ON INTERSECTION COHOMOLOGY WITH TORUS ACTION OF COMPLEXITY ONE, II

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ABSTRACT. We show that the components, appearing in the decomposition theorem for contraction maps of torus actions of complexity one, are intersection cohomology complexes of even codimensional subvarieties. As a consequence, we obtain the vanishing of the odd dimensional intersection cohomology for rational complete varieties with torus action of complexity one. The article also presents structural results on linear torus action in order to compute the intersection cohomology from the weight matrix. In particular, we determine the intersection cohomology Betti numbers of affine trinomial hypersurfaces in terms of their defining equation.

CONTENTS

1. Introduction 1
2. Preliminaries 5
3. Intersection cohomology and actions of finite groups 10
4. Linear torus actions of complexity one 16
5. Vanishing of odd dimensional intersection cohomology 31
6. Computing intersection cohomology 37
References 43

1. INTRODUCTION

We are studying intersection cohomology of complete complex algebraic varieties endowed with an action of an algebraic torus $\mathbb{T} = (\mathbb{C}^*)^n$. An important invariant in the classification of torus actions is the complexity. It is defined, for a $\mathbb{T}$-variety $X$, as the number $c(X) := \dim X - \dim \mathbb{T}/\mathbb{T}_0$, where $\mathbb{T}_0$ is the kernel of the torus action. Toric varieties are precisely normal varieties with torus action of complexity 0. They admit a combinatorial description by objects of convex geometry such as fans, polytopes, etc. Part of the geometry of a toric variety can be recorded from the combinatorial data. This was performed for the intersection cohomology [Sta87, DL91, Fie91, BBFK99, BM99, BBFK02, CMM18], where in the case of complete simplicial fans, this description is related to the $f$-vector of the toric variety (that is, the vector encoding the number of cones of the fan of each dimension).

For torus actions of complexity one, there is still a dictionary between $\mathbb{T}$-varieties and objects lying between geometry and combinatorics [AIPSV12]. This class of $\mathbb{T}$-varieties encompasses the $\mathbb{C}$*-surfaces, whose intersection cohomology has been studied by Fieseler and Kaup [FK86, FK91]. In [AL18, AL21], we started a program for determining the intersection cohomology Betti numbers in the complexity-one case.

More precisely, let $X$ be a complete variety with torus action of complexity one, let $\tilde{X}$ be the normalization of the graph of the rational quotient $\iota : X \to C$, where $C$ is the smooth
projective curve parameterizing the general orbits, and let \( \pi : \widetilde{X} \to X \) be the natural map, called the contraction map. Our approach consisted of the following two steps.

1. The determination of the intersection cohomology Betti numbers of \( \widetilde{X} \) (see [AL18], [AL21, Section 5.1]) from its toroidal structure and the toric decomposition theorem [CMM18].
2. The elaboration of an algorithm [AL21] expressing the intersection cohomology of \( X \) in terms of the one of \( \widetilde{X} \) via the decomposition theorem of the contraction map.

In this paper, we carry out the final step of our program for the complexity-one case, namely:

3. We provide a simple formula for the intersection cohomology Betti numbers. Moreover, we present structural results that describe these numbers from the defining equations.

Finally, we treat the example of affine trinomial hypersurfaces.

We now discuss the results of the paper. Our first upshot is an improvement of [AL21, Theorem 1.1]. It gives a simple form of the decomposition theorem for the contraction map. For a variety \( V \) we denote by \( IC_V \) its intersection cohomology complex and by \( IH^j(V; \mathbb{Q}) \) the intersection cohomology groups.

**Theorem A** (= Theorem 5.5). Let \( X \) be a normal variety with torus action of complexity one. Denote by \( E \) the image of the exceptional locus of the contraction map \( \pi : \widetilde{X} \to X \), and let \( \mathcal{O}_{even}(E) \) be the set of even codimensional torus orbits of \( X \) contained in \( E \). Then we have an isomorphism

\[
\pi_*IC_{\widetilde{X}} \cong IC_X \oplus \bigoplus_{O \in \mathcal{O}_{even}(E)} (\iota_O)_*IC^s_O,
\]

where \( \iota_O : \bar{O} \to X \) is the inclusion and \( s_O \in \mathbb{Z}_{\geq 0} \) for any \( O \in \mathcal{O}_{even}(E) \).

Next result, which is consequence of Theorem A, provides a cohomological criterion of rationality in the setting of torus actions of complexity one.

**Theorem B** (= Theorem 5.14). Let \( X \) be a complete variety with torus action of complexity one. Then the following are equivalent.

1. The variety \( X \) is rational.
2. We have \( IH^{2j+1}(X; \mathbb{Q}) = 0 \) for any \( j \in \mathbb{Z} \).

Application of Theorem A is a formula for the intersection cohomology of the description of [AIPSV12], in terms of divisorial fans, extending the 3-dimensional case [AL21, Theorem 1.2].

**Divisorial fans** are analogous to fans in toric geometry for complexity-one torus actions. They consist of finite collections \( \mathcal{E} \) of divisors with polyhedral coefficients (called polyhedral divisors) over Zariski open subsets of a smooth projective curve \( C \) with additional conditions. For a complexity-one \( T \)-variety \( X \) with defining divisorial fan \( \mathcal{E} \), the curve \( C \) represents the rational quotient of the torus action, while the polyhedral coefficients encode the geometry of the fibers of the global quotient \( \bar{X} \to C \) of the contraction space \( \widetilde{X} \).

Let us introduce further notations. We call Poincaré polynomial of a variety \( V \), the polynomial \( P_V(t) = \sum_{j \in \mathbb{Z}} b_j(V)t^j \), where \( b_j(V) := \dim IH^j(V; \mathbb{Q}) \). For a strictly convex polyhedral cone \( \sigma \) (resp. a fan \( \Sigma \)), let \( X_\sigma \) (resp. \( X_\Sigma \)) be the associated toric variety. We denote by \( g(\sigma; t^2) = \sum_{j \in \mathbb{Z}_{\geq 0}} g_j(\sigma)t^j \) the \( g \)-polynomial of \( \sigma \), where \( g_j(\sigma) \) is the \( j \)-th local intersection cohomology of \( X_\sigma \) along the closed orbit. Moreover, we write \( h(\Sigma; t^2) \) for the \( h \)-polynomial of a complete fan \( \Sigma \), which is the Poincaré polynomial of \( X_\Sigma \). Note that \( g(\sigma; t^2) \) and \( h(\Sigma; t^2) \) are combinatorial objects, see [DL91, Fie91, BBFK02]. Similarly, the \( h \)-polynomial \( h_{\tilde{\mathcal{E}}}(t) \) of a divisorial fan \( \tilde{\mathcal{E}} \) describing the contraction space \( \tilde{X} \) of a complete normal variety \( X \) with torus action of complexity one is the Poincaré polynomial of \( \tilde{X} \). The polynomial \( h_{\tilde{\mathcal{E}}}(t) \) is explicit from the divisorial fan of \( X \) (see the reminder in Section 5.1).
The formula for the Betti numbers can be expressed as follows (see Corollary 5.12 for the affine case).

**Theorem C** (= Corollary 5.11). Let $X$ be a complete normal $\mathbb{T}$-variety of complexity one with divisorial fan $\mathcal{E}$. Denote by $\mathcal{E}$ the divisorial fan of the contraction space $\bar{X}$ of $X$. Then the Poincaré polynomial of $X$ is given by the formula

$$P_X(t) = h_{\mathcal{E}}(t) - \sum_{\tau \in HF(\mathcal{E})} g_n(\tau) t^{c(\tau)} h(\text{Star}(\mathcal{E}, \tau); t^2),$$

where the set $HF(\mathcal{E})$ (see 5.3) consists of polyhedral cones that are in one-to-one correspondence with the orbits in the image of the exceptional locus of the contraction map of $X$. Moreover, for $\tau \in HF(\mathcal{E})$, the symbol $	ext{Star}(\mathcal{E}, \tau)$ stands for the fan of the normalization of the orbit closure associated with $\tau$. Finally, for any $\tau \in HF(\mathcal{E})$ the numbers $n(\tau)$ and $c(\tau)$ are respectively $\dim \tau - 1$ and $\dim \tau + 1$.

Part of the present work develops computation methods. More specifically, consider the projective space $\mathbb{P}_C^\ell$ with its natural toric structure for the torus $\mathbb{G} = (\mathbb{C}^*)^\ell$. Let $X \subset \mathbb{P}_C^\ell$ be a subvariety with linear torus action of complexity one, i.e. $X$ meets the open $\mathbb{G}$-orbit and there is a subtorus $\mathbb{T} \subset \mathbb{G}$ acting on $X$ with complexity one. The inclusion $\mathbb{T} \hookrightarrow \mathbb{G}$ induces a linear map $\mathbb{Z}^n \to \mathbb{Z}^{\ell}$ whose matrix $F$ is the weight matrix of the $\mathbb{T}$-action. In this article, we solve the following question.

**Question (\ast):** How do we compute the intersection cohomology of the variety $X$ from the matrix $F$?

Since the variety $X$ has same intersection cohomology Betti numbers as its normalization $\bar{X}$ (see [GS93, Section 5, Lemma 1]), answer of Question (\ast) means to build a divisorial fan of $\bar{X}$ from the matrix $F$ and apply Theorem C. For this, we use ingredients of [KSZ91], [AIPSV12, Section 4] that we now recall.

Start with the short exact sequence

$$0 \to \mathbb{Q}^n \xrightarrow{F} \mathbb{Q}^{\ell} \xrightarrow{P} \text{Coker}(F) \to 0,$$

a section $S : \mathbb{Z}^{\ell} \to \mathbb{Z}^n$ of $F$, and the fan $\Sigma$ generated by the images of $P$ of the faces of the first quadrant $\delta := \mathbb{Q}_{\geq 0}^{\ell}$. We say that the data $\theta = (\overline{\Sigma} = \mathbb{Z}^{\ell}, N = \mathbb{Z}^n, F, S, \Sigma)$ is a weight package. We attach to the weight package $\theta$ a piecewise linear map

$$\mathcal{D}_\theta : m \mapsto \sum_{\rho \in \Sigma(1)} \min_{v \in S(\delta \cap P^{-1}(\nu))} \langle m, v \rangle \cdot Z_{\rho},$$

which goes from the dual cone of $\sigma_\rho := S(\delta \cap F(N_{\mathbb{Q}}))$ to the vector space of Cartier $\mathbb{Q}$-divisors of $X_\Sigma$. Note that we are using toric notations, namely $\Sigma(1)$ is the set of rays of $\Sigma$, $\nu_\rho$ is the primitive generator of $\rho$ and $Z_\rho \subset X_\Sigma$ is the corresponding toric divisor.

Denoting by $\psi : \mathbb{G} \to \mathbb{T}_{\text{Coker}(F)}$ the quotient map onto the torus associated with $\text{Coker}(F)$, the pullback $\kappa^* (\mathcal{D}_\theta)$ is seen as a polyhedral divisor, where $\kappa$ is the composition of the normalization $\tilde{\mathcal{C}}_\theta \to C_\theta$ of the curve $C_\theta$ obtained as the Zariski closure of $\psi(X \cap \mathbb{G})$ in $X_\Sigma$ and the inclusion $C_\theta \subset X_\Sigma$. Precisely, write $x_0, \ldots, x_\ell$ for the homogeneous coordinates of $\mathbb{P}_C^\ell$ coming from the toric structure. Then the polyhedral divisor $\kappa^* (\mathcal{D}_\theta)$ describes the normalization of the variety $X^{(i)} = X \setminus V(x_i)$ (see Theorem 4.9). Via matrix operations (see Definition 4.16), one associates weight packages

$$\theta^{(i)} = (\overline{\Sigma}, N, F^{(i)}, S^{(i)}, \Sigma^{(i)}), \ 0 \leq i \leq \ell,$$

doing the $\mathbb{T}$-actions on the charts $X^{(i)} := X \setminus V(x_i)$. 

The following result, extending [AIPSV12, Section 4] to the non-normal case, gives an
inter relation between weight packages and defining equations of projective varieties with linear torus
action of complexity one.

**Theorem D** (= Theorem 4.22).  
(1) Let $X \subset \mathbb{P}^d_{\mathbb{C}}$ be a subvariety with linear torus action of
complexity one. Let $\theta$ be the weight package of $X$ and let $\theta^{(i)} = (N, N, F^{(i)}, S^{(i)}, \Sigma^{(i)})$ be
the weight package corresponding to the chart $X^{(i)}$. Let $\Sigma$ be a fan with support $\bigcup_{i=0}^{\ell} \Sigma^{(i)}$ such that
for $0 \leq i \leq \ell$ the set $\Sigma^{(i)} := \{ \sigma \in \Sigma | \sigma \subset | \Sigma^{(i)} | \}$ is a projective fan subdivision
of $\Sigma$. Denote by $\kappa^{(i)} : C_{\theta^{(i)}} \rightarrow X_{\Sigma^{(i)}}$ the map, which is the composition of the
projective modification $f^{(i)} : X_{\Sigma^{(i)}} \rightarrow X_{\Sigma^{(i)}}$, the inclusion $C_{\theta^{(i)}} \rightarrow X_{\Sigma^{(i)}}$, where $C_{\theta^{(i)}}$ is
the proper transform of $C_{\theta^{(i)}}$ under $f^{(i)}$, and the normalization $\overline{C_{\theta^{(i)}}} \rightarrow C_{\theta^{(i)}}$. Then the set
$$\{ \mathfrak{D}^{(i)}_{\theta} := \kappa^{(i)} \ast C_{\theta^{(i)}} \ | \ i = 0, 1, \ldots, \ell \}$$
generates a divisorial fan $\mathfrak{E}_{\theta}$ describing the normalization of $X$.

(2) Conversely, let $\theta = (N = \mathbb{Z}^d, N = \mathbb{Z}^n, F, S, \Sigma)$ be a weight package. Let $C \subset T_{\text{Coker}(F)} = (\mathbb{C}^*)^s$ be
an irreducible curve with defining equations $f_i (1 \leq i \leq a)$. Define the matrix
$$\hat{P} = \begin{bmatrix} b_{1,0} & b_{1,1} & \ldots & b_{1,\ell} \\ \vdots & \vdots & \ddots & \vdots \\ b_{s,0} & b_{s,1} & \ldots & b_{s,\ell} \end{bmatrix} \in \text{Mat}_{s \times \ell + 1}(\mathbb{Z}),$$
as the addition of $P$ of a first column so that the sum of the entries of each row is 0. Set
$$g_i(T_0, T_1, \ldots, T_r) := f_i \left( \prod_{j=0}^{\ell} T_j^{b_{1,j}}, \ldots, \prod_{j=0}^{\ell} T_j^{b_{s,j}} \right) \text{ for } 1 \leq i \leq a,$$
and assume that there exist Laurent monomials $u_i \in \mathbb{C}[T_0, T_0^{-1}, \ldots, T_r, T_r^{-1}]$ such that $X := \mathbb{V}(u_1 g_1, \ldots, u_a g_a) \subset \mathbb{P}^d_{\mathbb{C}}$ is irreducible. Then $X$ is a subvariety with linear torus action
of complexity one and its normalization is described by the divisorial fan $\mathfrak{E}_{\theta}$ obtained
from Contraction (1).

(3) If $X \subset \mathbb{P}^d_{\mathbb{C}}$ is a projective subvariety with linear torus action of complexity one with weight
package $\theta$, then $X$ admits a decomposition $X = \mathbb{V}(u_1 g_1, \ldots, u_a g_a)$ as in Assertion (2).

We illustrate our method with the example of affine trinomial hypersurfaces, which are zero
loci
$$X = \mathbb{V}(T_1^{a_{1,1}} + T_2^{a_{2,2}} + T_3^{a_{3,3}}) \subset \mathbb{A}^d_{\mathbb{C}}$$
such that $T_i^{a_{i,j}}$ is a monomial $\prod_{j=1}^{r_i} T_{i,j}^{n_{i,j}}$ for $i = 1, 2, 3$, with $r_i, n_{i,j} \in \mathbb{Z}_{>0}$ and $\ell = r_1 + r_2 + r_3$.
Set
$$u_i := \gcd(n_{i,1}, \ldots, n_{i,r_i}) \text{ for } i = 1, 2, 3, \ d = \gcd(u_1, u_2, u_3),$$
$$d_1 = \gcd(u_2/d, u_3/d), \ d_2 = \gcd(u_1/d, u_3/d), \ d_3 = \gcd(u_1/d, u_2/d) \text{ and } u = d d_1 d_2 d_3.$$  
From the trinomial equation, one associates a weight package $\theta$ (see 6.2).

Using the result of Kruglov [Kru19, Theorem 3.1] and Corollary 5.12 (the affine version of
Theorem C) we compute the intersection cohomology of any affine trinomial hypersurface.

**Theorem E** (= Corollary 6.6). Let
$$X = \mathbb{V}(T_1^{a_{1,1}} + T_2^{a_{2,2}} + T_3^{a_{3,3}}) \subset \mathbb{A}^d_{\mathbb{C}}$$
be an affine trinomial hypersurface with natural weight package $\theta = (N = \mathbb{Z}^d, N, F, S, \Sigma)$ and let
$(e_{1,1}, \ldots, e_{1,r_1}, e_{2,1}, \ldots, e_{2,r_2}, e_{3,1}, \ldots, e_{3,r_3})$ be the canonical basis of $N$. Set $\gamma = d(d_1 + d_2 + d_3)$
and
$$\Pi_i := \text{Cone} \left( (\sigma_{\theta} \times \{0\} ) \cup \left( S \left( \frac{d}{d_j n_{i,j}} e_{i,j} \right) | 1 \leq j \leq r_i \right) \times \{1\} \right) \)
for \(i = 1, 2, 3\). Then the Poincaré polynomial of the contraction space \(\tilde{X}\) of \(X\) is given by

\[
P_{\tilde{X}}(t) = (t^2 + (du - \gamma + 2)t - \gamma + 1) \cdot g(\sigma_0; t^2) + \sum_{i=1}^{3} dd_i \cdot g(\Pi_i; t^2).
\]

Furthermore, write

\[
H(\theta, \underline{n}_1, \underline{n}_2, \underline{n}_3) := \left\{ \tau \text{ face of } \sigma_0 \mid \tau \cap \left\{ \sum_{i=1}^{3} S \left( \frac{d}{d_j t_{i,j}} e_{i,j} \right) \mid (j_1, j_2, j_3) \in \prod_{i=1}^{3} \{1, \ldots, r_i\} \right\} \neq \emptyset \right\}.
\]

Then the Poincaré polynomial of \(X\) is obtained from the relation

\[
P_X(t) = P_{\tilde{X}}(t) - \sum_{\tau \in H(\theta, \underline{n}_1, \underline{n}_2, \underline{n}_3)} g_n(\tau) t^{c(\tau)},
\]

where \(n(\tau) = \dim \tau - 1\) and \(c(\tau) = \dim \tau + 1\).

Let us give a brief summary of the contents of each section. Section 2 gives preliminaries on intersection cohomology and torus actions. As preparation for Theorem A, we study in Section 3 intersection cohomology with finite group action. In 3.1 we obtain results on pullbacks of quotient maps of finite group actions for intersection cohomology complexes that might be of independent interest (see Propositions 3.6, 3.8). Section 4 introduces weight package theory, and describes the intersection cohomology with finite group action. In 4.30 we then show in 4.4 Theorem D. Moreover, we prove a similar result as Theorem D for contraction spaces of torus actions of complexity one (see Theorem 4.30). Section 5 is devoted to the proofs of Theorem A and Theorem B. Finally, in Section 6, we discuss some consequences of the results of Sections 4 and 5 and describe the intersection cohomology for trinomial hypersurfaces, where we treat the affine case and partially the projective case.

**Perspective.** It was conjectured [CHL18] that the decomposition theorem exists, in a strong form, over finite fields, and confirmed for toric varieties [Cat15] and convolution morphisms of partial affine flag varieties [CHL18, Section 6]. Can this be verified in the setting of torus actions of complexity one?

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1.1. **Convention.** Local systems are locally constant sheaves of \(\mathbb{Q}\)-vector spaces for the Euclidean topology with finite-dimensional stalks. Semi-projective means projective over affine.

2. **Preliminaries**

This section is devoted to the preliminaries on intersection cohomology and torus actions.

2.1. **Algebraic varieties.** We recall some basic notions on divisor theory. We write \(\leq\) for the coefficient-wise inequality between \(\mathbb{Q}\)-divisors. Given a \(\mathbb{Q}\)-divisor \(D\) on a normal variety \(Y\) we denote by \(H^0(Y, \mathcal{O}_Y(D))\) its space of global sections, and for a rational function \(f \in \mathbb{C}(Y)^*\) we write \(\text{div}(f)\) for its principal divisor and \(\text{ord}_Z(f)\) for its vanishing order along a prime divisor \(Z \subset Y\). For \(s \in H^0(Y, \mathcal{O}_Y(D)) \setminus \{0\}\), the zero locus \(Z_{Y,D}(s)\) is the union of the prime divisors that are in the support of \(\text{div}(s) + D\). We denote by \(Y_{D,s} := Y \setminus Z_{Y,D}(s)\) the complement. We say that the \(\mathbb{Q}\)-divisor \(D\) is semi-ample if for some \(r \in \mathbb{Z}_{>0}\), the open subsets \(Y_{rD,s}\) cover \(Y\), where \(s\) runs over \(H^0(Y, \mathcal{O}_Y(rD)) \setminus \{0\}\).
Definition 2.1. The \( \mathbb{Q} \)-divisor \( D \) on the normal variety \( Y \) is \textit{big} if there are \( r \in \mathbb{Z}_{>0} \) and \( s \in H^0(Y, \mathcal{O}_Y(rD)) \setminus \{0\} \) such that \( Y_{rD,s} \) is affine.

Later we will use the following observation.

Lemma 2.2. Let \( D, D' \) be two \( \mathbb{Q} \)-divisors on a normal variety \( Y \). Assume that \( D \) is semi-ample and that \( H^0(Y, \mathcal{O}_Y(rD)) \subset H^0(Y, \mathcal{O}_Y(rD')) \) for any \( r \in \mathbb{Z}_{>0} \). Then \( D \leq D' \).

Proof. Since \( D \) is semi-ample, there exist \( r \in \mathbb{Z}_{>0} \), an open covering \( (U_i)_{i \in I} \) of \( Y \) and a family of nonzero global sections \( (f_i)_{i \in I} \) of \( rD \) such that \( rD|_{U_i} = -\text{div}(f_i)|_{U_i} \) for any \( i \). Since each \( f_i \) is in \( H^0(Y, \mathcal{O}_Y(rD')) \), we have \( rD|_{U_i} = -\text{div}(f_i)|_{U_i} \leq rD'|_{U_i} \) for any \( i \), proving \( D \leq D' \). \( \square \)

2.2. Intersection cohomology. We set our convention on intersection cohomology theory (see [BBD82, GM83, CM09], [Wil17, Section 1], [Max19]). For a variety \( X \) we denote by \( D^b_{\text{const}}(X) \) the constructible derived category of sheaves of \( \mathbb{Q} \)-vector spaces on \( X \); this is a triangulated category with shift functor \( [1] \). Given a morphism \( f : X \to Y \) of varieties, we write

\[
\begin{align*}
&f_*, f : D^b_{\text{const}}(X) \to D^b_{\text{const}}(Y) \quad \text{and} \quad f^*, f^! : D^b_{\text{const}}(Y) \to D^b_{\text{const}}(X)
\end{align*}
\]

for the derived functors \( Rf_*, Rf_! \), etc. For a complex \( F \in D^b_{\text{const}}(X) \) we denote by \( \mathcal{H}^j(F) \) its \( j \)-th cohomology sheaf and the arrow \( \mathbb{D} : D^b_{\text{const}}(X) \to D^b_{\text{const}}(X) \) will be the Verdier duality.

Let \( X = \bigcup_{\lambda \in I} X_\lambda \) be an algebraic Whitney stratification, where \( X_\lambda \) is the open stratum and the \( i_\lambda : X_\lambda \to X \) are inclusions of strata. Let \( \mathcal{L} \) be a local system on \( X_0 \). We denote by \( IC_X(\mathcal{L}) \) the \textit{intersection cohomology complex} with coefficients in \( \mathcal{L} \). According to Deligne, it is uniquely determined by:

(1) the open stratum condition: \( i^*_\lambda IC_X(\mathcal{L}) = \mathcal{L}[\dim X] \),
(2) the support conditions: \( \mathcal{H}^j(i_\lambda^* IC_X(\mathcal{L})) = 0 \) for \( j \geq -\dim X_\lambda \) and \( \lambda \neq \lambda_0 \), and
(3) the co-support conditions: \( \mathcal{H}^j(i_\lambda^! IC_X(\mathcal{L})) = 0 \) for \( j \leq -\dim X_\lambda \) and \( \lambda \neq \lambda_0 \).

Intersection cohomology complexes belong to the category of \textit{perverse sheaves} for the middle perversity \( p \), i.e., it is an element of the heart of the category \( D^b_{\text{const}}(X) \) for the \( t \)-structure

\[
\begin{align*}
&t^0(X) := \{ F \in D^b_{\text{const}}(X) \mid \dim \text{Supp}(\mathcal{H}^j(F)) \leq -j \text{ for all } j \}, \\
&t^\geq(X) := \{ F \in D^b_{\text{const}}(X) \mid \dim \text{Supp}(\mathcal{H}^j(\mathbb{D}F)) \leq -j \text{ for all } j \}.
\end{align*}
\]

The \textit{intersection cohomology groups} with coefficients in \( \mathcal{L} \) are the hypercohomology groups

\[
IH^*(X; \mathcal{L}) := \mathbb{H}^*(X, IC_X(\mathcal{L})[-\dim X]).
\]

In particular, if \( \mathcal{L} = \mathbb{Q} \), then we set \( IC_X := IC_X(\mathcal{L}) \) and \( IH^*(X; \mathbb{Q}) := \mathbb{H}^*(X, IC_X[-\dim X]) \).

