Efficient time-splitting Hermite-Galerkin spectral method for the coupled nonlinear Schrödinger equations

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Abstract

The paper focuses on efficient time-splitting Hermite-Galerkin spectral approximation of the coupled nonlinear Schrödinger equations on the whole line. The original problem is decomposed into one nonlinear subproblem and one linear subproblem by time-splitting method. At each time step, the nonlinear subproblem is solved exactly. While the linear subproblem is efficiently solved by choosing suitable Hermite basis functions with matrix decomposition technique. Numerical experiments are carried out to demonstrate the effectiveness and efficiency of the proposed method.

Keywords

Coupled nonlinear Schrödinger equations, Hermite-Galerkin spectral method, discrete invariants, splitting method

Introduction

The coupled nonlinear Schrödinger equations are used to describe many physical problems (see, for instance, Chen and Zhang, Wang, Mu et al., and the references therein). There have been some numerical methods developed to compute their numerical solutions, for example, finite difference method, compact finite difference scheme, linearized energy-preserving compact finite difference scheme, symplectic and multi-symplectic scheme, splitting multi-symplectic method, local energy-preserving method, fourth order exponential time differencing method with local discontinuous Galerkin approximation, energy-preserving Galerkin method.

We consider the following coupled nonlinear Schrödinger equations on the whole line

\[
\begin{aligned}
&i \frac{\partial u}{\partial t} + \beta \frac{\partial^2 u}{\partial x^2} + \left( x_1 |u|^2 + (x_1 + 2x_2) |v|^2 \right) u \\
&\quad + \gamma u + \Gamma v = 0, \\
&i \frac{\partial v}{\partial t} + \beta \frac{\partial^2 v}{\partial x^2} + \left( x_1 |v|^2 + (x_1 + 2x_2) |u|^2 \right) v \\
&\quad + \gamma v + \Gamma u = 0,
\end{aligned}
\]

where \( i = \sqrt{-1} \), \( u(x, t) \) and \( v(x, t) \) are continuous complex value functions. The initial conditions and asymptotic boundary conditions of \( u \) and \( v \) in equation (1) are respectively

\[
\begin{aligned}
u(x, 0) &= u_0(x), \\
v(x, 0) &= v_0(x), \quad x \in (-\infty, \infty)
\end{aligned}
\]

and

\[
\lim_{|x| \to \infty} u(x, t) = \lim_{|x| \to \infty} v(x, t) = 0, \\ \quad t > 0
\]

The solutions of problem (1)–(3) have the following two important invariants

- Mass invariant

\[
M(t) = \int_{-\infty}^{\infty} \left( |u(x, t)|^2 + |v(x, t)|^2 \right) dx = M(0), \\ \quad t > 0
\]

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The aim of this paper is to develop efficient time-splitting Hermite-Galerkin spectral methods for problem (1)–(3). In fact, some time-splitting spectral methods have been applied to the numerical approximation of coupled nonlinear Schrödinger equations in literature. These methods enjoy both the advantages of the time-splitting methods and spectral methods. They are of high-order accuracy in time and spectral accuracy in space. However, the unbounded domain was truncated into a bounded one in these methods. The accuracy of these methods usually depends on how accurate the artificial boundary condition approximate the real far-field condition during the computational time. Furthermore, the resolution of the linear subproblem using non-Fourier type basis functions usually requires an equation solver after further time discretization.

We shall construct appropriate Hermite basis functions by matrix decomposition technique for the coupled nonlinear Schrödinger equation (1). The linear subproblem is reformulated into diagonal ordinary differential equations. Hence the total resolution process of (1) is efficient. Moreover, to achieve higher-accurate numerical solutions of some Schrödinger equations to an effective time, Hermite functions with scaling factors are used to generate the algebraic system.

The rest of the paper is organized as follows. In the next section, efficient second- and fourth-order time-splitting Hermite-Galerkin spectral methods are proposed to solve the coupled nonlinear Schrödinger equations. The construction of new modal basis functions and the efficient implementation strategy using these new basis functions are described in detail. Extensions of the splitting-methods using Hermite functions with scaling factors are presented in the subsequent section. In the penultimate section, ample numerical results are reported to demonstrate the efficiency and accuracy of the numerical methods. Some concluding remarks are given in the last section.

**Time-splitting Hermite-Galerkin spectral methods**

We will solve equation (1) by time-splitting method in time and by Hermite-Galerkin spectral method in space. First, we introduce the popular splitting methods for a nonlinear equation

\[ \varphi_t = f(\varphi) = \mathcal{L}\varphi + \mathcal{N}\varphi, \]

where $\mathcal{L}$ and $\mathcal{N}$ are linear and nonlinear operators, respectively.

