Triviality of Equivariant Maps in Crossed Products and Matrix Algebras

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Abstract
We consider a “twisted” noncommutative join procedure for unital C*-algebras which admit actions by a compact abelian group $G$ and its discrete abelian dual $\Gamma$, so that we may investigate an analogue of Baum-Dąbrowski-Hajac noncommutative Borsuk-Ulam theory in the twisted setting. Namely, under what conditions is it guaranteed that an equivariant map $\phi$ from a unital C*-algebra $A$ to the twisted join of $A$ and $\mathcal{C}^*(\Gamma)$ cannot exist? This pursuit is motivated by the twisted analogues of even spheres, which admit the same $K_0$ groups as even spheres and have an analogous Borsuk-Ulam theorem that is detected by $K_0$, despite the fact that the objects are not themselves deformations of a sphere. We find multiple sufficient conditions for twisted Borsuk-Ulam theorems to hold, one of which is the addition of another equivariance condition on $\phi$ that corresponds to the choice of twist. However, we also find multiple examples of equivariant maps $\phi$ that exist even under fairly restrictive assumptions. Finally, we consider an extension of unital contractibility (in the sense of Dąbrowski-Hajac-Neshveyev) “modulo $k$.”

Keywords: Borsuk-Ulam, twisted join, twisted suspension, free actions, torsion

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1 Introduction

If $X$ and $Y$ are compact Hausdorff spaces which admit free actions of a nontrivial finite group $G$, then continuous, equivariant maps from $X$ to $Y$ are severely restricted by dimension and homotopy invariants. Such claims are rooted in the Borsuk-Ulam theorem, which concerns spheres $S^n$ and the free $\mathbb{Z}/2\mathbb{Z}$ action $x \mapsto -x$. In this setting, an equivariant map $f$ is just an odd function, that is, a function that satisfies $f(-x) = -f(x)$ for each $x$ in the domain.

**Theorem 1.1** (Borsuk-Ulam Theorem). If $n$ is a positive integer, then the following hold.

1. Every continuous function from $S^n$ to $\mathbb{R}^n$ admits at least one pair of opposite points $x$ and $-x$ with the same image.

2. Every continuous, odd function from $S^n$ to $\mathbb{R}^n$ has a zero.

3. There is no continuous, odd function from $S^n$ to $S^{n-1}$.

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4. Every continuous, odd function from $S^{n-1}$ to $S^{n-1}$ is homotopically nontrivial.

5. Every continuous, odd function from $S^{n-1}$ to $S^{n-1}$ has odd degree.

Items 1-4 above are equivalent, and they follow from the stronger item 5. Generalizations and consequences of the Borsuk-Ulam theorem abound in combinatorics and algebraic topology, in which more general spaces, groups, and actions replace spheres, $\mathbb{Z}/2\mathbb{Z}$, and the antipodal action. More recently, significant progress has been made in generalizing these results to noncommutative topology, opening up many new potential problems. The reason for this is twofold: first, the behavior of noncommutative $C^*$-algebras can be quite different than the picture painted by compact Hausdorff topology, and second, the noncommutative setting allows discussion of compact quantum group actions in addition to actions of compact groups. The aim of such results is to restrict the existence or homotopy properties of equivariant homomorphisms between unital $C^*$-algebras. As such, the noncommutative join constructions in [5] play a major role in this story.

**Definition 1.2** ([5]). Let $A$ and $B$ be unital $C^*$-algebras. Then the noncommutative join of $A$ and $B$ is

$$A \bigodot B = \{ f \in C([0,1], A \otimes_{\min} B) : f(0) \in C \otimes B, f(1) \in A \otimes C \}.$$ 

If $(H, \Delta)$ is a compact quantum group and $\delta : A \to A \otimes H$ is a free coaction of $H$ on $A$, then the equivariant noncommutative join of $A$ and $H$ is

$$A \bigodot_\delta H = \{ f \in C([0,1], A \otimes_{\min} H) : f(0) \in C \otimes H, f(1) \in \delta(A) \}.$$ 

Further, $A \bigodot_\delta H$ admits a free coaction $\delta_\Delta$, which applies $id \otimes \Delta$ pointwise on $A \otimes_{\min} H$.

In the above definition, freeness is meant in the sense of [8] (see also [2]). When $H$ is equal to $C(G)$ for a compact group $G$, the two joins $A \bigotimes C(G)$ and $A \bigodot_\delta C(G)$ are isomorphic, and if $A \bigodot C(G)$ is given the diagonal action of $G$, the natural choice of isomorphism is equivariant for the diagonal action and $\delta_\Delta$. That is, the equivariant join is needed precisely when the quantum group $H$ is not commutative, to avoid applying a diagonal action. Further, the iterated join $\bigodot_{i=1}^n C(\mathbb{Z}/2\mathbb{Z})$ is isomorphic to $C(S^{n-1})$, and the induced diagonal action (at each additional join) is just the antipodal action of $\mathbb{Z}/2\mathbb{Z}$ on $C(S^{n-1})$. The conjectures of Baum, Dabrowski, and Hajac in [1] therefore directly generalize the Borsuk-Ulam theorem.

**Conjecture 1.3.** ([1] Conjecture 2.3)] Let $A$ be a unital $C^*$-algebra with a free action $\delta$ of a nontrivial compact quantum group $(H, \Delta)$. Also, let $A \bigodot_\delta H$ denote the equivariant noncommutative join of $A$ and $H$, with the induced action of $(H, \Delta)$ given by $\delta_\Delta$. Then both of the following hold.

1. There does not exist a $(\delta, \delta_\Delta)$-equivariant, unital $*$-homomorphism from $A$ to $A \bigodot_\delta H$.

2. There does not exist a $(\Delta, \delta_\Delta)$-equivariant, unital $*$-homomorphism from $H$ to $A \bigodot_\delta H$.

We note that when $H = C(\mathbb{Z}/2\mathbb{Z})$, $A \bigodot_\delta C(\mathbb{Z}/2\mathbb{Z})$ is equivariantly isomorphic to the unreduced suspension of $A$, $\Sigma A$, discussed in [4]. More specifically, $\Sigma A$ is the (non-equivariant) noncommutative join $A \bigodot C(\mathbb{Z}/2\mathbb{Z})$, and the action presented in [4] is the diagonal action of $\mathbb{Z}/2\mathbb{Z}$.

