GLOBAL, NON-SCATTERING SOLUTIONS TO THE QUINTIC, FOCUSING SEMILINEAR WAVE EQUATION ON $\mathbb{R}^{1+3}$

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Abstract

We consider the quintic, focusing semilinear wave equation on $\mathbb{R}^{1+3}$, in the radially symmetric setting, and construct infinite time blow-up, relaxation, and intermediate types of solutions. More precisely, we first define an admissible class of time-dependent length scales, which includes a symbol class of functions. Then, we construct solutions which can be decomposed, for all sufficiently large time, into an Aubin-Talenti (soliton) solution, re-scaled by an admissible length scale, plus radiation (which solves the free 3 dimensional wave equation), plus corrections which decay as time approaches infinity. The solutions include infinite time blow-up and relaxation with rates including, but not limited to, positive and negative powers of time, with exponents sufficiently small in absolute value. We also obtain solutions whose soliton component has oscillatory length scales, including ones which converge to zero along one sequence of times approaching infinity, but which diverge to infinity along another such sequence of times. The method of proof is similar to a recent wave maps work of the author, which is itself inspired by matched asymptotic expansions.

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1. Introduction

We consider the quintic, focusing semilinear wave equation on \( \mathbb{R}^{1+3} \), in the radially symmetric setting, namely

\[
-\partial_t^2 u + \partial_r^2 u + \frac{2}{r} \partial_r u + u^5 = 0, \quad r > 0
\]

The following energy is conserved for sufficiently regular solutions to (1.1).

\[
E_{SLW}(u, \partial_t u) = \int_0^\infty r^2 dr \left( \frac{(\partial_t u)^2}{2} + \frac{(\partial_r u)^2}{2} - \frac{u^6}{6} \right)
\]

If \( u \) is a solution to (1.1), then, so is the function \( u_\lambda(t,r) := \frac{1}{\sqrt{\lambda}} u \left( \frac{t}{\sqrt{\lambda}}, \frac{r}{\lambda} \right), \lambda > 0 \). We remark that the equation (1.1) is energy critical, since, if \( u \) is a sufficiently regular solution to (1.1), then, \( E_{SLW}(u, \partial_t u) = E_{SLW}(u_\lambda, \partial_t u_\lambda) \) (since \( E_{SLW}(u, \partial_t u) \) is independent of time). We denote the Aubin-Talentini soliton by

\[
Q_1(r) = \left( \frac{r^2}{3} + 1 \right)^{-1/2}
\]

For \( \lambda > 0 \), we let

\[
Q_\lambda(r) = \lambda^{-1/2} Q_1 \left( \frac{r}{\lambda} \right)
\]

The Cauchy problem associated to (1.1) is locally well-posed in \( H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3) \) (see, for instance, [12], [22], and references therein). The free wave equation

\[
-\partial_t^2 u + \partial_r^2 u + \frac{2}{r} \partial_r u = 0
\]

will also be of use to us, and we recall its formally conserved energy.

\[
E(u, \partial_t u) = \frac{1}{2} \left( \| \partial_t u \|_{L^2(r^2 dr)}^2 + \| \partial_r u \|_{L^2(r^2 dr)}^2 \right)
\]

Part of the work [4], of Duyckaerts, Kenig, and Merle, showed that if \( u \) is a global solution to (1.1) with \( E_{SLW}(u, \partial_t u) < 2E_{SLW}(Q_1,0) \), then, \( u \) either scatters, or, there exists \( v \) solving (1.3), \( n = \pm 1 \), and \( \lambda(t) \) defined for \( t \) sufficiently large, so that

\[
\lim_{t \to \infty} E(u - v - nQ_{\lambda(t)}, \partial_t u - \partial_t v) = 0
\]

Another characterization of solutions satisfying (1.4) was provided in a special case of one of the results of [15]. If \( (u, \partial_t u) \in C([0, \infty), H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)) \) satisfies (1.4), then, \( (u(0), \partial_t u(0)) \in M_D \), where \( M_D \) is a \( C^1 \) connected manifold constructed in [15].

The work [2], of Donninger and Krieger constructed infinite time blow-up and relaxation solutions to (1.1), with soliton length scale \( \lambda(t) = t^{-\mu} \), for \( |\mu| \) sufficiently small.

For each positive \( \lambda(t) \in C^\infty([50, \infty)) \) satisfying the following inequalities for all sufficiently large \( t \), and sufficiently small \( C_u, C_t, C_2 \geq 0 \) (the precise constraints are in (1.7) through (1.9)),

\[
-\frac{C_t}{t} \leq \frac{\lambda(t)}{t} \leq \frac{C_u}{t}, \quad \left| \frac{\lambda(k)}{t} \right| \leq \frac{C_k}{t^k}, \quad k \geq 2
\]
the current work constructs a finite energy solution to (1.1) which can be decomposed as
\[ u(t, r) = Q_\lambda(t)(r) + v_{rad}(t, r) + u_e(t, r) \]
where \( v_{rad} \) solves (1.3), \( E(u_e, \partial_t(Q_\lambda(t) + u_e)) \to 0 \) as \( t \to \infty \). (We remark that, aside from the precise smallness constraints on \( C_u, C_l, C_k \), the admissible class of \( \lambda \) of this work is the same as that of a wave maps work of the author, [20].) To the knowledge of the author, the existence of all of the solutions constructed in this work, aside from those with asymptotically constant \( \lambda(t) \), or \( \lambda(t) \) equal to precisely a power of \( t \) (as in [2]) is new. This includes new infinite time blow-up and relaxation solutions (see Remark 2 after Theorem 1.1), along with solutions that have oscillatory soliton length scales satisfying any combination of
\[ \xi \]
Here, \( C \) (in addition to oscillatory \( \lambda \) such that \( \lim_{t \to \infty} \lambda(t) = 0 \) or \( \infty \)) (see Remark 3 after Theorem 1.1). In addition, the procedure used to construct the ansatz of this work, and many aspects of the procedure used to complete the ansatz to an exact solution, are quite different than that used in [2].

We will now briefly summarize those aspects of the work [18] of Krieger, Schlag, and Tataru, which are necessary in order to precisely describe the admissible class of \( \lambda \) of the current work. In the process of completing our approximate solution to an exact solution, we use the distorted Fourier transform, \( \mathcal{F} \) from Proposition 4.3 of [18]. This is a Fourier transform associated to \( L \), which is the elliptic part of a conjugation of the linearization of (1.1) around \( Q_1 \). We also use the transference operator, \( \mathcal{K} \), defined in (2.7) of [18]. In the interest of brevity, we will not recall the definition of \( \mathcal{K} \) until it is used in the proof of this work. What is important to note here is that, by Proposition 5.2 of [18], for all \( \alpha \in \mathbb{R} \), \( \mathcal{K} \) and \( [\xi \partial_t, \mathcal{K}] \) are bounded on \( L^2_\rho, \alpha \), which has the norm
\[ ||f||^2_{L^2(\rho, \alpha)} = ||f(\xi_d)||^2 + ||f(\xi)d\xi||^2_{L^2((0, \infty), \rho(\xi)d\xi)} \]
Here, \( \xi_d \) is the negative eigenvalue of \( L \), and \( \rho \) is the density of the continuous part of the spectral measure. Finally, by Lemma 4.6 of [18], there exists \( C_\rho > 0 \) so that
\[ \frac{\rho(y)}{\rho(z)} \leq C_\rho \left( \sqrt{\frac{y}{z}} + \sqrt{\frac{z}{y}} \right), \quad y, z > 0 \]
Now, we can define our admissible class of \( \lambda \), which we denote \( \Lambda \). Let \( \Lambda \) be the set of positive functions \( \lambda \in C^\infty((50, \infty)) \) such that there exists \( T_\lambda > 50, C_u, C_l, C_k > 0 \) so that
\[ \frac{-C_l}{t} \leq \frac{\lambda'(t)}{\lambda(t)} \leq \frac{C_u}{t}, \quad \frac{|\lambda^{(k)}(t)|}{\lambda(t)} \leq \frac{C_k}{t^k}, \quad k \geq 2, t \geq T_\lambda \]
where, if \( M := \max\{C_u, C_l\} \),
\[ C_l + \frac{163}{2352} C_u < \frac{1}{588} \]
\[ 40 \left( \frac{3M^2}{16}(111 - 45 \log(4)) + \frac{15C_2}{16}(-12 + 12\sqrt{3} + 15\pi^2 - 4 \log(4)) \right) \]
\[ + 2(C_2(1 + ||\mathcal{K}||_{L^2(\rho, 0)}) + M^2(||\mathcal{K}||_{L^2(\rho, 0)} + 2||[\mathcal{K}, \xi \partial_t]|_{L^2(\rho, 0)} + ||\mathcal{K}||^2_{L^2(\rho, 0)}) + 2M(1 + ||\mathcal{K}||_{L^2(\rho, 0)}) \leq \frac{1}{12\sqrt{C_\rho}} \]
and

\[ M(1 + ||K||_{L^2(t, \infty)}) < \frac{1}{48\sqrt{C_\rho}} \]

**Remark** (Smallness constraints (1.7) through (1.9) can be achieved by re-scaling the exponent of \( t \)) If \( f \in C^\infty((0, \infty)) \) is any positive function satisfying, for all \( k \geq 1, \)

\[ \frac{|f^{(k)}(t)|}{f(t)} \leq \frac{D_k}{t^k}, \quad t \geq T_f \]

(for some \( T_f > 50 \)) then, the function defined for \( t > 50 \) by

\[ (1.10) \lambda(t) = f(t^{1/c}) \]

is in \( \Lambda \) for \( c > 0 \) sufficiently large. The main theorem of this work is the following.

**Theorem 1.1.** For each \( \lambda \in \Lambda \), there exists \( T_0 > 0 \) and a finite energy solution, \( u \), to (1.1), for \( t > T_0 \), satisfying

\[ u(t, r) = Q_\lambda(t)(r) + v_{rad}(t, r) + u_e(t, r) \]

where

\[ (-\partial_t^2 + \partial_r^2 + \frac{2}{r} \partial_r)v_{rad} = 0, \quad E(v_{rad}, \partial_t v_{rad}) < \infty \]

and

\[ E(u_e, \partial_t (Q_\lambda + u_e)) \leq \frac{C}{t} \left( \sup_{x \in [T_f, t]} \sqrt{\lambda(x)} \right)^2 \]

**Remark 1.** As in [20], the Cauchy data of \( v_{rad} \) is explicit in terms of \( \lambda \). In fact,

\[ v_{rad}(0, r) = 0, \quad \partial_t v_{rad}(0, r) = \frac{\psi(r)}{r} \left( \frac{-15\pi \sqrt{\lambda(r)} \lambda''(r)}{8} - \frac{\sqrt{3}\lambda'(r)}{2\lambda(r)} \right) + v_{3,0}(r) + v_{4,0}(r) \]

where \( v_{3,0} \) is defined in (4.57), \( v_{4,0} \) is defined in (4.71) (the function \( N_0 \) appearing in (4.71) is defined in (1.66)), and \( \psi \) is a relatively unimportant cutoff, defined in (4.11) (\( \psi(r) = 0 \) near \( r = 0 \), and \( \psi(r) = 1 \) for large \( r \)).

**Remark 2.** The solutions in Theorem 1.1 include infinite time blow-up and relaxation solutions including, but not limited to \( \lambda(t) \) being a power of \( t \), with exponent sufficiently small in absolute value. Firstly, if \( T_f > 0 \), and \( f \in C^\infty((T_f, \infty)) \) is a positive function satisfying

\[ \frac{|f^{(k)}(t)|}{f(t)} \leq C_{f,k} t^{-k}, \quad k \geq 1, \quad t > T_f \]

then, if \( g(t) = \log(f(t)) \), we have

\[ |g^{(n)}(t)| \leq C_n t^{-n}, \quad n \geq 1, \quad t > T_f \]

Therefore, a simple induction argument shows that for all \( k \in \mathbb{N} \), there exists \( T_k > 0 \) so that, if

\[ g_k(t) = (k\text{-th fold composition of log}) \ (t), \quad t > T_k \]
then, \( g_k \in C^\infty((T_k, \infty)) \), and, for \( t > T_k \),
\[
g_k(t) > 1, \quad |g_k^{(n)}(t)| \leq C_{n,k}t^{-n}, \quad n \geq 1
\]

Then, if \( k \in \mathbb{N} \) and \( \alpha \in \mathbb{R} \) Faa di Bruno’s formula, for instance, shows that if
\[
h_{k,\alpha}(t) = (g_k(t + T_k))^{\alpha}, \quad t > 50
\]
we have \( h_{k,\alpha} \in C^\infty((50, \infty)) \), and for \( t > 50 \),
\[
h_{k,\alpha}(t) > 0, \quad \frac{|h_{k,\alpha}^{(n)}(t)|}{h_{k,\alpha}(t)} \leq \frac{C_{n,k,\alpha}}{t^n g_k(t + T_k)}, \quad n \geq 1
\]
(recall that \( g_k(x) > 1 \) if \( x > T_k \)). Then, an induction argument shows that, for each \( N \in \mathbb{N} \), if
\( \beta_k \in \mathbb{N} \), and \( \alpha_k \in \mathbb{R} \), for all \( 1 \leq k \leq N \), and \( \epsilon \geq 0 \) is sufficiently small,
\[
\lambda(t) := t^{\pm \epsilon} \prod_{w=1}^{N} h_{\beta_w,\alpha_w}(t) \in \Lambda, \quad \lambda(t) := t^{\pm \epsilon} \in \Lambda
\]
This gives infinite time blow-up and relaxation solutions with a variety of rates, including powers of \( t \).

**Remark 3.** The solutions in Theorem 1.1 include solutions with \( \lambda(t) \) as in (1.10). It is relatively straightforward to verify that the following functions are in \( \Lambda \) (and the complete details of this verification for the following functions is given in the remarks following Theorem 1.1 of [21]). For
\[
0 < \lambda_0 < \lambda_1 \in \mathbb{R}, \quad 0 < \alpha_0, \alpha_1, \quad \alpha_0 + \alpha_1 < 1
\]
\[
\lambda(t) = \frac{\lambda_0 \log^{-\alpha_0}(t) + \lambda_1 \log^{\alpha_1}(t)}{2} + \frac{\left( \lambda_1 \log^{\alpha_1}(t) - \lambda_0 \log^{-\alpha_0}(t) \right) \sin(\log(\log(t)))}{2}
\]
For \( c > 0 \) sufficiently small, and \( |a| \) sufficiently small, depending on \( c \),
\[
\lambda(t) = t^a (2 + c \sin(\log(t)))
\]
As per (1.10), there are many more possible forms of \( \lambda \in \Lambda \).

**Remark 4.** The function \( u \) from Theorem 1.1 is of the form
\[
u = Q_{\lambda(t)}(r) + v_{\text{rad}}(t,r) + v_{\text{e}}(t,r) + v_f(t,r)
\]
where \( v_{\text{rad}}, v_{\text{e}} \) are fairly explicit, and
\[
(t,x) \mapsto v_f(t,|x|) \in C^0_l([T_0, \infty), H^2(\mathbb{R}^3)), \quad (t,x) \mapsto \partial_1 v_f(t,|x|) \in C^0([T_0, \infty), H^1(\mathbb{R}^3))
\]
This is due to the definition of the space \( Z \) (see (5.6)), the fact that dilation is continuous on \( L^2 \) and Lemma 2.7 of [18], which relates \( H^k(\mathbb{R}^3) \) to \( L^{\infty}_{\mu,\alpha} \) spaces.

As discussed previously, the work [2], of Donninger and Krieger constructed infinite time blow-up and relaxation solutions to (1.11), with soliton length scale \( \lambda(t) = t^{-\mu} \), for \( |\mu| \) sufficiently small. The aforementioned work [18], of Krieger, Schlag, and Tataru constructed finite time blow-up solutions to (1.1), with soliton length scale \( \lambda(t) = t^{1+\nu} \), for \( \nu > \frac{1}{2} \). The subsequent work of Krieger
and Schlag, [17], extended the range of $\nu$, to all $\nu > 0$. The stability of the solutions in [18] and [17] was studied in the work [14], of Krieger and Nahas. Work of Donninger, Huang, Krieger, and Schlag, [3], constructed other finite time blow-up solutions to (1.1), with soliton length scale given by $\lambda(t) = t^{1+\nu} e^{\epsilon_0 \sin(\log(t))}$, with $\nu > 3, |\epsilon_0| \ll 1$.

We also mention some constructions of solutions to the critical semilinear wave equation in other dimensions. The work [6] constructed finite time blow-up solutions to the critical semilinear wave equation in $4 + 1$ dimensions. Multisoliton solutions to the $5 + 1$ dimensional critical wave equation were constructed in [19], [23]. Infinite time blow-up solutions involving multiple dynamically rescaled solitons for the $5 + 1$ dimensional critical wave equation were constructed in [8]. Two bubble solutions to the $6 + 1$ dimensional critical wave equation were constructed in [7].

The work [11] studied the possible dynamical behavior of solutions to the critical wave equation in dimensions $3, 4$, and $5$, which have energy strictly less than that of the ground state soliton. The work [3] classified the possible dynamical behavior of solutions to the critical wave equation in spatial dimensions $3, 4$, and $5$, which have energy exactly equal to that of the ground state soliton. The work [5] studied the dynamics of solutions with energy bounded above by a quantity slightly larger than the ground state energy. Stable manifolds for the critical wave equation in $3$ spatial dimensions were constructed in [13] (studied further in [16]), [1], and [15]. Modulated soliton solutions emerging from randomized initial data were studied in [10].

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2. Notation

The equation (1.1), linearized around $Q_1(R)$ is

\begin{equation}
-\partial_t^2 u + \partial_R^2 u + \frac{2}{R} \partial_R u + \frac{45u}{(3 + R^2)^2} = 0
\end{equation}

Two linearly independent solutions to (2.1) are the following

$\phi_0(R) = -\sqrt{3} \frac{(R^2 - 3)}{2 (R^2 + 3)^{3/2}}, \quad e_2(R) = \frac{2 \left( R^4 - 18R^2 + 9 \right)}{\sqrt{3} R (R^2 + 3)^{3/2}}$

In particular, the function denoted by $\phi_0$ in [18], which we denote in this paper by $\tilde{\phi}_0$ satisfies

$\tilde{\phi}_0(R) = 2R\phi_0(R)$

The following function, which is (negative of) the potential obtained by linearizing (1.1) around $Q_{\lambda(t)}(r)$, will appear in many of our computations.

\begin{equation}
V(t, r) = \frac{-45}{\lambda(t)^2 \left( \frac{r^2}{\lambda(t)^2} + 3 \right)^2}
\end{equation}

3. Summary of the Proof

The method of proof of Theorem 1.1 is similar to that used in a wave maps work of the author, [20]. For completeness, we summarize the argument here. We choose $\lambda \in \Lambda$, restrict our attention to sufficiently large $t$, and start our ansatz with the soliton $Q_{\lambda(t)}(r)$. Inserting $Q_{\lambda(t)}(r)$ into (1.1)
produces the error term $\partial_t^2 Q_{\lambda(t)}(r)$. We improve this error term by finding an approximate solution to

$$\left( -\partial_t^2 + \partial_r^2 + \frac{2}{r} \partial_r + \frac{45\lambda(t)^2}{(r^2 + 3\lambda(t)^2)^2} \right) v = \partial_t^2 Q_{\lambda(t)}(r) \tag{3.1}$$

(The operator on the left hand side is the linearization of (1.1) around $Q_{\lambda(t)}$). Our starting point is a function of the form

$$u_c(t, r) = \chi_{\leq 1} \left( \frac{r}{h(t)} \right) u_e(t, r) + \chi_{\geq 1} \left( \frac{r}{h(t)} \right) u_w(t, r) \tag{3.2}$$

where $\chi_{\leq 1}, \chi_{\geq 1}$ are cutoffs satisfying (4.64), and we define a length scale $h(t) = \lambda(t) \left( \frac{r}{\lambda(t)} \right)^a$, where $a$ satisfies (4.3). (The function $h(t)$ is defined with the constraints (4.3) at the beginning of the argument, for convenience. One may not know at the beginning of the argument why $a$ is restricted to satisfy (4.3), and one could keep $a$ fairly general, and then impose the constraints (4.3) at various stages in the argument, as needed. We will give some intuition about the choice of $a$ in this section).

We will construct $u_c$ and $u_w$ to be good approximate solutions to (3.1) for small $r$ and large $r$, respectively, satisfying the following matching property: When one takes the asymptotic expansion of $u_c(t, r)$ and $u_w(t, r)$ for large $t$, in the region $r \sim h(t)$, one obtains functions of the form

$$\sum_{j=-n}^{n} \sum_{l=0}^{m} c_{j,l}(t) \log^l(r)$$

for various $n, m \geq 0$ (see sections 4.6, 4.7, and 4.8). We choose $u_c$ and $u_w$ so that sufficiently many of the $c_{j,l}(t)$ involved in the expansions of $u_c$ and $u_w$ are equal to each other. As we explain in more detail shortly, this is accomplished by essentially keeping enough degrees of freedom in $u_c$ and $u_w$, which are then fixed when imposing the matching conditions. We start with $u_{\ell,1}$, the first term in $u_c$, which solves the following equation.

$$\left( \partial_r^2 + \frac{2}{r} \partial_r + \frac{45\lambda(t)^2}{(r^2 + 3\lambda(t)^2)^2} \right) u_{\ell,1}(t, r) = \partial_t^2 Q_{\lambda(t)}(r) \tag{3.3}$$

(In other words, we remove the $\partial_t^2$ term from the left hand side of (3.1) when considering the small $r$ corrections). The most general solution to (3.3) which is not singular at the origin depends on one function of $t$, say $c_1(t)$ (see (1.4)). Recall that we must use sufficiently general solutions to the equations that we consider in order to be able to do the matching procedure, up to the order required, and we can regard $c_1(t)$ as a degree of freedom in $u_{\ell,1}$.

While the linear error term of $u_{\ell,1}$, which is $\partial_t^2 u_{\ell,1}$, is better than that of the soliton for small $r$, it is not sufficiently small for our purposes. So, we iterate the above procedure, obtaining first $u_{\ell,2}$ solving (3.3), with $u_{\ell,1}$ on the left hand side replaced by $u_{\ell,2}$, and $Q_{\lambda(t)}(r)$ on the right-hand side replaced by $u_{\ell,1}$, see (4.6). Then, we obtain $u_{\ell,3}$ by replacing $u_{\ell,2}$ with $u_{\ell,3}$, and replacing $u_{\ell,1}$ by $u_{\ell,2}$, see (4.8). We remark that $u_{\ell,2}$ and $u_{\ell,3}$ are not the most general solutions to their equations which are not singular at the origin, but, are sufficient for our purposes.

Next, we consider $u_w$, the large $r$ approximate solution. To understand the first few terms in $u_w$, one can start with a general (finite energy) solution to

$$\left( -\partial_t^2 + \partial_r^2 + \frac{2}{r} \partial_r \right) u = \partial_t^2 Q_{\lambda(t)}(r) \tag{3.4}$$
behavior of each of these functions shows that a particular choice of $u$ which leads to an eventual matching between $u_w$ and $u_e$ to first order. In order to explain why the particular choice of $v_{2,0}$ is made in the proof, we leave it general in this discussion). Also, when $\lambda'(t) \neq 0$, $Q_{\lambda(t)}$ and $w_{1}(t)$ both have infinite kinetic energy, but, an exact cancellation of the large $r$ behavior of each of these functions shows that $\hat{v}_{i}(Q_{\lambda(t)}(r) + \chi_{\geq 1}(\frac{r}{\lambda(t)})w_{1}(t, r)) \in L^{2}(r^{2}dr)$, see (5.14).

We can now do a first order matching of $u_e$ and $u_w$. (This is actually only one part of what is called “first order matching” in section 4.6). Note that the leading behavior of $u_{\ell,\text{main}}$, $w_{1,\text{lm}}$ and $v_{2,\text{lm}}$ is called "first order matching" in section 4.6). Note that the leading behavior of $u_{\ell,\text{main}}$, $w_{1,\text{lm}}$ and $v_{2,\text{lm}}$ is given by $u_{\ell,\text{main}}$, $w_{1,\text{lm}}$ and $v_{2,\text{lm}}$.

$$u_{\ell,\text{main}}(t, r) = -\frac{\sqrt{3} c_{1}(t) \lambda(t)}{2r} + \sqrt{\lambda(t) \lambda''(t)} \left( \frac{675 \pi \lambda(t)^{2}}{16 r^{2}} - \frac{3 \sqrt{3} \lambda(t) \left( 37 - 20 \log \left( \frac{r^{2}}{3 \lambda(t)^{2}} \right) \right)}{8 r} + \frac{\sqrt{3} r}{4 \lambda(t)} - \frac{15 \pi}{8} \right)$$

$$+ \frac{\left( \frac{15 \sqrt{3} \lambda(t)}{16 r} - \frac{\sqrt{3} r}{8 \lambda(t)} \right) \lambda'(t)^{2}}{\sqrt{\lambda(t)}}$$

$$w_{1,\text{lm}}(t, r) = \frac{1}{2} r \left( \frac{\sqrt{3} \lambda''(t)}{2 \sqrt{\lambda(t)}} - \frac{\sqrt{3} \lambda'(t)^{2}}{4 \lambda(t)^{3/2}} \right)$$

$$v_{2,\text{lm}}(t, r) = tv_{2,0}(t)$$

Here, we use the fact that $\lambda(t) \ll h(t) \ll t$, and work under the assumption of sufficiently regular $v_{2,0}$ (we will end up choosing $v_{2,0}$ to be a smooth function). Note that our freedom to choose $v_{2,0}$ only affects terms of the form $f_{0}(t)$, while there are larger terms of the form $rf_{1}(t)$ involved in both $u_{\ell}$ and $w_{1}$. On the other hand, the terms of the form $rf_{1}(t)$ in $u_{\ell}$ and $w_{1,\text{lm}}$ automatically match (in other words, they are the same). A similar kind of automatic matching of terms that are larger than those involving the degrees of freedom is observed in [20]. Therefore, we can choose $v_{2,0}$ so that the terms of the form $f_{0}(t)$ in the difference $u_{\ell,\text{main}} - w_{1,\text{lm}} - v_{2,\text{lm}}$ vanish. Doing so requires that

$$v_{2,0}(r) = -\frac{15 \pi \sqrt{\lambda(r) \lambda''(r)}}{8 r}, \quad r \geq T_{0}$$

(since we only consider $t \geq T_{0}$). This explains why, in the proof, $v_{2}$ is defined in (4.33) with the particular Cauchy data $v_{2,0}(r) = \frac{\psi(r)}{r} \left( \frac{15}{8} \pi \sqrt{\lambda(r) \lambda''(r)} \right)$, where $\psi$ is defined in (4.11). In this discussion, we now fix $v_{2,0}$ by the above formula.

