Quark magnetar in three-flavor Nambu–Jona-Lasinio model with vector interaction and magnetized gluon potential

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We investigate properties of strange quark matter in the framework of SU(3) Nambu–Jona-Lasinio(NJL) model with vector interaction under strong magnetic fields. The effects of vector-isoscalar and vector-isovector interaction on the equation of state of strange quark matter are investigated, and it is found that the equation of state is not sensitive to the vector-isovector interaction, however, a repulsive interaction in the vector-isoscalar channel gives a stiffer equation of state for cold dense quark matter. In the presence of magnetic field, gluons will be magnetized via quark loops, and the contribution from magnetized gluons to the equation of state is also estimated. The sound velocity square is a quantity to measure the hardness or softness of dense quark matter, and in the NJL model without vector interaction at zero magnetic field the sound velocity square is always less than 1/3. It is found that a repulsive vector-isoscalar interaction and a positive pressure contribution from magnetized gluons can enhance the sound velocity square, which can even reach 1. To construct quark magnetars under strong magnetic fields, we consider anisotropic pressures and use a density-dependent magnetic field profile to mimic the magnetic field distribution in a quark star. We also analyze the parameter region for the magnitude of vector-isoscalar interaction and the contribution from magnetized gluons in order to produce 2 solar mass quark magnetars.

I. INTRODUCTION

Investigating properties of strong interaction matter is one of the main topics of quantum chromodynamics (QCD). It is believed that there will be deconfinement phase transition from hadronic matter to quark-gluon plasma (QGP) at sufficiently high temperatures and from nuclear matter to quark matter (or color superconductor) at high baryon densities. The hot quark-gluon plasma can be created by heavy ion collisions at the Relativistic Heavy Ion collider (RHIC) and the Large Hadron Collider (LHC). The hot and dense quark matter might be created at FAIR in GSI and the Nuclotron-based Ion Collider Facility (NICA) at JINR, while the cold and dense quark matter may exist in the inner core of compact stars.

In the inner core of compact stars, the baryon density can reach or even be larger than about 6 times the normal nuclear matter density $n_0 = 0.16 \text{ fm}^{-3}$ so there might exist "exotic" matter like hyperons $^\text{[1]}$ $^\text{[2]}$, meson condensations $^\text{[3]}$ $^\text{[4]}$ and quark matter (normal quark matter or strange quark matter $^\text{[5]}$ $^\text{[6]}$, and color superconductor $^\text{[7]}$ $^\text{[8]}$). Strange quark matter has been conjectured to be the true ground state of QCD $^\text{[9]}$ $^\text{[10]}$, and many efforts have been put on investigating the conversion from neutron star to quark star that consists of strange quark matter which is made purely by $u, d, s$ quarks and some leptons like electron and muon due to charge neutrality and $\beta$-equilibrium $^\text{[11]}$ $^\text{[12]}$. There are also hybrid star conjectures on a transition from nuclear phase to quark phase at such high baryon density, and several authors have studied the phase transition in hybrid star $^\text{[13]}$ $^\text{[14]}$ $^\text{[15]}$ $^\text{[16]}$ $^\text{[21]}$ $^\text{[22]}$.

The equation of state (EoS) plays a central role in investigating properties of strong interaction matter, which is one of fundamental issues in nuclear physics, astrophysics and cosmology. On the one hand, the EoS generates unique mass versus radius (M-R) relations for neutron stars and the ultra-dense remnants of stellar evolution. On the other hand, the mass-radius relation of compact stars put strong constraints on the EoS for strong interaction matter at high baryon density and low temperature.

Recently, two heaviest neutron stars have been measured with high accuracy. One is the radio pulsar J1614-2230 $^\text{[23]}$ with a mass of $1.97 \pm 0.04 M_{\odot}$, and the other is J0348+0432 $^\text{[24]}$ with mass $2.01 \pm 0.04 M_{\odot}$. Even heavier neutron stars were discussed in the literatures $^\text{[25]}$ $^\text{[26]}$. It is suggested that the established existence of two-solar-mass neutron stars would prefer hard EoS based entirely on conventional nuclear degrees of freedom, and many soft equations of state, including hybrid stars containing significant "exotic" proportions of hyperons, Bose condensates or quark matter would be ruled out.

However, it is pointed out that the repulsive vector interaction in the Nambu–Jona-Lasinio (NJL) model can...
produce a stiff EoS for dense quark matter thus can generate two-solar-mass compact star. The role of the vector interaction in QCD vacuum and medium has been discussed in many literatures and . The introduction of the vector interaction within the NJL model is necessary to describe vector mesons, and the coupling constant is determined by the vector spectra . In order to describe vector bound states within the NJL model, the vector interaction must be attractive in the space-like components, thus it is repulsive in the time-like components, which is relevant for the number density in the mean field . The QCD phase diagram and the critical end point is sensitive to the sign of vector coupling constant as shown in .

Furthermore, the presence of external magnetic field can even harden the equation of state of dense quark matter as shown in Ref. . Many efforts have been taken on investigating the existence of quark core in neutron stars under strong magnetic field and . In this work, we will use SU(3) Nambu-Jona-Lasinio(NJL) model with vector interaction to investigate the magnetar, and we will also consider the pressure contribution from polarized gluons under magnetic field.

