Recent Progress in Regge Calculus

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While there has been some advance in the use of Regge calculus as a tool in numerical relativity, the main progress in Regge calculus recently has been in quantum gravity. After a brief discussion of this progress, attention is focussed on two particular, related aspects. Firstly, the possible definitions of diffeomorphisms or gauge transformations in Regge calculus are examined and examples are given. Secondly, an investigation of the signature of the simplicial supermetric is described. This is the Lund–Regge metric on simplicial configuration space and defines the distance between simplicial three–geometries. Information on its signature can be used to extend the rather limited results on the signature of the supermetric in the continuum case. This information is obtained by a combination of analytic and numerical techniques. For the three–sphere and the three–torus, the numerical results agree with the analytic ones and show the existence of degeneracy and signature change. Some “vertical” directions in simplicial configuration space, corresponding to simplicial metrics related by gauge transformations, are found for the three–torus.

This article is dedicated to Tullio Regge on the occasion of his sixty–fifth birthday.

1. INTRODUCTION

1996 is a year in which we celebrate not only the sixty–fifth birthday of Tullio Regge, but also the thirty–fifth birthday of one of his brain–children, Regge calculus [1], a discretization of general relativity which has provided an amazing source of fascination and fun for those who have worked on it, as well as the occasional moment of frustration!

Some of the recent progress in Regge calculus is described elsewhere in this volume [2–4]. Rather than attempting a general review of other advances, I shall mention a few areas and then concentrate on one, that of gauge transformations and the simplicial supermetric [5,6].

Although much of the current work in Regge calculus involves attempts at setting up a quantum theory of gravity, there has also been some progress on the classical front. An algorithm for the parallel evolution of sets of disconnected vertices in a spacelike hypersurface has been formulated [7] and should prove an invaluable tool in numerical relativity, in studies of evolution of model universes and calculations of gravitational radiation, providing predictions for measurements by LIGO. Other classical work includes an investigation of the use of area variables in four–dimensional Regge calculus but this issue is not fully resolved as yet.

A very interesting and promising development in the last five years has been an understanding of the connection between the work of Ponzano and Regge [8] and the state sum or manifold invariant of Turaev and Viro [9], which uses 6j–symbols for quantum groups to provide a regularized version of the Ponzano–Regge invariant. Just as the semi–classical limit of the Ponzano–Regge invariant is related to the Feynman path integral for three–dimensional gravity with the Regge calculus action, the Turaev–Viro invariant is related to Chern–Simons gravity [10]. Thus it is possible to write down a finite theory of quantum gravity, with cosmological constant, in three dimensions [11]. Barrett and Crane [12] have shown recently that the Ponzano–Regge wave function satisfies the Wheeler–DeWitt equation. There have been many attempts at extending these ideas to four
dimensions (see for example [12] and references in [13]) but there are still a number of open questions and unresolved issues.

The other main area of activity in quantum Regge calculus is in numerical simulations in two, three and four dimensions [14–16]. In particular there has been extensive investigation of the role of the measure (see also [3]) and the inclusion of matter represented by a scalar field. There is still some controversy over the relationship between the Regge calculus simulations and the alternative method known as dynamical triangulations of random surfaces (see eg [17]).

Numerical simulations are best guided and complemented by analytical results. One of the main tools here is the weak field expansion about flat space or some other classical solution. This has been used recently to study gauge transformations in simplicial gravity [3] and this work will be described more fully in section 2.

The existence of gauge transformations or diffeomorphisms is crucially important in another area of current research, the study of the simplicial supermetric. This is the metric on the space of simplicial three geometries and a knowledge of its signature is crucial for determining spacelike hypersurfaces in superspace, important in some formulations of quantum gravity. There are so-called “vertical directions” in simplicial configuration space or superspace, which correspond to metrics which are related by diffeomorphisms and one objective of current work is to identify these directions. Sections 3, 4.1 and 4.2 consist of discussions of the known results on the signature of the supermetric in the continuum, and analytic and numerical results in the discrete case.

