Tracking Influential Nodes in Dynamic Networks

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ABSTRACT

In this paper, we tackle a challenging problem inherent in many applications: tracking influential nodes in dynamic networks. Specifically, we model a dynamic network as a stream of edge weight updates. This general model embraces many practical scenarios as special cases, such as edge and node insertions, deletions as well as evolving weighted graphs. Under the popularly adopted linear threshold model and independent cascade model, we consider two essential versions of the problem: finding the nodes whose influences passing a user specified threshold and finding the top-k most influential nodes. Our key idea is to use the polling-based methods and maintain a sample of random RR sets so that we can approximate the influence of nodes with provable quality guarantees. We develop an efficient algorithm that incrementally updates the sample random RR sets against network changes. We also design methods to determine the proper sample sizes for the two versions of the problem so that we can provide strong quality guarantees and, at the same time, be efficient in both space and time. In addition to the thorough theoretical results, our experimental results on three real network data sets clearly demonstrate the effectiveness and efficiency of our algorithms.

1. INTRODUCTION

More and more applications are built on dynamic networks and need to track influential nodes. For example, consider cold-start recommendation in a dynamic social network—we want to recommend to a new commer some existing users in a social network. A new user may want to subscribe to the posts of some users in order to obtain hot posts (posts that are widely spread in the social network) at the earliest time. Clearly for such a new user we should recommend her some influential users in the current network. Traditional Influence Maximization cannot find those influential users we want here because it is for marketing in which all seed users have to be synchronized to spread the same content, while in reality online influential individuals often produce and spread their own contents in an asynchronous manner. The influential users we want are those who have high individual influence.

More often than not, the underlying network is highly dynamic, where each node is a user and an edge captures the interaction from a user to another. User interactions evolve continuously over time. In an active social network, such as Twitter, Facebook, LinkedIn, Tencent WeChat, and Sina Weibo, the evolving dynamics, such as rich user interactions over time, is the most important value. It is critical to capture the most influential users in an online manner. To address the needs, we have to tackle two challenges at the same time, influence computation and dynamics in networks.

Influence computation is very costly, technically #P-hard under most influence models. Most existing studies have to compromise and consider the influence maximization problem only on a static network. Here, influence maximization in a network is to find a set of vertices S such that the combined influence of the nodes in the set is maximized and S satisfies some constraints such as the size of S is within a budget. The incapability of handling dynamics in large evolving networks seriously deprives many opportunities and potentials in applications.

Although influence maximization and finding influential nodes are highly related since they both need to compute influence in one way or another, these two problems serve very different application scenarios and face different technical challenges. For example, influence maximization is a core technique in viral marketing [13]. At the same time, influence maximization is not useful in the cold-start recommendation scenario discussed above, since a user is interested in being connected with individual users of great potential influence and may follow them in interaction.

To the best of our knowledge, our study is the first to tackle the problem of tracking influential nodes in dynamic networks. Specifically, we model a dynamic network as a stream of edge weight updates. Our model is general and embraces many practical scenarios as special cases. Under the popularly adopted linear threshold model and independent cascade model, we consider two essential versions of the problem: (1) finding the nodes whose influences passing a user specified threshold; and (2) finding the top-k most influential nodes. Our key idea is to use the polling-based methods and maintain a sample of random RR sets so that we can approximate the influence of nodes with provable quality guarantees.

Recently, there is encouraging progress in influence maximization on dynamic networks [10, 2]. Due to the difference between influence maximization and finding influential nodes, the methods in those studies [10, 2] cannot be applied directly to find influential nodes. Moreover, in terms of specific techniques, our study is also very different from [10, 2]. Most importantly, the methods in [10, 2] are heuristic, and do not provide any provable quality guarantees. In addition, the influence model considered in [10] is the Independent Cascade model. The one in [2] is a non-linear system. We address both the Linear Threshold model and the Independent...
Cascade model in this study. To the best of our knowledge, we are the first to tackle influence computation with provable quality guarantee under the two most widely adopted influence models on dynamic networks.

To tackle the novel and challenging problem of finding influential nodes in dynamic networks, we make several technical contributions. We develop an efficient algorithm that incrementally updates the sample random RR sets against network changes. We also design methods to determine the proper sample sizes for the two versions of the problem so that we can provide strong quality guarantees and at the same time be efficient in both space and time. In addition to the thorough theoretical results, our experimental results on three real data sets clearly demonstrate the effectiveness and efficiency of our algorithms. The large data set used contains over 1.6 million nodes and over 30 million edges.

The rest of the paper is organized as follows. We review the related work in Section 2. In Section 3, we recall the Linear Threshold model and the Independent cascade model, review the polling-based method for computing influence spread, and formulate influence in dynamic networks. In Section 4, we present methods updating random RR sets over a stream of edge weight updates. In Section 5 we tackle the problem of tracking nodes whose influence spreads pass a user-defined threshold. In Section 6 we present methods to find the top-k influential nodes is settled. We report the experimental results in Section 7. We conclude the paper in Section 8.

2. RELATED WORK

Domingos et al. [13] proposed to take advantage of peer influence between users in social networks for marketing. Kempe et al. [19] formulated the problem using two discrete influence models, namely Independent Cascade model and Linear Threshold model. Since then, influence computation, especially influence maximization, has drawn much attention from both academia and industry [6, 14, 4, 30, 9, 17, 25]. Some heuristic methods were designed for computing influence spread under the Linear Threshold model [17, 9, 20]. For the Independent Cascade Model, [12] proposed approximations of influence spread estimations. Note that there are still gaps between estimations of influence spread and real influence spreads, which were not clearly quantified in [12]. Consequently, [12] cannot compute influence spread with provable quality guarantees. Recently, a polling-based method [4, 30, 31] was proposed for influence maximization under general triggering models. The key idea is to use some "Reversely Reachable" (RR) sets to approximate the real influence spread of nodes. The error of approximation can be bounded with a high probability if the number of RR sets is large enough.

Extracting influential nodes in social networks is also an important problem in social network analysis and has been extensively investigated [15, 1] [32, 5]. In addition to the marketing value, influential individuals are also useful in recommender systems in online web services [4, 32]. Due to the computational hardness of influence spread [3, 7], most methods did not use influence models to measure a user’s influence, but adopted measures like PageRank which can be efficiently computed.

In many applications, the underlying networks are evolving all the time [23, 24]. Rather than re-computing from scratch, incremental algorithms are more desirable in graph analysis tasks on dynamic networks. Maintaining PageRank values of nodes on an evolving graph was studied in [3, 25]. Hayashi et al. [15] proposed to utilize a sketch of all shortest paths to dynamically maintain the edge betweenness value. The dynamics considered by the above work is a stream of edge insertions/deletions, which is not suitable for influence computation. The dynamics of influence network is more complicated, because besides edge insertions/deletions, influence probabilities of edges may also evolve over time [23].

Aggarwal et al. [2] explored how to find a set of nodes that has the highest influence within a time window [t0, t0+h]. They modeled influence propagation as a non-linear system which is very different from triggering models like the Linear Threshold model or the Independent Cascade model. The algorithm in [2] is heuristic and the results produced do not come with any provable quality guarantee.

Chen et al. [10] investigated incrementally updating the seed set for influence maximization under the Independent Cascade model. They proposed an algorithm which utilizes the seed set mined from the former network snapshot to efficiently find the seed set of the current snapshot. An Upper Bound Interchange heuristic is applied in the algorithm. However, the algorithm in [10] is costly in processing updates, since updating the Upper Bound vector for filtering non-influential nodes takes \(O(m)\) time where \(m\) is the number of edges. Moreover, the SP1M heuristic [21], which does not have any approximation quality guarantee, was adopted in [10] for estimating influence spread of nodes. Thus, the set of influential nodes, even when the size of the seed set is set to 1, does not have any provable quality guarantee.

3. PRELIMINARIES

In this section, we recall the Linear Threshold influence model and the Independent Cascade Model [19]. We also review the polling method for computing influence spread [4, 31, 30]. We then formulate influence in dynamic networks. For readers’ convenience, Table 1 lists the frequently used notations.

### Table 1: Frequently used notations.

| Notation | Description |
|----------|-------------|
| \(G = (V, E, w)\) | A social network, where each edge \((u, v) \in E\) is associated with an influence weight \(w_{uv}\) |
| \(w_{uv}\) | weight of the edge \((u, v)\) (LT model); propagation probability of the edge \((u, v)\) (IC model) |
| \(n = \lvert V \rvert\) | The number of nodes in \(G\) |
| \(m = \lvert E \rvert\) | The number of edges in \(G\) |
| \(N^\text{in}(u)\) | The set of in-neighbors of \(u\) |
| \(w_u\) | Self-weight of \(u\) |
| \(W_u = w_u + \sum_{v \in N^\text{in}(u)} w_{uv}\) | the total weight of \(u\) |
| \(p_{uv}\) | \(p_{uv} = \frac{w_{uv}}{w_u}\), the probability that \(v\) is influenced by its neighbor \(u\) (LT Model) |
| \(I_u\) | The influence spread of node \(u\) |
| \(I\) | The average influence spread of individual nodes |
| \(M\) | The number of random RR sets |
| \(H\) | The hyper-graph consists of \(M\) random RR sets |
| \(D[u]\) | The degree of \(u \in V\) in \(H\) |
| \(F^\text{LT}(u)\) | \(F^\text{LT}(u) = \frac{\sum_{u \in V} w_{uv}}{M}\), the fraction of random RR sets containing \(u\) |
| \(I_{\text{max}}\) | Influence threshold set by users |
| \(F^\text{IC}(u)\) | Influence spread of the \(k\)-th most influential individual node |
| \(F^\text{IC}_k\) | The \(k\)-th highest \(F^\text{IC}(u)\) value for \(u \in V\) |

Influence threshold set by users
v. Denote by $W_v = w_v + \sum_{u \in N^+(v)} w_{uv}$ the total weight of v, where $N^+(v)$ is the set of v’s in-neighbors.