We provide some intuition regarding the constraints on $a$, (4.3). The linear error term of any of our small $r$ corrections, say $u_{\text{small}}$ is $\bar{c}_{1}^{2}u_{\text{small}}$, while that of a large $r$ correction is $V(t, r)u_{\text{large}}(t, r)$. In order to identify a region of the form $r \sim h(t)$ where $u_{\text{small}}$ and $u_{\text{large}}$, as well as their linear error terms, are of comparable size, we note that $u_{\text{small}}$ is a symbol in $t$, and recall the definition of $V$, given in (2.2). Therefore, the error terms of $u_{\text{small}}$ and $u_{\text{large}}$ are of comparable size when
h(t) \gg \lambda(t), \text{ and } t^{-2} \sim \lambda(t)^2 h(t)^{-4}. \text{ This would suggest that } h(t) \sim \sqrt{t \lambda(t)}. \text{ Our actual definition of } h(t) \text{ is larger than } \sqrt{t \lambda(t)} \text{ simply because increasing } h(t) \text{ improves the error terms associated to } u_w \text{ in the matching region (though it does make the error terms associated to } u_c \text{ larger) and it turns out that some error terms associated to } u_w \text{ are more delicate than those associated to } u_c. \text{ On the other hand, } a \text{ can not be too large, since this would make some of the error terms of } u_c \text{ too large.}

Next, we consider the higher order terms in } u_w. \text{ The linear error term associated to } w_1 + v_2 \text{ is } V(t,r)(w_1 + v_2) := RHS_2(t,r), \text{ where we recall the notation (2.22). Therefore, we consider a general, finite energy solution to (3.4) (with the right-hand side replaced by } RHS_2) \text{ given by } u_{w,2} + v_3, \text{ where } v_3 \text{ is a finite energy free wave with Cauchy data not yet specified, and } u_{w,2} \text{ is the particular solution with zero Cauchy data at infinity. As before, } v_3 \text{ can be regarded as a degree of freedom in our second correction in } u_w. \text{ Then, we compute the leading behavior of } v_{ex} \text{ and } u_{w,2} \text{ in the matching region (see Section 4.6). As shown in Section 4.6, } u_{ell,main} \text{ as well as the leading parts of } v_{ex} \text{ and } u_{w,2} \text{ contain terms of the form } f_0(t)r^{-2} \text{ and } f_1(t) \log(r)r^{-1}, \text{ which can not be fixed by an appropriate choice of the degrees of freedom of our solutions. However, exactly as before, and analogously to (2.21), these terms happen to exactly match each other, so that it is possible to choose } c_1(t), \text{ the degree of freedom in } u_{ell}, \text{ so that } u_{ell} \text{ matches } w_1 + v_2 + v_{ex} + u_{w,2} \text{ to an appropriate order (see again Section 4.6).}

Higher order matching, done in Section 4.7 fixes the Cauchy data of } v_3. \text{ In Section 4.8, we verify an “automatic” matching of certain higher order terms. At this stage, we have fully constructed } u_c, \text{ given in (3.22). Recalling that } u_w \text{ appears in (3.22) with a factor of } \chi_{\geq 1}(\frac{r}{h(t)}), \text{ we add another term to the ansatz, } \psi_2(\frac{r}{h(t)})u_3, \text{ in order to eliminate the linear error term } V(t,r)v_3(t,r)\chi_{\geq 1}(\frac{r}{h(t)}). \text{ (} \psi_2 \text{ is defined in (4.63), see also (4.59)).}

At this stage, we have the preliminary ansatz } Q_{\lambda(t)}(r) + u_c(t,r) + \psi_2(\frac{r}{h(t)})u_3(t,r) \text{ whose linear error term is small (see Lemmas 4.26, 4.28, 4.29, 4.30, 4.31). The nonlinear interactions of our preliminary ansatz are decomposed into a perturbative term } \tilde{N}_1, \text{ and a non-perturbative part, } \tilde{N}_0, \text{ see section 4.10. We eliminate } \tilde{N}_0 \text{ by first considering } u_{\tilde{N}_0}, \text{ the solution to}

\[-c_t^2 u_{\tilde{N}_0} + \partial_r^2 u_{\tilde{N}_0} + \frac{2}{r} \partial_r u_{\tilde{N}_0} = \tilde{N}_0\]

with zero Cauchy data at infinity. The linear error term of } u_{\tilde{N}_0}, \text{ namely } V(t,r)u_{\tilde{N}_0}, \text{ is worst near the origin, due to the fast decay of } V(t,r) \text{ for large } r, \text{ and turns out to be too large near the origin. We therefore add a free wave, } v_4, \text{ to } u_{\tilde{N}_0}, \text{ which cancels the leading part of } u_{\tilde{N}_0}(t,r) \text{ in the region } r \leq \frac{1}{2}, \text{ thereby allowing the linear error term of } u_{\tilde{N}_0} + v_4, \text{ which is } e_1 := V(t,r)(u_{\tilde{N}_0} + v_4)(t,r) \text{ to be much smaller than } V(t,r)u_{\tilde{N}_0}(t,r), \text{ for instance, in the region } r \leq \frac{1}{2}. \text{ There are still two parts of } e_1 \text{ which are not quite perturbative. These are eliminated by } u_4 \text{ (4.83) and } u_{\tilde{N}_0,ell}(4.85), \text{ see also (4.82). At this stage, we have the improved ansatz } Q_{\lambda(t)}(r) + u_c(t,r) + \psi_2(\frac{r}{h(t)})u_3(t,r) + u_{\tilde{N}_0}(t,r) + v_4(t,r) + u_{\tilde{N}_0,ell}(t,r)\psi_1(\frac{r}{h(t)}). \text{ The self interactions of the new free wave, } v_3(t,r), \text{ are not quite perturbative for large } r, \text{ though the rest of the nonlinear interactions of our ansatz are (see (4.86)). We therefore add a final term, } u_{\tilde{N}_2}, \text{ to the ansatz in order to eliminate the } v_4 \text{ self-interactions for large } r, \text{ see (4.88).}

Denoting our final ansatz by } Q_{\lambda(t)}(r) + u_{\tilde{N}_2}, \text{ we are now ready to substitute } u = Q_{\lambda(t)}(r) + u_{\tilde{N}_2} + v \text{ into (1.1), and solve the resulting equation for } v \text{ (5.1) using a fixed point argument.
More precisely, we formally derive the equation (5.4) solved by \( y \) defined by
\[
y(t, \xi) = \langle y_0(t), y_1(t, \xi \lambda(t)^{-2}) \rangle = \mathcal{F}(\langle \cdot, \cdot \rangle v(t, \lambda(t))(\xi)
\]
where \( \mathcal{F} \) is the distorted Fourier transform from [18], which we regard as a two-component vector, as in Proposition 4.3 of [18]. Then, we prove that (5.3) has a solution, say \((y_{0,0}, y_{0,1})\), which is regular enough to allow us to justify the statement that the function \( v \) defined by the following expression, with \( y_0 = y_{0,0}, y_1 = y_{0,1} \)
\[
v(t, r) = \frac{\lambda(t)}{r} \left( y_0(t) \phi_d \left( \frac{r}{\lambda(t)} \right) + \int_0^\infty \phi \left( \frac{r}{\lambda(t)}, \xi \right) y_1(t, -\frac{\xi}{\lambda(t)^2}) \rho(\xi) d\xi \right)
\]
is a solution to (5.1). The equation (5.4) is solved with a fixed point argument in the space \( Z \), defined in Section 5.1. (The map \( T \), whose fixed point is a solution to (5.4), is defined in (5.11)). The fixed point argument essentially only uses Minkowski’s inequality, and energy-type estimates. The full details are given just after Lemma 5.5.

4. Construction of the Ansatz

Let \( \lambda \in \Lambda \). By definition of \( \Lambda \) and (1.7), we have
\[
(x - t)^{-C_l} \leq \frac{\lambda(x)}{\lambda(t)} \leq (x - t)^{-C_u}, \quad x \geq t \geq T_\lambda, \quad x \mapsto \frac{\lambda(x)^{5/2}}{x} \text{ is decreasing on } [T_\lambda, \infty)
\]
In particular, by (1.7), there exists \( T_{\lambda,0} > 0 \) so that, for all \( t \geq T_{\lambda,0} \), \( t - \lambda(t) \geq 2T_\lambda \). Let \( T_0 > 0 \) satisfy the following, but be otherwise arbitrary.
\[
T_0 \geq e^{200}(1 + T_\lambda + T_{\lambda,0}) := T_{0,1}
\]
For the rest of the paper, we restrict \( t \) to satisfy \( t \geq T_0 \). Throughout the proof, the letter \( C \) will denote a constant, whose value may change from line to line, but which is independent of \( T_0 \), unless otherwise stated. Define
\[
h(t) = \lambda(t) \left( \frac{t}{\lambda(t)} \right)^a
\]
where \( a \) is any real number satisfying
\[
2(2 + \frac{5}{2}C_u + 8C_l) \leq a < \min \left\{ \frac{2(4 - 7C_u - 12C_l)}{13(1 - C_u)}, \frac{2(1 - 4C_u - 12C_l)}{3(1 - C_u)} \right\}
\]
In particular, \( a < \frac{2}{5} \), and thus the lower bound on \( a \) implies that \( x \mapsto h(x) \) is increasing, and that there exists \( C > 0 \) so that \( h(t) \geq C \sqrt{t}, \quad t \geq T_\lambda \). It is possible to choose \( a \) as in (1.3), by (1.7). We start with the near origin corrections.

4.1. First small \( r \) correction. We define \( u_{\text{el}} \) to be the following solution to
\[
\partial_t^2 u_{\text{el}} + \frac{2}{r} \partial_r u_{\text{el}} + \frac{45 u_{\text{el}}}{\lambda(t)^2 \left( \frac{r^2}{\lambda(t)^2} + 3 \right)^2} = \partial_t^2 Q_{\lambda(t)}(r)
\]
\[
u_{\text{el}}(t, R\lambda(t)) = \phi_0(R) \int_0^R s^2 \lambda(t)^2 f(s \lambda(t)) e_2(s) ds - e_2(R) \int_0^R s^2 \lambda(t)^2 f(s \lambda(t)) \phi_0(s) ds + c_1(t) \phi_0(R)
\]
\[
= \frac{f_1(R) \lambda'(t)^2}{\sqrt{\lambda(t)}} + f_2(R) \sqrt{\lambda(t) \lambda''(t)} + c_1(t) \phi_0(R)
\]
where

\[ f_1(R) = -\frac{\sqrt{3} R^2 (R^2 - 3)}{8 (R^2 + 3)^{3/2}} \]

\[ f_2(R) = \frac{\sqrt{3} R (R^4 - 66 R^2 + 30 (R^2 - 3) \log \left( \frac{1}{3} (R^2 + 3) \right) + 45 - 15 (R^4 - 18 R^2 + 9) \tan^{-1} \left( \frac{R}{\sqrt{3}} \right)}{4 R (R^2 + 3)^{3/2}} \]

and \( c_1 \) will be chosen later. Let

\[ f_{1,0}(R) = \frac{15 \sqrt{3}}{16 R} - \frac{\sqrt{3} R}{8}, \quad f_{1,1}(R) = f_1(R) - f_{1,0}(R) \]

\[ f_{2,0}(R) = \frac{675 \pi}{16 R} - \frac{3 \sqrt{3} (37 - 20 \log \left( \frac{R^2}{9} \right))}{8 R} + \frac{\sqrt{3} R}{4} - \frac{15 \pi}{8}, \quad f_{2,1}(R) = f_2(R) - f_{2,0}(R) \]

Then, for \( k = 0, 1 \),

\[ \lim_{s \to \infty} \left( \frac{s^3}{\log(s)} f_{k,1}(s) \right) < \infty \]

Then, the leading part of \( u_{\text{ell}} \) in the matching region \( r \sim h(t) \) will turn out to be:

\[ u_{\text{ell,main}}(t, r) = -\frac{\sqrt{3} c_1(t) \lambda(t)}{2r} + \frac{f_{1,0}(\frac{r}{\lambda(t)}) \lambda'(t)^2}{\sqrt{\lambda(t)}} + f_{2,0}(\frac{r}{\lambda(t)}) \sqrt{\lambda(t)} \lambda''(t) \]  \hspace{1cm} (4.5)

### 4.2 Second small \( r \) correction.

The second correction for small \( r \) eliminates the linear error term of \( u_{\text{ell}} \). In particular, we define \( u_{\text{ell,2}} \) to be the following solution to

\[ \partial_r^2 u_{\text{ell,2}} + \frac{45 u_{\text{ell,2}}}{\lambda(t)^2} \left( \frac{r^2}{\lambda(t)^2} + 3 \right)^2 \partial_t^2 u_{\text{ell,2}} = \partial_t^2 u_{\text{ell}}(t, r) \]

(4.6)

\[ u_{\text{ell,2}}(t, r) = \phi_0\left( \frac{r}{\lambda(t)} \right) \int_0^{\frac{r}{\lambda(t)}} s^2 \lambda(t)^2 \partial_t^2 u_{\text{ell}}(t, s \lambda(t)) e_2(s)ds - 2 \phi_0 \left( \frac{r}{\lambda(t)} \right) \int_0^{\frac{r}{\lambda(t)}} s^2 \lambda(t)^2 \partial_t^2 u_{\text{ell}}(t, s \lambda(t)) \phi_0(s)ds \]

In order to compute the relevant terms associated to \( u_{\text{ell,2}} \) for the eventual matching process (see Section 4.7), we first decompose \( u_{\text{ell}} \) as follows.

\[ u_{\text{ell}}(t, r) = \frac{f_1(\frac{r}{\lambda(t)}) \lambda(t)^2}{\sqrt{\lambda(t)}} + f_2(\frac{r}{\lambda(t)}) \sqrt{\lambda(t)} \lambda''(t) + c_1(t) \phi_0 \left( \frac{r}{\lambda(t)} \right) = u_{\text{ell,main}}(t, r) + e_{\text{ell,1}}(t, r) \]

where

\[ e_{\text{ell,1}}(t, r) = \frac{f_{1,1}(\frac{r}{\lambda(t)}) \lambda'(t)^2}{\sqrt{\lambda(t)}} + f_{2,1}(\frac{r}{\lambda(t)}) \sqrt{\lambda(t)} \lambda''(t) + c_1(t) \left( \phi_0 \left( \frac{r}{\lambda(t)} \right) + \frac{\sqrt{3} \lambda(t)}{2r} \right) \]
Then, for \( r \geq \lambda(t) \), we re-write the formula for \( u_{ell,2} \) as follows (the first four integrals contain the terms which will be required in the matching process).

\[
(4.7) \\
\begin{align*}
\frac{d}{dr}u_{ell,2}(t, r) &= \phi_0\left(\frac{r}{\lambda(t)}\right) \int_1^{\frac{r}{\lambda(t)}} s^2 \lambda(t)^2 \varphi_1^2 u_{ell,main}(t, s\lambda(t))e_2(s)ds - e_2\left(\frac{r}{\lambda(t)}\right) \int_0^{1} s^2 \lambda(t)^2 \varphi_1^2 u_{ell,main}(t, s\lambda(t))\phi_0(s)ds \\
&- e_2\left(\frac{r}{\lambda(t)}\right) \int_1^{\infty} s^2 \lambda(t)^2 \varphi_1^2 u_{ell,1}(t, s\lambda(t))\phi_0(s)ds - e_2\left(\frac{r}{\lambda(t)}\right) \int_0^{1} s^2 \lambda(t)^2 \varphi_1^2 u_{ell}(t, s\lambda(t))\phi_0(s)ds \\
&+ \phi_0\left(\frac{r}{\lambda(t)}\right) \int_1^{\frac{r}{\lambda(t)}} s^2 \lambda(t)^2 \varphi_1^2 u_{ell,1}(t, s\lambda(t))e_2(s)ds + \phi_0\left(\frac{r}{\lambda(t)}\right) \int_0^{1} s^2 \lambda(t)^2 \varphi_1^2 u_{ell}(t, s\lambda(t))e_2(s)ds \\
&+ e_2\left(\frac{r}{\lambda(t)}\right) \int_1^{\infty} s^2 \lambda(t)^2 \varphi_1^2 u_{ell,1}(t, s\lambda(t))\phi_0(s)ds
\end{align*}
\]

4.3. **Third small \( r \) correction.** We eliminate the error term of \( u_{ell,2} \) with the correction \( u_{ell,3} \), which the following solution to

\[
\frac{d^2}{dr^2}u_{ell,3} + \frac{2}{r} \frac{d}{dr}u_{ell,3} + \frac{45 u_{ell,3}}{\lambda(t)^2 \left( \frac{r^2}{\lambda(t)^2} + 3 \right)} = \varphi_1^2 u_{ell,2}(t, r)
\]

\[ \tag{4.8} \]

4.4. **First large \( r \) correction.** We first note that

\[
\varphi_1^2 Q_{\lambda(t)}(r) = \frac{\sqrt{3} \left( 2\lambda(t)\lambda''(t) - \lambda'(t)^2 \right)}{4r\lambda(t)^{3/2}} + O\left( \frac{1}{r^3} \right), \quad r \to \infty
\]

Then, we let

\[ \tag{4.9} w_1(t, r) = w_{1,0}(t, r) + \frac{t}{2} \int_0^{\pi} \sin(\theta) \hat{v}_{2,0}(\sqrt{r^2 + t^2 + 2rt \cos(\theta)})d\theta := w_{1,0}(t, r) + \hat{v}_2(t, r) \]

where

\[ \hat{v}_{2,0}(r) = \frac{\psi(r)}{r} \left( -\frac{\sqrt{3} \lambda'(r)}{2\sqrt{\lambda(r)}} \right), \quad w_{1,0}(t, r) = \frac{\sqrt{3}}{r} \left( \sqrt{\lambda(r+t) - \sqrt{\lambda(t)}} \right) \]

and \( \psi \in C^\infty(\mathbb{R}) \),

\[ \psi(x) = \begin{cases} 
0, & x \leq T_\lambda \\
1, & x \geq 2T_\lambda 
\end{cases} \]

By direct computation, \( w_1 \) solves

\[ \tag{4.12} -\varphi_1^2 w_1 + \frac{d^2}{dr^2}w_1 + \frac{2}{r} \frac{d}{dr}w_1 = \frac{\sqrt{3} \left( 2\lambda(t)\lambda''(t) - \lambda'(t)^2 \right)}{4r\lambda(t)^{3/2}} \]
The main contribution of $w_1$ in the matching region is

$$w_{1,\text{main}}(t, r) = \sum_{j=0}^{5} \frac{r^j \partial_t^j \left( \frac{\sqrt{\lambda(t)}}{2 \sqrt{\lambda(t)}} \right)}{(j + 1)!} + t \tilde{v}_{2,0}(t) + \frac{1}{6} r^2 \partial_t^2 \left( t \tilde{v}_{2,0}(t) \right) + \frac{1}{120} r^4 \partial_t^4 \left( t \tilde{v}_{2,0}(t) \right)$$

$$:= \frac{r^5}{720} \partial_t^5 \left( \frac{\sqrt{\lambda(t)}}{2 \sqrt{\lambda(t)}} \right) + w_{1,\text{cm}}(t, r)$$

We also define

$$w_{1,\text{im}}(t, r) := \sum_{j=0}^{1} \frac{r^j \partial_t^j \left( \frac{\sqrt{\lambda(t)}}{2 \sqrt{\lambda(t)}} \right)}{(j + 1)!} + t \tilde{v}_{2,0}(t)$$

A direct application of the fundamental theorem of calculus (or Taylor’s theorem) gives

**Lemma 4.1.** For $0 \leq j, k \leq 2$ or $j = 3, k = 0,$

$$|\partial_t^j \partial_r^k (w_1 - w_{1,\text{main}})(t, r)| \leq \frac{C_j r^{j-k}}{r^{j+1}} \sqrt{\lambda(t)}, \quad r \leq \frac{t}{2}$$

For any $j \geq 0,$

$$|\partial_t^j w_{1,\text{im}}(t, r)| \leq \frac{C_j r \sqrt{\lambda(t)}}{t^{j+2}}, \quad |\partial_t^j (w_{1,\text{cm}} - w_{1,\text{im}})(t, r)| \leq \frac{C_j r^{j+3} \sqrt{\lambda(t)}}{t^{j+1}}$$

$$|\partial_t^j (w_1 - w_{1,\text{cm}})(t, r)| \leq \begin{cases} \frac{C_j r^{j+3} \sqrt{\lambda(t)}}{t^{j+1}} & r \leq \frac{t}{2} \\ \frac{C_j r \sqrt{\lambda(t)}}{t^{j+2}} \sup_{x \in [t, t+r]} \sqrt{\lambda(x)} \log(t+r) \left( \frac{1}{r} + \frac{1}{r^{j+1}} \right) + \frac{C_j r \sqrt{\lambda(t)}}{t^{j+2}} & r \geq \frac{t}{2} \end{cases}$$

Recalling (4.10), we also have the following estimates.

**Lemma 4.2.** For all $j, k \geq 0,$ there exists $C_{j,k} > 0$ such that for $r \leq \frac{t}{2},$

$$|\partial_t^j \partial_r^k w_1(t, r)| \leq \begin{cases} C_{j,k} \sqrt{\lambda(t)} \frac{r}{t^{j+2}} & \text{for } k = 0 \\ \frac{1}{t^{j+2}} & \text{for } k \geq 1 \end{cases}$$

and for $r \geq \frac{t}{2},$

$$|\partial_t^j \partial_r^k w_{1,0}(t, r)| \leq \frac{C_{j,k} \sup_{x \in [t, t+r]} \sqrt{\lambda(x)}}{t^{j+1}}, \quad |\partial_t^j \partial_r^k \tilde{v}_2(t, r)| \leq \frac{C_{j,k} \left( \sup_{x \in [T, t+r]} \sqrt{\lambda(x)} \right) \log(t+r)}{r \langle r-t \rangle^{j+k}}$$

$$|\langle \partial_t + \partial_r \rangle \tilde{v}_2| \leq \frac{C \left( \sup_{x \in [T, t]} \sqrt{\lambda(x)} \right) \log(t+r)}{tr}, \quad r \geq \frac{t}{2}$$

$$t |\langle \partial_t \tilde{v}_2(t, r) \rangle + |\partial_r \tilde{v}_2(t, r)| + |\tilde{v}_2(t, r)| | \leq \frac{C \left( \sup_{x \in [T, t]} \sqrt{\lambda(x)} \right)}{t}, \quad r \leq \frac{t}{2}$$
Proof. For $r \leq \frac{t}{2}$, we start with the definition of $w_1$, (4.9), and get

\begin{equation}
(4.16) \quad w_1(t, r) = \frac{\sqrt{3}}{2} \int_0^1 \frac{\lambda'(t + ry)}{\sqrt{\lambda(t + ry)}} dy + \frac{t}{2} \int_{-1}^1 \tilde{v}_{2,0} \left( \frac{\sqrt{r^2 + t^2 + 2rtu}}{\sqrt{y^2 + t^2 + 2yitu}} \right) du
\end{equation}

\begin{align*}
&= \frac{\sqrt{3}}{2} \int_0^1 dy \int_t^{t+ry} ds \partial_s \left( \frac{\lambda'(s)}{\sqrt{\lambda(s)}} \right) + \frac{t}{2} \int_{-1}^1 du \int_0^r dy \frac{\tilde{v}_{2,0} \left( \frac{\sqrt{r^2 + t^2 + 2yitu}}{\sqrt{y^2 + t^2 + 2yitu}} \right)}{\sqrt{y^2 + t^2 + 2yitu}} (y + tu) \\
&= \frac{\sqrt{3}}{2} \int_0^1 r dw(1 - w) \partial_s \left( \frac{\lambda'(s)}{\sqrt{\lambda(s)}} \right) \bigg|_{s=t+ry} + \frac{t}{2} \int_{-1}^1 dy \int_0^r du \frac{\tilde{v}_{2,0} \left( \frac{\sqrt{r^2 + t^2 + 2yitu}}{\sqrt{y^2 + t^2 + 2yitu}} \right)}{\sqrt{y^2 + t^2 + 2yitu}} (y + tu)
\end{align*}

The symbol-type estimates on $\lambda$, (1.6), allow for the differentiation under the integral sign in $t$ and $r$, for the first term of the last line of the equation above, and give, for $r \leq \frac{t}{2}$,

\begin{align*}
|\partial_t^j \partial_r^k \left( \frac{\sqrt{3}}{2} \int_0^1 r dy (1 - y) \partial_s \left( \frac{\lambda'(s)}{\sqrt{\lambda(s)}} \right) \bigg|_{s=t+ry} \right)| &\leq C_{j,k} \frac{\sqrt{\lambda(t)}}{t^{2+j}} \left( r, \quad k = 0 \right) \\
&= \frac{C_{j,k} \sqrt{\lambda(t)}}{t^{2+j}} \left( r, \quad k = 0 \right)
\end{align*}

where we used (4.11). To estimate the second term on the last line of (4.16) in the region $r \leq \frac{t}{2}$, we first note that (4.10) implies that

\begin{equation}
(4.17) \quad |\tilde{v}_{2,0}^{(k)} (r)| \leq \frac{C_k \sqrt{\lambda(r)}}{r^{2+k}} 1_{\{r \geq T_\lambda\}}
\end{equation}

Direct estimation gives, for $r \leq \frac{t}{2}$,

\begin{align*}
|\partial_t^j \partial_r^k \left( \frac{t}{2} \int_0^r dy \int_{-1}^1 du \frac{\tilde{v}_{2,0} \left( \frac{\sqrt{r^2 + t^2 + 2yitu}}{\sqrt{y^2 + t^2 + 2yitu}} \right)}{\sqrt{y^2 + t^2 + 2yitu}} (y + tu) \right) | &\leq C_{j,k} \frac{\sqrt{\lambda(t)}}{t^{2+j}} \left( r, \quad k = 0 \right) \\
&= \frac{C_{j,k} \sqrt{\lambda(t)}}{t^{2+j}} \left( r, \quad k = 0 \right)
\end{align*}

Finally, (4.15) is obtained by directly estimating (4.9). For $w_{1,0}$ in the region $r \geq \frac{t}{2}$, we simply directly estimate (4.10). For $\tilde{v}_2$, we use the following. If $f \in C^\infty([0, \infty))$, then, for $k \geq 1$, and $u \in \mathbb{R}$,

\begin{equation}
(4.18) \quad \partial_t^k \left( f \left( \sqrt{r^2 + t^2 + 2ru} \right) \right) = \sum_{l=0}^{k} \sum_{n=1}^{k} (r + u)^l c_{l,k,n} f^{(n)} \left( \frac{\sqrt{r^2 + t^2 + 2ru}}{r^2 + t^2 + 2ru} \right) \left( \frac{1}{r} \right)^{l-1}
\end{equation}

Then, directly differentiating the definition of $\tilde{v}_2$ in (4.9), we get

\begin{equation}
(4.19) \quad |\partial_t^k \tilde{v}_2(t, r)| \leq \frac{C_k \left( \sup_{x \in \mathbb{R}} \lambda(x) \right) \log(r + t)}{(r - t)^k}, \quad r \geq \frac{t}{2}
\end{equation}

Next, we let $X = t \partial_t + r \partial_r$, and get

\begin{align*}
X(\tilde{v}_2)(t, r) &= \tilde{v}_2(t, r) + \frac{t}{2} \int_0^{\pi} \sin(\theta) \sqrt{r^2 + t^2 + 2rt \cos(\theta)} \tilde{v}_{2,0} \left( \frac{\sqrt{r^2 + t^2 + 2rt \cos(\theta)}}{\sqrt{y^2 + t^2 + 2yitu}} \right) d\theta
\end{align*}

Using $X(\tilde{v}_2) = t(\partial_t + r \partial_r) \tilde{v}_2 + (r - t) \partial_r \tilde{v}_2$, we get (4.14). Next, again using (4.18) and (4.19), we get a first estimate:

\begin{equation}
(4.20) \quad |\partial_r^k \partial_t^r \tilde{v}_2| \leq \frac{C_k \left( \sup_{x \in \mathbb{R}} \lambda(x) \right) \log(r + t)}{r^{(r-t)^k}} \left( \frac{1}{t} + \frac{r}{t(t - r)} \right), \quad r \geq \frac{t}{2}
\end{equation}
Next, we use
\[
\tilde{v}_2(t,r) = \frac{1}{2} \int_{-t}^{t} dw \tilde{w}_{2,0}(\sqrt{r^2 + t^2 + 2rtu})
\]
and directly differentiate to get
\[
|\partial_r^k \partial_t \tilde{v}_2| \leq \frac{C_k \left( \sup_{x \in [T_s, t+r]} \sqrt{\lambda(x)} \right)}{\langle r - t \rangle^{2+k}}, \quad r \geq \frac{t}{2}
\]
Combining this with (4.20), we get
\[
|\partial_r^k \partial_t \tilde{v}_2(t,r)| \leq \frac{C_k \left( \sup_{x \in [T_s, t+r]} \sqrt{\lambda(x)} \log(t) \right)}{r \langle t - r \rangle^{k+1}}, \quad r \geq \frac{t}{2}
\]
Finally, we differentiate the equation solved by \( \tilde{v}_2 \), namely
\[
\left( -\partial_t^2 + \partial_r^2 + \frac{2}{r} \partial_r \right) \tilde{v}_2 = 0
\]
to get
\[
|\partial_r^k \partial_t^2 \tilde{v}_2| \leq \frac{C_{k,j} \left( \sup_{x \in [T_s, t+r]} \sqrt{\lambda(x)} \log(t + r) \right)}{r \langle t - r \rangle^{2+k+j}}, \quad r \geq \frac{t}{2}
\]
An induction argument then finishes the proof of the lemma \( \square \)

Next, let
\[
(4.21) \quad \text{RHS}(t,r) = \partial_t^2 Q_{\lambda(t)}(r) - \left( \frac{\sqrt{3} \left( 2\lambda(t)\lambda''(t) - \lambda'(t)^2 \right)}{4r\lambda(t)^{3/2}} \right)
\]
Note that the solution to
\[
\begin{cases}
-\partial_t^2 v_{ex,s} + \partial_r^2 v_{ex,s} + \frac{2}{r} \partial_r v_{ex,s} = 0 \\
v_{ex,s}(s,r) = 0 \\
\partial_t v_{ex,s}(s,r) = \text{RHS}(s,r)
\end{cases}
\]
is
\[
v_{ex,s}(t,r) = \frac{f_3(s,r - s-t) - f_3(s,|r-(s-t)|)}{\pi r}
\]
where
\[
(4.22) \quad f_3(s,x) = \frac{\pi}{2} \int_x^{\infty} r \text{RHS}(s,r) dr
\]
\[
= -\frac{\sqrt{3}\pi \left( 3x \lambda(s)^2 \left( \sqrt{3\lambda(s)^2 + x^2} + 6x \right) + x^3 \left( \sqrt{3\lambda(s)^2 + x^2} - x \right) + 27\lambda(s)^4 \right) \lambda'(s)^2}{8 \left( \lambda(s) \left( 3\lambda(s)^2 + x^2 \right) \right)^{3/2}}
\]
\[
- \frac{\pi \left( x \left( x - \sqrt{3\lambda(s)^2 + x^2} \right) + 9\lambda(s)^2 \right) \lambda''(s)}{4\sqrt{3}x^2 \lambda(s) + \lambda(s)^3}
\]
Then, we let $v_{ex}$ denote the following solution to
\begin{equation}
-\partial_t^2 v_{ex} + \partial_r^2 v_{ex} + \frac{2}{r} \partial_r v_{ex} = \text{RHS}(t, r)
\end{equation}
\begin{equation}
v_{ex}(t, r) = \int_t^\infty v_{ex,s}(t, r) ds
\end{equation}

Lemma 4.3. For all $j \geq 0$, there exists $C_j > 0$ such that
\begin{equation}
|\partial_t^j \partial_r^k v_{ex}(t, r)| \leq \begin{cases} 
\frac{C_j \lambda(t)^{3/2 - k} \log(t)}{t^{2+j}}, & r \leq \lambda(t) \\
\frac{C_j \sup_{x \in [t, t+1]} \{\lambda(x)^{3/2}\} \log(t+r)}{t^{3+j}}, & r \geq \lambda(t)
\end{cases}, \quad k = 0, 1
\end{equation}

In addition,
\begin{equation}
||\partial_t v_{ex}||_{L^2(\rho^2 dr)} + ||\partial_r v_{ex}||_{L^2(\rho^2 dr)} \leq \frac{C \lambda(t)}{t}
\end{equation}

