Spectral covers

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1 Introduction

Spectral curves arose historically out of the study of differential equations of Lax type. Following Hitchin’s work [H1], they have acquired a central role in understanding the moduli spaces of vector bundles and Higgs bundles on a curve. Simpson’s work \[\tilde{S}\] suggests a similar role for spectral covers \(\tilde{S}\) of higher dimensional varieties \(S\) in moduli questions for bundles on \(S\).

The purpose of these notes is to combine and review various results about spectral covers, focusing on the decomposition of their Picards (and the resulting Prym identities) and the interpretation of a distinguished Prym component as parameter space for Higgs bundles. Much of this is modeled on Hitchin’s system, which we recall in section 1, and on several other systems based on moduli of Higgs bundles, or vector bundles with twisted endomorphisms, on curves. By peeling off several layers of data which are not essential for our purpose, we arrive at the notions of an abstract principal Higgs bundle and a cameral (roughly, a principal spectral) cover. Following [D3], this leads to the statement of the main result, theorem 12, as an equivalence between these somewhat abstract ‘Higgs’ and ‘spectral’ data, valid over an arbitrary complex variety and for a reductive Lie group \(G\). Several more familiar forms of the equivalence can then be derived in special cases by adding choices of representation, value bundle and twisted endomorphism. This endomorphism is required to be regular, but not semisimple. Thus the theory works well even for Higgs bundles which are everywhere nilpotent. After touching briefly on the symplectic side of the story in section 6, we discuss some of the issues involved in removing the regularity assumption, as well as some applications and open problems, in section 7.

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We work throughout over $\mathbf{C}$. The total space of a vector bundle (=locally free sheaf) $K$ is denoted $K$. Some more notation:

Groups: \( G \ B \ T \ N \ C \)

algebras: \( g \ b \ t \ n \ c \)

Principal bundles: \( G \ B \ T \ N \ C \)

bundles of algebras: \( g \ b \ t \ n \ c \)

2 Hitchin’s system

Let $\mathcal{M} := \mathcal{M}_C(n, d)$ be the moduli space of stable vector bundles of rank $n$ and degree $d$ on a smooth projective complex curve $C$. It is smooth and quasiprojective of dimension

\[ \tilde{g} := n^2(g - 1) + 1. \]

Its cotangent space at a point $E \in \mathcal{M}$ is given by

\[ T^*_E \mathcal{M} := H^0(\text{End}(E) \otimes \omega_C) \]

where $\omega_C$ is the canonical bundle of $C$. Our starting point is:

**Theorem 1 (Hitchin[H1])** The cotangent bundle $T^* \mathcal{M}$ is an algebraically completely integrable Hamiltonian system.

*Complete integrability* means that there is a map

\[ h : T^* \mathcal{M} \to B \]

to a $\tilde{g}$-dimensional vector space $B$ which is Lagrangian with respect to the natural symplectic structure on $T^* \mathcal{M}$ (i.e. the tangent spaces to a general fiber $h^{-1}(a)$ over $a \in B$ are maximal isotropic subspaces with respect to the symplectic form). In this situation one gets, by contraction with the symplectic form, a trivialization of the tangent bundle:

\[ T_{h^{-1}(a)} \to \mathcal{O}_{h^{-1}(a)} \otimes T^*_a B. \]

In particular, this produces a family of (‘Hamiltonian’) vector fields on $h^{-1}(a)$ which is parametrized by $T^*_a B$, and the flows generated by these on $h^{-1}(a)$ all commute. *Algebraic complete integrability* means additionally that the fibers $h^{-1}(a)$ are Zariski open subsets of abelian varieties on which the Hamiltonian flows are linear, i.e. the vector fields are constant.

We describe the idea of the proof in a slightly more general setting, following [BNR]. Let $K$ be a line bundle on $C$, with total space $K$. (In Hitchin’s situation, $K$ is $\omega_C$ and $K$ is $T^* C$.) A $K$-valued Higgs bundle is a pair

\[ (E, \phi : E \to E \otimes K) \]
consisting of a vector bundle $E$ on $C$ and a $K$-valued endomorphism. One imposes an appropriate stability condition, and obtains a good moduli space $M_K$ parametrizing equivalence classes of $K$-valued semistable Higgs bundles, with an open subset $M^s_K$ parametrizing isomorphism classes of stable ones, cf. [8].

Let $B := B_K$ be the vector space parametrizing polynomial maps

$$p_a : \mathbb{K} \rightarrow \mathbb{K}^n$$

of the form

$$p_a(x) = x^n + a_1 x^{n-1} + \cdots + a_n, \quad a_i \in H^0(K^\otimes i).$$

in other words,

$$B := \bigoplus_{i=1}^n H^0(K^\otimes i).$$

The assignment

$$(E, \phi) \mapsto \text{char}(\phi) := \det (xI - \phi)$$

provides a morphism

$$h_K : M_K \rightarrow B_K.$$  \hfill (5)

In Hitchin’s case, the desired map $h$ is the restriction of $h_{\omega_C}$ to $T^*M$, which is an open subset of $M_{\omega_C}^*$. Note that in this case $\dim B$ is, in Hitchin’s words, ‘somewhat miraculously’ equal to $g = \dim M$.

The spectral curve $\tilde{C}_a$ defined by $a \in B_K$ is the inverse image in $\mathbb{K}$ of the 0-section of $K^\otimes n$ under $p_a : \mathbb{K} \rightarrow \mathbb{K}^n$. It is finite over $C$ of degree $n$. The general fiber of $h_K$ is given by:

**Proposition 2** [BNR] For $a \in B$ with integral spectral curve $\tilde{C}_a$, there is a natural equivalence between isomorphism classes of:

1. Rank-1, torsion-free sheaves on $\tilde{C}_a$.
2. Pairs $(E, \phi : E \rightarrow E \otimes K)$ with $\text{char}(\phi) = a$.

When $\tilde{C}_a$ is non-singular, the fiber is thus $\text{Jac}(\tilde{C}_a)$, an abelian variety. In $T^*M$ the fiber is an open subset of this abelian variety. One checks that the missing part has codimension $\geq 2$, so the symplectic form, which is exact, must restrict to 0 on the fibers, completing the proof.
3 Some related systems

Polynomial matrices
One of the earliest appearances of an ACIHS (algebraically completely integrable Hamiltonian system) was in Jacobi’s work on the geodesic flow on an ellipsoid (or more generally, on a nonsingular quadric in $\mathbb{R}^k$). Jacobi discovered that this differential equation, taking place on the tangent (=cotangent!) bundle of the ellipsoid, can be integrated explicitly in terms of hyperelliptic theta functions. In our language, the total space of the flow is an ACIHS, fibered by (Zariski open subsets of) hyperelliptic Jacobians. We are essentially in the special case of Proposition 3 where

$$ C = \mathbb{P}^1, \quad n = 2, \quad K = \mathcal{O}_{\mathbb{P}^1}(k). $$

A variant of this system appeared in Mumford’s solution [Mu1] of the Schottky problem for hyperelliptic curves.

