SPECTRAL ESTIMATES FOR RUELLE OPERATORS
WITH TWO PARAMETERS AND SHARP LARGE DEVIATIONS

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Abstract. We obtain spectral estimates for the iterations of Ruelle operators
$L_{f + (a + ib)\tau + (c + id)g}$ with two complex parameters and Hölder continuous func-
tions $f, g$ generalizing the case $Pr(f) = 0$ studied in [9]. As an application
we prove a sharp large deviation theorem concerning exponentially shrinking
intervals which improves the result in [8].

1. Introduction. Let $M$ be a $C^2$ complete Riemannian manifold, and let $\varphi_t : M \to M$ be a $C^2$ weak mixing Axiom A flow ([5], [7]). Let $\Lambda$ be a basic set for $\varphi_t$,
that is, $\Lambda$ is a compact locally maximal invariant subset of $M$ and $\varphi_t$ is hyperbolic
and transitive on $\Lambda$.

As in [9], we will use a symbolic coding of the flow on $\Lambda$ provided by a fixed
Markov family $\{R_i\}_{i=1}^k$. More precisely, we consider a Markov family of pseudo-
rectangles $R_i = [U_i, S_i] = \{[x, y] : x \in U_i, y \in S_i\}$ (see section 2 for more details).
Denote by $P : R = \cup_{i=1}^k R_i \to R$ the related Poincaré map, by $\tau(x) > 0$ the first
return time function on $R$, and by $\sigma : U = \cup_{i=1}^k U_i \to U$ the shift map given by
$\sigma = \pi(U) \circ P$, where $\pi(U) : R \to U$ is the projection along stable leaves. The
flow $\varphi_t$ on $\Lambda$ is naturally related to the suspension flow $\sigma^T$ on the suspension space
$R^T$ (see section 2 for details). There exists a natural semi-conjugacy projection
$\pi(x, t) : R^T \to \Lambda$ which is one-to-one on a residual set (see [2]). For $x \in R$ set
$\tau^n(x) := \tau(x) + \tau(\sigma(x)) + \ldots + \tau(\sigma^{n-1}(x))$.

Given Hölder continuous functions $F, G : \Lambda \to \mathbb{R}$, define $f, g : R \to \mathbb{R}$ by
$f(x) = \int_0^{\tau(x)} F(\pi(x, t))dt$ , $g(x) = \int_0^{\tau(x)} G(\pi(x, t))dt$.

The main object of study in this paper are the Ruelle transfer operators of the form
$L_{f + s\tau + zg}v(x) = \sum_{\sigma y = x} e^{f(y) - s\tau(y) + z g(y)} v(y)$ , $s, z \in \mathbb{C}$ , $x \in U$. 

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depending on two complex parameters \( s \) and \( z \). Under certain assumptions, strong spectral estimates for such operators have been established in [9] and some significant applications to the study of zeta functions depending on two complex parameters have been made.

We denote by \( m_H \) the equilibrium state corresponding to \( H : R^* \to \mathbb{R} \) and by \( \mu_k \) the equilibrium state corresponding to \( k : R \to \mathbb{R} \). More precisely,

\[
\Pr_\sigma(k) = h(\sigma, \mu_k) + \int_R kd\mu_k = \sup_{\mu \in M_\sigma} \left\{ h(\sigma, \mu) + \int_R kd\mu \right\},
\]

and

\[
\Pr_\sigma(H) = h(\sigma^t, m_H) + \int_{R^*} Hdm_H = \sup_{m \in M_{\sigma^t}} \left\{ h(\sigma^t, m) + \int_{R^*} Hdm \right\},
\]

where \( h(\sigma, \mu) \) is the metric entropy of \( \sigma \) with respect to \( \mu \) and \( h(\sigma^t, m) \) is the metric entropy of the suspended flow \( \sigma^t \) with respect to \( m \). Let \( P = \Pr_\sigma(f) \).

Let \( \|h\|_0 \) denote the standard sup norm of \( h \) on \( \mathcal{U} \). For \( |b| \geq 1 \) and \( \beta > 0 \), as in [4], define the norm

\[
\|h\|_{\beta,b} = \|h\|_\infty + \frac{|h|_b}{|b|}
\]
on the space \( C^\beta(U) \) of \( \beta \)-Hölder functions on \( U \). Let \( \hat{U} \) be the set of those points \( x \in U \) such that \( \mathcal{P}^m(x) \) is not a boundary point of a rectangle for any integer \( m \). In a similar way define \( \hat{R} \).

Our first aim in this paper is to prove the following theorem.

**Theorem 1.** Let \( \varphi_t : M \to M \) satisfy the Standing Assumptions (see section 4) over the basic set \( \Lambda \), and let \( 0 < \beta < \alpha \). Let \( \mathcal{R} = \{R_i\}_{i=1}^k \) be a Markov family for \( \varphi_t \) over \( \Lambda \) as in section 2. Then for any real-valued functions \( f, g \in C^\infty(\hat{U}) \) and any constants \( \epsilon > 0 \) and \( B > 0 \) there exist constants \( 0 < \rho < 1 \), \( a_0 > 0 \), \( b_0 \geq 1 \) and \( C = C(B, \epsilon) > 0 \) such that if \( a, c \in \mathbb{R} \) satisfy \( |a|, |c| \leq a_0 \) then

\[
\|I_{f-a}^n + (c+b)xw\|_{\beta,b} \leq C e^{Pm} \rho^m |b|^\nu \|h\|_{\beta,b} \tag{1.1}
\]

for all \( h \in C^\beta(U) \), all integers \( m \geq 1 \) and all \( b, w \in \mathbb{R} \) with \( |b| \geq b_0 \) and \( |w| \leq B |b| \).

In Theorem 5.1 in [9] the above estimate has been proved in the case \( P = 0 \) assuming \( |w| \leq B |b|^{\nu} \) for some constant \( \nu \in (0, 1) \). The present results is significantly stronger. See also Remark 1 below.

In the proof of Theorem 1 we will use some arguments from the proof of Theorem 5.1 in [9] with necessary modifications. Notice that in Theorem 1 above we do not assume that pressure \( P \) of \( f \) is zero, unlike what has been done in previous papers. This contributes the term \( e^{Pm} \) in the right-hand-side of (1.1) which is significant especially in the case \( P < 0 \) which occurs in the applications concerning large deviations (see section 3). In previous papers the authors consider the case \( P = 0 \) and remark that the general case follows from this. However a more careful argument shows that an estimate of the form (1.1) does not follow immediately from a similar estimate with \( P = 0 \). See Remark 4 in section 4 below.

**Remark 1.** In the proof of Theorem 2 in section 3 we apply Theorem 1 with \( b = Cw \) for some constant \( C > 0 \); then \( |w| = \frac{1}{C} |b| \). The relevant part of Theorem 5.1 in [9] assumes \( |w| \leq B |b|^{\nu} \) for some \( \nu \in (0, 1) \) and this is clearly not sufficient for the proof of Theorem 2 below.
Let \( F, G : \Lambda \rightarrow \mathbb{R} \) be Hölder continuous functions and let again \( f, g : R \rightarrow \mathbb{R} \) be defined by
\[
f(x) = \int_0^{\tau(x)} F(\pi(x,t))dt, \quad g(x) = \int_0^{\tau(x)} G(\pi(x,t))dt.
\]
We will now assume that \( G > 0 \) everywhere on \( \Lambda \). Consider a number \( 0 < p \leq \int_R Gdm_{F+\xi(p)G} \in \left\{ \int_R Gdm_{F+tG}, \ t \in \mathbb{R} \right\} \), where \( \xi(p) \) is determined by the equation
\[
dPr_{\pi_\sigma}(F + tG) \bigg|_{t=\xi(p)} - p = 0.
\]
Let \( 0 < \rho < 1 \) be the constant from Theorem 1, and let \( 0 < \alpha_0 = -\frac{\log \rho}{2} \). Fix an arbitrary \( 0 < \delta \leq \alpha_0 \) and consider the sequence \( \{\delta_n\}_{n \in \mathbb{N}} \), where
\[\delta_n = e^{-\delta n}.\]
Set \( g_p = g - \tau p \). Then
\[g^n_p(x) = g^n(x) - \tau^n(x)p = \int_0^{\tau^n(x)} G(\pi(t,x))dt - \tau^n(x)p.\]
Clearly the property
\[\frac{g^n}{\tau^n} - p \in \left(-\frac{\delta_n}{\tau^n(x)}, \frac{\delta_n}{\tau^n(x)}\right),\]
is equivalent to
\[g^n(x) - \tau^n(x)p \in (-\delta_n, \delta_n).\]
On the other hand, since \( cn \leq \tau^n(x) \leq c_1 n \), \( \forall x \in R, \forall n \in \mathbb{N} \) with some constants \( 0 < c \leq c_1 \) for every \( x \), the interval \( \left(-\frac{\delta_n}{\tau^n(x)}, \frac{\delta_n}{\tau^n(x)}\right) \) is exponentially shrinking to 0 as \( n \rightarrow \infty \).

Our second problem concerns the analysis of the asymptotic of
\[
\mu \{ x : g^n(x) - \tau^n(x)p \in (-\delta_n, \delta_n) \}, \ n \rightarrow \infty,
\]
and for \( p \neq \int_R Gdm_F \) we obtain a large deviation result. On the other hand, as in the previous paper [8], we examine the measure of points \( x \in R \) for which the difference \( \frac{\xi'(x)}{\xi'(x)} - p \) stays in an exponentially shrinking interval.

Next, assume that the flow \( \sigma^x_t \) is topologically weak mixing and the function \( G \) is not cohomologous to a constant function. This implies that the set
\[
\left\{ \int_R Gdm_{F+tG}, \ t \in \mathbb{R} \right\}
\]
has a non-empty interior and setting \( \beta(t) = \Pr_{\sigma^x_t}(F + tG) \), one has
\[
\beta'(t) = \frac{d^2Pr_{\sigma^x_t}(F + tG)}{dt^2} = \sigma^2_{m^F+tG}(G)
\]
with
\[
\sigma^2_{m^F}(G) = \lim_{T \rightarrow \infty} \frac{1}{T} \left\{ \int_{R^*} \int_0^T G(\sigma^t_{F})(y)dm - T \int_{R^*} Gdm \right\}^2 < \infty.
\]
Moreover, \( \beta'(\xi(p)) = p \) and \( \xi(p) \) is differentiable with \( \xi'(p) = \frac{1}{\beta''(\xi(p))} > 0 \). Without loss of generality, by adding a constant, we may assume that \( \Pr_{\sigma^x_t}(F) = 0 \). Then
\(m_F \) and \( \Pr_{\sigma_*}(F + tG) \) do not change and \( \Pr_{\sigma}(f - \Pr_{\sigma_*}(F)\tau) = 0 \) yields \( \Pr_{\sigma}(f) = 0 \).

Introduce the rate function
\[
\gamma(p) =: \Pr_{\sigma_*}(F + \xi(p)G) - \xi(p)p.
\]

Then
\[
\gamma'(p) = \beta'((\xi(p)))\xi'(p) - \xi'(p)p - \xi(p) = -\xi(p),
\]
and \( \gamma(p) \leq 0 \) is a concave function with strict maximum 0 at \( p = \int_{R^r} Gdm_F \). Since \( G \) is not cohomologous to a constant function, the function \( g_p = g - p\tau \) is not cohomologous to a function in \( C(R : 2\pi\mathbb{Z}) \), and this yields
\[
\frac{d^2\Pr(f + t\xi p)}{dt^2}
\bigg|_{t=\xi(p)} = \omega(p) > 0. \tag{1.3}
\]

From now on for simplicity of the notation we will write \( \Pr \) instead of \( \Pr_{\sigma} \).

Consider the rate function
\[
J(p) =: \inf_{t \in \mathbb{R}} \{\Pr(f + t(g - \tau p))\} = \Pr(f + \eta(p)(g - \tau p)),
\]
where \( \eta(p) \) is the unique real number such that
\[
0 = \frac{d\Pr(f + t(g - \tau p))}{dt}
\bigg|_{t=\eta(p)} = \int_{R} g\,dm_{f+\eta(p)(g-\tau p)} - p \int_{R} \tau\,dm_{f+\eta(p)(g-\tau p)}.
\]

Notice that
\[
p = \int_{R} \int_{0}^{\tau} G(\pi(x,t))\,dt\,dm_{f+\eta(p)g_p} = \int_{R} \tau\,dm_{f+\eta(p)(g-p)} = \int_{R} G(\pi(x,t))\,dm_{F+\eta(p)(g-p)} + \frac{d\Pr_{\sigma_*}(F + tG)}{dt}
\bigg|_{t=\eta(p)}. \]

Here we have used the fact that \( F + \eta(p)G \) and \( F + \eta(p)G - \eta(p)p \) have the same equilibrium state in \( R^r \). Since \( \frac{d\Pr_{\sigma_*}(F + tG)}{dt} \) is increasing, there exists an unique \( \xi(p) \) such that
\[
\frac{d\Pr_{\sigma_*}(F + \xi(p)G)}{dt} = p,
\]
therefore \( \xi(p) = \eta(p) \). Hence
\[
J(p) = \Pr(f + \xi(p)(g - \tau p)).
\]

In section 2 we show that
\[
J(p) = \gamma(p) \int_{R} \tau\,d\mu_{f+\xi(p)(g-\tau p)} \tag{1.4}.
\]

This implies \( J(p) \leq 0 \) and \( J(p) = 0 \) if and only if \( p = \int_{R^r} Gdm_F \) and \( \xi(p) = 0 \).

We prove the following large deviation theorem which is also a local limit result.

