Periodic solutions from Lie symmetries for the
generalized Chen–Lee–Liu equation

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Abstract The nonlinear generalized Chen–Lee–Liu 1+1 evolution equation which describes the propagation of an optical pulse inside a monomode fiber is studied by using the method of Lie symmetries and the singularity analysis. Specifically, we determine the Lie point symmetries of the Chen–Lee–Liu equation and we reduce the equation by using the Lie invariants in order to determine similarity solutions. The solutions that we found have periodic behavior and describe optical solitons. Furthermore, the singularity analysis is applied in order to write algebraic solutions of the Chen–Lee–Liu with the use of Laurent expansions. The latter analysis supports the result for the existence of periodic behavior of the solutions.

1 Introduction

Lie symmetry analysis is a powerful method for the study of nonlinear differential equations. The main feature of Sophus Lie approach is that the existence of a symmetry vector for a given differential equation indicates the existence of an invariant surface which can be applied for the construction of a similarity transformation in order to simplify the differential equation under the so-called reduction process [1–4]. In addition, Lie symmetries can be used to determine algebraic equivalent systems [5] and give linearization criteria for nonlinear differential equations [6–8]. Furthermore, Lie symmetries can be applied to construct conservation laws [9–11], to determine new solutions from old solutions [12] and many other [4]. There are various important results in the literature on the application of Lie’s theory in mathematical physics and applied mathematics [13–26].

On the other hand, singularity analysis is an alternative way to study the integrability of nonlinear differential equations. The main requirement for the singularity analysis is the existence of movable singularities in the differential equation. Singularity analysis is associated with the French school led by Painlevé [27–29], and their approach was actually inspired by the successful application to the determination of the third integrable case of Euler’s equations for a spinning top by Kowalevskaya [30]. For modern approaches of the singularity analysis, we refer the reader in [31–34] and references therein, while some applications on partial differential equations can be found in [35–37].
In this work, we are interested in the determination of exact and analytic solutions for the generalized Chen–Lee–Liu equation (gCLL) is [38]

\[ i q_t + \frac{1}{2} q_{xx} - |q|^2 q + i \delta |q|^2 q_x = 0, \]  

with the application of the Lie’s theory and of the singularity analysis. Equation (1) under the Madelung transformation \( q(t, x) = \rho(t, x) \exp(i \int u(t, x') \, dx') \) the latter equation can be written as the following system of evolution equations

\[ \rho_t + \left( \rho u + \frac{1}{2} \delta \rho^2 \right)_x = 0, \]  
\[ u_t + uu_x + \rho_x + \delta (\rho u)_x + \left( \frac{\rho_x^2}{8 \rho^2} - \frac{\rho_{xx}}{4 \rho} \right)_x = 0, \]

in which \( \rho(t, x) \) is the intensity variable and chirp variables, while the positive parameter \( \delta \) is associated with the self-steepening phenomena [39]. The nonlinear evolution equation has been proposed for the description of an optical pulse inside a monomode fiber. For other applications of Eq. (1), we refer the reader in [40]. The solitons provided by the gCLL equation have been found to be essential for the description of phenomena in optical fiber theory [38,41–43], while some experimental evidences are presented in [44].

In the following sections, we determine the Lie point symmetries for the real function system (2), (3). For the admitted Lie point symmetries, we constructed the algebraic structure and we find the one-dimensional optimal system. Moreover, we find all possible similarity transformations which are used to reduce the differential equation and write the equivalent system. We find four independent similarity solutions. In particular, we determine static solutions, stationary solutions, travel-wave solutions and scaling solutions. We observe that the solution is periodic in the similarity variables which indicate the existence of kink solutions in the original variables. Furthermore, we apply the singularity analysis such that to write the analytic solution of the resulting system with the use of Laurent expansions. While the original system (2), (3) is investigated if possess the Painlevé property. At this point, it is important to mention some previous studies of Lie’s theory on optical physics [45–47], while only recently the algebraic properties of the Chen–Lee–Liu were studied in [48]. The plan of the paper is as follows.

