The role of entanglement in dynamical evolution

Vittorio Giovannetti\textsuperscript{1}, Seth Lloyd\textsuperscript{1,2}, and Lorenzo Maccone\textsuperscript{1}

\textsuperscript{1}Massachusetts Institute of Technology – Research Laboratory of Electronics
\textsuperscript{2}Massachusetts Institute of Technology – Department of Mechanical Engineering
77 Massachusetts Ave., Cambridge, MA 02139, USA

Entanglement or entanglement generating interactions permit to achieve the maximum allowed speed in the dynamical evolution of a composite system, when the energy resources are distributed among subsystems. The cases of pre-existing entanglement and of entanglement-building interactions are separately addressed. The role of classical correlations is also discussed.

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The problem of determining how to exploit the available resources to achieve the highest evolution speed is relevant to deriving physical limits in a variety of contexts. Of particular interest are, for example, the maximum rate for information processing \cite{1,2} or for information exchange through communication channels \cite{3,4}.

The time-energy uncertainty relation imposes a lower limit on the time interval $T_{\perp}$ that it takes for a quantum system to evolve through two orthogonal states \cite{1,2,3}. This bound on $T_{\perp}$ is related to the spread in energy of the system. Recently, Margolus and Levitin have linked such a quantity also to the average energy of the system \cite{1}. These two conditions together establish the quantum speed limit time, i.e. the minimum time $T(E, \Delta E)$ required for a system with energy $E$ and energy spread $\Delta E$ to evolve through two orthogonal states (see Sec. I).

In this paper we analyze the $T(E, \Delta E)$ of systems composed of $M$ subsystems, focusing on the role of correlations between the $M$ components and separately addressing the cases of non-interacting and interacting subsystems. When no interactions are present, we first show that for initially-separable pure states, the quantum speed limit is achievable only in the asymmetric situation in which only one of the subsystems evolves in time and carries all the system’s energy resources. We then provide an example that shows that the presence of entanglement in the initial state allows a dynamical speedup also when the energy resources are homogeneously distributed among the subsystems. In this way, showing that homogeneous separable states cannot exhibit speedup while at least one homogeneous entangled case that exhibits speedup exists, we prove that entanglement is necessary to achieve the quantum speed limit, at least in the case of pure states (see Sec. II). When classical mixtures are taken into account, the situation is more complex: energy-homogeneous separable states that reach the bound do exist. However, their ensemble realizations are either mixtures of entangled configurations or mixtures of energy-asymmetric configurations: more precisely, each separable unraveling of the state $\rho = \sum_{n} p_{n} \rho_{1}^{(n)} \otimes \ldots \otimes \rho_{M}^{(n)}$ must be composed by product states in which only a single subsystem evolves rapidly to an orthogonal configuration while the other ones do not evolve at all (see Sec. II.A).

In the case of interacting subsystems (see Sec. III), homogeneous pure unentangled states can still achieve the $T(E, \Delta E)$ bound. It will be shown that the reason for this behavior is the entanglement built up during the interaction.

I. QUANTUM SPEED LIMIT TIME

Consider a system in an initial state $|\Psi \rangle$ of mean energy $E = \langle \Psi | H | \Psi \rangle$ (where $H$ is the Hamiltonian and where we assume zero ground state energy). The Margolus-Levitin theorem \cite{1} asserts that it takes at least a time $T_{\perp} \geq \pi \hbar / (2E)$ for the system to evolve from $|\Psi \rangle$ to an orthogonal state. This result complements the time-energy uncertainty relation, which requires $T_{\perp} \geq \pi \hbar / (2\Delta E)$, where $\Delta E = \sqrt{\langle \Psi | (H - E)^{2} | \Psi \rangle}$ is the energy spread of the state $|\Psi \rangle$. The Margolus-Levitin theorem gives a better bound on $T_{\perp}$ than the uncertainty relations when an asymmetric energy distribution yields $\Delta E > E$. Joining the two above inequalities one obtains the quantum speed limit time, i.e. the minimum time $T(E, \Delta E)$ required for the evolution to an orthogonal state, as

\begin{equation}
T_{\perp} \geq T(E, \Delta E) = \max \left( \frac{\pi \hbar}{2E}, \frac{\pi \hbar}{2\Delta E} \right).
\end{equation}

In \cite{1} it has been shown that states that saturate this bound do exist. In the appendix the bound \cite{1}, derived only for pure states in \cite{1}, is shown to apply also for mixed states.

