The $M^X/M/c$ queue with state-dependent control at idle time and catastrophes

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Abstract. In this paper, we consider an $M^X/M/c$ queue with state-dependent control at idle time and catastrophes. Properties of the queues which terminate when the servers become idle are firstly studied. Recurrence, equilibrium distribution and equilibrium queue-size structure are studied for the case of resurrection and no catastrophes. All of these results and the first effective catastrophe occurrence time are then investigated for the case of resurrection and catastrophes. In particular, we can obtain the Laplace transform of the transition probability for the absorptive $M^X/M/c$ queue.

Keywords. Markovian bulk-arriving queues, equilibrium distribution, recurrence, queue size, effective catastrophe

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1. Introduction

Markovian queue theory is a basic and important branch of queueing theory. It interweaves general theory of queueing models and general theory and applications of continuous time Markov chains and has become a very successful and fruitful research field. Many research works can be referenced, for example, Gross and Harris [15], Asmussen [3] for the former and Anderson [1] for the latter. See also Chen [9] and [10], which contain many new materials for continuous time Markov chains.

Markovian queueing models with state-independent and state-dependent controls have also attracted considerable research interests. A most recent work on this direction could be seen in Chen et al [5]. In these models, arbitrary inputs are allowed when the queue is empty. Usually, this is due to the consideration of improving working efficiency. Gelenbe [13] and Gelenbe et al [14] introduced the particularly interesting concept of negative arrivals, whilst other related papers include Jain and Sigman [16] and Bayer and Boxma [4]. Parthasarathy and Krishna Kumar [19] allowed arbitrary input when the queue is empty, and Chen and Renshaw [7, 8] introduced the possibility of removing the entire workload. Di Crescenzo et al [11] considered the first effective catastrophe occurrence time of a birth-death process. Dudin and Karolik [12] investigated a BMAP/SM/1 system which is exposed to disasters arrivals. The $M_t/M_t/N$ queue with catastrophes can be seen in Zeifman and Korotysheva.

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From practical point of view, models with disasters are quite interesting, e.g., the work of hardware influenced by breaks and occasional power disappearance, the work of communication systems influenced by computer viruses or intentional external interventions, deletion of transactions in databases, the operation of air defense radars, etc. More detailed information about the real and potential applications and corresponding descriptions can be found in [2].

It is worth noting that the papers [5, 7, 8] discussed the queues with single server in the system. The present paper considers $c$-servers in the system, it is a natural generalization of the models considered in [5, 7, 8]. Since there are more than one servers in the system, the method in [5, 7, 8] fails and we have to find some other techniques and methods to treat it. Moreover, in order to consider the recurrence properties, equilibrium distribution, equilibrium queue-size structure and the first effective catastrophe occurrence time of the modified $M^X/M/c$ queue, we have to first show all the Laplace transforms of the transition probability for the absorptive $M^X/M/c$ queue, as usual, this is very difficult to get the transition probability or transition function for a general Markov process.

We shall define our model by specifying the infinitesimal characteristic, i.e., the so called $q$-matrix. Throughout this paper, let $E = \{0, 1, 2, \cdots \}$ denote the set of nonnegative integers.

**Definition 1.1.** A conservative $q$-matrix $Q = (q_{ij}; i, j \in E)$ is called the $M^X/M/c$ queueing $q$-matrix with state-dependent control at idle time and catastrophes (henceforth referred to as the modified $M^X/M/c$ $q$-matrix), if

$$Q = Q^* + Q_s + Q_d,$$

where $Q^* = (q^*_{ij}; i, j \in E)$, $Q_s = (q_s^{(s)}_{ij}; i, j \in E)$ and $Q_d = (q_d^{(d)}_{ij}; i, j \in E)$ are all conservative $q$-matrices which are given as follows

$$q_{ij}^* = \begin{cases} \min(i, c)b_0, & \text{if } i \geq 1, \ j = i - 1, \\ b_1 - [\min(i, c) - 1]b_0, & \text{if } i \geq 1, \ j = i, \\ b_{j-i+1}, & \text{if } i \geq 1, \ j \geq i + 1, \\ 0, & \text{otherwise}, \end{cases} \tag{1.2}$$

$$q_{ij}^{(s)} = \begin{cases} -h, & \text{if } i = 0, \ j = 0, \\ h_j, & \text{if } i = 0, \ j \geq 1, \\ 0, & \text{otherwise}, \end{cases} \tag{1.3}$$

$$q_{ij}^{(d)} = \begin{cases} \beta, & \text{if } i \geq 1, \ j = 0, \\ -\beta, & \text{if } i \geq 1, \ j = i, \\ 0, & \text{otherwise}, \end{cases} \tag{1.4}$$

respectively. Here $\beta \geq 0$, $h_j \geq 0$ ($j \geq 1$) and $b_j \geq 0$ ($j \neq 1$) with

$$0 \leq h := \sum_{j=1}^{\infty} h_j < \infty \text{ and } 0 < -b_1 = \sum_{j \neq 1} b_j < \infty.$$
we recover the model considered in Chen and Renshaw [8]. Whilst if
\( c \neq 0 \) (i.e., no catastrophe) and \( b_j = 0 \) \(( j \geq 2)\), we obtain the ordinary \( M^X/M/1 \) queue. If \( c = 1 \),
\( \beta = 0 \), \( h_1 = b_2 \) and \( h_j = 0 \) \(( j \geq 2)\), we obtain the simple \( M/M/1 \) queue. Whilst
taking \( c = 1 \) and \( b_j = 0 \) \(( j \geq 3)\) generates the model considered in Chen and Renshaw [7]. Furthermore, if \( \beta = 0 \),
\( h_1 = b_2 \) and \( h_j = 0 \) \(( j \geq 2)\), we obtain the simple \( M/M/c \) queue.

The structure of this paper is organized as follows. We first introduce some key lemmas
and consider the properties of the stopped \( M^X/M/c \) queue in Section 2, since our later
discussion depends heavily upon them. The \( M^X/M/c \) queue, modified to ensure that
arbitrary mass arrivals may occur when the system is empty, is fully discussed in Section 3.
Whilst the \( M^X/M/c \) queue with resurrection and catastrophes is studied in Section 4.

2. The stopped \( M^X/M/c \) queue

We first consider the \( M^X/M/c \) queue, modified by terminating the process when it first
becomes empty. The associated \( q \)-matrix \( Q^* \) is given by (1.2). Define the generating
functions
\[
B(s) = \sum_{j=0}^{\infty} b_j s^j \quad \text{and} \quad B_i(s) = B(s) + (i - 1)b_0(1 - s), \quad i = 1, 2, \ldots, c.
\]

These are all well defined for \( s \in [-1, 1] \), whilst as power-series they are \( C^\infty \) functions
with respect to \( s \) on \((-1, 1)\). Also, \( B(s) \) and \( B_i(s) \) are convex functions on \([0, 1]\). The convex
property of \( B_c(s) \) immediately yields the following simple, yet important, result.

**Lemma 2.1.** The equation \( B_c(s) = 0 \) has a smallest root \( u \) on \([0, 1]\) with \( u = 1 \) if \( B_c'(1) \leq 0 \)
and \( u < 1 \) if \( B_c'(1) > 0 \). More specifically,

(i) if \( B_c'(1) \leq 0 \), then \( B_c(s) > 0 \) for all \( s \in [0, 1] \), whence \( B_c(s) = 0 \) has exactly one root
\( s = 1 \) in \([0, 1]\); and,

(ii) if \( B_c'(1) > 0 \), then \( B_c(s) = 0 \) has exactly two roots, \( u \) and \( 1 \), in \([0, 1]\) such that
\( B_c(s) > 0 \) when \( 0 \leq s < u \) and \( B_c(s) < 0 \) when \( u < s < 1 \).