In Sect. 2, we determine the Lie symmetries and the similarity transformations for the gCLL equations, while we construct the similarity solutions. In Sect. 3, we show that the gCLL equation possess the Painlevé property and we write the generic solution by using a Right Painlevé Series. Finally, in Sect. 4 we discuss our results and we draw our conclusions. In “Appendices A, B and C,” we present the main mathematical theory of the tools that are applied in this work.

2 Lie symmetries and similarity transformations

For the system of the 1+1 evolution partial differential equations (2), (3) we apply the Lie theory in order to determine the generator [1,2,4]

\[ X = \xi_t (t, x, \rho, u) \, \partial_t + \xi_x (t, x, \rho, u) \, \partial_x + \eta_\rho (t, x, \rho, u) \, \partial_\rho + \eta_u (t, x, \rho, u) \, \partial_u, \]  

of the infinitesimal one parameter point transformation

\[ t' = t + \varepsilon \xi_t (t, x, \rho, u), \quad x' = x + \varepsilon \xi_x (t, x, \rho, u), \]
Table 1 Commutators of the admitted Lie symmetries

| $[X_I, X_J]$ | $X_1$ | $X_2$ | $X_3$ |
|--------------|-------|-------|-------|
| $X_1$        | 0     | 0     | $2X_1 - X_2$ |
| $X_2$        | 0     | 0     | $X_2$ |
| $X_3$        | $-2X_1 + X_2$ | $-2X_2$ | 0 |

Table 2 Adjoint representation of the admitted Lie algebra

| $\text{Ad} \left( e^{(eX_i)} \right) X_j$ | $X_1$ | $X_2$ | $X_3$ |
|------------------------------------------|-------|-------|-------|
| $X_1$                                    | $X_1$ | $X_2$ | $2\varepsilon \delta X_1 - \varepsilon X_2 + X_3$ |
| $X_2$                                    | $X_1$ | $X_2$ | $\varepsilon \delta X_2 + X_3$ |
| $X_3$                                    | $e^{-2\delta \varepsilon} X_1 - \frac{e^{-\delta \varepsilon}}{\delta} \left( e^{-\delta \varepsilon - 1} \right) X_2$ | $e^{-\delta \varepsilon} X_2$ | $X_3$ |

\[ \rho' = \rho + \varepsilon \eta^\rho \left( t, x, \rho, u \right), \quad u' = u + \varepsilon \eta^u \left( t, x, \rho, u \right), \]  

which keeps invariant the system (2), (3).

The possible generators are derived to be the following three
\[ X_1 = \partial_t, \quad X_2 = \partial_x \quad \text{and} \quad X_3 = 2t \partial_t + \left( \delta x - t \right) \partial_x - \rho \partial_{\rho} - \left( 1 + u \delta \right) \partial_u. \]  

The latter Lie point symmetries form the $A_{3,2}$ Lie algebra in the Morozov–Mubarakzyanov Classification Scheme [49–52]. The commutators and the adjoint representation of the admitted Lie point symmetries are presented in Tables 1 and 2.

In order to proceed with the derivation of all the possible independent similarity transformations which reduce the system (2), (3) we should determine the one-dimensional optimal system. Straightforward, from Table 2 we find the one-dimensional optimal system [3]
\[ \{X_1\}, \quad \{X_2\}, \quad \{X_1 + \alpha X_2\} \quad \text{and} \quad \{X_3\}. \]  

Thus, we continue our analysis by applying the Lie point symmetries in order to reduce the system of partial differential equations (2), (3) into a system of ordinary differential equations.

2.1 Similarity transformations

We proceed with the application of the similarity transformations.

2.1.1 Lie symmetry $X_1$

The application of the Lie point symmetry $X_1$ leads to the static solution $u = u \left( t \right), \quad \rho = \rho \left( t \right)$ where the reduced equations are determined to be
\[ \left( \rho u + \frac{1}{2} \delta \rho^2 \right)_x = 0, \]  
\[ uu_x + \rho_x + \delta \left( \rho u \right)_x + \left( \frac{\rho^2_x}{8 \rho^2} - \frac{\rho_{xx}}{4 \rho} \right)_x = 0, \]
or equivalently

\[ \rho u + \frac{1}{2} \delta \rho^2 = c_1, \quad (11) \]

\[ \frac{1}{2} u^2 + \rho + \delta (\rho u) + \left( \frac{\rho_x^2}{8 \rho^2} - \frac{\rho_{xx}}{4 \rho} \right) = c_2, \quad (12) \]

where now \( c_1, c_2 \) are two integration constants.

