Rotating Black Branes in Brans-Dicke-Born-Infeld Theory

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In this paper, we present a new class of charged rotating black brane solutions in the higher dimensional Brans-Dicke-Born-Infeld theory and investigate their properties. Solving the field equations directly is a non-trivial task because they include the second derivatives of the scalar field. We remove this difficulty through a conformal transformation. Also, we find that the suitable Lagrangian of Einstein-Born-Infeld-dilaton gravity is not the same as presented in [12]. We show that the given solutions can present black brane, with inner and outer event horizons, an extreme black brane or a naked singularity provided the parameters of the solutions are chosen suitably. These black brane solutions are neither asymptotically flat nor (anti)-de Sitter. Then we calculate finite Euclidean action, the conserved and thermodynamic quantities through the use of counterterm method. Finally, we argue that these quantities satisfy the first law of thermodynamics, and the entropy does not follow the area law.

I. INTRODUCTION

Lately there have been some renewed interest in the Brans-Dicke (BD) theory of gravitation [1]. On one hand, it is important for cosmological inflation models [2], in which the scalar field allows the inflationary epoch to end via bubble nucleation without the need for fine-tuning cosmological parameters (the "graceful exit" problem). Also, it was found that in the low-energy regime, the theory of fundamental strings can be reduced to an effective BD one [3].

Because scalar-tensor gravitation can agree with general relativity (GR) in the post-Newtonian limit, it is important to study strong field examples in which the two theories may give different predictions. These examples may not only provide further experimental and observational tests that might distinguish between GR and scalar-tensor gravitation, but they may also illuminate the structure of both theories.

The BD theory incorporates the Mach principle, which states that the phenomenon of inertia must arise from accelerations with respect to the general mass distribution of the universe. This theory is self-consistent, complete and for $|\omega| \geq 500$ in accord with solar system observations and

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experiments [4], where $\omega$ is an adjustable parameter. In this theory, the matter couples minimally to the metric and not directly to scalar field. Indeed, the scalar field does not exert any direct influence on matter, its only role is that of participating in the field equations that determine the geometry of the spacetime. More recently, many authors have investigated the gravitational collapse and black hole formation in the BD theory [5, 6].

Till now, nonlinear charged rotating black hole solutions for an arbitrary value of $\omega$ has not been constructed. In this paper, we want to construct exact rotating black brane solutions in BD-Born-Infeld (BDBI) theory for an arbitrary value of $\omega$ and investigate their properties. One can find that solving the field equations directly is a non-trivial task, because they include the second derivatives of the scalar field. We remove this difficulty through a conformal transformation. By using this transformation, the BDBI action reduce to Einstein-Born-Infeld-dilaton (EBId) action, and one can solve their field equations analytically.

The idea of the non-linear electrodynamics (BI) was first introduced in 1934 by Born and Infeld in order to obtain a finite value for the self-energy of point-like charges [7]. Although it become less popular with the introduction of QED, in recent years, the BI action has been occurring repeatedly with the development of superstring theory, where the dynamics of $D$-branes is governed by the BI action [8, 9]. Lately, black hole solutions in BI gravity with or without a cosmological constant have been considered by many authors [10, 11]. Both of the Lagrangians of EBId gravity presented here and in [12] show similar asymptotic behavior but only the one we considered is consistent with BD theory.

The outline of this paper is organized as follows. In Sec. II, we give a brief review of the field equations of BDBI theory in Jordan (or string) and Einstein frames. In Sec. III, we obtain charge rotating solution in $(n + 1)$-dimensions with $k$ rotation parameters and investigate their (asymptotic) properties. Sec. IV is devote to calculation of the finite action, the conserved and thermodynamic quantities of the $(n + 1)$-dimensional black brane solutions with a complete set of rotational parameters. We finish our paper with some concluding remarks.

