A NOTE ON WEAK $\mathcal{H}$ - SPACES, HEAVENLY EQUATIONS
AND THEIR EVOLUTION WEAK HEAVENLY EQUATIONS

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It is find a non-linear partial differential equation which we show contains the first Heavenly equation of Self-dual gravity and generalize the second one. This differential equation we call ”Weak Heavenly Equation” ($WH$-equation). For the two-dimensional case the $WH$-equation is brought into the evolution form (Cauchy-Kovalevski form) using the Legendre transformation. Finally, we find that this transformed equation (”Evolution Weak Heavenly Equation”) does admit very simple solutions.

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1. Introduction

Complex Self-dual (SD) gravity has been a great arena to understand new advances in other branches of mathematical physics; namely, Quantum Gravity, Conformal Field Theory, Integrable Models, Topological Field Theories, String Theory, etc. For this reason, any advance in this line is of great importance in the development of these branches.

Very recently has been a series of papers on SD gravity [1-5], based in the seminal Grant’s paper [6]. In they, the first and the second heavenly equations and all hyper-heavenly equations with (and without) cosmological constant were broughted into an evolution form called Cauchy-Kovalevski form. Some solutions for all these evolution equations also has been found. In all cases the relation between the heavenly equations and that of evolution has been a Legendre transformation applied on suitable coordinates.

On the other hand, the richness of structure in SD gravity allow us to explore what another possibilities might exists there. In this paper we explore one of these possibilities. We find that working with the spinorial formulation of SD gravity we find a non-linear partial differential equation of the second order. This equation we call ”Weak Heavenly equation” (WH-equation) and involves a holomorphic function of its arguments. This equation contains the first Heavenly equation and in fact generalize the second one. Taking the dimensional reduction of the WH-equation from 4 dimensions to the 2 dimensional case, we were able to find simple evolution equations and also very simple solutions for they.

The organization of this paper is as follows. In section 2 we find the differential equation of ”weak H”-spaces using the basic spinorial formalism. This derivation is based completely on an unpublished work by one of us [7]. We show how the first Heavenly equation results in this context and how the WH-equation is a generalization of the second Heavenly equation. The section 3 is devoted to perform a dimensional reduction of the WH-equation considering only the two dimensional case. We write the WH-equation in the Cauchy-Kovalevski form and we find some solutions by direct integration. Finally, in
section 4, we give our concluding remarks.

2. Differential Equations of Weak $\mathcal{H}$-Spaces

(i).- Generalities

In this section we start from the usual spinorial formulation for the 2-form formalism for SD gravity on a four dimensional complex Riemannian manifold $\mathcal{M}$ [8,9]. We would like to integrate the equations:

$$dS^\hat{A}\hat{B} + 2\alpha \wedge S^\hat{A}\hat{B} = 0$$ (2.1)

for $S^\hat{A}\hat{B}$ the anti-self-dual 2-form on $\mathcal{M}$ given by

$$S^\hat{A}\hat{B} = \frac{1}{2} \epsilon_{RS} g^R{}^A \wedge g^S{}^B,$$ (2.2)

where $\hat{A}, \hat{B} \in \{1, 2\}$, $\alpha$ is a 1-form on $\mathcal{M}$, $\epsilon_{RS}$ is the usual Levi-Civita’s matrix $(\epsilon_{RS}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = (\epsilon^{RS})$ and $g^{A\hat{B}}$ is a 1-form on $\mathcal{M}$ which is the spinorial tetrad defining the Riemannian metric as

$$g = -\frac{1}{2} g_{A\hat{B}} \otimes g^{A\hat{B}}.$$ (2.3)

Here $\otimes$ denotes the tensor product.

