Grand Lebesgue Spaces norm estimates for multivariate functional operations

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Abstract

We intend to derive the moment and exponential tail estimates for the so-called bivariate or more generally multivariate functional operations, not necessary to be linear or even multilinear. We will show also the strong or at least weak (i.e. up to multiplicative constant) exactness of obtained estimates.

Key words and phrases.

Binary and multivariate operators and operations; Banach, Lebesgue-Riesz, Orlicz and Grand Lebesgue Spaces (GLS); moment and tail estimates, ordinary convolution and infimal convolution, examples, exact or weak exact value of constants, upper and lower ordinary and exponential estimates, generating function, tail function, singular integral operators, multivariate multiplicative, tensor, Haussdorf, Hilbert, maximal, pseudo-differential, Hardy-Littlewood and other operations.

1 Introduction. Statement of problem.

Let \((X, F, \mu), (X_i, F_i, \mu_i), \ i = 1, 2, \ldots, d\) be measurable spaces equipped with non-trivial measures \(\mu, \mu_i\), not necessary to be probabilistic or bounded. We denote by \(L(p, x_i)\) the classical Lebesgue-Riesz spaces of measurable functions \(f_i : X_i \to R\) having a finite norm

\[
|f_i|L(p, X_i) = |f_i|_p := \left[ \int_{X_i} |f_i(x_i)|^p \mu_i(dx_i) \right]^{1/p}, \ \ p \in [1, \infty).
\] (1.0)
Let also \( V : \otimes_{i=1}^d L(q_i, X_i) \rightarrow L(p, X) \) be multivariate functional operation which maps the tensor product \( \otimes_{i=1}^d L(q_i, X_i), \ p_i \in [1, \infty) \) into the Lebesgue-Riesz space \( L(p, X) \):

\[
g = g(x) = V[f_1, f_2, \ldots, f_d](x) \in L(p, X).
\] (1.1)

It will be presumed more precisely that there exists a non-trivial set \( D \subset R_+^d \) such that

\[
\forall \vec{q} \in D \Rightarrow g(\cdot) \in L(p, X).
\]

Let \( \tau_i = \tau_i(q_i) \) be some continuous inside its domain of definition functions and strictly monotonically increasing functions, for instance

\[
\tau_i(q_i) = \beta_i(q_i)^{\gamma_i} \quad \beta_i, \gamma_i = \text{const} \in (0, \infty), \ q_i \in [1, \infty).
\]

Denote \( \vec{q} := (q_1, q_2, \ldots, q_d) = \{q_i\}, \ q_i \in [1, \infty) \) and define analogously a vector \( \vec{\alpha} := \{\alpha(i)\} = (\alpha_1, \alpha_2, \ldots, \alpha_d), \ \alpha_i \geq 0; \)

\[
\vec{q}^{\vec{\alpha}} \overset{\text{def}}{=} \prod_{i=1}^d (q_i)^{\alpha_i}, \quad |f|_{\vec{q}^{\vec{\alpha}}} \overset{\text{def}}{=} \prod_{i=1}^d (|f_i|_{q_i})^{\alpha_i},
\]

or more generally

\[
\vec{\tau}(\vec{q}) \overset{\text{def}}{=} \{\tau_i(q_i)\}, \quad \vec{\tau}(\vec{q})^{\vec{\alpha}} \overset{\text{def}}{=} \prod_{i=1}^d \left[ \tau_i(q_i) \right]^{\alpha_i},
\]

\[
|f|_{\vec{\tau}(\vec{q})} \overset{\text{def}}{=} \prod_{i=1}^d \left( |f_i|_{\tau_i(q_i)} \right)^{\alpha_i},
\]

or

\[
\vec{\tau}(\vec{q}) \overset{\text{def}}{=} \{\tau_i(q_i)\}, \quad \vec{\tau}(\vec{q})^{\vec{\alpha}} \overset{\text{def}}{=} \prod_{i=1}^d \left[ \tau_i(q_i) \right]^{\alpha_i},
\]

We assume in the sequel that this operation satisfies the following condition.

**Condition 1.1.** There exists a non-trivial domain \( D \subset \otimes_{i=1}^d [1, \infty) \subset R_+^d \) and a certain function \( \Theta = \Theta(\vec{q}) \in [1, \infty), \ \vec{q} \in D \) such that for the value \( p = \Theta(\vec{q}) \in [1, \infty) \) the following inequality holds true

\[
|g|_p = |V(f_1, f_2, \ldots, f_d)|_{\Theta(\vec{q})} \leq K(\vec{q}) \cdot |f|^{\vec{\alpha}}_{\vec{\tau}(\vec{q})},
\]

where

\[
K(\vec{q}) = K_{\alpha, \beta, \gamma, \tau}(\vec{q}) = \text{const} < \infty \iff \vec{q} \in D.
\] (1.2)

One can extend formally the definition this function as follows:

\[
K_{\alpha, \beta, \gamma, \tau}(\vec{q}) := +\infty, \ \vec{q} \notin D.
\]

As for the function \( K(\vec{q}) \): it may consists on the factors of the form for instance

\[
K_{1,i}(q_i) := \beta_i(q_i)^{\gamma_i}, \quad \beta_i, \gamma_i = \text{const} \in (0, \infty), \ q_i \in [1, \infty);
\]

or
\[ K_{2,j}(q_j) := \frac{\beta_j(q_j)^{\gamma_j}}{(q_j - 1)^{\delta_j}}, \quad \beta_j, \gamma_j, \delta_j = \text{const} \in (0, \infty), \quad q_j \in (1, \infty); \]
or at last
\[ K_{3,l}(q_l) = C \left( q_l - a_l \right)^{-c_l} \left( b_l - q_l \right)^{-s_l}, \quad q_l \in (a_l, b_l), \]

and so one.

