An infinite stratum of representability; some cylindric algebras are more representable than others

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Abstract. Let $2 < n < m \leq \omega$. Let $\mathsf{CA}_n$ denote the class of cylindric algebras of dimension $n$ and $\mathsf{RCA}_n$ denote the class of representable $\mathsf{CA}_n$s. We say that $\mathfrak{A} \in \mathsf{RCA}_n$ is representable up to $m$ if $\mathfrak{CmAt}\mathfrak{A}$ has an $m$-square representation. An $m$-square representation is locally relativized representation that is classical locally only on so called $m$-squares. Roughly if we zoom in by a movable window to an $m$-square representation, there will become a point determined and depending on $m$ where we mistake the $m$-square-representation for a genuine classical one. When we zoom out the non-representable part gets more exposed. For $2 < n < m < l \leq \omega$, an $l$-square representation is $m$-square; the converse however is not true. The variety $\mathsf{RCA}_n$ is a limiting case coinciding with $\mathsf{CA}_n$s having $\omega$-square representations. Let $\mathsf{RCA}_m^n$ be the class of algebras representable up to $m$. We show that $\mathsf{RCA}_m^{n+1} \subset \mathsf{RCA}_m^n$ for $m \geq n + 2$.

1 Introduction

Fix finite $n > 2$. Let $\mathsf{CRCA}_n$ denote the class of completely representable $\mathsf{CA}_n$s and $\mathsf{LCA}_n = \mathsf{EICRCA}_n$ be the class of algebras satisfying the Lyndon conditions. For a class $\mathcal{K}$ of Boolean algebras with operators, let $\mathcal{K} \cap \mathsf{At}$ denote the class of atomic algebras in $\mathcal{K}$. By modifying the games coding the Lyndon conditions allowing $\forall$ to reuse the pebble pairs on the board, we will show that $\mathsf{LCA}_n = \mathsf{EICRCA}_n = \mathsf{Els}_n \mathsf{NR}_n \mathsf{CA}_n \cap \mathsf{At}$. Define an $A \in \mathsf{CA}_n$ to be strongly representable $\iff$ $\mathfrak{A}$ is atomic and the complex algebra of its atom structure, equivalently its Dedekind-MacNeille completion, in symbols $\mathfrak{CmAt}\mathfrak{A}$ and $\mathsf{RCA}_n$ is a variety, a fortiori closed under forming subalgebras.

We denote the class of strongly representable atomic algebras of dimension $n$ by $\mathsf{SRCA}_n$. Nevertheless, there are atomic simple countable algebras that are representable, but not strongly representable. In fact, we shall see that there is a countable simple atomic algebra in $\mathsf{RCA}_n$ such that $\mathfrak{CmAt}\mathfrak{A} \not\subseteq \mathsf{SNr}_n \mathsf{CA}_{n+3}(\cup \mathsf{RCA}_n)$. So in a way some algebras are more representable than others. In fact, the following inclusions are known to hold:

$$\mathsf{CRCA}_n \subset \mathsf{LCA}_n \subset \mathsf{SRCA}_n \subset \mathsf{RCA}_n \cap \mathsf{At}.$$ 

In this paper we delve into a new notion, that of degrees of representability. Not all algebras are representable in the same way or strength. If $\mathfrak{C} \subset \mathfrak{D}$, with $\mathfrak{D} \in \mathsf{CA}_m$ for some ordinal (possibly infinite) $m$, we say that $\mathfrak{D}$ is an $m$-dilation of $\mathfrak{C}$ or simply a dilation if $m$ is clear from context. Using this jargon of ‘dilating algebras’ we say that $\mathfrak{A} \in \mathsf{RCA}_n$ is strongly representable up to $m > n \iff \mathfrak{CmAt}\mathfrak{A}$ admits an $m$-dilation

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equivalently $\text{cmAt}\mathfrak{A} \in \text{SNr}_{n}\text{CA}_{m}$. This means that, though $\mathfrak{A}$ itself is in $\text{RCA}_{n}$, the Dedekind-MacNeille completion of $\mathfrak{A}$ is not representable, but nevertheless it has some neat embedding property; it is ‘close’ to being representable. The bigger the dimension of the dilation of the representable algebra, the more representable the algebra is, the closer it is to being strongly representable. The representability of an atomic algebra does not force its Dedekind-MacNeille completion to be representable too, if it does then this algebra is strongly representable. A compelling question in this context is that if we let $(K_m : 2 < n < m \leq \omega)$ be the sequence whose $m$th entry $K_m$ is the class of algebras that are strongly representable up to $m$, it is obvious that this is a decreasing sequence, but is it strictly decreasing? In other words, are there $2 < n < l < j \leq \omega$ such that $K_l = K_j$? This question is far from being trivial, and will be answered below. Through the unfolding of this paper, we will investigate and make precise the notion of an algebra being more representable than another.

2 Preliminaries

We follow the notation of [2] which is in conformity with the notation in the monograph [3].

Definition 2.1. Assume that $\alpha < \beta$ are ordinals and that $\mathfrak{B} \in \text{CA}_\beta$. Then the $\alpha$–$\text{neat}$ $\text{reduct}$ of $\mathfrak{B}$, in symbols $\mathfrak{Nr}_\alpha \mathfrak{B}$, is the algebra obtained from $\mathfrak{B}$, by discarding cylindrifiers and diagonal elements whose indices are in $\beta \setminus \alpha$, and restricting the universe to the set $\mathfrak{Nr}_\alpha \mathfrak{B} = \{ x \in \mathfrak{B} : \{ i \in \beta : c_i x \neq x \} \subseteq \alpha \}$.

It is straightforward to check that $\mathfrak{Nr}_\alpha \mathfrak{B} \subseteq \mathfrak{CA}_\alpha$. Let $\alpha < \beta$ be ordinals. If $\mathfrak{A} \subseteq \mathfrak{CA}_\alpha$ and $\mathfrak{A} \subseteq \mathfrak{Nr}_\alpha \mathfrak{B}$, with $\mathfrak{B} \in \mathfrak{CA}_\beta$, then we say that $\mathfrak{A}$ neatly embeds in $\mathfrak{B}$, and that $\mathfrak{B}$ is a $\beta$–$\text{dilation}$ of $\mathfrak{A}$, or simply a dilation of $\mathfrak{A}$ if $\beta$ is clear from context. For $\mathfrak{K} \subseteq \mathfrak{CA}_\beta$, we write $\mathfrak{Nr}_\alpha \mathfrak{K}$ for the class $\{ \mathfrak{Nr}_\alpha \mathfrak{B} : \mathfrak{B} \in \mathfrak{K} \}$.

Following [3], $\text{Cs}_n$ denotes the class of cylindric set algebras of dimension $n$, and $\text{Gs}_n$ denotes the class of generalized cylindric set algebra of dimension $n$; $\mathcal{C} \in \text{Gs}_n$, if $\mathcal{C}$ has top element $V$ a disjoint union of cartesian squares, that is $V = \bigcup_{i \in I} nU_i$, $I$ is a non-empty indexing set, $U_i \neq \emptyset$ and $U_i \cap U_j = \emptyset$ for all $i \neq j$. The operations of $\mathcal{C}$ are defined like in cylindric set algebras of dimension $n$ relativized to $V$. It is known that $\text{IGs}_n = \text{RCA}_n = \text{SNr}_n \text{CA}_\omega = \bigcap_{k \in \omega} \text{SNr}_n \text{CA}_{n+k}$. We often identify set algebras with their domain referring to an injection $f : \mathfrak{A} \rightarrow \wp(V)$ ($\mathfrak{A} \in \text{CA}_n$) as a complete representation of $\mathfrak{A}$ (via $f$), where $V$ is a $\text{Gs}_n$ unit.

Definition 2.2. An algebra $\mathfrak{A} \in \text{CA}_n$ is completely representable $\iff$ there exists $\mathcal{C} \in \text{Gs}_n$, and an isomorphism $f : \mathfrak{A} \rightarrow \mathcal{C}$ such that for all $X \subseteq \mathfrak{A}$, $f(\sum X) = \bigcup_{x \in X} f(x)$, whenever $\sum X$ exists in $\mathfrak{A}$. In this case, we say that $\mathfrak{A}$ is completely representable via $f$.

It is known that $\mathfrak{A}$ is completely representable via $f : \mathfrak{A} \rightarrow \mathcal{C}$, where $\mathcal{C} \in \text{Gs}_n$, has top element $V$ say $\iff$ $\mathfrak{A}$ is atomic and $f$ is atomic in the sense that $f(\text{At}\mathfrak{A}) = \bigcup_{x \in \text{At}\mathfrak{A}} f(x) = V$ [4]. We denote the class of completely representable $\text{CA}_n$s by $\text{CRCA}_n$.

To define certain deterministic games to be used in the sequel, we recall the notions of atomic networks and atomic games [5, 6]. Let $i < n$. For $n$-ary sequences $\bar{x}$ and $\bar{y}$, $\bar{y}(j) = \bar{x}(j)$ for all $j \neq i$.

Definition 2.3. Fix finite $n > 2$ and assume that $\mathfrak{A} \in \text{CA}_n$ is atomic.
An $n$–dimensional atomic network on $\mathfrak{A}$ is a map $N : \Delta^n \rightarrow \mathrm{At}\mathfrak{A}$, where $\Delta$ is a non–empty set of nodes, denoted by $\mathrm{nodes}(N)$, satisfying the following consistency conditions for all $i < j < n$:

- If $\bar{x} \in \Delta^n \mathrm{nodes}(N)$ then $N(\bar{x}) \leq d_{ij} \iff x_i = x_j$,
- If $\bar{x}, \bar{y} \in \Delta^n \mathrm{nodes}(N), i < n$ and $\bar{x} \equiv_i \bar{y}$, then $N(\bar{x}) \leq c_i N(\bar{y})$.

For $n$–dimensional atomic networks $M$ and $N$, we write $M \equiv_i N \iff M(\bar{y}) = N(\bar{y})$ for all $\bar{y} \in \Delta^n (n \sim \{i\})$.

(2) Assume that $m, k \leq \omega$. The atomic game $G_k^m(\mathrm{At}\mathfrak{A})$, or simply $G_k^m$, is the game played on atomic networks of $\mathfrak{A}$ using $m$ nodes and having $k$ rounds [6, Definition 3.3.2], where $\forall i$ is offered only one move, namely, a cylindrifier move: Suppose that we are at round $t > 0$. Then $\forall i$ picks a previously played network $N_i (\mathrm{nodes}(N_i) \subseteq m), i < n$, $a \in \mathrm{At}\mathfrak{A}, x \in \Delta^n \mathrm{nodes}(N_i)$, such that $N_i(x) \leq c_i a$. For her response, $\exists$ has to deliver a network $M$ such that $\mathrm{nodes}(M) \subseteq m, M \equiv_i N$, and there is $\bar{y} \in \Delta^n \mathrm{nodes}(M)$ that satisfies $\bar{y} \equiv_i \bar{x}$ and $M(\bar{y}) = a$. We write $G_k(\mathrm{At}\mathfrak{A})$, or simply $G_k$, for $G_k^m(\mathrm{At}\mathfrak{A})$ if $m \geq \omega$.

