Holomorphic and anti-holomorphic conductivity flows in the quantum Hall effect

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Abstract
It is shown that the flow diagrams for the conductivities in the quantum Hall effect, arising from two ostensibly very different proposals based on modular symmetry, are in fact identical. The $\beta$-functions are different, the rates at which the flow lines are traversed are different, but the tangents to the flow lines are the same in both cases; hence, the flow diagrams are same in all aspects.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

The notion that the modular group, or a sub-group thereof, is relevant to a description of the quantum Hall effect is now nearly 20 years old, [1–3]. The basic idea is that a sub-modular group is an emergent symmetry that maps between different phases of the two-dimensional electron gas. Denote the Hall conductivity in a homogeneous quantum Hall sample by $\sigma_{xy}$ and the dc ohmic conductivity by $\sigma_{xx} \geq 0$ (throughout this paper we use natural units for the conductivity, with $\frac{e^2}{h} = 1$). The modular group acts on the complex conductivity $\sigma = \sigma_{xy} + i\sigma_{xx}$ via

$$\gamma(\sigma) = \frac{a\sigma + b}{c\sigma + d}$$

for any four integers $a$, $b$, $c$ and $d$ satisfying $ad - bc = 1$; these properties ensure that $\text{Im}(\gamma(\sigma)) > 0$ if $\text{Im}(\sigma) > 0$. Group multiplication is the same as matrix multiplication for $\gamma = \left(\begin{array}{cc}a & b \\ c & d \end{array} \right)$, though $\gamma$ and $-\gamma$ have the same effect in (1).

The description so far allows for $\gamma$ in (1) to be any element of the full modular group $\Gamma(1) \cong SL(2, \mathbb{Z})/\mathbb{Z}_2$. However $\Gamma(1)$ is not the group relevant to the quantum Hall effect as a general element will not preserve the property of odd denominators for the Hall conductivity.
Figure 1. A fundamental domain for $\Gamma_0(2)$.

at quantum Hall plateaux (exotic even denominator states require special consideration in the modular setting [4]). The sub-group $\Gamma_0(2) \subset \Gamma(1)$, defined by demanding that $c$ be even, does preserve the parity of the denominators and was identified as the correct group, at least for spin split samples, in [2, 3]. A key ingredient in the understanding of a sub-modular group as an emergent symmetry is particle-vortex duality [4–6].

The group $\Gamma_0(2)$ is an infinite discrete group generated by repeated actions of two elements which can be chosen to be

$$L : \sigma \rightarrow \sigma + 1 \quad \text{and} \quad F^2 : -\frac{1}{\sigma} \rightarrow -\frac{1}{\sigma} + 2; \quad (2)$$

an arbitrary element of the group can be obtained by repeated actions of these two maps (not necessarily uniquely). The map $L$ is a generalization of Landau level addition to include ohmic conductivity and $F^2$ is a similar generalization of Jain’s flux attachment transformation, transforming between ground state wavefunctions for the fractional quantum Hall effect [7]. These two maps are the basis of the ‘law of corresponding states’ [8], although that reference did not use complex notation. Using modular symmetry the whole upper-half complex conductivity plane can be generated from a single domain consisting of a vertical strip of width 1, with $1 \leq \sigma_{xy} \leq 2$, with the interior of the disc of radius 0.5, whose boundary is the semi-circle spanning 1–2, removed. This strip, a fundamental domain for $\Gamma_0(2)$, is shown in figure 1. This has powerful implications because it implies that information about the conductivity at any point can be obtained from a knowledge of the physical properties in the fundamental domain.

Modular symmetry is an emergent symmetry, it is only manifest in the low-energy, long-wavelength regime and as such is only useful in describing the static infrared physics of the quantum Hall effect. Even then it is still a mathematical idealization and is not universally applicable: one expects the quantum Hall effect to break down at very weak and also very strong magnetic fields. At weak fields (large $\sigma_{xy}$) the cyclotron energy becomes of the same order as the thermal energy, and thermal excitations make it easy for states to jump from one Landau level to another. At very strong fields, one expects a Wigner crystal to form and again the physics changes (at zero temperature this is expected to occur for $\sigma_{xy} < \frac{1}{2}$, but at finite temperature the Wigner crystal melts and filling factors less than $\sigma_{xy} < \frac{1}{2}$ have been observed). In the real world, one cannot access the whole upper-half conductivity plane by mapping the fundamental domain around, only a limited region will be accessible, but it is sufficient for
this region to be significantly larger than the fundamental domain itself for modular symmetry to be a useful concept. For example we can get detailed information about the fractional quantum Hall effect in the first Landau level, the region below the semi-circle in figure 1.