Conjecture 1.3 type 1 holds when $H$ is a compact quantum group with a torsion character other than a counit, as in [3 Corollary 2.7], from a direct application of the case $H = C(\mathbb{Z}/k\mathbb{Z})$ in [17 Corollary 2.4]. Conjecture 2 is false, and counterexamples exist for compact groups acting on nuclear $C^*$-algebras from [3 Theorem 2.6]. However, there are certain key examples for which the type 2 conjecture holds, as in [3 10 11 6]. In [17], we proposed a different type of join (and similarly, unreduced suspension) for $C^*$-algebras with free actions of $\mathbb{Z}/k\mathbb{Z}$, replacing the tensor product with a crossed product. We generalize this definition and adopt new terminology/notation to avoid confusion with Definition 1.2.
If \( \Gamma \) is a discrete abelian group and \( G \) is its compact abelian Pontryagin dual, then an action \( \alpha \) of \( G \) on \( A \) is equivalent to a grading of \( A \) by \( \Gamma \), or a coaction \( \delta : A \rightarrow A \otimes C^*(\Gamma) \) of the compact quantum group \( C^*(\Gamma) = C(G) \). For \( \gamma \in \Gamma \), which we may view as a character on \( G \), the spectral subspace
\[
A_\gamma = \{ a \in A : \alpha_g(a) = \gamma(g)a \text{ for all } g \in G \}
\]
is written in coaction form as the \( \gamma \)-isotypic subspace
\[
A_\gamma = \{ a \in A : \delta(a) = a \otimes \gamma \},
\]
and the subspaces \( A_\gamma \) produces a grading of \( A \). When \( \gamma = 1 \), \( A_\gamma \) is called the fixed-point subalgebra. Moreover, freeness of the (co)action is equivalent to a saturation property
\[
\forall \gamma \in \Gamma, \quad 1 \in \overline{A_\gamma A_\gamma^*}.
\]  
(1.4)

See [2] Theorem 0.4 for the equivalence of freeness and saturation in greater generality, as well as [19, 11, 9] for some useful discussion of saturation properties in the context of group actions, and [14] for related conditions when \( \Gamma \) is finite (but not necessarily abelian).

Definition 1.5. Let \( \Gamma \) be a discrete abelian group with \( G = \hat{\Gamma} \). Suppose \( A \) is a unital \( C^* \)-algebra and \( \beta \) is an action of \( \Gamma \) on \( A \). Then the \( \beta \)-twisted join of \( A \) and \( C^*(\Gamma) \cong C(G) \) is
\[
J(A, \beta) = \{ f \in C([0,1], A \rtimes_\beta \Gamma) : f(0) \in C^*(\Gamma), f(1) \in A \}.
\]  
(1.6)

When \( \Gamma = \mathbb{Z}/2\mathbb{Z} \), we call \( J(A, \beta) \) the \( \beta \)-twisted unreduced suspension of \( A \) and \( C^*(\mathbb{Z}/2\mathbb{Z}) \cong C(\mathbb{Z}/2\mathbb{Z}) \).

If \( \beta \) is the trivial action, then \( J(A, \beta) \) is just \( A \otimes C(G) \).

There is a natural grading of \( A \rtimes_\beta \Gamma \) by \( \Gamma \), which extends to a grading of \( J(A, \beta) \) pointwise, that places each \( \gamma \in \Gamma \) in the \( \gamma \)-isotypic subspace and each \( a \in A \) in the fixed-point subalgebra. This grading is determined from the action \( \beta \) of the compact group \( G = \hat{\Gamma} \) on \( A \rtimes_\beta \Gamma \), given by
\[
\beta_\gamma(a) = a, \quad a \in A \quad \quad \beta_\gamma(\gamma) = g(\gamma), \quad \gamma \in \Gamma.
\]

If \( \alpha \) is an action of \( G \) on \( A \) which commutes with \( \beta \), then we extend \( \alpha \) to \( A \rtimes_\beta \Gamma \) so that \( \alpha \) fixes all elements of the group \( \Gamma \). The composition \( \alpha \beta = \beta \alpha \) produces an action of \( G \) on \( A \rtimes_\beta \Gamma \), and hence a grading of \( A \rtimes_\beta \Gamma \) by \( C^*(\Gamma) \), generated by the following rule: if \( a \) is in the \( \tau \)-isotypic subspace of \( A \) from \( \alpha \), and \( \gamma \in \Gamma \), then \( a \gamma = \gamma \beta(a) \) is in the \( \tau \gamma \)-isotypic subspace of \( A \rtimes_\beta \Gamma \). In particular, if \( \Gamma = \mathbb{Z}/2\mathbb{Z} \) is generated by \( \mu \), then \( \alpha \beta \) is generated by the automorphism
\[
a_0 + a_1 \mu \quad \mapsto \quad \alpha(a_0) - \alpha(a_1) \mu.
\]

Similarly, if \( \Gamma = \mathbb{Z}/k\mathbb{Z} \) is generated by \( \mu \) and \( \omega = e^{2\pi i/k} \), then \( \alpha \beta \) is generated by the automorphism
\[
a_0 + a_1 \mu + \ldots + a_{k-1} \mu^{k-1} \quad \mapsto \quad \alpha(a_0) + \omega \alpha(a_1) \mu + \ldots + \omega^{k-1} \alpha(a_{k-1}) \mu^{k-1}.
\]

Definition 1.7. If \( \alpha \) is an action of a compact abelian group \( G \) on \( A \) that commutes with an action \( \beta \) of \( \Gamma = \hat{G} \) on \( A \), then let \( \alpha \beta \) denote the pointwise application of the action \( \alpha \beta \) on \( J(A, \beta) \).

In the non-twisted setting, when \( \beta \) is trivial, \( \alpha \beta \) corresponds to the diagonal action of \( G \) on \( A \otimes C(G) \). Hence, if \( G = \mathbb{Z}/k\mathbb{Z} \), \( \alpha \) is free, and \( \beta \) is trivial, then there are no \( (\alpha, \beta) \)-equivariant unital \( * \)-homomorphisms from \( A \) to \( J(A, \beta) \), as seen in [17] Corollary 2.4. To extend nonexistence results about equivariant maps \( A \rightarrow J(A, \beta) \) to cases where \( \beta \) is nontrivial, we know from [17] Example 3.7 that some assumption on \( \beta \), or the equivariant map in question, is still necessary. Here we consider two assumptions that insist \( \beta \) is not too far removed from the trivial action, in an attempt to generalize known examples and theorems.
Question 1.8. Suppose $\alpha$ and $\beta$ are commuting actions of $\mathbb{Z}/k\mathbb{Z}$ on a unital $C^*$-algebra $A$, and $\alpha$ is free. Consider the following conditions on $\beta$.