Proof. By directly estimating (4.22) and its derivatives, we get
\begin{equation}
|\partial_t^j \partial_r^k f_3(s, x)| \leq \frac{C_{j,k} \lambda(s)^{5/2}}{s^{2+j}} \frac{1}{(\max\{x, \lambda(s)\})^{k+1}}
\end{equation}

Then, from the definition of $v_{ex}$ (4.21), we get
\begin{equation}
v_{ex}(t, r) = \int_t^\infty dy \frac{f_3(t+y-r, y)}{\pi r} - \int_0^r dy \frac{f_3(r+y, y) - f_3(t+r, y)}{\pi r}
\end{equation}

Differentiation under the integral sign is possible by (4.27), and direct estimation of the integrals gives (4.25) for $r \geq \lambda(t)$. Direct estimation of the above integral also gives (4.25) for $r \leq \lambda(t)$, and $k = 0$. Finally, direct estimation of the $r$ derivative of
\begin{equation}
\partial_r v_{ex}(t, r) = -\frac{1}{\pi} \int_0^1 dz \partial_r^j f_3(t+r(1-z), rz) - \frac{1}{\pi} \int_0^1 dz \partial_j^1 f_3(t+r(1+z), rz) + \int_r^\infty dy \frac{1}{\pi} \int_0^1 \partial_j^{j+1} f_3(rz+r+y, y) dz
\end{equation}
finishes the proof of (4.25). For the energy estimate, (4.26), we note that
\begin{equation}
\partial_r v_{ex}(t, r) + \frac{v_{ex}(t, r)}{r} = \int_t^\infty ds \left( -\frac{1}{2r} \right) ((r+s-t)\text{RHS}(s, r+s-t) + (s-t-r)\text{RHS}(s, |s-t-r|))
\end{equation}
\begin{equation}
\partial_t v_{ex}(t, r) = \int_t^\infty ds \left( -\frac{1}{2r} \right) (-r-s-t)\text{RHS}(s, r+s-t) + (s-t-r)\text{RHS}(s, |s-t-r|)
\end{equation}

Then, Minkowski’s inequality and direct estimation of (4.21) give
\begin{equation}
||\partial_r v_{ex} + \frac{v_{ex}}{r}||_{L^2(\rho^2 dr)} + ||\partial_t v_{ex}(t, r)||_{L^2(\rho^2 dr)} \leq \frac{C \lambda(t)}{t}
\end{equation}

Finally, since $||\partial_r v_{ex} + \frac{v_{ex}}{r}||_{L^2(\rho^2 dr)} < \infty$, the dominated convergence theorem shows that
\begin{equation}
||\partial_r v_{ex} + \frac{v_{ex}}{r}||_{L^2(\rho^2 dr)} = \lim_{M \to \infty} \left( \int_0^M r^2 \partial_r r v_{ex}^2 dr \right)
\end{equation}

Then, (4.25) and the monotone convergence theorem, show that
\begin{equation}
||\partial_r v_{ex} + \frac{v_{ex}}{r}||_{L^2(\rho^2 dr)} = \int_0^\infty r^2 \partial_r r v_{ex}^2 dr
\end{equation}

which completes the proof of the lemma. \qed
We define $v_{\text{ex},\ell}$ to be the following solution to

$$\partial_t^2 v_{\text{ex},\ell} + \frac{2}{r} \partial_r v_{\text{ex},\ell} = \text{RHS}$$

$$v_{\text{ex},\ell}(t,r) = -\frac{1}{r} \int_0^r y^2 \text{RHS}(t,y) dy - \int_r^\infty y \text{RHS}(t,y) dy$$

$$= \sqrt{3} \lambda'(t)^2 \left( r \left( r - \sqrt{3 \lambda(t)^2 + r^2} \right) + \lambda(t)^2 \left( 45 \sinh^{-1} \left( \frac{r}{\sqrt{3 \lambda(t)}} \right) - \frac{24r}{\sqrt{3 \lambda(t)^2 + r^2}} \right) \right)$$

$$+ \frac{8r \lambda(t)^{3/2}}{4r \sqrt{\lambda(t)}}$$

It will be convenient to define the following principal part of $v_{\text{ex},\ell}$ in the matching region:

$$v_{\text{ex},\ell,0}(t,r) = 3 \sqrt{3} \lambda(t)^{3/2} \left( \lambda'(t)^2 \left( 15 \log \left( \frac{r}{4 \lambda(t)^2} \right) + 30 \log(r) - 17 \right) + 2 \lambda(t) \lambda''(t) \left( 5 \log \left( \frac{r}{4 \lambda(t)^2} \right) + 10 \log(r) + 1 \right) \right)$$

A direct computation gives

**Lemma 4.4.** For $0 \leq j + k \leq 5$,

$$|\partial_t^j \partial_r^k (v_{\text{ex},\ell}(t,r) - v_{\text{ex},\ell,0}(t,r))| \leq C \lambda(t)^{3/2} \left( \frac{(1+|\log(\frac{r}{\lambda(t)}))| \lambda(t)}{\lambda(t)^3 + r^{j+k}}, \quad r \leq \lambda(t) \right)$$

$$v_{\text{ex},\text{sub}}(t,r) := v_{\text{ex}}(t,r) - v_{\text{ex},\ell}(t,r)$$

solves

$$-\partial_t^2 v_{\text{ex},\text{sub}} + \frac{2}{r} \partial_r v_{\text{ex},\text{sub}} = \partial_t^2 v_{\text{ex},\ell}$$

In fact, we have the following formula for $v_{\text{ex},\text{sub}}$:

** Lemma 4.5.**

$$v_{\text{ex},\text{sub}}(t,r) = \int_0^\infty dx \int_{|r-x|}^{r+x} ds \left( -\frac{x}{2r} \right) \partial_t^2 v_{\text{ex},\ell}(t+s,x)$$

**Proof.**

$$v_{\text{ex}}(t,r) = \int_t^{\infty} ds \left( -\frac{1}{2r} \int_{|r-(s-t)|}^{r+s-t} y \text{RHS}(s,y) dy \right) = \int_0^\infty dy \int_{|r-y|}^{r+y} dw \left( -\frac{y}{2r} \text{RHS}(t+w,y) \right)$$
Integrating by parts in $w$, we get
\[
v_{ex}(t, r) = \int_0^\infty dy \left( \frac{-y}{2r} \right) (RHS(t + r + y, y)(r + y) - RHS(t + |r - y|, y)|r - y|) + \frac{1}{2r} \int_0^\infty dy \int_{|r - y|}^{r + y} w \partial_1 RHS(t + w, y) dw
\]
\[
= \int_0^\infty dy \left( \frac{-y}{2r} \right) (RHS(t, y)(r + y) - RHS(t, y)|r - y|) + \frac{1}{2r} \int_0^\infty dy \int_{|r - y|}^{r + y} u \partial_1 RHS(t + w, y) dw
\]
\[
= \int_0^\infty dy \left( \frac{-y}{2r} \right) ((RHS(t + r + y, y) - RHS(t, y))(r + y) - (RHS(t + |r - y|, y) - RHS(t, y)|r - y|)
\]
\[
+ \frac{1}{2r} \int_0^\infty dy \int_{|r - y|}^{r + y} w \partial_1 RHS(t + w, y) dw
\]

The first term on the right-hand side of the last equality above is $v_{ex, ell}$. So,

\[
v_{ex, sub}(t, r)
= \int_0^\infty dy \left( \frac{-y}{2r} \right) \left( \int_{|r - y|}^{r + y} \partial_1 RHS(t + w, y)(r + y - w) dw - \int_0^{|r - y|} \partial_1 RHS(t + w, y)(|r - y| - w) dw \right)
\]
\[
= \int_0^\infty dy \left( \frac{-y}{2r} \right) \left( \partial_1 RHS(t + r + q, y) - \partial_1 RHS(t + |r - q|, y)(q + y - |q - y|) \right)
\]
\[
= \int_0^\infty dx \int_{|x - r|}^{r + x} ds \left( -\frac{x}{2r} \right) \int_0^\infty dy \left( \frac{-y}{2r} \right) \partial_1^2 RHS(t + s, y)(x + y - |x - y|)
\]
\[
= \int_0^\infty dx \int_{|x - r|}^{r + x} ds \left( -\frac{x}{2r} \right) \partial_1^2 v_{ex, ell}(t + s, y)
\]

\( \square \)

The leading part of $v_{ex, sub}$ in the matching region is $v_{ex, sub, ell}$ (as the next lemma shows), given by

\[
(4.30) \quad v_{ex, sub, ell}(t, r) = -\frac{1}{r} \int_{\lambda(t)}^r x^2 \partial_1^2 v_{ex, ell, 0}(t, x) dx + \int_{\lambda(t)}^r x \partial_1^2 v_{ex, ell, 0}(t, x) dx
\]

In particular, $v_{ex, sub, ell}$ solves

\[
\partial_r^2 v_{ex, sub, ell} + \frac{2}{r} \partial_r v_{ex, sub, ell} = \partial_1^2 v_{ex, ell, 0}
\]

**Lemma 4.6.** For $j = 0, 0 \leq k \leq 2$ and $j = 1, 0 \leq k \leq 1$, and in the region $\lambda(t) \leq r \leq t$,

\[
|\partial_1^j \partial_r^k (v_{ex, sub}(t, r) - v_{ex, sub, ell}(t, r))| \leq \int_t^\infty ds \left| g_3(s) + \int_t^\infty ds \left| g_2(s) \log \left( \frac{4(s - t)^2}{3\lambda(s)^2} \right) \right| \right|
\]

\[
\leq \begin{cases} 
C\lambda(t)^{5/2} \log(t) + C\lambda(t)^{7/2} \log^2(t), & j = k = 0 \\
C\lambda(t)^{5/2} \log^2(t) t^4, & j = 1, k = 0 \\
C\lambda(t)^{5/2} \log(t) + C\lambda(t)^{7/2} \log^2(t), & j = 0, k = 1 \\
C\lambda(t)^{5/2} \log^2(t) t, & j + k = 2
\end{cases}
\]
where

\[ g_1(s) = \frac{3\sqrt{3} \sqrt{\lambda(s)} ((-17) \lambda'(s)^2 + 2 \lambda(s) \lambda''(s))}{16}, \quad g_2(s) = \frac{3\sqrt{3} \sqrt{\lambda(s)} (15 \lambda'(s)^2 + 10 \lambda(s) \lambda''(s))}{16} \]

\[ g_3(s) = \frac{2g_2(s)((\lambda'(s))^2 - \lambda(s) \lambda''(s)) + \lambda(s)(-4 \lambda'(s)g_2(s) + \lambda(s)g''_2(s))}{\lambda(s)^2} \]

Proof. We start with the definition (recall (4.28))

\[ v_{\text{ex,sub},0}(t, r) := \int_t^\infty ds \int_{s - t - r}^{s - t + r} dy \left( \frac{-y}{2r} \right)^2 c_s^2 v_{\text{ex, ell},0}(s, y) = \int_t^\infty ds \left( \frac{-1}{2r} \right) \int_{s - t - r}^{s - t + r} dy (g_3(s) + \log(\frac{4y^2}{3\lambda(s)^2}) g''_2(s)) \]

where \( g_1, g_2, g_3 \) are as in the lemma statement. We compute the \( y \) integrals, and decompose the \( s \) integrals in the following form, which makes it clear which terms will cancel when \( v_{\text{ex,sub,ell}} \) is subtracted. (Whenever \( y \) appears in integrals below, it arises from the change of variable \( s = t + ry \)).

\[ v_{\text{ex,sub},0}(t, r) = \frac{r}{4} \left( 2 + \log(\frac{16r^4}{9\lambda(t)^4}) \right) g''_2(t) + \frac{r}{2} g_3(t) - \int_t^\infty ds g_3(s) + \int_t^{t+r} ds (g_3(s) - g_3(t)) \left( 1 - \frac{(s-t)}{r} \right) \]

\[- \frac{1}{2} \int_0^1 dy (g''_2(t + r y) - g''_2(t)) r \left( (1 + y)(-2 + \log(\frac{4}{3\lambda(t)^2}(1 + y)^2)) \right. \]

\[ + (1 - y)(2 + \log(\frac{3\lambda(t)^2}{4r^2(1 - y)^2})) \]

\[- \frac{1}{2} \int_0^1 dy g''_2(t + r y) r(2y) \log(\frac{\lambda(t)^2}{\lambda(t + r y)^2}) - \int_t^\infty ds g''_2(s)(2 + \log(\frac{4(s-t)^2}{3\lambda(s)^2})) \]

\[ + \int_t^{t+r} ds (g''_2(s) - g''_2(t)) \log(\frac{4(s-t)^2}{3\lambda(s)^2}) - 2 \int_t^{t+r} ds g''_2(t) \log(\frac{\lambda(s)}{\lambda(t)}) \]

\[ + r (g''_2(t + r) - g''_2(t))(-1 + \log(4)) \]

\[ + r^2 \int_1^\infty \left( y \log(1 - \frac{1}{y^2}) + \coth^{-1}(y) - y + \frac{y^2}{2} \log(1 + \frac{2}{y - 1}) \right) g''_2(t + r y) dy - 2(g''_2(t) + r g''_2(t)) \]

We remark that the last line of the above expression was obtained using integration by parts. We note that

\[ \int_0^r c_t^2 v_{\text{ex, ell},0}(t, x)x dx - \frac{1}{r} \int_0^r c_t^2 v_{\text{ex, ell},0}(t, x)x^2 dx = \frac{1}{4} rg''_2(t) \left( \log\left(\frac{16r^4}{9\lambda(t)^4}\right) + 2 \right) - 2rg''_2(t) + \frac{1}{2} r g_3(t) \]
Therefore, recalling (4.30), we have

\[ v_{ex,sub,0} - v_{ex,sub,ell} + \int_t^{t+r} ds g_3(s) + \int_t^{t+r} ds g_2''(s) \log(\frac{4(s-t)^2}{3\lambda(s)^2}) \]

\[ = \int_t^{t+r} ds (g_3(s) - g_3(t))(1 - \frac{(s-t)}{r}) \]

\[ - \frac{r}{2} \int_0^1 dy (g''_2(t + ry) - g''_2(t)) \left( (1 + y)(-2 + \log(\frac{4r^2}{3\lambda(t)^2}(1 + y)^2)) + (1 - y)(2 + \log(\frac{3\lambda(t)^2}{4r^2(1 - y)^2})) \right) \]

\[ - r \int_0^{t+r} dy (g''_2(t + ry)\log(\frac{\lambda(t)^2}{\lambda(t+y)^2}) + \int_t^{t+r} ds (g''_2(s) - g''_2(t)) \log(\frac{4(s-t)^2}{3\lambda(s)^2}) \]

\[ - 2 \int_t^{t+r} ds g''_2(t) \log(\frac{\lambda(s)}{\lambda(t)}) + r(g''_2(t + r) - g''_2(t))(-1 + \log(4)) \]

\[ + r^2 \int_1^\infty \left( y \log(1 - \frac{1}{y}) + \coth^{-1}(y) - y + \frac{y^2}{2} \log(1 + \frac{2}{y-1}) \right) g''_2(t + ry) dy \]

\[ + \int_0^{\lambda(t)} \frac{\lambda(t)}{t q} \xi^2 v_{ex,ell,0}(t,x) dx - \frac{1}{r} \int_0^{\lambda(t)} \xi^2 v_{ex,ell,0}(t,x) x^2 dx \]

From Lemma 4.4 we thus get for \( j = 0, 0 \leq k \leq 2 \) and \( j = 1, 0 \leq k \leq 1 \),

\[ |\xi^j v_r^k (v_{ex,sub,0} - v_{ex,sub,ell}) + \int_t^\infty ds g_3(s) + \int_t^\infty ds g_2''(s) \log(\frac{4(s-t)^2}{3\lambda(s)^2})| \]

\[ \leq \frac{C r^{2-k} \lambda(t)^{5/2} \log(t)}{t^{5+j}} + \frac{C \lambda(t)^{7/2} \log(t)}{t^{5+j}}, \quad \lambda(t) \leq r \leq t \]

Next, for \( F(s,y) = \xi^2 v_{ex,ell}(s,y) - v_{ex,ell,0}(s,y) \), we have

\[ v_{ex,sub}(t,r) - v_{ex,sub,0}(t,r) = \int_t^\infty ds \int_{|s-t-r|}^{s-t+r} dy \left( \frac{-y}{2r} \right) \xi^2 \left( v_{ex,ell}(s,y) - v_{ex,ell,0}(s,y) \right) \]

\[ = \int_0^{r} dy \left( \frac{-y}{2} \right) F(t,y) + \int_r^{2r} dy \left( 2 - \frac{y}{r} \right) \left( \frac{-y}{2} \right) F(t,y) \]

\[ - \frac{1}{2} \int_0^1 dz \int_{r(1-z)}^{r(1+z)} dy (F(t + rz, y) - F(t,y)) \]

\[ - \frac{1}{2r} \int_t^{t+r} ds \int_{s-t-r}^{s-t+r} dy (F(s,y) - 2r(s-t)F(s,s-t)) \]

\[ - \int_t^{t+r} ds (s-t)F(s,s-t) + \int_t^{t+r} ds(s-t)F(s,s-t) \]

(4.31)

Direct estimation gives, in the region \( \lambda(t) \leq r \leq t \),

\[ |v_{ex,sub}(t,r) - v_{ex,sub,0}(t,r) + \int_t^\infty ds(s-t)F(s,s-t)| \leq \frac{C r^{2} \lambda(t)^{5/2} \log(t)}{t^{5}} + \frac{C \lambda(t)^{7/2} \log^2(t)}{t^{4}} \]

We now consider the derivatives of \( v_{ex,sub}(t,r) - v_{ex,sub,0}(t,r) + \int_t^\infty ds(s-t)F(s,s-t) \). For the first \( r \) derivative, we directly differentiate (4.31). To estimate the first time derivative, we start with

\[ v_{ex,sub}(t,r) - v_{ex,sub,0}(t,r) = \int_0^{\infty} dw \int_{|w-r|}^{w+r} dy \left( \frac{-y}{2r} \right) F(t+w, y) \]
and note that Lemma 4.4 justifies the differentiation under the integral sign in \( t \), leading to

\[
\partial_t (v_{ex, sub} - v_{ex, sub, 0}) = \int_t^\infty ds \int_{s-t-r}^{s-t+r} dy \left( -\frac{y}{2r} \right) \hat{c}_1 F(s, y)
\]

This expression, as well as its \( r \) and \( t \) derivatives are then directly estimated. Finally, we can infer an estimate on \( \partial_r^2 (v_{ex, sub} - v_{ex, sub, 0}) \) from our work up to this point, as soon as we justify the statement that

\[
(4.32) \quad \left( -\partial_t^2 + \partial_r^2 + \frac{2}{r} \partial_r \right) (v_{ex, sub} - v_{ex, sub, 0}) = \partial_t^2 (v_{ex, ell} - v_{ex, ell, 0})
\]

From (4.29), it suffices to study \( v_{ex, sub, 0} \). We note that

\[
v_{ex, sub, 0}(t, r) = \frac{-1}{2} \int_0^1 dy (g_3(t + ry) \cdot 2ry + g_2^2(t + ry) \left( r(1 + y)(-2 + \log(\frac{4r^2(1 + y)^2}{3\lambda(t + ry)^2})) \right)
+ r(1 - y)(2 + \log(\frac{3\lambda(t + ry)^2}{4r^2(1 - y)^2})))
- \frac{1}{2} \int_1^\infty dy (g_3(t + ry) \cdot 2ry + g_2^2(t + ry) \left( r(1 + y)(-2 + \log(\frac{4r^2(1 + y)^2}{3\lambda(t + ry)^2})) \right)
+ r(y - 1)(2 + \log(\frac{3\lambda(t + ry)^2}{4r(1 - y)^2})))
\]

Now, we can directly differentiate under the integral sign in \((t, r)\), given the symbol type estimates and smoothness of \( \lambda \), and then integrate by parts, to show that \( v_{ex, sub, 0} \) solves

\[
\left( -\partial_t^2 + \partial_r^2 + \frac{2}{r} \partial_r \right) (v_{ex, sub, 0}) = \partial_t^2 (v_{ex, ell, 0})
\]

which establishes (4.32). Finally, a direct estimation of \( \partial_t \left( \int_t^\infty ds (s-t) \partial_t^2 (v_{ex, ell}(s, y) - v_{ex, ell, 0}(s, y)) \right) \big|_{y=s-t} \)
gives: for \( 0 \leq k \leq 2 \) and \( j = 0 \) or \( 0 \leq k \leq 1 \) and \( j = 1 \),

\[
|\partial_t^j \partial_r^k \left( v_{ex, sub}(t, r) - v_{ex, sub, 0}(t, r) \right) + \int_t^\infty ds (s-t) \partial_t^2 (v_{ex, ell}(s, y) - v_{ex, ell, 0}(s, y)) \big|_{y=s-t} | \leq \begin{cases} \frac{C \lambda(\frac{1}{2}) \log(t)}{\sqrt{t}} & j = 0, k = 0 \\
\frac{C \lambda(\frac{1}{2}) \log(t) \sqrt{t}}{t} & j = 1, k = 0 \\
\frac{C \lambda(\frac{1}{2}) \log(t) \sqrt{t}}{rt} & j = 0, k = 1 \\
\frac{C \lambda(\frac{1}{2}) \log(t) \sqrt{t}}{rt^4} & j + k = 2 \end{cases}
\]

which finishes the proof of the lemma. \( \square \)

Finally, we consider the free wave \( v_2 \) solving

\[
(4.33) \quad \begin{cases} -\partial_t^2 v_2 + \partial_r^2 v_2 + \frac{2}{r} \partial_r v_2 = 0 \\
v_2(0, r) = 0, \quad \partial_r v_2(0, r) := v_{2, 0}(r) = \frac{\psi(r)}{r} \left( -15 \pi \sqrt{\lambda(r) \lambda''(r)} \right)
\end{cases}
\]

(This specific choice of \( v_{2, 0} \) is made so that \( w_1 + v_2 \) matches \( u_{ell} \) to an appropriate order, as will be shown in Section 4.6, see also the discussion following (3.3), and the comment after (4.54)). We have

\[
v_2(t, r) = \frac{t}{2} \int_0^\pi \sin(\theta) v_{2, 0}(\sqrt{r^2 + t^2 + 2rt \cos(\theta)}) d\theta
\]
We define \( v_{2,\text{main}} \) to be the first three terms in a small \( r \) expansion of \( v_2 \): (recall that \( t \geq T_0 > 2T_\lambda \), so \( \psi(t) = 1 \))

\[
v_{2,\text{main}}(t, r) = tv_{2,0}(t) + \frac{1}{6} r^2 \left( 2v'_{2,0}(t) + tv'''_{2,0}(t) \right) + \frac{1}{120} r^4 \left( 4v'''_{2,0}(t) + tv'''_{2,0}(t) \right)
\]

It will also be convenient to consider various pieces of \( v_{2,\text{main}} \):

\[
(4.34)
\]

\[
v_{2,\text{qm}}(t, r) = -\frac{15}{8} \left( \pi \sqrt{\lambda(t)} \lambda''(t) \right) + \frac{1}{6} r^2 \left( 2v_{2,0}(t) + tv''_{2,0}(t) \right) := v_{2,\text{lm}}(t, r) + \frac{1}{6} r^2 \left( 2v_{2,0}(t) + tv''_{2,0}(t) \right)
\]

We also note that

\[
(4.35)
\]

\[
|v^{(k)}_{2,0}(r)| \leq \frac{C_{k,\lambda}(r)}{r^{3+k}}
\]

Using the identical procedures as in Lemmas \ref{lem:4.1} \ref{lem:4.2} we get:

**Lemma 4.7.** For \( 0 \leq j, k \leq 2 \), or \( j = 3, k = 0 \)

\[
|\partial_t^j \partial_t^k (v_2 - v_{2,\text{main}})(t, r)| \leq \frac{C_{r,\lambda} t^j}{r^{3+j}}, \quad r \leq \frac{t}{2}
\]

For \( j \geq 0 \),

\[
|\partial_t^j v_{2,\text{lm}}(t, r)| \leq \frac{C_{r,\lambda} t^j}{t^{j+1}}, \quad \frac{t}{2} \leq r \leq \frac{t}{2}
\]

\[
|\partial_t^j (v_2 - v_{2,\text{qm}})| \leq \begin{cases} C_{r,\lambda} t^j / t^{j+1}, & \frac{t}{2} \leq r \leq \frac{t}{2} \\ C_{r,\lambda} t^j / t^{j+1}, & r \geq \frac{t}{2} \end{cases}
\]

**Lemma 4.8.** For all \( j, k \geq 0 \), there exists \( C_{j, k} > 0 \), such that

\[
|\partial_t^k \partial_t^j (v_2(t, r))| \leq \frac{C_{j, k}}{r^{j+k+1}}, \quad r \leq \frac{t}{2}
\]

\[
|\partial_t^k \partial_t^j v_2(t, r)| \leq \frac{C_{j, k}}{r^{j+k+1}}, \quad r \geq \frac{t}{2}
\]

\[
|\partial_t^k \partial_t^j (v_2(t, r))| \leq \frac{C_{j, k}}{r^{j+k+1}}, \quad r \geq \frac{t}{2}
\]

Letting \( v_1 = w_1 + v_\psi + v_2 \), we thus have a particular solution to

\[
-\partial_t^2 v_1 + \partial_r^2 v_1 + \frac{2}{r} \partial_r v_1 = \partial_t^2 Q_\lambda(t)(r)
\]

**4.5. Second large \( r \) correction.** Now, we need to correct the linear error term from \( v_1 \), which is (recall (2.2))

\[
V(t, r)(w_1 + v_2) := RHSS_2(t, r)
\]

From Lemmas \ref{lem:4.2} and \ref{lem:4.8} we get

\[
(4.36)
\]

\[
|\partial_t^r RHSS_2(t, r)| \leq \begin{cases} \frac{C_{j, k}}{r^{j+k+1}}, & r \leq \frac{t}{2} \\ \frac{C_{j, k}}{r^{j+k+1}}, & r \geq \frac{t}{2} \end{cases}
\]

\[
\left( \sup_{x \in [T_\lambda, t + \tau]} \sqrt{\lambda(x)} + \left( \sup_{x \in [T_\lambda, t + \tau]} \lambda(x) \right)^{3/2} \left( \frac{1}{\tau^2} + \frac{1}{\tau} \right) \right),
\]

\[
\geq \frac{t}{2}
\]
We consider the particular solution to

\[ -\partial_t^2 u_{w,2} + \partial_r^2 u_{w,2} + \frac{2}{r} \partial_r u_{w,2} = RHS_2(t, r) \]

given by

\[ u_{w,2}(t, r) = \int_t^\infty ds u_{w,2,s}(t, r) \]

where \( u_{w,2,s} \) solves

\[
\begin{cases}
-\partial_t^2 u_{w,2,s} + \partial_r^2 u_{w,2,s} + \frac{2}{r} \partial_r u_{w,2,s} = 0 \\
 u_{w,2,s}(s, r) = 0 \\
 \partial_t u_{w,2,s}(s, r) = RHS_2(s, r)
\end{cases}
\]

Therefore,

\[ u_{w,2}(t, r) = \int_t^\infty ds \left( \frac{-1}{2r} \int_{|r-(s-t)|}^{r+s-t} yRHS_2(s, y) dy \right) \]

**Lemma 4.9.** For \( r \geq \lambda(t) \),

\[ |u_{w,2}(t, r)| \leq \frac{C}{rt^2} \sup_{x \in [T \lambda, t+r]} \left( \lambda(x)^{5/2} \right) \log^2(t + r) \]

\[ |\partial_r u_{w,2}(t, r)| \leq \frac{C}{rt^2} \sup_{x \in [T \lambda, t+r]} \left( \lambda(x)^{5/2} \right) \log^2(t + r) \left( \frac{1}{r} + \frac{1}{t} \right) + \frac{C}{rt^2} \sup_{x \in [T \lambda, t+r]} \left( \lambda(x)^{3/2} \right) \log^2(t + r) \]

\[ ||\partial_t u_{w,2}(t, r)||_{L^2(r^2 dr)} + ||\partial_r u_{w,2}(t, r)||_{L^2(r^2 dr)} \leq \frac{C\lambda(t)}{t} \]

**Proof.** A direct estimation using (4.36) gives (4.39) and (4.40). The energy estimate, (4.41) is proven with the same procedure used to establish (4.26). The only detail to note is that the following rough estimate

\[ |u_{w,2}(t, r)| \leq \frac{C \sqrt{\lambda(t)}}{t}, \quad r > 0 \]

which results from directly inserting (4.36) into (4.38), verifies that

\[ ru_{w,2}(t, r)^2 \to 0, \quad r \to 0 \]

The leading part of \( u_{w,2} \) in the matching region is:

\[ u_{w,2,ell}(t, r) = -\frac{1}{r} \int_0^r dy y^2 RHS_2(t, y) - \int_r^\infty dy y RHS_2(t, y) \]

which solves

\[ \partial_r^2 u_{w,2,ell} + \frac{2}{r} \partial_r u_{w,2,ell} = RHS_2(t, r) \]
Lemma 4.10. We have the following estimates on $u_{w,2,ell}$. For $k \geq 0$,

$$|\partial_t^k u_{w,2,ell}(t, r)| \leq \begin{cases} \frac{C_k \lambda(t)^{3/2}}{t^{r+k}}, \quad r \leq \lambda(t) \\ \frac{C_k \log^2(t)(1+\log(\frac{t}{r}))\lambda(t)^2 \sup_{x \in [T_x,t]} \sqrt{\lambda(x)}}, \quad \lambda(t) \leq r \leq \frac{1}{2} \end{cases}$$