2.1 Clique guarded semantics

Fix $2 < n < \omega$, We study three approaches to approximating the class $\mathrm{RCA}_n$ by (a) basis, (b) existence of dilations and finally (c) (locally well–behaved) relativized representations, in analogy to the relation algebra case dealt with in [5, Chapter 13]. Examples include $m$–flat and $m$–square representations, where $2 < n < m < \omega$. It will always be the case, unless otherwise explicitly indicated, that $1 < n < m < \omega; n$ denotes the dimension. But first we recall certain relativized set algebras. A set $V (\subseteq \Delta^n)$ is diagonalizable if $s \in V \implies s \circ [i/j] \in V$. We say that $V \subseteq \Delta^n$ is locally square if whenever $s \in V$ and $\tau : n \rightarrow n$, then $s \circ \tau \in V$. Let $D_n (G_n)$ be the class of set algebras whose top elements are diagonalizable (locally square) and operations are defined like cylindric set algebra of dimension $n$ relativized to the top element $V$. We identify notionally a set algebra with its universe. Let $M$ be a relativized representation of $\mathfrak{A} \in \mathrm{CA}_n$, that is, there exists an injective homomorphism $f : \mathfrak{A} \rightarrow \varphi(V)$ where $V \subseteq \Delta^n$ and $\bigcup_{s \in V} \mathrm{rng}(s) = M$. For $s \in V$ and $a \in \mathfrak{A}$, we may write $a(s)$ for $s \in f(a)$. This notation does not refer to $f$, but whenever used then either $f$ will be clear from context, or immaterial in the context. We may also write $1^M$ for $V$. Let $\mathcal{L}(\mathfrak{A})^m$ be the first order signature using $m$ variables and one $n$–ary relation symbol for each element of $\mathfrak{A}$. Allowing infinitary conjunctions, we denote the resulting signature taken in $L_{\infty, \omega}$ by $\mathcal{L}(\mathfrak{A})^m_{\infty, \omega}$.

An $n$–clique, or simply a clique, is a set $C \subseteq M$ such $(a_0, \ldots, a_{n-1}) \in V = 1^M$ for all distinct $a_0, \ldots, a_{n-1} \in C$. Let

$$C^m(M) = \{s \in M^m : \mathrm{rng}(s) \text{ is an } n \text{ clique}\}.$$ 

Then $C^m(M)$ is called the $n$–Gaifman hypergraph, or simply Gaifman hypergraph of $M$, with the $n$–hyperedge relation $1^M$. The $n$–clique–guarded semantics, or simply clique–guarded semantics, $\models_{c}$, are defined inductively. Let $f$ be as above. For an atomic $n$–ary formula $a \in \mathfrak{A}, i \in n^m$, and $s \in M^m, M, s \models_c a(x_{i_0}, \ldots, x_{i_{m-1}}) \iff (s_{i_0}, \ldots, s_{i_{m-1}}) \in f(a)$. For equality, given $i < j < m$, $M, s \models_c x_i = x_j \iff s_i = s_j$. Boolean connectives, and infinitary disjunctions, are defined as expected. Semantics for existential quantifiers (cylindrifiers) are defined inductively for $\phi \in \mathcal{L}(A)^m_{\infty, \omega}$ as follows: For $i < m$ and $s \in M^m, M, s \models_c \exists x_i \phi \iff \text{there is a } t \in C^m(M), t \equiv_i s \text{ such that } M, t \models_c \phi$. 


Definition 2.4. Let \( A \in CA_n \), \( M \) a relativized representation of \( A \) and \( \mathcal{L}(A)^m \) be as above.

1. Then \( M \) is said to be \( m \)-square, if witnesses for cylindrifiers can be found on \( n \)-cliques. More precisely, for all \( \bar{s} \in C^m(M), a \in A, i < n \), and for any injective map \( l : n \to m \), if \( M \models \psi_i(a(s_{l(0)}), \ldots, s_{l(n-1)}) \), then there exists \( \bar{t} \in C^m(M) \) with \( \bar{t} \equiv_i \bar{s} \), and \( M \models a(t_{l(0)}, \ldots, t_{l(n-1)}) \).

2. \( M \) is said to be \((\text{infinitary}) \ m \)-flat if it is \( m \)-square and for all \( \phi \in (\mathcal{L}(A)^m)^{\omega})\mathcal{L}(A)^m \), for all \( \bar{s} \in C^m(M) \), if \( \phi \) is true, we have \( M \models [\exists x_1 \exists x_3 \phi \leftrightarrow \exists x_3 \exists x_1 \phi](\bar{s}) \).

We also need the notion of \( m \)-dimensional hyperbasis. This hyperbasis is made up of \( m \)-dimensional hypernetworks. An \( m \)-dimensional hypernetwork on the atomic algebra \( A \) is an \( n \)-dimensional network \( N \), with \( \text{nodes}(N) \subseteq m \), endowed with a set of labels \( \Lambda \) for hyperedges of length \( \leq m \), not equal to \( n \) (the dimension), such that \( \Lambda \cap \text{At}A = \emptyset \). We call a label in \( \Lambda \) a non-atomic label. Like in networks, \( n \)-hyperedges are labelled by atoms. In addition to the consistency properties for networks, an \( m \)-dimensional hypernetwork should satisfy the following additional consistency rule involving non-atomic labels: If \( \bar{x}, \bar{y} \in \Sigma^m \), \( |\bar{x}| = |\bar{y}| \neq n \) and \( \exists \bar{z} \), such that \( \forall i < |\bar{x}|, N(x_i, y_i, z) \leq d_{01}, \) then \( N(\bar{x}) = N(\bar{y}) \in \Lambda \).

Definition 2.5. Let \( 2 < n < m < \omega \) and \( A \in CA_n \) be atomic.

1. An \( m \)-dimensional basis \( B \) for \( A \) consists of a set of \( n \)-dimensional networks whose nodes \( \subseteq m \), satisfying the following properties:

   - For all \( a \in \text{At}A \), there is an \( N \in B \) such that \( N(0, 1, \ldots, n-1) = a \).

   - The cylindrifier property: For all \( N \in B, all i < n, all \bar{x} \in \text{nodes}(N) \subseteq n \), all \( a \in \text{At}A \), such that \( \text{nodes}(\bar{x}) \leq c_i a \), there exists \( M \in B, M \equiv_i N, \bar{y} \in \text{nodes}(M) \) such that \( \bar{y} \equiv_i \bar{x} \) and \( M(\bar{y}) = a \). We can always assume that \( \bar{y} \equiv_i \bar{x} \) and \( M(\bar{y}) = a \).

2. An \( m \)-dimensional hyperbasis \( H \) consists of \( m \)-dimensional hypernetworks, satisfying the above two conditions reformulated the obvious way for hypernetworks, in addition, \( H \) has an amalgamation property for overlapping hypernetworks; this property corresponds to commutativity of cylindrifiers:

   For all \( M, N \in H \) and \( x, y < m \), with \( M \equiv_{xy} N \), there is \( L \in H \) such that \( M \equiv_x L \equiv_y N \). Here \( M \equiv S N \), means that \( M \) and \( N \) agree off of \( S \) [5 Definition 12.11].

Definition 2.6. Let \( m \) be a finite ordinal \( > 0 \). An \( \bullet \) word is a finite string of substitutions \((s_i^j) \ (i, j < m)\), a \( c \) word is a finite string of cylindrifications \((c_i) \ (i < m)\); an \( sc \) word \( w \), is a finite string of both, namely, of substitutions and cylindrifications. An \( sc \) word induces a partial map \( \bar{w} : m \to m \):

   - \( \bar{e} = \text{Id} \),
   - \( \bar{w}^i_j = \bar{w} \circ [i,j] \),
   - \( \bar{w}c_i = \bar{w} \upharpoonright (m \setminus \{i\}) \).

If \( \bar{a} \in \Sigma^{m-1} \), we write \( s_\bar{a} \), or \( s_{a_0 \ldots a_{k-1}} \), where \( k = |\bar{a}| \), for an arbitrary chosen \( sc \) word \( w \) such that \( \bar{w} = \bar{a} \). Such a \( w \) exists by [5] Definition 5.23 Lemma 13.29].
The proof of the following lemma can be distilled from its RA analogue [5, Theorem 13.20], by reformulating deep concepts originally introduced by Hirsch and Hodkinson for RAs in the CA context, involving the notions of hypernetworks and hyperbasis. This can (and will) be done. In the coming proof, we highlight the main ideas needed to perform such a transfer from RAs to CAs [5, Definitions 12.1, 12.9, 12.10, 12.25, Propositions 12.25, 12.27]. In all cases, the $m$–dimensional dilation stipulated in the statement of the theorem, will have top element $C_m(M)$, where $M$ is the $m$–relativized representation of the given algebra, and the operations of the dilation are induced by the $n$-clique–guarded semantics. For a class $K$ of BAOs, $K \cap \text{At}$ denotes the class of atomic algebras in $K$.

**Lemma 2.7.** ([5, Theorems 13.45, 13.36].) Assume that $2 < n < m < \omega$ and let $\mathfrak{A} \in \text{CA}_n$. Then $\mathfrak{A} \in \text{SN}_\mathfrak{n}\text{CA}_m \iff \mathfrak{A}$ has an infinitary $m$–flat representation $\iff \mathfrak{A}$ has an $m$–flat representation. Furthermore, if $\mathfrak{A}$ is atomic, then $\mathfrak{A}$ has a complete infinitary $m$–flat representation $\iff \mathfrak{A} \in \text{S}_n\text{SN}_\mathfrak{n}(\text{CA}_m \cap \text{At})$. We can replace infinitary $m$–flat and $\text{CA}_m$ by $m$-square and $D_m$, respectively.

**Proof.** We give a sketchy sample. More details can be found in [14]. We start from representations to dilations. Let $M$ be an $m$–flat representation of $\mathfrak{A}$. For $\phi \in \Sigma(\mathfrak{A})^m$, let $\phi^M = \{ \bar{a} \in C_m(M) : M \models_c \phi(\bar{a}) \}$, where $C_m(M)$ is the $n$–Gaifman hypergraph. Let $\mathcal{D}$ be the algebra with universe $\{ \phi^M : \phi \in \Sigma(\mathfrak{A})^m \}$ and with cylindric operations induced by the $n$-clique–guarded (flat) semantics. For $r \in \mathfrak{A}$, and $\bar{x} \in C_m(M)$, we identify $r$ with the formula it defines in $\Sigma(\mathfrak{A})^m$, and we write $r(\bar{x})^M \iff M, \bar{x} \models_c r$. Then $\mathcal{D}$ is a set algebra with domain $\varphi(C_m(M))$ and with unit $1_{\mathcal{D}} = C_m(M)$. Since $M$ is $m$–flat, then cylindrifiers in $\mathcal{D}$ commute, and so $\mathcal{D} \in \text{CA}_m$. Now define $\theta : \mathfrak{A} \to \mathcal{D}$, via $r \mapsto r(\bar{x})^M$. Then exactly like in the proof of [5, Theorem 13.20], $\theta$ is an injective neat embedding, that is, $\theta(\mathfrak{A}) \subseteq \text{Nr}_\mathfrak{n}\mathcal{D}$. The relativized model $M$ itself might not be infinitary $m$–flat, but one can build an infinitary $m$–flat representation of $\mathfrak{A}$, whose base $M$ is an $\omega$–saturated model of the consistent first order theory, stipulating the existence of an $m$–flat representation, cf. [5, Proposition 13.17, Theorem 13.46 items (6) and (7)].