II. FIELD EQUATION AND CONFORMAL TRANSFORMATION

In $n + 1$ dimensions, the action of the BDBI theory with one scalar field $\Phi$ and a self-interacting potential $V(\Phi)$ can be written as

$$I_G = -\frac{1}{16\pi} \int_{\mathcal{M}} d^{n+1}x \sqrt{-g} \left( \Phi R - \frac{\omega}{\Phi} (\nabla \Phi)^2 - V(\Phi) + L(F) \right),$$ (1)
where $\mathcal{R}$ is the Ricci scalar, $\omega$ is the coupling constant, $\Phi$ denotes the BD scalar field and $V(\Phi)$ is a self-interacting potential for $\Phi$ and $L(F)$ is the Lagrangian of BI

$$L(F) = 4\beta^2 \left(1 - \sqrt{1 + \frac{F^2}{2\beta^2}}\right),$$

(2)

In Eq. (2), the constant $\beta$ is called BI parameter with dimension of mass, $F^2 = F^{\mu\nu}F_{\mu\nu}$ where $F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}$ is electromagnetic tensor field and $A_\mu$ is the vector potential. In the limit $\beta \to \infty$, $L(F)$ reduces to the standard Maxwell form $L(F) = -F^2$, while $L(F) \to 0$ as $\beta \to 0$.

Varying the action (1) with respect to the metric, scalar and vector fields give the field equations as

$$G_{\mu\nu} = \frac{\omega}{\Phi^2} \left(\nabla_\mu \Phi \nabla_\nu \Phi - \frac{1}{2} g_{\mu\nu}(\nabla \Phi)^2\right) - \frac{V(\Phi)}{2\Phi} g_{\mu\nu} + \frac{1}{\Phi} \left(\nabla_\mu \nabla_\nu \Phi - g_{\mu\nu}\nabla^2 \Phi\right)
+ \frac{2}{\Phi} \left(\frac{1}{4} g_{\mu\nu} L(F) + \frac{F_{\mu\lambda}F^\lambda_{\nu}}{\sqrt{1 + F^2/2\beta^2}}\right),$$

(3)

$$\nabla^2 \Phi = \frac{1}{2[(n-1)\omega + n] \left[\left((n+1)L(F) + \frac{4F^2}{\sqrt{1 + F^2/2\beta^2}}\right)
+ \left((n-1)\Phi \frac{dV(\Phi)}{d\Phi} - (n+1)V(\Phi)\right)\right],$$

(4)

$$\partial_\mu \left(\frac{\sqrt{-g}F^{\mu\nu}}{\sqrt{1 + F^2/2\beta^2}}\right) = 0,$$

(5)

where $G_{\mu\nu}$ and $\nabla_\mu$ are the Einstein tensor and covariant differentiation corresponding to the metric $g_{\mu\nu}$ respectively. Solving the field equations (3)-(5) directly is a non-trivial task because the right hand side of Eq. (3) includes the second derivatives of the scalar. We can remove this difficulty by the conformal transformation

$$\bar{g}_{\mu\nu} = \Phi^{2/(n-1)} g_{\mu\nu},$$

$$\bar{\Phi} = \frac{n-3}{4\alpha} \ln \Phi,$$

(6)

where

$$\alpha = \frac{n-3}{\sqrt{4(n-1)\omega + 4n}}$$

(7)

One may note that $\alpha$ goes to zero as $\omega$ goes to infinity and the BD theory reduces to Einstein theory. By this transformation, the action (1) transforms to

$$\bar{I}_G = -\frac{1}{16\pi} \int_{\mathcal{M}} d^{n+1}x \sqrt{-g} \left\{\bar{\mathcal{R}} - \frac{4}{n-1}(\nabla \bar{\Phi})^2 - \bar{V}(\bar{\Phi}) + \bar{L}(\bar{F}, \bar{\Phi})\right\},$$

(8)
where \( \bar{R} \) and \( \bar{\nabla} \) are the Ricci scalar and covariant differentiation corresponding to the metric \( \bar{g}_{\mu\nu} \), and \( \bar{V}(\bar{\Phi}) \) is

\[
\bar{V}(\bar{\Phi}) = \Phi^{-(n+1)/(n-1)} V(\Phi)
\]

The Born-Infeld Lagrangian coupled to a dilaton field, \( \bar{L}(\bar{F}, \bar{\Phi}) \) corresponding to the metric \( \bar{g}_{\mu\nu} \) is given by

\[
\bar{L}(\bar{F}, \bar{\Phi}) = 4\beta^2 e^{-4\alpha/2} \bar{F}^2
\]

In the limit \( \beta \to \infty \), \( \bar{L}(\bar{F}, \bar{\Phi}) \) reduces to the standard Maxwell field coupled to a dilaton field

\[
\bar{L}(\bar{F}, \bar{\Phi}) = -e^{-4\alpha/2} \bar{F}^2.
\]