2. GAUGE TRANSFORMATIONS OR Diffeomorphisms?

The first point that needs to be understood here is that there are differing points of view about how to define gauge transformations or diffeomorphisms in discrete gravity. One definition, which tends to be adopted by those approaching the subject from classical relativity, is transformations of the edge–lengths which leave the geometry invariant. The other, favoured more by those wishing to use results from lattice gauge theories, is transformations of the edge–lengths which leave the action invariant. Some consider the difference between the two a matter of semantics for, after all, how would one check that the discrete geometry is left invariant other than by considering the change in the action? It is not quite as simple as that, and depends to some extent on whether one thinks of the lattice as being embedded in some continuum manifold or whether one regards the lattice itself as existing independently of any background embedding.

If one adopts the “invariance of the geometry” definition, then a sensible implementation is to require that all local curvatures (and hence deficit angles) be unchanged under the transformation of edge–lengths. This is a very strong requirement and will not be met in general. The only exception is flat space where, in general, an infinite number of choices of edge–length will correspond to the same flat geometry. In the neighbourhood of flat space there will be approximate diffeomorphisms [18] where, when the edge–lengths change by order $\epsilon$, the deficit angles will change by smaller than order $\epsilon$.

On the other hand, in the “invariance of the action” definition, it is easy to imagine changes in the edge–lengths which could decrease the deficit angles in one region and compensatingly increase them in another region, producing no overall change in the action. This invariance could even be local in the sense that changes in the lengths of edges meeting at one vertex could be made so that the action (restricted to scalar curvature, curvature–squared and volume contributions, say) could be unchanged locally, ie: when evaluated over the simplices meeting at that vertex. Before mentioning recent results demonstrating precisely this locality property, let us consider in more detail why local gauge invariance is important in lattice gravity [20]. In ordinary gauge theories, local gauge invariance plays a central role as it gives rise to the Taylor–Slavnov identities, which ensure for example that the gauge bosons in the theory remain massless to all orders of perturbation theory. Similar results hold for lattice regularized versions of these theories. A small gauge breaking in such a lat-
tice theory would invalidate its usefulness as a representation of the original quantum theory; in general small gauge breakings are amplified by loop corrections which pick up contributions from all momenta and, in particular, short distance artifacts in the lattice model. In continuum quantum gravity, identities analogous to the Taylor–Slavnov identities can be written down by exploiting the local invariance properties of the functional integral and, clearly, it is desirable that lattice analogues exist here as for other gauge theories. Since it is the lattice action which appears in the functional integral, invariance of the action is the relevant consideration here.

In the weak field expansion of Regge gravity about flat space, gauge transformations, exact zero modes, were found in four dimensions in investigations of the lattice propagator [21]. Similar results have been found in three [22] and in two [1] dimensions. The form of these zero modes is precisely the discretized form of the diffeomorphisms in the continuum theory [1,22]. The weak field expansion about flat space is quite relevant for the full quantum theory, because in the neighbourhood of the ultraviolet fixed point found in three dimensions [22], the average curvature in zero and locally the curvature is small on the scale of the lattice spacing.

Recent work on edge–length variations about non–flat background lattices [1] indicates that the local gauge invariance is still there. It holds also in the presence of a scalar field.

In the remainder of this paper the main emphasis will be on transformations of the edge–lengths leaving the geometry invariant. These will be discussed in the context of the simplicial supermetric. A much more detailed account of this work can be found in [1] (see also references therein).

3. THE CONTINUUM SUPERMETRIC

The superspace of three–geometries on a fixed three–manifold is important in various approaches to quantum gravity. For example, in Dirac quantization the states are represented by wave functions on superspace and, in quantum cosmology, such wave functions define initial and final conditions.

