Tame semicascades and cascades generated by affine self-mappings of the $d$-torus

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Abstract. We give a complete characterization of the affine self-mappings $\varphi$ of the torus $\mathbb{T}^d$ that generate Köhler-tame semicascades and cascades. Namely, we show that the semicascade generated by $\varphi$ is tame if and only if the matrix $A$ of $\varphi$ satisfies $A^p = A^q$, where $p$ and $q$ are some nonnegative integers, $p \neq q$. For cascades the corresponding condition has the form $A^m = I$, where $m$ is some positive integer and $I$ is the identity matrix.

Key words: affine endomorphisms of the torus, semicascades, cascades, tame dynamical systems.

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1. Introduction and statement of results. A sequence $f_n$, $n = 1, 2, \ldots$, of real-valued functions on a set $X$ is said to be independent if there exist $a, b \in \mathbb{R}$, $a < b$, such that

$$\bigcap_{n \in P} \{x : f_n(x) < a\} \cap \bigcap_{n \in Q} \{x : f_n(x) > b\} \neq \emptyset$$

for all finite disjoint subsets $P, Q$ of indices.

Let $X$ be a compact Hausdorff space and let $G$ be a certain group or a semigroup whose elements are continuous mappings of $X$ into itself. The dynamical system $(X, G)$ is called tame if for every real-valued continuous function $f$ on $X$ the family $\{f_g : g \in G\}$ does not contain an independent sequence [2, Def. 3.1]. Here $f_g$ stands for the function $f_g(x) = f(gx)$, $x \in X$.

This definition of a tame (originally called regular) dynamical system goes back to Köhler, who considered semicascades and obtained first results on tameness [3, Sec. 5]. The study continued by Glasner, Megrelishvili, and other authors, see the recent survey [2] and the references therein, showed that the tame/untame dichotomy is closely connected with such properties of a system as minimality, distality, nonsensitivity, almost periodicity, chaotic behaviour, certain ergodic properties, and the structure of the Ellis enveloping semigroup. We note that it is usually difficult to decide whether a given system is tame.
Several conditions are known to be equivalent to tameness. In the important case when $X$ is a compact metric space the above K"{o}hler definition of a tame system reduces to a very simple equivalent definition below which reflects the structure of orbits $O(x) = \{gx : g \in G\}$, $x \in X$, of a system, and is especially meaningful for semicascades and cascades. (We provide a short explanation of the equivalence in the concluding Section 3, see Remark 1.)

**Definition** (an equivalent version in the metric case). Let $X$ be a compact metric space. We say that a system $(X, G)$ is tame if each sequence $g_n \in G, n = 1, 2, \ldots$ has a pointwise convergent subsequence $g_{n_k}, k = 1, 2, \ldots$ (i.e., a subsequence such that for every $x \in X$ the sequence $g_n x, g_{n_1} x, \ldots$ converges in $X$).

Let $\varphi : X \to X$ be a continuous mapping. We recall that the semicascade generated by $\varphi$ is the system $(X, G^+_\varphi)$, where $G^+_\varphi = \{\varphi^n, n = 0, 1, 2, \ldots\}$ is the semigroup of iterations of $\varphi$. Here $\varphi^0$ is the identity mapping and $\varphi^{n+1} = \varphi \circ \varphi^n, n = 0, 1, 2, \ldots$ When $\varphi$ is a self-homeomorphism of $X$ we can consider the cascade generated by $\varphi$, i.e., the system $(X, G_\varphi)$, where $G_\varphi = \{\varphi^n, n \in \mathbb{Z}\}$ is the group of iterations of $\varphi$ (naturally $\varphi^n$ for $n < 0$ are the iterations of the inverse mapping $\varphi^{-1}$).

In this note we consider semicascades and cascades generated by affine self-mappings $\varphi$ of the torus $\mathbb{T}^d = \mathbb{R}^d/(2\pi\mathbb{Z})^d$ (as usual $\mathbb{R}$ denotes the real line and $\mathbb{Z}$ is its additive subgroup of integers.). Given a $\varphi$, we have $\varphi(x) = Ax + b$, where $A$ is an integer $d \times d$ matrix and $b \in \mathbb{T}^d$. Certainly one can consider a cascade generated by $\varphi$ only in the case when $A$ is nonsingular and its inverse $A^{-1}$ is integer, i.e., when $\det A = \pm 1$.

