Solution of the Monoenergetic Neutron Transport Equation in a Half Space

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The analytical solution of neutron transport equation has fascinated mathematicians and physicists alike since the Milne half-space problem was introduce in 1921 [1]. Numerous numerical solutions exist, but understandably, there are only a few analytical solutions, with the prominent one being the singular eigenfunction expansion (SEE) introduced by Case [2] in 1960. For the half-space, the method, though yielding, an elegant analytical form resulting from half-range completeness, requires numerical evaluation of complicated integrals. In addition, one finds closed form analytical expressions only for the infinite medium and half-space cases. One can find the flux in a slab only iteratively. That is to say, in general one must expend a considerable numerical effort to get highly precise benchmarks from SEE. As a result, investigators have devised alternative methods, such as the CN [3], FN [4] and Greens Function Method (GFM) [5] based on the SEE have been devised. These methods take the SEE at their core and construct a numerical method around the analytical form. The FN method in particular has been most successful in generating highly precise benchmarks. No method yielding a precise numerical solution has yet been based solely on a fundamental discretization until now. Here, we show for the albedo problem with a source on the vacuum boundary of a homogeneous medium, a precise numerical solution is possible via Lagrange interpolation over a discrete set of directions.

Since this is an initial progress report of a new solution, we will consider only the simplest case in the half-space. In particular, the source will be isotropic and the medium isotropically scattering.

1. General solution to 1D transport equation in a half-space with source

We begin with the 1D transport equation for general anisotropic scattering

\[
\left[ \mu \frac{\partial}{\partial x} + 1 \right] \psi(x, \mu) = \frac{c}{2} \int_{-1}^{1} d\mu' f(\mu, \mu') \psi(x, \mu') + S(x, \mu),
\]  

(1)
with a general volume source and subject to an incoming source at the free surface
\( \psi(0, \mu), 0 \leq \mu \leq 1 \). One can represent the solution by a combination of the
solutions to the homogeneous form

\[
\left[ \mu \frac{\partial}{\partial x} + 1 \right] \psi_h(x, \mu) = \frac{c}{2} \int_{-1}^{1} d \mu' f(\mu, \mu') \psi_h(x, \mu')
\]

(2a)

and any particular solution including the source

\[
\left[ \mu \frac{\partial}{\partial x} + 1 \right] \psi_p(x, \mu) = \frac{c}{2} \int_{-1}^{1} d \mu' f(\mu, \mu') \psi_p(x, \mu') + S(x, \mu)
\]

(2b)

as

\[
\psi(x, \mu) = \psi_h(x, \mu) + \psi_p(x, \mu).
\]

(2c)

The boundary condition then applies to the combination.

The particular solution is therefore

\[
\psi_p(x, \mu) = \int_{-\infty}^{x} dx' \int_{-1}^{1} d \mu' G(x - x', \mu, \mu') S(x', \mu'),
\]

(3a)

where the Green’s function [2] is for \( x > x' \)

\[
G_+ (x - x', \mu, \mu') = \phi_+ (\mu') \phi_+ (\mu) \frac{e^{-|x-x'|\nu_0}}{N_{0+}} + \int_{0}^{1} dv' \frac{e^{-|x-x'|\nu'}}{N_{\nu'}} \phi_{v'} (\mu') \phi_v (\mu)
\]

(3b)

and for \( x < x' \)

\[
G_- (x - x', \mu, \mu') = -\phi_- (\mu') \phi_- (\mu) \frac{e^{-|x-x'|\nu_0}}{N_{0-}} + \int_{0}^{1} dv' \frac{e^{-|x-x'|\nu'}}{N_{-\nu'}} \phi_{v'} (\mu') \phi_{-v'} (\mu).
\]

(3c)
Starting from the SEE and with considerable algebra, we now derive an expression for the general solution noting

\[ N_{0\pm} \equiv \frac{c}{2} v_0^3 \left[ \frac{c}{v_0^2 - 1} - \frac{1}{v_0^2} \right] \]

\[ N_v \equiv v \left[ \left( 1 - cv \tanh^{-1} v \right)^2 + \left( c\pi v / 2 \right)^2 \right] \]

in the equations above.

