A SELF-SIMILAR APERIODIC SET OF 19 WANG TILES

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Abstract. We define a Wang tile set $\mathcal{U}$ of cardinality 19 and show that the set $\Omega_{\mathcal{U}}$ of all valid Wang tilings $\mathbb{Z}^2 \to \mathcal{U}$ is self-similar, aperiodic and is a minimal subshift of $\mathcal{U}^{\mathbb{Z}^2}$. Thus $\mathcal{U}$ is the second smallest self-similar aperiodic Wang tile set known after Ammann’s set of 16 Wang tiles. The proof is based on the unique composition property. We prove the existence of an expansive, primitive and recognizable 2-dimensional morphism $\omega : \Omega_{\mathcal{U}} \to \Omega_{\mathcal{U}}$ that is onto up to a shift. The proof of recognizability is done in two steps using at each step the same criteria (the existence of marker tiles) for proving the existence of a recognizable one-dimensional substitution that sends each tile either on a single tile or on a domino of two tiles.

1. Introduction

Wang tiles are unit square tiles with colored edges as in Figure 1. Given a finite set of Wang tiles, we consider tilings of the Euclidean plane using arbitrarily many copies of the tiles. Tiles are placed on the integer lattice points of the plane with their edges oriented horizontally and vertically. The tiles may not be rotated. The tiling is valid if every contiguous edges have the same color. Given a finite set $\mathcal{T}$ of Wang tiles, we denote by $\Omega_{\mathcal{T}}$ the set of all valid tilings $f : \mathbb{Z}^2 \to \mathcal{T}$. The set $\Omega_{\mathcal{T}}$ is a 2-dimensional subshift as it is invariant under translations and closed under taking limits. Hence $\Omega_{\mathcal{T}}$ is called the Wang shift of $\mathcal{T}$. A nonempty Wang shift $\Omega_{\mathcal{T}}$ is aperiodic if none of the tilings in $\Omega_{\mathcal{T}}$ have a nontrivial period. Chapters 10 and 11 of [GS87] and the more recent book on aperiodic order [BG13] give an excellent overview of what is known on aperiodic tilings (in $\mathbb{Z}^d$ as in $\mathbb{R}^d$) together with their applications to physics and crystallography.

A nonexhaustive list of aperiodic Wang tile sets is shown in Table 1 Examples of small aperiodic Wang tile sets include Ammann’s 16 tiles [GS87, p. 595], Kari’s 14 tiles [Kar96] and Culik’s 13 tiles [Cul96]. The question of finding the smallest aperiodic set of Wang tiles was open until Jeandel and Rao proved [JR15] the existence of an aperiodic set of 11 Wang tiles and that no set of Wang tiles of cardinality $\leq 10$ is aperiodic. The proofs of their two results is based on transducers, which has proven to be an excellent approach for the search of the smallest aperiodic Wang tile set. A transducer is a finite-state machine with two memory tapes: an input tape and an output tape.

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There is a natural way to construct a transducer from a set of Wang tiles (see Figure 4) which computes some row of tiles that can be placed above some other row of tiles.

In this contribution we introduce a set $\mathcal{U}$ of 19 Wang tiles (see Figure 1). The main result of the current contribution is that $\mathcal{U}$ is aperiodic and that it generates tilings (Figure 3) of $\mathbb{Z}^2$ with a self-similar structure. With 19 tiles, this makes $\mathcal{U}$ the second smallest self-similar aperiodic set after Ammann’s tile set which has 16 only tiles. The tile set $\mathcal{U}$ comes from the study of the structure of Jeandel-Rao aperiodic tilings. The link with Jeandel-Rao tilings needs more tools and space and will be done in a forthcoming paper [Lab18b]. In this contribution, we prove the following result.

**Theorem 1.** The Wang shift $\Omega_\mathcal{U}$ is self-similar, aperiodic and minimal.

To prove aperiodicity, we use the *composition-decomposition method* also called the *unique composition property* for tilings of $\mathbb{R}^d$ described in [GS87, Thm 10.1.1], [BG13, Sec. 5.7.1] and [Sol97, Lemma 2.7]. Let us cite [AGS92] which summarizes informally the composition-decomposition method as two conditions to be satisfied for a tile set $\mathcal{T}$: if

(C1) in every tiling admitted by $\mathcal{T}$ there is a unique way in which the tiles can be grouped into patches which lead to a tiling by supertiles; and

(C2) the markings on the supertiles, inherited from the original tiles, imply a matching condition for the supertiles which is exactly equivalent to that originally specified for the tiles,

then $\mathcal{T}$ is aperiodic. Condition (C1) was formalized with many vocabularies in the literature including the term *unambiguous* in [Oll08]. In this contribution (see Proposition 6), we use the notion of *recognizability*. This method was used to prove aperiodicity of the first discovered aperiodic examples (see Table 1). The proof of aperiodicity of 104 Berger’s tile set is obtained by the use of two-by-two substitutions of the form $\square \mapsto \square$, mapping each tile to a two-by-two square of 4 tiles. The same holds for Knuth’s and Robinson’s tile sets, their constructions being obtained as a simplification of Berger’s tile set (see also [Oll08]). Note that the composition-decomposition method does not apply to all aperiodic tile set, e.g. aperiodicity of Kari and Culik tile sets [Kar96, Cul96] follows from arithmetic properties.

| Author        | Size | Year | Proof of aperiodicity | s.s. | $\lambda$ | References       |
|--------------|------|------|------------------------|------|----------|------------------|
| Berger       | 104  | 1964 | unique composition property | yes | 2        | [Ber65, Oll08]   |
| Knuth        | 92   | 1968 |                         | yes | 2        | [Knu69, 2.3.4.3] |
| Robinson     | 56   | 1971 |                         | yes | 2        | [Rob71]         |
| Grünbaum et al. | 24  | 1987 |                         | yes | $\frac{1+\sqrt{5}}{2}$ | [GS87]         |
| Ammann       | 16   | 1971 |                         | yes | $\frac{1+\sqrt{5}}{2}$ | [GS87, p. 595] |
| Kari         | 14   | 1996 | arithmetic properties   | no   | –        | [Kar96]         |
| Culik        | 13   | 1996 |                         | no   | –        | [Cul96]         |
| Jeandel, Rao | 11   | 2015 | transducers and Fibonacci word | no   | –        | [JR15]          |

**Table 1.** A list of small aperiodic sets of Wang tiles. Whether the tile set is self-similar (s.s.) is indicated. When applicable, $\lambda$ denotes the expansion factor of the similarity involved in the composition (see Section 7 on stone inflations). Ammann’s tile set is the smallest known self-similar aperiodic Wang tile set. The history includes many unpublished results and a more complete story can be found in [JR15, Sec. 1.2].

As it is the case for Ammann set of 16 Wang tiles [GS87, p. 595], the substitutive structure of tilings in $\Omega_\mathcal{U}$ have the form $\square \mapsto \square$, $\square \mapsto \square$, $\square \mapsto \square$, $\square \mapsto \square$, that is, the image of a tile is a tile, a vertical or horizontal domino of two tiles, or a two-by-two square of 4 tiles. More precisely, in this contribution, we show that any tiling of $\mathbb{Z}^2$ by the 19 tiles in $\mathcal{U}$ can be decomposed uniquely into a...
or of the form □ → □, □ → □ or of the form □ → □, □ → □. Each step has the effect of identifying special rows (or columns) made of marker tiles (see Definition 9) that have to be adjacent in one direction but can not be adjacent in the

Figure 2. Any tiling in $\Omega_4$ can be decomposed uniquely into a tiling by these 19 supertiles. Each of these supertiles is equivalent to one of the tiles in $\mathcal{U}$ which explains the self-similar structure of tilings in $\Omega_4$. It is reminiscent of Figure 11.1.16 in [GS87].

...tilings by the supertiles shown in Figure 2. It can be seen by comparing Figure 1 and Figure 2 that the markings on the supertiles imply a matching condition which is equivalent to those specified for $\mathcal{U}$ (horizontal color KO in the supertiles plays the role of color $P$ in the tile set $\mathcal{U}$, etc.), thus satisfying (C1) and (C2).

Figure 3. A $13 \times 8$ rectangle tiled by tiles from $\mathcal{U}$. It corresponds to $\omega^5(u_4)$ where $\omega = \alpha \beta \gamma$ is defined in Section 6. Notice that the sequence of colors at the top correspond to the sequence at the bottom allowing a periodic tiling of an infinite vertical strip of width 13. Dashed lines identify the decomposition into supertiles. Solid lines identify the decomposition into supertiles of the next level.

The particularity and the importance of this contribution stands in breaking down the proof of unique composition property into smaller steps. Thus, each step consists in proving recognizability and surjectivity up to a shift for a substitution of the form □ → □, □ → □ or of the form □ → □, □ → □. Each step has the effect of identifying special rows (or columns) made of marker...
other direction. The wider applicability of this method (Theorem 10) allowing to be automated
by computer check will be used in Lab18b.

Following the theory of $S$-adic systems on $\mathbb{Z}$ [BD14] and hierarchical tilings of $\mathbb{R}^d$ [FS14], we
desubstitute tilings in $\Omega_\mathcal{U}$ into tilings in $\Omega_\mathcal{V}$ by a 2-dimensional morphism $\alpha : \Omega_\mathcal{V} \to \Omega_\mathcal{U}$ for some
Wang tile set $\mathcal{V}$ and we desubstitute tilings of $\Omega_\mathcal{V}$ into tilings in $\Omega_\mathcal{W}$ by a 2-dimensional morphism $\beta : \Omega_\mathcal{W} \to \Omega_\mathcal{V}$ for some Wang tile set $\mathcal{W}$. Finally we prove that $\alpha$ and $\beta$ are recognizable thus
satisfying (C1) and that there is a letter to letter bijection $\gamma : \Omega_\mathcal{U} \to \Omega_\mathcal{W}$ that satisfies (C2)
\[\Omega_\mathcal{U} \xrightarrow{\alpha : \square \mapsto \square \square \mapsto \square \circ \square} \Omega_\mathcal{V} \xrightarrow{\beta : \circ \square \mapsto \square \circ \square \mapsto \square \circ \square} \Omega_\mathcal{W} \xleftarrow{\gamma : \square \mapsto \square \square \mapsto \square \circ \square} \Omega_\mathcal{U}\]

A key result in this contribution is Theorem 10 since it is used twice to prove that we can desubstitute
uniquely. It can be seen as a 2-dimensional generalization to Wang subshifts of the notion of
derived sequences introduced in Dur98 for 1-dimensional substitutive sequences. Remark that an-
other generalization of derived sequences to tilings of $\mathbb{R}^2$ based on Voronoï tessellations is presented in Fra03. The advantage of markers and Theorem 10 is that the rectangular lattice structure of
Wang tilings is preserved under each derivation.

The paper is structured as follows. In Section 2, we present the necessary definitions and
notations on Wang tiles including self-similarity, recognizability and aperiodicity. In Section 3, we
present a sufficient condition for the existence of a recognizable 2-dimensional morphism allowing to
desubstitute any tiling of a Wang shift. In Section 4 and Section 5, we prove the existence of tile
notations on Wang tiles including self-similarity, recognizability and aperiodicity. In Section 3, we
prove self-similarity, aperiodicity and minimality of $\mathcal{U}$. The advantage of markers and Theorem 10 is that the rectangular lattice structure of
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2. Preliminaries

In this section, we introduce subshifts, shifts of finite type, Wang tiles, fusion of Wang tiles,
the transducer representation of Wang tiles, $d$-dimensional words, morphisms and languages. We
recall the notions of self-similarity and recognizability for proving aperiodicity.

We denote by $\mathbb{Z} = \{\ldots, -1, 0, 1, 2, \ldots\}$ the integers and by $\mathbb{N} = \{0, 1, 2, \ldots\}$ the nonnegative
integers. If $d \geq 1$ is an integer and $1 \leq k \leq d$, we denote by $e_k = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{Z}^d$ the
vector of the canonical basis of $\mathbb{Z}^d$ with a 1 as position $k$ and 0 elsewhere.

2.1. Subshifts and shifts of finite type. We follow the notations of Sch01. Let $\mathcal{A}$ be a finite
set, $d \geq 1$, and let $\mathcal{A}^{\mathbb{Z}^d}$ be the set of all maps $x : \mathbb{Z}^d \to \mathcal{A}$, furnished with the compact product
topology. We write a typical point $x \in \mathcal{A}^{\mathbb{Z}^d}$ as $x = (x_m) = (x_m : m \in \mathbb{Z}^d)$, where $x_m \in \mathcal{A}$ denotes the value of $x$ at $m$. The shift action $\sigma : \mathbb{N} \mapsto \mathcal{A}^{\mathbb{Z}^d}$ on $\mathcal{A}^{\mathbb{Z}^d}$ is defined by
\[
(\sigma^n(x))_m = x_{m+n}
\]
for every $x = (x_m) \in \mathcal{A}^{\mathbb{Z}^d}$ and $n \in \mathbb{Z}^d$. A subset $X \subset \mathcal{A}^{\mathbb{Z}^d}$ is shift-invariant if $\sigma^n(X) = X$ for
every $n \in \mathbb{Z}^d$, and a closed, shift-invariant subset $X \subset \mathcal{A}^{\mathbb{Z}^d}$ is a subshift. If $X \subset \mathcal{A}^{\mathbb{Z}^d}$ is a subshift
we write $\sigma = \sigma^X$ for the restriction of the shift-action (1) to $X$. If $X \subset \mathcal{A}^{\mathbb{Z}^d}$ is a subshift it
will sometimes be helpful to specify the shift-action of $\mathbb{Z}^d$ explicitly and to write $(X, \sigma)$ instead of $X$. A subshift $(X, \sigma)$ is called minimal if $X$ does not contain any nonempty, proper, closed
shift-invariant subset.

