Review on rationality problems of algebraic $k$-tori

Youngjin Bae

Abstract

Rationality problems of algebraic $k$–tori are closely related to rationality problems of the invariant field, also known as Noether’s Problem. We describe how a function field of algebraic $k$–tori can be identified as an invariant field under a group action and that a $k$–tori is rational if and only if its function field is rational over $k$. We also introduce character group of $k$–tori and numerical approach to determine rationality of $k$–tori.

Contents

1 Introduction 2

2 Algebraic $k$–tori 3

3 Character group of $k$–tori 8

4 Flabby resolution and numerical approach 9
1 Introduction

Let $k$ be a field and $K$ is a finitely generated field extension of $k$. $K$ is called *rational over $k$* or *$k$-rational* if $K$ is isomorphic to $k(x_1, \ldots, x_n)$ where $x_i$ are transcendental over $k$ and algebraically independent. There are also relaxed notions of rationality. $K$ is called *stably $k$-rational* if $K(y_1, \ldots, y_m)$ is $k$–rational for some transcendental and algebraically independent $y_i$. $K$ is called *$k$-unirational* if $k \subset K \subset k(x_1, \ldots, x_n)$ for some pure transcendental extension $k(x_1, \ldots, x_n)/k$.

The Noether’s Problem is the question of rationality of the invariant field under finite group action. For example, if $K = \mathbb{Q}(x_1, x_2)$ and $G = \{1, \sigma\} \cong C_2$ and $G$ acts on $K$ as permutation of variables $x_1, x_2$ (i.e. $\sigma$ fixes $\mathbb{Q}$, $\sigma(x_1) = x_2$ and $\sigma(x_2) = x_1$), then the invariant field $K^G$ is $\mathbb{Q}$–rational.

**Example 1.1** $K = \mathbb{Q}(x, y)$ and $G \cong C_2$, acting on $K$ as permutation of variables. Let $\frac{f}{g} \in K^G$, $f, g$ are coprime. We have

$$\frac{f(x, y)}{g(x, y)} = \sigma\left(\frac{f(x, y)}{g(x, y)}\right) = \frac{f(y, x)}{g(y, x)}$$

By observing that $\gcd(f(x, y), g(x, y)) = \gcd(f(y, x), g(y, x)) = 1$, we have $f(x, y) = f(y, x)$ and $g(x, y) = g(y, x)$.

Therefore, $K^G = \{\frac{f(x, y)}{g(x, y)} | f, g \text{ are symmetric}\}$, field of fractions (quotient field) of $S = \{f \in \mathbb{Q}[x, y] | f(x, y) = f(y, x)\}$. It is easy to see that $\psi : S \to \mathbb{Q}[s, t]$ is isomorphism, where

$$\psi(x + y) = s, \quad \psi(xy) = t$$

Therefore, $S \cong \mathbb{Q}[x, y]$ and $K^G \cong \mathbb{Q}(x, y)$, $\mathbb{Q}$–rational.

We can also consider case of $G$ acting on both of coefficients and variables.

**Example 1.2** $K = \mathbb{C}(x, y)$ and $G = \text{Gal}(\mathbb{C}/\mathbb{R}) = \{1, \sigma\} \cong C_2$. Suppose $G$ acts on $K$ by permuting $x, y$ and as complex conjugation on coefficients.

For example, $\sigma(ix + (1-i)xy + y^2) = -iy + (1+i)yx + x^2$. Then, $K^G \cong \mathbb{R}(x, y)$, is $\mathbb{R}$–rational.
\textbf{Proof.} For \( \frac{f(z,w)}{g(z,w)} \in K^G \), where \( f, g \) are coprime, \( \sigma(f) \) and \( \sigma(g) \) are also coprime. From \( \frac{f}{g} = \frac{\sigma(f)}{\sigma(g)} \), we have \( f = \sigma(f) \) and \( g = \sigma(g) \). Thus, \( K^G \) is quotient field of \( S \) where \( S := \{ f(z,w) \in \mathbb{C}[z,w] \mid f = \sigma(f) \} \).

