Analytical $r$-mode solutions are investigated within the linearized theory in the case of a slowly rotating, Newtonian, barotropic, non-magnetized, perfect-fluid star in which the gravitational radiation (GR) reaction force is present. For the GR reaction term we use the 3.5 post-Newtonian order expansion of the GR force, in order to include the contribution of the current quadrupole moment. We find the explicit expression for the $r$-mode velocity perturbations and we conclude that they are sinusoidal with the same frequency as the well-known GR force-free linear $r$-mode solution, and that the GR force drives the $r$-modes unstable with a growth timescale that agrees with the expression first found by Lindblom, Owen and Morsink. We also show that the amplitude of these velocity perturbations is corrected, relatively to the GR force-free case, by a term of order $\Omega^6$, where $\Omega$ is the angular velocity of the star.

PACS numbers: 04.40.Dg, 95.30.Lz, 97.10.Sj, 97.10.Kc

I. INTRODUCTION

Stars are not rigid systems; they naturally oscillate and the non-radial pulsations generate gravitational radiation (GR) that removes energy and angular momentum from the star. In non-rotating stars this process is always dissipative and, inevitably, the oscillation of the star dies off. However, if the star rotates this is not necessarily the case: under certain conditions the amplitude of the pulsation mode grows and a GR instability sets in. In this paper we are interested on a special class of stellar pulsation modes, called Chandrasekhar-Friedman-Schutz (CFS) instabilities, after the pioneering works of Refs. [1, 2]. The generating mechanism of these CFS instabilities is well understood. Indeed, consider a mode that counter-rotates, i.e., whose propagation direction is opposite to the star’s rotation in the co-rotating frame of the star. This mode has then negative angular momentum. When analyzing the properties of the mode in the inertial frame, one sees that the mode is dragged forward by the stellar rotation and can become prograde, i.e., it can rotate in the same sense of the star. Then, the gravitational waves emitted due to the mode oscillation carry away positive angular momentum from the star. Since positive angular momentum is extracted from a mode with negative angular momentum, the amplitude of the mode increases and an instability appears. So, although the GR emission lowers the inertial frame energy, under the above conditions it induces an increase in the mode energy as measured in the co-rotating frame. These two events can occur simultaneously because the energy in the co-rotating frame $E$ and the energy in the inertial frame $E_{\text{in}}$ are related by $E = E_{\text{in}} - \Omega J$, where $\Omega$ is the angular velocity of the star and $J$ is the angular momentum of the mode. Therefore, $E$ can grow if both $E_{\text{in}}$ and $\Omega J$ decrease [2]. The condition for the appearance of a CFS instability can be stated as follows. Given a mode with an angular pattern speed $\sigma$ in the inertial frame that propagates in a star that rotates with angular velocity $\Omega$, a CFS instability sets in if $0 < \sigma < \Omega$. In general, this means that for a given mode with fixed $\sigma$ the instability will appear only if the star is rapidly rotating with an angular velocity greater than the critical value $\Omega_{\text{crit}} = \sigma$. This requirement of a large stellar rotation, together with the unavoidable viscosity contributions that damp the instability, has diminished the astrophysical interest in the CFS instabilities [3].

This pessimistic scenario changed when the properties of a special class of stellar pulsation modes, called $r$-modes, were investigated in detail. The $r$-modes, which were first studied [4–7] more than twenty years ago, are pulsation modes of rotating stars that have the Coriolis force as their restoring force. These modes induce perturbations mainly in the fluid’s velocity and cause very small disturbances in the star’s density. More precisely, the velocity perturbations are of order $\Omega$, while the density perturbations are of order $\Omega^2$, where $\Omega$ is the star’s angular velocity. The unique feature that clearly distinguishes the $r$-modes from other stellar modes and that turn them potentially interesting for astrophysics is the fact that they are driven unstable by GR reaction in all
perfect-fluid rotating stars, no matter how slowly they rotate. This property was discovered by Andersson [8], and afterward confirmed more generally by Friedman and Morsink [9]. So, unlike other modes, r-modes are always retrograde in the star’s co-rotating frame and prograde in the inertial frame, i.e., the sign of the r-mode frequencies is always opposite in the two frames. In other words, the CFS instability always occurs for the r-modes because the angular pattern speed of a r-mode with angular quantum number \( l \) in an inertial frame is given by \( \sigma = \frac{(l - m + 1) \cdot \Omega}{l + 1} \), and so it satisfies the CFS condition, \( 0 < \sigma < \Omega \), for any value of the stellar rotation and for \( l \geq 2 \).

Two questions are then naturally raised, namely, what are the timescales associated with the growth of the r-mode CFS instability, and are the several viscosity processes and damping mechanisms present in a star sufficient to suppress the GR instability? These questions were first addressed in Ref. [10], where, for the evaluation of the growth timescale \( \tau \) of the instability, the small density perturbations induced by the r-mode were neglected and, in the framework of Newtonian hydrodynamics, the formalism developed in Ref. [11] was used to compute the GR reaction that induces the GR instability. The coupling to r-modes primarily through the current multipoles rather than the usual mass multipoles (this result follows straightforwardly from an order count in the multipoles: the perturbations in the mass multipoles are of order \( \Omega^2 \), while the perturbations in the current multipoles are of order \( \Omega \)). In Ref. [10] it was found that the energy \( E \) of the r-mode perturbation in the co-rotating frame varies, to lowest order in \( \Omega \), according to \( E = E_0 e^{-2l/\tau} \), where \( E_0 \) is the initial energy. Moreover, for \( l = 2 \), the timescale \( \tau_{GR} \) of the GR-driven instability is given by [10]

\[
\tau_{GR} = - \left( \frac{2^{17} \pi}{32} \frac{G}{c^7} \frac{\bar{J}}{3} \right)^{\frac{1}{2}},
\]

where \( G \) is the Newton’s constant, \( c \) is the velocity of light, and \( \bar{J} = \frac{1}{3} \int_0^R \int_0^2 r^3 \rho \, dr \, d\rho \) with \( \rho \) and \( R \) being the density and the surface’s radius of the unperturbed star, respectively. In parallel, a similar programme as the one of Ref. [10] was carried in Ref. [12], but with the important difference that, in order to include the small density perturbations, the calculation was performed up to second order in the stellar rotation (\( \Omega^2 \)). The density perturbation contribution is important to correctly evaluate the bulk viscosity effects, but has no influence on the GR timescale \( \tau_{GR} \) [12]. The calculations performed in Refs. [10, 12] indicate that, for a wide range of relevant temperatures and angular velocities of the star, \( \tau_{GR} \ll \tau_{\nu} \), with the gravitational timescale \( \tau_{GR} \) being several orders of magnitude smaller than the viscous timescale \( \tau_{\nu} \). Thus, for this range of temperatures and angular velocities of the star, viscosity cannot suppress the growth of the GR-driven instability of r-modes.