**Theorem 2.** Let the assumptions of Theorem 1 be satisfied. Assume that \( G : \Lambda \rightarrow (0, \infty) \) is a H"older continuous function for which there exists a Markov family \( \mathcal{R} = \{R_t\}_{t=1}^k \) for the flow \( \varphi_t \) on \( \Lambda \) such that \( G \) is constant on the stable leaves of all "rectangular boxes"
\[
B_t = \{\varphi_t(x) : x \in R_t, 0 \leq t \leq \tau(x)\},
\]
\[ i = 1, \ldots, k. \] Assume in addition that the flow \( \sigma^n \) is topologically weak mixing and the function \( G \) is not cohomologous to a constant function. Let \( 0 < \rho < 1 \) be the constant in Theorem 1 and let \( \delta_n = e^{-\delta_n}, 0 < \delta \leq -\frac{\log \rho}{2} \). Then
\[
\mu\{x : g^n(x) - \pi^n(x)p \in (-\delta_n, \delta_n)\} \sim \frac{2\delta_n}{\sqrt{2\pi\omega(p)n}}e^{nJ(p)}, \ n \to \infty. \tag{1.5}
\]

A similar result for the measure of points \( x \in R \) for which the difference
\[
\frac{1}{n} \int_0^{\tau^n(x)} G(\pi(t, x))dt - p
\]
stays in an exponentially shrinking interval has been obtained in [8] under the conditions that \( G \) is a Lipschitz function on \( \Lambda \) and \( \frac{\text{Lip} G}{\min G} < \mu_0 \) with a suitable positive constant \( \mu_0 \). In the present paper we improve the result in [8] assuming that \( G \) is only Hölder. Moreover, here we study the more natural difference
\[
\frac{1}{\tau^n(x)} \int_0^{\tau^n(x)} G(\pi(t, x))dt - p.
\]

This progress is essentially based on the spectral estimates for the Ruelle operator with two complex parameters established in Theorem 1. A further improvement would be the analysis of the asymptotic of
\[
\mu\{x : 1 \frac{1}{T} \int_0^T G(\pi(t, x))dt - p \in \left(-\frac{e^{-\delta T}}{T}, \frac{e^{-\delta T}}{T}\right)\}, \ \delta > 0,
\]
as \( T \to +\infty \) and this is an interesting open problem. On the other hand, the case when the interval \( \left(-\frac{e^{-\delta T}}{T}, \frac{e^{-\delta T}}{T}\right) \) is replaced by an interval \( (p, \frac{p}{T}), \alpha < \beta, \) has been studied in [13]. Comparing (1.5) with Theorem 1 in [13], one observes that in the case we deal with, by scaling, the variable tending to \( +\infty \) can take the form \( T_n = n \int_R \gamma d\mu_{\gamma+\xi(p)(g-p)}. \) Setting
\[
\omega(p) = \frac{1}{C^2(p)}\beta^n(p)) \int_R \gamma d\mu_{\gamma+\xi(p)(g-p)}, C(p) \neq 0,
\]
one may write the leading term in (1.5) as
\[
\frac{2\delta_n C(p)}{\sqrt{2\pi\beta^n(p)T_n}}e^{T_n\gamma(p)}
\]
which modulo the constant \( C(p) \) is similar to the asymptotic in [13] with \( T_n \to \infty, \) where the rate function is precisely \( \gamma(p) \).

**Remark 2.** The condition \( G > 0 \) in Theorem 2 is not a restriction since we can replace \( G \) by \( G + C > 0 \) for some large constant \( C > 0 \). Then \( p = \int_{R^n} G dm_{E} + C, \) and the asymptotic (1.5) is independent of the constant \( C. \) The assumption that \( G \) is constant on stable leaves of rectangular boxes \( B_i \) is significant, however it seems difficult to remove when very sensitive asymptotics such as (1.5) are obtained. For “standard” large deviation results, this assumption is not necessary, since one can use Sinai’s Lemma (see e.g. Proposition 1.2 in [7]) to replace an arbitrary Hölder function \( G \) by a cohomologous function which is constant on stable leaves. In [13] and [10], where instead of \( (-e^{-\delta n}, e^{-\delta n}) \) the authors deal with significantly larger intervals, however still smaller than \( (c/n, c/n) \) for a constant \( c > 0, \) claims have been made that the general case of Hölder functions on two-sided shifts is easily derived from the one for one-sided shifts. However in both papers there are no proofs of these
claims. For sharp estimates similar to (1.5), it is tempting to believe that such claims would be difficult to justify.

The paper is organised as follows. In section 2 we collect some facts about hyperbolic flows necessary for our exposition. In section 3 we prove Theorem 2 exploiting the spectral estimates (1.1). In the proof an integral involving a Ruelle operators is used with two complex parameters tending to infinity. The Standing Assumptions are stated in section 4 where also some crucial Lemmas are established. Theorem 1 is proved in section 5. The Appendix contains the proof of a technical lemma.

2. Preliminaries. As in section 1, let $\varphi_t : M \to M$ be a $C^2$ Axiom A flow on a Riemannian manifold $M$, and let $\Lambda$ be a basic set for $\varphi_t$. The restriction of the flow on $\Lambda$ is a hyperbolic flow [7]. For any $x \in M$ let $W^s(x), W^u(x)$ be the local stable and unstable manifolds through $x$, respectively (see [2], [5], [7]). When $M$ is compact and $M$ itself is a basic set, $\varphi_t$ is called an Anosov flow. It follows from the hyperbolicity of $\Lambda$ that if $\epsilon_0 > 0$ is sufficiently small, there exists $\epsilon_1 > 0$ such that if $x, y \in \Lambda$ and $d(x, y) < \epsilon_1$, then $W^s_{\epsilon_0}(x)$ and $\varphi_{[-\epsilon_0, \epsilon_0]}(W^u_{\epsilon_0}(y))$ intersect at exactly one point $[x, y] \in \Lambda$ (cf. [5]). That is, there exists a unique $t \in [-\epsilon_0, \epsilon_0]$ such that $\varphi_t([x, y]) \in W^u_{\epsilon_0}(y)$. Setting $\Delta(x, y) = t$, defines the so called temporal distance function.

We will use the set-up and some arguments from [11] and [9]. As in these papers, fix a (pseudo) Markov family $R = \{R_i\}_{i=1}^k$ of pseudo-rectangles

$$R_i = [U_i, S_i] = \{[x, y] : x \in U_i, y \in S_i\}.$$  

Set

$$R = \bigcup_{i=1}^k R_i , \quad U = \bigcup_{i=1}^k U_i.$$  

Consider the Poincaré map $P : R \to R$, defined by $P(x) = \varphi_{\tau(x)}(x) \in R$, where $\tau(x) > 0$ is the smallest positive time with $\varphi_{\tau(x)}(x) \in R$ (first return time function). The shift map $\sigma : U \to U$ is given by $\sigma = \pi(U) \circ P$, where $\pi(U) : R \to U$ is the projection along stable leaves.

The hyperbolicity of the flow on $\Lambda$ implies the existence of constants $c_0 \in (0, 1]$ and $\gamma_1 > \gamma_0 > 1$ such that

$$c_0 \gamma_0^m d(u_1, u_2) \leq d(\sigma^m(u_1), \sigma^m(u_2)) \leq \frac{\gamma_1^m}{c_0} d(u_1, u_2) \quad (2.1)$$

whenever $\sigma^j(u_1)$ and $\sigma^j(u_2)$ belong to the same $U_{i_j}$ for all $j = 0, 1 \ldots, m$.

Define a $k \times k$ matrix $A = \{A(i, j)\}_{i,j=1}^k$ by

$$A(i, j) = \begin{cases} 1 & \text{if } P(\text{Int } R_i) \cap \text{Int } R_j \neq \emptyset, \\ 0 & \text{otherwise}. \end{cases}$$

It is possible to construct a Markov family $R$ so that $A$ is irreducible and aperiodic (see [2]).

Consider the suspension space

$$R^T = \{(x, t) \in R \times \mathbb{R} : 0 \leq t \leq \tau(x)\}/ \sim,$$

where by $\sim$ we identify the points $(x, \tau(x))$ and $(\sigma x, 0)$. The corresponding suspension flow is defined by $\sigma^T_t(x, s) = (x, s + t)$ on $R^T$ taking into account the
identification \sim. For a Hölder continuous function \( f \) on \( R \), the \textit{topological pressure} \( \Pr(f) \) with respect to \( \sigma \) is defined by

\[
\Pr(f) = \sup_{m \in \mathcal{M}_\sigma} \{ h(\sigma, m) + \int f \, dm \},
\]

where \( \mathcal{M}_\sigma \) denotes the space of all \( \sigma \)-invariant Borel probability measures and \( h(\sigma, m) \) is the entropy of \( \sigma \) with respect to \( m \). We say that \( f \) and \( g \) are \textit{cohomologous} and we denote this by \( f \sim g \) if there exists a continuous function \( w \) such that \( f = g + w \circ \sigma - w \).

The proof of (1.4) follows from the definitions of \( \gamma(p) \) and \( \xi(p) \) and the following computation:

\[
\gamma(p) = h(\sigma, m_{F+\xi(p)G}) + \int_{R^\sigma} (F + \xi(p)G - \xi(p)p) \, d\mu_{F+\xi(p)G}
= h(\sigma, m_{F+\xi(p)G - \xi(p)p}) + \int_{R^\sigma} (F + \xi(p)G - \xi(p)p) \, d\mu_{F+\xi(p)G - \xi(p)p}
= \frac{\int_R \tau d\mu_{f + \xi(p)(g-p\tau)}}{\int_R \tau d\mu_{f + \xi(p)(g-p\tau)}} \cdot \frac{J(p)}{\int_R \tau d\mu_{f + \xi(p)(g-p\tau)}}.
\]

3. **Proof of Theorem 2.** In this section we prove Theorem 2 exploiting the spectral estimates obtained in Theorem 1. We work under the assumptions of Theorem 2, in particular, \( G \) is constant on stable leaves of rectangular boxes \( B_i \) for a certain Markov family \( \mathcal{R} = \{ R_i \}_{i=1}^k \). Then the function \( g(x) \) depends only on \( x \in U \).

We may replace \( f \) by a Hölder function \( \tilde{f} \) depending only on \( x \in U \) so that with some Hölder function \( z(x) \) we have

\[
f(x) = \tilde{f}(x) + z(\sigma(x)) - z(x).
\]

Therefore for all \( t \in R \) we have \( \Pr(f + t(g-p\tau)) = \Pr(\tilde{f} + t(g-p\tau)) \) and \( \mu_f = \mu_{\tilde{f}} \). Below we use again the notation \( f \) assuming that \( f(x) \) depends only on \( x \in U \).

We will examine the sequence

\[
\rho(n) = \int_{U} \chi_n(g^n_p(x)) \, d\mu,
\]

where \( \chi \in C^\infty_c(R : R^+) \) is a \textit{fixed cut-off function} and

\[
\chi_n(t) = \chi(\delta_n^{-1}t) \quad , \quad x \in R.
\]

**Remark 3.** In [8] the asymptotic of the function \( z(n) = \int_U \chi_n(g^n(x) - np) \, d\mu \) has been studied, where instead of the function \( g_p(x) = g(x) - p\tau(x) \) we have \( g(x) - p \). Passing to the Fourier transform in the proof of Proposition 1 below, we must deal with the Ruelle operator \( L_{f+\xi(p)+iu}g_p \) instead of the operator \( L_{f+\xi(p)+iu}(g(x)-p) \) involved in the analysis in [8]. In Theorem 1 the Ruelle operator \( L_{f+\xi(p)+iu}(g(x)-p) \) involves \textit{two phase functions} \( \tau(x) \) and \( g(x) \) with parameters \( b = pu \) and \( w = u \). Certain spectral estimates for the iterations of the Ruelle operator are necessary to cover the asymptotic when \( |u| \to \infty \). This was one of the motivation to examine Ruelle operators with two complex parameters.
Proposition 1. Under the assumptions of Theorem 2 we have the asymptotic
\[
\rho(n) \sim \frac{\delta_n}{\sqrt{2\pi \omega(p)n}} \left( \int \chi(t) dt \right) e^{nJ(p)}, \quad n \to \infty. \tag{3.3}
\]

Proof. The Ruelle operator \( L_{f+\xi(p)g_p} \) has a simple eigenvalue
\[
\lambda_p = e^{Pr(f+\xi(p)g_p)} = e^{J(p)},
\]
and so for all sufficiently small \( u \in \mathbb{C} \) the operator \( L_{f+(\xi(p)+iu)g_p} \) has a simple eigenvalue \( e^{Pr(f+(\xi(p)+iu)g_p)} \) and the rest of the spectrum of \( L_{f+(\xi(p)+iu)g_p} \) is contained in a disk of radius \( \theta \lambda_p \) with some \( 0 < \theta < 1 \). Note that
\[
d^2Pr(f+(\xi(p)+iu)g_p)\bigg|_{u=0} = -\omega(p) < 0.
\]
Clearly for the Fourier transform \( \hat{\chi}_n \) of \( \chi_n \) we get \( \hat{\chi}_n(u) = \delta_n \hat{\chi}(\delta_n u) \). Set \( \omega_n(y) = e^{-\xi(p)y} \chi_n(y) \). Since \( Pr(f) = 0 \), the Ruelle operator \( L_f \) has a simple eigenvalues \( 1 \) and the adjoint operator \( L_f^* \) satisfies
\[
L_f^* \mu = \mu,
\]
where we denote \( \mu = \mu_f \) as in section 1.