The extension to all values of $n$ is studied in [3] and, somewhat more analytically, in [AHP] and [AHH]. Beauville considers, for fixed $n$ and $k$, the space $B$ of polynomials:

$$ p = y^n + a_1(x)y^{n-1} + \cdots + a_n(x), \quad \deg(a_i) \leq ki $$

(7) in variables $x$ and $y$. Each $p$ determines an $n$-sheeted branched cover

$$ \tilde{C}_p \to \mathbb{P}^1. $$

The total space is the space of polynomial matrices

$$ M := H^0(\mathbb{P}^1, \operatorname{End}(\mathcal{O}^{\oplus n}) \otimes \mathcal{O}(d)), $$

(8) the map $h : M \to B$ is the characteristic polynomial, and $M_p$ is the fiber over a given $p \in B$. The result is that for smooth spectral curves $\tilde{C}_p$, $\mathbf{PGL}(n)$ acts freely and properly on $M_p$; the quotient is isomorphic to $J(\tilde{C}_p) \setminus \Theta$. (In order to obtain the entire $J(\tilde{C}_p)$, one must allow all pairs $(E, \phi)$ with $E$ of given degree, say 0. Among those, the ones with $E \cong \mathcal{O}_{\mathbb{P}^1}^{\oplus n}$ correspond to the open set $J(\tilde{C}_p) \setminus \Theta$.) This system is an ACIHS, in a slightly weaker sense than before: instead of a symplectic structure, it has a Poisson structure, i.e. a section $\beta$ of $\wedge^2 T$, such that the $\mathbb{C}$-linear sheaf map given by contraction with $\beta$

$$ \mathcal{O} \to \mathcal{T} $$

$$ f \mapsto df \beta $$

sends the Poisson bracket of functions to the bracket of vector fields. Any Poisson manifold is naturally foliated, with (locally analytic) symplectic leaves. For a Poisson ACIHS, we want each leaf to inherit a (symplectic) ACIHS, so
the symplectic foliation should be pulled back via \( h \) from a foliation of the base \( B \).

The result of \([2BNR]\) suggests that analogous systems should exist when \( \mathbb{P}^1 \) is replaced by an arbitrary base curve \( C \). The main point is to construct the Poisson structure. This was achieved by Bottacin \([Bn]\) and Markman \([M1]\), cf. section 6. In the case of the polynomial matrices though, everything (the commuting vector fields, the Poisson structure, etc.) can be written very explicitly. What makes these explicit results possible is that every vector bundle over \( \mathbb{P}^1 \) splits. This of course fails in genus \( > 1 \), but for elliptic curves the moduli space of vector bundles is still completely understood, so here too the system can be described explicitly:

For simplicity, consider vector bundles with fixed determinant. When the degree is 0, the moduli space is a projective space \( \mathbb{P}^{n-1} \) (or more canonically, the fiber over 0 of the Abel-Jacobi map

\[
C[n] \rightarrow J(C) = C.
\]

The ACIHS which arises is essentially the Treibich-Verdier theory \([TV]\) of elliptic solitons. When, on the other hand, the degree is 1 (or more generally, relatively prime to \( n \)), the moduli space is a single point; the corresponding system was studied explicitly in \([RS]\).

**Reductive groups**

In another direction, one can replace the vector bundles by principal \( G \)-bundles \( G \) for any reductive group \( G \). Again, there is a moduli space \( M_{G,K} \) parametrizing equivalence classes of semistable \( K \)-valued \( G \)-Higgs bundles, i.e. pairs \( (G, \phi) \) with \( \phi \in K \otimes \text{ad}(G) \). The Hitchin map goes to

\[
B := \bigoplus_i H^0(K^{\otimes d_i}),
\]

where the \( d_i \) are the degrees of the \( f_i \), a basis for the \( G \)-invariant polynomials on the Lie algebra \( \mathfrak{g} \):

\[
h : (G, \phi) \rightarrow (f_i(\phi))_i.
\]

When \( K = \omega_C \), Hitchin showed \([H1]\) that one still gets a completely integrable system, and that it is algebraically completely integrable for the classical groups \( GL(n), SL(n), SP(n), SO(n) \). The generic fibers are in each case (not quite canonically; one must choose various square roots! cf. sections 5.2 and 5.3) isomorphic to abelian varieties given in terms of the spectral curves \( C \):
$GL(n)$ $\tilde{C}$ has degree $n$ over $C$, the AV is $\text{Jac}(\tilde{C})$.

$SL(n)$ $\tilde{C}$ has degree $n$ over $C$, the AV is $\text{Prym}(\tilde{C}/C)$.

$SP(n)$ $\tilde{C}$ has degree $2n$ over $C$ and an involution $x \mapsto -x$.

The map factors through the quotient $\overline{C}$.

The AV is $\text{Prym}(\tilde{C}/\overline{C})$.

(9) $SO(n)$ $\tilde{C}$ has degree $n$ and an involution, with:

- a fixed component, when $n$ is odd.
- some fixed double points, when $n$ is even.

One must desingularize $\tilde{C}$ and the quotient $\overline{C}$, and ends up with the Prym of the desingularized double cover.

The algebraic complete integrability was verified in [KP1] for the exceptional group $G_2$. A sketch of the argument for any reductive $G$ is in [BK], and a complete proof was given in [F1]. We will outline a proof in section 5 below.

**Higher dimensions**

Finally, a sweeping extension of the notion of Higgs bundle is suggested by the work of Simpson [S]. To him, a Higgs bundle on a projective variety $S$ is a vector bundle (or principal $G$-bundle . . . ) $E$ with a symmetric, $\Omega^1_S$-valued endomorphism

$$\phi : E \longrightarrow E \otimes \Omega^1_S.$$ 

Here *symmetric* means the vanishing of:

$$\phi \wedge \phi : E \longrightarrow E \otimes \Omega^2_S,$$

a condition which is obviously vacuous on curves. He constructs a moduli space for such Higgs bundles (satisfying appropriate stability conditions), and establishes diffeomorphisms to corresponding moduli spaces of representations of $\pi_1(S)$ and of connections.

## 4 Decomposition of spectral Picards

### 4.1 The question

Let $(G, \phi)$ be a $K$-valued principal Higgs bundle on a complex variety $S$. Each representation

$$\rho : G \longrightarrow \text{Aut}(V)$$

determines an associated $K$-valued Higgs bundle

$$(V := G \times^G V, \rho(\phi),$$
which in turn determines a spectral cover $\tilde{S}_V \to S$.

The question, raised first in [AvM] when $S = \mathbb{P}^1$, is to relate the Picard varieties of the $\tilde{S}_V$ as $V$ varies, and in particular to find pieces common to all of them. For Adler and van Moerbeke, the motivation was that many evolution DEs (of Lax type) can be linearized on the Jacobians of spectral curves. This means that the "Liouville tori", which live naturally in the complexified domain of the DE (and hence are independent of the representation $V$) are mapped isogenously to their image in Pic($\tilde{S}_V$) for each nontrivial $V$; so one should be able to locate these tori among the pieces which occur in an isogeny decomposition of each of the Pic($\tilde{S}_V$). There are many specific examples where a pair of abelian varieties constructed from related covers of curves are known to be isomorphic or isogenous, and some of these lead to important identities among theta functions.

