Metric spaces and homotopy types

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Abstract

By analogy with methods of Spivak, there is a realization functor which takes a persistence diagram $Y$ in simplicial sets to an extended pseudo-metric space (or ep-metric space) $\text{Re}(Y)$. The functor $\text{Re}$ has a right adjoint, called the singular functor, which takes an ep-metric space $Z$ to a persistence diagram $\text{S}(Z)$. We give an explicit description of $\text{Re}(Y)$, and show that it depends only on the 1-skeleton $\text{sk}_1 Y$ of $Y$. If $X$ is a totally ordered ep-metric space, then there is an isomorphism $\text{Re}(\text{V}_*(X)) \cong X$, between the realization of the Vietoris-Rips diagram $\text{V}_*(X)$ and the ep-metric space $X$. The persistence diagrams $\text{V}_*(X)$ and $\text{S}(X)$ are section-wise equivalent for all such $X$.

Introduction

An extended pseudo-metric space, here called an ep-metric space, is a set $X$ together with a function $d : X \times X \to [0, \infty]$ such that the following conditions hold:

1) $d(x, x) = 0$,
2) $d(x, y) = d(y, x)$,
3) $d(x, z) \leq d(x, y) + d(y, z)$.

There is no condition that $d(x, y) = 0$ implies $x$ and $y$ coincide — this is where the adjective “pseudo” comes from, and the gadget is “extended” because we are allowing an infinite distance.

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A metric space is an ep-metric space for which \( d(x, y) = 0 \) implies \( x = y \), and all distances \( d(x, y) \) are finite.

The traditional objects of study in topological data analysis are finite metric spaces \( X \), and the most common analysis starts by creating a family of simplicial complexes \( V_s(X) \), the Vietoris-Rips complexes for \( X \), which are parameterized by a distance variable \( s \).

To construct the complex \( V_s(X) \), it is harmless at the outset to list the elements of \( X \), or give \( X \) a total ordering — one can always do this without damaging the homotopy type. Then \( V_s(X) \) is a simplicial complex (and a simplicial set), with simplices given by strings

\[
x_0 \leq x_1 \leq \cdots \leq x_n
\]

of elements of \( X \) such that \( d(x_i, x_j) \leq s \) for all \( i, j \). If \( s \leq t \) then there is an inclusion \( V_s X \subset V_t(X) \), and varying the distance parameter \( s \) gives a diagram (functor) \( V_s(X) : [0, \infty] \to \mathbf{sSet} \), taking values in simplicial sets.

Following Spivak [4] (sort of), one can take an arbitrary diagram \( Y : [0, \infty] \to \mathbf{sSet} \), and produce an ep-metric space \( \text{Re}(Y) \), called its realization. This realization functor has a right adjoint \( S \), called the singular functor, which takes an ep-metric space \( Z \) and produces a diagram \( S(Z) : [0, \infty] \to \mathbf{sSet} \) in simplicial sets.

One needs good cocompleteness properties to construct the realization functor \( \text{Re} \). Ordinary metric spaces are not well behaved in this regard, but it is shown in the first section (Lemma 3) that the category of ep-metric spaces has all of the colimits one could want. Then \( \text{Re}(Y) \) can be constructed as a colimit of finite metric spaces \( U^n_s \), one for each simplex \( \Delta^n \to Y_s \) of some section of \( Y \).

The metric space \( U^n_s \) is the set \( \{0, 1, \ldots, n\} \), equipped with a metric \( d \), where \( d(i, j) = s \) for \( i \neq j \). A morphism \( U^n_s \to Z \) of ep-metric spaces is a list \((x_0, x_1, \ldots, x_n)\) of elements of \( Z \) such that \( d(x_i, x_j) \leq s \) for all \( i, j \). Such lists have nothing to with orderings on \( Z \), and could have repeats.

With a bit of categorical homotopy theory, one shows (Proposition 7) that \( \text{Re}(Y) \) is the set of vertices of the simplicial set \( Y_\infty \) (evaluation of \( Y \) at \( \infty \)), equipped with a metric that is imposed by the proof of Lemma 3.

One wants to know about the homotopy properties of the counit map \( \eta : Y \to S(\text{Re}(Y)) \), especially when \( Y \) is an old friend such as the Vietoris-Rips system \( V_s(X) \). But \( \text{Re}(V_s(X)) \) is the original metric space \( X \) (Example 13), the object \( S(X) \) is the diagram \([0, \infty] \to \mathbf{sSet}\) with \( (S(X))_n = \text{hom}(U^n_s, X) \), and the counit \( \eta : V_s(X) \to S_t(X) \) in simplicial sets takes an \( n \)-simplex \( \sigma : \Delta^n \to V_t(X) \) to the list \((\sigma(0), \sigma(1), \ldots, \sigma(n))\) of its vertices.

We show in Section 3 (Theorem 10) the main result of this paper) that the map \( \eta : V_t(X) \to S_t(X) \) is a weak equivalence for all distance parameter values \( t \). The proof proceeds in two main steps, and involves technical results from the theory of simplicial approximation. The steps are the following:

1) We show (Lemma 1) that the map \( \eta \) induces a weak equivalence \( \eta_* : BNV_t(X) \to BNS_t(X) \), where \( \eta_* : NV_t(X) \to NS_t(X) \) is the induced comparison of posets of non-degenerate simplices. Here, \( V_t(X) \) is a simplicial complex,
so that $BNV_t(X)$ is a copy of the subdivision $sd(V_t(X))$, and is therefore weakly equivalent to $V_t(X)$.

2) There is a canonical map $\pi : sdS_t(X) \to BNS_t(X)$, and the second step in the proof of Theorem 16 is to show (Lemma 15) that this map $\pi$ is a weak equivalence.

It follows that the map $\eta$ induces a weak equivalence $sd(V_t(X)) \to sd(S_t(X))$, and Theorem 16 is a consequence.

