We develop the kinematics in Matrix Gravity, which is a modified theory of gravity obtained by a non-commutative deformation of General Relativity. In this model the usual interpretation of gravity as Riemannian geometry is replaced by a new kind of geometry, which is equivalent to a collection of Finsler geometries with several Finsler metrics depending both on the position and on the velocity. As a result the Riemannian geodesic flow is replaced by a collection of Finsler flows. This naturally leads to a model in which a particle is described by several mass parameters. If these mass parameters are different then the equivalence principle is violated. In the non-relativistic limit this also leads to corrections to the Newton’s gravitational potential. We find the first and second order corrections to the usual Riemannian geodesic flow and evaluate the anomalous nongeodesic acceleration in a particular case of static spherically symmetric background.
1 Introduction

Gravity is one of the most universal physical phenomenon. It is this universality that leads to a successful geometric interpretation of gravity in terms of Riemannian geometry in General Relativity. General Relativity is widely accepted as a pretty good approximation to the physical reality at large range of scales.

We would like to make two points here. First of all, the experimental evidence points to the fact that all matter exhibits quantum behavior at microscopic scales. Thus, it is generally believed that the classical general relativistic description of gravity is inadequate at short distances due to quantum fluctuations. However, despite the enormous efforts to unify gravity and quantum mechanics during the last several decades we still do not have a consistent theory of quantum gravity. There are, of course, some promising approaches, like the string theory, loop gravity and non-commutative geometry. But, at the time, none of them provides a complete consistent theory that can be verified by existing or realistic future experiments.

Secondly, in the last decade or so it became more and more evident that there might be a few problems in the classical domain as well. In addition to the old problem of gravitational singularities in General Relativity these gravitational anomalies include such effects as dark matter, dark energy, Pioneer anomaly, flyby anomaly, and others [8]. They might signal to new physics not only at the Planckian scales but at very large (galactic) scales as well.

This suggests that General Relativity, that works perfectly well at macroscopic scales, should be modified (or deformed) both at microscopic and at galactic (or cosmological) scales (or, in the language of high energy physics, both in the ultraviolet and the infrared). It is very intriguing to imagine that these effects (that is, the quantum origin of gravity and gravitational anomalies at large scales) could be somehow related. Of course, this modification should be done in such a way that at the usual distances the usual General Relativity is recovered. This condition puts some constraints (experimental bounds) on the deformation parameters; in the case of non-commutative field theory such bounds on the non-commutativity parameter were obtained in [5].

In this paper we investigate the motion of test particles in an extended theory of gravity, called Matrix Gravity, proposed in a series of recent papers [1, 2, 3]. The motivation for such a deformation of General Relativity is explained in detail in [2]. The very basic physical concepts are the notions of event and the space-time. An event is a collection of variables that specifies the location of a point in space at a certain time. To assign a time to each point in space one needs to place clocks at every point (say on a lattice in space) and to synchronize these clocks.
Once the position of the clocks is fixed the only way to synchronize the clocks is by transmitting the information from a fixed point (say, the origin of the coordinate system in space) to all other points. This can be done by sending a signal through space from one point to another. Therefore, the synchronization procedure depends on the propagation of the signal through space, and, as a result, on the properties of the space it propagates through, in particular, on the presence of any physical background fields in space. The propagation of signals is described by a wave equation (a hyperbolic partial differential equation of second order). Therefore, the propagation of a signal depends on the matrix of the coefficients (a symmetric 2-tensor) $g^{\mu\nu}(x)$ of the second derivatives in the wave equation which must be non-degenerate and have the signature $(−+\ldots+)$. This matrix can be interpreted as a pseudo-Riemannian metric, which defines the geodesic flow, the curvature and, finally, the Einstein equations of General Relativity (for more details, see [2]).

The picture described above applies to the propagation of light, which is described by a single wave equation. However, now we know that at microscopic scales there are other fields that could be used to transmit a signal. In particular, the propagation of a multiplet of $N$ gauge fields is described not by a single wave equation but by a hyperbolic system of second order partial differential equations. The coefficients at the second derivatives of such a system are not given by just a 2-tensor like $g^{\mu\nu}(x)$ but by a $N \times N$ matrix-valued symmetric 2-tensor $a^{\mu\nu}(x)$. If $a^{\mu\nu}$ does not factorize as $a^{\mu\nu} \neq E g^{\mu\nu}$, where $E$ is some non-degenerate matrix, then there is no geometric interpretation of this hyperbolic system in terms of a single Riemannian metric. Instead, we obtain a new kind of geometry that we call Matrix Geometry, which is equivalent to a collection of Finsler geometries. In this theory, instead of a single Riemannian geodesic flow, there is a system of $N$ Finsler geodesic flows. Moreover, a gravitating particle is described not by one mass parameter but by $N$ mass parameters (which could be different). Note that because the tensor $a^{\mu\nu}$ is matrix-valued, various components of this tensor do not commute, that is, $[a^{\mu\nu}, a^{\rho\theta}] \neq 0$. In this sense such geometry may be also called non-commutative Riemannian geometry. In the commutative limit, $a^{\mu\nu} \rightarrow g^{\mu\nu}$ and we recover the standard Riemannian geometry with all its ingredients. Only the total mass of a gravitating particle is observed. For more details and discussions see [2, 3].

As we outlined above, Matrix Gravity is a non-commutative modification of the standard General Relativity in which the metric tensor $g^{\mu\nu}$ is replaced by a
Hermitian $N \times N$ matrix-valued symmetric two-tensor
\[ a^{\mu\nu} = g^{\mu\nu} I + \kappa h^{\mu\nu}, \quad (1.1) \]
where $I$ is the identity matrix, $h^{\mu\nu}$ is a Hermitian matrix-valued traceless symmetric tensor, i.e.
\[ g^{\mu\nu} = \frac{1}{N} \text{tr} a^{\mu\nu}, \quad \text{tr} h^{\mu\nu} = 0, \quad (1.2) \]
and $\kappa$ is a deformation parameter.

The dynamics of the tensor field $a^{\mu\nu}$ is described by a diffeomorphism invariant action,
\[ S(a) = \int dx \mathcal{L}(a, \partial a), \quad (1.3) \]
where $dx$ is the standard Lebesgue measure on the spacetime manifold and $\mathcal{L}(a, \partial a)$ is the Lagragian density. Of course, as $\kappa \to 0$ this action should reproduce the usual Einstein-Hilbert action functional
\[ S(g) = \frac{1}{16\pi G} \int dx \ g^{1/2}(R - 2\Lambda), \quad (1.4) \]
where $g_{\mu\nu}$ is a pseudo-Riemannian metric, $g = |\det g_{\mu\nu}|$, $R$ is the scalar curvature of the metric $g$, $G$ is the Newton constant and $\Lambda$ is the cosmological constant.

The action of matrix gravity can be constructed in two different ways. One approach, developed in [1, 2], is to try to extend all standard concepts of differential geometry to the non-commutative setting and to construct a matrix-valued connection and a matrix-valued curvature.

The second approach, developed in [3], is based on the spectral asymptotics of a self-adjoint elliptic partial differential operator $L$ of second order with a positive definite leading symbol $\sigma_L(x, \xi) = a^{\mu\nu}(x) \xi_\mu \xi_\nu$. It is well known that there is an asymptotic expansion as $t \to 0$ of the $L^2$-trace of the heat semigroup of the operator $L$
\[ \text{Tr}_{L^2} \exp(-tL) \sim (4\pi t)^{-n/2} \sum_{k=0}^{\infty} t^k A_k, \quad (1.5) \]
where $A_k$ are spectral invariants of the operator $L$. For the Laplace-Beltrami operator $L = -g^{\mu\nu} \nabla_\mu \nabla_\nu$ these coefficients are well known, and, it turns out that the Einstein-Hilbert action is nothing but a linear combination of the first two coefficients, that is,
\[ S(g) = \frac{1}{16\pi G} \left(6A_1 - 2\Lambda A_0\right), \quad (1.6) \]
Therefore, a similar functional, which is automatically diffeomorphism-invariant, can be constructed by computing the same heat kernel coefficients for a more general partial differential operator $L$ of non-Laplace type (for more details, see [3]).

The field equations for the tensor $a^{\mu\nu}$, that we call non-commutative Einstein equations are obtained by varying the action with respect to $a^{\mu\nu}$; in the vacuum we have,

$$\frac{\partial L}{\partial a^{\alpha\beta}} - \partial_\mu \frac{\partial L}{\partial a^{\alpha\beta \mu}} = 0,$$

(1.7)

where $a^{\alpha\beta \mu} = \partial_\mu a^{\alpha\beta}$.

The action has an additional new global gauge symmetry

$$a^{\mu\nu}(x) \mapsto Ua^{\mu\nu}(x)U^{-1},$$

(1.8)

where $U$ is a constant unitary matrix (for more details, see the papers cited above).

