On Combinatorial Expansions of Conformal Blocks

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In a recent paper [1] the representation of Nekrasov partition function in terms of nontrivial two-dimensional conformal field theory has been suggested. For non-vanishing value of the deformation parameter $\epsilon = \epsilon_1 + \epsilon_2$ the instanton partition function is identified with a conformal block of Liouville theory with the central charge $c = 1 + 6\epsilon_2^2/\epsilon_1\epsilon_2$. If reversed, this observation means that the universal part of conformal blocks, which is the same for all two-dimensional conformal theories with non-degenerate Virasoro representations, possesses a non-trivial decomposition into sum over sets of the Young diagrams, different from the natural decomposition studied in conformal field theory. We provide some details about this intriguing new development in the simplest case of the four-point correlation functions.

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1 Introduction

Recent developments in the study of multidimensional nontrivial quantum field theories are strongly influenced by their resemblance, at least, in particular issues, to certain features of two-dimensional (2d) quantum field theories. A well-known example of such correspondence is the 2d free field representation of tau-functions, which appear as effective actions of multidimensional quantum field theories. The universality class of multidimensional quantum field theories, which can be reformulated in terms of tau functions of certain integrable systems, include a very interesting set of supersymmetric gauge theories, where the effective actions are described by Seiberg-Witten prepotentials [2]-[4].

Nekrasov partition functions [5]-[12] generalize the Seiberg-Witten prepotentials and originally arose from evaluation of instantonic sums in deformed $\mathcal{N} = 2$ supersymmetric gauge theories. In the language of integrable systems they look quite similar to combinatorial or character representations for the tau-functions [13], whose quasiclassical counterparts appeared in description of the Seiberg-Witten theory in terms of integrable systems [3, 14, 15]. A non-trivial feature of Nekrasov’s functions from this perspective is their dependence on extra parameters $\epsilon_1, \epsilon_2$ (it is common also to introduce $\epsilon = \epsilon_1 + \epsilon_2$), which come from the so called $\Omega$-background, providing the IR regularization of integrals over instanton moduli spaces [16, 17], but still lacking a clear group-theoretical interpretation. At $\epsilon = 0$, the Nekrasov partition functions can be represented in terms of free fermions, or two-dimensional conformal theory with the central charge $c = 1$ [7, 8, 18, 19], and overlap partially with particular cases of the Hurwitz-Kontsevich functions [20, 21], where sums over the Young diagrams come from combinatorics of coverings and appear finally in the well known form of group characters. However, the group theory meaning of the deformation to $\epsilon \neq 0$ remained obscure, though some formulas were found that preserve the nice properties of the $\epsilon = 0$ case (see [9] for the most interesting examples), and signs of relation to nontrivial conformal theories with $c \neq 1$ can be seen already in the relatively old papers like [22].

Therefore, a recent observation of [1] which makes this relation explicit is extremely important. It reveals deep connections between the Seiberg-Witten theory, Nekrasov partition functions and two-dimensional conformal theories in a very transparent, but somewhat mysterious form. The claim of [1] relates the Liouville CFT model with central charge $c = 1 + 6\epsilon_2^2/\epsilon_1\epsilon_2$ to a peculiar quiver Nekrasov partition function for the gauge theories with the powers of $U(2)$ gauge group. Particular quiver diagrams are associated with particular (multipoint) conformal blocks, while the gauge group $SU(2)$ is somehow related to the chiral algebra: the Virasoro algebra or actually $\hat{S}L(2)$ [23, 24] in the case of Liouville theory. Larger gauge groups are presumably related to CFT models with more free fields and higher $W$-like symmetries [25].

Unfortunately, while the statements in [1] are very explicit and clear, the underlying checks that were actually made are not described in an equally transparent form. It took us some effort to reproduce the argument in the simplest case of the four-point conformal block, and we present this calculation here, in a hope that this can help others to easier join this new promising line of research.

We begin from reminding the classical results [26, 27] about conformal blocks in $2d$ conformal theory and then turn to consideration of the observation of [1] about their relation to Nekrasov’s partition functions. Exact statement is that conformal block for generic (i.e.non-degenerate) Verma modules depending on five arbitrary dimensions $\Delta_i = \alpha_i(\epsilon - \alpha_i)/\epsilon_1\epsilon_2$ is exactly equal to certain linear combination of Nekrasov’s functions, see eq.(51). This generalizes non-trivially the well-known statement in the case of the free fields (see [7] and eq.(49)), where dimensions are restricted by two linear constraints: $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = \epsilon$ and intermediate-channel $\alpha = \alpha_1 + \alpha_2$. In other words, a switch to the $\alpha$-parametrization of dimensions appears to convert conformal blocks for Virasoro-Verma modules into a deformation of character-decomposition formulas, establishing a one more realization of relation between the Virasoro and $S\hat{L}(2)$ algebras.
2 Conformal block [27]

2.1 Conformal block from the operator expansion

The main ingredients of two-dimensional conformal field theory (2d CFT) are the states in the Hilbert space, their scalar product, and the structure constants of the operator algebra. The states are associated with the operators $V_\alpha(z)$, and the scalar product is determined by the two-point functions

$$K_{\bar{\alpha}\beta} = \langle \bar{\alpha}|\beta \rangle \sim \langle V_\alpha(0)V_\beta(\infty) \rangle,$$

(1)

(where $V_\beta(\infty) = \lim_{R\to\infty} R^{2\Delta_\beta}V_\beta(R)$ with $\Delta_\beta$ being dimension of the operator, or quantum number of the corresponding state). The operator product expansion (OPE) is defined as

$$V_\alpha(z)V_\beta(z') = \sum_\gamma \frac{C_{\bar{\alpha}\beta}^\gamma V_\gamma(z')}{(z-z')^{\Delta_\alpha+\Delta_\beta-\Delta_\gamma}}$$

(2)

Because of the freedom to choose the argument of $V_\gamma$ at the r.h.s. the structure constants $C$ are actually defined modulo triangular transformation in the space of $V_\gamma$ and their derivatives. Of course once the choice is made, like in [2], $C_{\bar{\alpha}\beta}^\gamma$ are defined unambiguously. Advantage of the asymmetric choice made in [2] will be seen in eq.(11) below.

In 2d CFT the Hilbert space of states can be described in terms of representations of conformal symmetry, generated by the conserved stress-energy tensor $\partial T(z) = 0$, i.e. the set of Verma modules, growing from distinguished (primary or highest-weight) states [26]. Actually $\hat{\alpha} = \{\alpha, Y\}$ is a multi-index, where $\alpha$ labels the primary states while $Y$ – the Young diagram (integer partition) $Y = \{k_1 \geq k_2 \geq \ldots \geq k_l > 0\}$ labels their descendants, obtained by the action of components of the stress-energy tensor, or the Virasoro generators

$$T(z) = \sum_{n=-\infty}^{\infty} \frac{L_n}{z^{n+2}}$$

(3)

on the primary state $V_\alpha = V_{\alpha,\emptyset}$:

$$V_{\alpha, Y} = L_{-Y}V_\alpha = L_{-k_1} \cdots L_{-k_l}L_{-k_1}V_\alpha$$

(4)

or the highest weight state: $L_kV_\alpha = 0$, for $k > 0$, and $L_0V_\alpha = \Delta_\alpha V_\alpha$, so that the generators $L_n$ from (3) with non-negative $n \geq 0$ do not show up in (1). Similarly, $[L_{-1}, V_\alpha(z)] = \partial V_\alpha(z)$. Dimension of the descendant is

$$\Delta_{\bar{\alpha}} = \Delta_{\alpha, Y} = \Delta_\alpha + |Y|$$

(5)

where $|Y| = k_1 + k_2 + \ldots + k_l$ is the size, i.e. the number of boxes of the Young diagram. Any combination of Virasoro generators can be brought to the ordered form (4) with the help of the Virasoro commutation relation

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}n(n^2-1)\delta_{m+n,0}$$

(6)

where $c$ is the central charge of the theory.

Virasoro symmetry implies that the norms and structure constants are uniquely defined by those for primary states:

$$K_{\bar{\alpha}\bar{\alpha}'\bar{\alpha}''} = K_{\alpha, Y; \alpha', Y'} = K_\alpha \delta_{\alpha,\alpha'} \delta_{|Y|,|Y'|} Q_{\Delta}(Y, Y'),$$

$$C_{\bar{\alpha}\bar{\alpha}'}^{\bar{\alpha}''} = C_{\alpha\alpha'}^{\alpha''} \delta_{\Delta,\Delta'}(Y, Y'; Y'')$$

(7)

\[1\] We shall concentrate in this text, only on the holomorphic (chiral) constituents (conformal blocks) of the correlation functions. Physical correlators are bilinear combinations of conformal blocks and possess global (modular) symmetry.
and \( Q \) and \( \beta \) are objects from representation theory of the Virasoro algebra, entirely independent of the properties of particular conformal model – those properties are fully concentrated in \( K \) and \( C \). \( Q \) and \( \beta \) depend only on dimensions and the central charge of the model and are important special functions of conformal field theory.

