Applications of $q$-Umbral Calculus to Modified Apostol Type $q$-Bernoulli Polynomials

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Abstract- This article aims to identify the generating function of modified Apostol type $q$-Bernoulli polynomials. With the aid of this generating function, some properties of modified Apostol type $q$-Bernoulli polynomials are given. It is shown that aforementioned polynomials are $q$-Appell. Hence, we make use of these polynomials to have applications on $q$-Umbral calculus. From those applications, we derive some theorems in order to get Apostol type modified $q$-Bernoulli polynomials as a linear combination of some known polynomials which we stated in the paper.

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1. Introduction

Throughout this paper, we make use of the following standard notations: $\mathbb{N} := \{1, 2, 3, \ldots \}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Also, as usual, $\mathbb{Z}$ denotes the set of integers, $\mathbb{R}$ denotes the set of real numbers and $\mathbb{C}$ denotes the set of complex numbers.

We now begin with the fundamental properties of $q$-calculus. Let $q$ be chosen as a fixed real number between 0 and 1. The $q$-analogue of any number $n$ is given by

$$[n]_q = \frac{1 - q^n}{1 - q}.$$ 

The expression

$$[n]_q! = [n]_q[n-1]_q \cdots [2]_q[1]_q$$

means the $q$-factorial of $n$, and also let $n, k \in \mathbb{N}_0$, for $k \leq n$

$$\binom{n}{k}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}$$

is called $q$-binomial coefficient. Note that $[0]_q! := 1$. The $q$-derivative of $f(x)$ is defined by

$$D_q f(x) = \frac{d_q f(x)}{d_q x} = \frac{f(x) - f(qx)}{(1 - q)x} \quad (0 < q < 1).$$ (1.1)

If $q \to 1^-$, it becomes

$$\lim_{q \to 1^-} D_q f(x) = \frac{df(x)}{dx}$$
represents familiar derivative of a function $f$, with respect to $x$. The Jackson definite $q$-integral of a function $f$ is also defined by
\[
\int_0^a f(x) dq x = a(1-q) \sum_{j=0}^{\infty} f(q^j a) q^j.
\]

The $q$-exponential functions are given by
\[
e_q (t) = \sum_{n=0}^{\infty} \frac{t^n}{[n]_q!} \quad \text{and} \quad E_q (t) = \sum_{n=0}^{\infty} \frac{q^{(2)} t^n}{[n]_q!} \quad (t \in \mathbb{C} \text{ with } |t| < 1)
\]
with the following equality
\[
e^{-1} (t) = E_q (t).
\]

These fundamental properties of $q$-calculus listed above are taken from the book [3].

By using an exponential function $e_q (x)$, Kupershmidt [10] defined the following $q$-Bernoulli polynomials
\[
\sum_{n=0}^{\infty} B_{n,q}(x) \frac{t^n}{[n]_q!} = \frac{t}{e_q(t) - 1} e_q(x t).
\]

In the case $x = 0$, $B_{n,q}(0) = B_n$ means the $n$-th $q$-Bernoulli number.

Very recently, Kurt [8] defined Apostol type $q$-Bernoulli polynomials of order $\alpha$ by making use of the following generating function:
\[
\sum_{n=0}^{\infty} B_{n,q}^{(\alpha)}(x, y, \lambda) \frac{t^n}{[n]_q!} \lambda^\alpha = \left( \frac{t}{\lambda e_q(t) - 1} \right)^\alpha e_q(x t) E_q(y t) \quad (1.2)
\]
where $\lambda \in \mathbb{C}$ and $\alpha \in \mathbb{N}_0$. In this paper, we will study on the following polynomial $B_{n,q}^{(1)}(x, \lambda) := B_{n,q}(x, \lambda)$ which is given by special cases $\alpha = 1$ and $y = 0$ in (1.2):
\[
\sum_{n=0}^{\infty} B_{n,q}(x, \lambda) \frac{t^n}{[n]_q!} = \frac{t}{\lambda e_q(t) - 1} e_q(x t). \quad (1.3)
\]

When $q \to 1$ in (1.3), it reduces to Apostol-Bernoulli polynomials, see [2,11].

We now review briefly the concept of $q$-umbral calculus. For the properties of $q$-umbral calculus, we refer the reader to see the references [1-4,7,13,14].