Observe that \( IH^*(X; \mathbb{Q}) = H^*(X; \mathbb{Q}) \) whenever \( X \) is smooth.

Definition 2.3. An object in \( D^b_{\text{const}}(X) \) is \textit{semi-simple} or \textit{pure} [BBD82, 5.4] if it is a finite direct sum of objects \( \iota_* IC_Z(\mathcal{L})[r] \), where \( r \in \mathbb{Z} \), \( \iota : Z \to X \) is the inclusion of a Zariski closed irreducible subvariety, and \( \mathcal{L} \) is a simple local system on a smooth Zariski open subset of \( Z \).

For a morphism of varieties \( f : X \to Y \) let \( Y^i := \{ y \in Y \mid \dim f^{-1}(y) = i \} \). The \textit{defect} of \( f \) is the number \( \text{def}(f) := \max \{ 2i + \dim Y^i - \dim X \mid Y^i \neq \emptyset \} \). The map \( f \) is \textit{semi-small} if \( \text{def}(f) = 0 \). Recall that \( X \) is \textit{rationally smooth} if for any \( x \in X \) the cohomology \( H^*_x(X; \mathbb{Q}) \) with support in \( \{ x \} \) is 0 when \( i \neq \dim(X) \). The following is the \textit{decomposition theorem}.

Theorem 2.4. [BBD82, Theorem 6.25], [Wil17, Theorem 2.4] Let \( f : X \to Y \) be a proper morphism of varieties. Then \( f_* IC_X \) is semi-simple. If further \( f \) is birational, then \( IC_Y \) is a summand of \( f_* IC_X \). Moreover, if \( X \) is rationally smooth and \( f \) is semi-small, then \( f_* IC_X \) is semi-simple and perverse.

Next results collect properties of intersection cohomology.
Lemma 2.5. Let \( f : X \to Y \) be a smooth morphism of varieties of relative dimension \( r \). Then we have \( f^*IC_Y[v] \cong IC_X \).

Proof. Same argument as [AL18, Lemma 2.4]. \( \square \)

Lemma 2.6. [GS93, Section 5, Lemma 1]. If \( f : X \to Y \) is a finite birational morphism of varieties, then we have \( f_*IC_X \cong IC_Y \). In particular, if \( \hat{Y} \) is the normalization of \( Y \), then \( IH^*(\hat{Y}; \mathbb{Q}) = IH^*(\hat{Y}; \mathbb{Q}) \).

Lemma 2.7. Let \( X \) be a quasi-projective variety with action of a connected algebraic group \( \Gamma \) and let \( G \subset \Gamma \) be a finite subgroup. Then \( IH^*(X/G; \mathbb{Q}) \cong IH^*(X; \mathbb{Q}) \).

Proof. The group \( G \) acts on \( IH^*(X; \mathbb{Q}) \) and \( IH^*(X; \mathbb{Q})^G \cong IH^*(X/G; \mathbb{Q}) \) [Kir86, 2.12]. Using decomposition theorem for equivariant desingularizations as in [AW97], it suffices to prove that \( G \) trivially acts on \( IH^*(X; \mathbb{Q}) \) when \( X \) is smooth, which follows from the proof of [DL76, 6.4]. \( \square \)

Lemma 2.8. Let \( X \) be an affine variety with a non-hyperbolic \( \mathbb{C}^* \)-action (i.e. all the weights have the same sign), let \( Y := X^{\mathbb{C}^*} \) be the fixed point set, and let \( d \in \mathbb{N} \) such that the \( \mathbb{C}^* \)-action on \( X[d] := X/\mu_d(\mathbb{C}) \) is free outside \( Y[d] := Y/\mu_d(\mathbb{C}) \). If \( \iota : Y[d] \to X[d] \) is the inclusion and \( \mathcal{F} \in D_{\text{const}}^b(X[d]) \) is \( \mathbb{C}^* \)-equivariant, then \( \mathbb{H}^*(X[d], \mathcal{F}) \cong \mathbb{H}^*(Y[d], \iota^*\mathcal{F}) \).

Proof. Same proof as in [DL91, Lemma 6.5], [CMM18, Lemma 4.2]. \( \square \)

2.3. Algebraic torus actions. We recall basic notions on torus actions from the perspective of Altmann-Hausen’s theory [AIPSV12]. We refer to [Tim97, LT16, Lan20] for generalization to reductive group actions.

Classification of torus actions is intimately related to questions of convex geometry. Let \( N \cong \mathbb{Z}^n \) be a lattice, let \( M \) be its dual, and let \( N_Q := \mathbb{Q} \otimes \mathbb{Z} N \) and \( M_Q := \mathbb{Q} \otimes \mathbb{Z} M \) be the associated \( \mathbb{Q} \)-vector spaces. We denote by

\[
M_Q \times N_Q \to \mathbb{Z}, (m, v) \mapsto \langle m, v \rangle
\]

the natural pairing and by \( T = T_N := \mathbb{C}^* \otimes \mathbb{Z} N \cong (\mathbb{C}^*)^n \) the associated torus. By Polyhedral cones in a finite dimensional \( \mathbb{Q} \)-vector space we mean sets of non-negative linear combinations of finitely many vectors. The polyhedral cone \( \sigma \subset N_Q \) is strictly convex if \( \{0\} \) is a face, or equivalently, if the dual cone \( \sigma^\vee := \{ m \in M_Q \mid \langle m, v \rangle \geq 0 \text{ for any } v \in \sigma \} \) is full-dimensional.

Fix a strictly convex polyhedral cone \( \sigma \subset N_Q \). Let \( \text{Pol}_\sigma(N_Q) = \{ \sigma + Q \mid Q \text{ polytopes of } N_Q \} \) be the set of \( \sigma \)-polyhedra and let \( Y \) be a normal semi-projective variety. A \( \sigma \)-polyhedral divisor \( \mathcal{D} \) is a formal sum

\[
\mathcal{D} = \sum_{Z \subset Y} \mathcal{D}_Z \cdot [Z],
\]

where \( Z \subset Y \) runs over the set of prime divisors of \( Y \). \( \mathcal{D}_Z \in \text{Pol}_\sigma(N_Q) \) and \( \mathcal{D}_Z = \sigma \) for all but finitely many prime divisors \( Z \). The evaluation at \( m \in \sigma^\vee \) of the polyhedral divisor \( \mathcal{D} \) is the \( \mathbb{Q} \)-divisor

\[
\mathcal{D}(m) := \sum_{Z \subset Y} \min_{v \in \mathcal{D}_Z} \langle m, v \rangle \cdot [Z],
\]

and its \( M \)-graded algebra is the subalgebra

\[
A[Y, \mathcal{D}] := \bigoplus_{m \in \sigma^\vee \cap M} H^0(Y, \mathcal{O}_Y(\mathcal{D}(m))) \chi^m
\]

of the group algebra \( \mathbb{C}(Y)[M] \), where \( \chi^m \) is the Laurent monomial corresponding to \( m \in M \). A \( \sigma \)-polyhedral divisor \( \mathcal{D} \) is proper if for any \( m \in \sigma^\vee \),

(i) \( \mathcal{D}(m) \) is \( \mathbb{Q} \)-Cartier and semi-ample, and
(ii) $\mathcal{D}(m)$ is big whenever $m$ is in the relative interior of the cone $\sigma^\vee$.

Note that an action of the torus $\mathbb{T} = \mathbb{T}_N$ on the affine variety $X = \text{Spec} \ A$ is equivalent to endow $A = \mathbb{C}[X]$ with an $M$-grading, and that the action is faithful if the weights of $A$ generate the lattice $M$. Passing to the quotient, we may always transform torus actions into effective ones. The following is due to Altmann-Hausen [AIPS12, Theorem 7].

**Theorem 2.9.** (1) If $\sigma \subset N_\mathbb{Q}$ is a strictly convex polyhedral cone, $Y$ is a normal semi-projective variety, and $\mathcal{D}$ is a proper $\sigma$-polyhedral divisor on $Y$, then the $\mathbb{T}$-scheme $X(\mathcal{D}) = X(Y, \mathcal{D}) := \text{Spec} \ A[Y, \mathcal{D}]$ is a normal affine variety equivariantly birational to the product $\mathbb{T} \times Y$.

(2) Any normal affine variety with an effective torus action arise from a proper polyhedral divisor.

Let $\xi = (f_1 \chi^{m_1}, \ldots, f_r \chi^{m_r})$ be a sequence of homogeneous elements of $\mathbb{C}(Y)[M]$, where $Y$ is a normal semi-projective variety, $f_1, \ldots, f_r \in \mathbb{C}(Y)^*$, and $m_1, \ldots, m_r \in M$ generate the dual $\xi^\vee \subset M_\mathbb{Q}$ of a strictly convex polyhedral cone $\xi$ of $\sigma \subset N_\mathbb{Q}$. We denote by $\xi \mathcal{D}$ the $\xi$-$\sigma$-polyhedral divisor

$$
\xi \mathcal{D} := \sum_{Z \subset Y} \xi \mathcal{D}_Z \cdot [Z],
$$

where $\xi \mathcal{D}_Z := \{ v \in N_\mathbb{Q} | \langle m_i, v \rangle \geq -\text{ord}_Z(f_i) \text{ for } i = 1, \ldots, r \}$. The following explains the relation between the homogeneous generators of $\xi$ and the polyhedral divisor $\xi \mathcal{D}$.

**Lemma 2.10.** Let $\mathcal{D}$ be a proper $\sigma$-polyhedral divisor on a normal semi-projective variety $Y$. Let $\xi = (f_1 \chi^{m_1}, \ldots, f_r \chi^{m_r})$ be a sequence of homogeneous elements of generating the algebra $A[Y, \mathcal{D}]$, where $f_1, \ldots, f_r \in \mathbb{C}(Y)^*$. Then $\xi \mathcal{D} = \mathcal{D}$.

**Proof.** We follow the proof of [Lan13, Théorème 2.4]. By construction we have

$$
\xi \mathcal{D}(m_i) + \text{div}(f_i) \geq 0 \text{ for } 1 \leq i \leq r.
$$

So $A[Y, \mathcal{D}] \subset A[Y, \xi \mathcal{D}]$. Moreover, for any $m \in \sigma^\vee = \xi^\vee$, one has

$$
H^0(Y, \mathcal{O}_Y(\mathcal{D}(m))) \subset H^0(Y, \mathcal{O}_Y(\xi \mathcal{D}(m))).
$$

Using Lemma 2.2, it follows that $\mathcal{D}(m) \leq \xi \mathcal{D}(m)$ for any $m \in \sigma^\vee$. But we also have $\mathcal{D}(m_i) + \text{div}(f_i) \geq 0 \text{ for } 1 \leq i \leq r$ since $f_i \chi^{m_i} \in A[Y, \mathcal{D}]$. Hence $\langle m_i, v \rangle \geq -\text{ord}_Z(f_i)$ for all $v \in \mathcal{D}_Z$ and $i \in \{1, \ldots, r\}$, that is $\mathcal{D}_Z \subset \xi \mathcal{D}_Z$. This implies that $\mathcal{D}(m) = \xi \mathcal{D}(m)$ for any $m \in \sigma^\vee$, and so $\mathcal{D} = \xi \mathcal{D}$, ending the proof of the lemma. $\square$

**Definition 2.11.** Let $Y$ be a normal variety. A divisorial fan over $(Y, N)$ is a finite set $\mathcal{E} = \{ \mathcal{D}^i \mid i \in I \}$ of proper polyhedral divisors defined over semi-projective Zariski subsets of $Y$ with lattice $N$, stable by (component-wise) intersections and such that the natural maps $X(\mathcal{D}^i \cap \mathcal{D}^j) \to X(\mathcal{D}^i)$ are open immersions for all $i, j$, where $\mathcal{D}^i \cap \mathcal{D}^j$ is the intersection between $\mathcal{D}^i$ and $\mathcal{D}^j$.

A divisorial fan $\mathcal{E}$ over $(Y, N)$ defines a finite type normal $\mathbb{T}$-scheme $X(\mathcal{E}) = X(\mathcal{E}, \mathcal{E})$ by gluing the charts $X(\mathcal{D})$ for $\mathcal{D} \in \mathcal{E}$. When $Y$ is one-dimensional, the scheme $X(\mathcal{E})$ is a variety. Furthermore, any normal variety with effective torus action arises from a divisorial fan [AIPS12, Section 5].

Next is the pullback operation for polyhedral divisors.

**Definition 2.12.** Let $Y, Y'$ be normal semi-projective varieties, let $\mathcal{D}$ be a proper $\sigma$-polyhedral divisor over $Y$ and write $\mathcal{D} = \sum_{i=1}^r \mathcal{D}_{E_i} \cdot E_i$, where $\mathcal{D}_{E_i} \in \text{Pol}_n(N_\mathbb{Q})$ and $E_i$ is a Cartier divisor over $Y$. Let $\varphi: Y' \to Y$ be a generically finite morphism such that $Z \not\subset \varphi(Y')$ for any prime divisor $Z \subset Y$ in the support of some $E_i$. Then the pullback of $\mathcal{D}$ under $\varphi$ is the proper polyhedral divisor $\varphi^*(\mathcal{D}) = \sum_{i=1}^r \mathcal{D}_{E_i} \cdot \varphi^*(E_i)$ over $Y'$. 
2.4. Toric varieties. (cf. [CLS11]) A fan in $N_{\mathbb{Q}}$ is a non-empty set of strictly convex polyhedral cones of $N_{\mathbb{Q}}$ stable by face relation and such that the intersection of any two elements is a mutual face of both. For a fan $\Sigma$ of $N_{\mathbb{Q}}$ (respectively, a polyhedron $Q$) and $a \in \mathbb{Z}_{\geq 0}$ we denote by $X_{\Sigma}$ the associated toric variety, by $\Sigma(a)$ (respectively, $Q(a)$) the set of its $a$-dimensional cones of $\Sigma$ (respectively, $a$-dimensional faces of $Q$), and by $\Sigma_{\text{max}}$ the set of maximal cones. Elements of $\Sigma(1)$ are called rays of $\Sigma$. For a ray $\rho \in \Sigma(1)$, we write $v_\rho$ for the associated primitive generator.

Moreover, if $\Sigma$ is simplicial, then the map $Z_\rho = O(\rho) \subset X_\Sigma$ for the associated toric divisor. Let $X_\sigma$ be the affine toric variety of the cone $\sigma \in \Sigma$. If $D$ is a torus invariant Cartier divisor, then we call trivialization of the line bundle $O_{X_\Sigma}(D)$ the data $(X_\sigma, \chi^{m_\sigma})_{\sigma \in \Sigma_{\text{max}}}$ such that $D|_{X_{\sigma}} = \div \chi^{m_\sigma}$.

We write $|\Sigma| := \bigcup_{\sigma \in \Sigma} \sigma \subset N_{\mathbb{Q}}$ for the support of $\Sigma$. Note that $X_\Sigma$ is complete if and only if $|\Sigma| = N_{\mathbb{Q}}$. We say that $\Sigma$ is simplicial if each cone $\sigma \in \Sigma$ is generated by a subset of a basis of $N_{\mathbb{Q}}$. Given any two fans $\Sigma, \Sigma'$ with respect to lattices $N, N'$, we call fan morphism between $\Sigma$ and $\Sigma'$ a linear map $\phi : N \to N'$ such that for any $\sigma \in \Sigma$ there is $\sigma' \in \Sigma'$ such that $\phi(\sigma \cap N) \subset \sigma'$. Note that fan morphisms from $\Sigma$ to $\Sigma'$ corresponds totoric morphisms from $X_\Sigma$ to $X_{\Sigma'}$, i.e. morphisms that are equivariant.

Homogeneous coordinate theory for a toric variety $X_\Sigma$ (without torus factor) is a presentation $X_\Sigma = (A^*_\Sigma \setminus Z_\Sigma)/G_\Sigma$, where $G_\Sigma$ is a diagonalizable group and $Z_\Sigma$ is a $G_\Sigma$-invariant closed subset. Example to have in mind is the projective space $\mathbb{P}^{n\shortminus 1}$, meaning that the action on $\mathbb{P}^{n\shortminus 1}$ is defined as $g \cdot x_{r} := g(\rho)X_\rho$ for any $g \in G_\Sigma$. The ring $R$ together with the graduation induced by the $G_\Sigma$-action is the homogeneous coordinate ring of $X_\Sigma$. The following is due to Cox [Cox95].

Theorem 2.13. [CLS11, Theorem 5.1.11] Consider the fan $\Sigma_{\text{Cox}}$ of $Q^r = \bigoplus_{\rho \in \Sigma(1)} Q \cdot e_\rho$ generated by the cones

$$\sigma_{\text{Cox}} = \sum_{\rho \in \Sigma(1)} Q_{\geq 0} \cdot e_\rho.$$ 

Then $X_{\Sigma_{\text{Cox}}} = A^*_\Sigma \setminus Z_\Sigma$ and the morphism $\varepsilon : A^*_\Sigma \setminus Z_\Sigma \to X_\Sigma$, inducing by the linear map $e_\rho \mapsto v_\rho$, is constant on the $G_\Sigma$-orbits for the diagonal $G_\Sigma$-action on the total coordinate space $A^*_\Sigma = \text{Spec } R$. More precisely, it factors through an isomorphism

$$(A^*_\Sigma \setminus Z_\Sigma)/G_\Sigma = \{ \text{closed } G_\Sigma\text{-orbits of } A^*_\Sigma \setminus Z_\Sigma \} \simeq X_\Sigma.$$ 

Moreover, the fan $\Sigma$ is simplicial if and only if all the $G_\Sigma$-orbits of $A^*_\Sigma \setminus Z_\Sigma$ are closed.

A consequence of Theorem 2.13 is the toric Hilbert’s Nullstellensatz.

Theorem 2.14. [CLS11, Propositions 5.2.4, 5.2.6] For a homogeneous ideal $I \subset R$, the subset

$$V_\lambda(I) := \{ y \in X_\Sigma \mid \text{there is } x \in \varepsilon^{-1}(y) \text{ with } f(x) = 0 \text{ for all } f \in I \}$$ 

is a Zariski closed subset of $X_\Sigma$ and all the Zariski closed subsets of $X_\Sigma$ arises in this way. Moreover, if $\Sigma$ is simplicial, then the map $I \mapsto V_\lambda(I)$ induces a one-to-one correspondence between the radical homogeneous ideals $I \subset B(\Sigma) \subset R$ and the Zariski closed subsets of $X_\Sigma$. 

We say that \( x, y \in \mathbb{A}^r_C \setminus Z_\Sigma \) are equivalent if \( \varepsilon^{-1}(x) = \varepsilon^{-1}(y) \). After numbering the rays of \( \Sigma \), write \( x = (x_1, \ldots, x_r) \) for a point of \( \mathbb{A}^r_C \setminus Z_\Sigma \) and denote by \( [x_1 : \ldots : x_r]_\Sigma \) its equivalence class called the Cox’s coordinates. In this way,
\[
V_h(I) = \{[x_1 : \ldots : x_r]_\Sigma \in \mathcal{X}_\Sigma | f(x_1, \ldots, x_r) = 0 \text{ for any } f \in I\}.
\]

2.5. Complexity-one case. Torus actions of complexity one are described by divisorial fans over curves. In this context, the globalization simplifies since a finite set \( \mathcal{E} = \{\mathcal{D}^i | i \in I\} \) of proper polyhedral divisors \( \mathcal{D}^i \) defined over dense open subsets \( Y_i \) of a smooth projective curve \( Y \) and stable by intersection is a divisorial fan if and only if:

1. For all \( i, j \in I \) and for any \( z \in Y \) the polyhedron \( \mathcal{D}^i_z \cap \mathcal{D}^j_z \) is either empty or a common face of \( \mathcal{D}^i_z \) and \( \mathcal{D}^j_z \), and

2. We have \( \deg \mathcal{D}^i \cap \mathcal{D}^j = \sigma_i \cap \sigma_j \cap \deg \mathcal{D}^i \) for all \( i, j \in I \), where \( \sigma_i \) and \( \sigma_j \) are the strictly convex polyhedral cones associated with \( \mathcal{D}^i \) and \( \mathcal{D}^j \). Here for a \( \sigma \)-polyhedral divisor \( \mathcal{D} \) over a curve \( Y_0 \), we set \( \deg \mathcal{D} := \sum_{z \in Y_0} \mathcal{D}_z \subset N_\mathbb{Q} \) if \( Y_0 \) is complete and \( \deg \mathcal{D} = 0 \) otherwise.

Moreover, using valuative criterion of properness, the \( \mathbb{T} \)-variety \( X(\mathcal{E}) \) is complete if and only if
\[
\bigcup_{i \in I} \bigcup_{z \in Y_i} \{z\} \times \mathcal{D}^i_z = Y \times N_\mathbb{Q}.
\]

Note that the relative spectra \( \text{Spec}_{X_i} \mathcal{A}_{\mathcal{D}^i} \) for \( \mathcal{D}^i \in \mathcal{E} \), where
\[
\mathcal{A}_{\mathcal{D}^i} := \bigoplus_{m \in \sigma \cap N_M} \mathcal{O}_{X_i}(\mathcal{D}^i(m))x^m,
\]
glue together into a \( \mathbb{T} \)-variety \( \tilde{X} \). The natural map \( \pi : \tilde{X} \to X \), where \( X = X(\mathcal{E}) \), is the contraction map. By [Vol10, Section 3, Lemma 1] the normalization of the graph of the rational map \( \iota : X \dashrightarrow Y \), induced by the inclusion \( \mathbb{C}(X)^\mathbb{T} \subset \mathbb{C}(X) \) identifies with the contraction space \( \tilde{X} \) and the natural projection to \( X \) with \( \pi \).

Next definition formulates the passage to contraction spaces via divisorial fans.

**Definition 2.15.** Given a divisorial fan \( \mathcal{E} = \{\mathcal{D}^i | i \in I\} \) over \( (Y, N) \), where \( Y \) is a smooth projective curve, a contraction divisorial fan of \( \mathcal{E} \) is a divisorial fan of the form
\[
\tilde{\mathcal{E}} = \{\mathcal{D}^i_{U_j \cap Y_i} := \iota^{U_j}_{i,j}(\mathcal{D}^i) | i \in I, j \in J\}.
\]

Here \( (U_j)_{j \in J} \) is a finite affine open covering of \( Y \) and \( \iota_{i,j} : Y_i \cap U_j \to Y \) is the inclusion. From the previous discussion, \( X(\tilde{\mathcal{E}}) \) is exactly the contraction space of \( X(\mathcal{E}) \).

We end this section with the description of the integral closure of affine varieties with torus action of complexity one. See [Lan15] for a version over arbitrary fields.

**Theorem 2.16.** [FZ03, Proposition 3.9], [Lan13, Théorème 2.4] Let \( Y \) be a smooth curve and consider the subalgebra \( A := \mathbb{C}[Y][f_1x^{m_1}, \ldots, f_rx^{m_r}] \subset \mathbb{C}(Y)[M] \), where \( m_1, \ldots, m_r \in M \) and \( f_1, \ldots, f_r \in \mathbb{C}(Y)^* \). Set \( \xi = (f_1x^{m_1}, \ldots, f_rx^{m_r}) \). Assume that \( A \) and \( \mathbb{C}(Y)[M] \) have same fraction field. Then \( \xi\mathcal{D} \) is proper and the normalization of \( A \) identifies with \( A[\mathbb{Y}, \xi\mathcal{D}] \).

3. Intersection cohomology and actions of finite groups

In this section, we study the pullback on intersection cohomology complexes of quotient maps of finite group actions.
3.1. Maps preserving intersection cohomology complexes.

**Definition 3.1.** We say that a finite dominant morphism of varieties \( f : X \to Y \) preserves the intersection cohomology complexes if there are proper birational morphisms \( \varphi : Y' \to Y \) and \( \bar{\varphi} : X' \to X \) with \( X' \) smooth and a Cartesian square

\[
\begin{array}{ccc}
X' & \xrightarrow{\bar{\varphi}} & X \\
\downarrow{f} & & \downarrow{f} \\
Y' & \xrightarrow{\varphi} & Y
\end{array}
\]

such that \( \bar{f}^*IC_{Y'} \simeq IC_{X'} \). We say further that \( f \) strictly preserves the intersection cohomology complexes if we can choose \( \bar{\varphi} \) semi-small.

**Lemma 3.2.** If \( F, G \in D^b_{\text{const}}(X) \) and \( F \oplus G \) is semi-simple, then so is \( F \). Moreover, if \( F \oplus G \) is perverse, then so is \( F \).

**Proof.** First claim is [BM99, Lemma 15]. Second claim is consequence of [Wil17, Proposition 1.8]. \( \square \)

**Proposition 3.3.** Let \( f : X \to Y \) be a finite dominant morphism of varieties. If \( f \) preserves the intersection cohomology complexes, then \( f^*IC_Y \) is semi-simple. If further \( f \) strictly preserves the intersection cohomology complexes, then \( f^*IC_Y \simeq IC_X \).

