Constrained dynamics of an inertial particle in a turbulent flow

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Abstract. Most of theoretical and numerical works for free advected particles in a turbulent flow, which only consider the drag force acting on the particles, fails to predict recent experimental results for the transport of finite size particles. These questions have motivated a series of experiments trying to emphasize the actual role of the drag force by imposing this one as an unambiguous leading forcing term acting on a particle in a turbulent background. This is achieved by considering the constrained dynamics of towed particles in a turbulent environment. In the present work, we focus on the influence of particles inertia on its velocity and acceleration Lagrangian statistics and energy spectral density. Our results are consistent with a filtering scenario resulting from the viscous response time of an inertial particle whose dynamics is coupled to the surrounding fluid via strong contribution of drag.

1. Introduction

Turbulent transport of material inclusions plays an important role in many natural and industrial situations. Being able to accurately model and predict the dynamics of dispersed particles transported by a turbulent carrier flow, remains a challenge. A critical point consists in finding a model which would describe correctly particles dynamics over a wide range of sizes and densities. From a quantitative point of view, no general and reliable model to describe and accurately predict the statistical properties of inclusions advected by a turbulent flow has emerged yet. Even writing an appropriate equation of motion for a particle transported by a turbulent flow remains a theoretical challenge, which has mostly been approached in some limit cases, assuming generally point like particles, as the celebrated Maxey-Riley-Gatignol equation [6, 4] which assumes point particles as it neglects non-uniformities of the flow field around the particles.

In most theoretical and numerical studies, this equation is further simplified and inertial effects are dominantly modeled by the Stokes drag force resulting from the finite response time \( \tau_p \) of the particle. In such models, dynamics of the particle is simply coupled to that of the carrier flow by a Stokesian equation of motion:

\[
\frac{d\vec{v}}{dt} = \frac{1}{\tau_p}(\vec{u} - \vec{v}),
\]

where \( \tau_p \) is the particle’s viscous response time, \( \vec{v} \) is the particle’s velocity and \( \vec{u} \) is the fluid’s velocity at the position of the particle. The actual range of validity of this approximation remains

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Figure 1. Turbulence is generated downstream a grid in a wind tunnel with a mean streamwise velocity $U = 10$ m/s. The measurement plane is located about 2 m downstream the grid. The sphere is towed by a cable attached to the grid.

unclear. Recent experimental and numerical investigations have shown that this assumption fails to predict simple dynamical features of small though finite size inertial particles [10]. For instance, while the dominant drag force assumption predicts a low-pass filtering effect of particles velocity fluctuations with increasing inertia, measurements show that the fluctuation level of inertial particles velocity remains identical to that of the carrier flow, even when the particle inertia is large. Similarly, Stokesian models predict a gaussianization of particles acceleration statistics as inertia is increased [1], while no such trend is observed experimentally for finite size inertial particles [10, 5]. These observations address the question of the exact role of the drag force in the equation of motion of advected particles. When is the drag term really dominant? When can the particle dynamics indeed be related to that of the carrier flow by a simple filtering mechanism?

These questions have motivated a series of experiments trying to emphasize the role of the filtering effects by forcing somehow the drag force acting on the particle to become indeed an unambiguous leading forcing term. Therefore the idea of considering a semi-constrained particle with an imposed dominant motion relative to the surrounding fluid. In the present study, this is achieved by considering a particle attached to a thin longitudinal cable in a wind tunnel experiment (see figure 1); as the particle is attached its streamwise motion is blocked what imposes a strong streamwise slippage velocity $(u_z - v_z)$ (where $\vec{u}$ and $\vec{v}$ are, respectively, the flow and the particle velocity) and therefore a strong drag force. We choose to use a long cable (about 2 m long, in order to reduce the constraint on the transverse motion, so that the particle is relatively free to move in the transverse directions $x$ and $y$, with almost a two-dimensional trajectory. Besides, we note that in the reference frame moving at the mean wind velocity $U$, this situation is equivalent to that of a particle towed with an average velocity $U$. The results presented here are therefore also relevant to this class of problems, with important practical issues for towed objects in water and air [3]. In the present work, we focus on the influence of inertia on the dynamics of velocity and acceleration of the towed particle and on its power spectral density.

