Higher Order Moments Dynamics for Some Multimode Quantum Master Equations

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Abstract—We derive Heisenberg equations for arbitrary high order moments of creation and annihilation operators in the case of the quantum master equation with a multimode generator which is quadratic in creation and annihilation operators and obtain their solutions. Based on them we also derive similar equations for the case of the quantum master equation, which occur after averaging the dynamics with a quadratic generator with respect to the classical Poisson process. This allows us to show that dynamics of arbitrary finite-order moments of creation and annihilation operators is fully defined by finite number of linear differential equations in this case.

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1. INTRODUCTION

This paper is further development of the series of works [1–5] devoted to quantum master equations for which dynamics of moments can be obtained explicitly. Namely, in [2, 3] we have shown that the master equations, which arise as averaging of unitary evolution with a quadratic (in creation and annihilation operators) Hamiltonian with respect to classical Poisson process, lead to closed linear ordinary differential equations for moments of creation and annihilation operators of fixed order. In [5] something similar was done for the Wiener stochastic processes. The main of this article is to generalize the results of [2] and consider the Gorini–Kossakowski–Sudarshan–Lindblad (GKSL) generators which arise as averaging of initially non-unitary dynamics by the classical Poisson process. This initial non-unitary dynamics is also assumed to have a quadratic generator in creation and annihilation operators. In contrast to [2, 3, 5] we obtain the closed linear ordinary differential equations for the moments of creation and annihilation operators up to fixed order, not for fixed order moments only. Thus, in this case the equations for the moments of creation and annihilation operators of given order depend on solutions of equations for lower-order moments. So they can be solved iteratively. To derive explicit form of such equations we consider several auxiliary problems, which we think can be interesting by themselves.

In Section 2 we derive a formula which allows one to calculate a GKSL generator of a product of operators. It can be viewed as a generalization of the Leibniz formula for a commutator, so we call it the Leibniz type formula.

In Section 3 we use this formula to derive the Heisenberg equations in the case of GKSL generators, which are quadratic in bosonic creation and annihilation operators. Despite the fact that these

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generators are well studied [6–11], we have not found the explicit form of the Heisenberg equations for higher-order (third and higher) moments in literature in this case. On the one hand, there are some reasons for it. Namely, for Gaussian states the higher moments could be defined by the Isserlis–Wick theorem (as we discuss in Section 5). Moreover, even for non-Gaussian states the dynamics is fully defined by dynamics of first and second moments as a special case of quantum Gaussian channels [12, Sec. 12.4]. On the other hand, for our purposes we need these general formulae as an auxiliary result (in particular, we use them in section 6 in the case, when the Isserlis–Wick theorem is not applicable) and the higher moments are important for calculation of covariances of such widespread properties as intensities of electromagnetic field [13, Subsec. 12.12.2], so its explicit description could be useful even for quadratic generators.

In Section 4 we derive the solutions of such equations. As the form of the solution is not simplified much in the case of time-independent coefficients we assume a general time-dependent case in this section despite the fact that it is not necessary for the main aim of our article. Moreover, this general time-independent situation could be interesting for the purposes of incoherent control [14, 15] and tomography in the case of time-dependent generators [16]. Let us remark that although the GKSL equations with a quadratic generator are well-studied, there is still keen current interest in further research into their properties and applications [17–23].

In Section 5 we prove the Isserlis–Wick theorem in our notation. Despite the fact that this section mostly just checks our results form the previous sections, namely, we testify their consistency with this theorem, there is still interest in the Wick theorem in literature [24] as well as in related combinatorial aspects [25]. So possibly our results can be useful for such a discussion. There are some generalizations of the Isserlis theorem in classical probability [26, 27] and our results could be relevant for their quantum generalizations.

In Section 6 we derive the Heisenberg equations for products of creation and annihilation operators in case of generators which occur after averaging the dynamics with a quadratic generator with respect to the classical Poisson process. For this case we show that dynamics of arbitrary finite-order moments of creation and annihilation operators is fully defined by a finite number of linear differential equations.

In Section 7 we discuss possible generalizations and directions for further development.

2. LEIBNIZ TYPE FORMULA FOR GKSL GENERATOR

We consider the GKSL equation for the density matrix of the form

$$\frac{d}{dt}\rho(t) = \mathcal{L}(\rho(t)),$$

where

$$\mathcal{L}(\rho) = -i[H, \rho] + \sum_j \left( \hat{C}_j \rho \hat{C}_j^\dagger - \frac{1}{2} \hat{C}_j^\dagger \hat{C}_j \rho - \frac{1}{2} \rho \hat{C}_j^\dagger \hat{C}_j \right).$$

Let us remark that following the tradition (see e.g., [20]) we call it a GKSL equation for unbounded generators too despite the fact that theorems from [28, 29] are not applicable in this case and moreover other generators could occur [30] in the unbounded case. Let also us remark that we consider the unbounded operators only formally.

In the Heisenberg representation

$$\frac{d}{dt}X(t) = \mathcal{L}^*(X(t)),$$

where

$$\mathcal{L}^*(X) = i[H, X] + \frac{1}{2} \sum_j \left( [\hat{C}_j, X] \hat{C}_j^\dagger + \hat{C}_j^\dagger [X, \hat{C}_j] \right).$$

If \( \hat{C}_j = 0 \), then one has a Leibniz rule for a commutator [31, Sec. 40]

$$\mathcal{L}^*(XY) = X\mathcal{L}^*(Y) + \mathcal{L}^*(X)Y.$$
and it can be generalized to
\[
\mathcal{L}^*(X_1 \ldots X_m) = X \mathcal{L}^*(Y) + \mathcal{L}^*(X)Y = \sum_{k=1}^m X_1 \ldots X_{k-1} \mathcal{L}^*(X_k)X_{k+1} \ldots X_m. \tag{2}
\]

For \( \dot{C}_j \neq 0 \) corrections to Leibniz rule occur [32, p. 85]
\[
\mathcal{L}^*(XY) = X \mathcal{L}^*(Y) + \mathcal{L}^*(X)Y + \sum_j [\dot{C}_j, X][Y, \dot{C}_j]. \tag{3}
\]

Let us generalize formula (2) for this case.