The astrophysical implications of the GR-driven instability in the r-modes are then very promising. Indeed, a long-standing unsolved problem of pulsar astrophysics is associated with the observed slow rotation rates of neutron stars. Neutron stars are believed to be formed through stellar gravitational collapse. The angular momentum of the system is conserved during this process and so the natural expectation is that the final neutron star should have a rotation rate close to the Keplerian velocity, i.e., the maximum velocity above which matter starts escaping through the equatorial plane (see nevertheless Ref. [13] for a different view). However, all the observed neutron stars present a rotation rate well below this maximum value. The r-modes with their peculiar properties provide a possible explanation for these small rotation rates of young pulsars in supernova remnants. Indeed, as we just saw, in a newly born, hot, rapidly-rotating neutron star the GR reaction force dominates bulk and shear viscosities for enough time to allow most of the star’s angular momentum to be radiated away as gravitational waves [10, 12]. As a result, the neutron star might spin down to just a small fraction of its initial angular velocity. The r-modes may also play an important astrophysical role in low-mass X-ray binaries. Indeed, GR emission due to the r-mode instability could balance the spin-up torque due to accretion of neutron stars in low-mass X-ray binaries, limiting in this way the maximum angular velocity of these pulsars to values consistent with observations [14–16]. GR released during the r-mode evolution seems also to be a very promising source for the detection of gravitational waves, even though the perturbations in the density induced by r-modes are small and thus, in a first intuitive analysis, they should not be associated with strong GR emission. A first attempt to develop a time-dependent model of the evolution of the r-mode instability, including an approximate discussion on the nonlinear phase where most of the angular momentum of the star is radiated away, has shown that the GR emitted can probably be detected in enhanced versions of laser interferometer detectors [17]. Although recent results on the nonlinear saturation of the r-mode energy are not so optimistic, they still point to the possible detection of gravitational waves if the unstable neutron stars are close enough [18].

A quite interesting feature that has emerged from recent investigations on r-modes is the presence of differential rotation induced by the r-mode oscillation in a background star that is initially uniformly rotating. That differential rotation drifts of kinematical nature could be induced by r-mode oscillations of the stellar fluid was first suggested in Refs. [19, 20], where an approximate analytical evaluation of nonlinear effects was performed. Differential rotation drifts induced by r-modes were also found in numerical simulations of nonlinear r-modes carried out both in general relativistic hydrodynamics [21], and in Newtonian hydrodynamics [22–24]. Differential rotation was also reported in a model of a thin spherical shell of a rotating incompressible fluid [25]. Recently, an analytical solution, representing differential rotation of r-modes that produce large scale drifts of fluid elements
along stellar latitudes, was found within the nonlinear Newtonian theory up to second order in the mode amplitude and in the absence of GR reaction [26]. This work is exact to second order, improving the approximate analytical computation performed in Refs. [19, 20]. As shown in Ref. [27], this differential rotation plays a relevant role in the nonlinear evolution of the $r$-mode instability. Indeed, in a newly born, hot, non-magnetized, rapidly rotating neutron star, $r$-modes saturates a few hundred seconds after the mode instability sets in; the saturation amplitude depends on the amount of differential rotation at the time the instability becomes active and can take values much smaller than unity [27]. Now, the question that remains unsolved is whether GR reaction also induces differential rotation, making an extra contribution. One of the aims of the present paper is to initiate a programme that hopefully will allow to answer this question.

In this paper, we will perform an analytical study of $r$-mode solutions within the linearized theory in the case of a slowly rotating, Newtonian, barotropic, non-magnetized, perfect-fluid star in which a gravitational radiation reaction force is present. The GR reaction contribution is assumed to be given by the 3.5 post-Newtonian order expansion that includes the current quadrupole moment, which is the main responsible for the GR instability that sets in. The detailed formalism of the 3.5 post-Newtonian order expansion is given in Refs. [28, 29]. It is known that the most unstable mode is the $l = m = 2$ $r$-mode, where $l$ and $m$ are the angular momentum and the azimuthal numbers, respectively. Hence, we will focus our attention on the GR-driven instability of this mode. We will find an analytical expression for the $r$-mode velocity perturbations, which is physically consistent with the energy evolution studied in Refs. [10, 12].

The plan of the paper is the following. In Section II we write the full Newtonian hydrodynamic equations with the GR reaction force. We review the main properties of the GR reaction force and we give the general formulas needed to compute this reaction force. The explicit computation of the GR reaction force in the 3.5 post-Newtonian order expansion is then performed in Appendix A. Section III is devoted to the linearized theory. The perturbed Newtonian hydrodynamic equations with the GR reaction force are written in Subsection III A. In Subsection III B we briefly review the well-known GR force-free linear $r$-mode solution that is used in Subsection III C to compute the first-order Eulerian change in the GR force. In Subsection III D we finally solve the Newtonian hydrodynamic equations with the GR force. In Section IV our results are discussed and future directions are pointed. In Appendix B, we identify explicitly the connection between some of our intermediary formulas for the GR force and those of previous numerical works on the subject.

**II. NEWTONIAN HYDRODYNAMIC EQUATIONS WITH GR REACTION FORCE**

The Newtonian hydrodynamic equations for a uniformly rotating, barotropic, non-magnetized, perfect-fluid star in the presence of the gravitational radiation (GR) reaction force are the Euler, continuity, and Poisson equations given, respectively, by

$$\partial_t \vec{v} + (\vec{v} \cdot \nabla) \vec{v} = -\rho^{-1} \nabla P - \nabla \Phi + \vec{F}^{GR},$$  

(2)

$$\partial_t \rho + \nabla \cdot (\rho \vec{v}) = 0,$$  

(3)

$$\nabla^2 \Phi = 4\pi G \rho,$$  

(4)

where $\rho$ is the density of the fluid, $\vec{v}$ is its velocity, $P$ is its pressure, $\Phi$ is the Newtonian potential, and $\vec{F}^{GR}$ is the GR reaction force per unit mass (or, simply, GR reaction force). The general expression for the GR reaction force $\vec{F}^{GR}$, that includes contributions from the time-varying mass multipole moments and from the time-varying current multipole moments, has been found and progressively discussed in detail up to the 3.5 post-Newtonian order. A review of this path and the 3.5 post-Newtonian expansion of the GR force can be found in Refs. [28, 29]. In this paper we are interested on the contribution of the GR reaction force to the $r$-mode oscillations. Contrary to what happens in other mode oscillations, the $r$-mode instability is predominantly excited by the time-varying current multipole moments rather than by the usual time-varying mass multipole moments [10] (see also a detailed discussion in Ref. [29]). Now, in order to include the contribution of the current multipole moments, the post-Newtonian expansion of the GR reaction force must be done at least up to the 3.5 order [28, 29]. This post-Newtonian order includes the contribution of the current quadrupole moment, which is the dominant current multipole moment. Since the main contribution for the $r$-mode instability comes from the current quadrupole moment, we will henceforth neglect all the terms coming from the time-varying mass multipole moments, i.e., we will not consider terms that appear in post-Newtonian orders below 3.5 neither the terms of the 3.5 post-Newtonian order that have their origin in the time-varying mass multipole moments. These terms are clearly identified in Refs. [28, 29]. So, in the so-called Blanchet’s gauge, the contribution of the current multipole moments to the 3.5 post-Newtonian order GR force, henceforth labelled as $\vec{F}^{GR}$ for simplicity, is \(^1\)

$$\vec{F}^{GR} = -\partial_t \vec{\beta} + \vec{v} \times (\nabla \times \vec{\beta}),$$  

(5)

where $\vec{\beta}$ is a vector whose components are given by

$$\beta_i = \frac{16G}{45c^7} ijk x_j x_q c_q^{[5]} k_q,$$  

(6)

\(^1\) See equations (12) and (17) of Ref. [29]. Our notation is slightly different from the one of Refs. [28, 29]. The correspondence is $\vec{\beta} \equiv \tilde{\beta}$. 

with $i = 1, 2, 3$ and similarly for the other Latin indices.