Using this property and applying the Fourier transform, we have
\[
\rho(n) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left( \int e^{iu g_p^n(x)} d\mu(x) \right) \hat{\chi}_n(u) du
\]
\[
= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left( \int e^{(\xi(p)+iu) g_p^n(x)} d\mu(x) \right) \hat{\omega}_n(u) du
\]
\[
= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left( \int L_f^n (\xi(p)+iu)g_p 1(x) d\mu(x) \right) \hat{\omega}_n(u) du
\]
\[
= \frac{\delta_n}{2\pi} \int_{-\infty}^{\infty} \left( \int L_f^n (\xi(p)+iu)g_p 1(x) d\mu(x) \right) \hat{\chi}(\delta_n(u-i\xi(p))) du.
\]
By Taylor expansion for small \(|u|\) one gets
\[
e^{Pr(f+(\xi(p)+iu)g_p)} = \lambda_p \left( 1 - \frac{\omega(p)}{2} u^2 + O(|u|^3) \right).
\]
We choose \( \epsilon_0 > 0 \) sufficiently small and changing the coordinates on \((-\epsilon_0, \epsilon_0)\) by \( u = \frac{\sqrt{2n}}{\sqrt{\omega(p)}} \), we write
\[
I_1(n) = \frac{\delta_n}{\sqrt{2\omega(p)\pi}} \lambda_p \int_{-\epsilon_1}^{\epsilon_1} \left( (1 - u^2 + iQ(v))^n (1 + O(v)) \hat{\chi}(\delta(v)) + O(\delta_n v) \right) dv
\]
\[
+ \delta_n \lambda_p \theta^n
\]
with \( \epsilon_1 = \sqrt{\frac{\omega(p)}{2}} \epsilon_0 \) and a real valued function \( Q(v) = O(|v|^3) \). The analysis of the asymptotic of this integral is given in section 4.1 in [10]. The leading term has the form
\[
\frac{\delta_n}{\sqrt{2\omega(p)\pi}} \hat{\chi}(0) \lambda_p \int_{-\epsilon_1}^{\epsilon_1} (1 - u^2)^n dv = \frac{\delta_n \lambda_p}{\sqrt{2\omega(p)\pi}} \hat{\chi}(0) \frac{\pi}{n} + \delta_n O\left( \frac{\lambda_p}{n} \right), \quad n \to \infty.
\]
Thus we deduce

\[ I_1(n) \sim \frac{\delta_n}{\sqrt{2\pi \omega(p)n}} \left( \int \chi(t) dt \right) e^{n^J(p)} + O\left( \frac{\delta_n e^{n^J(p)}}{n} \right). \] (3.4)

Next consider the integral

\[ I_2(n) = \frac{\delta_n}{2\pi} \int_{|u| \leq \frac{\pi}{p}} \left( \int L_{f+(\xi(p)+iu)g_p}^n 1(x) d\mu(x) \right) \hat{\chi}(\delta_n(u - i\xi(p))) du \]

with \( c \gg 1 \) sufficiently large. Since \( g_p \) is non-lattice, for \( 0 < \epsilon_0 \leq |u| \leq \frac{\pi}{p} \) the operator \( L_{f+(\xi(p)+iu)g_p} \) has no eigenvalues \( \lambda \) with \( |\lambda| = \lambda_p \) and the spectral radius of \( L_{f+(\xi(p)+iu)g_p} \) is strictly less than \( \lambda_p \). Thus, there exists \( \alpha = \alpha(p, c) \), \( 0 < \alpha < 1 \), such that for \( n \geq N(p, c) \) we have

\[ \| L_{f+(\xi(p)+iu)g_p}^n \| \leq \alpha^n \lambda_p^n. \] (3.5)

On the other hand,

\[ |\hat{\chi}(\delta_n(u - i\xi(p)))| \leq C_k e^{\epsilon_0 \delta_n |\xi(p)|} \frac{\delta_k}{\delta_n^k |u|^k}, \quad |u| \geq \epsilon_0, \quad \forall k \in \mathbb{N}, \] (3.6)

with \( c_0 > 0 \) depending on the support of \( \chi \). Applying (3.5) and (3.6) with \( k = 0 \), for large \( n \) we get

\[ I_2(n) = O\left( \frac{\delta_n e^{n^J(p)}}{n} \right). \]

Now consider

\[ I_3(n) = \frac{\delta_n}{2\pi} \int_{|u| > \frac{\pi}{p}} \left( \int L_{f+(\xi(p)+iu)g_p}^n 1(x) d\mu(x) \right) \hat{\chi}(\delta_n(u - i\xi(p))) du. \]

We are going to use the spectral estimates established in Theorem 1 for the Ruelle operator

\[ L_{f-(\xi(p)+iu)\tau+(\xi(p)+iu)g} = L_{f+\xi(p)(g-pr)-ipr+iu}. \]

Then \( |u| \leq \frac{1}{|p|} |pu| \) and for sufficiently large \( |u| \geq \frac{\pi}{p} \) and for every \( \epsilon > 0 \) we are in situation to apply the spectral estimates

\[ \| L_{f-(\xi(p)(g-pr)-ipr+iu)}^n \|_{\infty} \leq C e^{\rho^n |pu|^\epsilon}, \quad 0 < \rho < 1, \quad |pu| \geq c, \quad n \in \mathbb{N}. \] (3.7)

Fix \( 0 < \epsilon \leq 1/2 \) and apply the estimate (3.6) with \( k = 2 \) and (3.7) for \( \epsilon \). This gives

\[ |I_3(n)| \leq \delta_n \lambda_p^n A e^{\epsilon \delta_n |\xi(p)|} e^{\rho^n |pu|^{\epsilon}} \int_{|u| > \frac{\pi}{p}} |u|^{\epsilon - 2} du = D \delta_n \lambda_p^n \left( \frac{\rho^n}{\delta_n^2} \right). \]

Recall that we have the condition

\[ 0 < \delta \leq \alpha_0 \leq -\frac{\log \rho}{2}, \]

from which we deduce the inequality

\[ n \log \rho + 2n \delta - n \log \rho \leq 0. \]

This leads to

\[ \frac{\rho^n}{\delta_n^2} \leq \frac{1}{n}, \quad n \geq 1. \]
Thus, we conclude that
\[ I_3(n) = \mathcal{O}\left(\frac{\delta_n e^{nJ(p)}}{n}\right). \]
Consequently,
\[ \rho(n) = I_1(n) + I_2(n) + I_3(n) = \frac{\delta_n}{\sqrt{2\pi\omega(p)n}} \left( \int \chi(t) dt \right) e^{nJ(p)} \left( 1 + \mathcal{O}(1/\sqrt{n}) \right), \ n \to \infty \]
and this completes the proof of Proposition 1.

To establish Theorem 2, as in [10], [8], we approximate the characteristic function
\[ \mathbf{1}_{[-1,1]} \]
of the interval \([-1,1]\) by cut-off functions.

4. Ruelle operators – definitions and assumptions. Assume as in section 1
that \( \varphi_t : M \to M \) is a \( C^2 \) weak mixing Axiom A flow and \( \Lambda \) is a basic set for \( \varphi_t \).
Here we work under the same assumptions as these in [9]. One of these is:

\( \text{(LNIC): There exist } z_0 \in \Lambda, \epsilon_0 > 0 \text{ and } \theta_0 > 0 \text{ such that for any } \epsilon \in (0, \epsilon_0], \text{ any } \hat{z} \in \Lambda \cap W^u(z_0) \text{ and any tangent vector } \eta \in E^u(\hat{z}) \to \Lambda \text{ at } \hat{z} \text{ with } \|\eta\| = 1 \text{ there exist } \hat{z} \in \Lambda \cap W^u(\hat{z}), \hat{y}_1, \hat{y}_2 \in \Lambda \cap W^s(\hat{z}) \text{ with } \hat{y}_1 \neq \hat{y}_2, \delta = \delta(\hat{z}, \hat{y}_1, \hat{y}_2) > 0 \text{ and } \epsilon' = \epsilon'(\hat{z}, \hat{y}_1, \hat{y}_2) \in (0, \epsilon] \text{ such that} \]
\[ |\Delta(\exp_\epsilon^u(v), \pi_{\hat{y}_1}(z)) - \Delta(\exp_\epsilon^u(v), \pi_{\hat{y}_2}(z))| \geq \delta \|v\| \]
for all \( z \in W^u(\hat{z}) \cap \Lambda \) and \( v \in E^u(z; \epsilon') \) with \( \exp_\epsilon^u(v) \in \Lambda \) and \( \langle \frac{v}{\|v\|}, \eta_z \rangle \geq \theta_0, \) where \( \eta_z \) is the parallel translate of \( \eta \) along the geodesic in \( W^u_{\epsilon_0}(z_0) \) from \( \hat{z} \) to \( z. \)

The above condition may seem complicated at a first glance, however a careful look at it shows that it is just a rather natural non-integrability condition.

Given \( x \in \Lambda, T > 0 \) and \( \delta \in (0, \epsilon] \) set
\[ B^u_T(x, \delta) = \{ y \in W^u_T(x) : d(\varphi_t(x), \varphi_t(y)) \leq \delta, \ 0 \leq t \leq T \}. \]

We will say that \( \varphi_t \) has a regular distortion along unstable manifolds over the basic set \( \Lambda \) if there exists a constant \( \epsilon_0 > 0 \) with the following properties:

(a) For any \( 0 < \delta \leq \epsilon \leq \epsilon_0 \) there exists a constant \( R = R(\delta, \epsilon) > 0 \) such that
\[ \text{diam}(\Lambda \cap B^u_T(z, \delta)) \leq R \text{diam}(\Lambda \cap B^u_T(z, \delta)) \]
for any \( z \in \Lambda \) and any \( T > 0. \)

(b) For any \( \epsilon \in (0, \epsilon_0] \) and any \( \rho \in (0, 1) \) there exists \( \delta \in (0, \epsilon] \) such that for any \( z \in \Lambda \) and any \( T > 0 \) we have\[ \text{diam}(\Lambda \cap B^u_T(z, \delta)) \leq \rho \text{ diam}(\Lambda \cap B^u_T(z, \delta)). \]

In this paper we work under the following Standing Assumptions:

(A) \( \varphi_t \) has Lipschitz local holonomy maps over \( \Lambda, \)
(B) the local non-integrability condition (LNIC) holds for \( \varphi_t \) on \( \Lambda, \)
(C) \( \varphi_t \) has a regular distortion along unstable manifolds over the basic set \( \Lambda. \)

A rather large class of examples satisfying the conditions (A) – (C) is provided by imposing the following pinching condition:

(P): There exist constants \( C > 0 \) and \( \beta \geq \alpha > 0 \) such that for every \( x \in M \) we have
\[ \frac{1}{C} e^{\alpha t} \|u\| \leq \|d\varphi_t(x) \cdot u\| \leq C e^{\beta t} \|u\|, \quad u \in E^u(x), t > 0. \]
for some constants $\alpha_x, \beta_x > 0$ with $\alpha \leq \alpha_x \leq \beta_x \leq \beta$ and $2\alpha_x - \beta_x \geq \alpha$ for all $x \in M$.

We should note that (P) holds for geodesic flows on manifolds of strictly negative sectional curvature satisfying the so called $\frac{1}{2}$-pinching condition. (P) always holds when dim$(M) = 3$.

**Simplifying Assumptions:** $\varphi_t$ is a $C^2$ contact Anosov flow satisfying the condition (P).

By [12], the pinching condition (P) implies that $\varphi_t$ has Lipschitz local holonomy maps and regular distortion along unstable manifolds. This and Proposition 6.1 in [12] show that:

the Simplifying Assumptions imply the Standing Assumptions.

Throughout we work under the Standing Assumptions. In what follows we will use arguments similar to those in section 4 in [9], however technically they will be more complicated, since the numbers of parameters involved will increase. E.g. where we had functions $f_{at}, h_{at}$, etc., depending on two parameters, now we have to deal with functions $f_{ate}, h_{ate}$, etc., depending on three parameters. While some of the arguments we use here are almost the same as corresponding arguments in [9] (and we omit them), there are others that require more significant modification and we do them in some detail.

Let $R = \{R_i\}_{i=1}^k$ be a fixed Markov family for the flow $\varphi_t$ on $\Lambda$ consisting of rectangles $R_i = [U_i, S_i]$ and let $U = \bigcup_{i=1}^k U_i$ (see section 2). Then (2.1) hold for some constants $c_0 \in (0,1]$ and $\gamma_1 > \gamma_0 > 1$.

**Fix a number $\alpha > 0$ and two real-valued functions $f$ and $g$ in $C^\alpha(\hat{U})$.** Let $P = P_f$ be the unique real number so that $P(\tau - P) = 0$, where $P$ is the topological pressure with respect to $\sigma$. For any $t \in \mathbb{R}$ with $t \geq 1$, let $f_t$ be the average of $f$ over balls in $U$ of radius $1/t$ obtained as follows: fix an arbitrary extension $f \in C^\alpha(V)$ (with the same Hölder constant), where $V$ is an open neighbourhood of $U$ in $M$, and then take the averages in question. Then $f_t \in C^\alpha(V)$ and:

(a) $||f - f_t||_\infty \leq |f|/t^\alpha$;
(b) Lip$(f_t) \leq \text{Const} \cdot ||f||_\infty$;
(c) For any $\beta \in (0,\alpha)$ we have $|f - f_t|_{\beta} \leq 2 |f|/t^{\alpha-\beta}$.

**Remark 4.** As we remarked immediately after the statement of Theorem 1, in this theorem we do not assume that the pressure $P$ of $f$ is zero, and thus we get the term $e^{Pm}$ in the right-hand-side of (1.1). Assuming that the latter holds with $P = 0$ implies a weaker estimate than (1.1) in the general case. Indeed, assume we have proved (1.1) in the case $P = 0$, and then deal with the general case using the standard approach. Given $a, b$ as in the theorem and $h \in C^\beta(\hat{U})$, we have

\[
(\ast) \quad (L_{f-(a+ib)\tau}^m h)(u) = \sum_{\sigma^m(v) = u} e^{(f-(a+ib)\tau)^m(v)} h(v)
= (L_{f-(a+ib)\tau}^m (e^{P\tau^m(v)} h))(u).
\]

We can now apply (1.1) in the case $P = 0$ replacing $h$ by $e^{P\tau^m(v)} h$. Since $0 < c \leq \tau(u) \leq c_1$ for some constants $c$ and $c_1$, assuming e.g. $P < 0$ (the other case is similar), we get $\|e^{P\tau^m} h\|_\infty \leq e^{mPc} \|h\|_\infty$, and

\[
|e^{P\tau^m} h|_{\beta} \leq e^{mPc} |h|_{\beta} + |e^{P\tau^m}|_{\beta} \|h\|_\infty.
\]
Given $u, v \in U_1$, assuming e.g. $e^{P \tau_m(u)} > e^{P \tau_m(v)}$ and using (2.1), we deduce
\[
|e^{P \tau_m(u)} - e^{P \tau_m(v)}| \leq e^{P \tau_m(u)} |P \tau_m(u) - P \tau_m(v)|
\]
\[
\leq |P| e^{m P c} \sum_{j=0}^{m-1} |\tau(P^j(u)) - \tau(P^j(v))|
\]
\[
\leq |P| e^{m P c} \sum_{j=0}^{m-1} \text{Const} \gamma_1 \|\tau(d(u, v))\|^\beta
\]
\[
\leq \text{Const} e^{m P c} \|d(u, v)\|^\beta.
\]
Replacing $c$ by an appropriate $c_0 < c$, we get $|e^{P \tau_m}|_\beta \leq \text{Const} e^{m P c_0}$. This implies
\[
\|e^{P \tau_m} h\|_{\beta, b} \leq e^{m P c} \|h\|_\infty + \frac{1}{|b|} (e^{m P c} \|h\|_\beta + \text{Const} e^{m P c_0} \|h\|_\infty)
\]
\[
\leq \text{Const} e^{m P c_0} \|h\|_{\beta, b}.
\]
Combining the latter with (⋆) gives
\[
\|L^m_{f_{\alpha+ib} \tau} h\|_{\beta, b} \leq C e^{m P c} \|e^{P \tau_m(v)} h\|_{\beta, b} \leq C e^{m P c_0} \|h\|_{\beta, b}.
\]
As one can see this estimate is a bit worse than (1.1), since $c > c_0 > 0$ can be rather small constants.

