Novel soliton in dipolar BEC caused by the quantum fluctuations

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Received 27 October 2020 / Accepted 1 February 2021 / Published online 15 February 2021
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Abstract. Solitons in the extended hydrodynamic model of the dipolar Bose–Einstein condensates with the quantum fluctuations are considered. This model includes the continuity equation for the scalar field of concentration, the Euler equation for the vector field of velocity, the pressure evolution equation for the second rank tensor of pressure, and the evolution equation for the third rank tensor. Large amplitude soliton solution caused by the dipolar part of the quantum fluctuations is found. It appears as the bright soliton. Hence, it is the area of compression of the number of particles. Moreover, it exists for the repulsive short-range interaction.

1 Introduction

Dipolar Bose–Einstein condensates (BECs) demonstrate the formation of quantum droplets [1–21]. Each droplet is the small cloud of atoms, which can be considered as the soliton-like area of the increased concentration (the local density of the particle number). Systems of droplets are the highly nonlinear structures explained via the dipolar part of the quantum fluctuations. The interpretation is made via the generalized Gross–Pitaevskii (GP) equation including the fourth order nonlinearity.

The traditional GP equation contains one nonlinear term which is proportional to the third degree of the macroscopic wave function (the order parameter) [22]. This nonlinearity is caused by the major part of the short-range interaction contribution (the mean field part). In spite the fact that the BECs is the collection of particles in the quantum state with the lowest energy, a part of particles can exist in the excited states even at the zero temperature. It happens via the interaction beyond the mean-field approximation. This phenomenon is called the BEC depletion caused by the quantum fluctuations. It is studied both theoretically [23–26] and experimentally [27–29]. If one considers the quantum fluctuations caused by the short-range interaction within the Bogoliubov–de Gennes theory, one can find additional nonlinear term in the GP equation which is also caused by the short-range interaction, but it is proportional to the fourth degree of the macroscopic wave function. It contains some interaction constant as the traditional GP equation.

The dipolar BECs have been intensively studied over the last 20 years [30–39]. The non-superfluid fermionic dipolar gases are also considered in literature [40,41]. Traditionally the condensate depletion is studied in terms of Bogoliubov–de Gennes theory [42–44]. It includes the depletion of the dipolar BECs. However, here we present the microscopic many-particle quantum hydrodynamic theory of the quantum fluctuations. A model of the quantum fluctuations in dipolar BEC at finite temperatures are considered in Ref. [45].

However, the dipole–dipole interaction modifies properties of nonlinear structures in BECs including the width and the amplitude of the bright and dark solitons in the BECs [46]. To the best of our knowledge, the dipole–dipole interaction brings no novel soliton formation. In Ref. [46], the analytical solutions are demonstrated for the dipole–dipole interaction of the point-like objects. It causes the question: Does dipole–dipole interaction create novel soliton-like structures? In this paper, we give positive answer. Below, it is demonstrated that the dipolar part of quantum fluctuations causes novel soliton solution.

BEC is a quantum phenomenon. BECs are the collections of atoms creating the quantum nonlinear mediums which demonstrate existence of solitons [47–49]. However, the dark and bright solitons appearing from the mean field nonlinear Schrodinger equation (the GP equation) have a lot in common with the solitons in classical systems. The quantum fluctuations appear as...
the beyond mean field effect and impose more quantum insights in the observed phenomena. The quantum solitons are the fundamental example of these phenomena. Solitons obtained beyond the mean field regime are called the quantum solitons [50–55].

Recently, the microscopically justified quantum hydrodynamic model describing the quantum fluctuations via the equations additional to the Euler and continuity equations is derived [56, 57]. The interparticle interaction creates the source of the third rank tensor \( Q^{\alpha \beta \gamma} \), which gives the nonzero value of kinetic pressure \( p^{\alpha \beta} \). Both the kinetic pressure and the third rank tensor \( Q^{\alpha \beta \gamma} \) are related to the occupation of the excited states. Their nonzero values for the BECs are the consequence of the quantum fluctuations in BECs. The pressure evolution equation contains no trace of the interaction. Hence, its value depends on the third rank tensor \( Q^{\alpha \beta \gamma} \) only. The third rank tensor evolution equation contains the gradient of the concentration square \( n_i^2 \) multiplied by the additional interaction constant for the short-range interaction. The long-range dipole–dipole interaction leads to the third derivative of the macroscopic potential of the dipole–dipole interaction. However, no contribution of the external field, such as the trapping potential, is present in this equation.

