The Relativistic Rotation

Z.X. Cao

China Institute of Atomic Energy, P. O. Box 275(18), Beijing, 102413 China

Ch.L. Chen and L. Liu

Department of Physics, Beijing Normal University, Beijing 100875, China

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Abstract

The classical rotation is not self-consistent in the framework of the special theory of relativity. the Relativistic rotation is obtained, which takes the relativistic effect into account. It is demonstrated that the angular frequency of classical rotation is only valid in local approximation. The properties of the relativistic rotation and the relativistic transverse Doppler shift are discussed in this work.

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*Electronic address: caozx@iris.ciae.ac.cn
I. INTRODUCTION

The special and general theory of relativity which was developed in the beginning of the twentieth century, especially through Einstein’s work [1, 2, 3], has its roots far back in the past. But the rotational reference systems utilized in the present researches of physics are all classical [4, 5], although there exists the special and general theory of relativity. In the classical rotation, the velocity \( v \) of one point relative to the center of circle is proportional to its distance \( r \) from the center of circle and its angular frequency \( \omega_0 \). However, this relationship is not suitable within the special relativistic theory.

The velocity formula of classical rotation is given as \( v = r\omega_0 \). It could be understood from the following three aspects. First, the foundation of experiments, i.e., the experience of daily life. Second, Galilean transformations [4]. Third, the general conclusion derived from the first and the second points. Following the inference of Galilean transformations, the velocity formula of classical rotation is generally come into existence under any rotational conditions, such as the condition of long-distance and high-velocity. It means that the formula \( v = r\omega_0 \) is self-consistent in classical.

However, the application of Galilean transformations in high-velocity condition is proved early to be not reliable [4, 5]. It is not correct for the object with high velocity and must be replaced with the principle that could embodies the relativistic properties. Therefore, the extending velocity formula \( v = r\omega_0 \) of classical rotation to arbitrary is suspicious, and it is need to reevaluate. We should start from the special relativistic theory to reconsider the rotation with high velocity.

the proper relativistic metric of rotational reference system should be confirmed. The four-dimension metric of the uniform rotational system is classical in the general theory of relativity now [4, 5]. It means the proper relativistic present of the rotational system couldn’t be given in the general theory of relativity. And the uniform rigid rotation is instantaneous and local inertial, so we could research it in the frame of the special theory of relativity.

To perform the relativistic analysis of uniform rotation, the Galilean transformations and its deduced conclusion, i.e., the formula \( v = r\omega_0 \) is extended to be correct in infinite distances, should be discarded. Nevertheless, we hold the opinion as usual experience that the classical formula of rotation is applicable within short distances. In every observational point, what the observer measures is just the rotational velocities in very close distances,
while for the rotational velocities far away from the observers, the calculation using the transformation of velocity in the special theory of relativity is needed.

We analyze the classical rotation firstly, and point out its non-self-consistent disadvantage. A rotational formula of angular velocity with relativistic character is obtained, which approach to classical velocity in the limit of non-relativity. It shows that the classical rotation is only local and slow-velocity approximation of the relativistic one.

II. THE RELATIVISTIC ROTATION

In classical rotation the velocity $V$ of one point relative to another point is equal to the product of its angular frequency $\omega_0$ and its distance from center point $R$.

$$V = R\omega_0$$

(1)

We show in Fig.1 the rotation of a long pole around point $O$ with frequency $\omega_0$. We assume that the pole is in a part of space so far from all masses that all gravitational effects can be neglected. The distance of point $A$ and $B$ from center point $O$ is $R$ and $2R$, respectively. Their velocities relative to the center point $O$ are $V_{AO} = R\omega_0$ and $V_{BO} = 2R\omega_0$, respectively, in the classical limit. However, if one considers the transformation formula of the velocity in the special theory of relativity, the velocity of point $B$ over point $O$ becomes

$$V'_{BO} = \frac{2R\omega_0}{1 + \left(R\omega_0/c\right)^2} \neq 2R\omega_0.$$  

(2)

We can see from formula (2) that the velocity obtained from classical limit can not coincide with the one derived from the relativistic one due to the use of the transformation formula of the velocity in the special theory of relativity. It implies that the classical rotation is not self-consistent in the framework of the special theory of relativity.