To define the geometry of superspace one needs the notion of distance. This is defined on the larger space of three–metrics by the DeWitt supermetric [23], the properties of which are well–understood. Now the “points” of superspace are classes of diffeomorphically equivalent three–metrics and the DeWitt supermetric induces a supermetric on this superspace. The properties of this induced supermetric are only partially understood, as we shall see.

Consider a fixed three–manifold $M$ and let $\mu(M)$ be the space of three–metrics on $M$, with typical element $h_{ab}(x)$. The distance in $\mu$ between two metrics separated by an infinitesimal displacement $\delta h_{ab}$ is defined by

$$\delta S_{\mu}^2 = \int_M d^3 x N(x) \bar{G}^{abcd}(x) \delta h_{ab}(x) \delta h_{cd}(x)$$

where $N(x)$ is the lapse function and $\bar{G}^{abcd}$ the inverse DeWitt supermetric given by

$$\bar{G}^{abcd}(x) = \begin{cases} \frac{1}{2} h^{ac}(x) h^{bd}(x) + h^{ad}(x) h^{bc}(x) - 2 h^{ab}(x) h^{cd}(x) \end{cases}$$

This has signature $(-,+,+,,+,+,+,+,+,+,+,+,+,+)$ and so the signature of the metric on $\mu$ has an infinite number of both negative and positive signs.

Denote by $\text{Riem}(M)$ the superspace of three–geometries on $M$. A supermetric on $\text{Riem}(M)$ is induced from the DeWitt supermetric on $\mu(M)$ by choosing $\delta h_{ab}$ to represent the displacement between nearby three–geometries. Since each three–geometry can be represented by different three–metrics, $\delta h_{ab}$ is not unique: for any vector $\xi^a(x)$,

$$\delta h_{ab}'(x) = \delta h_{ab}(x) + D_{(a} \xi_{b)}(x) ,$$

represents the same displacement in superspace as $\delta h_{ab}$. Thus there are so–called “vertical” directions in $\mu(M)$, pure gauge directions,

$$k^a_{\text{vertical}}(x) = D_{(a} \xi_{b)}(x)$$

and “horizontal” directions, which are orthogonal to all of the vertical ones, in the metric on $\mu(M)$. 
The non-uniqueness of $\delta h_{ab}$ means that there are different notions of distance between three-geometries in $\text{Riem}(M)$. The conventional choice is to define $\delta S^2$ to be the minimum value of $\delta S^2_\mu$ for all $\delta h_{ab}$ representing the displacement between metrics on the nearly three-geometries. This means that distance is measured in “horizontal” directions in superspace, which is equivalent to choosing a gauge specified by the conditions

$$D^b (k_{ab} - h_{ab} k_c^c) = 0.$$  

(5)

In order to define spacelike hypersurfaces in superspace it is necessary to know the signature of the supermetric. This is a non-trivial problem because superspace is infinite-dimensional. The known results [24,25] include the following:

(i) there is at least one negative direction at each point, corresponding to constant conformal displacements

$$k_{ab} = \delta \Omega^2 h_{ab}(x).$$  

(6)

This is horizontal because it satisfies the gauge choice above;

(ii) if $M$ is the three-sphere, then in the neighbourhood of the round metric, the signature has just one negative direction, the conformal one, and all other directions are positive;

(iii) every manifold admits geometries with negative Ricci curvatures; in the open region of superspace defined by such geometries the signature has infinite numbers of both positive and negative signs. This means that, for the sphere, there must be points in superspace where the metric is degenerate.

These results cover only a small part of superspace, so the procedure is now to try to obtain more complete information on the signature by looking at simplicial approximations to superspace based on Regge calculus.

