Rigidity for dicritical germ of foliation in $\mathbb{C}^2$.

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Abstract

We study some kind of rigidity property for dicritical foliation in $\mathbb{C}^2$ thanks to a new meromorphic function associated to any dicritical foliation playing the role of equation of separatrix. To be more specific, we compute infinitesimal obstructions for realizing the holonomy pseudo-group of a dicritical foliation $F$ on any source manifold of a blowing-up morphism with same dual tree as the desingularization of $F$. Using two examples, we remark that these obstructions come from some analytical invariants shared by the space of leaves and the ambient space. This situation never occurs in the non-dicritical case.

Introduction and main statements

Considering the problem of moduli for a germ of singular holomorphic foliation $F$ in $\mathbb{C}^2$ leads naturally to point out some topological and analytical invariants. The invariants of first kind come from the reduction of singularities $E: (\mathcal{M}, \mathcal{D}) \to (\mathbb{C}^2, 0)$ of the foliation: a combinatorial invariant given by the topological class of the manifold $\mathcal{M}$ and a analytical one given by the analytical class of the manifold $\mathcal{M}$, called the resolution space of $\hat{F}$. The invariants of second kind are more related to the foliation itself: the collection of projective holonomy representations defined over each component of the divisor $\mathcal{D}$ and Dulac applications, in other words the holonomy pseudo-group. A natural problem is to know whether coherent data of above invariants correspond to a concrete foliation. A germ of formal foliation $\hat{F}$ in $\mathbb{C}^2$ is the data of a formal germ of 1-form at $0 \in \mathbb{C}^2$

$$\omega = a(x, y)dx + b(x, y)dy$$  \hspace{1cm} (1)

up to a unity and a conjugacy with $a, b \in \mathbb{C}[[x, y]]$. When $\omega$ is convergent, $\hat{F}$ is said to be convergent and will be denoted simply by $F$. A formal separatrix of $\hat{F}$ is a formal irreducible curve $\{ f = 0 \}$ given by a formal germ of reduced function $f$ such that $f$ divides $df \wedge \omega$ in $\mathbb{C}[[x, y]]$. If $F$ is convergent, a separatrix is nothing but a germ of analytical curve which is a formal separatrix of $F$ seen as a formal foliation. In [2], C. Camacho and P. Sad proved that any singular germ of foliation in $(\mathbb{C}^2, 0)$ admits at least one separatrix. A foliation is said dicritical when it has not a infinite number of separatrix.
Let us denote by $E : (\mathcal{M}, \mathcal{D}) \to (\mathbb{C}^2, 0)$ the reduction process (10)(8) of $\hat{\mathcal{F}}$ where $\mathcal{D}$ refers to the exceptional divisor $E^{-1}(0)$. The singularities of $E^*\hat{\mathcal{F}}$ are reduced, which means that, in some local coordinates, they belong to the following list:

1. $\lambda xdy + ydx + \cdots$ terms of higher order, $\lambda \not\in \mathbb{Q}$;
2. $xdy + \cdots$ terms of higher order.

In the second case, the singularity is called saddle-node. The formal normal forms for such singularity are given in [9]: there exist some coordinates so that the singularity is given by the following 1-form:

$$(\zeta x^p - p)ydx + x^{p+1}dy, \quad p \in \mathbb{N}^*, \zeta \in \mathbb{C}.$$}

The invariant curve $\{x = 0\}$ is called the strong invariant curve and $\{y = 0\}$ the weak one. If the germ of the divisor $\mathcal{D}$ is the weak invariant curve, then the singularity is called tangent saddle-node. We recall that $\mathcal{F}$ is said to be of second kind when $\hat{\mathcal{F}}$ is non-dicritical and when none singularity of $E^*\mathcal{F}$ is a tangent saddle-node. One have same definition in the convergent context with obvious transposition.

In this article, we study the extended notion of foliation of second kind. The definition is the same as foliation of second kind defined in [7] but, of course, we do not require the foliation to be non-dicritical. The construction of a balanced equation of the separatrix allows us to study this class of foliations. In the non-dicritical case, the balanced equation coincide with an equation of the finite number of separatrix and we prove a criterion for a foliation to be of second kind, which is a standard criterion in the non-dicritical case [7].

