Hybrid fault diagnosis capability analysis of highly connected graphs

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Abstract Zhu et al. [Theoret. Comput. Sci. 758 (2019) 1–8] introduced the $h$-edge tolerable diagnosability to measure the fault diagnosis capability of a multiprocessor system with faulty links. This kind of diagnosability is a generalization of the concept of traditional diagnosability. A graph is called a maximal connected graph if its minimum degree equals its vertex connectivity. It is well-known that many irregular networks are maximal connected graphs and the $h$-edge tolerable diagnosabilities of these networks are unknown, which is our motivation for research. In this paper, we obtain the lower bound of the $h$-edge tolerable diagnosability of a $t$-connected graph and establish the $h$-edge tolerable diagnosability of a maximal connected graph under the PMC model and the MM$^*$ model, which extends some results in [IEEE Trans. Comput. 23 (1974) 86–88], [IEEE Trans. Comput. 53 (2004) 1582–1590] and [Theoret. Comput. Sci. 796 (2019) 147–153]. As applications, the $h$-edge tolerable diagnosability of an exchanged hypercube is determined under the PMC model and the MM$^*$ model.

Keywords Maximal connected graph; Exchanged hypercube; Fault diagnosability; PMC model; MM$^*$ model

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1 Introduction

Processor failure has become an ineluctable event in a large-scale multiprocessor system. To keep the multiprocessor system performing its functions efficiently and economically, recognizing faulty processors correctly is a task of top priority. The process of recognizing faulty processors in a multiprocessor system is called *fault diagnosis*, and the *diagnosability* of a system is the maximum number of faulty processors the system can recognize. The PMC model and the MM model are two major models to investigate fault diagnosis in previous researches. The PMC model, proposed by Preparata, Metze and Chien [10], assumes that all adjacent processors of a system can test one another. The MM model which is the development of the MM model [9], proposed by Sengupta and Dahbura [12], assumes that each processor has to test two processors if the processor is adjacent to the latter two processors. Some references related to fault diagnosis studies under the PMC model or MM model can be seen in [2, 5, 6, 13–23].

In the real situation, both node and link faults can appear in a system. However, the traditional diagnosability for a multiprocessor system assumes that the system is without link faults. On the other hand, it is a natural question to ask how the diagnosability decreases if some links are missing for a multiprocessor system [13]. To address the deficiency of the traditional diagnosability for a multiprocessor system and answer the above question, the concept of the *h*-edge tolerable diagnosability $t_h^e(G)$, introduced by Zhu et al. [23], generalizes the theories of diagnosability and can better measure the diagnosis capability of a multiprocessor system $G$. In fact, this diagnosability is the worst-case diagnosability when the number of faulty links of $G$ does not exceed $h$. Briefly, $t_h^e(G)$ is the minimum diagnosability of graphs $G - F_e$ which satisfy that $F_e \subseteq E(G)$ and $|F_e| \leq h$. Note that if a processor $u$ has no fault-free neighbors, it is impossible to determine whether $u$ is faulty or not in the fault diagnosis. Then $t_h^e(G) = 0$ for $h \geq \delta(G)$, where $\delta(G)$ is the minimum degree of a graph $G$. Hence, a key issue for the
$h$-edge tolerable diagnosability of a graph $G$ study is the case of $0 \leq h \leq \delta(G)$.

In 2019, Zhu et al. [23] determined the $h$-edge tolerable diagnosabilities of hypercubes under the PMC model and the MM$^*$ model. Wei and Xu [17, 20] established the $h$-edge tolerable diagnosabilities of $k$-regular triangle-free graphs and balanced hypercubes under the PMC model and the MM$^*$ model. Recently, Lian et al. [6] established the $h$-edge tolerable diagnosability of a $k$-regular $k$-connected graph under the PMC model and the MM$^*$ model. Zhang et al. [22] determined the $h$-edge tolerable diagnosabilities of triangle-free graphs under the PMC model and the MM$^*$ model, which extends the results of triangle-free regular graphs [20]. Zhang et al. [21] determined the $h$-edge tolerable diagnosabilities of $k$-regular 2-cn graphs under the PMC model for $h \leq k - 5$.

A graph is called a maximal connected graph if its minimum degree equals its vertex connectivity. In this paper, we obtain the lower bound of the $h$-edge tolerable diagnosability of a $t$-connected graph and establish the $h$-edge tolerable diagnosability of a maximal connected graph under the PMC model and the MM$^*$ model, which provides a more precise characterization for the fault diagnosis capability of networks and generalizes some results in [2, 4, 6]. As applications, the $h$-edge tolerable diagnosability of an exchanged hypercube is determined under the PMC model and the MM$^*$ model.

The remainder of this paper is organized as follows. Some terminology and preliminaries are introduced in Section 2. The main results are given in Sections 3 and 4. The $h$-edge tolerable diagnosability of an exchanged hypercube is determined in Section 5. Finally, we concludes the paper in Section 6.

2 Terminology and preliminaries

A graph $G = (V(G), E(G))$ is used to represent a system (or a network), where each vertex of $G$ represents a processor and each edge of $G$ represents a link. The connectivity $\kappa(G)$ is the minimum cardinality of all vertex subsets $S \subseteq V(G)$ satisfying that $G - S$
A graph $G$ is said to be $t$-connected, if $\kappa(G) \geq t$. The neighborhood $N_G(v)$ of a vertex $v$ in $G$ is the set of vertices adjacent to $v$. We call $\min_{v \in V(G)} \{|N_G(v)|\}$ the minimum degree of a graph $G$, denoted by $\delta(G)$. A graph $G$ is said to be $t$-regular (or regular), if $|N_G(v)| = t$ for any vertex $v$ of $G$. We refer readers to [1] for terminology and notation unless stated otherwise.