Modular symmetry has very important implications for the way in which the conductivities change as the intrinsic scale of the microscopic physics is changed [9–20]. It allows one to identify fixed points of the flow unambiguously as any fixed point of \( \Gamma_0(2) \) must be a fixed point of the flow. By a fixed point of \( \Gamma_0(2) \) we mean here a complex conductivity, \( \sigma_* \), for which there exists an element \( \gamma \in \Gamma_0(2) \) such that \( \gamma(\sigma_*) = \sigma_* \), for example \( \sigma_* = \frac{2n}{2} \) are fixed points for any integer \( n \). Such points are isolated in the upper-half plane. An example of the power of modular symmetry is the selection rule that transitions between two quantum Hall plateaux with filling fractions \( \nu = \frac{p}{q} \) and \( \nu' = \frac{p'}{q'} \) are only allowed if \( |pq' - qp'| = 1 \) [10].

The first appearance of the modular group as an emergent symmetry in a two-dimensional system was [22], in the context of a model chosen for the properties similar to those expected of QCD. Some 7 years later it was suggested that the modular group, or a sub-group thereof, should be a low-energy emergent symmetry in the quantum Hall effect [23]; however, the sub-group identified in [23] was not correct for a quantitative description of the quantum Hall effect. A more detailed analysis was carried out in [1] and the particular sub-group of the modular group, \( \Gamma_0(2) \) defined above, was identified as the correct sub-group in [2, 3]. At almost the same time as [2] appeared, the ‘law of corresponding states’, based on effective field theory arguments, was put forward in [8]—this generates \( \Gamma_0(2) \) symmetry, but complex notation and the language of modular symmetry were not used in [8]. An alternative inhomogeneous action of \( \Gamma(1) \) was given in [24]. The relevance of other sub-groups of the modular group to the quantum Hall effect was investigated in [25] and extensions to other systems (such as 2D superconductors [4, 5], quantum Hall bi-layers [26] and graphene [27]). The effects of electron spin and Zeeman splitting were examined from the modular group point of view in [15, 19]. A review of modular symmetry in the quantum Hall effect, and the relation to \( N = 2 \) supersymmetric Yang–Mills theory, is given in [21].

A flow diagram for the quantum Hall effect, as the scattering length associated with electron transport is varied, was conjectured in [28], but the normalization of \( \sigma_{xx} \) was not determined. The scaling properties were further investigated theoretically in [29] and experimentally in [30, 31]. Macroscopically, the electron scattering length, or in quantum language the quantum coherence length of the electron wavefunction, can be controlled by varying the temperature, at least until the point where the temperature is so low that the coherence length becomes of the order of, or larger than, the sample size.

The first flow diagram compatible with \( \Gamma_0(2) \) symmetry appeared in [2]. The first quantitative investigations of the form of flow implied by sub-modular symmetry, in [9, 13], used gradient flow and the \( \epsilon \)-theorem for two-dimensional renormalization group flow and required the introduction of a metric on the upper-half conductivity plane. An alternative suggestion used holomorphic \( \beta \)-functions to model the flow [11, 15] and figure 2 is taken from [15]. The flow presented in [11] was compared with experimental data in [32–36] with encouraging results; figures 3–5 are reproduced from [32, 33]. Gradient flow was revisited in [16–18] using an anti-holomorphic potential and the resulting flow diagram compared with the experimental data [32, 33] in [18], with equally good agreement; see figures 6 and 7 reproduced from [18]. Indeed the similarity between the flow diagrams in [11, 15] and [18] is remarkable, but this should perhaps not be so surprising as they both rely on the same underlying symmetry, \( \Gamma_0(2) \), which is very restrictive. In this paper, it will be shown that in fact these two flow diagrams are identical, despite the fact that the underlying \( \beta \)-functions are different. The only difference in the integrated flow is the rate at which the flow lines are traversed as the length scale is changed; the flow diagram itself is identical in both cases.
Figure 2. Holomorphic $\Gamma_0(2)$ flow, first derived in [11] (figure from [15]). The horizontal axis is the Hall conductivity $\sigma_{xy}$ and the vertical axis is the ohmic conductivity $\sigma_{xx}$. Reproduced with permission from Dolan B P 2000 Phys. Rev. B 62 10278. Copyright (2000) by the American Physical Society.

