Breakdown of Conformal Invariance at Strongly Random Critical Points

M. B. Hastings\textsuperscript{1} and S. L. Sondhi\textsuperscript{2}

\textsuperscript{1} CNLS, MS B258, Los Alamos National Laboratory, Los Alamos, NM 87545, hastings@cnls.lanl.gov
\textsuperscript{2} Department of Physics, Princeton University, Princeton, NJ 08544

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We consider the breakdown of conformal and scale invariance in random systems with strongly random critical points. Extending previous results on one-dimensional systems, we provide an example of a three-dimensional system which has a strongly random critical point. The average correlation functions of this system demonstrate a breakdown of conformal invariance, while the typical correlation functions demonstrate a breakdown of scale invariance. The breakdown of conformal invariance is due to the vanishing of the correlation functions at the infinite disorder fixed point, causing the critical correlation functions to be controlled by a dangerously irrelevant operator describing the approach to the fixed point. We relate the computation of average correlation functions to a problem of persistence in the RG flow.

I. INTRODUCTION

The principle of conformal invariance\textsuperscript{[1]} has proven immensely powerful in the study of continuous phase transitions. While it is believed to hold at critical points of systems with short ranged interactions in all dimensions\textsuperscript{[2]}, it is especially powerful in two dimensions, where it has enabled a substantial classification of critical theories based on the representation theory of the Virasoro algebra and its extensions\textsuperscript{[3]}.

Recent interest has focused on applying this analysis to systems with quenched randomness in two dimensions; see for examples Refs.\textsuperscript{[4–6]}. The formal device that makes this possible is the construction of translationally invariant field theories whose correlation functions are the disorder averaged correlators of the random problem, quite generally via the replica trick and in Gaussian problems, via the supersymmetry method. (There is at least one notable example where neither is needed\textsuperscript{[7]}, and conformal invariance follows straightforwardly.)

Our interest in this paper, is in asking whether the average field theories are necessarily conformally invariant at their critical points. We will find that this is not the case in two instances of what we term strongly random critical points, a category that may well have considerable overlap with the “infinite disorder” critical points studied by means of real space renormalization group techniques\textsuperscript{[8]}. Both are localization problems, the first a well studied problem in \(d = 1\) where conformal invariance is even more powerful than in \(d = 2\), and the second is an analogous construction on our part in \(d = 3\). We also comment on another well studied member of this family of problems in \(d = 2\) which does appear to have a region where conformal invariance holds. We should note before proceeding further, that while we were motivated by the absence of conformal invariance in the one dimensional example, what we find is a breakdown of scale invariance itself in that the operator product expansions are anomalous even though two point correlators are algebraic.

Another question appears to be closely connected to the above issues: namely, how one might recover the distributions for correlation functions that should characterize the universal content of random fixed points, from computations of averages and higher moments. In both of our examples, the typical correlations are very different from the average ones and are not even algebraic, which suggest both the difficulty of reconstructing them from the averaged field theory and why the latter may have to be anomalous. This raises the question of whether the breakdown of conformal invariance we report may have echoes at other, less strongly random critical points. We will speculate briefly on this at the end of the paper.

One rationalization of our results will be that the averaged correlation functions vanish at the infinite disorder fixed point, causing the behavior of the critical correlation functions to be controlled by a dangerously irrelevant operator describing the approach to the fixed point. For the typical correlation functions, scale invariance breaks down, while the average correlation functions have power law behavior at the level of two point correlators, but break scale invariance for three and higher point correlations.

Within this interpretation, the power law behavior of two-point correlation functions is not the result of a non-vanishing scaling dimension for the given operators at the critical point, as in this case one would find scaling for multi-point correlations as well. Instead, the power law behavior is a feature of the leading corrections to scaling.

We begin by reviewing a one-dimensional system exhibiting the given breakdown in scale and conformal invariance, and show the anomalous operator product expansion. Next, we introduce a model system in three dimensions. We present evidence that the system is at a critical point, with relevant perturbations that introduce a localization length that diverges as the critical point is approached; however, we will be unable to describe fully the system off-criticality. We then compute typical correlation functions in this system and show that they violate scale invariance. Using a Liouville field theory to compute average correlation functions, we show that while the two point functions are power law, indicative of a critical point and compatible with scale invariance...
ance, multi-point correlation functions violate conformal invariance.

In an Appendix, we note that there exists a SUSY field theory which reproduces the correlation functions of the Liouville field theory. Although the correlation functions are easier to compute in the Liouville theory, the advantage of the SUSY theory is that it provides a purely local field theory which exhibits a breakdown of conformal invariance; the Liouville field theory will involve one global integral which effectively introduces a long-range interaction.

II. BREAKDOWN OF CONFORMAL INVARIANCE IN ONE DIMENSION

The conformal group is the group of transformations which leaves angles unchanged. In three, or more, dimensions, the conformal group is generated by translations, rotations, dilatations, and inversions \((x^\mu \rightarrow \mathbb{m} x^\mu)\). By composing inversion-translation-inversion, one generates the so-called special conformal transformations. In two dimensions, the conformal group is supplemented by all analytic transformations of the complex plane, while in one dimension any diffeomorphism is a conformal transformation.