The concept of the traditional diagnosability of a graph is presented as follows.

**Definition 2.1** ([3]) A graph $G = (V,E)$ of $n$ vertices is $t$-diagnosable if all faulty vertices can be detected without replacement, provided that the number of faults does not exceed $t$. The diagnosability $t(G)$ of a graph $G$ is the maximum value of $t$ such that $G$ is $t$-diagnosable.

For any two sets $A$ and $B$, we use $A - B$ to denote a set obtained by removing all elements of $B$ from $A$. The symmetric difference of two sets $F_1$ and $F_2$ is defined as the set $F_1 \triangle F_2 = (F_1 - F_2) \cup (F_2 - F_1)$. The following lemmas give necessary and sufficient conditions for a graph to be $t$-diagnosable under the PMC model and the MM* model.

**Lemma 2.2** ([3]) A graph $G = (V,E)$ is $t$-diagnosable under the PMC model if and only if for any two distinct subsets $F_1$ and $F_2$ of $V$ with $|F_1| \leq t$ and $|F_2| \leq t$, there exists an edge from $V - (F_1 \cup F_2)$ to $F_1 \triangle F_2$ (see Figure 1).

![Figure 1: The illustration of Lemma 2.2.](image)

**Lemma 2.3** ([12]) A graph $G = (V,E)$ is $t$-diagnosable under the MM* model if and only if for any two distinct subsets $F_1$ and $F_2$ of $V$ with $|F_1| \leq t$ and $|F_2| \leq t$, at least one of the following conditions is satisfied (see Figure 2):
(1) There are two vertices \( u, w \in V - (F_1 \cup F_2) \) and there is a vertex \( v \in F_1 \triangle F_2 \) such that \( uv \in E \) and \( uw \in E \).

(2) There are two vertices \( u, v \in F_1 - F_2 \) and there is a vertex \( w \in V - (F_1 \cup F_2) \) such that \( uw \in E \) and \( vw \in E \).

(3) There are two vertices \( u, v \in F_2 - F_1 \) and there is a vertex \( w \in V - (F_1 \cup F_2) \) such that \( uw \in E \) and \( vw \in E \).

![Figure 2: The illustration of Lemma 2.3.](image)

We call sets \( F_1 \) and \( F_2 \) distinguishable under the PMC (resp. MM*) model if they satisfy the condition of Lemma 2.2 (resp. at least one of the conditions of Lemma 2.3). Otherwise, \( F_1 \) and \( F_2 \) are said to be indistinguishable.

To better adapt to the real circumstances that link faults may happen [23], Zhu et al. introduced the definition of the \( h \)-edge tolerable diagnosability of graphs as follows.

**Definition 2.4** Given a diagnosis model and a graph \( G \), \( G \) is \( h \)-edge tolerable \( t \)-diagnosable under the diagnosis model if for any edge subset \( F_e \) of \( G \) with \( |F_e| \leq h \), the graph \( G - F_e \) is \( t \)-diagnosable under the diagnosis model. The \( h \)-edge tolerable diagnosability of \( G \), denoted as \( t^h_t(G) \), is the maximum integer \( t \) such that \( G \) is \( h \)-edge tolerable \( t \)-diagnosable.

Obviously, for a graph \( G \), the \( h \)-edge tolerable diagnosability is the traditional diagnosability when \( h = 0 \). We have \( t^0_t(G) = t(G) \).
A family of paths in $G$ is said to be internally-disjoint if no vertex of $G$ is an internal vertex of more than one path of the family. The following lemmas are important to the proof of our main results.

**Lemma 2.5 (Whitney(1932) [1])** A graph $G$ with at least three vertices is 2-connected if and only if any two vertices of $G$ are connected by at least two internally-disjoint paths.

**Lemma 2.6 ([6])** Let $G = (V, E)$ be a connected graph and $S \subseteq E$. If $|S| \leq \kappa(G)$, then $\kappa(G - S) \geq \kappa(G) - |S|$.

**Lemma 2.7 ([22])** Let $G = (V, E)$ be a connected graph with minimum degree $\delta(G)$. Then $t^c_h(G) \leq \delta(G) - h$ under the PMC model and the MM$^*$ model for $0 \leq h \leq \delta(G)$. Particularly, $t^c_{\delta(G)}(G) = 0$ under both the PMC model and the MM$^*$ model.

### 3 Hybrid fault diagnosis capability analysis of highly connected graphs under the PMC model

In this section, we will discuss the $h$-edge tolerable diagnosability of a highly connected graph under the PMC model.

First, we give a lower bound of the $h$-edge tolerable diagnosability of a $t$-connected graph under the PMC model.

**Theorem 3.1** Let $G = (V, E)$ be a $t$-connected graph with $|V| \geq 2(t - h) + 1$. Then $t^c_h(G) \geq t - h$ under the PMC model for $0 \leq h \leq t$.

**Proof.** If $h = t$, then $t^c_h(G) \geq 0 = t - h$ holds obviously.

Now, we assume that $h \leq t - 1$. For an arbitrary edge subset $F_e \subseteq E$ with $|F_e| \leq h$, suppose that there exist two distinct vertex subsets $F_1, F_2 \subseteq V$ such that $F_1$ and $F_2$ are indistinguishable in $G - F_e$ under the PMC model. We will prove this theorem by showing that $|F_1| \geq t - h + 1$ or $|F_2| \geq t - h + 1$ for $0 \leq h \leq t - 1$. 
If $|F_1 \cap F_2| \geq t - h$, then $|F_1| \geq t - h + 1$ or $|F_2| \geq t - h + 1$.