The solution to the homogeneous equation by SEE is

\[ \psi_h(x, \mu) = a_{0+} \phi_{0+}(\mu)e^{-x/v_0} + a_{0-} \phi_{0-}(\mu)e^{x/v_0} + \int_{-1}^{1} dv' e^{-x/v'} \phi_{v'}(\mu) A(v') \] (4)

giving the general solution from Eqs(3) and (4)

\[ \psi(x, \mu) = a_{0+} \phi_{0+}(\mu)e^{-x/v_0} + a_{0-} \phi_{0-}(\mu)e^{x/v_0} + \int_{-1}^{1} dv' e^{-x/v'} \phi_{v'}(\mu) A(v') + \int_{-\infty}^{1} dx' \int_{-1}^{1} d\mu' \left[ \phi_{0+}(\mu') \phi_{0+}(\mu) \frac{e^{-|x-x'|v_0}}{N_{0+}} + \int_{0}^{1} d\nu' e^{-|x-x'|\nu'} S(x', \mu') \right] + \int_{x}^{\infty} dx' \int_{-1}^{1} d\mu' \left[ -\phi_{0-}(\mu') \phi_{0-}(\mu) \frac{e^{-|x-x'|v_0}}{N_{0-}} + \int_{0}^{1} d\nu' e^{-|x-x'|\nu'} \phi_{-\nu'}(\mu') \phi_{-\nu'}(\mu) \right] S(x', \mu') \]

(5)

From orthogonality at \( x = 0 \)

\[ \int_{-1}^{1} d\mu \mu \phi_{0+}(\mu) \psi(0, \mu) = a_{0+} N_{0+} + \int_{-\infty}^{1} dx' \int_{-1}^{1} d\mu' \phi_{0+}(\mu') e^{-|x-x'|v_0} S(x', \mu') \] (6a)

\[ \int_{-1}^{1} d\mu \mu \phi_{0-}(\mu) \psi(0, \mu) = a_{0-} N_{0-} - \int_{0}^{1} dx' \int_{-1}^{1} d\mu' \phi_{0-}(\mu') e^{-|x-x'|v_0} S(x', \mu') \] (6b)

\[ \int_{-1}^{1} d\mu \mu \phi_{\nu}(\mu) \psi(0, \mu) = N_{\nu} A(\nu) + \int_{-\infty}^{1} dx' \int_{-1}^{1} d\mu' e^{-|x-x'|\nu} \phi_{\nu}(\mu') S(x', \mu') \] (6c)

\[ \int_{-1}^{1} d\mu \mu \phi_{-\nu}(\mu) \psi(0, \mu) = N_{-\nu} A(-\nu) + \int_{0}^{1} dx' \int_{-1}^{1} d\mu' e^{-|x-x'|\nu} \phi_{-\nu}(\mu') S(x', \mu') \] (6d)
Then solving for the expansion coefficients gives

\[
a_{0+} = \frac{1}{N_{0+}} \left\{ \int_{-1}^{1} d\mu \mu\phi_{0+}(\mu)\psi(0,\mu) - \int_{-\infty}^{0} dx' \int_{-1}^{1} d\mu' \phi_{0+}(\mu')e^{-|x'|\nu_0}S(x',\mu') \right\} (7a)
\]

\[
a_{0-} = \frac{1}{N_{0-}} \left\{ \int_{-1}^{1} d\mu \mu\phi_{0-}(\mu)\psi(0,\mu) + \int_{0}^{\infty} dx' \int_{-1}^{1} d\mu' \phi_{0-}(\mu')e^{-|x'|\nu_0}S(x',\mu') \right\} (7b)
\]

\[
A(\nu) = \frac{1}{N_{\nu}} \left\{ \int_{-1}^{1} d\mu \mu\phi_{\nu}(\mu)\psi(0,\mu) - \int_{-\infty}^{0} dx' \int_{-1}^{1} d\mu' e^{-|x'|\nu}\phi_{\nu}(\mu')S(x',\mu') \right\} (7c)
\]

\[
A(-\nu) = \frac{1}{N_{-\nu}} \left\{ \int_{-1}^{1} d\mu \mu\phi_{-\nu}(\mu)\psi(0,\mu) - \int_{0}^{\infty} dx' \int_{-1}^{1} d\mu' e^{-|x'|\nu}\phi_{-\nu}(\mu')S(x',\mu') \right\}. (7d)
\]