**Lemma 1.** Let \( \mathcal{L}^* \) be defined by (1), then
\[
\mathcal{L}^*(X_1 \ldots X_m) = \sum_{k=1}^m X_1 \ldots X_{k-1} \mathcal{L}^*(X_k)X_{k+1} \ldots X_m + \sum_{1 \leq k < l \leq m} X_1 \ldots X_{k-1}[\dot{C}_j^+, X_k]X_{k+1} \ldots X_{l-1}[X_l, \dot{C}_j]X_{l+1} \ldots X_m. \tag{4}
\]

**Proof.** Let us prove formula (4) by induction. Equation (3) is the base of induction. Let us assume that (4) is proved for \( m \) and let us prove it for \( m + 1 \). First of all let us apply (3) assuming \( X = X_1 \ldots X_m \) and \( Y = X_{m+1} \), then we have
\[
\mathcal{L}^*(X_1 \ldots X_{m+1}) = X_1 \ldots X_m \mathcal{L}^*(X_{m+1}) + \mathcal{L}^*(X_1 \ldots X_m)X_{m+1} + \sum_j [\dot{C}_j^+, X_1 \ldots X_m][X_{m+1}, \dot{C}_j].
\]

Now let us apply Eq. (4) for the \( m \)-fold case and the Leibniz rule for a commutator, then we have
\[
\begin{align*}
\mathcal{L}^*(X_1 \ldots X_{m+1}) &= X_1 \ldots X_m \mathcal{L}^*(X_{m+1}) + \sum_{k=1}^m X_1 \ldots \mathcal{L}^*(X_k)X_{k+1} \ldots X_{m+1} \\
+ &\sum_{1 \leq k < l \leq m} \sum_j X_1 \ldots X_{k-1}[[\dot{C}_j^+, X_k]X_{k+1} \ldots X_{l-1}[X_l, \dot{C}_j]X_{l+1} \ldots X_{m+1} \\
+ &\sum_{k=1}^m \sum_{j} X_1 \ldots X_{k-1}[\dot{C}_j^+, X_k]X_{k+1} \ldots X_m[X_{m+1}, \dot{C}_j] \\
= &\sum_{k=1}^{m+1} X_1 \ldots X_{k-1} \mathcal{L}^*(X_k)X_{k+1} \ldots X_{m+1} \\
+ &\sum_{1 \leq k < l \leq m+1} \sum_j X_1 \ldots X_{k-1}[[\dot{C}_j^+, X_k]X_{k+1} \ldots X_{l-1}[X_l, \dot{C}_j]X_{l+1} \ldots X_{m+1}.
\end{align*}
\]

Thus, we obtain Eq. (4) for the \( (m + 1) \)-fold case. \( \square \)

3. **HEISENBERG EQUATIONS**

We need to briefly present some notation from [1, 33–35] to formulate our result. In this section we consider Hilbert space \( \mathbb{C}^{2n} \). Let us define the \( 2n \)-dimensional vector of annihilation and creation operators \( \mathbf{a} = (\hat{a}_1, \ldots, \hat{a}_n, \hat{a}_1^+, \ldots, \hat{a}_n^+)^T \) satisfying canonical commutation relations [36, Sec. 1.1.2]
\[
[\hat{a}_i, \hat{a}_j] = \delta_{ij}, \quad [\hat{a}_i, \hat{a}_j^+] = [\hat{a}_i^+, \hat{a}_j] = 0.
\]

One could denote linear and quadratic forms in such operators by \( f^T \mathbf{a} \) and \( \mathbf{a}^T K \mathbf{a} \), respectively. Here \( f \in \mathbb{C}^{2n} \) and \( K \in \mathbb{C}^{2n \times 2n} \). Let us define \( 2n \times 2n \)-dimensional matrices as
\[
J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}, \quad E = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix},
\]
where \( I_n \) is the identity matrix from \( \mathbb{C}^{n \times n} \). Then the canonical commutation relations take the form
\[
[f^T a, a^T g] = -f^T J g, \quad \forall g, f \in \mathbb{C}^{2n}.
\] (5)

We also define the \(~\)-conjugation of vectors and matrices by the formulae
\[
\tilde{g} = E\bar{g}, \quad g \in \mathbb{C}^{2n}, \quad \tilde{K} = E\bar{KE}, \quad K \in \mathbb{C}^{2n \times 2n},
\]
where the overline is an (elementwise) complex conjugation.

**Lemma 2.** For \( H = \frac{1}{2}a^T H a + f^T a \), where \( H = H^T = \tilde{H} \), \( f = \tilde{f} \), and \( \tilde{C}_j = \gamma_j^T a \) Eq. (1) takes the form
\[
\mathcal{L}_{H, f}^*(X) = i \left[ \frac{1}{2}a^T H a + f^T a, X \right] + a^T \Gamma^T a - \frac{1}{2}a^T \Gamma^T a X - \frac{1}{2}X a^T \Gamma^T a,
\] (6)
where \( \Gamma = \sum_j \gamma_j \gamma_j^T \).

Let us formulate a part of Lemma 10 from [35] which is needed for purposes of this paper.

**Lemma 3.** Let \( \mathcal{L}_{H, f}^* \) be defined by (6), then
\[
\mathcal{L}_{H, f}^*(a) = B a + \varphi,
\] (7)
where
\[
B \equiv J \left( iH + \frac{\Gamma^T - \Gamma}{2} \right), \quad \varphi \equiv iJ f
\] (8)
and \( \mathcal{L}_{H, f}^*(a) \) is understood as application of \( \mathcal{L}_{H, f}^* \) to each element of \( a \).

Now let us generalize it to the \( m \)-fold case.

**Lemma 4.** Let \( g_1, \ldots, g_m \in \mathbb{C}^{2n} \), then
\[
\mathcal{L}_{H, f}^*(g_1^T a \ldots g_m^T a) = \sum_{k=1}^{m} g_1^T a \ldots g_{k-1}^T a g_k^T (B a + \varphi) g_{k+1}^T a \ldots g_m^T a
\]
\[
+ \sum_{1 \leq l < m} (g_l^T g_l a)^T \ldots g_{l-1}^T a \tilde{C}_i, g_l^T a \ldots g_{l-1}^T a \varphi, g_l^T a, \tilde{C}_i g_{l+1}^T a \ldots g_m^T a,
\] (9)
\[
\Xi \equiv J \Gamma^T J.
\] (10)

**Proof.** By Lemmas 1 and 3 and taking into account (5) we have
\[
\mathcal{L}^*(g_1^T a \ldots g_m^T a) = \sum_{k=1}^{m} g_1^T a \ldots g_{k-1}^T a \mathcal{L}^*(g_k^T a) g_{k+1}^T a \ldots g_m^T a
\]
\[
+ \sum_{1 \leq l < m} \sum_{i} g_1^T a \ldots g_{k-1}^T a [\tilde{C}_i, g_k^T a] g_{k+1}^T a \ldots g_l^T a [g_l^T a, \tilde{C}_i] g_{l+1}^T a \ldots g_m^T a
\]
\[
= \sum_{k=1}^{m} g_1^T a \ldots g_{k-1}^T a g_k^T (B a + \varphi) g_{k+1}^T a \ldots g_m^T a
\]
\[
+ \sum_{1 \leq l < m} \sum_{i} g_1^T a \ldots g_{k-1}^T a (g_k^T J \tilde{C}_i) g_{k+1}^T a \ldots g_l^T a (\gamma_i J \gamma_l) g_{l+1}^T a \ldots g_m^T a
\]
\[
= \sum_{k=1}^{m} g_1^T a \ldots g_{k-1}^T a g_k^T (B a + \varphi) g_{k+1}^T a \ldots g_m^T a
\]
\[
+ \sum_{1 \leq l < m} (g_l^T g_l a)^T g_1^T a \ldots g_{l-1}^T a g_{l+1}^T a \ldots g_m^T a.
\]