The inverse implication from dilations to representations harder. One constructs from the given $m$–dilation, an $m$–dimensional hyperbasis (that can be defined similarly to the RA case, cf. [5, Definition 12.11]) from which the required $m$-relativized representation is built. This can be done in a step–by step manner treating the hyperbasis as a saturated set of mosaics’, cf. [5, Proposition 13.37]. We show how an $m$–dimensional hyperbasis for the canonical extension of $\mathfrak{A} \subseteq \mathfrak{A}_n\mathcal{D}$ is obtained from an $m$–dilation of $\mathfrak{A}$ [5, Definition 13.22, lemmata 13.33-34-35, Proposition 36]. Suppose that $\mathfrak{A} \subseteq \text{Nr}_n\mathcal{D}$ for some $\mathcal{D} \in \text{CA}_m$. Then $\mathfrak{A}^+ \subseteq_c \text{Nr}_m\mathcal{D}^+$, and $\mathcal{D}^+$ is atomic. We show that $\mathcal{D}^+$ has an $m$–dimensional hyperbasis. First, it is not hard to see that for every $n \leq l \leq m$, $\text{Nr}_l\mathcal{D}^+$ is atomic. The set of non–atomic labels $\Lambda$ is the set $\bigcup_{k<m-1} \text{At}\text{Nr}_k\mathcal{D}^+$. For each atom $a$ of $\mathcal{D}^+$, define a labelled hypergraph $N_a$ as follows. Let $b \in \leq^m_m$. Then if $|b| = n$, so that $b$ has to get a label that is an atom of $\mathcal{D}^+$, one sets $N_a(b)$ to be the unique $r \in \text{At}\mathcal{D}^+$ such that $a \leq s_b r$; notation here is given in definition 2.6 If $n \neq |b| < m - 1$, $N_a(b)$ is the unique atom $r \in \text{Nr}_{|b|}\mathcal{D}^+$ such that $a \leq s_b r$. Since $\text{Nr}_{|b|}\mathcal{D}^+$ is atomic, this is well defined. Note that this label may be a non–atomic one; it might not be an atom of $\mathcal{D}^+$. But by definition it is a permitted label. Now fix $\lambda \in \Lambda$. The rest of the labelling is defined by $N_a(b) = \lambda$. Then $N_a$ as an $m$–dimensional hypernetwork, for each such chosen $a$, and $\{ N_a : a \in \text{At}\mathcal{D}^+ \}$ is the required $m$–dimensional hyperbasis. The rest of the proof consists of a fairly straightforward adaptation of the proof [5, Proposition 13.37], replacing edges by $n$–hyperedges.
For results on complete m–flat representations, one works in $L^n_{\infty, \omega}$ instead of first order logic. With $\mathcal{D}$ formed like above from (the complete m–flat representation) $M$, using $\mathcal{L}(\mathcal{A})^n_{\infty, \omega}$ instead of $L_n$, let $\phi^M$ be a non–zero element in $\mathcal{D}$. Choose $\bar{a} \in \phi^M$, and let $\tau = \bigwedge \{ \psi \in \mathcal{L}(\mathcal{A})^n_{\infty, \omega} : M \models \psi(\bar{a}) \}$. Then $\tau \in \mathcal{L}(\mathcal{A})^m_{\infty, \omega}$, and $\tau^M$ is an atom below $\phi^M$. The rest is entirely analogous, cf. [5, p.411].

\[ \square \]

The following lemma is proved in [13, Lemma 5.8]

**Lemma 2.8.** Let $2 < n < m$.

If $\mathfrak{A} \in \mathcal{C}A_n$ is finite and $\forall$ has a winning strategy in $G^n_\omega(\text{At}\mathfrak{A})$, then $\mathfrak{A}$ does not have an m–square representation.

In our next proof we use a rainbow constructions; in this we follow [4, 5]. Fix $2 < n < \omega$. Given relational structures $G$ (the greens) and $R$ (the reds) the rainbow atom structure of a $\mathcal{C}A_n$ consists of equivalence classes of surjective maps $a : n \to \Delta$, where $\Delta$ is a coloured graph. A *coloured graph* is a complete graph labelled by the rainbow colours, the greens $g \in G$, reds $r \in R$, and whites; and some $n – 1$ tuples are labelled by ‘shades of yellow’. In coloured graphs certain triangles are not allowed for example all green triangles are forbidden. A red triple $(r_{ij}, r_{jk'}, r_{i+k^*})$ $i,j,k \in R$ is not allowed, unless $i = i^*$, $j = j'$ and $k' = k^*$, in which case we say that the red indices match, cf.[4, 4.3.3]. The equivalence relation relates two such maps $\iff$ they essentially define the same graph [4, 4.3.4]. We let $[\bar{a}]$ denote the equivalence class containing $\bar{a}$. For $2 < n < \omega$, we use the graph version of the usual atomic $\omega$–rounded game $G^n_\omega(\alpha)$ with $m$ nodes, played on atomic networks of the $\mathcal{C}A_n$ atom structure $\alpha$. The game $G^m(\beta)$ where $\beta$ is a $\mathcal{C}A_n$ atom structure is like $G^n_\omega(\text{At}\mathfrak{A})$ except that $\forall$ has the option to reuse the $m$ nodes in play. We use the ‘graph versions’ of these games, cf. [4, 4.3.3]. The (complex) rainbow algebra based on $G$ and $R$ is denoted by $\mathfrak{A}_{G,R}$. The dimension $n$ will always be clear from context.

### 3 Degrees of representability

We let $S_c$ denotes the operation of forming complete sublgebras and $S_d$ denotes the operation of forming dense subalgebras. We let $I$ denote the operation of forming isomorphic images. For any class of BAOs $IK \subseteq S_dK \subseteq S_cK$. (It is not hard to show that for Boolean algebras the inclusion are proper).

**Definition 3.1.** Let $2 < n \leq l \leq m \leq \omega$. Let $O \in \{S, S_d, S_c, I\}$.

1. An algebra $\mathfrak{A} \in \mathcal{C}A_n$ has the $O$ neat embedding property up to $m$ if $\mathfrak{A} \in O\mathcal{N}r_n\mathcal{C}A_m$. If $m = \omega$ and $O = S$, we say simply that $\mathfrak{A}$ has the neat embedding property. (Observe that the last condition is equivalent to that $\mathfrak{A} \in \mathcal{R}C\mathcal{A}_n$).

2. An atomic algebra $\mathfrak{A} \in \mathcal{C}A_n$ has the complex $O$ neat embedding property up to $m$, if $\mathfrak{c}m\text{At}\mathfrak{A} \in O\mathcal{N}r_n\mathcal{C}A_m$. The word ‘complex’ here refers to the involvement of the complex algebra in the definition.

3. An atomic algebra $\mathfrak{A} \in \mathcal{R}C\mathcal{A}_n$ is strongly representable up to $l$ and $m$ if $\mathfrak{A} \in \mathcal{N}r_n\mathcal{C}A_l$ and $\mathfrak{c}m\text{At}\mathfrak{A} \in S\mathcal{N}r_n\mathcal{C}A_m$. If $l = n$ and $m = \omega$, we say that $\mathfrak{A}$ is strongly representable.
Theorem 3.2. Then is an atomic simple countable $\mathcal{A} \in \mathcal{CA}_n$ (i.e. has the neat embedding property) but not the complex $\mathcal{S}$ neat embedding property up to $m$ for any $m \geq n + 3$.

Proof. We show that there is a countable atomic $\mathcal{A} \in \mathcal{RCA}_n$ such that $\mathcal{CmAt}\mathcal{A}$ does not have an $n + 3$–square representation. This is proved in [13] in the context of omitting types. Here we give a direct shorter more streamlined proof. The idea however is essentially the same. Take the finite rainbow cylindric algebra $R(\Gamma)$ as defined in [6] Definition 3.6.9, where $\Gamma$ (the reds) is taken to be the complete irreflexive graph $m$, and the greens are $\{g_i : 1 \leq i \leq n - 1\} \cup \{g'_i : 1 \leq i \leq n + 1\}$ so that $G$ is the complete irreflexive graph $n + 1$.

Call this finite rainbow $n$–dimensional cylindric algebra, based on $G = n + 1$ and $R = n$, $\mathcal{CA}_{n+1,n}$ and denote its finite atom structure by $\mathcal{At}_r$. One then replaces each red colour used in constructing $\mathcal{CA}_{n+1,n}$ by infinitely many with superscripts from $\omega$, getting a weakly representable atom structure $\mathcal{At}$, that is, the term algebra $\mathcal{TmAt}$ is representable. The resulting atom structure (with $\omega$–many reds), call it $\mathcal{At}$, is the rainbow atom structure that is like the atom structure of the (atomic set) algebra denoted by $\mathcal{A}$ in [8] Definition 4.1 except that we have $n + 1$ greens and not infinitely many as is the case in [8]. Everything else is the same. In particular, the rainbow signature [6] Definition 3.6.9 now consists of $g_i : 1 \leq i < n - 1$, $g'_i : 1 \leq i \leq n + 1$, $w_i : i < n - 1$, $r_{kl}^{t} : k < l < n$, $t \in \omega$, binary relations, and $n - 1$ ary relations $y_S$, $S \subseteq n + 1$. There is a shade of red $\rho$; the latter is a binary relation that is outside the rainbow signature.

But $\rho$ is used as a label for coloured graphs built during a ‘rainbow game’, and in fact, $\exists$ can win the rainbow $\omega$–rounded game and she builds an $n$–homogeneous (coloured graph) model $M$ as indicated in the above outline by using $\rho$ when she is forced a red [8] Proposition 2.6, Lemma 2.7. Then, it can be shown exactly as in [8], that $\mathcal{TmAt}$ is representable as a set algebra with unit $\mathcal{m}M$. We give more details. In the present context, after the splitting ‘the finitely many red colours’ replacing each such red colour $r_{kl}^{t}$, $k < l < n$ by $\omega$ many $r_{kl}^{i}$, $i \in \omega$, the rainbow signature for the resulting rainbow theory as defined in [5] Definition 3.6.9 call this theory $T_{ra}$, consists of $g_i : 1 \leq i < n - 1$, $g'_i : 1 \leq i \leq n + 1$, $w_i : i < n - 1$, $r_{kl}^{i} : k < l < n$, $t \in \omega$, binary relations, and $n - 1$ ary relations $y_S$, $S \subseteq n + k - 2$ or $S = n + 1$. The set algebra $\mathcal{DB}(\mathcal{A}_{n+1,n}, \rho, \omega)$ of dimension $n$ has base an $n$–homogeneous model $M$ of another theory $T$ whose signature expands that of $T_{ra}$ by an additional binary relation (a shade of red) $\rho$. In this new signature $T$ is obtained from $T_{ra}$ by some axioms (consistency conditions) extending $T_{ra}$. Such axioms (consistency conditions) specify consistent triples involving $\rho$. We call the models of $T$ extended coloured graphs. In particular, $M$ is an extended coloured graph. To build $M$, the class of coloured graphs is considered in the signature $L \cup \{\rho\}$ like in usual rainbow constructions as given above with the two additional forbidden triples $(\rho, \rho, \rho)$ and $(\rho, \rho^\ast, \rho)$, where $\rho, \rho^\ast$ are any reds. This model $M$ is constructed as a countable limit of finite models of $T$ using a game played between $\exists$ and $\forall$. Here, unlike the extended $L_{\omega_1,\omega}$ theory dealt with in [8], $T$ is a first order one because the number of greens used are finite. In the rainbow game [4, 5] $\forall$ challenges $\exists$ with cones having green tints ($g'_i$), and $\exists$ wins if she can respond to such moves. This is the only way that $\forall$ can force a win. $\exists$ has to respond by labelling apexes of two successive cones, having the same base played by $\forall$. By the rules of the game, she has to use a red label. She resorts to $\rho$ whenever she is forced a red while using the rainbow reds will lead to an inconsistent triangle of reds; [8] Proposition 2.6, Lemma 2.7.