The description of collisions of solitons in BECs requires models obtained beyond the mean-field approximation [58–61]. In this paper, we demonstrate that the described above beyond mean-field model provides a novel soliton solution in dipolar BECs.

This paper is organized as follows. In Sect. 2, the extended quantum hydrodynamic model of dipolar BECs including the quantum fluctuations via the quantum Bohm potential (the pressure) evolution equation and the third rank tensor evolution equations is presented. Basic definitions are given along with the many-particle Schrödinger equation: 

\[
\hat{H}\Psi(R,t) = \left[ \sum_{i,j=1}^{N} \left( \frac{\hat{p}_i^2}{2m} + V_{\text{ext}}(r_i,t) \right) + \frac{1}{2} \sum_{i,j \neq i} U_{ij} + \frac{1}{2} \mu^2 \sum_{i,j \neq i} \frac{3 \alpha^2 z_{ij}^2}{r_{ij}^3} \right] \Psi(R,t),
\]

(1)

where \( m \) is the mass of the atom, \( \hat{p}_i = -i\hbar \nabla_i \) is the momentum of \( i \)-th particle, \( \hbar \) is the Planck constant, \( \mu \) is the magnetic moment of the atom (for the separated atoms), \( \Psi(R,t) \) is the wave function for the system of \( N \) quantum particles, \( R = \{ r_1, \ldots, r_N \} \), \( V_{\text{ext}} \) is the external potential. The short-range part of boson–boson interaction is presented via potential \( U_{ij} = U(r_{ij}) \), where \( r_{ij} = |r_i - r_j| \), and \( r_{ij} = r_i - r_j \). The long-range dipole–dipole interaction of align dipoles is presented by the last term of the Schrödinger Eq. (1).

The Feshbach resonance is presented in literature via the scattering theory. So, the interaction process of two particles of the dilute system is considered. This interaction process is described via the Schrödinger equation. If two particles under consideration (with numbers \( i = i_0 \) and \( j = j_0 \)) are at the large distance from other particles of the system, their variables can be separated in the many-particle wave function \( \Psi(R,t) = \psi(r_{i_0},r_{j_0}) \cdot \Psi(R_{N-2},t) \). Hence, the required two particle Schrödinger equation follows from the many-particle Schrödinger Eq. (1). Formal separation is possible since the potentials of interaction go to zero \( U_{ij} + (1 - 3 \alpha^2 z_{ij}^2/r_{ij}^3)/r_{ij}^3 \to 0 \) for pairs \( i = i_0 \) and \( j \neq j_0 \), and \( i \neq i_0 \) and \( j = j_0 \) due to the long distances. In reality, this separation of coordinates is possible for the finite time (the time of collision). However, the scattering of the independent pair of particles requires that the time of collision goes to infinity.

At this stage, we can consider the stationary collision process via the two-particle Schrödinger equation

\[
\left( \frac{\hat{p}_{i_0}^2}{2m} + \frac{\hat{p}_{j_0}^2}{2m} + U(|r_{i_0} - r_{j_0}|) \right) \psi(r_{i_0},r_{j_0}) = E\psi(r_{i_0},r_{j_0})
\]

(2)

where \( \psi(r_{i_0},r_{j_0}) = \psi(R) \phi(R_{CM}) \) with \( R = r_{i_0} - r_{j_0} \) is the coordinate of the relative motion, and \( R_{CM} = (r_{i_0} + r_{j_0})/2 \) is the coordinate of the center of mass (for the identical particles). Hence, there is the separation of variables \( R \) and \( R_{CM} \).

The single particle Schrödinger equation for relative motion \( \psi(R) \) is obtained (see for instance Eq. 3 in Ref. [70]). Further discussion of properties of wave function of the relative motion \( \psi(R) \) (the scattering solutions, the scattering phase shifts, etc.) can be found in Ref. [70] after Eq. 3.