Let us define $\hat{\mathcal{X}}_{\hat{\mathcal{F}}=0}$, the sheaf of base $\mathcal{D}$ whose fibre is the space of germs of vector field tangent to the total transform of the zero of a balanced equation $\hat{\mathcal{F}}$. Let $\hat{\mathcal{X}}_{\hat{\mathcal{F}}}$ be the sub-sheaf of vector field tangent to the foliation $\hat{\mathcal{F}}$. The sheaf $\hat{\mathcal{I}}_{\hat{\mathcal{F}}=\infty}$ refers to the germs of functions vanishing along the pole of $\hat{\mathcal{F}}$. The first

**Theorem.** Let $\hat{\mathcal{F}}$ be a balanced equation. The following properties are equivalent:

1. $\hat{\mathcal{F}}$ is of second kind.
2. $\nu_0(\hat{\mathcal{F}}) = \nu_0(\hat{\mathcal{F}}) + 1$.
3. The sequence of sheaves:

$$0 \to \hat{\mathcal{X}}_{\hat{\mathcal{F}}} \to \hat{\mathcal{X}}_{\hat{\mathcal{F}}=0} \xrightarrow{E^*\hat{\mathcal{F}}(\cdot)} \hat{\mathcal{I}}_{\hat{\mathcal{F}}=\infty} \to 0$$

is exact.

Thanks to the above result, we compute the infinitesimal obstructions for the following problem of existence:

Can any deformation of the resolution space of $\hat{\mathcal{F}}$ be followed by an isoholonomical deformation of $\hat{\mathcal{F}}$?
This question is the local version of an harder problem: can any manifold topologically equivalent to the resolution space of \( \hat{\mathcal{F}} \) carry a foliation with same holonomy group as \( \hat{\mathcal{F}} \). In the second part of this article, we will give a precise statement for this kind of problem. Finally, we are able to prove the following result:

**Theorem.** For a generic dicritical formal foliation \( \hat{\mathcal{F}} \), there exists an analytical topologically trivial deformation of the space of resolution of \( \hat{\mathcal{F}} \), which cannot carry any isoholonomical deformation of \( \hat{\mathcal{F}} \).

## 1 Multiplicity of a formal dicritical foliation.

Our first aim is to establish a formula, which links the multiplicity of \( \nu_0(\hat{\mathcal{F}}) \) and some invariants produced by the desingularization of \( \hat{\mathcal{F}} \).

The valence \( \nu(D) \) of an irreducible component \( D \) of \( \mathcal{D} \) is the number of irreducible components of \( \mathcal{D} \), which intersect \( D \). In the same way, the integer \( \nu_{\text{d}}(D) \) refers to the non-dicritical valence, which means the number of non-dicritical components, which intersect \( D \). Let us denote by \( \mathcal{D} \) the sheaf over \( \mathcal{D} \) of germs of holomorphic functions, which vanish along the component \( D \) of \( \mathcal{D} \). Let us denote by \( \mathfrak{M} \) the sheaf over \( \mathcal{D} \) of \( \mathcal{O} \)-module generated by the functions \( E^h \) with \( h \in \mathcal{O}_2 \) and \( h(0) = 0 \). By induction on the number of blowing-up in \( E \), one can prove that there exist numbers \( \nu(D) \), called the multiplicity of \( D \) such that, one has the following decomposition

\[
\mathfrak{M} = \prod_{D \in \text{Comp}(\mathcal{D})} \mathfrak{I}^{\nu(D)}_D.
\]

The integer \( \nu(D) \) is also the multiplicity of a curve whose strict transform is smooth and attached to a regular point of \( D \).

Let us consider the following definition introduced by C. Hertling in [4].

**Definition 1.1.** Let \( \hat{\mathcal{F}} \) be a germ of formal foliation given by a 1-form

\[
\omega = a(x, y)dx + b(x, y)dy
\]

1. Let \( (\hat{\mathcal{S}}, p) \) a germ of smooth formal invariant curve. If, in some coordinates, \( \hat{\mathcal{S}} \) is the curve \( \{ y = 0 \} \) and \( p \) the point \( (0,0) \), then the integer \( \text{ord}_p b(x,0) \) is called the index of \( \hat{\mathcal{F}} \) at \( p \) with the respect of \( \hat{\mathcal{S}} \) and is denoted by \( \text{Ind}(\hat{\mathcal{F}}, \hat{\mathcal{S}}, 0) \).

2. Let \( (\hat{\mathcal{S}}, p) \) a germ of smooth formal non-invariant curve. If, in some coordinates, \( \hat{\mathcal{S}} \) is the curve \( \{ y = 0 \} \) and \( p \) the point \( (0,0) \), then the integer \( \text{ord}_p a(x,0) \) is called the tangency order of \( \hat{\mathcal{F}} \) with the respect of \( \hat{\mathcal{S}} \) and is denoted \( \text{Tan}(\hat{\mathcal{F}}, \hat{\mathcal{S}}, p) \).

