VOLUME, FACETS AND DUAL POLYTOPES OF TWINNED CHAIN POLYTOPES

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Abstract. Let $P$ and $Q$ be finite partially ordered sets with $|P| = |Q| = d$, and $C(P) \subset \mathbb{R}^d$ and $C(Q) \subset \mathbb{R}^d$ their chain polytopes. The twinned chain polytope of $P$ and $Q$ is the normal Gorenstein Fano polytope $\Gamma(C(P), -C(Q)) \subset \mathbb{R}^d$ which is the convex hull of $C(P) \cup (-C(Q))$. In this paper, we study combinatorial properties of twinned chain polytopes. First, we will give the formula of the volume of twinned chain polytopes in terms of the underlying partially ordered sets. Second, we will characterize the facets of twinned chain polytopes in terms of the underlying partially ordered sets. Finally, we will provide the dual polytopes of twinned chain polytopes.

INTRODUCTION

A convex polytope $P \subset \mathbb{R}^d$ is integral if all vertices belong to $\mathbb{Z}^d$. An integral convex polytope $P \subset \mathbb{R}^d$ is normal if, for each integer $N > 0$ and for each $a \in NP \cap \mathbb{Z}^d$, there exist $a_1, \ldots, a_N \in P \cap \mathbb{Z}^d$ such that $a = a_1 + \cdots + a_N$, where $NP = \{N\alpha \mid \alpha \in P\}$. Furthermore, an integral convex polytope $P \subset \mathbb{R}^d$ is Fano if the origin of $\mathbb{R}^d$ is a unique integer point belonging to the interior of $P$. A Fano polytope $P \subset \mathbb{R}^d$ is Gorenstein if its dual polytope $P^\vee := \{x \in \mathbb{R}^d \mid \langle x, y \rangle \leq 1 \text{ for all } y \in P\}$ is integral as well. A Gorenstein Fano polytope is also said to be a reflexive polytope. In recent years, the study of Gorenstein Fano polytopes has been more vigorous. It is known that Gorenstein Fano polytopes correspond to Gorenstein toric Fano varieties, and they are related with mirror symmetry (see, e.g., [1 2]).

We recall some terminologies of partially ordered sets. Let $P = \{p_1, \ldots, p_d\}$ be a partially ordered set. A linear extension of $P$ is a permutation $\sigma = i_1i_2\cdots i_d$ of $[d] = \{1, 2, \ldots, d\}$ which satisfies $i_a < i_b$ if $p_i < p_j$ in $P$. A subset $I$ of $P$ is called a poset ideal of $P$ if $p_i \in I$ and $p_j \in P$ together with $p_j \leq p_i$ guarantee $p_j \in I$. Note that the empty set $\emptyset$ and $P$ itself are poset ideals of $P$. Let $J(P)$ denote the set of poset ideals of $P$. A subset $A$ of $P$ is called an antichain of $P$ if $p_i$ and $p_j$ belonging to $A$ with $i \neq j$ are incomparable. In particular, the empty set $\emptyset$ and each 1-element subsets $\{p_j\}$ are antichains of $P$. Let $A(P)$ denote the set of antichains of $P$. For
each subset \( I \subset P \), we define the \((0, 1)\)-vectors \( \rho(I) = \sum_{i \in I} e_i \), where \( e_1, \ldots, e_d \) are the canonical unit coordinate vectors of \( \mathbb{R}^d \). In particular \( \rho(\emptyset) \) is the origin \( \emptyset \) of \( \mathbb{R}^d \).