2 Model

2.1 The background of the model

Novel soliton solution in dipolar BECs is found in terms of the extended quantum hydrodynamic model, where the continuity equation for the scalar field of concentration \( n \), the Euler equation for the vector field of velocity \( v \), the pressure evolution equation for the second rank tensor of pressure \( p^{\alpha \beta} \) (or the quantum Bohm potential \( T^{\alpha \beta} \) which is the quantum part of the pressure), and the evolution equation for the third rank tensor \( Q^{\alpha \beta \gamma} \) are used. In this model, the quantum fluctuations are presented in the equation for the third rank tensor evolution. This model represents the microscopic motion of the quantum particles via the functions describing the collective dynamics [62–69]. The evolution of the collective hydrodynamic variables is obtained in accordance with the many-particle Schrödinger equation:

\[
\hat{H}\Psi(R,t) = \left[ \sum_{i,j=1}^{N} \left( \frac{\hat{p}_i^2}{2m} + V_{\text{ext}}(r_i,t) \right) + \frac{1}{2} \sum_{i,j \neq i} U_{ij} + \frac{1}{2} \mu^2 \sum_{i,j \neq i} \frac{3 \alpha^2 z_{ij}^2}{r_{ij}^3} \right] \Psi(R,t),
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where \( m \) is the mass of the atom, \( \hat{p}_i = -i\hbar \nabla_i \) is the momentum of \( i \)-th particle, \( \hbar \) is the Planck constant, \( \mu \) is the magnetic moment of the atom (for the separated atoms), \( \Psi(R,t) \) is the wave function for the system of \( N \) quantum particles, \( R = \{ r_1, \ldots, r_N \} \), \( V_{\text{ext}} \) is the external potential. The short-range part of boson–boson interaction is presented via potential \( U_{ij} = U(r_{ij}) \), where \( r_{ij} = |r_i - r_j| \), and \( r_{ij} = r_i - r_j \). The long-range dipole–dipole interaction of align dipoles is presented by the last term of the Schrödinger Eq. (1).

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At this stage, we can consider the stationary collision process via the two-particle Schrödinger equation

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\]

(2)

where \( \psi(r_{i_0},r_{j_0}) = \psi(R) \phi(R_{CM}) \) with \( R = r_{i_0} - r_{j_0} \) is the coordinate of the relative motion, and \( R_{CM} = (r_{i_0} + r_{j_0})/2 \) is the coordinate of the center of mass (for the identical particles). Hence, there is the separation of variables \( R \) and \( R_{CM} \).

The single particle Schrödinger equation for relative motion \( \psi(R) \) is obtained (see for instance Eq. 3 in Ref. [70]). Further discussion of properties of wave function of the relative motion \( \psi(R) \) (the scattering solutions, the scattering phase shifts, etc.) can be found in Ref. [70] after Eq. 3.
It shows that the scattering of pairs of atoms is included in the quantum model presented by Eq. (1). This model is used for the derivation of equations for the evolution of the collective variables. Hence, the equations given below include all necessary information about the scattering.

The hydrodynamic model derived below and the well-known Gross–Pitaevskii equation [22] contain the interaction constant (or several interaction constants as it is found below). The interaction constant is the integral of the potential of interaction \( U \) and the interaction constant \( g \) included in the quantum model presented by Eq. (1).

The evolution of the current (number density) \([65,72–74]\):

\[
n = \int dR \sum_{i=1}^{N} \delta(r - r_i) \Psi^*(R,t)\Psi(R,t).
\]

The integral in Eq. (3) contains the element of volume in \(3N\) dimensional space \(dR = \prod_{i=1}^{N} dr_i\).

The derivation [73] shows that the concentration (3) obeys the continuity equation

\[
\partial_t n + \nabla \cdot (nv) = 0.
\]

The current of particles is proportional to the momentum density and presented by the following equation

\[
j = \int dR \sum_{i=1}^{N} \delta(r - r_i) \times \frac{1}{2m_i} (\Psi^*(R,t)\hat{p}_i\Psi(R,t) + c.c.),
\]

with c.c. is the complex conjugation. The current allows to define the velocity vector field: \(v = \frac{j}{n}\).

