A FAMILY OF PARTIALLY ORDERED SETS WITH SMALL BALANCE CONSTANT

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Abstract. Given a finite poset \( P \) and two distinct elements \( x \) and \( y \), we let \( pr_P(x \prec y) \) denote the fraction of linear extensions of \( P \) in which \( x \) precedes \( y \). The balance constant \( \delta(P) \) of \( P \) is then defined by

\[
\delta(P) = \max_{x \neq y \in P} \min \{ pr_P(x \prec y), pr_P(y \prec x) \}.
\]

The 1/3-2/3 conjecture asserts that \( \delta(P) \geq \frac{1}{3} \) whenever \( P \) is not a chain, but except from certain trivial examples it is not known when equality occurs, or even if balance constants can approach 1/3.

In this paper we make some progress on the conjecture by exhibiting a sequence of posets with balance constants approaching \( \frac{1}{3} \left( \frac{93 - \sqrt{6697}}{2} \right) \approx 0.3488999 \), answering a question of Brightwell. These provide smaller balance constants than any other known nontrivial family.

1. Introduction

1.1. Definitions. Given a finite poset (partially ordered set) \( P = (P, \leq) \), and distinct elements \( x, y \in P \), we let \( pr_P(x \prec y) \) denote the proportion of linear extensions of \( P \) in which \( x \) precedes \( y \). In particular, \( pr_P(x \prec y) + pr_P(y \prec x) = 1 \), and if \( x \leq y \) in \( P \) then \( pr_P(x \prec y) = 1 \).

The balance constant \( \delta(P) \) is then defined by

\[
\delta(P) = \max_{x \neq y \in P} \min \{ pr_P(x \prec y), pr_P(y \prec x) \}.
\]

(If \( P \) consists of one element, we let \( \delta(P) = 0 \).) Thus \( \delta(P) \in [0, \frac{1}{2}] \) for any finite poset \( P \); in fact \( \delta(P) = 0 \) exactly when \( P \) is a chain.

1.2. The 1/3-2/3 Conjecture. The main conjecture about balance constants is the famous 1/3-2/3 conjecture.

Conjecture 1.1 (1/3-2/3 conjecture). If \( P \) is a finite poset which is not a chain, then \( \delta(P) \geq \frac{1}{3} \).
This conjecture was first proposed in 1968 by Kislitsyn [5], then again by Fredman in 1976 [3] and Linial [6]. All three were motivated by the information-theoretic context of comparison sorting, but the problem is of course interesting in its own right.

The $1/3$-$2/3$ conjecture has been studied extensively. The best bound which has been shown for all posets is due to Brightwell, Felsner, and Trotter [2] in 1995, who showed that

$$\delta(P) \geq \frac{5 - \sqrt{5}}{10} \approx 0.276393$$

whenever $P$ is not a chain. This improved a result of Kahn and Saks [4] in 1984 which showed the weaker estimate $\delta(P) \geq \frac{2}{7} \approx 0.272727$.

While still open for general partially ordered sets, the conjecture has been proven for several other families of partially ordered sets, for example posets of width 2 by Linial [6] and posets of height 2 by Trotter, Gehrlein, Fishburn [10]. In 2006, Peczarski described an even stronger conjecture, the so-called “gold partition conjecture”, which implies the $1/3$-$2/3$ conjecture; Peczarski proved this conjecture for posets with at most 11 elements [8], and later for 6-thin posets [9].

An extensive survey on the problem is given by Brightwell [1], which describes it as “one of the major open problems in the combinatorial theory of partial orders”.

1.3. Posets with small balance constant. The following example shows that the constant $1/3$ in best possible.

**Example 1.2.** Consider the poset $T$ with three elements $\{a, b, c\}$ with the single relation $a \leq b$ (shown in Figure 1). Then $\delta(T) = \frac{1}{3}$. It follows that linear sums of $T$ and the singleton poset have balance constant $1/3$.

![Figure 1. The poset $T$ with $\delta(T) = 1/3$.](image)

However, other than this example, little is known about the possible sets of balance constants. For example, it is not known whether there are any other posets which achieve a balance constant of exactly $1/3$, other than those in the example above. It is not even known whether balance constants can be arbitrarily close to $1/3$.