For any given time step $\tau > 0$, let $t_n = n\tau, n = 0, 1, 2, \ldots$. Denote $\varphi^n$ the approximation of $\varphi(t_n)$. Then, the second-order Strang splitting technique takes the following form \(^1^6,^1^7\)

\[ \varphi^{n+1} = e^{\mathcal{N}^\tau} \cdot e^{\mathcal{L}^\tau} \cdot e^{\mathcal{N}^\tau} \varphi^n \]  \hspace{1cm} (4)

A fourth-order sympletic time integrator for the nonlinear equation is as follows \(^1^8^-^2^0\)

\[ \varphi^{(1)} = e^{2\omega_1 \mathcal{N} \tau} \varphi^n, \varphi^{(2)} = e^{2\omega_2 \mathcal{L} \tau} \varphi^{(1)}, \]
\[ \varphi^{(3)} = e^{2\omega_3 \mathcal{N} \tau} \varphi^{(2)}, \varphi^{(4)} = e^{2\omega_4 \mathcal{L} \tau} \varphi^{(3)}, \]
\[ \varphi^{(5)} = e^{2\omega_3 \mathcal{N} \tau} \varphi^{(4)}, \varphi^{(6)} = e^{2\omega_2 \mathcal{L} \tau} \varphi^{(5)}, \]
\[ \varphi^{n+1} = e^{2\omega_1 \mathcal{N} \tau} \varphi^{(6)} \]  \hspace{1cm} (5)

where

\[ \omega_1 = 0.33780179798991440851, \]
\[ \omega_2 = 0.67560359597882881702, \]
\[ \omega_3 = -0.08780179798991440851, \]
\[ \omega_4 = -0.85120719795965763405 \]

Now, we split (1) into the following two subproblems:

\[ \begin{align*}
\frac{i}{\tau} \frac{\partial u}{\partial t} + (\alpha_1 |u|^2 + (\alpha_1 + 2\alpha_2)|\nu|^2)u &= 0, \\
\frac{i}{\tau} \frac{\partial \nu}{\partial t} + (\alpha_1 |\nu|^2 + (\alpha_1 + 2\alpha_2)|u|^2)\nu &= 0
\end{align*} \]  \hspace{1cm} (6)

and

\[ \begin{align*}
\frac{i}{\tau} \frac{\partial \tilde{u}}{\partial t} + \beta \frac{\partial^2 \tilde{u}}{\partial x^2} + \gamma \tilde{u} + \Gamma \nu &= 0, \\
\frac{i}{\tau} \frac{\partial \tilde{\nu}}{\partial t} + \beta \frac{\partial^2 \tilde{\nu}}{\partial x^2} + \gamma \tilde{\nu} + \Gamma u &= 0
\end{align*} \]  \hspace{1cm} (7)

On the one hand, the nonlinear subproblem (6) can be solved exactly. In fact, multiplying $\tilde{u}$ and $\tilde{\nu}$ on the
two equations of (6) respectively and subtracting them from their conjugates, we obtain that \(|u(x, t)| = |u(x, t_0)|\) and \(|v(x, t)| = |v(x, t_0)|\) for any \(t \in [t_0, t_{n+1}]\). Therefore, the subproblem (6) reduces to

\[
\begin{align*}
\left\{ \begin{array}{l}
\frac{\partial u(x, t)}{\partial t} + (x_1 |u(x, t_0)|^2 + (x_1 + 2x_2)|v(x, t_0)|^2)u(x, t) = 0, \\
\frac{\partial v(x, t)}{\partial t} + (x_1 |v(x, t_0)|^2 + (x_1 + 2x_2)|u(x, t_0)|^2)v(x, t) = 0.
\end{array} \right.
\]

Integrating the above two equations in time yields the solutions to (6):

\[
\begin{align*}
u(x, t) &= e^{i(x_1 |u(x, t_0)|^2 + (x_1 + 2x_2)|v(x, t_0)|^2)(t-t_0)} u(x, t_0), \\
v(x, t) &= e^{i(x_1 |v(x, t_0)|^2 + (x_1 + 2x_2)|u(x, t_0)|^2)(t-t_0)} v(x, t_0).
\end{align*}
\]

On the other hand, subproblem (7) can also be efficiently solved by constructing suitable spectral basis functions. First, we introduce some basic notations which are useful in defining the weak formulation of problem (7). For any nonnegative integer \(m\), we use \(H^m(\mathbb{R})\) to denote the usual Sobolev spaces with norm \(\| \cdot \|_{m, \mathbb{R}}\). In cases where no confusion would arise, \(\mathbb{R}\) may be dropped from the notations. Let \(H_n(x)\) be the Hermite polynomials defined on the whole line. We define the Hermite functions \(\hat{H}_n(x)\) as follow

\[
\hat{H}_n(x) = \frac{1}{\pi^{1/4} \sqrt{2^n n!}} e^{-x^2/2} H_n(x), \quad n \geq 0, x \in \mathbb{R}
\]

Moreover, let \(V_N = \text{span}\{H_n(x) : n = 0, 1, 2, \ldots, N\}\). Then, it is obvious that \(V_N \in H^1(\mathbb{R})\). The semi-discrete Hermite-Galerkin spectral approximation of (7) is:

Find \((u_N, v_N) \in V_N \times V_N\), such that

\[
\begin{align*}
\frac{\partial u_N}{\partial t}, \varphi_N - \beta \left( \frac{\partial u_N}{\partial x}, \frac{\partial \varphi_N}{\partial x} \right) + \gamma (u_N, \varphi_N) + \Gamma(v_N, \varphi_N) &= 0, \quad \forall \varphi_N \in V_N, \\
\frac{\partial v_N}{\partial t}, \phi_N - \beta \left( \frac{\partial v_N}{\partial x}, \frac{\partial \phi_N}{\partial x} \right) + \gamma (v_N, \phi_N) + \Gamma(u_N, \phi_N) &= 0, \quad \forall \phi_N \in V_N.
\end{align*}
\]

