THE FULL GROUP OF ISOMETRIES OF SOME COMPACT LIE GROUPS ENDOWED WITH A BI-INVARIANT METRIC

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Abstract. We describe the full group of isometries of absolutely simple, compact, connected real Lie groups, of $SO(4)$ and of $U(n)$, endowed with suitable bi-invariant Riemannian metrics.

Keywords. (absolutely simple, compact, connected) real Lie group, Lie algebra, Killing metric, Frobenius metric, (special) orthogonal group, compact symplectic group, (special) unitary group.

Mathematics Subject Classification (2020): 53C35, 22E15.

Grants: This research has been partially supported by GNSAGA-INdAM (Italy).

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Introduction

In this paper we describe the full group of isometries of some classes of real Lie groups, endowed with suitable bi-invariant Riemannian metrics: the Killing metric both on any absolutely simple, compact, connected Lie group and on the special orthogonal group $SO(4)$, and also the metric induced, on the unitary group $U(n)$, by the flat Frobenius metric of $M_n(C)$.

In [Dolcetti-Pertici 2015] and in [Dolcetti-Pertici 2018] we have already studied another relevant example of (semi-Riemannian) metric: the so-called trace metric, which is bi-invariant on $GL_n(\mathbb{R})$ and on its Lie subgroups. Some techniques, used in the present paper,

1In the present paper, a real Lie group is said to be absolutely simple if the complexification of its Lie algebra is a simple Lie algebra.

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have been developed in those papers and in [Dolcetti-Pertici 2019, Dolcetti-Pertici 2020, Dolcetti-Pertici 2021].

Given any Lie group \( G \), the Killing form of its Lie algebra extends, on the whole \( G \), to a bi-invariant symmetric \((0, 2)\)-tensor, denoted by \( \mathcal{K} \), and called Killing tensor of \( G \).

Some further properties of \( G \) give some relevant consequences. For instance, as well-known, \( G \) is semi-simple if and only if \( \mathcal{K} \) (and also \( -\mathcal{K} \)) is a semi-Riemannian metric on \( G \) (Cartan’s criterion) and, if \( G \) is semi-simple and compact, then the tensor \( -\mathcal{K} \) is a Riemannian metric on \( G \), we call Killing metric of \( G \). Furthermore, if \( G \) is connected, compact and simple, then \((G, -\mathcal{K})\) is a globally symmetric Riemannian manifold with non-negative sectional curvature and, moreover, if \( G \) is also absolutely simple, then \((G, -\mathcal{K})\) is an Einstein manifold. The Killing tensor of \( G \) is more than an example of bi-invariant tensor on \( G \), indeed if \( G \) is connected and absolutely simple, then every bi-invariant real \((0, 2)\)-tensor on \( G \) is a constant multiple of \( \mathcal{K} \). These results are essentially matter of Section 1.

Section 2 is devoted to the general result of this paper:

**Theorem 2.3** Let \( G \) be a absolutely simple, compact, connected real Lie group and let \( -\mathcal{K} \) be its Killing metric. Then \( F : (G, -\mathcal{K}) \to (G, -\mathcal{K}) \) is an isometry if and only if there exist an element \( a \in G \) and an automorphism \( \Phi \) of the Lie group \( G \) such that either \( F = L_a \circ \Phi \) or \( F = L_a \circ \Phi \circ j \), where \( L_a \) is the left translation associated to \( a \) and \( j \) is the inversion map.

Many classical groups satisfy all conditions of the above Theorem, precisely: the special orthogonal groups \( SO(n) \), with \( n \geq 3 \) and \( n \neq 4 \), the special unitary groups \( SU(n) \), with \( n \geq 2 \), the compact symplectic groups \( Sp(n) \), with \( n \geq 1 \).

A careful analysis of the automorphisms of each group allows us to deduce the complete lists of the isometries of \((G, -\mathcal{K})\), where \( G \) is one of the previous classical groups, (Theorem 2.5).

The manifold \((SO(4), -\mathcal{K})\) is not included in the previous result: indeed \( SO(4) \) is semi-simple but not simple, however \( -\mathcal{K} \) is still a Riemannian metric on it. Section 3 is devoted to this particular case. The key points are the following: \((SO(4), -\mathcal{K})\) is isometric to the Lie group \( \frac{SU(2) \times SU(2)}{\{\pm (I_2, I_2)\}} \) (endowed with its Killing metric) and the natural covering projection of \( SU(2) \times SU(2) \) (endowed with the product of the Killing metrics) onto the previous quotient, is clearly a local isometry. All isometries of \( SU(2) \times SU(2) \) are obtained by means of the analysis of Section 2 via a classical result of De Rham. Since these ones project as isometries of the quotient, we can obtain the main result of the section:

**Theorem 3.5** The isometries of \((SO(4), -\mathcal{K})\) are precisely the following maps:
$X \to AXB$, $X \to AX^T B$, $X \to A\tau(X)B$, $X \to A\tau(X)^T B$, where $A, B$ are matrices both in $SO(4)$ or both in $O(4) \setminus SO(4)$ (and $\tau$ is a suitable map constructed by means of the Cayley’s factorization of $SO(4)$).

Finally, Section 4 is devoted to $U(n)$, endowed with the bi-invariant Riemannian metric $\phi$, restriction to $U(n)$ of the flat Frobenius metric of $M_n(C)$. This metric is not multiple of the Killing tensor, because $U(n)$ is not semi-simple (and so its Killing tensor is degenerate).

Analogously to Section 3, we get a covering map (which is also a local isometry) from $SU(n) \times \mathbb{R}$ (endowed with a suitable product metric) onto $(U(n), \phi)$. This allows to get the following main results of this Section, with a difference between the cases $n = 2$ and $n \geq 3$, due to the fact that all isometries of $SU(2) \times \mathbb{R}$ project as isometries of $(U(2), \phi)$, whereas, for $n \geq 3$, $SU(n) \times \mathbb{R}$ has more types of isometries, not all projecting as maps of $U(n)$.

**Theorem 4.7.** The isometries of $(U(2), \phi)$ are precisely the following maps:

$X \to AXB$, $X \to AX^*B$, $X \to \frac{AXB}{\det(X)}$, $X \to \det(X)AX^*B$, with $A, B \in U(2)$.

**Theorem 4.8.** The isometries of $(U(n), \phi)$, with $n \geq 3$, are precisely the following maps:

$X \to AXB$, $X \to AX^*B$, $X \to AXB$, $X \to AX^T B$, with $A, B \in U(n)$.

**Acknowledgement.** We wish to express our gratitude to Fabio Podestà for his help and for many discussions about the matter of this paper.

1. **Notations and preliminary facts.**

1.1. **Notations.**

In this paper we will use many standard notations from the matrix theory, which should be clear from the context, among these: $M_n(\mathbb{R})$ for the vector space of real square matrices, $O(n)$ for the group of real orthogonal matrices, $SO(n)$ for the the group of real special orthogonal matrices, $Sp(n)$ for the compact symplectic group, $M_n(C)$ for the vector space of complex square matrices, $U(n)$ for the group of unitary matrices, $SU(n)$ for the group of special unitary matrices (all matrices of order $n$). If $A$ is a matrix, then $A^T$, $A^{-1}$, $\bar{A}$, $A^* := \bar{A}^T$ denote its transpose, its inverse (when it exists), its conjugate and its transpose conjugate. $I_n$ is the identity matrix of order $n$ and $i \in \mathbb{C}$ is the unit imaginary number.