On substitution into Eq(5)

\[
\psi(x,\mu) = \frac{1}{N_{0+}} \left\{ \int_{-1}^{1} d\mu \mu' \phi_{0+}(\mu')\psi(0,\mu') - \int_{-\infty}^{0} dx' \int_{-1}^{1} d\mu' \phi_{0+}(\mu')e^{x'/\nu_0}S(x',\mu') \right\} \phi_{0+}(\mu)e^{-x/\nu_0} +
\]

\[
+ \frac{1}{N_{0-}} \left\{ \int_{-1}^{1} d\mu \mu' \phi_{0-}(\mu')\psi(0,\mu') + \int_{0}^{\infty} dx' \int_{-1}^{1} d\mu' \phi_{0-}(\mu')e^{-x'/\nu_0}S(x',\mu') \right\} \phi_{0-}(\mu)e^{x/\nu_0} +
\]

\[
+ \int_{-\infty}^{\infty} dx' \int_{-1}^{1} d\mu' \phi_{0+}(\mu')\phi_{0-}(\mu)\frac{e^{-(x-x')/\nu_0}}{N_{0+}}S(x',\mu') +
\]

\[
- \int_{x}^{\infty} dx' \int_{-1}^{1} d\mu' \phi_{0-}(\mu')\phi_{0+}(\mu)\frac{e^{-(x-x')/\nu_0}}{N_{0-}}S(x',\mu') +
\]

\[
+ \int_{0}^{\infty} dx' \int_{-1}^{1} d\mu' e^{x'/\nu}\phi_{\nu}(\mu')S(x',\mu') \left\{ \int_{-1}^{1} d\mu \mu' \phi_{\nu}(\mu')\psi(0,\mu') - \int_{-\infty}^{0} dx' \int_{-1}^{1} d\mu' e^{x'/\nu}\phi_{\nu}(\mu')S(x',\mu') \right\} e^{-x'/\nu}\phi_{\nu}(\mu) +
\]

\[
+ \int_{0}^{\infty} dx' \int_{-1}^{1} d\mu' e^{-x'/\nu}\phi_{-\nu}(\mu')S(x',\mu') \left\{ \int_{-1}^{1} d\mu \mu' \phi_{-\nu}(\mu')\psi(0,\mu') - \int_{0}^{\infty} dx' \int_{-1}^{1} d\mu' e^{-x'/\nu}\phi_{-\nu}(\mu')S(x',\mu') \right\} e^{x'/\nu}\phi_{-\nu}(\mu) +
\]

\[
+ \int_{-\infty}^{\infty} dx' \int_{-1}^{1} d\mu' \int_{0}^{\infty} dx' e^{-(x-x')/\nu'}\phi_{\nu'}(\mu')\phi_{\nu}(\mu)S(x',\mu') +
\]

\[
+ \int_{x}^{\infty} dx' \int_{-1}^{1} d\mu' \int_{0}^{\infty} dx' e^{-(x-x')/\nu'}\phi_{-\nu'}(\mu')\phi_{-\nu}(\mu)S(x',\mu')
\]

(8)
and on combination of terms, we finally find

\[
\psi(x, \mu) = \frac{1}{N_{0+}} \left\{ \int_{-1}^{1} d\mu' \left\{ \mu' \psi(0, \mu') + \int_{0}^{x} dx' \psi(x', \mu') \right\} \phi_{0+}(\mu') \right\} \phi_{0+}(\mu) e^{-x/v_0} + \\
+ \frac{1}{N_{0-}} \left\{ \int_{-1}^{1} d\mu' \left\{ \mu' \psi(0, \mu') + \int_{0}^{x} dx' \psi(x', \mu') \right\} \phi_{0-}(\mu') \right\} \phi_{0-}(\mu) e^{x/v_0} + \\
+ \int_{-1}^{1} d\nu' \frac{1}{N_{\nu'}} \left\{ \int_{-1}^{1} d\mu' \left\{ \mu' \psi(0, \mu') + \int_{0}^{x} dx' \psi(x', \mu') \right\} \phi_{\nu'}(\mu') \right\} e^{-x/v'_{\nu'}} \phi_{\nu'}(\mu)
\]

(9)
to give on re-arrangement

\[
\psi(x, \mu) = \alpha_{0+} \phi_{0+}(\mu) e^{-x/v_0} + \alpha_{0-} \phi_{0-}(\mu) e^{x/v_0} + \int_{-1}^{1} d\nu' e^{-x/v'_{\nu'}} \phi_{\nu'}(\mu) A(\nu') + \psi_p(x, \mu)
\]