At this level we can invoke one aspect of the Frobenius theorem which implies the existence of the scalar spinorial functions on $\mathcal{M}$

$$\lambda^A{}^B, \tilde{\lambda}^A{}^B, q^A, \tilde{q}^A$$ (2.4a)

being $\lambda^A{}^B$ and $\tilde{\lambda}^A{}^B$ constrained by
\[ \det(\lambda^A_B), \det(\tilde{\lambda}^A_B) \neq 0. \] (2.4b)

Moreover, we can put the spinorial tetrad as

\[ g^{A\dot{1}} = \phi^{-1} e^\sigma dq^A; \quad g^{A\dot{2}} = \phi^{-1} e^{-\sigma} d\tilde{q}^A \] (2.5)

where we have used a convenient parametrization and the fact that this tetrad is obviously meaningful only modulo the SL(2,\(\mathbb{C}\)) transformations.1

It is easy to see that substituting Eqs. (2.5) into (2.1) we obtain for \( \dot{A}\dot{B} = \dot{1}\dot{1} \) and \( \dot{2}\dot{2} \), respectively

\[ (\alpha - d \ln \phi + d\sigma) \wedge dq^1 \wedge dq^2 = 0 \] (2.6a)

and

\[ (\alpha - d \ln \phi - d\sigma) \wedge d\tilde{q}^1 \wedge d\tilde{q}^2 = 0. \] (2.6b)

The pair \( \{q^A, \tilde{q}^A\} \) constitute a local chart on \( M \). So, Eqs. (2.6a, b) and the equation for the volume form \( \omega = -\phi^{-4}dq^1 \wedge dq^2 \wedge d\tilde{q}^1 \wedge d\tilde{q}^2 \) imply that \( \alpha \) can be written as

\[ \alpha = d \ln \phi + \frac{\partial \sigma}{\partial q^A} dq^A - \frac{\partial \sigma}{\partial \tilde{q}^A} d\tilde{q}^A. \] (2.7)

Taking the gauge on the spinorial tetrad (2.5) to be

\[ g^{A\dot{1}} = \phi^{-1} e^\sigma l^A_B dq^B; \quad g^{A\dot{2}} = \phi^{-1} e^{-\sigma} \tilde{l}^A_B d\tilde{q}^B \] (2.8)

with \( \det(l^A_B) = 1 = \det(\tilde{l}^A_B) \), it is possible to express the metric (2.3) in the form

1 Notice that the parametrization can be written as \( g^{A\dot{1}} = \phi^{-1} e^\sigma l^A_B dq^B \) and \( g^{A\dot{2}} = \phi^{-1} e^{-\sigma} \tilde{l}^A_B d\tilde{q}^B \). We can take \( g^{AB} \rightarrow g'AB = L^A_Rg^{RB} \) with \( (L^A_B) \in \text{SL}(2,\mathbb{C}) \) and \( \det(L^A_B) = 1 \). Moreover, taking \( L^A_B = -l^A_B \) we have \( g'\dot{A}\dot{1} = \phi^{-1} e^\sigma dq^A \), similarly for \( g'\dot{A}\dot{2} \) we take \( L^A_B = -\tilde{l}^A_B \).
\[ g = -\phi^{-2} \Omega_{AB} \, dq^A \otimes d\tilde{q}^B \] (2.9)

where \( \Omega_{AB} := \epsilon_{RS} l^R_A \, \tilde{l}^S_B \). This definition satisfies

\[ \det (\Omega_{AB}) = \frac{1}{2} \Omega_{AB} \, \Omega^{AB} = 1. \] (2.10)

Considering Eq. (2.1) with \( \dot{A} \dot{B} = \dot{1} \dot{2} \) the integrability conditions are given by

\[ d\alpha \wedge S^{12} = 0, \] (2.11)

it can be reduced to the formula:

\[ \Omega^{AB} \, \frac{\partial^2 \sigma}{\partial q^A \partial \tilde{q}^B} = 0. \] (2.12)

On the other hand, Eq. (2.2), for \( \dot{A} \dot{B} = \dot{1} \dot{2} \) it is found to be equivalent to a pair of scalar conditions

\[ \frac{\partial}{\partial q^A} (e^{2\sigma} \Omega^B_A) = 0, \quad \frac{\partial}{\partial \tilde{q}^B} (e^{-2\sigma} \Omega^A_B) = 0. \] (2.13)

Manipulating these equations we obtain the following set of differential equations

\[ \frac{\partial^2 \Omega^{AB}}{\partial q^A \partial \tilde{q}^B} + 4 \, \Omega^{AB} \, \frac{\partial \sigma}{\partial q^A} \frac{\partial \sigma}{\partial \tilde{q}^B} = 0 \] (2.14a)

\[ \Omega^{AB} \, \frac{\partial^2 \sigma}{\partial q^A \partial \tilde{q}^B} = 0. \] (2.14b)

Therefore Eq. (2.12) is implicitly contained in Eqs. (2.13) and does not represent an independent condition.