The basic notations and notions on real Lie groups and algebras are the following:

- $G$ is a real Lie group with identity $e$ and inversion map $j : x \mapsto x^{-1}$, $\mathfrak{g}$ is its Lie algebra (identified with its tangent space at $e$), $\exp : \mathfrak{g} \to G$ is the exponential map and $\text{Aut}(G)$ denotes the Lie group of all (smooth) automorphisms of $G$;
- if $\mathfrak{g}$ is a real Lie algebra, $\mathfrak{g}^\mathbb{C} := \mathfrak{g} \oplus i\mathfrak{g} = \mathfrak{g} \otimes \mathbb{C}$ will denote its complexification, which turns to be a complex Lie algebra, having $\mathfrak{g}$ as real subalgebra;
- if $\mathfrak{h}$ is a complex Lie algebra, $\mathfrak{h}^\mathbb{R}$ will denote its realification, i. e. $\mathfrak{h}^\mathbb{R}$ is simply $\mathfrak{h}$, regarded as real Lie algebra;
for every \( a \in G \), \( L_a \) and \( R_a \) are, respectively, the left and right translations in \( G \) associated to \( a \) and \( C_a := L_a \circ R_a^{-1} \) is the inner automorphism of \( G \) associated to \( a \);

- for every \( a \in G \), \( Ad_a \) is the automorphism of \( g \), defined as the differential at \( e \) of \( C_a \). It is well-known that \( \exp \circ Ad_a = C_a \circ \exp \);

- \( K \) is the left-invariant symmetric \((0,2)\)-tensor on the whole \( G \), extending the Killing form of \( g \), so that the Killing form of \( g \) agrees with \( K_e \). We call \( K \) the Killing tensor of the Lie group \( G \).

1.2. Lemma. The Killing tensor \( K \) of the Lie group \( G \) is bi-invariant on \( G \) and it is preserved by every \( \phi \in \text{Aut}(G) \) and by the inversion map \( j \) (i.e. \( \phi^*(K) = K \) and \( j^*(K) = K \)).

Proof. \( K_e \) is invariant with respect to all automorphisms of \( g \), hence the left-invariant tensor \( K \) is preserved by all smooth automorphisms of \( G \) (in particular by all inner automorphisms) and so \( K \) is right-invariant too. For the assertion on \( j \) see, for instance, [Helgason 1978 pp. 147-148]. □

1.3. Remarks-Definitions. We say that a (finite dimensional) Lie algebra \( g \) is simple, if it is non-abelian and has no ideals except 0 and \( g \); while we say that \( g \) is semi-simple, if it splits into the direct sum of simple Lie algebras; by the well-known Cartan’s criterion, \( g \) is semi-simple if and only if its Killing form is non-degenerate (see for instance [EOM - LA]).

A Lie group is said to be simple (respectively semi-simple), if its Lie algebra is simple (respectively semi-simple). Hence a simple Lie group is semi-simple too.

Note that if \( G \) is a semi-simple Lie group, then \((G, K)\) and \((G, -K)\) are semi-Riemannian manifolds. We refer to \(-K\) (the opposite of the Killing tensor \( K \)) as the Killing metric of the (semi-simple) Lie group.

1.4. Proposition. Let \( G \) be a semi-simple connected Lie group. Then
a) the geodesics of the semi-Riemannian manifold \((G, -K)\) are precisely the curves of the form \( t \mapsto x \exp(tv) \), for every \( t \in \mathbb{R} \), with \( x \) generic in \( G \) and \( v \) generic in the Lie algebra \( g \) of \( G \) (so \((G, -K)\) is geodesically complete);

b) the Levi-Civita connection \( \nabla \) of \((G, -K)\) is the 0-connection of Cartan-Schouten, defined by \( \nabla_X(Y) := \frac{1}{2}[X,Y] \), where \( X \) and \( Y \) are left-invariant vector fields on \( G \);

c) the curvature tensor of type \((1,3)\) of \((G, -K)\) is \( R_{XYZ} := \nabla_{[X,Y]}Z - [\nabla_X, \nabla_Y]Z = \frac{1}{4}[[X,Y],Z] \), where \( X, Y, Z \) are left-invariant vector fields on \( G \);

d) the curvature tensor of type \((0,4)\) of \((G, -K)\) is the bi-invariant tensor, defined by \( R_{XYZW} := -K(R_{XY}Z,W) = -\frac{1}{4}K([X,Y],[Z,W]) \), where \( X, Y, Z, W \) are left-invariant vector fields on \( G \).
Proof. Part (a), part (b) and part (c) follow directly from the results contained in [Helgason 1978, p. 148 and p. 548-550] (our tensor $R$ is the opposite of the corresponding tensor of [Helgason 1978]). Part (c) implies that $R_{XYZW} = -\frac{1}{4}K([X,Y],Z,W)$. By the skew-symmetry, with respect to the Killing form, of every operator $\text{ad}_v: x \to [v,x]$ (see for instance [EoM - KF]), we have $K([X,Y],Z,W) = K([X,Y],[Z,W])$ and this concludes (d). □

1.5. Remark-Definition. We say that a real Lie group $G$ is a complex Lie group, if it possesses a complex analytic structure, compatible with the real one, such that multiplication and inversion are holomorphic. It is known that a real Lie group $G$ with Lie algebra $\mathfrak{g}$ is complex if and only if there exists a complex Lie algebra $\mathfrak{h}$ such that $\mathfrak{h}^\mathbb{C} = \mathfrak{g}$ (see [Knapp 2002, Prop. 1.110 p. 95]).

1.6. Lemma. Let $G$ be a real Lie group and let $\mathfrak{g}$ be its Lie algebra with $\mathfrak{g}^\mathbb{C}$ as complexification. Then the complex Lie algebra $\mathfrak{g}^\mathbb{C}$ is simple if and only if $G$ is simple and not complex.

Proof. It follows from [Knapp 2002, Thm. 6.94 p. 407], remembering that if $\mathfrak{g}^\mathbb{C}$ is a simple complex Lie algebra, then $\mathfrak{g}$ is a simple real Lie algebra. □

1.7. Definition. We say that a real Lie group is absolutely simple if it is simple and not complex or, equivalently by Lemma 1.6, if the complexification of its Lie algebra is a simple, complex Lie algebra.

A standard consequence of Schur’s Lemma is the following

1.8. Proposition. Let $G$ be a real Lie group and assume that $G$ is connected and absolutely simple. Then every bi-invariant real $(0,2)$-tensor on $G$ is a constant multiple of the Killing metric $-K$ of $G$.

1.9. Lemma. Let $G$ be a real Lie group and assume that $G$ is semi-simple and compact. Then the Killing tensor $K$ of $G$ is negative definite at every point (i.e. the Killing metric $-K$ is a Riemannian metric on $G$).

Proof. It follows from [Helgason 1978, Prop. 6.6(i) p. 132, Cor. 6.7 p. 133]. □

1.10. Remark. Let $G$ be a simple, compact, connected, real Lie group and let $\mathfrak{g}$ be its Lie algebra; denote by $\Delta$ the diagonal of $G \times G$ and by $Z$ the center of $G$. $Z$ is a closed subgroup of $G$ and it is finite. Indeed the center of $\mathfrak{g}$ is zero (since $G$ is simple, see [Helgason 1978, Cor. 6.2 p. 132]). Since the Lie algebra of $Z$ agrees with the center of $\mathfrak{g}$, then $Z$ is a discrete subgroup of the compact group $G$ and therefore $Z$ is finite. Now we denote by $U$ the semisimple compact connected Lie group defined by $U := \frac{G \times G}{(Z \times Z) \cap \Delta}$ and consider the map $T : U \times G \to G$, $T(\{(g,h)\}, x) = gxh^{-1}$
where \{(g, h)\} is the class of \((g, h)\) in \(\frac{G \times G}{(Z \times Z) \cap \Delta}\). 

\(T\) is an effective and transitive left action of \(U\) on \(G\) and its isotropy subgroup at the identity is \(\tilde{\Delta} := \frac{\Delta}{(Z \times Z) \cap \Delta}\). Therefore \(G\) is diffeomorphic to the homogeneous space \(\frac{U}{\tilde{\Delta}}\). Moreover, for every \(\{(g, h)\} \in U\), the map \(x \mapsto T(\{(g, h)\}, x)\) is an isometry with respect to \(-K\) (and to \(K\)). Finally the pair \((U, \tilde{\Delta})\) is a Riemannian symmetric pair (in the sense of [Helgason 1978, p. 209]) with involutive automorphism given by \(\sigma(\{(g, h)\}) = \{(h, g)\}\).

