, i.e., $\|\hat{\theta}\| = \|\theta\| \cos \phi$. It can be divided into two parts:

\[
g(\|\hat{\theta}\|, \|\theta\|, \phi) = f_1(\|\hat{\theta}\|, \|\theta\|) + f_2(\|\hat{\theta}\|, \|\theta\|, \phi),
\]

where

\[
f_1(\|\hat{\theta}\|, \|\theta\|) \overset{\text{def}}{=} g(\|\hat{\theta}\|, \|\theta\|, 0)
\]

and

\[
f_2(\|\hat{\theta}\|, \|\theta\|, \phi) \overset{\text{def}}{=} g(\|\hat{\theta}\|, \|\theta\|, \phi) - g(\|\hat{\theta}\|, \|\theta\|, 0).
\]

For example, the square of the Bures’ distance is described as

\[
b^2(\rho_\theta, \rho_{\tilde{\theta}}) = 1 - F(\rho_\theta, \rho_{\tilde{\theta}})
= \frac{1}{2}(1 - \sqrt{1 - \|\theta\|^2 \|\hat{\theta}\|^2 - ||\hat{\theta}\|| \|\theta\|})
= \frac{1}{2}(1 - \sqrt{1 - \|\theta\|^2 \|\hat{\theta}\|^2 - ||\hat{\theta}\|| \|\theta\|})
+ \frac{1}{2} ||\hat{\theta}\|| \|\theta\| (1 - \cos \phi).
\]

This risk function can be approximated as

\[
b^2(\rho_\theta, \rho_{\tilde{\theta}}) \approx \frac{1}{4} \sum_{k,l} J_{\theta,k,l}(\theta^k - \hat{\theta}^k)(\theta^l - \hat{\theta}^l).
\]

Thus, the relations (13), (17), and (50) yield that

\[
C_{(0,0,r)}(b^2) \geq C_{(0,0,r)}^H(b^2) = \frac{3 + 2r}{4}.
\]

Therefore, the covariance guarantees that

\[
C_\rho(b^2) \geq \frac{3 + 2\|\theta\|}{4}.
\]

As another example, we can simplify the square of the Euclidean distance $\|\theta - \tilde{\theta}\|$ as follows.

\[
\|\theta - \tilde{\theta}\|^2 = \|\hat{\theta}\|^2 + \|\theta\|^2 - 2\|\theta\|\|\theta\| \cos \phi
= (\|\hat{\theta}\| - \|\theta\|)^2 + 2\|\theta\|\|\theta\| (1 - \cos \phi).
\]

Concerning this risk function, we obtain

\[
C_\rho(I) \leq C_\rho^H(I) = 3 + 2\|\theta\| - \|\theta\|^2.
\]

In the following, we construct a suitable estimator for the family $\mathcal{S}_{\text{null}}^{\otimes n}$. When we focus on the tensor representation on $(\mathbb{C}^2)^{\otimes n}$ of SU(2), we obtain its irreducible decomposition as

\[
(\mathbb{C}^2)^{\otimes n} = \bigoplus_{j=0 \text{ or } 1/2}^{n/2} \mathcal{H}_j \otimes \mathcal{H}_{n,j},
\]

\[
\mathcal{H}_{n,j} \overset{\text{def}}{=} C_{(n/2-j,1/2-j)}^{-1}(n,1/2-j-1).
\]

Using this decomposition, we perform the projection measurement $E_n = \{E_n\}$ on the system $(\mathbb{C}^2)^{\otimes n}$, where $E_n$ is the projection to $\mathcal{H}_j \otimes \mathcal{H}_{n,j}$. Then, we obtain the data $j$ and the final state $U_{j,e^{i\psi} \sin \phi} \rho_{j,p(\|\theta\|)} U_{j,e^{i\psi} \sin \phi}^*$ with the probability

\[
P_{n,\|\theta\|}(j) \overset{\text{def}}{=} \left( \frac{n}{2} - j \right) \left( \frac{n}{2} - j - 1 \right)
\]

\[
\cdot \left( \frac{1 - \|\theta\|}{2} \right)^{2-j} (1 + \|\theta\|)^{2+j}
\]

\[
+ \cdots + \left( \frac{1 - \|\theta\|}{2} \right)^{2-j} (1 + \|\theta\|)^{2+j} (1 - \frac{1}{2} \|\theta\|)^{2-j}
\]

\[
= \left( \frac{n}{2} - j \right) \left( \frac{n}{2} - j - 1 \right)
\]

\[
\cdot \left( \frac{1 - \|\theta\|}{2} \right)^{2-j} \left( 1 + \|\theta\| \right)^{2+j+1} (1 - \frac{1}{2} \|\theta\|)^{2+j+1},
\]

where $\rho_{\text{mix},n,j}$ is the completely mixed state on the space $\mathcal{H}_{n,j}$. Next, we take the partial trace with respect to the space $\mathcal{H}_{n,j}$, and perform the covariant measurement:

\[
M_{j}(\phi, \psi) \overset{\text{def}}{=} (2j+1)|j, e^{i\psi} \sin \phi/2 \rangle (j, e^{i\psi} \sin \phi/2 \sin \phi/4\pi
\]

Note that the measure $\sin \phi/4\pi \, d\phi \, d\psi$ is the invariant probability measure with parameter $\phi \in [0, \pi), \psi \in [0, 2\pi)$. 
When true parameter is $(0, 0, r)$, the distribution of data can be calculated as

$$\text{Tr} \rho_j M_j^i(\phi, \psi) = (2j + 1) \frac{1 - p}{1 - p^{2j + 1}} \left( 1 - (1 - p) \sin^2 \frac{\phi}{2} \right)^{2j} \sin \phi \frac{1}{4\pi},$$

where $p = p(r)$.

Finally, based on the data $j$ and $(\phi, \psi)$, we decide the estimate as

$$\hat{\theta}^1 = \frac{2j}{n} \cos \phi \sin \phi, \quad \hat{\theta}^2 = \frac{2j}{n} \sin \phi \sin \phi, \quad \hat{\theta}^3 = \frac{2j}{n} \cos \phi.$$

Hence, our measurement can be described by the POVM $M_{\text{cov}}^n \equiv \{ M_j^i(\phi, \psi) \otimes I_{\mathcal{H}_n} \}$ with the outcome $(j, \phi, \psi)$.

Next, we discuss the average error of the square of the Euclidean distance $\| \theta - \hat{\theta} \|^2$ except for the origin $(0,0,0)$. For the symmetry, we can assume that the true parameter is $(0,0,r)$. In this case, the average error of $\| \theta - \hat{\theta} \|^2$ equals

$$\sum_{j=0 \text{ or } 1/2}^{n/2} P_n(j) \left( \frac{2j}{n} - r \right)^2 + \frac{2r}{n} F_{j, \frac{1}{1+r}},$$

where

$$F_{j,p} \equiv \int_0^{2\pi} \int_0^{\pi} (1 - \cos \phi) \text{Tr} \rho_j M_j^i(\phi, \psi) d\phi d\psi.$$

As is proved in Appendix E2, the average error of the square of the Euclidean distance can be approximated as

$$\frac{1}{n} \text{E}_{\rho,j} \left( 1 - \sqrt{1 - r^2} \sqrt{1 - \left(\frac{2j}{n}\right)^2} \right) \geq 1 \left( \frac{3 + r}{2} \right) \frac{1}{n} C_\theta(b_r^2).$$

where we use the following approximation

$$1 - \sqrt{1 - r^2} \sqrt{1 - \left(\frac{2j}{n}\right)^2} \geq 1 \left( \frac{3 + r}{2} \right) \frac{1}{n} C_\theta(b_r).$$

for the case when $\frac{2j}{n}$ is close to $r$. Thus,

$$C_\theta(b_r^2) = \frac{3}{4} + \frac{r}{2}.$$

As a byproduct, we see that

$$\frac{2j}{n} \rightarrow r \text{ as } n \rightarrow \infty$$

in probability $P_{n,r}$.

Next, we proceed to the asymptotic behavior at the origin $(0,0,0)$. In this case the data $j$ obeys the distribution $P_{n,0}$:

$$P_{n,0}(j) = \frac{1}{2^n} \left( \left( \frac{n}{2} - j \right) - \left( \frac{n}{2} - j - 1 \right) \right) (2j + 1).$$

As is proved in Appendix E2, the average error of the square of the Euclidean distance can be approximated as

$$\sum_j P_{n,0}(j) \left( \frac{2j}{n} \right)^2 \geq \frac{3}{n} - \frac{4\sqrt{2}}{3n\sqrt{n}} + \frac{2}{n^2}. (57)$$

Since

$$\int \| \hat{\theta} - (0,0,0) \|^2 \text{Tr} M_{\text{cov}}^n (d\hat{\theta}) f_{(0,0,0)}^n = \sum_j P_{n,0}(j) \left( \frac{2j}{n} \right)^2,$$

we obtain $C_{(0,0,0)}(I) = C_{(0,0,0)}^H(I) = 3$, i.e., the equation (50) holds at the origin $(0,0,0)$. For a proof of (55), see Appendix E1.
On the other hand, by using (61), the quasi quantum CR bound can be calculated

\[ C_{\theta}(I) = (2 + \sqrt{1 - \|	heta\|^2})^2 = 5 - \|	heta\|^2 + 4\sqrt{1 - \|	heta\|^2}, \]  

(59)

Since 5 - \|	heta\|^2 + 4\sqrt{1 - \|	heta\|^2} - (3 + 2\|	heta\| - \|	heta\|^2) = 2(1 - \|	heta\|) + 4\sqrt{1 - \|	heta\|^2} is strictly greater than 0 in the mixed states case, using quantum correlation in measuring apparatus can improve the estimation error.

Remark 1 The equation (53) gives the asymptotic behavior of the error of \( M^J(\phi, \psi) \): \( F_{j,p} \cong \frac{1}{(1-p_j)} \). It can be checked from another viewpoint. First, we focus on another parameterization:

\[ M^J(z) dz \overset{\text{def}}{=} (2j + 1)|j, z)(j, z|dz. \]

The equation (53) of Theorem 4 guarantees that the POVM \( M^J(z) \) goes to the POVM \( |z\rangle\langle z| \). Thus, the equation (53) guarantees that its error goes to 0 with the rate \( \frac{1}{(1-p_j)} \). This fact indicates the importance of approximation mentioned by Theorem 4. Indeed, it plays an important role for the general weight matrix case.