The fact that the space $S_t(X)$ is weakly equivalent to $V_t(X)$ for each $t$ means that we have yet another system of spaces $S_*(X)$ that models persistent homotopy invariants for a data set $X$.

One should bear in mind, however, that $S_t(X)$ is an infinite complex. To see this, observe that if $x_0$ and $x_1$ are distinct points in $X$ with $d(x_0, x_1) \leq t$, then all of the lists

$$(x_0, x_1, x_0, x_1, \ldots, x_0, x_1)$$

define non-degenerate simplices of $S_t(X)$.

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1 ep-metric spaces

An extended pseudo-metric space [2] (or an uber metric space [4]) is a set $Y$, together with a function $d : Y \times Y \to [0, \infty]$, such that the following conditions hold:

a) $d(x, x) = 0$,

b) $d(x, y) = d(y, x)$,

c) $d(x, z) \leq d(x, y) + d(y, z)$.

Following [3], I use the term ep-metric spaces for these objects, which will be denoted by $(Y, d)$ in cases where clarity is required for the metric.

Every metric space $(X, d)$ is an ep-metric space, by composing the distance function $d : X \times X \to [0, \infty]$ with the inclusion $[0, \infty) \subset [0, \infty]$. 

3
A morphism between ep-metric spaces \((X, d_X)\) and \((Y, d_Y)\) is a function \(f : X \rightarrow Y\) such that
\[
d_Y(f(x), f(y)) \leq d_X(x, y).
\]
These morphisms are sometimes said to be non-expanding \([2]\).

I shall use the notation \(\text{ep} - \text{Met}\) to denote the category of ep-metric spaces and their morphisms.

**Example 1** (Quotient ep-metric spaces). Suppose that \((X, d)\) is an ep-metric space and that \(p : X \rightarrow Y\) is a surjective function.

For \(x, y \in Y\), set
\[
D(x, y) = \inf_P \sum_i d(x_i, y_i),
\]
where each \(P\) consists of pairs of points \((x_i, y_i)\) with \(x = x_0\) and \(y_k = y\), such that \(p(y_i) = p(x_{i+1})\).

Certainly \(D(x, x) = 0\) and \(D(x, y) = D(y, x)\). One thinks of each \(P\) in the definition of \(D(x, y)\) as a “polygonal path” from \(x\) to \(y\). Polygonal paths concatenate, so that \(D(x, z) \leq D(x, y) + D(y, z)\), and \(D\) gives the set \(Y\) an ep-metric space structure. This is the quotient ep-metric space structure on \(Y\).

If \(x, y\) are elements of \(X\), the pair \((x, y)\) is a polygonal path from \(x\) to \(y\), so that \(D(p(x), p(y)) \leq d(x, y)\). It follows that the function \(p\) defines a morphism \(p : (X, d) \rightarrow (Y, D)\) of ep-metric spaces.

**Example 2** (Dividing by zero). Suppose that \((X, d)\) is an ep-metric space. There is an equivalence relation on \(X\), with \(x \sim y\) if and only if \(d(x, y) = 0\).

Write \(p : X \rightarrow X/\sim =: Y\) for the corresponding quotient map.

Given a polygonal path \(P = \{(x_i, y_i)\}\) from \(x\) to \(y\) in \(X\) as above, \(d(y_i, x_{i+1}) = 0\), so the sum corresponding to \(P\) in \((1)\) can be rewritten as
\[
d(x, y) = d(x_0, y_0) + d(y_0, x_1) + d(x_1, y_1) + \cdots + d(x_k, y).
\]
It follows that \(d(x, y) \leq D(p(x), p(y))\), whereas \(D(p(x), p(y)) \leq d(x, y)\) by construction.

Thus, if \(D(p(x), p(y)) = 0\), then \(d(x, y) = 0\) so that \(p(x) = p(y)\).

**Lemma 3.** The category \(\text{ep} - \text{Met}\) of ep-metric spaces is cocomplete.

**Proof.** The empty set is the initial object for this category,

Suppose that \((X_i, d_i), i \in I\), is a list of ep-metric spaces. Form the set theoretic disjoint union \(X = \sqcup_i X_i\), and define a function
\[
d : X \times X \rightarrow [0, \infty]
\]
by setting \(d(x, y) = d_i(x, y)\) if \(x, y\) belong to the same summand \(X_i\) and \(d(x, y) = \infty\) otherwise. Any collection of morphisms \(f_i : X_i \rightarrow Y\) in \(\text{ep} - \text{Met}\) defines a unique function \(f = (f_i) : X \rightarrow A\), and this function is a morphism of \(\text{ep} - \text{Met}\) since
\[
d(f(x), f(y)) = d(f_i(x), f_j(y)) \leq \infty = d(x, y)
\]
if \( x \in X_i \) and \( y \in X_j \) with \( i \neq j \).

Suppose given a pair of morphisms

\[
A \xrightarrow{f} X \xrightarrow{g} B
\]

in \( \text{cp-Met} \), and form the set theoretic coequalizer \( \pi : X \to C \). The function \( p \) is the canonical map onto a set of equivalence classes of \( X \), which classes are defined by the relations \( f(a) \sim g(a) \) for \( a \in A \). We give \( C \) the quotient ep-metric space structure, as in Example 1.

Suppose that \( \alpha : (X, d_X) \to (Z, d_Z) \) is an morphism of ep-metric spaces such that \( \alpha \cdot f = \alpha \cdot g \). Write \( \alpha : C \to Z \) for the unique function such that \( \alpha \cdot p = \alpha \).

Suppose given a polygonal path \( P = \{(x_i, y_i)\} \) from \( x \) to \( y \) in \( X \). Then \( \alpha(y_i) = \alpha(x_{i+1}) \), so that

\[
d_Y(\alpha(x), \alpha(y)) \leq \sum_i d_Y(\alpha(x_i), \alpha(y_i)) \leq \sum_i d_X(x_i, y_i).
\]

This is true for every polygonal path from \( x \) to \( y \) in \( X \), so that

\[
d_Y(\alpha \cdot p(x), \alpha \cdot p(y)) \leq d_C(p(x), p(y)).
\]

It follows that \( \alpha : (C, d_C) \to (Z, d_Z) \) is a morphism of ep-metric spaces.