If $r \geq \frac{1}{2}$,

$$|\partial_t^k u_{w,2,ell}(t, r)| \leq \frac{C_k \lambda(t)^2 \log^2(r)}{rt^2} \left( \frac{\sup_{x \in [T_x,t+r]} \sqrt{\lambda(x)}}{t^k} + \sup_{x \in [T_x,t+r]} \lambda(x)^{3/2} \right) \left( \frac{1}{(t-r)^{1+k}} + \frac{1}{r} \right), \quad k = 0, 1$$

$$\left( \sup_{x \in [T_x,t+r]} \sqrt{\lambda(x)} + \sup_{x \in [T_x,t+r]} \lambda(x)^{3/2} \left( \frac{1}{(r-t)^{1+k}} + \frac{1}{r} \right) \right), \quad k \geq 2$$

Proof. If $r \leq \frac{1}{2}$,

$$\partial_t^k u_{w,2,ell}(t, r) = -\frac{1}{r} \int_0^r dyy^2 \partial_t^k RHS_2(t, y) - \int_{\frac{t}{2}}^r dyy^2 \partial_t^k RHS_2(t, y) - \int_\frac{r}{2}^\infty dyy^2 \partial_t^k (V(t, y)w_{1,0}(t, y))$$

$$- \int_\frac{r}{2}^\infty dyy^2 \partial_t^k (V(t, y)(v_2(t, y) + \tilde{v}_2(t, y)))$$

We directly estimate all integrals except for the last one, using Lemmas 4.2, 4.8. Since the estimates on $t$ derivatives of $v_2$ and $\tilde{v}_2$ in Lemmas 4.8 and 4.2, respectively, are not a power of $t$ better than the corresponding estimates on $v_2(t, r)$ and $\tilde{v}_2(t, r)$, in the region $r \geq \frac{1}{2}$, we treat the last integral term as follows. We have

$$\partial_t^k (V(t, y)(v_2 + \tilde{v}_2)) = \sum_{j=0}^k \binom{k}{j} \partial_t^{k-j} (V(t, y)) \partial_t^j (v_2 + \tilde{v}_2)$$

Let $H(f) = \partial_y^2 f + \frac{2}{y} \partial_y f$. If $j \geq 1$, we write (for both $v_2$ and $\tilde{v}_2$)

$$\frac{\partial_t^j v_2(t, y)}{H^{\frac{j}{2}} v_2(t, y)}, \quad j \text{ is even}$$

$$\frac{\partial_t^j v_2(t, y)}{H^{\frac{j-1}{2}} (\partial_t v_2)(t, y)}, \quad j \text{ is odd}$$

Then, we integrate by parts in $y$, integrating the terms in (4.45), using the following observation. If $f$ and $g$ are smooth functions of $y \in (0, \infty)$, $0 < a < b$, and $n \geq 1$, then,

$$\int_a^b dyy^2 g(y) H^n(f)(y) = \sum_{j=1}^n (-a^2 H^{j-1}(g)(a) \partial_y H^{n-j} f(a) + b^2 H^{j-1}(g)(b) \partial_y H^{n-j} f(b) + a^2 \partial_y (H^{j-1}(g)(a) H^{n-j} f(a) - b^2 \partial_y (H^{j-1}(g)(b) H^{n-j} f(b)) + \int_a^b f H^n(g) y^2 dy$$

In the case when $j$ is odd in (4.45), this process results in an integral involving $\partial_t v_2$, which we further re-write as $\partial_t v_2(t, y) = (\partial_t + \partial_y) v_2(t, y) - \partial_y v_2(t, y)$, and integrate by parts in $y$ for the $\partial_y v_2$.
term. (The point is that \((\partial_y + \partial_v) v_2(t, y)\) has better estimates near \(y = t\) than just \(\partial v_2(t, y)\), as per Lemmas 4.8 and 4.2). In the region \(r \geq \frac{t}{2}\), we start with

\[
\partial_t^k u_{w, 2, \text{ell}}(t, r) = -\frac{1}{r} \int_0^r dy y^2 \partial_t^k RHS_2(t, y) - \frac{1}{r} \int_0^r dy y^2 \partial_t^k (V(t, y)w_{1, 0}) - \frac{1}{r} \int_r^\infty dy y^2 \partial_t^k (V(t, y)(v_2 + \tilde{v}_2)) - \int_r^\infty dy y^2 \partial_t^k (V(t, y)v_2)
\]

Then, we use a similar procedure as above, integrating by parts for the sum of the third and fifth integrals above. Here, the only extra detail to note is that the boundary term at \(y = r\) from one integration by parts for the third integral cancels the boundary term at \(y = r\) from one integration by parts for the fifth integral. This allows for slightly better estimates than otherwise, and finishes the proof of the lemma.

We define \(u_{w, 2, \text{sub}}(t, r) = u_{w, 2}(t, r) - u_{w, 2, \text{ell}}(t, r)\). Given the equations that \(u_{w, 2}\) and \(u_{w, 2, \text{ell}}\) solve, namely (4.37) and (4.44), \(u_{w, 2, \text{sub}}\) solves

\[-\partial_t^2 u + \partial_r^2 u + \frac{2}{r} \partial_r u = \partial_t^k u_{w, 2, \text{ell}}\]

As in Lemma 4.13 we have

**Lemma 4.11.**

\[u_{w, 2, \text{sub}}(t, r) = \int_0^\infty dx \int_0^{r+x} ds \left( -\frac{x}{2r} \right) \partial_t^2 u_{w, 2, \text{ell}}(t + s, x)\]

Let

\[RHS_{2,0}(t, r) = V(t, r) (w_{1,lm} + v_{2,lm}), \quad RHS_{2,0} = V(t, r) (w_{1,cm} + v_{2,qm})\]

Then, the part of \(u_{w, 2, \text{ell}}\) which will be important for the matching procedure is \(u_{w, 2, \text{ell}, 0}\), which is defined by (4.46)

\[u_{w, 2, \text{ell}, 0}(t, r) = -\frac{1}{r} \int_0^r dy y^2 RHS_{2,0}(t, y) - \int_0^{\infty} dy y RHS_{2,0}(t, y) + \int_0^r dy (RHS_{2,0}(t, y) - RHS_{2,0}(t, y))\]

The following lemma gives the precise sense in which \(u_{w, 2, \text{ell}, 0}\) is the leading part of \(u_{w, 2, \text{ell}}\).

**Lemma 4.12.** For \(0 \leq j \leq 3, 0 \leq k \leq 2\), and in the region \(\lambda(t) \leq r \leq \frac{t}{2}\),

\[|\partial_t^j \partial_r^k \left( u_{w, 2, \text{ell}} - u_{w, 2, \text{ell}, 0} + \int_0^{\infty} xV(t, x)(w_1 + v_2 - w_{1,lm} - v_{2,lm})(t, x)dx \right) | \leq \frac{C_t 3^{-k} \lambda(t)^{5/2}}{r^{6+j}}\]

Also, for \(0 \leq j \leq 3\) and \(r \leq t\),

\[|\partial_t^j \left( u_{w, 2, \text{ell}} - u_{w, 2, \text{ell}, 0} \right)(t, r) | \leq \frac{C \lambda(t)^2 \sup_{t \in [T, t]} \sqrt{\lambda(x)}}{t^3} \left( \frac{1}{t} \right) \left( \frac{1}{r} \right) \left( \frac{1}{(t-r)^j} \right)\]

For \(k = 1, 2\),

\[|\partial_t^k \partial_r^k \left( u_{w, 2, \text{ell}} - u_{w, 2, \text{ell}, 0} \right)(t, r) | \leq \left\{ \begin{array}{ll}
\frac{C_0 3^{-k}}{\sqrt{\lambda(t) t^3}}, & r \leq \lambda(t) \\
\frac{C_0 (t)^{2} \sup_{t \in [T, t]} \sqrt{\lambda(x) \log(t)}}{t^3(t-r)^k} + \frac{C_0 \lambda(t)^2 \sup_{t \in [T, t]} \lambda(x)^{3/2} \log^2(t)}{t^3(t-r)^{1+k}}, & \frac{t}{2} \leq r \leq t
\end{array} \right.\]
Finally, for $0 \leq j \leq 3$,
\[
|\hat{c}_j^2\left(-\int_0^\infty dx V(t,x)(w_1 + v_2 - w_{1,lm} - v_{2,lm})\right)| \leq \frac{C\lambda(t)^2 \sup_{x \in [T,\ell]} \sqrt{\lambda(x)} \log(t)}{t^{3+j}}
\]
and for $0 \leq j \leq 5$, and $s \geq t$,
\[
|\hat{c}_j^2\left(u_{w,2,ell}(s,y) - u_{w,2,ell,0}(s,y) + \int_0^\infty dx V(s,x)(w_1 + v_2 - w_{1,lm} - v_{2,lm})(s,x)\right)| \leq \frac{|s-t|^6}{\sqrt{\lambda(s)^{6+j}}} \left\{
\begin{array}{ll}
\frac{C_j}{\sqrt{\lambda(s)^{6+j}}}, & s - t \leq \lambda(s) \\
\frac{C_j (s-t)^3 \lambda(s)^{5/2}}{s^{6+j}}, & \lambda(s) \leq s - t \leq \frac{s}{2} \\
\frac{C_j \log^2(s) \lambda(s)^2 \sup_{x \in [T,\ell]} \sqrt{\lambda(x)}}{s^{4+j}}, & \frac{s}{2} \leq s - t \leq s
\end{array}\right.
\]
Proof. From (1.43), (1.46), we get
\[
u_{w,2,ell}(t,r) - u_{w,2,ell,0}(t,r) = -\frac{1}{r} \int_0^r dy y^2 V(t,y) (w_1 + v_2 - w_{1,cm} - v_{2,cm}) + \int_0^r dy y^2 V(t,y) (w_1 - w_{1,cm} + v_2 - v_{2,cm}) - \int_0^\infty dy y^2 V(t,y) (w_1 + v_2 - w_{1,lm} - v_{2,lm})
\]
We start with the last term in the above expression. From integration by parts,
\[
-\int_0^\infty dy y^2 V(t,y)w_{1,0}(t,y) = \frac{15\sqrt{3} \lambda(t)}{4 \sqrt{\lambda(t)}} - \frac{45\pi}{16}\left(-\lambda''(t)\sqrt{\lambda(t)} + \frac{\lambda'(t)^2}{2\sqrt{\lambda(t)}}\right) + \frac{5\sqrt{3}}{8\lambda(t)} \int_0^\infty \left(\sqrt{3} \arctan\left(\frac{\sqrt{3}\lambda(t)}{x}\right)(x^2 + 3\lambda(t)^2) - 3x\lambda(t)\right) \hat{c}_x^2 \left(\frac{\lambda'(t) + x}{\sqrt{\lambda(t) + x}}\right) dx
\]
Directly evaluating the integrals gives the following equations:
\[
\int_0^\infty dy y^2 V(t,y)w_{1,lm}(t,y) = \frac{45\pi((\lambda'(t))^2 - 2\lambda(t)\lambda''(t))}{32\sqrt{\lambda(t)}}, \quad \int_0^\infty dy y^2 V(t,y)v_{2,lm}(t,y) = \frac{225\pi \sqrt{\lambda(t)} \lambda''(t)}{16}
\]
\[
-\int_0^\infty dy y^2 V(t,y)(\hat{v}_2 + v_2)(t,y) = -\frac{15\sqrt{3}}{4} \frac{\lambda'(t)}{\sqrt{\lambda(t)}} - \frac{225\pi \sqrt{\lambda(t)} \lambda''(t)}{16} - \frac{45\sqrt{3}}{4} \lambda(t)^2 \int_0^\infty wdw \int_{w-t}^{w+t} (y^2 + 3\lambda(t)^2)^2 (h(w) - h(t)) dy
\]
where
\[
h(x) = \frac{\psi(x)}{x} \left(\frac{\lambda'(x)}{\sqrt{\lambda(x)}} + \frac{5\sqrt{3}\pi}{4} \lambda''(x)\sqrt{\lambda(x)}\right)
\]
Therefore,
\[
-\int_0^\infty dy y^2 V(t,y)(w_1 + v_2 - w_{1,lm} - v_{2,lm}) = \frac{5\sqrt{3}}{8\lambda(t)} \int_0^\infty \left(\sqrt{3} \arctan\left(\frac{\sqrt{3}\lambda(t)}{x}\right)(x^2 + 3\lambda(t)^2) - 3x\lambda(t)\right) \hat{c}_x^2 \left(\frac{\lambda'(t) + x}{\sqrt{\lambda(t) + x}}\right) dx
\]
\[
- \frac{45\sqrt{3}}{4} \lambda(t)^2 \int_0^\infty wdw \int_{w-t}^{w+t} (h(w) - h(t)) dy
\]

\begin{equation}
(4.47)
\end{equation}
We directly estimate the first term on the right-hand side of (4.47) (and its derivatives). On the other hand, since \( v_2, \tilde{v}_2 \) are not symbols globally, we decompose the second term as

\[
\int_0^\infty wdw \int_{|t-w|}^{t+w} \frac{(h(w) - h(t))dy}{(y^2 + 3\lambda(t)^2)^2} = \int_{-\frac{t}{2}}^{t} (t + z)dz \int_{|z|}^{\infty} \frac{dy(h(t + z) - h(t))}{(y^2 + 3\lambda(t)^2)^2} + \int_{t}^{\infty} \frac{dy(h(w) - h(t))}{(y^2 + 3\lambda(t)^2)^2} + \int_{0}^{\frac{t}{2}} wdw \int_{t-w}^{t+w} \frac{dy(h(w) - h(t))}{(y^2 + 3\lambda(t)^2)^2} - \int_{\frac{t}{2}}^{\infty} \frac{dy(h(w) - h(t))}{(y^2 + 3\lambda(t)^2)^2}
\]

Then, we differentiate in \( t \), and directly estimate. This procedure does lead to a large number of terms to estimate, but, each one can be estimated directly with an elementary argument. All other estimates in the lemma statement follow from a direct computation.

Finally, it will be convenient to consider the following leading part of \( u_{w,2,\ell,0} \):

\[
(4.48) \quad u_{w,2,\ell,0,0}(t, r) = \frac{675\pi \lambda(t)^{5/2} \lambda''(t)}{16r^2} + \frac{45\sqrt{3} \sqrt{\lambda(t)}}{32r} \left( \lambda'(t)^2 \left( 4 \log \left( \frac{\sqrt{3} \lambda(t)}{r} \right) - 2 \right) - \lambda(t) \lambda''(t) \left( 8 \log \left( \frac{\sqrt{3} \lambda(t)}{r} \right) + 5r^2 - 4 \right) \right)
\]

We will also need the leading part of \( u_{w,2,\ell,0} - u_{w,2,\ell,0,0} \) in the matching region.

\[
u_{w,2,\ell,0,1}(t, r) := \frac{45}{2} a_1(t) \lambda(t)^2 r + \frac{45}{4} \lambda(t)^2 (6b_1(t) + 3\sqrt{3} \pi a_1(t) \lambda(t) + b_1(t) \log(\frac{9\lambda(t)^4}{r^4}))
\]

where

\[
(4.49) \quad a_1(t) = \frac{1}{4!} \epsilon_1^3 \left( \frac{\sqrt{3} \lambda(t)}{2 \sqrt{\lambda(t)}} \right), \quad b_1(t) = \frac{1}{6} \left( 2v_2'\ell(t) + tv_2''\ell(t) \right)
\]

Now, we prove the sense in which \( u_{w,2,\ell,0,0}, u_{w,2,\ell,0,1} \) are the leading and subleading parts of \( u_{w,2,\ell,0} \) in the matching region.

**Lemma 4.13.** For \( 0 \leq j, k \leq 2, j = 3, k = 0 \), and \( r \geq \lambda(t) \)

\[
|c_t^j c_r^k (u_{w,2,\ell,0} - u_{w,2,\ell,0,0} - u_{w,2,\ell,0,1})\ (t, r)| \leq \frac{C\lambda(t)^{9/2} (|\log(r)| + \log(t)) (1 + r^2/t^2)}{r^{k+3} t'^{2+j}}
\]

For \( j = 2, 3 \),

\[
|c_t^j (u_{w,2,\ell,0,0}(t, r) - u_{w,2,\ell,0,0}(t, r))| \leq \left\{ \begin{array}{ll} \frac{C\lambda(t)^{7/2}}{t^{7-2+j}} & r \leq \lambda(t) \\
C r^{1/2} \log(t) \left( \frac{2}{t^{2+j}} \right) & t \geq \lambda(t) \end{array} \right.
\]

For \( 0 \leq j \leq 5 \) and \( k = 0 \), or \( j = 2, k = 1, 2 \),

\[
|c_t^j c_r^k \left( u_{w,2,\ell,0} - \left( u_{w,2,\ell,0,0} - \frac{675\pi \lambda(t)^{5/2} \lambda''(t)}{16r^2} \right) \right) | \leq \left\{ \begin{array}{ll} \frac{C\lambda(t)^{5/2}}{t^{1+j} t'^{2+k}} (1 + |\log(\frac{r}{\lambda(t)})|) & r \leq \lambda(t) \\
C \frac{\lambda(t)^{7/2}}{t^{2+j} r^{2+k}} + C \frac{\lambda(t)^{5/2} \log(t) \log(t)}{t^{4+j}} & \lambda(t) \leq r \leq t \end{array} \right.
\]
Lemma 4.15. Directly from (4.14), we get

\[ -\frac{1}{r} \int_0^r dyy^2 RHS_{2,0,0}(t, y) - \int_0^\infty dyy RHS_{2,0,0}(t, y) = u_{w,2,ell,0,0}(t, r) \]

\[ = \frac{45}{32} \sqrt{\lambda(t)} \lambda''(t) \left( \frac{\lambda(t) \sqrt{3} (10\pi g_1(\frac{r}{\lambda(t)}) + 4g_2(\frac{r}{\lambda(t)}))}{r} + 4g_1(\frac{r}{\lambda(t)}) \right) - \frac{45}{32} \sqrt{\lambda(t)} \left( 2g_1(\frac{r}{\lambda(t)}) + \frac{2\sqrt{3}\lambda(t)g_2(\frac{r}{\lambda(t)})}{r} \right) \]

where

\[ g_1(R) = \tan^{-1}\left( \frac{\sqrt{3}}{R} \right) - \frac{\sqrt{3}}{R}, \quad g_2(R) = \log\left( \frac{3}{R^2} + 1 \right) \]

and

\[ -\frac{1}{r} \int_0^r dyy^2 (RHS_{2,0}(t, y) - RHS_{2,0,0}(t, y)) + \int_0^r dyy (RHS_{2,0}(t, y) - RHS_{2,0,0}(t, y)) - u_{w,2,ell,0,1}(t, r) \]

\[ = \frac{135\sqrt{3}\lambda(t)^3}{4r} \left( -2a_1(t) \tan^{-1}\left( \frac{\sqrt{3}\lambda(t)}{r} \right) - 2b_1(t) \tan^{-1}\left( \frac{r}{\sqrt{3}\lambda(t)} \right) \right) \]

\[ + \frac{45\lambda(t)^2}{4r} \left( 12a_1(t)\lambda(t)^2 \log\left( \frac{3\lambda(t)^2}{3\lambda(t)^2 + r^2} \right) - 2rb_1(t) \log\left( \frac{3\lambda(t)^2}{r^2} + 1 \right) \right) \]

Direct estimation also gives the following lemma.

Lemma 4.14. For \( r \geq \lambda(t) \), \( j \geq 0 \), and \( k = 0,1 \),

\[ |\partial_t^k \partial^j u_{w,2,ell,0,0}(t, r)| \leq C_j \lambda(t)^{5/2} \left( 1 + |\log\left( \frac{\lambda(t)}{r} \right)| \right), \quad |\partial_t^k u_{w,2,ell,0,1}(t, r)| \leq \frac{C_j \lambda(t)^{5/2} r}{r^{4+j}} \]

\[ |\partial_t^n \left( u_{w,2,ell,0} - \left( u_{w,2,ell,0,0} - \frac{675\pi \lambda(t)^{5/2} \lambda''(t)}{16r^2} \right) \right) | \leq \begin{cases} \frac{C\lambda(t)^{5/2} (1 + |\log(\frac{r}{\lambda(t)})|)}{r^{4+k}}, & r \leq \lambda(t) \\ \frac{C\lambda(t)^{7/2} r^{2+n}}{r^{2+n}} + \frac{C\lambda(t)^{5/2}}{r^{2+n}}, & t \geq r \geq \lambda(t) \end{cases}, \quad 0 \leq n \leq 3 \]

Recall that \( u_{w,2,sub} \) is a solution to

\[ -\partial_t^2 u + \partial_r^2 u + \frac{2}{r} \partial_r u = \partial_t^2 u_{w,2,ell}(t, r) \]

The following part of \( u_{w,2,sub} \) will also be important for the matching procedure:

\[ (4.50) \quad u_{w,3,ell,0}(t, r) := -\frac{1}{r} \int_0^r dyy^2 \partial_t^2 u_{w,2,ell,0,0}(t, y) + \int_0^r dyy \partial_t^2 u_{w,2,ell,0,0}(t, y) \]

Directly from (4.14), we get

Lemma 4.15.

\[ |\partial_t^k u_{w,3,ell,0}(t, r)| \leq \frac{Cr \lambda(t)^{5/2} (1 + |\log(\frac{r}{\lambda(t)})|)}{r^{4+k}}, \quad r \geq \lambda(t), \quad 0 \leq k \leq 2 \]

\( u_{w,3,ell,0} \) is the leading part of \( u_{w,2,sub,0} \) in the matching region, in the following sense.
Lemma 4.16. For \( \lambda(t) \leq r \leq \frac{t}{4} \),

\[
|c^j_l c^k_r (u_{w,2,sub} - u_{w,3,ell,0}) - \left( \int_t^\infty u_2(s)ds + \int_t^\infty dsq'_1(s) \log \left( \frac{s-t}{\sqrt{3} \lambda(s)} \right) \right) + \int_t^\infty ds(s-t) \hat{c}^2_s \left( u_{w,2,ell}(s,y) - (u_{w,2,ell,00}(s,y) - \frac{675\pi \lambda(s)^{5/2} \lambda''(s)}{16y^2}) \right) |_{y=s-t} \]

\[
\leq \left\{ \begin{array}{ll}
Cr^2 \lambda(t)^2 \sup_{x \in [T,t]} \sqrt{\lambda(x) \log^2(t)} & + \frac{C\lambda(t)^{7/2} \log^2(t)}{r^4}, \quad j = k = 0 \\
Cr \lambda(t)^2 \sup_{x \in [T,t]} \sqrt{\lambda(x) \log^2(t)} & + \frac{C\lambda(t)^{7/2} \log^2(t)}{r^4}, \quad k = 1, j = 0 \\
C\lambda(t)^2 \sup_{x \in [T,t]} \sqrt{\lambda(x) \log^2(t)} & + \frac{C\lambda(t)^{7/2} \log^2(t)}{r^4}, \quad j = 1, k = 0 \\
C\lambda(t)^2 \sup_{x \in [T,t]} \sqrt{\lambda(x) \log^2(t)} & + \frac{C\lambda(t)^{7/2} \log^2(t)}{r^4}, \quad j = 1, k = 1 \text{ or } j = 0, k = 2 \\
\end{array} \right.
\]

where

\[
q_1(t) = \frac{-45\sqrt{3} \lambda(t)^2}{2} c^2_t \sqrt{\lambda(t)}, \quad q_2(t) = \frac{-45\sqrt{3}}{32} \left( -8c^2_t \left( \sqrt{\lambda(t)} \right) - 5\pi^2 \lambda(t)^{3/2} \lambda''(t) \right)
\]

and

\[
w_2(t) = \frac{q_1(t) \lambda'(t)^2}{\lambda(t)^2} - q_2''(t) - \frac{(2q_1(t) \lambda'(t) + q_1(t) \lambda''(t))}{\lambda(t)}
\]

Proof. We recall the definition of \( u_{w,2,ell,0} \) in (4.48), and start by defining

\[
u_{w,2,sub,0}(t,r) = \int_t^\infty ds \int_{|s-t-r|}^{s-t+r} dy \left( \frac{-y}{2r} \right) \hat{c}^2_s \left( u_{w,2,ell,00}(s,y) - \frac{675\pi \lambda(s)^{5/2} \lambda''(s)}{16y^2} \right)
\]

Let \( q_1, q_2, w_2 \) be as in the lemma statement. Then,

\[
u_{w,2,sub,0}(t,r) = \int_t^\infty ds \int_{|s-t-r|}^{s-t+r} dy \left( w_2(s) - \log \left( \frac{\sqrt{3} \lambda(s)}{y} \right) q_1''(s) \right)
\]

\[
= \int_t^\infty ds \frac{w_2(s)(s-t+r-|s-t-r|)}{2r} \left( q_1''(s) (|s-t-r| \log((s-t-r)) - (s-t+r) \log(s-t+r) \right.
\]

\[
- (|s-t-r| - (s-t+r))(1 + \log(\sqrt{3} \lambda(s)))) \right) \bigg) \bigg)
\]
We now write this expression in such a way as to make manifest those terms (which are roughly on the order of $\frac{r\lambda(t)^{5/2}}{t^4}$) which will cancel in the combination $u_{w,2,\text{sub},0} - u_{w,3,\text{ell},0}$. This leads to

\begin{equation}
\frac{1}{2} \int_{t+r}^{\infty} ds q''_1(s) \log\left(\frac{s - t - r}{s - t + r}\right) - \frac{1}{2r} \int_{t+r}^{\infty} ds q''_1(s) \left(\log\left(\frac{s - t - r}{s - t + r}\right) + \frac{2r}{s - t}\right) \\
= \frac{-r}{2} q''_1(t) (-1 + \log(4)) \\
- \frac{r}{2} q''_1(t + r) - q''_1(t))(-1 + \log(4)) - \frac{r^2}{2} \int_{1}^{\infty} q'''_1(t + ry)((1 + y^2) \coth^{-1}(y) + y(-1 + \log(1 - \frac{1}{y^2})))dy
\end{equation}

We remark that the sixth integral term in the expression above comes from integration by parts:

\begin{equation}
\frac{1}{2} \int_{t+r}^{\infty} ds q''_1(s) \log\left(\frac{s - t - r}{s - t + r}\right) - \frac{1}{2r} \int_{t+r}^{\infty} ds q''_1(s) \left(\log\left(\frac{s - t - r}{s - t + r}\right) + \frac{2r}{s - t}\right) \\
= \frac{-r}{2} q''_1(t) (-1 + \log(4)) \\
- \frac{r}{2} q''_1(t + r) - q''_1(t))(-1 + \log(4)) - \frac{r^2}{2} \int_{1}^{\infty} q'''_1(t + ry)((1 + y^2) \coth^{-1}(y) + y(-1 + \log(1 - \frac{1}{y^2})))dy
\end{equation}

We recall the definition of $u_{w,3,\text{ell},0}$ \textbf{(1.50)}, and note that

\begin{equation}
\frac{-1}{r} \int_{0}^{r} y^2 c''_t \left(u_{w,2,\text{ell},00}(t, y) - \frac{675\pi\lambda(t)^{5/2}\lambda''(t)}{16y^2}\right) dy + \int_{0}^{r} y c''_t \left(u_{w,2,\text{ell},00}(t, y) - \frac{675\pi\lambda(t)^{5/2}\lambda''(t)}{16y^2}\right) dy \\
= \frac{r}{2} (-w_2(t) + \frac{3}{2} q''_1(t) + q''_1(t) \log\left(\frac{\sqrt{3}\lambda(t)}{r}\right))
\end{equation}
which are precisely the first non-integral terms in (1.51). Therefore,

\[
\begin{align*}
&u_{w,2,sub,0} - u_{w,3,ell,0} \\
&= \frac{1}{r} \int_r^\infty dy y^2 675\pi^2 c_t^2 (\lambda(t)^{5/2} \lambda''(t)) - \int_0^r dy 675\pi^2 c_t^2 (\lambda(t)^{5/2} \lambda''(t)) \\
&\quad - \frac{1}{r} \int_0^{\lambda(t)} dy y^2 2\left( u_{w,2,ell,00} - \frac{675\pi^2 \lambda(t)^{5/2} \lambda''(t)}{16y^2} \right) dy \\
&\quad + \int_t^\infty w_2(s) ds + \int_t^\infty ds q''_1(s) \log\left( \frac{s-t}{\sqrt{3}\lambda(s)} \right) + r \int_0^1 dy \left( q''_1(t+ry) \log\left( \frac{\lambda(t+ry)}{r} \right) - q''_1(t) \log\left( \frac{\lambda(t)}{r} \right) \right) \\
&\quad + r \int_0^1 dy \left( (w_2(t+ry) - w_2(t))2y \right) \\
&\quad - (q''_1(t+ry) - q''_1(t)) \left( (1+y) \log(r(1-y)) - (1+y) \log(r(1+y)) \right) \\
&\quad + 2y(1 + \log(\sqrt{3}\lambda(t+ry)))) \\
&\quad + \frac{1}{2} \int_0^1 dy (-q''_1(t))2y(\log(\lambda(t+ry)) - \log(\lambda(t))) \\
\end{align*}
\]