Define a map \( \psi : S \to \mathbb{R}[x,y] \) as

\[ z = x + yi, \quad w = x - yi \]

and

\[ \psi(f)(x,y) = f(z,w) \]

The coefficients of \( \psi(f) \) are real numbers. This is because, if we let \( f(z,w) = \sum_{n,m} a_{n,m}z^n w^m \), we have that

\[ \psi(f)(x,y) = f(z,w) = \sigma(f(z,w)) = \sigma(\sum_{n,m} a_{n,m}z^n w^m) = \sum_{n,m} \overline{a_{n,m}}w^n z^m \]

\[ = \sum_{n,m} a_{n,m}(x + iy)^n(x - iy)^m = \psi(f)(x,y). \]

Therefore, \( \psi(f) = \overline{\psi(f)} \), \( \psi(f) \in \mathbb{R}[x,y] \). It is easy to see that \( \psi \) is actually isomorphism, \( S \cong \mathbb{R}[x,y] \), and \( K^G \cong \mathbb{R}(x,y) \).

Another perspective to view this change of variables is identifying the field with rational function field of algebraic \( k \) – tori. (see Example 2.5 and Example 2.6)

\section{Algebraic \( k \) – tori}

Let \( k \) be a field. Then \( \mathbb{A}^n_k \) is \( n \)-dimension affine space over the field \( k \), simply \( k^n \) with usual vector space structure on it. A subset \( X \) of \( \mathbb{A}^n_k \) is an algebraic \( k \)-variety (\( k \)-variety in short) if it is a set of zeros of a system of equations with \( n \) variables \( x_1, ... x_n \) over \( k \). The ideal of polynomials that vanish on every points of \( X \) will be denoted by \( I(X) \). The coordinate ring of a variety \( X \) is defined to be the quotient

\[ A(X) := k[x_1, ..., x_n]/I(X) \]
Projective varieties can be similarly defined as the set of zeros of a system of homogeneous equations. Projective $n$-space $\mathbb{P}^n_k$ is defined as set of lines passing the origin in $\mathbb{A}^{n+1}_k$.

If $X, Y$ are varieties, a map $f : X \to Y$ is called regular if it can be presented as fraction of polynomials $p/q$, where $q$ does not vanishes in $X$. A map $f : X \to Y$ is called rational if it is regular on Zariski open dense set. (Formally, a regular map is defined as an equivalence class of pairs $<U, f_U>$ where $U$ is Zariski open subset of $U$. See [1]) Let $X$ be a variety, $K(X)$ is the rational function field, or function field in short, the set of rational maps $f : X \to \mathbb{A}^k$. For example, if $X$ is an affine variety over algebraically closed field $k$, $K(X)$ is quotient field of $\mathbb{A}(X)$.

**Example 2.1** Let $X = \{(x, y) \in \mathbb{A}^2 \mid xy = 1\}$ be a variety over $\mathbb{C}$. Then, $\mathbb{A}(X) = \mathbb{C}[x, y]/(xy - 1) \cong \mathbb{C}[x, \frac{1}{x}]$ and $K(X) \cong \mathbb{C}(x)$.

Two varieties $X, Y$ are isomorphic (resp. birationally isomorphic) if there is a bijective regular map (resp. rational map) $f : X \to Y$ and its inverse is also regular (resp. rational).

A variety $X$ in $\mathbb{A}^n_k$ is an algebraic group if it has a group structure on it, where the group operation and inversions are regular maps. (i.e. $\ast : X \times X \to X$ and $-1 : X \to X$ are regular)

Algebraic $k$-tori, or algebraic $k$-torus, is a special type of algebraic group over $k$. We call an algebraic group as $k$-tori when it is isomorphic to some power of multiplicative group over $\overline{k}$, the algebraic closure of $k$.

**Definition 2.1 (Multiplicative Group)** Let $k$ be a field, the multiplicative group $\mathbb{G}_m(k)$ is algebraic group in $\mathbb{A}^2_k$, defined as $\{(x, y) \in \mathbb{A}^2_k \mid xy = 1\}$, with operation $\cdot : \mathbb{G}_m(k) \times \mathbb{G}_m(k) \to \mathbb{G}_m(k)$ of $(x, \frac{1}{x}) \cdot (y, \frac{1}{y}) = (xy, \frac{1}{xy})$

**Example 2.2** $\mathbb{G}_m(\mathbb{R})$ is the curve $xy = 1$ on the real affine plane. It is isomorphic to $\mathbb{R}^\times$ as a group. ($(x, y) \to x$ is group isomorphism.)