Remark 2 One may think that the right hand side (R.H.S.) of (53) is strange because it is better than \( (1 - r^2)^\frac{1}{4} \), i.e., the error of the efficient estimator the binomial distribution. That is, when data \( k \) obeys \( n \)-binomial distribution with parameter \( \frac{1-r}{2}, \frac{1+r}{2} \) and we choose the estimator of \( \theta \) as \( k/n \) (it is called the efficient estimator), the error equals \( (1 - r^2)^\frac{1}{4} \), which is larger than the right hand side of (53). However, in mathematical statistics, it is known that we can improve the efficient estimator except for one point in the asymptotic second order. In our estimator, the right hand side of (53) at \( r = 0 \) is given in (57), and is larger than \( (1 - r^2)^\frac{1}{4} \).

B. General weight matrix

Next, we proceed to the general weight matrix. For the SU(2) symmetry, we can focus only on the point \( (0, 0, r) \) without of loss of generality. Concerning the RLD bound, we obtain the following lemma.

Lemma 5 For the weight matrix \( G = \left( \begin{array}{cc} \hat{G} & g \\ g^T & s \end{array} \right) \), the RLD bound at \( (0, 0, r) \) can be calculated as

\[ C_{0,0,r}^R(G) = \text{tr} G - r^2 s + 2r \sqrt{\text{det} \hat{G}}, \]

(60)

where \( \hat{G} \) is a \( 2 \times 2 \) symmetric matrix and \( g \) is a \( 2 \)-dimensional vector.

For a proof, see Appendix E.5.3 The main purpose of this subsection is the following theorem

Theorem 8 Assume the same assumption as Lemma 4 then

\[ C_{(0,0,r)}(G) = C_{(0,0,r)}^R(G) = \text{tr} G - r^2 s + 2r \sqrt{\text{det} \hat{G}}. \]

(61)

Furthermore, as is shown in Appendix E.4, the inequality

\[ C_\theta(G) = C_\theta^R(G) < \hat{C}_\theta(G) \]

(62)

holds. Thus, using quantum correlation in measuring apparatus can improve estimation error in the asymptotic setting.

As the first step of our proof of Theorem 8, we characterize the MSE matrix attaining the RLD bound \( C_{0,0,r}^R(G) \). The matrix

\[ V_{\hat{G},r} \overset{\text{def}}{=} \left( \begin{array}{cc} I + r \sqrt{\text{det} \hat{G}} \cdot \hat{G}^{-1} & 0 \\ 0 & 1 - r^2 \end{array} \right) \]

satisfies \( V_{\hat{G},r} \geq \hat{J}_{(0,0,r)}^{-1} \) and

\[ \text{tr} G V_{\hat{G},r} = \text{tr} G - r^2 s + r \sqrt{\text{det} \hat{G}} \cdot \hat{G}^{-1} \hat{G} = \text{tr} G - r^2 s + 2r \sqrt{\text{det} \hat{G}} = C_{0,0,r}^R(G). \]

Thus, when there exists a locally unbiased estimator with the covariance matrix \( V_{\hat{G}} \), the RLD bound \( C_{0,0,r}^R(G) \) can be attained.

In the following, we construct an estimator \( M^n \) locally unbiased at \( (0,0,r_0) \) for the \( n \)-fold tensor product family \( S_n \otimes n \) such that \( nV_{(0,0,r_0)}(M^n) \to V_{\hat{G}} \). In the family \( S_n \otimes n \), the SLDs can be expressed as

\[ \sqrt{nL_{(0,0,r),k}} = 2\sqrt{n\sigma_k^{(n)}} = 2 \bigoplus_j J_j,k \otimes I_{H_{n,j}}, \]

for \( k = 1, 2 \), and

\[ \sqrt{nL_{(0,0,r),3}} = \frac{1}{1 - r^2} \left( 2\sqrt{n\sigma_3^{(n)}} - rI \right) = \frac{1}{1 - r^2} \left( \bigoplus_j 2J_{j,3} \otimes I_{H_{n,j}} - rI_{(C^2)^{\otimes n}} \right). \]

First, we perform the projection-valued measurement \( E^n = \{ E^n_i \} \). Based only on this data \( j \), we decide the estimate of the third parameter \( \hat{\theta}_r^3 \) as

\[ \hat{\theta}_r^3(j) \overset{\text{def}}{=} \frac{1}{J_{n,r}} \frac{d\log P_{n,r}(j)}{dr} + r, \]

(63)

where

\[ J_{n,r} \overset{\text{def}}{=} \sum_j P_{n,r}(j) \left( \frac{d\log P_{n,r}(j)}{dr} \right)^2. \]
Then, we can easily check that this estimator $\hat{\theta}^2_r$ satisfies the following conditions:

$$\begin{align*}
\text{Tr} \frac{\partial \rho_{\theta}^{\otimes n}}{\partial \theta^k} \bigg|_{\theta=(0,0,r)} &= \left( \sum_j \partial_j^3(j) E_j^n \right) = \begin{cases} 1 & k = 3 \\ 0 & k = 1, 2 \end{cases} \\
\text{Tr} \rho_{\theta}^{\otimes n} \left( \sum_j \partial_j^3(j) E_j^n \right) &= 0.
\end{align*}$$

(64) (65)

The definition guarantees the equation (64) and the equation (65) for $k = 3$. The rest case can be checked as follows. The derivative of $\rho_{\theta}$ with respect to the first or second parameter at the point $(0,0,r)$ can be replaced by the derivative of $U_{x+iy}\rho_{(0,0,r)}U_{x+iy}^*$ with respect to $x$ or $y$. Since the probability $\text{Tr} U_{x+iy}\rho_{(0,0,r)}(U_{x+iy})^* M_j$ is independent of $x + iy$, we have

$$\frac{\partial \text{Tr} \rho_{\theta}^{\otimes n} E_j^n}{\partial \theta^k} = 0 \quad \text{for } k = 1, 2,$$

(66)

which implies (64) in the case of $k = 1, 2$.

Next, we take the partial trace with respect to $\mathcal{H}_{n,j}$, and perform the POVM $V^*_j M_j \rho_{\theta}^{\otimes n} (dx^1 dx^2)^* V_j$ on the space $\mathcal{H}_j$. After this measurement, we decide the estimate of the parameters $\hat{\theta}^1, \hat{\theta}^2$ as

$$\left( \hat{\theta}^1, \hat{\theta}^2 \right) = B_{j,r}^{-1} \left( \frac{x^1}{x^2} \right),$$

where

$$B_{j,r} \overset{\text{def}}{=} \left( \begin{array}{c}
\text{Tr}(\rho_{\theta} \circ 2J_{j,1} V^*_j Q V_j) \\
\text{Tr}(\rho_{\theta} \circ 2J_{j,1} V^*_j P V_j) \\
\text{Tr}(\rho_{\theta} \circ 2J_{j,2} V^*_j Q V_j) \\
\text{Tr}(\rho_{\theta} \circ 2J_{j,2} V^*_j P V_j)
\end{array} \right).$$

As is shown in Appendix E5, the relations

$$\begin{align*}
\text{Tr} \frac{\partial \rho_{\theta}^{\otimes n}}{\partial \theta^k} (\bigoplus_j \left( \int_{\mathbb{R}} \hat{\theta}^l M_{j,G}(d\hat{\theta}) \right) \otimes I_{\mathcal{H}_{n,j}}) &= \delta^l_k \\
\text{Tr} \rho_{\theta}^{\otimes n} (\bigoplus_j \left( \int_{\mathbb{R}} \hat{\theta}^l M_{j,G}(d\hat{\theta}) \right) \otimes I_{\mathcal{H}_{n,j}}) &= 0
\end{align*}$$

(67) (68)

hold for $l = 1, 2, k = 1, 2, 3$. Therefore, we see that our estimator $(\hat{\theta}^1, \hat{\theta}^2, \hat{\theta}^3)$ is locally unbiased at $(0,0,r)$.

Next, we prove that its covariance matrix $V_n$ satisfies

$$V_n \approx \begin{pmatrix}
I + r \sqrt{\det G^{-1}} & 0 \\
0 & 1 - r^2
\end{pmatrix} \frac{1}{n} = V_{j,G,r} \frac{1}{n}. $$

(69)

Using the equation (68) in Appendix E5, we have

$$\text{Tr} \rho_{\theta}^{\otimes n} \left( \bigoplus_j \left( \int_{\mathbb{R}} \hat{\theta}^3(j - r) M_{j,G}(d\hat{\theta}) \right) \otimes I_{\mathcal{H}_{n,j}} \right) = 0$$

for $l = 1, 2$. The definition of $\hat{\theta}^3(j)$ guarantees that

$$\text{Tr} \rho_{\theta}^{\otimes n} \left( \bigoplus_j \left( \int_{\mathbb{R}} \hat{\theta}^3(j - r)^2 M_j \otimes I_{\mathcal{H}_{n,j}} \right) \right) = \sum_j P_{n,r}(j) \left( \frac{1}{J_{j,n}} \frac{d \log P_{n,r}(j)}{dr} \right)^2 = J_{n,r}^{-1}.$$

As is shown in Appendix E6, the above value can be approximated by

$$J_{n,r} \approx (1 - r^2) \frac{1}{n} + \frac{1 - r^2}{n^2}. $$

(70)

In order to discuss other components of covariance matrix, we define the $2 \times 2$ matrix $V_{j,G,r}$:

$$[V_{j,G,r}^{k,l}] \overset{\text{def}}{=} [\text{Tr} \rho_{\theta} \int x^k x^l V_j M_{j,G}(dx)V_j^*].$$

By use of Theorem E6 this matrix can be calculated as

$$\begin{pmatrix}
\text{Tr} \rho_{\theta} V_j Q^2 V_j^* & \text{Tr} \rho_{\theta} V_j (Q \circ P) V_j^* \\
\text{Tr} \rho_{\theta} V_j (Q \circ P) V_j^* & \text{Tr} \rho_{\theta} V_j P^2 V_j^*
\end{pmatrix} + \sqrt{\det G} \frac{1}{2} \tilde{G}^{-1},$$

then the covariance matrix of the estimator $M_{j,G}$ on the state $\rho_{\theta}$ is

$$B_{j,r}^{-1} V_{j,G,r} (B_{j,r}^{-1})^T.$$


C. **Holevo bound in submodel**

Since the Holevo bound is attained in the asymptotic sense in the full model, Theorem 3 guarantees that the Holevo bound can be attained in the asymptotic sense in any submodel \( S = \{ \rho(\Theta) | \Theta \in \Theta \subset \mathbb{R}^d \} \). In the following, we calculate the Holevo bound in this case. Since the Holevo bound equals the SLD Fisher information in the one-dimensional case, we treat the two-dimensional case in the following. First, we suppose that the true state is \( \rho(0,0,r) \). Without loss of generality, by choosing a suitable coordinate, we can assume that the derivatives can be expressed as

\[
D_1 \equiv \frac{\partial \rho(\eta)}{\partial \eta} \bigg|_{\eta=(0,0,r)} = \sigma_1 \\
D_2 \equiv \frac{\partial \rho(\eta)}{\partial \eta^2} \bigg|_{\eta=(0,0,r)} = \cos \phi \sigma_2 + \sin \phi \sqrt{1 - r^2} \sigma_3,
\]

where \( 0 \leq \phi \leq \frac{\pi}{2} \). In the above assumption, we have the following theorem.