Each term on the right-hand side of this expression (and its derivatives) can be straightforwardly estimated now. We thus get: for \(0 \leq k \leq 2\) and \(j = 0\) or \(0 \leq k \leq 1\) and \(j = 1\),

\[
|\partial_t^j \partial_r^k \left( u_{w,2,sub,0} - u_{w,3,ell,0} - \left( \int_t^\infty w_2(s) ds + \int_t^\infty ds q''_1(s) \log\left( \frac{s-t}{\sqrt{3}\lambda(s)} \right) \right) \right) | \\
\leq \frac{C}{t^{4+j} \lambda(t)^{5/2}} \left( \frac{r^2}{t} + \lambda(t) \right), \quad \lambda(t) \leq r \leq t
\]

The next step is to estimate \(u_{w,2,sub} - u_{w,2,sub,0}\). Let

\[
F(s,y) = c_s^2 \left( u_{w,2,ell}(s,y) - (u_{w,2,ell,00}(s,y) - \frac{675\pi^2 \lambda(s)^{5/2} \lambda''(s)}{16y^2}) \right)
\]

Using the definitions of \(u_{w,2,sub}\) and \(u_{w,2,sub,0}\), we get

\[
\begin{align*}
u_{w,2,sub}(t,r) - u_{w,2,sub,0}(t,r) &= \int_t^\infty ds \int_{[s-t-r]}^{s-t+r} dy \left( \frac{-y}{2r} \right) F(s,y) \\
&= \int_t^\infty ds \int_{[s-t-r]}^{s-t+r} dy \left( \frac{-y}{2r} \right) c_s^2 \left( \frac{675\pi^2 \lambda(s)^{5/2} \lambda''(s)}{16y^2} \right) \\
&\quad + \int_t^\infty ds \int_{[s-t-r]}^{s-t+r} dy \left( \frac{-y}{2r} \right) c_s^2 (u_{w,2,ell}(s,y) - u_{w,2,ell,00}(s,y))
\end{align*}
\]

Direct estimation gives, for \(0 \leq k \leq 2\) and \(j = 0\) or \(0 \leq k \leq 1\) and \(j = 1\), and in the region \(\lambda(t) \leq r \leq \frac{t}{4}\),

\[
|\partial_t^j \partial_r^k \left( \int_t^\infty ds \int_{[s-t-r]}^{s-t+r} dy \left( \frac{-y}{2r} \right) c_s^2 \left( \frac{675\pi^2 \lambda(s)^{5/2} \lambda''(s)}{16y^2} \right) \right) | \leq \frac{C\lambda(t)^{7/2} \log(t)}{r^{4+j} \lambda(t)^{5/2}}
\]
Therefore, it suffices to estimate the following integral (with \(G(s, y) = \varepsilon_s^2 (u_{w,2,\ell}(s, y) - u_{w,2,\ell,00}(s, y))\))

\[
\int_t^{s} ds \int_{s-t}^{s-t+r} dy \left( -\frac{y}{2r} \right) G(s, y) \\
= \int_0^r \frac{dy}{r} \left( r - (y-r) \right) \left( -\frac{y}{2r} \right) G(t, y) + \int_r^{2r} \frac{dy}{r} \left( r - (y-r) \right) \left( -\frac{y}{2r} \right) G(t, y) \\
+ \int_0^1 r dz \int_{r-z}^{r+z} dy \left( -\frac{y}{2r} \right) (G(rz+t, y) - G(t, y)) - \int_t^{s} (s-t)G(s, s-t)ds \\
- \frac{1}{2r} \int_t^{s} ds \left( \int_{s-t}^{s-t+r} dy y G(s, y) - 2r(s-t)G(s, s-t) \right)
\]

Each integral is directly estimated, using Lemmas 4.12 and 4.13 except for the following. We write (4.52)

\[
\int_t^{s} G(s, s-t)ds = \int_t^{s} (s-t) \varepsilon_s^2 \left( -\frac{675\pi \lambda(s)^{5/2} \lambda''(s)}{16y^2} \right) |_{y=s-t} ds \\
+ \int_t^{s} (s-t) \varepsilon_s^2 \left( u_{w,2,\ell}(s, y) - (u_{w,2,\ell,00}(s, y) - \frac{675\pi \lambda(s)^{5/2} \lambda''(s)}{16y^2}) \right) |_{y=s-t} ds \\
- \int_t^{s} (s-t) \varepsilon_s^2 \left( u_{w,2,\ell}(s, y) - (u_{w,2,\ell,00}(s, y) - \frac{675\pi \lambda(s)^{5/2} \lambda''(s)}{16y^2}) \right) |_{y=s-t} ds
\]

The point is that the second term in (4.52) is not quite perturbative, while the others can be directly estimated. This gives rise to

\[
|u_{w,2,sub} - u_{w,2,sub,0} - \int_t^{s} ds (s-t) \varepsilon_s^2 \left( u_{w,2,\ell}(s, y) - (u_{w,2,\ell,00}(s, y) - \frac{675\pi \lambda(s)^{5/2} \lambda''(s)}{16y^2}) \right) |_{y=s-t} | \\
\leq C \varepsilon^2 \lambda^2 \left( \sup_{x \in [T_{\lambda}, t]} \sqrt{\lambda(x)} \log^2(t) \right) \frac{C \lambda(t)^{7/2} \log^2(t)}{t^4}, \quad \lambda(t) \leq r \leq \frac{t}{4}
\]

We estimate the derivatives of

\[
u_{w,2,sub} - u_{w,2,sub,0} - \int_t^{s} ds (s-t) \varepsilon_s^2 \left( u_{w,2,\ell}(s, y) - (u_{w,2,\ell,00}(s, y) - \frac{675\pi \lambda(s)^{5/2} \lambda''(s)}{16y^2}) \right) |_{y=s-t} \\
\]

exactly as in the proof of Lemma 4.6. This gives rise to the following. For \(j = 0, 0 \leq k \leq 2\) or \(j = 1, 0 \leq k \leq 2\) and in the region \(\lambda(t) \leq r \leq \frac{t}{4},

\[
|\varepsilon_r^k \varepsilon_t^j \left( u_{w,2,sub} - u_{w,2,sub,0} - \int_t^{s} ds (s-t) \varepsilon_s^2 \left( u_{w,2,\ell}(s, y) - (u_{w,2,\ell,00}(s, y) - \frac{675\pi \lambda(s)^{5/2} \lambda''(s)}{16y^2}) \right) \right) |_{y=s-t} | \\
\leq \frac{C \varepsilon^2 \lambda^2 \left( \sup_{x \in [T_{\lambda}, t]} \sqrt{\lambda(x)} \log^2(t) \right) \frac{C \lambda(t)^{7/2} \log^2(t)}{t^4}}{t^j}, \quad k = 1, j = 0 \\
\leq \frac{C \varepsilon^2 \lambda^2 \left( \sup_{x \in [T_{\lambda}, t]} \sqrt{\lambda(x)} \log^2(t) \right) \frac{C \lambda(t)^{7/2} \log^2(t)}{t^4}}{t^j}, \quad j = 1, k = 0 \\
\leq \frac{C \varepsilon^2 \lambda^2 \left( \sup_{x \in [T_{\lambda}, t]} \sqrt{\lambda(x)} \log^2(t) \right) \frac{C \lambda(t)^{7/2} \log^2(t)}{t^4}}{t^j}, \quad j + k = 2
\]

□
Finally, let $v_3$ be the solution to

$$
\begin{align*}
(4.53) \\
\begin{cases}
-\frac{\partial^2}{\partial t^2} v_3 + \frac{\partial^2}{\partial r^2} v_3 + \frac{2}{r} \frac{\partial}{\partial r} v_3 = 0 \\
v_3(0, r) = 0, \quad \partial_r v_3(0, r) = v_{3,0}(r)
\end{cases}
\end{align*}
$$

where $v_{3,0}$ is to be specified later. In particular, we have

$$
v_3(t, r) = \frac{t}{2} \int_0^\pi \sin(\theta)v_{3,0}(\sqrt{r^2 + t^2 + 2rt\cos(\theta)})d\theta
$$

Then, $u = u_{w,2} + v_3$ is a particular solution to the equation

$$
-\frac{\partial^2}{\partial t^2} u + \frac{\partial^2}{\partial r^2} u + \frac{2}{r} \frac{\partial}{\partial r} u = RHS_2(t, r)
$$

and $v_{3,0}$ will be chosen soon so that $u_{w,2} + v_3$ has a desired behavior in the matching region. The leading part of $v_3$ in the matching region will turn out to be

$$
v_{3,\text{main}}(t, r) = tv_{3,0}(t)
$$

4.6. Matching, Part 1. We recall the leading parts of $u_{\text{ell}}, w_1, v_2$, and $v_{\text{ex}}$ (from (4.5), (4.13), (4.34), (4.28), respectively) in the matching region.

$$
u_{\text{ell,main}}(t, r)
$$

$$
= -\frac{\sqrt{3}c_1(t)\lambda(t)}{2r} + \sqrt{\lambda(t)}\lambda''(t)\left(\frac{675\pi\lambda(t)^2}{16r^2} - \frac{3\sqrt{3}\lambda(t)\left(37 - 20\log\left(\frac{r^2}{3\lambda(t)^2}\right)\right)}{8r} + \frac{\sqrt{3}r - 15\pi}{8}\right)
$$

$$
+ \left(\frac{15\sqrt{3}\lambda(t)}{16r} - \frac{\sqrt{3}r}{8\lambda(t)}\right)\lambda'(t)^2
$$

$$
w_{1,\text{lm}}(t, r) = \frac{1}{2r}\left(\frac{\sqrt{3}\lambda''(t)}{2\sqrt{\lambda(t)} - 4\lambda(t)^{3/2}}\right)
$$

$$
v_{2,\text{lm}}(t, r) = \frac{15}{8}\pi\sqrt{\lambda(t)\lambda''(t)}
$$

$$
v_{\text{ex,ell,0}}(t, r) = \frac{3\sqrt{3}\sqrt{\lambda(t)}\left(37\log\left(\frac{4r^2}{3\lambda(t)^2}\right) - 17\right) + 2\lambda(t)\lambda''(t)\left(5\log\left(\frac{4r^2}{3\lambda(t)^2}\right) + 1\right)}{16r}
$$

Recalling the expression for $u_{w,2,\text{ell,0}}$ from (4.48), direct computation gives

$$
(4.54)
$$

$$
u_{\text{ell,main}} - (u_{w,2,\text{ell,0}} + v_{2,\text{lm}} + v_{\text{ex,ell,0}} + w_{1,\text{lm}})
$$

$$
= -\frac{\sqrt{3}\lambda(t)}{32r}\left(16c_1(t)\sqrt{\lambda(t)} + 6(15\log(4) - 37)\lambda'(t)^2 + 3(-75\pi^2 + 212 + 20\log(4))\lambda(t)\lambda''(t)\right)
$$

Note that $v_{2,\text{lm}}$ and $w_{1,\text{lm}}$ each contain terms of the form $f_k(t)r^k$ for $k = 0, 1$ which exactly cancel with terms of the same form from $u_{\text{ell,main}}$ when these functions are combined as in (4.54). This is (partly) due to the specific choice of $v_{2,\text{lm}}$ from (4.33). Also, $u_{\text{ell,main}}, u_{w,2,\text{ell,0}}$, and $v_{\text{ex,ell,0}}$ each separately contain terms of the form $f(t)\frac{\log(r)}{r}$, but, all such terms exactly cancel in the combination
in (4.54). This is reminiscent of a similar “automatic” matching of logarithmically higher order terms observed in [20]. We choose
\[ c_1(t) = \frac{3((74 - 30 \log(4)) \lambda'(t)^2 + (75\pi^2 - 4(53 + 5 \log(4))) \lambda(t) \lambda''(t))}{16\sqrt{\lambda(t)}} \]
so that
\[ (4.55) \quad u_{ell,\text{main}} - (u_{w,2,ell,00} + v_{2,lm} + v_{ex,ell,0} + w_{1,lm}) = 0 \]
Now that we have fixed the function \( c_1 \), we can directly estimate \( u_{ell} - u_{ell,\text{main}} \), and \( u_{ell} \). Note that the last estimate in the lemma statement is not sharp in the sense that \( \partial_t \lambda u_{ell}(t, r) \) has no singularity as \( r \to 0 \), but, the estimate suffices for its use later on.

**Lemma 4.17.** For \( 0 \leq j, k \leq 2 \),
\[ |\partial^j_t \partial^k_r (u_{ell}(t, R\lambda(t)) - u_{ell,\text{main}}(t, R\lambda(t)))| \leq C \frac{\lambda(t)^{3/2}(1 + \log(R))}{t^{2+j} R^{3+k}}, \quad R \geq 1 \]
For \( 0 \leq j, k \leq 4 \),
\[ |\partial^j_t \partial^k_r (u_{ell}(t, R\lambda(t)))| \leq C \frac{\lambda(t)^{3/2}}{t^{2+j} R^{3+k}} \begin{cases} 1, & k = 0 \\ R^{-k}, & 1 \leq k \leq 2 \\ R^{1-k}, & 3 \leq k \leq 4 \end{cases}, \quad R \leq 1 \]
\[ |\partial^j_t \partial^k_r u_{ell}(t, r)| \leq C \frac{\lambda(t)^{3/2}}{t^{2+j} R^{3+k}} \begin{cases} 1, & r \leq \lambda(t) \\ \frac{r}{\lambda(t)}, & r \geq \lambda(t) \end{cases}, \quad 0 \leq j + k \leq 6 \]

4.7. **Matching, Part 2.** Next, we consider matching of higher order terms. We recall the decomposition of \( u_{ell,2} \) given in (4.7), and define \( u_{ell,2,\text{main}} \) to be the sum of the terms arising in a large \( r \) expansion of \( u_{ell,2}(t, r) \) which do not decay as \( r \to \infty \). In particular, we have the following. Let
\[ a(t) = \frac{1}{2} \partial_t \left( \frac{\sqrt{3} \lambda'(t)}{2\sqrt{\lambda(t)}} \right), \quad b(t) = -\frac{15}{8} \pi \sqrt{-\lambda(t)} \lambda''(t) \]
We use (4.55), recall the definitions of \( u_{w,3,ell,0} \) and \( v_{ex,ell,0} \) in (4.50) and (4.28), respectively, inspect (4.7), and claim that the sum of the terms arising in a large \( r \) expansion of \( u_{ell,2}(t, r) \) which do not decay as \( r \to \infty \) is given by

\[
u_{ell,2,\text{main}}(t, r) \\
= v_{ex, sub, ell}(t, r) + u_{w,3,ell,0}(t, r) - \frac{2}{\lambda(t) \sqrt{3}} \int_{\lambda(t)}^{\infty} x^2 \partial^2_t (u_{w,2,ell,0,0} + v_{ex,ell,0})(t, x) \left( \phi_0 \left( \frac{x}{\lambda(t)} \right) + \frac{\sqrt{3} \lambda(t)}{2x} \right) dx \\
- \frac{15}{2} \frac{\lambda(t)^2 \log \left( \frac{r}{\lambda(t)} \right)}{\lambda(t)} b''(t) + \frac{1}{12} a''(t) x^3 + \frac{1}{6} b''(t) r^3 - \frac{15}{8} \frac{\lambda(t)^2 a''(t) r}{\lambda(t)} + \frac{43}{6} \lambda(t)^3 a''(t) + \frac{53}{4} \lambda(t)^2 b''(t) \\
- \frac{\lambda(t)^2}{12} \left( -206 \lambda(t) a''(t) - 3(13 + 30 \log \left( \frac{3}{2} \right)) b''(t) \right) \\
- \frac{2}{\sqrt{3}} \int_{1}^{\infty} s^2 \lambda(t)^2 \partial^2_t e_{ell,1}(t, s\lambda(t)) \phi_0(s) ds - \frac{2}{\sqrt{3}} \int_{0}^{1} s^2 \lambda(t)^2 \partial^2_t u_{ell}(t, s\lambda(t)) \phi_0(s) ds \]

Then, \( u_{ell,2,\text{main}} \) is the leading part of \( u_{ell,2} \) in the matching region in the following sense.
Lemma 4.18. We have the following estimates. For \( j = 0 \) and \( 0 \leq k \leq 2 \) or \( j = 1 \) and \( 0 \leq k \leq 1 \)

\[
|\partial_t^j \partial_r^k (u_{\text{ell},2} - u_{\text{ell},2,\text{main}})(t, r)| \leq \frac{C \lambda(t)^{9/2} (1 + \log^2 \frac{r}{\lambda(t)})}{r^{1+k} t^{4+j}}, \quad r \geq \lambda(t)
\]

\[
|\partial_t^j \partial_r^k u_{\text{ell},2}(t, r)| \leq \frac{C r^{2-k} (r + \lambda(t)) \sqrt{\lambda(t)}}{t^{4+j}}, \quad j, k = 0, 1
\]

\[
|\partial_t^{2+j} \partial_r^k u_{\text{ell},2}(t, r)| \leq \frac{C r^{2-k} \lambda(t)^{3/2}}{t^{6+j}}, \quad r \leq \lambda(t), \quad 0 \leq j, k \leq 2
\]

\[
|\partial_t^{2+j} u_{\text{ell},2}(t, r)| \leq \frac{C r^3 \lambda(t)}{t^{6+j}}, \quad r \geq \lambda(t), \quad 0 \leq j \leq 2
\]

(The estimates on \( u_{\text{ell},2} \) in the above lemma follow directly from Lemma 4.17, and the definition of \( u_{\text{ell},2} \).) Next, recalling the definitions of \( a_1 \) and \( b_1 \) (4.49), a direct computation gives

\[
(4.56)
\]

\[
e_{m,2} := u_{\text{ell},2,\text{main}} - (v_{\text{ex},\text{sub},\text{ell}} + u_{w,3,\text{ell},0} + w_{1,cm} - w_{1,lm} + v_{2,qm} - v_{2,lm} + u_{w,2,\text{ell},0,1})
\]

\[
e = -\frac{2}{\lambda(t)^{\sqrt{3}}} \int_{\lambda(t)}^{\infty} x^2 \partial_t^2 (u_{w,2,\text{ell},0,0} + v_{\text{ex},\text{ell},0})(t, x) \left( \phi_0 \left( \frac{x}{\lambda(t)} \right) + \frac{\sqrt{3} \lambda(t)}{2 x} \right) dx + \frac{43}{6} \lambda(t)^3 a''(t) + \frac{53}{4} \lambda(t)^2 b''(t)
\]

\[
+ \frac{2}{\sqrt{3}} \int_{\lambda(t)}^{\infty} x^2 \partial_t^2 \partial_s^2 \phi_0 \left( \frac{s}{\lambda(t)} \right) dx - \frac{2}{\sqrt{3}} \int_{0}^{1} s^2 \lambda(t)^2 \partial_t^2 \partial_s^2 \phi_0 \left( \frac{s}{\lambda(t)} \right) ds
\]

\[
- \frac{\lambda(t)^2}{12} \left( -206 \lambda(t) a''(t) - 3(13 + 30 \log \frac{3}{2}) b''(t) \right) - \frac{45}{4} \lambda(t)^2 ((6 + \log(9)) b_1(t) + 3 \sqrt{3} \pi a_1(t) \lambda(t))
\]

In particular, the \( r^3, r^2, r \), and \( \log(r) \) terms from the functions on the left-hand side all exactly cancel in the combination \( e_{m,2} \), which is independent of \( r \), and turns out to be perturbative, as per the following lemma.

Lemma 4.19. For \( 0 \leq j \leq 2 \),

\[
|\partial_t^j e_{m,2}(t)| \leq \frac{C \lambda(t)^{7/2}}{t^{1+j}}
\]
All of the estimates in the entire proof thus far are valid for all \( t \geq T_0 \) for any \( T_0 \geq T_{0,1} \) (recall \((4.12)\), \( T_{0,1} \) is some fixed, absolute constant). In particular, they are valid for all \( t \geq T_{0,1} \). Then, we choose

\[
(4.57) \quad v_{3,0}(r) = \frac{\psi(r)}{T_{0,1}} + \left( -\int_{t}^{\infty} w_2(s)ds - \int_{t}^{\infty} dsq'_1(s)\log\left(\frac{s-t}{\sqrt{3\lambda(s)}}\right) + \int_{t}^{\infty} dsq_2(s) + \int_{t}^{\infty} dsq''_2(s)\log\left(\frac{4(s-t)^2}{3\lambda(s)^2}\right) \right. \]

\[
- \int_{t}^{\infty} ds(s-t)\frac{\partial^2 q}{\partial s^2}(u_{w,2,ell}(s,y) - (u_{w,2,ell,00}(s,y) - \frac{675\pi\lambda(s)^{5/2}e^{y^2}}{16y^2})|_{y=s-t} \]

\[
+ \int_{t}^{\infty} ds(s-t)\frac{\partial^2 q}{\partial s^2}(v_{x,2,ell}(s,y) - v_{x,2,ell,0}(s,y))|_{y=s-t} \]

\[
+ \int_{0}^{\infty} xV(t,x)(w_1 + v_2 - w_1,lm - v_{2,lm})(t,x)dx\left|_{t=r} \right. \]

where \( \psi \) is defined in \((4.11)\), and further restrict \( T_0 \) to satisfy \( T_0 \geq 2T_{0,1} \), but is otherwise arbitrary. Lemmas \((4.4)\) \((4.12)\) \((4.13)\) thus imply

**Lemma 4.20.** For \( 0 \leq j \leq 3 \),

\[
|v_{3,0}^{(j)}(r)| \leq C\frac{1_{\{r \geq T_{0,1}\}}}{r^{4+j}}\lambda(r)^2 \sup_{x \in [T_\lambda, r]} \sqrt{\lambda(x) \log^2(r)} \]

Now that we have defined and estimated \( v_{3,0} \), we can estimate \( v_3 \), defined in \((4.53)\).

**Lemma 4.21.** For \( j = 0 \), \( 0 \leq k \leq 2 \) or \( j = 1 \), and \( 0 \leq k \leq 1 \),

\[
|\partial^k \partial_t v_3(t,r) - v_{3,\text{main}}(t,r)| \leq C r^{2-k} \lambda(t)^2 \sup_{x \in [T_\lambda, t]} \sqrt{\lambda(x) \log^2(t)} , \quad r \leq \frac{t}{2} \]

\[
|\partial_t^2 v_3(t,r) - v_{3,\text{main}}(t,r)| \leq C r \lambda(t)^2 \sup_{x \in [T_\lambda, t]} \sqrt{\lambda(x) \log^2(t)} , \quad r \leq \frac{t}{2} \]

For \( j + k \leq 2 \),

\[
|\partial^j \partial_t \partial_t v_3(t,r)| \leq C \left( \sup_{x \in [T_\lambda, r]} \sqrt{\lambda(x)} \right)^5 \log^2(r) \left( \frac{1}{(t-r)^{j+k}} + \frac{1}{(t-k)^{j+k}} \right) , \quad r \geq \frac{t}{2} \]

\[
|\partial_{tt} v_3(t,r)| + |\partial^2 v_3(t,r)| \leq \begin{cases} 
\frac{C \lambda(t)^2 \left( \sup_{x \in [T_\lambda, t]} \sqrt{\lambda(x)} \right) \log^2(t)}{r^{5/2}} , & r \leq \frac{t}{2} \\
\frac{C \left( \sup_{x \in [T_\lambda, t+r]} \sqrt{\lambda(x)} \right)^5 \log^2(t+r)}{r^{5/2}} \left( \frac{1}{(t-r)^{j+k}} + \frac{1}{(t-k)^{j+k}} \right) , & r \geq \frac{t}{2} 
\end{cases} \]

**Proof.** We note that

\[
(v_3 - v_{3,\text{main}})(t,r) = \frac{t}{2} \int_{0}^{\pi} \sin(\theta) \int_{0}^{r} dy(y - r)\frac{\partial^2 q}{\partial y^2}(v_{3,0}(\sqrt{y^2 + t^2 + 2yt \cos(\theta)})) \int_{0}^{\pi} \sin(\theta) \int_{0}^{r} dy(y + t \cos(\theta)) \frac{v_{3,0}'(\sqrt{t^2 + y^2 + 2yt \cos(\theta)})}{\sqrt{y^2 + t^2 + 2yt \cos(\theta)}} \]

This immediately leads to the estimates in the lemma statement for \( 0 \leq j, k \leq 1 \). Next, we use

\[
(v_3 - v_{3,\text{main}})(t,r) = \frac{t}{2} \int_{0}^{\pi} \sin(\theta) \int_{0}^{r} dy(y + t \cos(\theta)) \frac{v_{3,0}'(\sqrt{t^2 + y^2 + 2yt \cos(\theta)})}{\sqrt{y^2 + t^2 + 2yt \cos(\theta)}} \]

for the remaining estimates in the lemma statement, since we only estimated three derivatives of \( v_{3,0} \) in \((4.20)\). We then finish the proof by using the same procedure as in Lemma \((4.8)\) \(\square\)
4.8. Matching, Part 3. Here, we need to match the leading part of $u_{ell,3}$ (defined in (4.3)) with appropriate parts of $w_{1,main}$ and $v_{2,main}$. The leading part of $u_{ell,3}$ in the matching region is given by the following function, as per the following lemma.

$$u_{ell,3,main}(t, r) := \int_{\lambda(t)}^{r} \left( \frac{x^2}{r} + x \right) \tilde{c}_1^2(w_{1,cm} - w_{1,lm} + v_{2,qm} - v_{2,lm})(t, x) dx$$

**Lemma 4.22.** For $j = 0, 0 \leq k \leq 2$ or $j = 1, 0 \leq k \leq 1$,

$$|\tilde{c}_1^j \tilde{c}_1^k (u_{ell,3} - u_{ell,3,main})(t, r)| \leq \frac{Cr^{3-k}\lambda(t)^{5/2}}{\mu^2 + j}(1 + \log(\frac{r}{\lambda(t)})), \quad r \geq \lambda(t)$$

**Proof.** We recall that $u_{ell,2}$ is defined in (4.6), use the first order matching, (4.55), and get

(4.58)

$$u_{ell,2}(t, r) = \int_{0}^{1} s^2 \lambda(t)^2 \left( \phi_0(\frac{r}{\lambda(t)})e_2(s) + \lambda(t) \frac{\phi_0(s)}{r} - \left( e_2(\frac{r}{\lambda(t)})\phi_0(s) + \frac{1}{s} \right) \right) \tilde{c}_1^2(w_{1,lm} + v_{2,lm})(t, s\lambda(t))ds$$

$$+ w_{1,cm}(t, r) - w_{1,lm}(t, r) + v_{2,qm}(t, r) - v_{2,lm}(t, r)$$

$$+ \int_{1}^{\lambda(t)} s^2 \lambda(t)^2 \left( \phi_0(\frac{r}{\lambda(t)})e_2(s) + \lambda(t) \frac{\phi_0(s)}{r} - \left( e_2(\frac{r}{\lambda(t)})\phi_0(s) + \frac{1}{s} \right) \right) \tilde{c}_1^2(w_{1,lm} + v_{2,lm})(t, s\lambda(t))ds$$

$$+ \int_{1}^{\lambda(t)} (\phi_0(\frac{r}{\lambda(t)})e_2(s) - e_2(\frac{r}{\lambda(t)})\phi_0(s)) s^2 \lambda(t)^2 \tilde{c}_1^2(u_{w,2,ell,0} + v_{e,ell,0})(t, s\lambda(t))ds$$

$$+ \int_{1}^{\lambda(t)} (\phi_0(\frac{r}{\lambda(t)})e_2(s) - e_2(\frac{r}{\lambda(t)})\phi_0(s)) s^2 \lambda(t)^2 \tilde{c}_1^2(u_{ell} - v_{2,lm} - w_{1,lm})(t, s\lambda(t))ds$$

where we used

$I := \int_{0}^{\lambda(t)} s^2 \lambda(t)^2 \left( \frac{1}{s} - \lambda(t) \frac{\phi_0(s)}{r} \right) \tilde{c}_1^2(w_{1,lm} + v_{2,lm})(t, s\lambda(t))ds = w_{1,cm}(t, r) - w_{1,lm}(t, r) + v_{2,qm}(t, r) - v_{2,lm}(t, r)$

Therefore,

$$u_{ell,3}(t, r) - u_{ell,3,main}(t, r)$$

$$= \int_{0}^{1} (\phi_0(\frac{r}{\lambda(t)})e_2(s) - e_2(\frac{r}{\lambda(t)})\phi_0(s)) s^2 \lambda(t)^2 \tilde{c}_1^2(u_{ell,2})(t, s\lambda(t))ds$$

$$+ \int_{\lambda(t)}^{r} \left( \phi_0(\frac{r}{\lambda(t)})x^2 e_2(\frac{x}{\lambda(t)}) \right) \frac{\lambda(t)}{(\lambda(t))^2} - e_2(\frac{r}{\lambda(t)})x^2 \phi_0(\frac{x}{\lambda(t)}) + \frac{x^2}{r} - x \right) \tilde{c}_1^2(w_{1,cm} - w_{1,lm} + v_{2,qm} - v_{2,lm})(t, x)dx$$

$$+ \int_{\lambda(t)}^{r} \left( e_2(\frac{r}{\lambda(t)})x^2 \phi_0(\frac{x}{\lambda(t)}) \right) \frac{\lambda(t)}{(\lambda(t))^2} + \frac{\lambda(t)}{(\lambda(t))^2} - e_2(\frac{r}{\lambda(t)})x^2 \phi_0(\frac{x}{\lambda(t)}) \right) \tilde{c}_1^2(u_{ell,2} - I)(t, x)dx$$

We then insert each term of (4.58) into the last term in the above expression, and estimate the resulting integrals, and their derivatives directly. 

We also use Lemma 4.18 to directly estimate $u_{ell,3}$. 