It is known that the NJL model only considers the quark contribution to the pressure, therefore, the pressure from NJL model is smaller than that from lattice calculation at high temperature and zero chemical potential . In order to consider the gluon contribution to the pressure, the Polyakov-loop potential was firstly introduced in the framework of NJL model in and then in , where the Polyakov-loop potential is temperature dependent. The PNJL model can fit the equation of state at high temperature very well with lattice QCD results. However, to extend the Polyakov-loop potential to finite chemical potential is not trivial. In order to use the PNJL model to describe neutron stars, the Polyakov potential is modified to have chemical potential dependent in Refs. and . In order to describe magnetars, one has to consider the contribution from magnetized gluons. However, there is still no extension of Polyakov-loop potential under magnetic field. In this work, we will give an ansatz on the potential from magnetized gluons.

The paper is organized as follows. In Sec. II, we give a general description of the SU(3) NJL model with vector interaction under magnetic field with equilibrium, we make an ansatz of thermodynamical potential from magnetized gluons, and derive the equation of state of the strange quark matter with -equilibrium. Our numerical results are shown in Sec. III and the conclusion and discussion is given in Sec. IV.

II. THREE-FLAVOR NJL MODEL WITH VECTOR INTERACTION UNDER MAGNETIC FIELD

We study properties of three-flavor system with external strong magnetic fields under equilibrium condition, which is described by the Lagrangian density

$$\mathcal{L} = \mathcal{L}_q + \mathcal{L}_e - \frac{1}{4} F_{\mu\nu} F^{\mu\nu},$$

(1)

where and are the Lagrangian densities for quarks and electrons, respectively. is the strength tensor for external electromagnetic field. The magnetic field is a static magnetic field along direction, and . In this work, we do not
consider contributions from the anomalous magnetic moments \cite{94}.

The electron Lagrangian density is given as

$$L_e = \bar{e} [i \gamma \mu (i \partial^{\mu} - e A^{\mu}_{ext})] \gamma^\nu e. \quad (2)$$

The Lagrangian density for quarks is described by the gauged $N_f = 3$ NJL model with vector interaction,

$$L_q = \bar{\psi}_f [\gamma_{\mu} (i \partial^{\mu} - q_f A^{\mu}_{ext}) - \bar{m}_c] \psi_f + L_4 + L_6, \quad (3)$$

where $L_4$ indicates four-fermion interaction compatible with QCD symmetries $SU(3)_{color} \otimes SU(3)_L \otimes SU(3)_R$, and $L_6$ is the six-point interaction which is required to break the axial $U(1)_A$ symmetry. $\psi = (u, d, s)^T$ represents a quark field with three flavors, $\bar{m}_c = \text{diag}(m_u, m_d, m_s)$ is the current quark mass matrix, and $q_f$ the quark electric charge. The four-fermion interaction includes scalar, pseudoscalar, vector, axial-vector channels and takes the form of

$$L_4 = L_S + L_V + L_{I, V}. \quad (4)$$

The scalar part takes the form of

$$L_S = G_S \sum_{a = 0}^{8} [(\bar{\psi}_f \lambda_a \psi_f)^2 + (\bar{\psi}_f i \gamma_5 \lambda_a \psi_f)^2], \quad (5)$$

and the vector part is given as

$$L_V = -G_V \sum_{a = 0}^{8} [(\bar{\psi} \gamma^\mu \lambda_a \psi)^2 + (\bar{\psi} i \gamma_5 \gamma^\mu \lambda_a \psi)^2], \quad (6)$$

where $G_S$ and $G_V$ are the coupling constants in the scalar and vector channels, respectively. $\lambda_a (a = 1, \cdots, 8)$ are the Gell-Mann matrices, and the generators of the SU(3) flavor groups, and $\lambda_0 = \sqrt{2/3} I$ with $I$ the $3 \times 3$ unit matrix. In order to describe the nonet of scalars, pseudo-scalars, vectors and axial-vectors, a convenient representation is obtained by changing from $\{\lambda_0, \lambda_1, \cdots, \lambda_8\}$ to the set $\{\lambda_0, \lambda^+_1, \lambda^-_3, \lambda^+_4, \lambda^-_5, \lambda_6, \lambda_8\}$ with

$$\lambda^+_1 = \sqrt{\frac{1}{2} (\lambda_1 + i \lambda_2)},$$

$$\lambda^-_4 = \sqrt{\frac{1}{2} (\lambda_4 - i \lambda_5)},$$

$$\lambda^-_6 = \sqrt{\frac{1}{2} (\lambda_6 + i \lambda_7)}.$$

Hadrons in the u, d sector exhibit $SU(2)_I$ isospin symmetry. Up and down quarks have isospin $I = 1/2$, and isospin 3-components $(I_3)$ of 1/2 and $-1/2$, respectively. All other quarks have $I = 0$. For scalars, the coupling constant in the scalar-isoscalar ($\sigma$) and pseudoscalar-isovector ($\pi$) interactions have to be equal, which is constrained by chiral symmetry. However, the coupling constants for the vector-isoscalar ($\omega$) and vector-isovector ($\rho$) interaction can be separately invariant, thus can be chosen independently. The ratio of the coupling constants of the vector-isosinglet channel $\omega$ and vector-isovector channel $\rho$ to nucleons is empirically given by $g_{\omega NN}/g_{\rho NN} \approx 3$ in the chirally broken phase, and $g_{\omega QQ}/g_{\rho QQ} = 1$ in the chiral symmetric phase \cite{27,21,02}. To distinguish the isoscalar and isovector for vectors, we introduce an extra term for vector-isoscalar channel with the form of

$$L_{IV} = -G_{IV} [(\bar{\psi} \gamma^{\mu} \gamma^{5} \psi_f)^2 + (\bar{\psi} \gamma^{5} \gamma^{\mu} \gamma^{5} \psi_f)^2]. \quad (7)$$

The coupling constant for vector-isoscalar $G_{IV} = G_V$, and the coupling constant for vector-isovector $G_{IV} = G_V + G_{IV}$. In this work, we will investigate the role of vector-isovector interaction on the equation of state, therefore in our numerical calculations, we choose $G_{IV}$ as a free parameter.