4. THE SIMPLICIAL SUPERMETRIC

4.1. Analytic Results on the Signature

A simplicial three-manifold $M$ is constructed from tetrahedra joined together so that the neighbourhood of each point is homeomorphic to a region of $\text{R}^3$. A simplicial geometry is “fixed” by specifying a metric with signature $(+++)$; this is done by assigning (a) values to the $n_1$ squared edge-lengths; these must be positive and satisfy the triangle and tetrahedral inequalities, and (b) a flat metric, consistent with these values, to the interior of each tetrahedron. The region of $\text{R}^{n_1}$, with axes the squared edge-lengths $\ell_i$, $i = 1, 2, ..., n_1$, where the inequalities mentioned in (a) are satisfied, is called simplicial configuration space, $K(M)$. Distinct points in $K(M)$, corresponding to different assignments of edge-lengths in $M$ will, in general, correspond to distinct three-geometries. As discussed in section 2, this is not always true; the displacements of vertices of a flat geometry in a flat embedding space give a new assignment of edge-lengths corresponding to the same flat geometry. These are the simplicial analogues of diffeomorphisms and there are also approximate simplicial diffeomorphisms where the geometry is almost flat locally [19]. Thus the continuum limit of $K(M)$ is the space of three-metrics, not the superspace of three-geometries, which is why $K(M)$ has been called simplicial configuration space rather than simplicial superspace.

In order to define “vertical” and “horizontal” directions in $K(M)$ it is necessary to define a metric:

$$\delta S^2 = G_{mn}(t^i) \delta t^m \delta t^n,$$  

(7)

where $\delta S^2$ is the infinitesimal displacement between two points in $K(M)$. The supermetric $G_{mn}$ can be induced from the DeWitt supermetric on the space of continuum three-metrics as follows. A simplicial geometry in $K(M)$ can be represented (not uniquely) by a point in $\mu(M)$ corresponding to a piecework flat metric, and a displacement $\delta t^n$ in $K(M)$ can be represented by a perturbation $\delta h_{ab}$ in $\mu(M)$. Then we define

$$G_{mn}(t^i) \delta t^m \delta t^n = \int_M d^3x \, N(x) \text{G}^{abcd}(x) \times \delta h_{ab}(x) \delta h_{cd}(x)$$  

(8)

where $\text{G}^{abcd}$ is evaluated at the the piecewise flat metric. To make this definition meaningful we need to specify the following:
(i) the value of $N(x)$. Take $N(x) = 1$;
(ii) the gauge inside each tetrahedron. We call this choice the “Regge gauge freedom” and a natural choice is to take $\delta h_{ab}$ to be constant inside each tetrahedron:

$$D_c \delta h_{ab}(x) = 0, \quad \text{inside each } \tau \in \Sigma_3,$$

where $\Sigma_3$ is the set of tetrahedra in $K(M)$.

Of course, there may still be variations of the edge–lengths which preserve the geometry, so this type of gauge freedom remains.

Evaluation of the integral above leads to the expression

$$G_{mn}(t^t) \delta t^m \delta t^n = \sum_{\tau \in \Sigma_3} V(\tau) \{ \delta h_{ab}(\tau) \delta h^{ab}(\tau) - [\delta h^a(\tau)]^2 \},$$

where $V(\tau)$ is the volume of tetrahedron $\tau$.

The relation between $h_{ab}$ and the squared edge–lengths $t_{ab}$, linking vertices $a$ and $b$, may be obtained by picking a vertex $0$ in a tetrahedron and taking basis vectors along the edges from that vertex. Then

$$h_{ab}(\tau) = \frac{1}{2} (t_{0a} + t_{0b} - t_{ab})$$

and hence

$$\delta h_{ab}(\tau) = \frac{1}{2} (\delta t_{0a} + \delta t_{ab} - \delta t_{0b}).$$

This fixes the Regge gauge for each tetrahedron.