For any subset $S \subset \mathbb{Z}^d$ we denote by $\pi_S : \mathcal{A}^{\mathbb{Z}^d} \to \mathcal{A}^S$ the projection map which restricts every
$x \in \mathcal{A}^{\mathbb{Z}^d}$ to $S$. A pattern is a function $p : S \to \mathcal{A}$ for some finite subset $S \subset \mathbb{Z}^d$. A subshift
$X \subset \mathcal{A}^{Z^d}$ is a shift of finite type (SFT) if there exists a finite set $\mathcal{F}$ of forbidden patterns such that
\begin{equation}
X = \{ x \in \mathcal{A}^{Z^d} \mid \pi_S \cdot \sigma^n(x) \notin \mathcal{F} \text{ for every } n \in Z^d \text{ and } S \subset Z^d \}.
\end{equation}
In this case, we write $X = \text{SFT} (\mathcal{F})$.

### 2.2. Wang tiles.

A Wang tile $\tau = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is a unit square with colored edges formally represented as a tuple of four colors $(a, b, c, d) \in I \times J \times I \times J$ where $I$, $J$ are two finite sets (the vertical and horizontal colors respectively). For each Wang tile $\tau = (a, b, c, d)$, we denote by $\text{RIGHT}(\tau) = a$, $\text{TOP}(\tau) = b$, $\text{LEFT}(\tau) = c$, $\text{BOTTOM}(\tau) = d$ the colors of the right, top, left and bottom edges of $\tau$ [Wan61, Rob71].

Let $\mathcal{T}$ be a set of Wang tiles. A tiling of $Z^2$ by $\mathcal{T}$ is an assignation of $x$ to tiles of each position of $Z^2$ so that contiguous edges have the same color, that is, it is a function $x : Z^2 \rightarrow \mathcal{T}$ satisfying
\begin{align}
\text{RIGHT} \circ x(n) &= \text{LEFT} \circ x(n + e_1) \\
\text{TOP} \circ x(n) &= \text{BOTTOM} \circ x(n + e_2)
\end{align}
for every $n \in Z^2$. We denote by $\Omega_\mathcal{T} \subset \mathcal{T}^{Z^2}$ the set of all Wang tilings of $Z^2$ by $\mathcal{T}$ and we call it the Wang shift of $\mathcal{T}$. It is a SFT of the form [2].

A set of Wang tiles $\mathcal{T}$ tiles the plane if $\Omega_\mathcal{T} \neq \emptyset$ and does not tile the plane if $\Omega_\mathcal{T} = \emptyset$. A tiling $x \in \Omega_\mathcal{T}$ is periodic if there is a nonzero period $n \in Z^2 \setminus \{(0,0)\}$ such that $x = \sigma^n(x)$ and otherwise it is said nonperiodic. A set of Wang tiles $\mathcal{T}$ is periodic if there is a tiling $x \in \Omega_\mathcal{T}$ which is periodic. A Wang tile set $\mathcal{T}$ is aperiodic if $\Omega_\mathcal{T} \neq \emptyset$ and every tiling $x \in \Omega_\mathcal{T}$ is nonperiodic. As explained in the first page of [Rob71] (see also [BG13, Prop. 5.9]), if $\mathcal{T}$ is periodic, then there is a tiling $x$ by $\mathcal{T}$ with two linearly independent translation vectors (in particular a tiling $x$ with vertical and horizontal translation vectors).

We say that two Wang tile sets $\mathcal{T}$ and $\mathcal{S}$ are equivalent if there exist two bijections $i : I \rightarrow I'$ and $j : J \rightarrow J'$ such that
\begin{equation}
\mathcal{S} = \{(i(a), j(b), i(c), j(d)) \mid (a, b, c, d) \in \mathcal{T}\}.
\end{equation}

If $\mathcal{T}$ is a set of Wang tiles, then we define the dual tile set $\mathcal{T}^*$ as its image under a reflection through the positive diagonal, i.e.,
\begin{equation}
\mathcal{T}^* = \left\{(b, a, d, c) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid (a, b, c, d) \in \mathcal{T} \right\}.
\end{equation}

### 2.3. Fusion of Wang tiles.

Now, we introduce a fusion operation on Wang tiles that can be adjacent in a tiling. Let $u = \begin{bmatrix} B & X \\ A & Y \end{bmatrix}$ and $v = \begin{bmatrix} D & W \\ C & Z \end{bmatrix}$ be two Wang tiles. For $i \in \{1, 2\}$, we define a binary operation on Wang tiles denoted $\boxplus^i$ as
\begin{equation}
u \boxplus^1 v = \begin{bmatrix} BD & X \\ AC & Y \end{bmatrix} \quad \text{if } Y = W \quad \text{and} \quad u \boxplus^2 v = \begin{bmatrix} D & X \\ C & A \end{bmatrix} \quad \text{if } B = C.
\end{equation}

If $i = 1$ and $Y \neq W$ or if $i = 2$ and $B \neq C$, we say that $u \boxplus^i v$ is not well-defined. If $u \boxplus^i v$ is well-defined for some $i \in \{1, 2\}$, it means that $u$ and $v$ can appear at position $n$ and $n + e_i$ in a tiling for some $n \in Z^d$. When appropriate, we also use the following more visual notation for denoting $u \boxplus^i v$:
\begin{equation}u \boxplus^1 v = \begin{bmatrix} u & v \\ v & u \end{bmatrix} \quad \text{and} \quad u \boxplus^2 v = \begin{bmatrix} v \\ u \end{bmatrix}.
\end{equation}

For each $i \in \{1, 2\}$, one can define a new tile set from two Wang tile sets $\mathcal{T}$ and $\mathcal{S}$ as
\begin{equation}\mathcal{T} \boxplus^i \mathcal{S} = \{ u \boxplus^i v \text{ well-defined} \mid u \in \mathcal{T}, v \in \mathcal{S} \}.
\end{equation}
The fusion operation together with taking the dual of a tile set satisfy the following equations:

\[(T \boxdot_1 S)^* = T^* \boxdot_2 S^* \quad \text{and} \quad (T \boxdot_2 S)^* = T^* \boxdot_1 S^*.\]

2.4. **Transducer representation of Wang tiles.** A transducer \(M\) is a labeled directed graph whose nodes are called states and edges are called transitions. The transitions are labeled by pairs \(a|b\) of letters. The first letter \(a\) is the input symbol and the second letter \(b\) is the output symbol. There is no initial nor final state. A transducer \(M\) computes a relation \(\rho(M)\) between bi-infinite sequences of letters.

As observed in [Kar96] and extensively used in [JR15], any finite set of Wang tiles may be interpreted as a transducer. To a given tile set \(T\), the states of the corresponding transducer \(M_T\) are the colors of the vertical edges. The colors of horizontal edges are the input and output symbols. There is a transition from state \(s\) to state \(t\) with label \(a|b\) if and only if there is a tile \((t,b,s,a) \in T\) whose left, right, bottom and top edges are colored by \(s, t, a\) and \(b\), respectively:

\[M_T = \left\{ s \xrightarrow{a|b} t : \begin{array}{c} b \\ a \end{array} = (t,b,s,a) \in T \right\}.\]

The transducer \(M_U\) for the tile set \(U\) is shown in Figure 4. Sequences \(x\) and \(y\) are in the relation \(\rho(M_T)\) if and only if there exists a row of tiles, with matching vertical edges, whose bottom edges form sequence \(x\) and top edges sequence \(y\). For example, the sequence of bottom colors seen on the lowest row of the tiling shown at Figure 3 is \(KOKPOKOKPOKPO\). Starting in state \(G\) and reading that word as input in the transducer \(M_U\) we follow the transitions:

\[
\begin{align*}
G &\xrightarrow{K|P} I \xrightarrow{O|L} A \xrightarrow{K|K} I \xrightarrow{P|P} E \xrightarrow{O|L} G \xrightarrow{K|P} I \xrightarrow{O|L} A \xrightarrow{K|K} I \xrightarrow{P|P} E \xrightarrow{O|L} G \xrightarrow{K|P} I \xrightarrow{P|P} E \xrightarrow{O|L} G.
\end{align*}
\]

We finish in state \(G\) and we get \(PLKPLPKPLPPL\) as output which corresponds to the sequence of top colors of the lowest row of the tiling at Figure 3. Then, starting in state \(F\) and reading the previously output word \(PLKPLPKPLPPL\) as input we compute the second row of the tiling.

**Figure 4.** The transducer \(M_U\). Each tile of \(U\) corresponds to a transition. For example, the first tile of \(U\) is associated to \(J \xrightarrow{O|O} F\), etc.
at Figure 3, etc. In general, there is a one-to-one correspondence between valid tilings of the plane, and the iterated execution of the transducer.

Notice that the result of the usual composition of transducers corresponds to the transducer of $\mathcal{T} \circ \mathcal{S}$:

$$M_T \circ M_S = M_{T \circ S}.$$  

As done in [JR15], the result of the composition can be filtered by recursively removing any source or sink state from the transducer reducing the size of the tile set. This is very helpful for doing computations (see the proof Lemma 23).

2.5. $d$-dimensional word. In this section, we recall the definition of $d$-dimensional word that appeared in [CKR10] and we keep the notation $u \odot^j v$ they proposed for the concatenation.

If $i \leq j$ are integers, then $[i, j]$ denotes the interval of integers $\{i, i + 1, \ldots, j\}$. Let $n = (n_1, \ldots, n_d) \in \mathbb{N}^d$ and $A$ be an alphabet. We denote by $A^n$ the set of functions

$$u : [0, n_1 - 1] \times \cdots \times [0, n_d - 1] \to A.$$  

An element $u \in A^n$ is called a $d$-dimensional word $u$ of shape $n = (n_1, \ldots, n_d) \in \mathbb{N}^d$ on the alphabet $A$. The set of all finite $d$-dimensional word is $A^* = \{A^n \mid n \in \mathbb{N}^d\}$. A $d$-dimensional word of shape $e_k + \sum_{i=1}^d e_i$ is called a domino in the direction $e_k$. When the context is clear, we write $A$ instead of $A^{(1, \ldots, 1)}$. When $d = 2$, we represent a $d$-dimensional word $u$ of shape $(n_1, n_2)$ as a matrix with Cartesian coordinates:

$$u = \left( \begin{array}{ccc} u_{n_1, n_2 - 1} & \cdots & u_{n_1, 1, n_2 - 1} \\ \vdots & \ddots & \vdots \\ u_{0, 0} & \cdots & u_{0, n_2 - 1} \end{array} \right).$$  

Let $n, m \in \mathbb{N}^d$ and $u \in A^n$ and $v \in A^m$. If there exists an index $i$ such that the shapes $n$ and $m$ are equal except at index $i$, then the concatenation of $u$ and $v$ in the direction $e_i$ is well-defined: it is the $d$-dimensional word $u \odot^i v$ of shape $(n_1, \ldots, n_i-1, n_i + m_i, n_{i+1}, \ldots, n_d) \in \mathbb{N}^d$ defined as

$$(u \odot^i v)(a) = \begin{cases} u(a) & \text{if } 0 \leq a_i < n_i, \\ v(a - n_i e_i) & \text{if } n_i \leq a_i < n_i + m_i. \end{cases}$$  

If the shapes $n$ and $m$ are not equal except at index $i$, we say that the concatenation of $u \in A^n$ and $v \in A^m$ in the direction $e_i$ is not well-defined.

Let $n, m \in \mathbb{N}^d$ and $u \in A^n$ and $v \in A^m$. We say that $u$ occurs in $v$ at position $p \in \mathbb{N}^d$ if $v$ is large enough, i.e., $m - p - n \in \mathbb{N}^d$ and

$$v(a + p) = u(a)$$  

for all $a = (a_1, \ldots, a_d) \in \mathbb{N}^d$ such that $0 \leq a_i < n_i$ with $1 \leq i \leq d$. If $u$ occurs in $v$ at some position, then we say that $u$ is a $d$-dimensional subword or factor of $v$.

2.6. $d$-dimensional morphisms. In this section, we generalize the definition of $d$-dimensional morphisms [CKR10] to the case where the domain and codomain are different as for $S$-adic systems.

Let $A$ and $B$ be two alphabets. Let $X \subseteq A^d$. A function $\omega : X \to B^d$ is a $d$-dimensional morphism if for every $i$ with $1 \leq i \leq d$, and every $u, v \in X$ such that $u \odot^i v \in X$ is well-defined we have that the concatenation $\omega(u) \odot^i \omega(v)$ in direction $e_i$ is well-defined and

$$\omega(u \odot^i v) = \omega(u) \odot^i \omega(v).$$

The next lemma can be deduced from the definition. It says that when $d \geq 2$ every $d$-dimensional morphism defined on the whole space $X = A^d$ is uniform in the sense that it maps every letter to a $d$-dimensional word of the same shape. These are called block-substitutions in [Fra17].
Lemma 2. If $d \geq 2$ and $\omega : A^\omega \rightarrow B^\omega$ is a $d$-dimensional morphism, then there exists a shape $n \in \mathbb{N}^d$ such that $\omega(a) \in A^n$ for every letter $a \in A$.