Let $\mathbb{C}$ be a field of characteristic zero, and let $\mathcal{F}$ be the set of all formal power series in the variable $t$ over $\mathbb{C}$ with
\[
\mathcal{F} = \left\{ f \mid f(t) = \sum_{k=0}^{\infty} a_k \frac{t^k}{[k]_q!}, \quad (a_k \in \mathbb{C}) \right\}.
\]

Let $\mathbb{P}$ be the algebra of polynomials in the single variable $x$ over the field complex numbers and let $\mathbb{P}^*$ be the vector space of all linear functionals on $\mathbb{P}$. In the $q$-Umbral calculus, $\langle L|p(x) \rangle$ means the action of a linear functional $L$ on the polynomial $p(x)$. This operator has a linear property on $\mathbb{P}^*$ given by
\[
\langle L + M|p(x) \rangle = \langle L|p(x) \rangle + \langle M|p(x) \rangle
\]
and
\[
\langle cL|p(x) \rangle = c\langle L|p(x) \rangle
\]
for any constant $c$ in $\mathbb{C}$.

The formal power series
\[
f(t) = \sum_{k=0}^{\infty} a_k \frac{t^k}{[k]_q!} \quad (1.4)
\]
defines a linear functional on $\mathbb{P}$ by setting
\[
\langle f(t)|x^n \rangle = a_n \quad (n \geq 0). \quad (1.5)
\]
Taking \( f(t) = t^k \) in Eq. (1.4) and Eq. (1.5) gives
\[
\langle t^k | x^n \rangle = [n]_q! \delta_{n,k}, \quad (n, k \geq 0)
\]
(1.6)
where
\[
\delta_{n,k} = \begin{cases} 
1, & \text{if } n = k \\
0, & \text{if } n \neq k 
\end{cases} .
\]

Actually, any linear functional \( L \) in \( \mathbb{P}^* \) has the form (1.4). That is, since
\[
f_L(t) = \sum_{k=0}^\infty \langle L | x^k \rangle \frac{t^k}{[k]_q!},
\]
we have
\[
\langle f_L(t) | x^n \rangle = \langle L | x^n \rangle,
\]
and so as linear functionals \( L = f_L(t) \). Moreover, the map \( L \to f_L(t) \) is a vector space isomorphism from \( \mathbb{P}^* \) onto \( \mathcal{F} \). Henceforth, \( \mathcal{F} \) will denote both the algebra of formal power series in \( t \) and the vector space of all linear functionals on \( \mathbb{P} \), and so an element \( f(t) \) of \( \mathcal{F} \) will be thought of as both a formal power series and a linear functional. From (1.5), we have
\[
\langle c_q(yt) | x^n \rangle = y^n
\]
and so
\[
\langle c_q(yt) | p(x) \rangle = p(y) \quad (p(x) \in \mathbb{P}).
\]

The order \( o(f(t)) \) of a power series \( f(t) \) is the smallest integer \( k \) for which the coefficient of \( t^k \) does not vanish. If \( o(f(t)) = 0 \), then \( f(t) \) is called an invertible series. A series \( f(t) \) for which \( o(f(t)) = 1 \) will be called a delta series (c.f. [1-4, 7, 13, 14]).

If \( f_1(t), \ldots, f_m(t) \) are in \( \mathcal{F} \), then
\[
\langle f_1(t) \ldots f_m(t) | x^n \rangle = \sum_{i_1 + i_2 + \ldots + i_m = n} \binom{n}{i_1, \ldots, i_m} \langle f_1(t) | x^{i_1} \rangle \ldots \langle f_m(t) | x^{i_m} \rangle,
\]
where
\[
\binom{n}{i_1, \ldots, i_r}_q = \frac{[n]_q!}{[i_1]_q! \cdots [i_r]_q!}.
\]

We use the notation \( t^k \) for the \( k \)-th \( q \)-derivative operator on \( \mathbb{P} \) as follows:
\[
t^k x^n = \begin{cases} 
[n]_q! \cdot x^{n-k}, & k \leq n \\
0, & k > n
\end{cases}.
\]

If \( f(t) \) and \( g(t) \) are in \( \mathcal{F} \), then
\[
\langle f(t) g(t) | p(x) \rangle = \langle f(t) | g(t) p(x) \rangle = \langle g(t) | f(t) p(x) \rangle
\]
for all polynomials \( p(x) \). Notice that for all \( f(t) \) in \( \mathcal{F} \), and for all polynomials \( p(x) \)
\[
f(t) = \sum_{k=0}^\infty \langle f(t) | x^k \rangle \frac{t^k}{[k]_q!} \quad \text{and} \quad p(x) = \sum_{k=0}^\infty \langle t^k | p(x) \rangle \frac{x^k}{[k]_q!}.
\]
(1.7)

