ON THE CHOW GROUPS OF CERTAIN EPW SEXTICS

ROBERT LATERVEER

ABSTRACT. This note is about the Hilbert square $X = S^{[2]}$, where $S$ is a general $K3$ surface of degree 10, and the anti–symplectic birational involution $\iota$ of $X$ constructed by O’Grady. The main result is that the action of $\iota$ on certain pieces of the Chow groups of $X$ is as expected by Bloch’s conjecture. Since $X$ is birational to a double EPW sextic $X'$, this has consequences for the Chow ring of the EPW sextic $Y \subset \mathbb{P}^5$ associated to $X'$.

1. Introduction

For a smooth projective variety $X$ over $\mathbb{C}$, let $A^i(X) := CH^i(X)_{\mathbb{Q}}$ denote the Chow groups (i.e. the groups of codimension $i$ algebraic cycles on $X$ with $\mathbb{Q}$–coefficients, modulo rational equivalence). Let $A^i_{\text{hom}}(X)$ and $A^i_{AJ}(X) \subset A^i(X)$ denote the subgroup of homologically trivial (resp. Abel–Jacobi trivial) cycles. It seems fair to say that Chow groups of codimension $i > 1$ cycles are still poorly understood. To cite but one example, there is Bloch’s famous conjecture (which famously is still open for surfaces of general type with geometric genus 0):

**Conjecture 1.1 (Bloch [5]).** Let $S$ be a smooth projective surface. Let $\Gamma \in A^2(S \times S)$ be a correspondence such that

$$
\Gamma^* = 0 : H^2(X, \mathcal{O}_X) \to H^2(X, \mathcal{O}_X).
$$

Then

$$
\Gamma^* = 0 : A^2_{AJ}(X) \to A^2_{AJ}(X).
$$

For varieties of higher dimension, versions of conjecture [1,1] can be stated for 0–cycles and for codimension 2–cycles:

**Conjecture 1.2.** Let $X$ be a smooth projective variety of dimension $n$. Let $\Gamma \in A^n(X \times X)$ be a correspondence such that

$$
\Gamma^* = 0 : H^n(X, \mathcal{O}_X) \to H^n(X, \mathcal{O}_X).
$$

Then

$$
\Gamma^* = 0 : F^n A^n(X) \to A^n(X).
$$

Here $F^n A^n(\cdot)$ denotes the conjectural Bloch–Beilinson filtration, expected to exist for all smooth projective varieties [16], [17], [26], [27].

2010 Mathematics Subject Classification. Primary 14C15, 14C25, 14C30.

Key words and phrases. Algebraic cycles, Chow groups, motives, hyperkähler varieties, anti–symplectic involution, K3 surfaces, (double) EPW sextics, Beauville’s splitting principle, multiplicative Chow–Künneth decomposition, spread of algebraic cycles.
Conjecture 1.3. Let $X$ be a smooth projective variety of dimension $n$. Let $\Gamma \in A^n(X \times X)$ be a correspondence such that

$$\Gamma^* = 0 : H^2(X, \mathcal{O}_X) \to H^2(X, \mathcal{O}_X).$$

Then

$$\Gamma^* = 0 : A^2_{\text{AJ}}(X) \to A^2_{\text{AJ}}(X).$$

Let us now restrict attention to hyperkähler varieties $X$ (by which we mean: projective irreducible holomorphic symplectic manifolds \[2\], \[1\]). For the purposes of this introduction, we will optimistically assume the Chow ring of reducible holomorphic symplectic manifolds \[2\], \[1\]). For the conjectural Bloch–Beilinson filtration mentioned above. (This is expected to be the case for all hyperkähler varieties \[4\].)

Here is what conjectures 1.2 and 1.3 predict for the action of an anti–symplectic involution (i.e., an involution acting as $-1$ on the symplectic form) on the Chow groups of $X$:

Conjecture 1.4. Let $X$ be a hyperkähler variety of dimension 4. Let $\iota$ be an anti–symplectic involution of $X$. Then

$$\iota^* = -\text{id} : A^i_{(2)}(X) \to A^i_{(2)}(X) \quad \text{for } i = 2, 4;$$

$$\iota^* = \text{id} : A^4_{(4)}(X) \to A^4_{(4)}(X).$$

The statement for $A^4_{(4)}(X) = F^4 A^4(X)$ is conjecture 1.2 applied to the graph of $\iota$. The statement for $A^2_{(2)}(X) = A^2_{\text{AJ}}(X)$ is conjecture 1.3 applied to the graph of $\iota$. The statement for $A^4_{(4)}(X)$ then follows from the expected “hard Lefschetz” isomorphism $A^4_{(2)}(X) \cong A^4_{(2)}(X)$.

The main result of this note establishes a weak form of conjecture 1.4 for a 19–dimensional family of hyperkähler fourfolds:

**Theorem** (=theorem 3.1). Let $X$ be the Hilbert scheme $S^{[2]}$, where $S$ is a very general $K3$ surface of degree $d = 10$. Let $\iota \in \text{Bir}(X)$ be the anti–symplectic involution constructed by O’Grady \[29\]. Then

$$\iota^* = \text{id} : A^4_{(0)}(X) \to A^4_{(0)}(X);$$

$$\iota^* = -\text{id} : A^4_{(2)}(X) \to A^4_{(2)}(X);$$

$$(\Pi^2_X)_* \iota^* = -\text{id} : A^2_{(2)}(X) \to A^2_{(2)}(X);$$

$$(\Pi^4_X)_* \iota^* = \text{id} : A^4_{(4)}(X) \to A^4_{(4)}(X).$$

The birational involution $\iota$ of \[29\] is briefly explained in proposition 2.20 below. The notation $A^i_{(s)}(X)$ in theorem 3.1 refers to the bigraded ring structure constructed unconditionally for all Hilbert squares of $K3$ surfaces by Shen–Vial, using their version of the Fourier transform on the Chow ring \[35\] (cf. also section 2.2 below). The $\Pi^j_X$ refer to the Chow–Künneth decomposition of \[35\]; by construction, these have the property that $(\Pi^j_X)_* A^i(X) = A^i_{(2j-j)}(X)$. 

Let Conjecture 1.4.
It is known that a variety $X$ as in theorem 3.1 has a birational model $X'$ which is a hyperkähler variety; $X'$ is a so-called double EPW sextic \[30, 29, 32\]. The variety $X'$ has a generically $2 : 1$ morphism to a slightly singular sextic hypersurface $Y \subset \mathbb{P}^5$, called an EPW sextic \[10, 30\].

**Theorem 3.1** has interesting consequences for the Chow ring of this EPW sextic:

**Corollary (corollary 4.2).** Let $X$ be as in theorem 3.1, and let $Y \subset \mathbb{P}^5$ be the associated EPW sextic. For any $r \in \mathbb{N}$, let

$$E^r(\mathbb{Y}^r) \subset A^*(\mathbb{Y}^r)$$

be the subring generated by (pullbacks of) $A^1(Y)$ and $A^2(Y)$. The cycle class map

$$E^k(\mathbb{Y}^r) \to Gr_{W^{2k}}^H(\mathbb{Y}^r)$$

is injective for $k \geq 4r - 1$.

(Here, the Chow ring $A^*(\mathbb{Y}^r)$ is taken to mean the operational Chow cohomology of Fulton–MacPherson \[13\].)

In particular, taking $r = 1$, we find that the subspaces

$$\text{Im} \left( A^2(Y) \otimes A^1(Y) \to A^3(Y) \right),$$

$$\text{Im} \left( A^2(Y) \otimes A^2(Y) \to A^4(Y) \right)$$

are of dimension 1 (corollary 4.6). This is analogous to known results for 0–cycles on $K3$ surfaces \[3\] and on certain Calabi–Yau varieties \[41, 11\] (cf. remark 4.7 below).

**Theorem 3.1** is proven using the technique of “spread” of algebraic cycles in a family, as developed by Voisin in her seminal work on the Bloch/Hodge equivalence for complete intersections \[42, 43, 44, 45\].

In a final section (section 5), some questions related to theorem 3.1 are stated, which we hope may spur further research.

**Conventions.** In this article, the word variety will refer to a reduced irreducible scheme of finite type over $\mathbb{C}$. A subvariety is a (possibly reducible) reduced subscheme which is equidimensional.

**All Chow groups will be with rational coefficients:** we will denote by $A_j(X)$ the Chow group of $j$–dimensional cycles on $X$ with $\mathbb{Q}$–coefficients; for $X$ smooth of dimension $n$ the notations $A_j(X)$ and $A^{n-j}(X)$ are used interchangeably.

The notations $A_{hom}^j(X)$, $A_{AJ}^j(X)$ will be used to indicate the subgroups of homologically trivial, resp. Abel–Jacobi trivial cycles. For a morphism $f : X \to Y$, we will write $\Gamma_f \in A_*(X \times Y)$ for the graph of $f$. The contravariant category of Chow motives (i.e., pure motives with respect to rational equivalence as in \[34, 27\]) will be denoted $\mathcal{M}_{\text{rat}}$.

If $\tau : Y \to X$ is an inclusion of smooth varieties and $b \in A^j(Y)$, we will often write

$$b|_Y \in A^j(Y)$$

to indicate the class $\tau^*(b)$.

We use $H^j(X)$ to indicate singular cohomology $H^j(X, \mathbb{Q})$.

We write Aut($X$) and Bir($X$) to denote the group of automorphisms, resp. of birational automorphisms, of $X$. 
Given an involution \( \iota \in \text{Bir}(X) \), we will write \( A^j(X)^\iota \) (and \( H^j(X)^\iota )\) for the subgroup of \( A^j(X) \) (resp. \( H^j(X) \)) invariant under \( \iota \).

2. Preliminaries

2.1. MCK decomposition.

Definition 2.1 (Murre [26]). Let \( X \) be a smooth projective variety of dimension \( n \). We say that \( X \) has a CK decomposition if there exists a decomposition of the diagonal
\[
\Delta_X = \Pi_X^0 + \Pi_X^1 + \cdots + \Pi_X^{2n} \quad \text{in} \quad A^n(X \times X),
\]
such that the \( \Pi_X^i \) are mutually orthogonal idempotents and \( (\Pi_X^i)_\ast H^i(X) = H^i(X) \).

(NB: “CK decomposition” is shorthand for “Chow–K"unneth decomposition”.)

Remark 2.2. The existence of a CK decomposition for any smooth projective variety is part of Murre’s conjectures [26, 16].

Definition 2.3 (Shen–Vial [35]). Let \( X \) be a smooth projective variety of dimension \( n \). Let \( \Delta_X^{sm} \in A^{2n}(X \times X \times X) \) be the class of the small diagonal
\[
\Delta_X^{sm} := \{(x, x, x) \mid x \in X\} \subset X \times X \times X.
\]
An MCK decomposition is a CK decomposition \( \{\Pi_X^i\} \) of \( X \) that is multiplicative, i.e. it satisfies
\[
\Pi_X^k \circ \Delta_X^{sm} \circ (\Pi_X^i \times \Pi_X^j) = 0 \quad \text{in} \quad A^{2n}(X \times X \times X) \quad \text{for all} \quad i + j \neq k.
\]

(NB: “MCK decomposition” is shorthand for “multiplicative Chow–K"unneth decomposition”.)

Remark 2.4. The small diagonal (seen as a correspondence from \( X \times X \) to \( X \)) induces the multiplication morphism
\[
\Delta_X^{sm} : h(X) \otimes h(X) \to h(X) \quad \text{in} \quad \mathcal{M}_{\text{rat}}.
\]
Suppose \( X \) has a CK decomposition
\[
h(X) = \bigoplus_{i=0}^{2n} h^i(X) \quad \text{in} \quad \mathcal{M}_{\text{rat}}.
\]
By definition, this decomposition is multiplicative if for any \( i, j \) the composition
\[
h^i(X) \otimes h^j(X) \to h(X) \otimes h(X) \xrightarrow{\Delta_X^{sm}} h(X) \quad \text{in} \quad \mathcal{M}_{\text{rat}}
\]
factors through \( h^{i+j}(X) \). It follows that if \( X \) has an MCK decomposition, then setting
\[
A^i_{(j)}(X) := (\Pi_X^{i-j})_\ast A^i(X),
\]
one obtains a bigraded ring structure on the Chow ring: that is, the intersection product sends \( A^i_{(j)}(X) \otimes A^j_{(j')}^i(X) \) to \( A^{i+j}_{(j+j')}(X) \).