The six-fermion interaction $L_6$, i.e. the ’t Hooft term takes the form of

$$L_{det} = -K \{ det_f (\bar{\psi}_f (1 + \gamma_5) \psi_f) + det_f (\bar{\psi}_f (1 - \gamma_5) \psi_f) \}, \quad (8)$$

which is to break the $U(1)_A$ symmetry.

### A. The pressure from quark contribution

The equation of state is the most important aspect for physicists to acquire the properties of quark matter. In order to get the pressure and energy density of quark matter, we should firstly derive the thermodynamical potential $\Omega_f$. In procedure, we can calculate the thermodynamical quantities by using finite temperature field theory. In the mean-field approximation, the lagrangian density for quark part under mean-field approximation is

$$L_M = \bar{\psi}_f [\gamma_{\mu} (i \partial^{\mu} - q_f A^{\mu}_{ext})] - \hat{M} - 2G_{IV} \gamma_0 \tau_3 n_f \psi_f$$

$$-2G_S (\sigma_n^\mu + \sigma_n^5 + \sigma_n^6) + 4K \sigma_n \sigma_d \sigma_s - 4G_V \gamma_0 \hat{n}$$

$$+ 2G_V (n_n^u + n_n^d + n_n^s) + G_{IV} (n_n^u - n_n^d)^2,$$

where $\hat{n} = \left( \begin{array}{ccc} n_u & 0 & 0 \\ 0 & n_d & 0 \\ 0 & 0 & n_s \end{array} \right)$ and $\hat{M} = \left( \begin{array}{ccc} M_u & 0 & 0 \\ 0 & M_d & 0 \\ 0 & 0 & M_s \end{array} \right)$.

The quark mass is determined by the gap equation of

$$M_f = m_f - 4G_S \sigma_f + 2K \sigma_j \sigma_k,$$

with $f, j, k = u, d, s$ and the chiral condensate is given as

$$\sigma_f = (\bar{\psi}_f \psi_f) = -i \int \frac{d^4 p}{(2\pi)^4} \text{Tr} \left[ \frac{1}{(p - M_f + i\epsilon)} \right]. \quad (11)$$

After introducing Landau quantization and several steps of finite temperature field theory calculations, we can get the thermodynamical potential $\Omega_q$ of quark matter under magnetic fields, and the pressure density $p_q = \ldots$
with the logarithmic contribution

\[
\Omega_{\text{ln}}^f = -i \int_0^{\pi} \frac{d^4p}{(2\pi)^4} \ln \left\{ \frac{1}{T} \hat{p} - M_f + \gamma_0 \tilde{\mu}_f \right\},
\]

(13)

Here, \( \tilde{\mu}_f = \mu_f - 4G_V n_f - 2G_1 V \tau_3 f (n_u - n_d) \).

(14)

where \( \mu_f \) is the chemical potential for each flavor of quarks and \( \tau_3 f \) is the isospin quantum number for quarks:

\( \tau_3 u = 1, \tau_3 d = -1 \) and \( \tau_3 s = 0 \).

Following Ref. [13], one can get the condensates and pressure for quarks. The logarithmic contribution to the thermodynamical potential is given by

\[
\Omega_{\text{ln}}^f = \Omega_{\text{ln}}^{f, \text{vac}} + \Omega_{\text{ln}}^{f, \text{mag}} + \Omega_{\text{ln}}^{f, \text{med}}.
\]

(15)

The first term is the vacuum contribution

\[
\Omega_{\text{ln}}^{f, \text{vac}} = -\frac{N_c}{8\pi^2} \left\{ M_f^4 \ln \left\{ \frac{\Lambda + \epsilon_A}{M_f} \right\} - \epsilon_A \Lambda (\Lambda^2 + \epsilon_A^3) \right\}
\]

with \( \epsilon_A^2 = \Lambda^2 + M_f^2 \) and \( \Lambda \) the noncovariant cutoff. The magnetic field contribution takes the form of

\[
\Omega_{\text{ln}}^{f, \text{mag}} = \frac{N_c}{2\pi^2} (|q_f| B)^2 \left[ \frac{x_f^2}{4} + \zeta'(-1, x_f) \right],
\]

(17)

where \( \zeta(z, x) \) is the Riemann-Hurwitz function and

\[
\zeta'(-1, x_f) = d\zeta(z, x)/dz|_{z=-1}
\]

(18)

with \( x_f = \frac{M_f^2}{2|q_f| B} \). The medium contribution reads

\[
\Omega_{\text{ln}}^{f, \text{med}} = \sum_{k=0}^{k_{f, \text{max}}} \alpha_k \left\{ q_f |B| n_c \right\} \left\{ \tilde{\mu}_f \sqrt{\tilde{\mu}_f^2 - s_f(k, B)^2} \right\} - s_f(k, B)^2 \ln \left[ \frac{\tilde{\mu}_f + \sqrt{\tilde{\mu}_f^2 - s_f(k, B)^2}}{s_f(k, B)} \right],
\]

(19)

where

\[
s_f(k, B) = \sqrt{M^2 + 2|q_f| B k},
\]

(20)

and

\[
k_{f, \text{max}} = 2q_f^2 B = \frac{p_{f, \text{med}}^2}{2|q_f| B}.
\]

(21)

is the upper Landau level with \( \alpha_k = 2 - \delta_{k0} \).