**Example 3** Take $G = SL(4)$. The standard representation $V$ gives a branched cover $\tilde{S}_V \to S$ of degree 4. On the other hand, the 6-dimensional representation $\wedge^2 V$ (=the standard representation of the isogenous group $SO(6)$) gives a cover $\tilde{S} \to S$ of degree 6, which factors through an involution:

$$\tilde{S} \to \overline{S} \to S.$$ 

One has the isogeny decompositions:

$$\text{Pic}(\tilde{S}) \sim \text{Prym}(\tilde{S}/S) \oplus \text{Pic}(S)$$

$$\text{Pic}(\overline{S}) \sim \text{Prym}(\overline{S}/S) \oplus \text{Prym}(\overline{S}/S) \oplus \text{Pic}(S).$$

It turns out that

$$\text{Prym}(\tilde{S}/S) \sim \text{Prym}(\overline{S}/S).$$

For $S = \mathbb{P}^1$, this is Recillas' trigonal construction [R]. It says that every Jacobian of a trigonal curve is the Prym of a double cover of a tetragonal curve, and vice versa.

**Example 4** Take $G = SO(8)$ with its standard 8-dimensional representation $V$. The spectral cover has degree 8 and factors through an involution, $\tilde{S} \to \overline{S} \to S$. The two half-spin representations $V_1, V_2$ yield similar covers

$$\tilde{S}_i \to \overline{S}_i \to S, \quad i = 1, 2.$$ 

The tetragonal construction [D1] says that the three Pryms of the double covers are isomorphic. (These examples, as well as Pantazis’ bigonal construction and constructions based on some exceptional groups, are discussed in the context of spectral covers in [K] and [D2].)
It turns out in general that there is indeed a distinguished, Prym-like isogeny component common to all the spectral Picards, on which the solutions to Lax-type DEs evolve linearly. This was noticed in some cases already in \([\text{AvM}]\), and was greatly extended by Kanev’s construction of Prym-Tyurin varieties. (He still needs \(S\) to be \(P^1\) and the spectral cover to have generic ramification; some of his results apply only to minuscule representations.) Various parts of the general story have been worked out recently by a number of authors, based on either of two approaches: one, pursued in \([\text{D2, Me, MS}]\), is to decompose everything according to the action of the Weyl group \(W\) and to look for common pieces; the other, used in \([\text{BK, D3, F1, Sc}]\), relies on the correspondence of spectral data and Higgs bundles. The group-theoretic approach is described in the rest of this section. We take up the second method, known as \textit{abelianization}, in section 5.

4.2 Decomposition of spectral covers

The decomposition of spectral Picards arises from three sources. First, the spectral cover for a sum of representations is the union of the individual covers \(\tilde{S}_V\). Next, the cover \(\tilde{S}_V\) for an irreducible representation is still the union of subcovers \(\tilde{S}_\lambda\) indexed by weight orbits. And finally, the Picard of \(\tilde{S}_\lambda\) decomposes into Pryms. We start with a few observations about the dependence of the covers themselves on the representation. The decomposition of the Picards is taken up in the next subsection.

\textbf{Spectral covers}

There is an \textit{infinite} collection (of irreducible representations \(V := V_\mu\), hence) of spectral covers \(\tilde{S}_V\), which can be parametrized by their highest weights \(\mu\) in the dominant Weyl chamber \(C\), or equivalently by the \(W\)-orbit of extremal weights, in \(\Lambda/W\). Here \(T\) is a maximal torus in \(G\), \(\Lambda := \text{Hom}(T, \mathbb{C}^*)\) is the weight lattice (also called \textit{character lattice}) for \(G\), and \(W\) is the Weyl group. Each of these \(\tilde{S}_V\) decomposes as the union of its subcovers \(\tilde{S}_\lambda\), parametrizing eigenvalues in a given \(W\)-orbit \(W\lambda\). (\(\lambda\) runs over the weight-orbits in \(V_\mu\)).

\textbf{Parabolic covers}

There is a \textit{finite} collection of covers \(\tilde{S}_P\), parametrized by the conjugacy classes in \(G\) of parabolic subgroups (or equivalently by arbitrary dimensional faces \(F_P\) of the chamber \(C\)) such that (for general \(S\)) each eigenvalue cover \(\tilde{S}_\lambda\) is birational to some parabolic cover \(\tilde{S}_P\), the one whose open face \(F_P\) contains \(\lambda\).

\textbf{The cameral cover}

There is a \(W\)-Galois cover \(\tilde{S} \to S\) such that each \(\tilde{S}_P\) is isomorphic to \(\tilde{S}/W_P\), where \(W_P\) is the Weyl subgroup of \(W\) which stabilizes \(F_P\). We call \(\tilde{S}\) the \textit{cameral cover}, since, at least generically, it parametrizes the chambers determined by \(\phi\)
(in the duals of the Cartans), or equivalently the Borel subalgebras containing \( \phi \). This is constructed as follows: There is a morphism \( g \rightarrow t/W \) sending \( g \in \mathfrak{g} \) to the conjugacy class of its semi-simple part \( g_{ss} \). (More precisely, this is \( \text{Spec} \) of the composed ring homomorphism \( C[t]^W \rightarrow C[\mathfrak{g}]^G \rightarrow C[\mathfrak{g}] \).) Taking fiber product with the quotient map \( t \rightarrow t/W \), we get the cameral cover \( \tilde{g} \) of \( g \). The cameral cover \( \tilde{S} \rightarrow S \) of a \( K \)-valued principal Higgs bundle on \( S \) is glued from covers of open subsets in \( S \) (on which \( K \) and \( G \) are trivialized) which in turn are pullbacks by \( \phi \) of \( \tilde{g} \rightarrow \mathfrak{g} \).

### 4.3 Decomposition of spectral Picards

The decomposition of the Picard varieties of spectral covers can be described as follows:

**The cameral Picard**

From each isomorphism class of irreducible \( W \)-representations, choose an integral representative \( \Lambda_i \). (This can always be done, for Weyl groups.) The group ring \( Z[W] \) which acts on \( \text{Pic}(\tilde{S}) \) has an isogeny decomposition:

\[
Z[W] \sim \bigoplus_i \Lambda_i \otimes Z \Lambda_i^*,
\]

which is just the decomposition of the regular representation. There is a corresponding isotypic decomposition:

\[
\text{Pic}(\tilde{S}) \sim \bigoplus_i \Lambda_i \otimes Z \text{Prym}_{\Lambda_i}(\tilde{S}),
\]

where

\[
\text{Prym}_{\Lambda_i}(\tilde{S}) := \text{Hom}_W(\Lambda_i, \text{Pic}(\tilde{S})).
\]

**Parabolic Picards**

There are at least three reasonable ways of obtaining an isogeny decomposition of \( \text{Pic}(\tilde{S}_P) \), for a parabolic subgroup \( P \subset G \):

- The ‘Hecke’ ring \( \text{Corr}_P \) of correspondences on \( \tilde{S}_P \) over \( S \) acts on \( \text{Pic}(\tilde{S}_P) \), so every irreducible integral representation \( M \) of \( \text{Corr}_P \) determines a generalized Prym

\[
\text{Hom}_{\text{Corr}_P}(M, \text{Pic}(\tilde{S}_P)),
\]

and we obtain an isotypic decomposition of \( \text{Pic}(\tilde{S}_P) \) as before.