In this paper we analyze the quantum speed limit time \cite{1} of systems composed of $M$ parts. The Hamiltonian is of the form

\begin{equation}
H = \sum_{k=1}^{M} H_{k} + H_{\text{int}},
\end{equation}

where the $H_{k}$ are the free Hamiltonians of the subsystems and $H_{\text{int}}$ is a non-trivial interaction Hamiltonian between them. When $H_{\text{int}} = 0$, the Hamiltonian is not able to generate correlations between the subsystems so that they evolve independently. This case is analyzed in the following section, where it is shown that, unless correlations are present in the initial state of the system, the
energetic resources available cannot be efficiently used to achieve the bound \(1\) when they are distributed among the \(M\) parts.

II. NON INTERACTING SUBSYSTEMS

In this section we show that, for non-interacting subsystems, pure separable states cannot reach the quantum speed limit unless all energy resources are devoted to one of the subsystems. This is no more true if the initial state is entangled: a cooperative behavior is induced such that the single subsystems cannot be regarded as independent entities.

A separable pure state has the form

\[
|\Psi_{\text{sep}}\rangle = |\psi_1\rangle_1 \otimes \cdots \otimes |\psi_M\rangle_M ,
\]

where \(|\psi_k\rangle_k\) is the state of the \(k\)-th subsystem which has energy \(E_k\) and spread \(\Delta E_k\). Since there is no interaction \((H_{\text{int}} = 0)\), the vector \(|\Psi_{\text{sep}}\rangle\) remains factorizable at all times. It becomes orthogonal to its initial configuration if at least one of the subsystems evolves to an orthogonal state. The time employed by this process is limited by the energy and the energy spread of each subsystem, through Eq. \(4\). By choosing the time corresponding to the “fastest” subsystem, the time \(\tau_{\text{sep}}\) for the state \(|\Psi_{\text{sep}}\rangle\) is

\[
\tau_{\text{sep}} \geq \max \left( \frac{\pi \hbar}{2E_{\text{max}}}, \frac{\pi \hbar}{2\Delta E_{\text{max}}} \right) ,
\]

where \(E_{\text{max}}\) and \(\Delta E_{\text{max}}\) are the maximum values of the energy and energy spread of the \(M\) subsystems. For the state \(|\Psi_{\text{sep}}\rangle\), the total energy is \(E = \sum_k E_k\) and the total energy spread is \(\Delta E = \sqrt{\sum_k \Delta E_k^2}\). This implies that the bound imposed by Eq. \(4\) is always greater or equal than \(\tau(E, \Delta E)\) of Eq. \(1\), being equal only when \(E_{\text{max}} = E\) or \(\Delta E_{\text{max}} = \Delta E\), e.g. when one of the subsystems has all the energy or all the energy spread of the whole system.

This means that only such subsystem is evolving in time: the remaining \(M - 1\) are stationary. The gap between the bound \(4\) for separable pure states and the bound \(4\) for arbitrary states reaches its maximum value for systems that are homogeneous in the energy distribution, i.e. such that \(E_{\text{max}} = E/M\) and \(\Delta E_{\text{max}} = \Delta E/\sqrt{M}\). In this case, Eq. \(4\) implies that, for factorizable states, one has at least \(\tau_{\text{sep}} \geq \sqrt{M} \tau(E, \Delta E)\). In fact, if \(E \geq \Delta E\), \(\tau_{\text{sep}}\) is always greater than \(M\) times the quantum speed limit time. On the other hand, if \(\Delta E \geq E\), we find that

\[
\tau_{\text{sep}} \geq \begin{cases} \sqrt{M} \tau(E, \Delta E) & \text{for } M \leq M^* \\ \frac{M}{\sqrt{M^*}} \tau(E, \Delta E) & \text{for } M \geq M^* \end{cases} ,
\]

where \(M^* \equiv (E/\Delta E)^2\).