Lemma 1. Let us denote

\[
m_{jk} = \int_{-\infty}^{\infty} \hat{H}_j(x) \hat{H}_k(x) \, dx, \quad a_{jk} = \int_{-\infty}^{\infty} \hat{H}_j(x) \hat{H}_k'(x) \, dx
\]

Then we have

\[
m_{jk} = \delta_{jk} \begin{cases} 1, & j = k, \\ 0, & j \neq k, \end{cases}
\]

\[
a_{jk} = \begin{cases} -\frac{\sqrt{k(k-1)}}{2}, & j = k - 2, \\ \frac{k + 1}{2}, & j = k, \\ -\frac{\sqrt{(k+1)(k+2)}}{2}, & j = k + 2. \end{cases}
\]

Expanding \(u_N(x, t)\) and \(v_N(x, t)\) in the form

\[
u_N(x, t) = \sum_{k=0}^{N} \tilde{u}_k(t) \hat{H}_k(x), \quad v_N(x, t) = \sum_{k=0}^{N} \tilde{v}_k(t) \hat{H}_k(x)
\]

and choosing the test function \(\varphi_N\) and \(\phi_N\) to be \(H_i(x)(j = 0, 1, \ldots, N)\), we arrive at the following matrix form of problem (8)

\[
\begin{align*}
iM \frac{dU}{dt} - \beta AU + \gamma MU + \Gamma MV &= 0, \\
iM \frac{dV}{dt} - \beta AV + \gamma MV + \Gamma MU &= 0
\end{align*}
\]

where

\[
U = (\tilde{u}_0(t), \tilde{u}_1(t), \ldots, \tilde{u}_N(t))^T, \quad V = (\tilde{v}_0(t), \tilde{v}_1(t), \ldots, \tilde{v}_N(t))^T, \\
M = (m_{jk})_{k,j=0,1,\ldots,N}, \quad A = (a_{jk})_{k,j=0,1,\ldots,N}
\]

In order to solve equation (9) efficiently, we construct a new basis of \(V_N\). Notice that \(A\) is a real symmetric positive matrix. The eigenvalues of \(A\) are all real. Denote them by \(\lambda_0, \lambda_1, \ldots, \lambda_N\). Let \(Q\) be the matrix formed by the orthogonal eigenvectors of \(A\) and \(\Lambda = \text{diag}\{\lambda_0, \lambda_1, \ldots, \lambda_N\}\). They satisfy

\[
AQ = QA, \quad Q^T Q = E_{N+1}
\]

where \(E_{N+1}\) denotes the identity matrix of order \(N + 1\). Denote \(Q = (q_{mn})_{0 \leq m, n \leq N}\) and let

\[
\tilde{H}_k(x) = \sum_{m=0}^{N} q_{mk} \hat{H}_m(x), \quad k = 0, 1, 2, \ldots, N
\]
It is obvious that \( \{ \tilde{H}_0(x), \tilde{H}_1(x), \ldots, \tilde{H}_N(x) \} \) form a new basis of \( V_N \). Moreover, they satisfy

\[
(\tilde{H}_k(x), \tilde{H}_l(x)) = \sum_{m,n=0}^{N} q_{mk} q_{nl} (\tilde{H}_m(x), \tilde{H}_n(x)) = \langle Q^T M Q \rangle_{lk} = \delta_{lk},
\]

\[
\left( \frac{d\tilde{H}_k(x)}{dx}, \frac{d\tilde{H}_l(x)}{dx} \right) = \sum_{m,n=0}^{N} q_{mk} q_{nl} \left( \frac{dH_k(x)}{dx}, \frac{dH_l(x)}{dx} \right) = \langle Q^T A Q \rangle_{lk} = \lambda_l \delta_{lk}
\]

Expanding \( u_N(x,t) \) and \( v_N(x,t) \) in the form

\[
u_N(x,t) = \sum_{k=0}^{N} \tilde{u}_k(t) \tilde{H}_k(x), \quad v_N(x,t) = \sum_{k=0}^{N} \tilde{v}_k(t) \tilde{H}_k(x)
\]

and choosing the test functions \( \phi_N \) and \( \phi_N \) to be \( H_j(x) (j = 0, 1, \ldots, N) \), we arrive at the following matrix form of problem (8)

\[
\begin{cases}
  i \frac{d\tilde{U}}{dt} - \beta \Lambda \tilde{U} + \gamma \tilde{V} + \Gamma \tilde{V} = 0, \\
  i \frac{d\tilde{V}}{dt} - \beta \Lambda \tilde{V} + \gamma \tilde{V} + \Gamma \tilde{U} = 0
\end{cases}
\]

where

\[
\tilde{U} = (\tilde{u}_0(t), \tilde{u}_1(t), \ldots, \tilde{u}_N(t))^T, \quad \tilde{V} = (\tilde{v}_0(t), \tilde{v}_1(t), \ldots, \tilde{v}_N(t))^T
\]