By the Noether theorem this symmetry leads to the conserved currents (vector densities)

$$J_\mu = \left[ a^{\alpha\beta}, \frac{\partial L}{\partial a^{\alpha\beta \mu}} \right], \quad \partial_\mu J_\mu = 0.$$

In other words, this suggests the existence of new physical charges

$$Q = \int d\hat{x} J_0,$$

where $d\hat{x}$ denotes the integration over the space coordinates only. These charges have purely noncommutative origin and vanish in the commutative limit.

One can easily localize this global symmetry by introducing the local gauge transformations

$$a^{\mu\nu}(x) \mapsto U(x)a^{\mu\nu}(x)U^{-1}(x),$$

(1.9)

where $U(x) = \exp \omega(x)$, and $\omega$ is an anti-Hermitian matrix-valued function of compact support, i.e. vanishing at infinity, and a new Yang-Mills field $B_\mu$ that transforms as one-form under diffeomorphisms and as a connection under the gauge transformation. All the geometric structures, including the connection coefficients, the curvature etc, become covariant under the local gauge transformations if one simply replaces the partial derivatives $\partial_\mu$ by $\partial_\mu + B_\mu$. This leads to a gauged version of the above functionals (for more details see [1, 2, 3]).

This model may be viewed as a “noncommutative deformation” of Einstein gravity (coupled to a Yang-Mills model in the gauged version), which describes,
in the weak deformation limit, as \( \kappa \to 0 \), General Relativity, Yang-Mills fields (in the gauged version), and a multiplet of self-interacting massive two-tensor fields of spin 2 that interact also with gravity and the Yang-Mills fields. One should stress here that \( \kappa \) is a formal parameter that does not have a particular physical value; it is just a tool to develop the perturbation theory in \( h^{\mu \nu} \). At the end of the derivation we can just set \( \kappa = 1 \). What is measured and describes the extent of the non-commutative deformation is the tensor \( h^{\mu \nu}(x) \) and its derivatives, which can be parametrized by the invariants of this tensor like \( g_{\mu \alpha} g_{\nu \beta} \frac{1}{N} \text{tr} h^{\mu \nu} h^{\alpha \beta} \).

Our approach should be contrasted with the non-commutative extensions of gravity on non-commutative spaces with non-commutative coordinates

\[
[x^\mu, x^\nu] = \Theta^{\mu \nu},
\]

where \( \Theta^{\mu \nu} \) is a constant anti-symmetric matrix, and the Moyal product

\[
f(x) \star g(x) = \exp\left(\frac{i}{2} \Theta^{\mu \nu} \frac{\partial}{\partial y^\mu} \frac{\partial}{\partial z^\nu}\right) f(x+y)g(x+z) \bigg|_{y=z=0}.
\]

This approach immediately raises the question on the nature of the coordinates \( x^\mu \). The non-commutativity condition can only be true for some privileged coordinates, like Cartesian (or inertial) coordinates. The condition that the matrix \( \Theta^{\mu \nu} \) is constant, that is,

\[
\partial_\alpha \Theta^{\mu \nu} = 0,
\]

is not covariant. It breaks the diffeomorphism invariance of the theory, and, as the result, Lorentz invariance. Thus any such theory cannot be diffeomorphism invariant. One could try to replace it by the covariant condition

\[
\nabla_\alpha \Theta^{\mu \nu} = 0,
\]

where \( \nabla_\alpha \) are covariant derivatives with respect to some background metric. Then the matrix \( \Theta^{\mu \nu} \) could be viewed as a covariantly constant antisymmetric 2-tensor. However, the integrability conditions for this equation lead to very strong algebraic constraints on the curvature of the metric. Thus, the breaking of the diffeomorphism invariance (and as a result of Lorentz invariance) is an unavoidable feature of this approach to non-commutative gravity. Therefore, such theories can be ruled out by very restrictive experimental bounds on the non-commutativity parameter [5].

By contrast, the status of diffeomorphism invariance (and the Lorentz invariance) in Matrix Gravity is exactly the same as in General Relativity, namely, both
theories are diffeomorphism invariant, so, there are no preferred coordinates, and a condition like (1.10) is impossible. In our model it is not the coordinates that do not commute, but the metric! Therefore, the recent strong experimental constraints on the violation of Lorentz invariance do not apply to Matrix Gravity. It is rather the violation of the Equivalence Principle that is critical for Matrix Gravity. This feature could be used for an experimental test of the theory in the future.

One should also mention the relation of our approach to so called “analog models of gravity”. In particular, the analysis in [4] is surprisingly similar to the analysis of our papers [1, 2]. The authors of [4] consider a hyperbolic system of second order partial differential equations, the corresponding Hamilton-Jacobi equations and the Hamiltonian system as we did in [1,2]. In fact, their $f^{\mu\nu}$ is equivalent to our matrix-valued tensor $a^{\mu\nu}$. However, their goal was very different—they impose the commutativity conditions on $f^{\mu\nu}$ (eq. (44)) to enforce a unique effective metric for the compatibility with the Equivalence Principle. They barely mention the general geometric interpretation in terms of Finsler geometries as it “does not seem to be immediately relevant for either particle physics or gravitation” The motivation of the authors of [4] is also very different from our approach. Their idea is that gravity is not fundamental so that the effective metric simply reflects the properties of an underlying physics (such as fluid mechanics and condensed matter theory). They just need to have enough fields to be able to parametrize an arbitrary effective metric. In our approach, the matrix-valued field $a^{\mu\nu}$ is fundamental; it is: i) non-commutative and ii) dynamical.

The main goal of the present paper is to investigate the motion of test particles in a simple model of matrix gravity and study the non-geodesic corrections to general relativity.

The outline of this work is as follows. In Sect. 2. we develop the kinematics in Matrix Gravity. In Sect. 3. we compute the first and second order non-commutative corrections to the usual Riemannian geodesic flow. In Sect. 4 we find a static spherically symmetric solution of the dynamical equations of Matrix Gravity in a particular case of commutative $2 \times 2$ matrices. In Sect. 5 we evaluate the anomalous acceleration of test particles in this background. In Sect. 6 we discuss our results.
2 Kinematics in Matrix Gravity

2.1 Riemannian Geometry

Let us recall how the geodesic motion appears in General Relativity, that is, in Riemannian geometry (for more details, see [2]). First of all, let

\[ F(x, \xi) = \sqrt{-|\xi|^2}, \tag{2.1} \]

where \( \xi_\mu \) is a non-vanishing cotangent vector at the point \( x \), and \( |\xi|^2 = g^{\mu\nu}(x)\xi_\mu\xi_\nu \) (recall that the signature of our metric is \((- + \ldots +))\). Obviously, this is a homogeneous function of \( \xi \) of degree 1, that is,

\[ F(x, \lambda\xi) = \lambda F(x, \xi). \tag{2.2} \]

Let

\[ H(x, \xi) = -\frac{1}{2}F^2(x, \xi) = \frac{1}{2}|\xi|^2. \tag{2.3} \]

This is, of course, a homogeneous polynomial of \( \xi_\mu \) of order 2, and, therefore, the Riemannian metric can be recovered by

\[ g^{\mu\nu}(x) = \frac{\partial^2}{\partial \xi_\mu \partial \xi_\nu} H(x, \xi). \tag{2.4} \]

Now, let us consider a Hamiltonian system with the Hamiltonian \( H(x, \xi) \)

\[ \frac{dx^\mu}{dt} = \frac{\partial H(x, \xi)}{\partial \xi_\mu} = g^{\mu\nu}(x)\xi_\nu, \tag{2.5} \]

\[ \frac{d\xi_\mu}{dt} = \frac{\partial H(x, \xi)}{\partial x^\mu} = -\frac{1}{2}\partial_\mu g^{\alpha\beta}(x)\xi_\alpha\xi_\beta. \tag{2.6} \]

The trajectories of this Hamiltonian system are, then, nothing but the geodesics of the metric \( g_{\mu\nu} \). Of course, the Hamiltonian is conserved, that is,

\[ g^{\mu\nu}(x(t))\xi_\mu(t)\xi_\nu(t) = -E, \tag{2.7} \]

where \( E \) is a constant parameter.
2.2 Finsler Geometry

As it is explained in [2, 3] Matrix Gravity is closely related to Finsler geometry [9] rather than Riemannian geometry. In this section we follow the description of Finsler geometry outlined in [9]. To avoid confusion we should note that we present it in a slightly modified equivalent form, namely, we start with the Finsler function in the cotangent bundle rather than in the tangent bundle.

Finsler geometry is defined by a Finsler function \( F(x, \xi) \) which is a homogeneous function of \( \xi_\mu \) of degree 1 and the Hamiltonian

\[
H(x, \xi) = -\frac{1}{2} F^2(x, \xi). \tag{2.8}
\]

Such Hamiltonian is still a homogeneous function of \( \xi_\mu \) of degree 2, that is,

\[
\xi_\mu \frac{\partial}{\partial \xi_\mu} H(x, \xi) = 2H(x, \xi), \tag{2.9}
\]

but not necessarily a polynomial in \( \xi_\mu \).