1. The action $\beta$ is not free.

2. The individual automorphisms of $\beta$ are connected within $\text{Aut}(A)$ to the trivial automorphism.

Is either condition sufficient to guarantee that there are no $(\alpha, \tilde{\alpha})$-equivariant, unital $*$-homomorphisms from $A$ to $J(A, \beta)$?

The conditions were determined through the computation of various examples, as in [17] [16], chief among them odd-dimensional $\theta$-deformed spheres (defined in [13] [15]) and twisted versions thereof. In section [2] we consider more restrictive assumptions that guarantee nonexistence of equivariant maps from $A$ to $J(A, \beta)$. However, neither condition of Question 1.8 is actually sufficient in general, as shown in Theorems 3.1 and 3.3. Finally, in the remainder of section 3 we apply the usual embedding $A \rtimes_\beta \mathbb{Z}/k\mathbb{Z} \hookrightarrow M_k(A)$ to see consequences of the aforementioned results for deforming the diagonal inclusion $A \hookrightarrow M_k(A)$ to finite-dimensional and one-dimensional representations. Extending the definition of unital contractibility to unital contractibility “modulo $k$”, we find that the connection between contractibility properties and examples of equivariant maps $A \rightarrow J(A, \beta)$ is not as direct as in [6] Corollary 2.7.

2 Nonexistence

The noncommutative Borsuk-Ulam theorem in [17] Corollary 2.4] can be proved using an iteration procedure, which is not as immediate in the twisted setting. Specifically, while morphisms $A \rightarrow B$ induce morphisms $A \oplus C(G) \rightarrow B \oplus C(G)$, the same does not generally hold for twisted joins without additional equivariance requirements. In this section, we consider sufficient conditions that guarantee there are no equivariant morphisms $A \rightarrow J(A, \beta)$. To avoid unnecessary repetition, $\Gamma$ will always refer to a discrete abelian group, and $G$ will be its compact abelian Pontryagin dual group $\hat{G}$.

If $\beta$ is an action of $\Gamma$ on $A$, then we may extend $\beta$ to $A \rtimes_\beta \Gamma$ so that each group element is in the fixed-point subalgebra. We will also refer to this action, as well as its pointwise application on $J(A, \beta)$, by $\beta$.

Lemma 2.1. Suppose $\beta_A$ and $\beta_B$ are actions of $\Gamma$ on unital $C^*$-algebras $A$ and $B$. If $\phi : A \rightarrow B$ is a $(\beta_A, \beta_B)$-equivariant, unital $*$-homomorphism, then the rule

$$a \in A \mapsto \phi(a), \quad \gamma \in \Gamma \mapsto \gamma$$

produces a homomorphism $A \rtimes_{\beta_A} \Gamma \rightarrow B \rtimes_{\beta_B} \Gamma$. If $J_\phi : J(A, \beta_A) \rightarrow J(B, \beta_B)$ denotes the pointwise application of this map, then $J_\phi$ is $(\beta_A, \beta_B)$-equivariant.

If, in addition, $\alpha_A$ and $\alpha_B$ are actions of $G = \hat{\Gamma}$ on $A$ and $B$ which commute with $\beta_A$ and $\beta_B$, respectively, and $\phi$ is also $(\alpha_A, \alpha_B)$-equivariant, then $J_\phi$ is pointwise $(\alpha_A, \alpha_B)$-equivariant.

Proof. The function $\psi : A \rtimes_{\beta_A} \Gamma \rightarrow B \rtimes_{\beta_B} \Gamma$ defined by (2.2) is a unital $*$-homomorphism by the universal property of crossed products. The associated homomorphism

$$\text{id} \otimes \psi : C([0, 1]) \otimes (A \rtimes_{\beta_A} \Gamma) \rightarrow C([0, 1]) \otimes (B \rtimes_{\beta_B} \Gamma)$$

respects the boundary conditions of the twisted join, as $\psi(C^*(\Gamma)) = C^*(\Gamma)$ and $\psi(A) \subset B$. Therefore, it induces a homomorphism $J_\phi : J(A, \beta_A) \rightarrow J(B, \beta_B)$, which is pointwise $(\beta_A, \beta_B)$-equivariant by design. Finally, the actions $\tilde{\alpha_A}$ and $\tilde{\alpha_B}$ are pointwise applications of $\alpha_A\beta_A$ and $\alpha_B\beta_B$, so it suffices to prove that $\psi = (\alpha_A\beta_A, \alpha_B\beta_B)$-equivariant. For $g \in G$, 4
\[(\alpha_B \beta_B)_g \left( \psi \left( \sum a_\gamma \cdot \gamma \right) \right) = (\alpha_B \beta_B)_g \left( \sum \phi(a_\gamma) \cdot g(\gamma) \right) = \sum (\alpha_B)_g(\phi(a_\gamma)) \cdot g(\gamma) = \sum \phi((\alpha_A)_g(a_\gamma)) \cdot g(\gamma) = \psi \left( \sum (\alpha_A)_g(a_\gamma) \cdot g(\gamma) \right) = \psi \left( (\alpha_A \beta_A)_g \left( \sum a_\gamma \cdot \gamma \right) \right),\]

so \(\psi\) is \((\alpha_A \beta_A, \alpha_B \beta_B)\)-equivariant, and \(J_\phi\) is \((\widetilde{\alpha}_A, \widetilde{\alpha}_B)\)-equivariant. \(\square\)

The additional equivariance demanded in Lemma 2.1 is automatic if \(\beta\) is trivial. It follows that the result below generalizes [17, Corollary 2.4].

**Theorem 2.3.** Let \(\Gamma = G = \mathbb{Z}/k\mathbb{Z}, k \geq 2\), and suppose \(A\) is a unital \(C^*\)-algebra with two commuting actions \(\alpha\) and \(\beta\) of \(\mathbb{Z}/k\mathbb{Z}\), where \(\alpha\) is free. Then there is no unital \(*\)-homomorphism \(\phi: A \to J(A, \beta)\) which is both \((\alpha, \alpha)\)-equivariant and \((\beta, \beta)\)-equivariant.