Hence, from (10) it follows

\[ u = -\frac{1}{2} \delta \rho^2 + \frac{c_1}{\rho}, \quad (13) \]

that is, Eq. (12) reads

\[ \varrho_{xx} + \frac{3}{4} \delta^2 \varrho^5 - 2 \varrho^3 + \frac{c_2}{\varrho^2} - \frac{c_1}{\varrho^3} = 0, \quad (14) \]

where \( \varrho = \varrho^2 (x) \) and \( C_2 = \left( \frac{c_2}{\varrho^2} - \delta c_1 \right) \).

Equation (14) admits one Lie point symmetry, the vector field \( X_2 \), which is a reduced symmetry vector. Moreover, Eq. (14) can be integrated by quadratures, that is

\[ \frac{1}{2} (\varrho_x)^2 - \frac{\delta}{8} \varrho^6 - \frac{1}{4} \varrho^4 + \frac{1}{2} \left( \frac{c_2}{2} - \delta c_1 \right) \varrho^2 + \frac{c_1}{2 \varrho^2} = c_3, \quad (15) \]

where \( c_3 \) is a third integration constant.

In Fig. 1, we present the phase portrait of Eq. (14) from where it is clear that there are attractors in the dynamical system which provides a periodic behavior.

In order to write the analytic solution of Eq. (14) and understand the periodic behavior of the solution, we apply the singularity analysis. Specifically we apply the ARS algorithm. We search for the singular behavior \( \varrho (x) = \varrho_0 x^p \) from where we follow that the leading-order behavior has \( p = -\frac{1}{2} \) and \( (\varrho_0)^2 = \pm \frac{i}{\delta} \), it is clear from here that a periodic behavior will follow.

The second step of the ARS algorithms is the determination of the resonances, we do that by replacing \( \varrho (x) = \varrho_0 x^{-\frac{1}{2}} + m x^{-\frac{1}{2} + S} \) in (14) and linearize around \( m = 0 \), it follows the algebraic equation \((S + 1)(S - 3) = 0\), which gives the resonances \( S = -1 \) and \( S = 3 \).

Finally we write the Laurent expansion

\[ \varrho (x) = \varrho_0 x^{-\frac{1}{2}} + \varrho_1 x^{\frac{1}{2}} + \varrho_2 x^{\frac{3}{2}} + \cdots, \quad (16) \]

from where we test that it is a solution of Eq. (12) with integration constant the \( \varrho_3 \) and

\[ (\varrho_0)^2 = \frac{i}{\delta}, \quad \varrho_1 = -\frac{(1)^{\frac{3}{2}}}{2 \delta^{\frac{3}{2}}}, \quad \varrho_2 = -\frac{(1)^{1/4} (9 - 8 C_2 \delta)}{24 \delta^{\frac{5}{2}}}, \ldots. \]

We remark that the second integration constant is the position of the singularity \( x_0 \).

2.1.2 Lie symmetry \( X_2 \)

Reduction with respect to the vector field \( X_2 \) provides the stationary solutions \( u = u (t), \rho = \rho (t) \) where \( u_t = 0 \) and \( \rho_t = 0 \), that is \( u (t, x) = u_0, \rho (t, x) = \rho_0 \).
2.1.3 Lie symmetry $X_1 + \alpha X_2$

The application of the Lie symmetry vector $X_1 + \alpha X_2$ provides a travel-wave solution. Indeed, we find $u = u(z), \ \rho = \varrho^2(z)$ with $z = x - \alpha t$, where now the reduced system is

$$\varrho_{zz} + \frac{1}{2} \delta \varrho^4 = c_1,$$

$$\varrho_{zz} + 2(\alpha - \delta \varrho^2) \rho u - u^2 \varrho - \varrho^3 + c_2 \varrho = 0,$$

where $c_1, c_2$ are two integration constants.

With the use of (17), Eq. (18) reads as

$$\varrho_{zz} + \frac{3}{4} \delta^2 \varrho^5 - 2(1 + \alpha \delta) \varrho^3 + \tilde{C}_2 \varrho - \frac{c_1^2}{\varrho^3} = 0,$$

with $\tilde{C}_2 = (c_2 - \alpha - \delta c_1)$.

