Higher Rédei reciprocity and integral points on conics

Peter Koymans\textsuperscript{1} and Carlo Pagano\textsuperscript{1}

\textsuperscript{1}Max Planck Institute for Mathematics, Bonn

May 29, 2020

Abstract

Fix an integer \( l \) such that \( |l| \) is a prime 3 modulo 4. Let \( d > 0 \) be a squarefree integer and let \( N_d(x, y) \) be the principal binary quadratic form of \( \mathbb{Q}(\sqrt{d}) \). Building on a breakthrough of Smith [33], we give an asymptotic formula for the solubility of \( N_d(x, y) = l \) in integers \( x \) and \( y \) as \( d \) varies among squarefree integers divisible by \( l \).

As a corollary we give, in case \( l > 0 \), an asymptotic formula for the event that the Hasse Unit Index of the field \( \mathbb{Q}(\sqrt{-l}, \sqrt{d}) \) is 2 as \( d \) varies over all positive squarefree integers. We also improve the results of Fouvry and Klüners [11, 12] and recent results of Chan, Milovic and the authors [2] on the solubility of the negative Pell equation. Our main new tool is a generalization of a classical reciprocity law due to Rédei [31].

1 Introduction

The study of integral points on conics goes back to at least the ancient Greeks. Much later significant progress was made by the Indian mathematicians Brahmagupta and Bhaskara II around the years 650 and 1150 respectively. Brahmagupta was able to solve the Pell equation

\[ x^2 - dy^2 = 1 \] in \( x, y \in \mathbb{Z} \] (1.1)

in special cases, while Bhaskara II was the first to give a method to solve the Pell equation in full generality.

Let \( l \) be an integer such that \( |l| \) is a prime 3 modulo 4 or let \( l = -1 \). For a squarefree integer \( d > 0 \), we define

\[ N_d(x, y) = \begin{cases} x^2 + xy - \frac{d-1}{4} y^2 & \text{if } d \equiv 1 \text{ mod } 4 \\ x^2 - dy^2 & \text{otherwise}, \end{cases} \]

which is the principal binary quadratic form of \( \mathbb{Q}(\sqrt{d}) \). In this paper we look at the equation

\[ N_d(x, y) = l \in x, y \in \mathbb{Z} \] (1.2)

with \( d \) squarefree. Unlike equation (1.1) it is not always possible to find \( x, y \in \mathbb{Z} \) that satisfy the above equation. We denote by \( H(K) \) the narrow Hilbert class field of a number field \( K \),

\textsuperscript{1}Vivatsgasse 7, 53111 Bonn, Germany, koymans@mpim-bonn.mpg.de

\textsuperscript{1}Vivatsgasse 7, 53111 Bonn, Germany, carlein90@gmail.com
The relation in \( \text{Cl}(\mathbb{Q}(\sqrt{d})) \) of Equation (1.2) is soluble. In fact, for a fixed squarefree integer \( d \) it is easy to determine necessary and sufficient conditions on \( d \). In this case we know that \( d \) splits in the genus field of \( \mathbb{Q}(\sqrt{d}) \) if and only if there is an ideal in \( \mathbb{Q}(\sqrt{d}) \) with norm \( d \) and trivial Artin symbol in the narrow Hilbert class field of \( \mathbb{Q}(\sqrt{d}) \). We will now focus on \( l \) a prime 3 modulo 4, and shall later discuss the very classical case \( l = -1 \) known as the negative Pell equation.

Given \( d \), there exists an algorithm to compute the Hilbert class field of \( \mathbb{Q}(\sqrt{d}) \) both in the narrow and ordinary sense. Hence it is possible to decide given \( l \) and \( d \) whether equation (1.2) is soluble. In fact, for a fixed squarefree integer \( d \), an appeal to the Chebotarev Density Theorem gives an asymptotic for the number of primes \( l \) such that equation (1.2) is soluble.

In this paper we ask the opposite question. Instead of fixing \( d \), we shall treat \( l \) as fixed and vary \( d \). Equivalently, we ask how often there is some ideal with norm \( d \) which is the maximal abelian extension of \( \mathbb{Q} \) that is unramified at all finite places, while the ordinary Hilbert class field must also be unramified at the infinite places.

If \( l = -1 \), equation (1.2) is soluble if and only if the narrow and ordinary Hilbert class fields of \( \mathbb{Q}(\sqrt{d}) \) coincide. Instead, if \( l > 0 \) is a prime 3 modulo 4, then equation (1.2) is soluble if and only if there is an ideal in \( \mathbb{Q}(\sqrt{d}) \) with norm \( l \) and trivial Artin symbol in the narrow Hilbert class field of \( \mathbb{Q}(\sqrt{d}) \). We will now focus on \( l \) a prime 3 modulo 4, and shall later discuss the very classical case \( l = -1 \) known as the negative Pell equation.

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Theorem 1.1. Let \( l \) be an integer such that \( |l| \) is a prime 3 modulo 4. Then we have

\[
\lim_{x \to \infty} \frac{|S_{\mathbb{Q},X,l}|}{|S_{\mathbb{Q},X,l}|} = \gamma.
\]

We remark that \( \gamma \) has a very natural interpretation. Informally speaking, the quantity \( 2^{-j^2} \eta_{\infty} \eta_j^{-2} \) represents the probability that the 4--rank of a random element in the set \( S_{\mathbb{Q},X,l} \)
is equal to \( j \). This will be made precise in Theorem 5.17. Note that if the 4–rank of \( \text{Cl}(K) \) is \( j \), we have a natural generating set, coming from Gauss genus theory, of size \( j + 1 \) for \( 2\text{Cl}(K) \)[4]. Furthermore, Gauss genus theory says that there is exactly one relation between the generators. Hence \( 1/(2^{j+1} - 1) \) represents the probability that the ideal above \( l \) is the relation, if one thinks of the relation as being “random”. This is very much in spirit of Stevenhagen’s conjecture [34] on the solubility of the negative Pell equation. Although we shall not prove it, our techniques readily give the distribution of \( 2\text{Cl}(\mathbb{Q}(\sqrt{d}))^{2} \) as \( d \) varies in \( S_{\mathbb{Q}, \infty, l} \).

By classical techniques one can give an asymptotic formula for \( |S_{\mathbb{Q},X,l}| \); this requires only slight modifications of [25, Exercise 21, Section 6.2], see also [27, Section 3]. Indeed, we have

\[
|S_{\mathbb{Q},X,l}| \sim \frac{1}{\sqrt{\pi}} \cdot \frac{C(l) \cdot \delta(l)}{|l|} \cdot \frac{X}{\sqrt{\log X}},
\]

where

\[
C(l) = \lim_{s \to 1} \left( \sqrt{s - 1} \cdot \prod_{p \text{ odd}} \left( 1 + \frac{1}{p^s} \right) \right), \quad \delta(l) = \begin{cases} 
3/2 & \text{if } l \equiv 1 \text{ mod } 8 \\
3/4 & \text{if } l \equiv 3 \text{ mod } 8 \\
1 & \text{if } l \equiv 5 \text{ mod } 8 \\
3/4 & \text{if } l \equiv 7 \text{ mod } 8.
\end{cases}
\]

This yields the following corollary of Theorem 1.1.

**Corollary 1.2.** Take \( l \) to be an integer such that \(|l|\) is a prime 3 modulo 4. Then

\[
|S_{\mathbb{Z},X,l}| \sim \frac{\gamma}{\sqrt{\pi}} \cdot \frac{C(l) \cdot \delta(l)}{|l|} \cdot \frac{X}{\sqrt{\log X}}.
\]

Earlier work was done by Milovic [24], who showed that \( S_{\mathbb{Z},X,\pm 2} \) has the same order of magnitude as \( S_{\mathbb{Q},X,\pm 2} \). It is plausible that our methods can be adapted to the case \( l = \pm 2 \) as well.

An immediate application is the following result. For a biquadratic field \( \mathbb{Q}(\sqrt{a}, \sqrt{b}) \), the Hasse Unit Index is defined to be

\[
H_{a,b} := \left[ O_{\mathbb{Q}(\sqrt{a}, \sqrt{b})}^{\ast} : O_{\mathbb{Q}(\sqrt{a})}^{\ast} O_{\mathbb{Q}(\sqrt{b})}^{\ast} O_{\mathbb{Q}(\sqrt{ab})}^{\ast} \right].
\]

If the biquadratic field is totally complex, then it is known that \( H_{a,b} \in \{1, 2\} \), see for example the work of Lemmermeyer [23]. Our next theorem determines the distribution of the Hasse Unit Index in many cases.

**Corollary 1.3.** Let \( l > 3 \) be a prime 3 modulo 4. Then we have

\[
\left| \{0 < d < X \text{ squarefree} : H_{-l,d} = 2 \} \right| \sim |S_{\mathbb{Z},X,l}| + |S_{\mathbb{Z},X,-l}| \sim \left( \frac{\gamma}{\sqrt{\pi}} \cdot \frac{C(l) \cdot \delta(l)}{l} + \frac{\gamma}{\sqrt{\pi}} \cdot \frac{C(-l) \cdot \delta(-l)}{l} \right) \cdot \frac{X}{\sqrt{\log X}}.
\]

From a more geometric perspective, Theorem 1.1 counts how often there exists an integral point in a family of conics. As such, it is natural to view this result from the perspective of the integral Brauer–Manin obstruction. The seminal work [4] was the first to systematically study the integral Brauer–Manin obstruction.
We shall now return to the case \( l = -1 \). In this case Stevenhagen \cite{Stevenhagen} heuristically predicted how often equation (1.2) is soluble. His heuristical framework can be adjusted to also predict the constant appearing in Theorem 1.1, and we shall do so in Appendix A.

The first major result towards Stevenhagen’s conjecture is that of Fouvry–Klüners \cite{FouvryKluners1}. They showed that

\[
\alpha - o(1) \leq \left| \frac{S_{Z,X,-1}}{S_{Q,X,-1}} \right| \leq \frac{2}{3} + o(1)
\]
as \( X \to \infty \). Here \( \alpha \) is known as Stevenhagen’s constant and equals

\[
\alpha := \prod_{j=1}^{\infty} (1 + 2^{-j})^{-1} \approx 0.4194.
\]

Fouvry–Klüners \cite{FouvryKluners2} later improved the lower bound to \( \frac{5}{4} \alpha \), which was recently improved to \( \beta \alpha \) in \cite{Braak}, where

\[
\beta := \sum_{n=0}^{\infty} 2^{-n(n+3)/2} \approx 1.2832, \quad \beta \alpha \approx 0.53823.
\]

In this paper we improve both the upper and lower bounds.

**Theorem 1.4.** We have

\[
0.54302 - o(1) \leq \left| \frac{S_{Z,X,-1}}{S_{Q,X,-1}} \right| \leq 0.59944 + o(1)
\]
as \( X \to \infty \).

In Theorem 1.4 we have only given the first five decimals. The full constants can be found in Subsection 6.2 including a detailed explanation where the improvement comes from.

Proving the full Stevenhagen conjecture seems to be very hard. The reason for this is that the algebraic results in Smith break down due to the fact that all odd prime divisors are 1 modulo 4 in this family. For a more elaborate discussion on this topic, see Subsection 6.2 or \cite{Braak} p. 3-4. It is also for this reason that we restrict our attention to \(|l| \equiv 3 \text{ mod } 4\).

Theorem 1.1 and Theorem 1.4 make crucial use of a generalization of a reciprocity law due to Rédei \cite{Redei}. This generalization is proven in Section 3. An extensive treatment of the classical Rédei reciprocity law can be found in Corsman \cite{Corsman}, and was one of the main ingredients in Smith’s work on 4-Selmer groups and 8-torsion of class groups \cite{Smith}. Corsman’s and Smith’s formulations of the Rédei reciprocity law are not correct as stated, and this flaw was discovered and corrected by Stevenhagen \cite{Stevenhagen2}.

We will now roughly explain how we make use of our new reciprocity law. Following Smith’s method, we need to prove equidistribution of

\[
\text{Frob}_{K_{x_1,\ldots,x_m,y}/\mathbb{Q}(l)}
\]
as we vary \( y \), where \( K_{x_1,\ldots,x_m,y} \) is a completely explicit field depending only on \( x_1, \ldots, x_m \) and \( y \). Our reciprocity law implies that under suitable conditions

\[
\text{Frob}_{K_{x_1,\ldots,x_m,y}/\mathbb{Q}(l)} = \text{Frob}_{K_{x_1,\ldots,x_m,1}/\mathbb{Q}(y)}.
\]

This allows us to apply the Chebotarev Density Theorem to obtain the desired equidistribution. In the case \( m = 1 \), the fields \( K_{x_1,y} \) are constructed by Rédei, and one recovers the Rédei
reciprocity law. In the case \( m = 2 \), the field \( K_{x_1, \ldots, x_m, y} \) first appears in Amano \[1\] for special values of \( x_1, x_2 \) and \( y \), while the fields \( K_{x_1, \ldots, x_m, y} \) are constructed in full generality by Smith \[33\].

In the language of Smith, these fields are the field of definition of certain maps from \( G_{\mathbb{Q}} \) to \( \mathbb{F}_2 \) that Smith calls \( \phi_{x_1, \ldots, x_m, y} \) or simply \( \phi_y \). The field of definition is an unramified multiquadratic extension of a multiquadratic extension of \( \mathbb{Q} \). As such, they are intimately related to the \( 2 \)-torsion of the class groups of multiquadratic fields. This connection is explored in recent work of the authors \[20\].

We finish the introduction by mentioning some other important results related to class groups. A lot of attention has recently been given to providing non-trivial upper bounds for \( \text{Cl}(K)[l] \) for a fixed prime \( l \). This was initiated by Pierce \[28, 29\] for \( l = 3 \) and continued by Ellenberg and Venkatesh \[8\], Ellenberg, Pierce and Wood \[7\], Frei and Widmer \[13\], Pierce, Turnage-Butterbaugh and Wood \[30\].

Instead of studying class groups of quadratic extensions of \( \mathbb{Q} \), one can study the distribution of class groups in the family of degree \( l \) cyclic extensions of \( \mathbb{Q} \). This was explored by Gerth \[15\] and Klys \[18\], whose work was later generalized by the authors \[19\] using the Smith method \[33\]. It is natural to wonder if the methods in this paper can also be used to study norm forms coming from degree \( l \) cyclic extensions.

Acknowledgements

We are most grateful to Alexander Smith for explaining his work to us on several occasions. Peter Stevenhagen kindly explained his proof of Rédei reciprocity to us, which inspired us to prove a more general version of the Rédei reciprocity law. We thank Vladimir Mitankin for showing us a useful reference and we thank Stephanie Chan and Djordjo Milovic for our many insightful conversations about negative Pell. Both authors are grateful to the Max Planck Institute for Mathematics in Bonn for its hospitality and financial support.

2 Algebraic criteria

In this section we collect the algebraic lemmas that link our theorems to questions about the narrow class group. These lemmas are valid for arbitrary non-zero integers \( l \), and we shall only later restrict to \( l \) with \(|l| \equiv 3 \mod 4 \) a prime. For a non-zero integer \( l \), we define \( \text{sign}(l) = 0 \) if \( l > 0 \) and \( \text{sign}(l) = 1 \) if \( l < 0 \).

Lemma 2.1. Let \( l \) be a non-zero integer and let \( d > 0 \) be a squarefree integer. Then there are \( x, y \in \mathbb{Z} \) with \( N_d(x, y) = l \) if and only if there is an integral ideal \( I \) of \( \mathcal{O}_{\mathbb{Q}(\sqrt{d})} \) with norm \( |l| \) such that \( I \cdot (\sqrt{d})^{\text{sign}(l)} \) has trivial Artin symbol in \( H(\mathbb{Q}(\sqrt{d})) \).

Proof. Suppose that there are \( x, y \in \mathbb{Z} \) with \( N_d(x, y) = l \). In case \( d \equiv 1 \mod 4 \), we look at the ideal \( I = (x + y\sqrt{d} + 1) \). It has norm \(|l|\), and furthermore the element \( x + y\sqrt{d} + 1 \) has norm \( l \). Then \( I \cdot (\sqrt{d})^{\text{sign}(l)} \) is a principal ideal that has an element with positive norm. This implies that \( I \cdot (\sqrt{d})^{\text{sign}(l)} \) is a principal ideal with a totally positive generator, and hence it has trivial Artin symbol in \( H(\mathbb{Q}(\sqrt{d})) \). In case \( d \neq 1 \mod 4 \), we use a similar argument with the ideal \( I = (x + y\sqrt{d}) \).

For the other direction suppose that there is an integral ideal \( I \) of \( \mathcal{O}_{\mathbb{Q}(\sqrt{d})} \) with norm \(|l|\) and \( I \cdot (\sqrt{d})^{\text{sign}(l)} \) has trivial Artin symbol in \( H(\mathbb{Q}(\sqrt{d})) \). Then \( I \cdot (\sqrt{d})^{\text{sign}(l)} \) is a principal
ideal with a totally positive generator $\alpha$, so $N_{Q(\sqrt{d})/Q}(\alpha) = d^{\text{sign}(l)}|l|$. Hence we have

$$I = \left( \frac{\alpha}{\sqrt{d}^{\text{sign}(l)}} \right) \quad \text{and} \quad N_{Q(\sqrt{d})/Q} \left( \frac{\alpha}{\sqrt{d}^{\text{sign}(l)}} \right) = l.$$  

Expanding $\alpha/\sqrt{d}^{\text{sign}(l)}$ as $x + y\sqrt{d}$ if $d \equiv 1 \mod 4$ and $x + y\sqrt{d}$ otherwise, we get the desired $x, y \in \mathbb{Z}$ with $N_d(x, y) = l$. \hfill \Box

In case that $l \mid d$, we see that every prime dividing $l$ ramifies in $Q(\sqrt{d})$. Hence there is exactly one ideal $I$ of $Q(\sqrt{d})$ with norm $|l|$. Furthermore, since $l \in \text{Cl}(Q(\sqrt{d}))[2]$, we see that it is enough to demand that $I$ has trivial Artin symbol in the narrow $2\infty$-Hilbert class field of $Q(\sqrt{d})$, denoted $H_2(Q(\sqrt{d}))$, which is the maximal abelian extension of $Q(\sqrt{d})$ that is unramified at all finite places and has degree a power of 2. This yields the following criterion.

**Lemma 2.2.** Take a non-zero integer $l$ and take a squarefree integer $d > 0$ divisible by $l$. Then there exist $x, y \in \mathbb{Z}$ with $N_d(x, y) = l$ if and only if there is an integral ideal $I$ of $O_{Q(\sqrt{d})}$ with norm $|l|$ such that $I \cdot (\sqrt{d})^{\text{sign}(l)}$ has trivial Artin symbol in $H_2(Q(\sqrt{d}))$.

It is a well-known result that there are $x, y \in \mathbb{Z}$ such that $N_d(x, y) = -1$ if and only if there are $x, y \in \mathbb{Z}$ such that $x^2 - dy^2 = -1$, see for example [24, p. 122]. In this way we can also apply the above lemma to study the negative Pell equation. Our final lemma allows us to deduce Corollary 1.3 directly from Theorem 1.1.

**Lemma 2.3.** Suppose that $l > 3$ is an odd squarefree integer and let $d > 0$ be a squarefree integer with $d \neq l$ and $d \neq 3l$. Then we have $H_{-l, d} = 2$ if and only if $l \mid d$ and there are $x, y \in \mathbb{Z}$ with $N_d(x, y) = l$ or $N_d(x, y) = -l$.

**Proof.** By our assumptions on $l$ and $d$ we have that the roots of unity of $Q(\sqrt{-l}, \sqrt{d})$ are $\{\pm 1\}$. Let $\epsilon$ be the fundamental unit of $Q(\sqrt{d})$. Then Kubota’s work [22, Satz 2] shows that $H_{-l, d} = 2$ if and only if $\pm \epsilon$ is a square in $Q(\sqrt{-l}, \sqrt{d})$. By Kummer theory this is equivalent to $\epsilon = \pm z^2$ for some $z \in Q(\sqrt{d})^{\times}$. This is in turn equivalent to the requirements that every prime dividing $l$ must ramify in $Q(\sqrt{d})$, and furthermore that the unique ideal with norm $|l|$ is principal in $Q(\sqrt{d})$. These last two conditions are equivalent to $l \mid d$ and the existence of $x, y \in \mathbb{Z}$ with $N_d(x, y) = \pm l$. \hfill \Box

## 3 Higher Rédei reciprocity

This section contains the main algebraic innovation of this paper, which is a generalization of the classical Rédei reciprocity law (in turn a generalization of quadratic reciprocity). Fix an algebraic closure $\overline{Q}$ of $Q$ for the rest of the paper. All our number fields are implicitly taken inside this fixed algebraic closure $\overline{Q}$. If $K$ is a number field, we define $G_K := \text{Gal}(\overline{Q}/K)$.

Throughout, we view $E_2$ as a discrete $G_Q$-module with trivial action. If $\phi : G_Q \to X$ is a continuous map with $X$ a discrete topological space, we define $L(\phi)$ to be the smallest Galois extension $K$ of $Q$ through which $\phi$ factors via the canonical projection map $G_Q \to \text{Gal}(K/Q)$. This is well-defined by [19, Lemma 2.3]. For us an unramified extension $L/K$ shall always mean unramified at all finite places of $K$.
3.1 Statement of the reciprocity law

Let \( n \in \mathbb{Z}_{\geq 1} \) and let \( A \subseteq \Gamma_{\mathbb{P}_2}(\mathbb{Q}) := \text{Hom}_{\text{top.-gr.}}(G_\mathbb{Q}, \mathbb{F}_2) \) with \( |A| = n \). Let \( \chi_1, \chi_2 \) be two distinct elements of \( \Gamma_{\mathbb{P}_2}(\mathbb{Q}) - A \). Write

\[
A_1 := A \cup \{\chi_1\}, \quad A_2 := A \cup \{\chi_2\}.
\]

For a finite extension \( \mathbb{L}/\mathbb{Q} \), we denote by \( \text{Ram}(\mathbb{L}/\mathbb{Q}) \) the set of places of \( \mathbb{Q} \) that ramify in \( \mathbb{L}/\mathbb{Q} \). Furthermore, for a collection of characters \( T \subseteq \Gamma_{\mathbb{P}_2}(\mathbb{Q}) \), we denote by \( \mathbb{Q}(\{\chi\}_{\chi \in T}) \) the corresponding multiquadratic extension of \( \mathbb{Q} \). We assume that as \( \chi \) varies in \( A_1 \cup A_2 \) the \( n + 2 \) sets \( \text{Ram}(\mathbb{Q}(\chi)/\mathbb{Q}) \) are non-empty and pairwise disjoint. This in particular implies that \( A_1 \cup A_2 \) is a set of \( n + 2 \) linearly independent characters over \( \mathbb{F}_2 \). We now recall a definition from [20] Definition 3.21.

**Definition 3.1.** Let \( X \subseteq \Gamma_{\mathbb{P}_2}(\mathbb{Q}) \) be linearly independent and let \( \chi_0 \in X \). An expansion map with support \( X \) and pointer \( \chi_0 \) is a continuous group homomorphism

\[
\psi : G_\mathbb{Q} \to \mathbb{F}_2[\mathbb{P}_2^{X-\{\chi_0\}}] \times \mathbb{F}_2^{X-\{\chi_0\}}
\]

such that \( \pi \circ \psi = \chi \) for every \( \chi \in X - \{\chi_0\} \), where \( \pi_x : \mathbb{F}_2[\mathbb{P}_2^{X-\{\chi_0\}}] \times \mathbb{F}_2^{X-\{\chi_0\}} \to \mathbb{F}_2 \) is the natural projection, and \( \pi \circ \psi = \chi_0 \), where \( \pi : \mathbb{F}_2[\mathbb{P}_2^{X-\{\chi_0\}}] \to \mathbb{F}_2 \) is the unique non-trivial character that sends the subgroup \( \{0\} \times \mathbb{F}_2^{X-\{\chi_0\}} \) to 0.