On the other hand, \( \bar{L}(\bar{F}, \bar{\Phi}) \to 0 \) as \( \beta \to 0 \). It is convenient to set

\[
\bar{L}(\bar{F}, \bar{\Phi}) = 4\beta^2 e^{-4\alpha/2} \bar{F}^2
\]

where

\[
\bar{L}(\bar{\Phi}) = 1 - \sqrt{1 + \bar{\Phi}}, \quad \bar{\Phi} = \frac{e^{16\alpha/2}}{(n-1)(n-3)} \bar{F}^2.
\]

It is notable that this action is different from the action of EBId that has been presented in [12]. Due to the fact that the action of EBId in [12] is not consistent with conformal transformation, one can find that Eq. (8) is the suitable action for EBId gravity.

Varying the action \( \bar{L} \) with respect to \( \bar{g}_{\mu\nu} \), \( \bar{F}_{\mu\nu} \), and \( \bar{\Phi} \), we obtain equations of motion as

\[
\bar{R}_{\mu\nu} = \frac{4}{n-1} \left( \bar{\nabla}_{\mu} \bar{\Phi} \bar{\nabla}_{\nu} \bar{\Phi} + \frac{1}{4} \bar{V} \bar{g}_{\mu\nu} \right) - \frac{1}{(n-1)} \bar{L}(\bar{F}, \bar{\Phi}) \bar{g}_{\mu\nu}
\]

\[
-2e^{-4\alpha/2} \frac{\partial \bar{L}(\bar{Y})}{\partial \bar{\Phi}} \left( \bar{F}_{\mu\lambda} \bar{F}^{\lambda}_{\nu} - \frac{2\bar{F}^2}{(n-1)} \bar{g}_{\mu\nu} \right),
\]

\[
\bar{\nabla}^2 \bar{\Phi} = \frac{n-1}{8} \frac{\partial \bar{V}}{\partial \bar{\Phi}} + \frac{\alpha}{2(n-3)} \left( (n+1) \bar{L}(\bar{F}, \bar{\Phi}) - 8e^{-4\alpha/2} \bar{F}^2 \right),
\]

\[
\partial_{\mu} \left[ \sqrt{-\bar{g}} e^{-4\alpha/2} \bar{\Phi} \bar{L}(\bar{Y}) \bar{F}^{\mu\nu} \right] = 0.
\]

Therefore, if \( (\bar{g}_{\mu\nu}, \bar{F}_{\mu\nu}, \bar{\Phi}) \) is the solution of Eqs. (14)-(16) with potential \( \bar{V}(\bar{\Phi}) \), then

\[
[g_{\mu\nu}, F_{\mu\nu}, \Phi] = \left[ \exp \left( -\frac{8\alpha \bar{\Phi}}{(n-1)(n-3)} \right) \bar{g}_{\mu\nu}, \bar{F}_{\mu\nu}, \exp \left( \frac{4\alpha \bar{\Phi}}{n-3} \right) \right]
\]

is the solution of Eqs. (13)-(15) with potential \( V(\Phi) \).
III. CHARGED ROTATING SOLUTIONS IN \( n + 1 \) DIMENSIONS WITH \( k \) ROTATION PARAMETERS

Here we construct the \((n + 1)\)-dimensional solutions of BD theory with \( n \geq 4 \) and the quadratic potential

\[
V(\Phi) = 2\Lambda \Phi^2
\]

Applying the conformal transformation (15), the potential \( \bar{V}(\bar{\Phi}) \) becomes

\[
\bar{V}(\bar{\Phi}) = 2\Lambda \exp\left(\frac{4\alpha \bar{\Phi}}{n-1}\right),
\]

which is a Liouville-type potential. Thus, the problem of solving Eqs. (13)-(15) with quadratic potential reduces to the problem of solving Eqs. (11)-(16) with Liouville-type potential.