Therefore, to consider non-uniform $d$-dimensional morphism when $d \geq 2$, we need to restrict the domain to a strict subset $X \subset A^\omega$. For example, let $d = 2$ and $A = \{a, b, x, y\}$. The map $\omega : A \rightarrow A^\omega$ defined by $\omega : a \mapsto aaa, b \mapsto bbb, x \mapsto xx, y \mapsto yy$ is not a well-defined 2-dimensional morphism $A^\omega \rightarrow A^\omega$, but it is a well-defined 2-dimensional morphism $X \rightarrow A^\omega$, for the strict subset $X \subset A^\omega$ such that every column of every $w \in X$ is over $\{a, b\}$ or $\{x, y\}$. For instance, we have

$$\omega :\begin{array}{cccc | ccc | ccc}
  x & a & b & y & xx & aaa & bbb & yy \\
  y & a & a & y & yy & aaa & aaa & yy \\
  x & b & a & x & xx & bbb & aaa & xx \\
  y & b & b & y & yy & bbb & bbb & yy \\
  y & b & a & x & yy & bbb & aaa & xx \\
  x & a & b & x & xx & aaa & bbb & xx
\end{array}$$

A $d$-dimensional morphism $\omega : X \rightarrow B^\omega$ with $X \subset A^\omega$ can be extended to a $d$-dimensional morphism $\omega : Y \rightarrow B^\omega$ with $Y \subset A^\omega$.

In [CKR10] and [Moz89, p.144], they consider the case $A = B$ and they restrict the domain of $d$-dimensional morphisms to the language they generate.

Suppose now that $A = B$. We say that a $d$-dimensional morphism $\omega : A \rightarrow A^\omega$ is expansive if for every $a \in A$ and $K \in \mathbb{N}$, there exists $m \in \mathbb{N}$ such that $\min(\text{SHAPE}(\omega^m(a))) > K$. We say that $\omega$ is primitive if there exists $m \in \mathbb{N}$ such that for every $a, b \in A$ the letter $b$ occurs in $\omega^m(a)$.

The definition of prolongable substitutions [BR10, Def. 1.2.18-19] can be adapted in the case of $d$-dimensional morphisms. Let $d$-dimensional morphism $\omega : X \rightarrow A^\omega$ with $X \subset A^\omega$. Let $s \in \{+1,-1\}^d$. We say that $\omega$ is prolongable on letter $a \in A$ in the hyperoctant of sign $s$ if the letter $a$ appears in the appropriate corner of its own image $\omega(a)$ more precisely at position $p = (p_1, \ldots, p_d) \in \mathbb{N}^d$ where

$$p_i = \begin{cases} 0 & \text{if } s_i = +1, \\ n_i - 1 & \text{if } s_i = -1. \end{cases}$$

where $n = (n_1, \ldots, n_d) \in \mathbb{N}^d$ is the shape of $\omega(a)$. If $\omega$ is prolongable on letter $a \in A$ in the hyperoctant of sign $s$ and if $\lim_{m \rightarrow \infty} \min(\text{SHAPE}(\omega^m(a))) = \infty$, then $\lim_{m \rightarrow \infty} \omega^m(a)$ is a well-defined $d$-dimensional infinite word $s_0 \mathbb{N} \times \cdots \times s_{d-1} \mathbb{N} \rightarrow A$.

2.7. $d$-dimensional language. A subset $L \subset A^\omega$ is called a $d$-dimensional language. The factorial closure of a language $L$ is

$$L^{\text{Fact}} = \{ u \in A^\omega \mid u \text{ is a } d\text{-dimensional subword of some } v \in L \}.$$ 

A language $L$ is factorial if $L^{\text{Fact}} = L$. Every languages considered in this contribution are factorial. Given a tiling $x \in A^\omega$, the language $L(x)$ defined by $x$ is

$$L(x) = \{ u \in A^\omega \mid u \text{ is a } d\text{-dimensional subword of } x \}.$$ 

The language of a subshift $X \subset A^\omega$ is $L_X = \bigcup_{x \in X} L(x)$. Conversely, given a factorial language $L \subset A^\omega$ we define the subshift

$$X_L = \{ x \in A^\omega \mid L(x) \subseteq L \}.$$ 

A language $L \subset A^\omega$ is forbidden in a subshift $X \subset A^\omega$ if $L \cap L_X = \emptyset$. By extension, a $d$-dimensional subword $u \in A^\omega$ is forbidden in a subshift $X \subset A^\omega$ if the singleton language $\{u\}$ is forbidden in $X$. 


We say that a word $u \in \mathcal{A}^n$, with $n = (n_1, \ldots, n_d) \in \mathbb{N}^d$, admits a surrounding of radius $r \in \mathbb{N}$ in a language $L \subseteq \mathcal{A}^d$ (resp. in a subshift $X \subseteq \mathcal{A}^d$) if there exists $w \in \mathcal{A}^{n+2(r,\ldots,r)}$ such that $w \in L$ (resp. $w \in \mathcal{L}_X$) and $u$ occurs in $w$ at position $(r, \ldots, r)$.

Now we consider notions of languages related to morphisms. Given a $d$-dimensional morphism $\omega : \mathcal{A} \to \mathcal{B}^d$ and a language $L \subseteq \mathcal{A}^d$, we define the image of the language $L$ under $\omega$ as the language $L^\omega = \{ \omega(v) \mid v \in L \} \subseteq \mathcal{B}^d$ and the image of a subshift $X \subseteq \mathcal{A}^d$ under $\omega$ as the subshift $\omega(X) = \{ \omega(x) \mid x \in X \} \subseteq \mathcal{B}^d$.

The next lemma shows that the fact that a subshift is the image of another subshift under a $d$-dimensional morphism can be restated equivalently in terms of their languages.

**Theorem 3.** Let $\omega : X \to Y$ be a $d$-dimensional morphism for some subshifts $X \subseteq \mathcal{A}^d$ and $Y \subseteq \mathcal{B}^d$. The following conditions are equivalent:

(i) $Y = \omega(X)^\sigma$,

(ii) $\mathcal{L}_Y = \omega(\mathcal{L}_X)^{Fact}$.

**Proof.** (i) $\implies$ (ii) By definition of $\omega$, we have $\omega(\mathcal{L}_X) \subseteq \mathcal{L}_Y$ and $\omega(\mathcal{L}_X)^{Fact} \subseteq \mathcal{L}_Y$ since $\mathcal{L}_Y$ is a factorial language. Let $u \in \mathcal{L}_Y$, i.e., $u$ is a factor of some word $y \in Y$. There exists $k \in \mathbb{Z}^d$ and $x \in X$ such that $y = \sigma^k \omega(x)$. Therefore $u$ is a subword of $\omega(v)$ for some $v \in \mathcal{L}_X$ and we conclude $\mathcal{L}_Y \subseteq \omega(\mathcal{L}_X)^{Fact}$.

(ii) $\implies$ (i) We have $\omega(X) \subseteq Y$. Since $Y$ is shift-invariant, we also have $\omega(X)^\sigma \subseteq Y$. Let $y \in Y$. Thus $y$ is in the subshift generated by the language $\mathcal{L}_Y = \omega(\mathcal{L}_X)^{Fact}$. Since $X$ is closed, there exists $x \in X$ and $k \in \mathbb{Z}^d$ such that $y = \sigma^k \omega(x)$. Therefore $Y \subseteq \omega(X)$.

2.8. **Self-similar subshifts.** In this section we consider languages and subshifts defined from substitutions leading to self-similar structures. A subshift $X \subseteq \mathcal{A}^d$ (resp. a language $L \subseteq \mathcal{A}^d$) is **self-similar** if there exists an expansive $d$-dimensional morphism $\omega : \mathcal{A} \to \mathcal{A}^d$ such that $X = \omega(X)^\sigma$ (resp. $L = \omega(L)^{Fact}$).

Self-similar languages and subshifts can be constructed by iterative application of the morphism $\omega$ starting with the letters. The **language** $\mathcal{L}_\omega$ defined by an expansive $d$-dimensional morphism $\omega : \mathcal{A} \to \mathcal{A}^d$ is

$$\mathcal{L}_\omega = \{ u \in \mathcal{A}^d \mid u \text{ is a } d\text{-dimensional subword of } \omega^n(a) \text{ for some } a \in \mathcal{A} \text{ and } n \in \mathbb{N} \}.$$ 

It satisfies $\mathcal{L}_\omega = \omega(\mathcal{L}_\omega)^{Fact}$ and thus is self-similar. The **substitutive shift** $X_\omega = X_{\mathcal{L}_\omega}$ defined from the language of $\omega$ is a self-similar subshift. If $\omega$ is primitive then $X_\omega$ is the smallest nonempty subshift $X \subseteq \mathcal{A}^d$ satisfying $X = \omega(X)^\sigma$.

**Remark 4.** In some references, the definition of self-similar is more restrictive. For example, in [Sol98, p.268], the definition of self-similarity for tilings of $\mathbb{R}^d$ would be restated in terms of tilings of $\mathbb{Z}^d$ as: a subshift $X \subseteq \mathcal{A}^d$ is self-similar if there exists an expansive $d$-dimensional morphism $\omega : \mathcal{A} \to \mathcal{A}^d$ such that $X = X_\omega$. Observe that a nonempty subshift $X$ that satisfies $X = \omega(X)^\sigma$ for some expansive and primitive $d$-dimensional morphism $\omega$ contains $X_\omega$ but as illustrated in the following example, it needs not be $X_\omega$. 

Consider the 1-dimensional morphism $\mu : a \mapsto ca, b \mapsto bc, c \mapsto cbac$ over the alphabet $A = \{a, b, c\}$. It is expansive and primitive. Consider the language

$$L = \{ u \in A^d \mid u \text{ is a } d\text{-dimensional subword of } \mu^n(ab) \text{ for some } n \in \mathbb{N}\}$$

that satisfies $L \supseteq L_\omega$ and $L = \overline{\omega(L)_{Fact}}$. We have $ab \in L$ but $ab \notin L_\omega$ thus $L \supseteq L_\omega$.

The next lemma shows that there is a unique language $L$ satisfying $L = \overline{\omega(L)_{Fact}}$ and a unique subshift $X$ satisfying $X = \overline{\omega(X)}^\sigma$ whenever $\omega$ is primitive and the words of shape $(2, \ldots, 2)$ of $L$ or of $L_X$ are in $L_\omega$.

**Lemma 5.** Let $\omega : A \to A^d$ be an expansive and primitive $d$-dimensional morphism. Let $L \subseteq A^d$ be a $d$-dimensional language such that $L = \overline{\omega(L)_{Fact}}$. Then $L_\omega \subseteq L$ and the following conditions are equivalent:

1. $L = L_\omega$,
2. $X_L$ is minimal,
3. $X_L = X_\omega$,
4. $L \cap A^{(2, \ldots, 2)} = L_\omega \cap A^{(2, \ldots, 2)}$.

**Proof.** ($L_\omega \subseteq L$). Notice that, recursively, $L = \overline{\omega^m(L)_{Fact}}$ for every $m \geq 1$. For every $a \in A$ and $m \geq 1$, the $d$-dimensional word $\omega^m(a)$ is in the language $L$.

(i) $\implies$ (ii). We have that $X_{L_\omega} = X_\omega$ is the substitutive shift of $\omega$ which is minimal since $\omega$ is primitive. (ii) $\implies$ (iii). We know that $L_\omega \subseteq L$. Therefore $X_\omega \subseteq X_L$. Since $X_L$ is minimal, we conclude that $X_\omega = X_L$. (iii) $\implies$ (iv). If $X_\omega = X_L$ then $L = L_\omega$. Therefore, $L \cap A^{(2, \ldots, 2)} = L_\omega \cap A^{(2, \ldots, 2)}$.

(iv) $\implies$ (i). ($L \subseteq L_\omega$). Let $z \in L$. Since $\omega$ is expansive, let $m \in \mathbb{N}$ such that the image of every letter $a \in A$ by $\omega^m$ is larger than $z$, that is, $\text{SHAPE}(\omega^m(a)) \geq \text{SHAPE}(z)$ for all $a \in A$. We have $z \in \overline{\omega^m(L)_{Fact}}$. By the choice of $m$, $z$ can not overlap more than two block $\omega^m(a)$ in the same direction. Then, there exists a word $u \in L$ of shape $(2, \ldots, 2)$ such that $z$ is a subword of $\omega^m(u)$. From the hypothesis, every word of shape $(2, \ldots, 2)$ that appear in $L$ also appear in $L_\omega$. Therefore $u \in L_\omega$. Since $\omega$ is primitive, there exists $\ell$ such that every word $u$ of shape $(2, \ldots, 2)$ that appear in $L_\omega$ is also a subword of $\omega^\ell(a)$ for every $a \in A$. Therefore, $z$ is a subword of $\omega^{m+\ell}(a)$ for every $a \in A$. Then $z \in L_\omega$ and $L \subseteq L_\omega$. □

2.9. $d$-dimensional recognizability and aperiodicity. The definition of recognizability dates back to the work of Host, Quéffelec and Mossé [Mos92]. See also [ATY17] who propose a completion of the statement and proof for B. Mossé’s unilateral recognizability theorem. The definition introduced below is based on work of Berthé, Steiner and Yassawi [BSTY17] on the recognizability in the case of S-adic systems where more than one substitutions are involved.