If \( \Gamma = 0 \), then \( \tilde{U} \) and \( \tilde{V} \) in (10) will be independent. Thus, equation (10) can be solved directly by

\[
\tilde{u}_k(t) = e^{i(\gamma-\beta\lambda_k)(t-t_0)} \tilde{u}_k(t_0), \quad \tilde{v}_k(t) = e^{i(\gamma-\beta\lambda_k)(t-t_0)} \tilde{v}_k(t_0), \quad k = 0, 1, 2, \ldots, N
\]

Thus, the solution of equation (8) can be expressed in the form

\[
u_N(x,t) = \sum_{k=0}^{N} e^{i(\gamma-\beta\lambda_k)(t-t_0)} \tilde{u}_k(t_0) \tilde{H}_k(x), \quad v_N(x,t) = \sum_{k=0}^{N} e^{i(\gamma-\beta\lambda_k)(t-t_0)} \tilde{v}_k(t_0) \tilde{H}_k(x)
\]

Combining splitting steps via the standard second-order Strang splitting scheme (4), we obtain a second-order time-splitting Hermite-Galerkin spectral method for the coupled nonlinear Schrödinger equation (1).

The detailed computation process includes the following three steps:

**Step 1:**

\[
\begin{align*}
  u_k^* &= \exp \left[ i \frac{\tau}{2} (x_1|u_k^n|^2 + (x_1 + 2x_2)|v_k^n|^2) \right] u_k^n, \\
  v_k^* &= \exp \left[ i \frac{\tau}{2} (x_1|v_k^n|^2 + (x_1 + 2x_2)|u_k^n|^2) \right] v_k^n
\end{align*}
\]

**Step 2:**

\[
\begin{align*}
  u_k &= \sum_{l=0}^{N} e^{i(\gamma-\beta\lambda_l)\tau} (\tilde{u}_l) \tilde{H}_l(x_k), \\
  v_k &= \sum_{l=0}^{N} e^{i(\gamma-\beta\lambda_l)\tau} (\tilde{v}_l) \tilde{H}_l(x_k)
\end{align*}
\]

with

\[
\begin{align*}
  \tilde{u}_l &= \sum_{k=0}^{N} u_k^n \tilde{H}_l(x_k) \eta_k, \\
  \tilde{v}_l &= \sum_{k=0}^{N} v_k^n \tilde{H}_l(x_k) \eta_k
\end{align*}
\]

**Step 3:**

\[
\begin{align*}
  u_k^{n+1} &= \exp \left[ i \frac{\tau}{2} (x_1|u_k^n|^2 + (x_1 + 2x_2)|v_k^n|^2) \right] u_k^n, \\
  v_k^{n+1} &= \exp \left[ i \frac{\tau}{2} (x_1|v_k^n|^2 + (x_1 + 2x_2)|u_k^n|^2) \right] v_k^n
\end{align*}
\]

where \( \{x_k\} \) and \( \{\eta_k\} \) are, respectively, the Hermite-Gauss points and weights.\(^1\) and

\[
u^n = (u^n_0, u^n_1, \ldots, u^n_N)^T, \quad v^n = (v^n_0, v^n_1, \ldots, v^n_N)^T
\]

with \( u^n_k \) and \( v^n_k \) the approximation of \( u(x_k, t_n) \) and \( v(x_k, t_n) \).

If \( \Gamma \neq 0 \), then \( \tilde{U} \) and \( \tilde{V} \) in (10) will be not independent. Hence, equation (10) cannot be solved directly as in the case of \( \Gamma = 0 \). To remedy this inefficiency, we can make a linear transformation to decouple the system of linear equations.\(^9\) Let

\[
\begin{align*}
  \tilde{R} &= \tilde{U} + \tilde{V}, \\
  \tilde{S} &= \tilde{U} - \tilde{V}
\end{align*}
\]

Then equation (10) can be rewritten as

\[
\begin{align*}
  i \frac{d\tilde{R}}{dt} - \beta \Lambda \tilde{R} + \gamma \tilde{R} + \Gamma \tilde{R} &= 0, \\
  i \frac{d\tilde{S}}{dt} - \beta \Lambda \tilde{S} + \gamma \tilde{S} - \Gamma \tilde{S} &= 0
\end{align*}
\]

It is obvious that the functions \( \tilde{R} \) and \( \tilde{S} \) are independent in equation (14). So they can be efficiently
solved by the similar process as the resolution of (10) with $\Gamma = 0$. In fact, let

$$\tilde{R} = (\tilde{r}_0(t), \tilde{r}_1(t), \ldots, \tilde{r}_N(t))^T, \quad \tilde{S} = (\tilde{s}_0(t), \tilde{s}_1(t), \ldots, \tilde{s}_N(t))^T$$

Then equation (10) can be solved directly by

$$\tilde{r}_k(t) = e^{i(\gamma - \beta_k + \Gamma)(t - \tilde{t})}\tilde{r}_k(\tilde{t}),$$
$$\tilde{s}_k(t) = e^{i(\gamma - \beta_k - \Gamma)(t - \tilde{t})}\tilde{s}_k(\tilde{t})$$

As a result, we get

$$\tilde{U} = \frac{1}{2} \tilde{R} + \frac{1}{2} \tilde{S}, \quad \tilde{V} = \frac{1}{2} \tilde{R} - \frac{1}{2} \tilde{S}$$

The second-order time-splitting Hermite-Galerkin spectral method for the coupled nonlinear Schrödinger equation (1) is similar to the computation process of $\Gamma = 0$. The only difference is that we replace (12) in step 2 by