Practical evaluation of \( Q \) and \( \beta \) is a straightforward but tedious problem. By definition, \( Q_\Delta(Y,Y') \) is the pair correlator of two descendants:

\[
Q_\Delta(Y,Y') = \langle \Delta | L_\Delta L_{-\Delta} | \Delta \rangle \sim \langle L_{-\Delta} V_\Delta (0) L_{-\Delta} V_\Delta (\infty) \rangle
\]

As for \( \beta \), the simplest way is to express it through \( \gamma \), a counterpart of \( \beta \) for the three-point function:

\[
\gamma_{\Delta,\Delta',\Delta''}(Y,Y',Y'') = \sum_{\tilde{Y}} \beta_{\Delta,\Delta'}^{\Delta''}(Y,Y';\tilde{Y}) Q_{\Delta''}(\tilde{Y},Y'') = \langle L_{-\Delta} V_\Delta (0) L_{-\Delta} V_\Delta (1) L_{-\Delta} V_\Delta (\infty) \rangle
\]

Moreover, in what follows for practical calculations we only need the particular cases \( \beta_{\Delta_1,\Delta_2}^{\Delta_3}(Y) = \beta_{\Delta_1,\Delta_2}^{\Delta_3}(0,0;Y) \) and

\[
\gamma_{\Delta_1,\Delta_2}(Y) = \gamma_{\Delta_1,\Delta_2}(0,0;Y) = \sum_{\tilde{Y}} \beta_{\Delta,\Delta}^{\Delta_1,\Delta_2}(\tilde{Y}) Q_\Delta(\tilde{Y},Y)
\]

Given (2), the four-point function can be written as follows:

\[
\langle V_{\tilde{\alpha}_1}(z_1)V_{\tilde{\alpha}_2}(z_2)V_{\tilde{\alpha}_3}(z_3)V_{\tilde{\alpha}_4}(z_4) \rangle \overset{2}{=} \sum_{\tilde{\alpha}_{12},\tilde{\alpha}_{34}} \frac{\beta_{\tilde{\alpha}_{12}}^{\tilde{\alpha}_{34}}}{(z_1 - z_2)^{\tilde{\alpha}_{12} + \tilde{\alpha}_{34}}(z_3 - z_4)^{\tilde{\alpha}_{34} - \tilde{\alpha}_{12}}} \langle V_{\tilde{\alpha}_{12}}(z_2)V_{\tilde{\alpha}_{34}}(z_4) \rangle
\]

If we now put \( z_2 = \infty \) and \( z_4 = 0 \), then the two-point function at the r.h.s. turns into the scalar product (2) – this is what justifies the asymmetric definition in (2). It remains to put \( z_1 = 1 \), \( z_3 = x \), take primaries for the four original operators, i.e. put \( \tilde{\alpha}_i = \{ \alpha_i, \emptyset \} \) for \( i = 1, 2, 3, 4 \), and make use of the factorization formulas (7). Finally we pick up a contribution of particular intermediate channel \( \alpha_{12} = \alpha_{34} = \alpha \) in the sum at the r.h.s. of (11) – this gives the definition of the 4-point conformal block:

\[
F_\Delta(\Delta_1,\Delta_2;\Delta_3,\Delta_4|x) \equiv x^\sigma (C_{\alpha_1,\alpha_2}^\alpha C_{\alpha_3,\alpha_4}^\alpha) B_\Delta(\Delta_1,\Delta_2;\Delta_3,\Delta_4|x)
\]

where \( \sigma \) is a simple combination of dimensions which we do not need in his paper, and \( B \) is a pure representation-theory quantity:

\[
B_\Delta(\Delta_1,\Delta_2;\Delta_3,\Delta_4|x) = \sum_k x^k B_\Delta^{(k)} = \sum_{|Y|=|Y'|} x^{|Y|} B_{Y,Y'} =
\]

\[
= \sum_{|Y|=|Y'|} x^{|Y|} \beta_{\Delta_1,\Delta_2}^{\Delta} Q_{\Delta}(Y,Y') \beta_{\Delta_3,\Delta_4}^{\Delta}(Y')
\]

\[
= \sum_{|Y|=|Y'|} x^{|Y|} \gamma_{\Delta_1,\Delta_2}(Y) Q_{\Delta}^{-1}(Y,Y') \gamma_{\Delta_3,\Delta_4}(Y')
\]

\[\text{\footnote{This correlator behaves as } \Lambda^{2\Delta-2|Y'|}}, \text{ \footnote{when the argument } \Lambda \text{ of } V_\Delta(\Lambda) \text{ tends to infinity. One can multiply both sides of the equation by } \Lambda^{2\Delta-2|Y'|} \text{ and then take the limit, to make the correlator well defined. Similarly are defined other correlators involving } V(\infty), \text{ only the correction factor has to be properly adjusted in each case.}}\]

\[\text{\footnote{Note that because of asymmetry in the definitions (2) and (9), } \beta \text{ and } \gamma \text{ are not symmetric functions even of } Y \text{ and } Y' \text{ (since three points in these formulas are fixed in a special way). This asymmetry is, however, convenient for the next step: it simplifies formulas for correlators and conformal blocks.}}\]

[2]
Note that even when the channel is fixed, the sums survive over Virasoro descendants $V_{\hat{\alpha}}$ of the primary field $V_\alpha$, moreover these descendants can be different in $\hat{\alpha}_{(12)} = \{\hat{\alpha}, Y\}$ and $\hat{\alpha}_{(34)} = \{\hat{\alpha}, Y'\}$, therefore \((13)\) is actually a \textit{double} sum over Young diagrams. We emphasize once again that $\beta$ and $\gamma$ are highly asymmetric functions of the three dimensions in their arguments.

Similarly one can define multi-point conformal blocks. For this one needs to know a more general quantity than $\gamma(Y)$ from the Virasoro-group theory: $\gamma(Y_1, Y_2, \emptyset)$ which depends on two different Young diagrams. \textit{Two} reduces to \textit{one} for the four-point functions. We leave generic conformal blocks beyond consideration in this short text (but note that AGT conjecture \[\Pi\] is made in full generality and involves multi-point conformal blocks).

Many considerations in CFT involve Verma modules with null-vectors, when relevant representations of the Virasoro algebra are actually factors of Verma modules – then formula like \((13)\) involve sums over these factors only and modified quantities appear there instead of $K$ and $\beta$. As in \[\Pi\], we consider here only the generic case of non-degenerate Verma modules.

### 2.2 The 4-point conformal block for arbitrary Verma module

In order to evaluate $B$ from \((13)\) one needs to know just two ingredients: the triple vertex $\gamma$ and Shapovalov form $Q$ for Virasoro algebra. They are described by the simple formulas (in a slightly abbreviated notation for $Q_\Delta([k_1 k_2 \ldots],[k'_1 k'_2 \ldots])$):

\[
Q_\Delta(Y, Y') = \quad (14)
\]

| $Y/Y'$ | $\emptyset$ | $[1]$ | $[11]$ | $[3]$ | $[21]$ | $[111]$ | \ldots |
|--------|------------|-------|--------|-------|--------|--------|-------|
| $\emptyset$ | 1 | \multicolumn{5}{c|}{} |
| $[1]$ | $2\Delta$ | \multicolumn{5}{c|}{} |
| $[2]$ | $\frac{1}{2}(8\Delta + c)$ | $6\Delta$ | \multicolumn{3}{c|}{} |
| $[11]$ | $6\Delta$ | $4\Delta(1 + 2\Delta)$ | \multicolumn{5}{c|}{} |
| $[3]$ | \multicolumn{3}{c|}{$6\Delta + 2c$} | $2(8\Delta + c)$ | $24\Delta$ | \multicolumn{2}{c|}{} |
| $[21]$ | \multicolumn{3}{c|}{$2(8\Delta + c)$} | $8\Delta^2 + (34 + c)\Delta + 2c$ | $36\Delta(\Delta + 1)$ | \multicolumn{2}{c|}{} |
| $[111]$ | \multicolumn{3}{c|}{} | $24\Delta$ | $36\Delta(\Delta + 1)$ | $24\Delta(\Delta + 1)(2\Delta + 1)$ | \multicolumn{2}{c|}{} |
| \ldots | \multicolumn{7}{c|}{} |

and for arbitrary three primaries

\[
\gamma_{\Delta_1, \Delta_2;\Delta}(Y) = \prod_i \left( \Delta + k_i \Delta_1 - \Delta_2 + \sum_{j<i} k_j \right) \sim \langle L_{-Y} V_{\Delta}(0) V_{\Delta_1}(1) V_{\Delta_2}(\infty) \rangle \quad (15)
\]
In particular,
\[ \gamma_{\Delta_1, \Delta_2; \Delta}[1] = \Delta + \Delta_1 - \Delta_2, \]
\[ \gamma_{\Delta_1, \Delta_2; \Delta}[2] = \Delta + 2\Delta_1 - \Delta_2, \]
\[ \gamma_{\Delta_1, \Delta_2; \Delta}[11] = (\Delta + \Delta_1 - \Delta_2)(\Delta + \Delta_1 - \Delta_2 + 1), \]
\[ \gamma_{\Delta_1, \Delta_2; \Delta}[3] = \Delta + 3\Delta_1 - \Delta_2, \]
\[ \gamma_{\Delta_1, \Delta_2; \Delta}[21] = (\Delta + 2\Delta_1 - \Delta_2)(\Delta + \Delta_1 - \Delta_2 + 2), \]
\[ \gamma_{\Delta_1, \Delta_2; \Delta}[111] = (\Delta + \Delta_1 - \Delta_2)(\Delta + \Delta_1 - \Delta_2 + 1)(\Delta + \Delta_1 - \Delta_2 + 2), \]
\[ \vdots \]
\[ \gamma_{\Delta_1, \Delta_2; \Delta}[n] = \Delta + n\Delta_1 - \Delta_2, \]
\[ \vdots \]
\[ \gamma_{\Delta_1, \Delta_2; \Delta}[1^n] = (\Delta + \Delta_1 - \Delta_2)(\Delta + \Delta_1 - \Delta_2 + 1) \cdots (\Delta + \Delta_1 - \Delta_2 + n - 1) \]