Figure 3. Experimental temperature flow for the integer quantum Hall effect from [32], superimposed on the prediction (dotted lines) from holomorphic flow in [11]. Reproduced with permission from Murzin S S et al 2002 Phys. Rev. B 66 233314. Copyright (2002) by the American Physical Society.
Figure 4. Experimental temperature flow for the fractional quantum Hall effect in the $\nu = \frac{1}{3}$ to quantum Hall insulator transition, from [33]. Reproduced with permission from Murzin S S et al 2005 Phys. Rev. B 72 195317. Copyright (2005) by the American Physical Society.

Figure 5. Experimental temperature flow for the fractional quantum Hall effect, from [33], superimposed on the prediction (dotted lines) from holomorphic flow in [11]. Reproduced with permission from Murzin S S et al 2005 Phys. Rev. B 72 195317. Copyright (2005) by the American Physical Society.
2. $\beta$-functions

Any $\beta$-function 

$$\beta(\sigma, \bar{\sigma}) = \frac{d\sigma}{ds}$$

(3)
compatible with $\Gamma_0(2)$ must transform as
\[ \frac{dy}{ds} = \beta(y, \gamma(\delta)) = \frac{1}{(c\sigma + d)^2} \beta(\sigma, \delta), \] (4)
where $\gamma(\sigma) = \frac{a\sigma + b}{c\sigma + d}$ is a $\Gamma_0(2)$ transformation. The real parameter $s$ here is the logarithm of some length associated with the underlying physics, such as the electron scattering length (a function of temperature). $\beta$-functions compatible with $\Gamma_0(2)$ symmetry were first discussed in [9], in the context of gradient flow. Some general properties of $\beta$-functions satisfying (4), including the semi-circle law, were derived in [14].

The function
\[ f(\sigma) = \frac{\vartheta_4^2(\sigma)}{\vartheta_4^4(\sigma)} - \frac{1}{256q^2} \prod_{n=1}^{\infty} \frac{1}{(1 + q^{2n})^2}, \] (5)
where $q = e^{i\sigma}$, is invariant under $\Gamma_0(2)$, [38] (definitions and relevant properties of Jacobi $\vartheta$-functions are summarized in the appendix). Since $d(\gamma(\sigma)) \rightarrow (c\sigma + d)^2 d\sigma$ under $\Gamma_0(2)$ transformation it is immediate that $f'$ must transform as
\[ \frac{df}{d\sigma} \rightarrow (c\sigma + d)^2 \frac{df}{d\sigma}, \] (6)
i.e. it is a modular function of weight 2. The first use of such modular functions in the context of (4) was in [9] where the function
\[ E(\sigma) := \frac{1}{2\pi i} f' = 1 + 24 \sum_{n=1}^{\infty} \frac{nq^{2n}}{1 + q^{2n}} \] (7)
was considered.

A second function, with the same transformation properties as $f'$ but which is not analytic, was also considered in [9]:
\[ H(\sigma, \bar{\sigma}) := \frac{1}{\pi \text{Im}(\sigma)} + 16 \sum_{n=1}^{\infty} \frac{nq^{2n}}{1 - q^{2n}} = \frac{i}{\pi} \frac{d}{d\sigma} \ln \left( (\sigma - \bar{\sigma})^2 \vartheta_3^2 \vartheta_4^2 \right). \] (8)

All attempts at constructing $\beta$-functions compatible with $\Gamma_0(2)$ symmetry to date have focused on (7) and (8).

2.1. Holomorphic $\beta$-function

A $\beta$-function compatible with (4) is the holomorphic form of weight $-2$:
\[ \tilde{\beta}(\sigma) = -\frac{f'}{f} = \frac{1}{\pi} \left( \frac{\vartheta_3^2(\sigma)}{\vartheta_4^2(\sigma)} \right), \] (9)
where $f' = \frac{df}{d\sigma}$ [11, 15].