In \([8]\), Stanley introduced the order polytope \( O(P) \) and the chain polytope \( C(P) \) arising from a partially ordered set \( P \). It is known that both \( O(P) \) and \( C(P) \) are \( d \)-dimensional convex polytopes, and

\[
\{\text{the sets of vertices of } O(P)\} = \{\rho(I) \mid I \in J(P)\},
\]

\[
\{\text{the sets of vertices of } C(P)\} = \{\rho(A) \mid A \in \mathcal{A}(P)\}
\]

follows \([8\text{ Corollary 1.3, Theorem 2.2}]\). Moreover, \( O(P) \) and \( C(P) \) have the same volume \([8\text{ Theorem 4.1}]\). In particular, the volume of \( O(P) \) and \( C(P) \) are equal to \( e(P)/d! \), where \( e(P) \) is the number of linear extensions of \( P \) \([8\text{ Corollary 4.2}]\).

Let \( P = \{p_1, \ldots, p_d\} \) and \( Q = \{q_1, \ldots, q_d\} \) be partially ordered sets with \( |P| = |Q| = d \). We write \( \Gamma(C(P), -C(Q)) \subset \mathbb{R}^d \) for the convex hull which is the convex hull of \( C(P) \cup -C(Q) \). We call \( \Gamma(C(P), -C(Q)) \) the twinned chain polytope of \( P \) and \( Q \). Similarly, we define \( \Gamma(O(P), -O(Q)) \) and \( \Gamma(O(P), -C(Q)) \) as the convex hull of \( O(P) \cup -O(Q) \) and \( O(P) \cup -C(Q) \), respectively. Then it is known \( \Gamma(C(P), -C(Q)) \) and \( \Gamma(O(P), -C(Q)) \) are normal Gorenstein Fano polytopes \([5, 7]\), and if \( P \) and \( Q \) have a same linear extension, \( \Gamma(O(P), -O(Q)) \) is also normal Gorenstein Fano polytope \([4]\). Moreover, \( \Gamma(C(P), -C(Q)) \) and \( \Gamma(O(P), -C(Q)) \) have the same volume, and if \( P \) and \( Q \) have a same linear extension, \( \Gamma(O(P), -O(Q)) \), \( \Gamma(O(P), -C(Q)) \) and \( \Gamma(C(P), -C(Q)) \) have the same volume \([6]\). Hence one of the main problem is to determine the volume of the above polytopes in terms of partially ordered sets \( P \) and \( Q \).

In this paper, we study combinatorial properties of twinned chain polytopes. At first, we will show the structure of twinned chain polytopes is special \( (\text{Proposition 1.2}) \). This proposition is important in this paper. By using this, in section 1, we will give the formula of the volume of a twinned chain polytope \( \Gamma(C(P), -C(Q)) \) in terms of partially ordered sets \( P \) and \( Q \) \( (\text{Theorem 1.3}) \). Moreover, by this theorem, we can compute the volume of \( \Gamma(O(P), -O(Q)) \) and \( \Gamma(O(P), -C(Q)) \) \( (\text{Corollary 1.4}) \). By using Proposition 1.2 again, in Section 2, we will characterize the facets of twinned chain polytopes in terms of the underlying partially ordered sets \( (\text{Theorem 2.2}) \). Finally, we will provide the dual polytopes of twinned chain polytopes \( (\text{Corollary 2.5}) \).

**1. THE FORMULA OF THE VOLUME OF TWINNED CHAIN POLYTOPES**

For partially ordered sets \( P \) and \( Q \) with \( P \cap Q = \emptyset \), the *ordinal sum* of \( P \) and \( Q \) is the partially ordered set \( P \oplus Q \) on the union \( P \cup Q \) such that \( s \leq t \) in \( P \oplus Q \) if (a) \( s, t \in P \) and \( s \leq t \) in \( P \), or (b) \( s, t \in Q \) and \( s \leq t \) in \( Q \), or (c) \( s \in P \) and \( t \in Q \). Then we have \( \mathcal{A}(P \oplus Q) = \mathcal{A}(P) \cup \mathcal{A}(Q) \).
Let $P = \{ p_1, \ldots, p_d \}$ and $Q = \{ q_1, \ldots, q_d \}$ be partially ordered sets. Given a subset $W$ of $[d]$ we define the induced subposet of $P$ on $W$ to be the partially ordered set $P_W = \{ p_i \mid i \in W \}$ such that $p_i \leq p_j$ in $P_W$ if and only if $p_i \leq p_j$ in $P$. For $W \subseteq [d]$, we set $\Delta_W(P, Q) = P_W \oplus Q_{\overline{W}}$, where $\overline{W} = [d] \setminus W$. Note that $\Delta_W(P, Q)$ is a $d$-element partially ordered set and we have $\mathcal{A}(\Delta_W(P, Q)) = \mathcal{A}(P_W) \cup \mathcal{A}(Q_{\overline{W}})$.