Suppose that $|F_1 \cap F_2| \leq t - h - 1$. Then by Lemma 2.6,

$$\kappa(G - F_e) \geq \kappa(G) - |F_e| \geq t - h > t - h - 1 \geq |F_1 \cap F_2|.$$  

Therefore, $G - F_e - (F_1 \cap F_2)$ is connected.

If $V = F_1 \cup F_2$, then $|F_1 \cup F_2| = |V| \geq 2(t - h) + 1$. Thus, $|F_1| \geq t - h + 1$ or $|F_2| \geq t - h + 1$. Otherwise, $V - (F_1 \cup F_2) \neq \emptyset$. Since $G - F_e - (F_1 \cap F_2)$ is connected, there is an edge between $F_1 \triangle F_2$ and $V - (F_1 \cup F_2)$ in $G - F_e$, a contradiction by Lemma 2.2.

Let $h = 0$. By Theorem 3.1, we can obtain the following result.

**Corollary 3.2** ([4]) Let $G = (V, E)$ be a $t$-connected network with $N$ nodes and $t \geq 2$. $G$ is $t$-diagnosable under the PMC model if $N \geq 2t + 1$.

Note that a maximal connected graph $G$ is $\delta(G)$-connected. By Lemma 2.7 and Theorem 3.1, we obtain the following result.

**Theorem 3.3** Let $G = (V, E)$ be a maximal connected graph with $|V| \geq 2(\delta(G) - h) + 1$. Then $t^*_h(G) = \delta(G) - h$ under the PMC model for $0 \leq h \leq \delta(G)$.

Note that a $k$-regular $k$-connected graph is a maximal connected graph. By Theorem 3.3, we immediately obtain the following result.

**Corollary 3.4** ([6]) Let $G = (V, E)$ be a $k$-regular $k$-connected graph with $|V| \geq 2(k - h) + 1$. Then $t^*_h(G) = k - h$ under the PMC model for $0 \leq h \leq k$.

**4 Hybrid fault diagnosis capability analysis of highly connected graphs under the MM$^*$ model**

In this section, we will discuss the $h$-edge tolerable diagnosability of a highly connected graph under the MM$^*$ model.
In the following statements, we use \( \overline{G} \) to denote the complement graph of a simple graph \( G \), whose vertex set is \( V(G) \) and whose edges are the pairs of nonadjacent vertices of \( G \). By starting with a disjoint union of two graphs \( G \) and \( H \) (i.e., \( G \cup H \)) and adding edges joining every vertex of \( G \) to every vertex of \( H \), one obtains the join of \( G \) and \( H \), denoted by \( G \vee H \) [1]. For two graphs \( G \) and \( H \), we use \( G \ast_r H \) to denote a graph with the vertex set \( V(G) \cup V(H) \) and the edge set \( E(G) \cup E(H) \cup E[\delta(G), \delta(H)] \), where \( E[\delta(G), \delta(H)] \) is an edge subset and any edge in \( E[\delta(G), \delta(H)] \) has one endpoint in \( V(G) \) and the other endpoint in \( V(H) \). Clearly, \( 0 \leq |E[\delta(G), \delta(H)]| \leq |V(G)| \cdot |V(H)| \). We also use \( G \ast_1 H \) to denote a graph with the vertex set \( V(G) \cup V(H) \) and the edge set \( E(G) \cup E(H) \cup E[\delta(G), \delta(H)] \) such that any vertex of \( G \) is adjacent to only one vertex of \( H \). A spanning subgraph of a complete graph \( K_m \) is denoted by \( H_m \) (or \( H'_m \)). Given a graph \( G \) with \( \delta(G) \geq 3 \), let a collection of graphs

\[
\mathcal{F}(\delta(G)) = \{ \Gamma_i(\delta(G), l) \mid l \geq \delta(G) + 1, 1 \leq i \leq 4 \} \cup \{ \Gamma_5(\delta(G), l) \mid l \geq \delta(G) + 2 \},
\]

where \( \Gamma_1(\delta(G), l) = H_{\delta(G)} \vee \overline{K}_l \), \( \Gamma_2(\delta(G), l) = (H_{\delta(G)-1} \ast_1 K_l) \cup (\overline{K}_{l} \cup (K_{l} \ast_1 K_{l})) \), \( \Gamma_3(\delta(G), l) = (\overline{K}_{l} \ast_1 H_{2}) \cup (H_{\delta(G)-2} \vee \overline{K}_l) \cup (\overline{K}_{l} \ast_1 K_{l}) \cup (K_{l} \ast_1 H_{2}'), \) \( \Gamma_4(\delta(G), l) = (H_{\delta(G)} \vee \overline{K}_l) + e - E_0 \) with \( e = uv, u, v \in V(\overline{K}_l), E_0 \subseteq E[V(H_{\delta(G)}), \{u, v\}] \cap E[H_{\delta(G)}] \vee \overline{K}_l] \) and \( 0 \leq |E_0| \leq 2 \), \( \Gamma_5(\delta(G), l) = (V(H_{\delta(G)+1}) \cup V(\overline{K}_l), E[H_{\delta(G)+1}] \cup \{uv \mid u \in V(\overline{K}_l), v \in V(H_{\delta(G)+1}), \delta(G) \leq |E[\{u\}, V(H_{\delta(G)+1})]| \leq \delta(G) + 1}) \) (see Figure 3).

Let \( C(G) \) be the maximum number of common neighbors of any two vertices in the graph \( G \). We give some properties of graphs in \( \mathcal{F}(\delta(G)) \) as follows.