We now define, for any \( \lambda > 0 \),
\[
U_\lambda(s) := B_c(s) - \lambda s.
\]

It is clear that \( U_\lambda(\cdot) \) is also \( C^\infty \) with respect to \( s \) on \((-1, 1)\) and that it is convex on \([0, 1]\).

The following result can be easily proved.
Lemma 2.2. For any fixed $\lambda > 0$, the equation $U_\lambda(s) = 0$ has exactly one root $u(\lambda)$ on $[0, 1]$, and $0 < u(\lambda) < 1$.

Similar to the proof of Lemma 2.3 in [8], we can obtain the following Lemma.

**Lemma 2.3.** For $u(\cdot)$ as defined in Lemma 2.2 (i) $u(\lambda) \in C^\infty(0, \infty)$; (ii) $u(\lambda)$ is a decreasing function of $\lambda > 0$; (iii) $u(\lambda) \downarrow 0$ and $\lambda u(\lambda) \to b_0$ as $\lambda \to \infty$; (iv) when $\lambda \to 0^+$,

$$u(\lambda) \uparrow u \left\{
\begin{array}{ll}
= 1 & \text{if } B'_c(1) \leq 0, \\
< 1 & \text{if } B'_c(1) > 0,
\end{array}
\right. \tag{2.3}$$

where $u$ is the smallest root of $B_c(s) = 0$ on $[0, 1]$;

(v) for any positive integer $k$,

$$\lim_{\lambda \to 0^+} \frac{1 - u(\lambda)^k}{\lambda} = \begin{cases} 
\infty & \text{if } B'_c(1) \geq 0, \\
\frac{k}{-B'_c(1)} & \text{if } B'_c(1) < 0.
\end{cases} \tag{2.4}$$

Let $(p_{ij}^*(t); i, j \geq 0)$ and $(\phi_{ij}^*(\lambda); i, j \geq 0)$ be the $Q^*$-function and $Q^*$-resolvent, respectively.

**Theorem 2.1.** For any $i \geq 0$, $(\phi_{ij}^*(\lambda); 0 \leq j \leq c-1)$ is the unique solution of the following linear equations

$$\begin{cases} 
-\lambda \phi_{i0}^*(\lambda) - \sum_{k=1}^{c-1} u(\lambda)^{k-1}[B(c(u(\lambda))) - B_k(u(\lambda))])\phi_{ik}^*(\lambda) = -u(\lambda)^i, \\
-\lambda \phi_{i0}^*(\lambda) + b_0 \phi_{i1}^*(\lambda) = -\delta_{i0}, \\
(b_1 - \lambda)\phi_{i1}^*(\lambda) + 2b_0 \phi_{i2}^*(\lambda) = -\delta_{i1}, \\
\vdots \\
\sum_{k=1}^{j-1} \phi_{ik}^*(\lambda)b_{j-k-1} + [b_1 - (j-1)b_0 - \lambda]\phi_{ij}^*(\lambda) + (j+1)b_0 \phi_{ij+1}^*(\lambda) = -\delta_{ij}, \\
\vdots \\
\sum_{k=1}^{c-3} \phi_{ik}^*(\lambda)b_{c-k-1} + [b_1 - (c-3)b_0 - \lambda]\phi_{ic-2}^*(\lambda) + (c-1)b_0 \phi_{ic-1}^*(\lambda) = -\delta_{ic-2},
\end{cases} \tag{2.5}$$

where $u(\lambda)(\lambda > 0)$ is the unique root of $U_\lambda(s) = 0$ on $[0, 1]$.

**Proof.** By the Kolmogorov forward equations we have

$$\begin{cases} 
-\lambda \phi_{i0}^*(\lambda) + b_0 \phi_{i1}^*(\lambda) = -\delta_{i0}, \\
(b_1 - \lambda)\phi_{i1}^*(\lambda) + 2b_0 \phi_{i2}^*(\lambda) = -\delta_{i1}, \\
\vdots \\
\sum_{k=1}^{c-1} \phi_{ik}^*(\lambda)b_{c-k-1} + [b_1 - (c-1)b_0 - \lambda]\phi_{ic-1}^*(\lambda) + c_0 \phi_{ic}^*(\lambda) = -\delta_{ic}, \\
\sum_{k=1}^{j-1} \phi_{ik}^*(\lambda)b_{j-k-1} + [b_1 - (c-1)b_0 - \lambda]\phi_{ij}^*(\lambda) + c_0 \phi_{ij+1}^*(\lambda) = -\delta_{ij}, \quad (j \geq c+1).
\end{cases} \tag{2.6}$$

Thus,

$$\lambda \sum_{j=0}^\infty \phi_{ij}^*(\lambda)s^j - s^i = \sum_{k=1}^{c-1} s^{k-1}B_k(s)\phi_{ik}^*(\lambda) + B_c(s) \sum_{j=c}^\infty \phi_{ij}^*(\lambda)s^{j-1},$$
Since \( \lambda u(\lambda) = B_c(u(\lambda)) \), we have

\[
- \lambda \phi_{00}^*(\lambda) - \sum_{k=1}^{c-1} u(\lambda)^{k-1}[B_c(u(\lambda)) - B_k(u(\lambda))]\phi_{ik}^*(\lambda) = -u(\lambda)^i. \tag{2.7}
\]

Combining \eqref{2.7} with the first \( c-1 \) equations of \eqref{2.6} we can obtain the unique \((\phi_{ij}^*(\lambda); i \geq 0, 0 \leq j \leq c-1)\). The proof is complete.

Remark 2.1. As usual, it is very difficult to get the transition probability or transition function \((p^*_{ij}(t); i, j \in E)\) for a general Markov process. We find it particularly noteworthy that we can obtain the resolvent \((\phi_{ij}^*(\lambda); i \geq 0, 0 \leq j \leq c-1)\) of the transition probability \((p^*_{ij}(t); i \geq 0, 0 \leq j \leq c-1)\) from Theorem 2.1. Furthermore, by carefully checking the proof of Theorem 2.1 and using \eqref{2.6}, we can obtain all the resolvent \((\phi_{ij}^*(\lambda); i, j \in E)\) of the transition probability \((p^*_{ij}(t); i, j \in E)\).

By the Kolmogorov forward equation \(\Phi^*(\lambda)(\lambda I - Q^*) = I\) and Lemma 2.2, we can obtain the following theorem.

**Theorem 2.2.** The generating functions of the \(Q^*-\)resolvent take the form

\[
L_i(\lambda, s) := \sum_{j=0}^{\infty} \phi_{ij}^*(\lambda)s^j = \frac{B_c(s)\phi_{00}^*(\lambda) + \sum_{k=1}^{c-1} \phi_{ik}^*(\lambda)s^k (sB_c(s) - sB_k(s)) - s^{i+1}}{U_\lambda(s)} \quad (i \geq 0), \tag{2.8}
\]

where \(U_\lambda(s)\) is defined by \eqref{2.2}, and \((\phi_{ik}^*(\lambda); i \geq 0, 1 \leq k \leq c-1)\) is given by Theorem 2.1.

In particular, \(\phi_{00}^*(\lambda) = \frac{1}{\lambda}\), whilst, for \(i, j \geq 1\),

\[
\phi_{00}^*(\lambda) = \frac{u(\lambda)^i - \sum_{k=1}^{c-1} \phi_{ik}^*(\lambda)u(\lambda)^{k-1}(c-k)b_0(1-u(\lambda))}{\lambda} \quad \text{and} \quad \phi_{0j}^*(\lambda) = 0, \tag{2.9}
\]

where \(u(\lambda)(\lambda > 0)\) is the unique root of \(U_\lambda(s) = 0\) on \([0, 1]\).

Similar to that considered in Lemma 3.1 in Li and Chen [17], we immediately get the following lemma.