In Brightwell’s survey [1, Section 4], an example of partially ordered set with $A$ with $\delta(A) = \frac{16}{45} \approx 0.355556$ is given. Brightwell also gives a family of partially ordered sets with balance constant approaching $\frac{7 - \sqrt{17}}{8} \approx 0.359612$, and asks the following two questions.
Question 1.3. Is there a poset with balance constant between $\delta(T) = \frac{1}{3}$ and $\delta(A) = \frac{16}{45}$?

Question 1.4. Is $\frac{7 - \sqrt{17}}{8} \approx 0.359612$ the lowest possible limit point other than $1/3$?

Olson and Sagan [7] resolve the first question by finding a poset $C$ with

$$\delta(C) = \frac{37}{106} \approx 0.34905660$$

which to the author’s knowledge is the smallest balance constant exceeding $1/3$ which appears in the literature. This poset is shown in Figure 2.

Figure 2. The poset $C$ from [7, Figure 13], with $\delta(C) = 37/106$.

The aim of this paper is to answer both questions with a certain infinite family of partially ordered sets. We will prove the following theorem.

Theorem 1.5. There exists a sequence of posets whose balance constants approach

$$\kappa = \frac{1}{32} \left( 93 - \sqrt{6697} \right) \approx 0.34889999.$$

1.4. Roadmap. The rest of the paper is divided as follows. In Section 2 we introduce the main players in our proof, and introduce the notation which we will need for the construction. Section 3 then provides explicit formulas for the number of linear extensions of our family of posets, and finally in Section 4 we compile these results together to prove the main theorem.

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2. Setup

**Definition 2.1.** Throughout the paper let \( \kappa = \frac{1}{32} (93 - \sqrt{6697}) \).

We first define a “master poset” from which our construction will derive.

**Definition 2.2.** Let \( P_\infty \) denote the partially ordered set whose elements consist of two infinite \( \mathbb{N} \)-indexed chains
\[
 a_1 < a_2 < a_3 < \cdots \\
 b_1 < b_2 < b_3 < \cdots
\]
together with the additional covering relations that
\[
 \bullet \quad a_i \leq b_{i+1} \text{ whenever } i \equiv 1, 2, 3, 4 \pmod{5}, \text{ and} \\
 \bullet \quad b_j \leq a_{j+2} \text{ whenever } j \equiv 0, 2, 4 \pmod{5}.
\]

All our constructions will be obtained by taking the bottom-most elements of either chain.

**Definition 2.3.** For positive integers \( m \) and \( n \) we let \( P(m,n) \) denote the sub-poset \( P_\infty \) induced by taking the elements \( \{a_1, \ldots, a_m, b_1, \ldots, b_n\} \).

The example \( P(15,15) \) is shown in Figure 3.

Our main result is the following.

**Theorem 2.4.** As \( k \to \infty \),
\[
 \delta(P(5k,5k)) \to \kappa.
\]

To approach this result, we introduce further notation.

**Definition 2.5.** Let \( E(m,n) \) denote the number of linear extensions of \( P(m,n) \). For convenience we let \( E(0,n) = E(m,0) = 1 \) for positive integers \( m \) and \( n \), but we leave \( E(0,0) \) undefined.

Then \( E(m,n) \) may be computed recursively in the following way.

**Proposition 2.6.** For positive integers \( m \) and \( n \), we have
\[
 E(m,n) = \begin{cases} 
 E(m-1,n) & a_m > b_n \\
 E(m,n-1) & a_m < b_n \\
 E(m-1,n) + E(m,n-1) & \text{otherwise}
\end{cases}
\]

Proof. In a linear extension of \( P(m,n) \), either \( a_m \) or \( b_n \) must be the maximal element, and so the recursion follows by considering cases on this. \( \Box \)

According to Proposition 2.6, the interesting cases are those for which \( P(m,n) \) has no maximal element. To this end, we introduce the following terminology.

**Definition 2.7.** We say the pair \( (m,n) \) of positive integers is admissible if \( P(m,n) \) has no maximal element.

