Almost Sure Well-Posedness and Scattering of the 3D Cubic Nonlinear Schrödinger Equation

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Abstract: We study the random data problem for 3D, defocusing, cubic nonlinear Schrödinger equation in \(H^s_x(\mathbb{R}^3)\) with \(s < \frac{1}{2}\). First, we prove that the almost sure local well-posedness holds when \(\frac{1}{6} \leq s < \frac{1}{2}\) in the sense that the Duhamel term belongs to \(H^{1/2}_x(\mathbb{R}^3)\). Furthermore, we prove that the global well-posedness and scattering hold for randomized, radial, large data \(f \in H^s_x(\mathbb{R}^3)\) when \(\frac{17}{40} < s < \frac{1}{2}\). The key ingredient is to control the energy increment including the terms where the first order derivative acts on the linear flow, and our argument can lower down the order of derivative more than \(\frac{1}{2}\). To our best knowledge, this is the first almost sure large data global result for this model.

Contents

1. Introduction .......................................................... 548
   1.1 Almost sure local well-posedness ................................ 552
   1.2 Almost sure scattering ........................................... 555
     1.2.1 Sketch the proof of Theorem 1.4 .......................... 555
   1.3 Organization of the paper ..................................... 557
2. Preliminary .......................................................... 558
   2.1 Notation .......................................................... 558
   2.2 Atom space and bounded variation space .................. 560
   2.3 Useful lemmas .................................................... 561
   2.4 Maximal function estimates and Littlewood-Paley theory .. 563
   2.5 Probabilistic theory ............................................. 563
3. Almost Sure Strichartz Estimates ................................. 564
   3.1 Non-radial data .................................................. 564
   3.2 Radial data ....................................................... 567
4. Local Well-Posedness ............................................... 571
1. Introduction

In this paper, we consider the nonlinear Schrödinger equations (NLS):

\[
\begin{cases}
    i \partial_t u + \Delta u = \mu |u|^p u, \\
    u(0, x) = u_0(x),
\end{cases}
\]  

(1.1)

where \( p > 0 \), \( \mu = \pm 1 \), and \( u(t, x) : \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{C} \) is a complex-valued function. The positive sign “+” in nonlinear term of (1.1) denotes defocusing source, and the negative sign “−” denotes the focusing one.

The equation (1.1) has conserved mass

\[
M(u(t)) := \int_{\mathbb{R}^d} |u(t, x)|^2 \, dx = M(u_0),
\]

(1.2)

and energy

\[
E(u(t)) := \int_{\mathbb{R}^d} \frac{1}{2} |\nabla u(t, x)|^2 \, dx + \mu \int_{\mathbb{R}^d} \frac{1}{p+2} |u(t, x)|^{p+2} \, dx = E(u_0).
\]

(1.3)

The class of solutions to Eq. (1.1) is invariant under the scaling

\[
u(t, x) \rightarrow u_\lambda(t, x) = \lambda^{\frac{2}{p}} u(\lambda^2 t, \lambda^2 x) \quad \text{for} \quad \lambda > 0,
\]

(1.4)

which maps the initial data as

\[
u(0) \rightarrow u_\lambda(0) := \lambda^{\frac{2}{p}} u_0(\lambda x) \quad \text{for} \quad \lambda > 0.
\]

(1.5)

Denote

\[
s_c = \frac{d}{2} - \frac{2}{p},
\]

then the scaling leaves \( \dot{H}^s \) norm invariant, that is,

\[
\| u(0) \|_{\dot{H}^s} = \| u_\lambda(0) \|_{\dot{H}^s}.
\]

This gives the scaling critical exponent \( s_c \). Let

\[
2^* = \infty, \text{ when } d = 1 \text{ or } d = 2; \quad 2^* = \frac{4}{d-2}, \text{ when } d \geq 3.
\]
Therefore, according to the conservation law, the equation is called mass or $L^2_x$ critical when $p = \frac{4}{d}$, and energy or $\dot{H}^1_x$ critical when $p = \frac{4}{d-2}$. Moreover, when $\frac{4}{d} < p < 2^*$, we say that the equation is inter-critical.

Let us now take a brief overview on the well-posedness and scattering theory of NLS (1.1). Kato [48] first proposed a method based on the contraction mapping and the Strichartz estimate, and obtained the local well-posedness when $p < \frac{4}{d-2}$ in $H^1_x$. See also [77] by Tsutsumi for the $L^2_x$-solution when $p < \frac{4}{d}$. Note that the above two results concerned the sub-critical cases when $s > s_c$. The local well-posedness in the critical sense was solved by Cazenave and Weissler, see [22]. Moreover, we refer the readers to Cazenave’s textbook [21] for more complete local results of NLS.

The global well-posedness and scattering are basic topics for the long time behaviour of NLS. Lin and Strauss [63] obtained the large data scattering for the 3D, defocusing, cubic NLS with decaying data. Their argument relied on the Morawetz estimate, which was first discovered by Morawetz [66] for the Klein-Gordon equations. The global well-posedness and scattering in energy space were solved by Ginibre and Velo [44] in the defocusing inter-critical cases for $d \geq 3$. In this paper, we mainly focus on the results in $L^2_x$-based Sobolev spaces, thus we do not mention the vast scattering theory for NLS with decaying data.

The main breakthrough of the energy critical NLS was owed to Bourgain [10]. He introduced the powerful induction-on-energy method and the localised Morawetz estimate to study the defocusing equations with radial data for $d = 3, 4$. Bourgain’s method was then further exploited extensively: Nakanishi [67] introduced a modified version of Morawetz estimate for low dimensions, and solved the energy scattering in the inter-critical cases for $d = 1, 2$; Bourgain’s 3D result was extended to non-radial by Colliander, Keel, Staffilani, Takaoka, and Tao [27], based on a localised version of their interaction Morawetz estimate [26]. The results for defocusing energy critical NLS in higher dimensions were obtained by Ryckman and Visan [71,79]. Very few large data global results were available for supercritical models, however, see Li’s recent ground-breaking work [62] on a supercritical wave model.

For the focusing equations, Kenig and Merle [50] introduced the concentration compactness method to give a complete dynamical characterization below the energy of ground state, for the energy critical NLS in $d = 3, 4, 5$ with radial data. Their study opened a way to study the scattering of focusing equations below the ground state. Then, Duyckaerts, Holmer, and Roudenko [42,47] gave the result for 3D, focusing, cubic NLS, which is a typical model in the inter-critical cases. For the non-radial focusing energy-critical NLS, Killip and Visan [54] solved the $d \geq 5$ case, and Dodson [36] solved the 4D case. The scattering of 3D, focusing, energy-critical NLS in the non-radial case remains open.

The concentration compactness method also enlighten the development of mass critical NLS. Killip, Tao, Visan, and Zhang [55,56,76] studied the mass critical NLS in the radial case. Dodson then remove the radial assumption, and completely solved the global well-posedness and scattering of mass critical NLS in the defocusing case [32,34,35], and in the focusing case below the mass of ground state [33].

Next, we focus on the well-posedness results of 3D, defocusing, cubic NLS, for which the critical regularity exponent $s_c = 1/2$. We have learned that the equation is local well-posed in $\dot{H}^{1/2}_x$, while the global well-posedness and scattering hold in a smaller space $H^1_x$. A natural question is to ask the weakest space $X$ to guarantee the global well-posedness in $\dot{H}^{1/2}_x \cap X$. Bourgain [9] used the high-low decomposition method (introduced in [8]) to give $X = H^s_x$ with $s > \frac{11}{13}$. The lower bound was then
improved by “I-method” gradually in [25, 26, 74], and so far, the best result is $s > \frac{5}{7}$.

Under the radial assumption, Dodson [37] showed that the result holds for almost critical space $s > \frac{1}{2}$.

Note that $X$ spaces in the above mentioned results are all $\dot{H}^{1/2}$ super-critical. Recently, Dodson [38] gave a result in the critical space $X = \dot{W}^{11/7, 7/6}$, based on the observation that linear solution becomes more regular with initial data in $L^p_x$ with $p < 2$. Using this observation and the method in [1], the authors [72] obtained that $X = \dot{W}^{s, 1}$ for $s > \frac{12}{13}$, which is a sub-critical space with the order of derivative less than 1.

Currently, there is no result for the global well-posedness of 3D defocusing cubic NLS merely in $\dot{H}^{1/2}$ or $H^{1/2}$. Kenig and Merle [51] initiated another approach towards this problem. They proposed the concept of “conditional scattering”, namely the global well-posedness and scattering hold for the solution that is uniformly bounded in the critical space on the maximal existence interval. Generally for the inter-critical NLS, no global well-posed result is known in the critical space. See [2, 38] for some related results.

Now, we turn to the probability theory of NLS. Although it is ill-posedness for NLS below the critical regularity due to the result of Christ, Colliander, and Tao [24], Bourgain [6, 7] first introduced a probabilistic method to study the well-posedness problem for periodic NLS for “almost” all the initial data in super-critical spaces, based on the Gibbs measure constructed by Lebowitz–Rose–Speer [61]. The probabilistic well-posedness result for super-critical wave equations on compact manifolds was also studied by Burq and Tzvetkov [16, 17]. There have been extensive studies about such subject since then, and we refer the readers to [5] for more complete overviews.

Next, we only review the study of random data theory for NLS on $\mathbb{R}^d$. There are several ways of randomization for the initial data. We recall the one relying on the unit-scale decomposition in frequency, which is named as the Wiener randomization (see [83]). Then, under the Wiener randomization, Bényi, Oh, and Pocovnicu [3] studied the cubic NLS when $d \geq 3$. They proved the almost sure local well-posedness, small data scattering, and a “conditional” global well-posedness under some a priori hypothesis. Afterwards, the almost sure local results were improved by Bényi–Oh–Pocovnicu [4] and Pocovnicu–Wang [70]. The random data well-posedness for quintic NLS was studied by Brereton [11]. Later, Oh, Okamoto, and Pocovnicu [69] studied the almost sure global well-posedness for energy critical NLS on $d = 5, 6$. Here, we only focus on the NLS, and see Lührmann–Mendelson [64] for the first probabilistic result of non-linear wave equations on Euclidean space.

The large data almost sure scattering was first obtained by Dodson, Lührmann, and Mendelson [39] in the context of 4D, defocusing, energy-critical, nonlinear wave equation with randomized radial data, using a double bootstrap argument combining the energy and Morawetz estimates. The result was extended by Bringmann to non-radial 4D case [13], and to radial 3D case [12]. The related results on non-radial energy-critical nonlinear Klein-Gordon equations were studied by Chen and Wang [23]. The first almost sure scattering result for NLS was given by Killip, Murphy, and Visan [53]. They proved the result for 4D, defocusing, energy-critical case with almost all the radial initial data in $\dot{H}^s$ for $\frac{5}{6} < s < 1$. This result was then improved to $\frac{1}{2} < s < 1$ by Dodson, Lührmann, and Mendelson [40].

We remark that the Wiener randomization is closely related to the modulation space introduced by Feichtinger [43]. Such space has been applied to non-linear evolution equations before the development of Wiener randomization, dating back to the results of Wang, Zhao, Guo, and Hudzik [80, 81].
There are also other kinds of randomization for NLS on $\mathbb{R}^d$. Burq, Thomann, and Tzvetkov [15] constructed a Gibbs measure for NLS with harmonic potential, and proved almost sure $L^2$-scattering for 1D, defocusing NLS with $p \geq 5$, after changing the Schrödinger equations into the ones with harmonic oscillator potential by lens transform. Recently, Burq and Thomann [14] improved the result to all the short range exponents $p > 3$. See also [60] for higher dimensional extensions.

In addition, Murphy [65] introduced a new kind of randomization based on the physical space unit-scale decomposition, and studied the almost sure existence and uniqueness of wave operator for $L^2$ sub-critical NLS above the Strauss exponent. Then, Nakanishi and Yamamoto [68] extended the result below Strauss exponent, and applied the method on some quadratic Schrödinger models. We also mention that Bringmann’s almost sure scattering results [12,13] include other kinds of randomization for nonlinear wave equations on $\mathbb{R}^d$, involving the micro-local and the annuli decompositions of initial data.

To the best of our knowledge, the only study so far of global well-posedness and scattering for inter-critical NLS seems Burq, Thomann, and Tzvetkov’s 1D $L^2$-scattering result [15], based on the invariant measure for the Schrödinger equations with harmonic potential. Very recently, we learnt that Duerinckx [41] also studied the global well-posedness of cubic NLS adding a tiny dissipation with spatial inhomogeneous random initial data. In this paper, we intend to study a typical model of inter-critical NLS, namely the 3D, defocusing, cubic NLS under the Wiener randomization, at super-critical regularity.

Before stating the main result, we give the definition of the randomization:

**Definition 1.1** (Wiener randomization). Let $\tilde{\psi} \in C^\infty_0(\mathbb{R}^3)$ be a real-valued function such that $\tilde{\psi} \geq 0$, $\tilde{\psi}(-\xi) = \tilde{\psi}(\xi)$ for all $\xi \in \mathbb{R}^3$ and

$$
\tilde{\psi}(\xi) = \begin{cases} 
1, \text{ when } \xi \in [-\frac{1}{2}, \frac{1}{2}]^3, \\
0, \text{ when } \xi \not\in [-1, 1]^3.
\end{cases}
$$

Let

$$
\psi(\xi) := \frac{\tilde{\psi}(\xi)}{\sum_{k \in \mathbb{Z}^3} \tilde{\psi}(\xi - k)}.
$$

Then, $\psi \in C^\infty(\mathbb{R}^3)$ is a real-valued function, satisfying for all $\xi \in \mathbb{R}^3$, $0 \leq \psi \leq 1$, $\text{supp } \psi \subset [-1, 1]^3$, $\psi(-\xi) = \psi(\xi)$, and $\sum_{k \in \mathbb{Z}^3} \psi(\xi - k) = 1$.

For any $k \in \mathbb{Z}^3$, define $\psi_k(\xi) = \psi(\xi - k)$. Denote the Fourier transform by $\mathcal{F}$. Then, we define

$$
\Box_k f = \mathcal{F}^{-1}(\psi_k \mathcal{F} f).
$$

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space. Let $\{g_k\}_{k \in \mathbb{Z}^3}$ be a sequence of zero-mean, complex-valued Gaussian random variables on $\Omega$, where the real and imaginary parts of $g_k$ are independent. Then, for any function $f$, we define its randomization $f^{\omega}$ by

$$
f^{\omega} = \sum_{k \in \mathbb{Z}^3} g_k(\omega) \Box_k f. \quad (1.6)
$$
In the following, we use the statement “almost every $\omega \in \Omega$, $PC(\omega)$ holds” to mean that
\[ \mathbb{P}\left( \{ \omega \in \Omega : PC(\omega) \text{ holds} \} \right) = 1. \]

Now, we study the 3D, defocusing, cubic NLS with randomized initial data:
\[
\begin{align*}
    i \partial_t u + \Delta u &= |u|^2 u, \\
    u(0, x) &= f^\omega(x).
\end{align*}
\] (1.7)

For this model under the probabilistic setting, the local well-posedness, small data scattering, and conditional global well-posedness results have been established before.

We first recall the local results for (1.7). Bényi, Oh, and Pocovnicu [4] proved the local result with $f \in H^s_x$ when $\frac{2}{5} \sigma < s < \frac{1}{2}$ in the sense that Duhamel term belongs to $C(I; H^\sigma_x)$ for any fixed $\frac{1}{2} \leq \sigma \leq 1$. They also proved the improved local result when $\frac{1}{6} < s < \frac{1}{2}$ (except for the lower endpoint) by weakening the definition of local solution:
\[ u - z_1 - z_3 - \cdots - z_{2k-1} \in C(I; H^{1/2}_x), \]

where the function $z_k \in C(I; H^{s_k}_x)$ is defined by iteration with some $s_k < \frac{1}{2}$. Pocovnicu and Wang [70] also proved the local result in $L^2_x$ with Duhamel term in $C(I; L^4_x)$.

There are also global results of (1.7), either with small data restriction or with suitable a priori assumptions. Bényi, Oh, and Pocovnicu [3] proved the almost sure small data global well-posedness and scattering for $\frac{1}{4} < s < \frac{1}{2}$. Furthermore, they [3] also proved the random data global well-posedness when $\frac{1}{4} < s < \frac{1}{2}$ under two a priori assumptions:

- The Duhamel term is uniformly bounded in the critical space $H^{1/2}_x$ in the probabilistic setting.
- The 3D, defocusing, cubic NLS is globally well-posed with deterministic initial data in $H^{1/2}_x$.

Each of the above two a priori assumptions seems very difficult to verify.

In this paper, we improve the previous local results for (1.7). Moreover with the radial data, we prove the global well-posedness as well as scattering, without imposing any a priori assumption or size restriction, where the scattering result holds in the energy space.

1.1. Almost sure local well-posedness. The first main result in this paper concerns the almost sure local well-posedness. Previously for the random data local result, Bényi, Oh, and Pocovnicu [4] introduced the higher order expansion method; Pocovnicu and Wang’s argument [70] is based on the dispersive inequality; Dodson, Lührmann, and Mendelson [40] used the high dimensional version of smoothing effect and maximal function estimates. In this paper, we give some simple new approaches combining the atom space method by Koch–Tataru [57] and the variants of bilinear Strichartz estimate.

**Theorem 1.2** (Local well-posedness). Let $f \in H^s_x(\mathbb{R}^3)$. Then, for almost every $\omega \in \Omega$, it holds that:
(1) If $\frac{1}{6} \leq s < \frac{1}{2}$, then there exists $T > 0$ and a solution $u$ of (1.7) on $[0, T]$ such that
\[ u - e^{it\Delta} f^w \in C([0, T]; H^{\frac{1}{6}}_x(\mathbb{R}^3)). \]

(2) If $\frac{1}{3} < s < \frac{1}{2}$, then there exists $T > 0$ and a solution $u$ of (1.7) on $[0, T]$ such that
\[ u - e^{it\Delta} f^w \in C([0, T]; H^{\frac{1}{3}}_x(\mathbb{R}^3)). \]

Our result improves the local results in [4], where Bényi, Oh, and Pocovnicu [4] proved the same results in Theorem 1.2 (1) for $\frac{1}{5} < s < \frac{1}{2}$ and Theorem 1.2 (2) for $\frac{2}{5} < s < \frac{1}{2}$.

The following are some remarks concerning the theorem.

Remark 1.3. (1) For the first result in Theorem 1.2, the lower exponent $\frac{1}{6}$ is optimal for the non-linear estimate of
\[ \int_0^t e^{i(t-s)\Delta} (|e^{is\Delta} f^w|^2 e^{is\Delta} f^w) \, ds \]
with $f \in H^{1/6}_x(\mathbb{R}^3)$. In fact, this optimality can be verified by taking the Knapp counter-example $f$ such that $\hat{f}(\xi) = \langle k \rangle^{-s} \psi(\xi - k)$.

(2) It seems very difficult to extend the local solution obtained in Theorem 1.2 (1) to global directly. Therefore, we establish the local solution with higher regularity in Theorem 1.2 (2).

(3) Our second result can be compared to Dodson, Lührmann, and Mendelson’s local result [40] for 4D, cubic NLS, which is energy critical, since both results put the Duhamel term in $C([0, T]; H^\frac{1}{3}_x(\mathbb{R}^4))$. They proved local well-posedness for $\frac{1}{3} < s < 1$, also except the endpoint exponent $\frac{1}{2}$.

(4) Apparently, the local results in Theorem 1.2 also hold for focusing equations.

The proof of Theorem 1.2 (2) is more difficult than the first local result. We postponed here to illustrate the main idea. It reduces to consider the term
\[ \nabla (|e^{it\Delta} f^w|^2 e^{it\Delta} f^w) \]
with $f$ merely in $H^{\frac{1}{3}}_x$. The task is how to allocate the first order derivative to each $e^{it\Delta} f^w$. However, the use of bilinear Strichartz estimate or local smoothing can only lower down semi-derivative. Then, we overcome the difficulty by following two tools:

- We employ the $U^p - V^p$ method introduced by Koch-Tataru [57] to exploit the duality structure.
- We also apply the bilinear Strichartz estimate in the form of
\[ \left\| e^{it\Delta} \phi \right\| L^1_t L^\infty_x(\mathbb{R} \times \mathbb{R}^3) \]
with $1 \leq q, r \leq 2$, see Candy’s result [18]. Particularly, the use of (1.8) with $q < 2$ and $r = 2$ can reduce the loss of derivative, at the cost of lower time integration exponents.
Let \( v = e^{it\Delta} f^\omega \), and it suffices to control

\[
\int_0^T \int_{\mathbb{R}^3} g_{hi} \nabla v_{hi} v_{low}^2 \, dx \, dt,
\]

where \( e^{-it\Delta} g \in V^2(\mathbb{R}; L^2_\omega) \), and “hi”, “low” represent the size of frequency. Heuristically, by Hölder’s inequality,

\[
\int_0^T \int_{\mathbb{R}^3} g_{hi} \nabla v_{hi} v_{low}^2 \, dx \, dt \lesssim \| g_{hi} v_{low} \|_{L^3_t L^2_\omega} \| \nabla v_{hi} \|_{L^1_t L^\infty_x} \| v_{low} \|_{L^\infty_t L^2_\omega}.
\]

This can cut down the order of high-frequency derivative to \( \frac{1}{3} \) for \( v_{hi} \), with a total loss of low-frequency derivative of order \( \frac{2}{3} \), which can be assigned to each \( v_{low} \).

Note that the above observation is sharp with respect to the regularity. Furthermore, we have a logarithmic loss of derivative when passing \( g \) into \( V^2_{\Delta} \) by interpolation. That is the main reason why we need \( s > \frac{1}{3} \) to acquire additional regularity for summation.

Moreover, if we only requires

\[
u - e^{it\Delta} f^\omega \in C([0, T]; H^\sigma_x(\mathbb{R}^3))
\]

for \( \frac{1}{2} \leq \sigma < 1 \), the approach in above observation can provide enough additional regularity for summation. Thus, we expect that the argument works for the optimal lower endpoint, namely \( \frac{1}{3} \sigma \leq s < \frac{1}{2} \), which clearly includes the result in Theorem 1.2 (1). However, in this paper, we only consider two endpoint cases when \( \sigma = \frac{1}{2} \) or \( \sigma = 1 \), and present two different methods, respectively. For Theorem 1.2 (1), we provide another proof without exploiting the duality structure. In fact, there is only \( \frac{1}{3} \)-order derivative acting on the nonlinear term, and we can transfer it simply using the bilinear Strichartz estimate.

Next, we also compare our local result to the torus case, where there exists the invariant Gibbs measure for NLS, see [6,7,31,61]. The common key ingredient is to update the regularity of the first iteration

\[
u^{(1)} := \int_0^t e^{i(t-s)\Delta} (|e^{is\Delta} f^\omega|^2 e^{is\Delta} f^\omega) \, ds
\]

for rough data \( f^\omega \in H^\frac{s}{2} \). Theorem 1.2 indicates that in the Euclidean case, the first iteration \( \nu^{(1)} \) has improved regularity \( \sigma = 3s \) when \( s = \frac{1}{6} \), and \( \sigma = 3s - \) when \( s = \frac{1}{3} \). However, in the periodic case when the initial data is given by

\[
f^\omega(x) = \sum_{n \in \mathbb{Z}^d} \frac{g_n(\omega)}{|n|^{s+d/2}} e^{i n \cdot x},
\]

the first iteration has improved regularity \( \sigma = s + \frac{1}{2} - \), which is higher than the Euclidean case when \( s < \frac{1}{4} \), and lower when \( s \geq \frac{1}{4} \). Inspired by this, the result in Theorem 1.2 (2) seems better than the observation in periodic case, and there seems still some room to further improve Theorem 1.2 (1).
1.2. Almost sure scattering. Now we turn to our second main result for the global well-posedness and scattering:

**Theorem 1.4** (Global well-posedness and scattering). Let \( \frac{17}{40} < s \leq \frac{1}{2} \) and \( f \in H^{s}_{x}(\mathbb{R}^{3}) \) be radial. Then, for almost every \( \omega \in \Omega \), there exists a solution \( u \) of (1.7) on \( \mathbb{R} \) such that

\[
\left\| u - e^{it\Delta} f^{\omega} \right\|_{C(\mathbb{R}; H^{1}_{x}(\mathbb{R}^{3}))} < \infty.
\]

Moreover, the solution \( u \) scatters, in the sense that there exist \( u_{\pm} \in H^{1}_{x} \) such that

\[
\lim_{t \to \pm \infty} \left\| u - e^{it\Delta} f^{\omega} - e^{it\Delta} u_{\pm} \right\|_{H^{1}_{x}} = 0.
\]

The most significant point of this result is that we are able to control the energy increment containing the term \( \nabla e^{it\Delta} f^{\omega} \), under the assumption that \( f \) merely belongs to \( H^{s}_{x} \) with some \( s < \frac{1}{2} \).

Comparing to the energy-critical results in [40,53], for 3D, defocusing, cubic NLS, it is easier to derive space-time estimates, since the Morawetz type estimates are energy sub-critical. On the other hand, however, this problem seems more difficult, in the sense that we need to reduce the order of derivative more than \( \frac{1}{2} \) for \( \nabla e^{it\Delta} f^{\omega} \), while the current results for energy-critical NLS lower down at most \( \frac{1}{2} \)-order derivative, in the view of local smoothing effect.

Our method is different from the recent results on the almost sure scattering of nonlinear Schrödinger and wave equations [12,13,39,40,53]. To establish the almost energy conservation of \( u - e^{it\Delta} f^{\omega} \), we make a high-low frequency decomposition of the initial data, and keep track of the explicit increase of energy bound. Then, we implement a bootstrap argument for the energy, building upon a perturbed interaction Morawetz estimate, various nonlinear estimates and the bilinear Strichartz estimate.