No one will suspect the accuracy of classical rotation formula in daily life. In fact, the correction term in the denominator of Eq.2 is nearly equal to zero under the ordinary condition. Nevertheless, when the classical rotation formula is extended to infinite, i.e. $R\omega_0 \approx c$, the Galilean transformations is still applied but not the special relativity. It is shown that this kind of formula can not fulfill the special theory of relativity. This motivates us to improve the classical rotation.
because the general theory of relativity only gives the principle of equivalence, has nothing to do with the rotational system\[4, 5\], the relativistic rotational system can not be obtained through it. Therefore, we start from the special theory of relativity to study the rotational system, and to derive the kinematical equation of the relevant rotational system. From the kinematical point of view, the investigation of the rotation based on the special theory of relativity is suitable.

We can see from formula (2) that the velocity (Eq.1) of classical rotation is not correct for the whole description of rotation under the framework of special relativity. In order to obtain the suitable expression of the rotational velocity, we discard the assumption of formula (1) for the whole-description and only consider it as the local approximation. The formula (1) is only suitable in the limit of $R\omega_0 \ll c$. Furthermore, we think that there is not intrinsic discrepancy between the inertial and noninertial system if the gravitation is not taken into account, and the instantaneous localization of rotational system is approximately regarded as the inertial one. Moreover, the two adjacent systems of instantaneous localization fulfill the principle of special relativity.

![FIG. 1: The sketch map of the rotating pole](image)

In Fig the observers at each point would see that the object in very small area rotates
uniformly around the point. Assuming that the distance between point A and B is \( R \), we now calculate the velocity of point B relative to point A at the rotating long-pole. Obviously, we could not measure the velocity at point B directly. However, it could be measured via the following method: the distance between point A and B is divided into \( n \) equal parts, and each point in every equal parts could be considered as one instantaneous inertial reference system. Therefore, the velocity of the point, where the distance from the original point is \( \Delta R = R/n \), could be measured, and the velocity of this point is \( V_1 = \Delta R \omega_0 \). Then, we would measure the next \( \Delta R \) position, where the distance from the original point is \( 2\Delta R \), and the velocity of position \( 2\Delta R \) relative to the position \( \Delta R \) is also \( \Delta R \omega_0 \). The measurement is continued until reaching the position of \( R \), and then, we could calculate the velocity of the last point relative to the beginning point taking advantage of the relativistic velocity transfer formula. Based on the relations of velocity in these instantaneous inertial system, the velocity \( V_n \) is the very relativistic velocity \( V_{re} \) when \( n \) approaches infinite.

The velocity of each point relative to point A is

\[
\begin{align*}
\Delta R & \quad V_1 = \Delta R \omega_0 \\
2\Delta R & \quad V_2 = \frac{\Delta R \omega_0 + V_1}{1 + V_1 \Delta R \omega_0/c^2} \\
3\Delta R & \quad V_3 = \frac{\Delta R \omega_0 + V_2}{1 + V_2 \Delta R \omega_0/c^2} \\
& \quad \vdots \\
n\Delta R = R & \quad V_n = \frac{\Delta R \omega_0 + V_{n-1}}{1 + V_{n-1} \Delta R \omega_0/c^2}
\end{align*}
\]

The velocity of inertial system B relative to the one of A is the value of \( V_n \) in the limit of \( n \) approaching infinite. According to formula (3), we obtain

\[
V_n = \frac{c^2}{\Delta R \omega_0} \frac{V_{n-1} \Delta R \omega_0 + 1 + (\Delta R \omega_0)^2/c^2 - 1}{1 + V_{n-1} \Delta R \omega_0/c^2} = \frac{c^2}{\Delta R \omega_0} \left[ 1 - \frac{1 - (\Delta R \omega_0)^2/c^2}{1 + V_{n-1} \Delta R \omega_0/c^2} \right]
\]

