TWO-DIMENSIONAL METRIC SPACES WITH CURVATURE BOUNDED ABOVE I

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Abstract. We determine the local geometric structure of two-dimensional metric spaces with curvature bounded above as the union of finitely many properly embedded/branched immersed Lipschitz disks. As a result, we obtain a graph structure of the topological singular point set of such a singular surface.

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1. INTRODUCTION

Let $X$ be a locally compact, geodesically complete Alexandrov space with curvature bounded above. In this paper, we are concerned with the local structure of $X$. In general $X$ may have very complicated local geometry. For instance, $X$ may have no polyhedral structure even in local. There is such a two-dimensional space constructed by

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Kleiner (cf. [22]). In the present paper, we completely describe the local geometry of such spaces in dimension two.

The study of metric spaces with curvature bounded above began with the work of Alexandrov [3]. For the dimensions of such spaces $X$, Kleiner [16] proved that the topological dimension coincides with the maximal dimension of topological manifolds embedded in $X$. For \textit{geodesically complete} metric spaces $X$ with curvature bounded above, Otsu-Tanoue [26] implicitly showed that the topological dimension coincides with the Hausdorff dimension, which has been verified via a different method by a recent work due to Lytchak-Nagano [18]. [18] has also clarified that the local geometric properties of geodesically complete metric spaces $X$ with curvature bounded above have a lot of analogues to those of Alexandrov spaces with curvature bounded below (see also Remarks 1.5 and 1.7 below). Lytchak-Stadler [20] have recently proved that for every convex open ball in a CAT($\kappa$)-space there exists a complete CAT($-1$)-metric on the ball that is locally bi-Lipschitz to the original CAT($\kappa$)-metric; in particular, in local considerations on topological properties of CAT($\kappa$)-spaces, we may assume $\kappa$ to be $-1$.

For basic textbooks in this subject, there are several general references, and we refer to Ballmann [9], Bridson-Haefliger [11], Burago-Burago-Ivanov [12], Alexander-Kapovitch-Petrunin [2].

Now let us consider our main concern, the two-dimensional such spaces. The study in this particular dimension began with a classical deep work due to Alexandrov-Zalgeller [6] on two-dimensional topological manifolds with more general curvature bound, called the \textit{bounded curvature}. They constructed the curvature measure on such surfaces and established the Gauss-Bonnet theorem. See also Reshetnyak [29] for the work from an analytic point of view. Generalizing [6] and succeeding the works of Ballmann-Buyalo [10] and Arsinova-Buyalo [8], Burago-Buyalo [13] established the theory of two-dimensional polyhedra with curvature bounded above.

Here it should be emphasized that there were no general results determining local structure even in dimension two. The purpose of this paper is to determine the general local geometric structure of two-dimensional geodesically complete metric spaces with curvature bounded above.

Let $X$ be a two-dimensional locally compact, geodesically complete metric space with curvature $\leq \kappa$ for a constant $\kappa$. For every $p \in X$, the space of directions $\Sigma_p = \Sigma_p(X)$ is the disjoint union of finitely many points and connected finite graphs. Since we are interested in the local structure, we assume the most essential case when $\Sigma_p$ is a connected graph, called a CAT($1$)-graph (see Section 2). We shall determine the geometry of the closed $r$-ball $B(p, r)$ around $p$ for small enough $r > 0$ as follows.
Let $\mathcal{S}(X)$ denote the set of all topological singular points in $X$. For $\ell \geq 2\pi$ and $r > 0$, we denote by $D^2(\ell; r)$ the closed disk of radius $r$ around the vertex $O$ in the Euclidean cone over the circle of length $\ell$. A map $f : D^2(\ell; r) \to B(p, r)$ is called proper if $f^{-1}(\partial B(p, r)) = \partial D^2(\ell; r)$. Let $\tau_p(r)$ denote a function depending on $p$ and $r$ satisfying $\lim_{r \to 0} \tau_p(r) = 0$. Let $S(p, r)$ denote the metric sphere $\partial B(p, r)$.

The main result in this paper is stated as follows.

**Theorem 1.1.** For every $p \in X$ such that $\Sigma_p$ is a connected graph, there exists a positive number $r_0$ such that for every $0 < r \leq r_0$, $B(p, r)$ is a union of images $\text{Im} f_i$ of finitely many proper Lipschitz immersions $f_i : D^2(\ell_i; r) \to B(p, r)$ for some $\ell_i \geq 2\pi$, possibly with branch point $f_i^{-1}(p) = \{O\}$ satisfying the following:

1. With respect to the length metric induced from $X$, $\text{Im} f_i$ are CAT($\kappa$)-spaces;
2. Either $f_i$ is an embedding, or else $f_i(\partial D^2(\ell_i; r))$ is the union of two circles of length $\geq 2\pi r$ connected by an arc, which could be a point. In the latter case, $\ell_i \geq 4\pi$;
3. The bi-Lipschitz constant of $f_i$ is less than $1 + \tau_p(r)$ when $f_i$ is an embedding. If $f_i$ is a branched immersion, the local bi-Lipschitz constant of $f_i$ except at $\{O\}$ is less than $1 + \tau_p(r)$.

Moreover, $\mathcal{S}(X) \cap B(p, r)$ consists of finitely many simple Lipschitz arcs starting from $p$ and reaching $S(p, r)$.

**Remark 1.2.** One might ask if it is possible to fill the ball $B(p, r)$ with those $\text{Im} f_i$ that are convex in $X$ or properly embedded disks. However, both are impossible in general. For example, take the Euclidean cone $X$ over the union of two circles of length $2\pi$ joined by an arc. Note that any metric ball around the vertex of $X$ can not be written as a union of properly embedded disks as described in Theorem 1.1. For an example showing the impossibility of filling the ball via convex properly embedded CAT($\kappa$)-disks, see Example 4.5 for instance.

From the proof of Theorem 1.1, we actually have the following.

**Corollary 1.3.** Let $r = r_p$ be sufficiently small as in Theorem 1.1. Then for any locally injective continuous map $\zeta : [a, b] \to \Sigma_p(X)$, there is a closed subset $E$ of $X$ containing $p$ satisfying

1. $E$ is a CAT($\kappa$)-space with respect to the length metric;
2. $\Sigma_p(E) = \text{Im}(\zeta)$;
3. $\partial E \subset S(p, r)$ possibly except the segments from $p$ directing to the endpoints of $\zeta$. Here $\partial E$ denotes the set of points of $E$ where local geodesically completeness of $E$ fails.

The set $E$ is the image of a locally almost isometric, branched immersion, except at the branch locus $\{p\}$, from the closed disk of radius $r$ around the vertex in the Euclidean cone over the interval of length
L(ζ). When ζ is surjective in addition, this provides another description of \( B(p, r) \).

Using Theorem 1.1 we can define a metric graph structure on \( S(X) \) in a generalized sense (see Definition 6.7), and we have

**Corollary 1.4.** Suppose that \( \Sigma_p \) is a connected graph for every \( p \in X \). Then with respect to the induced length structure, \( S(X) \) is isometric to a metric graph having (possibly locally uncountably many vertices, but) the vertices of locally finite order.

**Remark 1.5.** In the general dimension, [18] has characterized the singular set in the \( k \)-dimensional part as a countably \((k - 1)\)-rectifiable set. Corollary 1.4 gives a refinement of this result in dimension two.

Recall that a compact metric graph \( \Sigma \) is a CAT(\( \mu \))-graph \((\mu > 0)\) if every non-contractible loop in \( \Sigma \) has length \( \geq 2\pi/\sqrt{\mu} \).

**Corollary 1.6.** For a given \( p \in X \) such that \( \Sigma_p \) is a connected graph, there exists a positive number \( r_p \) such that for every \( 0 < r \leq r_p, S(p, r) \) with the interior metric is a CAT(\( \mu_p(r) \))-graph having the same homotopy type as \( \Sigma_p \), where \( \mu_p(r) \) is given by the sharp constant

\[
\mu_p(r) = \begin{cases} 
\frac{\sin \sqrt{\kappa r}}{\sqrt{\kappa}} & \text{if } \kappa > 0, \\
\left( \frac{1}{r} \right)^2 & \text{if } \kappa = 0, \\
\left( \frac{\sinh \sqrt{-\kappa r}}{\sqrt{-\kappa}} \right)^2 & \text{if } \kappa < 0. 
\end{cases}
\]

**Remark 1.7.** A result in [18] shows that for every small \( r \), \( S(p, r) \) has the same homotopy type as \( \Sigma_p \) in the general dimension. Corollary 1.6 gives a refinement of this result in dimension two.

**Remark 1.8.** All the results in this paper are local. Therefore they are also valid under the assumption of local geodesic completeness of \( X \).

The idea of the proof of the main result is as follows. We know the structure of the space \( \Sigma_p \) of directions at \( p \), which is completely characterized as a CAT(1)-graph without endpoints. If we rescale the metric of \( X \) by the factor \( 1/r \), then \( (\frac{1}{r}X, p) \) converges to the tangent cone \( (K_p, o_p) \) at \( p \) as \( r \to 0 \) with respect to the pointed Gromov-Hausdorff topology. Let \( \Sigma_p^{\text{sing}} \) be a small neighborhood of the vertices of the graph \( \Sigma_p \), and \( \Sigma_p^{\text{reg}} \) the complement of \( \Sigma_p^{\text{sing}} \). Now the convergence theorem ([23]) applied to the unit cone \( K_1(\Sigma_p^{\text{reg}}) \) over \( \Sigma_p^{\text{reg}} \) yields the existence of a Lipschitz domain \( B^{\text{reg}}_1(p, r) \) of \( B(p, r) \) consisting of finitely many sectors corresponding to sectors of \( K_1(\Sigma_p^{\text{reg}}) \). One can
consider $B_{\text{reg}}(p, r)$ as a regular part of $B(p, r)$. The main problem is to determine the structure of the singular part $B_{\text{sing}}(p, r)$, the complement of $B_{\text{reg}}(p, r)$ in $B(p, r)$. To carry out this, we consider finitely many thin ruled surfaces, say $S_{ij}$ here, and fill $B_{\text{sing}}(p, r)$ using them. A key is to show that those ruled surfaces are CAT($\kappa$)-spaces with respect to the interior metrics and are homeomorphic to a disk. According to Alexandrov's result in [4], every ruled surface in a CAT($\kappa$)-space is also a CAT($\kappa$)-space with respect to the pull-back metric. Obviously, the interior metric and the pull-back metric are completely different from each other in general. Therefore we have to show that in our thin ruled surfaces pull-back metrics coincide with the interior metrics. After achieving this, it turns out that the topological singular point set $S(X)$ locally arises from the intersections of those thin ruled surfaces $S_{ij}$. We investigate how those ruled surfaces meet each other to get the structure of $S(X) \cap B(p, r)$ as the union of finitely many Lipschitz curves. Combining the structures of both $B_{\text{reg}}(p, r)$ and $B_{\text{sing}}(p, r)$ and considering the graph structure of $\Sigma_p$, we define the embeddings or the branched immersions $f_i : D^2(\ell_i; r) \to B(p, r)$ as described in Theorem 1.1.

As related studies on ruled surfaces, Petrunin-Stadler [27] have proved that for metric minimizing disks in CAT(0)-spaces, the pull-back metrics on the disks are CAT(0), which is a generalization of Alexandrov's result [4] on ruled surfaces in the CAT(0)-setting. According to Stadler [30, Theorem 2], for any Jordan triangle in a CAT(0)-space, every minimal disk filling of the triangle is an embedded disk that is CAT(0) with respect to the interior metric.

The organization of the paper is as follows. In Section 2, we recall and verify basic results for locally compact, geodesically complete Alexandrov spaces with curvature bounded above.

In Section 3, we give basic properties of a ruled surface $S$ in a CAT($\kappa$)-space. We discuss the pullback metric, the induced metric, the interior metric of $S$ and their relations. In the original argument in Alexandrov [4], there are several unclear points for the authors. For instance, there is no description in [4] about quasicontinuous monotone representations. We make clear all these points.

In Section 4, which is a key section, we investigate a thin ruled surface $S$ in a two dimensional space, and prove that $S$ actually admits the induced metric and therefore becomes a CAT($\kappa$)-space with respect to the interior metric. Then we obtain the crucial property that $S$ is homeomorphic to a disk.

In Section 5, we fill $B(p, r)$ via those embedded/branched immersed disks using thin ruled surfaces essentially. We prove Theorem 1.1(1), (2) and (3) except (1) for branched immersed disks.
In Section 6, we describe $S(X) \cap B(p, r)$ as a union of finitely many Lipschitz curves starting from $p$ and reaching points of $S(p, r)$. The structure of generalized metric graph of $S(X)$ is also discussed there.

In Section 7, we provide the proof of Theorem 1.1(1) for branched immersed disks as well as Corollary 1.3. In Appendix A following the basic idea of [4], we give the proof of Alexandrov’s result on ruled surfaces in CAT($\kappa$)-spaces based on the results proved in Section 3. Burago-Buyalo [13] gave a complete characterization of two-dimensional polyhedra of curvature bounded above. In the second part [25] of our works, we show the following:

(a) We provide sufficient conditions for two-dimensional metric spaces to have curvature bounded above, which shows that the results in this paper completely characterize the local structure of two-dimensional metric spaces with curvature bounded above.

(b) Every pointed two-dimensional geodesically complete locally CAT($\kappa$)-space $(X, p)$ can be approximated by a sequence of two-dimensional pointed geodesically complete, polyhedral locally CAT($\kappa$)-spaces $(X_n, p_n)$ having the same homotopy type as $X$ with respect to the pointed Gromov-Hausdorff topology. This solves a problem raised in Burago-Buyalo [13].

(c) We establish a Gauss-Bonnet type theorem for two-dimensional geodesically complete locally CAT($\kappa$)-spaces.

Most results in the present paper were announced in [32].

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2. Basic properties of CAT($\kappa$)-spaces

For some basic results in this section, we refer to [11], [12].

The distance between two points $x, y$ in a metric space $X$ is denoted by $|x, y|$ or $|x, y|_X$, and $d(x, y)$ or $d_X(x, y)$ sometimes. The metric $r$-ball around $p$ is denoted by $B(p, r)$. We sometimes use $B^X(p, r)$ to
emphasize the metric ball in $X$. Let $X$ be a locally compact, complete geodesic space with curvature $\leq \kappa$. By definition, for each point $p \in X$, there exists a positive number $r > 0$ with $r \leq \pi/2\sqrt{\kappa}$ when $\kappa > 0$ such that the ball $B(p, r)$ is convex and having the following properties: Let $M^2_\kappa$ be the simply connected complete surface of constant curvature $\kappa$, called the $\kappa$-plane in short. For any geodesic triangle $\triangle xyz$ in $B(p, r)$ with vertices $x, y$ and $z$, we denote by $\tilde{\triangle} xyz$ a comparison triangle in $M^2_\kappa$ having the same side lengths as $\triangle xyz$. Then the natural mapping $\tilde{\triangle} xyz \to \triangle xyz$ is non-expanding. A convex domain with this property is called a CAT($\kappa$)-domain. Such a space $X$ with curvature $\leq \kappa$ is called a locally CAT($\kappa$)-space, and $X$ is called a CAT($\kappa$)-space if $X$ itself is a CAT($\kappa$)-domain. Although all geodesics have constant speed by definition, most geodesics are assumed to have unit speed unless otherwise stated. For arbitrary $x$ and $y$ in $B(p, r)$, let $\gamma_{x,y} : [0, |x, y|] \to X$ denote a unique minimal geodesic joining $x$ to $y$. We say that a curve is shortest if its length is minimal among all curves joining the endpoints. The angle between the geodesics $\gamma_{y,x}$ and $\gamma_{y,z}$ is denoted by $\angle xyz$, and the corresponding angle of $\tilde{\triangle} xyz$ by $\tilde{\angle} xyz$. The space of directions and the tangent cone of $X$ at $p$ are denoted by $\Sigma_p = \Sigma_p(X)$ and $K_p = K_p(X)$ respectively. We shall occasionally use the identification $\Sigma_p = \Sigma_p \times \{1\} \subset K_p$. We denote by $\tilde{\gamma}_{x,y}(0)$ or $\tilde{\gamma}^y_x$, the direction at $x$ defined by $\gamma_{x,y}$. For every $\xi \in \Sigma_p(X)$, $\gamma_{\xi}$ denotes a geodesic with $\dot{\gamma}_{\xi}(0) = \xi$. For a path-connected subset $S \subset X$ and $x, y \in S$, we denote by $\gamma^S_{x,y}$ a shortest curve in $S$ joining $x$ to $y$ if it exists. Occasionally, we identify a geodesic with its image, and write as $x \in \gamma$ for instance. The length metric of $S$ induced from $X$ is denoted by $d_S$ or $|\cdot|$. For a closed subset $A$ of $X$ and for an accumulation point $p$ of $A$, the set of all directions $\xi \in \Sigma_p(X)$ such that there is a sequence $a_i$ in $A \setminus \{p\}$ satisfying $a_i \to p$ and $\tilde{\gamma}_{p,a_i}(0) \to \xi$ is denoted by $\Sigma(A)$ and called the space of directions of $A$ at $p$.

The upper semi-continuity of angle is fundamental in the geometry of spaces with curvature bounded above.

**Lemma 2.1.** Suppose that sequences $p_i, q_i$ and $r_i$ converge to $p, q$ and $r$ respectively in a CAT($\kappa$)-domain. Then we have $\limsup_{i \to \infty} \angle p_i q_i r_i \leq \angle pqr$.

Next we shortly discuss the connectivity of a small neighborhood of a given point in $X$. For each point $p \in X$, the set of components of $\Sigma_p$ are in one to one correspondence with the set of components of $B(p, r) \setminus \{p\}$ if $B(p, r)$ is a CAT($\kappa$)-domain (cf. [17]). We call the number of components of $\Sigma_p(X)$ the order of $p$.

Now we state the gluing theorem proved by [28], which is convenient to construct spaces with curvature bounded above. The proof is also found in [11, p.347].
Theorem 2.2. Let $D_i, i = 1, 2$, be a closed convex subset in an Alexandrov space $X_i$ with curvature $\leq \kappa$. If there is an isometry $f : D_1 \to D_2$, then the identification space $X_1 \cup_f X_2$ is an Alexandrov space with curvature $\leq \kappa$ with respect to the natural length metric.

From now, we assume $X$ to be geodesically complete. That is, every geodesic segment in $X$ can be extended to a geodesic defined on $\mathbb{R}$.

The following lemma follows from [13, Corollary 13.3].

Lemma 2.3. For a point $p \in X$, suppose that $\Sigma_p(X)$ has no isolated points. Then there exists a positive number $r$ such that every point $x$ in $B(p, r) \setminus \{p\}$ has order one.

Let $d_p$ denote the distance function from $p$. For every $x \neq p$, let us denote by $(\nabla d_p)(x)$ the set of all directions $\xi \in \Sigma_x(X)$ such that $\angle(\xi, \uparrow_p x) = \pi$. For simplicity, we set $-\nabla d_p(x) = \uparrow_p x$. The following lemma, which describes local geometry around a given point, is basic in our study of local structure of surfaces with curvature bounded above.

Lemma 2.4. For every $p \in X$, there exists a positive number $r_0$ such that for every $r$ with $0 < r \leq r_0$, $B(p, r)$ satisfies the following:

1. $\text{diam}((\nabla d_p)(x)) < \tau_p(r)$ for every $x \in B(p, r) \setminus \{p\}$;

2. For any two geodesics $\gamma_1$ and $\gamma_2$ starting at $p$ with angle $\theta$, and for every $s \in [0, r]$, the geodesic $\sigma_s(t)$ joining $\gamma_1(s)$ to $\gamma_2(s)$ satisfies that

$$|\angle(-\nabla d_p)(\sigma_s(t)), \dot{\sigma}_s(t)) - \pi/2| < \tau_p(\theta, s).$$

Proof. (1) is due to [23] (see also [13, Prop.7.3]). (2) easily follows from (1), and hence the proof is omitted.

The following lemma is fundamental, and plays an important role as in the case of Alexandrov space with curvature bounded below ([14]). For the proof, see [23, Lemma 3.6].

Lemma 2.5 (Jack Lemma). For every $p \in X$, there exists a positive number $r_0$ such that if $x \neq y \in B(p, r_0)$ and $q$ satisfy that $\check{\angle}pxq > \pi - \epsilon$ and $|x, y| < \epsilon \min\{|p, x|, |q, x|\}$, then we have

$$|\check{\angle}pxy - \check{\angle}pxy| < \tau_p(|p, x|, \epsilon).$$

In the study of spaces of curvature bounded below, the theory of the Gromov-Hausdorff convergence has been useful. We apply it in our case of curvature bounded above.

We denote by $\mathcal{H}^n$ the $n$-dimensional Hausdorff measure, and set $\omega_n := \mathcal{H}^n(S^n(1))$, where $S^n(1)$ is the unit $n$-sphere.
Theorem 2.6 (\cite{23}, Compare \cite{31}). For each positive integer $n$, there is a positive number $\epsilon_n$ satisfying the following: Let $X_i$, $i = 1, 2, \ldots$, and $X$ be $n$-dimensional locally compact, geodesically complete, pointed Alexandrov spaces with curvature $\leq \kappa$, and suppose that a compact CAT($\kappa$)-domain $U_i$ of $X_i$ converges to a compact CAT($\kappa$)-domain $U$ of $X$ with respect to the Gromov-Hausdorff distance. Then for every compact domain $V$ in int $U$ satisfying $H^{n-1}(\Sigma_x(X)) < \omega_{n-1} + \epsilon$ with $\epsilon \leq \epsilon_n$ for all $x \in V$, there is a compact domain $V_i$ in int $U_i$ and a $\tau(\epsilon, 1/i)$-almost isometry $\varphi_i : V_i \to V$ in the sense that

$$\left| \frac{|\varphi_i(x), \varphi_i(y)|}{|x, y|} - 1 \right| < \tau(\epsilon, 1/i),$$

for all $x, y \in V_i$.

Lemma 2.7. For every $p \in X$ and arbitrary $x, y \in B(p, r)$ we have

$$\hat{\beta}_{x,y} - \angle_{x,y} < \tau_p(r), \quad \hat{\beta}_{y,x} - \angle_{y,x} < \tau_p(r), \quad \hat{\beta}_{x,y} - \angle_{y,x} < \tau_p(r).$$

Proof. For the proof, it suffices to show that for every $\epsilon > 0$ there is an $r$ such that for arbitrary $x, y \in B(p, r)$ we have (2.1) for $\epsilon$ in place of $\tau_p(r)$. Fix a constant $C \geq 1$.

Case 1). $C^{-1} \leq |p, x|/|p, y| \leq C$.

We show (2.1) for $\epsilon = \tau_{p,C}(r)$, where $\tau_{p,C}(\cdot)$ is a function depending on $p, C$ with $\lim_{r \to 0} \tau_{p,C}(r) = 0$. Suppose $|x, y| < \zeta |p, x|$ for $\zeta > 0$.

Then Lemma 2.7 implies

$$\hat{\beta}_{x,y} - \angle_{x,y} < \tau_p(r, \zeta), \quad \hat{\beta}_{y,x} - \angle_{y,x} < \tau_p(r, \zeta).$$

Since $\hat{\beta}_{x,y} < \tau(\zeta)$, (2.1) holds when $r \leq r_0$ and $\zeta \leq \zeta_0$ for some $r_0, \zeta_0$.

Next, suppose $|x, y| \geq \zeta_0 |p, x|$. We proceed by contradiction. Suppose the lemma does not hold in this case. Then there are $x_n, y_n \to p$ with $|x_n, y_n| \geq \zeta_0 |p, x_n|$. Choose $z_n$ such that $x_n \in \gamma_{p, z_n}$ and $|z_n, x_n| = |p, x_n|$. Consider the rescaling limit

$$\left( \frac{1}{|p, x_n|} X, p \right) \to (K_p, o_p).$$

Since $\lim_{n \to \infty} (\angle_{px_n y_n} + \angle_{z_n x_n y_n}) = \pi$, we have

$$\lim_{n \to \infty} \angle_{px_n y_n} = \angle_{o_p x_\infty y_\infty} = \lim_{n \to \infty} \hat{\beta}_{px_n y_n},$$

where $x_\infty$ and $y_\infty$ are the limits of $x_n$ and $y_n$. Similarly, we have $\lim_{n \to \infty} \angle_{py_n x_n} = \lim_{n \to \infty} \hat{\beta}_{py_n x_n}$. Since obviously we have

$$\lim_{n \to \infty} \angle_{x_n y_n} = \angle_{x_\infty o_p y_\infty} = \lim_{n \to \infty} \hat{\beta}_{x_n y_n},$$

we derive a contradiction.

Case 2). $|p, x| < C^{-1} |p, y|$.
We show (2.1) for \( \epsilon = \tau_p(r) + \tau(C^{-1}) \). First note that
\[
\tilde{\angle} pyx < \tau(C^{-1}).
\]
Thus considering large \( C \), we only have to consider the angles at \( p \) and \( x \). Take \( z \) with \( x \in \gamma_{p,z} \) and \( |z, p| = |y, p| \). Then we have
\[
\tilde{\angle} ypz \geq \tilde{\angle} ypx \geq \angle ypx.
\]
From Case 1), we have
\[
\tilde{\angle} ypz - \angle ypz < \tau_p(r),
\]
Combining (2.3) and (2.4), we certainly have
\[
\tilde{\angle} ypx - \angle ypx < \tau_p(r).
\]
From (2.2) and the first inequality in (2.6), we have
\[
|\tilde{\angle} pxz - \tilde{\angle} ypz| < \tau(C^{-1}), \quad |\tilde{\angle} xzy - \tilde{\angle} pzy| < \tau(C^{-1}).
\]
From (2.5) and (2.4), we have
\[
|\tilde{\angle} xpy - \tilde{\angle} zpy| < \tau_p(r).
\]
Combining the last two inequalities and the second inequality in (2.6), we have
\[
|\tilde{\angle} pxz + \tilde{\angle} yzx - \pi| < \tau_p(r) + \tau(C^{-1}).
\]
Now consider the quadrangle \( \tilde{\Delta} xyz \) on \( M^2_\kappa \) which is the union of the triangles \( \Delta pxy \) and \( \Delta xyz \) glued along the edge \( \tilde{x} \tilde{y} \) corresponding to \( xy \). We estimate the deviation of the angle of the quadrangle \( \tilde{\Delta} xyz \) at \( \tilde{x} \) from \( \pi \). Combining the last two inequalities and the second inequality in (2.6), we have
\[
|\tilde{\angle} pxz + \tilde{\angle} yzx - \pi| < \tau_p(r) + \tau(C^{-1}).
\]
Since
\[
|\angle pxy + \angle yxz - \pi| < \tau_p(r),
\]
the last two inequalities yield \( \tilde{\angle} pxy - \angle pxy < \tau_p(r) + \tau(C^{-1}) \) as required. This completes the proof. \( \square \)

A point \( p \) in \( X \) is called a topological singular point of \( X \) if any neighborhood of \( p \) is not homeomorphic to a disk, and the set of all topological singular point of \( X \) is denoted by \( S(X) \). It is proved in [18] that if \( \dim X = n \), then \( \dim_H S(X) \leq n - 1 \). In particular \( X \setminus S(X) \) has full measure with respect to \( H^n \) ([26]).

**Two-dimensional case.** By a result of Otsu-Tanoue [26], the Hausdorff dimension of every relatively compact open domain of \( X \) is an integer. See [18] for a different proof. It is also known that \( \Sigma_p(X) \) is a compact geodesically complete CAT(1)-space for every \( p \in X \).

Obviously, if \( X \) is 1-dimensional, then it is a locally finite graph without endpoints. Now we assume \( X \) has dimension 2. Then any component \( \Sigma \) of \( \Sigma_p \) has dimension \( \leq 1 \). If \( \dim \Sigma = 1 \), then \( \Sigma \) has the
structure of a finite graph without endpoints. Furthermore $\Sigma$ is a so called CAT(1)-graph without endpoints in the sense that each simple closed curve in $\Sigma$ has length at least $2\pi$. If $\dim \Sigma = 0$, then $\Sigma$ is a point and the component of $B(p, r) \setminus \{p\}$ corresponding to $\Sigma$ is an arc for any small enough $r$. Thus, a small neighborhood of any point $p \in X$ is the gluing at $p$ of several purely 2-dimensional spaces with all links connected graphs and a ball around the vertex in the cone over finitely many points. Therefore the study of local structure around $p$ reduces to the case when $\Sigma_p$ is a connected CAT(1)-graph without endpoints.

**Lemma 2.8.** A neighborhood of $p \in X$ is homeomorphic to a two-dimensional disk if and only if $\Sigma_p(X)$ is a circle.

**Proof.** This follows from [24, Prop.3.1, Remark 3.4]. ⊓⊔

**Lemma 2.9.** Let $p \in S(X)$. Then $\Sigma_p(S(X))$ coincides with the set $V(\Sigma_p(X))$ of all vertices of the graph $\Sigma_p(X)$.

**Proof.** For every $v \in \Sigma_p(S(X))$, take a sequence $x_i$ in $S(X)$ converging to $p$ such that $\lim_{i \to \infty} \angle(\hat{\gamma}_{p,x_i}(0), v) = 0$. If $v$ is not a vertex of $\Sigma_p(X)$, choose $\epsilon > 0$ such that the $\epsilon$-neighborhood of $v$ contains no vertices of $\Sigma_p(X)$. Let $\delta_i := |x_i, p|$. Theorem 2.6 applied to the convergence $(\frac{1}{\epsilon}X, x_i) \to (K_p(X), v)$ yields that a small neighborhood of $x_i$ is almost isometric to a neighborhood in $\mathbb{R}^2$. This is a contradiction.