**Theorem 9** Assume that the weight matrix \( G \) is parameterized as

\( G = \begin{pmatrix} g_1 & g_2 & g_3 \end{pmatrix} \).

When \( \frac{g_1}{\sqrt{\det G}} < \frac{\cos \phi}{\sin \phi} \), the Holevo bound \( C_{(0,0,r)}^H(G) \) of the above subfamily can be calculated as

\[
C_{(0,0,r)}^H(G) = \text{tr} G + 2r \cos \phi \sqrt{\det G} - r^2 \sin^2 \phi g_1,
\]

and can be attained only by the following covariant matrix \( V_G \)

\[
V_G = I + r \sqrt{\det G} \cdot G^{-1} - \begin{pmatrix} r^2 \sin^2 \phi & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \end{pmatrix}. \tag{72}
\]

Otherwise, the Holevo bound \( C_{(0,0,r)}^H(G) \) and the covariant matrix \( V_G \) can be calculated by

\[
C_{(0,0,r)}^H(G) = \text{tr} G + \frac{\det G}{g_1} \left( \frac{\cos \phi}{\sin \phi} \right)^2 \tag{73}
\]

\[
V_G = I + \frac{\cos^2 \phi}{\sin^2 \phi} \begin{pmatrix} g_2^2 & -g_2 & g_1 \\
-g_2 & g_1 & 0 \\
-g_1 & 0 & 1 \end{pmatrix}. \tag{74}
\]

For a proof, see Appendix E.

On the other hand, the equation \( \Box \) guarantees that

\[
\tilde{C}_{(0,0,r)}^H(G) = (\text{tr} \sqrt{G})^2 = \text{tr} G + 2 \sqrt{\det G
\]

in this parameterization because \( J_\theta = I \). Since we can verify the inequality \( \text{tr} G + 2 \sqrt{\det G} > C_{(0,0,r)}^H(G) \) in the above two cases, we can check effectiveness of quantum correlation in the measuring apparatus in this case.

The set \( \{ V_G \} \text{det} G = 1 \) represents the optimal MSE matrixes. Its diagonal subset equals

\[
\left\{ I + \begin{pmatrix} r t^{-1} \cos \phi - r^2 \sin^2 \phi & 0 \\
0 & r t \cos \phi \end{pmatrix} : 0 < t \leq \frac{\cos \phi}{r \sin^2 \phi} \right\}. \tag{75}
\]

VI. **DISCUSSION**

We proved that the estimation error is evaluated by the Holevo bound in the asymptotic setting for estimators with quantum correlation in the measuring apparatus as well as for that without quantum correlation. We construct an estimator attaining the Holevo bound. In the covariant case, such an estimator is constructed as a covariant estimator. But, in the other case, it is constructed based on the approximation of the spin \( j \) system with sufficient large \( j \) to quantum Gaussian states family.

It is also checked based on the previous results that the Holevo bound cannot be attained by the individual measurement in the quantum two-level system. That is, using quantum correlation in the measuring apparatus can improve the estimation error in the asymptotic setting in the quantum two-level system.

Since the full parameter model of the quantum two-level system is D-invariant, its Holevo bound equals the RLD bound. Thus, its calculation is not so difficult. However, a submodel is not necessarily D-invariant. Hence, the calculation of its Holevo bound is not trivial. By comparing the previous result, we point out that this model is different from pure states model even in the limiting case \( r \rightarrow 1 \).

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**APPENDIX A: PROOFS OF THEOREMS AND LEMMAS IN SECTION III**

1. Proof of Theorem 1

For any complex valued vector \( \vec{b} = [b_j] \) and we define a complex valued vector \( \vec{a} = [a_j] = J_\theta^{-1} \vec{b} \) and matrixes \( X_\vec{b} \equiv \sum_j X_j b_j \) and \( L_\vec{a} \equiv \sum_j L_{0,j} a_j \). Since the assumption guarantees that \( \langle X_{\vec{b}}, L_{\vec{a}} \rangle = \langle \vec{b}, \vec{a} \rangle \), Schwarz inequality yields that

\[
\langle \vec{b} Z_{\theta} (\vec{X}) \rangle \langle \vec{b} | J_\theta^{-1} \vec{b} \rangle = \langle \vec{b} Z_{\theta} (\vec{X}) \rangle \langle \vec{b} | J_\theta^{-1} \vec{b} \rangle \geq \langle \vec{b} \rangle^2 \geq \langle \vec{b} \rangle^2 = | \langle \vec{b} \rangle |^2.
\]

Therefore, we obtain

\[
\langle \vec{b} Z_{\theta} (\vec{X}) \rangle \langle \vec{b} | J_\theta^{-1} \vec{b} \rangle \geq \langle \vec{b} \rangle | J_\theta^{-1} \vec{b} \rangle^2,
\]

which implies \( \Box \). Similarly we can prove \( \Box \).
2. Proof of Lemma 1

For any complex valued vector $\vec{b} = [b_j]$, we define matrix $X^*_{\vec{b},M} = \sum_j X^*_M b_j$. Since
$$\int_{\mathbb{R}^d} \langle \hat{\theta}, \hat{\theta} \rangle M(d\hat{\theta}) = X^*_{\vec{b},M},$$
we obtain
$$\langle \hat{\theta}, \hat{\theta} \rangle V_{\theta}(M)\hat{\theta} - \langle \hat{\theta}, \hat{\theta} \rangle Z_{\theta}(\hat{X}_M)|\hat{b}\rangle$$
$$= \int_{\mathbb{R}^d} \langle \hat{\theta}, \hat{b} \rangle^* M(d\hat{\theta}) - X^*_{\vec{b},M} X^*_{\vec{b},M}$$
$$= \int_{\mathbb{R}^d} (\langle \hat{\theta}, \hat{b} \rangle - X^*_{\vec{b},M})^* M(d\hat{\theta})(\langle \hat{\theta}, \hat{b} \rangle - X^*_{\vec{b},M}) \geq 0,$$
which implies (9). ■

3. Proof of Lemma 2

Since the real symmetric matrix $T \stackrel{\text{def}}{=} V - \text{Re} W$ satisfies
$$T \geq \text{Im} W,$$
we obtain
$$\text{tr} T \geq \min\{\text{tr} T' \mid T' \text{ real symmetric, } T \geq \text{Im} W\} = \text{tr} |\text{Im} W|.$$  
Therefore,
$$\text{tr} V \geq \text{tr} \text{Re} W + \text{tr} T \geq \text{tr} \text{Re} W + \text{tr} |\text{Im} W|.$$ ■

4. Proof of Theorem 2

Since
$$\rho_\theta \circ L_{\theta;j} = \rho_\theta \circ \tilde{L}_{\theta;j} = \rho_\theta + i/2 |\tilde{L}_{\theta;j}, \rho_\theta|$$
$$= \rho_\theta \circ \left( L_{\theta;j} + \frac{i}{2} D_\theta (L_{\theta;j}) \right),$$
we have
$$(I + i/2 D_\theta)(\tilde{L}_{\theta;j}) = L_{\theta;j},$$
which implies $\tilde{L}_{\theta;j} = (I + i/2 D_\theta)^{-1} L_{\theta;j}$. Since
$$\frac{\partial \rho_\theta}{\partial \theta_j} (\frac{\partial \rho_\theta}{\partial \theta_j})^* = (L_{\theta;j})^* \rho_\theta,$$
we have
$$J_{\theta;k,j} = \text{Tr} \rho_\theta L_{\theta;k} (\tilde{L}_{\theta;j})^* = \text{Tr} (L_{\theta;j})^* \rho_\theta L_{\theta;k}$$
$$= \text{Tr} \frac{\partial \rho_\theta}{\partial \theta_j} \tilde{L}_{\theta;k} = \text{tr} (\rho_\theta \circ L_{\theta;j}) \tilde{L}_{\theta;k} = \langle L_{\theta;j}, \tilde{L}_{\theta;k} \rangle \rho_\theta$$
$$= \langle L_{\theta;j}, (I + i/2 D_\theta)^{-1} L_{\theta;k} \rangle \rho_\theta.$$
Next, we define a linear map $\mathcal{L}$ from $\mathbb{C}^d$ to $T_\theta$ as follows,
$$\vec{b} \mapsto \sum_j b_j L_{\theta;j},$$
then its inverse $\mathcal{L}^{-1}$ and its adjoint $\mathcal{L}^*$ are expressed as
$$\mathcal{L}^{-1} : X \mapsto \sum_{k=1}^d \langle J_{\theta}^{-1} \rangle^{k,j} (L_{\theta;k}, X)$$
$$\mathcal{L}^* : X \mapsto \langle L_{\theta;j}, X \rangle \rho_\theta.$$  
Thus, the map $\tilde{J}_\theta$ can be described by
$$\mathcal{L}^* \circ (I + i/2 D_\theta)^{-1} \circ \mathcal{L} = \mathcal{L}^* \circ P_{\rho_\theta} (I + i/2 D_\theta)^{-1} P_{\rho_\theta} \circ \mathcal{L},$$
where $P_{\rho_\theta}$ is the projection to $T_\theta$. Since $T_\theta$ is invariant for $D_\theta$,
$$(P_{\rho_\theta} (I + i/2 D_\theta)^{-1} P_{\rho_\theta})^{-1} = P_{\rho_\theta} (I + i/2 D_\theta) P_{\rho_\theta}.$$  
Therefore, the inverse of $\tilde{J}_\theta$ equals
$$\mathcal{L}^{-1} \circ (P_{\rho_\theta} (I + i/2 D_\theta)^{-1} P_{\rho_\theta})^{-1} \circ (\mathcal{L}^*)^{-1}$$
$$= \mathcal{L}^{-1} \circ P_{\rho_\theta} (I + i/2 D_\theta) P_{\rho_\theta} \circ (\mathcal{L}^{-1})^*,$$
which implies
$$(J_{\theta}^{-1})^{k,j} = \sum_{l,l'} \langle J_{\theta}^{-1} \rangle^{k,l} \langle L_{\theta;l}, (I + i/2 D_\theta) L_{\theta;l'} \rangle \langle (J_{\theta}^{-1})^*, l' \rangle.$$ ■

