Investigation of the propagation of free waves in a rod in the design of building structures

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Abstract. This article investigates the dispersion dependences of waves propagating in a system consisting of an elastic rod immersed in a linear elastic medium. The phase velocities of a wave with a length $\lambda$, which propagate under stationary conditions, are investigated. It was found that for a sufficiently small $\eta (a/\lambda)$, i.e., for relatively large wavelengths, no waves can propagate in the system under stationary conditions, i.e. there are critical values of $\eta$ below which no real values of the phase velocity are possible.

1. Introduction

The system is presented (figure 1) using a model consisting of an elastic rod of radius $a$, immersed in an elastic medium, in which the waves $P$ and $S$ propagate with the velocity $C_P$ and $C_S$, respectively. It is known that the radial displacements of the bar will be small. The rod is assumed to be radially rigid. A similar model was used to investigate the dynamic response of an earthquake-prone buried pipeline system. The results obtained by us can lead to the determination of the parameters of the system at which resonance phenomena can occur [1].

2. Methods

The model under consideration consists of an infinite cylindrical rod of radius $a$, elastic modulus $E_0$ and density $\rho_0$, immersed in a linear isotropic elastic medium (figure 1). The movement of the bar in the longitudinal direction $Z$ is indicated by $U_p(z,t)$. The radial displacement of the bar is assumed to be zero.

![Figure 1. Design scheme.](image)
It is assumed that the environment behaves like a linear elastic material having a density $\rho$, Poisson's ratio $\nu$ and a shear modulus $\mu$. For the axisymmetric case considered here, the medium can be subject to radial and axial displacements depending on time, respectively, $U_r(r,z,t)$ and $U_z(r,z,t)$. The interaction between the rod and the environment is carried out by means of mechanical shear acting along the outer surface of the cylinder, which prevents between the rod and the medium. Then the equation of the rod will be

$$E_0 \frac{\partial^2 u_p(r,t)}{\partial z^2} + \frac{2r}{a} \frac{\partial u_p(a,r,t)}{\partial r} - \rho_0 \frac{\partial^2 u_p(r,t)}{\partial t^2} = 0$$

where $\tau_{rz}$ - represents the shear stress when interacting with the external surface. Under the above assumption, along with the requirements for the continuity of the displacements of the outer surface of the bar, the following conditions are fulfilled

$$U_r(a,z,t) = 0; \quad U_z(a,z,t) = U_z(z,t);$$

(2)

The problem that we will consider is the propagation of waves having a length $\lambda$, which propagates in the system with a speed $C$ in the $Z$ direction. Dynamic displacements of the environment at $r \geq a$ can be expressed using an incident wave [2].

$$U_r(r,z,t) = \Phi(r)e^{ik(z-ct)}$$
$$U_z(r,z,t) = \Psi(r)e^{ik(z-ct)}$$

$$\Phi(r) = A \left(\frac{h^*}{2\pi}\right) H_1^{(1)}(h^*r) + BH_1^{(1)}(k^*r)$$
$$\Psi(r) = AH_0^{(1)}(h^*r) - B\left(\frac{2k^*}{2\pi}\right)H_0^{(1)}(k^*r)$$

(3)

here $A$ and $B$ are arbitrary constants, $H_n^{(1)}$ - are Hankel functions of the first kind of the $n$-th order.

$$h^* = \left(\frac{2\pi}{\lambda}\right)^2 \left[\frac{c^2}{c_p^2} - 1\right]^\frac{1}{2}; \quad k^* = \left(\frac{2\pi}{\lambda}\right)^2 \left[\frac{c^2}{c_s^2} - 1\right]^\frac{1}{2}$$

$$C_p = \left[\frac{\mu}{\rho}\right]^\frac{1}{2}; \quad C_s = \left[\frac{2(1-\nu)}{(1-2\nu)}\right]\left[\frac{\mu}{\rho}\right]^\frac{1}{2}$$

(4)

$C_p$ and $C_s$ - are the velocities of propagation of $s$ and $p$ waves in an elastic medium. Thus, the terms associated with $A$ represent the waves P, and the terms associated with $B$ represent the propagation of the wave S. Then the constants $A$ and $B$ must satisfy the boundary conditions (2) and (3)

$$\frac{\lambda h^*}{2\pi} H_1^{(1)}(h^*a)A + H_1^{(1)}(k^*a)B = 0;$$

(5)

