Decay Constant Ratios $f_{\eta_c}/f_{J/\psi}$ and $f_{\eta_b}/f_{\Upsilon}$

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Abstract

We calculate the decay constant ratios $f_{\eta_c}/f_{J/\psi}$ and $f_{\eta_b}/f_{\Upsilon}$. In the calculation we take into account the mock meson structures of the mesons, as well as the difference of the wave functions at origin of the vector and pseudoscalar mesons studied by Ahmady and Mendel. We find that the different spin structures of the mesons much affect the ratios. We incorporate our results in the prediction of the branching ratios of $B \to K \eta_c$.

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The decay constants $f_{J/\psi}$ and $f_{\Upsilon}$ of the vector mesons are given experimentally from the decay rates to $e^+e^-$, but $f_{\eta_c}$ and $f_{\eta_b}$ of the pseudoscalar mesons are lack of experimental results. So it is necessary to calculate theoretically the ratios $f_{\eta_c}/f_{J/\psi}$ and $f_{\eta_b}/f_{\Upsilon}$ for various applications, for example, for the prediction of $B(B^+ \to K^+ \eta_c)$ from the experimental result $B(B^+ \to K^+ J/\psi) = [1.02 \pm 0.14] \times 10^{-3}$ [1, 2]. It has been widely assumed that the decay constants of the vector and pseudoscalar mesons are almost the same by considering their wave functions at origin to be the same and using the Van Royen-Weisskopf formula. However, Ahmady and Mendel [3] calculated the ratio of the wave functions at origin by considering the perturbation caused by the hyperfine splitting Hamiltonian, and found that the ratio is significantly different from unity. Then $f_{\eta_c}$ and $f_{\eta_b}$ become very different from $f_{J/\psi}$ and $f_{\Upsilon}$. In this paper we will further take into account the influence of the mock meson structure [1, 2] of the mesons on the calculation of the decay constants. We will consider its effects on $f_{\eta_c}$, $f_{\eta_b}$, and $f_{J/\psi}$, $f_{\Upsilon}$, which are originated from the different mock meson spin structures of the pseudoscalar and vector mesons. We find that the effects are very important contrary to the expectation that they are not important for the mesons composed of two heavy quarks. As a result, $f_{\eta_c}$ and $f_{\eta_b}$ become very close to $f_{J/\psi}$ and $f_{\Upsilon}$ again, as we will show in this paper.

The decay constants of pseudoscalar and vector mesons are defined by

$$<0| \bar{Q}_5 \gamma_\mu \gamma_5 Q' |M_P(K) >= f_P K^\mu, \quad <0| \bar{Q}_5 Q' |M_V(K, \varepsilon) >= f_V M_V \varepsilon^\mu.$$  (1)

The Van Royen-Weisskopf formula [4] for the decay constants is given by $f_M = \sqrt{12/m_M} |\Psi_M(0)|$, where $m_M$ and $\Psi_M(0)$ are the mass and the wave function at origin of the meson respectively. The Van Royen-Weisskopf formula is widely used for the calculation of the meson decay constants. This formula is obtained in the limit that the spinors of the quarks inside meson are approximated to two-component Pauli spinors [4, 5]. In this paper we are interested in the $J/\psi$ and $\Upsilon$
families. From the Van Royen-Weisskopf formula, we have

$$\left( \frac{f_{\eta_c}}{f_{J/\psi}} \right)^2 = \frac{m_{J/\psi}}{m_{\eta_c}} \frac{|\Psi_{\eta_c}(0)|^2}{|\Psi_{J/\psi}(0)|^2} = 1.040 \frac{|\Psi_{\eta_c}(0)|^2}{|\Psi_{J/\psi}(0)|^2}. \tag{2}$$

The approximation $|\Psi_{\eta_c}(0)| = |\Psi_{J/\psi}(0)|$ has been usually used, with which we get $(f_{\eta_c}/f_{J/\psi})^2 = 1.040$. However, recently Ahmady and Mendel calculated $|\Psi_{\eta_c}(0)|^2/|\Psi_{J/\psi}(0)|^2$ in their interesting work \[3\] based on the perturbation theory of quantum mechanics, and obtained the ratio as $1.4 \pm 0.1$. By incorporating this result in (2), Ahmady and Mendel obtained \[7\]