- \( \text{Pic}(\tilde{S}_P) \) maps, with torsion kernel, to \( \text{Pic}(\tilde{S}) \), so we obtain a decomposition of the former by intersecting its image with the isotypic components \( \Lambda_i \otimes Z \text{Prym}_{\Lambda_i}(\tilde{S}) \) of the latter.
• Since $\tilde{S}_P$ is the cover of $S$ associated to the $W$-cover $\tilde{S}$ via the permutation representation $\mathbb{Z}[W_P \setminus W]$ of $W$, we get an isogeny decomposition of $Pic(\tilde{S}_P)$ indexed by the irreducible representations in $\mathbb{Z}[W_P \setminus W]$.

It turns out ([D2], section 6) that all three decompositions agree and can be given explicitly as

$$
\oplus_i M_i \otimes \text{Prym}_{\Lambda_i}(\tilde{S}) \subset \oplus_i \Lambda_i \otimes \text{Prym}_{\Lambda_i}(\tilde{S}), \quad M_i := (\Lambda_i)^{W_P}.
$$

### Spectral Picards

To obtain the decomposition of the Picards of the original covers $\tilde{S}_V$ or $\tilde{S}_\lambda$, we need, in addition to the decomposition of $Pic(\tilde{S}_P)$, some information on the singularities. These can arise from two separate sources:

**Accidental singularities of the $\tilde{S}_\lambda$.** For a sufficiently general Higgs bundle, and for a weight $\lambda$ in the interior of the face $F_P$ of the Weyl chamber $C$, the natural map:

$$
i_\lambda : \tilde{S}_P \rightarrow \tilde{S}_\lambda
$$

is birational. For the *standard* representations of the classical groups of types $A_n, B_n$ or $C_n$, this is an isomorphism. But for general $\lambda$ it is *not*: In order for $i_\lambda$ to be an isomorphism, $\lambda$ must be a multiple of a fundamental weight, cf. [D2], lemma 4.2. In fact, the list of fundamental weights for which this happens is quite short; for the classical groups we have only: $\omega_1$ for $A_n$, $B_n$ and $C_n$, $\omega_n$ (the dual representation) for $A_n$, and $\omega_2$ for $B_2$. Note that for $D_n$ the list is *empty*. In particular, the covers produced by the standard representation of $SO(2n)$ are singular; this fact, noticed by Hitchin in [H1], explains the need for desingularization in his result (9).

**Gluing the $\tilde{S}_V$.** In addition to the singularities of each $i_\lambda$, there are the singularities created by the gluing map $\Pi_\lambda \tilde{S}_\lambda \rightarrow \tilde{S}_V$. This makes explicit formulas somewhat simpler in the case, studied by Kanev [K], of *minuscule* representations, i.e. representations whose weights form a single $W$-orbit. These singularities account, for instance, for the desingularization required in the $SO(2n + 1)$ case in [H].

### 4.4 The distinguished Prym

Combining much of the above, the Adler–van Moerbeke problem of finding a component common to the $Pic(\tilde{S}_V)$ for all non-trivial $V$ translates into:

*Find the irreducible representations $\Lambda_i$ of $W$ which occur in $\mathbb{Z}[W_P \setminus W]$ with positive multiplicity for all proper Weyl subgroups $W_P \subsetneq W$.***
By Frobenius reciprocity, or \((13)\), this is equivalent to

Find the irreducible representations \(\Lambda_i\) of \(W\) such that for every proper Weyl subgroup \(W_P \leq W\), the space of invariants \(M_i := (\Lambda_i)^{W_P}\) is non-zero.

One solution is now obvious: the reflection representation of \(W\) acting on the weight lattice \(\Lambda\) has this property. In fact, \(\Lambda^{W_P}\) in this case is just the face \(F_P\) of \(C\). The corresponding component \(\text{Prym}_A(S)\), is called the distinguished Prym. We will see in section 5 that its points correspond, modulo some corrections, to Higgs bundles.

For the classical groups, this turns out to be the only common component. For \(G_2\) and \(E_6\) it turns out (\cite{D2}, section 6) that a second common component exists. The geometric significance of points in these components is not known. As far as I know, the only component other than the distinguished Prym which has arisen ‘in nature’ is the one associated to the 1-dimensional sign representation of \(W\), cf. section 7 and \cite{KP2}.

5 Abelianization

5.1 Abstract vs. \(K\)-valued objects

We want to describe the abelianization procedure in a somewhat abstract setting, as an equivalence between principal Higgs bundles and certain spectral data. Once we fix a \(K\)-valued vector bundle \(K\), we obtain an equivalence between \(K\)-valued principal Higgs bundles and \(K\)-valued spectral data. Similarly, the choice of a representation \(V\) of \(G\) will determine an equivalence of \(K\)-valued Higgs bundles (of a given representation type) with \(K\)-valued spectral data.

As our model of a \(W\)-cover we take the natural quotient map

\[ G/T \rightarrow G/N \]

and its partial compactification

\[ G/T \rightarrow G/N. \]

Here \(T \subset G\) is a maximal torus, and \(N\) is its normalizer in \(G\). The quotient \(G/N\) parametrizes maximal tori (=Cartan subalgebras) \(t\) in \(g\), while \(G/T\) parametrizes pairs \(t \subset b\) with \(b \subset g\) a Borel subalgebra. An element \(x \in g\) is regular if the dimension of its centralizer \(c \subset g\) equals \(\dim T\) (=the rank of \(g\)). The partial compactifications \(G/N\) and \(G/T\) parametrize regular centralizers \(c\) and pairs \(c \subset b\), respectively.

In constructing the cameral cover in section 4.2, we used the \(W\)-cover \(t \rightarrow t/W\) and its pullback cover \(\hat{g} \rightarrow g\). Over the open subset \(g_{\text{reg}}\) of regular elements, the same cover is obtained by pulling back \((14)\) via the map \(\alpha : g_{\text{reg}} \rightarrow G/N\) sending an element to its centralizer:
When working with \( K \)-valued objects, it is usually more convenient to work with the left hand side of (15), i.e. with eigenvalues. When working with the abstract objects, this is unavailable, so we are forced to work with the eigenvectors, or the right hand side of (15). Thus:

**Definition 5** An abstract cameral cover of \( S \) is a finite morphism \( \tilde{S} \to S \) with \( W \)-action, which locally (etale) in \( S \) is a pullback of (14).