Now, we define a tangent vector \( u \) by

\[
u^\mu = \frac{\partial}{\partial \xi_\mu} H(x, \xi), \tag{2.10}
\]

and the Finsler metric

\[
G^{\mu\nu}(x, \xi) = \frac{\partial^2}{\partial \xi_\mu \partial \xi_\nu} H(x, \xi). \tag{2.11}
\]

The difference with the Riemannian metric is, obviously, that the Finsler metric does depend on \( \xi_\mu \), more precisely, it is a homogeneous function of \( \xi_\mu \) of degree 0, i.e.

\[
G^{\mu\nu}(x, \lambda \xi) = G^{\mu\nu}(x, \xi), \tag{2.12}
\]

so that it depends only on the direction of the covector \( \xi \) but not on its magnitude. This leads to a number of useful identities, in particular,

\[
H(x, \xi) = \frac{1}{2} G^{\mu\nu}(x, \xi) \xi_\mu \xi_\nu, \tag{2.13}
\]

and

\[
u^\mu = G^{\mu\nu}(x, \xi) \xi_\nu. \tag{2.14}
\]
Now, we can solve this equation for $\xi_\mu$ treating $u^\nu$ as independent variables to get

$$\xi_\mu = G_{\mu\nu}(x, u)u^\nu, \quad (2.15)$$

where $G_{\mu\nu}$ is the inverse Finsler metric defined by

$$G_{\mu\nu}(x, u)G^{\nu\alpha}(x, \xi) = \delta^\alpha_\mu. \quad (2.16)$$

By using the results obtained above we can express the Hamiltonian $H$ in terms of the vector $u^\mu$, more precisely we have

$$H(x, \xi(x, u)) = \frac{1}{2}G_{\mu\nu}(x, u)u^\mu u^\nu. \quad (2.17)$$

The derivatives of the Finsler metric obviously satisfy the identities

$$\frac{\partial}{\partial \xi_\alpha} G^{\beta\gamma}(x, \xi) = \frac{\partial}{\partial \xi_\beta} G^{\alpha\gamma}(x, \xi) = \frac{\partial}{\partial \xi_\gamma} G^{\alpha\beta}(x, \xi), \quad (2.18)$$

$$\xi_\mu \frac{\partial}{\partial \xi_\mu} G^{\nu\alpha}(x, \xi) = \xi_\mu \frac{\partial}{\partial \xi_\nu} G^{\mu\alpha}(x, \xi) = 0, \quad (2.19)$$

and, more generally,

$$\xi_\mu \frac{\partial^k}{\partial \xi_{v_1} \ldots \partial \xi_{v_k}} G^{\mu\alpha}(x, \xi) = 0. \quad (2.20)$$

This means, in particular, that the following relations hold

$$\frac{\partial u^\mu}{\partial \xi_\alpha} = G^{\mu\alpha}(x, \xi), \quad \frac{\partial \xi_\alpha}{\partial u^\mu} = G_{\mu\alpha}(x, u). \quad (2.21)$$

It is easy to see that the metric $G_{\mu\nu}(x, u)$ is a homogeneous function of $u$ of degree 0, that is,

$$u^\mu \frac{\partial}{\partial u^\mu} G_{\nu\alpha}(x, u) = 0, \quad (2.22)$$

and, therefore, $H(x, \xi(x, u))$ is a homogeneous function of $u$ of degree 2. This leads to the identities

$$\xi_\mu = \frac{1}{2} \frac{\partial}{\partial u^\mu} H(x, \xi(x, u)), \quad (2.23)$$

$$G_{\mu\nu}(x, u) = \frac{1}{2} \frac{\partial^2}{\partial u^\mu \partial u^\nu} H(x, \xi(x, u)). \quad (2.24)$$
Finally, this enables one to define the Finsler interval
\[ ds^2 = G_{\mu\nu}(x, \dot{x}) dx^\mu dx^\nu, \] (2.25)
so that
\[ d\tau = \sqrt{-ds^2} = \sqrt{-G_{\mu\nu}(x, \dot{x}) \dot{\xi}^\mu \dot{\xi}^\nu} \, dt = F(x, \xi(x), \dot{x}) \, dt, \] (2.26)
where
\[ \dot{\xi}^\mu = \frac{d\xi^\mu}{dt}, \quad \xi_{\mu} = G_{\mu\nu}(x, \dot{x}) \dot{x}^\nu. \] (2.27)

By treating \( H(x, \xi) \) as a Hamiltonian we obtain a system of first order ordinary differential equations
\[ \frac{d\xi^\mu}{dt} = \frac{\partial H(x, \xi)}{\partial \xi_{\mu}}, \] (2.28)
\[ \frac{d\xi_{\mu}}{dt} = -\frac{\partial H(x, \xi)}{\partial x^\mu}. \] (2.29)

The trajectories of this Hamiltonian system naturally replace the geodesics in Riemannian geometry. Again, as in the Riemannian case, the Hamiltonian is conserved along the integral trajectories
\[ H(x(t), \xi(t)) = -E. \] (2.30)

Of course, in the particular case, when the Hamiltonian is equal to \( H(x, \xi) = \frac{1}{2} |\xi|^2 \), all the constructions derived above reduce to the standard structure of Riemannian geometry.

### 2.3 Matrix Gravity

The kinematics in Matrix Gravity is defined as follows. In complete analogy with the above discussion we consider the matrix
\[ A(x, \xi) = a^{\mu\nu}(x) \xi_\mu \xi_\nu, \] (2.31)
where \( a^{\mu\nu} \) is the matrix-valued metric \([1,1]\). As we mentioned in the introduction this expression has been already encountered in physics, in particular, in \([4]\) it is shown that it is the most general structure describing “analog models” for gravity.

This is a Hermitian matrix, so it has real eigenvalues \( h_i(x, \xi), i = 1, 2, \ldots, N \).

We consider a generic case when the eigenvalues are simple. We note that the
eigenvalues $h_i(x, \xi)$ are homogeneous functions (but not polynomials!) of $\xi$ of degree 2. Thus, each one of them, more precisely $\sqrt{-h_i(x, \xi)}$, can serve as a Finsler function. In other words, we obtain $N$ different Finsler functions, and, therefore, $N$ different Finsler metrics. Thus, quite naturally, instead of a single Riemannian metric and a unique Riemannian geodesic flow there appears $N$ Finsler metrics and $N$ corresponding flows. In some sense, the noncommutativity leads to a “splitting” of a single geodesic to a system of close trajectories.

Now, to define a unique Finsler metric we need to define a unique Hamiltonian, which is a homogeneous function of the momenta of degree 2. It is defined in terms of the Finsler function as in (2.8) which is a homogeneous function of the momenta of degree 1. To define a unique Finsler function we can proceed as follows. Let $\mu_i, i = 1, \ldots, N$, be some dimensionless real parameters such that

$$\sum_{i=1}^{N} \mu_i = 1,$$

so that there are $(N - 1)$ independent parameters. Then we can define the Finsler function by

$$F(x, \xi) = \sum_{i=1}^{N} \mu_i \sqrt{-h_i(x, \xi)}.$$  

Notice that, in the commutative limit, as $\kappa \to 0$ and $a^{\mu\nu} = g^{\mu\nu}$, all eigenvalues of the matrix $A(x, \xi)$ degenerate to the same value, $h_i(x, \xi) = |\xi|^2$, and, hence, the Finsler function becomes $F(x, \xi) = \sqrt{-|\xi|^2}$. In this case the Finsler flow degenerates to the usual Riemannian geodesic flow.

Next, we define the Hamiltonian according to eq. (2.8)

$$H(x, \xi) = -\frac{1}{2} \left( \sum_{i=1}^{N} \mu_i \sqrt{-h_i(x, \xi)} \right)^2$$

$$= \frac{1}{2} \sum_{i=1}^{N} \mu_i^2 h_i(x, \xi) - \sum_{1 \leq i < j \leq N} \mu_i \mu_j \sqrt{h_i(x, \xi)h_j(x, \xi)}.$$  

In a particular case, when all parameters $\mu_i$ are equal, i.e. $\mu_i = 1/N$, the Finsler function reduces to

$$F(x, \xi) = \frac{1}{N} \sum_{i=1}^{N} \sqrt{-h_i(x, \xi)} = \frac{1}{N} \text{tr} \sqrt{-A(x, \xi)}.$$  

(2.35)
By using the decomposition of the matrix-valued metric $a^{\mu \nu}$ as in (1.1) one can see that
\[ \frac{1}{N} \text{tr} A(x, \xi) = |\xi|^2, \]  
and, therefore,
\[ \frac{1}{N} \sum_{i=1}^{N} h_i(x, \xi) = |\xi|^2. \]  
Thus, we conclude that in this particular case
\[ H(x, \xi) = \frac{1}{N} \left( \frac{1}{2} |\xi|^2 - \frac{1}{N} \sum_{1 \leq i < j \leq N} \sqrt{h_i(x, \xi) h_j(x, \xi)} \right). \]  

It is difficult to give a general physical picture of these models since the Hamiltonian is non-polynomial in the momenta. Hamiltonian systems with homogeneous Hamiltonians have not been studied as thoroughly as the usual systems with quadratic Hamiltonians and a potential.