**Lemma 2.4.** For any \(i \geq 0\), we have

\[
\int_0^\infty p^*_{ik}(t)dt < \infty, \quad k \geq 1. \tag{2.10}
\]

Let \(\{X^+_t; t \geq 0\}\) denote the \(Q^*-\)process. Define the extinction time as

\[
\tau^*_0 := \left\{\inf\{t > 0; X^+_t = 0\} \quad \text{if} \ X^*_t = 0 \text{ for some } t > 0, \right.
\]

\[
\infty \quad \text{if} \ X^*_t > 0 \text{ for all } t > 0,
\]

and \(w^*_k(t) := P(\tau^*_0 \leq t | X^*_0 = k)\). From \eqref{2.9}, we can immediately get the following conclusion.
Lemma 2.5. Suppose that the $Q^*$-process $\{X_t^*; \ t \geq 0\}$ starts from $X_0^* = k > 0$. Then the Laplace transform of $w_k^*(t)$ is

$$\int_0^\infty e^{-\lambda t} P(\tau_0^* \leq t | X_0^* = k) dt = \frac{u(\lambda)^k - \sum_{i=1}^{c-1} \phi_{ki}^*(\lambda)u(\lambda)^{i-1}(c-i)b_0(1-u(\lambda))}{\lambda},$$

where $u(\lambda)$ is the unique root of $U_\lambda(s) = 0$ on $[0, 1]$ and thereby possesses the properties in Lemma 2.3. $(\phi_{ki}^*(\lambda); \ k \geq 1, 1 \leq i \leq c-1)$ can be obtained by Theorem 2.1.

Theorem 2.3. For the $Q^*$-process $\{X_t^*; \ t \geq 0\}$, denote the extinction probability $e_k^* = P(\tau_0^* < \infty | X_0^* = k) = \lim_{t \to \infty} p_k^*(t) \ (k \geq 1)$ and $m_i^*(k) = \int_0^\infty p_{ki}^*(t) dt \ (i \geq 1)$.

(i) If $B'_c(1) \leq 0$, then for any $k \geq 1$, $e_k^* = 1$;

(ii) if $B'_c(1) > 0$, then for any $k \geq 1$, $e_k^*$ and $(m_i^*(k); \ 1 \leq i \leq c-1)$ is the unique solution of the following linear equations

$$\begin{cases}
e_k^* = u_k^k - \sum_{i=1}^{c-1} m_i^*(k)u^{i-1}(c-i)b_0(1-u), \\
b_0m_1^*(k) = e_k^*, \\
b_1m_1^*(k)+2b_0m_2^*(k) = -\delta_{k1}, \\
\ldots \\
\sum_{j=1}^{c-2} b_{j-i+1}m_{j}^*(k) + [b_1 - (j-1)b_0]m_{j+1}^*(k) + (j+1)b_0m_{j+2}^*(k) = -\delta_{kj}, \\
\ldots \\
\sum_{i=1}^{c-3} b_{c-i-1}m_{i}^*(k) + [b_1 - (c-3)b_0]m_{c-2}^*(k) + (c-1)b_0m_{c-1}^*(k) = -\delta_{kc-2},
\end{cases} \tag{2.11}$$

where $u$ is the smallest root of the equation $B_c(s) = 0$ on $[0, 1]$ with $u = 1$ if $B'_c(1) \leq 0$ and $u < 1$ if $B'_c(1) > 0$. Moreover, all the $(m_i^*(k); \ k \geq 1, i \geq 1)$ can be obtained.

(iii) The mean extinction time is

$$E(\tau_0^*|X_0^* = k) = \begin{cases} -\frac{1}{B'_c(1)} [k + \sum_{i=1}^{c-1} m_i^*(k)(c-i)b_0] & \text{if } B'_c(1) < 0, \\
\infty & \text{if } B'_c(1) \geq 0,
\end{cases}$$

where $(m_i^*(k); \ 1 \leq i \leq c-1)$ is given by (2.11).

Proof. Using Lemma 2.4, Lemma 2.5, (2.3), (2.10), in combination with the Tauberian theorem, yields

$$e_k^* = \lim_{\lambda \to 0^+} \lambda\phi_{k0}^*(\lambda) = \lim_{\lambda \to 0^+} \left(u(\lambda)^k - \sum_{i=1}^{c-1} \phi_{ki}^*(\lambda)u(\lambda)^{i-1}(c-i)b_0(1-u(\lambda))\right)$$

$$= \begin{cases} 1, & \text{if } B'_c(1) \leq 0, \\
u_k^k - \sum_{i=1}^{c-1} m_i^*(k)u^{i-1}(c-i)b_0(1-u) < 1, & \text{if } B'_c(1) > 0,
\end{cases} \tag{2.12}$$

Thus (i) is proven. By (2.3), (2.10) and letting $\lambda \to 0^+$ in every equation of (2.5), we can immediately obtain (2.11). Combining (2.11) with (2.6) we can obtain all the $(m_i^*(k); \ k \geq 1, i \geq 1)$, which completes the proof of (ii).
Using the Tauberian theorem once again and in conjunction with the result (2.3), (2.4) of Lemma 2.3 we obtain

\[ E(\tau_0^*|X_0^* = k) = \int_0^\infty P(\tau_0^* > t|X_0^* = k) \, dt = \lim_{\lambda \to 0^+} \frac{1 - \lambda \phi_{k0}^*(\lambda)}{\lambda} \]

\[ = \begin{cases} 
- \frac{1}{B_c'(1)} [k + \sum_{i=1}^{c-1} m_i^k(c-i)b_0] & \text{if } B_c'(1) < 0, \\
\infty & \text{if } B_c'(1) \geq 0,
\end{cases} \]

thus, (iii) is proven. The proof is complete.

3. The \(M^X/M/c\) queue with resurrection

We now extend the process by allowing mass arrivals of size \(j\) to occur at rate \(h_j\) when the queue is empty and \(\beta = 0\). So the \(q\)-matrix is \(\tilde{Q} = Q^* + Q_s\), where \(Q_s\) is given by (1.3). In addition to the generating functions \(B(s)\) and \(B_i(s)\) defined in (2.1), we need to define

\[ H(s) := \sum_{j=1}^\infty h_js^j. \]  

(3.1)

Obviously, \(H(s)\) is well defined on \([-1, 1]\) and \(H(1) = h > 0\). Moreover, denote

\[ \mu_1 = H'(1) = \sum_{j=1}^\infty jh_j, \]  

(3.2)

note that \(\mu_1\) is not currently presumed to be finite.

By Theorem 2.3 we have the following corollary.

**Corollary 3.1.** If the \(M^X/M/c\) queue with resurrection, the mean busy period is

\[ B = \begin{cases} 
- \frac{1}{hB_c'(1)} \sum_{k=1}^\infty h_k[k + \sum_{i=1}^{c-1} m_i^k(c-i)b_0] & \text{if } B_c'(1) < 0 \\
\infty & \text{if } B_c'(1) \geq 0.
\end{cases} \]

Let \(\tilde{P}(t) = (\tilde{p}_{ij}(t); i, j \geq 0)\) and \(\tilde{R}(\lambda) = (\tilde{r}_{ij}(\lambda); i, j \geq 0)\) be the \(\tilde{Q}\)-function and \(\tilde{Q}\)-resolvent, respectively. Similar to the proof of Theorem 3.1 in Chen and Renshaw [8], using the resolvent decomposition theorem (see Chen and Renshaw [8]), we have the following conclusion.