The rotation group in \((n + 1)\) dimensions is \( SO(n) \) and therefore the number of independent rotation parameters for a localized object is equal to the number of Casimir operators, which is \( [n/2] \equiv k \), where \( [x] \) is the integer part of \( x \). The solutions of the field equations (14)-(16) with \( k \) rotation parameter \( a_i \), and Liouville-type potential is

\[
d\bar{s}^2 = -f(r) \left( \Xi dt - \sum_{i=1}^{k} a_i d\varphi_i \right)^2 + \frac{r^2}{l^2} R^2(r) \sum_{i=1}^{k} \left( a_i dt - \Xi l^2 d\varphi_i \right)^2
\]

\[
- \frac{r^2}{l^2} R^2(r) \sum_{i<j}^{k} (a_i d\varphi_j - a_j d\varphi_i)^2 + \frac{dr^2}{f(r)} + \frac{r^2}{l^2} R^2(r) dX^2,
\]

\[
\Xi^2 = 1 + \sum_{i=1}^{k} \frac{a_i^2}{l^2},
\]

\[
\bar{F}_{tr} = \frac{q \Xi \beta e^{4\alpha \bar{\Phi}/(n-1)}}{\sqrt{q^2 e^{8\alpha \bar{\Phi}/(n-3)} + \beta^2 r^{2n-2} R^{2n-2}}},
\]

\[
\bar{F}_{\varphi r} = -\frac{a_i}{\Xi} \bar{F}_{tr},
\]

where \( l \) is a constant, called length scale and \( dX^2 \) is the flat Euclidean metric on \((n - k - 1)\)-dimensional submanifold with volume \( \omega_{n-k-1} \). Here \( f(r) \), \( R(r) \) and \( \bar{\Phi}(r) \) are

\[
f(r) = \left( \frac{(1 + \alpha^2) r^2}{(n-1)} \right) \left( \frac{2\Lambda (\frac{e}{r})^{-2\gamma}}{(\alpha^2 - n)} + \frac{4(n-3)\beta^2 (\frac{e}{r})^{2\gamma(n+1)/(n-3)}}{\lambda} \right) - \frac{m}{r^{(n-1)(1-\gamma)-1}} - \frac{4(1 + \alpha^2)^2 q^2 (\frac{e}{r})^{2\gamma(n-2)}}{\lambda r^{2(n-2)}} F(\eta),
\]

\[
F(\eta) = \frac{(n-3)\sqrt{1 + \eta}}{(n-1)\eta} - \frac{1}{2} \text{F} \left( \left[ \frac{1}{2}, \frac{(n-3)\Upsilon}{2(n-1)} \right], \left[ 1 + \frac{(n-3)\Upsilon}{2(n-1)} \right], -\eta \right),
\]

\[
\eta = \frac{q^2 (\frac{e}{r})^{2\gamma(n-1)(n-5)/(n-3)}}{\beta^2 r^{2(n-1)}},
\]

\[
\Upsilon = \frac{2n^2 + n - 5}{2n^2 - 3n + 1}.
\]
\[ \Upsilon = \frac{\alpha^2 + n - 2}{2\alpha^2 + n - 3} \]
\[ \lambda = (3n - 1)\alpha^2 + n(n - 3), \]
\[ R(r) = \exp\left(\frac{2\alpha \Phi}{n-1}\right) = \left(\frac{r}{c}\right)^{-\gamma}, \]
\[ \Phi(r) = -\frac{(n-1)\gamma}{2\alpha} \ln\left(\frac{r}{c}\right), \]

where \( c \) is an arbitrary constant and \( \gamma = \alpha^2/(\alpha^2 + 1) \). Using the conformal transformation (17), the \( (n + 1) \)-dimensional rotating solutions of BD theory with \( k \) rotation parameters can be obtained as

\[
ds^2 = -U(r) \left( \Xi dt - \sum_{i=1}^{k} a_i d\varphi_i \right)^2 + \frac{r^2}{l^2} H^2(r) \sum_{i=1}^{k} \left( a_i dt - \Xi l^2 d\varphi_i \right)^2 - \frac{r^2}{l^2} H^2(r) \sum_{i<j}^{k} (a_i d\varphi_j - a_j d\varphi_i)^2 + \frac{d\varphi_i}{V(r)} + \frac{r^2}{l^2} H^2(r) dX^2, \]

where \( U(r), V(r), H(r) \) and \( \Phi(r) \) are

\[
U(r) = \left(\frac{r}{c}\right)^{4\gamma/(n-3)} f(r), \]
\[
V(r) = \left(\frac{r}{c}\right)^{-4\gamma/(n-3)} f(r), \]
\[
H(r) = \left(\frac{r}{c}\right)^{-\gamma(n-5)/(n-3)}, \]
\[
\Phi(r) = \left(\frac{r}{c}\right)^{-2\gamma(n-1)/(n-3)}. \]