As field changes, same system of equations can define different varieties. For instance, the equation $xy = 1$ in previous example defines $\mathbb{G}_m(\mathbb{C})$ in $\mathbb{A}^2_\mathbb{C}$. 
which is different from $\mathbb{G}_m(\mathbb{R})$. If $E$ is a field and $F$ is its algebraic closure, an irreducible variety $V$ over $F$ entails the ring of equations, $I$. If $I$ happens to be in $E[x]$ (ring of polynomials over $E$), we can define $V(E)$, a variety over $E$ defined by equations in $I$. This can be viewed as restriction of scalar. Extension of scalar can be defined similarly.

**Definition 2.2 (Algebraic $k$-tori)** Let $k$ be a field with algebraic closure $\bar{k}$. If $T$ is an algebraic group over $k$, it is $k$-torus if and only if

$$T(\bar{k}) \cong (\mathbb{G}_m(\bar{k}))^r$$

for some $r$. The $r$ is called dimension of $T$.

**Example 2.3** $T = \mathbb{G}_m(\mathbb{R})$ is one dimensional $\mathbb{R}$-tori. This is because $T(\mathbb{C}) = \mathbb{G}_m(\mathbb{C})$.

From now, let $k^\times = \mathbb{G}_m(k)$ be the one dimensional torus over $k$. There are two one-dimensional $\mathbb{R}$-tori, one can be recognized as $\mathbb{R}^\times$, the other one can be recognized as $SO(2)$ as a group.

**Example 2.4** The norm one torus $N$ is a real algebraic group in $\mathbb{A}^2_\mathbb{R}$, defined by equation $x_1^2 + x_2^2 = 1$ (i.e. $N = \{(x_1, x_2) \in \mathbb{A}^2_\mathbb{R}|x_1^2 + x_2^2 = 1\}$), and operation $\cdot : N \times N \to N$ such that

$$(x_1, x_2) \cdot (y_1, y_2) = (x_1y_1 - x_2y_2, x_1y_2 + x_2y_1)$$

Indeed, $N$ is isomorphic to $SO(2)$ as a group.

Also, $N(\mathbb{C}) = \{(x_1, x_2) \in \mathbb{A}^2_\mathbb{C}|x_1^2 + x_2^2 = 1\}$ is isomorphic to $\mathbb{C}^\times$ as algebraic group. The map $\psi : N(\mathbb{C}) \to \mathbb{C}^\times$

$$\psi(x_1, x_2) = x_1 + ix_2$$

is isomorphism. Therefore, $N$ is one dimensional real torus.

If $T$ is a $k$-torus, $T$ is called split over $K$ if it satisfies $T(K) \cong (K^\times)^s$ for some extension $K/k$ and some $s$. For instance, $\mathbb{R}^\times$ is split over $\mathbb{R}$, $N$ is not.
It is easy to find split torus such as \((\mathbb{R}^\times)^2\) or \((\mathbb{R}^\times)^3\), being another torus. Also, for any integer \(r\), \(N^r\) is \(r\)-dimensional \(\mathbb{R} - \text{tori}\). Meanwhile, there are also some non-trivial (not a product of low-dimensional torus) torus.

**Example 2.5** Let \(P\) be a real algebraic group in \(\mathbb{A}^4_{\mathbb{R}}\), defined as
\[
P = \{(x_1, x_2, x_3, x_4) \in \mathbb{A}^4_{\mathbb{R}} | x_1 x_3 - x_2 x_4 = 1, x_1 x_4 + x_2 x_3 = 0\}
\]

Alternatively,
\[
P = \{A \in M_{2\times 2}(\mathbb{R}) | AA^t = \begin{pmatrix} s & 0 \\ 0 & s^{-1} \end{pmatrix}, s \in \mathbb{R} \setminus \{0\}\}
\]

and operation \(\cdot : P \times P \to P\) such that
\[
(x_1, x_2, x_3, x_4) \cdot (y_1, y_2, y_3, y_4) = (x_1 y_1 - x_2 y_2, x_1 y_2 + x_2 y_1, x_3 y_3 - x_4 y_4, x_3 y_4 + x_4 y_3)
\]