An example:
\[ K(q_1, q_2, q_3) = C(q_1)^{\gamma_1} \cdot \frac{(q_2)^{\gamma_2}}{(q_2 - 1)^{\delta_2}} \cdot (q_3 - a_3)^{-c_3} \left( b_3 - q_3 \right)^{-s_3}, \]
where
\[ C \in (0, \infty), \quad \gamma_1, \delta_2, c_3, s_3 > 0, \quad 1 \leq a_3 < b_3 < \infty, \]
and
\[ q_1 \in [1, \infty); \quad q_2 \in (1, \infty); \quad q_2 \in (a_3, b_3). \]

One can choose as the function \( K(q) \) its minimal value:
\[ |g|_p = |V(f_1, f_2, \ldots, f_d)|_{\Theta(q)} \leq \overline{K}(q) \cdot |f|^\overline{a}_{\tilde{I}(q)}, \quad (1.3) \]
so that
\[ \overline{K}(q) = \overline{K}_{\alpha, \beta, \gamma, \tau}(q) := \sup_{f \neq 0} \sup_{\varphi \in D} \left\{ \frac{|V(f_1, f_2, \ldots, f_d)|_{\Theta(q)}}{|f|^\overline{a}_{\tilde{I}(q)}} \right\}. \quad (1.3a) \]

Our goal in this preprint is a generalization of the estimate (1.2) into the more general spaces, namely, into a so-called Grand Lebesgue Spaces (GLS).

Hereafter we will denote by \( c_k = c_k(), \ C_k = C_k(), \ k = 1, 2, \ldots, \) with or without subscript, some positive finite non-essentially constructive constants, non necessarily at the same at each appearance.

We will denote also by the symbols \( K_j = K_j(d, n, p, p, q, \ldots) \) essentially positive finite functions depending only on the variables \( d, n, p, q, \ldots, \)

Let us bring some examples.

Example 1. Multiplicative bilinear operator.

In this example \( X = X_1 = X_2, \ \mu = \mu_1 = \mu_2 \) and
\[ g(x) = f_1(x) \cdot f_2(x). \]  

(1.4)

One can apply the classical Hölder’s inequality

\[ |g|_p \leq |f_1|_p \cdot |f_2|_p, \quad p \geq 1, \]  

(1.4a)

where

\[ \alpha, \beta > 1, \quad \frac{1}{\alpha} + \frac{1}{\beta} = 1. \]

Therefore

\[ |g|_p \leq \inf_{\alpha, \beta} \left[ |f_1|_\alpha \cdot |f_2|_\beta \right], \quad p \geq 1, \]  

(1.4b)

where \( \inf \) is calculated over all the values \( \alpha, \beta > 1 \), and such that \( 1/\alpha + 1/\beta = 1 \).

**Example 2. Tensor product.**

Here \((X, F, \mu) = (X_1, F_1, \mu_1) \otimes (X_2, F_2, \mu_2)\) and

\[ g = g(x_1, x_2) = f_1(x_1) \cdot f_2(x_2). \]  

(1.5)

On the other words, both the cofactors \( f_1, f_2 \) are independent in the probabilistic sense.

We conclude

\[ |g|_p = |f_1|_p \cdot |f_2|_p, \]  

(1.5a)

but the last relation is true even for all the non-negative values \( p; \quad p \geq 0 \).

**Example 3. Integral bilinear operator.**

Let us consider now the following integral bilinear (regular) operator

\[ g(x) := V_L[f_1, f_2](x) \overset{\text{def}}{=} \int_{X_1} \int_{X_2} L(x, x_1, x_2) f_1(x_1) f_2(x_2) \mu_1(dx_1) \mu_2(dx_2), \]  

(1.6)

\( p, p_1, p_2 \in (1, \infty). \) Put as ordinary \( p' = p/(p - 1), \quad p'_j = p_j/(p_j - 1), \quad j = 1, 2. \) Denote also by \( l(p, p_1, p_2) = l[L](p, p_1, p_2) \) the following mixed, or equally anisotropic norm of the kernel \( L(\cdot) : \)

\[ l[L](p, p_1, p_2) := | | | L |_{p_2, X_2} |_{p_1, X_1} |_{p, X} \overset{\text{def}}{=} \left\{ \int_X \left[ \int_{X_2} \left( \int_{X_1} |L(x, x_1, x_2)|^{p'_1} \mu_1(dx_1) \right)^{p'_2/p'_1} \mu_2(dx_2) \right]^{p/p'_2} \mu(dx) \right\}^{1/p}. \]
This notion was introduced at first by Benedek A. and Panzone R. [3]; see a detail investigation and applications in [5], [31], [32] etc.

It is not hard to obtain by means of Hölder’s inequality

\[ |g|_p \leq l[L](p, p_1, p_2) \cdot |f_1|_{p_1} |f_2|_{p_2}. \]

On the other words, in this example

\[ K_L(p, p_1, p_2) = l[L](p, p_1, p_2), \]

of course, for all the values of the parameters \((p, p_1, p_2)\) for which the right-hand side is finite.