Thus, we obtain (9). \( \square \)
Let us write in more compact notation. One can think about \( a_1^T \ldots a_n^T \) as components of tensor \( a \otimes \ldots \otimes a \). Similarly to many-body physics set-ups [37, Subsec. 3.7.2] let us use the subscripts for \( a, \varphi, \Xi \) to denote the number of the tensor multiplicand to which it corresponds, e.g. \( a_1 a_2 \equiv a \otimes a \), \( a_1 \varphi_2 \equiv a \otimes \varphi \), \( \Xi_2 a_3 \equiv \Xi \otimes a \) and so on. Similarly, the subscript for \( B \) denotes the number of the tensor multiplicand in which this matrix acts. Then, Eq. (9) takes the form

\[
L^*_{H, \Gamma, f}(\prod_{j=1}^{m} a_j) = \left( \sum_{k=1}^{m} B_k \right) \prod_{j=1}^{m} a_j + \sum_{k=1}^{m} \varphi_k \prod_{j=1}^{m} a_j + \sum_{1 \leq k \leq l \leq m} \Xi_{kl} \prod_{j=1}^{m} a_j.
\]

But let us make our notation even more compact by introducing set-valued subscripts

\[
a_{I} \equiv \prod_{j \in I} a_j,
\]

where \( I \) is some subset of a natural number. Similarly, let us denote by \( p \) a set of two natural numbers (pairs) and let \( P(I) \) be all possible pairs from \( I \). Then Eq. (9) takes the form

\[
L^*_{H, \Gamma, f}(a_{I}) = \left( \sum_{k \in I} B_k \right) a_{I} + \sum_{k \in I} \varphi_k a_{I \setminus \{k\}} + \sum_{p \in P(I)} \Xi_{p} a_{I \setminus p}.
\]

Let us consider the simplest cases. Namely, for \( I = \{1\} \) we revisit (7). For \( I = \{1, 2\} \) we have

\[
L^*_{H, \Gamma, f}(a_{12}) = (B_1 + B_2) a_{12} + \varphi_1 a_{2} + \varphi_2 a_{1} + \Xi_{12}.
\]

If one arranges the elements of the tensor \( a_{12} \equiv a_1 a_2 \equiv a \otimes a \) into the matrix \( a a^T \), then this equation takes the form

\[
L^*_{H, \Gamma, f}(a a^T) = B a a^T + a a^T B^T + \varphi a a^T + a^T \varphi + \Xi.
\]

Taking into account (8) and (10) we have

\[
L^*_{H, \Gamma, f}(a a^T) = J \left( iH + \frac{\Gamma T - \Gamma}{2} \right) a a^T + a a^T \left( -iH + \frac{\Gamma T - \Gamma}{2} \right) J + J \Gamma^T J + i J f a^T - i a^T J,
\]

which coincides with the other part of Lemma 10 from [35].

Then the Heisenberg equations for generator (6) with possibly time-dependent coefficients \( H(t), f(t), \Gamma(t) \)

\[
\frac{d}{dt} a_{I}(t) = \left( \sum_{k \in I} B_k(t) \right) a_{I}(t) + \sum_{k \in I} \varphi_k(t) a_{I \setminus \{k\}}(t) + \sum_{p \in P(I)} \Xi_{p}(t) a_{I \setminus p}(t), \tag{11}
\]

where definitions (8), (10) hold the same as for time-independent \( H(t), f(t), \Gamma(t) \).

4. SOLUTION OF HEISENBERG EQUATIONS

In this section we solve Eqs. (11).

**Theorem 1.** Let \( G(t) \) be a solution of the Cauchy problem

\[
\frac{d}{dt} G(t) = B(t) G(t), \quad G(0) = I_{2n} \tag{12}
\]

and

\[
\psi(t) = \int_0^t d\tau \ (G(\tau))^{-1} \varphi(\tau), \quad \beta_p(t) = \int_0^t d\tau \ (G_p(\tau))^{-1} \Xi_p(\tau), \tag{13}
\]

where \( p \) is a pair of indices (always in ascending order). Similarly to the previous section a natural subscript \( k \) for \( \psi_k(t) \) and \( G_k(t) \) means the number of the tensor multiplicand to which this vector or matrix corresponds. And similarly for set-valued indices we define

\[
\psi_{I}(t) \equiv \prod_{k \in I} \psi_k(t), \quad G_{I}(t) \equiv \prod_{k \in I} G_k(t), \quad \beta_{I}(t) \equiv \sum_{I = p_1 \cup \ldots \cup p_{|I|/2}} \beta_{p_1}(t) \ldots \beta_{p_{|I|/2}}(t), \tag{14}
\]

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where the last sum runs over all pairings of indices for \( I \) if \(|I|\) is even and equals zero for odd \(|I|\).

Then the solution of (11) takes the form
\[
a_I(t) = G_I(t) \sum_{I = I_1 \cup I_2 \cup I_3} \psi_{I_1}(t) \beta_{I_2}(t) a_{I_3}(0),
\]
where the sum runs over all possible expansion of \( I \) in a disjoint union of three sets \( I_1, I_2, I_3 \).

**Proof.** Let us mention that
\[
\frac{d}{dt} G_I(t) = \frac{d}{dt} \left( \prod_{i \in I} G_i(t) \right) = \sum_{k \in I} B_k(t) \prod_{i \in I} G_i(t) = \left( \sum_{k \in I} B_k(t) \right) G_I(t).
\]

Let us define
\[
\tilde{a}_I(t) \equiv (G_I(t))^{-1} a_I(t), \quad \tilde{\varphi}_I(t) \equiv (G_I(t))^{-1} \varphi_I(t), \quad \tilde{\Xi}_{kI}(t) \equiv (G_I(t))^{-1} \Xi_{kI}(t).
\]

Then
\[
\frac{d}{dt} \tilde{a}_I(t) = -(G_I(t))^{-1} \left( \frac{d}{dt} G_I(t) \right) (G_I(t))^{-1} a_I(t) + (G_I(t))^{-1} \left( \sum_{k \in I} B_k(t) \right) a_I(t)
\]
\[
\quad \quad + (G_I(t))^{-1} \sum_{k \in I} \varphi_k(t) a_{I \setminus \{k\}}(t) + (G_I(t))^{-1} \sum_{p \in P(I)} \Xi_{p}(t) a_{I \setminus p}(t)
\]
\[
= \sum_{k \in I} (G_{k}(t))^{-1} \varphi_k(t)(G_{I \setminus \{k\}}(t))^{-1} a_{I \setminus \{k\}}(t) + \sum_{p \in P(I)} (G_{p}(t))^{-1} \Xi_{p}(t)(G_{I \setminus p}(t))^{-1} a_{I \setminus p}(t)
\]
\[
= \sum_{k \in I} \tilde{\varphi}_k(t) \tilde{a}_{I \setminus \{k\}}(t) + \sum_{p \in P(I)} \tilde{\Xi}_{p}(t) \tilde{a}_{I \setminus p}(t).
\]