After the submission of this paper, we also learned that Camps [20] obtained an almost sure scattering result independently by a different approach.

Lastly, we remark that the lower bound of regularity \( \frac{17}{40} \) is not sharp. Here, we do not achieve this optimality, and only give a well-presented result. However, it is of great interest to improve the regularity’s lower bound down to \( \frac{1}{3} \), or even \( \frac{1}{6} \).

1.2.1. Sketch the proof of Theorem 1.4 The main ingredient of the proof is summarized as follows.

- **High-low frequency decomposition in the probabilistic setting.**

In probabilistic setting, we only have the boundedness in the almost every sense. Roughly speaking, in order to quantify the size of energy, we decompose the probability space \( \Omega \) by setting

\[
\widetilde{\Omega}_{M} = \left\{ \omega \in \Omega : \left\| f^{\omega} \right\|_{H^{s}} + N_{0} s^{\alpha} \left\| P^{\leq N_{0}} f^{\omega} \right\|_{H^{1}} + \left\| e^{it\Delta} f^{\omega} \right\|_{Y^{s}_{x}(\mathbb{R})} \leq M \left\| f \right\|_{H^{s}} \right\},
\]

where the \( \widetilde{Y}^{s} \)-norm is some required space-time norm defined by (3.11) and (5.2) below, and \( N_{0} \in \mathbb{Z}^{+} \) depends only on \( M \) and \( \left\| f \right\|_{H^{s}} \). See (5.5) for the precise definition of \( \widetilde{\Omega}_{M} \). Then it follows from the Borel-Cantelli Lemma that

\[
P\left( \bigcup_{M \geq 1} \widetilde{\Omega}_{M} \right) = 1.
\]
According to the decomposition above, we may consider ω ∈ \tilde{Ω}_M for each M separately. Now, we give the high-low frequency decomposition \( v = e^{it\Delta} P \geq N_0 f^\omega \) and \( w = u - v \). Then, for any ω ∈ \tilde{Ω}_M, there exists a constant \( C(M) > 0 \) such that

\[
E(w(0)) \leq C(M, \| f \|_{H^s}) N_0^{2(1-s)}.
\]

The application of Bourgain’s high-low decomposition method [8] to random Cauchy problem was first made by Colliander and Oh [28] for 1D NLS on \( \mathbb{T} \). However, in this paper, we do not intend to carry out Bourgain’s iteration procedure. We only make the decomposition in order for two benefits:

1. \( \hat{v} \) is supported on \( \{|\xi| \gtrsim N_0\} \).
2. We can explicitly keep track of the energy increment of \( N_0 \).

- **Strichartz estimates with \( \frac{1}{2} \)-derivative gain.**
  Note that \( v \) is not radial anymore under the Wiener randomization. However, due to the pioneering works [39, 40], we can prove that for the radial \( f \),

\[
\| (\nabla)^{s+\frac{1}{2}} v \|_{L^1_t L^\infty_x} < \infty, \tag{1.9}
\]

which is followed by combining a “radialish” Sobolev inequality for the square function and the local smoothing estimate. Note that the estimate (1.9) gains \( \frac{1}{2} \)-order derivative.

- **Global space-time bound for the nonlinear solution.**
  From the perturbed interaction Morawetz estimates, we can derive the bound of

\[
\| w \|_{L^4_t} \lesssim N_0^{\frac{1}{2}(1-s)},
\]

which is \( H^{\frac{1}{2}} \)-critical, under the a priori hypothesis of \( H^1 \)-bound. The high-low frequency decomposition also plays a crucial role for controlling the remainder. Using this estimate, we can obtain the following \( L^2 \)-critical estimate:

\[
\left( \sum_{N \in 2^\mathbb{N}} \| P_N w \|^2_{U^\infty_\Lambda(L^2_x)} \right)^{1/2} \lesssim N_0^{\frac{3}{2}(1-s)+}.
\]

However, these are far from sufficient for the estimates of energy bound.

Then, an observation is that combining the above \( L^4_t \)-estimate and the integral equation, for any \( L^2 \)-admissible \((q, r)\), we can further control \( \dot{H}^{\frac{1}{2}} \)-critical space-time norm:

\[
\left( \sum_{N \in 2^\mathbb{N}} N \| w_N \|^2_{L^q_t L^r_x} \right)^{1/2} \lesssim N_0^{\frac{3}{2}(1-s)}.
\]

Keeping in mind that the equation is \( \dot{H}^{\frac{1}{2}} \)-critical, the space-time estimate under the same scaling plays an important role throughout the whole argument.

Furthermore, applying the above \( \dot{H}^{\frac{1}{2}} \)-critical estimates, we can update the scaling up to \( \dot{H}^1_x \):

\[
\left( \sum_{N \in 2^\mathbb{N}} N^2 \| P_N w \|^2_{U^\infty_\Lambda(L^2_x)} \right)^{1/2} \lesssim N_0^{3(1-s)}.
\]
Almost Sure Well-Posedness and Scattering of the 3D Cubic Nonlinear Schrödinger Equation

For this purpose, we also need to use the maximal function techniques to deal with some critical cases. In the above argument, the use of \( U^2 \Delta \)-space has three advantages: we can transfer the derivative by duality formula, the \( U^2 \Delta \)-space can control the bilinear Strichartz estimate, and it allows estimates on any long-time interval.

- **Energy bound.**
  
The main goal is to prove
  \[
  \sup_{t \in \mathbb{R}} E(w(t)) \lesssim M N_0^{2(1-s)}.
  \]

It suffices to prove the bootstrap inequality
\[
\sup_{t \in I} \int_0^t \frac{d}{ds} E(w(s)) ds \lesssim M N_0^{-\alpha} N_0^{2(1-s)},
\]
for some \( \alpha > 0 \) under the assumption \( \sup_{t \in I} E(w(t)) \lesssim M N_0^{2(1-s)} \). Under this bootstrap hypothesis, we can also give the precise increase of \( N_0 \) for the various global space-time estimates on \( I \) obtained in the previous step, which are very useful for the control of energy increment. Now, the main term in the energy estimate is
\[
\bigg| \int_I \int_{\mathbb{R}^3} \nabla w_{hi} \nabla v_{hi} w_{low}^2 \, dx \, dt \bigg|,
\]
(1.10)
where “hi” and “low” represent the size of frequency.

We remark that the Morawetz estimate of the form \( \int \int |w|^4 |x| \, dx \, dt \) plays an important role in the former energy-critical results [39,40,53], but it is not sufficient for the 3D cubic case. First, the Morawetz estimate cannot yield the global space-time bounds, as in the previous step. That is the reason why the use of interaction Morawetz estimate is necessary. Second, using their method, (1.10) can be controlled by
\[
\| \nabla \nabla \|_{L^\infty_t L^2_x} \| \nabla v_{hi} \|_{L^\infty_t L^2_x} \left( \int_I \int_{\mathbb{R}^3} \frac{|w_{low}|^4}{|x|} \, dx \, dt \right)^{\frac{1}{2}}.
\]
Then, the energy bound can be followed by the “radialish” Sobolev inequality, local smoothing effect, and Morawetz estimate when \( s > \frac{1}{2} \). Unfortunately, this argument does not work here, since we lack the \( \nabla v \)-estimates in the view of (1.9), when \( s < \frac{1}{2} \) in our situation.

To overcome the difficulty, we observe that there is still some gap in the estimate
(1.10) \( \lesssim \| \nabla w_{hi} \|_{L^\infty_t L^2_x} \| \nabla v_{hi} \|_{L^2_t L^\infty_x} \| w_{low} \|_{L^4_{t,x}}^2 \),
towards the desired bound \( N_0^{2(1-s)} \). This gives us the room to use the global space-time estimates obtained above and the bilinear Strichartz estimate, which can further lower down the derivative for \( \nabla v_{hi} \).

1.3. Organization of the paper. In Sect. 2, we give some notation and useful results. In Sect. 3, we prove the almost sure space-time estimates for the linear solution. Then, we prove the local results in Theorem 1.2 in Sect. 4, and prove the global well-posedness and scattering results in Theorem 1.4 in Sect. 5.
2. Preliminary

2.1. Notation. For any $a \in \mathbb{R}$, $a \pm := a \pm \epsilon$ for some small $\epsilon > 0$. For any $z \in \mathbb{C}$, we define $\text{Re} z$ and $\text{Im} z$ as the real and imaginary part of $z$, respectively.

Let $C > 0$ denote some constant, and write $C(a) > 0$ for some constant depending on coefficient $a$. If $f \leq Cg$, we write $f \lesssim g$. If $f \leq Cg$ and $g \leq Cf$, we write $f \sim g$. Suppose further that $C = C(a)$ depends on $a$, then we write $f \lesssim_a g$ and $f \sim_a g$, respectively. If $f \leq 2^{-5}g$, we denote $f \ll g$ or $g \gg f$.

Moreover, we write “a.e. $\omega \in \Omega$” to mean “almost every $\omega \in \Omega$”.

We use $\hat{f}$ or $\mathcal{F}f$ to denote the Fourier transform of $f$:

$$\hat{f}(\xi) = \mathcal{F}f(\xi) := \int_{\mathbb{R}^d} e^{-ix \cdot \xi} f(x) dx.$$

We also define

$$\mathcal{F}^{-1}g(x) := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ix \cdot \xi} g(\xi) d\xi.$$

Using the Fourier transform, we can define the fractional derivative $|\nabla| := \mathcal{F}^{-1}|\xi|^s \mathcal{F}$ and $|\nabla|^p := \mathcal{F}^{-1}|\xi|^p \mathcal{F}$.

We next recall the unit-scale frequency decomposition in Definition 1.1. Let $\tilde{\psi} \in C^\infty_0(\mathbb{R}^3)$ be a real-valued function such that $\tilde{\psi} \geq 0$, $\tilde{\psi}(-\xi) = \tilde{\psi}(\xi)$ for all $\xi \in \mathbb{R}^3$ and

$$\tilde{\psi}(\xi) = \begin{cases} 1, & \text{when } \xi \in [-\frac{1}{2}, \frac{1}{2}]^3, \\ 0, & \text{when } \xi \notin [-1, 1]^3. \end{cases}$$

Let $\psi \in C^\infty_0(\mathbb{R}^d)$ be a real-valued function, satisfying that for all $\xi \in \mathbb{R}^d$, $0 \leq \psi \leq 1$, supp $\psi \subset [-1, 1]^d$, $\psi(-\xi) = \psi(\xi)$, and $\sum_{k \in \mathbb{Z}^d} \psi(\xi - k) = 1$. For any $k \in \mathbb{Z}^d$, define $\psi_k(\xi) = \psi(\xi - k)$. Then, we define

$$f_k = \Box_k f := \mathcal{F}^{-1}(\psi_k \mathcal{F} f).$$

We also introduce a fattening version of the unit-scale decomposition:

$$\Box_k f := \mathcal{F}^{-1}(\tilde{\psi}(2^{-1}(\xi - k)) \mathcal{F} f).$$

Therefore, by $\sum_{k \in \mathbb{Z}^d} \tilde{\psi}(2^{-1}(\xi - k)) \leq C$, for all $\xi \in \mathbb{R}^d$, we have orthogonality

$$\|\Box_k f\|_{L^2_k}^2 \sim \|f\|_{L^2_k}^2.$$

Note also that on the support of $\psi_k(\xi)$, we have $\tilde{\psi}(2^{-1}(\xi - k)) = 1$, which implies

$$\Box_k = \Box_k \Box_k.$$

We also need the usual inhomogeneous Littlewood-Paley decomposition for the dyadic number. Take a cut-off function $\phi \in C^\infty_0(\mathbb{R})$ such that $\phi(r) = 1$ if $0 \leq r \leq 1$ and $\phi(r) = 0$ if $r > 2$.

Then, we introduce the spatial cut-off function. Denote $\chi_0(r) = \phi(r)$, and $\chi_j(r) = \phi(2^{-j}r) - \phi(2^{-j+1}r)$ for $j \in \mathbb{N}^+$. We also define a fattening version $\tilde{\chi}_j := \phi(2^{-j-1}(|\xi|)) - \phi(2^{-j+2}r)$ with the property $\chi_j = \chi_j \tilde{\chi}_j$. 
Next, we give the definition of Littlewood-Paley dyadic projection operator. For dyadic $N \in 2^\mathbb{N}$, let $\phi_N(r) := \phi(N^{-1}r)$. Then, we define

$$\varphi_1(r) := \phi(r), \text{ and } \varphi_N(r) := \phi_N(r) - \phi_{N/2}(r), \text{ for any } N \geq 2.$$ 

We define the inhomogeneous Littlewood-Paley dyadic operator

$$f_1 = P_1 f := F^{-1}(\varphi_1(|\xi|) \hat{f}(\xi)),$$

and for any $N \geq 2$,

$$f_N = P_N f := F^{-1}(\varphi_N(|\xi|) \hat{f}(\xi)).$$

Then, by definition, we have $f = \sum_{N \in 2^\mathbb{N}} f_N$. Moreover, we also need the following: for any $N \in 2^\mathbb{N}$,

$$f_{\leq N} = P_{\leq N} f := F^{-1}(\varphi_1(|\xi|) \hat{f}(\xi)),$$

$$f_{\ll N} = P_{\ll N} f := F^{-1}(\varphi_N(2^5|\xi|) \hat{f}(\xi)),$$

$$f_{\lesssim N} = P_{\lesssim N} f := F^{-1}(\varphi_N(2^{-5}|\xi|) \hat{f}(\xi)),$$

and

$$f_{\sim N} = P_{\sim N} f := f_{\lesssim N} - f_{\ll N}.$$

We also denote that $f_{\gg N} = P_{\gg N} f := f - P_{\leq N} f$, $f_{\gg N} = P_{\gg N} f := f - P_{\lesssim N} f$, and $f_{\sim N} = P_{\sim N} f := f - P_{\ll N} f$.

Let $S(\mathbb{R}^d)$ be the Schwartz space, $S'(\mathbb{R}^d)$ be the tempered distribution space, and $C^\infty_0(\mathbb{R}^d)$ be the space of all the smooth compact-supported functions.

For Banach spaces $X$ and $Y$, we denote $X + Y$ as the sum space of $X$ and $Y$. Given $1 \leq p \leq \infty$, $L^p(\mathbb{R}^d)$ denotes the usual Lebesgue space. For any $0 \leq s < d/p$, we define the Sobolev space

$$\dot{W}^{s,p}(\mathbb{R}^d) := \{ f \in S'(\mathbb{R}^d) : \| f \|_{\dot{W}^{s,p}(\mathbb{R}^d)} := \| |\nabla|^s f \|_{L^p(\mathbb{R}^d)} < +\infty \}.$$

We denote that $\dot{H}^s(\mathbb{R}^d) := \dot{W}^{s,2}(\mathbb{R}^d)$. The inhomogeneous spaces are defined by

$$\dot{W}^{s,p}(\mathbb{R}^d) = \dot{W}^{s,p} \cap L^p(\mathbb{R}^d), \text{ and } \dot{H}^s(\mathbb{R}^d) = \dot{H}^s \cap L^2(\mathbb{R}^d).$$

We often use the abbreviations $H^s = \dot{H}^s(\mathbb{R}^d)$ and $L^p = L^p(\mathbb{R}^d)$. We also define $\langle \cdot, \cdot \rangle$ as real $L^2$ inner product:

$$\langle f, g \rangle = \text{Re} \int f(x) \overline{g}(x) \, dx.$$

For any $1 \leq p < \infty$, define $l^p_N = l^p_{N \in 2^\mathbb{N}}$ by its norm

$$\| c_N \|_{l^p_N} := \sum_{N \in 2^\mathbb{N}} |c_N|^p.$$
The space $l^p_k = l^p_{k \in \mathbb{Z}^d}$ is defined in a similar way. In this paper, we use the following abbreviations

$$
\sum_{N: N \leq N_1} := \sum_{N \in 2^{\mathbb{Z}^d}: N \leq N_1}, \quad \text{and} \quad \sum_{N \leq N_1} := \sum_{N, N_3 \in 2^{\mathbb{Z}^d}: N \leq N_1}.
$$

We then define the mixed norms: for $1 \leq q < \infty$, $1 \leq r \leq \infty$, and the function $u(t, x)$, we define

$$
\|u\|_{L^q_t L^r_x(\mathbb{R} \times \mathbb{R}^d)} := \int_{\mathbb{R}} \|u(t, \cdot)\|_{L^r_x}^q \, dt,
$$

and for the function $u_N(x)$, we define

$$
\|u_N\|_{L^q_{N} L^r_x(2^{2N} \times \mathbb{R}^d)} := \sum_{N} \|u_N(\cdot)\|_{L^r_x}^q.
$$

The $q = \infty$ case can be defined similarly.

For any $0 \leq \gamma \leq 1$, we call that the exponent pair $(q, r) \in \mathbb{R}^2$ is \(\dot{H}^\gamma\)-admissible, if

$$
\frac{2}{q} + \frac{d}{r} = \frac{d}{2} - \gamma, \quad 2 \leq q \leq \infty, \quad 2 \leq r \leq \infty, \quad \text{and} \quad (q, r, d) \neq (2, \infty, 2). \quad \text{If} \quad \gamma = 0, \quad \text{we say that} \quad (q, r) \quad \text{is} \quad L^2\text{-admissible}.
$$

2.2. Atom space and bounded variation space. We recall the definitions of $U^p$ and $V^p$, and some properties used in this paper. The $U^p-V^p$ method was first introduced by Koch–Tataru [57], and we also refer the readers to [19,46,58,59] for their complete theories.

**Definition 2.1.** Let $\mathcal{Z}$ be the set of finite partitions $-\infty < t_0 < t_1 < \ldots < t_K = \infty$.

(1) For $\{t_k\}_{k=0}^K \in \mathcal{Z}$ and $\{\phi_k\}_{k=0}^{K-1} \subset L^2_x$ with $\sum_{k=0}^{K-1} \|\phi_k\|_{L^2_x}^p = 1$, we call the function $a : \mathbb{R} \rightarrow L^2_x$ given by $a = \sum_{k=1}^{K} 1_{\{t_{k-1}, t_k\}} \phi_k - 1$ a $U^p$-atom. Furthermore, we define the atomic space

$$
U^p(\mathbb{R}; L^2_\mathcal{Z}) := \left\{ u = \sum_{j=1}^{\infty} \lambda_j a_j : a_j \text{ \(U^p\)-atom}, \lambda_j \in \mathbb{C} \text{ with} \sum_{j=1}^{\infty} |\lambda_j| < \infty \right\}, \quad (2.1)
$$

with norm

$$
\|u\|_{U^p(\mathbb{R}; L^2_\mathcal{Z})} := \inf \left\{ \sum_{j=1}^{\infty} |\lambda_j| : u = \sum_{j=1}^{\infty} \lambda_j a_j, a_j \text{ \(U^p\)-atom}, \lambda_j \in \mathbb{C} \right\}. \quad (2.2)
$$

(2) We define $V^p(\mathbb{R}; L^2_\mathcal{Z})$ as the normed space of all functions $v : \mathbb{R} \rightarrow L^2$ such that

$$
\|v\|_{V^p(\mathbb{R}; L^2_\mathcal{Z})} := \sup_{\{t_k\}_{k=0}^K \in \mathcal{Z}} \left( \sum_{k=1}^{K} \|v(t_k) - v(t_{k-1})\|_{L^2_x}^p \right)^{1/p} \quad (2.3)
$$

is finite, where we use the convention $v(t_K) = v(\infty) = 0$. $V^p_{rc}$ denotes the closed subspace of all right-continuous $V^p$ functions with $\lim_{t \rightarrow -\infty} v(t) = 0$. 
We define \( U^2_\Delta(\mathbb{R}; L^2_x) \) as the adapted normed space:
\[
U^2_\Delta(\mathbb{R}; L^2_x) := \{ u : \| u \|_{U^2_\Delta(\mathbb{R}; L^2_x)} := \| e^{-i\Delta} u \|_{U^2(\mathbb{R}; L^2_x)} < \infty \}.
\]

Similarly, \( V^2_\Delta(\mathbb{R}; L^2_x) \) denotes the adapted normed space
\[
V^2_\Delta(\mathbb{R}; L^2_x) := \{ u : \| u \|_{V^2_\Delta(\mathbb{R}; L^2_x)} := \| e^{-i\Delta} u \|_{V^2(\mathbb{R}; L^2_x)} < \infty, e^{-i\Delta} u \in V^2 \}.
\]

In this paper, we will use restriction spaces to some interval \( I \subset \mathbb{R} : U^p(I; L^2_x), V^p(I; L^2_x), U^p_\Delta(I; L^2_x) \), and \( V^p_\Delta(I; L^2_x) \). See Remark 2.23 in [46] for more details.

Note that for \( 1 \leq p < q < \infty \), the embeddings
\[
U^p(\mathbb{R}; L^2_x) \hookrightarrow L^\infty_I(\mathbb{R}; L^2_x), V^2(\mathbb{R}; L^2_x) \hookrightarrow L^\infty_I(\mathbb{R}; L^2_x),
\]
and \( U^p \hookrightarrow V^p \hookrightarrow U^q \) are continuous.

We need the following classical linear estimate and duality formula:

**Lemma 2.2** ([46]). Let \( I \) be an interval such that \( 0 = \inf I \). Then, for any \( f \in L^2_x \),
\[
\| e^{i\Delta} f \|_{U^2_\Delta(I; L^2_x)} \lesssim \| f \|_{L^2_x},
\]
and for \( F(t, x) \in L^1_t L^2_x(I \times \mathbb{R}^d)\),
\[
\int_0^t e^{i(t-s)\Delta} F(s) \, ds \|_{U^2_\Delta(I; L^2_x)} = \sup_{\| g \|_{V^2_\Delta(I; L^2_x)} = 1} \left| \int_I \int_{\mathbb{R}^3} F(t) g(t) \, dx \, dt \right|.
\]

We also need the following interpolation result to transfer from \( U^2_\Delta \) into \( V^2_\Delta \).

**Lemma 2.3** ([46]). Let \( q > 1 \), \( E \) be a Banach space and \( T : U^q_\Delta \to E \) be a bounded, linear operator with \( \| Tu \|_E \leq C_q \| u \|_{U^q_\Delta} \). In addition, assume that for some \( 1 \leq p < q \), there exists \( C_p \in (0, C_q] \) such that the estimate \( \| Tu \|_E \leq C_p \| u \|_{U^p_\Delta} \) holds true for all \( u \in U^p_\Delta \). Then, \( T \) satisfies the estimate for \( u \in V^p_\Delta \),
\[
\| Tu \|_E \leq \frac{4}{(1 - p/q) \ln 2} C_p (1 + 2(1 - p/q) \ln 2 + \ln \frac{C_q}{C_p}) \| u \|_{V^p_\Delta}.
\]

2.3. **Useful lemmas.** In this subsection, we gather some useful results.

**Lemma 2.4** (Schur’s test). For any \( a > 0 \), let sequences \( \{ a_N \}, \{ b_N \} \in l^2_{N \in \mathbb{N}} \), then we have
\[
\sum_{N_1 \leq N} \left( \frac{N_1}{N} \right)^a a_N b_N \lesssim \| a_N \|_{l^1_N} \| b_N \|_{l^2_N}.
\]

**Lemma 2.5** (Hardy’s inequality). For \( 1 < p < d \), we have that
\[
\| |x|^{-1} u \|_{L^p_{\mathbb{R}^d}} \lesssim \| \nabla u \|_{L^p_{\mathbb{R}^d}}.
\]
Lemma 2.6 (Local smoothing, [29, 45, 52]). We have that
\[
\sup_{R > 0} R^{-\frac{1}{2}} \left\| e^{it \Delta} f \right\|_{L^2_t(\mathbb{R}; L^2_{|x|<R})} \lesssim \left\| |\nabla|^{-\frac{1}{2}} f \right\|_{L^2_x}.
\]

Lemma 2.7 (Strichartz estimate, [49, 59]). Let \( I \subset \mathbb{R} \). Suppose that \((q, r)\) and \((\tilde{q}, \tilde{r})\) are \(L^2_x\)-admissible. Then,
\[
\left\| e^{it \Delta} \phi \right\|_{L^q_t L^r_x(\mathbb{R} \times \mathbb{R}^d)} \lesssim \| \phi \|_{L^2_x},
\]
and
\[
\left\| \int_0^t e^{i(t-s) \Delta} F(s) \, ds \right\|_{L^q_t L^r_x(\mathbb{R} \times \mathbb{R}^d)} \lesssim \| F \|_{L^q_t L^r_x(\mathbb{R} \times \mathbb{R}^d)}. \tag{2.7}
\]
Moreover, for \(2 \leq q < \infty\),
\[
\| u \|_{L^q_t L^r_x(I \times \mathbb{R}^3)} \lesssim \| u \|_{U^q_2(I; L^2_x)},
\]
and if we assume further \(q \neq 2\), then
\[
\| u \|_{L^q_t L^r_x(I \times \mathbb{R}^3)} \lesssim \| u \|_{V^2_2(I; L^2_x)}. \tag{2.9}
\]

In this paper, we need the following multi-scale bi-linear Strichartz estimate for Schrödinger equation, which is a particular case of Theorem 1.2 in [18]:

Lemma 2.8. Let \(1 \leq q, r \leq 2, \frac{1}{q} + \frac{2}{r} < 2\), and suppose that \(M, N \in \mathbb{Z}^+\) satisfy \(M \ll N\). Then for any \(\phi, \psi \in L^2_x(\mathbb{R}^3)\),
\[
\left\| e^{it \Delta} P_N \phi \right\|_{L^q_t L^r_x(\mathbb{R} \times \mathbb{R}^3)} \lesssim \left\| e^{it \Delta} P_N \phi \right\|_{L^2_x} \left\| P_M \psi \right\|_{L^2_x}. \tag{2.10}
\]

The bilinear Strichartz estimate was first introduced by Bourgain [8], and further extended in [27, 79], when \(q = r = 2\). The \(q, r < 2\) case was referred to bilinear restriction estimates for paraboloid, first obtained by Tao [75], based on the method developed by Wolff [82].