$$\left( \frac{f_{\eta_c}}{f_{J/\psi}} \right)^2 = 1.5 \pm 0.1. \tag{3}$$

Ahmady and Mendel also obtained $|\Psi_{\eta_b}(0)|^2/|\Psi_{\Upsilon}(0)|^2 = 1.16 \pm 0.06$ in their work \[3\]. Then by using $m_{\Upsilon} = 9.460$ GeV \[2\] and $m_{\eta_b} = 9.445$ GeV which is calculated from $(m_{\Upsilon} - m_{\eta_b}) = (m_c/m_b)^2 (m_{J/\psi} - m_{\eta_c})$ given by the hyperfine splitting Hamiltonian, we obtain

$$\left( \frac{f_{\eta_b}}{f_{\Upsilon}} \right)^2 = 1.16 \pm 0.06. \tag{4}$$

If we had taken $|\Psi_{\eta_b}(0)| = |\Psi_{\Upsilon}(0)|$, $(f_{\eta_b}/f_{\Upsilon})^2$ would have been given to be $1.002$ from a similar equation to (3). Therefore Ahmady and Mendel have taken into account the fact that the hyperfine splitting Hamiltonian makes the wave function at origin of the pseudoscalar meson bigger than that of the vector meson, and then have obtained the improved results given by (3) and (4) which are much larger than $1.040$ and $1.002$ which would have been given if the wave functions at origin had been taken to be equal.

The purpose of this paper is to improve further the results of (3) and (4) by taking into account the mock meson structures of the pseudoscalar and vector mesons, which are different by their spin structures, as well as the difference of the wave functions at origin which has been studied by Ahmady and Mendel. We work in the relativistic mock meson model of Godfrey, Isgur, and Capstick \[5, 8\], in which the meson state composed of a heavy quark $Q'$ and a heavy antiquark $\bar{Q}'$
is represented by

$$|M_P(0) > = \sqrt{2m_P} \int \frac{d^3p_{Q'}}{(2\pi)^{3/2}} \sqrt{2E_{Q'}} \Phi(p_{Q'}) \frac{1}{\sqrt{N_c}} \Phi(p_{Q'}') \Psi(r)$$

in the meson rest frame (where $p_{Q'} = -p_Q$) in which we work in this paper, where the arrow indicates a state with spin up (down) along a fixed axis and $c$ is the colour index which is summed. Whereas we wrote the pseudoscalar meson state in (5), we can also write the vector meson states in the same way with the spin combinations for the vector states, which are given by $(\uparrow\uparrow)$, $1/\sqrt{2}(\uparrow\downarrow + \downarrow\uparrow)$ and $(\downarrow\downarrow)$. In (5) we adopted the normalization of the creation and annihilation operators given by \{a(p, s), a^\dagger(p', s')\} = $(2\pi)^3 2E\delta_{ss'}\delta^3(p - p')$, and then the meson state in (5) is normalized by $< M_P(0)|M_P(0) > = 2m_P \delta^3(0)$, and also in the same way for the vector meson states. We take the momentum wave function $\Phi(p)$ in (5) as a Gaussian wave function

$$\Phi(p) = \frac{1}{(\sqrt{\pi}\beta)^{3/2}} e^{-p^2/2\beta^2}, \quad \Psi(r) = \left(\frac{\beta}{\sqrt{\pi}}\right)^{3/2} e^{-\beta^2 r^2/2},$$

where $\Psi(r)$ is the conjugate wave function in coordinate space. In Fig. 1, we display six different inter-quark potentials of the potential models in Refs. [9]–[14], which were obtained by fitting the data of the $J/\psi$ and $\Upsilon$ families (mainly their spectra). The mean square radii $\langle r^2 \rangle^{1/2}$ of the $J/\psi$ and $\Upsilon$ mesons are about 2.2 GeV$^{-1}$ and 1.3 GeV$^{-1}$ [15], respectively, which are in the confining long-distant linear potential range. Therefore, using the Gaussian wave function in (5) is appropriate. In particular, since we will calculate the ratios of the decay constants by using (11), the Gaussian wave function is reliable to use in our following calculations.