The electromagnetic field becomes:

\[
F_{\tau r} = \frac{q\Xi\beta \left(\frac{r}{c}\right)^{-4\gamma/(n-3)}}{\sqrt{q^2 + \beta^2 r^2(1-\gamma(n-5)/(n-3))c^{2\gamma(n-1)/(n-3)/}}}, \quad F_{\varphi \tau} = -\frac{a_i}{\Xi} F_{\tau r}. \]

It is worth to note that the scalar field \( \Phi(r) \) and electromagnetic field \( F_{\mu\nu} \) become zero as \( r \) goes to infinity. These solutions reduce to the solutions presented in Ref. [6] as \( \beta \) goes to infinity. In the absence of a nontrivial dilaton (\( \alpha = \gamma = 0 \) or \( \omega \to \infty \)), the above solutions reduce to those of Ref. [14] and in the limit \( \beta \to \infty \) and \( \omega \to \infty \), these solutions reduce to the solutions of Refs. [15, 16]. It is also notable to mention that these solutions are valid for all values of \( \omega \).

A. Properties of the solutions

In order to study the general structure of these solutions, we first look for the essential singularities. One can show that the Kretschmann scalar \( R_{\mu\nu\lambda\kappa} R^{\mu\nu\lambda\kappa} \) diverges at \( r = 0 \), and therefore
there is a curvature singularity located at \( r = 0 \). Seeking possible black hole solutions, we turn to look for the existence of horizons. Because of the presence of the hypergeometric function in the equation \( f(r) = 0 \), the radius of the event horizon cannot be found explicitly. The roots of the metric function \( f(r) \) are located at

\[
-\frac{r_+^{(n-1)-n} m}{4(1 + \alpha^2)^2} + \frac{(\alpha^2 - n)}{2(n - 1)(\alpha^2 - n)} + \frac{(n - 3)\beta^2 \left( \frac{r_+}{c} \right)^{2\gamma(n+1)/(n-3)} (1 - \sqrt{1 + \eta_+})}{(n - 1)\Lambda} \\
+ \frac{r_+^{2\gamma(n-2)-(n-1)} q^2 F_1 \left( \frac{1}{2}, \frac{(n-3)\Upsilon}{2(n-1)}, \frac{1}{2(n-1)} \right)}{\lambda c^{2\gamma(n-2)} \Upsilon} = 0
\]  

(30)

The angular velocities \( \Omega_i \) are

\[
\Omega_i = \frac{a_i}{\Xi c^n},
\]

(31)

and the temperature may be obtained through the use of definition of surface gravity, \( \kappa \),

\[
T_+ = \frac{1}{\beta_+} = \frac{\kappa}{2\pi} = \frac{1}{2\pi} \sqrt{-\frac{1}{2} (\nabla_\mu \chi_\nu) (\nabla^\mu \chi^\nu)},
\]

(32)

where \( \chi \) is the Killing vector given by

\[
\chi = \partial_t + \sum_{i=1}^k \Omega_i \partial_{\phi_i}.
\]

(33)

One obtains

\[
T_+ = -(1 + \alpha^2)r_+ \left[ (\alpha^2 - n)r_+^{\gamma(n-1)} m \right] - \frac{8\alpha^2 \beta^2 \left( \frac{r_+}{c} \right)^{2\gamma(n+1)/(n-3)}}{\lambda} \left( 1 - \sqrt{1 + \eta_+} \right) \\
- \frac{2(\alpha^2 - n)}{\lambda \Upsilon r_+^{2(n-1)}} q^2 F_1 \left( \frac{1}{2}, \frac{(n-3)\Upsilon}{2(n-1)}, \frac{1}{2(n-1)} \right) \left[ 1 + \frac{(n-3)\Upsilon}{2(n-1)}, -\eta_+ \right] \\
= -\frac{(1 + \alpha^2)r_+}{2\pi \Xi (n - 1)} \left[ \left( \frac{r_+}{c} \right)^{-2\gamma} \Lambda - 2\beta^2 \left( \frac{r_+}{c} \right)^{2\gamma(n+1)/(n-3)} \left( 1 - \sqrt{1 + \eta_+} \right) \right]
\]

(34)

which shows that the temperature of the solution is invariant under the conformal transformation \( \Xi \). This is due to the fact that the conformal parameter is regular at the horizon.

Asymptotic Behavior:

\( \alpha^2 = n \): The solution is ill-defined for \( \alpha^2 = n \) with a quadratic potential (\( \Lambda \neq 0 \)).