Let $G : \Lambda \rightarrow \mathbb{R}$ be a fixed α-Hölder function which is constant on the stable leaves of all “rectangular boxes” $B_i = \{\varphi_i(x) : x \in R_i, 0 \leq t \leq \tau(x)\}$, $i = 1, \ldots, k$.

Given a large parameter $t > 0$, define $G_t$ as above, so that $G_t$ is again constant on the stable leaves of all rectangular boxes $B_i$ and
\[
(\alpha') \|G - G_t\|_\infty \leq |G|_{\alpha}/t^\alpha;
\]
\[
(\beta') \text{Lip}(G_t) \leq \text{Const} \|G\|_\infty /t^\gamma;
\]
\[
(\gamma') \text{For any } \beta \in (0, \alpha) \text{ we have } |G - G_t|_\beta \leq 2 |G|_{\alpha}/t^{\alpha - \beta}.
\]
In particular, for some constant $C_0 > 0$ we have $\text{Lip}(G_t) \leq C_0 t$.

Then define $g_t : R \rightarrow \mathbb{R}$ by
\[
g_t(x) = \int_0^{\tau(x)} G_t(\pi(x, s)) \, ds.
\]
Clearly $g_t$ is α-Hölder and constant on stable leaves, so we can regard $g_t$ as a function on $U$. Thus, $g_t \in C^\alpha(U)$.

Let $\lambda_0 > 0$ be the largest eigenvalue of $L_f$, i.e. $\lambda_0 = e^P$, and let $\hat{\nu}_0$ be the (unique) probability measure on $U$ with $L^*_f \hat{\nu}_0 = \lambda_0 \hat{\nu}_0$. Fix a corresponding (positive) eigenfunction $h_0 \in C^\alpha(U)$ such that $\int_U h_0 \, d\hat{\nu}_0 = 1$. Then $d\nu_0 = h_0 \, d\hat{\nu}_0$ defines a $\sigma$-invariant probability measure $\nu_0$ on $U$. Setting
\[
f^{(0)} = f + \ln h_0(u) - \ln h_0(\sigma(u)) - \ln \lambda_0,
\]
we have $L^{*\nu_0} f^{(0)} = v_0$, i.e. $\int_U L^*_{f^{(0)}} H \, d\nu_0 = \int_U H \, d\nu_0$ for any $H \in C(U)$ and $L^{*\nu_0} 1 = 1$.

Given real numbers $a$, $c$ and $t$ (with $|a| + \frac{1}{|t|}$ small and $c \in I$), denote by $\lambda_{ate}$ the largest eigenvalue of $L_{f_{\alpha - \sigma t + cg}}$ on $C^\text{Lip}(U)$ and by $\hat{\nu}_{ate}$ the corresponding (positive) eigenfunction such that $\int_U h_{ate} \, d\hat{\nu}_{ate} = 1$, where $\hat{\nu}_{ate}$ is the unique probability
measure on \( U \) with \( L^*_{f_t - a \tau + cg_t} \hat{\nu}_{atc} = \lambda_{atc} \hat{\nu}_{atc} \). Setting \( d\nu_{atc} = h_{atc} \hat{d\nu}_{atc} \) defines a \( \sigma \)-invariant probability measure \( \nu_{atc} \) on \( U \).

Given \( \theta \in (0, 1) \), consider the metric \( d_\theta \) on \( \hat{U} \) defined by \( d_\theta(x, x) = 0 \) and \( d_\theta(x, y) = \theta^m \), where \( m \) is the largest integer such that \( x \neq y \) belong to the same cylinder of length \( m \). Taking \( \theta \in (0, 1) \) sufficiently close to 1 and \( \beta \in (0, \alpha) \) sufficiently close to 0 we have \( (d(x, y))^\alpha \leq \text{Const} \) for all \( x, y \in \hat{U} \). In what follows we assume that \( \theta \) and \( \beta \) satisfy these assumptions.

By the properties of the approximations \( f_t \) and \( g_t \) stated above, there exists a constant \( C_0 > 0 \), depending on \( f \) and \( \alpha \) but independent of \( \beta \), such that

\[ ||f_t - a \tau + cg_t - f||_\beta \leq C_0 (|a| + |c| + 1/t^{\alpha - \beta}) \]  

(4.2)

for all \( |a|, |c| \leq 1 \) and \( t \geq 1 \). Since the map \( f \mapsto \text{Pr}(f) \) is analytic (see e.g. Proposition 4.7 in [7]) and the related eigenfunction projection is also analytic in \( f \) (cf. e.g. Ch. 4 in [7]), it follows that the \textit{maximal eigenvalue} \( \lambda_{atc} = e^{\text{Pr}(f_t - a \tau + cg_t)} \) of the Ruelle operator \( L_{f_t - a \tau + cg_t} \) depends analytically on the parameters. Thus, there exists a constant \( a_0 > 0 \) such that, taking \( C_0 > 0 \) sufficiently large, we have

\[ \text{Pr}(f_t - a \tau + cg_t - P) \leq C_0 (|a| + |c| + 1/t^{\alpha - \beta}), \quad ||h_{atc} - h_0||_\beta \leq C_0 (|a| + |c| + 1/t^{\alpha - \beta}) \]  

(4.3)

for \( |a|, |c| \leq a_0 \) and \( 1/t \leq a_0 \). We take \( C_0 > 0 \) and \( a_0 > 0 \) so that

\[ \lambda_0/C_0 \leq \lambda_{atc} \leq C_0 \lambda_0, \]

\[ ||f_t||_\infty \leq C_0 \text{ and } 1/C_0 \leq h_{atc}(u) \leq C_0 \text{ for all } u \in U \text{ and all } |a|, |c|, 1/t \leq a_0. \]

Given real numbers \( a \), \( c \) and \( t \) with \( |a|, |c|, 1/t \leq a_0 \) consider the functions

\[ f_{atc} = f_t - a \tau + cg_t + \ln h_{atc} - \ln(h_{atc} \circ \sigma) - \ln \lambda_{atc} \]

and the operators \( \mathcal{M}_{atc} = L_{f_{atc}} : C(U) \rightarrow C(U) \). One checks that \( \mathcal{M}_{atc} \equiv 1 \).

Taking the constant \( C_0 > 0 \) sufficiently large, we may assume that

\[ ||f_{atc} - f^{(0)}||_\beta \leq C_0 \left[ |a| + |c| + \frac{1}{t^{\alpha - \beta}} \right], \quad |a|, |c|, 1/t \leq a_0. \]  

(4.4)

The proof of the following lemma is given in [9] when \( c = 0 \). In the case with three parameters the proof is almost the same, so we omit it.

**Lemma 1.** Taking the constant \( a_0 > 0 \) sufficiently small, there exists a constant \( T' > 0 \) such that for all \( a, t, c \in \mathbb{R} \) with \( |a|, |c| \leq a_0 \) and \( t \geq 1/a_0 \) we have \( h_{atc} \in C^{\text{Lip}}(\hat{U}) \text{ and Lip}(h_{atc}) \leq T't \).

Consequently, assuming \( a_0 > 0 \) is chosen sufficiently small, there exists a constant \( T > 0 \) (depending on \( |f|_\alpha \) and \( a_0 \)) such that

\[ ||f_{atc}||_\infty \leq T, \quad ||g_t||_\infty \leq T, \quad \text{Lip}(h_{atc}) \leq Tt, \quad \text{Lip}(f_{atc}) \leq Tt \]  

(4.5)

for \( |a|, |c|, 1/t \leq a_0 \). In what follows we assume that

\[ a_0, C_0, T \geq \max\{ ||\tau||_0, \text{Lip}(\tau|_\hat{U}) \}, \quad 1 < \gamma_0 < \gamma_1 \]

are fixed constants with (2.1) and (4.2) – (4.5).

We are going to study Ruelle operators of the form \( L_{f - s \tau + zg} \), where \( s = a + ib \) and \( z = c + iw \), \( a, b, c, w \in \mathbb{R} \), and \( |a|, |c| \leq a_0 \) for some constant \( a_0 > 0 \), approximating them by Ruelle operators of the form

\[ L_{abtz} = L_{f_{atc} - ib \tau + zg} : C^\alpha(\hat{U}) \rightarrow C^\alpha(\hat{U}). \]
Since \( f_{atc} - ib\tau + zg \) is Lipschitz, the operators \( \mathcal{L}_{atc} \) preserve each of the spaces \( C^\alpha(U) \) for \( 0 < \alpha' \leq 1 \) including the space \( C^{\text{Lip}}(U) \) of Lipschitz functions \( h : \tilde{U} \to \mathbb{C} \). For such \( h \) we will denote by \( \text{Lip}(h) \) the Lipschitz constant of \( h \). For \( |b| \geq 1 \), define the norm \( \|h\|_{\text{Lip},b} \) on \( C^{\text{Lip}}(U) \) by \( \|h\|_{\text{Lip},b} = \|h\|_0 + \frac{\text{Lip}(h)}{|b|} \). Recall the norm \( \|h\|_{\beta,b} = \|h\|_\infty + \frac{|h|_\beta}{|b|} \) on \( C^\beta(U) \) defined in section 1.

The main step in proving Theorem 1 is the following.

**Theorem 3.** Under the assumptions in Theorem 1 there exist constants \( 0 < \rho < 1 \), \( a_0 > 0 \), \( b_0 \geq 1 \), \( A_0 > 0 \) and \( C = C(B,e) > 0 \) such that if \( a, c \in \mathbb{R} \) satisfy \( |a|, |c| \leq a_0 \), then

\[
\left\| L^m_{f_{atc} - ib\tau + czg} h \right\|_{\text{Lip},b} \leq C \rho^m \|h\|_{\text{Lip},b}
\]

for all \( h \in C^{\text{Lip}}(U) \), all integers \( m \geq 1 \) and all \( b, w, t \in \mathbb{R} \) with \( |b| \geq b_0 \), \( te^{At} \leq |b| \) and \( |w| \leq B |b| \).

Throughout we work under the Standing Assumptions made above and with fixed real-valued functions \( f, g \in C^\alpha(U) \) as in section 1, where \( \alpha > 0 \) is a fixed number. Another fixed number \( \beta \in (0, \alpha) \) will be used later.

Assuming that all rectangles \( R_i \) are sufficiently small we have \( \text{diam}(U_i) < 1 \) for all \( i \). Recall the metric \( D \) on \( \tilde{U} \) defined in [11]:

\[
D(x,y) = \min\{\text{diam}(C) : x, y \in C, C \text{ is a cylinder contained in } U_i\}
\]

if \( x, y \in U_i \) for some \( i = 1, \ldots, k \), and \( D(x,y) = 1 \) otherwise. As shown in [11], \( d(x,y) \leq D(x,y) \) if \( x, y \in \tilde{U}; \) for some \( i \), and for any cylinder \( C \) in \( U \) the characteristic function \( \chi_C \) of \( \tilde{C} \) on \( \tilde{U} \) is Lipschitz with respect to \( D \) and \( \text{Lip}_D(\chi_C) \leq 1/\text{diam}(C) \). Let \( C^{\text{Lip}}(U) \) be the space of all Lipschitz functions \( h : \tilde{U} \to \mathbb{C} \) with respect to the metric \( D \) and let \( \text{Lip}_D(h) \) be the Lipschitz constant of \( h \) with respect to \( D \).

Given \( A > 0 \), denote by \( K_A(\tilde{U}) \) the set of all functions \( h \in C^{\text{Lip}}(U) \) such that \( h > 0 \) and

\[
\frac{|h(u) - h(u')|}{h(u')} \leq A D(u,u')
\]

for all \( u, u' \in \tilde{U} \) that belong to the same \( \tilde{U}; \) for some \( i = 1, \ldots, k \). For \( h \in K_A(\tilde{U}) \) we have \( |\ln h(u) - \ln h(v)| \leq A D(u,v) \) and so

\[
e^{-A D(u,v)} \leq \frac{h(u)}{h(v)} \leq e^{A D(u,v)}
\]

for all \( u, v \in \tilde{U}_i, \; i = 1, \ldots, k \).

**Fix an arbitrary constant** \( \gamma \) **with** \( 1 < \gamma < \gamma_0 \). The following lemma is similar to Lemma 5.2 in [9], however some technical details are different, so we sketch its proof in the Appendix.