In [4], one can find the following formula, which is an extension of [1] when there are dicritical components:
Proposition 1.1 (H). The multiplicity of $\hat{F}$ satisfies the equality

$$\nu_0(\hat{F}) + 1 = \sum_{D \in \text{Comp}(d)} \nu(D)\rho(D)$$

where

1. if $D$ is non-dicritical, $\rho(D) = -v_d(D) + \sum_{q \in D} \text{Ind}(E^*\hat{F}, D, q)$.
2. if $D$ is dicritical, $\rho(D) = 2 - v_d(D) + \sum_{q \in D} \text{Tan}(E^*\hat{F}, D, q)$.

One has to notice that this formula is true even if the application $E$ is not the exactly the reduction of singularities but any morphisms build over $E$ by a succession of elementary blowing-up.

We are going to define a notion of balanced equation of separatrix of $\hat{F}$. A separatrix of $\hat{F}$ is said to be isolated if its stricted transform with the respect of $E$ is attached to a non-dicritical component. When $D$ is dicritical, one call pencil of $D$ the set of invariant curves, which strict transform is attached to $D$.

Definition 1.2. A complete system of separatrix is the union of two germs of curve $Z \cup P$ where

1. $Z$ is the union of isolated separatrix and, for any dicritical component $D$ with valence smaller than 2, $2 - v(D)$ curves of the pencil of $D$.
2. $P$ is the union of $v(D) - 2$ curves of the pencil of any dicritical component $D$ with valence bigger than 3.

A balanced equation of separatrix is a germ of meromorphic function whose zeros and poles are the blown down at the origin of $\mathbb{C}^2$ of respectively the sets $Z$ and $P$.

Let us denote by $\mathcal{SN}\mathcal{T}(\hat{F})$ the set of singularity of $E^*\hat{F}$, which are tangent saddle-node. Let $\hat{F}$ be any balanced equation.

Proposition 1.2. We have the following equality

$$\nu_0(\hat{F}) = \nu_0(\hat{F}) + 1 + \sum_{s \in \mathcal{SN}\mathcal{T}(\hat{F})} \sum_{D \in V(s)} \nu(D) \left( \text{Ind}(E^*\hat{F}, D, s) - 1 \right)$$

where $V(s)$ refers to the set of irreducible components containing the point $s$. In particular, the multiplicity of $\hat{F}$ does not depend on the choice of $\hat{F}$.

Proof: Let us write $\hat{F} = \frac{\hat{N}}{\hat{P}}$ where $\hat{N}$ and $\hat{P}$ are holomorphic. The foliations $d\hat{N}$ and $d\hat{P}$ are desingularized by the morphism $E$. By applying
to \( \hat{F} \), \( d\hat{N} \) and \( d\hat{P} \), we get the next relations:

\[
\begin{align*}
\nu_0(\hat{F}) + 1 &= \sum_{D \in \text{Comp}(d)} \nu(D)\rho(D), \\
\nu_0(d\hat{N}) + 1 &= \sum_{D \in \text{Comp}(d)} \nu_\hat{\gamma}(D)\rho_\hat{\gamma}(D), \\
\nu_0(d\hat{P}) + 1 &= \sum_{D \in \text{Comp}(d)} \nu_\hat{\beta}(D)\rho_\hat{\beta}(D).
\end{align*}
\]

As components multiplicity depend only on \( E \), we have for any component \( D \)

\[
\nu(D) = \nu_\hat{\gamma}(D) = \nu_\hat{\beta}(D).
\]

Now, if \( D \) is non-dicritical for \( F \), we find

\[
\rho(D) = \text{Iso}(D) + \sum_{q \in D \cap \text{SN}(\hat{F})} \left( \text{Ind}(E^*\hat{F}, D, q) - 1 \right),
\]

where \( \text{Iso}(D) \) is the number of isolated separatrix attached to \( D \). If \( D \) is dicritical, \( \rho(D) \) is equal to \( 2 - v(D) \). By definition, if \( D \) is non-dicritical, the numbers \( \rho_\hat{\gamma}(D) \) and \( \text{Iso}(D) \) are equal. Moreover, if \( D \) is dicritical with valence smaller than 2, \( \rho_\hat{\gamma}(D) \) is equal to \( 2 - v(D) = \rho(D) \). On any other component, the foliation \( \{ d\hat{N} = 0 \} \) doesn’t have any isolated separatrix and the integer \( \rho_\hat{\beta}(D) \) vanishes. Hence, we have the relation

\[
\nu_0(\hat{N}) = \nu_0(d\hat{N}) + 1 = \sum_{D \in \text{Comp}(d)} \nu(D)\rho(D) + \sum_{D \in \text{Comp}(d)} \nu(D) \sum_{q \in D \cap \text{SN}(\hat{F})} \left( \text{Ind}(E^*\hat{F}, D, q) - 1 \right). 
\]