The property of having an MCK decomposition is severely restrictive, and is closely related to Beauville’s “weak splitting property” [4]. For more ample discussion, and examples of varieties with an MCK decomposition, we refer to [35, Section 8] and also [40, 36, 12, 23].
Lemma 2.5 (Vial [40]). Let $X, X'$ be birational hyperkähler varieties. Then $X$ has an MCK decomposition if and only if $X'$ has one.

Proof. This is noted in [40, Introduction]; the idea (as indicated in loc. cit.) is that Rieß’s result [33] implies that $X$ and $X'$ have isomorphic Chow motives and the isomorphism is compatible with the multiplicative structure.

2.2. MCK for $K3^2$.

Theorem 2.6 (Shen–Vial [35]). Let $S$ be a $K3$ surface, and $X = S[2]$. There exists an MCK decomposition $\{\Pi^X_j\}$ for $X$. In particular, setting

$$A^i_{(j)}(X) := (\Pi^X_{2i-j})_* A^i(X)$$

defines a bigraded ring structure $A^i_{(j)}(X)$ on $A^*(X)$. Moreover, $A^i_{(j)}(X)$ coincides with the bigrading on $A^*(X)$ defined by the Fourier transform.

Proof. The existence of $\{\Pi^X_j\}$ is a special case of [35, Theorem 13.4]. The “moreover” part is [35, Theorem 15.8].

Remark 2.7. The first statement of theorem 2.6 actually holds for $X = S^r$ for any $r \in \mathbb{N}$ [40].

Any $K3$ surface $S$ has an MCK decomposition [35, Example 8.17]. Since this property is stable under products [35, Theorem 8.6], $S^2$ also has an MCK decomposition. The following lemma records a basic compatibility between the bigradings on $A^4(S^2)$ and on $A^*(S^2)$:

Lemma 2.8. Let $S$ be a $K3$ surface, and $X = S[2]$. Let $\Psi \in A^4(X \times S^2)$ be the correspondence coming from the diagram

$$
\begin{array}{ccc}
S^2 & \leftarrow & \tilde{S}^2 \\
\downarrow h & & \downarrow \\
S^{(2)} & \leftarrow & S^2
\end{array}
$$

(the arrow labelled $h$ is the Hilbert–Chow morphism; the right vertical arrow is the blow–up of the diagonal). Then

$$((\Psi)_* R(X) \subset R(S^2),$$

$$(^t\Psi)_* R(S^2) \subset R(X),$$

where $R = A^4_{(4)}$ or $A^4_{(2)}, A^2_{(2)}$ or $A^2_{(0)} \cap A^2_{\text{hom}}$.

Proof. We prove the statement for $^t\Psi$ and $R = A^2_{(2)}$ or $A^2_{(0)} \cap A^2_{\text{hom}}$, which are the only cases we’ll be using (the other statements can be proven similarly). By construction of the MCK decomposition for $X$, there is a relation

$$\Pi^X_k = \frac{1}{2} ^t\Psi \circ \Pi^{S^2}_k \circ \Psi + \text{Rest} \quad \text{in } A^4(X \times X), \quad (k = 0, 2, 4, 6, 8),$$
where \( \{ \Pi^{S^2}_k \} \) is a product MCK decomposition for \( S^2 \), and “Rest” is a term coming from \( \Delta_S \subset S \times S \) which does not act on \( A^4(X) \) and on \( A^2_{\text{hom}}(X) \). Since \( \frac{1}{2} \Psi \circ \Psi \) is the identity on \( A^2_{\text{hom}}(X) = A^2_{\text{hom}}(X) \), we can write

\[
(t \Psi)_* (\Pi^{S^2}_k)_* = (t \Psi \circ \Pi^{S^2}_k)_* = \left( \frac{1}{2} t \Psi \circ \Psi \circ t \Psi \circ \Pi^{S^2}_k \right)_* : A^2_{\text{hom}}(S^2) \to A^2_{\text{hom}}(X) .
\]

In view of sublemma 2.9 below, this implies

\[
(2) \quad (t \Psi)_* (\Pi^{S^2}_k)_* A^2_{\text{hom}}(S^2) \subset (\Pi^{X^2}_k)_* A^2_{\text{hom}}(X) .
\]

But then, plugging in relation (1), we find

\[
(t \Psi)_* (\Pi^{S^2}_k)_* A^2_{\text{hom}}(S^2) \subset (\Pi^{X^2}_k)_* A^2_{\text{hom}}(X) .
\]

Taking \( k = 2 \), this proves

\[
(t \Psi)_* A^2_{(2)}(S^2) \subset A^2_{(2)}(X) .
\]

Taking \( k = 4 \), this proves

\[
(t \Psi)_* \left( A^2_{(0)}(S^2) \cap A^2_{\text{hom}}(S^2) \right) \subset A^2_{(0)}(X) \cap A^2_{\text{hom}}(X) .
\]

**Sublemma 2.9.** There is commutativity

\[
\Psi \circ t \Psi \circ \Pi^{S^2}_k = \Pi^{S^2}_k \circ \Psi \circ t \Psi \text{ in } A^4(S^4) .
\]

To prove the sublemma, we remark that \( h_* h^* = 2 \text{id} : A^i(S^{(2)}) \to A^i(S^{(2)}) \), and so

\[
(\Psi \circ t \Psi)_* = 2 g^* g_* = 2(\Delta_{S^2} + \Gamma_\tau)_* : A^i(S^2) \to A^i(S^2) ,
\]

where \( \tau \) denotes the involution switching the two factors. But \( \{ \Pi^{S^2}_k \} \), being a product decomposition, is symmetric and hence

\[
\Gamma_\tau \circ \Pi^{S^2}_k \circ \Gamma_\tau = (\tau \times \tau)^* \Pi^{S^2}_k = \Pi^{S^2}_k \text{ in } A^4(S^4) .
\]

This implies commutativity

\[
\Gamma_\tau \circ \Pi^{S^2}_k = \Pi^{S^2}_k \circ \Gamma_\tau \text{ in } A^4(S^4) ,
\]

which proves the sublemma.

**Remark 2.10.** Lemma 2.8 is probably true for any \((i, j)\) (i.e., \( \Psi \) should be “of pure grade 0” in the language of [36, Definition 1.1]). I have not been able to prove this.

2.3. Relative MCK for \( S^2 \).

**Notation 2.11.** Let \( S \to B \) be a family (i.e., a smooth projective morphism). For \( r \in \mathbb{N} \), we write \( S^{r/B} \) for the relative \( r \)-fold fibre product

\[
S^{r/B} := S \times_B S \times_B \cdots \times_B S \quad (r \text{ copies of } S).
\]
Proposition 2.12. Let $S \to B$ be a family of K3 surfaces. There exist relative correspondences
\[ \Pi_j^{S^{2/B}} \in A^4(S^{4/B}) \quad (j = 0, 2, 4, 6, 8), \]
such that for each $b \in B$, the restriction
\[ \Pi_j^{(S_b)^2} := \Pi_j^{S^{2/B}}|_{(S_b)^4} \in A^4((S_b)^4) \]
defines a self–dual MCK decomposition for $(S_b)^2$.

Proof. On any K3 surface $S_b$, there is the distinguished 0–cycle $o_{S_b}$ such that $c_2(S_b) = 24o_{S_b}$ [3]. Let $p_i : S \times_B S \to S$, $i = 1, 2$, denote the projections to the two factors. Let $T_{S/B}$ denote the relative tangent bundle. The assignment
\[ \Pi_0^S := (p_1)^*(\frac{1}{24}c_2(T_{S/B})) \quad A^2(S \times_B S), \]
\[ \Pi_4^S := (p_2)^*(\frac{1}{24}c_2(T_{S/B})) \quad A^2(S \times_B S), \]
\[ \Pi_2^S := \Delta_S - \Pi_0^S - \Pi_4^S \]
defines (by restriction) an MCK decomposition for each fibre:
\[ \Pi_j^{S_b} := \Pi_j^S|_{S_b \times S_b} \in A^2(S_b \times S_b) \quad (j = 0, 2, 4) \]
is an MCK decomposition for any $b \in B$ [35, Example 8.17].

Next, we consider the fourfold relative fibre product $S^{4/B}$. Let
\[ p_{ij} : S^{4/B} \to S^{2/B} \quad (1 \leq i < j \leq 4) \]
denote projection to the $i$-th and $j$-th factor. We set
\[ \Pi_j^{S^{2/B}} := \sum_{k+l=j} (p_{1k})^*(\Pi_k^S) \cdot (p_{2l})^*(\Pi_l^S) \in A^4(S^{4/B}), \quad (j = 0, 2, 4, 6, 8). \]

By construction, the restriction to each fibre induces an MCK decomposition (the “product MCK decomposition”)
\[ \Pi_j^{(S_b)^2} := \Pi_j^{S^{2/B}}|_{(S_b)^4} \sum_{k+l=j} \Pi_k^{S_b} \times \Pi_l^{S_b} \in A^4((S_b)^4), \quad (j = 0, 2, 4, 6, 8). \]

Proposition 2.13. Let $S \to B$ be a family of K3 surfaces. There exist relative correspondences
\[ \Theta_1, \Theta_2 \in A^4((S \times_B S) \times_B S), \quad \Xi_1, \Xi_2 \in A^2(S \times_B (S \times_B S)) \]
such that for each $b \in B$, the composition
\[ A^2(S_b \times S_b) \xrightarrow{([\Theta_1]_{(S_b)^3} + [\Theta_2]_{(S_b)^3})} A^2(S_b) \oplus A^2(S_b) \xrightarrow{([\Xi_1]_{(S_b)^3} + [\Xi_2]_{(S_b)^3})} A^2(S_b \times S_b) \]
acts as a projector on $A^2(S_b \times S_b)$. 

\[ \square \]
Proof. As before, let
\[ p_{ij} : S_i^{j/B} \rightarrow S_i^{k/B} \quad (1 \leq i < j \leq 4) \]
denote projection to the \( i \)-th and \( j \)-th factor, and let
\[ p_i : S_i^{k/B} \rightarrow S \quad (i = 1, 2) \]
denote projection to the \( i \)-th factor.