In the previous expressions, $S_{ij}(t)$ is the time-varying current quadrupole tensor,

$$S_{ij}(t) \equiv \int d^3x \epsilon_{kpi}(x_j x_k) \rho u_q,$$

(7)

$\epsilon_{ijk}$ is the Levi-Civita tensor, $x_i$ is the Cartesian coordinate of the point at which the tensor is evaluated, and

$$S_{ij}^{[n]}(t) \equiv \frac{d^n}{dt^n} S_{ij}(t).$$

(8)

The explicit computation of the GR reaction force is performed in Appendix A.

It is interesting to note that $\vec{\beta}$ can be appropriately identified as a gravitational vector potential [25]. This designation comes from the clear equivalence between the GR force per unit mass exerted on a moving fluid (5) and the Lorentz force per unit charge exerted by an electromagnetic field on a moving charged fluid, $F^\text{Lor} = -\partial_t \vec{A} + \vec{\phi} \times (\vec{\nabla} \times \vec{A})$, where $\vec{A}$ is the electromagnetic vector potential. In this analogy, the designation of gravitational vector potential for $\vec{\beta}$ follows straightforwardly if one identifies $\vec{\beta} \equiv \vec{A}$.

III. R-MODES IN THE LINEARIZED THEORY WITH THE GR REACTION CONTRIBUTION

In previous investigations [10, 12], the hydrodynamics equations with the GR force were used to obtain an expression for the time evolution of the physical energy of the $r$-mode perturbation, $dE/dt$, from which the gravitational radiation and viscous timescales were determined. In this section, we explicitly solve the hydrodynamics equations with the GR force, obtaining an expression for the first-order Eulerian change in the velocity of the $r$-modes $\delta^{(1)} \vec{v}$. It is shown that these velocity perturbations are sinusoidal with the same frequency as the well-known GR force-free linear $r$-mode solution; that the GR force drives the $r$-modes unstable with a growth timescale that agrees with the expression found in Refs. [10, 12]; and that the amplitude of these velocity perturbations is corrected, relatively to the GR force-free case, by a term of order $\Omega^6$.

A. Linearized hydrodynamic equations

The hydrodynamic equations (2)–(4) for a uniformly rotating, Newtonian, barotropic, non-magnetized, perfect-fluid star with the GR reaction force can be linearized yielding, in the inertial frame,

$$\partial_t \delta^{(1)} \vec{v} + (\delta^{(1)} \vec{v} \cdot \vec{\nabla}) \delta^{(1)} \vec{v} + \hat{\vec{v}} \cdot \vec{\nabla} \delta^{(1)} \vec{v} \equiv -\vec{\nabla} \delta^{(1)U} + \delta^{(1)} \vec{F}^{\text{GR}},$$

(9)

$$\partial_t \delta^{(1)} \rho + \hat{\vec{v}} \cdot \vec{\nabla} \delta^{(1)} \rho + \vec{\nabla} \cdot (\hat{\vec{v}} \delta^{(1)} \vec{v}) = 0,$$

(10)

$$\nabla^2 \delta^{(1)} \Phi = 4\pi G \delta^{(1)} \rho,$$

(11)

where we have defined

$$\delta^{(1)U} \equiv \frac{\delta^{(1)} P}{\rho} + \delta^{(1)} \Phi.$$  

(12)

In the above equations,

$$\hat{\vec{v}} = \Omega r \sin \theta \vec{e}_\phi$$

(13)

is the velocity of the unperturbed star and $\rho$ its mass density, while $\delta^{(1)} Q$ denotes the first-order Eulerian change in a quantity $Q$.

B. Linear $r$-mode solution without the GR reaction contribution

In our approach, we use the linear $r$-mode solution without the GR contribution to generate the GR reaction force. As already mentioned, the GR force drives $r$-modes unstable and the most unstable mode is the $l = m = 2$ $r$-mode. We will therefore concentrate our attention in this mode and in this subsection we briefly present this well-known solution.

In the absence of the GR reaction force, $\delta^{(1)} \vec{F}^{\text{GR}} = 0$, and to lowest order in $\alpha$ and in $\Omega$, where $\alpha$ is the dimensionless $r$-mode amplitude, the perturbed equations (9)–(11) allow $r$-mode solutions with velocity perturbations given by $\delta^{(1)} \vec{v} = \alpha \Omega R(r/R) \sqrt{5} \hat{\vec{v}} e^{i\omega t}$, where $R$ is radius of the unperturbed star. In spherical coordinates and for $l = 2$, the velocity perturbations are explicitly given by

$$\delta^{(1)} v_r = 0,$$

(14a)

$$\delta^{(1)} v_\theta = -\frac{i}{4} \sqrt{\frac{5 \alpha \Omega}{\pi R}} r^2 \sin \theta e^{i(2\phi + \omega t)},$$

(14b)

$$\delta^{(1)} v_\phi = \frac{1}{4} \sqrt{\frac{5 \alpha \Omega}{\pi R}} r^2 \sin \theta \cos \theta e^{i(2\phi + \omega t)},$$

(14c)

and from Eq. (9) one obtains

$$\delta^{(1)} U = \frac{1}{6} \sqrt{\frac{5 \alpha \Omega}{\pi R}} r^3 \sin^2 \theta \cos \theta e^{i(2\phi + \omega t)},$$

(15)

where $\omega = -4\Omega/3$ is the $l = 2$ $r$-mode frequency in the inertial frame [4].

C. Explicit computation of the first-order Eulerian change in the GR force

To find the linear $r$-mode solution of the perturbed hydrodynamic equations (9)–(11) with the GR reaction force we have to evaluate the first-order Eulerian change in the GR force, i.e., we have to compute $\delta^{(1)} \vec{F}^{\text{GR}}$. To do so we expand both the velocity vector, $\vec{v} = \hat{\vec{v}} + \delta^{(1)} \vec{v}$, and the current multipole tensors, $S_{ij} = \hat{S}_{ij} + \delta^{(1)} S_{ij}$, where $\hat{S}_{ij}$ describes the multipole tensor of the unperturbed star and $\delta^{(1)} S_{ij}$ is the first-order Eulerian change
in the multipole tensor. The first-order Eulerian change in the GR force \( \delta^{(1)}F_{\text{GR}} \) contains then terms of the type 
\[ \hat{v}_k \delta^{(1)} S^{[5]}_{ij}, \delta^{(1)} v_k, \text{ and } \delta^{(1)} S^{[6]}_{ij}. \]

Similarly, it is straightforward to find that the first-order Eulerian change in the GR force simply as a sum of terms of the type 
\[ \hat{v}_k \delta^{(1)} S^{[5]}_{ij} \text{ and } \delta^{(1)} S^{[6]}_{ij} . \]

The multipole tensors of the unperturbed star are easily computed from Eqs. (A3), with the use of Eq. (13), yielding
\[ \dot{S}_{ij} = 0; \quad i, j = x, y, z, \]
i.e., they all vanish. In practice, this result allows us to rewrite the first-order Eulerian change in the GR force simply as a sum of terms of the type 
\[ \hat{v}_k \delta^{(1)} S^{[5]}_{ij} \text{ and } \delta^{(1)} S^{[6]}_{ij} . \]

