Let $G$ be a reductive algebraic group over a local field $k$, and $K$ a quadratic extension of $k$. The aim of this section is to formulate a conjecture about representations of $G(K)$ distinguished by $G(k)$ in terms of Langlands parameters, or what is also called the Langlands-Vogan parametrization. A special case of this conjecture is for the degenerate case of $K = k + k$, in which case the question amounts to understanding the contragredient of a representation of $G(k)$ in terms of Langlands-Vogan parameters, which in fact we will take up first; results about $SL_2(k) \subset SL_2(K)$ from [AP03] in terms of Whittaker models (involving nontrivial $L$-packets for both groups) was an important guide, besides works of Jacquet on distinction of representations of $GL_n(K)$ by $U_n$.

Let $W_k$ be the Weil group of $k$, and $W'_k$ the Weil-Deligne group of $k$. Let $L^G(C) = \hat{G}(C) \rtimes W_k$, be the $L$-group of $G$ which comes equipped with a map onto $W_k$. An admissible homomorphism $\varphi_k : W'_k \to L^G$ is called a Langlands parameter for $G$. To an admissible homomorphism $\varphi_k$, is associated the group of connected components $C_{\varphi_k} = Z_{\varphi_k}/Z^0_{\varphi_k}$ where $Z_{\varphi_k}$ is the centralizer of $\varphi_k$ in $\hat{G}(C)$, and $Z^0_{\varphi_k}$ is its connected component of identity.

According to the Langlands-Vogan parametrization, to an irreducible admissible representation $\pi$ of $G(k)$, there corresponds a pair $(\varphi_k, \mu)$ consisting of an admissible homomorphism $\varphi_k : W'_k \to L^G$ and a representation $\mu$ of its group of connected components $C_{\varphi_k}$. The pair $(\varphi_k, \mu)$ determines $\pi$ uniquely, and one knows which pairs arise in this way. The Langlands-Vogan parametrization depends on fixing a base point, consisting of a pair $(\psi, N)$ where $\psi$ is a nondegenerate character on $N$ which is a maximal unipotent subgroup in a quasi-split innerform of $G(k)$.

Before we come to the description of the Langlands parameter of $\pi^\vee$ in terms of that of $\pi$, we recall that for a simple algebraic group $G$ over $C$, one has either:

1. $G(C)$ has no outer automorphism, or $G(C)$ is $D_{2n}$ for some $n$; in these cases, all irreducible finite dimensional representations of $G(C)$ are self-dual; or,
2. $G(C)$ has an outer automorphism of order 2 which has the property that it takes all irreducible finite dimensional representations of $G(C)$ to their contragredient.

The above results about simple algebraic groups (over $C$) can be extended to all semisimple algebraic groups, and then further to all reductive algebraic groups, and give in this generality of reductive algebraic groups an automorphism $\iota$ of $G(C)$ of order 1 or 2, unique up to inner automorphisms, which takes any finite dimensional representation of $G(C)$ to its dual. (For example, for a torus $T$, $\iota$ is just the inversion, $\iota : z \to z^{-1}, z \in T$.)
One can also define such an automorphism \( \iota \) for a quasi-split group \( G(k) \), for \( k \) any field, which is well-defined as an element of \( \text{Aut}(G)(k)/G(k) \) with \( G(k) = G(k)/\mathbb{Z}(k) \) sitting inside the automorphism group of \( G \) by inner-automorphisms. The element \( \iota \) is defined in the usual way using a based root datum for \( G(k) \). We will call this automorphism \( \iota \), the duality automorphism of \( G(k) \). For \( G = \hat{G} \), this automorphism of \( G(\mathbb{C}) = \hat{G}(\mathbb{C}) \) extends to an automorphism of \( ^kG(\mathbb{C}) = \hat{G}(\mathbb{C}) \rtimes W_k \), which will again be denoted by \( \iota \).