All this is very close to the description of \( \mathcal{H} \)-spaces theory via the \( \Omega \) formalism [8,9]. To see this we make \( \sigma = \text{constant} \), which implies that Eqs. (2.13) are equivalent to the existence of a scalar holomorphic function \( \Omega = \Omega(q_A, \tilde{q}_A) \) of its arguments such that
$$\Omega_{AB} = \frac{\partial^2 \Omega}{\partial q^A \partial q^B}. \quad (2.15)$$

The condition $\det(\Omega_{AB}) = 1$ reduces directly to the first heavenly equation and the metric becomes conformally equivalent to the $\mathcal{H}$-space.

(ii).- The Weak Heavenly Equations

We shall now seek for the equivalent formulation of the structure described by Eqs. (2.13) and (2.14) corresponding to the $\Theta$-formalism for the $\mathcal{H}$-spaces [8,9]. For this purpose, we observe first that Eqs. (2.13) are equivalent to the existence of functions $P_A, \tilde{P}_A$ such that they satisfy

$$e^{2\sigma} \Omega_{AB} = \frac{\partial}{\partial q^A} \tilde{P}_B, \quad e^{-2\sigma} \Omega_{AB} = \frac{\partial}{\partial \tilde{q}^B} P_A. \quad (2.16)$$

From Eq. (2.16) and the condition $\det(\Omega_{AB}) = 1$, we infer that

$$\det \left( \frac{\partial}{\partial q^A} \tilde{P}_B \right) = e^{4\sigma} \quad \text{and} \quad \det \left( \frac{\partial}{\partial \tilde{q}^B} P_A \right) = e^{-4\sigma}. \quad (2.17)$$

This means that the Jacobians

$$\frac{\partial(\tilde{P}_1, \tilde{P}_2)}{\partial(q^1, q^2)} \quad \text{and} \quad \frac{\partial(P_1, P_2)}{\partial(\tilde{q}_1, \tilde{q}_2)}$$

are different from zero. Thus, in place of the local chart $\{q_A, \tilde{q}_A\}$ we can, alternatively, employ the charts $\{q_A, P_A\}$ and $\{\tilde{q}_A, \tilde{P}_A\}$.

For the first chart $\{q_A, P_A\}$ the spinorial tetrad is

$$g^{A1} = \chi^{-1} dq^A \quad (2.18a)$$

$$g^{A2} = \chi^{-1}(dP^A - Q^A_B \, dq^B) \quad (2.18b)$$
with $P_A = P_A(q_A, \tilde{q}_A)$, $\chi^{-1} = \phi e^{-\sigma}$ and $Q^A_B = \frac{\partial P^A}{\partial q^B}$. For the second chart $\{\tilde{q}_A, \tilde{P}_A\}$ we have

$$g^{A_1} = -\tilde{\chi}^{-1}(d\tilde{P}^A - \tilde{Q}^A_B d\tilde{q}^B) \quad (2.19a)$$

$$g^{A_2} = \tilde{\chi}^{-1} d\tilde{q}^A \quad (2.19b)$$

where $\tilde{P}_A = \tilde{P}_A(q_A, \tilde{q}_A)$, $\tilde{\chi}^{-1} = \phi e^{\sigma}$ and $\tilde{Q}^A_B = \frac{\partial \tilde{P}^A}{\partial q^B}$.

