REDDUCTIVE HOMOGENEOUS LORENTZIAN MANIFOLDS

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ABSTRACT. We study homogeneous Lorentzian manifolds $M = G/L$ of a connected reductive Lie group $G$ modulo a connected reductive subgroup $L$, under the assumption that $M$ is (almost) $G$-effective and the isotropy representation is totally reducible. We show that the description of such manifolds reduces to the case of semisimple Lie groups $G$. Moreover, we prove that such a homogeneous space is reductive. We describe all totally reducible subgroups of the Lorentz group and divide them into three types. The subgroups of Type I are compact, while the subgroups of Type II and Type III are non-compact. The explicit description of the corresponding homogeneous Lorentzian spaces of Type II and III (under some mild assumption) is given. We also show that the description of Lorentz homogeneous manifolds $M = G/L$ of Type I reduces to the description of subgroups $L$ such that $M = G/L$ is an admissible manifold, i.e., an effective homogeneous manifold that admits an invariant Lorentzian metric. Whenever the subgroup $L$ is a maximal subgroup with these properties, we call such a manifold minimal admissible. We classify all minimal admissible homogeneous manifolds $G/L$ of a compact semisimple Lie group $G$ and describe all invariant Lorentzian metrics on them.

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INTRODUCTION

This paper is devoted to the investigation and classification of homogeneous $G$-effective Lorentzian manifolds $M = G/L$ of a reductive Lie group $G$ modulo a reductive subgroup $L$. Homogeneous $G$-effective Lorentzian manifolds $M = G/L$ are naturally divided into two classes:

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the class of proper manifolds, where the action of $G$ on $M$ is proper, or, equivalently, the stabilizer $L$ is compact, and the class of non-proper homogeneous manifolds, where $L$ is non-compact.

A remarkable result by N. Kowalski [Kow97] shows that the only non-proper homogeneous Lorentzian manifold $G/L$ of a simple Lie group $G$ is a space of constant curvature, that is (up to a covering) the Minkowski space or the (anti-)de Sitter space. This result had been generalized by M. Deffaf, K. Melnick and A. Zeghib [DMZ08] to the case of a semisimple Lie group $G$: Any non-proper homogeneous Lorentzian manifold of a semisimple Lie group $G$ is a local product of the Minkowski space or the (anti-)de Sitter space and a Riemannian homogeneous manifold.

Our starting point in this article is the classification of connected totally reducible Lie subgroups of the Lorentz group, which we present in Section 2. There we show that totally reducible subgroups of the Lorentz group fall into three types which we call Type I, II and III, respectively (see Theorem 2.4). Type I groups are compact subgroups, while Type II and Type III consist of non-compact subgroups. As a corollary, it is shown that any homogeneous manifold $M = G/L$ of a reductive Lie group $G$ with totally reducible isotropy subgroup admits a reductive decomposition. The explicit description of (non-proper) homogeneous Lorentzian reductive manifolds of Type II and Type III is given in Section 6, under some mild assumption. Note that all non-reductive (hence, non-proper) 4-dimensional homogeneous Lorentzian manifolds had been classified by Fels and Renner [FR06]. Proper homogeneous Lorentzian manifolds $M = G/L$ of semisimple Lie groups $G$ were studied in [Af12], where the structure of such manifolds had been elucidated. The classification problem of such manifolds is equivalent to the description of all admissible connected compact subgroups $L$. Recall that a connected Lie subgroup $L \subset G$ of $G$ is called admissible, if the homogeneous manifold $M = G/L$ admits an invariant Lorentzian metric. Since any closed subgroup $L' \subset L$ is also admissible, the problem reduces to the description of all maximal admissible subgroups. Proper homogeneous Lorentzian manifolds $M = G/L$, where $L$ is a maximally admissible subgroup of a semisimple Lie group $G$ have been described in [Af12]. For this case it is easy to see that a maximal admissible subgroup $L$ exists only when $G$ is simple.

In this paper we modify the notion of maximal admissible connected closed subgroups $L$ of a Lie group $G$, by demanding that $M = G/L$ admits an invariant Lorentz metric, such that there is no larger closed connected subgroup $\tilde{L} \supset L$ which makes the coset $\tilde{M} = G/\tilde{L}$ an effective homogeneous Lorentzian manifold. Then, the manifold $M = G/L$ is called minimal admissible. This more reasonable definition of the notion of minimality, allows us to extend the results of [Af12]. In particular, in Section 3.3 we give a necessary and sufficient condition for a proper homogeneous manifold $M = G/L$ of a reductive Lie group $G$ to be a minimal admissible homogeneous space, in terms of the reductive decomposition $\mathfrak{g} = \mathfrak{t} + \mathfrak{m}$. Then we divide such homogeneous spaces into three subclasses $\mathfrak{la}$, $\mathfrak{lb}$, and $\mathfrak{lc}$, depending on whether the centralizer $\mathfrak{m}^1$ is isomorphic to $\mathbb{R}$ (la), $\mathfrak{m}^1$ is three dimensional simple regular Lie algebra $\mathfrak{s}$ (lb), or $\mathfrak{m}^1$ is the commutative Lie algebra $\mathbb{R}^k$ (lc). In Section 4 we use the above result to describe all maximal admissible subgroups $L$ of a compact semisimple Lie group $G$ and classify all associated minimal admissible homogeneous manifolds $M = G/L$. Any such homogeneous space of Type la is a standard homogeneous contact manifold. Any indecomposable minimal admissible manifold $M = G/L$ of Type lb is the total space of the principal $\text{Sp}(1)$-bundle

$$M = G/L \to W = G/N_G(\mathfrak{g}(\alpha)) = G/\text{Sp}(1) \cdot L$$

over the symmetric quaternionic Kähler manifold $W$ corresponding to a simple compact Lie group $G$ (a Wolf space), and hence $M = G/L$ admits an invariant 3-Sasakian structure. Finally we show that there are no minimal admissible manifolds $M = G/L$ with compact $G$ of Type lc.
In Section 5 we generalize the results of Section 4 to the case of non-compact semisimple Lie groups $G$, and describe all proper minimal admissible homogeneous manifolds $M = G/L$ with compact stability subgroup $L$ of Type Ia and Type Ib. In particular, we show that such homogeneous space of Type Ia (respectively Type Ib) are non-compact real forms of para 3-Sasakian manifolds (respectively, pseudo 3-Sasakian manifolds). They are principal $\text{SL}(2, \mathbb{R})$-bundles (respectively $\text{Sp}(1)$-bundles) associated to the symmetric para-quaternionic Kähler manifolds (or splittable quaternionic Kähler manifolds), classified in [DJS04] (respectively, the symmetric pseudo-quaternionic Kähler manifolds, classified in [A/C05]).

1. Preliminaries

In this article we adopt the following notation and assumptions. If the opposite is not stated, we will assume that all Lie groups are connected. By a **homogeneous** $G$-**manifold** $M = G/L$ we understand a coset space of a connected Lie group $G$ modulo a closed connected subgroup $L \subset G$ (called the stability subgroup). If the opposite is not stated, we will assume that the homogeneous manifold $M = G/L$ is almost effective. This means that $G$ acts almost effectively on $M = G/L$, that is, $L$ contains no non-discrete normal subgroups of the Lie group $G$, or equivalently, that the isotropy representation $j : l \rightarrow \text{End}(T_o M)$ of the stability subalgebra $l$ is exact. Any almost effective homogeneous $G$-manifold $M = G/L$ becomes effective if we factorize $G$ by the maximal (discrete) normal subgroup $\Gamma$ which acts trivially on $M$. We say that a Lie subalgebra $l$ of a Lie algebra $g$ is **compact** if the adjoint subalgebra $\text{ad}_l|_g \subset \text{ad}_g$ generates a compact subgroup $L$ of the adjoint group $\text{Ad}_G$ corresponding to the Lie algebra $g$, and it is called **strongly compact** if it generates a compact subgroup of the simply connected Lie group $\hat{G}$ with $\text{Lie}(\hat{G}) = g$ (and hence, of any Lie group $G$ with $\text{Lie}(G) = g$).

Given a homogeneous $G$-manifold $G/L$, we will denote by $l := \text{Lie}(L) \subset g := \text{Lie}(G)$ the Lie algebras of $L \subset G$, respectively, and identify the tangent space $T_o G/L$ at the point $o = eL$ with the vector space $g/l$. The isotropy representation of the stability group $L$ will be denoted by $j : L \rightarrow \text{GL}(g/l)$. Recall that a homogeneous space $M = G/L$ is called **reductive** if the Lie algebra $g$ admits a **reductive decomposition**

$$g = l + m,$$

(1.1)
i.e., an $\text{Ad}_L$-invariant decomposition into a direct sum of subspaces. Hence we require that the subalgebra $l$ admits an $\text{Ad}_L$-invariant complement $m$ in $g$, which one can naturally identify with the tangent space $T_o G/L$. Then, the isotropy representation $j$ is identified with the adjoint representation $\text{Ad}_L|_m$, and the tangent bundle $TM$ is identified with the associated vector bundle $G \times_m m$. It turns out that $G$-invariant pseudo-Riemannian metrics bijectively correspond to $\text{Ad}_L$-invariant pseudo-Euclidean inner products on $m$. In fact, since by assumption we work with connected stability groups, the $\text{Ad}_L$-invariance can be equivalently replaced by $\text{ad}_j$-invariance. Note that the action of a Lie group $G$ on a reductive homogeneous manifold $M = G/L$ is almost effective if and only if the isotropy representation $\text{ad}_l|_m \rightarrow \mathfrak{gl}(m)$ is exact, and then $l$ has trivial intersection with the center $Z(g)$: $l \cap Z(g) = 0$. Next we will always assume this condition.

By a **homogeneous Lorentzian** $G$-**manifold** we understand a Lorentzian manifold $(M, g)$ with a transitive almost effective isometric action of a connected Lie group $G$. Then we identify $M$ with the coset space $M = G/L$, where $L = G_o$ is the stability subgroup of a point $o \in M$. Below we will always assume that $G \subset \text{ISO}(M, g)$ is a closed subgroup of the full isometry group, and that $L$ is connected. Then the isotropy group $j(L)$ is a closed connected subgroup of the Lorentz group. Next we are interested in the classification of homogeneous Lorentzian manifolds $M = G/L$ with a reductive decomposition as in (1.1) and with totally reducible isotropy representation. To simplify the exposition, we say that such reductive decomposition
is effective. Then, the classification of reductive homogeneous Lorentzian manifolds with totally reducible isotropy representation up to a covering, reduces to the description of all effective reductive decompositions (1.1), such that the isotropy subalgebra \( j(I) := \text{ad}_t |_{\mathfrak{m}} \) is a totally reducible subalgebra of the Lorentz Lie algebra \( \mathfrak{so}(\mathfrak{m}) \) (associated to some Lorentz inner product \( g_\mathfrak{m} \)).

