Derivations on Murray–von Neumann algebras

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For a given algebra \( \mathcal{A} \), a linear operator \( D: \mathcal{A} \to \mathcal{A} \) is called a derivation if \( D \) satisfies the Leibniz rule, that is, \( D(xy) = D(x)y +xD(y) \) for all \( x, y \in \mathcal{A} \). Each element \( a \in \mathcal{A} \) implements a derivation \( \text{ad}(a) \) on \( \mathcal{A} \) defined by \( \text{ad}(a)(x) = [a, x] = ax - xa, \ x \in \mathcal{A} \). Such derivations are said to be inner.

Let \( H \) be a Hilbert space, let \( B(H) \) be the \(*\)-algebra of all bounded linear operators on \( H \), and let \( \mathcal{M} \) be a von Neumann algebra, that is, a weakly closed unital \(*\)-subalgebra of \( B(H) \) (for details see [11]).

A densely defined closed linear operator \( x: \text{dom}(x) \to H \) (here the domain \( \text{dom}(x) \) of \( x \) is a linear subspace of \( H \)) is said to be affiliated with \( \mathcal{M} \) if \( yx \subset xy \) for all \( y \) in the commutant \( \mathcal{M}' \) of the algebra \( \mathcal{M} \).

Denote the set of all projections in \( \mathcal{M} \) by \( P(\mathcal{M}) \). Recall that two projections \( e, f \in P(\mathcal{M}) \) are said to be equivalent if there exists an element \( u \in \mathcal{M} \) such that \( u^*u = e \) and \( uu^* = f \). A projection \( p \in \mathcal{M} \) is said to be finite if the conditions \( q \leq p \) and \( q \sim p \) imply that \( q = p \). A linear operator \( x \) affiliated with \( \mathcal{M} \) is said to be measurable with respect to \( \mathcal{M} \) if \( \chi(\lambda, +\infty)(|x|) \) is a finite projection for some \( \lambda > 0 \). (Here \( \chi(\lambda, +\infty)(|x|) \) is the spectral projection of \( |x| \) corresponding to the interval \( (\lambda, +\infty) \)). We denote the set of all measurable operators by \( S(\mathcal{M}) \).

Let \( x, y \in S(\mathcal{M}) \). It is well known that \( x + y \) and \( xy \) are densely-defined and preclosed operators. Moreover, the closures of \( x + y, xy, \) and \( x^* \) are also in \( S(\mathcal{M}) \). The closures of \( x + y \) and \( xy \) are called the strong sum and strong product, respectively. When equipped with these operations, \( S(\mathcal{M}) \) becomes a unital \(*\)-algebra over \( \mathbb{C} \) (see [12] and [15]). It is clear that \( \mathcal{M} \) is a \(*\)-subalgebra of \( S(\mathcal{M}) \). In the case when \( \mathcal{M} \) is a finite von Neumann algebra, \( S(\mathcal{M}) \) is referred to as the Murray-von Neumann algebra associated with \( \mathcal{M} \) [9].

The hypothesis that all derivations of the algebra \( S(\mathcal{M}) \) associated with a von Neumann algebra \( \mathcal{M} \) of type II are inner was first conjectured by Ayupov (see [2] and [3]). As Kadison and Liu noted in [10], pp. 210–211 (see also [9], p. 2090), for type II\(_1\) algebras the “complete cohomological result would say that each derivation of \( S(\mathcal{M}) \) is inner. . . . The authors strongly feel that this is true; but it is still open”.

In this paper we announce the complete solution of this cohomological problem for type II\(_1\) von Neumann algebras \( \mathcal{M} \).

**Theorem 1.** Let \( \mathcal{M} \) be a type II\(_1\) von Neumann algebra, and let \( S(\mathcal{M}) \) be the Murray-von Neumann algebra of all operators affiliated with \( \mathcal{M} \). Then any derivation of \( S(\mathcal{M}) \) is inner.