Now, to compute the first-order Eulerian change in the multipole tensors \( \delta^{(1)} S_{ij} \) we use an approach in which we assume that the linear \( r \)-mode solution with \( \delta^{(1)}F_{\text{GR}} = 0 \) acts as a source for the current multipole tensor. In other words, and taking \( \delta^{(1)} S^{[5]}_{xx} \) as an example, we assume that
\[ \delta^{(1)} S^{[5]}_{xx} = -\int dV r^2 \sin \theta \cos \phi \]
\[ \times \left[ \delta^{(1)} \rho \left( \hat{v}_\theta \sin \phi + \hat{v}_\phi \cos \phi \cos \theta \right) \right. \]
\[ + \left. \rho \left( \delta^{(1)} v_\theta \sin \phi + \delta^{(1)} v_\phi \cos \phi \cos \theta \right) \right], \]

where \( \hat{v} \) is given by Eq. (13) and \( \delta^{(1)} \hat{v} \) is given by Eqs. (14) with \( \omega \) being now an arbitrary parameter to be determined. Since \( \delta^{(1)} \rho \) is of order \( \alpha \Omega^2 \), \( \delta^{(1)} \hat{v} \) is of order \( \alpha \Omega \), and \( \hat{v} \) is of order \( \Omega \), in Eq. (17) there are terms of order \( \alpha \Omega \), namely, \( \hat{v} \delta^{(1)} \rho \) and \( \rho \delta^{(1)} \hat{v} \), of order \( \alpha \Omega^3 \), namely, \( \hat{v} \delta^{(1)} \rho \). Since our interest is to find the \( r \)-mode solution induced by the GR force to lowest order in \( \alpha \) and in \( \Omega \), we will neglect in Eq. (17) the contribution coming from the terms \( \hat{v} \delta^{(1)} \rho \) and we will work only with the dominant terms \( \rho \delta^{(1)} \hat{v} \). This discussion applies similarly to the other current multipole tensors, whose first-order Eulerian change can be straightforwardly written from Eqs. (A3).

Under the above conditions, Eqs. (14) are inserted into Eq. (17) yielding
\[ \delta^{(1)} S^{[5]}_{xx} = -\alpha \Omega \sqrt{\frac{\pi}{5}} \frac{j}{R} e^{i \omega t} + O(\alpha \Omega^3), \]

where \( \omega \) is an arbitrary parameter, \( O(\alpha \Omega^3) \) denotes terms of order \( \alpha \Omega^3 \) or more, and \( j \) is defined as
\[ j = \int_0^R dr \hat{\rho} \hat{v} \delta^{(1)} \rho. \]

Similarly, it is straightforward to find that the first-order Eulerian change in the other multipole tensors satisfy the relations
\[ \delta^{(1)} S^{[6]}_{yy} = -\delta^{(1)} S^{[5]}_{xx}, \]
\[ \delta^{(1)} S^{[6]}_{xy} = \frac{i}{6} \delta^{(1)} S^{[5]}_{xx}, \]
\[ \delta^{(1)} S^{[6]}_{xz} = \delta^{(1)} S^{[6]}_{yz} = \delta^{(1)} S^{[6]}_{zz} = 0. \]

From Eqs. (18) and (20) it is then clear that the \( n^\text{th} \) time derivative of the multipole tensor perturbation is given by
\[ \delta^{(1)} S^{[n]}_{ij} = (i \omega)^n \delta^{(1)} S_{ij}. \]

In spite of the large number of terms that appear in the definition of the GR force [see Eqs. (A4) in the Appendix A], Eqs. (18), (20), and (21) allow a considerable simplification of the first-order Eulerian change in the GR force. Indeed, to lowest order in \( \Omega \), the first-order Eulerian change in the GR force is given simply by
\[ \delta^{(1)} F_{\text{GR}} \simeq -3i \kappa \frac{j}{R} \alpha \Omega^2 \omega_5 r^2 \sin^2 \theta \cos \theta e^{i(2 \phi + \omega t)}, \]
\[ \delta^{(1)} F_{\theta} \simeq i \kappa \frac{j}{R} \alpha \Omega \omega_5 (\omega_0 + 3 \Omega \sin^2 \theta) r^2 \sin \theta e^{i(2 \phi + \omega t)}, \]
\[ \delta^{(1)} F_{\phi} \simeq -\kappa \frac{j}{R} \alpha \Omega \omega_5^2 r^2 \sin \theta \cos \theta e^{i(2 \phi + \omega_0 t)}, \]
where the constant \( \kappa \) sets the strength of the GR reaction force and is defined as
\[ \kappa = \frac{16}{45} \sqrt{\frac{\pi}{5}} \frac{G}{c^3}. \]

D. Linear \( r \)-mode solution with the GR reaction contribution

We have now all that is needed to find the linear \( r \)-mode solution that solves, to lowest order in \( \Omega \), the linearized fluid equations (9)–(11) with the GR force.

It is well established that the GR drives the \( r \)-modes unstable. This means that the GR reaction induces a small complex imaginary part in the allowed \( r \)-mode frequency, i.e., one has
\[ \omega = \omega_0 + i \omega_1 \]

where \( \omega_0 \equiv \text{Re}[\omega] \) is the frequency of the \( r \)-mode and \( \omega_1 = \text{Im}[\omega] \) is the small imaginary part that is related to the growth timescale of the instability of the mode.

Assuming that \( \omega_1 / \omega_0 \ll 1 \), an assumption that will be checked at the end, the perturbed GR force (22) can finally be written as
\[ \delta^{(1)} F_{r} \simeq -3i \kappa \frac{j}{R} \alpha \Omega^2 \omega_0 \omega_5 r^2 \sin^2 \theta \cos \theta e^{-\omega_1 t} e^{i(2 \phi + \omega_0 t)}, \]
\[ \delta^{(1)} F_{\theta} \simeq i \kappa \frac{j}{R} \alpha \Omega \omega_0 (\omega_0 + 3 \Omega \sin^2 \theta) r^2 \sin \theta e^{-\omega_1 t} e^{i(2 \phi + \omega_0 t)}, \]
\[ \delta^{(1)} F_{\phi} \simeq -\kappa \frac{j}{R} \alpha \Omega \omega_0^2 r^2 \sin \theta \cos \theta e^{-\omega_1 t} e^{i(2 \phi + \omega_0 t)}. \]
The perturbed Euler equation (9), in the inertial frame, is given by

\begin{equation}
(\partial_t + \Omega \partial_\phi) \delta^{(1)} v_r - 2\Omega \sin \theta \delta^{(1)} v_\phi + \partial_r \delta^{(1)} U = \delta^{(1)} F^{GR}_r,
\end{equation}

\begin{equation}
(\partial_t + \Omega \partial_\phi) \delta^{(1)} v_\theta - 2\Omega \cos \theta \delta^{(1)} v_\phi + \frac{1}{r} \partial_\theta \delta^{(1)} U = \delta^{(1)} F^{GR}_\theta,
\end{equation}

\begin{equation}
(\partial_t + \Omega \partial_\phi) \delta^{(1)} v_\phi + 2\Omega \sin \theta \delta^{(1)} v_r + 2\Omega \cos \theta \delta^{(1)} v_\theta
+ \frac{1}{r \sin \theta} \partial_\phi \delta^{(1)} U = \delta^{(1)} F^{GR}_\phi,
\end{equation}

with \( \delta^{(1)} F^{GR} \) defined by Eqs. (25). In order to solve Eqs. (26), we try a solution that has the same structure as the GR force-free \( r \)-mode solution given by Eqs. (14) and (15). More specifically, we try an Ansatz of the type

\begin{equation}
\delta^{(1)} v_r = 0,
\end{equation}

\begin{equation}
\delta^{(1)} v_\theta = -\frac{i}{4} \sqrt{\frac{5}{\pi}} \frac{\alpha \Omega}{R} r^2 \sin \theta e^{-\omega t} e^{i(2\phi + \omega_0 t)}
+ \delta^{(1)} \tilde{v}_\theta(t, r, \theta, \phi),
\end{equation}

\begin{equation}
\delta^{(1)} v_\phi = \frac{1}{4} \sqrt{\frac{5}{\pi}} \frac{\alpha \Omega}{R} r^2 \sin \theta \cos \theta e^{-\omega t} e^{i(2\phi + \omega_0 t)}
+ \delta^{(1)} \tilde{v}_\phi(t, r, \theta, \phi),
\end{equation}

and

\begin{equation}
\delta^{(1)} U = \frac{1}{6} \sqrt{\frac{5}{\pi}} \frac{\alpha \Omega^2}{R^3} r^3 \sin^2 \theta \cos \theta e^{-\omega t} e^{i(2\phi + \omega_0 t)}
+ \delta^{(1)} \tilde{U}(t, r, \theta, \phi),
\end{equation}

where \( \delta^{(1)} \tilde{v} \) and \( \delta^{(1)} \tilde{U} \) are functions of \( t, r, \theta \) and \( \phi \) to be determined, and \( \varpi \) and \( \omega_0 \) are unknown constants to be fixed.