**Proof.** We let \(A_0 := A\), which admits actions \(\alpha_0 := \alpha\) and \(\beta\). The twisted join \(A_1 := J(A, \beta)\) admits the pointwise action of \(\beta\) (still denoted \(\beta\)), and we let \(\alpha_1 = \widetilde{\alpha}_0\). Iterating this procedure, we define \(A_n\) as an iterated twisted join of \(A\) via the rule

\[A_n = J(A_{n-1}, \beta), \quad \alpha_n = \widetilde{\alpha}_{n-1}.\]

Suppose \(\phi_0 := \phi: A_0 \to A_1\) is both \((\alpha_0, \alpha_1)\)-equivariant and \((\beta, \beta)\) equivariant. Then repeated applications of Lemma 2.1 produce \((\alpha_{n-1}, \alpha_n)\)-equivariant and \((\beta, \beta)\)-equivariant maps \(\phi_n: A_{n-1} \to A_n\). From composition of these maps in a chain, we find that for any \(n \in \mathbb{Z}^+\), there is an \((\alpha_0, \alpha_n)\)-equivariant, \((\beta, \beta)\)-equivariant map \(\Phi_n: A_0 \to A_n\).

Next, we apply a similar iteration procedure for maps into \(C^*(\mathbb{Z}/k\mathbb{Z})\). Let \(B_0 := C^*(\mathbb{Z}/k\mathbb{Z})\) be graded by \(\mathbb{Z}/k\mathbb{Z}\) in the usual way, inducing an action \(\rho_0\). Since \(\phi_0: A_0 \to J(A_0, \beta)\) is \((\alpha, \alpha)\)-equivariant and \((\beta, \beta)\)-equivariant, evaluation at the \(t = 0\) endpoint of (1.6) shows that there is a map \(\psi_0: A_0 \to B_0\) which is \((\alpha_0, \rho_0)\)-equivariant and \((\beta, \text{triv})\)-equivariant. Define

\[B_n = J(B_{n-1}, \text{triv}), \quad \rho_n = \widetilde{\rho}_{n-1},\]

and note that iteration of Lemma 2.1 once more establishes that for each \(n\), there is a \((\alpha_n, \rho_n)\)-equivariant, \((\beta, \text{triv})\)-equivariant map \(\psi_n: A_n \to B_n\). Since \(B_n\) is defined using a trivial twist in its iterated join procedure, we note that \(B_n \cong \otimes_{i=1}^{n+1} C(\mathbb{Z}/k\mathbb{Z}) = C((\mathbb{Z}/k\mathbb{Z})^{s+1})\), and \(\rho_n\) is the diagonal action.

Finally, the composition \(\psi_n \circ \Phi_n: A \to C(\mathbb{Z}/k\mathbb{Z}^{*n})\) is \((\alpha, \text{diag})\)-equivariant. Since the action \(\alpha\) is free, fix a generator \(\gamma \in \mathbb{Z}/k\mathbb{Z}\) and note that by (1.1), \(1 \in A_\gamma A_\gamma^*\). In particular, there is a finite \(N\) such that there exist \(a_1, b_1, \ldots, a_N, b_N \in A_\gamma\) with \(a_1 b_1^* \cdot \gamma\) invertible. On the other hand, from the increasing connectivity of the iterated joins of \(\mathbb{Z}/k\mathbb{Z}\), the number of elements in the \(\gamma\)-isotypic subspace of \(C(\mathbb{Z}/k\mathbb{Z}^{*n})\) required to generate an invertible grows without bound as \(n\) increases (see [12] and [12] Definition 4.3.1 and Proposition 4.4.3). It follows that for large \(n\), the existence of the \((\alpha, \text{diag})\)-equivariant map \(\psi_n \circ \Phi_n: A \to C((\mathbb{Z}/k\mathbb{Z})^{*n})\) leads to a contradiction. \(\square\)

Theorem 2.3 does not assume any condition on the freeness or non-freeness of \(\beta\). However, we note that the pointwise action \(\beta\) on \(J(A, \beta)\) is not free, as at the endpoint \(t = 0\) of (1.6), \(\beta\) corresponds to the trivial action on \(C^*(\mathbb{Z}/k\mathbb{Z})\). Therefore, if the original action \(\beta\) on \(A\) is free, there cannot be a \((\beta, \beta)\)-equivariant unital \(*\)-homomorphism from \(A\) to \(J(A, \beta)\), regardless of any other action \(\alpha\). It follows that Theorem 2.3 is useful when \(\beta\) is not free, so it may be safely viewed inside the framework of Question 1.8 condition 1.
If \( A = C(X) \) and \( k \in \mathbb{Z}^+ \) is prime, then any non-free action of \( \mathbb{Z}/k\mathbb{Z} \) has a nonempty fixed point set. When this set is an equivariant retract of \( X \), where \( X \) is acted upon freely, we may produce a twisted Borsuk-Ulam theorem as follows.

**Proposition 2.4.** Let \( k \in \mathbb{Z}^+ \) be prime, and let \( X \) be a compact Hausdorff space with two commuting \( \mathbb{Z}/k\mathbb{Z} \) actions \( \alpha \) and \( \beta \), where \( \alpha \) is free and \( \beta \) is not. Let \( Y = \text{Fix}(\beta) \neq \emptyset \) be equipped with the restricted action \( \gamma = \alpha|_Y \), and suppose there is an \((\alpha, \gamma)\)-equivariant continuous function \( f : X \to Y \). Then there is no unital, \((\alpha, \tilde{\alpha})\)-equivariant \(*\)-homomorphism from \( C(X) \) to \( J(C(X), \beta) \).

**Proof.** Suppose \( \phi : C(X) \to J(C(X), \beta) \) is \((\alpha, \tilde{\alpha})\)-equivariant. First, we have that the dual map \( f^* : C(Y) \to C(X) \) to \( f : X \to Y \) is \((\gamma, \alpha)\)-equivariant. Second, the restriction map \( C(X) \to C(Y) \) is certainly \((\alpha, \gamma)\)-equivariant and \((\beta, \text{triv})\)-equivariant, so it may be applied pointwise on the twisted joins by Lemma [2.1]. This gives

\[
\begin{align*}
C(Y) \xrightarrow{f^*} C(X) \xrightarrow{\phi} J(C(X), \beta) \xrightarrow{\text{pointwise } |_Y} J(C(Y), \text{triv}) \xrightarrow{\cong} C(Y) \oplus C(\mathbb{Z}/k\mathbb{Z}).
\end{align*}
\]

The composition is \((\gamma, \text{diag})\)-equivariant, contradicting [17] Corollary 2.4 (or, rather, the topological result [21] Corollary 3.1).