We define the influence probability $p_{uv}$ of an edge $(u,v)$ as $\frac{w_{uv}}{W_v}$. Clearly, for $v \in V$, $\sum_{u \in N^+(v)} p_{uv} \leq 1$.

In the Linear Threshold (LT) model [19], given a seed set $S \subseteq V$, the influence propagates in G as follows. First, every node $u$ randomly selects a threshold $\lambda_u \in [0,1]$, which reflects our lack of knowledge about users’ true thresholds. Then, influence propagates iteratively. Denote by $S_t$ the set of nodes that are active in step $i$ ($i = 0, 1, \ldots$) and $S_0 = S$. In each step $i \geq 0$, an inactive node $v$ becomes active if

$$\sum_{u \in N^+(v) \cap S_{i-1}} p_{uv} \geq \lambda_v$$

The propagation stops at step $t$ if $S_t = S_{t-1}$. Let $I(S)$ be the expected number of nodes that are finally active when the seed set is $S$. We call $I(S)$ the influence spread of $S$. Let $I_t$ be the influence spread of a single node $u$.

Kempe et al. [19] proved that the LT model is equivalent to a “live-edge” process where each node $v$ picks at most one incoming edge $(u,v)$ with probability $p_{uv}$. Consequently, $v$ does not pick any incoming edges with probability $1 - \sum_{u \in N^+(v)} p_{uv} = \frac{1}{w_v}$. All edges picked are “live” and the others are “dead”. Then, the expected number of nodes reachable from $S \subseteq V$ through live edges is $I(S)$, the influence spread of $S$.

It is worth noting that our description of the LT model here is slightly different from the original [19]: we use a function of edge weights and self-weight of nodes to represent influence probabilities. Representing influence probabilities in this way is widely adopted in the existing literature [9][17][31][30][16].

3.2 Independent Cascade Model

A social network in the Independent Cascade (IC) model is also a weighted graph $G = (V, E, w)$. Let $w_{uv}$ represent the propagation probability of the edge $(u,v)$, which is the probability that $v$ is activated by $u$ through the edge in the next step after $u$ is activated. Clearly for the IC model, all $w_{uv} \in [0, 1]$.

In the IC model [19], given a seed set $S \subseteq V$, the influence propagates in $G$ iteratively as follows. Denote by $S_t$ the set of nodes that are active in step $i$ ($i = 0, 1, \ldots$) and $S_0 = S$. At step $i+1$, each node $u$ in $S_i$ has a single chance to activate each inactive neighbor $v$ with an independent probability $w_{uv}$. The propagation stops at step $t$ if $S_t = \emptyset$. Similar to the LT model, the influence spread $I(S)$ denotes the expected number of nodes that are finally active when the seed set is $S$.

The “live-edge” process [19] of the IC model is to keep each edge $(u,v)$ with a probability $w_{uv}$ independently. All kept edges are “live” and the others are “dead”. Then, the expected number of nodes reachable from $S$ via live edges is the influence spread $I(S)$.

3.3 The Polling Method for Influence Computation

Chen et al. [9] proved that computing influence spread under the LT model is #P-hard. Recently, a polling-based method [3][31][30] was proposed for approximating influence spread of triggering models [19] like the LT model. Here we briefly review the polling method for computing influence spread in the LT model.

Given a social network $G = (V, E, w)$, a poll is conducted as follows: we pick a node $v \in V$ in random and then try to find out which nodes are likely to influence $v$. We run a Monte Carlo simulation of the equivalent “live-edge” process. The nodes that can reach $v$ via live edges are considered as the potential influencers of $v$. The set of influencers found by each poll is called a random RR (Reversely Reachable) set.

Let $R_1, R_2, ..., R_M$ be a sequence of random RR sets generated by $M$ polls, where $M$ can also be a random variable. The $M$ random RR sets form a random hyper-graph $H$ where the set of nodes is still $V$ and each random RR set is a hyper-edge. Denote by $D(S)$ the degree of a set of nodes $S$ in the hyper-graph, which is the number of hyper-edges containing at least one node in $S$. Let $\mathcal{F}_R(S) = \frac{D(S)}{M}$. By the linearity of expectation, it has been shown that $n\mathcal{F}_R(S)$ is an unbiased estimator of $I(S)$ [4][30]. Tang et al. [30] proved that the corresponding sequence $x_1, x_2, ..., x_M$ is a martingale [11], where $x_1 = 1$ if $S \cap R_1 \neq \emptyset$ and $x_i = 0$ otherwise. We have $E[\sum_{i=1}^M x_i] = E[D(S)] = \frac{M I(S)}{n}$. The following results [30] show how $E[\sum_{i=1}^M x_i]$ is concentrated around $\frac{M I(S)}{n}$.

**Corollary 1** [30]. For any $\xi > 0$,

$$\Pr\left[ \sum_{i=1}^M x_i - M p \geq \xi M p \right] \leq \exp\left( -\frac{\xi^2}{2} \frac{M p}{1 + \frac{\xi^2}{2} M p} \right)$$

$$\Pr\left[ \sum_{i=1}^M x_i - M p \leq -\xi M p \right] \leq \exp\left( -\frac{\xi^2}{2} \frac{M p}{1 + \frac{\xi^2}{2} M p} \right)$$

where $p = \frac{I(S)}{n}$.

Sections 3.1 and 3.2 will use the above results to analyze how many random RR sets are needed for extracting influential nodes. Note that since the problem we study in this paper is different from influence maximization, the results (theorems and lemmas) in [30] cannot be applied to our analysis.

3.4 Influence in Dynamic Networks

Real online social networks, such as the Facebook network and the Twitter network, change very fast and all the time. Relationships among users keep changing, and influence strength of relationships also varies over time. Lei et al. [22] pointed out that influence probabilities may change due to former inaccurate estimation or evolution of users’ relations over time. However, the traditional formulation of dynamic networks only considers the topological updates, that is, edge insertions and edge deletions [3][26][18]. Such a formulation is not suitable for realtime accurate analysis of influence.

According to the LT model reviewed in Section 3.1, the change of influence probabilities along edges can be reflected by the change of edge weights. For the IC model, since the weight of an edge is the propagation probability, the updates on edge weights are updates on propagation probabilities. Therefore, we model a dynamic network as a stream of weight updates on edges.

A weight update on an edge is a 5-tuple $(u,v,+,t,\Delta t)$, where $(u,v)$ is the edge updated, $+/-$ is a flag indicating whether the weight of $(u,v)$ is increased or decreased, $\Delta t$ is the amount of change to the weight and $t$ is the time stamp. The update is applied to the self-weight $w_{uv}$ if $u = v$. Clearly, edge insertions/deletions considered in the existing literature [3][26][10] can be easily written as weight increase/decrease updates. Moreover, node insertion/deletions can be written as edge insertions/deletions, too.

**Example 1.** A retweet network is a weighted graph $G = (V, E, w)$, where $V$ is a set of users. An edge $(u,v) \in E$ captures that user $v$ retweeted from user $u$. We can set $w_{uv}$ according to the propagation model adopted as follows.

**LT Model:** The edge weight $w_{uv}$ is the number of tweets that $v$ retweeted from $u$. The self-weight $w_v$ is the number of original
tweets posted by v. The weights reflect the influence in the social network. By intuition, if v retweeted many tweets from u, v is likely to be influenced by u. In contrast, if most of v’s tweets are original, v is not likely to be influenced by others.

IC Model: The edge weight w_{uv} is the probability that v retweets from u, which can be calculated according to v’s retweeting record in the past [3,16].

An essential task in online social influence analysis is to capture how the influence changes over time. For example, one may want to consider only the retweets within the past Δt time. Clearly, the set of edges E may change and the weights w_{uv} may increase or decrease over time. The dynamics of the retweet network can be depicted by a stream of edge weight updates \( \left\{ (u,v,+/−,Δ,t) \right\} \).

Given a dynamic network like the retweet network in Example[3], how can we keep track of influential users dynamically? In order to know the influential nodes, the critical point is to monitor influence of users. To solve this problem, we adopt the polling-based method for computing influence spread, and extend it to tackle dynamic networks. The major challenge is how to maintain a number of RR sets over a stream of edge weight updates, such that \( n_{FR}(S) \) is always an unbiased estimator of \( I(S) \). We propose a framework for updating RR sets that addresses various tasks of tracking influential nodes.