The evolution of the current (5) follows from the Schrödinger Eq. (1) and can be presented by the Euler equation:

\[
n n \partial_t v^\alpha + mn(v \cdot \nabla)v^\alpha - \frac{\hbar^2}{2m} n\nabla^\alpha \frac{\Delta \sqrt{n}}{\sqrt{n}} + \partial_\beta T^{\alpha\beta}_{df} + n\partial^\alpha V_{ext} = -gn\partial^\alpha n - n\partial^\alpha \Phi_d,
\]

where \(\Delta = \partial_\beta \partial_\beta\), and the Einstein’s rule for the summation on the repeating subindex is applied. The Euler Eq. (6) includes the macroscopic electrostatic potential of dipole–dipole interaction \(\Phi_d\). Tensor \(T^{\alpha\beta}_{df}\) is the part of the quantum Bohm potential (or the part of kinetic pressure) caused by the quantum fluctuations. Explanations for existence of nonzero tensor \(T^{\alpha\beta}_{df}\) and its interpretation can be seen from other equations presented below.

Major contribution of the short-range interaction appears in the mean-field approximation [22] corresponding to the first order by the interaction radius [73]. It is presented by the first term on the right-hand side of the Euler Eq. (6). It contains the interaction constant \(g\) expressed via the potential:

\[
g = \int dr U(r).
\]

The long range of interaction is presented in the self-consistent field approximation. Therefore, the twoparticle function of concentration appearing in the general equations is presented as the product of concentrations \(n(r,t)n(r’,t)\). It corresponds to the main contribution of the interaction via the macroscopic electrostatic potential (see for instance Eqs. (4.4) and (2.2) in Ref. [39]):

\[
\Phi_d = \mu^2 \int \frac{dr'}{[r - r']^3} \left(1 - 3 \left(\frac{z - z'}{|r - r'|^2}\right)^2\right) n(r',t).
\]

The dipole–dipole interaction presented by potential (8) is not full dipole–dipole interaction, but the longrange asymptotics of atom–atom interaction. Potential (8) satisfies the following differential equation

\[
\Delta \Phi_d = 4\pi \mu^2 (\partial_\xi^2 n - \triangle n),
\]

which is the modification of the Poisson equation. Equation (9) is derived for the arbitrary directions of the pair of dipoles with the further transition to the pair of aligned dipoles.
The evolution of the particles current \( j \) (5) leads to the flux of momentum \( \Pi^{\alpha\beta} \) which is defined as follows

\[
\Pi^{\alpha\beta} = \int dR \sum_{i=1}^{N} \delta(r - r_i) \frac{1}{4m^2} \left[ \Psi^*(R, t) \hat{p}^\alpha_i \hat{p}^\beta_i \Psi(R, t) + \hat{p}^*_{i\alpha} \Psi^*(R, t) \hat{p}^\beta_i \Psi(R, t) + c.c. \right].
\] (10)

The evolution of current \( j \) is presented within Eq. (6) in terms of the velocity field. The definition (10) appears during the derivation of Eq. (6). Equation (6) contains the momentum flux represented via the velocity field.

The continuity equation and the Euler equation are presented via the velocity field. Transition of the general equations to this form can be made by the representation of the macroscopic complex wave function \( \Psi(R, t) \) via the real functions \( \Psi(R, t) = a(R, t) \exp(\imath S(R, t)/\hbar) \). The real functions can be called the amplitude of the wave function \( a(R, t) \) and the phase of the wave function \( S(R, t) \). The gradient of the wave function gives the velocity of the quantum particle: \( v_i(R, t) = \nabla_i S(R, t)/m_i \). The derivation of the momentum flux represented via the velocity field gives the identity \( 0 = 0 \). However, it is the quasi-classical description of the pressure evolution equation. To understand the full quantum picture, we need to derive the equation for the second rank tensor evolution \( T^{\alpha\beta} \) (13) caused by the quantum fluctuations \( T_{qf}^{\alpha\beta} \) (it can be also interpreted as the part of pressure caused by the quantum fluctuations):

\[
\partial_t T^{\alpha\beta} + \partial_\gamma (v^\gamma T^{\alpha\beta}) + T^{\alpha\gamma} \partial_\gamma v^\beta + T^{\beta\gamma} \partial_\gamma v^\alpha + \partial_\gamma Q_{qf}^{\alpha\beta\gamma} = 0.
\] (14)