Conversely, suppose there is $v \in V(\Sigma_p(X))$ that is not contained in $\Sigma_p(S(X))$. Choose $\epsilon_0 > 0$ and $\delta_0 > 0$ such that the cone neighborhood

$$C(v, \delta_0, \epsilon_0) := \{x \mid \angle(\hat{\gamma}^v_x, v) \leq \delta_0, |p, x| \leq \epsilon_0\}$$

is included in $\mathcal{R}(X)$. Take three distinct directions $\xi_1, \xi_2, \xi_3 \in \Sigma_p(X)$ having angle $\delta_0/2$ with $v$, and set $x_i(\epsilon) := \gamma_{\xi_i}(\epsilon)$, where $\epsilon \leq \epsilon_0$, $1 \leq i \leq 3$. Note that the geodesic $[x_1(\epsilon), x_2(\epsilon)]$ converges to the geodesic $[\xi_1, \xi_2]$ in $K_p(X)$ under the convergence

$$\left(\frac{1}{\epsilon}X, p\right) \to (K_p(X), \sigma_p) \quad \text{as } \epsilon \to 0.$$

Let $y(\epsilon)$ be a nearest point of $[x_1(\epsilon), x_2(\epsilon)]$ from $x_3(\epsilon)$. Since $y(\epsilon) \in \mathcal{R}(X)$, $\Sigma_{y(\epsilon)}(X)$ must be a circle of length almost equal to $2\pi$ with $\angle(\nabla d_p(y(\epsilon)), -\nabla d_p(y(\epsilon))) = \pi$. However, Lemma 2.4 shows that the angle $\angle(\nabla d_p(y(\epsilon)), \eta_i(\epsilon))$ is almost $\pi/2$, where $\eta_i := x_i^{x_i(\epsilon)}$, $1 \leq i \leq 3$, which implies $\angle(\eta_1(\epsilon), \eta_2(\epsilon))$ is almost equal to $0$. This is a contradiction since $\angle(\eta_1(\epsilon), \eta_2(\epsilon)) = \pi$. ⊓⊔

**Remark 2.10.** In place of the above geometric argument of the second half of the proof of Lemma 2.9, we can also use more general topological result in [15, Theorem 2.1].

**Lemma 2.11.** Let $p \in S(X)$. For any $x \in S(X) \cap (B(p, r) \setminus \{p\})$, $V(\Sigma_x(X))$ is contained in the $\tau_p(r)$-neighborhood of $\{(-\nabla d_p)(x), (\nabla d_p)(x)\}$. 
Therefore there is a positive integer \( m \geq 3 \) such that the Gromov-Hausdorff distance between \( \Sigma_x(X) \) and the spherical suspension over \( m \) points is less than \( \tau_p(r) \).

**Proof.** This follows from [18, Proposition 6.6, Corollary 13.3]. \( \square \)

As an immediate consequence of Lemmas 2.9 and 2.11, we have

**Corollary 2.12.** Let \( p \in \mathcal{S}(X) \). For every \( x \in \mathcal{S}(X) \) \( \cap (B(p,r) \setminus \{p\}) \), \( \Sigma_x(\mathcal{S}(X)) \) is contained in a \( \tau_p(r) \)-neighborhood of \( \{(\nabla d_p)(x), (\nabla d_p)(x)\} \).

Finally in this subsection, we shortly discuss the cardinality of singular points in a two-dimensional manifold \( X \) with curvature \( \leq \kappa \). Let \( \epsilon > 0 \). We say that \( x \in X \) is an \( \epsilon \)-singular point if \( L(\Sigma_x(X)) \geq 2\pi + \epsilon \). We also say that \( x \) is a singular point if it is \( \epsilon \)-singular for some \( \epsilon > 0 \).

**Lemma 2.13.** (cf. [6], [13, Prop.4.5]) For a domain \( D \) of a two-dimensional manifold \( X \) with curvature \( \leq \kappa \), the set of all singular points contained in \( D \) is at most countable.

**Proof.** By Lemma 2.4(1), the set of all \( \epsilon \)-singular points contained in a bounded set is finite for every \( \epsilon > 0 \), which immediately yields the conclusion of the lemma. \( \square \)

## 3. Basic properties of ruled surfaces

We recall the notion of ruled surfaces in metric spaces introduced by Alexandrov [4]. The metric on a ruled surface discussed in [4] is the pull-back metric defined below, although an explicit definition was not given in [4]. See also Remark 3.4. In this section, we provide some fundamental properties of the pull-back metric, most of which are not contained in [4]. These are used in the proof of Alexandrov’s result (Theorem 3.17), which is presented in Appendix A. There are related results in [27, Section 2].

For our purpose, it is sufficient to consider ruled surfaces in spaces with curvature bounded above. Throughout this section, let \( X \) be a locally compact, complete geodesic space with curvature \( \leq \kappa \) with metric \( d_X \), where we do not need the dimension restriction, nor geodesic completeness.

We fix a rectangle \( R := [0, \ell] \times [0, 1] \) in this section.

**Ruled surfaces.**

**Definition 3.1.** A continuous map \( \sigma: R \to X \) is called a ruled surface in \( X \) if

1. for every \( s \in [0, \ell] \) the \( t \)-curve \( \lambda_s: [0, 1] \to X \) of \( \sigma \) defined as \( \lambda_s(t) := \sigma(s, t) \) is a minimal geodesic in \( X \) from \( \sigma(s, 0) \) to \( \sigma(s, 1) \);
(2) for some continuous function $\xi : [0, \ell] \to [0, 1]$, the curve $\Sigma(s) = \sigma(s, \xi(s))$ ($0 \leq s \leq \ell$) is rectifiable with respect to $dX$.

As usual, the subset $S$ of $X$ defined as $S := \sigma(R)$ is also called a ruled surface in $X$. For each $s \in [0, \ell]$, the minimal geodesic $\lambda_s : [0, 1] \to X$ is called a generator of $\sigma$, or a ruling geodesic of $\sigma$.

For each $t \in [0, 1]$, the curve $\sigma_t : [0, \ell] \to X$ is called a directrix of $\sigma$ at $t$.

**Pull-back metrics and induced metrics on ruled surfaces**

Let $\sigma : R \to X$ be a ruled surface in $X$ defined as above. We denote by $\text{Sing}(\sigma)$ (resp. by $\text{Reg}(\sigma)$) the set of all $s \in [0, \ell]$ such that $\lambda_s$ are constant (resp. nonconstant). For $s \in [0, \ell]$, we set

$$I_s := \{s\} \times [0, 1] \subset R.$$

**Definition 3.2.** We say that a (not necessarily continuous) map $c : [a, b] \to R$ is monotone if

- $p_1 \circ c$ is monotone non-decreasing or monotone non-increasing where $p_1 : R \to [0, \ell]$ is the projection to the first factor;
- if $p_1 \circ c(t) = p_1 \circ c(t') = s$ with $t < t'$, then $p_2 \circ c|_{[t, t']}$ is monotone, where $p_2 : R \to [0, 1]$ is the projection to the second factor.

Similarly, $c$ is said to be strictly monotone if $p_1 \circ c$ is strictly monotone.

We say that a monotone map $c : [a, b] \to R$ is a quasicontinuous curve if the following hold:

1. $p_1 \circ c([a, b])$ is a closed interval;
2. $c$ is continuous on the set of all $t$ with $p_1 \circ c(t) \in \text{Reg}(\sigma) \cup \text{int Sing}(\sigma)$.

We define the pull-back metric $e_\sigma$ on $R$ induced from $\sigma$ as

$$e_\sigma(u, u') := \inf_c L(\sigma \circ c),$$

where $c$ runs over all quasicontinuous curves in $R$ from $u$ to $u'$, and $L$ denotes the length of curves with respect to $dX$. Note that the metric $e_\sigma$ is certainly finite since our ruled surface $\sigma$ has the rectifiable curve $\Sigma$.

We denote by $R_*$ the quotient metric space $(R_*, e_\sigma) := (R, e_\sigma)/\{e_\sigma = 0\}$.

Let $\pi : R \to R_*$ be the projection.

**Example 3.3.** Let $\sigma_k : [0, 1/\pi] \to \mathbb{R}^2$ ($k = 0, 1$) be the curve defined by

$$\sigma_0(s) = \left(s, -s \cos \frac{1}{s}\right), \quad \sigma_1(s) = \left(s, s \sin \frac{1}{s}\right).$$

For $R := [0, 1/\pi] \times [0, 1]$, define the ruled surface $\sigma : R \to \mathbb{R}^2$ as in the above definition, where we have $\text{Sing}(\sigma) = \{0\}$. For $u = (0, 0)$,
\( u' = (1/\pi, 0) \), consider the map \( c : [0, 1/\pi] \to R \) by
\[
c(s) = \begin{cases} 
\sigma^{-1}(s, 0), & (0 < s \leq 1/\pi) \\
(0, 0), & (s = 0).
\end{cases}
\]

Since \( c \) oscillates infinitely many times near \( \{0\} \times [0, 1] \), it is quasicon- 
tinuous, but realizing the distance \( e_\sigma(u, u') \).

Remark that there is no continuous curve realizing \( e_\sigma(u, u') \) in Ex-
ample 3.3. Note also that \( \pi \circ c \) is always continuous for every quasi-
continuous curve \( c \). These are the reasons why we employ the notion 
of quasicontinuous curves in the definition (3.1) of the pull-back met-
ric \( e_\sigma \).

Obviously, \( e_\sigma(u, u') = 0 \) implies \( \sigma(u) = \sigma(u') \). Therefore, we can
define a map \( \sigma_* : R_* \to X \) such that \( \sigma = \sigma_* \circ \pi \). Note that \( \sigma_* : R_* \to X \)
is continuous. The properties of the projection \( \pi : R \to R_* \) depend 
on those of the end \( s \)-curves \( \sigma_0 \) and \( \sigma_1 \). If \( \sigma_0 \) and \( \sigma_1 \) are Lipschitz 
continuous, then so is \( \sigma \), and hence \( \pi : R \to R_* \) is continuous. However, 
in the general case, \( \pi : R \to R_* \) is not necessarily continuous (see Ex-
ample 3.5).}

Remark 3.4. Here are some remarks on the relation between the con-
ditions of ruled surfaces given in [4] and ours.

1. Some ruling geodesics \( \lambda_s \) of \( \sigma \) may be constant geodesics for all 
\( s \) in an interval of \([0, \ell]\). This case was excluded in [4] as the 
conditions of ruled surfaces defined there.

2. More restricted property, the existence of continuous arc in the 
preimage of any point of \( R_* \) by \( \pi \), than the existence of quasi-
continuous curve given in Corollary 3.8 was assumed in [4] as 
one of the conditions of the metric on the ruled surface under 
consideration.

Example 3.5. Let \( \sigma_1(s) \) \( (0 \leq s \leq \ell) \) be a continuous parametrization 
of a Koch curve on the unit sphere \( S^2(1) \subset \mathbb{R}^3 \). Letting \( \sigma_0(s) = O \), we 
define the ruled surface \( \sigma : [0, \ell] \times [0, 1] \to \mathbb{R}^3 \) by \( \sigma(s, t) = t\sigma_1(s) \). Note 
that
\[
e_\sigma((s, t), (s', t')) = \begin{cases} 
|t - t'| & \text{if } s = s' \\
t + t' & \text{if } s \neq s'.
\end{cases}
\]

Note that \( \pi : R \to R_* \) is continuous only at \( \{t = 0\} \).

For \( s \in [0, \ell] \), we set
\[
I^*_s := \pi(I_s).
\]

For a continuous curve \( c_* : [a_0, b_0] \to R_* \) (resp. \( \gamma : [a, b] \to S \)), 
we simply say that a quasicontinuous curve \( c : [a, b] \to R \) is a lift 
of \( c_* \) (resp. of \( \gamma \)) if \( c_* = \pi \circ c \) (resp. \( \gamma = \sigma \circ c \)) up to monotone 
parametrization.
From now, we fix arbitrary \( u, u' \in R \) with \( u = (s_0, t_0), u' = (s'_0, t'_0) \) and \( s_0 < s'_0 \). Take a sequence of quasicontinuous curves \( c_n : [0, 1] \to R \) from \( u \) to \( u' \) such that \( e_\sigma(u, u') = \lim_{n \to \infty} L(\sigma \circ c_n) \), where we may assume that \( c_n \) is monotone. By the Arzela-Ascoli theorem, passing to a subsequence we may assume that a Lipschitz parametrization \( \gamma_n \) of \( \sigma \circ c_n \) converges to a Lipschitz curve \( \gamma \) in \( S \) from \( \sigma(u) \) to \( \sigma(u') \). Note

\[
L(\gamma) \leq e_\sigma(u, u').
\]

We set

\[
J := [s_0, s'_0], \quad J_{\text{reg}} := J \cap \text{Reg}(\sigma), \quad J_{\text{sing}} := J \cap \text{Sing}(\sigma).
\]

In the following proposition, we show the equality in (3.2).

**Proposition 3.6.** Under the above situation, there is a lift \( c \) of \( \gamma \) in \( R \) from \( u \) to \( u' \).

In particular, \( \pi \circ c \) provides a (continuous) shortest curve \( c_* \) in \( R_* \) from \( \pi(u) \) to \( \pi(u') \), and we have

\[
L(c_*) = L(\gamma) = e_\sigma(\pi(u), \pi(u')).
\]

**Example 3.7.** Let \( \gamma : [0, 1] \to X \) be a minimal geodesic between distinct two points in a \( \text{CAT}(\kappa) \)-space. Consider the ruled surface \( \sigma : [0, 1] \times [0, 1] \to X \) defined as \( \sigma(s, t) = \gamma(t) \). Then \( e_\sigma((0,0), (1,1)) = L(\gamma) \). Note that any curve \( c(t) = (x(t), y(t)) \) such that \( x(t) \) and \( y(t) \) are monotone from 0 to 1 is a lift of \( \gamma \) from \( (0,0) \) to \( (1,1) \).

The above simple example suggests that in Proposition 3.6, one cannot construct a lift of the limit \( \gamma \) only from \( \gamma \), and one needs to take a subsequence of \( c_n \) properly to obtain a limit, which is expected as a lift of \( \gamma \). In the proof of Proposition 3.6 below, we proceed in this way.

**Proof of Proposition 3.6.** We show that the monotone curve \( c_n \) converges to a monotone quasicontinuous curve \( c \), up to monotone parametrization, except on \( \text{Sing}(\sigma) \times [0, 1] \). By the Arzela-Ascoli theorem, this is obvious if the length of \( c_n|_{J_{\text{reg}} \times [0, 1]} \) is uniformly bounded. However, when one of the end curves \( \sigma_0(s) \) and \( \sigma_1(s) \) is not rectifiable, one can not expect that the length of \( c_n|_{J_{\text{reg}} \times [0, 1]} \) is even finite.

In the argument below, we use the idea of the proof of the Arzela-Ascoli theorem taking the monotonicity of \( c_n \) into account. Since each \( c_n \) is continuous, for any \( s \in J \) there is \( t_n(s) \in [0, 1] \) satisfying \( c_n(t_n(s)) \in I_s \). Let \( J_0 \) be a countable dense subset of \( J \). Take a subsequence \( \{m\} \subset \{n\} \) such that \( c_m(t_m(s)) \) converges to a point \( x(s) \in I_s \) for every \( s \in J_0 \).

Roughly speaking, the limit curve \( c \) is defined via the limit set of the sequence \( \{\text{lim}(c_{m})\}_m \). For every \( s \in J_{\text{reg}} \), let us consider the subset \( E_s \subset I_s \) defined as the set of all points \( x \in I_s \) with \( \lim_{t_i \to \infty} c_{m_i}(t_i) = x \) for a subsequence \( \{m_i\} \subset \{m\} \) and \( t_i \in [0, 1] \). We set

\[
J_{\text{reg},1} := \{s \in J_{\text{reg}} \mid E_s \text{ is a single point}\}, \quad J_{\text{reg},2} := J_{\text{reg}} \setminus J_{\text{reg},1}.
\]
For $s \in J_{\text{reg},1}$, we define $x(s)$ by

$$E_s = \{x(s)\}.$$  

Note also that $J^0_{\text{reg},1}$ or $J^0_{\text{reg},2}$ may be empty. We begin with

1) $x(s)$ is continuous on $J_{\text{reg},1}$.

This is obvious since if $s_i \in J_{\text{reg},1}$ converges to $s \in J_{\text{reg},1}$, then any limit of $\{x(s_i)\}$ must belong to $E_s = \{x(s)\}$.

2) Next we show that $E_s$ is an interval for every $s \in J_{\text{reg},2}$.

For arbitrary $y, y' \in E_s$, choose subsequences $\{m_i\}$ and $\{m_{i'}\}$ of $\{m\}$ such that $c_{m_i}(t_i) \to y$ and $c_{m_{i'}}(t'_{i'}) \to y'$ as $i, i' \to \infty$ for some $t_i, t'_{i'} \in [0, 1]$. Take $s_j \in J^0_{\text{reg}}$ with $s < s_j$ converging to $s$. Note that $x(s_j) = \lim_{s \to \infty} c_{m_i}(t_{m_i}(s_j)) = \lim_{s \to \infty} c_{m_{i'}}(t_{m_{i'}}(s_j))$. Passing to a subsequence, we may assume that $x(s_j)$ converges to a point $z \in E_s$ as $j \to \infty$. As $i, i' \to \infty$ and then $j \to \infty$, the arcs $c_{m_i}([t_i, t_{m_i}(s_j)])$ and $c_{m_{i'}}([t_{i'}, t_{m_{i'}}(s_j)])$ converge to $[y, z]$ and $[y', z]$ respectively. Since $[y, z] \cup [y', z] \subset E_s$, we obtain $[y, y'] \subset E_s$.

3) For $s_i < s$ (resp. $s_i > s$) with $s_i \in J_{\text{reg},1}, s \in J_{\text{reg},2}$, let $s_i$ converge to $s$. In the below, we show that $x(s_i)$ converges to an endpoint of $E_s$ (resp. the other endpoint of $E_s$).

Let $\{y, y'\} = \partial E_s$.

(a) We assume $s_i < s$. The other case is similar. Suppose that $x(s_{i_k})$ converges to an interior point $v$ of $E_s$ as $k \to \infty$, for a subsequence $\{i_k\}$ of $\{i\}$. We also have subsequences $\{m_i\}$ and $\{m_{i'}\}$ of $\{m\}$ such that $c_{m_i}(t_i) \to y$ and $c_{m_{i'}}(t'_{i'}) \to y'$ for some $t_i, t'_{i'} \in [0, 1]$. As $\ell, \ell' \to \infty$ and then $k \to \infty$, the arcs $c_{m_i}([t_i, t_{m_i}(s_{i_k})])$ and $c_{m_{i'}}([t_{i'}, t_{m_{i'}}(s_{i_k})])$ converge to the subarcs $[v, y]$ and $[v, y']$ respectively. Now take a sequence $s_{\alpha} \in J^0_{\text{reg}}$ with $s_{\alpha} > s$ such that $x(s_{\alpha})$ converges to a point $w \in I$ as $\alpha \to \infty$. Note that

$$x(s_{\alpha}) = \lim_{\ell \to \infty} c_{m_{i}}(t_{m_{i}}(s_{\alpha})) = \lim_{\ell' \to \infty} c_{m_{i'}}(t_{m_{i'}}(s_{\alpha})).$$

Then we see that as $\ell, \ell' \to \infty$ and then $k \to \infty$, the arcs $c_{m_i}([t_{m_i}(s_{i_k}), t_{m_i}(s_{\alpha})])$ and $c_{m_{i'}}([t_{m_{i'}}(s_{i_k}), t_{m_{i'}}(s_{\alpha})])$ converge to the unions $[v, y] \cup [y, w]$ and $[v, y'] \cup [y', w]$ respectively. However, considering $\sigma \circ c_{m_i}$ or $\sigma \circ c_{m_{i'}}$, we have a contradiction since $\sigma \circ c_{m}$ is a sequence minimizing $c_{\sigma}(u, u')$.

(b) We show that as $s_{\alpha} < s$ converges to $s$, then $x(s_{\alpha})$ converges to a unique endpoint of $E_s$. Suppose that for subsequences $s_i \to s$ and $s_{i'} \to s$ with $s_i, s_{i'} < s$, $x(s_i)$ (resp. $x(s_{i'})$) converges to $y$ (resp. to $y'$). Choose large $i$ and $i' = i'(i)$ with $i' \gg i$. Then as $m \to \infty$, the arc $c_m([t_m(s_i), t_m(s_{i'}))]$ oscillates many times near $E_s$, which implies $\lim_{m \to \infty} L(\sigma \circ c_m) = \infty$. This is a contradiction.
(c) We show that as $s_i \to s$, $s_{i'} \to s$ with $s_i < s < s_{i'}$, if $x(s_i)$ converges to $y$, then $x(s_{i'})$ converges to $y'$. Otherwise, as $m \to \infty$, $i, i' \to \infty$, the arc $c_m([t_m(s_i), t_m(s_{i'})])$ converges to the union $[y, y'] \cup [y', y]$, which is a contradiction to the hypothesis that $\sigma \circ c_m$ is a minimizing sequence.

4) We show that $J_{\text{reg,2}}$ is at most countable, and

$$\sum_{s \in J_{\text{reg,2}}} L(\sigma(E_s)) \leq L(\gamma).$$

For arbitrary finite set $s_1 < s_2 < \cdots < s_k$ of $J_{\text{reg,2}}$, the argument in 3)- (c) shows that some subarcs of $c_m$ are so close to the union $E_{s_1} \cup \ldots \cup E_{s_k}$ for any large $m$. Thus, $\sigma(E_{s_1}) \cup \ldots \cup \sigma(E_{s_k})$ is the union of finite subarcs of $\gamma$. Therefore we have

$$\sum_{i=1}^{k} L(\sigma(E_s)) < L(\gamma).$$

The conclusion follows immediately.

5) For $s \in J_{\text{sing}}$, let $x(s) := (s, a) \in I_s$ for any fixed constant $a \in [0, 1]$, for instance. Let $L_0$ be the total sum of $L(\sigma(E_s))$ for all $s \in J_{\text{reg,2}}$. Now we consider the collection $C$ consisting of points $\{x(s) \mid s \in J_{\text{sing}} \cup J_{\text{reg,1}}\}$ and the intervals $E_s$ for all $s \in J_{\text{reg,2}}$. In view of 1) $\sim$ 4), it is possible to parametrize $C$ as a quasicontinuous curve $c : [s_0, s_0' + L_0] \to R$ from $u$ to $u'$. For the detail, see 2) in the proof of Proposition 3.14.

From construction, we see that $c$ is a lift of $\gamma$.

The second half of the assertion of the proposition is immediate. This completes the proof of Proposition 3.6. \qed

As an immediate consequence of Proposition 3.6, we have

**Corollary 3.8.** If $e_a(u, u') = 0$, then there is a strictly monotone quasicontinuous curve $c : [0, 1] \to R$ joining $u$ to $u'$ such that $\pi(c) = \pi(u) = \pi(u')$.

In particular, if Sing($\sigma$) is empty, $\pi^{-1}(\pi(u))$ is a strictly monotone (continuous) curve.

The following example shows that it is impossible to take a monotone (continuous) curve $c$ in Corollary 3.8 as well as in Proposition 3.6.

**Example 3.9.** Let $X$ be the one point union of two copies, say $\mathbb{R}^2_0$ and $\mathbb{R}^2_1$, of $\mathbb{R}^2$ at the origin $O$. Let $\sigma_k(u)$ be straight lines on $\mathbb{R}^2_k$ with $\sigma_k(0) = O$ ($k = 0, 1$). Consider strictly monotone continuous parametrizations $\sigma_k(\varphi_k(s))$ of $\sigma_k(t)$ with $\varphi_k(0) = 0$. Joining $\sigma_0(\varphi_0(s))$ and $\sigma_1(\varphi_1(s))$ by the minimal geodesics, we define a ruled surface $\sigma : \mathbb{R} \times [0, 1] \to X$. Note that Sing($\sigma$) = \{0\}. For each $s \in \mathbb{R} \setminus \{0\}$, let $t(s) \in (0, 1)$ be such that $\lambda_a(t(s)) = O$. Thus we have

$$\sigma^{-1}(O) = \{(s, t(s)) \mid s \in \mathbb{R} \setminus \{0\}\} \cup I_0.$$
Now choosing the two parameters $\varphi_0(s)$ and $\varphi_1(s)$ properly, we can let the function $t(s)$ oscillate as $s \to 0$. In that case, for arbitrary $u, u' \in \mathbb{R} \times [0, 1]$ with $\sigma(u) = \sigma(u') = O$ and $p_1(u) < p_1(u')$, there is no continuous curve in $\sigma^{-1}(O)$ joining $u$ and $u'$ but quasicontinuous one.

Next using the procedure in the proof of Proposition 3.10, we provide a condition for a curve $c_*$ in $R_*$ to have a lift $c$ in $R$.

**Definition 3.10.** For $x \in R_*$, we set

$$s(x) := \{s \in [0, \ell] \mid x \in I^*_s\} = p_1(\pi^{-1}(x)),$$

$$s_{\min}(x) := \min s(x), \quad s_{\max}(x) := \max s(x).$$

We write $s(x) < s(y)$ when $s_{\min}(x) < s_{\min}(y)$. For a subset $A$ of $R_*$, we also write $s(A) := \{s \in [0, \ell] \mid A \cap I^*_s \neq \emptyset\} = p_1(\pi^{-1}(A))$.

We need a lemma.

**Lemma 3.11.** For any continuous curve $c_* : [a, b] \to R_*$, $s(c_*([a, b]))$ is connected.

**Proof.** Choose $u \in \pi^{-1}(c_*(a))$ and $u' \in \pi^{-1}(c_*(b))$. We may assume $p_1(u) < p_1(u')$. Let $\gamma := \sigma_* \circ c_*$. Since $\pi^{-1}(c_*([a, b])) = \sigma^{-1}(\gamma([a, b]))$, $p_1(\pi^{-1}(c_*([a, b])))$ is closed. Suppose that $p_1(\pi^{-1}(c_*([a, b])))$ is not connected. Then there are some $s_- < s_+ \in [p_1(u), p_1(u')]$ satisfying

$$\pi^{-1}(c_*([a, b])) \subset [0, s_-] \times [0, 1] \cup [s_+, \ell] \times [0, 1].$$

Set $R_- := [0, s_-] \times [0, 1], R_+ := [s_+, \ell] \times [0, 1]$. In view of Corollary 3.8, we may assume that $\pi^{-1}(c_*(a)) \subset R_-$ and $\pi^{-1}(c_*(b)) \subset R_+$. Let us consider

$$t_- := \sup\{t \mid \pi^{-1}(c_*([a, t])) \subset R_\}.\]

Note that $\pi^{-1}(c_*(t_-)) \subset R_-$. Take $t_n > t_\to$ with $t_n \to t_\to$ such that $\pi^{-1}(c_*(t_n)) \subset R_+$. Choose a point $x_n \in \pi^{-1}(c_*(t_n))$. Passing to a subsequence, we may assume that $x_n$ converges to a point $x_\infty \in R_+$. This is a contradiction since $x_\infty \in \pi^{-1}(c_*(t_-))$.

**Definition 3.12.** For a continuous curve $c_* in R_*$, we say that a subset $A_* \subset I^*_s$ is $c_*$-convex if $c_*(t), c_*(t') \in A_*$ with $t \leq t'$, then $c_*([t, t']) \subset A_*$. For a continuous curve $\gamma$ in $S$, the notion of $\gamma$-convexity of a subset $\Lambda \subset \lambda_s$ is similarly defined.

Let $c_* : [a, b] \to (R_*, e_\sigma)$ be a continuous curve from $\pi(u)$ to $\pi(u')$, and put $s_0 = p_1(u), s'_0 = p_1(u')$.

For any $s \in [s_0, s'_0]$, we consider

$$E^*_s := I^*_s \cap c_*([a, b]),$$

which is nonempty by Lemma 3.11.
Lemma 3.13. For a continuous curve \( c_s : [a, b] \to (R_s, e_\sigma) \) with \( s_{\min}(c_s(a)) \leq s_{\max}(c_s(b)) \), suppose that \( E_s^* \) is \( c_s \)-convex for every \( s \in p_1(c_s([a, b])) \). Then we have the monotonicity for all \( t < t' \) in \([a, b] \),

\[
 s_{\min}(c_s(t)) \leq s_{\max}(c_s(t'))
\]

Proof. Suppose that there are \( t_1 < t_2 \) such that \( s_{\min}(c_s(t_1)) > s_{\max}(c_s(t_2)) \). If \( s_{\max}(c_s(b)) \geq s_{\min}(c_s(t_1)) \), then Lemma 3.11 shows the existence of \( t_3 \in [t_2, b] \) such that \( s(c_s(t_3)) \) meets \( s(c_s(t_1)) \). This contradicts the \( c_s \)-convexity of \( I_s^* \) for \( s \in s(c_s(t_1)) \cap s(c_s(t_3)) \), since \( c_s(t_1), c_s(t_3) \in I_s^* \) and \( c_s(t_2) \notin I_s^* \). If \( s_{\max}(c_s(b)) < s_{\min}(c_s(t_1)) \), then we have \( s(c_s(b)) < s(c_s(t_1)) \) with \( s_{\min}(c_s(a)) \leq s_{\max}(c_s(b)) \). Therefore similarly, we have a contradiction.

\[ \square \]

Proposition 3.14. Let \( c_s : [a, b] \to (R_s, e_\sigma) \) be a continuous curve with \( L(c_s) < \infty \) from \( \pi(u) \) to \( \pi(u') \). We assume the following: For each \( s \in \pi(c_s([a, b])) \),

1. \( E_s^* \) is \( c_s \)-convex;
2. the restriction \( c_s|_{E_s^*} \) is monotone for every \( E_s^* \) that is an interval;

Then there is a lift of \( c_s \) in \( R \) from \( u \) to \( u' \).

Proof. 1) Since we only need to construct a lift \( c \) on \( \text{Reg}(\sigma) \times [0, 1] \), we may assume \( \text{Sing}(\sigma) \) is empty. If \( s(c_s(a)) \) meets \( s(c_s(b)) \), then \( c_s \) is a geodesic subarc of \( I_s^* \) for \( s \in s(c_s(a)) \cap s(c_s(b)) \), and hence certainly has a lift in \( R \) by Corollary 3.3.

Thus we may assume \( s(c_s(a)) < s(c_s(b)) \). We denote by \( E_s^*_+ \) (resp. \( E_s^*_- \)) the collection of all \( E_s^* \) having positive length (resp. zero length, that is points). Since \( L(c_s) < \infty \), \( E_s^*_+ \) is at most countable, and

\[
 L_0 := \sum_{E_s^* \in E_s^*_+} L(E_s^*) \leq L(c_s).
\]

For each \( E_s^* \in E_s^*_0 \), by Corollary 3.3, \( \pi^{-1}(E_s^*) \) is a continuous strictly monotone arc, denoted by \( c_{E_s^*} \).

For \( E_s^* \in E_s^*+ \) with endpoints \( c_s(t), c_s(t') \) \( (t < t') \), from the convexity condition together with Lemma 3.13 we have

\[
 (3.3) \quad s_{\min}(c_s(t)) \leq s_{\max}(c_s(t')).
\]

Let \( a(t), b(t') \) be the endpoints of \( E_s := \pi^{-1}(E_s^*) \cap I_s \) corresponding to \( c_s(t), c_s(t') \) respectively. Let \( a_{\min}(t) \in \pi^{-1}(c_s(t)) \) and \( b_{\max}(t') \in \pi^{-1}(c_s(t')) \) be such that \( p_1(a_{\min}(t)) = s_{\min}(c_s(t)) \) and \( p_1(b_{\max}(t')) = s_{\max}(c_s(t')) \). Then let us denote by \( c_{E_s} \) the union of the subarc of \( \pi^{-1}(c_s(t)) \) from \( a_{\min}(t) \) to \( a(t) \), \( E_s^* \) and the subarc of \( \pi^{-1}(c_s(t')) \) from \( b(t') \) to \( b_{\max}(t') \).