We next recall that in [GGP], section 9, denoting \( G^{\text{ad}} \) the adjoint group of \( G \) (assumed without loss of generality at this point to be quasi-split since we want to construct something for the \( L \)-group), there is constructed a homomorphism from \( G^{\text{ad}}(k)/G(k) \rightarrow \mathring{C}_{\varphi_k} \), denoted \( g \mapsto \eta_g \). Let \( g_0 \) be the unique conjugacy class in \( G^{\text{ad}}(k) \) representing an element in \( T^{\text{ad}}(k) \) (with \( T^{\text{ad}} \) a maximally split, maximal torus in \( G^{\text{ad}}(k) \)) which acts by \(-1\) on all simple root spaces. Denote the corresponding \( \eta_{g_0} \) by \( \eta_{-1} \), a character on \( C_{\varphi_k} \), which will be the trivial character for example if \( g_0 \) can be lifted to \( G(k) \).

**Conjecture 1.** For an irreducible admissible representation \( \pi \) of \( G(k) \) with Langlands-Vogan parameter \( (\varphi_k, \mu) \), the Langlands-Vogan parameter of \( \pi^\vee \) is \((\iota \circ \varphi_k, (\iota \circ \mu)^\vee \otimes \eta_{-1}) \), where \( \iota \) is the duality automorphism of \( ^kG(\mathbb{C}) \).

For \( G \) quasi-split over \( k \), and \( \iota \) now an automorphism of \( G \) defined over \( k \) (well-defined up to inner-automorphisms by elements of \( G(k) \)), if \( C_{\varphi_k} \) is an elementary abelian 2-group (as is usually the case), the dual representation \( \pi^\vee \) is obtained by using the automorphism \( \iota \) of \( G(k) \), and then conjugating by the element \( g_0 \) in \( G^{\text{ad}}(k) \).

**Remark:** (a) Note that the conjecture allows for the possibility, for groups such as \( G_2(k), F_4(k), \) or \( E_8(k) \) to have non-selfdual representations arising out of component groups (which can be \( \mathbb{Z}/3, \mathbb{Z}/4, \) and \( \mathbb{Z}/5 \) in these respective cases). And indeed \( G_2(k), F_4(k), \) and \( E_8(k) \) are known to have non self-dual representations.

(b) A particular case of the conjecture is that the map \( \pi \rightarrow \pi^\vee \) takes \( L \)-packet of representations to \( L \)-packet of representations which if it stabilizes an \( L \)-packet, acts either as identity on it, or without fixed points.

(c) The conjecture above generalizes the [MVW] description of contragredient of representations of classical groups, such as for \( \text{Sp}(W), \text{U}(V) \), to all groups, and suggests an approach via analysis of conjugacy classes in \( G(k) \).

We now return to the context of \( G(k) \subset G(K) \). The \( L \)-group of \( G(K) \) is \((\hat{G}(\mathbb{C}) \ltimes \hat{G}(\mathbb{C})) \rtimes W_k \), leaving the precise description of the semi-direct product to the reader’s imagination. (If \( G \) is split over \( k \), this is of course standard.) The group \( \hat{G}(\mathbb{C}) \) comes equipped with an automorphism \( \iota \) of order 2, allowing us to define a subgroup of \((\hat{G}(\mathbb{C}) \ltimes \hat{G}(\mathbb{C})) \rtimes W_k \), to be

\[
\Delta^\iota(G)(\mathbb{C}) \rtimes W_k = \{(g, \iota(g))|g \in \hat{G}(\mathbb{C})\} \rtimes W_k.
\]