The framework is shown in Algorithm[1]. In Section[3] we discuss how to efficiently update the existing RR sets. How to decide if our current RR sets are insufficient, redundant or in proper amount depends on the specific task of tracking influential nodes. In Sections[5] and[6] respectively, we discuss this issue for two common tasks of tracking influential nodes, namely tracking nodes with influence greater than a threshold and tracking top-k influential nodes.

4. UPDATING RR SETS

In this section, we propose an incremental algorithm for updating existing RR sets over a stream of edge weight updates under both the LT model and the IC model. We prove that, by updating RR sets using our algorithm, \( n_{FR}(S) \) is always an unbiased estimation of \( I(S) \). We also analyze the cost of an update based on the assumption that we are maintaining in total \( M \) RR sets. Note that the value of \( M \) should be decided for specific tasks. In Sections[5] and[6] we discuss the value of \( M \) for two common tasks of tracking influential nodes.

4.1 Updating under the LT Model

First, we have a key observation about random RR sets for the LT model.

**FACT 1. A random RR set of the LT model is a simple path.**

**RATIONALE.** In the equivalent “live-edge” selection process of the LT model, each node selects at most one incoming edge as a live edge. In the polling process, a random RR set is the set of nodes that can be reversely reachable from a randomly picked node v via live edges. Thus the nodes in a random RR set together form a simple path.

Fig[1] illustrates a random RR set. The end point \( v_1 \) is picked in random at the beginning of the polling process. Then the path is generated by reversely propagating from \( v_1 \). The reverse propagation ends at \( v_i \) because \( v_i \) picks one of the nodes already in the path as its previous node. Note that the situation that \( v_i \) does not pick any previous nodes can be regarded as \( v_i \) picks itself as the previous node.

For a random RR set, suppose the starting node is \( v_1 \), we also store the previous node picked by \( v_1 \), which is useful in our algorithm for updating random RR sets maintained. We maintain an inverted index on all random RR sets so that we can access all the random RR sets passing a node. Moreover, we assume that the whole graph is stored and maintained in a way allowing random access to every node and its in-neighbors.

When there is an edge weight update \( (u,v,+/−,Δ,t) \) at time \( t \), our incremental algorithm works as follows. Denote by \( w'_{uv} \), the edge weight of \( (u,v) \) and \( W_v^t \) the total weight of v at time \( t \). We first update the edge weight of \( (u,v) \) and the total weight of v in the graph. Then, we consider the following two cases.

1. If the update is a weight increase \( (u,v,+,Δ,t) \), we retrieve all RR sets passing v using the inverted index. For each RR set retrieved, with probability \( \frac{Δ}{W_v^t} \), it is rerouted from v. If a RR set is rerouted, the previous node of v is set to u and we keep reversely propagating until no new nodes can be reversely reached.

2. If the update is a weight decrease \( (u,v,−,Δ,t) \), we retrieve all RR sets passing v where the previous node of v is u. Each retrieved RR set is rerouted from v with probability \( \frac{−Δ}{W_v^t} \). If a RR set is rerouted, we choose u among the in-neighbors of v at time \( t \) as the previous node of v with probability \( \frac{w'_{uv}}{W_v^t} \). We keep reversely propagating until no new nodes can be reversely reached.

When rerouting random RR sets, we use random access to obtain the nodes and the in-neighbors of them in the graph. We also update the inverted index.

Now we prove that, after our incremental maintenance of the random RR sets, \( n_{FR}(S) \) is still an unbiased estimator of \( I(S) \). To prove this, we only need to show that, at any time \( t \), the probability \( pp_{uv}^{t} \) that v selects its in-neighbor u is \( \frac{w'_{uv}}{W_v^t} \).

**THEOREM 1 (UNBIASED ESTIMATION (LT)).** At time \( t \), after the incremental maintenance of the random RR sets as described in this section, for any \( (u,v) \) where u is an in-neighbor of v at time \( t \), u is picked as the previous node of v with probability \( \frac{w'_{uv}}{W_v^t} \).
**Proof.** We only need to consider the basic case where, at time \( t \), there is at most one edge weight update. A general case of multiple weight updates can be simply treated as a series of the basic case.

We prove by induction. Apparently, at time 0, when the network has no edges, the theorem holds.

Assume when \( t = k - 1 \) \((k \geq 1)\), the probability that \( v \) selects its in-neighbor \( u \) is \( pp^k_{uv} = \frac{w_{uv}^{k-1}}{W_v^k} \).

When \( t = k \), three possible situations may arise.

**Case 1:** There is no update on any incoming edges of \( v \). In such a case, for each \( u \) that is an in-neighbor of \( v \), \( pp^k_{uv} = pp^{k-1}_{uv} = \frac{w_{uv}^{k-1}}{W_v^k} = \frac{w_{uv}^{k-1}}{W_v^k} \).

**Case 2:** An edge weight increase \((u, v, +, \Delta k)\) happens at time \( t = k \). So, \( w_{uv}^k = w_{uv}^{k-1} + \Delta k \) and \( W_v^k = W_v^{k-1} + \Delta k \). For \( u \), we have

\[
pp^k_{uv} = pp^{k-1}_{uv} \left(1 - \frac{\Delta k}{W_v^k}\right) + \frac{\Delta k}{W_v^k} \frac{w_{uv}^{k-1}}{W_v^{k-1}} \frac{w_{uv}^{k-1}}{W_v^{k-1}} = \frac{w_{uv}^{k-1}}{W_v^{k-1}} \frac{w_{uv}^{k-1}}{W_v^{k-1}} = \frac{w_{uv}^{k-1}}{W_v^{k-1}}
\]

For any other in-neighbor \( u' \) of \( v \), at time \( t = k \),

\[
pp^k_{u'v} = pp^{k-1}_{u'v} \left(1 - \frac{\Delta k}{W_v^k}\right) + \frac{\Delta k}{W_v^k} \frac{w_{u'v}^{k-1}}{W_v^{k-1}} \frac{w_{u'v}^{k-1}}{W_v^{k-1}} = \frac{w_{u'v}^{k-1}}{W_v^{k-1}} \frac{w_{u'v}^{k-1}}{W_v^{k-1}} = \frac{w_{u'v}^{k-1}}{W_v^{k-1}}
\]

**Case 3:** An edge weight decrease \((u, v, -, \Delta k)\) happens at time \( t = k \). Note that \( w_{uv}^k = w_{uv}^{k-1} - \Delta k \) and \( W_v^k = W_v^{k-1} - \Delta k \). For \( u \), we have

\[
pp^k_{uv} = pp^{k-1}_{uv} \left(1 + \frac{\Delta k}{W_v^k}\right) + \frac{\Delta k}{W_v^k} \frac{w_{uv}^{k-1}}{W_v^{k-1}} \frac{w_{uv}^{k-1}}{W_v^{k-1}} = \frac{w_{uv}^{k-1}}{W_v^{k-1}} \frac{w_{uv}^{k-1}}{W_v^{k-1}} = \frac{w_{uv}^{k-1}}{W_v^{k-1}}
\]

For any other in-neighbor \( u' \) of \( v \) other than \( u \),

\[
pp^k_{u'v} = pp^{k-1}_{u'v} \left(1 + \frac{\Delta k}{W_v^k}\right) + \frac{\Delta k}{W_v^k} \frac{w_{u'v}^{k-1}}{W_v^{k-1}} \frac{w_{u'v}^{k-1}}{W_v^{k-1}} = \frac{w_{u'v}^{k-1}}{W_v^{k-1}} \frac{w_{u'v}^{k-1}}{W_v^{k-1}} = \frac{w_{u'v}^{k-1}}{W_v^{k-1}}
\]

By treating \( v \) as also an in-neighbor of \( v \) itself and thus \( w_{v} = w_{vv} \), we can prove the case when the weight update is on \( w_v \). \( \square \)

We analyze the expected number of random RR sets needed to be rerouted if we have \( M \) RR sets. For a weight increase update \((u, v, +, \Delta t)\), before rerouting, the expected number of RR sets passing \( v \) is \( \frac{M \Delta t}{\mu_W v} \), where \( \mu_W v \) is the influence spread of \( v \) at time \( t - 1 \). Thus, the expected number of random RR sets rerouted is \( \frac{M \Delta t}{\mu_W v} \). Similarly, for a weight decrease update \((u, v, -, \Delta t)\), the expected number of random RR sets rerouted is \( \frac{M \Delta t}{\mu_W v} \). The number of RR sets rerouted in both cases is normally much smaller than \( M \).

Suppose at time \( t \), the expected cost of generating a RR set whose start point is \( v \) is \( C_v^t \), which is decided by the influence network \( G \) at time \( t \). Apparently, the cost of rerouting a random RR set at time \( t \) is no more than that of generating a new RR set. Thus, the expected cost of updating \( M \) random RR sets is \( \frac{M \Delta t}{\mu_W v} C_v^t \) for a weight increase update \((u, v, +, \Delta t)\), and \( \frac{M \Delta t}{\mu_W v} C_v^t \) for a weight decrease update \((u, v, -, \Delta t)\).