**Definition 6** A \( K \)-valued cameral cover (\( K \) is a vector bundle on \( S \)) consists of a cameral cover \( \pi: \tilde{S} \to S \) together with an \( S \)-morphism

\[
\tilde{S} \times \Lambda \to \mathbb{K}
\]

which is \( W \)-invariant (\( W \) acts on \( \tilde{S}, \Lambda \), hence diagonally on \( \tilde{S} \times \Lambda \)) and linear in \( \Lambda \).

We note that a cameral cover \( \tilde{S} \) determines quotients \( \tilde{S}_P \) for parabolic subgroups \( P \subset G \). A \( K \)-valued cameral cover determines additionally the \( \tilde{S}_\lambda \) for \( \lambda \in \Lambda \), as images in \( \mathbb{K} \) of \( \tilde{S} \times \{ \lambda \} \). The data of (16) is equivalent to a \( W \)-equivariant map \( \tilde{S} \to t \otimes C K \).

**Definition 7** A \( G \)-principal Higgs bundle on \( S \) is a pair \((\mathcal{G}, c)\) with \( \mathcal{G} \) a principal \( G \)-bundle and \( c \subset \text{ad}(\mathcal{G}) \) a subbundle of regular centralizers.

**Definition 8** A \( K \)-valued \( G \)-principal Higgs bundle consists of \((\mathcal{G}, c)\) as above together with a section \( \varphi \) of \( c \otimes K \).

A principal Higgs bundle \((\mathcal{G}, c)\) determines a cameral cover \( \tilde{S} \to S \) and a homomorphism \( \Lambda \to \text{Pic}(\tilde{S}) \). Let \( F \) be a parameter space for Higgs bundles with a given \( \tilde{S} \). Each non-zero \( \lambda \in \Lambda \) gives a non-trivial map \( F \to \text{Pic}(\tilde{S}) \). For \( \lambda \) in a face \( F_P \) of \( C \), this factors through \( \text{Pic}(\tilde{S}_P) \). The discussion in section 4.4 now suggests that \( F \) should be given roughly by the distinguished Prym,

\[
\text{Hom}_W(\Lambda, \text{Pic}(\tilde{S})).
\]

It turns out that this guess needs two corrections. The first correction involves restricting to a coset of a subgroup; the need for this is visible even in the simplest case where \( \tilde{S} \) is etale over \( S \), so \((\mathcal{G}, c)\) is everywhere regular and semisimple (i.e. \( c \) is a bundle of Cartans.) The second correction involves a twist along the ramification of \( \tilde{S} \) over \( S \). We explain these in the next two subsections.
5.2 The regular semisimple case: the shift

Example 9 Fix a smooth projective curve $C$ and a line bundle $K \in \text{Pic}(C)$ such that $K^\otimes 2 \approx \mathcal{O}_C$. This determines an etale double cover $\pi : \tilde{C} \to C$ with involution $i$, and homomorphisms

\[
\begin{align*}
\pi^* &: \text{Pic}(C) \to \text{Pic}(\tilde{C}), \\
\text{Nm} &: \text{Pic}(\tilde{C}) \to \text{Pic}(C), \\
i^* &: \text{Pic}(\tilde{C}) \to \text{Pic}(\tilde{C})
\end{align*}
\]

satisfying

\[
1 + i^* = \pi^* \circ \text{Nm}.
\]

• For $G = GL(2)$ we have $\Lambda = \mathbb{Z} \oplus \mathbb{Z}$, and $W = S_2$ permutes the summands, so

\[
\text{Hom}_W(\Lambda, \text{Pic}(\tilde{C})) \approx \text{Pic}(\tilde{C}).
\]

And indeed, the Higgs bundles corresponding to $\tilde{C}$ are parametrized by $\text{Pic}(\tilde{C})$: send $L \in \text{Pic}(\tilde{C})$ to $(\mathcal{G}, c)$, where $\mathcal{G}$ has associated rank-2 vector bundle $\mathcal{V} := \pi_* L$, and $c \subset \text{End}(\mathcal{V})$ is $\pi_* \mathcal{O}_{\tilde{C}}$.

• On the other hand, for $G = SL(2)$ we have $\Lambda = \mathbb{Z}$ and $W = S_2$ acts by $\pm 1$, so

\[
\text{Hom}_W(\Lambda, \text{Pic}(\tilde{S})) \approx \{ L \in \text{Pic}(\tilde{C}) \mid i^* L \approx L^{-1} \} = \ker(1 + i^*).
\]

This group has 4 connected components. The subgroup $\ker(\text{Nm})$ consists of 2 of these. The connected component of 0 is the classical Prym variety, cf. [Mu2]. Now the Higgs bundles correspond, via the above bijection $L \mapsto \pi_* L$, to

\[
\{ L \in \text{Pic}(\tilde{C}) \mid \det(\pi_* L) \approx \mathcal{O}_C \} = \text{Nm}^{-1}(K).
\]

Thus they form the non-zero coset of the subgroup $\ker(\text{Nm})$. (If we return to a higher dimensional $S$, it is possible for $K$ not to be in the image of $\text{Nm}$, so there may be no $SL(2)$-Higgs bundles corresponding to such a cover.)

This example generalizes to all $G$, as follows. The equivalence classes of extensions

\[
1 \to T \to N' \to W \to 1
\]

(in which the action of $W$ on $T$ is the standard one) are parametrized by the group cohomology $H^2(W, T)$. Here the 0 element corresponds to the semidirect product. The class $[N] \in H^2(W, T)$ of the normalizer $N$ of $T$ in $G$ may be 0, as it is for $G = GL(n)$, $PGL(n)$, $SL(2n + 1)$; or not, as for $G = SL(2n)$. 13
Assume first, for simplicity, that $S, \tilde{S}$ are connected and projective. There is then a natural group homomorphism

\[ c : \text{Hom}_W(\Lambda, \text{Pic}(\tilde{S})) \to H^2(W, T). \] (17)

Algebraically, this is an edge homomorphism for the Grothendieck spectral sequence of equivariant cohomology, which gives the exact sequence

\[ 0 \to H^1(W, T) \to H^1(S, C) \to \text{Hom}_W(\Lambda, \text{Pic}(\tilde{S})) \xrightarrow{c} H^2(W, T). \] (18)

where $C := \tilde{S} \times W T$. Geometrically, this expresses a Mumford group construction: giving $L \in \text{Hom}(\Lambda, \text{Pic}(\tilde{S}))$ is equivalent to giving a principal $T$-bundle $T$ over $\tilde{S}$; for $L \in \text{Hom}_W(\Lambda, \text{Pic}(\tilde{S}))$, $c(L)$ is the class in $H^2(W, T)$ of the group $N'$ of automorphisms of $T$ which commute with the action on $\tilde{S}$ of some $w \in W$.

To remove the restriction on $S, \tilde{S}$, we need to replace each occurrence of $T$ in (17, 18) by $\Gamma(\tilde{S}, T)$, the global sections of the trivial bundle on $\tilde{S}$ with fiber $T$. The natural map $H^2(W, T) \to H^2(W, \Gamma(\tilde{S}, T))$ allows us to think of $[N]$ as an element of $H^2(W, \Gamma(\tilde{S}, T))$.