As \( E^*\hat{F} \) is reduced, any point \( q \) in a dicritical component verifies the relation

\[
\text{Tan}(E^*\hat{F}, D, q) = 0.
\]

Hence, for any dicritical components with valence greater than 3, we have \( \rho_\hat{\beta}(D) = -\rho(D) \). In any other case, the integer \( \rho_\hat{\beta}(D) \) is zero. Therefore, we have

\[
\nu_0(\hat{P}) = \nu_0(d\hat{P}) + 1 = - \sum_{D \in \text{Comp}(d)} \nu(D)\rho(D). 
\]

The proposition is the combination of relations (2) and (3).
1.1 Dicritical foliation of second kind

The notion of balanced equation comes naturally with a notion of dicritical foliation of second kind. Known properties of second kind class will have analogues in our context.

Definition 1.3. \( \hat{F} \) is said to be of second kind when none singularity of \( E^* \hat{F} \) is a tangent saddle-node. When \( F \) is convergent, \( F \) is said to be of second kind when it is as formal foliation.

Of course, we do not require here \( \hat{F} \) to be non-dicritical. The following corollary of (1.2), which shows how these foliations naturally generalized the class of non-dicritical foliation of second kind:

Proposition 1.3. Let \( \hat{F} \) be a balanced equation. Then \( \hat{F} \) is of second kind if and only if \( \nu_0(\hat{F}) = \nu_0(\hat{F}) + 1 \)

This property is a clear analogue of a property for non-dicritical foliation of second kind established in [9].

2 Analysis of dicritical obstructions.

In [3], we prove the following result, which can be summary by saying that analytical invariants of the holonomy pseudo-group and analytical invariants of the ambient space are disconnected.

Theorem 2.1 ([3]). Let \( F_0 \) be a singular formal non-dicritical foliation of second kind at \( 0 \in \mathbb{C}^2 \) and \( E_0 : M \to \mathbb{C}^2 \) its desingularization. For any blowing-up process \( E_1 \) topologically equivalent to \( E_0 \), there exists a foliation \( F_1 \) at \( 0 \in \mathbb{C}^2 \) linked to \( F_0 \) by an isoholonomical deformation such that the desingularization of \( F_1 \) is exactly \( E_1 \).

For a precise definition of isoholonomical deformation, we refer to [6]; but basically, these are topologically trivial deformations along which the holonomy pseudo-group is constant. The main tool for the proof of the above theorem is the equivalent result at the infinitesimal level, which is a trivial consequence of the existence of the following exact sequence of sheaves

\[
0 \to \mathcal{X}_F \to \mathcal{X} \to \mathcal{O}_M \to 0. \tag{4}
\]

Here, all the sheaves have \( \mathcal{D} \) for base. The fibre of \( \mathcal{X} \) is the space of vector fields tangent to the total transform by \( E \) of the separatrix of \( F \) at the origin of \( \mathbb{C}^2 \). The sheaf \( \mathcal{X}_F \) is the sub-sheaf of \( \mathcal{X} \) whose fibre is the space of vector field tangent to the foliation \( E^*F \). These two sheaves correspond respectively to the modular space \( H^1(\mathcal{X}) \) of infinitesimal deformation of the ambient space and to the modular space \( H^1(\mathcal{X}_F) \) of infinitesimal isoholonomical deformation [6]. Finally, \( \mathcal{O}_M \) refers to the restriction at \( \mathcal{D} \) of the structural sheaf of \( M \). Since the first cohomology group of \( \mathcal{O}_M \) is trivial, one has the following sequence

\[
H^1(\mathcal{X}_F) \to H^1(\mathcal{X}) \to 0
\]

which is the starting point of the proof of (2.1).
2.1 Infinitesimal obstructions.