The value of \( C_v^t \) is not easy to figure out. However, we can analyze the expected value of \( C_v^t \) with the expectation taken over all

**4.2 Updating under the IC Model**

Rather than a simple path, a random RR set in the IC model is a random connected component. Fig. 2 illustrates an example. Suppose the start point (the randomly picked node at the beginning of a poll) of a RR set is \( v_1 \), then each node in this RR set can be reversely reachable from \( v_1 \) via live edges.

For a random RR set, we not only record the nodes in it but also all live edges among those nodes. We categorize live edges into two classes, namely BRS edges and cross edges. When a RR set is being generated by reversely propagating from the start point in a breadth-first search manner, if a live edge \((v_i, v_j)\) makes \( v_j \) propagated for the first time, \((v_i, v_j)\) is labeled as a BRS edge; otherwise it is labeled as a cross edge. For each node in a RR set, we use an adjacent list to store all live edges pointing to it. We also treat every node as a string and keep all nodes in a RR set in a prefix tree for fast retrieving a node and the address of its adjacent list of live edges.

Like the LT model, for the IC model, we also maintain an inverted index on all random RR sets so that we can access all RR sets containing a node. Since in the “live-edge” process of the IC model, every edge is picked independently, when there is an update \((u, v, +, \Delta t)\) at time \( t \), if \( u \) does not belong to this RR set at time \( t - 1 \), we further extend this RR set by reversely propagating from \( u \) in a breadth-first search manner.

Suppose \( t = k \), the probability that \( u \) appears in a random RR set is \( \frac{k}{n} \). Thus, the expected length of a random RR set at time \( t \) is \( \frac{1}{2} \sum_{v \in V} p_t^v = \bar{P} \), where \( \bar{P} \) is the average influence spread of individual nodes at time \( t \). Similarly, the expected cost of generating a random RR set at time \( t \) is \( \sum_{v \in V} \frac{1}{2} d_v = \frac{m}{n} \sum_{v \in V} \bar{P} \), where \( d_v \) is the number of in-neighbors of \( u \) at time \( t \). An upper bound of this cost is \( \frac{m}{n} \bar{P}_{\text{max}} \), where \( \bar{P}_{\text{max}} \) is the influence spread of the most influential node at time \( t \). Therefore, \( \frac{1}{2} \sum_{v \in V} C_v^t \leq \frac{m}{n} \bar{P}_{\text{max}} \).

![Figure 2: A random RR set of the IC model is a random connected component.](image-url)

At time \( t \), the probability that \( u \) appears in a random RR set is \( \frac{k}{n} \). Thus, the expected length of a random RR set at time \( t \) is \( \frac{1}{2} \sum_{v \in V} p_t^v = \bar{P} \), where \( \bar{P} \) is the average influence spread of individual nodes at time \( t \). Similarly, the expected cost of generating a random RR set at time \( t \) is \( \sum_{v \in V} \frac{1}{2} d_v = \frac{m}{n} \sum_{v \in V} \bar{P} \), where \( d_v \) is the number of in-neighbors of \( u \) at time \( t \). An upper bound of this cost is \( \frac{m}{n} \bar{P}_{\text{max}} \), where \( \bar{P}_{\text{max}} \) is the influence spread of the most influential node at time \( t \). Therefore, \( \frac{1}{2} \sum_{v \in V} C_v^t \leq \frac{m}{n} \bar{P}_{\text{max}} \).
Similar to the LT model, after updating the RR sets, we also update the inverted index.

Clearly, our incremental maintenance ensures that, for each edge \((u, v)\) at time \(t\), if \(v\) is a node of a RR set, the probability that \((u, v)\) is a live edge of this RR set is \(w^t_{uv}\).

**Theorem 2** *(Unbiased estimation (IC)).* At time \(t\), after the incremental maintenance of the random RR sets as described in this section, \(\forall (u, v) \in E^t\) where \(E^t\) is the edge set at time \(t\), if \(v\) is in a RR set, then \((u, v)\) is a live edge of this RR set with probability \(w^t_{uv}\).

According to Theorem 2 after our incremental maintenance of the random RR sets, \(nF_R(\mathcal{S})\) is still an unbiased estimator of \(I(\mathcal{S})\).

In our incremental maintenance, we need to find out if an edge \((u, v)\) is a live edge in a RR set. Suppose the number of nodes in a RR set is \(L\). Because normally the length of a node id is a constant, given an edge \((u, v)\), using the prefix tree we can find the address of \(v\)'s adjacent list in \(O(1)\) time. Then a linear search is performed to find out if \((u, v)\) is a live edge. Since in practice propagation probabilities are often small and \(\sum_{v \in N_u(v)N_v(u)}w_{uv}\) is often a small constant, in practice the average complexity of the linear search is \(O(1)\) and in total we only need \(O(1)\) time to decide if \((u, v)\) is a live edge in a RR set. Moreover, the space complexity of the RR set is \(O(L)\) in practice since every node only has a constant number of live edges pointing to it.

For the second situation when a live edge \((u, v)\) is deleted, it is not always necessary to traverse from the start point, which takes \(O(L)\) time if there are \(L\) nodes in the RR set. It is easy to see that removing cross edges does not change the connectivity of nodes in a RR set. Thus, if the removed live edge is labeled as a cross edge, we do not need to further update the RR set.

We analyze the expected number of random RR sets needed to be rerouted if we have \(M\) RR sets. Before updating, in expectation there are \(\frac{ME^{-1}}{n}\) RR sets containing \(v\). So the expected number of RR sets containing a live edge \((u, v)\) is \(\frac{ME^{-1}w^t_{uv}}{n}\) and the expected number of RR sets that do not contain \((u, v)\) as a live edge is \(\frac{ME^{-1}(1-w^t_{uv})}{n}\). Therefore, when there is an update on the edge \((u, v)\), no matter it is weight increase or weight decrease, the expected number of RR sets needed to be updated is \(\frac{ME^{-1}}{n}\).

For a weight increase update \((u, v, +, \Delta, t)\), adding a live edge \((u, v)\) only takes \(O(1)\) time. Similar to the incremental maintenance of the RR sets in the LT model, the expected cost of extending a RR set at time \(t\) is no more than generating a new one. Thus, the expected cost of processing a weight increase update \((u, v, +, \Delta, t)\) has an upper bound \(\frac{ME^{-1}\Delta}{n}C^t_u\), where \(C^t_u\) is the expected cost of generating a RR set with \(u\) as the start point. The average value of \(C^t_u\) (with the average taken over all \(u \in V\)) has an upper bound \(\frac{ME^{-1}\Delta}{n}\).

For a weight decrease update \((u, v, -, \Delta, t)\), removing a live edge \((u, v)\) takes \(O(1)\) time. As mentioned above, the cost of traversing from the start point via live edges is \(O(L)\) where \(L\) is the number of nodes in the RR set. Suppose the expected number of nodes of a RR set containing both \(u\) and \(v\) at time \(t\) is \(L^t_{uv}\), which is decided by the influence network \(G\) at time \(t\). The expected cost of processing a weight decrease update \((u, v, -, \Delta)\) is \(O\left(\frac{ME^{-1}L^t_{uv}}{n}\right)\). Similar to \(C^t_u\), the average value of \(L^t_{uv}\) has an upper bound \(L^t_{max}\).

A natural problem setting of finding influential nodes is to find all nodes whose influence spread is at least \(T\), where \(T\) is a user-specified threshold. In this section, we discuss how to use random RR sets to approximate the desired result.

Due to the \#P-hardness of computing influence spread under the LT model, it is not likely that we can find in polynomial time the exact set of nodes whose influence spread is at least \(T\). Thus, we turn to algorithms that allow controllable small errors. Specifically, we ensure that the recall of the set of nodes found by our algorithm is 100% and we tolerate some false positive nodes. Moreover, the influence spread of those false positive nodes should take a high probability to have a lower bound that is not much smaller than \(T\). We set the lower bound to \(T - \varepsilon n\), where \(\varepsilon\) controls the error.

According to Corollary 1, the larger \(M\), the more accurate the unbiased estimator \(nF_R(u)\). Thus, the intuition of deciding \(M\) is to make sure that, for each \(u\), \(nF_R(u)\) is large enough when \(I_u \geq T\), and small enough when \(I_u \leq T - \varepsilon n\).

We first show that \(nF_R(u)\) is not likely to be too much smaller than \(T\) if \(I_u \geq T\) and \(M\) is large enough.

**Lemma 1**. With \(M\) random RR sets, if \(I_u \geq T\), with probability at least \(1 - \exp(-\frac{M\varepsilon^2 n}{8T})\), \(nF_R(u) \geq T - \frac{\varepsilon n}{2}\).

