A NOTE ON INTEGRABLE MECHANICAL SYSTEMS ON SURFACES

LEO T. BUTLER
Department of Mathematics, Central Michigan University
Mount Pleasant, MI, 48859, USA

(Communicated by Kuo-Chang Chen)

ABSTRACT. Let \( \mathcal{S} \) be a compact, connected surface and \( H \in C^2(T^*\mathcal{S}) \) a Tonelli Hamiltonian. This note extends V. V. Kozlov’s result on the Euler characteristic of \( \mathcal{S} \) when \( H \) is real-analytically integrable, using a definition of topologically-tame integrability called semisimplicity. Theorem: If \( H \) is 2-semisimple, then \( \mathcal{S} \) has non-negative Euler characteristic; if \( H \) is 1-semisimple, then \( \mathcal{S} \) has positive Euler characteristic.

1. Introduction.

1.1. Kozlov’s Theorem. Say that a natural mechanical system is a Hamiltonian that is a sum of kinetic and potential energies. Let \( \mathcal{S} \) be a compact surface and \( H : T^*\mathcal{S} \to \mathbb{R} \) be an analytic natural mechanical system. Kozlov proved if \( H \) enjoys a second, independent analytic integral \( F \), then the Euler characteristic of \( \mathcal{S} \) is non-negative and so it is homeomorphic to \( S^2 \), \( T^2 \) or a non-orientable quotient thereof [10]. The hypotheses of Kozlov’s theorem can be relaxed as follows: (i) \( H \) need only be assumed to be fibre-wise strictly convex and super-linear (i.e. “Tonelli”); (ii) analyticity can be reduced to the combined hypotheses that \( H \) and \( F \) are \( C^2 \), and that there is an energy level \( H^{-1}(c) \) where \( c > \min \{ H(x, 0) \mid x \in \mathcal{S} \} \), such that the critical set of \( F \) intersects a fibre of the foot-point projection in only finitely many points [11].

The present note has two aims. First, it presents a proof of Kozlov’s theorem based on the theory of semisimplicity developed in [4]; see definition 1.1 below. Second, it extends Kozlov’s theorem to non-commutatively integrable Tonelli Hamiltonians. The latter is a non-trivial extension: in [5] there are Tonelli Hamiltonians that are constructed which are non-commutatively integrable and semisimple on the unit disk bundle, but which are not tangent to a semisimple singular Lagrangian fibration. In essence, the naïve trick of discarding extra integrals to achieve complete integrability necessarily expands the critical set, and in the above-quoted example, the critical set expands from a real-analytic set to a wild set analogous to the Fox wild arc.

2010 Mathematics Subject Classification. Primary: 37J35; secondary: 70H06.
Key words and phrases. Hamiltonian mechanics, integrability, topological obstructions.
The author thanks G. Knieper for his helpful comments and suggestions.
1.2. **Non-commutative integrability.** Let Σ be a smooth n-dimensional manifold. The canonical Poisson structure on the cotangent bundle $T^*\Sigma$ permits one to define a Poisson algebra structure on $C^\infty(T^*\Sigma)$ and consequently each smooth function $H : T^*\Sigma \to \mathbb{R}$ induces a Hamiltonian vector field $X_H$ defined by

$$X_H = \{H, \cdot\} = \sum_{i=1}^{n} \frac{\partial H}{\partial x^i} \frac{\partial}{\partial y^i} + \frac{\partial H}{\partial y^i} \frac{\partial}{\partial x^i},$$

where $(x^i, y^i)$ are canonical coordinates. A first integral of the Hamiltonian vector field $X_H$ is a smooth function $F$ which Poisson commutes with $H$: $\{H, F\} = 0$.

For a subspace $\mathfrak{A} \subset C^\infty(T^*\Sigma)$ and $p \in T^*\Sigma$, let $d_p\mathfrak{A} = \text{span}\{df(p) \mid f \in \mathfrak{A}\}$. Following [3], the differential dimension of $\mathfrak{A}$ is defined to be $\sup_p \dim d_p\mathfrak{A}$. Let $\mathfrak{A} \subset C^\infty(T^*\Sigma)$ be a subspace of first integrals of $H$ that contains $\hat{H}$ and let $Z(\mathfrak{A})$ be the subspace of $\mathfrak{A}$ which Poisson commutes with all of $\mathfrak{A}$. Let $k$ (resp. $l$) be the differential dimension of $\mathfrak{A}$ (resp. $Z(\mathfrak{A})$). Say that a point $p \in T^*\Sigma$ is regular for $\mathfrak{A}$ if $\dim d_p\mathfrak{A} = k$ and $\dim d_pZ(\mathfrak{A}) = l$ for all $q$ in the $\mathfrak{A}$-level set passing through $p$. We say that $H$ (or $\mathfrak{A}$) is non-commutatively integrable if $k + l = 2n$ and the set of regular points is dense.