2. Experimental Setup

The experiment has been run in a large wind tunnel (figure 1), with a measurement section of 0.75 m x 0.75 m where the turbulence is generated downstream a grid and reproduces almost ideal isotropic conditions. The cable used to tow the sphere has been chosen as thin and light as possible to minimize its inertia compared that of the particle. It has a maximum lineal density
of 20 $\mu g/cm$ and is composed of three filaments of Polyamide-Nylon with a diameter of 25 $\mu m$ each. The cable is 2.08 m long. It is attached to the grid at one end, and to a particle at the other end. As a consequence, particle's streamwise motion is blocked, but it is free to move in the transverse directions $x$ and $y$. Results reported here were obtained with a mean velocity of $10m/s$, corresponding to a Reynolds number of the carrier flow of $R_\lambda \sim 150$ (based on Taylor microscale) with typical scales: $L = 5.5$ cm (injection scale), $\eta = 260$ $\mu$m (dissipation scale) and $\tau_\eta = 4.4$ ms (dissipation time scale).

Seven classes of spherical particles, with different sizes and densities were investigated. Table 1 summarizes their main characteristics. Particles inertia is quantified in terms of their Stokes number $St = \tau_p/\tau_\eta$ where the viscous time $\tau_p$ is estimated based on empirical observations of drag coefficient dependence on particles Reynolds number $Re_p = |u - v|d/\nu$ [2], with $d$ the particle diameter.

Table 1. Particle classes studied in this work.

| Particle # | Material       | Diameter (mm) | Density (kg/m$^3$) | $St$ | $Re_p$ | $\tau_p$ (s) |
|------------|----------------|---------------|--------------------|------|--------|--------------|
| 1          | Expanded Polystyrene | 2.08          | 36.1               | 4.6  | 1370   | 0.036        |
| 2          | Expanded Polystyrene | 3.05          | 24.2               | 4.8  | 2010   | 0.026        |
| 3          | Expanded Polystyrene | 6.29          | 13.4               | 5.8  | 4150   | 0.027        |
| 4          | Expanded Polystyrene | 4.3           | 23.1               | 6.7  | 2840   | 0.030        |
| 5          | Plastic         | 3.11          | 1100               | 220  | 2050   | 0.966        |
| 6          | Plastic         | 2.43          | 1900               | 290  | 1180   | 2.18         |
| 7          | Lead            | 1.73          | 9130               | 950  | 1110   | 4.06         |

3. Post-processing

3.1. Trajectories

As the particles are large, their position is efficiently determined frame by frame with sub-pixel accuracy with a simple center of mass analysis. The conversion from pixels to length units is previously calibrated using a known printed patron (mask) which allows to determine the projective transformation between pixels and real world units. As only one particle is present in the field of view, tracking and trajectory reconstruction is trivial, leading for each recording to particle tracks as represented in the inset in figure 1.

3.2. Spectral analysis

Spectral analysis will be a key ingredient in the present work for the characterization of these trajectories. We present indeed in the following an original approach based on spectral analysis which allows to accurately estimate statistical quantities (variances of velocity and acceleration), minimizing biases from noise and avoiding data filtering with arbitrary parameters, what is a
Figure 2. (a) Spectra for particle with $St = 4.8$. The blue line is the position spectrum, the red and green lines are the velocity and acceleration spectra, respectively. The black line represents the velocity and accelerations spectra calculated from the position one multiplied by $(2\pi\nu)^2$ for the velocity and by $(2\pi\nu)^4$ for the acceleration. Dashed-lines represent power-law behaviors at high frequency before noise level is reached (exponent is 5.3 for position, 3.3 for velocity and 1.3 for acceleration) ; $\nu_c$ is the cut-off frequency above which the signal is modeled for the estimation of variances. (b) Raw trajectory (blue line) and trajectory low-pass filtered representing only the resolved dynamics below $\nu_c$ (red line).

recurrent issue when it comes to analyze Lagrangian data using for instance gaussian kernels [8] or local parabolic fits [12]. We also consider spectral analysis to access to the energy transfer function $H^2 = \frac{\langle \dot{\theta}^2 \rangle}{\langle \dot{\phi}^2 \rangle}$ between the dynamics of the particle and the carrier flow.

For a given experimental configuration Lagrangian spectrum of position, velocity and acceleration components are first estimated for each individual trajectory. At this point, velocity and acceleration are simply calculated as time differences of the position, without any filtering procedure. This is intentionally do so, in order to keep a maximum of spectral information, including information on noise level (which is very sensitive to differentiation), usually withdrawn by filtering the position. We then average all the individual spectra to obtain an estimation of the Lagrangian spectra of each class of particle. Figure 2a shows such typical spectra for particle position, velocity and acceleration. We also note that velocity and acceleration spectra can be consistently estimated from position’s spectrum by simply multiplying it by $(2\pi\nu)^2$ and $(2\pi\nu)^4$ respectively (from now on $\nu$ is the frequency variable in Hz in Fourier space). Physical results from these spectra will be discussed in the following sections, but for post-processing purposes we note here a common feature observed on the spectra for all the trajectories recorded in the present study : it has been observed in all the spectra that at high frequency they follow a power law energy dissipation cut-off, as represented by the dashed-line in figure 2a. Deviation at high frequencies from this power law appears when noise level is reached.