By integrating with respect to \( t \) we obtain a recurrent equation for \( \tilde{a}_I(t) \) (in terms of \( \tilde{a}_{I'}(t) \) with \( I' \) of lower cardinality than \( I \))
\[
\tilde{a}_I(t) = \tilde{a}_I(0) + \int_{0}^{t} d\tau \tilde{\varphi}_I(\tau) \tilde{a}_{I \setminus \{k\}}(\tau) + \int_{0}^{t} d\tau \tilde{\Xi}_{p}(\tau) \tilde{a}_{I \setminus p}(\tau).
\]

Let us prove that its solution has the form
\[
\tilde{a}_I(t) = \sum_{I = I_1 \cup I_2 \cup I_3} \psi_{I_1}(t) \beta_{I_2}(t) \tilde{a}_{I_3}(0)
\]
by induction. The base of induction
\[
\tilde{a}_I(t) = \tilde{a}_I(0) + \int_{0}^{t} d\tau \tilde{\varphi}_I(\tau) = \tilde{a}_I(0) + \psi_I(t) = \sum_{\{I\} = I_1 \cup I_3} \psi_{I_1}(t) \tilde{a}_{I_3}(0).
\]

So let (18) be proved for \( \tilde{a}_{I \setminus \{k\}}(t) \) and \( \tilde{a}_{I \setminus p}(t) \) and let us prove it for \( \tilde{a}_I(t) \). Namely, by Eq. (17) we have
\[
\tilde{a}_I(t) = \tilde{a}_I(0) + \sum_{k \in I} \sum_{I_1 \setminus \{k\} = I_1 \setminus I_2 \cup I_3} \int_{0}^{t} d\tau \tilde{\varphi}_k(\tau) \psi_{I_1}(\tau) \beta_{I_2}(\tau) \tilde{a}_{I_3}(0)
\]
\[
+ \sum_{p \in P(I)} \int_{0}^{t} d\tau \tilde{\Xi}_p(\tau) \sum_{I_1 \setminus p = I_1 \setminus I_2 \cup I_3} \psi_{I_1}(\tau) \beta_{I_2}(\tau) \tilde{a}_{I_3}(0).
\]
Let us remark that
\[
\frac{d}{dt} \psi_I(t) = \sum_{k \in I} \hat{\varphi}_k(t) \psi_{I \setminus \{k\}}(t)
\]
and
\[
\frac{d}{dt} \beta_I(t) = \sum_{I = p_1 \cup \ldots \cup p_{|I|/2}} \frac{d}{dt}(\beta_{p_1}(t) \ldots \beta_{p_{|I|/2}}(t))
\]
\[
= \sum_p \left( \frac{d}{dt} \beta_p(t) \right) \sum_{I \setminus p = p_1 \cup \ldots \cup p_{|I|/2-1}} \beta_{p_1}(t) \ldots \beta_{p_{|I|/2-1}}(t) = \sum_p \tilde{\Xi}_p(t) \beta_{I \setminus p}(t)
\]
for even $|I|$ and the sum is taken over all possible pairings (matchings). Thus, we have
\[
\hat{a}_I(t) = \hat{a}_I(0) + \sum_{I = I_1 \cup I_2 \cup I_3} \int_0^t d\tau \left( \frac{d}{d\tau} \psi_{I_1}(\tau) \right) \beta_{I_2}(\tau) \hat{a}_{I_3}(0)
\]
\[
+ \int_0^t d\tau \sum_{I = I_1 \cup I_2 \cup I_3} \psi_{I_1}(\tau) \frac{d}{d\tau} \beta_{I_2}(\tau) \hat{a}_{I_3}(0) = \sum_{I = I_1 \cup I_2 \cup I_3} \psi_{I_1}(t) \beta_{I_2}(t) \hat{a}_{I_3}(0).
\]
Thus, we have proved (18). Taking into account (16) we have (15).

Let us consider several special cases to illustrate formula (15). For $I = \{1\}$ we have
\[
a_1(t) = G_1(t)(a_1(0) + \psi_1(t)).
\]
If one defines the vector of first moments $\mu(t) \equiv \langle a_1(t) \rangle$, where $\langle \cdots \rangle \equiv \text{tr}(\cdots \rho_0)$ is the average taken with respect to the initial density matrix, then one has
\[
\mu_1(t) = G_1(t)(\mu_1(0) + \psi_1(t)).
\]
For $I = \{1, 2\}$ we have
\[
a_{12}(t) = G_{12}(t)(a_{12}(0) + \psi_1(t)a_2(0) + \psi_2(t)a_1(0) + \psi_1(t)\psi_2(t) + \beta_{12}(t)).
\]
After averaging with respect to the initial density matrix we have
\[
\langle a_{12}(t) \rangle = G_{12}(t)(\langle a_{12}(0) \rangle + \psi_1(t)\langle a_2(0) \rangle + \psi_2(t)\langle a_1(0) \rangle + \psi_1(t)\psi_2(t) + \beta_{12}(t)).
\]
If one defines the matrix of second central moments similarly to [35, Def. 5] as
\[
D_{12}(t) \equiv \langle a_{12}(t) \rangle - \langle a_1(t) \rangle \langle a_2(t) \rangle,
\]
then we have
\[
D_{12}(t) = G_{12}(t)(D_{12}(0) + \beta_{12}(t)).
\]
Equations (19) and (21) coincide with [35, Prop. 7].

For simplicity let us now assume $\varphi = 0$, then we have
\[
a_{1234}(t) = G_{1234}(t)(a_{1234}(0) + a_{12}(0)\beta_{34}(t) + a_{13}(0)\beta_{24}(t) + a_{14}(0)\beta_{23}(t) + \beta_{12}(t)\beta_{34}(t) + \beta_{13}(0)\beta_{24}(t) + \beta_{14}(0)\beta_{23}(t)).
\]
Averaging with respect to the initial density matrix and taking into account Eq. (20) we have
\[
\langle a_{1234}(t) \rangle - \langle a_{12}(t) \rangle \langle a_{34}(t) \rangle = G_{1234}(t)(\langle a_{1234}(0) \rangle - \langle a_{12}(0) \rangle \langle a_{34}(0) \rangle) + \langle a_{13}(0) \rangle \beta_{24}(t) + \langle a_{14}(0) \rangle \beta_{23}(t) + \beta_{13}(0)\beta_{24}(t) + \beta_{14}(0)\beta_{23}(t)).
\]
In particular, such a tensor contains the terms of the form $\langle \hat{a}_i^j \hat{a}_i^k \hat{a}_j \hat{a}_j \rangle - \langle \hat{a}_i^j \hat{a}_i^k \rangle \langle \hat{a}_j^l \hat{a}_j^l \rangle$, which describe the correlations of intensities of electromagnetic field [13, Subsec. 12.12.2].