By construction of \( \Pi_{2}^{S_{i/B}} \), for each \( b \in B \) we have equality
\[
\left( \Pi_{2}^{S_{i/B}} \right) |_{(S_b)^4} = \frac{1}{24^2} \left( t \Gamma_{p_1} \circ \Pi_{2}^{S} \circ \Gamma_{p_1} \circ ((p_{13})^*(\Delta_S) \cdot (p_2)c_2(T_{S/B}) \cdot (p_4)c_2(T_{S/B})) \\
+ t \Gamma_{p_2} \circ \Pi_{2}^{S} \circ \Gamma_{p_2} \circ ((p_{24})^*(\Delta_S) \cdot (p_1)c_2(T_{S/B}) \cdot (p_3)c_2(T_{S/B})) \right) |_{(S_b)^4} \quad \text{in } A^4((S_b)^4) .
\]

Indeed, using Lieberman’s lemma \([13, 16.1.1]\), we find that
\[
\left( \Gamma_{p_1} \circ \Pi_{2}^{S} \circ \Gamma_{p_1} \right) |_{(S_b)^4} = \left( \Gamma_{p_{13}} \circ (\Pi_{2}^{S}) \right) |_{(S_b)^4} = ((p_{13})^*(\Pi_{2}^{S})) |_{(S_b)^4} ,
\]
\[
\left( \Gamma_{p_2} \circ \Pi_{2}^{S} \circ \Gamma_{p_2} \right) |_{(S_b)^4} = \left( \Gamma_{p_{24}} \circ (\Pi_{2}^{S}) \right) |_{(S_b)^4} = ((p_{24})^*(\Pi_{2}^{S})) |_{(S_b)^4} ,
\]
and so both sides of (3) are equal to
\[
\Pi_{2}^{S_b} \times \Pi_{0}^{S_b} + \Pi_{2}^{S_b} \times \Pi_{2}^{S_b} \quad \in A^2((S_b)^4) .
\]

It follows that if we define
\[
\Theta_1 := \frac{1}{24^2} \Gamma_{p_1} \circ ((p_{13})^*(\Delta_S) \cdot (p_2)c_2(T_{S/B}) \cdot (p_4)c_2(T_{S/B})) \quad \in A^4((S \times_B S) \times_B S) ,
\]
\[
\Theta_2 := \frac{1}{24^2} \Gamma_{p_2} \circ ((p_{24})^*(\Delta_S) \cdot (p_1)c_2(T_{S/B}) \cdot (p_3)c_2(T_{S/B})) \quad \in A^4((S \times_B S) \times_B S) ,
\]
\[
\Xi_1 := \Gamma_{p_1} \circ \Pi_{2}^{S} \quad \in A^2(S \times_B (S \times_B S)) ,
\]
\[
\Xi_2 := \Gamma_{p_2} \circ \Pi_{2}^{S} \quad \in A^2(S \times_B (S \times_B S)) ,
\]
then we have
\[
\left((\Xi_1 \circ \Theta_1 + \Xi_2 \circ \Theta_2) |_{(S_b)^4}\right) \quad \left( \Pi_{2}^{S_b} \right) |_{(S_b)^4} = A^2(S_b \times S_b) \rightarrow A^2(2)(S_b \times S_b) \quad \forall b \in B .
\]
This proves the proposition. \( \square \)

2.4. Relative MCK for \( S^{[2]} \).

Proposition 2.14. Let \( S \rightarrow B \) be a family of K3 surfaces (i.e. each fibre \( S_b \) is a K3 surface), and let \( X \rightarrow B \) be the family of associated Hilbert schemes (i.e., a fibre \( X_b \) is \( (S_b)^{[2]} \)). There exist relative correspondences
\[
\Pi_X^j \quad \in A^4(\mathcal{X} \times_B \mathcal{X}) \quad (j = 0, 2, 4, 6, 8) ,
\]
such that for each \( b \in B \), the restrictions
\[
\Pi_X^{j_b} := \Pi_X^j |_{X_b \times X_b} \quad \in A^4(X_b \times X_b) \quad (j = 0, 2, 4, 6, 8)
\]
define an MCK decomposition for \( X_b \).
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Proof. The construction of an MCK decomposition for \( X_b \) given in [35, Theorem 13.4] can be done in a relative setting. That is, let \( \{ \Pi_j^S \} \) be a relative MCK decomposition for \( S \) as in proposition 2.12, and let \( \{ \Pi_j^{S/B} \} \) be the induced relative MCK decomposition for \( S^{2/B} \) as in proposition 2.12. Let

\[
Z \to B
\]

be the family obtained by blowing–up \( S \times_B S \) along the relative diagonal \( \Delta_S \). As in the proof of [35, Propositions 13.2 and 13.3], one can use \( \{ \Pi_j^{S/B} \} \) and \( \{ \Pi_j^S \} \) to define relative correspondences

\[
\Pi_j^Z \in A^4(Z \times_B Z) \quad (j = 0, 2, 4, 6, 8),
\]

which restrict to an MCK decomposition of each fibre \( Z_b \). Let \( p : Z \to X \) denote the morphism of \( B \)-schemes induced by the action of the symmetric group \( S_2 \), and let \( \Gamma_p \in A^4(Z \times_B X) \) be the graph of \( p \). We define

\[
\Pi_j^X := \frac{1}{2} \Gamma_p \circ \Pi_j^Z \circ \Gamma_p \in A^4(X \times_B X) \quad (j = 0, 2, 4, 6, 8).
\]

The restrictions \( \Pi_j^{X_b} := \Pi_j^X |_{X_b \times X_b} \) define an MCK decomposition for each fibre by [35, Theorem 13.4].

\[\square\]

2.5. Multiplicative structure of Chow ring of \( K3^{[2]} \).

Theorem 2.15 (Shen–Vial [35]). Let \( S \) be a \( K3 \) surface, and \( X = S^{[2]} \).

(i) Intersection product induces a surjection

\[
A^2_{(2)}(X) \otimes A^2_{(2)}(X) \twoheadrightarrow A^4_{(4)}(X).
\]

(ii) There is a distinguished class \( l \in A^2_{(0)}(X) \) such that intersection induces an isomorphism

\[
\cdot l : A^2_{(2)}(X) \xrightarrow{\cong} A^4_{(2)}(X).
\]

Proof. This is [35, Theorem 3].

\[\square\]

2.6. Refined CK decomposition.

Theorem 2.16 (Vial [38]). Let \( X \) be a smooth projective variety of dimension \( n \leq 5 \). Assume the Lefschetz standard conjecture \( B(X) \) holds (in particular, the K"unneth components \( \pi_i \in H^{2n}(X \times X) \) are algebraic). Then there is a splitting into mutually orthogonal idempotents

\[
\pi_i = \sum_j \pi_{i,j} \in H^{2n}(X \times X),
\]

such that

\[
(\pi_{i,j})_* H^i(X) = gr^j_N H^i(X),
\]

where \( wtN^* \) is the niveau filtration of [38].

---

\( ^1 \)The statement and proof of [35, Proposition 13.2] should be slightly modified, as noted in [36, Remark 2.8].
In particular,

\[(\pi_{2,1})_*H^3(X) = H^2(X) \cap F^1,\]
\[(\pi_{2,0})_*H^3(X) = H^2_{tr}(X).\]

(Here $F^*$ denotes the Hodge filtration, and $H^2_{tr}(X)$ is the orthogonal complement to $H^2(X) \cap F^1$ under the pairing

\[H^2(X) \otimes H^2(X) \to \mathbb{Q},\]
\[a \otimes b \mapsto a \cup h^{n-2} \cup b.\]

The projector $\pi_{2,1}$ is supported on $C \times D$, where $C \subset X$ is a curve and $D \subset X$ is a divisor.

Proof. This is [38, Theorem 1]. □

2.7. Mukai models.

Theorem 2.17 (Mukai [24]). Let $S$ be a general $K3$ surface of degree 10 (i.e. genus $g(S) = 6$). Let $G = G(2, 5)$ denote the Grassmannian of lines in $\mathbb{P}^4$. Then $S$ is isomorphic to the zero locus of a section of $\mathcal{O}_G(1)^{\oplus 3} \oplus \mathcal{O}_G(2)$.

This result can be exploited as follows:

Proposition 2.18 (Voisin [42]). Let $S \to B$ be the universal family of degree 10 $K3$ surfaces (i.e., $B$ is a Zariski open in a product of projective spaces parametrizing sections of $\mathcal{O}_G(1)^{\oplus 3} \oplus \mathcal{O}_G(2)$ that are smooth). We have

\[A^2_{hom}(S \times_B S) = 0.\]

Proof. The family $S \to B$ is the family of smooth complete intersections $S_b \subset G$ defined by the very ample line bundles $\mathcal{O}_G(1)$ (3 copies) and $\mathcal{O}_G(2)$. The Grassmannian $G$ has trivial Chow groups. The result is thus a special case of [42, Proposition 3.13] (NB: as explained in [42, Section 3.3], the hypothesis that [42, Conjecture 1.6] holds is satisfied in codimension 2, and so the result is unconditional in codimension 2). □

2.8. EPW sextics.

Definition 2.19 ([10]). Let $A \subset \wedge^3 \mathbb{C}^6$ be a subspace which is Lagrangian with respect to the symplectic form on $\wedge^3 \mathbb{C}^6$ given by the wedge product. The EPW sextic associated to $A$ is

\[Y_A := \left\{[v] \in \mathbb{P}(\mathbb{C}^6) \mid \dim(A \cap (v \wedge \wedge^2 \mathbb{C}^6)) \geq 1\right\} \subset \mathbb{P}(\mathbb{C}^6).\]

An EPW sextic is an $Y_A$ for some $A \subset \wedge^3 \mathbb{C}^6$ Lagrangian.

Proposition 2.20 (O’Grady [29]). Let $X = S^{[2]}$ where $S$ is a very general degree 10 $K3$ surface. There exists a non–trivial birational involution

\[\iota : X \dashrightarrow X.\]

There exists a $\iota$–invariant divisor $D \subset X$ (of Beauville–Bogomolov square 2), such that the action of $\iota$ on the Néron–Severi group $NS(X)$ is given by reflection in the span of $D$. 
Proof. This is \cite[Section 4.3]{29} (cf. also \cite[Section 3.1]{15}). The idea of the construction of \( \iota \) is as follows. Using Mukai’s work (theorem 2.17), the K3 surface \( S \) can be realized as a quadratic section \( S = V_5 \cap Q \) of the del Pezzo threefold \( V_5 := G \cap \mathbb{P}^6 \). Hence, a general unordered pair of points on \( S \) gives a general unordered pair of points \( (x, y) \) on \( V_5 \). One checks (by a dimension count) that there is a unique conic \( q = q_{x,y} \subset V_5 \) passing through the pair of points \( (x, y) \). Since \( S = V_5 \cap Q \) is a quadratic section, the conic \( q \) meets \( S \) in \( x, y \) plus 2 other points \( x', y' \). The involution is defined by this residual intersection, i.e.

\[
\iota(x, y) := (x', y') \quad \in \ X.
\]

\[\Box\]

**Theorem 2.21** (O’Grady \cite{30}). Let \( X \) and \( \iota \in \text{Bir}(X) \) be as in proposition 2.20. There exists a hyperkähler fourfold \( X' \) birational to \( X \), and a generically 2 : 1 morphism \( p : X' \to Y \) to an EPW sextic \( Y \).

Moreover, let \( \iota' \in \text{Bir}(X') \) be the birational involution induced by \( \iota \). Then \( Y \subset \mathbb{P}^5 \) is the closure of the quotient \( U' / \iota' \), where \( U' \subset X' \) is the open on which the involution \( \iota' \) is defined.

Proof. This is contained in \cite[Theorem 4.15]{32}. The idea is that there is a generically 2 : 1 rational map \( X \dashrightarrow Y_A \) to an EPW sextic with \( A \in \Delta \) in the notation of loc. cit. For \( S \) very general, the subspace \( A \) will be generic in \( \Delta \) and thus \( Y_A[3] \) will consist of a single point \( v_0 \).

Let \( X_A \to Y_A \) be the singular double cover of the EPW sextic as in loc. cit. According to \cite[Theorem 4.15]{32}, \( X_A \) has one singular point \( p_0 \) (lying over \( v_0 \in Y_A \)), and there exists a small resolution \( s : X'_A \to X_A \) with exceptional locus \( E := s^{-1}(p_0) \) isomorphic to \( \mathbb{P}^2 \), and such that \( X'_A \) is isomorphic to the Hilbert square of a certain K3 surface (the K3 surface denoted \( S_A(v_0) \) in loc. cit.). We define \( X' := X'_A \) and \( Y := Y_A \).

The singular variety \( X_A \) has an involution \( \iota_A \in \text{Aut}(X_A) \) (coinciding with \( \iota \in \text{Bir}(X) \) on an open) such that \( Y = X_A / \iota_A \). Since \( X'_A \to X_A \) is birational, this proves the “moreover” statement. \[\Box\]

### 3. Main Result

**Theorem 3.1.** Let \( X \) be the Hilbert scheme \( S^{(2)} \), where \( S \) is a very general K3 surface of degree \( d = 10 \). Let \( \iota \in \text{Bir}(X) \) be the anti–symplectic involution of proposition 2.20. Then

\[
\begin{align*}
\iota^* &= \text{id} : \quad A^4_{(0)}(X) \to A^4_{(0)}(X) ; \\
\iota^* &= -\text{id} : \quad A^4_{(2)}(X) \to A^4_{(2)}(X) ; \\
(\Pi^X_2)_{\iota^*} &= -\text{id} : \quad A^2_{(2)}(X) \to A^2_{(2)}(X) ; \\
(\Pi^X_4)_{\iota^*} &= \text{id} : \quad A^4_{(4)}(X) \to A^4_{(4)}(X) .
\end{align*}
\]

Proof. We first prove the statement for \( A^2_{(2)}(X) \). The statements for \( A^4_{(2)}(X) \) and for \( A^4_{(4)}(X) \) will be deduced from the statement for \( A^2_{(2)}(X) \) using theorem 2.15.