Using (1.7), we obtain
\[
p^{(k)}(x) = D^k_q p(x) = \sum_{l=k}^\infty \frac{\langle t^l | p(x) \rangle}{[l]_q!} x^{l-k} \prod_{s=1}^k [l-s+1]_q
\]
providing
\[
p^{(k)}(0) = \langle t^k | p(x) \rangle \quad \text{and} \quad \langle 1 | p^{(k)}(x) \rangle = p^{(k)}(0).
\]
(1.8)

Thus, from (1.8), we note that
\[
t^k p(x) = p^{(k)}(x) = D^k_q p(x).
\]
Let $f(t) \in \mathcal{F}$ be a delta series and let $g(t) \in \mathcal{F}$ be an invertible series. Then there exists a unique sequence $s_n(x)$ of polynomials satisfying the following property

$$\langle g(t) f(t)^k | s_n(x) \rangle = [n]_q \delta_{n,k} \quad (n, k \geq 0)$$

(1.9)

which is called an orthogonality condition for any $q$-Sheffer sequence, cf. [1-4, 7, 13, 14].

The sequence $s_n(x)$ is called the $q$-Sheffer sequence for the pair of $(g(t), f(t))$, or this $s_n(x)$ is $q$-Sheffer for $(g(t), f(t))$, which is denoted by $s_n(x) \sim (g(t), f(t))$.

Let $s_n(x)$ be $q$-Sheffer for $(g(t), f(t))$. Then for any $h(t)$ in $\mathcal{F}$, and for any polynomial $p(x)$, we have

$$h(t) = \sum_{k=0}^{\infty} \frac{\langle h(t) s_k(x) \rangle}{[k]_q!} g(t) f(t)^k \quad p(x) = \sum_{k=0}^{\infty} \frac{\langle g(t) f(t)^k p(x) \rangle}{[k]_q!} s_k(x)$$

(1.10)

and the sequence $s_n(x)$ is $q$-Sheffer for $(g(t), f(t))$ if and only if

$$\frac{1}{g(f(t))} c_q(x \mathcal{T}(t)) = \sum_{n=0}^{\infty} s_n(x) \frac{t^n}{[n]_q!}$$

(1.11)

for all $x$ in $\mathbb{C}$, where $\mathcal{T}(f(t)) = f(\mathcal{T}(t)) = t$.

An important property for the $q$-Sheffer sequence $s_n(x)$ having $(g(t), t)$ is the $q$-Appell sequence. It is also called $q$-Appell for $g(t)$ with the following consequence

$$s_n(x) = \frac{1}{g(t)} x^n \Leftrightarrow ts_n(x) = [n]_q s_{n-1}(x).$$

(1.12)

Further important property for $q$-Sheffer sequence $s_n(x)$ is as follows

$$s_n(x) \text{ is } q\text{-Appell for } g(t) \iff \frac{1}{g(t)} c_q(x t) = \sum_{k=0}^{\infty} s_n(x) \frac{t^n}{[n]_q!} \quad (x \in \mathbb{C}).$$

For having information about the properties of $q$-umbral theory, see [1-4, 7, 13, 14] and cited references therein.

Recently several authors have studied $q$-Bernoulli polynomials, $q$-Euler polynomials and various generalizations of these polynomials [1-15]. In the next section, we investigate modified Apostol type $q$-Bernoulli numbers and polynomials, and we apply these numbers and polynomials to $q$-umbral theory which is the systematic study of $q$-umbral algebra. Actually, we are motivated to write this paper from Kim’s systematic works on $q$-umbral theory [4-7].