Since we are concerned with the matrix elements in the left hand sides of (1) with the meson states in (5), it is convenient to represent the meson states by the matrix-valued representations given by

$$\Psi_{P, a\beta} \equiv - < 0| Q^\dagger_{\alpha} T_{\beta} | M_P(0) >, \quad \Psi_{V, a\beta} \equiv - < 0| Q^\dagger_{\alpha} T_{\beta} | M_V(0) >,$$

(7)
where \( \alpha, \beta \) are spinor indeces. With (7), the formulas in (1) are written as

\[
Tr(\gamma^0 \gamma_5 \Psi_P) = f_P m_P, \quad Tr(\gamma^\mu \Psi_V) = f_V m_V \varepsilon^\mu.
\]

If both two quarks inside the meson are static, the spinor combinat ions of \( u(0)\bar{v}(0) \) for the pseudoscalar and vector meson states are given respectively by [16, 17]

\[
P(0, 0) = -\frac{1}{\sqrt{2}} \frac{1 + \gamma^0}{2} \gamma^5, \quad V(0, 0, \varepsilon) = \frac{1}{\sqrt{2}} \frac{1 + \gamma^0}{2} \varepsilon,
\]

where the polarization vectors of the vector meson are given by \( \varepsilon^\mu_\pm = (1/\sqrt{2}) (0, 1, \pm i, 0) \) and \( \varepsilon^\mu_3 = (0, 0, 0, 1) \). By Lorentz boosting the static spinors we obtain \( \Psi_P \) and \( \Psi_V \) in (7) as

\[
\Psi_I = \sqrt{2m_I} \int \frac{d^3p_{Q'}}{(2\pi)^{3/2}} \Phi(p_{Q'}) \left( \frac{\sqrt{N_c} p_{Q'} + m_{Q'}}{\sqrt{2E_{Q'}^2}} \frac{\sqrt{2m_{Q'}(m_{Q'} + E_{Q'})}}{\sqrt{2m_{Q'}(m_{Q'} + E_{Q'})}} \right) S_I - \frac{\hat{p}_{Q'} + m_Q}{\sqrt{2m_Q(m_Q + E_{Q'})}},
\]

where \( I = P \) or \( V \), and \( S_P \) and \( S_V \) are respectively \( P(0, 0) \) and \( V(0, 0, \varepsilon) \) in (3). By incorporating (10) in (8), we obtain the following formula for the decay constants of pseudoscalar and vector mesons in the relativistic mock meson model:

\[
f_I = \frac{2\sqrt{3}}{\sqrt{m_I}} \int \frac{d^3p}{(2\pi)^{3/2}} \Phi(p) \left( \frac{E_{Q'} + m_{Q'}}{2E_{Q'}} \frac{E_{\bar{Q}} + m_{\bar{Q}}}{2E_{\bar{Q}}} \right)^{1/2} \left( 1 + a_I \frac{p^2}{(E_{Q'} + m_{Q'})(E_{\bar{Q}} + m_{\bar{Q}})} \right),
\]

where \( a_P = -1 \) and \( a_V = +1/3 \). We note that the above formulas for the decay constants have been derived by taking the four-component spinor into consideration, and it is reduced to the Van Royen-Weisskopf formula in the two-component spinor limit which corresponds to taking the \( p \to 0 \) limit in the last two factors of (11).

When the meson and quark masses are given, \( f_{\eta_c} \) and \( f_{J/\psi} \) can be calculated from (11) for a given value of the parameter \( \beta \) in (3). We obtained them numerically as functions of \( \beta \) by using \( m_{J/\psi} = 3.097 \) GeV, \( m_{\eta_c} = 2.979 \) GeV [2], and \( m_c = 1.78 \) GeV [18]. We present the results in Fig. 2. We performed the same calculations for \( f_{\eta_b} \) and \( f_{\Upsilon} \) with \( m_{\Upsilon} = 9.460 \) GeV [3], \( m_{\eta_b} = 9.445 \) GeV, and \( m_b = 5.17 \) GeV [18], and present the results in Fig. 3.
Using the experimental values $\Gamma(J/\psi \rightarrow e^+e^-) = 5.27 \pm 0.37$ keV, $\Gamma(\Upsilon \rightarrow e^+e^-) = 1.32 \pm 0.03$ keV \cite{2}, and the formula \cite{19}

$$\Gamma(V \rightarrow e^+e^-) = \frac{4\pi}{3} \frac{\alpha^2}{m_V} f_V^2 c_V,$$

where $c_{J/\psi} = \frac{4}{9}$ and $c_{\Upsilon} = \frac{1}{9}$, we get

$$f_{J/\psi} = 406 \pm 14 \text{ MeV}, \quad f_{\Upsilon} = 710 \pm 8 \text{ MeV}.$$  \hspace{1cm} (13)

Then, from (11) we obtain

$$\beta_{J/\psi} = 0.644 \pm 0.022 \text{ GeV}, \quad \beta_{\Upsilon} = 1.360 \pm 0.016 \text{ GeV}.$$  \hspace{1cm} (14)