**Example 4. Classical convolution.**

Let \( X, X_i = \mathbb{R}^d, i = 1, 2; d = 2 \) and \( \mu, \mu_i \) be as before Lebesgue measures. Consider a classical convolution operation

\[ g = f_1 \ast f_2, \iff g(x) = \int_{\mathbb{R}^d} f_1(x - y) f_2(y) \, dy. \quad (1.7) \]

Let also the values \( p_1, p_2 = \text{const} > 1, \) and \( r = \text{const} \in (1, \infty) \) are such that

\[ 1 + \frac{1}{r} = \frac{1}{p_1} + \frac{1}{p_2}. \quad (1.7a) \]

It will be presumed of course that (here and in the sequel) in this operation

\[ \forall(p_1, p_2) \in D \Rightarrow 1 \leq \frac{1}{p_1} + \frac{1}{p_2} \leq 2. \]

The classical Young’s inequality tell us that

\[ |g|_r \leq G(r, p_1, p_2) |f_1|_{p_1} |f_2|_{p_2}, \quad G(r, p_1, p_2) = \text{const} \leq 1. \]

The exact value of the "constant" \( G(r, p_1, p_2) \) was obtained at first by W. Beckner [2]; see also H.J. Brascamp and E.H. Lieb [6]:

\[ G(r, p_1, p_2) = \left[ \frac{v(p_1) v(p_2)}{v(r)} \right]^n, \quad (1.7b) \]

where

\[ v(p) := \left[ \frac{p^{1/p} (p')^{-1/p'}}{p'} \right]^{1/2}, \quad p' = p/(p - 1). \]

This fact may be easily generalized onto the unimodular local compact topological group. Let us consider the case \( X = X_1 = X_2 = [0, 2\pi]^d \) with ordinary Lebesgue measure. The convolution may be defined alike:

\[ g(x) = f_1 \ast f_2(x) := (2 \pi)^{-d} \int_X f_1(x - y) f_2(y) \, dy, \quad (1.7c) \]
where all the algebraic operations in (1.7b) are understood mod(2 \pi). The estimates (1.7a) and (1.7b) remains true.

**Example 5. Infimal convolution.**

Here \( x, y \in R^d; \mu, \mu_1, \mu_2 \) are usually Lebesgue measures.

\[ g = f_1 \Box f_2 \iff g(x) \overset{def}{=} \inf_{y \in \mathbb{R}^d} (f_1(x - y) + f_2(y)). \] \hspace{1cm} (1.8)

This operation appears in the theory of optimization, convex analysis etc.

Let \( p \) be arbitrary number from the set \([1, \infty)\); define the value

\[ K(d,p) \overset{def}{=} \sup_{|f_1|_p + |f_2|_p \in (0,\infty)} \left\{ \frac{|f_1 \Box f_2|_p}{|f_1|_p + |f_2|_p} \right\}. \] \hspace{1cm} (1.8a)

**Theorem 1.**

\[ K(d,p) = 2^{d/p}. \] \hspace{1cm} (1.8b)

**Proof. I. Upper bound.**

We can and will suppose without loss of generality that both the function \( f_1 \) and \( f_2 \) are non-negative, as well as the "common" one \( g \). We derive choosing \( y := x/2 \):

\[ g(x) \leq f_1(x/2) + f_2(x/2). \]

Define as ordinary the so-called dilation operator

\[ T_\lambda[f](x) \overset{def}{=} f(\lambda x), \lambda = \text{const} > 0; \]

then

\[ |T_\lambda[f](\cdot)|_p = \left[ \int_{\mathbb{R}^d} |f(\lambda x)|^p \, dx \right]^{1/p} = \left[ \int_{\mathbb{R}^d} |f(z)|^p \, \lambda^{-d} \, dz \right]^{1/p} = \lambda^{-d/p} |f|_p. \]

We have applying the last relation for the value \( \lambda = 1/2 \):

\[ |g|_p \leq 2^{d/p} \left[ |f_1|_p + |f_2|_p \right], \]

therefore \( K(d,p) \leq 2^{d/p}. \)

**II. Lower estimate.** We choose \( f_1 = f_2 =: f, \) where \( f = f(x), x \in R, \) is certain non-negative smooth with smooth strictly increasing on some finite non-trivial interval derivative function. We conclude

\[ t := \arg\min_y [f(x - y) + f(y)] = x/2; \]
then
\[ g(x) = [f \Box f](x) = 2f(x/2), \]

following
\[ |g|_p = 2|T_{0.5} f(\cdot)|_p = 2 \cdot 2^{d/p} |f|_p = 2^{d/p} [||f_1||_p + ||f_2||_p]. \]

Thus, \( K(d,p) \geq 2^{d/p} \).

The multidimensional case \( d \geq 2 \) may explored analogously: \( f_0(x_1, x_2) := f(x_1) \cdot f(x_2), \ d = 2. \)

The multivariate case
\[ g_m(x) := [ f_1 \Box f_2 \Box \ldots \Box f_m ](x) = [ \Box_{j=1}^m f_j ](x) \]
may be investigated quite analogously:
\[ |g_m|_p \leq m^{d/p} \sum_{j=1}^m |f_i|_p, \ p \geq 1, \]
herewith the constant \( m^{d/p} \) is the best possible.

As far as we know, see for example a recent review of Thomas Strömberg [38], this proposition is new.

**Example 6. Pseudo-differential product.**

A so-called pseudo-differential product (PDP) may be defined as follows. \( X = X_1 = X_2 = R, \) and as above \( \mu(dx) = \mu_1(dx) = \mu_2(dx) = dx; \)
\[ g(x) = PD[f_1, f_2](x) := \int R e^{ix(\alpha + \beta)} \sigma(x, \alpha, \beta) \tilde{f}_1(\alpha) \tilde{f}_2(\beta) \, d\alpha \, d\beta, \quad (1.9) \]
where the notation \( \tilde{f}(\cdot) \) stands for the Fourier transform:
\[ \tilde{f}(\cdot)(\gamma) \overset{def}{=} \int_R e^{i \gamma y} f(y) \, dy. \]

We impose on the "kernel - symbol" function \( \sigma = \sigma(x, \alpha, \beta) \) the classical Hörmanders condition [4].