Note that an expansion map is automatically surjective. There is another characterization of expansion maps that we give now, first given in Section 3.3 of [20]. We have an isomorphism

\[
\mathbb{F}_2[\mathbb{P}_2^{X-\{\chi_0\}}] \cong \mathbb{F}_2[\{t_x\}_{x \in X-\{\chi_0\}}]/(\{t_x^2\}_{x \in X-\{\chi_0\}})
\]

by sending \( t_x \) to \( 1 \cdot \text{id} + 1 \cdot e_x \), where \( e_x \) is the vector that is 1 exactly on the \( x \)-th coordinate. Note that the squarefree monomials \( t_Y := \prod_{y \in Y} t_y \) give a basis of \( \mathbb{F}_2[\{t_x\}_{x \in X-\{\chi_0\}}]/(\{t_x^2\}_{x \in X-\{\chi_0\}}) \), as \( Y \) varies through the subsets of \( X - \{\chi_0\} \). Therefore, projection on monomials gives rise to continuous 1-cochains

\[
\phi_Y(\psi) : G_\mathbb{Q} \to \mathbb{F}_2
\]

for every \( Y \subseteq X - \{\chi_0\} \). Together they allow us to reconstruct \( \psi \) by the formula

\[
\psi(g) = \left( \sum_{Y \subseteq X - \{\chi_0\}} \phi_Y(\psi)(g)t_Y, \{\chi(g)\}_{\chi \in X - \{\chi_0\}} \right).
\]

Now define \( \chi_S := \prod_{\chi \in S} \chi \), where the product is taken in \( \mathbb{F}_2 \). From equation (3.1) and the composition law for the semidirect product we deduce that

\[
(d\phi_Y(\psi))(g_1, g_2) = \sum_{S \subseteq Y} \chi_S(g_1)\phi_{Y - S}(\psi)(g_2),
\]

where \( d \) is the operator that sends \( \text{Map}(G_\mathbb{Q}, \mathbb{F}_2) \) to \( \text{Map}(G_\mathbb{Q} \times G_\mathbb{Q}, \mathbb{F}_2) \) with the rule

\[
(d\phi)(g_1, g_2) = \phi(g_1) + \phi(g_2) + \phi(g_1g_2).
\]
Equation (3.2) is simply equation (2.2) of Smith [33]. Conversely, if we are given a system of maps \( \{ \phi_y \}_{Y \subseteq X \setminus \{ x_0 \}} \) satisfying equation (3.2) and \( \phi_0 = \chi_0 \), we get an expansion map \( \psi \) with support \( X \) and pointer \( \chi_0 \).

Now suppose that we have two expansion maps

\[ \psi_1, \psi_2 : G_Q \rightarrow \mathbb{F}_2[\mathbb{F}_2^A] \times \mathbb{F}_2^A, \]

with supports \( A_1, A_2 \) and pointers \( \chi_1, \chi_2 \) respectively. Let \( \{ \phi_{1, B} \}_{B \subseteq A_1}, \{ \phi_{2, B} \}_{B \subseteq A_2} \) be the corresponding system of continuous 1-cochains from \( G_Q \) to \( \mathbb{F}_2 \) with \( \phi_{1,1} = \chi_1, \phi_{2,1} = \chi_2 \). Note that \( L(\psi_1), L(\psi_2) \) are central \( \mathbb{F}_2 \)-extensions of

\[ M(\psi_1) := \mathbb{Q}(\{ \chi \}_{\chi \in A_1}) \prod_{B \subseteq A_1} L(\phi_{1, B})/\mathbb{Q}, \quad M(\psi_2) := \mathbb{Q}(\{ \chi \}_{\chi \in A_2}) \prod_{B \subseteq A_2} L(\phi_{2, B})/\mathbb{Q}. \]

We need one more definition before stating our reciprocity law.

**Definition 3.2.** Let \( (A_1, A_2, \psi_1, \psi_2) \) be a 4-tuple as above. We say \( (A_1, A_2, \psi_1, \psi_2) \) is Rédei admissible if

- the extensions \( L(\psi_1)/\mathbb{Q}(\{ \chi \}_{\chi \in A_1}), L(\psi_2)/\mathbb{Q}(\{ \chi \}_{\chi \in A_2}) \) are unramified;
- each place of \( \text{Ram}(\mathbb{Q}(\chi_1)/\mathbb{Q}) \) splits completely in \( M(\psi_2)/\mathbb{Q} \) and similarly each place of \( \text{Ram}(\mathbb{Q}(\chi_2)/\mathbb{Q}) \) splits completely in \( M(\psi_1)/\mathbb{Q} \);
- if the infinite place \( \infty \) of \( \mathbb{Q} \) splits completely in \( \mathbb{Q}(\{ \chi \}_{\chi \in A_1 \cup A_2}) \), then \( \infty \) splits completely in \( M(\psi_1)M(\psi_2)/\mathbb{Q} \) as well.

We call \( (\chi_1, \chi_2) \) the pointer vector of the 4-tuple and we call \( A \) the base set of the 4-tuple.

Let now \( (A_1, A_2, \psi_1, \psi_2) \) be a Rédei admissible 4-tuple, as above, with pointer vector \( (\chi_1, \chi_2) \). Then it follows by the definition that each place \( v \) in \( \text{Ram}(\mathbb{Q}(\chi_1)/\mathbb{Q}) \) is unramified in \( L(\psi_2)/\mathbb{Q} \) and furthermore the consequently defined Artin class \( \text{Art}(v, L(\psi_2)/\mathbb{Q}) \) lands in \( \text{Gal}(L(\psi_2)/M(\psi_2)) \), which is a central subgroup of \( \text{Gal}(L(\psi_2)/\mathbb{Q}) \) of size equal to 2 and hence can uniquely be identified with \( \mathbb{F}_2 \). We conclude that \( \text{Art}(v, L(\psi_2)/\mathbb{Q}) \) is a well-defined element of \( \mathbb{F}_2 \). Symmetrically, the same holds if we swap the role of 1 and 2. Finally, for a quadratic extension \( \mathbb{Q}(\sqrt{d})/\mathbb{Q} \), we put \( \text{Ram}(\mathbb{Q}(\sqrt{d})/\mathbb{Q}) \) to be the set of places in \( \text{Ram}(\mathbb{Q}(\sqrt{d})/\mathbb{Q}) \) with the only exception of (2), which is excluded in case \( d \) has even 2-adic valuation. We can now state the reciprocity law.

**Theorem 3.3.** Let \( (A_1, A_2, \psi_1, \psi_2) \) be a Rédei admissible 4-tuple with pointer vector \( (\chi_1, \chi_2) \). Then we have

\[ \sum_{v \in \text{Ram}(\mathbb{Q}(\chi_1)/\mathbb{Q})} \text{Art}(v, L(\psi_2)/\mathbb{Q}) = \sum_{v' \in \text{Ram}(\mathbb{Q}(\chi_2)/\mathbb{Q})} \text{Art}(v', L(\psi_1)/\mathbb{Q}). \]

### 3.2 Proof of Theorem 3.3

Let \( (A_1, A_2, \psi_1, \psi_2) \) be a Rédei admissible 4-tuple with pointer vector \( (\chi_1, \chi_2) \) and base set \( A \). We start by observing that equation (3.2) implies that, for \( i \in \{1, 2\} \), the set of cochains \( \{ \phi_{i, B} \}_{B \subseteq A_i} \), when restricted to \( G_{\mathbb{Q}(\chi_i)} \), becomes a set of quadratic characters. By abuse of notation we shall use the same symbols for such quadratic characters. Furthermore, it is clear
from the definition of an expansion map that, for each \( i \in \{1, 2\} \), the character \( \phi_{i,A} \) generates a rank 1 free module over the ring \( \mathbb{F}_2[\text{Gal}(\mathbb{Q}((\chi)_{\chi \in A})/\mathbb{Q})] \) and that the corresponding Galois extension of \( \mathbb{Q} \) given by this module is precisely \( L(\chi)/\mathbb{Q} \).

For each \( B \subseteq A \) we denote by \( \alpha_{i,B} \in \frac{\mathbb{Q}((\chi)_{\chi \in B})}{\mathbb{Q}((\chi)_{\chi \in B})^*} \) the unique element, provided by Kummer theory, corresponding to \( \phi_{i,B} \). We have the following fact. The reader with some familiarity with Rédei symbols, as for instance treated in [35], will recognize that this fact is a generalization of the connection between Rédei fields and solution sets of certain attached conics, see [35] Section 5.

**Proposition 3.4.** Let \( i \in \{1, 2\} \). For each \( B \subseteq A \) we have that

\[
N_{\mathbb{Q}((\chi)_{\chi \in A})/\mathbb{Q}((\chi)_{\chi \in B})}(\alpha_{i,A}) = \alpha_{i,B},
\]

as elements of \( \frac{\mathbb{Q}((\chi)_{\chi \in B})}{\mathbb{Q}((\chi)_{\chi \in B})^*} \).

**Proof.** From the recursive formula (3.2) we see that it suffices to show the proposition when \( B \) is obtained from \( A \) by deleting a single element \( a \in A \); the full proposition then follows by applying this repeatedly. Hence we need to show that, for an element \( a \in A \), we have

\[
N_{\mathbb{Q}((\chi)_{\chi \in A})/\mathbb{Q}((\chi)_{\chi \in A - \{a\}})}(\alpha_{i,A}) = \alpha_{i,A-\{a\}} \in \frac{\mathbb{Q}((\chi)_{\chi \in A - \{a\}})}{\mathbb{Q}((\chi)_{\chi \in A - \{a\}})^*}.
\]

By Kummer theory, this is equivalent to showing that the co-restriction of the character \( \phi_{i,A} \) from \( G_{\mathbb{Q}((\chi)_{\chi \in A})} \) to \( G_{\mathbb{Q}((\chi)_{\chi \in A - \{a\}})} \) equals the character \( \phi_{i,A-\{a\}} \). Let us recall the following basic fact. Let \( G_1 \subseteq G_2 \) be a continuous index 2 inclusion of profinite groups, and let \( \chi : G_1 \to \mathbb{F}_2 \), \( \chi' : G_2 \to \mathbb{F}_2 \) be two continuous characters. Then the co-restriction of \( \chi \) to \( G_2 \) equals \( \chi' \) if and only \( \chi(\sigma^2) = \chi'(\sigma) \) and \( \chi(\sigma \tau \sigma^{-1}) + \chi(\tau) = \chi'(\tau) \) for each \( \sigma \in G_2 - G_1 \), \( \tau \in G_1 \).

The second relation only implies that the co-restriction of \( \chi \) equals \( \chi' \) as characters of the index 2 subgroup \( G_1 \). This leaves two possibilities for the character from the larger group \( G_2 \): these two possibilities constitute a single coset under the subgroup generated by the character \( \epsilon : G_2 \to G_2/G_1 = \mathbb{F}_2 \). In other words the second relation forces the co-restriction of \( \chi \) to be in the set \( \{\chi', \chi' + \epsilon\} \). This ambiguity is resolved by the first relation.

The second relation in our case follows precisely by the definition of expansion maps as coordinates of monomials in a semidirect product. In this language the fact that the monomial \( t_{A - \{a\}} \) maps under multiplication by the variable \( t_a \) to the monomial \( t_A \), translates precisely to the dual fact that the character \( \phi_{i,A} \) norms to \( \phi_{i,A-\{a\}} \), when viewed as characters of the group \( G_{\mathbb{Q}((\chi)_{\chi \in A})} \), which plays the role of \( G_1 \).

We now check the first relation, which forces the norm relation to hold as characters of the larger group \( G_{\mathbb{Q}((\chi)_{\chi \in A - \{a\}})} \), playing here the role of \( G_2 \). To this end, pick \( \sigma \) as above, and plug in \( (\sigma, \sigma) \) in equation (3.2): the left hand side gives \( \phi_{i,A}(\sigma^2) \), which is the quantity we are after, while the right hand side gives \( \phi_{i,A-\{a\}}(\sigma) \). This establishes the desired conclusion.

**Remark 1.** One could ask whether, as in the case of ordinary Rédei fields [26], it is possible to give a converse of the above Proposition 3.4. This is also discussed in [20], Section 5, Question 5). We hope to return to this topic in future work.

Let \( \Omega \) be the set of all places of \( \mathbb{Q}((\chi)_{\chi \in A}) \). An immediate consequence of Proposition 3.4 is the following crucial fact.
Corollary 3.5. Let \( i \in \{1, 2\} \).

(a) Let \( v \in \widetilde{\text{Ram}}(\mathbb{Q}(\chi_i)/\mathbb{Q}) \) be a finite place. Then the number of elements \( w \) of \( \Omega \) lying above \( v \) and with
\[
w(\alpha_{i,A}) \equiv 1 \mod 2
\]
has odd cardinality.

(b) Suppose that \( \infty \in \widetilde{\text{Ram}}(\mathbb{Q}(\chi_i)/\mathbb{Q}) \). Then the number of embeddings \( \sigma : \mathbb{Q}(\chi_{\chi}) \to \mathbb{R} \) with
\[
\sigma(\alpha_{i,A}) < 0
\]
has odd cardinality.

Proof. We shall explain the argument for part (a), part (b) can be obtained in the same way. Recall that for each \( i \in \{1, 2\} \) we have that \( \phi_{i,\emptyset} = \chi_i \). Therefore Proposition 3.4 applied to \( B := \emptyset \) yields
\[
N_{\mathbb{Q}(\chi_{\chi})/\mathbb{Q}}(\alpha_{i,A}) = \alpha_{i,\emptyset},
\]
which gives
\[
\sum_{w \in \Omega : w | v} w(\alpha_{i,A}) \equiv v(\alpha_{i,\emptyset}) \mod 2. \tag{3.3}
\]
On the other hand by definition \( \mathbb{Q}(\chi_i) = \mathbb{Q}(\sqrt{\alpha_{i,\emptyset}}) \). By the definition of \( \widetilde{\text{Ram}}(\mathbb{Q}(\chi_i)/\mathbb{Q}) \), this quadratic extension locally at \( v \) is obtained by adding the square root of a uniformizer. It follows that \( v(\alpha_{i,\emptyset}) \equiv 1 \mod 2 \). Hence the desired conclusion follows immediately from equation (3.3).

We shall make use of the following general lemma. For a local field \( K \) we denote by \((-, -)_K\) the Hilbert pairing on \( K^* \). If \( v \) denotes a place of a number field \( L \), we shall denote, by abuse of notation, by \((-, -)_v\) the pairing \((-, -)_{L_v}\): in our context the choice of \( L \) will be clear so the abuse of notation will not cause ambiguities. We recall the following fundamental fact.

Lemma 3.6. Let \( p \) be a rational prime and let \( K/\mathbb{Q}_p \) be a finite extension. Let \( \alpha \) be the unique class of \( K^*/K^* \) giving the unique unramified quadratic extension \( K(\sqrt{\alpha})/K \). Then the linear functional \((\alpha, -)_K : \frac{K^*}{K^{*2}} \to \mathbb{F}_2\)
equals \([v_K(-)] \mod 2\).

Proof. Let \( \pi \) be a uniformizer in \( K \). The local Artin map for \( K \) will send \( \pi \) to the generator of \( \text{Gal}(K(\sqrt{\alpha})/K) \), which shows that the Hilbert symbol \((\alpha, \pi)_K\) is non-trivial. Since this holds for all uniformizers, one immediately obtains the desired conclusion.

We need one final ingredient. Let \( w \) be a real place of a number field \( L \), i.e. a place corresponding to an embedding \( \sigma : L \to \mathbb{R} \). We put \( w(\alpha) \) to be 0 if \( \sigma(\alpha) > 0 \) and 1 otherwise.

Proposition 3.7. We have the following facts.
(a.1) Let \( v \) be a finite place outside \( \text{Ram}(Q(\chi_1)/Q) \cup \text{Ram}(Q(\chi_2)/Q) \). Then
\[
(\alpha_{1,A}, \alpha_{2,A})_w = 0
\]
for each element \( w \) of \( \Omega \) lying above \( v \).

(a.2) Suppose that the infinite place is outside \( \text{Ram}(Q(\chi_1)/Q) \cup \text{Ram}(Q(\chi_2)/Q) \). Then the value of \( (\alpha_{1,A}, \alpha_{2,A})_w \) is the same for all \( w \in \Omega \) lying above \( \infty \).

(b) Suppose \( (2) \in \text{Ram}(Q(\chi_1)/Q) \cup \text{Ram}(Q(\chi_2)/Q) \). Then
\[
(\alpha_{1,A}, \alpha_{2,A})_w = 0
\]
for each element \( w \) of \( \Omega \) lying above \( (2) \).

(c) Let \( \{1, 2\} = \{i, j\} \) and let \( v \) be in \( \widehat{\text{Ram}}(Q(\chi_i)/Q) \). Then
\[
(\alpha_{1,A}, \alpha_{2,A})_w = w(\alpha_{i,A}) \cdot \text{Art}(v, L(\psi_j)/Q)
\]
for each element \( w \) of \( \Omega \) lying above \( v \), where the product is taken in \( F_2 \).

Proof of Proposition 3.7 part (a.1). By definition, the extension \( L(\psi_1)/Q(\{\chi\}_{\chi \in A_1}) \) and the extension \( L(\psi_2)/Q(\{\chi\}_{\chi \in A_2}) \) are unramified above all finite places. It follows that for each \( v \) as in the statement we must have that \( L(\psi_1)/Q(\{\chi\}_{\chi \in A}) \) and \( L(\psi_2)/Q(\{\chi\}_{\chi \in A}) \) are unramified at any place \( w \in \Omega \) above \( v \).

Now, since the fields \( L(\psi_1), L(\psi_2) \) are respectively equal to the Galois closure (over \( Q \)) of the quadratic extensions \( Q(\{\chi\}_{\chi \in A})/(\sqrt{\alpha_{1,A}}/Q(\{\chi\}_{\chi \in A}), Q(\{\chi\}_{\chi \in A})/(\sqrt{\alpha_{2,A}}/Q(\{\chi\}_{\chi \in A})) \), it follows that the classes of \( \alpha_{1,A}, \alpha_{2,A} \) are unramified classes in \( Q(\{\chi\}_{\chi \in A})^\times \). But, thanks to Lemma 3.6, the Hilbert symbol between two unramified classes is always trivial and so we obtain the desired conclusion.

Proof of Proposition 3.7 part (a.2). In this case \( \infty \) splits completely in \( M(\psi_1)M(\psi_2) \). Since \( \alpha_{1,A}, \alpha_{2,A} \) are \( G_Q \)-invariants of, respectively, \( M(\psi_1)^+ \) and \( M(\psi_2)^+ \), it follows that \( \sigma(\alpha_{i,A}) \) has constantly the same sign as we vary \( \sigma : Q(\{\chi\}_{\chi \in A}) \to \mathbb{R} \) for \( i \in \{1, 2\} \). Indeed, the conjugates of \( \alpha_{i,A} \) are equal to \( \alpha_{i,A} \) times a square in \( M(\psi_i) \), therefore \( \sigma(\alpha_{i,A}) \) changes by the square of a real number, which is positive. Hence the conclusion follows at once, since the Hilbert symbol in the local field \( \mathbb{R} \) is entirely determined by the sign of the two entries.

Proof of Proposition 3.7 part (b). Suppose, without loss of generality, that \( (2) \) ramifies in \( Q(\chi_1) \), and thus not in \( Q(\chi_2) \). By assumption we additionally know that \( Q(\chi_1)/Q \) is obtained, locally at 2, by adding the square root of a unit. We then claim that for each place \( w \) in the statement, we must have that \( w(\alpha_{1,A}) \) is even. If not we would have that the extension \( L(\psi_1)/Q(\{\chi\}_{\chi \in A_1}) \) ramifies at the unique place above \( w \) in \( Q(\{\chi\}_{\chi \in A_1}) \), which would contradict that this extension is unramified at all finite places. On the other hand we know that \( L(\psi_2)/Q(\{\chi\}_{\chi \in A}) \) is unramified at \( w \): this follows from the fact that \( Q(\chi_2)/Q \) is unramified at 2 and the reasoning in part (a.1). Therefore we obtain the desired conclusion as an immediate consequence of Lemma 3.6.
Proof of Proposition 3.7 part (c). Let us firstly suppose that $v$ is a finite place and let $w \in \Omega$ be above $v$. We claim that $L(\psi_j)/\mathbb{Q}(\{\chi\}_{\chi \in A})$ is unramified at $w$. It is certainly true that any $w'$ above $w$ in $\mathbb{Q}(\{\chi'\}_{\chi' \in A_j})/\mathbb{Q}(\{\chi\}_{\chi \in A})$ will be unramified in $L(\psi_j)/\mathbb{Q}(\{\chi'\}_{\chi' \in A_j})$, since this last extension is unramified at all finite places. But then our claim follows immediately from the fact that $v$ is unramified in $\mathbb{Q}(\chi_j)/\mathbb{Q}$, as we already argued in part (a.1).

Furthermore, we know that $v$ splits completely in $M(\psi_j)/\mathbb{Q}$ and $\alpha_{j,A}$ is a $G_{\mathbb{Q}}$-invariant class in $\frac{M(\psi_j)^*}{\mathbb{Q}(\psi_j)^*}$. It follows that among all $2^n$ embeddings of $\mathbb{Q}(\{\chi\}_{\chi \in A}) \to \mathbb{Q}$, we have that $\alpha_{j,A}$ always lands in the same unramified class of $\frac{\mathbb{Q}^*}{\mathbb{Q}(\psi_j)^*}$. Recalling that

$$L(\psi_j) = M(\psi_j)\left(\sqrt{\alpha_{j,A}}\right),$$

we see that this class is trivial if and only if $\text{Art}(v, L(\psi_j)/\mathbb{Q})$ is trivial. On the other hand, Lemma 3.6 tells us that $(\alpha_{i,A},\alpha_{j,A}) = w(\alpha_{i,A})$ in case $\text{Art}(v, L(\psi_j)/\mathbb{Q})$ is non-trivial and equals 0 in case $\text{Art}(v, L(\psi_j)/\mathbb{Q})$ is trivial. This is precisely the desired conclusion.

The case that $v$ equals the place $\infty$ of $\mathbb{Q}$ goes as follows. The formula is correct as soon as $w(\alpha_{i,A}) = 0$, because the Hilbert symbol on the reals vanishes as soon as one of the two entries is positive. But we know that $\infty$ splits completely in $M(\psi_j)/\mathbb{Q}$ and that $L(\psi_j)$ is obtained by adding the square root of $\alpha_{j,A}$. Hence, recalling one more time that $\alpha_{j,A}$ is a $G_{\mathbb{Q}}$-invariant class in $\frac{M(\psi_j)^*}{\mathbb{Q}(\psi_j)^*}$, we have that the sign of $\sigma(\alpha_{j,A})$ is independent of the embedding $\sigma : \mathbb{Q}(\{\chi\}_{\chi \in A}) \to \mathbb{R}$, and it is positive if and only if $\text{Art}(\infty, L(\psi_j)/\mathbb{Q})$ is trivial. Hence the desired conclusion holds also when $w(\alpha_{i,A}) = 1$. This ends the proof.

We are now ready to prove the main result of this section.

Proof of Theorem 3.3. Recall that $\Omega$ denotes the set of all places of $\mathbb{Q}(\{\chi\}_{\chi \in A})$. Hilbert’s reciprocity law yields

$$\sum_{w \in \Omega} (\alpha_{1,A}, \alpha_{2,A})_w = 0. \quad (3.4)$$

Thanks to Proposition 3.7 part (a.1) and part (b) we obtain that each of the Hilbert symbols with $w$ lying above a finite place outside of $\widehat{\text{Ram}}(\mathbb{Q}(\chi_1)/\mathbb{Q}) \cup \widehat{\text{Ram}}(\mathbb{Q}(\chi_2)/\mathbb{Q})$ is 0. From Proposition 3.7 part (a.2) we deduce that if $\infty$ is outside $\widehat{\text{Ram}}(\mathbb{Q}(\chi_1)/\mathbb{Q}) \cup \widehat{\text{Ram}}(\mathbb{Q}(\chi_2)/\mathbb{Q})$ (which happens if and only if it is outside $\widehat{\text{Ram}}(\mathbb{Q}(\chi_1)/\mathbb{Q}) \cup \widehat{\text{Ram}}(\mathbb{Q}(\chi_2)/\mathbb{Q})$), then the total contribution coming from all the places above $\infty$ in $\Omega$ equals $2^n$ times the same number, and, hence, since $n \geq 1$, we get 0.

Let now $v$ be a place in $\widehat{\text{Ram}}(\mathbb{Q}(\chi_1)/\mathbb{Q})$. Then thanks to Proposition 3.7 part (c), we obtain that for each $w$ above $v$ in $\mathbb{Q}(\{\chi\}_{\chi \in A})/\mathbb{Q}$ we have

$$(\alpha_{1,A}, \alpha_{2,A})_w = w(\alpha_{1,A}) \cdot \text{Art}(v, L(\psi_2)/\mathbb{Q})).$$

Therefore Corollary 3.8 allows us to conclude that

$$\sum_{w \in \Omega : w \mid v} (\alpha_{1,A}, \alpha_{2,A})_w = \text{Art}(v, L(\psi_2)/\mathbb{Q}))$$

and similarly

$$\sum_{w \in \Omega : w \mid v} (\alpha_{1,A}, \alpha_{2,A})_w = \text{Art}(v, L(\psi_1)/\mathbb{Q}))$$
for a place \( v \in \text{Ram}(Q(\chi_2)/Q) \). Hence in total equation (3.3) becomes

\[
\sum_{v \in \text{Ram}(Q(\chi_1)/Q)} \text{Art}(v, L(\psi_2)/Q) + \sum_{v' \in \text{Ram}(Q(\chi_2)/Q)} \text{Art}(v', L(\psi_1)/Q) = 0,
\]

which can be rewritten as

\[
\sum_{v \in \text{Ram}(Q(\chi_1)/Q)} \text{Art}(v, L(\psi_2)/Q) = \sum_{v' \in \text{Ram}(Q(\chi_2)/Q)} \text{Art}(v', L(\psi_1)/Q),
\]

and this is precisely the desired conclusion. \( \square \)

4 A reflection principle

The material in this section is based directly on [33, Section 2], and therefore we shall go over it rather quickly. We start by introducing some important notation, which will be similar to Smith’s notation [33, p. 8] and [19, p. 42-43]. Then we shall develop the theory of expansions. Once this is done, we introduce the notions of minimality and agreement. The section ends with two reflection principles, which relate the class group structure of different fields. This will serve as the algebraic input for our analytic machinery.