**Proof.** Take the commutative diagram

\[
\begin{array}{ccc}
X' & \xrightarrow{\bar{\varphi}} & X \\
\downarrow{f} & & \downarrow{f} \\
Y' & \xrightarrow{\varphi} & Y
\end{array}
\]

of Definition 3.1. By base change [Dim04, Theorem 2.3.26] we have \( f^* \varphi_*IC_{Y'} \simeq \bar{\varphi} \bar{f}^*IC_{Y'} \simeq \bar{\varphi}_* IC_{X'} \). The decomposition theorem gives

\[
\varphi_*IC_{Y'} \simeq IC_Y \oplus \bigoplus_{a \in I} (t_a)_* IC_{Y_a}(L_a)[r_a],
\]

where \( t_a : Y_a \to Y' \) are inclusions of closed subvarieties and \( L_a \) are local systems defined on the smooth locus. Set

\[
F := f^* IC_Y \text{ and } G := f^* \left( \bigoplus_{a \in I} (t_a)_* IC_{Y_a}(L_a)[r_a] \right).
\]

Then \( F \oplus G \simeq \bar{\varphi}_* IC_{X'} \), is semi-simple. So by Lemma 3.2, \( F \) is semi-simple. Assume further that \( \bar{\varphi} \) is semi-small. Then the examination of the supports gives

\[
F \simeq IC_X \oplus \bigoplus_{\beta \in J} (t_\beta)_* IC_{W_\beta}(M_\beta)^{\oplus \kappa_\beta},
\]

where \( t_\beta : W_\beta \to X \) are strict inclusions of closed subvarieties. Assume, toward a contradiction, that each \( M_\beta \) is nonzero. Take a subvariety \( W_{\beta_0} \) for \( \beta_0 \in J \) of maximal dimension. Definitions of \( IC_X \) and \( IC_Y \) come with algebraic Whitney stratifications

\[
X = \bigcup_{\lambda \in \Lambda} X_\lambda \text{ and } Y = \bigcup_{\alpha \in \Delta} Y_\alpha.
\]

Let \( \lambda_1 \in \Lambda \) (respectively, \( \alpha_1 \in \Delta \)) be the unique index such that the generic point (respectively, the image by \( f \) of the generic point) of \( W_{\beta_0} \) is contained in the stratum \( X_{\lambda_1} \) (respectively, \( Y_{\alpha_1} \)). Since \( f \) is finite, we have \( \dim W_{\beta_0} \leq \dim X_{\lambda_1} \). Note that \( \dim W_{\beta_0} \leq \dim X_{\lambda_1} \). Moreover, the fact that \( W_{\beta_0} \) is of maximal dimension implies that there is \( x \in W_{\beta_0} \) such that \( x \in X_{\lambda_1}, f(x) \in Y_{\alpha_1} \).
and with the condition that \( x \) is not contained in any irreducible subvarieties \( W_\beta \) with \( \beta \neq \beta_0 \). The open stratum and support conditions imply

\[
\mathcal{H}^{-\dim W_{\beta_0}}(IC_Y)_{f(x)} = 0 = \mathcal{H}^{-\dim W_{\beta_0}}(IC_X)_x.
\]

Therefore

\[
\mathcal{M}^{\oplus_{\beta_0}}_{\beta_0, x} = \mathcal{H}^{-\dim W_{\beta_0}}(IC_X)_x \oplus \bigoplus_{\beta \in J} \mathcal{H}^{-\dim W_{\beta_0}}((\iota_\beta)_*IC_{W_\beta}(\mathcal{M}^{\oplus_{\beta_0}}_{\beta, x})_x = \mathcal{H}^{-\dim W_{\beta_0}}(\mathcal{F})_x
\]

\[
= \mathcal{H}^{-\dim W_{\beta_0}}(f^*IC_Y)_x = \mathcal{H}^{-\dim W_{\beta_0}}(IC_Y)_{f(x)} = 0.
\]

So \( \mathcal{M}_{\beta_0} = 0 \), yielding a contradiction. We conclude that \( f^*IC_Y \simeq \mathcal{F} \simeq IC_X \), as required. \( \square \)

**Lemma 3.4.** Let \( X \) be a smooth quasi-projective variety with action of a finite group \( G \). Set \( Y = X/G \). Then \( IC_Y \simeq \mathbb{Q}V[\dim Y] \).

**Proof.** See [GS93, Section 5, Proposition 3]. \( \square \)

**Lemma 3.5.** Let \( X \) be a quasi-projective variety with action of a finite group \( G \). Then the quotient map \( f : X \to Y \) preserves the intersection cohomology complexes.

**Proof.** By [AW97] there is a \( G \)-equivariant resolution of singularities \( \bar{\varphi} : X' \to X \). Denote by \( \bar{f} : X' \to Y' \) the quotient map. Then we have a Cartesian square

\[
\begin{array}{ccc}
X' & \xrightarrow{\bar{\varphi}} & X \\
\downarrow{\bar{f}} & & \downarrow{f} \\
Y' & \xrightarrow{\varphi} & Y \\
\end{array}
\]

with \( \bar{\varphi} \) proper since \( \varphi \) is. Moreover, if \( U \subset X \) is a dense Zariski open subset such that \( \bar{\varphi}_{|\varphi^{-1}(U)} : \varphi^{-1}(U) \to U \) is an isomorphism, then \( \varphi_{|\varphi^{-1}(V)} : \varphi^{-1}(V) \to V \) is an isomorphism, where \( W = \bigcup_{g \in G} g \cdot U \) and \( V = W/G \). Thus \( \varphi \) is birational. Finally, by Lemma 3.4,

\[
\bar{f}^*IC_{Y'} \simeq \bar{f}^*\mathbb{Q}V'[\dim Y'] \simeq \mathbb{Q}V[\dim X'] \simeq IC_{X'},
\]

proving that \( f \) preserves the intersection cohomology complexes. \( \square \)

**Proposition 3.6.** Let \( X \) be a quasi-projective variety with action of a finite group \( G \). Denote by \( f : X \to Y \) the quotient map. Then \( f^*IC_Y \) is semi-simple. Assume further that \( X \) has an equivariant semi-small resolution of singularities. Then \( f^*IC_Y \simeq IC_X \).

**Proof.** This follows from Proposition 3.3 and Lemma 3.5. \( \square \)

**Example 3.7.** For \( f : X \to Y \) dominant finite, \( f^*IC_Y \) is generally not isomorphic to \( IC_X \). Indeed, let \( X \) be an affine cone over an elliptic curve and denote by \( x \) its vertex. Let \( f : X \to Y = \mathbb{A}^2_\mathbb{C} \) be the finite morphism given by Noether normalization. Then from [Fie91, Lemma 2.1], [DL91, Lemma 6.5], the stalk at \( x \) is

\[
\mathcal{H}^{-2}(IC_X)_x = \mathbb{Q}, \quad \mathcal{H}^{-1}(IC_X)_x = \mathbb{Q}^2, \quad \text{and} \quad \mathcal{H}^j(IC_X)_x = 0 \text{ for } j \neq -2, -1,
\]

while one has

\[
\mathcal{H}^{-2}(\mathbb{Q}X[2])_x = \mathbb{Q} \quad \text{and} \quad \mathcal{H}^j(\mathbb{Q}X[2])_x = 0 \text{ for } j \neq -2.
\]

\(^1\)Example based on an idea by Williamson communicated on MathOverflow.
3.2. Finite group actions on toric varieties. Our aim is the following.

**Proposition 3.8.** Let $X$ be a toric variety for the torus $\mathbb{T}$, let $G \subset \mathbb{T}$ be a finite subgroup and denote by $f : X \to Y = X/G$ the quotient map for the natural $G$-action. Then $f^*IC_X \cong IC_Y$.

Proof of Proposition 3.8 uses the toric decomposition theorem [CMM18].

**Theorem 3.9.** [CMM18, Theorem D] Let $\Sigma$ be a fan, let $\Sigma'$ be a fan subdivision of $\Sigma$ and let $q : X_{\Sigma'} \to X_\Sigma$ be the corresponding toric morphism. Then we have

$$q_*IC_{X_{\Sigma'}} \cong \bigoplus_{b \in \mathbb{Z}} \bigoplus_{\tau \in \Sigma} (\tau)_*IC_{V(\tau)}^{(b,\tau)}[-b],$$

where $\tau : V(\tau) \to X_\Sigma$ is the inclusion of the closure of the orbit $O(\tau) \subset X_\Sigma$.

**Lemma 3.10.** Let $\Sigma$ be a fan of $N_{\mathbb{Q}}$ and let $\Sigma'$ be a simplicial fan subdivising $\Sigma$. Consider a lattice $N_0 \subset N_{\mathbb{Q}}$ containing $N$ with $[N_0 : N]$ finite. Denote by $X_{\Sigma,N_0}, X_{\Sigma',N_0}$ the associated toric varieties with respect to the lattice $N_0$. If $q : X_{\Sigma'} \to X_\Sigma$ and $q_0 : X_{\Sigma',N_0} \to X_{\Sigma,N_0}$ are the toric modifications and

$$q_*IC_{X_{\Sigma'}} \cong \bigoplus_{b \in \mathbb{Z}} \bigoplus_{\tau \in \Sigma} (\tau)_*IC_{V(\tau)}^{(b,\tau)}[-b]$$

and

$$(q_0)_*IC_{X_{\Sigma',N_0}} \cong \bigoplus_{b \in \mathbb{Z}} \bigoplus_{\tau \in \Sigma} ((\tau)_*IC_{V(\tau)}^{(b,\tau)}[-b],$$

are the decompositions of Theorem 3.9. Then $s_{b,\tau} = s_{b,\tau,N_0}$ for all $b, \tau$. In other words, the multiplicities of the toric decomposition theorem do depend on the ambient lattice $N$.

Proof. Consequence of [CMM18, Corollary 7.5] and the fact that the polynomials $R_{\tau}^\sigma(T) = \sum_{j \in \mathbb{Z}} \dim \mathcal{H}^j(IC_{V(\tau)}) x_j T^j$ for $x_\sigma \in O(\sigma)$ do not depend on the lattice $N$ (see [Fie91, Section 1, Theorem 1.2]).

Proof of Proposition 3.8. We proceed by induction on the dimension. The result is true for dimensions 0, 1. Assume that this holds for $< \dim X$ and write $X = X_\Sigma$ for a fan $\Sigma$ of $N_{\mathbb{Q}}$. Note that $Y = X_{\Sigma,N_0}$, where $N_0 \subset N_{\mathbb{Q}}$ is a lattice containing $N$ with $[N_0 : N]$ finite. Consider $q_0 : X_{\Sigma',N_0} \to X_{\Sigma,N_0}$ the morphism induced by a fan subdivision $\Sigma'$ with $X_{\Sigma'}$ smooth. We have a Cartesian commutative diagram

$$
\begin{array}{ccc}
X_{\Sigma'} & \xrightarrow{q} & X_\Sigma \\
\downarrow f_0 & & \downarrow f \\
X_{\Sigma',N_0} & \xrightarrow{q_0} & X_{\Sigma,N_0}.
\end{array}
$$

By base change [Dim04, Theorem 2.3.26],

$$f^*(q_0)_*IC_{X_{\Sigma',N_0}} \cong q_*f^*IC_{X_{\Sigma',N_0}}.$$ \)

Write $\gamma_\tau : V(\tau)/G \to X_{\Sigma,N_0}$ for the inclusion and let $f_\tau : V(\tau) \to V(\tau)/G$ be the quotient. On one hand Theorem 3.9 yields

$$f^*(q_0)_*IC_{X_{\Sigma',N_0}} \cong f^* \left( \bigoplus_{b \in \mathbb{Z}} \bigoplus_{\tau \in \Sigma} ((\tau)_*IC_{V(\tau)}^{(b,\tau)}[-b],$$

Note that the commutative diagram

$$
\begin{array}{ccc}
V(\tau) & \xrightarrow{\iota_\tau} & X_\Sigma \\
\downarrow f & & \downarrow f \\
V(\tau)/G & \xrightarrow{\gamma_\tau} & X_{\Sigma,N_0}
\end{array}
$$

implies that $f^* IC_{X_{\Sigma',N_0}} \cong f^* IC_{X_{\Sigma'}}$.
is Cartesian. By base change and induction we have
\[ f^*(\gamma_{\tau})_*IC_{V(\tau)/G} \simeq (\tau)_* f^* IC_{V(\tau)/G} \simeq (\tau)_* IC_{V(\tau)} \]
for any nonzero \( \tau \) and hence
\[ f^*(q_{0})_*IC_{X_{\Sigma},N_{0}} \simeq f^* IC_{X_{\Sigma},N_{0}} \oplus \bigoplus_{b \in \mathbb{Z}} \bigoplus_{\tau \in \Sigma} (\tau)_* IC_{V(\tau)}^{\oplus b, \tau} [-b]. \]
On the other hand, by Lemma 3.4, \( f_{0}^* IC_{X_{\Sigma},N_{0}} \simeq IC_{X_{\Sigma}} \) and thus by Theorem 3.9 and Lemma 3.10,
\[ q_* f_{0}^* IC_{X_{\Sigma},N_{0}} \simeq q_* IC_{X_{\Sigma}} \simeq \bigoplus \bigoplus_{b \in \mathbb{Z}} (\tau)_* IC_{V(\tau)}^{\oplus b, \tau} [-b]. \]
This implies that
\[ f^* IC_{X_{\Sigma},N_{0}} \simeq \bigoplus \bigoplus (\tau)_* IC_{V(\tau)}^{\oplus b, \tau} [-b] \]
and so
\[ \bigoplus \bigoplus (\tau)_* IC_{V(\tau)}^{\oplus b, \tau} [-b] \simeq q_* IC_{X_{\Sigma}} \]
\[ \simeq \bigoplus \bigoplus (\tau)_* IC_{V(\tau)}^{\oplus b, \tau} [-b] \oplus \bigoplus \bigoplus (\tau)_* IC_{V(\tau)}^{\oplus b, \tau} [-b]. \]
The uniqueness of the decomposition gives
\[ d_{b, \tau} = \begin{cases} 0 & \text{if } b \in \mathbb{Z} \setminus \{0\} \text{ or } \tau \in \Sigma \setminus \{0\}, \\ 1 & \text{if } b = 0 \text{ and } \tau = 0. \end{cases} \]
So \( f^* IC_{X_{\Sigma},N_{0}} \simeq IC_{X_{\Sigma}} \), as required. \( \square \)

3.3. Seifert torus bundles. In this section, we prove that the local systems in the decomposition theorem for contraction maps of torus actions of complexity one are trivial.

Definition 3.11. [AL21, Section 2.3] A Seifert torus bundle with fiber \( \mathbb{T} := (\mathbb{C}^*)^n \) is a homogeneous fiber space \( E := \mathbb{T} \times^G X \), where \( G \subset \mathbb{T} \) is a finite group and \( X \) is a quasi-projective \( G \)-variety, together with the projection \( \varepsilon : E \to B = X/G \).

For a Seifert torus bundle
\[ \varepsilon : E := \mathbb{T} \times^G X \to B := X/G, \]
the \( G \)-action on \( \mathbb{T} \times X \) by translation on the first factor \( \mathbb{T} \) induces a \( G \)-action on \( E \). We have \( E/G \simeq \mathbb{T} \times X/G \) and a commutative diagram
\[
\begin{array}{ccc}
E & \xrightarrow{f} & E/G \\
\varepsilon \downarrow & & \downarrow p \\
B, & & 
\end{array}
\]
where \( f : E \to E/G \) is the quotient and \( p : E/G \to B \) is, after identification, the projection \( \mathbb{T} \times B \to B \) on the second factor [AL21, Lemma 2.7].

Let \( X \) be a normal variety with an effective complexity-one \( \mathbb{T} \)-action. Let \( \pi : \tilde{X} \to X \) be the contraction map. Assume that \( \pi \) is not an isomorphism. Let \( q : \tilde{X} \to C \) be the global quotient onto a smooth projective curve \( C \). Let \( E \subset X \) be the image of the exceptional locus of \( \pi \). We fix an orbit \( O \subset E \). For any \( z \in C \) let \( O_z \) be the unique orbit contained in the subset \( \pi^{-1}(O) \cap q^{-1}(\{z\}) \). The following is a relative version of Luna’s slice theorem.
Lemma 3.12. [AL21, Lemma 4.4] Set

$$\widetilde{X}_O := \{ x \in \tilde{X} \mid O_{q(x)} \subset \mathbb{T} \cdot x \}.$$  

Then $\widetilde{X}_O$ is a Zariski open subset containing $\pi^{-1}(O)$, which is affine over $C$. The image $X_O = \pi(\widetilde{X}_O)$ is a Zariski affine open subset of $X$ in which all the orbit closures contain $O$ as closed orbit. Moreover, the map $\pi$ induces a Cartesian square

$$\begin{array}{c}
\widetilde{X}_O \cong T_O \times^G \tilde{X}_1 \\
\quad \downarrow \pi \\
X_O \cong T_O \times^G X_1
\end{array}$$

where the horizontal arrows are Seifert torus bundles and the vertical ones are proper morphisms. The torus $T_O$ is the quotient of $\mathbb{T}$ by the neutral connected component of the stabilizer $T_x$ at a point $x \in O$, and the group $G$ is the group of connected components of $T_x$. Finally, $X_1/G$ has a unique fixed point under the action of $T/T_x$, whose preimage under the map $X_O \to X_1/G$ is $O$.

Next we discuss on the toroidal structure of contraction spaces.

Remark 3.13. Any $\mathbb{T}$-variety $X(\mathcal{D})$ associated with a $\sigma$-polyhedral divisor $\mathcal{D}$ over a smooth affine curve $Y$ is locally toric for the étale topology. Indeed, take $y \in Y$ and a uniformizer $\varpi_y$ of the local ring $O_{Y,y}$. Then $\psi : V \to \mathbb{A}^1_C, z \mapsto \varpi_y(z)$ is étale on a Zariski open subset $V$ of $Y$ containing $y$. Considering

$$\mathcal{D}_0 := \sum_{z \in V} [\psi(z)] \text{ and } \mathcal{D}_V := \sum_{z \in V} \mathcal{D}_z \cdot z$$

and assuming $\psi^{-1}(0) = \{ y \}$ and $\{ z \in V \mid \mathcal{D}_z \neq \sigma \} = \{ y \}$, the map $\psi$ induces an isomorphism $X(\mathcal{D}_V) \cong V \times_{\mathbb{A}^1_C} X(\mathcal{D}_0)$. Hence the composition of the projection $V \times_{\mathbb{A}^1_C} X(\mathcal{D}_0) \to X(\mathcal{D}_0)$ and the inclusion $X(\mathcal{D}_0) \hookrightarrow X_{\sigma_y}$, where $X_{\sigma_y}$ is the affine toric variety associated with $\sigma_y = \text{Cone}(\sigma \times \{ 0 \}) \cup (\mathcal{D}_y \times \{ 1 \})$, gives an étale map $\phi : X(\mathcal{D}_V) \to X_{\sigma_y}$ and the desired toroidal structure.

Lemma 3.14. With the notation of Lemma 3.12, we have $\varepsilon^* IC_{\tilde{X}_{1/G}}[r] \cong IC_{\widetilde{X}_O}$, where $r = \dim O$.

Proof. By Remark 3.13, the variety $\widetilde{X}_O$ has an étale open covering $(\phi_i : U_i \to X_{\sigma_i})_{1 \leq i \leq s}$. Consider the $G$-action on $\widetilde{X}_O \cong T_O \times^G \tilde{X}_1$ given by translation on the factor $T_O$. Then this action is obtained via an inclusion $G \subset \mathbb{T}$ [AL21, Lemma 4.2]. From the inclusions $G \subset \{ 1 \} \times \mathbb{T} \subset G_m \times \mathbb{T}$, the group $G$ also acts on $X_{\sigma_i}$ and $\phi_i$ is $G$-equivariant. To sum-up we have a Cartesian square

$$\begin{array}{c}
U_i \\
\downarrow \phi_i \\
X_{\sigma_i}
\end{array}$$

for $i = 1, 2, \ldots, s$, where the vertical arrows are quotient maps. It follows from the description in Remark 3.13 of $\phi_i$ that $\varphi_i$ is étale. Using Lemma 2.5 and Proposition 3.8 we deduce that

$$\begin{align*}
g_i^* IC_{U_i/G} & \cong g_i^* \varphi_i^* IC_{X_{\sigma_i}/G} \\
& \cong (\varphi_i \circ g_i)^* IC_{X_{\sigma_i}/G} \\
& \cong (f_i \circ \phi_i)^* IC_{X_{\sigma_i}/G}
\end{align*}$$

$$\cong \phi_i^* f_i^* IC_{X_{\sigma_i}/G} \cong \phi_i^* IC_{X_{\sigma_i}} \cong IC_{U_i}.$$  

So $(f^* IC_{\widetilde{X}_{O/G}})_{U_i} \cong (IC_{\widetilde{X}_O})_{U_i}$ for any $i \in \{ 1, \ldots, s \}$, where $f : \widetilde{X}_O \to \widetilde{X}_O/G$ is the quotient map. As $f^* IC_{\widetilde{X}_{O/G}}$ is semi-simple (see Proposition 3.6), we have $f^* IC_{\widetilde{X}_O/G} \cong IC_{\widetilde{X}_O}$. Finally
consider the commutative diagram

\[
\begin{array}{ccc}
\bar{X}_O & \xrightarrow{f} & \bar{X}_O/G \\
\varepsilon \downarrow & & \downarrow p \\
\bar{X}_1/G & & 
\end{array}
\]

given by the Seifert torus bundle $\varepsilon$, where $p : \bar{X}_O/G \simeq \mathbb{T}_O \times \bar{X}_1/G \to \bar{X}_1/G$ is the projection on the second factor. Since $p$ is smooth, by Lemma 2.5 we have

\[
\varepsilon^*IC_{\bar{X}_1/G}[r] \simeq (p \circ f)^*IC_{\bar{X}_1/G}[r] \simeq f^*p^*IC_{\bar{X}_1/G}[r] \simeq f^*IC_{\bar{X}_O/G} \simeq IC_{\bar{X}_O},
\]

proving the lemma.

The following result improves [AL21, Theorem 1.1].

**Proposition 3.15.** Let $X$ be a normal $\mathbb{T}$-variety of complexity one and let $\pi : \bar{X} \to X$ be the contraction map. Then we have

\[
\pi_*IC_{\bar{X}} \simeq IC_X \oplus \bigoplus_{O \in \mathcal{O}(E)} \bigoplus_{b \in \mathbb{Z}} IC_{O}^{\boxplus_{b,O}}[-b],
\]

where $\mathcal{O}(E)$ is the set of orbits of $E$ and the $s_{b,O}$ are nonnegative integers.

**Proof.** Following the proof of [AL21, Theorem 1.1 (ii)], we need to have $\mathcal{H}^j(\pi_*IC_{\bar{X}})|_O$ constant for any orbit $O \subset E$ and any $j \in \mathbb{Z}$. Using Lemma 3.14 and base change [Dim04, Theorem 2.3.26] from the diagram of Lemma 3.12,

\[
\mathcal{H}^j(\pi_*IC_{\bar{X}})|_O \simeq \mathcal{H}^j(\pi_*\varepsilon^*IC_{\bar{X}_1/G}[r])|_O \simeq \mathcal{H}^j(\varepsilon_1^*(\pi_1)_*IC_{\bar{X}_1/G}[r])|_O \simeq \mathcal{H}^j((\pi_1)_*IC_{\bar{X}_1/G})_{x_0} \otimes \mathbb{Q}_O,
\]

where $r = \dim O$ and $x_0$ is the unique fixed point of $X_1/G$. This shows the proposition.

4. LINEAR TORUS ACTIONS OF COMPLEXITY ONE

This section develops weight package theory in order to describe linear torus actions of complexity one on possibly non-normal varieties.

4.1. Linear torus actions: the affine case. Consider a torus $\mathbb{G} = (\mathbb{C}^*)^\ell$ with character and one-parameter subgroup lattices $\mathcal{M}$ and $\mathcal{N}$, and endow $\mathbb{P}_\mathbb{C}^\ell$ with the toric $\mathbb{G}$-action

\[
(\lambda_1, \ldots, \lambda_\ell) \cdot [x_0 : \ldots : x_\ell] = [x_0 : \lambda_1 x_1 : \ldots : \lambda_\ell x_\ell], \text{ where } (\lambda_1, \ldots, \lambda_\ell) \in \mathbb{G}.
\]

Let $\mathbb{T} \subset \mathbb{G}$ be a subtorus.

**Definition 4.1.** A subvariety of $\mathbb{P}_\mathbb{C}^\ell$ with linear $\mathbb{T}$-action is an irreducible Zariski closed subset of $\mathbb{P}_\mathbb{C}^\ell$ intersecting the open $\mathbb{G}$-orbit and stable by $\mathbb{T}$-action. A subvariety of $\mathbb{A}_\mathbb{C}^\ell$ with linear $\mathbb{T}$-action is an irreducible Zariski closed subset $X \subset \mathbb{A}_\mathbb{C}^\ell$ such that there is subvariety $Z$ of $\mathbb{P}_\mathbb{C}^\ell$ with linear $\mathbb{T}$-torus action with $X = Z \setminus H$, where $H$ is the coordinate hyperplane $\mathcal{V}(x_0) \subset \mathbb{P}_\mathbb{C}^\ell$.

The following characterizes linear torus actions when the singularities are mild.

**Proposition 4.2.** [KCLV89, 2.4] Any projective normal variety with faithful torus action can be equivariantly embedded as a subvariety with linear torus action.

Let $X$ be a subvariety of $\mathbb{A}_\mathbb{C}^\ell$ with linear $\mathbb{T}$-action and set $n = \dim \mathbb{T}$. Let $E = (e_1, \ldots, e_\ell)$ be the canonical basis of $\mathcal{N} = \mathbb{Z}^\ell$ and fix a basis $B$ of $N$. 