We next embed $\mathcal{CA}_{n+1,n}$ into the complex algebra $\mathcal{CmAt}$, the Dedekind-MacNeille
completion of \( \Sigma mAt \). Let \( CRG_f \) denote the class of coloured graphs on \( At_f \) and \( CRG \) be the class of coloured graph on \( At \). We can assume that \( CRG_f \subseteq CRG \). Write \( M_a \) for the atom that is the (equivalence class of the) surjection \( a : n \to M, M \in CRG \). Here we identify \( a \) with \([a]\); no harm will ensue. We define the (equivalence) relation \( \sim \) on \( At \) by \( M_b \sim M_a \), \((M,N \in CRG) \iff \text{they are everywhere identical except possibly at red edges:} \)

\[
M_a(a(i),a(j)) = r_l \iff N_b(b(i),b(j)) = r_k, \text{for some } l,k \in \omega.
\]

We say that \( M_a \) is a copy of \( N_b \) if \( M_a \sim N_b \). Now we define a map \( \Theta : CA_{n+1,n} = \Sigma mAt_f \) to \( \Sigma mAt \), by specifying first its values on \( At_f \), via \( M_a \mapsto \sum j M_a^{(j)} \); where \( M_a^{(j)} \) is a copy of \( M_a \); each atom maps to the suprema of its copies. (If \( M_a \) has no red edges, then by \( \sum j M_a^{(j)} \), we understand \( M_a \).) This map is extended to \( CA_{n+1,n} \) the obvious way. The map \( \Theta \) is well-defined, because \( \Sigma mAt \) is complete. It is not hard to show that the map \( \Theta \) is an injective homomorphism. We check preservation of all the \( QEA_n \) operations. The Boolean join is obvious.

- For complementation: It suffices to check preservation of complementation ‘at atoms’ of \( At_f \). So let \( M_a \in At_f \) with \( a : n \to M, M \in CGR_f \subseteq CGR \). Then:

\[
\Theta(\sim M_a) = \Theta( \bigcup_{[b] \neq [a]} M_b ) = \bigcup_{[b] \neq [a]} \Theta(M_b) = \bigcup_{[b] \neq [a]} \sum_j M_b^{(j)} = \bigcup_{[b] \neq [a]} \sum_j (\sim M_b)^{[j]} = \bigcup_{[b] \neq [a]} \bigwedge_j M_b^{(j)} = \bigwedge_j (\sim M_a)^{[j]} = \sim (\sum_j M_a^{[j]}) = \sim \Theta(a)
\]

- Diagonal elements. Let \( l < k < n \). Then:

\[
M_x \leq \Theta(d_{lk}^{\Sigma mAt_f}) \iff M_x \leq \sum_j \bigcup_{a_i = a_k} M_a^{(j)} \\
\iff M_x \leq \bigcup_{a_i = a_k} \sum_j M_a^{(j)} \\
\iff M_x = M_a^{(j)} \text{ for some } a : n \to M \text{ such that } a(l) = a(k) \\
\iff M_x \in d_{lk}^{\Sigma mAt}.
\]

- Cylindrifiers. Let \( i < n \). By additivity of cylindrifiers, we restrict our attention to atoms \( M_a \in At_f \) with \( a : n \to M, M \in CGR_f \subseteq CGR \). Then:

\[
\Theta(c_i^{\Sigma mAt_f} M_a) = f( \bigcup_{[c] \equiv_i [a]} M_c ) = \bigcup_{[c] \equiv_i [a]} \Theta(M_c) \\
= \bigcup_{[c] \equiv_i [a]} \sum_j M_c^{(j)} = \sum_j \bigcup_{[c] \equiv_i [a]} M_c^{(j)} = \sum_j c_i^{\Sigma mAt} M_a^{(j)} \\
= c_i^{\Sigma mAt} \sum_j M_a^{(j)} = c_i^{\Sigma mAt} \Theta(M_a).
\]

8
It is straightforward to show that \( \forall \) has winning strategy first in the Ehrenfeucht–Fraïssé forth private game played between \( \exists \) and \( \forall \) on the complete irreflexive graphs \( n + 1 \) and \( n \) in \( n + 1 \) rounds \( \text{EF}^{n+1}_{n+1}(n+1, n) \) [6] Definition 16.2] since \( n + 1 \) is 'longer' than \( n \). Here \( r \) is the number of rounds and \( p \) is the number of pairs of pebbles on board. Using (any) \( p > n \) many pairs of pebbles available on the board \( \forall \) can win this game in \( n + 1 \) many rounds. In each round \( 0, 1 \ldots n, \exists \) places a new pebble on a new element of \( n + 1 \). The edge relation in \( n \) is irreflexive so to avoid losing \( \exists \) must respond by placing the other pebble of the pair on an unused element of \( n \). After \( n \) rounds there will be no such element, so she loses in the next round. \( \forall \) lifts his winning strategy from the private Ehrenfeucht–Fraïssé forth game \( \text{EF}^{n+1}_{n+1}(n+1, n) \) to the graph game on \( \text{At}_I = \text{At}(\mathfrak{A}_{n+1,n}) \) [4] pp. 841 forcing a win using \( n + 3 \) nodes. He bombards \( \exists \) with cones having common base and distinct green tints until \( \exists \) is forced to play an inconsistent red triangle (where indices of reds do not match). Thus \( \forall \) has a winning strategy for \( \exists \) in \( G^{n+3}\text{At}(\mathfrak{C}A_{n+1,n}) \) using the usual rainbow strategy by bombarding \( \exists \) with cones having the same base and distinct green tints. He needs \( n + 3 \) nodes to implement his winning strategy. In fact he need \( n + 3 \) nodes to force a win in the weaker game \( G^{n+3}_\omega \) without the need to resue the nodes in play. Then by Lemma 2.8 this implies that \( \mathfrak{C}A_{n+1,n} \) does not have an \( n + 3 \)–square representation. Since \( \mathfrak{C}A_{n+1,n} \) embeds into \( \mathfrak{CmAt} \), hence \( \mathfrak{CmAt} \) does not have an \( n + 3 \)–square representation, too. \( \square \)

The following definition to be used in the sequel is taken from [1]:

**Definition 3.3.** [1] Definition 3.1] Let \( \mathfrak{R} \) be a relation algebra, with non–identity atoms \( I \) and \( 2 < n < \omega \). Assume that \( J \subseteq \wp(I) \) and \( E \subseteq 3^\omega \).

1. We say that \((J, E)\) is an \( n \)-blur for \( \mathfrak{R} \), if \( J \) is a complex \( n \)-blur defined as follows:
   
   - (1) Each element of \( J \) is non–empty,
   - (2) \( \bigcup J = I \),
   - (3) \( (\forall P \in I)(\forall W \in J)(I \subseteq P; W) \),
   - (4) \( (\forall V_1, \ldots, V_n, W_1, \ldots, W_n \in J)(\exists T \in J)(\forall 2 \leq i \leq n)\text{safe}(V_i, W_i, T) \), that is there is for \( v \in V_i, w \in W_i \) and \( t \in T \), we have \( v; w \leq t \),
   - (5) \( (\forall P_1, \ldots, P_n, Q_2, \ldots, Q_n \in I)(\forall W \in J)W \cap P_2; Q_n \cap \ldots P_n; Q_n \neq \emptyset \).

   and the tenary relation \( E \) is an index blur as in item (ii) of [1] Definition 3.1.

2. We say that \((J, E)\) is a strong \( n \)-blur, if it \((J, E)\) is an \( n \)-blur, such that the complex \( n \)-blur satisfies:

   \( (\forall V_1, \ldots, V_n, W_1, \ldots, W_n \in J)(\forall T \in J)(\forall 2 \leq i \leq n)\text{safe}(V_i, W_i, T) \).

**Theorem 3.4.** For every \( 2 < n < l < \omega \), there is an algebra \( \mathfrak{B} \) in \( \mathfrak{N}_n\mathfrak{C}A_l \cap \mathfrak{RCA}_n \), but is not strongly representable up to \( l \) and \( \omega \). In particular, \( \mathfrak{B} \) is not strongly representable.

**Proof.** We give an example of a blowing up and blurring a finite relation algebra \( \mathfrak{R} \) getting an infinite countable atomic \( \mathcal{R} \in \mathcal{R}A \) such that that \( \text{At}\mathcal{R} \) is weakly but not strongly representable. Furthermore \( \mathcal{R} \) has an \( n \) dimensional cylindric basis, and \( \text{Mat}_n(\text{At}\mathfrak{R}) \) is a weakly but not strongly representable \( \mathfrak{C}A_n \) atom structure. This example is based on
a generalization of the construction in \cite{1}. Our exposition of the construction in \cite{1} will be addressing an (abstract) finite relation algebra $\mathfrak{A}$ having an $l$–blur in the sense of definition \cite{1} Definition 3.1, with $3 \leq l \leq k < \omega$ and $k$ depending on $l$. Occasionally we use the concrete Maddux algebra $\mathfrak{C}_k(2,3)$ to make certain concepts more tangible. Here $k$ is the number of non–identity atoms is concrete example of $\mathfrak{A}$. In this algebra a triple $(a,b,c)$ of non–identity atoms is consistent $\iff |\{a,b,c\}| \neq 1$, i.e only monochromatic triangles are forbidden.

We use the notation in \cite{1}. Let $2 < n \leq l < \omega$. One starts with a finite relation algebra $\mathfrak{R}$ that has only representations, if any, on finite sets (bases), having an $l$–blur $(J,E)$ as in \cite{1} Definition 3.1 recalled in definition \S 3.3. After blowing up and bluring $\mathfrak{R}$, by splitting each of its atoms into infinitely many, one gets an infinite atomic representable relation algebra $\mathfrak{Bb}(\mathfrak{R},J,E)$ \cite{1} p.73, whose atom structure $\mathfrak{At}$ is weakly but not strongly representable. The atom structure $\mathfrak{At}$ is not strongly representable, because $\mathfrak{R}$ is not blurred in $\mathfrak{CmAt}$. The finite relation algebra $\mathfrak{R}$ embeds into $\mathfrak{CmAt}$, so that a representation of $\mathfrak{CmAt}$, necessarily on an infinite base, induces one of $\mathfrak{R}$ on the same base, which is impossible. The representability of $\mathfrak{Bb}(\mathfrak{R},J,E)$ depend on the properties of the $l$–blur, which blurs $\mathfrak{R}$ in $\mathfrak{Bb}(\mathfrak{R},J,E)$. The set of blurs here, namely, $J$ is finite. In the case of $\mathfrak{C}_k(2,3)$ used in \cite{1}, the set of blurs is the set of all subsets of non–identity atoms having the same size $l < \omega$, where $k = f(l) \geq l$ for some recursive function $f$ from $\omega \to \omega$, so that $k$ depends recursively on $l$. One (but not the only) way to define the index blur $E \subseteq ^3\omega$ is as follows \cite{10} Theorem 3.1.1: $E(i,j,k) \iff (\exists p,q,r)\{\{p,q,r\} = \{i,j,k\}$ and $r-q = q-p$. This is a concrete instance of an index blur as defined in \cite{1} Definition 3.1(iii)](recall definition \S 3.3 above), but defined uniformly, it does not depends on the blurs. The underlying set of $\mathfrak{At}$, the atom structure of $\mathfrak{Bb}(\mathfrak{R},J,E)$ is the following set consisting of triplets: $\mathfrak{At} = \{(i,P,W) : i \in \omega, P \in \mathfrak{At} \mathfrak{R} \sim \{1\}, W \in J\} \cup \{\{1\}$. When $\mathfrak{R} = \mathfrak{C}_k(2,3)$ (some finite $k > 0$), composition is defined by singling out the following (together with their Peircean transforms), as the consistent triples: $(a,b,c)$ is consistent $\iff$ one of $a,b,c$ is $1$ and the other two are equal, or if $a = (i,P,S), b = (j,Q,Z), c = (k,R,W)$

\[ S \cap Z \cap W \neq \emptyset \implies E(i,j,k) \&\{P,Q,R\} \neq 1. \]