Then we can also give the condensates for each flavor of quarks:

\[
\sigma_f = \sigma_f^{\text{vac}} + \sigma_f^{\text{mag}} + \sigma_f^{\text{med}}
\]

(22)

with

\[
\sigma_f^{\text{vac}} = -\frac{M_f N_c}{2\pi^2} \left\{ \Lambda \sqrt{\Lambda^2 + M_f^2} - M_f^2 \ln \left[ \frac{(\Lambda + \sqrt{\Lambda^2 + M_f^2})^2}{M_f^2} \right] \right\},
\]

(23)

\[
\sigma_f^{\text{mag}} = -\frac{M_f N_c}{2\pi^2} (|q_f| B \left\{ \ln |f(x_f)| \right\} - \frac{1}{2} \ln(2\pi) + \ln(x_f) - x_f \ln(x_f)),
\]

(24)

\[
\sigma_f^{\text{med}} = \sum_{k=0}^{k_{f, \text{max}}} \alpha_k \frac{M_f |q_f| B n_c}{\pi^2} \left\{ \ln \left[ \frac{\tilde{\mu}_f + \sqrt{\tilde{\mu}_f^2 - s_f(k, B)^2}}{s_f(k, B)} \right] \right\}.
\]

(25)

B. The pressure from leptons

For SQM, we assume it is neutrino-free and composed of \( u, d, s \) quarks and \( e^- \) in beta-equilibrium with electric charge neutrality. The weak beta-equilibrium condition can then be expressed as

\[
\mu_u + \mu_e = \mu_d = \mu_s,
\]

(26)

where \( \mu_i \) (\( i = u, d, s \) and \( e^- \)) is the chemical potential of the particles in SQM. Furthermore, the electric charge neutrality condition can be written as

\[
\frac{2}{3} n_u = \frac{1}{3} n_d + \frac{1}{3} n_s + n_e.
\]

(27)

where

\[
n_f = \sum_{k=0}^{k_{f, \text{max}}} \alpha_k \frac{|q_f| B n_c}{2\pi^2} k_{F, f}
\]

(28)

is the number density for each flavor of quarks with \( k_{F, f} = \sqrt{\mu_f^2 - s_f(k, B)^2} \), and

\[
n_l = \sum_{k=0}^{k_{l, \text{max}}} \alpha_k \frac{|q_l| B n_c}{2\pi^2} k_{F, l}
\]

(29)

the number density of electrons.

We can also write the leptonic contribution to the pressure density, which takes the form of:

\[
p_l = \sum_{k=0}^{k_{l, \text{max}}} \alpha_k \left\{ \frac{|q_l| B n_c}{4\pi^2} \left\{ \mu_l \sqrt{\mu_l^2 - s_l(k, B)^2} - s_l(k, B)^2 \ln \left[ \frac{\mu_l + \sqrt{\mu_l^2 - s_l(k, B)^2}}{s_l(k, B)} \right] \right\} \right\}.
\]

(30)
C. The pressure from magnetized gluons

It is well known that the NJL model only considers the quark contribution to the pressure, which is smaller than that from lattice calculation at high temperature and zero chemical potential \[85, 86\]. In the presence of strong magnetic field, not only quarks are polarized along the direction of \( B \), but also gluons will be polarized via the quark loop. There are still no calculations of pressure contributed from magnetized gluon degrees of freedom.

In order to consider the quark contribution to the pressure, on the one hand, the Polyakov-loop potential was introduced in the framework of NJL model in \[87, 88\], where the Polyakov-loop potential is temperature dependent. The PNJL model can fit the equation of state at high temperature very well with lattice QCD results. However, to extend the Polyakov-loop potential to finite temperature, on the one hand, the Polyakov-loop potential was contributed from magnetized gluon degrees of freedom. The corresponding Debye mass of the longitudinal gluon fields \( A_\mu \) has the screening mass

\[
m_g^2(eB) = \sum_f |q_f|^2 \frac{g_{eff}^2}{4\pi^2} |eB|,
\]

at zero temperature and density. One can guess that at nonzero temperature and density, the longitudinal gluons take the screening mass of

\[
m_g^2(T, \mu, eB) = g^2(aT^2 + b\mu^2 + ceB)
\]

where \( a, b, c \) are constants. Taking into account the transverse gluons, the pressure density from magnetized gluons can be estimated as

\[
p_g(T, \mu; eB) = a_0\mu^2 eB + b_0\mu^4 + c_0 T^2 eB + d_0\mu^2 T^2 + e_0 T^4,
\]

with \( a_0, b_0, c_0, d_0, e_0 \) being free parameters. At zero temperature,

\[
p_g(T = 0, \mu; eB) = a_0\mu^2 eB + b_0\mu^4.
\]

Furthermore, for the screening mass of gluons, considering the coefficients in front of \( eB \) is almost the same magnitude as that of \( \mu^2 \), and we neglect the anisotropy pressure density caused by the magnetic field with magnitude \( eB < 10^{19}G \). For simplicity, in this work, we use the following ansatz of the pressure density of magnetized gluons

\[
p_g(T = 0, \mu; eB) = a_0(\mu^2 eB + \mu^4).
\]

Under the strong magnetic field, a massive resonance \( g\sqrt{eB} \) is excited in the longitudinal \((1+1)\) component of the gluon propagator under strong magnetic fields \[102\]. The corresponding Debye mass of the longitudinal gluon fields \( A_\mu \) has the screening mass

\[
m_g^2(eB) = \sum_f |q_f|^2 \frac{g_{eff}^2}{4\pi^2} |eB|,
\]

and the perturbation theory at one-loop gives the pressure density of gluons as

\[
p_g^{PT} = p_g^{SB} - N_g \frac{g_{eff}^2}{32} \left[ \frac{5}{9} T^4 + \frac{2}{\pi^2} \mu^2 T^2 + \frac{1}{\pi^4} \mu^4 \right],
\]

the gluon pressure become chemical potential dependent via quark loop, though the coefficient in front of \( \mu^4 \) is only 1/45 of that of \( T^4 \).