4.1 Notation

In this paper \( X \) will always be a product set \( X_1 \times \cdots \times X_r \), where each \( X_i \) is a finite, non-empty set of primes intersecting trivially with all the other \( X_j \). This allows us to identify \((x_1, \ldots, x_r) \in X\) with the squarefree integer \( x_1 \cdot \ldots \cdot x_r \), and we shall often do so implicitly. For \( a \in \mathbb{Z}_{\geq 0} \), we will write \([a]\) for the set \( \{1, \ldots, a\} \). If \( S \subseteq [r] \), we define

\[
\overline{X}_S := \prod_{i \in S} (X_i \times X_i) \times \prod_{i \in [r]-S} X_i,
\]

and we let \( \pi_i \) be the projection to \( X_i \times X_i \) if \( i \in S \) and to \( X_i \) if \( i \notin S \). The natural projection maps from \( X_i \times X_i \) to \( X_i \) are denoted by \( \text{pr}_1 \) and \( \text{pr}_2 \). For two subsets \( S, S_0 \subseteq [r] \), we let \( \pi_{S,S_0} \) be the projection map from \( \overline{X}_S \) to

\[
\prod_{i \in S} (X_i \times X_i) \times \prod_{i \in ([r]-S)\cap S_0} X_i
\]

given by \( \pi_i \) on each \( i \in S_0 \). The set \( S \) shall often be clear from context, in case we will simply write \( \pi_{S_0} \) for \( \pi_{S,S_0} \). Finally, take some \( \bar{x} \in \overline{X}_S \) and \( T \subseteq S \subseteq [r] \). Then we define \( \bar{x}(T) \) to be the following multiset

\[
\{ \bar{y} \in \overline{X}_T : \pi_{[r]-(S-T)}(\bar{y}) = \pi_{[r]-(S-T)}(\bar{x}) \text{ and } \forall i \in S-T \exists j \in [2] : \pi_i(\bar{y}) = \text{pr}_j(\pi_i(\bar{x})) \}
\]

with the multiplicity of \( \bar{y} \in \bar{x}(T) \) being

\[
\prod_{i \in S-T} |\{ j \in [2] : \pi_i(\bar{y}) = \text{pr}_j(\pi_i(\bar{x})) \}|
\]

With these notations, we can now start deriving a reflection principle, which will be almost identical to [33, Theorem 2.8]. To do so, we will need to introduce expansions, governing expansions, minimality and agreement.
4.2 Expansions

In this subsection, we shall quickly recall the facts about expansions that we will need. These are treated more elaborately in [33, Section 2.1], [19, Section 7]. The paper [20] is entirely devoted to a careful study of the properties of expansions. We start by defining pre-expansions and expansions.

**Definition 4.1.** For any integer $x$, we let $\chi_x : G_Q \rightarrow \mathbb{F}_2$ be the character corresponding to $\mathbb{Q}(\sqrt{x})$. Let $X := X_1 \times \cdots \times X_r$ with $|X_i| = 2$ for $i \in [r]$. For a subset $U \subseteq [r]$, we declare $\chi_U : G_Q \rightarrow \mathbb{F}_2$ to be

$$\chi_U(\sigma) := \prod_{i \in U} \chi_{pr_i(\pi_i(x))} \cdot \phi(x)(\sigma).$$

A pre-expansion for $X$ is a sequence $\{\phi_T\}_{T \subseteq [r]}$ where each $\phi_T : G_Q \rightarrow \mathbb{F}_2$ is a continuous 1-cochain satisfying

$$(d\phi_T)(\sigma, \tau) = \sum_{\emptyset \neq U \subseteq T} \chi_U(\sigma) \phi_T - U(\tau). \quad (4.1)$$

Recall that $d\phi_T(\sigma, \tau) := \phi_T(\sigma) + \phi_T(\tau) + \phi_T(\sigma\tau)$. For the remainder of the paper, we shall always assume that $\phi_0$ is linearly independent from the space of characters spanned by $\{\chi_{\{i\}}\}_{i \in [r]}$.

We say that a pre-expansion is promising if for every $i \in [r]$, every prime $p \in X_i$ splits completely in $L(\phi_{[r]-\{i\}})$. Furthermore, a pre-expansion is said to be good if $L(\phi_T)$ is an unramified extension of $L(\chi_T \cdot \phi_0)$ for all $T \subseteq [r]$.

An expansion for $X$ is a sequence $\{\phi_T\}_{T \subseteq [r]}$ satisfying the recursive equation $d\phi_T$ for each $T \subseteq [r]$. An expansion is good if $L(\phi_T)$ is an unramified extension of $L(\chi_T \cdot \phi_0)$ for all $T \subseteq [r]$.

Finally, we say that $X$ is cooperative if for each distinct $i, j \in [r]$, we have that the character $\chi_{\{i\}}$ is locally trivial at 2 and at each prime $p \in X_j$.

The following result is the key result regarding expansions. It is a rephrasing of [33, Proposition 2.1], see also [19, Proposition 7.3] for a similar statement. Informally, it shows that a pre-expansion can be completed to a good expansion under favorable circumstances.

**Proposition 4.2.** Let $X := X_1 \times \cdots \times X_r$ with $|X_i| = 2$ for $i \in [r]$. Suppose that $X$ is cooperative and let $\{\phi_T\}_{T \subseteq [r]}$ be a promising, good pre-expansion for $X$. Then there is a continuous map $\phi_{[r]} : G_Q \rightarrow \mathbb{F}_2$ such that

$$(d\phi_{[r]})(\sigma, \tau) = \sum_{\emptyset \neq U \subseteq [r]} \chi_U(\sigma) \phi_{[r]} - U(\tau)$$

with $L(\phi_{[r]})$ an unramified extension of $L(\chi_{[r]} \cdot \phi_0)$.

In Section 5 we work with many expansions simultaneously. For a box $X = X_1 \times \cdots \times X_r$, $S \subseteq [r]$ and $\bar{x} \in \overline{X}_S$, we shall use the shorthand $\phi_{x,a}$ for an expansion map $\phi_S$ associated to the box $\prod_{i \in S} \pi_i(\bar{x})$ with $\phi_0 = \chi_a$. Observe that $\phi_{x,a}$ only depends on $a$ and the sets $\pi_i(\bar{x})$ with $i \in S$. In case that $\text{pr}_1(\pi_i(\bar{x})) \neq \text{pr}_2(\pi_i(\bar{x}))$ for all $i \in S$, we can naturally view each $\pi_i(\bar{x})$ as a set with two primes as required for Definition 4.1. If instead $\text{pr}_1(\pi_i(\bar{x})) = \text{pr}_2(\pi_i(\bar{x}))$ for some $i \in S$, we set $\phi_{x,a}$ to be zero.
4.3 Governing expansions

Since we have to work with many expansions simultaneously in the final section, we abstract
the essential properties in the notion of governing expansions.

**Definition 4.3.** Let \( X := X_1 \times \cdots \times X_r \), let \( S \subseteq [r] \) and let \( a \in \mathbb{Z} \) be squarefree. We say that there exists a governing expansion \( \mathcal{G} \) on \((X,S,a)\) if

- we have for all \( T \subseteq S \) and all \( \bar{x} \in \overline{X}_T \) a good expansion \( \phi_{\bar{x},a} \) satisfying
  \[
  (d\phi_{\bar{x},a})(\sigma,\tau) = \sum_{\emptyset \subsetneq T' \subseteq T} \chi_{T'}(\sigma)\phi_{\pi_{T'\setminus T}(\bar{x}),a}(\tau);
  \]

- take \( T \subseteq S, i \in T \) and \( \bar{x}_0, \bar{x}_1, \bar{x}_2 \in \overline{X}_T \). Suppose that
  \[
  \pi_{T \setminus \{i\}}(\bar{x}_0) = \pi_{T \setminus \{i\}}(\bar{x}_1) = \pi_{T \setminus \{i\}}(\bar{x}_2)
  \]
  and that there are primes \( p_0, p_1, p_2 \) satisfying
  \[
  \text{pr}_j(\pi_i(\bar{x}_k)) = p_{k+j-1}
  \]
  for all \( j \in \{1, 2\} \) and \( k \in \{0, 1, 2\} \), where the indices are taken modulo 3. Then we have
  \[
  \phi_{\bar{x}_0,a} + \phi_{\bar{x}_1,a} = \phi_{\bar{x}_2,a}. \tag{4.2}
  \]

These conditions are rather stringent, and typically there does not exist a governing expansion \( \mathcal{G} \) on \((X,S,a)\). To construct governing expansions, we introduce additive systems.

**Definition 4.4.** Let \( X := X_1 \times \cdots \times X_r \). An additive system \( \mathfrak{A} \) on \( X \) is a tuple

\[
(\mathcal{Y}_S, \mathcal{Y}_S^0, F_S, A_S)_{S \subseteq [r]}
\]

satisfying

- for each \( S \subseteq [r], \) we have that \( A_S \) is a finite \( \mathbb{F}_2 \) vector space, \( \overline{Y}_S \) and \( \overline{Y}_S^0 \) are sets satisfying
  \[
  \overline{Y}_S^0 \subseteq \overline{Y}_S \subseteq \overline{X}_S
  \]
  and \( F_S : \overline{Y}_S \to A_S \) is a function such that
  \[
  \overline{Y}_S^0 := \{ \bar{y} \in \overline{Y}_S : F_S(\bar{y}) = 0 \};
  \]

- we have for all non-empty \( S \subseteq [r] \) that
  \[
  \overline{Y}_S = \{ \bar{x} \in \overline{X}_S : \bar{x}(T) \subseteq \overline{Y}_T^0 \text{ for all } T \subseteq S \}.
  \]

Here we view \( \bar{x}(T) \) as a set by forgetting the multiplicities;
• take \( i \in S \subseteq [r] \) and take \( \bar{x}_0, \bar{x}_1, \bar{x}_2 \in \bar{Y}_S \) satisfying

\[
\pi_{S-\{i\}}(\bar{x}_0) = \pi_{S-\{i\}}(\bar{x}_1) = \pi_{S-\{i\}}(\bar{x}_2)
\]

such that there are primes \( p_0, p_1, p_2 \) with

\[
pr_j(\pi_i(\bar{x}_k)) = p_{k+j-1}
\]

for all \( j \in \{1, 2\} \) and all \( k \in \{0, 1, 2\} \), where the indices are taken modulo 3. Then we have

\[
F_S(\bar{x}_0) + F_S(\bar{x}_1) = F_S(\bar{x}_2).
\]

(4.3)

We will sometimes write \( Y_S(A), Y'_S(A), F_S(A) \) and \( A_S(A) \) to stress that this data is associated to the additive system \( A \).

We remark that equation (4.3) implies that

\[
F_S(\bar{x}) = 0
\]

in case \( pr_1(\pi_i(\bar{x})) = pr_2(\pi_i(\bar{x})) \). We will now construct an additive system that will help us find governing expansions.

**Lemma 4.5.** Let \( r \geq 2 \) be an integer and let \( X := X_1 \times \cdots \times X_r \) be such that \( X_j \) contains only odd primes. Take an odd squarefree integer \( a = q_1 \cdots q_t \) such that

\[
\left( \frac{a}{p} \right) = 1 \quad \text{and} \quad \left( \frac{p}{q_i} \right) = 1 \text{ for all } i \in [t]
\]

(4.4)

for all \( p \in X_j \) with \( j \in [r] \). Let \( \Omega \) be a set of places of \( \mathbb{Q} \) disjoint from the \( X_j \) and \( q_i \). We assume that every \( v \in \Omega \) splits in \( \mathbb{Q}(\sqrt{a}) \). Let \( W \subseteq X \) be a subset such that for all \( w_1, w_2 \in W \), for all distinct \( i, j \in [r] \) and for all \( v \in \Omega \)

\[
\left( \frac{\pi_i(w_1)}{\pi_j(w_1)} \right) = \left( \frac{\pi_i(w_2)}{\pi_j(w_2)} \right), \quad \pi_i(w_1)\pi_i(w_2) \equiv 1 \bmod 8, \quad \pi_i(w_1)\pi_i(w_2) \equiv \square \bmod v.
\]

Then there exists an additive system \( A \) on \( X \) such that

• we have \( Y'_0(A) = W \);

• we have \( |A_S(A)| \leq 2^{r-1+|\Omega|} \) for all \( S \subseteq [r] \);

• suppose that \( Z := Z_1 \times \cdots \times Z_r \) satisfies \( Z_i \subseteq X_i \) and suppose that

\[
Z_{[r]} \subseteq Y_{[r]}(A).
\]

Then there exists a governing expansion \( \mathcal{G} \) on \( (Z, [r], a) \) such that every \( v \in \Omega \) splits completely in \( \phi_{\bar{z}, a} \) for \( \bar{z} \in \bar{Z}_{[r]} \).
Proof. Note that an additive system $\mathfrak{A}$ is uniquely specified by the maps $F_S(\mathfrak{A})$ and the set $Y_0(\mathfrak{A})$. We take $Y_0(\mathfrak{A}) = W$ and we will inductively construct the maps $F_S(\mathfrak{A})$.

If $S = \emptyset$, we take $F_0(\mathfrak{A})$ to be the zero map.

Now suppose that $S = \{i\}$. For $\bar{x} \in \overline{Y}_0(\mathfrak{A})$, Proposition 4.2 and equation (4.1) imply that there is a good expansion $\phi_{\bar{x},a}$. Define

$$K := \mathbb{Q}(\{\sqrt{q_1}, \ldots, \sqrt{q_t}\} \cup \{\sqrt{p} : p \in X_j \text{ for some } j \in [r]\})$$

and let $M$ be the narrow Hilbert class field of $K$. For every prime $p$ ramifying in $K$, we choose once and for all an inertia subgroup $I_p$ of $\text{Gal}(M/\mathbb{Q})$, which has size $2$, and we denote by $\sigma_p$ the non-trivial element of $I_p$.

Then by twisting $\phi_{\bar{x},a}$ with characters, we can ensure that

$$\phi_{\bar{x},a}(\sigma_p) = 0 \quad (4.5)$$

for all $p$ ramifying in $K$. Observe that such a twist is then still a good expansion. Furthermore, with this choice of $\phi_{x,a}$ we claim that equation (4.2) holds. Indeed, suppose that $\bar{x}_0, \bar{x}_1, \bar{x}_2 \in \overline{Y}_i(\mathfrak{A})$ satisfy the assumptions for equation (4.2). Then we have

$$d(\phi_{\bar{x}_0,a} + \phi_{\bar{x}_1,a} + \phi_{\bar{x}_2,a}) = 0.$$ 

Since $\phi_{\bar{x}_j,a}$ is a good expansion for $j \in \{0, 1, 2\}$, this shows that $\phi_{\bar{x}_0,a} + \phi_{\bar{x}_1,a} + \phi_{\bar{x}_2,a}$ is a quadratic character with field of definition inside $\text{Gal}(M/\mathbb{Q})$. Then equation (4.3) follows from equation (4.2).

Now we proceed inductively. We see that Proposition 4.2 implies that there is a good expansion $\phi_{\bar{x},a}$ for $\bar{x} \in \overline{Y}_S(\mathfrak{A})$. As before, we twist $\phi_{\bar{x},a}$ to guarantee that

$$\phi_{\bar{x},a}(\sigma_p) = 0.$$ 

Again we have that

$$d(\phi_{\bar{x}_0,a} + \phi_{\bar{x}_1,a} + \phi_{\bar{x}_2,a}) = 0,$$

which follows from the fact that equation (4.2) holds for all the $\phi_{\pi_T(\bar{x}),a}$ for $T \subseteq S$ by the induction hypothesis. From this, we deduce just like before that

$$\phi_{\bar{x}_0,a} + \phi_{\bar{x}_1,a} + \phi_{\bar{x}_2,a} = 0.$$ 

We define $F_S(\mathfrak{A})$ by sending $\bar{x} \in \overline{Y}_S(\mathfrak{A})$ to

$$\phi_{\bar{x},a}(\text{Frob } k)$$

as $k$ runs through $\pi_j(\bar{x})$ for $j \in [r] - S$ and $\Omega$. This defines our additive system $\mathfrak{A}$, which indeed satisfies the listed properties. \qed
4.4 Raw cocycles and minimality

Let \( N := \mathbb{Q}_2 / \mathbb{Z}_2 \), which we endow with the discrete topology. We view \( N \) as a \( G_Q \)-module with trivial action. For any \( x \in X \), we let \( N(x) \) be the \( G_Q \)-module \( N \) twisted with the action of \( \mathbb{Q}(\sqrt{r}) \), i.e. \( \sigma_x n = n \) if \( \chi_x(\sigma) = 0 \) and \( \sigma_x n = -n \) if \( \chi_x(\sigma) = 1 \).

**Definition 4.6.** We define \( \text{Cocy}(Q, N(x)[2^k]) \) to be the set of continuous 1-cocycles \( \psi \) such that \( \mathbb{Q}(\sqrt{r})/\mathbb{Q}(\sqrt{ra}) \) is unramified.

**Remark 2.** If \( k > 1 \), then \( L(\psi) \) automatically contains \( \mathbb{Q}(\sqrt{r}) \). However, this need not be the case if \( k = 1 \).

By definition of \( \text{Cocy}(Q, N(x)[2^k]) \), restriction of cocycles induces an isomorphism

\[
\text{Cocy}(Q, N(x)[2^k]) \cong \text{Cocy}(\text{Gal}(\mathbb{Q}(\sqrt{r})/\mathbb{Q}), N(x)[2^k]).
\]

Of fundamental importance is the split exact sequence

\[
0 \rightarrow N(x)[2^k] \rightarrow \text{Cocy}(\text{Gal}(\mathbb{Q}(\sqrt{r})/\mathbb{Q}), N(x)[2^k]) \rightarrow \text{Cl}(\mathbb{Q}(\sqrt{r}))[2^k] \rightarrow 0.
\]

Fix an element \( \sigma \in \text{Gal}(\mathbb{Q}(\sqrt{r})/\mathbb{Q}) \) projecting non-trivially in \( \text{Gal}(\mathbb{Q}(\sqrt{r})/\mathbb{Q}) \). The first map is given by sending \( n \) to the unique cocycle that sends \( \sigma \) to \( n \) and sends the group \( \text{Gal}(\mathbb{Q}(\sqrt{r})/\mathbb{Q}(\sqrt{r})) \) to zero, while the second map is simply restriction of cocycles. The exact sequence is split, since all the groups appearing are killed by \( 2^k \) and \( N(x)[2^k] \cong \mathbb{Z}/2^k\mathbb{Z} \) as abelian groups. This allows us to work with cocycles instead of the class group.

If \( x \) is a squarefree integer, we have a natural map

\[
f: \{ d \text{ squarefree} : d \mid \Delta_{\mathbb{Q}(\sqrt{r})} \} \rightarrow \text{Cl}(\mathbb{Q}(\sqrt{r}))[2],
\]

where \( \Delta_K \) denotes the discriminant of \( K \). We now define the \( m \)-th Artin pairing

\[
\text{Art}_{m,x} : f^{-1}(2^{m-1}\text{Cl}(\mathbb{Q}(\sqrt{r}))[2^m]) \times 2^{m-1}\text{Cocy}(\text{Gal}(\mathbb{Q}(\sqrt{r})/\mathbb{Q}), N(x)[2^m]) \rightarrow \mathbb{F}_2
\]

by sending \((b, \chi) \) to \( \psi(\text{Frob } b) \), where \( b \) is the unique ideal of norm \( b \) and furthermore \( \psi \in \text{Cocy}(\text{Gal}(\mathbb{Q}(\sqrt{r})/\mathbb{Q}), N(x)[2^m]) \) is any lift of \( \chi \) satisfying \( 2^{m-1}\psi = \chi \). The left kernel of this pairing is \( f^{-1}(2^{m-1}\text{Cl}(\mathbb{Q}(\sqrt{r}))[2^{m+1}]) \) and the right kernel of this pairing is \( 2^{m}\text{Cocy}(\text{Gal}(\mathbb{Q}(\sqrt{r})/\mathbb{Q}), N(x)[2^{m+1}]) \).

We introduce the notion of a raw cocycle and minimality.

**Definition 4.7.** A raw cocycle for \( x \) is a sequence \( \{ \psi_i \}_{i=0}^k \) satisfying \( \psi_i \in \text{Cocy}(G_Q, N(x)[2^k]) \) and \( 2\psi_i = \psi_{i-1} \) for all \( 1 \leq i \leq k \). We call \( k \) the order of the raw cocycle.

**Definition 4.8.** Let \( X := X_1 \times \cdots \times X_r \), let \( S \subseteq [r] \) and let \( \bar{x} \in X_S \). Suppose that we are given for each \( x \in \bar{x}(0) \) a raw cocycle \( \{ \psi_{i,x} \}_{i=0}^{[S]} \). We say that the set of raw cocycles is minimal at \( \bar{x} \) if

\[
\sum_{y \in \bar{y}(0)} \psi_{T,y} = 0
\]

for all \( T \subseteq S \) and all \( \bar{y} \in \bar{x}(T) \). Note that the sum here has to be taken with multiplicities.

We have all the necessary notation to state our first reflection principle, which is directly based on part (i) of Theorem 2.8 of Smith [33].
Theorem 4.9. Let $X := X_1 \times \cdots \times X_r$, let $S \subseteq [r]$ with $|S| \geq 2$ and let $\bar{x} \in \overline{X}_S$. Take $x_0 \in \bar{x}(0)$. Let $\{\psi_{i,x}\}_{i=0}^{S_0}$ be a raw cocycle for all $x$ in $\bar{x}(0)$ except $x_0$. We assume that for all $T \subseteq S$ and all $\bar{y} \in \bar{x}(T)$ not containing $x_0$, we have that $\{\psi_{i,y}\}_{i=0}^{S_0}$ is minimal at $\bar{y}$. Then there is a raw cocycle $\{\psi_{i,x_0}\}_{i=0}^{S_0}$ such that $\psi_{1,x_0} = \psi_{1,x}$ for all $x \in \bar{x}(0)$.

Additionally, suppose that there is an integer $b$ such that $b \in f^{-1}(2^{m-1}\text{Cl}(\mathbb{Q}(\sqrt{x}))[2^m])$ for all $x \in \bar{x}(0)$. Also suppose that for every $i \in S$ we have that $\text{pr}_1(\pi_i(\bar{x})) \cdot \text{pr}_2(\pi_i(\bar{x}))$ is a square locally at 2 and at all primes in $\pi_j(\bar{x})$ with $i \neq j$. Then we also have

$$\sum_{x \in \bar{x}(0)} \text{Art}_{[S],x}(b, \psi_{1,x}) = 0.$$  

Proof. We note that minimality implies that $\psi_{1,x} = \psi_{1,x'}$ for all $x, x' \in \bar{x}(0)$ not equal to $x_0$. For the first part, we put

$$\psi_{[S],x_0} := - \sum_{x \in \bar{x}(0)} \psi_{[S],x}.$$  

Then it is easily seen that $2^{[S]-1}\psi_{[S],x_0} = \psi_{1,x}$ for any $x \in \bar{x}(0)$. Smith \cite{Smith} Proposition 2.5 shows that $\psi_{[S],x_0}$ is a cocycle from $G_{\mathbb{Q}}$ to $N(x_0)[2^{k}]$. It remains to deal with the ramification locus of $L(\psi_{[S],x_0})$.

We first claim that $2\psi_{[S],x_0} \in \text{Cocy}(\text{Gal}(H(\mathbb{Q}(\sqrt{x_0})/\mathbb{Q}), N(x_0)[2^{[S]-1}])$. But observe that the minimality assumptions and equation (4.6) imply that

$$2\psi_{[S],x_0} = - \sum_{x \in \bar{x}(0)} \psi_{[S]-1,x}$$  

for any $y \in \bar{x}(T)$ not containing $x_0$ such that $|T| = |S| - 1$. From this we immediately deduce the claim, since for any prime $p$ not dividing $\Delta_{\mathbb{Q}(\sqrt{x_0})}$ we can take $\bar{y}$ such that $p$ is unramified in every $L(\psi_{[S]-1,x})$ for $x \in \bar{x}(0) - \bar{y}(0)$. Since $L(\psi_{[S],x_0})/L(2\psi_{[S],x_0})$ is a central $\mathbb{F}_2$-extension, we see that $L(\psi_{[S],x_0})/L(2\psi_{[S],x_0})$ can be made unramified over $\mathbb{Q}$ at all primes, except those that ramify in $L(2\psi_{[S],x_0})$, by twisting with a character $\chi$, see for example \cite{Artin} Proposition 4.8.