Here
\[ K(p, p_1, p_2) < \infty \iff 1/p = 1/p_1 + 1/p_2 < 3/2, \ 1 < p, p_1, p_2 \leq \infty, \quad (1.9a) \]
see [4] etc.

The case of another pseudo - differential bilinear operators, for instance, Bochner-Riesz average, may be found in [28], [29].

**Example 7. Hilbert’s bilinear operator.**
In this example under at the same restrictions as in previous considerations
\[
g(x) = H_{\lambda_1}, \lambda_2[f_1, f_2](x) := v.p. \int_R f_1(x - \lambda_1 y) f_2(x - \lambda_2 y) \, dy/y, \quad (1.10)
\]
\[
\lambda_1, \lambda_2 = \text{const} \in \mathbb{R}.
\]
This Hilbert’s bilinear operator is the particular case of the last one, namely
\[
\sigma(x, \alpha, \beta) = i \pi \text{sign}(\lambda_1 \alpha + \lambda_2 \beta).
\]

Example 8. Maximal operator.

Define the following \textit{maximal multilinear} operator
\[
g(x) = M_{R}[f_1, f_2, \ldots, f_d](x) \overset{\text{def}}{=} \sup_{R \times R} \prod_{i=1}^{d} \left\{ \frac{1}{|R|} \int_R |f_i(y_i)| \, dy_i \right\}, \quad (1.11)
\]
where \( x \in R^d \) and \( R \) denotes the family of all rectangles in \( R^d \) with sides parallel to the axes.

Here \( p_i \in (1, \infty) \),
\[
\frac{1}{p} = \sum_{i=1}^{d} \frac{1}{p_i}, \quad (1.11a)
\]
\[
K_R(p, p_1, p_2, \ldots, p_d) := C^d \prod_{i=1}^{d} \frac{p_i}{p_i - 1}, \quad (1.11b)
\]
so that
\[
|g(\cdot)|_p = |M_{R}[f_1, f_2, \ldots, f_d](\cdot)|_p \leq K_R(p, p_1, p_2, \ldots, p_d) \prod_{i=1}^{d} |f_i|_{p_i}, \quad (1.11c)
\]
see [28], [29]; see also the reference therein.

Example 9. Hausdorff’s operation.

The operator of a form
\[
g(x) = H_{\Phi, \tilde{A}}[\tilde{f}](x) := \int_{R^n} \frac{\Phi(t)}{|t|^n} \prod_{i=1}^{m} f_i(A_i(t)) \, dt, \; x \in R^n \quad (1.12)
\]
is said to be \textit{Hausdorff operator}.

It is bounded under some natural conditions, see [12], [13], [7]; and as before
\[
K_H(p, \tilde{p}) < \infty \iff \frac{1}{p} = \sum_{i=1}^{m} \frac{1}{p_i}, \; p, p_1 \in (1, \infty).
\]
More precisely, under these conditions
\[ K_H(p, \tilde{p}) \leq C(\Phi, \tilde{A}, m, n) \cdot \prod_{j=1}^{m} \frac{p_j^2}{p_j - 1}. \] (1.12a)

**Example 10. Bounded multiplicative Toeplitz operators on sequence spaces.**

The so-called Toeplitz operator acting on the numerical infinite sequences \( \{x_1, x_1, \ldots\} \) and in the values also in ones by the formulae
\[ g_n = M[f](x) \overset{def}{=} \sum_{k=1}^{\infty} f \left( \frac{n}{k} \right) x_k. \] (1.13)

Here the function \( f(\cdot) \) is defined on the set of all positive rational numbers \( Q_+ \), equipped with usually uniform measure: \( \mu_1\{t/m\} = 1 \); the correspondent measures \( \mu; \mu_2 \) defined on the set of all positive natural numbers is ordinary countable measure with unit value of each number.

Nicola Thorn in the recent article [39] has proved the following bilateral estimate
\[ |y|_p \leq "1" \cdot |f|_{p_2} \cdot |x|_{p_1}, \] (1.13a)
iff
\[ \frac{1}{p} = 1 - \frac{1}{p_1} - \frac{1}{p_2}, \quad p, p_1, p_2 \in (1, \infty). \] (1.13b)

Herewith the constant "1" in the Thorn estimate is in general case the best possible [39].

Alike assertions holds true for the multilinear fractional operator, Bochner-Riesz averages, multiple Riesz transform and so one.

### 2 Grand Lebesgue Spaces.

We recall here for reader convenience some used further facts about the so-called Grand Lebesgue Spaces (GLS); more information about this GLS may be found in articles and monographs [35], [25], [26], [8], [9], [10], [15], [16], [20], [22], [23], [34], and so one.