**Theorem 3.1.** For \(\tilde{R}(\lambda) = (\tilde{r}_{ij}(\lambda); i, j \geq 0)\), we have

\[ \tilde{r}_{00}(\lambda) = \left[ \lambda + \lambda \sum_{i=1}^\infty \sum_{j=1}^\infty h_i \phi_{ij}^*(\lambda) \right]^{-1}, \]  

(3.3)

\[ \tilde{r}_{io}(\lambda) = \tilde{r}_{00}(\lambda)b_0\phi_{i1}^*(\lambda) \quad (i \geq 1), \]  

(3.4)

\[ \tilde{r}_{0j}(\lambda) = \tilde{r}_{00}(\lambda)\sum_{i=1}^\infty h_i \phi_{ij}^*(\lambda) \quad (j \geq 1), \]  

(3.5)

\[ \tilde{r}_{ij}(\lambda) = \phi_{ij}^*(\lambda) + \tilde{r}_{io}(\lambda) \sum_{k=1}^\infty h_k \phi_{kj}^*(\lambda) \quad (i, j \geq 1), \]  

(3.6)

where \(\Phi^*(\lambda) = (\phi_{ij}^*(\lambda); i, j \geq 0)\) is the \(Q^*\)-resolvent given in Theorem 2.2.
Remark 3.1. The Laplace transforms \((\tilde{r}_{ij}(\lambda); i, j \geq 0)\) of the transition probability \((\tilde{p}_{ij}(t), i, j \geq 0)\) can be obtained. Indeed, by Theorem 2.1 and Remark 2.1 we can get \((\phi_{ij}^*(\lambda); i, j \geq 1)\). Then by (3.3) and (3.4), we have
\[
\tilde{r}_{00}(\lambda) = \left[ \lambda + h - H(u(\lambda)) + \sum_{k=1}^{c-1} u(\lambda)^{k-1}(c-k)b_0(1-u(\lambda)) \sum_{i=1}^{\infty} h_i \phi_{ik}^*(\lambda) \right]^{-1} \tag{3.7}
\]
and
\[
\tilde{r}_{10}(\lambda) = \frac{b_0 \phi_{11}^*(\lambda)}{\lambda + h - H(u(\lambda)) + \sum_{k=1}^{c-1} u(\lambda)^{k-1}(c-k)b_0(1-u(\lambda)) \sum_{i=1}^{\infty} h_i \phi_{ik}^*(\lambda)}, \quad i \geq 1. \tag{3.8}
\]
Substituting \(\tilde{r}_{00}(\lambda)\) and \((\phi_{ij}^*(\lambda); i, j \geq 1)\) into (3.5), we can obtain \(\tilde{r}_{ij}(\lambda) (j \geq 1)\). Finally, making use of \(\tilde{r}_{i0}(\lambda) (i \geq 1)\) and \((\phi_{ij}^*(\lambda); i, j \geq 1)\) in (3.6), we can get \((\tilde{r}_{ij}(\lambda); i, j \geq 1)\).

We now consider the recurrence properties of the modified \(M^X/M/c\) queue determined by q-matrix \(\tilde{Q}\).

Theorem 3.2. For the modified \(M^X/M/c\) queueing process with q-matrix \(\tilde{Q}\), we have

(i) the process is recurrent if and only if \(B'_c(1) \leq 0\),

(ii) the process is positive recurrent if and only if \(B'_c(1) < 0\) and \(\mu_1 < \infty\), where \(\mu_1\) is defined by (3.2).

Proof. (i) Since the process is irreducible, it is recurrent if and only if \(\lim_{\lambda \to 0^+} \tilde{r}_{00}(\lambda) = \infty\), and so by (3.3) if and only if
\[
\lim_{\lambda \to 0^+} \sum_{i=1}^{\infty} h_i \sum_{j=1}^{\infty} \lambda \phi_{ij}^*(\lambda) = 0. \tag{3.9}
\]
So from (2.9) we see that (3.9) holds true if and only if
\[
\lim_{\lambda \to 0^+} \left( u(\lambda)i - \sum_{k=1}^{c-1} \phi_{ik}^*(\lambda)u(\lambda)^{k-1}(c-k)b_0(1-u(\lambda)) \right) = 1 \text{ for all } i \geq 1,
\]
which, by (2.12), is equivalent to \(B'_c(1) \leq 0\).

(ii) Again using irreducibility and (3.3), the process is positive recurrent if and only if \(\lim_{\lambda \to 0^+} \lambda \tilde{r}_{00}(\lambda) > 0\), i.e., \(\lim_{\lambda \to 0^+} \sum_{i=1}^{\infty} h_i \sum_{j=1}^{\infty} \phi_{ij}^*(\lambda) < \infty\). Comparison with (2.3), (2.4), (2.9) and (2.10) then shows that the process is positive recurrent if and only if \(B'_c(1) < 0\) and \(\mu_1 = \sum_{i=1}^{\infty} i h_i < \infty\). The proof is complete.

Having determined conditions for our modified \(M^X/M/c\) queue to be positive recurrent, we are now in a position to determine the equilibrium distribution by giving the generating function \(\tilde{\Pi}(s) := \sum_{j=0}^{\infty} \tilde{\pi}_j s^j\).

Theorem 3.3. Suppose that \(B'_c(1) < 0\) and \(\mu_1 < \infty\). The equilibrium generating function \(\tilde{\Pi}(s)\) takes the form
\[
\tilde{\Pi}(s) = \tilde{\pi}_0 \left[ 1 + \frac{s(h - H(s))}{B'_c(s)} \right] + \frac{1}{B'_c(s)} \sum_{k=1}^{c-1} \tilde{\pi}_k s^k (c-k)b_0(1-s), \tag{3.10}
\]

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where
\[
\begin{align*}
\bar{\pi}_0 &= -B'_c(1)\left[-B'_c(1) + \mu_1 + \sum_{k=1}^{c-1} r_k(c-k)b_0\right]^{-1}, \\
\bar{\pi}_k &= \bar{\pi}_0 r_k \quad (k \geq 1),
\end{align*}
\]
and \( r_k := \sum_{i=1}^{\infty} h_i m_k^*(i) \) \((k \geq 1)\) satisfies
\[
\begin{align*}
\begin{cases}
b_0 r_1 = h, \\
b_1 r_1 + 2b_0 r_2 = -h_1, \\
\vdots \\
\sum_{i=1}^{c-1} b_{c-i+1} r_i + [b_1 - (c-1)b_0]r_c + cb_0 r_{c+1} = -h_c, \\
\sum_{i=1}^{j-1} b_{j-i+1} r_i + [b_1 - (c-1)b_0]r_j + cb_0 r_{j+1} = -h_j \quad (j \geq c+1).
\end{cases}
\end{align*}
\]

Proof. Noting that \( \bar{\pi}_j = \lim_{\lambda \to 0^+} \lambda \tilde{r}_{0j}(\lambda) \) for all \( j \geq 0 \), let us first consider \( j = 0 \). Paralleling the proof of Theorem 3.2 we see that
\[
\bar{\pi}_0 = \lim_{\lambda \to 0^+} \lambda \tilde{r}_{00}(\lambda) = -B'_c(1)\left[-B'_c(1) + \mu_1 + \sum_{k=1}^{c-1} r_k(c-k)b_0\right]^{-1},
\]
whilst for \( j \geq 1 \), it follows from (3.3) that
\[
\bar{\pi}_j = \lim_{\lambda \to 0^+} \lambda \tilde{r}_{0j}(\lambda) = \bar{\pi}_0 \sum_{i=1}^{\infty} h_i \int_{0}^{\infty} p_{ij}^*(t) dt = \bar{\pi}_0 r_j,
\]
whence by (2.10) and letting \( \lambda \to 0^+ \) in every equation of (2.6), we can immediately obtain (3.12). Thus, on applying (2.8) and (2.9), we have
\[
\Pi(s) = \bar{\pi}_0 \left[1 + \lim_{\lambda \to 0^+} \sum_{i=1}^{\infty} h_i \sum_{j=1}^{\infty} \phi_{ij}^*(\lambda) s^j \right]
= \bar{\pi}_0 \left[1 + \frac{s(h-H(s))}{B_c(s)} \right] + \frac{1}{B_c(s)} \sum_{k=1}^{c-1} \tilde{\pi}_k s^k(c-k)b_0(1-s).
\]
The proof is complete.