But from the shape of $\psi_{[S],x_0}$, it follows that the primes that already ramify in $L(2\psi_{[S],x_0})$ can not ramify further in $L(\psi_{[S],x_0})/L(2\psi_{[S],x_0})$. Indeed, from equation (4.6) it follows that the ramification degree of any prime $p$ in $L(\psi_{[S],x_0})$ is at most 2. Therefore $L(\psi_{[S],x_0} + \chi)$ is an unramified extension of $\mathbb{Q}(\sqrt{x_0})$ for some character $\chi$, proving the first part.

For the second part, recall that $\text{Art}_{[S],x}(b, \psi_{1,x})$ does not depend on the choice of the lift $\psi$ with $2^{[S]-1}\psi = \psi_{1,x}$, and hence we may choose the lift $\psi_{[S],x_0} + \chi$ for $\psi_{1,x_0}$. By definition we have that

$$\sum_{x \in \bar{x}(0)} \text{Art}_{[S],x}(b, \psi_{1,x}) = \sum_{x \in \bar{x}(0)} \psi_{[S],x}(\text{Frob } b).$$  

Now locally at any prime $p$ dividing $b$, we know that $\mathbb{Q}_p(\sqrt{x}) = \mathbb{Q}_p(\sqrt{x'})$ for all $x, x' \in \text{Frob}(\bar{x}(0))$ by our assumptions. Since $\psi_{[S],x}$ becomes a character when restricted to $\mathbb{Q}_p(\sqrt{x})$, the relation

$$\sum_{x \in \bar{x}(0)} \psi_{[S],x} = \chi$$
yields
\[ \sum_{x \in \bar{x}(\emptyset)} \psi_{|S|,x}(\text{Frob } b) = \chi(\text{Frob } b). \]

We claim that the last expression is trivial. Indeed \( b \) is in \( 2\text{Cl}(\mathbb{Q}(\sqrt{x})) \)[4] for all \( x \in \bar{x}(\emptyset) \), and therefore pairs trivially with any character that is unramified outside the union of the primes dividing \( \Delta_{\mathbb{Q}(\sqrt{x})} \) as \( x \) ranges through \( \bar{x}(\emptyset) \). But equation [4.6] shows that the character \( \chi \) is of this shape, concluding the proof of our theorem.

4.5 Agreement

Our second reflection principle is based on the notion of agreement.

Definition 4.10. Let \( X := X_1 \times \cdots \times X_r \), let \( S \subseteq [r] \), let \( i_a \in S \) and let \( \bar{x} \in \overline{X}_S \) be given. Take for each \( x \in \bar{x}(\emptyset) \) a raw cocycle \( \{\psi_{i_a,x}\}_{i_a}^{[S]} \). We further assume that we have an expansion \( \phi_{\pi_{S-(i_a)}}(\bar{x}) \cdot pr_1(\pi_{i_a}(\bar{x})) \cdot pr_2(\pi_{i_a}(\bar{x})) \). We say that the set of raw cocycles agrees with the expansion at \( \bar{x} \) if
\[ \sum_{y \in \bar{y}(\emptyset)} \psi_{|T|,y} \begin{cases} \phi_{\pi_{T-(i_a)}}(\bar{y}) \cdot pr_1(\pi_{i_a}(\bar{y})) \cdot pr_2(\pi_{i_a}(\bar{y})) & \text{if } i_a \in T \\ 0 & \text{if } i_a \notin T \end{cases} \]
for all \( T \subseteq S \) and all \( \bar{y} \in \bar{x}(T) \).

We are now ready to state a reflection principle that is very similar to part (ii) of Theorem 2.8.

Theorem 4.11. Let \( X := X_1 \times \cdots \times X_r \), let \( S \subseteq [r] \) with \(|S| \geq 2\), let \( i_a \in S \) and let \( \bar{x} \in \overline{X}_S \) be given. Take \( x_0 \in \bar{x}(\emptyset) \). Let \( \{\psi_{i_a,x}\}_{i_a=0}^{[S]} \) be a raw cocycle for all \( x \in \bar{x}(\emptyset) \) except \( x_0 \) such that there is a character \( \chi \) with the property
\[ \psi_{1,x} = \chi + \chi_{\pi_{i_a}(x)} \]
for all \( x \).

Assume that there is a good expansion \( \phi_{\pi_{S-(i_a)}}(\bar{x}) \cdot pr_1(\pi_{i_a}(\bar{x})) \cdot pr_2(\pi_{i_a}(\bar{x})) \). We further assume that for all \( T \subseteq S \) and all \( \bar{y} \in \bar{x}(T) \) not containing \( x_0 \), we have that \( \{\psi_{i_a,x}^{[T]} \}_{i_a=0}^{[S]} \) agrees with the expansion at \( \bar{y} \). Then there is a raw cocycle \( \{\psi_{i_a,x_0}\}_{i_a=0}^{[S]} \) such that \( \psi_{1,x_0} = \psi_{1,x} \) for all \( x \in \bar{x}(\emptyset) \) with \( \pi_{i_a}(x_0) = \pi_{i_a}(x) \).

Further suppose that for every \( i \in S \), \( pr_1(\pi_i(\bar{x})) \cdot pr_2(\pi_i(\bar{x})) \) is a square locally at 2 and at all primes in \( \pi_i(\bar{x}) \) with \( i \neq j \). Moreover, assume that there exists an integer \( b \) satisfying \( b \in f^{-1}(2^{m-1}\text{Cl}(\mathbb{Q}(\sqrt{x}))[2^m]) \) for all \( x \in \bar{x}(\emptyset) \). Then we have
\[ \sum_{x \in \bar{x}(\emptyset)} \text{Art}_S|_{x}(b, \psi_{1,x}) = \sum_{p \mid b} \phi_{\pi_{S-(i_a)}}(\bar{x}) \cdot pr_1(\pi_{i_a}(\bar{x})) \cdot pr_2(\pi_{i_a}(\bar{x}))(\text{Frob } p). \] (4.7)

If instead \( x/b \in f^{-1}(2^{m-1}\text{Cl}(\mathbb{Q}(\sqrt{x}))[2^m]) \) for all \( x \in \bar{x}(\emptyset) \)
\[ \sum_{x \in \bar{x}(\emptyset)} \text{Art}_S|_{x/b}(b, \psi_{1,x}) = \sum_{p \mid b} \phi_{\pi_{S-(i_a)}}(\bar{x}) \cdot pr_1(\pi_{i_a}(\bar{x})) \cdot pr_2(\pi_{i_a}(\bar{x}))(\text{Frob } p). \] (4.8)

Proof. This can be proven in the same way as Theorem 4.9. \( \square \)

Remark 3. Write \( \phi = \phi_{\pi_{S-(i_a)}}(\bar{x}) \cdot pr_1(\pi_{i_a}(\bar{x})) \cdot pr_2(\pi_{i_a}(\bar{x})) \). The agreement assumptions imply that \( p \) splits completely in \( M(\phi) \), so that \( \text{Frob}(p) \) lands in \( \text{Gal}(L(\phi)/M(\phi)) \cong \mathbb{F}_2 \).
5 Analytic prerequisites

Throughout the paper our implied constants may depend on \( l \). We shall not record this dependence. The material in this section is rather similar to [33, Section 5] and [33, Section 6], but there is one major hurdle to overcome. Indeed, Smith does not prove the analogue of Corollary 6.11 for class groups. To do so, one needs to make the Markov chain analysis of Gerth effective.

Another significant complicating factor is that the 4-rank distribution in our family is different due to the fact that \( N_d(x, y) = l \) is soluble over \( \mathbb{Q} \) by assumption. This requires some changes to be made to the Markov chains appearing in Gerth [14]. These problems are dealt with in our companion paper [21].

5.1 Combinatorial results

In Section 6 we will make essential use of the following two combinatorial results first proven in Smith [33] with slightly different notation.

Let \( X := X_1 \times \cdots \times X_r \), let \( S \subseteq [r] \) and let \( Z \subseteq X \) with \( |\pi_r[\cdot]^{-1}(Z)| = 1 \). We define the \( \mathbb{F}_2 \)-vector spaces

\[
V := \text{Map}(Z, \mathbb{F}_2), \quad W := \text{Map}(\text{Cube}_S(Z), \mathbb{F}_2),
\]

where \( \text{Cube}_S(Z) \) is the set of \( \bar{x} \in X_S \) such that \( \bar{x}(\emptyset) \subseteq Z \). We define a linear map \( d : V \to W \), not to be confused with the map \( d \) on 1-cochains, by

\[
dF(\bar{x}) = \sum_{x \in \bar{x}(\emptyset)} F(x),
\]

where the sum has to be taken with multiplicities. This has the effect that

\[
dF(\bar{x}) = 0
\]
as soon as there exists some \( i \in S \) with \( \text{pr}_1(\pi_i(\bar{x})) = \text{pr}_2(\pi_i(\bar{x})) \) just like in Smith [33, Definition 4.2]. Define \( \mathcal{G}_S(Z) \) to be the image of \( d \).

Lemma 5.1. Let \( X := X_1 \times \cdots \times X_r \), let \( S \subseteq [r] \) and suppose that \( |X_i| = 1 \) for \( i \in [r] - S \). We have that

\[
\dim_{\mathbb{F}_2} \mathcal{G}_S(X) = \prod_{i \in S} (|X_i| - 1).
\]

Proof. See [19, Proposition 9.3].

Recall the definition of an additive system given in Definition 4.4. Given an additive system \( \mathfrak{A} \), we set

\[
C(\mathfrak{A}) := \bigcap_{i \in S} \left\{ \bar{x} \in X_S : \bar{x}(S - \{i\}) \cap \text{Y}_{S-\{i\}}^{\infty}(\mathfrak{A}) \neq \emptyset \right\}.
\]

We call an additive system \( \mathfrak{A} \) on \( X \) \((a, S)\)-acceptable if

- \( |A_T(\mathfrak{A})| \leq a \) for all subsets \( T \) of \( S \);
- \( \bar{x} \in C(\mathfrak{A}) \) implies \( \bar{x}(\emptyset) \subseteq \text{Y}_{\emptyset}^{\infty}(\mathfrak{A}) \).
Proposition 5.2. There exists an absolute constant $A > 0$ such that the following holds. Let $r > 0$ be an integer, let $X_1, \ldots, X_r$ be finite non-empty sets and let $X$ be their product. Take $S \subseteq [r]$ with $|S| \geq 2$, $|\pi_{[r]-S}(X)| = 1$ and put $n := \min_{i \in S} |X_i|$. Let $a \geq 2$ and $\epsilon > 0$ be given. Assume that $\epsilon < a^{-1}$ and

$$\log n \geq A \cdot 6^{|S|} \cdot \log \epsilon^{-1}.$$ 

Then there exists $g \in \mathcal{G}_S(X)$ such that for all $(a,S)$-acceptable additive systems $\mathfrak{A}$ on $X$ and for all $F : Y_0(\mathfrak{A}) \to \mathbb{F}_2$ satisfying $dF(\bar{x}) = g(\bar{x})$ for all $\bar{x} \in C(\mathfrak{A})$, we have

$$\frac{|Y_0(\mathfrak{A})|}{2} - |X| \cdot \epsilon \leq |F^{-1}(0)| \leq \frac{|Y_0(\mathfrak{A})|}{2} + |X| \cdot \epsilon.$$

Proof. This is Proposition 4.4 in Smith [33] and reproven in a slightly more general setting in Proposition 8.7 of [19]. \qed

5.2 Prime divisors

Let $l$ be an integer such that $|l|$ is prime and congruent to 3 modulo 4 or let $l = -1$. In Theorem 5.1 we are only interested in those squarefree integers $d > 0$ with the properties

$$l \mid d, \quad p \mid \frac{d}{l} \text{ implies } \left(\frac{l}{p}\right) = 1 \text{ or } p = 2 \quad (5.1)$$

$$\left(\frac{-d/l}{|l|}\right) = 1. \quad (5.2)$$

Indeed, this is equivalent to $l \mid d$ and the solubility of the equation

$$x^2 - dy^2 = l \text{ in } x, y \in \mathbb{Q}.$$ 

We remark that equation (5.2) is equivalent to a set of congruence conditions for $d$ modulo 8. Hence we need to insert congruence conditions in Section 5 of Smith [33]. This has already been done in Section 10 of [19] for squarefree integers $d$ such that $p \mid d$ implies $p \equiv 0, 1 \text{ mod } l$, and in Section 4 of [2] with completely different techniques for squarefree integers $d$ such that $p \mid d$ implies $p \equiv 1, 2 \text{ mod } 4$. Both these techniques are straightforward to generalize to obtain the following results, here we shall follow [33, Section 5].

Define

$$S(N, l) := \{1 \leq d < N : d \text{ squarefree and satisfies equation (5.1) and (5.2)}\}$$

and

$$S_r(N, l) := \{d \in S(N, l) : \omega(d) = r\}.$$ 

We list the distinct prime divisors of $d$ as $p_1 < p_2 < \cdots < p_r$. With these notations we can state our next theorem.

**Theorem 5.3.** Fix an integer $l$ such that $|l|$ is prime and congruent to 3 modulo 4 or $l = -1$. Let $N$ be a real number and put $\mu := \frac{1}{2} \log \log N$. Then there are absolute constants $A_1, A_2 > 0$ such that

$$\frac{A_1 N}{\log N} \cdot \frac{\mu^{r-1}}{(r-1)!} \leq |S_r(N, l)| \leq \frac{A_2 N}{\log N} \cdot \frac{\mu^{r-1}}{(r-1)!} \quad (5.3)$$
for all $1 \leq r \leq 200\mu$ and
\[
\frac{|\{d \in S(N, l) : |\omega(d) - \mu| > \mu^{2/3}\}|}{|S(N, l)|} \ll \exp \left( -\frac{1}{3} \mu^{1/3} \right).
\]
\[(5.4)\]

Now assume that $r$ is such that
\[
|r - \mu| \leq \mu^{2/3}
\]
and take $D_1 > 3$ and $C_0 > 1$. In this case we have

(i) the bound
\[
1 - \frac{|\{d \in S_r(N, l) : 2D_1 < p_i < p_{i+1}/2 \text{ for all } p_i > D_1\}|}{|S_r(N, l)|} \ll \frac{1}{\log D_1} + \frac{1}{(\log N)^{1/4}};
\]
(ii) the bound
\[
1 - \frac{|\{d \in S_r(N, l) : \frac{1}{2} \log \log p_i - i < C_0^{1/5} \max(i, C_0)^{4/5} \text{ for all } i < \frac{1}{2}r\}|}{|S_r(N, l)|} \ll \exp(-kC_0)
\]
for some absolute constant $k$;

(iii) the bound
\[
\frac{|\{d \in S_r(N, l) : \log \frac{p_i}{\log \log p_i} \leq (\log \log \log N)^{1/2} \cdot \sum_{j=1}^{i-1} \log p_j \text{ for all } \frac{1}{2}r^{1/2} < i < \frac{1}{2}r\}|}{|S_r(N, l)|} \ll \exp \left( -(\log \log \log N)^{1/4} \right).
\]

Proof. Condition (5.1) is incorporated in Smith’s argument just as in [19] by inserting a congruence condition in the definition of $F(x)$ [33, p. 51]. To deal with the congruence condition (5.2), we simply impose further congruence conditions on the primes for each invertible residue class in $(\mathbb{Z}/8\mathbb{Z})^*$ and then sum up the contributions.

More explicitly, we define for a congruence class $a \in (\mathbb{Z}/8\mathbb{Z})^*$ the sum
\[
F_a(x) = \sum_{\substack{p \leq x \hspace{1cm} \log p_i - i < \frac{1}{2} \log \log x \hspace{1cm} \text{for all } i \leq r \hspace{1cm} 1 \leq i \leq r \hspace{1cm} (p/p) = 1 \hspace{1cm} p \equiv a_i \ (\text{mod } 8)}} \frac{1}{p}
\]
so that there exist constants $B_a, A, c > 0$ such that
\[
\left| F_a(x) - \frac{\log \log x}{8} - B_a \right| \leq A \cdot e^{-c\sqrt{\log x}}.
\]

Let $(a_1, \ldots, a_r) \in (\mathbb{Z}/8\mathbb{Z})^r$ be a vector. Then for $T$ any set of tuples of primes $(p_1, \ldots, p_r)$ of length $r$ with $p_i \equiv a_i \ (\text{mod } 8)$ and $(l/p) = 1$, we define the grid
\[
\text{Grid}(T) = \bigcup_{(p_1, \ldots, p_r) \in T} \prod_{1 \leq i \leq r} \left[ 8 \cdot \left( F_{a_i}(p_i) - \frac{1}{p_i} - B_{a_i} \right) , 8 \cdot (F_{a_i}(p_i) - B_{a_i}) \right].
\]

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Following the proof of Smith [33, p. 52], we compare the quantity

\[ \sum_{p_1, \ldots, p_r < N} \frac{8^r}{p_1 \cdot \ldots \cdot p_r} \]

against the integral \( I_r \) for each vector \((a_1, \ldots, a_r) \in (\mathbb{Z}/8\mathbb{Z})^r\). Then equation (5.3) follows from a version of [33, Proposition 5.5] for the set of squarefree integers \( d \) satisfying equation (5.1) and the condition \( p_i \equiv a_i \mod 8 \) after summing over all possible vectors \((a_1, \ldots, a_r) \in (\mathbb{Z}/8\mathbb{Z})^r\) such that \( x^2 - a_1 \cdot \ldots \cdot a_r y^2 = l \) is soluble in \( \mathbb{Q}_2 \).

The assertion (5.4) is deduced from equation (5.3), from standard bounds on the tails of the Poisson distribution and from a good bound for the number of integers with more than \( 100 \log \log N \) prime divisors. Such a bound follows immediately when one computes the average of \( \tau(n) \). The claims (i), (ii) and (iii) are a straightforward generalization of the material in Section 5 of Smith [33].

5.3 Graph theory

In our final section we shall use some results in graph theory that we prove here. This subsection may be skipped on first reading.

**Definition 5.4.** We say that an undirected graph \( G \) on \( n \) vertices is \((\kappa, m)\)-bad if there is an induced subgraph with at least \( \kappa \) vertices and no clique of size \( m \).

**Lemma 5.5.** Let \( \kappa \) be an integer with \( \kappa \geq n/\log n \) and let \( m \) be an integer satisfying

\[ 3 \leq m \leq (\log \kappa)^{1/4}. \]

Then there is an absolute constant \( C \) such that for all \( n > C \) there are no more than

\[ 2^{\binom{n}{2}} \cdot 2^{-\frac{1}{m} \binom{n}{2}} \]

\((\kappa, m)\)-bad graphs \( G \).

**Proof.** Theorem 1 of [26] states that under our conditions the number of graphs on \( n \) vertices without a clique of size \( m \) is at most

\[ 2^{(1 - \frac{1}{m-1}) \binom{n}{2} + o(n^2/m)}. \]

Hence the number of bad graphs \( G \) is bounded by

\[ 2^n \cdot 2^{\binom{n}{2} - \binom{2}{2}} \cdot 2^{(1 - \frac{1}{m-1}) \binom{2}{2} + o(n^2/m)} \]

For sufficiently large \( n \), we have that

\[ 2^n \cdot 2^{\left(\frac{1}{m-1}\right) \binom{2}{2} + o(n^2/m)} \leq 2^{\frac{1}{4m} \binom{2}{2}} \]

and this proves the lemma. \( \square \)
Lemma 5.6. Let $V$, $W$ be sets and let $m$ be an integer. There are at most
\[
\binom{|V|}{m} \cdot \left(1 - \frac{1}{2^{m}}\right)^{|W|} \cdot 2^{|V|\cdot|W|}
\]
functions $f : V \times W \to \mathbb{F}_2$ with the property that there are pairwise distinct $v_1, \ldots, v_m \in V$ such that for all $w \in W$ there is some $i \in [m]$ with $f(v_i, w) = 0$.

Proof. We start by fixing a subset $\{v_1, \ldots, v_m\} \subseteq V$ of cardinality $m$ and some $w \in W$. Then the probability that $f(v_i, w) = 0$ for some $i \in [m]$ is
\[
1 - \frac{1}{2^{m}}.
\]
This probability is independent as we vary $w$, hence the probability that for all $w \in W$ there is some $i \in [m]$ with $f(v_i, w) = 0$ is given by
\[
\left(1 - \frac{1}{2^{m}}\right)^{|W|}.
\]
Since there are at most $\binom{|V|}{m}$ subsets of cardinality $m$, the probability, that there are pairwise distinct $v_1, \ldots, v_m \in V$ such that for all $w \in W$ there is some $i \in [m]$ with $f(v_i, w) = 0$, is at most
\[
\binom{|V|}{m} \cdot \left(1 - \frac{1}{2^{m}}\right)^{|W|}.
\]
This implies the lemma. □

5.4 Equidistribution of Legendre symbol matrices

In this subsection we state several equidistribution results pertaining to matrices of Legendre symbols. These results are straightforward modifications of the material in [2, 33]. We start with two definitions.

Definition 5.7. Suppose that $l$ is a non-zero integer. A prebox is a pair $(X, P)$ satisfying

- $P$ consists entirely of prime numbers such that the images of $P$, $l$ and $-1$ are linearly independent in $\mathbb{Q}_l^*$;
- $X = X_1 \times \cdots \times X_r$, where each $X_i$ consists entirely of prime numbers with $X_i \cap P = \emptyset$;
- there exists a sequence of real numbers
  \[
  0 < s_1 < t_1 < s_2 < t_2 < \cdots < s_r < t_r
  \]
  such that every prime $p \in X_i$ satisfies $s_i < p < t_i$ and $(l/p) = 1$.

Define the (potentially infinite) sequence $d_1, d_2, \ldots$ as in Definition 6.2 of Smith [33]. Then we have $d_i^2 < |d_{i+1}|$. We say that $(X, P)$ is Siegel-less above $t$ if for all $x \in X$ we have that $d_i \mid lx \prod_{p \in P} p$ implies $|d_i| < t$. 

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Definition 5.8. Write $A \sqcup B$ for the disjoint union of two sets $A$ and $B$. Let $(X, P)$ be a prebox. Put
\[
M := \{(i, j) : 1 \leq i < j \leq r\}, \quad M_P := [r] \times (P \sqcup \{-1\}).
\]
Let $\mathcal{M} \subseteq M$ and let $\mathcal{N} \subseteq M_P$. Given a map $a : \mathcal{M} \sqcup \mathcal{N} \to \{\pm 1\}$, we define $X(a)$ to be the subset of tuples $(x_1, \ldots, x_r) \in X$ with
\[
\left(\frac{x_i}{x_j}\right) = a(i, j) \quad \text{for all} \quad (i, j) \in \mathcal{M}, \quad \left(\frac{p}{x_j}\right) = a(i, p) \quad \text{for all} \quad (i, p) \in \mathcal{N}.
\]

Ideally we would like to show that every $X(a)$ is of the expected size. Although we are not able to prove this in full generality, we will prove slightly weaker results that still suffice for our application. If $S \subseteq [r]$, $Q \in \prod_{i \in S} X_i$ and $j \in S$, we write
\[
X_j(a, Q) := \left\{ x_j \in X_j : \left(\frac{\pi_i(Q)}{x_j}\right) = a(i, j) \quad \text{for} \quad i \in S \quad \text{and} \quad \left(\frac{p}{x_j}\right) = a(j, p) \quad \text{for} \quad p \in P \right\}.
\]
Here we use the convention that for $i > j$
\[
a(i, j) := a(j, i) \cdot (-1)^{a(i-j-1) - \frac{a(i-j-1)}{2}}.
\]
We also define $X(a, Q)$ to be the subset of $x \in X(a)$ with $\pi_S(x) = Q$.

Proposition 5.9. Let $l$ be a non-zero integer. For every choice of positive constants $c_1, \ldots, c_8$ satisfying $c_3 > 1$, $c_5 > 3$ and
\[
\frac{1}{8} > c_8 + \frac{c_7 \log 2}{2} + \frac{1}{c_1} + \frac{c_2 c_4}{2},
\]
there exists a constant $A$ such that the following holds.