Besides,

$$U_r(a,z) = 0;$$

$$\tau_{rz}(a,z,t) = \mu \frac{\partial U_z(a,z,t)}{\partial r}$$

(6)

Substituting this expression and the continuity condition into equations (1) and (2), we obtain

$$\frac{\partial^2 u_p(a,z,t)}{\partial t^2} - \rho_0 \frac{\partial^2 u_p(a,z,t)}{\partial z^2} - \left(\frac{2\mu}{a\rho_0}\right)\frac{\partial u_p(a,z,t)}{\partial r} = 0;$$

$$c_0 = \sqrt{E_0/\rho_0}$$

The speed of propagation of longitudinal waves in a free cylindrical rod with $U_r(r,z,t)$, given by equations (2) and (5) are satisfied under stationary conditions if [3].
\[ \Psi(a) = \frac{2\mu}{a\rho_0 c_0^2 \left( \frac{2\pi}{\lambda} \right)^2 \left[ 1 - \left( \frac{c_0}{c_s} \right)^2 \right]} \frac{\partial \Phi(r)}{\partial r} |_{r=a} \]  

(8)

By substituting this expression instead of \( \Psi(r) \), we obtain the second required condition for the constants A and B

\[ a \left( \frac{2\pi}{\lambda} \right)^2 \left[ 1 - \left( \frac{c_0}{c_s} \right)^2 \right] \left[ A H_0^{(1)}(h' a) - B \left( \frac{\lambda k_0^*}{2\pi} \right) H_0^{(1)}(k_0^* a) \right] = - \left( \frac{2\rho c_s^2}{\rho_0 c_0^2} \right) \left[ A h^2 H_1^{(1)}(h' a) - B \left( \frac{4k_0^*}{2\pi} \right) H_1^{(1)}(k_0^* a) \right] \]  

(9)

Here it is convenient to express the solution in terms of dimensionless quantities, especially in terms of dimensionless coefficients related to the velocities of propagation of waves P and S in the medium and the velocity of propagation of c0 waves in a free rod.

\[ \eta = \frac{a}{c_s}; \quad \Gamma = \frac{c}{c_0}; \]

\[ S_c = \frac{c_p}{c_s} = \frac{2(1-v)/(1-2v)}{\pi}; \quad S_p = \frac{c_p}{c_0}; \quad R_D = \frac{\rho}{\rho_0} \]

Using these parameters in relations (5) and (9) and after some transformations, we obtain the following system of equations for A and B:

\[ |E| \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \]  

(10)

where

\[ \begin{bmatrix} Y_p H_1^{(1)}(Y_p) & 2\pi \xi H_1^{(1)}(Y_s) \\ 2\pi \xi \left( 2\pi \xi \right)^2 (1-\Gamma^2) H_0^{(1)}(Y_p) + \left( \frac{2S_p R_D}{S_c^2} \right) Y_p H_1^{(1)}(Y_p) & -Y_p \left( \frac{2R_D S_p}{S_c^2} \right) Y_s H_1^{(1)}(Y_s) + (2\pi \xi)^2 (1-\Gamma^2) H_0^{(1)}(Y_s) \end{bmatrix} \]

\[ \gamma_p = 2\pi \xi \left( \frac{S_c \Gamma}{S_p} \right)^2 - 1 \right]^{1/2}; \quad \gamma_s = 2\pi \xi \left( \frac{S_c \Gamma}{S_p} \right)^2 - 1 \right]^{1/2}; \]

For non-trivial solutions, the determinant \( |E| \) in equation (10) must be equal to zero. Decomposition of the determinant leads to the frequency equation

\[ (1-\Gamma^2) \left[ H_0^{(1)}(Y_p) H_1^{(1)}(Y_s) + \frac{Y_p Y_s}{(2\pi \xi)^2} H_0^{(1)}(Y_s) H_1^{(1)}(Y_p) \right] + \left( \frac{r_D}{\rho_0} \right)^2 \left( \frac{\Gamma}{s_p} \right)^2 - 1 \right]^{1/2} H_1^{(1)}(Y_s) H_1^{(1)}(Y_p) = 0 \]  

(11)

The roots of equation (11) \( \Gamma = c/c_0 \) give the corresponding phase velocities [4,5].