Let us consider the case of constant coefficients. Then the solution of Eq. (12) has the form
\[
G(t) = e^{B_t}.
\]
Then Eqs. (13) take the form
\[ \psi(t) = \int_0^t d\tau \, e^{-B\tau} \varphi = \frac{1 - e^{-Bt}}{B} \varphi, \]
(23)
\[ \beta_{12}(t) = \int_0^t d\tau \, e^{-(B_1 + B_2)\tau} \Xi_{12} = \frac{1 - e^{-(B_1 + B_2)t}}{B_1 + B_2} \Xi_{12}. \]
(24)

Here and below we understand function of \( B \) as a Taylor series

\[ \frac{1 - e^{-Bt}}{B} = \sum_{j=1}^{\infty} (-1)^{j+1} \frac{1}{j!} B_j \varphi. \]

In particular, it is well-defined even if \( B \) is degenerate. Then Eqs. (19) take the form

\[ \mu_1(t) = e^{B_1 t} \mu_1(0) + \frac{e^{B_1 t} - 1}{B_1} \varphi, \quad \mu_{12}(t) = e^{(B_1 + B_2)t} \mu_{12}(0) + \frac{e^{(B_1 + B_2)t} - 1}{B_1 + B_2} \beta_{12}, \]

which coincides with [35, Prop. 9].

5. CONSISTENCY WITH ISSERLIS–WICK THEOREM

Several different (but deeply related) statements are usually called the Wick theorem [38, Paragraph 17]. Namely, some of them are about normal order of operators and valid independently of state. The others are about higher moments of creation and annihilation operators for Gaussian states (often about only some special cases of them like vacuum or thermal states for quadratic Hamiltonians). So the latter ones are quantum versions of the Isserlis theorem for higher moments of classical Gaussian states. To highlight that we speak about the statement of the second kind we refer to it as the Isserlis–Wick theorem.

**Lemma 5.** Let \( g \in \mathbb{C}^{2n} \) and \( z \in \mathbb{C}^{2n} \) have the form \( z = (z_1, \ldots, z_n, \bar{z}_1, \ldots, \bar{z}_n)^T \), then
\[ g^T a e^{iz^T a} = g^T \left( \frac{\partial}{\partial (iz)} - \frac{1}{2} J(iz) \right) e^{iz^T a}. \]
(25)

**Proof.** By Feynman–Wilcox formula [39]
\[ \frac{\partial}{\partial (iz)} e^{iz^T a} = \int_0^1 ds e^{iz^T a} \frac{\partial}{\partial (iz)} (iz^T a) e^{-iz^T a} = \int_0^1 ds e^{iz^T a} ae^{-iz^T a} e^{iz^T a} = \int_0^1 ds (a + sJ(iz)) e^{iz^T a}, \]
where \( e^{iz^T a} ae^{-iz^T a} = a + sJ(iz) \) due to [35, Lemma 4]. Multiplying by \( g \) we obtain (25).

Applying Eq. (25) iteratively we obtain the following lemma.

**Lemma 6.** Let \( g_1, \ldots, g_m \in \mathbb{C}^{2n} \) and \( z \in \mathbb{C}^{2n} \) have the form \( z = (z_1, \ldots, z_n, \bar{z}_1, \ldots, \bar{z}_n)^T \), then
\[ g_1^T a \ldots g_m^T a e^{iz^T a} = g_m^T \left( \frac{\partial}{\partial (iz)} - \frac{1}{2} J(iz) \right) \ldots g_1^T \left( \frac{\partial}{\partial (iz)} - \frac{1}{2} J(iz) \right) e^{iz^T a}. \]
(26)

The definition [12, Subsec. 12.3.1] of the characteristic function of the state \( \rho \) in our notation takes the form
\[ h(z) \equiv \text{tr}(e^{iz^T a} \rho), \quad z = (z_1, \ldots, z_n, \bar{z}_1, \ldots, \bar{z}_n)^T. \]
(27)

Lemma 6 allows one to calculate moments of creation and annihilation operators in terms of \( h(z) \), namely, averaging both sides of Eq. (26) we obtain the following corollary.
Corollary 1. Let \( g_1, \ldots, g_m \in \mathbb{C}^{2n} \) and \( h(z) \) be defined by Eq. (27)
\[
\langle g_1^T a \cdots g_m^T a \rangle = g_m^T \left( \frac{\partial}{\partial (iz)} - \frac{1}{2} J(i z) \right) \cdots g_1^T \left( \frac{\partial}{\partial (iz)} - \frac{1}{2} J(i z) \right) h(z) \bigg|_{z=0},
\]
where \( (\cdot) = \text{tr}(\cdot \rho_0) \).

Now let us consider a Gaussian state with zero mean and the covariance matrix \( C \), i.e., let us assume \[12, \text{Subsec. 12.3.2}\] \( h(z) = e^{\frac{1}{2} (iz)^T C (iz)} \). The covariance matrix \( C \) is a symmetric part of the matrix of second central moments \[35, \text{Sec. 3.3}\]
\[
C = \frac{1}{2} (D + D^T).
\]
Due to the fact that the skew-symmetric part of the matrix \( D \) is defined by canonical commutation relations (5), we have
\[
D = C - \frac{1}{2} J.
\]

Lemma 7. For a Gaussian state with zero mean and the matrix of second central moments \( D \), we have
\[
\langle a_{\{1,2\}} | I \rangle = \sum_{j,k \in I, j < k} (D_{1j} D_{2k} + D_{1k} D_{2j}) \langle a_{I \setminus \{1,2,j,k\}} \rangle + D_{12}(a_I).
\]

Proof. Let us calculate
\[
g_1^T \left( \frac{\partial}{\partial (iz)} - \frac{1}{2} J(i z) \right) e^{\frac{1}{2} (iz)^T C (iz)} = g_1^T \left( C - \frac{1}{2} J \right) (iz) e^{\frac{1}{2} (iz)^T C (iz)} = g_1^T D(i z) e^{\frac{1}{2} (iz)^T C (iz)},
\]
then
\[
g_2^T \left( \frac{\partial}{\partial (iz)} - \frac{1}{2} J(i z) \right) g_1^T \left( \frac{\partial}{\partial (iz)} - \frac{1}{2} J(i z) \right) e^{\frac{1}{2} (iz)^T C (iz)}
= g_2^T \left( \frac{\partial}{\partial (iz)} - \frac{1}{2} J(i z) \right) \left( g_1^T D(i z) e^{\frac{1}{2} (iz)^T C (iz)} \right)
= (g_1^T D g_2 + g_1^T D(i z) g_2^T D(i z)) e^{\frac{1}{2} (iz)^T C (iz)}.
\]