The weight matrix of $X$ is the matrix $\text{Mat}_{E,D}(F) \in \text{Mat}_{\ell \times n}(\mathbb{C})$ for the linear map $F : N \to \overline{N}$ corresponding to the inclusion $T \hookrightarrow G$. The matrix factorization is the short exact sequence

$$0 \to N \xrightarrow{F} \overline{N} \xrightarrow{P} \text{Coker}(F) = \overline{N}/F(N) \to 0$$

together with a splitting $S : \overline{N} \to N$, that is, $S$ is linear and $S \circ F = \text{id}_N$. Note that $F(N)$ must be a saturated, i.e.

$$\overline{N} \cap F(N)_Q \subset \overline{N}_Q$$

is equal to $F(N)$. Moreover, the map

$$\phi : \overline{N} \to N \oplus \text{Coker}(F), \ v \mapsto (S(v), P(v))$$

is a $\mathbb{Z}$-module isomorphism. The quotient fan $\Sigma$ of $X$ is the coarsest fan of $\text{Coker}(F)_Q$ containing all the strictly convex polyhedral cones $P(\delta_0)$, where $\delta_0$ runs over the face of the first quadrant

$$\delta := \left\{ \sum_{i=1}^\ell \alpha_i e_i \mid \alpha_i \in \mathbb{Q}_{\geq 0}, \ 1 \leq i \leq \ell \right\} \subset \overline{N}_Q.$$

The weight package of $X$ is the data $\theta = (\overline{N}, N, F, S, \Sigma)$.

The following result is known [AIPS12, Section 4]. For the convenience of the reader, we give a proof.

**Lemma 4.4.** Let $X \subset \mathbb{A}^\ell_\mathbb{C}$ be a subvariety with linear $T$-action and let $\theta = (\overline{N}, N, F, S, \Sigma)$ be its weight package. Consider the strictly convex polyhedral cone $\sigma_\theta := S(\delta \cap F(N)_Q)$ and the $\sigma_\theta$-polyhedral divisor

$$\mathcal{D}_\theta := \sum_{Z \subset X_\Sigma} \mathcal{D}_{\theta,Z} \cdot [Z]$$

defined as follows. If $Z_\rho = O(\rho)$ is the divisor corresponding to $\rho \in \Sigma(1)$, then set

$$\mathcal{D}_{\theta,Z_\rho} = S(\delta \cap P^{-1}(v_\rho)).$$

Otherwise, set $\mathcal{D}_{\theta,Z} = \sigma_\theta$. Then $\mathcal{D}_\theta$ is proper and $X(X_\Sigma, \mathcal{D}_\theta)$ is $T$-isomorphic to $\mathbb{A}^\ell_\mathbb{C}$.

**Proof.** Identify $\overline{N}$ with $N \oplus \text{Coker}(F)$. Denote by $x^w$ the monomial associated with $w \in \text{Coker}(F)^\vee := \text{Hom}_\mathbb{Z}(\text{Coker}(F), \mathbb{Z})$ and let $m \in M$. Observe that $x^w x^m \in A[X_\Sigma, \mathcal{D}_\theta]$ if and only if

$$\langle w, v_\rho \rangle_{\text{Coker}(F)} + \langle m, S(v) \rangle_N \geq 0$$

and

$$\langle m, v_0 \rangle_N \geq 0$$

for all $\rho \in \Sigma(1)$, $v \in \delta$ such that $P(v) = v_\rho$ and $v_0 \in \delta \cap N$. This is equivalent to $\langle (w, m), e \rangle_{\overline{N}} \geq 0$ for any $e \in \delta$, showing that $X(X_\Sigma, \mathcal{D}_\theta)$ is $T$-isomorphic to $\mathbb{A}^\ell_\mathbb{C}$.

Let us prove that $\mathcal{D}_\theta$ is proper. Denote by $(m_i, w_i)$ ($1 \leq i \leq \ell$) the lattice generators of the rays of the dual cone $\delta^\vee \subset \overline{M}_\mathbb{Q}$. In particular, the elements $x^{w_i} x^{m_i}$ generate the algebra $A[X_\Sigma, \mathcal{D}_\theta]$ and

$$\mathcal{D}_{\theta,Z_\rho} = \{ v \in N_Q \mid \langle m_i, v \rangle_N + \langle w_i, v_\rho \rangle_{\text{Coker}(F)} \geq 0 \text{ for } i = 1, \ldots, \ell \}.$$
Note that for any \( s = (s_1, \ldots, s_l) \in \mathcal{H}_L \) there is a unique \( \lambda(s) \in \mathbb{Z}_{>0} \) such that \( \sum_{i=1}^l s_i m_i = \lambda(s) \cdot m \). By [Lan15, Lemma 2.11],

\[
\min_{v \in \mathcal{D}_\theta} \langle m, v \rangle = - \min_{s = (s_1, \ldots, s_l) \in \mathcal{H}_L} \frac{\sum_{i=1}^l s_i \langle w_i, v_p \rangle}{\lambda(s)}, \text{ that is } \mathcal{D}_\theta(m) = - \min_{s = (s_1, \ldots, s_l) \in \mathcal{H}_L} \frac{\text{div } f_s}{\lambda(s)},
\]

where \( f_s := \prod_{i=1}^l (x^{w_i})^{s_i} \). Let \( \Sigma_0 \) be the subfan of \( \Sigma \) generated by the rays of \( \Sigma \). Note that the complement of \( X_{\Sigma_0} \subset X_\Sigma \) is of codimension 2. By the previous equality, \( (\mathcal{D}_\theta(m))_{X_{\Sigma_0}} \) is semi-ample. Let \( A = \bigoplus_{r \in \mathbb{Z}_{>0}} A_r \) be the graded algebra, where

\[
A_r = H^0(X_\Sigma, \mathcal{O}_{X_\Sigma}(\mathcal{D}_\theta(rm))),
\]

and let \( V := \text{Proj}(A) \). Since \( X_\Sigma \) is the Chow quotient [KSZ91, Section 4], there is a surjective projective morphism \( p : X_\Sigma \to V \). Let \( d \in \mathbb{Z}_{>0} \) such that the \( d \)-th Veronese subalgebra

\[
A(d) = \bigoplus_{d \geq 0} A_{dr}
\]

of \( A \) is generated by its degree-one elements. Let \( E \) be the \( \mathbb{Q} \)-divisor on \( X_\Sigma \) such that \( dE \) is the pullback by \( p \) of the \( \mathcal{O}(1) \) of \( A(d) \). In this way,

\[
\Gamma(X_{\Sigma_0}, \mathcal{O}_{X_\Sigma}(rE)) = \Gamma(X_{\Sigma_0}, \mathcal{O}_{X_\Sigma}(\mathcal{D}_\theta(rm))) \text{ for any } r \in \mathbb{Z}_{>0}.
\]

By Lemma 2.2, \( \mathcal{D}_\theta(m) = E \) is semi-ample.

Assume that \( m \) is in the relative interior of \( \sigma^\vee \). Since the Chow quotient is the normalization of the canonical component of the projective limit of GIT quotients of the form \( V \), the map \( p \) is birational. Hence choosing \( f \in A_r \setminus \{0\} \) for some \( r \in \mathbb{Z}_{>0} \) such that \( p^{-1}(D_+(f)) \to D_+(f), x \mapsto p(x) \) is an isomorphism, we have

\[
(X_\Sigma)_{f, \mathcal{D}_\theta(rm)} = p^{-1}(D_+(f))
\]

affine. Thus \( \mathcal{D}_\theta(m) \) is big, proving the lemma. \( \square \)

**Definition 4.5.** The polyhedral divisor \( \mathcal{D}_\theta \) is the *associated polyhedral divisor of \( \theta \).*

From now on assume that the \( T \)-action on \( X \subset \mathbb{A}_C^\ell \) is of complexity one. We define the curve \( \widehat{C}_\theta \) as follows.

1. If \( \mathbb{C}[X]^T \neq \mathbb{C} \), then \( \widehat{C}_\theta \) is the normalization of \( \text{Spec } \mathbb{C}[X]^T \);
2. Otherwise \( \widehat{C}_\theta \) is the smooth projective curve associated with the one-variable function field \( \mathbb{C}(X)^T/\mathbb{C} \).

The next result relates the curve \( \widehat{C}_\theta \) with the closed embedding \( X \hookrightarrow \mathbb{A}_C^\ell \).

**Lemma 4.6.** With the notation of 4.3, the rational quotient \( \psi : \mathbb{A}_C^\ell \dashrightarrow X_\Sigma \) for the \( T \)-action is regular on the open \( G \)-orbit. Furthermore, \( \widehat{C}_\theta \) is the normalization of the Zariski closure \( C_\theta \) of \( \psi(X \cap G) \) in \( X_\Sigma \).

**Proof.** First claim is clear since \( \psi|_G : G \to G/T \) is the quotient. The restriction \( \psi|_{X \cap G} : X \cap G \to \psi(X \cap G) \) is thus a geometric quotient, and the rational map \( \psi|_X : X \dashrightarrow C_\theta \) induces an isomorphism \( \mathbb{C}(X)^T \simeq \mathbb{C}(C_\theta) \). Let

\[
q : \mathbb{A}_C^\ell \to Y_0 := \mathbb{A}_C^\ell // T
\]

be the morphism induced by the inclusion \( \mathbb{C}[\mathbb{A}_C^\ell]^T \subset \mathbb{C}[\mathbb{A}_C^\ell] \). Since \( X_\Sigma \) is the Chow quotient [KSZ91], there is a projective surjective morphism \( r : X_\Sigma \to Y_0 \) such that \( r \circ \psi = q \). This
induces a commutative diagram

$$X \xrightarrow{q_1} C_\theta^1 \xrightarrow{r_\theta} X / \mathbb{T}.$$ 

Consequently, $r_\theta : C_\theta \rightarrow X / \mathbb{T}$ is dominant projective. If $A_0 := \mathbb{C}[X]^\mathbb{T} \neq \mathbb{C}$, then $r_\theta$ is finite birational. So $\hat{C}_\theta$ is the normalization of $C_\theta$.

**Definition 4.7.** The pullback polyhedral divisor of $(X, \theta)$ is $\hat{D}_\theta := \kappa^*(\hat{D}_\theta)$, where $\kappa : \hat{C}_\theta \rightarrow X_\Sigma$ is the composition of the inclusion $C_\theta \subset X_\Sigma$ (see Lemma 4.6) and the normalization.

**Lemma 4.8.** Let $\mathcal{D}$ be a proper $\sigma$-polyhedral divisor over a semi-projective normal variety $Y$. Let $\varphi : S \rightarrow Y$ be a morphism, where $S$ is a smooth curve and $\varphi$ is the composition of a finite birational morphism $S \rightarrow S'$ and a closed immersion $S' \rightarrow Y$. Assume that the image of $\varphi$ is not contained in the union of the prime divisors of $\{Z \subset Y | D_Z \neq \sigma\}$. Let $(g_1 \chi^{m_1}, \ldots, g_r \chi^{m_r})$ be a system of generators of $A[Y, \mathcal{D}]$ with $g_i \in \mathcal{C}(Y)^*$ and such that $f_i = g_i \circ \varphi \in \mathcal{C}(S)^*$ for $1 \leq i \leq r$. Then the pullback $\varphi^*(\mathcal{D})$ is given by

$$\varphi^*(\mathcal{D})_z = \{v \in \mathbb{N}_0 \mid \langle m_i, v \rangle \geq -\text{ord}_z(f_i) \text{ for } i = 1, \ldots, r\} \text{ for any } z \in S.$$ 

**Proof.** Let $B$ be the integral closure of $\mathbb{C}[S][f_1 \chi^{m_1}, \ldots, f_r \chi^{m_r}]$ in $\mathbb{C}(S)[M]$. By Theorem 2.16, $B = A[S, \xi \mathcal{D}]$, where $\xi = (f_1 \chi^{m_1}, \ldots, f_r \chi^{m_r})$. Fix a primitive vector $m \in M$ in the dual of $\xi \sigma = \sigma$ and set $L = \mathbb{Q}_{>0} m$. Denote by $\mathcal{H}_L$ the Hilbert basis in $\mathbb{Z}^r$ of $\gamma^{-1}(L) \cap \mathbb{Q}_{>0}^r$, where $\gamma : \mathbb{Q}^r \rightarrow \mathbb{Q}_+$ is the linear map sending the canonical basis to $(m_1, \ldots, m_r)$. Consider $\mathcal{H}_L^*$ and $\lambda(s) \in \mathbb{Z}_{>0}$ for $s \in \mathcal{H}_L^*$ as in the proof of Lemma 4.4. Using [Lan15, Lemma 2.11] we have

$$\xi \mathcal{D}(m) = -\min_{s = (s_1, \ldots, s_r) \in \mathcal{H}_L^*} \frac{\text{div}(f_s)}{\lambda(s)}, \text{ where } f_s := \prod_{i=1}^r f_i^{s_i}.$$ 

As $A[Y, \mathcal{D}] = \mathbb{C}[Y][g_1 \chi^{m_1}, \ldots, g_r \chi^{m_r}]$, Lemma 2.10 implies $\eta \mathcal{D} = \mathcal{D}$, where $\eta = (g_1 \chi^{m_1}, \ldots, g_r \chi^{m_r})$, and

$$\mathcal{D}(m) = \eta \mathcal{D}(m) = -\min_{s = (s_1, \ldots, s_r) \in \mathcal{H}_L^*} \frac{\text{div}(g_s)}{\lambda(s)} \text{ for } g_s := \prod_{i=1}^r g_i^{s_i}.$$ 

For any prime divisor $Z \subset Y$, let $U_Z \subset Y$ be an open subset such that $\eta \mathcal{D}(m)|_{U_Z} = a_Z \cdot [Z \cap U_Z]$ for some $a_Z \in \mathbb{Q}$, $U_Z \cap Z \neq \emptyset$. Moreover, we ask these $U_Z$ cover $Y$. Note that for any $Z$, there is $s(Z) \in \mathcal{H}_L^*$ such that

$$\mathcal{D}(m)|_{U_Z} = -\left(\frac{\text{div}(g_{s(Z)})}{\lambda(s(Z))}\right)|_{U_Z}.$$ 

So

$$\varphi^*(\mathcal{D}(m)|_{U_Z}) = -\left(\frac{\text{div}(\varphi^*g_{s(Z)})}{\lambda(s(Z))}\right)|_{\varphi^{-1}(U_Z)} = -\left(\frac{\text{div}(f_{s(Z)})}{\lambda(s(Z))}\right)|_{\varphi^{-1}(U_Z)} \leq \xi \mathcal{D}(m)|_{\varphi^{-1}(U_Z)}$$

for any $Z$, that is $\varphi^*(\mathcal{D})(w) \leq \xi \mathcal{D}(w)$ for any $w \in \sigma^\ast$. Finally, since $\text{div}(f_i) + \varphi^*(\mathcal{D})(m_i) \geq 0$ for $i = 1, \ldots, r$ we have $\varphi^*(\mathcal{D})_z \leq \xi \mathcal{D}_z$ for any $z \in S$, and therefore $\varphi^*(\mathcal{D}) = \xi \mathcal{D}$ as required.

The following describes normalization of affine subvarieties with linear torus action of complexity one (see [AIPS12, Section 4.2] for the normal case).

**Theorem 4.9.** Let $X \subset \mathbb{A}^n_\mathbb{C}$ be a subvariety with linear torus action of complexity one and with weight package $\theta = (N, F, S, \Sigma)$. Then the normalization of $X$ is equivariantly isomorphic to $X(\hat{C}_\theta, \hat{D}_\theta)$, where $\hat{D}_\theta$ is the pullback polyhedral divisor of $\theta$. 
We conclude that \( \nu \) is isomorphic to \( \gamma \).

**Proof.** Let \( \gamma \) be a diagram. So \( f \) and \( \eta \) are distinct and \( \nu \) is the normalization of \( X \). Since \( \kappa \) is the normalization of \( X \), each \( \nu \) is an isomorphism. To sum-up we have a commutative square

\[
\begin{array}{ccc}
\wedge \gamma & \xrightarrow{f} & \wedge X \\
\nu \downarrow & & \downarrow \eta \\
\hat{\nu} \downarrow & & \downarrow \theta \\
\wedge \theta & \xrightarrow{f} & \wedge \theta
\end{array}
\]

The matrix factorization is \( f \)-orbit, each \( \nu \) is the normalization. Here \( \kappa \) is obtained from the composition of the rational maps \( \eta_0 \), \( \hat{f} \) and \( \nu^{-1} \). As \( \wedge \gamma \) and \( \wedge \theta \) are smooth, \( \gamma \) is an open immersion. Moreover, \( \gamma \) is bijective from the diagram. So \( \gamma \) is an isomorphism. To sum-up we have a commutative square

\[
\begin{array}{ccc}
\wedge \theta & \xrightarrow{\eta} & X_{\Sigma} \\
\gamma \downarrow & & \downarrow f \\
\hat{\nu} \downarrow & & \downarrow \theta \\
\wedge \theta & \xrightarrow{\nu} & \wedge \theta
\end{array}
\]

and therefore \( \nu \gamma = \gamma \nu \hat{f} = \eta f \). This proves the proposition. \( \square \)

**Example 4.10.** Let \( P(z) = \prod_{i=1}^s (z - z_i)^{m_i} \in \mathbb{C}[z] \) be a polynomial, where the \( z_i \in \mathbb{C} \) are distinct and \( m_i \in \mathbb{Z}_{>0} \). Consider the \( \mathbb{C}^* \)-surface

\[ X = \{ (x, y, z) \mid \lambda \cdot (x, y, z) = (\lambda x, \lambda^d y, z) \} \text{ for } \lambda \in \mathbb{C}^*. \]

The weight package \( \theta \) of \( X \) is described as follows. The matrix factorization is

\[ P = \begin{pmatrix} -d & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \]

for \( S \) take \( (1 \ 0 \ 0) \) and \( X_{\Sigma} = \mathbb{P}^1 \times \mathbb{A}^1 \). Moreover, denoting by \( Z_{(-1,0)} \) the divisor of \( X_{\Sigma} \) corresponding to the ray \( \mathbb{Q}_{>0} (-1, 0) \), we have \( \mathcal{D}_\theta = (\mathbb{Q}_0 + \mathbb{Q}_{>0}) \cdot Z_{(-1,0)} \). Also

\[ \mathcal{D}_\theta = \{ ([x_0 : x_1], t) \in \mathbb{P}^1 \times \mathbb{A}^1 \mid x_1 P(t) - x_0 = 0 \} \simeq \mathbb{A}^1. \]

Since \( \kappa^*[Z_{(-1,0)}] = \sum_{i=1}^s m_i [z_i] \), we deduce that \( \mathcal{D}_\theta = \sum_{i=1}^s (\mathbb{Q}_0 + \mathbb{Q}_{>0}) \cdot [z_i] \) over \( \mathbb{A}^1 \) describes the normalization of \( X \), recovering [FZ03, Example 3.10].

Finally, we study the pullback divisor \( \mathcal{D}_\theta \) under subdivision of the fan \( \Sigma \).

**Proposition 4.11.** Let \( \Sigma' \) be a fan subdivision of \( \Sigma \) such that \( X_{\Sigma'} \) is semi-projective and let \( f : X_{\Sigma'} \rightarrow X_{\Sigma} \) be the corresponding toric modification. Let \( C_{\theta} \subset X_{\Sigma} \) be the proper transform of the curve \( C_{\theta} \subset X_{\Sigma} \) under \( f \). Let \( \eta : \hat{C}_{\theta} \rightarrow X_{\Sigma'} \) be the composition of the normalization \( \eta_0 : \hat{C}_{\theta} \rightarrow C_{\theta} \) and the closed immersion \( C_{\theta} \rightarrow X_{\Sigma} \). Then \( X(\hat{C}_{\theta}, \eta^* f^* \mathcal{D}_\theta) \) is equivariantly isomorphic to \( X(\hat{C}_{\theta}, \mathcal{D}_\theta) \).

**Proof.** Let \( \hat{f} : C_{\theta} \rightarrow C_{\theta} \) be the morphism induced by \( f \). Then we have commutative square

\[
\begin{array}{ccc}
\hat{C}_{\theta} & \xrightarrow{\eta} & C_{\theta} \\
\gamma \downarrow & & \downarrow \hat{f} \\
\hat{C}_{\theta} & \xrightarrow{\nu} & C_{\theta}
\end{array}
\]

where \( \nu \) is the normalization. Here \( \gamma \) is obtained from the composition of the rational maps \( \eta_0 \), \( \hat{f} \) and \( \nu^{-1} \). As \( \hat{C}_{\theta} \) and \( C_{\theta} \) are smooth, \( \gamma \) is an open immersion. Moreover, \( \gamma \) is bijective from the diagram. So \( \gamma \) is an isomorphism. To sum-up we have a commutative square

\[
\begin{array}{ccc}
\hat{C}_{\theta} & \xrightarrow{\eta f} & X_{\Sigma'} \\
\gamma \downarrow & & \downarrow f \\
\hat{C}_{\theta} & \xrightarrow{\nu} & X_{\Sigma}
\end{array}
\]

and therefore \( \gamma^* \mathcal{D}_\theta = \gamma^* \eta^* \mathcal{D}_\theta = \eta^* f^* \mathcal{D}_\theta \). This proves the proposition. \( \square \)
4.2. Curves on toric surfaces. The next two sections (4.2 and 4.3) study examples illustrating the theory. Computing pullback of polyhedral divisors of weight packages $\theta = (N, F, S, \Sigma)$ of hypersurfaces with complexity-one torus action involves considering the following problem.

Assume that $X_\Sigma$ is a surface, take $C := \{(x, y) \in T_{\text{Coker}(F)} = (\mathbb{C}^*)^2 \mid f(x, y) = 0\}$ for some $f \in \mathbb{C}[x, x^{-1}, y, y^{-1}]$ irreducible, let $C_\theta$ be the Zariski closure of $C$ in $X_\Sigma$ and let $\kappa$ be the composition of the normalization $\tilde{C}_\theta \to C_\theta$ and the inclusion $C_\theta \subset X_\Sigma$. Given a toric divisor $Z_\rho \subset X_\Sigma$, how do we compute $\kappa^*((Z_\rho))$ in terms of $\Sigma$ and $f$?

Composing with toric modifications, we assume $X_\Sigma$ smooth.

Example 4.12. Consider the fan $\Sigma$ of $\mathbb{Q}^2$ with maximal cones

|   | Generators          |
|---|---------------------|
| $\sigma_1$ | $(1, 0), (1, 1)$   |
| $\sigma_2$ | $(1, 1), (0, 1)$   |
| $\sigma_3$ | $(0, 1), (-1, -2)$ |
| $\sigma_4$ | $(-1, -2), (0, -1)$ |
| $\sigma_5$ | $(0, -1), (1, 0)$ |

and the curve $C = \{(x, y) \in (\mathbb{C}^*)^2 \mid y + x + x^2 = 0\}$.

We now provide a three-step calculation method for calculating the pullbacks $\kappa^*((Z_\rho))$.

Step 1. Determine Laurent monomials $x_\sigma, y_\sigma$ such that $\mathbb{C}[\sigma^\vee \cap M] = \mathbb{C}[x_\sigma, y_\sigma]$ for any $\sigma \in \Sigma_{\text{max}}$. Then write $x = x_{\sigma_1}^{m(\sigma_1)}y_{\sigma_1}^{n(\sigma_1)}$ and $y = x_{\sigma_2}^{m(\sigma_2)}y_{\sigma_2}^{n(\sigma_2)}$ with $n(\sigma), m(\sigma) \in \mathbb{Z}$ and substitute: $f(x, y) = f(x_{\sigma_1}^{m(\sigma_1)}y_{\sigma_1}^{n(\sigma_1)}, x_{\sigma_2}^{m(\sigma_2)}y_{\sigma_2}^{n(\sigma_2)})$. Multiply by a Laurent monomial $x_{\sigma}^{s(\sigma)}y_{\sigma}^{t(\sigma)}$ in order to have $f_\sigma(x_\sigma, y_\sigma) := x_\sigma^{s(\sigma)}y_\sigma^{t(\sigma)}f(x, y) \in \mathbb{C}[\sigma^\vee \cap M]$ irreducible. In conclusion $C_\theta \cap X_\sigma = \text{Spec} \mathbb{C}[x_\sigma, y_\sigma]/(f_\sigma)$.

Example 4.13. Returning to Example 4.12, the local equations are:

| $C_\theta \cap X_\sigma$ | Equation $f_\sigma$ |
|-------------------------|---------------------|
| $1 + x_1 y_1 + y_1$    |                     |
| $1 + x_2 + y_2$        |                     |
| $1 + x_3 + y_3$        |                     |
| $1 + x_4 y_4 + y_4$    |                     |
| $1 + x_5 y_5 + x_5^2 y_5$ |                 |

Step 2. We compute the trivializations of $\mathcal{O}_{X_\Sigma}(Z_\rho)$ for $\rho \in \Sigma(1)$. Let $\rho^-, \rho^+ \in \Sigma(1) \setminus \{\rho\}$ distinct such that $\sigma^- := \rho^- + \rho$, $\sigma^+ := \rho^+ + \rho \in \Sigma_{\text{max}}$. Consider $m^-_\rho, m^+_\rho \in M$ satisfying $\langle m^-_\rho, v_\rho \rangle = 1, \langle m^-_\rho, v^-_\rho \rangle = 0, \langle m^+_\rho, v_\rho \rangle = 1, \langle m^+_\rho, v^+_\rho \rangle = 0$. Then the trivialization of $Z_\rho$ is given by

- $(X_\sigma, 1)$ for any $\sigma \in \Sigma_{\text{max}} \setminus \{\sigma^-, \sigma^+\}$;
- $(X^-_{\sigma^-}, \chi^{m^-_\rho})$ and $(X^+_{\sigma^+}, \chi^{m^+_\rho})$.
Example 4.14. Non-trivial trivializations for Example 4.12 are given by:

| Z(1,0) | (X_{\sigma_1}, \chi^{(1,-1)}), (X_{\sigma_2}, \chi^{(1,0)}) |
| Z(1,1) | (X_{\sigma_1}, \chi^{(0,1)}), (X_{\sigma_2}, \chi^{(1,0)}) |
| Z(0,1) | (X_{\sigma_2}, \chi^{(-1,1)}), (X_{\sigma_3}, \chi^{(-2,1)}) |
| Z(-1,-2) | (X_{\sigma_1}, \chi^{(-1,0)}), (X_{\sigma_4}, \chi^{(-1,0)}) |
| Z(0,-1) | (X_{\sigma_4}, \chi^{(2,-1)}), (X_{\sigma_5}, \chi^{(0,-1)}) |

Step 3. Let \( \varphi : \hat{C}_\theta \to C_\theta \) be the desingularization. With the notation of Step 2, set \( \Omega^\pm := \varphi^{-1}(X_{\sigma^\pm} \cap C_\theta) \) and consider the sheaf of ideals \( \mathcal{J}_\rho \subset \mathcal{O}_{C_\theta} \) generated by the restricted trivializations, i.e.