**Lemma 4.1** Given a graph \( G \) with \( \delta(G) \geq 3 \), the graphs in \( \mathcal{F}(\delta(G)) \) satisfy the following properties:

1. The graphs in \( \mathcal{F}(\delta(G)) \) are all irregular;

2. \( \min\{C(F) \mid F \in \mathcal{F}(\delta(G))\} \geq \delta(G) - 1. \)
Proof. (1) Suppose \( x \in V(H_{\delta(G)-i+1}) \) and \( y \in V(K_l) \) for \( 1 \leq i \leq 3 \). Since \( |N_{\Gamma_i(\delta(G),l)}(x)| \geq l \geq \delta(G) + 1 > \delta(G) = |N_{\Gamma_i(\delta(G),l)}(y)| \), we know the graph \( \Gamma_i(\delta(G),l) \) is irregular for \( 1 \leq i \leq 3 \).

Since \( l > \delta(G) \geq 3 > 2 > |E_0| \), there exist two vertices \( x \in V(H_{\delta(G)}) \) and \( y \in V(K_l) \) such that \( V(K_l) \subseteq N_{\Gamma_4(\delta(G),l)}(x) \). Note that \( |N_{\Gamma_4(\delta(G),l)}(x)| \geq l \geq \delta(G) + 1 > \delta(G) = |N_{\Gamma_4(\delta(G),l)}(y)| \). We know that the graph \( \Gamma_4(\delta(G),l) \) is irregular.

Assume to the contrary that \( \Gamma_5(\delta(G),l) \) is a regular graph. Note that there exists a vertex \( x \in V(H_{\delta(G)+1}) \) such that

\[
|N_{\Gamma_5(\delta(G),l)}(x)| \geq \frac{|E[V(H_{\delta(G)+1}), V(K_l)]|}{|V(H_{\delta(G)+1})|} \geq \frac{\delta(G)|V(K_l)|}{|V(H_{\delta(G)+1})|} \geq \frac{\delta(G)(\delta(G) + 2)}{\delta(G) + 1} > \delta(G).
\]

If \( |N_{\Gamma_5(\delta(G),l)}(x)| = \delta(G) + 1 \), then \( |N_{\Gamma_5(\delta(G),l)}(y)| = \delta(G) + 1 \) for any \( y \in V(K_l) \). Thus there exists a vertex \( z \in V(H_{\delta(G)+1}) \) such that

\[
|N_{\Gamma_5(\delta(G),l)}(z)| \geq \frac{|E[V(H_{\delta(G)+1}), V(K_l)]|}{|V(H_{\delta(G)+1})|} = \frac{(\delta(G) + 1)|V(K_l)|}{\delta(G) + 1} = l \geq \delta(G) + 2,
\]

which contradicts that \( \Gamma_5(\delta(G),l) \) is a regular graph. If \( |N_{\Gamma_5(\delta(G),l)}(x)| \geq \delta(G) + 2 \), then \( |N_{\Gamma_5(\delta(G),l)}(y)| \geq \delta(G) + 2 \) for any \( y \in V(K_l) \), which contradicts \( N_{\Gamma_5(\delta(G),l)}(y) \subseteq V(H_{\delta(G)+1}) \).
(2) Suppose \( x, y \in V(H_{\delta(G)}) \) (see Figure 3(a)). Since \( |N_{1}(\delta(G), l)(x) \cap N_{1}(\delta(G), l)(y)| \geq \delta(G) + 1 \), we have \( C(\Gamma_{1}(\delta(G), l)) \geq \delta(G) + 1 \).

Note that \( |N_{1}(\delta(G), l)(z)| = \delta(G) \) for any \( z \in V(K_{l}) \) and \( i \in \{2, 3\} \) (see Figure 3(b) and Figure 3(c)). Since \( l \geq \delta(G) + 1 > 2 \), there exist two distinct vertices \( x, y \in V(K_{l}) \) such that \( |N_{1}(\delta(G), l)(x) \cap N_{1}(\delta(G), l)(y)| \geq \delta(G) + i \). Thus, we have \( C(\Gamma_{1}(\delta(G), l)) \geq \delta(G) - i + 2 \), where \( i \in \{2, 3\} \).

Since \( |V(H_{\delta(G)})| = \delta(G) \geq 3 > 2 \geq |E_{0}| \), there exist \( x, y \in V(H_{\delta(G)}) \) such that \( V(K_{l}) \subseteq N_{1}(\delta(G), l)(x) \) and \( |N_{1}(\delta(G), l)(y) \cap V(K_{l})| \geq l - 1 \geq \delta(G) \) (see Figure 3(d)). Then \( C(\Gamma_{1}(\delta(G), l)) \geq \delta(G) \).

Note that \( |N_{2}(\delta(G), l)(z)| \geq \delta(G) \) and \( N_{2}(\delta(G), l)(z) \subseteq V(H_{\delta(G)+1}) \) for any \( z \in V(K_{l}) \) (see Figure 3(e)). Since \( l \geq \delta(G) + 2 > |V(H_{\delta(G)+1})| \), there exist two distinct vertices \( x, y \in V(K_{l}) \) such that \( |N_{2}(\delta(G), l)(x) \cap N_{2}(\delta(G), l)(y)| \geq \delta(G) \). Thus, we have \( C(\Gamma_{2}(\delta(G), l)) \geq \delta(G) \).

As mentioned above, we get the desired results. \( \square \)

Now, we give a lower bound of the \( h \)-edge tolerable diagnosability of a \( t \)-connected graph \( G \) under the MM* model. For a given vertex \( x \in V(G) \), we use \( E(x) \) to denote the edges incident with \( x \) in \( G \).