**Proof.** If \(I_u \geq T\), we have
\[
Pr\{nF_R(u) \leq T - \frac{\varepsilon n}{2}\} = Pr\{nF_R(u) \leq I_u - (I_u - T + \frac{\varepsilon n}{2})\} \\
= Pr\{nF_R(u) \leq 1 - I_u - T + \frac{\varepsilon n}{2} I_u\} \\
\leq \exp\left\{ - \frac{M(I_u - T + \frac{\varepsilon n}{2})^2}{2nI_u} \right\}
\]
\[(I_u - T + \frac{\varepsilon n}{2})\] is non-decreasing with respect to \(I_u\) when \(I_u \geq T\). Thus,
\[
Pr\{nF_R(u) \leq T - \frac{\varepsilon n}{2}\} = \exp\left(-\frac{M\varepsilon^2 n}{8T}\right)
\]

Similarly, if \(I_u \leq T - \varepsilon n\), the probability that \(nF_R(u)\) is abnormally large is pretty small when \(M\) is large.

**Lemma 2**. With \(M\) random RR sets, if \(I_u \leq T - \varepsilon\), with probability at least \(1 - 2\exp(-\frac{M\varepsilon^2 n}{12T})\), \(nF_R(u) \leq T - \frac{\varepsilon n}{2}\).

**Proof.** We prove that if \(I_u \leq T - \varepsilon\), \(Pr\{nF_R(u) - I_u \geq \frac{\varepsilon n}{2}\} \leq 2\exp(-\frac{M\varepsilon^2 n}{12T})\). Note that \(nF_R(u) - I_u \leq \frac{\varepsilon n}{2}\) is a sufficient condition for \(nF_R(u) \leq T - \frac{\varepsilon n}{2}\) when \(I_u \leq T - \varepsilon\).

First, suppose \(T \geq \frac{\varepsilon n}{2}\), which means \(\frac{\varepsilon n}{2} \leq T - \varepsilon n\). There are two possible cases.

**Case 1:** \(\frac{\varepsilon n}{2} \leq I_u \leq T - \varepsilon n\). Then,
\[
Pr\{|nF_R(u) - I_u| \geq \frac{\varepsilon n}{2}\} \leq Pr\{|M|F_R(u) - \frac{MI_u}{n}| \geq \frac{\varepsilon M}{2}\} \\
\leq 2\exp\left(-\frac{1}{3} \frac{MI_u \varepsilon^2 n^2}{4MI_u}\right) \\
\leq 2\exp\left(-\frac{M\varepsilon^2 n}{12I_u}\right) \leq 2\exp(-\frac{M\varepsilon^2 n}{12T})
\]

**5. TRACKING THRESHOLD-BASED INFLUENTIAL NODES**
Case 2: \( I_u \leq \frac{\varepsilon}{2} \). Then,

\[
\Pr\{nF_R(u) - I_u \geq \frac{\varepsilon n}{2}\} = \Pr\{M_F(u) - \frac{MI_u}{n} \geq \frac{\varepsilon M}{2}\} \\
\leq \exp(-\frac{1}{2 + \varepsilon^2\frac{2n}{M^2}} \frac{M^2I_u^2}{n^2} \frac{n}{4I_u^2}) \\
\leq \exp(-\frac{1}{2 + \varepsilon^2\frac{2n}{M^2}} \frac{M^2I_u^2}{n^2} \frac{n}{4I_u^2}) \\
\leq \exp(-\frac{3M^2}{16}) \leq 2\exp(-\frac{M^2\varepsilon^2n}{12T})
\]

Second, if \( T \leq \frac{3\varepsilon n}{2} \), for all \( I_u \leq T - \varepsilon n, I_u \leq \frac{\varepsilon n}{2} \). Then, all \( I_u \leq T - \varepsilon n \) fall into Case 2 above and the lemma still holds.

Because \( \exp(-\frac{M^2\varepsilon^2n}{12T}) \leq 2\exp(-\frac{M^2\varepsilon^2n}{12T}) \), by applying Boole’s inequality (that is, the Union bound), with probability at least \( 1 - 2n \cdot \exp(-\frac{M^2\varepsilon^2n}{12T}) \), every \( nF_R \) satisfies the conditions in Lemmas 1 and 2. Therefore, we have the following theorem on the sample size \( M \) for finding nodes whose influence spread is at least \( T \).

**Theorem 3.** By setting the number of random RR sets \( M = \frac{127}{2\delta^4} \ln \frac{2T}{\varepsilon} \), with probability at least \( 1 - \delta \) the following conditions hold for every node \( u \).

1. If \( I_u \geq T \), then \( nF_R(u) \geq T - \frac{\varepsilon n}{T} \).
2. If \( I_u < T - \varepsilon n \), then \( nF_R(u) < T - \frac{\varepsilon n}{T} \).

One nice property of \( M \) in Theorem 3 is that, given \( n, T, \varepsilon \) and \( \delta \), \( M \) is a constant. Therefore, when we track nodes of influence spread at least \( T \) in a dynamic network, no matter how the network changes, the sample size \( M \) remains the same.

6. TRACKING TOP-K MOST INFLUENTIAL NODES

Another useful problem setting is to find the top-\( k \)-influential nodes, where \( k \) is a user-specified parameter.

Denote by \( I^k \) the influence spread of the \( k \)-th most influential node. Extracting top-\( k \) influential individual nodes equals extracting all nodes whose influence spread is at least \( I^k \). Again, due to the \#P-harness of influence computation, we probably have to tolerate errors in the result when designing algorithms. Similar to the task in Section 5, we hope the result returned by our algorithm contains all real top-\( k \)-nodes, and for each false-positive node returned, its influence spread is no smaller than \( I^k - \varepsilon n \) with a high probability.

In this section, we first analyze the number of random RR sets \( M \) we need to achieve the above goal with a high probability. We show that \( M \) is proportional to the maximum individual influence spread \( I_{\text{max}} \) and devise an algorithm that can give a really good estimation of \( I_{\text{max}} \) with a high probability. Then, combining the theoretical results in Section 6, we propose a method that improves the precision of the result set of nodes, that is, reducing the number of false-positive nodes.

6.1 Sample Size

Unlike the task in section 5, we do not know the threshold \( I^k \) in advance. Thus when selecting nodes according to values of \( nF_R(u) \), we do not have a threshold value. This is similar to mining top-\( k \)-itemsets using sampled transactions [28, 29]. The intuition of our idea to solve the problem is that, if we have enough samples, we can bound the threshold value within a small range.

To collect all real top-\( k \) influential nodes and filter out all nodes whose influence spreads are smaller than \( I^k - \varepsilon n \), we sample enough random RR sets such that for every \( u \in V \), \( |nR(u) - I_u| \leq \frac{\varepsilon n}{T} \) with a high probability. Denote by \( F_R(k) \) the \( k \)-th highest \( F_R \) value. We have the following result.

**Lemma 3.** If for all \( u \in V \), \( |nF_R(u) - I_u| \leq \frac{\varepsilon n}{T} \), then the following conditions holds, (1) if \( I_u \geq I^k \), then \( nF_R(u) \geq nF_R^k - \frac{\varepsilon n}{T} \); and (2) if \( I_u \leq I^k - \varepsilon n \), then \( nF_R(u) \leq nF_R^k - \frac{\varepsilon n}{T} \).

**Proof.** First, \( nF_R(k) \geq I^k - \varepsilon n \) because there are at least \( k \) nodes having the value of \( nF_R \) at least \( I^k - \varepsilon n \). Second, \( nF_R^k \leq I^k + \frac{\varepsilon n}{T} \) because there are at most \( k \) nodes having the value of \( nF_R \) at least \( I^k - \frac{\varepsilon n}{T} \). Thus, we have \( nF_R(k) - \frac{\varepsilon n}{T} \leq I^k \leq nF_R^k + \frac{\varepsilon n}{T} \).

If \( I_u \geq I^k \), we have \( nF_R(u) \geq I_u - \frac{\varepsilon n}{T} \geq I^k - \frac{\varepsilon n}{T} \geq nF_R^k - \frac{\varepsilon n}{T} \). If \( I_u \leq I^k - \varepsilon n \), we have \( nF_R(u) \leq I_u + \frac{\varepsilon n}{T} \leq I^k - \frac{\varepsilon n}{T} \leq nF_R^k - \frac{\varepsilon n}{T} \).

So we need to derive a lower bound of \( M \) to make sure \( |nF_R(u) - I_u| \leq \frac{\varepsilon n}{T} \) for every \( u \in V \) with a high probability. Denote by \( I_{\text{max}} \), the maximum individual influence spread.

**Lemma 4.** When the number of random RR sets is \( M \), with probability at least \( 1 - 2\exp(-\frac{M^2\varepsilon^2n}{48T}) \), \( |nF_R(u) - I_u| \leq \frac{\varepsilon n}{T} \).