Invariance under translation, dilatation, and rotation fixes the two point function to be a power law. Invariance under inversion fixes the form of the three-point correlation function. For example, consider a real scalar field, \(\phi\), in \(d > 2\) dimensions with action

\[
S = -\frac{1}{2} (\partial_\mu \phi)^2.
\]

Consider the connected two-point function of \(\phi^2\), \(\langle \phi^2(0) \phi^2(x) \rangle \propto |x|^{2-2d}\), a power law as expected. The connected three-point function is given by

\[
\langle \phi^2(0) \phi^2(x) \phi^2(y) \rangle \propto |x|^{2-d} |y|^{2-d} |x-y|^{2-d},
\]

in agreement with inversion symmetry.

In one and two dimensions, conformal invariance imposes even more stringent requirements on the correlation functions. In one dimension, conformal invariance requires that, at a critical point, all correlation functions are constant. An example of this behavior in statistical mechanics is the one-dimensional Yang-Lee edge.

Another example is the following quantum system. Consider a one-dimensional Dirac particle with Hamiltonian

\[
H = \begin{pmatrix}
0 & m(x) + \partial_x \\
m(x) - \partial_x & 0
\end{pmatrix}.
\]

If we take \(m(x) = \overline{m}\), then, looking at the zero energy Green’s function, the point \(\overline{m} = 0\) is a critical point. In accord with conformal invariance, the Green’s function is a constant. For non-zero \(\overline{m}\), the system acquires a correlation length proportional to \(\overline{m}^{-1}\).

Turning to a random version of this critical point, however, we find a breakdown of conformal invariance. Let \(m(x) = \overline{m} + \delta m(x)\), with \(\delta m(x)\) a quenched Gaussian random variable with vanishing mean and

\[
\overline{\delta m(x) \delta m(x')} = g \delta(x - x'),
\]

where the overline denotes averaging over disorder. With this random mass, the Dirac equation is the continuum limit of a tight-binding Hamiltonian on a one-dimensional chain, with randomly chosen hopping elements and no on-site potential \(E_0\). The zero energy eigenfunctions of this problem provide an exception to exponential localization in one-dimension.

For this problem, we will be summarizing old results on the two-point functions. We will note that the two-point function, while invariant under translation, dilatation, and inversion, is not invariant under arbitrary diffeomorphisms, thus violating conformal invariance in one dimension. Then, we will show how the correlation functions may be obtained from the exact zero energy eigenfunctions of the problem, and use this to compute the three-point function. This function will not be invariant under inversion or special conformal transformations, thus breaking conformal invariance in a way that generalizes to higher dimensions.

For \(\overline{m} = 0\), it has been shown \[13, 15\] that the average zero energy Green’s function decays as a power law

\[
\overline{G(x, y)} \propto |x-y|^{-3/2}.
\]

All positive moments of the average Green’s function behave similarly. For convenience, in this paper we choose to focus on a density of states correlation function. Let \(\psi_a(x)\) be an eigenfunction of Hamiltonian \(H\), with energy \(E_a\). Introduce an infinitesimal parameter \(\eta\). Define the density of states \(\rho(x)\) to be equal to

\[
\rho(x) = \text{Im}(G(x, x)) = \text{Im}(\sum_a \frac{1}{E_a + i\eta} |\psi_a(x)|^2).
\]

The two point correlation function of the density of states scales as Eq. (5). On the other hand, if we look at the full probability distribution of the density of states at two points, we find that if \(\rho(x)\) is of order unity, then the typical \(\rho(y)\) is of order

\[
e^{-\sqrt{|x-y|}}.
\]

Eq. (7) is not invariant under general diffeomorphisms, while Eq. (5) is not scale invariant. However, we emphasize that the point \(\overline{m} = 0\) is a critical point for the random system, as for \(\overline{m} \neq 0\), the average Green’s function acquires a diverging correlation length proportional to \(\overline{m}^{-2}\), while the typical correlation function is proportional to
Similarly, away from zero energy one finds a finite localization length, diverging as $\ln^2(1/E)$ or $\ln(1/E)$ for average and typical correlation functions, respectively.

To derive the above results, consider the exact zero energy eigenfunctions of the problem:

$$\begin{pmatrix} e^{\beta(x)} & 0 \\ 0 & e^{-\beta(x)} \end{pmatrix} \left( \begin{array}{c} 0 \\ e^{-\beta(x)} \end{array} \right)$$

where $\beta(x) = \int m(y) \, dy$. Representing the zero energy Green’s function as a resolvent

$$G(x, y) = \langle x | \frac{1}{H + i\eta} | y \rangle,$$

we note that, for a finite system in the limit $\eta \to 0$, the Green’s function and density of states are controlled by the zero energy eigenfunction; the imaginary part of the Green’s function is equal to $\frac{1}{\beta(x)} \psi(x) \psi(y)$ where $\psi$ is the zero energy eigenfunction. The zero energy Green’s function is a matrix; the different components of this matrix provide information on the two different zero energy eigenfunctions. In the Appendix, a supersymmetric field theory is constructed such that the zero energy Green’s function is a correlation function of the form