We consider the universal family \( S \to B \)
of all smooth degree 10 \( K3 \) surfaces \( S_b \). Here the base \( B \) is a Zariski–open in a product of projective spaces
\[
B \subset \tilde{B} := \mathbb{P}H^0(G, \mathcal{O}(1))^{\times 3} \times \mathbb{P}H^0(G, \mathcal{O}(3))
\]
corresponding to theorem 2.17.

We will write \( X \to B \) for the universal family of Hilbert squares of degree 10 \( K3 \) surfaces, and \( X_b \) for a fibre of \( X \to B \) over \( b \in B \). This family is obtained from the family \( S \times_B S \) (whose fibres are products \( S_b \times S_b \)) by a “hat” of morphisms over \( B \)

\[
\begin{array}{ccc}
S \times_B S & \hookrightarrow & X \\
\tilde{S} \times_B S & \hookrightarrow & S \times_B S
\end{array}
\]

where \( \tilde{S} \times_B S \) is the blow–up of \( S \times_B S \) with centre the relative diagonal, and the southwest arrow is the quotient morphism for the natural action of the symmetric group on 2 elements. This diagram \((4)\) gives rise to relative correspondences

\[
\Psi \in A^4(X \times_B S \times_B S), \quad \iota \Psi \in A^4(S \times_B S \times_B X).
\]

(For details on relative correspondences, cf. [27], and also [9], [8], [28].) Restricting to a fibre over \( b \in B \), diagram \((4)\) induces the familiar diagram

\[
\begin{array}{ccc}
\tilde{S} \times_B S & \hookrightarrow & \tilde{S}_b \times S_b \\
\tilde{S} \times_B S & \hookrightarrow & S_b \times S_b
\end{array}
\]

(where \( \tilde{S}_b \times S_b \) is the blow–up of \( S_b \times S_b \) along the diagonal), and the (absolute) correspondences

\[
\Psi_b \in A^4(X_b \times S_b \times S_b), \quad \iota \Psi_b \in A^4(S_b \times S_b \times X_b)
\]

Since the construction of the birational involution \( \iota_b \in \text{Bir}(X_b) \) of proposition 2.20 is geometric in nature, it naturally extends to the relative setting. More precisely, let

\[
V \to B' \to B
\]
denote the family of smooth codimension 3 linear sections of the Grassmannian \( G = G(2, 5) \) of lines in \( \mathbb{P}^4 \) (so \( B' \) is an open in \((\mathbb{P}H^0(G, \mathcal{O}(1)))^{\times 3}\), and each fibre \( V_b \) of the family \( V \to B \) is the del Pezzo threefold usually denoted \( V_5 \)). Let \( \mathcal{F} \to B \) denote the family of Fano varieties of conics contained in \( V_b \) (so the family \( \mathcal{F} \to B \) is isotrivial with fibre \( F(V_5) \) according to the previous parenthesis). Associating to a general unordered pair of 2 points on \( S_b \) the unique conic in \( V_b \) containing this pair of points defines a rational map of \( B \)–schemes

\[
X \dashrightarrow \mathcal{F}.
\]

Taking the residual intersection of the conic with the surface \( S_b \), we get a birational involution of \( B \)–schemes

\[
\iota : X \to X,
\]

such that restriction to a fibre gives the birational involution \( \iota_b : X_b \to X_b \) of proposition 2.20.
Let $\Gamma_\ell \in A^4(\mathcal{X} \times_B \mathcal{X})$ denote the closure of the graph of the birational map $\iota$. The fact that $\iota_b$ acts as $-1$ on $H^{2,0}(X_b)$ for all $b \in B$, combined with the fact that $H^2(X_b) \subset H^2(X_b)$ is the smallest Hodge substructure containing $H^{2,0}$, implies that

$$(\iota\Gamma + \Delta_{\mathcal{X}}) \circ (\pi_2 X_b) = 0 \quad \text{in} \ H^8(X_b \times X_b), \forall b \in B.$$

In view of the refined Chow–Künneth decomposition (theorem 2.16), this implies that

$$(\iota\Gamma + \Delta_{\mathcal{X}}) \circ (\pi_2 X_b) = 0 \quad \text{in} \ H^8(X_b \times X_b), \forall b \in B,$$

where $\gamma_b$ is some cycle supported on $Y_b \times Y_b$, for $Y_b \subset X_b$ a divisor.

Let $\{\Pi_j\}$ be a relative MCK decomposition as in proposition 2.14. The relation (5) implies the following: the relative correspondence

$$\Gamma_0 := (\iota\Gamma + \Delta_{\mathcal{X}}) \circ \Pi_2 \in A^4(\mathcal{X} \times_B \mathcal{X})$$

has the property that for each $b \in B$, there exists a divisor $Y_b \subset X_b$ and a cycle $\gamma_b$ supported on $Y_b \times Y_b$ such that

$$(\Gamma_0)|_{X_b \times X_b} = \gamma_b \quad \text{in} \ H^8(X_b \times X_b).$$

At this point, we recall Voisin’s “spread–out” result:

**Proposition 3.2** (Voisin [42]). Let $\mathcal{X} \rightarrow B$ be a smooth projective morphism of relative dimension $n$. Let $\Gamma \in A^n(\mathcal{X} \times_B \mathcal{X})$ be a cycle such that for all $b \in B$, there exists a closed algebraic subset $Y_b \subset X_b$ of codimension $c$, and a cycle $\gamma_b \in A_n(Y_b \times Y_b)$ such that

$$\Gamma|_{X_b \times X_b} = \gamma_b \quad \text{in} \ H^{2n}(X_b \times X_b).$$

Then there exists a closed algebraic subset $Y \subset \mathcal{X}$ of codimension $c$, and a cycle $\gamma \in A_n(Y \times_B Y)$ such that

$$\Gamma|_{X_b \times X_b} = \gamma|_{X_b \times X_b} \quad \text{in} \ H^{2n}(X_b \times X_b) \quad \forall b \in B.$$

**Proof.** This is a Hilbert schemes argument [42 Proposition 3.7].

Applying proposition 3.2 to $\Gamma_0$, it follows there exists a divisor $Y \subset \mathcal{X}$ and a cycle $\gamma \in A_n(Y \times_B Y)$ such that

$$(\Gamma_0 - \gamma)|_{X_b \times X_b} = 0 \quad \text{in} \ H^8(X_b \times X_b), \forall b \in B.$$

That is, the relative correspondence

$$\Gamma_1 := \Gamma_0 - \gamma \in A^4(\mathcal{X} \times_B \mathcal{X})$$

has the property of being homologically trivial on every fibre:

$$(\Gamma_1)|_{X_b \times X_b} = 0 \quad \text{in} \ H^8(X_b \times X_b), \forall b \in B.$$

At this point, it is convenient to consider the family $S \times_B S$ (of products of surfaces $S_b \times S_b$), rather than the family $\mathcal{X}$ (of Hilbert schemes $(S_b)^{[2]}$). That is, we consider the relative correspondence

$$\Gamma_2 := \Psi \circ \Gamma_1 \circ \iota^! \Psi \in A^4(S^{4/B}),$$

where

$$S^{4/B} := S \times_B S \times_B S \times_B S.$$
Since
\[(\Gamma_2)|_{(S_b)^d} = (\Psi_b) \circ ((\Gamma_1)|_{X_b \times X_b}) \circ \delta \in A^4((S_b)^4)\]
(restriction and composition commute), the relative correspondence \(\Gamma_2\) has the property of being homologically trivial on every fibre:

\[(\Gamma_2)|_{(S_b)^d} = 0 \quad \text{in} \quad H^4((S_b)^4), \quad \forall b \in B.\]

Let us now define four relative correspondences

\[\Gamma_{3,k,\ell} := \Theta_k \circ \Gamma_2 \circ \Xi_{\ell} \quad \in A^2(S^2/B), \quad k, \ell \in \{1, 2\},\]

where \(\Xi_{\ell}, \Theta_k\) are as in proposition 2.13.

It follows from (6) there is fibrewise homological vanishing

\[(\Gamma_{3,k,\ell})|_{S_b \times S_b} = 0 \quad \text{in} \quad H^4(S_b \times S_b) \quad \forall b \in B \quad (k, \ell \in \{1, 2\}).\]

Applying the Leray spectral sequence argument of [42, Lemmas 3.11 and 3.12], one finds that there exist

\[\delta_{k,\ell} \in \text{Im}\left(A^i(G \times G \times B) \rightarrow A^2(S \times B S)\right) \quad (k, \ell \in \{1, 2\})\]

such that (after replacing \(B\) by a smaller Zariski open subset) there is global homological vanishing

\[\Gamma_{3,k,\ell} + \delta_{k,\ell} \quad \in A^2_{hom}(S \times B S) \quad (k, \ell \in \{1, 2\}).\]

But then, in view of proposition 2.18, we have that

\[\Gamma_{3,k,\ell} + \delta_{k,\ell} = 0 \quad \in A^2(S \times B S) \quad (k, \ell \in \{1, 2\}).\]

Composing on both sides, this implies there are also rational equivalences

\[(7) \quad \Xi_k \circ \Gamma_{3,k,\ell} \circ \Theta_{\ell} + \delta'_{k,\ell} = 0 \quad \in A^4(S^{4/B}) \quad (k, \ell \in \{1, 2\}),\]

where we define \(\delta'_{k,\ell} := \Xi_k \circ \delta_{k,\ell} \circ \Theta_{\ell}\).

We note that the action of the restricted correspondences \(\delta_{k,\ell}|_{S_b \times S_b}\) on \(A^i(S_b)\) factors over \(A^{i+2}(G)\). Since the Grassmannian \(G\) has trivial Chow groups, this implies that

\[(\delta_{k,\ell}|_{S_b \times S_b})_* = 0: \quad A^i_{hom}(S_b) \rightarrow A^i_{hom}(S_b) \quad \forall b \in B \quad (k, \ell \in \{1, 2\}).\]

As \(\delta'_{k,\ell}\) is composed with \(\delta_{k,\ell}\), the same property holds for \(\delta'_{k,\ell}\):

\[(\delta'_{k,\ell}|_{S_b \times S_b})_* = 0: \quad A^i_{hom}(S_b \times S_b) \rightarrow A^i_{hom}(S_b \times S_b) \quad \forall b \in B \quad \forall i \quad (k, \ell \in \{1, 2\}).\]

Plugging this in the restriction of equality (7) to a fibre, we see that

\[(\Xi_k \circ \Gamma_{3,k,\ell} \circ \Theta_{\ell})(S_b)^d)_* = 0: \quad A^i_{hom}(S_b \times S_b) \rightarrow A^i_{hom}(S_b \times S_b), \quad \text{for all} \ i \ \text{and} \ b \in B \quad (k, \ell \in \{1, 2\}).\]
In view of the definition of the $\Gamma_{3}^{k,\ell}$, this implies that
\[
\left((\Xi_{1} \circ \Theta_{1} + \Xi_{2} \circ \Theta_{2}) \circ \Gamma_{2} \circ (\Xi_{1} \circ \Theta_{1} + \Xi_{2} \circ \Theta_{2})\right|_{(S_{b})^{4}}\ast \ast = \sum_{k,\ell \in \{1,2\}} \left((\Xi_{k} \Theta_{k} \circ \Gamma_{2} \circ \Xi_{\ell} \circ \Theta_{\ell})\right|_{(S_{b})^{4}}\ast \ast = 0: \ A_{h\text{om}}^{i}(S_{b} \times S_{b}) \to A_{h\text{om}}^{i}(S_{b} \times S_{b}) , \text{ for all } i \text{ and for all } b \in B .
\]