\( \alpha^2 > n \): In this case, as \( r \) goes to infinity the dominant term in Eq. \( (25) \) is the second term (\( m \) term), and therefore the spacetime has a cosmological horizon for positive values of the mass parameter, despite the sign of the cosmological constant \( \Lambda \).

\( \alpha^2 < n \): For \( \alpha^2 < n \), as \( r \) goes to infinity the dominant term is the first term (\( \Lambda \) term), and therefore there exist a cosmological horizon for \( \Lambda > 0 \), while there is no cosmological horizons if
\( \Lambda < 0 \). Indeed, in the latter case \( (\alpha^2 < n \text{ and } \Lambda < 0) \) the spacetimes associated with the solution (25)-(28) exhibit a variety of possible causal structures depending on the values of the metric parameters \( \alpha, m, q \) and \( \Lambda \). One can obtain the causal structure by finding the roots of \( V(r) = 0 \).

Unfortunately, because of the nature of the exponents and hypergeometric function in (26), it is not possible to find explicitly the location of horizons. But, we can obtain some information by considering the temperature of the horizons. Here, we draw the Penrose diagram to show that the casual structure is asymptotically well behaved. For reason of economy, we draw the Penrose diagram only for the solution that presents a black brane with inner and outer horizons (negative \( \Lambda \) and \( \alpha < \sqrt{n} \)). The causal structure can be constructed following the general prescriptions indicated in [18]. The Penrose diagram is shown in Figs. 1 and 2 for \( \alpha < 1 \) and \( 1 \leq \alpha < \sqrt{n} \) respectively.

Also it is worth to write down the asymptotic behavior of the Ricci scalar. Indeed, the form of the Ricci scalar for large values of \( r \) is:

\[
R = -\frac{n^2}{(n-3)^2l^2} \left(\frac{2\alpha^2 + n - 3}{n - \alpha^2}\right) \left(\frac{c}{r}\right)^{2(n-1)\gamma/(n-3)}
\]  

(35)
which does not approach a nonzero constant as in the case of asymptotically AdS spacetimes. It is worth to mention that the Ricci scalar of the solution (24) - (27) goes to zero as \( r \) goes to infinity, but with a slower rate than that of an asymptotically flat spacetimes in the absence of the scalar field.

Equation (34) shows that the temperature is negative for the two cases of (i) \( \alpha > \sqrt{n} \) despite the sign of \( \Lambda \), and (ii) positive \( \Lambda \) despite the value of \( \alpha \). As we argued above in these two cases we encounter with cosmological horizons, and therefore the cosmological horizons have negative temperature. Indeed, the metric of Eqs. (24)-(28) has two inner and outer horizons located at \( r_- \) and \( r_+ \), provided the mass parameter \( m \) is greater than \( m_{\text{ext}} \), an extreme black brane in the case of \( m = m_{\text{ext}} \), and a naked singularity if \( m < m_{\text{ext}} \) only for negative \( \Lambda \) and \( \alpha < \sqrt{n} \) where

\[
m_{\text{ext}} = \frac{4(1 + \alpha^2)^2}{r_{\text{ext}}^{\gamma(n-1)-n}} \left( \frac{r_{\text{ext}}}{c} \right)^{-2\gamma} \frac{\Lambda}{2(n-1)(\alpha^2 - n)} + \frac{(n-3)2\beta^2}{(n-1)\lambda} \left( \frac{r_{\text{ext}}}{c} \right)^{2\gamma(n+1)/(n-3)} \left( 1 - \sqrt{1 + \eta_{\text{ext}}} \right)
\]

(36)

in Eq. (36), \( r_{\text{ext}} \) is the root of temperature relation (34) such that

\[
1 - \frac{\Lambda}{2\beta^2} \left( \frac{r_{\text{ext}}}{c} \right)^{-8\gamma/(n-3)} = 1 - \eta_{\text{ext}},
\]

(37)
where
\[
\eta_{\text{ext}} = \frac{q^2 \left( \frac{r_{\text{ext}}}{c} \right)^{2\gamma(n-1)(n-5)/(n-3)}}{\beta^2 r_{\text{ext}}^{2(n-1)}}.
\]

Note that in the absence of scalar field \((\alpha = \gamma = 0)\) \(m_{\text{ext}}\) reduces to that obtained in [16].