Clearly, \( \Delta^\iota(G)(\mathbb{C}) \rtimes W_k \) is the \( L \)-group of either \( G \) if \( \iota \) is trivial, or is the \( L \)-group of the unique quasi-split but not split outer form of \( G \) which splits over \( K \); in either case, we have defined a companion form of \( G \), to be denoted by \( G^{\text{out}} \); for example, for \( G = \text{GL}_n, G^{\text{out}} = \text{U}_n \).
A character of $G(k)$ of order 2: In an earlier paper of the 2nd author [P01], there is the construction of a character $\chi_K : G(k) \to \mathbb{Z}/2$ associated to any quadratic extension $K$ of $k$ which plays an important role in questions about distinction; the character $\chi_K$ is functorial under maps of reductive algebraic groups with abelian kernel and cokernel. We will review the construction of $\chi_K$ here. Let $G^{\text{ad}}$ denote the adjoint group of $G$, i.e., $G$ modulo center, and $G^{\text{sc}}$ the simply connected cover of $G^{\text{ad}}$. Let $Z_{sc}$ be the center of $G^{\text{sc}}$. Then we have an exact sequence of groups,

$$1 \to Z_{sc}(k) \to G^{\text{sc}}(k) \to G^{\text{ad}}(k) \to H^1(\text{Gal}(\bar{k}/k), Z_{sc}) \to \cdots$$

The character $\chi_K$ factors through a character of $G^{\text{ad}}(k)$ via the natural map, $G(k) \to G^{\text{ad}}(k)$, so we need to construct one for $G^{\text{ad}}(k)$, which arises from the previous exact sequence from a character of $H^1(\text{Gal}(\bar{k}/k), Z_{sc})$, which by Tate-Nakayama duality amounts to constructing an element of $H^1(\text{Gal}(\bar{k}/k), \hat{Z}_{sc})$, where $\hat{Z}_{sc}$ is the Cartier dual of $Z_{sc}$.

Let $\hat{G}'$ be the connected component of the $L$-group of $G'G^{\text{ad}}$. It is clear that one can choose a regular unipotent in $\hat{G}'$ such that the corresponding Jacobson-Morozov embedding of $\text{SL}_2(\mathbb{C})$ into $\hat{G}'$ is invariant under (pinned) outer automorphisms of $\hat{G}'$. The center of $\text{SL}_2(\mathbb{C})$ under this embedding goes to the center of $\hat{G}'$ which is nothing but $\hat{Z}_{sc}$, inducing a map $H^1(\text{Gal}(\bar{k}/k), Z_{sc}) \to H^1(\text{Gal}(\bar{k}/k), \hat{Z}_{sc})$. Since $H^1(\text{Gal}(\bar{k}/k), Z_{sc})$ parametrizes quadratic etale extensions of $k$ of degree 2, we have finally constructed the character $\chi_K : G(k) \to Z/2$ associated to any quadratic extension $K$ of $k$.

Example: (a) For $G = \text{GL}_n$, $\chi_K = \omega_{K/k} \circ \det$ for $n$ even and trivial for $n$ odd.

(b) For $G = \text{U}_n$, defined using a hermitian form over $K$, $\chi_K$ is trivial for all $n$.

Conjecture 2. An irreducible admissible representation $\pi$ of $G(K)$ with Langlands-Vogan parameter $(\varphi_k, \mu)$ is distinguished by the character $\chi_K$ of $G(k)$ if and only if

1. The parameter $\varphi_k : W'_k \to (\hat{G}(\mathbb{C}) \times \hat{G}(\mathbb{C})) \times W_k$, factors through $\Delta^i(G)(\mathbb{C}) \times W_k$.
2. $\iota \circ \mu = \mu' \otimes \eta_{-1}$.
3. For representations $\pi$ inside an $L$-packet of representations of $G(K)$ with character of the component group $\mu$ satisfying (2) above, the dimension of $\text{Hom}_{G(K)}(\pi, \mathbb{C})$ is independent of $\mu$, and is equal to the number of inequivalent ways of lifting the parameter $\varphi_k : W'_k \to (\hat{G}(\mathbb{C}) \times \hat{G}(\mathbb{C})) \times W_k$, to a parameter $\varphi'_k : W'_k \to \Delta^i(G)(\mathbb{C}) \times W_k$:

\[
\begin{array}{ccc}
\Delta^i(G)(\mathbb{C}) \times W_k, & \text{mod} & \text{the equivalence relation on these parameters} \\
W'_k & \longrightarrow & (\hat{G}(\mathbb{C}) \times \hat{G}(\mathbb{C})) \times W_k,
\end{array}
\]