As already mentioned above [see Eq. (1)], it is known from previous investigations [10, 12] on the time evolution of the physical energy of the r-mode that \( \varpi = 1/\gamma_{GR} \) is proportional to \( \Omega^6 \). This result follows straightforwardly from an inspection of Eqs. (26) with \( \delta^{(1)} F^{GR} \) and \( \delta^{(1)} \tilde{v} \) defined by Eqs. (25) and (27), respectively. Indeed, the time derivative of \( \delta^{(1)} \tilde{v} \) on the left-hand side of Eqs. (26) yields terms proportional to \( \Omega \varpi \), while the right-hand side of these equations contains, to leading order in \( \Omega \), terms proportional to \( \Omega^7 \). Similarly, the \( r \)-mode frequency \( \omega_0 \) could also have a correction (relatively to the GR force-free value \( \omega_0 = -4\Omega/3 \)) proportional to \( \Omega^6 \). Therefore, we write \( \varpi \) and \( \omega_0 \) in the form

\begin{equation}
\varpi = -\frac{8}{9} \sqrt{\frac{\pi}{5}} \gamma R \Omega^6 a
\end{equation}

and

\begin{equation}
\omega_0 = -\frac{4\Omega}{3} + \frac{8}{9} \sqrt{\frac{\pi}{5}} \gamma R \Omega^6 b,
\end{equation}

where \( a \) and \( b \) are (by now arbitrary) real dimensionless constants and we have defined

\begin{equation}
\gamma = \frac{210}{3^4} \frac{J}{R} \kappa.
\end{equation}

Note that for \( a = 1 \) the growth timescale of the GR-induced \( r \)-mode instability, \( \tau_{GR} = 1/\varpi \), coincides with the one obtained in Refs. [10, 12]. For \( b = 0 \), the frequency \( \omega_0 \) coincides with that of the GR force-free linear \( r \)-mode solution.

For such an Ansatz the perturbed Euler equation reduces to the following system of differential equations for \( \delta^{(1)} \tilde{v}_\phi \), \( \delta^{(1)} \tilde{v}_\theta \) and \( \delta^{(1)} \tilde{U} \),

\begin{equation}
-2\Omega \sin \theta \delta^{(1)} \tilde{v}_\phi + \partial_r \delta^{(1)} \tilde{U}
= i\gamma \alpha \Omega^2 r^2 \sin^2 \theta \cos \theta e^{-\omega t} e^{i(2\phi + \omega_0 t)},
\end{equation}

\begin{equation}
(\partial_t + \Omega \partial_\phi) \delta^{(1)} \tilde{v}_\theta - 2\Omega \cos \theta \delta^{(1)} \tilde{v}_\phi + \frac{1}{r} \partial_\theta \delta^{(1)} \tilde{U}
= i\gamma \left[ \frac{2}{9} (a + b + 2) - \sin^2 \theta \right] \alpha \Omega^2 r^2 \sin \theta
\times e^{-\omega t} e^{i(2\phi + \omega_0 t)},
\end{equation}

\begin{equation}
(\partial_t + \Omega \partial_\phi) \delta^{(1)} \tilde{v}_\phi + 2\Omega \cos \theta \delta^{(1)} \tilde{v}_\theta + \frac{1}{r \sin \theta} \partial_\phi \delta^{(1)} \tilde{U}
= -\frac{2\gamma}{9} (a + b + 2) \alpha \Omega^2 r^2 \sin \theta \cos \theta e^{-\omega t} e^{i(2\phi + \omega_0 t)}.
\end{equation}

The right-hand side of the above system induces a solution of the form

\begin{equation}
\delta^{(1)} \tilde{v}_\phi = i\gamma \alpha \Omega^2 r^2 f(\theta) e^{-\omega t} e^{i(2\phi + \omega_0 t)},
\end{equation}

\begin{equation}
\delta^{(1)} \tilde{v}_\theta = -\gamma \alpha \Omega^2 r^2 g(\theta) e^{-\omega t} e^{i(2\phi + \omega_0 t)},
\end{equation}

and

\begin{equation}
\delta^{(1)} \tilde{U} = \frac{i}{3} \gamma \alpha \Omega^2 r^3 h(\theta) e^{-\omega t} e^{i(2\phi + \omega_0 t)},
\end{equation}

where \( f(\theta), g(\theta) \) and \( h(\theta) \) are functions of \( \theta \) determined by the system of equations

\begin{equation}
2 \sin \theta g(\theta) + ib(\theta) = i \sin^2 \theta \cos \theta,
\end{equation}

\begin{equation}
2 f(\theta) - 6 \cos \theta g(\theta) - i \partial_\theta h(\theta)
= -3i \sin \theta \left[ \frac{2}{9} (a + b + 2) - \sin^2 \theta \right],
\end{equation}

\begin{equation}
3i \sin \theta \cos \theta f(\theta) - i \sin \theta g(\theta) - h(\theta)
= -\frac{1}{3} (a + b + 2) \sin^2 \theta \cos \theta.
\end{equation}
From the system of equations (35) it is straightforward to obtain a linear first-order differential equation for \( h(\theta) \), namely,

\[
\frac{dh(\theta)}{d\theta} + \frac{1 - 3 \cos^2 \theta}{\sin \theta \cos \theta} h(\theta) = q(\theta),
\]

where the source term \( q(\theta) \) is given by

\[
q(\theta) = \frac{8}{9} (a + bi - 1) \sin \theta.
\]  

(37)

This equation has the solution

\[
h(\theta) = A \sin^2 \theta \cos \theta + \frac{8}{9} (a + bi - 1) \sin \theta \cos \theta \ln(\tan \theta),
\]

(38)

where \( A \) is a (complex) constant determined by initial data. In Eq. (38), the first term of the right-hand side corresponds to the general solution of the homogeneous equation [Eq. (36) with \( q(\theta) = 0 \)], and the second term on the right-hand side corresponds to the particular solution of Eq. (36) with the source term \( q(\theta) \). From Eqs. (35) and (38) is then straightforward to obtain:

\[
f(\theta) = -\frac{i}{2} (A - 1) \sin \theta + \frac{i}{9} (a + bi - 1) \sin \theta
\]

\[
- \frac{4i}{9} (a + bi - 1) \sin \theta \ln(\tan \theta),
\]

(39)

\[
g(\theta) = -\frac{i}{2} (A - 1) \sin \theta \cos \theta
\]

\[
- \frac{4i}{9} (a + bi - 1) \sin \theta \cos \theta \ln(\tan \theta).
\]

(40)

The above solution to the perturbed Euler equation must also satisfy the continuity equation (10), which at order \( \alpha \Omega^5 \) is simply given by

\[
\vec{\nabla} \cdot \vec{\delta}^{(1)} \tilde{\vec{v}} = 0.
\]

(41)

Inserting \( \delta^{(1)} \vec{v} \) from Eq. (33), with \( f(\theta) \) and \( g(\theta) \) given by Eqs. (39) and (40), into the continuity equation (41) we obtain:

\[
\partial_\theta [ \sin \theta f(\theta) ] - 2g(\theta) = 0
\]

(42)

or

\[
- \frac{4i}{9} (a + bi - 1) \sin \theta \cos \theta + \frac{2i}{9} (a + bi - 1) \sin \theta \cos \theta = 0.
\]

(43)

implying that \( a = 1 \) and \( b = 0 \).