In both cases the metric takes the form

$$g = -\chi^{-2} dq^A \otimes (dP_A - Q_{AB} dq^B), \quad (2.20a)$$

and

$$g = -\tilde{\chi}^{-2} d\tilde{q}^A \otimes (d\tilde{P}_A - \tilde{Q}_{AB} d\tilde{q}^B). \quad (2.20b)$$

Thus, apart from the conformal factors the metric is determined entirely by the symmetric functions $Q_{(AB)}$ or $\tilde{Q}_{(AB)}$, respectively.

On the other hand, one can write Eqs. (2.1) and (2.2) for $\dot{A}\dot{B} = \dot{1}\dot{1}$ in the context of the $\Theta$-formalism. These equations leads to

$$(\alpha - d \ln \chi) \wedge dq^1 \wedge dq^2 = 0 \quad (2.21)$$

which implies that $\alpha$ takes the form

$$\alpha = d \ln \chi + \alpha_A dq^A. \quad (2.22)$$

Working with $S^{12}$ from (2.2) we obtain

$$(2\alpha_A + \frac{\partial}{\partial P^A} Q^S_S) dp^A \wedge dq^1 \wedge dq^2 = 0. \quad (2.23)$$
From this equation, it can be observed that

$$\alpha_A = -\frac{1}{2} \frac{\partial}{\partial P^A} Q^S_S,$$  \hspace{1cm} (2.24)

and consequently, Eq. (2.22) amounts to

$$\alpha = d \ln \chi - \frac{1}{2} \frac{\partial}{\partial P^A} Q^S_S \ dq^A.$$  \hspace{1cm} (2.25)

After some manipulations of Eq. (2.1) for $\dot{A}\dot{B} = \dot{2}\dot{2}$ and cancelling terms involving $d \ln \chi$, the factor $\chi^{-2}$ and considering the fact that $\det(Q^A_B) = 0$, it can be expressed as

$$-dQ_{AB} \wedge dP^A \wedge dq^B + d(\det(Q^A_B)) \wedge dq^1 \wedge dq^2$$

$$-\frac{\partial}{\partial P^c} Q^S_S dq^c \wedge [dP^1 \wedge dp^2 - Q_{AB}dP^A \wedge dq^B] = 0.$$  \hspace{1cm} (2.26)

Or, equivalently

$$\left\{ \frac{\partial Q^B_A}{\partial P^B} - \frac{\partial}{\partial P^A} Q^B_B \right\} dq^A \wedge dP^1 \wedge dP^2$$

$$+\left\{ \frac{\partial}{\partial P^A} \det (Q^R_S) + Q^R_A \frac{\partial}{\partial P^R} Q^S_S - \frac{\partial Q^B_A}{\partial q^B} \right\} dP^A \wedge dq^1 \wedge dq^2 = 0.$$  \hspace{1cm} (2.27)

After some manipulations we can conclude that Eqs. (2.1) are fulfilled by $S\dot{A}\dot{B}$ in the chart $\{q_A, P_A\}$ if and only if the structural functions $Q^A_B$ fulfill the differential conditions

$$\frac{\partial}{\partial P^B} Q^B_A = 0,$$  \hspace{1cm} (2.28a)

$$Q^B_S \frac{\partial}{\partial P^B} Q^S_A + \frac{\partial}{\partial q^B} Q^B_A = 0.$$  \hspace{1cm} (2.28b)

with the corresponding 1-form $\alpha$ given by Eq. (2.25).

A similar procedure can be exactly realized with the other equivalent chart $\{\tilde{q}_A, \tilde{P}_A\}$ leading to similar set of equations.
A direct consequence of Eqs. (2.28a, b) is the existence of the scalar holomorphic function of its arguments $\theta_A = \theta_A(q_A, P_A)$, which satisfies

$$Q_{AB} = \frac{\partial}{\partial P^B} \theta_A. \quad (2.29)$$

At this level it is convenient to introduce the notational conventions

$$\partial_A : = \frac{\partial}{\partial P^A}, \quad \partial^A : = \frac{\partial}{\partial P_A},$$

$$\partial_A : = \frac{\partial}{\partial q^A}, \quad \partial^A : = \frac{\partial}{\partial q_A}. \quad (2.30)$$