Let \( E_s \) be the union of the collections \( E_s^*_0 \) and \( E_s^*_+ \). Note that from construction, the family of \( p_1 \)-images \( \{ p_1(c_{E_s}) \mid E_s \in E_s^* \} \) is pairwise
disjoint, and all the union coincides with \([s_0, s'_0]\). In particular, we can define the natural order on the set \(\mathcal{E}^s\).

2) We are now ready to parametrize the union of all those arcs \(c_{E_s^*} (E_s^* \in \mathcal{E}^s)\), to construct a lift \(c : [s_0, s_0 + L_0] \to R\) of \(c_*\). For each \(E_s^* \in \mathcal{E}^s\), let \(\mathcal{E}_s^+ (s)\) denote the set of all \(E_{s'}^* \in \mathcal{E}_s^+\) with \(E_{s'}^* < E_s^*\). We set

\[
\ell(E_s^*) := \sum_{E_{s'}^* \in \mathcal{E}_s^+ (s)} L(E_{s'}^*).
\]

For \(E_s^* \in \mathcal{E}_0^s\), let \(a, b\) be the endpoints of the arc \(c_{E_s^*}\) with \(p_1(a) \leq p_1(b)\). We parametrize \(c_{E_s^*}\) on \([\ell(E_s^*) + p_1(a), \ell(E_s^*) + p_1(b)]\) by the condition

\[
p_1(c_{E_s^*}(\ell(E_s^*) + t)) = t \quad \text{for} \quad t \in [p_1(a), p_1(b)].
\]

For \(E_s^* \in \mathcal{E}_s^+\) with endpoints \(c_*(t), c_*(t') \quad (t < t')\), let \(a(t), b(t') \in \partial E_s^*, a_{\min}(t), b_{\max}(t')\) be defined as in the previous paragraph. Then we parametrize \(c_{E_s^*}\) on \([\ell(E_s^*) + p_1(a_{\min}(t)), \ell(E_s^*) + p_1(b_{\max}(t'))]\) by the conditions that

\[
p_1(c_{E_s^*}(\ell(E_s^*) + t)) = t
\]

for \(t \in [p_1(a_{\min}(t)), p_1(a(t))]) \cup [L(E_s^*) + p_1(b(t'))], L(E_s^*) + p_1(b_{\max}(t'))]\) and

\[
c_{E_s^*}(\ell(E_s^*) + p_1(a(t)) + t) = E_s^*(t)
\]

for \(t \in [0, L(E_s^*)]\), where \(E_s^*(t)\) is the arc-length parameter from \(a(t)\) to \(b(t')\).

Finally we observe the continuity of the family \(\{c_{E_s^*} \mid E_s^* \in \mathcal{E}^s\}\) in the following sense: Let \(\{E_{s_i}^*\} \in \mathcal{E}_0^s\) be a Cauchy sequence in \(R_*\) satisfying \(E_{s_i}^* < E_s^*\) (resp. \(E_{s_i}^* > E_s^*\)) such that its limit meets \(E_s^*\). Let \(a, b\) be the initial and terminal points of \(c_{E_s^*}\) respectively. Then \(c_{E_{s_i}^*}\) converges to \(a\) (resp. to \(b\)). This follows from the conditions (1), (2) and (5.3), and the detail is omitted here.

Thus we can define the curve \(c : [s_0, s_0 + L_0] \to R\) as the union of all \(c_{E_s^*} (E_s^* \in \mathcal{E}^s)\). It is easy to see that \(c\) is a continuous and monotone lift of \(c_*\). This completes the proof. \(\square\)

**Remark 3.15.** To consider the problem of lifting a curve \(\gamma\) in \(S\), we need an extra condition on \(\sigma\) or \(\gamma\), which will be discussed later in Proposition 3.24.

By Proposition 3.14 we immediately have the following.

**Proposition 3.16.** Let \(c_* : [a, b] \to (R_*, e_\sigma)\) be a shortest curve from \(\pi(u)\) to \(\pi(u')\). Then there is a lift \(c\) of \(c_*\) from \(u\) to \(u'\).

In [4, Theorem 2], Alexandrov proved the following result, which plays a crucial role in the present paper.

**Theorem 3.17 ([4]).** Let \(S\) be a ruled surface in a CAT(\(\kappa\))-space \(X\) with parametrization \(\sigma : R \to X\). Then \((R_*, e_\sigma)\) is a CAT(\(\kappa\))-space.
The proof of Theorem 3.17 is deferred to Appendix A.

One might expect to define the induced “metric” $d_\sigma$ on $S$ along $\sigma$ as

$$d_\sigma(x, y) := \inf \{ e_\sigma(u, v) \mid \sigma(u) = x \text{ and } \sigma(v) = y \}. $$

However, $d_\sigma$ does not necessarily satisfy the triangle inequality. See Remark 4.3. Even if $(S, d_\sigma)$ becomes a metric space, in certain cases, it could be far from the notion of “induced metric”, as described in the following example.

Example 3.18. Let us consider the following curve $\alpha : [0, 5\pi] \to \mathbb{C}$ on $\mathbb{C} = \mathbb{R}^2$ defined as

$$\alpha(s) = \begin{cases} e^{\sqrt{-1} s} & (0 \leq s \leq \pi/2), \\ (0, 2s/\pi) & (\pi/2 \leq s \leq \pi), \\ (0, 4 - 2s/\pi) & (\pi \leq s \leq 3\pi/2), \\ e^{\sqrt{-1}(s-\pi)} & (3\pi/2 \leq s \leq 5\pi). \end{cases}$$

We define the ruled surface $\sigma : [0, 5\pi] \times [0, 1] \to \mathbb{R}^3$ by $\sigma(s, t) = (\alpha(s), t)$. In this case, $d_\sigma$ is a distance on the image $S$ of $\sigma$. Actually $d_\sigma$ coincides with the interior metric of $S$ defined in Definition 3.23.

On the other hand, if we consider the restriction $\sigma'$ of $\sigma$ to $[0, 3\pi] \times [0, 1]$, then $d_{\sigma'}$ is not the distance on the ruled surface $S'$ defined by $\sigma'$.

Lemma 3.19. Suppose that we have for all $u, v \in R$,

(3.4) \hspace{1cm} \sigma(u) = \sigma(v) \iff e_\sigma(u, v) = 0.

Then $(S, d_\sigma)$ is a metric space, and $\sigma_* : (R, e_\sigma) \to (S, d_\sigma)$ is an isometry.

Proof. First note that $e_\sigma(u, u') = 0$ implies $\sigma(u) = \sigma(v)$. Suppose (3.4) holds for all $u, v \in R$. Then we have $d_\sigma(x, y) = e_\sigma(u, v)$ for all $x, y \in S$ and $u \in \sigma^{-1}(x), v \in \sigma^{-1}(y)$. This implies that $d_\sigma$ is a metric on $S$. It is also obvious that $\sigma_* : (R, e_\sigma) \to (S, d_\sigma)$ is an isometry. \qed

Definition 3.20. We say that $S$ has the induced metric from $\sigma$ if $\sigma_* : R \to S$ is injective. This is the case when (3.4) holds for all $u, v \in R$, and therefore $\sigma_* : (R, e_\sigma) \to (S, d_\sigma)$ is an isometry by Lemma 3.19. In this case, $d_\sigma$ is called the induced metric from $\sigma$.

Corollary 3.21. Let $S$ be a ruled surface in a CAT($\kappa$)-space $X$ with parametrization $\sigma : R \to X$. If $S$ has the induced metric from $\sigma$, then $(S, d_\sigma)$ is a CAT($\kappa$)-space.

From now, in the rest of this section, we consider curves $\gamma$ in $S$ with respect to the topology of $S$ induced from $X$.

Lemma 3.22. If $S$ has the induced metric from $\sigma$, then $s(\gamma([a, b]))$ is an interval for any continuous curve $\gamma : [a, b] \to S$. 
Proof. Let \( J := [a, b] \). If the conclusion does not hold, we have \( s_- < s_+ \) in \( s(\gamma(J)) \) such that \( (s_-, s_+) \) does not meet \( s(\gamma(J)) \). Set \( R_- := [0, s_-] \times [0, 1], \ R_+ := [s_+, 1] \times [0, 1] \). Let \( J_- \) and \( J_+ \) be the set of all \( t \in J \) such that the arc \( \sigma^{-1}(\gamma(t)) \) is contained in \( R_- \) and \( R_+ \) respectively. Since \( S \) has the induced metric from \( \sigma \), Corollary 3.3 implies that \( J = J_+ \cup J_- \). We show that \( J_- \) and \( J_+ \) are open, yielding a contradiction. Suppose \( J_- \) is not open for instance, and choose \( t \in J_- \backslash \text{int} J_- \) and a sequence \( t_n \) in \( J_+ \) converging to \( t \). Choose any \( x_n \in \sigma^{-1}(\gamma(t_n)) \) converging to a point \( x_\infty \in R_+ \). Since \( \sigma(x_n) = \gamma(t_n) \rightarrow \sigma(x_\infty) \) as \( n \rightarrow \infty \), we have \( \gamma(t) = \sigma(x_\infty) \). It turns out that \( \sigma^{-1}(\gamma(t)) \in R_+ \). This is a contradiction to \( t \in R_- \).

Interior metrics on ruled surfaces
Let \( S \) be a ruled surface in \( X \) with parametrization \( \sigma: R \rightarrow X \).

Definition 3.23. We denote by \( d_S \) the interior metric on \( S \) associated with \( d_X \) defined as
\[
d_S(x_0, x_1) := \inf \{ L(\gamma) \mid \gamma \text{ is a curve in } S \text{ from } x_0 \text{ to } x_1 \}.
\]
Due to the Arzela-Ascoli theorem, \( (S, d_S) \) is a geodesic space.

We discuss the problem of lifting curves in \( S \). For a subset \( A \subset S \), we set
\[
s(A) := \{ s \mid \lambda_s \cap A \neq \emptyset \}.
\]
Note that \( s(A) = p_1(\sigma^{-1}(A)) \). In particular, for every \( x \in S \), we define \( s(x), s_{\max}(x), s_{\min}(x) \) in this way as in Definition 3.10.

Proposition 3.24. Let \( \gamma: [a, b] \rightarrow S \) be a continuous curve of finite length from \( \sigma(u) \) to \( \sigma(u') \) with \( p_1(u) < p_1(u') \). Set \( J := [p_1(u), p_1(u')] \), and \( \Lambda_s^* := \sigma_s^{-1}(\gamma([a, b])) \cap I_s^* \) (\( s \in J \)).

We assume the following:
1. For arbitrary \( t < t' \) in \([a, b] \), \( s(\gamma([t, t'])) \) is connected;
2. \( \sigma_s(\Lambda_s^*) \) is \( \gamma \)-convex for each \( s \in J \);
3. \( \gamma \) is monotone on \( \sigma_s(\Lambda_s^*) \) that is an interval.

Then there is a lift of \( \gamma \) in \( R \) from \( u \) to \( u' \).

Lemma 3.25. For a continuous curve \( \gamma: [a, b] \rightarrow S \) with \( s_{\min}(\gamma(a)) \leq s_{\max}(\gamma(b)) \), suppose (1), (2) of Proposition 3.24 for \( \gamma \). Then we have the monotonicity for all \( t < t' \) in \([a, b] \),
\[
s_{\min}(\gamma(t)) \leq s_{\max}(\gamma(t')).
\]

Proof. Using the condition (1) of Proposition 3.24 in place of Lemma 3.11, we can proceed in the same manner as the proof of Lemma 3.13 in our setting, to get the conclusion. \( \square \)
Proof of Proposition 3.24. In view of the conditions (2),(3) and Lemma 3.25, using \( \Lambda^*_s \) in place of \( \hat{E}^*_s \), we construct the family of continuous arcs \( c_{\Lambda^*_s} \) by the same manner as in Proposition 3.14. Then parametrize them and take the union of those arcs to obtain a lift of \( \gamma \) in \( R \). Since the procedure is the same, we omit the detail. □

Theorem 3.26. Let \( S \) be a ruled surface in a \( \text{CAT}(\kappa) \)-space \( X \) with parametrization \( \sigma : R \to X \). If \( S \) has the induced metric from \( \sigma \), then we have
\[
\text{dist}_S = \text{dist}_\sigma,
\]
and \( (S, \text{dist}_S) \) is a \( \text{CAT}(\kappa) \)-space.

Proof. Since \( \text{dist}_S \leq \text{dist}_\sigma \), to see \( \text{dist}_S = \text{dist}_\sigma \), it suffices to show \( \text{dist}_S(x, x') \geq \text{dist}_\sigma(x, x') \) for arbitrary \( x, x' \in S \). Take a \( \text{dist}_S \)-shortest curve \( \gamma : [a, b] \to S \) from \( x \) to \( x' \). Since \( \gamma \) is \( \text{dist}_S \)-shortest, the conditions (2),(3) in Proposition 3.24 certainly hold for \( \gamma \). By Lemma 3.22, \( s(\gamma([t_1, t_2])) \) is an interval, and the condition (1) in Proposition 3.24 holds too. Therefore by Proposition 3.24, we have a lift of \( \gamma \) in \( R \). Thus we have \( \text{dist}_S(x, x') = L(\gamma) \geq \text{dist}_\sigma(x, x') \). Finally Corollary 3.21 implies that \( (S, \text{dist}_S) \) is a \( \text{CAT}(\kappa) \)-space. This completes the proof. □

4. Thin ruled surfaces

Let \( X \) be a locally compact, geodesically complete two-dimensional space with curvature \( \leq \kappa \), and fix \( p \in X \). It is known that \( \Sigma_p(X) \) is a finite metric graph without endpoints. For a vertex \( v \) of \( \Sigma_p(X) \), take \( \nu_1, \nu_2 \in \Sigma_p(X) \) with equal distance to \( v \) such that \( \angle(\nu_1, v) + \angle(v, \nu_2) = \angle(\nu_1, \nu_2) \) and \( v \) is the unique vertex contained in the shortest geodesic joining \( \nu_1 \) and \( \nu_2 \) in \( \Sigma_p(X) \). We set
\[
\delta := \angle(\nu_1, v) = \angle(\nu_2, v),
\]
where \( \delta \) is assumed to be small enough and will be determined later on in Section 4. Let \( \alpha_i : [0, \ell] \to X \) be geodesics in the directions \( \nu_i \), \( i = 1, 2 \). Joining \( \alpha_1(s) \) to \( \alpha_2(s) \) by the minimal geodesic \( \lambda_s : [0, 1] \to X \), we have a ruled surface \( S \) in \( X \). Let \( B(p, r) \) be a small ball, and we assume \( \ell = 2r \). Set \( R = [0, \ell] \times [0, 1] \). Let \( \sigma : R \to S \) be the map that defines \( S \):
\[
\sigma(s, t) = \lambda_s(t).
\]

We define the boundary and the interior of \( S \) as
\[
\partial S := \alpha_1 \cup \alpha_2 \cup \lambda_\ell, \quad \text{int } S := S \setminus \partial S.
\]
The purpose of this section is to prove the following

**Theorem 4.1.** There exists an \( r_p > 0 \) such that for every \( r \in (0, r_p] \), 
\( S \) with length metric is a \( \text{CAT}(\kappa) \)-space homeomorphic to a two-disk.

The proof of Theorem 4.1 is completed in Subsection 4.4. As shown in the following example, Theorem 4.1 does not hold for a general ruled surface even in a two-dimensional ambient space.

**Example 4.2.** For any \( 0 < a < \pi/2 \), let \( X_0 \) be the complement of the domain \( \{(x, y) \mid |y| < (\tan a)x\} \) on the \( xy \)-plane. For \( b \) with \( a < b < a + \pi/4 \), consider the Euclidean cone \( K(I) \) over a closed interval \( I \) of length \( 2b \). Let \( X_1 \) be the gluing of \( X_0 \) and \( K(I) \) along their boundaries, where the origin \( o \) of \( X_0 \) is identified with the vertex of \( K(I) \). Let \( \xi \) be the midpoint of \( I \) and \( \gamma_\xi \) denote the geodesic ray of \( X_1 \) from \( o \) in the direction \( \xi \). Next consider the Euclidean cone \( K(J) \) over an interval \( J \) of length \( \theta \) with \( \pi - (b - a) < \theta < \pi \). Let \( X \) be the gluing of \( X_1 \) and \( K(J) \) in such a way that \( \partial K(J) \) is identified with \( \gamma_\xi \) and \( L := \{(x, 0) \mid x \leq 0\} \subset X_0 \) in an obvious way. It is easy to see that \( X \) is a locally compact, geodesically complete, two-dimensional \( \text{CAT}(0) \)-space. Let \( p = (0, -10) \in X_0 \subset X \), and let \( \sigma_+ \) (resp. \( \sigma_- \)) be the geodesic ray starting from \( o \) defined by the ray \( y = (\tan a)x \) (resp. by the ray \( y = -(\tan a)x \)). Note that the geodesic in \( X \) joining \( p \) and \( \sigma_+(1) \) intersect \( \sigma_- \setminus \{o\} \) because of (4.3). Let \( \ell := 2d(p, \sigma_+(1)) \), and let \( \alpha_1 : [0, \ell] \to X \) be the geodesic starting from \( p \) through \( \sigma_+(1) \). Let \( q_1 \) be the intersection point of \( \alpha_1 \) with \( \gamma_\xi \). Let \( q_2 \) be the point of \( L \) such that \( d(p, q_1) = d(p, q_2) \). Letting \( \alpha_2 : [0, \ell] \to X \) be the geodesic starting from \( p \) through \( q_2 \), consider the ruled surface \( S = S(\alpha_1, \alpha_2) \) in \( X \). Let \( \Delta_1 \) (resp. \( \Delta_2 \)) be the geodesic triangle region in \( X_1 \) (resp in \( K(J) \)) with vertices \( p \), \( \alpha_1(\ell) \) and \( \alpha_2(\ell) \) (resp. \( o \), \( q_1 \) and \( q_2 \)). Obviously, \( S \) is the gluing of \( \Delta_1 \) and \( \Delta_2 \) along the geodesic segments \( oq_1 \) and \( oq_2 \). In particular \( S \) is not homeomorphic to a disk.

**Remark 4.3.** (1) Note that in Example 4.2, \( \text{diam} \left( (\nabla d_p)(o) \right) = \pi \), which never happens in a small neighborhood of \( p \) by Lemma 2.4. This suggests the validity of Theorem 4.1 which is verified in the argument below.
(2) In Example 4.2, take two points \( x, y \) from the distinct components of \( S \setminus \Delta_2 \) respectively. Then if \( x \) and \( y \) are sufficiently close to the point \( o \), then we have \( d_x(x, y) > d_x(x, o) + d_x(o, y) \). Thus \( d_x \) is not a distance for Example 4.2.

4.1. Behavior of ruling geodesics. In this subsection, we start the study of the behavior of ruling geodesics of \( S \). We begin with two examples, which help us to understand the argument in the rest of the paper.

**Example 4.4** (Kleiner). *(cf. [23]*) First consider a smooth nonnegative function \( f : \mathbb{R} \to \mathbb{R}_+ \) such that \( \{ f = 0 \} = \{ 1/n \mid n = 1, 2, \ldots \} \cup [1, \infty) \cup (-\infty, 0] \). Let \( \Omega := \{(x, y) \mid |y| \leq f(x), x \in \mathbb{R} \} \), equipped with the natural length metric induced from that of \( \mathbb{R}^2 \). We set \( I_n^+ := \{(x, +f(x)) \mid 1/(n+1) \leq x \leq 1/n \}, I_n^- := \{(x, -f(x)) \mid 1/(n+1) \leq x \leq 1/n \} \), and \( L_+ := \{(x, 0) \mid x \geq 1 \}, L_- := \{(x, 0) \mid x \leq 0 \} \). Let \( \ell_n \) denote the length of \( I_n^+ \), and let \( \kappa_n \) be the maximum of absolute geodesic curvature of \( I_n^\pm \). We choose \( f \) satisfying

\[
\sum \ell_n < \infty, \quad \sum \kappa_n \ell_n < 2\pi.
\]

By these conditions, one can take a closed domain \( H \) in \( \mathbb{R}^2 \) such that

1. \( \partial H \) is smooth, connected and concave in the sense that the geodesic curvature is nonpositive everywhere;
2. there are consecutive points \( p_1, p_2, \ldots, \) on \( \partial H \) such that the subarc \( K_n \) between \( p_n \) and \( p_{n+1} \) of \( \partial H \) has length equal to \( \ell_n \);
3. if we denote \( p_\infty \) the limit of \( p_n \), the closure of the complement of the arc between \( p_1 \) and \( p_\infty \) in \( \partial H \) consists of two geodesic rays, say \( R_+ \) and \( R_- \), in \( \mathbb{R}^2 \) with \( p_1 \in R_+ \) and \( p_\infty \in R_- \);
4. the absolute geodesic curvature of \( K_n \) is greater than or equal to \( \kappa_n \) everywhere.

Take four copies \( H_1, \ldots, H_4 \) of \( H \), and denote \( K_n, R_\pm \subset \partial H_\alpha \) by \( K_n^{(\alpha)}, R_\pm^{(\alpha)} \) \( (1 \leq \alpha \leq 4) \) respectively. We put

\[
\partial_+ \Omega := \left( \bigcup_{n=1}^\infty I_n^+ \right) \cup L_+ \cup L_-,
\]

\[
\partial_- \Omega := \left( \bigcup_{n=1}^\infty I_n^- \right) \cup L_+ \cup L_-.
\]

Now glue \( H_1, H_2 \) and \( \Omega \) along their boundaries \( \partial H_1, \partial H_2, \partial_+ \Omega \) in such a way that \( I_n, L_+ \) and \( L_- \) are glued with \( K_n^{(1)}, R_+^{(1)} \) and \( R_-^{(1)} \) \( (\alpha = 1, 2) \) respectively in an obvious way. Similarly glue \( H_3, H_4 \) and \( \Omega \) along their boundaries \( \partial H_3, \partial H_4 \) and \( \partial_- \Omega \).
Let \(X\) be the result of these gluings equipped with natural length metric, which is a two-dimensional locally compact, geodesically complete space. Let \(\iota: \Omega \to X\) be the natural inclusion, and let \(O = (0, 0) \in \Omega\). Note that no neighborhood of \(p := \iota(O)\) in \(X\) has triangulation. Approximating \(f\) by functions \(f_k (k = 1, 2, \ldots)\) that are 0 near \(0\), we have polyhedral spaces \(X_k\) in a similar way which approximate \(X\) in the sense of Gromov-Hausdorff distance. Applying a result in [13], we see that \(X_k\) are CAT(0)-spaces. Thus the limit space \(X\) is also a CAT(0)-space. Note that \(S(X)\) consists of the two curves \(\iota(\partial_+ \Omega)\) and \(\iota(\partial_- \Omega)\).

Example 4.5. This example is based on Example 4.4. The point is we make different gluings. This time we glue \(H_1, H_2, H_3, H_4\) and \(\Omega\) along their boundaries as follows:

1. \(K_n^{(1)}\) is glued with \(I_n^+\) for all \(n\);
2. \(K_n^{(2)}\) is glued with \(I_n^+\) if \(n \not\equiv 2 \pmod{4}\) and with \(I_n^-\) if \(n \equiv 2 \pmod{4}\);
3. \(K_n^{(3)}\) is glued with \(I_n^+\) if \(n \equiv 0, 1 \pmod{4}\) and with \(I_n^-\) if \(n \equiv 2, 3 \pmod{4}\);
4. \(K_n^{(4)}\) is glued with \(I_n^+\) if \(n \equiv 3 \pmod{4}\) and with \(I_n^-\) if \(n \not\equiv 3 \pmod{4}\).

Here, \(R^{+}_\alpha\) and \(R^{-}_\alpha\) \((1 \leq \alpha \leq 4)\) are glued with \(L_+\) and \(L_-\) respectively in these gluings. The result \(Y\) of these gluings equipped with natural length metric is a two-dimensional locally compact, geodesically complete CAT(0)-space. Let \(\iota: (\Pi_{i=1}^4 H_i) \amalg \Omega \to Y\) be the identification map. Note that

\[
(4.5) \quad \text{for all } 1 \leq \alpha \neq \beta \leq 4, \ i(\kappa^{\alpha}_n) = i(\kappa^{\beta}_n) \quad \text{for some } n.
\]

Let \(p := \iota(O)\), where \(O\) is the origin of \(\Omega\), and let \(v\) denote the direction at \(p\) defined by the union of all \(I_n^+\) \((n = 1, 2, \ldots)\). For small \(\epsilon > 0\), take sufficiently small \(r > 0\) and choose \(a_i \in S(p, r) \cap \iota(H_i)\), \(1 \leq i \leq 4\), such that \(\angle(\hat{\gamma}_{p,a_i}(0), v) = \epsilon\). Let \(S(a_i, a_j)\) be the ruled surface defined by the geodesic segments \(\gamma_{p,a_i}\) and \(\gamma_{p,a_j}\). Then it follows from (4.5) that \(S(a_i, a_j)\) are not convex in \(Y\) for \(\| i \neq j\).
Remark 4.6. (1) In Example 4.4 if we take $a_i$ in a way similar to Example 4.5, then $S(a_i,a_j)$ for $i = 1, 2$ and $j = 3, 4$ are convex in $X$ while $S(a_1,a_2)$ and $S(a_3,a_4)$ are not convex. Considering the other vertex of $\Sigma_p(X)$, it is possible to fill a neighborhood of the singular set $B(p,r) \cap S$ via those convex ruled surfaces. This is not the case of Example 4.5.

(2) In Example 4.5 it is impossible to fill the ball $B(p,r)$ for any $r > 0$ via a properly embedded convex disks. More strongly, there is no such a convex disk properly embedded in $B(p,r)$. If there is such a convex disk $D$, from the convexity of $D$, we can take some $a_i \neq a_j$ in $\partial D$. The convexity of $D$ also implies that $S(a_i,a_j) \subset D$, and hence $S(a_i,a_j)$ must be convex. However this is impossible as indicated in Example 4.5.

For $x \in S$ with $x = \lambda_x(t)$, from Lemma 2.4, we have

\begin{equation}
|\angle(\pm \dot{\lambda}_x(t), (\nabla d_p)(x)) - \pi/2| < \tau_p(|p,x|, \delta).
\end{equation}

For $x \in S$, let $\Sigma_x(S)$ denote the set of all directions $\xi \in \Sigma_x(X)$ such that $\xi = \lim_{i \to \infty} \frac{\dot{x}_i}{|\dot{x}_i|}$ for some sequence $x_i \in S$ with $|x, x_i|_X \to 0$, as in Section 2. We call $\Sigma_x(S)$ the extrinsic space of directions of $S$ at $x$.

In this paper, we use the following terminology. We call a direction $\xi \in \Sigma_x(X)$

\begin{equation*}
\begin{cases}
\text{horizontal} & \text{if } \angle(\xi, \pm (\nabla d_p)(x)) \leq 3\pi/10, \\
\text{vertical} & \text{if } \angle(\xi, \pm (\nabla d_p)(x)) \geq \pi/5, \\
\text{medial} & \text{if it is horizontal and vertical.}
\end{cases}
\end{equation*}

We also call a direction $\xi \in \Sigma_x(X)$

\begin{equation*}
\begin{cases}
\text{negative} & \text{if } \angle(\xi, -(\nabla d_p)(x)) < \pi/2 \\
\text{positive} & \text{if } \angle(\xi, (\nabla d_p)(x)) < \pi/2.
\end{cases}
\end{equation*}

We say that an open subset $\Omega \subset \Sigma_x(X)$ is in the positive side (resp. negative side) of $\Sigma_x(X)$ if every element of $\Omega$ is positive (resp. negative).
Assume that a Lipschitz curve \( c : [a, b] \to B(p, r) \setminus \{p\} \) has the right and left directions \( \dot{c}_+(t) \) and \( \dot{c}_-(t) \) respectively at every \( t \in [a, b] \). We say that such a curve \( c \) is vertical (resp. horizontal or medial) if both \( \dot{c}_+(t) \) and \( \dot{c}_-(t) \) are vertical (resp. horizontal or medial) for every \( t \in [a, b] \).

Recall that for every \( x \in S \),

\[
\begin{align*}
s(x) &= \{ s \in [0, \ell] \mid x \in \lambda_s \}, \\
s_{\text{max}}(x) &= \max s(x), \\
s_{\text{min}}(x) &= \min s(x).
\end{align*}
\]

For every \( x \in \text{int} S \), we set

\[
\begin{align*}
\dot{\lambda}(x) &= \{ \uparrow_{x}^{\lambda_s(1)} \mid s \in s(x) \}, \\
-\dot{\lambda}(x) &= \{ \uparrow_{x}^{\lambda_s(0)} \mid s \in s(x) \}.
\end{align*}
\]

We show that \( s(x) \) is a closed interval later in Lemma 4.28.

**Lemma 4.7.** For every \( x \in \text{int} S \), we have

1. \( \text{diam}(s(x))/|p, x| < \tau_p(|p, x|) \);
2. \( \text{diam}(\pm \dot{\lambda}(x)) < \tau_p(|p, x|) \).

**Proof.** Suppose that the conclusion does not hold. Then we have a sequence \( x_i \in \text{int} S \) and a positive constant \( c \) such that one of the following holds:

1. \( \text{diam}(s(x_i))/|p, x_i| \geq c \);
2. \( \text{diam}(\pm \dot{\lambda}(x_i)) \geq c \).

Let \( x_i = \lambda_{s_i}(t_i) \). Note that \( 0 < t_i < \ell \).

We may assume \( t_i = \min\{t_i, 1 - t_i\} \) since the other case is similar. For any other \( s_i' \in s(x_i) \), from \( |x_i, p| \to 0 \), we have

\[
\lim_{i \to \infty} \angle x_i \lambda_{s_i}(0)p = \pi/2 - \delta, \quad \lim_{i \to \infty} \angle x_i \lambda_{s_i}'(0)p = \pi/2 - \delta.
\]
We may assume that $s'_i < s_i$ without loss of generality. Note also that

$$
\lim_{i \to \infty} \angle x_i \lambda'_i(0) \lambda_i(0) = \pi/2 + \delta,
$$

$$
\lim_{i \to \infty} (\hat{\lambda}_i(0) x_i \lambda'_i(0) + \hat{\lambda}_i(0) \lambda'_i(0)) = \pi.
$$

It follows that

$$
\lim_{i \to \infty} \angle \lambda_i(0) x_i \lambda'_i(0) \leq \lim_{i \to \infty} \hat{\lambda}_i(0) x_i \lambda'_i(0)
$$

$$
= \lim_{i \to \infty} (\pi - \hat{\lambda}_i(0) \lambda'_i(0)) = \pi.
$$

Thus we conclude that $\text{diam}(-\hat{\lambda}(x_i)) \to 0$. Therefore the assumption $(ii)$ does not hold. Note that

$$
|s_i - s'_i| \leq |p, x_i| t_i \hat{\lambda}_i(0) x_i \lambda'_i(0),
$$

Therefore, from $\lim_{i \to \infty} \hat{\lambda}_i(0) x_i \lambda'_i(0) = 0$, we see that the assumption $(i)$ does not hold either.