Next, we calculate the electric charge of the solutions. According to the Gauss theorem, the electric charge is the projections of the electromagnetic field tensor on special hypersurfaces. Denoting the volume of the hypersurface boundary at constant \(t\) and \(r\) by \(V_{n-1} = (2\pi)^k \omega_{n-k-1}\), the electric charge per unit volume \(V_{n-1}\) can be found by calculating the flux of the electric field at infinity, yielding
\[
Q = \frac{\Xi q}{4\pi l^{n-2}}.
\]

Comparing the above charge with the charge of black brane solutions of Einstein-Born–Infeld–dilaton gravity, one finds that charge is invariant under the conformal transformation (6). The electric potential \(U\), measured at infinity with respect to the horizon, is defined by [19]
\[
U = A_\mu \chi^\mu \big|_{r \to \infty} - A_\mu \chi^\mu \big|_{r = r_+},
\]
where \(\chi\) is the null generators of the event horizon. One can easily show that the vector potential \(A_\mu\) corresponding to electromagnetic tensor can be written as
\[
A_\mu = \frac{qc^{(3-n)\gamma}}{\Gamma r^\Gamma} \, 2F_1 \left( \left[ \frac{1}{2}, \frac{(n-3)\Upsilon}{2(n-1)} \right], \left[ 1 + \frac{(n-3)\Upsilon}{2(n-1)} \right], -\eta \right) (\Xi \delta^i_\mu - a_i \delta^i_\mu) \quad \text{(no sum on \(i\))},
\]
where \(\Gamma = (n-3)(1-\gamma) + 1\). Therefore the electric potential is
\[
U = \frac{qc^{(3-n)\gamma}}{2\Gamma r_+^\Gamma} \, 2F_1 \left( \left[ \frac{1}{2}, \frac{(n-3)\Upsilon}{2(n-1)} \right], \left[ 1 + \frac{(n-3)\Upsilon}{2(n-1)} \right], -\eta_+ \right).
\]

**IV. ACTION AND CONSERVED QUANTITIES**

The action does not have a well-defined variational principle, we should add the boundary action to it for ensuring well-defined Euler-Lagrange equations. The suitable boundary action is
\[
I_b = -\frac{1}{8\pi} \int_{\partial M} d^n x \sqrt{-\gamma K \Phi},
\]
where \(\gamma\) and \(K\) are the determinant of the induced metric and the trace of extrinsic curvature of boundary. In general the action \(I_G + I_b\) is divergent when evaluated on the solutions, as is the Hamiltonian and other associated conserved quantities. For asymptotically (A)dS solutions of
Einstein gravity, the way that one deals with these divergences is through the use of counterterm method inspired by (A)dS/CFT correspondence [20]. Although, in the presence of a non-trivial BD scalar field with potential \( V(\Phi) = 2\Lambda \Phi^2 \), the spacetime may not behave as either dS (\( \Lambda > 0 \)) or AdS (\( \Lambda < 0 \)). In fact, it has been shown that with the exception of a pure cosmological constant potential, where \( \alpha = 0 \), no AdS or dS static spherically symmetric solution exist for Liouville-type potential [21]. But, as in the case of asymptotically AdS spacetimes, according to the domain-wall/QFT (quantum field theory) correspondence [22], there may be a suitable counterterm for the action which removes the divergences. Since our solutions have flat boundary \([R_{abcd}(h) = 0]\), there exists only one boundary counterterm

\[
I_{ct} = -\frac{1}{8\pi} \int_{\partial M} d^n x \sqrt{-\gamma} \frac{(n-1)}{l_{\text{eff}}},
\]

where \( l_{\text{eff}} \) is given by

\[
l_{\text{eff}}^2 = \frac{(n-1)(\alpha^2 - n)}{2\Lambda \Phi^3}.
\]

One may note that as \( \alpha \) goes to zero, the effective \( l_{\text{eff}}^2 \) of Eq. (44) reduces to \( l^2 = -(n-1)/2\Lambda \) of the (A)dS spacetimes. The total action, \( I \), can be written as

\[
I = I_G + I_b + I_{ct}.
\]

