Proximity to a Nearly Superconducting Quantum Critical Liquid

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The coupling between superconductors and a quantum critical liquid that is nearly superconducting provides natural interpretation for the Josephson effect over unexpectedly long junctions, and the remarkable stripe-spacing dependence of the critical temperature in LSCO and YBCO superconductors.

We first consider the general behavior of the QCL. It is known that the Coulomb interaction is strong and long-ranged in cuprates, which frustrates the phase separation of holes to form stripes. Therefore, we believe that the dynamical exponent of the QCL should be $z = 1$. An immediate consequence of this exponent is that the inverse of a length has the dimension of energy. On the other hand, since the spin pairs are well formed, the amplitude fluctuations of the pairs are massive and can be integrated out in principle. The important dynamics comes from the phase $\theta$ degrees of freedom of the pairs. Because of the isotropy in space and (imaginary) time, a generalized Josephson scaling relation for the phase stiffness can be obtained as follows. The Green’s function for $\theta$ is, in general,

$$G(q) = q^{-(2-\eta)} Y(q\xi),$$

where $q = (k, \omega)$ is a three-vector ($k$ is the spatial wave vector and $\omega$ is the Matsubara frequency), $\eta$ is the anomalous dimension, $\xi$ is the correlation length, and finally $Y(\zeta)$ is a scaling function of $\zeta$. The phase stiffness $\rho$ (including the superfluid density and the compressibility) is determined by

$$\lim_{q \to 0} G(q) = b^2 / q \rho q^2,$$

where $b = |\langle \sqrt{x} e^{i\theta} \rangle| \sim \xi^{-\beta/\nu}$ is the order parameter with $\beta$ and $\nu$ being the scaling exponent for the order parameter and correlation length, respectively. Here $x$ is the hole density in the QCL. Thus

$$\rho = b^2 \lim_{q \to 0} q^{-\eta} Y^{-1}(q\xi) \sim \xi^{\eta-2\beta/\nu} = \xi^{-(d+z-2)},$$

where in the last step we have used the generalized hyperscaling relation $2\beta = \nu(d+z-2+\eta)$ with $d = 2$ being the spatial dimension in our case. Thus the desired Josephson scaling relation reads $\rho \sim \xi^{-1}$. This result could also be obtained by a simple power counting as follows. Near criticality, one expects that

$$\int_{\xi^{d+z}} \rho(\partial_\mu \theta)^2 \sim 1$$
where the integration is over a volume of $\xi^{d+z}$ due to the isotropy in space and time. The ansatz $\partial_s \theta \sim 1/\xi$ gives immediately the Josephson scaling. The above analysis implies that both the superfluid density and the compressibility scale as energy and/or inverse length. This statement is important to discuss the finite-size scaling in the following. With the general behavior of the QCL in mind, let us now discuss the relevance of the QCL in the cuprates.

(I) Photodoping experiment: In a very recent and elegant experiment, superconducting wires are generated by photo-doping a film with $x$ slightly lower than $x_c$. The unilluminted region serves as a junction. The critical current $I_c$ is measured as a function of the junction length $s$ (in the direction of current flow). It turns out that a relation

$$I_c \propto 1/s$$

holds from $s = 45nm$ up to $s = 100nm$, a length much larger than the coherence length $\xi_s \sim (1 - 5nm)$ in the superconducting leads. This phenomenon is difficult to understand if the junction is a normal insulator or metal; as one would expect $I_c(s) \propto e^{-s/l}$ where $l = \xi_s$ for a normal insulator and $l = v_F/T$ for a ballistic metal. According to the experimental setup, $l \ll 100nm$ for both cases.

Therefore, the junction may be in a special state of matter, which is arguably a QCL. We note that the geometry of the experimental setup is two one-dimensional (1D) superconducting wires immersed in a QCL. This is an important aspect of the experiment. The Cooper pairs can tunnel through an extended area connecting the two wires. Because the junction length is the only relevant length scale, the Josephson energy scales as

$$E_J = F(s/\xi)/s$$

with a scaling function $F(s/\xi)$ if the junction is not exactly at criticality. It is expected that $F(s/\xi)$ remains a constant at $s \ll \xi$. The above result can also be understood as follows. Since $s$ is the only length scale, the superfluid density $\rho_s$ in the junction scales as $1/s$ and the transverse extension of the supercurrent flow is $X_s \sim s$, up to corrections from a scaling function. The Josephson energy scale is thus given by

$$E_J \sim \int_{sX_s} dx dy \rho_s(\nabla \theta)^2 \sim F(s/\xi)/s,$$

where the integration is over the area $sX_s$, where supercurrent flows. We have used $\nabla \theta \sim 1/s$ for a rough estimate, and have included correction from the finiteness of $s/\xi$ by the scaling function. Since $E_J$ determines the critical current $I_c$, in suitable units we have

$$I_{c,s} = F(s/\xi).$$

At $s < \xi$ we expect that $F$ is roughly a constant, so that Eq. (5) is recovered. The plot of $I_{c,s}$ is nothing but a scaling function. From the data $I_{c,s}$ drops to zero quickly at $s > 100nm$. This suggests that $\xi \sim 100nm$. The decrease of $I_{c,s}$ may be related to the thermal activation of vortices in the junction. More efforts should be devoted to explain the sharp drop of the scaling function, which is beyond the scope of this scaling argument. On the other hand, because $z = 1$, we expect that $\xi \sim 1/T$ in the critical regime, so that the scaling relation can be rewritten as

$$I_c = F(T_s).$$

An important prediction follows from this function: Instead of varying $s$, one could vary the temperature $T$ to find the scaling behavior, provided that the superconductivity within the wires is hardly changed. This situation is perhaps not feasible by photodoping, but is plausible by other techniques.