$$\langle \bar{\psi}(x) \psi(y) \rangle.$$

If the infinite volume limit is taken before the $\eta \to 0$ limit, the resolvent and density of states depend on other eigenfunctions beyond the exact zero energy eigenfunction. We now show, however, that the scaling properties of correlation functions for both these quantities are the same for either order of limits. Consider eigenfunction $\psi_a$ at energy $E_a$. The eigenfunction is a zero energy eigenfunction of Hamiltonian

$$H = \begin{pmatrix} -E & m(x) + \partial_x \\ m(x) - \partial_x & -E \end{pmatrix}.$$

By performing an axial transformation,

$$\tilde{\psi}_a = \begin{pmatrix} e^{-\beta(x)} & 0 \\ 0 & e^{\beta(x)} \end{pmatrix} \psi_a,$$

we find that $\tilde{\psi}_a$ is a zero energy eigenfunction of Hamiltonian

$$H = \begin{pmatrix} -Ee^{2\beta(x)} & \partial_x \\ -\partial_x & -Ee^{-2\beta(x)} \end{pmatrix}.$$

This Hamiltonian has diagonal disorder, and one expects to find all the eigenfunctions localized; however, for small $E$, the diagonal terms can be ignored for lengths such that $Ee^{2\beta} << 1$. Since $\beta$ has rms fluctuations of order $\sqrt{L}$, the diagonal terms in Hamiltonian (13) can be ignored for $L << \ln^2(1/E)$, and the exponential factor from the axial transformation (12) controls the magnitude of $\psi_a$ (we note that the exponential behavior of $\beta$ ensures that the dominant contribution to the magnitude of $\psi_a$ arises from the axial transformation, rather than any fluctuations in magnitude of $\psi_a$). Thus, we find that over short length scales all the eigenfunctions track the zero energy eigenfunction in relative magnitude; over longer length scales, the non-zero energy eigenfunctions can become localized.

Now consider correlation functions of density of states for finite $\eta$. Since the density of states includes eigenfunctions with energy $E$ of order $\eta$, we find that $\eta$ sets a length scale in the system (this length is identical to the average localization length discussed above). However, on lengths shorter than this scale, the magnitude of the eigenfunctions contributing to the density of states can be obtained from that of the zero energy eigenfunction, up to constant factors. Consider a correlation function of the density of states, $\prod_i \rho(x_i)$. We argue that these correlation functions, in the thermodynamic limit with fixed $\eta$, are equivalent, up to constant factors, to correlation functions taken in a finite system as $\eta \to 0$, provided that in the first case $|x_i - x_j| << \ln^2(\frac{1}{\eta})$. A similar argument may be made for average Green’s functions.

Given this equivalence, for the remainder of the paper we consider only the behavior of the zero-energy eigenfunction. The log of the exact zero-energy eigenfunction, $\beta(x)$, executes a random walk, such that the mean-square fluctuations in the log are proportional to the length scale $L$. If we focus on the first of the two eigenfunctions, we see that it is strongly peaked at a given point, the maximum of $\beta(x)$ (the other eigenfunction is peaked at the maximum of $-\beta(x)$).

After normalizing the eigenfunction, it is exponentially small away from the maximum. Typically the eigenfunction decays as a stretched exponential, giving Eq. (6). It is possible, though, for the eigenfunction to have a secondary maximum. This possibility dominates the average two-point function. Eq. (6) can be obtained as follows: we wish to compute the probability that $\beta(x)$ has global maxima at $0, x$. Since the eigenfunction normalization can be absorbed by a shift in $\beta$, we fix $\beta(0) = \beta(x) = 0$. Everywhere else, $\beta(y) < 0$. Thus, the probability to have maxima at $0, x$ scales as the return probability of a random walk required to persist below $0$. This scales as $|x|^{-3/2}$, giving Eq. (6). Further, the random walk technique shows that all positive moments have the same scaling, as any positive moment of the zero energy eigenfunction is sharply peaked at the maxima of $\beta$.

Tuning away from the critical point by taking $\mathcal{M} \neq 0$, we find that the zero energy eigenfunction is normalizable only for a finite system, where it describes eigenfunctions decaying into the system. The decay of the eigenfunctions is governed by the diverging correlation length discussed above.

We now turn to the three-point correlation function.
of the density. This function can be represented in the supersymmetric field theory as
\[ \langle \bar{\psi}_1(0)\psi_1(0)\bar{\psi}_2(x)\psi_2(x)\bar{\psi}_3(y)\psi_3(y) \rangle. \] (14)

The average three-point correlation function can be computed by an extension of the random walk argument. Assume that \( 0 < x < y \). Then, we consider a random walk that starts at 0, returns to zero at \( x \), and returns again to 0 at \( y \). This gives the average three-point correlation function
\[ \rho(0)\rho(x)\rho(y) \sim |x|^{-3/2}|x-y|^{-3/2}. \] (15)

As stated above, Eq. (15) is not invariant under inversion or special conformal transformations.

