A NOTE ON HARMONIC FORMS AND THE
BOUNDARY OF THE KÄHLER CONE

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Abstract. Motivated by the results of Wu-Yau-Zheng [3], we show that under a certain curvature assumption the harmonic representative of any boundary class of the Kähler cone is nonnegative.

The purpose of this note is to prove the following:

**Theorem 1.** Let \((M^n, g)\) be a compact complex \(n\)-dimensional Kähler manifold satisfying the following curvature condition: for any \(x \in M\), unitary frame \(\{e_1, \ldots, e_n\}\) of \(T_{x(1,0)}^1(M)\) and any real numbers \(\xi_1, \ldots, \xi_n\) we have

\[
\sum_{i,j=1}^n R_{ii,jj}(\xi_i - \xi_j)^2 \geq 0.
\]

Let \(\alpha\) be in the closure of the Kähler cone of \(M\) and \(\eta\) be the unique harmonic representative in \(\alpha\). Then \(\eta\) is nonnegative. Moreover, \(\eta\) is positive if and only if \(\alpha^n [M] > 0\).

Our motivation comes from the results of Wu, Yau and Zheng in [3] where the authors studied a degenerate complex Monge Ampère equation to better understand the boundary of the Kähler cone under the curvature condition in (1). We will give two proofs of Theorem 1, and one purpose here is to point out that a rather straightforward observation made on the proof in [3] leads to a proof of Theorem 1. After showing this, we then present another short self-contained proof of Theorem 1.

Before we begin, let us first recall some basic definitions and notation. Given a complex manifold \(M\), recall that a real class \(\alpha \in H^{(1,1)}(M)\) is called a Kähler class if \(\alpha\) contains a smooth positive definite representative \(\eta\). The space of Kähler classes is a convex cone in \(H^{(1,1)}(M)\) referred to as the Kähler cone which we denote by \(\mathcal{K}\). We say that \(\alpha\) is

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*After posting the first version of this note, it was pointed out to us that the nonnegativity of \(\eta\) in Theorem 1 had been obtained by Zhang in [5].
in the closure of $\mathcal{K}$ if $[(1-t)\omega + t\eta] \in \mathcal{K}$ for any smooth $\eta \in \alpha$, $\omega \in \mathcal{K}$ and $t \in [0, 1)$. Finally, given any real $\alpha \in H^{(1,1)}(M)$ we use $\alpha^n[M]$ to denote the integral $\int_M \alpha^n$.

The following is proved in [3]:

**Theorem 2.** [Wu-Yau-Zheng] Let $(M^n, g)$ be a compact complex $n$-dimensional Kähler manifold satisfying the curvature condition in Theorem 1. Then any boundary class of the Kähler cone can be represented by a $C^\infty$ closed $(1,1)$ form that is everywhere nonnegative.

In particular, they prove that: if $\omega_0$ is the Kähler form of $(M, g)$ and $\Phi \in H^{(1,1)}(M)$ is real and satisfies $[\omega_0 + t\Phi] \in \mathcal{K}$ for $0 \leq t < 1$ and

$$\int_M (\omega_0 + \Phi)^n = 0,$$

then there exists a smooth solution $v$ to

$$\begin{cases} (\omega_0 + \Phi + dd^c v)^n = 0, \\ \omega_0 + \Phi + dd^c v \geq 0 \end{cases}$$

on $M$. Note that $[\omega_0 + t\Phi] \in \mathcal{K}$ for $0 \leq t < 1$ iff $[\omega_0 + \Phi]$ is in the closure of $\mathcal{K}$. Thus the solvability of (2) for any $\Phi$ above is equivalent to the statement of Theorem 2 for boundary classes $\alpha$ satisfying $\alpha^n[M] = 0$. Their proof is to consider smooth solutions $v_t$ to

$$\begin{cases} (\omega_0 + t\Phi + dd^c v_t)^n = \gamma(t)\omega_0^n, \\ \omega_0 + t\Phi + dd^c v_t > 0 \end{cases}$$

for each $0 \leq t < 1$ where $\gamma(t)$ is the normalizing factor

$$\gamma(t) = \frac{1}{V(M,g)} \int_M (\omega_0 + t\Phi)^n,$$

and the existence of each $v_t$ is guaranteed by the results of Yau [4]. The solution to (2) is then obtained by letting $t \to 1$. In fact, under the curvature assumptions in Theorem 1 they show that $v_t = tv$ where $v$ is a fixed function independent of $t$ and hence solves (2) by letting $t \to 1$ in (3). We are now ready to present:

**First proof of Theorem 1** Let $v_t = tv$ be as above. We begin by showing that $v$ satisfies a rather special property. Since

$$(\omega_0 + (t\Phi + dd^c v))^n = \gamma(t)\omega_0^n$$
for all $0 \leq t < 1$ by (3), if we let $a_1(x), \ldots, a_n(x)$ be the ordered eigenvalues of $\Phi + dd^c v$ at $x \in M$ (with respect to $\omega_0$) we obtain

$$\prod_{i=1}^{n}(1 + ta_i(x)) = \gamma(t)$$

for all $0 \leq t < 1$. Since the RHS does not depend on $x$, it is not hard to show that the $a_i(x)'s$ are constant functions on $M$ for each $i$, in other words the eigenvalues of $\Phi + dd^c v$ with respect to $\omega_0$ must be the same at each point on $M$. In particular, the trace of $\omega_0 + \Phi + dd^c v$ is constant. Suppose now that $\Phi$ had been chosen as the unique harmonic representative in $[\Phi]$. Thus $\omega_0 + \Phi$ is also harmonic and as pointed out in [3], it follows by the curvature assumption in (1) that $\omega_0 + \Phi$ must also be parallel. In particular, the trace of $\omega_0 + \Phi$ is constant and hence the trace of $dd^c v$ is constant as well. Since $M$ is compact, it follows that $v$ must also be constant on $M$. Equivalently, we can summarize this as: in general (without assuming $\Phi$ is harmonic), (2) is always satisfied by any $v$ such that $\omega_0 + \Phi + dd^c v$ is harmonic. In other words, we have shown that the conclusion of Theorem 2 is always satisfied by the harmonic representative of any boundary class, and thus we have proved the first statement in Theorem 1. The proof of the second statement of Theorem 1 is the same as that in the Second proof of Theorem 1 below.