2. Totally reducible subalgebras of the Lorentz algebra

In this section we classify all totally reducible subalgebras of the Lorentz algebra, and next specify the maximal ones. In the following it is convenient to work with the \((n + 2)\)-dimensional Minkowski vector space, which we shall denote by

\[
(V := \mathbb{R}^{1,n+1}, \ g(u, v) = \langle u, v \rangle_{\text{Min}} = -u_0 v_0 + \sum_{i=1}^{n+1} u_i v_i).
\]

The corresponding connected Lorentz group will be denoted by \( \text{SO}^0(V) \equiv \text{SO}^0(1, n + 1) \) and we will write \( \mathfrak{so}(V) \equiv \mathfrak{so}(1, n + 1) \) for its Lie algebra, which is also referred to as the Lorentz algebra. Let us fix a basis \( p, e_1, \ldots, e_n, q \) of \( V \) (Witt basis), where \( p, q \) are isotropic vectors with \( g(p, q) = 1 \), and \( e_1, \ldots, e_n \) is the orthonormal basis of the orthogonal complement to \( \mathbb{R} p + \mathbb{R} q \) in \( V \). The 2-dimensional Minkowski subspace spanned by \( p, q \) will be denoted by

\[
U = \mathbb{R} p + \mathbb{R} q,
\]

while the vector space which is orthogonal to \( U \) will be denoted by \( E \). Obviously, \( E \) is spanned by \( e_1, \ldots, e_n \) and endowed with the Euclidean metric \( g|_E \), coincides with the Euclidean space \( \mathbb{R}^n \).

**Definition 2.1.** The direct sum decomposition

\[
(2.1) \quad V = \mathbb{R} p + E + \mathbb{R} q = U + E.
\]

is called a standard decomposition of \( V \).

In terms of Lie algebras, we can identify skew-symmetric endomorphisms with bivectors, and so the Lorentz algebra \( \mathfrak{so}(1, n + 1) \equiv \mathfrak{so}(V) \) with \( \Lambda^2 V \). Then,

**Lemma 2.2.** The Lorentz algebra \( \mathfrak{so}(V) \) admits a depth-1 grading

\[
\mathfrak{so}(V) = \mathfrak{g}^{-1} + \mathfrak{g}^0 + \mathfrak{g}^1 = q \wedge E + (\mathbb{R} p \wedge q + \mathfrak{so}(E)) + p \wedge E,
\]

which is the eigenspace decomposition of the endomorphism \( d = \text{ad}_{p \wedge q} \).

**Proof.** This claim immediately follows by (2.1) and the identification \( \mathfrak{so}(V) = \Lambda^2 V \). \( \square \)

Note that the parabolic subalgebra

\[
\mathfrak{p} := \mathfrak{g}^0 + \mathfrak{g}^1 = \mathbb{R} p \wedge q + \mathfrak{so}(E) + p \wedge E,
\]

coincides with the Lie algebra of the Lie group \( \text{SO}^0(V)_{\mathbb{R} p} \) preserving the isotropic line \( \mathbb{R} p \).

Consider now an orthogonal direct sum decomposition

\[
E = H + H^\perp
\]

of the Euclidean vector space \( (E \cong \mathbb{R}^n, g|_E) \) such that \( \dim H = k, \dim H^\perp = n - k, 0 \leq k \leq n - 1 \) and let us denote by

\[
(2.2) \quad V(H) := \mathbb{R} p + H + \mathbb{R} q \subseteq V
\]

the Lorentzian vector space of dimension \( k + 2 \). Note that \( V \) can be expressed as the direct sum decomposition

\[
(2.3) \quad V = (\mathbb{R} p + H + \mathbb{R} q) + H^\perp = V(H) + H^\perp.
\]
Moreover,

- for \( k = n \) we have \( V(H) = V \), while
- for \( k = 0 \), \( E = H^\perp \) and \( V(H) = U = \mathbb{R}p + \mathbb{R}q \).

Set \( h_k := \mathfrak{so}(V(H)) + \mathfrak{so}(H^\perp) \). Then \( h_k \) is a direct sum of a Lorentz algebra and an orthogonal Lie algebra, in particular

\[
(2.4) \quad h_k = \mathfrak{so}(V(H)) + \mathfrak{so}(H^\perp) \cong \mathfrak{so}(1, k + 1) + \mathfrak{so}(n - k).
\]

**Proposition 2.3.** A maximal subalgebra \( h \) of \( \mathfrak{so}(V) \) is conjugated to one of the following subalgebras:

- the maximal compact subalgebra \( \mathfrak{so}(n + 1) \);
- \( h_k = \mathfrak{so}(V(H)) + \mathfrak{so}(H^\perp) \cong \mathfrak{so}(1, k + 1) + \mathfrak{so}(n - k) \), \( 1 \leq k \leq n - 1 \);
- the parabolic subalgebra \( p = \mathbb{R}p \wedge q + p \wedge E + \mathfrak{so}(E) \)

**Proof.** It is clear that the algebras listed above are maximal subalgebras of \( \mathfrak{so}(V) \). Let \( h \) be a maximal subalgebra of \( \mathfrak{so}(V) \). It is well-known that there are no proper irreducible subalgebras in \( \mathfrak{so}(V) \). Hence \( h \) preserves a proper subspace of \( V \). If \( h \) preserves a proper non-degenerate subspace of \( V \), then \( h \) preserves also a Minkowski subspace \( W \subset V \). If \( \dim W = 1 \), then \( h \) is conjugated to \( \mathfrak{so}(n + 1) \). If \( \dim W = 2 \), then \( h \) is conjugated to the parabolic subalgebra \( p \) (note that \( \mathfrak{so}(1, 1) + \mathfrak{so}(n) \) is not a maximal subalgebra of \( \mathfrak{so}(1, n + 1) \)). Finally, if \( \dim W \geq 3 \), then \( h \) is conjugated to \( h_k \) for some \( 1 \leq k \leq n - 1 \). Suppose now that \( h \) does not preserve any non-degenerate subspace of \( V \). Then \( h \) preserves a degenerate subspace \( W \subset V \) and the isotropic line \( W \cap W^\perp \). Let \( p \) be a generator of \( W \cap W^\perp \) and fix an isotropic vector \( q \in V \) as above. Then we obtain the decomposition \( V = \mathbb{R}p + E + \mathbb{R}q \), and conclude that

\[
h = \mathbb{R}p \wedge q + \mathfrak{so}(E) + p \wedge E = p.
\]

This completes the proof. \( \square \)

Let us now pose the main theorem of this section.

**Theorem 2.4.** A totally reducible subalgebra \( h \) of the Lorentz algebra \( \mathfrak{so}(V) \) is conjugated to a subalgebra of one of the following types:

- **Type I:** a subalgebra of the maximal compact Lie algebra \( \mathfrak{so}(n + 1) \);
- **Type II:** a subalgebra of the form \( \mathfrak{so}(W) + \mathfrak{k} \), where \( W = V(H) \) is a Lorentzian subspace of \( V \), \( \dim W \geq 3 \), and \( \mathfrak{k} \) is a compact subalgebra of \( \mathfrak{so}(W^\perp) \);
- **Type III:** a subalgebra of the form \( \mathbb{R}d \oplus \mathfrak{k} \), where \( d = p \wedge q + C_0 \) with \( C_0 \in \mathfrak{so}(E) = \mathfrak{so}(n) \) (which is possibly zero), and \( \mathfrak{k} \) is a subalgebra of the centralizer \( C_{\mathfrak{so}(n)}(C_0) \).

**Proof.** Note that the Lie algebra \( \mathfrak{so}(V) \) is of **Type II**. First suppose that \( h \) is different from \( \mathfrak{so}(V) \) and \( h \) does not preserve any proper non-degenerate subspace of \( V \). By Proposition 2.3, \( h \) is conjugated to a subalgebra of the parabolic algebra and it preserves the line \( \mathbb{R}p \). Since \( h \) is totally reducible, it preserves a vector subspace \( V_0 \subset V \) complementary to \( \mathbb{R}p \). By the assumption, \( V_0 \) is degenerate. Consequently \( h \) preserves the isotropic line \( \ell = V_0 \cap V_0^\perp \), and the vector subspace \( \mathbb{R}p + \ell \subset V \). It is clear that the vector subspace \( \mathbb{R}p + \ell \subset V \) is Lorentzian, and this gives a contradiction.

Suppose now that \( h \) preserves a non-degenerate subspace of \( V \). Then \( h \) preserves a Lorentzian subspace \( W \subset V \), and we may assume that \( h \) does not preserve any proper non-degenerate subspace of \( W \). As we have seen just above, this implies that the projection of \( h \) to \( \mathfrak{so}(W) \) coincides with \( \mathfrak{so}(W) \). If \( \dim W = 1 \), then \( h \) is of **Type I**. If \( \dim W = 2 \), then \( h \) is of **Type III**. If \( \dim W \geq 3 \), then \( h \) is of **Type II**. \( \square \)
Corollary 2.5. Connected totally reducible subgroups of the Lorentz group $SO(V)$ are divided into the following 3 types:

Type I: connected subgroups of the maximal compact Lie group $SO(n+1)$;

Type II: connected subgroups of the form $SO(W) \cdot K$, where $W = V(H)$ is a Lorentzian subspace of $V$, $\dim W \geq 3$, and $K$ is a connected subgroup of $SO(W^\perp)$;

Type III: subgroups of the form $T \cdot K$, where $K \subset SO(E)$ is a connected subgroup and $T$ is a 1-parameter subgroup of $SO(1,1) \times SO(E)$ not contained in $SO(E)$ and commuting with $K$.

Remark 2.6. It is not hard to show that reductive subalgebras $\mathfrak{h}$ of $so(V)$ are exhausted by the subalgebras specified in Theorem 2.4, together with subalgebras $\mathfrak{h}$ of the following form. Consider an orthogonal decomposition $E = E' + E''$ and the corresponding decomposition

$$V = (\mathbb{R} p + E' + \mathbb{R} q) + E''.$$

Let $\mathfrak{k} \subset so(E'')$ be a subalgebra and let $\varphi : E' \to so(E'')$ be a linear map such that $\varphi(E') \subset so(E'')$ is commutative, it commutes with $\mathfrak{k}$ and has trivial intersection with $\mathfrak{k}$. Then, the Lie subalgebra

$$\mathfrak{h} = \{ p \wedge X + \varphi(X) | X \in E' \} + \mathfrak{k} \subset so(V)$$

is reductive, but not totally reducible.

As it is already indicated, the above theorem motivates us to introduce the following definition.

Definition 2.7. (a) We refer to the subalgebras specified by Theorem 2.4 as the totally reducible Lie subalgebras of the Lorentz algebra $so(V) = so(1, n+1)$ of Type I, II, and III, respectively, and similarly for the corresponding connected closed Lie subgroups of $SO^0(V) \equiv SO^0(1, n+1)$.

(b) A homogeneous Lorentzian manifold $M = G/L$ is called a $G$-manifold of Type I (respectively, II, III) if the isotropy Lie algebra $j(1)$ is of Type I (respectively, II, III).