The three-point function enables us to find the operator product expansion for the theory. Given a set of operators, \( O_A \), with dimensions, \( d_A \), one expects that
\[ O_A(0)O_B(x) \rightarrow f_{ABC}x^{-(d_A+d_B-d_C)}O_C. \] (16)

Since all positive moments of the density of states scale equivalently, we need consider only two operators. One operator is the identity operator, \( I \), while the other operator \( O \) is the density of states operator, with dimension \( d \). Eq. (3) suggests \( d = -3/4 \). Then, consistency with the operator product expansion would give
\[ \rho(0)\rho(x)\rho(y) \sim |x|^{-3/4}|x-y|^{-3/4}|y|^{-3/4}, \] (17)

which, like Eq. (2), obeys inversion symmetry. However, this is not what we have found. Thus, we rationalize our results as follows: the scaling dimension of the operator \( O \) vanishes, as indeed it must in one dimension. However, the rms fluctuations in \( \beta \) scale as \( \sqrt{L} \) and the inverse rms fluctuations in \( \beta \) tend to zero as \( L \rightarrow \infty \); equivalently, the statistical properties of the zero energy eigenfunction are unchanged under scaling \( x \rightarrow x/L, g \rightarrow Lg \). If, however, we take \( g \) divergent, we find that the zero-energy eigenfunction is localized at a point and all correlation functions vanish. Thus, \( g^{-1} \) is a dangerously irrelevant variable. In the scaling limit, defined as the limit in which the separation between points in correlation functions tends to infinity, all correlation functions are controlled by the vanishing of this variable as the fixed point is approached. Similarly, \( f_{ABC} \) vanishes as the fluctuations in \( \beta \) diverge. However, we have found above is that \( f \) vanishes as a power law as the fixed point is approached. Thus, the power law (3) is not the result of \( d \neq 0 \), but the result of \( f \rightarrow 0 \) as the fixed point is approached.

We note that the behavior we have found is similar to that found using a real-space RG approach on the discrete tight-binding version of this model. In that approach, one renormalizes the distribution of hopping elements, reducing a cutoff on the strength of the hopping.

In the scaling limit, almost all the hopping elements become vanishingly small compared to the cutoff; the fraction of hopping elements (vanishing as a power law in the scaling limit) that are of order the cutoff control the correlation functions.

## III. A MODEL SYSTEM IN THREE DIMENSIONS

To demonstrate the generality of this breakdown of conformal invariance, we will look for higher dimensional examples. Using a real-space renormalization group, some examples have been found of infinite disorder quantum critical points two or more dimensions [10]. However, these are interacting quantum systems and can only be described by 2+1 or 3+1 dimensional theories with a privileged time coordinate, so they would not be expected to have conformal invariance, although the question of scale invariance in these systems remains an interesting problem for future work. There is also some evidence [17] that a two-dimensional random hopping model could have an infinite disorder critical point. However, there is as yet no evidence for breakdown of conformal invariance in this model. Therefore, we will construct a model system in three-dimensions that provides a strongly random critical point without conformal invariance.

After presenting the Hamiltonian and eigenfunctions, we will present evidence that the system is indeed at a critical point between two different localized phases. Further evidence for the criticality of our model will result from behavior of typical and average correlation functions that we find below.

Consider the following Hamiltonian
\[ H = \begin{pmatrix} 0 & (\partial_\mu - A_\mu)^2 \\ (\partial_\mu + A_\mu)^2 & 0 \end{pmatrix}, \] (18)

where
\[ A_\mu(x) = \partial_\mu \beta(x). \] (19)

The Hamiltonian has two exact zero energy eigenfunctions. They are
\[ \begin{pmatrix} e^{\beta(x)} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ e^{-\beta(x)} \end{pmatrix}. \] (20)

We pick \( \beta(x) \) with probability distribution \( e^{-S[\beta]} \) where
\[ S[\beta] = \frac{1}{2g} \int d^3x (\partial^2 \beta)^2. \] (21)

We pick this distribution of disorder as the best balance between having too little and too much disorder, while retaining a local distribution of disorder. Let us show that for less disorder, the critical point would not be
strongly random, while for more disorder we might lose the critical point completely.

With this distribution for $\beta$, the mean-square fluctuations in $\beta$ are proportional to $L$, the length scale of the system, just as in the one-dimensional system the fluctuations in $\int m(y) \, dy$ were proportional to the length of the system. This increase in fluctuations with length scale enables us to reach a strongly random fixed point.

Suppose instead that we had chosen a disorder distribution with smaller fluctuations in $\beta$. If we had chosen a two-dimensional system with $S[\beta] = \int d^2x \, (\partial_x \beta)^2$, the fluctuations in $\beta$ would only grow logarithmically with $L$, the exponential of $\beta$ would behave as a power law, and we find a conformally invariant fixed point [18]. If we had considered a three-dimensional system with $S[\beta] = \int d^3x \, (\partial_x \beta)^2$, we would have found a fixed point with vanishing disorder.