Substituting these explicit formulas into (13), we obtain:

\[ B^{(0)} = 1 \]
\[ B^{(1)}_\Delta = \frac{(\Delta + \Delta_1 - \Delta_2)(\Delta + \Delta_3 - \Delta_4)}{2\Delta} \]
\[ B^{(2)}_\Delta = \frac{(\Delta + \Delta_1 - \Delta_2)(\Delta + \Delta_1 - \Delta_2 + 1)(\Delta + \Delta_3 - \Delta_4)(\Delta + \Delta_3 - \Delta_4 + 1) +}{4\Delta(2\Delta + 1)} + \frac{[(\Delta_2 + \Delta_1)(2\Delta + 1) + \Delta(\Delta - 1) - 3(\Delta_2 - \Delta_1)^2][((\Delta_3 + \Delta_4)(2\Delta + 1) + \Delta(\Delta - 1) - 3(\Delta_3 - \Delta_4)^2]}{2(2\Delta + 1)(2\Delta(8\Delta - 5) + (2\Delta + 1)c)} \]
\[ B^{(3)}_\Delta = \frac{1}{2\Delta(3\Delta^2 + c\Delta - 7\Delta + 2 + c)} \left[ (\Delta + 3\Delta_1 - \Delta_2)(\Delta^2 + 3\Delta + 2)(\Delta + 3\Delta_3 - \Delta_4) - 2(\Delta + 3\Delta_1 - \Delta_2)(\Delta + 1)(\Delta + 2\Delta_3 - \Delta_4)(\Delta + \Delta_3 - \Delta_4 + 2) + + (\Delta + 3\Delta_1 - \Delta_2)(\Delta + \Delta_3 - \Delta_4)(\Delta + \Delta_3 - \Delta_4 + 1)(\Delta + \Delta_3 - \Delta_4 + 2) - 2(\Delta + 2\Delta_1 - \Delta_2)(\Delta + \Delta_1 - \Delta_2 + 2)(\Delta + 1)(\Delta + 3\Delta_3 - \Delta_4) + + (\Delta + \Delta_1 - \Delta_2)(\Delta + \Delta_1 - \Delta_2 + 1)(\Delta + \Delta_1 - \Delta_2 + 2)(\Delta + 3\Delta_3 - \Delta_4) + + 2(\Delta + 2\Delta_1 - \Delta_2)(\Delta + \Delta_1 - \Delta_2 + 2)(6\Delta_3^3 + 9\Delta_2^2 - 9\Delta + 2c\Delta^2 + 3c\Delta + c)(\Delta + 2\Delta_3 - \Delta_4)(\Delta + \Delta_3 - \Delta_4 + 2)}{16\Delta^2 + 2(c - 5)\Delta + c} \]
\[ - \frac{(\Delta + 2\Delta_1 - \Delta_2)(\Delta + \Delta_1 - \Delta_2 + 2)(9\Delta_2^2 - 7\Delta + 3c\Delta + c)(\Delta + \Delta_3 - \Delta_4)(\Delta + \Delta_3 - \Delta_4 + 1)(\Delta + \Delta_3 - \Delta_4 + 2)}{16\Delta^2 + 2(c - 5)\Delta + c} \]
\[ - \frac{(\Delta + \Delta_1 - \Delta_2)(\Delta + \Delta_1 - \Delta_2 + 1)(\Delta + \Delta_1 - \Delta_2 + 2)(9\Delta_2^2 - 7\Delta + 3c\Delta + c)(\Delta + 2\Delta_3 - \Delta_4)(\Delta + \Delta_3 - \Delta_4 + 2)}{16\Delta^2 + 2(c - 5)\Delta + c} \]
\[ + (\Delta + \Delta_1 - \Delta_2)(\Delta + \Delta_1 - \Delta_2 + 1)(\Delta + \Delta_1 - \Delta_2 + 2)(\Delta + \Delta_3 - \Delta_4)(\Delta + \Delta_3 - \Delta_4 + 1)(\Delta + \Delta_3 - \Delta_4 + 2) \times \]
Generally, for example

\[ Q = \frac{(24\Delta^2 - 26\Delta + 11c\Delta + 8c + c^2)}{12(16\Delta^2 + 2(c - 5)\Delta + c)} \]

\[
\text{\ldots}
\]

Eqs. (16) and (17) are exactly the well-known formulas from [27] (with misprints corrected).

2.3 Comments on the derivation of \( Q \) and \( \gamma \)

In this section we follow the standard basic of conformal theory [26, 27]. Evaluation of the Shapovalov form \( Q_\Delta \) is absolutely straightforward. By definition (8) it is non-vanishing only for \( |Y| = |Y'| \). The first few elements of the matrix \( Q_\Delta(Y,Y') \) for (14) can be easily calculated

\[
Q_\Delta(\emptyset, \emptyset) = \langle \Delta | \Delta \rangle = 1;
\]

\[
Q_\Delta([1], [1]) = \langle \Delta | L_1 L_{-1} | \Delta \rangle = \langle \Delta | L_{-1} L_1 + 2L_0 | \Delta \rangle = (0 + 2\Delta) \langle \Delta | \Delta \rangle = 2\Delta
\]

for \( |Y| = 0 \) and \( |Y| = 1 \). Next,

[2], [2] : 

\[ L_2 L_{-2} = L_{-2} L_2 + 4L_0 + \frac{c}{2} \rightarrow 4\Delta + c/2 = \frac{1}{2}(8\Delta + c), \]

[2], [11] : 

\[ L_2^2 L_{-1} = (L_{-1} L_2 + 3L_1) L_{-1} \rightarrow 3L_1 L_{-1} = 3L_{-1} L_1 + 6L_0 \rightarrow 6\Delta, \]

[11], [11] : 

\[ L_1^2 L_{-2} = L_1(L_{-1} L_1 + 2L_0)L_{-1} = L_1 L_{-1}(L_{-1} L_1 + 2L_0) + 2L_1(L_{-1} L_0 + L_{-1}) \rightarrow
\]

\[ \rightarrow 0 + ((2 + 2\Delta + 2)L_1 L_{-1} \rightarrow 4\Delta(1 + 2\Delta) \]

for \( |Y| = 2 \), and

[3], [3] : 

\[ L_3 L_{-3} \rightarrow 6\Delta + 2c, \]

[3], [21] : 

\[ L_3 L_{-1} L_{-2} \rightarrow 4L_2 L_{-2} \rightarrow 2(8\Delta + c), \]

[3], [111] : 

\[ L_3 L_3^2 L_{-1} \rightarrow 4L_2^2 L_{-1} \rightarrow 24\Delta, \]

[21], [21] : 

\[ L_2 L_1 L_{-1} L_{-2} = L_2(L_{-1} L_1 + 2L_0)L_{-2} \rightarrow 3L_2^2 L_{-1} + 2L_2 L_{-2}(\Delta + 2) \rightarrow \]

\[ \rightarrow 18\Delta + (8\Delta + c)(\Delta + 2) = 8\Delta^2 + (34 + c)\Delta + 2c, \]

[21], [111] : 

\[ L_2^2 L_3 L_{-1} = L_2(L_{-1} L_1 + 2L_0)L_1^2 = 2L_2(L_{-1} L_1 + 2L_0)L_{-1} + 2L_2(L_{-1} L_0 + L_{-1}) \rightarrow
\]

\[ \rightarrow 2L_2^2 L_1(\Delta + (\Delta + 1) + (\Delta + 1)) \rightarrow 36\Delta(\Delta + 1), \]

[111], [111] : 

\[ L_3^2 L_3 L_{-1} = L_3^2 L_{-1}(L_{-1} L_1 + 2L_0)L_{-1} = L_3^2 L_{-1}(L_{-1} L_1 + 2L_0) + 2L_3^2 L_{-1}(L_{-1} L_0 + L_{-1}) \rightarrow
\]

\[ \rightarrow 2L_3^2 L_1(\Delta + (\Delta + 1) + (\Delta + 1)) \rightarrow 24\Delta(1 + \Delta)(1 + 2\Delta) \]

Generally, for example

\[
Q_\Delta([n], [n]) = \langle \Delta | L_n L_{-n} | \Delta \rangle = \langle \Delta | L_{-n} L_n + \frac{c}{12}n(n^2 - 1) + 2nL_0 | \Delta \rangle = \frac{c}{12}n(n^2 - 1) + 2n\Delta,
\]

\[
Q_\Delta([n], [n - 1, 1]) = \langle \Delta | L_n L_{-1} L_{-(n-1)} | \Delta \rangle = (n + 1)\langle \Delta | L_{n-1} L_{-(n-1)} | \Delta \rangle =
\]

\[ = (n + 1)Q([n - 1], [n - 1]|\Delta) = \frac{c}{12}(n + 1)n(n - 1)(n - 2) + 2(n^2 - 1)\Delta
\]
and so on, up to

\[ Q_\Delta([1^n], [1^n]) = \langle \Delta | L_1^n L_{-1}^n | \Delta \rangle = n(2\Delta + n - 1)(\Delta | L_1^{n-1} L_{-1}^{n-1} | \Delta) = n! \frac{\Gamma(2\Delta + n)}{\Gamma(2\Delta)} \] (22)

The last formula follows from two recurrent relations:

\[ L_0 L_{-1}^n = L_{-1} L_0 L_{-1}^{n-1} + L_{-1}^n = \ldots = L_{-1}^n (L_0 + n) = L_{-1}^n (\Delta + n) \] (23)

and

\[ L_1 L_{-1}^n = L_{-1} L_1 L_{-1}^{n-1} + 2L_0 L_{-1}^{n-1} = \ldots \]

\[ + 2L_{-1}^{n-1} (\Delta + n - 1 + (\Delta + n - 2) + \ldots + \Delta) \rightarrow 0 + n(2\Delta + n - 1)L_{-1}^{n-1} \]

The nice factorized formula (15) to the three-point function \( \gamma \) can be derived [28] using the general theory [26] [27]. The simplest way to obtain (15) for generic \( Y \), is to use explicit expression for the action of the Virasoro generators on the primary fields

\[ [L_n, V_\Delta(z)] = \left( z^{n+1} \frac{d}{dz} + (n + 1) z^n \Delta \right) V_\Delta(z) \] (24)

and the manifest expression for the holomorphic part of the 3-point correlator of the primary fields

\[ \langle V_{\Delta_1}(z_1)V_{\Delta_2}(z_2)V_{\Delta_3}(z_3) \rangle = \frac{C_{\Delta_1, \Delta_2, \Delta_3}}{(z_1 - z_2)^{\Delta_1 + \Delta_2 - \Delta_3}(z_1 - z_3)^{\Delta_1 + \Delta_3 - \Delta_2}(z_2 - z_3)^{\Delta_2 + \Delta_3 - \Delta_1}} \] (25)

taken at points \( z_1 = 0, z_2 = z, z_3 = \infty \) (\( z \) is put 1 after calculating all derivatives).

The simplest example is

\[ \langle V_{\Delta_1}(z_1)V_{\Delta_2}(z_2)L_{-1} V_\Delta(z) \rangle = \frac{\partial}{\partial z} \langle V_{\Delta_1}(z_1)V_{\Delta_2}(z_2)V_\Delta(z) \rangle = \]

\[ = \frac{\partial}{\partial z} (z - z_1)^{\Delta + \Delta_1 - \Delta_2}(z - z_2)^{\Delta + \Delta_2 - \Delta_1}(z_1 - z_2)^{\Delta_1 + \Delta_2 - \Delta} \]

Only derivative of the first factor with \( (z - z_1) \) contributes when we put \( z_2 = \infty \), and this explains why \( \gamma(\Delta_1, \Delta_2, \Delta) = \Delta + \Delta_1 - \Delta_2 \).