To appreciate the importance of the special geometry of the junction we addressed so far, consider another situation. Suppose the superconducting leads are wide enough so that effectively the QCL only lives within the junction. Neglecting boundary corrections the Josephson energy should be extensive in the junction width $L$. The only other length scale is $s$. By dimension analysis, we obtain

$$E_J \sim \int_0^L dy \int_0^s dx \rho_s(\nabla \theta)^2 \sim (L/s)F(s/\xi, s/L)/s.$$  

In the last expression, the first factor $L/s$ is extensive in $L$ but is dimensionless. The remaining part is the characteristic energy. Here $F(s/\xi, s/L)$ is another scaling function that is finite at vanishing arguments. In this geometry, one would expect that

$$I_{c,s} = (L/s)F(s/\xi, s/L).$$

At $s \ll \xi$ and $s \ll L$, one would expect that $I_c \propto L/s^2$, which should be contrasted with Eq. (5). This scaling form leads us to propose the experimental test of the idea of QCL in the junction. If one can create a wide junction by photodoping the wide stripes of material then the critical current should follow Eq. (11). The data of Decca et al. (8) begin with $s > L$, and are better described by the first geometry. (In fact, in their case, the width $L$ can not be well defined as the wires become rounded at the tips).

The QCL should also manifest itself in the magnetic field dependence of the Josephson critical current. Since the proliferation of vortices, each carrying a flux quantum, would wash out superconductivity, we expect that the magnetic field $H$ defines another length scale through $1/\sqrt{H}$. Thus in the presence of a weak magnetic field, we expect that in a given geometry of the junction the field dependence of the critical current should be qualitatively given by

$$I_c(H)/I_c(0) = 1 - a(\xi \sqrt{H})^n$$
where \( n \) is a universal positive exponent and \( a \) a non-universal positive constant. In the critical regime where \( \xi > 1/T \) the above expression should be replaced by \( I_c(H)/I_c(0) = 1 - a(\sqrt{H}/T^2)^n \). From the experiment of Decca et al, it turns out that \( n = 4 \). In order to see whether their results support the QCL scenario, it is desirable to vary both \( H \) and \( T \) and to see whether \( I_c(H)/I_c(0) \) is a scaling function of \( H/T^2 \) (in the critical regime).

(II) Stripes in underdoped cuprates: Let us now discuss the relevance of QCL in underdoped cuprates with hole-rich stripes. It was found that in LSCO materials there exists a striking linear relation between the superconducting critical temperature \( T_c \) the incommensurate width \( \delta \) near the peak at \( (\pi, \pi) \) in the neutron scattering spectrum. Recently, a similar relation was discovered from the inelastic neutron scattering data in YBCO samples. More interestingly, it was further argued that \( T_c \propto \delta \) translates to

\[
T_c = \frac{hv^*}{s},
\]

where \( h = 2\pi\hbar \) is the Planck constant, \( s \) the spacing between stripes, and \( v^* \) a material-dependent constant with the dimension of velocity. Two aspects of Eq. (12) are quite unusual. First, for a given material, \( v^* \) should be constant and small \( (v^* \ll v_F \) with \( v_F \) being the Fermi velocity). And second, the robustness of the scaling up to the optimum doping, where the stripes, if they exist at all, would be densely spaced \( (s \sim 1\text{nm}) \). It was conjectured that \( v^* \) may be related to an unknown collective-mode of the stripes and/or the motion of heavy quasiparticles near the flat band around \( (\pi, 0) \). Given the experimental results, this conjecture is an important clue to the underlying mechanism. Here we argue that Eq. (12) follows from dynamic stripes coupled by QCL.