1.11. Proposition. Let \(G\) be a simple, compact, connected, real Lie group and let \(-K\) be its Killing metric.

Then \((G, -K)\) is a globally symmetric Riemannian manifold with non-negative sectional curvature; furthermore every connected component of the Lie group of its isometries is diffeomorphic to \(\frac{G \times G}{(Z \times Z) \cap \Delta}\), where \(Z\) is the center of \(G\) and \(\Delta\) is the diagonal of \(G \times G\). Moreover, if \(G\) is absolutely simple too, then \((G, -K)\) is an Einstein manifold.

Proof. By [Helgason 1978, Prop. 3.4 p. 209], \((G, -K)\) is a globally symmetric Riemannian manifold, via Remark 1.10. By Proposition 1.11 (d), the sectional curvature of the space generated by two left-invariant and \(R\)-independent vector fields \(X, Y\) of \(G\), agrees with \(-\frac{1}{4}K([X, Y], [X, Y])\), which is non-negative and equal to 0 if and only if \([X, Y] = 0\). The assertion about the connected components of the Lie group of the isometries, follows from [Helgason 1978, Thm. 4.1 (i) p. 243] and from the fact that in a Lie group all connected components are diffeomorphic to the component containing the identity.

The last statement is a consequence of Proposition 1.12 taking into account that the Ricci tensor of \((G, -K)\) is bi-invariant. \(\square\)

1.12. Remark. For further details and information on Lie groups with bi-invariant metrics, we refer, for instance, to [Alexandrino-Bettiol 2015, Ch. 2].

2. Isometries of a Compact Lie Group

2.1. Lemma. Let \(g\) be a real Lie algebra, whose complexification \(g^C\) is a simple, complex Lie algebra and let \(L\) be an isometry with respect to the Killing form \(B\) of \(g\), such that \(L([v, w]) = [v, L(w)]\) for every \(v, w \in g\).

Then \(L = \pm Id_g\).

Proof. The killing form \(B^C\) of \(g^C\) is the extension by \(C\)-linearity of the Killing form \(B\) of \(g\); by \(C\)-linearity too, \(L\) can be extended to a map \(L^C : g^C \to g^C\) which is an isometry, with respect to the Killing form \(B^C\) of \(g^C\), satisfying again the analogous condition \(L^C([v, w]) = [v, L^C(w)]\) for every \(v, w \in g^C\). Let \(\lambda \in \mathbb{C}\) be an eigenvalue of \(L^C\) and let \(V_\lambda \neq \{0\}\) be the corresponding eigenspace. If \(v \in g^C\) and \(w \in V_\lambda\), then \(L^C([v, w]) = [v, \lambda w] = \lambda[v, w]\), so \([v, w] \in V_\lambda\), which turns out to be a non-zero ideal of \(g^C\) and therefore \(V_\lambda = g^C\), i. e. \(L^C = \lambda Id_{g^C}\). Since \(L^C\) is an isometry with respect to the Killing form \(B^C\), which is non-degenerate by Cartan’s criterion, the map \(L^C\) agrees with \(\pm Id_{g^C}\), so that \(L = \pm Id_g\). \(\square\)
2.2. Proposition. Let $G$ be an absolutely simple, compact, connected, real Lie group and let $-K$ be its Killing metric. Then $F : (G, -K) \to (G, -K)$ is an isometry, fixing the identity $e \in G$, if and only if there exists an automorphism $\Phi$ of the Lie group $G$ such that either $F = \Phi$ or $F = \Phi \circ j$, where $j$ is the inversion map.

Proof. Lemma 1.2 implies that the automorphisms and the inversion map of the Lie group $G$ are isometries with respect to $-K$, fixing $e$.

For the converse, let $J$ be the group of isometries of $(G, -K)$, let $J_0$ be the corresponding subgroup of isotropy at $e$ and let $J^0, J_0^0$ be their connected components containing the identity. In Remark 1.10 we observed that $(U, \tilde{\Delta})$ is a Riemannian symmetric pair and so, by [Helgason 1978, Thm. 4.1(i) p. 243], we have $J^0 \simeq U$ (as Lie groups). From this we get that $\dim(J) = \dim(J^0) = \dim(U) = 2 \dim(G)$ and therefore $\dim(J_0^0) = \dim(J_0) = \dim(J) - \dim(G) = \dim(G)$.

Let us consider the adjoint representations of $G$ and of its Lie algebra $\mathfrak{g}$, denoted by $Ad : G \to GL(\mathfrak{g})$ and by $ad : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$, respectively; we indicate with $Ad(G)$ and with $ad(\mathfrak{g})$ their images. Note that $Ad(G)$ is a closed Lie subgroup of $GL(\mathfrak{g})$ and $ad(\mathfrak{g})$ is its Lie algebra; moreover, since the kernel of the map $ad$ agrees with the center of $\mathfrak{g}$ and this last is zero, we get that $ad : \mathfrak{g} \to ad(\mathfrak{g})$ is an isomorphism of Lie algebras; this implies that $Ad(G)$ and $G$ have the same dimension.

Let us consider also the representation $d : J_0 \to GL(\mathfrak{g})$ defined as the differential at $e$ of every element of $J_0$. By [O’Neill 1983, Prop. 62 p. 91], $d$ is a faithful representation and so: $d(J_0^0) = (d(J_0))^0$ (the component of the image $d(J_0)$ containing the identity). Hence $\dim(d(J_0))^0 = \dim(J_0^0) = \dim(G)$. Since $G$ is connected, we have the inclusion $Ad(G) \subseteq (d(J_0))^0$. Now these manifolds have the same dimension, hence, by the Theorem of invariance of domain, $Ad(G)$ is open in $(d(J_0))^0$; moreover, $Ad(G)$ is compact and $(d(J_0))^0$ is connected and this allows to get that $Ad(G) = (d(J_0))^0$.

For any fixed $F \in J_0$, the previous equality gives: $dF Ad(G) dF^{-1} = Ad(G)$. Hence there exists a unique automorphism $\alpha$ of $Ad(G)$ such that:

\[(*) \quad dF \circ Ad_X \circ dF^{-1} = \alpha(Ad_X), \quad \text{for every } X \in G.\]

We denote by $\exp : \mathfrak{g} \to G$ and by $\tilde{\exp} : ad(\mathfrak{g}) \to Ad(G)$ the two usual exponential maps. It is well-known that $Ad \circ \exp = \tilde{\exp} \circ ad$ (see for instance [Hall 2015, Thm. 3.28 p. 60]). For every $t \in \mathbb{R}$ and every $v \in \mathfrak{g}$, the (*) above becomes:

\[(***) \quad dF \circ Ad_{\exp(tv)} \circ dF^{-1} = \alpha(Ad_{\exp(tv)}).\]

Now let $\tilde{\alpha}$ be the unique automorphism of $ad(\mathfrak{g})$ such that $\alpha \circ \tilde{\exp} = \tilde{\exp} \circ \tilde{\alpha}$. The map $\overline{\alpha} := ad^{-1} \circ \tilde{\alpha} \circ ad$ is an automorphism of the Lie algebra $\mathfrak{g}$, satisfying $Ad \circ \exp \circ \overline{\alpha} = \alpha \circ Ad \circ \exp$. Hence, for every $t \in \mathbb{R}$ and every $v \in \mathfrak{g}$, the (***) becomes:

\[(****) \quad dF \circ Ad_{\exp(tv)} \circ dF^{-1} = Ad_{\exp(\overline{\alpha}(v))}.\]
Now, if we derive the identity (***), with respect to $t$, for $t = 0$, we get:

$$dF \circ ad_e \circ dF^{-1} = ad_{\alpha(e)}.$$ 

Since $ad_e(w) = [v, w]$ for every $v, w \in g$ and remembering that $\alpha$ is an automorphism of the Lie algebra $g$, we get $dF([v, w]) = [\alpha(v), dF(w)] = \alpha([v, dF(w)])$ and so $(\alpha^{-1} \circ dF)([v, w]) = [v, (\alpha^{-1} \circ dF)(w)]$, for every $v, w \in g$. Note that $dF$ and $\alpha$ are both isometries of $g$, with respect to its Killing form; moreover, since $G$ is absolutely simple, its Lie algebra $g$ satisfies the hypotheses of Lemma 2.24, thus we obtain $dF = \pm \alpha$.