□
Lemma 4.23. \n|\partial_t^j \partial_r^k u_{ell,3}(t, r)| \leq \frac{C \sqrt{\lambda(t)} r^{4-k}(r + \lambda(t))}{t^{6+j}}, \quad j, k = 0, 1

For 0 \leq k \leq 1,
|\partial_t^k \partial_r^2 u_{ell,3}(t, r)| \leq \begin{cases} C r^{4-k} \lambda(t)^{3/2}, & r \leq \lambda(t) \\ C r^{-k} \lambda(t), & r \geq \lambda(t) \end{cases}

By direct computation, we get
\[ u_{ell,3,main} - (w_{1,main} - w_{1,cm} + v_{2,main} - v_{2,qm}) = - \int_0^{\lambda(t)} \left( \frac{-x^2}{r} + x \right) \partial_t^2 (w_{1,cm} - w_{1,lm} + v_{2,qm} - v_{2,tm}) (t, x) dx \]
Thus, the terms in $u_{ell,3,main}$ which grow fastest as $r$ approaches infinity match $w_{1,main} - w_{1,cm} + v_{2,main} - v_{2,qm}$. A direct estimation gives

Lemma 4.24. For $j = 0$, 0 \leq k \leq 2 or $j = 1$, 0 \leq k \leq 1, and for $r \geq \lambda(t)$,
\[ |\partial_t^j \partial_r^k (u_{ell,3,main} - (w_{1,main} - w_{1,cm} + v_{2,main} - v_{2,qm}))| \leq \frac{\lambda(t) \sqrt{t^{11/2}}}{t^{6+j}} \begin{cases} 1, & k = 0 \\ \lambda(t)^{4/11}, & k \geq 1 \end{cases} \]

4.9. Preliminary Ansatz. We need to add another correction into our ansatz, in order to eliminate the linear error term associated to $v_3$, namely
\[ e_3(t, r) := \frac{-\chi_{\geq 1} \left( \frac{r}{\lambda(t)} \right) \cdot 45 \lambda(t)^2 v_3(t, r)}{(3\lambda(t)^2 + r^2)^2} \]
We let $u_3$ be the solution to
\[ -\partial_t^2 u_3 + \partial_r^2 u_3 + \frac{2}{r} \partial_r u_3 = e_3(t, r) \]
with zero Cauchy data at infinity. In other words,
\[ u_3(t, r) = \int_t^{\infty} ds \left( -\frac{1}{2r} \int_{[r-(s-t)]} y e_3(s, y) dy \right) \]
Then, we have

Lemma 4.25. For $r \geq \frac{h(t)}{t}$, the following estimates are true.
\[ |u_3(t, r)| \leq \frac{C \lambda(t)^2 \left( \sup_{x \in [T_\lambda, t]} \sqrt{\lambda(x)} \right)^5 ( \log^2(t) + \log^2(r))}{t^4} \]
\[ |u_3(t, r)| \leq \frac{C \lambda(t)^2 \left( \sup_{x \in [T_\lambda, t]} \sqrt{\lambda(x)} \right)^5 \log^2(t)}{rt^3} \]
\[ |\partial_r u_3(t, r)| \leq \frac{C \lambda(t)^2 \left( \sup_{x \in [T_\lambda, t]} \sqrt{\lambda(x)} \right)^5 \log^2(t)}{rt^3} \left( \frac{1}{t} + \frac{1 + \frac{r}{(t-r)^2} + \frac{r}{t^3}}{t} \right) \]
\[ |\partial_t u_3(t, r)| \leq C \frac{\lambda(t)^2 \left( \sup_{x \in [T_\lambda, t]} \sqrt{\lambda(x)} \right)^5 \log^2(t)}{t^3}, \quad \frac{h(t)}{4} \leq r \leq \frac{t}{2} \]
We estimate again insert (4.62) into (4.59), and estimate the integrals by:

\[
|\partial_t u_3(t, r)| + |\partial_r^2 u_3(t, r)| \leq \frac{C \lambda(t)^2 \left( \sup_{x \in [T, t]} \sqrt{\lambda(x)} \right)^5 \log^2(t)}{rt^4 h(t)^2} \left( 1 + \frac{r h(t)^2}{t} \right), \quad \frac{h(t)}{4} \leq r \leq \frac{t}{2}
\]

\[
|\partial_r u_3| \leq \frac{C \lambda(t)^2 \left( \sup_{x \in [T, t]} \sqrt{\lambda(x)} \right)^5 \log^2(t)}{rt^3 h(t)^2} \cdot \left( \frac{h(t)^2}{r^3} + \frac{h(t)^2}{r} + \frac{1}{t} \right), \quad \frac{h(t)}{4} \leq r \leq \frac{t}{2}
\]

Finally,

\[
(4.61) \quad \|\partial_t u_3\|_{L^2(\{r^2 dr\})} + \|\partial_r u_3\|_{L^2(\{r^2 dr\})} \leq \frac{C \lambda(t)^2 \left( \sup_{x \in [T, t]} \sqrt{\lambda(x)} \right)^5 \log^2(t)}{r^3}
\]

**Proof.** We start with the following estimates on $e_3$, which are a direct consequence of Lemma 4.21

\[
(4.62) \quad |e_3(t, r)| \leq \frac{C 1_{\{r \geq \bar{h}(t)\}} \lambda(t)^2}{(h(t)^2 + r^2)^2} \left( \frac{\lambda(t)^2 \left( \sup_{x \in [T, t]} \sqrt{\lambda(x)} \right)^5 \log^2(r)}{t} \right), \quad r \leq \frac{t}{2}
\]

Then, we insert these into (4.59), to get

\[
u_3(t, r) \leq C \int_t^{t+2r} \frac{ds}{r} \int_{|r-(s-t)|}^{r+s-t} \frac{y \lambda(s)^4 \left( \sup_{x \in [T, s]} \sqrt{\lambda(x)} \right) \log^2(s)}{s^3 (h(s)^2 + y^2)^2} dy + C \int_t^{t+2r} \frac{ds}{r} \int_{|r-(s-t)|}^{r+s-t} \frac{\lambda(s)^2 \left( \sup_{x \in [T, s]} \sqrt{\lambda(x)} \right)^5 \log^2(y)dy}{(s - y)^2 s^4}
\]

\[
+ C \int_t^{\infty} \frac{ds}{r} \int_{|r-(s-t)|}^{r+s-t} \frac{y \lambda(s)^2 \left( \sup_{x \in [T, s]} \sqrt{\lambda(x)} \right) \log^2(s)}{(h(s)^2 + y^2)^2} dy + C \int_t^{\infty} \frac{ds}{r} \int_{|r-(s-t)|}^{r+s-t} \frac{\lambda(s)^2 \left( \sup_{x \in [T, s]} \sqrt{\lambda(x)} \right) \log^2(s)}{(s - y)^2 s^4} dy
\]

which gives the first estimate in the lemma statement. For (4.60) (which is useful when $r \geq t$) we again insert (4.62) into (4.59), and estimate the integrals by:

\[
u_3(t, r) \leq C \int_t^{\infty} \frac{ds}{r} \int_{h(s)}^{\infty} \frac{\lambda(s)^4 \left( \sup_{x \in [T, s]} \sqrt{\lambda(x)} \right) \log^2(s)}{y^3 s^5} dy + C \int_t^{\infty} \frac{ds}{r} \int_{\frac{h(s)}{2}}^{\infty} \frac{\lambda(s)^2 \left( \sup_{x \in [T, s]} \sqrt{\lambda(x)} \right) \log^2(s)}{s^2 y^2 \left( s - y \right)^2} dy
\]

We estimate $\partial_r u_3$ similarly. For $\partial_t u_3$, we note that

\[
u_3(t, r) = \int_0^\infty dw \left( \frac{-1}{2r} \int_{|r-w|}^{r+w} y e_3(t + w, y) dy \right)
\]

so

\[
\partial_t u_3(t, r) = \int_0^\infty dw \left( \frac{-1}{2r} \int_{|r-w|}^{r+w} y \partial_1 e_3(t + w, y) dy \right)
\]
Then, we estimate $\partial_t u_3$ similarly to $u_3$. $\partial_r u_3$ and $\partial_t^2 u_3$ are estimated similarly to $\partial_r u_3$. Next, we use the equation

$$\partial_t^2 u_3 = e_3(t, r) - \frac{2}{r} \partial_r u_3 + \partial_t^2 u_3$$

to estimate $\partial_t^2 u_3$ from our previous estimates. Finally, the energy estimate, (4.61), is proven with the same procedure used to establish (4.41), with the analog of (4.42) being

$$|u_3(t, r)| \leq \frac{C\lambda(t)^4 \left( \sup_{x \in [T, t]} \sqrt{\lambda(x)} \right) \log^2(t)}{h(t)^4} + \frac{C\lambda(t)^2 \left( \sup_{x \in [T, t]} \sqrt{\lambda(x)} \right)^5 \log^2(t)}{t^3}$$

(which results from directly inserting (4.62) into (4.59)). \hfill \square

Let

$$\psi_2(x) = \begin{cases} 1, & x \geq \frac{1}{2} \\ 0, & x \leq \frac{1}{4} \end{cases}$$

We will insert the function $\psi_2 \left( \frac{r}{h(t)} \right) u_3(t, r)$ into our ansatz, and define its error term as

$$e_{u_3}(t, r) := - \left( -\partial_t^2 + \partial_r^2 - \frac{2}{r} \partial_r - V(t, r) \right) \left( \psi_2 \left( \frac{r}{h(t)} \right) u_3(t, r) \right)$$

where we recall the definition of $V$ in (2.2). Using the support properties of $e_3$ and $\psi_2$,

$$(1 - \psi_2 \left( \frac{r}{h(t)} \right)) e_3(t, r) = 0$$

and this gives

$$e_{u_3}(t, r) := -2\partial_r u_3(t, r) \partial_r \left( \psi_2 \left( \frac{r}{h(t)} \right) \right) + 2\partial_t u_3(t, r) \partial_t \left( \psi_2 \left( \frac{r}{h(t)} \right) \right) + V(t, r) u_3(t, r) \psi_2 \left( \frac{r}{h(t)} \right)$$

$$+ u_3(t, r) \left( -\partial_r^2 \left( \psi_2 \left( \frac{r}{h(t)} \right) \right) + \partial_t^2 \left( \psi_2 \left( \frac{r}{h(t)} \right) \right) - \frac{2}{r} \partial_r \left( \psi_2 \left( \frac{r}{h(t)} \right) \right) \right)$$

A direct application of Lemma 4.25 gives

**Lemma 4.26.**

$$\|e_{u_3}(t, \cdot)\|_{H^1(\mathbb{R}^3)} \leq \frac{C\lambda(t)^2 \left( \sup_{x \in [T, t]} \sqrt{\lambda(x)} \right)^5 \log^2(t)}{h(t)^4}$$

We now assemble our previous corrections into a function, $u_a$, as follows. Let

$$u_e(t, r) = u_{eH}(t, r) + u_{eH, 2}(t, r) + u_{eH, 3}(t, r), \quad u_w(t, r) = w_1(t, r) + v_{ex}(t, r) + v_2(t, r) + u_{w, 2}(t, r) + v_3(t, r)$$

$$\chi_{\leq 1} \in C^\infty([0, \infty)), \quad \chi_{\leq 1}(x) = \begin{cases} 1, & x \leq 1 \\ 0, & x \geq 2 \end{cases}, \quad 0 \leq \chi_{\leq 1}(x) \leq 1, \quad \chi_{\geq 1}(x) = 1 - \chi_{\leq 1}(x)$$

$$\chi_{\leq 1} \left( \frac{r}{h(t)} \right) u_e(t, r) + (1 - \chi_{\leq 1} \left( \frac{r}{h(t)} \right)) u_w(t, r)$$

(4.64)
Then, we define
\[ u_q(t, r) = u_e(t, r) + u_3(t, r)\psi_2\left(\frac{r}{h(t)}\right) \]
If we substitute \( u(t, r) = Q_\lambda(t)(r) + u_q(t, r) + v(t, r) \) into (1.1), we obtain
\[
(4.65) \quad - c_t^2 v + \dot{c}_r v + \frac{2}{r} \dot{c}_t + \frac{45\lambda(t)^2}{(3\lambda(t)^2 + r^2)^2} v(t, r) = e_{\text{match}}(t, r) + e_{\text{ell}, 3}(t, r) + e_{\text{ex}}(t, r) + e_{w, 2}(t, r) + \frac{45\lambda(t)^5}{Q_\lambda(t)^5} - \frac{5Q_\lambda(t)^4}{Q_\lambda(t)^4}(u_a + v) \]
where
\[
e_{\text{match}}(t, r) = \left( c_t^2(\chi \leq 1(\frac{r}{h(t)})) - \frac{2}{r} \dot{c}_t(\chi \leq 1(\frac{r}{h(t)})) \right) (u_e - u_w) + 2c_t(\chi \leq 1(\frac{r}{h(t)}))\dot{c}_t(u_e - u_w) - 2\dot{c}_t(\chi \leq 1(\frac{r}{h(t)}))\dot{c}_t(u_e - u_w) \]
\[
e_{\text{ell}, 3}(t, r) = \chi \leq 1(\frac{r}{h(t)}) c_t^2 u_{\text{ell}, 3} \]
\[
e_{\text{ex}}(t, r) := -\chi \geq 1(\frac{r}{h(t)}) \frac{45\lambda(t)^2 v_{\text{ex}}(t, r)}{(3\lambda(t)^2 + r^2)^2} \]
\[
e_{w, 2}(t, r) = V(t, r) u_{w, 2}(t, r) \chi \geq 1(\frac{r}{h(t)}) \]
We now estimate each of the linear error terms, starting with \( e_{\text{match}} \). Using (4.55) and (4.56), we get
\[
u_e(t, r) - u_w(t, r) = u_{\text{ell}} - u_{\text{ell}, \text{main}} + u_{\text{ell}, 2} - u_{\text{ell}, 2, \text{main}} + u_{\text{ell}, 3} - u_{\text{ell}, 3, \text{main}} - (v_3 - v_{3, \text{main}}) + e_{m, 2} - (w_1 - w_{1, \text{main}}) - (v_2 - v_{2, \text{main}}) - (u_{w, 2} - u_{w, 2, \text{ell}, 00} - u_{w, 2, \text{ell}, 0, 1} - u_{w, 3, \text{ell}, 0}) - (v_{\text{ex}} - v_{\text{ex, ell}, 0} - v_{\text{ex, sub, ell}}) + u_{\text{ell}, 3, \text{main}} - (w_{1, \text{main}} - w_{1, \text{cm}} + v_{2, \text{main}} - v_{2, qm}) - v_{3, \text{main}} \]
By directly combining Lemmas 4.17, 4.18, 4.2, 4.21, 4.19, 4.16, 4.7, 4.13, 4.12, 4.6, 4.4, 4.24 we get

**Lemma 4.27.** For \( j = 0, 0 \leq k \leq 2 \) or \( j = 1, 0 \leq k \leq 1 \), and in the region \( \lambda(t) \leq r \leq \frac{4}{12} \), we have
\[
|c_t^j c_r^k ((u_e - u_w + w_1 - w_{1, \text{main}})(t, r))| \leq C \frac{\sqrt{\lambda(t)}}{r^3 + k t^2 + j} \left( \lambda(t)^4 \log^2(t) + \frac{r^9 \lambda(t)}{t^6} \right) \]
\[
+ C \left\{ \begin{aligned}
&\frac{r^2 \lambda(t)^2 \text{sup}_{e \in [T, t]} \sqrt{\lambda(x) \log^2(t)}}{t^3} + \frac{\lambda(t)^2 \log^2(t)}{t^3} & j = k = 0 \\
&\frac{r \lambda(t)^2 \text{sup}_{e \in [T, t]} \sqrt{\lambda(x) \log^2(t)}}{t^3} + \frac{\lambda(t)^{7/2} \log^2(t)}{t^4} & j = 1, k = 0 \\
&\frac{\lambda(t)^2 \text{sup}_{e \in [T, t]} \sqrt{\lambda(x) \log^2(t)}}{t^3} & j = 1, k = 0 \\
&\frac{\lambda(t)^2 \text{sup}_{e \in [T, t]} \sqrt{\lambda(x) \log^2(t)}}{t^3} & j + k = 2
\end{aligned} \right\}
\]
Lemmas 4.1 and 4.27 directly give
Lemma 4.28.
\[ \|e_{\text{match}}(t, | \cdot |)\|_{H^1(\mathbb{R}^3)} \leq \frac{C \log^2(t)}{t^{4+\delta}} \]
where
\[ \delta = \frac{1}{2} \min \{4 - 7C_u - \frac{13}{2}a(1 - C_u), -2 - C_u + \frac{7}{2}a(1 - C_u), 1 - 4C_u + \frac{3}{2}a(-1 + C_u), \frac{1}{2}a(1 - C_u) - 3C_u \} > 0 \]
Note that the positivity of \( \delta \) is due to (4.3). Next, we consider \( e_{w,2} \). By Lemma 4.29, we get

Lemma 4.29.
\[ \|e_{w,2}(t, | \cdot |)\|_{H^1(\mathbb{R}^3)} \leq \frac{C \lambda(t)^2 \sup_{x \in [T_{\lambda}, t]}(\lambda(x)^{5/2}) \log^2(t)}{h(t)^{7/2} t^2} + \frac{C \lambda(t)^2 \log^2(t) \sup_{x \in [T_{\lambda}, t]}(\lambda(x)^{3/2})}{t^2 h(t)^{7/2}} \]
Similarly, Lemma 4.3 gives

Lemma 4.30.
\[ \|e_{ex}(t, | \cdot |)\|_{H^1(\mathbb{R}^3)} \leq \frac{C \lambda(t)^{9/2} \log(t)}{t^2 h(t)^{7/2}} \left(1 + \frac{1}{h(t)} \right) \]

The next linear error term to consider is \( e_{ell,3} \). From Lemma 4.23, we get

Lemma 4.31.
\[ \|e_{ell,3}(t, | \cdot |)\|_{H^1(\mathbb{R}^3)} \leq \frac{C h(t)^{13/2} \sqrt{\lambda(t)}}{t^5} \]

4.10. Nonlinear error terms, part 1. We study the nonlinear error terms in (4.65). In particular, we define

\[ u_{a,0}(t, r) := \chi_{\leq 1}(\frac{r}{h(t)})u_{\text{ell}}(t, r) + (1 - \chi_{\leq 1}(\frac{r}{h(t)}))(w_1 + v_2 + v_3) \]

\[ u_{a,1}(t, r) := \chi_{\leq 1}(\frac{r}{h(t)})(u_{w,2}(t, r) + u_{ell,3}(t, r)) + (1 - \chi_{\leq 1}(\frac{r}{h(t)}))(v_{ex} + u_{w,3}) + u_3(t, r)\psi_2(\frac{r}{h(t)}) \]

so that \( u_a(t, r) = u_{a,0}(t, r) + u_{a,1}(t, r) \). Next, we split the nonlinear terms into \( N_0 \), which is not quite perturbative, and \( N_1 \) which is.

\[ (4.66) \]
\[ N_0(t, r) := - \left(10Q_{\lambda(t)}^3 u_{a,0}^2 + 10Q_{\lambda(t)}^2 u_{a,0}^3 + 5Q_{\lambda(t)} u_{a,0}^4 + u_{a,0}^5 \right) \]

\[ N_1(t, r) := - \left(10Q_{\lambda(t)}^3 (u_{a,1}^2 - u_{a,0}^2) + 10Q_{\lambda(t)}^2 (u_{a,1}^3 - u_{a,0}^3) + 5Q_{\lambda(t)} (u_{a,1}^4 - u_{a,0}^4) + u_{a,1}^5 - u_{a,0}^5 \right) \]

We begin by showing that \( N_1 \) is perturbative:

Lemma 4.32.
\[ ||N_1(t, | \cdot |)||_{H^1(\mathbb{R}^3)} \leq \frac{C \log^{10}(t)}{t^{4+\delta_1}} \]
where \( \delta_1 > 0 \) is given by
\[ \delta_1 = \frac{1}{2} \min \{2 + \frac{5}{2}a(-1 + C_u) - 5C_u, \frac{3}{2}a(1 - C_u) - 3C_u, \frac{3}{2}a - \frac{3}{2}(1 - a)C_l - \frac{7}{2}C_u, \frac{3}{2}C_l - \frac{9}{2}C_u, 2 - C_l - \frac{25}{2}C_u \} \]
Proof. From the definition of $N_1$, we get
\begin{equation}
|N_1(t, r)| \leq C|u_{a,1}|(|Q_{\Lambda}|^3(|u_a| + |u_{a,0}|) + u_{a,0}^4 + u_a^4)
\end{equation}

Let
\begin{equation}
\begin{split}
|u_{a,0,est}(t, r)| &= \begin{cases} 
\frac{\lambda(t)^{3/2}}{t^2} & r \leq \lambda(t) \\
\frac{r \sup_{x \leq T_{\lambda, t}^r} \sqrt{\lambda(x)}}{r} & r \geq \lambda(t)
\end{cases}
\end{split}
\end{equation}

Then, using Lemmas 4.17, 4.21, 4.23, 4.25 and 4.25 to get,
\begin{equation}
|\partial_r^k u_{a,0}(t, r)| \leq \begin{cases} 
\frac{C_r}{\sqrt{\lambda(t)} t^2} & k = 1 \\
C \left( \frac{1}{r^k} + \frac{1}{\langle t - r \rangle^k} \right) \cdot u_{a,0,est}(t, r) & k = 0 \\
C \left( \frac{1}{r^k} + \frac{1}{\langle t - r \rangle^k} \right) \cdot u_{a,0,est}(t, r) & k = 1
\end{cases}
\end{equation}

For later use, we also note that, for $0 \leq j \leq 3$,
\begin{equation}
|\partial_t^j u_{a,0}(t, r)| \leq C \left( \frac{1}{t^j} + \frac{1}{\langle t - r \rangle^j} \right) \cdot u_{a,0,est}(t, r)
\end{equation}

and the following two estimates are true:
\begin{equation}
|\partial_t u_{a,0}(t, r)| \leq C u_{a,0,est}(t, r) \left( \frac{1}{t} + \frac{1}{\langle t - r \rangle^2} + \frac{1}{r^2} \right)
\end{equation}
\begin{equation}
|\partial_{tt} u_{a,0}(t, r)| \leq C u_{a,0,est}(t, r) \left( \frac{1}{t^2} + \frac{1}{\langle t - r \rangle^3} + \frac{1}{r^3} \right)
\end{equation}

Next, we use Lemmas 4.18, 4.23, 4.19, 4.21, 4.25 to get, for $k = 0, 1$,
\begin{equation}
|\partial_r^k u_{a,1}(t, r)| \leq C \left( \frac{1}{r^k} + \frac{1}{t^k} \right) \cdot \left( \frac{r^2 (r + \lambda(t)) \sqrt{\lambda(t)}}{t^4} \right)
\end{equation}
\begin{equation}
+ C \left( \frac{\sup_{x \leq T_{\lambda, t}^r} \lambda(x)^{5/2} \log^2(t + r)}{rt^2} \right) + C \left( \frac{\sup_{x \leq T_{\lambda, t}^r} \lambda(x)^{5/2} \log^2(t + r)}{t^3 \max \{r, t\}} \right)
\end{equation}

We obtain the lemma statement by inserting the above into (4.67), and estimating directly, using Lemma 4.25 to estimate $\|\partial_r u_3(t, r)\|_{L^2(t^2 dr)}$. \hfill \Box

Next, we consider $N_0$, which we recall is defined in (4.66). We let $u_{N_0}$ be the solution to the following equation with 0 Cauchy data at infinity:
\[-\partial_t^2 u_{N_0} + 2 \partial_r u_{N_0} + \frac{2}{r} \partial_r u_{N_0} = N_0\]
We have

\[ u_{N_0}(t, r) = \int_t^\infty ds \left( \frac{-1}{2r} \int_{|r-(s-t)|}^{r+s-t} y N_0(s, y) dy \right) \]  

(4.70)

Note that all of the estimates in the entire proof thus far are valid for all \( t \geq T_0 \) for any \( T_0 \geq 2T_{0,1} \) (recall that we restricted \( T_0 \) after (4.57)). In particular, they are valid for all \( t \geq 2T_{0,1} \). We then define

\[ v_{4,0}(r) = \frac{1}{r} \int_r^\infty ds (s-r) N_0(s, s-r) \psi \left( \frac{r T_{\lambda}}{2T_{0,1}} \right), \quad r > 0 \]

(4.71)

for \( \psi \) as in (4.11), restrict \( T_0 \) to satisfy \( T_0 \geq 4T_{0,1} \), but is otherwise arbitrary, and we let \( v_4 \) solve

\[ \begin{cases} -c_4^2 v_4 + c_4^2 v_4 + \frac{2}{\lambda} \partial_r v_4 = 0 \\ v_4(0, r) = 0, \quad \partial_t v_4(0, r) = v_{4,0}(r) \end{cases} \]

Then, we have the following estimates

**Lemma 4.33.** For \( j = 0, 1, 2 \),

\[ |c_t^j (u_{N_0}(t, r) + tv_{4,0}(t))| \leq C \left( \min\{1, r\} \left( \sup_{x \in [T_{\lambda}, t]} \sqrt{\lambda(x)} \right)^{25} \log^{10}(t) \right. \]

\[ \left. + \frac{r \left( \sup_{x \in [T_{\lambda}, t]} \sqrt{\lambda(x)} \right)^5 \left( \log^{10}(t) + \log^{10}(r) \right)}{t^4} + \frac{r^2 \left( \sup_{x \in [T_{\lambda}, t]} \sqrt{\lambda(x)} \right)^3}{t^4} \right), \quad r \leq \frac{t}{2} \]

(4.72)

\[ |u_{N_0}(t, r)| \leq \frac{C \left( \sup_{x \in [T_{\lambda}, t]} \sqrt{\lambda(x)} \right)^5 \log^{10}(t)}{rt^2}, \quad r \geq \frac{t}{2} \]

(4.73)

\[ |\partial_r u_{N_0}(t, r)| \leq \frac{C \left( \sup_{x \in [T_{\lambda}, t]} \sqrt{\lambda(x)} \right)^5 \log^{10}(t) \min\{1, r\}}{rt^4} \]

\[ + \frac{C \left( \sup_{x \in [T_{\lambda}, t]} \sqrt{\lambda(x)} \right)^5 \left( \log^{10}(t) + \log^{10}(r) \right)}{t^4} + \frac{C \left( \sup_{x \in [T_{\lambda}, t]} \sqrt{\lambda(x)} \right)^3}{t^4}, \quad r \leq \frac{t}{2} \]

(4.74)

\[ |\partial_r^2 u_{N_0}(t, r)| \leq \frac{C \left( \sup_{x \in [T_{\lambda}, t+r]} \sqrt{\lambda(x)} \right)^5 \log^{10}(t+r)}{rt^3} \left( 1 + \left( \sup_{x \in [T_{\lambda}, t+r]} \sqrt{\lambda(x)} \right)^{20} \left( \frac{1}{t} + \frac{1}{(t-r)^{10}} \right) \right), \quad r \geq \frac{t}{2} \]

For \( 0 \leq k \leq 2 \),

\[ |c_r^k (v_4(t, r) - tv_{4,0}(t))| \leq \frac{C r^{2-k} \left( \sup_{x \in [T_{\lambda}, t]} \sqrt{\lambda(x)} \right)^5 \log^{10}(t)}{t^3}, \quad r \leq \frac{t}{2} \]

(4.75)
For $j = 1, 2,$

$$|\partial_t^j (v_4 - tv_{4,0}(t))| \leq \frac{C r^{3-j} \left( \sup_{x \in [T, t]} \sqrt{\lambda(x)} \right)^5 \log^{10}(t)}{t^6}, \quad r \leq \frac{t}{2}$$

For $0 \leq j + k \leq 2,$

$$|\partial_t^j \partial_r^k v_4(t, r)| \leq \frac{C}{r(t-r)^2} \left( \sup_{x \in [T, t+r]} \sqrt{\lambda(x)} \right)^5 \log^{10}(t+r) \left( \frac{1}{(t-r)^{2j+k}} + \frac{1}{t^{j+k}} \right), \quad r \geq \frac{t}{2}$$

Finally,

$$||\partial_t u_{N_0}(t, r)||_{L^2(r^2dr)} + ||\partial_r u_{N_0}||_{L^2(r^2dr)} \leq \frac{C \left( \sup_{x \in [T, t]} \sqrt{\lambda(x)} \right)^5 \log^{10}(t)}{t^{5/2}}$$

Proof. We start with estimates on $N_0$. Define

$$N_{0,\text{est}}(t, r) = \begin{cases} \frac{\lambda(t)^{3/2}}{r^{3/2}}, & r \leq \lambda(t) \\ \frac{\lambda(t)^{5/2}}{r^{3/2}}, & \lambda(t) \leq r \leq h(t) \\ \frac{\left( \sup_{x \in [T, t]} \sqrt{\lambda(x)} \right)^5}{r^3} \log^{10}(t+r) \left( 1 + \frac{\left( \sup_{x \in [T, t+r]} \sqrt{\lambda(x)} \right)^{10}}{(t-r)^{10}} \right), & r \geq \frac{t}{2} \end{cases}$$

Directly estimating the definition of $N_0$, we get, for $k = 0, 1$,

$$|\partial_r^k N_0(t, r)| \leq C \left( \frac{1}{r} + \frac{1}{(t-r)^j} + \frac{1}{t} \right)^k \cdot N_{0,\text{est}}(t, r)$$

We also note that

$$|\partial_t N_0(t, r)| \leq C(Q_\lambda(r)^2 u_{a,0}^2 + u_{a,0}^4) |\partial_t Q_\lambda| + C(|Q_\lambda|^3 u_{a,0} + u_{a,0}^4) |\partial_t u_{a,0}|$$

Similarly directly estimating $\partial_r^2 N_0(t, r)$, and using (4.68), we get, for $j = 1, 2, 3$,

$$|\partial_r^j N_0(t, r)| \leq C N_{0,\text{est}}(t, r) \left( \frac{1}{t^j} + \frac{1}{(t-r)^j} \right)$$

Also with the same procedure, for $j = 1, 2$,

$$|\partial_t^j \partial_r N_0(t, r)| \leq C \left( \frac{1}{t^j} + \frac{1}{(t-r)^j} \right) \left( \frac{1}{r} + \frac{1}{(t-r)^j} + \frac{1}{t} \right) \cdot N_{0,\text{est}}(t, r)$$

Inserting (4.79) into (4.70) gives (4.73). To obtain (4.72) for $j = 0$, we first decompose (4.70) as

$$u_{N_0}(t, r) = \int_t^{t+r} ds \left( \frac{-1}{2r} \right) \int_{r-s-t}^{r+s-t} y N_0(s, y) dy - \int_t^\infty ds(s-t) N_0(s, s-t) + \int_t^{t+r} ds(s-t) N_0(s, s-t) + \int_{t+r}^\infty ds \left( \frac{-1}{2r} \right) \int_{s-t-r}^{s+t-r} (y N_0(s, y) - y N_0(s, s-t)) dy$$
To estimate the last term in \( \text{(4.80)} \), we use the mean value theorem for the function \( x \mapsto N_0(s, x) \), and directly estimate the remaining terms, to get \( \text{(4.72)} \) for \( j = 0 \). Next, we directly estimate the definition of \( v_{4,0} \), \( \text{(4.71)} \), to get, for \( 0 \leq j \leq 3 \),

\[
\tag{4.81}
|v_{4,0}^{(j)}(r)| \leq C \frac{\left( \sup_{x \in [T,t]} \lambda(x) \right)^{5} \log^{10}(r) 1_{\{r \geq T \}}}{r^{4+j}}
\]

For \( \text{(4.75)} \), we write

\[
v_{4}(t, r) - tv_{4,0}(t) = \frac{t}{2} \int_{0}^{\pi} \sin(\theta) \left( v_{4,0}(\sqrt{r^2 + t^2 + 2rt \cos(\theta)}) - v_{4,0}(t) \right) d\theta
\]

\[
= \frac{t}{2} \int_{0}^{\pi} \sin(\theta) \int_{0}^{r} dy(\theta - y) \frac{\partial^2_y}{\partial y^2} \left( v_{4,0}(\sqrt{y^2 + t^2 + 2yt \cos(\theta)}) \right) d\theta
\]

Directly estimating this gives \( \text{(4.75)} \), and \( \text{(4.76)} \) for \( j = 1 \). Since we only have three derivatives of \( v_{4,0} \) estimated in \( \text{(4.81)} \), we obtain \( \text{(4.76)} \) for \( j = 2 \) by directly differentiating and estimating

\[
v_{4}(t, r) - tv_{4,0}(t) = \frac{t}{2} \int_{0}^{\pi} \sin(\theta) \int_{0}^{r} \partial_y \partial^2_y \left( v_{4,0}(\sqrt{y^2 + t^2 + 2yt \cos(\theta)}) \right) dy d\theta
\]

Next, we obtain \( \text{(4.77)} \) with the same procedure as for the analogous estimates in Lemma \( \text{4.21} \).