Now, for any $x \in S$ and $s \in (0, \ell)$ with $x \notin \lambda_s$, let $y \in \lambda_s$ be such that $|x, y| = |x, \lambda_s|$. By Lemma 2.4 we have either

$$
\angle \binom{\uparrow_x, \downarrow d_y} \t < \tau_{\downarrow}(\delta, r) \ or \ \angle \binom{\uparrow_y, \downarrow dx} \t < \tau_{\downarrow}(\delta, r).
$$

**Remark 4.8.** From the proof of Lemma 4.7 by contradiction, one might think the function $\tau_{\downarrow}(\cdot, r)$ depends also on $S$. However the number of possible $S$ at $p$ is finite. Therefore it is possible to find such a function depending only on $p$. From now on, we use this convention.

**Lemma 4.9.** Under the above situation, if $\angle \binom{\uparrow_y, \downarrow d_y} \t \binom{\uparrow_x, \downarrow dx} \t < \tau_{\downarrow}(\delta, r)$ (resp. $\angle \binom{\uparrow_y, \downarrow dx} \t < \tau_{\downarrow}(\delta, r)$), then we have

$$
\angle \binom{\uparrow_x, \downarrow d_x} \t < \tau_{\downarrow}(\delta, r) \ (\text{resp.} \ \angle \binom{\uparrow_x, \downarrow dx} \t < \tau_{\downarrow}(\delta, r)).
$$

**Proof.** We assume $\angle \binom{\uparrow_y, \downarrow d_y} \t \binom{\uparrow_x, \downarrow dx} \t < \tau_{\downarrow}(\delta, r)$. The proof of the other case is similar. By Lemma 2.4, we have $\angle pxy \t < \tau_{\downarrow}(\delta, r)$. Since $\angle xpy \t < 2\delta$, it follows from Lemma 2.7 that $\angle pxy > \pi - \tau_{\downarrow}(\delta, r)$, which implies $\angle \binom{\uparrow_x, \downarrow dx} \t \binom{\uparrow_y, \downarrow dx} \t < \tau_{\downarrow}(\delta, r)$.

**Lemma 4.10.** For $x \in \text{int} S$, fix $s_0$ and $t_0$ with $x = \lambda_{s_0}(t_0)$. Then for every $u \in \Sigma_x(S)$ with

$$
\angle(u, \pm \lambda_{s_0}(t_0)) \geq \pi/3,
$$

there exists a shortest path $\xi_x : [-1, 1] \to \Sigma_x(S)$ satisfying

$$
\begin{cases}
  u \in \xi_x([-1, 1]), \\
  \angle_x(\pm \lambda_{s_0}(t_0)) \leq \tau_{\downarrow}(r), \\
  \xi_x([-1, 1]) \subset \Sigma_x(S).
\end{cases}
$$
We need a sublemma.

**Sublemma 4.11.** There is a uniform positive constant \( c \) satisfying the following: For any \( x \in S \) and any horizontal direction \( \xi \in \Sigma_x(S) \), let \( y_i \in S \) be a sequence such that \( y_i \to x \) and \( \uparrow_x y_i \to \xi \). There is an \( i_0 \) such that if \( \lambda_{s_i} \) meets \( y_i \) for some \( i \geq i_0 \), then we have

\[
|x, \lambda_{s_i}| \geq c|x, y_i|.
\]

**Proof.** Note that \( \lambda_{s_i} \) does not pass through \( x \) for any large enough \( i \) since \( \xi \) would be a direction tangent to \( \lambda_{s_i} \) at \( x \) otherwise, which is a contradiction. Take \( z_i \in \lambda_{s_i} \) such that \(|x, z_i| = |x, \lambda_{s_i}|\). By Lemmas 2.4 and 2.11, we have

\[
\frac{\angle xz_i y_i - \pi/2}{\tau_p(\delta, r)} < \frac{\angle xz_i y_i - \pi/2}{\tau_p(\delta, r)}.
\]

It follows that \( \angle(\uparrow_{z_i} (-\nabla d_p)) < \tau_p(\delta, r) \) or \( \angle(\uparrow_{z_i} \nabla d_p) < \tau_p(\delta, r) \). We assume the former since the latter case is similar. It follows from Lemma 4.9 that \( \angle(\uparrow_{z_i} \nabla d_p) < \tau_p(\delta, r) \). Lemma 2.5 implies that

\[
\frac{\angle xz_i y_i - \pi/2}{\tau_p(\delta, r)} < \frac{\angle xz_i y_i - \pi/2}{\tau_p(\delta, r)}.
\]

By Lemma 2.7, we have

\[
\frac{\angle xz_i y_i - \pi/2}{\tau_p(\delta, r)} < \frac{\angle xz_i y_i - \pi/2}{\tau_p(\delta, r)}.
\]

Let \( y_i \) be the point on the geodesic \( z_i p \) satisfying \( |z_i, x_i| = |z_i, x| \). By (4.9), we have \( |\angle xz_i y_i - \pi/2| < \tau_p(\delta, r) \). It follows from (4.10) that

\[
|\angle xz_i y_i - \pi/2| < \tau_p(\delta, r).
\]

Now let us consider the convergence

\[
\frac{1}{|x, z_i|} \to (K_x(X), o_x),
\]

as \( i \to \infty \). Let \( z_\infty \in \Sigma_x \subset K_x(X) \) be the limit of \( z_i \) under this convergence. Since \( \angle (z_\infty, \nabla d_p) < \tau_p(\delta, r) \) and since \( \uparrow_x y_i \to v \), the limit \( y_\infty \) of \( y_i \) under the above convergence certainly exists, and we have

\[
\angle y_\infty o_x z_\infty < \pi/2 - \pi/3 + \tau_p(\delta, r) < \pi/5.
\]
By (4.11), we also have
\begin{equation}
|\angle oxyz - \pi/2| < \tau_p(\delta, r),
\end{equation}
and (4.12) imply that $|o_x, z_\infty| \geq c|o_x, y_\infty|$ for some uniform constant $c > 0$. This yields the conclusion of the lemma via contradiction.

Proof of Lemma 4.10. Let $y_i \in S$ be such that $|x, y_i|_X \to 0$ and $\uparrow y_i$ converges to $u$. Take $s_i \in (0, \ell)$, $t_i \in (0, 1)$ such that $y_i = \lambda_{s_i}(t_i)$. Let $\epsilon_i := |x, y_i|_X$, and consider the convergence
\[
\left( \frac{1}{\epsilon_i}X, x \right) \to (K_x(X), o_x),
\]
as $i \to \infty$. Note that the minimal geodesic $\hat{\lambda}_{s_i}(t) := \lambda_{s_i}(t_i + \epsilon_i t)$, $(-t_i/\epsilon_i < t < (1 - t_i)/\epsilon_i)$, has a uniformly bounded speed for $\frac{1}{\epsilon_i}X$ independent of $i$. Therefore passing to a subsequence, we may assume that $\lambda_{s_i}(t)$ converges to a minimal geodesic $\hat{\lambda}_\infty(t)$ in $K_x(X)$ defined on $(-\infty, \infty)$, where this convergence is uniform on every bounded interval. Note that $\hat{\lambda}_\infty(0) = u$. From Sublemma 4.11 and 4.17, the geodesic $\hat{\lambda}_\infty$ does not pass through $o_x$. Consider the curve
\begin{equation}
\hat{\xi}_\infty(t) := \hat{\lambda}_\infty(t)/|\hat{\lambda}_\infty(t)|.
\end{equation}
Obviously, $\hat{\xi}_\infty$ is a shortest path in $\Sigma_x(X)$ and $\hat{\xi}((-\infty, \infty)) \subset \Sigma_x(S)$.

Let $\xi_\infty : [-1, 1] \to \Sigma_x(S)$ be a reparametrization of the extension $\hat{\xi}_\infty : [-\infty, \infty] \to \Sigma_x(S)$ of $\xi_\infty$.

Take an arbitrary $w_+ \in (\nabla d_p)(x)$ and set $w_- = -(\nabla d_p)(x)$. Consider the two sets $\{w_+, \hat{\lambda}_{s_0}(t_0), w_-, -\hat{\lambda}_{s_0}(t_0)\}$ and $\{w_+, \xi_\infty(1), w_-, \xi_\infty(-1)\}$. They are on a circle $C$ in $\Sigma_x(X)$ in these orders, where
\begin{equation}
|L(C) - 2\pi| < \tau_p(r).
\end{equation}
Since $|\angle (\pm w, \xi_\infty(\pm 1)) - \pi/2| < \tau_p(\delta, r)$ and $|\angle (\pm w, \pm \hat{\lambda}_{s_0}(t_0)) - \pi/2| < \tau_p(\delta, r)$, we have the conclusion (4.8).

This completes the proof.

A direction $\xi \in \Sigma_x(X)$ is called regular if $\xi \notin \mathcal{S}(\Sigma_x(X))$.

Lemma 4.12. For $x \in \text{int} S$, let $\xi_1, \xi_2 \in \Sigma_x(S)$ be positively horizontal (resp. negatively horizontal). Assume that $\xi_1$ is regular. Take an $X$-geodesic $\gamma_1$ such that $\gamma_1(0) = \xi_1$, and a sequence $x_i \in S$ such that $|x, x_i|_X \to 0$ and $\uparrow x_i \to \xi_2$. Then for $s_0$ with $\lambda_{s_0} \ni x$, there exists an $\epsilon > 0$ such that if some ruling geodesic $\lambda_s$ with $|s - s_0| < \epsilon$ passes through $x_i$ for a sufficiently large $i$, then it passes through $\gamma_1$, too.
Proof. Suppose that the conclusion does not hold. Then there exists a sequence \( s_i \to s_0 \) such that \( \lambda_{s_i} \) meets \( x_i \) while \( \lambda_{s_i} \) does not pass through \( \gamma_1 \). Applying Lemma 4.10 to \( x = \lambda_{s_0}(t_0) \) and \( \xi_2 \), we have a shortest arc \( \xi_\infty : [-1, 1] \to \Sigma_x(X) \) joining two points close to \( \pm \lambda_{s_0}(t_0) \) such that \( \xi_2 \in \xi_\infty([-1, 1]) \). Similarly, applying Lemma 4.10 to \( \xi_1 \), we have a shortest arc \( \xi_\infty : ([{-1, 1}] \to \Sigma_x(X) \) joining two points close to \( \pm \lambda_{s_0}(t_0) \) such that \( \xi_1 \in \xi_\infty([-1, 1]) \).

Note that both \( \xi_\infty \) and \( \xi_\infty \) pass through the positive side of \( \Sigma_x(X) \) and connect points close to \( \pm \lambda_{s_0}(t_0) \). From construction, \( \xi_\infty \) and \( \xi_\infty \) pass through horizontal directions \( \xi_2 \) and \( \xi_1 \) respectively. Furthermore both intersections \( \xi_\infty([-1, -1 + \epsilon_1]) \cap \xi_\infty([-1, -1 + \bar{\epsilon}_1]) \) and \( \xi_\infty([-1 - \epsilon_2, 1]) \cap \xi_\infty([-1 - \bar{\epsilon}_2, 1]) \) are not empty for some small \( \epsilon_i, \bar{\epsilon}_i > 0 \) (\( i = 1, 2 \)) since they are in the regular parts of \( \Sigma_x(X) \) by Corollary 2.12. Therefore by the uniqueness of geodesics in the CAT(1)-space \( \Sigma_x(X) \), we conclude that \( \xi_\infty([-1 + \epsilon_3, 1 - \epsilon_3]) = \xi_\infty([-1 + \bar{\epsilon}_3, 1 - \bar{\epsilon}_3]) \) some small \( \epsilon_3, \bar{\epsilon}_3 > 0 \), and in particular \( \xi_\infty \) and \( \xi_\infty \) pass through both \( \xi_1 \) and \( \xi_2 \).

Take \( \xi_3, \xi_4 \in \Sigma_x(X) \) close to \( \xi_1 \) such that every element of the arc \([\xi_3, \xi_4]\) in \( \Sigma_x(X) \) is regular and \( \xi_1 \) is the midpoint of \([\xi_3, \xi_4]\). Let \( \gamma_i : [0, \epsilon] \to X \) be \( X \)-geodesics with \( \gamma(0) = \xi_i \) (\( i = 3, 4 \)), and \( \gamma_5 \) the \( X \)-minimal geodesic joining \( \gamma_3(\epsilon) \) to \( \gamma_4(\epsilon) \). If \( \epsilon > 0 \) is small enough, then the triangle \( \Delta(\gamma_3, \gamma_4, \gamma_5) \) bounds a domain in \( X \) homeomorphic to a two-disk \( D \). Let \( \text{int} D \) denote the interior of the disk \( D \). Note that \( \text{int} D \) is open in \( X \) and that \( \gamma_1([0, \epsilon_1]) \subset D \) for a small \( \epsilon_1 < \epsilon \). Since \( \text{int} D \) is open in \( X \) and since \( \xi_\infty \) is constructed by (4.14), \( \lambda_{s_0} \) really passes through \( \gamma_1 \) for large \( i \), which is a contradiction. This completes the proof. \( \square \)

4.2. Canonical balls. In this subsection, we introduce the notion of canonical balls, which turns out to be useful to have better understanding of the behavior of ruling geodesics of \( S \).

We denote by \( \mathcal{R}(X) \) the set of topological regular points, \( \mathcal{R}(X) = X \setminus \mathcal{S}(X) \).

Definition 4.13. For \( x \in B(p, r) \), a ball \( B(x, \epsilon) \) is called canonical if for every \( y \in B(x, \epsilon) \setminus \{ x \} \) with vertical \( \uparrow_y \), we have \( y \in \mathcal{R}(X) \).

Lemma 4.14. There exists an \( r = r_p > 0 \) such that there is a canonical ball around every point in \( B(p, r) \setminus \{ p \} \).
Lemma 4.14 is a direct consequence of the following Lemma 4.15, which is immediate from Corollary 2.12.

**Lemma 4.15.** For every $x \in \mathcal{S}(X) \cap B(p, r) \setminus \{p\}$, we have

$$\sup \{ \angle (\xi, \nabla d_p) \mid \xi \in \Sigma_x(S(X)) \text{ is positive} \} < \tau_p(|p, x|),$$

$$\sup \{ \angle (\xi, -\nabla d_p) \mid \xi \in \Sigma_x(S(X)) \text{ is negative} \} < \tau_p(|p, x|).$$

**Definition 4.16.** For $x \in \text{int} \ S$, let $B(x, \epsilon)$ be a canonical ball. We set

$$U_+(x, \epsilon) := \{ y \in B(x, \epsilon) \mid \angle(\gamma_{y, \epsilon}, \dot{\lambda}(x)) < \pi/4 \}$$

$$U_-(x, \epsilon) := \{ y \in B(x, \epsilon) \mid \angle(\gamma_{y, \epsilon}, -\dot{\lambda}(x)) < \pi/4 \}.$$

Note that both $U_+(x, \epsilon)$ and $U_-(x, \epsilon)$ are convex in $X$ for small $\epsilon > 0$.

In Lemma 4.21 we show that $U_\pm(x, \epsilon) \subset S$ for a small $\epsilon > 0$.

We denote by $|A|$ the cardinality of a set $A$.

**Lemma 4.17.** Let $\gamma : [0, 1] \rightarrow X$ be a vertical $X$-geodesic in $B(p, r)$. Then we have $|\gamma \cap \mathcal{S}(X)| < \infty$.

**Proof.** Suppose that the lemma does not hold. Then we would have an accumulation point $x = \gamma(t_0)$ of $\gamma \cap \mathcal{S}(X)$. It turns out that either $\dot{\gamma}(t_0)$ or $-\dot{\gamma}(t_0)$ is in $\Sigma_x(S(X))$, which is a contradiction to the existence of a canonical ball around $x$. \qed

**Remark 4.18.** At this stage, we do not know yet a uniform bound on $|\gamma \cap \mathcal{S}(X)|$ for all the vertical geodesics $\gamma$. In Section 6, we give a uniform bound (see Sublemma 6.8).

The following is a key lemma.
Lemma 4.19. (no-return lemma) For every \( s_0 \), there exists an \( \epsilon > 0 \) such that for any \( s_1 \in (s_0 - \epsilon, s_0) \) (resp. \( s_1 \in (s_0, s_0 + \epsilon) \)), there are no \( t_0, t_1 \in [0,1] \) satisfying that \( \lambda_{s_1(t_1)} \) is positively horizontal (resp. negatively horizontal) of \( \Sigma_{\lambda_{s_0}(t_0)}(X) \).

Proof. Suppose that the conclusion does not hold. Then we have some sequence \( s_i < s_0 \) with \( \lim_{i \to \infty} s_i = s_0 \) such that

\[
\lambda_{s_i}((0, u_i)) \text{ is positively horizontal for some } t_i, u_i \in (0,1).
\]

(see Figure 2). We show that both \( \lambda_{s_i}((0, u_i)) \) and \( \lambda_{s_i}((u_i, 1)) \) meet \( \lambda_{s_0} \), which yields a contradiction to the minimality of \( \lambda_{s_0} \).

From Lemmas 4.14 and 4.17, it is possible to cover \( \lambda_{s_0} \) by finitely many canonical balls \( B(x_\alpha, \epsilon_\alpha), \ 1 \leq \alpha \leq N \), where \( x_\alpha = \lambda_{s_0}(t_\alpha) \) and \( t_\alpha < t_{\alpha + 1} \). Taking smaller \( \epsilon_\alpha \) if necessary, we may further assume that for any \( i \)

\[
\begin{align*}
(1) & \quad \lambda_{s_i} \subset \bigcup_{\alpha=1}^{N} B(x_\alpha, \epsilon_\alpha); \\
(2) & \quad B(x_\alpha, \epsilon_\alpha) \cap B(x_{\alpha+1}, \epsilon_{\alpha+1}) \cap \lambda_{s_i} \subset U_+(x_\alpha, \epsilon_\alpha) \cap U_-(x_{\alpha+1}, \epsilon_{\alpha+1})
\end{align*}
\]

for each \( \alpha \).

Note that \( \lambda_{s_0} \cap S_X \subset \{x_\alpha\}_{\alpha=1}^{N} \), and that \( U_+(x_\alpha, \epsilon_\alpha) \cap U_-(x_{\alpha+1}, \epsilon_{\alpha+1}) \) is convex in \( X \) and homeomorphic to a disk. Suppose that \( \lambda_{s_i}((0, u_i)) \) does not meet \( \lambda_{s_0} \). Take a maximal interval \( I_\alpha \) in \([0,1]\) such that \( \lambda_{s_0}(I_\alpha) \subset B(x_\alpha, \epsilon_\alpha) \), and set

\[
\xi_{\alpha}^i(t) := \lambda_{s_i}^\uparrow(t) \quad (t \in I_\alpha).
\]

From the assumption, \( \xi_{\alpha}^i(I_\alpha) \) is in either the negative side or the positive side of \( \Sigma_{x_\alpha}(X) \). Note that \( \xi_{\alpha}^i(I_\alpha) \) is in the negative side of \( \Sigma_{x_\alpha}(X) \). Let \( \alpha_0 = \alpha_0(i) \) be such that \( \lambda_{s_0}(t_i) \in B(x_{\alpha_0}, \epsilon_{\alpha_0}) \). From (4.16), \( \xi_{\alpha_0}^i(I_{\alpha_0}) \) is in the positive side of \( \Sigma_{x_{\alpha_0}}(X) \). Therefore for some \( \alpha \leq \alpha_0 \), \( \xi_{\alpha}^i(I_\alpha) \) is in the negative side of \( \Sigma_{x_{\alpha-1}}(X) \) and \( \xi_{\alpha}^i(I_\alpha) \) is in the positive side of \( \Sigma_{x_\alpha}(X) \). Now \( \lambda_{s_0} \) divides the disk domain \( U_+(x_{\alpha-1}, \epsilon_{\alpha-1}) \cap U_-(x_\alpha, \epsilon_\alpha) \) into two disk domains \( D_- \) and \( D_+ \), where we may assume that \( \lambda_{s_i}(t_-) \in \)
$D_-$ and $\lambda_\alpha(t_+) \in D_+$ for some $t_- \in I_{\alpha-1}$ and $t_+ \in I_\alpha$. Thus $\lambda_\alpha([t_-, t_+])$ must meet $\lambda_\alpha_0$.

Similarly, we would have another intersection point of $\lambda_\alpha((u_1, 1))$ and $\lambda_\alpha_0$. This completes the proof. □

The following lemma is a global version of Lemma 4.19.

**Lemma 4.20.** For arbitrary $s_1 < s_2$, there are no $t_1, t_2 \in [0, 1]$ satisfying that $\lambda_{s_2}(t_2)$ (resp. $\lambda_{s_1}(t_1)$) is negatively horizontal in $\Sigma_{s_1}(t_1)(X)$ (resp. positively horizontal in $\Sigma_{s_2}(t_2)(X)$).

**Proof.** Let $I(s_1)$ be the set of all $s \in (s_1, s_2]$ such that there are no $t_1, t \in [0, 1]$ satisfying that $\lambda_{s_2}(t_2)$ is negatively horizontal in $\Sigma_{s_1}(t_1)(X)$. By Lemma 4.19 $(s_1, s_0) \subset I(s_1)$ for some $s_0 \in (s_1, s_2)$. Let $u$ be the supremum of those $s_0$. From the continuity of the map $\sigma : R \to S$, $(s_1, s_2] \setminus I(s_1)$ is open in $(s_1, s_2]$. It follows that $u \in I(s_1)$. Suppose that $u < s_2$. Then we have a sequence of positive numbers $\epsilon_i$ with $\epsilon_i \to 0$ such that $u_i := u + \epsilon_i \notin I(s_1)$. Namely we have sequences $t_i$ and $t_i'$ satisfying that $\lambda_{u_i}(t_i)$ is negatively horizontal in $\Sigma_{u_i}(t_i)(X)$. Set $x_i := \lambda_{u_i}(t_i)$, and let $y_i := \lambda_{u_i}(t_i)$. Take $z_i \in \lambda_{u_i}$ and $w_i \in \lambda_u$ such that

$$|y_i, z_i| = |y_i, \lambda_{u_i}|, \quad |z_i, w_i| = |z_i, \lambda_u|.$$

Since $\lambda_{u_i}$ is horizontal, we have $y_i \neq z_i$. By (4.6), we obtain

$$\angle(\uparrow_{z_i}^y, \nabla d_p) < \tau_p(\delta, r) \quad \text{or} \quad \angle(\uparrow_{z_i}^y, -\nabla d_p) < \tau_p(\delta, r).$$

We show that

$$(4.17) \quad \angle(\uparrow_{z_i}^y, \nabla d_p) < \tau_p(\delta, r).$$

Otherwise, we have $\angle(\uparrow_{z_i}^y, -\nabla d_p) < \tau_p(\delta, r)$. By view of Lemma 4.9 it turns out that

$$\angle x_i y_i z_i > 2\pi/3 - \tau_p(\delta, r),$$

and hence

$$\angle x_i z_i y_i < \pi - \angle x_i y_i z_i - \angle y_i x_i z_i + \tau_p(r) < \pi/3 + \tau_p(\delta, r).$$

This is a contradiction to the choice of $z_i$.

Next note that $w_i \neq z_i$. Because if $w_i = z_i$, then $\uparrow_{y_i}^{w_i}$ must be negatively horizontal by (4.17), which contradicts $u \in I(s_1)$. Now by Lemma 4.19 $\uparrow_{z_i}^w$ is positively horizontal. In view of Lemma 4.9 we have

$$(4.18) \quad \angle(\uparrow_{z_i}^y, -\nabla d_p) < \tau_p(\delta, r),$$

It follows from (4.17) and (4.18) that $\angle y_i z_i w_i > \pi - \tau_p(\delta, r)$, which implies $\angle(\uparrow_{y_i}^{w_i}, -\nabla d_p) < \tau_p(\delta, r)$. In particular $\uparrow_{y_i}^{w_i}$ is negatively horizontal. This contradicts $u \in I(s_1)$. Thus we conclude $u = s_2$.

Similarly we see that there are no $t_1, t_2$ satisfying that $\lambda_{s_1}(t_1)$ is positively horizontal. This completes the proof. □
Lemma 4.21. For every $x \in \text{int } S$, there exists an $\epsilon > 0$ such that
\[ U_+(x, \epsilon) \subset S, \quad U_-(x, \epsilon) \subset S. \]

Proof. Let $B(x, \epsilon_0)$ be a canonical ball. Take the positively horizontal $v_+ \in \Sigma_x(X)$ (resp. negatively horizontal $v_- \in \Sigma_x(X)$) such that $\angle(v_\pm, \lambda(x)) = \pi/4$. Let $\gamma_\pm$ be $X$-geodesics starting from $x$ with $\dot{\gamma}_\pm(0) = v_\pm$.

Sublemma 4.22. For any $0 < \epsilon < \epsilon_0$, there are $s_- \in (0, s_{\min}(x))$ and $s_+ \in (s_{\max}(x), r)$ such that $\lambda_{s_\pm}$ pass through $\gamma_\pm((0, \epsilon])$ respectively.

\[ \gamma_- \quad \gamma_+ \]
\[ \lambda_{s_-} \quad \lambda_{s_+} \]

Proof. Suppose that there is no such $s_- < s_{\min}(x)$. Then we have a sequence $s_i < s_{\min}(x)$ converging to $s_{\min}(x)$ such that $\lambda_{s_i}$ does not pass through $\gamma_-((0, \epsilon])$ for some $\epsilon > 0$ and all $i$. As in the proof of Lemma 4.10 together with Lemma 4.19, the curves $\xi_i(t) = \lambda_{x_i}(t)$ ($t \in [0, 1]$) in $\Sigma_x(X)$ pass through $\dot{\gamma}_-(0)$ for all $i$. This shows in particular that $\dot{\gamma}_-(0) \in \Sigma_x(S)$. Since $\dot{\gamma}_-(0)$ is regular, it follows from Lemma 4.12 that $\lambda_{s_i}$ meets $\gamma_-(0, \epsilon]$ for every large enough $i$. This is a contradiction. Similarly, we see that $\lambda_s$ meets $\gamma_+(0, \epsilon]$ for any $s > s_+$ sufficiently close to $s_+$.

Take a sufficiently small $0 < \epsilon_1 < \epsilon_0$ such that
\[ \Delta \gamma_+(\epsilon_1)x\gamma_-(\epsilon_1) \text{ spans a disk domain in } X. \]

Let $s_\pm$ be as in Sublemma 4.22 for $\epsilon_1$, and set $I = [s_-, s_+]$ and $\gamma_{\epsilon_1} = \gamma_+(0, \epsilon_1) \cup \gamma_-(0, \epsilon_1)$. It follows from the continuity of $\sigma$, Lemma 4.19 and (4.19) that $\lambda_s$ meets $\gamma_{\epsilon_1}$ for all $s \in I$. Now define $\varphi : I \to \gamma_{\epsilon_1}$ by $\varphi(s) = \lambda_s \cap \gamma_{\epsilon_1}$. Let $\gamma_{\epsilon}(\epsilon) := \varphi(s \pm)$. Since $\varphi$ is continuous, the intermediate-value theorem implies that
\[ \gamma_{\epsilon_2} \subset \text{Im } \varphi \subset S, \]

where $\epsilon_2 := \min\{\epsilon_+, \epsilon_-\}$.

For any $0 < \epsilon \leq \epsilon_2$, let $\mu_{\epsilon}$ be the $X$-geodesic joining $\gamma_-({\epsilon})$ to $\gamma_+({\epsilon})$. Put
\[ \hat{\gamma}_{\epsilon, \epsilon_2} := \gamma_-([\epsilon, \epsilon_2]) \cup \mu_{\epsilon} \cup \gamma_+([\epsilon, \epsilon_2]). \]

Similarly, we can define the map $\psi : I \to \hat{\gamma}_{\epsilon, \epsilon_2}$ by $\psi(s) = \lambda_s \cap \hat{\gamma}_{\epsilon, \epsilon_2}$. Again, since $\psi$ is continuous, the intermediate-value theorem implies
that $\psi$ is surjective, and hence $\mu_\epsilon \subset S$. Now we can take $\epsilon_3^+ > 0$ such that
\[ U_+(x, \epsilon_3^+) \subset \bigcup_{0 \leq \epsilon \leq \epsilon_2} \mu_\epsilon \subset S. \]
Similarly we have $U_-(x, \epsilon_3^-) \subset S$ for some $\epsilon_3^- > 0$. This completes the proof of Lemma 4.21. \hfill \Box

4.3. Spaces of directions. In this subsection, we determine the structure of the space of directions of $S$ at each point of $S$.

**Lemma 4.23.** For every $x \in S$, let $\xi \in \Sigma_x(S)$ be regular in $\Sigma_x(X)$, and let $\gamma$ be an $X$-geodesic with $\dot{\gamma}(0) = \xi$. Then $\gamma([0, \epsilon]) \subset S$ for a small $\epsilon > 0$. Furthermore, $\epsilon$ can be taken locally uniformly for $\xi$.

**Proof.** First assume $x \in \text{int} S$, and let $x = \lambda_s(t)$. From Lemma 4.12, we may assume that $\angle(\xi, \pm \lambda_s(t)) \geq \pi/3$. If $\xi$ is positive, Lemmas 4.12 and 4.19 imply that for some $s_1 > s_{\max}(x)$, $\lambda_s$ meets $\gamma$ at, say $\gamma(t(s))$ for every $s \in [s_{\max}(x), s_1]$. Since $\xi$ is horizontal, $t(s)$ is unique and continuous in $s$, and $t(s) > 0$ if $s > s_{\max}(x)$. Therefore $\gamma([0, t(s_1)]) \subset S$. From this argument, the local uniformness of $t(s_1)$ for $\xi$ is clear. The case when $\xi$ is negative is similar. If $x \in \partial S \setminus \{p\}$, the proof is similar.