**Step 2.1:**

$$\begin{align*}
\langle \tilde{r} \rangle_i &= (u^*)_i + (v^*)_i, \\
\langle s \rangle_i &= (u^*)_i - (v^*)_i
\end{align*}$$

**Step 2.2:**

$$\begin{align*}
\tilde{r}^{(n)}_k &= e^{i(\gamma - \beta_k + \Gamma)t} \langle \tilde{r}^{(n)} \rangle_i, \\
\tilde{s}^{(n)}_k &= e^{i(\gamma - \beta_k - \Gamma)t} \langle \tilde{s}^{(n)} \rangle_i
\end{align*}$$

**Step 2.3:**

$$u^{(n)}_k = \sum_{j=0}^{N} \frac{\tilde{r}^{(n)}_j + \tilde{s}^{(n)}_j}{2} \tilde{H}_j(x_k), \quad v^{(n)}_k = \sum_{j=0}^{N} \frac{\tilde{r}^{(n)}_j - \tilde{s}^{(n)}_j}{2} \tilde{H}_j(x_k).$$

Moreover, combining the splitting steps via the fourth-order symplectic time integrator (5), we obtain a fourth-order time-splitting Hermite-Galerkin spectral method for the coupled nonlinear Schrödinger equation (1). The detailed computation process for the resolution of equation (1) with $\Gamma = 0$ includes the following steps:

**Step 1:**

$$\begin{align*}
u^{(1)}_k &= \exp[i2\omega_1 \tau(x_1|u^{(1)}|^2 + (x_1 + 2x_2)|v^{(1)}|^2)]u^{(1)}_k, \\
v^{(1)}_k &= \exp[i2\omega_1 \tau(x_1|v^{(1)}|^2 + (x_1 + 2x_2)|u^{(1)}|^2)]v^{(1)}_k
\end{align*}$$

**Step 2:**

$$\begin{align*}
u^{(2)}_k &= \sum_{l=0}^{N} e^{i2\omega_2 \tau(x_1|u^{(1)}|^2)} \tilde{H}_l(x_k), \\
v^{(2)}_k &= \sum_{l=0}^{N} e^{i2\omega_2 \tau(x_1|v^{(1)}|^2)} \tilde{H}_l(x_k)
\end{align*}$$

Then equation (20) can be solved directly by

$$\begin{align*}
u^{(4)}_k &= \sum_{l=0}^{N} e^{i2\omega_4 \tau(x_1|u^{(3)}|^2)} \tilde{H}_l(x_k), \\
v^{(4)}_k &= \sum_{l=0}^{N} e^{i2\omega_4 \tau(x_1|v^{(3)}|^2)} \tilde{H}_l(x_k)
\end{align*}$$

Moreover, combining the splitting steps via the fourth-order symplectic time integrator (5), we obtain a fourth-order time-splitting Hermite-Galerkin spectral method for the coupled nonlinear Schrödinger equation (1). The detailed computation process for the resolution of equation (1) with $\Gamma = 0$ includes the following steps:

**Step 1:**

$$\begin{align*}
u^{(1)}_k &= \exp[i2\omega_1 \tau(x_1|u^{(1)}|^2 + (x_1 + 2x_2)|v^{(1)}|^2)]u^{(1)}_k, \\
v^{(1)}_k &= \exp[i2\omega_1 \tau(x_1|v^{(1)}|^2 + (x_1 + 2x_2)|u^{(1)}|^2)]v^{(1)}_k
\end{align*}$$

**Step 2:**

$$\begin{align*}
u^{(2)}_k &= \sum_{l=0}^{N} e^{i2\omega_2 \tau(x_1|u^{(1)}|^2)} \tilde{H}_l(x_k), \\
v^{(2)}_k &= \sum_{l=0}^{N} e^{i2\omega_2 \tau(x_1|v^{(1)}|^2)} \tilde{H}_l(x_k)
\end{align*}$$

**Step 3:**

$$\begin{align*}
u^{(3)}_k &= \exp[i2\omega_3 \tau(x_1|u^{(2)}|^2 + (x_1 + 2x_2)|v^{(2)}|^2)]u^{(2)}_k, \\
v^{(3)}_k &= \exp[i2\omega_3 \tau(x_1|v^{(2)}|^2 + (x_1 + 2x_2)|u^{(2)}|^2)]v^{(2)}_k
\end{align*}$$

**Step 4:**

$$\begin{align*}
u^{(4)}_k &= \sum_{l=0}^{N} e^{i2\omega_4 \tau(x_1|u^{(3)}|^2)} \tilde{H}_l(x_k), \\
v^{(4)}_k &= \sum_{l=0}^{N} e^{i2\omega_4 \tau(x_1|v^{(3)}|^2)} \tilde{H}_l(x_k)
\end{align*}$$

**Step 5:**

$$\begin{align*}
u^{(5)}_k &= \exp[i2\omega_5 \tau(x_1|u^{(4)}|^2 + (x_1 + 2x_2)|v^{(4)}|^2)]u^{(4)}_k, \\
v^{(5)}_k &= \exp[i2\omega_5 \tau(x_1|v^{(4)}|^2 + (x_1 + 2x_2)|u^{(4)}|^2)]v^{(4)}_k
\end{align*}$$