Next, for \( \text{(4.74)} \), we note that

\[
\partial_r u_{N_0}(t, r) = \partial_r \left( u_{N_0}(t, r) + \int_{t}^{\infty} ds(s-t)N_0(s, s-t) \right)
\]

and we write \( u_{N_0}(t, r) + \int_{t}^{\infty} ds(s-t)N_0(s, s-t) \) using the expression \( \text{(4.80)} \). \( \text{(4.74)} \) follows from direct differentiation. If \( r' \geq \frac{t}{2} \), we simply directly differentiate \( \text{(4.70)} \), to get

\[
\partial_r u_{N_0}(t, r) = \frac{-u_{N_0}(t, r)}{r} + \int_{t}^{\infty} ds \left( \frac{-1}{2r} \right) ((r + s - t)N_0(s, r + s - t) + (s - t - r)N_0(s, |s - t - r|))
\]

To estimate \( \partial_t^2(u_{N_0}(t, r) + \int_{t}^{\infty} ds(s-t)N_0(s, s-t)) \), we use \( \text{(4.80)} \) to get

\[
\partial_t^2 \left( u_{N_0}(t, r) + \int_{t}^{\infty} ds(s-t)N_0(s, s-t) \right)
\]

\[
= \partial_t^2 \left( \int_{t}^{r} dw \left( \frac{-1}{2r} \right) \int_{\lfloor r-w \rfloor}^{\lceil r+w \rceil} yN_0(w+t, y)dy \right) + \partial_t^2 \left( \int_{0}^{r} dw wN_0(t+w, w) \right)
\]

\[
+ \partial_t^2 \left( \int_{r}^{\infty} dw \left( \frac{-1}{2r} \right) \int_{w-r}^{w+r} y(N_0(t+w, y) - N_0(t+w, w))dy \right)
\]

We then directly estimate this expression.

Finally, a direct computation using \( \text{(4.79)} \) gives

\[
||N_0(t, r)||_{L^2(v^{2}dr)} \leq C \left( \sup_{x \in [T, t]} \lambda(x) \right)^{5} \log^{10}(t) \left( \frac{\left( \sup_{x \in [T, t]} \lambda(x) \right)^{20}}{t^{7/2}} \left( 1 + \frac{\left( \sup_{x \in [T, t]} \lambda(x) \right)^{20}}{\sqrt{t}} \right) \right)
\]

The same procedure used to establish \( \text{(4.61)} \) then gives \( \text{(4.78)} \). \( \square \)
We define the linear error terms associated to \( u_{N_0} \) and \( v_4 \) by
\[
(4.82) \quad e_{u_{N_0}}(t, r) := -45\lambda(t)^2 u_{N_0}(t, r) \quad \text{and} \quad e_{v_4}(t, r) := -45\lambda(t)^2 v_4(t, r)
\]
Then, let \( \psi_1 \in C^\infty([0, \infty)) \) satisfy \( 0 \leq \psi_1(x) \leq 1 \),
\[
\psi_1(x) = \begin{cases} 1, & x \leq \frac{1}{4} \\ 0, & x \geq \frac{3}{4} \end{cases}
\]
First, a direct application of Lemma 4.33 shows that the following piece of the linear error terms of \( u_{N_0} \) and \( v_4 \) is perturbative.

**Lemma 4.34.**
\[
\| (e_{u_{N_0}} + e_{v_4} \psi_1(\frac{|\cdot|}{t})) \chi_{t} \|_{\dot{H}^1(\mathbb{R}^3)} \leq \frac{C \lambda(t)^2}{t^{9/2}} \left( \sup_{x \in [T_x, t]} \sqrt{\lambda(x)} \right)^3
\]

Next, we let \( u_4 \) solve the following equation with zero Cauchy data at infinity:
\[
(4.83) \quad -\partial_t^2 u_4 + \partial_r^2 u_4 + \frac{2}{r} \partial_r u_4 = e_{v_4}(t, r)(1 - \psi_1(\frac{r}{t})) \chi_{t} \left( \frac{4r}{t} \right)
\]
We have
\[
(4.84) \quad u_4(t, r) = \int_{t}^{\infty} ds \left( \frac{-1}{2r} \int_{|r-(s-t)|}^{r+s-t} y \left( e_{v_4}(s, y)(1 - \psi_1(\frac{y}{s})) \chi_{\frac{1}{1}} \left( \frac{4y}{s} \right) \right) dy \right)
\]
Then, we get

**Lemma 4.35.**
\[
|u_4(t, r)| \leq \begin{cases} \frac{C \lambda(t)^2 \left( \sup_{x \in [T_x, t]} \sqrt{\lambda(x)} \right)^5 \log^{10}(t)}{t^{9/2}}, & r \leq \frac{t}{2} \\ \frac{C \lambda(t)^2 \left( \sup_{x \in [T_x, t]} \sqrt{\lambda(x)} \right)^5 \log^{10}(t)}{t^{13}}, & r \geq \frac{t}{2} \end{cases}
\]
\[
|\partial_r u_4(t, r)| \leq \begin{cases} \frac{C \lambda(t)^2 \left( \sup_{x \in [T_x, t]} \sqrt{\lambda(x)} \right)^5 \log^{10}(t)}{t^{10}}, & r \leq \frac{t}{2} \\ \frac{C \lambda(t)^2 \left( \sup_{x \in [T_x, t]} \sqrt{\lambda(x)} \right)^5 \log^{10}(r)}{t^{13}}, & r \geq \frac{t}{2} \end{cases}
\]
\[
||\partial_r u_4(t, r)||_{L^2(r^2 dr)} + ||\partial_r u_4(t, r)||_{L^2(r^2 dr)} \leq \frac{C \lambda(t)^2 \left( \sup_{x \in [T_x, t]} \sqrt{\lambda(x)} \right)^5 \log^{10}(t)}{t^{13}}
\]

**Proof.** We directly estimate (4.84), using Lemma 4.33. We then directly estimate the \( r \) derivative of the \( u_4 \)-analog of (1.80). The energy estimate is obtained with the same procedure as in Lemma 4.25.

We define the linear error term of \( u_4 \) by
\[
(4.85) \quad e_{u_4}(t, r) := -45\lambda(t)^2 u_4(t, r) \quad \text{and} \quad e_{v_4}(t, r) := -45\lambda(t)^2 v_4(t, r)
\]
A direct application of Lemma 4.33 gives
Lemma 4.36.

$$||e_{u_4}(t, \cdot)||_{H^1(\mathbb{R}^3)} \leq \frac{C \sqrt{\lambda(t)} (1 + \lambda(t)) \left( \sup_{x \in [T_\lambda, t]} \sqrt{\lambda(x)} \right)^5 \log^{10}(t)}{t^5}$$

Finally, we eliminate the remaining piece of the $v_4$ and $u_{N_0}$ linear error terms. In particular, we let $u_{N_0, \text{ell}}$ be the following particular solution to

$$\frac{\partial^2}{\partial r^2} u + \frac{2}{r} \frac{\partial}{\partial r} u - Vu = F(t, r) := (e_{u_{N_0}} + e_{v_4}) \chi_{\leq \frac{4r}{t}}$$

$$u_{N_0, \text{ell}}(t, r) = \phi_0 \left( \frac{r}{\lambda(t)} \right) \int_0^{\frac{\lambda r}{t}} s^2 \lambda(t)^2 F(t, s \lambda(t)) e_2(s) ds - e_2 \left( \frac{r}{\lambda(t)} \right) \int_0^{\frac{\lambda r}{t}} s^2 \lambda(t)^2 F(t, s \lambda(t)) \phi_0(s) ds$$

Another direct computation using Lemma 4.33 gives

Lemma 4.37. For $j = 0$ and $0 \leq k \leq 2$ or $j = 1, 2$ and $0 \leq k \leq 1$,

$$|\partial^2_t \partial^k_r u_{N_0, \text{ell}}(t, r)| \leq \frac{C}{t^{j+k}}$$

We will insert $\psi_1(\frac{r}{t}) u_{N_0, \text{ell}}(t, r)$ into our ansatz, and define its error term as

$$e_{u_{N_0, \text{ell}}}(t, r) := - \left( -\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} - V(t, r) \right) \left( \psi_1(\frac{r}{t}) u_{N_0, \text{ell}}(t, r) - F(t, r) \right)$$

By the support properties of $\chi_{\leq 1}$, $F(t, r) = 0$ for $r \geq \frac{t}{2}$, which means that $\psi_1(\frac{r}{t}) F(t, r) = F(t, r)$.

Using this observation, we get

Lemma 4.38.

$$||e_{u_{N_0, \text{ell}}}(t, \cdot)||_{H^1(\mathbb{R}^3)} \leq \frac{C \left( \sup_{x \in [T_\lambda, t]} \sqrt{\lambda(x)} \right)^5 \log^{10}(t)}{t^{9/2}} \left( 1 + \frac{1}{\lambda(t)} \right)$$

4.11. Nonlinear error terms, part 2. The new function added to our ansatz last section is

$$u_n(t, r) := u_{N_0}(t, r) + v_4(t, r) + u_4(t, r) + u_{N_0, \text{ell}}(t, r) \psi_1(\frac{r}{t}) := u_{\text{new}}(t, r) + v_4(t, r) (1 - \psi_1(\frac{r}{t}))$$

The corresponding new nonlinear interactions which need to be studied are contained in $N_2$, defined by

$$N_2(t, r) := - \left( 5 \left( (Q_{\lambda} + u_{a})^4 - Q_{\lambda}^4 \right) u_n + 10 (Q_{\lambda} + u_{a})^3 u_n^2 + 10 (Q_{\lambda} + u_{a})^2 u_n^3 + 5 (Q_{\lambda} + u_{a}) u_n^4 + u_n^5 \right)$$

This is split into a perturbative piece, $N_{2,1}$, and the remainder $N_{2,0}$.

$$N_{2,0} = N_2 \bigg|_{u_n \to \psi_1(1 - \psi_1(\frac{r}{t}))}, \quad N_{2,1} = N_2 - N_{2,0}$$

For ease of notation, we let $\tilde{v}_4(t, r) := v_4(t, r) (1 - \psi_1(\frac{r}{t}))$. Then, we have
Lemma 4.39.

\[ ||N_{2,1}(t, \cdot)||_{H^1(\mathbb{R}^3)} \leq C \frac{\log^{21}(t)}{t^{4+\delta_2}}, \quad \delta_2 = \frac{1}{2} \min\{\frac{1}{2} - \frac{7}{2}C_u - C_l, \frac{3}{2} - \frac{29}{2}C_u - C_l\} > 0 \]

Proof. We note that

\[ u_{new}(t,r) = u_{N_0}(t,r) + \psi_1(\frac{r}{t})u_4(t,r) + u_{N_0,ell}(t,r) \psi_1(\frac{r}{t}) \]

and recall that \( \psi_1(\frac{r}{t}) = 1 \) if \( \frac{r}{t} \leq \frac{1}{2} \). Then, using Lemmas 4.43, 4.35, and 4.37 we get

\[ |u_{new}(t,r)| \leq C \left\{ \begin{array}{ll}
\min\{1,r\} \left( \sup_{x \in [T_{\lambda},t]} \sqrt{\lambda(x)} \right)^{25} \log^{10}(t) + r \left( \sup_{x \in [T_{\lambda},t]} \sqrt{\lambda(x)} \right)^{5} \log^{10}(t) + \log^{10}(r) \\
+ r^2 \left( \sup_{x \in [T_{\lambda},t]} \sqrt{\lambda(x)} \right)^{4} \log(t) + \left( \sup_{x \in [T_{\lambda},t]} \sqrt{\lambda(x)} \right)^{5} \log^{11}(t), & r \leq \frac{t}{2} \\
+ \left( \left( \sup_{x \in [T_{\lambda},t]} \sqrt{\lambda(x)} \right)^{25} \log^{10}(t) + \left( \sup_{x \in [T_{\lambda},t]} \sqrt{\lambda(x)} \right)^{5} \log^{11}(t), & r \geq \frac{t}{2}
\end{array} \right. \]

Using the above estimate on \( u_{new} \) and Lemma 4.35 along with

\[ |N_{2,1}| \leq C |u_{new}| (|Q_\lambda|^3 |u_a| + |u_{new}| + |\tilde{v}_4|) + u_{new}^4 + \tilde{v}_4^4 + u_a^4 \]

a straightforward, but slightly long calculation gives

\[ ||N_{2,1}||_{L^2(t^2 dr)} \leq C \frac{\log^{21}(t)}{t^{4+\epsilon_2}}, \quad \epsilon_2 = \frac{1}{2} \min\{\frac{1}{2} - \frac{7}{2}C_u, \frac{3}{2} - \frac{29}{2}C_u\} > 0 \]

\( \partial_r N_{2,1} \) is estimated with the same procedure (which is a direct, but slightly long computation).

Next, we need to eliminate \( N_{2,0} \). We let \( u_{N_2} \) be the solution to the following equation, with 0 Cauchy data at infinity.

\[ (4.88) \quad -\partial_t^2 u + \frac{2}{r} \partial_r u = N_{2,0} \]

Then, we have

Lemma 4.40.

\[ |u_{N_2}(t,r)| \leq \left\{ \begin{array}{ll}
C \left( \sup_{x \in [T_{\lambda},t]} \sqrt{\lambda(x)} \right)^{9} \log^{18}(t), & r \leq \frac{t}{2} \\
C \left( \sup_{x \in [T_{\lambda},t]} \sqrt{\lambda(x)} \right)^{25} \log^{50}(t), & r \geq \frac{t}{2}
\end{array} \right. \]

\[ |\partial_r u_{N_2}(t,r)| \leq C \left\{ \begin{array}{ll}
\left( \sup_{x \in [T_{\lambda},t]} \sqrt{\lambda(x)} \right)^{9} \log^{50}(t), & r \leq \frac{t}{2} \\
\left( \sup_{x \in [T_{\lambda},t]} \sqrt{\lambda(x)} \right)^{9} \log^{50}(r) + \left( \sup_{x \in [T_{\lambda},r]} \sqrt{\lambda(x)} \right)^{16} \log^{50}(r), & r \geq \frac{t}{2}
\end{array} \right. \]

\[ ||\partial_t u_{N_2}||_{L^2(t^2 dr)} + ||\partial_r u_{N_2}||_{L^2(t^2 dr)} \leq C \left( \sup_{x \in [T_{\lambda},t]} \sqrt{\lambda(x)} \right)^{25} \log^{50}(t) \]
Proof. We recall (4.86) and the fact that \( \psi_1(t) = 1 \) if \( \frac{r}{t} \leq \frac{1}{2} \), and use Lemma 4.33 to get

\[
|N_{2,0}(t, r)| \leq CN_{2,0,est}(t, r)
\]

for

\[
N_{2,0,est}(t, r)
= \frac{1}{r(\frac{r}{t} - 1)} \log^10(r) \left( \sup_{x \in [T, r]} \sqrt{\lambda(x)} \right)^5 \cdot \left( \frac{\sup_{x \in [T, r]} \sqrt{\lambda(x)} 8 \log^2(r)}{r^{4t^2}} + \frac{\sup_{x \in [T, r]} \sqrt{\lambda(x)}^{8} \log^{10}(r)}{r^{4}(\frac{t}{r} - 1)^2} + \frac{\sup_{x \in [T, r]} \sqrt{\lambda(x)}^{4} \log^{8}(r)}{r^{4}} \right) + \left( \frac{\sup_{x \in [T, r]} \sqrt{\lambda(x)}^{20} \log^{8}(r)}{r^{4}t^8} + \frac{\sup_{x \in [T, r]} \sqrt{\lambda(x)}^{20} \log^{40}(r)}{r^{4}(\frac{t}{r} - 1)^8} \right)
\]

We also get

\[
|\partial_r N_{2,0}(t, r)| \leq CN_{2,0,est}(t, r) \left( \frac{1}{r} + \frac{1}{\frac{t}{r} - 1} + \frac{1}{t} \right) + C \sup_{x \in [T, r]} \sqrt{\lambda(x)}^{23} \log^{42}(r) \frac{\sup_{x \in [T, r]} \sqrt{\lambda(x)}^{26} \log^{40}(t)}{t^4} |\partial_r u_3(t, r)|
\]

We note that

\[
(4.89) \quad u_{N_2}(t, r) = \int_{t}^{\infty} ds \left( \frac{1}{2r} \int_{|r-(s-t)|}^{r+s-t} y N_{2,0}(s, y) dy \right)
\]

We then directly estimate (4.89) for all \( r > 0 \), and the \( r \)-derivative of the \( u_{N_2} \)-analog of (4.80) for \( r \leq \frac{t}{2} \). When \( r \geq \frac{t}{2} \), we directly estimate the \( r \)-derivative of (4.89). The energy estimate is obtained with the same procedure as in Lemma 4.25.

Next, we estimate \( e_{u_{N_2}} \), the linear error term associated to \( u_{N_2} \), defined by

\[
e_{u_{N_2}}(t, r) := \frac{-45\lambda(t)^2 u_{N_2}(t, r)}{3\lambda(t)^2 + r^2)
\]

A direct application of Lemma 4.40 gives

**Lemma 4.41.**

\[
\|e_{u_{N_2}}(t, \cdot)\|_{H^1(\mathbb{R}^3)} \leq C \left( \sup_{x \in [T, r]} \sqrt{\lambda(x)} \right)^9 \log^{50}(t) \frac{\sqrt{\lambda(t)^5}}{1 + \frac{1}{\lambda(t)}}
\]
At this stage, the only error terms that remain are the nonlinear interactions between \( u_{N_2} \) and the previous corrections, which we collect together in \( N_3 \):

\[
-N_3 = 5 \left( (Q_\lambda + u_a + u_n)^4 - Q_\lambda^4 \right) u_{N_2} + 10 (Q_\lambda + u_a + u_n)^3 u_{N_2}^2 + 10 (Q_\lambda + u_a + u_n)^2 u_{N_2}^3 + 5 (Q_\lambda + u_a + u_n) u_{N_2}^4 + u_{N_2}^5
\]

Then, we have

Lemma 4.42.

\[
||N_3(t, \cdot)||_{H^1(\mathbb{R}^3)} \leq C \log^{90}(t) \frac{1}{t^{4 + \frac{1}{2} - 6C_u}}
\]

Proof. We start with

\[
|N_3(t, r)| \leq C |u_{N_2}|(|Q_\lambda|^3(|u_a| + |u_n| + |u_{N_2}|) + u_a^4 + u_n^4 + u_{N_2}^4)
\]

Using (4.68), (4.69), (4.87) and Lemmas 4.33, 4.40, we get

(4.90)

\[
|u_a| + |u_n| + |u_{N_2}| \leq C \begin{cases}
\log(t) \left( \sup_{\mathbb{R}^3} |\lambda(x)|^4 \right)^{-\frac{1}{2}} + \min[1, r] \left( \sup_{\mathbb{R}^3} |\lambda(x)|^{25} \log^{10}(t) \right)^{-\frac{1}{2}}, & r \leq \lambda(t) \\
\left( \sup_{\mathbb{R}^3} |\lambda(x)| \right)^{\frac{1}{2}}, & \lambda(t) \leq r \leq \frac{t}{2} \\
\left( \sup_{\mathbb{R}^3} |\lambda(x)| \right)^{\frac{1}{2}}, & r \geq \frac{t}{2}
\end{cases}
\]

Then, a direct computation gives

\[
||N_3(t, r)||_{L^2(\mathbb{R}^3)} \leq C \log^{90}(t) \frac{1}{t^{4 + \delta_{3,0}}}, \quad \delta_{3,0} = \frac{1}{2} \min \frac{5}{2} - \frac{29}{2} C_u, 3 - \frac{45}{2} C_u > 0
\]

\( \partial_r N_3 \) is estimated similarly. The only detail worth noting is that we use the energy estimates from Lemmas 4.40 and 4.25 to estimate \( ||\partial_r u_{N_2}||_{L^2(\mathbb{R}^3)} \) and \( ||\partial_r u_3||_{L^2(\mathbb{R}^3)} \), and pointwise estimates on derivatives otherwise.

The estimates from Lemma 4.42 show that \( N_3 \) can be treated perturbatively in the next section of the argument. Therefore, our final ansatz, which we denote by \( u_{\text{ansatz}} \) is the sum of all terms added thus far:

\[
u_{\text{ansatz}}(t, r) = u_a(t, r) + u_n(t, r) + u_{N_2}(t, r)
\]

Now, we estimate the error term (which we denote by \( F_5 \)) of \( u_{\text{ansatz}} \):

\[
F_5(t, r) = e_{\text{match}} + e_{\text{ell},2} + e_{\text{ex}} + e_{w,2} + e_{u_3} + (e_{u_{N_0}} + e_{u_{N_1}} + (e_{u_{N_2}} + e_{u_{N_3}} + e_{u_{N_4}} + e_{u_{N_5}} + N_1 + N_2 + N_3
\]

Lemma 4.43.

\[
||F_5(t, \cdot)||_{H^1(\mathbb{R}^3)} \leq C \log^{90}(t) \frac{1}{t^{4 + 2\epsilon}}
\]

where \( \epsilon > 3C_l \) and is given by

\[
\epsilon = \frac{1}{2} \min \{ \delta, 4 - \frac{13}{2} a(1 - C_u) - 7 C_u, -2 + \frac{7}{2} a(1 - C_u) - \frac{5}{2} C_u - \frac{3}{2} C_l, 1 - 4 C_u, 1 - 25 \frac{2}{2} C_u, \delta_1, \delta_2, \frac{1}{2} (1 - 6 C_u) \}.
\]
For later use, we decompose $\partial_r u_{\text{ansatz}}$ as follows.

$$\partial_r u_{\text{ansatz}}(t, r) = u_{an,r,0}(t, r) + u_{an,r,1}(t, r)$$

where

$$u_{an,r,1}(t, r) = \partial_r u_{N_0}(t, r) + \partial_r u_{N_2}(t, r)$$

**Lemma 4.44.**

(4.91)  
$$\|u_{\text{ansatz}}(t, r)\|_{L^{p,r}} \leq C \left( \sup_{x \in [T, t]} \sqrt{\lambda(x)} \right)^{5} \log^{10}(t)$$

(4.92)  
$$\|\partial_r (Q_{\lambda}(r)) Q_{\lambda}(r)^2 u_{\text{ansatz}}\|_{L^{p,r}} + \|Q_{\lambda}(r)^3 u_{an,r,0}(t, r)\|_{L^{p,r}} \leq \frac{C}{t^2 \lambda(t)}$$

(4.93)  
$$\|\frac{1}{\lambda(t)} u_{an,r,0}\|_{L^{p,r}} \leq \frac{C \left( \sup_{x \in [T, t]} \sqrt{\lambda(x)} \right)}{t^2}$$

(4.94)  
$$\|Q_{\lambda}(r)^2 u_{\text{ansatz}}(t, r)^2\|_{L^{p,r}} \leq C \left( \log^2(t) \left( \sup_{x \in [T, t]} \sqrt{\lambda(x)} \right)^6 + \lambda(t) \left( \sup_{x \in [T, t]} \sqrt{\lambda(x)} \right)^{10} \log^{22}(t) \right)$$

(4.95)  
$$\|u_{an,r,1}(t, r)\|_{L^{2, r^2 dr}} \leq \frac{C \left( \sup_{x \in [T, t]} \sqrt{\lambda(x)} \right)^5 \log^{10}(t)}{t^{5/2}}$$

(4.96)  
$$\|Q_{\lambda}(r)^3 u_{\text{ansatz}}(t, r)\|_{L^{p,r}} \leq \frac{(\lambda(t))^2 3(111 - 45 \log(4))}{16} + \frac{\sqrt{\lambda}(t) \left( \sup_{x \in [T, t]} \sqrt{\lambda(x)} \right)^4}{t^3 \lambda(t)^{3/2}}$$

**Proof.** The proof is a direct calculation. In particular, we use (4.90) for (4.91) and (4.94). We use (4.3) (which gives the precise constants in (4.96)) along with Lemmas 4.18, 4.23, 4.24, 4.3, 4.8, 4.9, 4.10, 4.21, 4.33, 4.37, 4.40 for (4.96), (4.92), (4.93). Lemmas 4.33 and 4.40 give (4.95). □
5. Constructing the Exact Solution

If we substitute $u(t, r) = Q_\lambda(t)(r) + u_{\text{ansatz}}(t, r) + v(t, r)$ into (1.1), we get

\begin{equation}
-c_t^2 v + c_r^2 v + \frac{2}{r} c_r v + \frac{45\lambda(t)^2}{(3\lambda(t)^2 + r^2)^2} v = F_5(t, r) + F_3(t, r)
\end{equation}

where

\begin{equation}
F_3 = L_1(v) + N(v), \quad L_1(v) = (-5(Q_\lambda + u_{\text{ansatz}})^4 + 5Q_\lambda^4)v
\end{equation}

\begin{equation}
N(v) = -10(Q_\lambda + u_{\text{ansatz}})^3 v^2 - 10(Q_\lambda + u_{\text{ansatz}})^2 v^3 - 5(Q_\lambda + u_{\text{ansatz}}) v^4 - v^5
\end{equation}

We proceed as follows. First, we formally derive the equation solved by $y$ defined by

\[ y(t, \xi) = (y_0(t), y_1(t, \xi(\lambda(t))^{-2})) = F (\cdot) (v(t, \lambda(t))) (\xi) \]

where $F$ is the distorted Fourier transform from [18], which we regard as a two-component vector, as in Proposition 4.3 of [18]. Then, we prove that this formally derived equation has a solution, say $(y_{0,0}, y_{0,1})$, which is regular enough to allow us to justify the statement that the function $v$ defined by the following expression, with $y_0 = y_{0,0}, y_1 = y_{0,1}$

\begin{equation}
v(t, r) = \frac{\lambda(t)}{r} \left( y_0(t) \phi_d\left( \frac{r}{\lambda(t)} \right) + \int_0^\infty \phi\left( \frac{r}{\lambda(t)}, \xi \right) y_1(t, \frac{\xi}{\lambda(t)^2}) \rho(\xi) d\xi \right)
\end{equation}

is a solution to (5.1). Letting $v$ be as in (5.3), we formally get that (5.1) is equivalent to the system consisting of the following two equations. We remind the reader that the first component of the distorted Fourier transform of a function is a real number, while the second component is a function of frequency.