In obtaining the above results, we took the quark masses as $m_c = 1.30 - 1.85$ GeV and $m_b = 4.70 - 5.20$ GeV, which cover the quark masses in the six potential models in Refs. \cite{3}–\cite{14}. We note that the errors in (14) came from the quark mass ranges considered, as well as from the experimental errors of the vector meson decay constants in (13). From the results of Ahmady and Mendel \cite{3},

$$|\Psi_{J/\psi}(0)|^2/|\Psi_{J/\psi}(0)|^2 = 1.4 \pm 0.1 \text{ and } |\Psi_{\Upsilon}(0)|^2/|\Psi_{\Upsilon}(0)|^2 = 1.16 \pm 0.06,$$

we get

$$\beta_{J/\psi} = \beta_{J/\psi} \times (1.4 \pm 0.1)^{1/3} = 0.721 \pm 0.042 \text{ GeV},$$

$$\beta_{\Upsilon} = \beta_{\Upsilon} \times (1.16 \pm 0.06)^{1/3} = 1.428 \pm 0.041 \text{ GeV},$$  \hspace{1cm} (15)

since $|\Psi_{M}(0)|^2 = (\beta_{M}/\pi)^3$ from (13). Then by using these values of $\beta_{J/\psi}$ and $\beta_{\Upsilon}$ in (11), we obtain

$$f_{J/\psi} = 420 \pm 52 \text{ MeV}, \quad f_{\Upsilon} = 705 \pm 27 \text{ MeV},$$  \hspace{1cm} (16)

and then the ratios are given by

$$\left(\frac{f_{J/\psi}}{f_{J/\psi}}\right)^2 = 1.06 \pm 0.14, \quad \left(\frac{f_{\Upsilon}}{f_{\Upsilon}}\right)^2 = 0.99 \pm 0.04,$$  \hspace{1cm} (17)

instead of (3) and (4). We obtained the results (17) by taking into account the different mock meson spin structures of the pseudoscalar and vector mesons, as well as their different values of the wave functions at origin caused by the hyperfine
splitting Hamiltonian. We note that the difference between the results in (I7) and those in (3) and (4) has come from the relativistic correction (obtained by taking the four-component spinor into consideration) which is calculated by assuming a Gaussian wave function for the quarkonium. The obtained ratios in (I7) are much close to unity compared to the results of (3) and (4) which were obtained by Ahmady and Mendel [3, 7] by taking into account only the difference of the wave functions at origin.

The errors in the above results (I4)–(I7) come from the ranges of the values of the quark masses \( m_c \) and \( m_b \) which we took as \( m_c = 1.30 - 1.85 \) GeV and \( m_b = 4.70 - 5.20 \) GeV, as well as the experimental errors in the values of \( f_{J/\psi} \) and \( f_{\Upsilon} \) in (I3). In order to see the magnitude of the error in the results induced by the sensitivity to the quark masses, we present the results which are obtained when we use fixed values of \( m_c \) and \( m_b \) in the calculation: When we use \( m_c = 1.78 \) GeV and \( m_b = 5.17 \) GeV [18], (I4) becomes \( \beta_{J/\psi} = 0.638 \pm 0.015 \) GeV, \( \beta_{\Upsilon} = 1.357 \pm 0.013 \) GeV, (I5) \( \beta_{\eta_c} = 0.714 \pm 0.034 \) GeV, \( \beta_{\eta_b} = 1.426 \pm 0.038 \) GeV, (I6) \( f_{\eta_c} = 424 \pm 25 \) MeV, \( f_{\eta_b} = 709 \pm 20 \) MeV, and finally (I7) becomes \( (f_{\eta_c}/f_{J/\psi})^2 = 1.09 \pm 0.07 \), \( (f_{\eta_b}/f_{\Upsilon})^2 = 1.00 \pm 0.04 \). By comparing these values with those in (I6) and (I7), we can see the sensitivity of the results to the values of the quark masses \( m_c \) and \( m_b \).