On the other hand, for the given distribution of disorder, we can make an argument that the system is at a critical point. For the given distribution of disorder, by letting $A_{\mu} = \partial_{\mu} \beta(x) + J_{\mu}$, for $J_{\mu} \neq 0$, we can tune away from criticality. As in the one-dimensional example, we find that the eigenfunction is normalizable only for a finite system and describes eigenfunctions decaying into the system. As now the zero energy eigenfunctions are
\[ \left( e^{\beta(x) + J_{\mu} x^\mu} \right), \left( e^{-\beta(x) - J_{\mu} x^\mu} \right), \] (22)
the typical eigenfunction decays with a length proportional to $|J|^{-1}$. This demonstrates that there is a diverging length scale associated with the given problem, implying that we are at a critical point. If we consider a system in a square box, with $J_{\mu} = j(0,0,0)$, as we tune $j$ from positive to negative there is a phase transition between eigenfunctions localized on opposite faces of the box. It should also be possible to tune away from criticality by considering Green’s functions at non-zero energy. This is a more difficult task, for future work.

However, if we had considered a two-dimensional system with $S[\beta] = \int d^2x \, (\partial^2 \beta)^2$, the mean-square fluctuations in $\beta$ would grow as $L^2$. We would find that the typical correlation functions for eigenfunction $[20]$ would decay exponentially. In the case of the system in a box, even for vanishing $J$ there would be eigenfunctions exponentially localized on the faces of the box, and the phase transition would occur at a non-vanishing value of $J$. Therefore, for this two-dimensional distribution we would not have such strong evidence that the system is critical.

Finally, let us emphasize that the distribution of disorder we have chosen in Eq. (21) is local. As a result, the SUSY field theory in the Appendix is also local, and so the breakdown of conformal invariance is not a result of long-range interactions.

**IV. TYPICAL CORRELATION FUNCTIONS AND ONE-POINT FUNCTIONS**

Let us focus on the first zero energy eigenfunction of Eq. (21) and write $\psi(x) = e^{\beta(x)}$. We normalize $\psi$ such that $\int d^2x \, \psi^2(x) = 1$. In exact analogy with the one-dimensional system, we find that if $\psi(x)$ is of order unity, then
\[ \psi_{\text{typ}}(y) \propto e^{-\sqrt{g|x-y|}}, \] (23)

We can also consider a typical one-point function, the magnitude $|\psi(x)|^2$. Of course, on average, this function is equal to $\frac{1}{L^2}$. Typically, however, it is of order $e^{-\sqrt{gL}}$. (24)

Clearly, Eqs. (23,24) violate scale invariance. We will next turn to the average correlation functions. They will not violate scale invariance, but they will violate conformal invariance. The advantage of considering average correlation functions is that, as shown in the Appendix, they can be obtained as correlation functions in a purely local field theory.

**V. MULTIFRACTALS AND LIOUVILLE FIELD THEORY**

Let us look at average correlation functions of the form
\[ W_q(x, L) = \langle |\psi(x)|^q \psi(0)|^q \rangle, \] (25)

where the overline denotes ensemble averaging. From the scaling behavior of $W_q$, we can compute the scaling of $\rho(x)^{q/2} \rho(0)^{q/2}$.

The inverse participation ratio, $W_q(L) = W_q(0, L)$, is expected to vary as a power law
\[ W_q(L) \propto \frac{1}{L^{3+\tau(q)}}, \] (26)
while the two-point function is expected to vary as
\[ W_q(x, L) \propto \frac{1}{L^{3+\tau(q)} e^{x^2(q)}}. \] (27)

For localized eigenfunctions, $\tau(q) = 0$, while for a plane wave eigenfunction $\tau(q) = 3(q - 1)$. In some localization problems, one finds multifractal scaling for the $\tau(q)$ at criticality; see ref. [19] for a review.

The typical correlation functions of the previous section can be considered as exponentials of ensemble averages of logarithms of correlation functions. They can be obtained from $q \to 0$ limits of the average correlation functions.

We will now introduce a Liouville field theory [20,14] to compute the functions $W_q$. The correct normalization of eigenfunction $\psi$ can be obtained by
\[ \psi(x) = N^{-1/2} e^{\beta(x)}, \]  
(28)

where
\[ N = \int d^3 x \ e^{2\beta(x)}. \]  
(29)

The correlation functions \( W_q \) can be written as
\[ W_q(x, L) = \frac{1}{Z_0} \int [d\beta] N^{-q} e^{q\beta(0)} e^{q\beta(x)} e^{-S[\beta]}, \]  
(30)

where \( Z_0 = \int [d\beta] e^{-S[\beta]} \). The normalization can be exponentiated by
\[ N^{-q} = \frac{1}{\Gamma(q)} \int_0^\infty d\omega \omega^{-1} e^{-\omega N}. \]  
(31)

As a result,
\[ W_q(x, L) = \frac{1}{Z_0 \Gamma(q)} \int_0^\infty d\omega [d\beta] \omega^{-1} e^{q\beta(0)} e^{q\beta(x)} e^{-S[\beta]-\omega e^{2\beta}}. \]  
(32)

This provides a three-dimensional Liouville field theory, with the strange feature that the action involves fourth derivative terms instead of second derivative terms.