Let $A < t < s_1$ and suppose that $(X, P)$ is a prebox that is Siegel-less above $t$. Let $\mathcal{M} \subseteq M$ and let $\mathcal{N} \subseteq M_P$. Let $1 \leq k \leq r$ be an integer such that $(i, p) \in \mathcal{M}$ implies $i > k$. Furthermore, if $i > k$, we assume that $X_i$ equals the set of primes in $s_i < q < t_i$ satisfying
\[
\left(\frac{l}{q}\right) = 1 \quad \text{and} \quad \left(\frac{p}{q}\right) = a(i, p) \quad \text{for} \quad (i, p) \in \mathcal{N}.
\]
Finally assume that
\[
(i) \quad p \in P \quad \text{implies} \quad p < s_1 \quad \text{and} \quad |P| \leq \log t_i - i \quad \text{for all} \quad 1 \leq i \leq r;
\]
\[
(ii) \quad \log t_k < t_1^{c_3} \quad \text{and if} \quad k < r, \quad \text{we assume that} \quad \log t_k + 1 > \max((\log t_1)^{c_5}, t^{c_6});
\]
\[
(iii) \quad \text{we assume that for all} \quad 1 \leq i \leq r \quad |X_i| \geq \frac{2^{c_3 i} \cdot t_i}{(\log t_i)^{c_4}};
\]
\[
(iv) \quad r^{c_1} < t_1;
\]
\[
(v) \quad \text{putting} \quad j_i := i - 1 + \lfloor c_7 \log t_i \rfloor, \quad \text{we assume that} \quad j_1 > k. \quad \text{Furthermore,} \quad j_i \leq r \quad \text{implies} \quad (\log t_i)^{c_5} < \log t_{j_i}.
\]
Then we have for all $a : M \sqcup N \to \{\pm 1\}$
\[
\left| |X(a)| - \frac{|X|}{2|M|} \right| \leq t_1^{-c_8} \cdot \frac{|X|}{2|M|}.
\]

Now additionally assume that $M = M$ and $N$ contains $(i, p)$ for every $i > k$ and every $p \in P \cup \{-1\}$. Furthermore, suppose that $U, V \subseteq [r]$ are disjoint subsets such that $U \cup V = [r']$ and suppose that $u \in U$ implies $u > k$ and
\[
\log \log s_u > \frac{1}{5} \max(r, \log \log t_r).
\]

We say that $Q \in \pi_V(X)$ is poor if there is some $u \in U$ such that
\[
\left| |X_u(a, Q)| - \frac{|X_u|}{2|V|} \right| > \frac{|V| \cdot |X_u|}{t_1^{c_8}}. \]

Then we also have
\[
\sum_{Q \in \pi_V(X) \text{ poor}} |X(a, Q)| \leq \frac{r \cdot |X|}{t_1^{c_8} \cdot 2|M|}.
\]

**Proof.** This is a straightforward generalization of [33, Proposition 6.3] and [2, Proposition 5.10].

Our next proposition deals with the small primes but at the cost of introducing permutations. We define $P(k_2)$ to be the set of permutations $\sigma : [r] \to [r]$ that fix every $k_2 < i \leq r$. Furthermore, if $a : M \sqcup M_P \to \{\pm 1\}$, we define $\sigma(a)$ to be
\[
\sigma(a)(i, j) = a(\sigma(i), \sigma(j)), \quad \sigma(a)(i, p) = a(\sigma(i), p).
\]

**Proposition 5.10.** Let $l$ be a non-zero integer. For every choice of positive constants $c_1, \ldots, c_{12}$, satisfying $c_3 > 1$, $c_5 > 3$ and
\[
\frac{1}{8} > c_8 + \frac{c_7 \log 2}{2} + \frac{1}{c_1} + \frac{c_2 c_4}{2}, \quad c_{10} \log 2 + 2 c_{11} + c_{12} < 1 \quad \text{and} \quad c_{12} + c_{11} < c_9,
\]
there exists a constant $A$ such that the following holds.

Let $A < t$ and suppose that $(X, P)$ is a prebox that is Siegel-less above $t$ such that $X_i$ equals the set of primes $p$ in the interval $(s_i, t_i)$ satisfying $(l/p) = 1$. Let $k_0, k_1, k_2$ be integers such that $0 \leq k_0 < k_1 < k_2 \leq r$. We assume that

(i) $p \in P$ implies $p < s_{k_0 + 1}$ and $|P| \leq \log t_i - i$ for all $i > k_0$;

(ii) $\log t_{k_1} < t_{k_0 + 1}^{c_5}$ and $\log t_{k_1 + 1} > \max((\log t_{k_0 + 1})^{c_5}, t_{ce})$;

(iii) for all $i > k_0$
\[
|X_i| \geq \frac{2^{|P| + c_4} \cdot k_2^{c_9} \cdot t_i}{(\log t_i)^{c_4}};
\]

(iv) $r^{c_1} < t_{k_0 + 1}$;
(v) we assume that $k_1 - k_0 < c_7 \log t_{k_0 + 1}$. Furthermore, $i > k_0$ and $i - 1 + [c_7 \log t_i] \leq j \leq r$ implies $$(\log t_i)^{c_5} < \log t_j;$$

(vi) $k_2 > A$ and $s_{k_0 + 1} > t$;

(vii) $c_{10} \log k_2 > |P| + k_0$ and $c_{11} \log k_2 > \log k_1$.

Then we have

$$
\sum_{a \in \mathbb{F}_{2^\mathbb{M} \sqcup \mathbb{M}_P}} 2^{-|\mathbb{M} \sqcup \mathbb{M}_P|} \cdot k_2! \cdot |X| - \sum_{\sigma \in \mathcal{P}(k_2)} |X(\sigma(a))| \leq \left( t_{k_0 + 1}^{-c_{12}} + t_{k_0 + 1}^{-c_8} \right) \cdot k_2! \cdot |X|.
$$

Proof. This is a straightforward generalization of [33, Proposition 6.4].

5.5 Boxes

We now define boxes. Boxes are product spaces of the shape $X := X_1 \times \cdots \times X_r$, where $X_1, \ldots, X_r$ are “nice” sets of primes. Then we state an important proposition that allows us to transition from squarefree integers to boxes.

**Definition 5.11.** Let $l$ be such that $|l|$ is a prime 3 modulo 4 or $l = -1$. Suppose that $D_1 > \max(100, |l|)$ is a real number and let $1 \leq k \leq r$ be integers. Let $t := (p_1, \ldots, p_k, t_{k+1}, \ldots, t_r)$ be a tuple satisfying the following properties

- the $p_i$ are prime numbers satisfying $p_1 < \cdots < p_k < D_1$ and the $t_j$ are real numbers with $D_1 < t_{k+1} < \cdots < t_r$;

- we have $|l| \in \{p_1, \ldots, p_k\}$ and we have for all $i = 1, \ldots, k$ that $\gcd(2l, p_i) > 1$ or $\left( \frac{l}{p_i} \right) = 1$.

To $t$ we associate a box $X := X_1 \times \cdots \times X_r$ as follows; we set $X_i := \{p_i\}$ for $1 \leq i \leq k$, while for $i > k$ we let $X_i$ be the set of prime numbers $p$ with $\left( \frac{l}{p} \right) = 1$ in the interval $$(t_i, \left(1 + \frac{1}{e^{1-k} \log D_1} \right) \cdot t_i).$$

Note that for $l = -1$ this is the same definition as Definition 5.11 in [2]. Furthermore, if $l \neq -1$, we can turn any box into a prebox by removing $\{ |l| \}$ and taking $\mathcal{P} = \emptyset$. We define

$$S^*(N, l) := \{ 1 \leq d < N : d \text{ squarefree and satisfies equation (5.1)} \}$$

and

$$S^*_r(N, l) := \{ d \in S^*(N, l) : \omega(d) = r \}.$$ 

Then there is a natural injective map $i : X \to S^*_r(\infty, l)$, which is a superset of $S_r(\infty, l)$. Hence it makes sense to speak of the intersection $i(X) \cap V$ for $V$ a subset of $S_r(\infty, l)$. We can now state our analogue of Proposition 6.9 in Smith [33].
Proposition 5.12. Take \( l \) to be an integer such that \(|l|\) is a prime 3 modulo 4 or \( l = -1 \). Let \( N \geq D_1 > \max(100, |l|) \) and \( \log \log N \geq 2 \log \log D_1 \). Take any \( r \) satisfying equation (5.5). Let \( V, W \) be subsets of \( S_r(N, l) \) with the additional requirement that

\[
W \subseteq \{ d \in S_r(N, l) : 2D_1 < p_i < p_{i+1}/2 \text{ for all } p_i > D_1 \}.
\]

Take any \( \epsilon > 0 \) with

\[
|W| > (1 - \epsilon) \cdot |S_r(N, l)|.
\]

Assume that there exists a real number \( \delta > 0 \) such that for all boxes \( X \) with \( i(X) \subseteq S_r^*(N, l) \) and \( i(X) \cap W \neq \emptyset \) we have

\[
(\delta - \epsilon) \cdot |i(X) \cap S_r(N, l)| \leq |i(X) \cap V| \leq (\delta + \epsilon) \cdot |i(X) \cap S_r(N, l)|.
\]

Then

\[
|V| = \delta \cdot |S_r(N, l)| + O \left( \frac{1}{\log D_1} \cdot |S_r(N, l)| \right).
\]

Proof. This is a straightforward adaptation of Proposition 6.9 in Smith [33]. \( \square \)

When we apply Proposition 6.3 and Theorem 6.4 of Smith [33], we need to ensure the Siegel-less condition, i.e. we need to avoid all boxes \( X \) such that there are \( x \in X \) and some \( i \) with \( |d_i| > D_1 \) and \( d_i \mid x \). To do so, we shall add the union of all such boxes \( X \) to \( W \). Therefore it is important to show that this union is small, and this is exactly what the following proposition does.

Proposition 5.13. Let \( l \) be an integer such that \(|l|\) is a prime or \( l = -1 \). Take \( N \) and \( r \) satisfying equation (5.5). Also take \( N \geq D_1 > \max(100, |l|) \) with \( \log \log N \geq 2 \log \log D_1 \). Let \( f_1, f_2, \ldots \) be any sequence of squarefree integers greater than \( D_1 \) satisfying \( f_i \leq f_{i+1} \). Define

\[
W_i := \{ d \in S_r(N, l) : \text{there is a box } X \text{ with } d \in X \text{ and } f_i \mid x \text{ for some } x \in X \}.
\]

Then we have

\[
\left| \bigcup_{i=1}^{\infty} W_i \right| \ll \frac{|S_r(N, l)|}{\log D_1}.
\]

Proof. This is a small generalization of Theorem 5.13 in [2], which is based on Proposition 6.10 in Smith [33]. \( \square \)

5.6 Rédei matrices

The previous subsections provide us with enough tools to deal with the 4-rank distribution in our family of discriminants. We only handle the case where \( l \) is such that \(|l|\) is a prime 3 modulo 4. The analogous results for \( l = -1 \) can be found in [2, Section 5]. We now define the Rédei matrix associated to a squarefree integer \( d > 0 \).

Definition 5.14. Let \( d > 0 \) be a squarefree integer and suppose that \( \Delta_{Q(\sqrt{d})} \) has \( t \) prime divisors, say \( p_1, \ldots, p_t \). We can uniquely decompose \( \chi_d \) as

\[
\chi_d = \sum_{i=1}^{t} \chi_{p_i}.
\]
where \( \chi_i : G_Q \to \mathbb{F}_2 \) has conductor a power of \( p_i \). In case \( p_i \neq 2 \), we have \( \chi_i = \chi_{p_i^2} \), where \( p_i^2 \) has the same absolute value as \( p_i \) and is 1 modulo 4. When \( p_i = 2 \), we have \( \chi_i \in \{ \chi_4, \chi_{-8}, \chi_8 \} \).

The Rédéi matrix \( R(d) \) is a \( t \times t \) matrix with entry \( (i,j) \) equal to

\[
\chi_j(\text{Frob } p_i) \text{ if } i \neq j, \quad \sum_{k \neq i} \chi_k(\text{Frob } p_i) \text{ if } i = j,
\]

so the sum of every row is zero.

It is a classical fact that

\[
\text{rk}_4 \text{ Cl}(\mathbb{Q}(\sqrt{d})) = t - 1 - \text{rk } R(d).
\]

One of the pleasant properties of \( X(a) \) is that all \( x \in X(a) \) have the same Rédéi matrix, and hence the same 4-rank. There are several constraints for the possible shapes of the Rédéi matrix. First of all, there is quadratic reciprocity that relates the entry \( (i,j) \) with \( (j,i) \).

Second of all, if \( d \in S(N,l) \), then there are further constraints coming from equation (5.1) and equation (5.2). We will now indicate what conditions this forces on \( a \).

**Definition 5.15.** Let \( X \) be a box corresponding to \( t = (p_1, \ldots, p_k, t_{k+1}, \ldots, t_r) \) and let \( \tilde{j} \) be the index for which \( X_{\tilde{j}} = \{ || \} \). We define \( \text{Map}(M \sqcup M_{\emptyset}, \{ \pm 1 \}) \) to be the set of maps from \( M \sqcup M_{\emptyset} \) to \( \{ \pm 1 \} \). Put \( \text{Map}(M \sqcup M_{\emptyset}, \{ \pm 1 \}, \tilde{j}, l) \) to be the subset of \( \text{Map}(M \sqcup M_{\emptyset}, \{ \pm 1 \}) \) satisfying

- if \( X_1 \neq \{ 2 \} \) and \( l > 0 \), then \( a(i, \tilde{j}) = a(i, -1) \) for all \( i < \tilde{j}, a(\tilde{j}, i) = 1 \) for all \( i > \tilde{j} \) and

\[
\prod_{i=1}^r a(i, -1) = 1;
\]

- if \( X_1 \neq \{ 2 \} \) and \( l < 0 \), then \( a(i, \tilde{j}) = 1 \) for all \( i < \tilde{j}, a(\tilde{j}, i) = a(i, -1) \) for all \( i > \tilde{j} \);

- if \( X_1 = \{ 2 \} \) and \( l > 0 \), then \( a(i, \tilde{j}) = a(i, -1) \) for all \( 2 \leq i < \tilde{j}, a(\tilde{j}, i) = 1 \) for all \( i > \tilde{j} \) and

\[
\prod_{i=1}^r a(i, -1) = \left( \frac{2}{|l|} \right);
\]

- if \( X_1 = \{ 2 \} \) and \( l < 0 \), then \( a(i, \tilde{j}) = 1 \) for all \( 2 \leq i < \tilde{j}, a(\tilde{j}, i) = a(i, -1) \) for all \( i > \tilde{j} \) and \( l \equiv 1 \mod 8 \).

We will now describe exactly the kind of boxes that we will be working with for the rest of the paper.

**Definition 5.16.** Let \( X \) be a box and let \( N \) be a real number. Put

\[
D_1 := e^{(\log \log N) \frac{100}{\log N}}, \quad C_0 := \frac{\log \log \log N}{100}, \quad C_0' := \sqrt{\log \log \log N}.
\]

We let \( W \) be the largest subset of \( S_r(N,l) \) satisfying

- the requirement \( W \cap W_i = \emptyset \) for all \( i \geq 1 \), where \( W_i \) is the set as constructed in Proposition 5.13.
• the requirement
\[ W \subseteq \{ d \in S_r(N, l) : 2D_1 < p_i < p_{i+1}/2 \text{ for all } p_i > D_1 \}; \quad (5.7) \]

• and the requirement
\[ W \subseteq \left\{ d \in S_r(N, l) : \left| \frac{1}{2} \log \log p_i - i \right| < C_0^{1/5} \max(i, C_0)^{4/5} \right\}. \quad (5.8) \]

We say that \( X \) is \( N \)-decent if \( r \) satisfies equation (5.5), \( i(X) \subseteq S^*_r(N, l) \) and \( i(X) \cap W \neq \emptyset \).

Now let \( W' \) be the largest subset of \( W \) satisfying

• the requirement
\[ W' \subseteq \left\{ d \in S_r(N, l) : \left| \frac{1}{2} \log \log p_i - i \right| < C'_0^{1/5} \max(i, C'_0)^{4/5} \right\}; \quad (5.9) \]

• and the requirement that for every \( d \in W' \) there is some \( i \) with \( \frac{1}{2}r^{1/2} < i < \frac{1}{2} \) and
\[ \frac{\log p_i}{\log \log p_i} > (\log \log \log N)^{1/2} \cdot \sum_{j=1}^{i-1} \log p_j. \quad (5.10) \]

We say that \( X \) is \( N \)-good if \( X \) is \( N \)-decent and \( i(X) \cap W' \neq \emptyset \).

The main point of Definition 5.16 is that we can apply the results in Subsection 5.4 to these boxes provided that \( N \) is sufficiently large. Let \( P(m, n, j) \) be the probability that a randomly chosen \( m \times n \) matrix with coefficients in \( \mathbb{F}_2 \) has right kernel of rank \( j \). Then we have the explicit formula
\[ P(m, n, j) = \frac{1}{2^{nm}} \prod_{i=0}^{n-j-1} \frac{(2^m - 2^i)(2^n - 2^i)}{2^{m-j} - 2^i}, \]
which we will use throughout the paper. For the remainder of this paper, \( \iota \) denotes the unique group isomorphism between \( \{ \pm 1 \} \) and \( \mathbb{F}_2 \). To prove the next theorem, it suffices to work with \( N \)-decent boxes \( X \), while we will work with \( N \)-good boxes in Section 6.

**Theorem 5.17.** Let \( l \) be such that \( |l| \) is a prime 3 modulo 4. Then we have for all \( k \geq 0 \)
\[ \lim_{s \to \infty} P(s, s, k) \cdot |S(N, l)| - \left| \left\{ d \in S(N, l) : \mathrm{rk}_4 \mathrm{Cl}(\mathbb{Q}(\sqrt{d})) = k \right\} \right| = O \left( \frac{|S(N, l)|}{(\log \log N)^c} \right) \]
for some absolute constant \( c > 0 \).

**Proof.** Let \( r \) be an integer satisfying equation (5.5). Since one easily bounds the differences
\[ \lim_{s \to \infty} P(s, s, k) - P(r-1, r-1, k), \]
we may work with \( P(r-1, r-1, k) \) instead. We now follow the proof of Smith [33 Corollary 6.11]. The first step is to reduce to \( N \)-decent \( X \), for which we use our Theorem 5.16 Proposition 5.12 and Proposition 5.13.
Now let \( X = X_1 \times \cdots \times X_r \) be an \( N \)-decent box, so in particular \( i(X) \subseteq S_r^*(N, l) \). It suffices to prove that
\[
|P(r - 1, r - 1, k) \cdot |i(X) \cap S_r(N, l)| - \left| \left\{ d \in i(X) \cap S_r(N, l) : \text{rk}_4 \Cl(\Q(\sqrt{d})) = k \right\} \right| = O \left( \frac{|X|}{(\log \log N)^c} \right) \tag{5.11}
\]
for some absolute constant \( c > 0 \), since \( |X| \ll |i(X) \cap S_r(N, l)| \). We now apply Proposition \ref{prop:5.10} to the box \( X' \) with \( X_j \) removed. Then we obtain an absolute constant \( c' \) such that
\[
\sum_{a \in \Map(M' \sqcup M'_0, \{\pm 1\})} 2^{-|M'| - |M'_0|} \cdot (r - 1)! \cdot |X'| - \sum_{\sigma \in \P(r - 1)} |X'(\sigma(a))| \leq \frac{(r - 1)! \cdot |X'|}{(\log \log N)^{c'}} \tag{5.12}
\]
where \( M' = \{(i, j) : 1 \leq i < j \leq r - 1\} \) and \( M'_0 = [r - 1] \times \{-1\} \). Let \( S \) be the set of permutations of \( [r] \) that fix \( j \). Then equation \ref{eq:5.12} implies
\[
\sum_{a \in \Map(M \sqcup M_0, \{\pm 1\}, j, l)} 2^{-|M'| - |M'_0|} \cdot (r - 1)! \cdot |X| - \sum_{\sigma \in S} |X(\sigma(a))| \leq \frac{(r - 1)! \cdot |X|}{(\log \log N)^{c'}} \tag{5.13}
\]
Note that if \( a \in \Map(M \sqcup M_0, \{\pm 1\}, j, l) \), then so is \( \sigma(a) \) for any permutation \( \sigma \in S \). Also observe that \( a \in \Map(M \sqcup M_0, \{\pm 1\}, j, l) \) implies \( i(X(a)) \subseteq S_r(N, l) \). Furthermore, if \( i(X(a)) \cap S_r(N, l) \neq \emptyset \), then we certainly have \( a \in \Map(M \sqcup M_0, \{\pm 1\}, j, l) \).

Set
\[
Q(X, k, l) := \frac{\left| \{a \in \Map(M \sqcup M_0, \{\pm 1\}, j, l) : \dim_{\F_2} \ker(A) = k + 1\} \right|}{\left| \Map(M \sqcup M_0, \{\pm 1\}, j, l) \right|}.
\]
Here \( A \) is the Rédei matrix associated to \( a \) in the obvious way. Note that the matrix \( A' \) associated to \( \sigma(a) \) has the same rank as the matrix \( A \) associated to \( a \). Then, because of equation \ref{eq:5.13}, it is enough to show that there is an absolute constant \( c > 0 \) such that
\[
|P(r - 1, r - 1, k) - Q(X, k, l)| = O \left( \frac{1}{(\log \log N)^c} \right)
\]
for every \( r \) satisfying equation \ref{eq:5.15}. But this follows from \cite[Theorem 4.8]{21}.

\section{Proof of main theorems}

The aim of this section is to prove Theorem \ref{thm:1.1} and Theorem \ref{thm:1.4}. We start with the proof of Theorem \ref{thm:1.1}. The proof of Theorem \ref{thm:1.4} is almost identical and we shall only indicate the necessary changes in Subsection \ref{subsec:6.2}.

\subsection{Proof of Theorem \ref{thm:1.1}}

Let \( D_{l,k}(n) \) be the set of squarefree integers \( d \) divisible by \( l \) such that \( \text{rk}_{2k} \Cl(\Q(\sqrt{d})) = n \) and furthermore \( l \in 2^{k-1} \Cl(\Q(\sqrt{d})) | 2^k \) if \( l > 0 \) and \( l(\sqrt{d}) \in 2^{k-1} \Cl(\Q(\sqrt{d})) | 2^k \) if \( l < 0 \), where \( l \) is the unique ideal with norm \( l \). Then we have the decomposition
\[
\bigcup_{n=0}^{\infty} D_{l,k}(n) = S(\infty, l).
\]
Our next theorem is very much in spirit of the heuristic assumptions that led to Stev-
hagen’s conjecture [34]. Theorem 5.17 is an immediate consequence of Theorem 5.16 and the theorem below.

**Theorem 6.1.** Let \( l \) be an integer such that \( |l| \) is a prime 3 modulo 4. There are \( c, A, N_0 > 0 \) such that for all integers \( N > N_0 \), all integers \( m \geq 2 \) and all sequences of integers \( n_2 \geq \cdots \geq n_m \geq n_{m+1} \geq 0 \)

\[
\left| \frac{[N]}{2n_m} \prod_{i=2}^{m+1} D_{l,i}(n_i) \right| = \frac{P(n_m,n_{m},n_{m+1})}{(\log \log \log N)^{m+\gamma}}. 
\]

To prove Theorem 6.1, our first step is to reduce to boxes with some nice properties. Definition 5.16 precisely pinpoints the boxes for which we will prove the desired equidistribution. We will now state a proposition and prove that the proposition implies Theorem 6.1, so that it remains to prove the proposition.

**Proposition 6.2.** Let \( l \) be an integer such that \( |l| \) is a prime 3 modulo 4. There are \( c, A, N_0 > 0 \) such that for all integers \( N > N_0 \), all integers \( m \geq 2 \), all sequences of integers \( n_2 \geq \cdots \geq n_m \geq n_{m+1} \geq 0 \) and all \( N \)-good boxes \( X \)

\[
\left| i(X) \cap \bigcap_{i=2}^{m+1} D_{l,i}(n_i) \right| = \frac{P(n_m,n_{m},n_{m+1})}{(\log \log \log N)^{m+\gamma}}. 
\]

**Proof that Proposition 6.2 implies Theorem 6.1.** Due to equation 5.14 we may restrict to \( S_r(N,l) \) with \( r \) satisfying equation 5.5. Let \( D_1 \) and \( W \) be as in Definition 5.16. Part (i), (ii) and (iii) of Theorem 5.13 give upper bounds for the complements of the sets appearing in equation 5.7, equation 5.9 and equation 5.10 respectively. Furthermore, Proposition 5.13 shows that most \( d \in W \) are outside the union of the \( W_i \). Therefore we see that there is an absolute constant \( C > 0 \) with

\[
|W| > \left( 1 - \frac{C}{\exp \left( (\log \log \log N)^{1/4} \right) } \right) \cdot |S_r(N,l)|. 
\]

We now apply Proposition 5.12 two times; in both case with our \( D_1 \) and \( W \), and

\[
V_1 := |N| \prod_{i=2}^{m+1} D_{l,i}(n_i), \quad V_2 := |N| \prod_{i=2}^{m} D_{l,i}(n_i) 
\]

respectively. Theorem 5.17 and Proposition 5.2 ensure that equation 5.16 is satisfied. Then we get

\[
V_1 = \lim_{s \to \infty} P(s, s, n_2) \prod_{i=2}^{m} \frac{P(n_i,n_{i},n_{i+1})}{2n_i} \cdot |S_r(N,l)| + O \left( \frac{|S_r(N,l)|}{(\log \log \log N)^{m+\gamma}} \right) 
\]
and

\[
V_2 = \lim_{s \to \infty} P(s, s, n_2) \prod_{i=2}^{m-1} \frac{P(n_i,n_{i},n_{i+1})}{2n_i} \cdot |S_r(N,l)| + O \left( \frac{|S_r(N,l)|}{(\log \log \log N)^{m+\gamma}} \right). 
\]

This quickly implies Theorem 6.1.

\[\square\]
Our next goal is to fix the first Rédei matrix. In other words, we split $X$ into the union $X(a)$ with $a$ running over all maps from $M \sqcup M \emptyset$ to $\{\pm 1\}$. Smith’s method does not prove equidistribution for all $a$, but only for most $a$. This prompts our next definition.