2.4 Free fields

The simplest and most well known example of conformal theory with arbitrary central charge \( c = 1 + 6Q^2 \) is the theory of free massless field \( \phi \) with background charge \( Q \), considered to be a generic real or even complex number. The primaries \( V_\alpha = :e^{\alpha \phi} : \) in this theory have conformal dimensions

\[ \Delta_\alpha = \alpha (Q - \alpha) \] (27)

The OPE in this theory is very simple:

\[ :e^{\alpha_1 \phi(z_1)} : \cdot e^{\alpha_2 \phi(z_2)} : = \frac{e^{\alpha_1 \phi(z_1) + \alpha_2 \phi(z_2)}}{(z_1 - z_2)^{2\alpha_1 \alpha_2}} = \]

\[ = z_2^{-2\alpha_1 \alpha_2} \left( 1 + z_2 \alpha_1 \partial \phi(z_2) + z_2^2 \frac{\alpha_1^2}{2} (\partial \phi(z_2))^2 + z_2^2 \frac{\alpha_1^2}{2} (\partial^2 \phi(z_2)) + \ldots \right) e^{(\alpha_1 + \alpha_2) \phi(z_2)} : = \]

\[ = z_2^{-2\alpha_1 \alpha_2} \left( 1 + z_2 \beta_{\text{free}[1]} L_{-1} + z_2^2 (\beta_{\text{free}[2]} L_{-2} + \beta_{\text{free}[1]} L_{-1}^2) + \ldots \right) e^{(\alpha_1 + \alpha_2) \phi(z_2)} : \]

and leads to the following selection rule for the four-point conformal block

\[ \alpha \equiv \alpha_{(12)} = \alpha_1 + \alpha_2 = Q - \alpha_3 - \alpha_4, \] (29)
It means that in the intermediate channel there is a single primary operator with the conformal dimension
\[
\Delta = \alpha (Q - \alpha) = (\alpha_1 + \alpha_2)(Q - \alpha_1 - \alpha_2) = (Q - \alpha_3 - \alpha_4)(\alpha_3 + \alpha_4) = (\alpha_1 + \alpha_2)(\alpha_3 + \alpha_4)
\] (30)

The four-point conformal block in this theory just equals
\[
\langle e^{\alpha_1 \phi(z_1)} e^{\alpha_2 \phi(z_2)} e^{\alpha_3 \phi(z_3)} e^{\alpha_4 \phi(z_4)} \rangle = \prod_{i<j} z_{ik}^{-2\alpha_i \alpha_j}
\] (31)
with \(\sum_i \alpha_i = Q\). Putting \(z_1 = 1, z_2 = \infty, z_3 = x, z_4 = 0\) as requested in (12), then the r.h.s. turns into
\[
B_{\text{free}}(x) = x^{-2\alpha_3 \alpha_4} (1 - x)^{-2\alpha_1 \alpha_3} = x^{-2\alpha_3 \alpha_4} \sum_{k=0}^{\infty} x^k \frac{\Gamma(k + 2\alpha_1 \alpha_3)}{k! \Gamma(2\alpha_1 \alpha_3)} =
\]
\[
x^{-2\alpha_3 \alpha_4} \left(1 + 2\alpha_1 \alpha_3 x + \alpha_1 \alpha_3 (2\alpha_1 \alpha_3 + 1)x^2 + \frac{2\alpha_1 \alpha_3 (\alpha_1 \alpha_3 + 1)(2\alpha_1 \alpha_3 + 1)}{3} x^3 + \ldots \right)
\] (32)

In particular, the coefficient in the first term of expansion in brackets is \(2\alpha_1 \alpha_3\), what should be compared with (16) which in this case is equal to
\[
B_{\text{free}}^{(1)} = \frac{(\Delta + \Delta_1 - \Delta_2)(\Delta + \Delta_3 - \Delta_4)}{2\Delta} = \frac{2\alpha_1 (Q - \alpha_1 - \alpha_2) \cdot 2\alpha_3 (Q - \alpha_3 - \alpha_4)}{2(\alpha_1 + \alpha_2)(\alpha_3 + \alpha_4)} \equiv 2\alpha_1 \alpha_3
\] (33)

In a similar way one can check that the next two coefficients coincide with the values of \(B_{\text{free}}^{(2)}\) and \(B_{\text{free}}^{(3)}\) obtained when the free-field dimensions and (30) are substituted into (17) and (18):
\[
B_{\text{free}}^{(2)} = \frac{2\alpha_1 \alpha_3 (2\alpha_1 \alpha_3 + 1)}{2!},
\]
\[
B_{\text{free}}^{(3)} = \frac{2\alpha_1 \alpha_3 (2\alpha_1 \alpha_3 + 1)(2\alpha_1 \alpha_3 + 2)}{3!}
\] (34)

Note that they are independent of \(Q\), despite particular dimensions \(\Delta_{\alpha_i}\) are \(Q\)-dependent. Naturally, one expects that if explicit expressions like (16)-(18) are found for higher terms of the \(x\)-expansion, then substitution of (30) would give
\[
B_{\text{free}}^{(k)} = \frac{\Gamma(k + 2\alpha_1 \alpha_3)}{k! \Gamma(2\alpha_1 \alpha_3)}
\] (35)

To summarize, the representation theory formula (13) provides an amusing expansion of the free field correlator (32) into a bilinear sum over Young diagrams of equal sizes.

### 2.5 Character decomposition

Expression (32) (for further convinience in this section we denote \(\sqrt{2}\alpha_1 = m_1, \sqrt{2}\alpha_3 = m_2\); we shall see below that they play the role of the rescaled mass parameters \(\mu\) in 4d theory) has an obvious alternative expansion into the single sum over partitions (see e.g. [29])
\[
(1 - x)^{-m_1 m_2} = \sum_k x^k \frac{\Gamma(m_1 m_2 + k)}{\Gamma(m_1 m_2)} =
\]
\[
= \sum_{Y} x^{|Y|} \prod_{(i,j) \in Y} \frac{(m_1 + j - i)(m_2 + j - i)}{h(i,j)^2} \equiv \sum_{Y} x^{|Y|} z^Y \quad \text{U(1)}
\] (36)

\(^4\) This rescaling is just an artefact of inconvenient normalizations, chosen originally in [26, 27] and can be absorbed by renormalization of the scalar field \(\phi(z) = \sqrt{2}\varphi(z)\) (cf. e.g. normalizations in [31] and [32]).
where for the \((i, j) \in Y = (k_1 \geq k_2 \geq \ldots \geq k_l)\) with co-ordinates \((i, j)\), such that \(i = 1, \ldots, l\) and \(j = 1, \ldots, k_i\), the "hook" length \(h(i, j)\) is

\[
    h(i, j) = k_i(Y) - j + k_j(Y^T) - i + 1
\]

so that

\[
    \prod_{(i, j) \in Y} (m_f + j - i) = \prod_{i=1}^{l} \prod_{j=1}^{k_i} (m_f + j - i) = \prod_{i=1}^{l} (m_f + 1 - i) \ldots (m_f + k_i - i) = \prod_{i=1}^{l} \frac{\Gamma(m_f + k_i - i)}{\Gamma(m_f + 1 - i)}, \quad f = 1, 2
\]

Coefficients of expansion (36) have already typical shape of Nekrasov’s formulas, where (in the notations of [1])

\[
    Z_U^{(1)} = \prod_{(i, j) \in Y} \frac{\phi_Y(m_1; i, j)\phi_Y(m_2; i, j)}{E_Y^2(0; i, j)}
\]

with

\[
    \phi_Y(m; i, j) = m + i - j,
    \quad E_Y(\alpha; i, j) = \alpha + (k_j^T - i + 1) + (k_i - j)
\]

Above \(\{k_i\}\) and \(\{k_j^T\}\) are respectively the lengths of rows and heights of columns in the Young diagram \(Y\) and its transposed \(Y^T\).

Formulas (36)-(39) follow immediately from the Cauchy formula for the Schur functions (see e.g. [29], for recent reviews of the character decompositions see also [13, 7, 30])

\[
    \prod_{i,j} \frac{1}{1 - \lambda_i \lambda_j'} = \sum_Y s_Y(\lambda) s_Y(\lambda')
\]

which is, when written in terms of the Miwa variables

\[
    t_k = \frac{1}{k} \sum_{i=1}^{N} \chi_i^k
\]

the decomposition formula

\[
    \exp \left( \sum_{k=1}^{\infty} k t_k t_k' \right) = \sum_Y \chi_Y(t) \chi_Y(t'),
\]

for the charachers (due to the first Weyl formula)

\[
    s_Y(\lambda) = \frac{\det_{ij} \lambda_j^{k_i+N-i}}{\det_{ij} \lambda_j^{N-i}} = \chi_Y(t)
\]

associated with the Young diagrams \(Y\) so that \(\chi_Y = \chi[k_1 k_2 \ldots k_l]\), e.g.

\[
    \chi[0] = 1, \quad \chi[1] = t_1, \quad \chi[2] = t_2 + \frac{t_1^2}{2}, \quad \chi[11] = -t_2 + \frac{t_1^2}{2}, \quad \chi[3] = t_3 + t_1 t_2 + \frac{t_1^3}{6}, \quad \chi[21] = -t_3 + \frac{t_1^3}{3}, \quad \chi[111] = t_3 - t_1 t_2 + \frac{t_1^3}{6}, \quad \ldots
\]
Literally, one has to take in \( \{ \lambda_i \} = \left\{ \sqrt{x}, \sqrt{x}, \ldots, \sqrt{x}, 0, 0, \ldots \right\} \) 

(and analytically continue further from integer values of \( m_f \), in terms of Miwa variables \( \text{[42]} \) this corresponds to \( t_k \sim \frac{m}{k} x^{k/2} \)), then

\[
 s_Y(\lambda) = x^{|Y|/2} \prod_{(i,j) \in Y} \frac{m + j - i}{h(i,j)} \quad \text{(46)}
\]

which can be treated as the case of coincident \( \lambda \)'s in the Weyl formula \( \text{[44]} \), after applying the l'Hôpital rule. Putting further \( x = \frac{1}{m^*} \), and taking the limit \( m \to \infty \), one gets further

\[
d_Y \equiv \lim_{m \to \infty} m^{|Y|} \prod_{(i,j) \in Y} \frac{m + j - i}{h(i,j)} = \prod_{(i,j) \in Y} \frac{1}{h(i,j)} = \dim R_Y = |Y|!
\]

whose square \( \mu_Y = d_Y^2 \) is known also as the Plancherel measure. Summing over partitions in \( \text{[36]} \) is performed, using particular cases of the formulas \( \text{[41]}, \text{[43]} \) (the Burnside theorems)

\[
 \sum_Y d_Y \chi_Y(t) = e^{t^1}, \quad \text{at} \ t'_k = \delta_{k,1}
\]

\[
 \sum_Y \mu_Y x^{|Y|} = \sum_Y d_Y^2 x^{|Y|} = e^x, \quad \text{if also} \ t_k = x \delta_{k,1}
\]

We hereby conclude, that in the free field case, there exists a combinatorial re-interpretation of the decomposition for conformal block

\[
 \sum_{Y,Y'} x^{|Y|} B_Y^{\alpha_1, \alpha_2} (\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (1 - x)^{-2\alpha_1 \alpha_3} = (1 - x)^{-m_1 m_2} = \sum_Y x^{|Y|} \mathcal{Z}_Y^{U(1)}
\]

where the coefficients \( \mathcal{Z}_Y^{U(1)} \) are defined in \( \text{[39]} \).