(We are avoiding monochromatic triangles). That is if for $W \in J$, $E^W = \{(i,P,W) : i \in \omega, P \in W\}$, then

\[ (i,P,S); (j,Q,Z) = \bigcup\{E^W : S \cap Z \cap W = \emptyset\} \]

\[ \bigcup\{(k,R,W) : E(i,j,k), |\{P,Q,R\}| \neq 1\}. \]

More generally, for the $\mathfrak{R}$ as postulated in the hypothesis, composition in $\mathfrak{At}$ is defined as follow. First the index blur $E$ can be taken to be like above. Now the triple \[(i,P,S), (j,Q,Z), (k,R,W)\] in which no two entries are equal, is consistent if either $S, Z, W$ are safe, briefly safe$(S, Z, W)$, witness item (4) in definition \S 3.3 (which vacuously hold if $S \cap Z \cap W = \emptyset$), or $E(i,j,k)$ and $P; Q \leq R$ in $\mathfrak{R}$. This generalizes the above definition of composition, because in $\mathfrak{C}_k(2,3)$, the triple of non–identity atoms $(P,Q,R)$ is consistent $\iff$ they do not have the same colour $\iff |\{P,Q,R\}| \neq 1$. Having specified its atom structure, its timely to specify the relation algebra $\mathfrak{Bb}(\mathfrak{R},J,E) \subseteq \mathfrak{CmAt}$. The relation algebra $\mathfrak{Bb}(\mathfrak{R},J,E)$ is $\mathfrak{TmAt}$ (the term algebra). Its universe is the set $\{X \subseteq H \cup \{1\} : X \cap E^W \in \text{Cof}(E^W), \text{ for all } W \in J\}$, where $\text{Cof}(E^W)$ denotes the
set of co–finite subsets of $E^W$, that is subsets of $E^W$ whose complement is infinite, with $E^W$ as defined above. The relation algebra operations are lifted from $At$ the usual way. The algebra $\mathfrak{Bb}(\mathfrak{R},J,E)$ is proved to be representable [1] as shown next. For brevity, denote $\mathfrak{Bb}(\mathfrak{R},J,E)$ by $\mathfrak{R}$, and its domain by $R$. For $a \in At$, and $W \in J$, set $U^a = \{X \in R : a \in X\}$ and $U^W = \{X \in R : |X \cap E^W| \geq \omega\}$. Then the principal ultrafilters of $\mathfrak{R}$ are exactly $U^a$, $a \in H$ and $U^W$ are non-principal ultrafilters for $W \in J$ when $E^W$ is infinite. Let $J' = \{W \in J : |E^W| \geq \omega\}$, and let $Uf = \{U^a : a \in F\} \cup \{U^W : W \in J'\}$. $Uf$ is the set of ultrafilters of $\mathfrak{R}$ which is used as colours to represent $\mathfrak{R}$, cf. [1, pp. 75–77]. The representation is built from coloured graphs whose edges are labelled by elements in $Uf$ in a fairly standard step–by–step construction.

Now we show why the Dedekind–MacNeille completion $\mathfrak{CmAt}$ is not representable. For $P \in I$, let $H^P = \{(i,P,W) : i \in \omega, W \in J, P \in W\}$. Let $P_1 = \{H^P : P \in I\}$ and $P_2 = \{E^W : W \in J\}$. These are two partitions of $At$. The partition $P_2$ was used to represent, $\mathfrak{Bb}(\mathfrak{R},J,E)$, in the sense that the ternary relation corresponding to composition was defined on $At$, in a such a way so that the singletons generate the partition $(E^W : W \in J)$ up to “finite deviations.” The partition $P_1$ will now be used to show that $\mathfrak{Cm}(\mathfrak{Bb}(\mathfrak{R},J,E)) = \mathfrak{Cm}(At)$ is not representable. This follows by observing that composition restricted to $P_1$ satisfies: $H^P; H^Q = \bigcup\{H^Z : Z; P \leq Q \text{ in } \mathfrak{R}\}$ which means that $\mathfrak{R}$ embeds into the complex algebra $\mathfrak{CmAt}$ prohibiting its representability, because $\mathfrak{R}$ allows only representations having a finite base. So far we have been dealing with relation algebras. The construction lifts to higher dimensions expressed in $\mathfrak{CA}_n$s, $2 < n < \omega$, as shown next. Let $\mathfrak{R}$ be as in the hypothesis. Let $3 < n \leq l$. We blow up and blur $\mathfrak{R}$. $\mathfrak{R}$ is blown up by splitting all of the atoms each to infinitely many defining an (infinite atoms) structure $At$. $\mathfrak{R}$ is blurred by using a finite set of blurs (or colours) $J$. The term algebra $\mathfrak{Bb}(\mathfrak{R},J,E)$ over $At$, is representable using the finite number of blurs. Such blurs are basically non–principal ultrafilters; they are used as colours together with the principal ultrafilters (the atoms) to represent completely the canonical extension of $\mathfrak{Bb}(\mathfrak{R},J,E)$. Because $(J,E)$ is a complex set of $l$–blurs, this atom structure has an $l$–dimensional cylindric basis, namely, $At_{ca} = Mat_1(At)$. The resulting $l$–dimensional cylindric term algebra $\mathfrak{Smat}_l(At)$, and an algebra $\mathfrak{C}$ having atom structure $At_{ca}$ (denoted in [1] by $\mathfrak{Bb}(\mathfrak{R},J,E)$) such that $\mathfrak{Smat}_l(At) \subseteq \mathfrak{C} \subseteq \mathfrak{CmAt}(At)$ is shown to be representable. Assume that the $m$–blur $(J,E)$ is strong, then by definition $(J,E)$ is a strong $j$ blur for all $n \leq j \leq m$. Furthermore, by [1, item (3) pp. 80], $\mathfrak{Bb}(\mathfrak{R},J,E) = \mathfrak{R}a\mathfrak{Bb}(\mathfrak{R},J,E)$ and $\mathfrak{Bb}(\mathfrak{R},J,E) \cong \mathfrak{Nr}\mathfrak{Bb}_m(\mathfrak{R},J,E)$. ∎

$LCA_n$ denotes the elementary class of $\mathfrak{RCA}_n$s satisfying the Lyndon conditions [3, Definition 3.5.1].

**Theorem 3.5.** Let $2 < n < m \leq \omega$. Then $\mathfrak{ElNr}_n\mathfrak{CA}_\omega \cap At \subseteq LCA_n$. Furthermore, for any elementary class $K$ between $\mathfrak{ElNr}_n\mathfrak{CA}_\omega \cap At$ and $LCA_n$, $\mathfrak{RCA}_n$ is generated by $AtK$.

**Proof.** It suffices to show that $\mathfrak{Nr}_n\mathfrak{CA}_\omega \cap At \subseteq LCA_n$, since the last class is elementary. This follows from Lemma 2.8 since if $\mathfrak{A} \in \mathfrak{Nr}_n\mathfrak{CA}_\omega$ is atomic, then $\exists$ has a winning strategy in $G^\omega(At2)$, hence in $G_\omega(At2)$, a fortiori, $\exists$ has a winning strategy in $G_k(At2)$ for all $k < \omega$, so (by definition) $\mathfrak{A} \in LCA_n$. To show strictness of the last inclusion, let $V = nQ$ and let $\mathfrak{A} \in \mathfrak{C}_n$ have universe $\varphi(V)$. Then $\mathfrak{A} \in \mathfrak{Nr}_n\mathfrak{CA}_\omega$. Let $y = \{s \in V : s_0 + 1 = \sum_{i \geq 0} s_i\}$ and $\mathfrak{B} = \mathfrak{Eg}^\mathfrak{A}(\{y\} \cup X)$, where $X = \{s : s \in V\}$. Now $\mathfrak{B}$ and $\mathfrak{A}$ having same top element $V$, share the same atom structure, namely, the singletons, so
\( \text{CmAt}\mathfrak{B} = \mathfrak{A} \). Furthermore, plainly \( \mathfrak{A}, \mathfrak{B} \in \text{CRCA}_n \). So \( \mathfrak{B} \in \text{CRCA}_n \subseteq \text{LCA}_n \), and as proved in [12], \( \mathfrak{B} \not\in \text{EIN}_n \text{CA}_{n+1} \), hence \( \mathfrak{B} \) witnesses the required strict inclusion.

Now we show that \( \text{AtEIN}_n \text{CA}_\omega \) generates \( \text{RCAs}_n \). Let \( \text{FC}_n \) denote the class of \( \text{full CS}_{n,s} \), that is \( \text{CS}_{n,s} \) having universe \( \varphi(U) \) \((U \text{ non-empty set})\). First we show that \( \text{FC}_n \subseteq \text{CmAtN}_n \text{CA}_\omega \). Let \( \mathfrak{A} \in \text{FC}_n \). Then \( \mathfrak{A} \in \text{N}_n \text{CA}_\omega \cap \text{At} \), hence \( \text{AtA} \in \text{AtN}_n \text{CA}_\omega \) and \( \mathfrak{A} = \text{CmAtA} \in \text{CmAtN}_n \text{CA}_\omega \). The required now follows from the following chain of inclusions: \( \text{RCAs}_n = \text{SPFC}_n \subseteq \text{SPCmAtN}_n \text{CA}_\omega \subseteq \text{SP} \text{CmAt} (\text{EIN}_n \text{CA}_\omega) \subseteq \text{SP} \text{CmAtK} \subseteq \text{SPCmLCAS}_n \subseteq \text{RCAs}_n \), where \( K \) is given above.

\( \square \)

Let \( 2 < n \leq l \leq m \leq \omega \). Denote the class of \( \text{CA}_n \)'s having the complex \( \text{O} \) neat embedding property up to \( m \) by \( \text{CNPCA}_{n,m}^\text{O} \), and let \( \text{RCA}_{n,m} = \text{CNPCA}_{n,m}^\text{O} \cap \text{RCA}_n \). Denote the class of strongly representable \( \text{CA}_n \)'s up to \( l \) and \( m \) by \( \text{RCA}_{n,m}^\text{S} \). Observe that \( \text{RCA}_{n,m}^\text{S} = \text{RCA}_{n,m}^\text{S} \) and that when \( m = \omega \) both classes coincide with the class of strongly representable \( \text{CA}_n \)'s. For a class \( K \) of \( \text{BAO}_s \), \( K \cap \text{Count} \) denotes the class of countable algebras in \( K \), and recall that \( K \cap \text{At} \) denotes the class of atomic algebras in \( K \).