So, it is possible to give a more concise form for Eqs. (2.29) and (2.28b), they are respectively

$$Q^B_A = -\partial^B \theta_A \quad (2.31)$$

and

$$\partial^B \theta_R \cdot \partial_R \theta_A - \partial^B \partial_B \theta_A = 0. \quad (2.32)$$

We shall consider they as the fundamental equations of the weak heavens ($\mathcal{WH}$) in the $\theta$-formalism. Eq. (2.32) can be expressed as

$$\partial^B \{ \partial_B \theta^R \cdot \partial_R \theta_A + \partial_B \theta_A \} = 0, \quad (2.33)$$

implying the existence of a scalar function $\phi_A$ which satisfies the conservation law $\partial^B \partial_B \phi_A - 0$ with

$$\partial_B \theta^R \cdot \partial_R \theta_A + \partial_B \theta_A = \partial_B \phi_A. \quad (2.34)$$
The structure of the WH (2.32) is intrinsically related to the properties of the 1-form (2.25) which satisfies the proposition

\[ d\alpha = 0 \iff Q_s^S = -\partial^* \theta_s = \frac{\partial}{\partial q^A} \mathcal{R}(q) \cdot P^A + S, \]  

where \( \mathcal{R} \) and \( S \) are arbitrary functions.

(iii).- Reduction to the First Heavenly Equation

This case corresponds to the metrics conformally equivalent to a right-flat space. To see this we consider the \( \Omega \)-formalism [8,9]. According with Eq. (2.7) and (2.11), \( \alpha \) can be expressed as

\[ d\alpha = -2 \frac{\partial^2 \sigma}{\partial q^A \partial \tilde{q}^B}. \]  

So, we have of course

\[ d\alpha = 0 \iff \frac{\partial^2 \sigma}{\partial q^A \partial \tilde{q}^B} = 0, \]

which implies that \( \sigma = \rho(q) + \tilde{\rho}(\tilde{q}) \). With \( \sigma \) of this form, we can equivalently spell out (2.13) in the form

\[ \frac{\partial \Omega'_A}{\partial q^A} = 0, \quad \frac{\partial \Omega'_A}{\partial \tilde{q}^B} = 0 \]  

where \( \Omega'_{AB} = \exp(2(\rho - \tilde{\rho}))\Omega_{AB} \). This equation implies the existence of the scalar functions \( \Omega \) such that

\[ \Omega'_{AB} = \frac{\partial^2 \Omega}{\partial q^A \partial \tilde{q}^B} \Rightarrow \Omega = \exp(2(\tilde{\rho} - \rho)) \frac{\partial^2 \Omega}{\partial q^A \partial \tilde{q}^B}. \]

The metric is now

\[ g = -\phi^{-2} \exp(2(\rho - \tilde{\rho})) \frac{\partial^2 \Omega}{\partial q^A \partial \tilde{q}^B} dq^A \otimes d\tilde{q}^B \]  

\[ (2.40) \]
while \( \det(\Omega_{\partial^2 q^A \partial \tilde{q}^B}) = 4\exp(4(\rho - \tilde{\rho})) \).

In the last step, we execute the coordinate transformation in the re-parametrization of the two congruences of the null strings

\[
q^A = q^A(q'), \quad \tilde{q}^A = \tilde{q}^A(\tilde{q}')
\]

with Jacobians so adjusted that

\[
\frac{\partial(q')}{\partial(q)} = \exp(4\rho), \quad \frac{\partial(\tilde{q}')}{\partial(q)} = \exp(-4\tilde{\rho}).
\]

They are different from zero. This reduces (2.40) to the first Heavenly equation

\[
\det(\frac{\partial^2 \Omega}{\partial q^A \partial \tilde{q}^B}) = 1.
\]

The metric in the new chart \( \{q'^A, \tilde{q}'^A\} \) is now

\[
g = -\phi'^{-2} \frac{\partial^2 \Omega}{\partial q'^A \partial \tilde{q}'^B} dq'^A \otimes d\tilde{q}'^B, \quad \phi' = \phi \exp(\rho - \tilde{\rho}).
\]

\[iv).- \text{Generalization of the Second Heavenly Equation}\]