One can in fact characterize all the admissible pairs exactly. We obtain, essentially by definition, the following characterization.
Lemma 2.8. The pair \((m,n)\) is admissible if and only if it is one of the following forms:

1. \(m = 5k + 4\) and \(n \in \{5k + 3, 5k + 4\}\).
2. \(m = 5k + 3\) and \(n \in \{5k + 1, 5k + 2, 5k + 3\}\).
3. \(m = 5k + 2\) and \(n \in \{5k + 1, 5k + 2\}\).
4. \(m = 5k + 1\) and \(n \in \{5k, 5k + 1\}\).
5. \(m = 5k\) and \(n \in \{5k - 2, 5k - 1, 5k, 5k + 1\}\).

Remark 2.9. Note that this means that \((m,n)\) is admissible if and only if \(|m - n| \leq 5\), in which case only the residues \(m \pmod{5}\), \(n \pmod{5}\) are relevant. In particular, if \((m,n)\) is admissible then so is \((m+5,n+5)\).
3. Enumeration

We now proceed to give explicitly compute $E(m, n)$ using induction. Several base cases are needed for this proof; we do not address these here, but simply record the results in Appendix A.

In order to make this possible, we make the following observation.

**Lemma 3.1.** If $(m, n)$ is admissible then

$$E(m + 10, n + 10) = 164E(m + 5, n + 5) - 27E(m, n).$$

**Proof.** One may verify this manually for $m, n \leq 15$. Once this is done the result follows by induction on $m + n$ owing to Proposition 2.6. □

This implies that the values of $E(m, n)$ satisfy a linear recurrence. Thus it makes sense to introduce the roots of the corresponding characteristic polynomial.

**Definition 3.2.** Throughout this paper, let

$$\theta = 82 + \sqrt{6697} \approx 163.8352$$

$$\overline{\theta} = 82 - \sqrt{6697} \approx 0.1648$$

be the two roots of the polynomial $t^2 - 164t + 27$.

Then, a direct computation using the results of Appendix A allows us to compute explicit closed forms:

**Proposition 3.3.** We have the following twelve closed forms.

1. $$E(5k + 4, 5k + 4) = \frac{3025}{2\sqrt{6697}} \left( \theta^k - \overline{\theta}^k \right) + \frac{37}{2} \left( \theta^k + \overline{\theta}^k \right)$$
2. $$E(5k + 4, 5k + 3) = \frac{1883}{2\sqrt{6697}} \left( \theta^k - \overline{\theta}^k \right) + \frac{23}{2} \left( \theta^k + \overline{\theta}^k \right)$$
3. $$E(5k + 3, 5k + 3) = \frac{571}{\sqrt{6697}} \left( \theta^k - \overline{\theta}^k \right) + 7 \left( \theta^k + \overline{\theta}^k \right)$$
4. $$E(5k + 3, 5k + 2) = \frac{741}{2\sqrt{6697}} \left( \theta^k - \overline{\theta}^k \right) + \frac{9}{2} \left( \theta^k + \overline{\theta}^k \right)$$
5. $$E(5k + 3, 5k + 1) = \frac{170}{\sqrt{6697}} \left( \theta^k - \overline{\theta}^k \right) + 2 \left( \theta^k + \overline{\theta}^k \right)$$
6. $$E(5k + 2, 5k + 2) = \frac{401}{2\sqrt{6697}} \left( \theta^k - \overline{\theta}^k \right) + \frac{5}{2} \left( \theta^k + \overline{\theta}^k \right)$$
7. $$E(5k + 2, 5k + 1) = \frac{247}{2\sqrt{6697}} \left( \theta^k - \overline{\theta}^k \right) + \frac{3}{2} \left( \theta^k + \overline{\theta}^k \right)$$
8. $$E(5k + 1, 5k + 1) = \frac{77}{\sqrt{6697}} \left( \theta^k - \overline{\theta}^k \right) + \left( \theta^k + \overline{\theta}^k \right)$$
9. $$E(5k + 1, 5k) = \frac{93}{2\sqrt{6697}} \left( \theta^k - \overline{\theta}^k \right) + \frac{1}{2} \left( \theta^k + \overline{\theta}^k \right)$$
A FAMILY OF POSETS WITH SMALL BALANCE CONSTANT