5. Proof of Lemma 3

Let $P$ be the projection to $T_\theta$ with respect to the inner product $\langle \ , \ \rangle_\theta$, and $P^c$ be the the projection to its orthogonal space with respect to the inner product. When $X$ satisfies the condition (9), $\langle P(X^k), L_j \rangle_\theta = \langle X^k, L_j \rangle_\theta = \delta_j^k$. Thus, $P(\tilde{X}) = [P(X^i)]$ satisfies the condition (9). Moreover,
$$\text{Tr} \rho_\theta P(X^k)P^c(X^j)$$
$$= \text{Tr} \left( \rho_\theta \circ P(X^k) + \frac{1}{2} \rho_\theta \circ D_\theta (P(X^k)) \right) P^c(X^j)$$
$$= \text{Tr} \left( \rho_\theta \circ P(X^k) + \frac{1}{2} \rho_\theta \circ D_\theta (P(X^k)) \right) P^c(X^j)$$
$$= \text{Tr} \left( \rho_\theta \circ \left( P(X^k) + \frac{1}{2} D_\theta (P(X^k)) \right) \right) P^c(X^j)$$
$$= \langle P(X^k) + \frac{1}{2} D_\theta (P(X^k)), P^c(X^j) \rangle_\theta = 0.$$  
Thus, we obtain
$$Z_\theta(\tilde{X}) = Z_\theta(P(\tilde{X})) + Z_\theta(P^c(\tilde{X})) \geq Z_\theta(P(\tilde{X})).$$
which implies that
\[
\sqrt{G} \text{Re} Z_\theta(\bar{X}) \sqrt{G} + |\sqrt{G} \text{Im} Z_\theta(\bar{X}) \sqrt{G}| \\
\geq \sqrt{G} Z_\theta(\bar{X}) \sqrt{G} \geq \sqrt{G} Z_\theta(P(\bar{X})) \sqrt{G}.
\]

Since the matrix $\sqrt{G} \text{Im} Z_\theta(\bar{X}) \sqrt{G}$ is imaginary Hermite matrix, $|\sqrt{G} \text{Im} Z_\theta(\bar{X}) \sqrt{G}|$ is real symmetric matrix. Therefore, Lemma 2 guarantees that
\[
Z_\theta(\bar{X}) \geq Z_\theta(P(\bar{X})�)
\]
which implies (20).

Next, we proceed to a proof of (21). Since the basis $\langle L_1, \ldots, L_m \rangle$ is normally orthogonal concerning SLD, the equation (15) guarantees that
\[
\text{Tr} \rho_\theta L_k L_j = \text{Tr} \rho_\theta L_k \circ L_j + \frac{1}{2} \text{Tr} \rho_\theta [L_k, L_j] = \delta_{k,j} - \frac{1}{2} D_{\theta,k,j}
\]
(A1)

Hence, when we choose the vector $\nu^k = (v_1^k, \ldots, v_m^k)$ satisfying that $X^k = \sum_j v_j^k L_j$,
\[
\text{Tr} \frac{\partial \rho_\theta}{\partial \theta^k} X^k = \text{Re} \langle d_k | j \rangle \quad \text{(A2)}
\]

\[
\text{Tr} \rho_\theta X^k X^j = \langle v_j^k | j \rangle \quad \text{(A3)}
\]

Therefore, we obtain (21).

6. Proof of Theorem 4

There exists $d_1 \times d_2$ matrix $O$ such that $\sqrt{P^t_X} GP_X = O^T \sqrt{G} P_X$ and $O^T O = I_{d_1}$. Since $X^k = \sum_{i=1}^{d_2} P^t_{X,i} L_{\theta,i}$,
\[
C_{\theta,1}(G, \bar{X}) \\
= \text{tr} \sqrt{G} \text{Re} Z_\theta(\bar{X}) \sqrt{G} + \text{tr} |\sqrt{G} \text{Im} Z_\theta(\bar{X}) \sqrt{G}| \\
= \text{tr} \sqrt{G} P_X \text{Re} Z_\theta(\bar{L})^T P_X \sqrt{G} \\
+ \text{tr} |\sqrt{G} P_X \text{Im} Z_\theta(\bar{L})^T P_X \sqrt{G}| \\
= \text{tr} O^T \sqrt{G} P_X \text{Re} Z_\theta(\bar{L})^T P_X \sqrt{G} \\
+ \text{tr} |O^T \sqrt{G} P_X \text{Im} Z_\theta(\bar{L})^T P_X \sqrt{G}| \\
= C_{\theta,2}(P_X^t GP_X, \bar{L}) = C_{\theta,2}^R(P_X^t GP_X).
\]

7. Proof of Lemma 4

Let $T^m_\theta$ be the linear space spanned by the orbit of the SLD tangent space of $S^{\otimes n}$. Since any element $X$ of $T_\theta$ satisfies
\[
\sqrt{n} \left( D_\theta \circ \cdots \circ D_\theta(X) \right)_{\theta}^{(n)} = \sqrt{n} D_\theta \circ \cdots \circ D_\theta(X^{(n)}),
\]
the $T^m_\theta$ equals
\[
\{ \sqrt{n} X^{(n)} | X \in T_\theta \}.
\]

Furthermore, the vector $\sqrt{n} \bar{X}^{(n)} = [\sqrt{n}(X^{(i)})^{(n)}]$ satisfies
\[
C_\theta(G, \sqrt{n} \bar{X}^{(n)}) = n C_\theta(G, \bar{X}).
\]

Therefore, Lemma 4 guarantees that
\[
C_{\theta,H,n}^R(G) \\
= \min_{X, X' \in T_\theta} \left\{ C_\theta(G, \sqrt{n} X^{(n)}) \left| \langle \sqrt{n} L_{\theta,k}^{(n)} \sqrt{n} (X^{(i)})^{(n)} \rangle_\theta = \delta_k^i \right. \right\} \\
= \min_{X, X' \in T_\theta} \left\{ n C_\theta(G, \bar{X}) \left| \langle L_{\theta,k} \bar{X} \rangle_\theta = \delta_k^i \right. \right\} \\
= \frac{1}{n} \min_{Y, Y' \in T_\theta} \left\{ C_\theta(G, Y) \left| \langle L_{\theta,k} Y \rangle_\theta = \delta_k^i \right. \right\},
\]
where we put $Y = \frac{1}{n} \bar{X}$. Therefore, we obtain (31).

8. Proof of Theorem 5

Lemma 4 guarantees that
\[
V_\theta(M^n) \geq Z_\theta(\bar{X}M^n).
\]

Since the vector $\bar{Y}_M = (Y_{M,i}^n) = \sum_{j} (A_{\theta}(M)^{-1})_{i,j} X_M$ of Hermitian matrixes satisfies
\[
\langle \sqrt{n} L_{\theta,j} X_M \rangle_\theta = \delta_j^i,
\]
\[
Z_\theta(\bar{X} M^n) = A_{\theta}(M^n) Z_\theta(\bar{Y}_M^n) A_{\theta}^T(M^n),
\]
the relations
\[
\text{tr} G V_\theta(M^n) \\
\geq \text{tr} \sqrt{G} \text{Re} A_{\theta}(M^n) Z_\theta(\bar{Y}_M^n) A_{\theta}^T(M^n) \sqrt{G} \\
+ \text{tr} |\sqrt{G} \text{Im} A_{\theta}(M^n) Z_\theta(\bar{Y}_M^n) A_{\theta}^T(M^n) \sqrt{G}| \\
\geq C_{\theta,H,n}^R(A_{\theta}^T(M^n) GA_{\theta}(M^n)) \\
= \sum_{n=1}^{d_1} C_{\theta,R}^R(A_{\theta}^T(M^n) GA_{\theta}(M^n))
\]
hold. Taking the limit, we obtain
\[
\lim_{n \to \infty} n \text{tr} G V_\theta(M^n) \geq \lim_{n \to \infty} C_{\theta,H}^R(A_{\theta}^T(M^n) GA_{\theta}(M^n)) = C_{\theta}^R(G),
\]
which implies (32).
APPENDIX B: PROOF OF THEOREM [8]

Let $E$ be the joint measurement of $P \otimes I + I \otimes P$ and $Q \otimes I - I \otimes Q$ on the space $L^2(\mathbb{R}) \otimes L^2(\mathbb{R})$. As was proved in Holevo [3], the POVM $M_G$ satisfies

$$\text{Tr} M_G(dx \, dy) \rho = \text{Tr} E(dx \, dy)(\rho \otimes |\phi_G\rangle \langle \phi_G|).$$

(B1)

Thus,

$$\text{Tr} x^2 M_G(dx \, dy) \rho = \text{Tr} x^2 E(dx \, dy)(\rho \otimes |\phi_G\rangle \langle \phi_G|)
= \text{Tr}(Q^2 \otimes I + I \otimes Q^2)(\rho \otimes |\phi_G\rangle \langle \phi_G|)
= (\text{Tr} Q^2 \rho) + \hat{G}^{1,1},$$

which implies equation (36) regarding the $(1, 1)$ element.

Since $\langle \phi_G | P | \phi_G \rangle = \langle \phi_G | Q | \phi_G \rangle = 0$, Concerning $(1, 2)$ element, we have

$$\text{Tr} x y M_G(dx \, dy) \rho = \text{Tr} x y E(dx \, dy)(\rho \otimes |\phi_G\rangle \langle \phi_G|)
= \text{Tr}(Q \circ P \otimes I + I \otimes Q \circ P - P \otimes Q + Q \otimes P)
\cdot (\rho \otimes |\phi_G\rangle \langle \phi_G|)
= (\text{Tr}(Q \circ P) \rho) + \langle \phi_G | Q \circ P | \phi_G \rangle = \text{Tr}(Q \circ P) \rho + \hat{G}^{1,2}.$$

We can similarly prove equation (36) for other elements.