We will frequently use the version of bi-linear estimate for general functions, see Candy’s result [18].

Lemma 2.9. Let \(I \subset \mathbb{R}, a \in I, 1 \leq q, r \leq 2, \frac{1}{q} + \frac{2}{r} < 2\), and suppose that \(M, N \in \mathbb{Z}^+\) satisfy \(M \ll N\). Then,
\[
\left\| P_N u P_M v \right\|_{L^q_t L^r_x(I \times \mathbb{R}^3)} \lesssim \left\| P_N u \right\|_{U^q_2(I; L^2_x(\mathbb{R}^3))} \left\| P_M v \right\|_{U^2_2(I; L^2_x(\mathbb{R}^3))}, \tag{2.11}
\]

The bilinear Strichartz estimate was first introduced by Bourgain [8], and further extended in [27, 79], when \(q = r = 2\). The \(q, r < 2\) case was referred to bilinear restriction estimates for paraboloid, first obtained by Tao [75], based on the method developed by Wolff [82].

We will frequently use the version of bi-linear estimate for general functions, see Candy’s result [18].
2.4. Maximal function estimates and Littlewood-Paley theory. Let $\mathcal{M}$ be the Hardy-Littlewood maximal operator:

$$\mathcal{M}f(x) := \sup_{r > 0} \frac{1}{|B(0, r)|} \int_{B(0, r)} |f(x - y)| \, dy,$$

where $B(0, r) = \{x \in \mathbb{R}^d : |x| \leq r\}$. $\mathcal{M}$ is bounded on $L^p_x$ for $1 < p < \infty$. Furthermore, we have the vector-valued version of the boundedness:

**Lemma 2.10** $(L^p_t)^2$-boundedness for maximal function, see [73]). Let $1 < p < \infty$ and $\{f_j\}_{j \in \mathbb{N}^+}$ be a sequence of functions such that $\|f_j\|_{L^p_t}^2 \in L^p_x$. Then, we have

$$\|\mathcal{M}(f_j)\|_{L^p_t}^2 \lesssim \|f_j\|_{L^p_t}^2.$$ 

We also gather some useful classical results about the Littlewood-Paley projection operator.

**Lemma 2.11** (Maximal Littlewood-Paley estimates). Let $1 < p < \infty$ and $f \in L^p_x(\mathbb{R}^d)$. Then, we have

$$\|\sup_{N \in 2\mathbb{N}} |P_N f|\|_{L^p} + \|\sup_{N \in 2\mathbb{N}} |P_{\leq N} f|\|_{L^p} \lesssim \|f\|_{L^p}.$$

**Proof** Note that $\mathcal{F}^{-1}(\phi_N)$ is a $L^1$-renormalised, radial Schwartz function, we have that for any $x \in \mathbb{R}^d$,

$$|P_N f(x)| = |\mathcal{F}^{-1}(\phi_N) \ast f(x)| \lesssim \mathcal{M}(f)(x),$$

where $\mathcal{M}$ is the Hardy-Littlewood maximal operator. Then, by the $L^p$ boundedness of $\mathcal{M}$,

$$\|\sup_{N \in 2\mathbb{N}} |P_N f|\|_{L^p} \lesssim \|\mathcal{M}(f)\|_{L^p} \lesssim \|f\|_{L^p}.$$ 

The proof for $P_{\leq N} f$ follows similarly. $\square$

**Lemma 2.12** (Littlewood-Paley estimates). Let $1 < p < \infty$ and $f \in L^p_x(\mathbb{R}^d)$. Then, we have

$$\|f_N\|_{L^p_t}^2 \sim_p \|f\|_{L^p}.$$ 

2.5. Probabilistic theory. We recall the large deviation estimate, which holds for the random variable sequence $\{\text{Reg}_k, \text{Im}_k\}$ in the Definition 1.1.

**Lemma 2.13** (Large deviation estimate, [16]). Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space. Let $\{g_n\}_{n \in \mathbb{N}^+}$ be a sequence of real-valued, independent, zero-mean random variables with associated distributions $\{\mu_n\}_{n \in \mathbb{N}^+}$ on $\Omega$. Suppose $\{\mu_n\}_{n \in \mathbb{N}^+}$ satisfies that there exists $c > 0$ such that for all $\gamma \in \mathbb{R}$ and $n \in \mathbb{N}^+$

$$\left| \int_{\mathbb{R}} e^{\gamma x} \, d\mu_n(x) \right| \leq e^{c\gamma^2},$$

where $\mu_n$ denotes the $n$-th element of the sequence $\{\mu_n\}_{n \in \mathbb{N}^+}$.
then there exists $\alpha > 0$ such that for any $\lambda > 0$ and any complex-valued sequence $\{c_n\}_{n \in \mathbb{N}^+} \in l_2^n$, we have
\[
P\left( \left\{ \omega : \sum_{n=1}^{\infty} |c_ng_n(\omega)| > \lambda \right\} \right) \leq 2 \exp \left\{ -\alpha \lambda \|c_n\|_{l_2^n}^{-2} \right\}.
\]
Furthermore, there exists $C > 0$ such that for any $2 \leq p < \infty$ and complex-valued sequence $\{c_n\}_{n \in \mathbb{N}^+} \in l_2^n$, we have
\[
\left\| \sum_{n=1}^{\infty} c_ng_n(\omega) \right\|_{L^p(\Omega)} \leq C \sqrt{p} \|c_n\|_{l_2^n}.
\] (2.12)

The following lemma can be proved by the method in [78], see also [39,40].

**Lemma 2.14** Let $F$ be a real-valued measurable function on a probability space $(\Omega, \mathcal{A}, P)$. Suppose that there exists $C_0 > 0, K > 0$ and $p_0 \geq 1$ such that for any $p \geq p_0$, we have
\[
\|F\|_{L^p(\Omega)} \leq \sqrt{p} C_0 K.
\]
Then, there exist $c > 0$ and $C_1 > 0$, depending on $C_0$ and $p_0$ but independent of $K$, such that for any $\lambda > 0$,
\[
P\left( \left\{ \omega \in \Omega : |F(\omega)| > \lambda \right\} \right) \leq C_1 e^{-c\lambda^2 K^{-2}}.
\]

3. Almost Sure Strichartz Estimates

3.1. Non-radial data.

**Lemma 3.1.** Let $s \in \mathbb{R}$ and $f \in H_s^x$. Suppose that the randomization $f^\omega$ is defined in Definition 1.1. Then, we have the following estimates:

1. For any $2 \leq q, r < \infty$ with $\frac{2}{q} + \frac{3}{r} \leq \frac{3}{2}$, and for any $p \geq 2$,
\[
\left\| \left\langle \nabla \right\rangle^s e^{it\Delta} P_N f^\omega \right\|_{L^p_t L^q_x(\Omega \times 2^N \times \mathbb{R} \times \mathbb{R}^3)} \lesssim q \sqrt{p} \|f\|_{H^s}.
\] (3.1)

2. For any $p \geq 2$,
\[
\left\| \left\langle \nabla \right\rangle^s e^{it\Delta} P_N f^\omega \right\|_{L^{p}_t L^{\infty}_x(\Omega \times 2^N \times \mathbb{R} \times \mathbb{R}^3)} \lesssim \sqrt{p} \|f\|_{H^s}.
\] (3.2)

3. For any $2 \leq q < \infty$ and $p \geq 2$,
\[
\left\| \left\langle \nabla \right\rangle^s e^{it\Delta} P_N f^\omega \right\|_{L^{p}_t L^{q}_x(\Omega \times 2^N \times \mathbb{R} \times \mathbb{R}^3)} \lesssim q \sqrt{p} \|f\|_{H^s}.
\] (3.3)

4. For any $2 < r \leq \infty$ and $p \geq 2$,
\[
\left\| \left\langle \nabla \right\rangle^s e^{it\Delta} P_N f^\omega \right\|_{L^{p}_t L^{\infty}_x(\Omega \times 2^N \times \mathbb{R} \times \mathbb{R}^3)} \lesssim r \sqrt{p} \|f\|_{H^s}.
\] (3.4)
Remark 3.2. We remark that for example, by Minkowski’s inequality and Lemma 2.12, (3.1) also gives that

\[
\left\| \langle \nabla \rangle^s e^{it\Delta} f^{(s)} \right\|_{L^p_t L^q_x(L^r_x)} \lesssim \sqrt{p} \left\| f \right\|_{H^s_x}.
\]

**Proof.** In the proof of this lemma, we restrict the variables on \( \omega \in \Omega, N \in 2\mathbb{N}, t \in \mathbb{R}, x \in \mathbb{R}^3, \) and \( k \in \mathbb{Z}^3. \) The implicit constant \( C(q, r) \) may depend on some fixed exponents \( q, r, \) and we abbreviate it to \( C \) for short.

We first prove (3.1). Let \( p_0 = \max\{q, r\}. \) If \( 2 \leq p \leq p_0, \) then \( p \sim p_0 \sim 1. \) By Hölder’s inequality in \( \omega, \) Minkowski’s inequality, and Lemma 2.13,

\[
\left\| \langle \nabla \rangle^s e^{it\Delta} P_N f \right\|_{L^p_t L^q_x(L^r_x)} \lesssim \left\| \langle \nabla \rangle^s e^{it\Delta} P_N f \right\|_{L^p_t L^q_x(L^r_x)} \lesssim \sqrt{p} \left\| \langle \nabla \rangle^s e^{it\Delta} P_N f \right\|_{L^p_t L^q_x(L^r_x)}.
\]

If \( p \geq p_0, \) simply using Minkowski’s inequality and Lemma 2.13,

\[
\left\| \langle \nabla \rangle^s e^{it\Delta} P_N f \right\|_{L^p_t L^q_x(L^r_x)} \lesssim \left\| \langle \nabla \rangle^s e^{it\Delta} P_N f \right\|_{L^p_t L^q_x(L^r_x)} \lesssim \sqrt{p} \left\| \langle \nabla \rangle^s e^{it\Delta} P_N f \right\|_{L^p_t L^q_x(L^r_x)}.
\]

Therefore, we have that for any \( p \geq 2, \)

\[
\left\| \langle \nabla \rangle^s e^{it\Delta} P_N f \right\|_{L^p_t L^q_x(L^r_x)} \lesssim \sqrt{p} \left\| \langle \nabla \rangle^s e^{it\Delta} P_N f \right\|_{L^p_t L^q_x(L^r_x)}.
\]

Now, let \( 2 \leq r_0 \leq r \) such that \( (q, r_0) \) is \( L^2 \)-admissible. For any \( k \in \mathbb{Z}^3, \) by the support property of \( \psi_k \) and Bernstein’s inequality, we have

\[
\left\| \langle \nabla \rangle^s e^{it\Delta} P_N f \right\|_{L^q_t L^r_x} \lesssim \left\| \langle \nabla \rangle^s e^{it\Delta} P_N f \right\|_{L^q_t L^r_x} \lesssim \sqrt{p} \left\| \langle \nabla \rangle^s e^{it\Delta} P_N f \right\|_{L^q_t L^r_x}.
\]

Then, by (3.5), (3.6), Lemma 2.7, and orthogonality, we have

\[
\left\| \langle \nabla \rangle^s e^{it\Delta} P_N f \right\|_{L^p_t L^q_x(L^r_x)} \lesssim \sqrt{p} \left\| \langle \nabla \rangle^s e^{it\Delta} P_N f \right\|_{L^p_t L^q_x(L^r_x)} \lesssim \sqrt{p} \left\| \langle \nabla \rangle^s e^{it\Delta} P_N f \right\|_{L^p_t L^q_x(L^r_x)}.
\]

This gives (3.1).

Next, we prove (3.2). By Plancherel’s identity, we have

\[
\left\| \langle \nabla \rangle^s e^{it\Delta} P_N f \right\|_{L^p_t L^q_x(L^r_x)} \lesssim \left\| \langle \nabla \rangle^s P_N f \right\|_{L^p_t L^q_x(L^r_x)} \lesssim \left\| \langle \nabla \rangle^s f \right\|_{L^p_t L^q_x}.
\]
Then, by Minkowski’s inequality and Lemma 2.13,
\[
\left\| (\nabla)^s f^{\omega} \right\|_{L_x^p L_t^q} \lesssim \left\| (\nabla)^s f^{\omega} \right\|_{L_x^p L_t^{q'}} \lesssim \sqrt{P} \left\| (\nabla)^s \Box k f \right\|_{L_x^{p'} L_t^{q'}} \lesssim \sqrt{P} \| f \|_{H^s_x}.
\] (3.8)

Then, (3.7) and (3.8) imply (3.2).

We then prove (3.3). Let \( 0 < \varepsilon \ll 1 \). Using the Sobolev’s embedding \( W^{2s, \frac{4}{\varepsilon}} \hookrightarrow L^\infty \) in \( x \), we have
\[
\left\| (\nabla)^{s-2\varepsilon} e^{it\Delta} P_N f^{\omega} \right\|_{L_{t,x}^{p} L_{t,x}^{q} L_{r,x}^{\infty}} \lesssim \left\| (\nabla)^{s-4\varepsilon} e^{it\Delta} P_N f^{\omega} \right\|_{L_{t,x}^{p} L_{t,x}^{q} L_{r,x}^{\infty}} \lesssim \left\| (\nabla)^{s-4\varepsilon} (\partial_t)^{2\varepsilon} e^{it\Delta} P_N f^{\omega} \right\|_{L_{t,x}^{p} L_{t,x}^{q} L_{r,x}^{\infty}} \lesssim \left\| (\nabla)^s e^{it\Delta} P_N f^{\omega} \right\|_{L_{t,x}^{p} L_{t,x}^{q} L_{r,x}^{\infty}} \lesssim \left\| (\nabla)^s e^{it\Delta} P_N f^{\omega} \right\|_{L_{t,x}^{p} L_{t,x}^{q} L_{r,x}^{\infty}}.
\]
By (3.9) and (3.1), we have that (3.3) holds.

Finally, we prove (3.4). We only consider the \( Z \)-norm by
\[
\left\| (\nabla)^{s-6\varepsilon} e^{it\Delta} P_N f^{\omega} \right\|_{L_{t,x}^{p} L_{t,x}^{q} L_{r,x}^{\infty}} \lesssim \left\| (\nabla)^{s-6\varepsilon} e^{it\Delta} P_N f^{\omega} \right\|_{L_{t,x}^{p} L_{t,x}^{q} L_{r,x}^{\infty}} \lesssim \left\| (\nabla)^s e^{it\Delta} P_N f^{\omega} \right\|_{L_{t,x}^{p} L_{t,x}^{q} L_{r,x}^{\infty}} \lesssim \left\| (\nabla)^s e^{it\Delta} P_N f^{\omega} \right\|_{L_{t,x}^{p} L_{t,x}^{q} L_{r,x}^{\infty}}.
\]
Then, it follows from (3.10) and (3.1) that (3.4) holds.

Next, we gather all the space-time norms that will be used below. Let \( \frac{1}{6} \leq s < \frac{1}{2} \), and given any suitably small constant \( \varepsilon > 0 \). Define the \( Y^s(I) \) space by its norm
\[
\| u \|_{Y^s(I)} := \left\| (\nabla)^s u_N \right\|_{L_{t,x}^{p} L_{t,x}^{q} L_{r,x}^{\infty}(2^N \times I \times \mathbb{R}^3)} + \left\| (\nabla)^s e^{it\Delta} P_N L_{t,x}^{q} L_{r,x}^{\infty}(2^N \times I \times \mathbb{R}^3)} \right\| \)
and the \( Z \)-norm by
\[
\| u \|_{Z^s(I)} := \left\| (\nabla)^s e^{it\Delta} P_N L_{t,x}^{q} L_{r,x}^{\infty}(2^N \times I \times \mathbb{R}^3)} + \left\| (\nabla)^s e^{it\Delta} P_N L_{t,x}^{q} L_{r,x}^{\infty}(2^N \times I \times \mathbb{R}^3)} \right\|.
\]
Note that (2, 6) and \( (1/\varepsilon, 6/(3 - 4\varepsilon)) \) are \( L^2_{t,r} \)-admissible.

Since \( \varepsilon \) is a fixed small constant, by Lemmas 3.1 and 2.14, we have

**Corollary 3.3.** Let \( \frac{1}{6} \leq s < \frac{1}{2} \) and \( f \in H^s_x \). Then, there exist constants \( C, c > 0 \) such that for any \( \lambda \),
\[
\mathbb{P}\left( \{ \omega \in \Omega : \left\| e^{it\Delta} f^{\omega} \right\|_{Y^s(\mathbb{R})} + \left\| e^{it\Delta} f^{\omega} \right\|_{Z^s(\mathbb{R})} > \lambda \} \right) \leq C \exp \left\{ -c\lambda^2 \| f \|_{H^s_x(\mathbb{R})}^{-2} \right\}.
\]
Moreover,
\[
\left\| e^{it\Delta} f^{\omega} \right\|_{Y^s(\mathbb{R})} + \left\| e^{it\Delta} f^{\omega} \right\|_{Z^s(\mathbb{R})} < +\infty, \ a.e. \ \omega \in \Omega.
\]
Proposition 3.4. Let $\mathbb{P}$ be almost sure well-posed and scattering of the 3D cubic nonlinear Schrödinger equation. Here, we derive a super-critical estimate for the randomized radial data that can acquire $1 \over \varepsilon$. For any $s < s_0 + {1 \over 2}$, there exist constants $C, c > 0$ such that for any $\lambda > 0$,

$$\mathbb{P}\{ (\omega \in \Omega : \| (\nabla)^s e^{it\Delta} f^\omega \|_{L_l^2 L_s^r(\mathbb{R} \times \mathbb{R}^3)} > \lambda \} \leq C \exp \{- c\lambda^2 \| f \|_{H_{s_0}^1(\mathbb{R}^3)}^{-2} \}. \quad (3.13)$$

Moreover,

$$\| (\nabla)^s e^{it\Delta} f^\omega \|_{L_l^2 L_s^r(\mathbb{R} \times \mathbb{R}^3)} < \infty, \quad \text{a.e. } \omega \in \Omega. \quad (3.14)$$

To prove Proposition 3.4, we need the following 3D version of radial Sobolev estimate for the square function.

Lemma 3.5. Suppose that the function $f$ is radial and that $2 \leq r \leq \infty$. Then, for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that,

$$\| |x|^{1 - {r \over 2}} \| f_k \|_{L^r_\varepsilon(\mathbb{R}^3)} \leq C_\varepsilon \| f \|_{H_s^r(\mathbb{R}^3)}. \quad (3.15)$$

Proof. It suffices to prove the $r = \infty$ case, since the general case can be obtained by interpolation with

$$\| f_k \|_{L^r_\varepsilon(\mathbb{R}^3)} \sim \| f \|_{L^r_\varepsilon(\mathbb{R}^3)}.$$

Since $f$ is radial, we can write $\hat{f}(x) = \hat{f}(|x|)$. Assume without loss of generality that $x = (0, 0, |x|)$. Then, by integration-by-parts and the spherical coordinate

$$\xi(\rho, \theta, \alpha) = (\rho \sin \theta \cos \alpha, \rho \sin \theta \sin \alpha, \rho \cos \theta),$$

we have

$$f_k = \int_0^{2\pi} \int_0^\pi \int_0^\infty \psi_k(\xi(\rho, \theta, \alpha)) \hat{f}(\rho) e^{i|x|\rho \cos \theta} \rho^2 \sin \theta d\rho d\theta d\alpha$$

$$= \frac{1}{i|x|} \int_0^{2\pi} \int_0^\pi \int_0^\infty \psi_k(\xi(\rho, \theta, \alpha)) \partial_\theta \left( e^{i|x|\rho \cos \theta} \hat{f}(\rho) \right) \rho \sin \theta d\rho d\theta d\alpha$$

$$= \frac{1}{i|x|} \int_0^{2\pi} \int_0^\pi \partial_\theta \left( \psi_k(\xi(\rho, \theta, \alpha)) e^{i|x|\rho \cos \theta} \hat{f}(\rho) \right) \rho d\theta d\rho d\rho \quad (3.16a)$$
\[
- \frac{1}{i|x|} \int_0^\infty \int_0^{2\pi} \psi_k(\xi(\rho, \pi, \alpha)) e^{-i|x|\rho} \hat{f}(\rho) \rho \, d\alpha \, d\rho \\
+ \frac{1}{i|x|} \int_0^\infty \int_0^{2\pi} \psi_k(\xi(\rho, 0, \alpha)) e^{i|x|\rho} \hat{f}(\rho) \rho \, d\alpha \, d\rho.
\] (3.16b)

Denote that
\[k = (k_1, k_2, k_3) = (|k| \sin \theta_k \cos \alpha_k, |k| \sin \theta_k \sin \alpha_k, |k| \cos \theta_k).\]

By the support of \(\psi_k\), we have that \(\rho\) is supported on the set \(\{\rho : |\rho - |k|| \lesssim 1\}\), \(\theta\) is supported on \(\{\theta : |\theta - \theta_k| \lesssim \frac{1}{|k|}\}\), and \(\alpha\) is supported on \(\{\alpha : |\alpha - \alpha_k| \lesssim \frac{1}{|k| \sin \theta_k}\}\). Furthermore, we also have
\[|\partial_\theta (\psi_k(\xi(\rho, \theta, \alpha)))| \lesssim |\nabla_\xi \psi_k(\xi(\rho, \theta, \alpha))|.
\]

Therefore, we have
\[
|\text{(3.16a)}| \lesssim \frac{1}{|x|} \frac{1}{|k| \sin \theta_k} \|\hat{f}(\rho)\|^2 \|L^1_{|\rho - |k|| \leq 1}\| \lesssim \frac{1}{|x|} \frac{1}{|k| \sin \theta_k} \|\hat{f}(\rho)\|^2 \|L^2_{|\rho - |k|| \leq 1}\|. \tag{3.17}
\]

Similarly, we also have
\[
|\text{(3.16b)}| + |\text{(3.16c)}| \lesssim \frac{1}{|x|} \frac{1}{|k| \sin \theta_k} \|\hat{f}(\rho)\|^2 \|L^2_{|\rho - |k|| \leq 1}\|. \tag{3.18}
\]

Then, by (3.17) and (3.18),
\[
|x|^2 \sum_{k \in \mathbb{Z}^3} |f_k|^2 \lesssim \sum_{k \in \mathbb{Z}^3} \frac{1}{|k| \sin \theta_k} \|\hat{f}(\rho)\|^2 \|L^2_{|\rho - |k|| \leq 1}\|
\sim \sum_{N \in \mathbb{N}} \sum_{k_1, k_2 \in \mathbb{Z}^2 \atop |k_1| \leq N, |k_2| \leq N} \frac{1}{k_1^2 + k_2^2} \|\hat{f}(\rho)\|^2 \|L^2_{|\rho - N| \leq 1}\|
\lesssim \sum_{N \in \mathbb{N}} \ln N \|\hat{f}(\rho)\|^2 \|L^2_{|\rho - N| \leq 1}\| \lesssim \|f\|^2_{H^s_x(\mathbb{R}^3)}.
\]

This finishes the proof of (3.15). \(\square\)

We also need the following mismatch estimates concerning the commutator of \(\chi_j\) and \(\Box_k\). The same result was already proved in [40] for 4D, and their argument can be easily extended to general dimensions. Therefore, we omit the details of proof.

**Lemma 3.6** (Mismatch estimates). Let \(2 \leq r \leq \infty\), \(j, l \geq 0\) and \(k, m \in \mathbb{Z}^3\). Suppose that \(l > j + 5\) and \(|k - m| > 100\). Then for any integer \(M > 0\), we have
\[
\|\chi_j \Box_k \chi_l\|_{L^r_\xi(\mathbb{R}^3) \rightarrow L^r_\xi(\mathbb{R}^3)} \leq C_M 2^{-M l}, \tag{3.19}
\]
and
\[
\|\Box_k \chi_l \Box_m\|_{L^r_\xi(\mathbb{R}^3) \rightarrow L^r_\xi(\mathbb{R}^3)} \leq C_M 2^{-M l} |k - m|^{-M}. \tag{3.20}
\]
Now, we give the proof of Proposition 3.4 using Lemmas 3.5 and 3.6.