Therefore, we arrive at the conclusion that the \( r \)-mode velocity perturbations are sinusoidal with the same frequency as the well-known GR force-free linear \( r \)-mode solution, and that the GR force drives the \( r \)-modes unstable with a growth timescale that agrees with the expression found in Refs. [10, 12].

Inserting solutions (33) and (34) [with \( f(\theta) \), \( g(\theta) \) and \( h(\theta) \) given by Eqs. (38)–(40), \( a = 1 \) and \( b = 0 \)] into Eqs. (27) and (28) we finally get that the \( r \)-mode solution to the linearized fluid equations with the GR force is given by

\[
\delta^{(1)} v_r = 0,
\]

(44a)

\[
\delta^{(1)} v_\theta = -\frac{i}{2} \alpha \Omega \left[ \frac{1}{2} \sqrt{\frac{5}{\pi R}} + i \gamma (A - 1) \Omega^5 \right] r^2 \sin \theta
\]

\[
\times e^{-\sqrt{\frac{\pi}{5}} \gamma R \Omega^6 t},
\]

(44b)

\[
\delta^{(1)} v_\phi = \frac{1}{2} \alpha \Omega \left[ \frac{1}{2} \sqrt{\frac{5}{\pi R}} + i \gamma (A - 1) \Omega^5 \right] r^2 \sin \theta \cos \theta
\]

\[
\times e^{-\sqrt{\frac{\pi}{5}} \gamma R \Omega^6 t},
\]

(44c)

and

\[
\delta^{(1)} U = \frac{1}{3} \alpha \Omega^2 \left[ \frac{1}{2} \sqrt{\frac{5}{\pi R}} + i \gamma A \Omega^5 \right] r^3 \sin^2 \theta \cos \theta
\]

\[
\times e^{-\sqrt{\frac{\pi}{5}} \gamma R \Omega^6 t},
\]

(45)

with \( \varpi \) and \( \omega_0 \) given by

\[
\varpi = -\frac{8}{9} \sqrt{\frac{\pi}{5}} \gamma R \Omega^6
\]

(46)

and

\[
\omega_0 = -\frac{4 \Omega}{3}.
\]

(47)

The velocity perturbations given by Eqs. (44) have a piece similar to the GR force-free solution, the difference being the factor \( e^{-\sqrt{\varpi} t} \) responsible for the exponential growth of the \( r \)-mode amplitude due to the presence of a GR reaction force, and another piece proportional to \( \alpha \gamma (A - 1) \Omega^{5/2} \), where \( A \) is a constant fixed by the choice of initial data. If this constant is chosen to be of order unity, \( A \sim \mathcal{O}(1) \), then the second term in the square brackets of Eqs. (44) and (45) is much smaller than the first term. Indeed, let us assume that the mass density \( \rho \) and the pressure \( P \) of the perfect-fluid star are related by the usual polytropic equation of state \( P = k \rho^\gamma \), with \( k \) such that \( M = 1.4M_\odot \) and \( R = 12.53 \). For such a choice of the equation of state, \( \tilde{J} = 1.12 \times 10^{45} \) kg m\(^4\) and the Keplerian angular velocity at which the star starts shedding mass through the equator is \( \Omega_K = \frac{2 \mu}{3} \sqrt{G \rho} = 5612 \) sec\(^{-1}\), where \( \rho = 3.38 \times 10^{17} \) kg/m\(^3\) is the average mass density of the star and \( G \) is the gravitational constant. Then, in Eqs. (44) [or similarly in Eq. (45)] the expression inside the square brackets can be written as \( \sqrt{5/(4\pi \tilde{R}^2)}[1 + 1.08 \times 10^{-5}i(A - 1)/\Omega(\Omega_K)^5] \). For \( A \sim \mathcal{O}(1) \) and \( \Omega \lesssim \Omega_K \), the second term in the square bracket can be neglected.

Note that in the GR force-free limit, the constant \( \kappa \) that sets the strength of the GR reaction force goes to zero. Therefore, in this limit \( \gamma \) defined in Eq. (31) goes to zero and the \( \delta^{(1)} \vec{v} \) perturbations vanish as they should, since they are proportional to \( \gamma \).
IV. DISCUSSION OF THE RESULTS

Analytical $r$-mode solutions were investigated, within the linearized theory, in the case of a slowly rotating, Newtonian, barotropic, non-magnetized, perfect-fluid star in which a gravitational radiation (GR) reaction force is present. For the GR reaction term we have used the 3.5 post-Newtonian order expansion of the GR force, in order to include the contribution of the current quadrupole moment. The evaluation of the GR force perturbation requires the knowledge of the multipole moment perturbation and its time derivatives. To compute this Eulerian change in the multipole moment we have used an approach in which we assumed that the GR force-free linear $r$-mode solution acts as a source for the current multipole moment. We have carried our analysis of the perturbed fluid equations only to lowest order in $\Omega$. This order is enough to evaluate the GR reaction contribution. A higher order analysis would include perturbations in the density which would be important if we were interested in the accurate evaluation of the viscosity effects.

We have found the analytical expression of the $r$-mode velocity perturbations that solves the Newtonian hydrodynamic equations with the GR reaction force. From this solution, given by Eqs. (44) and (45), four important features of the $r$-mode instability driven by the GR reaction can be read:

(i) The velocity perturbations $\delta v^{(1)}$ are proportional to $e^{i(2\phi + \omega_0 t)}$, with $\omega_0 = -4\Omega/3$. Thus, they have the same sinusoidal behavior and the same frequency $\omega_0$ as the solution of the GR force-free linear $r$-mode perturbation. To this limit, $\tau$ is significantly from the evolution in slowly-rotating stars [30].

(ii) The amplitude of the velocity perturbations is proportional to $\exp\{-\pi t\}$. Since $\pi < 0$, the GR force induces then an exponential growth in the $r$-mode amplitude. The e-folding growth timescale $\tau_{GR} = 1/\pi$ agrees with the GR timescale (1) first found in Ref. [10]. Thus, these velocity perturbations are consistent with the expression found in Ref. [10] for the time evolution of the energy of the $r$-mode perturbation, $dE/dt$.

(iii) The velocity perturbations $\delta v^{(1)}$ contain a piece proportional to $\gamma(A - 1)\alpha\Omega^6$, where $A$ is an arbitrary constant fixed by the choice of initial data. If we choose this constant $A$ to be of order unity, then this part of the solution could be neglected.

(iv) The parameter $\kappa$, defined in Eq. (23), sets the strength of the GR reaction force. Thus, the GR force-free limit is obtained when we set $\kappa$ equal to zero. In this limit, $\pi$ and $\gamma$ defined, respectively, by Eqs. (29) and (31) go to zero. Then, from Eqs. (44) and (45) we recover the GR force-free linear $r$-mode solution given by Eqs. (14) and (15).