From the derived frequency equation it can be seen that the roots \( \Gamma = c/c_0 \), which determine the phase velocities \( c \), are dimensionless.

\[ \Gamma = \eta \left( \eta, S_c, S_p, R_D \right) \text{, where } S_c \geq \sqrt{2} \]

is a parameter explicitly dependent on Poisson’s ratio D. For real bounded values of \( \eta \), it can be proved that no real roots exist for the values

\[ \Gamma \geq \frac{S_p}{S_c} \]  

(12)

3. Results

Thus, the phase velocities satisfy inequality (10.a), i.e. phase velocities are always less than the
velocities of S waves that can propagate in the medium. It follows that \( \gamma_p \) and \( \gamma_s \), as defined in equations (10), are always complex [6].

Proof that no real roots of \( \Gamma \) exist for \( \Gamma > S_p / S_c \). First, one can prove that no roots of \( \Gamma \) exist in the domain \( S_p^2 < S_c^2 < \Gamma^2 < S_p^2 \).

If \( \Gamma \) exists in this region, then from their definition the equation (12)

\[
\gamma_p = i b_p; \quad \gamma_s = b_s;
\]

where \( b_p > 0; b_s > 0 \) are real.

After some transformations, we get the following equations:

\[
\begin{pmatrix} H_0^{(1)}(i b_p) / H_1^{(1)}(i b_p) \end{pmatrix} + i \alpha \begin{pmatrix} H_0^{(1)}(b_s) / H_0^{(1)}(b_s) \end{pmatrix} + B_i = 0; \quad (13)
\]

\[
\alpha = \frac{b_p b_s}{(1 - \Gamma^2)(2 \pi \eta)^2};
\]

\[
\beta = [R_0 \Gamma^2 / (1 - \Gamma^2 \pi \eta)] \left[(1 - \Gamma^2) S_p^2 \right]^{1/2}; \quad (14)
\]

\( \alpha \) and \( \beta \) will be real. As

\[
H_0^{(1)}(i b_p) / H_1^{(1)}(i b_p) = -i k_0(b_p) / k_1(b_p) = i d
\]

where \( d \) - is real, equation (13) will take the following form:

\[
H_0^{(1)}(b_s) / H_1^{(1)}(b_s) = -(1/\beta) [B + d]
\]

To satisfy this condition, we use the basic definition of the Hankel function [5].

\[
H_n^{(1)}(x) = J_n(x) + i Y_n(x)
\]

\[
I m \left[ (J_0(b_s) + i Y_0(b_s)) / (J_1(b_s) + i Y_1(b_s)) \right] = 0
\]

then the required condition takes the form:

\[
Y_0(b_s) J_1(b_s) - J_0(b_s) Y_1(b_s) = 0
\]

which is called the Vronskaya condition:

\[
W(b_s) = \begin{bmatrix} Y_0(b_s) & J_0(b_s) \\ J_1(b_s) & Y_1(b_s) \end{bmatrix}
\]

Since \( W(b_s) \) cannot be zero, so no roots

\[
S_p^2 < S_c^2 < \Gamma^2 < S_p^2
\]

cannot exist.

Thus, we have shown that there are no real roots in these areas.

When using standard dependencies

\[
H_0^{(1)}(ix) = \left( \frac{2}{\pi \eta} \right) K_0(x); \quad H_1^{(1)}(ix) = -\left( \frac{2}{\pi \eta} \right) K_1(x); \quad (15)
\]

where \( K_n(x) \) - are modified Bessel functions. In (11), substituting (14), we obtain the following relation [7]:

\[
(1 - \Gamma^2) \left[ K_0(\theta b_p) K_1(\theta b_s) - \beta_0 \beta_5 K_0(\theta b_s) K_1(\theta b_p) \right] + [R_0 \beta_5 \Gamma^2 / (2 \pi \eta)] K_1(\theta b_p) K_1(\theta b_s) = 0; \quad (16)
\]

Here \( \beta_p = [1 - \left( \frac{\Gamma}{S_p} \right)^2]^{1/2} > 0; \quad \beta_0 = [1 - \left( \frac{S_c}{S_p} \right)^2]^{1/2} > 0; \]
are all real. Studying equation (14), one can prove that for all bounded \( \eta \) (for \( R_D > 0 \)) all roots must lie in the domain

\[ 1 < \Gamma < S_p / S_c; \]

i.e. phase velocities take values within

\[ C_0 < C < C_s \]  \hspace{1cm} (17)

For \( \eta \to \infty \) or \( R_D = 0 \), the root of the frequency equation is \( \Gamma^2 = 1 \), i.e. \( C = C_0 \).