Defining \( \eta = 1 - (\Delta R \omega_0)^2/c^2 \), \( U_i = \frac{V \Delta R \omega_0}{c} \) we could rewrite formula (4) as

\[
U_n = 1 + \frac{-\eta}{1 + U_{n-1}}, \quad U_2 = 1 + \frac{-\eta}{1 + U_1}, \quad U_1 = 1 - \eta
\]

The \( U_n \) could be expressed in terms of continued fraction

\[
U_n = 1 + \frac{-\eta}{1 + \frac{-\eta}{2 + \frac{2}{2 + \frac{2}{2 - \eta}}}}
\]
The recursive properties of the continued fraction is as follow

\[ U_{n,1} = \frac{p_1}{q_1} = \frac{1}{1}, U_{n,2} = \frac{p_2}{q_2} = \frac{2-\eta}{2}, \ldots, \]

\[ U_{n,i} = \frac{p_i}{q_i}, \ldots, U_n = \frac{p_n}{q_n} \quad (7) \]

\[ p_i = 2p_{i-1} - \eta p_{i-2}, q_i = 2q_{i-1} - \eta q_{i-2} \quad (i = 3, 4, \ldots, n) \]

where the series of \( p_i \) and \( q_i \) could be solved in terms of the characteristic equation of recursive series

\[ x^2 - 2x + \eta = 0 \quad (8) \]

The two solutions of formula (8) are

\[ x_1 = 1 + \sqrt{1 - \eta}, \quad x_2 = 1 - \sqrt{1 - \eta} \quad (9) \]

Then, the general solutions of \( p_i \) and \( q_i \) are

\[ p_i = A_1(1 + \sqrt{1 - \eta})^i + A_2(1 - \sqrt{1 - \eta})^i \]
\[ q_i = A_3(1 + \sqrt{1 - \eta})^i + A_4(1 - \sqrt{1 - \eta})^i \quad (10) \]

It is obtained with the original condition of formula (7).

\[ A_1 = \frac{\sqrt{1 - \eta}}{2}, A_2 = -\frac{\sqrt{1 - \eta}}{2}, \]
\[ A_3 = \frac{1}{2}, A_4 = \frac{1}{2} \quad (11) \]

Inserting formula (11) into formula (10) and considering the formula (7), we can get

\[ U_n = \sqrt{1 - \eta}(1 + \sqrt{1 - \eta})^n - (1 - \sqrt{1 - \eta})^n \quad (12) \]

Inserting \( \eta = 1 - (\Delta R_0/c)^2 \) and \( \Delta R = R/n \) into the former formula (12), the relativistic velocity in the limit of \( n \to \infty \) is

\[ \frac{V_{re}}{c} = \lim_{n \to \infty} \frac{U_n}{\sqrt{1 - \eta}} = \lim_{n \to \infty} \frac{(1 + \frac{R_0}{nc})^n - (1 - \frac{R_0}{nc})^n}{(1 + \sqrt{1 - \eta})^n + (1 - \sqrt{1 - \eta})^n} = \tanh\left(\frac{R_0}{c}\right) \]

i.e.,

\[ V_{re}(R) = c \tanh\left(\frac{R_0}{c}\right) \quad (13) \]

which is the expecting formula of relativistic rotation velocity. We calculate numerically the results of formula (3), and the consistent results are reached with formula (13). Comparing with the classical rotation formula (1), we can see that the classical rotation is the
approximation of formula (13) in the limit of $c \to \infty$. For the case far from classical rotation, an additional correction term $-R^3\omega_0^3/c^2$ is obtained from formula (13) in comparison with formula (1), i.e.,

$$V_{re} \simeq R\omega_0 - R^3\omega_0^3/c^2 \tag{14}$$

Considering the three points $(A, B, \text{and } C)$ in the same line, we could calculate easily the their velocity using the relativistic transfer formula of velocity and formula (13)

$$V_{CA} = \frac{V_{CB} + V_{BA}}{1 + V_{CB}V_{BA}/c^2} = c \tanh\left(\frac{R_{CA}\omega_0}{c}\right) \tag{15}$$