**Theorem 4.2** Let \( G = (V, E) \) be a \( t \)-connected graph with \( |V| \geq 2(t - h) + 3 \) and \( \delta(G) \geq 3 \). If \( G \notin \mathcal{F}(\delta(G)) \), then \( t_{h}^{\ast}(G) \geq t - h \) under the MM* model for \( 0 \leq h \leq t \).

**Proof.** If \( h = t \), then \( t_{h}^{\ast}(G) \geq 0 = t - h \) holds obviously.

Now, we assume that \( h \leq t - 1 \). For an arbitrary edge subset \( F_{e} \subseteq E \) with \( |F_{e}| \leq h \), suppose that there exist two distinct vertex subsets \( F_{1}, F_{2} \subseteq V \) such that \( F_{1} \) and \( F_{2} \) are indistinguishable in \( G - F_{e} \) under the MM* model. We will prove the lemma by showing that \( |F_{1}| \geq t - h + 1 \) or \( |F_{2}| \geq t - h + 1 \) for \( 0 \leq h \leq t - 1 \) and \( \delta(G) \geq 3 \).

If \( |F_{1} \cap F_{2}| \geq t - h \), then \( |F_{1}| \geq t - h + 1 \) or \( |F_{2}| \geq t - h + 1 \).
Suppose that $|F_1 \cap F_2| \leq t - h - 1$. Then by Lemma 2.6,

$$\kappa(G - F_e) \geq \kappa(G) - |F_e| \geq t - h > t - h - 1 \geq |F_1 \cap F_2|.$$  

Therefore, $G - F_e - (F_1 \cap F_2)$ is connected.

If $V = F_1 \cup F_2$, then $|F_1 \cup F_2| = |V| \geq 2(t - h) + 3$. Thus, $|F_1| \geq t - h + 1$ or $|F_2| \geq t - h + 1$. Now, we assume that $V - (F_1 \cup F_2) \neq \emptyset$.

**Claim 1**: $V - (F_1 \cup F_2)$ is an independent set of $G - F_e$.

Otherwise, $E(G[V - (F_1 \cup F_2)]) \neq \emptyset$. Since $G - F_e - (F_1 \cap F_2)$ is connected, there exist three distinct vertices $x, y \in V - (F_1 \cup F_2)$ and $w \in F_1 \Delta F_2$ such that $xy, yw \in E(G - F_e)$. By Lemma 2.3, $F_1$ and $F_2$ are distinguishable in $G - F_e$ under the MM* model, which is a contradiction.

Pick a vertex $u \in V - (F_1 \cup F_2)$. Note that $F_1$ and $F_2$ are indistinguishable in $G$ under the MM* model. By Claim 1, $|F_1 \cap F_2| \geq |N_G(u)| - h - 2 \geq \delta(G) - h - 2 \geq t - h - 2$ (see Figure 4).

![Figure 4: Illustration of $|F_1 \cap F_2| \geq t - h - 2$.](image)

**Case 1**: $|F_1 \cap F_2| = t - h - 2$.

In this case, we have $|F_e| = h$, $F_e \subseteq E(u)$ and $|N_G(u)| = \delta(G) = t$. Thus, $|N_{G - F_e}(u) \cap (F_1 - F_2)| = |N_{G - F_e}(u) \cap (F_2 - F_1)| = 1$. If $|F_1 - F_2| \geq 3$ or $|F_2 - F_1| \geq 3$, then $|F_1| \geq t - h + 1$ or $|F_2| \geq t - h + 1$. Therefore, $1 \leq |F_1 - F_2| \leq 2$ and $1 \leq |F_2 - F_1| \leq 2$. 

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Claim 2: $|V - (F_1 \cup F_2)| \geq 2.$

$|V - (F_1 \cup F_2)| = |V| - |F_1 \cap F_2| - |F_1 \triangle F_2| \geq 2(t - h) + 3 - (t - h - 2) - 4 = t - h + 1 \geq 2.$

Claim 3: $|N_G(v)| = \delta(G)$ for any $v \in V - (F_1 \cup F_2)$. Otherwise, by Claim 2, $|N_G(v)| > \delta(G)$ for some $v \in V - (F_1 \cup F_2)$ with $v \neq u$ (see Figure 5). By Claim 1 and Lemma 2.3, we have $F_1$ and $F_2$ are distinguishable in $G - F_e$

![Figure 5: Illustration of the proof of Claim 3.](image)

under the MM$^*$ model, which is a contradiction.

Since $F_1$ and $F_2$ are indistinguishable in $G - F_e$ under the MM$^*$ model, by Claim 2 we have $|N_G(v) \cap (F_1 - F_2)| \leq 1$ and $|N_G(v) \cap (F_2 - F_1)| \leq 1$ for any $v \in V - (F_1 \cup F_2) - \{u\}$.

If $h \geq 1$, then by Claim 3, $|N_G(v) \cap (F_1 \cap F_2)| \geq |N_G(v)| - 2 = \delta(G) - 2$, which contradicts $|N_G(v) \cap (F_1 \cap F_2)| \leq |F_1 \cap F_2| = \delta(G) - h - 2 \leq \delta(G) - 3$.

Now, we consider the case of $h = 0$. Then $|F_1 \cap F_2| = \delta(G) - 2$ and $|V - (F_1 \cup F_2)| = |V| - |F_1 \cap F_2| - |F_1 \cap F_2| \geq 2\delta(G) + 3 - (\delta(G) - 2) - 4 = \delta(G) + 1$. Note that $\delta(G) \leq |F_1 \cup F_2| \leq \delta(G) + 2$. Thus, we distinguish the following cases.

Case 1.1: $|F_1 \cup F_2| = \delta(G)$.