2. Modified Apostol type $q$-Bernoulli numbers and polynomials

Recall from (1.3) that

$$\sum_{n=0}^{\infty} B_{n,q}(x, \lambda) \frac{t^n}{[n]_q!} = \frac{t}{\lambda e_q(t) - 1} c_q(x t) \quad (\lambda \neq 1).$$

(2.1)

Taking $t \to 0$ on the above gives $B_{0,q}(x, \lambda) = 0$. This shows that the generating function of these polynomials is not invertible. Therefore, we need to modify slightly Eq. (2.1) as follows

$$F^*_q(x, t) = \sum_{n=0}^{\infty} B^*_q(x, \lambda) \frac{t^n}{[n]_q!} = \frac{1}{\lambda e_q(t) - 1} c_q(x t)$$

representing

$$\frac{B_{n+1,q}(x, \lambda)}{[n+1]_q} = B^*_q(x, \lambda).$$

(2.2)

Here we called $B^*_q(x, \lambda)$ modified Apostol type $q$-Bernoulli polynomials. Now

$$\lim_{t \to 0} F^*_q(x, t) = B^*_q(x, \lambda) = \frac{1}{\lambda - 1} \neq 0 \quad (\lambda \neq 1).$$
This modification yields to being invertible for generating function of modified Apostol type $q$-Bernoulli polynomials. As a traditional for some special polynomials to be a number, in the case when $x = 0$, $B_{n,q}^*(0, \lambda) = B_{n,q}^*(\lambda)$ is called the modified Apostol type $n$-th $q$-Bernoulli number. Now we list some properties of modified Apostol type $q$-Bernoulli polynomials as follows.

From (2.2), we obtain

$$B_{n,q}^*(x, \lambda) = \sum_{k=0}^{n} \binom{n}{k}_q B_{k,q}^*(\lambda) x^{n-k} = \sum_{k=0}^{n} \binom{n}{k}_q B_{n-k,q}^*(\lambda) x^k. \quad (2.3)$$

By (2.2), the modified Apostol type $q$-Bernoulli numbers can be found by means of the following recurrence relation:

$$B_{0,q}^*(x, \lambda) = 1 \quad \text{and} \quad \lambda B_{n,q}^*(1, \lambda) - B_{n,q}^*(\lambda) = \delta_{0,n}. \quad (2.4)$$

A few numbers are listed below:

$$B_{0,q}^*(\lambda) = \frac{1}{\lambda-1}, \quad B_{1,q}^*(\lambda) = \frac{-\lambda}{(\lambda-1)^2}, \quad B_{2,q}^*(\lambda) = \frac{\lambda(1+\lambda q)}{(\lambda-1)^3}, \quad B_{3,q}^*(\lambda) = \frac{-\lambda(1+2\lambda q + 2\lambda^2 q^2 + \lambda^3 q^3)}{(\lambda-1)^4}.$$

From (1.11) and (1.12), we have

$$B_{n,q}^*(x, \lambda) \sim (\lambda e_q(t) - 1, t) \quad (2.5)$$

and

$$t B_{n,q}^*(x, \lambda) = [n]_q B_{n-1,q}^*(x, \lambda) = B_{n,q}(x, \lambda). \quad (2.6)$$

It follows from (2.6) that $B_{n,q}^*(x, \lambda)$ is $q$-Appell for $\lambda e_q(t) - 1$.

We now have the following theorem.

**Theorem 1.** Let $p(x) \in \mathbb{P}$. We have

$$\left\langle \frac{\lambda e_q(t) - 1}{t} \mid p(x) \right\rangle = \lambda \int_0^1 p(u) d_q u.$$

**Proof.** From Eq. (2.5) and Eq. (2.6), we write

$$B_{n,q}^*(x, \lambda) = \frac{1}{\lambda e_q(t) - 1} x^n \quad (n \geq 0).$$

By (1.1) and (1.6), we obtain the following calculations

$$\left\langle \frac{\lambda e_q(t) - 1}{t} \mid x^n \right\rangle = \frac{1}{[n+1]_q} \left\langle \frac{\lambda e_q(t) - 1}{t} \mid t x^{n+1} \right\rangle$$

$$= \frac{1}{[n+1]_q} \left(\lambda e_q(t) - 1 \mid x^{n+1}\right)$$

$$= \frac{\lambda}{[n+1]_q} = \lambda \int_0^1 x^n d_q x.$$

Thus, from (2.7), we arrive at

$$\left\langle \frac{\lambda e_q(t) - 1}{t} \mid p(x) \right\rangle = \lambda \int_0^1 p(u) d_q u \quad (p(x) \in \mathbb{P})$$

which is desired result.
Example 1. If we take $p(x) = B_{n,q}^*(x,\lambda)$ in Theorem 1, on the one hand, we derive
\[
\lambda \int_0^1 B_{n,q}^*(x,\lambda) dq x = \left\langle \frac{\lambda e_q(t) - 1}{t} | B_{n,q}^*(x,\lambda) \right\rangle \\
= \left\langle 1 | \frac{\lambda e_q(t) - 1}{t} B_{n+1,q}^*(x,\lambda) \right\rangle \\
= \frac{1}{[n+1]_q} [x^{n+1}] = [n]_q! \delta_{n+1,0}.
\]