Now the limit of possible values of the phase velocity given in equation (16) can be determined. First, it is obvious that the condition \( C > C_0 \) must be satisfied, i.e. the stiffness of a buried bar is always greater than the stiffness of a free bar. Consequently, the speed of wave propagation must be greater than \( C_0 \).

It is important to consider the form of displacements for the permissible region \( C_0 < C < C_s \). In this region, \( h^* \) and \( k^* \) that occur in equations (3a) are both imaginary, and therefore, using dependences (13), the displacements \( U_r \) and \( U_z \) take the following form [8,9].

\[
U_r = -(2/\pi)[iA \beta_p K_1(\theta \beta_p r / a) + B \beta_s K_1(\theta \beta_s r / a)]e^{[i2\pi/\lambda(z-ct)]}
\]

\[
U_z = -(2/\pi)[iAK_0(\theta \beta_p r / a) + B \beta_s K_0(\theta \beta_s r / a)]e^{[i2\pi/\lambda(z-ct)]}
\]

Thus, the displacement decreases monotonically at \( r (r > a) \) (figure 2).

4. Conclusion

Numerical results showing dispersion curves \( C / C_0 \) versus \( \eta = \alpha / \lambda \) are shown in figure 3. (a) - (d); for the medium at \( v = 0.25 \); and several values of the parameters \( S_p = C_p / C_0 \) and \( R_D = \rho / \rho_0 \). \( R_D = 0.05 \) represents a relatively soft environment, while \( R_D = 5.0 \) represents a relatively high density environment. Obviously, all curves tend to unity, as \( \eta \) becomes large. The ongoing asymptotic approximation to unity, in particular, is much faster for relatively small values of \( R_D \). Thus, for other small coefficients \( \rho / \rho_0 \) or \( \eta \geq 1 \), the phase velocity approaches the velocity of longitudinal waves in a free rod and, therefore, it is obvious that the interaction between the rod and the environment will be small.
Figure 3 also shows that for a sufficiently small $\eta$, i.e. for relatively large wavelengths $\lambda$, under stationary conditions in the system, no free waves can propagate, i.e. there is a critical value $\eta$, $\eta_c$, below which no real solutions are possible. These critical values in figure 4 at values of $\eta$ are indicated by vertical lines. At these critical values, the phase velocity will be

$$\Gamma = \frac{C}{C_0} = \frac{S_v}{S_c},$$

those $C=C_s$, which has been proven to be the upper bound for the roots of the frequency
equation.
Since it was noted that $C = C_S$ is in all cases the upper boundary of the solution and it is easy to represent the dispersion dependence of $C/C_s$ on $\eta = a/\lambda$. Since it is obvious from the equation $C_0 < C < C_s$ that $R_p/R_c = C_s/C_0 \geq 1$ is the admissible region of this second parameter. The dispersion dependences for $C/C_s$ are presented as functions of $\eta$ and $C_s/C_0$ for some values of the parameter $R_D$. These dependences in figure 4 occur as dispersion surfaces for the medium at $v = 0.25$. It can be seen from the figures that for a large $R_D$, the $C$ value approaches $C_s$.

Figure 4. Dispersion surfaces.

Figure 5 shows the dependence of the $C_s/C_0$ functional for different values of $R_D$. It can be seen from this figure that for a small $R_D (R_D = 0.05)$, waves with relatively long lengths ($\lambda > 10a$) can propagate in the system. For a relatively dense material $R_D = 2$, waves with a length $\lambda > 1.5a$ can propagate in the system.

Thus, it can be concluded that the change in the phase velocity is more noticeable for the values of $C_s/C_0$ in the region approaching unity, and the most perceptible for relatively small values of $\eta$.

Below are the dispersion curves for:
$v = 0.25$
(a)$R_v = 2.0$; (b)$R_v = 3.0$
(c)$R_v = 6.0$ (d)$R_v = 10.0$
$R_D = 5.0$___________
$R_D = 1.0$ __________
$R_D = 0.05$ __________

Figure 5. Dependence $\eta$ on $C_s / C_0$. 
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