Let $X \subseteq A^{2d}$ and $\omega : X \to B^{2d}$ be a $d$-dimensional morphism. If $y \in \overline{\omega(X)}^\sigma$, i.e., $y = \sigma^k\omega(x)$ for some $x \in X$ and $k \in \mathbb{Z}^d$, where $\sigma$ is the $d$-dimensional shift map, we say that $(k, x)$ is a $\omega$-representation of $y$. We say that it is centered if $y_0$ lies inside of the image of $x_0$, i.e., if $0 \leq k < \text{SHAPE}(\omega(x_0))$ coordinate-wise. We say that $\omega$ is recognizable in $X \subseteq A^{2d}$ if each $y \in B^{2d}$ has at most one centered $\omega$-representation $(k, x)$ with $x \in X$.

The unique composition property and conditions (C1) and (C2) can be stated in terms of subshifts, $d$-dimensional morphisms and recognizability.

**Proposition 6.** Let $\omega : A \to A^{2d}$ be an expansive $d$-dimensional morphism. Let $X \subseteq A^{2d}$ be a self-similar subshift such that $\overline{\omega(X)}^\sigma = X$. If $\omega$ is recognizable in $X$, then $X$ is aperiodic.
Proof. Suppose that there exists a periodic tiling \( y \in X \) with period \( p \in \mathbb{Z}^d \setminus \{(0,0)\} \) satisfying \( \sigma^p y = y \). Since \( \omega \) is expansive, let \( m \in \mathbb{N} \) such that the shape of the image of every letter \( a \in \mathcal{A} \) by \( \omega^m \) is large enough, that is, \( \text{SHAPE}(\sigma^m(a)) \geq p \) for every letter \( a \in \mathcal{A} \). By hypothesis, every \( y \in X \) has a \( \omega \)-representation. Recursively, there exists a \( \omega^m \)-representation \((k,x)\) of \( y \) satisfying \( y = \sigma^k \omega^m(x) \). We may assume that it is centered since \( X \) is shift-invariant. By definition of centered representation, for every \( u \in \mathbb{Z}^d \) such that \( 0 \leq u < \text{SHAPE}(\omega^m(x_0)) \), \((u,x)\) is a centered \( \omega^m \)-representation of \( \sigma^u \omega^m(x) = \sigma^{u-k} y \). By the choice of \( m \), there exists \( u \in \mathbb{Z}^d \) such that \( 0 \leq u < \text{SHAPE}(\omega^m(x_0)) \) and \( 0 \leq u + p < \text{SHAPE}(\omega^m(x_0)) \). Therefore \((u,x)\) is a centered \( \omega^m \)-representation of \( \sigma^u \omega^m(x) = \sigma^{u-k} y \) and \((u+p,x)\) is a centered \( \omega^m \)-representation of \( \sigma^{u+p} \omega^m(x) = \sigma^{u-k+p} y = \sigma^{u-k} y \). Therefore, \( \omega^m \) is not recognizable which implies that \( \omega \) is not recognizable which is a contradiction. We conclude that there is no periodic tiling \( y \in X \). \( \square \)

Note that the converse of Proposition \ref{prop:recognizable} for tilings was proved in \cite{Sol98} as a generalization of a result of Mossé \cite{Mos92} who showed that recognizability and aperiodicity are equivalent for primitive substitutive sequences.

**Lemma 7.** Let \( \omega : X \to Y \) be some \( d \)-dimensional morphism between two subshifts \( X \) and \( Y \). If \( Y \) is aperiodic, then \( X \) is aperiodic.

**Proof.** If \( X \) contains a periodic tiling \( x \), then \( \omega(x) \in Y \) is periodic. \( \square \)

3. A SUFFICIENT CONDITION FOR RECOGNIZABILITY AND SURJECTIVITY

The goal of this section is to show that under some hypothesis made on a Wang tile set \( T \), namely the existence of markers, there exists another set \( S \) of Wang tiles and a nontrivial recognizable 2-dimensional morphism \( \Omega_S \to \Omega_T \) that is onto up to a shift. More precisely, every Wang tiling by \( T \) is up to a shift the image under a nontrivial 2-dimensional morphism \( \omega \) of a unique Wang tiling in \( \Omega_S \).

3.1. A sufficient condition for recognizability. The next lemma provides a sufficient condition for recognizability of \( d \)-dimensional morphism. It is weak in the sense that it applies to morphism whose images of letters are letters or a domino in a given direction, but it is sufficient for our needs.

**Lemma 8.** Let \( d \geq 1 \) and \( i \) such that \( 1 \leq i \leq d \). Let \( X \subseteq \mathcal{A}^{2d} \) and \( \omega : X \to \mathcal{B}^{2d} \) be a \( d \)-dimensional morphism such that the image of letters are letters or dominoes in the direction \( e_i \). If \( \omega|_{\mathcal{A}} \) is injective and there exists a subset \( M \subset \mathcal{B} \) such that

\[
\omega(\mathcal{A}) \subseteq (\mathcal{B} \setminus M) \cup \left( (\mathcal{B} \setminus M) \ominus e^i M \right),
\]

or

\[
\omega(\mathcal{A}) \subseteq (\mathcal{B} \setminus M) \cup \left( M \ominus e^i (\mathcal{B} \setminus M) \right),
\]

then \( \omega \) is recognizable in \( X \).

Informally, the letters \( m \in M \) have the role of markers: when we see them in the image under \( \omega \), we know that they must appear as the left (resp. right) part of a domino. A formal definition of markers is given in the next section (Definition \ref{def:markers}).

**Proof.** Let \((k,x)\) and \((k',x')\) be two centered \( \omega \)-representations of \( y \in \mathcal{B}^{2d} \) with \( k,k' \in \mathbb{Z}^d \) and \( x,x' \in X \). We want to show that they are equal.

Since the image of a letter under \( \omega \) is a letter or a domino in the direction \( e_i \), then \( k,k' \in \{0,e_i\} \).

If \( y_0 \in M \), then \( y_0 \) appears as the left or right part of a domino and thus \( k = k' = e_i \) if Equation (6) holds or \( k = k' = 0 \) if Equation (7) holds.
Suppose now that \( y_0 \in B \setminus M \). If Equation (6) holds, then \( k = k' = 0 \). Suppose that Equation (7) holds. By contradiction, suppose that \( k \neq k' \) and assume without lost of generality that \( k = 0 \) and \( k' = e_i \). This means that \( \omega(x'_0) = y_{-e_i} \circ^i y_0 \) is a domino in the direction \( e_i \). Since \( y_0 \in B \setminus M \), we must have that \( y_{-e_i} \in M \) is a left marker. This is impossible as \( \omega(x_{-e_i}) = y_{-e_i} \in B \setminus M \) or \( \omega(x_{-e_i}) = y_{-e_i} \circ^i y_{-e_i} \in M \circ^i (B \setminus M) \). Therefore, we must have \( k = k' \) and \( \omega(x) = \omega(x') \).

Suppose by contradiction that \( x \neq x' \). Let \( a = (a_1, \ldots, a_d) \in \mathbb{Z}^d \) be some minimal vector with respect to \( \|a\|_\infty \) such that \( x_a \neq x'_a \). Injectivity of \( \omega \) implies that \( \omega(x_a) \) and \( \omega(x'_a) \) must have different shapes. Suppose without lost of generality that \( \omega(x_a) \in B \) and \( \omega(x'_a) = b \circ^i c \in B \circ^i B \).

We need to consider two cases: \( a_i \geq 0 \) and \( a_i < 0 \).

Suppose \( a_i \geq 0 \). We must have that Equation (6) holds. We have \( \omega(x_a) = b \in B \setminus M \) and \( c \in M \). But then \( \omega(x_a + e_i) = c \) or \( \omega(x_a + e_i) = c \circ^i d \) for some \( c \in B \setminus M \) and \( d \in B \) which is a contradiction.

Suppose \( a_i < 0 \). We must have that Equation (7) holds. We have \( \omega(x_a) = c \in B \setminus M \) and \( b \in M \). But then \( \omega(x_a - e_i) = b \) or \( \omega(x_a + e_i) = d \circ^i b \) for some \( b \in B \setminus M \) and \( d \in B \) which is a contradiction. We conclude that \( x = x' \). \( \square \)

### 3.2. A sufficient condition for surjectivity up to a shift.

Theorem [10] is the key result in this contribution since it is used in Section 4 to desubstitute tilings in \( \Omega_\mathcal{T} \) into tilings in \( \Omega_\mathcal{V} \) for some Wang tile set \( \mathcal{V} \) and in Section 5 to desubstitute tilings of \( \Omega_\mathcal{V} \) into tilings in \( \Omega_\mathcal{W} \) for some Wang tile set \( \mathcal{W} \). It can be seen as a 2-dimensional generalization of the notion of derived sequence introduced in [Dur98] for the 1-dimensional substitutive case. While for the 1-dimensional substitutive case the derived sequence is obtained by considering the return words to a single letter, here it gives sufficient condition for the existence of a derived tiling by considering the return words to a subset of letters \( \mathcal{T} \setminus M \) for some \( M \subset \mathcal{T} \).

**Definition 9.** Let \( \mathcal{T} \) be a Wang tile set and let \( \Omega_\mathcal{T} \) be its Wang shift. A nonempty proper subset \( M \subset \mathcal{T} \) is called markers in the direction \( e_1 \) if

\[
(8) \quad M \circ^i M, \quad M \circ^2 (\mathcal{T} \setminus M), \quad (\mathcal{T} \setminus M) \circ^2 M
\]

are forbidden in \( \Omega_\mathcal{T} \). It is called markers in the direction \( e_2 \) if

\[
(9) \quad M \circ^2 M, \quad M \circ^1 (\mathcal{T} \setminus M), \quad (\mathcal{T} \setminus M) \circ^1 M
\]

are forbidden in \( \Omega_\mathcal{T} \).

The markers in the direction \( e_1 \) (resp. \( e_2 \)) appear as nonadjacent columns (resp. rows) of tiles in a tiling. The existence of markers allows to desubstitute tilings uniquely by 2-dimensional morphism that are essentially 1-dimensional.

**Theorem 10.** Let \( \mathcal{T} \) be a Wang tile set and let \( \Omega_\mathcal{T} \) be its Wang shift. If there exists a subset \( M \subset \mathcal{T} \) of markers in the direction \( e_i \in \{e_1, e_2\} \), then there exists a Wang tile set \( \mathcal{S} \) and a 2-dimensional morphism \( \omega : \Omega_\mathcal{S} \to \Omega_\mathcal{T} \) such that

\[
(10) \quad \omega(\mathcal{S}) \subseteq (\mathcal{T} \setminus M) \cup ((\mathcal{T} \setminus M) \circ^i M)
\]

which is recognizable in the Wang shift \( \Omega_\mathcal{S} \) and surjective up to a shift, i.e., \( \omega(\Omega_\mathcal{S}) \cup \sigma^e \omega(\Omega_\mathcal{S}) = \Omega_\mathcal{T} \).

**Proof.** For every \( r \in \mathbb{N} \), we define the set of dominoes in the direction \( e_i \) that admits a surrounding of radius \( r \) in \( \Omega_\mathcal{T} \):

\[
D_{\mathcal{T}, e_i, r} = \{ u \circ^i v \mid u \circ^i v \text{ admits a surrounding of radius } r \text{ in } \Omega_\mathcal{T} \}.
\]
Notice that since $M$ is a set of markers, there exists $r \in \mathbb{N}$ such that $D_{T,e_i,r} \cap (M \odot^i M) = \emptyset$. Now let $r \in \mathbb{N}$ be any nonnegative integer and let $P_r$, $K_r$ be the following sets:

$$
P_r = \left\{ u \boxdi v \mid u \in T \setminus M, v \in M \text{ and } u \odot^i v \in D_{T,e_i,r} \right\},
$$

$$
K_r = \left\{ u \in T \setminus M \mid \text{ there exists } v \in T \setminus M \text{ such that } u \odot^i v \in D_{T,e_i,r} \right\}.
$$

Let $S = K_r \cup P_r$ be a Wang tile set and $\Omega_S$ be its Wang shift. Let $\omega : \Omega_S \to \Omega_T$ be the 2-dimensional morphism defined by

$$
\omega : \begin{cases} 
  u \boxdi v & \mapsto u \odot^i v \text{ if } u \boxdi v \in P_r, \\
  u & \mapsto u \text{ if } u \in K_r.
\end{cases}
$$

From now on, we suppose that the markers $M$ are in the direction $e_i = e_1$, the argument being symmetric for $i = 2$. First we show that $\omega$ is well-defined. Let $x \in \Omega_S$ be a Wang tiling. We show that $\omega(x)$ is a valid Wang tiling. It is sufficient to prove it for all dominoes $p \odot^j q$ that appear in $\omega(x)$ with $p, q \in T$ and $j \in \{1, 2\}$. Any domino $p \odot^j q$ that appears in $\omega(x)$ appears in the image under $\omega$ of a letter or of a domino in $x$. The first case can happen only if $j = 1$ and there exists $u \boxdi v \in P_r$ such that $p \odot^1 q = u \odot^1 v = \omega(u \boxdi v)$, in which case the right color of $p$ is equal to the left color of $q$ since $r(p) = r(u) = \ell(v) = \ell(q)$. In the second case, $p$ and $q$ belong to the image of distinct letters. The case $j = 1$ (horizontal dominoes) is easy since $\omega$ preserves the right and left colors of tiles, i.e., $r(\omega(s)) = r(s)$ and $\ell(\omega(s)) = \ell(s)$ for all $s \in S$ so that $r(p) = \ell(q)$. Consider now $j = 2$ (vertical dominoes). Any vertical domino $p \odot^2 q$ that appears in $\omega(x)$ appears in the image under $\omega$ of a vertical domino $p' \odot^2 q'$ in $x$. There are two cases to consider. If $p', q' \in K_r$, then $t(p) = t(\omega(p')) = t(p') = b(q') = b(\omega(q')) = b(q')$. If $p', q' \in P_r$, then there exists $a, b, c, d \in T$ such that $p \odot^2 q$ appears in $(a \odot^1 b) \odot^2 (c \odot^1 d) = \omega(p' \odot^2 q')$. Then

$$
(t(a), t(b)) = t(a \odot^1 b) = t(p') = b(q') = b(\omega(q')) = b(c \odot^1 d) = (b(c), b(d)).
$$

Then $t(a) = b(c)$ and $t(b) = b(d)$ from which we conclude that $t(p) = b(q)$. Therefore $\omega(x)$ is a valid Wang tiling of the plane and belongs to $\Omega_T$.

