BI-IN Variant Schwartz multipliers and local solvability on nilpotent Lie groups

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Let $X$ denote a finite-dimensional vector space with a fixed positive definite inner product, and let $\mathcal{S}(X)$ denote the Schwartz space on $X$. We let $\mathcal{M}(X)$ denote the space of continuous endomorphisms of $\mathcal{S}(X)$ that commute with the action of $X$ on $\mathcal{S}(X)$. The elements of $\mathcal{M}(X)$ are given by convolution by tempered distributions; i.e., for $E \in \mathcal{M}(X)$ there is a $D_E \in \mathcal{S}^*(X)$ such that $Ef(x) = \langle D_E, l_x f \rangle := D_E * f(x)$, where $l_x f(y) = f(y - x)$. Conversely, if $D \in \mathcal{S}^*(X)$, then one can easily see that $E_D : f \mapsto D * f$ is a mapping of $\mathcal{S}(X)$ into the smooth functions on $X$ that commutes with translation. Schwartz [S] shows that $E_D \in \mathcal{M}(X)$ if and only if $D$, the Fourier transform of $D$, is given by a smooth function on $X^*$ which has polynomial bounds on all derivatives. In this note we announce analogues of these results for arbitrary nilpotent Lie groups. Complete proofs will appear elsewhere.

Let $N$ denote a connected, simply connected nilpotent Lie group, with Lie algebra $n$. The exponential mapping, $\exp : n \to N$, is a diffeomorphism, and in terms of the corresponding coordinates left and right translation on $N$ are polynomial mappings. Thus, if $\mathcal{S}(N)$ denotes the image under composition with $\exp$ of $\mathcal{S}(n)$, the right and left action of $N$ on $\mathcal{S}(N)$ are continuous endomorphisms, where $\mathcal{S}(N)$ is topologized so that composition with $\exp$ is an isomorphism from $\mathcal{S}(n)$ to $\mathcal{S}(N)$. We denote by $\mathcal{S}^*(N)$ the dual of $\mathcal{S}(N)$, the space of tempered distributions on $N$.

For $f \in \mathcal{S}(N)$, the Fourier transform of $f$, $\hat{f}$, is defined on $n^*$, the dual of $n$, by

$$\hat{f}(\xi) = \int_n f(\exp X) e^{-2\pi i \langle \xi, X \rangle} dX.$$  

One has that $f \mapsto \hat{f}$ is an isomorphism from $\mathcal{S}(N)$ onto $\mathcal{S}(n^*)$. For $D \in \mathcal{S}^*(N)$, $\hat{D}$ is defined on $\mathcal{S}(n^*)$ by $\langle \hat{D}, f \rangle = \langle D, \hat{f} \circ \log \rangle$, where log denotes the inverse of $\exp$.

Let $\text{Ad}^*$ denote the coadjoint representation of $N$ on $n^*$. A tempered distribution $D$ on $n^*$ is said to be $\text{Ad}^*$-invariant if $\langle D, f \circ \text{Ad}^* x \rangle = \langle D, f \rangle$ for all $x \in N$ and $f \in \mathcal{S}(n^*)$. A tempered distribution $D$ on $N$ is said to be bi-invariant if $\langle D, \tau_x f \rangle = \langle D, l_x f \rangle$ for all $f \in \mathcal{S}(N)$, where $\tau_x f(y) = f(yx)$ and $l_x f(y) = f(x^{-1}y)$ for all $x, y \in N$. A straightforward computation shows that an element $D \in \mathcal{S}(N)$ is bi-invariant if and only if $\hat{D}$ is $\text{Ad}^*$-invariant.

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Let $\mathcal{M}(N)$ denote the space of continuous endomorphisms on $\mathcal{S}(N)$ that commute with both right and left translations by elements of $N$. As in the Euclidean case, one has that for each $E \in \mathcal{M}(N)$ there is a $D_E \in \mathcal{S}^*(N)$ such that $Ef = D_E \ast f$, where, as before, $D_E \ast f(x) := \langle D_E, f \rangle$. If $D \in \mathcal{S}^*(N)$ we denote by $E_D$ the mapping defined on $\mathcal{S}(N)$ by $E_D f = D \ast f$.

Let $PB_N^\infty(n^*)$ denote the space of smooth, Ad*-invariant functions defined on $n^*$ with polynomial bounds on all derivatives. This space is topologized using the seminorms $\nu_{ij}$ defined on $PB_N^\infty(n^*)$ by

$$\nu_{ij}(\theta) = \sup_{|\alpha| \leq j} \sup_{\xi \in n^*} |\partial^\alpha \theta(\xi)| / (1 + ||\xi||^2)^i,$$

where $\partial^\alpha$ denotes the standard differential operator corresponding to the multi-index $\alpha$, and some fixed basis of $n^*$. A sequence $\{E_n\} \subset \mathcal{M}(N)$ converges to 0 if $E_n f \to 0$ in $\mathcal{S}(N)$ for each $f \in \mathcal{S}(N)$.