**Theorem 3.6.** Let \( 2 < n \leq l < m \leq \omega \) and \( \text{O} \in \{ \text{S}, \text{S}_c, \text{S}_d, \text{I} \} \). Then the following hold:

1. \( \text{RCA}_{n,m}^\text{O} \subseteq \text{RCA}_{n,l}^\text{O} \) and \( \text{RCA}_{n,l}^\text{I} \subseteq \text{RCA}_{n,d}^\text{S} \subseteq \text{RCA}_{n,l}^\text{S} \subseteq \text{RCA}_{n,l}^\text{S} \). The last inclusion is proper for \( l \geq n + 3 \),

2. For \( \text{O} \in \{ \text{S}, \text{S}_c, \text{S}_d \} \), \( \text{CNPCA}_{n,l}^\text{O} \subseteq \text{ON}_n \text{CA}_l \) (that is the complex \( \text{O} \) neat embedding property is stronger than the \( \text{O} \) neat embedding property), and for \( \text{O} = \text{I} \), the inclusion is proper for \( l \geq n + 3 \). But for \( \text{O} = \text{I} \), \( \text{CNPCA}_{n,l}^\text{I} \not\subseteq \text{N}_n \text{CA}_l \) (so the complex \( \text{I} \) neat embedding property does not imply the \( \text{I} \) neat embedding property),

3. If \( \mathfrak{A} \) is finite, then \( \mathfrak{A} \in \text{CNPCA}_{n,l}^\text{O} \iff \mathfrak{A} \in \text{ON}_n \text{CA}_l \) and \( \mathfrak{A} \in \text{RCA}_{n,l}^\text{O} \iff \mathfrak{A} \in \text{RCA}_n \cap \text{ON}_n \text{CA}_l \). Furthermore, for any positive \( k \), \( \text{CNPCA}_{n,n+k+1}^\text{O} \subseteq \text{CNPCA}_{n,n+k}^\text{O} \), and finally \( \text{CNPCA}_{n,\omega}^\text{O} \subseteq \text{RCA}_n \),

4. \( (\exists \mathfrak{A} \in \text{RCA}_n \cap \text{At} \sim \text{CNPCA}_{n,l}^\text{S}) \implies \text{SN}_n \text{CA}_k \) is not atom–canonical for all \( k \geq l \).

In particular, \( \text{SN}_n \text{CA}_k \) is not atom–canonical for all \( k \geq n + 3 \),

5. If \( \text{SN}_n \text{CA}_k \) is atom–canonical, then \( \text{RCA}_{n,l}^\text{S} \) is first order definable. There exists a finite \( k > n + 1 \), such that \( \text{RCA}_{n,k} \) is not first order definable.

6. Let \( 2 < n < l \leq \omega \). Then \( \text{RCA}_{n}^\text{I,} \omega \cap \text{Count} \neq \emptyset \iff l < \omega \).

**Proof.** (1): The inclusions in the first item are by definition. To show the strictness of the last inclusion, we proceed in this way. We show that there an \( \text{RCA}_n \) with countably many atoms outside \( \text{S}_n \text{N}_n \text{CA}_{n+3} \). Take the a rainbow-like \( \text{CA}_n \), call it \( \mathfrak{C} \) based on the ordered structure \( \mathbb{Z} \) and \( \mathbb{N} \). The reds \( R \) is the set \( \{ r_{ij} : i < j < \omega (= \mathbb{N}) \} \) and the green colours used constitute the set \( \{ g_{ij} : 1 \leq i < n - 1 \} \cup \{ g_{i0} : i \in \mathbb{Z} \} \). In complete coloured graphs the forbidden triples are like the usual rainbow constructions based on \( \mathbb{Z} \) and \( \mathbb{N} \), but now the triple \( (g_{i0}, g_{i1}, r_{ij}) \) is also forbidden if \( \{(i,k), (j,l)\} \) is not an order preserving partial function from \( \mathbb{Z} \to \mathbb{N} \). It can be shown that \( \forall \) has a winning strategy in the graph version of the game \( \text{G}^{n+3}(\text{At}\mathfrak{C}) \) played on coloured graphs \( [4] \). The rough idea here, is that, as is the case with winning strategy’s of \( \forall \) in rainbow constructions, \( \forall \) bombards \( \exists \) with cones having distinct green tints demanding a red label from \( \exists \) to
apexes of successive cones. The number of nodes are limited but ∀ has the option to re-use them, so this process will not end after finitely many rounds. The added order preserving condition relating two greens and a red, forces ∃ to choose red labels, one of whose indices forms a decreasing sequence in N. In ω many rounds ∀ forces a win, so E ∈ S_nNr_nCA_{n+3}. More rigorously, ∀ plays as follows: In the initial round ∀ plays a graph M with nodes 0, 1, . . . , n−1 such that M(i, j) = w_0 for i < j < n−1 and M(i, n−1) = g_i (i = 1, . . . , n−2), M(0, n−1) = g_0 and M(0, 1, . . . , n−2) = y_2. This is a 0 cone. In the following move ∀ chooses the base of the cone (0, . . . , n−2) and demands a node n with M_2(i, n) = g_i (i = 1, . . . , n−2), and M_2(0, n) = g_0. ∃ must choose a label for the edge (n + 1, n) of M_2. It must be a red atom r_{mk}, m, k ∈ N. Since −1 < 0, then by the ‘order preserving’ condition we have m < k. In the next move ∀ plays the face (0, . . . , n−2) and demands a node n + 1, with M_3(i, n) = g_i (i = 1, . . . , n−2), such that M_3(0, n + 2) = g_0−2. Then M_3(n + 1, n) and M_3(n + 1, n − 1) both being red, the indices must match. M_3(n + 1, n) = r_{lm} and M_3(n + 1, n − 1) = r_{km} with l < m ∈ N. In the next round ∀ plays (0, . . . , n−2) and re-uses the node 2 such that M_4(0, 2) = g_0−3. This time we have M_4(n, n−1) = r_{ji} for some j < l < m ∈ N. Continuing in this manner leads to a decreasing sequence in N. We have proved the required. Since EmAtE = E and E /∈ S_nNr_nCA_{n+3} we are done.

(2): Let O ∈ {S, S_n, S_d}. If EmAtA ∈ ONr_nCA_I, then A ⊆ ONr_nCA_I, so A ∈ S_dONr_nCA_I ⊆ ONr_nCA_I. This proves the first part. The strictness of the last inclusion follows from Theorem 3.22 since the atomic countable algebra A constructed in op.cit is in RCA_n, but EmAtA does not have an n + 3-square representation, least is in SNr_nCA_I for any l ≥ n + 3. For the last non–inclusion in item (2), we use the set algebras A and E in Theorem 3.5. Now B ⊆ d A, A ∈ S_n, and clearly EmAtB = A ∈ (Nr_nCA_{ω}).

Follows by definition observing that if A is finite then A = EmAtA. The strictness of the first inclusion follows from the construction in [11] where it shown that for any positive k, there is a finite algebra A in Nr_nCA_{n+k} ∼ SNr_nCA_{n+k}. The inclusion CNPCA_{n,ω} ⊆ RCA_n holds because if B ∈ CNPCA_{n,ω}, then B ⊆ EmAtB ∈ ONr_nCA_{ω} ⊆ RCA_n. The A used in the last item of theorem 3.22 witnesses the strictness of the last inclusion proving the last required in this item.

(3): Follows from the definition and the construction used above.

(4): Follows from that SNr_nCA_I is canonical. So if it is atom–canonical too, then At(SNr_nCA_I) = {S : EmAtA ⊆ SNr_nCA_I}, the former class is elementary [3], and the latter class is elementray ↔ RCA_{n,l} is elementary. Non–elementarity follows from [6] Corollary 3.7.2 where it is proved that RCA_{n,ω} is not elementary, together with the fact that ∩_{n<k<ω} SNr_nCA_k = RCA_n. In more detail, let A_i be the sequence of strongly representable CA_{n,s} with EmAtA_i = A_i and A = ∏_{i/U} A_i is not strongly representable. Hence EmAtA /∈ SNr_nCA_{ω} = ∩_{i∈ω} SNr_nCA_{n+i}, so EmAtA /∈ SNr_nCA_I for all l > k, for some k ∈ ω, k > n. But for each such l, A_i ⊆ SNr_nCA_{l} (≠ RCA_n), so A_i is a sequence of algebras such that EmAtA_i = A_i ∈ SNr_nCA_I, but EmAt(∏_{i/U} A_i) = EmAtA /∈ SNr_nCA_I, for all l ≥ k. That k has to be strictly greater than n + 1, follows because SNr_nCA_{n+1} is atom–canonical.

(5): ⇐=: Let l < ω. Then the required follows from Theorem 3.22 namely, there exists a countable A ∈ Nr_nCA_I ∩ RCA_n such that EmAtA /∈ RCA_n. Now we prove === : Assume for contradiction that there is an A ∈ RCA_{n,ω} ∩ Count. Then by definition A ⊆ Nr_nCA_{ω}, so A ∈ CRCA_n. But this complete representation induces an(n ordinary) representation of EmAtA which is a contradiction.
4 Complete and other forms of representations

Theorem 4.1. Let \( \alpha \) be any countable ordinal (possibly infinite) and \( \mathfrak{A} \in \text{CA}_\alpha \). If \( \mathfrak{A} \) is atomic with countably many atoms, then \( \mathfrak{A} \) is completely representable \( \iff \mathfrak{A} \in S_r \text{NR}_\alpha \text{CA}_\alpha \cap \text{At} \). The implication \( \Rightarrow \) holds with no restriction on the cardinality of atoms.

Proof. Assume that \( \mathfrak{A} \subseteq \mathfrak{D} \). We can assume that \( \mathfrak{A} \) is countable and \( \mathfrak{D} \in \text{DC}_\alpha + \omega \). Now we use exactly the argument [10] Theorem 3.2.4, replacing \( \text{FM}_T \) in op.cit by \( \mathfrak{B} \). Omitting the one non–principal type of co–atoms, we get the required complete representation. Assume that \( \mathfrak{M} \) is the base of a complete representation of \( \mathfrak{A} \), whose unit is a weak generalized space, that is, \( 1^\mathfrak{M} = \bigcup \mathfrak{a}_i \), where \( \mathfrak{a}_i \cap \mathfrak{a}_j = \emptyset \) for distinct \( i \) and \( j \), in some index set \( I \), that is, we have an isomorphism \( t: \mathfrak{B} \rightarrow \mathfrak{C} \), where \( \mathfrak{C} \in \text{GS}_\alpha \) has unit \( 1^\mathfrak{M} \), and \( t \) preserves arbitrary meets carrying them to set–theoretic intersections. For \( i \in I \), let \( E_i = \mathfrak{a}_i \). Take \( f_i \in \mathfrak{a}_i \) where \( q_i \uparrow \alpha = p_i \) and let \( W_i = \{ f \in \mathfrak{a}_i : |k \in \alpha + \omega : f(k) \neq f_i(k)| < \omega \} \). Let \( C_i = \varphi(W_i) \). Then \( C_i \) is atomic; indeed the atoms are the singletons. Let \( x \in \text{NR}_\alpha C_i \), that is, \( x = x \) for all \( \alpha \leq i < \alpha + \omega \). Now if \( f \in x \) and \( g \in W_i \) satisfy \( g(k) = f(k) \) for all \( k < \alpha \), then \( g \in x \). Hence \( \text{NR}_\alpha C_i \) is atomic; its atoms are \( \{ g \in W_i : \{ g(i) : i < \alpha \} \subseteq U_i \} \). Define \( h_i: \mathfrak{A} \rightarrow \text{NR}_\alpha C_i \) by \( h_i(a) = \{ f \in W_i : \exists a' \in \text{At} \mathfrak{A}, a' \leq a; (f(i) : i < \alpha) \in t(a') \} \). Let \( \mathfrak{D} = \text{PL} C_i \). Let \( \pi_i: \mathfrak{D} \rightarrow C_i \) be the \( i \)th projection map. Now clearly \( \mathfrak{D} \) is atomic, because it is a product of atomic algebras, and its atoms are \( \{ \pi_i(\beta) : \beta \in \text{At}(C_i) \} \). Now \( \mathfrak{A} \) embeds into \( \text{NR}_\alpha \mathfrak{D} \) via \( J: a \mapsto (\pi_i(a) : i \in I) \). If \( x \in \text{NR}_\alpha \mathfrak{D} \), then for each \( i \), we have \( \pi_i(x) \in \text{NR}_\alpha C_i \), and if \( x \) is non–zero, then \( \pi_i(x) \neq 0 \). By atomicity of \( C_i \), there is an \( \alpha \)–ary tuple \( y \), such that \( \{ g \in W_i : g(k) = y_k \} \subseteq \pi_i(x) \). It follows that there is an atom of \( b \in \mathfrak{A} \), such that \( x \cdot J(b) = 0 \), and so the embedding is atomic, hence complete. We have shown that \( \mathfrak{A} \in S_r \text{NR}_\alpha \text{CA}_\alpha + \omega \) and we are done.