1. Case $b_{5t-1} \prec a_{5t+1} \prec b_{5t}$. Then if we add the relation $b_{5t-1} \leq a_{5t+1} \leq b_{5t}$ to $\mathcal{P}$, the resulting poset is isomorphic to the linear sum of $\mathcal{P}(5t, 5t - 1)$ and an inverted copy of $\mathcal{P}(5s + 1, 5s - 1)$. An example with $(k, t) = (3, 1)$ is shown in Figure [4].
The number of linear extensions in this case is then

\[ E(5t, 5t - 1)E(5s + 1, 5s - 1) = E(5t, 5t - 1)E(5s, 5s - 1) \]

\[ = \left[ \frac{1}{6} \left( \frac{125}{\sqrt{6697}} + 1 \right) + o(1) \right]^2 \theta^k \]

\[ = \left[ \frac{1161 + 125\sqrt{6697}}{12 \cdot 6697} + o(1) \right] \theta^k \]

\[ \approx (0.17745 + o(1)) \theta^k. \]

**4.2. Case** \( b_{5t} < a_{5t+1} < b_{5t+1} \). Then if we add the relation \( b_{5t} \leq a_{5t+1} \leq b_{5t+1} \) to \( \mathcal{P} \), the resulting poset is isomorphic to the linear sum of \( \mathcal{P}(5t, 5t) \) and an inverted copy of \( \mathcal{P}(5s, 5s - 1) \).
A FAMILY OF POSETS WITH SMALL BALANCE CONSTANT

The number of linear extensions in this case is then \( E(5t, 5t)E(5s, 5s - 1) \), which equals

\[
E(5t, 5t)E(5s, 5s - 1) = \left[ \frac{1}{3} \left( \frac{77}{\sqrt{6997}} + 1 \right) + o(1) \right] \left[ \frac{1}{6} \left( \frac{125}{\sqrt{6997}} + 1 \right) + o(1) \right] \theta^k
\]

\[
= \left[ \frac{8161 + 101\sqrt{6997}}{9 \cdot 6697} + o(1) \right] \theta^k
\]

\[
\approx (0.27253 + o(1))\theta^k.
\]

4.3. Case \( b_{5t+1} < a_{5t+1} < b_{5t+2} \). Then if we add the relation \( b_{5t} \leq a_{5t+1} \leq b_{5t+1} \) to \( \mathcal{P} \), the resulting poset is isomorphic to the linear sum of \( \mathcal{P}(5t, 5t+1) \) and an inverted copy of \( \mathcal{P}(5s - 1, 5s - 1) \).

Thus the number of linear extensions in this case is equal to

\[
E(5t, 5t+1)E(5s - 1, 5s - 1) = \left[ \frac{1}{2} \left( \frac{61}{\sqrt{6997}} + 1 \right) + o(1) \right] \left[ \frac{1}{2} \left( \frac{3025}{\sqrt{6997}} + 37 \right) + o(1) \right] \theta^{k-1}
\]

\[
= \left[ \frac{1411 + 15\sqrt{6697}}{2 \cdot 6697} + o(1) \right] \theta^k
\]

\[
\approx [0.19699 + o(1)] \theta^k.
\]

4.4. Collating the cases. On the other hand, the total number of linear extension of \( \mathcal{P} \) is

\[
E(5t, 5t) = \left[ \frac{1}{3} \left( \frac{77}{\sqrt{6697}} + 1 \right) + o(1) \right] \theta^k
\]

\[
\approx (0.64697 + o(1))\theta^k.
\]

So, division gives

\[
\operatorname{pr}_{\mathcal{P}} (b_{5t} < a_{5t+1} < b_{5t}) = \frac{1}{6} \left( \frac{29}{\sqrt{6697}} + 2 \right) + o(1)
\]

\[
\approx 0.27427 + o(1)
\]

\[
\operatorname{pr}_{\mathcal{P}} (b_{5t} < a_{5t+1} < b_{5t+1}) = \frac{1}{6} \left( \frac{125}{\sqrt{6697}} + 1 \right) + o(1)
\]

\[
\approx 0.42124 + o(1)
\]

\[
\operatorname{pr}_{\mathcal{P}} (b_{5t+1} < a_{5t+1} < b_{5t+1}) = \frac{1}{2} \left( \frac{-32}{\sqrt{6697}} + 1 \right) + o(1)
\]

\[
\approx 0.30449 + o(1).
\]

It follows that

\[
\min (\operatorname{pr}_{\mathcal{P}}(a_{5t+1} < b_j), \operatorname{pr}_{\mathcal{P}}(b_j < a_{5t+1})) < \frac{1}{3} < \kappa
\]
for \( j \in \{5t, 5t + 1\} \), assuming \( t > 0 \). Hence it holds for all \( j \), since for \( j \notin \{5t, 5t + 1\} \) the left-hand side vanishes.