2.4 Kinematics

The problem is, now, how to use these mathematical tools to describe the motion of physical massive test particles in Matrix Gravity. The motion of a massive particle in the gravitational field is determined in General Relativity by the action which is proportional to the interval, that is,
\[ S_{\text{particle}} = - \int_{P_1}^{P_2} m \sqrt{-g_{\mu \nu}(x) dx^\mu dx^\nu} = - \int_{t_1}^{t_2} m \sqrt{-|\dot{x}|^2} dt, \]  
where $m$ is the mass of the particle, $P_1$ and $P_2$ are the initial and the final position of the particle in the spacetime, $t$ is a parameter, $t_1$ and $t_2$ are the initial and the final values, $\dot{x}^\mu = \frac{d x^\mu}{dt}$ and $|\dot{x}|^2 = g_{\mu \nu}(x) \dot{x}^\mu \dot{x}^\nu$. This action is, of course, reparametrization-invariant. So, as always, there is a freedom of choosing the parameter $t$. We can always choose the parameter to be the affine parameter such that $|\dot{x}|^2$ is constant, for example, if the parameter is the proper time $t = \tau$, then $|\dot{x}|^2 = -1$. The Euler-Lagrange equations for this functional are, of course,
\[ \frac{D \dot{x}^\nu}{dt} = \frac{d^2 \dot{x}^\nu}{dt^2} + \Gamma^\nu_{\alpha \beta}(x) \dot{x}^\alpha \dot{x}^\beta = 0, \]
where $\Gamma^\alpha_{\mu\beta}$ are the standard Christoffel symbols of the metric $g_{\mu\nu}$. Of course, the equivalence principle holds since these equations do not depend on the mass.

In Matrix Gravity a particle is described instead of one mass parameter $m$ by $N$ different mass parameters

$$m_i = m_\mu i,$$

where

$$m = \sum_{i=1}^{N} m_i. \quad (2.42)$$

The parameters $m_i$ describe the “tendency” for a particle to move along the trajectory determined by the corresponding Hamiltonian $h_i(x, \xi)$. In the commutative limit we only observe the total mass $m$.

We define the Finsler function $F(x, \xi)$ and the Hamiltonian $H(x, \xi)$ as in eqs. (2.33) and (2.34). Then the action for a particle in the gravitational field has the form

$$S_{\text{particle}} = - \int_{t_1}^{t_2} m F(x, \xi(x, \dot{x})) \, dt.$$  \hspace{1cm} (2.43)

Thus, the Finsler function $F(x, \xi(x, \dot{x}))$ (with the covector $\xi_\mu$ expressed in terms of the tangent vector $\dot{x}_\mu$) plays the role of the Lagrangian. To study the role of non-commutative corrections, it is convenient to rewrite this action in the form that resembles the action in General Relativity.

$$S_{\text{particle}} = - \int_{t_1}^{t_2} m_{\text{eff}}(x, \dot{x}) \sqrt{-|\dot{x}|^2} \, dt,$$  \hspace{1cm} (2.44)

with some “effective mass” $m_{\text{eff}}(x, \dot{x})$ that depends on the location and on the velocity of the particle

$$m_{\text{eff}}(x, \dot{x}) = \sum_{i=1}^{N} m_i \sqrt{|h_i(x, \xi(\dot{x}))| \, |\dot{x}|^2}.$$  \hspace{1cm} (2.45)

This action is again reparametrization-invariant. Therefore, we can choose the natural arc-length parameter so that $F(x, \xi(x, \dot{x})) = 1$. Then the equations of motion determined by the Euler-Lagrange equations have the same form

$$\frac{d^2 x^\mu}{dt^2} + \Gamma^\mu_{\alpha\beta}(x, \dot{x}) \dot{x}^\alpha \dot{x}^\beta = 0,$$  \hspace{1cm} (2.46)
where $\gamma^\mu_{\alpha\beta}(x, \dot{x})$ are the Finsler Christoffel coefficients defined by the equations that look identical to the usual equations but with the Finsler metric instead of the Riemannian metric, that is,

$$\gamma^\mu_{\alpha\beta}(x, \dot{x}) = \frac{1}{2} G^{\mu
u}(x, \xi(x, \dot{x})) \left( \frac{\partial}{\partial x^\alpha} G_{\nu\beta}(x, \dot{x}) + \frac{\partial}{\partial x^\beta} G_{\nu\alpha}(x, \dot{x}) - \frac{\partial}{\partial x^\nu} G_{\alpha\beta}(x, \dot{x}) \right).$$

(2.47)

To study the role of non-commutative corrections it is convenient to rewrite these equations in a covariant form in the Riemannian language. In the commutative limit, as $\kappa \to 0$, we can expand all our constructions in power series in $\kappa$ so that the non-perturbed quantities are the Riemannian ones. In particular, we have

$$\gamma^\mu_{\alpha\beta}(x, \dot{x}) = \Gamma^\mu_{\alpha\beta}(x) + \theta^\mu_{\alpha\beta}(x, \dot{x}),$$

(2.48)

where $\theta^\mu_{\alpha\beta}$ are some tensors of order $\kappa$. Then the equations of motion can be written in the form

$$\frac{D\dot{x}^\nu}{dt} = A^\nu_{\text{anom}}(x, \dot{x}),$$

(2.49)

where

$$\frac{D\dot{x}^\nu}{dt} = \frac{d^2 x^\mu}{dt^2} + \Gamma^\mu_{\alpha\beta}(x) \dot{x}^\alpha \dot{x}^\beta,$$

(2.50)

and

$$A^\nu_{\text{anom}}(x, \dot{x}) = -\theta^\nu_{\alpha\beta}(x, \dot{x}) \dot{x}^\alpha \dot{x}^\beta,$$

(2.51)

is the anomalous nongeodesic acceleration.

### 3 Perturbation Theory

We see that the motion of test particles in matrix Gravity is quite different from that of General Relativity. The most important difference is that particles exhibit a non-geodesic motion. In other words, there is no Riemannian metric such that particles move along the geodesics of that metric. It is this anomalous acceleration that we are going to study in this paper.

In the commutative limit the action of a particle in Matrix Gravity reduces to the action of a particle in General Relativity with the mass $m$ determined by the sum of all masses $m_i$. In this paper we consider two different cases. In the first case, that we call the nonuniform model, we assume that all mass parameters are different, and in the second case, that we call the uniform model, we discuss what happens if they are equal to each other.
3.1 Nonuniform Model: First Order in $\kappa$

So, in this section we study the generic case when the parameters $\mu_i$ are different. As we already mentioned above, in this case the Finsler function $F(x, \xi)$ is given by (2.33). By using the decomposition (1.1) of the matrix-valued metric $a_{\mu\nu}$ we have

$$A(x, \xi) = a^{\mu\nu}(x)\xi_{\mu}\xi_{\nu} = |\xi|^2 I + \kappa h^{\mu\nu}(x)\xi_{\mu}\xi_{\nu}. \quad (3.52)$$

Therefore, the eigenvalues of the matrix $A(x, \xi)$ are

$$h_i(x, \xi) = |\xi|^2 + \kappa \lambda_i(x, \xi), \quad (3.53)$$

where $\lambda_i(x, \xi)$ are the eigenvalues of the matrix $h^{\mu\nu}(x)\xi_{\mu}\xi_{\nu}$. In the first order in $\kappa$ we get the Finsler function

$$F(x, \xi) = \sqrt{-|\xi|^2 \left(1 + \frac{1}{2} P(x, \xi) \right)} + O(\kappa^2), \quad (3.54)$$

and the Hamiltonian

$$H(x, \xi) = \frac{1}{2} |\xi|^2 + \frac{1}{2} P(x, \xi) + O(\kappa^2), \quad (3.55)$$

where

$$P(x, \xi) = \sum_{i=1}^{N} \mu_i \lambda_i(x, \xi). \quad (3.56)$$

By using the fact that $P(x, \xi)$ is a homogeneous function of $\xi$ of order 2, we find the Finsler metric

$$G^{\mu\nu}(x, \xi) = g^{\mu\nu}(x) + \kappa q^{\mu\nu}(x, \xi) + O(\kappa^2), \quad (3.57)$$

and its inverse

$$G_{\mu\nu}(x, u) = g_{\mu\nu}(x) - \kappa q_{\mu\nu}(x, \xi(x, u)) + O(\kappa^2), \quad (3.58)$$

where

$$q^{\mu\nu}(x, \xi) = \frac{1}{2} \frac{\partial^2}{\partial \xi_{\mu} \partial \xi_{\nu}} P(x, \xi). \quad (3.59)$$