Remark 3.2. Theorem 3.3 gives the generating function of equilibrium distribution for \( M^X/M/c \) queue with resurrection. If \( h_1 = b_2, h_j = b_{j+1} = 0 \) \((j \geq 2)\) and let \( \rho = \frac{b_2}{b_0} \), then we recover the ordinary \( M/M/c \) queue, in this case, (3.10) reduces to
\[
\Pi(s) = \frac{b_0}{cb_0 - b_2 s} \sum_{k=0}^{c-1} \tilde{\pi}_k s^k(c-k),
\]
where
\[
\bar{\pi}_0 = \left[\sum_{k=0}^{c} \frac{\rho^k}{k!} \right]^{-1} \quad \text{and} \quad \bar{\pi}_k = \begin{cases} \frac{\rho^k}{k!} \bar{\pi}_0 & k = 1, 2, \ldots, c-1, \\ \frac{\rho^k}{c!(c-k)} \bar{\pi}_0 & k \geq c. \end{cases}
\]
If \( c = 1 \), then we recover the corresponding result in Chen and Renshaw [8].
From (3.10), we can obtain the following corollary which illustrates other important queueing features.

**Corollary 3.2.** The equilibrium queue size, \(N\), has expectation

\[
E(N) = \bar{\pi}_0 \left[\frac{B''(1)\mu_1}{2(B'_c(1))^2} - \frac{2\mu_1 + H'(1)}{2B'_c(1)}\right] + \sum_{k=1}^{c-1} \bar{\pi}_k(c-k)b_0 \left[\frac{B''(1)}{2(B'_c(1))^2} - \frac{k}{B'_c(1)}\right] \tag{3.13}
\]

if both \(B''(1)\) and \(H''(1)\) are finite, and \(E(N) = \infty\) otherwise. The equilibrium waiting queue size, \(L_w\), has expectation

\[
E(L_w) = \bar{\pi}_0 \left[\frac{B''(1)\mu_1}{2(B'_c(1))^2} - \frac{2\mu_1 + H'(1)}{2B'_c(1)} + \epsilon\right] + \sum_{k=1}^{c-1} \bar{\pi}_k(c-k) \left[\frac{B''(1)}{2(B'_c(1))^2} - \frac{k}{B'_c(1)}\right] + 1 - \epsilon \tag{3.14}
\]

if both \(B''(1)\) and \(H''(1)\) are finite, and \(E(L_w) = \infty\) otherwise. Here \((\bar{\pi}_k; 0 \leq k \leq c - 1)\) is given in Theorem 3.3.

4. The \(M^X/M/c\) queue with resurrection and catastrophes

In this section, we consider the general case that \(\beta > 0\), the \(q\)-matrix \(Q\) now takes the form \((1.1)\), i.e., \(Q = Q^* + Q_s + Q_d\), where \(Q_s\) and \(Q_d\) are defined by \((1.3)\) and \((1.4)\). Let \(\{X_t; t \geq 0\}\) denote the \(Q\)-process, \(P(t) = (p_{ij}(t); i, j \geq 0)\) and \(R(\lambda) = (r_{ij}(\lambda); i, j \geq 0)\) be the \(Q\)-function and \(Q\)-resolvent, respectively. Noting that the properties of this process are substantially different from that in the case \(\beta = 0\) considered in Section 3. Using the resolvent decomposition theorem (see Chen and Renshaw [6]), we have the following theorem.

**Theorem 4.1.** For \(R(\lambda) = (r_{ij}(\lambda); i, j \geq 0)\), we have

\[
r_{00}(\lambda) = \left[\lambda + \lambda \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} h_{ij} \phi_{ij}^*(\lambda + \beta)\right]^{-1}, \tag{4.1}
\]

\[
r_{i0}(\lambda) = r_{00}(\lambda) \left[b_0 \phi_{i1}^*(\lambda + \beta) + \beta \sum_{k=1}^{\infty} \phi_{ik}^*(\lambda + \beta)\right] \quad (i \geq 1), \tag{4.2}
\]

\[
r_{0j}(\lambda) = r_{00}(\lambda) \sum_{i=1}^{\infty} h_{ij} \phi_{ij}^*(\lambda + \beta) \quad (j \geq 1), \tag{4.3}
\]

\[
r_{ij}(\lambda) = \phi_{ij}^*(\lambda + \beta) + r_{00}(\lambda) \sum_{k=1}^{\infty} h_{kj} \phi_{kj}^*(\lambda + \beta) \quad (i, j \geq 1), \tag{4.4}
\]

where \(\Phi^*(\lambda) = (\phi_{ij}^*(\lambda); i, j \geq 0)\) is the \(Q^*\)-resolvent given in Theorem 2.2.

Replacing \(\lambda\) with \(\lambda + \beta\) in \((2.9)\), we have

\[
\sum_{k=1}^{\infty} \phi_{ik}^*(\lambda + \beta) = \frac{1 - u(\lambda + \beta)^i + \sum_{k=1}^{c-1} \phi_{ik}^*(\lambda + \beta)u(\lambda + \beta)^{k-1}(c - k)b_0(1 - u(\lambda + \beta))}{\lambda + \beta}. \tag{4.5}
\]

Since

\[
b_0 \phi_{i1}^*(\lambda + \beta) = (\lambda + \beta)^i M_i(\lambda),
\]
where
\[ M_i(\lambda) = \sum_{k=1}^{c-1} \phi_{ik}^*(\lambda + \beta)u(\lambda + \beta)^{k-1}(c - k)b_0(1 - u(\lambda + \beta)). \]

It follows from (4.2) that
\[ r_{i0}(\lambda) = r_{00}(\lambda)\frac{\lambda u(\lambda + \beta)^i - \lambda M_i(\lambda) + \beta}{\lambda + \beta}. \tag{4.6} \]

Moreover, (4.1) can be rewritten as
\[ r_{00}(\lambda) = \left[ \lambda + \frac{\lambda}{\lambda + \beta} \left( H(1) - H(u(\lambda + \beta)) \right) \right. \]
\[ \left. + \sum_{k=1}^{c-1} u(\lambda + \beta)^{k-1}(c - k)b_0(1 - u(\lambda + \beta)) \sum_{i=1}^{\infty} h_i \phi_{ik}^*(\lambda + \beta) \right]^{-1}. \tag{4.7} \]

Paralleling Section 2, define the hitting time \( \tau_0 = \inf\{ t > 0; X_t = 0 \} \) with \( \tau_0 = \infty \) if \( X_t > 0 \) for all \( t > 0 \), and let \( e_k = P(\tau_0 < \infty|X_0 = k) \) and \( w_k(t) = P(\tau_0 \leq t|X_0 = k) \) denote the hitting probability and distribution function of \( \tau_0 \), starting from \( X_0 = k \), respectively.