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Fig. 1 Phase portrait for Eq. (14) for various values of the free parameters $[\delta, c_1, C_2]$
Equation (19) is of the form of the static solution (14) and can be integrated by quadratures, the only main difference is the coefficient of the $\varrho^3$, where now it is $(1 + \alpha \delta)$. The latter quantity can be positive or negative, either if the travel wave travel of the left to the right or from the right to the left.

In Fig. 2, we present the phase portrait for Eq. (19) and $(1 + \alpha \delta) < 0$, specifically for $(1 + \alpha \delta) = -1$, and for the rest of the constants to have the same values as in Fig. 1. Again we observe a similar behavior and the existence of periodic solutions which correspond to travel-wave solutions.

As in the case of the static solution, the application of the singularity analysis provides the analytic solution expressed by the Laurent expansion

$$\varrho (x) = \varrho_0 x^{-\frac{1}{2}} + \varrho_1 x^{\frac{1}{2}} + \varrho_2 x^{\frac{3}{2}} + \cdots,$$

(20)
Fig. 3 Numerical simulation of Eq. (25) for various values of the free parameters \( \{c_1, C_2, (1 + \alpha \delta)\} \) and initial conditions \( \varphi(0) = 1 \) and \( \varphi_z(0) = 0 \). The plots of the first row are for (left to right) \( \{1, 1, 0.5\} \), \( \{1, 1, 1\} \), \( \{1, 1, 3\} \), the plots of the second row are for \( \{2, 0, 0.5\} \), \( \{2, 0, 1\} \), \( \{2, 0, 3\} \) and the plots of the third row are for \( \{0.1, 0.1, 0.5\} \), \( \{0.1, 0.1, 1\} \) and \( \{0.1, 0.1, 3\} \)

where now

\[
(\varphi_0)^2 = \pm \frac{i}{\delta}, \quad \varphi_1 = -\frac{(-1)^{3/2}}{2\delta^{9/2}} \left(1 + \alpha \delta\right), \quad \varphi_2 = -\frac{(-1)^{1/4} \left(9 + 9\alpha \delta \right)}{24\delta^{5/2}}, \quad \ldots
\]

and \( \varphi_3 \) is an arbitrary constant.

In Fig. 3, we present the numerical solution of (19) where we observe the existence of travel-waves, i.e., a kink solutions, for the gCLL equation.

2.1.4 Lie symmetry \( X_3 \)

We proceed with the application of the similarity transformation provided by the scaling symmetry \( X_3 \). The similarity transformation is

\[
\cdots \]
\[ u(t, x) = -\frac{1}{\delta} + \frac{1}{\sqrt{t}} U(\sigma), \quad \rho(t, x) = \frac{1}{\sqrt{t}} R^2(\sigma) \text{ with } \sigma = \left(\frac{x}{\sqrt{t}} + \frac{\sqrt{t}}{\delta}\right). \] (22)

Therefore, the reduced system after the application of the latter similarity transformation is derived to be

\[ 2RU - \sigma R + \delta R^2 = c_1, \] (23)

\[ R_{\sigma\sigma} - U^2 R + (\sigma - 2\delta R^2) UR + c_2 R = 0, \] (24)

where \( c_1, c_2 \) are two integration constants.

Therefore, we end with only one differential equation

\[ R_{\sigma\sigma} + \frac{3}{4} \delta^2 R^2 - 2(1 + \alpha \delta) R^3 + \left(C_2 + \frac{\sigma^2}{4}\right)R - \frac{c_1^2}{R^3} = 0 \] (25)

where now \( C_2 = \left(\frac{\alpha}{2} - \delta c_1\right) \). The main difference with the previous reduction is that the coefficient of the linear term is not a constant.

Easily, we observe that the resulting solution is again periodic solution but in this case around a central which moves. In Fig. 4, we present numerical simulation for Eq. (25) for the initial condition \( R(0) = 1, R(0) = 0 \) and for various values of the free parameters \( \{c_1, C_2, (1 + \alpha \delta)\} \), from the figures it is clear the periodic behavior of the scaling solution, that is nothing else than a kink solution.