Now we show that $\omega$ is surjective up to a shift. Let $y \in \Omega_T$ be a Wang tiling which can be seen as a function $y : \mathbb{Z}^2 \to T$. Consider the set $y^{-1}(M) \subset \mathbb{Z}^2$ of positions of marker tiles in $y$. From the definition of markers in the direction $e_i = e_1$, markers appear in nonadjacent columns in a tiling. Formally, there exists a set $H \subset \mathbb{Z}$ such that $y^{-1}(M) = H \times \mathbb{Z}$ and $1 \not\in H - H$. Since $1 \not\in H - H$, there exists a strictly increasing sequence $(a_k)_{k \in \mathbb{Z}}$ such that $\mathbb{Z} \setminus H = \{a_k \mid k \in \mathbb{Z}\}$. We assume that $a_0 = 0$ if $0 \in \mathbb{Z} \setminus H$ and $a_0 = -1$ if $0 \in H$ which makes the sequence $(a_k)_{k \in \mathbb{Z}}$ uniquely defined. Let

$$
x : \mathbb{Z}^2 \to S \begin{cases} 
  (k, \ell) & \mapsto y(a_k, \ell) \text{ if } a_k + 1 \not\in H, \\
  (k, \ell) & \mapsto y(a_k, \ell) \boxdi y(a_k + 1, \ell) \text{ if } a_k + 1 \in H.
\end{cases}
$$

The function $x$ is well-defined since $x(k, \ell) \in S$ for all $(k, \ell) \in \mathbb{Z}^2$. Indeed, if $a_k + 1 \not\in H$, then $x(k, \ell) = y(a_k, \ell) \in T \setminus M$. Also $y(a_k + 1, \ell) \in T \setminus M$. Since $y$ is a valid Wang tiling $y(a_k, \ell) \odot^1 y(a_k + 1, \ell)$ admits arbitrarily large surrounding in $\Omega_T$. Therefore, $x(k, \ell) = y(a_k, \ell) \in K_{\infty} \subseteq K_r \subset S$ and $x(k, \ell) \not\in H$. Moreover, we deduce that $x \in \Omega_S$, that is it is a valid Wang tiling of the plane, from the fact that $y \in \Omega_T$ is a valid Wang tiling of the plane.

We may now finish the proof of surjectivity. If $0 \not\in H$, then the tiling $y$ is exactly the image under $\omega$ of the tiling $x$ that we constructed: $y = \omega(x)$. If $0 \in H$, then the tiling $y$ is a shift of the image under $\omega$ of the tiling $x$: $y = \sigma^{e_1} \omega(x)$. Thus $\omega$ is surjective up to a shift.
The function $\omega$ is of the form

$$\omega(S) \subseteq (T \setminus M) \cup ((T \setminus M) \odot^1 M)$$

and its restriction on $S$ is injective by construction. Therefore, we conclude from Lemma 8 that $\omega$ is recognizable in $\Omega_S$. \hfill $\Box$

In Equation (11), given two Wang tiles $u$ and $v$ such that $u \sqcap^i v$ is well-defined for $i \in \{1, 2\}$, we consider a 2-dimensional morphism of the form

$$u \sqcap^i v \mapsto u \odot^i v$$

which can be seen as

$$\begin{array}{c}
\text{BD} \\
X \\
Z \\
AC
\end{array} \mapsto
\begin{array}{c}
\text{B} \\
Y \\
D \\
C
\end{array}
$$

or

$$\begin{array}{c}
\text{D} \\
N \\
A
\end{array} \mapsto
\begin{array}{c}
\text{W} \\
Z \\
B
\end{array}$$

whether $i = 1$ or $i = 2$.

**Remark 11.** In Equation (11), the set $P_r$ can be computed with the help of transducers since the fusion operation $\sqcap^i$ on Wang tiles can be seen as the product of transducers (see Equation (5)). The algorithmic aspects of Theorem 10 will be considered in a forthcoming work [Lab18b].

**Remark 12.** The existence of a 2-dimensional morphism $\omega : \Omega_S \to \Omega_T$ which is recognizable in the Wang shift $\Omega_S$ and surjective up to a shift implies the existence of a homeomorphism between $\Omega_S$ and $\Omega_T$ where each tile have specific real width and height and tilings are defined in $\mathbb{R}^2$ rather than in $\mathbb{Z}^2$. We have chosen in this contribution the point of view of tilings of $\mathbb{Z}^2$ which makes some aspects look easier and other more intricate.

### 4. Desubstitution of $\Omega_U$

In this section, we prove that tilings in $\Omega_U$ can be desubstituted uniquely into tilings of $\mathbb{Z}^2$ over another set $\mathcal{V}$ of Wang tiles. The proof follows from the computation of the set of vertical dominoes appearing in $\Omega_U$ (Lemma 13) whose proof is done with Sage.

The tile set $\mathcal{U}$ seen in Figure 1 is defined over an alphabet $\{A, B, C, D, E, F, G, H, I, J\}$ of 10 vertical colors and an alphabet $\{K, L, M, N, O, P\}$ of 6 horizontal colors. We identify each tile $u_i$ in $\mathcal{U}$ with an index $i \in \{0, \ldots, 18\}$ in the following way and we draw this index in the center of each tile to ease their identification.

$$\mathcal{U} = \left\{ u_0 = \begin{array}{c}
O \\
1 \\
0 \\
F \\
O
\end{array}, u_1 = \begin{array}{c}
O \\
H \\
1 \\
F \\
L
\end{array}, u_2 = \begin{array}{c}
M \\
2 \\
J \\
P
\end{array}, u_3 = \begin{array}{c}
M \\
3 \\
D \\
K
\end{array}, u_4 = \begin{array}{c}
P \\
4 \\
H \\
P
\end{array}, u_5 = \begin{array}{c}
H \\
5 \\
H \\
N
\end{array},
\right.$$  
$$u_6 = \begin{array}{c}
K \\
F \\
6 \\
H \\
P
\end{array}, u_7 = \begin{array}{c}
K \\
D \\
7 \\
P
\end{array}, u_8 = \begin{array}{c}
O \\
I \\
8 \\
B \\
O
\end{array}, u_9 = \begin{array}{c}
L \\
E \\
9 \\
G \\
O
\end{array}, u_{10} = \begin{array}{c}
L \\
C \\
10 \text{G} \\
O \\
L
\end{array}, u_{11} = \begin{array}{c}
L \\
I \\
11 \text{A} \\
O \\
L
\end{array}, u_{12} = \begin{array}{c}
P \\
G \text{12} \\
P
\end{array},
\right.$$  
$$u_{13} = \begin{array}{c}
P \\
I \text{13} \\
E \\
P
\end{array}, u_{14} = \begin{array}{c}
P \\
G \text{14} \\
I \\
K
\end{array}, u_{15} = \begin{array}{c}
P \\
I \text{15} \\
I \\
K
\end{array}, u_{16} = \begin{array}{c}
K \\
B \\
16 \\
I \\
M
\end{array}, u_{17} = \begin{array}{c}
K \\
A \\
17 \\
I \\
K
\end{array}, u_{18} = \begin{array}{c}
N \\
I \\
18 \\
C \\
P
\end{array} \right\}.$$

The transducer representation of the tile set $\mathcal{U}$ was shown in Figure 4.
Lemma 13. The set of dominoes in direction $e_2$ allowing a surrounding of radius 2 in $\Omega_\mathcal{U}$ is

$$D_{\mathcal{U},e_2,2} = \{(u_0, u_8), (u_1, u_8), (u_1, u_9), (u_1, u_{11}), (u_2, u_{16}), (u_3, u_{16}), (u_4, u_{13}), (u_5, u_{13}),
\quad (u_6, u_{14}), (u_6, u_{17}), (u_7, u_{15}), (u_8, u_0), (u_8, u_9), (u_9, u_{11}), (u_9, u_1), (u_9, u_{10}),
\quad (u_{10}, u_1), (u_{11}, u_1), (u_{11}, u_{10}), (u_{12}, u_6), (u_{13}, u_4), (u_{13}, u_7), (u_{13}, u_{18}), (u_{14}, u_2),
\quad (u_{14}, u_6), (u_{14}, u_{12}), (u_{15}, u_7), (u_{15}, u_{13}), (u_{15}, u_{18}), (u_{16}, u_3), (u_{16}, u_{14}), (u_{16}, u_{17}),
\quad (u_{17}, u_3), (u_{17}, u_{14}), (u_{18}, u_5)\}.$$  

The proof of Lemma 13 is done with Sage in Section 8.

Lemma 14. The set $R = \{u_0, u_1, u_2, u_3, u_4, u_5, u_6, u_7\}$ is a set of markers in the direction $e_2$ for $\mathcal{U}$.

Proof. First we show that $R \odot^2 R$ is forbidden in $\Omega_\mathcal{U}$. The language of dominoes in direction $e_2$ in $\Omega_\mathcal{U}$ is a subset of the set of dominoes in direction $e_2$ allowing a surrounding of radius 2 in $\Omega_\mathcal{U}$:

$$\{(u, v) \in \mathcal{U}^2 \mid u \odot^2 v \in \mathcal{L}(\Omega_\mathcal{U})\} \subseteq D_{\mathcal{U},e_2,2}$$

computed in Lemma 13. Since $D_{\mathcal{U},e_2,2} \cap R^2 = \emptyset$, we deduce that $(R \odot^2 R) \cap \mathcal{L}(\Omega_\mathcal{U}) = \emptyset$, that is $R \odot^2 R$ is forbidden in $\Omega_\mathcal{U}$.

Finally, we remark that

$$\text{LEFT}(\mathcal{U} \setminus R) = \text{RIGHT}(\mathcal{U} \setminus R) = \{A, B, C, E, G, I\},$$

$$\text{LEFT}(R) = \text{RIGHT}(R) = \{D, F, H, J\},$$

which are disjoint. Thus $R \odot^1 (\mathcal{U} \setminus R)$ and $(\mathcal{U} \setminus R) \odot^1 R$ are forbidden in $\Omega_\mathcal{U}$. We conclude that $R$ is a set of markers in the direction $e_2$ for $\mathcal{U}$. \hfill \Box

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure5.png}
\caption{The morphism $\alpha : \Omega_\mathcal{V} \rightarrow \Omega_\mathcal{U}$.}
\end{figure}
Proposition 15. There exists a tile set $\mathcal{V}$ of cardinality 21 and a $d$-dimensional morphism $\alpha : \Omega_{\mathcal{V}} \to \Omega_{\mathcal{U}}$ that is recognizable in $\Omega_{\mathcal{V}}$ and $\alpha(\Omega_{\mathcal{V}}) \cup \sigma^{e_2} \alpha(\Omega_{\mathcal{V}}) = \Omega_{\mathcal{U}}$.