3.3. A new estimator of variances and correlation functions

Classically, noise is removed by low-pass filtering the signal at an optimal frequency cut-off $\nu_n$ which preserves most of the signal and eliminates most of the noise. An usual way is to filter the signal using a gaussian kernel (derivatives of gaussian kernels are used to filter and differentiate the signal simultaneously), where the kernel width is spanned and the optimal value is choosen based on the estimation of the variance of the filtered acceleration [8] (in the present data, such optimal filtering frequency would lie somewhere between 200 Hz and 300 Hz). In spite of the
optimization of the choice of this cut-off frequency, the filtering process necessarily results in an underestimation of fluctuation level, as energy at high frequency is removed simultaneously with the noise. Though this is generally negligible for position or velocity fluctuations which are essentially large scale quantities (with a sharply decreasing spectrum at small scales), it can become a significant bias for the estimation of acceleration variance requiring empirical compensations. For instance, the rule of the thumb to estimate the variance of Lagrangian acceleration of tracers from particle tracking measurements is to take 80% of the value of the linear extrapolation to zero of the trend with kernel width of the variance of filtered acceleration. Such empirical procedures for the estimation of Lagrangian variances are a recurrent problem in the community.

Here we propose a new estimation of variances of position, velocity and acceleration, based on the analysis of the signal spectrum. We first recall that the variance of a given quantity $q$ can be estimated from its spectrum as $\sigma_q^2 = \int |\hat{q}^2(\nu)|d\nu$. Therefore, instead of simply filtering arbitrarily the signal what drastically eliminates the high frequency contribution to the total variance, we propose to model the high frequency behavior of the spectra using the best available empirical trend, based for instance on the measured spectrum at high frequency but before the onset of noise. This approach is similar to LES (Large Eddy Simulation): we consider the scales above the experimental noise level as being resolved and we choose to model the unresolved small scales. In practice the actual cut-off frequency $\nu_c$ defining the resolved and modeled scales will be chosen for scales slightly larger than the noise limit, as illustrated in figure 2a. In the present study, the small scale model for unresolved scales is empirically chosen based on the power-law behavior observed for the smallest resolved scales. Compared to simple classical filtering approach, the benefit here is that, though we cannot completely warranty the extent of validity of the small scale model, we do not simply withdraw high frequency contributions but we model them with an empirical and non arbitrary correction. This approach is also convenient as the estimation of variances for velocity and acceleration can be mostly obtained from information on position only, as the spectra of velocity and acceleration are trivially obtained from that of position, thus reducing errors from signal differentiation.

In the end, the variance of position, velocity and acceleration is estimated as the contribution of the resolved scales (which can be equivalently obtained either as the integral of the spectrum

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**Figure 3.** (a) Correlation function obtained with the non-filtered spectra (blue line) and with the filtered one (red line). (b) Typical velocity spectrum of a particle (blue line), flow spectrum from hot-wire anemometry (green line) and energy transfer function (red line) obtained as the ration of the previous.
up to $\nu_c$ or as the variance of the resolved temporal signal low-pass filtered below $\nu_c$ (see figure 2b) to which we add the contribution of the small scales estimated as the integral above $\nu_c$ of the model for the high frequency unresolved spectrum. When the high frequency model for the position spectrum is a power law $A\nu^\alpha$, the standard deviations can be calculated using the following formulas:

$$\sigma_x = \sqrt{\langle x^2_\text{f} \rangle + \frac{A}{\alpha + 1} \left( \left( \frac{\nu_s}{2} \right)^{\alpha+1} - \nu_c^{\alpha+1} \right) - \langle x \rangle^2}, \quad (2)$$

$$\sigma_{vx} = \sqrt{\langle v_x^2 \rangle + \frac{A}{\alpha + 3} \left( \left( \frac{\nu_s}{2} \right)^{\alpha+3} - \nu_c^{\alpha+3} \right) - \langle v_x \rangle^2}, \quad (3)$$

$$\sigma_{ax} = \sqrt{\langle a_x^2 \rangle + \frac{A}{\alpha + 5} \left( \left( \frac{\nu_s}{2} \right)^{\alpha+5} - \nu_c^{\alpha+5} \right) - \langle a_x \rangle^2}, \quad (4)$$

where the subindex $f$ is used to indicate the resolved low-pass filtered signal and $\nu_s$ is the sampling frequency of the acquisitions. The values $A$ and $\alpha$ are obtained fitting high frequency behavior the spectrum of position modeled as $\|\hat{x}\|^2(\nu) = A\nu^\alpha$.