D. Total pressure of SQM with \( \beta \)-equilibrium under magnetic field

Under strong magnetic fields, the \( O(3) \) rotational symmetry in SQM is broken and the pressure for SQM might become anisotropic, i.e., the longitudinal pressure \( P_{||} \) which is parallel to the magnetic field orientation is different from the transverse pressure \( P_{\perp} \) which is perpendicular to the orientation of magnetic field. The analytic
forms for longitudinal and transverse pressure densities of the system are given by [74]

\[ p_\| = p - \frac{1}{2}B^2, \quad (41) \]

\[ p_\perp = p + \frac{1}{2}B^2 - MB, \quad (42) \]

where we have defined

\[ p = p_u + p_d + p_s - p_0, \quad (43) \]

with \( p_0 = -\Omega_0 = -\Omega(T = 0, \mu = 0, B = 0) \) the vacuum pressure density, which ensures \( p = 0 \) in the vacuum. \( M \) is the system magnetization, and takes the form of

\[ M = -\frac{\partial \Omega}{\partial B} = \sum_{i=u,d,s,l} M_i, \quad (44) \]

The energy density for SQM is given by

\[ \epsilon = -p + \sum_{i=u,d,s,l} \mu_i n_i + \frac{1}{2}B^2. \quad (45) \]

One can find that the longitudinal pressure density \( p_\| \) satisfies the Hugenholtz-Van Hove (HVH) theorem for the transverse pressure density \( p_\perp \) does not because of the extra contributions from the magnetic field. We can see that the magnetic energy density term \( B^2/2 \) contributes oppositely to the longitudinal and transverse pressures under magnetic fields, which will lead to a tremendous pressure anisotropy when the magnetic field is very strong.

III. NUMERICAL RESULTS AND CONCLUSIONS

For our numerical calculations, following [3], the set of parameters we used is: \( \Lambda = 631.4 \text{MeV}, \quad m_u = m_d = 5.5 \text{MeV}, \quad m_s = 135.7 \text{MeV}, \quad G\Lambda^2 = 1.835 \) and \( K\Lambda^5 = 9.29. \)

A. The effect of vector-isovector interaction and vector-isoscalar interaction under magnetic field

Firstly, we analyze the effect of vector-isovector and vector-isoscalar interaction on the equation of state of strange quark matter Eq. [12] under zero magnetic field \( B = 0 \). For zero magnetic field, there is no anisotropy in the system, the longitudinal pressure density is equal to the transverse pressure density, i.e., \( p_\| = p_\perp = p(B = 0) \).

In Fig. 1, we show the pressure density of SQM as a function of energy density for three cases: 1) \( G_V = G_{IV} = 0 \), 2) \( G_V = 0.8G_S, G_{IV} = 0 \), and 3) \( G_V = G_{IV} = 0.8G_S \). Comparing the cases of 1) \( G_V = G_{IV} = 0 \) and 2) \( G_V = 0.8G_S, G_{IV} = 0 \), one can find that the repulsive vector-isoscalar interaction \( G_V \) gives a stiffer equation of state. However, comparing the cases of 2) \( G_V = 0.8G_S, G_{IV} = 0 \) and 3) \( G_V = G_{IV} = 0.8G_S \), it is observed that the equation of states for these two cases are almost the same, which indicates that the vector-isovector interaction does not affect the equation of state too much in SQM. One can find from Eq. (12) that the contribution from the vector-isovector interaction is mainly dependent on the u-d quark isospin asymmetry \( (n_u - n_d) \) and the coupling constant \( G_{IV} \). Since the isospin asymmetry in SQM is small, the vector-isovector interaction is very tiny with the parameter set \( G_V = G_{IV} = 0.8G_S \).

It can be understood like that, the repulsive vector-isoscalar interaction shifts the chemical potential to a larger value, which makes the equation of state stiffer. However, the interaction in the vector-isovector channel shifts the isospin chemical potential, and this effect is negligible for the equation of state under \( \beta \)-equilibrium. Therefore, in the following numerical calculations, we simply take \( G_{IV} = 0 \).

Fig. 2 shows the chemical potentials for u, d and s quarks as functions of the magnetic field \( B \) with \( G_V = 0 \) and \( G_V = 0.8G_S \) at fixed baryon number density \( n_b = 10n_0 \) in SQM. The chemical potential for each flavor with \( G_V = 0.8G_S \) is enhanced magnificently comparing the case of \( G_V = 0 \), which implies that a stiffer EoS can be generated once considering large vector-isoscalar coupling constant. One can also observe that, for both cases with \( G_V = 0 \) and \( G_V = 0.8G_S \), the chemical potential for quarks keeps a constant below the magnitude of \( 10^{19} \text{G} \), and decreases with the constant magnetic field above that.

