Compositions of states and observables in Fock spaces

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Abstract

This article is concerned with compositions in the context of three standard quantizations in the Fock spaces framework, namely, anti-Wick, Wick and Weyl quantizations. The first one is a composition of states and is closely related to the standard scattering identification operator encountered in Quantum Electrodynamics for time dynamics issues (see [15], [7]). Anti-Wick quantization and Segal-Bargmann transforms are implied here for that purpose. The other compositions are for observables (operators in some specific classes) for the Wick and Weyl symbols. For the Wick symbol of the composition of two operators, we obtain an absolutely converging series and for the Weyl symbol, the remainder term of the expansion is absolutely converging, still in the Fock spaces framework.

Keywords: Scattering identification operator, composition, quantization, Fock spaces, infinite dimensional analysis, composition of states, composition of operators, anti-Wick quantization, Wick quantization, Wick symbol, Husimi function, Weyl symbol, Mizrahi series, Wiener spaces, Segal-Bargmann transform, heat operator, symbolic calculus, semiclassical analysis, QED, quantum electrodynamics.

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Contents

1 Introduction. 2

2 Anti-Wick quantization and the scattering identification operator. 6
  2.1 Wiener extension and Segal isomorphism. 6
  2.2 Segal Bargmann transform. 7
  2.3 Power series expansions. 9
  2.4 Connection between $T^f_{FH}$ and $T^f_{FW}$. 11
  2.5 Segal Bargmann transform and the product law $I$. 13

3 Absolute convergence of the Mizrahi series. 15

4 Weyl symbol composition. 18
1 Introduction.

The purpose of this article is to present within the Fock space framework three different results concerning three common quantizations, that is, anti-Wick, Wick and Weyl quantizations.

In the first part of this work, we are concerned with the anti-Wick quantization together with the operator commonly called the scattering identification operator for some spin boson model (see below). Namely, our first aim is to show that the scattering identification can also be defined using the anti-Wick quantization.

For any infinite dimensional real separable Hilbert space $H$, $\mathcal{F}_s(H_C)$ denotes the symmetrized Fock space over $H_C$ where $H_C$ stands for the complexification of $H$. (See [20], vol. I, sect II.4.) For all $X$ in $H$, $a(X)$ and $a^*(X)$ denote annihilation and creation operators (unbounded operators in $\mathcal{F}_s(H_C)$) associated with $X$. (See [20], vol. II, sect X.7.) The state $\Psi_0$ stands for the vacuum. The subspace $\mathcal{F}_s^{\text{fin}}(H_C)$ denotes the set of all elements written as finite linear combinations of $a^*(X_1)\ldots a^*(X_m)\Psi_0$ with $X_j$ in $H$. For some special Hilbert space $H$, the identification operator is an unbounded operator in $\mathcal{F}_s(H_C)$ defined with the help of some internal composition law in $\mathcal{F}_s^{\text{fin}}(H_C)$ introduced in [15]. See also [7] and [23]. This internal composition law can be defined for any Hilbert space $H$. Let us recalled below the definition as it is written in [23] in formula (17.92). See also [7].

**Definition 1.1.** One denotes by $I$ the bilinear map from $\mathcal{F}_s^{\text{fin}}(H_C) \times \mathcal{F}_s^{\text{fin}}(H_C)$ to $\mathcal{F}_s(H_C)$ satisfying for all $X_1, \ldots, X_m$ and all $Y_1, \ldots, Y_p$ in $H$:

$$I\left(a^*(X_1)\ldots a^*(X_m)\Psi_0, a^*(Y_1)\ldots a^*(Y_p)\Psi_0\right) = a^*(X_1)\ldots a^*(X_m)a^*(Y_1)\ldots a^*(Y_p)\Psi_0.$$  \hspace{1cm} (1.1)

The above product is defined for $f$ and $g$ in $\mathcal{F}_s^{\text{fin}}(H_C)$ and we shall see in Proposition 1.2 that its domain of definition can be extended. Also recall here that creation operators are commuting.

In Section 2.5 we study the connection between this composition law $I$ and the Segal Bargmann transform. Preliminary, we present in Section 2.2 three variants of the Segal Bargmann transform of $f$ (Definition 2.3), for all $f$ in $\mathcal{F}_s(H_C)$. The first one, denoted $T^{\text{FH}}_h f$, is an element of the Fock space $\mathcal{F}_s(H^2_C)$. The two others are depending on a (semiclassical) parameter $h > 0$. The second one, denoted by $T^{\text{FW}}_h f$, is a function on $H^2$ being Gateaux anti-holomorphic if identifying $(q,p)$ with $q + ip$. The third one is denoted by $T^{\text{FW}}_h f$ and is a $L^2$ function on some space denoted $B^2$ equipped with a Gaussian measure $\mu_{B^2,h}$. The properties of the space $B$ and of the measures $\mu_{B,h}$ on $B$ and $\mu_{B^2,h}$ on $B^2$ are recalled in Section 2.1. One may say that spaces $B$ suitable for this construction are Wiener extensions of $H$.

We prove in Theorem 2.12 that, when $f$ and $g$ are in some suitable domain included in $\mathcal{F}_s(H_C)$:

$$T^{\text{FH}}_h I(f,g)(q,p) = T^{\text{FH}}_h f(q,p) T^{\text{FH}}_h g(q,p) \quad (q,p) \in H^2$$ \hspace{1cm} (1.2)

$$T^{\text{FW}}_h I(f,g)(q,p) = T^{\text{FW}}_h f(q,p) T^{\text{FW}}_h g(q,p) \quad a.e. \ (q,p) \in B^2.$$ \hspace{1cm} (1.3)

Therefore, via Segal Bargmann transform, the composition law $I$ is reduced to nothing else than the usual multiplication for functions.

We are now extending the domain of definition of the law $I$.  

\hspace{1cm}
For any $R \geq 1$, denote by $\mathcal{F}_s^R(H_C)$ the subspace of all $f = (f_n)$ in $\mathcal{F}_s(H_C)$ such that the following series is converging:
\[
\|f\|^2_R = \sum_{n \geq 0} R^n \|f_n\|^2 < \infty.
\] (1.4)

In particular, $\mathcal{F}_s^1(H_C) = \mathcal{F}_s(H_C)$.

**Proposition 1.2.** For all $R$, $R'$ and $R''$ satisfying $R \geq 1$, $R' \geq 1$, $R'' \geq 1$ and $(1/R'') = (1/R) + (1/R')$, the composition law $I$ in Definition [74] is extended to a continuous bilinear map from $\mathcal{F}_s^R(H_C) \times \mathcal{F}_s^{R'}(H_C)$ into $\mathcal{F}_s^{R''}(H_C)$. For all $f$ in $\mathcal{F}_s^R(H_C)$ and $g$ in $\mathcal{F}_s^{R'}(H_C)$, we have:
\[
\|I(f, g)\|_{R''} \leq \|f\|_R \|g\|_{R'}.
\] (1.5)

This is proved is Section [75].

We now turn to the scattering identification operator itself. First, let us give the form of this operator, and we shall give more details below. In this theory, we have an infinite dimensional Hilbert space $H$ and a finite dimensional Hilbert space $\mathcal{H}_{sp}$ describing the spins of the particles. Then we are interested in some element $U$ of $\mathcal{F}_s(H_C) \otimes \mathcal{H}_{sp}$. More precisely, for each $R > 1$, $U$ is in $\mathcal{F}_s^R(H_C) \otimes \mathcal{H}_{sp}$. One now chooses a Hermitian basis $S^\lambda$ $(1 \leq \lambda \leq d)$ of $\mathcal{H}_{sp}$. We denote by $U_\lambda$ the components of the ground state $U$ in this basis,
\[
U = \sum_{\lambda=1}^d U_\lambda \otimes S^\lambda.
\] (1.6)

The scattering identification $J$ is defined for all $\varphi$ in $\mathcal{F}_s^R(H_C)$ (with $R > 1$) by:
\[
J_\varphi = \sum_{\lambda=1}^d J_\lambda \varphi \otimes S^\lambda, \quad J_\lambda \varphi = I(U_\lambda, \varphi).
\] (1.7)

By Proposition 1.2, $J$ is bounded from $\mathcal{F}_s^R(H_C)$ into $\mathcal{F}_s(H_C) \otimes \mathcal{H}_{sp}$ for all $R > 1$.