Nehoro˘sev [13], who generalized the Liouville-Arnol’d theorem [1], proved that if $H$ is non-commutatively integrable and $p$ is a regular point, then there is a neighbourhood $U$ with coordinate chart $(\theta, I, x, y) : U \to T^k \times \mathbb{R}^k \times \mathbb{R}^{2(n-k)}$, where the Poisson bracket is canonical and $H = H(I)$. Dazord and Delzant [7] generalized this result and showed that the regular point set $X \subset T^*\Sigma$ is fibred by isotropic $k$-dimensional tori $F = T^k$ and the quotient of $X$ by these fibres, $P$, is a Poisson manifold with a foliation $\zeta$ by symplectic leaves. When this foliation is a fibration, one has the following diagram

$$\begin{align*}
F & \xleftarrow{i_\ast} X \xrightarrow{i_X} T^*\Sigma \\
S & \xleftarrow{i_P} P \xrightarrow{\pi_P} Q
\end{align*}$$

where $i_\ast$ is an inclusion map, $g$ is the fibration map, $S$ is a symplectic leaf of $P$, $Q$ is an integral affine manifold of dimension $k$ and $\pi_P C^\infty(Q)$ is the centre of $C^\infty(P)$ which induces the Hamiltonian vector fields that are tangent to the isotropic fibres of $g$.

1.3. **Geometric semisimplicity.** Let us abstract the notion of complete and non-commutative integrability. A smooth flow $\varphi : M \times \mathbb{R} \to M$ is integrable if there is an open, dense subset $R \subset M$ that is covered by angle-action charts which conjugate $\varphi$ to a translation-type flow on the tori of $T^k \times \mathbb{R}^l$. There is an open dense subset $L \subset R$ fibred by $\varphi$-invariant tori; let $f : L \to B$ be the induced smooth quotient map and let $\Gamma = M - L$ be the singular set. If $\Gamma$ is a tamely-embedded polyhedron, then $\varphi$ is said to be $k$-semisimple with respect to $(f, L, B)$, or just semisimple [4]. Of most interest is when $\varphi$ is a Hamiltonian flow on a cotangent bundle or possibly a regular iso-energy surface.

**Definition 1.1** (c.f. [14, 4]). A Hamiltonian flow is geometrically $k$-semisimple if it is $k$-semisimple with respect to $(f, L, B)$ and $f$ is an isotropic fibration.

In this case, because the fibres of $f$ are isotropic, $\varphi$ is non-commutatively integrable, so geometric semisimplicity is a topologically-tame type of non-commutative
integrability. Taimanov [14] introduced a related notion of geometric simplicity, see sections 2.2-2.3 of [4] for further discussion. If $\varphi$ is real-analytically non-commutatively integrable, then the triangulability of real-analytic sets implies that $\varphi$ is geometrically semisimple; and $B$ may be taken to be a disjoint union of open balls. On the other hand, geometric semisimplicity is a weaker property than real-analytic non-commutative integrability [4]. A basic question is:

**Question 1.** What are the obstructions to the existence of a geometrically semisimple (resp. semisimple, completely integrable) flow?

**1.4. Results.** Here are the two main theorems of this note. In both cases, $\mathcal{G}$ is a compact, connected surface and $H : T^*\mathcal{G} \to \mathbb{R}$ is a $C^2$ Tonelli Hamiltonian.

**Theorem 1.2** (c.f. Kozlov [10]). If $H$ is 1- or 2-semisimple, then $\mathcal{G}$ has non-negative Euler characteristic.

**Theorem 1.3.** If $H$ is 1-semisimple, then $\mathcal{G}$ is homeomorphic to $S^2$ or $\mathbb{R}P^2$.

**Remark 1.** In two degrees of freedom, as here, 1- or 2-semisimplicity implies the fibres of the fibration are isotropic, so 1- or 2-semisimplicity implies geometric 1- or 2-semisimplicity.