All terms in Eq. (14) are proportional to \( T_{qf}^{\alpha\beta} \) and the flux of pressure in the comoving frame \( Q_{qf}^{\alpha\beta\gamma} \). Hence, tensor \( Q_{qf}^{\alpha\beta\gamma} \) goes to zero together with the kinetic pressure at the zero temperature. Consequently, Eq. (14) gives the identity \( 0 = 0 \). However, it is the quasi-classical description of the pressure evolution equation. To understand the full quantum picture, we need to derive the equation for the third rank tensor evolution \( Q_{qf}^{\alpha\beta\gamma} \) at the arbitrary temperature. Afterward, we make the transition to the zero temperature in the derived equation. As the result we get Eq. (19) presented below. Equation (19) shows that \( \partial_\gamma Q_{qf}^{\alpha\beta\gamma} \neq 0 \) at the zero temperature due to the nonzero contribution of the interaction. Hence, there is the interaction related source of \( Q_{qf}^{\alpha\beta\gamma} \) and consequently it gives the source of \( T_{qf}^{\alpha\beta} \) via the pressure evolution equation. Therefore, the interaction causes the occupation of the quantum states with nonminimal energies. This description corresponds to the well-known nature of the quantum fluctuations [24, 42–44].

No interaction gives contribution in the pressure evolution Eq. (14). The form of the trapping potential does not affect the pressure evolution either.

The evolution of the second rank tensor of pressure leads to the flux of the momentum fluctuation:

\[
M^{\alpha\beta\gamma} = \int dR \sum_{i=1}^{N} \delta(r - r_i) \frac{1}{8m^3} \left[ \Psi^*(R, t) \hat{p}^\alpha_i \hat{p}^\beta_i \hat{p}^\gamma_i \Psi(R, t) + \hat{p}^*_{i\alpha} \Psi^*(R, t) \hat{p}^\beta_i \hat{p}^\gamma_i \Psi(R, t) + \hat{p}^*_{i\beta} \Psi^*(R, t) \hat{p}^\alpha_i \hat{p}^\gamma_i \Psi(R, t) + c.c. \right].
\] (15)
The method of the introduction of the velocity field described after Eq. (10) gives the representation of the momentum flux (10) in form of Eq. (11). Same procedure can be used for the representation of the third rank tensor (15). It gives the following representation of the third rank tensor $M^\alpha\beta\gamma$ via the velocity field and other hydrodynamic functions:

$$
M^\alpha\beta\gamma = n v^\alpha v^\beta v^\gamma + v^\alpha p^\beta + v^\beta p^\alpha + v^\gamma p^\alpha + Q^{\alpha\beta\gamma} + T^\alpha\beta\gamma,
$$

(16)

where we have two new functions. One of them the quasi-classic third rank tensor in the comoving frame:

$$
Q^{\alpha\beta\gamma}(r, t) = \int dR \sum_{i=1}^{N} \delta(r - r_i) B_i^\alpha(R, t) u_i^\alpha u_i^\beta u_i^\gamma.
$$

(17)

The quantum part of the tensor $M^\alpha\beta\gamma$ is found:

$$
T^\alpha\beta\gamma = - \frac{\hbar^2}{12m^2} n(\delta^\alpha\delta^\beta v^\gamma + \delta^\alpha \partial^\gamma v^\beta + \delta^\beta \partial^\gamma v^\alpha) + T^\beta\gamma \cdot v^\alpha + T^\alpha\beta \cdot v^\gamma + T^\alpha\gamma \cdot v^\beta.
$$

(18)

It is called the quantum part since it is proportional to $\hbar^2$. Hence, it is equal to zero in the classic limit, while other parts of tensor $M^\alpha\beta\gamma$ have nonzero classic limit.