Which is compatible with complex multiplication of
\[
(x_1 + x_2 i, x_3 + x_4 i) \cdot (y_1 + y_2 i, y_3 + y_4 i)
\]

Moreover, \(P(\mathbb{C})\) is isomorphic to \((\mathbb{C}^\times)^2\), by sending
\[
(x_1, x_2, x_3, x_4) \to ((x_1 + x_2 i, x_3 + x_4 i), (x_1 - x_2 i, x_3 - x_4 i)) = ((z, \frac{1}{z}), (w, \frac{1}{w}))
\]

Therefore, \(P\) is 2-dimensional \(\mathbb{R} - \text{tori}\).

By tracking the function fields of \(P(\mathbb{R})\) and \(P(\mathbb{C})\), we have the same trick of change of variables as in **Example 1.2**.

**Example 2.6** In the previous example, the coordinate ring of \(P(\mathbb{C})\) is
\[
A(P(\mathbb{C})) = \mathbb{C}[x_1, x_2, x_3, x_4]/(x_1 x_3 - x_2 x_4 - 1, x_1 x_4 + x_2 x_3) \cong \mathbb{C}[z, \frac{1}{z}, w, \frac{1}{w}]
\]
where \(z = x_1 + x_2 i\) and \(w = x_1 - x_2 i\). The function field of \(P(\mathbb{C})\) is
\[
K(P(\mathbb{C})) \cong \mathbb{C}(z, w)
\]
Let $G = \text{Gal}(\mathbb{C}/\mathbb{R})$ acts on $K(P(\mathbb{C}))$ as in Example 1.2. Observe that the coordinate ring of $P(\mathbb{R})$ is $A(P(\mathbb{R})) = A(P(\mathbb{C}))^G$ and the function field of $P(\mathbb{R})$ is $K(P(\mathbb{R})) = K(P(\mathbb{C}))^G \cong \mathbb{C}(z,w)^G$ (note that $G$ actions on $K(P(\mathbb{C}))$ and $\mathbb{C}(z,w)$ are equivalent through the isomorphism). In short, we have that

$$K(P(\mathbb{R})) \cong \mathbb{C}(z,w)^G$$

Therefore, when $G = \text{Gal}(\mathbb{C}/\mathbb{R})$ action on $C(z,w)$ is given, we can convert the rationality problem to the rationality problem of $K(P(\mathbb{R}))$, the function field of $P(\mathbb{R})$. In this sense, the following definition and theorem are natural.

**Definition 2.3 (Rationality of $k$ - variety)** We say that a variety $X$ over $k$ is rational if, equivalently,

1. $X$ is birationally isomorphic to $\mathbb{P}^n_k$ for some $n$.
2. $K(X) \cong k(x_1, ..., x_n)$

If $K/k$ is Galois extension, a $k$ - tori $T$ is $K$ - rational if it is rational as a $K$-variety $T(K)$. If $k$ is algebraically closed, there is unique $n$-dimension tori $T_n = (k^*)^n$. Since the function field of $T_n$ is $k(x_1, ..., x_n)$, thus $T_n$ is $k$-rational.

**Theorem 2.1** The following two problems are equivalent.

1. The rationality problem of $n$ dimensional $k$ - tori $T$
2. The rationality problem of invariant field $K^G$

where $G = \text{Gal}(\overline{k}/k)$ and $K = k(x_1, ..., x_n)$.

There is a connection between the $G$ action on $K$ and $k$ - tori $T$, connecting the two rationality problems given in the previous theorem. To be specific, the character group of $T$ determines both the $G$ action and $T$ uniquely.
3 Character group of \( k - \text{tori} \)

Definition 3.1 (Character group of \( k - \text{tori} \)) Let \( T \) be \( k - \text{tori} \). Then \( X(T) \), the character group of \( T \) is the set of algebraic group homomorphisms (a regular map preserving the group structure) from \( T \) to \( \mathbb{G}_m \), denoted by \( \text{Hom}(T, \mathbb{G}_m) \) or \( \text{Hom}(T, \mathbb{G}_m^\times) \).