Let $W = \{ i_1, \ldots, i_k \} \subseteq [d]$ and $\overline{W} = \{ i_{k+1}, \ldots, i_d \} \subseteq [d]$ with $W \cup \overline{W} = [d]$. Then we have $\Delta_W(P, Q) = \{ p_{i_1}, \ldots, p_{i_k}, q_{i_{k+1}}, \ldots, q_{i_d} \}$. Also, let $R = \{ r_1, \ldots, r_d \}$ be the partially ordered set such that $r_i \leq r_j$ if (a) $i, j \in W$ and $p_i \leq p_j$ in $\Delta_W(P, Q)$, or (b) $i, j \in \overline{W}$ and $q_i \leq q_j$ in $\Delta_W(P, Q)$, or (c) $i \in W, j \in \overline{W}$ and $p_i \leq q_j$ in $\Delta_W(P, Q)$. We call a permutation $\sigma$ of $[d]$ a linear extension of $\Delta_W(P, Q)$, if $\sigma$ is a linear extension of $R$, and we write $e(\Delta_W(P, Q))$ for the number of linear extensions of $\Delta_W(P, Q)$, i.e., $e(\Delta_W(P, Q)) = e(R)$.

For $A \subseteq \Delta_W(P, Q)$, we define the $(-1,0,1)$-vectors $\rho'(A) = \sum_{p \in A} \mathbf{e}_i - \sum_{q \in A} \mathbf{e}_j$ and we set

$$C'(\Delta_W(P, Q)) = \text{conv}(\{ \rho(A) \mid A \in \mathcal{A}(\Delta_W(P, Q)) \}).$$

First, we will show the following lemma.

**Lemma 1.1.** Let $P = \{ p_1, \ldots, p_d \}$ and $Q = \{ q_1, \ldots, q_d \}$ be partially ordered sets, and let $W \subseteq [d]$. Then $C'(\Delta_W(P, Q))$ is an integral convex polytope of dimension $d$ and the volume equals $e(\Delta_W(P, Q))/d!$.

Before proving Lemma 1.1, we recall properties of integral convex polytopes. Let $\mathbb{Z}^{d \times d}$ denote the set of $d \times d$ integral matrices. A matrix $A \in \mathbb{Z}^{d \times d}$ is unimodular if $\det(A) = \pm 1$. Given integral convex polytopes $P$ and $Q$ in $\mathbb{R}^d$ of dimension $d$, we say that $P$ and $Q$ are unimodularly equivalent if there exists a unimodular matrix $U \in \mathbb{Z}^{d \times d}$ and an integral vector $w$, such that $Q = f_U(P) + w$, where $f_U$ is the linear transformation in $\mathbb{R}^d$ defined by $U$, i.e., $f_U(v) = vU$ for all $v \in \mathbb{R}^d$. Clearly, if $P$ and $Q$ are unimodularly equivalent, then $\text{vol}(P) = \text{vol}(Q)$, where $\text{vol}(\cdot)$ denotes the usual volume.

Now, we prove Lemma 1.1.