In this section, we establish a equivalent of the sequence (4) in order to compute the formal infinitesimal obstructions, which prevent the theorem (2.1) from being true for dicritical foliations.

**Proposition 2.1.** The following propositions are equivalent:

1. \( F \) is second kind.
2. The sequence of sheaves

\[
0 \rightarrow \hat{\mathcal{F}}_F \rightarrow \hat{\mathcal{F}}_{F=0} \xrightarrow{E^* \hat{\mathcal{F}}_{(1)}} \mathcal{I}_{F=\infty} \rightarrow 0
\]

is exact.

First, we show that the order of multiplicity of the blown-up balanced equation along any irreducible component of the divisor behave well with respect to the multiplicity of the foliation.

**Lemma 2.2.** For any component \( D \), we have the following alternative:

1. if \( D \) is non-dicritical, \( \nu_D(\hat{F}) = \nu_D(F) + 1 \),
2. if \( D \) is dicritical, \( \nu_D(\hat{F}) = \nu_D(F) \),

where \( \nu_D(*) \) refers to the multiplicity of the blown-up in a generic point of \( D \).

**Proof:** The proof is an induction on the height of the component \( D \) in the blowing-up process. At height 1, one can see that: the integers \( \nu_{D_0}(F) \) and \( \nu_0(F) \) are equal; if \( D_0 \) is non-dicritical, \( \nu_{D_0}(\hat{F}) = \nu_0(\hat{F}) \) else \( \nu_{D_0}(\hat{F}) = \nu_0(F) + 1 \). Hence, the lemma is the proposition (\).

For the induction, we look at the blowing-up process at height \( i \) and consider a component \( D \) of \( D^{i+1} \) obtained by blowing-up of \( c \in D^i \). Let \( \hat{F}^i \) by the divided blown-up of \( \hat{F} \) by \( E^i \). We have the following relations:

\[
\nu_D(\hat{F}) = \nu_c(E^i \hat{F}) + \sum_{D_c \in V(c)} \nu_{D_c}(\hat{F}) + \epsilon(D), \quad (5)
\]

\[
\nu_D(\hat{F}) = \nu_c(\hat{F}^i) + \sum_{D_c \in V(c)} \nu_{D_c}(\hat{F}), \quad (6)
\]

where \( \epsilon(D) = 0 \) when \( D \) is non-dicritical, 1 else. From now on, one has to look at each different cases. We recall that \( V(s) \) refers to the set of irreducible components of \( D \) that contain \( s \).

1. \( V(c) \) consists of one component \( D_0 \).

   (a) \( D_0 \) is non-dicritical: let \( \hat{F}_c \) the germ of meromorphic function near the point \( c \) product of \( \hat{F}^i \) and of a germ of equation of \( D_0 \). By definition, \( \hat{F}_c \) is a balanced equation for \( E^i \hat{F} \). In view of (\), we have

\[
\nu_c(\hat{F}_c) = \nu_c(E^i \hat{F}) + 1. \quad (7)
\]
Now, the above construction ensures the equality
\[ \nu_c(\tilde{F}_c) = 1 + \nu_c(\tilde{F}_c^i). \] (8)

Moreover, the induction hypothesis shows the relation
\[ \nu_{D_0}(\tilde{F}) = \nu_{D_0}(\tilde{F}) + 1. \] (9)

The association of (5), (6), (7), (8) and (9) gives the checked result for \( D \).

\[ \nu_{D}(\tilde{F}) = \nu_{D}(\tilde{F}) + 1 - \epsilon(D). \]

(b) \( D_0 \) is dicritical: in that case, one has to choose for \( \tilde{F}_c \) the germ \( \tilde{F}_c^i \). Once again, \( \tilde{F}_c \) is a balanced equation for \( E^*\tilde{F}_c \). Hence, \( \nu_c(\tilde{F}_c) = \nu_c(\tilde{F}_c^i) \). Therefore, one gets the relation \( \nu_{D_0}(\tilde{F}) = \nu_{D_0}(\tilde{F}_c^i) \). Under the induction hypothesis, \( \nu_{D_0}(\tilde{F}) \) and \( \nu_{D_0}(\tilde{F}) \) are equal. The association of previous relation ensures the equality

\[ \nu_{D}(\tilde{F}) = \nu_{D}(\tilde{F}) + 1 - \epsilon(D), \]

\( D_0 \) being dicritical, \( \epsilon(D) \) vanishes.

2. \( V(c) \) consists of two components \( D_0 \) and \( D_1 \).