Fig. 3 shows the constituent mass of u quark and s quark as functions of baryon density in charge neutral SQM for \( B = 0 \) and \( B = 2 \times 10^{19} \text{G} \) with \( G_V = 0 \) and \( G_V = 0.8G_S \), respectively. In the case of zero magnetic
field $B = 0$ and $G_V = 0$, the constituent quark mass for $u$ quark decreases from the vacuum mass almost linearly to $50\text{MeV}$ in the region of baryon number density below $n_b \simeq 0.35\text{fm}^{-3} \simeq 2n_0$, and then slowly decreases with baryon number density. The constituent quark mass for $s$ quark also drops almost linearly from its vacuum mass to around $475\text{MeV}$ in the region of baryon number density below $n_b \simeq 0.35\text{fm}^{-3} \simeq 2n_0$, and almost keeps as a constant in the region of $0.35 < n_b < 0.7\text{fm}^{-3}$ ($2 < n_b/n_0 < 4$), and then decreases quickly with baryon number density. This behavior is similar to Fig. 3.9 in Ref.\textsuperscript{32}. When the magnetic field is turned on, it is found that under the magnitude of $B = 2 \times 10^{19}$G, only a tiny magnetic catalysis effect can be observed for $u$ quark and $s$ quark in the low baryon density region, and at high baryon density region, the magnetic field with magnitude of $B = 2 \times 10^{19}$G almost has no effect on constituent quark mass at fixed baryon density. However, it is noticed that the repulsive vector-isoscalar interaction can smooth away the saturation region and make the strange quark mass decreases linearly with the baryon number density.

As shown in Fig. 3 the magnetic field in the magnitude of $B = 2 \times 10^{19}$G does not essentially affect the constituent quark mass. In Fig. 4 we investigate the vacuum constituent mass of $u$, $d$ and $s$ quarks as functions of magnetic field in SQM, and one can find that the masses of three different flavors of quarks do not change so much when the magnetic field is smaller than $10^{19}$G, while the $u$, $d$ quark masses increase drastically when the magnetic field is bigger than $3 \times 10^{19}$G, which indicates the magnetic catalysis phenomenon. It should also be noticed that the masses of $d$ quark and $s$ quark increase more slowly with the magnetic field compared to the $u$ quark mass case.

### B. The equation of state, sound velocity and magnetar mass

As it is accepted, the magnetic field strength in the inner core region of compact stars could be much larger than the magnetic field at the surface, then a density-dependent magnetic field distribution inside the compact star is usually used to describe this behavior. We use the following popular parametrization for the density-dependent magnetic field profile in QSs as Refs.\textsuperscript{77, 113–114}.

$$B = B_{\text{surf}} + B_0\left[1 - \exp\left(-\beta_0(n_b/n_0)^\gamma\right)\right],$$  \hspace{1cm} (46)$$

where $B_{\text{surf}}$ is the magnetic field strength at the surface of compact stars and its value is fixed at $B_{\text{surf}} = 10^{15}$G in this work, $n_0 = 0.16\text{fm}^{-3}$ is the normal nuclear matter density, $B_0$ is the constant magnetic field, which is a parameter with dimension of $B$, and $\beta_0$ and $\gamma$ are two
dimensionless parameters that control how exactly the magnetic field strength decays from the center to the surface. In order to reproduce the magnetic field which is weak below the nuclear saturation point while getting stronger at higher density, we take $B_0 = 4 \times 10^{18} G$, $\beta = 0.003$ and $\gamma = 3$ as the set of parameters in the following calculations, and this magnetic field distribution is already proved to be a gentle magnetic field distribution for SQM inside QSs, which can lead to a small pressure anisotropy and small maximum mass splitting for QSs in a density-dependent quark model [116].

The anisotropic pressure densities in Eq. (11) and Eq. (12) are calculated in cases of $B_0 = 0$ and $B_0 = 4 \times 10^{18} G$ with $a_0 = -0.01, 0, 0.01$ and $G_V = 0$, $G_V = 0.4G_S$, $G_V = 0.8G_S$ and $G_V = 1.1G_S$, respectively. We calculate the transverse pressure density as a function of energy density for SQM in Fig. 5 while the longitudinal pressure case in Fig. 6.

One can see from Fig. 5 that: 1) With fixed $G_V$ and fixed $a_0$, the transverse pressure density for $B_0 = 4 \times 10^{18} G$ is higher than that for $B_0 = 0$; 2) With fixed $G_V$, the positive magnetized gluon pressure density ($a_0 = 0.01$) and the case $B_0 = 0$ gives the hardest equation of state, while the negative magnetized gluon pressure ($a_0 = -0.01$) and the case $B_0 = 0$ always gives the softest equation of state; 3) For the case of negative magnetized gluon pressure ($a_0 = -0.01$), it is found that the equations of state for both $B_0 = 0$ and $B_0 = 4 \times 10^{18} G$ are not sensitive to the value of $G_V$. However, for the case of positive magnetized gluon pressure ($a_0 = 0.01$), the equations of state for both $B_0 = 0$ and $B_0 = 4 \times 10^{18} G$ are very sensitive to the value of $G_V$. Compared to Fig. 5, we can find that the pressure anisotropy for longitudinal and transverse pressure is small when $G_V$ is as big as $G_V = 0.8G_S, G_V = 1.1G_S$ for $a_0 = 0.01$, while the pressure anisotropy gets larger as the decrement of $G_V$ for $a_0 = -0.01$. Compared to the result from Fig. 5, we find that the magnetized gluon pressure contribution is more important to stiffen the EoS for SQM than the contribution from magnetic field, and by using this contribution from magnetized gluon one can describe a heavy QS (about $2M_\odot$) with small pressure anisotropy (like $G_V = 0.8G_S, G_V = 1.1G_S$ for $a_0 = 0.01$ cases) under a reasonable magnetic field distribution inside QSs.