The following proposition is motivated by a topological picture: if \( A = C(X) \) is commutative and \( \alpha \) is a free action of \( \mathbb{Z}/k\mathbb{Z} \) which permutes the components of \( X \), then for non-free actions \( \beta \) on \( X \), there might not exist equivariant maps \( A \to J(A, \beta) \) due to the existence of finite-order unitaries. The arguments depend upon a standard matrix expansion map

\[
E : A \rtimes_\beta \mathbb{Z}/k\mathbb{Z} \to M_k(A),
\]

defined by mapping \( a \in A \) to the diagonal matrix with entries \( a, \beta_1(a), \ldots, \beta_k(a) \) and mapping a generator \( \mu \in \mathbb{Z}/k\mathbb{Z} \) to a \( \{0, 1\} \)-valued matrix \( S \) which induces the shift \( e_n \mapsto e_{n+1} \) in the standard basis of \( \mathbb{C}^k \). The action \( \beta_0 \) on \( A \rtimes_\beta \mathbb{Z}/k\mathbb{Z} \) then corresponds to the entrywise application of \( \beta \) on the subalgebra \( E(A \rtimes_\beta \mathbb{Z}/k\mathbb{Z}) \subseteq M_k(A) \).

**Proposition 2.6.** Fix \( k \geq 2 \) and a generator \( \mu \) of \( \mathbb{Z}/k\mathbb{Z} \cong \mathbb{Z}/k\mathbb{Z} \). Suppose \( A \) is a unital \( C^* \)-algebra with a free action \( \alpha \) of \( \mathbb{Z}/k\mathbb{Z} \), such that there is a unitary \( x \) in the \( \mu \)-isotypic subspace with \( x^k = 1 \). Further, let \( \beta \) be an action of \( \mathbb{Z}/k\mathbb{Z} \) on \( A \) which commutes with \( \alpha \), such that the ideal \( I \) generated by terms \( \beta_m(a) - a, m \in \mathbb{Z}/k\mathbb{Z}, a \in A \) is proper, and the \( K_0(A/I) \)-class of the unit 1 is not divisible by \( k \). Then there is no \((\alpha, \tilde{\alpha})\)-equivariant unital \(*\)-homomorphism from \( A \) to \( J(A, \beta) \).

**Proof.** If \( \phi : A \to J(A, \beta) \) is \((\alpha, \tilde{\alpha})\)-equivariant, then \( \phi(x) \) determines a path of unitaries \( u_t \in A \rtimes_\beta \mathbb{Z}/k\mathbb{Z} \) with \( u_t^k = 1 \), connecting \( u_0 \in C^*(\mathbb{Z}/k\mathbb{Z}) \) to \( u_1 \in A \). Now, \( u_t \) is also in the \( \mu \)-isotypic subspace of \( A \rtimes_\beta \mathbb{Z}/k\mathbb{Z} \) for \( \alpha \tilde{\beta} \), so \( u_0 \) is of the form \( c u \mu \) for some \( c \in \mathbb{C} \) with \( c^k = 1 \), and \( u_1 \) is in the \( \mu \)-isotypic subspace of \( A \) for \( \alpha \). Let \( v_t = c^{-1} u_t \) and define the projections

\[
P_t = \frac{1 + v_t + v_t^2 + \ldots + v_t^{k-1}}{k}
\]

for \( t \in [0, 1] \). Apply the expansion map \( E \) of (2.5) and the entrywise quotient \( A \to A/I \), where \( I \) identifies any \( a \in A \) with \( \beta_0(a) \) for any \( m \). Then the quotient of \( E(P_t) \) produces a path of projections in \( M_k(A/I) \) connecting a matrix \( T \in M_k(\mathbb{C}) \) to a diagonal matrix with entries \( a + I, a + I, \ldots, a + I \) for some \( a \in A \). The matrix \( T \) is of the form \( \frac{1}{k} \sum_{n=0}^{k-1} S^k \) for an order \( k \) shift \( S \in U_k(\mathbb{C}) \). Since \( S \) has eigenvalues \( 1, \omega, \ldots, \omega^{k-1} \) for a primitive \( k \)th root of unity \( \omega \), it follows that \( T \) has rank 1, and \( T \) is equivalent in \( K_0(A/I) \) to the unit 1. On the other hand, the diagonal matrix with entries \( a + I, \ldots, a + I \) is equivalent in \( K_0(A/I) \) to the sum of \( a + I \) with itself \( k \) times. Therefore, \( 1 \in K_0(A/I) \) is divisible by \( k \). \( \square \)
In Proposition 2.6, the assumptions imply that $\alpha$ is free and $\beta$ is not free. Below we write a commutative subcase of the result.

**Corollary 2.7.** Fix $k \geq 2$ and let $X$ be a disconnected compact Hausdorff space. Let $\alpha$ and $\beta$ be two commuting actions of $\mathbb{Z}/k\mathbb{Z}$ on $X$ such that Fix$(\beta)$ is nonempty and there is a clopen set $Y \subseteq X$ such that $Y \cup \alpha_1(Y) \cup \ldots \cup \alpha_{k-1}(Y) = X$ and the union is disjoint. Then there is no $(\alpha, \beta)$-equivariant unital $*$-isomorphism from $C(X)$ to $J(C(X), \beta)$.

**Proof.** Fix a primitive $k$th root of unity $\mu$, and define a function $f$ on $X$ by assigning $f(p) = \mu^{m}$ if and only if $m \in \alpha^m(Y)$. Then $f \in C(X)$ is a unitary in the $\mu$-isotypic subspace of $\alpha$, such that $f^k = 1$. Since $Z := \text{Fix}(\beta)$ is not empty, the ideal in $C(X)$ generated by $\beta_m(a) - a$, $0 \leq m \leq k - 1$, $a \in C(X)$ is proper. Specifically, $C(X)/I \cong C(Z)$. Next, there is a map $K_0(C(Z)) \to K_0(C) \cong \mathbb{Z}$ induced by evaluation at a point $q \in Z$, which sends the unit $1 \in C(Z)$ to $1$, so is indivisible by $k$ in $K_0(C(Z))$. Finally, we may apply Proposition 2.6. 

If $k$ is prime, note that Fix($\beta$) is empty if and only if $\beta$ is free. Therefore, these results also fall under the umbrella of Question 1.8 condition 1. Finally, following the $K$-theoretic computations in [13][15][16], we see that $\theta$-defomed spheres, and certain twisted unreduced suspensions thereof, admit twisted Borsuk-Ulam theorems. First, we recall the definitions, as in [13][15][16].