The character group \( X(T) \) of \( T \) has a group structure defined by component-wise multiplication. Also, if \( T \) is split over \( L \) for finite Galois extension of base field \( k \), \( G = \text{Gal}(L/k) \) acts on \( X(T) \). Moreover, it is known that \( X(T) \) is torsion-free \( \mathbb{Z} \)-module (i.e. isomorphic to \( \mathbb{Z}^n \) for some \( n \)). Therefore, \( X(T) \) is a \( G \)-lattice (a free \( \mathbb{Z} \)-module with \( G \)-action).

Example 3.1 If \( T = \mathbb{C}^\times \) is multiplicative group of \( \mathbb{C} \), then \( X(T) \) is set of regular functions \( f : \mathbb{C}^\times \to \mathbb{C}^\times \) such that \( f(xy) = f(x)f(y) \) for \( x, y \in \mathbb{C}^\times \). Since \( f \) is a rational function, it is a meromorphic function over \( \mathbb{C} \). Also, we have \( f(\mathbb{C}^\times) \subset \mathbb{C}^\times \), which implies \( 0 \) is the only point where \( f \) can have zeros or poles. Therefore, \( f(t) = t^n \) for some \( n \in \mathbb{Z} \). If we write a function \( t \to t^n \) as \( t^n \), we have

\[
X(T) = \{t^n | n \in \mathbb{Z} \} \cong \mathbb{Z}^1
\]

as a group. \( G = \text{Gal}(\mathbb{C}/\mathbb{C}) = \{\text{id}\} \) acts trivially on \( X(T) \).

In general, if \( k \) is algebraically closed, the character group of \((k^\times)^n = \mathbb{G}_m^n\) is

\[
X(\mathbb{G}_m^n) = \{f_{t_1,\ldots,t_n} : \mathbb{G}_m^n \to \mathbb{G}_m | f_{t_1,\ldots,t_n}(x_1,\ldots,x_n) = \prod_i x_i^{t_i}, t_i \in \mathbb{Z} \} \\
= \prod_{i=1}^n \{f_t : \mathbb{G}_m \to \mathbb{G}_m | f_t(x_i) = x_i^{t}, t \in \mathbb{Z} \} \cong \mathbb{Z}^n
\]

Example 3.2 Let \( P \) be the 2-dimension \( \mathbb{R} - \text{tori} \) in Example 2.5. Then, the character group of \( P \) is

\[
X(P) = \{f_{t_1,t_2} : P \to \mathbb{C}^\times | f_{t_1,t_2}(x_1, x_2, x_3, x_4) = (x_1 + x_2i)^{t_1}(x_1 - x_2i)^{t_2} \}
\]

Let \( z = x_1 + x_2i, w = x_1 - x_2i \), then we have the natural extension of \( X(P) \) to \( X(P(C)) \)
Observe that the complex conjugation \( \sigma \in G \) exchanges \( z \) and \( w \), thus acting on \( \mathbb{Z}^2 \) as 2 \times 2 matrix \[
\begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}.
\]

It is known that when a \( G = \text{Gal}(K/k) \) action (as \( \mathbb{Z} \)-linear function) on \( \mathbb{Z}^n \) is given, there exists unique \( n \)-dimensional \( k - \text{tori} \) which has the given \( G - \text{lattice} \) as its character group. Furthermore, there are conditions of \( G - \text{lattice} \) corresponding to the rationality conditions of \( k - \text{tori} \) and of invariant fields.

4 Flabby resolution and numerical approach

This section contains many results in [2]. Let \( G \) be a group and \( M \) be a \( G - \text{lattice} \) (\( M \cong \mathbb{Z}^n \) as group and has \( G \)-linear action on it). \( M \) is called a permutation \( G \)-lattice if \( M \cong \bigoplus_{1 \leq i \leq m} \mathbb{Z}[G/H_i] \) for some subgroups \( H_1, ..., H_m \) of \( G \) (equivalently, there exists a \( \mathbb{Z} \)-basis of \( M \) such that \( G \) acts on \( M \) as permutation of the basis). \( M \) is called stably permutation \( G \)-lattice if \( M \bigoplus P \cong Q \) for some permutation \( G - \text{lattices} \) \( P \) and \( Q \). \( M \) is called invertible if it is a direct summand of a permutation \( G \)-lattice, i.e. \( P \cong M \bigoplus M' \) for some permutation \( G \)-lattice \( P \) and \( M' \).