**Example 4** ("Bad" filtered colimit). If one starts with a diagram of metric spaces, the colimit \( C \) that is produced by Lemma 3 is an ep-metric space, and it may be that \( d(x, y) = 0 \) in the coequalizer \( C \) for some elements \( x \), \( y \) with \( x \neq y \).

In particular, suppose that \( X_s = \{(\frac{1}{s\sqrt{2}}, 0), (0, \frac{1}{s\sqrt{2}})\} \subset \mathbb{R}^2 \) for \( 0 < s < \infty \).

Write \( p_s = (\frac{1}{s\sqrt{2}}, 0) \) and \( q_s = (0, \frac{1}{s\sqrt{2}}) \) in \( X_s \). Then \( d(p_s, q_s) = \frac{1}{s} \). For \( s \leq t \) there is an ep-metric space map \( X_s \to X_t \) which is defined by \( p_s \mapsto p_t \) and \( q_s \mapsto q_t \).

The filtered colimit \( \lim_{\leftarrow s} X_s \) has two distinct points, namely \( p_{\infty} \) and \( q_{\infty} \), and \( d(p_{\infty}, q_{\infty}) \leq d(p_s, q_s) = \frac{1}{s} \) for all \( s > 0 \). It follows that \( d(p_{\infty}, q_{\infty}) = 0 \), whereas \( p_{\infty} \neq q_{\infty} \).

**Lemma 5.** Suppose that \( X \) is an ep-metric space. Then there is an isomorphism of ep-metric spaces

\[
\psi : \lim_{\rightarrow F} F \rightarrow X,
\]

where \( F \) varies over the finite subsets of \( X \), with their induced ep-metric space structures.

**Proof.** The collection of finite subsets of \( X \) is filtered, and the set \( X \) is a filtered colimit of its finite subsets, so the function defining the ep-metric space map \( \psi \) is a bijection. Write \( d_{\infty} \) for the metric on the filtered colimit.

If \( x, y \in X \) and \( d(x, y) = s \leq \infty \) in \( X \), then there is a finite subset \( F \) with \( x, y \in F \) such that \( d(x, y) = s \) in \( F \). The list \((x, y)\) is a polygonal path from \( x \) to \( y \) in \( F \), so that \( d_{\infty}(x, y) \leq d(x, y) \). It follows that \( d(x, y) = d_{\infty}(x, y) \), and so \( \psi \) is an isomorphism. \( \square \)
An ep-metric space \((X, d)\) has an associated system of posets \(P_s(X) : [0, \infty] \to s\text{Set}\), where \(P_s(X)\) is the collection of finite subsets \(F\) of \(X\) such that \(d(x, y) \leq s\) for any two members \(x, y\) of \(X\).

This construction defines a system of abstract simplicial complexes \(V_s(X)\), which can be constructed entirely within simplicial sets when \(X\) has a total ordering. In that case, the \(n\)-simplices of the simplicial set \(V_s(X)\) are the strings \(x_0 \leq x_1 \leq \cdots \leq x_n\) such that \(d(x_i, x_j) \leq s\). The diagram \(V_s(X) : [0, \infty] \to s\text{Set}\) is the Vietoris-Rips system. The spaces \(V_s(X)\) are independent up to weak equivalence of the ordering on \(X\), because there is a canonical weak equivalence \((\text{a “last vertex map”}) \gamma : BP_s(X) \to V_s(X)\) of systems, while the spaces \(BP_s(X)\) are defined independently from the ordering. In classical terms, the nerve \(BP_s(X)\) of the poset \(P_s(X)\) (non-degenerate simplices of the Vietoris-Rips complex \(V_s(X)\)) is the barycentric subdivision of \(V_s(X)\).

**Example 6** (Excision for path components). Suppose that \(X\) and \(Y\) are finite subsets of an ep-metric space \(Z\), with the induced ep-metric space structures. Consider the inclusions of finite ep-metric spaces \(X \cap Y \to Y \to X \cup Y\) inside \(Z\). Write \(X \cup_m Y\) for the corresponding pushout in the category of ep-metric spaces. The unique map

\[
X \cup_m Y \to X \cup Y
\]

of ep-metric spaces is the identity on the underlying point set. Write \(d_m\) for the metric on \(X \cup_m Y\). Then \(d_m(x, y)\) is the minimum of sums

\[
\sum d(x_i, x_{i+1}),
\]

indexed over paths \(P : x = x_0, x_1, \ldots, x_n = y\), such that for each \(i\) the points \(x_i, x_{i+1}\) are either both in \(X\) or both in \(Y\).

All sums in (2) are finite, and \(d_m(x, y)\) is realized by a particular path \(P\) since \(X\) and \(Y\) are finite. Note that \(d(x, y) \leq d_m(x, y)\), by construction, and that \(d(x, y) = d_m(x, y)\) if \(x, y\) are both in either \(X\) or \(Y\).

There are induced simplicial set maps

\[
V_s(X) \cup V_s(Y) \to V_s(X \cup_m Y) \to V_s(X \cup Y),
\]

all of which are the identity on vertices. There is a 1-simplex \(\sigma = \{x, y\}\) of \(V_s(X \cup_m Y)\) if and only if there is a path

\[
P : x = x_0, x_1, \ldots, x_n = y
\]
consisting of 1-simplices in either $X$ or $Y$, such that
\[
\sum d(x_i, x_{i+1}) \leq s.
\]
Then all $d(x_i, x_{i+1}) \leq s$, so that $x$ and $y$ are in the same path component of $V_s(X) \cup V_s(Y)$. It follows that there is an induced isomorphism
\[
\pi_0(V_s(X) \cup V_s(Y)) \cong \pi_0 V_s(X \cup_m Y). \tag{3}
\]
The isomorphisms induce isomorphisms
\[
\pi_0(V_s(X) \cup V_s(Y)) \cong \pi_0 V_s(X \cup_m Y). \tag{4}
\]
for arbitrary subsets $X$ and $Y$ of an ep-metric space $Z$, by an application of Lemma 5.