Let \((Z, B, \nu)\) be certain measure space with some non-trivial measure \( \nu \). Let also \( \psi = \psi(p), \quad p \in (a, b), \quad \exists a \geq 1, \quad \exists b = \text{const} \in (a, \infty) \) (or \( p \in [a, b) \)) be bounded from below: \( \inf \psi(p) > 0 \) continuous inside the semi-open interval \( p \in (a, b) \) numerical valued function such that the auxiliary function
\[ h(p) = h[\psi](p) \overset{def}{=} p \ln \psi(p) \] (2.0)
is convex. The set of all such a functions will be denoted by \( \Psi; \Psi = \cup_{a,b: 1 \leq a<b<\infty} \Psi(a,b). \)

As ordinary, in this section for arbitrary measurable function \( f : Z \to R \)

\[ |f|_p = |f|_{p,\nu} \overset{def}{=} \left( \int_Z |f(z)|^p \nu(dz) \right)^{1/p}, \ p \in [1, \infty). \]

An important example. Let \( \eta \) be a measurable function such that there exists \( b = \text{const} > 1 \) so that \( |\xi|_b < \infty \). The so-called natural \( G\Psi_\eta \) function \( \psi_\eta = \psi^{(\eta)}(p) \)

for the r.v. \( \eta \) is defined by a formula

\[ \psi^{(\eta)}(p) \overset{def}{=} |\eta|_p. \]

Then \( \eta \in G\psi_\eta \)

and

\[ ||\eta||_{G\psi_\eta} = 1. \]

We can and will suppose \( a = \inf\{p, \psi(p) < \infty \} \) and correspondingly \( b = \sup\{p, \psi(p) < \infty \}, \) so that \( \text{supp } \psi = [a,b) \) or \( \text{supp } \psi = [a,b] \) or \( \text{supp } \psi = (a,b] \) or at last \( \text{supp } \psi = (a,b) \). The set of all such a functions will be denoted by \( \Psi(a,b) = \{\psi(\cdot)\}; \Psi = \Psi(1,\infty). \)

By definition, the (Banach) Grand Lebesgue Space (GLS) space \( G\psi = G\psi(a,b) \) consists on all the numerical valued (real or complex) measurable functions \( \zeta \) defined on our measurable space \( Z = (Z, B, \nu) \) and having a finite norm

\[ ||\zeta|| = ||\zeta||_{G\psi} \overset{def}{=} \sup_{p \in (a,b)} \left\{ \frac{|\zeta|_p}{\psi(p)} \right\}. \tag{2.1} \]

The function \( \psi = \psi(p) \) is named as a generating function for this Grand Lebesgue Spaces.

These spaces are Banach functional space, are complete, and rearrangement invariant in the classical sense, and were investigated in particular in many works, see the aforementioned works.

We refer here some used in the sequel facts about these spaces and supplement more.

Define as usually for any measurable function \( \zeta : Z \to R \) its tail function

\[ T\zeta(y) \overset{def}{=} \max \{ \nu\{z : f(z) > y\}, \nu\{z : f(z) < -y\} \}, \ y > 0. \]

It is known that by virtue of Tchebychev - Markov inequality: if \( \zeta \neq 0, \) and \( \zeta \in G\psi(a,b), \) then

\[ T\zeta(y) \leq \exp \left( -h_\psi(\ln(y/||\zeta||)) \right), \ y \geq ||\zeta||, \tag{2.2} \]
where
\[ h(p) = h[\psi](p) \overset{def}{=} p \ln \psi(p), \ a \leq p < b; \]
and \( h^*(\cdot) \) denotes a famous Young-Fenchel, or Legendre transform for the function \( h(\cdot) : \)
\[ h^*(v) \overset{def}{=} \sup_{p \in (a,b)} (pv - h(p)). \]
This assertion is alike to the famous Chernoff’s estimate. It allows us to deduce the exponential tails bounds for the function \( \zeta = \zeta(z) \).

Let us introduce a very popular example. Let \( \gamma = \text{constin}(0, \infty) \); define the following \( \Psi \) function
\[ \psi_\gamma(p) \overset{def}{=} p^\gamma, \ p \in [1, \infty). \]
If \( f \in G\psi_\gamma : ||f||_{G\psi_\gamma} = K \in (0, \infty) \), i.e.
\[ \sup_{p \geq 1} \left\{ \frac{|f|_p}{p^\gamma} \right\} = K < \infty, \]
then
\[ T_f(y) \leq \exp \left\{ -\gamma \, e^{-1} \, (y/K)^{1/\gamma} \right\}, \ y \geq K. \]  \hspace{1cm} (2.3)
For instance, the case \( \gamma = 1/2 \) correspondent with classical subgaussian functions.

If the measure \( \nu \) is bounded, for instance \( \nu(Z) = 1 \), then the inverse conclusion to the (2.3) is also true: the any (measurable) function \( f : Z \to R \) satisfies the estimate (2.3), then \( f \in G\psi_\gamma \) and moreover \( ||f||_{G\psi_\gamma} = C(\gamma) \cdot K \in [0, \infty) \).

These Grand Lebesgue Spaces (GLS) are also closely related under simple natural conditions with the so-called exponential Orlicz ones. Namely, introduce the following exponential Young-Orlicz function
\[ N_{\psi}(u) = \exp \left( h^*_\psi(\ln |u|) \right), \ |u| \geq 1; \ N_{\psi}(u) = Cu^2, \ |u| < 1, \]
and the correspondent Orlicz norm will be denoted by \( || \cdot ||L(N_{\psi}) = || \cdot ||L(N) \). It was done
\[ ||\zeta||_{G\psi} \leq C_1||\zeta||L(N) \leq C_2||\zeta||_{G\psi}, \ 0 < C_1 < C_2 < \infty. \]  \hspace{1cm} (2.4)
Note for instance that for the \( \psi_\gamma \) function the correspondent Young-Orlicz one has a form
\[ N_{\psi_\gamma}(u) = \exp( C \, |u|^{1/\gamma} ), \ |u| \geq 1. \]
Let us introduce the following example, with a following degenerate $\psi$ – function. Define

$$\psi_{(r)}(p) := 1, \quad p = r, \quad \psi_{(r)}(p) = +\infty, \quad p \neq r.$$ \hspace{1cm} (2.5)

Here $r = \text{const} \in [1, \infty)$.