**Step 6:**

$$\begin{align*}
u^{(6)}_k &= \sum_{l=0}^{N} e^{i2\omega_6 \tau(x_1|u^{(5)}|^2)} \tilde{H}_l(x_k), \\
v^{(6)}_k &= \sum_{l=0}^{N} e^{i2\omega_6 \tau(x_1|v^{(5)}|^2)} \tilde{H}_l(x_k)
\end{align*}$$

**Step 7:**

$$\begin{align*}
u^{(7)}_k &= \exp[i2\omega_7 \tau(x_1|u^{(6)}|^2 + (x_1 + 2x_2)|v^{(6)}|^2)]u^{(6)}_k, \\
v^{(7)}_k &= \exp[i2\omega_7 \tau(x_1|v^{(6)}|^2 + (x_1 + 2x_2)|u^{(6)}|^2)]v^{(6)}_k
\end{align*}$$

where $\bar{\Phi}_l$ are the Hermite expansion coefficients of the function $\Phi$, defined by

$$\bar{\Phi}_l = \sum_{k=0}^{N} \Phi(x_k) \tilde{H}_l(x_k) \eta_k, \quad l = 0, 1, 2, \ldots, N$$

In the case $\Gamma \neq 0$, we replace equation (19) (similarly (21), (23)) with a process similar to (15)–(17). Then the fourth-order time-splitting Hermite-Galerkin spectral method for the coupled nonlinear Schrödinger equation (1) still works.

**Time-splitting methods using Hermite functions with scaling factors**

In some actual computation, if the solution is still far from negligible at the effective interval with large
polynomial degree, proper scaling factors are necessary. Let \( L \) be a scaling factor, we first make the linear transformation \( \bar{x} = Lx \).\(^{21,22} \) We then turn to solve the scaled Schrödinger equations as follows

\[
\begin{aligned}
& \frac{\partial u(\bar{x}, t)}{\partial t} + \beta \frac{\partial^2 u(\bar{x}, t)}{\partial \bar{x}^2} + \gamma u(\bar{x}, t) + \Gamma v(\bar{x}, t) \\
& + (z_1 |u(\bar{x}, t)|^2 + (z_1 + 2z_2) |v(\bar{x}, t)|^2)u(\bar{x}, t) = 0,

& \frac{\partial v(\bar{x}, t)}{\partial t} + \beta \frac{\partial^2 v(\bar{x}, t)}{\partial \bar{x}^2} + \gamma v(\bar{x}, t) + \Gamma u(\bar{x}, t) \\
& + (z_1 |v(\bar{x}, t)|^2 + (z_1 + 2z_2) |u(\bar{x}, t)|^2)v(\bar{x}, t) = 0,

& (\bar{x}, 0) = u_0(\bar{x}), v(\bar{x}, 0) = v_0(\bar{x}), \bar{x} \in (-\infty, \infty)
\end{aligned}
\]

(25)

In order to keep the efficient computation process of the time-splitting Hermite-Galerkin spectral method, let \( \mathcal{H}_n(\bar{x}) = \frac{1}{\sqrt{L}} \hat{\mathcal{H}}_n(\bar{x}/L) \), and expand

\[
\begin{aligned}
u_N(\bar{x}, t) &= \sum_{k=0}^{N} \hat{u}_k(t) \hat{\mathcal{H}}_n(\bar{x}), \\
v_N(\bar{x}, t) &= \sum_{k=0}^{N} \hat{v}_k(t) \hat{\mathcal{H}}_n(\bar{x})
\end{aligned}
\]

Then, we arrive at the following formulation similar to (9), i.e.,

\[
\begin{aligned}
iM \frac{dU^L}{dt} - \beta \frac{1}{\bar{L}^2} AU^L + \gamma MU^L + \Gamma MV^L = 0,

iM \frac{dV^L}{dt} - \beta \frac{1}{\bar{L}^2} AV^L + \gamma MV^L + \Gamma MU^L = 0
\end{aligned}
\]

(26)

where

\[
\begin{aligned}
U^L &= (\hat{u}_k(t), \hat{u}_k(t), \ldots, \hat{u}_k(t))^T, \\
V^L &= (\hat{v}_k(t), \hat{v}_k(t), \ldots, \hat{v}_k(t))^T
\end{aligned}
\]

Assume that

\[
\left( \frac{1}{\bar{L}^2} A \right) \tilde{Q} = \tilde{Q} \tilde{\Lambda}, \quad \tilde{Q}^T \tilde{Q} = E_{N+1},
\]

where \( \tilde{\Lambda} = \text{diag}\{\tilde{\lambda}_0, \tilde{\lambda}_1, \ldots, \tilde{\lambda}_N\} \) are the \( N + 1 \) eigenvalues of \( \frac{1}{\bar{L}^2} A \).