**Proof of Proposition 3.4.** Take some $\epsilon > 0$ that will be defined later. We first consider the $r < \infty$ case. Then, for any $p \geq r$, by Lemma 2.13,

$$
\| (\nabla)^s e^{it \Delta} f \|_{L^p_{t,x} L^2_t L^p_x} \lesssim \sum_{N \in 2^\mathbb{N}} N^s \| e^{it \Delta} P_N f \|_{L^2_t L^p_x L^p_x} \\
\lesssim \sqrt{p} \sum_{N \in 2^\mathbb{N}} N^s \left\| e^{it \Delta} \Box_k P_N f \right\|_{L^2_t L^2_x L^2_k} \\
\lesssim \sqrt{p} \sum_{N \in 2^\mathbb{N}} N^s \left\| \chi_j \Box_k \chi \lesssim_j e^{it \Delta} P_N f \right\|_{L^2_t L^2_x L^2_k} + \sqrt{p} \sum_{N \in 2^\mathbb{N}} N^s \left\| \chi_j \Box_k \chi \gtrsim_j e^{it \Delta} P_N f \right\|_{L^2_t L^2_x L^2_k}.
$$

(3.21a) (3.21b)

We first bound the term (3.21a). When $j = 0$, by Minkowski’s, Bernstein’s inequalities, Lemmas 3.5 and 2.6,

$$
\sum_{N \in 2^\mathbb{N}} N^s \left\| \chi_0 \Box_k \chi \lesssim_5 e^{it \Delta} P_N f \right\|_{L^2_t L^2_x L^2_k} \lesssim \sum_{N \in 2^\mathbb{N}} N^s \left\| \chi_0 \Box_k \chi \lesssim_5 e^{it \Delta} P_N f \right\|_{L^2_t L^2_x L^2_k} \lesssim \sum_{N \in 2^\mathbb{N}} N^s \left\| \chi \lesssim_5 e^{it \Delta} P_N f \right\|_{L^2_t L^2_x L^2_k} \lesssim \sum_{N \in 2^\mathbb{N}} N^s \left\| \chi \lesssim e^{it \Delta} P_N f \right\|_{L^2_t L^2_x} \lesssim \left\| f \right\|_{H^s_x}^{s+\frac{1}{2}}.
$$

(3.22)

For $j \geq 1$, also by Lemmas 3.5 and 2.6,

$$
\sum_{N \in 2^\mathbb{N}} \sum_{j \geq 1} N^s \left\| \chi_j \Box_k \chi \lesssim_j e^{it \Delta} P_N f \right\|_{L^2_t L^2_x L^2_k} \lesssim \sum_{N \in 2^\mathbb{N}} \sum_{j \geq 1} N^s \left\| \chi_j \chi \lesssim_j e^{it \Delta} P_N f \right\|_{L^2_t L^2_x L^2_k} \lesssim \sum_{N \in 2^\mathbb{N}} \sum_{j \geq 1} N^s \left\| \chi \lesssim_j e^{it \Delta} P_N f \right\|_{L^2_t L^2_x} \lesssim \sum_{N \in 2^\mathbb{N}} \sum_{j \geq 1} N^s \left\| \chi \lesssim j e^{it \Delta} P_N f \right\|_{L^2_t L^2_x} \lesssim \sum_{N \in 2^\mathbb{N}} \sum_{j \geq 1} N^s \left\| \chi \lesssim j e^{it \Delta} P_N f \right\|_{L^2_t L^2_x} \lesssim \left\| f \right\|_{H^s_x}^{s+2e-\frac{1}{2}}.
$$

(3.23)

Then, combining (3.22) and (3.23), we have

$$
(3.21a) \lesssim \sqrt{p} \left\| f \right\|_{H^s_x}^{s+2e-\frac{1}{2}}.
$$

(3.24)
Next, we consider (3.21b). We decompose that

\[
(3.21b) \lesssim \sqrt{p} \sum_{N \in \mathbb{N}} \sum_{j \geq 0} N^s \left\| \chi_j \square_k \chi_{j+5} e^{it\Delta} P_N f \right\|_{L^2_t L^r_x k_k^2}^2 \\
\lesssim \sqrt{p} \sum_{N \in \mathbb{N}} \sum_{j \geq 0} \sum_{l > j+5} N^s \left\| \chi_j \square_k \chi_{l} e^{it\Delta} P_N f \right\|_{L^2_t L^r_x k_k^2}^2 \\
\lesssim \sqrt{p} \sum_{N \in \mathbb{N}} \sum_{j \geq 0} \sum_{l > j+5} N^s \left\| \sum_{m \in \mathbb{Z}^3: \vert m - k \vert \leq 100} \chi_j \square_k \chi_{l} \square_m \chi_{l} e^{it\Delta} P_N f \right\|_{L^2_t L^r_x k_k^2}^2 \\
+ \sqrt{p} \sum_{N \in \mathbb{N}} \sum_{j \geq 0} \sum_{l > j+5} N^s \left\| \sum_{m \in \mathbb{Z}^3: \vert m - k \vert > 100} \chi_j \square_k \chi_{l} \square_m \chi_{l} e^{it\Delta} P_N f \right\|_{L^2_t L^r_x k_k^2}^2. 
\]

(3.25a)

Now, we take some \( M \gg 1 \). For (3.25a), by Minkowski’s inequality, Lemmas 3.6, and 2.6,

\[
(3.25a) \lesssim \sqrt{p} \sum_{N \in \mathbb{N}} \sum_{j \geq 0} \sum_{l > j+5} N^s \left\| \chi_j \square_k \chi_{l} e^{it\Delta} P_N f \right\|_{L^2_t L^r_x k_k^2}^2 \\
\lesssim \sqrt{p} \sum_{N \in \mathbb{N}} \sum_{j \geq 0} \sum_{l > j+5} N^s \left\| \chi_j \square_k \chi_{l} e^{it\Delta} P_N f \right\|_{L^2_t L^r_x k_k^2}^2 \\
\lesssim \sqrt{p} \sum_{N \in \mathbb{N}} \sum_{j \geq 0} \sum_{l > j+5} N^s 2^{-Ml} \left\| \sum_{m \in \mathbb{Z}^3: \vert m - k \vert \leq 100} \chi_j \square_k \chi_{l} e^{it\Delta} P_N f \right\|_{L^2_t L^r_x k_k^2}^2 \\
\lesssim \sqrt{p} \sum_{N \in \mathbb{N}} \sum_{j \geq 0} \sum_{l > j+5} N^s 2^{-Ml} \left\| \chi_l e^{it\Delta} P_N f \right\|_{L^2_t L^r_x k_k^2}^2 \\
\lesssim \sqrt{p} \sum_{N \in \mathbb{N}} \sum_{j \geq 0} \sum_{l > j+5} N^s \left\| \chi_l e^{it\Delta} P_N f \right\|_{L^2_t L^r_x k_k^2}^2 \\
\lesssim \sqrt{p} \sum_{N \in \mathbb{N}} \sum_{j \geq 0} \sum_{l > j+5} N^s \left\| \chi_l e^{it\Delta} P_N f \right\|_{H^s}^2. 
\]

(3.26)

Similarly by Lemmas 3.6, 2.6, and Young’s inequality in \( k \),

\[
(3.25b) = \sqrt{p} \sum_{N \in \mathbb{N}} \sum_{j \geq 0} \sum_{l > j+5} N^s \left\| \chi_j \square_k \chi_{l} e^{it\Delta} P_N f \right\|_{L^2_t L^r_x k_k^2}^2 \\
\lesssim \sqrt{p} \sum_{N \in \mathbb{N}} \sum_{j \geq 0} \sum_{l > j+5} N^s \left\| \chi_j \square_k \chi_{l} e^{it\Delta} P_N f \right\|_{L^2_t L^r_x k_k^2}^2 \\
\lesssim \sqrt{p} \sum_{N \in \mathbb{N}} \sum_{j \geq 0} \sum_{l > j+5} N^s 2^{-Ml} \left\| \chi_l e^{it\Delta} P_N f \right\|_{L^2_t L^r_x k_k^2}^2 \\
\lesssim \sqrt{p} \sum_{N \in \mathbb{N}} \sum_{j \geq 0} \sum_{l > j+5} N^s 2^{-Ml} \left\| \chi_l e^{it\Delta} P_N f \right\|_{L^2_t L^r_x k_k^2}^2 \\
\lesssim \sqrt{p} \sum_{N \in \mathbb{N}} \sum_{j \geq 0} \sum_{l > j+5} N^s 2^{-Ml} \left\| \chi_l e^{it\Delta} P_N f \right\|_{L^2_t L^r_x k_k^2}^2. 
\]
Almost Sure Well-Posedness and Scattering of the 3D Cubic Nonlinear Schrödinger Equation

\[ \lesssim \sqrt{p} \sum_{N \in 2^j} \sum_{j \geq 0} \sum_{j \geq 1} N^{s-1} 2^{-2(M-j)} l \| P_N f \|_{L_x^2} \lesssim \sqrt{p} \| f \|_{H_x^{s-\frac{1}{2}}}. \quad (3.27) \]

Therefore, we have

\[ (3.21) \lesssim \sqrt{p} \| f \|_{H_x^{s-\frac{1}{2}}}. \quad (3.28) \]

By (3.24) and (3.28),

\[ \left\| \langle \nabla \rangle^s e^{it\Delta} f^{\omega} \right\|_{L_t^p L_x^p L^\infty_x} \lesssim \sum_{N \in 2^j} N^s \left\| e^{it\Delta} P_N f^{\omega} \right\|_{L_t^p L_x^p L^\infty_x} \lesssim (3.21) + (3.21) \lesssim \sqrt{p} \| f \|_{H_x^{s-\frac{1}{2}+3\epsilon}}. \]

Let \( \epsilon \leq \frac{1}{2} (s_0 - s + \frac{1}{2}) \), then we have (3.13) holds for \( r < \infty \).

When \( r = \infty \), using the similar argument above with \( r = \frac{3}{2} \),

\[ \left\| \langle \nabla \rangle^s e^{it\Delta} f^{\omega} \right\|_{L_t^p L_x^p L^\infty_x} \lesssim \sum_{N \in 2^j} N^{s+\epsilon} \left\| e^{it\Delta} P_N f^{\omega} \right\|_{L_t^p L_x^p L^\infty_x} \lesssim \sqrt{p} \| f \|_{H_x^{s-\frac{1}{2}+3\epsilon}}. \]

Let \( \epsilon \leq \frac{1}{3} (s_0 - s + \frac{1}{2}) \), then we finish the proof of the \( r = \infty \) case. \qed

4. Local Well-Posedness

4.1. Reduction to the deterministic problem. Suppose that \( u = v + w \) with \( u_0 = v_0 + w_0 \), and \( v, w \) satisfy

\[
\begin{align*}
  &i \partial_t w + \Delta w = |w + v|^2 (w + v), \\
  &v = e^{it\Delta} v_0, \\
  &w(0, x) = w_0(x).
\end{align*}
\]

This decomposition is referred as Bourgain’s trick [6] or Da Prato-Debussche’s trick [30]. For \( l \in [0, 1] \), define the working space

\[ \| w \|_{X^l(I)} = \left( \sum_{N \in 2^j} N^{2l} \| w_N \|_{L_x^2(I; L^2_x)}^2 \right)^{\frac{1}{2}}. \quad (4.2) \]

Then, we have the local results for \( H_x^{\frac{1}{2}} \)-data and \( H_x^{\frac{1}{2}+} \)-data, separately:

**Proposition 4.1.** Let \( \frac{1}{6} \leq s < \frac{1}{2} \), \( v \in Y^s \cap Z^s (\mathbb{R}) \), and \( w_0 \in H_x^{\frac{1}{2}} \). Then, there exists some \( T > 0 \) depending on \( s \), \( w_0 \), and \( v_0 \) such that there uniquely exists a solution \( w \) of (4.1) on \([0, T]\) with

\[ w \in C([0, T]; H_x^{\frac{1}{2}}) \cap X^\frac{1}{2} ([0, T]). \]

**Proposition 4.2.** Let \( \frac{1}{3} < s \leq \frac{1}{2} \), \( v \in Y^s \cap Z^s (\mathbb{R}) \), and \( w_0 \in H_x^1 \). Then, there exists some \( T > 0 \) depending on \( s \), \( \| w_0 \|_{H_x^1} \), and \( v_0 \) such that there uniquely exists a solution \( w \) of (4.1) on \([0, T]\) with

\[ w \in C([0, T]; H_x^1) \cap X^1 ([0, T]). \]
In fact, the \( s = \frac{1}{2} \) case is trivial. However, we need it for the global result in Proposition 5.1.

Now, we give the proof of Theorem 1.2, assuming that Propositions 4.1 and 4.2 hold.

**Proof of Theorem 1.2.** Let

\[
    u(t) = e^{it\Delta} f^\omega + w(t),
\]

then \( w \) satisfies the equation (4.1) with

\[
    v_0 = f^\omega, \quad w_0 = 0, \quad \text{and} \quad v = e^{it\Delta} f^\omega.
\]

We first prove Theorem 1.2 (1), using the result in Proposition 4.1. By Corollary 3.3, we have for almost every \( \omega \in \Omega \),

\[
    \|v\|_{Y^1(\mathbb{R})} + \|v\|_{Z^1(\mathbb{R})} < \infty.
\]

Since \( w_0 = 0 \), we can apply Proposition 4.1 to obtain the existence and uniqueness of \( w \in C([0, T]; H^s_x) \) for almost every \( \omega \in \Omega \).

The proof of Theorem 1.2 (2) by Proposition 4.2 is similar as above, so we omit the details. \( \square \)

**4.2. Proof of Proposition 4.1.** We make the choices of some parameters and define the auxiliary working space:

1. Let \( C_0 > 0 \) be the constant such that

\[
    \left\| e^{it\Delta} w_0 \right\|_{X^s_1([0, +\infty))} \leq C_0 \left\| w_0 \right\|_{H^s_x}.
\]

2. Let

\[
    R := \max \left\{ \frac{C_0 \left\| w_0 \right\|_{H^s_x}}{\frac{1}{2}}, 1 \right\}.
\]

3. Let \( \delta \) and \( \varepsilon \) be some constants such that \( 0 < \delta, \varepsilon \ll 1 \).

4. Define the following space:

\[
    \left\| w \right\|_{\tilde{X}^s_1(I)} = \left\| \langle \nabla \rangle^{\frac{1}{2}} w_N \right\|_{L^2_t L^6_x L^6_{xyz}(2\mathbb{N} \times I \times \mathbb{R}^3)} + \left\| \langle \nabla \rangle^{\frac{3}{2}} w_N \right\|_{L^2_t L^6_x L^6_{xyz}(2\mathbb{N} \times I \times \mathbb{R}^3)} + \left\| \langle \nabla \rangle^{\frac{1}{2}} w \right\|_{L^2_t L^6_x L^6_{xyz}(I \times \mathbb{R}^3)} + \left\| w \right\|_{L^2_{\omega} L^2_{xyz}(I \times \mathbb{R}^3)} + \left\| w \right\|_{L^6_{\omega} L^6_{xyz}(I \times \mathbb{R}^3)},
\]

5. Let \( T > 0 \) satisfy the smallness conditions

\[
    \left\| e^{it\Delta} w_0 \right\|_{\tilde{X}^s_1([0, T])} + \left\| v \right\|_{Y^s([0, T])} \leq \delta,
\]

and

\[
    \delta T^2 \left\| v \right\|_{Z^1(\mathbb{R})} + T^\frac{1}{100} R^2 \left\| v \right\|_{Z^1(\mathbb{R})} \leq \delta^3.
\]
We remark that
\[ X^\frac{1}{2}([0, T]) \hookrightarrow \widetilde{X}^\frac{1}{2}([0, T]), \]
and \( T \) depends on \( s, \delta, \varepsilon, v_0, \) and \( w_0 \). Let the working space be defined by
\[ B_{R, \delta, T} := \left\{ w \in C([0, T]; H^\frac{1}{2}) : \|w\|_{X^\frac{1}{2}([0, T])} \leq 2R, \|w\|_{\widetilde{X}^\frac{1}{2}([0, T])} \leq 2\delta \right\}. \]
Define
\[ \Phi_{w_0, v}(w) = e^{it\Delta} w_0 - i \int_0^t e^{i(t-s)\Delta} (|u|^2 u) \, ds. \]
Now, we are going to prove that \( \Phi_{w_0, v} \) is a contraction mapping on \( B_{R, \delta, T} \), which is reduced to prove the following nonlinear estimate
\[ \left\| \int_0^t e^{i(t-s)\Delta} (|u|^2 u) \, ds \right\|_{X^\frac{1}{2}([0, T])} \leq \delta. \]
In fact, we can use similar argument to prove
\[ \left\| \Phi_{w_0, v}(w_1) - \Phi_{w_0, v}(w_2) \right\|_{X^\frac{1}{2}([0, T])} \leq \frac{1}{2} \left\| w_1 - w_2 \right\|_{X^\frac{1}{2}([0, T])}, \]
and then finish the proof of contraction mapping.

Therefore, we reduce the proof of Proposition 4.1 to the following lemma:

**Lemma 4.3.** Let \( \frac{1}{6} \leq s < \frac{1}{2} \), and \( \delta, \varepsilon, C_0, R, T \) be defined as above. Assume that the following estimates hold:
\[ \|v\|_{Y^s([0, T])} \leq \delta, \|w\|_{X^\frac{1}{2}([0, T])} \lesssim R, \text{ and } \|w\|_{\widetilde{X}^\frac{1}{2}([0, T])} \lesssim \delta. \]
Then
\[ \left\| \int_0^t e^{i(t-s)\Delta} (|u|^2 u) \, ds \right\|_{X^\frac{1}{2}([0, T])} \leq \delta. \] \hspace{1cm} (4.3)

**Proof.** In the following, we shall slightly abuse notation and write \( u \) for both itself and its complex conjugate, and all the space-time norms are taken over \([0, T] \times \mathbb{R}^3\) without writing its integral region. First, to prove (4.3), by Lemma 2.2, Hölder’s inequality, and embedding \( V^2_\Delta \hookrightarrow L^\infty_t L^2_x \), we are reduced to consider
\[ \left\| \int_0^t e^{i(t-s)\Delta} (|u|^2 u) \, ds \right\|_{X^\frac{1}{2}([0, T])} \lesssim \left( \sum_{N \in 2^N} N \sup_{\|g\|_{V^2_\Delta} = 1} \left| \int_0^T \left\langle P_N(|u|^2 u), g \right\rangle \, dt \right|^2 \right)^{\frac{1}{2}} \]
\[ \lesssim \left( \sum_{N \in 2^N} N \sup_{\|g\|_{V^2_\Delta} = 1} \left\| P_N(|u|^2 u) \right\|_{L^1_t L^2_x} \left\| g \right\|_{L^\infty_t L^2_x} \right)^{\frac{1}{2}} \] \hspace{1cm} (4.4)
\[ \lesssim \|N^\frac{1}{2} P_N(|u|^2 u)\|_{L^1_t L^2_x}. \]
Noting that
\[ |P_N(|u|^2u)| \lesssim \sum_{N_1: N_1 \geq N} P_N(u_{N_1}u_{N_1}^2), \]
by Cauchy–Schwarz’s inequality in \( N_1 \) and Lemma 2.12,
\begin{align*}
\| N^{1/2} P_N(|u|^2u) \|_{L^1_t L^2_x} & \leq \| \sum_{N_1: N_1 \geq N} N^{1/2} \frac{N^{1/2}}{N_1^2} P_N(u_{N_1}u_{N_1}^2) \|_{L^1_t L^2_x} \\
& \leq \| N^{1/2} P_N(u_{N_1}u_{N_1}^2) \|_{L^1_t L^2_x} \\
& \leq \| N^{1/2} u_{N_1}u_{N_1}^2 \|_{L^1_t L^2_x} \lesssim I + II,
\end{align*}
where we denote
\[ I := \| N^{1/2} w_{N_1}u_{N_1}^2 \|_{L^1_t L^2_x}, \quad \text{and} \quad II := \| N^{1/2} v_{N_1}u_{N_1}^2 \|_{L^1_t L^2_x}. \]

First, we deal with the term \( I \), where the \( 1/2 \)-order derivative acts on \( w \). This is the simpler case, since \( w \) allows estimates with the derivative of order \( 1/2 \). By frequency support property,
\begin{align*}
I & \lesssim \| N^{1/2} w_{N_1}u_{N_1} \|_{L^1_t L^2_x} \| L^2_x \|_{N_1} \\
& \quad + \| N^{1/2} w_{N_1}u_{N_1}^2 \|_{L^1_t L^2_x} \| L^2_x \|_{N_1}. \tag{4.6a}
\end{align*}
Now, we estimate (4.6a). By Hölder’s inequality and Lemma 2.4, it holds that
\begin{align*}
(4.6a) & \lesssim \sum_{N_1} N^{1/2} \| w_{N_1} \|_{L^2_x}^{9/2} N^{1/2} \| u_{N_1} \|_{L^2_x}^{9/2} \| u_{N_1} \|_{L^6_x} \\
& \lesssim (\langle \nabla \rangle^{1/2} w_{N_1} \|_{L^2_x}^{9/2} \| u_{N_1} \|_{L^2_x}^{9/2} \| u_{N_1} \|_{L^6_x}) \\
& \lesssim \| w \|_{X^{1/2}} \| w \|_{X^{1/2}} + \| v \|_{Y^0}. \tag{4.7}
\end{align*}

For (4.6b), by Hölder’s inequality, Lemmas 2.11 and 2.12,
\begin{align*}
(4.6b) & \lesssim \| N^{1/2} w_{N_1} \|_{L^2_x}^{9/2} \| u_{N_1} \|_{L^6_x} \| u_{N_1} \|_{L^6_x} \\
& \lesssim \| (\langle \nabla \rangle^{1/2} w_{N_1} \|_{L^2_x}^{9/2} \| u_{N_1} \|_{L^6_x} \| u_{N_1} \|_{L^6_x}) \\
& \lesssim \| w \|_{X^{1/2}} \| w \|_{X^{1/2}} + \| v \|_{Y^0}. \tag{4.8}
\end{align*}
Then, by (4.7) and (4.8), we have
\[ I \lesssim (4.6a) + (4.6b) \lesssim \| w \|_{X^{1/2}} \| w \|_{X^{1/2}} + \| v \|_{Y^0} \lesssim \delta^3. \tag{4.9} \]
Next, we consider the term $II$, where the $1/2$-order derivative acts on $v$. However, the function $v$ can only have $1/6$-order derivative. Therefore, we need to transfer the additional fractional order derivative to other functions. We make a frequency decomposition:

$$II \lesssim \| N_1^{1/2} v_{N_1} u_{\leq N_1} \|_{L_t^1 L_x^2}^2 \tag{4.10a}$$

$$+ \| N_1^{1/2} v_{N_1} u_{\leq N_1} \|_{L_t^1 L_x^2}^2 \tag{4.10b}$$

$$+ \| N_1^{1/2} v_{N_1} w_{\leq N_1} \|_{L_t^1 L_x^2}^2 \tag{4.10c}$$

By Lemma 2.12, Hölder’s inequality, and embedding $I_N^2 \hookrightarrow I_N^3$, we have

$$\| \langle \nabla \rangle^{6/5} v_{N_1} \|_{L_t^2 L_x^6} \| \langle \nabla \rangle^{6/5} u_{\leq N_1} \|_{L_t^6 L_x^6} \lesssim \| v \|_{Y_x}^2 \| u \|_{Y_x}^2 \lesssim \delta^3. \tag{4.11}$$

Next, we consider (4.10b) and (4.10c). The proof is more difficult, where we use the bilinear Strichartz estimate to transfer derivative. However, this approach will create the term $\| v \|_{Z_x}$, which cannot get smallness by letting the interval small. Therefore, we also need some $T$ to control the $Z_x$-norm.

Now, we consider the term (4.10b). By Hölder’s inequality,

$$\| v_{N_1} v_{N_2} \|_{L_t^{12} L_x^2} \lesssim N_2^{\frac{1}{12}} N_1^{\frac{1}{2}} \| P_{N_1} v_0 \|_{L_x^2} \| P_{N_2} v_0 \|_{L_x^2} \lesssim N_1^{-\frac{1}{2}} \| v \|_{Z_x}^2. \tag{4.13}$$

Note that

$$\| u \|_{L_t^{\frac{3}{1-\epsilon}} L_x^{6/3\epsilon}} \lesssim \| u \|_{X_{\frac{1}{2}}} + \| v \|_{Y_x} \lesssim \delta. \tag{4.14}$$

By (4.12), (4.13), (4.14), and Hölder’s inequality,

$$\| v \|_{Z_x} \lesssim \sum_{N_2 \ll N_1} N_2^{\frac{1}{12}} N_1^{\frac{1}{2}} \| P_{N_1} v_0 \|_{L_x^2} \| P_{N_2} v_0 \|_{L_x^2} \lesssim N_1^{-\frac{1}{2}} \| v \|_{Z_x}^2. \tag{4.15}$$
Finally, we consider the term (4.10c). By Hölder’s inequality,

\begin{align}
(4.10c) & \lesssim \sum_{N_2 \ll N_1} N_1^{\frac{1}{2}} \| v_{N_1} w_{N_2} u \|_{L_x^4 L_t^2} \\
& \lesssim \sum_{N_2 \ll N_1} N_1^{\frac{1}{2}} \| v_{N_1} w_{N_2} \|_{L_x^{\frac{3}{4} + \epsilon} L_t^2} \| v_{N_1} \|_{L_x^{\frac{3}{5} - \epsilon} L_t^1} \| w_{N_2} \|_{L_x^{\frac{3}{5} - \epsilon} L_t^1} \| u \|_{L_x^{q_1} L_t^{r_1}},
\end{align}

where \((q_1, r_1)\) is defined by

\[ \frac{2 - 3\epsilon}{4} = \frac{4 - 3\epsilon}{q_1}, \text{ and } \frac{1 - 3\epsilon}{6} = \frac{4 - 3\epsilon}{r_1}. \]

Noting that \(q_1 = \frac{4(4 - 3\epsilon)}{3(2 - 3\epsilon)} = \frac{8}{3} + \) and \(r_1 = \frac{2(4 - 3\epsilon)}{1 - 3\epsilon} = 8 +\), we have \(\frac{3}{2} - \frac{2}{q_1} - \frac{3}{r_1} = \frac{3}{8} +\), and there exists \(q_2 = \frac{4(4 - 3\epsilon)}{5 + 3\epsilon}\) such that

\[ \frac{2}{q_2} + \frac{3}{r_1} = 1. \]

Then, we have

\[ \| u \|_{L_x^{q_2} L_t^{r_1}} \lesssim \| w \|_{L_x^{q_2} L_t^{r_1}} + \| v \|_{L_x^{q_2} L_t^{r_1}} \lesssim \delta. \]

By Lemma 2.9, for \(N_2 \ll N_1\),

\[ \| v_{N_1} w_{N_2} \|_{L_x^{\frac{3}{4} + \epsilon} L_t^2} \lesssim N_1^{\frac{1}{2}} N_1^{-\frac{1}{2}} \| P_{N_1} v_0 \|_{L_x^2} \| w_{N_2} \|_{U^2} \]
\[ \lesssim N_1^{-\frac{2}{3}} \| v \|_{Z^s} \| w \|_{X^2} \lesssim N_1^{-\frac{2}{3}} R \| v \|_{Z^s}. \]

We remark that for (4.18), if we do not invoke the \(U^p-V^p\) method, by Lemma 2.9, it reduces to deal with the term

\[ N_1^{\frac{1}{2}} \left( \| P_{N_1} u_0 \|_{L_x^2} + \| P_{N_1} (|u|^2 u) \|_{L_x^{1/2} L_t^1} \right). \]

Thus, the argument would be more complex, especially when \(N_1^{1/2}\) acts on \(|v|^2 v\).