Let $\pi : \tilde{G} \to G$ be the universal covering group of $G$ and let $\tilde{F} : \tilde{G} \to \tilde{G}$ be such that $F \circ \pi = \pi \circ \tilde{F}$, with $\tilde{F}(\tilde{e}) = \tilde{e}$, where $\tilde{e}$ is the identity of $\tilde{G}$; from this we get $\tilde{F}_* = \pi_*^{-1} \circ dF \circ \pi_* = \pi_*^{-1} \circ (\pm \alpha) \circ \pi_*$, where $\tilde{F}_*$, $\pi_*$ denote the differentials at the identity $\tilde{e}$ of $\tilde{F}$ and $\pi$, respectively. If we denote by $\beta$ the automorphism of the Lie algebra $\tilde{g}$ of $\tilde{G}$, given by $\beta = \pi_*^{-1} \circ \alpha \circ \pi_*$, we can write $\tilde{F}_* = \pm \beta$.

By [Warner 1985, Thm. 3.27 p. 101], there exists a unique automorphism $\Psi$ of the simply connected Lie group $\tilde{G}$, whose differential at the identity $\tilde{e}$, $\Psi_*$, agrees with $\beta$. Hence $\tilde{F}_* = \pm \Psi_*$. Since $\Psi$ is an automorphism of $\tilde{G}$, it is an isometry of $(\tilde{G}, -\tilde{K})$, where $-\tilde{K}$ is the Killing metric of $\tilde{G}$ (remember Lemma 1.22). It is easy to check that $\pi : (\tilde{G}, -\tilde{K}) \to (G, -K)$ is a local isometry and this implies that $\tilde{F} : (\tilde{G}, -\tilde{K}) \to (G, -K)$ is an isometry too.

If $\tilde{F}_* = \Psi_*$, then $\tilde{F} = \Psi$ (see, for instance, [O’Neill 1983, Prop. 62 p. 91]) and hence $F \circ \pi = \pi \circ \Psi$. The surjectivity of $\pi$, together with the fact that $\pi$ and $\Psi$ are Lie group homomorphisms, implies that $F$ is a (bijective) endomorphism of $G$. This allows to conclude that $F \in Aut(G)$.

Suppose now that $\tilde{F}_* = -\Psi_*$. We denote by $\tilde{j}$ the inversion map of $\tilde{G}$ and by $\tilde{j}_*$ its differential at the identity $\tilde{e}$. By Lemma 1.22 $\tilde{j}$ is an isometry of $(\tilde{G}, -\tilde{K})$, furthermore $\tilde{j}_*$ agrees with the opposite of the identity map (see, for instance, [Helgason 1978] p. 147).

Now $\tilde{F}_* = \tilde{j}_* \circ \Psi_* = (\tilde{j} \circ \Psi)_*$ and, arguing as in the previous case, we get that $\tilde{F} = \tilde{j} \circ \Psi$ and so, $F \circ \pi = \pi \circ \tilde{j} \circ \Psi = j \circ \pi \circ \Psi$. Hence $j \circ F \circ \pi = \pi \circ \Psi$ and, as above, we obtain that $\Phi := j \circ F \in Aut(G)$; therefore we conclude that $F = j \circ \Phi = \Phi \circ j$, with $\Phi \in Aut(G)$. □

2.3. Theorem. Let $G$ be an absolutely simple, compact, connected real Lie group and let $-K$ be its Killing metric. Then $F : (G, -K) \to (G, -K)$ is an isometry if and only if there exist an element $a \in G$ and an automorphism $\Phi$ of the Lie group $G$ such that either $F = L_a \circ \Phi$ or $F = L_{a^{-1}} \circ \Phi \circ j$, where $L_a$ is the left translation associated to $a$ and $j$ is the inversion map.

Proof. Note that $L_a \circ \Phi$ and $L_a \circ \Phi \circ j$ are both isometries, because they are compositions of isometries (remember again Lemma 1.22).

The converse follows from Proposition 2.24 because, for $a = F(e)$, $L_a^{-1} \circ F$ is an isometry fixing the identity $e \in G$. □
2.4. Remark. As known, relevant examples of absolutely simple, compact, connected, real Lie groups are:

- the special orthogonal group $SO(n)$, $n \geq 3, n \neq 4$;
- the special unitary group $SU(n)$, $n \geq 2$;
- the compact symplectic group $Sp(n)$, $n \geq 1$.

The automorphisms of $SO(n)$, with $n \geq 3$ odd, of $SU(2)$ and of $Sp(n)$, with $n \geq 1$, are precisely the inner automorphisms of the corresponding group.

Furthermore the automorphisms of $SO(n)$, with $n \geq 6$ even, are precisely the maps $X \mapsto AXA^T$, with $A \in O(n)$.

Finally the automorphisms of $SU(n)$, with $n \geq 3$, are the inner automorphisms and all the maps $X \mapsto CXC^*$, where $C \in SU(n)$.

From these facts and from Theorem 2.3 we can easily get the following

2.5. Theorem.

a) The isometries of $(SO(n), -\mathcal{K})$, with $n \geq 3$ odd, are precisely the following maps:

$$X \mapsto AXB \text{ and } X \mapsto AX^TB, \text{ with } A, B \in SO(n).$$

b) The isometries of $(SO(n), -\mathcal{K})$, with $n \geq 6$ even, are precisely the following maps:

$$X \mapsto AXB \text{ and } X \mapsto AX^TB, \text{ with } A, B \text{ both in } SO(n) \text{ or both in } O(n) \setminus SO(n).$$

c) The isometries of $(SU(2), -\mathcal{K})$ are precisely the following maps:

$$X \mapsto AXB \text{ and } X \mapsto AX^*B, \text{ with } A, B \in SU(2).$$

d) The isometries of $(SU(n), -\mathcal{K})$, with $n \geq 3$, are precisely the following maps:

$$X \mapsto AXB, \ A \mapsto AX^*B, \ X \mapsto AXB \text{ and } X \mapsto AX^TB, \text{ with } A, B \in SU(n).$$

e) The isometries of $(Sp(n), -\mathcal{K})$, with $n \geq 1$, are precisely the following maps:

$$X \mapsto AXB \text{ and } X \mapsto AX^{-1}B, \text{ with } A, B \in Sp(n).$$

2.6. Remark. The Lie groups of isometries of $(SO(n), -\mathcal{K})$, with $n \geq 3$ odd, of isometries of $(SU(2), -\mathcal{K})$, and of isometries of $(Sp(n), -\mathcal{K})$, with $n \geq 1$, have two connected components, while the Lie groups of isometries of $(SO(n), -\mathcal{K})$, with $n \geq 6$ even, and of isometries of $(SU(n), -\mathcal{K})$, with $n \geq 3$, have four connected components.

2.7. Remark. If $G$ is one of the groups $SO(n)$, $n \geq 3$ and $n \neq 4$, $SU(n)$, $n \geq 2$, or $Sp(n)$, $n \geq 1$, then $\mathcal{K}_A(X, Y) = c \cdot tr(A^{-1}XA^{-1}Y)$, for some strictly positive constant $c$, for every $A \in G$ and for every $X, Y \in T_A(G)$ (as we can deduce, for instance, from [Sepanski 2007 Ex. 6.19 p. 129]).