**Proof of Lemma 1.1.** Let $U = (u_{ij})_{1 \leq i,j \leq d} \in \mathbb{Z}^{d \times d}$ be a unimodular matrix such that $u_{ij} = \begin{cases} 1 & (i = j \text{ and } i \in W), \\ -1 & (i = j \text{ and } i \in \overline{W}), \\ 0 & (i \neq j). \end{cases}$

Then $C'(\Delta_W(P, Q)) = f_U(C(R))$, where $R$ is the partially ordered set defined by the above. This says that $C'(\Delta_W(P, Q))$ and $C(R)$ are unimodularly equivalent. Hence since the volume of $C(R)$ is equal to $e(R)/d!$, We have

$$\text{vol}(C'(\Delta_W(P, Q))) = \text{vol}(C(R)) = e(R)/d! = e(\Delta_W(P, Q))/d!,$$

as desired.

$\square$
Let $\mathcal{P} \subset \mathbb{R}^d$ be an integral convex polytope of dimension $d$. For $W \subset [d]$, we set
\[
\mathcal{P}_W = \{(x_1, \ldots, x_d) \in \mathcal{P} \mid \text{if } i \in W, x_i \geq 0 \text{ and if } j \notin W, x_j \leq 0\}.
\]

The following is the key proposition in this paper.

**Proposition 1.2.** Let $P = \{p_1, \ldots, p_d\}$ and $Q = \{q_1, \ldots, q_d\}$ be partially ordered sets. Then we have
\[
\Gamma(\mathcal{C}(P), -\mathcal{C}(Q)) = \bigcup_{W \subset [d]} \mathcal{C}'(\Delta_W(P, Q)).
\]

In particular, for any subset $W \subset [d]$, we have
\[
\Gamma(\mathcal{C}(P), -\mathcal{C}(Q)) = \mathcal{C}'(\Delta_W(P, Q)).
\]

**Proof.** For an integral convex polytope $\mathcal{P}$, we write $V(\mathcal{P})$ for the vertex set of $\mathcal{P}$, and for $W \subset [d]$, we set
\[
V_W(\mathcal{P}) = \{(x_1, \ldots, x_d) \in V(\mathcal{P}) \mid \text{if } i \in W, x_i \geq 0 \text{ and if } j \notin W, x_j \leq 0\}
\]
Then for any $W \subset [d]$, we have
\[
V_W(\Gamma(\mathcal{C}(P), -\mathcal{C}(Q))) = V(\mathcal{C}'(\Delta_W(P, Q))) \setminus \{(0, \ldots, 0)\}
\]
since $\mathcal{A}(\Delta_W(P, Q)) = \mathcal{A}(P_W) \cup \mathcal{A}(Q_{\overline{W}})$. Hence it follows that
\[
\Gamma(\mathcal{C}(P), -\mathcal{C}(Q)) = \text{conv}(V_W(\Gamma(\mathcal{C}(P), -\mathcal{C}(Q))) \cup \{(0, \ldots, 0)\}) = \mathcal{C}'(\Delta_W(P, Q)).
\]
Moreover, we obtain
\[
\Gamma(\mathcal{C}(P), -\mathcal{C}(Q)) \supset \bigcup_{W \subset [d]} \mathcal{C}'(\Delta_W(P, Q)).
\]