**Proof.** We need to consider two possible cases.

**Case 1:** If \( I_u \geq \frac{\varepsilon n}{T} \)

\[
\Pr\{|nF_R(u) - I_u| \geq \frac{\varepsilon n}{4}\} \leq 2\exp(-\frac{(\frac{M^2\varepsilon^2n}{48T})}{3}\frac{MI_u}{n}) \\
\leq 2\exp(-\frac{M^2\varepsilon^2n}{48T})
\]

**Case 2:** If \( I_u \leq \frac{\varepsilon n}{T} \)

\[
\Pr\{|nF_R(u) - I_u| \geq \frac{\varepsilon n}{4}\} = \Pr\{nF_R(u) - I_u \geq \frac{\varepsilon n}{4}\} \\
\leq \exp(-\frac{3\varepsilon}{8} \frac{MI_u}{n}) = \exp(-\frac{3M\varepsilon}{32}) \\
\leq 2\exp(-\frac{M^2\varepsilon^2n}{48T})
\]

By applying the Union Bound, with probability at least \( 1 - 2n \cdot \exp(-\frac{M^2\varepsilon^2n}{48T}) \), we have \( |nF_R(u) - I_u| \leq \frac{\varepsilon n}{T} \) for \( u \in V \). In sequel, we have the following theorem setting the value of \( M \).

**Theorem 4.** By setting the number of random RR sets \( M = \frac{48T}{\varepsilon^2} \ln \frac{2T}{\varepsilon} \), with probability at least \( 1 - \delta \) the following conditions hold. (1) If \( I_u \geq I^k \), then \( nF_R(u) \geq nF_R^k - \frac{\varepsilon n}{T} \); and (2) If \( I_u \leq I^k - \varepsilon n \), then \( nF_R(u) < nF_R^k - \frac{\varepsilon n}{T} \).

Unlike [28, 29, 27], the sample size in Theorem 4 not only depends on the confidence level \( 1 - \delta \) and the error \( \varepsilon \), but also is proportional to \( I_{\text{max}} \), which varies over different datasets. This is meaningful in practice, because for a social network, \( I_{\text{max}} \) is normally very small comparing to \( n \) [8, 17, 30]. One may link finding influential nodes with finding frequent itemsets remotely due to the intuition that a node frequent in many RR sets is likely influential. In sampling based frequent itemsets mining [28, 29, 27], the sample size is decided by \( \varepsilon \) and \( \delta \) only, and thus is in general larger than ours here.
6.2 Estimating $I_{max}$ while Sampling

The sample size $M$ in Theorem 4 depends on $I_{max}$, which is unknown and hard to compute in exact. In this subsection, we devise a sampling algorithm that gives a tight upper bound of $I_{max}$ with a high probability and at the same time samples enough random RR sets we need.

Our algorithm sets $M = \frac{48\ln 2n}{\delta^2}$ and progressively increases $x_n$ until it is enough larger than $I_{max}$. The intuition is that, if $nF_{R^*}$ is sufficiently smaller than $x_n$, probably the current $M$ is large enough.

Algorithm 2 shows our sampling method. We prove that the final random RR sets are enough and $x_n$ the upper bound of $I_{max}$ is tight. Note that the first 3 lines of Algorithm 2 ensure that $x \geq 4\epsilon$ and $\epsilon' \leq \frac{1}{4}$.  

**Lemma 5.** When $M = \frac{48\ln 2n}{\delta^2}$ and $\epsilon'x = \epsilon$, if $I_{max} \geq x_n$, with probability at least $1 - \delta_1$, $nF_{R^*} \geq (1 - \epsilon')x_n$, where $\delta_1 = \left(\frac{\delta}{2n}\right)^{24}$.

**Proof.** Suppose $u$ is a node with the maximum influence. Since $x_n \leq I_{max}$, we have

$$\Pr\{nF_{R^*} \leq (1 - \epsilon')x_n\} \leq \Pr\{nF_{R}(u) \leq (1 - \epsilon')I_{max}\}$$

$$= \exp(-\frac{\epsilon'^2M_{I_{max}}}{2})$$

$$\leq \exp(-\frac{\epsilon'^2M_{I}}{2})$$

$$= \exp(-\frac{\epsilon'^x48\ln 2n}{2\epsilon'^2})$$

$$= \exp(-24\ln \frac{2n}{\delta}) = \left(\frac{\delta}{2n}\right)^{24}$$

Lemma 6 shows that with a high probability, if $F_{R^*} \leq (1 - \epsilon')x$, then the current random RR sets are enough.

**Lemma 6.** When $x_n \geq I_{max}$, $M = \frac{48\ln 2n}{\delta^2}$ and $\epsilon'x = \epsilon$, with probability at least $1 - \delta_2$, if $I_u \leq (1 - 2\epsilon')x_n$, then $nF_{R}(u) \leq (1 - \epsilon')x_n$, where $\delta_2 = n\left(\frac{\delta}{2n}\right)^{16}$.

**Proof.** We have two possible cases.

1. If $\frac{(1 - \epsilon')x_n}{2} \leq I_u \leq (1 - 2\epsilon')x_n$,

$$\Pr\{nF_{R}(u) \geq (1 - \epsilon')x_n\}$$

$$= \Pr\{nF_{R}(u) \leq 1 + \frac{((1 - \epsilon')x_n - I_u)\cdot|I_u|}{I_{max}}\}$$

$$\leq \exp(-\frac{3}{8}\frac{((1 - \epsilon')x_n - I_u)\cdot|I_u|}{I_{max}})$$

$$\leq \exp(-\frac{3}{8}\frac{(1 - \epsilon')M_{I}}{16})$$

$$\leq \exp(-16\ln \frac{2n}{\delta}) = \left(\frac{\delta}{2n}\right)^{18}$$

**Case 2:** If $I_u \leq \frac{(1 - \epsilon')x_n}{2}$,

$$\Pr\{nF_{R}(u) \geq (1 - \epsilon')x_n\}$$

$$= \Pr\{nF_{R}(u) \leq 1 + \frac{((1 - \epsilon')x_n - I_u)\cdot|I_u|}{I_{max}}\}$$

$$\leq \exp(-\frac{3}{8}\frac{((1 - \epsilon')x_n - I_u)\cdot|I_u|}{I_{max}})$$

$$\leq \exp(-\frac{3}{8}\frac{(1 - \epsilon')M_{I}}{16})$$

$$\leq \exp(-\frac{9\ln 2n\delta}{2\epsilon'^2})$$

$$\leq \exp(-16\ln \frac{2n}{\delta}) = \left(\frac{\delta}{2n}\right)^{18}$$

Applying the Union Bound, we have that, with probability at least $1 - n\left(\frac{\delta}{2n}\right)^{16}$, $nF_{R}(u) \leq (1 - \epsilon')x_n$ for any $u$ such that $I_u \leq (1 - 2\epsilon')x_n$.

Lemma 6 implies that when $x_n \geq \frac{I_{max}}{2n\delta}$, $nF_{R^*} \leq (1 - \epsilon')x_n$ with a high probability. The first time in Algorithm 2 when $x_n \geq \frac{I_{max}}{2n\delta}$, we have $x_n \leq \max(4\epsilon n, \frac{I_{max}}{2n\delta} + \epsilon n)$. If we set $\epsilon$ small enough (i.e., smaller than $\frac{I_{max}}{4n\delta}$), the upper bound $x_n$ is at most $\frac{I_{max}}{2n\delta} + \epsilon n \leq \frac{2}{3}I_{max}$. This is achievable in practice since $I_{max}$ has some trivial lower bounds, such as max over $\sum_i \sum_j P_{uv}$.

**Theorem 5.** If $\epsilon = \Omega(1)$, then with probability $1 - o\left(\frac{1}{n^2}\right)$, Algorithm 2 returns $M = \frac{48\ln 2n}{\delta^2}$ $\geq \frac{48I_{max}}{n\epsilon^2}$ random RR sets, and $x_n \leq \max(4\epsilon n, \frac{I_{max}}{2n\delta})$.

**Proof.** Apparently an upper bound of $x$ is 1. So when $\epsilon = \Omega(1)$, the number of the test on whether $F_{R^*} \leq (1 - \epsilon')x$ is conducted at most $O(1)$ times. Each time the failure probability is at most $n\left(\frac{\delta}{2n}\right)^{16}$ and $\left(\frac{\delta}{2n}\right)^{24} = o\left(\frac{1}{n^2}\right)$. Therefore, in total the failure probability is $o\left(\frac{1}{n^2}\right)$.

When the network is updated, the value of $I_{max}$ may change. Thus, after we update the random RR sets, we called Algorithm 2 to ensure that we have enough random RR sets. In addition, if $I_{max}$ decreases dramatically, which means the current sample size is too large, we also have a similar test to decide how many random RR sets we can abandon. The intuition is that if the current $x$ is too much bigger than $F_{R^*}$, probably we can use a smaller $x$.

**Lemma 7.** When $M = \frac{48\ln 2n}{\delta^2}$ and $y \leq x$, with probability $1 - \delta_3$ the following statement holds: If $F_{R^*} \leq (1 - \epsilon_3)y$ where $\epsilon_3 = \sqrt{\frac{(1 - \epsilon')x_n}{24\epsilon y \ln \frac{2n}{\delta}}}$, then $I_{max} \leq y$. 