The Euclidean actions (45) per unit volume \( V_{n-1} \) can be obtained as

\[
I = \frac{\beta^\gamma}{4\pi l^{n-2}} \frac{c^{(n-1)\gamma}}{\Lambda^\gamma(\gamma - 1)} \left[ 4\gamma + n - 3 \right] r_+^{2n-1} \frac{r_+^{2\gamma(n-2)}}{\alpha^2(n-2) \gamma(n-1)} \left[ \frac{1}{2} \left( \frac{n-3}{2(n-1)} \right), \left[ 1 + \frac{(n-3)}{2(n-1)} \right], -\eta_+ \right)
\]

\[
+ \frac{(1 + \alpha^2)r_+^{n-\gamma(n-1)}}{2(n-1)} \left[ \frac{(\alpha^2 - 1)}{\alpha^2 - n} \left[ \frac{r_+}{c} \right]^{-2\gamma} \right] \Lambda - \frac{2[\lambda + (n-1)(n-3)] \beta^2}{\lambda \left[ \frac{r_+}{c} \right]^{-2\gamma(n+1)/(n-3)} \left[ 1 - \sqrt{1 + \eta_+} \right]}
\]

It is easy to show that the mass \( M \) and the angular momentum \( J_i \) calculated in Jordan (or string) frame is the same as Einstein frame and they are remain unchanged under conformal transformations, that is

\[
J_i = \frac{c^{(n-1)\gamma}}{16\pi l^{n-2}} \left( \frac{n - \alpha^2}{1 + \alpha^2} \right) \Xi m a_i,
\]

\[
M = \frac{c^{(n-1)\gamma}}{16\pi l^{n-2}} \left( \frac{(n - \alpha^2)\Xi^2 + \alpha^2 - 1}{1 + \alpha^2} \right) m
\]

For \( a_i = 0 \ (\Xi = 1) \), the angular momentum per unit length vanishes, and therefore \( a_i \) is the \( i \)th rotational parameter of the spacetime.
We calculate the entropy through the use of Gibbs-Duhem relation
\[ S = \frac{1}{T} (\mathcal{M} - \Gamma_i C_i) - I, \tag{47} \]
where \( I \) is the finite total action \[46] evaluated on the classical solution, and \( C_i \) and \( \Gamma_i \) are the conserved charges and their associate chemical potentials respectively. It is straightforward to show that
\[ S = \Xi c (n - 1) \gamma 4 l (n - 2) r (n - 1)(1 - \gamma) + \]
for the entropy per unit volume \( V_{n - 1} \). It is worth to note that the area law is no longer valid in Brans-Dicke theory \[23\]. Nevertheless, the entropy remains unchanged under conformal transformations.

Comparing the conserved and thermodynamic quantities calculated in this section with those obtained in EBId gravity, one finds that they are invariant under the conformal transformation \[17\]. It is easy to show that these quantities calculated satisfy the first law of thermodynamics,
\[ dM = TdS + \sum_{i=1}^{k} \Omega_i dJ_i + UdQ \tag{49} \]

V. CLOSING REMARKS

The main goal of this paper is solving the field equations of BD theory in the presence of non-linear electromagnetic field for an arbitrary value of \( \omega \). As one can find, solving the field equations directly is a non-trivial task because they include the second derivatives of the scalar field. We could remove this difficulty through a conformal transformation. Indeed, after conformal transformation, the BDBI action is reduced to EBId action. We found that the suitable Lagrangian of EBId gravity is not the same as the one presented in \[12\], because it is not consistent with conformal transformation. We also found analytical solutions of BDBI theory, using conformal transformation \[6\] and investigated their properties. Then we found that these solutions which exist only for \( \alpha^2 \neq n \), have a cosmological horizon for (i) \( \alpha^2 > n \) despite(regardless of) the sign of \( \Lambda \), and (ii) positive values of \( \Lambda \), despite(disregarding) the magnitude of \( \alpha \). For \( \alpha^2 < n \), the solutions present black branes with inner and outer horizons if \( m > m_{ext} \), an extreme black brane if \( m = m_{ext} \), and a naked singularity otherwise. Also we presented Penrose diagrams and showed that the black brane solutions are neither asymptotically flat nor (anti)-de Sitter. We computed the finite Euclidean action through the use of counterterm method and obtained the thermodynamic
and conserved quantities of the solutions. We found that the entropy does not follow the area law in BDBI theory but one can show that it trace the area law in EBI gravity. One can find that the conserved and thermodynamic quantities are invariant under the conformal transformation and satisfy the first law of thermodynamics. The study of spherical symmetric solutions of BDBI theory with non-zero curvature boundary remains to be carried out in the future.

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