Here the indices are raised and lowered with the Riemannian metric, and

$$u^{\mu}(x, \xi) = G^{\mu\nu}(x, \xi)\xi_{\nu}, \quad \xi_{\mu}(x, u) = G_{\mu\nu}(x, u)u^\nu. \quad (3.60)$$
Since $P(x, \xi)$ is a homogeneous function of $\xi$ of order 2 we have

$$P(x, \xi) = q^{\mu\nu}(x, \xi)\xi_\mu\xi_\nu.$$  \hspace{1cm} (3.61)

Note that since $\text{tr} h^{\mu\nu} = 0$ the matrix $h^{\mu\nu}\xi_\mu\xi_\nu$ is traceless, which implies that the sum of its eigenvalues is equal to zero. Thus, in the uniform case, when all mass parameters $\mu_i$ are the same, the function $P(x, \xi)$ vanishes. In this case the effects of non-commutativity are of the second order in $\kappa$; we study this case in the next section.

We also note that

$$|\xi|^2 = |u|^2 - 2\kappa P(x, \xi(x, u)) + O(\kappa^2).$$  \hspace{1cm} (3.62)

Thus, our Lagrangian is

$$F(x, \xi(x, \dot{x})) = \sqrt{-|\dot{x}|^2 \left(1 - \frac{\kappa}{2} \frac{1}{|\dot{x}|^2} P(x, \xi(x, \dot{x}))\right)} + O(\kappa^2),$$  \hspace{1cm} (3.63)

Finally, we compute the Christoffel symbols to obtain

$$\theta^\mu_{\alpha\beta}(x, \dot{x}) = -\frac{1}{2} \kappa g^{\mu\nu} \left(\nabla_\alpha q_{\beta\nu}(x, \dot{x}) + \nabla_\beta q_{\alpha\nu}(x, \dot{x}) - \nabla_\nu q_{\alpha\beta}(x, \dot{x})\right) + O(\kappa^2),$$  \hspace{1cm} (3.64)

and the covariant derivatives are defined with the Riemannian metric.

Thus, the anomalous acceleration is

$$A^\mu_{\text{anom}} = \frac{\kappa}{2} g^{\mu\nu} \left(2\nabla_\alpha q_{\beta\nu}(x, \dot{x}) - \nabla_\nu q_{\alpha\beta}(x, \dot{x})\right)\dot{x}^\alpha\dot{x}^\beta + O(\kappa^2),$$  \hspace{1cm} (3.65)

### 3.2 Uniform Model: Second Order in $\kappa$

So, in this section we will simply assume that all mass parameters are equal, that is,

$$m_i = \frac{m}{N}.$$  \hspace{1cm} (3.66)

In this case the Finsler function $F(x, \xi)$ is given by (2.35). By using the decomposition of the matrix-valued metric and the fact that $\text{tr} h^{\mu\nu} = 0$ we get the Finsler function

$$F(x, \xi) = \sqrt{-|\dot{\xi}|^2 \left(1 - \kappa^2 \frac{1}{|\dot{\xi}|^2} S^{\mu\nu\alpha\beta}(x)\xi_\mu\xi_\nu\xi_\alpha\xi_\beta\right)} + O(\kappa^3),$$  \hspace{1cm} (3.67)
and the Hamiltonian

\[
H(x, \xi) = \frac{1}{2} |\xi|^2 \left( 1 - \kappa^2 \frac{1}{4} S^{\mu \nu \alpha \beta}(x) \frac{\xi_\mu \xi_\nu \xi_\alpha \xi_\beta}{|\xi|^4} \right) + O(\kappa^3),
\]

where

\[
S^{\mu \nu \alpha \beta} = \frac{1}{N} \text{tr} \left( h^{\mu \nu} h^{\alpha \beta} \right).
\]

By using the above, we compute the Finsler metric

\[
G^{\mu \nu}(x, \xi) = g^{\mu \nu}(x) - \kappa^2 \frac{1}{4} S^{\mu \nu \alpha \beta}(x) \frac{u^\alpha u^\beta}{|u|^2} + O(\kappa^3),
\]

and its inverse

\[
G_{\mu \nu}(x, u) = g_{\mu \nu}(x) + \kappa^2 \frac{1}{4} S_{\mu \nu \alpha \beta}(x) \frac{u^\alpha u^\beta}{|u|^2} + O(\kappa^3).
\]

We also note that

\[
|\xi|^2 = |u|^2 + \kappa^2 \frac{1}{2} S_{\mu \nu \alpha \beta}(x) \frac{u^\mu u^\nu u^\alpha u^\beta}{|u|^2} + O(\kappa^3).
\]

Thus, our Lagrangian is

\[
F(x, \xi(x, \dot{x})) = \sqrt{|\dot{x}|^2} \left( 1 + \kappa^2 \frac{1}{8} S^{\mu \nu \alpha \beta}(x) \frac{\dot{x}^\mu \dot{x}^\nu \dot{x}^\alpha \dot{x}^\beta}{|\dot{x}|^4} \right) + O(\kappa^3),
\]

Finally, we compute the Christoffel symbols to obtain

\[
\theta^{\mu \nu \alpha}(x, \dot{x}) = \kappa^2 \frac{1}{8} g^{\mu \nu} \left( \nabla_\alpha S_{\beta \gamma \rho \sigma} + \nabla_\beta S_{\gamma \rho \sigma \alpha} - \nabla_\gamma S_{\rho \alpha \beta \sigma} \right) \frac{\dot{x}^\rho \dot{x}^\sigma}{|\dot{x}|^2} + O(\kappa^3).
\]

Thus, the anomalous acceleration is

\[
A^\mu_{\text{anom}} = -\kappa^2 \frac{1}{8} g^{\mu \nu} \left( 2 \nabla_\alpha S_{\beta \gamma \rho \sigma} - \nabla_\gamma S_{\alpha \beta \rho \sigma} \right) \frac{\dot{x}^\rho \dot{x}^\sigma}{|\dot{x}|^2} \dot{x}^\alpha \dot{x}^\beta + O(\kappa^3),
\]

Notice that with our choice of the parameter \( t \) we have \( F(x, \xi(x, \xi)) = 1 \), and, therefore, in the equations of motion we can substitute with the same accuracy

\[
|\xi|^2 = -1 + O(\kappa^2), \quad |\dot{x}|^2 = -1 + O(\kappa^2).
\]

Therefore, we obtain finally

\[
A^\mu_{\text{anom}} = -\kappa^2 \frac{1}{8} g^{\mu \nu} \left( 2 \nabla_\alpha S_{\beta \gamma \rho \sigma} - \nabla_\gamma S_{\alpha \beta \rho \sigma} \right) \dot{x}^\rho \dot{x}^\sigma \dot{x}^\alpha \dot{x}^\beta + O(\kappa^3).
\]
3.3 Non-commutative Corrections to Newton’s Law

Now, we will derive the non-commutative corrections to the Newton’s Law. We label the coordinates as

\[ x^0 = t, \quad x^1 = r, \quad x^2 = \theta, \quad x^3 = \varphi, \]  

and consider the static spherically symmetric (Schwarzschild) metric

\[ ds^2 = -U(r)dt^2 + U^{-1}(r)dr^2 + r^2(d\theta^2 + \sin^2 \theta d\varphi^2), \]  

where

\[ U(r) = 1 - \frac{r_g}{r}, \quad r_g = 2GM, \]  

and \( M \) is the mass of the central body. It is worth recalling that here \( t \) is the coordinate time. In the previous sections we used \( t \) to denote an affine parameter of the trajectory that we agreed to choose to be the proper time. In the present section we use \( \tau \) to denote the proper time and \( t \) to denote the coordinate time.

The motion of test particles in Schwarzschild geometry is very well studied in General Relativity, see, for example [10]. Assuming that the particle moves in the equatorial plane \( \theta = \pi/2 \) away from the center, that is, \( dr/d\tau > 0 \), the equations of motion have the following integrals [10].

\[ \dot{x}^0 = \frac{dt}{d\tau} = \frac{E}{mU(r)}, \]  
\[ \dot{x}^1 = \frac{dr}{d\tau} = \sqrt{\frac{E^2}{m^2} - \left(1 + \frac{L^2}{m^2r^2}\right)U(r)}, \]  
\[ \dot{x}^2 = \frac{d\theta}{d\tau} = 0, \quad \theta = \frac{\pi}{2}, \]  
\[ \dot{x}^3 = \frac{d\varphi}{d\tau} = \frac{L}{mr^2}, \]  

where \( m, L, \) and \( E \) are the mass of the particle, its orbital momentum and the energy.