(a) \( D_0 \) and \( D_1 \) are non-dicritical: let \( \tilde{F}_c \) the product of \( \tilde{F}_c^i \) and of a germ of equation for \( D_0 \cup D_1 \). Since the function \( \tilde{F}_c \) is a balanced equation for \( E^*\tilde{F}_c \), \( \nu_c(\tilde{F}_c) \) is equal to \( \nu_c(E^*\tilde{F}_c) + 1 \). Therefore, one gets the relation \( \nu_c(\tilde{F}_c) = \nu_c(\tilde{F}_c^i) \). Under the induction hypothesis, \( \nu_{D_0}(\tilde{F}) \) and \( \nu_{D_0}(\tilde{F}) \) are equal. The association of previous relation ensures the equality

\[ \nu_{D}(\tilde{F}) = \nu_{D}(\tilde{F}) + 1 - \epsilon(D). \]

(b) \( D_0 \) is dicritical and \( D_1 \) is non-dicritical: this case can be treated in the same way.

\[ \square \]

Proof: Let us suppose \( F \) of second kind and let \( c \) be any point of \( D \). The kernel of \( E^*\tilde{F}_c \) clearly intersects \( \tilde{X} \) along the sheaf \( \tilde{X}_F \). Now, we compute the co-kernel over \( c \). In each following case, we give a local expression of \( E^*\tilde{F}_c \) in adapted coordinates. The existence of such coordinates is proved in [9] and uses the second kind hypothesis. We give also a solution \( X \) in \( \tilde{X}_{F_0} \) of the equation \( E^*\tilde{F}_c(X) = g \) for \( g \) in \( \tilde{F}_{\infty} \):

1. \( c \) is a regular point of a non-dicritical component, which is neither a zero nor a pole of \( \tilde{F} \):

\[ E^*\tilde{F}_c = \frac{u}{x} dx , \quad X = \frac{g}{u} \frac{\partial}{\partial x} , \quad (\tilde{F}_{\infty} \mid_0) = O_c. \]
2. \( c \) is a regular point of a dicritical component, which is neither a zero nor a pole of \( \hat{F} \):

\[
E^{*}\frac{\hat{\omega}}{\hat{F}} = u \, dy, \quad X = \frac{g}{u} \frac{\partial}{\partial y}, \quad \left( \tilde{\mathcal{I}}_{F=\infty} \right)_{c} = \mathcal{O}_{c}.
\]

3. \( c \) is a singular point of the divisor:

\[
E^{*}\frac{\hat{\omega}}{\hat{F}} = u \frac{d}{dx}, \quad X = \frac{g}{u} \frac{\partial}{\partial y}, \quad \left( \tilde{\mathcal{I}}_{F=\infty} \right)_{c} = \mathcal{O}_{c}.
\]

4. \( c \) is a zero of \( \hat{F} \):

\[
E^{*}\frac{\hat{\omega}}{\hat{F}} = u \, dy, \quad X = \frac{g}{u} \frac{\partial}{\partial y}, \quad \left( \tilde{\mathcal{I}}_{F=\infty} \right)_{c} = \mathcal{O}_{c}.
\]

5. \( c \) is a pole of \( \hat{F} \):

\[
E^{*}\frac{\hat{\omega}}{\hat{F}} = u \, dy, \quad X = \frac{g}{uy} \frac{\partial}{\partial y}, \quad \left( \tilde{\mathcal{I}}_{F=\infty} \right)_{c} = (y).
\]

Hence, in any case, the morphism \( E^{*}\frac{\hat{\omega}}{\hat{F}}(\cdot) \) is onto the sheaf \( \tilde{\mathcal{I}}_{F=\infty} \), which ensures the exactness of the sequence.

\[ \square \]

**Corollary 2.1.** The space of formal infinitesimal obstructions to the problem \((\mathcal{P})\) is \( H^{1}(\mathcal{D}, \tilde{\mathcal{I}}_{F=\infty}) \). This is a \( \mathbb{C} \)-space of finite dimension. This dimension is a topological invariant.

In order to prove the above corollary, let us first consider \( P \) a germ of curve in \((\mathbb{C}^{2}, 0)\) and \( \tilde{\mathcal{I}}_{P} \) the subsheaf of \( \mathcal{O}_{\mathcal{M}} \) of functions vanishing along the strict transform of \( P \) by \( E \). The map \( E \) is a composition \( E = E_{0} \circ \cdots \circ E_{N} \) where \( E_{i} \) is the standard blowing-up of one point. For any point \( c \) in a divisor \((E_{0} \circ \cdots \circ E_{j})^{-1}(0), 0 \geq j \geq N, \nu_{c}(P) \) refers to the multiplicity at \( c \) of the strict transform of \( P \) with respect to \( E_{0} \circ \cdots \circ E_{j} \).