By (4.16), (4.17), (4.18), and Hölder’s inequality in \(t\),

\begin{align}
(4.10c) & \lesssim \sum_{N_2 \ll N_1} N_1^{\frac{1}{2}} \| v_{N_1} w_{N_2} \|_{L_x^{\frac{3}{4} + \epsilon} L_t^2} \| v_{N_1} \|_{L_x^{\frac{3}{5} - \epsilon} L_t^1} \| w_{N_2} \|_{L_x^{\frac{3}{5} - \epsilon} L_t^1} \| u \|_{L_x^{q_1} L_t^{r_1}} \\
& \lesssim T^{(\frac{1}{3} - \epsilon)(\frac{5}{9}) - (\frac{1}{q_1})} \sum_{N_2 \ll N_1} N_1^{\frac{1}{18}} R^{\frac{2}{3} + \epsilon} \| v \|_{Z^s} \| v_{N_1} \|_{L_x^{\frac{3}{5} - \epsilon} L_t^1} \| w_{N_2} \|_{L_x^{\frac{3}{5} - \epsilon} L_t^1} \| u \|_{L_x^{q_2} L_t^{r_1}} \\
& \lesssim T^{\frac{1}{10} \delta^{\frac{2}{3} - \epsilon} R^{\frac{2}{3} + \epsilon} \| v \|_{Z^s} \| v \|_{L_x^{\frac{3}{5} - \epsilon} L_t^1} \| u \|_{L_x^{q_2} L_t^{r_1}}} \\
& \lesssim T^{\frac{1}{10} \delta^{\frac{2}{3} - \epsilon} R^{\frac{2}{3} + \epsilon}} \| v \|_{Z^s} \| v \|_{L_x^{\frac{3}{5} - \epsilon} L_t^1} \| u \|_{L_x^{q_2} L_t^{r_1}}. \]
\end{align}

Therefore, by (4.11), (4.15), and (4.19),

\[ II \lesssim (4.10a) + (4.10b) + (4.10c) \lesssim \delta^3 + \delta T^{\epsilon^2} \| v \|_{Z^s}^2 + T^{\frac{1}{10} \delta^2} R^2 \| v \|_{Z^s}. \]
Then, combining (4.5), (4.9), and (4.20), we have
\[
\left\| \int_0^t e^{i(t-s)\Delta} (|u|^2 u) \right\|_{L_t^\infty H_x^1 \cap X_x^1} \leq I + II
\]
\[
\leq C \left( \delta^3 + \delta T \epsilon^2 \|v\|_{L_t^2}^2 + T \frac{1}{100} R^2 \|v\|_{L_t^2}^2 \right)
\]
\[
\leq C \delta^3 \leq \delta.
\]
This completes the proof of the lemma.

4.3. Proof of Proposition 4.2.
In the following, we write
\[
U_{\Lambda}^2 = U_{\Lambda}^2(I; L_x^2) \quad \text{and} \quad V_{\Lambda}^2 = V_{\Lambda}^2(I; L_x^2)
\]
for short. We make the choices of some parameters:

(1) Let \( C_0 > 0 \) be the constant such that
\[
\left\| e^{it\Delta} w_0 \right\|_{X^1([\mathbb{R}])} \leq C_0 \|w_0\|_{H^1_x}.
\]

(2) Let \( 0 < \epsilon < \frac{1}{100}(s - \frac{1}{3}) \) and
\[
R := \max \left\{ C_0 \|w_0\|_{H^1_x}, 1 \right\}.
\]

(3) Let \( T > 0 \) satisfy the smallness conditions:
\[
\|v\|_{Y^s([0, T])} \leq R, \quad \text{and} \quad CT \frac{\epsilon}{100} \left( \|v\|_{Z^s([\mathbb{R}])}^3 + R^3 \right) \leq \frac{1}{2} R.
\]

Note that \( T \) depends on \( s, \|w_0\|_{H^1_x}, v, \) and \( \|v\|_{Z^s([\mathbb{R}])} \). Let the working space be defined by
\[
B_{R, T} := \left\{ w \in C([0, T]; H^1_x) : \|w\|_{X^1([0, T])} \leq 2R \right\}.
\]
Define
\[
\Phi_{w_0, v}(w) = e^{it\Delta} w_0 - i \int_0^t e^{i(t-s)\Delta} (|u|^2 u) \, ds.
\]
Similar as in the former section, in order to get that \( \Phi_{w_0, v} \) is a contraction mapping on \( B_{R, T} \), it suffices to prove:

**Lemma 4.4.** Let \( \frac{1}{3} < s \leq \frac{1}{2}, 0 < \epsilon < \frac{1}{100}(s - \frac{1}{3}), \) and \( C_0, R \) be defined as above. Assume that \( 0 < T < 1 \) satisfies the smallness condition (4.21), and let
\[
\|w\|_{X^1([0, T])} \lesssim R.
\]
Then
\[
\left\| \int_0^t e^{i(t-s)\Delta} (|u|^2 u) \, ds \right\|_{X^1([0, T])} \leq R.
\]
Proof. Again, we do not distinguish $u$ and $\bar{u}$, and all the space-time norms are restricted on $[0, T] \times \mathbb{R}^3$. By Lemma 2.2 and frequency decomposition, we have

\[
\| \int_0^T e^{i(t-x)\Delta} |u|^2 u \, ds \|_{X^1} \lesssim \left( \sum_{N \in \mathbb{N}} N^2 \sup_{\|g\|_{V^2_\Delta} = 1} \left| \int_0^T \langle P_N(|u|^2 u), g \rangle dt \right|^2 \right)^{1/2} \lesssim \left( \sum_{N \in \mathbb{N}} N^2 \sup_{\|g\|_{V^2_\Delta} = 1} \left| \int_0^T \int P_N \left( \sum_{N_1} u_{N_1} u_{L_\Delta}^2 \right) g \, dx \, dt \right|^2 \right)^{1/2} \lesssim I + II
\]

where

\[I := \left( \sum_{N \in \mathbb{N}} N^2 \sup_{\|g\|_{V^2_\Delta} = 1} \left| \int_0^T \int P_N \left( \sum_{N_1} w_{N_1} u_{L_\Delta}^2 \right) g \, dx \, dt \right|^2 \right)^{1/2},\]

and

\[II := \left( \sum_{N \in \mathbb{N}} N^2 \sup_{\|g\|_{V^2_\Delta} = 1} \left| \int_0^T \int P_N \left( \sum_{N_1} v_{N_1} u_{L_\Delta}^2 \right) g \, dx \, dt \right|^2 \right)^{1/2}.
\]

We first consider the term $I$, where the first order derivative acts on $w$. By Hölder’s inequality and embedding $V^2_\Delta \hookrightarrow L^\infty_t L^2_x$, we have

\[
I \lesssim \left( \sum_{N \in \mathbb{N}} N^2 \sup_{\|g\|_{V^2_\Delta} = 1} \| P_N \left( \sum_{N_1 : N \lesssim N_1} w_{N_1} u_{L_\Delta}^2 \right) \|_{L^1_t L^2_x} \| g \|_{L^\infty_t L^2_x}^2 \right)^{1/2} \lesssim \left( \sum_{N \in \mathbb{N}} N^2 \sup_{\|g\|_{V^2_\Delta} = 1} \| P_N \left( \sum_{N_1 : N \lesssim N_1} w_{N_1} u_{L_\Delta}^2 \right) \|_{L^1_t L^2_x} \| g \|_{V^2_\Delta}^2 \right)^{1/2} \lesssim \left( \sum_{N \in \mathbb{N}} N^2 \| P_N \left( \sum_{N_1 : N \lesssim N_1} w_{N_1} u_{L_\Delta}^2 \right) \|_{L^2_t L^2_x} \right)^{1/2}.
\]

Then, by (4.23), Minkowski’s inequality, Hölder’s inequality in $N_1$, and Lemma 2.12,

\[
I \lesssim \left( \sum_{N \in \mathbb{N}} N^2 \| P_N \left( \sum_{N_1 : N \lesssim N_1} w_{N_1} u_{L_\Delta}^2 \right) \|_{L^2_t L^2_x} \right)^{1/2} \lesssim \left( \sum_{N \in \mathbb{N}} N^2 \| P_N \left( \sum_{N_1 : N \lesssim N_1} w_{N_1} u_{L_\Delta}^2 \right) \|_{L^2_t L^2_x} \right)^{1/2} \lesssim \left( \sum_{N \in \mathbb{N}} N^2 \| P_N \left( \sum_{N_1 : N \lesssim N_1} w_{N_1} u_{L_\Delta}^2 \right) \|_{L^2_t L^2_x} \right)^{1/2} \lesssim \left( \sum_{N \in \mathbb{N}} N^2 \| P_N \left( \sum_{N_1 : N \lesssim N_1} w_{N_1} u_{L_\Delta}^2 \right) \|_{L^2_t L^2_x} \right)^{1/2}.
\]

(4.24)
By (4.24), Hölder’s inequality, Lemmas 2.11 and 2.7,
\[ I \lesssim \| N_1 w N_1 u_{\leq N_1} \|_{L^1_t L^2_x}^2 \]
\[ \lesssim \| N_1 w N_1 \|_{L^6_x} \| u \|_{L^2_t L^6_x} \| u_{\leq N_1} \|_{L^1_t L^2_x} \]
\[ \lesssim \| \langle \nabla \rangle w \|_{L^2_t L^6_x} \| u \|_{L^4_t L^6_x} \]
\[ \lesssim T^{1/2} \| \langle \nabla \rangle w \|_{L^2_t L^6_x} \| u \|_{L^\infty_t L^6_x} \]
\[ \lesssim T^{1/2} \| w \|_{X^1} \left( \| v \|_{Z^1} + \| w \|_{X^1} \right)^2. \] (4.25)

Therefore, by (4.25) and the choice of \( T \), we have
\[ I \leq C T^{1/2} \| w \|_{X^1} \left( \| v \|_{Z^1} + \| w \|_{X^1} \right)^2 \leq C T^{1/2} \| v \|_Z^3 + R^3 \leq \frac{1}{2} R. \] (4.26)

We next consider the term \( II \), where the first order derivative acts fully on \( v \). However, \( v \) can only have estimates with the derivative of order \( \frac{1}{3} \), thus there is a gap of \( \frac{2}{3} \)-order derivative. Note that the bilinear Strichartz estimate can only lower down \( 1 \)-order derivative. Therefore, this is the main case where we need to exploit the duality structure. To this end, we make a frequency decomposition:

\[ II \lesssim \left( \sum_{N \in 2\mathbb{N}} N^2 \sup_{\| g \|_{L^\infty}_x = 1} | \int_0^T \int P_N \left( \sum_{N_1} v_{N_1} u_{\leq N_1} \right) g \, dx \, dt \right)^2 \] (4.27a)

\[ + \left( \sum_{N \in 2\mathbb{N}} N^2 \sup_{\| g \|_{L^\infty}_x = 1} | \int_0^T \int P_N \left( \sum_{N_1} v_{N_1} u_{\leq N_1} \right) g \, dx \, dt \right)^2 \] (4.27b)

\[ + \left( \sum_{N \in 2\mathbb{N}} N^2 \sup_{\| g \|_{L^\infty}_x = 1} | \int_0^T \int P_N \left( \sum_{N_1} v_{N_1} u_{\leq N_1} \right) g \, dx \, dt \right)^2 \] (4.27c)

We first estimate (4.27a). Using the same method as in (4.23) and (4.24),

\[ (4.27a) \lesssim \left( \sum_{N \in 2\mathbb{N}} N^2 \sup_{\| g \|_{L^\infty}_x = 1} | \int_0^T \int P_N \left( \sum_{N_1} v_{N_1} u_{\leq N_1} \right) g \, dx \, dt \right)^2 \]
\[ \lesssim \sum_{N_1 \in 2\mathbb{N}} \| N_1 v_{N_1} u_{\leq N_1} \|_{L^1_t L^2_x}. \] (4.28)

Then, by (4.28), Lemma 2.7, Hölder’s inequality in \( N_1 \), and \( l^2_{N_1} \hookrightarrow l^3_{N_1} \), we have

\[ (4.27a) \lesssim \sum_{N_1 \in 2\mathbb{N}} \| N_1 v_{N_1} u_{\leq N_1} \|_{L^1_t L^2_x} \]
\[ \lesssim T^{1/2} \sum_{N_1 \in 2\mathbb{N}} \| \langle \nabla \rangle^{1/3} v_{N_1} \|_{L^6_{t,x}} \| \langle \nabla \rangle^{1/3} u_{\leq N_1} \|_{L^6_{t,x}} \| \langle \nabla \rangle^{1/3} u_{\leq N_1} \|_{L^6_{t,x}} \]
\[ \lesssim T^{1/2} \| \langle \nabla \rangle^{1/3} v_{N_1} \|_{L^6_{t,x}} \| \langle \nabla \rangle^{1/3} u_{N_1} \|_{L^6_{t,x}} \]
\[ \lesssim T^{1/2} \| v \|_{Y^1} \left( \| v \|_{Y^1} + \| w \|_{X^1} \right)^2 \lesssim T^{1/2} R^3. \] (4.29)
Next, we consider (4.27b). Similar as above, by Hölder’s inequality and Lemma 2.7,

\[(4.27b) \lesssim \left( \sum_{N \in 2^N} N^2 \sup_{\|g\|_{L^\infty}} \left| \int_0^T \int P_N \left( \sum_{N_1} v_{N_1} u_{\ll N_1} u_{\sim N_1} \right) g \, dx \, dt \right|^2 \right)^{1/2} \]

\[\lesssim \left( \sum_{N \in 2^N} N^2 \left\| P_N \left( \sum_{N_1: N \ll N_1} v_{N_1} u_{\ll N_1} u_{\sim N_1} \right) \right\|_{L_t^2 L_x^2}^2 \right)^{1/2} \]

\[\lesssim \sum_{N_1} N_1 \left\| v_{N_1} u_{\ll N_1} u_{\sim N_1} \right\|_{L_t^1 L_x^2} \]

\[\lesssim \sum_{N_1} N_1 \left\| v_{N_1} u_{\ll N_1} u_{\sim N_1} \right\|_{L_t^1 L_x^2}^2 + \sum_{N_1} N_1 \left\| v_{N_1} u_{\ll N_1} w_{\sim N_1} \right\|_{L_t^1 L_x^2}^2. \]  

For the first term in the right hand side of (4.30), by Hölder’s inequality,

\[\sum_{N_1} N_1 \left\| v_{N_1} u_{\ll N_1} u_{\sim N_1} \right\|_{L_t^1 L_x^2} \lesssim \sum_{N_1 \gg N_2} N_1 \left\| v_{N_1} u_{N_2} \right\|_{L_t^1 L_x^2} \left\| u_{\sim N_1} \right\|_{L_t^{\infty}}. \]  

By the bilinear Strichartz estimate in Lemma 2.9, for $N_2 \ll N_1$,

\[ \left\| v_{N_1} u_{N_2} \right\|_{L_t^{12} L_x^{12}} \lesssim N_2^{\frac{1}{6}} N_1^{\frac{1}{6}} \left( \left\| P_{N_1} v_0 \right\|_{L_x^3} + \left\| P_{N_2} u \right\|_{U^2} \right) \]

\[\lesssim N_2^{-\frac{1}{6}} N_1^{\frac{5}{6}} \left\| \nabla \right\|_{L_x^\infty}^\frac{1}{3} P_{N_1} v_0 \left\|_{L_x^3} + N_2^\frac{1}{3} \left\| P_{N_2} u \right\|_{U^2} \right) \]

\[\lesssim N_2^{-\frac{1}{6}} N_1^{\frac{5}{6}} \left\| v \right\|_{Z^3} \left( \left\| v \right\|_{Z^3} + \left\| w \right\|_{X_1} \right), \]  

which implies

\[\sum_{N_1 \gg N_2} N_1 \left\| v_{N_1} u_{N_2} \right\|_{L_t^{12} L_x^{12}} \left\| u_{\sim N_1} \right\|_{L_t^{\infty}} \]

\[\lesssim T^{\frac{1}{12}} \sum_{N_1 \gg N_2} N_1 \left\| v_{N_1} u_{N_2} \right\|_{L_t^{12} L_x^{12}} \left\| u_{\sim N_1} \right\|_{L_t^{\infty}} \]  

\[\lesssim T^{\frac{1}{12}} \sum_{N_1 \gg N_2} N_1^{\frac{1}{6}} N_2^{\frac{1}{6}} \left\| v \right\|_{Z^3} \left( \left\| v \right\|_{Z^3} + \left\| w \right\|_{X_1} \right) N_1^{\frac{1}{3} + k \epsilon} \left\| \nabla \right\|_{L_x^\infty}^\frac{1}{3} \left\| v_{\sim N_1} \right\|_{L_t^{\infty}} \]  

\[\lesssim T^{\frac{1}{12}} \sum_{N_1 \gg N_2} N_1^{-\frac{1}{6} + k \epsilon} N_2^{\frac{1}{6}} \left\| v \right\|_{Z^3} \left( \left\| v \right\|_{Z^3} + \left\| w \right\|_{X_1} \right) \]

\[\lesssim T^{\frac{1}{12}} \left\| v \right\|_{Z^3} \left( \left\| v \right\|_{Z^3} + \left\| w \right\|_{X_1} \right). \]

Then,

\[\sum_{N_1} N_1 \left\| v_{N_1} u_{\ll N_1} u_{\sim N_1} \right\|_{L_t^1 L_x^2} \lesssim T^{\frac{1}{12}} \left( \left\| v \right\|_{Z^3} + R^3 \right). \]  

For the second term in the right hand side of (4.30), by Hölder’s inequality,

\[ \sum_{N_1} N_1 \| v_{N_1} u_{N_1} w_{N_1} \|_{L_t^1 L_x^2} \lesssim \sum_{N_1} N_1 \| v_{N_1} \|_{L_t^\infty L_x^\infty} \| u_{N_1} \|_{L_t^1 L_x^\infty} \| w_{N_1} \|_{L_t^\infty L_x^2} \]

\[ \lesssim T^{\frac{1}{2}} \sum_{N_1} N_1^{-\frac{1}{2}+\varepsilon} \| \langle \nabla \rangle^{\frac{1}{2}-\varepsilon} v_{N_1} \|_{L_t^\infty L_x^\infty} \| u \|_{L_t^1 L_x^\infty} \| w_{N_1} \|_{U_\Lambda^2} \]

\[ \lesssim T^{\frac{1}{2}} \sum_{N_1} N_1^{-\frac{1}{2}+\varepsilon} \| v \|_{Z^1} (\| v \|_{L_t^2 L_x^\infty} + \| w \|_{L_t^2 L_x^\infty}) \| w \|_{X^1} \]

\[ \lesssim T^{\frac{1}{2}} \| v \|_{Z^1} (\| v \|_{Y^1} + \| w \|_{X^1}) \| w \|_{X^1} \]

\[ \lesssim T^{\frac{1}{2}} (\| v \|_{Z^1}^3 + R^3). \]  

(4.35)

Therefore, by (4.34) and (4.35),

\[ (4.27b) \lesssim T^{\frac{1}{12}} (\| v \|_{Z^1}^3 + R^3) + T^{\frac{1}{2}} (\| v \|_{Z^1}^3 + R^3) \]

\[ \lesssim T^{\frac{1}{12}} (\| v \|_{Z^1}^3 + R^3). \]  

(4.36)

Finally, we consider the main term (4.27c). By frequency support property,

\[ (4.27c) = \left( \sum_{N_1 \in 2^N} N^2 \sup_{\| g \|_{V_\Delta^2} = 1} | \int_0^T \int \sum_{N_1} v_{N_1} u_{N_1}^2 g \ dx dt |^2 \right)^{\frac{1}{2}} \]

\[ \lesssim \sum_{N_1 \in 2^N} \frac{1}{2} \sup_{\| g \|_{V_\Delta^2} = 1} | \int_0^T \int \sum_{N_1} v_{N_1} u_{N_1}^2 g_N \ dx dt | \]

\[ \lesssim \sum_{N_1 \in 2^N} \frac{1}{2} \sup_{\| g \|_{V_\Delta^2} = 1} | \int_0^T \int v_{N_1} u_{N_1}^2 g_N \ dx dt | \]

\[ \lesssim \sum_{N_1 \leq N_2} \frac{1}{2} \sup_{\| g \|_{V_\Delta^2} = 1} | \int_0^T \int v_{N_1} g_N u_{N_1} u_{N_2} \ dx dt |. \]  

(4.37)

To estimate this, we need to use the bilinear Strichartz estimate for both \( g_N u_{N_1} \) and \( \nabla v_{N_1} u_{N_2} \). Now, we give the estimate for \( g_N u_{N_1} \), where we also need to pass \( g_N \) into \( V_\Delta^2 \) by interpolation. By Lemma 2.9, for \( N_1 \ll N \),

\[ \| g_N u_{N_1} \|_{L_t^{\frac{3}{2}} L_x^2} \lesssim N_1^{\frac{3}{2} - \frac{2}{3} \varepsilon} \| g_N \|_{U_\Lambda^2} (\| P_{N_1} v_0 \|_{L_t^2} + \| w_{N_1} \|_{U_\Lambda^2}). \]  

(4.38)

and by Hölder’s inequality and Lemma 2.7,

\[ \| g_N u_{N_1} \|_{L_t^{\frac{3}{2}} L_x^2} \lesssim \| g_N \|_{L_t^{\frac{3}{2}} L_x^2} \| u_{N_1} \|_{L_t^{\infty} L_x^4} \]

\[ \lesssim N_1^{\frac{3}{2} - \frac{2}{3} \varepsilon} \| g_N \|_{U_\Lambda^2} \left( \| P_{N_1} v_0 \|_{L_t^2} + \| w_{N_1} \|_{U_\Lambda^2} \right). \]  

(4.39)
Then, noting that $\varepsilon < \frac{1}{100} (s - \frac{1}{3})$, by (4.38), (4.39), and Lemma 2.3, for $N_1 \ll N$,

\[
\|g_N u_{N_1}\|_{L^\alpha_t L^\infty_x} \lesssim \frac{N^{\frac{1}{2} - \frac{2}{3} \varepsilon}}{N_1^{\frac{1}{2} - \frac{2}{3} \varepsilon}} \|g_N\|_{L^\alpha_t} \left( \|P_{N_1} v_0\|_{L^2_x} + \|w_{N_1}\|_{U^{2}_\Lambda} \right)
\lesssim \frac{N^{\frac{1}{2} - \frac{2}{3} \varepsilon}}{N_1^{\frac{1}{2} - \frac{2}{3} \varepsilon}} \left( N_1^{-100\varepsilon} \left\| P_{N_1} v_0 \right\|_{L^2_x} + N_1^{\frac{1}{2} + 100\varepsilon} \left\| w_{N_1} \right\|_{U^{2}_\Lambda} \right)
\lesssim \frac{N^{\frac{1}{2} - \frac{2}{3} \varepsilon}}{N_1^{\frac{1}{2} - \frac{2}{3} \varepsilon}} \left( \|v\|_{Z^s} + \|w\|_{X^1} \right).
\]

(4.40)

Next, we give the estimate for $\nabla v_N u_{N_2}$. Using Lemma 2.9 again, for $N_2 \ll N$, we also have

\[
\|\nabla v_N u_{N_2}\|_{L^\alpha_t L^\infty_x} \lesssim \frac{N^{\frac{1}{2} - \frac{2}{3} \varepsilon}}{N_1^{\frac{1}{2} - \frac{2}{3} \varepsilon}} \|\nabla v_N u_{N_2}\|_{L^\alpha_t L^\infty_x} \left( \|P_{N_2} v_0\|_{L^2_x} + \|w_{N_2}\|_{U^{2}_\Lambda} \right)
\lesssim N_2^{-\frac{1}{2} \varepsilon} N_1^{-\frac{1}{3} + 20\varepsilon} \|v\|_{Z^s} \left( N_2^{-\frac{1}{6} + 20\varepsilon} \left\| P_{N_2} v_0 \right\|_{L^2_x} + N_2^{-\frac{1}{6} + 20\varepsilon} \left\| w_{N_2} \right\|_{U^{2}_\Lambda} \right)
\lesssim N_2^{-\frac{1}{2} \varepsilon} N_1^{-\frac{1}{3} + 20\varepsilon} \|v\|_{Z^s} \left( \|v\|_{Z^s} + \|w\|_{X^1} \right).
\]

(4.41)

Then, we are ready to bound (4.27c). By (4.40), (4.41), and Hölder’s inequality,

\[
N \int_0^T \int g_N v_N u_{N_1} u_{N_2} \, dx \, dt \lesssim \|g_N u_{N_1}\|_{L^\alpha_t L^\infty_x} \|\nabla v_N u_{N_2}\|_{L^\alpha_t L^\infty_x} \|u_{N_2}\|_{L^\infty_t L^2_x} \lesssim \frac{N^{\frac{1}{3} - 100\varepsilon}}{N_1^{\frac{1}{3} - 100\varepsilon}} \left( N_1^{-\frac{1}{3} - 100\varepsilon} N_2^{-\frac{1}{3} - 20\varepsilon} \|v\|_{Z^s} + \|w\|_{X^1} \right) \|v\|_{Z^s} \left( \|v\|_{Z^s} + \|w\|_{X^1} \right) \]

(4.42)

\[
\lesssim T \frac{N^{\frac{1}{3} - 100\varepsilon}}{N_1^{\frac{1}{3} - 100\varepsilon}} \left( N_2^{-\frac{1}{3} - 100\varepsilon} N_1^{\frac{1}{3} - 20\varepsilon} \|v\|_{Z^s} + \|w\|_{X^1} \right) \|v\|_{Z^s} \left( \|v\|_{Z^s} + \|w\|_{X^1} \right) \]

(4.43)

Note that

\[
\sum_{N_1 \leq N_2 \leq N} N_1^{-\frac{1}{3} - 100\varepsilon} N_2^{-\frac{1}{3} - 20\varepsilon} \lesssim 1,
\]

then by (4.37) and (4.42),

\[
(4.27c) \lesssim \sum_{N_1 \leq N_2 \leq N} \frac{N}{\|g\|_{L^\alpha_t}^2} \left| \int_0^T \int v_N g_N u_{N_1} u_{N_2} \, dx \, ds \right|
\lesssim T \frac{N^{\frac{1}{3}}}{N_1^{\frac{1}{3}}} \sum_{N_1 \leq N_2 \leq N} N_1^{\frac{1}{3} - 100\varepsilon} N_2^{-\frac{1}{3} - 20\varepsilon} \|v\|_{Y^s} \|v\|_{Z^s} \left( \|v\|_{Z^s} + \|w\|_{X^1} \right)^2
\lesssim T \frac{N^{\frac{1}{3}}}{N_1^{\frac{1}{3}}} \|v\|_{Z^s} \left( \|v\|_{Z^s} + R \right)^2 \lesssim T \frac{N^{\frac{1}{3}}}{N_1^{\frac{1}{3}}} \left( \|v\|_{Z^s} + R^3 \right).
\]
Therefore, by (4.29), (4.36), and (4.43), noting also that $T < 1$,

$$ II \leq (4.27a) + (4.27b) + (4.27c) $$

$$ \leq C \left( T^\frac{1}{2} + T^\frac{1}{12} + T^\frac{e}{100} \right) \left( \|v\|_{Z^r}^3 + R^3 \right) $$

$$ \leq CT^\frac{e}{100} \left( \|v\|_{Z^r}^3 + R^3 \right) \leq \frac{1}{2} R. $$

(4.44)

Then, by (4.26) and (4.44), we have

$$ \| \int_0^t e^{i(t-s)\Delta} (|u|^2 u) \|_{X^1} \leq I + II \leq R. $$

This proves (4.22).