APPENDIX C: PROOF OF THEOREM [6]

First, we prove (37). Since

$$\rho_{j,p} - \rho_0, \frac{z}{\sqrt{2j}}$$

$$= \frac{p^{2j+1}}{1 - p^{2j+1}} (1 - p) \sum_{n=0}^{2j} p^n |n\rangle \langle n| - (1 - p) \sum_{n=2j+1}^{\infty} p^n |n\rangle \langle n|,$$

we have

$$\|\rho_{j,p} - \rho_0, \frac{z}{\sqrt{2j}}\|
= \frac{p^{2j+1}}{1 - p^{2j+1}} (1 - p) \sum_{n=0}^{2j} p^n + (1 - p) \sum_{n=2j+1}^{\infty} p^n
\leq \frac{p^{2j+1}}{1 - p^{2j+1}} + p^{2j+1} \to 0,$$

which implies (37). Next, we prepare a lemma for our proof of (38).

**Lemma 6** Assume that a sequence of normalized vector $a^n = \{a_i^n\}_{i=0}^\infty$ and a normalized vector $a = \{a_i\}_{i=0}^\infty$ satisfies

$$a_i^n \to a_i \text{ as } n \to \infty,$$

then

$$\sum_{i=0}^{\infty} |a_i^n - a_i|^2 \to 0.$$

**Proof:** For any real number $\epsilon > 0$, there exists an integers $N_1$ such that

$$\sum_{i=N_1}^{\infty} |a_i|^2 \leq \epsilon.$$

Furthermore, we can choose another integer $N_2$ such that

$$\sum_{i=0}^{N_1-1} |a_i^n - a_i|^2 < \epsilon, \quad \sum_{i=0}^{N_1-1} |a_i^n|^2 - |a_i|^2 < \epsilon, \quad \forall n \geq N_2.$$

Hence, we have

$$\sum_{i=N_1}^{\infty} |a_i^n|^2 = 1 - \sum_{i=0}^{N_1-1} |a_i^n|^2 \leq 1 - \left( \sum_{i=0}^{N_1-1} |a_i|^2 - \epsilon \right) \leq 2\epsilon.$$

Therefore,

$$\sum_{i=0}^{\infty} |a_i^n - a_i|^2 \leq \sum_{i=0}^{N_1-1} |a_i^n - a_i|^2 + 2 \sum_{i=N_1}^{\infty} (|a_i^n|^2 + |a_i|^2)
\leq 2(\epsilon + \epsilon) = 7\epsilon.$$

Then, our proof is completed. \[\square\]

We can calculate $|j, \frac{z}{\sqrt{2j}}\rangle$ as

$$|j, \frac{z}{\sqrt{2j}}\rangle = \sum_{n=0}^{2j} \sqrt{\binom{2j}{n}} \left( \frac{\alpha}{\sqrt{2}} \right)^n \left( 1 - |\alpha|^2 \right)^{\frac{2j-n}{2}} |n\rangle.$$

Its coefficient converges as

$$\sqrt{\binom{2j}{n}} \left( \frac{\alpha}{\sqrt{2}} \right)^n \left( 1 - |\alpha|^2 \right)^{\frac{2j-n}{2}}
= \sqrt{\frac{(2j)!}{(2j-n)!}(2j)^n} \left( 1 - |\alpha|^2 \right)^{-n/2} \left( 1 - |\alpha|^2 \right)^{\frac{2j-n}{2}} \frac{|\alpha|^2}{\sqrt{n!}}
\to e^{-|\alpha|^2 \frac{\alpha^n}{\sqrt{n!}}\frac{1}{\sqrt{2j}}}$

as $j \to \infty$.

Thus, Lemma 6 guarantees that

$$\|z - |j, \frac{z}{\sqrt{2j}}\rangle\| \to 0,$$

which implies (38).

$$J_{j,+}|n\rangle = \sqrt{n+1}(2j-n+1)|n+1\rangle \quad (n = 1, \ldots, 2j)$$

$$J_{j,0} = 0$$

$$J_{j,-}|n\rangle = \sqrt{n}(2j-n)|n\rangle \quad (n = 0, \ldots, 2j-1)$$

$$J_{j,-2j}|0\rangle = 0$$
\[
(a - \frac{1}{\sqrt{2} j} J_{j,+})\rho_{j,p}(a - \frac{1}{\sqrt{2} j} J_{j,+})^*\\
= \frac{1 - p}{1 - p^{2j+1}} \sum_{n=1}^{2j} \left( \sqrt{n\left( \sqrt{\frac{2j-n+1}{2j}} - 1 \right)} \right)^2 p^n |n-1\rangle \langle n-1| \]
\]
Since the inequality \(1 - \sqrt{1 - x} \leq \sqrt{x}\) holds for \(0 \leq x \leq 1\), we have
\[
\text{Tr}(a - \frac{1}{\sqrt{2} j} J_{j,+})\rho_{j,p}(a - \frac{1}{\sqrt{2} j} J_{j,+})^*\\
\leq \frac{1 - p}{1 - p^{2j+1}} \sum_{n=1}^{2j} \frac{n(n-1)}{2j} p^n\\
\leq (1 - p) \sum_{n=0}^{\infty} \frac{n^2}{2j} p^n = \frac{1 - p}{2} \frac{p(1+p)}{1-p^2} \rightarrow 0,
\]
which implies (39). Since
\[
\text{Tr}(a^* - \frac{1}{\sqrt{2} j} J_{j,-})\rho_{j,p}(a^* - \frac{1}{\sqrt{2} j} J_{j,-})^*\\
= \frac{1 - p}{1 - p^{2j+1}} \sum_{n=0}^{2j-1} \left( \sqrt{n+1\left( \sqrt{\frac{2j-n}{2j}} - 1 \right)} \right)^2 p^n |n+1\rangle \langle n+1|\\
+ \frac{1 - p}{1 - p^{2j+1}} (2j+1)^2 p^{2j}|2j+1\rangle \langle 2j+1|,
\]
Since the inequality \(1 - \sqrt{1 - x} \leq \sqrt{x}\) holds for \(0 \leq x \leq 1\), we have
\[
\text{Tr}(a^* - \frac{1}{\sqrt{2} j} J_{j,-})\rho_{j,p}(a^* - \frac{1}{\sqrt{2} j} J_{j,-})^*\\
\leq \frac{1 - p}{1 - p^{2j+1}} \sum_{n=0}^{2j-1} \frac{(n+1)n}{2j} p^n + \frac{1 - p}{1 - p^{2j+1}} (2j+1)^2 p^{2j}\\
\leq (1 - p) \sum_{n=0}^{\infty} \frac{(n+1)^2}{2j} p^n + \frac{1 - p}{1 - p^{2j+1}} (2j+1)^2 p^{2j}\\
= \frac{1 - p}{2j} \frac{1+p}{(1-p)^3} + \frac{1 - p}{1 - p^{2j+1}} (2j+1)^2 p^{2j} \rightarrow 0,
\]
which implies (40). Since
\[
(Q - \frac{1}{\sqrt{2} j} J_{j})^2 + (P - \frac{1}{\sqrt{2} j} J_{j})^2\\
= (a - \frac{1}{\sqrt{2} j} J_{j,+})^* (a - \frac{1}{\sqrt{2} j} J_{j,+})\\
+ (a - \frac{1}{\sqrt{2} j} J_{j,+}) (a - \frac{1}{\sqrt{2} j} J_{j,+})^*,
\]
the relations (39) and (40) guarantee the relation (41). Also, we obtain (42).
Similarly, we can show

\[ \left| \text{Tr} \frac{p^{2j+1}}{1-p^{2j+1}} (1-p) \sum_{n=0}^{2j} p^n |n\rangle \langle n| (Q \circ P) \right| \to 0. \]

Thus, we obtain (45). By using Schwarz inequality of the inner product \((X, Y) \to \text{Tr} \rho_{j,p}(X \circ Y)\), we obtain

\[ \left| \text{Tr} \rho_{j,p}(Q - \frac{1}{\sqrt{J}} J_x) \circ Q \right| ^2 \leq \text{Tr} \rho_{j,p}(Q - \frac{1}{\sqrt{J}} J_x)^2 \text{Tr} \rho_{j,p} Q^2. \]

Thus, the relations (44) and (43) guarantee the relation (40). Similarly, we obtain (47) - (49).

**APPENDIX D: USEFUL FORMULA FOR FISHER INFORMATION**

In this section, we explain a useful formula for Fisher information, which are applied to our proof of (40) and (41). Let \( \mathcal{S} = \{p_\theta(\omega_1, \omega_2) | \theta \in \Theta \subset \mathbb{R} \} \) be a family of probability distributions on \( \Omega_1 \times \Omega_2 \). We define the marginal distribution \( p_\theta(\omega_1) \) and conditional distribution as

\[ p_\theta(\omega_1) \overset{\text{def}}{=} \sum_{\omega_2 \in \Omega_2} p_\theta(\omega_1, \omega_2), \quad p_\theta(\omega_2 | \omega_1) \overset{\text{def}}{=} \frac{p_\theta(\omega_1, \omega_2)}{p_\theta(\omega_1)}. \]

Then, the following theorem holds for the family of distributions \( \mathcal{S}, \mathcal{S}_1 \overset{\text{def}}{=} \{p_\theta(\omega_1) | \theta \in \Theta \subset \mathbb{R} \}, \) and \( \mathcal{S}_{\omega_1} \overset{\text{def}}{=} \{p_\theta(\omega_2 | \omega_1) | \theta \in \Theta \subset \mathbb{R} \} \).

**Theorem 10** The Fisher information \( J_\theta \) of the family \( \mathcal{S} \) satisfies

\[ J_\theta = J_{1,\theta} + \sum_{\omega_1 \in \Omega_1} p_\theta(\omega_1) J_{\omega_1, \theta}, \tag{D1} \]

where \( J_{1,\theta} \) is the Fisher information of \( \mathcal{S}_1 \) and \( J_{\omega_1, \theta} \) is the Fisher information of \( \mathcal{S}_{\omega_1} \). Moreover, the information less has another form:

\[ J_{\omega_1, \theta} = \sum_{\omega_2 \in \Omega_2} p_\theta(\omega_2 | \omega_1) \left( \frac{d \log p_\theta(\omega_2 | \omega_1)}{d \theta} \right)^2 - \left( \sum_{\omega_2 \in \Omega_2} p_\theta(\omega_2 | \omega_1) \frac{d \log p_\theta(\omega_2 | \omega_1)}{d \theta} \right)^2. \tag{D2} \]

Thus, the average \( \sum_{\omega_1 \in \Omega_1} p_\theta(\omega_1) J_{\omega_1, \theta} \) can be regarded as information loss by losing the data \( \omega_2 \).