Proof. From Lemma 14, $R = \{u_0, u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8, u_9, u_{10}, u_{11}, u_{12}, u_{13}, u_{14}, u_{15}, u_{16}, u_{17}, u_{18}\}$ is a set of markers in the direction $e_2$ for $\mathcal{U}$. Therefore, Theorem 10 applies. Let $P_2$ be the set

$$P_2 = \{ u \sqcup v \mid u \in \mathcal{U} \setminus R, v \in R \text{ and } (u, v) \in D_{\mathcal{U}, e_2, 2} \}$$

according to Lemma 13. Let $K_2$ be the set

$$K_2 = \{ u \in \mathcal{U} \setminus R \mid \text{there exists } v \in \mathcal{U} \setminus R \text{ such that } (u, v) \in D_{\mathcal{U}, e_2, 2} \} = \{u_8, u_9, u_{11}, u_{13}, u_{14}, u_{15}, u_{16}, u_{17}\}.$$ 

Let $\Omega_{\mathcal{V}}$ be the Wang shift of the Wang tile set $\mathcal{V} = K_2 \cup P_2$ of cardinality 21:

$$\mathcal{V} = \{ v_0 = \begin{array}{c} \text{L} \text{I} \text{O} \\ \text{O} \text{A} \text{O} \end{array}, v_1 = \begin{array}{c} \text{O} \text{I} \text{B} \\ \text{O} \text{O} \text{B} \end{array}, v_2 = \begin{array}{c} \text{P} \text{I} \text{E} \\ \text{G} \text{K} \text{O} \end{array}, v_3 = \begin{array}{c} \text{L} \text{E} \text{G} \\ \text{I} \text{K} \text{O} \end{array}, v_4 = \begin{array}{c} \text{A} \text{J} \text{E} \\ \text{K} \text{M} \text{O} \end{array}, v_5 = \begin{array}{c} \text{K} \text{B} \text{I} \\ \text{P} \text{O} \text{M} \end{array}, v_6 = \begin{array}{c} \text{P} \text{G} \text{I} \\ \text{K} \text{M} \text{O} \end{array},
\begin{array}{c} \text{O} \text{I} \text{B} \\ \text{K} \text{O} \text{M} \end{array}, v_7 = \begin{array}{c} \text{O} \text{I} \text{B} \\ \text{K} \text{O} \text{M} \end{array}, v_8 = \begin{array}{c} \text{O} \text{I} \text{B} \\ \text{K} \text{O} \text{M} \end{array}, v_9 = \begin{array}{c} \text{O} \text{I} \text{B} \\ \text{K} \text{O} \text{M} \end{array}, v_{10} = \begin{array}{c} \text{O} \text{I} \text{B} \\ \text{K} \text{O} \text{M} \end{array}, v_{11} = \begin{array}{c} \text{O} \text{I} \text{B} \\ \text{K} \text{O} \text{M} \end{array}, v_{12} = \begin{array}{c} \text{O} \text{I} \text{B} \\ \text{K} \text{O} \text{M} \end{array}, v_{13} = \begin{array}{c} \text{O} \text{I} \text{B} \\ \text{K} \text{O} \text{M} \end{array}, v_{14} = \begin{array}{c} \text{O} \text{I} \text{B} \\ \text{K} \text{O} \text{M} \end{array}, v_{15} = \begin{array}{c} \text{O} \text{I} \text{B} \\ \text{K} \text{O} \text{M} \end{array}, v_{16} = \begin{array}{c} \text{O} \text{I} \text{B} \\ \text{K} \text{O} \text{M} \end{array}, v_{17} = \begin{array}{c} \text{O} \text{I} \text{B} \\ \text{K} \text{O} \text{M} \end{array}, v_{18} = \begin{array}{c} \text{O} \text{I} \text{B} \\ \text{K} \text{O} \text{M} \end{array}, v_{19} = \begin{array}{c} \text{O} \text{I} \text{B} \\ \text{K} \text{O} \text{M} \end{array}, v_{20} = \begin{array}{c} \text{O} \text{I} \text{B} \\ \text{K} \text{O} \text{M} \end{array} \}.$$ 

Let $\alpha : \Omega_{\mathcal{V}} \to \Omega_{\mathcal{U}}$ be the 2-dimensional morphism defined by

$$\alpha : \begin{cases} u \sqcup v & \mapsto u \odot v \quad \text{if } u \sqcup v \in P_2 \\ k & \mapsto k \quad \text{if } k \in K_2. \end{cases}$$

Then $\alpha : \Omega_{\mathcal{V}} \to \Omega_{\mathcal{U}}$ is recognizable in $\Omega_{\mathcal{V}}$ and $\alpha(\Omega_{\mathcal{V}}) \cup \sigma^{e_2} \alpha(\Omega_{\mathcal{V}}) = \Omega_{\mathcal{U}}$. \qed

The morphism $\alpha$ in terms of the $\mathcal{U} = \{u_i\}_{0 \leq i \leq 18}$ and $\mathcal{V} = \{v_i\}_{0 \leq i \leq 20}$ is in Figure 5. The tile set $\mathcal{V}$ uses the following alphabet of 13 vertical colors:

$$\{A, B, E, G, I, AF, BF, CH, EH, GF, ID, IH, IJ\}$$

and the following alphabet of 5 horizontal colors:

$$\{K, L, M, O, P\}.$$ 

Notice that the set of vertical left and right colors has increased in size compared to $\mathcal{U}$ and that the set of horizontal bottom and top colors has decreased in size. Indeed, color N has disappeared.

5. Desubstitution of $\Omega_{\mathcal{V}}$

In this section, we prove that tilings in $\Omega_{\mathcal{V}}$ can be desubstituted uniquely into tilings of $\mathbb{Z}^2$ over another set $\mathcal{W}$ of Wang tiles. The proof follows from the computation of the set of horizontal dominoes appearing in $\Omega_{\mathcal{V}}$ (Lemma 16) whose proof is done with Sage.

Lemma 16. The set of dominoes in direction $e_1$, allowing a surrounding of radius 1 in $\Omega_{\mathcal{V}}$ is

$$D_{\mathcal{V}, e_1, 1} = \{ (v_0, v_4), (v_1, v_5), (v_2, v_3), (v_3, v_6), (v_4, v_1), (v_4, v_2), (v_5, v_1), (v_5, v_2), (v_5, v_7), (v_6, v_1), (v_7, v_0), (v_7, v_1), (v_8, v_{16}), (v_9, v_{17}), (v_{10}, v_{14}), (v_{11}, v_{15}), (v_{12}, v_{15}), (v_{13}, v_{15}), (v_{14}, v_{11}), (v_{14}, v_{18}), (v_{15}, v_{18}), (v_{15}, v_{20}), (v_{16}, v_{12}), (v_{17}, v_{12}), (v_{17}, v_{19}), (v_{18}, v_8), (v_{19}, v_8), (v_{20}, v_9), (v_{20}, v_{13}) \}.$$
The proof of Lemma 16 is done with Sage in Section 8.

Lemma 17. The set \( R = \{ v_0, v_1, v_3, v_8, v_9, v_{14}, v_{15} \} \) is a set of markers in the direction \( e_1 \) for \( \mathcal{V} \).

Proof. First we show that \( R \odot^1 R \) is forbidden in \( \Omega_\mathcal{V} \). The language of dominoes in direction \( e_1 \) in \( \Omega_\mathcal{V} \) is a subset of the set of dominoes in direction \( e_1 \) allowing a surrounding of radius 1 in \( \Omega_\mathcal{V} \):

\[
\{(u, v) \in \mathcal{V}^2 \mid u \odot^1 v \in \mathcal{L}(\Omega_\mathcal{V})\} \subseteq D_{\mathcal{V}, e_1, 1}
\]

computed in Lemma 16. Since \( D_{\mathcal{V}, e_1, 1} \cap \mathcal{V}^2 = \emptyset \), we deduce that \( (R \odot^1 R) \cap \mathcal{L}(\Omega_\mathcal{V}) = \emptyset \), that is \( R \odot^1 R \) is forbidden in \( \Omega_\mathcal{V} \).

Finally, we remark that

\[
\text{BOTTOM}(\mathcal{V} \setminus R) = \text{TOP}(\mathcal{V} \setminus R) = \{K, M, P\},
\]

\[
\text{BOTTOM}(R) = \text{TOP}(R) = \{L, O\},
\]

which are disjoint. Thus \( R \odot^2 (\mathcal{V} \setminus R) \) and \( (\mathcal{V} \setminus R) \odot^2 R \) are forbidden in \( \Omega_\mathcal{V} \). We conclude that \( R \) is a set of markers in the direction \( e_1 \) for \( \mathcal{V} \).

\[\square\]

Figure 6. The morphism \( \beta : \Omega_\mathcal{W} \to \Omega_\mathcal{V} \).

Proposition 18. There exists a tile set \( \mathcal{W} \) of cardinality 19 and a d-dimensional morphism \( \beta : \Omega_\mathcal{W} \to \Omega_\mathcal{V} \) that is recognizable in \( \Omega_\mathcal{W} \) and \( \beta(\Omega_\mathcal{W}) \cup \sigma^1 \beta(\Omega_\mathcal{W}) = \Omega_\mathcal{V} \).

Proof. From Lemma 17, \( R = \{ v_0, v_1, v_3, v_8, v_9, v_{14}, v_{15} \} \) is a set of markers in the direction \( e_1 \) for \( \mathcal{V} \). Therefore, Theorem 10 applies. Let \( P_1 \) be the set

\[
P_1 = \left\{ u \odot^1 v \mid u \in \mathcal{V} \setminus R, v \in R \text{ and } u \odot^1 v \in D_{\mathcal{V}, e_1, 1} \right\}
\]

\[
= \left\{ v_2, v_3, v_4, v_1, v_5, v_1, v_6, v_1, v_7, v_0, v_7, v_1, v_10, v_{14}, v_{11}, v_{15}, v_{12}, v_{15}, v_{13}, v_{15}, v_{18}, v_8, v_{19}, v_8, v_{20}, v_9 \right\}
\]

according to Lemma 16. Let \( K_1 \) be the set

\[
K_1 = \{ u \in \mathcal{V} \setminus R \mid \text{there exists } v \in \mathcal{V} \setminus R \text{ such that } (u, v) \in D_{\mathcal{V}, e_1, 1} \}
\]

\[
= \{ u_8, u_9, u_{11}, u_{13}, u_{14}, u_{15}, u_{16}, u_{17} \}.
\]
Let $\Omega_W$ be the Wang shift of the Wang tile set $\mathcal{W} = K_1 \cup P_1$ of cardinality 19: 
\[ \mathcal{W} = \left\{ w_0 = K, w_1 = M, w_2 = P, w_3 = K, w_4 = M, w_5 = K, w_6 = P \right\}. \]

Let $\beta : \Omega_W \to \Omega_\mathcal{W}$ be the 2-dimensional morphism defined by 
\[ \beta : \begin{cases} u \sqcup^1 v &\mapsto u \downarrow^1 v & \text{if } u \sqcup^1 v \in P_1 \\ k &\mapsto k & \text{if } k \in K_1. \end{cases} \]

Then $\beta : \Omega_W \to \Omega_\mathcal{W}$ is recognizable in $\Omega_W$ and $\beta(\Omega_W) \cup \sigma^e_1 \beta(\Omega_W) = \Omega_\mathcal{W}$. \hfill \Box

The morphism $\beta$ in terms of the $\mathcal{V} = \{v_i\}_{0 \leq i \leq 20}$ and $\mathcal{W} = \{w_i\}_{0 \leq i \leq 18}$ is in Figure 6. The tile set $\mathcal{W}$ uses the following alphabet of 10 vertical colors: 
\[ \{A, B, G, I, AF, BF, GF, ID, IH, IJ\} \]
and the following alphabet of 6 horizontal colors: 
\[ \{K, M, KO, MO, PL, PO\}. \]

Notice that the set of horizontal left and right colors has increased in size compared to $\mathcal{V}$ and that the set of vertical bottom and top colors has decreased in size. Indeed, colors E, CH and EH have disappeared.

### 6. Self-similarity and aperiodicity of $\Omega_d$

Let $\gamma : \mathcal{U} \to \mathcal{W}$ be the map defined by the rule $u_i \mapsto w_i$ for every $i \in \{0, 1, \ldots, 18\}$. On tiles, the effect is shown below where the position of tile $u_i$ in the table on the left is the same as the position of the tile $w_i$ in the table on the right:

\[ \gamma : \begin{cases} O \mapsto K & P \mapsto KO \\ J \mapsto P & H \mapsto M \\ F \mapsto L & \vdots \\ G \mapsto I \end{cases} \]

\[ \begin{align*} \begin{array}{cccccccc} O \mapsto K & P \mapsto KO & M \mapsto L \mapsto M & N \mapsto MO & M \mapsto PL \mapsto K \mapsto PO \\ J \mapsto A & E \mapsto AF & H \mapsto B & C \mapsto BF & D \mapsto G & I \mapsto GF & F \mapsto I & G \mapsto ID \\ \end{array} \end{align*} \]

Lemma 19. $\mathcal{U}$ and $\mathcal{W}$ are equivalent Wang tile sets. The map $\gamma : \mathcal{U} \to \mathcal{W}$ defines a $d$-dimensional morphism $\gamma : \Omega_\mathcal{U} \to \Omega_\mathcal{W}$ which is a bijection.