In order to show that the bound is indeed achievable when entanglement is present consider the following entangled state

\[
|\Psi_{\text{ent}}\rangle = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} |n\rangle_1 \otimes \cdots \otimes |n\rangle_M ,
\]

where \(|n\rangle_k\) is the energy eigenstate (of energy \(n \hbar \omega_0\)) of the \(k\)-th subsystem. The state \(|\Psi_{\text{ent}}\rangle\) is homogeneous since each of the \(M\) subsystems has energy \(E = \hbar \omega_0 (N - 1)/2\) and \(\Delta E = \hbar \omega_0 \sqrt{N^2 - 1}/(2\sqrt{3})\). The total energy and energy spread are given by \(E = ME\) and \(\Delta E = M\Delta E\) respectively. The scalar product of \(|\Psi_{\text{ent}}\rangle\) with its time evolved \(|\Psi_{\text{ent}}(t)\rangle\) is

\[
\langle \Psi_{\text{ent}} | \Psi_{\text{ent}}(t) \rangle = \frac{1}{N} \sum_{n=0}^{N-1} e^{-iM\omega_0 t} ,
\]

where the factor \(M\) in the exponential is a peculiar signature of the energy entanglement. The value of \(\tau_{\text{ent}}\) for the state \(|\Psi_{\text{ent}}\rangle\) is given by the smallest time \(t \geq 0\) for which this quantity is zero, i.e. \(2\pi/(NM\omega_0)\). It is smaller by a factor \(\sim \sqrt{M}\) than what it would be for homogeneous separable pure states with the same value of \(E\) and \(\Delta E\), as can be checked through Eq. \(4\). The above example can be easily extended to the general case in which all \(H_k\) are not necessarily identical. The effect shown here can be exploited where it is necessary to increase the speed of systems while equally sharing the energy resources among the subsystems. States of the type \(|\Psi_{\text{ent}}\rangle\) have been used in \(8\) in order to increase the time resolution of traveling pulses.

In summary, pure separable states can reach the quantum speed limit only in the case of highly asymmetric configurations where one of the subsystems evolves to an orthogonal configuration at the maximum speed allowed by its energetic resources, while the other subsystems do not evolve. In all other cases entanglement is necessary to achieve the bound. This, of course, does not imply that all entangled states evolve faster than their unentangled counterparts.

A. Classical mixtures.

What happens when classical correlations among subsystems are considered? A separable state of an \(M\)-parts composite system can be always expressed by the following convex convolution

\[
\varrho = \sum_n p_n \varrho^{(n)}_1 \otimes \cdots \otimes \varrho^{(n)}_M ,
\]

where \(p_n\) are positive coefficients which sum up to one and where the normalized density matrix \(\varrho^{(n)}_k\) describes a state of the \(k\)-th subsystem with energy \(E^{(n)}_k\) and energy spread \(\Delta E^{(n)}_k\). Equation \(8\) is a mixture of independent product state configurations, labeled by the parameter \(n\), which occur with probability \(p_n\): it displays classical
correlations but no entanglement between the \(M\) subsystems. For non-interacting systems, the energy \(E\) and spread in energy \(\Delta E\) of \(\varrho\) are given by

\[
E = \sum_n p_n \sum_{k=1}^N E_k^{(n)} \tag{9}
\]

\[
\Delta E^2 = \sum_n p_n \left[ \left( \sum_{k=1}^M (E_k^{(n)})^2 \right) + \left( \sum_{k=1}^M E_k^{(n)} - E \right)^2 \right],
\]

and the state \(\varrho(t)\) at time \(t\) is always of the form \(\mathcal{S}\) where \(g_k^{(n)}\) are replaced by their evolved \(\hat{g}_k^{(n)}(t)\). If \(\varrho\) reaches the quantum speed limit then it is orthogonal to its evolved at time \(T(E, \Delta E)\), i.e.