**Definition 4.1 (1st Group Cohomology)** Let \( G \) be a group and \( M \) be a \( G - \text{lattice} \). For \( g \in G \) and \( m \in M \), let \( g.m = m^g \) be \( g \) acting on \( m \). The first group cohomology \( H^1(G, M) \) is a group defined as

\[
H^1(G, M) = Z^1(G, M)/B^1(G, M)
\]

where \( Z^1(G, M) = \{ f : G \to M | f(gh) = f(g)^hf(h) \} \) and \( B^1(G, M) = \{ f : G \to M | f(g) = m_f^g m_f^{-1} \text{ for some } m_f \in M \} \)
$H^1(G, M) = 0$ simply implies that if $f : G \to M$ satisfies $f(gh) = f(g)^hf(h)$, then there exists $m \in M$ such that $f(g) = m^g m^{-1}$. $M$ is called coflabby if $H^1(G, M) = 0$.

**Definition 4.2 (-1st Tate Cohomology)** Let $G$ be finite group of order $n$ and $M$ be a $G$-lattice. The -1st group cohomology $\hat{H}^{-1}(G, M)$ is a group defined as

$$\hat{H}^{-1}(G, M) = Z^{-1}(G, M)/B^{-1}(G, M)$$

where

$$Z^{-1}(G, M) = \{m \in M | \sum_{g \in G} m^g = 0\}$$

and

$$B^{-1}(G, M) = \{\sum_{g \in G} m^g - \text{id} | m^g \in M\}$$

Similarly, $M$ is called flabby if $\hat{H}^{-1}(G, M) = 0$. It is clear that a $k$-tori is rational if and only if $X(T)$ is permutation $G$-lattice. Thus, the rationality problems of $k$-tori and invariant fields can be reduced into problem of finding permutation $G$-lattice (equivalent to find finite subgroup of $GL(n, \mathbb{Z})$. However, this problem is not solved yet, even though there are many results in weakened problems.

Let $C(G)$ be the category of all $G$-lattices and $S(G)$ be the category of all permutation $G$-lattices. Define equivalence relation on $C(G)$ by $M_1 \sim M_2$ if and only if there exist $P_1, P_2 \in S(G)$ such that $M_1 \bigoplus P_1 \cong M_2 \bigoplus P_2$. Let $[M]$ be equivalence class containing $M$ under this relation.

**Theorem 4.1** (Endo and Miyata [3, Lemma 1.1], Colliot-Thélène and Sansuc [4, Lemma 3]) For any $G$-lattice $M$, there is a short exact sequence of $G$-lattices

$$0 \to M \to P \to F \to 0$$

where $P$ is permutation and $F$ is flabby.

In the previous theorem, $[F]$ is called the flabby class of $M$, denoted by $[M]^f$. 
Theorem 4.2 (Akinari and Aiichi [2, 17pp]) If $M$ is stably permutation, then $[M]^f_l$. If $M$ is invertible, $[M]^f_l$ is invertible.

It is not difficult to see that

$$M \text{ is permutation } \Rightarrow M \text{ is stably permutation}$$

Furthermore, it is true that

$$M \text{ is stably permutation } \Rightarrow M \text{ is invertible } \Rightarrow M \text{ is flabby and coflabby}$$

In [2], they gave the complete list of stably permutation lattices for dimension 4 and 5 by computing $[M]^f_l$ for finite subgroup of $GL(n, \mathbb{Z})$, which is equivalent to classifying stably rational tori. Thus, the rationality problems for low dimensional $k$-tori can be resolved by finding conditions which can determine a stably permutation $M$ is permutation or not.
References

[1] Robin Hartshorne *Algebraic Geometry*. Springer, New York, 24-25, 1977.

[2] Akinari Hoshi, Aiichi Yamasaki *Rationality Problem for Algebraic Tori* *(Memoirs of the American Mathematical Society)* American Mathematical Society, 2017.

[3] S.Endo, T.Miyata *On a classification of the function fields of algebraic tori* Nagoya Math, 85-104, 1975.

[4] J.-L. Colliot-Thélène, J.-J. Sansuc *La R-équivalence sur les tores* 175-229, 1977.