It was pointed out in the work [117] that to construct a hybrid star with mass heavier than $2M_\odot$, a large sound velocity square for quark matter, say larger than 1/3 is
preferred. It is known that the sound velocity square for ideal gas or for strongly coupled conformal theory can be 1/3. However, for strongly interacting liquid, the sound velocity square is normally smaller than 1/3. Therefore, it is interesting to investigate the sound velocity in our current model, and the results of sound velocity

\[ c_s^2 = \frac{dp}{d\epsilon} \tag{47} \]

directly derived from equations of state in Fig. 7 and Fig. 8. Because the main purpose of this work is to explore the properties of SQM under strong magnetic field by considering the magnetized gluon contribution, we choose \( B_0 = 4 \times 10^{18} G \) with \( a_0 = 0.0, 0.01, -0.01 \) and \( B_0 = 0 \) with \( a = 0 \) by considering the vector-isoscalar interaction as \( G_V = 0 \), \( G_V = 0.8G_S \) and \( G_V = 1.1G_S \) in the following parts.

The sound velocity square of dense quark matter for transverse pressure density in Fig. 7 in the NJL model without vector interaction (\( G_V = 0 \)) and at \( B_0 = 0 \) is always less than 1/3, which is in agreement with common sense. It is also understood that when the system is more strongly coupled, the sound velocity square is getting smaller, and reach the smallest value at the phase transition point, where the system is regarded as most strongly coupled \[118].

However, it is observed that under magnetic field with magnitude of \( B_0 = 4 \times 10^{18} G \), even at \( G_V = 0 \), the sound velocity square for transverse pressure density case can increase to 0.6 at most. Also in the case of \( B_0 = 0 \), if one switches on a repulsive interaction in the vector-isoscalar channel, the sound velocity square for transverse pressure density case also increases and can become bigger than 1/3. The stronger the \( G_V \) is, the larger sound velocity square will be.

Another factor to increase the sound velocity is from the magnetized gluons. With a positive contribution from magnetized gluon pressure, and also taken into account the repulsive interaction in the vector-isoscalar channel, the sound velocity for transverse pressure case can be even larger than 1 (for \( G_V = 1.1G_S \), \( B_0 = 4 \times 10^{18} G \) and \( a_0 = 0.01 \) case), i.e., larger than the speed of light, which is of course not physical. So we can use the condition \( c_s^2 < 1 \) to constrain the EoS.

For the sound velocity square corresponding to longitudinal pressure from Fig. 8 we can find that the sound velocity square under \( B_0 = 4 \times 10^{18} G \) with different \( G_V \) and \( a_0 \) are all smaller than those \( c_s^2 \) corresponding to the transverse pressure case, which is self-consisted with Fig. 7 and Fig. 8. One can also find that all the sound velocity square corresponding to longitudinal pressure are smaller than 1.

Since we have calculated that the pressure anisotropy for SQM with the contribution from magnetized gluon pressure under the magnetic field distribution inside the QSs within \( B_0 = 4 \times 10^{18} G \), \( \beta_0 = 0.003 \) and \( \gamma = 3 \) is not big, we can approximately use the EoS from longitudinal or transverse pressure case to construct the magnetar under density dependent magnetic field. We should firstly introduce the TOV equations \[119] which can give the quark star with isotropic pressure:

\[ \frac{dM}{dr} = 4\pi r^2 \epsilon(r), \tag{48} \]

\[ \frac{dp}{dr} = \frac{-Ge(r)M(r)}{r^2} \left[1 + \frac{p(r)}{\epsilon(r)}\right] \frac{1}{[1 + 4\pi p(r)r^3]^{3/2}} \frac{[2GM(r)]^{1/2}}{r}. \tag{49} \]
We give the maximum mass of quark star in Figure 9 by using transverse pressure and longitudinal pressure respectively, and we use $B_0 = 4 \times 10^{18} G$ with $a_0 = 0.01, -0.01$ and $G_V = 0$, $G_V = 0.4G_S$, $G_V = 0.8G_S$ and $G_V = 1.1G_S$ for transverse pressure case while $B_0 = 4 \times 10^{18} G$, $a_0 = 0.01$ with $G_V = 0.8G_S$ and $G_V = 1.1G_S$ for longitudinal pressure case. We can read the following information from Figure 9: 1) At $B_0 = 0$ and $G_V = 0$, the three-flavor NJL model gives the maximum mass of quark star is about $1.4M_\odot$; 2) At $B_0 = 0$ but increases the repulsive interaction $G_V$ in the vector-isoscalar channel, the the maximum mass of quark star can reach $1.75M_\odot$ for $G_V = 1.1G_S$; 3) In the case of $G_V = 0$, when put quark matter under the magnetic field, the maximum mass of quark magnetar for transverse pressure case can be as heavy as $1.65M_\odot$, and if one takes into account a positive pressure density contributed from magnetized quasi-gluons, the mass of quark magnetar can reach $1.8M_\odot$; 4) For the magnitude of $B_0 = 4 \times 10^{18} G$, the sound velocity square for transverse pressure case reaches almost 0.9 for $G_V = 0.8G_S$ and $a_0 = 0.01$, and the magnetar mass is $2.17M_\odot$, while the maximum mass of quark star is $2.01M_\odot$ for longitudinal pressure within $G_V = 0.8G_S$ and $a_0 = 0.01$ under $B_0 = 4 \times 10^{18} G$, which is consistent with the recently discovered large mass pulsar J0348+0432 ($2.01 \pm 0.04)M_\odot$.