In some spin boson models, an important role is played by some Hamiltonian acting in the Hilbert space $\mathcal{F}_s(H_C) \otimes \mathcal{H}_{sp}$ where $H$ is the configuration space for the one photon particle states, that is to say, the space of all $f$ in $L^2(\mathbb{R}^3, \mathbb{R}^3)$ satisfying $k \cdot f(k) = 0$ almost everywhere for $k \in \mathbb{R}^3$, and $\mathcal{H}_{sp}$ is some finite dimensional space. Some Hamiltonians of this type are defined in [22], [15], [8]. For a model related to NMR, see [5] or [21] in a more physical viewpoint. One shows (see [8] or [22]) that for this Hamiltonian, there exists an eigenfunction $U$ with unit norm, (this fact comes from [13] or [22]), with an eigenvalue located at the bottom of the spectrum of the Hamiltonian. If some coupling constant appearing in the operator is small enough, this ground state is unique up to a constant multiplicative factor. One also shows in [15] that $U$ belongs to $\mathcal{F}_s^R(H_C)$ for every $R \geq 1$. Perhaps these techniques could be applied in other situations.

Our next purpose is to show that the operators $J_\lambda$ are defined by the anti-Wick quantization. First, let us recall the definition.

With some bounded functions $F$ on $H^2$, satisfying suitable conditions, we can usually define an anti-Wick operator $O_{\phi_k}^W(F)$, bounded on $\mathcal{F}_s(H_C)$ and depending of $h > 0$ in the following way. First, we choose a Wiener extension $B$ of $H$. See Theorem [22] below for definition and existence. For each $h > 0$,
this space is provided with a Gaussian measure \( \mu_{B,h} \) with variance \( h \), and \( B^2 \) is provided with a Gaussian measure \( \mu_{B^2,h} \). The definition of the anti-Wick quantization uses one of the three variants of the Segal transformation of Definition 2.3, namely the transformation \( T_{h}^{FW} \). If \( f \) is an element of \( \mathcal{F}_s(H_C) \), \( T_{h}^{FW} f \) is an element of \( L^2(B^2, \mu_{B^2,h}) \). If \( F \) is a bounded function on \( H^2 \), we can define a related anti-Wick operator only if \( F \) has a stochastic extension \( \tilde{F} \), in the sense of Definition 2.4 below, in \( L^2(B^2, \mu_{B^2,h}) \). If \( F \) is bounded then the stochastic extension \( \tilde{F} \) is also bounded. Then, the operator \( Op_{h}^{AW}(F) \) is the unique operator such that, for each \( f \) and \( g \) in \( \mathcal{F}_s(H_C) \):

\[
< Op_{h}^{AW}(F) f, g > = \int_{B^2} \tilde{F}(x) T_{h}^{FW} f(x) \overline{T_{h}^{FW} g(x)}d\mu_{B^2,h}(x) \tag{1.8}
\]

The main result of the first part of this article, proved in Section 2.5, is the following one (with \( h = 1 \)). In the statement, one uses another variant of the Segal Bargmann transformation, denoted by \( T_{h}^{FH} \), also defined in Definition 2.4. For each \( U \) in \( \mathcal{F}_s(H_C) \), \( T_{h}^{FH} U \) is a regular function on \( H^2 \). It has a stochastic extension, which is \( T_{h}^{FW} U \) (Proposition 2.10).

**Theorem 1.3.** If \( U \) is of the form \( I(U) \), with \( U_{\lambda} \) in \( \mathcal{F}_s^R(H_C) \) for all \( R > 1 \), one has:

\[
J_{\lambda} = Op_{1}^{AW}(f_{\lambda}), \quad \text{with} \quad F_{\lambda} = T_{1}^{FH} U_{\lambda}. \tag{1.9}
\]

The specificity of this operator is that \( T_{1}^{FH} U_{\lambda} \) is not bounded. However, the integral (1.8) makes sense if \( f \) is in \( \mathcal{F}_s^R(H_C) \) for some \( R > 1 \), because the product \( \tilde{F}_{\lambda}(x) T_{h}^{FW} f(x) \) is the image by \( T_{h}^{FW} \) of \( I(U_{\lambda}, f) \) (Theorem 2.12), and \( I((U_{\lambda}, f) \) is a well defined element of \( \mathcal{F}_s(H_C) \) by Proposition 1.2. Therefore the operator \( J_{\lambda} \) defined in that way is unbounded: its domain contains the union of the \( \mathcal{F}_s^R(H_C) \).

There are two other ways to make a relation between an operator and a function \( F \) on \( H^2 \). the Weyl and the Wick symbols. Coherent states \( \Psi_{X,h} \) are elements of the Fock space \( \mathcal{F}_s(H_C) \) indexed by \( X = (q,p) \in H^2 \) and depending on \( h > 0 \), defined by:

\[
\Psi_{X,h} = \sum_{m \geq 0} \frac{e^{-|q|^2+|p|^2}}{4\pi^m (2h)^{m/2} \sqrt{m!}} (q+ip) \otimes \cdots \otimes (q+ip). \tag{1.10}
\]

We remind that we call Wick symbol of an operator \( A \) in \( \mathcal{F}_s(H_C) \) with a domain containing the coherent states \( \Psi_{X,h} \) (\( X \in H^2 \)), the following function defined on \( H^2 \) by the product is linear with respect to the first variable:

\[
\sigma_{h}^{Wick}(A) = < A \Psi_{X,h}, \Psi_{X,h} >. \tag{1.11}
\]

For the Weyl symbol, see \([2]\) and \([3]\). Let us denote by \( \sigma_h \text{Weyl}(A) \) and \( \sigma_h \text{Wick}(A) \) the symbols of an operator \( A = Op_{h}^{AW}(F) \), for a suitable given function \( F \) on \( H^2 \). When \( F \) is a suitable function, (see below), let \( Op_{h}^{Weyl}(F) \) be the corresponding Weyl operator. The relation between these three functions is given by the heat operator.

The heat operator is defined for each measurable bounded function \( F \) in \( H^2 \) admitting a stochastic extension \( \tilde{F} \) in \( L^1(B^2, \mu_{B,h}) \) (see Definition 2.4) and for each \( h > 0 \) by:

\[
(H_h F)(X) = \int_{B^2} \tilde{F}(X+Y)d\mu_{B^2,h}(Y) \tag{1.12}
\]
for $X \in H^2$.

It is proved in [3] (formula (2.13)) that, if $A = O_{h}^{AW}(F)$, where the function $F$ on $H^2$ must have a stochastic extension $\tilde{F}$, as we saw:

$$\sigma_{h}^{\text{Wick}}(A) = H_{h}F$$

It is proved in [4] (formula (33)) that, for each suitable function $G$ admitting a stochastic extension $\tilde{G}$:

$$\sigma_{h}^{\text{Weyl}}\left(\text{Op}_{h}^{\text{weyl}}(G)\right)(X) = H_{h/2}G(X), \quad X \in H^2.$$  \hfill (1.13)

Therefore, if $A = O_{h}^{AW}(F)$, we have also:

$$\sigma_{h}^{\text{Weyl}}(A) = H_{h/2}G$$

The function $F_{\lambda} = T_{1}^{FH}U_{\lambda}$ is Gateaux holomorphic (Proposition 2.6). Since any holomorphic function remains fixed by this operator then these three symbols of $J_{\lambda}$ are equal.

The product $I$ of two coherent states, defined in (1.10) is another formula that can be useful. We prove the following equality in Section 2.5, for all $X$ and $Y$ in $H^2$:

$$I(\Psi_{X,h}, \Psi_{Y,h}) = e^{\frac{1}{2}X \cdot Y} \Psi_{X+Y,h}.$$  \hfill (1.14)

In the second part of this article, we consider Wick symbols and Section 3 is devoted to the Wick symbol for compositions. We remind that we call Wick symbol of an operator $A$ in $\mathcal{F}_s(H_{\mathbb{C}})$ with a domain containing the coherent states $\Psi_{X,h}$ ($X \in H^2$), the function defined on $H^2$ by (1.11).

In Section 3, we shall state and prove a formula giving the Wick symbol of the composed $A \circ B$ of two bounded operators $A$ and $B$ in $\mathcal{F}_s(H_{\mathbb{C}})$, as the sum of an absolutely convergent series. In finite dimension, this is a result of Mizrahi [19] and Appleby [6].

In the third part of this work, in Section 4 we study the composition for the Weyl quantization. The Weyl calculus is one of the possibilities to define an operator in the Fock space $\mathcal{F}_s(H_{\mathbb{C}})$, depending on a parameter $h > 0$ and denoted by $O_{h}^{\text{weyl}}(F)$, with a suitable function $F$ defined in the space $H^2$. The simplest way to define this operator, at least implicitly, is to give its Wick symbol. In order to define the Weyl operator $O_{h}^{\text{weyl}}(F)$ and in order for it to be bounded, it is necessary that $F$ has stochastic extension $F$ in $B^2$ (in the sense of the Definition 2.9). Then one has:

This step is described with more details in [2] and [4].

We now recall a set of functions $F$ for which this calculus can be applied. See [3] and [11].