Finally, we point that the same LES approach also gives an appropriate way to obtain clean estimations of Lagrangian correlation functions, in particular for acceleration. Such estimation is indeed generally polluted by the small scale noise, which produces spurious oscillations at short time lags (see figure 3a showing the correlation function of acceleration for the raw signal, corresponding to the acceleration spectrum in figure 2). Removing such oscillations is usually achieved by filtering the signal in order to eliminate the noise, what generally also results in a widening of the correlation peak what may bias the estimation of the correlation time of acceleration. In the LES approach, we can estimate the correlation function as the inverse Fourier transform of the corrected spectrum built as the actual resolved spectrum for frequencies below $\nu_c$ and extrapolated based on the adopted model for frequencies above $\nu_c$. The resulting correlation function, shown in figure 3a does not exhibit any more the spurious oscillations and preserves the characteristic correlation time scales.

3.4. Transfer functions

As already mentioned, the present study is highly motivated by the fact that the effect of particles inertia is usually described as a simple filtering effect, where the dynamics of the particle only experiences turbulent solicitations from the carrier flow at time scales typically slower than the viscous response time of the particle $\tau_p$. This is for instance a direct theoretical result of the Tchen-Hinze theory [11], where the Stokesian dynamics of inertial particles freely advected by a turbulent velocity field $\vec{u}$, modeled by eq. 1 directly acts as a first-order low-pass filter where particles energy spectrum is related to fluid’s via an energy transfer function $H^2_{TH} = \frac{\|\hat{v}\|^2}{\|\hat{u}\|^2} \propto 1/(1 + \nu^2 \tau_p^2)$ with a low-pass cut-off frequency $\nu_{lp} = \tau_p^{-1}$. In a more general context, the transfer function approach gives an interesting insight in the coupling between particle and carrier flow dynamics [7].

In the coming sections, we will therefore discuss the experimental energy transfer function

$$H^2 = \frac{\|\hat{v}\|^2}{\|\hat{u}\|^2}, \quad (5)$$

where $\|\hat{v}\|^2$ is the particle Lagrangian velocity spectrum and $\|\hat{u}\|^2$ is temporal spectrum of the the background flow measured with classical hot-wire anemometry in the plane of motion of
4. Results and discussion

4.1. Statistical Analysis

Figures 4a&b represent velocity and acceleration probability density functions (PDFs) centered and normalized to variance one for all the particles investigated. Remarkably, all the PDFs collapse onto a single curve which is found to be well fitted by a Gaussian distribution. While gaussianity of velocity statistics was already observed for freely advected particles, gaussianization of acceleration is an interesting finding. Recent investigations made for freely advected finite size particles [10, 13] have shown that the acceleration PDFs remains in that case highly non-Gaussian even for particles with large inertia. This observation has been noted as being in contrast with the prediction from Stokesian models (with a dominant drag force due to particles/fluid relative motion) of the turbulent dynamics of particles which predict a gaussianization of acceleration PDFs for highly inertial particles [1]. Our present measurements show that such gaussianization is indeed observed when a strong dominant drag force is exerted on the particle, therefore in qualitative agreement with Stokesian models.

As a result of their gaussianity, velocity and acceleration fluctuations are therefore simply characterized by their standard deviations $\sigma_v$ and $\sigma_a$. Figure 5a represents the velocity turbulent fluctuations rate for the horizontal and vertical components of the velocity $\tau_{v_{x,y}} = \sigma_{v_{x,y}}/U$ as a function of particles Stokes number (note that both component exhibit almost the same trend, as a result of the good isotropy level of the grid generated turbulence). Interestingly, we observe that the turbulence fluctuation level $\tau_{v_{x,y}}$ decreases with increasing inertia. This is in contrast with measurements for freely advected particles [10] where the turbulence fluctuation level has been measured not to depend on particles inertia. Moreover the decrease of turbulence level is in qualitative agreement with a low-pass filtering effect due to particle inertia as predicted with drag dominated models in the spirit of the Tchen-Hinze model [11]. We note that acceleration variance is also observed to decrease with increasing inertia (figure 5b), though this was also observed for freely advected particles.
4.2. Transfer functions analysis

In order to probe further the filtering scenario, we investigate the power spectra of particles velocity as well as the transfer functions between particle’s and carrier flow’s energy. Figure 6a shows the spectra for all the 7 different particle classes investigated in the present work. We observe that at low frequencies spectra are not significantly dependent on particles inertia while on the contrary high frequency cut-off occurs earlier as particles Stokes number increases, consistently with a filtering scenario due to particles inertia. For very large inertia, the spectra becomes relatively peaked with a dominant frequency, corresponding to a pendulum mode for the heaviest towed particles. Such oscillatory modes have also been observed in numerical studies of heavy towed objects in laminar conditions [9].