**Theorem 4.2.** If \( h = 0 \) and \( \beta > 0 \). Then for any \( k \geq 1 \),
\[ \int_0^\infty e^{-\lambda} P(\tau_0 \leq t|X_0 = k) dt = \frac{1}{\beta} \left[ 1 - u(\beta)^k - \sum_{i=1}^{c-1} \phi_{ki}^*(\lambda + \beta)u(\lambda + \beta)^{i-1}(c - i)b_0(1 - u(\lambda + \beta)) \right]. \tag{4.8} \]

where \( u(\lambda) \) is the unique root of \( U_\lambda(s) = 0 \) on \([0, 1]\) and \( (\phi_{ki}^*(\lambda); \ k \geq 1, \ 1 \leq i \leq c - 1) \) is given by Theorem 2.1. Moreover, \( e_k = 1 \ (k = 1, 2, \ldots) \) and the mean extinction time is finite and given by
\[ E(\tau_0|X_0 = k) = \frac{1}{\beta} \left[ 1 - u(\beta)^k + \sum_{i=1}^{c-1} \phi_{ki}^*(\beta)u(\beta)^{i-1}(c - i)b_0(1 - u(\lambda + \beta)) \right]. \tag{4.9} \]

**Proof.** (4.8) is just (4.6), since \( r_{00}(\lambda) = \frac{1}{\lambda} \) in the current case. Hence,
\[ e_k = \lim_{\lambda \to 0^+} \frac{\beta + \lambda u(\lambda + \beta)^k - \lambda \sum_{i=1}^{c-1} \phi_{ki}^*(\lambda + \beta)u(\lambda + \beta)^{i-1}(c - i)b_0(1 - u(\lambda + \beta))}{\lambda + \beta} = 1. \]

By using (4.8) and the Tauberian theorem, we can obtain (4.9). The proof is complete.

We now consider the case \( h > 0 \). We first give the following important lemma which follows from Theorem 2.1.
Lemma 4.1. For \((p_{ij}^\ast(t); i, j \geq 0)\) and \((\phi_{ij}^\ast(\lambda); i, j \geq 0)\) given in Section 2, denote \(L_j(\lambda) = \sum_{i=1}^{\infty} h_i \phi_{ij}^\ast(\lambda)\) \((j \geq 0)\). Then \((L_j(\lambda); 0 \leq j \leq c - 1)\) is the unique solution of the following linear equations

\[
\begin{align*}
-\lambda L_0(\lambda) - \sum_{k=1}^{c-1} u(\lambda)^{k-1}[B_c(u(\lambda)) - B_k(u(\lambda))]L_k(\lambda) &= -H(u(\lambda)), \\
-\lambda L_0(\lambda) + b_0 L_1(\lambda) &= 0, \\
(b_1 - \lambda)L_1(\lambda) + 2b_0 L_2(\lambda) &= -h_1, \\
\ldots \\
\sum_{k=1}^{j-1} L_k(\lambda)b_{j-k+1} + [b_1 - (j - 1)b_0 - \lambda]L_j(\lambda) + (j + 1)b_0 L_{j+1}(\lambda) &= -h_j, \\
\ldots \\
\sum_{k=1}^{c-3} L_k(\lambda)b_{c-k-1} + [b_1 - (c - 3)b_0 - \lambda]L_{c-2}(\lambda) + (c - 1)b_0 L_{c-1}(\lambda) &= -h_{c-2},
\end{align*}
\]

where \(u(\lambda)(\lambda > 0)\) is the unique root of \(U_\lambda(s) = 0\) on \([0, 1]\). Moreover, all the \((L_j(\lambda); j \geq 0)\) can be obtained.

Theorem 4.3. If \(h, \beta > 0\), then the \(Q\)-process is always positive recurrent.

Proof. It follows from (4.7) that \(\lim_{\lambda\to 0^+} r_{00}(\lambda) = \infty\), and so the \(Q\)-process is recurrent. Noting \(\lim_{t\to\infty} p_{00}(t) = \lim_{\lambda\to 0^+} \lambda r_{00}(\lambda)\) and again using (4.7) yield

\[
\lim_{t\to\infty} p_{00}(t) = \beta \left[\beta + H(1) - H(u(\beta)) + \sum_{k=1}^{c-1} L_k(\beta)u(\beta)^{k-1}(c - k)b_0(1 - u(\beta))\right]^{-1} > 0. \tag{4.10}
\]

The proof is complete.

The following theorem further gives the equilibrium distribution \(\{\pi_j; j \geq 0\}\) of the \(Q\)-process.

Theorem 4.4. The equilibrium distribution of the \(Q\)-process is given by

\[
\Pi(s) = \pi_0 \left[1 + \frac{s(H(u(\beta)) - H(s))}{U_\beta(s)}\right] + \sum_{k=1}^{c-1} \pi_k(c - k)b_0[s^k(1 - s) - su(\beta)^{k-1}(1 - u(\beta))] \tag{4.11}
\]

where \(\Pi(s) = \sum_{k=0}^{\infty} \pi_k s^k\), \(U_\beta(s)\) is defined in (2.2). Furthermore,

\[
\pi_0 = \beta \left[\beta + H(1) - H(u(\beta)) + \sum_{k=1}^{c-1} L_j(\beta)u(\beta)^{k-1}(c - k)b_0(1 - u(\beta))\right]^{-1}, \tag{4.12}
\]

\[
\pi_j = \pi_0 L_j(\beta) \quad (j \geq 1)
\]

and \((L_j(\beta); j \geq 1)\) is given by Lemma 4.1.

Proof. First note that (4.12) is just (4.10), whilst, it follows from (4.3) that for \(j \geq 1\),

\[
\pi_j = \lim_{\lambda\to 0^+} \lambda r_{0j}(\lambda) = \pi_0 L_j(\beta). \tag{4.13}
\]

Hence, using (2.8) and the above two equality we can get (4.11). The proof is comlete.
As a direct consequence of Theorem 4.4, we have the following corollary regarding queue size.

**Corollary 4.1.** The equilibrium queue size, $N$, has expectation

$$E(N) = \pi_0 \frac{(H(u(\beta)) - h - \mu_1)(-\beta) - (H(u(\beta)) - h)(B_c'(1) - \beta)}{\beta^2} \sum_{j=1}^{c-1} \pi_k(c-k)b_0[\beta + u(\beta)^{k-1}(1 - u(\beta))]B_c'(1)$$

and the equilibrium waiting queue size, $L_w$, has expectation

$$E(L_w) = \pi_0 \left[ \frac{(H(u(\beta)) - h - \mu_1)(-\beta) - (H(u(\beta)) - h)(B_c'(1) - \beta)}{\beta^2} \sum_{j=1}^{c-1} \pi_k(c-k)[b_0(\beta + u(\beta)^{k-1}(1 - u(\beta))]B_c'(1) + \beta^2 \right] - c,$$

where $(\pi_k; 0 \leq k \leq c-1)$ is given in Theorem 4.4.

From now on, we consider the first effective catastrophe occurrence time of the $Q$-process \{X; t \geq 0\}.

For all $j, n \in E$ and $t > 0$, $p_{jn}(t)$ satisfy the following system of forward equations

$$
\begin{align*}
\begin{cases}
p'_{j0}(t) &= -(h + \beta)p_{j0}(t) + b_0p_{j1}(t) + \beta, \\
p'_{jn}(t) &= h_np_{j0}(t) + \sum_{k=1}^{n-1} b_{n-k+1}p_{jk}(t) \\
&\quad + [b_1 - (n-1)b_0 - \beta]p_{jn}(t) + (n+1)b_0p_{j,n+1}(t), \quad 1 \leq n \leq c-1, \\
p'_{jn}(t) &= h_np_{j0}(t) + \sum_{k=1}^{n-1} b_{n-k+1}p_{j,n+1}(t) + [b_1 - (c-1)b_0 - \beta]p_{jn}(t) + cb_0p_{j,n+1}(t), \quad n \geq c.
\end{cases}
\end{align*}
$$

Here $\sum_{k=1}^{n-1} = 0$ if $n = 1$ as a notation. Obviously, for all $j, n \in E$ and $t > 0$, the $\tilde{Q}$-function $\tilde{p}_{jn}(t)$ satisfy the system of forward equations (4.14) by setting $\beta = 0$.