2 Metric space realizations

Write $U^n_s$ for the collection of axis points $x_i = \frac{s \sqrt{2}}{2} e_i$, where
\[
e_i = (0, \ldots, i+1, \ldots, 1, \ldots, 0) \in \mathbb{R}^{n+1}.
\]
for $0 \leq i \leq n$. Observe that $d(x_i, x_j) = s$ in $\mathbb{R}^{n+1}$ for $i \neq j$. Another way of looking at it: $U^n_s$ is the set $n = \{0, 1, \ldots, n\}$ with $d(i, j) = s$ for $i \neq j$.

An ep-metric space morphism $f : U^n_s \rightarrow Y$ consists of points $f(x_i), 0 \leq i \leq n$, such that $d_Y(f(x_i), f(x_j)) \leq s$ for all $i, j$.

Write $\textbf{sSet}^{[0, \infty]}$ for the category of diagrams (functors) $X : [0, \infty] \rightarrow \textbf{sSet}$ and their natural transformation, which take values in simplicial sets and are defined on the poset $[0, \infty]$. I usually write $s \mapsto X_s$ for such a diagram $X$. In particular, $X_\infty$ is the value that the diagram $X$ takes at the terminal object of $[0, \infty]$.

Suppose that $K$ is a simplicial set. The representable diagram $L_s K$ satisfies the universal property
\[
\text{hom}(L_s K, X) \cong \text{hom}(K, X_s).
\]
One shows that
\[
(L_s K)_t = \begin{cases} 
\emptyset & \text{if } t < s, \\
K & \text{if } t \geq s.
\end{cases}
\]
The set of maps $L_s \Delta^n \rightarrow X$ can be identified with the set of $n$-simplices of the simplicial set $X_s$.

A morphism $L_t \Delta^m \rightarrow L_s \Delta^n$ consists of a relation $s \leq t$ and a simplicial map $\theta : \Delta^m \rightarrow \Delta^n$. In the presence of such a morphism, the function $\theta : m \rightarrow n$ defines an ep-metric space morphism $U^n_s \rightarrow U^n_t$, since $s = d(\theta(i), \theta(j)) \leq d(i,j) = t$. 

2.1 The realization functor

Suppose that $X : [0, \infty] \to s\text{Set}$ is a diagram. The category $\Delta/X$ of simplices of $X$ has maps $L_s \Delta^n \to X$ as objects and commutative diagrams

$$
\begin{align*}
L_t \Delta^m & \xrightarrow{\theta} L_s \Delta^n \\
\tau & \downarrow \\
X & \xrightarrow{\sigma}
\end{align*}
$$
as morphisms.

Equivalently, a simplex of $X$ is a simplicial set map $\Delta^n \to X$, and a morphism of simplices is a diagram

$$
\begin{align*}
\Delta^m & \xrightarrow{\theta} \Delta^n \\
\tau & \downarrow \\
X_t & \xrightarrow{\sigma} X_s
\end{align*}
$$

(5)

Every simplex $\Delta^n \to X_s$ determines a simplex

$$
\Delta^n \to X_s \to X_\infty,
$$

and we have a functor $r : \Delta/X \to \Delta/X_\infty$, where $\Delta/X_\infty$ is the simplex category of the simplicial set $X_\infty$.

There is an inclusion $i : \Delta/X_\infty \to \Delta/X$, and the composite $r \cdot i$ is the identity. The maps

$$
\begin{align*}
\Delta^n & \xrightarrow{1} \Delta^n \\
r(\sigma) & \downarrow \\
X_\infty & \xrightarrow{\sigma} X_s
\end{align*}
$$

(6)
define a natural transformation $h : i \cdot r \to 1$.

There is a functor

$$
\Delta/X \to s\text{Set}
$$

(7)

which takes a morphism [5] to the map $\theta : \Delta^m \to \Delta^n$.

The translation category $E_X$ for the functor [7] is a simplicial category that has objects consisting of pairs $(\sigma, x)$ where $\sigma : \Delta^n \to X_s$ and $x \in \Delta^n$ (of a fixed dimension). A morphism $(\tau, y) \to (\sigma, x)$ of $E_X$ is a morphism $\theta : \tau \to \sigma$ as in [5] such that $\theta(y) = x$. The path component simplicial set $\pi_0 E_X$ of the category $E_X$ is isomorphic to the colimit

$$
\lim_{L_s \Delta^n \to X} \Delta^n.
$$

There is a corresponding translation category $E_{X_\infty}$ for the functor which takes the simplex $\Delta^n \to X_\infty$ to the simplicial set $\Delta^n$, and there is an induced
functor \( i_* : E_{X_{\infty}} \subset E_X \). The functor \( r : \Delta/X \rightarrow \Delta/X_{\infty} \) induces a functor \( r_* : E_X \rightarrow E_{X_{\infty}} \). The composite \( r_* \cdot i_* \) is the identity on \( E_{X_{\infty}} \), and the map identifies a natural transformation \( i_* \cdot r_* \rightarrow 1 \) of functors \( E_{X_{\infty}} \rightarrow E_{X_{\infty}} \).

The translation categories \( E_X \) and \( E_{X_{\infty}} \) are therefore homotopy equivalent, and thus have isomorphic simplicial sets of path components. It follows that there are isomorphisms of simplicial sets

\[
X_{\infty} \leftarrow \lim_{\Delta^n \rightarrow X_{\infty}} \Delta^n \rightarrow \lim_{L_s \Delta^n \rightarrow X} \Delta^n
\]

Suppose that \( X : [0, \infty] \rightarrow s\text{Set} \) is a diagram, and set

\[
\text{Re}(X) = \lim_{L_s \Delta^n \rightarrow X} U^n_s
\]

in the category of ep-metric spaces.