1. \( \mathcal{J}_\rho(\Omega^\pm) = \chi_{\rho^\pm} \cdot \mathbb{C}[\Omega^\pm] \);
2. \( \mathcal{J}_\rho(\varphi^{-1}(X_{\sigma} \cap C_\theta)) = \mathbb{C}[\varphi^{-1}(X_{\sigma} \cap C_\theta)] \) for any \( \sigma \in \Sigma_{\max} \setminus \{\sigma^-_\rho, \sigma^+_\rho\} \).

Denote by \( \kappa^\pm_\rho \) the restriction \( \kappa_{|X_{\sigma^\pm}} \). There is a unique decomposition

\[
\mathcal{J}_\rho(\Omega^\pm) = \prod_{j=1}^{s^\pm_\rho} \mathfrak{m}^\pm_{\rho, j}, \text{ where } \mathfrak{m}^\pm_{\rho, j} \subset \mathbb{C}[\Omega^\pm_\rho]
\]

are maximal ideals. We compute \( \kappa^*([Z_\rho]) \) via \( \kappa^*_{\rho^\pm}([Z_\rho]|_{X_{\sigma^\pm}}) = \sum_{j=1}^{s^\pm_\rho} \zeta^\pm_{\rho, j} \cdot [\zeta^\pm_{\rho, j}] \), where \( \zeta^\pm_{\rho, j} \in \hat{C}_\theta \)

Example 4.15. The computation for Example 4.12 of the pullbacks is given by:

| Z(1,0) | \( \kappa^*([Z(1,0)]) = 0 \) |
| Z(1,1) | \( \kappa^*([Z(1,1)]) = [\kappa(1,1)] \) |
| Z(0,1) | \( \kappa^*([Z(0,1)]) = [\kappa(0,1)] \) |
| Z(1,0) | \( \kappa^*([Z(-1,-2)]) = [\kappa(-1,-2)] \) |
| Z(0,-1) | \( \kappa^*([Z(0,-1)]) = 0 \) |

4.3. The \( \mathbb{C}^* \)-surface \( z_1z_2^2 + z_2z_0^2 + z_3^2 = 0 \). We now study an example of a projective variety with linear torus action of complexity one.

The toric variety \( \mathbb{P}^3_\mathbb{C} \). We are in the case \( \ell = 3 \). Let \( e_1 = (1, 0, 0), e_2 = (0, 1, 0), e_3 = (0, 0, 1) \) and \( e = -e_1 - e_2 - e_3 \). The fan of \( \mathbb{P}^3_\mathbb{C} \) is generated by the cones \( \delta = \delta(0) = \text{Cone}(e_1, e_2, e_3) \) and \( \delta^{(i)} = \text{Cone}(e, e_j \mid j \neq i), \ i = 1, 2, 3 \).

Goal. Consider the \( \mathbb{C}^* \)-surface

\[
X := \mathbb{V}(z_1z_2^2 + z_2z_0^2 + z_3^2) \subset \mathbb{P}^3_\mathbb{C} \text{ with action } \lambda \cdot [z_0 : \ldots : z_3] = [z_0 : \lambda^{-3}z_1 : \lambda^3z_2 : \lambda z_3].
\]
Our goal is constructing a divisorial fan for $X$ from weight packages $\theta^{(i)}$ of the charts $X^{(i)} := X_{\delta^{(i)}} \cap X$.

The chart $X^{(0)}$. Let us describe $\theta^{(0)} = (\mathbb{Z}^3, \mathbb{Z}, F^{(0)}, S^{(0)}, \Sigma^{(0)})$. The weight matrix is $F^{(0)} = t \left( \begin{array}{ccc} -3 & 3 & 1 \\ 0 & -3 & 3 \\ 1 & 0 & 3 \end{array} \right)$, the $P$-matrix is $P^{(0)} = \left( \begin{array}{ccc} 1 & 1 & 0 \\ 1 & 0 & 0 \end{array} \right)$, for $S^{(0)}$ we choose $(0 \ 1 \ -2)$ and the quotient fan $\Sigma^{(0)}$ is generated by the cones Cone($(1,0),(1,1)), Cone((0,1),(1,1))$.

The chart $X^{(1)}$. Next, we determine $\theta^{(1)} = (\mathbb{Z}^3, \mathbb{Z}, F^{(1)}, S^{(1)}, \Sigma^{(1)})$ by enhancing $F^{(0)}$ to $\hat{F} := t \left( \begin{array}{ccc} 0 & -3 & 3 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{array} \right)$, where $-3$ is the weight of $z_1$. Removing the zero in the second entry we get $F^{(1)} = t \left( \begin{array}{ccc} 3 & 6 & 4 \end{array} \right)$. Furthermore, we enhance the matrix factorization

$$0 \to \mathbb{Z} \to \hat{F} \to \mathbb{Z}^4 \to \mathbb{Z}^2 \to 0,$$

where $\hat{P} = \left( \begin{array}{ccc} -2 & 1 & 1 & 0 \\ -4 & 1 & 0 & 3 \end{array} \right)$.

Observe that we add a new column in the first position of $P^{(0)}$ in order that the lines of $\hat{P}$ are orthogonal to $(1,1,1)$ and $t\hat{F}$. Now $P^{(1)}$ is obtained by removing the second column of $\hat{P}$. Similarly, we enhance $S^{(0)}$ to $\hat{S} := (1 \ 0 \ 1 \ -2)$ so that $S^{(1)} = (1 \ 1 \ -2)$. The quotient fan $\Sigma^{(1)}$ is the complete fan with rays $Q_{\geq 0}(-1,-2), Q_{\geq 0}(1,0), Q_{\geq 0}(0,1)$. It can be obtained from two manners: as the fan generated by $P^{(1)}(\delta_0)$, where $\delta_0$ runs over the faces of $\delta = \delta^{(0)}$, or as the fan generated by $P^{(0)}(\delta)$, where $\delta$ runs over the faces of $\delta^{(1)}$.

The other charts. We construct the other weight packages $\theta^{(i)}$ by removing the $(i + 1)$-th column of $\hat{P}$ and $\hat{S}$. To sum-up we obtain:

| Weight package $\theta^{(i)}$ | Quotient fan $\Sigma^{(i)}$ |
|--------------------------------|-----------------------------|
| $\theta^{(0)} = (\mathbb{Z}^3, \mathbb{Z}, t \left( \begin{array}{ccc} -3 & 3 & 1 \\ 0 & 1 & -2 \end{array} \right), \Sigma^{(0)})$ | Cone($(1,0),(1,1)), Cone((0,1),(1,1))$ |
| $\theta^{(1)} = (\mathbb{Z}^3, \mathbb{Z}, t \left( \begin{array}{ccc} 3 & 6 & 4 \\ 1 & 1 & -2 \end{array} \right), \Sigma^{(1)})$ | Cone($(1,0),(0,1)), Cone((1,0),(-1,2))$, Cone((0,1),(-1,-2)) |
| $\theta^{(2)} = (\mathbb{Z}^3, \mathbb{Z}, t \left( \begin{array}{ccc} -3 & -6 & -2 \\ 1 & 0 & -2 \end{array} \right), \Sigma^{(2)})$ | Cone($(1,1),(0,1)), Cone((1,1),(-1,2))$, Cone((0,1),(-1,-2)) |
| $\theta^{(3)} = (\mathbb{Z}^3, \mathbb{Z}, t \left( \begin{array}{ccc} -1 & -4 & 2 \\ 1 & 0 & 1 \end{array} \right), \Sigma^{(3)})$ | Cone($(1,1),(1,0)), Cone((1,0),(-1,-2))$ |

Constructing the divisorial fan. Let $\Sigma$ be the fan of Example 4.12. For each $i$ there is an open subset $X_{\Sigma^{(i)}} \subset X_{\Sigma}$ and a modification $f^{(i)} : X_{\Sigma^{(i)}} \to X_{\Sigma^{(i)}}$. The proper transform $C_{\theta^{(i)}} \subset X_{\Sigma^{(i)}}$ of $C_{\theta^{(i)}}$ have local equations as in Example 4.13. Denote by $\kappa^{(i)} : \tilde{C}_{\theta} \to X_{\Sigma^{(i)}}$ the morphism obtained by composing the normalization of $C^{\prime}_{\theta^{(i)}}$, the closed immersion $C_{\theta^{(i)}} \hookrightarrow X_{\Sigma^{(i)}}$ and the modification $f^{(i)}$. Set $\mathcal{D}_{\theta}^{(i)} = \kappa^{(i)} \mathcal{D}_{\theta^{(i)}}$ and denote by $\tilde{C}$ the smooth completion of the affine plane curve $\mathbb{V}(y + x + x^2)$. Using Example 4.15, the divisorial fan $\mathcal{D}_{\theta}$ of the $\mathbb{C}^*$-surface $X$ is:
4.4. Linear torus actions: the projective case. Inspired by Example 4.3, we now treat the projective case. Main result of this section, Theorem 4.22, is an extension of Theorem 4.9.

**Definition 4.16.** Let \( \theta = (N, \mathbb{Z}^n, N, F, S, \Sigma) \) be an abstract weight package. This means that \( F : N \to N \) is an injective morphism, \( S : N \to N \) is a section. The symbol \( \Sigma \) stands for the fan of \( \text{Coker}(F) \) generated by the cones \( P(\delta_0) \), where \( \delta_0 \) is a face of the first quadrant \( \delta = \mathbb{Q}_{\geq 0} \subset N_{\mathbb{Q}} \) and \( P : N \to \text{Coker}(F) \) is the quotient. We define the sequence

\[
\theta := (\theta^{(0)}, \theta^{(1)}, \ldots, \theta^{(\ell)})
\]

of weight packages by enhancing \( \theta \) into a weight package \( \hat{\theta} \). Fix basis so that we identify \( N \) and \( \mathbb{N} \) with \( \mathbb{Z}^n, \mathbb{Z}^\ell \), see \( F \) as an \( \ell \times n \)-matrix \( (a_{i,j}) \) and \( P \) as an \( s \times \ell \)-matrix \( (b_{i,j}) \), where \( s \) is the rank of \( \text{Coker}(F) \). The weight package \( \hat{\theta} \) is \( (\mathbb{Z} \oplus \mathbb{N} = \mathbb{Z}^{\ell+1}, N = \mathbb{Z}^n, \hat{F}, \hat{S}, \hat{\Sigma}) \), where:

1. \( \hat{F} = \begin{bmatrix} 0 & \cdots & 0 \\
    a_{1,1} & a_{1,2} & \cdots \\
    \vdots & \ddots & \ddots \\
    a_{\ell,1} & \cdots & a_{\ell,n} \end{bmatrix} \) and \( \hat{P} = \begin{bmatrix} b_{1,0} & b_{1,1} & \cdots & b_{1,\ell} \\
    \vdots & \vdots & \ddots & \vdots \\
    b_{s,0} & b_{s,1} & \cdots & b_{s,\ell} \end{bmatrix} \in \text{Mat}_{s \times (\ell+1)}(\mathbb{Z}) \)

with the condition \( \sum_{j=0}^\ell b_{i,j} = 0 \) for any \( i \in \{1, \ldots, s\} \); In this way, we get the matrix factorization

\[
0 \to N \xrightarrow{\hat{F}} \mathbb{Z} \oplus \mathbb{N} \xrightarrow{\hat{P}} \text{Coker}(F) = \text{Coker}(\hat{F}) \to 0.
\]

2. We define the fan \( \hat{\Sigma} \) as the fan generated by the cones \( \hat{P}(\delta_0) \), where \( \delta_0 \) runs over the faces of the first quadrant \( \hat{\delta} = \mathbb{Q}_{\geq 0} \subset \mathbb{Q} \oplus \mathbb{N}_{\mathbb{Q}} \).

3. Similarly, the section \( \hat{S} \) is the matrix

\[
\begin{bmatrix}
    s_{1,0} & s_{1,1} & \cdots & s_{1,\ell} \\
    \vdots & \vdots & \ddots & \vdots \\
    s_{n,0} & s_{n,1} & \cdots & s_{n,\ell} \\
\end{bmatrix} \in \text{Mat}_{n \times (\ell+1)}(\mathbb{Z}), \quad \text{where } S = (s_{i,j})
\]

and with the condition \( \sum_{j=0}^\ell s_{i,j} = 0 \) for any \( i \in \{1, \ldots, n\} \).

Now define \( \theta^{(0)} \) as \( \theta \) and for \( v \in \{1, \ldots, \ell\} \),

\[
\theta^{(v)} = (N, N, F^{(v)}, S^{(v)}, \Sigma^{(v)}) \text{ where: }
\]

(I) \( F^{(v)} \) is obtained by omitting the line of index \( v \) of the matrix

\[
\hat{F} - \begin{bmatrix} a_{v,1} & \cdots & a_{v,n} \\
    \vdots & \ddots & \vdots \\
    a_{v,1} & \cdots & a_{v,n} \end{bmatrix} \in \text{Mat}_{(\ell+1) \times n}(\mathbb{Z});
\]
(II) The linear map $F^{(v)}$ induces the matrix factorization

$$0 \rightarrow N \xrightarrow{F^{(v)}} \overline{N} \xrightarrow{P^{(v)}} \text{Coker}(F^{(v)}) \rightarrow 0,$$

where $P^{(v)}$ is obtained by omitting the column of index $v$ of the matrix $\hat{P}$. The fan of $\Sigma^{(v)}$ of Coker$(F^{(v)})$ is fan generated by the cones $P^{(v)}(\delta_0)$, where $\delta_0$ runs over the set of faces of $\delta = \mathbb{Q}^\ell_\geq 0 \subset \overline{N}_{\Phi}$.

(III) The section $S^{(v)}$ is obtained by omitting the column of index $v$ of the matrix $\hat{S}$.

The following builds affine subvarieties with linear torus action from weight packages.

**Lemma 4.17.** Let $\theta = (\overline{N} = \mathbb{Z}^\ell, N = \mathbb{Z}^n, F, S, \Sigma)$ be a weight package. Let $C \subset \mathbb{T}_{\text{Coker}(F)}$ be an irreducible curve with defining ideal $I_C = (f_i(x_1, \ldots, x_n) \mid 1 \leq i \leq a)$. Let $C_\theta$ be Zariski closure of $C$ in $X_{\Sigma}$. Consider the matrix

$$P = \begin{bmatrix} b_{1,1} & \cdots & b_{1,\ell} \\ \vdots & & \vdots \\ b_{n,1} & \cdots & b_{n,\ell} \end{bmatrix} \in \text{Mat}_{n \times \ell}(\mathbb{Z})$$

coming from the matrix factorization of $\theta$. Set

$$g_i(T_1, \ldots, T_\ell) := f_i \left( \prod_{j=1}^\ell T_j^{b_{i,j}}, \ldots, \prod_{j=1}^\ell T_j^{b_{i,j}} \right) \text{ for } 1 \leq i \leq a,$$

and assume that there are Laurent monomials $u_i \in \mathbb{C}[T_1, T_1^{-1}, \ldots, T_\ell, T_\ell^{-1}]$ such that $X := \mathbb{V}(u_1, \ldots, u_a g_a) \subset \mathbb{A}^\ell_{\mathbb{C}}$ is irreducible. Then $X \subset \mathbb{A}^\ell_{\mathbb{C}}$ is a subvariety with linear torus action of complexity one and its normalization is equivariantly isomorphic to $X(\overline{C}_\theta, \mathcal{D}_\theta)$, where $\overline{C}_\theta$ is the normalization of $C_\theta$ and $\mathcal{D}_\theta$ is pullback polyhedral divisor as in Definition 4.7.

**Proof.** Set $F = (a_{i,j})_{1 \leq i \leq \ell, 1 \leq j \leq n}$. Then the action on the coordinates is given by

$$\mu \cdot (x_1, \ldots, x_n) = \left( \prod_{j=1}^n \mu_j^{a_{1,j}} \right) \cdot x_1, \ldots, \left( \prod_{j=1}^n \mu_j^{a_{\ell,j}} \right) \cdot x_\ell$$

for any $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{T}_N = \left( \mathbb{C}^* \right)^n$. Remarking that $g_i$ is a $\mathbb{T}_N$-invariant function (since $P \circ F = 0$) and $u_i$ is a $\mathbb{T}_N$-eigenfunction, the subset $X \subset \mathbb{A}^\ell_{\mathbb{C}}$ is stable $\mathbb{T}_N$-action. Let $\psi : \mathbb{G} \rightarrow \mathbb{T}_{\text{Coker}(F)}$ be the quotient. Note that $X$ intersects $\mathbb{G}$ because an element of $C$ lifts via $\psi$ to solution of the equations $g_i$. Also the map $X \cap \mathbb{G} \rightarrow C$, $x \mapsto \psi(x)$ is a geometric quotient for the $\mathbb{T}_N$-action. So $X \subset \mathbb{A}^\ell_{\mathbb{C}}$ is a subvariety with linear torus action of complexity one and we conclude by Theorem 4.9. \hfill \Box

**Example 4.18.** Lemma 4.17 for $C$ smooth does not give, in general, normal varieties. Since for the weight package $\theta = (\overline{N}, N, F, S, \Sigma)$ with $\overline{N} = \mathbb{Z}^4, N = \mathbb{Z}^2$,

$$F = \begin{pmatrix} 4 & 3 & 0 & 12 \\ 0 & 0 & 1 & -1 \end{pmatrix}, \quad S = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

the fan $\Sigma$ of $\mathbb{P}^2_{\mathbb{C}}$, and the elliptic curve $C = \mathbb{V}(z_0 z_2^2 + z_1^3 + z_0^2 z_1) \subset \mathbb{P}^2_{\mathbb{C}}$, we get the non-normal hypersurface

$$X = \mathbb{V}(x_1^8 x_3 x_4 + x_2^{12} + x_2^4 x_3^2 x_4) \subset \mathbb{A}^4_{\mathbb{C}}.$$

**Proposition 4.19.** Let $\theta = (\overline{N} = \mathbb{Z}^\ell, N = \mathbb{Z}^n, F, S, \Sigma)$ be a weight package. Consider a Laurent polynomial of the form

$$g_1(T_1, \ldots, T_\ell) := f_1 \left( \prod_{j=1}^\ell T_j^{b_{1,j}}, \ldots, \prod_{j=1}^\ell T_j^{b_{\ell,j}} \right),$$
where \( f_1 \in \mathbb{C}[\text{Coker}(F)] = \mathbb{C}[x_1, x_1^{-1}, \ldots, x_s, x_s^{-1}] \) is irreducible and the \( b_{i,j} \) are the coefficients of the \( P \)-matrix of \( \theta \). Then there is a Laurent monomial \( u_1 \in \mathbb{C}[T_1, T_1^{-1}, \ldots, T_\ell, T_\ell^{-1}] \) such that \( u_1 g_1 \in \mathbb{C}[T_1, \ldots, T_\ell] \) is irreducible.

**Proof.** Let \( \ell_1, \ldots, \ell_s \) be the vectors of \( \mathbb{Z}^\ell \) corresponding to the first \( s \) lines of \( P \). We may find \( \ell_{s+1}, \ldots, \ell_\ell \) such that \( (\ell_1, \ldots, \ell_\ell) \) is a basis of \( \mathbb{Z}^\ell \). Denote by \( v_i = (\ell_{i,1}, \ldots, \ell_{i,\ell}) \) the coordinates of \( \ell_i \) and set \( X_i := \prod_{j=1}^{\ell_i} T_j^{i,j} \). Then

\[
\mathbb{C}[T_1, T_1^{-1}, \ldots, T_\ell, T_\ell^{-1}] = \mathbb{C}[X_1, X_1^{-1}, \ldots, X_\ell, X_\ell^{-1}] \quad \text{and} \quad g_1(T_1, \ldots, T_\ell) = f_1(X_1, \ldots, X_s).
\]

Set \( R = \mathbb{C}[X_1, \ldots, X_\ell] \). Let \( S, V \in R \) such that \( f_1(X_1, \ldots, X_s) = SV \). Since \( \deg_X(S) + \deg_X(f_1(X_1, \ldots, X_s)) = 0 \) for \( i = s + 1, \ldots, \ell \), we have \( S, V \in \mathbb{C}[X_1, \ldots, X_s] \). But \( f_1(X_1, \ldots, X_s) \) is irreducible in \( \mathbb{C}[X_1, X_1^{-1}, \ldots, X_s, X_s^{-1}] \). So we may assume that \( S \) is a monomial and \( V \) is a product of a monomial and an irreducible element of \( R \), whence the irreducibility of \( g_1 \).

\[ \blacksquare \]

**Corollary 4.20.** Any subvariety \( X \subset \mathbb{A}_e \) with linear torus action of complexity one has a decomposition \( X = \bigvee(u_1 g_1, \ldots, u_a g_a) \) as in Lemma 4.17.

**Proof.** The \( T_N \)-action on \( \mathbb{A}_e \) induces an \( M \)-grading on the ring \( R := \mathbb{C}[x_1, \ldots, x_\ell] \). Declare an element of \( R \) homogeneous if it is homogeneous with respect to the \( M \)-grading. The vanishing ideal of \( X \) in \( R \) is generated by non-constant homogeneous polynomials \( h_1, \ldots, h_a \in R \). Set \( X_i := \bigvee(h_i) \subset \mathbb{A}_e \) and consider the decomposition

\[
X_i = \bigcup_{j=1}^{s_i} X_{i,j}, \quad 1 \leq i \leq a
\]

in irreducible components. Since \( X_i \) is \( T_N \)-stable, each component \( X_{i,j} \) is of the form \( \bigvee(h_{i,j}) \subset \mathbb{A}_e \) for some homogeneous irreducible \( h_{i,j} \in R \). Let \( E \) be the set of pairs \((i, j)\) with \( 1 \leq i \leq a \) and \( 1 \leq j \leq s_i \) such that \( h_{i,j} \) is not a scalar multiplication of a coordinate \( x_k \). By Proposition 4.19 we have for \((i, j) \in E\) a decomposition \( X_{i,j} = \bigvee(u_{i,j} g_{i,j}) \) obtained by pulling back \( \psi(\mathbb{G} \cap X_{i,j}) = \bigvee(f_{i,j}) \subset \mathbb{C}[\text{Coker}(F)] \) by the quotient \( \psi : \mathbb{G} \to \mathbb{C}[\text{Coker}(F)] \), where \( u_{i,j} \) is a monomial. For \((i, j) \notin E\) with \( 1 \leq i \leq a \) and \( 1 \leq j \leq s_i \) we set \( u_{i,j} = h_{i,j} \) and \( g_{i,j} = 1 \). Furthermore, for \( 1 \leq i \leq a \) write

\[
u_i = \prod_{j=1}^{s_i} u_{i,j}, \quad g_i = \prod_{j=1}^{s_i} g_{i,j} \quad \text{and} \quad f_i = \prod_{j=1}^{s_i} f_{i,j}.
\]

Then \( X = \bigvee(u_1 g_1, \ldots, u_a g_a) \). Moreover, the zero locus \( \bigvee(f_1, \ldots, f_\ell) \subset \mathbb{C}[\text{Coker}(F)] \) is the image \( \psi(X \cap \mathbb{G}) \), whence the result.

\[ \blacksquare \]

**Lemma 4.21.** Let \( \theta = (\overline{N}, N, F, S, \Sigma) \) be a weight package and consider the sequence \( \theta^{(i)} = (\overline{N}, N, F^{(i)}, S^{(i)}, \Sigma^{(i)}) \) that comes from the enhanced structure (see Definition 4.16). Let

\[
\delta = \delta^{(0)} = \sum_{i=0}^\ell \mathbb{Q}_{\geq 0} e_i, \quad \delta^{(j)} = \mathbb{Q}_{\geq 0} e + \sum_{i \neq j} \mathbb{Q}_{\geq 0} e_i \quad \text{for} \quad j = 1, 2, \ldots, \ell,
\]

be the maximal cones generated the fan of \( \mathbb{P}^\ell \), where \((e_1, \ldots, e_\ell)\) is the standard basis of \( \overline{N}_\mathbb{Q} = \mathbb{Q}_\ell \) and \( e = -\sum_{i=1}^{\ell} e_i \). Then \( \Sigma^{(i)} \) is the fan generated by the cones \( P(\delta') \), where \( \delta' \) runs over the faces of \( \delta^{(i)} \). Moreover, we have the identity

\[
S^{(i)}(P^{(i)-1}(v) \cap \delta^{(0)}) = S(P^{-1}(v) \cap \delta^{(i)})
\]

for all \( i \in \{1, 2, \ldots, \ell\} \) and \( v \in |\Sigma^{(i)}| \).
Proof. First claim is consequence of the identities
\[ P^{(a)}(e_i) = P(e); \quad P^{(a)}(e_i+1) = P(e_i) \text{ for } 1 \leq i \leq a; \]
\[ P^{(a)}(e_j) = P(e_j) \text{ for } a < j \leq \ell. \]
Similarly, consider the map
\[ f : \delta^{(i)} \to \delta^{(0)}, \quad \lambda e + \sum_{j \neq i} \lambda_j e_j \mapsto \lambda e_1 + \sum_{1 \leq j \leq i} \lambda_j e_{j+1} + \sum_{i < j \leq \ell} \lambda_j e_j (\lambda_j, \lambda \in \mathbb{Q}_{\geq 0}). \]
Then
\[ f(P^{-1}(v) \cap \delta^{(i)}) = P^{(i)-1}(v) \cap \delta^{(0)}. \]
As \( S^{(i)}(f(w)) = S(w) \) for any \( w \in \delta^{(i)} \), we conclude that
\[ S^{(i)}(P^{(i)-1}(v) \cap \delta^{(0)}) = S^{(i)}(f(P^{-1}(v) \cap \delta^{(i)})) = S(P^{-1}(v) \cap \delta^{(i)}), \]
finishing the proof of the lemma. \( \square \)

**Theorem 4.22.**

1. Let \( X \subset \mathbb{P}_{\ell}^\ell \) be a subvariety with linear torus action of complexity one. Let \( \theta \) be the weight package of \( X \) and consider the sequence
\[ \theta^{(i)} = (\overline{N}, N, F^{(i)}, S^{(i)}, \Sigma^{(i)}), \quad 0 \leq i \leq \ell, \]
that comes from the enhanced structure (see Definition 4.16). Let \( \Sigma \) be a fan with support \( \bigcup_{i=0}^{\ell} |\Sigma^{(i)}| \) such that for \( 0 \leq i \leq \ell \) the set \( \overline{\Sigma}^{(i)} = \{ \sigma \in \Sigma \mid \sigma \subset |\Sigma^{(i)}| \} \) is a projective fan subdivision of \( \Sigma^{(i)} \). Denote by \( \kappa^{(i)} : \overline{C}_{\theta^{(i)}} \to X_{\overline{\Sigma}^{(i)}} \) the map, which is the composition of the projective modification \( f^{(i)} : X_{\overline{\Sigma}^{(i)}} \to X_{\Sigma^{(i)}} \), the inclusion \( C'_{\theta^{(i)}} \to X_{\overline{\Sigma}^{(i)}} \), where \( C'_{\theta^{(i)}} \) is the proper transform of \( C_{\theta^{(i)}} \) under \( f^{(i)} \), and the normalization \( \overline{C}_{\theta^{(i)}} \to \overline{C}_{\theta^{(i)}} \). Then the set
\[ \{ \overline{\Theta}^{(i)}_{\theta^{(i)}} = \kappa^{(i)}*\Delta^{(i)}_{\theta^{(i)}} \mid i = 0, 1, \ldots, \ell \} \]
generates a divisorial fan \( \mathcal{E}_\theta \) over a smooth projective curve \( \overline{C} \). Moreover the normalization of \( X \) is equivariantly isomorphic to \( X(\overline{C}, \mathcal{E}_\theta) \).