Note that an effective homogeneous Lorentz manifold $M = G/L$ of Type I (respectively, II, III) has a compact (respectively, non-compact) stabilizer $L$. Let us now prove the following result.

Proposition 2.8. Each homogeneous Lorentzian manifold $M = G/L$ of a reductive Lie group $G$ with a totally reducible isotropy subgroup is a reductive homogeneous manifold.

Proof. We may assume that the homogeneous Lorentz manifold $M = G/L$ is effective. Then the isotropy representation $j$ is exact. Since the isotropy group $j(L)$ of Type I is compact and the isotropy group of Type II is an almost direct product $j(L) = SO^0(V(H)) \cdot K$ of (a) Lorentz Lie group and a compact Lie group $K$, the adjoint representation $Ad_L | \mathfrak{k}$ is totally reducible. Moreover, there is an $Ad_L$-invariant subspace $\mathfrak{m}$ complementary to $\mathfrak{l}$. It remains to check that the isotropy group $j(L) = T \cdot K$, generated by the isotropy Lie algebra $j(l) = \mathbb{R} d \oplus \mathfrak{t}$ admits a reductive decomposition. Since $\mathfrak{t} \subset \mathfrak{g}$ is a compact subalgebra, we may chose an $j(\mathfrak{t})$-invariant subspace $\mathfrak{m}$ complementary to $\mathfrak{l}$ and write the Lie algebra $\mathfrak{g}$ in the form

$$\mathfrak{g} = \mathfrak{l} + \mathfrak{m} = \mathfrak{l} + \mathbb{R} p + \mathbb{R} q + E_0 + E'$$

such that $j(d) = p \wedge q + C_0$,

$$C_{\mathfrak{g}}(\mathfrak{t}) = \mathbb{R} d + Z(\mathfrak{t}) + \mathbb{R} p + \mathbb{R} q + E_0,$$

and $j(\mathfrak{t})E' = E'$. Since $j(d)p$, $j(d)q \in C_{\mathfrak{g}}(\mathfrak{t})$, we have

$$j(d)p = p + \alpha d + z_1, \quad j(d)q = -q + \beta d + z_2, \quad z_1, z_2 \in Z(\mathfrak{t}).$$

Taking $p' := p + \alpha d + z_1$, $q' := -q - \beta d - z_2 \in C_{\mathfrak{g}}(\mathfrak{t})$, we get the $j(l)$-invariant subspace $\mathfrak{m'} := \mathbb{R} p' + \mathbb{R} q' + E_0 + E'$ complementary to $j(l)$ and such that $j(d)p' = p'$, $j(d)q' = -q'$.
3. Admissible homogeneous manifolds

In this section we introduce a small but important modification of the notion of admissible subgroups, introduced by the first author in [A/12].

**Definition 3.1.** Let $G$ be a connected Lie group. Then

(i) A closed connected subgroup $L$ of $G$ is called admissible if the homogeneous space $M = G/L$ is almost effective and admits an invariant Lorentz metric $g$. Then $M = G/L$ is said to be an admissible homogeneous space.

(ii) An admissible subgroup $L \subset G$ is said to be maximal admissible and then the manifold $M = G/L$ is said to be a minimal admissible, if there is no larger almost effective admissible subgroup $L \supset L$ with $\dim \tilde{L} > \dim L$.

(iii) The Lie algebra $l = \text{Lie}(L)$ of a (maximal) admissible subgroup $L \subset G$ is called a (maximal) admissible subalgebra of $g$.

**Remark 3.2.** Definition 3.1 provides a modification of the notion of a minimal admissible homogeneous manifold given in [A/12], where the effectivity condition of the action of $G$ on $G/\tilde{L}$ is not required. This modification is useful by the following reason. Let $M_1 = G/L_1$ be an almost effective minimal admissible manifold. Consider the direct product of $M_1$ with the sphere, $M = G/L = G_1/L_1 \times SO(m + 1)/SO(m)$. Then $M$ is a minimal admissible manifold. However, it is not a minimal admissible manifold in the sense of [A/12]. Indeed, $L_1 \times SO(m + 1)$ is an admissible (but not locally effective) subgroup of $G = L_1 \times SO(m + 1)$, which contains $L = L_1 \times SO(m)$. Below we will construct a large new series of minimal admissible homogeneous manifolds $G/L$ of compact semisimple Lie groups $G$.

3.1. Admissible homogeneous manifolds of reductive non-semisimple Lie groups. Let $M = G/L$ be a homogeneous manifold of a reductive Lie group $G = G^s \cdot Z(G)$, where $G^s$ is the semisimple part of $G$, $Z(G)$ is the connected non-trivial center of $G$, and $L \subset G$ is a reductive subgroup. Then the Lie algebra $g = g^s + Z(g)$ is a direct sum of the semisimple ideal $g^s$ and the center $Z(g)$ of $g$. Due to the effectivity it holds $l \cap Z(g) = 0$. Denote by 

\[
pr_{g^s} : l \rightarrow l_s := pr_{g^s}(l) \subset g^s
\]

the natural projection of $l$ to $g^s$, which is an isomorphism between the Lie algebras $l$ and $l_s$. The adjoint actions of $l$ and $l_s$ on $g$ coincide. Since $l_s$ is reductive, there is a reductive decomposition $g^s = l_s + m_s$ of $g^s$. This gives the reductive decomposition

\[
g = l + m = l + (m_s + Z(g)).
\]

The centralizer $m^l$ is given by

\[
m^l = m_s^l + Z(g) \neq 0.
\]

As a consequence we obtain the following.

**Theorem 3.3.** Let $M = G/L$ be a homogeneous manifold of a reductive group $G = G^s \cdot Z(G)$ with a non-trivial connected center $Z(G)$ and a reductive stability subgroup $L$, and let $M^s = G^s/L_s$ be the associated homogeneous manifold of the semisimple Lie group $G^s$ described above. Then any invariant Riemannian or Lorentzian metric on $M^s = G^s/L_s$ may be extended to an invariant Lorentz metric on $M = G/L$.

**Proof.** Let $g = l + (m_s + z)$ be a reductive decomposition as above. Then any $j(L)$-invariant Euclidean or Lorentzian inner product $g_m$ in $m_s$ is extended to the $j(L)$-invariant Lorentzian inner product $g_m = g_{m_s} \oplus g_{Z(g)}$ in $m$, where $g_{Z(g)}$ is any Lorentzian or Euclidean inner product in $Z(g)$.
3.2. Proper admissible homogeneous manifolds. Let \( M = G/L \) be a proper homogeneous manifold, i.e., the stabilizer \( L \) is compact. We fix a reductive decomposition and identify the isotropy representation \( j : L \to \text{GL}(T_0M) = \text{GL}(m) \) with the restriction \( \text{Ad}_L|_m \) to \( m \) of the adjoint representation \( \text{Ad}_L|_g \). Since the compact linear group \( \text{Ad}_L \subseteq \text{GL}(g) \) is totally reducible, there exists an \( \text{Ad}_L \) invariant complement \( m \) to \( l \) in \( g \). The homogeneous space \( M \) admits an invariant Riemannian metric \( g_m \), which is determined by an \( j(L) \)-invariant Euclidean metric \( g_m \) on \( m \). Such a metric \( g_m \) can be defined as the barycentre of the compact orbit \( j(L)^*g_0 \subset \text{Met}(m) \) of the natural action of the group \( j(L) \) in the convex cone \( \text{Met}(m) \) of Euclidean inner products on \( m \), where \( g_0 \) is a fixed Euclidean metric. Assume that \( j(L) \) preserves a line \( \mathbb{R}Z \subset m \) on \( m \).

Then, for sufficiently big number \( \lambda \),

\[ g_\lambda := g_m - \lambda Z^* \otimes Z^*, \quad Z^* := g_m \circ Z \]
defines an \( j(L) \)-invariant Lorentzian scalar product, which defines an invariant Lorentz metric on \( M = G/L \). Conversely, any invariant Lorentzian metric can be obtained by this construction.

**Proposition 3.4.** ([A/12]) Let \( M = G/L \) be a homogeneous manifold with compact connected stability subgroup \( L \). The manifold \( M = G/L \) is admissible if and only if the centralizer \( m^l \) is non-trivial. Moreover, let \( g_m \) be an \( j(L) \)-invariant Euclidean metric in \( m \), and \( 0 \neq Z \in m^l \). Then for sufficiently large number \( \lambda \), the formula \( g_\lambda := g_m - \lambda Z^* \otimes Z^* \), where \( Z^* := g_m \circ Z \), defines a Lorentzian \( j(L) \)-invariant inner product on \( m \), which is extended to a \( G \)-invariant Lorentzian metric on \( M \).

**Corollary 3.5.** If \( M = G/L \) is an admissible homogeneous manifold with compact stabilizer, then any closed subgroup \( L' \subset L \) is admissible, that is, the manifold \( M' = G/L' \) admits a \( G \)-invariant Lorentzian metric.

**Proof.** Denote by \( l = l' + q \) a reductive decomposition for \( L/L' \). Then \( g = l' + m' = l' + (q + m) \) is a reductive decomposition for \( G/L' \) and the action of \( G \) on \( G/L' \) is almost effective. It holds \( C_{m'}(l') \supseteq C_m(l) \). Then, by Proposition 3.4 it follows that \( M' \) is an admissible homogeneous manifold of \( G \).

3.3. Condition for minimal admissibility. Next our aim is to describe sufficient conditions for a homogeneous manifold \( M = G/L \) of a reductive group \( G \) with compact stability subgroup \( L \) to be a minimal admissible homogeneous space.

**Lemma 3.6.** (Key lemma) Let \( M = G/L \) be a minimal admissible homogeneous manifold of a reductive Lie group \( G \) with a compact stabilizer \( L \). Then, the reductive decomposition can be chosen in such a way that

\[ g = l + m = l + m^l + m', \quad m' = [l, m] \]

and \( m^l \) is a reductive subalgebra of \( g \). Moreover, there are three possibilities:

(a) \( m^l = \mathbb{R}Z \) is a compact 1-dimensional Lie algebra which is not in the center of \( g \), and \( C_g(Z) = l + \mathbb{R}Z \).

(b) \( m^l = s \) is a rank-one regular simple Lie algebra, i.e., \( \text{sp}(1) \) or \( \text{sl}(2, \mathbb{R}) \), such that \( C_g(s) = l \).

For any element \( Z \in s \) which defines a compact subalgebra \( \mathbb{R}Z \), it holds \( C_g(Z) = l + \mathbb{R}Z \).

(c) \( m^l = \mathbb{R}^k \) is a commutative subalgebra which does not contain any compact non-central subalgebra \( \mathbb{R}Z \).