And let us calculate the commutation relations
\[
\left[ g_3^T \left( \frac{\partial}{\partial (iz)} - \frac{1}{2} J(i z) \right), g_1^T D g_2 + g_1^T D(i z) g_2^T D(i z) \right] = \left[ g_3^T \frac{\partial}{\partial (iz)}, g_1^T D(i z) g_2^T D(i z) \right]
= g_1^T D g_3 g_2^T D(i z) + g_1^T D(i z) g_2^T g_3
\]
and
\[
\left[ g_4^T \left( \frac{\partial}{\partial (iz)} - \frac{1}{2} J(i z) \right), g_1^T D g_3 g_2^T D(i z) + g_1^T D(i z) g_2^T g_3 \right]
= g_1^T D g_4 g_3 g_2^T D g_4 + g_1^T D g_4 g_2^T g_3.
\]
Then by applying Eq. (29) and this commutation relations we have
\[
g_m^T \left( \frac{\partial}{\partial (iz)} - \frac{1}{2} J(i z) \right) \cdots g_1^T \left( \frac{\partial}{\partial (iz)} - \frac{1}{2} J(i z) \right) e^{\frac{1}{2} (iz)^T C (iz)}
= g_m^T \left( \frac{\partial}{\partial (iz)} - \frac{1}{2} J(i z) \right) \cdots g_3 \left( \frac{\partial}{\partial (iz)} - \frac{1}{2} J(i z) \right) \left( g_1^T D g_2 + g_1^T D(i z) g_2^T D(i z) \right) e^{\frac{1}{2} (iz)^T C (iz)}
= (g_1^T D g_2 + g_1^T D(i z) g_2^T D(i z)) g_m^T \left( \frac{\partial}{\partial (iz)} - \frac{1}{2} J(i z) \right) \cdots g_3 \left( \frac{\partial}{\partial (iz)} - \frac{1}{2} J(i z) \right) e^{\frac{1}{2} (iz)^T C (iz)}
+ \sum_{j,k} \left( g_1^T D g_k g_2^T D g_j + g_1^T D g_j g_2^T D g_k \right) \prod_{l \neq 1,2,j,k} g_l^T \left( \frac{\partial}{\partial (iz)} - \frac{1}{2} J(i z) \right) e^{\frac{1}{2} (iz)^T C (iz)}.
\]
Assuming \( z = 0 \) at both sides of these equations and taking into account Corollary 1 we obtain Eq. (28).

In particular, for \( I = \{1, 2, 3, 4\} \) we have
\[
\langle a_{1234} \rangle = D_{13}D_{24} + D_{14}D_{23} + D_{12}\langle a_{34} \rangle,
\]
and taking into account \( \langle a_{34} \rangle = D_{34} \) we obtain
\[
\langle a_{1234} \rangle = D_{12}D_{34} + D_{13}D_{24} + D_{14}D_{23}.
\]
And for \( I = \{1, 2, 3\} \) we have
\[
\langle a_{123} \rangle = D_{12}\langle a_3 \rangle = 0.
\]

In general, applying recurrence relation (28) iteratively we obtain Isserlis–Wick theorem.

**Theorem 2** (Isserlis–Wick). For the Gaussian state with zero mean and the matrix of second central moments \( D \), we have
\[
\langle a_I \rangle = D_I,
\]
where \( D_I \) is defined similarly to Eq. (14) for \( \beta_I(t) \)
\[
D_I = \begin{cases} 
\sum_{I=p_1|\ldots|p_{|I|/2}} D_{p_1} \ldots D_{p_{|I|/2}} & \text{for even } |I|, \\
0 & \text{for odd } |I|,
\end{cases}
\]
and the sum is taken over all pairings of elements of \( I \).

**Corollary 2.** For the Gaussian state with mean \( \mu \) and the matrix of second central moments \( D \), we have
\[
\langle a_I \rangle = \sum_{I=I_1\sqcup I_2} \mu_{I_1} D_{I_2}.
\]

**Proof.** For the Gaussian state with non-zero mean \( \mu \), Theorem 2 can be applied to
\[
\prod_{k \in I}(a_k - \mu_k) = D_I,
\]
then
\[
\langle a_I \rangle = \prod_{k \in I}(a_k - \mu_k + \mu_k) = \sum_{I=I_1\sqcup I_2} \prod_{k \in I_1} (a_k - \mu_k) \prod_{l \in I_2} \mu_l = \sum_{I=I_1\sqcup I_2} \mu_{I_1} D_{I_2}.
\]

In particular, we have
\[
\langle a_1 \rangle = \mu_1,
\]
\[
\langle a_{12} \rangle = \mu_1\mu_2 + D_{12},
\]
\[
\langle a_{123} \rangle = D_{12}\mu_3 + D_{13}\mu_2 + D_{23}\mu_1 + \mu_1\mu_2\mu_3,
\]
\[
\langle a_{1234} \rangle = \mu_1\mu_2 + \mu_1\mu_2D_{34} + \mu_1\mu_3D_{24} + \mu_1\mu_4D_{23}
+ \mu_3\mu_4D_{12} + \mu_2\mu_4D_{13} + \mu_2\mu_3D_{14} + D_{12}D_{34} + D_{13}D_{24} + D_{14}D_{23}.
\]

Now let us assume that the initial state is Gaussian, i.e., the moments of creation and annihilation operators satisfy Eq. (30) and the evolution of \( a_I \) is defined by Eq. (15). Then Eq. (19) leads to
\[
\mu_I(t) = \prod_{k \in I}(G_k(t)(\mu_k(0) + \psi_k(t))) = G_I(t) \sum_{I=I_1\sqcup I_2} \mu_{I_1}(0) \psi_{I_2}(t)
\]
and Eq. (21) leads to
\[
D_I(t) = \sum_{I=p_1|\ldots|p_{|I|/2}} G_{p_1}(t)(D_{p_1}(0) + \beta_{p_1}(t)) \ldots G_{p_{|I|/2}}(t)(D_{p_{|I|/2}}(0) + \beta_{p_{|I|/2}}(t))
\]
\[
= G_I(t) \sum_{I=I_1\sqcup I_2} D_{I_1}(0) \beta_{I_2}(t).
\]
Then by Eq. (15) we have
\[
\langle a_I(t) \rangle = G_I(t) \sum_{I=I_1\cup I_2\cup I_3} \psi_I(t) \beta_{I_2}(t) \langle a_{I_3}(0) \rangle = G_I(t) \sum_{I=I_1\cup I_2\cup I_3} \psi_I(t) \beta_{I_2}(t) \mu_{I_3}(0) D_{I_4}(0)
\]
\[
= \sum_{I=I_1'\cup I_2'} \left( G_{I_1'}(t) \sum_{I_3'=I_1'\cup I_3'} \psi_{I_3'}(t) \mu_{I_3'}(0) \right) \left( G_{I_2'}(t) \sum_{I_4'=I_2'\cup I_4'} \beta_{I_4'}(t) D_{I_4'}(0) \right)
\]
\[
= \sum_{I=I_1'\cup I_2'} \mu_{I_1'}(t) D_{I_2'}(t) = \sum_{I=I_1\cup I_2} \mu_{I_1}(t) D_{I_2}(t)
\]
Hence, the dynamics preserves the Isserlis–Wick theorem. This is not surprising, since the dynamics with a quadratic generator preserves the Gaussian states [7, 11].