\[
\sum_{nm} p_n p_m \chi_n^{(m,n)}(t) \cdots \chi_M^{(m,n)}(t) \big|_{t = T(E, \Delta E)} = 0
\]

where \(\chi_n^{(m,n)}(t) = \text{Tr}[\hat{g}_n^{(m)}(t) \hat{g}_k^{(m)}]\). Using the spectral decomposition, one immediately sees that all the terms \(\chi\) are non-negative real quantities. This means that Eq. (10) is satisfied if and only if each of the summed terms is equal to zero independently on \(n\) and \(m\). In particular, focusing on the case \(n = m\), at least one subsystem must exist (say the \(k_n\)-th) for which \(\chi_n^{(n,n)}(T(E, \Delta E)) = 0\). Applying the quantum speed limit \(\mathcal{T}\) to the state \(\hat{g}_k^{(n)}\) of this subsystem the following inequality results

\[
T(E, \Delta E) \geq T(E_k^{(n)}, \Delta E_k^{(n)}).
\]

Suppose now that \(E \geq \Delta E\), i.e. \(T(E, \Delta E) = \pi \hbar/(2\Delta E)\). In this case, from Eqs. (9) and (10) one can show that for each \(n\), \(\Delta E_k^{(n)} = 0\) for all \(k \neq k_n\), while \(E_k^{(n)} \geq \Delta E_k^{(n)} = \Delta E\). On the other hand, if \(\Delta E \geq E\) then one can analogously obtain that for each \(n\), \(E_k^{(n)} = 0\) for all \(k \neq k_n\), while \(\Delta E_k^{(n)} \geq E_k^{(n)} = E\). In both cases the inequality (11) becomes an identity, i.e. the quantum speed limit time \(T(E_k^{(n)}, \Delta E_k^{(n)})\) of the state \(\hat{g}_k^{(n)}\) coincides with the quantum speed limit time \(T(E, \Delta E)\) of the global state \(\varrho\).

Moreover, for all \(k \neq k_n\) the states \(\hat{g}_k^{(n)}\) are eigenstates of the Hamiltonians \(H_k\), i.e. they cannot evolve to orthogonal configurations.

This proves that the separable state \(\mathcal{S}\) achieves the quantum speed limit only if, in any statistical realization \(n\) of the system, a single subsystem evolves to an orthogonal configuration at its own maximum speed limit time (which coincides with \(T(E, \Delta E)\) of the whole system). All the other subsystems do not evolve.

Classical correlations among subsystems, however, can produce configurations \(\varrho\) that achieve the speed limit and are statistically energy-homogeneous: i.e. in average all subsystems share the same resources. As an example, consider the separable state \(\varrho = (\varrho_0 \otimes \varrho_0 + \varrho_0 \otimes \varrho_0)/2\) of a bipartite system composed by two identical subsystems (e.g. two spins), where \(\varrho_0\) is the zero energy ground state and \(\varrho_a\) is a normalized density matrix which saturates its own quantum speed limit. Since the energy \(E\) and the energy spread \(\Delta E\) of the \(\varrho_a\) coincide with those of \(\varrho_0\), these two matrices have the same value of \(T(E, \Delta E)\). The density operator \(\varrho\), describes a mixture where half of the times the first spin is in the state \(\varrho_a\) and the second spin is in the ground state, and in the other half their roles are exchanged: of course in this configuration in average the two spins are in the same state \((\varrho_a + \varrho_0)/2\). Assume now that \(\text{Tr}[\varrho_a(t) \varrho_b] = \text{Tr}[\varrho_b(t) \varrho_a] = 0\): in this case \(\varrho\) will saturate the quantum speed limit.

Since all the above derivation applies for separable unravellings of the form \(\mathcal{S}\), one can say that in each experimental run only one of the subsystems evolves to an orthogonal state. Of course the state \(\varrho\) allows also unravellings that are not of the form \(\mathcal{S}\) in which the statistical realizations may contain entanglement between subsystems (e.g. a fully mixed state can be obtained from a statistical mixture of maximally entangled states). The above derivation does not apply to these entangled decompositions of \(\varrho\), yet the role of entanglement is self-evident in this case. Hence, practically, there are two different ways to build “fast” separable states through classical correlations: either starting from the separable configurations \(\mathcal{S}\) in which only one of the subsystems evolves, or starting from entangled configurations. What is definitely impossible is to build a \(\varrho\) that reaches the bound by mixing separable configurations in which the energy is not concentrated in one of the subsystems.