For any subalgebra of type (a) and (b), the normalizer \( N_g(m^l) = l \oplus m^l \) of the subalgebra \( m^l \) is a reductive subalgebra of \( g \) of maximal rank.
Proof. Let \( \mathfrak{g} = \mathfrak{l} + \mathfrak{m} \) be a reductive decomposition associated to the manifold \( M \). It is clear that for the centralizer \( C_\mathfrak{g}(\mathfrak{l}) \) of the stability subalgebra \( \mathfrak{l} \) in \( \mathfrak{g} \) it holds \( C_\mathfrak{g}(\mathfrak{l}) = \mathbb{Z}(\mathfrak{l}) + \mathfrak{m}' \), where \( \mathfrak{m}' = C_\mathfrak{g}(\mathfrak{m}) \) is non-trivial by Proposition 3.4. Since the centralizer of a compact Lie algebra in a reductive Lie algebra is reductive, \( C_\mathfrak{g}(\mathfrak{m}) \) is a reductive Lie algebra. Since \( \mathbb{Z}(\mathfrak{l}) \subset C_\mathfrak{g}(\mathfrak{m}) \) is an ideal, there is a complementary reductive ideal, which we denote again by \( \mathfrak{m}' \). We may extend \( \mathfrak{m}' \) to an \( \text{adj} \)-invariant subspace \( \mathfrak{m} \) complementary to \( \mathfrak{l} \). Then, the new reductive decomposition \( \mathfrak{g} = \mathfrak{l} + \mathfrak{m} \) satisfies the desired properties.

Now we prove that if the reductive Lie algebra \( \mathfrak{m}' \) has a compact non-central subalgebra \( \mathbb{R} Z \), then it has rank one. Indeed, the Lie subalgebra \( \mathfrak{i} : = \mathfrak{l} + \mathbb{R} Z \) is compact and effective if \( Z \) is not in the center of \( \mathfrak{g} \). If \( \text{rk}(\mathfrak{m}') > 1 \), then there is an element \( Z' \in \mathfrak{m}' \) non-proportional to \( Z \), which commutes with \( Z \), and hence with \( \mathfrak{i} \). Then, \( \mathfrak{i} \) generates a compact admissible subgroup of \( G \), a fact which contradicts the maximality of \( L \). Therefore, if the reductive Lie algebra \( \mathfrak{m}' \) has a compact non-central subalgebra \( \mathbb{R} Z \), then it is either isomorphic to \( \mathbb{R} \) or to a simple rank-one subalgebra, and hence to \( \mathfrak{sp}(1) \) or to \( \mathfrak{sl}(2, \mathbb{R}) \). If there is no compact non-central subalgebra \( \mathbb{R} Z \), then the reductive subalgebra \( \mathfrak{m}' \) is commutative.

Let us now check that if \( \mathbb{R} Z \subset \mathfrak{m}' \) is a compact non-central subalgebra, then the normalizer satisfies \( N_\mathfrak{g}(Z) = \mathfrak{l} + \mathbb{R} Z \). The normalizer can be decomposed as
\[
N_\mathfrak{g}(Z) = \mathfrak{l} + \mathbb{R} Z + C_\mathfrak{m}'(Z),
\]
where \( \mathfrak{m}' = [\mathfrak{i}, \mathfrak{m}] \) is the complementary to \( \mathfrak{m}' \) subspace of \( \mathfrak{m} \). If \( C_\mathfrak{m}'(Z) \neq 0 \), then the compact subgroup \( \hat{L} \) generated by \( \mathfrak{i} = \mathfrak{l} + \mathbb{R} Z \) will be admissible, which is impossible. \( \square \)

4. Minimal compact homogeneous Lorentzian manifolds with compact stabilizer

Here we describe compact homogeneous Lorentzian manifolds \( M = G/L \) with compact stabilizer. Note that for a compact group \( G \) there is no minimal admissible manifolds \( M = G/L \) of Type Ic. Hence, below we will discuss only the homogeneous manifolds of Type Ia and Ib, and extend the results of [A/12].

4.1. Type Ia. The classification problem for the Type Ia homogeneous manifolds \( M = G/L \) of a compact semisimple group \( G \) reduces to the description of all effective reductive decompositions of a compact semisimple Lie algebra \( \mathfrak{g} \) of the form
\[
\mathfrak{g} = \mathfrak{l} + \mathfrak{m} = \mathfrak{l} + (\mathbb{R} Z + \mathfrak{m'}),
\]
where \( \mathfrak{h} = \mathfrak{l} + \mathbb{R} Z \) (direct sum) is the centralizer of a non-trivial element \( Z \in \mathfrak{m} \), that is, \( \mathfrak{h} = C_\mathfrak{g}(Z) \), such that the subalgebra \( \mathbb{R} Z \) is compact. Without loss of generality, we may assume that \( Z \) is \( B \)-orthogonal to \( \mathfrak{l} \) and \( B(Z, Z) = -1 \), where we denote by \( B \) the Killing form. In [A/S03] the vector \( Z \in \mathfrak{g} \) is called the contact element, and it is shown that it generates a closed subgroup \( T_2^1 \).

Let \( H = C_G(T_2^1) \) be the (closed) subgroup of \( G \), generated by \( \mathfrak{h} \). It is the centralizer of the 1-parameter subgroup \( T_2^1 = \exp \mathbb{R} Z \). Then the homogeneous space \( F = G/H \) is a so-called (generalized) flag manifold. It is well-known that in this case \( H \) is connected. Since \( \mathfrak{h} = \mathfrak{l} + \mathbb{R} Z \), the group \( H = L \cdot T_2^1 \) is a product of the normal subgroups \( L, T_2^1 \) and the intersection \( L \cap T_2^1 \) is a finite cyclic normal central subgroup. Hence the natural projection
\[
\pi : M = G/L \to F = G/H
\]
is a circle bundle over the flag manifold \( F \). Moreover, the \( \text{Ad}_L \)-invariant subspace \( \mathfrak{m}' \) defines the invariant contact structure \( \mathcal{D} \subset TM \) on \( M = G/L \). The corresponding invariant contact 1-form \( \theta \) is the extension of the 1-form \( \theta_0 := B \circ Z \).
**Definition 4.1.** [AfS03] The manifold \( M = G/L \) which is the total space of the principal circle bundle \( \pi : M = G/L \rightarrow F = G/H \) over the flag manifold \( F = G/H \), endowed with the invariant contact structure \( D \subset TM \) associated to the subspace \( m' \) of (4.1), is called a **standard homogeneous contact manifold**. Moreover, \( D \) is referred to as the **standard homogeneous contact structure**.

For a given compact semisimple Lie group \( G \), standard homogeneous contact manifolds \( M = G/L \) are in one-to-one correspondence with elements \( Z \in \mathfrak{g} \) (defined up to a scaling), generating closed 1-parameter subgroups \( T^1_Z = \exp \mathbb{R}Z \) of \( G \). As above, denote by \( \mathfrak{h} = \mathfrak{C}_g(Z) = \mathfrak{l} + \mathbb{RZ} \) the centralizer of \( Z \) in \( \mathfrak{g} \), where \( \mathfrak{l} \) is the \( B \)-orthogonal complement of \( \mathbb{RZ} \) in \( \mathfrak{h} \). In addition, consider the \( B \)-orthogonal standard decomposition (4.1) associated to \( Z \). In such terms one can prove the following

**Theorem 4.2.** [AfS03] The subalgebras \( \mathfrak{l} \) and \( \mathfrak{h} = \mathfrak{l} + \mathbb{RZ} \) (direct sum) generate closed subgroups \( L \) and \( H \) of \( G \), and the natural \( G \)-equivariant projection \( \pi : M = G/L \rightarrow F = G/H \) is a principal \( T^1_Z \)-bundle over the flag manifold \( F = G/H \), with the invariant contact structure associated to the subspace \( m' \). The action of \( G \) on \( M = G/L \) is effective if \( G \) has trivial center.

Now we are ready to present the theorem which describes all compact homogeneous Lorentzian manifolds \( M = G/L \) of **Type la**.

**Theorem 4.3.** Let \( G \) be a semisimple compact Lie group. Then all minimal admissible homogeneous \( G \)-manifolds are exhausted by the standard homogeneous contact manifolds \( M = G/L \), which are the circle bundles \( \pi : G/L \rightarrow F = G/H \) over flag manifolds associated to the standard decomposition (4.1). Let \( g_{m'} \) be any \( \text{ad}_l \)-invariant Euclidean metric on \( m' \). Then any invariant Lorentzian metric on \( M = G/L \) is the invariant extension of the pseudo-Euclidean metric on \( T^1_M = m = \mathbb{RZ} + m' \) of the form \( g_m = -\lambda Z^* \otimes Z^* + g_{m'} \), where \( Z^* := B \circ Z \) and \( \mathbb{R} \ni \lambda > 0 \).

**Proof.** It is sufficient to show that \( M = G/L \) is a minimal admissible homogeneous manifold. It follows from the relation \( C_{g}(\mathfrak{l}) = \mathfrak{l} + \mathbb{RZ} \), which implies \( [\mathfrak{l}, m'] = m' \) and \( C_{g}(Z) = \mathfrak{l} + \mathbb{RZ} \). Using these relations, one can check that any admissible Lie algebra

\[
\tilde{\mathfrak{l}} = \tilde{\mathfrak{l}} \cap \mathbb{RZ} + \tilde{\mathfrak{l}} \cap m'
\]

coincides with \( \mathfrak{l} \). This completes the proof. \( \square \)

4.2. **Type lb.** Now we give a classification of minimal admissible homogeneous manifolds \( M = G/L \) of **Type lb** with simple compact \( G \). Each such manifold has a reductive decomposition of the form

\[
g = \mathfrak{l} + m = \mathfrak{l} + (\mathfrak{sp}(1) + m')
\]

such that

\[
m' = \mathfrak{sp}(1), \quad C_{g}(\mathfrak{sp}(1)) = \mathfrak{l} + \mathfrak{sp}(1)
\]

and

\[
C_{g}(Z) = \mathfrak{l} + \mathbb{RZ}, \quad \text{for any non-zero } Z \in \mathfrak{sp}(1).
\]

It is easy to see that for any non-zero \( Z \in \mathfrak{sp}(1) \), the \( B \)-orthogonal complement \( \mathfrak{d} = Z^\perp \cap m \) defines an invariant contact structure on \( M = G/L \). In particular, the manifold \( M = G/L \) admits more than one invariant contact structures. According to [AfS03] such homogeneous manifolds \( M = G/L \) are called **homogeneous contact manifolds of special type**. It is proven that all such manifolds are exhausted by homogeneous spaces of the form \( M = G/C_G(\mathfrak{sp}(1)) \), that is, quotients of a compact simple Lie group modulo the centralizer of a regular 3-dimensional subalgebra \( \mathfrak{sp}(1) \). We get the following
Proposition 4.4. Minimal admissible homogeneous manifolds $M = G/L$ of a compact simple Lie group $G$ of Type I b are exactly the contact homogeneous manifolds of special type.

The following theorem provides a classification of all such homogeneous spaces.