But
\[
\left((\Xi_{1} \circ \Theta_{1} + \Xi_{2} \circ \Theta_{2})\right|_{(S_{b})^{4}}\ast \ast = (\Pi_{2}^{(S_{b})^{2}})\ast : \ A^{2}(S_{b} \times S_{b}) \to A^{2}(S_{b} \times S_{b})
\]
(proposition [2.13], and $A_{2}^{2} \subset A_{h\text{om}}^{2}$, and so this simplifies to
\[
(\Pi_{2}^{(S_{b})^{2}})\ast (\Gamma_{2})\ast \ast = 0: \ A_{2}^{2}(S_{b} \times S_{b}) \to A_{2}^{2}(S_{b} \times S_{b}) \text{ for } b \in B \text{ general .}
\]

To finish the proof of the $A_{2}^{2}(X)$ part of theorem [3.1] it remains to connect the action of (the restriction of) $\Gamma_{2}$ and the action of (the restriction of) the relative correspondence $\Gamma_{0}$ that we started out with. We make this connection in the next two lemmas:

**Lemma 3.3.** Notation as above. There is equality
\[
(\Gamma_{2})\ast = ((\Psi \circ \Gamma_{0} \circ \iota \Psi)|_{(S_{b})^{4}})\ast : \ A_{h\text{om}}^{2}(S_{b} \times S_{b}) \to A_{h\text{om}}^{2}(S_{b} \times S_{b}) \text{ for } b \in B \text{ general .}
\]

**Proof.** Unravelling the various definitions we made, we find
\[
\Gamma_{2} = \Psi \circ \Gamma_{1} \circ \iota \Psi \circ \Pi_{2}^{S_{b}^{2}/B} = \Psi \circ (\Gamma_{0} - \gamma) \circ \iota \Psi \circ \Pi_{2}^{S_{b}^{2}/B} = \Psi \circ \Gamma_{0} \circ \iota \Psi \circ \Pi_{2}^{S_{b}^{2}/B} - \gamma' \text{ in } A^{4}(S_{b}^{4}/B) ,
\]
where $\gamma' := \Psi \circ \gamma \circ \iota \Psi \circ \Pi_{2}^{S_{b}^{2}/B}$ is a completely decomposed cycle. The restriction of $\gamma'$ to a general fibre $(S_{b})^{4}$ will be a completely decomposed cycle, and as such will not act on $A_{h\text{om}}^{2}(S_{b} \times S_{b})$. This proves the lemma. \hfill \square

**Lemma 3.4.** Notation as above. There is equality
\[
(\iota \Psi_{b})\ast (\Psi_{b})\ast = 2 \text{ id} : \ A_{h\text{om}}^{2}(X_{b}) \to A_{h\text{om}}^{2}(X_{b}) \forall b \in B .
\]

**Proof.** This is noted in the proof of lemma [2.8] \hfill \square

Obviously, lemmas [3.3] and [3.4] suffice to prove the $A_{2}^{2}(X)$ part of theorem [3.1] indeed, the combination of lemma [3.3] with (8) implies that
\[
(\Pi_{2}^{(S_{b})^{2}})\ast (\Psi \circ \Gamma_{0} \circ \iota \Psi)|_{(S_{b})^{4}}\ast = 0: \ A_{2}^{2}(S_{b} \times S_{b}) \to A^{2}(S_{b} \times S_{b}) \text{ for } b \in B \text{ general .}
\]

Applying relation (2) and lemma [3.4] it follows that
\[
(\Pi_{2}^{X_{b}})\ast (\Gamma_{0})\ast = 0: \ A_{2}^{2}(X_{b}) \to A^{2}(X_{b}) \text{ for } b \in B \text{ general .}
\]
By virtue of the definition of $\Gamma_0$ (and the fact that $A^2_{(2)}(X_b) = (\Pi^X_2)_*A^2_{\text{hom}}(X_b)$), it follows that 

$$(\Pi^X_2)_*(\Gamma_0 + \Delta_{X_b})_* = 0: \quad A^2_{(2)}(X_b) \rightarrow A^2_{(2)}(X_b) \quad \text{for } b \in B \text{ general},$$

as asserted by theorem \ref{3.1}.

We have now proven the $A^2_{(2)}(X)$ part of theorem \ref{3.1}. The statement for $A^4_{(4)}(X)$ follows easily from this. Indeed, Shen–Vial have proven the multiplication map 

$$A^2_{(2)}(X) \otimes A^2_{(2)}(X) \rightarrow A^4_{(4)}(X)$$

is surjective (theorem \ref{2.15}). Given $b \in A^4_{(4)}(X)$, we can thus write 

$$b = a_1 \cdot a_2 \quad \text{in } A^4(X),$$

where $a_1, a_2 \in A^2_{(2)}(X)$. The statement we have just proven for $A^2_{(2)}$ implies that 

$$\iota^*(a_j) = -a_j + r_j \quad \text{in } A^2(X) \quad (j = 1, 2),$$

where $r_j \in A^2_{(0)}(X) \cap A^2_{\text{hom}}(X)$. It follows that 

$$\iota^*(b) = \iota^*(a_1 \cdot a_2) = \iota^*(a_1) \cdot \iota^*(a_2) = (-a_1 + r_1) \cdot (-a_2 + r_1) = a_1 \cdot a_2 - r_1 \cdot a_2 - r_2 \cdot a_1 \quad \text{in } A^4(X).$$

(NB: note that $\iota$ is not a morphism, and so the second equality is not trivial. The second equality happens to be true since $a_1, a_2 \in A^2_{\text{gen}}(X)$; this is \cite{35} Proposition B.6.) But then, since $-r_1 \cdot a_2 - r_2 \cdot a_1 \in A^4_{(2)}(X)$, we have 

$$(\Pi^X_4)_*\iota^*(b) = a_1 \cdot a_2 = b \quad \text{in } A^4_{(4)}(X)$$

as requested.

It remains to prove theorem \ref{3.1} for $A^4_{(0)}(X)$ and $A^4_{(2)}(X)$. As we have seen (theorem \ref{2.15}), Shen–Vial have shown there exists a class $l \in A^2_{(0)}(X)$ inducing an isomorphism 

$$(9) \quad \cdot l: \quad A^2_{(2)}(X) \xrightarrow{\cong} A^4_{(2)}(X). \quad$$

We need to understand the action of $\iota$ on the class $l \in A^2(X)$. To this end, we will prove the following:

**Proposition 3.5.** Let $X$ and $\iota$ be as in theorem \ref{3.7} Let $l \in A^2_{(2)}(X)$ be the class as in theorem \ref{2.15} Then 

$$\iota^*(l) = \pm l \quad \text{in } A^2(X).$$

Proposition \ref{3.5} suffices to complete the proof of theorem \ref{3.1} Indeed, one has 

$$A^4_{(0)}(X) = \mathbb{Q}[l^2]$$

\cite[Theorem 4.6]{35}. The action of $\iota$ on $l^2$ satisfies 

$$\iota^*(l^2) = \iota^*(l) \cdot \iota^*(l) + \iota^*\left((l - \iota\iota^*l)^2 \cdot (l - \iota\iota^*l)^2\right) = l^2 \quad \text{in } A^4(X).$$

(Here, for the first equality we have used \cite{35} Lemma B.4, and the second equality follows from proposition \ref{5.5}). This proves the $A^4_{(0)}$ part of theorem \ref{3.1}.
It remains to prove the $A^4_{(2)}$ part of theorem 3.1. For this, let us suppose for a moment that proposition 3.5 is true with a minus sign, i.e.

$$\iota^*(l) = -l \quad \text{in } A^2(X).$$

Using the isomorphism $\iota^*(9)$, [35, Proposition B.6], and the fact that (as proven above) $\iota^* = -\text{id}$ on $A^2_{(2)}(X)$, this would imply

$$\iota^* = \text{id}: \quad A^4_{(2)}(X) \rightarrow A^4(X).$$

(10)

The statement for $A^4_{(4)}(X)$ we have just proven is that for any $b \in A^4_{(4)}(X)$ we have

$$\iota^*(b) = b + r \quad \text{in } A^4(X)$$

where $r \in A^4_{(2)}(X)$. Since $\iota^*\iota^*(b) = b$, this implies that $\iota^*(r) = -r$, and so (using equality (10)) $r = 0$. That is, we find that

$$\iota^* = \text{id}: \quad A^4(X) \rightarrow A^4(X).$$

Applying [35, Lemma 3.1] to $\cdot \Gamma_t - \Delta_X$, this would imply that

$$\iota^* \cdot \Gamma_t - \Delta_X = \gamma \quad \text{in } A^4(X \times X),$$

where $\gamma$ is a cycle supported on $X \times D$ for $D \subset X$ a divisor. In particular, this would imply

$$\iota^* = \text{id}: \quad H^{2,0}(X) \rightarrow H^{2,0}(X),$$

which is absurd since we know that $\iota$ is non–symplectic. The minus sign in proposition 3.5 can thus be excluded; assuming proposition 3.5 is true, we must have $\iota^*(l) = l$.

Now let $c \in A^4_{(2)}(X)$. Using the isomorphism (9), we can find $a \in A^2_{(2)}(X)$ such that

$$c = l \cdot a \quad \text{in } A^4(X).$$

We know (from the $A^2_{(2)}(X)$ statement proven above) that $\iota^*(a) = -a + r$, where $r \in A^2_{(0)}(X) \cap A^2_{\text{hom}}(X)$. But then

$$\iota^*(c) = \iota^*(l \cdot a) = \iota^*(l) \cdot \iota^*(a) = l \cdot (-a + r) = -l \cdot a = -c \quad \text{in } A^4(X).$$

Here, the second equality holds thanks to [35, Proposition B.6], and the third equality comes from proposition 3.5 and the statement for $A^2_{(2)}$. The fourth equality uses $A^2_{(0)}(X) \cdot (A^2_{(0)}(X) \cap A^2_{\text{hom}}(X)) = 0$, which is a consequence of $A^4_{(0)}(X) \cap A^4_{\text{hom}}(X) = 0$. This proves theorem 3.1 assuming proposition 3.5.

We now proceed with the proof of proposition 3.5. The first step is to prove the corresponding statement in homology:

**Lemma 3.6.** Let $S$ be any K3 surface and let $X = S^{[2]}$. Let $l \in A^2(X)$ be the class of theorem 2.75 and let $\iota \in \text{Bir}(X)$ be any birational involution. Then we have

$$\iota^*(l) = \pm l \quad \text{in } H^4(X).$$

**Proof.** Shen and Vial have constructed a distinguished cycle $L \in A^2(X \times X)$ (whose cohomology class is the Beauville–Bogomolov class denoted $\mathcal{B}$ in loc. cit.), and an eigenspace decomposition

$$A^2(X) = \Lambda^2_{25} \oplus \Lambda^2_{2} \oplus \Lambda^2_{0},$$

$$A^2(X) = \Lambda^2_{25} \oplus \Lambda^2_{2} \oplus \Lambda^2_{0},$$

(11)
where
\[ \Lambda^i_\lambda := \{a \in A^i(X) \mid (L^2)_*(a) = \lambda a\}, \]
and
\[ \Lambda^2_{25} = \mathbb{Q}[t] \]
(This is [35, Theorem 14.5, Propositions 14.6 and 14.8], combined with [35, Theorem 2.2]).

We now observe the following commutativity relation in cohomology:

**Lemma 3.7.** Set–up as in lemma [3.6] Then
\[ (L^2)_*t^* = t^*(L^2)_* : \quad H^i(X) \to H^i(X). \]

**Proof.** Let \( L \in A^2(X \times X) \) be the Shen–Vial cycle as above. As proven in [35, Proposition 1.3(i)], the Shen–Vial cycle satisfies a quadratic relation
\[ L^2 = 2\Delta_X - \frac{2}{25}(l_1 + l_2)L - \frac{1}{23 \cdot 25}(2l_1^2 - 23l_1l_2 + 2l_2^2) \quad \text{in} \quad H^8(X \times X), \]
where \( l := (i_\Delta)^*(L) \) (and \( i_\Delta : X \to X \times X \) is the diagonal embedding) and \( l_i := (p_i)^*(l) \) (and \( p_i \) are the obvious projections).