### 2.6 Beyond free fields

Arbitrary conformal model can be effectively described in terms of free fields \( \text{[31, 32]} \), however the number of fields can be greater than one and the constraints \( \text{[29]} \) on the intermediate state should be released. The natural question is what happens then to eqs.\( \text{[33]-[35]} \). The AGT conjecture \( \text{[1]} \) is that for generic \( \alpha \), not obligatory equal to \( \alpha_1 + \alpha_2 \), the coefficients

\[
 B^{(k)} = \sum_{|Y|=|Y'|=k} B_{Y,Y'}
\]

are equal to the expansion coefficients of Nekrasov’s functions (to be explicitly defined in s.4.2):

\[
 \sum_{|Y|=|Y'|} x^{|Y|} B_Y^{\alpha_1, \alpha_2; \alpha_3, \alpha_4} = \sum_{|Y|=|Y'|} x^{|Y|} B_Y^{\alpha_1+\alpha_2, \alpha_3, \alpha_4} \sum_{|Y|=|Y'|} x^{|Y|+|Y'|} \mathcal{Z}_{Y,Y'}^{SU(2)} = (1 - x)^{-\nu} \sum_{|Y|=|Y'|} x^{|Y|+|Y'|} \mathcal{Z}_{Y,Y'}^{SU(2)} = \sum_{|Y|=|Y'|} x^{|Y|+|Y'|} \mathcal{Z}_{Y,Y'}^{SU(2)}
\]

(51)
where the $U(1)$ factor $(1 - x)^{1-\nu}$ is itself represented by the similar pattern formula (49). Among other things the first line of (51) implies that the combination of $SU(2)$ $Z$-functions is strictly unity whenever $\alpha = \alpha_1 + \alpha_2$. Restriction to $SU(2)$ at the r.h.s. of (51) looks to be related to restriction to a single free field at the l.h.s. Note that (51) is a universal group-theory relation with no reference to particular conformal model at the l.h.s. and particular SUSY Yang-Mills theory at the r.h.s.: it relates explicit group-theoretical quantities, canonically associated with the Young diagrams. The only point is that though canonical, these associations still look rather sophisticated and lack clear interpretation in representation theory of linear and symmetric groups.

3 AGT relations

The claim of [1] is actually more general than (51). The observation is that Nekrasov partition functions factorize exactly in the same way as conformal blocks in (12):

$$Z_{\text{Nek}} = Z_{\text{cl}}Z_{\text{pert}}Z_{\text{inst}}$$

where

$$Z_{\text{inst}} = B_\Delta(\Delta_1, \Delta_2; \Delta_3, \Delta_4|x)$$

for some choice of the dimensions $\Delta_1, \Delta_2, \Delta_3, \Delta_4$ and $\Delta$,

$$Z_{\text{cl}} = x^\sigma$$

and

$$Z_{\text{pert}} = C_a^{\alpha_1 \alpha_2}K_\alpha C_a^{\alpha_3 \alpha_4}$$

for some choice of conformal theory and its primary states.

More than that, this conformal theory has been claimed to be the Liouville model with the central charge

$$c = 1 + 6Q^2 = 1 + \frac{6(\epsilon_1 + \epsilon_2)^2}{\epsilon_1 \epsilon_2}$$

and dimensions\(^5\)

$$\Delta_\alpha = \frac{\alpha(\epsilon - \alpha)}{\epsilon_1 \epsilon_2}$$

with $\epsilon = \epsilon_1 + \epsilon_2$. The $\alpha$’s are linear combinations of parameters - gauge fields $a$ and masses $\mu$ - of Nekrasov’s partition functions, see below.

Furthermore, this identification has been generalized to higher-point conformal blocks: to be related to Nekrasov’s functions for non-trivial patterns of branes (quiver gauge theories). The relations of [1] involves only Nekrasov’s functions for $SU(2)$ and $U(2)$ “gauge” groups. As already mentioned, for arbitrary groups it should presumably involve conformal blocks of the reduced WZNW model [32] (Affine Toda, to be concrete, [25]), expressed through more than a single scalar field.

Note that while (53) is just the universal relation (51), eq. (55) essentially involves sophisticated expressions for the structure constants in the Liouville theory (the DOZZ vertices [33]), which look like non-trivial quantities from representation theory of the quantum Kac-Moody algebras (see also [24]). We leave (54) and (55) beyond the scope of this paper: instead, we are going to concentrate on (53) in the form of (51).

\(^5\) Note rescaling by $\epsilon_1 \epsilon_2$ as compared to the standard notation (27), it is innocent, but simplifies the formulas. If $Q$ is parameterized in the CFT-standard way as $Q = b + 1/b$ then $b = \sqrt{\epsilon_1 / \epsilon_2}$. We emphasize that we do not see any need to require that $\epsilon_1 \epsilon_2 = 1$ for identification of conformal blocks and Nekrasov’s partition functions for instanton sums.
4 Nekrasov partition functions

These are the newly discovered special functions $Z_{Y,Y'}[\epsilon_1,\epsilon_2]$ with increasing number of applications in modern mathematical physics. For $\epsilon_1 = -\epsilon_2 \equiv \hbar$ they are closely related to characters of symmetric and linear groups, if further $\hbar \rightarrow 0$ they reproduce the Seiberg-Witten prepotentials. The conjecture of [1] can be considered as one more application: Nekrasov instanton partition functions describe the conformal blocks with Virasoro in the role of the chiral algebra. The number of $\epsilon$-parameters can be increased in this way, and these extra parameters can actually play a role in generalizations to other chiral algebras. Representation-theory interpretation of generic Nekrasov functions remains obscure, and they can be (temporarily?) considered as providing combinatorial rather than character decompositions.

4.1 $U(1)$ case, $Z_Y$

For particular case $\epsilon = 0$, for the $c = 1$ free conformal theory, eq.(49) is the well known $U(1)$ precursor of the AGT relation (51). It describes Nekrasov partition function of the $U(1)$ gauge theory with $N_f = 2$ flavours as a conformal block [7]

$$Z_{\text{inst}}^{U(1),N_f=2}(x; m_1, m_2) = \langle e^{i(\alpha+m_2)\varphi(\infty)}e^{-im_2\varphi(1)x}L_0 e^{im_1\varphi(1)}e^{-i(\alpha+m_1)\varphi(0)} \rangle =$$

$$= x^{m_1^2+(\alpha+m_1)^2} \langle e^{i(\alpha+m_2)\varphi(\infty)}e^{-im_2\varphi(1)x}e^{im_1\varphi(x)}e^{-i(\alpha+m_1)\varphi(0)} \rangle = x^{\alpha^2} (1-x)^{-m_1 m_2}$$

(58)

where for simplicity $\epsilon_1 = -\epsilon_2 = \hbar = 1$. As expected, the intermediate channel for the conformal block is projected to a single representation with dimension of the primary field $\Delta_\alpha \sim \alpha^2$. Conformal block (58) has an expansion (36) over a single set of Young diagrams, we already discussed in sect. 2.5.

Formula (58) is a matrix element in the $U(1)$ conformal theory with $c = 1$, which possesses not only a free-boson but also a free-fermion representations [7] [8] for the current

$$J(z) = \partial \varphi(z) =: \tilde{\psi}(z) \psi(z) := \sum_{n \in \mathbb{Z}} \frac{J_n}{z^{n+1}}$$

(59)

so that the stress-energy tensor (3) is just $T(z) \sim J(z)^2$.

The limit of infinite masses $m_{1,2} \rightarrow \infty$ in (58) corresponds to decoupling of matter from the gauge fields and leads to

$$Z^{U(1)}(\alpha, \Lambda) = \langle \alpha| e^{J_1} e^{J_2} L_0 e^{J_1-1} |\alpha\rangle = \Lambda^{\alpha^2/2} \sum_Y \mu_Y \Lambda^{|Y|} \equiv \Lambda^{\alpha^2/2} e^{\Lambda}$$

(60)

with $\Lambda^2 = m_1 m_2 x^2$ is fixed at $x \rightarrow 0$, $m_{1,2} \rightarrow \infty$ being the scale parameter of the pure $U(1)$ gauge theory.

As was already noticed in sect. 2.5 these formulas remain intact at arbitrary $\epsilon \neq 0$, since switching on $\epsilon$ corresponds to nothing more than the “twisting” of the stress-tensor of the $U(1)$ theory $T(z) \rightarrow T(z) + \epsilon J(z)$. This just shifts the weights of the twisted fermions and deforms the combinatorial formulas, whose summation leads basically to the same results (see e.g. [9]).