2. Conservely, let \( \theta = (\overline{N}, N, F, S, \Sigma) \) be a weight package. Let \( C \subset \mathbb{T}_{\text{Coker}(F)} = (\mathbb{C}^*)^s \) be an irreducible curve with defining ideal
\[ I_C = (f_i(x_1, \ldots, x_a) \mid 1 \leq i \leq a). \]
Consider the enhanced \( P \)-matrix
\[ \hat{P} = \begin{bmatrix} b_{1,0} & b_{1,1} & \cdots & b_{1,\ell} \\ \vdots & \vdots & \ddots & \vdots \\ b_{s,0} & b_{s,1} & \cdots & b_{s,\ell} \end{bmatrix} \in \text{Mat}_{s \times \ell+1}(\mathbb{Z}) \]
as in Definition 4.16. Set
\[ g_i(T_0, T_1, \ldots, T_\ell) := f_i \left( \prod_{j=0}^{\ell} T_j^{b_{1,j}}, \ldots, \prod_{j=0}^{\ell} T_j^{b_{s,j}} \right) \text{ for } 1 \leq i \leq a, \]
and assume that there exist Laurent monomials \( u_i \in \mathbb{C}[T_0, T_0^{-1}, \ldots, T_\ell, T_\ell^{-1}] \) such that \( X := \mathbb{V}(u_1g_1, \ldots, u_ag_a) \subset \mathbb{P}_\ell^\ell \) is irreducible. Then \( X \) is a subvariety with linear torus action of complexity one and its normalization is described by the divisorial fan \( \mathcal{E}_\theta \) obtained from Contraction (1).

3. If \( X \subset \mathbb{P}_\ell^\ell \) is a projective subvariety with linear torus of complexity one with weight package \( \theta \), then \( X \) admits a decomposition \( X = \mathbb{V}(u_1g_1, \ldots, u_ag_a) \) as in Assertion (2).
Proof. (1) Lemma 4.21 implies that the $X_{\mathcal{D}(i)}$ define a Zariski open covering of $X_{\Sigma}$, while Lemma 4.4 gives that $X(\mathcal{D}_{\theta(i)})$ is $T_N$-isomorphic to $X_{\theta(i)}$ for $0 \leq i \leq \ell$. So it follows from [AHS08, Proposition 4.3] (see the proof argument of [AHS08, Theorem 5.6]) that the set
\[
\{ f^{(i)}(\mathcal{D}_{\theta(i)}) \mid i = 0, 1, \ldots, \ell \}
\]
generates a divisorial fan $\mathcal{E}$ over $(X_{\Sigma}, N)$ describing the $T_N$-variety $\mathbb{P}^\ell_C$. Let $\eta: \tilde{X} \to X(\mathcal{E}) \simeq \mathbb{P}^\ell_C$ be the composition of the the normalization $\tilde{X} \to X$ with the closed immersion $X \to \mathbb{P}^\ell_C$. Note that the polyhedral divisorial $\mathcal{D}_{\theta(i)}$ encodes the normalization of the affine $T_N$-variety $X(i) := X_{\theta(i)} \cap X$ for $0 \leq i \leq \ell$ by virtue of Proposition 4.11 and Theorem 4.9. So for all $i, j \in \{0, 1, \ldots, \ell\}$ we have the natural identifications
\[
X(\mathcal{D}_{\theta(i)}) \cap X(\mathcal{D}_{\theta(j)}) \simeq \eta^{-1}(X(f^{(i)}(\mathcal{D}_{\theta(i)}) \cap f^{(j)}(\mathcal{D}_{\theta(j)}))) \simeq X(\kappa^{(i)}(\mathcal{D}_{\theta(i)} \cap \mathcal{D}_{\theta(j)})),
\]
showing that $\mathcal{E}_\theta$ generates a divisorial fan. We conclude that the normalization of $X$ is equivariantly isomorphic to $X(\mathcal{E}_\theta)$.

(2) Consequence of Lemma 4.17 applied to each chart $X(i) = X \cap X_{\theta(i)}$ and Construction (1).

(3) Follows from the affine case (see Corollary 4.20) via homogeneization. 

4.5. Geometry of the contraction space. From the suggestion in [AIPS12, Section 4.2, Remark 15], we now study contraction spaces of torus actions of complexity one via weight packages.

Lemma 4.23. Let $\theta = (N = \mathbb{Z}^\ell, N, S, F, \Sigma)$ be a weight package. Let $\Delta$ be the smallest fan of $\overline{N}_Q = \mathbb{Q}^\ell$ with support $\delta = \mathbb{Q}^\ell_{\geq 0}$ such that the map
\[
\gamma: \Delta \to \Sigma, \ \delta_0 \mapsto P(\delta_0),
\]
induced by the matrix factorization of $\theta$, is a fan morphism. Then the toric $G$-variety $X_\Delta$ is the normalization $Z$ of the Zariski closure of the graph of the $T_N$-invariant rational map $A^F_\mathbb{C} \dasharrow X_\Sigma$ in $A^F_\mathbb{C} \times X_\Sigma$.

Proof. Existence and unicity of $\Delta$ follow from that $\Delta$ must be the fan of $\overline{N}_Q$ generated by the cone $P^{-1}(\sigma) \cap \delta$, where $\sigma$ runs over $\Sigma$. The map $\gamma$ induces a toric morphism $p: X_\Delta \to X_\Sigma$. We see that for any $\sigma \in \Sigma_{\max}$, we have $p^{-1}(X(\sigma)) = X(p^{-1}(\sigma) \cap \delta)$. So $p$ is affine. The existence of two morphisms $p: X_\Delta \to X_\Sigma$ and $\pi_\Delta: X_\Delta \to A^F_\mathbb{C}$ induces a morphism $\varphi: X_\Delta \to Z$, which is birational. Since the composition $\pi_\Delta$ of $\varphi$ with the natural morphism $Z \to A^F_\mathbb{C}$ is proper, $\varphi$ is proper. Similarly, as the quotients $p: X_\Delta \to X_\Sigma$ and $Z \to X_\Sigma$ are affine, the morphism $\varphi$ is affine. So $\varphi$ is birational finite and by Zariski’s Main Theorem, $\varphi$ is an isomorphism, proving the lemma.

Definition 4.24. The fan $\Delta$ in Lemma 4.23 is the lifting fan of the weight package $\theta$.

Corollary 4.25. Let $X \subset A^F_\mathbb{C}$ be a subvariety with linear torus action of complexity one and let $\theta$ be its weight package. With the notation of Lemma 4.23, consider the toric map $\pi_\Delta: X_\Delta \to A^F_\mathbb{C}$, where $\Delta$ is the lifting fan of $\theta$. Then the normalization of the proper transform of $X$ under $\pi_\Delta$ is the contraction space $\tilde{X}$ of $X$ and the morphism $\pi: \tilde{X} \to X$ induced by restriction of $\pi_\Delta$ is the contraction map.

Proof. Let $S$ be the Zariski closure of the graph of $A^F_\mathbb{C} \dasharrow X_\Sigma$ in $A^F_\mathbb{C} \times X_\Sigma$. The proper transform $X_1$ of $X$ under the natural map $S \to A^F_\mathbb{C}$ is the closure of $\{(x, \psi(x)) \mid x \in X \cap \mathbb{G}\} \subset S$, where $\psi: \mathbb{G} \to \mathbb{T}_{\text{Coker}(F)}$ is the quotient. Thus the normalization of $X_1$ is $\tilde{X}$. 

Corollary 4.26. With the notation of Corollary 4.25, assume $X_\Sigma$ affine. Then the rational $\mathbb{T}_N$-invariant map $A^*_C \longrightarrow X_\Sigma$ is the global quotient $A^*_C \rightarrow A^*_C/\mathbb{T}_N$. In particular, the contraction map $\pi : \bar{X} \rightarrow X$ is the normalization of $X$.

Proof. Indeed, as $p : X_\Delta \rightarrow X_\Sigma$ is affine, $X_\Delta$ is affine. Moreover, the support of $\Delta$ is $\delta$. So $\pi_\Delta$ is the identity and by Corollary 4.25, $\pi$ is the normalization. \qed

Let us give a concrete example.

Example 4.27. For $\lambda \in \mathbb{C}^*$ take $X = \mathbb{V}(x_3 - \lambda) \subset A^3_C$ with weight package

$$\theta = (\mathbb{Z}^3, \mathbb{Z}, (1 \ 1 \ 0), (1 \ 0 \ 0), \Sigma)$$

and set

$$v_1 = (1,0,0), v_2 = (1,1,0), v_3 = (0,1,0), v_4 = (0,0,1).$$

The lifting fan $\Delta$ is generated by $\text{Cone}(v_1, v_2, v_4), \text{Cone}(v_2, v_3, v_4)$. Using homogeneous coordinates (see 2.4), we have

$$X_\Delta = (A^4_C \setminus \mathbb{V}(t_1, t_3))/\mathbb{G}_\Delta,$$

where $\mathbb{G}_\Delta = \mathbb{C}^*$ acts on $A^4_C$ via

$$\mu \cdot (t_1, \ldots, t_4) = (\mu t_1, \mu^{-1} t_2, \mu t_3, t_4)$$

and the variables $t_1, \ldots, t_4$ correspond to $v_1, \ldots, v_4$. In order to describe $\pi_\Delta$, consider the matrix

$$\mathcal{D} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

whose the columns are the vectors $v_1, \ldots, v_4$. Then

$$\pi_\Delta : (A^4_C \setminus \mathbb{Z}(\Delta))/\mathbb{G}_\Delta \rightarrow A^3_C, \ [t_1 : \ldots : t_4]_\Delta \mapsto \theta(t_1 t_2, t_2 t_3, t_4),$$

where the monomials correspond to the lines of $\mathcal{D}$. Moreover, the quotient for the $\mathbb{C}^*$-action is

$$X_\Delta \rightarrow \mathbb{P}^1_{A^*_C} \times A^1_C, \ [t_1 : \ldots : t_4]_\Delta \mapsto ([t_1 : t_3], t_4),$$

and the proper transform of $X$ under $\pi_\Delta$ is $X_1 = \{[t_1 : \ldots : t_4]_\Delta \mid t_4 = \lambda\} \subset X_\Delta$. So $X_1 = \bar{X}$ and the contraction map $\pi = \pi_\Delta|X_1$ is the blowing-up of $X$ at $(0,0,\lambda)$.

Let us formalize what we saw in Example 4.27.

Theorem 4.28. Let $X \subset A^*_C$ be a subvariety with linear torus action of complexity one. Let $\theta = (\mathcal{N} = \mathbb{Z}^t, N, F, S, \Sigma)$ be the weight package of $X$ and write $\Delta$ for its lifting fan. Let $(v_1, \ldots, v_r)$ be the list of primitive vectors of $\mathcal{N}$ generating the rays of the fan $\Delta$. Consider the matrix $\mathcal{D} = (g_{i,j}) \in \text{Mat}_{t \times r}(\mathbb{Z})$ where the columns are the vectors $v_1, \ldots, v_r$ in the canonical basis of $\mathcal{N}$. Consider $X$ with presentation $X = \mathbb{V}(g_1 = u_1 f_1, \ldots, g_a = u_a f_a)$ as in Lemma 4.17. Let $w_v$ be a monomial such that

$$h_v := w_v \cdot g_v \left( \prod_{j=1}^r T_{j,1}^{q_{i,j}}, \ldots, \prod_{j=1}^r T_{j,1}^{q_{i,j}} \right)$$

is homogenous in $C[T_1, \ldots, T_r]$ for the $\text{Cl}(X_\Delta)$-grading, where $T_i$ corresponds to ray associated with $v_i$ and $1 \leq v \leq a$. Assume that the subvariety $X_1 \subset X_\Delta$ defined in homogeneous coordinates by

$$X_1 = \{[z_1 : \ldots : z_r]_\Delta \mid h_v(z_1, \ldots, z_r) = 0 \text{ for } v = 1, \ldots, a\}$$

is irreducible. Then the normalization of $X_1$ is $\mathbb{T}_N$-isomorphic to the contraction space $\bar{X}$ and encoded by a contraction divisorial fan (see Definition 2.15) of $\{\mathcal{D}_\theta\}$. 
Theorem 4.30. Let \( \theta = (N, F, S, \Sigma) \) be a weight package. Let \((\theta^{(0)}, \ldots, \theta^{(t)})\) be the sequence of weight packages from the enhanced structure \( \hat{\theta} \) of \( \theta \) (see Definition 4.16). Each \( \theta^{(i)} \) corresponds to a maximal cone \( \delta^{(i)} \) of the fan of the projective space \( \mathbb{P}^t \) with here \( \delta^{(0)} = \delta = \mathbb{Q}^t_{\geq 0} \). We denote by \( \Delta^{(i)} \) the smallest fan of \( \bar{N}_Q \) with support \( \delta^{(i)} \) such that the map
\[
\Delta^{(i)} \to \Sigma^{(i)}, \ \delta_0 \mapsto P(\delta_0)
\]
is a fan morphism, where \( P: \bar{N}_Q \to \text{Coker}(F)Q \) is induced by the matrix factorization of \( \theta \). The enhanced lifting fan \( \hat{\Delta} \) of \( \theta \) is the fan of \( \bar{N}_Q \) generated by \( \Delta^{(0)}, \ldots, \Delta^{(t)} \).

The following describes up to normalization of contraction spaces of projective varieties with linear torus action of complexity one.

Theorem 4.30. Let \( X \subset \mathbb{P}^d \) be a projective variety with linear torus action of complexity one and let \( \theta = (N, F, S, \Sigma) \) be a weight package of \( X \). Let \( \hat{\Delta} \) be the enhanced lifting fan of \( \theta \) and let \((v_1, \ldots, v_r)\) be the list of primitive vectors generating the rays of the fan \( \hat{\Delta} \). Denote by \( \mathcal{Q} = (q_{ij}) \in \text{Mat}_{r 	imes r}(\mathbb{Z}) \) the matrix where the \( i \)-th columns is \( v_i \). Assume that \( X = \mathbb{V}(g_1 = u_1f_1, \ldots, g_a = u_af_a) \) arises from the construction of Theorem 4.22. Let \( w_\nu \) be a monomial such that
\[
h_\nu := w_\nu \cdot g_\nu \left( 1, \prod_{j=1}^r T_j^{q_{1j}}, \ldots, \prod_{j=1}^r T_j^{q_{rj}} \right)
\]
is homogeneous in \( \mathbb{C}[T_1, \ldots, T_r] \) for the \( \text{Cl}(X_{\hat{\Delta}}) \)-grading, where \( T_i \) corresponds to the ray associated with \( v_i \) and for \( 1 \leq v \leq a \). Assume that the subvariety \( X_1 \subset X_{\hat{\Delta}} \) defined in homogeneous coordinates by
\[
X_1 = \{ [z_1 : \ldots : z_r]_\hat{\Delta} | h_\nu(z_1, \ldots, z_r) = 0 \text{ for } v = 1, \ldots, a \}
\]
is irreducible. Then the normalization of \( X_1 \) is \( \mathbb{T}_N \)-isomorphic to the contraction space \( \hat{X} \) and encoded by a contraction divisorial fan (see 2.15) of the divisorial fan \( \mathcal{E}_0 \) of 4.22.

Proof. Same proof as Theorem 4.28.

Example 4.31. Consider the cubic \( \mathbb{C}^* \)-surface \( X = \mathbb{V}(z_1z_2^2 + z_2z_3^2 + z_3^3) \subset \mathbb{P}_C^3 \) of Section 4.3. Let \((e_1, e_2, e_3)\) be the canonical basis of \( \mathbb{Q}^3 \), set \( e = -e_1 - e_2 - e_3, v_0 = (0, 3, 1), v_1 = (-3, 3, 1), v_2 = (3, -3, -1), v_3 = (3, -1, -1) \). The enhanced lifting fan is

| \( \Delta_{\text{max}}^{(0)} \) | Cone(e_1, e_2, v_0), Cone(e_1, e_3, v_0) |
| \( \Delta_{\text{max}}^{(1)} \) | Cone(e, e_2, v_1), Cone(e_2, e_3, v_1), Cone(e, e_3, v_1) |
| \( \Delta_{\text{max}}^{(2)} \) | Cone(e_1, e, v_2), Cone(e_1, e_3, v_2), Cone(e, e_3, v_2) |
| \( \Delta_{\text{max}}^{(3)} \) | Cone(e_1, e_2, v_3), Cone(e_2, e, v_3) |

Note that \( G_{\Delta} = (\mathbb{C}^*)^5 \) acts on homogeneous coordinate space \( \mathbb{A}_C^8 \) of \( X_{\Delta} \) via
\[
(\alpha, \beta, \gamma, \lambda, \mu) \cdot (t_1, \ldots, t_8) = (\alpha\gamma^3\lambda^3\mu^3-t_1, \alpha\beta^3\gamma^3\lambda^3\mu t_2, \alpha\beta^{-1}\gamma^{-1}\lambda\mu t_3, \alpha t_4, \beta t_5, \gamma t_6, \lambda t_7, \mu t_8)
\]
where the variables \( t_1, \ldots, t_8 \) correspond to \( e_1, e_2, e_3, e, v_0, \ldots, v_3 \). Finally, the contraction space up to normalization is the variety
\[
X_1 = \{ [x_1, \ldots, x_8]_\Delta | x_1x_2^2x_3^3x_4^4 + x_2x_3^2x_8^2 + x_3^3 = 0 \}.\]
5. Vanishing of odd dimensional intersection cohomology

In this section, we prove Theorem 5.5, which gives a precise form of the decomposition theorem for contraction maps of torus actions of complexity one.

5.1. Combinatorics of \( h \)-vectors. We start by recalling notations for intersection cohomology of toric varieties. Take any strictly convex polyhedral cone \( \sigma \subset N_\mathbb{Q} \). Let \( N_\mathbb{Q} \) be the subspace of \( N_\mathbb{Q} \) generated by \( \sigma \) and denote by \( M(\sigma)_\mathbb{Q} \) its dual, seen as quotient of \( M_\mathbb{Q} \). The dual cone \( \omega(\sigma) \) of \( \sigma \) in \( M(\sigma)_\mathbb{Q} \) is strictly convex. We define the polytope \( Q(\sigma) \subset M(\sigma)_\mathbb{Q} \) as intersection of \( \omega(\sigma) \) with an affine hyperplane of \( M(\sigma)_\mathbb{Q} \) cutting all one-dimensional faces of \( \omega(\sigma) \). Actually, \( Q(\sigma) \) is defined up to combinatorial equivalence, meaning that another choice of a hyperplane gives a polytope with same poset of faces than \( Q(\sigma) \) and same dimension function on each face. We denote by \( \Sigma^*(\sigma) \) the normal fan of \( N(\sigma)_\mathbb{Q} \) of the polytope \( Q(\sigma) \).

For a complete fan \( \Sigma \) and a strictly convex polyhedral cone \( \sigma \subset N_\mathbb{Q} \) we define the \( h \)-polynomial \( h(\Sigma; t^2) \) and the \( g \)-polynomial \( g(\sigma; t^2) \) as

\[
h(\Sigma; t^2) = P_{X_\Sigma}(t) = \sum_{j \in \mathbb{Z}} \dim IH^j(X_\Sigma; \mathbb{Q}) t^j;
\]

\[
g(\sigma; t^2) = \sum_{j \in \mathbb{Z}} \mathcal{H}^{j-n}(IC_{X_\sigma})_x t^j,
\]

where \( n = \dim N_\mathbb{Q} \) and \( x \in O(\sigma) \subset X_\sigma \). The following computes \( h(\Sigma; t^2) \) and \( g(\sigma; t^2) \) via a double induction.

**Theorem 5.1.** [Fie91, DL91, Section 6, BBFK02, Section 5] The polynomial \( g(\sigma; t^2) \) does not depend on \( x \in O(\sigma) \), and for \( \Sigma \) a complete fan we have

\[
h(\Sigma; t^2) = \sum_{\sigma \in \Sigma} (t^2 - 1)^{\dim \sigma} g(\sigma; t^2);
\]

\[
g(\sigma; t^2) = \begin{cases}
\tau_{\leq d-1}((1-t^2)h(\Sigma^*(\sigma); t^2)) & \text{if } d \geq 3 \\
1 & \text{if } d \leq 2,
\end{cases}
\]

where \( n = \dim N_\mathbb{Q}, d = \dim N(\sigma)_\mathbb{Q} \), and \( \tau_{\leq d-1} \) is the truncation of polynomials to degrees \( \leq d-1 \).

Theorem 5.1 implies that \( g(\sigma; t^2) \) and \( h(\Sigma; t^2) \) are even polynomials. We define the \( g \)-number \( g_j(\sigma) \) via \( g(\sigma; t^2) = \sum_{0 \leq j \leq 2n} g_j(\sigma) t^j \).

**Remark 5.2.** [Fie91, Section 1, Remark (iii)] Let \( \sigma \) be a full-dimensional strictly convex polyhedral cone and let \( \Sigma \) be a complete fan of \( N_\mathbb{Q} \). Here are some computations of \( h(\Sigma; t^2), g(\sigma; t^2) \) in small dimensions.

| \( \dim N_\mathbb{Q} \) | \( h(\Sigma; t^2) \) | \( g(\sigma; t^2) \) |
|----------------------|------------------|------------------|
| 2                    | \( 1 + (|\Sigma(1)| - 2) t^2 + t^4 \) | 1                |
| 3                    | \( 1 + (|\Sigma(1)| - 3) t^2 + (|\Sigma(1)| - 3) t^4 + t^6 \) | \( 1 + (|\Sigma(1)| - 3) t^2 \) |
| 4                    | \( \sum_{i=0}^3 \sum_{\sigma \in \Sigma(4)} (1 + (|\sigma(1)| - i) t^2) (t - 1)^{4-i} \) | \( 1 + (|\sigma(1)| - 4) t^2 \) |

Notions of \( h \)-polynomials extend for divisorial fans.