### III. INTERACTING SUBSYSTEMS

For the sake of simplicity, in analyzing interacting subsystems, we focus only on the pure state case where the effects of entanglement are more evident. Two situations are possible. Either \(H_{\text{int}}\) does not introduce any entanglement in the initial state of the system or \(H_{\text{int}}\) builds up entanglement among subsystems. In the first case, no correlations among the subsystems are created so that each subsystem evolves independently as \(\varrho = \otimes_{j=1}^M |\psi_j(t)\rangle\langle \psi_j|\), unless entanglement was present initially. Since this type of evolution can always be described as determined by an interaction-free effective Hamiltonian, the results of the previous section apply. In the second case, when \(H_{\text{int}}\) builds up entanglement, the system may reach the bound even though no entanglement was already present initially. In fact, as will be shown through an example, one can tailor suitable entangling Hamiltonians that speed up the dynamical evolution even for initial homogeneous separable states.

An interaction capable of speeding up the dynamics is given by the following Hamiltonian for \(M\) qubits

\[
H = \hbar \omega_0 \sum_{k=1}^M \left( 1 - \sigma_x^{(k)} \right),
\]

where \(\sigma_x^{(k)}\) is the Pauli operator \(|1\rangle\langle 0| + |0\rangle\langle 1|\) for the
k-th qubit and where $S = \prod_{k=1}^{M} \sigma_z^{(k)}$. The first term in Eq. (12) is the free Hamiltonian which rotates independently each of the qubits at frequency $\omega_0$. The second term is a global interaction which rotates collectively all the qubits at frequency $\omega$, coupling them together. Consider an initial factorized state where all qubits are in eigenstates of the $\sigma_z^{(k)}$ Pauli matrices, i.e.

$$|\Psi\rangle \equiv |J_1\rangle_1 \cdots \otimes |J_M\rangle_M,$$

where $J_k$ are either 0 or 1. This is an homogeneous configuration of the system. Moreover, the energy $E = \hbar(\omega + M\omega_0)$ and the energy spread $\Delta E = \hbar \sqrt{\omega^2 + M\omega_0^2}$ of this state give $T(E, \Delta E) = \pi/(2\sqrt{\omega^2 + M\omega_0^2})$. The state $|\Psi\rangle$ evolves to the entangled configuration

$$|\Psi(t)\rangle = e^{-itE/\hbar} \left[ \cos(\omega t)|J(t)\rangle + i \sin(\omega t)|\overline{J}(t)\rangle \right],$$

where

$$|J(t)\rangle = \bigotimes_{k=1}^{M} \left[ \cos(\omega_0 t)|J_k\rangle_k + i \sin(\omega_0 t)|\overline{J}_k\rangle_k \right],$$

with the overbar denoting qubit negation (|0⟩ ≡ |1⟩, |1⟩ ≡ |0⟩). Imposing the orthogonality between $|\Psi\rangle$ and $|\Psi(t)\rangle$, we find that $T_\perp$ is the minimum value of $t$ for which

$$\cos(\omega t) \cos^M(\omega_0 t) + i^{M+1} \sin(\omega t) \sin^M(\omega_0 t) = 0.$$ (16)

From Eq. (16) it is easy to check that, for $\omega = 0$ (no interaction) $T_\perp = \sqrt{M}$ times bigger than $T(E, \Delta E)$. Instead, for $\omega_0 = 0$ (no free evolution) the system reaches the speed limit, i.e. $T_\perp = T(E, \Delta E)$. In Fig. 1 the value of $T_\perp$ is compared to the value of $T(E, \Delta E)$ for different values of $\omega$, showing that as the interaction becomes predominant, $T_\perp$ tends to $T(E, \Delta E)$.