Denote \( \tilde{Q} = (\tilde{q}_{mn})_{0 \leq m, n \leq N} \) and let

\[
\tilde{\mathcal{H}}_k(\bar{x}) = \sum_{m=0}^{N} \tilde{q}_{mk} \tilde{\mathcal{H}}_m(\bar{x}), \quad k = 0, 1, 2, \ldots, N
\]

Now, expanding \( u_N(Lx, t) \) and \( v_N(Lx, t) \) in the following form

\[
\begin{aligned}
u_N(Lx, t) &= \sum_{k=0}^{N} \hat{u}_k(t) \tilde{\mathcal{H}}_k(\bar{x}), \\
v_N(Lx, t) &= \sum_{k=0}^{N} \hat{v}_k(t) \tilde{\mathcal{H}}_k(\bar{x})
\end{aligned}
\]

we then arrive at the following formulation

\[
\begin{aligned}
iM \frac{d\tilde{U}^L}{dt} - \beta \tilde{\Lambda} \tilde{U}^L + \gamma \tilde{U}^L + \Gamma \tilde{V}^L = 0,

iM \frac{d\tilde{V}^L}{dt} - \beta \tilde{\Lambda} \tilde{V}^L + \gamma \tilde{V}^L + \Gamma \tilde{U}^L = 0
\end{aligned}
\]

(27)

---

Figure 1. Maximum errors of second-order time-splitting Hermite-Galerkin spectral method.
Figure 2. Maximum errors of fourth-order time-splitting Hermite-Galerkin spectral method.

Figure 3. Discrete mass and discrete energy and their residuals by second-order time-splitting method. (a) discrete mass, (b) discrete energy, (c) residual of mass, (d) residual of energy.
where
\[
\tilde{U}^L = (\tilde{U}_0^L(t), \tilde{U}_1^L(t), \ldots, \tilde{U}_N^L(t))^T,
\]
\[
\tilde{V}^L = (\tilde{V}_0^L(t), \tilde{V}_1^L(t), \ldots, \tilde{V}_N^L(t))^T
\]

Then, the detailed computation process of the time-splitting methods using Hermite functions with scaling factors for the coupled nonlinear Schrödinger equation (1) can be expressed in the similar processes in last section. For simplicity, we only show the process of the second-order time-splitting Hermite-Galerkin spectral method with \( \Gamma = 0 \). The other resolution processes can be expressed similarly.

Step 1:
\[
\begin{align*}
    u_k^* &= \exp\left[ \tau \left( a_1 |u_k^*|^2 + (x_1 + 2x_2) |v_k^*|^2 \right) \right] |u_k^*|, \\
    v_k^* &= \exp\left[ \tau \left( a_1 |v_k^*|^2 + (x_1 + 2x_2) |u_k^*|^2 \right) \right] |v_k^*|
\end{align*}
\]

(28)

Step 2:
\[
\begin{align*}
    u_k^{*+1} &= \sum_{l=0}^N e^{i \tau (\beta_l - \beta_k)} H_l(\tilde{x}_k) u_l, \\
    v_k^{*+1} &= \sum_{l=0}^N e^{i \tau (\gamma_l - \gamma_k)} H_l(\tilde{x}_k) v_l
\end{align*}
\]

with
\[
\tilde{u}_i^* = \sum_{k=0}^N u_k^* H_k(\tilde{x}_k) \tilde{y}_k, \quad \tilde{v}_i^* = \sum_{k=0}^N v_k^* H_k(\tilde{x}_k) \tilde{y}_k
\]

(29)

Step 3:
\[
\begin{align*}
    u_k^{*+1} &= \exp\left[ \tau \left( a_1 |u_k^{*+1}|^2 + (x_1 + 2x_2) |v_k^{*+1}|^2 \right) \right] u_k^*, \\
    v_k^{*+1} &= \exp\left[ \tau \left( a_1 |v_k^{*+1}|^2 + (x_1 + 2x_2) |u_k^{*+1}|^2 \right) \right] v_k^*
\end{align*}
\]

(30)

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure4}
\caption{Discrete mass and discrete energy and their residuals by fourth-order time-splitting method. (a) discrete mass, (b) discrete energy, (c) residual of mass, (d) residual of energy.}
\end{figure}
where $\tilde{x}_k = L x_k$ and $\tilde{y}_k = L y_k$, and

$$u^n = (u^n_0, u^n_1, \ldots, u^n_N)^T, \quad v^n = (v^n_0, v^n_1, \ldots, v^n_N)^T$$

with $u^n_k$ and $v^n_k$ the approximation of $u(x_k, t_n)$ and $v(x_k, t_n)$.

### Numerical validation

In this section, the numerical data is measured by the following discrete norm

Maximum Error: $e_{\text{max}}^n = \max_{0 \leq k \leq N}(|u(x_k, t_n) - u^n_k|, |v(x_k, t_n) - v^n_k|)$.

$$M^p = \sum_{0 \leq k \leq N} (|u^n_k|^2 + |v^n_k|^2)\tilde{n}_k,$$

$$E^p = \frac{1}{2} \sum_{0 \leq k \leq N} \left[ - \frac{\partial u^n_k}{\partial x_k} \right]^2 + \left[ \left( \frac{\partial v^n_k}{\partial x_k} \right)^2 \right] + \frac{\gamma}{2} \left[ |u^n_k|^2 + |v^n_k|^2 \right] + 2\Gamma \cdot Re(u^n_k v^n_k) + \frac{\gamma_1}{2} \left( |u^n_k|^4 + |v^n_k|^4 \right) + (\alpha_1 + 2\alpha_2) |u^n_k|^2 |v^n_k|^2 \tilde{n}_k.$$
$N = 512$ and let $\tau$ vary. Figure 1 (right) shows the Maximum errors at $T = 1$ as a function of $\tau$ with $L = 1$ and $L = 2$. One observes that the numerical solutions have a second-order convergence rate in time.