The following lemma reveals that $(p_{ij}(t))$ can be expressed in terms of $(\tilde{p}_{ij}(t))$.

**Lemma 4.2.** (see [18]) For all $j, n \in E$, $t > 0$, we have

$$p_{jn}(t) = e^{-\beta t} \tilde{p}_{jn}(t) + \beta \int_0^t e^{-\beta \tau} \tilde{p}_{jn}(\tau) d\tau,$$

or

$$r_{jn}(\lambda) = \tilde{r}_{jn}(\lambda + \beta) + \frac{\beta}{\lambda} \tilde{p}_{jn}(\lambda + \beta), \quad \lambda > 0.$$
absorbing state $-1$. In other words, \( \{M_t; t \geq 0\} \) is a modified \( M^X/M/c \) queuing process with catastrophes with state space \( S = \{-1, 0, 1, \ldots\} \). Its \( q \)-matrix is

\[
Q_M = \bar{Q}^* + \bar{Q}_s + \bar{Q}_d,
\]

where \( \bar{Q}^* = (\bar{q}_{ij}^*, i, j \in S) \) and \( \bar{Q}_s = (\bar{q}_{ij}^{(s)}, i, j \in S) \) are given by

\[
\bar{q}_{ij}^* = \begin{cases} q_{ij}^*, & \text{if } i, j \in E, \\ 0, & \text{otherwise} \end{cases}
\]

and

\[
\bar{q}_{ij}^{(s)} = \begin{cases} q_{ij}^{(s)}, & \text{if } i, j \in E, \\ 0, & \text{otherwise}, \end{cases}
\]

respectively. \( \bar{Q}_d = (\bar{q}_{ij}^{(d)}, i, j \in S) \) is given by

\[
\bar{q}_{ij}^{(d)} = \begin{cases} \beta, & \text{if } i \geq 1, j = -1, \\ -\beta, & \text{if } i \geq 1, j = i, \\ 0, & \text{otherwise.} \end{cases}
\]

Let \( H(t) = (h_{jn}(t), j, n \in S) \) and \( \eta(\lambda) = (\eta_{jn}(\lambda), j, n \in S) \) be the \( Q_M \)-function and \( Q_M \)-resolvent, respectively. For all \( j \in E, n \in S \) and \( t \geq 0 \), \( h_{jn}(t) \) satisfy the following system of forward equations

\[
\begin{cases}
    h'_{j,-1}(t) = \beta (1 - h_{j,-1}(t) - h_{j0}(t)), \\
    h'_{j0}(t) = -hh_{j0}(t) + b_0h_{j1}(t), \\
    h'_{jn}(t) = h_nh_{j0}(t) + \sum_{k=1}^{n-1} b_{n-k+1}h_{jk}(t) + [b_1 - (n - 1)b_0 - \beta]h_{jn}(t) + (n + 1)b_0h_{jn+1}(t), 1 \leq n \leq c - 1, \\
    h'_{jn}(t) = h_nh_{j0}(t) + \sum_{k=1}^{n-1} b_{n-k+1}h_{jk}(t) + [b_1 - (n - 1)b_0 - \beta]h_{jn}(t) + cb_0h_{jn+1}(t), n \geq c.
\end{cases}
\]

By the relation of \( \{X_t; t \geq 0\} \) and \( \{M_t; t \geq 0\} \), we have

\[
P(C_{j0} > t) = \int_t^{+\infty} d_{j0}(\tau) d\tau = \sum_{n=0}^{+\infty} h_{jn}(t) = 1 - h_{j,-1}(t), \quad j \in E.
\]

In the following theorem we shall express the Laplace transform \( (\eta_{jn}(\lambda); j \in E, n \in S) \).

**Theorem 4.5.** For all \( j \in E \) and \( \lambda > 0 \), we have

\[
\eta_{j,-1}(\lambda) = \frac{\beta}{\lambda + \beta} \left[ 1 - \frac{\tilde{\lambda}_{j0}(\lambda + \beta)}{1 - \beta \tilde{r}_{00}(\lambda + \beta)} \right],
\]

\[
\eta_{jn}(\lambda) = \tilde{\eta}_{jn}(\lambda + \beta) + \beta \tilde{r}_{0n}(\lambda + \beta) \frac{\tilde{\lambda}_{j0}(\lambda + \beta)}{1 - \beta \tilde{r}_{00}(\lambda + \beta)}, \quad n \geq 0,
\]

where \( (\tilde{\eta}_{jn}(\lambda + \beta); j, n \in E) \) is given by Remark 3.1.
Proof. For \( j = 0 \), taking the Laplace transform of the second to the fourth equations in (4.19), we obtain
\[
\begin{align*}
(\lambda + h)\eta_{00}(\lambda) - 1 &= b_0\eta_{01}(\lambda), \\
[\lambda - b_1 + (n - 1)b_0 + \beta]\eta_{0n}(\lambda) &= h_n\eta_{00}(\lambda) + \sum_{k=1}^{n-1} b_{n-k+1}\eta_{0k}(\lambda) + (n + 1)b_0\eta_{0,n+1}(\lambda), \quad 1 \leq n \leq c - 1,
\end{align*}
\]
(4.23)
Similarly, taking the Laplace transform of (4.14), we have
\[
\begin{align*}
(\lambda + h + \beta)r_{00}(\lambda) - 1 &= b_0r_{01}(\lambda) + \frac{\beta}{\lambda}, \\
[\lambda - b_1 + (n - 1)b_0 + \beta]r_{0n}(\lambda) &= h_nr_{00}(\lambda) + \sum_{k=1}^{n-1} b_{n-k+1}r_{0k}(\lambda) + (n + 1)b_0r_{0,n+1}(\lambda), \quad 1 \leq n \leq c - 1,
\end{align*}
\]
(4.24)
Let
\[
\eta_{0n}(\lambda) = A(\lambda)r_{0n}(\lambda), \quad n \geq 0.
\]
(4.25)
Substituting (4.25) into (4.23), and using (4.24), we obtain
\[
A(\lambda) = \frac{\lambda}{\lambda + \beta - \lambda\beta r_{00}(\lambda)}.
\]
(4.26)
Making use of (4.16) in (4.25), with \( A(\lambda) \) given in (4.26), we obtain (4.22) for \( j = 0 \).

For \( 1 \leq j \leq c - 1 \), taking the Laplace transform of the second to the fourth equations in (4.19), we have
\[
\begin{align*}
(\lambda + h)\eta_{j0}(\lambda) &= b_0\eta_{j1}(\lambda), \\
[\lambda - b_1 + (n - 1)b_0 + \beta]\eta_{jn}(\lambda) - \delta_{jn} &= h_n\eta_{j0}(\lambda) + \sum_{k=1}^{n-1} b_{n-k+1}\eta_{jk}(\lambda) + (n + 1)b_0\eta_{j,n+1}(\lambda), \quad 1 \leq n \leq c - 1,
\end{align*}
\]
(4.27)
and for \( j \geq c \),
\[
\begin{align*}
(\lambda + h)\eta_{j0}(\lambda) &= b_0\eta_{j1}(\lambda), \\
[\lambda - b_1 + (n - 1)b_0 + \beta]\eta_{jn}(\lambda) - \delta_{jn} &= h_n\eta_{j0}(\lambda) + \sum_{k=1}^{n-1} b_{n-k+1}\eta_{jk}(\lambda) + (n + 1)b_0\eta_{j,n+1}(\lambda), \quad 1 \leq n \leq c - 1,
\end{align*}
\]
(4.28)
Let
\[ \eta_{jn}(\lambda) = D_j(\lambda)r_{jn}(\lambda) + F_j(\lambda)r_{0n}(\lambda), \quad j \geq 1, \quad n \geq 0. \quad (4.29) \]
Recalling the Laplace transform of (4.14), and substituting (4.29) into (4.27), (4.28) respectively, one has
\[ D_j(\lambda) = 1, \quad F_j(\lambda) = \frac{\beta[\lambda r_{j0}(\lambda) - 1]}{\lambda + \beta - \lambda \beta r_{00}(\lambda)}, \quad j \geq 1. \quad (4.30) \]
Hence, making use of (4.16) in (4.29), with \( D_j(\lambda) \) and \( F_j(\lambda) \) given in (4.30), some straightforward calculations lead us to (4.22) for \( j \geq 1 \). Furthermore, taking the Laplace transform of the first equation in (4.19), we obtain \( \eta_{j,-1}(\lambda) = \frac{\beta}{\lambda + \beta} \left[ \frac{1}{\lambda} - \eta_{j0}(\lambda) \right] \). Hence, making use of (4.22) for \( n = 0 \), (4.21) finally follows. The proof is complete.