induced by twisting representations (or parameters) by characters of $Z_Q(k)$, where $Z_Q$ is the maximal abelian quotient of $G^{\text{out}}$ as an algebraic
group over \( k \), whose base change to \( \mathbb{Z}_Q(K) \) are trivial; in fact, one should consider the degree of the corresponding map from the set of homomorphisms \( W'_k \to \Delta'(G)(\mathbb{C}) \times W_k \), up to equivalence to its image in the set of homomorphisms up to equivalence \( W'_k \to (\hat{G}(\mathbb{C}) \times \hat{G}(\mathbb{C})) \times W_k \), taking the twisting by central characters into account.

Remark: (a) The condition (1) in the conjecture fixes those \( L \)-packets of \( G(K) \) for which there is a possible representation with \( G(k) \)-invariant form (by the quadratic character \( \chi_K \)). Condition (2) then works inside the \( L \)-packet to fix those that have \( G(k) \)-invariant form. Condition (3) gives the multiplicity, which is then either 0 or a number dependent only on the \( L \)-packet. Abelian twists have been used since one would like multiplicity one for \( T(\mathbb{Q}) \), for a torus \( T \), see below for some details; its necessity is also reminded by a conjecture of Jacquet about multiplicity formula for the pair \((\text{GL}_n(K), U_n(k))\) that we come to later.

(b) For \( \text{SL}_2(k) \subset \text{SL}_2(K) \), we know by [AP03] that a representation \( \pi \) of \( \text{SL}_2(K) \) is distinguished by \( \text{SL}_2(k) \) if and only if \( \pi \) has a Whittaker model with respect to a character of \( K/k \). This condition about Whittaker model can be written as \( \psi_\sigma(x) = \psi(-x) \), which is the condition (2) above.

(c) The mysterious role of the character \( \chi_K \) in the conjecture is related eventually to the symmetry of the bilinear form \( B : \pi \times \pi' \to \mathbb{C} \); indeed the character \( \chi_K \) is constructed via an element of the center of \( \hat{G} \) which determines symmetry property of selfdual representations of \( \hat{G} \).

(d) It seems curious that in this paper, we had many occasions to consider the question about fibers of functorial lifts.

Evidence for the multiplicity formula:

(1) For a torus \( T \) over \( k \), it is easy to see that \( T^{\text{out}} \) is the torus which sits in the exact sequence,

\[
1 \to T \to R_{K/k}(T) \to T^{\text{out}} \to 1.
\]

Thus, at the level of \( k \)-rational points, we have,

\[
1 \to T(k) \to T(K) \to T^{\text{out}}(k) \to H^1(\text{Gal}(\bar{k}/k), T) \to H^1(\text{Gal}(\bar{k}/K), T) \to \cdots
\]

It follows that characters of \( T(K) \) which are trivial on \( T(k) \) arise from restriction of characters of \( T^{\text{out}}(k) \), and that if \( T \) splits over \( K \), there are \( H^1(\text{Gal}(\bar{k}/k), T) \) many characters of \( T^{\text{out}}(k) \) giving rise to the same character of \( T(K) \).

(2) The pair \((\text{GL}_n(K), \text{GL}_n(k))\). In this case, \( G^{\text{out}} \), is the unitary group \( U_n \) defined by \( K/k \). Our conjecture above says that representations of \( \text{GL}_n(K) \) distinguished by \( \text{GL}_n(k) \) are precisely those which arise as base change of a representation of \( U_n(k) \). It has been known, cf. [GGP], that the base change map taking the Langlands parameter of a representation of \( U_n(k) \) to one of \( \text{GL}_n(K) \) is an injective map; since multiplicity one for the pair \((\text{GL}_n(K), \text{GL}_n(k))\) is well-known (an elementary result based on the method
of Gelfand pairs, cf. [Fli91]), our multiplicity formula matches well with known results in this case.