**VI. AVERAGE CORRELATION FUNCTIONS**

We will consider the average correlation functions. We will demonstrate that the exponents \( \tau(q) \) and \( \sigma(q) \) are independent of \( q \). We will be unable to calculate \( \sigma(q) \) exactly, but we will argue that the power law behavior of Eq. (27) is obeyed for a finite \( \sigma(q) \), leading to conformal invariance for the two point function. The behavior of the average correlation function is again related to problems of persistence in nonequilibrium systems. Then, as in one dimension, we will show using OPEs of the theory that the three point function breaks conformal invariance.

If we look at the scaling behavior of Eq. (32), we find that as the length scale \( L \) increases, \( \beta \) increases proportional to \( L^{1/2} \), and so \( e^{q\beta} \) becomes sharper. Effectively, \( q \) increases proportional to \( L^{1/2} \), and so for any \( q > 0 \), the scaling behavior of the correlation functions is the same, so that \( \tau(q) \) and \( \sigma(q) \) are independent of \( q \). The \( q \to 0 \) and \( L \to \infty \) limits do not commute.

In the large \( q \) limit, and therefore in the scaling limit, the two-point function is dominated by field configurations in which \( \beta \) reaches its maximum at 0 and \( x \). The one-point function is dominated by configurations in which \( \beta \) reaches its maximum at 0; the probability of this happening is proportional to \( 1/L^3 \) and so \( \tau(q) = 0 \), characteristic of a localized eigenfunction.

To find \( \sigma(q) \), we need to look at the scaling of the following functional integral
\[ \int [d\beta] e^{-S[\beta]-f(\beta)} \rho_1[\beta] \rho_2[\beta]. \]  
(33)

Here, \( \rho_1[\beta] = \delta(\beta(0)) \) and \( \rho_2[\beta] = \delta(\beta(x)) \), while \( f(\beta) = 0 \) for \( \beta \leq 0 \) and \( f(\beta) = \infty \) for \( \beta > 0 \), so that the potential \( f \) has a hard wall at \( \beta = 0 \). This reproduces the desired constraint that \( \beta \) have a maximum at both 0 and \( x \). Let us define an operator \( O(x) \) that inserts a factor of \( \delta(\beta(x)) \) in Eq. (33). Then \( W_q(x, L) = \langle O(0)O(x) \rangle \).

We will proceed with a renormalization group technique as follows: initially the functionals \( \rho_1, \rho_2 \) define \( \delta \)-function distributions for \( \beta \) at 0, \( x \). We will replace these with probability distribution functionals for \( \beta \), and for the normal derivative of \( \beta \), on the surface of a sphere at radius \( a \) around points 0 and \( x \). Initially \( a \) will be small, but we will integrate out the field \( \beta \) in a shell between \( a \) and \( a + da \), and compute the change in the probability distribution as \( a \) is increased. In one-dimension, this amounts to computing the probability distribution of a random walk, persisting below zero, at points 0 ± \( a \), \( x ± a \).

As the RG above is carried out with spheres centered on \( x, 0 \), at \( a = |x/2| \) the two spheres merge, and one must match boundary conditions. On the surface of the merged spheres, the field \( \beta \) is of order \( \sqrt{|x|} \). To compute the scaling of the two point function, one must compute the probability that \( \beta \) persists below zero on the surface of both spheres up to scale \( |x/2| \), with matched boundary conditions (we will discuss this computation below). After the spheres merge, one has a single surface propagating outwards as \( a \) is increased, but the persistence probability on this surface is independent of \( x \).

Given the two-point function, consider the three-point correlation function \( \langle O(0)O(x)O(y) \rangle \), with \( |x| << |y| \). At the scale \( |y| \), the field \( \beta \) is of order \( \sqrt{|y|} \ll \sqrt{|x|} \). Therefore, for the purpose of computing the scaling of the three-point function with respect to \( y \), we can simply assume that \( \beta \) is pinned at the value zero near the point 0. Thus, the two operators \( O(0), O(x) \) fuse into a single operator \( O \).

\[ \lim_{x \to 0} O(x)O(0) = x^{-\sigma} O(0). \]  
(34)

The exponent as \( \sigma \) in the above equation is exactly \( \sigma(q) \) for \( q > 0 \), as the two-point function can be computed by fusing the two operators \( O \) into a single operator, yielding a one-point function, for which
\[ \langle O(0) \rangle = 1/L^3. \]  
(35)

Therefore,
\[ \langle O(0)O(x) \rangle \propto x^{-\sigma}. \]  
(36)

We now discuss the procedure for computing \( \sigma \). First we consider the problem of persistence in \( \beta \) for a single
sphere, and then show how to obtain \( \sigma \) from the persistence problem by matching boundaries. However, the problem in persistence is sufficiently difficult that we will not attempt to obtain a numerical value for \( \sigma \). The related persistence problem will at least enable us to argue that \( \sigma \) is finite so that correlation functions are power