Equation for the evolution of quantum-thermal part of the third rank tensor is [56,57]:

$$
\partial_t Q_{rf}^{\alpha\beta\gamma} + \partial_\delta(v^\delta Q_{rf}^{\alpha\beta\gamma}) + Q_{rf}^{\alpha\gamma\delta} \partial_\delta v^\beta + Q_{rf}^{\alpha\beta\delta} \partial_\delta v^\gamma

+ Q_{rf}^{\alpha\beta\gamma} = \frac{\hbar^2}{4m^2 n} \left( g_2 I_0^{\alpha\beta\gamma\delta} n + \delta^\alpha \partial^\gamma \partial^\beta \Phi_d \right)

+ \frac{1}{mn} \left( T_{rf}^{\alpha\beta} T_{rf}^{\gamma\delta} + T_{rf}^{\alpha\gamma} \partial^\delta T_{rf}^{\beta\alpha} + T_{rf}^{\alpha\delta} \partial^\beta T_{rf}^{\gamma\alpha} \right),
$$

(19)

where

$$
I_0^{\alpha\beta\gamma\delta} = \delta^\alpha\beta \delta^\gamma\delta + \delta^\alpha\gamma \delta^\beta\delta + \delta^\alpha\delta \delta^\beta\gamma.
$$

(20)

The main contribution of the short-range interaction is proportional to the second interaction constant:

$$
g_2 = \frac{2}{3} \int dr U''(r).
$$

(21)

This term consists of two parts. The first part is proportional to parameter $g_2$, which is the integral of the second derivative of the short-range interaction potential $U$ (see Eq. (21)). It is related to the short-range interaction only. If we neglect the dipole–dipole interaction, it would be single source of the quantum fluctuations. If we include the dipole–dipole interaction, it would give the second source of the quantum fluctuations. The dipole–dipole interaction is presented by the second part of the first term on the right-hand side of Eq. (19). Let us to point out that the short-range interaction in the Euler Eq. (6) and the third rank tensor evolution Eq. (19) contains the first derivative of the concentration of bosons. The dipole–dipole interaction in the Euler Eq. (6) (presented by the last term) is the first derivative of the macroscopic potential (8). Therefore, we find that the role of the dipolar part of the quantum fluctuations increases at the smaller scales in comparison with the quantum fluctuations caused by the short-range interaction and in comparison with the interactions in the Euler Eq. (6). This specific behavior of the dipolar fluctuations can lead to some novel phenomena, an example of these phenomena is found below.

Different versions of the extended hydrodynamics for various physical systems are presented in Refs. [65,72,75–77]. Novel approaches to the development of hydrodynamics are recently presented in Refs. [78] and [79].

### 3 Large amplitude soliton in dipolar BECs

Let us apply the model presented above to the study the solitons in dipolar BECs. Our goal is to consider the influence of the quantum fluctuations. Moreover, novel soliton solution caused by the dipolar part of quantum fluctuations is analytically found below.

We consider solitons in the uniform boundless BEC with no restriction on the amplitude of soliton. Hence, we have no external potential $V_{ext} = 0$. Consider the simplified form of the hydrodynamic equations giving main contribution in the soliton solution. The continuity Eq. (4) requires no simplification. No simplification of the equation of field (9) is required too.

We drop the traditional part of the quantum Bohm potential in the Euler Eq. (6):

$$
\begin{align*}
mn \partial_t v^\alpha + mn(v \cdot \nabla) v^\alpha + \partial_\beta T_{rf}^{\alpha\beta} = - gn \partial^\alpha n - n \partial^\alpha \Phi_d.
\end{align*}
$$

(22)

The pressure evolution equation simplifies to two terms:

$$
\begin{align*}
\partial_t T_{rf}^{\alpha\beta} + \partial_\gamma Q_{rf}^{\alpha\beta\gamma} = 0,
\end{align*}
$$

(23)
that perturbations vanish at \( \xi \to \pm \infty \), following reduction of Eqs. (4), (9), (22), (23), (24).

where the quantum Bohm potential \( T_q^{\alpha \beta} \) is caused purely by the flux \( Q_q^{\alpha \beta \gamma} \).