5. Global Well-Posedness and Scattering

5.1. Reduction to the deterministic problem. Noting that $s > \frac{17}{40}$, fix a suitable small $\varepsilon > 0$ such that

$$ 0 < \varepsilon < \min \left\{ \frac{1}{3} (3s - 1), \frac{1}{6} (s - \frac{17}{40}) \right\}. $$

(5.1)

Let $\tilde{Y}^s(I)$ be defined by its norm

$$ \| v \|_{\tilde{Y}^s(I)} := \| v \|_{Y^s(I)} + \| \langle \nabla \rangle^{s+\frac{1}{2}-\varepsilon} v_N \|_{L^2_N L^\infty_t L^\infty_x (2^N \times I \times \mathbb{R}^3)}. $$

(5.2)

Recall that

$$ \| v \|_{Z^s(I)} = \| \langle \nabla \rangle^{s-\varepsilon} P_N v \|_{L^\infty_t L^\infty_x (2^N \times I \times \mathbb{R}^3)} + \| \langle \nabla \rangle^s P_N v \|_{L^\infty_t L^2_x (2^N \times I \times \mathbb{R}^3)}. $$

Proposition 5.1. Let $\frac{17}{40} < s \leq \frac{1}{2}$, $A > 0$, and $\varepsilon$ be sufficiently small satisfying (5.1). Then, there exists $N_0 = N_0(A) \gg 1$ such that the following properties hold. Let $u_0 \in H^s_x$, $v_0$ satisfy that $\text{supp} \ \hat{v}_0 \subset \{ \xi \in \mathbb{R}^3 : |\xi| \geq \frac{1}{2} N_0 \}$, and $w_0 = u_0 - v_0$. Moreover, let $v = e^{it\Delta} v_0$ and $w = u - v$. Suppose that $v \in \tilde{Y}^s \cap Z^s(\mathbb{R})$, $w_0 \in H^1$ such that

$$ \| u_0 \|_{H^s_x} + \| v \|_{\tilde{Y}^s \cap Z^s(\mathbb{R})} \leq A, \ \text{and} \ E(w_0) \leq AN_0^{2(1-s)}. $$

Then, there exists a solution $u$ of (1.7) on $\mathbb{R}$ with $w \in C(\mathbb{R}; H^1_x)$. Furthermore, there exists $u_\pm \in H^1_x$ such that

$$ \lim_{t \to \pm \infty} \left\| u(t) - v(t) - e^{it\Delta} u_\pm \right\|_{H^1_x} = 0. $$

We will give the proof of Proposition 5.1 in Sects. 5.2, 5.3, and 5.4. Now we prove Theorem 1.4 assuming that Proposition 5.1 holds.
Proof of Theorem 1.4. Let \( N_0 = N_0(M, \| f \|_{H^1}) \in 2^\mathbb{N} \) that will be defined later, and make a high-low frequency decomposition for the initial data

\[
    u(t) = e^{it\Delta} P_{\geq N_0} f^{oo} + w(t),
\]

then \( w \) satisfies the equation (4.1) with

\[
    u_0 = f^{oo}, \quad v_0 = P_{\geq N_0} f^{oo}, \quad w_0 = P_{\leq N_0} f^{oo}, \quad \text{and} \quad v = e^{it\Delta} P_{\geq N_0} f^{oo}.
\]

Since \( f \) is radial, by Corollary 3.3, boundedness of the operator \( P_{\geq N_0} \), Proposition 3.4, and Lemma 2.14, we have

\[
    \mathbb{P}( \{ \omega \in \Omega : \| u_0 \|_{H^1} + \| v \|_{\tilde{Y} \cap Z^r(\mathbb{R})} > \lambda \} ) \lesssim e^{-C\lambda^2 \| f \|^2_{H^1}}. \tag{5.3}
\]

For any \( p \geq 2 \), we have

\[
    \| w_0 \|_{L^p \dot{H}^1} \lesssim \sum_{k \in \mathbb{Z}^3} g_k(\omega) \| P_{\leq N_0} f_k \|_{L^2_x L^p_\omega} \lesssim \sqrt{p} \| \nabla P_{\leq N_0} f_k \|_{L^2_x L^2_\omega} \lesssim \sqrt{p} \| \nabla P_{\leq N_0} f \|_{L^2_x} \lesssim \sqrt{p} N_0^{1-s} \| f \|_{H^1}.
\]

For any \( p \geq 2 \), we also have

\[
    \| w_0 \|_{L^p_x \dot{H}^1} \lesssim \sqrt{p} \| P_{\leq N_0} f \|_{L^2_x} \lesssim \sqrt{p} \| f \|_{L^2_x}.
\]

Then, by Lemma 2.14,

\[
    \mathbb{P}( \{ \omega \in \Omega : \| u_0 \|_{\dot{H}^1} + \| w_0 \|_{L^4_x} \geq \lambda \} ) \lesssim e^{-C\lambda^2 \| f \|^2_{H^1}}. \tag{5.4}
\]

For any \( M \geq 1 \), let \( \widetilde{\Omega}_M \) be defined by

\[
    \widetilde{\Omega}_M = \left\{ \omega \in \Omega : \frac{1}{N_0^{1-s}} \| u_0 \|_{\dot{H}^1} + \| w_0 \|_{L^4_x} < M \| f \|_{H^1}, \quad \frac{1}{N_0^{1-s}} \| u_0 \|_{\dot{H}^1} + \| w_0 \|_{L^4_x} < M \| f \|_{H^1} \right\}. \tag{5.5}
\]

Therefore, by (5.3) and (5.4), we have

\[
    \mathbb{P}(\widetilde{\Omega}_M^c) \lesssim e^{-CM^2}. \tag{5.6}
\]

For any \( \omega \in \widetilde{\Omega}_M \), we have \( \| v \|_{\tilde{Y} \cap Z^r(\mathbb{R})} < M \| f \|_{H^1} \), and

\[
    E(w_0) \leq CM^2 N_0^{2(1-s)} \cdot \max\left\{ M^2 \| f \|_{H^1}^4, 1 \right\}.
\]

Therefore, for any \( M > 1 \) and any \( \omega \in \widetilde{\Omega}_M \), let

\[
    A = A(M, \| f \|_{H^1}) := \max\left\{ M \| f \|_{H^1}, CM^2 \cdot \max\{ M^2 \| f \|_{H^1}^4, 1 \} \right\},
\]

then
then we have \( v = e^{it\Delta} P_{\geq N_0} f^\omega \),

\[
\|u_0\|_{H^1_x} + \|v\|_{\tilde{\mathcal{X}}_{t,\mathbb{R}}} \leq A, \quad \text{and} \quad E(w_0) \leq AN_0^{2(1-s)}.
\]

Therefore, we can apply Proposition 5.1. Let \( N_0 \) depend on \( A \) as in the statement of Proposition 5.1, and we obtain a global solution \( w \) that scatters. Then, for any \( \omega \in \hat{\Omega} = \bigcup_{M \geq 1} \hat{\Omega}_M \), we can also derive that (4.1) admits a global solution \( w \) that scatters. By (5.6), we have that \( \mathbb{P}(\hat{\Omega}) = 1 \). Then for almost every \( \omega \in \Omega \), we obtain the global well-posedness and scattering for (4.1). This finishes the proof of Theorem 1.4. \( \square \)

5.2. Global space-time estimates.

**Lemma 5.2** (Interaction Morawetz). Let \( w \in C([0, T]; H^1_x) \) be the solution of perturbation equation (4.1). Then, we have

\[
\|w\|_{L^4_t L^4_x}^4 \lesssim \|w\|_{L^\infty_t L^2_x}^2 \|w\|_{L^4_t H^1_x} + \|\nabla w\|_{L^\infty_t L^2_x}^2 \|w\|_{L^2_t L^\infty_x}^2 + \|v\|_{L^4_t L^4_x}^4,
\]  

(5.7)

where all the space-time norms are taken over \([0, T] \times \mathbb{R}^3\).

**Proof.** Recall that \( w \) satisfies

\[
i\partial_t w + \Delta w = |w|^2 w + e,
\]

where \( e = |u|^2 u - |w|^2 w \). Denote that

\[
m(t, x) = \frac{1}{2} |w(t, x)|^2; \quad p(t, x) = \frac{1}{2} \operatorname{Im}(\overline{w}(t, x) \nabla w(t, x)).
\]

Then, we have

\[
\partial_t m = -2 \nabla \cdot p + \operatorname{Im}(e\overline{w}),
\]  

(5.8)

and

\[
\partial_t p = -\operatorname{Re} \nabla \cdot (\nabla \overline{w} \nabla w) - \frac{1}{4} \nabla (|w|^4) + \frac{1}{2} \nabla \Delta m + \operatorname{Re}(e\overline{w}) - \frac{1}{2} \operatorname{Re}(\overline{w}e).
\]  

(5.9)

Moreover, we note that

\[
\partial_j \left( \frac{x_k}{|x|} \right) = \delta_{jk} \frac{x_k}{|x|} - \frac{x_j x_k}{|x|^3}; \quad \nabla \cdot \frac{x}{|x|} = \frac{2}{|x|}; \quad \Delta \nabla \cdot \frac{x}{|x|} = -8\pi \delta(x).
\]

Let

\[
M(t) := \int_{\mathbb{R}^{3+3}} \frac{x - y}{|x - y|} \cdot p(t, x) m(t, y) \, dx \, dy,
\]

then by (5.8) and (5.9), we have the interaction Morawetz identity

\[
\partial_t M(t) = \int_{\mathbb{R}^{3+3}} \frac{x - y}{|x - y|} \cdot \partial_t p(t, x) m(t, y) \, dx \, dy
\]

\[
+ \int_{\mathbb{R}^{3+3}} \frac{x - y}{|x - y|} \cdot p(t, x) \partial_t m(t, y) \, dx \, dy
\]
\[
\begin{align*}
  & \left( - \text{Re} \nabla \cdot (\nabla \bar{w} \nabla w) - \frac{1}{4} \nabla(|w|^4) \right)(t, x) m(t, y) \, dx \, dy \\
  & - 2 \int_{\mathbb{R}^{3+}} \frac{x - y}{|x - y|} \cdot p(t, x) \nabla \cdot p(t, y) \, dx \, dy \\
  & + \frac{1}{2} \int_{\mathbb{R}^{3+}} \frac{x - y}{|x - y|} \cdot \nabla \Delta m(t, x) m(t, y) \, dx \, dy \\
  & + \int_{\mathbb{R}^{3+}} \frac{x - y}{|x - y|} \cdot p(t, x) \text{Im}(e\bar{w})(t, y) \, dx \, dy \\
  & + \int_{\mathbb{R}^{3+}} \frac{x - y}{|x - y|} \cdot \text{Re}(\partial_t \nabla w)(t, x) m(t, y) \, dx \, dy \\
  & + \int_{\mathbb{R}^{3+}} \frac{1}{|x - y|} \cdot \text{Re}(\partial_t \bar{w})(t, x) m(t, y) \, dx \, dy.
\end{align*}
\] (5.10a)

Note that by the classical argument in [26], we have

\[(5.10a) + (5.10b) \geq 0,\]

and

\[(5.10c) \gtrsim \|w(t)\|_{L_t^4}^4.\]

Moreover,

\[\sup_{t \in [0, T]} M(t) \lesssim \|w\|_{L_t^\infty L_x^2}^2 \|w\|_{L_t^\infty H_x^1}^2.\]

Then, integrating over \([0, T]\), it holds that

\[C \|w\|_{L_t^4}^4 \leq M(T) - M(0) + \int_0^T \left| (5.10d) \right| + \left| (5.10e) \right| + \left| (5.10f) \right| dt,\]

thus

\[\|w\|_{L_t^4}^4 \lesssim \|w\|_{L_t^\infty L_x^2}^2 \|w\|_{L_t^\infty H_x^1}^2 + \int_0^T \left| (5.10d) \right| + \left| (5.10e) \right| + \left| (5.10f) \right| dt.\]

By Hölder's inequality and Lemma 2.5, we have

\[\int_0^T \left| (5.10d) \right| + \left| (5.10e) \right| + \left| (5.10f) \right| dt \lesssim \|\nabla w\|_{L_t^\infty L_x^2}^2 \|w\|_{L_t^\infty L_x^2}^2 \|e\|_{L_t^1 L_x^2}^2 \lesssim \|\nabla w\|_{L_t^\infty L_x^2}^2 \|w\|_{L_t^\infty L_x^2}^2 \|v\|_{L_t^2 L_x^\infty} \left( \|v\|_{L_t^4 L_x^2}^2 + \|w\|_{L_t^4 L_x^2}^2 \right) .\]

Therefore, we have

\[\|w\|_{L_t^4}^4 \lesssim \|w\|_{L_t^\infty L_x^2}^2 \|w\|_{L_t^\infty H_x^1}^\frac{1}{2} + \|\nabla w\|_{L_t^\infty L_x^2}^2 \|w\|_{L_t^\infty L_x^2}^2 \|v\|_{L_t^2 L_x^\infty} \left( \|v\|_{L_t^4 L_x^2}^2 + \|w\|_{L_t^4 L_x^2}^2 \right) .\]

Then, by Young's inequality, we have

\[\|w\|_{L_t^4}^4 \lesssim \|w\|_{L_t^\infty L_x^2}^2 \|w\|_{L_t^\infty H_x^1}^\frac{1}{2} + \|\nabla w\|_{L_t^\infty L_x^2}^2 \|w\|_{L_t^\infty L_x^2}^4 \|v\|_{L_t^2 L_x^\infty}^2 + \|v\|_{L_t^4 L_x^2}^4 .\]

This completes the proof of this lemma. \(\square\)
Almost Sure Well-Posedness and Scattering of the 3D Cubic Nonlinear Schrödinger Equation

However, the use of $L_{t,x}^{1/4}$-norm is not enough for our argument. First, we can update our space-time estimates to $H_x^{1/2}$-level.

**Lemma 5.3** ($H_x^{1/2}$-regular promotion). Let $0 < \varepsilon \ll 1$ satisfy (5.1), and $w \in C([0, T]; H_x^{1/2})$ be the solution of the perturbation equation (4.1). Then, for any $0 \leq l \leq \frac{1}{2}$ and $L_x^{2}$-admissible $(q, r)$,

$$
\|N^l w_N\|_{L^3 t L^5_x}^2 \lesssim \|w_0\|_{H_x^l}^2 + \|N_{\frac{1}{4}}< \varepsilon \|w\|^2_{L_t^\infty H_x^{l+\frac{1}{2}}} + \|\langle \nabla \rangle^l v_N\|_{L_t^2 L_x^\infty}^2 \left( \|w\|_{L_t^{4} L_x^{4}}^2 + \|v\|_{Y^r}^2 \right),
$$

where all the space-time norms are taken over $[0, T] \times \mathbb{R}^3$.

**Proof.** By Minkowski’s inequality, Lemmas 2.7 and 2.12, we have

\begin{align*}
\|N^l P_N w\|_{L^3 t L^5_x}^2 & \lesssim \|w_0\|_{H_x^l}^2 + \|N_{\frac{1}{4}}< \varepsilon \|w\|^2_{L_t^\infty H_x^{l+\frac{1}{2}}} \\
& \quad + \|N_{\frac{1}{4}}< \varepsilon \|w\|^2_{L_t^\infty H_x^{l+\frac{1}{2}}} \\
& \lesssim \|w_0\|_{H_x^l}^2 + \|\langle \nabla \rangle^l (|w|^2 w)\|_{L_t^2 L_x^\infty}^2 \\
& \quad + \|N_{\frac{1}{4}}< \varepsilon \|w\|^2_{L_t^\infty H_x^{l+\frac{1}{2}}} \\
& \lesssim \|w_0\|_{H_x^l}^2 + \|\langle \nabla \rangle^l (|w|^2 w)\|_{L_t^2 L_x^\infty}^2 + I + II,
\end{align*}

where

$$
I := \|P_N \langle \nabla \rangle^l \left( \sum_{N_1: N \leq N_1} w_{N_1} u_{\leq N_1} O(u_{\leq N_1} + v_{\leq N_1}) \right)\|_{L_t^2 L_x^{\infty} L_x^2}^2.
$$

and

$$
II := \|P_N \langle \nabla \rangle^l \left( \sum_{N_1: N \leq N_1} u_{N_1} u_{\leq N_1} \right)\|_{L_t^2 L_x^{\infty} L_x^2}^2.
$$

Here, $O(u_{\leq N_1} + v_{\leq N_1})$ denotes a linear combination of $u_{\leq N_1}$ and $v_{\leq N_1}$. By Hölder’s inequality, we have

\begin{align*}
I & \lesssim \sum_{N_1} \|\langle \nabla \rangle^l (w_{N_1} u_{\leq N_1} O(u_{\leq N_1} + v_{\leq N_1}))\|_{L_t^2 L_x^{\infty} L_x^2}^2 \\
& \lesssim \sum_{N_1} \|N^l w_{N_1} u_{\leq N_1} O(u_{\leq N_1} + v_{\leq N_1})\|_{L_t^2 L_x^{\infty} L_x^2}^2 \\
& \lesssim \sum_{N_1} \|N^l w_{N_1} \|_{L_t^\infty L_x^2} \|v_{\leq N_1}\|_{L_t^4 L_x^{12}} \|O(u_{\leq N_1} + v_{\leq N_1})\|_{L_t^{12} L_x^4} \\
& \lesssim \|\langle \nabla \rangle^{l+\frac{1}{2}}\|_{L_t^\infty L_x^{4}} \|v\|_{Y^r} \left( \|w\|_{L_t^{4} L_x^{4}} + \|v\|_{Y^r} \right),
\end{align*}

(5.12)
and by Lemma 2.11,

\[
II \lesssim \|N_t^1 v_{N_1} u \leq N_1 u \leq N_1\|_{L^1_t L^2_x}^2 \|N_t^1 L^2_x \|_{L^1_t L^2_x}^2 \\
\lesssim \|N_t^1 v_{N_1} \|_{L^2_t L^\infty_x}^2 \|N_1 \|_{L^1_t L^2_x}^2 \\
\lesssim \|N_t^1 v_{N_1} \|_{L^2_t L^\infty_x}^2 \|N_1 \|_{L^1_t L^2_x}^2 (\|w\|_{L^4_t} + \|w\|_{Y^s})^2.
\]

(5.13)

Therefore, by (5.11), (5.12), and (5.13), we have the desired estimates. \(\square\)

Based on the space-time estimates in \(H^{1/2}\)-level, and keeping in mind that the equation is \(H^{1/2}\)-critical, we can further obtain the estimates in \(H^l\)-level with \(l > 1/2\). We need larger class of space-time norms, and it is more convenient to invoke the \(U^p-V^p\) method: recall that

\[
\|w\|_{X^l(I)} = \left( \sum_{N \in 2^N} N^{2l} \|w_N\|_{U^2_{\Delta}(I; L^2_{x})}^2 \right)^{1/2}.
\]

Lemma 5.4 (\(H^1\)-regular promotion). Let \(0 < \varepsilon \ll 1\) satisfy \((5.1)\), and \(w \in C([0, T]; H^1_x)\) be the solution of perturbation equation \((4.1)\). For any \(\frac{1}{3} < s \leq \frac{1}{2}\), we have

\[
\|w\|_{X^s} \lesssim \|w_0\|_{H^1_x} + \|w\|_{L^\infty_t H^1_x} \left( \|w\|_{Y^s \cap Z^s} + \|N^{1/2} w_N\|_{L^2_{x} L^2_{x}} \right) (\|w\|_{Y^s} + \|w\|_{L^6_t L^2_x}) \\
+ \|w\|_{L^\infty_t H^1_x} \left( \|w\|_{Y^s} + \|N^{1/2} w_N\|_{L^2_{x} L^2_{x}} \right) (\|w\|_{Y^s} + \|w\|_{L^6_t L^2_x}) \\
+ \|w\|_{\tilde{Y}_s} \left( \|w\|_{Y^s} + \|N^{1/2} w_N\|_{L^2_{x} L^2_{x}} \right) (\|w\|_{Z^s} + \|w\|_{L^6_t H^2_x}) \\
+ \|w\|_{\tilde{Y}_s} \left( \|w\|_{Z^s} + \|w\|_{X^s} \right) (\|w\|_{Z^s} + \|w\|_{L^6_t H^2_x}),
\]

where all the space-time norms are taken over \([0, T] \times \mathbb{R}^3\).