The classical Lebesgue - Riesz norm $|f|_r$ coincides with GLS one relative the $\psi_{(r)}(p)$ function:

$$|f|_r = ||f||_{G\psi_{(r)}},$$

if we take of course $C/(+\infty) = 0$.

Thus, the classical theory of Lebesgue - Riesz spaces may be embedded onto GLS one.

3 Main result. Upper estimate in the GLS norm. Exactness.

Let us suppose that each function $f_i(\cdot)$ belongs to some $G\psi_i$ space:

$$\exists (a_i, b_i), \quad 1 \leq a_i < b_i \leq \infty \; \forall q_i \in (a_i, b_i) \Rightarrow$$

$$|f_i|_{L(q_i, X_i)} \leq \psi_i(q_i) \; ||f_i||_{G\psi_i}, \; i = 1, 2, \ldots, d.$$ \hspace{1cm} (3.0)

Of course, each these $\Psi$ function $\psi_i(q_i)$ be choosed as a natural ones:

$$\psi_i(q_i) := |f_i|_{q_i},$$

if they are finite still for some values $q_i \in (1, \infty)$; then they are finite inside certain non-trivial interval $(a_i, b_i); \quad 1 \leq a_i < b_i \leq \infty$.

Define

$$G(\bar{q}) \overset{d e f}{=} K(\bar{q}) \cdot \prod_{i=1}^{d} \psi_i^{a_i}(q_i),$$ \hspace{1cm} (3.1)

and

$$F_{\alpha}[\bar{f}](\bar{q}) \overset{d e f}{=} |\bar{f}|_{\bar{q}} \bar{q}^{\alpha/(\bar{q})}.$$ \hspace{1cm} (3.2)

We will proceed from the obtained before estimate

$$|g|_p = |V(f_1, f_2, \ldots, f_d)|_{\Theta(\bar{q})} \leq K(\bar{q}) \cdot |\bar{f}|_{\bar{q}}^{\alpha/(\bar{q})}.$$ \hspace{1cm} (3.3)

We derive after substituting

$$|g|_p \leq G(\bar{q}) \cdot F_{\alpha}[\bar{f}](\bar{q}),$$
if of course \( p = \Theta(\bar{q}) \).

Let us introduce the set of "layers"

\[
R(p) \overset{\text{def}}{=} \{ \bar{q}, \bar{q} \in D; \Theta(\bar{q}) = p \}, \quad p \in (a, b), \quad 1 \leq a < b \leq \infty.
\] (3.4)

and define

\[
\kappa(p) = \kappa[\bar{f}, V(\cdot)](p) \overset{\text{def}}{=} \inf_{\bar{q} \in R(p)} \left[ G(\bar{q}) \cdot F_\alpha[\bar{f}](\bar{q}) \right].
\] (3.5)

We proved really the following main result of this report.

**Theorem 3.1.** We assert in fact that under formulated above restrictions and notations

\[
\|g[\bar{f}, V]\| G_\kappa \leq 1,
\] (3.6)

with correspondent exponential tail estimation.

*Let us discuss now the exactness of the estimate of theorem 3.1.* It is true still in the so-called "one-dimensional case" \( d = 1 \), see [32], [33], [34].

Note that the exactness of our estimates holds true if for instance the each function \( \psi_i(p) \) coincides correspondingly with natural function for the function \( f_i: \psi_i(p) = |f_i|_p \).

The multivariate case \( d \geq 2 \) may be investigated quite analogously. In detail, denote alike in [33]

\[
U(\psi, f) = \left[ \frac{\|V[f]\| G_\psi}{\|g\| G_\kappa} \right];
\]

and

\[
\overline{U} = \sup_{\psi \in \Psi} \sup_{0 \neq f \in G_\psi} U(\psi, f);
\] (3.7)

then

\[
\overline{U} = 1.
\] (3.8)

### 4 Examples.

We will use the following auxiliary facts.

**Lemma 4.1.**
\[
\min_{\alpha, \beta} \left[ \alpha^\gamma_1 \beta^\gamma_2 : \alpha, \beta > 0, 1/\alpha + 1/\beta = 1 \right] = \frac{(\gamma_1 + \gamma_2)^{\gamma_1 + \gamma_2}}{\gamma_1^{\gamma_1} \gamma_2^{\gamma_2}}.
\]

Here \(\gamma_1, \gamma_2 = \text{const} > 0\).

**Lemma 4.2.**

\[
\min_{p_1, p_2 \geq 1} \left[ p_1^\gamma_1 p_2^\gamma_2 : p_1^{-1} + p_2^{-1} = p^{-1} \right] = p^{\gamma_1 + \gamma_2} \cdot \frac{(\gamma_1 + \gamma_2)^{\gamma_1 + \gamma_2}}{\gamma_1^{\gamma_1} \gamma_2^{\gamma_2}}.
\]

As before, \(\gamma_1, \gamma_2 = \text{const} > 0\).

We return to the considered examples (1 - 10). In each case we assume that

\[
f_i \in G_{\psi_{\gamma_i}}, \exists \gamma_i \in (0, \infty),
\]

and that \(d \geq 2\); the case \(d = 1\) is investigated, e.g. in [32], [33].