Theorem 4.5. [AlS03] For any simple compact Lie group $G$ different from the exceptional Lie group $G_2$ there is a unique up to isomorphism minimal admissible homogeneous manifold

$$M = G/L = G/C_G(\mathfrak{sp}(1))$$

of Type I b. It is the 3-Sasakian manifold associated to the Wolf space corresponding to $G$, i.e., the homogeneous space

$$W = G/N_G(\mathfrak{sp}(1)) = G/\mathfrak{Sp}(1) \cdot L,$$

where $\mathfrak{sp}(1) \subset \mathfrak{g}^\mathbb{C}$ is the 3-dimensional subalgebra associated to a long root of $\mathfrak{g}^\mathbb{C}$. When $G = G_2$, in addition to the just described structure, there is the quotient $G/\mathfrak{Sp}(1)$, where $\mathfrak{Sp}(1) = \mathfrak{C}_{G_2}(\mathfrak{sp}(1))$, and $\mathfrak{sp}(1)$ is the 3-dimensional subalgebra, associated to a short root.

Proof. Let

$$\mathfrak{g}^\mathbb{C} = \mathfrak{c} + \sum_{\alpha \in \mathfrak{r}} \mathfrak{g}_\alpha$$

be the root space decomposition of the complexification $\mathfrak{g}^\mathbb{C}$ of the Lie algebra $\mathfrak{g}$ with respect to a Cartan subalgebra $\mathfrak{c}$ of $\mathfrak{g}^\mathbb{C}$. We denote by $\tau$ the antiinvolution (the complex conjugation with respect to $\mathfrak{g}$), which determines the real form $\mathfrak{g}$, that is, $\mathfrak{g} = (\mathfrak{g}^\mathbb{C})^\tau$ (see [GOV94]). Then, up to conjugation, the regular Lie subalgebra $\mathfrak{s}$ is the compact form $\mathfrak{g}(\alpha)^\tau$ of the subalgebra

$$\mathfrak{g}(\alpha) := \text{span}(H_\alpha, E_\alpha, E_{-\alpha})$$

associated to a root $\alpha \in \mathfrak{r}$ of $\mathfrak{g}^\mathbb{C}$. It is well-known that all roots of the same length are conjugated. Therefore, up to conjugation, there are either one or two subalgebras $\mathfrak{g}(\alpha)$.

Assume first that $\alpha$ is a long root of a simple complex Lie algebra $\mathfrak{g}^\mathbb{C}$. Then, the dual element $H_\alpha \in \mathfrak{c}$ defines a (depth 2) $\mathbb{Z}$-grading

$$\mathfrak{g}^\mathbb{C} = \mathfrak{g}^{-2} + \mathfrak{g}^{-1} + \mathfrak{g}^0 + \mathfrak{g}^1 + \mathfrak{g}^2$$

of $\mathfrak{g}^\mathbb{C}$, where $\mathfrak{g}^j$ is the eigenspace of $\text{ad}H_\alpha$ with the eigenvalue $j$. Moreover, $\mathfrak{g}^{-2} = \mathbb{C} E_{\pm\alpha}$, and $\mathfrak{g}^0 = \mathbb{C} H_\alpha \oplus \mathfrak{f}^\mathbb{C}$, where $\mathfrak{f}^\mathbb{C} = C_{\mathfrak{g}^\mathbb{C}}(\mathfrak{s}^\mathbb{C})$. The even subalgebra is given by

$$\mathfrak{g}^{-2} + \mathfrak{g}^0 + \mathfrak{g}^2 = \mathfrak{g}(\alpha) \oplus \mathfrak{f}^\mathbb{C}.$$

It follows that the condition (4.3) holds true, and consequently $M = G/L$ is a minimal homogeneous manifold.

It remains to study the case when $\mathfrak{g}(\alpha)$ is the subalgebra associated to a short root $\alpha$. Recall that there are roots of different length only for the simple Lie groups of type $B_n, C_n, F_4, G_2$. It is easy to check that the root systems $R$ of $B_n, C_n$ and $F_4$ satisfy the following property: for a short root $\alpha \in \mathfrak{r}$ there exists an orthogonal to $\alpha$ root $\beta$ such that $\alpha + \beta$ is a root. Then, the root vector $E_\beta$ is annihilated by $\text{ad}H_\alpha$, but it is not annihilated by the operator $\text{ad}E_\alpha$. In other words, $E_\beta \in C_{\mathfrak{g}^\mathbb{C}}(H_\alpha)$, but $E_\beta \notin \mathfrak{f}^\mathbb{C} = C_{\mathfrak{g}^\mathbb{C}}(\mathfrak{g}(\alpha))$. This gives a contradiction.

However, for the exceptional Lie algebra $\mathfrak{g}_2$ one can easily check that the sum $\alpha + \beta$ of a short root $\alpha$ and any other root $\beta$ is not a root. This implies that the total spaces of both principal bundles

$$\pi_1 : G_2/\mathfrak{sp}(1)^\alpha \rightarrow W = G_2/\mathfrak{sp}(1)^\alpha \times \mathfrak{sp}(1)^\beta,$$

$$\pi_2 : G_2/\mathfrak{sp}(1)^\beta \rightarrow W = G_2/\mathfrak{sp}(1)^\alpha \times \mathfrak{sp}(1)^\beta$$

over the Wolf space $W = G_2/\mathfrak{sp}(1)^\alpha \times \mathfrak{sp}(1)^\beta$ are minimally admissible homogeneous manifolds.
Remark 4.6. Using the classification of the Wolf spaces corresponding to compact simple Lie groups (see, e.g., [Bes86]), we list for the convenience of the reader the admissible decompositions $\mathfrak{g} = \mathfrak{l} + (\mathfrak{sp}(1) + \mathfrak{m}')$ corresponding to the minimal admissible homogeneous spaces associated to the Theorem 4.5:

$$
\begin{align*}
\mathfrak{su}(p+2) &= (\mathfrak{R} \mathfrak{l} + \mathfrak{su}(p)) + (\mathfrak{su}(2) + \mathfrak{C}^p \otimes \mathfrak{C}^2), \quad \mathfrak{l} = \text{diag} (2\mathfrak{i}E_p, -\mathfrak{p}E_2) \\
\mathfrak{so}(p+4) &= (\mathfrak{so}(p) + \mathfrak{so}(3)) + (\mathfrak{so}(3) + \mathfrak{R}^p \otimes \mathfrak{R}^4), \\
\mathfrak{sp}(p+1) &= \mathfrak{sp}(p) + (\mathfrak{sp}(1) + \mathfrak{H}^p), \\
\mathfrak{c}_6 &= \mathfrak{su}(6) + (\mathfrak{su}(2) + \wedge^3 \mathfrak{C}^6 \otimes \mathfrak{C}^2), \\
\mathfrak{c}_7 &= \mathfrak{so}(12) + (\mathfrak{su}(2) + \Delta_{12} \otimes \mathfrak{C}^2), \\
\mathfrak{c}_8 &= \mathfrak{c}_7 + (\mathfrak{su}(2) + \wedge^2 \mathfrak{R}^8 \otimes \mathfrak{C}^2), \\
\mathfrak{f}_4 &= \mathfrak{sp}(3) + (\mathfrak{su}(2) + \Delta^3 \mathfrak{H}^6 \otimes \mathfrak{C}^2), \\
\mathfrak{g}_2 &= \mathfrak{su}(2) + (\mathfrak{su}(2) + \otimes^3 \mathfrak{C}^2 \otimes \mathfrak{C}^2), \\
\mathfrak{g}_2 &= \mathfrak{su}(2) + (\mathfrak{su}(2) + \mathfrak{C}^2 \otimes \otimes^3 \mathfrak{C}^2),
\end{align*}
$$

where $U \otimes V$ denotes denote the highest irreducible component of $U \otimes V$, and $\Delta$ denotes the irreducible complex spin module.

5. Homogeneous non-compact Lorentzian manifolds of semisimple Lie groups with compact stabilizers

Let $G$ be a semisimple non-compact Lie group. We are interested in a description of homogeneous Lorentzian manifolds $M = G/L$ with compact stabilizers $L$. We may reduce their study to the description of all minimal admissible homogeneous manifolds $M = G/L$ of this type, or equivalently, of all maximal admissible Lie subgroups $L$ of the group $G$. We consider only manifolds of Type Ia and Ib. Note that there also exist manifolds of Type Ic for non-compact semisimple Lie groups.

5.1. Type Ia. The study of homogeneous space of this type reduces to the description of all reductive decompositions having the form

$$
\mathfrak{g} = \mathfrak{l} + \mathfrak{R} \mathfrak{Z} + \mathfrak{m}',
$$

where $\mathfrak{h} := \mathfrak{l} + \mathfrak{R} \mathfrak{Z} = C_0(\mathfrak{Z})$, and $[\mathfrak{h}, \mathfrak{m}'] = \mathfrak{m}'$, and hence to a description of compact one-dimensional subalgebras $\mathfrak{R} \mathfrak{Z}$ with compact centralizer $\mathfrak{h}$. Let

$$
\mathfrak{g} = \mathfrak{k} + \mathfrak{p}
$$

be a Cartan decomposition such that $\mathfrak{h} \subset \mathfrak{k}$. We fix a Cartan subalgebra $\mathfrak{c}_k$ which contains $\mathfrak{Z}$ and extend it to a Cartan subalgebra $\mathfrak{c} := \mathfrak{c}_k + \mathfrak{c}_p$ of $\mathfrak{g}$. If the relation (5.1) holds, then $\mathfrak{c}_p = 0$ and any compact subalgebra $\mathfrak{R} \mathfrak{Z}$ with $C_0(\mathfrak{Z}) = \mathfrak{h}$ defines a maximal admissible subalgebra $\mathfrak{l}$, which is the $B$-orthogonal complement to $\mathfrak{Z}$ in $\mathfrak{h}$. The condition $\mathfrak{c}_p = 0$ implies $\text{rk}(\mathfrak{k}) = \text{rk}(\mathfrak{g})$. First we consider the classical simple real Lie algebras with a compact Cartan subalgebra $\mathfrak{c} \subset \mathfrak{k}$, with aim to describe all elements $\mathfrak{Z} \in \mathfrak{c}$ having trivial centralizers $C_0(\mathfrak{Z}) = 0$. Then, $\mathfrak{l}$ will be the orthogonal complement of $\mathfrak{Z}$ in $C_0(\mathfrak{Z}) = 0$, $\mathfrak{m}'$ will be the orthogonal complement to $\mathfrak{l}$ in $\mathfrak{g}$, and moreover $\mathfrak{m} = \mathfrak{R} \mathfrak{Z} + \mathfrak{m}'$ (direct sum). We use these facts, to obtain a description for all classical semisimple real Lie algebras.
Remark 5.1. First it is convenient to pose the Cartan decompositions \( g = \mathfrak{k} + \mathfrak{p} \) of all classical real simple non-compact Lie algebras \( g \) with \( \text{rk} \mathfrak{k} = \text{rk} \mathfrak{g} \) (we follow [GOV94]).

\[
\begin{align*}
\mathfrak{su}(p, q) &= \mathfrak{s}(\mathfrak{u}(p) + \mathfrak{u}(q)) + \mathbb{C}^p \otimes \mathbb{C}^q, \quad p, q \geq 1, \\
\mathfrak{so}(p, q) &= \mathfrak{s}(\mathfrak{so}(p) + \mathfrak{so}(q)) + \mathbb{R}^p \otimes \mathbb{R}^q, \quad p, q \geq 1, \quad pq \text{ is even}, \\
\mathfrak{sp}(2n, \mathbb{R}) &= \mathfrak{u}(n) + S^2 \mathbb{C}^n, \quad n \geq 1, \\
\mathfrak{sp}(p, q) &= \mathfrak{s}(\mathfrak{sp}(p) + \mathfrak{so}(q)) + \mathbb{H}^p \otimes \mathbb{H}^q, \quad p, q \geq 1, \\
\mathfrak{so}(n, \mathbb{H}) &= \mathfrak{u}(n) + \wedge^2 \mathbb{C}^n, \quad n \geq 1.
\end{align*}
\]

Now we are ready to consider these algebras case-by-case.