**Proof.**
If we set \( \delta_1 = \frac{\delta}{2n} \) then \( \epsilon_1 = \frac{\epsilon}{\sqrt{2n}} \). So when \( F_{R^*} \) decreases because of the changes of network, we find the minimum \( y \) such that \( y \geq \frac{F_{R^*} - \epsilon_1}{\epsilon_1} \). If \( y \leq x \), we can substitute the current \( x \) with \( y \) and discard \( \frac{\delta (x-y)}{\epsilon} \) random RR sets.

**6.3 Improving Precision**

According to Theorem 4, we filter out nodes such that \( n F_{R^*}(u) < n F_R(u) < n F_{R^*}(u) \). Using Theorem 3 we can further improve the filtering threshold to make the precision higher.

In the proof of Lemma 3, we show that \( n F_{R^*}(u) < n F_R(u) \leq \frac{x}{\epsilon} \). Thus, we can filter out all nodes \( u \) such that \( I_0 \leq n F_{R^*}(u) < n F_R(u) \). If our current sample size is \( M = \frac{48 \sqrt{n \epsilon}}{\delta} \), by applying Theorem 6 we have a new filtering condition \( F_{R^*}(u) < \frac{F_{R^*} - \epsilon_1}{\epsilon} \leq \frac{F_{R^*}}{x} \) where \( \epsilon_1 = \sqrt{\frac{F_{R^*} - \epsilon_1}{\epsilon}} \). All nodes meeting this new condition should be filtered out. In this way more nodes can be filtered out and the precision of the result can be improved, meanwhile the failure probability is only increased by \( \delta \) at most.

**6.4 Maintaining \( F_{R^*} \) Dynamically**

When applying our sampling algorithm, it is important to find \( n F_{R^*} \) efficiently because the test on whether \( F_{R^*} \) is frequently called. A brute force solution is to perform an \( O(n) \) search but the cost may be unacceptably high in practice. To solve this problem, we devise a simple data structure that can retrieve \( n F_{R^*} \) in \( O(n \log n + n F_{R^*}) \) time. Note that \( F_{R^*} \) is a fraction substantially smaller than \( 1 \) in practice.

We use an array \( C \) of \( O(n) \) space for counting how many nodes \( u \) such that \( |n F_{R^*}(u)| = i \) for \( i = 1, 2, \ldots, n \). We use another array \( R_C \) of \( O(n \log n) \) space to count how many nodes meet the condition \( |n F_{R^*}(u)| \in \left[ \frac{n}{2}, \frac{n}{2} \right] \) for \( j = 1, 2, \ldots, \log n \), as illustrated in Fig. 3. When the random RR sets are updated and thus \( F_{R^*}(u) \) changes, we only need \( O(1) \) time to update \( C \) and \( R_C \). When we need to find \( n F_{R^*} \), we first use \( O(\log n) \) time to find the range \( \left[ \frac{n}{2}, \frac{n}{2} \right] \) containing it, which is the first range whose count is non-zero. Clearly \( \frac{n}{2} \) is a 2-approximation of \( \left[ n F_{R^*} \right] \). If we want the exact value of \( n F_{R^*} \), we issue a search on the part \( \left[ \frac{n}{2}, \frac{n}{2} \right] \) of \( C \) in \( O(n F_{R^*}) \) time to find \( n F_{R^*} \). The time complexity is \( O(n F_{R^*}) \) because the length of the range containing \( n F_{R^*} \) is obviously \( O(n F_{R^*}) \). Thus in total the cost of finding the exact value of \( n F_{R^*} \) is \( O(\log n + n F_{R^*}) \).

7. EXPERIMENTS

In this section, we report a series of experiments on three real networks to verify our algorithms. The experimental results demonstrate that our algorithms are both effective and efficient.

7.1 Experimental Settings

| Network   | Number of nodes | Number of edges | Average degree |
|-----------|-----------------|-----------------|----------------|
| wiki-Vote | 7,115           | 103,689         | 14.6           |
| Flixster  | 99,353          | 977,738         | 9.9            |
| soc-Pokec | 1,632,803       | 30,622,564      | 18.8           |

Table 2: The statistics of the data sets.

We ran our experiments on three real network data sets that are publicly available online (http://snap.stanford.edu and http://www.cs.ubc.ca/~wewu/). Table 2 shows the statistics of the three data sets.

To simulate dynamic networks, for each data set, we randomly partitioned all edges exclusively into 3 groups: \( E_1 \) (85% of the edges), \( E_2 \) (5% of the edges) and \( E_3 \) (10% of the edges). We used \( B = (V,E_1 \cup E_2) \) as the base network. \( E_2 \) and \( E_3 \) were used to simulate a stream of updates.

For the LT model, for each edge \((u,v)\) in the base network, we set the weight to be 1. For each edge \((u,v) \in E_3\), we generated a weight increase update \((u,v,+1)\) (timestamps ignored at this time). For each edge \((u,v) \in E_2\), we generated one weight decrease update \((u,v,-\Delta)\) and one weight increase update \((u,v,+\Delta)\) where \( \Delta \) was picked uniformly at random in \([0,1]\). We randomly shuffled those updates to form an update stream by adding random time stamps. For each data set, we generated 10 different instances of the base network and update stream, and thus ran the experiments 10 times. Note that for the 10 instances, although the base networks and update streams are different, the final snapshots of them are identical to the data set itself.

For the IC model, we first assigned propagation probabilities of edges in the final snapshot, i.e. the whole graph. We set \( w_{in} = \frac{1}{\text{in-degree}(v)} \), where in-degree \((v)\) is the number of in-neighbors of \( v \) in the whole graph. Then, for each edge \((u,v)\) in the base network, we set \( w_{in} \) to \( \frac{1}{\text{in-degree}(v)} \). For each edge \((u,v) \in E_3\), we generated a weight increase update \((u,v,+1)\) (timestamps ignored at this time). For each edge \((u,v) \in E_2\), we generated one weight decrease update \((u,v,-\Delta)\) and one weight increase update \((u,v,+\Delta, \frac{1}{\text{in-degree}(v)})\) where \( \Delta \) was picked uniformly at random in \([0,1]\). We randomly shuffled those updates to form an update stream by adding random time stamps. For each dataset we also generated 10 instances.

For the parameters of tracking nodes of influence at least \( T \), we set \( e = 0.0002 \) and \( \delta = 0.001 \) for all data sets, and set \( T = 0.001 \times n \) for wiki-Vote and Flixster, \( T = 0.005 \times n \) for soc-Pokec. This is because in large scale networks like soc-Pokec, there are not many nodes that can influence more than 0.1% of the nodes. Based on the results of this task, we set different parameters for top-k influential nodes tracking for different data sets. Specifically, we set \( K = 50 \) and \( \delta = 0.001 \) for all three data sets, and set \( e = 0.0005, 0.001, 0.0008 \) for wiki-Vote, Flixster and soc-Pokec, respectively.

We did not compare with the heuristic algorithms like Page-Rank, since those methods do not have provable quality guarantees.

All algorithms were implemented in Java and ran on a Linux machine of an Intel Xeon 2.00GHz CPU and 100GB main memory.

7.2 Effectiveness

A challenge in evaluating the effectiveness of our algorithms is that the ground truth is hard to obtain. The existing literature of influence maximization [19, 2, 17, 31, 30, 12] always use the influence spread estimated by 20,000 times Monte Carlo (MC) simulations as the ground truth. However, such a method is not suitable for our tasks, because the ranking of nodes really matters here. Even 20,000 times MC simulations may not be able to distinguish...
nodes with close influence spread. As a result, the ranking of nodes may differ much from the real ranking. Moreover, the effectiveness of our algorithms has theoretical guarantees while 20,000 times MC simulations is essentially a heuristic. It is not reasonable to verify an algorithm with a theoretical guarantee using the results obtained by a heuristic method without any quality guarantees.

In our experiments, we only used wiki-Vote and Flixster to run MC simulations and compare the results to those produced by our algorithms. We used 2,000,000 times MC simulations as the (pseudo) ground truth in the hope that we can get more accurate results. According to our experiments, even so many MC simulations may generate slightly different rankings of nodes in two different runs but the difference is acceptably small. We only compare results on the identical final snapshot shared by all instances because running MC simulations on multiple snapshots is unaffordable (10 days on the final snapshots of Flixster).

Tables 3-8 report on the wiki-Vote and Flixster data sets, respectively, the recall of the sets of influential nodes returned by our algorithms and the maximum errors of the false positive nodes in absolute influence value. The results on the 10 runs (Ins1-Ins10) are reported. Our methods achieved 100% recall every time as guaranteed theoretically. Moreover, the real errors in influence were substantially smaller than the maximum error bound provided by our theoretical analysis.

Table 7 and Table 8 report our estimation of the upper bound of $I_{\text{max}}$ on the final snapshot of each network when tracking top-k influential nodes. The results indicate that the upper bound estimated by our algorithm is only a little greater than the real $I_{\text{max}}$.