6. AVERAGING WITH RESPECT TO POISSON PROCESS

If one now averages time–evolution with respect to the Poisson process with the parameter \( \lambda > 0 \), then we obtain a new semigroup
\[
\sum_{n=0}^{\infty} e^{n\mathcal{L}_{\mu,\gamma,f}} \frac{\lambda^n}{n!} e^{-\lambda t} = e^{\lambda t (e^{\mathcal{L}_{\mu,\gamma,f}} - 1)}
\]
with the generator \( e^{\mathcal{L}_{\mu,\gamma,f}} - 1 \). Thus, we have the following Heisenberg equation
\[
\frac{d}{dt} X(t) = \lambda (e^{\mathcal{L}_{\mu,\gamma,f}} X(t) - X(t)).
\]
(31)

For \( e^{\mathcal{L}_{\mu,\gamma,f}} a_I(t) \) can be calculated by Theorem 1, which leads to the following theorem.

**Theorem 3.** For \( X(t) = a_I(t) \) Eq. (31) takes the form
\[
\frac{d}{dt} a_I(t) = \lambda (G_I(1) - 1) a_I(t) + G_I(1) \sum_{I=I_1\cup I_2\cup I_3, I_3 \neq I} \psi_{I_1}(1) \beta_{I_2}(1) a_{I_3}(t),
\]
(32)
where \( G_I(t), \psi_{I_1}(1), \beta_{I_2}(1) \) are defined by Eqs. (22)–(24).

In particular, for \( I = \{1\} \) and \( I = \{1, 2\} \) we have
\[
\frac{d}{dt} a_1(t) = (e^{B_1} - 1) a_1(t) + \frac{e^{B_1} - 1}{B_1} \varphi_1,
\]
\[
\frac{d}{dt} a_{12}(t) = (e^{B_1+B_2} - 1) a_{12}(t) + e^{B_1} a_1(t) \frac{e^{B_2} - 1}{B_2} \varphi_2 + \frac{e^{B_1} - 1}{B_1} \varphi_1 e^{B_2} a_2(t) + \frac{e^{B_1+B_2} - 1}{B_1 + B_2} \beta_{12}.
\]
By averaging these equations with respect to the initial density matrix one obtains
\[
\frac{d}{dt} D_{12}(t) = (e^{B_1+B_2} - 1) D_{12}(t) + \frac{e^{B_1+B_2} - 1}{B_1 + B_2} \beta_{12}
\]
\[
+ (e^{B_1} - 1)(e^{B_2} - 1) \mu_1(t) \mu_2(t) + (e^{B_1} - 1) \mu_1(t) \frac{e^{B_2} - 1}{B_2} \varphi_2 + \frac{e^{B_1} - 1}{B_1} \varphi_1 (e^{B_2} - 1) \mu_2(t).
\]
Thus, similarly to [4] the equation for \( D_{12}(t) \) is not closed and depends on the first moments \( \mu(t) \). Hence, such evolution does not preserve the Isserlis–Wick theorem. Actually, it is not a surprise due to the fact that only the GKSL generators leading to preservation of Gaussian states during evolution are quadratic ones [11, Prop. 4].

Equation (32) can be solved as
\[
a_I(t) = e^{\lambda (G_I(1)-1)} a_I(0) + \lambda \int_0^t d\tau e^{\lambda (G_I(1)-1)(t-\tau)} G_I(1) \sum_{I=I_1\cup I_2\cup I_3, I_3 \neq I} \psi_{I_1}(1) \beta_{I_2}(1) a_{I_3}(\tau),
\]
where $a_{I_3}(\tau)$ at the right-hand side of this equation has $I_3$ of lower cardinality than $I$. This allows one to solve the equation for $a_I(t)$ with $I$ of lower cardinality and substitute them into equations for $a_I(t)$ with $I$ of higher cardinality and obtain solutions for any given $I$.

Similarly to [3, 4] one can consider a master equation of the form

$$\frac{d}{dt}X(t) = \sum_k \lambda_k \left( e^{\mathcal{L}_k(t)} X(t) X(t) - X(t) X(t) \right), \quad \lambda_k > 0,$$

i.e., with the generator, which is a combination of the generators of (31). Then, analogously to Eq. (32), one has for $X(t) = a_I(t)$ the following Heisenberg equation

$$\frac{d}{dt}a_I(t) = \sum_k \lambda_k \left( (G^{(k)}_I(1) - 1) a_I(t) + G^{(k)}_I(1) \sum_{I = I_1 \sqcup I_2 \sqcup I_3, I_3 \neq I} \psi^{(k)}_{I_1}(1) \beta^{(k)}_{I_2}(1) a_{I_3}(t) \right),$$

where $G^{(k)}_I(t)$, $\psi^{(k)}_{I_1}(1)$, $\beta^{(k)}_{I_2}(1)$ are defined similarly to Eqs. (22)–(24).

After averaging with respect to the initial density matrix both Eqs. (32) and (33) allow one to obtain any finite-order moment dynamics by solving a system of a finite number of linear ordinary differential equations. Let us also remark that analogously to [3] Heisenberg equations are enough to define multitime correlations functions by the regression formula.

7. CONCLUSIONS

Similarly to the case of averaging unitary dynamics with respect to the Levy processes and fields [40–44] the GKSL equations arising from averaging with respect to the Possion process are a key ingredient for generalization of our result to the arbitrary Levy processes and fields due to the Levy-Khintchine theorem [45]. The other important ingredient is the GKSL equations arising from averaging with respect to the Wiener process which seems to be also manageable by methods developed here similarly to [5].

The most of the results of this work can be generalized for the fermionic case. But what is more important is that they could be generalized for the case, when lemma 1 leads to the closed Heisenberg equations for some operators only, but not all the moments of creation and annihilation operators. It could be interpreted as some kind of dissipative analog of dynamical symmetry [46]. And averaging with respect to the Possion processes or even the arbitrary Levy processes seems to preserve such a symmetry. We think that it is an important direction of further development.