We will show that for any $x, y \in V(\Gamma(\mathcal{C}(P), -\mathcal{C}(Q)))$ and $a, b \in \mathbb{R}$ with $a + b = 1, a \geq 0$ and $b \geq 0$, there exists $W \subset [d]$ such that $ax + by \in \mathcal{C}'(\Delta_W(P, Q))$. This shows $\bigcup_{W \subset [d]} \mathcal{C}'(\Delta_W(P, Q))$ is convex. When $x, y \in \mathcal{C}(P)$ or $x, y \in (-\mathcal{C}(Q))$, it clearly follows. Let
\[
A_1 = \{p_{i_1}, \ldots, p_{i_1}, p_{i_{1+1}}, \ldots, p_{i_m}\}
\]
and
\[
A_2 = \{q_{i_1}, \ldots, q_{i_1}, q_{i_{1+1}}, \ldots, q_{i_n}\}
\]
be antichains of $\mathcal{A}(P)$ and $\mathcal{A}(Q)$, and we set $x = \rho(A_1)$ and $y = -\rho(A_2)$. We should show the case $a \geq b$. Let $W = \{i_1, \ldots, i_m\} \subset [d]$ and $c = a - b$. Then
\[
A_1' = \{p_{i_1}, \ldots, p_{i_m}\}, \ A_2' = \{p_{i_{1+1}}, \ldots, p_{i_m}\} \text{ and } A_3' = \{q_{i_{m+1}}, \ldots, q_{i_n}\}
\]
are antichains of $\Delta_W(P, Q)$. We set $x' = \rho(A_1'), y' = \rho(A_2')$ and $z' = \rho(A_3')$. Then we have
\[
ax + by = cx' + by' + bz' \text{ and } c + 2b = 1.
\]
Hence $ax + by \in \mathcal{C}'(\Delta_W(P, Q))$.

Therefore, since $\bigcup_{W \subset [d]} \mathcal{C}'(\Delta_W(P, Q))$ is convex, we have
\[
\Gamma(\mathcal{C}(P), -\mathcal{C}(Q)) = \bigcup_{W \subset [d]} \mathcal{C}'(\Delta_W(P, Q)).
\]
In particular, 
\[ \Gamma(\mathcal{C}(P), -\mathcal{C}(Q))_W = C'(\Delta_W(P, Q)). \]
as desired. 

In this section, we give the formula of the volume of \( \Gamma(\mathcal{C}(P), -\mathcal{C}(Q)) \) in terms of partially ordered sets \( P \) and \( Q \). In fact,

**Theorem 1.3.** Let \( P = \{p_1, \ldots, p_d\} \) and \( Q = \{q_1, \ldots, q_d\} \) be partially ordered sets. Then we have

\[ \text{vol}(\Gamma(\mathcal{C}(P), -\mathcal{C}(Q))) = \sum_{W \subset [d]} \frac{e(\Delta_W(P, Q))}{d!}. \]

**Proof.** By Lemma 1.1 and Proposition 1.2 it immediately follows. \( \Box \)

We recall that the volume of \( \Gamma(\mathcal{C}(P), -\mathcal{C}(Q)) \) equals that of \( \Gamma(\mathcal{O}(P), -\mathcal{C}(Q)) \). Moreover, if \( P \) and \( Q \) have a same linear extension, the volume of \( \Gamma(\mathcal{C}(P), -\mathcal{C}(Q)) \) equals that of \( \Gamma(\mathcal{O}(P), -\mathcal{O}(Q)) \). Hence by Theorem 1.3 we obtain the following corollary.

**Corollary 1.4.** Let \( P = \{p_1, \ldots, p_d\} \) and \( Q = \{q_1, \ldots, q_d\} \) be partially ordered sets. Then we have

\[ \text{vol}(\Gamma(\mathcal{O}(P), -\mathcal{C}(Q))) = \sum_{W \subset [d]} \frac{e(\Delta_W(P, Q))}{d!}. \]

Moreover, if \( P \) and \( Q \) have a same linear extension, then we have

\[ \text{vol}(\Gamma(\mathcal{O}(P), -\mathcal{O}(Q))) = \sum_{W \subset [d]} \frac{e(\Delta_W(P, Q))}{d!}. \]

**Remark 1.5.** By the proof of Proposition 1.2 for any \( W \subset [d] \), \( \Gamma(\mathcal{C}(P), -\mathcal{C}(Q))_W \) is an integral convex polytope. However, \( \Gamma(\mathcal{O}(P), -\mathcal{C}(Q))_W \) and \( \Gamma(\mathcal{O}(P), -\mathcal{C}(Q))_W \) are not always integral. In fact, let \( P = \{p_1, p_2\} \) be a 2-element chain with \( p_1 \leq p_2 \) and \( Q = \{q_1, q_2\} \) a 2-element chain with \( q_1 \leq q_2 \). Then for \( W = \{1\} \), we know that \( \Gamma(\mathcal{O}(P), -\mathcal{O}(Q))_W \) and \( \Gamma(\mathcal{O}(P), -\mathcal{C}(Q))_W \) are not integral convex polytopes. This means that it is difficult to prove Corollary 1.4 without using \( \Gamma(\mathcal{C}(P), -\mathcal{C}(Q)) \).