**Remark 2.** In an earlier version of this note [6], Theorem 1.3 was proven under the additional hypothesis that $H$ is reversible. That proof uses a result of Glasmachers & Knieper [9, 8] concerning a zero-entropy geodesic flow of a reversible Finsler metric on $T^2$. The present note dispenses with the reversibility hypothesis.

**Remark 3.** Bangert has asked a series of questions in [12] concerning integrable Tonelli Hamiltonians which are integrable in a weaker sense—the additional integral need only be independent of $H$ on an open dense set. These questions are very interesting but beyond the scope and techniques of this paper. Likewise, the paper by Bialy proves an extension of Kozlov’s theorem for geodesic flows when the additional integral satisfies his condition \( \aleph \) [2]. That work relies on properties of minimizing geodesics.

### 2. Preliminaries

Let us recall a few items concerning a geometric semisimplicity.

Let $\Sigma$ be a compact smooth manifold and $H : T^*\Sigma \to \mathbb{R}$ be a $C^2$ Tonelli Hamiltonian that is geometrically semisimple with respect to $(f, L, B)$. The complement $\Gamma = T^*\Sigma - L$ is a tamely embedded polyhedron, so the number of components of $L \cap H^{-1}((-\infty, c])$ is finite for any $c$. [4, Lemma 15] implies that there is a component $L_i \subset L$ such that $\pi_1(L_i)$ has a finite-index image in $\pi_1(\Sigma)$:

$$
\pi_1(L_i) \xrightarrow{\pi_1(L_i, \ast)} \pi_1(T^*\Sigma) \xrightarrow{\pi_*} \pi_1(\Sigma).
$$

(3)

Suppose that $B_0 \subset B$ is a nowhere dense subset such that $f^{-1}(B_0) \cup \Gamma = \Gamma_1$ is tamely embedded polyhedron whose complement $L_1 = f^{-1}(B_1)$, $B_1 = B - B_0$, is dense. One calls $(f_1 = f|L_1, L_1, B_1)$ a refinement of $(f, L, B)$. In [4, Lemma 18] it is proven that

**Proposition 1.** If $\dim B \leq 2$, then $(f, L, B)$ has a refinement $(f_1, L_1, B_1)$ such that each component of $B_1$ is homotopy equivalent to either a point or $S^1$. 

Figure 1. Schematic proof of 1: We cut the base $B$ along the blue curves; the ends $E_j$ are cylinders and the “compact” part of $B_1$ is a union of disks.

Let $0(\Sigma) \subset T^*\Sigma$ be the zero section of the cotangent bundle of $\Sigma$. An exact Lagrangian graph $\Lambda \subset T^*\Sigma$ is the graph of an exact 1-form; $0(\Sigma)$ is an example. A Tonelli Hamiltonian $H : T^*\Sigma \to \mathbb{R}$ is fibre-wise strictly convex and grows super-linearly in the fibres. Consequently, there is a $c \in \mathbb{R}$ such that $\{ H \leq c \}$ contains $0(\Sigma)$ and therefore an exact Lagrangian graph. Let $c$ be the infimum of the set of $c$ such that $\{ H \leq c \}$ contains an exact Lagrangian graph; this is Mañé's critical value. For all $c > c$, the sublevel set $\{ H \leq c \}$ is fibre-wise strictly convex and contains an exact Lagrangian graph. The Hamiltonian flow of $H$ on an energy level $c$ above $c$ is, up to an orbit equivalence, the geodesic flow of a Finsler metric.

3. Proofs. Proposition 1 allows us to prove Theorem 1.2. The manifold $\Sigma$ in the previous section is the surface $\mathcal{S}$.

Proof of Theorem 1.2. Suppose that $H$ is geometrically 2-semisimple. We will deal with the case of 1-semisimplicity below. By Proposition 1 we can suppose that each component of the base of the fibration $f$ is homotopy equivalent to a point or a circle. Thus, each component of $L_i \subset L$ is homotopy equivalent to $\mathbb{T}^2$ or a $\mathbb{T}^2$-bundle over $\mathbb{T}^1$. In both cases, $\pi_1(L_i)$ is solvable, and so $\pi_1(\mathcal{S})$ contains a solvable subgroup of finite index. Since $\mathcal{S}$ is a surface, the theorem is proved.