In order to investigate the difference for the maximum mass of QSs by using longitudinal pressure and transverse pressure, we define a mass difference parameter

$$\delta_m = \frac{M_\perp - M_{||}}{(M_\perp + M_{||})/2},$$

(50)

where $M_\perp$ ($M_{||}$) represents the maximum mass of QSs with transverse (longitudinal) orientation pressure, respectively. We calculate the mass difference for the transverse and longitudinal pressure within $G_V = 0.8G_S$ and $a_0 = 0.01$ under $B_0 = 4 \times 10^{18} G$ is $\delta_m = 7.65\%$, which implies that the pressure anisotropy in this case is very small due to the tiny mass asymmetry, and this is the reason why we can use the isotropic TOV equation to calculate the properties of QSs approximately. We can also find the similar results from $G_V = 0.8G_S$, $a_0 = 0.01$ and $B_0 = 4 \times 10^{18} G$ case that the maximum mass of QS for transverse pressure case is $2.30M_\odot$ while $2.20M_\odot$ for longitudinal pressure case, and the mass difference for this case is $\delta_m = 4.44\%$, which implies a smaller pressure anisotropy than $G_V = 0.8G_S$ case. Therefore, our results indicate that we can get stiffer EoS by considering the contributions from density-dependent magnetic field, the repulsive interaction in the vector-isoscalar channel and magnetized gluon pressure. Since the pressure anisotropy from density-dependent magnetic field is not big, we can calculate the properties of QSs under magnetic field by using isotropic TOV equation approximately, and we find the mass difference of magnetars by using transverse and longitudinal pressure is also very small.

In Fig. 10 we show the parameter region for $a_0$ and $G_V$ to produce 2 solar mass QSs under $B_0 = 4 \times 10^{18} G$ and by using longitudinal pressure and transverse pressure, respectively. The parameter sets of $a_0$ and $G_V$ on this two lines can describe 2 solar mass QSs. One can see that in order to produce 2 solar mass QSs, if the repulsive vector interaction is stronger, the needed contribution from magnetized gluons is smaller, or vice versa. It is also observed that larger $a_0$ and $G_V$ parameters are needed to produce 2 solar mass QSs if one use the longitudinal pressure density. However, when the magnitude of repulsive vector interaction increases, it is noticed that the difference between using longitudinal pressure and
transverse pressure to produce 2 solar mass QSs becomes smaller, because the pressure anisotropy decreases when large contribution from the repulsive interaction in the vector-isoscalar channel is considered, which can be read from Fig. 5 and Fig. 6.

IV. CONCLUSION AND DISCUSSION

In this work, we construct quark magnetars in the framework of SU(3) NJL model with vector interaction under strong magnetic field.

We investigate the effect of vector-isoscalar and vector-isovector interaction on the equation of state, and it is found that the equation of state is not sensitive to the vector-isovector interaction, however, a repulsive interaction in the vector-isoscalar channel gives a stiffer equation of state for cold dense quark matter. The result is reasonable because the interaction in the vector-isovector channel shifts the isospin chemical potential, and this effect is negligible on the equation of state under $\beta$-equilibrium, while the repulsive vector-isoscalar interaction shifts the chemical potential to a larger value, which makes the equation of state stiffer.

In the presence of magnetic field, the pressure of the system is shown to be anisotrophic along and perpendicular to the magnetic field direction with the former being generally larger than the latter. Gluons will be magnetized via quark loops, and we also estimate the pressure contributed from magnetized gluons. Normally the NJL model only considers the contribution from quark degrees of freedom on the pressure, which is always underestimated. We estimate the pressure density contributed from magnetized gluons, which should be positive in order to compensate the pressure density of quasi-quarks described by the NJL model. It is found that magnetized quarks and gluons also give a stiffer equation of state.

The sound velocity square is one of fundamental properties of hot/dense matter, which measures the hardness or softness of the equation of state. It is known that hot and dense quark matter in the NJL model without vector interaction at zero magnetic field is always less than $1/3$. It is also understood that when the system is more strongly coupled, the sound velocity square is getting smaller, and reach the smallest value at the phase transition point, where the system is regarded as most strongly coupled. However, it is found that the sound velocity square can be larger than $1/3$ when a strong repulsive interaction is introduced, and the sound velocity square can change a lot when strong magnetic field is added. Furthermore, the sound velocity square corresponding to the transverse pressure density case can even reach 1 when the magnetized gluon contribution is taken into account.

Since the pressure anisotropy from density-dependent magnetic field is small, we can calculate the properties of QSs under magnetic field by using isotropic TOV equation approximately, and we find the magnetar mass difference by using transverse and longitudinal pressure cases are very small and the maximum mass of quark star can be enhanced by the contribution from the density-dependent magnetic field, the repulsive interaction in the vector-isovector channel and the magnetized gluon pressure. We also give the parameter region for $a_0$ and $G_V$ which can describe the 2 solar mass quark star by using longitudinal pressure and transverse pressure respectively.

Acknowledgement. — We thank valuable discussions with M. Alford and A. Schmitt on sound velocity of quark matter, with J. Schaffner-Bielich and A. Sedrakian on magnetars, and with A. Vuorinen on pressure from gluon degrees of freedom. M. Huang thanks the hospitality of Frankfurt University, Tours University and TU Vienna, where the final stage of this work is performed. This work is supported by the NSFC under Grant Nos. 11275213, 11275125, 11135011 and 11261130311 (CRC 110 by DFG and NSFC), also supported by CAS key project KJCX2-EW-N01, and Youth Innovation Promotion Association of CAS, the China-France collaboration Project “Cal Yuanpei 2013”, the Shanghai Rising-Star Program under grant No. 11QH1401100, the “Shu Guang” project supported by Shanghai Municipal Education Commission, the National Basic Research Program of China (973 Program) under Contract No. 2015CB856904, and Shanghai Education Development Foundation, the Program for Professor of Special Appointment (Eastern Scholar) at Shanghai Institutions of Higher Learning, and the Science and Technology Commission of Shanghai Municipality (11DZ2260700).

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