Proof. Similar to the proof of Lemma 5.3, we have

\[
\|w\|_{X^s} \lesssim \left( \sum_{N \in 2^N} N^{2} \|w\|_{L^2_t} \|g\|_{V^2_{\Delta}} \right) \int_0^T \int P_N(|u|^2 u) g \, dx \, dt \right)^{1/2}
\]

\[
\lesssim \|w_0\|_{H^1_x} + \left( \sum_{N \in 2^N} N^{2} \|g\|_{V^2_{\Delta}} \right) \left( \int_0^T \int P_N(\sum_{N_1} u_{N_1}^2 u_{N_1}^2) g \, dx \, dt \right)^{1/2}
\]

\[
\lesssim \|w_0\|_{H^1_x} + I + II,
\]

where

\[
I := \left( \sum_{N \in 2^N} N^{2} \|g\|_{V^2_{\Delta}} \right) \left( \int_0^T \int P_N(\sum_{N_1} w_{N_1}^2 u_{N_1}^2) g \, dx \, dt \right)^{1/2},
\]
and
\[
II := \left( \sum_{N \in 2^N} N^2 \sup_{\|g\|_{Y^s} = 1} \int_0^T \int P_N(\sum_{N_1} w_{N_1} u_{\leq N_1}) g \, dx \, dt \right)^{\frac{1}{2}}.
\]

We first consider the term \( I \), where the first order derivative acts on \( w \). By Hölder’s inequality, and Lemmas 2.7 and 2.12,
\[
I \lesssim \left( \sum_{N \in 2^N} N^2 \left\| P_N(\sum_{N_1} w_{N_1} u_{\leq N_1}) \right\|_{L^2_t L^\infty_x} \right)^{\frac{1}{2}}
\]
\[
\lesssim \left\| \nabla \left( \sum_{N_1} w_{N_1} u_{\leq N_1} \right) \right\|_{L^2_t L^{\frac{18}{15}}_x} \lesssim \left\| \nabla \left( \sum_{N_1} w_{N_1} u_{\sim N_1} u_{\leq N_1} \right) \right\|_{L^2_t L^\frac{18}{15}_x} \quad (5.14a)
\]
\[
\lesssim \left\| \nabla \left( \sum_{N_1} w_{N_1} u_{\leq N_1}^2 \right) \right\|_{L^2_t L^{\frac{18}{15}}_x} \quad (5.14b)
\]

Now, the main task is to update the summation of \( w_{N_1} \) to \( l_{N_1}^2 \). To this end, for (5.14a), we can simply use Hölder’s inequality in \( N_1 \) for \( w_{N_1} \) and \( u_{\sim N_1} \). Precisely, by (4.24), Hölder’s inequality, and Lemma 2.12,
\[
(5.14a) \lesssim \left\| \sum_{N_1} \| N_1 w_{N_1} u_{\sim N_1} u_{\leq N_1} \|_{L^\infty_t L^\frac{18}{15}_x} \right\|_{L^2_t} \lesssim \left\| \sum_{N_1} \| N_1 w_{N_1} \|_{L^2_t L^\infty_x} \| u_{\sim N_1} \|_{L^\infty_t L^\frac{9}{2}_x} \| u_{\leq N_1} \|_{L^\infty_t L^\frac{9}{2}_x} \right\|_{L^2_t} \quad (5.15)
\]
\[
\lesssim \left\| \sum_{N_1} \left( \| \nabla w \|_{L^\infty_t L^2_x} + \| u_{N_1} \|_{L^\infty_t L^\frac{9}{2}_x} \right) \right\|_{L^2_t} \lesssim \left\| \| \nabla w \|_{H^s_t} + \| N^\frac{1}{2} w_N \|_{L^\infty_t L^2_x} \right\|_{L^2_t} \lesssim \left( \| \| \nabla w \|_{H^s_t} + \| \nabla w \|_{L^2_t L^\frac{9}{2}_x} \right). \]

For the second term (5.14b), we need to invoke the vector-valued Hardy-Littlewood maximal function to cover the critical summation problem. Using Lemma 2.10 and Hölder’s inequality,
\[
(5.14b) \lesssim \left\| N \sum_{N_1: N_1 \sim N} w_{N_1} u_{\leq N_1}^2 \right\|_{L^2_t L^{\frac{18}{15}}_x} \lesssim \left\| N \sum_{N_1: N_1 \sim N} w_{N_1} u_{\leq N_1}^2 \right\|_{L^2_t L^{\frac{18}{15}}_x} \lesssim \left\| N \sum_{N_1: N_1 \sim N} w_{N_1} u_{\leq N_1}^2 \right\|_{L^2_t L^{\frac{18}{15}}_x} \lesssim \left\| N \sum_{N_1: N_1 \sim N} w_{N_1} u_{\leq N_1}^2 \right\|_{L^2_t L^{\frac{18}{15}}_x} \lesssim \left( \| \nabla w \|_{H^s_t} + \| \nabla w \|_{L^2_t L^\frac{9}{2}_x} \right). \]

For the second term (5.14b), we need to invoke the vector-valued Hardy-Littlewood maximal function to cover the critical summation problem. Using Lemma 2.10 and Hölder’s inequality,
By Lemmas 2.4, 2.12, and Hölder’s inequality,
\[
\| \sup_{N} |u_{\lesssim N}|^2 \|_{L_t^{5/3} L_x^{\infty}} \lesssim \| \sup_{N} \sum_{N_1, N_2 : N_1 \leq N_2 \lesssim N} |u_{N_1} u_{N_2}| \|_{L_t^{5/3} L_x^{9/4}} \\
\lesssim \| \sum_{N_1 \leq N_2} |u_{N_1} u_{N_2}| \|_{L_t^{5/3} L_x^{9/4}} \\
\lesssim \| \sum_{N_1 \leq N_2} \left( \frac{N_1}{N_2} \right)^{3\varepsilon} N_1^{-3\varepsilon} u_{N_1} N_2^{3\varepsilon} u_{N_2} \|_{L_t^{5/3} L_x^{9/4}} \\
\lesssim \sum_{N_1 \leq N_2} \left( \frac{N_1}{N_2} \right)^{3\varepsilon} u_{N_1} \|_{L_t^{5/3} L_x^{9/4}} \| N_2^{3\varepsilon} u_{N_2} \|_{L_t^{5/3} L_x^{9/4}} \\
\lesssim \| N_1^{-3\varepsilon} u_{N_1} \|_{L_t^{5/3} L_x^{9/4}} \| N_2^{3\varepsilon} u_{N_2} \|_{L_t^{5/3} L_x^{9/4}} \\
\lesssim \| N_1^{-3\varepsilon} u_{N_1} \|_{L_t^{5/3} L_x^{9/4}} \| N_2^{3\varepsilon} u_{N_2} \|_{L_t^{5/3} L_x^{9/4}} \\
\lesssim \| N_1^{-3\varepsilon} u_{N_1} \|_{L_t^{5/3} L_x^{9/4}} \| N_2^{3\varepsilon} u_{N_2} \|_{L_t^{5/3} L_x^{9/4}} \\
\lesssim \| N_1^{-3\varepsilon} u_{N_1} \|_{L_t^{5/3} L_x^{9/4}} \| N_2^{3\varepsilon} u_{N_2} \|_{L_t^{5/3} L_x^{9/4}} \\
\lesssim \left( \| \mathbb{V}_{N_1 \mathbb{I}_{L_t^{5/3} L_x^{9/4}}} + \| N_1^{-3\varepsilon} w_{N_1} \|_{L_t^{5/3} L_x^{9/4}} \right) \\
\cdot \left( \| \mathbb{V} \|_{Y^s} + \| \langle \nabla \rangle \mathbb{V} \|_{L_t^{6/5} L_x^{9/4}} \right) \\
\lesssim \left( \| \mathbb{V} \|_{Y^s} + \| N_1^{1/2 - 3\varepsilon} w_{N_1} \|_{L_t^{5/3} L_x^{9/4}} \right) \\
\cdot \left( \| \mathbb{V} \|_{Y^s} + \| \langle \nabla \rangle \mathbb{V} \|_{L_t^{6/5} L_x^{9/4}} \right). \tag{5.17} \\
\]
Combining (5.16) and (5.17), we have
\[
(5.14b) \lesssim \| N w_N \|_{L_t^{\infty} L_x^{2} L_{t,x}^{2}} \| \sup_{N} |u_{\lesssim N}|^2 \|_{L_t^{5/3} L_x^{9/4}} \\
\lesssim \| w \|_{L_t^{\infty} H_x^1} \left( \| \mathbb{V} \|_{Y^s} + \| N_1^{1/2 - 3\varepsilon} w_{N_1} \|_{L_t^{5/3} L_x^{9/4}} \right) \left( \| \mathbb{V} \|_{Y^s} + \| \langle \nabla \rangle \mathbb{V} \|_{L_t^{6/5} L_x^{9/4}} \right). \tag{5.18} \\
\]
Therefore, by (5.15) and (5.18),
\[
I \lesssim (5.14a) + (5.14b) \\
\lesssim \left( \| \mathbb{V} \|_{Y^s} + \| N_1^{1/2 - 3\varepsilon} w_{N_1} \|_{L_t^{5/3} L_x^{9/4}} \right) \left( \| \mathbb{V} \|_{Y^s} + \| \langle \nabla \rangle \mathbb{V} \|_{L_t^{6/5} L_x^{9/4}} \right) \\
+ \| w \|_{L_t^{\infty} H_x^1} \left( \| \mathbb{V} \|_{Y^s} + \| N_1^{1/2 - 3\varepsilon} w_{N_1} \|_{L_t^{5/3} L_x^{9/4}} \right) \left( \| \mathbb{V} \|_{Y^s} + \| \langle \nabla \rangle \mathbb{V} \|_{L_t^{6/5} L_x^{9/4}} \right). \tag{5.19} \\
\]
Next, we estimate the term II, where we need to use the bilinear Strichartz estimate and the duality structure. By frequency support property, we obtain
\[
II \lesssim \| N_1 v_{N_1} u_{\lesssim N_1} u_{\lesssim N_1} \|_{L_t^{1} L_x^{2} L_{t,x}^{2}} \tag{5.20a} \\
+ \left( \sum_{N \in 2^N} N^2 \| g \|_{Y^s_{\Delta}} \right) \| \int_0^T \int P_N \left( \sum_{N_1} v_{N_1} u_{\lesssim N_1}^2 \right) g \ dx dt \right)^{1/2}. \tag{5.20b} \\
\]
Now, we estimate (5.20a). Then, by $l_{N_1}^1 \rightarrow l_{N_1}^2$ and Hölder’s inequality,

\[
(5.20a) \lesssim \sum_{N_1} \left\| (\nabla)^{-s} v_{N_1} \right\|_{L_t^2 L_x^\infty} \left( \left\| (\nabla)^s u_{N_1} \right\|_{L_t^2 L_x^6} + \left\| u \right\|_{L_t^\infty L_x^3} \right) \lesssim \left\| (\nabla)^{-s} v_{N_1} \right\|_{L_t^2 L_x^\infty} \left( \left\| (\nabla)^s u_{N_1} \right\|_{L_t^2 L_x^6} + \left\| u \right\|_{L_t^\infty H_x^\frac{1}{2}} \right) \lesssim \left\| (\nabla)^{s+\frac{1}{2}-\varepsilon} v_{N_1} \right\|_{L_t^2 L_x^\infty} \left( \left\| (\nabla)^s u_{N_1} \right\|_{L_t^2 L_x^6} + \left\| (\nabla)^s w_{N_1} \right\|_{L_t^2 L_x^6} \right) \cdot \left( \left\| v \right\|_{L_t^\infty L_x^3} + \left\| w \right\|_{L_t^\infty H_x^\frac{1}{2}} \right) \lesssim \left\| v \right\|_{Y^s} \left( \left\| v \right\|_{Z^s} + \left\| w \right\|_{X^s} \right),
\]

(5.21)

Next, we consider (5.20b). To this end, we establish a bilinear Strichartz estimate before the proof for (5.20b). By Lemma 2.9, for $N_1 \ll N$,

\[
\left\| u_{N_1} g_N \right\|_{L_t^2 L_x^s} \lesssim \frac{N_1}{N} \left( \left\| P_{N_1} v_0 \right\|_{L_t^2} + \left\| w_{N_1} \right\|_{U^2_\Delta} \right) \left\| g \right\|_{U^2_\Delta},
\]

(5.22)

and by Bernstein’s, Hölder’s inequalities, Lemma 2.7, and embedding $U^4_\Delta \hookrightarrow L_t^\infty L_x^2$,

\[
\left\| u_{N_1} g_N \right\|_{L_t^2 L_x^s} \lesssim \left\| u_{N_1} \right\|_{L_t^2 L_x^\infty} \left\| g_N \right\|_{L_t^\infty L_x^2} \lesssim N_1^\frac{1}{s} \left\| u_{N_1} \right\|_{L_t^2 L_x^6} \left\| g \right\|_{U^4_\Delta} \lesssim N_1^\frac{1}{s} \left( \left\| P_{N_1} v_0 \right\|_{L_t^2} + \left\| w_{N_1} \right\|_{U^2_\Delta} \right) \left\| g \right\|_{U^4_\Delta}.
\]

(5.23)

By (5.22), (5.23), and Lemma 2.3, we have the bilinear Strichartz estimate

\[
\left\| u_{N_1} g_N \right\|_{L_t^2 L_x^s} \lesssim \frac{N_1^s}{N} \left( \left\| P_{N_1} v_0 \right\|_{L_t^2} + \left\| w_{N_1} \right\|_{U^2_\Delta} \right) \left\| g \right\|_{V^2_\Delta} \lesssim \frac{N_1^{1-s+\varepsilon}}{N} \left( \left\| P_{N_1} v_0 \right\|_{L_t^2} + \left\| w_{N_1} \right\|_{U^2_\Delta} \right) \left\| g \right\|_{V^2_\Delta}.
\]

(5.24)

Noting that $N_1 \leq N_2$, by the choice of $\varepsilon$ in (5.1), we have $N_1^{1-s+\varepsilon} < N_2^{2s-2\varepsilon} < N_1^{s-\varepsilon} N_2^{s-\varepsilon}$. Then, by (5.24),

\[
(5.20b) \lesssim \sum_{N_1, N_2: N_1 \leq N_2 \ll N} \sup \left\| \int_0^T \int_0^T \left\| u_{N_1} u_{N_2} g_N \right\|_{L_t^2} \right\|_{V^2_\Delta} \lesssim \sum_{N_1 \leq N_2} \left\| (\nabla)^{s+\frac{1}{2}-\varepsilon} u_{N_1} \right\|_{L_t^2 L_x^\infty} \left\| u_{N_2} \right\|_{L_t^\infty L_x^6} \left\| u_{N_1} g_N \right\|_{L_t^2 L_x^s} \lesssim \sum_{N_1 \leq N_2} \left\| (\nabla)^{s+\frac{1}{2}-\varepsilon} u_{N_1} \right\|_{L_t^2 L_x^\infty} \left\| u_{N_2} \right\|_{L_t^\infty L_x^6} \left\| N_1^{1-s+\varepsilon} \left( \left\| P_{N_1} v_0 \right\|_{L_t^2} + \left\| w_{N_1} \right\|_{U^2_\Delta} \right) \right\|_{V^2_\Delta} \lesssim \left\| v \right\|_{Y^s} \left( \left\| v \right\|_{Z^s} + \left\| w \right\|_{X^s} \right) \left( \left\| v \right\|_{Z^s} + \left\| w \right\|_{L_t^\infty H_x^\frac{1}{2}} \right).
\]

(5.25)
By (5.21) and (5.25),
\[
II \lesssim \|v\|_{\dot{Y}^{s}} \left( \|v\|_{\dot{Y}^{s}} + \|N_{1}^{l} w N_{1} \|_{L_{t}^{2} L_{x}^{2}} \|w\|_{L_{t}^{\infty} H_{x}^{s}}^{\frac{1}{2}} \right) + \|v\|_{\dot{Y}^{s}} \left( \|v\|_{Z^{s}} + \|w\|_{X^{s}} \right) \left( \|w\|_{Z^{s}} + \|w\|_{L_{t}^{\infty} H_{x}^{s}} \right).
\]
(5.26)

Then (5.19) and (5.26) give the desired estimates. \(\square\)

5.3. Energy bound.

**Proposition 5.5.** Let the assumptions in Proposition 5.1 hold. Take some \(T > 0\) such that \(w \in C([0, T]; H^{1}_{1})\). Then, there exists \(N_{0} = N_{0}(A) \gg 1\) with the following properties. Assume that \(\widehat{v_{0}}\) is supported on \(\{\xi \in \mathbb{R}^{3} : |\xi| \geq \frac{1}{2} N_{0}\}\),
\[
\|u_{0}\|_{H^{s}_{1}} + \|v\|_{\dot{Y}^{s} \cap Z^{s}(\mathbb{R})} \leq A, \text{ and } E(u_{0}) \leq AN_{0}^{2(1-s)}.
\]
Then, we have
\[
\sup_{t \in [0, T]} E(u(t)) \leq 2AN_{0}^{2(1-s)}.
\]
(5.27)

**Proof.** Let \(I = [0, T]\) and \(N_{0} = N_{0}(A)\) that will be defined later. From now on, all the space-time norms are taken over \(I \times \mathbb{R}^{3}\). We implement a bootstrap procedure on \(I\): assume an a priori bound
\[
\sup_{t \in I} E(u(t)) \leq 2AN_{0}^{2(1-s)},
\]
(5.28)
then it suffices to prove that
\[
\sup_{t \in I} E(u(t)) \leq \frac{3}{2} AN_{0}^{2(1-s)}.
\]
(5.29)

To start with, we collect useful estimates on \(I\). Now, we use the notation \(C = C(A)\) for short, and the implicit constants in “\(\lesssim\)” depend on \(A\). Moreover, we take all the space-time norms over \(I \times \mathbb{R}^{3}\). By interpolation
\[
\|v\|_{L_{x}^{\frac{3}{2}}} \lesssim \|v\|_{L_{x}^{\frac{3}{2}}} \|v\|_{L_{x}^{\infty}}, \text{ and } \|v\|_{L_{x}^{4}} \lesssim \|v\|_{L_{x}^{2}} \|v\|_{L_{x}^{\infty}}
\]
we have
\[
\|v\|_{L_{t}^{\infty} L_{x}^{\frac{3}{2}} \cap L_{t}^{\infty} L_{x}^{4}} + \| (\nabla)^{s} v \|_{L_{t}^{\infty} L_{x}^{2}} + \|v\|_{L_{t,x}^{4}} \lesssim \|v\|_{Y^{s} \cap Z^{s}} \lesssim 1.
\]
(5.30)

By the frequency support of \(v\), we have for any \(0 \leq l \leq s + \frac{1}{2} - \varepsilon\),
\[
\| (\nabla)^{l} v_{N} \|_{L_{t}^{2} L_{x}^{2}} \lesssim N_{0}^{l-s-\frac{3}{2}+\varepsilon} \|v\|_{\dot{Y}^{s}} \lesssim N_{0}^{l-s-\frac{1}{2}+\varepsilon} \lesssim 1.
\]
(5.31)

By the conservation of mass, we have \(\|u(t)\|_{L_{x}^{2}} = |u_{0}|_{L_{x}^{2}}\). Then, combining (5.30), we have for all \(t \in [0, T]\),
\[
\|w(t)\|_{L_{x}^{2}} \lesssim \|u(t)\|_{L_{x}^{2}} + \|v(t)\|_{L_{x}^{2}} \lesssim |u_{0}|_{L_{x}^{2}} + 1 \lesssim 1.
\]
(5.32)
By bootstrap hypothesis (5.28),
\[ \| w \|_{L_t^\infty L_x^4} \lesssim N_0^{\frac{1}{2}(1-s)}, \text{ and } \| w \|_{L_t^\infty \dot{H}_x^1} \lesssim N_0^{1-s}. \] (5.33)

Then, by interpolation, (5.32), and (5.33), we have for any \( 0 \leq l \leq 1, \)
\[ \| w \|_{L_t^\infty \dot{H}_x^l} \lesssim N_0^{l(1-s)}. \] (5.34)

Next, we derive various space-time bounds combining Lemmas 5.2, 5.3, and 5.4, under the above setting.

**Lemma 5.6.** Suppose that the assumptions in Proposition 5.5 and the estimate (5.28) hold for some \( N_0 \gg 1. \)

1. Then the following interaction Morawetz estimate holds:
\[ \| w \|_{L_t^4 L_x} \lesssim N_0^{\frac{1-s}{4}}. \] (5.35)

2. Let \( 0 < \varepsilon \ll 1 \) be defined in Proposition 5.1, then
\[ \| w \|_{X^0} \lesssim N_0^{(\frac{1}{2} + \varepsilon)(1-s)}. \] (5.36)

3. If \( 0 < l \leq \frac{1}{2} \) and \((q, r)\) is \( L_x^2\)-admissible, then
\[ \| N^l w_N \|_{L_t^q L_x^l L_x^q} \lesssim N_0^{l(1-s)}. \] (5.37)

4. Moreover,
\[ \| w \|_{X^1} \lesssim N_0^{3(1-s)}. \] (5.38)

**Remark 5.7.** Roughly speaking, the interaction Morawetz estimate in Lemma 5.2 yields
\[ \| w \|_{L_t^4 L_x}^4 \lesssim N_0^{1-s} + N_0^{2(1-s)} \| v \|_{L_t^2 L_x^\infty}^2. \]

Since \( v \) is high-frequency truncated, we are able to cover the additional increment for the remainder in the view of (5.31). This is the main reason why we implement the high-low frequency decomposition for the initial data.

**Proof.** Note that \( s > 0, \) by the perturbed Morawetz estimate in Lemma 5.2 and (5.34),
\[ \| w \|_{L_t^4 L_x}^4 \lesssim \| w \|_{L_t^\infty L_x^2}^2 \| w \|_{L_t^\infty \dot{H}_x^1}^2 + \| \nabla w \|_{L_t^\infty L_x^2}^2 \| w \|_{L_t^\infty L_x^2}^2 \| v \|_{L_t^2 L_x^\infty}^2 + \| v \|_{L_t^4 L_x}^4 \lesssim N_0^{1-s} + N_0^{2(1-s)} N_0^{-1-2s+\varepsilon} + 1 \lesssim N_0^{1-s}. \]
Next, we prove (5.36). Noting that \((\frac{2}{1-\varepsilon}, \frac{6}{1+2\varepsilon})\) is \(L^2_x\)-admissible, by embedding \(V^2_\Delta \hookrightarrow L^2_t L^6_x L^{\frac{6}{1+2\varepsilon}}_x\), Lemma 2.12, Hölder’s inequality, (5.35), (5.32), and (5.33),

\[
\|w\|_{X^0} \lesssim \|w_0\|_{L^2_t} + \|u\|_{L^2_t L^6_x} \lesssim 1 + \|u\|_{L^4_t L^6_x} \|u\|_{L^2_t L^{\frac{6}{1+2\varepsilon}}_x} \lesssim 1 + (\|w\|_{L^4_t L^6_x} + \|v\|_{Y^s}) (\|w\|_{L^6_x} + \|v\|_{Z^s}) = 1 + (N_0^{-1+s} + 1)^{2(1+\varepsilon)} (N_0^{\frac{2(1-s)}{3(1-2\varepsilon)}} + 1)^{1-2\varepsilon} \lesssim N_0^{\frac{5}{2} + \varepsilon}(1-s).
\]

Then, we prove (5.37). By Lemma 5.3 and (5.35), for any \(0 < l \leq \frac{1}{2}\) and \(L^2_x\)-admissible \((q, r)\),

\[
\|N^l w_N\|_{L^q_t L^r_x L^6_x} \lesssim \|w_0\|_{\dot{H}^l} + (\|w\|_{L^\infty_t H^l_x} + \|\langle \nabla \rangle^l v_N\|_{L^q_t L^r_x}) (\|v\|_{Y^s} + \|w\|_{L^4_t L^6_x}) \lesssim N_0^{l(1-s)} + (N_0^{l(1+s)}(1-s) + 1 + N_0^{\frac{1}{2}(1-s)}) \lesssim N_0^{(l+1)(1-s)}.
\]

Now, we prove the \(X^1\)-estimate. To this end, we first derive the interpolation

\[
\|w\|_{L^\infty_t L^6_x} \lesssim \|w\|_{L^4_t L^6_x} \|w\|_{L^\infty_t \dot{H}^1_x} \lesssim N_0^{\frac{1}{2}(1-s)}, \quad (5.39)
\]

and

\[
\|\langle \nabla \rangle^{3\varepsilon} w\|_{L^\infty_t L^6_x} \lesssim \|w\|_{L^4_t L^6_x} \|w\|_{L^\infty_t \dot{H}^{1+4\varepsilon}_x} \lesssim N_0^{\frac{1}{2}(1+3\varepsilon)(1-s)}. \quad (5.40)
\]

By (5.36),

\[
\|w\|_{X^s} \lesssim \|w\|_{X^0}^{1-s} \|w\|_{X^1}^{s} \lesssim N_0^{(1-s)^2} \|w\|_{X^1}^{s}. \quad (5.41)
\]

Then, by Lemma 5.4, (5.37), (5.39), (5.40), and (5.41),

\[
\|w\|_{X^1} \lesssim \|w_0\|_{H^l_x} + \|w\|_{L^\infty_t H^l_x} (\|v\|_{Y^s} + \|Z^s\|) + \|N_s^{\frac{1}{2}} w_N\|_{L^q_t L^r_x} \|w\|_{L^\infty_t L^6_x} \lesssim N_0^{1-s} + N_0^{\frac{3}{2}(1-s)} (1 + N_0^{\frac{1}{2}(1-s)}) + N_0^{\frac{3}{2}(1-s)} (1 + N_0^{\frac{1}{2}(1-s)}) + (1 + N_0^{\frac{3}{2}(1-s)}) (1 + N_0^{\frac{1}{2}(1-s)^2}) \lesssim N_0^{3(1-s)} + N_0^{1-s} \|w\|_{X^1}^{s}.
\]
Noting that \(3(1-s) \geq 1\), by Young’s inequality,
\[
\|w\|_{X^1} \lesssim N_0^{3(1-s)} + N_0 \lesssim N_0^{3(1-s)}.
\]
Hence, (5.38) holds. This finishes the proof of the lemma.