**Definition 5.3.** Let \( \sigma \subset N_\mathbb{Q} \) be a strictly convex polyhedral cone and let \( \Lambda \in \text{Pol}_\sigma(N_\mathbb{Q}) \). The Cayley cone \( \text{Cay}(\Lambda) \subset N_\mathbb{Q} \times \mathbb{Q} \) of \( \Lambda \) is the cone generated by \( (\sigma \times \{0\}) \cup (\Lambda \times \{1\}) \subset N_\mathbb{Q} \times \mathbb{Q} \).
Let $\mathcal{D}$ be a $\sigma$-polyhedral divisor over a smooth curve $C$. The $g$-polynomial $g_{\mathcal{D}}(t)$ of $\mathcal{D}$ is defined as follows. Let $\bar{C}$ be the smooth compactification of $C$ and let $\rho_g(\bar{C})$ be its genus. If $C = \bar{C}$, then we set
\[
g_{\mathcal{D}}(t) = (t^2 + 2\rho_g(\bar{C})t + 1 - a)g(\sigma; t^2) + \sum_{z \in \text{Supp}(\mathcal{D})} g(\text{Cay} (\mathcal{D}_z); t^2),
\]
where $a$ is the cardinality of $\text{Supp}(\mathcal{D}) := \{z \in C \mid \mathcal{D}_z = \sigma\}$. Otherwise we set
\[
g_{\mathcal{D}}(t) = ((2\rho_g(\bar{C}) + b - 1)t + 1 - a)g(\sigma; t^2) + \sum_{z \in \text{Supp}(\mathcal{D})} g(\text{Cay} (\mathcal{D}_z); t^2),
\]
where $b$ is the cardinality of $\bar{C} \setminus C$. Let $\mathcal{D}' = \{\bar{D}_i \mid i \in I\}$ be a contraction divisorial fan over $\bar{C}$ of a complete complexity-one $\mathbb{T}$-variety. For any $z \in \bar{C}$, we write $\Sigma_z(\mathcal{D}')$ for the complete fan generated
\[
\text{Cay} (\mathcal{D}_z') \quad \text{and} \quad \text{Cone}(\{(\sigma_i \times \{0\}) \cup (\sigma_i \times \{-1\})\}) \quad \text{for any} \ i \in I,
\]
where $\mathcal{D}_i' \in \mathcal{D}'$ is a $\sigma_i$-polyhedral divisor. The $h$-polynomial of $\mathcal{D}'$ is the polynomial
\[
h_{\mathcal{D}'}(t) = ((1 - c)t^2 + 2\rho_g(\bar{C})t + 1 - c)h(\Sigma(\mathcal{D}'); t^2) + \sum_{z \in \Sigma(\mathcal{D})} h(\Sigma_z(\mathcal{D}'); t^2),
\]
where $c$ is the cardinality of $\{z \in \bar{C} \mid \mathcal{D}_i' \neq \sigma \text{ for some } i \in I\}$ and $\Sigma(\mathcal{D}')$ is the fan generated by the cones $\sigma_i$, $i \in I$.

The following justifies the terminology of $h$-polynomials for divisorial fans. Note that calculation of the Poincaré polynomial of the contraction space of a complete complexity-one $\mathbb{T}$-variety can be obtained from results of [CMS08].

**Theorem 5.4.** [AL18, Theorem 1.1], [AL21, 5.1, 5.17] If $\mathcal{D}'$ is the contraction divisorial fan a complete complexity-one $\mathbb{T}$-variety, then the Poincaré polynomial $P_{X(\mathcal{D}')} (t)$ of $X(\mathcal{D}')$ is $h_{\mathcal{D}'} (t)$. Furthermore, if $\mathcal{D}$ is a proper $\sigma$-polyhedral divisor over a smooth curve and $\sigma$ is full-dimensional, then the Poincaré polynomial $P_{X(\mathcal{D})} (t)$ of the contraction space of $X(\mathcal{D})$ is $g_{\mathcal{D}} (t)$.

### 5.2. Topology of the contraction map

Our goal is to prove the following.

**Theorem 5.5.** Let $X$ be a normal variety with effective torus action of complexity one. Denote by $E$ the image of the exceptional locus of the contraction map $\pi : \tilde{X} \to X$, and let $\mathcal{O}_{\text{even}} (E)$ be the set of even codimensional torus orbits of $X$ contained in $E$. Then we have
\[
\pi_* \text{IC}_{\tilde{X}} \simeq \text{IC}_X \oplus \bigoplus_{O \in \mathcal{O}_{\text{even}} (E)} (\iota_O)_* \text{IC}^{\oplus s_O \sigma}_O,
\]
where $\iota_O : \tilde{O} \to X$ is the inclusion and $s_O \in \mathbb{Z}_{\geq 0}$ for any $O \in \mathcal{O}_{\text{even}} (E)$.

Let $X$ be as in Theorem 5.5. By Proposition 3.15 we have
\[
(1) \quad \pi_* \text{IC}_{\tilde{X}} \simeq \text{IC}_X \oplus \bigoplus_{O \in \mathcal{O}(E)} (\iota_O)_* \text{IC}^{\oplus s_{b,O}}_O [-b],
\]
where $\mathcal{O}(E)$ is the set of orbits of $E$ and $s_{b,O} \in \mathbb{Z}_{\geq 0}$. Set $S_O (t) = S_{X, O} (t) := \sum_{b \in \mathbb{Z}} s_{b,O} t^b \in \mathbb{Z}[t, t^{-1}]$. From Identity (1),
\[
P_X (t) = P_{\tilde{X}} (t) - \sum_{O \in \mathcal{O}(E)} \tilde{S}_O (t) P_O (t), \quad \text{where} \ \tilde{S}_O (t) = \tilde{S}_{X, O} (t) := S_O (t) t^{\dim X - \dim O}
\]
for any $O \in \mathcal{O}(E)$. Thus, we need prove that $\tilde{S}_O (t) = \lambda t^{\dim X - \dim O}$, where $\lambda \in \mathbb{Z}_{\geq 0}$ and $\lambda = 0$ provided that $\dim X - \dim O$ is odd.
Remark 5.6. [AL21, Sections 4.1, 4.2]. By Lemma 3.12 there exist, for any $O \in \mathcal{O}(E)$, a $T$-stable Zariski open subset $X_O \subset X$ containing $O$ as unique closed orbit and a fiber product decomposition $X_O \cong T_O \times^G X_1$, where $X_1 = X_{1,O}$ is a variety with complexity-one torus action having a unique attractive fixed point $x$ and $G = G_O \subset T$ is a finite subgroup. Let $\Gamma$ be the subgroup generated by the $G_O$ for $O \in \mathcal{O}(E)$. We will say that $X$ satisfies Condition $(\star)$ if all the $G_O$ can be chosen trivial. Note that Condition $(\star)$ is satisfied for $X/\Gamma$.

Lemma 5.7. With the notation of 5.6, we have $\tilde{S}_{X,O}(t) = \tilde{S}_{X/\Gamma,O/\Gamma}(t)$ for any $O \in \mathcal{O}(E)$.

Proof. Set

$$\gamma(X, O, O_2)(t) = \sum_{i \in \mathbb{Z}} \dim H^i(O_2, t_2^* IC_O)t^i,$$

where $O, O_2 \in \mathcal{O}(E)$ with $O \prec O_2$ and $\iota : O_2 \to X$ is the inclusion. The subset $X(O, O_2) := \bigcup_{O \prec O_1 \prec O_2} O_1 \subset \bar{O}$ is Zariski open and contains $O_2$ as unique closed orbit. Let $\delta : \hat{X}(O, O_2) \to X(O, O_2)$ be the normalization. Then the toric variety $\hat{X}(O, O_2)$ is affine and $\hat{O}_2 := \delta^{-1}(O_2)$ is its closed orbit. We have a Cartesian square

$$\begin{array}{cc}
\hat{O}_2 & \xrightarrow{\beta} & O_2 \\
\iota_{\hat{O}_2} & & \iota_{O_2} \\
\hat{X}(O, O_2) & \xrightarrow{\delta} & X(O, O_2),
\end{array}$$

where $\iota_{O_2}$ and $\iota_{\hat{O}_2}$ are inclusions. So by base change [Dim04, Theorem 2.3.26] and Lemma 2.6,

$$\beta_{\iota_{\hat{O}_2}}^* IC_{\hat{X}(O, O_2)} \simeq \iota_{O_2}^* \delta_* IC_{\hat{X}(O, O_2)} \simeq \iota_{O_2}^* IC_{\hat{X}(O, O_2)}.$$

Passing to hypercohomology gives

$$\gamma(X, O, O_2)(t) = \sum_{i \in \mathbb{Z}} \dim H^i(\hat{O}_2, t_{\hat{O}_2}^* IC_{\hat{X}(O, O_2)})t^i.$$

Note that $\hat{O}_2$ is the fixed point set of a non-hyperbolic $C^*$-action on $\hat{X}(O, O_2).$ Let $e \in \mathbb{Z}_{>0}$ such that the $C^*$-action on $\hat{X}(O, O_2)/\mu_e(\mathbb{C})$ is free outside $\hat{O}_2/\mu_e(\mathbb{C})$. Using Lemmata 2.8, 2.7 and [BM99, Theorems 10, 11],

$$P_{\hat{X}(O, O_2)}(t) = P_{\hat{X}(O, O_2)/\mu_e(\mathbb{C})}(t) = t^{\dim O\gamma(X/\mu_e(\mathbb{C}), O/\mu_e(\mathbb{C}), O_2/\mu_e(\mathbb{C}))(t)}.$$

Taking a lattice point in the relative interior of the cone $\sigma_{O_2} \subset N_\mathbb{Q}$, where $X_{O_2}$ is described by a $\sigma_{O_2}$-polyhedral divisor, defines a non-hyperbolic $C^*$-action on $X_{O_2}$ in which $O_2$ is the fixed point.

Finally, by restricting the decomposition theorem for the contraction maps of $X$ and $X/\Gamma$, to $O_2$ and $O_2/\Gamma$ gives

$$P_{\hat{X}_{O_2}}(t) - P_{X_{O_2}}(t) = \sum_{O \prec O_2} P_{\hat{X}(O, O_2)}(t)\tilde{S}_{X,O}(t)$$

and

$$P_{\hat{X}_{O_2}/\Gamma}(t) - P_{X_{O_2}/\Gamma}(t) = \sum_{O \prec O_2} P_{\hat{X}(O, O_2)/\Gamma}(t)\tilde{S}_{X/\Gamma,O/\Gamma}(t).$$

Finally, applying Lemma 2.7 for the $\Gamma$-actions on $X_{O_2}$, $\hat{X}_{O_2}$, $\hat{X}(O, O_2)$ gives

$$(2) \sum_{O \prec O_2} P_{\hat{X}(O, O_2)}(t)\tilde{S}_{X,O}(t) = \sum_{O \prec O_2} P_{\hat{X}(O, O_2)}(t)\tilde{S}_{X/\Gamma,O/\Gamma}(t).$$
We prove the lemma by induction on the codimension of $O_2 \in \mathcal{O}(E)$.

**Initial step.** Case $\dim X - \dim O_2 = 2$ follows from Identity (2) since $O \prec O_2$ implies $O = O_2$.

**Induction step.** Assume that this holds for codimension $d \geq 2$ and consider $O_2$ with $\dim X - \dim O_2 = d + 1$. By induction $\tilde{S}_{X,O}(t) = \tilde{S}_{X/T,O/\Gamma}(t)$ for any $O \in \mathcal{O}(E)$ such that $O \prec O_2$ and $O \neq O_2$. Therefore Identity (2) implies $\tilde{S}_{X,O_2}(t) = \tilde{S}_{X/T,O_2/\Gamma}(t)$, as required.

**Definition 5.8.** For all $O_1, O_2 \in \mathcal{O}(E)$ write $O_1 \prec O_2$ when $O_2 \subset \hat{O}_1$. Set

$$R_{O_1,O_2}(t) = \sum_{i \in \mathbb{Z}} \dim \mathcal{H}^i(\text{IC}_{O_1}) x^t t^i \in \mathbb{Z}[t,t^{-1}],$$

where $x^t \in O_2$ and $O_1 \prec O_2$. Note that $R_{O_1,O_2}(t)$ does not depend on the choice of $x \in O_2$ [AL21, Remark 5.21].

**Lemma 5.9.** Assume that $X$ satisfies Condition $(\ast)$ and take the notation of Remark 5.6. Then

$$P_{\bar{X}_{1,O_2}}(t) - P_{X,O_2}(t) = \sum_{O_1 \prec O_2} R_{O_1,O_2}(t) t^{\dim O_1} \tilde{S}_{O_1}(t),$$

where $\bar{X}_{1,O_2}$ is the contraction space of $X_{1,O_2}$.

**Proof.** Take the stalks at $x \in O_2$ on both sides of Equation (1), where on the left-hand side we use that $X_{O_2} \simeq \mathbb{T}_{O_2} \times X_{1,O_2}$ and $\bar{X}_{O_2} \simeq \mathbb{T}_{O_2} \times \bar{X}_{1,O}$ and Kunneth’s formula. \hfill \Box

**Proof of Theorem 5.5.** Changing $X$ by $X/\Gamma$, we may assume (see Lemma 5.7) that $X$ satisfies Condition $(\ast)$ of Remark 5.6. We show the result by induction on the dimension of $X$.

**Initial step.** The statement holds for surfaces [AL21, Example 5.3].

**Induction step.** Assume that this holds for dimension $\leq d_0$. Let $X$ be a normal complexity-one $\mathbb{T}$-variety of dimension $d = d_0 + 1$.

**Claim:** The polynomial $\tilde{S}_{O_0}(t)$ is of degree $\leq d$, where $O_0 = \{x\}$ and $x$ is a fixed point of $E$.

**Proof of the Claim.** For a subset $F \subset E$ let $\mathcal{O}(F)$ be the set of orbits of $F$ and let $E_x = E \cap X_{1,O_0}$. By Lemma 5.9,

$$P_{\bar{X}_{O_0}}(t) - P_{X,O_0}(t) = \tilde{S}_{O_0}(t) + R(t), \text{ where } R(t) := \sum_{O \in \mathcal{O}(E \setminus \{x\})} R_{O,O_0}(t) t^{\dim O} \tilde{S}_O(t).$$

First, observe that $P_{\bar{X}_{O_0}}(t) - P_{X,O_0}(t)$ is of degree $d$. Indeed, let $\tilde{D}_{O_0}$ be the $\sigma_{O_0}$-polyhedral divisor over a smooth projective curve $\tilde{C}$ describing the $\mathbb{T}$-variety $X_{O_0}$. By Theorem 5.4,

$$P_{\bar{X}_{O_0}}(t) = g_{\tilde{D}_{O_0}}(t) = (t^2 + 2\rho_2(\tilde{C})t + 1 - a) \cdot g(\sigma_{O_0}; t^2) + \sum_{z \in \text{Supp}(\tilde{D}_{O_0})} g(\text{Cay}(\tilde{D}_{O_0,z}); t^2),$$

where $a$ is the cardinality of $\text{Supp}(\tilde{D}_{O_0})$. By Theorem 5.1, the polynomials

$$g(\sigma_{O_0}; t^2) \text{ and } g(\text{Cay}(\tilde{D}_{O_0,z}); t^2) \text{ for } z \in \text{Supp}(\tilde{D}_{O_0})$$

are of degrees $\leq d - 2$ and $\leq d - 1$. Hence $P_{\bar{X}_{O_0}}(t)$ is of degree $\leq d$. Also, since $X_{O_0}$ has a unique attractive fixed point, $X_{O_0}$ is, after taking the quotient by a finite subgroup of $\mathbb{T}$, an affine cone over a projective variety $V$. So

$$P_{X,O_0}(t) = \tau_{d-1}((1 - t^2)P_V(t)),$$
where \( \tau_{\leq d-1} \) is the truncation to degrees \( \leq d-1 \) (see [Fie91, Lemma 2.1], [AL21, Proposition 5.6]), and \( P_{X_{O_0}}(t) - P_{X_{O_0}}(t) \) is of degree \( \leq d \).

Next, we estimate the degree of \( R(t) \). Let \( O \in \mathcal{O}(E_x \setminus \{x\}) \) and let \( \pi_O : \tilde{X}_O \to X_O \) be the contraction map. Note that, due to Condition \((\ast)\),

\[
X_O \simeq \mathbb{T}_O \times X_{1,O} \text{ and } \tilde{X}_O \simeq \mathbb{T}_O \times \tilde{X}_{1,O}.
\]

Let \( y \in X_{1,O} \) be the unique attractive fixed point and set \( O' = \{y\} \). As \( \pi_O \) is obtained from the product of \( \mathbb{T}_O \) and the contraction map of \( X_{1,O} \),

\[
\tilde{S}_{X,O}(t) = \tilde{S}_{X_{1,O},O'}(t).
\]

Now remark that \( \dim O \geq 1 \) implies \( \dim X_{1,O} < d \). Consequently, by induction, there exists \( \lambda_O \in \mathbb{Z}_{\geq 0} \) such that

\[
\tilde{S}_{X,O}(t) = \lambda_O t^{d-\dim O}
\]

with \( \lambda_O = 0 \) if \( d - \dim O \) is odd. In addition, using Theorem 5.1 and [AL21, Lemma 5.20], the degree of \( R_{O,O_0}(t)^{\dim O} \) is \( < \dim O \). Hence the degree of \( R(t) \) is \( < d \). Thus the degree of

\[
\tilde{S}_{O_0}(t) = P_{X_{O_0}}(t) - P_{X_{O_0}}(t) - R(t)
\]

is \( \leq d \). This proves the claim. \( \square \)

By induction,

\[
Q(t) := \sum_{O_0 \in \mathcal{O}(E), \dim(O_0) = 0} \tilde{S}_{O_0}(t) = P_{\tilde{X}}(t) - P_X(t) - \sum_{O \in \mathcal{O}(E \setminus X^\tau)} \tilde{S}_{X,O}(t)P_O(t)
\]

\[
= P_{\tilde{X}}(t) - P_X(t) - \sum_{O \in \mathcal{O}(E \setminus X^\tau)} \tilde{S}_{X_{1,O},O'}(t)P_O(t)
\]

\[
= P_{\tilde{X}}(t) - P_X(t) - \sum_{O \in \mathcal{O}(E \setminus X^\tau)} \lambda_O t^{d-\dim O}P_O(t)
\]

is Poincaré symmetric, i.e. \( Q(t) = t^{2d}Q(1/t) \). The claim implies that \( Q(t) \) is of degree \( \leq d \). So for any \( O_0 \in \mathcal{O}(E) \) with \( \dim O_0 = 0 \), there exists \( \lambda_{O_0} \in \mathbb{Z}_{\geq 0} \) such that \( \tilde{S}_{O_0}(t) = \lambda_{O_0} t^d \). Since from the proof of the claim the number \( \lambda_{O_0} \) is the \( (d - 2) \)-th coefficient of \( g(\sigma_{O_0}, t^2) \), we have \( \lambda_{O_0} = 0 \) when \( d \) is odd, proving the theorem. \( \square \)

5.3. Betti numbers via divisorial fans. We recall how to describe the image of the exceptional locus of the contraction map of torus actions of complexity one in terms of divisorial fans. Let \( \mathcal{E} = \{\mathcal{D}^i \mid i \in I\} \) be a divisorial fan over a smooth curve \( Y \), where \( \mathcal{D}^i \) is a \( \sigma_i \)-polyhedral divisor for \( i \in I \), and let \( \Sigma(\mathcal{E}) \) be the fan generated by the \( \sigma_i \). We call degree of \( \mathcal{E} \) the set

\[
\deg(\mathcal{E}) = \bigcup_{i \in I} \deg(\mathcal{D}^i) \subset \mathbb{N}_Q.
\]

Let \( HF(\mathcal{E}) = \{\tau \in \Sigma(\mathcal{E}) \mid \deg(\mathcal{E}) \cap \tau \neq \emptyset\} \) and let \( E \) be the image of the exceptional locus of the contraction map of \( X = X(\mathcal{E}) \). By [Tim97, Sections 3 and 4] we have a bijection

\[ HF(\mathcal{E}) \to \mathcal{O}(E), \quad \tau \mapsto O_\tau \]

between \( HF(\mathcal{E}) \) and the set of orbits of \( E \).

The correspondence \( \tau \mapsto O_\tau \) is seen as follows. Let \( \mathcal{D} \in \mathcal{E} \) be a \( \sigma \)-polyhedral divisor and let \( \tau \in HF(\mathcal{E}) \) such that \( \tau \cap \deg(\mathcal{D}) \neq \emptyset \). Then the vanishing of the ideal

\[
I_{\tau, \mathcal{D}} = \bigoplus_{m \in \sigma_i \cap M \setminus \tau} H^0(Y, \mathcal{O}_Y(\mathcal{D}(m)))\chi^m \subset A[Y, \mathcal{D}]
\]
is the orbit closure $O_\tau$ in $X(\mathfrak{D})$. In particular, the fan
\[
\text{Star}(\mathcal{E}, \tau) := \{ q_\tau(\gamma) \mid \gamma_0 \in HF(\mathcal{E}) \text{ and } \tau \text{ is a face of } \gamma_0 \}
\]
describes the normalization of the orbit closure $O_\tau$ in $X$, where $N(\tau) \subset N$ is the sublattice generated by $\tau \cap N$ and $q_\tau : N_\mathbb{Q} \to (N/N(\tau))_\mathbb{Q}$ is the canonical surjection.

Remark 5.10. By Lemma 3.12 each orbit $O \in \mathcal{O}(E)$ gives rise to an open set $X_O = \mathbb{T}_O \times^G X_1 \subset X$. Let $\tilde{\mathcal{D}}_O$ be the $\sigma_O$-polyhedral divisor describing $X_1 = X_{1,O}$ Then the map $O \mapsto \sigma_O$ is the inverse of $HF(\mathcal{E}) \to \mathcal{O}(E)$, $\tau \mapsto O_\tau$.

As a consequence of Theorem 5.5, one can give a precise formula of the intersection cohomology Betti numbers of any complete normal variety with torus action of complexity one.  

Corollary 5.11. Let $X = X(\mathcal{E})$ be a complete normal $\mathbb{T}$-variety of complexity one with defining divisorial fan $\mathcal{E}$. Denote by $\tilde{\mathcal{E}}$ the divisorial fan of the contraction space $\tilde{X}$ of $X$. For $\tau \in HF(\mathcal{E})$, set $n(\tau) := \dim \tau - 1$ and $c(\tau) := \dim \tau + 1$. Then the Poincaré polynomial of $X$ is given by the formula
\[
P_X(t) = h_{\tilde{E}}(t) - \sum_{\tau \in HF(\mathcal{E})} g_n(\tau) t^{c(\tau)} h(\text{Star}(\mathcal{E}, \tau); t^2).
\]

Proof. Regarding the proof of Theorem 5.5, we observe that $\tilde{S}_O(t) = \lambda_O t^{\dim X - \dim O}$, where $\lambda_O$ is leading coefficient of the polynomial $P_{X_{1,O}}(t) - P_{X_{1,O}}(t)$. This latter, is according to Theorem 5.1 and [AL21, Proposition 5.6], the number $g_{\dim \sigma_O - 1}(\sigma_O)$, whence the result. \qed

Another consequence is the affine case with a unique attractive fixed point. If $\mathcal{E}$ is the divisorial fan $\{ \mathfrak{D} \}$, then we will respectively write $HF(\mathfrak{D})$ and $\text{Star}(\mathfrak{D}, \tau)$ instead of $HF(\mathcal{E})$ and $\text{Star}(\mathcal{E}, \tau)$.

Corollary 5.12. Let $\mathfrak{D}$ be a proper $\sigma$-polyhedral divisor over a smooth projective curve. Assume that $X = X(\mathfrak{D})$ has a unique attractive fixed point, i.e. $\sigma \subset N_\mathbb{Q}$ is full-dimensional. For $\tau \in HF(\mathfrak{D})$, set $n(\tau) := \dim \tau - 1$ and $c(\tau) := \dim \tau + 1$. Then the Poincaré polynomial of $X$ is given by the formula
\[
P_X(t) = g_{\mathfrak{D}}(t) - \sum_{\tau \in HF(\mathfrak{D})} g_n(\tau) t^{c(\tau)} g(\text{Star}(\mathfrak{D}, \tau); t^2).
\]

Proof. Same proof as Corollary 5.11. \qed

Example 5.13. With the notation of Corollary 5.11, note that if $\Sigma(\mathcal{E})$ is simplicial, then
\[
\begin{cases}
g_n(\tau)(\tau) = 0 & \text{if } \dim(\tau) \neq 1 \\
g_n(\tau)(\tau) = 1 & \text{if } \dim(\tau) = 1
\end{cases}
\]
for any $\tau \in HF(\mathcal{E})$. By Theorem 5.5 we have
\[
\pi_* IC_{\tilde{X}} \simeq IC_X \oplus \bigoplus_{\mathcal{O} \in \mathcal{O}_2(E)} (\iota_\mathcal{O})_* IC_{\tilde{\mathcal{O}}},
\]
where $\mathcal{O}_2(E)$ is the set of codimension-two orbits of $X$ in $E$. In the case where $\tilde{X}$ is rationally smooth, the fan $\Sigma(\mathcal{E})$ is simplicial. Therefore, we recover [AL21, Theorem 1.1(iii)], which was proven via the decomposition theorem for semi-small maps.
5.4. Vanishing of odd dimensional intersection cohomology. Theorem 5.5 implies the following rationality criterion.

**Theorem 5.14.** Let $X$ be any complete variety with torus action of complexity one. Then the following are equivalent.

(i) $X$ is a rational variety.

(ii) We have $IH^{2j+1}(X; \mathbb{Q}) = 0$ for any $j \in \mathbb{Z}$.

**Proof.** By Theorem 5.5 we have

$$P_X(t) = P_{\bar{X}}(t) - \sum_{O \in O(E)} \tilde{S}_O(t)P_O(t)$$

with $\tilde{S}_O(t), P_O(t) \in \mathbb{Z}[t^2]$ for any $O \in O(E)$, and where $\bar{X}$ is the contraction space of $X$. Hence $P_X(t) \in \mathbb{Z}[t^2]$ if and only if $P_{\bar{X}}(t) \in \mathbb{Z}[t^2]$, and by Theorem 5.4, this is equivalent to that the genus of the smooth projective curve $\bar{C}$ such that $\mathbb{C}(X)^T \simeq \mathbb{C}(\bar{C})$ is 0. Since $X$ is birationally equivalent to $\bar{C} \times \mathbb{P}^n$ [Tim08, Section 1, Corollary 3], we conclude by Lüroth’s theorem. □

6. Computing intersection cohomology

We now relate the computation of intersection cohomology with linear torus action of complexity one from the defining equations and treat the case of trinomial hypersurfaces.