How to distinguish if the semicascade (cascade) generated by an affine self-mapping of the torus is tame? This question was suggested by A. V. Romanov (private communication). The results of this work are the following two theorems.

**Theorem 1.** The semicascade generated by an affine self-mapping $\varphi$ of $\mathbb{T}^d$ is tame if and only if the matrix $A$ of $\varphi$ satisfies the condition $A^p = A^q$, where $p$ and $q$ are some nonnegative integers with $p \neq q$.

**Theorem 2.** The cascade generated by an affine self-mapping $\varphi$ of $\mathbb{T}^d$ is tame if and only if the matrix $A$ of $\varphi$ satisfies the condition $A^m = I$, 

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where \( m \) is some positive integer and \( I \) is the identity matrix.

The proof is given in the next section. It is based on elementary Fourier analysis argument.

The results of this work were announced at the conference “Topology, Geometry, and Dynamics: Rokhlin – 100” [4].

2. Proof of the theorems. It is convenient to consider the general case of families (not necessarily semigroups or groups) of affine self-mappings of the torus. Theorems 1 and 2 immediately follow from the lemma below.

**Lemma.** Let \( \Phi \) be a family of affine self-mappings of \( \mathbb{T}^d \). Let \( M(\Phi) \) be the family of all matrices of the mappings in \( \Phi \). The following conditions are equivalent:

(i) each sequence in \( \Phi \) has a pointwise convergent subsequence;

(ii) the family \( M(\Phi) \) is finite.

**Proof of the Lemma.** The part (ii) \( \Rightarrow \) (i) is trivial. Let us show that (i) \( \Rightarrow \) (ii). Given a Lebesgue integrable function \( f \) on \( \mathbb{T}^d \) its Fourier transform \( \hat{f} \) is defined by

\[
\hat{f}(\lambda) = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} f(x) e^{-i(\lambda, x)} dx, \quad \lambda \in \mathbb{Z}^d,
\]

where \( (\lambda, x) \) is the usual inner product of vectors \( \lambda \in \mathbb{Z}^d \) and \( x \in \mathbb{T}^d \), i.e., \( (\lambda, x) = \sum_{j=1}^d \lambda_j x_j \) for \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_d) \), \( x = (x^1, x^2, \ldots, x^d) \).

For a vector \( \lambda \in \mathbb{Z}^d \) let \( e_\lambda \) stands for the exponential function with frequency \( \lambda \), i.e., \( e_\lambda(x) = e^{i(\lambda, x)} \), \( x \in \mathbb{T}^d \). Note, that if \( \gamma_1, \gamma_2, \ldots \) is a sequence of real numbers, and \( \lambda_1, \lambda_2, \ldots \) is an unbounded sequence of vectors in \( \mathbb{Z}^d \), then the sequence \( e^{i\gamma_1} e_{\lambda_1}, e^{i\gamma_2} e_{\lambda_2}, \ldots \) is not pointwise convergent on \( \mathbb{T}^d \). Indeed, assuming the contrary, let \( \lim_{n \to \infty} e^{i\gamma_n} e^{i(\lambda_n, x)} = \xi(x) \), \( x \in \mathbb{T}^d \). Using dominated convergence theorem we see that the function \( \xi \) is integrable on \( \mathbb{T}^d \) and, since \( \xi(x) e^{-i\gamma_n} e^{-i(\lambda_n, x)} \to 1 \) for all \( x \in \mathbb{T}^d \), we obtain

\[
\frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} \xi(x) e^{-i\gamma_n} e^{-i(\lambda_n, x)} dx \to 1, \quad \text{as } n \to \infty,
\]

so, \( e^{-i\gamma_n} \xi(\lambda_n) \to 1 \), which is impossible since the Fourier coefficients of every integrable function vanish at infinity.
Given a matrix $A$, denote its transpose by $A^*$. Consider an arbitrary vector $u \in \mathbb{Z}^d$ and the set $E(u) = \{e_u \circ \varphi : \varphi \in \Phi\}$. Certainly, (i) implies that each sequence in $E(u)$ has a pointwise convergent subsequence. At the same time for $\varphi(x) = Ax + b$ we have $e_u \circ \varphi(x) = e^{i(u, Ax+b)} = e^{i(u,b)}e^{i(u, Ax)} = e^{i(u,b)}e^{i(A^* u, x)} = e^{i(u,b)}e^{i(A^* u, x)}$. Thus we see that from (i) it follows that the set $\Lambda(u) = \{A^* u : A \in M(\Phi)\}$, which is the set of frequencies of functions in $E(u)$, is finite.