It follows from the identifications of \([8]\) that \( \text{Re}(X) \) is the set of vertices of \( X_{\infty} \), equipped with an ep-metric space structure.

If \( x \) and \( y \) are two such vertices, and are the boundary of a 1-simplex \( \Delta^1 \rightarrow X_s \rightarrow X_{\infty} \)
then \( x \) and \( y \) are in the image of a map \( U^1_s \rightarrow \text{Re}(X) \), so that \( d(x, y) \leq s \). If there is a sequence of 1-simplices \( \omega_i : \Delta^1 \rightarrow X_s \) that define a polygonal path

\[
P : x = x_0 \Leftrightarrow x_1 \Leftrightarrow \cdots \Leftrightarrow x_k = y
\]

of 1-simplices in \( X_{\infty} \), then \( d(x, y) \leq \sum_i s_i \) by definition. Formally, we set

\[
d(x, y) = \inf_P \{ \sum_i s_i \}.
\]

provided such polygonal paths exist. Otherwise, we set \( d(x, y) = \infty \).

The resulting metric \( d \) is the metric which is imposed on the set of vertices of \( X_{\infty} \) by the requirement that

\[
\text{Re}(X) = \lim_{L_s \Delta^n \rightarrow X} U^n_s
\]

in the category of ep-metric spaces — see Lemma\([8]\). We have shown the following:

**Proposition 7.** Suppose that \( X : [0, \infty] \rightarrow s\text{Set} \) is a diagram. Then the ep-metric space \( \text{Re}(X) \) has underlying set given by the set of vertices of \( X_{\infty} \), with metric defined within path components by \([8]\). Elements \( x \) and \( y \) that are in distinct path components have \( d(x, y) = \infty \).

**Example 8** (Realization of Vietoris-Rips systems). Suppose that \( X \) is a finite ep-metric space, and that \( X \) is totally ordered.
The realization $\text{Re}(V_\ast(X))$ has $X$ as its underlying set, and $V_\infty(X) = \Delta^X$ is a finite simplex, which is connected, so that there is a finite polygonal path in $X$ between any two points $x, y \in X$. We have a relation

$$d(x, y) \leq \sum_i s_i$$

in $X$ for any polygonal path $P$ which is defined by $1$-simplices $\omega_i \in V_i(X)$. This means that $d(x, y)$ in $X$ coincides with the distance between $x$ and $y$ in the metric space $\text{Re}(X)$. It follows that the identity on the set $X$ induces an isomorphism of ep-metric spaces

$$\phi : \text{Re}(V_\ast(X)) \xrightarrow{\cong} X.$$ 

This map $\phi$ is an isomorphism of ep-metric spaces, by Lemma 5 and the previous paragraphs.

**Example 9** (Degree Rips systems). Continue with a finite totally ordered ep-metric space $X$ as in Example 8, let $k$ be a positive integer, and consider the degree Rips system $L_{*,k}(X)$. We choose $k$ such that the system of complexes $L_{*,k}(X)$ is non-empty, ie. such that $k \leq |X|$. Then $L_t,k(X) = V_t(X)$ for $t$ sufficiently large, and $X$ is the underlying set of $\text{Re}(L_{*,k}(X))$.

The maps

$$\text{Re}(L_{*,k}(X)) \to \text{Re}(V_\ast(X)) \to X$$

are isomorphisms of metric spaces, by a cofinality argument.

Here is a special case:

**Lemma 10**. Suppose that $K$ is a simplicial complex and that $s > 0$. Then $\text{Re}(L_sK) = \text{Re}(L_s\text{sk}_1(K))$, and $\text{Re}(L_sK)$ is the set of vertices $K_0$ with a metric $d$ defined by

$$d(x, y) = \begin{cases} \infty & \text{if } [x] \neq [y] \text{ in } \pi_0(K), \\ \min_P s \cdot k & \text{if } [x] = [y]. \end{cases}$$

where $P$ varies through the polygonal paths

$$P : x = x_0 \Rightarrow x_1 \Rightarrow \cdots \Rightarrow x_k = y$$

of $1$-simplices between $x$ and $y$.

**Proof.** Write $\text{Re}(K) = \text{Re}(L_sK)$. The simplicial set $K$ is a colimit of its simplices, and so there is an isomorphism

$$\lim_{\Delta^n \to K} L_s \Delta^n \xrightarrow{\cong} L_sK.$$ 

It follows that there is an isomorphism

$$\lim_{\Delta^n \to K} U^n \xrightarrow{\cong} \text{Re}(L_sK).$$
Suppose that $n \geq 2$. Then $\partial \Delta^n$ and $\Delta^n$ have the same vertices, and any two vertices $x, y$ are on a common face $\Delta^{n-1} \subset \Delta^n$. It follows that $d(x, y) = s$ in $\text{Re}(\partial \Delta^n)$ and $\text{Re}(\Delta^n)$, and the induced map

\[ \text{Re}(\partial \Delta^n) \to \text{Re}(\Delta^n) \]

is an isomorphism for $n \geq 2$.

The displayed metric $d$ on the vertices of $K$ defines a metric space $\text{Re}(K)$, with maps $\sigma_* : U^n_s \to \text{Re}(K)$ for all simplices $\sigma : \Delta^n \to K$, which maps are natural with respect to the simplicial structure of $K$.

Any family of metric space morphisms $f_\sigma : U^n_s \to Y$ determines a unique function $f : K_0 \to Y$. Also, $d(f(x), f(y)) \leq s$ if $x, y$ are in a common simplex $\Delta^1 \to K$. If $P$ is a polygonal path between $x$ and $y$ as above, then $d(f(x), f(y)) \leq k \cdot s$. This is true for all such polygonal paths, so $d(f(x), f(y)) \leq d(x, y)$.

If $x$ and $y$ are in distinct components of $K$, then $d(f(x), f(y)) \leq d(x, y) = \infty$.

The following result says that the realization $\text{Re}(X)$ of a diagram $X : [0, \infty] \to \text{sSet}$ depends only on the associated diagram of graphs $\text{sk}_1(X)$.