**Definition 6.3.** Let $X$ be a $N$-good box and let $a \in \text{Map}(M \sqcup M \emptyset, \{\pm 1\})$. Set

$$r'(a,X) := \begin{cases} r & \text{if } X_1 = \{2\} \text{ or } \prod_{i=1}^r a(i,-1) = 1 \\ r + 1 & \text{otherwise.} \end{cases}$$

Recall that we associated a $r'(a,X) \times r'(a,X)$ matrix $A$ with coefficients in $\mathbb{F}_2$ to $a$ during the proof of Theorem 5.17, which is simply the Rédei matrix of $x$ for any choice of $x \in X(a)$. Let $V$ be the vector space $\mathbb{F}_2^{r'(a,X)}$. We define

$$D_{a,2} := \{ v \in V : v^T A = 0 \}, \quad D_{a,2}^\vee := \{ v \in V : A v = 0 \}.$$

Put

$$n_{\text{max}} := \left\lfloor c' \frac{m^2 \log \log \log \log N}{6m^2} \right\rfloor, \quad n_{a,2} := \dim_{\mathbb{F}_2} D_{a,2} - 1,$$

where $c'$ is a constant specified later. Let $X$ be a $N$-good box and let $\tilde{j}$ be the index such that $X_{\tilde{j}} = \{|l|\}$. We define the vectors $R := (1,1,\ldots,1) \in D_{a,2}^\vee$ and

$$C := \begin{cases} (1,1,\ldots,1) & \text{if } X_1 = \{2\} \text{ or } \prod_{i=1}^r a(i,-1) = 1 \\ (0,1,1,\ldots,1) & \text{otherwise.} \end{cases}$$

We next define the vector

$$L := \begin{cases} (0,\ldots,0,1,0,\ldots,0) \in D_{a,2} & \text{if } l > 0 \\ (0,\ldots,0,1,0,\ldots,0) + C \in D_{a,2} & \text{if } l < 0, \end{cases}$$

where $(0,\ldots,0,1,0,\ldots,0)$ has a 1 exactly on the $\tilde{j}$-th position. Since $l \mid d$, the solubility of $x^2 - dy^2 = l$ in $x,y \in \mathbb{Q}$ is precisely equivalent to $L$ being in $D_{a,2}$. We fix a choice of an index $i$ satisfying equation (5.10) and we call it $k_{\text{gap}}$. Then we say that $a \in \text{Map}(M \sqcup M \emptyset, \{\pm 1\})$ is $(N,m,X)$-acceptable if the following conditions are satisfied

- $n_{a,2} \leq n_{\text{max}}$;
- we have $a \in \text{Map}(M \sqcup M \emptyset, \{\pm 1\}, \tilde{j}, l)$, see Definition 5.15;
- we have for all $j > k$

$$|X_j(a,Q)| \geq \frac{|X_j|}{(\log t_{k+1})^{100}}, \quad (6.1)$$

where $Q$ is the unique point in $X_1 \times \cdots \times X_k$;
- putting

$$S_{\text{pre},4} := \left\{ i \in [r] : \frac{k_{\text{gap}}}{2} \leq i < k_{\text{gap}} \text{ and } a(i,-1) = 1 \right\}, \quad \alpha_{\text{pre}} := |S_{\text{pre},4}|$$
and

\[ S_{\text{post}, 4} := \{ i \in [r] : k_{\text{gap}} \leq i \leq 2k_{\text{gap}} \text{ and } a(i, -1) = 1 \}, \quad \alpha_{\text{post}} := |S_{\text{post}, 4}|, \]

we have

\[
\left| \frac{\alpha_{\text{pre}} - k_{\text{gap}}}{4} \right| \leq \frac{k_{\text{gap}}}{\log \log \log \log N}, \quad \left| \frac{\alpha_{\text{post}} - k_{\text{gap}}}{2} \right| \leq \frac{k_{\text{gap}}}{\log \log \log \log N}, \tag{6.2}
\]

and we further have for all \( T_1 \in D_{a, 2}, \ T_2 \in D_{a, 2}' \) such that \( T_1 \not\in \langle L \rangle \) or \( T_2 \not\in \langle R \rangle \)

\[
\left| \left| \{ i \in S_{\text{pre}, 4} : \pi_i(T_1 + T_2) = 0 \} \right| - \frac{\alpha_{\text{pre}}}{2} \right| \leq \frac{\alpha_{\text{pre}}}{\log \log \log \log N} \tag{6.3}
\]

and

\[
\left| \left| \{ i \in S_{\text{post}, 4} : \pi_i(T_1 + T_2) = 0 \} \right| - \frac{\alpha_{\text{post}}}{2} \right| \leq \frac{\alpha_{\text{post}}}{\log \log \log \log N}. \tag{6.4}
\]

Let us explain the second condition. Note that given \( a \in \text{Map}(M \sqcup M_9, \{ \pm 1 \}) \), \( i(X(a)) \) is entirely contained in \( S_r(N, l) \) or completely disjoint from \( S_r(N, l) \). Since we only care about the intersection \( i(X) \cap S_r(N, l) \), we only restrict to those \( a \) with \( i(X(a)) \subseteq S_r(N, l) \), and this is exactly what the second condition does. The importance of the fourth condition will be explained after our next two definitions.

Once we have fixed the first Rédei matrix, we are ready to study all the higher Rédei matrices. In fact, we can prove equidistribution of all the higher Rédei matrices. We formalize this as follows.

**Definition 6.4.** Let \( a \in \text{Map}(M \sqcup M_9, \{ \pm 1 \}) \) and \( m \in \mathbb{Z}_{\geq 2} \) be given. Choose filtrations of vector spaces

\[ D_{a, 2} \supseteq \cdots \supseteq D_{a, m}, \quad D_{a, 2}' \supseteq \cdots \supseteq D_{a, m}' \]

with \( L \in D_{a, m} \) and \( R \in D_{a, m}' \). Define for \( 2 \leq i \leq m \)

\[ n_{a,i} := \dim_{\mathbb{F}_2} D_{a,i} - 1. \]

If \( \text{Art}_{a,i} : D_{a,i} \times D_{a,i}' \to \mathbb{F}_2 \) are bilinear pairings for \( 2 \leq i \leq m \), we call the set \( \{ \text{Art}_{a,i} \}_{2 \leq i \leq m} \) a sequence of Artin pairings if for every \( 2 \leq i < m \) the left kernel of \( \text{Art}_{a,i} \) is \( D_{a, i+1} \) and the right kernel of \( \text{Art}_{a,i} \) is \( D_{a,i+1}' \). We say that a bilinear pairing

\[ \text{Art}_{a,i} : D_{a,i} \times D_{a,i}' \to \mathbb{F}_2 \]

is valid if \( L \) and \( R \) are respectively in the left and right kernel. We call a sequence of Artin pairings valid if every element of the sequence is.

Let \( X \) be an \( N \)-good box, let \( a \in \text{Map}(M \sqcup M_9, \{ \pm 1 \}) \) be \( (N, m, X) \)-acceptable and also let \( d \in i(X(a)) \). We can naturally associate an infinite sequence of Artin pairings to \( d \) as follows. Write the prime divisors of the discriminant of \( \mathbb{Q}(\sqrt{d}) \) as \( p_1, \ldots, p_{r'(a,X)} \) with \( p_1 < \cdots < p_{r'(a,X)} \). By construction, we have that for each \( v \in D_{a, 2} \) that the unique ideal in \( \mathbb{Q}(\sqrt{d}) \) with norm

\[
\prod_{i=1}^{r'(a,X)} p_i^{\pi_i(v)}
\]

is valid.
is in $\text{2Cl}(\mathbb{Q}(\sqrt{d}))[4]$. Similarly we have for each $v \in D_{a,2}^\vee$ that the character

$$r'(a,X) \sum_{i=1}^{\pi(v)} \chi_i$$

is in $\text{2Cl}^\vee(\mathbb{Q}(\sqrt{d}))[4]$, where $\chi_i$ is as in Definition 5.14. In other words, we have natural epimorphisms

$$D_{a,2} \rightarrow \text{2Cl}(\mathbb{Q}(\sqrt{d}))[4] \text{ and } D_{a,2}^\vee \rightarrow \text{2Cl}^\vee(\mathbb{Q}(\sqrt{d}))[4].$$

Now we declare $D_{a,i,d}$ and $D_{a,i,d}^\vee$ to be the inverse image of respectively $2^{i-1}\text{Cl}(\mathbb{Q}(\sqrt{d}))[2^i]$ and $2^{i-1}\text{Cl}^\vee(\mathbb{Q}(\sqrt{d}))[2^i]$ under these maps. Furthermore, we let $\text{Art}_{a,i,d}$ be the natural pairing

$$2^{i-1}\text{Cl}(\mathbb{Q}(\sqrt{d}))[2^i] \times 2^{i-1}\text{Cl}^\vee(\mathbb{Q}(\sqrt{d}))[2^i] \rightarrow \mathbb{F}_2$$

pulled back to $D_{a,i,d}$ and $D_{a,i,d}^\vee$.

This gives an infinite sequence of Artin pairings $\text{Art}_{a,i,d}$ for every $d$. Furthermore, the sequence is valid if and only if equation (1.2) is soluble. Finally, we define for a sequence of Artin pairings

$$X(a, \{\text{Art}_{a,i}\}_{2 \leq i \leq m}) := \{d \in X(a) : \text{Art}_{a,i,d} = \text{Art}_{a,i} \text{ for } 2 \leq i \leq m\}.$$
\* in case \( j_1 = n_{a,m} + 1 \), we have \( c_{j_1,j_2}(F) = 0 \) for all \( 1 \leq j_1, j_2 \leq n_{a,m} \), \( |S| = m \), \( i_1(j_1,j_2) = j \) and

\[
S - \{i_1(j_1,j_2), i_2(j_1,j_2)\} \subseteq \bigcap_{i=1}^{n_{a,2}} \{j \in [r] : \pi_j(v_i) = 0\} \cap \bigcap_{i=1}^{n_{a,2}} \{j \in [r] : \pi_j(w_i) = 0\}
\]

and

\[
i_2(j_1,j_2) \in \{j \in [r] : \pi_j(w_{j_2}) = 1\} \cap \bigcap_{i=1}^{n_{a,2}} \{j \in [r] : \pi_j(v_i) = 0\} \cap \bigcap_{i \neq j_2} \{j \in [r] : \pi_j(w_i) = 0\}
\]

and \( a(i,j) = 1 \) for all distinct \( i, j \in S \).

\* in case \( j_1 \neq n_{a,m} + 1 \), we have \( |S| = m + 1 \), \( i_1(j_1,j_2) \in S \) and

\[
S - \{i_1(j_1,j_2), i_2(j_1,j_2)\} \subseteq \bigcap_{i=1}^{n_{a,2}} \{j \in [r] : \pi_j(v_i) = 0\} \cap \bigcap_{i=1}^{n_{a,2}} \{j \in [r] : \pi_j(w_i) = 0\}
\]

and

\[
i_1(j_1,j_2) \in \{j \in [r] : \pi_j(w_{j_2}) = 1\} \cap \bigcap_{i=1}^{n_{a,2}} \{j \in [r] : \pi_j(v_i) = 0\} \cap \bigcap_{i \neq j_2} \{j \in [r] : \pi_j(w_i) = 0\}
\]

and

\[
i_2(j_1,j_2) \in \{j \in [r] : \pi_j(v_{j_1}) = 1\} \cap \bigcap_{i=1}^{n_{a,2}} \{j \in [r] : \pi_j(v_i) = 0\} \cap \bigcap_{i \neq j_1} \{j \in [r] : \pi_j(w_i) = 0\}.
\]

In simple words, the last row of \( \text{Art}_{a,m,d} \) corresponds to the Artin pairing with \( l \). This is exactly the case \( j_1 = n_{a,m} + 1 \) in the above definition. To prove equidistribution of these entries, we will use our higher Rédei reciprocity law. It is for this reason that in this case our choice of variable indices is different than Smith’s choice [33, p. 32], while if \( j_1 \leq n_{a,m} \) we make exactly the same choice as Smith.

In case \( j_1 = n_{a,m} + 1 \), it will be essential that we restrict our variable indices to always be 1 modulo 4 and also satisfy \( a(i,j) = 1 \) for all distinct \( i, j \in S \). Indeed, this is needed to apply our higher Rédei reciprocity law. In case \( j_1 \neq n_{a,m} + 1 \) neither of these facts will be important, just as in Smith’s original work.

It is a non-trivial task to show that we can find variable indices for most \( a \). It is here that the material in Subsection 5.3 and the fourth condition in Definition 6.3 turn out to be crucial. To find our variable indices, we start with a lemma.

**Lemma 6.6.** Suppose that \( a \in \widehat{\text{Map}}(M \sqcup M_0, \{\pm 1\}, \tilde{j}, l) \) satisfies equation (6.3). Assume that \( v_1, \ldots, v_d, L \in D_{a,2} \) and \( v_{d+1}, \ldots, v_e, R \in D_{a,2}^\vee \) are linearly independent. Then we have for all \( \mathbf{v} \in \mathbb{F}_2^e \) the estimate

\[
\left| \left\{ i \in S_{\text{pre},4} : \pi_i(v_j) = \pi_j(\mathbf{v}) \text{ for all } 1 \leq j \leq e \right\} - \frac{\alpha_{\text{pre}}}{2e} \right| \leq \frac{100^e \cdot k_{\text{gap}}}{\log \log \log \log N}.
\]
**Proof.** This is a small adjustment of Lemma 13.7 in [19]. We stress that the term generic in [19] Lemma 13.7] is an unfortunate clash of terminology, and refers to a satisfying the natural analog of our equation (6.3).

We have a completely similar result for the range $k_{\text{gap}} < i \leq 2k_{\text{gap}}$ using equation (6.4). Let us now construct a graph $G$ associated to $a$. The set of vertices of $G$ is $S_{\text{pre},4}$. Furthermore, for distinct $i,j \in S_{\text{pre},4}$ there is an edge between $i$ and $j$ if and only if $a(i,j) = 1$. Note that this is independent of the order of $i$ and $j$, since $a(i,-1) = 1$ for all $i \in S_{\text{pre},4}$. Hence $G$ is an undirected graph. To find our variable indices, we certainly need to be able to find a clique in $G$ of size $m$. This will follow, for most $G$, from Lemma 5.5. The final index $i_2(j_1,j_2)$ is then found using Lemma [5.6]

**Definition 6.7.** We say that $a \in \text{Map}(M \sqcup M_\emptyset, \{\pm1\})$ is very $(N,m,X)$-acceptable if $a$ is $(N,m,X)$-acceptable, the graph $G$ associated to $a$ is not $(n/\log n,m)$-bad and furthermore the function $f : S_{\text{pre},4} \times S_{\text{post},4} \rightarrow F_2$ given by $f(s,t) = \nu(a(s,t))$ is such that for all pairwise distinct $v_1, \ldots, v_m \in S_{\text{pre},4}$ there is some $w \in S_{\text{post},4}$ such that for all $i \in [m]$ we have $f(v_i,w) = 0$.

This brings us to our next reduction step.

**Proposition 6.8.** Let $l$ be an integer such that $|l|$ is a prime 3 modulo 4. There are $c, A, N_0 > 0$ such that for all integers $N > N_0$, all integers $m \geq 2$, all sequences of integers $n_2 \geq \cdots \geq n_m \geq 0$, all $N$-good boxes $X$, all very $(N,m,X)$-acceptable $a \in \text{Map}(M \sqcup M_\emptyset, \{\pm1\})$, all sequences of valid Artin pairings $\{\text{Art}_{a,i}\}_{2 \leq i \leq m-1}$ with $n_{a,i} = n_i$ for $2 \leq i \leq m$ and an Artin pairing $\text{Art}_{a,m} : D_{a,m} \times D_{a,m}^\vee \rightarrow F_2$ with $R$ in the right kernel

$$|X(a, \{\text{Art}_{a,i}\}_{2 \leq i \leq m-1})| - 2^{-n_m(n_m+1)} \cdot |X(a, \{\text{Art}_{a,i}\}_{2 \leq i \leq m})| \leq \frac{A \cdot |X(a)|}{(\log \log \log \log N)^{m+n}}.$$

**Remark 4.** We do not need to assume that $\{\text{Art}_{a,i}\}_{2 \leq i \leq m-1}$ is valid, but it suffices for our purposes and avoids some casework later on.

**Proof that Proposition 6.8 implies Proposition 6.8.** We observe that $i(X(a)) \cap S_r(N,l) \neq \emptyset$ implies $a \in \text{Map}(M \sqcup M_\emptyset, \{\pm1\})$. Hence we can bound

$$\left| i(X) \cap \bigcap_{i=2}^{m+1} D_{i,i}(n_i) \right| - \frac{P(n_m, n_m, n_m+1)}{2^{n_m}}, \quad \left| i(X) \cap \bigcap_{i=2}^{m} D_{i,i}(n_i) \right| \leq \sum_{a \in \text{Map}(M \sqcup M_\emptyset, \{\pm1\})} \left| i(X(a)) \cap \bigcap_{i=2}^{m+1} D_{i,i}(n_i) \right| - \frac{P(n_m, n_m, n_m+1)}{2^{n_m}} \cdot \left| i(X(a)) \cap \bigcap_{i=2}^{m} D_{i,i}(n_i) \right|.$$

We split this sum over the very $(N,m,X)$-acceptable $a \in \text{Map}(M \sqcup M_\emptyset, \{\pm1\})$ and the remaining $a$. For the very $(N,m,X)$-acceptable $a$ we may apply Proposition 6.8 by further splitting the sum over all possible sequences of valid Artin pairings and an Artin pairing $D_{a,m} \times D_{a,m}^\vee \rightarrow F_2$ with $R$ in the right kernel.

Note that in the set of bilinear pairings $D_{a,m} \times D_{a,m}^\vee \rightarrow F_2$ with $R$ in the right kernel, there are precisely

$$2^{n_m(n_m+1)} \cdot \frac{P(n_m, n_m, n_m+1)}{2^{n_m}}.$$
such that the left kernel has dimension \( n_{m+1} + 1 \) and \( L \) is in the left kernel. There are at most \( 2^m n_{\max}^2 \) sequences of Artin pairings, so we stay within the error term of Proposition 6.2 provided that we take the constant \( c' \) in the definition of \( n_{\max} \) smaller than the constant \( c \) guaranteed by Proposition 6.8.

Hence it suffices to bound

\[
\sum_{a \in \text{Map}(M \sqcup M_{\emptyset}, \{\pm 1\}, \bar{j}, \bar{l})} |i(X(a))|.
\]

We first tackle those \( a \) for which \( n_{a,2} > n_{\max} \). These \( a \) can easily be dealt with using equation (5.11) for \( k \leq n_{\max} \) inducing an error of size

\[
O \left( \frac{|i(X) \cap S_v(N \times \bar{l})|}{(\log \log \log \log N)^{\max_k c}} \right)
\]

for some absolute constant \( c > 0 \).

We will now dispatch those \( a \) that fail equation (5.1). We declare two maps \( a, a' \in \text{Map}(M \sqcup M_{\emptyset}, \{\pm 1\}, \bar{j}, \bar{l}) \) to be equivalent at some integer \( i > k \), written as \( a \sim_i a' \) if

\[
a(j, i) = a'(j, i) \text{ for all } 1 \leq j \leq k \text{ and } a(i, -1) = a'(i, -1).
\]

Observe that if \( a \) fails equation (5.1), then so does any \( a' \) with \( a \sim_i a' \). We call an equivalence class bad if there exists some \( a \in \) the equivalence classes failing equation (5.1). In a given bad equivalence class we clearly have the bound

\[
\left| \bigcup_{a' \sim_i a'} X(a') \right| \leq \frac{|X|}{(\log \ell_{k+1})^{100}}.
\]

A simple computation shows that we stay within the error term of Proposition 6.2 when we sum over all \( i \) and all bad equivalence classes.

We still have to deal with those \( a \) failing equation (5.2), equation (5.3) or equation (5.4). To do so, we will use the ideas introduced in [32, p. 25-26]. Call a generic if \( D_{a,2} \cap D'_{a,2} = \emptyset \), where we view \( D_{a,2} \) and \( D'_{a,2} \) as subspaces of \( V' \). Let us now suppose that \( r = r'(a, X) \), the other case can be dealt with similarly. Take a non-zero vector \( v \in \mathbb{F}_2^t \) with \( \lambda \) ones with \( v \neq L \) and \( v \neq R \). We claim that

\[
\frac{|\{a \in \text{Map}(M \sqcup M_{\emptyset}, \{\pm 1\}, \bar{j}, \bar{l}) : v \in D_{a,2} \cap D'_{a,2}, r = r'(a, X)\}|}{|\{a \in \text{Map}(M \sqcup M_{\emptyset}, \{\pm 1\}, \bar{j}, \bar{l}) : r = r'(a, X)\}|} = O(2^{-r-\lambda}).
\]

We have that the proportion of \( a \) with \( v \in D_{a,2} \) is equal to \( O(2^{-r}) \). Furthermore, the condition that also \( v \in D'_{a,2} \) implies that for every \( i \) with \( \pi_i(v) = 1 \) we have \( a(i, -1) = 1 \). These are \( O(2^{-\lambda}) \) independent extra conditions giving a total of \( O(2^{-r-\lambda}) \). This establishes the claim. For the case that \( v = L \) or \( v = R \), we make fundamental use of the fact that \( |\ell| \) is equivalent to 3 modulo 4 to show that the above proportion is still \( O(2^{-r}) \).

Summing over all non-zero vectors \( v \in V \) then gives that the proportion of \( a \in \text{Map}(M \sqcup M_{\emptyset}, \{\pm 1\}, \bar{j}, \bar{l}) \), which are not generic, is bounded by

\[
O \left( \sum_{\lambda=1}^{r} 2^{-r-\lambda} \binom{r}{\lambda} \right) = O \left( 0.75^r \right).
\]
Take some \( v, w \in V \). Recall that the proportion of \( a \) with \( v \in D_{a,2} \) is bounded by \( O(2^{-r}) \) provided that \( v \not\in (L) \). Similarly, the proportion of \( a \) with \( w \in D'_{a,2} \) is bounded by \( O(2^{-r}) \) if \( w \not\in (R) \). Finally, if \( a \) is generic, the proportion of \( a \) with \( (v, w) \in D_{a,2} \times D'_{a,2} \) is bounded by \( O(4^{-r}) \) as long as \( v \not\in (L) \) and \( w \not\in (R) \).

But Hoeffding’s inequality yields that the proportion of \( (v, w) \in V \times V \) satisfying

\[
\left| \left\{ i \in S_{\text{pre},4} : \pi_i(v + w) = 0 \right\} - \frac{\alpha_{\text{pre}}}{2} \right| > \frac{\alpha_{\text{pre}}}{\log \log \log \log N}
\]

is at most

\[
O\left( \exp \left( - (\log \log \log N)^{-2} \cdot \alpha_{\text{pre}} \right) \right).
\]

From the last two observations we quickly deduce that the proportion of generic \( a \) for which equation (6.3) fails is also bounded by

\[
O\left( \exp \left( - (\log \log \log N)^{-2} \cdot \alpha_{\text{pre}} \right) \right),
\]

and a similar argument applies for the proportion of \( a \) failing equation (6.4). For the proportion of \( a \) failing equation (6.2), it is even easier to get an upper bound.

We have now found an upper bound for the proportion of \( a \) failing equation (6.2), equation (6.3), or equation (6.4). From Lemma 6.5 and Lemma 5.6 we directly deduce an upper bound for the proportion of \( a \) that are \((N, m, X)\)-acceptable but not very \((N, m, X)\)-acceptable. To finish the proof, we merely need to bound the union of \( X(a) \) over these \( a \). This follows from Proposition 5.10.

We remark that we can always find variable indices as in Definition 6.5 if \( a \) is very \((N, m, X)\)-acceptable and \( N \) is sufficiently large. This is a simple computation once we use that

\[
m < \log \log \log \log \log N,
\]

since otherwise Theorem 6.1 is trivial. We now have all the required setup for our next proposition, where we fix one prime for all indices smaller than \( k_{\text{gap}} \) except the variable indices.