**Lemma 2.** Assuming \( a_0 > 0 \) is chosen sufficiently small, there exists a constant \( A_0 > 0 \) such that for all \( a, c, t, \in \mathbb{R} \) with \( |a|, |c| \leq a_0 \) and \( t \geq 1 \) the following hold:

(a) If \( H \in K_Q(\tilde{U}) \) for some \( Q > 0 \), then

\[
\frac{|(M^{m}_{atc} H)(u) - (M^{m}_{atc} H)(u')|}{(M^{m}_{atc} H)(u')} \leq A_0 \left[ \frac{Q}{\gamma^m} + e^{A_0 t} \right] D(u, u')
\]

for all \( m \geq 1 \) and all \( u, u' \in U_i, \; i = 1, \ldots, k \).
(b) If the functions \( h \) and \( H \) on \( \hat{U} \) and \( Q > 0 \) are such that \( H > 0 \) on \( \hat{U} \) and
\[
|h(v) - h(v')| \leq t Q H(v') D(v, v')
\]
for any \( v, v' \in \hat{U} \), \( i = 1, \ldots, k \), then for any integer \( m \geq 1 \) and any \( b, w \in \mathbb{R} \) with
\[
|b|, |w| \geq 1,
\]
for \( z = c + kw \) we have
\[
|L^m_{abc}(h(u)) - L^m_{abc}(h(u'))| \leq A_0 \left( t Q (M^m_{abc}(H))(u') + (|b| + e^{4\alpha t} + t |w|)(M^m_{abc}(H))(u') \right) D(u, u')
\]
whenever \( u, u' \in \hat{U} \), for some constant \( B > 0 \), then
\[
|L^m_{abc}(h(u)) - L^m_{abc}(h(u'))| \leq A_1 \left( t Q (M^m_{abc}(H))(u') + t |b|(M^m_{abc}(H))(u') \right) D(u, u')
\]
for some constant \( A_1 > 0 \), depending on \( B \).

From now on we will assume that \( a_0 \) and \( A_0 \) are fixed with the properties in Lemma 2 above and \( a, b, c, w, t \in \mathbb{R} \) are such that \( |a|, |c| \leq a_0 \), \( |b|, t, |w| \geq 1 \) and \( |w| \leq B |b| \). As before, set \( z = c + id \).

As in [9], we need the entire set-up and notation from section 4 in [11], so we will now recall some of it.

Following section 4 in [11], fix an arbitrary point \( z_0 \in \Lambda \) and constants \( \epsilon_0 > 0 \) and \( \theta_0 \in (0,1) \) with the properties described in (LNIC). Assume that \( z_0 \in \text{Int}_\Lambda(U_1) \), \( U_1 \subseteq \Lambda \cap W^{\epsilon_0}(z_0) \) and \( S_1 \subseteq \Lambda \cap W^{\epsilon_0}(z_0) \). Fix an arbitrary constant \( \theta_1 \) such that \( 0 < \theta_0 < \theta_1 < 1 \).

Next, fix an arbitrary orthonormal basis \( e_1, \ldots, e_n \in E^n(z_0) \) and a \( C^1 \) parameterization \( r(s) = \exp_{z_0}^n(s), s \in V_0' \), of a small neighborhood \( W_0 \) of \( z_0 \) in \( W^{\epsilon_0}(z_0) \) such that \( V_0' \) is a convex compact neighborhood of \( 0 \) in \( \mathbb{R}^n \approx \text{span}(e_1, \ldots, e_n) = E^n(z_0) \).

Then \( r(0) = z_0 \) and \( \frac{\partial r}{\partial s_i}(s)|_{s=0} = e_i \) for all \( i = 1, \ldots, n \). Set \( U_0' = W_0 \cap \Lambda \). Shrinking \( W_0 \) (and therefore \( V_0' \) as well) if necessary, we may assume that \( U_0' \subset \text{Int}_\Lambda(U_1) \) and
\[
\left| \left( \frac{\partial r}{\partial s_i}(s), \frac{\partial r}{\partial s_j}(s) \right) - \delta_{ij} \right|
\]
is uniformly small for all \( i, j = 1, \ldots, n \) and \( s \in V_0' \), so that
\[
\frac{1}{2} \langle \xi, \eta \rangle \leq \left( \frac{\partial r(s) \cdot \xi}{\partial s_i}(s), \frac{\partial r(s) \cdot \eta}{\partial s_j}(s) \right) \leq \frac{1}{2} \langle \xi, \eta \rangle
\]
for all \( \xi, \eta \in E^n(z_0) \) and \( s \in V_0' \), and \( \frac{1}{2} \|s - s'\| \leq d(r(s), r(s')) \leq 2 \|s - s'\| \) for all \( s, s' \in V_0' \).

**Definitions ([11]).** (a) For a cylinder \( C \subset U_0' \) and a unit vector \( \xi \in E^n(z_0) \) we will say that a separation by a \( \xi \)-plane occurs in \( C \) if there exist \( u, v \in C \) with \( d(u, v) \geq \frac{1}{2} \text{diam}(C) \) such that
\[
\left( r^{-1}(u) - r^{-1}(v), \frac{1}{\|r^{-1}(u) - r^{-1}(v)\|} \cdot \xi \right) \geq \theta_1.
\]

Let \( S_\xi \) be the family of all cylinders \( C \) contained in \( U_0' \) such that a separation by an \( \xi \)-plane occurs in \( C \).

(b) Given an open subset \( V \) of \( U_0' \) which is a finite union of open cylinders and \( \delta > 0 \), let \( C_1, \ldots, C_p \) \( (p = p(\delta) \geq 1) \) be the family of maximal closed cylinders in \( V \) with \( \text{diam}(C_j) \leq \delta \). For any unit vector \( \xi \in E^n(z_0) \) set
\[
M_\xi^{(\delta)}(V) = \cup \{ C_j : C_j \in S_\xi, 1 \leq j \leq p \}.
\]
In what follows we will construct, amongst other things, a sequence of unit vectors \( \xi_1, \xi_2, \ldots, \xi_{j_0} \in E^n(z_0) \). For each \( \ell = 1, \ldots, j_0 \) set \( R_\ell = \{ \eta \in \mathbb{S}^{n-1} : \langle \eta, \xi_\ell \rangle \geq \theta_0 \} \). For \( t \in \mathbb{R} \) and \( s \in E^n(z_0) \) set
\[
I_{\eta,t}g(s) = \frac{g(s + t \eta) - g(s)}{t},
\]
for \( t \neq 0 \) (increment of \( g \) in the direction of \( \eta \)).

**Lemma 3.** ([11]) There exist integers \( 1 \leq n_1 \leq N_0 \) and \( \ell_0 \geq 1 \), a sequence of unit vectors \( \eta_1, \eta_2, \ldots, \eta_{\ell_0} \in E^n(z_0) \) and a non-empty open subset \( U_0 \) of \( U' \) which is a finite union of open cylinders of length \( n_1 \) such that setting \( \mathcal{U} = \sigma^\eta_1(U_0) \) we have:

(a) For any integer \( N \geq N_0 \) there exist Lipschitz maps \( v_1(\ell), v_2(\ell) : U \to U \) \( (\ell = 1, \ldots, \ell_0) \) such that \( \sigma^N(v_1(\ell)(x)) = x \) for all \( x \in \mathcal{U} \) and \( v_1(\ell)(\mathcal{U}) \) is a finite union of open cylinders of length \( N \) \( (i = 1, 2; \ell = 1, 2, \ldots, \ell_0) \).

(b) There exists a constant \( \delta > 0 \) such that for all \( \ell = 1, \ldots, \ell_0 \), \( s \in r^{-1}(U_0) \), \( 0 < |h| \leq \delta \) and \( \eta \in R_\ell \) with \( s + h \eta \in r^{-1}(U_0 \cap \Lambda) \) we have
\[
| I_{\eta,h} \left( \tau^N(v_2(\ell)(\tilde{r}(\cdot))) - \tau^N(v_1(\ell)(\tilde{r}(\cdot))) \right) (s) | \geq \frac{\delta}{2}.
\]

(c) We have \( v_1(\ell)(U) \cap v_2(\ell)(U) = \emptyset \) whenever \( (i, \ell) \neq (j', \ell') \).

(d) For any open cylinder \( V \) in \( U_0 \) there exists a constant \( \delta' = \delta'(V) > 0 \) such that
\[
V \subset M(\eta_1(V)) \cup M(\eta_2(V)) \cup \ldots \cup M(\eta_{\ell_0}(V))
\]
for all \( \delta \in (0, \delta'] \).

Fix \( U_0 \) and \( \mathcal{U} \) with the properties described in Lemma 3; then \( \mathcal{U} = U \).

Set \( \delta = \min_{1 \leq \ell \leq \ell_0} \delta j, \) \( n_0 = \max_{1 \leq \ell \leq \ell_0} m_\ell \), and fix an arbitrary point \( \hat{z}_0 \in U_0^{(\ell_0)} \cap \hat{U} \).

Fix integers \( 1 \leq n_1 \leq N_0 \) and \( \ell_0 \geq 1 \), unit vectors \( \eta_1, \eta_2, \ldots, \eta_{\ell_0} \in E^n(z_0) \) and a non-empty open subset \( U_0 \) of \( W_0 \) with the properties described in Lemma 3. By the choice of \( U_0 \), \( \sigma^\eta_1 : U_0 \to \mathcal{U} \) is one-to-one and has an inverse map \( \psi : U \to U_0 \), which is Lipschitz.

Next, assume that \( B > 1 \), \( \beta \in (0, \alpha) \) and \( E \geq \max \left\{ 4A_0, BC_1, 2A_0 \frac{T}{\gamma - 1} \right\} \) are fixed constants, where \( A_0 \geq 1 \) is the constant from Lemma 2 and \( C_1 \) is the constant from the proof of Lemma 2 in the Appendix. **Fix an integer** \( N \geq N_0 \) such that
\[
\gamma^N \geq \max \left\{ \left( 200 \gamma_1 n_1 A_0 \right)^2, \frac{512 \gamma_1^{n_1} E}{c_0 \delta' \rho} \right\}.
\]

We will also assume that the parameter \( t = t(a_0, N) > 1 \) is fixed with
\[
a_0 \leq \frac{1}{t^{\alpha - \beta}} \leq 2a_0, \quad t \leq \frac{c_0 \delta' \rho \gamma^N}{500 E \gamma_1^{n_1}}.
\]

(Part of this condition will be needed for the proof of Theorem 1.) Clearly the above requires to assume that \( a_0 = a_0(N) \) satisfies \( a_0^{1/(\alpha - \beta)} \leq t \). Some other conditions on the small parameter \( a_0 = a_0(N) > 0 \) will be imposed later. We will also need to choose
\[
b_0 \geq te^{A_0 t}.
\]

Let the parameters \( b, w \in \mathbb{R} \) be so that \( |w| \leq B |b| \) and \( |b|, |w| \geq b_0 \).
Next, fix maps \( v_1^{(\ell)} : U \rightarrow U \) \( (\ell = 1, \ldots, \ell_0, i = 1, 2) \) with the properties (a), (b), (c) and (d) in Lemma 3. In particular, (c) gives \( v_1^{(\ell)}(U) \cap v_1^{(\ell')}(U) = \emptyset \) for all \( (i, \ell) \neq (i', \ell') \).

Since \( U_0 \) is a finite union of open cylinders, it follows from Lemma 3(d) that there exist a constant \( \delta' = \delta'(U_0) > 0 \) such that

\[
M^{(\delta')}_{\eta_1}(U_0) \cup \ldots \cup M^{(\delta')}_{\eta_0}(U_0) \supset U_0
\]

for all \( \delta \in (0, \delta'] \). **Fix \( \delta' \) with this property.** Set

\[
\epsilon_1 = \min \left\{ \frac{1}{32C_0}, c_1, \frac{1}{4E}, \frac{1}{\delta \rho^o + 2}, \frac{c_0 r_0}{\gamma_1^{\alpha_1}}, \frac{c_0^2 (\gamma - 1)}{16T \gamma_1^{\alpha_1}} \right\}.
\]

We will also assume that \( b_0 \) is chosen so that \( \frac{\epsilon_1}{b_0} \leq \delta' \).

Let \( C_m \) \( (1 \leq m \leq p) \) be the family of maximal closed cylinders contained in \( \overline{U_0} \) with \( \text{diam}(C_m) \leq \frac{\epsilon_1}{|b|} \) such that \( U_0 \subset \bigcup_{j=m}^{p} C_m \) and \( \overline{U_0} = \bigcup_{m=1}^{p} C_m \). As in [12],

\[
\rho \frac{\epsilon_1}{|b|} \leq \text{diam}(C_m) \leq \frac{\epsilon_1}{|b|}, \quad 1 \leq m \leq p.
\]

Fix an integer \( q_0 \geq 1 \) such that \( 32 \rho^o - 1 < \theta_1 - \theta_0 \), i.e. \( \theta_0 < \theta_1 - 32 \rho^o - 1 \). Next, let \( D_1, \ldots, D_q \) be the list of all closed cylinders contained in \( \overline{U_0} \) that are subcylinders of co-length \( p_0 q_0 \) of some \( C_m \) \( (1 \leq m \leq p) \). Then

\[
\overline{U_0} = C_1 \cup \ldots \cup C_p = D_1 \cup \ldots \cup D_q.
\]

Moreover,

\[
\rho \rho^o q_0 + 1, \quad \frac{\epsilon_1}{|b|} \leq \text{diam}(D_j) \leq \rho \rho^o \cdot \frac{\epsilon_1}{|b|}, \quad 1 \leq j \leq q.
\]