By Claim 1 and Claim 3, we have $G$ is isomorphic to $\Gamma_1(\delta(G), l)$ for some graph $H_{\delta(G)}$ and some integer $l$, a contradiction.

Case 1.2: $|F_1 \cup F_2| = \delta(G) + 1$.

Without loss of generality, we assume that $F_1 - F_2 = \{w_1, w_2\}$ and $F_2 - F_1 = \{w\}$. Since $G$ is a $t$-connected graph, we have that $\kappa(G) - |F_1 \cap F_2| \geq t - (t - 2) = 2$. Then
\( G - (F_1 \cap F_2) \) is 2-connected. If \( w_1w_2 \notin E(G) \), then by Lemma 2.5, there exist two internally-disjoint paths \( P \) and \( Q \) from \( w_1 \) to \( w_2 \). Since \( F_1 \) and \( F_2 \) are indistinguishable under the MM* model, for any vertex \( x \in V - (F_1 \cup F_2) \), \( xw_1 \in E(G) \) and \( xw_2 \notin E(G) \) (or \( xw_1 \notin E(G) \) and \( xw_2 \in E(G) \)) (see Figure 6). By Claim 3, we have \( xw \in E(G) \) for any vertex \( x \in V - (F_1 \cup F_2) \). Therefore, by Claim 1, \( V(P) \cap V(Q) \supseteq \{w_1, w_2, w\} \), a contradiction. Thus, we have \( w_1 \) is adjacent to \( w_2 \) and \( G \) is isomorphic to \( \Gamma_{l}^{2}(\delta(G)) \) for some graph \( H_{\delta(G)-1} \) and some integer \( l \), which contradicts \( G \notin F(\delta(G)) \).

**Case 1.3:** \( |F_1 \cup F_2| = \delta(G) + 2 \).

In this case, \( |F_1 - F_2| = |F_2 - F_1| = 2 \) (see Figure 7). By Claim 1 and Claim 3, we have \( G \) is isomorphic to \( \Gamma_{3}(\delta(G), l) \) for some graphs \( H_{\delta(G)-2}, H_2, H_{2}^{2} \) and some integer \( l \), a contradiction.

**Figure 6:** Illustration of \( w_1w_2 \in E(G) \).

**Figure 7:** Illustration of the proof of Case 1.3.

**Case 2:** \( |F_1 \cap F_2| = t - h - 1 \).
Since $F_1 \neq F_2$, without loss of generality, we assume that $F_2 - F_1 \neq \emptyset$. If $|F_1 - F_2| \geq 2$ or $|F_2 - F_1| \geq 2$, then $|F_1| \geq t - h + 1$ or $|F_2| \geq t - h + 1$. Therefore, $|F_1 - F_2| \leq 1$ and $|F_2 - F_1| = 1$. Thus, $|F_1 \cup F_2| = |F_1 \cap F_2| + |F_1 \Delta F_2| \leq (t - h - 1) + 2 = t - h + 1$ and so $|V - (F_1 \cup F_2)| \geq 2(t - h) + 3 - (t - h + 1) = t - h + 2 \geq 3$.

**Case 2.1: $F_1 - F_2 = \emptyset$.**

Note that $N_{G-F_e}(w) \subseteq F_1 \cup F_2$ for any $w \in V - (F_1 \cup F_2)$. Then, $\delta(G) - h \geq t - h = |F_1 \cup F_2| \geq |N_{G-F_e}(w)| \geq \delta(G) - h$. Therefore, $t = \delta(G)$ and $|N_{G-F_e}(w)| = \delta(G) - h$ for any $w \in V - (F_1 \cup F_2)$.

If $h \geq 1$, then $|N_{G-F_e}(w')| = \delta(G) - h \leq \delta(G) - 1$ and $F_e \subseteq E(w')$, where $w' \in V - (F_1 \cup F_2)$. Since $|V - (F_1 \cup F_2)| \geq 3$, there exists a vertex $w'' \in V - (F_1 \cup F_2) - \{w\}$ such that $|N_{G-F_e}(w'')| = |N_G(w'')| \geq \delta(G) > \delta(G) - 1 \geq |F_1 \cup F_2|$, a contradiction.

If $h = 0$, then $|F_1 \cup F_2| = \delta(G)$ and $|V - (F_1 \cup F_2)| \geq 2\delta(G) + 3 - \delta(G) = \delta(G) + 3$. By Claim 1, we know that $G$ is isomorphic to $\Gamma_1(\delta(G), l)$ for some graph $H_{\delta(G)}$ and some integer $l$, which is a contradiction.

**Case 2.2: $F_1 - F_2 \neq \emptyset$.**

In this case, $|F_1 - F_2| = |F_2 - F_1| = 1$ and $|F_1 \cup F_2| = t - h + 1$. Suppose $x \in V - (F_1 \cup F_2)$. If $|E(x) \cap F_e| \leq h - 2$, then $|N_{G-F_e}(x)| = |N_G(x)| - |E(x) \cap F_e| \geq \delta(G) - (h - 2) > \delta(G) - h + 1 \geq t - h + 1 = |F_1 \cup F_2|$, which contradicts $N_{G-F_e}(x) \subseteq F_1 \cup F_2$.

Thus, $|E(x) \cap F_e| \geq h - 1$. Hence, $|E(y) \cap F_e| \leq |F_e| - |E(x) \cap F_e| \leq 1$ for any $y \in V - (F_1 \cup F_2) - \{x\}$. Since $|V - (F_1 \cup F_2)| \geq 3$, pick $z \in V - (F_1 \cup F_2)$ with $|E_z(z)| = 0$.