On the other hand,
\[
\lambda [n+1]_q \int_0^1 B_{n,q}^*(x,\lambda) dq x = \lambda [n+1]_q \int_0^1 \sum_{k=0}^{n} \binom{n}{k}_q B_{n-k,q}^*(\lambda) x^k dq x \\
= \lambda [n+1]_q \sum_{k=0}^{n} \binom{n}{k}_q B_{n-k,q}^*(\lambda) \int_0^1 x^k dq x \\
= \lambda \sum_{k=0}^{n} \binom{n+1}{k+1}_q B_{n-k,q}^*(\lambda).
\]

Thus we have the following interesting property for modified Apostol type $q$-Bernoulli numbers derived from Theorem 1 for $n \geq 0$:
\[
\sum_{k=0}^{n} \binom{n+1}{k+1}_q B_{n-k,q}^*(\lambda) = 0
\]
which can be also generated by Eq.(2.3) and Eq.(2.4).

The following is an immediate result emerging from (1.10) and (2.5) that
\[
p(x) = \sum_{k=0}^{\infty} \frac{[n+1]_q}{[k]_q!} \left\langle \frac{\lambda e_q(t) - 1}{t} x^k p(x) \right\rangle B_{k,q}^*(x,\lambda) \\
= \sum_{k=0}^{\infty} \frac{[n+1]_q}{[k]_q!} \frac{\lambda e_q(t) - 1}{t} x^k p(x) B_{k,q}^*(x,\lambda) \\
= \lambda \sum_{k=0}^{\infty} \frac{[n+1]_q}{[k]_q!} B_{k,q}^*(x,\lambda) \int_0^1 x^k p(x) dq x.
\]

By choosing suitable polynomials $p(x)$, one can derive some interesting results. So we omit to give examples, and so we now take care of a fundamental property in $q$-umbral theory which is stated below by Theorem 2.

Theorem 2. Let $n$ be nonnegative integer. Then we have
\[
\left\langle \frac{e_q(t) - 1}{t} | B_{n,q}^*(u,\lambda) \right\rangle = \int_0^1 B_{n,q}^*(u,\lambda) dq u.
\]

Proof. From (2.3), we first obtain
\[
\int_x^{x+y} B_{n,q}^*(u,\lambda) dq u = \sum_{k=0}^{n} \binom{n}{k}_q B_{n-k,q}^*(\lambda) \frac{1}{[k+1]_q} (x+y)^{k+1} - x^{k+1} \\
= \frac{1}{[n+1]_q} \sum_{k=0}^{n} \binom{n+1}{k+1}_q B_{n-k,q}^*(\lambda) \{(x+y)^{k+1} - x^{k+1}\} \\
= \frac{1}{[n+1]_q} \left( B_{n+1,q}^*(x+y,\lambda) - B_{n+1,q}^*(x,\lambda) \right).
\]
Thus, by applying (2.8), we get

\[
\frac{e_q(t) - 1}{t} B_{n,q}^*(x, \lambda) = \frac{1}{[n+1]_q} \frac{e_q(t) - 1}{t} t B_{n+1,q}^*(x, \lambda)
\]

\[
= \frac{1}{[n+1]_q} \left\{ B_{n+1,q}^*(1, \lambda) - B_{n+1,q}^*(\lambda) \right\}
\]

\[
= \int_0^1 B_{n,q}^*(u, \lambda) d_q u.
\]

Comparing Eq. (2.8) with Eq. (2.9), we complete the proof of this theorem.

The following theorem is useful to derive any polynomial as a linear combination of modified Apostol type \(q\)-Bernoulli polynomials.