For small values of \( R \), the dominant terms are the

\[ R_{\sigma\sigma} + \left(C_2 + \frac{\sigma^2}{4}\right)R - \frac{c_1^2}{R^3} = 0 \] (26)

which is nothing else than the Ermakov–Pinney equation [53–55].

### 3 Singularity analysis

Let us now apply the ARS algorithm to study if the system (2), (3) possess the Painlevé property.

In order to determine the singular behavior we replace in (2), (3) the following expression

\[ \rho(t, x) = \rho_0(t, x) \phi(t, x)^p, \quad u(t, x) = u_0(t, x) \phi(t, x)^q \]

where \( \phi(t, x) \) is a singular function. Hence, it follows that the leading-order terms provide \( \{p, q\} = (-1, -1) \), where

\[ \rho_0(t, x) = -\frac{1}{\delta} \phi_x^2, \quad u_0(t, x) = -\frac{1}{4} \phi_x^2. \]

For the second step of the ARS algorithm, we determine the resonances which are \( \{s_1, s_2, s_3, s_4\} = \{-1, 2, 2, 3\} \) from where we conclude that the solution is given by Right Painlevé Series. Indeed the solutions is expressed as

\[ \rho(t, x) = \rho_0(t, x) \phi(t, x)^{-1} + \rho_1(t, x) + \rho_2(t, x) \phi(t, x) + \rho_3(t, x) \phi(t, x)^3 + \cdots \] (27)

\[ u(t, x) = u_0(t, x) \phi(t, x)^{-1} + u_1(t, x) + u_2(t, x) \phi(t, x) + u_3(t, x) \phi(t, x)^3 + \cdots \] (28)

where the consistency test gives that functions \( \rho_2(t, x), \rho_3(t, x) \) and \( u_2(t, x) \) are arbitrary.

We conclude that the generalized Chen–Lee–Liu equation possess the Painlevé property and its solution is expressed by Right Laurent expansions. The latter expressions can describe Kink solutions for specific initial conditions.
Fig. 4 Numerical simulation of Eq. (25) for various values of the free parameters \(\{c_1, C_2, (1 + \alpha \delta)\}\) and initial conditions \(R(0) = 1\) and \(R_\sigma(0) = 0\). The plots of the first row are for (left to right) \(\{1, 1, 0.5\}\), \(\{1, 1, 1\}\), \(\{1, 1, 3\}\), the plots of the second row are for \(\{2, 0, 0.5\}\), \(\{2, 0, 1\}\), \(\{2, 0, 3\}\) and the plots of the third row are for \(\{0.1, 0.1, 0.5\}\), \(\{0.1, 0.1, 1\}\) and \(\{0.1, 0.1, 3\}\).

### 4 Conclusion

In this work, we determine exact and analytic periodic solutions for the gCLL equation with the use of Lie symmetries. The gCLL equation admits a three-dimensional Lie algebra, which leads to four different similarity transformations. In particular, we found static, stationary, travel-wave and scaling similarity solutions.

Except from the stationary solution, which is not of interest, the other solutions are periodic solutions. It is interesting that in all cases for small values of the intensity variable \(\rho(t, x)\) the gCLL is reduced to the Ermakov–Pinney equation, which is a well-known integrable system.

Moreover, we investigated if the gCLL possess the Painlevé property, for that we applied the ARS algorithm where we were able to write the analytic solution of the gCLL with the use of Right Painlevé Series.
The periodic solutions, provided by the similarity transformations, are directly related with the existence of dark and bright solitons for the nonlinear differential equation [56–58]. Optical solitons are exact solutions of mathematical models with direct applications in the information transfer in optical fibers [59]. As far the results or our analysis, are concerned, the travel-wave solution which was found before for the gCLL describes a 1-soliton solution known as kink solution. Furthermore, the scaling solution is also a kink solution, where now the amplitude of the oscillation is not a constant.

The determination of these new kink solutions is essential for the physical viability of the model. A study of the properties of the kink solutions and their real-world applications extends the scope of this study and will be performed in a future study.