- Let \( g = \mathfrak{su}(p, q), \ p, q \geq 1 \). Then
  \[
  g = \mathfrak{k} + \mathfrak{p} = \mathfrak{s}(\mathfrak{u}(p) + \mathfrak{u}(q)) + \mathbb{C}^p \otimes \mathbb{C}^q,
  \]
  where for a subalgebra \( \mathfrak{f} \subset \mathfrak{u}(m) \), \( s(\mathfrak{f}) \) denotes the intersection \( \mathfrak{f} \cap \mathfrak{su}(m) \). The Lie algebra \( \mathfrak{su}(p, q) \) consists of the complex matrices
  \[
  \begin{pmatrix}
  X_1 & Y \\
  Y^T & X_2
  \end{pmatrix},
  \quad
  X_1^T = -X_1, \quad X_2^T = -X_2, \quad \text{tr} X_1 + \text{tr} X_2 = 0,
  \]
  here \( X_1, X_2, \) and \( Y \) are blocks of the size \( p \times p, q \times q, \) and \( p \times q, \) respectively. Elements of \( \mathfrak{k} \) are given by the matrices with \( Y = 0 \), and elements of \( \mathfrak{p} \) are given by the matrices with \( X_1 = X_2 = 0 \). The standard Cartan subalgebra \( \mathfrak{c} = \mathfrak{c}_p \) consists of elements of \( \mathfrak{k} \) with diagonal matrices \( X_1, X_2 \).
  
  Fix some \( Z \in \mathfrak{c} \) and let
  \[
  \mathbb{C}^p = \bigoplus_{i=1}^r V_i, \quad \mathbb{C}^q = \bigoplus_{\alpha=1}^s U_{\alpha}
  \]
  be the eigenspace decompositions of the endomorphisms \( Z|_{\mathbb{C}^p} \) and \( Z|_{\mathbb{C}^q} \), respectively. Then
  \[
  Z = \sum_{j=1}^r b_j I_j + \sum_{\alpha=1}^s c_\alpha I'_\alpha,
  \]
  where \( b_j, c_\alpha \in \mathbb{R} \), and \( I_j \) is an endomorphism acting on \( V_j \) as the multiplication by \( i \) and annihilating the orthogonal complement to \( V_j \); the endomorphisms \( I'_\alpha \) are defined in a similar way. Since \( Z \in \mathfrak{su}(p, q) \), it holds
  \[
  \sum_{j=1}^r b_j \dim \mathbb{C} V_j + \sum_{\alpha=1}^s c_\alpha \dim \mathbb{C} V_\alpha = 0.
  \]
  Since \( \exp(tX) \) is compact, the numbers \( b_1, \ldots, b_r, c_1, \ldots, c_s \) are commensurable, and we may assume that these numbers are rational. The condition \( C_{\mathfrak{w}}(Z) = 0 \) implies \( C_{\mathfrak{p}}(Z) = 0 \). Let an element \( A \in \mathfrak{p} \) be given by the matrix \( Y = E_{i\alpha} \), where \( E_{i\alpha} \) is the matrix with \( 1 \) at the position \( (i, \alpha) \) and zeros in the rest entries. Then \( [Z, A] \) is given by the matrix \( (b_j - c_\alpha) E_{j\alpha} \). This implies that the condition \( C_{\mathfrak{p}}(Z) = 0 \) is equivalent to the condition
  \[
  b_j \neq c_\alpha, \quad 1 \leq j \leq r, \quad 1 \leq \alpha \leq s.
  \]
  Now we see that
  \[
  C_{\mathfrak{k}}(Z) = s\left( \bigoplus_{i=1}^r u(V_i) + \bigoplus_{\alpha=1}^s u(U_\alpha) \right)
  \]
  and
  \[
  I = \bigoplus_{i=1}^r s\mathfrak{u}(V_i) + s\mathfrak{u}(U_\alpha) + \hat{I},
  \]
  where \( \hat{I} \) is the orthogonal complement to \( Z \) in \( s(\langle I_1, \ldots, I_r, I'_1, \ldots, I'_s \rangle) \) with respect to the Killing form. Finally we obtain the decomposition
  \[
  \mathfrak{g} = I + m = I + (\mathbb{R}Z + m'),
  \]
where
\[ m' = p + \sum_{1 \leq j < k \leq r} V_j \otimes_C V_k + \sum_{1 \leq \alpha < \beta \leq s} U_\alpha \otimes_C U_\beta. \]

- Let \( \mathfrak{g} = \mathfrak{so}(p, q) \), \( p, q \geq 1 \). Then
\[ \mathfrak{g} = \mathfrak{t} + p = (\mathfrak{so}(p) + \mathfrak{so}(q)) \oplus \mathbb{R}^p \oplus \mathbb{R}^q. \]
The condition \( \text{rk}(\mathfrak{g}) = \text{rk}(\mathfrak{t}) \) is fulfilled whenever at least on of the numbers \( p, q \) is even. Fix orthonormal bases \( e_1, \ldots, e_p \) and \( f_1, \ldots, f_q \) of the spaces \( \mathbb{R}^p \) and \( \mathbb{R}^q \), respectively. The Cartan subalgebra \( \mathfrak{c} \) of \( \mathfrak{so}(p, q) \) is the direct sum of the Cartan subalgebras of \( \mathfrak{so}(p) \) and \( \mathfrak{so}(q) \). The Cartan subalgebra of \( \mathfrak{so}(p) \) consists of the elements
\[ e_1 \wedge e_2, \ldots, e_{[\frac{p}{2}]} \wedge e_{[\frac{p}{2}]-1} \wedge e_{[\frac{p}{2}]+1}. \]

A similar structure has the Cartan subalgebra of \( \mathfrak{so}(q) \). For a given element \( Z \in \mathfrak{c} \) we may assume that the basis is adapted to the canonical form of the element \( Z \), i.e., there are decompositions
\[ \mathbb{R}^p = \bigoplus_{i=1}^r V_i, \quad \mathbb{R}^q = \bigoplus_{\alpha=0}^s U_\alpha \]
such that
\[ Z = \sum_{j=1}^r b_j I_j + \sum_{\alpha=1}^s c_\alpha I'_{\alpha}, \]
where \( b_j, c_\alpha \in \mathbb{R} \), and \( I_j, I'_{\alpha} \) are complex structures on \( V_j, U_\alpha \), respectively. As in the previous case, we may assume that the numbers \( b_1, \ldots, b_r, c_1, \ldots, c_s \) are rational; the condition \( c_p(Z) = 0 \)
implies
\[ b_j \neq c_\alpha, \quad 1 \leq j \leq r, \quad 1 \leq \alpha \leq s, \]
and one of the spaces \( V_0 \) or \( U_0 \) is trivial. We assume that \( U_0 = 0 \). Moreover, if \( q \) is odd, then \( V_0 = 0 \). It is clear that
\[ C_\mathfrak{t}(Z) = \mathfrak{so}(V_0) + \bigoplus_{i=1}^r u(V_i) + \bigoplus_{\alpha=1}^s u(U_\alpha), \]
and
\[ \mathfrak{I} = \mathfrak{so}(V_0) + \bigoplus_{i=1}^r \mathfrak{su}(V_i) + \bigoplus_{\alpha=1}^s \mathfrak{su}(U_\alpha) + \mathfrak{I}, \]
where \( \mathfrak{I} \) is the orthogonal complement to \( Z \) in \( \langle I_1, \ldots, I_r, I'_1, \ldots, I'_s \rangle \) with respect to the Killing form. Recall that there is a 2-grading
\[ \mathfrak{so}(2m) = \mathfrak{u}(m) + \wedge^2 \mathbb{C}^m. \]
This implies that
\[ \mathfrak{g} = \mathfrak{I} + \mathfrak{m} = \mathfrak{I} + (\mathbb{R} \mathfrak{Z} + \mathfrak{m'}), \]
where
\[ \mathfrak{m'} = p + \sum_{j=1}^r \wedge^2 V_j + \sum_{\alpha=1}^s \wedge^2 U_\alpha + \sum_{0 \leq j < k \leq r} V_j \otimes V_k + \sum_{1 \leq \alpha < \beta \leq s} U_\alpha \otimes U_\beta. \]
- Let \( \mathfrak{g} = \mathfrak{sp}(2n, \mathbb{R}) \), \( n \geq 1 \). Then
\[ \mathfrak{g} = \mathfrak{t} + p = \mathfrak{u}(n) + S^2 \mathbb{C}^n. \]
The Lie algebra \( \mathfrak{sp}(2n, \mathbb{R}) \) consists of the real matrices
\[ \begin{pmatrix} X & Y_1 \\ Y_2 & -X^T \end{pmatrix}, \quad Y_1^T = Y_1, \quad Y_2^T = Y_2, \]
here \( X, Y_1, \) and \( Y_2 \) are square \( n \times n \) matrices. Elements of \( \mathfrak{t} \) are given by the matrices with \( X^T = -X \) and \( Y_2 = -Y_1 \), while elements of \( p \) are given by the matrices with \( X^T = X \) and \( Y_2 = Y_1 \). The standard Cartan subalgebra \( \mathfrak{c} = \mathfrak{c}_p \) consists of the elements of \( \mathfrak{t} \) given by diagonal matrices \( Y_1 \). Let \( Z \in \mathfrak{c} \) be given by the matrix \( Y_1 = \text{diag}(z_1, \ldots, z_n) \). The condition \( C_\mathfrak{p}(Z) = 0 \)
implies easily that the numbers $z_i$ are non-zero and pairwise different. The center $C_{\ell}(Z)$ coincides with $c$. Then $l$ is made up of the elements from $c$ given by $Y = \text{diag}(y_1, \ldots, y_n)$ satisfying the condition $y_1z_1 + \cdots + y_nz_n = 0$. Finally,
\[ g = I + m = I + (\mathbb{R}Z + m'), \]
where
\[ m' = p + \left\{ \begin{pmatrix} X & Y \\ -Y & X \end{pmatrix} \in \mathfrak{u}(n)|Y \text{ has zeros on the diagonal} \right\}. \]