**Theorem 3.** For \(q(x) \in P_n\), let

\[
q(x) = \sum_{k=0}^{n} b_{k,q} B_{k,q}^*(x, \lambda).
\]

Then

\[
b_{k,q} = \frac{1}{[k]_q !} \left\{ \lambda q^{(k)}(1) - q^{(k)}(0) \right\}.
\]

**Proof.** It follows from (1.9) that

\[
\langle (\lambda e_q(t) - 1) t^k | B_{n,q}^*(x, \lambda) \rangle = [n]_q ! \delta_{n,k} \quad (n, k \geq 0).
\]

We now consider the following sets of polynomials of degree less than or equal to \(n\):

\[
P_n = \{ q(x) \in C[x] | \deg q(x) \leq n \}.
\]

For \(q(x) \in P_n\), we further consider that

\[
q(x) = \sum_{k=0}^{n} b_{k,q} B_{k,q}^*(x, \lambda).
\]

Combining (2.10) with (2.11), it becomes

\[
\langle (\lambda e_q(t) - 1) t^k | q(x) \rangle = \sum_{l=0}^{n} b_{l,q} \langle (\lambda e_q(t) - 1) t^k | B_{l,q}^*(x, \lambda) \rangle
\]

\[
= \sum_{l=0}^{n} b_{l,q} [l]_q ! \delta_{l,k} = [k]_q ! b_{k,q}.
\]

Thus, from (2.12), we have

\[
b_{k,q} = \frac{1}{[k]_q !} \langle (\lambda e_q(t) - 1) t^k | q(x) \rangle = \frac{1}{[k]_q !} \left\{ \lambda q^{(k)}(1) - q^{(k)}(0) \right\},
\]

where \(q^{(k)}(x) = D_q^n q(x)\). Thus the proof is completed.

When we choose \(q(x) = E_{n,q}(x)\), we have the following corollary which is given by its proof.

**Corollary 1.** Let \(n \geq 2\). Then

\[
E_{n,q}(x) = (\lambda q - 1) B_{n,q}^*(x, \lambda) + [n]_q \left( \frac{\lambda + 1}{2} \right) B_{n-1,q}^*(x, \lambda)
\]

\[
- (\lambda + 1) \sum_{k=0}^{n-2} \binom{n}{k} B_{k,q}^*(x, \lambda).
\]
Proof. Recall that the $q$-Euler polynomials $E_{n,q}(x)$ are defined by
\[
\sum_{n=0}^{\infty} E_{n,q}(x) \frac{t^n}{[n]_q!} = \frac{[2]_q}{e_q(t) + 1 - e_q(x)} \quad (c.f. [12, 15])
\]
which in turn yields to
\[
E_{n,q}(x) \sim \left( \frac{e_q(t) + 1}{[2]_q}, t \right) \quad (n \geq 0)
\]
and
\[
tE_{n,q}(x) = [n]_q E_{n-1,q}(x).
\]
Set
\[
q(x) = E_{n,q}(x) \in \mathbb{P}_n.
\]
Then it becomes
\[
E_{n,q}(x) = \sum_{k=0}^{n} b_{k,q} B_{k,q}^*(x, \lambda).
\]
Let us now evaluate the coefficients $b_{k,q}$ as follows
\[
b_{k,q} = \frac{1}{[k]_q!} \langle \langle \lambda e_q(t) - 1 \rangle \ | E_{n,q}(x) \rangle \rangle
\]
\[
= \frac{[n]_q [n-1]_q \cdots [n-k+1]_q}{[k]_q!} \langle \langle \lambda e_q(t) - 1 \rangle \ | E_{n-k,q}(x) \rangle \rangle
\]
\[
= \binom{n}{k} \langle \langle \lambda e_q(t) - 1 \rangle \ | E_{n-k,q}(x) \rangle \rangle
\]
\[
= \binom{n}{k} \langle \langle \lambda E_{n-k,q}(1) - E_{n-k,q} \rangle \rangle,
\]
where $E_{n,q} := E_{n,q}(0)$ are called $q$-Euler numbers satisfying the following property
\[
E_{n,q}(1) + E_{n,q} = [2]_q \delta_{0,n}
\]
with the conditions $E_{0,q} = 1$ and $E_{1,q} = -\frac{1}{2}$. By (2.13) and (2.14), we have
\[
E_{n,q}(x) = b_{n,q} B_{n,q}^*(x, \lambda) + b_{n-1,q} B_{n-1,q}^*(x, \lambda) + \sum_{k=0}^{n-2} b_{k,q} B_{k,q}^*(x, \lambda)
\]
\[
= (\lambda q - 1) B_{n,q}^*(x, \lambda) + [n]_q \left( \frac{\lambda + 1}{2} \right) B_{n-1,q}^*(x, \lambda)
\]
\[
- (\lambda + 1) \sum_{k=0}^{n-2} \binom{n}{k} \langle \langle \lambda E_{n-k,q}(1) - E_{n-k,q} \rangle \rangle.
\]
\[
\Box
\]
Recall from (1.2) that Apostol type $q$-Bernoulli polynomials of order $r$ are given by the following generating function, for $y = 0$ (see [8]):
\[
\sum_{n=0}^{\infty} B_{n,q}^{(r)}(x, \lambda) \frac{t^n}{[n]_q!} = \left( \frac{t}{e_q(t) - 1} \right)^r e_q(x),
\]
where $t \in \mathbb{C}$ and $r \in \mathbb{N}_0$. If $t$ approaches to 0 on the above, it yields to $B_{0,q}^{(r)}(x, \lambda) = 0$, which means that the generating function of $B_{n,q}^{(r)}(x, \lambda)$ is not invertible. So, we need to modify slightly Eq. (2.1), as follows
\[
F_{q}^{(r)}(x, t) = \sum_{n=0}^{\infty} B_{n,q}^{(r)}(x, \lambda) \frac{t^n}{[n]_q!} = \left( \frac{1}{e_q(t) - 1} \right)^r e_q(x).
\]
(2.15)
The polynomials $\widetilde{B}_{n,q}^{(r)}(x, \lambda)$ may be called as modified Apostol type $q$-Bernoulli polynomials of higher order. Notice that
$$\lim_{t \to 0} \frac{g^r(t)}{g^r(t) - 1} = \widetilde{B}_{n,q}^{(r)}(x, \lambda) = \left( \frac{1}{\lambda - 1} \right)^r \neq 0 \quad (\lambda \neq 1),$$
which implies an invertible for generating function of modified Apostol type $q$-Bernoulli polynomials of higher order. In the case $x = 0$, $\widetilde{B}_{n,q}^{(r)}(0, \lambda) := \widetilde{B}_{n,q}^{(r)}(\lambda)$ may be called the modified Apostol type $q$-Bernoulli numbers.