**Proof:** The Fisher information \( J_\theta \) equals

\[ \sum_{\omega_1 \in \Omega_1} \sum_{\omega_2 \in \Omega_2} p_\theta(\omega_1) p_\theta(\omega_2 | \omega_1) \left( \frac{d \log p_\theta(\omega_2 | \omega_1)}{d \theta} \right)^2 \]

\[ = \sum_{\omega_1 \in \Omega_1} p_\theta(\omega_1) \sum_{\omega_2 \in \Omega_2} p_\theta(\omega_2 | \omega_1) \left( \frac{d \log p_\theta(\omega_1)}{d \theta} + \frac{d \log p_\theta(\omega_2 | \omega_1)}{d \theta} \right)^2 \]

\[ = \sum_{\omega_1 \in \Omega_1} p_\theta(\omega_1) \left( \frac{d \log p_\theta(\omega_1)}{d \theta} \right)^2 + \sum_{\omega_2 \in \Omega_2} p_\theta(\omega_2 | \omega_1) \left( \frac{d \log p_\theta(\omega_2 | \omega_1)}{d \theta} \right)^2 + 2 \sum_{\omega_2 \in \Omega_2} p_\theta(\omega_2 | \omega_1) \frac{d \log p_\theta(\omega_1)}{d \theta} \frac{d \log p_\theta(\omega_1)}{d \theta}. \]

However, the second term is vanished as

\[ \sum_{\omega_1 \in \Omega_1} \sum_{\omega_2 \in \Omega_2} p_\theta(\omega_2 | \omega_1) \frac{d \log p_\theta(\omega_2 | \omega_1)}{d \theta} \frac{d \log p_\theta(\omega_2 | \omega_1)}{d \theta} \]

\[ = \frac{d \log p_\theta(\omega_1)}{d \theta} \sum_{\omega_1 \in \Omega_1} p_\theta(\omega_1) \frac{d \log p_\theta(\omega_1)}{d \theta} \sum_{\omega_2 \in \Omega_2} p_\theta(\omega_2 | \omega_1) \frac{d \log p_\theta(\omega_1)}{d \theta} \]

\[ = 0. \]

Thus, we obtain (41). Moreover, we can easily check (42).

**APPENDIX E: PROOFS FOR SECTION VII**

1. proof of (55)

The L.H.S. of (55) can be calculated as

\[ \sum_{j=0}^{n/2} \binom{n}{j} \left( \frac{2j}{n} - r \right)^2 \]

\[ = 4 \sum_{k=0}^{[n/2]} P_{n,r} \left( \frac{n}{2} - k \right) \left( \frac{k}{n} - q(r) \right)^2 \]

\[ = 4 \left( q(r)^2 + \sum_{k=0}^{[n/2]} P_{n,r} (n/2 - k) \left( \frac{k^2}{n^2} - 2q(r) \frac{k}{n} \right) \right) , \]

where \( q(r) \overset{\text{def}}{=} \frac{1-r}{2} \). Since the probability \( P_{n,r} (\frac{n}{2} - k) \) has another expression: \( P_{n,r} (\frac{n}{2} - k) = \frac{1}{r} \left( \binom{n}{k} - \binom{n}{k-1} \right) q(r)^k (1 - q(r))^n - 1 \left( 1 - \frac{1-r}{1+r} \right)^{n-k} , \)
we can calculate the expectations of \( k \) and \( k^2 \) as follows.

\[
\sum_{k=0}^{[n/2]} k^2 P_{n,r}(\frac{n}{2} - k)
\]

\[
= \sum_{k=0}^{[n/2]} k^2 \frac{1}{r} \left( \binom{n}{k} - \binom{n}{k-1} \right) q(r)^k (1 - q(r))^{n-k+1} + O\left(\left(\frac{1-r}{1+r}\right)^{n/2}\right)
\]

\[
= \frac{1}{r} \sum_{k=0}^{[n/2]-1} k(k-1) + k \binom{n}{k} q(r)^k (1 - q(r))^{n-k+1} + O\left(\left(\frac{1-r}{1+r}\right)^{n/2}\right)
\]

\[
= \sum_{k=0}^{[n/2]} kP_{n,r}(\frac{n}{2} - k)
\]

\[
= \sum_{k=0}^{[n/2]} k \left( \binom{n}{k} - \binom{n}{k-1} \right) q(r)^k (1 - q(r))^{n-k+1} + O\left(\left(\frac{1-r}{1+r}\right)^{n/2}\right)
\]

\[
= \frac{1}{r} \sum_{k=0}^{[n/2]-1} k(k+1) + k \binom{n}{k} q(r)^k (1 - q(r))^{n-k+1} + O\left(\left(\frac{1-r}{1+r}\right)^{n/2}\right)
\]

Furthermore, every term appearing in the above equation is calculated as

\[
\sum_{k=0}^{[n/2]-1} k \binom{n}{k} q(r)^k (1 - q(r))^{n-k}
\]

\[
=\sum_{k=0}^{n} k \binom{n}{k} q(r)^k (1 - q(r))^{n-k} + O((1 - r^2)^{n/2})
\]

\[= np + O((1 - r^2)^{n/2})\]

\[
\sum_{k=0}^{[n/2]-1} k(k-1) \binom{n}{k} q(r)^k (1 - q(r))^{n-k}
\]

\[
=\sum_{k=0}^{n} k(k-1) \binom{n}{k} q(r)^k (1 - q(r))^{n-k} + O((1 - r^2)^{n/2})
\]

\[= n(n-1)p^2 + O((1 - r^2)^{n/2}).\]

Note that \((1 - r^2) > \frac{1}{1+r}\). Using there formulas, we obtain

\[
q(r)^2 + \sum_{k=0}^{[n/2]} P_{n,r}(n/2 - k) \frac{k^2}{n^2} - 2q(r)\frac{k}{n}
\]

\[= \frac{1-r^2}{4} \frac{1}{n} - \frac{1-r}{2r} \frac{1}{n^2} + O((1 - r^2)^{n/2}),\]

which implies \((55)\).}

\[\blacksquare\]

2. proof of \((57)\)

The left hand side of \((57)\) is calculated as

\[
\sum_{j=1}^{n} \frac{1}{2^n} \left( \binom{n}{\frac{n}{2} - j} - \binom{n}{\frac{n}{2} - j - 1} \right) (2j + 1) \left(\frac{2j}{n}\right)^2
\]

\[
= \sum_{k=1}^{[\frac{n}{2}]} \frac{1}{2^n} \left( \binom{n}{k} - \binom{n}{k-1} \right) (n - 2k + 1) \left(\frac{n-2k}{n}\right)^2
\]

\[+ \frac{1}{2^n} \binom{n}{0} (n - 2 \cdot 0 + 1) \left(\frac{n-2 \cdot 0}{n}\right)^2\]

\[
= \frac{1}{n^{2\cdot n}} \sum_{k=0}^{[\frac{n}{2}]} \binom{n}{k} (n - 2k + 1)(n - 2k)^2
\]

\[= \sum_{k=0}^{[\frac{n}{2}]-1} \binom{n}{k} (n - 2k + 1)(n - 2k)^2 \]

\[= \sum_{k=0}^{[\frac{n}{2}]-1} \binom{n}{k} (n - 2k - 1)(n - 2k - 2) \quad (E1)\]
When $n$ is even, $(n - 2(\frac{n}{2}) + 1)(n - 2(\frac{n}{2}))^2 = 0$. Then, the above value are calculated

$$\sum \frac{1}{n} \left( \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor - 1} \frac{n}{k} \right)(n - 2k + 1)(n - 2k)^2$$

$$= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{k} (n - 2k - 1)(n - 2k - 2)^2$$

$$= \frac{1}{n^{2/2n}} \left( \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{k} \right) 6(n - 2k)^2 - 8(n - 2k) + 4 \right).$$

The first term is calculated as

$$\frac{1}{2n} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{k} 6(n - 2k)^2$$

$$= \frac{1}{2n+1} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{k} 6(2n(k/2)) - (\frac{n}{2}) 6(n - 2\frac{n}{2})^2$$

$$= \frac{1}{2} \cdot 6 \cdot 4n^2 \cdot \frac{1}{4} = 3n.$$

Since $\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{k} = \binom{n-1}{\lfloor \frac{n}{2} \rfloor}$, we have

$$\frac{1}{2n} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{k} = \frac{1}{2n+1} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{k} = \frac{n}{4}$$

$$\frac{1}{2n} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{k} = \frac{1}{2n+1} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{k} = \frac{n}{2} \cdot \frac{1}{2n+1}$$

$$\frac{1}{2n} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{k} = \frac{1}{2} + \frac{n}{2} \cdot \frac{1}{2n+1}.$$

Thus,

$$\frac{1}{2n} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{k} (-8(n - 2k) + 4)$$

$$= \frac{1}{2n} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{k} (-8(n - 2k)) \frac{1}{2} = \frac{n}{2} \cdot \frac{1}{2n+1}$$

$$= -8(n - 2) \frac{n}{2} + \frac{n}{2} \cdot \frac{1}{2n+1} + 4\left( \frac{n}{2} \right) \cdot \frac{n}{2} \cdot \frac{1}{2n+1}$$

$$= (-8n - 4) \frac{n}{2} \cdot \frac{1}{2n+1} + 2.$$

Since $\left( \frac{n}{2} \right) \sqrt{\pi n} \cong \sqrt{\frac{2\pi n}{n^{2/2n}}}$, we have

$$\frac{3}{n} - 4\left( \frac{1}{n} + \frac{1}{n^2} \right) \left( \frac{n}{2} \right) \cdot \frac{1}{2n+1} + \frac{2}{n^2} \cong \frac{3}{n} - \frac{4\sqrt{2}}{\sqrt{\pi n}} \cdot \frac{1}{n^{2/2n}} + \frac{2}{n^2}$$

When $n$ is odd, $(n - 2(\frac{n+1}{2}) - 1)(n - 2(\frac{n+1}{2}) - 2)^2 = 0$. Then, the above value are calculated

$$\sum \frac{1}{n} \left( \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{k} \right)(n - 2k + 1)(n - 2k)^2$$

$$= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{k} (n - 2k - 1)(n - 2k - 2)^2$$

$$= \frac{1}{n^{2/2n}} \left( \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{k} \right) 6(n - 2k)^2 - 8(n - 2k) + 4 \right).$$