(10a)

\[
\psi_p(x, \mu) \equiv \frac{1}{N_{0+}} \int_{-1}^{1} d\mu' \int_{0}^{x} dx' \psi(x', \mu') \phi_{0+}(\mu') \phi_{0+}(\mu) e^{-x/v_0} + \\
+ \frac{1}{N_{0-}} \int_{-1}^{1} d\mu' \int_{0}^{x} dx' \psi(x', \mu') \phi_{0-}(\mu') \phi_{0-}(\mu) e^{x/v_0} + \\
+ \int_{-1}^{1} d\nu' e^{-x/v'_{\nu'}} \left[ \int_{-1}^{1} d\mu \int_{0}^{x} dx' \phi_{\nu'}(\mu') \right] \phi_{\nu'}(\mu),
\]

(10b)

where

\[
\alpha_{0\pm} \equiv \frac{1}{N_{0\pm}} \int_{-1}^{1} d\mu \mu \phi_{0\pm}(\mu) \psi(0, \mu)
\]

(10c)

\[
A(\nu) \equiv \frac{1}{N_{\nu}} \int_{-1}^{1} d\mu \mu \phi_{\nu}(\mu) \psi(0, \mu).
\]

(10d)

For a half-space, the flux must also be finite at infinity, so
\[ \lim_{x \to \infty} \psi(x, \mu) < \infty, \quad (11) \]

which from Eqs(10) says

\[ \alpha_{0-} \equiv 0 \]

\[ A(\nu) \equiv 0; \quad -1 \leq \nu \leq 0 \quad (12a,b,c) \]

\[ \lim_{x \to \infty} \psi_P(x, \mu) < \infty. \]

This seems to indicate from the first term in Eq(10b)

\[
\lim_{x \to \infty} \left[ e^{-x/\nu_0} \int_0^x dx' e^{x'/\nu_0} S(x', \mu') \right] = \lim_{x \to \infty} \left[ \int_0^x dx' e^{x'/\nu_0} S(x', \mu') \right] e^{x/\nu_0} = v_0 \lim_{x \to \infty} S(x, \mu') < \infty
\]

and from the second term

\[
\lim_{x \to \infty} \left[ e^{x/\nu_0} \int_0^x dx' e^{-x'/\nu_0} S(x', \mu') \right] = \lim_{x \to \infty} \left[ \int_0^x dx' e^{-x'/\nu_0} S(x', \mu') \right] e^{-x/\nu_0} = v_0 \lim_{x \to \infty} S(x, \mu') < \infty
\]

and similarly from the third term for \(-1 \leq \nu \leq 0\).

2. **Lagrange interpolation**

Consider the solution to the following albedo problem

\[
\psi(x, \mu) = a_0 \phi_{0+} (\mu) e^{-x/\nu_0} + \int_0^1 dv e^{-x/\nu} \phi_v (\mu) A(\nu), \quad (14)
\]
where the incoming flux at the free surface is $\psi(0, \mu), \ 0 \leq \mu \leq 1$. Note that one constructs solution such that the solution vanishes at infinity. Consequently

$$\psi(0, -\mu) = a_{0+} \phi_{0+}(-\mu) + \int_{0}^{1} d\nu \phi_{+}(-\mu) A(\nu),$$  \hspace{1cm} (15a)$$

$$\psi(0, \mu) = a_{0+} \phi_{0+}(\mu) + \int_{0}^{1} d\nu \phi_{+}(\mu) A(\nu),$$  \hspace{1cm} (15b)$$

where

$$\phi_{0+}(\mu) = \frac{c v_{0}}{2} \frac{1}{v_{0} - \mu}$$  \hspace{1cm} (15c,d)$$

$$\phi_{+}(\mu) = \frac{c v}{2} P \frac{1}{v - \mu} + \lambda(\nu) \delta(\nu - \mu).$$

Thus

$$\lambda(\nu) A(\nu) + \frac{c}{2} \int_{-1}^{1} d\nu' \frac{\nu'}{\nu' - \nu} A(\nu') = \psi(0, \nu) - a_{0+} \frac{c v_{0}}{2} \frac{1}{v_{0} - \nu}.$$  \hspace{1cm} (16)$$