Proof of Theorem 1.3. To prove Theorem 1.3, we must adapt the diagram in (2) to our needs. In this case, the fibre $F = \mathbb{T}^1$, the base of the fibration $P(= B)$ is a 3-dimensional Poisson manifold with a foliation $\zeta$ by symplectic surfaces $S$. The foliation $\zeta$ is a fibration, in fact, because the Casimirs of $P$ are functionally dependent on the reduction of the Hamiltonian $H|X$. In this case, the quotient of $P$ by $\zeta$, $Q$, is a finite union of 1-manifolds: $Q = \cup_i Q_i$, where $Q_i \simeq \mathbb{R}$ or $\mathbb{T}^1$. Since $H|X = h \circ G$ for some $h \in C^2(Q)$, the Tonelli property of $H$ implies that no component of $Q$ is a circle.
It follows that there is a component $X_i = G^{-1}(q_i)$ such that
\[ \pi_1(X_i) \xrightarrow{i_{X_i,*}} \pi_1(T^*S) \xrightarrow{\pi_*} \pi_1(\mathcal{S}) \]
has a finite index image. Since $X_i$ is homotopy equivalent to an $F = T^1$-principal bundle over the symplectic surface $S_i$, a leaf of $\zeta|X_i$, it remains to examine the possibilities.

$S_i$ is compact. In this case, $G^{-1}(q)$ is compact for any $q \in Q_i$, and therefore it must be a connected component of an energy level. Since, above the critical value, the energy levels are connected, $G^{-1}(q)$ is an energy level. If the Euler characteristic of $S$ is negative, then $\pi_1(\mathcal{S})$ contains no non-trivial normal abelian subgroups. Therefore, the inclusion $F \hookrightarrow T^*S$ is null-homotopic; this implies that all orbits of the Tonelli Hamiltonian in a super-critical energy surface are contractible–absurd. Therefore, the Euler characteristic of $\mathcal{S}$ must be non-negative.

Since every orbit of the Tonelli Hamiltonian is closed on a super-critical energy level, the Euler characteristic of $\mathcal{S}$ must be positive.

$S_i$ is non-compact. Let $c$ be an energy level such that $\pi(S_i) = G(H^{-1}(c)) = q$. Let $X_c = G^{-1}(q)$, $g_c = g|X_c$ and $P_c = g(X_c)$. $X_c \subset H^{-1}(c)$ has a complement $\Gamma_c = \Gamma \cap H^{-1}(c)$ and is fibred by $F = T^1$. The Hamiltonian flow of $H$ restricted to $H^{-1}(c)$ is therefore 1-semisimple with respect to $(g_c, X_c, P_c)$. Now, $S_c$ is a symplectic leaf of the foliation $\zeta$ and therefore is a connected symplectic surface. By Proposition 1, there is a refinement $(g'_c, X'_c, P'_c)$ such that each component of $P'_c$ is homotopy equivalent to a point or $T^1$. Moreover, by [4, Lemma 15], the inclusion of one of the components of $X'_c$ in $T^*\mathcal{S}$ is almost surjective on $\pi_1$. But the components of $X'_c$ are homotopy equivalent to $T^1$ or $T^2$ (principal $T^1$-bundles over $*$ and $T^1$ respectively).

Therefore, $\pi_1(\mathcal{S})$ contains a finite-index abelian subgroup. Hence the Euler characteristic of $\mathcal{S}$ is non-negative (this completes the proof of Theorem 1.2). It therefore remains to prove that the Euler characteristic of $\mathcal{S}$ is positive. To do so, we will prove

**Lemma 3.1.** If $H$ is 1-semisimple, then $\dim H_1(\mathcal{S}; Q) = 0$.

**Proof of Lemma 3.1.** Let $\varphi^c$ be the Hamiltonian flow of $H$ restricted to the iso-energy set $H^{-1}(c)$.

Let $X' \subset X'_c$ be a component of $X'_c$ and let $\xymatrix{ F' \ar@{^(->}[r] & X' \ar@{^(->}[r] & P'}$ be the induced fibration of $X'$ by the closed orbits of $\varphi^c$. Each orbit is homologous to the homology class of the fibre $F'_p$.