This SL(2, C)-gauge transformations leave the tetrad (2.18a, b) form-invariant 2

\[
g^A_{\bar{i}} \Rightarrow g'^A_{\bar{i}} = \chi'^{-1} dq'^A
\]

\[
g^A_{\bar{2}} \Rightarrow g'^A_{\bar{2}} = \chi'^{-1}(dp'^A - Q'^A_B dq'^B)
\]

\[2 \text{ The SL}(2, \mathbb{C})\)-gauge transformations induced by } q^A = q^A(q'), \frac{\partial(q')}{\partial(q)} : \exp(4\rho) \neq 0, \text{ are } \frac{\partial q^A}{\partial q'^B} = \exp(-2\rho)l^A_B, \frac{\partial q'^A}{\partial q^B} = -\exp(2\rho)l^A_B, \det(l^A_B) = 1 \text{ where the matrix } (l^A_B) \in \text{SL}(2, \mathbb{C}).\]
if and only if the structural functions $\chi$ and $Q^A_B$ transforms as

$$\chi' = \chi \exp(2\rho) \quad (2.46a)$$

$$P'^A = \frac{\partial q'^A}{\partial q^B} P^B + \tau^A; \quad \frac{\partial \tau^A}{\partial P^B} = 0 \quad (2.46c)$$

and

$$Q'^A_B = \frac{\partial q'^A}{\partial q^R} Q^R_S \frac{\partial q^S}{\partial q'^B} + P^R \frac{\partial}{\partial q^B} \left( \frac{\partial q'^A}{\partial q^R} \right) + \frac{\partial \tau^A}{\partial q^B}. \quad (2.48)$$

Employing Eq. (2.25) in the primed version and Eq. (2.48) we can show that assuming that Eqs. (2.1) are fulfilled, the 1-form $\alpha$ is invariant under the discussed coordinate-tetrad transformation i.e.

$$\alpha' = \alpha. \quad (2.49)$$

Working in the $\{q_A, P_A\}$ chart, we take $d\alpha = 0$, then according with (2.35) we have

$$Q^S_S = -\frac{\partial \mathcal{R}(q)}{\partial q^S} \cdot P^S - S(q). \quad (50a)$$

Passing to the chart $\{q'_A, P'_A\}$ and using (2.48) we arrive to

$$Q'^S_S = P^S \frac{\partial}{\partial q^S} \left[ \ln \frac{\partial(q')}{\partial(q)} - \mathcal{R}(q) \right] + \frac{\partial \tau^R}{\partial q^R} - S(q). \quad (2.50b)$$

Since $\mathcal{R}(q)$ and $S(q)$ are arbitrary functions, without any lost of generality we can take

$$\mathcal{R}(q) = \ln \frac{\partial(q')}{\partial(q)}, \quad S(q) = \frac{\partial \tau^R}{\partial q^R}. \quad (2.51)$$

Therefore assuming that $d\alpha = 0$ this implies that $Q^S_S$ vanishes, or

$$d\alpha = 0 \iff Q^S_S = 0 \quad (2.52)$$
and therefore from Eq. (2.31)

\[ Q^S_S = \partial^S \theta_S = 0. \]  

(2.53)

Thus, there exists the holomorphic scalar function \( \Theta \) satisfying

\[ \theta_A = \partial_A \Theta = \frac{\partial}{\partial P^A} \Theta. \]  

(2.54)

Correspondingly

\[ Q_{AB} = \partial_A \partial_B \Theta \]  

(2.55)

is automatically symmetric. Substituting Eq. (2.55) in Eq. (2.28b) we get that the \( \mathcal{WH} \)-equation reduces to

\[-\partial^S \partial^R \Theta \cdot \partial_A (\partial_S \partial_R \Theta) - \partial^R \rho_R \partial_A \Theta = 0 \]  

(2.56)

or

\[-\partial_A \left\{ \frac{1}{2} \partial^S \partial^R \Theta \cdot \partial_S \partial_R \Theta + \partial^R \rho_R \Theta \right\} = 0 \]  