Equation for the evolution of quantum-thermal part of the third rank tensor is:

\[
\partial_t Q_q^{\alpha \beta \gamma} = \frac{\hbar^2}{4m^2} \left( g_2 r_0^{\alpha \beta \delta} \partial^\delta n + \partial^\alpha \partial^\beta \partial^\gamma \Phi_d \right), \tag{24}
\]

where \( \Phi_d \) is the interaction on the right-hand side represents the quantum fluctuations. It is assumed that the evolution of the tensor \( Q_q^{\alpha \beta \gamma} \) is mainly caused by the interaction.

There is no independent source of interaction in the pressure evolution Eq. (24). Hence, the third rank tensor \( Q_q^{\alpha \beta \gamma} \) is the single source of interaction in the pressure evolution equation. As it is mentioned above, we focus on the pressure or the quantum Bohm potential caused by the quantum fluctuations. Hence, we consider the interaction caused \( T_q^{\alpha \beta} \) and drop the kinematic terms (see Eq. (24)).

We consider the one-dimensional solution. We chose the direction of wave propagation perpendicular to the direction of titled dipoles. We seek the stationary solutions of the nonlinear equations. We assume the steady state in the comoving frame. Therefore, the dependence of the time and space coordinates is combined in the single variable \( \xi = x - ut \). Parameter \( u \) is the constant velocity of the nonlinear solution. Therefore, all hydrodynamic functions depend on \( \xi \) and \( u \). We also assume that perturbations vanish at \( \xi \to \pm \infty \). It gives the following reduction of Eqs. (4), (9), (22), (23), (24).

Figure 1 shows that there is the “potential gap” in the area of positive deviations \( \Delta \). It means that there is a bright soliton (the area of increased concentration)

Equation (39) can be integrated to obtain the “energy integral” in the following manner

\[
\frac{1}{2} \left( \partial_\xi \eta \right)^2 + \frac{m^2 \mu^2}{\pi \mu^2 \hbar^2} V_{\text{eff}}(n) = 0, \tag{25}
\]

where \( V_{\text{eff}}(n) \) is the Sagdeev potential \([80–86]\):

\[
V_{\text{eff}}(n) = \frac{1}{2} \left( \left( g - 4\pi \mu^2 - \frac{1}{u^2} \frac{\hbar^2}{4m^2} \right) g_2 - \frac{m\mu^2}{n} \right) (n - n_0)^2. \tag{26}
\]

Details of derivation of Eqs. (25) and (26) are presented in the Supplementary Materials. Moreover, the estimation of the area of applicability of Eqs. (22), (23) and (24) is discussed in Supplementary Materials either.

In order for the soliton to exist, the effective potential \( V_{\text{eff}}(n) \) should have a local maximum in the point \( n = 0 \). Moreover, equation \( V_{\text{eff}}(n) = 0 \) should have at least one real solution \( n' \neq 0 \). This value of concentration \( n' \) determines the amplitude \( n' \) of the soliton as the function of velocity \( u \).

Equation \( V_{\text{eff}} = 0 \) can be solved analytically for \( n \neq n_0 \):

\[
n' = \frac{m\mu^2}{g - 4\pi \mu^2 - \frac{1}{u^2} \frac{\hbar^2}{4m^2} 3g_2}. \tag{27}
\]

Equation (26) allows to introduce the effective dimensionless interaction constant \( G = (n_0/m\mu^2)(g - 4\pi \mu^2 - 3\hbar^2 g_2/4m^2 u^2) \).

Let us consider the limit of the small dipole–dipole interaction and the small short-range part of the quantum fluctuations in comparison with the mean-field of
the short-range interaction presented by the Gross–Pitaevskii interaction constant $g$. Hence, we have $G \approx 2g/mu^2$. We have soliton solution for the positive interaction constant $G \sim g > 0$ which corresponds to the repulsive short-range interaction.

Consider Eq. (25) in more details and justify that the soliton solution appears at $G > 0$. The first term in Eq. (25) is positive. Hence, Eq. (25) has solutions at the negative effective potential (26) $V_{eff} < 0$. We have $V_{eff}(n = n_0) = 0$. Next, we find $V_{eff} < 0$ at $n > n_0$. Moreover, the point exists, where $V_{eff}(n \neq n_0) = 0$. It creates the area for the “finite motion” of the effective quasi-particle in potential $V_{eff}$ at the negative effective energy. The concentration $n$ is always positive $n > 0$. Hence, we can obtain $V_{eff}(n \neq n_0) = 0$ for the positive $G$. It can be seen from the structure of effective potential $V_{eff}$ (26), where the last term is negative. Hence, the superposition of other terms $G$ should be positive.