Let us define a modified cycle
\[ L' := \Gamma_\iota \circ L \circ \Gamma_\iota \quad \in A^2(X \times X). \]
Using Lieberman’s lemma ([13, 16.1.1] or [39, Lemma 3.3]), plus the fact that \( ^t\Gamma_\iota = \Gamma_\iota \), we see that
\[ L' = (t \times \iota)^*(L) \quad \text{in} \quad A^2(X \times X). \]
Define also \( l_i' := (i_\iota)^*(L') \in A^2(X) \) and \( l_i' := (p_i)^*(l') \in A^2(X \times X), i = 1, 2. \) Since the diagram
\[
\begin{array}{ccc}
X \times X & \xrightarrow{p_i} & X \\
\downarrow t \times \iota & & \downarrow t \\
X \times X & \xrightarrow{p_i} & X \\
\end{array}
\]
commutes, we have the relations
\[ l_i' = (t \times \iota)^*(l_i) \quad \text{in} \quad A^2(X \times X), \quad i = 1, 2. \]
Let us apply \( (t \times \iota)^* \) to the quadratic relation (12). The result is a relation
\[ (t \times \iota)^*(L^2) = 2\Delta_X - \frac{2}{25}(t \times \iota)^*(l_1 + l_2)L - \frac{1}{23 \cdot 25}(t \times \iota)^*(2l_1^2 - 23l_1l_2 + 2l_2^2) \quad \text{in} \quad H^8(X \times X). \]
But
\[ (t \times \iota)^*(L^2) = ((t \times \iota)^*L)^2 = (L')^2 \quad \text{in} \quad A^4(X \times X). \]
Plugging this in equality (14), and also using the relations (13), we find that the cycle \( L' \) satisfies a quadratic relation
\[ (L')^2 = 2\Delta_X - \frac{2}{25}(l_1' - l_2')L' - \frac{1}{23 \cdot 25}(2(l_1')^2 - 23l_1'l_2' + 2(l_2')^2) \quad \text{in} \quad H^8(X \times X). \]
But then, applying the unicity result \cite[Proposition 1.3 (v)]{35}, we find there is equality
\[ L' = \pm L \quad \text{in} \quad H^4(X \times X). \]

In particular, there is equality
\[ (L')^2 = L^2 \quad \text{in} \quad H^8(X \times X). \]

In view of equality (15), this means
\[ \Gamma_i \circ (L^2) \circ \Gamma_i = L^2 \quad \text{in} \quad H^8(X \times X), \]
and so (by composing with \( \Gamma_i \))
\[ \Gamma_i \circ (L^2) = (L^2) \circ \Gamma_i \quad \text{in} \quad H^4(X \times X). \]

This proves lemma 3.7.

The eigenspace decomposition (11) induces an eigenspace decomposition modulo homological equivalence:

\[ \text{Im} \left( A^2(X) \to H^4(X) \right) = \Lambda^2_{25} + \frac{\Lambda^2_2}{A^2_{(0)}(X) \cap A^2_{\text{hom}}(X)} \]

(this is the algebraic part of the eigenspace decomposition of \( H^4(X) \) given in \cite[Proposition 1.3(iii)]{35}).

Lemma (3.7) implies \( \iota \) preserves this eigenspace decomposition modulo homological equivalence. In particular, \( \iota^* \Lambda^2_{25} \subset \Lambda^2_{25} \) (modulo homologically trivial cycles), and so
\[ \iota^*(l) = dl \quad \text{in} \quad H^4(X), \]
for some \( d \in \mathbb{Q} \). Since \( \iota \) is an involution, we must have \( d = \pm 1 \). This proves lemma 3.6.

The next step (in proving proposition 3.5) is to upgrade to rational equivalence. Here, we use again the method of “spread” developed in \cite{42}, \cite{43}. As above, let \( S \to B \) resp. \( \mathcal{X} \to B \) denote the family of all smooth degree 10 \( K3 \) surfaces \( S_b \subset \mathbb{P}^3 \), resp. of all Hilbert schemes \( X_b = (S_b)^2 \). We note that there exists a relative cycle
\[ \mathcal{L} \in A^2(\mathcal{X}) \]
such that restriction
\[ \mathcal{L}|_{X_b} = l_b \quad \in \quad A^2(X_b) \quad \forall b \in B \]
is the distinguished class (denoted \( l \) in theorem 2.15) for the fibre \( X_b \). Indeed, one can define \( \mathcal{L} \) as
\[ \mathcal{L} := \frac{5}{6} c_2(T_{\mathcal{X}/B}) \quad \in \quad A^2(\mathcal{X}), \]
where \( T_{\mathcal{X}/B} \) is the relative tangent bundle of the smooth morphism \( \mathcal{X} \to B \). Since for any \( b \in B \) there is a relation
\[ l_b = \frac{5}{6} c_2(X_b) \quad \in \quad A^2(X_b) \]
\cite[Equation (93)]{35}, this implies (17).
The relative cycle
\[ \Gamma_0 := \mathcal{L} \pm i^*(\mathcal{L}) \in A^2(\mathcal{X}) \]
is such that the restriction to each fibre is homologically trivial:
\[ (\Gamma_0)|_{X_b} = 0 \text{ in } H^4(X_b). \]
(Here, “±” is taken to mean + (resp. −) if lemma 3.6 is true with a − (resp. a +).) Thus, the relative cycle
\[ \Gamma_1 := \Psi_*(\Gamma_0) \in A^2(\mathcal{S} \times_B \mathcal{S}) \]
also is homologically trivial on each fibre. (Here, \( \Psi \) is the relative correspondence from \( \mathcal{X} \) to \( \mathcal{S} \times_B \mathcal{S} \) as in the proof of theorem 3.1.)

Applying [42, Lemma 3.12], up to shrinking \( B \) we can make \( \Gamma_1 \) globally homologically trivial. That is, there exists
\[ \psi \in \text{Im} \left( A^2(B \times \mathbb{P}^3 \times \mathbb{P}^3) \to A^2(\mathcal{S} \times_B \mathcal{S}) \right) \]
such that (after replacing \( B \) by a non–empty open subset \( B' \subset B \))
\[ \Gamma_2 := \Gamma_1 + \psi \in A^2(\mathcal{S} \times_{B'} \mathcal{S}) \]
is actually in \( A^2_{\text{hom}}(\mathcal{S} \times_{B'} \mathcal{S}) \).
But \( A^2_{\text{hom}}(\mathcal{S} \times_{B'} \mathcal{S}) = 0 \) (proposition 2.18), and so
\[ \Gamma_2 = 0 \text{ in } A^2(\mathcal{S} \times_{B'} \mathcal{S}). \]
Restricting to a fibre, we find
\[ (\Gamma_1)|_{S_b \times S_b} + \psi|_{S_b \times S_b} = 0 \text{ in } A^2(S_b \times S_b) \text{ } \forall b \in B'. \]
As \( \Gamma_1 \) is fibrewise homologically trivial, the same goes for \( \psi \):
(18) \[ \psi|_{S_b \times S_b} = 0 \text{ in } H^4(S_b \times S_b) \text{ } \forall b \in B'. \]
But \( A^2(\mathbb{P}^3 \times \mathbb{P}^3) = \bigoplus_i A^i(\mathbb{P}^3) \otimes A^{2-i}(\mathbb{P}^3) \) and so
\[ \psi|_{S_b \times S_b} = \lambda_0[S_b] \times H^2_b + \lambda_1 H_b \times H_b + \lambda_2 H^2_b \times [S_b] \text{ in } A^2(S_b \times S_b), \]
where \( \lambda_i \in \mathbb{Q} \) and \( H_b \in A^1(S_b) \) is an ample class on \( S_b \). It follows from the vanishing (18) that the \( \lambda_i \) must be 0, and so \( \psi|_{S_b \times S_b} \) is rationally trivial, and hence also
\[ (\Gamma_1)|_{S_b \times S_b} = 0 \text{ in } A^2(S_b \times S_b). \]
Composing with \( i^* \Psi_b \), it follows that also
\[ (i^* \Psi_b)_*( (\Gamma_1)|_{S_b \times S_b} ) = (i^* \Psi_b)_*( (\Psi_b)_*( (\Gamma_0)|_{X_b} ) ) = 0 \text{ in } A^2(X_b) \text{ } \forall b \in B'. \]
On the other hand, as we have seen above \( (\Gamma_0)|_{X_b} \in A^2_{\text{hom}}(X_b) \) and \( (i^* \Psi_b)_*( (\Psi_b)_* ) \text{ is the identity on } A^2_{\text{hom}}(X_b) \). It follows that
\[ (\Gamma_0)|_{X_b} = (t_b \pm (t_b)^*(h_b))|_{X_b} = 0 \text{ in } A^2(X_b) \text{ } \forall b \in B'. \]
This proves proposition 3.5 for general \( b \in B \). To extend to all \( b \in B \), one can invoke [45, Lemma 3.2]. Proposition 3.5 and hence theorem 3.1 are now proven.
\[ \square \]
Remark 3.8. Can one prove the commutativity of lemma 3.7 also modulo rational equivalence, i.e. can one prove
\[(L^2)_* \tau^* \cong \tau^*(L^2)_*: A^i(X) \to A^i(X)\]

This would imply that \( \iota \) respects the eigenspace decomposition \( \Lambda^i_\lambda \) of [35] (and in particular, that \( \iota \) respects the bigraded ring structure \( A^*_\lambda(X) \)).

The proof of lemma 3.7 given above does not extend to rational equivalence, for the following reason: The quadratic relation (12) still holds modulo rational equivalence [35, Theorem 14.5], and so \( L' \) satisfies the quadratic relation (10) modulo rational equivalence. However, the unicity result ([35, Proposition 1.3(v)]), that allowed us to conclude from this that \( L = \pm L' \), is only known modulo homological equivalence.

(\text{This unicity result modulo rational equivalence is conjecturally true, and would follow from the Bloch–Beilinson conjectures [35, Proposition 3.4]}.)

4. SOME CONSEQUENCES

4.1. A birational statement. Theorem 3.1 can be extended to other birational models:

Corollary 4.1. Let \( X \) and \( \iota \) be as in theorem 3.1. Let \( X' \) be a hyperkähler variety birational to \( X \), and let \( \iota' \in \text{Bir}(X') \) be the birational involution induced by \( \iota \). Then

\[
\begin{align*}
(\iota')^* &= \text{id}: A^4_{(0)}(X') \to A^4_{(0)}(X') ; \\
(\Pi^X_{(2)})_*(\iota')^* &= -\text{id}: A^4_{(2)}(X') \to A^4_{(2)}(X') ; \\
(\Pi^X_{(4)})_*(\iota')^* &= \text{id}: A^4_{(4)}(X') \to A^4_{(4)}(X') .
\end{align*}
\]

Proof. First, let us recall (lemma 2.5) that \( X' \) has an MCK decomposition (induced by an MCK decomposition for \( X \)), and the birational map \( \phi: X \dashrightarrow X' \) induces isomorphisms

\[
\phi_*: A^i_{(j)}(X) \xrightarrow{\cong} A^i_{(j)}(X') .
\]

To deduce corollary 4.1 from theorem 3.1 it only remains to establish commutativity of the diagram

\[
\begin{array}{ccc}
A^i_{(j)}(X) & \xrightarrow{\phi_*} & A^i_{(j)}(X') \\
\downarrow \iota^* & & \downarrow (\iota')^* \\
A^i(X) & \xrightarrow{\phi_*} & A^i(X')
\end{array}
\]

for the relevant \((i, j)\). Let \( U \subseteq X, U' \subseteq X' \) be opens such that \( \iota \) is everywhere defined on \( U \) and \( \phi \) induces an isomorphism between \( U \) and \( U' \). Any 0–cycle \( \alpha \in A^i(X) \) is represented by a cycle \( \alpha \) with support contained in \( U \). Then \( \phi_*(\alpha) \) is represented by the cycle with isomorphic support in \( U' \). This proves commutativity of the square (20) for \( i = 4 \). The Bloch–Srinivas argument [7] (or, more precisely, [35, Lemma 3.1]) applied to the correspondence

\[
\bar{\Gamma}_\phi \circ \bar{\Gamma}_\iota = \bar{\Gamma}_{\iota'} \circ \bar{\Gamma}_\phi \in A^4(X \times X)
\]
(where \( \tilde{\Gamma} \) indicates closure of the cycle \( \Gamma \subset X \times X \)) then shows commutativity for \( A^2_{A,i}(X) = A^2_{hom}(X) \), and so (since \( A^2_{(2)} \subset A^2_{hom} \)) the square (20) commutes for \((i, j) = (2, 2)\). \( \square \)

4.2. **EPW sextics.** Let \( X \) and \( \iota \) be as in theorem 3.1. As we have seen, there is a birational modification \( X' \) of \( X \) that is a hyperkähler fourfold, and such that there is a generically \( 2:1 \) morphism from \( X' \) to an EPW sextic \( Y \) [30], cf. theorem 2.21 above. The following result is about the Chow ring \( A^* \) (in the sense of operational Chow cohomology [13]) of the EPW sextic \( Y \). We note that for any variety \( M \), there exists a "cycle class" map \( A^i(M) \to Gr^W_{2i} H^{2i}(M) \) which is functorial, and agrees with the usual cycle class map for smooth \( M \) [37].