Our results are strictly valid only for the slow-rotation regime since our solution holds only to leading order. However, there are good indications that the evolution of $r$-modes in rapidly-spinning stars does not differ significantly from the evolution in slowly-rotating stars [30]. Hence, it is reasonable to expect that at least the qualitative features that we found are also valid in the rapid-rotation regime.

A natural extension of the present work is to try to find an analytical $r$-mode solution of the nonlinear hydrodynamic equations with the GR reaction force. The nonlinear GR force-free case was analyzed in Ref. [26], where an analytical $r$-mode solution in the nonlinear theory up to second order in the mode amplitude was found. This solution represents differential rotation that produces large scale drifts of fluid elements along stellar latitudes. It contains two separate pieces, one induced by first-order quantities and another determined by the choice of initial data. Since these two pieces cannot cancel each other, differential rotation is an unavoidable nonlinear kinematic feature of $r$-modes. The analysis of Ref. [26] can be extended to the case in which the GR reaction force is present. The main purpose of this extension is to find if the GR reaction force provides an extra source of differential rotation. This work is in progress [31].

Acknowledgements

It is a pleasure to acknowledge Shijun Yoshida, Kostas Kokkotas and Luciano Rezzolla for useful suggestions that improved the manuscript. We also acknowledge the anonymous referee for her/his valuable comments. This work was supported in part by Fundação para a Ciência e Tecnologia (FCT). OJCD acknowledges financial support from FCT through grant SFRH/BPD/2004.

APPENDIX A: EXPLICIT COMPUTATION OF THE GR REACTION FORCE

In the cartesian basis the explicit components of the gravitational vector potential (6) are

$$\beta_x = \frac{16G}{45c^7} \left[ yz \left( S_{zz}^5 - S_{yy}^5 \right) - xzS_{xy}^5 + xyS_{xx}^5 + \left( y^2 - z^2 \right) S_{yy}^5 \right], \quad (A1a)$$

$$\beta_y = \frac{16G}{45c^7} \left[ \right. \right.$$

$$\left. \left. + yzS_{xy}^5 - \left( x^2 - z^2 \right) S_{xx}^5 - xyS_{yy}^5 \right], \quad (A1b)$$

$$\beta_z = \frac{16G}{45c^7} \left[ xy \left( S_{yy}^5 - S_{xx}^5 \right) + \left( x^2 - y^2 \right) S_{xy}^5 - yzS_{xz}^5 + xzS_{yz}^5 \right]. \quad (A1c)$$

The use of the usual relations between cartesian $(x, y, z)$ and spherical $(r, \theta, \phi)$ coordinates yields the components of the gravitational vector potential $\beta$ in spherical coor-
where the current multipole tensor in spherical coordinates is given explicitly by

\[ S_{xx} = -\int dV \rho r^2 \sin \theta \cos \phi (v_\theta \sin \phi + v_\phi \cos \theta \cos \phi), \quad (A3a) \]

Inserting Eqs. (A2) into Eq. (5) yields the spherical components of the GR reaction force,

\[ F_{r}^{GR} = \frac{-16G}{15c^2} v_\theta \left\{ \cos \theta \left( \cos \phi S_{yz}^{[5]} + \sin \phi S_{xz}^{[5]} \right) + \sin \theta \left[ \frac{1}{2} \sin(2\phi) \left( -S_{xx}^{[5]} + S_{yy}^{[5]} + \cos(2\phi) S_{xy}^{[5]} \right) \right] \right\} \]

\[ + \frac{8G}{15c^2} v_\phi \left\{ \sin(2\theta) \sin(2\phi) S_{xy}^{[5]} + \frac{1}{2} \sin(2\theta) \left[ 3 \left( S_{xx}^{[5]} + S_{yy}^{[5]} \right) + \cos(2\phi) \left( S_{xx}^{[5]} - S_{yy}^{[5]} \right) \right] \right\} \]

\[ + 2 \cos(2\theta) \left( \cos \phi S_{yz}^{[5]} + \sin \phi S_{xz}^{[5]} \right) \}, \quad (A4a) \]

\[ F_{\theta}^{GR} = \frac{16G}{45c^2} r v_\theta \left\{ \cos \theta \left( \cos \phi S_{yz}^{[6]} + \sin \phi S_{xz}^{[6]} \right) + \sin \theta \left[ \cos(2\phi) S_{xy}^{[6]} + \frac{1}{2} \sin(2\phi) \left( S_{yy}^{[6]} - S_{xx}^{[6]} \right) \right] \right\} \]

\[ + \frac{4G}{45c^2} r v_\phi \left\{ 3 \left( S_{xy}^{[6]} + S_{xx}^{[6]} \right) (1 + 3 \cos(2\theta)) - 6 \sin^2 \theta \left[ \cos(2\phi) \left( S_{xx}^{[6]} - S_{yy}^{[6]} \right) + 2 \sin(2\phi) S_{xy}^{[6]} \right] \right\} \]

\[ - 12 \sin(2\theta) \left( \cos \phi S_{yz}^{[6]} + \sin \phi S_{xz}^{[6]} \right) \}

\[ + \frac{16G}{15c^2} r v_\theta \left\{ \cos \theta \left( \cos \phi S_{yz}^{[6]} - \sin \phi S_{xz}^{[6]} \right) + \sin \theta \left[ \frac{1}{2} \sin(2\phi) \left( -S_{xx}^{[6]} + S_{yy}^{[6]} + \cos(2\phi) S_{xy}^{[6]} \right) \right] \right\} \}, \quad (A4b) \]

\[ F_{\phi}^{GR} = \frac{-8G}{45c^2} \left\{ 2 \cos(2\theta) \cos \phi S_{xx}^{[6]} + \frac{1}{2} \sin(2\theta) \left[ 3 \left( S_{xx}^{[6]} + S_{yy}^{[6]} \right) + \cos(2\phi) \left( S_{xx}^{[6]} - S_{yy}^{[6]} \right) \right] \right\} \]

\[ + 2 \cos(2\theta) \sin \phi S_{yz}^{[6]} + \sin(2\phi) S_{xy}^{[6]} \}

\[ - \frac{8G}{15c^2} r v_\theta \left\{ \sin(2\theta) \sin(2\phi) S_{xy}^{[5]} + \frac{1}{2} \sin(2\theta) \left[ 3 \left( S_{xx}^{[5]} + S_{yy}^{[5]} \right) + \cos(2\phi) \left( S_{xx}^{[5]} - S_{yy}^{[5]} \right) \right] \right\} \]

\[ + 2 \cos(2\theta) \left( \cos \phi S_{yz}^{[5]} + \sin \phi S_{xz}^{[5]} \right) \}

\[ - \frac{4G}{45c^2} r v_\phi \left\{ 3 \left( S_{xy}^{[5]} + S_{xx}^{[5]} \right) (1 + 3 \cos(2\theta)) - 6 \sin^2 \theta \left[ \cos(2\phi) \left( S_{xx}^{[5]} - S_{yy}^{[5]} \right) + 2 \sin(2\phi) S_{xy}^{[5]} \right] \right\} \]

\[ - 12 \sin(2\theta) \left( \cos \phi S_{yz}^{[5]} + \sin \phi S_{xz}^{[5]} \right) \}, \quad (A4c) \]
where we have made use of the relation $S_{zz} = -(S_{xx} + S_{yy})$ [see Eqs. (A3)] to remove $S_{zz}$ from the final expression.