**Definition 1.4.** For each real separable Hilbert space $H$ and for each nonnegative quadratic form $Q$ on $H^2$, let $S(H^2,Q)$ be the class of functions $f \in C^\infty(H^2)$ such that there exists $C(f) > 0$ satisfying, for any integer $m \geq 0$,

$$|(d^m f)(x)(U_1, \ldots, U_m)| \leq C(f) Q(U_1)^{1/2} \ldots Q(U_m)^{1/2}.$$ \hfill (1.15)

The smallest constant satisfying (1.15) is denoted by $\|f\|_Q$. 

5
We denote by $A_Q$ the linear map in $H$ satisfying:

$$Q(X) = (A_QX) \cdot X$$

(1.16)

with $A_Q \in \mathcal{L}(H^2)$, $A_Q = A_Q^*$, $A_Q$ nonnegative. In what follows, $A_Q$ will also be trace class in $H^2$. The idea of defining a class of symbols in this way, in a quantization purpose, with the help of a quadratic form on the phase space, goes back to Hörmander [14] and Unterberger [24].

If $A_Q$ is trace class then any function $F$ lying in $S(H^2, Q)$ has a stochastic extension (in the sense of Definition 2.9), (see [16]), and one can associate with it a bounded operator $Op_{Weyl}^h(F)$ following [2] and [4]. This operator satisfies (1.13). For every function $F$ belonging to $S(H^2, Q)$, the heat operator can also be written as:

$$H_{h/2} F = \sum_{k=0}^{\infty} \frac{(h/4)^k}{k!} \Delta^k F.$$  (1.17)

The series is convergent and defines an element of $S(H^2, Q)$.

One shows in [4] that, if $F$ and $G$ are in $S(H^2, Q)$ and if $A_Q$ is trace class then the composition $Op_{Weyl}^h(F) \circ Op_{Weyl}^h(G)$ can be written as $Op_{Weyl}^h(K_{Weyl}^h(F, G))$ with $K_{Weyl}^h(F, G)$ belonging to $S(H^2, 4Q)$. The purpose of Section 4 (Theorem 4.1) is to give an expression of $K_{Weyl}^h(F, G)$ written as an absolutely convergent series.

2 Anti-Wick quantization and the scattering identification operator.

2.1 Wiener extension and Segal isomorphism.

Let us recall how the Fock space $F_s(H_C)$ is made isomorphic to some $L^2$ space for any Hilbert space $H$.

**Theorem 2.1.** (Gross [2], [22], Kuo [18]). Let $H$ be a real separable Hilbert space. Then, there exists a (non unique) Banach space $B$ containing $H$, such that $B' \subset H' = H \subset B$, each space being dense in the following, and for all $h > 0$, there exists a probability measure $\mu_{B,h}$ on the Borel $\sigma$-algebra of $B$ satisfying,

$$\int_B e^{\imath a(x)} d\mu_{B,h}(x) = e^{-\frac{h}{2} |a|^2}, \quad a \in B'.$$  (2.1)

Here, $|a|$ denotes the norm in $H$ and the notation $a(x)$ stands for the duality between $B'$ and $B$.

The exact conditions to be fulfilled by $B$ together with the properties involved in this paper are recalled in [2]. In the following, we say that $B$ is a Wiener extension of $H$. Then, $B^2$ is a Wiener extension of $H^2$.

We recall [18] that, for all $a$ in $B' \subset H$, the mapping $B \ni a \mapsto a(x)$ belongs to $L^2(B, \mu_{B,h})$, with a norm equal to $h^{\frac{1}{2}} |a|$. Thus, the mapping associating with every $a \in B'$, the above function considered as an element of $L^2(B, \mu_{B,h})$, can be extended by density to a mapping $a \mapsto \ell_a$ from $H$ into $L^2(B, \mu_{B,h})$. 6
We now recall the Segal isomorphism from $\mathcal{F}_s(H_C)$ to $L^2(B, \mu_{B,h})$.

One chooses a Hilbertian basis $\{e_j\}$ of $H_C$. For all multi-indices $\alpha$ (a map from $\mathbb{N}$ into $\mathbb{N}$ with $\alpha_j = 0$ except for a finite number of indices), one defines an element $u_\alpha$ of $\mathcal{F}_s(H_C)$ by:

$$u_\alpha = (\alpha!)^{-1/2} \prod_j \left( a^*(e_j) \right)^{\alpha_j} \Psi_0 = (\alpha!)^{-1/2}(a^*(e))^{\alpha}\Psi_0$$  \hspace{1cm} (2.2)

where $\Psi_0$ is the vacuum state. We know that the $u_\alpha$ constitute a Hilbertian basis of $\mathcal{F}_s(H_C)$. We also have:

$$u_\alpha = (\alpha!)^{-1/2} \sqrt{|\alpha|!} \Sigma \left( \bigotimes_j e_j^{\alpha_j} \right)$$

where $\Sigma$ denotes the orthogonal projection from the Fock space $\mathcal{F}(H_C)$ to the symmetrization $\mathcal{F}_s(H_C)$.

**Definition 2.2.** One denotes by $S_h$ the continuous linear mapping from $\mathcal{F}_{\text{fin}}(H_C)$ into $L^2(B, \mu_{B,h})$ satisfying for every multi-indices $\alpha$:

$$S_h u_\alpha = \prod_{\alpha_j \neq 0} H_{\alpha_j} \left( \ell_{e_j}(x) \right) \sqrt{h}$$  \hspace{1cm} (2.3)

where the $H_n$ are the Hermite polynomials on $\mathbb{R}$ (chosen as an orthonormal system in $L^2(\mathbb{R}, \mu_{\mathbb{R}, 1})$).

One shows (see [17]) that $S_h$ is extended to an unitary isomorphism (Segal isomorphism) from $\mathcal{F}_s(H_C)$ into $L^2(B, \mu_{B,h})$.

### 2.2 Segal Bargmann transform.

Let $H$ be a separable real Hilbert space of infinite dimension. Let $\mathcal{F}_s(H_C)$ be the associated Fock space. We now define three Segal Bargmann transforms for any element in $\mathcal{F}_s(H_C)$. The first one is an element of $\mathcal{F}_s(H_C \times H_C)$ and is denoted by $T_{HF}^F f$. The second one is a function on $H^2$ and is denoted by $T_{Fh}^{FW} f$. The third transform, denoted by $T_{Fh}^{FW} f$, is an element of $L^2(B^2, \mu_{B^2,h})$ where $B$ is a Wiener extension of $H$ (see Section 2.1) and $\mu_{B^2,h}$ is the Wiener measure with variance $h$ on the Borel $\sigma$–algebra of $B^2$.

The connection between the two first transforms is realized, in some sense, with the stochastic extension notion. The relation between the last two transforms is carried out with the reproducing kernel (see Section 2.4).

Coherent states (1.10) are involved in order to define $T_{HF}^F f$. For the sake of clarity, in notations, we dissociate the notions concerning the Fock space $\mathcal{F}_s(H_C \times H_C)$ and those for $\mathcal{F}_s(H_C)$ by using an index or an exponent 2. In particular, with $\mathcal{F}_s(H_C \times H_C)$, the scalar product is denoted by $\langle \cdot, \cdot \rangle_2$, the functor $\Gamma$ is written as $\Gamma_2$ and $S_{h,2}$ stands for the Segal isomorphism. Recall that it is linear with respect to the left variable.

**Definition 2.3.** Let $H$ be a real separable space of infinite dimension.

i) The mapping $T_{HF}^F$ acting from $\mathcal{F}_s(H_C)$ into $\mathcal{F}_s(H_C \times H_C)$ is defined by:

$$T_{HF}^F = \Gamma(U_\pm) \quad U_\pm(X) = (1/\sqrt{2})(X, \pm iX) \quad X \in H_C.$$  \hspace{1cm} (2.4)
Proposition 2.4.  i) The operator $T^F H$ is a partial isometry from $F_s(H_C)$ into $F_s(H_C \times H_C)$. For all $R > 1$ and for any $f$ in $F_s^H(H_C)$, we have with the notation (2.3):
\[
\|T^F H f\|_{R,2} = \|f\|_R
\]

ii) The map $T^F H$ is a partial isometry from $F_s(H_C)$ into $L^2(B^2, \mu_{B^2,h})$.

The next Proposition is used Section 2.3.