4.2 $SU(2)$ case, $Z_{Y,Y'}$

The $SU(2)$ Nekrasov’s functions $Z_{Y,Y'}$ are much more involved. They are manifestly given by the formulas

$$Z_{Y,Y'} = \frac{\eta(Y_1,Y_2)}{\xi(Y_1,Y_2)}$$

(61)

with

$$\eta(Y_1,Y_2) = \prod_{(i,j) \in Y_1} \prod_{\alpha=1}^4 \left( \phi(a_1,i,j) + \mu_\alpha \right) \prod_{(i,j) \in Y_2} \prod_{\alpha=1}^4 \left( \phi(a_2,i,j) + \mu_\alpha \right)$$

(62)
\[ \xi(Y_1, Y_2) = \prod_{(i,j) \in Y_1} E(a_1 - a_1, Y_1, i, j) \left( \epsilon - E(a_1 - a_1, Y_1, i, j) \right) \times \]
\[ \times \prod_{(i,j) \in Y_1} E(a_2 - a_2, Y_1, i, j) \left( \epsilon - E(a_2 - a_2, Y_1, i, j) \right) \times \]
\[ \times \prod_{(i,j) \in Y_2} E(a_2 - a_2, Y_2, i, j) \left( \epsilon - E(a_2 - a_2, Y_2, i, j) \right) \times \]
\[ \times \prod_{(i,j) \in Y_2} E(a_2 - a_2, Y_2, i, j) \left( \epsilon - E(a_2 - a_2, Y_2, i, j) \right) \]
\[ \text{where } a_2 = -a_1 \text{ and } \]
\[ \phi(a, i, j) = a + \epsilon_1(i - 1) + \epsilon_2(j - 1) \]
\[ E(a, Y_1, Y_2, i, j) = a + \epsilon_1 \left( k_{T}^2(Y_1) - i + 1 \right) - \epsilon_2 \left( k_{i}(Y_2) - j \right) \]
defined for two Young diagrams \( Y_1 \) and \( Y_2 \). These functions are natural generalization of the \( U(1) \) quantities in (40). Note that for \( U(1) \) \( \epsilon_1 = -\epsilon_2 \) and the scaling factor \( \epsilon_1 \epsilon_2 \) has not been introduced yet in (40). Note also that, in variance with the \( U(1) \) case, where \( \sqrt{2a} = m \) enters the product in the numerator of (40), in the \( SU(2) \) case, (62) there are no square roots in front of \( a_1 \).

Explicit expressions for the first few terms of their expansion are
\[ Z[0][0] = -\frac{1}{\epsilon_1 \epsilon_2} \cdot \frac{\prod_{r=1}^{4}(a + \mu_{r})}{2a(2a + \epsilon)}, \]
\[ Z[0][1] = -\frac{1}{\epsilon_1 \epsilon_2} \cdot \frac{\prod_{r=1}^{4}(a - \mu_{r})}{2a(2a - \epsilon)}; \]

for the instantonic charge \( k = |Y| + |Y'| = 1 \),
\[ Z[2][0] = \frac{1}{2! \epsilon_1 \epsilon_2} \cdot \frac{\prod_{r=1}^{4}(a + \mu_{r})(a + \mu_{r} + \epsilon_2)}{2a(2a + \epsilon_2)(2a + \epsilon)(2a + \epsilon + \epsilon_2)}, \]
\[ Z[0][2] = \frac{1}{2! \epsilon_1 \epsilon_2} \cdot \frac{\prod_{r=1}^{4}(a - \mu_{r})(a - \mu_{r} - \epsilon_2)}{2a(2a - \epsilon_2)(2a - \epsilon)(2a - \epsilon - \epsilon_2)}, \]
\[ Z[1][0] = -\frac{1}{2! \epsilon_1 \epsilon_2} \cdot \frac{\prod_{r=1}^{4}(a + \mu_{r})(a + \mu_{r} + \epsilon_1)}{2a(2a + \epsilon_1)(2a + \epsilon)(2a + \epsilon + \epsilon_1)}, \]
\[ Z[0][1] = -\frac{1}{2! \epsilon_1 \epsilon_2} \cdot \frac{\prod_{r=1}^{4}(a - \mu_{r})(a - \mu_{r} - \epsilon_1)}{2a(2a - \epsilon_1)(2a - \epsilon)(2a - \epsilon - \epsilon_1)}, \]
\[ Z[1][1] = \frac{1}{\epsilon_1 \epsilon_2} \cdot \frac{\prod_{r=1}^{4}(a + \mu_{r})(a - \mu_{r})}{(4a^2 - \epsilon_1^2)(4a^2 - \epsilon_2^2)}; \]

for \( k = 2 \), then
\[ Z[3][0] = -\frac{1}{3! \epsilon_1 \epsilon_2} \cdot \frac{\prod_{r=1}^{4}(a + \mu_{r})(a + \mu_{r} + \epsilon_2)(a + \mu_{r} + 2\epsilon_2)}{2a(2a + \epsilon_2)(2a + 2\epsilon_2)(2a + \epsilon)(2a + \epsilon + \epsilon_2)(2a + \epsilon + 2\epsilon_2)}, \]
\[ \text{etc.} \] (67)

Clearly, there is a symmetry
\[ Z_{Y', Y}(a, \epsilon_1, \epsilon_2) = Z_{Y, Y'}(-a, \epsilon_1, \epsilon_2) \]
(68)

and
\[ Z_{Y, Y'}(a, \epsilon_1, \epsilon_2) = Z_{Y', Y'}(a, \epsilon_2, \epsilon_1) \] (69)
The mnemonic rule to construct these expressions is rather simple: in transition \([n] \to [n+1]\) one introduces additional entries with an extra \(\epsilon_2\) both in the numerator and denominator, while in transition \([1^n] \to [1^{n+1}]\) one adds extra \(\epsilon_1\).

5 Beyond free fields, continued

5.1 The case of \(\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0\): the conformal block

We now return to consideration of conformal blocks \(B_\Delta\) with unconstrained dimension \(\Delta_\alpha\). The simplest case to begin with is when all the other dimensions are vanishing, i.e. the four “external” primaries are just unit operators with \(\Delta_1 = \Delta_2 = \Delta_3 = \Delta_4 = 0\), while the “intermediate” \(\Delta = \frac{\alpha(\epsilon - \alpha)}{\epsilon_1\epsilon_2}\) remains nontrivial and arbitrary. Then, formulas (16)-(18) reduce to:

\[
B^{(1)}(0, 0; 0, 0) = \frac{\alpha(\epsilon - \alpha)}{2\epsilon_1\epsilon_2} = -\frac{4a^2 - \epsilon^2}{8\epsilon_1\epsilon_2},
\]

\[
B^{(2)}(0, 0; 0, 0) = \frac{\Delta(8\Delta^3 + (c + 8)\Delta^2 + (2c - 8)\Delta + c)}{4(16\Delta^2 + (2c - 10)\Delta + c)} = \frac{(4a^2 - \epsilon^2)}{256\epsilon_1^2\epsilon_2^2(4a^2 - (2\epsilon_1 + \epsilon_2)^2)(4a^2 - (\epsilon_1 + 2\epsilon_2)^2)}.
\]

\[
(128a^6 - 48a^4(4\epsilon_1^2 + 11\epsilon_1\epsilon_2 + 4\epsilon_2^2) + 24a^2(3\epsilon_1^4 + 23\epsilon_1^2\epsilon_2 + 36\epsilon_1^2\epsilon_2^2 + 23\epsilon_1\epsilon_2^4 + 3\epsilon_2^4) - (8\epsilon_1^6 + 105\epsilon_1^5\epsilon_2 + 420\epsilon_1^4\epsilon_2^2 + 662\epsilon_1^3\epsilon_2^3 + 420\epsilon_1^2\epsilon_2^4 + 105\epsilon_1\epsilon_2^5 + 8\epsilon_2^6)),
\]

\[
B^{(3)}(0, 0; 0, 0) = \frac{\Delta(\Delta + 2)(8\Delta^3 + (c + 18)\Delta^2 + (3c - 14)\Delta + 2c)}{24(16\Delta^2 + (2c - 10)\Delta + c)} = \frac{(4a^2 - \epsilon^2)}{6144\epsilon_1^2\epsilon_2^2(4a^2 - (2\epsilon_1 + \epsilon_2)^2)(4a^2 - (2\epsilon_1 + 2\epsilon_2)^2)}.
\]

\[
(128a^6 - 16a^4(12\epsilon_1^2 + 43\epsilon_1\epsilon_2 + 12\epsilon_2^2) + 8a^2(9\epsilon_1^4 + 91\epsilon_1^3\epsilon_2 + 142\epsilon_1^2\epsilon_2^2 + 91\epsilon_1\epsilon_2^3 + 9\epsilon_2^4) - (8\epsilon_1^6 + 139\epsilon_1^5\epsilon_2 + 632\epsilon_1^4\epsilon_2^2 + 1034\epsilon_1^3\epsilon_2^3 + 632\epsilon_1^2\epsilon_2^4 + 139\epsilon_1\epsilon_2^5 + 8\epsilon_2^6)).
\]

where at the last step we made a shift

\[
\alpha = a + \frac{\epsilon}{2},
\]

which symmetrizes and slightly simplifies the formulas. Obviously, denominators (Kac determinants) in (16)-(18) are nicely consistent with the substitutions (56) and (57) – a fact, very well known from generic conformal field theory. Note also that denominators are independent of the other dimensions \(\Delta_1, \ldots, \Delta_4\), thus this consistency will persist for generic 4-point conformal blocks.

However, the numerators in (70)-(72) look pretty sophisticated. The question is: what is the appropriate analogue of character decomposition (49) for these quantities? The AGT answer is (51) and we now proceed to check the check and analysis of this relation.