Given \( j = 1, \ldots, q, \ell = 1, \ldots, \ell_0 \) and \( i = 1, 2 \), set \( \tilde{D}_j = D_j \cap \tilde{U}, \ Z_j = \sigma^{\alpha_1}(\tilde{D}_j) \), \( \tilde{Z}_j = Z_j \cap \tilde{U}, \ X_i^{(\ell)} = v_i^{(\ell)}(\tilde{Z}_j) \), and \( \tilde{X}_i^{(\ell)} = X_i^{(\ell)} \cap \tilde{U} \). It then follows that \( D_j = \psi(Z_j) \) and \( U = \bigcup_{j=1}^{q} Z_j \). Moreover, \( \sigma^{N-N_1}(v_1^{(\ell)}(x)) = \psi(x) \) for all \( x \in U \), and all \( X_i^{(\ell)} \) are cylinders such that \( X_i^{(\ell)} \cap X_i^{(\ell')} = \emptyset \) whenever \( (i, \ell) \neq (i', \ell') \), and

\[
\text{diam}(X_i^{(\ell)}) \geq \frac{c_0 \rho^o \rho^o q_0 + 1}{\gamma_1^N} \cdot \frac{\epsilon_1}{|b|}
\]

for all \( i = 1, 2, j = 1, \ldots, q \) and \( \ell = 1, \ldots, \ell_0 \). The characteristic function

\[
\omega_i^{(\ell)} = \chi_{X_i^{(\ell)}} : \tilde{U} \rightarrow [0, 1]
\]

of \( X_i^{(\ell)} \) belongs to \( C_D^{\text{Lip}}(\tilde{U}) \) and \( \text{Lip}_D(X_i^{(\ell)}) \leq 1/\text{diam}(X_i^{(\ell)}) \). Set

\[
\mu_0 = \mu_0(N) = \min \left\{ \frac{1}{4}, \frac{c_0 \rho^o q_0 + 2}{4 \gamma_1^N}, \frac{1}{4 e^{2TN} \sin^2 \left( \frac{\delta \rho \epsilon_1}{256} \right)} \right\}.
\]

Let \( J \) be a subset of the set

\[
\Xi = \{ (i, j, \ell) : 1 \leq i \leq 2, \ 1 \leq j \leq q, \ 1 \leq \ell \leq \ell_0 \}.
\]

Define the function \( \omega = \omega_J : \tilde{U} \rightarrow [0, 1] \) by

\[
\omega = 1 - \mu_0 \sum_{(i,j,\ell) \in J} \omega_i^{(\ell)}.
\]
Clearly \( \omega \in C^\operatorname{Lip}_D(\hat{U}) \) and \( 1 - \mu_0 \leq \omega(u) \leq 1 \) for any \( u \in \hat{U} \). Moreover,

\[
\operatorname{Lip}_D(\omega) \leq \Gamma = \frac{2\mu_0 \gamma^N_1}{c_0 \rho^N_0 \epsilon^2} \cdot \frac{|b|}{\epsilon_1}.
\]

Next, define the contraction operator \( N = N_j(a, b, t, c) : C^\operatorname{Lip}_D(\hat{U}) \longrightarrow C^\operatorname{Lip}_D(\hat{U}) \) by

\[
(Nh)(\omega) = \mathcal{M}_\omega(\omega, h).
\]

Using Lemma 2 above, the proof of the following lemma is very similar to that of Lemma 5.6 in [11] and we omit it.

**Lemma 4.** Under the above conditions for \( N \) and \( \mu \) the following hold:

(a) \( N \in K_{E|b}(\hat{U}) \) for any \( h \in K_{E|b}(\hat{U}) \);

(b) If \( h \in C^\operatorname{Lip}_D(\hat{U}) \) and \( H \in K_{E|b}(\hat{U}) \) are such that

\[
|h(v) - h(v')| \leq E t \cdot |b|H(v') \cdot D(v, v')
\]

for any \( v, v' \in U_j \), \( j = 1, \ldots, k \) and \( |h| \leq H \) on \( \hat{U} \), then for any \( i = 1, \ldots, k \) and any \( u, u' \in \hat{U}_i \) we have

\[
|\left( \mathcal{L}^N_{\operatorname{atc}, h} \right)(u) - \left( \mathcal{L}^N_{\operatorname{atc}, h} \right)(u') | \leq E t \cdot |b| \cdot (N H)(u') \cdot D(u, u').
\]

**Definition.** A subset \( J \) of \( \Xi \) will be called dense if for any \( m = 1, \ldots, p \) there exists \((i, j, \ell) \in J \) such that \( \mathcal{D}_j \subset C_m \).

Denote by \( J = J(a, b, z) \) the set of all dense subsets \( J \) of \( \Xi \).

Although the operator \( N \) here is different, the proof of the following lemma is very similar to that of Lemma 5.8 in [11] and we omit it.

**Lemma 5.** Given the number \( N \), there exist \( \rho_2 = \rho_2(N) \in (0, 1) \) and \( a_0 = a_0(N) > 0 \) such that

\[
\int_{\hat{U}} (N_j H)^2 \, d\nu \leq \rho_2 \int_{\hat{U}} H^2 \, d\nu
\]

everywhere \( |a|, |c| \leq a_0, t \geq 1/a_0 \), \( J \) is dense and \( H \in K_{E|b}(\hat{U}) \).

Until the end of this section we will assume that \( h, H \in C^\operatorname{Lip}_D(\hat{U}) \) are fixed functions such that

\[
H \in K_{E|b}(\hat{U}) \quad \text{and} \quad |h(u) - H(u)| \leq h(u) \leq H(u) \quad \text{in} \quad \hat{U},
\]

and

\[
|h(u) - h(u')| \leq E t \cdot |b|H(u') \cdot D(u, u') \quad \text{whenever} \quad u, u' \in \hat{U}_i, \quad i = 1, \ldots, k.
\]

Let again \( z = c + i w \). Define the functions \( \chi^{(i)}_{\ell} : \hat{U} \longrightarrow \mathbb{C} \) \( (i = 1, \ldots, j_0, \ i = 1, 2) \) by

\[
\chi^{(1)}_{\ell}(u) = \frac{e^{f_{\ell}^N(\cdot)(v_{\ell}^1(\cdot))(u)) h(v_{\ell}^1(u)) + e^{f_{\ell}^N(\cdot)(v_{\ell}^2(\cdot))(u)) h(v_{\ell}^2(u))}}{(1 - \mu) e^{f_{\ell}^N(\cdot)(v_{\ell}^1(\cdot))(u)) H(v_{\ell}^1(u)) + e^{f_{\ell}^N(\cdot)(v_{\ell}^2(\cdot))(u)) H(v_{\ell}^2(u))}},
\]

\[
\chi^{(2)}_{\ell}(u) = \frac{e^{f_{\ell}^N(\cdot)(v_{\ell}^1(\cdot))(u)) h(v_{\ell}^1(u)) + e^{f_{\ell}^N(\cdot)(v_{\ell}^2(\cdot))(u)) h(v_{\ell}^2(u))}}{e^{f_{\ell}^N(\cdot)(v_{\ell}^1(\cdot))(u)) H(v_{\ell}^1(u)) + (1 - \mu) e^{f_{\ell}^N(\cdot)(v_{\ell}^2(\cdot))(u)) H(v_{\ell}^2(u))}},
\]

and set

\[
\gamma_{\ell}(u) = |b| \cdot \left| \tau^{(1)}_{\ell}(v_{\ell}^1(u)) - \tau^{(1)}_{\ell}(v_{\ell}^2(u)) \right|
\]

and

\[
\gamma_{\ell}(u) = |b| \cdot \left| \tau^{(2)}_{\ell}(v_{\ell}^1(u)) - \tau^{(2)}_{\ell}(v_{\ell}^2(u)) \right|
\]

and

\[
\gamma_{\ell}(u) = |b| \cdot \left| \tau^{(3)}_{\ell}(v_{\ell}^1(u)) - \tau^{(3)}_{\ell}(v_{\ell}^2(u)) \right|
\]
for all $u \in \hat{U}$.

**Definitions ([11]).** We will say that the cylinders $D_j$ and $D_{j'}$ are *adjacent* if they are subcylinders of the same $C_m$ for some $m$. If $D_j$ and $D_{j'}$ are contained in $C_m$ for some $m$ and for some $\ell = 1, \ldots, \ell_0$ there exist $u \in D_j$ and $v \in D_{j'}$ such that $d(u, v) \geq \frac{1}{2} \text{diam}(C_m)$ and

$$\left\langle \frac{r^{-1}(v) - r^{-1}(u)}{\|r^{-1}(v) - r^{-1}(u)\|}, \eta_\ell \right\rangle \geq \theta_1$$

we will say that $D_j$ and $D_{j'}$ are $\eta_\ell$-separable in $C_m$.

As a consequence of Lemma 3(b) one gets the following whose proof is almost the same as that of Lemma 5.9 in [11], so we omit it.

**Lemma 6.** Let $j, j' \in \{1, 2, \ldots, q\}$ be such that $D_j$ and $D_{j'}$ are contained in $C_m$ for some $m = 1, \ldots, p$ and are $\eta_\ell$-separable in $C_m$ for some $\ell = 1, \ldots, \ell_0$. Then

$$|\gamma_\ell(u) - \gamma_\ell(u')| \geq A c_2 \epsilon_1$$

for all $u \in \hat{Z}_j$ and $u' \in \hat{Z}_{j'}$, where $c_2 = \frac{\delta \rho}{16}$.

The following lemma is the analogue of Lemma 5.10 in [11] and represents the main step in proving Theorem 1.

**Lemma 7.** Assume $|b| \geq b_0$ for some sufficiently large $b_0 > 0$, $|a|, |c| \leq a_0$, and let $|w| \leq B|b|$. Then for any $j = 1, \ldots, q$ there exist $i \in \{1, 2\}$, $j' \in \{1, \ldots, q\}$ and $\ell \in \{1, \ldots, \ell_0\}$ such that $D_j$ and $D_{j'}$ are adjacent and $\chi^{(i)}_\ell(u) \leq 1$ for all $u \in \hat{Z}_{j'}$.

To prove this we need the following lemma which is the analogue of Lemma 14 in [4] and its proof is very similar, so we omit it.

**Lemma 8.** If $h$ and $H$ satisfy (4.9)-(4.10), then for any $j = 1, \ldots, q$, $i = 1, 2$ and $\ell = 1, \ldots, \ell_0$ we have:

(a) $\frac{1}{2} \leq \frac{H(v^{(i)}_j(u))}{H(v^{(i)}_{j'}(u)))} \leq 2$ for all $u', u'' \in \hat{Z}_j$;

(b) Either for all $u \in \hat{Z}_j$ we have $|h(v^{(i)}_j(u))| \leq \frac{3}{4} H(v^{(i)}_j(u))$, or $|h(v^{(i)}_j(u))| \geq \frac{1}{4} H(v^{(i)}_j(u))$ for all $u \in \hat{Z}_j$.

**Sketch of proof of Lemma 7.** We use a modification of the proof of Lemma 5.10 in [11].

Given $j = 1, \ldots, q$, let $m = 1, \ldots, p$ be such that $D_j \subset C_m$. As in [11] we find $j', j'' = 1, \ldots, q$ such that $D_{j'}, D_{j''} \subset C_m$ and $D_j$ and $D_{j''}$ are $\eta_\ell$-separable in $C_m$.

Fix $\ell$, $j'$ and $j''$ with the above properties, and set $\hat{Z} = \hat{Z}_j \cup \hat{Z}_{j'} \cup \hat{Z}_{j''}$. If there exist $t \in \{j, j', j''\}$ and $i = 1, 2$ such that the first alternative in Lemma 8(b) holds for $\hat{Z}_t$, $\ell$ and $i$, then $\mu \leq 1/4$ implies $\chi^{(i)}_\ell(u) \leq 1$ for any $u \in \hat{Z}_t$.

Assume that for every $t \in \{j, j', j''\}$ and every $i = 1, 2$ the second alternative in Lemma 8(b) holds for $\hat{Z}_t$, $\ell$ and $i$, i.e. $|h(v^{(i)}_j(u))| \geq \frac{1}{4} H(v^{(i)}_j(u))$, $u \in \hat{Z}$.

Since $\psi(\hat{Z}) = \hat{D}_j \cup \hat{D}_{j'} \cup \hat{D}_{j''} \subset C_m$, given $u, u' \in \hat{Z}$ both $\sigma^{N-n_1}(v^{(i)}_j(u))$ and $\sigma^{N-n_1}(v^{(i)}_j(u')) \in C_m$. Moreover, $C' = v^{(i)}_j(\sigma^{n_1}(C_m))$ is a cylinder with $\text{diam}(C') \leq$
Now the estimate (6.2) in the Appendix below implies
\[
|g_i^N(v_i^f(u)) - g_i^N(v_i^f(u'))| \leq \frac{C_1 t \epsilon_1}{c_0 \gamma_0^{N-1} |b|}.
\]
Assume for example that
\[
e^{\gamma_0 N(v_i^f(u))}|h(v_i^f(u))| \geq e^{\gamma_0 N(v_i^f(u'))}|h(v_i^f(u'))|.
\]
Then\(^1\)
\[
\frac{e^{\gamma_0 N(v_i^f(u))}|h(v_i^f(u))| - e^{\gamma_0 N(v_i^f(u'))}|h(v_i^f(u'))|}{\text{min}\{|e^{\gamma_0 N(v_i^f(u))}|h(v_i^f(u))|, e^{\gamma_0 N(v_i^f(u'))}|h(v_i^f(u'))|\}}
\]
\[
\leq \frac{e^{\gamma_0 N(v_i^f(u))} - e^{\gamma_0 N(v_i^f(u'))}}{e^{\gamma_0 N(v_i^f(u'))}} + \frac{e^{\gamma_0 N(v_i^f(u))}h(v_i^f(u)) - h(v_i^f(u'))}{e^{\gamma_0 N(v_i^f(u'))}h(v_i^f(u'))}
\]
\[
\leq \frac{e^{\gamma_0 N(v_i^f(u))} - e^{\gamma_0 N(v_i^f(u'))}}{e^{\gamma_0 N(v_i^f(u'))}}
\]
\[
+ |e^{i\gamma_0 N(v_i^f(u))} - e^{i\gamma_0 N(v_i^f(u'))}| + 4E|b|\text{e}^{2a_0 N T} \text{diam}(C')
\]
\[
\leq \left( e^{C_1 t} + |\text{e}^{C_1 t} D(v_i^f(u), v_i^f(u')) + 4E|b|\text{e}^{2a_0 N T} \frac{\gamma_0^{n_1} \epsilon_1}{c_0 \gamma_0^N} \right)
\]
\[
\leq \frac{(B + A_0) \gamma_0^{n_1} \epsilon_1}{c_0 \gamma_0^N} + \frac{4E\gamma_0^{n_1} \epsilon_1}{c_0 (\text{e}^{-2a_0 N T} \gamma_0)^N} < \frac{\pi}{12}
\]
assuming \(a_0 > 0\) is chosen sufficiently small and \(N\) sufficiently large. So, the angle between the complex numbers \(e^{\gamma_0 N(v_i^f(u))}h(v_i^f(u))\) and \(e^{\gamma_0 N(v_i^f(u'))}h(v_i^f(u'))\) (regarded as vectors in \(\mathbb{R}^2\)) is \(\leq \pi/6\). In particular, for each \(i = 1, 2\) we can choose a real continuous function \(\theta_i(u), u \in \tilde{Z}\), with values in \([0, \pi/6]\) and a constant \(\lambda_i\) such that
\[
e^{\gamma_0 N(v_i^f(u))}h(v_i^f(u)) = e^{i(\lambda_i + \theta_i(u))}e^{\gamma_0 N(v_i^f(u))}|h(v_i^f(u))|
\]
for all \(u \in \tilde{Z}\). Fix an arbitrary \(u_0 \in \tilde{Z}\) and set \(\lambda = \gamma_0(u_0)\). Replacing e.g \(\lambda_2\) by \(\lambda_2 + 2m\pi\) for some integer \(m\), we may assume that \(|\lambda_2 - \lambda_1 + \lambda| \leq \pi\). Using the above, \(\theta \leq 2 \sin \theta\) for \(\theta \in [0, \pi/6]\), and some elementary geometry yields
\[
|\theta_i(u) - \theta_i(u')| \leq 2 \sin |\theta_i(u) - \theta_i(u')| < \frac{c_2 \epsilon_1}{8}.
\]
The difference between the arguments of the complex numbers
\[
e^{i b \tau^N(v_1^f(u))}e^{\gamma_0 N(v_1^f(u))}h(v_1^f(u)) \quad \text{and} \quad e^{i b \tau^N(v_1^f(u))}e^{\gamma_0 N(v_1^f(u))}h(v_1^f(u))
\]
\(\Gamma(u \tau^N(u^f)) = [b \tau^N(v_1^f(u)) + \theta_2(u) + \lambda_2] - [b \tau^N(v_1^f(u)) + \theta_1(u) + \lambda_1]
\]
\(\Gamma(u \tau^N(u^f)) = (-\lambda_2 - \lambda_1) + \gamma_0(u) + (\theta_2(u) - \theta_1(u)).
\]