In the non-relativistic limit for weak gravitational fields, assuming

\[ E = m + E', \]  

with \( E' << m \), and \( r >> r_s \) one can identify the coordinate time with the proper
time, so that
\[
\dot{x}^0 = \frac{dt}{d\tau} = 1. \tag{3.87}
\]
Further, for the non-relativistic motion we have \( \dot{r}, \dot{\theta}, \dot{\phi} \ll 1 \), and the radial
velocity reduces, of course, to the standard Newtonian expression
\[
\dot{x}^1 = \frac{dr}{d\tau} = \sqrt{\frac{2E'}{m} - \frac{L^2}{m^2 r^2} + \frac{r_s}{r}}, \tag{3.88}
\]
which for \( L = 0 \) becomes
\[
\dot{x}^1 = \frac{dr}{d\tau} = \sqrt{\frac{2E'}{m} + \frac{r_s}{r}}, \tag{3.89}
\]
It is worth stressing that the anomalous acceleration due to non-commutativity
in the non-relativistic limit can be interpreted as a correction to the Newton’s Law.
Assuming that a particle is moving in the equatorial plane, \( \theta = \pi/2 \), with zero
orbital momentum, \( \varphi = \text{const} \), the equation of motion is
\[
\frac{d^2 r}{dt^2} = -\frac{\partial}{\partial r}V_{\text{eff}}(r) = -\frac{GM}{r^2} + A'_{\text{anom}}, \tag{3.90}
\]
where in the uniform model
\[
A'_{\text{anom}} = \frac{\kappa^2}{8} \partial_r S^{0000} + O(\kappa^3), \tag{3.91}
\]
with \( S^{0000} = \frac{1}{8} \text{tr} h^{00} h^{00} \), and in the non-uniform model
\[
A'_{\text{anom}} = -\frac{\kappa}{2} \partial_r q^{00} + O(\kappa^2), \tag{3.92}
\]
with \( q^{00} \) being the component of the tensor \( q^{\mu\nu} \) defined by (3.59). This gives the
non-commutative corrections to Newton’s Law: in the uniform model,
\[
V_{\text{eff}}(r) = -\frac{GM}{r} - \frac{\kappa^2}{8} S^{0000}(r) + O(\kappa^3), \tag{3.93}
\]
and, in the nonuniform model,
\[
V_{\text{eff}}(r) = -\frac{GM}{r} + \frac{\kappa}{2} q^{00}(r) + O(\kappa^2). \tag{3.94}
\]
Here, of course, the tensor components \( S^{0000} \) and \( q^{00} \) should be obtained by the
solution of the non-commutative Einstein field equations (in the perturbation the-
ory).
4 Noncommutative Einstein Equations

The dynamics of the tensor field $a^{\mu\nu}$ is described by the action functional of Matrix Gravity. As was outlined in the Introduction, there is no unique way to construct such an action functional: there are at least two approaches, one [2] based on a non-commutative generalization of Riemannian geometry and another one [3] based on the spectral asymptotics of a non-Laplace type partial differential operator. The exact non-commutative Einstein equations for the action functional proposed in [2] were found in our recent work [6]. The equations of motion for the spectral approach were obtained within the perturbation theory in our paper [7]. In the present work we are using the approach of [2]. We should mention that in the perturbation theory the difference between these two approaches consists in just some numerical parameters of the action; the general structure of the terms is the same.

In the following we will give a very brief overview of the general formalism, more details can be found in [2, 6]. We define the matrix-valued tensor $b_{\mu\nu}$ by

$$a^{\mu\nu} b_{\nu\lambda} = \delta_{\lambda}^{\mu} \delta_{\lambda}^{\nu}, \quad (4.1)$$

the matrix-valued connection coefficients $\mathcal{A}^{\alpha}_{\mu\beta}$ by

$$\mathcal{A}^{\alpha}_{\mu\beta} = \frac{1}{2} b_{\mu\nu} (a^{\alpha\gamma} \partial_{\gamma} a^{\sigma\alpha} - a^{\beta\gamma} \partial_{\gamma} a^{\sigma\beta} - a^{\gamma\sigma} \partial_{\gamma} a^{\alpha\beta} - a^{\sigma\gamma} \partial_{\gamma} a^{\alpha\sigma}) b_{\rho\sigma}, \quad (4.2)$$

and the matrix-valued Riemann curvature tensor

$$R^{\lambda}_{\alpha\mu
u} = \partial_{\mu} \mathcal{A}^{\lambda}_{\alpha\nu} - \partial_{\nu} \mathcal{A}^{\lambda}_{\alpha\mu} + \mathcal{A}^{\lambda}_{\beta\mu} \mathcal{A}^{\beta}_{\alpha\nu} - \mathcal{A}^{\lambda}_{\beta\nu} \mathcal{A}^{\beta}_{\alpha\mu}. \quad (4.3)$$

Next, we define a matrix-valued density

$$\rho = \int_{\mathbb{R}^n} \frac{d\xi}{\pi^2} \exp[-a^{\mu\nu} \xi_{\mu} \xi_{\nu}] \cdot (4.4)$$

This enables us to define the action of Matrix Gravity as follows

$$S_{\text{MG}}(a) = \frac{1}{16\pi G} \int_{M} dx \text{Re} \frac{1}{N} \text{tr} \rho \left( a^{\mu\nu} R_{\mu\nu}^{\alpha} - 2\Lambda \right), \quad (4.5)$$

where $G$ is the Newtonian gravitational constant and $\Lambda$ is the cosmological constant.
Of course, in the commutative limit all these constructions become the standard geometric background of General Relativity. The tensors $a^{\mu\nu}$ and $b_{\mu\nu}$ become the contravariant and covariant Riemannian metrics, the coefficients $\mathcal{R}^\alpha_{\mu\nu}$ become the Christoffel symbols, the tensor $R^\alpha_{\beta\mu\nu}$ becomes the standard Riemann tensor and the action of Matrix Gravity becomes the Einstein action functional. In the presence of matter one should add to this functional the action of the matter fields and particles. The non-commutative Einstein equations were obtained in [6]. These equations, in full generality, are a complicated system of non-linear second-order partial differential equations. Their study is just beginning. Of course, it would be extremely interesting to obtain some simple exact solutions.

In the present paper we study the effects of these equations in the simplest possible case restricting ourselves to a commutative algebra. The commutativity assumption enormously simplifies the dynamical equations. In this case they look exactly as the Einstein equations in the vacuum

$$R_{\mu\nu} = \Lambda b_{\mu\nu} ,$$

where $R_{\mu\nu}$ is the matrix-valued Ricci tensor defined by

$$R_{\mu\nu} = R^\alpha_{\mu\nu\alpha} .$$

### 4.1 Static Spherically Symmetric Solutions

We study, now, the static spherically symmetric solution of the equation (4.6). We present the matrix-valued metric $a^{\mu\nu}$ by writing the “matrix-valued Hamiltonian”

$$a^{\mu\nu} \xi_\mu \xi_\nu = A(r)(\xi_0)^2 + B(r)(\xi_1)^2 + \frac{1}{r^2} \left[ (\xi_2)^2 + \frac{1}{\sin^2 \theta}(\xi_3)^2 \right] ,$$

or the “matrix-valued interval”

$$b_{\mu\nu} dx^\mu dx^\nu = A^{-1}(r) dt^2 + B^{-1}(r) dr^2 + \frac{2}{r^2} \left( d\theta^2 + \sin^2 \theta d\phi^2 \right) ,$$

where the coefficients $A(r)$ and $B(r)$ are commuting matrices that depend only on the radial coordinate $r$. This simply means that we choose the following ansatz

$$a^{00} = A , \quad a^{11} = B ,$$

$$a^{22} = \frac{1}{r^2} I , \quad a^{33} = \frac{1}{r^2 \sin^2 \theta} I .$$

(4.10)
Next, by computing the connection coefficients $\mathcal{A}^\alpha_{\mu\nu}$ and the matrix-valued Ricci tensor we obtain the equations of motion

$$
\mathcal{R}_{00} = A^{-1}B \left[ \frac{1}{2} A^{-1}A'' - \frac{3}{4} A^{-2} (A')^2 + \frac{1}{4} A^{-1}A'B^{-1}B' + \frac{1}{r} A^{-1}A' \right] = \Lambda A^{-1}, \quad (4.11)
$$

$$
\mathcal{R}_{11} = \frac{1}{2} A^{-1}A'' - \frac{3}{4} A^{-2} (A')^2 + \frac{1}{4} A^{-1}A'B^{-1}B' - \frac{1}{r} B^{-1}B' = \Lambda B^{-1}, \quad (4.12)
$$

$$
\mathcal{R}_{22} = -\frac{r}{2} B' - B + \frac{r}{2} B A^{-1}A' + I = \Lambda r^2 \cdot I, \quad (4.13)
$$

$$
\mathcal{R}_{33} = \sin^2 \theta \mathcal{R}_{22} = \Lambda r^2 \sin^2 \theta \cdot I. \quad (4.14)
$$

where the prime denotes differentiation with respect to $r$.