**Theorem A.** The mapping $\mathcal{M}(N) \to PB_N^\infty(n^*): E \to \tilde{D}_E$ is a homeomorphism and an algebra isomorphism, the products being composition on $\mathcal{M}(N)$ and pointwise multiplication on $PB_N^\infty(n^*)$.

For $\xi \in n^*$, let $\pi_\xi$ denote the irreducible unitary representation of $N$ that corresponds to the Ad*-orbit of $\xi$ by the Kirillov theory. For $\theta \in PB_N^\infty(n^*)$, let $D_\theta$ be the tempered distribution on $N$ with Fourier transform $\theta$.

**Theorem B.** For $\theta \in PB_N^\infty(n^*)$, $f \in \mathcal{S}(N)$, and $\xi \in n^*$,

$$\pi_\xi(D_\theta \ast f) = \theta(\xi) \pi_\xi(f).$$

As an application of these results, we consider the question of local solvability. Recall that a left invariant differential operator $L$ on $N$ is said to be locally solvable if there is an open set $U \subset N$ such that $C^\infty_0(U) \subset L(C^\infty(U))$.

Let $o(\xi)$ denote the Ad*-orbit in $n^*$ that contains $\xi$, and having fixed a norm on $n^*$, set $|o(\xi)| = \inf(||\xi'||: \xi' \in o(\xi))$. Suppose that $N$ contains a discrete, cocompact subgroup $\Gamma$. Then $L^2(\Gamma \backslash N)$ is a direct sum of subspaces $\mathcal{H}_\xi$ such that the restriction to $\mathcal{H}_\xi$ of right translation is a finite multiple of $\pi_\xi$. We denote by $(\Gamma \backslash N)_0^\infty$ the elements of $\hat{N}$ appearing in this decomposition that are in general position.

**Theorem C.** Let $L$ be a left invariant differential operator on $N$. Suppose that for each $\pi_\xi \in (\Gamma \backslash N)_0^\infty$, $\pi_\xi(L)$ has a bounded right inverse $A_\xi$ on $\mathcal{H}_\xi$, and that the norm of $A_\xi$ is bounded by a polynomial in $|o(\xi)|$. Then $L$ is locally solvable.

The proof of Theorem A requires the introduction of somewhat more general spaces. Let $\mathcal{A}$ be a subspace of the center of $\mathcal{A}$, and let $\lambda \in \mathcal{A}^\ast$. We define the unitary character $\chi_\lambda$ on $H := \exp(\mathcal{A})$ by $\chi_\lambda(\exp X) = e^{2\pi i \langle \lambda, X \rangle}$, and denote by $\mathcal{S}(N/H, \chi_\lambda)$ the space of all smooth functions $f$ defined on $N$ such that $f(xy) = \chi_\lambda(y)f(x)$ for all $x \in N$, $y \in H$, and such that $f \circ \exp_{\mathcal{A}} \in \mathcal{S}(\mathcal{A})$, where $\mathcal{A}$ is a complement to $\mathcal{A}$ in $\mathcal{A}$. The topology of $\mathcal{S}(N/H, \chi_\lambda)$ is defined by requiring that the mapping $f \to f \circ \exp_{\mathcal{A}}$ be a homeomorphism. Define $P_\lambda: \mathcal{S}(N) \to \mathcal{S}(N/H, \chi_\lambda)$ by

$$P_\lambda f(\exp X) = \int f(\exp(X + Y)) \chi_\lambda(-Y) dY.$$
$P_\lambda$ is an open surjection and thus its adjoint $P_\lambda^*$ is an isomorphism of $\mathcal{S}^*(N/H, x_\lambda)$ into $\mathcal{S}(N)$.

Let $k^\perp$ be the annihilator of $k$ in $\kappa^*$. For $\lambda \in k^\perp$ (identified as a subspace of $\kappa^*$), there is a natural Schwartz space on $k^\perp + \lambda$, $\mathcal{S}(k^\perp + \lambda)$, given by composing elements of $\mathcal{S}(k^\perp)$ with translation by $-\lambda$. Considering $\mathcal{S}(N/H, x_\lambda)$ and $\mathcal{S}(k^\perp + \lambda)$ as subspaces of $\mathcal{S}^*(N)$ and $\mathcal{S}^*(\kappa^*)$ respectively, the Fourier transform is defined on these spaces and one has that $f \mapsto \hat{f}$ is an isomorphism of $\mathcal{S}(N/H, x_\lambda)$ onto $\mathcal{S}(k^\perp + \lambda)$ and of $\mathcal{S}(k^\perp + \lambda)$ onto $\mathcal{S}(N/H, x_{-\lambda})$. Also one has that for $D \in \mathcal{S}^*(N/H, x_\lambda)$, $(P_\lambda^* D)^\wedge = R_\lambda^* \tilde{D}$, where $R_\lambda : \mathcal{S}(\kappa^*) \rightarrow \mathcal{S}(k^\perp + \lambda)$ is restriction, and $\tilde{D}$ is the element in $\mathcal{S}^*(k^\perp - \lambda)$ defined by $(\tilde{D}, f) = (D, \hat{f})$. Thus $(P_\lambda^* D)^\wedge$ is supported on $k^\perp + \lambda$ and has no normal derivatives.