Finally we consider the case $x = p$. For small enough $\epsilon > 0$, let $\xi_\pm \in \Sigma_p(X)$ be such that
\[ \angle(\xi_+, \xi_-) = \angle(\xi_+, \xi) + \angle(\xi, \xi_-) = 2\angle(\xi_+, \xi) = 2\epsilon, \]
and let $\gamma_\pm$ be $X$-geodesics with $\dot{\gamma}_\pm(0) = \xi_\pm$. For a small $\eta > 0$, let $U(\eta)$ denote the domain bounded by $\gamma_\pm$ and $S(p, \eta)$. If $\eta$ is small enough, then $U(\eta)$ is homeomorphic to a disk. Take $x_i \in S$ with $x_i \to p$ such that $\uparrow^p_{x_i} \to \xi$. Then for a large $N$, we have $x_i \in U(\eta)$ for all $i \geq N$. If $x_i = \lambda_{s_i}(t_i)$, $\lambda_{s_i}$ must meet $\gamma_\pm$. The intermediate-value theorem then yields that the subdomain of $U(\eta)$ bounded by $\gamma_\pm$ and $\lambda_{s_N}$ is contained in $S$. In particular $\gamma([0, \epsilon_1]) \subset S$ for small $\epsilon_1 > 0$. \hfill \Box

**Remark 4.24.** If $\xi \in \Sigma_x(S)$ is singular in $\Sigma_x(X)$, Lemma 4.23 does not hold in general. See Examples 4.4 and 4.5.

**Lemma 4.25.** Let $x \in S$.

1. If $x \in \text{int} S$, then $\Sigma_x(S)$ is a circle of length $< 2\pi + \tau_p(r)$.
2. If $x \in \partial S$, then $\Sigma_x(S)$ is an arc.

**Proof.** (1) First we show that $\Sigma_x(S)$ contains a circle $C$. Take an $s_0 \in s(x)$ and $t_0$ with $x = \lambda_{s_0}(t_0)$. Obviously
\[ C_0 := \{ \xi \in \Sigma_x(X) \mid \angle(\pm \lambda_{s_0}(t_0), \xi) \leq \pi/3 \} \]
consists of two arcs in the regular part of $\Sigma_x(X)$. It follows from Lemma 4.21 that $C_0$ is contained in $\Sigma_x(S)$. For a positively horizontal
direction \( v_+ \in \Sigma_x(S) \), we apply Lemma 4.10 to obtain a minimal arc \( C_+ \) in \( \Sigma_x(S) \) joining two points close to \( \pm \lambda_{s_0}(t_0) \) and containing \( v_+ \).

Similarly, for a negatively horizontal direction \( v_- \in \Sigma_x(S) \), we apply Lemma 4.10 to obtain a minimal arc \( C_- \) in \( \Sigma_x(S) \) joining two points close to \( \pm \lambda_{s_0}(t_0) \) and containing \( v_- \). Obviously the union of \( C_0, C_+ \) and \( C_- \) forms a circle \( C \) in \( \Sigma_x(S) \). It follows from Lemma 4.7 and (4.8) that

\[
|L(C_\pm) - \pi| < \tau_p(r), \quad |L(C \setminus (C_+ \cup C_-))| < \tau_p(r),
\]

which implies \( |L(C) - 2\pi| < \tau_p(r) \).

Suppose next that \( \Sigma_x(S) \setminus C \) is not empty, and take a \( w \) in \( \Sigma_x(S) \setminus C \). Since \( \angle(w, \pm \lambda_{s_0}(t_0)) \geq \pi/3 \), we can apply Lemma 4.10 to obtain a minimal arc \( C_1 \) in \( \Sigma_x(S) \) joining two points close to \( \pm \lambda_{s_0}(t_0) \) and containing \( w \). Note that the complement \( C_1' \) of a small neighborhood of \( \pm \lambda_{s_0}(t_0) \) in \( C_1 \) is contained in \( C \), and \( w \) must be contained in \( C_1' \), which is a contradiction.

(2) If \( x = \lambda_s(0) \) with \( 0 < s < \ell \) (resp. \( s = \ell \)), then \( \Sigma_x(S) \) is an arc with endpoints \( \pm \hat{\alpha}_1(s) \) (resp. \( -\hat{\alpha}_1(\ell) \) and \( \hat{\lambda}_s(0) \)) through \( \hat{\lambda}_s(0) \) (recall (4.2)). The case \( x = \lambda_s(1) \) with \( 0 < s \leq \ell \) is similar. Next consider the case \( x = p \). Let \( v \) and \( \nu_1, \nu_2 \) be as in (4.1). We show that \( \Sigma_p(S) \) coincides with the arc \( [\nu_1, \nu_2] \) in \( \Sigma_p(X) \). Let \( \eta_i \) be any interior point of \( [\nu_i, v] \), and let \( \sigma_i \) be \( X \)-geodesics with \( \hat{\sigma}_i(0) = \eta_i \). If \( s > 0 \) is small enough, then \( \lambda_s \) meets both \( \sigma_1 \) and \( \sigma_2 \). This implies that \( [\nu_1, \eta_1] \cup [\eta_2, \nu_2] \) is contained in \( \Sigma_p(S) \).

Definition 4.26. For \( x \in S \), let \( \Sigma_x(S^{\text{int}}) \) denote the intrinsic space of directions of \( S \) at \( x \), which is defined as the completion of the set of all equivalence classes of \( S \)-geodesics starting from \( x \) equipped with the upper angle \( \angle_s^X \) for the induced interior metric of \( S \).

Lemma 4.27. \( \Sigma_x(S) \) is isometric to \( \Sigma_x(S^{\text{int}}) \).

Proof. First assume \( x \in \text{int} S \). Let

\[
\Omega := \Sigma_x(S) \cap S(\Sigma_x(X)).
\]

Note that \( |\Omega| < \infty \). We first show that each component \( \Sigma \) of \( \Sigma_x(S) \setminus \Omega \) is isometrically embedded in \( \Sigma_x(S^{\text{int}}) \). Take \( \xi_1 \) and \( \xi_2 \) from \( \Sigma \) with \( |\xi_1, \xi_2| < \pi \). Let \( \mu_i : [0, \epsilon] \to X \) be an \( X \)-geodesic with \( \mu_i(0) = \xi_i \). Then for small \( \epsilon \), we have from Lemma 4.23

1. \( \mu_i \subset S \)
2. every \( X \)-geodesic joining \( \mu_1(t) \) and \( \mu_2(t) \) is contained in \( S \) for every \( t \in [0, \epsilon] \).
Thus we conclude that \( \angle^X(\xi_1, \xi_2) = \angle^S(\xi_1, \xi_2) \).

Next, for any \( v \in \Omega \), take \( \xi_3, \xi_4 \in \Sigma_\omega(S) \setminus \Omega \) close to \( v \) such that \( \xi_3, v \) and \( \xi_4 \) are in this order on the circle \( \Sigma_\omega(S) \). Take \( X \)-geodesics \( \gamma_i \) (\( i = 3, 4 \)) in the direction \( \xi_i \). By Lemma 4.12, we can find a sequence \( s_i \) such that \( s_i \to s_0 \in \sigma(x) \) and \( \lambda_{s_i} \) meets both \( \gamma_3 \) and \( \gamma_4 \). This implies that \( \angle^X(\xi_3, \xi_4) = \angle^S(\xi_3, \xi_4) \). This completes the proof.

The case \( x \in \partial S \) is similar, and hence we omit the proof. \( \square \)

4.4. **Proof of Theorem 4.1** In this subsection, we first prove Theorem 4.1. Then we control the difference between the geometries of \( S \) and \( X \).

**Lemma 4.28.** For every \( x \in S \), we have

1. \( s(x) \) is either a point or a closed interval;
2. \( \sigma^{-1}(x) \) is a strictly monotone arc in \( R \).

**Proof.** Suppose that the conclusion (1) does not hold. Then we would have \( s_- < s_+ \) such that \( s_\pm \in s(x) \) and \( (s_-, s_+) \) does not meet \( s(x) \). By Lemma 2.4, we may assume \( x \neq p \). Choose an \( S \)-geodesic \( \gamma : [0, a) \to S \) starting from \( x \) such that \( \dot{\gamma}(0) \) is a positive, horizontal and regular direction. Let us denote

\[
I = \{ s \in (s_-, s_+) \mid \lambda_s \text{ passes through } \gamma \setminus \{ x \} \}.
\]

From Lemmas 4.19 and 4.12 there is \( \epsilon_0 > 0 \) such that \( (s_-, s_+ + \epsilon) \subset I \) for every \( 0 < \epsilon \leq \epsilon_0 \). Since \( x \in \lambda_{s_+} \), this is a contradiction to Lemma 4.20.

(2) follows immediately from (1) and the injectivity of \( \sigma|_{I_s} \) for each \( s \in (0, \ell] \). \( \square \)

**Proof of Theorem 4.1** By Lemma 4.28, we have (3.4) for all \( u, v \in R \). Thus \( S \) has the induced metric from \( \sigma \). Theorem 3.26 then implies that \( (S, d_S) \) is a \( \text{CAT}(\kappa) \)-space.

We set \( S^{\text{int}} := (S, d_S) \).

**Lemma 4.29.** \( S^{\text{int}} \) is locally geodesically complete.

**Proof.** This is immediate from Lemma 4.25 in a straightforward way. See [11, Proposition II.5.12], [19, Theorem 1.5] together with [17, Theorem A] for general considerations. \( \square \)

We prove that \( S^{\text{int}} \) is a topological two-manifold with boundary. In view of Lemmas 2.8, 4.25, 4.27 and 4.29, it suffices to show that a small \( S \)-ball around any point \( x \in \partial S \) is homeomorphic to a half disk. Suppose \( x = p \). The other cases are similar. The argument is standard. Logically, we proceed as follows. For a positive integer \( m \) with \( m \geq \lceil \pi/2 \delta \rceil + 1 \), gluing \( m \) copies of \( S \) in order around \( p \), we have a sector \( T \) with sector angle \( \geq \pi \) at \( p \), which is a \( \text{CAT}(\kappa) \)-space by Theorem 2.2. Glue two copies of \( T \) along their edges to obtain a \( \text{CAT}(\kappa) \)-space \( W \).
for which \( L(\Sigma_p(W)) \geq 2\pi \). Then Lemma \[2.8\] shows that \( p \) has an open disk neighborhood, which implies that \( p \) has a half-disk neighborhood in \( S \).

Finally, from the CAT(\( \kappa \))-property of \( S \), the contractibility of \( S \) is immediate since we may assume that the diameter of \( S \) for the metric \( d_S \) is less than \( \pi/\sqrt{\kappa} \) when \( \kappa > 0 \). This completes the proof of Theorem \[4.1\]. □

In the rest of this section, we present a few results that control the difference between the geometries of \( X \) and \( S \). These will be needed in Sections 5 and 7.

**Lemma 4.30.** For arbitrary distinct \( x, y \in S \), let \( (-\nabla^S d_x)(y) \) denote \( \dot{\gamma}^S_{x,y}(0) \), where \( \dot{\gamma}^S_{x,y} \) is the \( S \)-geodesic from \( y \) to \( x \). Then we have

\[
\begin{align*}
(1) & \quad \angle(\dot{\gamma}^S_{x,y}(0), \dot{\gamma}^X_{x,y}(0)) < \tau_x(|x, y|_X); \\
(2) & \quad \angle((\nabla^S d_x)(y), (\nabla^X d_x)(y)) < \tau_x(|x, y|_X).
\end{align*}
\]

For the proof, we need a sublemma.

**Sublemma 4.31.** For every \( x \in S \), we have

\[
\sup_{y \in B^S(x, s)} \frac{|x, y|_S}{|x, y|_X} < 1 + \tau_x(s).
\]

When \( x \in S \setminus S(X) \), Sublemma \[4.31\] and Lemma \[4.30\] are clear.

**Proof of Sublemma \[4.31\]** If the sublemma does not hold, there would exist a sequence \( x_n \) in \( S \) converging to \( x \) such that

\[
|\dot{x}_n|_S > 1 + c > 1,
\]

for some constant \( c > 0 \) independent of \( n \). Passing to a subsequence, we may assume that \( \uparrow x_n \) converges to a direction \( v \in \Sigma_x(X) \). It is easily seen from \[4.20\] that \( v \) is a vertex of \( \Sigma_x(X) \). Take a small enough \( \epsilon > 0 \) compared with \( c \) and an \( s_n \in s(x_n) \). Let \( y_n \) be an element of \( \lambda_n \) with \( |x_n, y_n| = \epsilon |x, x_n|_X \). From Lemma \[4.25\], the \( X \)-geodesic joining \( x \) and \( y_n \) is contained in \( S \) for any large \( n \). It follows from triangle inequality that

\[
\frac{|x, x_n|_S}{|x, x_n|_X} > 1 + c > 1,
\]

which is a contradiction. □

**Proof of Lemma \[4.30\]** If Lemma \[4.30\] does not hold, there would be a sequence \( x_i \) of \( S \) converging to \( x \) such that

\[
\angle(\dot{\gamma}^S_{x_i, x}(0), \dot{\gamma}^X_{x_i, x}(0)) > c > 0, \quad \text{or}
\]

\[
\angle((\nabla^S d_x)(x_i), (\nabla^X d_x)(x_i)) > c > 0,
\]
where $c$ is a uniform positive constant. From now, we assume $x \in \text{int} S$. 
The other case is similar. We may assume that $\xi^X_i := \dot{\gamma}^X_{x,x_i}(0)$ and \[\xi^S_i := \dot{\gamma}^S_{x,x_i}(0)\] converge to $\xi^X \in \Sigma_x(X)$ and $\xi^S \in \Sigma_x(S) \subset \Sigma_x(X)$ respectively. Note that $\xi^X \in \Sigma_x(S)$.

(1) First we assume (4.21). Then we have $\angle(\xi^X,\xi^S) \geq c$. We show $\xi^X \in \Sigma_x(S) \cap V(\Sigma_x(X))$. Actually, by Lemma 4.23 if $\xi^X \in \Sigma_x(S) \setminus V(\Sigma_x(X))$, we have $\epsilon > 0$ such that $\gamma^X_{\xi^X}([0,\epsilon]) \subset S$ for any large $i$. It turns out $\gamma^X_{\xi^X_{x_i}} \subset S$, which is a contradiction to (4.21). Similarly, we have $\xi^S \in \Sigma_x(S) \cap V(\Sigma_x(X))$. Since $\Sigma_x(S) \cap V(\Sigma_x(X))$ is a point, it follows that $\xi^X = \xi^S$. This is a contradiction.

(2) Next assume (4.22). We set $\xi := \xi^X = \xi^S$, and $\Sigma := \Sigma_x(S)$. From the above argument of (1) and (4.22), we have $\xi \in \Sigma \cap V(\Sigma_x(X))$. Letting $t^X_i := [x_i,x|x_i$ consider the convergence $(\frac{1}{t}X,x_i) \to (K_x(X),\xi^X)$. Similarly, letting $t^S_i := [x_i,x_i|x_i$, from Lemma 4.27 we have the convergence $(\frac{1}{t}S_{y_i},x_i) \to (K(\Sigma),\xi^S)$. Let $\mu_1, \mu_2$ be elements of $\Sigma \setminus V(\Sigma_x(X))$ near $\xi$ such that $\xi$ is in the interior of the shortest arc between $\mu_1$ and $\mu_2$. Take any $s_i \in s(x_i)$, and let $y_i, z_i$ be the intersections of $\lambda_{x_i}$ and $\gamma_{s_{\mu_1}}, \gamma_{s_{\mu_2}}$ respectively. Let $y_{\infty} \in K(\Sigma)$ and $z_{\infty} \in K(\Sigma)$ be the limit of $y_i$ and $z_i$ under the above rescaling limit respectively. By Lemma 2.1 we have

$$\lim_{i \to \infty} \angle y_i x_i x \leq \angle y_{\infty} x_{\infty} x, \quad \lim_{i \to \infty} \angle z_i x_i x \leq \angle z_{\infty} x_{\infty} x,$$

$$\lim_{i \to \infty} \angle y_i x_i x \leq \angle y_{\infty} x_{\infty} x, \quad \lim_{i \to \infty} \angle z_i x_i x \leq \angle z_{\infty} x_{\infty} x.$$

It follows from

$$\angle y_i x_i x + \angle z_i x_i x \geq \pi, \quad \angle y_i x_i x + \angle z_i x_i x \geq \pi,$$

$$\angle y_{\infty} x_{\infty} x + \angle z_{\infty} x_{\infty} x = \pi$$

that

$$\lim_{i \to \infty} \angle y_i x_i x = \angle y_{\infty} x_{\infty} x = \lim_{i \to \infty} \angle y_i x_i x,$$

$$\lim_{i \to \infty} \angle z_i x_i x = \angle z_{\infty} x_{\infty} x = \lim_{i \to \infty} \angle z_i x_i x.$$
Lemma 4.32. For any\( x, y \in S \), suppose that the \( S \)-geodesic \( \gamma_{x,y}^S : [0,1] \to S \) from \( x \) to \( y \) is vertical. Then \( \gamma_{x,y}^S \) is an \( X \)-geodesic.

Proof. For any \( t \in [0,1] \), let \( \varepsilon > 0 \) be chosen as in Lemma 4.21 for \( z := \gamma_{x,y}^S(t) \). Choose \( t_n \to t \), and set \( z_n := \gamma_{x,y}^S(t_n) \). Let \( \gamma_{n}^X \) be the \( X \)-geodesic joining \( x \) to \( z_n \). In view of Lemmas 4.26 and 4.21, we have
\[
\gamma_{n}^X \subset U(\varepsilon) \subset S.
\]
This implies that \( \gamma_{n}^X \) must be a subarc of \( \gamma_{x,y}^S \), and hence \( \gamma_{x,y}^S \) is an \( X \)-geodesic. \( \Box \)

Lemma 4.33. For \( x, y \in S \) with \( x \in \alpha_1 \) and \( y \in \alpha_2 \) satisfying
\[
||p, x||_X - ||p, y||_X < |x, y|_X/100,
\]
the \( S \)-geodesic joining \( x \) and \( y \) is an \( X \)-geodesic.

Proof. It follows from the assumption that
\[
||p, x||_S - ||p, y||_S < |x, y|_X/100 \leq |x, y|_S/100.
\]
Using Lemma 2.5 in \( S \), we have
\[
|\angle p_{x,y}^S(t)x - \pi/2| < \pi/3, \quad |\angle p_{x,y}^S(t)y - \pi/2| < \pi/3,
\]
for all \( t \in (0,1) \), where \( \gamma_{x,y}^S : [0,1] \to S \) is the \( S \)-geodesic joining \( x \) to \( y \). This implies that \( \gamma_{x,y}^S \) is vertical. The lemma then follows from Lemma 4.32. \( \Box \)

In a way similar to Lemma 4.33, we have the following.

Lemma 4.34. For arbitrary \( x, y \in S \) such that
\[
||p, x||_S - ||p, y||_S < |x, y|_S/100,
\]
the \( S \)-geodesic joining \( x \) and \( y \) is an \( X \)-geodesic.

5. Filling via \( \text{CAT}(k) \)-disks

Let \( v \) be a vertex of \( \Sigma_p \) of order \( N \), and let \( \nu_1, \ldots, \nu_N \) be the set of all points of \( \Sigma_p \) with \( d(\nu_i, v) = \delta \) for a sufficiently small positive number \( \delta \). Take a small enough \( r > 0 \) and points \( a_1, \ldots, a_N \) of \( S(p, 2r) \) with \( \gamma_{p,a_i}(0) = \nu_i \) and \( r \leq r_p \). For simplicity, we denote by \( S(a_i, a_j) \) the ruled surface \( S(\gamma_{p,a_i}, \gamma_{p,a_j}) \) spanned by \( \gamma_{p,a_i} \) and \( \gamma_{p,a_j} \). Let \( V(\Sigma_p) \) be the set of all vertices of the graph \( \Sigma_p \). Since \( V(\Sigma_p) \) is finite, we have a positive number \( r_p \) such that for any \( 0 < r \leq r_p \), all the \( S(a_i, a_j) \),
when \( v \) runs over \( V(\Sigma_p) \), satisfy the conclusion of Theorem 4.1. Then obviously \( S(X) \cap B(p, r) \) is contained in the union of all \( S(a_i, a_j) \) when \( v \) runs over \( V(\Sigma_p) \).

**Sector correspondence.** We fix \( S := S(a_i, a_j) \) for a moment, and set

\[
\Omega(S, r)^X := B^X(p, r) \cap S, \quad C^X := S^X(p, r) \cap S.
\]

From here on, we use the symbols \( B^X(p, r) \) and \( S^X(p, r) \) to emphasize the metric ball and the metric circle in \( X \). Note that \( \Omega(S, r)^X \) is bounded by the two geodesics \( \gamma_{p,a_i}, \gamma_{p,a_j} \) and \( C^X \).

To show Theorem 1.1(3), we need the following lemma.

**Lemma 5.1.** For any small enough \( r \leq r_p \), the sector \( \Omega(S, r)^X \) is \( \tau_p(r) \)-almost isometric to a Euclidean sector.

**Proof.** By Lemma 4.30, for every \( x \in C^X \), we have \( \angle(\hat{\gamma}_{x,p}^X(0), \hat{\gamma}_{x,p}^S(0)) < \tau_p(r) \). Since \( \angle(\hat{\gamma}_{x,p}^X(0), \hat{C}^X) = \pi/2 \), it follows that

\[
|\angle(\hat{\gamma}_{x,p}^S(0), \hat{C}^X) - \pi/2| < \tau_p(r).
\]

Consider the rescaling limit of the CAT(\( k \))-space: \( (\frac{1}{r} S(p) \rightarrow (K_p(S), o_p) \) as \( r \rightarrow 0 \). By Theorem 2.6, we have a \( \tau_p(r) \)-almost isometry \( \varphi : \Omega(S, r)^X \rightarrow \text{image}(\varphi) \subset \mathbb{R}^2 \). It suffices to show that image(\( \varphi \)) is \( \tau_p(r) \)-almost isometric to a Euclidean sector. Although the argument below is elementary and standard, we present a proof for completeness since we do not find a reference.

We may assume \( \varphi(p) = (0, 0) = O \). Let \( L_k \) be the line segment from \( O \) to \( \varphi(\gamma_k(r)) \) (\( k = 1, 2 \)). We express \( L_k \) in the polar coordinates as

\[
L_k(x) = (x, \theta_k) \quad (0 \leq x \leq x_k(r)),
\]

where \( \theta_k \) is a constant and \( x_k(r) := |\varphi(\gamma_k(r)), O| \). Let \( \theta_0 \) be the direction representing the midpoint of \( L_1(0) \) and \( L_2(0) \). We may assume that \( \theta_1 < \theta_0 = 0 < \theta_2 \), and let \( L_0 \) be the line segment from \( O \) in the direction \( \theta_0 : L_0(x) = (x, 0) \). Let \( \varphi(C^X) \) intersect \( L_0 \) with \( (r_0, 0) \). Set \( q_k := L_k(x_k(r)) \).

Let \( U_k \) (resp. \( D_k \)) be the domain bounded by \( L_0, \varphi(\gamma_k) \) and \( \varphi(C^X) \) (resp. by \( L_0, L_k \) and \( \varphi(C^X) \)). Let \( \Omega(L_1, L_2; r) \) denote the Euclidean sector bounded by the rays in the direction \( L_1, L_2 \) and the circle of radius \( r \). In the first step, we deform image(\( \varphi \)) = \( U_1 \cup U_2 \) to \( D_1 \cup D_2 \) via a \( \tau_p(r) \)-almost isometry. In the second step, we deform \( D_1 \cup D_2 \) to \( \Omega(L_1, L_2; r) \) via a \( \tau_p(r) \)-almost isometry.

Step 1). Choose a point \( q_0 \in L_0 \) such that \( \angle \pi/4 \leq \varphi q_0 q_0 \leq \pi/3 \) for \( k = 1, 2 \). Note that \([q_k, q_0] \subset U_k \). Let \( J_k \) denote the union \([O, q_0] \cup [q_0, q_k] \). Let \( \hat{U}_k \) (resp. \( \hat{D}_k \)) be the domain bounded by \( L_0, \varphi(\gamma_k) \) and \([q_k, q_0] \) (resp. by \( L_0, L_k \) and \([q_k, q_0] \)). We first show that \( \hat{U}_k \) is \( \tau_p(r) \)-almost isometric to \( \hat{D}_k \).
Let $J_k(x)$ ($0 \leq x \leq L(J_k)$) be the arc-length parameter of $J_k$ with $J_k(0) = O$. For every $x \in [0, L(J_k)]$, let $\zeta_k(x, s)$ ($0 \leq s \leq 2$) be the segment such that

- $\zeta_k(x, 0) = J_k(x)$, $\zeta_k(x, 1) \in L_k$;
- $|O, \zeta_k(x, s)| = |O, \zeta_k(x, 0)|$ for all $s \in [0, 2]$;
- $s \mapsto \zeta_k(x, s)$ is proportional to arc-length.

Then $\zeta_k(x, s)$ ($0 \leq x \leq L(J_k), 0 \leq s \leq 1$) defines a parametrization of $\hat{D}_k$, and differentiable except at $x = x_0$, where $J_k(x_0) = q_0$. Take a unique $t_k(x) \in (0, 2)$ such that

$$
\zeta_k(x, t_k(x)) \in \text{Im} (\varphi_k \circ \gamma_k).
$$

Now, we define $\psi_k : \hat{U}_k \to \hat{D}_k$ ($k = 1, 2$) by

$$
\psi_k(\zeta_k(x, s)) := \zeta_k \left( x, \frac{s}{t_k(x)} \right).
$$

Obviously $t_k(x)$ is locally Lipschitz, and hence differentiable on a set $\Omega \subset [0, L(J_k)]$ with full measure since $\zeta_k(x, s)$ defines a locally bi-Lipschitz embedding.

**Sublemma 5.2.** Each $\psi_k : \hat{U}_k \to \hat{D}_k$ is a $\tau_p(r)$-almost isometry.

**Proof.** In the expression $\psi_k(x, s) := \psi_k \circ \zeta_k(x, s) = \zeta_k(x, s/t_k(x))$, we have on $\Omega \times [0, 1]$

$$
(5.2) \quad \frac{\partial \psi_k}{\partial s} = \frac{1}{t_k(x)} \frac{\partial \zeta_k}{\partial s}, \quad \frac{\partial \psi_k}{\partial x} = \frac{\partial \zeta_k}{\partial x} + \left( -s t_k'(x) \right) \frac{\partial \zeta_k}{\partial s}.
$$

It is easily checked that

$$
(5.3) \quad \begin{cases} 
0 < c_1 < \left| \frac{\partial \zeta_k}{\partial x} \right| < c_2, \\
0 < c_3 < \left( \frac{\partial \zeta_k}{\partial s}, \frac{\partial \zeta_k}{\partial x} \right) < \pi - c_4,
\end{cases}
$$

for some uniform positive constants $c_1, \ldots, c_4$. Note also that

$$
(5.4) \quad |t_k(x) - 1| < \tau_p(r).
$$
By the property of $\varphi$, we see that any tangent vector to $\varphi \circ \gamma_k$ is $\tau_p(r)$-almost parallel to the radial direction. Now consider the curve $\eta_k(x) = \zeta_k(x, t_k(x))$ parametrizing $\varphi \circ \gamma_k$. It follows from the expression 

$$
\frac{d\eta_k}{dx}(x) = \frac{\partial \zeta_k}{\partial x}(x, t_k(x)) + \frac{\partial \zeta_k}{\partial s}(x, t_k(x)) t_k'(x),
$$

that

$$
(5.5) \quad \left| \frac{\partial \zeta_k}{\partial s}(x, s) t_k'(x) \right| < \tau_p(r).
$$

Let

$$
v := \frac{\partial \zeta_k}{\partial x}, \quad V := d\psi_k(v),
$$

$$
w := \frac{\partial \zeta_k}{\partial s} \left| \frac{\partial \zeta_k}{\partial s} \right|, \quad W := d\psi_k(w).
$$

Combining (5.2), (5.3), (5.4) and (5.5), we have

$$
||V - v|| < \tau_p(r), \quad ||W - w|| < \tau_p(r),
$$

$$
|\langle V, W \rangle - \langle v, w \rangle| < \tau_p(r).
$$

Together with (5.3), this implies $||d\psi_k(u)| - 1| < \tau_p(r)$ for each unit tangent vector $u$ on $\Omega \times [0, 1]$. This completes the proof of Sublemma 5.2.

Obviously, the $\tau_p(r)$-almost isometry $\psi_k : \hat{U}_k \to \hat{D}_k$ extends to a $\tau_p(r)$-almost isometry $\psi_k : U_k \to D_k$. Combining $\psi_1$ and $\psi_2$, we obtain a $\tau_p(r)$-almost isometry $\psi$ between the image of $\varphi$ and $D_1 \cup D_2$:

$$
\psi : \text{Im}(\varphi) \to D_1 \cup D_2 \subset \mathbb{R}^2.
$$

Step 2). Finally we deform $D_1 \cup D_2$ to the Euclidean sector $\Omega(L_1, L_2; r)$. Let $\varphi(C^X)$ be parametrized as $\varphi(C^X) = (r(t), \theta(t)) \quad 0 \leq t \leq 1$. For every $0 \leq r' \leq r(t)$, let us define

$$
\phi(r', \theta(t)) = \left( \frac{r}{r(t)} r', \theta(t) \right),
$$

which defines a $\tau_p(r)$-almost isometry

$$
\phi : D_1 \cup D_2 \to \Omega(L_1, L_2; r).
$$

Thus the composition $\phi \circ \psi \circ \varphi : \Omega(S, r)^X \to \Omega(L_1, L_2; r)$ is a $\tau_p(r)$-almost isometry. This completes the proof of Lemma 5.1.

Lemma 5.3. $S \cap B(p, r)$ is a $\text{CAT}(\kappa)$-space with respect to the interior metric.
**Proof.** It suffices to show that every point \( q \in S \cap S(p, r) \) has a neighborhood \( U \) in \( S \cap B(p, r) \) such that any \( S \)-geodesic triangle region whose vertices are in \( U \) is contained in \( S \cap B(p, r) \). To achieve this, we only have to show that \( S \cap B(p, r) \) is boundary convex, in the sense that for arbitrary \( x, y \in S \cap S(p, r) \), any \( S \)-minimal geodesic \( \gamma_{x,y}^S \) joining them is contained in \( S \cap B(p, r) \). We may assume that \( \gamma_{x,y}^S \) is vertical, and therefore it is an \( X \)-geodesic (see also Lemma 6.3). Hence the conclusion follows from the \( X \)-convexity of \( B(p, r) \). □

**Filling ball.** Now we fill the ball \( B(p, r) \) via properly embedded/branched immersed \( \text{CAT}(\kappa) \)-disks. For a vertex \( v \) of \( \Sigma_p \) of order \( N \), let \( \nu_1, \ldots, \nu_N \) and \( a_1, \ldots, a_N \) be as in the beginning of Section 5. For every pair \((i, j)\) with \( 1 \leq i < j \leq N \), we want to take a simple loop in \( \Sigma_p(X) \) passing through \( \nu_i, v \) and \( \nu_j \). Since this is not possible in general, we consider the two cases.