Proof. We consider the following bijection between the colors of $\mathcal{U}$ and the colors of $\mathcal{W}$:

\[ h : \begin{cases} O \mapsto K & P \mapsto KO \\ L \mapsto M & N \mapsto MO \\ M \mapsto PL \mapsto K \mapsto PO \\ \end{cases} \]

\[ k : \begin{cases} J \mapsto A & E \mapsto AF \\ H \mapsto B & C \mapsto BF \\ D \mapsto G & I \mapsto GF \\ F \mapsto I & G \mapsto ID \\ B \mapsto IH & A \mapsto IJ \\ \end{cases} \]
where $h$ relabels the horizontal (bottom and top) colors and where $k$ relabels the vertical (left and right) colors. For every $i$ in \{0, 1,\ldots, 18\}, we observe that the application of the map $h$ on the bottom and left colors and $k$ on the left and right colors transforms bijectively $u_i$ into $w_i$:

$$\gamma(\mathcal{U}) = \{(k(a), h(b), k(c), h(d)) \mid (a, b, c, d) \in \mathcal{U}\} = \mathcal{W}.$$ 

Thus $\gamma$ is a bijection and $\mathcal{U}$ and $\mathcal{W}$ are equivalent Wang tile sets.\hfill $\Box$

Consider the $d$-dimensional morphism $\omega : \Omega_\mathcal{U} \to \Omega_\mathcal{U}$ defined as $\omega = \alpha \circ \beta \circ \gamma$ shown in Figure 7.

![Figure 7. The morphism $\omega : \Omega_\mathcal{U} \to \Omega_\mathcal{U}$. The image of each tile $\omega(u_i)$ corresponds to the supertiles shown in Figure 2.]

**Lemma 20.** The morphism $\omega$ is primitive. The characteristic polynomial of its incidence matrix $M$ is

$$\chi_M(x) = x^3 \cdot (x - 1)^4 \cdot (x + 1)^4 \cdot (x^2 - 3x + 1) \cdot (x^2 + x - 1)^3.$$

The dominant eigenvalue is $\varphi^2 = \varphi + 1 = (3 + \sqrt{5})/2$. The associated positive right eigenvector is

$$\left(1, 3\varphi^3, \varphi^2, \varphi^2, \varphi^2, \varphi^2, \varphi^2, \varphi^2, \varphi^2, \varphi^2, \varphi^2, \varphi^2, \varphi^2, \varphi^2, \varphi^2, \varphi^2, \varphi^2, \varphi^2\right)^t$$

and associated positive left eigenvector is

$$\left(1, 1, \varphi, \varphi, \varphi, \varphi, \varphi, \varphi, \varphi, \varphi, \varphi, \varphi, \varphi, \varphi, \varphi, \varphi, \varphi, \varphi\right).$$
Proof. The incidence matrix $M$ of $\omega$ is

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

whose 7-th power $M^7$ is positive. Therefore $M$ is a primitive matrix and the morphism is primitive. The computation of characteristic polynomial, eigenvalues and eigenvectors of $M$ is done using Sage in Section 8.

We now prove the main result using the composition-decomposition approach. Informally, the morphism $\alpha\beta : \Omega_W \to \Omega_U$ allows to group tiles in any tilings admitted by $U$ in a unique way into supertiles of the form $\alpha\beta(w_i)$ for $i \in \{0, 1, \ldots, 18\}$, thus satisfying condition (C1). The morphism $\gamma : \Omega_U \to \Omega_W$ implies that the markings on the supertiles $W$ imply a matching condition for the supertiles which is exactly equivalent to that originally specified for the tiles $U$, thus satisfying condition (C2).

Proposition 21. The $d$-dimensional morphism $\omega : \Omega_U \to \Omega_U$ defined as $\omega = \alpha \circ \beta \circ \gamma$ is expansive, recognizable in $\Omega_U$ and satisfies

\[
\Omega_U = \omega(\Omega_U) \cup \sigma^{e_1}\omega(\Omega_U) \cup \sigma^{e_2}\omega(\Omega_U) \cup \sigma^{e_1+e_2}\omega(\Omega_U).
\]

Proof. From Proposition 13, Proposition 18 and Lemma 19, we have the sequence of recognizable 2-dimensional morphisms:

\[
\Omega_U \xleftarrow{\alpha} \Omega_V \xleftarrow{\beta} \Omega_W \xleftarrow{\gamma} \Omega_U.
\]

and the equations $\alpha(\Omega_V) \cup \sigma^{e_2}\alpha(\Omega_V) = \Omega_U$ and $\beta(\Omega_W) \cup \sigma^{e_1}\beta(\Omega_W) = \Omega_V$. Consider the $d$-dimensional morphism $\omega = \alpha\beta\gamma$ which is recognizable in $\Omega_U$. Since $\alpha\sigma^{e_1} = \sigma^{e_1}\alpha$, we deduce Equation (12). As shown in Lemma 20, $\omega$ is primitive and sends at least one letter to a 2-dimensional word of shape $(2, 2)$. Thus it is expansive.

We can now prove the aperiodicity of $U$ from the recognizability of $\omega$.

Corollary 22. $\omega(\Omega_U)^\sigma = \Omega_U$, $\Omega_U$ is self-similar and aperiodic.

Proof. From Proposition 21, $\omega$ is expansive and recognizable in $\Omega_U$. By taking the closure under the shift $\sigma$ on both sides of Equation (12), we deduce that $\Omega_U = \overline{\omega(\Omega_U)^\sigma} = \omega(\Omega_U)^\sigma$. Thus $\Omega_U$ is self-similar. We conclude aperiodicity of $\Omega_U$ from Proposition 6.
The next lemma shows that the subwords of shape \((2, 2)\) in \(\Omega_\mathcal{U}\) and \(\mathcal{X}_\omega\) are the same.

**Lemma 23.** We have \(\mathcal{L}(\mathcal{X}_\omega) \cap \mathcal{U}^{(2, 2)} = \mathcal{L}(\Omega_\mathcal{U}) \cap \mathcal{U}^{(2, 2)} = S\) where \(S\) is

\[
\begin{pmatrix}
  u_8 u_{15} & u_8 u_{16} & u_8 u_{16} & u_9 u_{14} & u_{11} u_{17} & u_{16} u_{13} & u_{16} u_{15} \\
  u_0 u_3 & u_1 u_2 & u_1 u_2 & u_1 u_6 & u_1 u_6 & u_2 u_4 & u_3 u_7 \\
  u_{13} u_9 & u_{13} u_9 & u_{14} u_8 & u_{14} u_{11} & u_{14} u_{13} & u_{17} u_{13} & u_{15} u_8 \\
  u_4 u_1 & u_5 u_1 & u_6 u_1 & u_6 u_1 & u_6 u_5 & u_6 u_1 & u_7 u_1 \\
  u_{15} u_{11} & u_0 u_3 & u_9 u_{14} & u_{11} u_{17} & u_{11} u_{17} & u_{14} u_{12} & u_{19} u_{12} \\
  u_7 u_1 & u_8 u_{16} & u_8 u_{16} & u_8 u_{16} & u_9 u_{14} & u_9 u_{14} & u_9 u_{14} \\
  u_1 u_6 & u_1 u_3 & u_{10} u_{14} & u_{11} u_{17} & u_{12} u_9 & u_{14} u_1 & u_{14} u_1 \\
  u_{10} u_{14} & u_{11} u_1 & u_{11} u_{17} & u_{12} u_9 & u_{13} u_9 & u_{13} u_9 & u_{13} u_9 \\
  u_2 u_4 & u_6 u_1 & u_6 u_5 & u_{12} u_9 & u_7 u_1 & u_{13} u_9 & u_{13} u_9 \\
  u_{14} u_{13} & u_{14} u_{11} & u_{14} u_{18} & u_{14} u_{18} & u_{15} u_{11} & u_{15} u_{11} & u_{15} u_{11} \\
  u_7 u_1 & u_{14} u_{11} & u_{16} u_8 & u_{16} u_{13} & u_{14} u_{13} & u_{17} u_{13} & u_{17} u_8 \\
  u_{16} u_{15} & u_{16} u_8 & u_{16} u_{13} & u_{16} u_{13} & u_{16} u_{15} & u_{16} u_{15} & u_{16} u_{15} \\
  u_{14} u_{18} & u_5 u_1 & u_{18} u_{10} & u_{16} u_{15} & u_{16} u_{15} & u_{17} u_{13} & u_{17} u_8 \\
  u_{17} u_{13} & u_{18} u_{10} & u_{18} u_{10} & u_{18} u_{10} & u_{18} u_{10} & u_{18} u_{10} & u_{18} u_{10}
\end{pmatrix}
\]

The proof of Lemma 23 is done using Sage in Section 8.

**Proposition 24.** \(\Omega_\mathcal{U}\) is minimal. More precisely, \(\Omega_\mathcal{U} = \mathcal{X}_\omega\) where \(\omega = \alpha \beta \gamma\).

**Proof.** From Corollary 22 we have \(\overline{\omega(\Omega_\mathcal{U})} = \Omega_\mathcal{U}\). From Lemma 3 this is equivalent to \(\mathcal{L}(\Omega_\mathcal{U}) = \overline{\mathcal{L}(\Omega_\mathcal{U})}\) Fact. From Lemma 23 we have \(\mathcal{L}(\mathcal{X}_\omega) \cap \mathcal{U}^{(2, 2)} = \mathcal{L}(\Omega_\mathcal{U}) \cap \mathcal{U}^{(2, 2)}\). From Lemma 20 and Proposition 21, \(\omega\) is primitive and expansive. From Lemma 5 we conclude that \(\mathcal{L}(\mathcal{X}_\omega) = \mathcal{L}(\Omega_\mathcal{U})\), \(\mathcal{X}_\omega = \Omega_\mathcal{U}\) and \(\Omega_\mathcal{U}\) is minimal.

**Proof of Theorem 7.** It is proved that \(\mathcal{U}\) is self-similar and aperiodic in Corollary 22 and that \(\Omega_\mathcal{U}\) is minimal in Proposition 24.

We deduce also that \(\mathcal{V}\) is aperiodic from Lemma 7 since \(\alpha\) sends periodic tilings onto periodic tilings.

**Proposition 25.** The vector of frequency of the tiles \(\{u_i\}_{0 \leq i \leq 18}\) in any tiling of plane by \(\mathcal{U}\) is

\[
\frac{1}{2} \left( \frac{1}{\varphi^8}, \frac{3}{\varphi^5}, \frac{1}{\varphi^6}, \frac{1}{\varphi^{10}}, \frac{1}{\varphi^7}, \frac{1}{\varphi^8}, \frac{1}{\varphi^4}, \frac{1}{\varphi^5}, \frac{1}{\varphi^6}, \frac{1}{\varphi^5}, \frac{1}{\varphi^7}, \frac{1}{\varphi^8}, \varphi^4, \varphi^5, \varphi^4, \varphi^5, \varphi^6 \right)^t.
\]

where

\[
\frac{1}{2\varphi^4} \approx 0.0729, \quad \frac{1}{2\varphi^5} \approx 0.0451, \quad \frac{1}{2\varphi^6} \approx 0.0279, \quad \frac{1}{2\varphi^7} \approx 0.0172, \quad \frac{1}{2\varphi^8} \approx 0.0106,
\]

and

\[
\frac{2}{2\varphi^5} \approx 0.0902, \quad \frac{3}{2\varphi^5} \approx 0.1353, \quad \frac{1}{2\varphi^4} + \frac{1}{2\varphi^6} \approx 0.1008.
\]

**Proof.** The sum of the values of the right eigenvector given in Lemma 20 is \(2\varphi^8 = 42\varphi + 26\). Therefore, the frequency vector is \(\frac{1}{2\varphi^8}\) times the right eigenvector.

We remark that the morphism \(\omega\) on \(\mathcal{U}\) is not prolongable on any Wang tile but its square \(\omega^2\) is prolongable on many of them (see Figure 8).
Figure 8. The gray background color identify tiles where the morphism $\omega^2 : \Omega_U \to \Omega_U$ is prolongable on the initial tile $u_{16}$.

7. A stone inflation

In this short section, we illustrate how the substitution $\omega$ can be seen as a stone inflation \cite{BG13} p. 147–148. We consider the following correspondence between colors of the tile set $U$ and length of edges:

- $\{A, B, C, E, G, I\} \mapsto$ vertical edge of length 1
- $\{D, F, H, J\} \mapsto$ vertical edge of length $\varphi^{-1}$
- $\{K, M, N, P\} \mapsto$ horizontal edge of length 1
- $\{L, O\} \mapsto$ horizontal edge of length $\varphi^{-1}$

To each tile in $U$ we associate a rectangle $\subset \mathbb{R}^2$ according to the rule

$$\text{RECTANGLE}(u_i) = \begin{cases} [0, \varphi^{-1}] \times [0, \varphi^{-1}] & \text{if } i \in \{0, 1\}, \\ [0, 1] \times [0, \varphi^{-1}] & \text{if } i \in \{2, 3, 4, 5, 6, 7\}, \\ [0, \varphi^{-1}] \times [0, 1] & \text{if } i \in \{8, 9, 10, 11\}, \\ [0, 1] \times [0, 1] & \text{if } i \in \{12, 13, 14, 15, 16, 17, 18\}. \end{cases}$$

The relative area of each rectangle is proportional to the left eigenvector of the incidence matrix of $\omega$ (Lemma \ref{lem:eigen}). Then the substitution $\omega : \Omega_U \to \Omega_U$ defines a stone inflation on the finite set of tiles $\{\text{RECTANGLE}(u_i)\}_{0 \leq i \leq 18}$ with expansion factor $\varphi = (1 + \sqrt{5})/2$ (see Figure 9).

8. Proofs of lemmas based on Sage

This section gathers proofs of Lemma \ref{lem:wang}, Lemma \ref{lem:transducer}, Lemma \ref{lem:eigen} and Lemma \ref{lem:24} made using version 8.2 of Sage \cite{Sag18} together with the optional Sage package slabbe-0.4.2 \cite{Lab18a} which can be installed by running the command `sage -pip install slabbe`.