(2.57)

and so the \( \mathcal{WH} \)-equation reduces to the scalar condition

\[ \frac{1}{2} \partial^A \partial^B \Theta \cdot \partial_A \partial_B \Theta + \partial^A \rho_A \Theta = \Xi(q). \]  

(2.58)

But with \( Q_{AB} = \partial_A \partial_B \Theta \), we can send \( \Theta \Rightarrow \Theta + \chi^A(q)P_A \), where \( \chi^A(q) \) is chosen to be a solution to \( \rho_{AX^A} = \Xi(q) \). This maintains the basic \( Q_{AB} = \partial_A \partial_B \Theta \), reducing (2.58) to the second heavenly equation in its standard form [8,9]

\[ \frac{1}{2} \partial^A \partial^B \Theta \cdot \partial_A \partial_B \Theta + \partial^A \rho_B \Theta = 0. \]  

(2.57)

The metric is now
being thus conformally equivalent to the (strong) \( \mathcal{H} \)-spaces [8,9].

The basic point of this argument is that the \( \mathcal{W} \mathcal{H} \)-equation constitute a legitimate generalization of the second heavenly equation, containing it as a special case.

Detailed computations about connections, curvature, Bianchi identities, etc. can be found in Ref. [7].

3. The Evolution Weak Heavenly Equations

In this section we first consider the \( \mathcal{W} \mathcal{H} \)-equation given by (2.28). After performing the dimensional reduction of this equation, we have

\[
\frac{\partial \Theta}{\partial p} \cdot \frac{\partial^2 \Theta}{\partial p^2} + \frac{\partial^2 \Theta}{\partial p \partial q} = 0
\]

defined on a two-dimensional manifold \( \Sigma \) with coordinates \( \{p, q\} \) and where \( p = P^1, \ q = q^1 \) and \( \theta^1 = \theta_1 = \theta(p, q) \).

Under this dimensional reduction the metric is

\[
g = -\chi^{-2} dq \otimes dp, \quad (3.2)
\]

the conformally flat metric, because \( Q_1^1 = \frac{\partial p}{\partial q} = 0 \).

For our purpose is more convenient to write (3.1) in the form

\[
\theta_{,p} \cdot \theta_{,pp} - \theta_{,pq} = 0.
\]

(3.3)

where \( \theta_{,pq} \equiv \frac{\partial^2 \theta}{\partial p \partial q} \).

In terms of differential forms the above equation is as follows
\[ dθ - θ_p dp - θ_q dq = 0, \]  
\text{(3.4a)}

\[ \frac{1}{2} d(θ^2_p) ∧ dq - dθ_p ∧ dp = 0. \]  
\text{(3.4b)}

Writing (3.4a) as

\[ d(θ - pθ_p) + pdθ_p - θ_q dq = 0. \]  
\text{(3.5)}

Now, performing the following Legendre transformation in the similar spirit of [1]

\[ t = -θ_p \Rightarrow p = p(t, q) \]

\[ H = H(t, q) = θ(p(t, q), q) + t · p(t, q), \]  
\text{(3.6)}

from (3.4b) and (3.5) we easily find

\[ dH - H_t dt - H_q dq = 0, \]  
\text{(3.7a)}

\[ t dt ∧ dq + H_{tq} dt ∧ dq = 0. \]  
\text{(3.7b)}

Thus, the above equations lead to

\[ H_{tq} = -t. \]  
\text{(3.8)}

a second order partial differential equation for a holomorphic function \( H(q, t) \) of its arguments. This is a very simple equation which can be directly integrated out

\[ H(t, q) = -\frac{1}{2} qt^2 + \int f(q) dq \]  
\text{(3.9)}
where \( f(q) \) is a function which only depends on \( q \).