If

$$A(\nu) = A_{1}(\nu) - a_{0+} A_{2}(\nu),$$  \hspace{1cm} (17a)$$

then

$$\lambda(\nu) A_{1}(\nu) + \frac{c}{2} \int_{-1}^{1} d\nu' \frac{\nu'}{\nu' - \nu} A_{1}(\nu') = \psi(0, \nu)$$  \hspace{1cm} (17b,c)$$

$$\lambda(\nu) A_{2}(\nu) + \frac{c}{2} \int_{-1}^{1} d\nu' \frac{\nu'}{\nu' - \nu} A_{2}(\nu') = \frac{c v_{0}}{2} \frac{1}{v_{0} - \nu},$$

which gives
\[ \psi(0,-\mu) = a_{0+} \left[ \phi_{0+}(-\mu) - \int_0^1 dv \phi_v(-\mu) A_2(v) \right] + \int_0^1 d\nu \phi_v(-\mu) A_1(\nu) \] (18a)

with

\[ a_{0+} = \frac{1}{N_{0+}} \int_{-1}^{1} d\mu \mu \phi_{0+}(\mu) \psi(0,\mu) \]

\[ = \frac{1}{N_{0+}} \left[ \int_0^1 d\mu \mu \phi_{0+}(\mu) \psi(0,\mu) - \int_0^1 d\mu \mu \phi_{0+}(-\mu) \psi(0,-\mu) \right] \] (18b)

Introducing \( \psi(0,-\mu) \) from Eq(18a) gives

\[ a_{0+} = \frac{1}{N_{0+}} \left\{ \int_0^1 d\mu \mu \phi_{0+}(\mu) \psi(0,\mu) - \right\}

\[ \left\{ -a_{0+} \left[ J_{0+} - \int_0^1 dv J_v A_2(v) \right] - \int_0^1 d\nu J_v A_1(\nu) \right\} \] (19a)

with

\[ J_v = \int_0^1 d\mu \mu \phi_{0+}(-\mu) \phi_v(-\mu) \] (19b)

or

\[ J_v = \begin{cases} \left( \frac{cv}{2} \right) \phi_{0+}(v) \left[ v_0 \ln \left( \frac{v_0+1}{v_0} \right) - v \ln \left( \frac{v+1}{v} \right) \right], & 0 \leq v \leq 1 \\ \left( \frac{cv_0}{2} \right) \ln \left( \frac{v_0+1}{v_0} \right) - \frac{1}{v_0+1}, & v = 0+. \end{cases} \] (19c)

Solving for \( a_{0+} \) in Eq(19a)
\[ a_{0+} = \left\{ N_{0+} + \left[ J_{0+} - \int_0^1 d\nu J_\nu A_2 (\nu) \right] \right\}^{-1} \left\{ \int_0^1 d\mu \mu \phi_{0+} (\mu) \psi (0, \mu) - \int_0^1 d\nu J_\nu A_1 (\nu) \right\} \]  

(20)

then provides \( a_{0+} \) in terms of the solution to the singular integral equations of Eqs(17bc) to be numerically solved in the next section.

3. Numerical considerations
Consider Eqs(17bc) as

\[ \lambda (\nu) A_i (\nu) + \frac{c}{2} \int_{-1}^1 d\nu' \frac{\nu'}{\nu' - \nu} A_i (\nu') = f_i, \]  

(21a)

where

\[ f_i (\nu) = \begin{cases} \psi (0, \nu), & i = 1 \\ \phi_{0+} (\nu), & i = 2. \end{cases} \]  

(21b,c)

The desired solution is in terms of discrete values of the variable \( A_i (\nu) \). On discretization of Eqs(21), one must use caution because of the principal value integration. The principal value integral is most efficiently accommodated however through Lagrange interpolation

\[ A_i (\nu) = \sum_{j=1}^N l_j (\nu) A_j \]  

(22a)

with

\[ l_j (\nu) = \frac{P_N^* (\nu)}{(\nu - \nu_j) P_N^* (\nu_j)} \]  

(22b)

and \( \nu \) are the zeros of the polynomial \( P_N^* (\nu) = 0; \ \nu = \nu_j, j = 1, \ldots, N \). WE consider only half-range Legendre polynomials where \( P_N^* (\nu) \equiv P_N (2\nu - 1) \).
When introduced into Eq (21a), there results

\[ \lambda (v) A_i (v) + \sum_{j=1}^{N} I_j (v) A_{ij} = f_i (v) \]  