For $f \in \mathcal{S}(N/H, x_\lambda)$ and $D \in \mathcal{S}^*(N/H, x_{-\lambda})$, the convolution $D * f$ is defined by setting $D * f(x) = (D, l_x(f))$ for each $x \in N$. Suppose now that $D \in \mathcal{S}^*(N)$ and $f \in \mathcal{S}(N)$. One can use Abelian Fourier analysis to study the mapping defined on $x$, the center of $\kappa$, by $Y \mapsto D * f(\exp(X + Y))$. If this mapping is in $\mathcal{S}(\kappa)$, then

$$D * f(\exp X) = \int_{x^*} P_\lambda(D * f)(\exp X) \, d\lambda,$$

for appropriately normalized Lebesgue measure $d\lambda$. Furthermore, $P_\lambda(D * f) = D_\lambda * P_\lambda f$, where $D_\lambda$ is the element of $\mathcal{S}^*(N/H, x_{-\lambda})$ whose Fourier transform, $\tilde{D}_\lambda$, agrees with the restriction to $k^\perp + \lambda$ of $\tilde{D}$. Thus, convolution between elements of $\mathcal{S}^*(N)$ and $\mathcal{S}(N)$ decomposes into convolutions between elements of $\mathcal{S}^*(N/H, x_{-\lambda})$ and $\mathcal{S}(N/H, x_\lambda)$ in such a way that smoothness and growth conditions on $\tilde{D}$, $D \in \mathcal{S}^*(N)$ are inherited by $\tilde{D}_\lambda$, $D_\lambda \in \mathcal{S}^*(N/H, x_{-\lambda})$. One then proceeds by induction on the dimension of $N/H$. Of course, this requires maintaining considerable control of the various seminorm estimates that appear in the decompositions.

The proof of Theorem B follows along the usual induction argument lines with the Plancherel Theorem being used to reduce the dimension.

For Theorem C, one constructs a $\theta$ on $\kappa^*$ such that both $\theta$ and $1/\theta$ are in $PB_\infty^2(\kappa^*)$, and such that $\sum ||A(\xi)||_0(\xi) < \infty$, the sum being over $\left(\Gamma \setminus N\right)_0^\bullet$. One then uses the fact that $(D_1/\theta * f) * (D_\theta * g) = f * g$ and the Dixmier and Mallivan [DM] factorization to complete the proof.

REMARKS. The fact that $D_\theta \in \mathcal{M}\mathcal{S}(N)$ was proved by R. Howe in [H], and indeed, the ideas presented there are the foundation of this work. Theorem B was proved for the case where $\theta$ is a polynomial by A. Kirillov in [K]. In [CG], L. Corwin and F. Greenleaf proved Theorem C with the additional assumption that all the representations in general position were induced from a common, normal subgroup. One-sided Schwartz multipliers have been studied by L. Corwin in [C].
REFERENCES

[C] L. Corwin, *Tempered distributions on Heisenberg groups whose convolution with Schwartz class functions is Schwartz class*, J. Funct. Anal. **44** (1981), 328–347.

[CG] L. Corwin and F. Greenleaf, *Solvability of certain left-invariant differential operators by nilmanifold theory*, Comm. Pure Appl. Math. **36** (1983), 755–765.

[DM] J. Dixmier et P. Mallivan, *Factorisations de fonctions et de vecteurs indéfiniment différentiables*, Bull. Sci. Math. **102** (1978), 305–330.

[H] R. Howe, *On a connection between nilpotent groups and oscillatory integrals associated to singularities*, Pacific J. Math. **73** (1977), 329–364.

[K] A. Kirillov, *Unitary representations of nilpotent Lie groups*, Uspekhi Mat. Nauk. **17** (1962), 57–110.

[R] M. Rais, *Solutions élémentaires des opérateurs différentiels bi-invariants sur un groupe de Lie nilpotent*, C. R. Acad. Sci. Paris Sér. A-B **273** (1971), A495–498.

[S] L. Schwartz, *Théorie des distributions*, tome II, Hermann, Paris, 1959.

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