**Lemma 11.** Suppose that $X : [0, \infty] \to \text{sSet}$ is a diagram. Then the inclusion $\text{sk}_1 X \subset X$ induces an isomorphism

\[ \text{Re}(\text{sk}_1 X) \cong \text{Re}(X). \]

**Proof.** The diagram of 1-skeleta $\text{sk}_1 X$ is a colimit

\[ \lim_{L, \Delta^n \to X} L_* \text{sk}_1 \Delta^n, \]

since the functor $\text{sk}_1$ preserves colimits. There are commutative diagrams

\[ \begin{array}{ccc}
\text{Re}(L_* \text{sk}_1 \Delta^n) & \to & \text{Re}(\text{sk}_1 X) \\
\cong \downarrow & & \downarrow \\
\text{Re}(L_* \Delta^n) & \to & \text{Re}(X)
\end{array} \]

that are natural in the simplices of $X$, and it follows that the induced map $\text{Re}(\text{sk}_1 X) \to \text{Re}(X)$ is an isomorphism, as required.

**2.2 Partial realizations**

Suppose again that $X : [0, \infty] \to \text{sSet}$ is a diagram in simplicial sets. We construct partial realizations by writing

\[ \text{Re}(X)_s = \lim_{L, \Delta^n \to X, t \leq s} U^n_t. \]
This is the colimit of a functor taking values in ep-metric spaces, which is defined on the full subcategory $\Delta / X \leq s$ of $\Delta / X$ having objects $Lt\Delta^n \to X$ with $t \leq s$. A map $Lt\Delta^n \to X$ can be identified with a simplex $\Delta^n \to X_t$, and the relation $t \leq s$ defines a simplex $\Delta^n \to X_t \to X_s$, so that we have a functor

$$r : \Delta / X \leq s \to \Delta / X_s,$$

along with an inclusion $i : \Delta / X_s \subset \Delta / X \leq s$. The composite $r \cdot i$ is the identity, and the composite $i \cdot r$ is homotopic to the identity, just as before.

There is a functor $\Delta / X \leq s \to sSet$ which takes a simplex $\Delta^n \to X_t$ to the simplicial set $\Delta^n$. By manipulating path components of homotopy colimits, one finds isomorphisms

$$X_s \xrightarrow{\approx} \lim_{\Delta^n \to X_s} \Delta^n \xrightarrow{\approx} \lim_{Lt\Delta^n \to X, t \leq s} \Delta^n.$$

that are analogous to the isomorphisms of (9). It follows, as in Theorem 7, that the set underlying the metric space $\text{Re}(X)_s$ is the set of vertices of the simplicial set $X_s$.

The metric $d$ on $(X_s)_0$ is defined as before: $d(x, y) = \infty$ if $x$ and $y$ not in the same path component of $X_s$. Otherwise

$$d(x, y) = \inf_P \{ \sum t_i \},$$

indexed over all polygonal paths

$$P : x = x_0 \xleftarrow{\sim} x_1 \xleftarrow{\sim} \cdots \xleftarrow{\sim} x_k = y$$

that are defined by 1-simplices $\omega : \Delta^1 \to X_t$, with $t_i \leq s$.

We then have the following analogue of Proposition 7:

**Proposition 12.** Suppose that $X : [0, \infty] \to sSet$ is a functor. Then the ep-metric space $\text{Re}(X)_s$ has underlying set given by the set of vertices of $X_s$, with metric defined within path components by (10). Elements $x, y$ in distinct path components have $d(x, y) = \infty$.

The map $X_s \to X_\infty$ defines a map $\text{Re}(X)_s \to \text{Re}(X)$ and $\text{Re}(X)_\infty = \text{Re}(X)$. There is an isomorphism of ep-metric spaces

$$\lim_s \text{Re}(X)_s \xrightarrow{\approx} \text{Re}(X),$$

since the element $\infty$ is terminal in $[0, \infty]$ and $\text{Re}(X)_\infty = \text{Re}(X)$.

**Example 13** (Partial metrics for Vietoris-Rips complexes). Suppose that $X$ is a finite totally ordered ep-metric space. Consider the associated functor $V_\epsilon(X) : [0, \infty] \to sSet$.

The associated ep-metric space $\text{Re}(X)_\epsilon$ has underlying set $X$. We have $d(x, y) = \infty$ if $x, y$ are in distinct path components of $V_\epsilon(X)$. Otherwise

$$d(x, y) = \inf_P \{ \sum d(x_i, x_{i+1}) \},$$

where $P$ is a polygonal path from $x$ to $y$. This is analogous to the isomorphisms of (9). It follows, as in Theorem 7, that the set underlying the metric space $\text{Re}(X)_\epsilon$ is the set of vertices of the simplicial set $X_\epsilon$.

The metric $d$ on $(X_\epsilon)_0$ is defined as before: $d(x, y) = \infty$ if $x, y$ are in distinct path components of $X_\epsilon$. Otherwise

$$d(x, y) = \inf_P \{ \sum d(x_i, x_{i+1}) \},$$

indexed over all polygonal paths

$$P : x = x_0 \xleftarrow{\sim} x_1 \xleftarrow{\sim} \cdots \xleftarrow{\sim} x_k = y$$

that are defined by 1-simplices $\omega : \Delta^1 \to X_t$, with $t_i \leq \epsilon$.
indexed over all polygonal paths

\[ P : x = x_0, x_1, \ldots, x_n = y, \]

with \( d(x_i, x_{i+1}) \leq s. \)

If \( d(x, y) = t \leq s \) in \( X \) then \( d(x, y) = t \) in \( \text{Re}(X)_s \). Otherwise, the distance between \( x \) and \( y \) in the same path component of \( \text{Re}(X)_s \) is more interesting — it is achieved by a particular path \( P \) since \( X \) is finite, and \( d(x, y) \) is a type of weighted path length.