**Definition 2.8.** For an antisymmetric matrix $\theta \in M_n(\mathbb{R})$, let

$$C(S^{2n-2}_\theta) := C^*(z_1, \ldots, z_n \mid z_k \text{ normal, } z_k z_j = e^{2\pi i \theta_{jk}} z_j z_k, z_1 z_1^* + \ldots + z_n z_n^* = 1)$$

denote the $\theta$-sphere of dimension $2n - 1$. If $\theta$ is such that $z_n$ is central, then for $\rho$ the top left $(n - 1) \times (n - 1)$ minor of $\theta$, let

$$C(S^{2n-2}_\rho) := C(S^{2n-2}_\theta)/\langle z_n - z_n^* \rangle$$

denote the $\rho$-sphere of dimension $2n - 2$. If $\theta$ is instead such that $z_n$ anticommutes with all of the other $z_i$, then let

$$\mathcal{R}^{2n-2}_\rho := C(S^{2n-2}_\theta)/\langle z_n - z_n^* \rangle$$

denote the “twisted” analogue of the $\rho$-sphere of dimension $2n - 2$. Each of the above objects admits an antipodal action of $\mathbb{Z}/2\mathbb{Z}$, denoted $\alpha$, which negates each generator in the presentation. For appropriate choices of $\rho$ and $\omega$, we have that

$$C(S^{2n-2}_\rho) \cong C(S^{2n-3}_\omega) \oplus C(\mathbb{Z}/2\mathbb{Z}) \cong J(C(S^{2n-3}_\omega), \text{triv})$$

and

$$\mathcal{R}^{2n-2}_\rho \cong J(C(S^{2n-3}_\omega), \alpha).$$

In both cases, the antipodal action on the $(2n - 3)$-dimensional $\omega$-sphere induces the antipodal action on the (twisted) join, in the usual way.

The antipodal action $\alpha$ is free, and both the trivial action and the antipodal action satisfy condition 2 of Question 1.8.

**Example 2.9.** (Consequences of [13] Corollary 4.12 and [15] Theorem 1.8) Let $A$ be an algebra in Definition 2.8 of dimension $k$, and let $B$ be an algebra in Definition 2.8 of dimension $k + 1$. If $\mathbb{Z}/2\mathbb{Z}$ acts on $A$ and $B$ via the antipodal action, then there is no equivariant, unital $*$-homomorphism from $A$ to $B$.

Let $C$ be another algebra in Definition 2.8 of dimension $k$, and suppose $\phi : A \to C$ is equivariant. If $k$ is odd, then the $K_1$-groups of both algebras are isomorphic to $\mathbb{Z}$, and the induced map $\phi_*$ on $K_1$ is nontrivial. If $k$ is even, then the $K_0$-groups of both algebras are isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$, with the first component generated by the unit 1, and the induced map $\phi_*$ on $K_0$ is such that $\text{Ran}(\phi_*)$ is not cyclic.
3 Existence

Recall the following assumptions of Question 1.8 for commuting \( \mathbb{Z}/k\mathbb{Z} \)-actions \( \alpha \) and \( \beta \) on a unital \( C^* \)-algebra \( A \), where \( \alpha \) is free.

1. The action \( \beta \) is not free.

2. The individual automorphisms of \( \beta \) are connected within \( \text{Aut}(A) \) to the trivial automorphism.

When we discuss continuous paths in \( \text{Aut}(A) \) or \( \text{Hom}(A, B) \), we will always mean continuous with respect to the pointwise norm topology. Both conditions assert that \( \beta \) is in some sense similar to the trivial action, and the conditions are motivated by examples and counterexamples from \cite{17, 16} and the previous section. In particular, condition 2 may be thought of as the demand that \( \beta \) is “orientation-preserving.” However, we find that neither condition is sufficient to rule out \((\alpha, \tilde{\alpha})\)-equivariant maps \( A \to J(A, \beta) \).

**Theorem 3.1.** Let \( A = C(S^1) \) be generated by the coordinate unitary \( z \), and equip \( A \) with the antipodal action \( \alpha : z \mapsto -z \) and the conjugation action \( \beta : z \mapsto z^* \) of \( \mathbb{Z}/2\mathbb{Z} \). There is an \((\alpha, \tilde{\alpha})\)-equivariant, unital \(*\)-homomorphism from \( A \) to \( J(A, \beta) \). Since \( \alpha \) is free and \( \beta \) is not free, condition 1 of Question 1.8 is insufficient.

**Proof.** Let \( z = x + iy \), so that \( \alpha \) negates both \( x \) and \( y \), but \( \beta \) fixes \( x \) and negates \( y \). Also, let \( C^*(\mathbb{Z}/2\mathbb{Z}) \) be generated by the self-adjoint unitary \( \mu \). It follows that in \( A \times_{\beta} \mathbb{Z}/2\mathbb{Z} \), \( \mu y = -\mu y \) and \( x\mu = \mu x \). The points \( a_t, b_t \in A \times_{\beta} \mathbb{Z}/2\mathbb{Z} \) defined by

\[
a_t = tx, \quad b_t = ty + \sqrt{1-t^2}\mu,
\]

for \( t \in [0, 1] \) are self-adjoint, commute with each other, and satisfy

\[
a_t^2 + b_t^2 = t^2 x^2 + \left( t^2 y^2 + t\sqrt{1-t^2}\delta y + t\sqrt{1-t^2}\beta y + (1-t^2) \right)
= t^2(x^2 + y^2) + (1-t^2)
= 1.
\]

Since \( a_0 = 0 \) and \( b_0 = \mu \) belong to \( C^*(\mathbb{Z}/2\mathbb{Z}) \), and similarly \( a_1 = x \) and \( b_1 = y \) belong to \( A \), it follows that \( f(t) = a_t + ib_t \) is a unitary element in \( J(A, \beta) \). Further, \( \tilde{\alpha}(f) = -f \), since \( \tilde{\alpha} \) is defined as the pointwise application of \( \alpha \beta \), which negates \( x, y \), and \( \mu \) in \( A \times_{\beta} \mathbb{Z}/2\mathbb{Z} \). Finally, the unital \(*\)-homomorphism \( \phi : A \to J(A, \beta) \) defined by \( \phi(z) = f \) is \((\alpha, \tilde{\alpha})\)-equivariant.