First we import the necessary libraries from slabbe:

```
sage: from slabbe import WangTileSet, TikzPicture
```

We create the tile set $U$:

```
sage: L = ['FOJO', 'FOHL', 'JMFP', 'DMFK', 'HPJP', 'HPHN', 'HKFP', 'HKDP', 'BOIO',
       'GLEO', 'GLCL', 'ALIO', 'EPGP', 'EPIP', 'IPGK', 'IPIK', 'IKBM', 'IKAK', 'CNIP']
sage: U = WangTileSet(L); U
Wang tile set of cardinality 19
```

This allows to create the transducer graph of $U$ shown at Figure 4.

```
sage: G = U.to_transducer_graph(); G
Looped digraph on 10 vertices
```

```python
sage: TikzPicture.from_graph(G).pdf()
```
Proof of Lemma 13. We compute the number of dominoes in the direction $e_2$ allowing a surrounding of radius 1, 2, 3 in $\Omega_\omega$. We obtain 35 dominoes in the direction $e_2$ allowing a surrounding of radius 2 and 3.

```sage
sage: [len(U.dominoes_with_surrounding(i=2,radius=r)) for r in [1,2,3]]
[37, 35, 35]
sage: U.dominoes_with_surrounding(i=2,radius=2)
{(0, 8), (1, 8), (1, 9), (1, 11), (2, 16), (3, 16), (4, 13), (5, 13), (6, 14), (6, 17), (7, 15), (8, 0), (8, 9), (8, 11), (9, 1), (9, 10), (10, 1), (11, 1), (11, 10), (12, 6), (13, 4), (13, 7), (13, 18), (14, 2), (14, 6), (15, 7), (15, 13), (15, 18), (16, 3), (16, 14), (16, 17), (17, 3), (17, 14), (18, 5)}
```

Now we create the tile set $V$:

```sage
sage: L = [('A','L','I','O'), ('B','O','I','O'), ('E','P','I','P'), ('G','L','E','O'), ('I','K','A','K'), ('I','K','B','M'), ('I','P','G','K'), ('I','P','I','K'), ('AF','O','IH','O'), ('BF','O','IJ','O'), ('CH','P','IH','P'), ('EH','K','GF','P'), ('EH','K','ID','P'), ('EH','P','IJ','P'), ('GF','O','CH','L'), ('GF','O','EH','O'), ('ID','M','AF','K'), ('ID','M','BF','M'), ('IH','K','GF','K'), ('IH','K','ID','K'), ('IJ','M','GF','K')]
sage: V = WangTileSet(L); V
Wang tile set of cardinality 21
```

Proof of Lemma 16. We compute the number of dominoes in the direction $e_1$ allowing a surrounding of radius 1, 2 in $\Omega_V$. We obtain 30 dominoes in the direction $e_1$ allowing a surrounding of radius 1 and 2.

```sage
sage: [len(V.dominoes_with_surrounding(i=1,radius=r)) for r in [1,2]]
[30, 30]
sage: V.dominoes_with_surrounding(i=1,radius=1)
{(0, 4), (1, 5), (2, 3), (3, 6), (4, 1), (4, 2), (5, 1), (5, 2), (5, 7), (6, 1), (7, 0), (7, 1), (8, 16), (9, 17), (10, 14), (11, 15), (12, 15), (13, 15), (14, 2), (14, 6), (15, 7), (15, 13), (15, 18), (16, 3), (16, 14), (16, 17), (17, 3), (17, 14), (18, 5)}
```
Proof of Lemma 20. We create the matrix $M$ in Sage. The characteristic polynomial, the eigenvalues and the eigenvectors of $M$ can be computed easily in Sage. After changing the ring of the matrix to the number field containing $\varphi$, we obtain the result stated in the proposition. First we create the substitution $\omega$:

```python
sage: from slabbe import Substitution2d
sage: da = {0: [[11]], 1: [[8]], 2: [[13]], 3: [[9]], 4: [[17]], 5: [[16]], 6: [[14]],
       ..., 7: [[18]], 8: [[11, 1]], 9: [[8, 0]], 10: [[18, 5]], 11: [[12, 6]],
       ..., 12: [[13, 7]], 13: [[13, 4]], 14: [[10, 1]], 15: [[9, 1]], 16: [[17, 3]],
       ..., 17: [[16, 3]], 18: [[14, 6]], 19: [[15, 7]], 20: [[14, 2]]
```

```python
sage: alpha = Substitution2d(da)
```

```python
sage: db = {0: [[4]], 1: [[5]], 2: [[7], [0]], 3: [[2], [3]], 4: [[4], [1]],
       ..., 5: [[6], [1]], 6: [[7], [1]], 7: [[6], [1]], 8: [[18]], 9: [[16]],
       ..., 10: [[17]], 11: [[20]], 12: [[19], [8]], 13: [[18], [8]], 14: [[12], [15]],
       ..., 15: [[11], [15]], 16: [[10], [14]], 17: [[13], [16]], 18: [[20], [9]]
```

```python
sage: beta = Substitution2d(db)
```

```python
sage: omega = alpha*beta
```

Then we create its incidence matrix, we check it is primitive, we compute it characteristic polynomial, its eigenvalues and left and right eigenvectors:

```python
sage: M = matrix(omega)
```

```python
sage: is_primitive(M)
```

```python
sage: M.charpoly().factor()
```

```python
sage: z = polygen(QQ, 'z')
```

```python
sage: K.<phi> = NumberField(z**2-z-1, 'phi', embedding=RR(1.6))
```

```python
sage: MK = M.change_ring(K)
```

```python
sage: MK.eigenvalues()
```

```python
sage: MK.eigenvectors_right()[0][1]
```

```python
sage: MK.eigenvectors_left()[0][1]
```


Proof of Lemma 23. From Corollary 22, we know that $\mathcal{L}(\Omega_{2}) = \omega(\mathcal{L}(\Omega_{2}))$. Therefore from Lemma 5 we have $\mathcal{L}_\omega \subseteq \mathcal{L}(\Omega_{2})$.

Now we show using Sage that $\mathcal{L}(\Omega_{2}) \cap U^{(2,2)} \subseteq S$. First we compute $U \boxplus_{1} U$. It contains 35 tiles (after recursively removing any source or sink state). Then we compute $(U \boxplus_{1} U) \boxplus_{2} (U \boxplus_{1} U)$ for which the resulting transducer after recursively removing any source or sink state contains 55 transitions (or tiles). Among them, 5 can not be surrounded, that is, there is no valid tiling of a $3 \times 3$ rectangle with them in the middle. For each tile $t$ of the remaining 50 tiles, we compute all valid ways of writing $t = (u \boxplus_{1} v) \boxplus_{2} (w \boxplus_{1} z)$ with $u, v, w, z \in U$. It turns out that there is a unique way in each case leading to a set $S \subseteq U^{(2,2)}$ of 50 subwords of shape $(2,2)$ satisfying (14, 11), (14, 18), (15, 18), (15, 20), (16, 12), (17, 12), (17, 19), (18, 8),

(18, 10), (19, 8), (20, 9), (20, 13)}
\( L(\Omega_U) \cap U^{(2,2)} \subseteq S \). The following takes 4s if using solver='Gurobi' \[\text{GO18}\] and 4 min if using solver='dancing_links':

```
sage: tilings = U.tiling_with_surrounding(2,2,radius=1,solver='Gurobi')
sage: len(tilings)
50
sage: S = sorted(t.table() for t in tilings)
sage: [matrix.column([col[::-1] for col in s]) for s in S]
[[ 8 16], [ 8 16], [ 9 14], [11 17], [16 8], [16 13], [16 15],
  [ 0 3], [ 1 2], [ 1 3], [ 1 6], [ 1 6], [ 2 0], [ 2 4], [ 3 7],
  [13 9], [13 9], [14 8], [14 11], [14 13], [17 8], [17 13], [15 8],
  [ 4 1], [ 5 1], [ 6 1], [ 6 1], [ 6 5], [ 6 1], [ 6 5], [ 7 1],
  [15 11], [ 0 3], [ 9 14], [11 17], [ 1 2], [ 1 6], [10 12], [ 1 6],
  [ 7 1], [15 11], [ 8 16], [ 8 16], [ 9 14], [ 9 14], [ 9 14], [10 12],
  [ 1 6], [ 1 3], [10 14], [ 6 1], [ 4 1], [ 7 1], [18 10], [ 2 0],
  [10 14], [11 17], [11 17], [12 9], [13 9], [13 9], [13 9], [14 8],
  [ 2 4], [ 6 1], [ 6 5], [12 9], [ 7 1], [13 9], [18 10], [ 3 7],
  [14 13], [14 11], [14 18], [14 8], [15 11], [15 8], [15 11], [16 13],
  [ 3 7], [14 11], [14 18], [14 13], [14 18], [17 13], [ 3 7], [14 11],
  [16 15], [16 8], [16 13], [16 15], [16 15], [16 15], [17 13], [17 8],
  [14 18], [ 5 1],
  [17 13], [18 10]]
```

Now we show using Sage that \( S \subseteq L_\omega \). We compute the set \( L_\omega \cap U^{(2,2)} \) using Sage.

```
sage: F = omega.list_2x2_factors()
sage: len(F)
50
sage: sorted(F) == S
True
```

Thus we have shown \( S \subseteq L_\omega \cap U^{(2,2)} \subseteq L(\Omega_U) \cap U^{(2,2)} \subseteq S \). \( \square \)

9. Perspectives

In this contribution, we have chosen to study the substitutive structure of \( \Omega_U \) as \( \alpha \circ \beta \circ \gamma \) where \( \alpha \) involves vertical dominoes, \( \beta \) involves horizontal dominoes and \( \gamma \) is a bijection. We could have done it the other way around as \( \alpha' \circ \beta' \circ \gamma' \) where \( \alpha' \) involves horizontal dominoes, \( \beta' \) involves vertical dominoes and \( \gamma' \) is some other bijection as shown below. It would be interesting to investigate all of these possible sequences of substitutions describing \( \Omega_U \).

\[
\begin{array}{cccccc}
\alpha & \beta & \gamma \\
\Omega_U & \Omega_V & \Omega_W & \Omega_{U'} & \Omega_{V'} & \Omega_{W'} \\
\alpha' & \beta' & \gamma'
\end{array}
\]

In \[\text{GS87}, \text{p. 595}\], an aperiodic set of 16 Wang tiles is deduced from the Ammann tiling \( A2 \) \[\text{AGS92, Aki12}\] without much details. A substitution showing its self-similar structure is also given with a structure very close to the 2-dimensional morphism \( \omega : \Omega_U \to \Omega_U \). Is there a link between
Ammann tilings $A2$ and $\Omega_U$? Can we find a concave hexagonal shape with decorations related to $\Omega_U$? Can we factor the substitution as product of morphisms sending letters to letters or dominoes? The previous question is related to the notion of elementary substitutions in D0L systems.

Many one-dimensional properties of $\Omega_U$ can be extracted from the morphism $\omega$. For example, considering the tiles $u_1$, $u_5$ and $u_6$, one can see that the effect of $\omega^2$ on the indices of tiles is

$$1 \mapsto \begin{pmatrix} 5 & 1 \\ 18 & 10 \end{pmatrix}, 5 \mapsto \begin{pmatrix} 5 & 1 & 6 \\ 18 & 10 & 14 \end{pmatrix}, 6 \mapsto \begin{pmatrix} 6 & 1 & 6 \\ 12 & 9 & 14 \end{pmatrix}$$

which contains the one-dimensional morphism: $\mu : 1 \mapsto 51, 5 \mapsto 516, 6 \mapsto 616$. This can be used to show that there exist tilings in $\Omega_U$ that contains rows using only the tiles in the set $\{u_1, u_5, u_6\}$.

As we have seen $\omega^2$ is prolongable on many letters thus it admits many fixed point $x \in \Omega_U \subset \mathbb{U}^\mathbb{Z}$. Such fixed point were called shape-symmetric $d$-dimensional word by Arnaud Maes when the image of every letter on the diagonal are squares. In [CKR10] (see also [Cha09, Chapter 4]), it is proved that a multidimensional infinite word $x : \mathbb{N}^d \to \mathcal{A}$ over a finite alphabet $\mathcal{A}$ is the image by a coding of a shape-symmetric infinite word if and only if $x$ is $S$-automatic for some abstract numeration system $S$ built on a regular language containing the empty word. Recall that an infinite word is $S$-automatic if, for all $n \geq 0$, its $(n+1)$-th letter is the output of a deterministic automaton fed with the representation of $n$ in the numeration system $S$. Can we find the numeration system $S$ and the deterministic automaton associated to $\Omega_U$?

As noticed by the anonymous referee, the factor complexity of the language of $\Omega_U$ is quadratic, the first values for the number of distinct 2-dimensional word of shape $(n,n)$ for $n \geq 1$ being: 19, 50, 94, 154, 229, 317, 420. Tilings generated by expansive and primitive are linearly repetitive (which means that there is some $C > 0$ for which every patch of radius $r$ is found somewhere in every ball of radius $Cr$) and one can show that a linearly repetitive tiling of $\mathbb{R}^d$ must have factor complexity bounded by a constant times $r^d$.

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