We denote by $\phi$ the (flat) Frobenius hermitian metric of $M_m(\mathbb{C})$ $(m \geq 2)$, defined by $\phi(A, B) = Re(tr(AB^*))$, for every $A, B \in M_m(\mathbb{C})$. To simplify the notations, we denote also by $\phi$ its restriction to each submanifold $N$ of $M_m(\mathbb{C})$ and we call it Frobenius metric of $N$. It is just a computation that, if $A \in U(m)$, then the maps $L_A$ and $R_A$, are isometries of $(M_m(\mathbb{C}), \phi)$ and, therefore, the Frobenius metric of $U(m)$ is bi-invariant. Moreover,
arguing as in [Dolcetti-Pertici 2018] Recall 4.1, it is simple to verify that the expression of the Frobenius metric \( \phi \) of \( U(m) \) is the following: \( \phi_\mathcal{A}(X,Y) = -\text{tr}(A^*X^*Y) \), for every \( A \in U(m) \) and for every \( X, Y \in T_A(U(m)) \).

In each of the above cases, \( G \) is a (closed) Lie subgroup of \( U(n) \) or of \( U(2n) \), i.e. \( G \) is a submanifold of some \( U(m) \) \((m \geq 2)\); hence, on \( G \), the metric \( \phi \) is bi-invariant and \( \phi = -\frac{1}{c}K \) (with \( c > 0 \)). Therefore, if \( G \) is one of the above groups, then Proposition 1.4 Proposition 1.11 and Theorem 2.3 hold also with \( \phi \) instead of \( -K \).

2.8. Remark. Parts (a) and (b) of Theorem 2.5 can be compared with an analogous result, obtained in [Abe-Akiyama-Hatori 2013] Thm. 1.

3. ISOMETRIES OF \( SO(4) \)

3.1. Remark. By Lemma 1.39 the Killing metric of the semi-simple compact Lie group \( SO(4) \) is a Riemannian metric on \( SO(4) \). It is easy to check that the Killing form of the special orthogonal Lie algebra \( \mathfrak{so}(4) \), evaluated at \( U, V \), agrees with \( 2 \text{tr}(U,V) \) (this extend to the case \( n = 2 \) the formula (3) of [Sepanski 2007] Ex. 6.19 p. 129)). Hence the Killing metric \( -K \) of \( SO(4) \) agrees with the double of the Frobenius metric \( \phi \) of \( SO(4) \).

Therefore, for the Lie group \( SO(4) \), Proposition 1.4 holds for \( \phi \) as well as for \( -K \). However, in [Dolcetti-Pertici 2018] Prop. 4.3], we have already proved that \((SO(4), \phi)\) (and so, also \((SO(4), -K))\) is an Einstein globally symmetric Riemannian manifold with non-negative sectional curvature.

3.2. Remarks-Definitions.

a) The map \( \rho : \mathbb{C} \to M_2(\mathbb{R}) \), given by \( \rho(z) := \begin{pmatrix} \text{Re}(z) & -\text{Im}(z) \\ \text{Im}(z) & \text{Re}(z) \end{pmatrix} \) is a monomorphism of \( \mathbb{R} \)-algebras between \( \mathbb{C} \) and \( M_2(\mathbb{R}) \).

More generally, for any \( h \geq 1 \), we still denote by \( \rho \) the monomorphism of \( \mathbb{R} \)-algebras: \( M_h(\mathbb{C}) \to M_{2h}(\mathbb{R}) \), which maps the \( h \times h \) complex matrix \( Z = (z_{ij}) \) to the \( (2h) \times (2h) \) block real matrix \( (\rho(z_{ij})) \), having \( h^2 \) blocks of order \( 2 \times 2 \). We refer to \( \rho \) as the decomplexification map of \( M_h(\mathbb{C}) \) into \( M_{2h}(\mathbb{R}) \).

It is known that, for every \( Z \in M_h(\mathbb{C}) \), the map \( \rho \) satisfies:

\[ \text{tr}(\rho(Z)) = 2\text{Re}((Z)), \quad \text{det}(\rho(Z)) = |\text{det}(Z)|^2 \quad \text{and} \quad \rho(Z^*) = \rho(Z)^T. \]

For simplicity, we still denote by \( \rho \) all its restrictions to any subset of \( M_h(\mathbb{C}) \). Hence, for instance, \( \rho(U(h)) = \rho(M_h(\mathbb{C})) \cap SO(2h) \) is a Lie subgroup of \( SO(2h) \) (isomorphic to \( U(h) \)) and, in particular, \( \rho(SU(2)) \) is a Lie subgroup of \( SO(4) \), isomorphic to \( SU(2) \).

b) We consider the matrix \( J = J^T = J^{-1} \in \mathcal{O}(4) \), defined by \( J := \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \)

and the Lie subgroup of \( SO(4) \), conjugate to \( \rho(SU(2)) \) in \( \mathcal{O}(4) \), defined by \( J\rho(SU(2))J \).
It is easy to check that $\rho(SU(2)) \cap \{J \rho(SU(2)) J\} = \{\pm I_4\}$ and that $X$ commutes with $JY$, for every $X,Y \in \rho(SU(2))$. Moreover, it is known that every matrix of $SO(4)$ has a Cayley’s factorization as commutative product of a matrix of $\rho(SU(2))$ and of a matrix of $J \rho(SU(2)) J$ and that such factorization is unique up to the sign of both matrices (see, for instance, [Hunt-Mullineux-Cripps-Cross 2016, Thm. 3.2] and also [Mebius 2005, Thomas 2014, [Perez-Gracia-Thomas 2017]).

c) Let us consider $X = \rho(X_1) [J \rho(X_2) J] \times (X_1, X_2 \in SU(2))$ a matrix in $SO(4)$, together with its Cayley’s factorization. The map $\tau : SO(4) \to SO(4)$, given by $X = \rho(X_1) [J \rho(X_2) J] \to \tau(X) := \rho(X_1) [J \rho(X_2) J]^T = \rho(X_1) [J \rho(X_2^T) J]$, is well-defined and bijective; moreover $\tau^2 = Id$ and $\tau \circ j = j \circ \tau$ (where $j$ is the inversion map (i.e. the transposition map) of $SO(4)$).

d) The map $\tilde{\chi} : SU(2) \times SU(2) \to SO(4)$, defined by $\tilde{\chi}(X,Y) = \rho(X) J \rho(Y) J$, is an epimorphism of Lie groups, whose kernel is $\{\pm (I_2, I_2)\}$. Then the map $\tilde{\chi}$ induces a Lie group isomorphism $\chi : \frac{SU(2) \times SU(2)}{\{\pm (I_2, I_2)\}} \to SO(4)$.

Then $(SO(4), -K)$ is a Riemannian manifold isometric to $\left(\frac{SU(2) \times SU(2)}{\{\pm (I_2, I_2)\}}, -K'\right)$, where $-K'$ is the Killing metric of the Lie group $\frac{SU(2) \times SU(2)}{\{\pm (I_2, I_2)\}}$.

e) The Killing tensor of $SU(2) \times SU(2)$ is $K_2 \times K_2$, where $K_2$ denotes the Killing tensor of $SU(2)$. We denote by $\sigma : SU(2) \times SU(2) \to SU(2) \times SU(2)$ the map which interchanges the two factors of $SU(2) \times SU(2)$. By a classical result due to de Rham (see de Rham 1952, Thm. III p. 341), the isometries of $(SU(2) \times SU(2), -(K_2 \times K_2))$ are precisely the maps $\psi_1 \times \psi_2 : (X,Y) \to (\psi_1(X), \psi_2(Y))$ and the maps $(\psi_1 \times \psi_2) \circ \sigma : (X,Y) \to (\psi_1(Y), \psi_2(X))$, where $\psi_1, \psi_2$ are isometries of $(SU(2), -K_2)$. In particular, the map $\sigma$ in an isometry of $(SU(2) \times SU(2), -(K_2 \times K_2))$.