Now, we are prepared to give the proof of Proposition 5.5. By (4.1) and integration-by-parts, we have
\[
\frac{d}{dt} E(w(t)) = \text{Im} \int \Delta \overline{w} (|u|^2 u - |w|^2 w) \, dx + \text{Im} \int |u|^2 u (|u|^2 u - |w|^2 w) \, dx
\]
\[
= - \text{Im} \int \nabla \overline{w} \cdot \nabla (|u|^2 u - |w|^2 w) \, dx + \text{Im} \int |u|^2 u (|u|^2 u - |w|^2 w) \, dx.
\]
Again, we do not distinguish between \(u\) and \(\overline{w}\). Then, we have
\[
\sup_{t \in I} E(t) \lesssim E(w_0)
\]
\[
+ |\int \int \nabla w \cdot \nabla (v + w)^2 \, dx \, dt| \quad (5.42)
\]
\[
+ |\int \int \nabla w \cdot \nabla w v(v + w) \, dx \, dt| \quad (5.43)
\]
\[
+ |\int \int |u|^2 u (|u|^2 u - |w|^2 w) \, dx \, dt|. \quad (5.44)
\]

**Estimate on (5.42).** This is the main case, where we need the restriction \(s > \frac{17}{40}\). We first make a frequency decomposition:

\[
(5.42) \lesssim \sum_{N \in 2^N} \left| \int \int \nabla w \cdot \nabla v_N v_{\geq N} (v + w) \, dx \, dt \right| \quad (5.45a)
\]
\[
+ \sum_{N \in 2^N} \left| \int \int \nabla w_N \cdot \nabla v_N \, dx \, dt \right| \quad (5.45b)
\]
\[
+ \sum_{N \in 2^N} \left| \int \int \nabla w_N \cdot \nabla w_{\leq N} (v + w) \, dx \, dt \right| \quad (5.45c)
\]
\[
+ \sum_{N \in 2^N} \left| \int \int \nabla w_N \cdot \nabla w_{\leq N} v_{\leq N} \, dx \, dt \right|. \quad (5.45d)
\]

For (5.45a), we can directly transfer the derivative from \(v_N\) to \(v_{\geq N}\). By Hölder’s inequality, Lemma 2.12, and (5.31),

\[
(5.45a) \lesssim \sum_{N \leq N_1} \left| \int \int \nabla w \cdot \nabla v_N v_{N_1} (v + w) \, dx \, dt \right|
\]
\[
\lesssim \sum_{N \leq N_1} \|w\|_{L^\infty_t H^s_x} \|\nabla v_N\|_{L^2_t L^\infty_x} \|v_{N_1}\|_{L^2_t L^\infty_x} \left( \|v\|_{L^\infty_t L^2_x} + \|w\|_{L^\infty_t L^2_x} \right)
\]
\[
\lesssim N_0^{1-s} \sum_{N \leq N_1} \frac{N^s}{N^1} \|\nabla v_N\|_{L^2_t L^\infty_x} \|v_{N_1}\|_{L^2_t L^\infty_x}
\]
\[
\lesssim N_0^{1-s} \|\nabla v_N\|_{L^2_t L^\infty_x} \|v_{N_1}\|_{L^2_t L^\infty_x} \lesssim N_0^{1-s}.
\]
Next, we bound (5.45b), where we use the bilinear Strichartz estimate for $\nabla w_Nv_{\ll N}$ to lower down the derivative of $\nabla v_N$. From Lemma 2.9, (5.38), and (5.30), for $N_1 \ll N$, we have that
\[
\left\| \nabla w_Nv_{N_1} \right\|_{L^2_t L^\infty_x} \lesssim \frac{N_1}{N^{\frac{1}{2}}} \left\| w_N \right\|_{L^2_t L^\infty_x} \left\| v_{N_1} \right\|_{L^2_t L^\infty_x} \lesssim N_1^{1-s} N^{-\frac{1}{2}} \left\| v_{N_1} \right\|_{H^s_t} \lesssim N_1^{1-s} N^{-\frac{1}{2}} N_0^{3(1-s)}.
\] (5.47)

Note that $\frac{17}{40} < s$ gives
\[
\left\| \nabla v\right\|_{L^2_t L^\infty_x} \lesssim N^{\frac{3}{10}} \left\| \nabla v \right\|_{L^2_t L^\infty_x}, \quad \text{and} \quad \left\| v_{N_1} \right\|_{L^2_t L^\infty_x} \lesssim N_1^{\frac{37}{30}} \left\| \nabla v_{N_1} \right\|_{L^2_t L^\infty_x}.
\] (5.48)

Then, combining Hölder’s inequality, (5.47), (5.48), (5.30), (5.31), and (5.34), it holds that
\[
(5.45b) \lesssim \sum_{N_1 \ll N} \left| \int \int \nabla w_N \cdot \nabla v_{N_1} (v_{\ll N} + w_{\ll N}) \, dx \, dr \right|
\lesssim \sum_{N_1 \ll N} \left\| \nabla w_N \right\|_{L^2_t L^\infty_x} \left\| \nabla v_{N_1} \right\|_{L^2_t L^\infty_x} \left( N_1^{1-s} N^{-\frac{1}{2}} N_0^{3(1-s)} \right)^{\frac{1}{2}}
\lesssim N_0^{\frac{3}{2}(1-s)} \sum_{N_1 \ll N} \left\| \nabla v_{N_1} \right\|_{L^2_t L^\infty_x} \left\| v_{N_1} \right\|_{L^2_t L^\infty_x} \left( N_1^{1-s} N^{-\frac{1}{2}} N_0^{3(1-s)} \right)^{\frac{1}{2}}
\lesssim N_0^{\frac{3}{2}(1-s)} \sum_{N_1 \ll N} N^{\frac{3}{10}} N_1^{\frac{11}{100}} N_1^{\frac{1}{2}(1-s)} N^{-\frac{1}{8}}
\lesssim N_0^{\frac{3}{2}(1-s)} \sum_{N_1 \ll N} N_1^{-\frac{11}{50}} N^{-\frac{1}{8}} \lesssim N_0^{\frac{1}{2}(1-s)} N_0^{2(1-s)}.
\] (5.49)

Next, we deal with the term (5.45c), where we can directly transfer the derivative from $v_N$ to $w_{\gg N}$. Note that by $s > \frac{17}{40}$,
\[
\sum_{N, N_1: N_0 \lesssim N \lesssim N_1} \frac{N_1^{1-s+\varepsilon}}{N_1^{\frac{7\theta}{10}}} \lesssim N_0^{-\frac{1}{100}}.
\] (5.50)

By interpolation, (5.35), and (5.37), we also have
\[
\left\| N_1^{\frac{1}{10}} w_{N_1} \right\|_{L^2_t L^\infty_x} \lesssim \left\| w \right\|_{L^3_t L^\infty_x} \left\| \nabla \right\|_{L^2_t L^\infty_x} \lesssim N_0^{\frac{3}{2}(1-s)}.
\] (5.51)
Therefore, by Hölder’s inequality, (5.31), (5.50), (5.51), and (5.35),

\[
(5.45\text{e}) \lesssim \sum_{N, N_1 \in 2^\mathbb{N}, N \leq N_1} \left| \int I \int \nabla w_N \cdot \nabla v_N w_{N_1} (w + v) \, dx \, dt \right|
\]

\[
\lesssim \sum_{N, N_1 \in 2^\mathbb{N}, N \leq N_1} \frac{N^{\frac{1}{2} - s + \epsilon}}{N_1^{\frac{1}{10}}} \| \nabla w_N \|_{L_t^\infty L_x^2} \| (\nabla)^{s + \epsilon} v_N \|_{L_t^2 L_x^{\infty}}
\]

\[
\cdot \| \nabla w_{N_1} \|_{L_t^4 L_x^4} \left( \| w \|_{L_t^4 L_x^4} + \| v \|_{L_t^4 L_x^4} \right)
\]

\[
\lesssim N_0^{-\frac{1}{100}} N_0^{1-s} N_0^{\frac{3}{2}(1-s)} N_0^{\frac{3}{2}(1-s)} \lesssim N_0^{-\frac{1}{100}} N_0^{2(1-s)}.
\]

Now, we deal with the term (5.45d), which is the main part of the whole argument. We postponed here to illustrate the key idea. Roughly speaking, by Hölder’s inequality, (5.54), and (5.35),

\[
\left| \int I \int \nabla w_N \cdot \nabla v_N w_{\ll N} \, dx \, dt \right| \lesssim \| \nabla w \|_{L_t^\infty L_x^2} \| \nabla v \|_{L_t^2 L_x^\infty} \| w \|_{L_t^4 L_x^4}^2
\]

\[
\lesssim N_0^{\frac{3}{2}(1-s)} \| \nabla v \|_{L_t^2 L_x^\infty}^2.
\]

Although we lack the \( \| \nabla v \|_{L_t^2 L_x^\infty} \)-estimate, the bilinear Strichartz estimate for \( \nabla w_N w_{\ll N} \) can be introduced to lower down the derivative of \( \nabla v_N \). In the view of (5.37) and (5.38), this procedure will cause the increase of \( N_0 \). This is allowed, since there is still \( N_0^{\frac{1}{2}(1-s)} \)-gap towards the energy increment \( N_0^{2(1-s)} \) in (5.53).

Now, we give the concrete argument for the estimate of (5.45d). By Lemma 2.9, (5.37), (5.38), and (5.36), noting that \( N_1 \ll N \),

\[
\| \nabla w_N w_{N_1} \|_{L_t^2 L_x^4} \lesssim \frac{N_1}{N^{\frac{1}{2}}} N \| w_N \|_{U_\Delta^2} \| w_{N_1} \|_{U_\Delta^2}
\]

\[
\lesssim N_1 N^{-\frac{1}{2}} \| w_N \|_{X^1} \| w_{N_1} \|_{X^0}
\]

\[
\lesssim N_1 N^{-\frac{1}{2}} N_0^{\frac{3}{2}(1-s)} N_0^{\left(\frac{5}{6} + \epsilon\right)(1-s)}
\]

\[
\lesssim N_1 N^{-\frac{1}{2}} N_0^{\frac{3}{2}(1-s)} N_0^{\left(\frac{23}{6} + \epsilon\right)(1-s)}
\]

Therefore, by Hölder’s inequality, (5.54) and (5.31),

\[
(5.45d) \lesssim \sum_{N_1 \leq N \ll N} \left| \int I \int \nabla w_N \cdot \nabla v_N w_{N_1} w_{N_2} \, dx \, dt \right|
\]

\[
\lesssim \sum_{N_1 \leq N \ll N} \| \nabla w_N \|_{L_t^\infty L_x^2} \| \nabla v_N \|_{L_t^2 L_x^\infty} \| \nabla w_N w_{N_1} \|_{L_t^2 L_x^4} \| w_{N_2} \|_{L_t^4 L_x^4}
\]

\[
\cdot \| w_{N_1} \|_{L_t^4 L_x^4} \| w_{N_2} \|_{L_t^4 L_x^4} \| w_{N_1} \|_{L_t^2 L_x^4} \| w_{N_2} \|_{L_t^2 L_x^4}
\]

\[
\lesssim \sum_{N_1 \leq N \ll N} N_0^{\left(\frac{17}{10} + 10\epsilon\right)(1-s)} N_0^{\frac{1}{2} - s + \epsilon} \| v \|_{X^1} \left[ N_1 N^{-\frac{1}{2}} N_0^{\left(\frac{23}{6} + \epsilon\right)(1-s)} \right]^{\frac{1}{2} - 10\epsilon}
\]
\[ \cdot N_{0}^{\frac{1}{2} (\frac{2}{\alpha} + 20\varepsilon) (1-s)} N_{0}^{\frac{1}{2} (1-s)} N_{1}^{-\frac{3}{\alpha} (1-10\varepsilon)} N_{0}^{-\frac{3}{\alpha} (10\varepsilon) (1-s)} \]

\[ \lesssim N_{0}^{(2-20\varepsilon) (1-s)} \sum_{N_{1} \leq N_{2} \ll N} N_{1}^{170} - s + 6\varepsilon. \]  

(5.55)

By the choice of \( \varepsilon \) in (5.1), \( \frac{17}{40} - s + 6\varepsilon < 0 \), then we obtain that

\[ (5.45d) \lesssim N_{0}^{-20\varepsilon (1-s)} N_{0}^{2(1-s)}. \]  

(5.56)

Combining (5.46), (5.49), (5.52), and (5.56), we have

\[ (5.42) \lesssim (5.45a) + (5.45b) + (5.45c) + (5.45d) \]

\[ \lesssim \left( N_{0}^{-1 (1-s)} + N_{0}^{-\frac{1}{2} (1-s)} + N_{0}^{-\frac{1}{10} (1-s)} + N_{0}^{-20\varepsilon (1-s)} \right) N_{0}^{2(1-s)} \]  

(5.57)

\[ \lesssim N_{0}^{-20\varepsilon (1-s)} N_{0}^{2(1-s)}. \]

**Estimate on (5.43).** The proof for (5.43) is easier, since there is no derivative acting on \( v \). However, the integration contains two \( \nabla w \) terms, which already leads to the increment of \( N_{0}^{2(1-s)} \). Therefore, we need to cover the additional \( N_{0} \). By Hölder’s inequality,

\[ (5.43) \lesssim \| \nabla w \|_{L_{t}^{\infty} L_{x}^{2}}^{2} \| \nabla \|_{L_{t}^{1} L_{x}^{\infty}} \| u \|_{L_{t}^{2} L_{x}^{\infty}} \]

\[ \lesssim \| \nabla w \|_{L_{t}^{2} L_{x}^{\infty}} \| \nabla \|_{L_{t}^{1} L_{x}^{\infty}} \left( \| w \|_{L_{t}^{2} L_{x}^{\infty}} + \| v \|_{L_{t}^{2} L_{x}^{\infty}} \right). \]  

(5.58)

Due to the failure of endpoint Strichartz estimate for \( L_{t}^{2} L_{x}^{\infty} \), we use the interpolation, Bernstein’s inequality, Lemma 2.7, (5.37), and (5.38)

\[ \| w \|_{L_{t}^{2} L_{x}^{\infty}} \lesssim \| N^{\varepsilon} P_{N} w \|_{L_{t}^{2} L_{x}^{\infty}} \]

\[ \lesssim \| P_{N} w \|_{L_{t}^{2} L_{x}^{\infty}}^{1-2\varepsilon} \| N^{\frac{1}{2}} P_{N} w \|_{L_{t}^{2} L_{x}^{\infty}}^{2\varepsilon} \]

\[ \lesssim \| N^{\frac{1}{2}} P_{N} w \|_{L_{t}^{2} L_{x}^{\infty}}^{1-2\varepsilon} \| N^{\frac{1}{2}} P_{N} w \|_{L_{t}^{2} L_{x}^{\infty}}^{2\varepsilon} \]

\[ \lesssim \| N^{\frac{1}{2}} P_{N} w \|_{L_{t}^{2} L_{x}^{\infty}}^{1-2\varepsilon} \| w \|_{L_{t}^{2} L_{x}^{\infty}}^{2\varepsilon} \]

\[ \lesssim N_{0}^{(\frac{5}{2}+3\varepsilon) (1-s)}. \]

Since \( s > \frac{17}{40} \) implies \( 1 - \frac{5}{2} s + (4 - 3s) \varepsilon < 1 - \frac{5}{2} s + 3 \varepsilon < -\frac{1}{100} \), by (5.58), (5.59), and (5.31),

\[ (5.43) \lesssim \| \nabla w \|_{L_{t}^{2} L_{x}^{\infty}}^{2} \| v \|_{L_{t}^{2} L_{x}^{\infty}} \left( \| w \|_{L_{t}^{2} L_{x}^{\infty}} + \| v \|_{L_{t}^{2} L_{x}^{\infty}} \right) \]

\[ \lesssim N_{0}^{2(1-s)} N_{0}^{-s-\frac{1}{2} + \varepsilon} \left( N_{0}^{(\frac{5}{2}+3\varepsilon) (1-s)} + N_{0}^{-s-\frac{1}{2} + \varepsilon} \right) \]

\[ \lesssim N_{0}^{-s-\frac{1}{2} + \varepsilon} N_{0}^{1 (1-s)} N_{0}^{3 (1-s)} \lesssim N_{0}^{1-\frac{5}{2} s + \varepsilon} N_{0}^{2 (1-s)} \lesssim N_{0}^{-\frac{1}{100}} N_{0}^{2(1-s)}. \]  

(5.60)

**Estimate on (5.44).** This is a simple case, where no derivative appears. By Hölder’s inequality, (5.35), and (5.34),

\[ (5.44) \lesssim \int \int |u|^{2} u (|u|^{2} u - |w|^{2} w) \mathrm{d}x \mathrm{d}t \]

\[ \lesssim \| u \|_{L_{t}^{2} L_{x}^{\infty}} \left( \| u \|_{L_{t}^{2} L_{x}^{\infty}}^{2} + \| w \|_{L_{t}^{4} L_{x}^{\infty}}^{2} \right) \left( \| u \|_{L_{t}^{2} L_{x}^{\infty}}^{3} + \| w \|_{L_{t}^{2} L_{x}^{\infty}}^{3} \right) \]

\[ \lesssim N_{0}^{-s-\frac{1}{2} + \varepsilon} N_{0}^{1 (1-s)} N_{0}^{3 (1-s)} \lesssim N_{0}^{1-\frac{5}{2} s + \varepsilon} N_{0}^{2 (1-s)} \lesssim N_{0}^{-\frac{1}{100}} N_{0}^{2(1-s)}. \]
Then, by choosing $N_0 = N_0(A)$ suitably large, and combining (5.57), (5.60), and (5.61), we have

$$
\sup_{t \in I} E(t) \leq E(w_0) + (5.42) + (5.43) + (5.44)
$$

$$
\leq AN_0^{2(1-s)} + C(A) \cdot \left( N_0^{-20\epsilon(1-s)} + N_0^{-\frac{1}{100}} \right) N_0^{2(1-s)}
$$

(5.61)

Then, by the standard bootstrap argument, we finish the proof of (5.27).

5.4. Proof of Proposition 5.1. We first prove the global well-posedness. Since $v \in Y^s \cap Z^s(\mathbb{R})$ and $w_0 \in H^1_x$, by Proposition 4.2, there exists $T_1$ depending on $\|w_0\|_{H^1_x}$ and $\|v\|_{Y^s(\mathbb{R}) \cap Z^s(\mathbb{R})}$, such that $w \in C([0, T_1]; H^1_x)$ solves (4.1). By Proposition 5.5, we have

$$
E(w(T_1)) \leq \sup_{t \in [0, T_1]} E(w(t)) \leq 2AN_0^{2(1-s)}.
$$

Then, we have $\|w(T_1)\|_{H^1_x}^2 \leq 2AN_0^{2(1-s)}$, and can apply Proposition 4.2 again starting from $T_1$. Since the energy bound in (5.27) does not rely on $T$, we can extend the solution on $\mathbb{R}$ by induction, and get

$$
\sup_{t \in \mathbb{R}} E(w(t)) \leq 2AN_0^{2(1-s)}.
$$

(5.62)

Next, we prove the scattering statement. Since the global well-posedness already holds, we do not care about the explicit expression of $A$ and $N_0$. We only consider the forward-in-time case, and it suffices to prove that

$$
\lim_{T \to +\infty} \left\| \int_T^{+\infty} e^{-it\Delta} (|u|^2 u) dt \right\|_{H^1_x} = 0.
$$

(5.63)

First note that by (5.62), we have Lemmas 5.2, 5.3, and 5.4 hold on $[0, \infty)$. Then,

$$
\|w\|_{X^1([0, +\infty))} \leq C(A, N_0).
$$

In particular,

$$
\lim_{T \to +\infty} \|w\|_{L^4_t L^6_x([T, +\infty) \times \mathbb{R}^3)} = 0.
$$

Moreover, we also have

$$
\lim_{T \to +\infty} \|v\|_{\tilde{Y}^s([T, +\infty))} = 0.
$$

Now, all the space-time norms are taken over $[T, +\infty) \times \mathbb{R}^3$. We split

L.H.S. of (5.63) $\lesssim \left\| \int_T^{+\infty} e^{-it\Delta} ((\nabla) w u^2) dt \right\|_{L^2_t}$

(5.64a)

$$
+ \left\| \int_T^{+\infty} e^{-it\Delta} ((\nabla) v u^2) dt \right\|_{L^2_t}.
$$

(5.64b)
The proof for the first term (5.64a) is easy. By Lemma 2.7 and Hölder’s inequality,

\[(5.64a) \lesssim \| \langle \nabla \rangle w u \|^2_{L^2_t L^6_x} \]

\[\lesssim \| \langle \nabla \rangle w \|^2_{L^\infty_t L^2_x} \left( \| w \|^2_{L^4_t L^6_x} + \| v \|^2_{L^2_t L^6_x} \right) \]

\[\lesssim C(A, N_0) \left( \| w \|^2_{L^4_t L^6_x([T, +\infty))} + \| v \|^2_{\tilde{Y}^3([T, +\infty))} \right).\]

Then, we have

\[\lim_{T \to +\infty} (5.64a) = 0. \quad (5.65)\]

Next, we deal with the term (5.64b). By frequency decomposition,

\[(5.64b) \lesssim \sum_{N \in 2N} \left\| \int_T^{+\infty} e^{-it\Delta} \langle \nabla \rangle v_N u_{\geq N} u \right\|_{L^2_t} \]

\[+ \sum_{N \in 2N} \left\| \int_T^{+\infty} e^{-it\Delta} \langle \nabla \rangle v_N u^2_{\leq N} \right\|_{L^2_t}. \quad (5.66a)\]

By Hölder’s inequality and Lemma 2.4, we have

\[(5.66a) \lesssim C \sum_{N \leq N_1} \left\| \langle \nabla \rangle v_N u_{N_1} u \right\|_{L^1_t L^2_x} \]

\[\lesssim C \sum_{N \leq N_1} \frac{N^\frac{1}{2}}{N_1^\frac{1}{2}} \left\| \langle \nabla \rangle \frac{1}{2} v_N \right\|_{L^2_t L^\infty_x} \left\| \langle \nabla \rangle \frac{1}{2} v_{N_1} \right\|_{L^2_t L^\infty_x} \| u \|_{L^\infty_t L^2_x} \]

\[+ C \sum_{N \leq N_1} \frac{N^\frac{1}{2}}{N_1^\frac{1}{2}} \left\| \langle \nabla \rangle \frac{1}{2} v_N \right\|_{L^2_t L^\infty_x} \left\| \langle \nabla \rangle \frac{1}{2} w_{N_1} \right\|_{L^4_t L^{\infty}_x} \| u \|_{L^4_t L^{\infty}_x} \]

\[\lesssim C \left\| \langle \nabla \rangle \frac{1}{2} v_N \right\|_{L^2_t L^\infty_x} \left\| \langle \nabla \rangle \frac{1}{2} v_{N_1} \right\|_{L^2_t L^\infty_x} \| u \|_{L^\infty_t L^2_x} \]

\[+ C \left\| \langle \nabla \rangle \frac{1}{2} v_N \right\|_{L^2_t L^\infty_x} \left\| \langle \nabla \rangle \frac{1}{2} w_{N_1} \right\|_{L^4_t L^{\infty}_x} \| u \|_{L^4_t L^{\infty}_x} \]

\[\lesssim C \| v \|^2_{\tilde{Y}^3([T, +\infty))} + C \| v \|^2_{\tilde{Y}^3([T, +\infty))} \| u \|_{X^1} (\| v \|_{Y^3} + \| w \|_{X^1}) \]

\[\lesssim C \| v \|^2_{\tilde{Y}^3([T, +\infty))} + C(A, N_0) \| v \|_{Y^3([T, +\infty))}. \]

Then,

\[\lim_{T \to +\infty} (5.66a) = 0. \quad (5.67)\]

Finally, we estimate (5.66b), where we need to exploit the duality structure as in the proof of Proposition 4.2. In this case, it is unnecessary to invoke the $U^p-V^p$ method as before for two reasons: first, we are considering the dual operator of $e^{i\Delta}$; second, we
have estimate for $\| (\nabla)^{\frac{5}{6} + \epsilon} v \|_{L^2_t L^\infty_x}$ under the radial assumption. We can simply use the duality representation of the $L^2_\xi$-norm:

\begin{equation}
(5.66b) \leq C \sum_{N_1 \ll N \ll N} \left\| \int_{T}^{+\infty} e^{-it\Delta} ( (\nabla) v u_{N_1} u_{N_2} ) dt \right\|_{L^2_\xi} \leq C \sum_{N_1 \ll N \ll N} \sup_{\| g \|_{L^2_\xi}} \left\| \int_{T}^{+\infty} (g, e^{-it\Delta} ( (\nabla) v u_{N_1} u_{N_2} )) dt \right\| \leq C \sum_{N_1 \ll N \ll N} \sup_{\| g \|_{L^2_\xi}} \left\| \int_{T}^{+\infty} (e^{it\Delta} g_{\sim N}) (\nabla) v u_{N_1} u_{N_2} \, dx \, dt \right\|.
\end{equation}

By Lemma 2.9, for $N_1 \ll N$,

\begin{equation}
\left\| (e^{it\Delta} g_{\sim N}) u_{N_1} \right\|_{L^2_{t,x}} \lesssim \frac{N_1}{N^{1/2}} \left\| g_{\sim N} \right\|_{L^2_\xi} \left( \| P_{N_1} v_0 \|_{L^2_\xi} + \| w_{N_1} \|_{U^2_\Delta} \right) \lesssim \frac{N^{1/3}}{N^{1/4}} \left( \frac{N_1}{N} \left\| P_{N_1} v_0 \right\|_{L^2_\xi} + \frac{N_1}{N} \left\| w_{N_1} \right\|_{U^2_\Delta} \right) \lesssim \frac{N^{1/3}}{N^{1/4}} C(A, N_0).
\end{equation}

By (5.69) and Hölder’s inequality, for $N_1 \leq N_2 \ll N$,

\begin{equation}
\int_{T}^{+\infty} \int_{T}^{+\infty} (e^{it\Delta} g_{\sim N}) (\nabla) v u_{N_1} u_{N_2} \, dx \, dt \leq C \left( \left\| (e^{it\Delta} g_{\sim N}) u_{N_1} \right\|_{L^2_{t,x}} \right) \left\| (\nabla) v \right\|_{L^2_t L^\infty_x} \left\| u_{N_2} \right\|_{L^\infty_t L^2_\xi} \leq C(A, N_0) \frac{N^{1/3}}{N^{1/4}} \left\| (\nabla) v \right\|_{L^2_t L^\infty_x} \left\| u_{N_2} \right\|_{L^\infty_t L^2_\xi} \leq C(A, N_0) \frac{N^{1/3}}{N^{1/4}} N^{-\varepsilon} \left\| v \right\|_{\tilde{Y}^{s}_{T, (+\infty)}}.
\end{equation}

Then, by (5.68) and (5.70),

\begin{equation}
(5.66b) \leq C \sum_{N_1 \ll N \ll N} \sup_{\| g \|_{L^2_\xi}} \left\| \int_{T}^{+\infty} (e^{it\Delta} g_{\sim N}) (\nabla) v u_{N_1} u_{N_2} \, dx \, dt \right\| \leq C(A, N_0) \sum_{N_1 \ll N \ll N} \frac{N^{1/3}}{N^{1/4}} N^{-\varepsilon} \left\| v \right\|_{\tilde{Y}^{s}_{T, (+\infty)}} \lesssim C(A, N_0) \left\| v \right\|_{\tilde{Y}^{s}_{T, (+\infty)}}.
\end{equation}

Then,

\begin{equation}
\lim_{T \to +\infty} \frac{1}{T} (5.66b) = 0.
\end{equation}
It follows from (5.65), (5.67), and (5.71) that (5.63) holds. This finishes the proof of Proposition 5.1.

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