Proposition 2.5.  For any $V$ in $H$ being identified with the element $(V, 0)$ and for all $X = (q, p)$ in $H^2$, we have:
\[
T^F H (a^*(V)f)(q, p) = \frac{1}{\sqrt{2h}} (V \cdot q - iV \cdot p)(T^F H f)(q, p), \tag{2.8}
\]
\[
T^F H (a(V)f)(q, p) = \sqrt{h} \left( V \cdot (\partial_q + i\partial_p) \right) (T^F H f)(q, p). \tag{2.9}
\]

Proof. Let us set $(q + ip)^{\otimes m} = (q + ip) \otimes \cdots \otimes (q + ip)$ ($m$ times). Using the formulas (X.63) of [20], we have:
\[
a(V)(q + ip)^{\otimes m} = \sqrt{m} \left( V \cdot (q + ip) \right) (q + ip)^{\otimes m-1},
\]
\[
a^*(V)(q + ip)^{\otimes m} = \frac{1}{\sqrt{m+1}} \sum_{k=0}^{m} (q + ip)^{\otimes k} \otimes V \otimes (q + ip)^{\otimes m-k}.
\]

In other words:
\[
V \cdot (\partial_q - i\partial_p)(q + ip)^{\otimes m} = 2\sqrt{m} a^*(V)(q + ip)^{\otimes m-1}.
\]

By (1.10), it follows that:
\[
a(V)\Psi_{q,p,h} = \frac{1}{\sqrt{2h}} V \cdot (q + ip)\Psi_{q,p,h}
\]
and also:
\[
a^*(V)\Psi_{q,p,h} = \sqrt{\frac{h}{2}} e^{-(|q|^2 + |p|^2)/4h} V \cdot (\partial_q - i\partial_p) \left( e^{(|q|^2 + |p|^2)/4h} \Psi_{q,p,h} \right).
\]

Equalities (2.8) and (2.9) follow easily by (2.5).
2.3 Power series expansions.

We give here the power series expansions of the two functions \( T^{FH}_h f \) and \( T^{FW}_h f \) with two different convergence senses and for any \( f \) in \( \mathcal{F}_s(H_C) \).

Fix a Hilbertian basis \((e_j)\) of \( H \). Then define the basis \((u_\alpha)\) of \( \mathcal{F}_s(H_C) \) as in (2.2). For each multi-index \( \alpha \), also define a function \( \Phi_{\alpha,h} \) on \( H^2 \) by:

\[
\Phi_{\alpha,h}(q,p) = (2h)^{-|\alpha|/2}(\alpha!)^{-1/2}\prod_j (e_j \cdot (q - ip))^{\alpha_j}.
\] (2.10)

The convergence concerning the expansion of the function \( T^{FH}_h f \) is pointwise and not absolute due to the infinite dimensional setting. Let \( \mathcal{M} \) be the set of all multi-indices \( \alpha \) and let \((I_N)\) be an increasing sequence of finite subsets of \( I \) with union \( \mathcal{M} \).

**Proposition 2.6.** i) For any \( f \) in \( \mathcal{F}_s(H_C) \) and for all \( X \) in \( H^2 \), we have:

\[
(T^{FH}_h f)(X) = \lim_{N \to \infty} \sum_{\alpha \in I_N} \langle f, u_\alpha \rangle \Phi_{\alpha,h}(X).
\] (2.11)

ii) The function \( T^{FH}_h f \) is Gateaux anti-holomorphic if \( H^2 \) is identified with \( H_C \) (when identifying \( X = (q,p) \) with \( q + ip \)). That is, for any complex subspace of \( H_C \) of finite dimension, the restriction of \( T^{FH}_h f \) to \( E \) is anti-holomorphic.

iii) For any square-summable sequence \((a_\alpha)\), there is a function \( F \) on \( H^2 \) satisfying for all \( X \) in \( H^2 \)

\[
F(X) = \lim_{N \to \infty} \sum_{\alpha \in I_N} a_\alpha \Phi_{\alpha,h}(X).
\] (2.12)

Moreover, there exists \( f \) in \( \mathcal{F}_s(H_C) \) satisfying \( T^{FH}_h f = F \).

**Proof.** Let us first prove that :

\[
T^{FH}_h u_\alpha(X) = e^{\frac{|X|^2}{4h}} \langle u_\alpha, \Psi_{X,h} \rangle = \Phi_{\alpha,h}(X) \quad X = (q,p) \in H^2.
\] (2.13)

This point is proved by induction on \(|\alpha|\). For \( \alpha = 0 \), namely for \( u_0 = \Psi_0 \) and \( \Phi_{0,h} = 1 \), it is a consequence of (2.3). It is then obtained by iteration using (2.8).

Concerning point i), set for any integers \( N \):

\[
f_N = \sum_{\alpha \in I_N} \langle f, u_\alpha \rangle u_\alpha.
\] (2.14)

According to (2.13), one has for any \( X \) in \( H^2 \):

\[
\sum_{\alpha \in I_N} \langle f, u_\alpha \rangle \Phi_{\alpha,h}(X) = e^{\frac{|X|^2}{4h}} \langle f_N, \Psi_{X,h} \rangle.
\]

Since \( f_N \) converges to \( f \) in \( \mathcal{F}_s(H_C) \) then point i) follows, for any fixed \( X \).
ii) Let $E$ be a complex subspace of $H_C$ of finite dimension $m$. Set $(w_1, ..., w_m)$ a complex orthonormal basis of $E$. Thus, any element $Z$ of $E$ is written as $Z = z_1w_1 + \cdots + z_mw_m$, with $z_j$ in $\mathbb{C}$. Thanks to (2.10) and (2.11), we can then write, for all $Z$ in $E$:

$$T^F_H f(Z) = \sum c_\gamma z_\gamma$$

where the series in the right hand side is converging for all $z$ in $\mathbb{C}^m$. The anti-holomorphic property of the restriction of $T^F_H f$ to $E$ then follows.

iii) If the sequence $(a_\alpha)$ is square-summable, let $f_N$ be the element of $\mathcal{F}_s(H_C)$ defined by:

$$f_N = \sum_{\alpha \in I_N} a_\alpha u_\alpha.$$ 

The sequence $(f_N)$ tends to an element $f$ of $\mathcal{F}_s(H_C)$ and the above reasoning shows that:

$$\lim_{N \to \infty} \sum_{\alpha \in I_N} a_\alpha \Phi_{\alpha,h}(X) = T^F_H f(X).$$

□

Let us now consider the series expansion of $T^F_H f$. One defines a function $\tilde{\Phi}_{\alpha,h}$ almost everywhere on $B^2$ by:

$$\tilde{\Phi}_{\alpha,h}(q, p) = (2h)^{-|\alpha|/2}(\alpha!)^{-1/2} \prod (\ell_{e_j}(q) - i\ell_{e_j}(p))^{\alpha_j}.$$ (2.15)

The functions $\tilde{\Phi}_{\alpha,h}$ constitute an orthonormal system in $L^2(B^2, \mu_{B^2,h})$.

**Proposition 2.7.** One has:

$$T^F_W u_\alpha = \tilde{\Phi}_{\alpha,h}. \quad (2.16)$$

**Proof.** If $(e_j)$ is a Hilbertian basis of $H$ then we use the Hilbertian basis of $H^2$ constituted of the $(e_j, 0)$ and $(0, e_j)$. The elements of $\mathcal{F}_s(H_C \times H_C)$ constructed with this basis defined as in (2.2) are denoted by $u_{\alpha \beta}$. We observe that:

$$\Gamma(U_-)u_\alpha = 2^{-|\alpha|/2} \sum_{\beta+\gamma=\alpha} \sqrt{\beta! \gamma!} (-i)^{|\gamma|} u_{\beta \gamma}. $$

According to the Segal isomorphism definition in (2.3), considering $(q,p)$ as the variable of $B^2$ and setting $\ell_{(e_j, 0)}(q,p) = \ell_{e_j}(q)$ and $\ell_{(0,e_j)}(q,p) = \ell_{e_j}(p)$, one checks:

$$S_{h,2}(u_{\beta \gamma})(q,p) = \prod_{j, \beta_j + \gamma_j \neq 0} H_{\beta_j} \left( \ell_{e_j}(q)/\sqrt{h} \right) H_{\gamma_j} \left( \ell_{e_j}(p)/\sqrt{h} \right)$$

where the $H_k$ are the Hermite polynomials chosen being an orthonormal system in $L^2(\mathbb{R}, \mu_{\mathbb{R},1})$. Then we use the following identity, probably standard, for all $(x,y)$ in $\mathbb{R}^2$:

$$\frac{(x - iy)^m}{\sqrt{m!}} = \sum_{p+q=m} \sqrt{m! / p! q!} (-i)^q H_q(x)H_q(y).$$

The proof of the Proposition is then completed. □

The result below will be used in Section 3.
Proposition 2.8. For any $f$ and $g$ in $\mathcal{F}_s(H_{\mathbb{C}})$, one has in the sense of the $L^2(B^2, \mu_{B^2,h})$ Hilbert space convergence:

$$T_h^{FW} f = \lim_{N \to \infty} \sum_{\alpha \in I_N} < f, u_\alpha > \tilde{\Phi}_{\alpha,h}$$

(2.17)

$$< f, g >= \sum < T_h^{FW} f, \tilde{\Phi}_{\alpha,h} >_h < \tilde{\Phi}_{\alpha,h}, T_h^{FW} g >_h$$

(2.18)

where the series is absolutely converging.