We now use the expression

$$V^2(\tau) = \frac{1}{(3!)^2} \det(h_{ab}(\tau)).$$

and consider perturbations $\delta t^t$ corresponding to $\delta h_{ab}$. Expansion of

$$V^2(t^t + \delta t^t) = \frac{1}{(3!)^2} \det(h_{ab} + \delta h_{ab})$$

via

$$V^2(t^t + \delta t^t) = \frac{1}{(3!)^2} \exp[T r \log(h_{ab} + \delta h_{ab})],$$

leads, at first order, to

$$\delta h^a(\tau) = \frac{1}{V^2} \frac{\partial V^2(\tau)}{\partial t^m} \delta t^m,$$

and, at second order, to

$$G_{mn}(t^t) = - \sum_{\tau \in \Sigma_3} \frac{1}{V(\tau)} \frac{\partial V^2(\tau)}{\partial t^m \partial t^n}.$$

This is the expression written down by Lund and Regge [2] for the metric on $K(M)$. It can be shown that this gives the shortest distance between simplicial three–metric among all choices of Regge gauge which vanish on the triangles. It is not exactly “horizontal” in the sense of the continuum because of the possibility of simplicial diffeomorphisms.

We are particularly interested in the signature of the Lund–Regge supermetric $G_{mn}$ and we give here several analytic results which give limited information about it.

(i) The conformal direction is always timelike. Suppose that

$$\delta t^m = \delta \Omega^2 t^m.$$  

Now $V^2$ is a homogeneous polynomial of degree three in $t^t$, so Euler’s theorem applied to the expression for $G_{mn}$ above leads to

$$G_{mn}(t^t) t^m t^n = - 6 V_{TOT}(t^t) < 0.$$  

Also, the conformal direction is orthogonal to any gauge direction $\delta t^t$ because

$$G_{mn} = t^m \delta t^n = - 4 \frac{\partial V}{\partial t^n} T O T \delta t^n = - 4 V_{TOT}$$

which is zero for a gauge transformation (by which we mean here any change in edge–lengths which does not change the geometry).

(ii) There are at least $n_1 - n_3$ spacelike directions where $n_3$ is the number of tetrahedra in $K(M)$. This follows from showing that

$$G_{mn} \delta t^m \delta t^n = - 4 \sum_{\tau} \frac{1}{V(\tau)} \frac{\partial V(\tau)}{\partial t^m} \frac{\partial V(\tau)}{\partial t^n} \delta t^m \delta t^n$$

$$+ \sum_{\tau} V(T) \delta h_{ab}(\tau) \delta h^{ab}(\tau)$$
which is non–negative if
\[
\frac{\partial V(\tau)}{\partial \tau^m} \delta t^m = 0 \quad \text{for all } \tau . \tag{22}
\]

Therefore variations of the edge–lengths leaving the volumes of all tetrahedra unchanged correspond to spacelike directions. There are \(n_3\) conditions on \(n_1\) edge–length changes so there are at least \(n_1 - n_3\) independent spacelike directions.

(iii) Diffeomorphism modes: consider a flat simplicial geometry embedded locally in \(R^3\), with vertices at \(x_A, A = 1, 2, ..., n_0\). Displacements of these vertices through \(\delta x_A\) produce changes in the edge–lengths \(\delta t^m\) which correspond to gauge directions since the flat geometry is unchanged. It can be shown (see \cite{6} for details) that

\[
\delta S^2 = 2 \frac{\partial V_{TOT}}{\partial t_{CD}} \delta x_{CD}^2 - \sum_{\tau} \frac{2}{V(\tau)} \left( \frac{\partial V(\tau)}{\partial t_{CD}} \delta x_{CD} \right)^2 \tag{23}
\]

Although the second term gives a negative contribution and the first does not appear to have a definite sign, I conjecture that gauge modes are positive (see further comments on this in the numerical section).

4.2. Numerical Results on the Signature

The limited analytic information on the signature of the simplicial supermetric described above may be checked and supplemented by numerical evaluation of the supermetric over the whole of simplicial configuration space or at least a non–trivial region of it. The general method employed here is as follows:

(i) choose a simple triangulation of a chosen manifold;

(ii) assign edge–lengths (consistent with the triangle and tetrahedral inequalities);

(iii) calculate the Lund–Regge supermetric \(G_{mn}\);

(iv) find the eigenvalues of the matrix \(G_{mn}\);

(v) count the numbers of positive, negative and zero eigenvalues to determine the signature;

(vi) update the edge–lengths and repeat the procedure.