**Case I.** There is a simple loop \( \zeta \) in \( \Sigma_p(X) \) through \( \nu_i, v \) and \( \nu_j \).

Consider the ruled surface \( S(a_i, a_j) \) as well as the other ruled surfaces defined around other points of \( \zeta \) which are vertices of \( \Sigma_p(X) \) (if they exist). By Lemma 5.3, considering the regular part of \( \zeta \) as well, we can define a proper Lipschitz embedding \( f_{ij}^\nu : D^2(\ell; r) \to B(p, r) \) with \( f_{ij}^\nu(O) = p \) satisfying \( \Sigma_p(\text{Im}(f_{ij}^\nu)) = \zeta \), where \( \ell \) is the length of \( \zeta \).

**Proof of Theorem 6.1 for embedded disks.** Lemma 5.3 together with the gluing procedure as discussed after Lemma 4.29 implies that \( \text{Im}(f_{ij}^\nu) \) is a \( \text{CAT}(\kappa) \)-space. Note that \( f_{ij}^\nu \) has bi-Lipschitz constant \( < 1 + \tau_p(r) \). □

**Case II.** There are no simple loops in \( \Sigma_p(X) \) containing \( \nu_i, v \) and \( \nu_j \).

**Claim 5.4.** There is an immersion \( g : S^1 \to \Sigma_p(X) \) such that

1. if \( W \) is the set of multiple points of \( g \), then \( g^{-1}(W) \) consists of two arcs \( W_1, W_2 \) (they may be points), and each restriction \( g|_{W_a} : W_a \to W \ (a = 1, 2) \) is injective;
2. there is an arc \( I \) of \( S^1 \) such that \( g(I) \) coincides with the arc between \( \nu_i \) and \( \nu_j \) containing \( v \).

**Proof.** In view of the present case, there are non-contractible loops \( C_i \) and \( C_j \) at \( v \), freely homotopic to a circle, such that \( \nu_i \in C_i, \nu_j \in C_j, \nu_j \notin C_i, \nu_i \notin C_j \). If both \( C_i \) and \( C_j \) are simple, we can define a desired immersion \( g : S^1 \to \Sigma_p(X) \) with \( W = \{v\} \). Suppose \( C_i \) is not simple. Then \( C_i \) contains a simple loop \( \tilde{C}_i \) at a point \( u_i \) such that \( C_i \) is the union of \( \tilde{C}_i \) and the arc \([v, u_i]\). If \( C_j \) is also not simple, then we consider the union of simple loops \( \tilde{C}_i \), \( \tilde{C}_j \) and the arc \([u_i, u_j]\). If only \( C_i \) is not
simple, then we consider the union of simple loops $\bar{C}_i, C_j$ and the arc $[u_t, v]$. This observation provides a desired immersion $g : S^1 \to \Sigma_p(X)$ with $W = [u_t, u_j]$ or $W = [u_t, v]$.

First suppose $W = \{v\}$ and find $\nu_k \in C_i$ and $\nu_\ell \in C_j$, $1 \leq k, \ell \leq N$, $k, \ell \neq i, j$. Chasing on $g(I)$ in the order

$$\nu_i \to v \to \nu_j \to \nu_\ell \to v \to \nu_k \to \nu_i,$$

we consider the ruled surfaces $S(a_i, a_j)$, $S(a_k, a_\ell)$ as well as the other ruled surfaces defined around other points of $g(I)$ which are vertices of $\Sigma_p(X)$ (if they exist). By Lemma 5.1, considering the regular part of $g(I)$ as well, we can define a proper Lipschitz immersion $f_{ij}^v : D^2(\ell; r) \to B(p, r)$ with branched point $(f_{ij}^v)^{-1}(p) = \{O\}$ satisfying $\Sigma_p(\text{Im}(f_{ij}^v)) = g(S^1)$, in a way similar to Case I. Note that any multiple point $q \in \text{Im}f_{ij}^v$ lies in a direction close to $v$.

Next suppose $W = [u_t, v]$ and find $\nu_\ell \in C_j$ with $1 \leq \ell \leq N$, $\ell \neq i, j$. Chasing on $g(I)$ in the order

$$\nu_i \to v \to \nu_j \to \nu_\ell \to v \to \nu_i \to u_t \to \bar{C}_i \to \nu_i,$$

we similarly consider the ruled surfaces $S(a_i, a_j)$, $S(a_j, a_\ell)$ as well as the other ruled surfaces defined around other points of $g(I)$ which are vertices of $\Sigma_p(X)$ (if they exist). In a way similar to the previous case, we can define a desired proper Lipschitz immersion $f_{ij}^v : D^2(\ell; r) \to B(p, r)$ branched at the point $(f_{ij}^v)^{-1}(p) = \{O\}$ satisfying $\Sigma_p(\text{Im}(f_{ij}^v)) = g(S^1)$.

The other case is similar, and hence omitted.

Note that $f_{ij}^v$ has bi-Lipschitz constant (resp. local bi-Lipschitz constant except the origin) $< 1 + \tau_p(r)$ in Case I (resp. in Case II).

**Lemma 5.5.**

$$B(p, r) = \bigcup_{v \in V(\Sigma_p(X))} \left( \bigcup_{1 \leq i < j \leq N} \text{Im} f_{ij}^v \right).$$

**Proof.** First note that from construction, $\Sigma_p(X)$ coincides with all the union of $\Sigma_p(\text{Im} f_{ij}^v)$. Suppose there is a point $x \in B(p, r)$ which is not contained in any image $\text{Im} f_{ij}^v$. Let $\xi := \frac{x}{x}$. Take some $\text{Im} f_{ij}^v$ such that $\xi \in \Sigma_p(\text{Im} f_{ij}^v)$. We may assume that $\xi$ is close to the vertex $v$, since if $\xi$ is far from any vertex of $\Sigma_p(X)$, then $x = \gamma_\xi([p, x])$ is certainly contained in the union of all the images $\text{Im} f_{ij}^v$, which is a contradiction.

Let $\gamma$ be a geodesic in $\text{Im} f_{ij}^v$ starting from $p$ in the direction $\xi$. Note that $\gamma$ reaches the metric sphere $S(p, r)$ (see also Sublemma 4.31). Let $x'$ be the point of $\gamma_\xi$ such that $[p, x'] = [p, x]$. Consider the geodesic $\gamma^X_{x, x'}$. If we extend $\gamma^X_{x, x'}$ through $x'$, it meets $\gamma_{p, a_k}$ for some $k$. Similarly, if we extend $\gamma^X_{x, x}$ through $x$, it meets $\gamma_{p, a_\ell}$ for some $\ell$. Lemma 4.33 yields that $x \in S(a_k, a_j)$. This is a contradiction. □
Combining Lemma \[\text{Lemma 5.3}\] and the above discussion, we complete the proof of Theorem \[\text{Theorem 1.1(1), (2), (3)}\] except (1) for the branched immersed disks that occur from the above Case II.

The proof of Theorem \[\text{Theorem 1.1(1)}\] for the branched immersed disks is deferred to Section \[\text{7}\].

6. Graph structure of singular set

Our next step is to characterize $S(X) \cap B(p, r)$ as a union of finitely many Lipschitz curves.

For a subset $A$ of $X$, we denote by $\partial A$ the complement in $\bar{A}$ of the set of all points $a$ of $A$ such that there is a neighborhood of $a$ homeomorphic to an open disk and contained in $A$.

For distinct $1 \leq i, j, k \leq N$, we set

$$C_{ij;k} := (\partial(S(a_i, a_j) - S(a_j, a_k)) - \partial S(a_i, a_j)) \cap B(p, r).$$

**Lemma 6.1.** $C_{ij;k}$ is a simple Lipschitz arc in $S(X)$ such that

1. it starts from $p$ and reaches a point of $\partial B(p, r);$  
2. its length is less than $(1 + \tau_p(r))r;$  
3. each point of $\Sigma_x(C_{ij;k})$ is a vertex of $\Sigma_x(X)$ for every $x \in C_{ij;k}$. In particular, $C_{ij;k}$ has definite directions everywhere, and $\frac{|d_x(x) - d_y(y)|}{|x - y|} \geq 1 - \tau_p(r)$ for all $x, y \in C_{ij;k}$.

**Proof.** For each $s \in [0, 2r]$, consider the ruling geodesic $\lambda_s(t)(0 \leq t \leq 1)$ of $S(a_k, a_j)$ joining $\gamma_{p,a_k}(s)$ to $\gamma_{p,a_j}(s)$ in $X$. Let $t_0 \in (0, 1)$ be the first parameter at which $\lambda_s$ meet $S(a_i, a_j)$.

We claim that

$$\lambda_s([t_0, 1]) \subset S(a_i, a_j).$$

Since $z_s := \lambda_s(t_0)$ is a topological singular point of $X$, by Lemma \[\text{Lemma 4.33}\] we can take a direction $\xi_0 \in \Sigma_{z_s}(S(a_i, a_j))$ with $\angle(\xi_0, \lambda_s(t_0)) = \pi$. A geodesic $\gamma_{\xi_0}$ in $S(a_i, a_j)$ with direction $\xi_0$ reaches $\gamma_{p,a_i}$. Take $\xi_1 \in \Sigma_{z_s}(S(a_i, a_j))$ with $\angle(\xi_0, \xi_1) = \pi$. Similarly a geodesic $\gamma_{\xi_1}$ in $S(a_i, a_j)$ with direction $\xi_1$ reaches $\gamma_{p,a_j}$. It follows from Lemma \[\text{Lemma 4.34}\] that $\gamma_{\xi_0}$ and $\gamma_{\xi_1}$ form a geodesic in $X$. In particular, $\gamma_{\xi_0}$ is a geodesic in $X$, and therefore $\gamma_{\xi_0}$ and $\lambda_s([t_0, 1])$ form a geodesic, say $\gamma$, in $X$, Lemma \[\text{Lemma 4.33}\] implies that $\gamma$ is contained in $S(a_i, a_j)$, and so is $\lambda_s([t_0, 1])$.

Since the curve $c(s) := z_s$ is continuous, its image coincides with $C_{ij;k}$. By Corollary \[\text{2.12}\] we have

$$\angle((\nabla d_p)(c(s)), \Sigma_{c(s)}(C_{ij;k})) < \tau_p(r), \ \angle((\nabla d_p)(c(s)), \Sigma_{c(s)}(-C_{ij;k})) < \tau_p(r),$$

where $\Sigma_{c(s)}(C_{ij;k})$ (resp. $\Sigma_{c(s)}(-C_{ij;k})$) denote the space of directions of $C_{ij;k}$ at $c(s)$ in the positive direction (resp. negative direction).

Now we take another parametrization $\varphi(s)$ of $C_{ij;k}$ defined as $\varphi(s) = C_{ij;k} \cap S(p, s)$, where $S(p, s)$ denotes the metric circle of radius $s$ with respect to $d_X$. If $s'$ is close enough to $s$, then we have $\angle(\varphi(s'), \nabla d_p(\varphi(s))) < \tau_p(r).$
Lemma 6.3. \( \varphi(s), \varphi(s') \) is not a direct consequence of Lemma 6.2.

\[
\lim_{s' \to s} \frac{|\varphi(s), \varphi(s')|}{|s - s'|} \leq 1 + \tau_p(r).
\]

Thus \( \varphi \) is Lipschitz with Lipschitz constant \( 1 + \tau_p(r) \), and therefore having length \( L(\varphi) = L(C_{ijk}) \leq (1 + \tau_p(r))r \). \( \Box \)

Lemma 6.1 claims that the closure of \( S(a_j, a_k) - S(a_i, a_j) \) “transversally” intersects \( S(a_i, a_j) \) with the Lipschitz curve \( C_{ijk} \). In particular we have

**Lemma 6.2.** \( C_{ijk} = C_{jik} = C_{jki} \).

In view of Lemma 6.2 we use the notation

\[ C_{ijk} := C_{ijk}. \]

Using the discussion in the proof of Lemma 6.1, we show the following refined version of Lemma 6.3 which is not a direct consequence of Lemma 6.3.

**Lemma 6.3.** For arbitrary \( x, y \in S = S(a_i, a_j) \) such that

\[ ||p, x|| - ||p, y|| < |x, y|/100, \]

the \( X \)-geodesic joining \( x \) and \( y \) is an \( S \)-geodesic.

**Proof.** Consider the geodesic \( \gamma^X_{x,y} \) and extend it in the both directions until it reaches \( \gamma_{p,a_i} \) and \( \gamma_{p,a_j} \) for some \( k, \ell \) at \( w_k \in \gamma_{p,a_i} \) and \( w_{\ell} \in \gamma_{p,a_j} \) respectively. That is,

\[ [w_k, w_{\ell}]_X = [w_k, x]_X \cup [x, y]_X \cup [y, w_{\ell}]_X. \]

Let \( z \) (resp. \( u \)) be the first point at which \([w_k, x]\) (resp. \([w_{\ell}, y]\)) meets \( R(a_i, a_j) \). As in the proof of Lemma 6.1 we have points \( w_i \in \gamma_{p,a_i} \) and \( w_j \in \gamma_{p,a_j} \) such that

\[ [w_i, w_j]_X = [w_i, z]_X \cup [z, x]_X \cup [x, y]_X \cup [y, w_j]_X. \]

From the hypothesis, we have \(||p, w_i|| - ||p, w_j|| < |w_i, w_j|_X/100\). Lemma 6.3 then implies that \([w_i, w_j]_X \) is an \( S \)-geodesic. Thus we conclude that \([x, y]_X \) is an \( S \)-geodesic as required. \( \Box \)

For a vertex \( v \) of \( \Sigma_p(X) \), let \( a_1, \ldots, a_N \in S(p, 2r) \) be as in Section 5 where \( N = N_v \). Let \( S(a_1, \ldots, a_N; r) \) be the closed domain of \( B(p, r) \) bounded by \( \gamma_{\varphi, s} \) (1 \( \leq i \leq N \)), and \( S(p, r) \). Note that \( S(a_1, \ldots, a_N; r) \) is the union of all the ruled surfaces \( S(a_i, a_j) \) and \( B(p, r) \).

**Corollary 6.4.** For a vertex \( v \) of \( \Sigma_p(X) \), the union of all \( C_{ijk} \) as above coincides with \( S(X) \cap S(a_1, \ldots, a_N; r) \).
Proof. Since every element of $S(X) \cap S(a_1, \ldots, a_N; r)$ comes from the intersection of some $S(a_i, a_j)$ and $S(a_k, a_\ell)$, it suffices to show that $\partial(S(a_i, a_j) \cap S(a_k, a_\ell)) \setminus S(p, r)$ is contained in $C_{ijk} \cup C_{ij\ell}$. For every $x \in \partial(S(a_i, a_j) \cap S(a_k, a_\ell))$, take an $s$ such that the ruling geodesic $\lambda_s$ joining $\gamma_{p,a_i}(s)$ to $\gamma_{p,a_j}(s)$ goes through $x$, say at $\lambda_s(t_0) = x$. Since $x \in S(X)$, Lemma 2.11, Theorem 4.1 and Corollary 2.12 imply the existence of a direction $\xi \in \Sigma_x(S(a_k, a_\ell))$ such that $\angle(\xi, \lambda_s'(t_0)) = \pi$. Then a geodesic $\gamma_\xi$ in $S(a_k, a_\ell)$ with direction $\xi$ must reach $\gamma_{p,a_k}$ or $\gamma_{p,a_\ell}$. Suppose it reaches $\gamma_{p,a_k}$ for instance: $\gamma_\xi(t_1) \in \gamma_{p,a_k}$ for some $t_1 > 0$. An argument similar to that in the proof of Lemma 6.3 then implies that $\gamma_\xi([0, t_1])$ does not meet $S(a_i, a_j)$ except for $x$, and that the union $\gamma_\xi([0, t_1]) \cup \lambda_s([t_0, 1])$ forms a geodesic in $S(a_k, a_\ell)$. This shows $x \in C_{ijk}$. \hfill \Box

Proof of the second half of Theorem 1.1. It is now an immediate consequence of Lemma 6.1 and Corollary 6.4. \hfill \Box

We call a curve $C$ in $S(X)$ a singular curve.

Remark 6.5. Each singular curve $C$ contained in $S(a_1, \ldots, a_N; r)$ has the direction $v$ at $p$. From now on, we always consider the case when $d_p$ is strictly increasing along $C$. In that case, for each interior point $q$ of $C$, $C$ has definite directions $\Sigma_q(C)$ consisting of two vertices of $\Sigma_q(X)$.

Structure of metric circles. Next we discuss the structure of $S(p, r)$.

Let $b_i := \gamma_{p,a_i}(r)$. For $0 < t \leq r$, set

$$S(v; t) := \left( \bigcup_{1 \leq i < j \leq N} S(a_i, a_j) \right) \cap S^X(p, t).$$

Lemma 6.6. For each $0 < t \leq r$, $S(v; t)$ is a tree with endpoints $\gamma_{p,a_i}(t)$ $(1 \leq i \leq N)$.

Proof. For $3 \leq k \leq N$, put

$$S_k(v; t) := \left( \bigcup_{1 \leq i < j \leq k} S(a_i, a_j) \right) \cap S^X(p, t).$$

Inductively we show that $S_k(v; t)$ is a tree for every $3 \leq k \leq N$. This is certainly true for $k = 3$. Assume that $S_{k-1}(v; t)$ is a tree. Set

$$S(a_i, a_k)(t) := S(a_i, a_k) \cap S^X(p, t).$$

Let $p_k(t) := \gamma_{p,a_k}(t)$. Let $q(t)$ be the point of $S_{k-1}(v, t)$ where the arc starting from $p_k(t)$ in $S_k(v, t)$ first meets $S_{k-1}(v, t)$. For every $1 \leq i \neq j \leq k - 1$ with $q(t) \in S(a_i, a_j)(t)$, (6.1) implies that

$$S(a_i, a_k)(t) \setminus [p_k(t), q(t)] \subset S(a_i, a_j),$$

(6.3)
where \([p_k(t), q(t)]\) denotes the arc between \(p_k(t)\) and \(q(t)\) in \(S_k(v, t)\). (3.3) implies that \(S_k(v, t) = S_{k-1}(v, t) \cup [p_k(t), q(t)]\). Thus \(S_k(v, t)\) is a tree.

**Proof of Corollary 1.6.** Let \(r_p \geq r > 0\) be as in Theorem 1.4. By Lemma 6.6, for every vertex \(v\) of \(\Sigma_p(X)\), \(S(v, r)\) is a tree with endpoints \(b_i\) (1 \(\leq i \leq N\)). Therefore \(S(p, r)\) has the same homotopy type as \(\Sigma_p(X)\).

Let \(\alpha\) be any non-contractible simple closed loop in \(S(p, r)\). From the discussion in Case I and the proof of Theorem 1.4 in Section 5 there is a non-contractible simple closed loop \(\zeta\) in \(\Sigma_p(X)\) of length, say \(\ell \geq 2\pi\), and a properly embedded \(\text{CAT}(\kappa)\)-disk \(f : D^2(\ell; r) \rightarrow B(p, r)\) associated with \(\zeta\) such that \(\Sigma_p(\text{Im}(f)) = \zeta\) and \(f(\partial D^2(\ell; r)) = \alpha\). For \(v \in \zeta \cap V(\Sigma_p(X))\), let \(\xi_i, \xi_j \in \zeta\) be points nearby \(v\) such that \(v\) is the midpoint of the arc \([\xi_i, \xi_j]\). Let \(S_{ij}\) be the ruled surface defined by \(\xi_i, \xi_j\). Note that \(B_{S_{ij}}(p, r) \subseteq S_{ij} \cap B^X(p, r)\). Since we may assume \(r < \pi/\sqrt{\kappa}\), the nearest point map \(S_{ij} \cap B^X(p, r) \rightarrow S_{ij}(p, r)\) is distance non-increasing. Let \(\tilde{S}_{ij}\) be a sector in the model \(M_2^\kappa\) with vertex \(\tilde{p}\) bounded by two geodesics of length \(r\) and \(S(\tilde{p}, r)\) such that the sector angle at \(\tilde{p}\) is equal to \(\angle(\xi_1, \xi_2)\). From the curvature condition, we have

\[
L(S_{ij}(p, r)) \geq L(\tilde{S}_{ij} \cap S(\tilde{p}, r)) = \angle(\xi_1, \xi_2)/\sqrt{\mu(\kappa, r)},
\]

yielding

\[
L(S_{ij} \cap S^X(p, r)) \geq \angle(\xi_1, \xi_2)/\sqrt{\mu(\kappa, r)}.
\]

Applying a similar argument to the other parts of \(\alpha\) and \(\zeta\), we conclude that

\[
L(\alpha) \geq L(\zeta)/\sqrt{\mu(\kappa, r)} \geq 2\pi/\sqrt{\mu(\kappa, r)}.
\]

This completes the proof. \(\square\)

Now we define a metric graph structure of \(S(X)\) in a generalized sense as follows.

**Definition 6.7.** We consider the relative topology of \(S(X)\) with length metric. Let \(I\) be an open set of \(S(X)\). We call \(I\) an open arc in \(S(X)\) if it is open in \(S(X)\) and is isometric to an open interval. A maximal open arc \(I\) with respect to the inclusion is called an open edge of \(S(X)\). We denote by \(E(S(X))\) (resp. \(|E(S(X))|\)) the set (resp. the union) of all open edges in \(S(X)\). We call each element of \(S(X) \setminus |E(S(X))|\) a vertex of \(S(X)\). We denote by \(V(S(X))\) the set of all vertices of \(S(X)\). Let us denote by \(V_\ast(S(X)) \subseteq V(S(X))\) the set of all accumulation points of \(V(S(X))\). The case \(V_\ast(S(X)) = V(S(X)) \cup H^1(V_\ast(S(X))) > 0\) may happen (see Example 6.9). As usual, two vertices \(v_1\) and \(v_2\) of \(S(X)\) are adjacent if there is at least one open edge joining them. The order of a vertex \(v\) is defined as the limit of the number of components of \(B^{S(X)}(v, \epsilon) \setminus \{v\}\) as \(\epsilon \rightarrow 0\).
Proof of Corollary 1.4. First note that by Theorem 1.1, $S(X)$ is locally path-connected. For a given point $p \in S(X)$ and $v \in V(\Sigma_p(X))$, let $N = N_v$ be the branching number of $v$ in $\Sigma_p(X)$, and take $r = r_p$ as in Theorem 1.1. For $\delta > 0$ with $\delta \ll \min \{ \angle(v, v') | v \neq v' \in V(\Sigma_p(X)) \}$, let $\gamma_1, \ldots, \gamma_N$ be geodesics from $p$ with $\angle(\dot{\gamma}_i(0), v) = \delta$ and $\angle(\dot{\gamma}_i(0), \dot{\gamma}_j(0)) = 2\delta$ for $1 \leq i \neq j \leq N$. By Corollary 6.4, we have $S(X) \cap U(v) = \bigcup_{1 \leq i < j < k \leq N} C_{ijk}$, where $U(v) := C(v, \delta, r)$ is the cone neighborhood around $v$ (see [2,7]). By Lemma 6.1 (3), the distance function $d_p$ is strictly monotone on each $C_{ijk}$. It follows from Corollary 6.4 that $V(S(X))$ has locally finite order.

In what follows, we give an explicit sharp bound on the orders at the vertices in $S(X) \cap B(p, r)$.

Sublemma 6.8. $S(X) \cap U(v)$ can be written as the union of at most $N_v - 2$ singular curves starting from $p$ in the direction $v$, and reaching $S(p, r)$ such that $d_p$ is strictly increasing along $C$.

Proof. By Lemma 6.4, $S(X) \cap U(v)$ coincides with the set of all topological singular points resulting from the intersections of distinct ruled surfaces $S_{ij}$ and $S_{i'j'}$ for all $1 \leq i < j \leq N$, $1 \leq i' < j' \leq N$ with $(i, j) \neq (i', j')$. For $2 \leq k \leq N$, let $E_k$ be the union of all $S_{ij}$ with $1 \leq i < j \leq k$. We inductively define singular curves $C_j$ ($2 \leq j \leq N - 1$) as the set of all points of $E_j$ where geodesics almost perpendicularly starting from points of $\gamma_{j+1}$ intersect $E_j$ for the first time. Then it is obvious to see that $S(X) \cap U(v) = C_2 \cup \cdots \cup C_{N-1}$. From Lemma 6.1, $d_p$ is strictly increasing along $C$. \qed

Let $\Gamma := S(X) \cap B(p, r)$. It follows from Sublemma 6.8 that

- the order at the vertex $p$ of the graph $\Gamma \cap U(v)$ is at most $N_v - 2$;
- the order at any vertex $y$ in $\Gamma \cap U(v) \setminus \{p\}$ is at most $2(N_v - 2)$.

Therefore the maximum of orders of vertices contained in $\Gamma$ is at most

$$\max \left\{ \sum_{v \in V(\Sigma_p(X))} (N_v - 2), \max_{v \in V(\Sigma_p(X))} 2(N_v - 2) \right\}.$$ 

This completes the proof of Corollary 1.4. \qed

We exhibit the following example, which is another version of Example 4.4. Here we use the notion of $\epsilon$-Cantor set (cf. [7]) to produce a two-dimensional CAT(0)-space $X$ such that $V_*(S(X))$ is one-dimensional. Similar construction for a boundary singular set of a limit space of manifolds with boundary was made in [33].
Example 6.9. For any $0 < \epsilon < 1$, set $\delta := 1 - \epsilon$. We define the so-called $\epsilon$-Cantor set of $[0, 1]$ inductively as follows: We start with $I_0 := [0, 1]$, and remove from $I_0$ the open interval of length $\delta/2$ around the center of $I_0$. We denote by $I_1$ the result of this removing. Note that $I_1$ consists of $2^1$ disjoint closed intervals $I_{1,j}$ ($j = 1, 2$) having the same length and that $L(I_1) = 1 - \delta/2$. Suppose that we have constructed $I_k$ consisting of $2^k$ disjoint closed intervals $I_{k,j}$ ($1 \leq j \leq 2^k$) of the same length such that $L(I_k) = 1 - \delta/2 - \cdots - \delta/2^k$. Remove from each $I_{k,j}$ the open interval of length $\delta/2^{k+1}$ around the center of $I_{k,j}$. We denote by $I_{k+1}$ the result of this removing. Thus, inductively we have constructed $I_n$ for every $n$. Finally we set

$$I_\infty := \bigcap_{n=0}^{\infty} I_n, \quad J_n := [0, 1] \setminus I_n, \quad J_\infty := \bigcup_{n=0}^{\infty} J_n = [0, 1] \setminus I_\infty.$$

Note that $\mathcal{H}^1(I_\infty) = \lim_{n \to \infty} L(I_n) = 1 - \delta = \epsilon$. The set $I_\infty$ is called an $\epsilon$-Cantor set.

Next, inductively we define smooth functions $f_n : \mathbb{R} \to [0, 1]$ ($n \in \mathbb{N}$) such that

- $\text{supp}(f_n) = J_n$;
- $f_n = f_{n-1}$ on $J_{n-1}$;
- if we set $\hat{J}_n^\pm := \{(x, \pm f_n(x)) \mid x \in J_n\}$, then the length $\ell_n$ and the maximum $\kappa_n$ of absolute geodesic curvature of $\hat{J}_n^\pm$ satisfy (4.4).

Now we define the limit $f := \lim_{n \to \infty} f_n : \mathbb{R} \to [0, 1]$, which satisfies $\text{supp}(f) = J_\infty$. Using $f$, we define the closed subset $\Omega$ of $\mathbb{R}^2$ by

$$\Omega := \{(x, y) \mid |y| \leq f(x), x \in \mathbb{R}\},$$

equipped with the length metric. Set

$$\partial_2 \Omega := \{(x, y) \mid y = \pm f(x), x \in \mathbb{R}\}.$$

Take closed concave domains $H_\pm$ in $\mathbb{R}^2$ homeomorphic to the half plane such that for certain isometries $g_\pm : \partial_2 \Omega \to \partial H_\pm$ the absolute geodesic curvature of $g_\pm(J_n^\pm)$ is greater than $\kappa_n$. Take two copies $H_1^\pm, H_2^\pm$ of $H_\pm$, and make a gluing of $H_1^+, H_2^+, H_1^-, H_2^-$ and $\Omega$ along their boundaries via $g_\pm$ as in Example 4.4 to get a two-dimensional locally compact, geodesically complete CAT(0)-space $X$. Note that $V_\epsilon(S(X)) = V(S(X)) = I_\infty$ and therefore $\mathcal{H}^1(V_\epsilon(S(X))) = \epsilon > 0$.

7. Approximations by polyhedral spaces

In this section, we give the proof of Theorem 1.1(1) for branched immersed disks. We need to recall the notion of turn, which was first defined in the context of surfaces with bounded curvature in [6] (see also [29]).
Definition 7.1. For a moment, let $X$ be a surface with bounded curvature. In $X$, we have the notion of angles between geodesics starting from a point, and use the same notations for spaces of directions, etc (\cite[Theorem II.10]{6}).

Let $F$ be a domain in $X$ with boundary $C$. For an open arc $e$ of $C$, we assume that $e$ has definite directions at the endpoints $a, b$ and the spaces of directions of $F$ at $a$ and $b$ have positive lengths. Then the turn (rotation) $\tau_F(e)$ of $e$ (\cite[Chapter VI]{6}) from the side of $F$ is defined as follows: Let $\gamma_n$ be a broken geodesic in $F \setminus e$ except the endpoints joining $a$ and $b$ and converging to $e$ as $n \to \infty$. Let $\Gamma_n$ be the domain bounded by $e$ and $\gamma_n$. We denote by $\alpha_n$ and $\beta_n$ the sector angle of $\Gamma_n$ at $a$ and $b$ respectively. Let $\theta_{ni}$, $(1 \leq i \leq N_n)$, denote the sector angle at the break points of $\gamma_n$, viewed from $F \setminus \Gamma_n$. Let

$$\tilde{\tau}_F(\gamma_n) := \sum_{i=1}^{N_n} (\pi - \theta_{ni}) + \alpha_n + \beta_n.$$ 

Then the turn $\tau_F(e)$ is defined as

$$\tau_F(e) := \lim_{n \to \infty} \tilde{\tau}_F(\gamma_n),$$

where the existence of the above limit is shown in \cite[Theorem VI.2]{6}.