The above scalar product is the canonical $L^2(B^2, \mu_{B^2,h})$ scalar product.

Again, we emphasize that the convergence in (2.17) is the $L^2(B^2, \mu_{B^2,h})$ Hilbert space convergence whereas it is a pointwise convergence in (2.11), for all $X$ in $H^2$.

Proof. Let $f_N$ be the element defined in (2.14). One notices in view of (2.16) that:

$$T_h^{FW} f_N = \sum_{\alpha \in I_N} < f, u_\alpha > \tilde{\Phi}_{\alpha,h}.$$  

According to point ii) in Proposition 2.4, one sees that $T_h^{FW} f_N$ tends to $T_h^{FW} f$ in $L^2(B^2, \mu_{B^2,h})$. One then gets equality (2.17). The second equality therefore holds true since the set of functions $\tilde{\Phi}_{\alpha,h}$ is an orthonormal system in $L^2(B^2, \mu_{B^2,h})$.

2.4 Connection between $T_h^{FH} f$ and $T_h^{FW} f$.

In the finite dimensional framework, the Bargmann transform of a function $f \in L^2(\mathbb{R}^n)$ is a function belonging to $L^2(\mathbb{R}^{2n})$ (with suitable measures) holomorphic when $\mathbb{R}^{2n}$ is identified with $\mathbb{C}^n$. Here, in infinite dimension, the function $T_h^{FH} f$ is Gateaux anti-holomorphic when $(q, p)$ is identified with $q + ip$ (Proposition 2.6) and $T_h^{FW} f$ belongs to an $L^2$ space on a suitable $B^2$ for a convenient mesure. One then should specify the relation between these two functions. We underline that one cannot consider the restriction to $H^2$ of a measurable function defined on $B^2$ since the measure of $H^2$ in $B^2$ is zero.

We prove that $T_h^{FW} f$ is a stochastic extension of $T_h^{FH} f$ in the sense of the definition below.

With each finite dimensional space $E \subset H$, one can associate a map $\pi_E : B \to E$ defined almost everywhere by:

$$\pi_E(x) = \sum_{j=1}^{\dim(E)} \ell_{u_j}(x) u_j$$

where the $u_j (1 \leq j \leq \dim(E))$ form an orthonormal basis of $E$ and $\ell_{u_j}$ is defined in Section 2.1. This map does not depend on the choice of the orthonormal basis.

Definition 2.9. Let $H$ be a separable real Hilbert space of infinite dimension. Set $B$ a Wiener extension of $H$ and set $\mu_{B,h}$ the measure defined in Section 2.1. One says that a measurable function $f$ on $H$ admits a stochastic extension in $L^p(B, \mu_{B,h})$ ($1 \leq p < \infty$) if the sequence of functions $F \circ \pi_{E_n} : B \to \mathbb{C}$ is a Cauchy sequence in $L^p(B, \mu_{B,h})$ for any increasing sequence of finite dimensional complex subspaces $(E_n)$ of $H$ having a dense union in $H$. The limit, which is independent of the sequence $(E_n)$, is called the stochastic extension of $F$ and is here often denoted by $\tilde{F}$.
We also prove in the following that $T^F \mathcal{H} f$ is the image of $T^F \mathcal{W} f$ using the reproducing kernel defined below.

For all $X = (q,p)$ in $H^2$, one defines a function $\ell_X$ almost everywhere on $B^2$ by $\ell_X(Y) = \ell_{q-\imath p}(y) + \imath \ell_{q-\imath p}(\eta)$ with $Y = (y, \eta)$. One also defines almost everywhere a function $Y = (y, \eta) \to B_h(X, Y)$ by:

$$B_h(X, Y) = e^{\frac{\imath}{2h} \ell_X(Y)}.$$ 

One knows that the function $Y \to B_h(X, Y)$ belongs to $L^2(B^2, \mu_{B^2})$ for any $X$ in $H^2$ in such a way that, for all $F$ in $L^2(B^2, \mu_{B^2})$, the following integral called reproducing kernel:

$$(B_h F)(X) = \int_{B^2} B_h(X, Y) F(Y) d\mu_{B^2}(Y)$$

defines a function $B_h F$ on $H^2$.

The link between $T^F \mathcal{H} f$ and $T^F \mathcal{W} f$ is given by the proposition below.

**Proposition 2.10.** i) There exists a stochastic extension in $L^2(B^2, \mu_{B^2})$ of the function $T^F \mathcal{H} f$ defined on $H^2$, for all $f \in \mathcal{F}_s(H_C)$. This stochastic extension is $T^F \mathcal{W} f$.

ii) One has,

$$(T^F \mathcal{H} f)(X) = \int_{B^2} B_h(X, Y) T^F \mathcal{W} f(Y) d\mu_{B^2}(Y)$$

for every $f$ in $\mathcal{F}_s(H_C)$ and for all $X$ in $H^2$.

The proof of point i) relies on the next theorem, which is a variant of Theorem 8.8 in [2] adapted to our current situation.

**Theorem 2.11.** Let $H$ be an infinite dimensional separable complex Hilbert space and set $B$ a Wiener extension of $H$. The space $B$ is considered as a real space. Fix a continuous function $F$ defined on $H$. We assume that:

i) The function $F$ restricted to $E$ is anti-homomorphic where $E$ is any finite dimensional complex subspace of $H$.

ii) For every finite dimensional complex subspace $E$ of $H$, the function $F$ restricted to $E$ belongs to $L^2(E, \mu_{E,h})$ and its norm is bounded independently of $E$ ($\mu_{E,h}$ denotes the Gaussian measure with variance $h$ on $E$ considered as a real space).

Then, there exists $\tilde{F}$ a stochastic extension in $L^2(B, \mu_{B,h})$ of the function $F$ (in the sense of Definition 2.9). Moreover, we have:

$$\|\tilde{F}\|_{L^2(B, \mu_{B,h})} \leq \sup_E \|R_E F\|_{L^2(E, \mu_{E,h})}$$

where $R_E F$ stands for $F$ restricted to $E$ and where the supremum is running over all the finite dimensional complex subspaces $E$ in $H$.

**Proof of Proposition 2.10** i) For all $f$ in $\mathcal{F}_s(H_C)$, we know (Proposition 2.6) that the function $T^F \mathcal{H} f$, defined on $H^2$ is identified to a function $F$ being Gateaux anti-holomorphic on $H_C$ when $H^2$ and $H_C$ are
identified. One also knows that this function satisfies hypothesis ii) of Theorem 2.11 and, for all complex subspaces $E$ of $HC$:

$$\|R_E F\|_{L^2(E, \mu_{E,h})} \leq \|f\|.$$  

This fact comes from the standard properties of the Segal Bargmann transform in finite dimension. According to Theorem 2.11, $T_h^{FW} f$ has a stochastic extension $\tilde{F}$ in $L^2(B^2, \mu_{B^2,h})$. Let $(f_N)$ be a sequence in $F_s(H_C)$ converging to $f$ where each $f_N$ is a linear combination of the $u_\alpha$. From (2.13) and (2.16), $T_h^{FW} f_N$ is the stochastic extension of $T_h^{FH} f_N$. The sequence $T_h^{FW} f_N$ tends to $T_h^{FW} f$ in $L^2(B^2, \mu_{B^2,h})$. For all complex subspaces $E$ of finite dimension in $H^2$, one has:

$$\|R_E T_h^{FH} (f_N - f)\|_{L^2(E, \mu_{E,h})} \leq \|f_N - f\|.$$  

Consequently, in view of Theorem 2.11, the sequence of the $T_h^{FW} f_N$, stochastic extensions of the $T_h^{FH} f_N$, converges to the stochastic extension of $T_h^{FH} f$, which is therefore equal to $T_h^{FW} f$.

ii) With the above notations, the Segal Bargmann standard properties within the finite dimensional framework show that, for all $X$ in $H^2$ and for any $N$:

$$T_h^{FH} f_N(X) = \int_{B^2} B_h(X,Y) T_h^{FW} f_N(Y) d\mu_{B^2,h}(Y).$$

From Definition (2.13), for all $X$ in $H^2$, the sequence $T_h^{FH} f_N(X)$ converges to $T_h^{FH} f(X)$. Besides, the sequence $(T_h^{FW} f_N)$ tends vers $T_h^{FW} f$ in $L^2(B^2, \mu_{B^2,h})$ and we consequently have, since the function $Y \to B_h(X,Y)$ belongs to $L^2(B^2, \mu_{B^2,h})$:

$$\lim_{N \to \infty} \int_{B^2} B_h(X,Y) T_h^{FW} (f_N - f)(Y) d\mu_{B^2,h}(Y) = 0.$$  

One then obtains (2.19). \hfill \Box

### 2.5 Segal Bargmann transform and the product law $I$.

We first start with:

**Proof of Proposition 2.2.** We consider the Hilbertian basis $(u_\alpha)$ of (2.2), and equality (2.22). Setting $\varphi = I(f,g)$, we have, for all multi-indices $\gamma$:

$$< \varphi, u_\gamma > = \sum_{\alpha+\beta=\gamma} < f, u_\alpha > < g, u_\beta > \left[ \frac{\gamma!}{\alpha!\beta!} \right]^{1/2}.$$  

From Cauchy-Schwarz:

$$(R'')^{\gamma} | < \varphi, u_\gamma > |^2 \leq \sum_{\alpha+\beta=\gamma} R^\alpha | < f, u_\alpha > |^2 (R')^\beta | < g, u_\beta > |^2 \sum_{\alpha+\beta=\gamma} \frac{\gamma!}{\alpha!\beta!} \frac{(R'')^{\gamma}}{R^{\alpha}(R')^\beta}.$$  

If $(1/R'') = (1/R) + (1/R')$ then the latter sum equals to 1. One verifies that $f$ belongs to $F_s^R(H_C)$ if and only if

$$\|f\|_{F_s}^2 = \sum | < f, u_\alpha > |^2 R^\alpha < \infty.$$  

Therefore, the Proposition is proved. \hfill \Box
Theorem 2.12. i) For all \( h > 0 \), for any \( f \) in \( \mathcal{F}^R_s(H_C) \) and \( g \) in \( \mathcal{F}^{R'}_s(H_C) \), \( (R \geq 1, R' \geq 1, (1/R) + 1/R' \leq 1) \), one has:
\[
T_h^{FH} I(f, g) = (T_h^{FH} f) (T_h^{FH} g)
\]
where the composition in the right hand side refers to the ordinary product of functions defined on \( H^2 \).

ii) One also has, for any Wiener extension \( B \) of \( H \):
\[
T_h^{FW} I(f, g)(X) = (T_h^{FW} f)(X) (T_h^{FW} g)(X) \quad \text{a.e.} X \in B^2.
\]

Proof. Point i) We use the Hilbertian basis \((u_\alpha)\) of \( H^2 \). From (2.22) and (1.1), one sees:
\[
I(u_\alpha, u_\beta) = \left( \frac{(\alpha + \beta)!}{\alpha!\beta!} \right)^{1/2} u_{\alpha+\beta}.
\]
Therefore, if \( f \) and \( g \) belong to \( \mathcal{F}^\infty_s(H_C) \) (defined in Section 1),
\[
T_h^{FH} I(f, g)(X) = \sum_{\alpha, \beta} \langle f, u_\alpha \rangle < \varphi, u_\alpha > \langle g, u_\beta \rangle \left( \frac{(\alpha + \beta)!}{\alpha!\beta!} \right)^{1/2} T_h^{FH} u_{\alpha+\beta}(X).
\]
According to Proposition 2.6
\[
T_h^{FH} f(X) = \sum_\alpha < f, u_\alpha > \Phi_{\alpha,h}(X).
\]
From (2.10), one verifies:
\[
\sqrt{\alpha + \beta)!\Phi_{\alpha+\beta,h}(X) = \sqrt{\alpha!\Phi_{\alpha,h}(X)} \sqrt{\beta!\Phi_{\beta,h}(X)}.
\]

Point ii) One knows by Proposition 2.11 that \( T_h^{FH} I(f, g), T_h^{FH} f \) and \( T_h^{FH} g \) have the stochastic extensions \( T_h^{FW} I(f, g), T_h^{FW} f \) and \( T_h^{FW} g \) in the \( L^2(B^2, d\mu_{B^2,h}) \) sense. Let \((E_n)\) be an increasing sequence of finite dimensional subspaces of \( H \), with a dense union. Since \( T_h^{FH} f \circ \tilde{\pi}_{E_n} \) and \( T_h^{FH} g \circ \tilde{\pi}_{E_n} \) converge to \( T_h^{FW} f \) and \( T_h^{FW} g \) in \( L^2(B^2, d\mu_{B^2,h}) \), the product \( T_h^{FH} f T_h^{FH} g \circ \tilde{\pi}_{E_n} \) converges to \( T_h^{FW} f T_h^{FW} g \) in \( L^1(B^2, d\mu_{B^2,h}) \). The convergence is almost sure for a subsequence indexed by \( \varphi(n) \). By point i), \((T_h^{FH} f T_h^{FH} g) \circ \tilde{\pi}_{E_\varphi(n)} = T_h^{FH} I(f, g) \circ \tilde{\pi}_{E_\varphi(n)} \). Moreover, it converges to \( T_h^{FW} I(f, g) \) in \( L^2 \). Extracting a further subsequence gives an almost sure convergence and the equality \( T_h^{FW} I(f, g) = (T_h^{FW} f)(T_h^{FW} g) \) a.e.

\[\square\]

Proof of Theorem 1.3. We prove equality (1.9). We use Definition (1.8). We have, for all \( f \) in \( \mathcal{F}^R_s(H_C) \) \( (R > 1) \) and for all \( g \) in \( \mathcal{F}_s(H_C) \), according to Proposition 2.4 and Definition (1.7):
\[
< J_\lambda f, g > = \int_{B^2} (T_1^{FW} I(U_\lambda, f))(x) \overline{(T_1^{FW} g)(x)} d\mu_{B^2,1}(x).
\]
In view of Theorem 2.12
\[
< J_\lambda f, g > = \int_{B^2} (T_1^{FW} U_\lambda)(x) (T_1^{FW} f)(x) \overline{(T_1^{FW} g)(x)} d\mu_{B^2,1}(x).
\]
From Theorem 2.10 the function $T^F_{1}U_\lambda$ is indeed the stochastic extension of $F_\lambda = T^F_{1}U_\lambda$, which formally proves equality (1.18). The convergence of the integral is handled by the fact that, for all $f$ in $F^R_s(H_C)$ ($R > 1$), if $R'$ satisfies $(1/R) + (1/R') = 1$ then we know that $U_\lambda$ is in $F^R_{s'}(H_C)$. From Proposition 1.2, $I(U_\lambda, f)$ is well defined as an element in $F_s(H_C)$. Following Proposition 2.4, $T^F_{1}I(U_\lambda, f)$ is well defined as an element of $L^2(B^2, \mu_{BZ, 1})$, as $T^F_{1}w^g$, ensuring the convergence of the integral.

\[ \square \]

Proof of equality (1.14). We can write, if $X = (q, p)$:

\[ \Psi_{X, h} = \sum_{m \geq 0} e^{-\frac{|X|^2}{2\hbar}} (a^*(q + ip))^m \Psi_0. \]

One checks that the coherent states belong to $F^R_s(H_C)$ for every $R > 1$. Let $\Psi^N_{X, h}$ be a truncated coherent state, the sum running on $\{m \leq N\}$ instead of $\mathbb{N}$.

Proposition 1.2 with $R = R' = 2$ shows that $I(\Psi^N_{X, h}, \Psi^N_{Y, h})$ converges to $I(\Psi_{X, h}, \Psi_{Y, h})$ in $F^1_s(H_C) = F_s(H_C)$. The computation below shows that $I(\Psi^N_{X, h}, \Psi^N_{Y, h}) = \Psi^N_{X+Y, h} + V_N$ and one easily proves that the rest $V_N$ converges to 0 when $N$ converges to infinity.

\[ I(\Psi^N_{X, h}, \Psi^N_{Y, h}) = \sum_{0 \leq m, n \leq N} e^{-\frac{|X|^2 + |Y|^2}{4\hbar}} I((a^*(q + ip))^m \Psi_0, (a^*(q' + ip'))^n \Psi_0) \]

\[ = \sum_{0 \leq m, n \leq N} e^{-\frac{|X|^2 + |Y|^2}{4\hbar}} (a^*(q + ip))^m (a^*(q' + ip'))^n \Psi_0 \]

\[ = e^{-\frac{|X|^2 + |Y|^2}{4\hbar}} \sum_{s=0}^{N} \frac{1}{s!(2\hbar)^{s/2}} \sum_{m=0}^{s} \frac{s!}{m!(s-m)!} (a^*(q + ip))^m (a^*(q' + ip'))^{s-m} \Psi_0 + \ldots \]

\[ \ldots + e^{-\frac{|X|^2 + |Y|^2}{4\hbar}} \frac{1}{s!(2\hbar)^{s/2}} \sum_{m=s-N}^{N} \frac{s!}{m!(s-m)!} (a^*(q + ip))^m (a^*(q' + ip'))^{s-m} \Psi_0 \]

\[ = e^{-\frac{|X|^2 + |Y|^2}{4\hbar}} \sum_{s=0}^{N} \frac{1}{s!(2\hbar)^{s/2}} (a^*(q + q' + i(p + p')))^s \Psi_0 + V_N = \Psi^N_{X+Y, h} + V_N. \]

One then deduces (1.14).

\[ \square \]

3 Absolute convergence of the Mizrahi series.

If $A$ is a bounded operator on $F_s(H_C)$ or an unbounded operator ($A, D(A)$) such that the domain $D(A)$ contains the coherent states, then the Wick symbol of $A$ is the the function $\sigma^w_\hbar(A)$ defined on $H^2$ by:

\[ \sigma^w_\hbar(A)(X) = \langle A\Psi_{X, h}, \Psi_{X, h} \rangle, \]

where the coherent states $\Psi_{X, h}$ are defined in (1.10).
We shall state and prove a formula giving the Wick symbol of the composed \( A \circ B \) of two bounded operators \( A \) and \( B \) in \( \mathcal{F}_s(H_C) \), as the sum of a convergent series. In finite dimension, it is a result of Mizrahi [19] and Appleby [6].