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REFERENCES

1. A. E. Teretenkov, “Dynamics of moments for quadratic GKSL generators,” Math. Notes 106, 151–155 (2019).
2. A. E. Teretenkov, “Dynamics of moments of arbitrary order for stochastic poisson compressions,” Math. Notes 107, 695–698 (2020).
3. Iu. A. Nosal and A. E. Teretenkov, “Exact dynamics of moments and correlation functions for GKSL fermionic equations of poisson type,” Math. Notes 108, 911–915 (2020).
4. T. Linowski, A. Teretenkov and L. Rudnicki, “Dissipative evolution of covariance matrix beyond quadratic order to probability and statistics,” arXiv: 2105.12644 (2021).
5. D. D. Ivanov and A. E. Teretenkov, “Moments dynamics and stationary states for classical diffusion-type GKSL equations,” arXiv: 2203.01472 (2022).
6. R. Bausch and A. Stahl, “On the description of noise in quantum systems,” Zeitschr. Phys. A 204, 32–46 (1967).
7. P. Vanheuverzijn, “Generators for quasi-free completely positive semigroups,” Ann. Inst. H. Poincare, Sect. A 29, 123–138 (1978).
8. V. V. Dodonov and O. V. Manko, “Quantum damped oscillator in a magnetic field,” Phys. A (Amsterdam, Neth.) 130, 353–366 (1985).
9. V. V. Dodonov and V. I. Man’ko, “Evolution equations for the density matrices of linear open systems,” in Classical and Quantum Effects in Electrodynamics, Proc. Lebedev Phys. Inst. 176, 53–60 (1988).
10. T. Prosen and T. H. Seligman, “Quantization over boson operator spaces,” J. Phys. A: Math. Theor. 43, 392004 (2010).
11. T. Heinosaari, A. S. Holevo, and M. M. Wolf, “The semigroup structure of Gaussian channels,” Quantum Inf. Comput. 10, 619–635 (2010).
12. A. S. Holevo, Quantum Systems, Channels, Information: A Mathematical Introduction (De Gruyter, Berlin, 2012).
13. L. Mandel and E. Wolf, Optical Coherence and Quantum Optics (Cambridge Univ. Press, Cambridge, 1995).
14. O. V. Morzhin and A. N. Pechen, “Minimal time generation of density matrices for a two-level quantum system driven by coherent and incoherent controls,” Int. J. Theor. Phys. 60, 576–584 (2021).
15. O. V. Morzhin and A. N. Pechen, “Maximization of the overlap between density matrices for a two-level open quantum system driven by coherent and incoherent controls,” Lobachevskii J. Math. 40, 1532–1548 (2019).
16. O. V. Man’ko, “Coherent states of a free particle with varying mass in the probability representation of quantum mechanics,” J. Russ. Laser Res. 43, 90–95 (2022).
17. J. R. Bolanos-Servin, R. Quezada, and J. I. Rios-Cangas, “A characterization of quantum gaussian states in terms of annihilation moments,” arXiv:2111.06570 (2021).
18. V. P. Flynn, E. Cobanera, and L. Viola, “Topology by dissipation and Majorana bosons in metastable quadratic Markovian dynamics,” arXiv:2104.03985 (2021).
19. J. Agredo, F. Fagnola, and D. Poletti, “Gaussian quantum Markov semigroups on a one-mode fock space: Irreducibility and normal invariant states,” Open Sys. Inform. Dyn. 28, 2150001 (2021).
20. J. Agredo, F. Fagnola, and D. Poletti, “The decoherence-free subalgebra of Gaussian quantum Markov semigroups,” arXiv:2112.13781 (2021).
21. T. Barthel and Y. Zhang, “Solving quasi-free and quadratic Lindblad master equations for open fermionic and bosonic systems,” arXiv:2112.08344 (2021).
22. A. Gaidash, A. Kozubov, G. Miroshnichenko, and A. D. Kiselev, “Quantum dynamics of mixed polarization states: Effects of environment-mediated intermode coupling,” J. Opt. Soc. Am. B 38, 2603–2611 (2021).
23. S. Medvedeva, A. Gaidash, A. Kozubov, and G. Miroshnichenko, “Dynamics of field observables in quantum channels,” J. Phys.: Conf. Ser. 1894, 012007 (2021).
24. L. Ferialdi and L. Diosi, “General Wick’s theorem for bosonic and fermionic operators,” Phys. Rev. A 104, 052209 (2021).
25. M. Schork, “Recent developments in combinatorial aspects of normal ordering,” Enumer. Combin. Appl. 1 (2021).
26. J. V. Michalowicz, J. M. Nichols, F. Bucholtz, and C. C. Olson, “A general Isserlis theorem for mixed-Gaussian random variables,” Stat. Prob. Lett. 81, 1233–1240 (2011).
27. C. Vignat, “A generalized Isserlis theorem for location mixtures of Gaussian random vectors,” Stat. Prob. Lett. 82, 67–71 (2012).
28. G. Lindblad, “On the generators of quantum dynamical semigroups,” Commun. Math. Phys. 48, 119–130 (1976).
29. V. Gorini, A. Kossakowski, and E. C. G. Sudarshan, “Completely positive dynamical semigroups of N-level systems,” J. Math. Phys. 17, 821–825 (1976).
30. A. S. Holevo, “On singular perturbations of quantum dynamical semigroups,” Math. Notes 103, 133–144 (2018).
31. V. V. Prasolov, Problems and Theorems in Linear Algebra (Am. Math. Soc., Providence RI, 1994).
32. A. S. Holevo, Statistical Structure of Quantum Theory (Springer, Berlin, 2003).
33. A. E. Teretenkov, “Quadratic dissipative evolution of gaussian states,” Math. Notes 100, 642–646 (2016).
34. A. E. Teretenkov, “Quadratic dissipative evolution of gaussian states with drift,” Math. Notes 101, 341–351 (2017).
35. A. E. Teretenkov, “Irreversible quantum evolution with quadratic generator: Review,” Inf. Dimens. Anal. Quant. Prob. Rel. Top. 22, 1930001 (2019).
36. M. O. Scully and M. S. Zubairy, Quantum Optics (Cambridge Univ. Press, Cambridge, 1997).
37. H.-P. Breuer and F. Petruccione, The Theory of Open Quantum Systems (Oxford Univ. Press, Oxford, 2002).
38. N. N. Bogolubov and D. V. Shirkov, *Introduction to the Theory of Quantized Fields* (Fizmatlit, Moscow, 1995; Wiley, New York, 1980).
39. A. M. Chebotarev and A. E. Teretenkov, “Operator-valued ODEs and Feynman’s formula,” Math. Notes 92, 837–842 (2012).
40. E. B. Davies, “Some contraction semigroups in quantum probability,” Zeitschr. Wahrscheinlichk. Verwand. Gebiete 23, 261–273 (1972).
41. A. Kossakowski, “On quantum statistical mechanics of non-Hamiltonian systems,” Rep. Math. Phys. 3, 247–274 (1972).
42. B. Kummerer and H. Maassen, “The essentially commutative dilations of dynamical semigroups on $M_n$,” Commun. Math. Phys. 109, 1–22 (1987).
43. A. S. Holevo, “Covariant quantum Markovian evolutions,” J. Math. Phys. 37, 1812–1832 (1996).
44. A. S. Holevo, “Covariant quantum dynamical semigroups: Unbounded generators,” in *Irreversibility and Causality Semigroups and Rigged Hilbert Spaces* (Springer, Berlin, 1998), pp. 67–81.
45. K. I. Sato, “Basic results on Levy processes,” in *Levy Processes* (Birkhauser, Boston, MA, 2001), pp. 3–37.
46. I. A. Malkin and V. I. Man’ko, *Dynamic Symmetries and Coherent States of Quantum Systems* (Nauka, Moscow, 1979) [in Russian].