This example shows that a suitable $H_{\text{int}}$ can allow a homogeneous pure state to reach the quantum speed limit. In order to reach this bound, however, the interaction must i) connect all the qubits and ii) be sufficiently strong (see Fig. 1). A simple counterexample for i) can be obtained by considering the case in which the $M$ qubits are divided in $G$ non-interacting groups which have an Hamiltonian of the same form of (12) and contain $Q = M/G$ qubits each. In this case entanglement cannot build up between qubits of different groups and it is immediate to see that $T_\perp$ is at least $\sqrt{M/Q} T(E, \Delta E)$. The order $K$ of the interaction (i.e. the number of subsystems that are involved in a single vertex of interaction) also plays an important role: the example illustrated by the Hamiltonian (12) describes an $M$-th order case. For any given $K$, a rich variety of cases are possible depending on how the interaction is capable of constructing entanglement between subsystems. The simplest example is an Ising-like model where there is a $K$-th order coupling between neighbors in a chain of qubits. Here one only has a $K^{1/2}$ improvement over the non-interacting case (16).

A recent proposal [4] uses the effect described in this section to increase the communication rate by a factor $\sqrt{M}$ over a communication channel composed of $M$ independent parallel channels which uses the same resources.

FIG. 1: Plot of $T_\perp$ from Eq. (16) (asterisks) and of $T(E, \Delta E)$ of the state (13) (dashed line) as a function of the relative intensity of the interaction Hamiltonian $\omega/\omega_0$ for $M = 9$. The lower shaded region is the area forbidden by Eq. (1).

IV. CONCLUSIONS

In conclusion we have studied the quantum speed limit for composite systems. We have analyzed the role of correlations (quantum and classical) among subsystems emphasizing the role of entanglement. The Hamiltonians that do not create quantum correlations need to operate on initially entangled states in order to speed up the dynamics (except for the special case in which only one subsystem evolves). On the other hand, entanglement-generating Hamiltonians are capable of speeding up the dynamics even starting from separable configurations.

APPENDIX A: THE QUANTUM SPEED LIMIT TIME FOR MIXED STATES

Here the quantum speed limit, which was proved for pure states in [1], is extended to mixed states. The quantity $T_\perp$ is defined as the minimum time $t$ for which the evolved density matrix $\bar{\rho}(t)$ of a system of energy $E$ and energy spread $\Delta E$ becomes orthogonal to the initial state $\rho$, i.e. $\text{Tr}[\bar{\rho}(t)\rho] = 0$. Using the spectral decomposition, $\bar{\rho}$ can be written as $\bar{\rho} = \sum_n \lambda_n \phi_n \langle \phi_n |$, where $\lambda_n$ are positive coefficients which sum up to one and where $\{|\phi_n\rangle\}$ is an orthonormal set. From the definition of $T_\perp$ it then follows that

$$T_\perp \geq \min_{\epsilon \geq 0} \left( \langle \phi_n(t) | \phi_m \rangle = 0 \quad \forall n, m \right)$$

$$\geq \min_{\epsilon \geq 0} \left( \langle \phi_n(t) | \phi_n \rangle = 0 \right) \quad \forall n,$$ (A1)
where $|\phi_n(t)\rangle$ is the time-evolved of $|\phi_n\rangle$. Applying Eq. (1) to the pure states $|\phi_n\rangle$, one finds
\[
T_{\perp} \geq \max \left( \frac{\pi \hbar}{2 E_{\text{min}}}, \frac{\pi \hbar}{2 \Delta E_{\text{min}}} \right), 
\]
(A2)

where $E_{\text{min}}$ and $\Delta E_{\text{min}}$ are respectively the minima on $n$ of the energy $E_n$ and of the spread $\Delta E_n$ of the state $|\phi_n\rangle$. Since for the state $\rho$ the energy and the energy spread are such that
\[
E = \sum_n \lambda_n E_n \geq E_{\text{min}} \tag{A3}
\]

from Eq. (A2) the quantum speed limit follows.

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\[
\Delta E = \sqrt{\sum_n \lambda_n [\Delta E_n^2 + (E - E_n)^2]} \geq \Delta E_{\text{min}},
\]