(3) The pair \((GL_n(K), U_n(k))\). It is known that if a representation \(\pi\) of \(GL_n(K)\) is distinguished by \(U_n(k)\), then it must arise as a base change of a representation of \(GL_n(k)\). It has been conjectured by Jacquet in [Jac01] that if \(n\) is odd, then \(\dim \text{Hom}_{U_n(k)}[\pi, C]\) is equal to half the number of representations of \(GL_n(k)\) which base change to the representation \(\pi\) of \(GL_n(K)\). Our conjectures fit well with this, and suggest that for \(n\) even, \(\dim \text{Hom}_{U_n(k)}[\pi, C]\) is equal to the number of equivalence classes of representations of \(GL_n(k)\) under the equivalence \(V \sim V \otimes \omega_{K/k}\) which base change to \(\pi\) (note that \(V\) and \(V \otimes \omega_{K/k}\) have the same base change to \(GL_n(K)\)).

(4) In the case of \((SL_2(K), SL_2(k))\), the multiplicity of the space of \(SL_2(k)\)-invariant linear forms on a representation of \(SL_2(K)\) was studied in [AP03] in detail, and it was found that \(\dim \text{Hom}_{SL_2(k)}[\pi, C]\), as \(\pi\) runs over an \(L\)-packet of representations of \(SL_2(K)\), is either \(d_\pi\) or 0 for an integer \(d_\pi\) which depends only on the \(L\)-packet of \(\pi\), and in the notation used earlier in the paper, is given by

\[
d_\pi = \frac{X_\pi}{Z_\pi/Y_\pi}.
\]

It suffices then to note the following lemma.

**Lemma 1.** The basechange map \(\pi \rightarrow \Pi = \text{BC}(\pi)\) from irreducible admissible representations of \(GL_2(k)\) to representations of \(GL_2(K)\), for \(K/k\) quadratic, descends to a map — call it \(TBC\) (for twisted basechange) — from irreducible admissible representations of \(GL_2(k)\), considered up to twists by characters, to irreducible admissible representations of \(GL_2(K)\) considered up to twists by characters. Then the fibers of \(TBC\), which is nothing but the fibers of the base change map for \(SL_2\), given in terms of the \(L\)-group as liftings:

\[
\begin{array}{ccc}
PGL_2(\mathbb{C}) & \rightarrow & (PGL_2(\mathbb{C}) \times PGL_2(\mathbb{C})) \rtimes \text{Gal}(K/k), \\
W_k & \downarrow & \\
\end{array}
\]

has order given by,

\[
\frac{X_\Pi}{Z_\Pi/Y_\Pi}.
\]

**Proof.** : Consider the group,

\[
A_\Pi = \left\{ \chi : K^\times \rightarrow \mathbb{C}^\times \mid (\Pi \otimes \chi)^\sigma \cong \Pi \otimes \chi \right\} / \left\{ \chi : K^\times \rightarrow \mathbb{C}^\times \mid \chi = \chi^\sigma \right\}.
\]

For \(\chi \in A_\Pi\), let \(\pi_{\chi^{-1}}\chi^\sigma\) denote the class of pair of representations \(\{\pi', \pi' \otimes \omega_{E/F}\}\) of \(GL_2(k)\) such that \(\Pi \otimes \chi = \text{BC}(\pi')\). Note that \(Z_\Pi/Y_\Pi\) acts freely on \(\{\pi_{\chi^{-1}}\chi^\sigma \mid \chi \in A_\Pi\}\). The fibers of \(TBC\) is given by orbits under the above action:

\[
\frac{\{\pi_{\chi^{-1}}\chi^\sigma \mid \chi \in A_\Pi\}}{Z_\Pi/Y_\Pi}.
\]
We note that $A_{\Pi}$ is in bijection with $Y_{\Pi}$ under the map $\chi \mapsto \chi^{-1}\chi^\sigma$, and that if $\Pi$ is in the discrete series, then

(a) The set on the numerator above is in bijection with $A_{\Pi}$, and
(b) $Y_{\Pi}$ is in bijection with $X_{\Pi}$.