**Proposition 6.9.** Let \( l \) be an integer such that \(|l|\) is a prime 3 modulo 4. There are \( c, A, N_0 > 0 \) such that for all integers \( N > N_0 \), all integers \( m \geq 2 \), all \( N \)-good boxes \( X \), all very \((N, m, X)\)-acceptable \( a \in \text{Map}(M \cup M_0, \{\pm 1\}) \), all sequences of valid Artin pairings \( \{\text{Art}_{a,i}\}_{2 \leq i \leq m-1} \), all non-zero multiplicative characters \( F : \text{Mat}(n_{a,m} + 1, n_{a,m}, F_2) \to \{\pm 1\} \), all sets of variable indices \( S \) for \( F \) and all \( Q \in \prod_{i \in [k_{\text{gap}}] - S} X_i \) such that

\[
|X_j(a, Q)| \geq 4^{-k_{\text{gap}}} \cdot |X_j|
\]

for all \( j \in S \), we have

\[
\left| \sum_{x \in X(a, Q, \{\text{Art}_{a,i}\}_{2 \leq i \leq m-1})} F(\text{Art}_{a,m,j}(x)) \right| \leq \frac{A \cdot |X(a, Q)|}{(\log \log \log \log N)^{\frac{1}{m+1}}}.
\]

Here \( X(a, Q, \{\text{Art}_{a,i}\}_{2 \leq i \leq m-1}) \) is defined as the subset of \( x \in X(a, \{\text{Art}_{a,i}\}_{2 \leq i \leq m-1}) \) with \( \pi_i(x) \) equal to \( \pi_i(Q) \) for \( i \in [k_{\text{gap}}] - S \).
Proof that Proposition 6.9 implies Proposition 6.8. The proof is almost identical to the one given in [2]. Proof that Proposition 6.10 implies Proposition 6.6. We have to show that equation (6.6) is typically satisfied. We apply Proposition 5.9 to
\[(X_{k+1}(a, Q') \times \cdots \times X_r(a, Q'), Q'),\]
where \(Q'\) is the unique element of \(X_1 \times \cdots \times X_k\). Crucially, all the required conditions for Proposition 5.9 are satisfied due to equation (6.1), completing our reduction step.

It is time for our final reduction step. If \(c_{j_1,j_2}(F) \neq 0\) for some \(1 \leq j_1, j_2 \leq n_{a,m}\), Smith’s method applies without any significant changes. If however \(c_{j_1,j_2}(F) = 0\) for all \(1 \leq j_1, j_2 \leq n_{a,m}\), Smith’s method breaks down. It is here that we make essential use of our generalized Rödèi reciprocity law.

We shall now add the algebraic structure needed to apply our reflection principles. The required equidistribution will then be a consequence of the Chebotarev Density Theorem and Proposition 5.2. From now on we shall make heavy use of the notation introduced in Subsections 4.1 and 4.3.

**Definition 6.10.** Take a \(N\)-good box \(X\), a very \((N, m, X)\)-acceptable \(a \in \text{Map}(M \cup M_{\emptyset}, \{\pm 1\})\) and a non-zero multiplicative character \(F : \text{Mat}(n_{a,m} + 1, n_{a,m}, \mathbb{F}_2) \to \{\pm 1\}\). Let \(S\) be a set of variable indices for \(F\). Fix a choice of \(j_1\) and \(j_2\) with \(c_{j_1,j_2}(F) \neq 0\) as in Definition 6.5.

Put \(S' := S \cap [k_{\text{gap}}]\). For each \(i \in S'\), let \(Z_i\) be subsets of \(X_i\) with cardinality
\[M_{\text{box}} := \left\lfloor (\log \log \log N)^{\frac{1}{n(n+1)}} \right\rfloor.\]

Note that \(M_{\text{box}} \geq 2\) for \(N\) greater than an absolute constant by equation (6.2). Put
\[Z := \prod_{i \in S'} Z_i, \quad Z' := \prod_{i \in S' \setminus \{i_1(j_1,j_2)\}} Z_i.\]

If \(j_1 \leq n_{a,m}\), we say that \(Z\) is well-governed for \((a, F)\) if for every distinct \(a_1, a_2 \in Z_{i_1(j_1,j_2)}\) there is a governing expansion \(\mathcal{G}\) on \((Z', S' \setminus \{i_1(j_1,j_2)\}, a_1a_2)\). Put
\[M_0(Z) := \prod_{a_1, a_2 \in Z_{i_1(j_1,j_2)}} \prod_{a_1 \neq a_2} L(\phi_{x,a_1a_2})\]
and
\[M(Z) := \prod_{a_1, a_2 \in Z_{i_1(j_1,j_2)}} \prod_{a_1 \neq a_2} L(\phi_{x,a_1a_2}).\]

If \(j_1 = n_{a,m} + 1\), we say that \(Z\) is well-governed for \((a, F)\) if there is a governing expansion \(\mathcal{G}_j\) on \((Z, S', l)\) and furthermore for each \(j \in S'\) and each \(q \in X_j\) there is a governing expansion \(\mathcal{G}_{j,q}\) on \((\prod_{i \in S' \setminus \{j\}} X_i, S' \setminus \{j\}, q)\). Put
\[M_0(Z) := \prod_{T \subseteq S'} \prod_{x \in X_T} \prod_{j \in S' \setminus \{j\}} \prod_{q \in X_j} L(\phi_{x,q})\]
and
\[M(Z) := M_0(Z) \prod_{x \in X_{S'}} L(\phi_{x,l}).\]
so $M(Z)$ is a central Galois extension of $M_0(Z)$ in both cases.

Take some $Q \in \prod_{i \in [k_{\text{gap}}] - S} X_i$. Then we define, for $i > k_{\text{gap}}$, $X_i(a, Q, M_0(Z))$ to be the subset of primes $p \in X_i$ such that $p$ splits completely in $M_0(Z)$, $p \in X_i(a, Q)$ and

$$\left( \frac{z}{p} \right) = a(j, i)$$

for all $j \in S'$ and all $z \in Z_j$.

Note that these conditions are equivalent to $\text{Frob}_p$ being equal to a given central element in the Galois group of the compositum of $M_0(Z)$ and $\mathbb{Q} (\sqrt{x})$ with $x$ running through $-1$, the prime divisors of $Q$ and the primes in $Z_j$ for $j \in S'$.

We let

$$\tilde{Z} := Q \times Z \times \prod_{i > k_{\text{gap}}} X_i(a, Q, M_0(Z)).$$

We call $\tilde{Z}$ a satisfactory product space for $(X, a, F, Q)$ if $Z$ is well-governed for $(a, F)$, the primes in $Q$ split completely in $M_0(Z)$ and if we have for all $i < j$ with $i, j \in S'$, all $z_i \in Z_i$ and all $z_j \in Z_j$

$$\left( \frac{z_i}{z_j} \right) = a(i, j) = 1$$

and $Z_i \subseteq X_i(a, Q)$.

Once we added the necessary algebraic structure to our box, we can construct a suitable additive system $\mathfrak{A}$ to which we apply Proposition 5.2. This is the goal of the next lemma, which provides the critical link between our algebraic results and Proposition 5.2.

**Lemma 6.11.** Let a very $(N, m, X)$-acceptable $a \in \text{Map}(M \cup M_0, \{ \pm 1 \})$, a sequence of valid Artin pairings $\{ \text{Art}_{a, i} \}_{2 \leq i \leq m - 1}$, a non-zero multiplicative character $F : \text{Mat}(n_{a, m} + 1, n_{a, m}, \mathbb{F}_2) \to \{ \pm 1 \}$ and a set of variable indices $S$ for $F$ be given. Take $\tilde{Z}$ to be a satisfactory product space for $(X, a, F, Q)$. Then there is a $(2^{n_{\text{max}} (n_{\text{max}} + m + 2)}, S)$-acceptable additive system $\mathfrak{A}$ with $\nabla_0^\circ(\mathfrak{A}) = \tilde{Z} \cap X(a, \{ \text{Art}_{a, i} \}_{2 \leq i \leq m - 1})$ such that

$$\sum_{\bar{x} \in C(\mathfrak{A})} \iota(F(\text{Art}_{a, m, z})) = \phi_{\pi_S - (i_1(j_1, j_2))}(\bar{x}) \cdot (\text{Frob}(p_1) \cdot \text{Frob}(p_2))$$

(6.7)

for all $\bar{x} \in C(\mathfrak{A})$, where $(p_1, p_2) := \pi_{i_2(j_1, j_2)}(\bar{x})$. Here $c$ equals $\text{pr}_1(\pi_{i_1(j_1, j_2)}(\bar{x})) \cdot \text{pr}_2(\pi_{i_1(j_1, j_2)}(\bar{x}))$ if $j_1 \leq n_{a, m}$ and equals $1$ otherwise.

**Proof.** We shall proceed to explicitly construct $\mathfrak{A}$ by induction. We start by introducing some notation. Let $w \in D_{a, 2}^\vee$ be one of the chosen basis vectors and let $x \in X(a)$ be given. A raw cocycle for $(x, w)$ is a sequence $\{ \psi_{x, w, i} \}_{i = 0}^k$ of maximal length with $\psi_{x, w, i} \in \text{Cocyc}(G_Q, N(x)[2^i])$, $2\psi_{x, w, i+1} = \psi_{x, w, i}$, $L(\psi_{x, w, i})/\mathbb{Q}(\sqrt{f})$ unramified and

$$\psi_{x, w, 1} = \sum_{i=1}^{r'(a, X)} \pi_i(w) \chi_i$$

with $\chi_i$ as in Definition 5.11. A raw cocycle $\mathfrak{R}(w)$ for $(x, a, w)$ is a choice of raw cocycle for every $(x, w)$ with $x \in X(a)$. Recall that $i_1(j_1, j_2)$, $i_2(j_1, j_2)$, $j_1$ and $j_2$ are the integers associated to our set of variable indices $S$ as in Definition 5.5. Set

$$\nabla_0^\circ(\mathfrak{A}) := \tilde{Z} \cap X(a, \{ \text{Art}_{a, i} \}_{2 \leq i \leq m - 1}).$$
If \( j_1 = n_{a,m} + 1 \), we claim that there is a governing expansion \( \mathcal{G}' \) on \((Z, S', q_1 q_2)\) for any two distinct elements \( q_1, q_2 \in X_{i_2(j_1, j_2)}(a, Q, M_0(Z))\). To prove the claim, take any prime \( q \) in \( X_{i_2(j_1, j_2)}(a, Q, M_0(Z))\). We shall prove by induction on \( T \subseteq S' \) that there is a governing expansion \( \mathcal{G}'(T) \) on

\[
\prod_{j \in T} Z_i(T, q)\]

Once this is proven, the claim follows upon taking \( T = S' \) and using equation (4.2).

In the base case \( T = \emptyset \) our inductive statement is clear. Now suppose that \(|T| > 0\). By Proposition 4.2 and induction, it suffices to show that for every \( i \in T \), every \( p \in \mathbb{Z}_i \) and every \( \bar{\lambda} \in \mathbb{X}_{T-i} \), we have that \( p \) splits completely in \( L(\phi_{x,q}) \). We have already checked, earlier in our induction, that \( p \) splits completely in

\[
\prod_{j \in T-i} L(\phi_{\sigma_{T-j}(\bar{\lambda}),q}).
\]

By Theorem 3.3 the splitting of \( p \) in \( L(\phi_{x,q}) \) is equivalent to \( q \) splitting completely in \( L(\phi_{x,p}) \), which is ensured by the fact that \( q \in X_{i_2(j_1, j_2)}(a, Q, M_0(Z)) \). Note that \( p \) and \( q \) are 1 modulo 4, so that the sets \( \text{Ram}(Q(\chi_p)/Q) \) and \( \text{Ram}(Q(\chi_q)/Q) \) are disjoint. This establishes the claim.

Let \( K \) be the field obtained by adjoining \( \sqrt[i]{q} \) and all the \( \sqrt{z_i} \) and \( \sqrt{q} \) to \( Q \), where \( z_i \in \mathbb{Z}_i \) for some \( i \in S' \) and \( q \in X_{i_2(j_1, j_2)}(a, Q, M_0(Z)) \). Take \( M \) to be the narrow Hilbert class field of \( K \). For each prime \( p \) ramifying in \( M \), any inertia subgroup at \( p \) has size 2 and hence precisely one non-trivial element. Choose such an element \( \sigma_p \) for every \( p \) that ramifies in \( M \).

First suppose that \( j_1 \leq n_{a,m} \). To shorten our formulas, we define for \( \bar{\lambda} \in \mathbb{X}_S \) and \( i \in S \)

\[
\text{prp}(\bar{\lambda}, i) = \text{pr}_1(\pi_i(\bar{\lambda})) \cdot \text{pr}_2(\pi_i(\bar{\lambda})).
\]

Then we can always choose our maps \( \phi_{x,\text{prp}(\bar{\lambda}, i_1(j_1, j_2))} : G_Q \to \mathbb{F}_2 \) in such a way that \( \phi_{x,\text{prp}(\bar{\lambda}, i_1(j_1, j_2))}(\sigma_p) = 0 \) for all \( p \), where \( \bar{\lambda} \) runs over all elements of \( \mathbb{Z}_T \) with \( T \subseteq S' \) not containing \( i_1(j_1, j_2) \). Proposition 2.3 of Smith [33] shows that with this choice the \( \phi \) maps are additive, in other words equation (4.2) holds. Let \( T \subseteq S \). We shall construct our maps \( F_{T'} \) with \( T' \subseteq T \) in such a way that \( \mathbb{Y}_{T'}(\mathfrak{A}) \) is precisely the set of cubes \( \bar{\lambda} \) satisfying \( \bar{\lambda}(0) \subseteq \mathbb{Y}_{T'}(\mathfrak{A}) \) and the following properties

- we have for all \( T' \subseteq T \) with \( i_2(j_1, j_2) \notin T' \), all \( \bar{\lambda} \in \bar{\lambda}(T') \) and all \( j \neq j_2 \)

\[
\sum_{y \in \bar{\lambda}(T')} \psi_{y, w_j, |T'|} = 0;
\]

- we have for all \( T' \subseteq T \) with \( i_2(j_1, j_2) \notin T' \) and all \( \bar{\lambda} \in \bar{\lambda}(T') \)

\[
\sum_{y \in \bar{\lambda}(T')} \psi_{y, w_2, |T'|} = \begin{cases} 
\phi_{\sigma_{T'-\{i_1(j_1, j_2)\}}}(\bar{\lambda}, \text{prp}(\bar{\lambda}, i_1(j_1, j_2))) & \text{if } i_1(j_1, j_2) \in T' \\
0 & \text{if } i_1(j_1, j_2) \notin T';
\end{cases}
\]

- we have for all \( T' \subseteq T \) with \( i_2(j_1, j_2) \notin T' \), \( \bar{\lambda} \in \bar{\lambda}(T') \), all \( j \) and \( i \in S - T' \)

\[
\sum_{y \in \bar{\lambda}(T')} \psi_{y, w_j, |T'|+1(\sigma_{\pi_i(\bar{\lambda})})} = 0.
\]
Now suppose that $j_1 = n_{a,m} + 1$. We now choose our maps $\phi_{x,q_1,q_2} : G_Q \to F_2$ in such a way that $\phi_{x,q_1,q_2}(\sigma_p) = 0$ for all $p$, for all $x \in \mathbb{Z}_T$ with $T \subseteq S'$ and all $q_1, q_2 \in X_{i_2(j_1,j_2)}(a, Q, M_0(Z))$. Let $T \subseteq S$. In this case we construct our maps $F_T(\mathfrak{A})$ such that $\mathcal{Y}_T(\mathfrak{A})$ equals the cubes $\bar{x}$ with $\bar{x}(\emptyset) \subseteq \mathcal{Y}_0(\mathfrak{A})$ and

- we have for all $T' \subseteq T$, all $\bar{y} \in \bar{x}(T')$ and all $j \neq j_2$

\[
\sum_{y \in \bar{y}(\emptyset)} \psi_{y,w_j,|T'|} = 0;
\]

- we have for all $T' \subseteq T$ and all $\bar{y} \in \bar{x}(T')$

\[
\sum_{y \in \bar{y}(\emptyset)} \psi_{y,w_{j_2},|T'|} = \begin{cases} 
\phi_{\pi_{T'-(i_2(j_1,j_2))}(\bar{y}),\mathrm{pp}(\bar{x},i_2(j_1,j_2))} & \text{if } i_2(j_1,j_2) \in T' \\
0 & \text{if } i_2(j_1,j_2) \notin T'; 
\end{cases}
\]

- we have for all $T' \subseteq T$, $\bar{y} \in \bar{x}(T')$, all $j$ and $i \in S - T'$

\[
\sum_{y \in \bar{y}(\emptyset)} \psi_{y,w_j,|T'|+1}(\sigma_{\pi_i(\bar{x})}) = 0.
\]

Let us prove by induction that $\mathcal{Y}_T(\mathfrak{A})$ is as claimed. We shall construct the map $F_T(\mathfrak{A})$ during the induction. Until otherwise stated, we shall treat the case $j_1 \leq n_{a,m}$. At the end we indicate the modifications necessary to deal with the case $j_1 = n_{a,m} + 1$. Take $\bar{x} \in \mathcal{Y}_T(\mathfrak{A})$.

If $i_2(j_1,j_2) \in T$ or $T = S - \{i_2(j_1,j_2)\}$, we simply let $F_T$ be the zero map. Henceforth we will assume that $i_2(j_1,j_2) \notin T$ and $|T| < |S| - 1$. Then we define

\[
\psi_j := \begin{cases} 
\sum_{x \in \mathcal{Y}(\emptyset)} \psi_{x,w_j,|T|} & \text{if } j \neq j_2 \text{ or } i_1(j_1,j_2) \notin T \\
\phi_{\pi_{T-\{i_1(j_1,j_2)\}}(\bar{x}),\mathrm{pp}(\bar{x},i_1(j_1,j_2))} + \sum_{x \in \mathcal{Y}(\emptyset)} \psi_{x,w_{j_2},|T|} & \text{otherwise}. 
\end{cases}
\]

In the former case Proposition 2.5 of Smith [33] implies that $\psi_j$ is a quadratic character of $G_Q$, while in the latter case Proposition 2.6 of Smith [33] demonstrates that $\psi_j$ is a quadratic character. Take a point $x \in \bar{x}(\emptyset)$. We claim that $\psi_j$ is an unramified character of $\mathbb{Q}(\sqrt{x})$.

If $p = \pi_i(\bar{x})$ with $i \not\in T$, this is clear. So suppose that $i \in T$ and write $\pi_i(\bar{x}) = \{p_1,p_2\}$ with $p_1 = \pi_i(x)$. It is clear that $\psi_j$ does not ramify at $p_1$, so it suffices to show that $\psi_j$ does not ramify at $p_2$. Let $\bar{y}_k \in \bar{x}(T - \{i\})$ be the cube with $\pi_i(\bar{y}_k) = p_k$. Then we have

\[
\psi_j(\sigma_{p_2}) = \sum_{x \in \mathcal{Y}(\emptyset)} \psi_{x,w_j,|T|}(\sigma_{p_2}) = \sum_{y \in \mathcal{Y}(\emptyset)} \psi_{y,w_j,|T|}(\sigma_{p_2}) + \sum_{y \in \mathcal{Y}(\emptyset)} \psi_{y,w_j,|T|}(\sigma_{p_2}) = 0 + 0 = 0.
\]

The first sum is clearly zero, since all the $\psi_{y,w_j,|T|}$ with $y \notin \mathcal{Y}(\emptyset)$ are unramified at $p_2$. Furthermore, the second sum is zero by equation (68) with $T' := T - \{i\}$. This proves our claim.

Next we claim that $\pi_i(\bar{x})$ splits completely in $L(\psi_j)$ for all $i \not\in T$. But indeed, we even have that $\pi_i(\bar{x})$ has residue field degree 1 in every $\psi_{x,w_j,|T|}$ for $x \in \bar{x}(\emptyset)$ because $2\psi_{x,w_j,|T|+1} = \psi_{x,w_j,|T|}$. Pick some $x \in \bar{x}(\emptyset)$ and let $p \in \pi_i(x)$ for some $i \in T$. It is straightforward to deduce from $\bar{x}(\emptyset) \subseteq X(a)$ that

\[
\psi_j|_{G_{\mathbb{Q}(\sqrt{x})}}(\mathrm{Frob}(p))
\]

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Finally, we define an additive map \( F_{T,j,1} \) to \( \mathbb{F}_2^{|T|} \).

It follows from Lemma 6.6 that there exists a set \( A \subseteq [r] \) and a bijection \( f : [n_{a,2} + 1] \to A \) such that \( A \cap S = \emptyset \) and

\[
\pi_{f(i)}(w_k) = \delta_{i,k}
\]

for \( 1 \leq i, k \leq n_{a,2} \) and furthermore

\[
\pi_{f(n_{a,2} + 1)}(w_k) = 0
\]

for all \( 1 \leq k \leq n_{a,2} \). Then we define an additive map \( F_{T,j,2} \) to \( \mathbb{F}_2^{n_{a,2} + 1} \) by

\[
(\psi_j(\sigma_i(x)))_{i \in A}.
\]

Finally, we define an additive map \( F_{T,j,3} \) to \( \mathbb{F}_2^{|S| - |T|} \) by sending \( \bar{x} \) to

\[
\left( \sum_{x \in \bar{x}(\emptyset)} \psi_{x,w_j,|T| + 1}(\sigma_{\pi_i(\bar{x})}) \right)_{i \in S - T}.
\]

We define our map \( F_T(\mathfrak{A}) \) to be \( (F_{T,j,1}, F_{T,j,2}, F_{T,j,3})_{1 \leq j \leq n_{a,2}} \). Note that the maps \( F_{T,j,1} \) and \( F_{T,j,2} \) encode precisely when \( \psi_j = 0 \). From this it becomes obvious that \( \nabla_T(\mathfrak{A}) \) has the claimed shape.

Our next task is to verify that our additive system is \( (2^{n_{max}(n_{max} + m + 2), S}) \)-acceptable. For the first requirement, this is obvious from the construction of \( F_T \) above and the inequality \( n_{a,2} \leq n_{max} \). We still need to deal with the second requirement. Take \( \bar{x} \in C(\mathfrak{A}) \). If there is some \( i \in S \) such that

\[
|\bar{x}(S - \{i\}) \cap \nabla_{S - \{i\}}(\mathfrak{A})| = 2,
\]

then we are done. Henceforth we assume that

\[
|\bar{x}(S - \{i\}) \cap \nabla_{S - \{i\}}(\mathfrak{A})| = 1
\]

for all \( i \in S \) and let \( x_0 \) be the unique element in \( \bar{x}(\emptyset) \) outside \( \bar{x}(S - \{i\}) \cap \nabla_{S - \{i\}}(\mathfrak{A}) \) for all \( i \in S \). Then we need to prove that \( x_0 \in \bar{Y}_0(\mathfrak{A}) \). Clearly, \( x_0 \in \bar{Z} \cap X(a) \). Take an integer \( 2 \leq m' \leq m - 1 \), integers \( 1 \leq j'_1 \leq n_{a,m'} + 1 \) and \( 1 \leq j'_2 \leq n_{a,m'} \). It suffices to prove that

\[
i(E_{j'_1,j'_2}(\text{Art}_{a,m',x_0})) = i(E_{j'_1,j'_2}(\text{Art}_{a,m'})).
\]

Choose a subset \( T \) of \( S \) of size \( m' \) not containing \( i_1(j_1,j_2) \) and \( i_2(j_1,j_2) \). Then the above identity follows from Theorem 4.9 applied to any cube in \( \bar{x}(T) \) containing \( x_0 \).

We still need to prove equation (6.17). Recall that \( j_1 \leq n_{a,m} \). Take some indices \( (j_3,j_4) \) with \( (j_3,j_4) \neq (j_1,j_2) \). We claim that

\[
\sum_{x \in \bar{x}(\emptyset)} i(E_{j_3,j_4}(\text{Art}_{a,m,x})) = 0.
\]

First suppose that \( j_3 \leq n_{a,m} \). Then this follows from two applications of Theorem 4.9. In case \( j_3 = n_{a,m} + 1 \) we apply Theorem 4.11 twice to obtain

\[
\sum_{x \in \bar{x}(\emptyset)} i(E_{j_3,j_4}(\text{Art}_{a,m,x})) = 0.
\]
Here we use equation (4.7), if \( l > 0 \), and equation (4.8), if \( l < 0 \). We deduce that
\[
\sum_{x \in \mathcal{I}(0)} \iota(E_{j_1, j_2}(\text{Art}_{a,m,x})) = \phi_{\pi_{S'}}(\bar{x}), \text{prp}(\bar{x}, i_1(j_1, j_2))(Frob(p_1) \cdot Frob(p_2)).
\]
Adding these identities together yields equation (6.7). This proves the lemma for \( j_1 \leq n_{a,m} \).