Equation (9) can immediately be integrated to give
\[ \frac{f'}{f} d\sigma = \frac{df}{f} = -d\sigma \quad \Rightarrow \quad f(\sigma(s)) = Ce^{-s}, \] (10)
where $C$ is a (complex) constant. The integral curves of the flow are curves on which the complex phase of $f(\sigma)$ is constant. These are easily plotted as a contour plot of the phase of $f(\sigma)$ [11, 15], and the relevant part of figure 2 from [15] is shown in figure 2 here.

This holomorphic flow was compared with experimental data for temperature flow in the integral quantum effect in [32] and the fractional effect in [33]: figure 3 is taken from [32] and figures 4 and 5 are from [33]. For a short review see [37].
2.2. Anti-holomorphic gradient flow

Following on from [9], non-holomorphic gradient flow $\beta$-functions, compatible with $\Gamma_0(2)$ symmetry, were further explored in [13] and a specific form was proposed in [16] and further investigated in [17, 18, 20]. The $\beta$-function proposed in [16] was integrated numerically and plotted in [18].

This $\beta$-function is obtained by first considering the holomorphic function for $\Gamma_0(2)$ of weight +2 in (7), $E(\sigma)$, which satisfies

$$E(\gamma(\sigma)) = (c\sigma + d)^2 E(\sigma).$$

This is then used to generate anti-holomorphic gradient flow

$$\beta^\sigma(\sigma, \overline{\sigma}) = -i G_{\sigma\sigma} \overline{E}(\sigma) = -\frac{1}{2\pi} G^{\sigma\sigma} (\partial_{\sigma} \ln f),$$

(11)

where $G_{\sigma\sigma}$ is a metric on the $\sigma$-plane.

A natural choice of metric is the modular invariant line element

$$d\sigma d\overline{\sigma} \sim d\sigma d\overline{\sigma} (\Im \sigma)^2,$$

(12)

but the choice of metric does not actually affect the flow diagrams, provided only that it is regular, as we shall see. Under modular transformations

$$(\Im \sigma)^2 \rightarrow \frac{(\Im \sigma)^2}{|c\sigma + d|^2},$$

(13)

so, since

$$E(\gamma(\sigma)) = (c\sigma + d)^2 E(\sigma),$$

(14)

(11) transforms correctly as

$$\beta^\sigma(\gamma(\sigma), \overline{\gamma(\sigma)}) = \frac{1}{(c\sigma + d)^2} \beta^\sigma(\sigma, \overline{\sigma}).$$

Equation (11) was integrated numerically in [18] and the resulting flow is plotted, together with the experimental data from [32] and [33], in figures 6 and 7.

2.3. Comparison of $\beta$-functions

The similarity with the flow plotted from (9) in figure 2, and from (11) in figures 6 and 7, is clear and we now show that these flows are identical, even though the two underlying $\beta$-functions differ. The crucial point is that the shape of the flow lines depends only on their tangents at any point, not on the individual values of $\sigma_{\sigma\sigma} = \Re \beta$ and $\sigma_{\sigma\overline{\sigma}} = \Im \beta$ separately, but only on the ratio

$$\frac{\sigma_{\sigma\overline{\sigma}}}{\sigma_{\sigma\sigma}} := \frac{\Im \beta}{\Re \beta},$$

(16)

and this ratio is the same for both (9) and (11). To see this first note that $G^{\sigma\overline{\sigma}}$, being real, drops out in the ratio for (11) so we only need to consider $\frac{\Im (\frac{f'}{f})}{\Re (\frac{f'}{f})}$ for (11) and compare this with $\frac{\Im (\frac{f'}{f})}{\Re (\frac{f'}{f})}$ for (9). But for any complex number $w$

$$\frac{\Im (w)}{\Re (w)} = \frac{\Im (\frac{1}{w})}{\Re (\frac{1}{w})},$$

(17)

and hence the ratio is the same for both (9) and (11). The tangent to the resulting flow lines is therefore the same at every point in the upper-half $\sigma$-plane and so the plots of the two flows
are necessarily identical. Of course integrating the equations gives different solutions, but the solutions only differ in the rate at which the flow lines are traversed, not in the shape of the plots.