The solitary waves exist due to the balance between the nonlinearities caused by the GP interaction, dipole–dipole interaction, and short-range part of quantum fluctuations and the dispersion induced by the dipolar part of quantum fluctuations. The dipole–dipole interaction and short-range part of quantum fluctuations introduce an additional negative contribution to the interaction constant of the GP approximation.

The contribution of the short-range part of quantum fluctuations in the effective interaction constant $G$ might be minor. However, the soliton exists due to the existence of the first term in Eq. (25) (the term proportional to the second derivative of the concentration). This term is caused by the dipolar part of the quantum fluctuations. Therefore, the obtained soliton has no simplification in the mean-field limit. It exists in the quantum regime only.

To understand the possibility of the soliton existence in the regime under consideration, we study the form of the Sagdeev potential (42).

The Sagdeev potential (42) is plotted in Fig. 1 for the positive effective interaction constant. This regime is chosen since it shows the existence of the soliton solution.

Figures 1 and 2 show that the amplitude of soliton (the value $n'$, where $V(n') = 0$) decreases with the increase of the effective interaction constant $G$. The increase of the velocity of soliton $u$ decreases constant $G$ and, consequently, decreases the amplitude. The increase of the soliton velocity $u$ diminish the role of the short-range interaction part of the quantum fluctuations in the effective interaction constant.

The traditional bright and dark solitons in neutral atomic BEC are caused by the nonlinearity created by the short-range interaction in the Gross–Pitaevskii approximation. There are different generalizations of the Gross–Pitaevskii model include effects beyond the mean-field approximation. An example of the beyond mean-field model of BEC has been derived in this paper. The quantum fluctuations have been included here via the extended hydrodynamic model which includes the continuity equation for the scalar field of concentration, the Euler equation for the vector field of velocity, the pressure evolution equation for the second rank tensor of pressure, and the evolution equation for the third rank tensor.

The dipolar part of quantum fluctuations is presented by the term proportional to the third derivative of the electrostatic potential. For one-dimensional perturbations in the single fluid species, the potential is proportional to the variation of the concentration from the equilibrium value.

Hence, the high derivatives of the concentration appears in the term presenting the dipolar part of quantum fluctuations.

4 Conclusions

The extended quantum hydrodynamic model has been applied to study the contribution of the quantum fluctuations in the properties of solitons in the dipolar BECs. The extended quantum hydrodynamic model includes the equations for the evolution of the pressure and the flux of pressure. The flux of pressure is the third rank tensor, but we need one element of this tensor to study the plane waves and the plane solitons in the uniform medium. Necessity of the application of the extended quantum hydrodynamics is in the fact that the quantum fluctuations (both the short-range part and the dipolar part) appear in the equation for the third rank tensor evolution.

It has been found that the dipolar part of the quantum fluctuations leads to the formation of the novel soliton solution. This solution has no limit in the mean field regime, where the continuity and Euler equations are used without the contribution of the higher rank tensors. It appears as the bright soliton since it is the area of the increased concentration. However, it exists at the repulsive short-range interaction. The dipolar part of the quantum fluctuations creates the anomalous dispersion since it gives the negative contribution in the dispersion of waves in the short-wavelength limit. It leads to the compression of the wave packet. This compression is balanced by the repulsive nonlinearity caused by the short-range interaction (the mean field part). The large amplitude soliton solution has been obtained from the hydrodynamic equations by the Sagdeev potential method.

Acknowledgements We acknowledge that the work is supported by the Russian Foundation for Basic Research (Grant No. 20-02-00476). This paper has been supported by the RUDN University Strategic Academic Leadership Program.

Author contributions

This paper has one author, who made the preparation of the manuscript and carried on the research presented in the manuscript.
Data Availability Statement This manuscript has no associated data or the data will not be deposited. [Authors’ comment: Data sharing is not applicable to this article as no new data were created or analyzed in this study, which is a purely theoretical one.]

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