We now describe the results for two manifolds, \(S^3\) and \(T^3\).

4.2.1. The three–sphere

We chose the simplest triangulation of \(S^3\), the surface of a four–simplex, which has 5 vertices, 10 edges, 10 triangles and 5 tetrahedra. Thus, in this case, simplicial configuration space is 10–dimensional, each point in it corresponding to an assignment of 10 squared edge–lengths.

Since the signature is scale invariant, the longest edge can be fixed at limit length. To sample the whole space by dividing the limit interval into 10, it looks at first as though there will be \(10^{10}\) points to investigate. However, the triangle and tetrahedral inequalities reduce this to 102,160 points. The results on the signature are the same everywhere; the signature is \((-,+ +,+ +,+ +,+ +,+ +,+ +)\). It is easy to see that this is consistent with the analytic results.

The results on the eigenvalues of \(G_{mn}\) are also consistent with symmetry considerations. The symmetry group of the triangulation is \(S_5\), the permutation group on five vertices. For the case of equal edge–lengths in the triangulation, the eigenvalues of \(G_{mn}\) may be classified by the irreducible representations of \(S_5\) and their degeneracies given by the dimensions of those representations. This is because a permutation of the vertices corresponds to a matrix acting on the 10–dimensional space of edges. These matrices give a 10–dimensional reducible representation of \(S_5\) which can be decomposed into irreducible representations as

\[
10 = 1 + 4 + 5 . \tag{24}
\]

This can be compared with the numerical results when all the \(t_i\) are one. The eigenvalues of \(G_{mn}\) are \(-1/\sqrt{2}\) with degeneracy 1, \(1/3\sqrt{2}\) with degeneracy 4, and \(5/6\sqrt{2}\) with degeneracy 5. In general, these degeneracies are broken by departure from the completely symmetric assignment of edge–lengths.

4.2.2. The three–torus

To obtain a triangulation of \(T^3\), we take a lattice of cubes, with \(n_x, n_y\) and \(n_z\) cubes in the \(x–, y–\) and \(z–\)directions respectively. Each cube
is divided into 6 tetrahedra by drawing in face diagonals and a body diagonal [21]. Then there are \( n_0 = n_x n_y n_z \) vertices, \( 7n_0 \) edges, \( 12n_0 \) triangles and \( 6n_0 \) tetrahedra, and simplicial configuration space is \( 7n_0 \)-dimensional. We considered two particular flat triangulations, one with right–angled tetrahedra and one with isosceles tetrahedra (see [6] for details). The computations ranged from \( 3 \times 3 \times 3 \) vertices and 189 edges to \( 6 \times 6 \times 7 \) vertices and 1764 edges. Unlike the three–sphere case, the numbers of edges, even in the smallest case, mean that it is impossible to sample the whole of simplicial configuration space, so the investigation was done in two ways, firstly by making random variations in the edge–lengths about flat space and secondly by making perturbations along the flat–space eigenvectors \( v^j \):

\[
t_i^\text{new} = t_i^\text{flat} + \epsilon v^j ,
\]

for all \( t^i \) and each \( v^j \) in turn, with \( \epsilon \) small.

The results described here are for the 189–dimensional case. For flat space, the signature has 13 negative signs and 176 positive ones. This changes away from flat space and there are even some zero eigenvalues which are presumably finite.