From these facts and from Theorem 2.5 (c), if we denote by $j$ the inversion map of $SU(2)$, we get the following

3.3. Proposition. The isometries of $(SU(2) \times SU(2), -(K_2 \times K_2))$ are precisely the maps of the form:

$(L_{A_1} \circ R_{A_2}) \times (L_{B_1} \circ R_{B_2})$;
$(L_{A_1} \circ R_{A_2} \circ j) \times (L_{B_1} \circ R_{B_2})$;
$(L_{A_1} \circ R_{A_2}) \times (L_{B_1} \circ R_{B_2} \circ j)$;
$(L_{A_1} \circ R_{A_2} \circ j) \times (L_{B_1} \circ R_{B_2} \circ j)$;
$((L_{A_1} \circ R_{A_2}) \times (L_{B_1} \circ R_{B_2})) \circ \sigma$;
$((L_{A_1} \circ R_{A_2} \circ j) \times (L_{B_1} \circ R_{B_2})) \circ \sigma$;
$((L_{A_1} \circ R_{A_2}) \times (L_{B_1} \circ R_{B_2} \circ j)) \circ \sigma$;
$((L_{A_1} \circ R_{A_2} \circ j) \times (L_{B_1} \circ R_{B_2} \circ j)) \circ \sigma$

where $A_1, A_2, B_1, B_2$ are generic elements of $SU(2)$. 


In particular the isometries of \((SU(2) \times SU(2), -(K_2 \times K_2))\), fixing the identity \((I_2, I_2)\), are the previous ones, with \(A'_1 = A_2\) and \(B'_1 = B_2\).

3.4. Proposition. Let \(\pi : SU(2) \times SU(2) \to \frac{SU(2) \times SU(2)}{\pm(I_2, I_2)}\) be the natural covering projection.

If \(\Psi\) is an isometry of \(\left\{\frac{SU(2) \times SU(2)}{\pm(I_2, I_2)}\right\}\), fixing the identity of the group, then there exists a unique isometry \(\tilde{\Psi}\) of \(\left\{\frac{SU(2) \times SU(2)}{\pm(I_2, I_2)}\right\}\), fixing the identity \((I_2, I_2)\), such that \(\Psi \circ \pi = \pi \circ \tilde{\Psi}\).

Conversely, if \(\tilde{\Psi}\) is an isometry of \(\left\{\frac{SU(2) \times SU(2)}{\pm(I_2, I_2)}\right\}\), fixing the identity \((I_2, I_2)\), then there exists a unique isometry \(\Psi\) of \(\left\{\frac{SU(2) \times SU(2)}{\pm(I_2, I_2)}\right\}\), fixing the identity of the group, such that \(\Psi \circ \pi = \pi \circ \tilde{\Psi}\).

Proof. Let \(\Psi\) be an isometry of \(\left\{\frac{SU(2) \times SU(2)}{\pm(I_2, I_2)}\right\}\), fixing the identity of the group. Since \(SU(2) \times SU(2)\) is simply connected, there exists a unique homeomorphism \(\tilde{\Psi} : SU(2) \times SU(2) \to SU(2) \times SU(2)\), fixing the identity \((I_2, I_2)\) and such that \(\Psi \circ \pi = \pi \circ \tilde{\Psi}\).

If \(\pi\) is a local isometry from \((SU(2) \times SU(2)), -(K_2 \times K_2))\) onto \(\left\{\frac{SU(2) \times SU(2)}{\pm(I_2, I_2)}\right\}, \left\{-K\right\}\), the map \(\tilde{\Psi}\) is an isometry of \((SU(2) \times SU(2), -(K_2 \times K_2))\).

For the converse, we denote by \(\mu\) the isometry of \((SU(2) \times SU(2), -(K_2 \times K_2))\) defined by \(\mu(A, B) = (-A, -B)\). From Theorem 3.3 (c) and from Remarks-Definitions 3.2 (e), the map \(\mu\) commutes with all isometries \(\tilde{\Psi}\) of \((SU(2) \times SU(2), -(K_2 \times K_2))\), fixing the identity of the group, and so, all these last project as isometries of the quotient.

3.5. Theorem. The isometries of \((SO(4), -K)\) are precisely the following maps:

\[X \to AXB,\]
\[X \to AX^TB,\]
\[X \to A\tau(X)B,\]
\[X \to A\tau(X)^TB,\]

where \(A, B\) are matrices both in \(SO(4)\) or both in \(O(4) \setminus SO(4)\).

Proof. By Propositions 3.3 and 3.4 all isometries of \(\left\{\frac{SU(2) \times SU(2)}{\pm(I_2, I_2)}\right\}\), fixing the identity, are obtained by projecting onto the quotient the following isometries of \((SU(2) \times SU(2), -(K_2 \times K_2))\):

\[C_A \times C_B, \ (C_A \times C_B) \circ (j \times id), \ (C_A \times C_B) \circ (id \times j), \ (C_A \times C_B) \circ \sigma, \ (C_A \times C_B) \circ (j \times id) \circ \sigma, \ (C_A \times C_B) \circ (id \times j) \circ \sigma, \ (C_A \times C_B) \circ (j \times j) \circ \sigma,\]

with \(A, B \in SU(2)\). Here \(id\) and \(j\) denote, respectively, the identity and the inversion map of \(SU(2)\), whereas \(C_X\) denotes, as usual, the inner automorphism of \(SU(2)\) associated to a generic element \(X\) of \(SU(2)\).

By Remarks-Definitions 3.2 (d), the isometries of \((SO(4), -K)\), fixing the identity \(I_4\), are of the form \(\chi \circ \Phi \circ \chi^{-1}\), where \(\Phi\) is one of the above isometries of \(\left\{\frac{SU(2) \times SU(2)}{\pm(I_2, I_2)}\right\}, \left\{-K\right\}\).

Standard computations show that \(\chi \circ (C_A \times C_B) \circ \chi^{-1} = C_{\chi(A, B)}\), for every \(A, B \in SU(2)\), \(\chi \circ (id \times j) \circ \chi^{-1} = \tau\) (and so, \(\tau\) is an isometry of \((SO(4), -K)\)), \(\chi \circ (id \times j) \circ \chi^{-1} = \pi \circ j = j \circ \pi\).
ISOMETRIES OF SOME COMPACT LIE GROUPS

Let \( \chi \circ (j \times j) \circ \chi^{-1} = \tilde{j} \), where \( \tilde{j} \) denotes the inversion map of \( SO(4) \), and \( \chi \circ \sigma \circ \chi^{-1} = C_j \), where \( J \) is the matrix of \( O(4) \setminus SO(4) \), defined in Remarks-Definitions\ref{4}\((b)\). From this, we get that the complete list of the isometries of \((SO(4), -K)\), fixing the identity \( I_4 \), is the following: \( C_M \), \( C_M \circ \tilde{j} \circ \tau \), \( C_M \circ \tau \), \( C_M \circ \tilde{j} \), where \( M \) is a generic matrix of \( O(4) \).

To get the full group of isometry of \((SO(4), -K)\), it suffices to compose these isometries with a left translation \( L_A \), with \( A \in SO(4) \). This allows to conclude the proof. \(\square\)

3.6. Remark. The full group of isometries of \((SO(4), -K)\) has 8 connected components, all diffeomorphic to \( SO(4) \times SO(4) \) \(\{\pm(I_4, I_4)\} \).