- Let $g = \mathfrak{sp}(p, q)$, $p, q \geq 1$. Then
\[ g = \mathfrak{k} + p = (\mathfrak{sp}(p) + \mathfrak{so}(q)) + \mathbb{H}^p \otimes \mathbb{H}^q. \]
The elements of the Lie algebra $\mathfrak{sp}(p, q)$ may be identified with the quaternionic matrices
\[ \begin{pmatrix} X_1 & Y \\ Y^T & X_2 \end{pmatrix}, \quad X_1^T = -X_1, \quad X_2^T = -X_2, \]
here $X_1$, $X_2$, and $Y$ are blocks of the size $p \times q$, $q \times q$, and $p \times q$, respectively. Elements of $\mathfrak{k}$ are given by the matrices with $Y = 0$, and elements of $p$ are given by the matrices with $X_1 = X_2 = 0$. The standard Cartan subalgebra $\mathfrak{c} = \mathfrak{c}_p$ consists of the elements of $\mathfrak{k}$ with diagonal matrices $X_1, X_2$ with imaginary complex numbers on the diagonal. Let $Z \in \mathfrak{c}$. It is given by the diagonal matrix
\[ \text{diag}(0E_{p_0}, z_1iE_{p_1}, \ldots, z_r iE_{p_r}, 0E_{q_0}, z'_1 iE_{q_1}, \ldots, z'_s iE_{q_s}), \]
where $z_i$, $z'_\alpha$ are real numbers; the numbers $z_1, \ldots, z_r$ are pairwise different; the same holds for the numbers $z'_1, \ldots, z'_s$. The condition $C_{\mathfrak{sp}}(Z) = 0$ implies
\[ z_j \neq z'_\alpha, \quad 1 \leq j \leq r, \quad 1 \leq \alpha \leq s, \]
and at least one of the numbers $p_0$, $q_0$ is zero. We assume that $q_0 = 0$. Recall that there is a decomposition
\[ \mathbb{H}^m = \mathbb{C}^m + j \mathbb{C}^m, \]
and it holds
\[ C_{\mathfrak{sp}}(\text{Op}(iE_m)) = \mathfrak{u}(m), \]
where $\text{Op}(iE_m)$ is an element of $\mathfrak{sp}(m)$ with the matrix $iE_m$, and $\mathfrak{u}(m)$ acts diagonally in $\mathbb{H}^m = \mathbb{C}^m + j \mathbb{C}^m$. Moreover, there is a $\mathbb{Z}$-grading
\[ \mathfrak{sp}(m) = \mathfrak{u}(m) + S^2 \mathbb{C}^m. \]
This implies that
\[ C_{\ell}(Z) = \mathfrak{sp}(p_0) + \bigoplus_{i=1}^r \mathfrak{u}(p_i) + \bigoplus_{\alpha=1}^s \mathfrak{u}(q_i) \]
and
\[ l = \mathfrak{sp}(p_0) + \bigoplus_{i=1}^r \mathfrak{su}(p_i) + \bigoplus_{\alpha=1}^s \mathfrak{su}(q_i) + \hat{\mathfrak{l}}, \]
where $\hat{\mathfrak{l}}$ is the orthogonal complement to $Z$ in the space spanned by operators corresponding to the matrices $iE_{p_1}, \ldots, iE_{p_r}, iE_{q_1}, \ldots, iE_{q_s}$. We obtain the decomposition
\[ g = I + m = I + (\mathbb{R}Z + m'), \]
where
\[ m' = p + \sum_{j=1}^r S^2 \mathbb{C}^{p_j} + \sum_{\alpha=1}^s S^2 \mathbb{C}^{q_\alpha} + \sum_{0 \leq j < k \leq r} \mathbb{H}^{p_j} \otimes \mathbb{H}^{p_k} + \sum_{1 \leq \alpha < \beta \leq s} \mathbb{H}^{q_\alpha} \otimes \mathbb{H}^{q_\beta}. \]

- Let $g = \mathfrak{so}(n, \mathbb{H})$, $n \geq 1$. Then
\[ g = \mathfrak{k} + p = \mathfrak{u}(n) + \wedge^2 \mathbb{C}^n. \]
The Lie algebra \( so(n, \mathbb{H}) \) consists of the complex matrices
\[
\left( \begin{array}{cc}
X & Y \\
-\overline{Y} & \overline{X}
\end{array} \right), \quad X^T = -X, \quad \overline{Y}^T = Y,
\]
here \( X, Y \) are square \( n \times n \) matrices. Elements of \( \mathfrak{k} \) are given by the matrices with \( \overline{X} = X \) and \( Y = Y \), while elements of \( \mathfrak{p} \) are given by the matrices with \( \overline{X} = -X \) and \( \overline{Y} = -Y \). The standard Cartan subalgebra \( \mathfrak{c} = \mathfrak{c}_p \) consists of the elements of \( \mathfrak{k} \) given by diagonal matrices \( X \) with imaginary elements at the diagonal. Let \( Z \in \mathfrak{c} \) be given by the matrix
\[
Z = \text{diag}(z_1 i, \ldots, z_n i).
\]
The condition \( C_p(Z) = 0 \) easily implies that the numbers \( z_i \) are non-zero and pairwise different. The center \( C_l(Z) \) coincides with \( \mathfrak{c} \). Then \( \mathfrak{l} \) is made up of the elements from \( \mathfrak{c} \) given by \( V = \text{diag}(v_1 i, \ldots, v_n i) \) satisfying the condition \( v_1 z_1 + \cdots + v_n z_n = 0 \). Finally,
\[
\mathfrak{g} = \mathfrak{l} + \mathfrak{m} = \mathfrak{l} + (\mathbb{R} Z + \mathfrak{m}'),
\]
where
\[
\mathfrak{m}' = \mathfrak{p} + \left\{ \left( \begin{array}{cc}
X & Y \\
-\overline{Y} & \overline{X}
\end{array} \right) \in \mathfrak{u}(n)|Y \text{ has zeros at the diagonal} \right\}.
\]

Suppose now that \( \mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_r \) is a real semisimple Lie algebra such that its simple ideals \( \mathfrak{g}_i \) are classical non-compact Lie algebras. As before consider the decomposition
\[
\mathfrak{g} = \mathfrak{l} + \mathfrak{m} = \mathfrak{l} + (\mathbb{R} Z + \mathfrak{m}').
\]
The vector \( Z \) may be represented as
\[
Z = Z_1 + \cdots + Z_r, \quad Z_i \in \mathfrak{g}_i.
\]
The maximality condition implies that all \( Z_i \) are non-zero. We immediately conclude that
\[
\mathfrak{g} = \mathfrak{l} + \mathfrak{m} = \left( \sum_{i=1}^r \mathfrak{l}_i + \mathfrak{i} \right) + \left( \mathbb{R} Z + \sum_{i=1}^r \mathfrak{m}'_i \right),
\]
where \( \mathfrak{g}_i = \mathfrak{l} + (\mathbb{R} Z_i + \mathfrak{m}'_i) \) are the just obtained decompositions, and \( \mathfrak{i} \) is the orthogonal complement to \( Z \) in \( \langle Z_1, \ldots, Z_r \rangle \).

5.2. Type Ib. In this case the problem reduces to the description of homogeneous manifolds \( M = G/L \) of non-compact simple Lie groups \( G \) with compact stabilizers \( L \), which admit a reductive decomposition of the form
\[
\mathfrak{g} = \mathfrak{l} + \mathfrak{m} = \mathfrak{l} + (\mathfrak{s} + \mathfrak{m}')
\]
where \( \mathfrak{h} = \mathfrak{l} + \mathfrak{s} \) (direct sum) is the normalizer \( N_{\mathfrak{g}}(\mathfrak{s}) \) of a 3-dimensional simple regular subalgebra isomorphic to \( \mathfrak{sp}(1) \) or \( \mathfrak{sl}(2, \mathbb{R}) \). The complexification \( \mathfrak{g}^C = \mathfrak{l}^C + \mathfrak{s}^C + (\mathfrak{m}')^C \) is the standard decomposition associated to the 3-dimensional regular subalgebra
\[
\mathfrak{s}^C = \mathfrak{g}(\alpha) = \text{span}(H_\alpha, E_{\pm \alpha})
\]
of \( \mathfrak{g}^C \), associated to a long root \( \alpha \in \mathfrak{r} \), and for the case of the Lie algebra of \( \mathfrak{g}_2 \) also to a short root, see Section 4. The complex subalgebra \( \mathfrak{g}(\alpha) \) generates a subgroup \( G(\alpha) \) of the Lie group \( G^C \), and the homogeneous manifold
\[
M^C = G^C/N_{G^C}(G(\alpha)) = G^C/H^C
\]
is the complexification of the Wolf space \( (G^C)^\tau / (H^C)^\tau \) corresponding to the compact form \( (G^C)^\tau \) of \( G^C \). The classification of such decompositions for real simple Lie algebras \( \mathfrak{g} \) reduces to the
classification of all real forms \( \mathfrak{g} \) of the complex simple Lie algebras \( \mathfrak{g}^C \), defined by an anti-involution \( \sigma \) which preserves the subalgebra \( \mathfrak{g}(\alpha) \). Then, the fixed point set \((\mathfrak{g}(\alpha))^\sigma\) of the subalgebra \( \mathfrak{g}(\alpha) \) is either the compact real form \( \mathfrak{sp}(1) \) or the non-compact real form \( \mathfrak{sl}(2, \mathbb{R}) \). Hence, our task is equivalent to the classification of pseudo-Riemannian symmetric quaternionic-Kähler manifolds \( Q = G/\mathfrak{sp}(1) \cdot L \) solved in [A/C05], and para-quaternionic Kähler symmetric manifolds \( P = G/\mathfrak{sl}(2, \mathbb{R}) \cdot L \), solved in [Ch89] and [DJS04].