Consider the vectors $u_j = (0,0,\ldots,0,1,0,\ldots,0) \in \mathbb{Z}^d$, where $j = 1, 2, \ldots, d$. According to what we have shown, for every $j$ the set $\Lambda(u_j) = \{A^* u_j : A \in M(\Phi)\}$ is finite. Since $A^* u_j$ is the $j$th column of the matrix $A^*$, i.e., the $j$th row of the matrix $A$, we see that for every $j$, $j = 1, 2, \ldots, d$, the set of the $j$th rows of the matrices $A \in M(\Phi)$ is finite. Hence, $M(\Phi)$ is finite. The Lemma is proved. Theorems 1 and 2 follow.

3. Remarks. 1. Given a Hausdorff compact space $X$ let $C(X)$ be the Banach space of all complex-valued continuous functions on $X$ (with the usual sup-norm). Let $F$ be a bounded set of real-valued functions in $C(X)$. It is known that the following two conditions are equivalent:

(a) $F$ does not contain an independent sequence;
(b) each sequence in $F$ has a pointwise convergent subsequence.

The equivalence of these conditions as well as their equivalence to certain other ones originates from the famous Rosenthal’s $l^1$ theorem [5], [6]. For a thorough discussion see [1] (especially [1, Theorem 3.11]). This equivalence plays a significant role in investigations related to tameness, see [2, Sec. 2 and 3]. Applied to the families $\{f_g : g \in G\}$, where $f \in C(X)$ are real functions, it yields the corresponding equivalent properties of $(X, G)$.

Consider a family $\Phi$ of self-mappings of $X$. Certainly if each sequence in $\Phi$ has a pointwise convergent subsequence, then for every real-valued $f \in C(X)$ each sequence in $\{f \circ \varphi : \varphi \in \Phi\}$ has a pointwise convergent subsequence. One can easily verify that in the metric case the converse is also true. Indeed, if $X$ is a compact metric space, then $C(X)$ is separable, so, we can chose a countable family of real functions $f_1, f_2, \ldots$ in $C(X)$ that distinguishes between the points of $X$, i.e., such that for every $u, v \in X$, $u \neq v$, there exists $j$ with $f_j(u) \neq f_j(v)$. Let $\{\varphi_n\}$ be a sequence in $\Phi$. We apply the diagonal process, namely, starting from the sequence $\{\varphi_n\} \overset{def}{=} \{\varphi_n\}$ we construct inductively a family of sequences $\{\varphi_n^0, n =
1, 2, . . . }, j = 1, 2, . . . , so that for each j the sequence \( \{ \varphi^j_n, n = 1, 2, . . . \} \)

is a subsequence of \( \{ \varphi^{j-1}_n, n = 1, 2, . . . \} \) and the sequence \( \{ f_j \circ \varphi^j_n, n = 1, 2, . . . \} \) is pointwise convergent. Clearly \( \{ \varphi^{(n)}_n, n = 1, 2, . . . \} \) is a pointwise convergent subsequence of \( \{ \varphi_n \} \).

This argument, combined with the equivalence of (a) and (b), shows that in the metric case the definition of tameness given in terms of pointwise convergent sequences in \( G \) is equivalent to the original one given in terms of independent sequences (see Introduction).

2. In relation with Theorems 1 and 2 let us indicate two conditions in terms of the Jordan form \( A_J \) of a matrix \( A \) equivalent to the \( A^p = A^q \) condition. Clearly, if \( A^p = A^q \), where \( p, q \) are certain nonnegative integers, \( p \neq q \), then there exists an integer \( s > 0 \) such that each eigenvalue of \( A \) is either 0 or is an \( s \)th root of unity and all Jordan blocks which correspond to nonzero eigenvalues are degenerate, i.e., of size \( 1 \times 1 \). This in turn implies that there exists a positive integer \( m \) such that \( (A_J)^m \) is a diagonal matrix with only ones and zeroes on the diagonal. Obviously the converse is also true: every matrix \( A \) whose Jordan form satisfies one of the two conditions above satisfies the \( A^p = A^q \) condition.

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