\begin{equation}
- y''_0(t) - \frac{\lambda'(t)}{\lambda(t)} y'_0(t) - \frac{\xi_d}{\lambda(t)^2} y_0 = F_{2,0}(t) - \frac{\lambda'(t)}{\lambda(t)} y'_0(t) + F (\cdot) (F_5 + F_3)(t, \lambda(t)) \bigg|_0
- c_t^2 y_1 - \omega y_1 = F_{2,1}(t, \omega \lambda(t)^2) + F (\cdot) (F_5 + F_3)(t, \lambda(t)) \bigg|_1 (\omega \lambda(t)^2)
\end{equation}

where

\begin{equation}
F_2(t, \eta) = \begin{bmatrix}
\frac{y_0(t)}{y_1(t, \frac{\eta}{\lambda(t)^2})} \\
\frac{y'_0(t)}{y_1(t, \frac{\eta}{\lambda(t)^2})}
\end{bmatrix} \frac{\lambda''(t)}{\lambda(t)} + 2 \frac{\lambda'(t)}{\lambda(t)} \begin{bmatrix}
\frac{y'_0(t)}{y_1(t, \frac{\eta}{\lambda(t)^2})} \\
\frac{\eta_d}{\lambda(t)^2}
\end{bmatrix} - 2 \left( \frac{\lambda'(t)}{\lambda(t)} \right) \mathcal{K} \left( \begin{bmatrix}
\frac{y'_0(t)}{y_1(t, \frac{\eta}{\lambda(t)^2})} \\
\frac{\eta_d}{\lambda(t)^2}
\end{bmatrix} \right)
+ \left( \frac{\lambda'(t)}{\lambda(t)} \right)^2 \mathcal{K} \left( \begin{bmatrix}
\frac{y_0(t)}{y_1(t, \frac{\eta}{\lambda(t)^2})} \\
\frac{\eta_d}{\lambda(t)^2}
\end{bmatrix} \right)
\end{equation}

and we use the notation $x_i$ to denote the $i + 1$ entry of the vector $x = \begin{bmatrix} x_0 \\ x_1 \end{bmatrix}$. 
5.1. **The iteration space.** The space in which we will solve (5.4) is defined as follows. We define $Z$ to be the set of elements $\begin{bmatrix} y_0(t) \\ y_1(t, \omega) \end{bmatrix}$ such that $y_0 : [T_0, \infty) \to \mathbb{R}$, and $y_1$ is an (equivalence class) of measurable functions, $y_1 : [T_0, \infty) \times (0, \infty) \to \mathbb{R}$ satisfying

$$y_0(t) \in C^1_t([T_0, \infty))$$

$$y_1(t, \omega) \frac{t^{2+\epsilon - \frac{2}{3}C_1}}{ \log ^{90}(t)} \lambda(t) (\omega \lambda(t)^2) \sqrt{\rho(\omega \lambda(t)^2)} \in C^0_t([T_0, \infty), L^2(\omega \lambda(t)^2))$$

$$\partial_t y_1(t, \omega) \frac{t^{3+\epsilon - \frac{4}{3}C_1}}{ \log ^{90}(t)} \lambda(t) (\sqrt{\omega \lambda(t)^2}) \sqrt{\rho(\omega \lambda(t)^2)} \in C^0_t([T_0, \infty), L^2(\omega \lambda(t)^2))$$

and $\left\| \begin{bmatrix} y_0 \\ y_1 \end{bmatrix} \right\|_Z < \infty$ where

(5.6)

$$\left\| \begin{bmatrix} y_0 \\ y_1 \end{bmatrix} \right\|_Z = \sup_{t \geq T_0} \left( \frac{t^{2+\epsilon - \frac{2}{3}C_1}}{ \log ^{90}(t)} \left( |y_0(t)| + \lambda(t) |\omega \lambda(t)^2| y_1(t, \omega) \right) \left\| L^2(\rho(\omega \lambda(t)^2)) \right\| \right)$$

$$+ \frac{t^{3+\epsilon - \frac{4}{3}C_1}}{ \log ^{90}(t)} \left( |y_0'(t)| + \lambda(t) |\partial_t y_1(t, \omega)| \left\| L^2(\rho(\omega \lambda(t)^2)) \right\| + C_Z^{-1} \lambda(t) \left\| \sqrt{\omega \lambda(t)^2} \partial_t y_1(t, \omega) \right\| \left\| L^2(\rho(\omega \lambda(t)^2)) \right\| \right)$$

and $C_Z > 0$ is otherwise arbitrary, and will be further constrained later on. Then, $(Z, \left\| \cdot \right\|_Z)$ is a normed vector space. We remark that this space is similar to the iteration space used in a Yang-Mills paper of the author [21]. By directly estimating (5.5), and using the boundedness of $\mathcal{K}$ and $[\mathcal{K}, \xi \partial_t]$ from Proposition 5.2 of [18], we obtain the following lemma.

**Lemma 5.1.** There exists $C > 0$, independent of $T_0$ and $C_Z$, such that, for all $y \in Z$,

$$|F_{2,0}(t)| + \lambda(t) |F_{2,1}(t, \omega \lambda(t)^2)| \left\| L^2(\rho(\omega \lambda(t)^2)) \right\|$$

$$\leq 2 \cdot \frac{\log ^{90}(t)}{t^{1+\epsilon - \frac{2}{3}C_1}} \left( C_2(1 + ||\mathcal{K}||_{L^2(\lambda^2 \rho \xi^2)}) + M^2 \left( ||\mathcal{K}||_{L^2(\lambda^2 \rho \xi^2)} + 2 ||[\mathcal{K}, \xi \partial_t]\right) \right)$$

$$+ 2M \left( 1 + ||\mathcal{K}||_{L^2(\lambda^2 \rho \xi^2)} \right)$$

$$\lambda(t) |\omega \lambda(t)^2| F_{2,1}(t, \omega \lambda(t)^2) \left\| L^2(\rho(\omega \lambda(t)^2)) \right\| \leq \frac{C \log ^{90}(t)}{t^{1+\epsilon - \frac{2}{3}C_1}}$$

Next, we understand the relation between $v$ and $y$ in (5.3) by proving the following.

**Lemma 5.2.** For all $y_d \in \mathbb{R}$, $y : (0, \infty) \to \mathbb{R}$ satisfying $y(\xi) \xi \in L^2(\rho(\xi) d\xi)$, if

(5.7)

$$w(R) = \frac{1}{R} \left( y_d \phi_d(R) + \int_0^\infty \phi(R, \xi) y(\xi) \rho(\xi) d\xi \right), \quad R > 0$$

then, for $R > 0$,

(5.8)

$$|w(R)| \leq \frac{C}{R} \left( |y_d| + ||\xi y(\xi)|| \left\| L^2(\rho(\xi) d\xi) \right\| \right)$$
Lemma 5.3. There exists $C > 0$ so that for all $y_1, y_2 \in B_1(0) \subset Z$ of the form

$$y_i := \begin{bmatrix} y_{0,i}(t) \\ y_{1,i}(t, \omega) \end{bmatrix}, \quad i = 1, 2$$

if

$$v_i(t, r) = \frac{\lambda(t)}{r} \left( y_{0,i}(t) \phi_d \left( \frac{r}{\lambda(t)} \right) + \int_0^{\infty} \phi \left( \frac{r}{\lambda(t)} \right) y_{1,i}(t, \frac{\xi}{\lambda(t)^2}) \rho(\xi) d\xi \right), \quad r > 0$$

then,

$$\| F_3(v_2) - F_3(v_1) \|_{L^2(\mathbb{R}^2 \times \mathbb{R})} \leq \frac{2 \cdot \log^{60}(t)}{t^{2+\epsilon}} \frac{\| y_1 - y_2 \|_Z}{2C_1} \cdot \left( 20 \| Q \rho \|_{L^\infty} \right)$$

$$+ C \left( \| Q \rho \|_{L^\infty}^2 + \| u_{\text{ansatz}} \|_{L^\infty}^4 + \frac{\log^{60}(t)}{t^{8+\epsilon - \frac{3}{2}C_1}} + \frac{\log^{60}(t)}{\lambda(t)^{3/2}t^{2+\epsilon - \frac{3}{2}C_1}} \right)$$

where

$$\phi(R, \xi) = 2 \text{Re} (a(\xi)f_+(R, \xi))$$

Using Proposition 4.5 of [18], we have

$$|\phi(R, \xi)| \leq C|a(\xi)|, \quad R^2 \xi \geq C_1$$

When $R^2 \xi \leq C_1$, we use Proposition 4.4 of [18] to write

$$\phi(R, \xi) = \tilde{\phi}_0(R) + \frac{1}{R} \sum_{j=1}^{\infty} (R^2 \xi)^j \phi_j(R^2)$$

where

$$|\phi_j(R)| \leq \frac{C^j R}{(j-1)! \langle R \rangle^{1/2}}$$

This gives, for $R^2 \xi \leq C_1$,

$$|\phi(R, \xi)| \leq \frac{CR}{\langle R \rangle}$$

Next, we use the discussion preceding Lemma 4.2 of [18], which notes that $\phi_d \in C^\infty([0, \infty))$, $\phi_d(0) = 0$, and $\phi_d$ is exponentially decaying. Directly inserting these estimates into (5.7), gives (5.8). Finally, (5.9) is the $L^2$ isometry property of $F$, and (5.10) follows from Lemma 2.7 of [18].

Now, we can estimate $F_3$ (defined in (5.2)).

Lemma 5.3. There exists $C > 0$ so that for all $y_1, y_2 \in B_1(0) \subset Z$ of the form

$$\| w(R) \|_{L^2(\mathbb{R}^2 \times dR)} = \left( |y_d|^2 + |y(\xi)|^2 \right)^{1/2}$$

$$C^{-1} \| w(| \cdot |) \|_{H^1(\mathbb{R}^3)} \leq \left( |y_d|^2 + \| y(\xi) \sqrt{\langle \xi \rangle} \|_{L^2(\rho(\xi) d\xi)}^2 \right)^{1/2} \leq C \| w(| \cdot |) \|_{H^1(\mathbb{R}^3)}$$
Lemma 5.4. There exists \( C_\rho > 0 \) (independent of \( T_0 \)) so that

\[
\frac{\rho(\omega\lambda(t)x^2)}{\rho(\omega\lambda(t))^2} \leq C_\rho \left( \frac{\lambda(x)}{\lambda(t)} + \frac{\lambda(t)}{\lambda(x)} \right), \quad x, t \geq T_0
\]

Define (5.11)

\[
T \left( \begin{array}{c} y_0 \\ y_1 \end{array} \right) (t, \omega) = \left[ e_+(t) \int_t^\infty \frac{F_0(\omega)\lambda(\omega)}{2\sqrt{-\xi_d}} e_-(w) dw + e_-(t) \int_t^\infty \frac{F_0(\omega)\lambda(\omega)}{2\sqrt{-\xi_d}} e_+(w) dw \right], \quad \left[ y_0 \ y_1 \right] \in \overline{B}_1(0) \subset \mathbb{Z}
\]

where

\[
F_i(t, \omega) = \left\{ \begin{array}{ll}
\mathcal{F} \left( \cdot \right) (F_5 + F_3)(t, \cdot \lambda(t))_0 + F_{2,0}(t) - \frac{\lambda(t)}{\lambda(0)} y'_0(t), & i = 0 \\
\mathcal{F} \left( \cdot \right) (F_5 + F_3)(t, \cdot \lambda(t))_1 (\omega\lambda(t)^2) + F_{2,1}(t, \omega\lambda(t)^2), & i = 1
\end{array} \right.
\]

and \( F_3 \) is given by (5.2), with \( v \) given by (5.3). Lemmas 5.1, 5.3, 4.43 and (1.8) directly give

Lemma 5.5. There exists \( C > 0 \) (independent of \( T_0 \)) so that, for all \( y \in \overline{B}_1(0) \), and \( s, t \geq T_0 \),

\[
|F_0(t)| \leq C \frac{\log^{.0}(t)}{t^{4 + \epsilon - \frac{3}{2}C_1}}
\]

\[
\lambda(s)||F_1(s, \omega)||_{L^2(\rho(\omega\lambda(t)s^2)d\omega)} \leq \frac{\log^{.0}(s)}{s^{4 + \epsilon - \frac{3}{2}C_1}} \left( \frac{C \log^{.0}(s)}{s^{1 - \frac{3}{2}C_1}} + \frac{C \log^{.0}(s)}{s^{1 - 3C_1}} + \frac{1}{12 \sqrt{C_\rho}} \right)
\]

There exists \( C_1 > 0 \) (independent of \( T_0 \) and \( C_Z \)) so that, for all \( y \in \overline{B}_1(0) \), and \( s \geq T_0 \),

\[
\lambda(s)||\sqrt{\omega\lambda(t)s^2} F_1(s, \omega)||_{L^2(\rho(\omega\lambda(t)s^2)d\omega)} \leq \frac{C_1 \log^{.0}(s)}{s^{4 + \epsilon - \frac{3}{2}C_1}} + \frac{C_Z \log^{.0}(s)}{24 \sqrt{C_\rho s^{4 + \epsilon - \frac{3}{2}C_1}}}
\]
We proceed to show that $T$ maps $\overline{B_1(0)} \subset Z$ into itself. If $\left[ \begin{smallmatrix} y_0 \\ y_1 \end{smallmatrix} \right] \in \overline{B_1(0)} \subset Z$, we have

$$|T(\left[ \begin{smallmatrix} y_0 \\ y_1 \end{smallmatrix} \right])_0| \leq e_+(t) \int_t^\infty \frac{|F_0(w)|\lambda(w)}{2\sqrt{-\xi_d}} e_-(w)dw + e_-(t) \int_{T_0}^t \frac{|F_0(w)|\lambda(w)}{2\sqrt{-\xi_d}} e_+(w)dw$$

From Lemma 5.5 we thus get

$$|T(\left[ \begin{smallmatrix} y_0 \\ y_1 \end{smallmatrix} \right])_0| \leq C \left( e_+(t) \int_t^\infty \frac{\log_9(w)\lambda(w)}{w^{4+\frac{\epsilon}{2}C_1}} e_-(w)dw + e_-(t) \int_{T_0}^t \frac{\log_9(w)\lambda(w)}{w^{4+\frac{\epsilon}{2}C_1}} e_+(w)dw \right) := CI(t)$$

Integrating by parts (where we integrate $\frac{\sqrt{-\xi_d}}{\lambda(w)} e_+(w)$ and differentiate the rest of the integrand), we get

$$I(t) \leq e_+(t) \left( \frac{\log_9(t)\lambda(t)^2}{\sqrt{-\xi_d}} e_-(w) \int_t^\infty \frac{C}{w^{1-C_u}} \frac{\log_9(w)\lambda(w)}{w^{4+\frac{\epsilon}{2}C_1}} dw \right) + e_-(t) \left( \frac{\log_9(t)\lambda(t)^2}{\sqrt{-\xi_d}} e_+(w) \int_{T_0}^t \frac{C}{w^{1-C_u}} \frac{\log_9(w)\lambda(w)}{w^{4+\frac{\epsilon}{2}C_1}} dw \right)$$

where $C$ is independent of $T_0$. To get this, we also used

$$\frac{\log_9(w)\lambda(w)^2}{w^{4+\frac{\epsilon}{2}C_1}} e_+(w) \bigg|_{w=T_0} \leq \frac{\log_9(t)\lambda(t)^2}{\sqrt{-\xi_d}} e_+(t)$$

In other words, the boundary term at $T_0$ is negative. We therefore get, for $C$ independent of $T_0$,

$$I(t) \leq \frac{C \log_9(t)\lambda(t)^2}{t^{4+\frac{\epsilon}{2}C_1}} + \frac{CI(t)}{T_0^{1-C_u}}, \quad t \geq T_0$$

Since $1 - C_u > 0$, there exists $T_3 > 0$ so that, for all $T_0 \geq T_3$,

$$I(t) \leq \frac{C_2 \log_9(t)\lambda(t)^2}{t^{4+\frac{\epsilon}{2}C_1}}, \quad t \geq T_0$$

(where $C_2$ is independent of $T_0$). From here on, we further restrict $T_0$ to satisfy $T_0 \geq T_3$. Since

$$T(\left[ \begin{smallmatrix} y_0 \\ y_1 \end{smallmatrix} \right])_0(t) = \frac{\sqrt{-\xi_d}}{\lambda(t)} e_+(t) \int_t^\infty \frac{F_0(w)\lambda(w)e_-(w)dw}{2\sqrt{-\xi_d}} - e_-(t) \int_{T_0}^t \frac{F_0(w)\lambda(w)e_+(w)dw}{2\sqrt{-\xi_d}}$$

we get

$$|T(\left[ \begin{smallmatrix} y_0 \\ y_1 \end{smallmatrix} \right])_0 + \lambda(t)||T(\left[ \begin{smallmatrix} y_0 \\ y_1 \end{smallmatrix} \right])_0(t)| \leq \frac{C \log_9(t)\lambda(t)^2}{t^{4+\frac{\epsilon}{2}C_1}}$$

Using Lemma 5.4 and (4.1.), we get

$$\lambda(t)||T(\left[ \begin{smallmatrix} y_0 \\ y_1 \end{smallmatrix} \right])_1(t,\omega)||_{L^2(\rho(\omega\lambda(t)^2)\omega d\omega)} \leq \sqrt{C_\rho} \int_t^\infty d(s-t)||\lambda(s)F_1(s,\omega)||_{L^2(\rho(\omega\lambda(s)^2)\omega d\omega)} \left( \frac{\lambda(t)}{\lambda(s)} + (\frac{\lambda(t)^{3/2}}{\lambda(s)^{3/2}} \right)$$

$$\leq 2\sqrt{C_\rho} \int_t^\infty d(s-t)||\lambda(s)F_1(s,\omega)||_{L^2(\rho(\omega\lambda(s)^2)\omega d\omega)} \left( \frac{\lambda(t)}{\lambda(s)} \right)^{3/2}$$
Using Lemma 5.5, we get
\[
\lambda(t)||T(\begin{bmatrix} y_0 \\ y_1 \end{bmatrix})||_{1(t,\omega)}||_{L^2(\rho(\omega\lambda(t)^2)\,d\omega)} \leq \frac{2\sqrt{C_p}}{t^{3+\frac{3}{2}C_1}} \left( \frac{C \log^{100}(t)}{t^{2+\epsilon}} + \frac{1}{12t^2} + \frac{\log^{90}(t)}{12t^2} \right)
\]

We remark that the constant 2 multiplying \(\sqrt{C_p}\) is not sharp. For instance, the integral was, in part, estimated by using the fact that \(s \mapsto \frac{\log^{100}(s)}{s}\) is decreasing on \([T_0, \infty)\). We get
\[
(5.12) \quad \frac{t^{2+\epsilon-\frac{3}{2}C_1} \lambda(t)}{\log^{90}(t)} ||T(\begin{bmatrix} y_0 \\ y_1 \end{bmatrix})||_{1(t,\omega)}||_{L^2(\rho(\omega\lambda(t)^2)\,d\omega)} \leq \frac{1}{6} + C \left( \frac{\log^{100}(t)}{t^{1-\frac{3}{2}C_1}} + \frac{\log^{90}(t)}{t^{2-3C_1}} \right)
\]

Also,
\[
\lambda(t)||\hat{c}_T(\begin{bmatrix} y_0 \\ y_1 \end{bmatrix})||_{1(t,\omega)}||_{L^2(\rho(\omega\lambda(t)^2)\,d\omega)} \leq 2\sqrt{C_p} \int_{t}^{\infty} \lambda(s)||\hat{c}_T(\begin{bmatrix} y_0 \\ y_1 \end{bmatrix})||_{1(t,\omega)}||_{L^2(\rho(\omega\lambda(s)^2)\,d\omega)} \left( \frac{\lambda(t)}{\lambda(s)} \right) \frac{1}{t} \frac{\log^{90}(t)}{24/C_p} ds
\]

Therefore, the identical argument used to obtain (5.12) gives
\[
\frac{t^{3+\epsilon-\frac{3}{2}C_1} \lambda(t)}{\log^{90}(t)} ||\hat{c}_T(\begin{bmatrix} y_0 \\ y_1 \end{bmatrix})||_{1(t,\omega)}||_{L^2(\rho(\omega\lambda(t)^2)\,d\omega)} \leq \frac{1}{6} + C \left( \frac{\log^{100}(t)}{t^{1-\frac{3}{2}C_1}} + \frac{\log^{90}(t)}{t^{2-3C_1}} \right)
\]

Next, using Lemmas 5.4 and 5.5, we get
\[
C_Z^{-1} \lambda(t)||\sqrt{\omega\lambda(t)^2} \hat{c}_T(\begin{bmatrix} y_0 \\ y_1 \end{bmatrix})||_{1(t,\omega)}||_{L^2(\rho(\omega\lambda(t)^2)\,d\omega)} \leq \sqrt{C_p} C_Z^{-1} \int_{t}^{\infty} \lambda(s)||\sqrt{\omega\lambda(s)^2} F_1(s,\omega)||_{L^2(\rho(\omega\lambda(s)^2)\,d\omega)} \left( \frac{\lambda(t)}{\lambda(s)} \right) \frac{1}{t} \frac{\log^{90}(t)}{24/C_p} ds
\]

where we remind the reader that the constant \(C_1\) from Lemma 5.5 is independent of \(C_Z, T_0\). Finally, with the same procedure,
\[
\lambda(t)||\omega\lambda(t)^2 \hat{T}(\begin{bmatrix} y_0 \\ y_1 \end{bmatrix})||_{1(t,\omega)}||_{L^2(\rho(\omega\lambda(t)^2)\,d\omega)} \leq \frac{2\sqrt{C_p}}{t^{3+\epsilon-\frac{3}{2}C_1}} \left( C_1 + 2MCZ(1 + ||K||_{L^2(Z)}) \right)
\]

We therefore get that there exists \(C > 0, C_1 > 0\), independent of \(C_Z, T_0\), such that, for all \(y \in B_1(0) \subset Z\),
\[
||T(y)|| \leq \frac{5}{12} + \frac{2\sqrt{C_p}C_1}{C_Z} + \sup_{t \geq T_0} \left( C \left( \frac{\log^{100}(t)}{t^{1-\frac{3}{2}C_1}} + \frac{\log^{90}(t)}{t^{2-3C_1}} \right) + \frac{2\sqrt{C_p}}{t} \lambda(t)(C_1 + \frac{C \lambda(t)}{24\sqrt{C_p}}) \right)
\]
Using the fact that $F_2$ is linear in $\left[ \begin{array}{c} y_0 \\ y_1 \end{array} \right]$ and Lemmas 5.1, 5.3, we get that there exists $C > 0, C_3 > 0$, independent of $C_Z, T_0$, such that, for all $y, z \in B_1(0) \subset Z$,

$$\|T(y) - T(z)\|_Z \leq C \|y - z\|_Z \left( \sup_{t \geq T_0} \left( \frac{1}{t^{1-C_u}} \right) + \sup_{t \geq T_0} \left( \frac{\log^9(t)}{t^3} \right) + \sup_{t \geq T_0} \left( \frac{\log^{10}(t)}{t^{1-\frac{3}{2}C_u - \frac{3}{2}C_t}} \right) \right)$$

$$+ ||y - z||_Z \left( \frac{5}{12} + \frac{2\sqrt{C_P C_3}}{C_Z} \right) + \sup_{t \geq T_0} \left( \frac{1}{t^{1-C_u}} \right)$$

Now, we fix $C_Z = 24\sqrt{C_P \max\{C_1, C_3\}}$. Since $\epsilon > 3C_t$, there exists $T_4 > 0$ so that, for all $T_0 \geq T_4$, $T$ is a strict contraction on $B_1(0) \subset Z$. By Banach's fixed point theorem, there thus exists a fixed point of $T$, say $\overline{y} \in B_1(0) \subset Z$. By the definition of $T$, $\overline{y}$ is a solution to (5.4).

5.2. Decomposition of the solution as in Theorem 1.1. We write $\overline{y} = \left[ \begin{array}{c} y_0 \\ y_1 \end{array} \right]$ and define $v_f$ by

$$v_f(t, r) = \frac{\lambda(t)}{r} \left( \overline{y}_0(t) \phi_d \left( \frac{r}{\lambda(t)} \right) + \int_0^\infty \phi \left( \frac{r}{\lambda(t)}, \xi \right) \overline{y}_2(t, \frac{\xi}{\lambda(t)^2}) \rho(\xi) d\xi \right), \quad r > 0$$

By our derivation of (5.1) from (5.1), $v_f$ is a solution to (5.1). Let

$$v_{\text{rad}}(t, r) := v_2(t, r) + v_3(t, r) + v_4(t, r) + \tilde{v}_2(t, r)$$

so that

$$(-\tilde{\alpha}_t^2 + \tilde{\alpha}_r^2 + \frac{2}{r} \tilde{\alpha}_r) v_{\text{rad}}(t, r) = 0$$

and let

$$v_e(t, r) := u_{\text{ansatz}}(t, r) - v_{\text{rad}}(t, r)$$

Defining $u_e(t, r) = v_e(t, r) + v_f(t, r)$, we get that $u$ given by

$$u(t, r) := Q_{\lambda(t)}(r) + v_{\text{rad}}(t, r) + u_e(t, r)$$

is a solution to (1.1). We conclude the proof of Theorem 1.1 by showing

$$E_{\text{SLW}}(u, \partial_t u) < \infty, \quad E(u_e, \partial_t (Q_{\lambda(t)} + u_e)) \leq C \left( \sup_{x \in [T_{\lambda}, t]} \sqrt{\lambda(x)} \right)^2 \left( \frac{\log(t)}{t} \right)$$

We use Lemmas 4.17 (for $u_{\text{ell}}$), 4.18 (for $v_2$), 4.20, 4.21 (for $v_3$), 4.18 (for $u_{\text{ell}2}$), 4.23 (for $u_{\text{ell}3}$), 4.24 (for $v_2, w_1$), 4.25 (for $v_3$), 4.35 (for $v_{w2}$), 4.35 (for $v_3$), 4.37 (for $u_n, v_{n0}$), 4.39 (for $u_m, v_{n2}$), 4.40 (for $u_{n2}$) to get

$$\|\partial_r v_e(t, r)\|_{L^2(r^2 \text{d}r)} + ||v_e(t, r)||_{L^6(r^2 \text{d}r)} \leq C \sup_{x \in [T_{\lambda}, t]} \left( \frac{\sqrt{\lambda(x)}}{\sqrt{t}} \right)$$

We also need to capture an exact cancellation between $\partial_t w_{1,0}(t, r)$ and $\partial_t Q_{\lambda(t)}(r)$ for large $r$, since each of these functions are not separately in $L^2(r^2 \text{d}r)$ (unless $\lambda'(t) = 0$). For this, we recall the definitions (1.2) and (4.3), and note that

$$\partial_t (w_{1,0}(t, r) + Q_{\lambda(t)}(r)) = \frac{3}{2} \left( \frac{\lambda'(t)}{\sqrt{\lambda(t)}} \left( \frac{1}{1 + \frac{3\lambda(t)^2}{r^2}} \right) - 3\lambda(t)^3/2 \lambda'(t) \left( \frac{r^2 + 3\lambda(t)^2}{r^2 + 3\lambda(t)^2} \right) \right)$$

$$\left( \frac{\lambda'(t)}{r \sqrt{\lambda(t)}} \right)$$
Then, by the transference identity of \[18\], namely, we get verification of (5.13). By Lemma 5.2, we have

\[
\lambda(t) \frac{\partial}{\partial t} Q_\lambda(r) + v_1(t, r) = C \sqrt{\frac{\lambda(t)}{r}} \sup_{x \in [T_\lambda, t]} \left( \sqrt{\frac{\lambda(x)}{r}} \right)
\]

Next, we have

\[
v_f(t, r) = \lambda(t) \frac{\partial}{\partial t} F^{-1} \left( \frac{y(t)}{\sqrt{t}}, \frac{1}{\lambda(t)} \right) \frac{r}{\lambda(t)}, \quad r > 0
\]

Then, by the transference identity of \[18\], namely,

\[
\mathcal{F}(r \partial_r u)(\xi) = \left[ -2 \xi \mathcal{F}(u)'_1(\xi) \right] + \mathcal{K}(\mathcal{F}(u))(\xi), \quad \mathcal{F}(u)(\xi) = \left[ \frac{\mathcal{F}(u)_0}{\mathcal{F}(u)_1(\xi)} \right]
\]

we get

\[
||\partial_t v_f||_{L^2(\cdot^2, dr)} = C \frac{\lambda(t)}{\lambda(t)} ||v_f||_{L^2(\cdot^2, dr)} + \frac{\lambda(t)}{r} \mathcal{F}^{-1} \left( \frac{y(t)}{\sqrt{t}}, \frac{1}{\lambda(t)} \right) \frac{r}{\lambda(t)} ||v_f||_{L^2(\cdot^2, dr)}
\]

By Lemma 5.2,

\[
||v_f||_{L^2(\cdot^2, dr)} + \lambda(t) ||\partial_r v_f||_{L^2(\cdot^2, dr)} \leq C \frac{\lambda(t)^{3/2} \log^{90}(t)}{t^{3+\varepsilon - \frac{2}{2}C_1}}
\]

In particular, using the Sobolev embedding $H^1(\mathbb{R}^3) \subset L^6(\mathbb{R}^3)$, we get

\[
||v_f(t, r)||_{L^6(\cdot^2, dr)} \leq C \frac{\lambda(t)^{3/2} \log^{90}(t)}{t^{2+\varepsilon - \frac{2}{2}C_1 - C_2}}
\]

From (4.17), (4.31), Lemma 4.20 and (4.35), we get that $E(v_{rad}, \partial_t v_{rad}) < \infty$, which finishes the verification of (5.13).

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