Let
$$g^r(t, \lambda) = (\lambda e_q(t) - 1)^r.$$
It is clear that $g^r(t, \lambda)$ is an invertible series. It follows from (2.15) that $\widetilde{B}_{n,q}^{(r)}(x, \lambda)$ is $q$-Appell for $(\lambda e_q(t) - 1)^r$. So, by (1.12), we have
$$\widetilde{B}_{n,q}^{(r)}(x, \lambda) = \frac{1}{g^r(t, \lambda)} x^n,$$
and
$$\frac{i \widetilde{B}_{n,q}^{(r)}(x, \lambda)}{\lambda} = [n]_q \widetilde{B}_{n-1,q}^{(r)}(x, \lambda).$$
Thus, we have
$$\widetilde{B}_{n,q}^{(r)}(x, \lambda) \sim ((\lambda e_q(t) - 1)^r, t).$$
By (1.5) and (2.15), we get
$$\left( \frac{1}{\lambda e_q(t) - 1} \right)^r e_q(x^t) = \sum_{r=0}^{n} \frac{n!}{t^n} \widetilde{B}_{n,q}^{(r)}(\lambda) y^r. \quad (2.16)$$
Here we find that
$$\left( \frac{1}{\lambda e_q(t) - 1} \right)^r x^n = \sum_{i_1 + \cdots + i_r = n} \frac{n!}{t^n} \prod_{j=1}^{r} \widetilde{B}_{i_j,q}^{*} (\lambda) \cdots \widetilde{B}_{i_r,q}^{*} (\lambda). \quad (2.17)$$
By using (2.16), we have
$$\left( \frac{1}{\lambda e_q(t) - 1} \right)^r x^n = \widetilde{B}_{n,q}^{(r)}(\lambda). \quad (2.18)$$
Therefore, by (2.17) and (2.18), we obtain the following theorem.