We see in Example 8 that there is an isomorphism of ep-metric spaces \( \phi : \text{Re}(V_\ast(X)) \cong X \). It follows that there is an ep-metric space map \( \phi_s : \text{Re}(X)_s \to X \) which is the identity on the underlying point set \( X \), and compresses distances.

3 The singular functor

The right adjoint \( S \) of the realization functor \( \text{Re} \) is defined for an ep-metric space \( Y \) by

\[ S(Y)_{s,n} = \text{hom}(U^n_s, Y), \]

where \( \text{hom}(U^n_s, Y) \) is the collection of ep-metric space morphisms \( U^n_s \to Y \).

Equivalently, \( S(Y)_{s,n} \) is the set of families of points \( (x_0, x_1, \ldots, x_n) \) in \( Y \) such that \( d(x_i, x_j) \leq s \).

A simplex \( (x_0, x_1, \ldots, x_n) \) is alternatively a function \( n \to Y \) (a “bag of words”), with a distance restriction. There is no requirement that the elements \( x_i \) are distinct. This simplex is non-degenerate if and only if \( x_i \neq x_{i+1} \) for \( 0 \leq i \leq n - 1 \).

Suppose that an ep-metric space \( X \) is totally ordered, as in Example 8 above. Then, in view of the discussion of Example 8, the canonical map \( \eta : V_\ast(X) \to S \text{Re}(V_\ast(X)) \) consists of functions \( \eta : V_t(X) \to S_t(X) \) which send simplices \( \sigma : x_0 \leq x_1 \leq \cdots \leq x_n \) with \( d(x_i, x_j) \leq t \) to the list of points \( (x_0, x_1, \ldots, x_n) \).

If \( \sigma \) is non-degenerate, so that the vertices \( x_i \) are distinct, then \( \eta(\sigma) \) is a non-degenerate simplex of \( S_t(X) \).

The poset \( N_Z \) of non-degenerate simplices of a simplicial set \( Z \) has \( \sigma \leq \tau \) if there is a subcomplex inclusion \( \langle \sigma \rangle \subset \langle \tau \rangle \), where \( \langle \sigma \rangle \) is the subcomplex of \( Z \) which is generated by the simplex \( \sigma \). Equivalently, \( \sigma \leq \tau \) if there is an ordinal number map \( \theta \) such that \( \theta^\ast(\tau) = \sigma \).

The map \( \eta \) induces a morphism \( \eta_* : NV_t(X) \to NS_t(X) \) of posets of non-degenerate simplices.

**Lemma 14.** Suppose that \( X \) is a totally ordered ep-metric space. Then the induced simplicial set map

\[ \eta_* : \text{BNV}_t(X) \to \text{BNS}_t(X) \]

of associated nerves is a weak equivalence.
Proof. Given a non-degenerate simplex $\sigma \in S_t(X)$, write $L(\sigma)$ for its list of distinct elements.

Suppose that $\langle \tau \rangle \subset \langle \sigma \rangle$, where $\tau$ and $\sigma$ are non-degenerate simplices of $S_t(X)$. Then $\tau = s \cdot d(\sigma)$ for an (iterated) face map $d$ and degeneracy $s$. Then

$$L(\tau) = L(s \cdot d(\sigma)) = L(d(\sigma)) \subset L(\sigma).$$

It follows that the assignment $\sigma \mapsto L(\sigma)$ defines a poset morphism

$$L : NS_t(X) \to NV_t(X).$$

The composite

$$NV_t(X) \xrightarrow{\eta} NS_t(X) \xrightarrow{L} NV_t(X)$$

is the identity on $NV_t(X)$.

Consider the composite poset morphism

$$NS_t(X) \xrightarrow{L} NV_t(X) \xrightarrow{\eta} NS_t(X). \quad (12)$$

Given a non-degenerate simplex $\tau = (y_0, \ldots, y_r)$ of $S_t(X)$, write $L(\tau) = (s_0, \ldots, s_k)$ for the list of distinct elements of $\tau$, in the order specified by the total order for $X$. Then the list

$$V(\tau) = (y_0, \ldots, y_r, s_0, \ldots, s_k)$$

is a simplex of $S_t(X)$, since each $s_j$ is some $y_{ij}$, and there are relations

$$\langle \tau \rangle \leq \langle V(\tau) \rangle \geq \langle L(\tau) \rangle$$

as subcomplexes of $S_t(X)$.

The simplex $V(\tau)$ has the form $V(\tau) = s(V_*(\tau))$ for a unique iterated degeneracy $s$ and a unique non-degenerate simplex $V_*(\tau)$ (see Lemma 18), and $\langle V(\tau) \rangle = \langle V_*(\tau) \rangle$.

Suppose that $\gamma$ is non-degenerate in $S_t(X)$ and that $\gamma \in \langle \tau \rangle$. Then $\gamma = d(\tau)$ for some face map $d$, and $\gamma = (x_0, \ldots, x_k)$ is a sublist of $\tau = (y_0, \ldots, y_r)$. The ordered list $L(\gamma)$ of distinct elements of $\gamma$ is a sublist of $L(\tau)$, and $V(\gamma)$ is a sublist of $V(\tau)$. There is a diagram of relations

$$\langle \tau \rangle \xrightarrow{} \langle V_*(\tau) \rangle \xleftarrow{} \langle L(\tau) \rangle$$

$$\langle \gamma \rangle \xrightarrow{} \langle V_*(\gamma) \rangle \xleftarrow{} \langle L(\gamma) \rangle$$

It follows that the composite (12) is homotopic to the identity on the poset $NS_t(X)$, and the Lemma follows.

The subdivision $sd(Z)$ of a simplicial set $Z$ is defined by

$$sd(Z) = \lim_{\Delta^n \to Z} BN\Delta^n.$$
The poset morphisms $N\Delta^n \to NZ$ that are induced by simplices $\Delta^n \to Z$ together induce a map

$$\pi : \text{sd}(Z) \to BNZ.$$  

It is known [1] (and not difficult to prove) that the map $\pi$ is a bijection for simplicial sets $Z$ that are polyhedral.