Alternatively, we can choose the other coordinate in (3.4) to perform the Legendre transformation

\[
t = -\theta_q \quad \Rightarrow \quad q = q(t, p)
\]

\[
\mathcal{H} = \mathcal{H}(t, p) = \theta(p, q, (t, p)) + t \cdot q(t, p).
\] (3.10)

Using (3.4ab) and

\[
d(\theta - q\theta_q) + qd\theta_q - \theta_x dp = 0
\] (3.11)

we find (taking \( p \equiv x \))

\[
d\mathcal{H} - \mathcal{H}_tdt - \mathcal{H}_xdx = 0,
\] (3.12a)

\[
\frac{1}{2}d(\mathcal{H}_x^2) \wedge d\mathcal{H}_t - d\mathcal{H}_x \wedge dx = 0.
\] (3.12b)

Equivalently we obtain

\[
\mathcal{H}_{xt} [\mathcal{H}_{xt} \cdot \mathcal{H}_{xx} - \mathcal{H}_{xx} \cdot \frac{\mathcal{H}_{tt}}{\mathcal{H}_{xt}} + 1] = 0.
\] (3.13)

Assuming that \( \mathcal{H}_{xt} \) is different from zero we find that (3.13) reduces to

\[
\mathcal{H}_{xx} \mathcal{H}_{tt} - \mathcal{H}_{xt}^2 - (log \mathcal{H}_x)_t = 0.
\] (3.14)

This equation appears to be the second heavenly equation for the two variables \( \{x, t\} \) with an additional logarithmic term.

It is possible to find formal solutions to (3.14) in the sense of [1, 6], but instead of this, we would like to perform a second Legendre transformation on it. This as an attempt
to put this equation in a more simple appearance and so, try to find simple solutions for it.

In terms of differential forms (3.14) gives

\[ dH - H_x dx - H_t dt = 0, \quad (3.15a) \]

\[ dH_x \wedge dH_t + d(logH_x) \wedge dx = 0. \quad (3.15b) \]

Performing the Legendre transformation

\[ z = -H_x \quad \Rightarrow \quad x = x(z,t) \]

\[ F = F(z,t) = H(x(z,t),t) + z \cdot x(z,t) \quad (3.16) \]

From (3.15b) and

\[ d(H - xH_{xx}) + xdH_x - H_t dt = 0 \quad (3.17) \]

one gets

\[ dF - F_z dz - F_t dt = 0 \quad (3.18a) \]

\[ d(-z) \wedge dF_t + d(log(-z)) \wedge dF_z = 0. \quad (3.18b) \]

The logarithmic term imposes a constriction on the possible values of the coordinate $z$. So, restricting our procedure to $z < 0$ we find

\[ F_{tt} + \frac{1}{z} F_{zt} = 0. \quad (3.19) \]
This equation has very simple solutions given by

$$\mathcal{F}(z, t) = \exp[k \left( \frac{z^2}{2} + t \right)]$$

(3.20)

where $k$ is a constant.

It is remarkable that the Cauchy-Kovalevski form of the $W\mathcal{H}$-equation, coincides with a modification of the second heavenly equation (3.14).

Also it is a very remarkable that the simplicity found in the solutions for the $W\mathcal{H}$-equation in both cases.

4. Concluding Remarks

In this paper we found a non-linear partial differential equation of the second order that we have call Weak Heavenly equation $W\mathcal{H}$. We showed that this equation is connected with the first and second Heavenly equations in a close way. In fact, the $W\mathcal{H}$ equation appears as a natural generalization for the second Heavenly equation. It is claimed that the $W\mathcal{H}$-equation is a more fundamental equations of SD gravity than the Heavenly ones.

We perform the dimensional reduction for the $W\mathcal{H}$-equation defined from the four dimensional manifold $\mathcal{M}$ to the two-manifold $\Sigma$. On $\Sigma$ we proved that the metric is conformally flat. After the dimensional reduction we perform a Legendre transformation on the coordinate $p$ and we found a very simple evolution $W\mathcal{H}$-equation which admits very simple solutions. The choice of the coordinate $q$ to perform the Legendre transformation gave us a differential equation which looks like the second Heavenly equation for two dimensions. This differential equation has an additional logarithmic term. Making a new Legendre transformation on it we finally found a very simple differential equation. This equation also admits simple solutions.
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