In the end of this section, we give examples.

**Example 1.6.** Let \( P = \{p_1, p_2, p_3\} \) and \( Q = \{q_1, q_2, q_3\} \) be partially ordered sets as follows.

\[ P: \begin{array}{c}
  p_1 \\
  \downarrow \\
  p_2 \\
  \downarrow \\
  p_3 \\
\end{array} \quad Q: \begin{array}{c}
  q_1 \\
  \downarrow \\
  q_2 \\
  \downarrow \\
  q_3 \\
\end{array} \]
Then $\Gamma(C(P), -C(Q))$ is centrally symmetric, i.e., for each facet $F$ of $\Gamma(C(P), -C(Q))$, $-F$ is a facet of $\Gamma(C(P), -C(Q))$. For each subset $W$ of $\{1, 2, 3\}$, $\Delta_W(P, Q)$ is the following:

\[
\Delta_{\{1,2,3\}}(P, Q): \quad \Delta_{\{1,2\}}(P, Q): \quad \Delta_{\{1,3\}}(P, Q): \quad \Delta_{\{2,3\}}(P, Q):
\]

\[
\Delta_{\{1\}}(P, Q): \quad \Delta_{\{2\}}(P, Q): \quad \Delta_{\{3\}}(P, Q): \quad \Delta_{\emptyset}(P, Q):
\]

Hence we have

$$\text{vol}(\Gamma(C(P), -C(Q))) = 4 \times \frac{1}{6} + 4 \times \frac{2}{6} = 2.$$  

**Example 1.7.** Let $P = \{p_1, \ldots, p_d\}$ be a $d$-element antichain and $Q = \{q_1, \ldots, q_d\}$ a $d$-element chain with $q_1 \leq \cdots \leq q_d$. For $W \subset [d]$, we will compute the volume of $C'(\Delta_W(P, Q))$. We set $W = \{1, \ldots, k\}$. Then $P_W$ is a $k$-element antichain and $Q_W$ is a $(d-k)$-element chain. Hence we have

$$C'(\Delta_W(P, Q)) = \text{conv}(\{[0, 1]^k \times \{0\}^{d-k}, -e_{k+1}, \ldots, -e_d\})$$

and $\text{vol}(C'(\Delta_W(P, Q))) = k!/d!$. Therefore, we obtain

$$\text{vol}(\Gamma(C(P), -C(Q))) = \sum_{k=0}^{d} \binom{d}{k} \frac{k!}{d!} = \sum_{k=0}^{d} \frac{1}{k!}$$

Moreover, by Corollary 1.4, we have

$$\text{vol}(\Gamma(O(P), -O(Q))) = \text{vol}(\Gamma(O(P), -C(Q))) = \sum_{k=0}^{d} \frac{1}{k!}.$$

**Remark 1.8.** For a positive integer $d$, We write $a(d)$ for total number of arrangements of a $d$-element set. Then

$$\text{vol}(\Gamma(C(P), -C(Q))) = \frac{a(d-1)}{d!}.$$
where \( P \) and \( Q \) are partially ordered sets in Example 1.7.

2. FACETS AND DUAL POLYTOPES OF TWINNED CHAIN POLYTOPES

We recall properties of facets of chain polytopes. Let \( P = \{p_1, \ldots, p_d\} \) be a partially ordered set. Then there are two types of the supporting hyperplanes of facets for the chain polytope \( C(P) \),

- for each element \( p_i \) of the partially ordered set \( P \), \( x_i = 0 \),
- for each maximal chain \( C \) of the partially ordered set \( P \), \( \sum_{p_i \in C} x_i = 1 \).