A polyhedral simplicial set is a subobject of the nerve of a poset. All oriented simplicial complexes are polyhedral in this sense. Examples include the Vietoris-Rips systems $V_s(X)$ associated to a totally ordered ep-metric space $X$, since

$$V_s(X) \subset V_\infty(X) = BX,$$

where $BX$ is the nerve of the totally ordered poset $X$.

**Lemma 15.** Suppose that $X$ is an ep-metric space. Then the map

$$\pi : \text{sd}(S_t(X)) \to BNS_t(X)$$

is a weak equivalence.

**Proof.** We show that all subcomplexes $\langle \sigma \rangle$ which are generated by non-degenerate simplices $\sigma$ of $S_t(X)$ are contractible. Then Lemma 4.2 of [1] implies that the map $\pi$ is a weak equivalence.

A non-degenerate simplex $\sigma$ has the form $\sigma = (x_0, x_1, \ldots, x_k)$ with $x_i \neq x_{i+1}$. The simplices $\tau$ of $\langle \sigma \rangle$ have the form

$$\tau = \theta^* \sigma = (x_{\theta(0)}, \ldots, x_{\theta(k)}),$$

where $\theta : k \to n$ is an ordinal number morphism.

For each such $\theta$, the list

$$(x_0, x_{\theta(0)}, \ldots, x_{\theta(k)})$$

defines a simplex $\tau_\ast$ of $\langle \sigma \rangle$, since $\tau_\ast$ is a face of the simplex

$$s_0(\sigma) = (x_0, x_0, \ldots, x_k).$$

The simplices $\tau_\ast$ define functors

$$x_0 \to x_0 \to \cdots \to x_0$$

or homotopies, that consist of simplices of $\langle \sigma \rangle$ that patch together to give a contracting homotopy $\langle \sigma \rangle \times \Delta^1 \to \langle \sigma \rangle$.  

$\Box$
Theorem 16. Suppose that $X$ is a totally ordered ep-metric space. Then there is a diagram of weak equivalences

$$
\begin{array}{c}
BNV_t(X) \xrightarrow{\pi} \text{sd}(V_t(X)) \xrightarrow{\gamma} V_t(X) \\
\downarrow \eta \downarrow \quad \quad \quad \quad \downarrow \eta \\
BNS_t(X) \xrightarrow{\pi} \text{sd}(S_t(X)) \xrightarrow{\gamma} S_t(X)
\end{array}
$$

In particular, the map $\eta : V_t(X) \rightarrow S_t(X)$ is a weak equivalence.

This diagram is natural in $t$.

Proof. The map $\eta^* : BNV_t(X) \rightarrow BNS_t(X)$ is a weak equivalence by Lemma 14. The map $\pi : \text{sd}(S_t(X)) \rightarrow BNS_t(X)$ is a weak equivalence by Lemma 15. The instances of the maps $\gamma$ are weak equivalences. It follows that the maps $\eta^* : \text{sd}(V_t(X)) \rightarrow \text{sd}(S_t(X))$ and $\eta : V_t(X) \rightarrow S_t(X)$ are weak equivalences. \qed

Remark 17. The total ordering on the ep-metric space $X$ in Theorem 16 is intimately involved in the definition of the Vietoris-Rips system $V_*(X)$, the morphism

$$\eta : V_t(X) \rightarrow S_t(\text{Re}(V_*(X))) = S_t(X),$$

and all induced maps $\eta^*$.

The counit $\eta : Z \rightarrow S(\text{Re}(Z))$ is not a sectionwise weak equivalence in general. One can show that $S_\infty(\text{Re}(Z))$ is the nerve of the trivial groupoid on the vertex set of $Z_\infty$ (Proposition 7), and is therefore contractible, whereas the space $Z_\infty$ may not be contractible. For example, if $K$ is a simplicial set, then there is an identification $K = (L_s K)_\infty$.

It may be that the map

$$\eta : BP_t(X) \rightarrow S_t(\text{Re}(BP_*(X)))$$

is a weak equivalence for arbitrary ep-metric spaces $X$, but this has not been proved. Such a result would give a non-oriented version of Theorem 16.

The following is a classical result, which is included here for the sake of completeness. This result is usually neither expressed nor proved in the form displayed here.

Lemma 18. Suppose that $\sigma$ is an $n$-simplex of a simplicial set $X$. Then there is a unique iterated degeneracy and a non-degenerate simplex $x$ such that $\sigma = s(x)$.

An iterated degeneracy is a surjective ordinal number map $s : n \rightarrow k$. Such a map induces a function $s : X_k \rightarrow X_n$ for a simplicial set $X$. Lemma 18 says that $\sigma = s(x)$ for some iterated degeneracy $s$ and a non-degenerate simplex $x$, and that this representation is unique.

Proof of Lemma 18. Suppose that $\sigma = s(x) = s'(x')$ where $s, s'$ are iterated degeneracies and $x, x'$ are non-degenerate.
The $x = d(\sigma)$ for some face map $d$ such that $d \cdot s = 1$, and so $d(s'(x')) = s''(d''(x))$ for some iterated degeneracy $s''$ and face map $d''$. But $x$ is non-degenerate, so that $s'' = 1$ and $x = d''(x')$. Similarly, $x' = d(x)$ for some face map $d$. But then $x$ and $x'$ have the same dimension, and $d = 1$, so that $x = x'$.

If $s \neq s'$ there is a face map $d$ such that $d \cdot s = 1$ but $d \cdot s' \neq 1$. Then $\sigma = s(x) = s'(x)$ for $s \neq s'$ and $x$ non-degenerate, then

$$d(\sigma) = x = d(s'(x)) = s''(d''(x))$$

for some degeneracy $s''$ and face map $d''$, at least one of which is non-trivial.

But $x$ is non-degenerate, so that $s'' = 1$ and $x = d''(x)$ only if $d'' = 1$. This contradicts the assumption that $s \neq s'$. $\square$

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