Fix \( 2 < n < \omega \). Call an atomic \( \mathfrak{A} \in \text{CA}_\alpha \) weakly (strongly) representable \( \iff \text{At} \mathfrak{A} \) is weakly (strongly) representable. Let \( \text{WRCA}_n \) (\( \text{SRCA}_n \)) denote the class of all such \( \text{CA}_\alpha \)'s, respectively. Then the class \( \text{SRCA}_n \) is not elementary and \( \text{LCA}_n \subseteq \text{SRCA}_n \subseteq \text{WRCA}_n \) [6].

Theorem 4.2. Let \( 2 < n < \omega \). Then the following hold:

1. \( S_r \text{NR}_n \text{CA}_\omega \cap \text{Count} = \text{CRCA}_n \cap \text{Count} \), and \( \text{EI} \text{S}_\epsilon \text{NR}_n \text{CA}_\omega \cap \text{At} = \text{LCA}_n \).

2. \( S \text{NR}_n \text{CA}_\omega \cap \text{At} = \text{WRCA}_n \), and \( \text{PEI} \text{S}_\epsilon \text{NR}_n \text{CA}_\omega \cap \text{At} \subseteq \text{SRCA}_n \).

Proof. For the first required one uses [9] Theorem 5.3.6. For the second required, show that \( \text{LCA}_n = \text{EI} \text{CRCA}_n = \text{EI}(S_r \text{NR}_n \text{CA}_\omega \cap \text{At}) \). Assume that \( \mathfrak{A} \in \text{LCA}_n \). Then, by definition, for all \( k < \omega \), \( \exists \) has a winning strategy in \( G_k (\text{At} \mathfrak{A}) \). Using ultrapowers followed by an elementary chain argument like in [9] Theorem 3.3.5, \( \exists \) has a winning strategy in \( G_\omega (\text{At} \mathfrak{B}) \) for some countable \( \mathfrak{B} \equiv \mathfrak{A} \), and so by [6] Theorem 3.3.3 \( \mathfrak{B} \) is completely representable. Thus \( \mathfrak{A} \in \text{EI} \text{CRCA}_n \). One shows that \( \text{EI}(S_r \text{NR}_n \text{CA}_\omega \cap \text{At}) \subseteq \text{LCA}_n \) exactly like in item (1) of Theorem 3.5.5. So \( \text{LCA}_n = \text{EI} \text{CRCA}_n \subseteq \text{EI}(S_r \text{NR}_n \text{CA}_\omega \cap \text{At}) \subseteq \text{LCA}_n \), and we are done. The first part of item (2) follows from the definition and the last part follows from that \( \text{LCA}_n \subseteq \text{SRCA}_n \), and that (it is easy to check that) \( \text{SRCA}_n \) is closed under \( P \).
Theorem 4.3. For $2 < n < \omega$. Then CRCA$_n$ is not elementary [4]. Furthermore, CRCA$_n \subseteq S_{\le} \text{Nr}_n(CA_\omega \cap \text{At}) \cap \text{At} \subseteq S_{<} \text{Nr}_nCA_\omega \cap \text{At}$. At least two of the previous three classes are distinct but the elementary closure of each coincides with LCA$_n$. Furthermore, all three classes coincide on the class of atomic algebras having countably many atoms.

Proof. We use the following uncountable version of Ramsey’s theorem due to Erdos and Rado: If $r \geq 2$ is finite, $k$ an infinite cardinal, then $exp_r(k)+ \rightarrow (k^+)^+\kappa+1$ where $exp_0(k) = k$ and inductively $exp_{r+1}(k) = 2^{exp_r(k)}$. The above partition symbol describes the following statement. If $f$ is a coloring of the $r+1$ element subsets of a set of cardinality $exp_r(k)+$ in $k$ many colors, then there is a homogeneous set of cardinality $k+$ (a set, all whose $r+1$ element subsets get the same $f$-value). Let $\kappa$ be a given infinite cardinal. We shall construct an atomless algebra $\mathcal{C} \in CA_\omega$ such that for all $n < \omega$, $\mathfrak{N}_{\kappa^n} \mathcal{C}$ is atomic having uncountably many atoms, but lacks a complete representation. An application of Lemma 2.8 will finish the proof. We use a simplified more basic version of a rainbow construction where only the two predominant colours, namely, the reds and blues are available. The algebra $\mathfrak{C}$ will be constructed from a relation algebra possessing an $\omega$-dimensional cylindric basis. To define the relation algebra we specify its atoms and the forbidden triples of atoms. The atoms are $\text{id}$, $g_0^i : i < 2^\kappa$ and $r_j : 1 \leq j < \kappa$, all symmetric. The forbidden triples of atoms are all permutations of $(\text{id}, x, y)$ for $x \neq y$, $(r_j, r_j, r_j)$ for $1 \leq j < \kappa$ and $(g_0^i, g_0^j, g_0^j)$ for $i, j, j^* < 2^\kappa$. Write $g_0$ for $\{g_0^i : i < 2^\kappa\}$ and $r_+$ for $\{r_j : 1 \leq j < \kappa\}$. Call this atom structure $\alpha$. Consider the term algebra $\mathfrak{R}$ defined to be the subalgebra of the complex algebra of this atom structure generated by the atoms. We claim that $\mathfrak{R}$, as a relation algebra, has no complete representation, hence any algebra sharing this atom structure is not completely representable, too. Indeed, it is easy to show that if $\mathfrak{A}$ and $\mathfrak{B}$ are atomic relation algebras sharing the same atom structure, so that $\text{At}\mathfrak{A} = \text{At}\mathfrak{B}$, then $\mathfrak{A}$ is completely representable $\iff$ $\mathfrak{B}$ is completely representable.

Assume for contradiction that $\mathfrak{R}$ has a complete representation $\mathfrak{M}$. Let $x, y$ be points in the representation with $\mathfrak{M} \models r_1(x, y)$. For each $i < 2^\kappa$, there is a point $z_i \in \mathfrak{M}$ such that $\mathfrak{M} \models g_0^i(x, z_i) \wedge r_1(z_i, y)$. Let $Z = \{z_i : i < 2^\kappa\}$. Within $Z$, each edge is labelled by one of the $\kappa$ atoms in $r_+$. The Erdos-Rado theorem forces the existence of three points $z^1, z^2, z^3 \in Z$ such that $\mathfrak{M} \models r_j(z^1, z^2) \wedge r_j(z^2, z^3) \wedge r_j(z^3, z_1)$, for some single $j < \kappa$. This contradicts the definition of composition in $\mathfrak{R}$ (since we avoided monochromatic triangles). Let $S$ be the set of all atomic $\mathfrak{R}$-networks $N$ with nodes $\omega$ such that $\{r_j : 1 \leq i < \kappa : r_i$ is the label of an edge in $N\}$ is finite. Then it is straightforward to show $S$ is an amalgamation class, that is for all $M, N \in S$ if $M \equiv_i L \equiv_j N$, witness [5] Definition 12.8] for notation. Now let $X$ be the set of finite $\mathfrak{R}$-networks $N$ with nodes $\subset \kappa$ such that:

1. each edge of $N$ is either (a) an atom of $\mathfrak{R}$ or (b) a cofinite subset of $r_+ = \{r_j : 1 \leq j < \kappa\}$ or (c) a cofinite subset of $g_0 = \{g_0^i : i < 2^\kappa\}$ and
2. $N$ is ‘triangle-closed’, i.e. for all $l, m, n \in \text{nodes}(N)$ we have $N(l, n) \leq N(l, m); N(m, n)$.

That means if an edge $(l, m)$ is labelled by $\text{id}$ then $N(l, n) = N(m, n)$ and if $N(l, m), N(m, n) \leq g_0$ then $N(l, n) \cdot g_0 = 0$ and if $N(l, m) = N(m, n) = r_j$ (some $1 \leq j < \omega$) then $N(l, n) \cdot r_j = 0$.