**Corollary 4.2.** Let \( X \) be as in theorem 3.1, and let \( Y \subset \mathbb{P}^5 \) be the associated EPW sextic. For any \( r \in \mathbb{N} \), let \( E^*(Y)^r \subset A^*(Y)^r \) be the subring generated by (pullbacks of) \( A^1(Y) \) and \( A^2(Y) \). The cycle class map

\[
E^k(Y)^r \to Gr^W_{2k} H^{2k}(Y)^r
\]

is injective for \( k \geq 4r - 1 \).

**Proof.** Let \( X' \) and \( \iota' \) be as in theorem 2.21. The point is that \( X' \) and hence also \( (X')^r \), has an MCK decomposition [35]. Let \( p: X' \to Y \) denote the morphism of theorem 2.21.

**Lemma 4.3.** We have

\[
p^* A^2(Y) \subset A^2_{(0)}(X').
\]

**Proof.** As explained in the proof of theorem 2.21 the morphism \( p \) decomposes as

\[
p: X' := X'_A \to X_A \to Y,
\]

where \( s \) is a small resolution of the singular variety \( X_A \), and \( q \) is a double cover with covering involution \( \iota_A \) (and \( \iota_A \) agrees with \( \iota' \) on the open where \( \iota' \) is defined). Because of the equality \( q \circ \iota_A = q \circ X_A \to Y \), one has an inclusion

\[
q^* A^i(Y) \subset A^i(X_A)^{\iota_A} \quad \forall i.
\]

Because of the equality \( \iota_A \circ s = s \circ \iota' \): \( X' \to X_A \), one has an inclusion

\[
s^* (A^i(X_A)^{\iota_A}) \subset A^i(X')^{\iota'} \quad \forall i.
\]

Combining these two inclusions and taking \( i = 2 \), we find in particular that

\[
p^* A^2(Y) \subset A^2(X')^{\iota'}.
\]

Given \( b \in A^2(Y) \), let us write

\[
p^*(b) = c_0 + c_2 \in A^2_{(0)}(X') \oplus A^2_{(2)}(X').
\]

Applying \( \iota' \), we find

\[
(\iota')^* p^*(b) = p^*(b) = c_0 + c_2 \in A^2_{(0)}(X') \oplus A^2_{(2)}(X').
\]

(21)
On the other hand, corollary 4.1 implies that

\[(\iota')^*(c_2) = -c_2 + r \in A^2(X'),\]

where \(r \in A^2_{(0)}(X').\) It follows that

\[(22) \quad (\iota')^* p^*(b) = (\iota')^*(c_0) + (\iota')^*(c_2) = (\iota')^*(c_0) + r - c_2 \in A^2_{(0)}(X') \oplus A^2_{(2)}(X')
\]

(where we have used lemma 4.4 below to obtain that \((\iota')^*(c_0) \in A^2_{(2)}(X')).\) Comparing expressions (21) and (22), we find

\[(\iota')^*(c_0) + r = c_0 \in A^2_{(0)}(X'), \quad -c_2 = c_2 \in A^2_{(2)}(X'),\]

proving lemma 4.3.

**Lemma 4.4.** Set–up as above. Let \(b \in A^2(Y),\) and write

\[p^*(b) = c_0 + c_2 \in A^2_{(0)}(X') \oplus A^2_{(2)}(X').\]

Then

\[(\iota')^*(c_0) \in A^2_{(0)}(X').\]

**Proof.** Suppose

\[(\iota')^*(c_0) = d_0 + d_2 \in A^2(X'),\]

with \(d_0 \in A^2_{(0)}(X')\) and \(d_2 \in A^2_{(2)}(X').\)

Let \(\gamma \in A^4(X \times X')\) be the correspondence of \([33]\) (cf. also lemma 2.5) inducing an isomorphism of bigraded rings

\[(23) \quad \gamma_* : A^*_{(*)}(X) \xrightarrow{\cong} A^*_{(*)}(X').\]

Let \(l \in A^2_{(0)}(X)\) be the distinguished class of theorem 2.15 and define

\[l' := \gamma_*(l) \in A^2_{(0)}(X').\]

The 0–cycle \(c_0 \cdot l'\) is in \(A^4_{(0)}(X'),\) and so corollary 4.1 implies that

\[(24) \quad (\iota')^*(c_0 \cdot l') = c_0 \cdot l' \in A^4_{(0)}(X').\]

On the other hand, we have

\[(25) \quad (\iota')^*(c_0 \cdot l') = (\iota')^*(c_0) \cdot (\iota')^*(l') = (d_0 + d_2) \cdot l' = d_0 \cdot l' + d_2 \cdot l' \in A^4(X').\]

(Here in the first equality, we have used sublemma 4.5 below, and in the second equality we have used proposition 3.5, which we have seen must be true with a + sign.)

Since \(d_0 \cdot l' \in A^4_{(0)}(X')\) and \(d_2 \cdot l' \in A^4_{(2)}(X'),\) comparing expressions (25) and (24), we see that we must have

\[d_0 \cdot l' = c_0 \cdot l' \in A^4_{(0)}(X'), \quad d_2 \cdot l' = 0 \in A^4_{(2)}(X').\]

In view of the isomorphism (23) and the injectivity part of theorem 2.15, this implies that \(d_2 = 0,\) thus proving lemma 4.4.\(\blacksquare\)

Above, we have used the following sublemma:
Sublemma 4.5. Let \( c_0 \in A^2_{(0)}(X') \) be as above. For any \( e \in A^2(X') \), there is equality
\[
(i'_e)^*(c_0 \cdot e) = (i'_e)^*(c_0) \cdot (i'_e)^*(e).
\]

Proof. As we have seen above, the morphism \( p : X' \to X_A \to Y \), where \( X_A \) is the “singular double EPW sextic” on which there exists an involution \( \iota_A \in \text{Aut}(X_A) \) extending \( i \). We thus have
\[
p^*A^2(Y) \subset s^*A^2(X_A)
\]
(where we recall that \( A^*(\cdot) \) of the singular varieties \( Y \) and \( X_A \) means the operational Chow cohomology of [13]). Let \( b_A := q^*(b) \in A^2(X_A) \), so that \( p^*(b) = s^*(b_A) \) in \( A^2(X') \). We claim that there is equality
\[
(i'_e)^*(s^*(b_A) \cdot e) = (i'_e)^*s^*(b_A) \cdot (i'_e)^*(e) \quad \forall e \in A^2(X').
\]
Since there is also equality
\[
(i'_e)^*(c_2 \cdot e) = (i'_e)^*(c_2) \cdot (i'_e)^*(e) \quad \forall e \in A^2(X')
\]
(because \( c_2 \in A^2_{(2)}(X') \subset A^2_{A_X}(X') \)), this follows from [35, Proposition B.6]), and the definitions imply that \( c_0 = s^*(b_A) - c_2 \), the claim suffices to prove the sublemma.

To prove the claim, we exploit the fact that there is a commutative square
\[
\begin{array}{ccc}
X' & \xrightarrow{i'_e} & X' \\
\downarrow s & & \downarrow s \\
X_A & \xrightarrow{i_A} & X_A
\end{array}
\]
Since \( s_* : A_0(X') \to A_0(X_A) \) is an isomorphism, it suffices to prove equality (26) holds after pushing forward under \( s \). The pushforward of the left-hand side of (26) is
\[
s_*((i'_e)^*(s^*(b_A) \cdot e)) = ((i_A)^{-1})_*s_*((i'_e)^*(s^*(b_A) \cdot e))
\]
\[
= (i_A)^*s_*(s^*(b_A) \cdot e)
\]
\[
= (i_A)^*(b_A \cap s_*(e))
\]
\[
= (i_A)^*(b_A) \cap (i_A)_*s_*(e) \quad \text{in } A_0(X_A).
\]
(Here the notation \( \alpha \cap \beta \) means the action of an operational Chow cohomology class \( \alpha \in A^*(X_A) \) on \( \beta \in A_*(X_A) \). The third equality is an application of the projection formula as given in [13, Definition 17.3].)

The pushforward of the right-hand side of (26) is
\[
s_*((i'_e)^*s^*(b_A) \cdot (i'_e)^*(e)) = s_*((i_A)^*(b_A) \cdot (i'_e)^*(e))
\]
\[
= (i_A)^*(b_A) \cap s_*(i'_e)^*(e)
\]
\[
= (i_A)^*(b_A) \cap (i_A)^{-1}_*s_*(e)
\]
\[
= (i_A)^*(b_A) \cap (i_A)_*s_*(e) \quad \text{in } A_0(X_A).
\]
This proves the claim, and hence sublemma 4.5.  \( \square \)
Lemma 4.3, combined with the obvious fact that \( A^1(X') = A^1_{(0)}(X') \), implies that 
\[
(p')^*E^*(Y') \subset A^*_0((X')^r).
\]
Since there is a commutative diagram
\[
\begin{array}{ccc}
A^k_{(0)}((X')^r) & \rightarrow & H^{2k}((X')^r) \\
\uparrow (p')^* & & \uparrow (p')^* \\
E^k(Y^r) & \rightarrow & \text{Gr}^W H^{2k}(Y^r),
\end{array}
\]
and the cycle class map
\[
A^k_{(0)}((X')^r) \rightarrow H^{2k}((X')^r)
\]
is known to be injective for \( k \geq 4r - 1 \) ([40 Introduction]; this follows for instance from [38, Section 4.3]), this establishes corollary 4.2.

We single out a particular case of corollary 4.2:

\textbf{Corollary 4.6.} Let \( Y \subset \mathbb{P}^5 \) be an EPW sextic as in corollary 4.2. The subspaces
\[
\text{Im} \left( A^2(Y) \otimes A^1(Y) \rightarrow A^3(Y) \right),
\]
\[
\text{Im} \left( A^2(Y) \otimes A^2(Y) \rightarrow A^4(Y) \right)
\]
are of dimension 1.

\textit{Proof.} This follows from corollary 4.2 combined with the fact that
\[
N^3(Y) := \text{Im} \left( A^3(Y) \rightarrow \text{Gr}^W H^6(Y) \right)
\]
is of dimension 1. To see this, since the pairing
\[
NS(X)^t \otimes N^3(X)^t \rightarrow N^4(X)^t \cong \mathbb{Q}
\]
is non–degenerate, it suffices to prove
\[
\text{(27)} \quad \dim NS(Y) = \dim NS(X)^t = 1.
\]
But \( \dim \text{Gr}^W H^2(Y) = 1 \) (weak Lefschetz for the hypersurface \( Y \subset \mathbb{P}^5 \)), and so
\[
\dim H^2(X)^t = 1,
\]
proving (27).

(Alternatively, (27) can also be proven directly: \( \iota \) acts on \( NS(X) \) as reflection in the span of \( D \) (proposition 2.20), and so \( NS(X)^t = \mathbb{Q}[D] \) is of dimension 1.)