If \( H \) is a separable, real Hilbert space, if \( (e_j) \) is a Hilbert basis of \( H \), we define, for all multi-index \( \alpha = (\alpha_j) \) (which means \( \alpha_j = 0 \) except for a finite number of values of \( j \)), two differential operators on \( H^2 \), denoting by \( (q,p) \) the variable of \( H^2 \):

\[
(\partial_q \pm i \partial_p)^\alpha = \prod_j \left( \frac{\partial}{\partial q_j} \pm i \frac{\partial}{\partial p_j} \right)^{\alpha_j}.
\]

**Theorem 3.1.** For every bounded operators \( A \) and \( B \) in \( \mathcal{F}_s(H_C) \), the Wick symbols \( F \) and \( G \) of \( A \) and \( B \) are \( C^\infty \) functions on \( H^2 \). For each \( k \geq 0 \), the following series, a priori defined using a Hilbertian basis \( (e_j) \) of \( H \):

\[
C_k^{\text{wick}}(F,G) = 2^{-k} \sum_{|\alpha|=k} \left( \frac{1}{|\alpha|!} \right) (\partial_q - i \partial_p)^\alpha F (\partial_q + i \partial_p)^\alpha G
\]

is absolutely convergent and its sum \( C_k^{\text{wick}}(F,G) \) is independent of the basis. We have, for all \( X \) in \( H^2 \):

\[
\sigma_k^{\text{wick}}(A \circ B)(X) = \sum_{k=0}^\infty h^k C_k^{\text{wick}}(\sigma_k^{\text{wick}}(A),\sigma_k^{\text{wick}}(B))(X),
\]

where the series is absolutely convergent.

See [6] or [19] in the case of finite dimension. See also [1], formula (15)(i). If the operators \( A \) and \( B \) act in \( \mathcal{F}_s(H_C) \otimes E \) where \( E \) is a Hilbert of finite dimension then the functions \( F \) and \( G \) are defined on \( H^2 \), taking values in \( \mathcal{L}(E) \) and the product in the right hand side of (3.3) is a composition of operators. Note that, if one of the functions \( F \) or \( G \) is scalar valued, one has

\[
C_1^{\text{wick}}(F,G) - C_1^{\text{wick}}(G,F) = i^{-1}\{F,G\}.
\]

One denotes by \( \sigma \) the symplectic form on \( H^2 \) defined by \( \sigma((x,\xi),(y,\eta)) = y \cdot \xi - x \cdot \eta \) and by \( \{F,G\} \) the Poisson bracket of two functions \( F \) and \( G \) in \( C^1(H^2) \), defined by \( \{F,G\}(X) = -\sigma(dF(X),dG(X)) \).

**Proof of Theorem 3.1.** It suffices to prove the following equality, for all \( X \) in \( H^2 \):

\[
< B\Psi_{Xh}, A^*\Psi_{Xh} >= \sum \frac{h^{|\alpha|}}{2^{|\alpha|} \alpha!} (\partial_q - i \partial_p)^\alpha \sigma^{\text{wick}}_h(A)(X) (\partial_q + i \partial_p)^\alpha \sigma^{\text{wick}}_h(B)(X)
\]

(3.4)

**Proof of (3.4) for \( X = 0 \).** By Proposition 2.8, we have:

\[
< B\Psi_{0h}, A^*\Psi_{0h} >= \sum < T^F_{hW} B\Psi_{0h}, \tilde{\Psi}_{\alpha,h} > < \tilde{\Psi}_{\alpha,h}, T^F_{hW} A^*\Psi_{0h} >.
\]

Applying Proposition 2.10 to \( B\Psi_{0h} \), we obtain, for \( X \) in \( H^2 \):

\[
T^F_{hW}(B\Psi_{0h})(X) = \int_{B^2} \mathcal{B}_h(X,Y) T^F_{hW}(B\Psi_{0h})(Y) d\mu_{B^{2,h}}(Y).
\]
Differentiating the integral,

\[(\partial_x + i\partial_\xi) T^F_h(B\Psi_\infty)(X) = \hbar^{-1} \int_{\mathcal{B}_2} \ell_{e_\xi}(y + i\eta) B_h(X, Y) T^F_h(B\Psi_\infty)(Y) \, d\mu_{B^2_h}(Y).\]

Iterating and then setting \(X = 0\), one obtains

\[(\partial_x + i\partial_\xi)^\alpha T^F_h(B\Psi_\infty)(0) = \hbar^{-|\alpha|} \int_{\mathcal{B}_2} \Pi(\ell_{e_\xi}(y) + i\ell_{e_\xi}(\eta))^{\alpha_1} T^F_h(B\Psi_\infty)(Y) \, d\mu_{B^2_h}(Y).\]

Equivalently,

\[< T^F_h(B\Psi_\infty), \tilde{\Phi}_{\alpha, h} > = \left( \frac{\hbar}{2} \right)^{|\alpha|/2} (\alpha!)^{-1/2} (\partial_x + i\partial_\xi)^\alpha T^F_h(B\Psi_\infty)(0) \]

and the same equality holds for \(T^F_h(A^*\Psi_\infty)\). Consequently,

\[< B\Psi_\infty, A^*\Psi_\infty > = \sum \left( \frac{\hbar}{2} \right)^{|\alpha|} (\alpha!)^{-1/2} (\partial_x + i\partial_\xi)^\alpha T^F_h(B\Psi_\infty)(0) \frac{1}{(\partial_x + i\partial_\xi)^\alpha T^F_h(A^*\Psi_\infty)(0)}.\]

For any bounded operator \(B\) in \(\mathcal{F}_s(H_G)\), for all \(X\) and \(Y\) in \(H^2\), set:

\[(S_h B)(X, Y) = \frac{\langle B\Psi_{X, h}, \Psi_{Y, h} \rangle}{\langle \Psi_{X, h}, \Psi_{Y, h} \rangle}.\] (3.5)

By (2.7) and (2.5), we have:

\[(S_h B)(X, Y) = e^{\frac{Y^2}{\hbar}} e^{-\frac{1}{\hbar}(y + i\xi)\cdot(y - i\eta)} T^F_h(B\Psi_{X, h})(Y)\]

Therefore, when \(Y = (y, \eta)\) is identified with \(y + i\eta\), the function \(Y \rightarrow (S_h B)(X, Y)\) is Gateaux anti-holomorphic. Similarly, the function \(X \rightarrow (S_h B)(X, Y)\) is Gateaux holomorphic. Since the restriction of this function to the diagonal is the Wick symbol of \(B\), one has,

\[(\partial_q + i\partial_p)^\alpha T^F_h(B\Psi_\infty)(0) = (\partial_q + i\partial_p)^\alpha \sigma^{\text{wick}}_h(B)(0).\]

The equality (3.4) is then proved for \(X = 0\).

**Proof of (3.4) for arbitrary \(X\).** The coherent states \(\Psi_{X, h}\) defined in (1.10) satisfy classically

\[\Psi_{X, h} = V_h(X) \Psi_0 \quad V_h(X) = e^{-\frac{1}{\hbar} \Phi_S(\tilde{X})}\]

where \(\Psi_0\) is the vacuum (independent of \(h\)), \(\tilde{X} = (-b, a)\) for \(X = (a, b)\), and \(\Phi_S(\tilde{X})\) is the Segal field associated with \(\tilde{X} \in H^2\). We know that, for all \(U\) and \(V\) in \(H^2\):

\[e^{i\Phi_S(U)} e^{i\Phi_S(V)} = e^{\frac{i}{2} \sigma(U, V)} e^{i\Phi_S(U + V)}\]

where \(\sigma\) is the symplectic form \(\sigma((a, b), (q, p)) = b \cdot q - a \cdot p\). Using (3.6) and (3.7), we obtain:

\[\sigma^{\text{wick}}_h \left( V_h(X) A V_h(-X) \right)(Y) = \left( \sigma^{\text{wick}}_h(A) \right) (Y - X).\]

One then deduces equality (3.4) for any arbitrary \(X\), and Theorem 3.1 is proved.
4 Weyl symbol composition.