A Lie symmetries

We briefly discuss the main definition and algorithm for the determination of Lie point symmetries. Consider the one-dimensional parameter point transformation

\[ \tilde{x}^k = x^k + \varepsilon \xi^i \left(x^k, u\right) \], \hspace{1cm} \text{(29)}

\[ \tilde{\eta} = \eta + \varepsilon \eta \left(x^k, u\right) \], \hspace{1cm} \text{(30)}

with generator \( X = \xi^i \left(x^k, u\right) \partial_i + \eta \left(x^k, u\right) \partial_u \), then the differential equation \( H \left(x^k, u, u_i, u_{ij}, \ldots, u_{ij\ldots i_n}\right) \) is invariant under the action of the one parameter point transformation if and only if

\[ \lim_{\varepsilon \to 0} \frac{\tilde{H} \left(\tilde{x}^k, \tilde{u}, \tilde{u}_i, \tilde{u}_{ij}, \ldots, \tilde{u}_{ij\ldots i_n}\right) - H \left(x^k, u, u_i, u_{ij}, \ldots, u_{ij\ldots i_n}\right)}{\varepsilon} = 0, \]

or equivalently, if there exists a function \( \lambda \) such that the following condition to be true [1,2,4]

\[ X^{[n]} H - \lambda H = 0 \] \hspace{1cm} \text{(31)}

where \( X^{[n]} \) is called the n-th prolongation/extension of \( X \) in the jet-space defined as

\[ X^{[n]} = X + \left( D_i \eta - u_k D_i \xi^k \right) \partial_{u_i} + \left( D_i \eta^{[i]} - u_{jk} D_i \xi^k \right) \partial_{u_{ij}} + \cdots \]

\[ + \left( D_i \eta^{[i]} \cdots \partial_{u_{ij\ldots i_{n-1}}} - u_{ij\ldots i_k} D_i \xi^k \right) \partial_{u_{ij\ldots i_n}}. \] \hspace{1cm} \text{(32)}

If \( X \) is a symmetry vector for the differential equation \( H \), then we can always find a coordinate transformation such that the symmetry vector to be written in the canonical coordinates, i.e., \( X = \partial_{x^n} \), where the differential equation is

\[ H = H \left(x^\mu, u, u_i, u_{ij}, \ldots, u_{ij\ldots i_n}\right), \mu \neq n, \]

clear from the last expression it follows \( \partial_{x^n} H = 0 \). The coordinate transformation which leads to the canonical coordinates is called similarity transformation and it is mainly applied for the reduction of the differential equation.

B One-dimensional optimal system

Let a given differential equation admit as Lie symmetries the elements \( \{X_1, X_2, \ldots, X_n\} \) of the n-dimensional Lie algebra \( G_n \) with structure constants \( C_{jk}^i \). The two symmetry vectors \( Z, W \) defined as
we shall say that are equivalent if and only if \([3]\)

\[
W = \sum_{j=i}^{n} \text{Ad} \left( \exp(\epsilon_i X_i) \right) Z
\]

or

\[
W = cZ, \; c = \text{const} \quad \text{that is} \quad b_i = ca_i. \quad (35)
\]

Operator \(\text{Ad}(\exp(\epsilon X_i)) X_j\) defined as

\[
\text{Ad}(\exp(\epsilon X_i)) X_j = X_j - \epsilon [X_i, X_j] + \frac{1}{2} \epsilon^2 [X_i, [X_i, X_j]] + \cdots \quad (36)
\]

is called the adjoint representation.

The determination of all the one-dimensional subalgebras of \(G_n\) which are not related through the adjoint representation is necessary in order to perform a complete classification of all the possible similarity transformations, i.e., similarity solutions, for a given differential equation. This classification is known as the one-dimensional optimal system.

### C Singularity analysis

The development of the Painlevé test for the determination of integrability of a given equation or system of equations and its systematic use has been succinctly summarized by Ablowitz, Ramani and Segur in the so-called ARS algorithm [60–62]. The ARS algorithm is constructed by three basic steps, they are: (a) determine the leading-order term which describes the behavior of the solution near the singularity, (b) find the position of the resonances which shows the existence and the position of the integration constants and (c) write a Laurent expansion with leading-order term determined in the first step in order to perform the consistency test and the solution, for a review on the ARS algorithm and various applications we refer the reader in [63], while in [64] a discussion between the Lie’s approach and the singularity analysis is given.

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