**Lemme 2.3.**

\[
\dim_{\mathbb{C}} H^{1}(\mathcal{D}, \tilde{\mathcal{I}}_{P}) = \sum_{c} \left( \frac{\nu_{c}(P)(\nu_{c}(P) - 1)}{2} \right)
\]

**Proof:** The proof is an induction on the length of the blowing-up process. Let \( E_{0} \) be the blowing-up of the origin. Let us consider the canonical system of coordinates \((x_{1}, y_{1})\) and \((x_{2}, y_{2})\) in adapted neighborhood of \( E_{0}^{-1}(0) \) such that the change of coordinates is written

\[
y_{2} = y_{1}x_{1}, \quad x_{2} = \frac{1}{y_{1}}.
\]

Let \( p \) be a reduced equation of \( P \) and \( p_{1} \) and \( p_{2} \) defined by

\[
E^{*}p = x_{1}^{\nu_{0}(p)}p_{1} \quad E^{*}p = y_{2}^{\nu_{0}(p)}p_{2}.
\]
Hence, we can describe the space of global sections

\[ H^0(V_1, \tilde{\mathcal{F}}_P) \cong p_1 \mathbb{C}[x_1, y_1] \]
\[ H^0(V_1 \cap V_2, \tilde{\mathcal{F}}_P) \cong p_1 \mathbb{C}[x_1](y_1) \]
\[ H^0(V_2, \tilde{\mathcal{F}}_P) \cong \left\{ y_1^{-\nu_0(p)} \sum_{i,j \in \mathbb{N}^2} a_{ij} x_1^iy_1^{-i} \mid a_{ij} \in \mathbb{C} \right\} \]

An simple computation shows that the following isomorphism

\[ H^0(V_1 \cap V_2, \tilde{\mathcal{F}}_P) / H^0(V_1, \tilde{\mathcal{F}}_P) \bigoplus H^0(V_2, \tilde{\mathcal{F}}_P) \cong \mathbb{C}^{\nu_1(p)(\nu_2(p) - 1)} \]

We decompose the desingularization morphism \( E = E_0 \circ E_1 \) where \( E_0 \) is the first blowing-up of the origin. Let \( \{s_1, \ldots, s_n\} \) refer to the intersection of \( D_0 \) and the strict transform of \( P \). We denote the components of the exceptional divisor \( D_i = E_1^{-1}(s_i) \). For \( i = 1, \ldots, n \), consider \( U_i(\epsilon) = B(s_i, \epsilon), \epsilon > 0 \) be a disc for any smooth metric on \( D_0 \) such that \( U_i \) does not meet \( U_j \) for \( j \neq i \). Let \( U_0 \) be the complementary of \( \bigcup_{i=1}^{4} \bar{B}(s_i, \epsilon/2) \). Finally, let us denote by \( U_i \) the open set \( E_1^{-1}(U_i) \).

The system \( \{U_0, U_1, \ldots, U_i\} \) provides a covering of the divisor \( \mathcal{D} \) and the Mayer-Vietoris sequence for the sheaf \( \tilde{\mathcal{F}}_P \) is written

\[ 0 \to N \to H^1(D, \tilde{\mathcal{F}}_P) \to \bigoplus_i H^1(U_i, \tilde{\mathcal{F}}_P) \to \bigoplus_{ij} H^1(U_i \cap U_j, \tilde{\mathcal{F}}_P) \to 0. \]

where \( N \) is given by

\[ \bigoplus_i H^0(U_i, \tilde{\mathcal{F}}_P) \to \bigoplus_{ij} H^0(U_i \cap U_j, \tilde{\mathcal{F}}_P) \to N \to 0. \]

Since \( U_i \cap U_j \) is Stein and \( \tilde{\mathcal{F}}_P \) a coherent sheaf, the module \( H^1(U_i \cap U_j, \tilde{\mathcal{F}}_P) \) is trivial. Moreover, the morphism \( E_1 \) and the Hartogs argument induce following isomorphisms

\[ H^0(U_i, \tilde{\mathcal{F}}_P) \cong H^0(U_i, \tilde{\mathcal{F}}_P), \]
\[ H^0(U_i \cap U_j, \tilde{\mathcal{F}}_P) \cong H^0(U_i \cap U_j, \tilde{\mathcal{F}}_P). \]