3.7. Remark. Also Theorem 4.2.5 can be compared with the analogous result, obtained in \[\text{Abe-Akiyama-Hatori 2013} \text{ Thm. 1}, \text{ for } n = 4.\]

4. ISOMETRIES OF \(U(n)\)

In this Section we describe the full group of isometries of the Riemannian manifold \((U(n), \phi) \ (n \geq 2)\), where \( \phi \) is the Frobenius metric of \( U(n) \), defined by \( \phi_A(X,Y) = -tr(A^*X A^*Y) \), for every \( A \in U(n) \) and for every \( X,Y \in T_A(U(n)) \). By the way, note that \( \phi \) can be also obtained, by the Frobenius metric \( \phi_0 \) of \( SO(2n) \), as \( \phi = \frac{1}{2} \rho^*(\phi_0) \), where \( \rho \) is the decomplexification map of \( U(n) \) into \( SO(2n) \).

4.1. Remarks-Definitions.

a) The pair \((SU(n) \times \mathbb{R}, p)\), where \( p \) is the map \( p : SU(n) \times \mathbb{R} \rightarrow U(n) \), defined by \( p(B,x) = e^{ix}B \), is the (analytic) universal covering group of \( U(n) \).

Indeed \( p \) is clearly an analytic homomorphism of Lie groups, whose differential, at the point \((B,x) \in SU(n) \times \mathbb{R})\), maps the tangent vector \((W,\lambda)\) to \( e^{ix}(W + i\lambda B) \). At the identity \((I_n, 0)\), this map has kernel zero and so it is an isomorphism, hence, by \[\text{Warner 1983}\text{ Prop.3.26 p.100}], it is a covering map.

b) From (a), we get easily that, if \( \mathcal{K} \) and \( \widetilde{\mathcal{K}} \) are the Killing tensors of \( U(n) \) and of \( SU(n) \times \mathbb{R} \), respectively, then we have: \( p^*(\mathcal{K}) = \widetilde{\mathcal{K}} \). Since \( \widetilde{\mathcal{K}} \) is the product of the Killing tensors of \( SU(n) \) and of \( \mathbb{R} \) (and this last is zero), then, remembering again \[\text{Sepanski 2007}\text{ Ex. 6.19 p.129}], we have:

\[\widetilde{\mathcal{K}}_{(B,x)}((W,\lambda),(W',\lambda')) = 2n tr(B^* W B^* W')\]

for every \( B \in SU(n) \), for every \( W,W' \in T_B(SU(n)) \) and for every \( x,\lambda,\lambda' \in \mathbb{R} \).

Let \( A := e^{ix}B = p(B,x) \) (with \( B \in SU(n) \) and \( x \in \mathbb{R} \)). If \( Y,Z \in T_A(U(n)) \), then, by (a), \( Y \) and \( Z \) are the images, through the tangent map of \( p \), of \( (e^{-ix}Y - \frac{1}{n}tr(A^* Y)B, -\frac{1}{n}tr(A^* Y)B) \) and of \( (e^{-ix}Z - \frac{1}{n}tr(A^* Z)B, -\frac{1}{n}tr(A^* Z)B) \), respectively (note that \( tr(A^* Y) \) and \( tr(A^* Z) \) are purely imaginary, because \( A^* Y \) and \( A^* Z \) are skew-hermitian matrices). Since \( p^*(\mathcal{K}) = \widetilde{\mathcal{K}} \), we get that:

\[\mathcal{K}_A(Y,Z) = \mathcal{K}_{(B,x)}((e^{-ix}Y - \frac{1}{n}tr(A^* Y)B, -\frac{1}{n}tr(A^* Y)B), (e^{-ix}Z - \frac{1}{n}tr(A^* Z)B, -\frac{1}{n}tr(A^* Z)B)) = 2n tr(B^*(e^{-ix}Y - \frac{1}{n}tr(A^* Y)B) B^*(e^{-ix}Z - \frac{1}{n}tr(A^* Z)B)) =\]
2n \((\text{tr}(A^*Y A^*Z) - \frac{1}{n}\text{tr}(A^*Y)\text{tr}(A^*Z)) = 2n\text{tr}(A^*Y A^*Z) - 2\text{tr}(A^*Y)\text{tr}(A^*Z) = -2n\phi_A(Y, Z) - 2\text{tr}(A^*Y)\text{tr}(A^*Z).

Therefore we can state the following

4.2. Lemma. The Killing tensor \(K\) of \(U(n)\) has the following expression:

\[K_A(Y, Z) = 2n\text{tr}(A^*Y A^*Z) - 2\text{tr}(A^*Y)\text{tr}(A^*Z) = -2n\phi_A(Y, Z) - 2\text{tr}(A^*Y)\text{tr}(A^*Z)\]

for every \(A \in U(n)\) and every \(Y, Z \in T_A(U(n))\).

4.3. Remark. The Killing tensor \(K\) of \(U(n)\) is a (degenerate) negative semi-definite tensor (and so \(U(n)\) is not semi-simple).

It suffices to check it at the identity \(I_n \in U(n)\). For, by Lemma 4.2, we have \(K_{I_n}(I_n, I_n) = 0\) and furthermore, if \(Y\) is a skew-hermitian matrix with purely imaginary eigenvalues \(iy_1, \ldots, iy_n\), then

\[K_{I_n}(Y, Y) = -2n\sum_{j=1}^n y_j^2 + \sum_{j,k=1}^n 2y_jy_k \leq -2n\sum_{j=1}^n y_j^2 + \sum_{j,k=1}^n (y_j^2 + y_k^2) = 0.\]

4.4. Remark-Definition. On the product manifold \(SU(n) \times \mathbb{R}\), we consider the metric \(\mathcal{H}\) defined in the following way:

\[\mathcal{H}_{(B,x)}((W,\lambda), (W',\lambda')) = -\text{tr}(B^*WB'^*W') + n\lambda\lambda', \text{ for every } B \in SU(n), \text{ for every } W, W' \in T_B(SU(n)) \text{ and for every } x, \lambda, \lambda' \in \mathbb{R}.

Note that the metric \(\mathcal{H}\) is the product of a constant positive multiple of the Killing metric of \(SU(n)\) and of a constant positive multiple of the euclidean metric of \(\mathbb{R}\).

By [de Rham 1952] Thm. III p. 341], the isometries of \((SU(n) \times \mathbb{R}, \mathcal{H})\) are precisely the maps of the form \(\Phi \times \alpha\), where \(\Phi\) is an isometry of \(SU(n)\), endowed with its Killing metric, and \(\alpha\) is an euclidean isometry of \(\mathbb{R}\).

4.5. Lemma.

a) The map \(p : (SU(n) \times \mathbb{R}, \mathcal{H}) \to (U(n), \phi)\) is a local isometry.

b) For every isometry \(F\) of \((U(n), \phi)\), fixing the identity \(I_n\) of \(U(n)\), there is a unique isometry \(\tilde{F}\) of \((SU(n) \times \mathbb{R}, \mathcal{H})\), fixing the the identity \((I_n, 0)\) of \(SU(n) \times \mathbb{R}\), such that \(p \circ \tilde{F} = F \circ p\).

Proof. If \(x, \lambda, \lambda' \in \mathbb{R}, B \in SU(n), W, W' \in T_B(SU(n))\) (so \(\text{tr}(B^*W) = \text{tr}(B^*W') = 0\), by Remarks-Definitions 4.1(a), we have:

\[p^*(\phi)_{(B,x)}((W,\lambda), (W',\lambda')) = \phi_{(e^{i\lambda}B)}(e^{i\lambda'}(W + i\lambda B), e^{i\lambda'}(W' + i\lambda'B)) = -\text{tr}((B^*W + i\lambda B)(B^*W' + i\lambda' B')) = -\text{tr}(B^*WB'^*W') + n\lambda\lambda' = \mathcal{H}_{(B,x)}((W,\lambda), (W',\lambda')),\]

i.e. \(p^*(\phi) = \mathcal{H}\) and the proof of (a) is complete.