Next, consider the universal \( C^* \)-algebra

\[
A \cong C^*(x, y \mid x = x^*, y = y^*, x^2 + y^2 = 1).
\]

While \( A \) is itself noncommutative, there is an obvious surjection from \( A \) onto \( C(S^1) \). Moreover, \( A \) admits a \( \mathbb{Z}/2\mathbb{Z} \) action generated by

\[
\alpha : \begin{array}{ccc} x & \mapsto & -x \\ y & \mapsto & -y \end{array}, \tag{3.2}
\]

analogous to the antipodal action on the quotient \( C(S^1) \). Using a rotation argument motivated by the commutative quotient, we find that the automorphism which generates \( \alpha \) is connected within \( \text{Aut}(A) \) to the trivial automorphism. Specifically, if \( s, t \in \mathbb{R} \) have \( s^2 + t^2 = 1 \), then

\[
(sx + ty)^2 + (-tx + sy)^2 = (s^2x^2 + stxy + styx + t^2y^2) + (t^2x^2 - stxy - styx + s^2y^2)
= (s^2 + t^2)(x^2 + y^2)
= 1,
\]
so there is an endomorphism $R_{s,t} : A \to A$ defined by $R_{s,t}(x) = sx + ty$ and $R_{s,t}(y) = -tx + sy$. The inverse of $R_{s,t}$ is $R_{s,-t}$, as

$$s(sx - ty) + t(tx + sy) = (s^2 + t^2)x = x,$$

and similarly for the reverse composition. Therefore, $R_{s,t}$ is an automorphism for each $(s,t) \in \mathbb{S}^1$. Finally, $\alpha_1 = R_{(-1,0)}$ is connected via a path $R_{s,t} \in \text{Aut}(A)$ to the trivial automorphism $R_{(1,0)}$.

**Theorem 3.3.** Let $A = C^*(x, y \mid x = x^*, y = y^*, x^2 + y^2 = 1)$, equip $A$ with the action $\alpha$ of $\mathbb{Z}/2\mathbb{Z}$ that negates $x$ and $y$, and let $\alpha = \beta$. Then there is an $(\alpha, \tilde{\alpha})$-equivariant, unital $\ast$-homomorphism from $A$ to $J(A, \beta)$. Since $\alpha$ is free and the automorphism generating $\beta$ is connected within $\text{Aut}(A)$ to the trivial automorphism, condition 2 of Question 1.8 is insufficient.

**Proof.** Let $C^*(\mathbb{Z}/2\mathbb{Z})$ be generated by the self-adjoint unitary $\mu$, and define the self-adjoint elements $a_t, b_t \in A \rtimes_\beta \mathbb{Z}/2\mathbb{Z}$ by

$$a_t = tx + \frac{\sqrt{1 - t^2}}{\sqrt{2}} \mu, \quad b_t = ty + \frac{\sqrt{1 - t^2}}{\sqrt{2}} \mu.$$

Since $\mu$ anticommutes with both $x$ and $y$, it follows that

$$a_t^2 + b_t^2 = \left(t^2 x^2 + \frac{1 - t^2}{2}\right) + \left(t^2 y^2 + \frac{1 - t^2}{2}\right) = t^2(x^2 + y^2) + (1 - t^2) = 1.$$

We also have that $a_0 = b_0 = \frac{1}{\sqrt{2}} \mu \in C^*(\mathbb{Z}/2\mathbb{Z})$, and similarly, $a_1 = x$ and $b_1 = y$ belong to $A$. Therefore $f(t) = a_t$ and $g(t) = b_t$ define elements $f, g \in J(A, \beta)$. Since $f$ and $g$ are negated by $\tilde{\alpha}$, the map $\phi : A \to J(A, \beta)$ defined by $\phi(x) = f$, $\phi(y) = g$ is an $(\alpha, \tilde{\alpha})$-equivariant, unital $\ast$-homomorphism.

Next, we expand upon some consequences of the two previous theorems.

**Remark 3.4.** In both Theorem 3.1 and Theorem 3.3, the chosen equivariant morphism $A \to J(A, \beta)$ is such that evaluation at the $t = 1$ endpoint of the twisted join produces the usual embedding $A \hookrightarrow A \rtimes_\beta \mathbb{Z}/2\mathbb{Z}$. Further, since evaluation at $t = 0$ produces a map $A \to C^*(\mathbb{Z}/2\mathbb{Z})$, which has a commutative codomain, and the largest commutative quotient of $A$ is $C(\mathbb{S}^1)$, there is a factorization

$$A \to C(\mathbb{S}^1) \xrightarrow{\Lambda} C^*(\mathbb{Z}/2\mathbb{Z}).$$

The morphism $\Lambda$ is dual to a continuous function $\lambda : \mathbb{Z}/2\mathbb{Z} \to \mathbb{S}^1$, i.e. a selection of two points in $\mathbb{S}^1$. Since $\mathbb{S}^1$ is path connected, we may apply a path $\lambda_t$ connecting $\lambda = \lambda_0$ to $\lambda_1$, which selects two identical points. This produces a path connecting $\Lambda$ to a one-dimensional representation $A \to \mathbb{C}$. We conclude that the inclusion map $A \hookrightarrow A \rtimes_\beta \mathbb{Z}/2\mathbb{Z}$ may be connected to a one-dimensional representation.

**Remark 3.5.** Let $A$ be as in Theorem 3.3, so $A$ again has $C(\mathbb{S}^1)$ as its largest commutative quotient. The commutator ideal of $A$ is invariant, and the induced action on the quotient $C(\mathbb{S}^1)$ is the antipodal action. However, if $\alpha = \beta$ is the antipodal action on $C(\mathbb{S}^1)$, then there is no $(\alpha, \tilde{\alpha})$-equivariant unital $\ast$-homomorphism $C(\mathbb{S}^1) \to J(C(\mathbb{S}^1), \beta)$. In particular, $J(C(\mathbb{S}^1), \beta)$ is a twisted analogue of a 2-sphere that appears in Definition 2.8 and Example 2.9 Therefore, the failure of Borsuk-Ulam theorems is not preserved in quotient procedures.
In the non-twisted setting, if a compact group $G$ acts on $A$, then existence of an equivariant morphism $A \to A \otimes C(G)$ is equivalent to the existence of a path in $\text{Hom}(A,A)$ between an equivariant endomorphism and a one-dimensional representation of $A$ (see, e.g., the more general result \textit{[17] Lemma 2.5]). Since the identity endomorphism is certainly equivariant for any action, it is of perhaps independent interest whether or not $\text{id}_A$ may be connected to a one-dimensional representation, regardless of the presence of an action. If such a path does exist, then $A$ is called \textit{unitally contractible}, as in \textit{[17] Definition 2.6}. In the commutative case $A = C(X)$, this property corresponds to contractibility of $X$. Moreover, it is immediate that if equivariant maps $A \to A \otimes C(G)$ cannot exist, then $A$ cannot be unitally contractible. To consider analogous concepts in the twisted setting, we extend the notion of unital contractibility “modulo $k$” by consideration of finite-dimensional representations.