Next, we test the convergence rate of the fourth-order time-splitting Hermite-Galerkin spectral method (FTSHGSM). To this end, fix $\tau = 0.005$ and let $N$ vary. Figure 2 (left) shows the Maximum errors at $T = 1$ as a function of $N$ with $L = 1$ and $L = 2$. One observes again that the errors behave like $e^{-cv\sqrt{N}}$. Now, fix $N = 512$ and let $\tau$ vary. Figure 2 (right) indicates that the numerical solutions have a fourth-order convergence rate in time.

We also find from Figures 1 and 2 that the numerical results with $L = 2$ are better than those with $L = 1$.

Finally, let $N = 512$, $L = 2$, and $\tau = 0.1$. Figures 3 and 4 display, respectively, the discrete mass and discrete energy and their residuals by second-order time-splitting method and fourth-order time-splitting method. It is obvious that both schemes conserve the discrete mass. And fourth-order time-splitting method conserves discrete energy better than second-order time-splitting method.

**Example 2.** The initial conditions in this example are chosen as

\begin{align*}
u(x,0) &= \sqrt{2}r_1 \text{sech}(r_1 x + 10) e^{iV_1 x}, \\
v(x,0) &= \sqrt{2}r_2 \text{sech}(r_2 x - 10) e^{iV_2 x}
\end{align*}

where $r_i$ and $V_i$ ($i = 1, 2$) are amplitudes and velocities respectively.
Figure 8. Evolution of the modulus of numerical solution with different $\Gamma$. 
We fix $\beta = 1, x_1 = 1, \gamma = \Gamma = 0$, and then solve (1) by FTSHGSM. The discretization parameters are $N = 512, L = 2, \tau = 0.1$. The other parameters are select as following

- **case 1**: $r_1 = 0.6, r_2 = 0.5, V_1 = 1/8, V_2 = -1/4, x_2 = 0$.
- **case 2**: $r_1 = 1.2, r_2 = 1, V_1 = 0.1, V_2 = -0.1, x_2 = -1/6$.
- **case 3**: $r_1 = 1.2, r_2 = 1, V_1 = 0.1, V_2 = -0.1, x_2 = -0.35$.
- **case 4**: $r_1 = 1.2, r_2 = 1, V_1 = 0.4, V_2 = -0.4, x_2 = 1$.

Figure 5 shows the corresponding computational results. These results are consistent with those in Chen and Zhang. However, we use 513 points in the whole line, while 801 points are used in a bounded domain in Chen and Zhang.

**Example 3.** In this example, the initial conditions are chosen as

$$u(x, 0) = \sqrt{2} r_1 \text{sech} \left( r_1 x + \frac{1}{2} D_0 \right) e^{iV_0 x / 4},$$

$$v(x, 0) = \sqrt{2} r_2 \text{sech} \left( r_2 x - \frac{1}{2} D_0 \right) e^{-iV_0 x / 4}$$

where $V_0, D_0, r_1,$ and $r_2$ are known constants. We first test the convergence rate of time. To this end, let $V_0 = 1, D_0 = 25, r_1 = r_2 = 1,$ and $\beta = 1, x_1 = 1, x_2 = -1/6, \gamma = 1, \Gamma = 0.05$. Since the exact solution of the problems is unknown, we take the numerical solution computed with $\tau = 0.0001$ as the “exact solution” ($u^*, v^*$). The other computational parameters are $N = 512$ and $L = 2$. Figure 6 shows the maximum errors at $T = 1$ as a function of $\tau$. One observes that the numerical solution has second-order and fourth-order accuracy in time for the two splitting methods respectively. Moreover, for the same time step, the errors of fourth-order time-splitting method are smaller than those of second-order time-splitting method.

Next, let $x_1 = 1, x_2 = 0, \beta = 1, \gamma = 0$. The discretization parameters in the fourth-order time-splitting method are taken as $N = 512, L = 2, \tau = 0.01$. The elastic collisions of $|u|$ and $|v|$ with different $\Gamma$ are displayed in Figure 7. The time evolution of the two solitons $|u|$ and $|v|$ during a long time with different $\Gamma$ is also displayed in Figure 8. From the two figures, one can see that the collisions with different $\Gamma$ are elastic, the reason is that the parameter $x_2$ is chosen to be 0. Moreover, the amplitude of the densities has some jumps at the collision point, this implies that energy exchange occurs between $u$ and $v$. And, the larger the linear coupling parameter $\Gamma$ is, the stronger the jumps are. These results are in full consistent with those in Wang.

**Concluding remarks**

We presented efficient time-splitting Hermite-Galerkin spectral methods for the coupled nonlinear Schrödinger equation on the whole line. The time discretization errors are the splitting errors. By introducing a family of new basis functions by matrix decomposition technique, the linear subproblem reduced to diagonal ordinary equations, which can be solved efficiently. Several numerical results were presented to confirm the accuracy and efficiency of the method. It is expected that the proposed scheme can be applied to high dimensional coupled nonlinear Schrödinger equations.

**Research data**

The data used to support the findings of this study are included within the article.

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