By using the equations (4.11) and (4.12) we find

$$
A^{-1}A' + B^{-1}B' = 0; \quad (4.15)
$$

the general solution of this equation is

$$
A(r)B(r) = C_1, \quad (4.16)
$$

where $C_1$ is an arbitrary constant matrix from our algebra. We require that at the spatial infinity as $r \to \infty$ the matrices $A$ and $B$ and, therefore, the matrix $C$ as well, are non-degenerate.

By using this relation we obtain further from eqs. (4.12) and (4.13) two compatible equations for the matrix $B$

$$
B'' + \frac{2}{r} B' + 2\Lambda = 0, \quad (4.17)
$$

and

$$
rB' + B = (1 - \Lambda r^2)I. \quad (4.18)
$$

The general solution of the eq. (4.18) is

$$
B(r) = \left( 1 - \frac{1}{3} \Lambda r^2 \right) I + \frac{1}{r} C_2, \quad (4.19)
$$

where $C_2$ is another arbitrary constant matrix from our algebra. It is not difficult to see that this form of the matrix $B$ also satisfies the eq. (4.17). The matrix $A$ is now obtained from the equation (4.16)

$$
A(r) = C_1 \left[ \left( 1 - \frac{1}{3} \Lambda r^2 \right) I + \frac{1}{r} C_2 \right]^{-1}. \quad (4.20)
$$
We will also require that in the limit $\kappa \to 0$ we should get the standard Schwarzschild solution with the cosmological constant

$$B(r) = -A^{-1}(r) = \left( 1 - \frac{1}{3} \Lambda r^2 - \frac{r_g}{r} \right) I,$$  \hspace{1cm} (4.21)

where $r_g$ is the gravitational radius of the central body of mass $M$,

$$r_g = 2GM,$$  \hspace{1cm} (4.22)

that is, in that limit the matrices $C_1$ and $C_2$ should be

$$C_1 = -I, \quad C_2 = -r_g I.$$  \hspace{1cm} (4.23)

### 4.2 2 × 2 Matrices

To be specific, we restrict ourselves further to real symmetric $2 \times 2$ matrices generated by

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{and} \quad \tau = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$  \hspace{1cm} (4.24)

In this case the constant matrices $C_1$ and $C_2$ can be expressed in terms of four real parameters

$$C_1 = \alpha I + \theta \tau, \quad C_2 = \mu I + L \tau,$$  \hspace{1cm} (4.25)

where $\theta = \kappa \bar{\theta}$ and $L = \kappa \bar{L}$ are the parameters of first order in the deformation parameter $\kappa$. Here the parameters $\alpha$ and $\theta$ are dimensionless and the parameters $\mu$ and $L$ have the dimension of length.

Then the matrix $B(r)$ has the form

$$B(r) = \left( 1 - \frac{1}{3} \Lambda r^2 + \frac{\mu}{r} \right) I + \frac{L}{r} \tau.$$  \hspace{1cm} (4.26)

Next, noting that $\tau^2 = I$, and by using the relation

$$(aI + b\tau)^{-1} = \frac{1}{a^2 - b^2} (aI - b\tau),$$  \hspace{1cm} (4.27)

we obtain the matrix $A(r)$

$$A(r) = \varphi(r) I + \psi(r) \tau,$$  \hspace{1cm} (4.28)
where
\[
\varphi(r) = \alpha \left(1 - \frac{1}{3} \Lambda r^2 + \frac{\alpha - \theta L}{r}\right), \quad (4.29)
\]
\[
\psi(r) = \theta \left(1 - \frac{1}{3} \Lambda r^2 + \frac{\theta - \alpha L}{r}\right). \quad (4.30)
\]

The parameters $\alpha, \theta, \mu$ and $L$ should be determined by the boundary conditions at spatial infinity. The question of boundary conditions is a subtle point since we do not know the physical nature of the additional degrees of freedom. We will simply require that the diagonal part of the metric is asymptotically De Sitter. This immediately gives
\[
\alpha = -1. \quad (4.31)
\]

Now, we introduce a new parameter
\[
r_0 = |\Lambda|^{-1/2}, \quad (4.32)
\]
and require that for $r_g << r << r_0$, the diagonal part of the metric, more precisely, the function $\varphi(r)$ is asymptotically Schwarzschild, that is,
\[
\varphi(r) = -1 - \frac{r_g}{r} + O\left(\frac{r_g^2}{r^2}\right) + O\left(\frac{r_g^2}{r_0^2}\right). \quad (4.33)
\]

This fixes the parameter $\mu$
\[
\mu = -r_g + \theta L. \quad (4.34)
\]
The parameters $\theta$ and $L$ remain undetermined.
Finally, by introducing new parameters
\[
\rho = (1 + \theta^2) L - \theta r_g \quad (4.35)
\]
\[
r_{\pm} = r_g - (\theta \pm 1)L \quad (4.36)
\]
we can rewrite our solution in the form
\[
\varphi(r) = \frac{-r \left(r_+ - \frac{1}{3} \Lambda r^3 - r_+ + 2\theta L\right)}{\left(r - \frac{1}{3} \Lambda r^3 - r_-\right)\left(r - \frac{1}{3} \Lambda r^3 - r_+\right)}, \quad (4.37)
\]
\[
\psi(r) = \frac{r \left[\theta \left(r - \frac{1}{3} \Lambda r^3\right) + \rho\right]}{\left(r - \frac{1}{3} \Lambda r^3 - r_-\right)\left(r - \frac{1}{3} \Lambda r^3 - r_+\right)}. \quad (4.38)
\]
Of course, as $\kappa \to 0$ both parameters $L = \kappa \bar{L}$ and $\theta = \kappa \bar{\theta}$ vanish and we get the standard Schwarzschild solution with the cosmological constant.

Notice that the matrix-valued metric $\delta^{\mu\nu}$ becomes singular when the matrices $A$ and $B$ are not invertible, that is, when

$$\det A(r) = 0.$$ (4.39)

The solutions of this equation are the roots of the cubic polynomials

$$r - \frac{1}{3} \Lambda r^3 - r_- = 0 \quad \text{and} \quad r - \frac{1}{3} \Lambda r^3 - r_+ = 0 \quad (4.40)$$

Recall that the standard Schwarzschild coordinate singularity, which determines the position of the event horizon, is located at $r = r_g$. The presence of singularities depends on the values of the parameters. We analyze, now, the first eq. in (4.40). In the case $\Lambda \leq 0$ the polynomial has one root if $r_- > 0$ and does not have any roots if $r_- < 0$. In the case $\Lambda > 0$ it is easy to see that: i) if $r_- > (2/3)r_0$, then there are no roots, ii) if $0 < r_- < (2/3)r_0$, then the polynomial has two roots, and ii) if $r_- < 0$, then the polynomial has one root. The same applies to the second eq. in (4.40).

We emphasize that there are two cases without any singularities at any finite value of $r$. This happens if either: a) $\Lambda \leq 0$ and $r_\pm < 0$, or b) $\Lambda > 0$ and $r_\pm > (2/3)r_0$. This can certainly happen for large values of $|\theta|$ and $|L|$. In particular, if $\theta$ and $L$ have the same signs and

$$|\theta| > 1 + \frac{r_g}{|L|}, \quad (4.41)$$

then both $r_\pm$ are negative, $r_\pm < 0$, and if $\theta$ and $L$ have opposite signs and

$$|\theta| > 1 + \frac{2r_0 - r_g}{|L|}, \quad (4.42)$$

then $r_\pm > (2/3)r_0$. This is a very interesting phenomenon which is entirely new and due to the additional degrees of freedom.

We would like to clarify some points. The parameters $\mu_i$ introduced in the previous sections describe the properties of the test particle, that is, the matter. The parameters $\theta$ and $\rho$ introduced in the static and spherically symmetric solution of non-commutative Einstein equations describe the properties of the gravitational field, that is, the properties of the source of the gravitational field, that is, the central body. The parameters $\theta$ and $\rho$ are not related to the parameters $\mu_i$. 
5 Anomalous Acceleration

In this section we are going to evaluate the anomalous acceleration of non-relativistic test particles in the static spherically symmetric gravitational field of a massive central body.