Hence, \( N \) is identified with \( H^1(D_0, \tilde{\mathcal{F}}_P) \). All these remarks and an inductive limit on the neighborhood of \( D_1 \) provide the next isomorphisms

\[ H^1(D, \tilde{\mathcal{F}}_P) \cong H^1(D_0, \tilde{\mathcal{F}}_P) \oplus \bigoplus_i H^1(D_i, \tilde{\mathcal{F}}_P). \]

Therefore, the lemma is a straightforward computation from the hypothesis of induction and the formula above.
Proof: (2.1) Applying the previous lemma with $P = \{\hat{F} = \infty\}$, once gets the finite dimension statement. Furthermore, looking at the formula, one can see that this dimension depends only the topology of the morphism of desingularization which is a topological invariant of the foliation [1].

\[\square\]

2.2 A generic example

Let $r$ and $n \geq 2$ be positive integers and $\mathcal{F}_{n,r_1\ldots r_n}$ the foliation given by the one form
\[x^{r+2}d\left(\frac{x^{r+2} - \sum_{j=1}^{r+1} q_j x^{r+1-j} y^j}{x^{r+1}}\right).\]

Let us suppose that $Q(t) = \sum_{j=1}^{r+1} q_j t^j$ satisfies $Q'(t) = \prod_{i=1}^{n} (t - t_i)^{r_i}$. In ([5]), M. Klughertz proved that these foliations are topological normal forms for $\mathcal{M}$-foliation, i.e., foliations regular after one blowing-up such that all its invariant curves are separatrix. To be more specific, any foliation regular after one blowing-up such that all its invariant curves are separatrix and satisfying

- there is $n$ invariant curves tangent to the divisor.
- these $n$ curves are tangent to the divisor with orders $r_1, \ldots, r_n$.

is topologically to a foliation $\mathcal{F}_{n,r_1\ldots r_n}$. The position of the tangency point $t_1, \ldots, t_n$ does not have any importance. Since, the dimension of $H^1(D, \mathcal{I}_{\hat{F}=\infty})$ is a topological invariant, one can compute this dimension for this family of foliations in order to extend the result to any $\mathcal{M}$-foliation.

By a direct computation, one has $\nu_0(\mathcal{F}_{n,r_1\ldots r_n}) = r + 1$. The foliation is regular after one blowing and the exceptionnal divisor $D_0$ is not invariant. There are $n$ integral curves $S_i$, $i = 1..n$, which are tangent to the divisor. In the canonical coordinates $y = tx, x = x$, the points of tangency are given by $t = t_j$ and the $r_j$ are the respective order of tangency. Hence, the foliation is completely reduced once one has reduced the germ of curves $S_i \cup (D_0)_{t_i}$. Therefore, the foliation has $n$ isolated separatrix, which are the curves $S_i$ if viewed after one blowing.

Let us denote by $h_i$ a reduced equation of $S_i$ at the origin. One can see that $\nu_0(h_i) = r_i + 1$. Futhermore, the component $D_0$ is dicritical with valence $n$. Let $\{\alpha_i\}_{i=1..n-2}$ be $n-2$ equations of curves of the pencil of $D_0$. By definition, the meromorphic function
\[F = \frac{h_1(x,y)h_2(x,y)\cdots h_n(x,y)}{\alpha_1(x,y)\alpha_2(x,y)\cdots\alpha_{n-2}(x,y)}\]

is a balanced equation for $\mathcal{F}$. In particular, since $\nu_0(\alpha_i) = 1$,

\[\nu_0(F) = \sum_{i=1}^{n} \nu_0(h_i) - (n-2) = \sum_{i=1}^{n} (r_i + 1) - n + 2 = r + 2\]
Hence,
\[ \nu_0(F) = \nu_0(F_{n,r_1,\ldots,r_n}) + 1 \]
which was predicted by (1.3). Moreover, in view of (2.1), the space of obstructions is of dimension \( \frac{(n-2)(n-3)}{2} \). Hence, for any \( \mathcal{M} \)-simple foliation \( F \) topologically equivalent to one \( F_{n,r_1,\ldots,r_n} \) with \( n \) greater than 4, there exists a deformation of the resolution space, which cannot carry any isoholonomical deformation of \( F \). As already mentioned, this situation never occurs in the non-dicritical case and suggests that the space of leaves and the resolution space share some analytical invariants.

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