**Example 4.1.** Suppose \(f_1(\cdot) \in G_{\psi_{\gamma_1}}, f_2(\cdot) \in G_{\psi_{\gamma_2}},\) and as in the example 1 \(g(x) = f_1(x) \cdot f_2(x)\). Denote

\[
\theta := \gamma_1 + \gamma_2.
\]

We deduce by virtue of Lemma 4.1 that \(g(\cdot) \in G_{\psi_{\theta}}\) and herewith

\[
||g||_{G_{\psi_{\theta}}} \leq ||f_1||_{G_{\psi_{\gamma_1}}} \cdot ||f_2||_{G_{\psi_{\gamma_2}}}. \tag{4.1}
\]

**Example 4.2. Tensor product.** Suppose that \(f_1(\cdot) \in G_{\psi_{\gamma_1}}, f_2(\cdot) \in G_{\psi_{\gamma_2}},\) and that as in the example 2 \(g(x_1, x_2) = f_1(x_1) \cdot f_2(x_2), x_1 \in X_1, x_2 \in X_2\). We apply the relations (1.5), (1.5a) and deduce again \(g(\cdot) \in G_{\psi_{\theta}}\) and furthermore

\[
||g||_{G_{\psi_{\theta}}} \leq ||f_1||_{G_{\psi_{\gamma_1}}} \cdot ||f_2||_{G_{\psi_{\gamma_2}}} \tag{4.2}
\]

with the exact value of the coefficient "1."

More generally, if for certain \(\Psi\) functions \(\nu_1 = \nu_1(p), \nu_2 = \nu_2(p) \Rightarrow f_i \in G_{\nu_i}, \ i = 1, 2,\) then

\[
||g||_{G(\nu_1 \cdot \nu_2)} \leq ||f_1||_{G\nu_1} \cdot ||f_2||_{G\nu_2}. \tag{4.2a}
\]

Note that the equality in the last estimate (4.2a) may be attained if for instance both the \(\Psi\) functions are correspondingly the natural functions for \(f_1, f_2:\)

\[
\nu_1(p) = |f_1|_p, \nu_2(p) = |f_2|_p.
\]

**Example 4.3. Integral bilinear operator.**
We return to the integral bilinear (regular) operator (1.6). Suppose for simplicity that \( \mu(X) = \mu_j(X_j) = 1 \) and that
\[
\bar{L} := \sup_{x, x_1, x_2} |L(x, x_1, x_2)| < \infty;
\]
then
\[
|g|_p \leq \bar{L} \cdot |f_1|_p \cdot |f_2|_p.
\]

One can repeat the previous considerations: if for certain \( \Psi \) – functions \( \nu_1 = \nu_1(p) \), \( \nu_2 = \nu_2(p) \) \( \Rightarrow f_i \in G\nu_i \), \( i = 1, 2 \), then
\[
||g||_{G(\nu_1 \cdot \nu_2)} \leq \bar{L} \cdot ||f_1||_{G\nu_1} \cdot ||f_2||_{G\nu_2}.
\]  
(4.3)

**Example 4.4. Classical convolution.**

Let \( f_1 \in G\psi_{\gamma_1} \), \( f_2 \in G\psi_{\gamma_2} \), \( X = X_1 = X_2 = R^n \) or \( X = X_1 = X_2 = [0, 2\pi]^n \) and \( g = f_1 \ast f_2 \). One can apply the proposition of Lemma 4.2 and Beckner’s estimate:
\[
||g||_{G\psi_\theta} \leq \frac{(\gamma_1 + \gamma_2)^{1+\gamma_2}}{\gamma_1^{1/\gamma_1} \gamma_2^{1/\gamma_2}} ||f_1||_{G\psi_{\gamma_1}} \cdot ||f_2||_{G\psi_{\gamma_2}}.
\]  
(4.4)

A slight generalization: let \( f_1 \in G\zeta_1 \), \( f_2 \in G\zeta_2 \) for some \( \Psi \) – functions \( \zeta_1 \), \( \zeta_2 \); define a new such a function
\[
\zeta(p) = \zeta[\zeta_1, \zeta_2](p) := \inf \{G(p, p_1, p_2) \zeta_1(p_1) \zeta_2(p_2) : p_1, p_2 \in (1, \infty), 1/p_1 + 1/p_2 = 1 + 1/p \}.
\]

We deduce a non-refined up to multiplicative constant in general case convolution estimate in the GLS terms
\[
||g||_{G\zeta} \leq ||f_1||_{G\zeta_1} \cdot ||f_2||_{G\zeta_2}.
\]  
(4.4a)

Recall that in this example
\[
1 \leq \frac{1}{p_1} + \frac{1}{p_2} \leq 2.
\]

**Example 4.5. Infimal convolution, see example 5.**

A very simple estimate:
\[
|g(x)| = |f_1 \square f_2|(x) \leq |f_1(x/2)| + |f_2(x/2)|, \ x \in R^d.
\]  
(4.5)

Further, suppose \( f_1, f_2 \in G\psi \) for some \( \psi \in \Psi \). It follows immediately from (1.8a) and (1.8b) that
Thus, $g(\cdot) \in G\psi$ and herewith

$$
||g||G\psi \leq 2^d \psi(p) \left(||f_1||G\psi + ||f_2||G\psi\right).
$$

(4.5a)

**Examples 4.6, 4.7. Pseudo - differential product and Hilbert operation.**

These cases may be investigated quite analogously to the one for ordinary convolution operation and may be omitted.

**Example 4.8. Maximal operator, see (1.11).**

Suppose

$$
\exists \gamma \in (0, \infty) \Rightarrow f_i \in G\psi_\gamma,
$$

(4.8a)

and recall that here $d \geq 2$.