\textbf{Definition 3.6.} Let $A$ be a unital $C^*$-algebra $A$. Then $A$ is called \textit{unitally contractible modulo $k$} if the embedding $a \in A \mapsto a \otimes 1_k \in A \otimes M_k(\mathbb{C})$ is connected within $\text{Hom}(A,A \otimes M_k(\mathbb{C}))$ to a $k$-dimensional representation, that is, a map $\rho \in \text{Hom}(A,\mathbb{C} \otimes M_k(\mathbb{C}))$. Similarly, $A$ is called \textit{strongly unitally contractible modulo $k$} if the embedding $a \mapsto a \otimes 1_k$ may be connected within $\text{Hom}(A,A \otimes M_k(\mathbb{C}))$ to a one-dimensional representation $\rho \in \text{Hom}(A,\mathbb{C} \otimes \mathbb{C})$.

\textbf{Example 3.7.} Fix $k \geq 2$. The matrix algebra $M_k(\mathbb{C})$ admits a free $\mathbb{Z}/k\mathbb{Z}$ action, and it is contractible modulo $k$, but it is not strongly unitally contractible modulo $k$.

\textbf{Proof.} Let $V$ be diagonal with entries $1,\omega,\ldots,\omega^{k-1}$, where $\omega$ is a primitive $k$th root of unity. Conjugation by $V$ is a free $\mathbb{Z}/k\mathbb{Z}$ action of $M_k(\mathbb{C})$, as the matrix $W$ which acts on the standard basis of $\mathbb{C}^n$ by $e_i \mapsto e_{i+1}$ is unitary and in the $\omega$-isotypic subspace of the action. Further, the embeddings $\text{id} \otimes 1$ and $1 \otimes \text{id}$ of $M_k(\mathbb{C})$ into $M_k(\mathbb{C}) \otimes M_k(\mathbb{C}) \cong M_{k^2}(\mathbb{C})$ are conjugate. That is, there is a unitary $U \in M_{k^2}(\mathbb{C})$ such that for each $M \in M_k(\mathbb{C})$, $U(M \otimes 1)U^* = 1 \otimes M$. Because the unitary group of $M_{k^2}(\mathbb{C})$ is path connected, the two embeddings are connected via $M \mapsto U_i(M \otimes 1)U_i^*$, where $U_0 = I$, $U_1 = U$, and each $U_i$ is unitary. Therefore, $M_k(\mathbb{C})$ is unitally contractible modulo $k$. However, $M_k(\mathbb{C})$ has no one-dimensional representations, so it cannot be strongly unitally contractible modulo $k$. \hfill $\Box$

In analogy with \textit{[17] Corollary 2.7], we seek a connection between Borsuk-Ulam theorems and (strong) unital contractibility of $A$ modulo the same $k$. Certainly, if $A$ has $K$-theory invariants which remain nontrivial under the quotient $G/kG$, then $A$ is not unitally contractible modulo $k$.

\textbf{Example 3.8.} The circle $C(S^1)$ is not unitally contractible modulo $k$ for any $k$. This holds even though Theorem \textit{3.3} produces an equivariant map $C(S^1) \to J(C(S^1),\beta)$ such that evaluation at $t = 1$ gives the usual inclusion $A \hookrightarrow A \rtimes_{\beta} \mathbb{Z}/2\mathbb{Z}$. In particular, it is crucial for this example that $\beta$ is orientation-reversing.

We may, however, adapt the counterexample in Theorem \textit{3.3} to show the $C^*$-algebra $A$ used therein is \textit{strongly} unitally contractible modulo 2.

\textbf{Example 3.9.} The $C^*$-algebra $A = C^*(x,y \mid x = x^*, y = y^*, x^2 + y^2 = 1)$ is strongly unitally contractible modulo 2.

\textbf{Proof.} It is known from Theorem \textit{3.3} that if $\alpha = \beta$ is the action generated by $x \mapsto -x$, $y \mapsto -y$, then there is an $(\alpha,\tilde{\alpha})$-equivariant map $\phi : A \to J(A,\alpha)$. An examination of the proof shows that $ev_{t = 1}(\phi(a)) = a$ for each $a \in A$. Expanding the crossed product $A \rtimes_{\alpha} \mathbb{Z}/2\mathbb{Z}$ via $E : a_0 + a_1\mu \mapsto \left(\begin{array}{cc} a_0 & a_1 \\ \alpha(a_1) & \alpha(a_0) \end{array}\right)$ then shows that the embedding

$$\psi_1 : a \in A \mapsto \left(\begin{array}{c} a \\ \alpha(a) \end{array}\right) \in A \otimes M_2(\mathbb{C})$$
may be connected via a path $\psi_t \in \text{Hom}(A, A \otimes M_2(\mathbb{C}))$ to a homomorphism $\psi_0$ such that $\text{Ran}(\psi_0) \subseteq E(C^*(\mathbb{Z}/2\mathbb{Z}))$.

Both endpoints of the above path must be adjusted to show strong unital contractibility modulo 2. First, the automorphism generating the action $\alpha$ is connected in $\text{Aut}(A)$ via a path $\alpha_t \in \text{Aut}(A)$ to $\alpha_0 = \text{id}$. Therefore

$$a \in A \mapsto \begin{pmatrix} a & \alpha_t(a) \\ \alpha_t(a) & a \end{pmatrix} \in A \otimes M_2(\mathbb{C})$$

may be used to connect $\psi_1$ to the diagonal embedding $A \hookrightarrow A \otimes M_2(\mathbb{C})$. Second, $\psi_0$ maps $A$ maps to a subalgebra of $M_2(\mathbb{C})$ isomorphic to $C^*(\mathbb{Z}/2\mathbb{Z})$, which is commutative. Using the factorization technique of Remark 3.4, we see that $\psi_0$ may be connected to a one-dimensional representation. Gluing all of the constructed paths together shows that $A$ is indeed strongly unitaly contractible modulo 2.

Finally, we have seen that the nontriviality of equivariant maps proved in [17, Corollary 2.4] for $\mathbb{Z}/k\mathbb{Z}$ actions is removed when crossed products replace tensor products, or when a matrix algebra is introduced, even under a fairly stringent assumption on the twist.

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