Suppose that $\dim H_1(\mathcal{S}; Q) > 0$. Let $\Omega \in H_1(\mathcal{S}; Q)$ be an integral homology class. Each such class $\Omega$ contains a closed geodesic, and so contains the projection of a closed orbit $\gamma$ of the Hamiltonian flow of $H$ restricted to $H^{-1}(c)$. If $\gamma \subset X'_c$, then $\Omega = [\gamma] \in Z[F'_p]$ for some regular fibre $F'_p$, as in the preceding paragraph. If $\gamma \not\subset X'_c$, then, by the density of $X'_c$ and continuity in initial conditions of $\varphi^c$, each multiple of $\gamma$ is approximated by a broken integral curve in $X'_c$, i.e. a curve $w$ of the form
\[
 w(t) = \begin{cases} 
 \varphi^c_{2T^1}(p) & \text{if } 0 \leq t \leq \frac{1}{2} \\
 \delta_p(t) & \text{if } \frac{1}{2} \leq t \leq 1.
\end{cases}
\]
where $\delta_p : [\frac{1}{2}, 1] \to H^{-1}(c) \cap T^*_{\pi(p)} \mathcal{G}$, $p \in X'_c$ and $\phi_T^c(p) = p$. It follows that some multiple of the homology class $\Omega$ is homologous to a closed orbit of $\phi^c|X'_c$ and therefore that $Q\Omega \subset Q[F_p]$.

This proves that $H_1(\hat{\mathcal{G}}; Q)$ is a finite union of 1-dimensional sub-spaces and therefore it is at most 1-dimensional.

If $\dim H_1(\hat{\mathcal{G}}; Q) = 1$, then $\mathcal{G}$ is a Klein bottle and has a double cover $\hat{\mathcal{G}} \to \mathcal{G}$ where $\hat{\mathcal{G}}$ is a 2-torus. The pulled-back Hamiltonian $\tilde{H}$ is 1-semisimple. The conclusion of the previous paragraph implies that $\dim H_1(\hat{\mathcal{G}}; Q) \leq 1$--an absurdity. Therefore, $\dim H_1(\hat{\mathcal{G}}; Q)$ must be 0.

Given Lemma 3.1, the proof of Theorem 1.3 is complete.

REFERENCES

[1] V. I. Arnol’d, Mathematical Methods of Classical Mechanics, Translated from the 1974 Russian original by K. Vogtmann and A. Weinstein. Corrected reprint of the second (1989) edition. Graduate Texts in Mathematics, 60. Springer-Verlag, New York, 1997.

[2] M. Bialy, Integrable geodesic flows on surfaces, Geom. Funct. Anal., 20 (2010), 357–367.

[3] A. V. Bolsinov and B. Jovanović, Complete involutive algebras of functions on cotangent bundles of homogeneous spaces, Math. Z., 246 (2004), 213–236.

[4] L. T. Butler, Invariant fibrations of geodesic flows, Topology, 44 (2005), 769–789.

[5] ______, An optical Hamiltonian and obstructions to integrability, Nonlinearity, 19 (2006), 2123–2135.

[6] ______, A generalization of Kozlov’s theorem on integrable mechanical systems on surfaces, Preprint arXiv:1208.1460v1 (2012), 1–7.

[7] P. Dazord and T. Delzant, Le problème général des variables actions-angles, J. Differential Geom., 26 (1987), 223–251.

[8] E. Glasmachers and G. Knieper, Characterization of geodesic flows on $T^2$ with and without positive topological entropy, Geom. Funct. Anal., 20 (2010), 1259–1277.

[9] ______, Minimal geodesic foliation on $T^2$ in case of vanishing topological entropy, J. Topol. Anal., 3 (2011), 511–520.

[10] V. V. Kozlov, Topological obstructions to the integrability of natural mechanical systems, Dokl. Akad. Nauk SSSR, 249 (1979), 1299–1302.

[11] ______, Symmetries, Topology and Resonances in Hamiltonian Mechanics, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 31, Springer-Verlag, Berlin, 1996, Translated from the Russian manuscript by S. V. Bolotin, D. Treshchev and Yuri Fedorov.

[12] Y. Long, Collection of problems proposed at International Conference on Variational Methods, Front. Math. China, 3 (2008), 259–273.

[13] N. N. Nehorošev, Action-angle variables, and their generalizations, Trudy Moskov. Mat. Obšč., 26 (1972), 181–198.

[14] I. A. Taǐmanov, Topological obstructions to the integrability of geodesic flows on nonsimply connected manifolds, Izv. Akad. Nauk SSSR Ser. Mat., 51 (1987), 429–435, 448.

Received April 2013; revised July 2013.

E-mail address: L. Butler@cmich.edu