Let us now denote by \( \Delta_{j0}(\lambda) \) the Laplace transform of \( d_{j0}(t) \) (\( j \in \mathbb{E} \)). Since (4.20) implies \( d_{j0}(t) = b'_{j,-1}(t) \) (\( j \in \mathbb{E} \)), i.e., \( \Delta_{j0}(\lambda) = \lambda \eta_{j,-1}(\lambda) \), recalling (4.21) for \( j \in \mathbb{E} \) and \( \lambda > 0 \), we immediately have the following corollary.

**Corollary 4.2.** For all \( j \in \mathbb{E} \), there holds
\[ \Delta_{j0}(\lambda) = \frac{\beta}{\lambda + \beta} - \frac{\lambda}{\lambda + \beta} \cdot \frac{\beta \tilde{r}_{j0}(\lambda + \beta)}{1 - \beta \tilde{r}_{00}(\lambda + \beta)}, \quad (4.31) \]
where \( \tilde{r}_{j0}(\lambda + \beta) \) (\( j \in \mathbb{E} \)) are given by Remark 3.1.

Next we shall study the expectation and variance of the first catastrophe time \( C_{j0} \). Since \( E(C_{j0}) = -\lim_{\lambda \to 0^+} \Delta'_{j0}(\lambda) \) and \( E(C^2_{j0}) = \lim_{\lambda \to 0^+} \Delta''_{j0}(\lambda) \), by (4.31) and after some easy algebraic computation, we can immediately obtain the following theorem.

**Theorem 4.6.** For all \( j \in \mathbb{E} \), there holds
\[ E(C_{j0}) = \frac{1}{\beta} + \frac{\tilde{r}_{j0}(\beta)}{1 - \beta \tilde{r}_{00}(\beta)}, \quad (4.32) \]
\[ \text{Var}(C_{j0}) = \frac{1}{\beta^2} \left\{ 1 - \frac{\beta^2 \tilde{r}_{j0}^2(\beta)}{[1 - \beta \tilde{r}_{00}(\beta)]^2} - \frac{2 \beta^2}{1 - \beta \tilde{r}_{00}(\beta)} \frac{d}{d\beta} \tilde{r}_{j0}(\beta) - \frac{2 \beta^3 \tilde{r}_{j0}(\beta)}{[1 - \beta \tilde{r}_{00}(\beta)]^2} \frac{d}{d\beta} \tilde{r}_{00}(\beta) \right\}, \quad (4.33) \]
where \( \tilde{r}_{j0}(\beta) \) (\( j \in \mathbb{E} \)) are given by Remark 3.1.

The following theorems will show the asymptotic behavior of the mean first catastrophe time for \( \beta \downarrow 0 \) and for \( \beta \to +\infty \).

**Theorem 4.7.** (i) If \( B'_c(1) < 0 \) and \( \mu_1 < \infty \), then
\[ \lim_{\beta \downarrow 0} \beta E(C_{j0}) = \frac{-B'_c(1) + \mu_1 + \sum_{k=1}^{c-1} r_k(c-k)b_0}{\mu_1 + \sum_{k=1}^{c-1} r_k(c-k)b_0}, \quad j \in \mathbb{E}; \quad (4.34) \]
(ii) if \( B'_c(1) > 0 \), then
\[ \lim_{\beta \downarrow 0} \left\{ E(C_{00}) - \frac{1}{\beta} \right\} = \frac{1}{h - H(u) + \sum_{k=1}^{c-1} u^k(c-k)b_0(1-u)r_k}, \quad (4.35) \]
\[ \lim_{\beta \downarrow 0} \left\{ E(C_{j0}) - \frac{1}{\beta} \right\} = \frac{b_0m^*_1(j)}{h - H(u) + \sum_{k=1}^{c-1} u^k(c-k)b_0(1-u)r_k}, \quad j \geq 1, \quad (4.36) \]
where \( (m^*_1(j); \quad j \geq 1) \) and \( (r_k; \quad 1 \leq k \leq c - 1) \) are given by (2.11) and (3.12), respectively.
Proof. (i) If $B'_c(1) < 0$ and $\mu_1 < \infty$, then from Theorem 3.2 we know that $\{X_t; \ t \geq 0\}$ is positive recurrent. Hence, using (4.32) and Tauberian theorem, we have

$$\lim_{\beta \downarrow 0} \beta E(C_{j0}) = 1 + \frac{\lim_{t \to +\infty} \tilde{p}_{j0}(t)}{1 - \lim_{t \to +\infty} \tilde{p}_{00}(t)} = \frac{1}{1 - \pi_0}.$$  

Then, (4.34) immediately follows from (3.11).

(ii) If $B'_c(1) > 0$, then $\{\tilde{X}_t; \ t \geq 0\}$ is transient. Hence, by (2.3), (3.7), (3.8) and (4.32), we can obtain (4.35) and (4.36). The proof is complete.

**Theorem 4.8.** For $E(C_{j0})$ given by (4.32), there holds

$$\lim_{\beta \to +\infty} E(C_{00}) = \frac{1}{h},$$

$$\lim_{\beta \to +\infty} \beta E(C_{j0}) = \begin{cases} 1 + \frac{b_0}{h}, & j = 1, \\ 1, & j \geq 2. \end{cases}$$

Proof. Since

$$(\beta + h)\tilde{r}_{j0}(\beta) = \delta_{j0} + b_0\tilde{r}_{j1}(\beta), \ j \in E,$$

from (4.32) we have

$$E(C_{00}) = \frac{1}{\beta} + \frac{\tilde{r}_{00}(\beta)}{h\tilde{r}_{00}(\beta) - b_0\tilde{r}_{01}(\beta)},$$

$$E(C_{j0}) = \frac{1}{\beta} \left[ 1 - \frac{h\tilde{r}_{j0}(\beta) - b_0\tilde{r}_{j1}(\beta)}{h\tilde{r}_{00}(\beta) - b_0\tilde{r}_{01}(\beta)} \right], \ (j = 1, 2, \ldots) .$$

By Tauberian theorem, we obtain

$$\lim_{\beta \to +\infty} E(C_{00}) = \frac{\lim_{t \to 0} \tilde{p}_{00}(t)}{h \lim_{t \to 0} \tilde{p}_{00}(t) - b_0 \lim_{t \to 0} \tilde{p}_{01}(t)} = \frac{1}{h}$$

and

$$\lim_{\beta \to +\infty} \beta E(C_{j0}) = 1 - \frac{h \lim_{t \to 0} \tilde{p}_{j0}(t) - b_0 \lim_{t \to 0} \tilde{p}_{j1}(t)}{h \lim_{t \to 0} \tilde{p}_{00}(t) - b_0 \lim_{t \to 0} \tilde{p}_{01}(t)}$$

$$= \begin{cases} 1 + \frac{b_0}{h}, & j = 1, \\ 1, & j \geq 2. \end{cases}$$

The proof is complete.

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