We write \( M(P) \) for the set of maximal chains of \( P \). Then the number of facets of \( C(P) \) equals \( |M(P)| + d \).

At first, we show the following lemma.

**Lemma 2.1.** Let \( P = \{p_1, \ldots, p_d\} \) and \( Q = \{q_1, \ldots, q_d\} \) be partially ordered sets, and let \( W \subset [d] \). Then there are three types of the supporting hyperplanes of facets for \( C'((\Delta_W(P, Q))) \),

- for each element \( p_i \) of the partially ordered set \( \Delta_W(P, Q) \), \( x_i = 0 \),
- for each element \( q_j \) of the partially ordered set \( \Delta_W(P, Q) \), \( -x_j = 0 \),
- for each maximal chain \( C \) of the partially ordered set \( \Delta_W(P, Q) \), \( \sum_{p_i \in C} x_i - \sum_{q_j \in C} x_j = 1 \).

**Proof.** By the proof of Lemma 1.1 it immediately follows. \( \square \)

In this section, we characterize facets of \( \Gamma(C(P), -C(Q)) \) in terms of partially ordered sets \( P \) and \( Q \). In fact,

**Theorem 2.2.** Let \( P = \{p_1, \ldots, p_d\} \) and \( Q = \{q_1, \ldots, q_d\} \) be partially ordered sets. Then there exists just one type of the supporting hyperplanes of facets for \( \Gamma(C(P), -C(Q)) \),

- for each \( W \subset [d] \) and for each maximal chain \( C \) of the partially ordered set \( \Delta_W(P, Q) \), \( \sum_{p_i \in C} x_i - \sum_{q_j \in C} x_j = 1 \).

Moreover, the number of facets of \( \Gamma(C(P), -C(Q)) \) equals \( |\bigcup_{W \subset [d]} M(\Delta_W(P, Q))| \).

**Proof.** We let \( W \) be a subset of \([d]\) and let \( C \) be a maximal chain of \( \Delta_W(P, Q) \). Then by Lemma 2.1 \( F_C = H_C \cap C'(\Delta_W(P, Q)) \) is a facet of \( C'(\Delta_W(P, Q)) \), where \( H_C \) is the hyperplane

\[
\{(x_1, \ldots, x_d) \in \mathbb{R}^d \mid \sum_{p_i \in C} x_i - \sum_{q_j \in C} x_j = 1\}
\]

in \( \mathbb{R}^d \). We let \( y = (y_1, \ldots, y_d) \) be an interior point of \( F_C \). Then by Lemma 2.1 we know \( y_i > 0 \) if \( i \in W \) and \( y_j < 0 \) if \( j \in W^c \). Hence for any \( W' \subset [d] \) with \( W \neq W' \), we have \( y \notin C'(\Delta_W'(P, Q)) \). Therefore, it follows that \( y \) doesn’t belong to the interior of \( \Gamma(C(P), -C(Q)) \). By Proposition 1.2 \( H_C \cap \Gamma(C(P), -C(Q)) \) is a facet of \( \Gamma(C(P), -C(Q)) \).
Since $\Gamma(C(P), -C(Q))$ is Gorenstein Fano, the supporting hyperplane of each facet of $\Gamma(C(P), -C(Q))$ is of the form $a_1x_1 + \cdots + a_dx_d = 1$ with each $a_i \in \mathbb{Z}$. Hence there is just one type of the supporting hyperplanes of facets for $\Gamma(C(P), -C(Q))$, for each $W \subseteq [d]$ and for each maximal chain $C$ of the partially ordered set $\Delta_W(P, Q)$, \[
abla \sum_{i \in C} x_i - \sum_{j \in C} x_j = 1, \text{ as desired.} \] □