It remains to indicate the necessary changes in case \( j_1 = n_{a,m} + 1 \). In this case we let \( F_T \) be the zero map if \( T = S \). Otherwise we define
\[
\psi_j := \begin{cases} 
\sum_{x \in \mathcal{I}(0)} \psi(x, w_j, [T]) & \text{if } j \neq j_2 \text{ or } i_2(j_1, j_2) \not\in T \\
\phi_{\pi_{T - (i_2(j_1, j_2))}}(\bar{x}), \text{prp}(\bar{x}, i_2(j_1, j_2)) + \sum_{x \in \mathcal{I}(0)} \psi(x, w_j, [T]) & \text{otherwise.}
\end{cases}
\]
Now we proceed by defining the maps \( F_T, j, i \) just as in the case \( j_1 \leq n_{a,m} \). Then we see that \( \mathfrak{A} \) is certainly \( (2^{n_{\max}(n_{a,m}+m+2)}, S) \)-acceptable. Now we have for all \( (j_3, j_4) \) with \( j_3 \leq n_{a,m} \)
\[
\sum_{x \in \mathcal{I}(0)} \iota(E_{j_3, j_4}(\text{Art}_{a,m,x})) = 0
\]
simply because \( c_{j_3, j_4}(F) = 0 \) by our choice of variable indices. Furthermore, Theorem 4.9 shows that for all \( (j_3, j_4) \) with \( j_3 = n_{a,m+1} \) and \( j_2 \neq j_4 \)
\[
\sum_{x \in \mathcal{I}(0)} \iota(E_{j_3, j_4}(\text{Art}_{a,m,x})) = 0.
\]
Finally, Theorem 4.11 and Theorem 3.3 imply that
\[
\sum_{x \in \mathcal{I}(0)} \iota(E_{j_1, j_2}(\text{Art}_{a,m,x})) = \phi_{\pi_{S'}}(\bar{x}), \text{prp}(\bar{x}, i_2(j_1, j_2))(Frob(l)) = \phi_{\pi_{S'}}(\bar{x}), \iota(Frob(p_1) \cdot Frob(p_2))
\]
with \( (p_1, p_2) := \pi_{i_2(j_1, j_2)}(\bar{x}) \). Here \( \text{Frob}(l) \) is to be interpreted as \( \text{Frob}(|l|) \cdot \text{Frob}(\infty) \) if \( l < 0 \). Hence we conclude that
\[
\sum_{x \in \mathcal{I}(0)} \iota(F(\text{Art}_{a,m,x})) = \phi_{\pi_{S'}}(\bar{x}), \iota(Frob(p_1) \cdot Frob(p_2)),
\]
which completes the proof of our lemma because \( i_1(j_1, j_2) = j \not\in S' \) in this case. \( \Box \)

**Proposition 6.12.** Let \( l \) be an integer such that \( |l| \) is a prime 3 modulo 4. There are \( c, A, N_0 > 0 \) such that for all integers \( N > N_0 \), all integers \( m \geq 2 \), all \( N \)-good boxes \( X \), all very \( (N, m, X) \)-acceptable \( a \in \text{Map}(M \sqcup M_0, \{\pm 1\}) \), all sequences of valid Artin pairings \( \{\text{Art}_{a,i}\}_{2 \leq i \leq m-1} \), all non-zero multiplicative characters \( F : \text{Mat}(n_{a,m}+1, n_{a,m}; \mathbb{F}_2) \to \{\pm 1\} \), all sets of variable indices \( S \) for \( F \), all \( Q \in \prod_{i \in [k_{\text{sep}}]-S} X_i \) and all satisfactory product spaces \( \tilde{Z} \) for \( (X, a, F, Q) \)
\[
\sum_{x \in \tilde{Z} \cap X(a, Q) \cap \{\text{Art}_{a,i}\}_{2 \leq i \leq m-1}} |F(\text{Art}_{a,m,i}(x))| \leq \frac{A \cdot |\tilde{Z} \cap X(a, Q)|}{(\log \log \log \log N)^{m-1}}.
\]
Proof that Proposition 6.12 implies Proposition 6.9. The proof is very similar to the proof of Proposition 7.5 implies Proposition 7.4 in Smith [33]. We only indicate the necessary changes here. There is a small gap in Smith’s argument, namely when he applies the Chebotarev Density Theorem on page 81. Indeed, Smith does not argue why there are no Siegel zeroes. Fortunately, this can be easily overcome by an appeal to the classical result of Heilbronn [17] and the fact that our box $X$ is Siegel-less.

We need to construct an additive system $\mathfrak{A}'$ on $S'$ that guarantees the existence of the governing expansions $\mathfrak{G}_1$ and $\mathfrak{G}_{j,q}$. This is done in Lemma 4.5 and Proposition 3.3 of Smith [33], but it is essential here that $a(i, -1) = 1$ for all $i \in S'$ and $a(i, j) = 1$ for all distinct $i, j \in S'$ to ensure the validity of equation (4.14).

Now let $Z$ and $Z'$ be well-governed for $(a, F)$ and suppose that $Z \cap Z' = \{x\}$. Let $K$ be the field obtained by adjoining $\sqrt[p]{p}$ to $\mathbb{Q}$ where $p$ runs over all the prime divisors of $x$. Then, for Smith’s reduction step to work, we need to prove that

$$[KM_o(Z)M_o(Z') : K] = [KM_o(Z) : K]^2 = [KM_o(Z') : K]^2,$$

which follows from Proposition 2.4 of Smith [33].

Proof of Proposition 6.12. This proof is similar to the proof of Proposition 7.5 in Smith [33] except that one needs to use the additive system constructed in Lemma 6.11 instead of the one constructed in Section 3 of Smith [33]. We will now give all the details.

Take $\sigma \in \text{Gal}(M(Z)/M_o(Z))$ and define

$$X_{i_2(j_1, j_2)}(a, Q, M_o(Z), \sigma)$$

to be the subset of $p \in X_{i_2(j_1, j_2)}(a, Q, M_o(Z))$ that map to $\sigma$ under Frobenius. By [33, Proposition 2.4] we have an isomorphism

$$\text{Gal}(M(Z)/M_o(Z)) \cong \mathfrak{G}_S(Z)$$

(6.9)

by sending $\sigma$ to the map

$$\bar{\sigma} \mapsto \begin{cases} \phi_{\mathfrak{A}, l}(\sigma) & \text{if } j_1 = n_{a,m} + 1 \\ \phi_{\pi^S_-(i_1(j_1, j_2)), p}(\bar{\sigma}) & \text{otherwise.} \end{cases}$$

The Chebotarev Density Theorem and Lemma 5.1 imply that

$$|X_{i_2(j_1, j_2)}(a, Q, M_o(Z), \sigma)| = \frac{|X_{i_2(j_1, j_2)}(a, Q, M_o(Z))|}{2(M_{\text{top}} - 1)^{|S|}} \cdot \left(1 + O(e^{-2k_{\text{gap}}})\right).$$

Then it follows from Proposition 5.9 that for almost all choices of

$$Q' \in \prod_{i \in |r| - |k_{\text{gap}}| - \{i_2(j_1, j_2)\}} X_i(a, Q, M_o(Z))$$

with

$$\left(\frac{\pi_i(Q')}{\pi_j(Q')}\right) = a(i, j),$$

we have

$$|X_{i_2(j_1, j_2)}(a, Q \times Q', M_o(Z), \sigma)| = \frac{|X_{i_2(j_1, j_2)}(a, Q \times Q', M_o(Z))|}{2(M_{\text{top}} - 1)^{|S|}} \cdot \left(1 + O(e^{-k_{\text{gap}}})\right)$$

(6.10)
for each \( \sigma \), where \( X_{i_2(j_1,j_2)}(a, Q \times Q', M_0(Z)) \) is the subset of \( X_{i_2(j_1,j_2)}(a, Q, M_0(Z)) \) projecting to \( Q' \), and similarly for \( X_{i_2(j_1,j_2)}(a, Q \times Q', M_0(Z), \sigma) \).

We now apply Proposition 5.2 to the space \( Z \times [M_{\text{box}}] \) with

\[
\epsilon = \frac{1}{(\log \log \log N)^{1+O(1)}}
\]

for some sufficiently small constant \( c \). Let \( g_0 \in \mathcal{G}'(Z \times [M_{\text{box}}]) \) be the function guaranteed by Proposition 5.2. If we pick primes \( x_1, \ldots, x_{M_{\text{box}}} \in X_{i_2(j_1,j_2)}(a, Q \times Q', M_0(Z)) \), then we have an obvious isomorphism

\[
\varphi : Z \times [M_{\text{box}}] \cong \{Q\} \times \{Q'\} \times Z \times \{x_1, \ldots, x_{M_{\text{box}}}\}.
\]

To the primes \( x_1, \ldots, x_{M_{\text{box}}} \in X_{i_2(j_1,j_2)}(a, Q \times Q', M_0(Z)) \), we can associate a function \( g_{x_1,\ldots,x_{M_{\text{box}}}} \in \mathcal{G}'(Z \times [M_{\text{box}}]) \) by setting

\[
(\bar{z},(i,j)) \mapsto \phi_{\bar{z}}(\text{Frob } x_i) + \phi_{\bar{z}}(\text{Frob } x_j),
\]

where \( \phi_{\bar{z}} \) is \( \phi_{\bar{z},l} \) or \( \phi_{\pi_{S',\cdots}} \) depending on the value of \( j_1 \). In case \( g = g_0 \), we get the desired oscillation from Proposition 5.2 applied to the function \( F(\text{Art}_{a,m,i}(x)) \) pulled back to \( Z \times [M_{\text{box}}] \) via \( \varphi \) and the additive system \( \mathfrak{A} \) from Lemma 6.11 also pulled back to \( Z \times [M_{\text{box}}] \) via \( \varphi \).

It remains to split the set \( X_{i_2(j_1,j_2)}(a, Q \times Q', M_0(Z)) \) in blocks of size \( M_{\text{box}} \) (and a small remainder) such that we have \( g_{x_1,\ldots,x_{M_{\text{box}}}} = g_0 \) for almost every block. For this we claim that given \( \text{Frob}(x_1) \), there is a unique choice of \( \text{Frob}(x_2), \ldots, \text{Frob}(x_{M_{\text{box}}}) \) such that

\[
g_{x_1,\ldots,x_{M_{\text{box}}}} = g_0,
\]

and furthermore \( \text{Frob}(x_2), \ldots, \text{Frob}(x_{M_{\text{box}}}) \) are linear functions of \( \text{Frob}(x_1) \). Once we establish the claim, we use equation (6.10) to partition \( X_{i_2(j_1,j_2)}(a, Q \times Q', M_0(Z)) \) in the desired way.

To prove the claim, we remark that there is an isomorphism between \( \mathcal{G}'(Z \times [M_{\text{box}}]) \) and the sets of maps \( g \) from \( [M_{\text{box}}] \times [M_{\text{box}}] \) to \( \mathcal{G}'(Z) \) satisfying

\[
g(i,j) + g(j,k) = g(i,k).
\]

Hence, thinking of \( g_0 \) as a map from \( [M_{\text{box}}] \times [M_{\text{box}}] \) to \( \mathcal{G}'(Z) \), we see that for any \( 1 < j \leq M_{\text{box}} \)

\[
\phi_{\bar{z}}(\text{Frob } x_1) + \phi_{\bar{z}}(\text{Frob } x_j) = g_0(1, j) \in \mathcal{G}'(Z),
\]

which uniquely specifies \( \text{Frob}(x_j) \) as linear function of \( \text{Frob}(x_1) \) and \( g_0 \) by equation (6.9). Finally, we see that with this choice of \( \text{Frob}(x_2), \ldots, \text{Frob}(x_{M_{\text{box}}}) \), we also have for all \( i,j \in [M_{\text{box}}] \)

\[
\phi_{\bar{z}}(\text{Frob } x_i) + \phi_{\bar{z}}(\text{Frob } x_j) = g_0(i,j)
\]

so that \( g_{x_1,\ldots,x_{M_{\text{box}}}} = g_0 \) as desired. \( \square \)
6.2 Proof of Theorem 1.4

The goal of this subsection is to sketch the proof of Theorem 1.4. The major issue is that there is no choice of variable indices as in Definition 6.5 for such discriminants. This comes from the fact that the Rédei matrix is symmetric in this case, and hence the left and right kernels coincide. For this reason one can not apply the reflection principles from Section 4, and Smith’s method breaks down.

Nevertheless, one can still make inroads for this problem. The 8-rank can be attacked using classical Rédei symbols. In this case one is actually able to prove analogues of the results in Section 4 by making substantial use of the classical Rédei reciprocity law. The key feature here is that the Rédei symbol is fully symmetric in all its entries, while this is not the case for higher Rédei symbols. This is the approach taken in [2], which leads to the following result. We write ClOrd for the ordinary class group.

**Theorem 6.13.** Define for any \( n \geq m \geq 0 \)

\[
\alpha := \prod_{j=1}^{\infty} (1 + 2^{-j})^{-1}, \quad f(n, m) := \frac{\alpha \cdot P(n, n, m)}{2^n \cdot \prod_{j=1}^{n} (2^j - 1)}.
\]

Put

\[
A_{n,m}(X) := \{ d \in S_{\mathbb{Q}, X, -1} : \text{rk}_4 \text{Cl}(\mathbb{Q}(\sqrt{d})) = \text{rk}_4 \text{ClOrd}(\mathbb{Q}(\sqrt{d})) = n \text{ and } \text{rk}_8 \text{Cl}(\mathbb{Q}(\sqrt{d})) = m \}.
\]

Then we have

\[
\lim_{X \to \infty} \frac{|A_{n,m}(X)|}{|S_{\mathbb{Q}, X, -1}|} = f(n, m).
\]

**Proof.** This is [2, Theorem 1.2] except we order by discriminants instead of radicands in [2]. It is straightforward to adjust the proof to account for this new ordering. \( \square \)

Note that \( \alpha/\prod_{j=1}^{n} (2^j - 1) \) is the probability that the 4-rank is \( n \) in the family \( S_{\mathbb{Q}, \infty, -1}. \) The term \( P(n, n, m)/2^n \) is already familiar, since it also appeared in the previous subsection.

We improve on this result in two rather distinct ways. Firstly, we still have equation (4.8) from Theorem 4.11. This allows us to deal with the Artin pairing of \( (\sqrt{d}) \) and therefore we can detect whether the 8-rank of the narrow and ordinary class group are different. Crucially, Theorem 6.13 ensures that we have the correct 8-rank distribution for the narrow class group. This allows us to substantially improve the known upper bounds for the solubility of the negative Pell equations.

Secondly, the pairing Art\(_2\) need no longer be symmetric. In case that the left and right kernel intersect trivially, we can use the ideas from the previous subsection. This gives only minor improvements to the known upper and lower bounds. The reason for this is that most discriminants have a small 4-rank, while this trick is especially effective when the 4-rank gets large. We believe that any further progress will require substantial new ideas.

The remainder of this subsection is devoted to computing the improvement. We start by giving precise statements for the results mentioned in the previous two paragraphs.

**Theorem 6.14.** We have for all \( n \geq m \geq 0 \)

\[
\lim_{X \to \infty} \frac{|\{ d \in A_{n,m}(X) : \text{rk}_4 \text{ClOrd}(\mathbb{Q}(\sqrt{d})) = m - 1 \}|}{|S_{\mathbb{Q}, X, -1}|} = f(n, m) \cdot \frac{2^m - 1}{2^m}.
\]
Let $g(n, m)$ be the probability that a uniformly chosen $n \times n$ matrix with coefficients in $\mathbb{F}_2$ is such that the left and right kernel intersect trivially, given that the kernel has dimension $m$.

**Theorem 6.15.** We have for all $n \geq m \geq 0$

$$\lim_{X \to \infty} \frac{|\{d \in A_{n,m}(X) : \text{L Ker}(\text{Art}_{2,d}) \cap \text{R Ker}(\text{Art}_{2,d}) = \langle (1, \ldots, 1) \rangle\}|}{|S_{\mathbb{Q}, X, -1}|} = f(n, m) \cdot g(n, m).$$

Here L Ker and R Ker denote respectively the left and right kernel. Note that these are both naturally subspaces of $\mathbb{F}_2^r$, where $r$ is the number of prime divisors of $d$. Hence it makes sense to intersect them.

**Proof.** This result would follow immediately once one proves that $\text{Art}_{2,d}$ is a random matrix. But this is precisely what is done in the proof of [2]. $\Box$

**Remark 5.** It is not very hard to give an explicit formula for $g(n, m)$. To do so, we will compute the probability that the left and right kernel intersect trivially, and the right kernel is some given subspace $W$ of dimension $m$. We will see soon that this probability is independent on the choice of subspace $W$ and hence equals $g(n, m)$.

Indeed, after right multiplying $A$ by an invertible matrix $X$ (and left multiplying by the transpose of $X$), it suffices to deal with the case that the right kernel is $\{e_1, \ldots, e_m\}$, where the $e_i$ are standard basis vectors. For such matrices, it is straightforward to count how many have do not have a non-zero vector in the span of $\{e_1, \ldots, e_m\}$ in the left kernel. This leads to the explicit formula

$$g(n, m) = \frac{P(n - m, m, 0) \cdot \prod_{j=0}^{n-2m-1} (1 - 2^{j-n+m})}{P(n, n-m, 0)}.$$

From Theorem 6.14 we deduce that the proportion of squarefree integers $d$ for which negative Pell is soluble is at most

$$\frac{2}{3} - \sum_{n \geq m \geq 1} f(n, m) \cdot \frac{2^m - 1}{2^m},$$

where the constant $\frac{2}{3}$ comes from squarefree integers $d$ for which the 4-rank of the narrow and ordinary class group are different. These are already dealt with in the work of Fouvry–Klüners [11].

We still need to compute the further improvement coming from Theorem 6.15 and the methods in the previous subsection. To compute the improvement to the upper and lower bounds that we get, first observe that we must restrict to squarefree integers $d$ for which the 4-ranks and 8-ranks of the narrow and ordinary class group coincide, and for which the 8-rank is at least 1. Indeed, these are precisely the squarefree integers $d$ that do not fall under the purview of the theorems in [11, 12, Theorem 6.13] or Theorem 6.14. Furthermore, we must obviously restrict to squarefree integers such that $\text{Art}_{2,d}$ has left and right kernel that intersect trivially. The total proportion of such squarefree integers $d$ is

$$\sum_{n \geq m \geq 1} \frac{f(n, m) \cdot g(n, m)}{2^m}.$$
From the material in Appendix A we see that
\[ \sum_{n \geq m \geq 1} f(n, m) \cdot g(n, m) \cdot \frac{2^m}{2^{m+1} - 1} \]
are such that negative Pell is soluble and
\[ \sum_{n \geq m \geq 1} f(n, m) \cdot g(n, m) \cdot \frac{2^m - 1}{2^{m+1} - 1} \]
are such that negative Pell is not soluble. This yields Theorem 1.4 after a numerical computation.

A Stevenhagen’s conjecture revisited

Let \( l \) be an integer such that \(|l|\) is a prime 3 modulo 4. Define for any integer \( n \geq 0 \) the quantities
\[ \text{Pr}_{l,2}(n) := \lim_{N \to \infty} \frac{|S_{Z,N,l} \cap D_{l,2}(n)|}{|[N] \cap D_{l,2}(n)|}, \]
where \( D_{l,k}(n) \) is defined at the beginning of Section 6. Let us first prove that the limit exists. To do so, we look at
\[ \liminf_{N \to \infty} \frac{|S_{Z,N,I} \cap D_{l,2}(n)|}{|[N] \cap D_{l,2}(n)|} \quad \text{and} \quad \limsup_{N \to \infty} \frac{|S_{Z,N,I} \cap D_{l,2}(n)|}{|[N] \cap D_{l,2}(n)|}. \]

Theorem 6.1 gives increasingly better lower bounds for \( \liminf \), and increasingly better upper bounds for \( \limsup \). We conclude that the \( \liminf \) and \( \limsup \) are equal, and hence the limit exists. From the Markov chain behavior in Theorem 6.1, we also see that
\[ \text{Pr}_{l,3}(m, n) := \lim_{N \to \infty} \frac{|S_{Z,N,I} \cap D_{l,2}(m) \cap D_{l,3}(n)|}{|[N] \cap D_{l,2}(m) \cap D_{l,3}(n)|} \]
exists and equals \( \text{Pr}_{l,2}(n) \) for every \( m \geq n \). Then we deduce from the identity
\[ \frac{|S_{Z,N,I} \cap D_{l,2}(n)|}{|[N] \cap D_{l,2}(n)|} = \sum_{i=0}^{n} \frac{|S_{Z,N,I} \cap D_{l,2}(i) \cap D_{l,3}(i)|}{|[N] \cap D_{l,2}(i) \cap D_{l,3}(i)|} \cdot \frac{|[N] \cap D_{l,2}(n) \cap D_{l,3}(i)|}{|[N] \cap D_{l,2}(n)|} \]
by taking \( N \to \infty \) that
\[ \text{Pr}_{l,2}(n) = \sum_{i=0}^{n} \text{Pr}_{l,3}(n, i) \cdot \frac{P(n, n, i)}{2^n} = \sum_{i=0}^{n} \text{Pr}_{l,2}(i) \cdot \frac{P(n, n, i)}{2^n}. \] (A.1)

We claim that
\[ \frac{1}{2^{n+1} - 1} = \sum_{i=0}^{n} \frac{1}{2^{i+1} - 1} \cdot \frac{P(n, n, i)}{2^n}. \] (A.2)

Let us first show that the claim implies Theorem 1.1. Since we clearly have \( \text{Pr}_{l,2}(0) = 1 \), the claim and equation (A.1) imply that
\[ \text{Pr}_{l,2}(n) = \frac{1}{2^{n+1} - 1}. \] (A.3)
Now consider the decomposition
\[
\frac{|S_{Z,N,l}|}{|S_{Q,N,l}|} = \sum_{n=0}^{\infty} \frac{|S_{Z,N,l} \cap D_{l,2}(n)|}{|[N] \cap D_{l,2}(n)|} \cdot \frac{|[N] \cap D_{l,2}(n)|}{|S_{Q,N,l}|}.
\]

Then equation (A.3), Theorem 5.17 and Fatou’s lemma imply
\[
\limsup_{N \to \infty} \frac{|S_{Z,N,l}|}{|S_{Q,N,l}|} \geq \liminf_{N \to \infty} \frac{|S_{Z,N,l}|}{|S_{Q,N,l}|} \geq \sum_{n=0}^{\infty} \frac{2^{-n^2} \eta_{n} \eta_{n}^{-2}}{2^{n+1} - 1}.
\]  

Similarly, we get
\[
\limsup_{N \to \infty} \frac{|S_{Q,N,l} \setminus S_{Z,N,l}|}{|S_{Q,N,l}|} \geq \liminf_{N \to \infty} \frac{|S_{Q,N,l} \setminus S_{Z,N,l}|}{|S_{Q,N,l}|} \geq \sum_{n=0}^{\infty} \frac{2^{-n^2} \eta_{n} \eta_{n}^{-2} \cdot (2^{n+1} - 2)}{2^{n+1} - 1}.
\]  

But it is a classical fact that
\[
\sum_{n=0}^{\infty} \frac{2^{-n^2} \eta_{n} \eta_{n}^{-2}}{2^{n+1} - 1} = 1.
\]  

Therefore equation (A.4) and equation (A.5) imply that
\[
\liminf_{N \to \infty} \frac{|S_{Z,N,l}|}{|S_{Q,N,l}|} = \limsup_{N \to \infty} \frac{|S_{Z,N,l}|}{|S_{Q,N,l}|} = \sum_{n=0}^{\infty} \frac{2^{-n^2} \eta_{n} \eta_{n}^{-2}}{2^{n+1} - 1},
\]  

and Theorem 1.1 follows.

It remains to prove the claim. Look at the probability space of pairs \((T, U)\) with the uniform measure, where \(T\) is a surjective linear map \(F_2^{[n+1]} \to F_2^{[n]}\) and \(U\) is a pairing \(F_2^{[n+1]} \times F_2^{[n]} \to F_2\). All our probabilities will be with respect to this probability space. Now fix a non-zero element \(x \in F_2^{[n+1]}\). Note that
\[
\mathbb{P}(x \in \ker(T)) = \frac{1}{2^{n+1} - 1}.
\]  

We write \(\text{leftker}(U)\) for the set of vectors \(v \in F_2^{[n+1]}\) such that \(U(v, w) = 0\) for all \(w \in F_2^{[n]}\). Write \(A_{i,x}\) for the event that \(x \in \text{leftker}(U)\) and \(\dim_{F_2} \text{leftker}(U) = i + 1\). Then we have
\[
\mathbb{P}(x \in \ker(T)) = \sum_{i=0}^{n} \mathbb{P}(x \in \ker(T) | A_{i,x}) \cdot \mathbb{P}(A_{i,x}).
\]  

Observe that \(\mathbb{P}(A_{i,x}) = P(n, n, i)/2^n\). Next we have
\[
\mathbb{P}(x \in \ker(T) | A_{i,x}) = \sum_{V} \mathbb{P}(x \in \ker(T) | A_{i,x}, T(\text{leftker}(U)) = V) \cdot \mathbb{P}(T(\text{leftker}(U)) = V | A_{i,x}),
\]  

where the sum is over \(i\)-dimensional subspaces \(V\) of \(F_2^{[n]}\). But we have
\[
\mathbb{P}(x \in \ker(T) | A_{i,x}, T(\text{leftker}(U)) = V) = \frac{1}{2^{i+1} - 1},
\]  

since restricting \(T\) to \(\text{leftker}(U)\) gives a random surjective linear map \(T'\) from \(\text{leftker}(U)\) to \(V\), with \(x \in \text{leftker}(U)\). Hence we conclude that
\[
\mathbb{P}(x \in \ker(T) | A_{i,x}) = \frac{1}{2^{i+1} - 1}.
\]  

Inserting this in equation (A.6) and (A.7), we obtain the desired identity.
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