For an interior point $c$ of $e$ having definite two directions $\Sigma_c(e)$, the turn of $e$ at $c$ from the side $F$ is defined as

$$\tau_F(c) := \pi - L(\Sigma_c(F)).$$

We now assume the following additional conditions for all $c \in e$:

1. $L(\Sigma_c(F)) > 0$;
2. $e$ has definite two directions $\Sigma_c(e)$.

Consider the constant $\mu_F(e) \in [0, \infty]$ defined by

$$\mu_F(e) := \sup_{\{a_i\}} \sum_{i=1}^{n-1} |\tau_F((a_i, a_{i+1}))| + \sum_{i=2}^{n-1} |\tau_F(a_i)|,$$

where $\{a_i\} = \{a_i\}_{i=1}^{n}$ runs over all the consecutive points on $e$. The constant $\mu_F(e)$ is called the turn variation of $e$ from the side $F$, and $e$ has finite turn variation when $\mu_F(e) < \infty$. For general treatments of curves with finite turn variation in CAT($\kappa$)-spaces, see \cite{Pi} and the references therein.

Let $e$ be a simple arc on $X$. One can define the notion of sides $F_+$ and $F_-$ of $e$. Under the corresponding assumptions, we define the turns $\tau_{F_+}(e)$, $\tau_{F_-}(e)$ of $e$ from $F_+$ and $F_-$ respectively, as above. Similarly, we define the turn variations $\mu_{F_+}(e)$, $\mu_{F_-}(e)$ from $F_+$ and $F_-$. We call $e$ to have finite turn variation if $\mu_{F_+}(e) < \infty$ and $\mu_{F_-}(e) < \infty$ (actually both are finite if one is so (\cite[Lemma IX.1]{6})). When $e$ has finite turn variation, $\tau_{F_+}$ and $\tau_{F_-}$ provide signed Borel measures on $e$ (\cite[Theorem IX.1]{6}).
The structure of the union of ruled surfaces. Let \( p \in \mathcal{S}(X) \), and \( r = r_p > 0 \) be as in Theorem 1.1. From now, we work on \( B(p, r) \). Fix any \( v \in V(\Sigma_p(X)) \), and let \( N = N_v \) be the branching number of \( \Sigma_p(X) \) at \( v \). For small enough \( \delta_p > 0 \), let \( \gamma_1, \ldots, \gamma_N \) be the geodesics from \( p \) with \( \angle(\dot{\gamma}_i(0), v) = \delta_p \), and \( \angle(\dot{\gamma}_j(0), \dot{\gamma}_j(0)) = 2\delta_p \) for \( 1 \leq i \neq j \leq N \). For \( 2 \leq k \leq N \), we define \( E_k \) as the union of ruled surfaces \( S_{ij} \) determined by \( \gamma_i \) and \( \gamma_j \) for all \( 1 \leq i \neq j \leq k \).

Let \( C \) denote the union of all singular curves \( C_{ij} \) (1 \( \leq i < j < \ell \leq k \)). By Lemma 6.4, \( C \) coincides with the set of all topological singular points resulting from the intersections of distinct ruled surfaces \( S_{ij} \) and \( S_{i'j'} \) for all \( 1 \leq i < j \leq k, 1 \leq i' < j' \leq k \) with \( (i, j) \neq (i', j') \).

Note that a singular curve in the direction \( v \) not included in \( C \) might meet \( E_k \). We consider the graph structure of \( C \) inherited from that of \( \mathcal{S}(X) \), which is not the one of \( C \) itself introduced as in Definition 6.7.

Thus we set

\[
E(C) := C \cap E(\mathcal{S}(X)), \quad V(C) := C \cap V(\mathcal{S}(X)), \quad V_s(C) := C \cap V_s(\mathcal{S}(X)),
\]

and call \( E(C) \) and \( V(C) \) the set of edges and the set of vertices of \( C \) respectively. Remember that all edges are assumed to be open (Definition 6.7).

**Definition 7.2.** We say that a vertex point \( x \in V(C) \) is **singular** if either \( x \in V_s(C) \) or there are two singular curves \( C_1, C_2 \) in \( C \) starting from \( x \) such that

1. \( \angle_x(C_1, C_2) = 0 \);
2. \( C_1 \) and \( C_2 \) have no intersections near \( x \) other than \( x \).

The direction \( v \in \Sigma_x(C) \) determined by the above \( C_1 \) and \( C_2 \) as well as \( v = \lim_{t \to \infty} \frac{x_t}{t} \) with \( V(C) \ni x_t \to x \) is also called **singular**. The set of singular vertices of \( C \) is denoted by \( V_{\text{sing}}(C) \). The set of singular directions at \( x \in V_{\text{sing}}(C) \) is denoted by \( \Sigma_{x,\text{sing}}(C) \).

We set \( r(x) := d_p(x) \) for simplicity.

For \( x \in C \setminus \{p\} \), let \( \Sigma_{x,+}(C) := \Sigma_x(C \setminus \text{int}B(p, r(x))) \) and \( \Sigma_{x,-}(C) := \Sigma_x(C \cap B(p, r(x))) \). By Lemma 2.11, we may assume that for all \( x \in C \setminus \{p\} \),

\[
\begin{align*}
\angle(\nabla d_p(x), \Sigma_{x,+}(C)) &< 10^{-10}, \quad \text{diam}(\Sigma_{x,+}(C)) < 10^{-10}, \\
\angle(-\nabla d_p(x), \Sigma_{x,-}(C)) &< 10^{-10}, \quad \text{diam}(\Sigma_{x,-}(C)) < 10^{-10}.
\end{align*}
\]

The following lemma is clear.

**Lemma 7.3.** For every \( x \in C \setminus \{p\} \), \( \Sigma_x(E_k)(\subseteq \Sigma_x(X)) \) coincides with the union of all circles \( \Sigma_x(S_{ij}) \) such that \( x \in S_{ij} \subset E_k \), where the circles \( \Sigma_x(S_{ij}) \) are attached at the points of \( \Sigma_{x,\pm}(C) \).

The following is a main result of this section.

**Theorem 7.4.** \( E_k \) is a \( \text{CAT}(\kappa) \)-space.
Remark 7.5. Recently, we learned that Theorem 7.4 is a direct consequence of the main result of Lytchak and Stadler \[21\]. However, in what follows, we present our original proof, which provides deep insights on the local geometry of $X$, and will also be used in \[25\] as one of key methods.

The basic strategy of the proof of Theorem 7.4 is to use the results \[13\] Theorems 0.5 and 0.6 on the characterizations for polyhedral spaces to be CAT($\kappa$)-spaces.

Let $F^\kappa$ be the family of two-dimensional polyhedral locally CAT($\kappa$)-spaces $F$ possibly with boundary $\partial F$ such that any edge of $\partial F$ has finite turn variation. For a collection $\{F_i\}$ of $F^\kappa$, let $X$ be the polyhedron resulting from certain gluing of $\{F_i\}$ along their edges. We always consider the intrinsic metric of $X$ induced from those of $F_i$. We consider the following two conditions:

(A) For any Borel subset $B$ of an arbitrary edge $e$ of $X$, and arbitrary faces $F_i, F_j$ adjacent to $e$, we have 
$$\tau_{F_i}(B) + \tau_{F_j}(B) \leq 0;$$

(B) For any vertex $x$ of $X$, $\Sigma_x(X)$ is CAT(1).

Theorem 7.6. (\[13\] Theorem 0.5) A polyhedron $X$ resulting from certain gluing of $\{F_i\} \subset F^\kappa$ along their edges belongs to $F^\kappa$ if and only if the conditions (A), (B) are satisfied.

Theorem 7.7. (\[13\] Theorem 0.6) Each polyhedron $X$ in $F^\kappa$ can be glued from the faces $\{F_i\}$ contained in $F^\kappa$ along their edges in such a way that the conditions (A), (B) are satisfied.

In particular, each edge of $S(X)$ has finite turn variation.

Note that $E_k$ is not a polyhedral space in general. Even in that case, we have some difficulty mentioned below. From these reasons, we shall do surgeries to get a polyhedral space $\tilde{E}_k$ which approximates $E_k$ in the Gromov-Hausdorff sense. The point is, we can apply Theorems 7.6 and 7.7 to $\tilde{E}_k$ to conclude that it is CAT($\kappa$). Finally taking the limit, we will obtain the conclusion.

From now on, we set $E := E_k$ for simplicity. We need some preliminary argument on the local geometry of $E$.

Lemma 7.8. For every $x \in E$, $\Sigma_x(E)$ is isometric to the intrinsic space of directions $\Sigma_x(E^\text{int})$ in the sense of Definition 4.26.

Proof. The basic idea of the proof is the same as that of Lemma 4.27. Obviously, we may assume $x \in C$. We only consider the case $x \neq p$. We first show that each component $\Sigma$ of $\Sigma_x(E) \setminus \Sigma_x(C)$ is isometrically embedded in $\Sigma_x(E^\text{int})$. For $\xi_1, \xi_2 \in \Sigma$ with $|\xi_1, \xi_2| < \pi$, let $\mu_n$ be an $X$-geodesic with $\mu_n(0) = \xi_n$ ($n = 1, 2$). Then for small $\epsilon$, we have $\mu_1([0, \epsilon]) \subset S_{ij}$ and $\mu_2([0, \epsilon]) \subset S_{kl}$ for some $S_{ij}, S_{kl}$ in $E$. Note that
the $X$-geodesic $\gamma_{\mu_1(t),\mu_2(t)}^X$ joining $\mu_1(t)$ and $\mu_2(t)$ does not meet $C$, and hence $\gamma_{\mu_1(t),\mu_2(t)}^X$ is contained in the same ruled surface $S_{ij} = S_{kl} \subset E$. This implies that $\angle X(\xi_1,\xi_2) = \angle^{S_{ij}}(\xi_1,\xi_2)$. From $\angle^X \leq \angle^E \leq \angle^{S_{ij}}$, we conclude that $\angle^X(\xi_1,\xi_2) = \angle^E(\xi_1,\xi_2)$ and the existence of an isometric embedding $\iota: \Sigma \to \Sigma_{\delta}(E^{\text{int}})$.

Next, for any $v \in \Sigma_2(C)$, take $\xi_3,\xi_4 \in \Sigma_2(E) \setminus \Sigma_2(C)$ close to $v$ such that the segment $[\xi_3,\xi_4]$ in $\Sigma_2(E)$ meets $\Sigma_2(C)$ only at $v$. Take $X$-geodesics $\alpha_3,\alpha_4$ in the direction $\xi_3,\xi_4$, and choose $S_{ij},S_{ij'}$ in $E$ such that $\alpha_3(t) \subset S_{ij}$ and $\alpha_4(t) \subset S_{ij'}$. Let $\alpha: [0,1] \to X$ be the $X$-geodesic from $\alpha_3(t)$ to $\alpha_4(t)$. By $\text{(4.2)}$, $\alpha$ is vertical. We extend $\alpha$ until it reaches $\partial E$. We can choose such an extension that $\alpha(t_-) \in \gamma_{\ell}$ and $\alpha(t_+) \in \gamma'_{\ell}$ with $\ell \in \{i,j\}$ and $\ell' \in \{i',j'\}$ for some $t_- < 0 < 1 < t_+$. Thus, we have $\alpha([t_-,t_+]) \subset S_{\ell\ell'}$.

By Lemma $7.12$, we can find a sequence $s_n$ such that the ruling geodesics $\lambda_{sn}$ of $S_{\ell\ell'}$ meets both $\alpha_3$ and $\alpha_4$ and $\lambda_{sn}$ converges to a ruling geodesic through $x$ as $n \to \infty$. This implies that $\angle^X(\xi_3,\xi_4) = \angle^{S_{ij}}(\xi_3,\xi_4)$, and hence $\angle^X(\xi_3,\xi_4) = \angle^E(\xi_3,\xi_4)$. This completes the proof.

**Lemma 7.9.** For arbitrary $x,y \in E$, let $\gamma := \gamma^E_{x,y}: [0,|x,y|_E] \to E$ be an $E$-shortest curve between $x$ and $y$. Suppose that the set of accumulation points of $\gamma \cap \mathcal{S}(X)$ is finite. Then $\gamma$ is an $X$-geodesic.

**Proof.** Set $\Gamma := \gamma \cap \mathcal{S}(X)$. We only have to consider the case when $\Gamma$ has a unique accumulation point $\gamma(u)$ with $\Gamma = \{\gamma(t_i),\gamma(s_j),\gamma(u)\mid i,j = 1,2,\ldots\}$ with $0 \leq t_1 < t_2 < \cdots < t_i < \cdots < u < \cdots < s_j < \cdots < s_2 < s_1 \leq |x,y|_E$ and $\lim_{i \to \infty} t_i = \lim_{j \to \infty} s_j = u$.

Note that for each $i$ and any small enough $\varepsilon > 0$, $\gamma([t_{i-1} + \varepsilon,t_i - \varepsilon])$ is contained in the surface $X \setminus \mathcal{S}(X)$. Therefore $\gamma|_{[t_{i-1} + \varepsilon,t_i - \varepsilon]}$ is locally $X$-minimizing, and hence $X$-minimizing. Thus $\gamma|_{[t_{i-1},t_i]}$ is $X$-minimizing.

By Lemma $7.8$, we have

$$\angle^X(\gamma(t_i),\gamma(t_{i-1}),\gamma(t_{i+1}),\gamma(t_i),\gamma(t_{i+1})) = \angle^E(\gamma^E(\gamma(t_i),\gamma(t_{i-1}),\gamma(t_{i+1}),\gamma(t_i),\gamma(t_{i+1}))) = \pi.$$ 

Therefore $\gamma|_{[t_{i-1},t_{i+1}]}$ is an $X$-geodesic, which implies that $\gamma|_{[0,u]}$ is an $X$-geodesic. Similarly, $\gamma|_{[u,|x,y|_E]}$ is an $X$-geodesic. In a way similar to the above, we have $\angle^X(\gamma(u),\gamma(0),\gamma(\max(|x,y|_E),\gamma(0))) = \pi$. It follows that $\gamma$ is an $X$-geodesic.

**Remark 7.10.** Lemma $7.9$ does not hold in case a subarc of $\gamma$ is contained in $\mathcal{S}(X)$ (see Example $1.4$).

**Lemma 7.11.** For a fixed $x \in E$, we have for every $y(\neq x) \in E$

$$\frac{|x,y|_E}{|x,y|_X} < 1 + \tau_x(|x,y|_X).$$

**Proof.** We may assume $x \in E \cap \mathcal{S}(X)$. Suppose the conclusion does not hold. Then we have a sequence $y_n \in E$ converging to $x$ such that
\[ \frac{|x, y_n|_p}{|x, y_n|_X} > 1 + c \] for some positive constant \( c \). Passing to a subsequence, we may assume that all \( y_n \in S_{ij} \) for some \( S_{ij} \). This is a contradiction to Lemma 4.31 since \( |x, y_n|_E \leq |x, y_n|_{S_{ij}} \).

\[ \square \]

Proof of Theorem 7.4. For each edge \( e \in E(C) \), let \( D_i \) (\( 1 \leq i \leq m(e) \)) be open half-disks in \( X \) with \( \partial D_i = e \) such that

(7.3) \[ \bigcup_{i=1}^{m(e)} D_i \] is an open neighborhood of \( e \) in \( X \).

Let \( \tau_{D_i} \) be the turn of \( e \) from the side \( D_i \). We want to apply Theorem 7.7 to the completion of the components of \( E \setminus C \). Let \( A \) be such a completion containing some \( D_i \). However here are some difficulties: The domain \( A \) might be too thin to define \( \tau_{D_i}(e) \) because of the presence of singular vertices. In particular, we do not know if \( e \) has finite turn variation in \( A \). We also have to care about \( V_\text{sing}(C) \). To overcome these difficulties, we do surgeries around points of \( V_\text{sing}(C) \). At this moment, we can apply Theorem 7.7 to \( e \) locally. Each point of \( e \) has a convex neighborhood \( P \in \bigcup_{i=1}^{m(e)} D_i \) such that \( \partial P \) consists of broken geodesics joining the endpoints of \( e \cap P \). It follows from Theorem 7.7 that we have for all \( 1 \leq i \neq j \leq m(e) \),

(7.4) \[ \tau_{D_i}(e \cap P) + \tau_{D_j}(e \cap P) \leq 0 \]

and \( e \) has locally finite turn variation in \( D_i \).

Let \( \epsilon_0 \) be any positive number. For \( x \in V_\text{sing}(C) \), we assume that the singularity of \( x \) occurs from the positive direction. Namely, there is \( v \in \Sigma_{x_+}^\text{sing}(C) \). The other case \( v \in \Sigma_{x_-}^\text{sing}(C) \) is similarly discussed. Let \( C(v) \) denote the union of singular curves in \( C \) starting at \( x \) in the direction \( v \).

Choose \( \delta = \delta_x > 0 \) and \( \epsilon = \epsilon_x > 0 \) with \( \delta, \epsilon \leq \epsilon_0 \) and \( \epsilon \ll \delta \) satisfying

(7.5) \[ \{ B_{2\delta}(x)(v, 2\delta) \} \ (v \in \Sigma_{x}^\text{sing}(C)) \] is mutually disjoint;

(7.6) \[ C(v, \delta, 2\epsilon) \] (see (2.7)) covers \( C(v) \cap B_+(x, 2\epsilon) \);

(7.7) \[ E \cap S(p, r(x) + \epsilon) \] does not meet \( V_\text{sing}(C) \),

where \( B_+(x, \epsilon) := B(x, \epsilon) \setminus \text{int} B(p, r(x)) \). By Lemma 5.6, \( C(v, \delta, \epsilon) \cap E \cap S(p, \delta(x) + \epsilon) \) is a tree, say \( \hat{T}(x, v) \). Replacing each edge of \( \hat{T}(x, v) \) by the \( X \)-geodesic between the endpoints, we obtain a geodesic tree \( T(x, v) \). By Lemma 4.32, we have \( T(x, v) \subset E \). Let \( K(x, v) \) be a closed domain of \( E \) bounded by \( T(x, v) \) and the \( X \)-geodesic segments between \( x \) and the endpoints of the tree \( T(x, v) \). Note that such \( X \)-geodesics between \( x \) and the endpoints of \( T(x, v) \) are contained in \( E \). Taking smaller \( \delta, \epsilon \) if necessary, we may assume

(7.8) \[ \tilde{Z}^X_{xy} y' < \epsilon_0 \] for all \( y, y' \in T(x, v) \).
For each vertex $y \in V(T(x,v))$, take the $X$-geodesic $\gamma_{x,y}^X$ between $x$ and $y$. For each edge $e \in E(T(x,v))$ with endpoints $y, y'$, let $\Delta_e^X$ denote the $X$-geodesic triangle consisting of $\gamma_{x,y}^X \cup \gamma_{x,y'}^X \cup \gamma_{y,y'}^X$.

Let $\tilde{\Delta}_e^X$ be the triangular region bounded by $\tilde{\Delta}_e^X$. Gluing $\{\tilde{\Delta}_e^X | e \in E(T(x,v))\}$ properly, we obtain a polyhedral space $\tilde{K}(x,v)$ corresponding to $K(x,v)$.

We provide a relation between $K(x,v)$ and $\tilde{K}(x,v)$.

**Lemma 7.12.** (1) Let $y, y'$ be arbitrary endpoints of $T(x,v)$. For arbitrary $z \in \gamma_{x,y}^X = \gamma_{x,y'}^E$ and $z' \in \gamma_{x,y'}^X = \gamma_{x,y'}^E$, assuming $|x, z|_X \leq |x, z'|_X$, we have

$$||z, z'|_E - |\tilde{z}, \tilde{z}'|| < \tau(\epsilon_0)|x, z|_X,$$

where $\tilde{z}, \tilde{z}' \in \partial \tilde{K}(x,v)$ are the points corresponding to $z, z'$ respectively.

(2) For arbitrary $y \in T(x,v)$ and $z \in \partial K(x,v)$, we have

$$||y, z|_E - |\tilde{y}, \tilde{z}|| < \tau(\epsilon_0)|x, y|_X,$$

where $\tilde{y}, \tilde{z}$ are the points of $\partial \tilde{K}(x,v)$ corresponding to $y, z$.

**Proof.** (1) Let $\tilde{z} \in \gamma_{x,z'}^E$ be the point such that $|x, \tilde{z}|_X = |x, z|_X$. Note that $\gamma_{x,z'}^E$ is vertical, and therefore contained in $E$. By triangle inequality, we have $||z, z'|_E - |\tilde{z}, \tilde{z}'|_E| \leq |\tilde{z}, \tilde{z}'|_E = |\tilde{z}, z|_X$. (7.8) implies $\tilde{z}xz\tilde{z} < \epsilon_0$, and hence $|\tilde{z}, z|_X < \tau(\epsilon_0)|x, z|_X$. In view of $\gamma_{x,y}^X = \gamma_{x,y}^E$ and $\gamma_{x,y'}^X = \gamma_{x,y'}^E$, we have

$$||z, z'|_E - (|x, z'|_X - |x, z|_X)| \leq \tau(\epsilon_0)|x, z|_X.$$  

Since $\angle \tilde{z}x\tilde{z} < \tau(\epsilon_0)$, similarly we have $||\tilde{z}, z'|_E - (|x, z'|_X - |x, z|_X)| \leq \tau(\epsilon_0)|x, z|_X$. Combining the last two inequalities, we obtain the required inequality.

(2) Choose $w \in \partial T(x,v)$ such that $z \in \gamma_{x,w}$. From (7.8), we have $|y, w|_E = |y, w|_X < \tau(\epsilon_0)|x, y|_X$. Therefore by triangle inequality, we obtain $||y, z|_E - |z, w|_E| \leq |y, w|_E < \tau(\epsilon_0)|x, y|_X$. Since $\angle \tilde{y}x\tilde{z} < \epsilon_0$, by a similar consideration on the triangle $\tilde{y}x\tilde{z}$ in $M^2_\kappa$, we have $||\tilde{y}, \tilde{z}| - |\tilde{z}, \tilde{w}|| < \tau(\epsilon_0)|x, y|_X$, where $\tilde{w}$ is the point of $\partial \tilde{T}(x,v)$ corresponding to $z$.  

\[\hspace{1cm} \]

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w. Since $|z, w|_E = |z, w|_X = |\tilde{z}, \tilde{w}|$, combining the last two inequalities, we obtain the required inequality. 

In $E$, we do surgeries by removing $K(x, v)$ from $E$, and gluing $E \setminus K(x, v)$ and $\tilde{K}(x, v)$ along their isometric boundaries to get a new space, say $\tilde{E}_{x,v}$.

**Proposition 7.13.** For each vertex $\tilde{y}$ of $\tilde{K}(x, v)$, $\Sigma_{\tilde{y}}(\tilde{E}_{x,v})$ is CAT(1).

We begin with

**Lemma 7.14.** For each vertex $y \in V(T(x, v))$, $\Sigma_y(\tilde{E}_{x,v})$ is CAT(1), where $\tilde{y} \in V(\tilde{T}(x, v))$ is the vertex corresponding to $y$.

**Proof.** The lemma is clear when $y$ is an endpoint of $T(x, v)$. From now, we assume that $y$ is an interior vertex of $T(x, v)$. Let us consider

$$\Sigma_y := \Sigma_y(K(x, v)) \subset \Sigma_y(X),$$

$$\Sigma_y^+ := \Sigma_y(E \setminus \text{int } K(x, v)) \subset \Sigma_y(X),$$

$$\tilde{\Sigma}_y := \Sigma_{\tilde{y}}(\tilde{K}(x, v)).$$

Note that $\Sigma_y(E) = \Sigma_y^- \cup \Sigma_y^+$ is a subgraph of $\Sigma_y(X)$ without endpoints, and hence it is CAT(1). Since $\Sigma_y(\tilde{E}_{x,v}) = \tilde{\Sigma}_y^- \cup \tilde{\Sigma}_y^+$, it suffices to show

**Claim 7.15.** There is an expanding map $\Sigma_y^- \to \tilde{\Sigma}_y^-$. 

**Proof.** Let $\gamma := \gamma_{y,x}^X : [0, |y, x|_X] \to X$, and set $u := \gamma(0)$. Choose any $\xi \in \Sigma_y(T(x, v)) \subset \Sigma_y^-$, and let $w \in \Sigma_y^-$ be the direction of $C$. Let $\tilde{\xi}, \tilde{u}$ be the directions in $\tilde{\Sigma}_y^-$ corresponding to $\xi, u$ respectively.

Case i) $u \in \Sigma_y^-$. 

Since $X$ is locally CAT($\kappa$), we have $\angle^X(\xi, u) \leq \angle(\tilde{\xi}, \tilde{u})$. Therefore the correspondence $\xi \to \tilde{\xi}, u \to \tilde{u}$ gives rise to the desired expanding map $\Sigma_y^- \to \tilde{\Sigma}_y^-$. 

Case ii) $u \notin \Sigma_y^-$. 

This is the case when $\gamma$ leaves $E$ after $y = \gamma(0)$ at least for a short time. From (7.7), $y$ is contained in an open edge in $E(C)$. Therefore, for small enough $t > 0$, the $X$-geodesic starting from $\gamma(t)$ to $\gamma(0)$ must meet $C$. This implies

$$\angle^X(\xi, w) \leq \angle^X(\xi, u) \leq \angle(\tilde{\xi}, \tilde{u}).$$

Therefore the correspondence $\xi \to \tilde{\xi}, w \to \tilde{w}$ gives rise to the desired expanding map $\Sigma_y^- \to \tilde{\Sigma}_y^-$. Note that by (7.3), $\Sigma_y(X)$ is homeomorphic to the suspension with vertices $\Sigma_x(C)$, from which (7.9) also follows. 

This completes the proof of Lemma 7.14. 

For the proof of Proposition 7.13, it suffices to show the following.
Lemma 7.16. $\Sigma_x(\tilde{E}_{x,v})$ is CAT(1).

Proof. Let $\sigma_i$ ($1 \leq i \leq m$) be the X-geodesics joining $x$ to the points of $\partial T(x,v)$, and set $\nu_i := \tilde{\sigma}_i(0)$ ($1 \leq i \leq m$). Remember that $\Sigma_x(K(x,v))$ consists of $m$ segments from the vertex $v$ to $\nu_i$ of length $\delta$. Since $\Sigma_x(X)$ is CAT(1), it suffices to show

$$\angle(\tilde{\nu}_i, \tilde{\nu}_j) \geq 2\delta$$

for all $1 \leq i \neq j \leq m$,

where $\tilde{\nu}_i$ denotes the direction at $\tilde{x}$ corresponding to $\nu_i$.

For arbitrary $y, y' \in V_{\text{int}}(T(x,v))$ adjacent to $\partial T(x,v)$, let us assume that $z_1, \ldots, z_t \in \partial T(x,v)$ (resp. $z_{t'}, \ldots, z_{n'} \in \partial T(x,v)$ with $1' < \cdots < n'$) are the set of $\partial T(x,v)$ adjacent to $y$ (resp. to $y'$) with $z_i \in \sigma_i$ (resp. $z_{t'} \in \sigma_{t'}$). Set $v_y := \gamma_{x,y}^X(0)$ and $v_y := \gamma_{x,y'}^X(0) \in V(\Sigma_x(\tilde{K}(x,v)))$. Using the angle comparison for $\Delta_X^X$, we have for any $1 \leq i \neq j \leq t$

$$\angle(\tilde{\nu}_i, \tilde{\nu}_j) = \angle(\tilde{\nu}_i, \tilde{v}_y) + \angle(\tilde{v}_y, \tilde{v}_j) \
\geq \angle^X(\nu_i, v_y) + \angle^X(v_y, \nu_j) = 2\delta.$$

Let $\bigcup_{a=1}^k [y_{a-1}, y_a]$ be the shortest path from $y$ to $y'$ in $T(x,v)$ with $y = y_0, y' = y_k$, $y_a \in V_{\text{int}}(T(x,v))$ and $[y_{a-1}, y_a] \in E(T(x,v))$. Then for arbitrary $1 \leq i \leq t$ and $1' \leq j' \leq n'$, we have

$$\angle(\tilde{\nu}_i, \tilde{v}_{j'}) = \angle(\tilde{\nu}_i, \tilde{v}_y) + \sum_{a=1}^k \angle(\tilde{v}_{y_{a-1}}, \tilde{v}_{y_a}) + \angle(\tilde{v}_{y_a}, \tilde{v}_{j'}) \
\geq \angle^X(\nu_i, v_y) + \sum_{a=1}^k \angle^X(v_{y_{a-1}}, v_{y_a}) + \angle^X(v_{y_a}, \nu_{j'}) \geq 2\delta.$$

This completes the proof of Lemma 7.16. \hfill \Box

Note that in $\tilde{E}_{x,v}$, the subarc $[x, y]$ of $C$ is replaced by the geodesic $[\tilde{x}, \tilde{y}] := \gamma_{\tilde{x}, \tilde{y}}$. On the singular locus $\tilde{C}(x,v)$ of $\tilde{E}_{x,v}$, we consider the graph structure inherited from $C$ (and hence from $S(X)$), except that $\tilde{x}, \tilde{y} \in V(\tilde{C}(x,v))$ and $(\tilde{x}, \tilde{y}) \in E(\tilde{C}(x,v))$.

After all the surgeries at $x$ possibly in the both positive and negative singular directions, we obtain a new space, denoted by $\tilde{E}_x$. Note that the point $\tilde{x} \in \tilde{E}_x$ replacing $x$ is no longer singular in the graph structure of the new singular locus $\tilde{C}(x) \subset \tilde{E}_x$.

In what follows, we shall perform such surgeries finitely many times consistently in the directions of $\Sigma_x^{\text{sing}}(X)$ at points $x \in V^{\text{sing}}(C)$ so that the surgery parts cover $V^{\text{sing}}(C)$.

First take $\epsilon = \epsilon_p > 0$ satisfying (7.5), (7.6), (7.7) and (7.8) for $x = p$, and set $\delta_0 = \epsilon_p$. Remember that $S(p, \delta_0)$ does not meet $V(C)$. We enlarge the radius of the ball $B(p, \delta_0)$, and choose $r_1 > \delta_0$ such that during the enlarging, $S(p, r_1)$ first meets $V^{\text{sing}}(C)$, say at $x$, after $S(p, \delta_0)$. We call $r_1$ a critical radius in the surgeries. Now we do the above surgery at $x$, either in the negative direction $-\nabla d_p(x)$, where
the surgeries should be carried out inside the annulus $A(p, \delta_0, r_1) = B(p, r_1) \setminus \text{int} B(p, \delta_0)$, or in the positive direction $\nabla d_p(x)$ to resolve the singularity at $x$.