In fact, we prove that any derivation of \( S(\mathcal{M}) \) is continuous in the topology of convergence in measure on \( S(\mathcal{M}) \), and then we use known results from [4], [5], and [7] giving us that any derivation of \( S(\mathcal{M}) \) which is continuous in this topology is necessarily inner.
When $\mathcal{M}$ is an arbitrary von Neumann algebra, Sankaran [14] and Yeadon [16] introduced the algebra $LS(\mathcal{M})$ of locally measurable operators affiliated with $\mathcal{M}$, with the operations of strong sum and strong multiplication. An operator $x$ affiliated with $\mathcal{M}$ is said to be locally measurable (with respect to $\mathcal{M}$) if there is a sequence $\{z_n\}_{n=0}^{\infty} \subset Z(\mathcal{M})$ of projections in the centre $Z(\mathcal{M})$ of $\mathcal{M}$ such that $z_n \uparrow 1$, $z_n(H) \subset \text{dom}(x)$, and $x z_n \in S(\mathcal{M})$ for all $n \geq 0$.

Using Theorem 1 and results from [1], [6], and [7], we obtain a necessary and sufficient condition for the existence of a non-inner derivation of the algebras $S(\mathcal{M})$ and $LS(\mathcal{M})$. This result provides a complete answer to the problem posed by Ayupov in [2] and an adaptation of the celebrated Kadison–Sakai theorem [8], [13] to algebras of unbounded operators.

**Corollary 2.** Let $\mathcal{M}$ be an arbitrary von Neumann algebra. Then the following assertions are equivalent:

(a) any derivation of $LS(\mathcal{M})$ (of $S(\mathcal{M})$) is inner;

(b) a type $I_{\text{fin}}$ direct summand of $\mathcal{M}$ is atomic.

In other words, the algebra $S(\mathcal{M})$ (or $LS(\mathcal{M})$) admits non-inner derivations if and only if the type $I_{\text{fin}}$ direct summand of $\mathcal{M}$ is non-trivial and non-atomic.

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**Bibliography**

[1] S. Albeverio, Sh. A. Ayupov, and K. K. Kudaybergenov, *J. Funct. Anal.* **256**:9 (2009), 2917–2943.

[2] Ш.А. Аюпов, *Докл. АН РУз*, 2000, № 3, 14–17. [Sh. A. Ayupov, *Dokl. Uzbek Akad. Nauk*, 2000, no. 3, 14–17.]

[3] Sh. A. Ayupov and K. K. Kudaybergenov, *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* **13**:2 (2010), 305–337.

[4] Sh. Ayupov and K. Kudaybergenov, *J. Math. Anal. Appl.* **408**:1 (2013), 256–267.

[5] А.Ф. Бер, *Матем. заметки* **93**:5 (2013), 658–664; English transl., A. F. Ber, *Math. Notes* **93**:5 (2013), 654–659.

[6] A. F. Ber, V. I. Chilin, and F. A. Sukochev, *Extracta Math.* **21**:2 (2006), 107–147.

[7] A. F. Ber, V. I. Chilin, and F. A. Sukochev, *Proc. Lond. Math. Soc.* (3) **109**:1 (2014), 65–89.

[8] R. V. Kadison, *Ann. of Math.* (2) **83**:2 (1966), 280–293.

[9] R. V. Kadison and Z. Liu, *Proc. Natl. Acad. Sci. USA* **111**:6 (2014), 2087–2093.

[10] R. V. Kadison and Z. Liu, *Math. Scand.* **115**:2 (2014), 206–228.

[11] R. V. Kadison and J. R. Ringrose, *Fundamentals of the theory of operator algebras*, vol. II: *Advanced theory*, Pure Appl. Math., vol. 100, Part 2, Academic Press, Inc., Orlando, FL 1986, i–xiv, 399–1074 pp.

[12] E. Nelson, *J. Funct. Anal.* **15**:2 (1974), 103–116.

[13] S. Sakai, *Ann. of Math.* (2) **83**:2 (1966), 273–279.

[14] S. Sankaran, *J. Lond. Math. Soc.* **34**:3 (1959), 337–344.

[15] I. E. Segal, *Ann. of Math.* (2) **57**:3 (1953), 401–457.
[16] F. J. Yeadon, Proc. Cambridge Philos. Soc. 74:2 (1973), 257–268.

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