\textbf{Remark 4.7.} It is instructive to compare corollary 4.6 with known results concerning the Chow ring of K3 surfaces and of Calabi–Yau varieties. For any K3 surface \( S \), it is known that
\[
\dim \text{Im} \left( A^1(S) \otimes A^1(S) \rightarrow A^2(S) \right) = 1
\]
For a generic Calabi–Yau complete intersection $X$ of dimension $n$, it is known that
\[ \dim \text{Im} \left( A^i(X) \otimes A^{n-i}(X) \rightarrow A^n(X) \right) = 1, \quad \forall 0 < i < n \]

The new part of corollary 4.6, with respect to these results, is the part about
\[ \text{Im} \left( A^2(X) \otimes A^1(X) \rightarrow A^3(X) \right). \]

This part is conjecturally true for all EPW sextics, but presumably not true for general Calabi–Yau varieties (or even general Calabi–Yau complete intersections). This is related to the question of determining which varieties satisfy Beauville’s weak splitting property.

5. SOME QUESTIONS

In this final section, we record some questions suggested by the results of this note.

**Question 5.1.** Let $X$ and $\iota$ be as in theorem 3.1. Can one prove that $\iota$ preserves the bigrading $A^*_\sigma(X)$ of the Chow ring (so that the $\Pi^I_\sigma$ in theorem 3.1 can be omitted)? It is a matter of regret that I have not been able to prove this (cf. remark 3.8).

**Question 5.2.** Let $X$ and $\iota$ be as in theorem 3.1. What can one say about the action of $\iota$ on $A^3(X)$? This seems more difficult than theorem 3.1. Indeed, the action of $\iota$ on $A^2_{\text{hom}}$ and on $A^3$ is determined by “behaviour up to codimension 1 phenomena”. The action of $\iota$ on $A^3_{(2)}$, on the other hand, should be determined by the action of $\iota$ on $H^{3,1}(X)$, which is not as neat as the action on $H^{2,0}(X)$ and $H^{4,0}(X)$. I am not even sure what the conjectural statement should be.

**Question 5.3.** Let $X'$ be a double EPW sextic as in theorem 2.21, and let $E \sim \mathbb{P}^2$ denote the exceptional locus of the small resolution $X' \rightarrow X_A$. Since $E \subseteq X'$ is a constant cycle subvariety, the ideas developed in [47] suggest that $E$ should lie in $A^2(0)(X')$. Can this actually be proven?

**Question 5.4.** Let $X$ be a Hilbert square $X = S^{[2]}$, where $S$ is a very general K3 surface of degree
\[ d = 2(4n^2 + 8n + 5), \quad n \in \mathbb{N}. \]

Generalizing O’Grady’s result (theorem 2.21 which corresponds to $n = 0$), Iliev–Madonna show that $X$ is birational to a double EPW sextic [15 Theorem 1.1], and so there exists an anti–symplectic birational involution $\iota$ on $X$. It follows from Shen–Vial’s work (theorem 2.6) that $X$ has an MCK decomposition. Is it possible to prove theorem 3.1 for $X$ and $\iota$?

The case $n = 1$ (i.e. $d = 34$ and $g(S) = 18$) can perhaps be done similarly to theorem 3.1 in this case, the general K3 surface $S$ is given as zero locus of some vector bundle $V$ on the orthogonal Grassmannian $OG(3, 9)$ [25]. The only problem is that I’m not sure whether the vector bundle $V$ is sufficiently ample for Voisin’s method of spread to work in this case.

The case $n > 1$ seems considerably more difficult, due to the absence of Mukai models for $S$.

**Question 5.5.** Double EPW sextics form a 20–dimensional family of hyperkähler fourfolds of K3[2] type (and the above–mentioned work [15] constructs a countably infinite number of codimension 1 subfamilies, elements of which are birational to Hilbert squares of K3 surfaces). More
ambitiously than the above question[5,4] can one somehow prove theorem[3,7] for all double EPW sextics?

One obvious problem is that first (in order for the statement of theorem[3,7] even to make sense), one would need to construct an MCK decomposition for a general double EPW sextic.

Acknowledgements. Thanks to all participants of the Strasbourg 2014/2015 “groupe de travail” based on the monograph[45] for a very pleasant atmosphere. Much thanks to Jiji and Baba for hospitably receiving me in Kokuryo, where this note was written ☺.

REFERENCES

[1] A. Beauville, Some remarks on Kähler manifolds with $c_1 = 0$, in: Classification of algebraic and analytic manifolds (Katata, 1982), Birkhäuser Boston, Boston 1983,
[2] A. Beauville, Variétés Kähleriennes dont la première classe de Chern est nulle, J. Differential Geom. 18 no. 4 (1983), 755—782,
[3] A. Beauville and C. Voisin, On the Chow ring of a K3 surface, J. Alg. Geom. 13 (2004), 417—426,
[4] A. Beauville, On the splitting of the Bloch–Beilinson filtration, in: Algebraic cycles and motives (J. Nagel and C. Peters, editors), London Math. Soc. Lecture Notes 344, Cambridge University Press 2007,
[5] S. Bloch, Lectures on algebraic cycles, Duke Univ. Press Durham 1980,
[6] S. Bloch and A. Ogus, Gersten’s conjecture and the homology of schemes, Ann. Sci. Ecole Norm. Sup. 4 (1974), 181—202,
[7] S. Bloch and V. Srinivas, Remarks on correspondences and algebraic cycles, American Journal of Mathematics Vol. 105, No 5 (1983), 1235—1253,
[8] A. Corti and M. Hanamura, Motivic decomposition and intersection Chow groups, I, Duke Math. J. 103 (2000), 459—522,
[9] C. Deninger and J. Murre, Motivic decomposition of abelian schemes and the Fourier transform. J. reine u. angew. Math. 422 (1991), 201—219,
[10] D. Eisenbud, S. Popescu and C. Walter, Lagrangian subbundles and codimension 3 subcanonical subschemes, Duke Math. J. 107(3) (2001), 427—467,
[11] L. Fu, Decomposition of small diagonals and Chow rings of hypersurfaces and Calabi–Yau complete intersections, Advances in Mathematics (2013), 894—924,
[12] L. Fu, Z. Tian and Ch. Vial, Motivic hyperkähler resolution conjecture for generalized Kummer varieties, arXiv:1608.04968,
[13] W. Fulton, Intersection theory, Springer–Verlag Ergebnisse der Mathematik, Berlin Heidelberg New York Tokyo 1984,
[14] B. Gordon, M. Hanamura and J. Murre, Relative Chow–Künneth projectors for modular varieties, J. reine u. angew. Math. 558 (2003), 1—14,
[15] A. Iliev and C. Madonna, EPW sextics and Hilbert squares of K3 surfaces, Serdica Math. Journal 41 no. 4 (2015), 343—354,
[16] U. Jannsen, Motivic sheaves and filtrations on Chow groups, in: Motives (U. Jannsen et alii, eds.), Proceedings of Symposia in Pure Mathematics Vol. 55 (1994), Part 1,
[17] U. Jannsen, On finite–dimensional motives and Murre’s conjecture, in: Algebraic cycles and motives (J. Nagel and C. Peters, editors), Cambridge University Press, Cambridge 2007,
[18] B. Kahn, J. Murre and C. Pedrini, On the transcendental part of the motive of a surface, in: Algebraic cycles and motives (J. Nagel and C. Peters, editors), Cambridge University Press, Cambridge 2007,
[19] S.-I. Kimura, Fractional intersection and bivariant theory, Comm. Alg. 20 (1992), 285-302,
[20] S. Kleiman, Algebraic cycles and the Weil conjectures, in: Dix exposés sur la cohomologie des schémas, North–Holland Amsterdam, 1968, 359—386,
[21] S. Kleiman, The standard conjectures, in: Motives (U. Jannsen et alii, eds.), Proceedings of Symposia in Pure Mathematics Vol. 55 (1994), Part 1,
[22] R. Laterveer, Algebraic cycles on a very special EPW sextic, to appear in Rend. Sem. Mat. Univ. Padova,
[23] R. Laterveer and Ch. Vial, On the Chow ring of Cynk–Hulek Calabi–Yau varieties and Schreieder varieties, arXiv:1712.03070,
[24] S. Mukai, Curves, K3 surfaces and Fano 3-folds of genus $\leq 10$, in: Algebraic Geometry and Commutative Algebra I (H.Hijikata et al., eds.), Kinokuniya Tokyo 1988,
[25] S. Mukai, Polarized K3 surfaces of genus 18 and 20, in: Complex Projective Geometry, Trieste 1989/Bergen 1989 (G. Ellingsrud et al., eds.), London Mathematical Society Lecture Note Series 179, Cambridge University Press 1992,
[26] J. Murre, On a conjectural filtration on the Chow groups of an algebraic variety, parts I and II, Indag. Math. 4 (1993), 177—201,
[27] J. Murre, J. Nagel and C. Peters, Lectures on the theory of pure motives, Amer. Math. Soc. University Lecture Series 61, Providence 2013,
[28] J. Nagel and M. Saito, Relative Chow–Künneth decompositions for conic bundles and Prym varieties, Int. Math. Res. Not. 2009, no. 16, 2978—3001.
[29] K. O’Grady, Involutiones and linear systems on holomorphic symplectic manifolds, Geom. Funct. Anal. 15 no 6 (2005), 1223—1274,
[30] K. O’Grady, Irreducible symplectic 4–folds and Eisenbud–Popescu–Walter sextics, Duke Math. J. 134(1) (2006), 99—137,
[31] K. O’Grady, EPW–sextics: taxonomy, Manuscripta Math. 138 1 (2012), 221—272,
[32] K. O’Grady, Double covers of EPW–sextics, Michigan Math. J. 62 (2013), 143—184,
[33] U. Rieß, On the Chow ring of birational irreducible symplectic varieties, Manuscripta Math. 145 (2014), 473—501,
[34] T. Scholl, Classical motives, in: Motives (U. Jannsen et alii, eds.), Proceedings of Symposia in Pure Mathematics Vol. 55 (1994), Part 1,
[35] M. Shen and Ch. Vial, The Fourier transform for certain hyperKähler fourfolds, Memoirs of the AMS 240 (2016), no.1139,
[36] M. Shen and Ch. Vial, The motive of the Hilbert cube $X^{[3]}$, Forum Math. Sigma 4 (2016),
[37] B. Totaro, Chow groups, Chow cohomology, and linear varieties, Forum of Mathematics, Sigma vol. 1 e1 (2014),
[38] Ch. Vial, Niveau and coniveau filtrations on cohomology groups and Chow groups, Proceedings of the LMS 106(2) (2013), 410—444,
[39] Ch. Vial, Remarks on motives of abelian type, Tohoku Math. J. 69 (2017), 195—220,
[40] Ch. Vial, On the motive of some hyperkähler varieties, J. für Reine u. Angew. Math. 725 (2017), 235—247,
[41] C. Voisin, Chow rings and decomposition theorems for $K3$ surfaces and Calabi–Yau hypersurfaces, Geom. Topol. 16 (2012), 433—473,
[42] C. Voisin, The generalized Hodge and Bloch conjectures are equivalent for general complete intersections, Ann. Sci. Ecole Norm. Sup. 46, fascicule 3 (2013), 449—475,
[43] C. Voisin, The generalized Hodge and Bloch conjectures are equivalent for general complete intersections, II, J. Math. Sci. Univ. Tokyo 22 (2015), 491—517,
[44] C. Voisin, Bloch’s conjecture for Catanese and Barlow surfaces, J. Differential Geometry 97 (2014), 149—175,
[45] C. Voisin, Chow Rings, Decomposition of the Diagonal, and the Topology of Families, Princeton University Press, Princeton and Oxford, 2014,
[46] C. Voisin, Hodge structures, Coniveau and Algebraic Cycles, in: “The Legacy of Bernhard Riemann After One Hundred and Fifty Years”, ALM35, Higher Education Press and International Press, Beijing Boston 2016.

[47] C. Voisin, Remarks and questions on coisotropic subvarieties and 0–cycles of hyper–Kähler varieties, in: K3 Surfaces and Their Moduli, Proceedings of the Schiermonnikoog conference 2014 (C. Faber, G. Farkas, G. van der Geer, editors), Progress in Maths 315, Birkhäuser 2016.

INSTITUT DE RECHERCHE MATHEMATIQUE AVANCEE, CNRS – UNIVERSITE DE STRASBOURG, 7 RUE RENÉ DESCARTES, 67084 STRASBOURG CEDEX, FRANCE.

E-mail address: robert.laterveer@math.unistra.fr