We note that the stripes are metallic. They behaves as anti-phase domain walls for the remaining antiferromagnetic background. (Recent mean-field calculations for the t-J model indicated that in-phase stripes are slightly more favorable energetically. However, it was believed that quantum fluctuations might stabilize anti-phase stripes.) The holes in the stripes are not strictly localized in the transverse direction. Rather, because of quantum fluctuations, the stripes are dynamic unless pinned by inhomogeneities. Such transverse fluctuations were believed to develop strong superconducting correlations along the longitudinal direction of the stripes. Also because of the transverse fluctuations of the stripes, the intervening region between the stripes may be driven to a quantum critical liquid, which is almost superconducting. Combining these considerations, we model the whole system as a superposition of superconducting stripes and quantum critical liquids plus coupling between them. This phenomenological model should be understood as an effective theory in the low energy limit. It does not answer why there are stripes and a QCL, which are the issue of a higher energy scale. Without loss of generality, let us assume that an isolated stripe \( \alpha \) can be described by an action

\[
S_\alpha = \frac{1}{2} \int d\tau \left[ u(\partial_\tau \theta_\alpha)^2 + v(\partial_\theta \theta_\alpha)^2 \right],
\]

where \( u \) and \( v \) are the bare stiffness components with respect to space \( y \) and time \( \tau \), and are assumed to be independent of the stripe spacing \( s \). (Because of transverse fluctuations of the stripes, the spacing is only defined in an average sense.) As we argued, the stripes are coupled by the QCL. We imagine to find the effective coupling between the stripes mediated by the QCL. This is virtually a real-space renormalization. In the long wave length and low energy limit, we need only to retain lowest order couplings. Clearly, there may be corrections to the bare stiffness components, which we believe is relatively small and is neglected in the following. Among the other effects, the Josephson-like coupling between nearest-estripes \( \alpha \) and \( \beta \),

\[
\frac{1}{2} \int d\tau \int dy j(\theta_\alpha - \theta_\beta)^2
\]

is the most relevant. The coefficient \( j \) can be obtained by the same scaling arguments as that for the wide-junction geometry described above,

\[
\tilde{F}(s/\xi, s/L/\bar{s}),
\]

where \( L \) now denotes the characteristic length of the stripes. Again in the long wave length and low energy limit, we can approximate \( (\theta_\alpha - \theta_\beta)^2/\bar{s}^2 \) as \( (\partial_\theta \theta)^2 \), so that the total action for the stripes can be put back into a continuum form in the \( r \equiv (x, y, \tau) \) space as,

\[
S = \frac{1}{2} \int d^3 r \rho(\partial_\mu \theta)^2,
\]

where \( \rho \) denotes the three components, \( \rho_x = \tilde{F}(s/\xi, s/L)/s \), \( \rho_y = u/s \) and \( \rho_\tau = v/s \). This is an anisotropic 3D XY-model. To leading order all of the stiffness components in the new action are inversely proportional to \( s \). If the system is macroscopically isotropic due to stripes in both directions, the global stiffness \( \rho \) is given by the geometrical mean of the three components, \( \rho_{\text{eq}} = \tilde{F}^{1/3} \) so that

\[
\rho = \tilde{F}(s/\xi, s/L)/s. \tag{13}
\]

Here \( H = \tilde{F}^{1/3} \) is a scaling function given by \( \tilde{F} \). The 3D XY-model develops long range ordering as long as \( \rho > 0 \) in thermodynamic limit (i.e., in an infinite plane and at zero temperature). This is again a quantum critical point with \( z = 1 \) due to isotropy in space and time. Since \( \rho \) has the dimension of energy, it naturally
defines the scale of the superconducting gap. Thus at $T > \rho$ the system is in a quantum critical state, while at $T < \rho$ it is a renormalized superconductor. As a result, the transition temperature is given by $T_c = \rho$, which was also indicated earlier in some stripe scenarios of high temperature superconductivity. \cite{20} In combination with the behavior of $\rho$ we obtain

$$T_c s = H(s/\xi, s/L). \quad (14)$$

For long stripes, we can simplify the expression by the approximation $T_c s = H(s/\xi, 0)$. This explains why $T_c$ should be determined by $s$. The scaling function takes into account the correction due to the finiteness of $s/\xi$. The above results follow essentially from a finite-size scaling with respect to a QCL. It is not clear a priori that such a scaling should work at all the more or less microscopic stripe spacing scales. Furthermore, the above relation does not guarantee that $T_c s$ will be a constant if $H$ varies significantly at moderate arguments. However, $\xi \sim 1/T_c$ at $T = T_c$ in the critical regime of the QCL. Consequently,

$$T_c s = H(T_c s, 0). \quad (15)$$

This is a remarkable result, indicating that the plot of $H(T_c s, 0)$ against $T_c s$ only accesses a unique point of the scaling function at $T_c s = hv^*$, with $hv^*$ determined by the equation

$$hv^* = H(hv^*, 0). \quad (16)$$

We believe that this explains the robustness of Eq. \cite{12}, even though the scaling function $H(\zeta, 0)$ may vary with $\zeta$ due to the finite size effect. Clearly the notion of a QCL plays an important role in our arguments.

In summary, proximity to a quantum critical liquid that is nearly superconducting provides appealing interpretation of the abnormal Josephson effect and the stripe spacing dependence of $T_c$ in underdoped cuprates. We also predicted the Josephson effect in a geometry different from that in Ref. \cite{8}. The field dependence of the Josephson current is also discussed. Naturally one would expect other properties of superconductors in proximity to a quantum critical liquid. These may include scaling behavior of the various stiffness as a function of temperature, probing frequency and length scale.

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