6.1. First consequences. We start with the affine case having a unique attractive fixed point.

**Corollary 6.1** (Consequence of Theorem 5.4, Theorem 4.9 and Corollary 5.12). Let $\theta = (N, N, F, S, \Sigma)$ be a weight package and let

$$X = \mathcal{V}(u_1f_1, \ldots, u_af_a) \subset \mathbb{A}^d_C$$

be the subvariety with torus action of complexity one arising from $\theta$ via Construction 4.17. Assume that $\sigma_\theta := S(F(N_\mathbb{Q}) \cap \delta)$ is full-dimensional, where $\delta = \mathbb{Q}_{\geq 0} \subset \overline{\mathbb{Q}} = \mathbb{Q}^f$. Then the following hold.

(i) If the associated curve $C_\theta$ of $\theta$ is projective, then the intersection cohomology Betti numbers of $X$ are described by the formula

$$P_X(t) = g_{\mathcal{D}_\theta}(t) - \sum_{\tau \in HF(\mathcal{D}_\theta)} g_{n(\tau)}(\tau) t^{c(\tau)} g(Star(\mathcal{D}_\theta, \tau); t^2),$$

where $n(\tau) = \dim \tau - 1$ and $c(\tau) = \dim \tau + 1$ for any $\tau \in HF(\mathcal{D}_\theta)$.

(ii) Assume that the associated curve $C_\theta$ is projective. Let

$$X_1 = \{[z_1 : \ldots : z_r]_\Delta \in X_\Delta \mid h_v(z_1, \ldots, z_r) = 0 \text{ for } v = 1, \ldots, a\}$$

be defined as in 4.28, where $\Delta$ is the lifting fan of $\theta$. Then the intersection cohomology Betti numbers of $X_1$ and of the contraction space $\bar{X}$ of $X$ is given by the formula

$$P_{\bar{X}}(t) = P_{X_1}(t) = g_{\mathcal{D}_\theta}(t).$$

(iii) Assume that $C_\theta$ is affine. Then $P_X(t) = g_{\mathcal{D}_\theta}(t)$.

Next we pass to the projective case.

**Corollary 6.2** (Consequence of Theorem 4.22 and Corollary 5.11). Let $\theta = (N, N, F, S, \Sigma)$ be a weight package and let

$$X = \mathcal{V}(u_1f_1, \ldots, u_af_a) \subset \mathbb{P}^d_C$$

be the subvariety with torus action of complexity one obtained from $\theta$ via Construction 4.22. Then the following hold.
(i) The intersection cohomology Betti numbers of $X$ are described by the formula

$$P_X(t) = h_{\mathcal{E}_\theta}(t) - \sum_{\tau \in HF(\mathcal{E}_\theta)} g_n(\tau) t^{c(\tau)} g(\mathcal{E}_\theta, \tau); t^2),$$

where $\mathcal{E}_\theta$ is the divisorial fan of $M$, $n(\tau) = \dim \tau - 1$ and $c(\tau) = \dim \tau + 1$ for any $\tau \in HF(\mathcal{E}_\theta)$.

(ii) Let $X_1$ be the subvariety

$$\{(z_1 : \ldots : z_r) \in X_\Delta \mid h_v(z_1, \ldots, z_r) = 0 \text{ for } v = 1, \ldots, a \}$$

arising in Construction 4.30, where $\Delta$ is the enhanced lifting fan of $\theta$. Then the intersection cohomology Betti numbers of $X_1$ and of contraction space $\bar{X}$ of $X$ are given by the formula

$$P_\bar{X}(t) = P_{X_1}(t) = h_{\mathcal{E}_\theta}(t).$$

6.2. Intersection cohomology of affine trinomial hypersurfaces. By affine trinomial hypersurface we mean a hypersurface

$$X = \mathbb{V}(T_{1}^{p_1} + T_{2}^{p_2} + T_{3}^{p_3}) \subset \mathbb{A}_{\mathbb{C}}^3$$

such that $T_{i}^{p_i} = \prod_{j=1}^{r_i} T_{i,j}^{n_{i,j}}$ for $i = 1, 2, 3$, where $r_i, n_{i,j} \in \mathbb{Z}_{>0}$ and $\ell = r_1 + r_2 + r_3$. Here the variables $T_{i,j}$ are independent. Note that trinomial hypersurfaces have singularity loci of codimension $\geq 2$ by Jacobian criterion. Hence they are normal. Set

$$R = \begin{pmatrix} -n_{1,1} & \ldots & -n_{1,r_1} & 0 & \ldots & 0 \\ -n_{1,1} & \ldots & -n_{1,r_1} & 0 & \ldots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots \\ 0 & \ldots & 0 & \ldots & 0 & 0 \\ n_{3,1} & \ldots & n_{3,r_3} & 0 & \ldots & 0 \end{pmatrix}.$$  

Write

$$u_i := \gcd(n_{1,1}, \ldots, n_{i,r_i}) \text{ for } i = 1, 2, 3, \quad d = \gcd(u_1, u_2, u_3),$$

$$d_1 = \gcd(u_2/d, u_3/d), \quad d_2 = \gcd(u_1/d, u_3/d) \quad \text{and} \quad d_3 = \gcd(u_1/d, u_2/d).$$

The matrix $R$ induces an exact sequence

$$0 \to N^F \to \overline{N} \to \mathbb{Z}^\ell \xrightarrow{R} \text{Coker}(F) \to 0.$$ 

Here we identify $\text{Coker}(F)$ with the image of $R$, that is the sublattice of $\mathbb{Z}^2$ generated by $(-u_1, -u_1), (u_2, 0), (0, u_3)$. Let $\Sigma$ be the fan generated by the cones $R(\delta_0)$, where $\delta_0$ runs over the faces of the first quadrant $\delta = \mathbb{Q}_{\geq 0}^\ell$. Then (see [Kru19, Section 4]) $X_\Sigma$ is the weighted projective plane $\mathbb{P}(d_1, d_2, d_3)$. Let $w_1, w_2, w_3$ be the homogeneous coordinates of $\mathbb{P}(d_1, d_2, d_3)$ and consider the smooth curve

$$C_{d_1,d_2,d_3} := \mathbb{V}(w_1^{u/d_1} + w_2^{u/d_2} + w_3^{u/d_3}) \subset \mathbb{P}(d_1, d_2, d_3),$$

where $u = dd_1d_2d_3$. Furthermore, if $S : \overline{N} \to N$ is a section of $F$, then the weight package $\theta = (\overline{N}, N, F, S, \Sigma)$ defines a linear torus action of complexity one on $X \subset \mathbb{A}_{\mathbb{C}}^3$. Kruglov proved the following.

**Theorem 6.3.** [Kru19, Theorem 3.1] Let

$$X = \mathbb{V}(T_{1}^{p_1} + T_{2}^{p_2} + T_{3}^{p_3}) \subset \mathbb{A}_{\mathbb{C}}^3$$

be an affine trinomial hypersurface with its natural weight package $\theta = (\overline{N}, N, F, S, \Sigma)$. Then $\overline{\mathcal{D}}_\theta$ is defined over the curve $C_{d_1,d_2,d_3} \subset \mathbb{P}(d_1, d_2, d_3) = X_\Sigma$ and given by the relation

$$\overline{\mathcal{D}}_\theta = \overline{\mathcal{D}}_{\theta,1} \cdot E_1 + \overline{\mathcal{D}}_{\theta,2} \cdot E_2 + \overline{\mathcal{D}}_{\theta,3} \cdot E_3,$$

where:
(i) The polyhedron $\tilde{\mathcal{D}}_{\theta,i}$ for $1 \leq i \leq 3$ is the Minkowski sum of $\sigma_\theta := S(\mathbb{Q}_{\geq 0}^3 \cap F(N_\mathbb{Q}))$ and the convex hull of the set
\[
\left\{ S \left( \frac{d}{d_j n_{i,j}} e_{i,j} \right) \mid 1 \leq j \leq r_i \right\}.
\]
Here
\[
(e_{1,1}, \ldots, e_{1,r_1}, e_{2,1}, \ldots, e_{2,r_2}, e_{3,1}, \ldots, e_{3,r_3})
\]
is the canonical basis of $\overline{N} = \mathbb{Z}^3$.
(ii) The divisors $E_1, E_2, E_3$ are defined by
\[
E_i := \tau^*(H_i) = \sum_{z \in C_{d_1,d_2,d_3}} (H_i, C_{d_1,d_2,d_3})_z \cdot [z],
\]
where $H_i = \mathbb{V}(w_i) \subset \mathbb{P}(d_1, d_2, d_3)$, $\iota : C_{d_1,d_2,d_3} \to \mathbb{P}(d_1, d_2, d_3)$ is the inclusion, and $(H_i, C_{d_1,d_2,d_3})_z$ is the local intersection number between $H_i$ and $C_{d_1,d_2,d_3}$ at $z$.

**Remark 6.4.** [Kru19, Remark 4.1] Let $\zeta \in \mu_{2d_3}(\mathbb{C})$ be a primitive root such that $\zeta^a = -1$. Then
\[
E_1 = \sum_{i=0}^{d_3-1} [\varsigma([0 : 1 : \zeta i])],
\]
$E_2 = \sum_{i=0}^{d_2-1} [\varsigma([1 : 0 : \zeta i])],
\]
$E_3 = \sum_{i=0}^{d_1-1} [\varsigma([1 : 1 : 0])],
\]
where $\eta_j \in \mu_{dd_j}(\mathbb{C})$ is a primitive root for $j = 1, 2, 3$ and $\varsigma : \mathbb{P}_C^3 \to \mathbb{P}(d_1, d_2, d_3)$ is the quotient.

**Remark 6.5.** [Kru19, Remark 4.2] The genus of the curve $C_{d_1,d_2,d_3}$ is
\[
\frac{d}{2} (u - (d_1 + d_2 + d_3)) + 1.
\]

The following corollary describes the intersection cohomology of affine trinomial hypersurfaces. We formulate the result in a way that we can directly compute from the defining equation.

**Corollary 6.6.** Let $X = \mathbb{V}(T_1^{d_1} + T_2^{d_2} + T_3^{d_3}) \subset \mathbb{A}_C^3$ be an affine trinomial hypersurface with its natural weight package $\theta = (\overline{N}, N, F, S, \Sigma)$. Set $\sigma_\theta = S(\mathbb{Q}_{\geq 0}^3 \cap F(N_\mathbb{Q}))$, $\gamma = d(d_1 + d_2 + d_3)$ and
\[
\Pi_i := \text{Cone} \left( (\sigma_\theta \times \{0\}) \cup \left\{ S \left( \frac{d}{d_j n_{i,j}} e_{i,j} \right) \mid 1 \leq j \leq r_i \right\} \times \{1\} \right)
\]
for $i = 1, 2, 3$. Then the intersection cohomology Betti numbers of the contraction space $\tilde{X}$ of $X$ is given by the formula
\[
P_{\tilde{X}}(t) = (t^2 + (du - \gamma + 2)t - \gamma + 1) \cdot g(\sigma_\theta; t^2) + \sum_{i=1}^{3} dd_i \cdot g(\Pi_i; t^2).
\]
Write
\[
H(\theta, n_1, n_2, n_3) := \left\{ \tau \text{ face of } \sigma_\theta \mid \tau \cap \left\{ \sum_{i=1}^{3} S \left( \frac{d}{d_j n_{i,j}} e_{i,j} \right) \mid (j_1, j_2, j_3) \in \prod_{i=1}^{3} \{1, \ldots, r_i\} \right\} \neq \emptyset \right\}
\]
Then the intersection cohomology Betti numbers of $X$ are given by the formula
\[
P_X(t) = P_{\tilde{X}}(t) - \sum_{\tau \in H(\theta, n_1, n_2, n_3)} g_n(\tau) t^{c(\tau)},
\]
where $n(\tau) = \dim \tau - 1$ and $c(\tau) = \dim \tau + 1$. 
Proof. Formulae for $P_X(t)$ and $P_X(t)$ are consequences of Corollary 6.1 and of Theorem 6.3 by observing that orbit closures of the image of the exceptional locus of the contraction map are affine spaces.

Example 6.7. Consider the affine trinomial hypersurface

$$X = \mathcal{V}(T_{1,1}T_{1,2} + T_{2,1}T_{2,2} + T_{3,1}^2) \subset \mathbb{A}^5_C$$

with weight matrix and section

$$F = \begin{pmatrix}
-1 & 1 & 0 & 0 & 0 \\
2 & 0 & 2 & 0 & 1 \\
0 & 0 & -1 & 1 & 0
\end{pmatrix} \text{ and } S = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0
\end{pmatrix}.$$ 

Set $v_1 = (2, 1, 0), v_2 = (2, 1, 2), v_3 = (0, 1, 2), v_4 = (0, 1, 0), b_0 = (0, 0, 0, 1), b_1 = (1, 0, 0, 1), b_2 = (0, 0, 1, 1), b_3 = (0, 1/2, 0, 1).$ The table

| Generators | $\sigma_\theta$ | $v_1, v_2, v_3, v_4$ | $\Pi_1$ | $\sigma_\theta \times \{0\}, b_0, b_1$ | $\Pi_2$ | $\sigma_\theta \times \{0\}, b_0, b_2$ | $\Pi_3$ | $\sigma_\theta \times \{0\}, b_3$ |
|------------|------------------|-----------------------|---------|------------------------------------|---------|------------------------------------|---------|------------------------------------|

implies that $g(\sigma_\theta; t^2) = g(\Pi_3; t^2) = 1 + t^2,$ and $g(\Pi_1; t^2) = 2g(\Pi_2; t^2) = 1 + 2t^2.$ So

$$P_X(t) = (t^2 - 2)g(\sigma_\theta; t^2) + \sum_{i=1}^{3} g(\Pi_i; t^2) = t^4 + 4t^2 + 1.$$ 

Moreover, $H(\theta, \{\Pi_1, \Pi_2, \Pi_3\}) = \{\text{faces of } \sigma_\theta\}.$ Denote by $\rho_1, \ldots, \rho_4$ the rays of $\sigma_\theta.$ Then $c(\rho_i) = 2, c(\sigma_\theta) = 4,$ $g_n(\sigma_\theta)(\rho_i) = g_n(\sigma_\theta)(\sigma_\theta) = 1$ and from Corollary 6.6,

$$P_X(t) = P_X(t) - g_n(\sigma_\theta)t^{c(\sigma_\theta)} - \sum_{i=1}^{4} g_n(\rho_i)(\rho_i)t^{c(\rho_i)} = 1,$$

which is expected since $X$ is an affine cone over a smooth quadric.

6.3. Intersection cohomology of relevant projective trinomial hypersurfaces. By projective trinomial hypersurface we mean a hypersurface

$$X = \mathcal{V}(T_1^{2n_1} + T_2^{2n_2} + T_3^{2n_3}) \subset \mathbb{P}^\ell_C$$

such that $T_i^{n_{i,j}} = \prod_{1 \leq j \leq r_i} T_{i,j}^{n_{i,j}}$ for $i = 1, 2, 3,$

where $n_{i,j}, r_i \in \mathbb{Z}_{>0}$ and $\ell + 1 = r_1 + r_2 + r_3.$ The defining equation of $X$ must be homogeneous, so $s = \sum_{1 \leq j \leq r_i} n_{i,j}$ does not depend on $i.$ We introduce the following technical condition.

Definition 6.8. The projective trinomial hypersurface $X$ is relevant if each monomial $T_i^{2n}$ has at least two distinct variables.

From now on we assume $X$ relevant\(^2\) Let

$$X^{(i,j)} = \mathcal{V}((T_1^{2n_1} + T_2^{2n_2} + T_3^{2n_3})_{|T_{i,j}=1}) \subset \mathbb{A}^\ell_C$$

be the localization with respect to $T_{i,j},$ and let $\theta = (N, F, S, \Sigma)$ be the weight package of $X^{(i,j)}$ as in 6.2. We denote by $\mathbb{\hat{\theta}} = (\theta^{(i,j)})_{1 \leq i \leq 3, 1 \leq j \leq r_i}$ the sequence of weight packages coming

\(^2\)The case of irrelevant projective trinomial hypersurfaces will be carried out in the article [ABL].
from the enhanced structure of \( \theta \). Concretely, the \( P \)-matrix of \( \tilde{\theta} \) corresponds to the linear map \( \mathbb{Z}^{\ell+1} \to \tilde{R}(\mathbb{Z}^{\ell+1}) \), \( w \mapsto \tilde{R}(w) \), where

\[
\tilde{R} = \left( \begin{array}{cccccc}
-n_{1,1} & \cdots & -n_{1,r_1} & n_{2,1} & \cdots & n_{2,r_2} \\
-n_{1,1} & \cdots & -n_{1,r_1} & 0 & \cdots & 0
\end{array} \right).
\]

The \( P \)-matrix \( P^{(i,j)} \) of \( \theta^{(i,j)} \) is obtained from \( \tilde{R} \) by omitting the column corresponding to the variable \( T_{i,j} \). Write

\[
\theta^{(i,j)} = (N, N, F^{(i,j)}, S^{(i,j)}, \Sigma^{(i,j)}).
\]

Since the sum of the components of each line of \( \tilde{R} \) is \( 0 \), the subspaces \( \text{Coker}(F^{(i,j)}) \subset \mathbb{Z}^{\ell} \) and the fan \( \Sigma^{(i,j)} \) does not depend on \((i, j)\). We will consider the notations \( d, d_1, d_2, d_3, u \) defined in Section 6.2 for \( \theta \) so that \( X_{\Sigma^{(i,j)}} = \mathbb{P}(d_1, d_2, d_3) \) for any \((i, j)\).

The next result is a direct consequence of Theorem 6.3.

**Corollary 6.9.** Let

\[
X = \mathbb{V}(T_1^{d_1} + T_2^{d_2} + T_3^{d_3}) \subset \mathbb{P}^\ell
\]

be a relevant projective trinomial hypersurface with weight package \( \theta \), and denote by \( \underline{\theta} = (\theta^{(i,j)}) \) the sequence from the enhanced structure of \( \theta \). Then the normal \( \mathbb{T}_N \)-variety \( X \) is described by the divisorial fan \( \mathcal{D}_\theta = \{ \mathcal{D}_{g(a,b)} \} \), where each element \( \mathcal{D}_{g(a,b)} \) is defined over the curve \( C_{d_1,d_2,d_3} \) and is given by the relation

\[
\mathcal{D}_{g(a,b)} = \mathcal{D}_{g(a,b)_1} \cdot E_1 + \mathcal{D}_{g(a,b)_2} \cdot E_2 + \mathcal{D}_{g(a,b)_3} \cdot E_3.
\]

The polyhedral divisor \( \mathcal{D}_{g(a,b)} \) has the following properties.

(i) The polyhedron \( \mathcal{D}_{g(a,b),i} \) for \( 1 \leq i \leq 3 \) is the Minkowski sum of \( \sigma_{g(a,b)} = S^{(a,b)}(Q_\geq 0 \cap F^{(a,b)}(N_\mathbb{Q})) \) and the convex hull of the set

\[
\Gamma_i^{(a,b)} := \left\{ S^{(a,b)} \left( \frac{d}{d_i n_{i,j}} e_{i,j} \right) \mid (i, j) \neq (a, b), 1 \leq j \leq r_i \right\}.
\]

Here

\[
(e_{1,1}, \ldots, e_{1,r_1}, e_{2,1}, \ldots, e_{(a,b)}, \ldots, e_{3,r_3})
\]

is the canonical basis of \( N = \mathbb{Z}^\ell \) (the symbol \( e_{(a,b)} \) means that we omit the index \( (a, b) \)).

(ii) The divisors \( E_1, E_2, E_3 \) satisfy Condition (ii) of Theorem 6.3.

Next gives a formula for the intersection cohomology of relevant projective trinomial hypersurfaces. We refer to [BC94, BB96] for another description using Hodge theory.

**Corollary 6.10.** Let

\[
X = \mathbb{V}(T_1^{d_1} + T_2^{d_2} + T_3^{d_3}) \subset \mathbb{P}^\ell
\]

be a relevant projective trinomial hypersurface with weight package \( \theta \), and denote by \( \underline{\theta} = (\theta^{(i,j)}) \) the sequence from the enhanced structure of \( \theta \). Set

\[
\sigma_{g(a,b)} = S^{(a,b)}(Q_\geq 0 \cap F^{(a,b)}(N_\mathbb{Q}))
\]

\[
\Pi_i^{(a,b)} := \text{Cone} \left( (\sigma_{g(a,b)} \times \{0\}) \cup (\Gamma_i^{(a,b)} \times \{1\}) \right) \quad \text{for } i = 1, 2, 3,
\]

\[
\Pi^{(a,b)} = \text{Cone} \left( (\sigma_{g(a,b)} \times \{0\}) \cup (\sigma_{g(a,b)} \times \{-1\}) \right),
\]

where the set \( \Gamma_i^{(a,b)} \) is defined in Corollary 6.9. Write \( \gamma = d(d_1 + d_2 + d_3) \) and let \( \Sigma_i(\theta) \) be the complete fan generated by

\[
\left\{ \Pi_i^{(a,b)}, \Pi^{(a,b)} \mid 1 \leq a \leq 3, 1 \leq b \leq r_a \right\}.
\]
Then the intersection cohomology Betti numbers of the contraction space $\tilde{X}$ of $X$ is given by

$$P_{\tilde{X}}(t) = ((1 - \gamma)t^2 + (du - \gamma + 2)t + 1 - \gamma) \cdot h(\Sigma(\theta); t^2) + \sum_{i=1}^{3} dd_i \cdot h(\Sigma_i(\theta); t^2),$$

where $\Sigma(\theta)$ is the fan generated by the cones $\sigma_{g(a,b)}$.

Furthermore, write

$$H(\theta, u_1, u_2, u_3) = \bigcup_{1 \leq a \leq 3, 1 \leq b \leq r_a} H(\theta^{(a,b)}, u_1, u_2, u_3).$$

Then the intersection cohomology Betti numbers of $X$ are given by the formula

$$P_X(t) = P_{\tilde{X}}(t) - \sum_{\tau \in H(\theta, u_1, u_2, u_3)} g(\tau)(\tau) \cdot P_{\tilde{X}^{n-c(\tau)}}(t),$$

where $n = \ell - 2$, $n(\tau) = \dim \tau - 1$ and $c(\tau) = \dim \tau + 1$.

**Proof.** Consequence of Corollary 6.2, Corollary 6.9 and the fact that orbit closures of the image of the exceptional locus of the contraction map of $X$ are projective spaces. \qed

**Example 6.11.** Consider the projective trinomial fourfold hypersurface

$$\mathcal{V}(T_{1,1}^3 T_{1,1}^9 + T_{2,1}^6 T_{2,2}^6 + T_{3,1}^3 T_{3,2}^9) \subset \mathbb{P}^5$$

with weight package $\theta = (Z^5, Z^3, F, S, \Sigma)$ such that

$$F = t \begin{pmatrix} 0 & 1 & -1 & 0 & 0 \\ 2 & 0 & 3 & 6 & 0 \\ 0 & 0 & 0 & -3 & 1 \end{pmatrix} \quad \text{and} \quad S = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Set

$$v_1 = (3, 1, 0), \quad v_2 = (0, 1, 0), \quad v_3 = (3, 1, 2), \quad v_4 = (0, 1, 2),$$

$$w_1 = (-3, -3, -4), \quad w_2 = (-1, -1, -2), \quad w_3 = (-6, -3, -4), \quad w_4 = (-2, -1, -2).$$

Then maximal cones of $\Sigma(\theta)$ are

| Cone | Generators |
|------|------------|
| $\sigma_{g(1,1)}$ | $v_1, v_2, v_3, v_4$ |
| $\sigma_{g(1,2)}$ | $w_1, w_2, w_3, w_4$ |
| $\sigma_{g(2,1)}$ | $v_2, v_4, w_3, w_4$ |
| $\sigma_{g(2,2)}$ | $v_1, v_3, w_1, w_2$ |
| $\sigma_{g(3,1)}$ | $v_3, v_4, w_1, w_3$ |
| $\sigma_{g(3,2)}$ | $v_1, v_2, w_2, w_4$ |

and thus $h(\Sigma(\theta); t^2) = 1 + 5t^2 + 5t^4 + t^6$. Moreover, below we get the $f$-vector of $\Sigma_4(\theta)$

| $\sigma \in \Sigma_4(\theta)$ | Number of cones |
|-----------------|----------------|
| $\dim(\sigma) = 1$ | 11 |
| $\dim(\sigma) = 2$ | 29 |
| $\dim(\sigma) = 3, \ | 20 |
| $\sigma(1)| = 3$ |
| $\dim(\sigma) = 3, \ | 10 |
| $\sigma(1)| = 4$ |
| $\dim(\sigma) = 4, \ | 8 |
| $\sigma(1)| = 5$ |
| $\dim(\sigma) = 4, \ | 4 |
| $\sigma(1)| = 6$ |

and by Remark 5.2, $h(\Sigma_i(\theta); t^2) = t^8 + 7t^6 + 12t^4 + 7t^2 + 1$ for $i = 1, 2, 3$. Note that in the above table, only cases $\dim(\sigma) = 1, 2, 4$ need to be computed, since the rest follows from the vanishing of the Euler characteristic of the polytope of $\Sigma_i(\theta)$ and Poincaré duality. We conclude that

$$P_{\tilde{X}}(t) = t^8 + t^7 + 15t^6 + 5t^5 + 28t^4 + 5t^3 + 15t^2 + t + 1$$

and
\[ P_X(t) = t^8 + t^7 + 7t^6 + 5t^5 + 14t^4 + 5t^3 + 7t^2 + t + 1. \]

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