\(^1\)Using some estimates as in the proof of Lemma 2(b) in the Appendix below and \(\|eg_i^N\|_0 \leq a_0 NT\) by (4.3).
Given $u' \in \tilde{Z}_j'$ and $u'' \in \tilde{Z}_j''$, since $\tilde{D}_j'$ and $\tilde{D}_j''$ are contained in $C_m$ and are $\eta$-separable in $C_m$, it follows from Lemma 6 and the above that

$$|\Gamma^{(\ell)}(u') - \Gamma^{(\ell)}(u'')| \geq |\xi(u') - \xi(u'')| - |\theta_1(u') - \theta_1(u'')| - |\theta_2(u') - \theta_2(u'')| \geq \frac{c_2 \epsilon_1}{2}.$$  

Thus, $|\Gamma^{(\ell)}(u') - \Gamma^{(\ell)}(u'')| \geq \frac{c_2 \epsilon_1}{2}$ for all $u' \in \tilde{Z}_j'$ and $u'' \in \tilde{Z}_j''$. Hence either $|\Gamma^{(\ell)}(u')| \geq \frac{c_2 \epsilon_1}{2}$ for all $u' \in \tilde{Z}_j'$ or $|\Gamma^{(\ell)}(u'')| \geq \frac{c_2 \epsilon_1}{2}$ for all $u'' \in \tilde{Z}_j''$.

Assume for example that $|\Gamma^{(\ell)}(u')| \geq \frac{c_2 \epsilon_1}{2}$ for all $u' \in \tilde{Z}_j'$. Since $\tilde{Z} \subset \sigma^{n_1}(C_m)$, as in [11], for any $u \in \tilde{Z}$ we get $|\Gamma^{(\ell)}(u)| < \frac{c_2 \epsilon_1}{2}$. Thus, $\frac{c_2 \epsilon_1}{2} \leq |\Gamma^{(\ell)}(u)| < \frac{c_2 \epsilon_1}{2}$ for all $u \in \tilde{Z}_j'$. Now as in [4] (see also [11]) one shows that $\chi^{(1)}(u) \leq 1$ and $\chi^{(2)}(u) \leq 1$ for all $u \in \tilde{Z}_j'$.

Parts (a) and (b) of the following lemma can be proved in the same way as the corresponding parts of Lemma 5.5 in [11], while part (c) follows from Lemma 4(b).

**Lemma 9.** There exist a positive integer $N$ and constants $\hat{\rho} = \hat{\rho}(N) \in (0,1)$, $a_0 = a_0(N) > 0$, $b_0 = b_0(N) > 0$ and $E \geq 1$ such that for every $a, b, c, t \geq 1, w \in \mathbb{R}$ with $|a|, |c| \leq a_0$, $|b| \geq b_0$ such that $|w| \leq B|b|$, there exists a finite family $\{N_j\}_{j \in J}$ of operators

$$N_j = N_j(a, b, t, c) : C^{Lip}_D(\tilde{U}) \to C^{Lip}_D(\tilde{U}),$$

where $J = J(a, b, t, c)$, with the following properties:

(a) The operators $N_j$ preserve the cone $K_{E|b|}(\tilde{U})$;

(b) For all $H \in K_{E|b|}(\tilde{U})$ and $J \in J$ we have

$$\int_0^1 \langle N_j H \rangle^2 \, d\nu_0 \leq \hat{\rho} \int_0^1 H^2 \, d\nu_0.$$

(c) If $h, H \in C^{Lip}_D(\tilde{U})$ are such that $H \in K_{E|b|}(\tilde{U})$, $|h(u)| \leq H(u)$ for all $u \in \tilde{U}$ and

$$|h(u) - h(u')| \leq Et |b| H(u') D(u, u')$$

whenever $u, u' \in \tilde{U}_i$ for some $i = 1, \ldots, k$, then there exists $J \in J$ such that $|\mathcal{L}_{abtz}^N h(u)| \leq (N_j H)(u)$ for all $u \in \tilde{U}$ and for $z = c + iw$ we have

$$|\langle \mathcal{L}_{abtz}^N h \rangle(u) - \langle \mathcal{L}_{abtz}^N h \rangle(u')| \leq Et |b| (N_j H)(u') D(u, u')$$

whenever $u, u' \in \tilde{U}_i$ for some $i = 1, \ldots, k$.

**Proof of Lemma 9.** Set $\hat{\rho} = 1 - \epsilon_2$.

Let $a \in \mathbb{R}$ and $b, w \in \mathbb{R}$ be such that $|a| \leq a_0$ and $|w| \leq B|b|$, $|b|, |w| \geq b_0$, and let $J \in J(a, b)$. Then (a) follows from Lemma 4(a), while (b) follows from Lemma 5.

To check (c), assume that $h, H \in C^{Lip}_D(\tilde{U})$ satisfy (4.9) and (4.10). Now define the subset $J$ of $J(a, b)$ in the following way. First, include in $J$ all $(1, j, \ell) \in \mathbb{Z}$ such that $\chi_j^{(1)}(u) \leq 1$ for all $u \in \tilde{Z}_j$. Then for any $j = 1, \ldots, q$ and $\ell = 1, \ldots, \ell_0$ include $(2, j, \ell)$ in $J$ if and only if $(1, j, \ell)$ has not been included in $J$ (that is, $\chi_j^{(1)}(u) > 1$ for some $u \in \tilde{Z}_j$) and $\chi_j^{(2)}(u) \leq 1$ for all $u \in \tilde{Z}_j$. It follows from Lemma 7 that $J$ is dense.

Consider the operator $\mathcal{N} = \mathcal{N}_j(a, b) : C^{Lip}_D(\tilde{U}) \to C^{Lip}_D(\tilde{U})$. Then Lemma 4(b) implies

$$|\langle \mathcal{L}_{abtz}^N h \rangle(u) - \langle \mathcal{L}_{abtz}^N h \rangle(u')| \leq Et |b| (N_j H)(u') D(u, u').$$
whenever $u, u' \in \hat{U}_i$ for some $i = 1, \ldots, k$. So, it remains to show that
\[
|\mathcal{L}_{ab\ell}^N h(u)| \leq (N\mathcal{H})(u), \quad u \in \hat{U}.
\] (4.11)

Let $u \in \hat{U}$. If $u \notin \hat{Z}_j$ for any $(i, j, \ell) \in J$, then $\omega(v) = 1$ whenever $\sigma^N v = u$ (since $v \in X_{i,j}^{(t)}$ implies $u = \sigma^N v \in Z_j$). Hence
\[
|\mathcal{L}_{ab\ell}^N h(u)| = \sum_{\sigma^N v = u} e^{(f_{ab\ell}^N - i\sigma \tau^N + i\mu g^N)(v)} h(v) 
\leq (\mathcal{M}_{ab\ell}(\omega H))(u) = (N\mathcal{H})(u).
\]

Assume that $u \in \hat{Z}_j$ for some $(i, j, \ell) \in J$. We will consider the case $i = 1$; the case $i = 2$ is similar. (Notice that by the definition of $J$, we cannot have both $(1, j, \ell)$ and $(2, j, \ell)$ in $J$.) Then $\chi_{(1)}(u) \leq 1$, and therefore
\[
|\mathcal{L}_{ab\ell}^N h(u)| \leq \sum_{\sigma^N v = u, v \notin v_1(u), v_2(u)} e^{(f_{ab\ell}^N - i\tau^N + i\mu g^N)(v)} h(v)
+ \sum_{\sigma^N v = u, v \notin v_1(u), v_2(u)} e^{(f_{ab\ell}^N - i\tau^N + i\mu g^N)(v_1(u))} h(v_1(u)) + e^{(f_{ab\ell}^N - i\tau^N + i\mu g^N)(v_2(u))} h(v_2(u))
\leq \sum_{\sigma^N v = u, v \notin v_1(u), v_2(u)} e^{f_{ab\ell}^N(v)} h(v)
+ \left[1 - \mu e^{f_{ab\ell}^N(v_1(u))} H(v_1(u)) + e^{f_{ab\ell}^N(v_2(u))} H(v_2(u))\right].
\]

Since $(1, j, \ell) \in J$ and $(2, j, \ell) \notin J$, the definition of the function $\omega$ gives $\omega(v_1(u)) \geq 1 - \mu$ and $\omega(v_2(u)) = 1$. This and (4.9) imply
\[
|\mathcal{L}_{ab\ell}^N h(u)| \leq \sum_{\sigma^N v = u, v \notin v_1(u), v_2(u)} e^{f_{ab\ell}^N(v)} \omega(v) H(v)
+ \left[e^{f_{ab\ell}^N(v_1(u))} \omega(v_1(u)) H(v_1(u)) + e^{f_{ab\ell}^N(v_2(u))} \omega(v_2(u)) H(v_2(u))\right]
= (N\mathcal{H})(u),
\]
which proves (4.11). \qed

5. Proofs of theorems. Proof of Theorem 3. We use an argument from [4].

Let $B > 0$ be a constant. Let $N, \rho, a_0, w_0$ and $E$ be as in Lemma 9. Given $a, b, c, w, t \in \mathbb{R}$ with $|a| \leq a_0$, $|b| \geq b_0$, $|w| \leq B|b|$, let $\{N_{kj}\}_{j \in J}$ be a finite family of operators having the properties (a), (b) and (c) in Lemma 9.

Let $h \in C^{Lip}(U)$ be such that $\|h\|_{Lip, b} \leq 1$. Then $|h(u)| \leq 1$ for all $u \in U$ and $\text{Lip}(h) \leq |b|$. Thus, for any $u, v \in \hat{U}_i$, $i = 1, \ldots, k$, we have
\[
|h(u) - h(v)| \leq |b| d(u, v) \leq |b| D(u, v),
\]
so $\text{Lip}_D(h) \leq |b|$. Set $h^{(m)} = L_{ab\ell}^N h$. Define the sequence of functions $\{H^{(m)}\}$ recursively by $H^{(0)} = 1$ and $H^{(m+1)} = N_{kj} H^{(m)}$, where $J_m \in J$ is chosen by induction so that the conclusions of Lemma 9(c) are satisfied with $h = h^{(m)}$, $H = H^{(m)}$ and $J = J_m$.

Since $H^{(0)} \in K_{E|b|}(U)$, it follows that $H^{(m)} \in K_{E|b|}(U)$ for all $m \geq 0$. Moreover, for $h^{(0)} = h$ we clearly have $|h^{(0)}| \leq H^{(0)}$ and
\[
|h^{(0)}(u) - h^{(0)}(u')| \leq |b| d(u, u') \leq Et|b| H^{(0)}(u') D(u, u')
\]
whenever $u, u' \in \widehat{U}_i$ for some $i = 1, \ldots, k$. Now Lemma 9(c) implies that $h^{(m)}$ and $H^{(m)}$ satisfy similar conditions for all $m \geq 0$. In particular, $|h^{(m)}| \leq H^{(m)}$ on $\widehat{U}$ for all $m$.

Using an induction on $m$ and property (b) in Lemma 9, we get

$$\int_{\widehat{U}} (H^{(m)})^2 \, d\nu \leq \hat{\rho} \int_{\widehat{U}} (H^{(m-1)})^2 \, d\nu \leq \hat{\rho}^m.$$ Hence

$$\int_{\widehat{U}} |\mathcal{L}^N_{abtz} h|^2 \, d\nu = \int_{\widehat{U}} |\mathcal{L}^N_{abtz} h’|^2 \, d\nu = \int_{\widehat{U}} |h^{(m)}|^2 \, d\nu \leq \int_{\widehat{U}} (H^{(m)})^2 \, d\nu \leq \hat{\rho}^m$$

for all $m \geq 1$. This proves the theorem.

As in [4] and [11] we need the following lemma whose proof is the same.