For the isosceles tetrahedral lattice, there are \( 189 \) vertices and \( 1764 \) edges. Unlike the three–sphere case, this group has a subgroup \( Z_2 \times S_3 \) (corresponding to “parity” transformations and permutations of the axes). The 189–dimensional reducible representation of this group decomposes as

\[
189 = 3 \times (1_1) + 2 \times (2_1) + 3 \times (2_2)
+ 2 \times (4) + 5 \times (6_1 + 6_2 + 6_3 + 6_4)
+ 2 \times (6_5 + 6_6 + 6_7 + 6_8)
\]

where \( 2_1 \) is the first irreducible representation of dimension 2, and so on. The multiplicities of the eigenvalues of \( G_{mn} \) for the right–tetrahedron lattice agree exactly with this complicated decomposition provided that the two apparent multiplicities of 8 are interpreted as 2 + 2, and the two multiplicities of 3 are interpreted as \( 2 + 1 \).

The conjecture about positive eigenvalues for the gauge modes has not been proved but has support from a number of special cases for the right–tetrahedral lattice, where it reduces to

\[
\delta S^2 = \sum_{C,D} \delta x_{CD}^2 - \frac{1}{12} \sum_\tau \left[ \sum_{C,D} x_{CD} \delta x_{CD} \right]^2
\]

for all \( t^i \) and each \( v^j \) in turn, with \( \epsilon \) small.

The results described here are for the 189–dimensional case. For flat space, the signature has 13 negative signs and 176 positive ones. This changes away from flat space and there are even some zero eigenvalues which are presumably finite analogues of the infinite number of non-gauge directions predicted by Giulini [24] for open regions with negative Ricci curvature.

To investigate the existence of gauge modes, the edges are varied in the directions of the eigenvectors as already described and the new deficit angles calculated to see whether the geometry remains flat. For the isosceles tetrahedral lattice, the results are that the eigenvalues with eigenvalues \( \lambda = \frac{1}{2} \) appear to be approximate diffeomorphisms, since the deficit angles are of order \( \epsilon^3 \) and, for an \( n_0 = n \times n \times n \) lattice, there are \( 6n - 4 \) eigenvectors corresponding to this eigenvalue.

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\[
\delta S^2 = \sum_{C,D} \delta x_{CD}^2 - \frac{1}{12} \sum_\tau \left[ \sum_{C,D} x_{CD} \delta x_{CD} \right]^2
\]

Thus \( \delta S^2 \) is positive in the following cases:

(i) if just one vertex moves through \( \delta r \),

\[
\delta S^2 = 8 \delta r^2
\]

(ii) if the edges in only one co–ordinate direction change,

\[
\delta S^2 \geq \frac{1}{2} \sum \delta x_{CD}^2
\]

(iii) if all the \( \delta x_{CD} \)'s have the same magnitude \( \delta L \),

\[
\delta S^2 > (\delta L)^2 (n_1 - \frac{3}{4} n_3) > 0
\]

since \( n_1 > n_3 \).

Let us briefly summarize the numerical results of this section. For triangulations of both manifolds there are negative modes, including the conformational direction. For the five–cell triangulation of \( S^3 \) there is a single negative mode, but preliminary results on the 600–cell triangulation indicate a total of 92 [30]. In all the cases studied there are at least \( n_1 - n_3 \) positive modes. The gauge
modes found for $T^3$ are positive but we have no proof that this is generally true. The results for $T^3$ show that the supermetric can become degenerate and the signature changes as simplicial configuration space is explored.

The main advantage of the discrete approach to the supermetric is that the continuum infinite dimensional superspace is reduced to simplicial configuration space which is finite dimensional but preserves elements of both the physical degrees of freedom and the diffeomorphisms. For larger and larger triangulations, more and more aspects of both should be recovered.

One obvious line of further study is to understand the vertical and horizontal directions in simplicial configuration space and relate them to the vertical and horizontal directions in the continuum. Another avenue is to extend the work on $S^3$ to a triangulation with an arbitrarily large number of vertices, which would then encode all the dynamic and gauge degrees of freedom, unlike the five vertex model studied here, which is too small to exhibit the expected number of (approximate) gauge modes (the number of edges 10 is less than $3n_0$ in this case). Of course, the methods described here could also be applied to triangulations of other manifolds.

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