**Proposition 10** [D3] Fix an etale $W$-cover $\pi : \tilde{S} \to S$. The following data are equivalent:

1. Principal $G$-Higgs bundles $(G, c)$ with cameral cover $\tilde{S}$.
2. Principal $N$-bundles $N$ over $S$ whose quotient by $T$ is $\tilde{S}$.
3. $W$-equivariant homomorphisms $L : \Lambda \to \text{Pic}(\tilde{S})$ with $c(L) = [N] \in H^2(W, \Gamma(\tilde{S}, T))$.

We observe that while the shifted objects correspond to Higgs bundles, the unshifted objects

\[ \mathcal{L} \in \text{Hom}_W(\Lambda, \text{Pic}(\tilde{S})), \quad c(\mathcal{L}) = 0 \]

come from the $C$-torsers in $H^1(S, C)$.

### 5.3 The regular case: the twist along the ramification

**Example 11** Modify example 9 by letting $K \in \text{Pic}(C)$ be arbitrary, and choose a section $b$ of $K \otimes^2$ which vanishes on a simple divisor $B \subset C$. We get a double cover $\pi : \tilde{C} \to C$ branched along $B$, ramified along a divisor

\[ R \subset \tilde{C}, \quad \pi(R) = B. \]

Via $L \mapsto \pi_* L$, the Higgs bundles still correspond to

\[ \{ L \in \text{Pic}(\tilde{C}) \mid \det(\pi_* L) \approx \mathcal{O}_C \} = \text{Nm}^{-1}(K). \]
But this is no longer in $\text{Hom}_W(\Lambda, \text{Pic}(\tilde{S}))$; rather, the line bundles in question satisfy

$$i^*L \approx L^{-1}(R).$$

For arbitrary $G$, let $\Phi$ denote the root system and $\Phi^+$ the set of positive roots. There is a decomposition

$$\overline{G/T} \smallsetminus G/T = \bigcup_{\alpha \in \Phi^+} R_\alpha$$

of the boundary into components, with $R_\alpha$ the fixed locus of the reflection $\sigma_\alpha$ in $\alpha$. (Via (13), these correspond to the complexified walls in $t$.) Thus each cameral cover $\tilde{S} \to S$ comes with a natural set of (Cartier) ramification divisors, which we still denote $R_\alpha$, $\alpha \in \Phi^+$.

For $w \in W$, set

$$F_w := \{ \alpha \in \Phi^+ \mid w^{-1}\alpha \in \Phi^- \} = \Phi^+ \cap w\Phi^-,$$

and choose a $W$-invariant form $\langle , \rangle$ on $\Lambda$. We consider the variety

$$\text{Hom}_{W;R}(\Lambda, \text{Pic}(\tilde{S}))$$

of $R$-twisted $W$-equivariant homomorphisms, i.e. homomorphisms $L$ satisfying

$$w^*L(\lambda) \approx L(w\lambda) \left( \sum_{\alpha \in F_w} \frac{-2\alpha, w\lambda}{(\alpha, \alpha)} R_\alpha \right), \quad \lambda \in \Lambda, \quad w \in W.$$  

(20)

This turns out to be the correct analogue of (19). (E.g. for a reflection $w = \sigma_\alpha$, $F_w$ is $\{ \alpha \}$, so this gives $w^*L(\lambda) \approx L(w\lambda) \left( \frac{(\alpha, 2\lambda)}{(\alpha, \alpha)} R_\alpha \right)$, which specializes to (19).) As before, there is a class map

$$c : \text{Hom}_{W;R}(\Lambda, \text{Pic}(\tilde{S})) \to H^2(W; \Gamma(\tilde{S}, T))$$

which can be described via a Mumford-group construction.

To understand this twist, consider the formal object

$$\frac{1}{2}\text{Ram} : \quad \begin{array}{c} \Lambda \\ \lambda \end{array} \to \quad \begin{array}{c} \mathbb{Q} \otimes \text{Pic}^{\tilde{S}} \\ \sum_{(\alpha \in \Phi^+)} \frac{(\alpha, \lambda)}{(\alpha, \alpha)} R_\alpha. \end{array}$$

In an obvious sense, a principal $T$-bundle $T$ on $\tilde{S}$ (or a homomorphism $L : \Lambda \to \text{Pic}(\tilde{S})$) is $R$-twisted $W$-equivariant if and only if $T(-\frac{1}{2}\text{Ram})$ is $W$-equivariant, i.e. if $T$ and $\frac{1}{2}\text{Ram}$ transform the same way under $W$. The problem with this is that $\frac{1}{2}\text{Ram}$ itself does not make sense as a $T$-bundle, because the coefficients $\frac{(\alpha, \lambda)}{(\alpha, \alpha)}$ are not integers. (This argument shows that if $\text{Hom}_{W;R}(\Lambda, \text{Pic}(\tilde{S}))$ is non-empty, it is a torser over the untwisted $\text{Hom}_W(\Lambda, \text{Pic}(\tilde{S}))$.)
Theorem 12 [D] For a cameral cover \( \tilde{S} \rightarrow S \), the following data are equivalent:

1. \( G \)-principal Higgs bundles with cameral cover \( \tilde{S} \).
2. \( R \)-twisted \( W \)-equivariant homomorphisms \( L \in c^{-1}(\mathcal{N}) \).

The theorem has an essentially local nature, as there is no requirement that \( S \) be, say, projective. We also do not need the condition of generic behavior near the ramification, which appears in [F1, Me, Sc]. Thus we may consider an extreme case, where \( \tilde{S} \) is ‘everywhere ramified’:

Example 13 In example 11, take the section \( b = 0 \). The resulting cover \( \tilde{C} \) is a ‘ribbon’, or length-2 non-reduced structure on \( C \): it is the length-2 neighborhood of \( C \) in \( K \). The \( \text{SL}(2) \)-Higgs bundles \((G, c)\) for this \( \tilde{C} \) have an everywhere nilpotent \( c \), so the vector bundle \( V := G \times_{\text{SL}(2)} \mathcal{V} \approx \pi_* L \) (where \( V \) is the standard 2-dimensional representation) fits in an exact sequence

\[
0 \rightarrow \mathcal{S} \rightarrow V \rightarrow \mathcal{Q} \rightarrow 0
\]

with \( \mathcal{S} \otimes K \approx \mathcal{Q} \). Such data are specified by the line bundle \( \mathcal{Q} \), satisfying \( \mathcal{Q} \otimes^2 K \approx K \), and an extension class in \( \text{Ext}^1(\mathcal{Q}, \mathcal{S}) \approx H^1(K^{-1}) \). The kernel of the restriction map \( \text{Pic}(\tilde{C}) \rightarrow \text{Pic}(C) \) is also given by \( H^1(K^{-1}) \) (use the exact sequence \( 0 \rightarrow K^{-1} \rightarrow \pi_* \mathcal{O}_\tilde{C} \rightarrow \mathcal{O}_C \rightarrow 0 \)), and the \( R \)-twist produces the required square roots of \( K \). (For more details on the nilpotent locus, cf. [L] and [DEL].)