We also report the recall of the top ranked nodes estimated by sampled random RR sets in Fig. 4 and Fig. 5. The measure $Recall@N$ is calculated by $\frac{T_P\supseteq R}{TP}$, where $TP$ is the number of nodes ranked top-$N$ by both our algorithms and the ground truth. The results show that the rankings of the top nodes generated by our algorithms have very good quality.

For the largest data set soc-Pokec, we did not run 2,000,000 times MC simulations to obtain the pseudo ground truth since the MC simulations are too costly. Instead, we compare the similarity between the results generated by different instances. Recall that the final snapshots of the 10 instances are the same. If the sets of influential nodes at the final snapshots of the 10 instances are similar, at least our algorithms are stable, that is, insensitive to the order of updates. To measure the similarity between two sets of influential nodes, we adopted the Jaccard similarity.

Fig. 6 and Fig. 7 show the results where $I_1, \ldots, I_{10}$ represent the results of the first, $\ldots$, tenth instances, respectively. We also ran the sampling algorithm directly on the final snapshot, that is, we computed the influential nodes directly from the final snapshot using sampling without any updates. The result is denoted by ST. The results show that the outcomes from different instances are very similar, and they are similar to the outcome from ST, too. The minimum similarity in all cases is 87%. The similarities in the top-k tasks are higher than those in the threshold-based tasks. This is because the sample size of top-k task is normally much larger. Thus, the estimated influence spread $nF_{\text{R}}(u)$ is generally more accurate.

### 7.3 Scalability

We also tested the scalability of our algorithms. Fig. 8 shows the average running time with respect to the number of updates processed. The average is taken on the running times of the 10 instances. The time spent when the number of updates is 0 reflects the computational cost of running the sampling algorithm on the base network. Clearly, the non-incremental algorithm (rerunning the sampling algorithm from scratch when the network changes) is not competent at all because the running time of processing the base network and the update stream is only several times larger than the running time of processing the base network, and the number of updates is huge, tens of thousands or even millions. This result shows that our incremental algorithm outperforms rerunning the sampling algorithm from scratch by several orders of magnitude.

For the LT model, our algorithm scales up roughly linearly. For the IC model, the running time increases more than linear. This is due to our experimental settings. For the LT model, the sum of propagation probabilities from all in-neighbors is roughly 0.9 but becomes 1 at the beginning for the IC model. At the beginning, the sum of propagation probabilities from all in-neighbors of a node is always 1, while in the IC model, the sum of propagation probabilities from all in-neighbors is roughly 0.9 but becomes 1 finally. Thus, the increases of nodes change more dramatically in the IC model than in the LT model. According to our analysis in Section 4.2, the cost of updating the RR sets is proportional to $I_{\text{max}}^{-1}$, the influence of $v$ at time $t-1$, and $M$, the sample size. In the top-k task, $M$ is decided by $I_{\text{max}}$. Thus the running time curves of the IC model are not linear.

Fig. 8 shows how $I_{\text{max}}$ and the sample size $M$ in the top-k task changes over time in the soc-Pokec dataset. Recall in Fig. 8 and

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**Table 7: Estimated upper bounds of $I_{\text{max}}$ and the experimental values. (LT model)**

| Data set | Instance | $xn$ | $I_{\text{max}}$ | Approximation Ratio |
|----------|----------|------|------------------|---------------------|
| wiki-Vote | I1-I5, I7, I9-I10 | 60.4775 | 51.770 | 1.128 |
| Flixster | I1-I10 | 64.035 | 1.194 |

**Table 8: Estimated upper bounds of $I_{\text{max}}$ and the experimental values. (IC model)**

| Data set | Instance | $xn$ | $I_{\text{max}}$ | Approximation Ratio |
|----------|----------|------|------------------|---------------------|
| wiki-Vote | I1-I10 | 56.92 | 53.669 | 1.062 |
| Flixster | I1-I10 | 495.265 | 372.207 | 1.331 |

**Figure 4: Recall@N (LT model).**

(a) wiki-Vote $N = 1, 2, \ldots, 100$  
(b) Flixster $N = 1, 2, \ldots, 1000$

**Figure 5: Recall@N (IC model).**

(a) wiki-Vote $N = 1, 2, \ldots, 100$  
(b) Flixster $N = 1, 2, \ldots, 1000$
Table 3: Recall and error on Wiki-Vote. The errors are measured in absolute influence value. (LT model)

| Recall (Threshold) | Theoretical Value | Ins1 | Ins2 | Ins3 | Ins4 | Ins5 | Ins6 | Ins7 | Ins8 | Ins9 | Ins10 |
|--------------------|-------------------|------|------|------|------|------|------|------|------|------|-------|
| Max. Error (Threshold) | 0.0002 + 7115 = 1.423 | 0.793 | 0.793 | 0.793 | 0.793 | 0.793 | 0.793 | 0.793 | 0.793 | 0.793 | 0.793 |
| Recall (Top-K) | 100% | 100% | 100% | 100% | 100% | 100% | 100% | 100% | 100% | 100% | 100% |
| Max. Error (Top-K) | 0.0005 + 7115 = 3.5575 | 1.299 | 1.299 | 1.336 | 1.336 | 1.299 | 1.190 | 1.190 | 1.190 | 1.190 | 1.190 |

Table 4: Recall and error on Flixster. The errors are measured in absolute influence value. (LT model)

| Recall (Threshold) | Theoretical Value | Ins1 | Ins2 | Ins3 | Ins4 | Ins5 | Ins6 | Ins7 | Ins8 | Ins9 | Ins10 |
|--------------------|-------------------|------|------|------|------|------|------|------|------|------|-------|
| Max. Error (Threshold) | 0.0002 + 99053 = 19.81 | 10.42 | 11.05 | 11.05 | 10.55 | 10.94 | 9.89 | 11.54 | 10.82 | 10.68 | 11.19 |
| Recall (Top-K) | 100% | 100% | 100% | 100% | 100% | 100% | 100% | 100% | 100% | 100% | 100% |
| Max. Error (Top-K) | 0.001 + 99053 = 99.05 | 41.95 | 40.44 | 38.83 | 44.29 | 43.44 | 42.54 | 38.83 | 41.88 | 42.05 | 42.54 |

Figure 6: Similarity among results in different instances. (LT model)

Fig. [9], for every short time segment when $M$ is stable, the corresponding segment of the running time curve is roughly linear.

8. CONCLUSIONS AND FUTURE WORK

In this paper, we proposed novel, effective and efficient polling-based algorithms for tracking influential individual nodes in dynamic networks under the Linear Threshold model and the Independent Cascade model. We modeled dynamics in a network as a stream of edge weight updates. We devised an efficient incremental algorithm for updating random RR sets against network changes. For two interesting settings of influential node tracking, namely, tracking nodes with influence above a given threshold and tracking top-k influential nodes, we derived the number of random RR sets we need to approximate the exact set of influential nodes. We reported a series of experiments on three real networks and demonstrated the effectiveness and efficiency of our algorithms.

There are a few interesting directions for future work. For example, can we apply similar techniques to other influence models such as the Continuous Time Diffusion Model [13]? Since the Continuous Time Diffusion model has an implicit time constraint, how to efficiently update RR sets according to the time constraint is a critical challenge. Another challenge is the influence maximization problem on dynamic networks with provable quality guarantees.

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Table 5: Recall and error on Wiki-Vote. The errors are measured in absolute influence value. (IC model)

| Recall (Threshold) | Theoretical Value | Ins1 | Ins2 | Ins3 | Ins4 | Ins5 | Ins6 | Ins7 | Ins8 | Ins9 | Ins10 |
|-------------------|-------------------|------|------|------|------|------|------|------|------|------|-------|
| Max. Error (Threshold) | 100% | 100% | 100% | 100% | 100% | 100% | 100% | 100% | 100% | 100% | 100% |
| Recall (Top-K) | 100% | 100% | 100% | 100% | 100% | 100% | 100% | 100% | 100% | 100% | 100% |
| Max. Error (Top-K) | 0.0005 × 7115 = 3.5575 | 1.271 | 1.187 | 1.377 | 1.377 | 1.271 | 1.271 | 1.187 | 1.187 | 1.119 |

Table 6: Recall and error on Flixster. The errors are measured in absolute influence value. (IC model)

| Recall (Threshold) | Theoretical Value | Ins1 | Ins2 | Ins3 | Ins4 | Ins5 | Ins6 | Ins7 | Ins8 | Ins9 | Ins10 |
|-------------------|-------------------|------|------|------|------|------|------|------|------|------|-------|
| Max. Error (Threshold) | 100% | 100% | 100% | 100% | 100% | 100% | 100% | 100% | 100% | 100% | 100% |
| Recall (Top-K) | 100% | 100% | 100% | 100% | 100% | 100% | 100% | 100% | 100% | 100% | 100% |
| Max. Error (Top-K) | 0.0005 × 99053 = 99.05 | 42.52 | 43.40 | 43.20 | 43.25 | 43.40 | 43.40 | 43.44 | 40.04 | 41.67 | 41.67 |

Figure 9: $I_{max}$ and $M$ (sample size) change over time of soc-Pokec data.

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