(23a)

with

\[ I_j (v) \equiv \frac{c}{2} \int_{0}^{1} dv' \frac{v'}{(v' - v)} I_j (v'). \]  

(23b)

or with Eq (22b)

\[ I_j (v) \equiv \frac{c}{2P_N^* (v_j)} \int_{0}^{1} dv' \left[ \frac{v'}{(v' - v)} \frac{P_N^* (v')}{(v' - v_j)} \right]. \]  

(23c)

The integral conveniently evaluates to

\[ I_j (v) = -\frac{c}{2P_N^* (v_j)} \begin{cases} 2, & \text{for } \nu = \nu_j, \\ 0, & \text{for other cases.} \end{cases} \]  

(24)

When evaluated at the zero \( \nu_m, m = 1, ..., N \), Eq (23a) becomes

\[ \sum_{j=1}^{N} \left[ \lambda (\nu_m) + I_m (\nu_m) \right] \delta_{jm} + \left( 1 - \delta_{jm} \right) I_j (\nu_m) A_{ij} = f_{im}, \quad m = 1, ..., N. \]  

(25)

Thus, the analytical form for \( a_{0+} \) is

\[ a_{0+} = \left\{ N_{0+} + [J_{0+} - K_1] \right\}^{-1} \left\{ \int_{0}^{1} d \mu \mu \phi_{0+} (\mu) \phi (0, \mu) - K_2 \right\}, \]  

(26a)
where

\[ K_i \equiv K_{i1} + K_{i2} \]  \hspace{1cm} (26b)

and

\[ K_{i1} \equiv \frac{cV_0}{2} \ln \left[ \frac{v_0 + 1}{v_0} \right] \int_0^1 d\nu \nu \phi_{0+} (\nu) A_i (\nu) \]

\[ K_{i2} \equiv \frac{cV_0}{2} \int_0^1 d\nu \nu^2 \phi_{0+} (\nu) \ln \left[ \frac{\nu + 1}{\nu} \right] A_i (\nu) \]  \hspace{1cm} (26c,d)

Seems that \( K_{i1} \) can be identified as

\[ K_{i1} = \left( \frac{cV_0}{2} \right)^2 \ln \left[ \frac{v_0 + 1}{v_0} \right] \sum_{j=1}^N A_{ij} \int_0^1 d\nu \frac{\nu}{v_0 - \nu} l_j (\nu) \]

to give

\[ K_{i1} = -\left( \frac{cV_0}{2} \right)^2 \ln \left[ \frac{v_0 + 1}{v_0} \right] \sum_{j=1}^N I_j (v_0) A_{ij} \]

\[ K_{i1} = 2 \left( \frac{cV_0}{2} \right)^2 \ln \left[ \frac{v_0 + 1}{v_0} \right] \sum_{j=1}^N \frac{A_{ij}}{P_N^* (v_j)} \frac{1}{v_j - v_0} \left[ v_j Q_N (2v_j - 1) + v_0 Q_N (2v_0 - 1) \right] \]. \hspace{1cm} (27)

\( K_{i2} \) has defied analytical efforts to evaluate and is perform via Gauss quadrature.

Once \( a_{0+} \) and \( A_{ij}, i = 1,2; j = 1,...,N \) are known, Eq(15a) gives the exiting flux as

\[ \psi (0,-\mu) = a_{0+} \left[ \phi_{0+} (-\mu) - \sum_{j=1}^N I_j (-\mu) A_{2j} \right] + \int_0^1 d\nu \sum_{j=1}^N I_j (-\mu) A_{ij} , \]  \hspace{1cm} (28)

with substitution of Eq(23b).
4. Numerical Results

We now consider a uniform source striking the free surface of a semi-infinite isotopically scattering homogeneous medium. We use iteration in \( N \) to find the exiting flux at 10 uniformly spaced directions to at least 7-places. The iteration advances in \( N \) with a stride of four until convergence. Wynn-epsilon (W-e) acceleration [6] enhances convergence. All digits shown in Table 1 are identical in comparison to the response matrix method [7]. Table 2 gives an addition benchmark where the precision is expected to be better than one unit in the last place.