The first term is calculated as

$$\frac{1}{2n} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{k} 6(n - 2k)^2$$

$$= \frac{1}{2n+1} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{k} 6(2n(k/2)) - (\frac{n}{2}) 6(n - 2\frac{n}{2})^2$$

$$= \frac{1}{2} \cdot 6 \cdot 4n^2 \cdot \frac{1}{4n} = 3n.$$

Since $\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{k} = \sum_{k=\lfloor \frac{n}{2} \rfloor+1}^{n-1} \binom{n}{k}$, we have

$$\frac{1}{2n} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{k} = \frac{1}{2n+1} \sum_{k=\lfloor \frac{n}{2} \rfloor+1}^{n-1} \binom{n}{k} = \frac{n}{2} \cdot \frac{n}{2n+1}$$

$$= \frac{1}{2} + \frac{n}{2} \cdot \frac{1}{2n+1}$$

$$\frac{1}{2n} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{k} = \frac{1}{2} + \frac{n}{2} \cdot \frac{1}{2n+1}.$$

Thus,

$$\frac{1}{2n} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{k} (-8(n - 2k) + 4)$$

$$= \frac{1}{2n} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{k} (-8(n - 2k)) \frac{1}{2} = \frac{n}{2} \cdot \frac{1}{2n+1}$$

$$= -8(n - 2) \frac{n}{2} + \frac{n}{2} \cdot \frac{1}{2n+1} + 4\left( \frac{n}{2} \right) \cdot \frac{n}{2} \cdot \frac{1}{2n+1}$$

$$= (-8n - 4) \frac{n}{2} \cdot \frac{1}{2n+1} + 2.$$

Since $\left( \frac{n}{2} \right) \sqrt{\pi n} \cong \sqrt{\frac{2\pi n}{n^{2/2n}}}$, we have

$$\frac{3}{n} - 4\left( \frac{1}{n} + \frac{1}{n^2} \right) \left( \frac{n}{2} \right) \cdot \frac{1}{2n+1} + \frac{2}{n^2} \cong \frac{3}{n} - \frac{4\sqrt{2}}{\sqrt{\pi n}} \cdot \frac{1}{n^{2/2n}} + \frac{2}{n^2}$$

3. Proof of Lemma 5

First, we parameterize the square root of $G$ as

$$\sqrt{G} = \begin{pmatrix} A & a \\ a^T & t \end{pmatrix}.$$
where $A$ is a $2 \times 2$ symmetric matrix and $a$ is a 2-dimensional vector.

$$C^R_{(0,0,r)}(G)$$

$$= \text{tr} \left( G - r^2 s^2 + r \text{tr} \begin{pmatrix} A & a \\ a^T & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} A & a \\ a^T & 0 \end{pmatrix} \right).$$

By putting $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, we can calculated the second term as:

$$\text{tr} \begin{pmatrix} A & a \\ a^T & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} A & a \\ a^T & 0 \end{pmatrix}$$

$$= \text{tr} \begin{pmatrix} \text{det}(A)J & AJa \\ (AJa)^T & 0 \end{pmatrix}$$

$$= 2\sqrt{\text{det}(A)^2} + \|AJa\|^2 = 2\sqrt{\text{det}(A)^2} + (Ja|A^2|Ja)$$

$$= 2\sqrt{\text{det}(A^2 + |a\rangle\langle a|)},$$

where the final equation can be checked by choosing a basis such that $a = \begin{pmatrix} |a\rangle \\ 0 \end{pmatrix}$. Since $\tilde{G} = A^2 + |a\rangle\langle a|$, we obtain (60).

4. Proof of (62)

First, we focus the following expressions of $C^R_{(0)}(G)$ and $\hat{C}_0(G)$

$$\hat{C}_0(G) = \left( \text{tr} \sqrt{GJ_0^{-1}\sqrt{G}} \right)^2$$

$$C^R_{(0)}(G) = \text{tr} \sqrt{GJ_0^{-1}\sqrt{G}} + \text{tr} \left[ 2\sqrt{GJ_0^{-1}D_0J_0^{-1}\sqrt{G}} \right].$$

When we put the real symmetric matrix $A \overset{\text{def}}{=} \sqrt{GJ_0^{-1}\sqrt{G}}$ and the real anti-symmetric matrix $B \overset{\text{def}}{=} 2\sqrt{GJ_0^{-1}D_0J_0^{-1}\sqrt{G}}$, the relation

$$A + iB \geq 0.$$  \hspace{1cm} (E5)

Here, we dragonize $A$ as

$$A = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$$

with $a \geq b \geq c$ and $c > 0$, where the final strict inequality follows from $G > 0$. Since $|B|$ is a constant times of a two-dimensional projection $P$. Hence,

$$PAP + iB = P(A + iB)P \geq 0.$$  \hspace{1cm} (E7)

If we regard $PAP$ as a two-dimensional matrix, $\text{tr}|B| \leq 2\sqrt{\text{det}PAP}$. Thus, by considering the maximum case of the minimum eigen value of $PAP$, we have

$$\text{tr}|B| \leq 2\sqrt{ab}$$

Therefore,

$$\hat{C}_0(G) - C^R_{(0)}(G) = (\text{tr} \sqrt{A})^2 - (\text{tr} A + \text{tr}|B|) \geq 2 \left( \sqrt{ab} + \sqrt{bc} + \sqrt{ca} - \sqrt{ab} \right) = 2 \left( \sqrt{bc} + \sqrt{ca} \right) > 0.$$  \hspace{1cm} \blacksquare

5. Proofs of (67) and (68)

Since

$$\int_{\mathbb{R}} \hat{\theta}^k V_j^* \hat{M}_{\hat{G}}(d\hat{\theta})V_j = \left\{ \begin{array}{ll} Q & k = 1 \\ P & k = 2 \end{array} \right.,$$

we have

$$\text{Tr}(\rho_{j,p} \circ 2J_{j,k}) \int_{\mathbb{R}} \hat{\theta}^k M_{j,G}(d\hat{\theta}) = \delta_{k}^{l}$$

for $k,l = 1,2$, where

$$M_{j,G}(d\hat{\theta}) \overset{\text{def}}{=} V_j^* M_G \circ B_j^{-1}(d\hat{\theta})V_j.$$

Since the matrices $\rho_{j,p} \circ \frac{1}{1 - r^2}(2J_{j,3} - rI)$ and $\rho_{j,p}$ are diagonal and all diagonal elements of $V_j^*QV_j$ and $V_j^*PV_j$ are 0, we have

$$\text{Tr}(\rho_{j,p} \circ \frac{1}{1 - r^2}(2J_{j,3} - rI)V_j^*QV_j = \text{Tr} \rho_{j,p}V_j^*QV_j = 0$$

$$\text{Tr}(\rho_{j,p} \circ \frac{1}{1 - r^2}(2J_{j,3} - rI)V_j^*PV_j = \text{Tr} \rho_{j,p}V_j^*PV_j = 0.$$  \hspace{1cm} (E9)

Thus,

$$\text{Tr}(\rho_{j,p} \circ \frac{1}{1 - r^2}(2J_{j,3} - rI)) \int_{\mathbb{R}} \hat{\theta}^k M_{j,G}(d\hat{\theta})$$

$$= \text{Tr} \rho_{j,p} \int_{\mathbb{R}} \hat{\theta}^k M_{j,G}(d\hat{\theta}) = 0$$

for $l = 1,2$. Therefore,

$$\text{Tr} \frac{\partial \rho_{n,r}^{\otimes n}}{\partial \theta^k} \left( \bigoplus_j \left( \int_{\mathbb{R}} \hat{\theta}^k M_{j,G}(d\hat{\theta}) \right) \otimes I_{H_{n,j}} \right)$$

$$= \sum_j P_{n,r}(j) \text{Tr}(\rho_{j,p} \circ 2J_{j,k}) \int_{\mathbb{R}} \hat{\theta}^k M_{j,G}(d\hat{\theta})$$

for $k,l = 1,2$. For the $k = 3$ case, the above quantity equals

$$\sum_j P_{n,r}(j) \text{Tr}(\rho_{j,p} \circ \frac{1}{1 - r^2}(2J_{j,3} - rI)) \int_{\mathbb{R}} \hat{\theta}^k M_{j,G}(d\hat{\theta}) = 0.$$  \hspace{1cm} (E10)

Furthermore, we have

$$\text{Tr} \rho_{(0,0,r)}^{\otimes n} \left( \bigoplus_j \left( \int_{\mathbb{R}} \hat{\theta}^k M_{j,G}(d\hat{\theta}) \right) \otimes I_{H_{n,j}} \right)$$

$$= \sum_j P_{n,r}(j) \text{Tr} \rho_{j,p} \int_{\mathbb{R}} \hat{\theta}^k M_{j,G}(d\hat{\theta}) = 0$$  \hspace{1cm} (E10)

for $l = 1,2$.  \hspace{1cm} \blacksquare
6. Proof of (70)

First, we focus on the following equation

\[
\frac{n}{1 - r^2} = \text{Tr} \rho_{(0,0,r)}^{\otimes n} \left( \bigoplus_j \frac{1}{1 - r^2} (2J_{j,3} - r I) \otimes I_{\mathcal{H}_{n,j}} \right)
\]

= \sum_j P_{n,r}(j) \sum_{m=-j}^j \frac{1 - p}{1 - p^{j+1}} p^{j-m} \left( \frac{1}{1 - r^2} \right)^2 (2m - r)^2.

Then, applying Theorem 10, we can see that the difference \( \frac{n}{1 - r^2} - J_{n,r} \) equals information loss. Thus,

\[
\frac{n}{1 - r^2} - J_{n,r} = \sum_j P_{n,r}(j) \tilde{J}_{j,r},
\]

where

\[
\tilde{J}_{j,r} = \sum_{m=-j}^j \frac{1 - p(r)}{1 - p(r)^{j+1}} p(r)^{j-m} \left( \frac{2m - r}{1 - r^2} \right)^2
\]

\[
- \left( \sum_{m=-j}^j \frac{1 - p(r)}{1 - p(r)^{j+1}} p(r)^{j-m} \left( \frac{2m - r}{1 - r^2} \right)^2 \right).
\]

\[
= \frac{4(1 - p(r))}{(1 - p(r)^{j+1})(1 - r^2)^2} \left( \frac{p(r) + p(r)^2}{1 - p(r)} \right)^3 - \left( 1 + p(r) \right) p(r)^{j+1} \frac{1}{(1 - p(r))^3}
\]

\[
- \frac{(4j + (2j)^2(1 - p(r))) p(r)^{j+1}}{(1 - p(r))^2}
\]

\[
- \frac{4(1 - p(r))^2}{(1 - p(r)^{j+1})^2(1 - r^2)^2} \times \left( \frac{p(r)(1 - p(r)^{2j})}{(1 - p(r))^2} - \frac{2jp(r)^{2j+1}}{1 - p(r)} \right)^2.
\]

\[
= \frac{1}{r^2(1 - r^2)} + O(p(r)^{2j}).
\]

which implies (70).