In [4], one shows that, if \( F \) and \( G \) are in \( S(H^2, Q) \) (see Definition 1.4) and when \( A_Q \) (see (1.16)) is trace class then the composition \( \Op_h^{\text{weyl}}(F) \circ \Op_h^{\text{weyl}}(G) \) is written as \( \Op_h^{\text{weyl}}(\Kweyl_h(F, G)) \) with \( \Kweyl_h(F, G) \) in \( S(H^2, 4Q) \). Our purpose is now to write \( \Kweyl_h(F, G) \) as an absolutely convergent series.

One defines a differential operator \( \sigma(\nabla_1, \nabla_2) \) on \( H^2 \times H^2 \) by:

\[
\sigma(\nabla_1, \nabla_2) F = \sum_j \frac{\partial^2 F}{\partial y_j \partial \xi_j} - \frac{\partial^2 F}{\partial x_j \partial \eta_j}
\]  

(4.1)

where \((x, \xi, y, \eta)\) is the variable in \( H^2 \times H^2 \). This operator is defined only for functions \( F \) for which the series converges. For all integers \( k \) and all functions \( F \) and \( G \) such that the series below makes sense, set:

\[
C_k^{\text{weyl}}(F, G)(X) = \frac{1}{(2i)^k k!} \sigma(\nabla_1, \nabla_2)^k (F \otimes G)(X, X).
\]  

(4.2)

**Theorem 4.1.** Let \( Q \) be a nonnegative quadratic form on \( H^2 \) where \( A_Q \) is trace class. Let \( F \) and \( G \) be two functions in \( S(H^2, Q) \). Then,

i) For each \( k \geq 0 \), we have:

\[
\|C_k^{\text{weyl}}(F, G)\|_{4Q} \leq \|F\|_Q \|G\|_Q (\text{Tr} A_Q)^k \frac{k!}{2^k k!}.
\]  

(4.3)

The series

\[
\Kweyl_h(F, G)(X) = \sum_{k=0}^{\infty} h^k C_k^{\text{weyl}}(F, G)
\]

is absolutely convergent and defines a function \( \Kweyl_h(F, G) \) in \( S(H^2, 4Q) \).

ii) One has:

\[
\Op_h^{\text{weyl}}(F) \circ \Op_h^{\text{weyl}}(G) = \Op_h^{\text{weyl}}(\Kweyl_h(F, G)).
\]  

(4.4)

iii) For all integers \( M \), one can write

\[
\Kweyl_h(F, G) = \sum_{k=0}^{M} h^k C_k^{\text{weyl}}(F, G) + h^{M+1} R_M^{\text{weyl}}(F, G, h)
\]  

(4.5)

with:

\[
\|R_M^{\text{weyl}}(F, G, h)\|_{4Q} \leq \|F\|_Q \|G\|_Q \frac{(\text{Tr} A_Q)^{M+1}}{(M + 1)!} e^{(h/2)\text{Tr} A_Q}.
\]

Proof. i) and iii) According to [11], we have, choosing a Hilbertian basis \((e_j)\):

\[
\|\sigma(\nabla_1, \nabla_2)^k (F \otimes G)\|_{\infty} \leq \|F\|_Q \|G\|_Q \left[ \sum_j 2Q(e_j, 0)^{1/2} Q(0, e_j)^{1/2} \right]^k
\]  

18
\[ \| F \|_Q \| G \|_Q \left[ \sum_j (Q(e_j, 0) + Q(0, e_j)) \right]^k \leq \| F \|_Q \| G \|_Q (\text{Tr} A_Q)^k. \]

From (4.2), one sees:
\[ \| C_{\text{weyl}}^k (F, G) \|_\infty \leq \frac{1}{2^k k!} \| \sigma(\nabla_1, \nabla_2)^k (F \otimes G) \|_\infty. \]

We estimate similarly the derivatives and we deduce (4.3). The series defining \( K_{\text{weyl}}^k (F, G) \) in Theorem 4.1 is then convergent and the asymptotic expansion iii) is valid.

ii) Equality (4.4) comes from the next Lemma. Indeed, this Lemma shows that the operators in the left and right hand sides of (4.4) share the same Wick symbol. Thus they are equal.

**Lemma 4.2.** For all \( F \) and \( G \) in \( S(H^2, Q_A) \), one has:
\[ \sum_{k=0}^{\infty} h^k C_k^\text{wick} (H_{h/2} F, H_{h/2} G) = H_{h/2} \left[ \sum_{k=0}^{\infty} h^k C_k^\text{weyl} (F, G) \right] \]  
(4.6)

where \( C_k^\text{weyl} (F, G) \) is defined in (4.2).

**Proof of the Lemma.** We denote by \( \Phi(h) \) the right hand side and by \( \Psi(h) \) the left hand side of (4.6). Since \( F \) belongs to \( S(H^2, Q) \), then we have (1.17), where the series is convergent and defines an element of \( S(H^2, Q) \). Looking at the powers of \( h \), we are led to compare the coefficient of \( h^k \) in \( \Phi(h) \)
\[ a_k = \sum_{s+\ell=k} \frac{1}{4^\ell \ell!} \Delta^\ell C_{s}^\text{weyl} (F, G) \]
with the coefficient of \( h^k \) in \( \Psi(h) \)
\[ b_k = \sum_{s+\ell+u=k} \frac{1}{4^{\ell+u} s! u!} C_u^\text{wick} (\Delta^\ell F, \Delta^u G). \]

Let \( \sigma(\nabla_1, \nabla_2) \) be the differential operator on \( H^2 \times H^2 \) defined in (4.1). We denote by \( R \) the restriction operator defined for all functions \( F \) on \( H^2 \times H^2 \) by:
\[ (RF)(X) = F(X, X). \]

We have
\[ \Delta R = \bar{R} \Delta \quad \bar{\Delta} = \Delta_1 + \Delta_2 + 2 \nabla_1 \cdot \nabla_2 \quad \nabla_1 \cdot \nabla_2 = \sum_j \frac{\partial^2}{\partial x_j \partial y_j} + \frac{\partial^2}{\partial \xi_j \partial \eta_j}. \]

We then have to compare the following coefficients (operators acting on functions defined on \( H^2 \times H^2 \))
\[ A_k = \sum_{s+\ell=k} \frac{1}{(2\ell)\times 4^{\ell} \ell!} (\bar{\Delta})^\ell (\sigma(\nabla_1, \nabla_2))^s \]
\( B_k = \sum_{s+\ell+n=k} \frac{1}{2^{n} \cdot \ell!} \sum_{|\alpha|=n} \frac{1}{\alpha!} (\partial_x - i\partial_\xi)^\alpha (\partial_y + i\partial_\eta)^\alpha \Delta_1^\ell \Delta_2^n. \)

Expanding \((\Delta)^\ell\) using the binomial formula, we are led to compare:

\[
A'_k = \sum_{\ell=0}^{k} \sum_{u+s=\ell} \frac{i^{\ell-k}}{(k-\ell)!u!s!(\ell-u-s)!} (\nabla_1 \cdot \nabla_2)^{\ell-u-s} (\sigma(\nabla_1, \nabla_2))^{k-\ell} \Delta_1^u \Delta_2^s.
\]

\[
B'_k = \sum_{s+u+|\alpha|=k} \frac{1}{2^{k+u+s} u!s!|\alpha|!} (\partial_x - i\partial_\xi)^\alpha (\partial_y + i\partial_\eta)^\alpha \Delta_1^u \Delta_2^s.
\]

Thus, we have to compare, for \(t = u + s\) being integers smaller or equal than \(k\):

\[
A_k(t) = \sum_{\ell=t}^{k} \frac{i^{\ell-k}}{(k-\ell)!|\ell-t|!} (\nabla_1 \cdot \nabla_2)^{\ell-t} (\sigma(\nabla_1, \nabla_2))^{k-\ell}
\]

\[
B_k(t) = \sum_{|\alpha|=k-t} \frac{1}{\alpha!} (\partial_x - i\partial_\xi)^\alpha (\partial_y + i\partial_\eta)^\alpha.
\]

The term \(B_k\) is directly expanded. The term \(A_k\) is less direct and contains the terms \(\partial_x^a \partial_y^b \partial_x^{-a} \partial_y^{-b}\), for \(|\alpha| = k-t\), with the coefficient

\[
\sum_c \frac{i^{t+|a+b+c-\alpha|+c}(-1)^{a-b-c}}{(a+b+c-\alpha)!c!(\alpha-b-c)!(\alpha-a-c)!}
\]

the sum running over all \(c\) such that all the multi-indices in the denominator have positive components. A standard fact (concerning the hypergeometric distribution) then allows to conclude that \(A_k(t) = B_k(t)\).

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