**Lemma 10.** Let $\beta \in (0, \alpha)$. There exists a constants $A_1 > 0$ such that for all $a, b, c, t, w \in \mathbb{R}$ with $|a|, |c|, 1/|b|, 1/t \leq a_0$ such that $|w| \leq B|b|$, and all positive integers $m$ and all $h \in C^\beta(U)$ we have

$$|\mathcal{L}^N_{abtz} h(u) - \mathcal{L}^N_{abtz} h(u’)| \leq A_1 \left[ \frac{|h|_{\beta}}{m^\beta} + |b| (\mathcal{M}^\alpha_{atc}(h))(u') \right] (d(u, u'))^\beta$$

for all $u, u’ \in U_i$.

We will derive Theorem 1 from Theorem 3 and Lemma 10 above.

**Proof of Theorem 1.** We essentially repeat the proofs of Corollaries 2 and 3 in [4] (see also the Appendix in [9]).

Let $\epsilon > 0$, $B > 0$ and $\beta \in (0, \alpha)$. Take $\rho \in (0, 1)$, $a_0 > 0$, $b_0 > 0$, $A_0 > 0$ and $N$ as in Theorem 2. We will assume that $\rho \geq \frac{1}{20}$. Let $a, b, c, w \in \mathbb{R}$ be such that $|a|, |c| \leq a_0$ and $|b| \geq b_0$. Let $t > 0$ be such that $1/t^{n-\beta} \leq a_0$. Assume that $|w| \leq B|b|$ and set $z = c + iu$.

First, as in [4] one derives from Theorem 2 and Lemma 10 (approximating functions $h \in C^\beta(\widehat{U})$ by Lipschitz functions) that there exist constants $C_6 > 0$ and $\rho_3 \in (0, 1)$ such that

$$\|\mathcal{L}^n_{abtz} h\|_{\beta, b} \leq C_6 |b|^\epsilon \rho_3^n, \quad n \geq 0,$$

for all $h \in C^\beta(\widehat{U})$.

Next, given $h \in C^\beta(\widehat{U})$, we have

$$\mathcal{L}^n_{abtz}(h/h_{atc}) = \frac{1}{\lambda_{atc}} \mathcal{L}^{n}_{h_{atc}} L_{f_1-s \tau + z g_1} h,$$

where again $s = a + ib$ and $z = c + iw$, so by (4.3) and (4.4) we get

$$\|L_{f_1-s \tau + z g_1} h\|_{\beta, b} \leq \lambda_{atc} \|h_{atc} \mathcal{L}^n_{abtz}(h/h_{atc})\|_{\beta, b} \leq \text{Const} \lambda^n_{atc} (e^{3C_\alpha a_0 \rho_3})^n |b|^\epsilon \|h/h_{atc}\|_{\beta, b} \leq \text{Const} \lambda^n_{atc} \rho_3^n |b|^\epsilon \|h\|_{\beta, b},$$

where $\lambda_{atc} \leq e^{3C_\alpha a_0 \lambda_0}$ and $\rho_3 e^{3C_\alpha a_0} = \rho_4 < 1$, provided $a_0 > 0$ is small enough.
We will now approximate $L_{f-st+zg}$ by $L_{f_t-st+cg_t}$ in two steps. First, the above implies
\[
\|L_{f-st+zg}^n h\|_{\beta,w} = \|L_{f_t-st+cg_t}^n \left( e^{(f^n-f_t^n)+c(g^n-g_t^n)h} \right) \|_{\beta,b} \\
\leq C \lambda_0^n \rho^n |b|^\tau \|e^{(f^n-f_t^n)+c(g^n-g_t^n)h}\|_{\beta,b}
\]
for some constant $C > 0$. Choosing $C$ appropriately, we have $\|f-f_t\|_0 \leq C \alpha_0$ and $|f-f_t|_{\beta} \leq C/t^{n-\beta} \leq C$, so $\|f^n-f_t^n\|_0 \leq n \|f-f_t\|_0 \leq C n \alpha_0$, and similarly $|f^n-f_t^n|_{\beta} \leq C n \alpha_0$. Similar estimates hold for $g^n-g_t^n$. Thus,
\[
\|e^{(f^n-f_t^n)+c(g^n-g_t^n)h}\|_0 \leq e^{C n \alpha_0} \|h\|_0,
\]
and
\[
|e^{(f^n-f_t^n)+c(g^n-g_t^n)h}|_{\beta} \leq \|e^{(f^n-f_t^n)+c(g^n-g_t^n)h}\|_0 |h|_{\beta} + |e^{(f^n-f_t^n)+c(g^n-g_t^n)h}|_{\beta} \|h\|_\infty \\
\leq e^{C n \alpha_0} |h|_{\beta} + e^{C n \alpha_0} \|f^n-f_t^n\| + c(g^n-g_t^n)|_{\beta} \|h\|_\infty \\
\leq C n e^{C n \alpha_0} \|h\|_{\beta},
\]
replacing $C$ by a larger constant where necessary. Combining this with the previous estimate gives
\[
\|e^{(f^n-f_t^n)+c(g^n-g_t^n)h}\|_{\beta,b} \leq C n e^{C n \alpha_0} \|h\|_{\beta},
\]
so
\[
\|L_{f-st+zg}^n h\|_{\beta,b} \leq C \lambda_0^n \rho^n \|h\|_{\beta,b}.
\]
Taking $\alpha_0 > 0$ sufficiently small, we may assume that $\rho_t e^{C \alpha_0} < 1$. Now take an arbitrary $\rho_t$ with $\rho_t e^{C \alpha_0} < \rho_5 < 1$. Then we can take the constant $C_7 > 0$ so large that $n \rho_t e^{C \alpha_0} \leq C_7 \rho_5^0$ for all integers $n \geq 1$. This gives
\[
\|L_{f-st+zg}^n h\|_{\beta,b} \leq C_7 \lambda_0^n \rho_5^n |b|^\tau \|h\|_{\beta,b}
\]
for all $n \geq 0$. Using the latter we can write
\[
\|L_{f-st+zg}^n h\|_{\beta,b} = \|L_{f_t-st+cg_t}^n \left( e^{iw(g^n-g_t^n)h} \right) \|_{\beta,b} \\
\leq C_7 \lambda_0^n \rho_5^n |b|^\tau \|e^{iw(g^n-g_t^n)h}\|_{\beta,b}.
\]
We have
\[
\|e^{iw(g^n-g_t^n)h}\|_0 = \|h\|_0 , \quad |g-g_t|_{\beta} \leq C \sqrt{t^{n-\beta}} \leq 1
\]
assuming $t > 1$ is sufficiently large, so
\[
|e^{iw(g^n-g_t^n)h}|_{\beta} \leq \|e^{iw(g^n-g_t^n)h}\|_0 |h|_{\beta} + \|e^{iw(g^n-g_t^n)h}|_{\beta} \|h\|_0 \leq |h|_{\beta} + \|h\|_\infty \|g^n-g_t^n|_{\beta} \|h\|_0.
\]
Therefore
\[
\|e^{iw(g^n-g_t^n)h}\|_{\beta,b} = \|e^{iw(g^n-g_t^n)h}\|_0 + \frac{|w|}{|b|} \|e^{iw(g^n-g_t^n)h}|_{\beta} \leq 2 B \|h\|_{\beta,b}.
\]
This yields
\[
\|L_{f-st+zg}^n h\|_{\beta,b} \leq C_8 \lambda_0^n \rho_5^n |b|^\tau \|h\|_{\beta,b}.
\]
Choosing $\rho_6$ with $\rho_5 < \rho_6 < 1$ and taking the constant $C_9 > C_8$ sufficiently large, we get
\[
\|L_{f-st+zg}^n h\|_{\beta,b} \leq C_9 \lambda_0^n \rho_6^n |b|^\tau \|h\|_{\beta,b}
\]
for all integers $n \geq 0$. \qed
6. Appendix: Proof of Lemma 2. (a) Let $u, u' \in \hat{U}$ for some $i = 1, \ldots, k$ and let $m \geq 1$ be an integer. For any $v \in \hat{U}$ with $\sigma^m(v) = u$, denote by $v' = v'(v)$ the unique $v' \in \hat{U}$ in the cylinder of length $m$ containing $v$ such that $\sigma^m(v') = u'$. Then
\[
|f^m_{ate}(v) - f^m_{ate}(v')| \leq \sum_{j=0}^{m-1} |f^m_{ate}(\sigma^j(v)) - f^m_{ate}(\sigma^j(v'))| \\
\leq \frac{Tt}{c_0(\gamma_0 - 1)} d(u, u') \leq C_1 t D(u, u')
\]
for some constant $C_1 > 0$. Similarly,
\[
|g^m_t(v) - g^m_t(v')| \leq C_1 t D(u, u').
\]
If $D(u, u') = \text{diam}(C')$ for some cylinder $C' = C[i_{m+1}, \ldots, i_p]$, then
\[
v, v'(v) \in C'' = C[i_0, i_1, \ldots, i_p]
\]
for some cylinder $C''$ with $\sigma^m(C'') = C'$, so
\[
D(v, v') \leq \text{diam}(C'') \leq \frac{1}{c_0 \gamma_0^m} \text{diam}(C') = \frac{D(u, u')}{c_0 \gamma_0^m}.
\]
We have
\[
\frac{\langle M^m_{ate} H \rangle(u) - \langle M^m_{ate} H \rangle(u')}{M^m_{ate} H (u')} = \frac{\sum_{\sigma^m v = u} e^{\int_{ate}^m (v) + cg^m_t(v) H(v)} - \sum_{\sigma^m v = u} e^{\int_{ate}^m (v') + cg^m_t(v') H(v')}}{M^m_{ate} H (u')}
\]
\[
\leq \frac{\sum_{\sigma^m v = u} \left| e^{\int_{ate}^m (v) + cg^m_t(v)} - e^{\int_{ate}^m (v') + cg^m_t(v')} \right| H(v')}{M^m_{ate} H (u')}
\]
\[
+ \frac{\sum_{\sigma^m v = u} \left| e^{\int_{ate}^m (v) + cg^m_t(v)} - e^{\int_{ate}^m (v') + cg^m_t(v')} \right| Q H(v') D(v, v')}{M^m_{ate} H (u')}
\]
\[
\leq \frac{\sum_{\sigma^m v = u} \left| e^{\int_{ate}^m (v) + cg^m_t(v)} - [f^m_{ate}(v') + cg^m_t(v')] \right| H(v')}{M^m_{ate} H (u')}
\]
By (6.1) and (6.2),
\[
|f^m_{ate}(v) + cg^m_t(v) - [f^m_{ate}(v') + cg^m_t(v')]| \leq 2C_1 t D(u, u') \leq 2C_1 t
\]
which implies
\[
|e^{\int_{ate}^m (v) + cg^m_t(v)} - [f^m_{ate}(v') + cg^m_t(v')] - 1| \leq e^{2C_1 t} 2C_1 t D(u, u').
\]
A more precise estimate follows from (4.4) and (4.5):
\[
|f^m_{ate}(v) + cg^m_t(v) - [f^m_{ate}(v') + cg^m_t(v')]| \\
\leq |f^m_{ate}(v) - f^m_t(v)| + |P - a||\tau^m(v) - \tau^m(v')|
\]
A for some constant \( \sigma \) and \( v \) notation.

(b) Let

\[
\frac{Q D(u, u')}{c_0 \gamma m} \sum_{\sigma m v = u} e^{f_{atc}(v)+cg_i^m(v)} h(v) + \frac{Q D(u, u')}{c_0 \gamma m} \sum_{\sigma m v = u} 2C_1 t e^{f_{atc}(v')(v')} H(v') \leq e^{C_2} e^{C_2 a_0} + 2C_1 t e^{2C_1 t} D(u, u') \leq A_0 \left[ \frac{Q}{\gamma m} + e^{A_0 t} t \right] D(u, u'),
\]

for some constant \( A_0 > 0 \) independent of \( a, c, t, m \) and \( Q \).

(b) Let \( m \geq 1 \) be an integer and \( u, u' \in \hat{U}_i \) for some \( i = 1, \ldots, k \). Using the notation \( v' = v'(v) \) and the constant \( C_2 > 0 \) from part (a) above, where \( \sigma^m v = u \) and \( \sigma^m v' = u' \), and some of the estimates from the proof of part (a), we get

\[
|L_{abt}^m h(u) - L_{abt}^m h(u')| = | \sum_{\sigma^m v = u} (e^{f_{atc}(v)+cg_i^m(v)} - e^{f_{atc}(v') + cg_i^m(v')}) h(v) |
\]

Using the constants \( C_1, C_2 > 0 \) from the proof of part (a) and \( e^{C_2 a_0} < \gamma_0 / \hat{\gamma} \) we get

\[
\sum_{\sigma^m v = u} e^{f_{atc}(v)+cg_i^m(v)} |h(v) - h(v')| \leq e^{C_2} e^{C_2 a_0} \frac{Q D(u, u')}{c_0 \gamma m} \sum_{\sigma^m v = u} e^{f_{atc}(v') + cg_i^m(v')} H(v').
\]
\[ \frac{e^{C_2 t} Q}{c_0 \gamma^m}(M_{\text{atc}}^m H)(u') D(u, u'). \]

This, implies

\[ |L_{\text{abtz}}^m h(u) - L_{\text{abtz}}^m h(u')| \leq \frac{e^{C_2 t} Q}{c_0 \gamma^m}(M_{\text{atc}}^m H)(u') D(u, u') + C_1 t D(u, u') + e^{A_0 t + t|w|}(M_{\text{atc}}^m |h|)(u') D(u, u'), \]

Thus, taking the constant \( A_0 > 0 \) sufficiently large we get

\[ |(L_{\text{abtz}}^N h)(u) - (L_{\text{abtz}}^N h)(u')| \leq A_0 \left( \frac{t Q}{\gamma^m}(M_{\text{atc}}^m H)(u') + (|b| + e^{A_0 t + t|w|})(M_{\text{atc}}^m |h|)(u') \right) D(u, u'), \]

which proves the assertion.

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