This proves the lemma if $\Pi$ is a discrete series representation. For principal series representations, both (1) and (2) are wrong in general, thus the lemma is more subtle; we verify it in a case-by-case check below. 

**Examples:** We illustrate the multiplicity formula with examples of principal series representations of $\text{SL}_2(K)$ taken from [AP03]. In what follows, we introduce the notation $\pi_1 \sim \pi_2$ for two representations of $\text{GL}_2(K)$ (or $\text{GL}_2(k)$) which are twists of each other by a character.

Let $V$ be an irreducible admissible representation of $\text{SL}_2(K)$ that occurs in the restriction of a principal series representation $\pi = \text{Ps}(\chi_1, \chi_2)$ of $\text{GL}_2(K)$. Suppose that $V$ is distinguished with respect to $\text{SL}_2(k)$ (and therefore $\chi_1\chi_2^{-1}|_{k^\times} = 1$ or $\chi_1\chi_2^{-1} = (\chi_1\chi_2^{-1})^\sigma$). Then we have,

(a) $\dim_{\mathbb{C}} \text{Hom}_{\text{SL}_2(k)}(V, 1) = 1$, if $\chi_1\chi_2^{-1}|_{k^\times} = 1$, and $\chi_1 \neq \chi_2$. In this case, since $\chi_1/\chi_2$ is trivial on $k^\times$, there is a character $\chi$ of $K^\times$ such that $\chi_1/\chi_2 = \chi/\chi^\sigma$, and so

$$\pi = \text{Ps}(\chi_1, \chi_2) \sim \text{Ps}(\chi_1/\chi_2, 1) = \text{Ps}(\chi/\chi^\sigma, 1) \sim \text{Ps}(\chi, \chi^\sigma),$$

comes as the base change of a unique representation of $\text{GL}_2(k)$ (up to twists) corresponding to the character $\chi$ of $K^\times$.

(b) $\dim_{\mathbb{C}} \text{Hom}_{\text{SL}_2(k)}(V, 1) = 1$, if $\chi_1\chi_2^{-1}|_{k^\times} = \omega_{K/k}$, $\chi_1^2 = \chi_2^2$, $\chi_1 \neq \chi_2$.

In this case, $\chi_1/\chi_2$ is a Galois invariant quadratic character of $K^\times$, so $\chi_1/\chi_2 = \mu \circ \mathbb{N}$ for a character $\mu$ of $k^\times$ with $\mu^2 = \omega_{K/k}$. Hence the representation $\pi$ (up to twists) is the base change of a unique principal series representation of $\text{GL}_2(k)$ (up to twists).

(c) $\dim_{\mathbb{C}} \text{Hom}_{\text{SL}_2(k)}(V, 1) = 2$, if either $\chi_1\chi_2^{-1} = (\chi_1\chi_2^{-1})^\sigma = \mu \circ \mathbb{N}$ and $\chi_1^2 \neq \chi_2^2$, or $\chi_1 = \chi_2$. In this case,

$$\pi = \text{Ps}(\chi_1, \chi_2) \sim \text{Ps}(\chi_1/\chi_2, 1) = \text{Ps}(\mu \circ \mathbb{N}, 1),$$

hence, $\pi = \text{Ps}(\chi_1, \chi_2)$ arises as base change of two principal series representations of $\text{GL}_2(k)$ which are $\text{Ps}(\mu, 1)$, and $\text{Ps}(\mu, \omega_{K/k})$.

(d) $\dim_{\mathbb{C}} \text{Hom}_{\text{SL}_2(k)}(V, 1) = 3$, if $\chi_1\chi_2^{-1}|_{k^\times} = 1$, $\chi_1^2 = \chi_2^2$, $\chi_1 = \chi_2$.

In this case, the representation $\pi$ (up to twists) is the base change of a unique discrete series representation of $\text{GL}_2(k)$, and two principal series representations of $\text{GL}_2(k)$ (up to twists).

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A RELATIVE LOCAL LANGLANDS CORRESPONDENCE

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