5.4 Adding values and representations

Fix a vector bundle \( K \), and consider the moduli space \( \mathcal{M}_{S,G,K} \) of \( K \)-valued \( G \)-principal Higgs bundles on \( S \). (It can be constructed as in Simpson’s [S], even though the objects we need to parametrize are slightly different than his. In this subsection we outline a direct construction.) It comes with a Hitchin map:

\[
h : \mathcal{M}_{S,G,K} \rightarrow B_K
\]

where \( B := B_K \) parametrizes all possible Hitchin data. Theorem [E] gives a precise description of the fibers of this map, independent of the values bundle \( K \). This leaves us with the relatively minor task of describing, for each \( K \), the corresponding base, i.e. the closed subvariety \( B_s \) of \( B \) parametrizing split Hitchin data, or \( K \)-valued cameral covers. The point is that Higgs bundles satisfy a symmetry condition, which in Simpson’s setup is

\[
\varphi \wedge \varphi = 0,
\]

and is built into our definition [F] through the assumption that \( c \) is regular, hence abelian. Since commuting operators have common eigenvectors, this gives a splitness condition on the Hitchin data, which we describe below. (When \( K \) is a line bundle, the condition is vacuous, \( B_s = B \).) The upshot is:
Lemma 14 The following data are equivalent:
(a) A $K$-valued cameral cover of $S$.
(b) A split, graded homomorphism $R' \to \text{Sym}^* K$.
(c) A split Hitchin datum $b \in B_s$.

Here $R'$ is the graded ring of $W$-invariant polynomials on $t$:

\begin{equation}
R' := (\text{Sym}^* t)^W \approx \mathbb{C}[\sigma_1, \ldots, \sigma_l], \quad \deg(\sigma_i) = d_i
\end{equation}

where $l := \text{Rank}(\mathfrak{g})$ and the $\sigma_i$ form a basis for the $W$-invariant polynomials. The Hitchin base is the vector space

\[ B := B_K := \bigoplus_{i=1}^l H^0(S, \text{Sym}^{d_i} K) \approx \text{Hom}(R', \text{Sym}^* K). \]

For each $\lambda \in \Lambda$ (or $\lambda \in t^*$, for that matter), the expression in an indeterminate $x$:

\begin{equation}
q_\lambda(x, t) := \prod_{w \in W} (x - w\lambda(t)), \quad t \in t,
\end{equation}

is $W$-invariant (as a function of $t$), so it defines an element $q_\lambda(x) \in R'[x]$. A Hitchin datum $b \in B \approx \text{Hom}(R', \text{Sym}^* K)$ sends this to

\[ q_{\lambda,b}(x) \in \text{Sym}^*(K)[x]. \]

We say that $b$ is split if, at each point of $S$ and for each $\lambda$, the polynomial $q_{\lambda,b}(x)$ factors completely, into terms linear in $x$.

We note that, for $\lambda$ in the interior of $C$ (the positive Weyl chamber), $q_{\lambda,b}$ gives the equation in $K$ of the spectral cover $\tilde{S}_\lambda$ of section 4.2: $q_{\lambda,b}$ gives a morphism $K \to \text{Sym}^N K$, where $N := \#W$, and $\tilde{S}_\lambda$ is the inverse image of the zero-section. (When $\lambda$ is in a face $F_P$ of $C$, we define analogous polynomials $q_P^{\lambda}(x, t)$ and $q_P^{\lambda,b}(x)$ by taking the product in (24) to be over $w \in W_P \setminus W$. These give the reduced equations in this case, and $q_\lambda$ is an appropriate power.)

Over $B_s$, there is a universal $K$-valued cameral cover
\[ \tilde{S} \to B_s \]
with ramification divisor $R \subset \tilde{S}$. From the relative Picard,
\[ \text{Pic}(\tilde{S}/B_s) \]
we concoct the relative $N$-shifted, $R$-twisted Prym
\[ \text{Prym}_{\Lambda,R}(\tilde{S}/B_s). \]

By Theorem 12, this can then be considered as a parameter space $\mathcal{M}_{S,G,K}$ for all $K$-valued $G$-principal Higgs bundles on $S$. (Recall that our objects are assumed to be everywhere regular!) It comes with a ‘Hitchin map’, namely the projection to $B_s$, and the fibers corresponding to smooth projective $\tilde{S}$ are abelian varieties. When $S$ is a smooth, projective curve, we recover this way the algebraic complete integrability of Hitchin’s system and its generalizations.
6 Symplectic and Poisson structures

The total space of Hitchin’s original system is a cotangent bundle, hence has a natural symplectic structure. For the polynomial matrix systems of \cite{B} and \cite{AH}, there is a natural Poisson structure which one writes down explicitly.

In \cite{Bn} and \cite{M1}, this result is extended to the systems \( \mathcal{M}_{C,K} \) of \( K \)-valued \( GL(n) \) Higgs bundles on \( C \), when \( K \approx \omega_C(D) \) for an effective divisor \( D \) on \( C \). There is a general-nonsense pairing on the cotangent spaces, so the point is to check that this pairing is ‘closed’, i.e. satisfies the identity required for a Poisson structure. Bottacin does this by an explicit computation along the lines of \cite{B}. Markman’s idea is to consider the moduli space \( \mathcal{M}_D \) of stable vector bundles on \( C \) with level-\( D \) structure. He realizes an open subset \( \mathcal{M}^0_{C,K} \) of \( \mathcal{M}_{C,K} \), parametrizing Higgs bundles whose covers are nice, as a quotient (by an action of the level group) of \( T^* \mathcal{M}_D \), so the natural symplectic form on \( T^* \mathcal{M}_D \) descends to a Poisson structure on \( \mathcal{M}^0_{C,K} \). This is identified with the general-nonsense form (wherever both exist), proving its closedness.

In \cite{Muk}, Mukai constructs a symplectic structure on the moduli space of simple sheaves on a \( K3 \) surface \( S \). Given a curve \( C \subset S \), one can consider the moduli of sheaves having the numerical invariants of a line bundle on a curve in the linear system \( |nC| \) on \( S \). This has a support map to the projective space \( |nC| \), which turns it into an ACIHS. This system specializes, by a ‘degeneration to the normal cone’ argument, cf. \cite{DEL}, to Hitchin’s, allowing translation of various results about Hitchin’s system (such as Laumon’s description of the nilpotent cone, cf. \cite{L}) to Mukai’s.