**Theorem 4.** Let $n$ be nonnegative integer. Then we have
$$\widetilde{B}_{n,q}^{(r)}(\lambda) = \sum_{i_1 + \cdots + i_r = n} \frac{n!}{t^n} \prod_{j=1}^{r} \widetilde{B}_{i_j,q}^{*} (\lambda).$$
Set
$$q(x) = \widetilde{B}_{n,q}^{(r)}(x, \lambda) \in \mathbb{P}_n.$$ Then, by Theorem 3, we write
$$\widetilde{B}_{n,q}^{(r)}(x, \lambda) = \sum_{k=0}^{n} b_{k,q} B_{k,q}^{(r)}(x, \lambda), \quad (2.19)$$
where the coefficient $b_{k,q}$ is given by
$$b_{k,q} = \frac{1}{[k]_q} \left[ (\lambda e_q(t) - 1)^k | q(1) \right]. \quad (2.20)$$
From the Eq. (2.15), we have
\[
\sum_{n=0}^{\infty} \left( \lambda \widetilde{B}_{n,q}^{(r)} (1, \lambda) - \widetilde{B}_{n,q}^{(r)} (\lambda) \right) \frac{t^n}{[n]_q!} = \left( \frac{1}{\lambda e_q (t)} - 1 \right)^r (\lambda e_q (t) - 1) \\
= \left( \frac{1}{\lambda e_q (t)} - 1 \right)^{r-1} \\
= \sum_{n=0}^{\infty} \widetilde{B}_{n,q}^{(r-1)} (\lambda) \frac{t^n}{[n]_q!}.
\]

By comparing the coefficients \( \frac{t^n}{[n]_q!} \) in the above equation, we get
\[
\lambda \widetilde{B}_{n,q}^{(r)} (1, \lambda) - \widetilde{B}_{n,q}^{(r)} (\lambda) = \widetilde{B}_{n,q}^{(r-1)} (\lambda).
\] (2.21)

From the Eqs. (2.19), (2.20) and (2.21), we get the following theorem.

**Theorem 5.** Let \( n \in \mathbb{N}_0 \) and \( r \in \mathbb{N}_0 \). Then
\[
\widetilde{B}_{n,q}^{(r)} (x, \lambda) = \sum_{k=0}^{n} \binom{n}{k} q_{n-k,q} (\lambda) B_{k,q}^* (x, \lambda).
\]

Let us assume that
\[
q(x) = \sum_{k=0}^{n} b_{k,q}^* \widetilde{B}_{k,q}^{(r)} (x, \lambda) \in \mathbb{P}_n. \tag{2.22}
\]

We use a similar method in order to find the coefficient \( b_{k,q}^* \) as same as Theorem 3. So we omit the details and give the following equality:
\[
b_{k,q}^* = \frac{1}{[k]_q!} \sum_{l=0}^{r} \binom{r}{l} \lambda^l (-1)^{r-l} \sum_{m \geq 0} \sum_{i_1, i_2, \ldots, i_l = m} \binom{m}{i_1, i_2, \ldots, i_l} \frac{1}{[m]_q!} q^{(m+k)} (0).
\]

By (2.22) and coefficient \( b_{k,q}^* \), we state the following theorem.

**Theorem 6.** For \( n \in \mathbb{N}_0 \), let
\[
q(x) = \sum_{k=0}^{n} b_{k,q}^* \widetilde{B}_{k,q}^{(r)} (x, \lambda) \in \mathbb{P}_n.
\]

Then
\[
b_{k,q}^* = \frac{1}{[k]_q!} \left( (\lambda e_q (t) - 1) t^k \mid q(x) \right) \\
= \frac{1}{[k]_q!} \sum_{m \geq 0} \sum_{l=0}^{r} \binom{r}{l} \lambda^l (-1)^{r-l} \sum_{i_1, i_2, \ldots, i_l = m} \binom{m}{i_1, i_2, \ldots, i_l} \frac{1}{[m]_q!} q^{(m+k)} (0),
\]
where \( q^{(k)} (x) = D_k^q q(x) \).

Let us consider \( q(x) = B_{n,q}^* (x, \lambda) \in \mathbb{P}_n \). Then, by Theorem 6, we have
\[
B_{n,q}^* (x, \lambda) = \sum_{k=0}^{n} b_{k,q}^* \widetilde{B}_{k,q}^{(r)} (x, \lambda). \tag{2.23}
\]

From Theorem 6 and (2.23), we acquire the following theorem.
Theorem 7. For \( n, r \in \mathbb{N}_0 \), the following equality holds true:

\[
B_{n,q}^r(x, \lambda) = \sum_{k=0}^{n} \left( \sum_{m=0}^{n-k} \sum_{l=0}^{r} \sum_{i_1+\cdots+i_l=m} (-1)^{r-l} \lambda^l \binom{r}{l} \binom{m}{q} \right) B_{n-k-m,q}^{m+k}(\lambda) \times \binom{m+k}{m} B_{n-k,q}^m(x, \lambda).
\]

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