5.2 The case of \(\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0\): Nekrasov’s functions

In this case we put \(\nu = 0\) in (51). Now we need to adjust the four parameters \(\mu_1, \ldots, \mu_4\) so that (51) is satisfied for explicit expressions (70)-(72) and (65)-(67).
First of all, we need to match (70) and (65):

\[
B^{(1)}(0, 0; 0, 0) = -\frac{4a^2 - \epsilon^2}{8\epsilon_1 \epsilon_2} = Z_{[1][0]} + Z_{[0][1]} = -\frac{1}{\epsilon_1 \epsilon_2} \cdot \prod_{r=1}^{4}(a + \mu_r) - \frac{1}{\epsilon_1 \epsilon_2} \cdot \prod_{r=1}^{4}(a - \mu_r)
\]  

(74)

In order to get rid of poles at the r.h.s., one of the \( \mu \) parameters should be equal to \( \epsilon/2 \), let it be \( \mu_4 = \epsilon/2 \). Then the r.h.s. of (74) turns into

\[
-\frac{a^2 + \mu_1 \mu_2 + \mu_1 \mu_3 + \mu_2 \mu_3}{2\epsilon_1 \epsilon_2}
\]  

(75)

and the matching condition with the l.h.s. is

\[
s_2 \equiv \mu_1 \mu_2 + \mu_1 \mu_3 + \mu_2 \mu_3 = -\frac{\epsilon^2}{4}
\]  

(76)

Far less trivial is the matching between (71) and (60):

\[
B^{(2)}(0, 0; 0, 0) = \frac{(4a^2 - \epsilon^2)}{256\epsilon_1^2 \epsilon_2^2 (4a^2 - (2\epsilon_1 + \epsilon_2)^2)(4a^2 - (2\epsilon_1 + \epsilon_2)^2)} \left( 128a^6 - 48a^4(4\epsilon_1^2 + 11\epsilon_1 \epsilon_2 + 4\epsilon_2^2) + 24a^2(3\epsilon_1^4 + 23\epsilon_1^3 \epsilon_2 + 36\epsilon_1^2 \epsilon_2^2 + 23\epsilon_1 \epsilon_2^3 + 3\epsilon_2^4) - (8\epsilon_1^6 + 105\epsilon_1^5 \epsilon_2 + 420\epsilon_1^4 \epsilon_2^2 + 662\epsilon_1^3 \epsilon_2^3 + 420\epsilon_1^2 \epsilon_2^4 + 105\epsilon_1 \epsilon_2^5 + 8\epsilon_2^6) \right) = Z_{[2][0]} + Z_{[0][2]} + Z_{[1][1]} + Z_{[0][1]} + Z_{[1][1]} = \frac{1}{\epsilon_1^2 \epsilon_2^2} \cdot \prod_{r=1}^{4}(a + \mu_r)(a - \mu_r)
\]  

\[
+ \frac{1}{2!} \cdot \frac{\prod_{r=1}^{4}(a + \mu_r)(a - \mu_r + \epsilon_2)}{2a(2a + \epsilon_2)(2a + \epsilon_2)(2a + \epsilon + \epsilon_2)} + \frac{1}{2!} \cdot \frac{\prod_{r=1}^{4}(a - \mu_r)(a - \mu_r - \epsilon_2)}{2a(2a - \epsilon_2)(2a - \epsilon_2)(2a - \epsilon - \epsilon_2)}
\]  

\[
- \frac{1}{2!} \cdot \frac{\prod_{r=1}^{4}(a + \mu_r)(a + \mu_r + \epsilon_1)}{2a(2a + \epsilon_1)(2a + \epsilon)(2a + \epsilon + \epsilon_1)} - \frac{1}{2!} \cdot \frac{\prod_{r=1}^{4}(a - \mu_r)(a - \mu_r - \epsilon_1)}{2a(2a - \epsilon_1)(2a - \epsilon)(2a - \epsilon - \epsilon_1)}
\]  

\[
= \frac{128a^4 + \ldots}{\epsilon_1^2 \epsilon_2^2 (4a^2 - \epsilon_1^2)(4a^2 - \epsilon_2^2)(4a^2 - \epsilon^2)(4a^2 - (2\epsilon_1 + \epsilon_2)^2)(4a^2 - (2\epsilon_1 + \epsilon_2)^2)}
\]  

(77)

Numerator at the r.h.s. is a polynomial of degree 7 in \( a^2 \) and it should match the l.h.s., which we multiply by \((4a^2 - \epsilon^2)(4a^2 - \epsilon_1^2)(4a^2 - \epsilon_2^2)\) in order to convert it into a similar polynomial of degree 14 with the first term \(128a^{14}\). The next coefficients of these two polynomials are:

| degree | l.h.s. | r.h.s. |
|--------|--------|--------|
| \(a^{14}\) | 128 | 128 |
| \(a^{12}\) | \( -288\epsilon^2 - 80\epsilon_1 \epsilon_2 \) | \( 256 \sum_{r<s} \mu_r \mu_s - 128 \epsilon \sum_{r=1}^{4} \mu_r - 160\epsilon^2 - 80\epsilon_1 \epsilon_2 \) |
| \(a^{10}\) | \( 256 \sum_{r<s} \mu_r \mu_s - 224\epsilon^2 - 80\epsilon_1 \epsilon_2 \) | \( 
\ldots \) |

Further lines in the right column are quite involved. For instance, the coefficient in front of \( a^{10} \) is

\[
16 \times [6s_4 + 8s_2^2 - 8\epsilon(s_1 s_2 + s_3) - (16\epsilon_1^2 + 16\epsilon_2^2 + 38\epsilon_1 \epsilon_2)s_2 + (2\epsilon_1^2 + 2\epsilon_2^2 + 3\epsilon_1 \epsilon_2)s_1^2 + (8\epsilon_1^3 + 8\epsilon_2^3 + 29\epsilon_1 \epsilon_2)s_1 + (30\epsilon_1 \epsilon_2 + 18\epsilon_1^2 + 18\epsilon_2^2)\epsilon_1 \epsilon_2]
\]  

(78)

where \( s_k \) denote the symmetric polynomials of four \( \mu_i \) of degree \( k \), \( s_4 \equiv \mu_1 \mu_2 \mu_3 \mu_4 \), \( s_3 = \sum_{a>b>c} \mu_a \mu_b \mu_c \) etc. We do not write down its left-column counterpart and the further lines here: expressions are getting pretty long, while the procedure is hopefully already clear.
Three more relations are obtained from level two - those being more involved, and also at level three. Common solutions (of course, modulo 24 permutations of $\mu$'s: $\mu_{1,2} = \pm \epsilon/2$, $\mu_{3,4} = \mp \epsilon/2$, or $\mu_{1,2} = -\epsilon/2$, $\mu_{3,4} = \epsilon/2$, or $\mu_1 = -\epsilon/2$, $\mu_2 = 3\epsilon/2$, $\mu_{3,4} = \epsilon/2$ (definitely, with all permutations of $\mu$'s)). The reason for existence of several solutions will become clear in the next subsection. With this choice the coefficients of all other powers of $a^2$ also match in the two columns.

Analogous check works also at level 3.

5.3 Restoring dimensions $\Delta_1, \ldots, \Delta_4$

It is now a simple exercise to switch on non-vanishing $\alpha_1, \ldots, \alpha_4$. Start with the first level:

$$B^{(1)}(\Delta_1, \Delta_2; \Delta_3, \Delta_4) = \frac{2a^4 + 2a^2 \left( e(-\alpha_1 + \alpha_2 - \alpha_3 + \alpha_4) + \alpha_1^2 - \alpha_2^2 + \alpha_3^2 - \alpha_4^2 - \epsilon^2/2 \right)}{\epsilon_1 \epsilon_2 (4a^2 - \epsilon^2)} - \frac{2 \left( \alpha_1 (\epsilon - \alpha_1) - \alpha_2 (\epsilon - \alpha_2) + \epsilon^2/4 \right) \left( \alpha_3 (\epsilon - \alpha_3) - \alpha_4 (\epsilon - \alpha_4) + \epsilon^2/4 \right)}{\epsilon_1 \epsilon_2 (4a^2 - \epsilon^2)}$$

(79)

to be compared with

$$Z_{[1][0]} + Z_{[0][1]} + \nu = \frac{2a^4 + (2s - \epsilon - 1 - 4\epsilon_1 \epsilon_2 \nu) a^2 + 2s_4 - \nu a^2 + \epsilon_1 \epsilon_2 \epsilon^2 \nu}{\epsilon_1 \epsilon_2 (4a^2 - \epsilon^2)}$$

(80)

This gives two relations to determine four $\mu$'s and $\nu$:

$$2 \left( e(-\alpha_1 + \alpha_2 - \alpha_3 + \alpha_4) + \alpha_1^2 - \alpha_2^2 + \alpha_3^2 - \alpha_4^2 - \epsilon^2/2 \right) = 2s_2 - \epsilon s_1 - 4\epsilon_1 \epsilon_2 \nu$$

$$2 \left( \alpha_1 (\epsilon - \alpha_1) - \alpha_2 (\epsilon - \alpha_2) + \epsilon^2/4 \right) \left( \alpha_3 (\epsilon - \alpha_3) - \alpha_4 (\epsilon - \alpha_4) + \epsilon^2/4 \right) = 2s_4 - \epsilon s_3 + \epsilon^2 \epsilon_1 \epsilon_2 \nu$$

(81)

Three more relations are obtained from level two - those being more involved, and also at level three.