If $h \geq 2$, then $|N_{G-F_e}(z)| = |N_G(z)| \geq \delta(G) > \delta(G) - h + 1 \geq |F_1 \cup F_2|$, which contradicts $N_{G-F_e}(z) \subseteq F_1 \cup F_2$.

If $h = 0$, then $|F_1 \cap F_2| = t - 1 \geq 0$ and $|V - (F_1 \cup F_2)| = |V| - |F_1 \cup F_2| \geq 2t + 3 - (t + 1) = t + 2 \geq 3$. Pick a vertex $x' \in V - (F_1 \cup F_2)$. Note that $N_G(x') \subseteq F_1 \cup F_2$ by Claim 1. Thus, $t + 1 = |F_1 \cup F_2| \geq |N_G(x')| \geq \delta(G) \geq t$. If $\delta(G) = t$, then $|F_1 \cup F_2| = \delta(G) + 1$.
and $|V - (F_1 \cup F_2)| \geq \delta(G) + 2$. Hence, $G$ is isomorphic to $\Gamma_5(\delta(G), l)$ for some graph $H_{\delta(G)+1}$ and some integer $l$, a contradiction. If $\delta(G) = t + 1$, then $|F_1 \cup F_2| = \delta(G)$ and $|V - (F_1 \cup F_2)| = |V| - |F_1 \cup F_2| \geq 2t + 3 - (t + 1) = t + 2 = \delta(G) + 1$. Hence, $G$ is isomorphic to $\Gamma_1(\delta(G), l)$ for some graph $H_{\delta(G)}$ and some integer $l$, a contradiction.

If $h = 1$, then $|F_1 \cap F_2| = t - 2 \geq 0$, $|F_1 \cup F_2| = t \leq \delta(G)$ and $|V - (F_1 \cup F_2)| = |V| - |F_1 \cup F_2| \geq 2(t-1) + 3 - t = t + 1 \geq 3$. Thus, there exists a vertex $z' \in V - (F_1 \cup F_2)$ such that $F_e \cap E(z') = \emptyset$. Note that $N_{G-F_e}(z') = N_G(z') \subseteq F_1 \cup F_2$ by Claim 1. We have $|F_1 \cup F_2| \geq \delta(G)$. Therefore, $t = \delta(G)$. Without loss of generality, we assume that $F_e = \{u_1u_2\}$.

If $F_e \cap E(G[F_1 \cup F_2]) \neq \emptyset$, then $|N_{G-F_e}(u)| = |N_G(u)| \geq \delta(G) = |F_1 \cup F_2|$ for any vertex $u \in V - (F_1 \cup F_2)$. Since $N_{G-F_e}(u) \subseteq F_1 \cup F_2$, we have $|N_{G-F_e}(u)| = \delta(G) = |F_1 \cup F_2|$ for any vertex $u \in V - (F_1 \cup F_2)$. By Claim 1, $G$ is isomorphic to $\Gamma_1(\delta(G), l)$ for some graph $H_{\delta(G)}$ and some integer $l$, a contradiction.

If $F_e \cap E[F_1 \cup F_2, V - (F_1 \cup F_2)] \neq \emptyset$, then we can suppose that $u_1 \in V - (F_1 \cup F_2)$ and $u_2 \in F_1 \cup F_2$. Note that $N_{G-F_e}(u_1) \subseteq F_1 \cup F_2 - \{u_2\}$. Then $\delta(G) - 1 \leq |N_G(u_1)| - |F_e| = |N_{G-F_e}(u_1)| \leq |F_1 \cup F_2 - \{u_2\}| = \delta(G) - 1$. Therefore, $|N_G(u_1)| = \delta(G)$ and $G$ is isomorphic to $\Gamma_1(\delta(G), l)$ for some graph $H_{\delta(G)}$ and some integer $l$, a contradiction.

If $F_e \cap E(G[V - (F_1 \cup F_2)]) \neq \emptyset$, then $u_1, u_2 \in V - (F_1 \cup F_2)$. Note that $N_{G-F_e}(u_i) \subseteq F_1 \cup F_2$ for $i \in \{1, 2\}$. Then $\delta(G) - 1 \leq |N_G(u_i)| - |F_e| = |N_{G-F_e}(u_i)| \leq |F_1 \cup F_2| = \delta(G)$. Therefore, $|N_G(u_i)| \in \{\delta(G), \delta(G) + 1\}$ for $i \in \{1, 2\}$ and $G$ is isomorphic to $\Gamma_4(\delta(G), l)$ for some graph $H_{\delta(G)}$ and some integer $l$ (see Figure 3(d)), a contradiction.

As mentioned above, we complete the proof of Theorem 4.2. \hfill \Box

Let $G$ be a $t$-regular $t$-connected graph and $h = 0$. By Lemma 4.1 (1) and Theorem 4.2, we have the following result.

**Corollary 4.3 ([2])** Let $G = (V, E)$ be a $t$-regular $t$-connected network with $N$ nodes and $t > 2$. $G$ is $t$-diagnosable under the MM* model if $N \geq 2t + 3$. 

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Note that a maximal connected graph $G$ is $\delta(G)$-connected. By Lemma 2.7 and Theorem 4.2, we obtain the following result.

**Theorem 4.4** Let $G = (V, E)$ be a maximal connected graph with $|V| \geq 2(\delta(G) - h) + 3$. If $G \notin \mathcal{F}(\delta(G))$, then $t^e_h(G) = \delta(G) - h$ under the MM$^*$ model for $0 \leq h \leq \delta(G)$ and $\delta(G) \geq 3$.

Note that a $k$-regular $k$-connected graph is a maximal connected graph. By Lemma 4.1 (1) and Theorem 4.4, we immediately obtain the following result.