The nature of the repulsive fixed points at $\sigma_\ast = \frac{1+i}{2}$, and its images under $\Gamma_0(2)$, can be investigated in detail by using the approximate form

$$f(\sigma_\ast + \epsilon) = \frac{1}{4} - a\epsilon^2 + o(\epsilon^4),$$

with $a = \left\{ r(\frac{1}{4}) \right\}^4$ positive [11, 12]. Then $-\frac{f}{f'} \approx \frac{1}{8\pi r_0} = \frac{\sigma}{8|\sigma|^4}$ giving the flow lines of a hyperbolic fixed point as shown in figure 8. Of course one obtains the same form by using gradient flow with $-\frac{f}{f'} \approx 8\sigma\tau$; they differ by an overall real factor but the geometry of the flow is the same. The fact that the former case has a divergent factor $\sim \frac{1}{|\epsilon|^2}$ is not a pathology—at zero temperature one expects a discrete jump from $\sigma_{xy} = 0$ to $\sigma_{xy} = 1$ as the magnetic field is varied [11].

Near the origin $\sigma \approx \epsilon$ one has $f \approx -16\epsilon^{-\frac{1}{2}}$ [11], and $-\frac{f}{f'} = \frac{i\pi}{\sigma^2}$ so the holomorphic $\beta$-function gives

$$\tilde{\beta} \approx \frac{f}{f'} = \frac{i\sigma^2}{\pi},$$

while the anti-holomorphic gradient flow gives

$$\beta \approx (Im(\sigma))^2 \frac{i\sigma^2}{2|\sigma|^4}.$$
3. Conclusion

In conclusion, it has been shown that the geometry of the two different conductivity flows presented in [11] and [16] is identical, despite the different $\beta$-functions. The only differences lie in the rate at which the flow lines are traversed, not in their shape.

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Appendix

Jacobi $\vartheta$-functions are defined as

\[
\vartheta_2 = 2 \sum_{n=0}^{\infty} q^{(n+\frac{1}{2})^2} = 2q^{\frac{1}{2}} \prod_{n=1}^{\infty} (1 - q^{2n})(1 + q^{2n})^2, \quad (A.1)
\]

\[
\vartheta_3 = \sum_{n=-\infty}^{\infty} q^{n^2} = \prod_{n=1}^{\infty} (1 - q^{2n})(1 + q^{2n-1})^2, \quad (A.2)
\]

\[
\vartheta_4 = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} = \prod_{n=1}^{\infty} (1 - q^{2n})(1 - q^{2n-1})^2, \quad (A.3)
\]

with $q := e^{i\pi \sigma}$ (the conventions are those of [39], except that $\tau$ there is replaced by $\sigma$ here).

The $\vartheta$-functions satisfy the relation

\[
\vartheta_3^4 = \vartheta_2^4 + \vartheta_4^4 \quad (A.4)
\]

and have the following transformations under $T : \sigma \to \sigma + 1$ and $S : \sigma \to -\frac{1}{\sigma}$:

\[
\vartheta_2(\sigma + 1) = e^{\pi \sigma} \vartheta_2(\sigma), \quad \vartheta_2\left(-\frac{1}{\sigma}\right) = \sqrt{-\pi \sigma} \vartheta_3(\sigma), \quad (A.5)
\]

\[
\vartheta_3(\sigma + 1) = \vartheta_4(\sigma), \quad \vartheta_3\left(-\frac{1}{\sigma}\right) = \sqrt{-\pi \sigma} \vartheta_2(\sigma), \quad (A.6)
\]

\[
\vartheta_4(\sigma + 1) = \vartheta_3(\sigma), \quad \vartheta_4\left(-\frac{1}{\sigma}\right) = \sqrt{-\pi \sigma} \vartheta_2(\sigma). \quad (A.7)
\]

Using these properties, it is not difficult to show that the function

\[
f(\sigma) = -\frac{\vartheta_3^4 \vartheta_4^4}{\vartheta_2^8} = -\frac{1}{256q^{2}} \prod_{n=1}^{\infty} \frac{(1 - q^{4n-2})^8}{(1 + q^{2n})^{16}}
\]

\[
= -\frac{1}{256q^{2}} \prod_{n=1}^{\infty} \frac{(1 - q^{2n})^8}{(1 + q^{2n})^{16}(1 - q^{4n})^8}
\]

\[
= -\frac{1}{256q^{2}} \prod_{n=1}^{\infty} \frac{1}{(1 + q^{2n})^{24}}
\]

is invariant under Landau level addition, $L$, and flux attachment, $F^2$, sending $\sigma \to \frac{1}{1 - \sigma}$.  

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Reference