However, despite the number of emerging matching constraints is large, they are all consistent(!), this is exactly the observation of [II], and they do have a common solution. In fact, there are eight common solutions (of course, modulo 24 permutations of $\mu_1, \ldots, \mu_4$):

| I : $\mu_1 = -\frac{\epsilon}{2} + \alpha_1 + \alpha_2$, $\mu_2 = \frac{\epsilon}{2} + \alpha_1 - \alpha_2$, $\mu_3 = -\frac{\epsilon}{2} + \alpha_3 + \alpha_4$, $\mu_4 = \frac{\epsilon}{2} + \alpha_3 - \alpha_4$, $\nu = \frac{2\alpha_1 \alpha_3}{\epsilon_1 \epsilon_2}$ |
|---|
| II : $\mu_1 = \frac{3\epsilon}{2} - \alpha_1 - \alpha_2$, $\mu_2 = \frac{\epsilon}{2} + \alpha_2 - \alpha_1$, $\mu_3 = \frac{3\epsilon}{2} - \alpha_3 - \alpha_4$, $\mu_4 = \frac{\epsilon}{2} + \alpha_4 - \alpha_3$, $\nu = \frac{2(\epsilon - \alpha_1)(\epsilon - \alpha_3)}{\epsilon_1 \epsilon_2}$ |
| III : $\mu_1 = -\frac{\epsilon}{2} + \alpha_1 + \alpha_2$, $\mu_2 = \frac{\epsilon}{2} + \alpha_1 - \alpha_2$, $\mu_3 = \frac{3\epsilon}{2} - \alpha_3 - \alpha_4$, $\mu_4 = \frac{\epsilon}{2} + \alpha_4 - \alpha_3$, $\nu = \frac{2\alpha_1 (\epsilon - \alpha_3)}{\epsilon_1 \epsilon_2}$ |
| IV : $\mu_1 = \frac{\epsilon}{2} + \alpha_2 - \alpha_1$, $\mu_2 = \frac{3\epsilon}{2} - \alpha_1 - \alpha_2$, $\mu_3 = -\frac{\epsilon}{2} + \alpha_3 + \alpha_4$, $\mu_4 = \frac{\epsilon}{2} + \alpha_3 - \alpha_4$, $\nu = \frac{2(\epsilon - \alpha_1) \alpha_3}{\epsilon_1 \epsilon_2}$ |
\[ V: \quad \mu_1 = -\frac{\epsilon}{2} + \alpha_1 + \alpha_2, \quad \mu_2 = \frac{\epsilon}{2} + \alpha_2 - \alpha_1, \quad \mu_3 = -\frac{\epsilon}{2} + \alpha_3 + \alpha_4, \quad \mu_4 = \frac{\epsilon}{2} + \alpha_4 - \alpha_3, \]
\[ \nu = \frac{(\alpha_2 + \alpha_4)^2 - \alpha_1^2 - \alpha_3^2 - (\alpha_2 + \alpha_4 - \alpha_1 - \alpha_3)\epsilon}{\epsilon_1\epsilon_2} \]
\[ VI: \quad \mu_1 = \frac{3\epsilon}{2} - \alpha_1 - \alpha_2, \quad \mu_2 = \frac{\epsilon}{2} + \alpha_1 - \alpha_2, \quad \mu_3 = -\frac{\epsilon}{2} + \alpha_3 + \alpha_4, \quad \mu_4 = \frac{\epsilon}{2} + \alpha_4 - \alpha_3, \]
\[ \nu = \frac{(\alpha_2 - \alpha_4)^2 - \alpha_1^2 - \alpha_3^2 + (\alpha_4 - \alpha_2 + \alpha_1 + \alpha_3)\epsilon}{\epsilon_1\epsilon_2} \]
\[ VII: \quad \mu_1 = \frac{\epsilon}{2} + \alpha_1 + \alpha_2, \quad \mu_2 = \frac{\epsilon}{2} + \alpha_2 - \alpha_1, \quad \mu_3 = \frac{3\epsilon}{2} - \alpha_3 - \alpha_4, \quad \mu_4 = \frac{\epsilon}{2} + \alpha_3 - \alpha_4, \]
\[ \nu = \frac{(\alpha_2 - \alpha_4)^2 - \alpha_1^2 - \alpha_3^2 + (\alpha_2 - \alpha_4 + \alpha_1 + \alpha_3)\epsilon}{\epsilon_1\epsilon_2} \]
\[ VIII: \quad \mu_1 = \frac{3\epsilon}{2} - \alpha_1 - \alpha_2, \quad \mu_2 = \frac{\epsilon}{2} + \alpha_1 - \alpha_2, \quad \mu_3 = \frac{3\epsilon}{2} - \alpha_3 - \alpha_4, \quad \mu_4 = \frac{\epsilon}{2} + \alpha_3 - \alpha_4, \]
\[ \nu = \frac{2\epsilon^2 + (\alpha_2 + \alpha_4)^2 - \alpha_1^2 - \alpha_3^2 + (\alpha_1 + \alpha_3 - 3\alpha_2 - 3\alpha_4)\epsilon}{\epsilon_1\epsilon_2} \]  

Given these values, it is easy to check consistency of the first four solutions (51) at levels two and three. One of these solutions (III) coincides with that in [11] (after appropriately shifting \(\mu\)'s and changing notations: \(\alpha_1 \leftrightarrow \alpha_2\)). Equality in the first line of (51) corresponds to choice of the boxed solution in (82), for other choices one should substitute \(\alpha_1 + \alpha_2\) appropriately.

Hence, it is indeed quite easy to believe that this correspondence survives at all higher levels, but the check requires either a tedious calculation or a clever theoretical proof - which should be the natural next step in the study of the AGT conjecture.

### 5.4 Symmetries, zeroes and poles

Now let us discuss what is the reason for existence of eight solutions, (82)-(83) for the correspondence (51). The first four solutions (82) are related to each other by transformations of the type \(\alpha \rightarrow \epsilon - \alpha\). Indeed, I turns to II under all \(\alpha_i \rightarrow \epsilon - \alpha_i\), II goes to III under \(\alpha_{1,2} \rightarrow \epsilon - \alpha_{1,2}\) and to IV under \(\alpha_{3,4} \rightarrow \epsilon - \alpha_{3,4}\). Similarly the second four solutions are related: \(V \rightarrow VI \rightarrow VII \rightarrow VIII\) is provided by the chain of transformations \(\alpha_{1,2} \rightarrow \epsilon - \alpha_{1,2}, \alpha_{1,2,3,4} \rightarrow \epsilon - \alpha_{1,2,3,4}, \alpha_{1,2} \rightarrow \epsilon - \alpha_{1,2}\).

Other possible reflections \(\alpha \rightarrow \epsilon - \alpha\) do not lead to new solutions. Indeed, say, changing just one \(\alpha_1 \rightarrow \epsilon - \alpha_1\) transforms I to IV etc.

Therefore, the AGT relation is invariant under reflecting any of \(\alpha_i\). This symmetry is evident at the conformal side of the AGT relation. Indeed, the dimensions of operators (27) are invariant under reflecting any of \(\alpha\).

More intriguing is the second, \(\mathbb{Z}_2\) symmetry, which relates the first four, (82) and the second four, (83) solutions. It is generated by the permutation \(\alpha_1 \leftrightarrow \alpha_2, \alpha_3 \leftrightarrow \alpha_4\). Note that under this transformation one should also change the “U(1) parameter” \(\nu\):

\[ \frac{2\alpha_1\alpha_3}{\epsilon_1\epsilon_2} \rightarrow \frac{(\alpha_2 + \alpha_4)^2 - \alpha_1^2 - \alpha_3^2 - (\alpha_2 + \alpha_4 - \alpha_1 - \alpha_3)\epsilon}{\epsilon_1\epsilon_2} \]  

This transformation of \(\nu\) is trivial only for the case of free fields, i.e. when conditions (29) are imposed on \(\alpha\)'s: then the r.h.s. in (54) coincides with the l.h.s. Generally, this looks like a non-trivial symmetry (duality) of the Nekrasov partition functions for conformal theories,

\[ \sum_{k=0}^{m} \frac{\Gamma(m-k+\nu)}{k!\Gamma(\nu)} \sum_{|Y|+|Y'|=k} Z_{Y,Y'} \{ \alpha \} = \sum_{k=0}^{m} \frac{\Gamma(m-k+\tilde{\nu})}{k!\Gamma(\tilde{\nu})} \sum_{|Y|+|Y'|=k} Z_{Y,Y'} \{ \tilde{\alpha} \} \]  

where \(\tilde{\alpha} = \alpha_1 + \alpha_2, \alpha_3 + \alpha_4 = \epsilon\). This can happen
if \( a + \mu_i = 0 \) and \( -a + \mu_j = 0 \) for some \( i \) and \( j \), what implies in turn that \( \mu_i + \mu_j = 0 \). Looking at the boxed solution in [82], we see that this is indeed the case: \( \mu_1 + \mu_3 = 0 \) if \( \alpha_1 + \ldots + \alpha_4 = \epsilon \). Note that all functions \( Z_{Y,Y'} \) but trivial vanish at this point.

Conformal blocks \( B^{(k)} \) have poles at the zeroes of the Kac determinants, which occur at \( a = \pm (s_1\epsilon_1 + s_2\epsilon_2) \) with all the positive half-integers \( s_1 \) and \( s_2 \) such that, at the level \( N, 4s_1s_2 \leq N \). It is easy to see that the denominators vanish at these points, and the functions \( Z_{Y,Y'} \) acquire the poles there - in accordance with (51).

6 Conclusion

To conclude, we explicitly checked the conjecture of [1] for the first terms of the \( x \)-expansion of the four-point conformal block. The statement is that the universal part of conformal block, which depends only on five dimensions (four external lines and one intermediate) and the central charge of 2d conformal theory can be expanded into a linear combination of Nekrasov partition functions for the conformally invariant 4d gauge \( U(2) \) theory with four fundamental multiplets. Relation between appropriately parameterized dimensions and parameters \( (a, \mu_1, \mu_2, \mu_3, \mu_4; \epsilon = \epsilon_1 + \epsilon_2) \) of Nekrasov’s functions is linear: if

\[
\Delta_i = \frac{\alpha_i(\epsilon - \alpha_i)}{\epsilon_1\epsilon_2}, \quad c = 1 + \frac{6\epsilon^2}{\epsilon_1\epsilon_2}
\]

then

\[
a = \alpha - \frac{\epsilon}{2}
\]

and \( \mu \)'s are given by any of the eight expressions (82). Since one can simultaneously rescale all the seven parameters \( a, \mu_1, \ldots, \mu_4, \epsilon_1, \epsilon_2 \) in the \( U(2) \) Nekrasov’s functions, and the common factor drops out of them in the conformal case (\( \beta \sim N_f - 2N_c = 0 \)), the number of free parameters is actually six – exactly the same as that of \( \Delta_1, \ldots, \Delta_4, \Delta \) and \( c \) on the other side of the AGT relation.

This check concerns only a small part of the AGT conjecture. In particular, we did not touch a technically trivial, but conceptually deep relation between perturbative part of Nekrasov’s functions and the structure constants of the Liouville theory, which is, perhaps, the most beautiful part of the conjecture. The generalizations are obvious, however explicit checks become increasingly complicated and - at the present level of understanding - can be performed only by computer simulations (reported in [1]). Still, explicit check ”by hands” in the simplest case is important to understand the statement, and our goal in this paper was to explain how it works in some detail, avoiding yet a discussion of underlying physics. We hope nevertheless that a performed explicit check sheds some light to the conjectured nontrivial relation between the two-dimensional conformal and four-dimensional gauge theory, and we are going to return to different aspects of this relation elsewhere.

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