Decomposing \( \beta \) into spherical harmonics,

\[
\beta(a, \theta, \phi) = \sum_{l,m} f_{lm}(a) Y_{lm}(\theta, \phi),
\]

the probability distribution functionals can be written as

\[
\rho[f_{lm}(a), \partial_a f_{lm}(a)].
\]

First, let us consider the probability that, for a single sphere, \( \beta \) does not cross the hard wall up to scale \( a \). Defining

\[
g_{lm}(T) = e^{-T/2} f_{lm}(e^T),
\]

we can write the action \( S[\beta] \) as

\[
\sum_{l,m} \int \frac{1}{2} \left( 3/4 + l(l + 1) + 2 \partial_T + \partial_T^2 \right) g_{lm}(T) \left. dT \right|_{T'}.
\]

With this scaling of field, and logarithmic transform of the RG scale, the problem becomes that of persistence in a Gaussian stationary process \[24\]. With action given by Eq. (40), the correlation function

\[
\langle g_{lm}(T)g_{lm}(T') \rangle
\]

is exponentially decaying in \( |T - T'| \). Therefore, for a finite number of fields \( g \), the persistence probability is exponentially decaying in \( T \), and decays as \( a^{-\theta} \) for some power \( \theta \). We have an infinite number of fields \( g_{lm} \), but for large \( l \) they have very small fluctuations, so it seems likely that \( \theta \) tends to a finite limit as we include fields with greater and greater \( l \).

In principle it is possible to compute \( \theta \) for a finite number of \( g \). However, since the process is non-Markovian due to the fourth derivatives it is quite difficult; even for the simpler problem of one field, with an action containing only fourth derivative and no other terms, this problem \[23\] was solved only relatively recently \[24\].

For a finite system, without periodic boundary conditions, the persistence exponent determines the probability that the maximum of the eigenfunction will be located at a given interior point, instead of at the boundary.

Given the persistence exponent, we can compute \( \sigma \). At scale \( a = |x/2| \), the spheres start to merge. Beyond this scale, we have only one surface. Immediately before the spheres touch, the probability that \( \beta \) is a maximum at 0 in the sphere centered around 0 is

\[
|x|^{-\theta}.
\]

The probability that \( \beta \) is a maximum at 0 in the sphere centered at \( x \) is the same. The product is \( |x|^{-2\theta} \).

However, at this point where the two spheres meet, the values of \( \beta \) and \( \partial \beta \) must be equal on both spheres. This is a matching of boundary conditions. To match boundary conditions, the scaled variables \( g \) must be equal on both spheres. Since \( g \) is stationary, the probability that the scaled variables are equal is independent of the RG scale \( T \). However, for \( \beta \) to be equal on both spheres, there is an additional Jacobian which gives rise to a further power law correction. In this case, the mean square fluctuations in \( \beta \) grow as \( \sqrt{L} \), while the fluctuations in \( \partial \beta \) do not grow in \( L \). Therefore, the probability that \( \beta \) is a maximum at both 0 and \( x \), and that the boundary conditions match (equivalently, that the maxima at 0 and \( x \) coincide) is \( |x|^{-2\theta-1/2} \).

This is to be compared to Eq. (42) for only one sphere, so that the relative probability to find two maxima is

\[
\sigma = 2\theta + 1/2 - \theta = \theta + 1/2.
\]

We see this explicitly in the problem of the random Dirac equation \[3\] in one-dimension, where the persistence exponent \( \theta = 1 \) (typically in the literature, this is quoted as \( \theta = 1/2 \) as one typically computes only the persistence probability for a nonequilibrium process going forward in time; our spheres move outwards in both directions and so the exponent is doubled). The average correlation function decays as \( x^{-3/2} = x^{-\theta-1/2} \).

Even without a precise value for \( \sigma \), we can still demonstrate the breakdown of conformal invariance. The two-point function calculated above exhibits scale and conformal invariance in 3 dimensions. However, as in one dimension, the operator product expansion is not consistent with scaling. From the two-point function, one would expect that \( O \) would have scaling dimension \( \sigma/2 \).

However, Eq. (34) gives a scaling dimension of \( \sigma \) for \( O \). The trouble is that Eqs. (34) are not simultaneously consistent with scaling.

To make the contradiction more concrete, consider the three point function \( W_q(x, y, L) = \langle \psi(0) \psi(x) \psi(y) \rangle \). In the limit when \( |x| << |y| \), although we were unable to compute \( \sigma \), we are able to use the operator product expansion to demonstrate a breakdown of conformal invariance.

In the limit \( |x| << |y| \), the operator product expansion gives

\[
W_q(x, y, L) \propto L^{-3} |x|^{-\sigma} |y|^{-\sigma}.
\]

This is similar to Eq. (3) found in one dimension. Eq. (44) violates inversion symmetry with respect to a point \( |z| \) where \( |z| << |x| \).