We deduce solving the following extremal problem

$$
\prod_{i=1}^{d} \left\{ \frac{p_i^{\gamma+1}}{p_i - 1} \right\} \rightarrow \min
$$

subject to the limitation

$$
l : \sum_{i=1}^{d} \frac{1}{p_i} = \frac{1}{p}:
$$

$$
Z := \min_{(i)} \prod_{i=1}^{d} \left\{ \frac{p_i^{\gamma+1}}{p_i - 1} \right\} = \left( \frac{dp}{dp-1} \right)^{d(\gamma+1)} \leq C_1(d) p^{d(\gamma+1)},
$$

so that

$$
|g|_p \leq C_2(d) p^{d(\gamma+1)} \prod_{i=1}^{d} ||f_i||G\psi_\gamma,
$$

(4.8b)

or equally

$$
||g||G\psi_{d(\gamma+1)} \leq C_2(d) \prod_{i=1}^{d} ||f_i||G\psi_\gamma.
$$

(4.8)

**Example 4.9. Hausdorff’s operation.**

We find repeating the considerations and notations of the previous example that under conditions of the example 9 for the function of the form
\[ g(x) = H_{\Phi, \vec{A}}[f](x) := \int_{R^n} \frac{\Phi(t)}{|t|^n} \prod_{i=1}^{m} f_i(A_i(t)) \ dt, \ x \in R^n \]

we have

\[ |g|_p \leq C_3(d, \Phi, \vec{A}) \ p^{d(\gamma + 2)} \prod_{i=1}^{d} ||f_i||G_{\psi}, \]

or equally

\[ ||g||G_{\psi(d\gamma + 2)} \leq C_3(d, \Phi, \vec{A}) \prod_{i=1}^{d} ||f_i||G_{\psi}. \] (4.9)

**Example 4.10.** Toeplitz operators on sequence spaces. See 1.13.

Suppose as above that \( f \in G_{\psi_{\gamma_1}}, \ x \in G_{\psi_{\gamma_2}}, \ \gamma_1, \ \gamma_2 = \text{const} > 0 \), and that

\[ \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p'}; \ p' := \frac{p}{p - 1} > 1. \]

Introduce a new \( \Psi \) – function

\[ \tau_{\gamma_1, \gamma_2}(p) \overset{\text{def}}{=} \left( \frac{(\gamma_1 + \gamma_2)^{\gamma_1 + \gamma_2}}{\gamma_1^{\gamma_1} \gamma_2^{\gamma_2}} \right) \left( \frac{p}{p - 1} \right)^{\gamma_1 + \gamma_2}. \]

We conclude on the basis of Lemma 4.2 and theorem 1

\[ ||g||G_{\tau_{\gamma_1, \gamma_2}} \leq ||f||G_{\psi_1} \cdot ||x||G_{\psi_2}. \] (4.10)

### 5 Tail description of obtained results.

Many of obtained estimates may be expressed in the terms of tail behavior of used functions. Namely, suppose for simplicity in this section that all our measures are bounded: \( \mu_i(X_i) = \mu(X) = 1 \).

Let us return at first to the example 4.1, i.e. to the multiplicative operation; assume that

\[ T_{f_i}(u) \leq \exp \left[ -(u/S_i)^{1/\gamma_i} \right], \ u \geq 1, \ S_i = \text{const} > 0, \]

\[ i = 1, 2, \ldots, d, \ \gamma_i \in (0, \infty); \] (5.0)

then
\[ T_{f_1, f_2}(u) \leq \exp \left\{ -C(\gamma_1, \gamma_2) \left[ \frac{u}{(S_1 S_2)^{\gamma_1/2}} \right]^{\gamma_2/(\gamma_1+\gamma_2)} \right\}, \quad u \geq 1. \] (5.1)

At the same estimation (5.1) holds true also in the examples (4.2), (4.3) and (4.4), of course with another constants instead \( C(\gamma_1, \gamma_2) \).

Let us pay our attention to the example 4.8, devoting to the maximal operations. The assumption (4.8a) may be rewritten in the case when \( \mu_i(X_i) = 1 \) as follows

\[ \exists \gamma \in (0, \infty), \exists k_i \in (0, \infty) \Rightarrow T_{f_i}(u) \leq \exp \left( -\left( \frac{u}{k_i} \right)^{1/\gamma} \right), \quad i = 1, 2, \ldots, d; \] (5.2)

and proposition of this example - as follows

\[ T_{g}(u) \leq \exp \left\{ -C_2(\kappa, \gamma_1, \gamma_2) \frac{u^{1/d(\gamma+1)}}{\gamma+1} \right\}, \quad u \geq 1. \] (5.3)

Alike result holds true for the Hausdorff operation, see example 4.9. Indeed, impose on the functions \( \{ f_i \} \) again the condition (5.2) as well as the conditions of boundedness of our measures \( \mu_i, \mu \). Then

\[ T_{g}(u) \leq \exp \left\{ -C_2(\kappa, \gamma_1, \gamma_2) \frac{u^{1/d(\gamma+2)}}{\gamma+2} \right\}. \] (5.3)

6 Concluding remarks.

A. It is interest by our opinion to investigate the feature of compactness of these multivariate operations, as well as to obtain the exact value of appeared constants in the Grand Lebesgue Spaces setting.

B. Analogous research may be provided for another singular multivariate operations, say, for the bilinear fractional Riesz operation [19]

\[ R[f_1, f_2](x) = \int_{\mathbb{R}^2n} \frac{f_1(y) f_2(z) \ dy \ dz}{\left( |x-y|^2 + |x-z|^2 \right)^{n-\beta}}, \quad \beta \in (0, n) \]

as well as oscillator multilinear integral operations [17]; commutators of singular integral operations [17], [40]; bilinear Bochner-Riesz means [24], [28], and so one.

C. One of interest application of the estimates for multivariate operation to a system of quadratic derivative nonlinear Schrödinger equations is represented in a recent article [21].
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