| \( \mu c \) | \( 0.4 \) | \( 0.5 \) | \( 0.6 \) | \( 0.7 \) | \( 0.8 \) | \( 0.9 \) |
|---|---|---|---|---|---|---|
| 1.0000E+00 | 8.3357277E-02 | 1.1522588E-01 | 1.5541466E-01 | 2.0867995E-01 | 2.8525450E-01 | 4.1494748E-01 |
| 9.0000E-01 | 8.8463080E-02 | 1.2198994E-01 | 1.6408263E-01 | 2.1951910E-01 | 2.9852634E-01 | 4.3054104E-01 |
| 8.0000E-01 | 9.4223842E-02 | 1.2963262E-01 | 1.7381852E-01 | 2.3159952E-01 | 3.1315688E-01 | 4.4742308E-01 |
| 7.0000E-01 | 1.0084825E-01 | 1.3834805E-01 | 1.8484644E-01 | 2.4516406E-01 | 3.2938544E-01 | 4.6578138E-01 |
| 6.0000E-01 | 1.0853464E-01 | 1.4839744E-01 | 1.9746445E-01 | 2.6053111E-01 | 3.4752006E-01 | 4.8585154E-01 |
| 5.0000E-01 | 1.1758562E-01 | 1.6014443E-01 | 2.1208237E-01 | 2.7813176E-01 | 3.6796976E-01 | 5.0793891E-01 |
| 4.0000E-01 | 1.2844760E-01 | 1.7411918E-01 | 2.2928957E-01 | 2.9857597E-01 | 3.9130218E-01 | 5.3245844E-01 |
| 3.0000E-01 | 1.4182526E-01 | 1.9114797E-01 | 2.4999130E-01 | 3.2278524E-01 | 4.1835924E-01 | 5.6001640E-01 |
| 2.0000E-01 | 1.5895774E-01 | 2.1266387E-01 | 2.7573436E-01 | 3.5230943E-01 | 4.5053604E-01 | 5.9161280E-01 |
| 1.0000E-01 | 1.8252201E-01 | 2.4172077E-01 | 3.0977054E-01 | 3.9036735E-01 | 4.9070973E-01 | 6.2933582E-01 |
| 0.0000E+00 | 2.2540333E-01 | 2.9289321E-01 | 3.6754446E-01 | 4.5227744E-01 | 5.5278640E-01 | 6.8377223E-01 |

Table 1 Seven place benchmark for albedo problem for \( 0.4 \leq c \leq 0.9 \)

| \( \mu c \) | \( 0.99 \) | \( 0.999 \) | \( 0.9999 \) | \( 0.99999 \) |
|---|---|---|---|---|
| 1.0000E+00 | 7.5272072E-01 | 9.1284533E-01 | 9.7141778E-01 | 9.9085481E-01 |
| 9.0000E-01 | 7.6430578E-01 | 9.1772948E-01 | 9.7311399E-01 | 9.9140750E-01 |
| 8.0000E-01 | 7.7629964E-01 | 9.2268485E-01 | 9.7482251E-01 | 9.9196288E-01 |
| 7.0000E-01 | 7.8873595E-01 | 9.2771823E-01 | 9.7654524E-01 | 9.9252152E-01 |
| 6.0000E-01 | 8.0166441E-01 | 9.3283900E-01 | 9.7828488E-01 | 9.9308429E-01 |
| 5.0000E-01 | 8.1513990E-01 | 9.3806068E-01 | 9.8004549E-01 | 9.9365243E-01 |
| 4.0000E-01 | 8.2924965E-01 | 9.4304513E-01 | 9.8183049E-01 | 9.9422798E-01 |
| 3.0000E-01 | 8.4412871E-01 | 9.4890800E-01 | 9.8365999E-01 | 9.9481442E-01 |
| 2.0000E-01 | 8.6002328E-01 | 9.5463979E-01 | 9.8554678E-01 | 9.9541861E-01 |
| 1.0000E-01 | 8.7751247E-01 | 9.6077432E-01 | 9.8754820E-01 | 9.9605770E-01 |
| 0.0000E+00 | 9.0000000E-01 | 9.6837722E-01 | 9.9000000E-01 | 9.9683772E-01 |

Table 1 Seven place benchmark for albedo problem for \( 0.99 \leq c \leq 0.99999 \)

While the method performs well for \( c > 0.4 \) with tables 1 and 2 taking less than 1.25 Min on a Precision Dell PC, the same cannot be said about \( c < 0.4 \). The computational time increases rapidly as \( c \) approaches zero. An effort will be made to resolve this defect in the method.

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