All we have to do is to evaluate the components of the anomalous acceleration (3.77). As we will see the only essential component of the anomalous acceleration is the radial one \( A_{\text{anom}} \). All other components of the anomalous acceleration are negligible in this limit. As we will see below, the anomalous acceleration is caused by the radial gradient of the component \( h^{00} \) of the matrix-valued metric, which is

\[
\kappa h^{00} = \psi(r)\tau, \tag{5.1}
\]

where \( \psi(r) \) is given by (4.38). Our analysis is restricted to the perturbation theory in the deformation parameter \( \kappa \) (first order in \( \kappa \) in the non-uniform model and second order in \( \kappa \) in the uniform model). That is, we should expand our result in powers of \( \rho \) and \( \theta \) and keep only linear terms in the non-uniform model and quadratic terms in the uniform model.

For future use we write the function \( \psi(r) \) in the first order in the parameter \( \kappa \)

\[
\psi(r) = r \left[ \theta \left( r - \frac{1}{2} \Lambda r^3 \right) + \rho \right] \frac{1}{\left( r - \frac{1}{2} \Lambda r^3 - r_g \right)^2} + O(\kappa^2), \tag{5.2}
\]

and for \( r < r_0 \)

\[
\psi(r) = \frac{r (\theta + \rho)}{(r - r_g)^2} + O(\kappa^2), \tag{5.3}
\]

and, finally, for \( r_g < r < r_0 \),

\[
\psi(r) = \theta + \frac{\rho}{r} + O(\kappa^2). \tag{5.4}
\]

We would like to emphasize at this point that the perturbation theory we are going to perform is only valid for small corrections. When the corrections become large we need to consider the exact equations of motion (2.51).

5.1 Uniform Model

In the non-relativistic limit the formula for the anomalous radial acceleration (3.91) gives

\[
A'_{\text{anom}} = \frac{1}{4} \psi(r)\psi'(r) + O(\kappa^3). \tag{5.5}
\]
The derivative of the function $\psi(r)$ is easily computed

$$\psi'(r) = \omega(r)\psi(r),$$

(5.6)

where

$$\omega(r) = \frac{1}{r} + \frac{\theta(1 - \Lambda r^2)}{\theta \left(r - \frac{1}{3} \Lambda r^3\right) + \rho} - \frac{1 - \Lambda r^2}{r - \frac{1}{3} \Lambda r^3 - r_g} - \frac{1 - \Lambda r^2}{r - \frac{1}{3} \Lambda r^3 - r_+}.\quad (5.7)$$

Thus, we obtain finally

$$A'_{\text{anom}} = \frac{1}{4} \psi^2(r)\omega(r) + O(\kappa^3).\quad (5.8)$$

Recall that the parameters $\rho$ and $\theta$ are of first order in $\kappa$. Strictly speaking we should expand this formula in $\rho$ and $\theta$ keeping only quadratic terms; we get

$$A'_{\text{anom}} = \frac{1}{4} \left[ \theta \left(r - \frac{1}{3} \Lambda r^3\right) + \rho \right] r \left( r - \frac{1}{3} \Lambda r^3 - r_g \right) \left[ \theta \left(2r - \frac{4}{3} \Lambda r^3\right) + \rho \right]$$

$$-2r(1 - \Lambda r^2) \left[ \theta \left(r - \frac{1}{3} \Lambda r^3\right) + \rho \right] + O(\kappa^3).\quad (5.9)$$

For $r << r_0$ (that is, $|\Lambda| r^2 << 1$) this becomes

$$A'_{\text{anom}} = -\frac{1}{4} \frac{r(\theta r + \rho)}{(r - \frac{1}{3} \Lambda r^3 - r_g)} \left( \rho + 2\theta r_g \right) + O(\kappa^3).\quad (5.10)$$

We need to keep the term linear in $\Lambda$ since we do not know the values of the parameters $\theta$ and $\rho$. Finally, for $r_g << r << r_0$ we obtain

$$A'_{\text{anom}} = -\frac{1}{4} \left( \theta + \frac{\rho}{r} \right) \left( \frac{\rho + 2\theta r_g}{r^2} - \frac{2}{3} \theta \Lambda r \right) + O(\kappa^3).\quad (5.11)$$

### 5.2 Non-uniform Model

Similarly, in the non-uniform model the anomalous acceleration is given by eq. (3.92). In the $2 \times 2$ matrix case considered above the eigenvalues of the matrix $h^{\mu\nu}\xi_\mu\xi_\nu$ are

$$\lambda_{1,2} = \pm \frac{1}{2} \text{tr} (h^{\mu\nu})\xi_\mu\xi_\nu.\quad (5.12)$$

Therefore,
\[ P(x, \xi) = \mu_1 \lambda_1 + \mu_2 \lambda_2 = \gamma \frac{1}{2} \text{tr} (h^{\mu \nu} \tau_\mu \xi_\nu), \]  
(5.13)
where
\[ \gamma = \mu_1 - \mu_2. \]  
(5.14)
Thus
\[ q^{\mu \nu} = \gamma \frac{1}{2} \text{tr} (h^{\mu \nu} \tau_\mu \xi_\nu). \]  
(5.15)
So, we obtain
\[ \kappa q^{00} = \gamma \psi(r). \]  
(5.16)
Thus
\[
A_{\text{anom}}^r = -\frac{1}{2} \gamma \psi'(r) + O(\kappa^2)
= -\frac{1}{2} \gamma \psi(r) \omega(r) + O(\kappa^2).
\]  
(5.17)

Now, we recall that \( \rho \) and \( \theta \) are of first order in \( \kappa \) and expand in powers of \( \rho \) and \( \theta \) keeping only linear terms

\[
A_{\text{anom}}^r = -\frac{1}{2} \gamma \left. \left[ \left( r - \frac{1}{3} \Lambda r^3 - r_g \right) \left[ \theta \left( \frac{2}{3} \Lambda r^3 \right) + \rho \right] - 2r(1 - \Lambda r^2) \left[ \theta \left( \frac{1}{3} \Lambda r^2 \right) + \rho \right] \right] + O(\kappa^2). \]  
(5.18)

In the case \( r << r_0 \) (when \( |\Lambda| r^2 << 1 \) this takes the form
\[
A_{\text{anom}}^r = \frac{1}{2} \gamma \left[ (\rho + 2\theta r_g) r + \rho r_g - \frac{2}{3} \theta \Lambda r^4 \right] \frac{1}{(r - r_g)^3} + O(\kappa^2). \]  
(5.19)

Finally, for \( r_g << r << r_0 \) we obtain
\[
A_{\text{anom}}^r = \frac{1}{2} \gamma \left[ \frac{(\rho + 2\theta r_g) r}{r^2} - \frac{2}{3} \theta \Lambda r \right] + O(\kappa^2). \]  
(5.20)
6 Conclusions

In this paper we described the kinematics of test particles in the framework of a recently developed modified theory of gravitation, called Matrix Gravity [1, 2, 3]. We outlined the motivation for this theory, which is a non-commutative deformation of General Relativity. Matrix Gravity can be interpreted in terms of a collection of Finsler geometries on the spacetime manifold rather than in terms of Riemannian geometry. This leads, in particular, to a new phenomenon of splitting of Riemannian geodesics into a system of trajectories (Finsler geodesics) close to the Riemannian geodesic. More precisely, instead of one Riemannian metric we have several Finsler metrics and different mass parameters which describe the tendency to follow a particular Finsler geodesics determined by a particular Finsler metric. As a result the test particles exhibit a non-geodesic motion which can be interpreted in terms of an anomalous acceleration.

By using a commutative algebra we found a static spherically symmetric solution of the modified Einstein equations. In this case a completely new feature appears due to the presence of additional degrees of freedom. The coordinate singularities of our model depend of additional parameters (constants of integration). Interestingly, there is a range of values for these free parameters in which no singularity occurs. This is just one of the intriguing differences between Matrix Gravity and General Relativity.

The description of matter in Matrix Gravity needs additional study. In this paper we studied just the behavior of classical test particles. We propose to describe a gravitating particle by several mass parameters rather than one parameter as in General Relativity. We considered two models of matter: a uniform one, in which all mass parameters are equal, and a non-uniform one, in which the mass parameters are different. The choice of one model over the other should be dictated by physical reasons. It is worth emphasizing that in the generic non-uniform model the equivalence principle is violated.

The interesting question whether the matter is described by only one mass parameter or more than one mass parameters as well as the more general question of the physical origin of multiple mass parameters requires further study. Since we do not know much about the physical origin of these mass parameters masses, we do not have to assume that they are positive. We do not exclude the possibility that some of the mass parameters can be negative or zero. This would imply, of course, that in this theory there is also gravitational repulsion (antigravity). This could help solve the problem of the gravitational collapse in General Relativity, which is caused by the infinite gravitational attraction.
The next step of our analysis of the phenomenological consequences of Matrix Gravity is to apply the kinematic model developed in the previous sections to the study of such effects as the motion of Pioneer spacecrafts (Pioneer anomaly) and galactic rotations (dark matter). It would be very interesting to understand if the anomalous acceleration of the spacecrafts and the flat rotation curves of galaxies can be explained without the concept of dark matter.

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