In higher dimensions, the moduli space \( \mathcal{M} \) of \( \Omega^1 \)-valued Higgs bundles carries a natural symplectic structure \( \mathcal{S} \). (Corlette points out in \cite{S} that certain components of an open subset in \( \mathcal{M} \) can be described as cotangent bundles.) It is not clear at the moment exactly when one should expect to have an ACIHS, with symplectic, Poisson or quasi symplectic structure, on the moduli spaces of \( K \)-valued Higgs bundles for higher dimensional \( S \), arbitrary \( G \), and arbitrary vector bundle \( K \). A beautiful new idea \cite{M2} is that Mukai’s results extend to the moduli of those sheaves on a (symplectic, Poisson or quasi symplectic) variety \( X \) whose support in \( X \) is Lagrangian. Again, there is a general-nonsense pairing. At points where the support is non-singular projective, this can be identified with another, more geometric pairing, constructed using the cubic condition of \cite{DM}, which is known to satisfy the closedness requirement. This approach is quite powerful, as it includes many non-linear examples such as Mukai’s, in addition to the line-bundle valued spectral systems of \cite{Bn} and also Simpson’s \( \Omega^1 \)-valued \( GL(n) \)-Higgs bundles: just take \( X := T^* S \to S \), with its natural symplectic form, and the support in \( X \) to be proper over \( S \) of degree \( n \); such sheaves correspond to Higgs bundles by \( \pi_\ast \).

The structure group \( GL(n) \) can of course be replaced by an arbitrary reductive group \( G \). Using Theorem \ref{T:structure}, this yields (in the analogous cases) a Poisson structure on the Higgs moduli space \( \mathcal{M}_{S,G,K} \) described at the end of
the previous section. The fibers of the generalized Hitchin map are Lagrangian with respect to this structure. Along the lines of our general approach, the necessary modifications are clear: $\pi_*$ is replaced by the equivalence of Theorem 12. One thus considers only Lagrangian supports which retain a $W$-action, and only equivariant sheaves on them (with the numerical invariants of a line bundle). These two restrictions are symplecticly dual, so the moduli space of Lagrangian sheaves with these invariance properties is a symplectic (respectively, Poisson) subspace of the total moduli space, and the fibers of the Hitchin map are Lagrangian as expected.

A more detailed review of the ACIHS aspects of Higgs bundles will appear in [DM2].

7 Some applications and problems

Some applications
In [H1], Hitchin used his integrable system to compute several cohomology groups of the moduli space $\mathcal{SM}$ (of rank 2, fixed odd determinant vector bundles on a curve $C$) with coefficients in symmetric powers of its tangent sheaf $\mathcal{T}$. The point is that the symmetric algebra $\text{Sym}^\bullet \mathcal{T}$ is the direct image of $\mathcal{O}_{\mathcal{T}^*\mathcal{SM}}$, and sections of the latter all pull back via the Hitchin map $h$ from functions on the base $B$, since the fibers of $h$ are open subsets in abelian varieties, and the missing locus has codimension $\geq 2$. Hitchin’s system is used in [BNR] to compute a couple of "Verlinde numbers" for GL(n), namely the dimensions $h^0(\mathcal{M}, \Theta) = 1$, $h^0(\mathcal{SM}, \Theta) = n^g$. These results are now subsumed in the general Verlinde formulas, cf. [F2], [BL], and other references therein.

A pretty application of spectral covers was obtained by Katzarkov and Pantev [KP2]. Let $S$ be a smooth, projective, complex variety, and $\rho : \pi_1(S) \to G$ a Zariski dense representation into a simple $G$ (over $\mathbb{C}$). Assume that the $\Omega^1$-valued Higgs bundle $(\mathcal{V}, \phi)$ associated to $\rho$ by Simpson is (regular and) generically semisimple, so the cameral cover is reduced. Among other things, they show that $\rho$ factors through a representation of an orbicurve if and only if the non-standard component $\text{Prym}_\epsilon(\tilde{S})$ is non zero, where $\epsilon$ is the one-dimensional sign representation of $W$. (In a sense, this is the opposite of $\text{Prym}_\Lambda(\tilde{S})$: while $\text{Prym}_\Lambda(\tilde{S})$ is common to $\text{Pic}(\tilde{S})$ for all proper Weyl subgroups, $\text{Prym}_\epsilon(\tilde{S})$ occurs in none except for the full cameral Picard.)

Another application is in [KoP]: the moduli spaces of SL(n)- or GL(n)-stable bundles on a curve have certain obvious automorphisms, coming from tensoring with line bundles on the curve, from inversion, or from automorphisms of the curve. Kouvidakis and Pantev use the dominant direct-image maps from spectral Picards and Pryms to the moduli spaces to show that there are no further, unexpected automorphisms. This then leads to a ‘non-abelian Torelli theorem’, stating that a curve is determined by the isomorphism class of the moduli space of bundles on it.
Compatibility?
Hitchin’s construction \[ \text{[H2]} \] of the projectively flat connection on the vector bundle of non-abelian theta functions over the moduli space of curves does not really use much about spectral covers. Nor do other constructions of Faltings \[ \text{[F1]} \] and Witten et al \[ \text{APW} \]. Hitchin’s work suggests that the ‘right’ approach should be based on comparison of the non-abelian connection near a curve \( C \) with the abelian connection for standard theta functions on spectral covers \( \tilde{C} \) of \( C \). One conjecture concerning the possible relationship between these connections appears in \[ \text{[A]} \], and some related versions have been attempted by several people, so far in vain. What’s missing is a compatibility statement between the actions of the two connections on pulled-back sections. If the expected compatibility turns out to hold, it would give another proof of the projective flatness. It should also imply projective finiteness and projective unitarity of monodromy for the non-abelian thetas, and may or may not bring us closer to a ‘finite-dimensional’ proof of Faltings’ theorem (=the former Verlinde conjecture).

Irregulars?
The Higgs bundles we consider in this survey are assumed to be everywhere regular. This is a reasonable assumption for line-bundle valued Higgs bundles on a curve or surface, but not in \( \text{dim} \geq 3 \). This is because the complement of \( g_{\text{reg}} \) has codimension 3 in \( g \). The source of the difficulty is that the analogue of \( \text{(15)} \) fails over \( g \). There are two candidates for the universal camera cover: \( \tilde{g} \), defined by the left hand side of \( \text{(15)} \), is finite over \( g \) with \( W \) action, but does not have a family of line bundles parametrized by \( \Lambda \). These live on \( \tilde{\gamma} \), the object defined by the right hand side, which parametrizes pairs \( (x, b) \), \( x \in b \subset g \). This suggests that the right way to analyze irregular Higgs bundles may involve spectral data consisting of a tower

\[
\tilde{S} \overset{\sigma}{\rightarrow} \tilde{S} \rightarrow S
\]

together with a homomorphism \( \mathcal{L} : \Lambda \rightarrow \text{Pic}(\tilde{S}) \) such that the collection of sheaves

\[
\sigma_*(\mathcal{L}(\lambda)), \quad \lambda \in \Lambda
\]

on \( \tilde{S} \) is \( R \)-twisted \( W \)-equivariant in an appropriate sense. As a first step, one may wish to understand the direct images \( R^i\sigma_*(\mathcal{L}(\lambda)) \) and in particular the cohomologies \( H^i(F, \mathcal{L}(\lambda)) \) where \( F \), usually called a Springer fiber, is a fiber of \( \sigma \). For regular \( x \), this fiber is a single point. For \( x = 0 \), the fiber is all of \( G/B \), so the fiber cohomology is given by the Borel-Weil-Bott theorem. The question may thus be considered as a desired extension of BWB to general Springer fibers.
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