Figures 6b&c show two typical energy transfer functions $H^2 = \left| \frac{\hat{\nu}_x}{\hat{\nu}_y} \right|^2 / \left| \frac{\hat{\nu}_x}{\hat{\nu}_y} \right|^2$, one for low inertia particles the other for large inertia. The global shape of this transfer function is relatively well approximated as a second order band-pass filter:

$$H^2 = \frac{A}{1 + Q^2 \left( \frac{\nu}{\nu_0} - \frac{\nu_0}{\nu} \right)}, \quad \text{(6)}$$

with $A$ the gain factor, $Q$ the quality factor and $\nu_0$ the central frequency. The second order filter can be seen as a combination of a first order high-pass and a first order low-pass filter and can be equivalently characterized by the corresponding low-pass and high-pass cut-off frequencies, $\nu_{lp}$ and $\nu_{hp}$ such as $\nu_0 = \sqrt{\nu_{hp}\nu_{lp}}$ and $Q = \nu_0 / (\nu_{lp} - \nu_{hp})$.

The fact that we observe a second-order band-pass behavior rather than just a first order low-pass behavior (as in classical Stokesian models for freely advected particles) is related to the fact that in the present configuration the motion of the particles is constrained by the cable. As a consequence the high-pass part of the transfer function (relative to the low frequency behavior) is to be related to the interaction of the particle with the towing cable which constrains the motion to be bounded and imposes the average velocity of the particle (and hence the density of energy at vanishing frequency $\nu \to 0$) to be null. On the other hand, the low-pass part of the transfer function (related to the high frequency behavior) is expected to be related to particles inertia.

This is supported by figure 7 which shows the trend with Stokes number of the high-pass and low-pass cut-off frequencies determined from the fit of the measured transfer function with a 2nd
order band-pass filter model as defined by eq. 6. We can observe that the high-pass frequency is independent of particle inertia, while the low-pass frequency does decrease with increasing particles inertia.

5. Conclusions

This work aims at investigating fundamental features of drag force acting on a particle in a turbulent environment, with the particular goal to emphasize the filtering mechanism due to particles inertia, whose role has not been clearly identified for the case of freely advected particles. This has been accomplished by considering a semi-constrained particle dynamics, where the particle cannot follow flow structures and where a dominant drag is imposed by a strong particle-fluid slippage velocity. At the same time, this arrangement is particularly interesting as it is analog to towed systems, which are of primary importance for instance in aeronautics or acoustic streamers in the ocean. As far as we know, our results give the first clear experimental observation of a filtering effect on the Lagrangian dynamics of finite size particles in turbulent flow induced by their inertia, consistent with usual Stokesian models. However, at this point the agreement is only qualitative as a more quantitative comparison would require a refinement of the usual model in eq. 1 to include the cable tension (and gravity) which constrains the motion of the particle. The fact that the cable itself is also submitted to turbulent fluctuations further complicates the problem. These interaction with the towing cable (which can for instance, on one hand enhance dissipation due to the cable stiffness, and on the other hand enhance fluctuations as it transmits to the particle part of its fluctuating energy) may explain for instance that at high frequencies the energy transfer function is not perfectly fitted by a first order low-pass filter and that the low-pass cut-off frequency represented in figure 7 does not simply scale as the inverse of particles response time $\tau_p^{-1}$ –or equivalently as $St^{-1}$ – as it would be expected for a simple Stokes drag. These questions are under investigation by ongoing experiments at different intensity of turbulence and trying also to capture the dynamics of the cable alone (without the towed particle).

References

[1] Jeremy Bec, Luca Biferale, Guido Boffetta, Antonio Celani, Massimo Cencini, Alessandra Lanotte, S. Musacchio, and Federico Toschi. Acceleration statistics of heavy particles in
Figure 7. (a) High-pass and low-pass frequencies obtained from the 2nd order band-pass model (eq. 6) fit of the energy transfer function for all particles investigated, plotted as a function of Stokes number.