**Corollary 4.5** ([6]) Let $G = (V, E)$ be a $k$-regular $k$-connected graph with $|V| \geq 2(k - h) + 3$. Then $t^e_h(G) = k - h$ under the MM$^*$ model for $0 \leq h \leq k$ and $k \geq 3$.

By Lemma 4.1 (2) and Theorem 4.4, we can also obtain the following result.

**Corollary 4.6** Let $G = (V, E)$ be a maximal connected graph with $|V| \geq 2(\delta(G) - h) + 3$ and $C(G) \leq \delta(G) - 2$. Then $t^e_h(G) = \delta(G) - h$ under the MM$^*$ model for $0 \leq h \leq \delta(G)$ and $\delta(G) \geq 3$.

### 5 Applications to the exchanged hypercube

In this section, we will obtain the $h$-edge tolerable diagnosability of an exchanged hypercube by the results in Sections 3 and 4.

The $n$-dimensional *hypercube* $Q_n$ [11] is a graph with vertex set $\{x_{n-1}x_{n-2}\ldots x_0 \mid x_i \in \{0, 1\}, 0 \leq i \leq n-1\}$ and two vertices are adjacent if and only if they differ exactly in one position. One useful property of $Q_n$ is listed below.

**Lemma 5.1** ([24]) Any two vertices in $V(Q_n)$ have exactly two common neighbors for $n \geq 3$ if they have any.

As a variant of the hypercube, the exchanged hypercube proposed by Loh et al. [7], is a graph obtained by systematically removing links from a hypercube.
Definition 5.2 ([7]) An exchanged hypercube is an undirected graph $EH(s, t) = (V, E)$, where $s \geq 1$ and $t \geq 1$,

$$V = \{a_1 \ldots a_i b_i \ldots b_j c \mid a_i, b_j, c \in \{0, 1\}, 1 \leq i \leq s, 1 \leq j \leq t\}$$

is the vertex set, and $E$ is an edge set composed of the following three types of disjoint sets $E_1, E_2$ and $E_3$:

$$E_1 = \{(v_1, v_2) \in V \times V \mid v_1[s + t : 1] = v_2[s + t : 1], v_1[0] \neq v_2[0]\},$$

$$E_2 = \{(v_1, v_2) \in V \times V \mid v_1[s + t : t + 1] = v_2[s + t : t + 1],$$

$$H(v_1[t : 1], v_2[t : 1]) = 1, v_1[0] = v_2[0] = 1\},$$

and

$$E_3 = \{(v_1, v_2) \in V \times V \mid v_1[t : 1] = v_2[t : 1],$$

$$H(v_1[s + t : t + 1], v_2[s + t : t + 1]) = 1, v_1[0] = v_2[0] = 0\},$$

where $v[x:y]$ denotes the bit pattern of $v$ from dimension $y$ to dimension $x$ and $H(u, w)$ denotes the Hamming distance between binary sequence $u$ and binary sequence $w$.

The exchanged hypercubes $EH(1, 1)$ and $EH(1, 2)$ are described in Figure 8. Suppose that $v = 1101$ and $u = 0111$ are two vertices of $EH(1, 2)$. Then $v[0] = 1$, $v[1] = 0$, $v[2] = 1$, $v[3] = 1$, $v[3 : 1] = 110$, $u[3 : 1] = 011$ and $H(v[3 : 1], u[3 : 1]) = 2$.

Lemma 5.3 ([7, 8]) Let $EH(s, t)$ be an exchanged hypercube. Then $\delta(EH(s, t)) = \kappa(EH(s, t)) = \min\{s + 1, t + 1\}$.

Since $EH(s, t)$ is a subgraph of $Q_n$, we have $C(EH(s, t)) \leq C(Q_n)$. By Lemma 5.1, we have the following corollary.

Corollary 5.4 The exchanged hypercube $EH(s, t)$ satisfies that $C(EH(s, t)) \leq 2$. 17
By Lemma 5.3 and Corollary 5.4, we have that $C(EH(s, t)) \leq 2 \leq \delta(EH(s, t)) - 2$ for $\min\{s, t\} \geq 3$. By Lemma 5.3, we know the exchanged hypercube is a maximal connected graph. Note that the exchanged hypercube $EH(s, t)$ has $2^{s+t+1}$ vertices and

$$2^{s+t+1} \geq 2^{2\min\{s, t\}+1} \geq 2(\min\{s + 1, t + 1\} - h) + 3 = 2(\delta(EH(s, t)) - h) + 3.$$ 

Thus, by Theorem 3.3 and Corollary 4.6, the following theorems hold.

**Theorem 5.5** Let $EH(s, t)$ be an exchanged hypercube. Then $t^e_h(EH(s, t)) = \min\{s + 1, t + 1\} - h$ under the PMC model for $0 \leq h \leq \min\{s + 1, t + 1\}$.

**Theorem 5.6** Let $EH(s, t)$ be an exchanged hypercube. Then $t^e_h(EH(s, t)) = \min\{s + 1, t + 1\} - h$ under the MM$^*$ model for $0 \leq h \leq \min\{s + 1, t + 1\}$ and $\min\{s, t\} \geq 3$.

### 6 Conclusions

In this paper, we obtain the lower bound of the $h$-edge tolerable diagnosability of a $t$-connected graph and establish the $h$-edge tolerable diagnosability of a maximal connected graph under the PMC model and the MM$^*$ model, which extends some results in [2, 4, 6]. As applications, the $h$-edge tolerable diagnosability of an exchanged hypercube is determined under the PMC model and the MM$^*$ model. In fact, by our main
results, the $h$-edge tolerable diagnosabilities of many well-known irregular networks can be determined under the PMC model and the MM* model.

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