7. Proof of (71)

For the covariance of the POVM \( M_{\text{cov}}^n \), the Fisher information matrix \( J_{(0,0,0),0}^M \) is a scalar times of the identical matrix. We apply Theorem 32 to the family of probability distributions \( p_r(j, \phi, \psi) \) \( \text{Tr} \rho_{(0,0,r)}^{\otimes n} M^j(\phi, \psi) \otimes I_{\mathcal{H}_{n,j}} = \)
On the other hand, Applying Theorem 10 to $P_{n,r}(j)\Tr\rho_{j,p}M^j(\phi,\psi)$. Then, we calculate the Fisher information:

$$J_{\text{cov}}^n = \sum_j P_{n,r}(j) \left( \frac{dP_{n,r}(j)}{dr} \right)^2 + \sum_j P_{n,r}(j) \int \left( \frac{d\Tr\rho_{j,p}M^j(\phi,\psi)}{dr} \right)^2 \Tr\rho_{j,p}M^j(\phi,\psi) \, d\phi \, d\psi.$$ 

On the other hand, Applying Theorem 10 to $P_{n,r}(j)\langle j, m|\rho_{j,p}|j, m\rangle$, we have

$$J_{\text{cov}}^n = \sum_j \left( \frac{dP_{n,r}(j)}{dr} \right)^2 P_{n,r}(j) + \sum_j P_{n,r}(j) \sum_{m=-j}^j \left( \frac{d\langle j, m|\rho_{j,p}|j, m\rangle}{dr} \right)^2 \langle j, m|\rho_{j,p}|j, m\rangle.$$ 

Thus,

$$J_{\text{cov}}^n = \frac{n}{1-r^2} - \sum_j P_{n,r}(j) \left( \int \left( \frac{d\Tr\rho_{j,p}M^j(\phi,\psi)}{dr} \right)^2 \Tr\rho_{j,p}M^j(\phi,\psi) \, d\phi \, d\psi \right) - \sum_{m=-j}^j \left( \frac{d\langle j, m|\rho_{j,p}|j, m\rangle}{dr} \right)^2 \langle j, m|\rho_{j,p}|j, m\rangle.$$ 

In the case of $r = 0$, Since $\Tr\rho_{j,p}M^j(\phi,\psi) = (2j+1)\frac{1-p(r)}{1-p(r)} \frac{1+r\cos\phi}{1+r} \frac{2j\sin\phi}{4\pi}$, we obtain

$$J_{\text{cov}}^n = \int_0^{2\pi} \int_0^\pi \left( \frac{d\log \frac{1+p(r)}{1-p(r)} \frac{1+r\cos\phi}{1+r} \frac{2j\sin\phi}{4\pi}}{dr} \right)^2 \Tr\rho_{j,p}M^j(\phi,\psi) \, d\phi \, d\psi.$$ 

Its first and second terms are calculated as

$$\int_0^{2\pi} \int_0^\pi \left( \frac{d\log \frac{1+r\cos\phi}{1+r}}{dr} \right)^2 \frac{2j}{2j+1} \frac{1-p(r)}{1-p(r)^{2j+1}} \frac{1+r\cos\phi}{1+r} \frac{2j\sin\phi}{4\pi} \, d\phi \, d\psi = \int_0^\pi \frac{1}{2} (2j+1) \frac{1-p(r)}{1-p(r)^{2j+1}} \frac{1+r\cos\phi}{1+r} \frac{2j\sin\phi}{4\pi} \, d\phi \, d\psi = \int_0^\pi \frac{1}{2} (2j+1) \frac{1-p(r)}{1-p(r)^{2j+1}} \frac{1+r\cos\phi}{1+r} \frac{2j\sin\phi}{4\pi} \, d\phi \, d\psi.$$ 

we have

$$J_{\text{cov}}^n = n - \sum_j P_{n,0}(j) \frac{4}{3} j.$$ 

Therefore, if the relation

$$- \sum_{m=-j}^j \left( \frac{d\langle j, m|\rho_{j,p}|j, m\rangle}{dr} \right)^2 \langle j, m|\rho_{j,p}|j, m\rangle = \frac{4}{3} j (j+1),$$

we have

$$P_{n,0}(j) \frac{4}{3} j \approx \frac{4\sqrt{2}}{3\sqrt{\pi}} \sqrt{n} + \frac{2}{3}$$

(E11)
holds, we obtain (E11). Hence, in the following, we will prove (E14).

\[
\sum_j P_{n,0}(j) \frac{4}{3} j
\]
\[
= \frac{2}{3} \sum_j \frac{1}{2^n} \left( \binom{n}{\frac{n}{2} - j} - \binom{n}{\frac{n}{2} - j - 1} \right) 2j(2j + 1)
\]
\[
= \frac{2}{3} \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \frac{1}{2^n} \left( \binom{n}{k} - \binom{n}{k-1} \right) (n - 2k + 1)(n - 2k)
\]
\[
+ \frac{2}{3} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{1}{2^n} \binom{n}{k} (0 - 2 \cdot 0 + 1)(n - 2 \cdot 0)
\]
\[
= \frac{2}{3} 2^n \left( \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{k} (n - 2k + 1)(n - 2k)
\]
\[
- \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor - 1} \binom{n}{k} (n - 2k - 1)(n - 2k - 2) \right)
\]
\[
= \frac{2}{3} 2^n \left( \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{k} 8(n - 2k - 4) \right) \approx \frac{4 \sqrt{2}}{3 \sqrt{n}} \sqrt{n} + \frac{2}{3}.
\]

When \( n \) is even, \((n - 2(\frac{n}{2}) + 1)(n - 2(\frac{n}{2}))^2 = 0 \). Then, the above value are calculated

\[
\sum_j P_{n,0}(j) \frac{4}{3} j
\]
\[
= \frac{2}{3} 2^n \left( \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{k} (n - 2k + 1)(n - 2k)
\]
\[
- \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor - 1} \binom{n}{k} (n - 2k - 1)(n - 2k - 2) \right)
\]
\[
= \frac{1}{32} 2^n \left( \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{k} 8(n - 2k - 4) \right) \approx \frac{4 \sqrt{2}}{3 \sqrt{n}} \sqrt{n} + \frac{2}{3}.
\]

When \( n \) is odd, \((n - 2(\frac{n}{2}) - 1)(n - 2(\frac{n}{2}) - 2)^2 = 0 \). Then, the above value are calculated

\[
\sum_j P_{n,0}(j) \frac{4}{3} j
\]
\[
= \frac{2}{3} 2^n \left( \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{k} (n - 2k + 1)(n - 2k)
\]
\[
- \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor - 1} \binom{n}{k} (n - 2k - 1)(n - 2k - 2) \right)
\]
\[
= \frac{1}{32} 2^n \left( \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{k} 8(n - 2k - 4) \right) \approx \frac{4 \sqrt{2}}{3 \sqrt{n}} \sqrt{n} + \frac{2}{3}.
\]

which implies (E11). Hence, by putting

\[
d_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad d_2 = \begin{pmatrix} 0 & \cos \phi \\ \sin \phi & \sin \phi \end{pmatrix},
\]

we obtain

\[
C_{s}^H (G) = \min_{v \in [0,1]} \left\{ \text{tr} | \sqrt{G}Z_{J}(v)\sqrt{G} | \Re \{d_k|J|v^k \} = \delta^k \right\},
\]

where

\[
J \equiv \begin{pmatrix} 1 & -ir & 0 \\ ir & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.
\]

Hence, from the condition

\[
\langle d_j | J | v^k \rangle = \delta_j^k.
\]

Then, \( v^1 \) and \( v^2 \) are parameterized as

\[
v^1 = L_1 - t \sin \phi L_2 + t \cos \phi L_3
\]
\[
v^2 = (\bar{s} \cos \phi + s \phi) L_2 + (\bar{s} \cos \phi + s \phi) L_3.
\]

The matrix \( Z_{J}(v) \) can be calculated as

\[
\begin{pmatrix} 1 + t^2 & ts - ir(-s \sin \phi + \cos \phi) \\ ts + ir(-s \sin \phi + \cos \phi) & 1 + s^2 \end{pmatrix}.
\]

Thus, the quantity \( \text{tr} | \sqrt{G}Z_{J}(v)\sqrt{G} | \) equals

\[
\text{tr} G + g_1 (t + \frac{g_2}{g_1})^2 + \frac{\text{det} G}{g_1} s^2 + 2r | \cos \phi - \sin \phi s | \sqrt{\text{det} G}.
\]

(E14)

In the following, we treat the case of \( \frac{g_2}{\sqrt{\text{det} G}} < \frac{\cos \phi}{\sin \phi} \).

The minimum value of (E14) equals \( \text{tr} G + 2r \cos \phi \det G - r^2 \sin^2 \phi g_1 \) which is attained by the parameters \( t = \frac{-2s g_1}{g_1}, s = \frac{\cos \phi}{\sin \phi} \). Thus, the discussion in subsection (E13) guarantees that the Holevo bound is attained only by the following covariance matrix

\[
\text{Re} \{Z_{\phi}(\bar{X}) + \sqrt{G}^{-1} | \sqrt{G} \text{Im} \{Z_{\phi}(\bar{X}) \sqrt{G} \} | + r | -s \sin \phi + \cos \phi | \sqrt{\text{det} G} G^{-1}
\]
\[
= \begin{pmatrix} 1 + t^2 & ts \\ ts & 1 + s^2 \end{pmatrix} + r | -s \sin \phi + \cos \phi | \sqrt{\text{det} G} G^{-1}
\]

=R.H.S. of (E12).

In the opposite case, the minimum value of (E14) equals R.H.S. of (E13), which is attained by the parameters \( t = -2s g_1, s = \frac{\cos \phi}{\sin \phi} \). Substituting these parameters into (E15), we obtain (E14).
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