These show that the velocity relation in formula (13) is self-consistent in the framework of special relativity, and now the relativistic space-time interval are different with the classical one[4]. It could be expressed in cylindrical coordinates

$$ds^2 = dr^2 + dz^2 + r^2 d\phi^2 + 2cr \tanh(r\omega_0/c)d\phi dt - c^2 \cosh^{-2}(r\omega_0/c)dt^2 \tag{16}$$

It is also observed from the comparison between formula (13) and (13) that the angular velocity in the classical rotation is only the value in token of the frequency of relativistic rotation, while not the real angular velocity of relativistic rotation. The relativistic angular velocity $\omega$ of two points in relativistic rotation is no longer the physical invariable in the classical rotation. It is with respect to not only the token angular velocity but also the distance of the two points, i.e.,

$$\omega(r) = \frac{c\tanh(r\omega_0/c)}{r} \tag{17}$$

It is obviously $\omega(r) \to \omega_0$ with $r \to 0$ and $\omega(r) \to 0$ with $r \to \infty$, which is different with the classical one. In classical rotation reference systems, what could be observed is only within the range from observation point extending to $c/\omega_0$. Otherwise, with larger distances, the observation velocity will greater than velocity of light that could not be achieved by realistic objects. Nevertheless, in the relativistic rotation reference systems, the former case could not exist. It could be found from formula (13) that all the velocity are less than velocity of light $c$. Furthermore, the discrepancy of angular velocity with the relativistic uniform rotation are generally not observed in the dailylife due to the condition $R\omega_0 \ll c$. 

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III. TRANSVERSE DOPPLER EFFECT IN RELATIVISTIC ROTATION

The expression of transverse Doppler effect can be obtained from special relativity in classical rotation \[5\]. For instance, there is a light-source with frequency $\nu_0$ at point B in Fig.1. Following the Doppler formula in special relativity and formula (1), the frequency of the light at point A in classical rotation is given as

$$\nu_{\text{classic}} = \nu_0 \sqrt{1 - \left(\frac{R\omega_0}{c}\right)^2} \quad (18)$$

However, in relativistic rotation, through the Doppler formula in special relativity and formula (13), the observed frequency of the light at point A is as follow

$$\nu_{\text{relativity}} = \nu_0 \sqrt{1 - \tanh^2\left(\frac{R\omega_0}{c}\right)} = \nu_0 \cosh^{-1}\left(\frac{R\omega_0}{c}\right) \quad (19)$$

For further understanding, we plot in Fig.2 the transverse Doppler shift for classical and relativistic rotations, respectively, for the comparison between formula (18) and (19). The product $R\omega_0$ in classical rotation can not greater than velocity of light, otherwise, imaginary number will occur in the frequency. However, in relativistic rotation, this product may greater than the velocity of light, and the properties of $\nu_{\text{relativity}}$ in equation (19) will not be affected. The difference of these two cases would only exhibit when the value of $R\omega_0$ approaches the velocity of light. Expanding the series of formula (18) and (19) to their
order with $R^4$, We can get their difference

$$\Delta \nu = \nu_{\text{relativity}} - \nu_{\text{classic}} = \nu_0 \frac{(R\omega_0/c)^4}{3} + O(R^4) \quad (20)$$

It is obviously showed in formula (20) that the discrepancy is hardly observed in ordinary conditions. However, we may observe this discrepancy in the large-scale rotations such as the rotation of galaxy.

IV. SUMMARY

The classical rotation which considers the rotation formula (1) as the whole-conditions is not self-consistent in the special relativistic description. We propose the velocity expression (13) of relativistic rotation taking advantage of formula (1) only as the local condition. This velocity is self-consistent for the whole description of rotation. The relativistic rotation can reduce to the classical one in the classical conditions, while greatly different properties are exhibited in comparison with the classical rotation in the relativistic condition, i.e. $R\omega_0 \sim c$. With the flourish development of astrophysics, the self-consistent relativistic rotation would make one understand the phenomena of celestial rotations more suitably than the classical one. Moreover, the relativistic rotation may provide a feasible way for the basic and theoretical description of rotation.

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