The decays of \( B \rightarrow K J/\psi \) are important for the check of the factorization hypothesis and the search of the \( CP \) violation phenomena in the \( B \) meson decays, therefore there have been continuous and intensive experimental improvements on their measurements. Recently CLEO reported the new results [1]:

\[
B(B^0 \rightarrow K^0 J/\psi) = [1.15 \pm 0.23(\text{stat}) \pm 0.17(\text{syst})] \times 10^{-3},
\]

\[
B(B^0 \rightarrow K^*(892)^0 J/\psi) = [1.36 \pm 0.27(\text{stat}) \pm 0.22(\text{syst})] \times 10^{-3}, \quad (18)
\]

\[
B(B^+ \rightarrow K^*(892)^+ J/\psi) = [1.58 \pm 0.47(\text{stat}) \pm 0.27(\text{syst})] \times 10^{-3}.
\]
by incorporating the world average \[\bar{\Gamma}(B \rightarrow K^+ J/\psi) = [1.02 \pm 0.14] \times 10^{-3}. \tag{19}\]

In connection with \(B \rightarrow K J/\psi\), the decays of \(B \rightarrow K \eta_c\) have been intensively studied theoretically [4, 20, 21], since the decays are expected to be measured in near future and by comparing the two decay modes we can get a lot of valuable informations on the hadronic structures. Studying the form factors phenomenologically in detail, Gourdin *et al.* obtained the following results [20]:

\[
T = \frac{\Gamma(B \rightarrow K \eta_c)}{\Gamma(B \rightarrow K J/\psi)} = \bar{T} \times \left(\frac{f_{\eta_c}}{f_{J/\psi}}\right)^2, \quad 0.957 \leq \bar{T} \leq 1.259, \tag{20}
\]

\[
T^* = \frac{\Gamma(B \rightarrow K^* \eta_c)}{\Gamma(B \rightarrow K^* J/\psi)} = \bar{T}^* \times \left(\frac{f_{\eta_c}}{f_{J/\psi}}\right)^2, \quad 0.456 \leq \bar{T}^* \leq 0.872. \tag{21}
\]

By incorporating our obtained decay constant ratios (17) in (20), from (18) and (19) we predict

\[
B(B^+ \rightarrow K^+ \eta_c) = \bar{T} \times [(1.11 \pm 0.17) \times 10^{-3}] = [0.90 \sim 1.61] \times 10^{-3},
\]

\[
B(B^0 \rightarrow K^0 \eta_c) = \bar{T} \times [(1.25 \pm 0.32) \times 10^{-3}] = [0.89 \sim 1.98] \times 10^{-3},
\]

\[
B(B^+ \rightarrow K^*(892)^+ \eta_c) = \bar{T}^* \times [(1.72 \pm 0.60) \times 10^{-3}] = [0.51 \sim 2.03] \times 10^{-3},
\]

\[
B(B^0 \rightarrow K^*(892)^0 \eta_c) = \bar{T}^* \times [(1.48 \pm 0.39) \times 10^{-3}] = [0.50 \sim 1.63] \times 10^{-3}.
\]

In obtaining (21) we combined the errors in (17), (18) and (19) by root mean square. For the predictions in (21) we used the range in (20) of the values of \(\bar{T}\) and \(\bar{T}^*\) obtained by Gourdin *et al.* [21]. However, \(\bar{T}\) and \(\bar{T}^*\) are very dependent (especially \(\bar{T}^*\)) on the model for the form factors of \((B \rightarrow K)\) and \((B \rightarrow K^*)\). Ahmady and Mendel applied the heavy quark effective theory and obtained \(\bar{T} = 1.12\) and \(\bar{T}^* = 0.27\) [7], whose \(\bar{T}\) is inside, but \(\bar{T}^*\) is outside of the range in (20). Particularly, Deshpande and Trampetic [22] emphasized that the value of \(\bar{T}^*\) is very much dependent on the model adopted and pointed out that the measurements of the decay \(B \rightarrow K^* \eta_c\) will provide a valuable criterion for the model for the hadronic form factors.
In conclusion, we calculated the decay constant ratios $f_{\eta_c}/f_{J/\psi}$ and $f_{\eta_b}/f_{\Upsilon}$ by taking into account the mock meson structures of the mesons, as well as the difference of the wave functions at origin of the pseudoscalar and vector mesons. The results have been significantly affected by the different mock meson spin structures of the mesons. We incorporated the obtained ratios of the decay constants in the prediction of the branching ratios of the $B \to K \eta_c$ decays.

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Figure Captions

Fig. 1. The inter-quark potentials of the potential models in [9–14]. The radial distance of the horizontal axis is in the unit of GeV\(^{-1}\) (1 GeV\(^{-1}\) = 0.197 fm), and the potential energy of the vertical axis is in the unit of GeV.

Fig. 2. \(f_\eta\) and \(f_{J/\psi}\) as functions of the parameter \(\beta\).

Fig. 3. \(f_\eta\) and \(f_\Upsilon\) as functions of the parameter \(\beta\).