This completes the proof of Proposition 4.1 when \( i \equiv 1 \pmod{5} \); the other four cases are analogous. \(\square\)

**Appendix A. Examples of values**

The following table lists the values of \( E(m, n) \) for \( \max(m, n) \leq 15 \) (except for \( E(0, 0) \) undefined). The pairs \((m, n)\) which are admissible are bolded.

| \( n = \) | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|-----------|---|---|---|---|---|---|---|---|---|---|----|
| \( m = 0 \) | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| \( m = 1 \) | 1 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 |
| \( m = 2 \) | 1 | 3 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 |
| \( m = 3 \) | 1 | 4 | 9 | 14 | 14 | 14 | 14 | 14 | 14 | 14 | 14 |
| \( m = 4 \) | 1 | 4 | 9 | 23 | 37 | 37 | 37 | 37 | 37 | 37 | 37 |
| \( m = 5 \) | 1 | 4 | 9 | 32 | 69 | 106 | 143 | 143 | 143 | 143 | 143 |
| \( m = 6 \) | 1 | 4 | 9 | 32 | 69 | 175 | 318 | 318 | 318 | 318 | 318 |
| \( m = 7 \) | 1 | 4 | 9 | 32 | 69 | 175 | 493 | 811 | 811 | 811 | 811 |
| \( m = 8 \) | 1 | 4 | 9 | 32 | 69 | 175 | 668 | 1479 | 2290 | 2290 | 2290 |
| \( m = 9 \) | 1 | 4 | 9 | 32 | 69 | 175 | 668 | 1479 | 3769 | 6059 | 6059 |
| \( m = 10 \) | 1 | 4 | 9 | 32 | 69 | 175 | 668 | 1479 | 5248 | 11307 | 17366 |
| \( m = 11 \) | 1 | 4 | 9 | 32 | 69 | 175 | 668 | 1479 | 5248 | 11307 | 28673 |
| \( m = 12 \) | 1 | 4 | 9 | 32 | 69 | 175 | 668 | 1479 | 5248 | 11307 | 28673 |
| \( m = 13 \) | 1 | 4 | 9 | 32 | 69 | 175 | 668 | 1479 | 5248 | 11307 | 28673 |
| \( m = 14 \) | 1 | 4 | 9 | 32 | 69 | 175 | 668 | 1479 | 5248 | 11307 | 28673 |
| \( m = 15 \) | 1 | 4 | 9 | 32 | 69 | 175 | 668 | 1479 | 5248 | 11307 | 28673 |

| \( n = \) | 11 | 12 | 13 | 14 | 15 |
|-----------|----|----|----|----|----|
| \( m = 0 \) | 1 | 1 | 1 | 1 | 1 |
| \( m = 1 \) | 2 | 2 | 2 | 2 | 2 |
| \( m = 2 \) | 5 | 5 | 5 | 5 | 5 |
| \( m = 3 \) | 14 | 14 | 14 | 14 | 14 |
| \( m = 4 \) | 37 | 37 | 37 | 37 | 37 |
| \( m = 5 \) | 328 | 365 | 402 | 439 | 476 |
| \( m = 6 \) | 318 | 318 | 318 | 318 | 318 |
| \( m = 7 \) | 811 | 811 | 811 | 811 | 811 |
| \( m = 8 \) | 2290 | 2290 | 2290 | 2290 | 2290 |
| \( m = 9 \) | 6059 | 6059 | 6059 | 6059 | 6059 |
| \( m = 10 \) | 23425 | 29484 | 35543 | 41602 | 47661 |
| \( m = 11 \) | 52098 | 52098 | 52098 | 52098 | 52098 |
| \( m = 12 \) | 80771 | 132869 | 132869 | 132869 | 132869 |
| \( m = 13 \) | 109444 | 242313 | 375182 | 375182 | 375182 |
| \( m = 14 \) | 138117 | 242313 | 617495 | 992677 | 992677 |
| \( m = 15 \) | 166790 | 242313 | 859808 | 1852485 | 2845162 |
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