As a check to our expressions for the GR reaction terms, we apply them to the spherical shell system discussed in detail in Ref. [25]. The authors have computed the instability timescale driven by the GR reaction and concluded that it is given by Eq. (1) with $\hat{J} \equiv \rho R^2$, and $\rho$ and $R$ being the surface density and the radius of the spherical shell. Now, if we introduce the $r$-mode velocity given by Eqs. (1) and (7) of Ref. [25] into our Eqs. (A2) and (A3), we obtain Eqs. (B14) and (55) of Ref. [25] that define the timescale of the instability on the shell.

**APPENDIX B: CONNECTION TO PREVIOUS WORKS**

In previous works, in which the effects of the GR force on the $r$-mode evolution were studied numerically, the final form of the GR force was always written as a function of the multipole moments $J_{lm}$, instead of the quadrupole tensors $J_{ij}$ [22–24]. To compare our expressions with those of Refs. [22–24] we establish the connection between the two approaches in this Appendix.

The current multipole moments are defined as [11]

$$J_{lm} = \int dV \rho \bar{v} \cdot \bar{Y}_{lm}^B,$$  \hspace{1cm} (B1)

where $l$ and $m$ are the angular momentum and the azimuthal numbers, respectively, $\bar{Y}_{lm}^B$ is a spherical harmonic vector of the magnetic type defined by [11]

$$\bar{Y}_{lm}^B = (-1)^m \bar{r}^m \bar{Y}_{lm}^B,$$  \hspace{1cm} (B2)

and

$$\bar{Y}_{lm}^B = [l(l+1)]^{-1/2} \bar{r} \times (\bar{\nabla} Y_{lm}).$$  \hspace{1cm} (B3)

The usual spherical harmonic functions $Y_{lm}$ are given by

$$Y_{lm} = (-1)^m \left[ \frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!} \right]^{1/2} \bar{r}^m P_l^m(\cos \theta),$$  \hspace{1cm} (B4)

where $\theta$ and $\phi$ are the angular spherical coordinates and $P_l^m(\cos \theta)$ are the associated Legendre functions defined by

$$P_l^m(\cos \theta) = \frac{1}{2^l l!} \sin^{l-m} \theta \frac{d^{l+m}}{d \cos \theta^{l+m}}(\cos^2 \theta - 1)^l.$$  \hspace{1cm} (B5)

Like in our case, the authors of Refs. [22–24] are interested only in the most unstable mode, the $l = 2$ $r$-mode. This mode excites the $J_{2m}$ current multipole moments which in a spherical coordinate system are written as

$$J_{22} = \frac{1}{4} \sqrt{\frac{5}{\pi}} \int dV \rho \bar{v} (r \sin \theta \cos \phi - i \bar{v}_\phi),$$  \hspace{1cm} (B6a)

$$J_{21} = \frac{1}{4} \sqrt{\frac{5}{\pi}} \int dV \rho \bar{v} (r \sin \theta \cos \phi + i \bar{v}_\phi),$$  \hspace{1cm} (B6b)

$$J_{20} = -\frac{1}{4} \sqrt{\frac{15}{2\pi}} \int dV \rho r^2 \sin (2\theta).$$  \hspace{1cm} (B6c)

From Eqs. (A3) and (B6) it is straightforward to verify the following correspondence between the current quadrupole tensor $S_{ij}$ and the current multipole moments $J_{2m}$ [23]:

$$S_{yy} - S_{xx} + 2i S_{xy} = 4 \sqrt{\frac{\pi}{5}} J_{22},$$  \hspace{1cm} (B7a)

$$S_{xx} - i S_{yz} = 2 \sqrt{\frac{\pi}{5}} J_{21},$$  \hspace{1cm} (B7b)

$$S_{xx} + S_{yy} = - S_{zz} = 2 \sqrt{\frac{2\pi}{15}} J_{20}.$$  \hspace{1cm} (B7c)

The most unstable $r$-mode is in fact the $l = m = 2$ mode which excites $J_{22}$ but not $J_{21}$ and $J_{20}$. Hence, in Refs. [22–24], the GR reaction force is computed assuming that, in Eq. (B7), $J_{21} \simeq 0$ and $J_{20} \simeq 0$, i.e.,

$$S_{yy} - S_{xx} + 2i S_{xy} = 4 \sqrt{\frac{\pi}{5}} J_{22},$$  \hspace{1cm} (B8a)

$$S_{xx} - i S_{yz} \simeq 0,$$  \hspace{1cm} (B8b)

$$S_{xx} + S_{yy} \simeq - S_{zz} \simeq 0.$$  \hspace{1cm} (B8c)

It is also assumed there that

$$J_{22}^{[n]} \simeq (i \omega)^n J_{22},$$  \hspace{1cm} (B9)

and it is found that the GR force in the inertial frame is given by

$$F_r^{GR} \simeq 3\kappa (\bar{v}_\theta - v_\phi \cos \theta) r \sin \theta e^{2i\phi} J_{22}^{[5]},$$  \hspace{1cm} (B10a)

$$F_{\theta}^{GR} \simeq -i \kappa r^2 \sin \theta e^{2i\phi} J_{22}^{[6]} + 3\kappa (\bar{v}_r + v_\phi \sin \theta) r \sin \theta e^{2i\phi} J_{22}^{[5]},$$  \hspace{1cm} (B10b)

$$F_{\phi}^{GR} \simeq \kappa r^2 \sin \theta \cos \theta e^{2i\phi} J_{22}^{[6]} + 3\kappa (\bar{v}_r \cos \theta - v_\phi \sin \theta) r \sin \theta e^{2i\phi} J_{22}^{[5]}.$$  \hspace{1cm} (B10c)

Now, let us apply the perturbative approach used in our paper to check if our results match with Eq. (B10). If we linearize the multipole moments, $J_{lm} = \hat{J}_{lm} + \delta(1) J_{lm}$, where $\hat{J}_{lm}$ describes the multipole moment of the unperturbed star and $\delta(1) J_{lm}$ is the first-order Eulerian change in the multipole moment, it is straightforward to show, using Eqs. (B6) and (13), that

$$\hat{J}_{lm} = 0; \hspace{1cm} l = 2; m = 0, 1, 2,$$  \hspace{1cm} (B11)
and the use of Eqs. (B6) and (14) yields

\[
\delta^{(1)}J_{22} = \alpha \Omega \frac{J}{\Re e^{i\omega t}} + \mathcal{O}(\alpha \Omega^3), 
\]

(B12a)

\[
\delta^{(1)}J_{21} = 0 + \mathcal{O}(\alpha \Omega^3),
\]

(B12b)

\[
\delta^{(1)}J_{22} = 0 + \mathcal{O}(\alpha \Omega^3),
\]

(B12c)

together with

\[
\delta^{(1)}J_{22}^{[n]} = (i\omega)^n \delta^{(1)}J_{22}. 
\]

(B13)

3 In fact, the value of \( \kappa \) in Refs. [22–24] is two times ours (compare our Eq. (23) with Eqs. (A.8) and (A.9) of Ref. [23]). Their extra factor of 2 appears to be a typographical mistake. Indeed, with our value of \( \kappa \) we find an exact agreement with the instability timescale of the \( r \)-mode in a spherical shell [see discussion after Eq. (A4)] and we also get for the Newtonian star a GR growth timescale that agrees with the one found in Refs. [10, 12]. If we use the value of \( \kappa \) defined in Refs. [22–24] we obtain a timescale that is twice the correct one in both geometries.