For $N \in X$ let $\hat{N} \in \mathcal{C}(S)$ be defined by

$$\{L \in S : L(m, n) \leq N(m, n) \text{ for } m, n \in \text{nodes}(N)\}.$$

For $i \in \omega$, let $N|_{-i}$ be the subgraph of $N$ obtained by deleting the node $i$. Then if
$N \in X, \ i < \omega$ then $c_i\widehat{N} = \widehat{N}\{i\}$. The inclusion $c_i\widehat{N} \subseteq (\widehat{N}\{i\})$ is clear. Conversely, let $L \in (\widehat{N}\{i\})$. We seek $M \equiv_i L$ with $M \in \widehat{N}$. This will prove that $L \in c_i\widehat{N}$, as required. Since $L \in S$ the set $T = \{t_i \notin L\}$ is infinite. Let $T$ be the disjoint union of two infinite sets $Y \cup Y'$, say. To define the $\omega$-network $M$ we must define the labels of all edges involving the node $i$ (other labels are given by $M \equiv_i L$). We define these labels by enumerating the edges and labeling them one at a time. So let $j \neq i < \kappa$. Suppose $j \in \text{nodes}(N)$. We must choose $M(i,j) \leq N(i,j)$. If $N(i,j)$ is an atom then of course $M(i,j) = N(i,j)$. Since $N$ is finite, this defines only finitely many labels of $M$. If $N(i,j)$ is cofinite subset of $g_0$ then let $M(i,j)$ be an arbitrary atom in $N(i,j)$. And if $N(i,j)$ is a cofinite subset of $r_\kappa$ then let $M(i,j)$ be an element of $N(i,j) \cap Y'$ which has not been used as the label of any edge of $M$ which has already been chosen (possible, since at each stage only finitely many have been chosen so far). If $j \notin \text{nodes}(N)$ then we can let $M(i,j) = r_k \in Y$ some $1 \leq k < \kappa$ such that no edge of $M$ has already been labelled by $r_k$. It is not hard to check that each triangle of $M$ is consistent (we have avoided all monochromatic triangles) and clearly $M \in \widehat{N}$ and $M \equiv_i L$. The labeling avoided all but finitely many elements of $Y'$, so $M \in S$. So $(\widehat{N}\{i\}) \subseteq c_i\widehat{N}$. Now let $\hat{X} = \{\hat{N} : N \in X\} \subseteq \mathfrak{c}(S)$. We claim that the subalgebra of $\mathfrak{c}(S)$ generated by $\hat{X}$ is simply obtained from $\hat{X}$ by closing under finite unions. Clearly all these finite unions are generated by $\hat{X}$. We must show that the set of finite unions of $\hat{X}$ is closed under all cylindric operations. Closure under unions is given. For $\hat{N} \in X$ we have $\neg \hat{N} = \bigcup_{m,n \in \text{nodes}(N)} N_{mn}$ where $N_{mn}$ is a network with nodes $\{m,n\}$ and labeling $N_{mn}(m,n) = -N(m,n)$. $N_{mn}$ may not belong to $X$ but it is equivalent to a union of at most finitely many members of $\hat{X}$. The diagonal $d_{ij} \in \mathfrak{c}(S)$ is equal to $\hat{N}$ where $N$ is a network with nodes $\{i,j\}$ and labeling $N(i,j) = 1d$. Closure under cylindrification is given. Let $\mathfrak{C}$ be the subalgebra of $\mathfrak{c}(S)$ generated by $\hat{X}$. Then $\mathfrak{R} = \mathfrak{Ra}(\mathfrak{C})$. To see why, each element of $\mathfrak{R}$ is a union of a finite number of atoms, possibly a co-finite subset of $g_0$ and possibly a co-finite subset of $r_\kappa$. Clearly $\mathfrak{R} \subseteq \mathfrak{Ra}(\mathfrak{C})$. Conversely, each element $z \in \mathfrak{Ra}(\mathfrak{C})$ is a finite union $\bigcup_{\hat{N} \in F} \hat{N}$, for some finite subset $F$ of $X$, satisfying $c_iz = z$, for $i > 1$. Let $i_0, \ldots , i_k$ be an enumeration of all the nodes, other than 0 and 1, that occur as nodes of networks in $F$. Then $c_{i_0} \ldots c_{i_k}z = \bigcup_{\hat{N} \in F} c_{i_0} \ldots c_{i_k}\hat{N} = \bigcup_{\hat{N} \in F} (\hat{N}\{i_0,1\}) \in \mathfrak{R}$. So $\mathfrak{Ra}(\mathfrak{C}) \subseteq \mathfrak{R}$. Thus $\mathfrak{R}$ is relation algebra reduct of $\mathfrak{C} \in \mathfrak{CA}_\omega$ but has no complete representation. Let $n > 2$. Let $\mathfrak{B} = \mathfrak{Na}_n\mathfrak{C}$. Then $\mathfrak{B}$ is relation algebra reduct of $\mathfrak{C} \in \mathfrak{CA}_\omega$ but has no complete representation. Let $n > 2$. Let $\mathfrak{B} = \mathfrak{Na}_n\mathfrak{C}$. Then $\mathfrak{B} \in \mathfrak{Na}_n\mathfrak{CA}_\omega$, is atomic, but has no complete representation for plainly a complete representation of $\mathfrak{B}$ induces one of $\mathfrak{R}$. In fact, because $\mathfrak{B}$ is generated by its two dimensional elements, and its dimension is at least three, its $\Delta f$ reduct is not completely representable. We show that the $\omega$-dilation $\mathfrak{C}$ is atomless. For any $N \in X$, we can add an extra node extending $N$ to $M$ such that $\emptyset \subseteq M' \subseteq N'$, so that $N'$ cannot be an atom in $\mathfrak{C}$. Then $\mathfrak{Na}_n\mathfrak{C}$ ($2 < n < \omega$) is atom, but has no complete representation. By observing from the proof of the previous Theorem that $\mathfrak{Na}_n\mathfrak{CA}_\omega \subseteq \mathfrak{CA}_n(= \mathfrak{EICRA}_n)$ and similarly for $\mathfrak{Ra}_s$, we have $\mathfrak{Ra}\mathfrak{CA}_\omega \subseteq \mathfrak{LRRA} = \mathfrak{EICRRA}$, we get: the classes $\mathfrak{CRCA}_n$ and $\mathfrak{CRRA}$ are not elementary. 

Consider the statement: There exists a countable, complete and atomic $L_n$ first order theory $T$ in a signature $L$ such that the type $\Gamma$ consisting of co-atoms in the cylindric Tarski-Lindenbaum quotient algebra $\mathfrak{fm}_T$ is realizable in every $m$-square model, but $\Gamma$ cannot be isolated using $\leq l$ variables, where $n \leq l < m \leq \omega$. A co-atom of $\mathfrak{fm}_T$ is the negation of an atom in $\mathfrak{fm}_T$, that is to say, is an element of the form $\Psi / \equiv_T$,
where $\Psi / \equiv_T = (\neg \phi / \equiv_T) \equiv (\phi / \equiv_T)$ and $\phi / \equiv_T$ is an atom in $\fml_T$ (for $L$-formulas, $\phi$ and $\psi$). Here the quotient algebra $\fml_T$ is formed relative to the congruence relation of semantical equivalence modulo $T$. An $m$-square model of $T$ is an $m$-square representation of $\fml_T$. The last statement denoted by $\not\VT(n,n + 3)$ and not $\VT(l,\omega)$ hold. Furthermore, if for each $n < \omega$, there exists a finite relation algebra $A_m$ having $m - 1$ strong blur and no $m$-dimensional relational basis, then for $2 < n < \omega$ and $l = m = \omega$, $\VT(l,m) \iff l = m = \omega$. In this direction we have the following strong partial result that seems to confirm our conjecture.

**Theorem 4.4.** For $2 < n < \omega$ and $n \leq l < \omega$, not $\VT(n,n + 3)$ and not $\VT(l,\omega)$ hold. Furthermore, if for each $n < \omega$, there exists a finite relation algebra $A_m$ having $m - 1$ strong blur and no $m$-dimensional relational basis, then for $2 < n < \omega$ and $l = m = \omega$, $\VT(l,m) \iff l = m = \omega$.

**Proof.** We start by the last part. Let $A_m$ be as in the hypothesis with strong $m - 1$-blur $(J,E)$ and $m$-dimensional relational basis. We ‘blow up and blur’ $A_m$ in place of the Maddux algebra $E_2(2,3)$ blown up and blurred in [1] Lemma 5.1, where $k < \omega$ is the number of non–identity atoms and $k$ depends recursively on $l$, giving the desired $l$–blurriness, cf. [1] Lemmata 4.2, 4.3. Now take $A = Bb_n(R_m, J, E)$ as defined in [1] to be the $C_A_n$ obtained after blowing up and blurring $A$ to a weakly representable atom structure $\mathcal{R}$. Here by [1] Theorem 3.2 9(iii), $\text{Mat}_n\mathcal{R}$ (the set of $n$-basic matrices on $\mathcal{R}$) is a $C_A_n$ atom structure and $A$ is an atomic subalgebra of $\text{CmMat}_n(\mathcal{R})$ containing $\text{CmMat}_n(\mathcal{R})$, cf. [1]. Then $A \in RCA_n \cap Nr_nCA_l$ but $A$ has no complete $m$-square representation. In fact, by [1] item (3) pp.80, $A \cong Nr_mBb_l(R_m,J,E)$. The last algebra $Bb_l(R_m,J,E)$ is defined and the isomorphism holds because $A_m$ has a strong $l$-blur. A complete $m$-square representation of an atomic $B \in CA_n$ induces an $m$-square representation of $\text{CmAt}B$. To see why, assume that $B$ has an $m$-square complete representation via $f : B \to D$, where $D = \phi(V)$ and the base of the representation $M = \bigcup_{s \in V} \text{rng}(s)$ is $m$-square. Let $\mathcal{C} \in \text{CmAt}B$. For $c \in C$, let $c \downarrow = \{a \in \text{At}\mathcal{C} : a \leq c\} = \{a \in \text{At}B : a \leq c\}$. Define, representing $\mathcal{C}$, $g : \mathcal{C} \to D$ by $g(c) = \sum_{x \in c} f(x)$, then $g$ is the required homomorphism into $\phi(V)$ having base $M$. But $\text{CmAt}A$ does not have an $m$-square representation, because $\mathcal{R}$ does not have an $m$-dimensional relational basis, and $\mathcal{R} \subseteq \text{RaCmAt}A$. So an $m$-square representation of $\text{CmAt}A$ induces one of $\mathcal{R}$ which that $\mathcal{R}$ has no $m$-dimensional relational basis, a contradiction.

We prove not $\VT(m - 1,m)$, hence the required. By [3] §4.3, we can (and will) assume that $A = \fml_T$ for a countable, simple and atomic theory $L_n$ theory $T$. Let $\Gamma$ be the $n$–type consisting of co–atoms of $T$. Then $\Gamma$ is realizable in every $m$–square model, for if $M$ is an $m$–square model omitting $\Gamma$, then $M$ would be the base of a complete $m$–square representation of $A$, and so by Theorem [2,7] $A \in S,Nr_mD_m$ which is impossible. Suppose for contradiction that $\phi$ is an $m - 1$ witness, so that $T \models \phi \rightarrow \alpha$, for all $\alpha \in \Gamma$, where recall that $\Gamma$ is the set of coatoms. Then since $A$ is simple, we can assume without loss that $A$ is a set algebra with base $M$ say. Let $M = (M,R_i)_{i \in \omega}$ be the corresponding model (in a relational signature) to this set algebra in the sense of [3] §4.3. Let $\phi^M$ denote the set of all assignments satisfying $\phi$ in $M$. We have $M \models T$ and $\phi^M \in A$, because $A \in Nr_nCA_{m-1}$. But $T \models \exists \alpha \phi$, hence $\phi^M \neq 0$, from which it follows that $\phi^M$ must intersect an atom $\alpha \in A$ (recall that the latter is atomic). Let $\psi$ be the formula, such
that $\psi^M = \alpha$. Then it cannot be the case that $T \models \phi \rightarrow \neg \psi$, hence $\phi$ is not a witness, contradiction and we are done. Finally, $\text{notVT}(n, n+3)$ and $\text{notVT}(l, \omega)$ ($n \leq l < \omega$) follow from Theorems $3.2$ and $3.4$.

References

[1] H. Andréka, I. Németi and T. Sayed Ahmed, Omitting types for finite variable fragments and complete representations. Journal of Symbolic Logic. 73 (2008) pp. 65–89.

[2] H. Andréka, M. Ferenczi and I. Németi, (Editors), Cylindric-like Algebras and Algebraic Logic, Bolyai Society Mathematical Studies and Springer-Verlag, 22 (2012).

[3] L. Henkin, J. D. Monk, J.D. and A. Tarski Cylindric Algebras, Part 1, Part 2. North Holland, 1970, 1985.

[4] R. Hirsch and I. Hodkinson Complete representations in algebraic logic, Journal of Symbolic Logic, 62(3)(1997) p. 816–847.

[5] R. Hirsch and I. Hodkinson, Relation algebras by games. Studies in Logic and the Foundations of Mathematics, 147 (2002).

[6] R. Hirsch and I. Hodkinson Completions and complete representations, in [2] pp. 61–90.

[7] R. Hirsch and I. Hodkinson Strongly representable atom structures of cylindric algebras, Journal of Symbolic Logic 74(2009), pp. 811–828

[8] I. Hodkinson, Atom structures of relation and cylindric algebras. Annals of pure and applied logic, 89(1997), p.117–148.

[9] T. Sayed Ahmed, Neat reducts and neat embeddings in cylindric algebras, in [2], pp. 105–134.

[10] T. Sayed Ahmed Completions, Complete representations and Omitting types, in [2], pp. 186–205.

[11] R. Hirsch and T. Sayed Ahmed, The neat embedding problem for algebras other than cylindric algebras and for infinite dimensions. Journal of Symbolic Logic 79(1) (2014) pp.208–222.

[12] T. Sayed Ahmed and I. Németi, On neat reducts of algebras of logic, Studia Logica. 68(2) (2001), pp. 229–262.

[13] T. Sayed Ahmed On notions of representability for cylindric polyadic algebras and a solution to the finitizability problem for first order logic with equality. Mathematical Logic quarterly (2015) 61(6) pp. 418–447.

[14] T. Sayed Ahmed Atom-canonicity in algebraic logic in connecting to omitting types in modal fragments of $L_{\omega, \omega}$ (2016). Archiv: 1608.03513