Select a non-compact homogeneous manifold \( M = G/L \) with compact stabilizer \( L = C_G(A_1) \), where \( A_1 \) is isomorphic to \( \mathfrak{sp}(1) \) or to \( \mathfrak{sl}(2, \mathbb{R}) \). Assume that \( \mathfrak{g}(\alpha) \) is the 3-dimensional Lie algebra associated to a long root. In the first case the homogeneous manifold \( Q = G/H \), where \( G \) is the Lie group with the Lie algebra \( \mathfrak{g} = (\mathfrak{g}^C)^\sigma \) and \( H = L \cdot \mathfrak{sp}(1) \) is the subgroup generated by \( \mathfrak{h} = (\mathfrak{t}^C)^\sigma + (\mathfrak{g}(\alpha))^\sigma \), is a symmetric pseudo-quaternionic Kähler manifold. All such manifolds are classified in [A/C05]. In the second case, the manifold \( Q = G/H = G/L \cdot \mathfrak{sl}(2, \mathbb{R}) \) is a symmetric para-quaternionic Kähler manifold. All such manifolds are classified in [DJS04]. Now, we need the classification of all anti-involutions \( \sigma \) of the complex simple Lie algebra \( \mathfrak{g}^C \), with the standard grading

\[
\mathfrak{g}^C = \mathfrak{g}^{-2} + \mathfrak{g}^{-1} + \mathfrak{g}^0 + \mathfrak{g}^1 + \mathfrak{g}^2
\]

preserved by the grading element \( H_\mu \in \mathfrak{c} \). Such real forms had been classified in [Ch89]. To get our classification, we have to examine the list of all pseudo-Riemannian symmetric quaternionic Kähler spaces \( G/\mathfrak{sp}(1) \cdot L \) presented in [A/S03] and the list of all para-quaternionic-Kähler symmetric spaces given in [DJS04] or [Ch89], and from the stated homogeneous spaces select the manifolds with compact stabilizer \( L \).

In the first case, we deduce that \( M = G/L \) is the total space of the principal \( \mathfrak{sp}(1) \)-bundle

\[
\pi : M = G/L \to Q = G/L \cdot \mathfrak{sp}(1)
\]

which is a pseudo-Riemannian 3-Sasakian manifold. In the second case, \( M \) is the total space of the principal \( \mathfrak{sl}(2, \mathbb{R}) \)-bundle

\[
\pi : M = G/L \to Q = G/L \cdot \mathfrak{sl}(2, \mathbb{R}),
\]

which is a para-3-Sasakian manifold.

**Remark 5.2.** Note that the complex exceptional Lie group \( \mathbb{G}_2^C \) has two real forms. The normal form \( G_{2(2)} \) with maximal compact subgroup \( K = \mathfrak{sp}(1)^{sh} \cdot \mathfrak{sp}(1)^t \) corresponding to short root and orthogonal to it long root, and the compact form \( G_2 \), which defines the quaternionic-Kähler symmetric space \( G_2 / \mathfrak{sp}(1)^{sh} \cdot \mathfrak{sp}(1)^t \). Hence, there are two minimal admissible manifolds of Type I b of the non-compact group \( G_{2(2)} \), namely: \( G_{2(2)} / \mathfrak{sp}(1)^{sh} \) and \( G_{2(2)} / \mathfrak{sp}(1)^t \).

We may summarize as follows:

**Theorem 5.3.** Let \( G \) be a non-compact simple Lie group. All minimal admissible non-compact homogeneous manifolds \( M = G/L \) of Type I b corresponding to \( G \) are exhausted by pseudo-3-Sasakian manifolds \( M = G/L = G/C_G(\mathfrak{sp}(1)) \), by para-3-Sasakian manifolds \( M = G/L = G/C_G(\mathfrak{sl}(2, \mathbb{R})) \), both associated to a long root, and also by the non-compact homogeneous manifold \( G_2 / \mathfrak{sl}(2, \mathbb{R}) \), where \( \mathfrak{sl}(2, \mathbb{R}) \) is the subgroup corresponding to a simple Lie algebra associated to a short root of the split Lie algebra \( \mathfrak{g}_{2(2)} \).
Remark 5.4. Using the classification given in [A/C05], we may present the admissible decompositions \( \mathfrak{g} = \mathfrak{l} + (\mathfrak{sp}(1) + \mathfrak{m'}) \) corresponding to the homogeneous spaces of Theorem 5.3:
\[
\begin{align*}
\mathfrak{su}(p, 2) &= (\mathbb{R} \mathfrak{l} + \mathfrak{su}(p)) + (\mathfrak{su}(2) + \mathbb{C}^p \otimes \mathbb{C}^2), \quad I = \text{diag} (2iE_p, -piE_2) \\
\mathfrak{so}(p, 4) &= (\mathfrak{so}(p) + \mathfrak{so}(3)) + (\mathfrak{so}(3) + \mathbb{R}^p \otimes \mathbb{R}^4), \\
\mathfrak{sp}(p, 1) &= \mathfrak{sp}(p) + (\mathfrak{sp}(1) + \mathbb{H}^{p-1}), \\
\mathfrak{e}_{0(2)} &= \mathfrak{su}(6) + (\mathfrak{su}(2) + \mathbb{C}^3 \otimes \mathbb{C}^2), \\
\mathfrak{e}_{7(-5)} &= \mathfrak{so}(12) + (\mathfrak{su}(2) + \Delta_{12} \otimes \mathbb{C}^2), \\
\mathfrak{e}_{8(-24)} &= \mathfrak{su}(2) + (\mathfrak{su}(2) + \Delta_{24} \otimes \mathbb{C}^2), \\
\mathfrak{f}_{4(4)} &= \mathfrak{sp}(3) + (\mathfrak{su}(2) + \Delta_{12} \otimes \mathbb{C}^2), \\
\mathfrak{g}_2(2) &= \mathfrak{su}(2) + (\mathfrak{su}(2) + \mathbb{C}^2 \otimes \mathbb{C}^2), \\
\mathfrak{g}_2(3) &= \mathfrak{su}(2) + (\mathfrak{su}(2) + \mathbb{C}^2 \otimes \mathbb{C}^3).
\end{align*}
\]

Based on the results in [DJS04], we deduce that there is only one admissible decomposition \( \mathfrak{g} = \mathfrak{l} + (\mathfrak{sl}(2, \mathbb{R}) + \mathfrak{m'}) \):
\[
\mathfrak{su}(p + 1, 1) = (\mathbb{R} \mathfrak{l} + \mathfrak{su}(p)) + (\mathfrak{su}(1, 1) + \mathbb{C}^p \otimes \mathbb{C}^{1,1}), \quad I = \text{diag} (2iE_p, -piE_2).
\]

6. Homogeneous Lorentzian Manifolds with a Reductive Stabilizer of Type II and Type III

In this final section we present the classification of homogeneous Lorentzian manifolds \( G/L \) with a reductive stabilizer of Type II and Type III (under certain assumptions). Let us first treat the case corresponding to Type II stabilizers.

Theorem 6.1. Let \( M = G/L \) be a simply connected (almost) effective homogeneous Lorentz manifold with isotropy subalgebra of Type II, that is \( j_x(\mathfrak{l}) = \mathfrak{so}(W) + \mathfrak{k}, \) where \( W \) is a Lorentzian subspace of dimension \( m = \dim W > 2 \) and \( \mathfrak{k} \) is a compact subalgebra of \( \mathfrak{so}(W^\perp) \). Suppose that \( \mathfrak{k} \) does not annihilate any non-zero vector in \( W^\perp \). Then, \( M \) must be a direct product of an \( m \)-dimensional homogeneous Lorentz space \( M_0 \) and of a homogeneous Riemannian manifold \( \mathcal{N} = G_1/N \). In particular,
- if \( m > 3 \), then \( M_0 \) is a space of constant curvature, that is the Minkowski space \( \mathbb{R}^{1,m-1} \), or the de Sitter space \( \text{dS}^m = SO(1, m)/SO(1, m - 1) \), or the anti de Sitter space \( \text{AdS}^m = SO(2, m - 1)/SO(1, m - 1) \).
- if \( m = 3 \), then \( M_0 \) is either the space of constant curvature, or the Lie group \( \text{SL}(2, \mathbb{R}) \).

Proof. When \( L \) is of Type II, then the reductive decomposition can be written as
\[
\mathfrak{g} = \mathfrak{l} + \mathfrak{m} = (\mathfrak{so}(W) + \mathfrak{k}) + (W + \mathcal{U})
\]
where \( \mathcal{U} = W^\perp \) is an Euclidean vector space and \( \mathfrak{k} \subset \mathfrak{so}(\mathcal{U}) \). The map
\[
\Lambda^2 (W + \mathcal{U}) = \Lambda^2 W + W \otimes \mathcal{U} + \Lambda^2 \mathcal{U} \to \mathfrak{g},
\]
given by the Lie brackets, is a \( (\mathfrak{so}(W) + \mathfrak{k}) \)-equivariant linear map. This implies the relations
\[
[W, W] \subset \mathfrak{so}(W) + W, \quad [W, \mathcal{U}] = 0, \quad [\mathfrak{k}, \mathcal{U}] \subset \mathcal{U}.
\]
Thus, \( \mathfrak{g} \) is the direct sum of two ideals, i.e.,
\[
\mathfrak{g} = (\mathfrak{so}(W) + W) + (\mathfrak{k} + \mathcal{U}).
\]
Hence, \( M \) must be a product of a homogeneous Lorentzian space \( M_0 \) and a homogeneous Riemannian manifold. This proves our first claim.
Remark 6.3. Let $M = G/L$ be a simply connected (almost) effective homogeneous Lorentz manifold with isotropy subalgebra of Type II, that is $j_a(l) = \mathbb{R}d + \mathfrak{t}$, $d = p \wedge q + C_0$, $C_0 \in \mathfrak{so}(E)$, and $\mathfrak{t}$ is a compact subalgebra of $\mathfrak{so}(E)$; $\mathfrak{t}$ commutes with $C_0$. Suppose that $\mathfrak{t}$ does not annihilate any non-zero vector in $E$. Then $M$ is a direct product of 2-dimensional constant curvature Lorentz space $M_0$ and of a homogeneous Riemannian manifold $N = G_1/N$.

Proof. Since the restriction of the Lie bracket $\Lambda^2\mathfrak{m} \to \mathfrak{g}$ is an $(\mathbb{R}d + \mathfrak{t})$-equivariant map, we conclude that

$$[p, q] = \lambda d, \quad (\lambda \in \mathbb{R}), \quad [p, E] = [q, E] = 0, \quad [E, E] \subset \mathbb{R}d + \mathfrak{t} + E.$$ 

Then, the Jacobi identity yields the relation $[[X, Y], p] = 0$, for any $X, Y \in E$, i.e., $[E, E] \subset \mathfrak{t} + E$. Similarly, $[[p, q], X] = 0$, i.e., either $\lambda = 0$ or $C_0 = 0$. This easily implies the proof.

Let us finally highlight the following remark about the subspace $E_0 \subset E$ consisting of vectors annihilated by $\mathfrak{t}$, and its triviality when one uses maximal admissible subgroups.

Remark 6.3. Let $M = G/L$ be a simply connected (almost) effective homogeneous Lorentz manifold with isotropy subalgebra of Type II or Type III. Suppose that the Lie subalgebra $\mathfrak{l} \subset \mathfrak{g}$ is maximal admissible. In this case we claim that the subspace $E_0 \subset E$ consisting of vectors annihilated by $\mathfrak{t}$ is trivial. Indeed, it is easy to see that the decomposition

$$\mathfrak{g} = (1 + E_0) + (\mathbb{R}p + E_1 + \mathbb{R}q)$$

contradicts the maximality assumption (here $E_1$ is the orthogonal complement to $E_0$ in $E$).

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