**Remark 2.3.** For some partially ordered sets $P$ and $Q$, it follows that \[
abla \sum_{W \subset [d]} |M(\Delta_W(P, Q))| \neq | \bigcup_{W \subset [d]} M(\Delta_W(P, Q))|. \]
For instance, let $P = \{p_1, p_2, p_3\}$ and $Q = \{q_1, q_2, q_3\}$ be 3-element antichains. For $W_1 = \{1\}$, $C_1 = \{p_1, q_3\}$ is a maximal chain of $\Delta_{W_1}(P, Q)$. Then for $W_2 = \{1, 2\}$, $C_1$ is also a maximal chain of $\Delta_{W_2}(P, Q)$. Hence we have \[
abla \sum_{W \subset [d]} |M(\Delta_W(P, Q))| > | \bigcup_{W \subset [d]} M(\Delta_W(P, Q))|. \]

We give an example of Theorem 2.2.

**Example 2.4.** Let $P$ and $Q$ be partially ordered sets in Example 1.7. We fix $W = \{i_1, \ldots, i_k\} \subset [d]$. Then we have \[
abla M(\Delta_W(P, Q)) = \\{p_{i_s}, q_{i_k+s}, \ldots, q_{i_k}\} \mid 1 \leq s \leq k\}
and $|M(\Delta_W(P, Q))| = k$. Hence \[
abla | \bigcup_{W \subset [d]} M(\Delta_W(P, Q))| = \sum_{k=1}^{d} \binom{d}{k} k + 1 \]
\[
abla = d \cdot 2^{d-1} + 1. \]

Let $\mathcal{P} \subset \mathbb{R}^d$ be a Gorenstein Fano polytope of dimension $d$. Then a point $a \in \mathbb{R}^d$ is a vertex of $\mathcal{P}^\vee$ if and only if $H \cap \mathcal{P}$ is a facet of $\mathcal{P}$, where $H$ is the hyperplane \[
abla \{x \in \mathbb{R}^d \mid \langle a, x \rangle = 1\}
\] in $\mathbb{R}^d$ ([3 Corollary 35.6]). Hence by Theorem 2.1, we obtain the following Corollary.

**Corollary 2.5.** Let $P = \{p_1, \ldots, p_d\}$ and $Q = \{q_1, \ldots, q_d\}$ be partially ordered sets. Then we have \[
abla V(\Gamma(C(P), -C(Q))^\vee) = \bigcup_{W \subset [d]} \{\rho'(C) \in \mathbb{R}^d \mid C \in \mathcal{M}(\Delta_W(P, Q))\},
\] namely, \[
abla \Gamma(C(P), -C(Q))^\vee = \text{conv} \left( \bigcup_{W \subset [d]} \{\rho'(C) \in \mathbb{R}^d \mid C \in \mathcal{M}(\Delta_W(P, Q))\} \right).
\]
Finally, we give an example of Corollary 2.5.
Example 2.6. Let $P$ and $Q$ be the partially ordered sets in Example 1.6. Then by Corollary 2.5, the vertices of $\Gamma(C(P), -C(Q))^\vee$ are the followings:

$$\pm(1, 1, 0), \pm(1, 0, 1), \pm(1, -1, 0), \pm(1, 1, -1), \pm(1, -1, 1), \pm(1, 0, -1).$$

Moreover, there don’t exist partially ordered sets $P'$ and $Q'$ with $|P'| = |Q'| = d$ such that $\Gamma(C(P'), -C(Q'))^\vee$ and $\Gamma(C(P'), -C(Q'))$ are unimodularly equivalent. Indeed, since $\Gamma(C(P), -C(Q))^\vee$ is centrally symmetric and the number of its vertices equals 12, each of $P'$ and $Q'$ needs to have just 7 antichains. However, there exists no 3-element partially ordered set which has just 7 antichains.

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