As in one-dimension, we interpret this to mean that \( O \) has vanishing scaling dimension, but that all the correlation functions of \( O \) vanish at the infinite disorder fixed
point. As the theory approaches the fixed point, the average correlation functions vanish as a power law, so that if one only considers the two-point functions one obtains the wrong scaling dimension for $O$. The inverse of the mean-square fluctuations in $\beta$ is a dangerously irrelevant operator that describes the approach to the fixed point; when this operator vanishes, so do the correlators.

VII. CONCLUSION

We have considered the breakdown of conformal invariance in certain strongly random fixed points. The breakdown of conformal invariance is related to a power law vanishing of correlation functions approaching the fixed point.

Although we were unable to compute the exponent $\sigma$ precisely, we were able to obtain some understanding of the operator product expansion, and to relate $\sigma$ to a persistence exponent in a nonequilibrium process. The connection to persistence is not surprising; the RG flow describing the approach to the fixed point is nonequilibrium, as the flow approaches the fixed point but never reaches it. In contrast to ordinary critical points, it is the approach to the fixed point, rather than the fixed point itself, that controls the exponents.

Within other strong disorder RGs, such as for the transverse field Ising model [13,16], average correlation functions are controlled by the probability that a given pair of sites will remain within the ordered phase as the RG is run up to the scale of the separation between the sites. This is also a question of persistence.

Interestingly, for other random fixed points, such as the problem of two-dimensional Dirac fermions in random magnetic field [18] or the problem of Anderson localization [14], it has been found that averages of low moments of correlation functions exhibit conformal invariance, but for sufficiently high moments conformal invariance breaks down. Shapiro showed for the Anderson localization problem [14] that the breakdown of conformal invariance could be described by a universal distribution of conductance with a broad tail, such that higher moments were divergent when evaluated with the fixed point distribution of conductance. As a result, the higher moments are controlled by the approach to the universal distribution.

We would like to close with a conjecture. For the strongly random systems we consider, all positive moments of the correlation functions have the same scaling behavior. For systems with weaker randomness, one observes different scaling dimensions for the various moments, $q$. However, there is an expectation that the scaling dimension will saturate, and become independent of $q$ for sufficiently large $q$. At this point, one may calculate correlation functions with $q = \infty$. Consider for definiteness a disordered Ising model. In this case, the only contribution to an $n$-point correlation function of spins will arise when all $n$ points are in a locally ferromagnetic region. In this case, we conjecture that the computation of correlation functions will again become a problem of persistence in the RG flow, that the system will stay within the ferromagnetic phase. Therefore, we conjecture that a breakdown of conformal invariance will be seen for sufficiently large moments in other systems as well.

VIII. APPENDIX: A SUSY FIELD THEORY

In this Appendix, we will show that there exists a supersymmetric field theory [21] which reproduces the average correlation functions computed above in the Liouville field theory. Variants of the supersymmetric field theory can be used to describe the one-dimensional and three-dimensional problems considered. The construction of the supersymmetric field theory will show that there exists a purely local field theory with a finite number of fields which violates conformal invariance at the critical point. In the Liouville theory, the integral over $\omega$ in Eq. (32) introduces a non-local interaction, which might otherwise appear to be responsible for the breakdown of conformal invariance.

We are interested in the resolvent

$$G(x, y) = \langle x | \frac{1}{H + \eta} | y \rangle. \quad (45)$$

If the $\eta \to 0$ limit is taken before the infinite volume limit, the resolvent is determined by the exact zero-energy eigenfunction. As discussed above, all the localization properties near zero energy are controlled by the zero energy eigenfunction.

The computation of averages of $G$ follows standard techniques [21]. Introducing a superfield $\Phi$, with bosonic component $\phi$ and fermionic component $\psi$, the resolvent is

$$\frac{1}{Z} \int [d\beta] e^{-S[\beta]} e^{-\overline{F}(iH-\eta)\Phi} \langle x \rangle \langle y \rangle, \quad (46)$$

where

$$Z = \int [d\beta] e^{-S[\beta]} \quad (47)$$

By introducing a set of $m$ superfields $\Phi_i$, we can compute $m$-th moments of the correlation functions. Let us emphasize that we can take $m$ to be fixed, but large, and then we have a single field theory from which high moments and multi-point correlation functions can be computed.

In order to compute the typical correlation functions of section IV, we would need to introduce an infinite set of superfields to compute all moments, and then obtain the
distribution of correlation functions from the moments. Alternately, we could compute them from the Liouville field theory as a limit of the $q$-th moment of the correlation functions as $q \to 0$. In any event, we do not know how to obtain them from a purely local field theory.

In the SUSY field theory, $\beta$ does not acquire any dynamics as $\Phi$ is integrated out. Physically, this is the statement that $\beta$ is quenched. If $\beta$ were not quenched, one would expect $\beta$ to acquire a second derivative term under the RG, and then the theory would flow to a fixed point with finite fluctuations in $\beta$. The fact that disorder is quenched is important for the existence of a strong disorder critical point.

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