Part (b) follows from part (a) and from the fact that \((SU(n) \times \mathbb{R}, p)\) is the universal covering of \(U(n)\).

\[\square\]

4.6. Proposition.

a) Every isometry of \((SU(2) \times \mathbb{R}, \mathcal{H})\), fixing the identity \((I_2, 0)\) of \(SU(2) \times \mathbb{R}\), projects (through the covering map \(p\)) as an isometry of \((U(2), \phi)\), fixing the identity \(I_2\) of \(U(2)\).
b) The isometries of \((U(2), \phi)\), fixing the identity \(I_2\) of \(U(2)\), are precisely the following maps:
\[ X \rightarrow BXB^*, \quad X \rightarrow B^* \quad \text{and} \quad X \rightarrow \frac{BXB^*}{\det(X)} \quad \text{with} \quad B \in SU(2). \]

Proof. We denote by \(id\) the identity map of \(\mathbb{R}\), by \(Id\) the identity map of \(SU(2)\), by \(j\) the inversion map of \(SU(2)\) and by \(C_B = L_B \circ R_B^*\) the inner automorphism of \(SU(2)\), associated to \(B\). By Remark-Definition 4.4 and by Theorem 2.5 (c), the isometries of \((SU(2) \times \mathbb{R}, \mathcal{H})\), fixing the identity \((I_2, 0) \in SU(2) \times \mathbb{R}\), are precisely the maps of the form:
\[ C_B \times (\pm id) = (C_B \times id) \circ (Id \times (\pm id)) \quad \text{and} \quad (C_B \circ j) \times (\pm id) = (C_B \times id) \circ (j \times (\pm id)), \]
with \(B \in SU(2)\).

Easy computations show that all the maps \(C_B \times id\) (with \(B \in SU(2)\)), \(Id \times id\), \(Id \times (-id)\), \(j \times id\) and \(j \times (-id)\) project as maps of \(U(2)\). More precisely, \(C_B \times id\) projects as the inner automorphism of \(U(2)\) associated to \(B\), \(Id \times id\) as the identity map of \(U(2)\), \(Id \times (-id)\) as the involution of \(U(2)\) given by \(A \mapsto \frac{A}{\det(A)} \quad j \times id\) as the involution of \(U(2)\) given by \(A \mapsto \det(A)A^*\), and \(j \times (-id)\) as the inversion of \(U(2)\).

By composition, the maps of \(U(2)\), obtained in this way, are those in the statement. They are isometries of \((U(2), \phi)\), by Lemma 4.3 (a). Part (b) of the same Lemma implies that there is no other isometries. \(\square\)

4.7. Theorem. The isometries of \((U(2), \phi)\) are precisely the following maps:
\[ X \rightarrow AXB, \quad X \rightarrow AX^*B, \quad X \rightarrow AXB \quad \frac{\det(X)}{\det(Y)}, \quad X \rightarrow \det(X)AX^*B, \]
with \(A, B \in U(2)\).

Proof. It follows directly from Proposition 4.6 (b), by left (or right) translation with a matrix of \(U(2)\). \(\square\)

4.8. Theorem. The isometries of \((U(n), \phi)\), with \(n \geq 3\), are precisely the following maps:
\[ X \rightarrow AXB, \quad X \rightarrow AX^*B, \quad X \rightarrow AX^TB, \quad X \rightarrow AXTB, \]
with \(A, B \in U(n)\).

Proof. We use the same notations as in the proof of Proposition 4.6 (with \(n \geq 3\) generic, instead of \(n = 2\)) and the further following: \(\mu(X) = X\) and \(\eta(X) = X^T\) for every \(X \in SU(n)\). Again, by Remark-Definition 4.4 and by Theorem 2.5 (d), the isometries of
(\text{SU}(n) \times \mathbb{R}, \mathcal{H})$, fixing the the identity \((I_n, 0)\) of \(\text{SU}(n) \times \mathbb{R}\), are precisely the maps of the form:

\[ C_B \times (\pm \text{id}) = (C_B \times \text{id}) \circ (Id \times (\pm \text{id})), \]

\[ (C_B \circ j) \times (\pm \text{id}) = (C_B \times \text{id}) \circ (j \times (\pm \text{id})), \]

\[ (C_B \circ \mu) \times (\pm \text{id}) = (C_B \times \text{id}) \circ (\mu \times (\pm \text{id})) \]

and

\[ (C_B \circ \eta) \times (\pm \text{id}) = (C_B \times \text{id}) \circ (\eta \times (\pm \text{id})), \]

with \(B \in \text{SU}(n)\).

Since the isometries of \((\text{SU}(n) \times \mathbb{R}, \mathcal{H})\), projecting (throughout \(p\)) as maps of \(U(n)\), form a group with respect to the composition, it suffices to examine the following isometries:

- \(C_B \times \text{id}\), which projects as the inner automorphism of \(U(n)\), associated to \(B \in \text{SU}(n)\),
- \(\text{id} \times \text{id}\), which projects as the identity map of \(U(n)\),
- \(j \times (-\text{id})\), which projects as the inversion map of \(U(n)\),
- \(\mu \times (-\text{id})\), which projects as the (complex) conjugation map of \(U(n)\),
- \(\eta \times \text{id}\), which projects as the transposition map of \(U(n)\),

and \(\text{id} \times (-\text{id})\), \(j \times \text{id}\), \(\mu \times \text{id}\), \(\eta \times (-\text{id})\), which, on the contrary, do not project as maps of \(U(n)\). The proofs of the first five cases are obvious. For the isometries, which do not project as maps of \(U(n)\), we consider, as example, only the case \(\text{id} \times (-\text{id})\); the other cases can be treated in the same way.

For, we have \(I_n = p(I_n, 0) = p(e^{\frac{2\pi}{n} I_n}, -\frac{2\pi}{n})\), \(p \circ (\text{id} \times (-\text{id})) (I_n, 0) = I_n\) and \(p \circ (\text{id} \times (-\text{id}) (e^{\frac{2\pi}{n} I_n}, -\frac{2\pi}{n}) = e^{\frac{2\pi}{n} I_n}\); these last two are different, because \(n \geq 3\) and so, the isometry \(\text{id} \times (-\text{id})\) does not project as map of \(U(n)\).

Therefore, taking into account Lemma 1.6, the isometries of \((U(n), \phi)\), fixing the identity \(I_n\), are the following maps: \(X \rightarrow BXB^*, X \rightarrow BX^*B^*, X \rightarrow BXB^*, X \rightarrow BX^TB^*\), with \(B \in \text{SU}(n)\). Now, by left (or right) translation with a matrix of \(U(n)\), we obtain all the isometries in the statement. \(\square\)

4.9. Remarks.

a) The full group of isometries of \((U(n), \phi)\), both for \(n = 2\) and for \(n \geq 3\), has 4 connected components, all diffeomorphic to

\[ \{ \lambda (I_n, I_n) : \lambda \in \mathbb{C}, |\lambda| = 1 \} \]

Indeed, arguing as in Remark 1.11, the group generated by left and right translations of \(U(n)\) is diffeomorphic to \(\frac{Z}{(Z \times \Delta) \cap \Delta} \), where \(Z\) and \(\Delta\) are, respectively, the center of \(U(n)\) and the diagonal of \(U(n) \times U(n)\). We conclude, because the center of \(U(n)\) is \(\{ I_n : \lambda \in \mathbb{C}, |\lambda| = 1 \}\).

b) For every \(n \geq 2\), \((U(n), \phi)\) is a (globally) symmetric Riemannian manifold.

Indeed, for every \(A \in U(n)\), the map \(X \mapsto AX^*A\) is an isometry of \((U(n), \phi)\), fixing \(A\), and whose differential at \(A\) is the opposite of the identity map of \(T_A(U(n))\).

4.10. Remark. Compare Theorems 4.14 and 4.15 with an analogous result, obtained in [Hatori-Molnár 2012 Thm. 8].
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