Remark 1. It has been known for some time that under the stronger assumption of nonnegative holomorphic bisectional curvature a harmonic $(1, 1)$ form must be parallel (see for example [1, 2] and references therein). This fact played a key role in the classification results for nonnegatively curved Kähler manifolds in [2] for example. The proof of parallelism uses the Bochner formula for $(1, 1)$ forms on Kähler manifolds and generalizes immediately to the curvature condition (1).

One may ask if it can directly be proved that the harmonic representative of a boundary class above is actually nonnegative. We present this in the following.

Second proof of Theorem 1. Let $\eta$ be as in Theorem 1 and let $\omega_0$ be the Kähler form for $(M, g)$. By the above remarks, $\eta$ is parallel and thus has constant real eigenvalues $a_1, \ldots, a_n$ on $M$ with respect to $\omega_0$. Also, $[(1 - t)\omega_0 + t\eta] \in K$ for every $t \in [0, 1]$. In other words, for each $t \in [0, 1]$ there exists $f_t \in C^\infty(M)$ and $\omega_t \in K$ such that $(1 - t)\omega_0 + t\eta = \omega_t + dd^c f_t$, giving

$$Vol_g(M) \prod_{i=1}^{n}(1 - t + ta_i) = \int_M ((1 - t)\omega_0 + t\eta)^n = \int_M (\omega_t + dd^c f_t)^n > 0$$
for all $t \in [0, 1)$. On the other hand, if $a_k < 0$ for some $k$ then $1 - t + ta_k$
and thus the product on the LHS above would vanish for some $t_0 \in (0, 1)$ giving a contradiction. Thus $a_i$ must be nonnegative for each $i$, in other words $\eta$ is nonnegative. In particular, we have $\int_M \eta^n \geq 0$
with strict inequality if and only if $\eta$ is positive. This completes the proof. □

The fact that the harmonic form $\eta$ is parallel allows for a corresponding decomposition of the universal cover $\widetilde{M}$ of $M$ by the de Rham decomposition Theorem for Kähler manifolds. This in turn will allow a further description of the boundary of $K$. Let $\widetilde{M}$ be the universal cover of $M$ with projection $\pi : \widetilde{M} \to M$. By the de Rham decomposition Theorem for Kähler manifolds, we may write

$$(\widetilde{M}, \widetilde{\omega}_0) = (\widetilde{M}_0, \widetilde{\sigma}_0) \times (\widetilde{M}_1, \widetilde{\sigma}_1) \times \cdots \times (\widetilde{M}_k, \widetilde{\sigma}_k)$$

where $\widetilde{\omega}_0 = \pi^*(\omega_0)$, each factor on the RHS is irreducible and Kähler and the decomposition is unique up to permutation. In the following we will identify $\pi_1(M)$, the first fundamental group of $M$, with the corresponding group of deck transformations of $\widetilde{M}$.

**Corollary 1.** With the above notations, the boundary of $K$ can be identified with the space of harmonic $(1, 1)$ forms $\tilde{\eta}$ on $\widetilde{M}$ satisfying: $\tilde{\eta}$ is equivariant with respect to $\pi_1(M)$ and $\tilde{\eta} = \prod_{i=1}^k a_i \widetilde{\sigma}_i$, where $a_i \geq 0$ for all $i$ with equality holding for some $i$.

*Proof.* Let $\tilde{\eta}$ be a harmonic form on $\widetilde{M}$ as above. Then $\tilde{\eta}$ descends to a harmonic form $\eta$ and it is easy to see that $[\eta]$ is in the boundary of $K$. Note that the map $\tilde{\eta} \to [\eta]$ is one-one.

On the other hand, if $\alpha$ is in the boundary of $K$ and $\eta$ is the unique harmonic representative in $\alpha$, then $\eta$ is nonnegative by Theorem and also parallel. Hence the eigenvalues of $\eta$ are nonnegative constants. Thus $\tilde{\eta} = \pi^*(\eta)$ is likewise harmonic with nonnegative constants. By the de Rham decomposition theorem for Kähler manifolds, $\widetilde{M}$ splits into a product $\widetilde{N}_0 \times \widetilde{N}_1 \times \cdots \times \widetilde{N}_l$ such that $\tilde{\eta}$ splits accordingly as $\tilde{\eta}_0 \times \tilde{\eta}_1 \times \cdots \times \tilde{\eta}_l$ where $\tilde{\eta}_0$ is the zero form on $\widetilde{N}_0$ and $\tilde{\eta}_i$ is a positive multiple of the Kähler form on each $\widetilde{N}_i$, $1 \leq i \leq l$. By the uniqueness of de Rham decomposition, by further decomposing $N_i$ into irreducible factors one can see that $\tilde{\eta}$ is a harmonic form on $\widetilde{M}$ as in the theorem. □

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