We again perform such surgeries at all points $x \in S(p, r_1) \cap V_{\text{sing}}(C)$. Here, taking the smallest constant $\epsilon = \epsilon_x$ among all $x$ and all singular directions there, we may assume that those surgeries are carried out based on a common metric sphere around $p$. More precisely, for some $0 < \delta_1 < r_1 - \delta_0$, we have $V(T(x, v)) \subset S(p, r_1 + \delta_1)$ (resp. $V(T(x, v)) \subset S(p, r_1 - \delta_1)$) for all $x \in S(p, r_1) \cap V_{\text{sing}}(C)$ and $v \in \Sigma^\text{sing}_x(C)$ (resp. $v \in \Sigma^\text{sing}_{x,-}(C)$). We call $\delta_1$ (resp. $\delta_0$) the surgery radius at $S(p, r_1)$ (resp. at $p$).

Then we again enlarge the radius of $B(p, r_1 + \delta_1)$ until the next critical radius $r_2$. Repeating this procedure, we have a possibly infinite sequence of critical radii $r_i$,

$$0 < r_1 < r_2 < \cdots < r_i < \cdots,$$

and surgery radii $\delta_i$ at $S(p, r_i)$ with $r_i + \delta_i < r_{i+1} - \delta_{i+1}$ such that the $X$-annulus $A^X(p, r_i + \delta_i, r_{i+1} - \delta_{i+1})$ does not meet $V_{\text{sing}}(C)$. Note also that the number of surgeries at points of $S(p, \delta_i)$ is bounded by the uniform constant $N_v - 2$.

We show that one can cover $V_{\text{sing}}(C)$ after performing surgeries as above finitely many times. Suppose $r_* = \lim_{i \to \infty} r_i < r$. From construction, $S(p, r_*)$ meets $V_{\text{sing}}(C)$. We again do surgeries at points of $S(p, r_*) \cap V_{\text{sing}}(C)$. For the surgeries in the negative direction at those points, we can make them consistent with the previous surgeries since our procedure is done based on metric spheres around $p$. This shows that after finitely many such surgeries, we can resolve all singular vertices in $V(C)$. Let $0 < r_1 < r_2 < \cdots < r_j < r$ be critical radii, and $\delta_i$ ($0 \leq i \leq J$) surgery radii, where we may assume that $A(p, r_j + \delta_j, r)$ does not meet $V_{\text{sing}}(C)$ by taking slightly larger $r$ if necessary.

Let $\mathcal{K} := \{K_{n,i} := K(x_n, v_{n,i}) \mid 1 \leq n \leq M, 1 \leq i \leq L_n\}$ be the set of all cone-like domains in $\tilde{E}$ constructed as above for $x_n \in V_{\text{sing}}(C)$ and $v_{n,i} \in \Sigma^\text{sing}_x(C)$ which arise in the course of the surgeries. Set $I_0 := [0, \delta_0]$, $I_j := [r_j - \delta_j, r_j + \delta_j]$ and $A_j := d_p^{-1}(I_j)$ ($1 \leq j \leq J$). From construction, we have the following for every $K_n \in \mathcal{K}$.

- $K_n \in \mathcal{K}$ is convex in $E$;
- $K_n$ is contained in some $A_j$;
- $K_n$ and $K_{n'}$ do not have intersection in their interiors for all $n \neq n'$;
- the number of $K_n$ contained in $A_j$ is at most $N_v - 2$ for each $1 \leq j \leq J$.

Let $\tilde{E}$ be the result of those surgeries, and let $\tilde{C}$ be the singular locus of $\tilde{E}$, with graph structure $V(\tilde{C})$, $E(\tilde{C})$ defined as above. Note that $V(\tilde{C})$ is finite and $V_{\text{sing}}(\tilde{C})$ is empty.
Lemma 7.17. \( \hat{E} \) is a \( \text{CAT}(\kappa) \)-space.

Proof. From the construction and Proposition 7.13 we have

- for every edge \( e \) of \( E(\hat{C}) \), the condition (A) holds and \( e \) has finite turn variation;
- \( \Sigma_{\tilde{g}}(\hat{E}) \) is \( \text{CAT}(1) \) for every \( \tilde{g} \in V(\hat{C}) \).

Consider any triangulation of \( \hat{E} \) extending \( V(\hat{C}) \) and \( E(\hat{C}) \) by adding geodesic edges if necessary. Now, we are ready to apply Theorem 7.6 to this triangulation to conclude that \( \hat{E} \) is \( \text{CAT}(\kappa) \). \( \square \)

Now we are going to show the Gromov-Hausdorff convergence \( \hat{E} \to E \) as \( \epsilon_0 \to 0 \).

For each \( K(x_n, v_{n,i}) \in \mathcal{K} \), we fix any element \( y_{n,i} \in V(T(x_n, v_{n,i})) \), and let \( \gamma_{n,i} : [0,|x_n, y_{n,i}|_E] \to E \) be an \( E \)-geodesic from \( x_n \) to \( y_{n,i} \).

Define \( \varphi : \hat{E} \to E \) as follows. Let \( \varphi \) be identical outside the surgery part. For every \( \hat{z} \in \text{int} \hat{K}(x_n, v_{n,i}) \), we let

\[
\varphi(\hat{z}) := \gamma_{n,i}(|\hat{x}_n, \hat{z}|).
\]

Since \( \text{diam}(K(x_n, v_{n,i})) < \tau(\epsilon_0) \), the image of \( \varphi \) is \( \tau(\epsilon_0) \)-dense in \( E \).

For arbitrary \( \hat{z}, \hat{z}' \in \hat{E} \), set \( z = \varphi(\hat{z}), z' = \varphi(\hat{z}') \), and choose an \( E \)-shortest curve \( \gamma : [0, |z, z'|_E] \to E \) between \( z \) and \( z' \). Suppose first that \( d_p(\gamma(t)) \) takes a local minimum or local maximum. Then we see that \( \gamma \) is vertical, and hence an \( X \)-geodesic. Moreover, \( \gamma \) intersects \( C \) almost perpendicularly with at most \( N_v - 2 \) points (Sublemma 6.8). This implies that \( \gamma \) meets at most \( N_v - 2 \) elements of \( \mathcal{K} \). Therefore from Lemma 7.12, we have

\[ ||z, z'|_E - |\hat{z}, \hat{z}'|_E < \tau(\epsilon_0)(r + N_v - 2). \]

Now we assume that \( d_p(\gamma(t)) \) is strictly monotone. Let \( \mathcal{K}_\gamma \) be set of all \( K(x_n, v_{n,i}) \in \mathcal{K} \) meeting \( \gamma \). For simplicity, we relabel \( \mathcal{K}_\gamma \) as \( \mathcal{K}_\gamma = \{K_i | 1 \leq i \leq l \} \). Let \( \mathcal{K}_j \) be the set of all \( K_n \in \mathcal{K}_\gamma \) contained in \( A_j \). If \( \gamma \) meets \( K_n \in \mathcal{K}_j \) with \( \{z_n', z'_n\} = \gamma \cap \partial K_n \), then from Lemma 7.12 we have

\[ |\varphi^{-1}(z_n), \varphi^{-1}(z'_n)|_E - |z_n, z'_n|_E < 2\tau(\epsilon_0)\delta_j. \]

It follows that

\[ ||z, z'|_E - |\hat{z}, \hat{z}'|_E < 2r(N_v - 1)\tau(\epsilon_0). \]

In this way, we conclude that \( \hat{E} \) converges to \( E \) as \( \epsilon_0 \to 0 \) with respect to the Gromov-Hausdorff distance, which yields that \( E \) is a \( \text{CAT}(\kappa) \)-space. This completes the proof of Theorem 7.4. \( \square \)

Proof of Theorem 7.4 (1) in Case II. We consider Case II in the subsection of filling ball of Section 5. We only have to apply Theorem 7.4 for \( k = 4 \) to Case II. The rest of the argument is similar to that in Case I given in Section 5 and hence omitted. \( \square \)
The proof of Corollary 1.3 is similar to that of Theorem 1.1(1) in Case II, and hence omitted.

Using Theorem 7.18 we also have the following.

**Theorem 7.18.** In Theorem 1.1 every union $\text{Im} f_{i_1} \cup \cdots \cup \text{Im} f_{i_k}$ is a CAT$(\kappa)$-space.

**Proof.** The basic idea of the proof of Theorem 7.18 is the same as that of Theorem 1.1(1) for branched immersed disks. Set $\Sigma := \Sigma_{p,i}$ and consider $\Sigma := \Sigma_{p,i_1} \cup \cdots \cup \Sigma_{p,i_k}$. For each $v \in V(\Sigma(X))$ contained in $\Sigma$, we construct a ruled surface $S$ for which we may assume CAT$(\kappa)$ by taking smaller $r$. Let $S(v)$ denote the union of all such ruled surfaces $S$. By Theorem 7.18 $S(v)$ is CAT$(\kappa)$. The rest of the argument is the same as before, and hence omitted. □

**APPENDIX A. ALEXANDROV’S RESULT ON RULED SURFACES**

Following the ideas of Alexandrov in [4], we prove Theorem 3.17. As mentioned in Section 1, it also follows from [27] in the CAT(0)-setting. We denote by $D_{\kappa}$ the diameter of $M_{\kappa}^2$. Recall that a CAT$(\kappa)$-space is defined as a $D_{\kappa}$-geodesic space in which every triangle with perimeter $< 2D_{\kappa}$ is not thicker than its comparison triangle in $M_{\kappa}^2$ with the same side lengths, where a $D_{\kappa}$-geodesic space means a metric space in which any two points with distance $< D_{\kappa}$ can be joined by a minimal geodesic. Throughout this appendix, let $X$ be a CAT$(\kappa)$-space.

A.1. **Finite sequences of ruling geodesics.** Let $S$ be a ruled surface in $X$ with parametrization $\sigma : R \to X$, where $R = [0, \ell] \times [0, 1]$. Let $\pi : R \to R_\ast$ and $p_1 : R \to [0, \ell]$ be as in Section 3.

We give an explicit formulation of the pullback metric $e_\sigma$. For $u = (s_0, t_0)$ and $u' = (s_0, t_0')$ with $s_0 < s_0'$ in $R$, let $\Delta : s_0 \leq s_1 \leq \cdots \leq s_n = s_0'$ be a decomposition of $[s_0', s_0]$, and set $|\Delta| = \max\{|s_i - s_{i-1}| | 1 \leq i \leq n\}$. We consider

$$e_\sigma^\Delta(\pi(u), \pi(u')) := \inf \left\{ \sum_{i=1}^n |x_{i-1} - x_i| \mid x_0 = \sigma(u), x_n = \sigma(u'), x_i \in \lambda_{s_i} \right\}.$$ 

Choose a sequence $\{x_i\}_{i=0,1,\ldots,n}$ in $X$ such that $x_0 = \sigma(u)$, $x_n = \sigma(u')$, $x_i \in \lambda_{s_i}$ for all $i \in \{1, \ldots, n-1\}$, and

$$e_\sigma^\Delta(\pi(u), \pi(u')) = \sum_{i=1}^n |x_{i-1} - x_i|.$$ 

We call such a sequence $\{x_i\}_{i=0,1,\ldots,n}$ a $\Delta$-minimizing chain along $S$ from $\sigma(u)$ to $\sigma(u')$. Notice that possibly we have $x_{i-1} = x_i$ for some $i \in \{1, \ldots, n\}$. We set $\gamma^\Delta := \bigcup x_{i-1}x_i$, and call it a $\Delta$-minimizing broken geodesic in $X$ from $\sigma(u)$ to $\sigma(u')$, which realizes $L(\gamma^\Delta) = e_\sigma^\Delta(\pi(u), \pi(u'))$. 

Lemma A.1. Under the above situation, we have the following:

1. $e_\sigma(\pi(u), \pi(u')) = \sup_{\Delta} e_\sigma^\Delta(\pi(u), \pi(u'))$, where $\Delta$ runs over all decompositions of $[s_0, s'_0]$;
2. For any sequence $\Delta_n$ of decompositions of $[s_0, s'_0]$ satisfying $\lim_{n \to \infty} |\Delta_n| = 0$, we have $e_\sigma(\pi(u), \pi(u')) = \lim_{n \to \infty} e_{\sigma}^\Delta(\pi(u), \pi(u'))$.

Proof. By Proposition 3.6, there is a shortest curve $c_{0*} : [0, 1] \to (R^2, e_\sigma)$ from $\pi(u)$ to $\pi(u')$ together with its lift $c_0$. Set $\gamma_0(t) := \sigma \circ c_0(t)$. For any decompositon $\Delta = \{s_i\}_{i=1}^N$ of $[s_0, s'_0]$, take $t_i \in [0, 1]$ such that $\gamma_0(t_i) \in \lambda_{s_i}$. Then in view of Proposition 3.10, we have

$$e_\sigma^\Delta(\pi(u), \pi(u')) \leq \sum_{i=1}^N |\gamma_0(t_{i-1}), \gamma_0(t_i)| \leq L(\gamma_0) = L(c_{0*}) = e_\sigma(\pi(u), \pi(u')).$$

Thus we have $\sup_{\Delta} e_\sigma^\Delta(\pi(u), \pi(u')) \leq e_\sigma(\pi(u), \pi(u'))$.

Let $\{\Delta_n\}$ be a sequence of decompositions of $[s_0, s'_0]$ with $\lim_{n \to \infty} |\Delta_n| = 0$ such that

$$\lim_{n \to \infty} e_{\sigma}^\Delta(\pi(u), \pi(u')) = \liminf_{|\Delta| \to 0} e_\sigma^\Delta(\pi(u), \pi(u')).$$

Let $\gamma : [0, 1] \to X$ be a $\Delta_n$-minimizing broken geodesic in $X$. Passing to a subsequence, we may assume that $\gamma_n$ converges to a curve $\gamma : [0, 1] \to X$. From $|\Delta_n| \to 0$, it follows that $\gamma([0, 1]) \subset S$.

Sublemma A.2. $\gamma$ has a lift in $R$ from $u$ to $u'$. Proof. We may assume $\text{Sing}(\sigma)$ is empty. Let $\Delta_n : s_0 = s_{n,0} \leq s_{n,1} \leq \cdots \leq s_{n,k_n} = s'_0,$ and $\gamma_n = \gamma_{\Delta_n} := \bigcup_{i=1}^{k_n} x_{n,i-1} x_{n,i}$ with $x_{n,i} \in \lambda_{s_{n,i}}$. Choose $a_{n,i} \in I_{s_{n,i}}$ with $\sigma(a_{n,i}) = x_{n,i}$ for $1 \leq i \leq k_n - 1$, and consider the Euclidean broken geodesic $c_n := \bigcup_{i=1}^{k_n} a_{n,i-1} a_{n,i}$. Note that $c_n$ is monotone, and $\sigma \circ c_n$ also converges to $\gamma$ as $n \to \infty$. We show that a subsequence of $c_n$ converges to a curve $c$, which is a lift of $\gamma$. We do not know if $L(\sigma \circ c_n)$ is uniformly bounded or even if it is finite, which is the only difference from Proposition 3.6.

Since the basic strategy is the same as the proof of Proposition 3.6 we present only an outline. Let $J_0$ be a countable dense subset of $J = [s_0, s'_0]$. For each $s \in J$, choose a point $c_n(t_n(s))$ of $c_n$ with $c_n(t_n(s)) \in I_s$. Now we have a subsequence $\{m\}$ of $\{n\}$ such that $c_m(t_m(s))$ converges to a point $x(s) \in I_s$ for every $s \in J_0$. We consider the limit set, say $LS(\{c_m\})$, of the sequence $\{\text{Im}(c_m)\}$, and set $E_s := LS(\{c_m\}) \cap I_s,$
as in the proof of Proposition 3.6. Then we have the decomposition \( J = J_1 \cup J_2 \), where
\[
J_1 = \{ s \in J \mid E_s \text{ is a single point} \}, \quad J_2 = J \setminus J_1.
\]
In the same way, we have the conclusions (1) \( \sim \) (4) in the proof of Proposition 3.6. Here it should be remarked that the following holds as well,
\[
\sum_{s \in J_2} \sigma(E_s) \leq L(\gamma).
\]
Thus as before, we obtain a monotone continuous parametrization on the union of points and segments \( \{E_s \mid s \in J \} \), which provides a lift of \( \gamma \) from \( u \) to \( u' \).

By Sublemma A.2, we conclude
\[
e_\sigma(\pi(u), \pi(u')) \leq L(\gamma) \leq \lim_{n \to \infty} L(\gamma_n) = \liminf_{n \to \infty} e_{\sigma_n}(\pi(u), \pi(u')).
\]
This completes the proof.

From the choice of a \( \Delta \)-minimizing chain along \( S \), we derive the following:

**Lemma A.3.** In the setting discussed above, let \( \{x_i\}_{i=0}^{n} \) be a \( \Delta \)-minimizing chain along \( S \) from \( \sigma(u) \) to \( \sigma(u') \). Then for each \( i \in \{1, \ldots, n-1\} \) and for each \( t \in \{0, 1\} \) we have
\[
\angle x_{i-1} x_i \lambda_{s_i}(t) + \angle \lambda_{s_i}(t)x_ix_{i+1} \geq \pi,
\]
whenever \( |x_{i-1}, x_i|, |x_i, x_{i+1}| < D_\kappa \), and the angles \( \angle x_{i-1} x_i \lambda_{s_i}(t) \) and \( \angle \lambda_{s_i}(t)x_ix_{i+1} \) can be defined.

**Proof.** First we show the conclusion in the case \( t = 0 \). Set \( \theta_i^- := \angle x_{i-1} x_i \lambda_{s_i}(0) \) and \( \theta_i^+ := \angle \lambda_{s_i}(0)x_ix_{i+1} \). Take \( t_i \in \{0, 1\} \) with \( x_i = \lambda_{s_i}(t_i) \), where we may assume \( t_i \neq 0 \). If we put \( h(\epsilon) := |\lambda_{s_i}(t_i - \epsilon), x_{i-1}| + |\lambda_{s_i}(t_i - \epsilon), x_{i+1}| \) for small \( \epsilon > 0 \), then by the first variation formula (see e.g., [11] Corollary II.3.6) together with the \( \Delta \)-minimizing property of \( \{x_i\}_{i=0}^{n} \), we have
\[
0 \leq \lim_{\epsilon \to 0^+} \frac{h(\epsilon) - h(0)}{\epsilon} = -(\cos \theta_i^- + \cos \theta_i^+) - \pi.
\]
This implies \( \theta_i^- + \theta_i^+ \geq \pi \). Similarly, we see the inequality for \( t = 1 \).

Let \( u_* := \pi(u), v_* := \pi(v), w_* := \pi(w) \) be distinct points in \( R_* \). Assume for a while that
\[
p_1(u) \leq p_1(v) \leq p_1(w),
\]
and choose a decomposition \( \Delta = \{ s_i \}_{i=0}^{n} \) of \( [p_1(u), p_1(v)] \) such that for some \( m \in \{1, \ldots, n-1\} \) we have \( p_1(v) = s_m \). Let \( \Delta' := \{ s_i \}_{i=0}^{m} \) be the decomposition of \( [p_1(u), p_1(v)] \), and \( \Delta'' := \{ s_{m+i} \}_{i=0}^{n-m} \) the decomposition of \( [p_1(v), p_1(w)] \). Take a \( \Delta' \)-minimizing chain \( \{y_i\}_{i=0}^{m} \) along \( S \) from \( \sigma(u) \) to \( \sigma(v) \), and a \( \Delta'' \)-minimizing chain \( \{y_{m+i}\}_{i=0}^{n-m} \) along \( S \) from \( \sigma(v) \) to \( \sigma(w) \).
along $S$ from $\sigma(v)$ to $\sigma(w)$, and a $\Delta$-minimizing chain $\{z_i\}_{i=0,1,\ldots,n}$ along $S$ from $\sigma(u)$ to $\sigma(w)$. Assume in addition that we have

$$e^\Delta_{\sigma}(u_*, v_*) + e^\Delta_{\sigma}(w_*, u_*) < 2D_\kappa.$$ 

Set $x := \sigma(u)$, $y := \sigma(v)$ and $z := \sigma(w)$. Let $B^\Delta(xy)$ be the broken geodesic $\bigcup_{i=1}^m y_{i-1}y_i$ in $X$ joining $x$ and $y$, $B^\Delta(yz)$ the broken geodesic $\bigcup_{i=1}^{n-m} y_{m+i-1}y_{m+i}$ in $X$ joining $y$ and $z$, $B^\Delta(zx)$ the broken geodesic $\bigcup_{i=1}^n z_{i-1}z_i$ in $X$ joining $z$ and $x$. We denote by $P^\Delta(xyz)$ the polygon in $X$ defined by

$$P^\Delta(xyz) := B^\Delta(xy) \cup B^\Delta(yz) \cup B^\Delta(zx),$$

and we call $P^\Delta(xyz)$ the $\Delta$-minimizing chain triple along $S$. We denote by $\theta^\Delta_x(y, z)$ the angle at $x$ in $X$ between $B^\Delta(xy)$ and $B^\Delta(zx)$, by $\theta^\Delta_y(z, x)$ the angle at $y$ in $X$ between $B^\Delta(yz)$ and $B^\Delta(xy)$, by $\theta^\Delta_z(x, y)$ the angle at $z$ in $X$ between $B^\Delta(zx)$ and $B^\Delta(yz)$.

![Diagram](image_url)

In the model surface $M^2$, we define a comparison polygon $\tilde{P}^\Delta(xyz)$ for $P^\Delta(xyz)$ as follows: Let $\Delta \tilde{x}y_1 \tilde{z}_1$ and $\Delta \tilde{y}_{n-1} \tilde{z}_{n-1} \tilde{z}$ be comparison triangles in $M^2_\kappa$ for $\Delta xy_1z_1$ and for $\Delta y_{n-1}z_{n-1}z$, respectively. For each $i \in \{1, \ldots, n-1\}$, take comparison triangles $\Delta \tilde{y}_i \tilde{y}_{i+1} \tilde{z}_i$ and $\Delta \tilde{y}_{i+1} \tilde{z}_i \tilde{z}_{i+1}$ in $M^2_\kappa$ for $\Delta y_iy_{i+1}z_i$ and for $\Delta y_{i+1}z_i\tilde{z}_{i+1}$, respectively, and then glue all the comparison triangles in $M^2_\kappa$ along $\tilde{y}_i \tilde{z}_i$, and along $\tilde{y}_{i+1} \tilde{z}_i$, for all $i \in \{1, \ldots, n-1\}$. Let $\tilde{B}^\Delta(xy)$ be the broken geodesic $\bigcup_{i=1}^m \tilde{y}_{i-1} \tilde{y}_i$ in $M^2_\kappa$ joining $\tilde{x}$ and $\tilde{y}$, $\tilde{B}^\Delta(yz)$ the broken geodesic $\bigcup_{i=1}^{n-m} \tilde{y}_{m+i-1} \tilde{y}_{m+i}$ in $M^2_\kappa$ joining $\tilde{y}$ and $\tilde{z}$, $\tilde{B}^\Delta(zx)$ the broken geodesic $\bigcup_{i=1}^n \tilde{z}_{i-1} \tilde{z}_i$ in $M^2_\kappa$ joining $\tilde{z}$ and $\tilde{x}$. Then we put

$$\tilde{P}^\Delta(xyz) := \tilde{B}^\Delta(xy) \cup \tilde{B}^\Delta(yz) \cup \tilde{B}^\Delta(zx),$$

and we call $\tilde{P}^\Delta(xyz)$ a comparison $\Delta$-minimizing chain triple in $M^2_\kappa$ for $P^\Delta(xyz)$. We denote by $\theta^\Delta_x(y, z)$ the angle at $\tilde{x}$ in $M^2_\kappa$ between $\tilde{B}^\Delta(xy)$ and $\tilde{B}^\Delta(zx)$, by $\theta^\Delta_y(z, x)$ the angle at $\tilde{y}$ in $M^2_\kappa$ between $\tilde{B}^\Delta(yz)$
and \( \tilde{B}^\Delta(xy) \), by \( \tilde{\theta}_x^\Delta(x,y) \) the angle at \( \tilde{z} \) in \( M_\kappa^2 \) between \( \tilde{B}^\Delta(xz) \) and \( \tilde{B}^\Delta(yz) \). Note that

\[
\theta_x^\Delta(y,z) \leq \tilde{\theta}_x^\Delta(y,z), \quad \theta_y^\Delta(z,x) \leq \tilde{\theta}_y^\Delta(z,x), \quad \theta_z^\Delta(x,y) \leq \tilde{\theta}_z^\Delta(x,y).
\]

From Lemma A.3 we derive the following concavity of \( \Delta \) that the broken points are distinct to each other. Choose \( \tilde{y}_i \in \tilde{P}^\Delta(xy) \) at \( \tilde{z}_i \) at \( \tilde{P}^\Delta(xy) \) at the concave vertices, we obtain a triangle \( \Delta \bar{x} \bar{y} \bar{z} \) in \( M_\kappa^2 \) whose side-lengths satisfy

\[
|\bar{x}, \bar{y}| = c_\sigma^\Delta(u_*, v_*), \quad |\bar{y}, \bar{z}| = c_\sigma^\Delta(v_*, v_*), \quad |\bar{z}, \bar{x}| = c_\sigma^\Delta(v_*, u_*).
\]

We call \( \Delta \bar{x} \bar{y} \bar{z} \) a comparison \( \Delta \)-minimizing stretched triangle in \( M_\kappa^2 \) for \( P^\Delta(xy) \), and we denote it by \( \bar{P}^\Delta(xy) \). We denote by \( \tilde{\theta}_x^\Delta(y,z) \) the angle \( \angle \tilde{y} \bar{x} \tilde{z} \) at \( \tilde{x} \) in \( M_\kappa^2 \) between \( \tilde{x} \bar{y} \) and \( \tilde{x} \bar{z} \), by \( \tilde{\theta}_y^\Delta(z,x) \) the angle \( \angle \tilde{z} \bar{y} \tilde{x} \) at \( \tilde{y} \) in \( M_\kappa^2 \) between \( \tilde{y} \bar{z} \) and \( \tilde{y} \bar{x} \), and by \( \tilde{\theta}_z^\Delta(x,y) \) the angle \( \angle \tilde{x} \bar{z} \tilde{y} \) at \( \tilde{z} \) in \( M_\kappa^2 \) between \( \tilde{z} \bar{x} \) and \( \tilde{z} \bar{y} \). Let \( \tilde{y}_i \in \tilde{x} \bar{y} \tilde{z}, \bar{z}_i \in \bar{x} \tilde{z} \), \( i \in \{1, \ldots, n-1\} \), be the points corresponding to \( \tilde{y}_i \) and to \( \tilde{z}_i \), respectively. Since \( \tilde{P}^\Delta(xy) \) is concave except the vertices, the Alexandrov stretching lemma (see e.g., [31] Lemma I.2.16]) leads to the following:

**Lemma A.4.** Under the setting discussed above, we have

\[
\tilde{\theta}_x^\Delta(y,z) \leq \tilde{\theta}_x^\Delta(y,z), \quad \tilde{\theta}_y^\Delta(z,x) \leq \tilde{\theta}_y^\Delta(z,x), \quad \tilde{\theta}_z^\Delta(x,y) \leq \tilde{\theta}_z^\Delta(x,y).
\]

Moreover, for all \( i \in \{1, \ldots, n-1\} \) we have \( |y_i, z_i| \leq |\tilde{y}_i, \tilde{z}_i| \).

Let \( y_j \in \bar{B}^\Delta(xy) \setminus \{x, y\} \) be a broken point for \( j \in \{1, \ldots, m-1\} \), \( y_k \in \bar{B}^\Delta(yz) \setminus \{y, z\} \) a broken point for \( k \in \{m+1, \ldots, n-1\} \), and \( z_l \in \bar{B}^\Delta(xz) \setminus \{x, z\} \) a broken point for \( l \in \{1, \ldots, n-1\} \). Assume that the broken points \( y_j, y_k \), and \( z_l \) are distinct to each other. Choose four \( \Delta \)-minimizing chain triples \( \bar{P}^\Delta(xy_jz_l), \bar{P}^\Delta(y_jy_yk), \bar{P}^\Delta(z_ly_kz_l) \), and \( \bar{P}^\Delta(y_jy_yk) \) along \( \bar{S} \), and take comparison \( \Delta \)-minimizing stretched triangles \( \bar{P}^\Delta(xy_jz_l), \bar{P}^\Delta(y_jy_yk), \bar{P}^\Delta(z_ly_kz_l), \) and \( \bar{P}^\Delta(y_jy_yk) \) in \( M_\kappa^2 \) for \( \bar{P}^\Delta(xy_jz_l), \bar{P}^\Delta(y_jy_yk), \bar{P}^\Delta(z_ly_kz_l), \) and \( \bar{P}^\Delta(y_jy_yk) \), respectively.

From Lemma A.4 we derive the following monotonicity:

**Lemma A.5.** Under the setting discussed above, we have

\[
\tilde{\theta}_x^\Delta(y_j, z_l) \leq \tilde{\theta}_x^\Delta(y_j, z_l), \quad \tilde{\theta}_y^\Delta(y_j, y_k) \leq \tilde{\theta}_y^\Delta(z_l, y_k), \quad \tilde{\theta}_z^\Delta(z_l, y_k) \leq \tilde{\theta}_z^\Delta(x, y_k).
\]

**Proof.** Gluing the triangles \( \bar{P}^\Delta(xy_jz_l) = \Delta \bar{x} \bar{y} \bar{z}, \bar{P}^\Delta(y_jy_yk) = \Delta \bar{y} \bar{y} \bar{k}, \bar{P}^\Delta(z_ly_kz_l) = \Delta \bar{z} \bar{y} \bar{k} \), and \( \bar{P}^\Delta(y_jy_yk) = \Delta \bar{y} \bar{y} \bar{k} \) in \( M_\kappa^2 \) along the edges \( \bar{y}_j \bar{y}_k, \bar{y}_k \bar{z}_l, \) and \( \bar{z}_l \bar{y}_j \), we obtain a hexagon \( \bar{x} \bar{y}_j \bar{y}_k \bar{z}_l \bar{z}_l \bar{y}_j \) in \( M_\kappa^2 \) whose side-lengths satisfy \( |\bar{x}, \bar{y}_j| + |\bar{y}_j, \bar{y}_k| = c_\sigma^\Delta(u_*, v_*), |\bar{y}_k, \bar{z}_l| + |\bar{z}_l, \bar{y}_j| = c_\sigma^\Delta(v_*, u_*) \),...
and $|\tilde{z}, \tilde{z}_l| + |\tilde{z}_l, \tilde{x}| = e^\Delta_\sigma(w_*, u_*)$. By Lemmas A.3 and A.4 we have
\[
\pi \leq \theta^\Delta_{y_j}(x, z_l) + \theta^\Delta_{y_j}(z_l, y_k) + \theta^\Delta_{y_j}(y_k, y)
\leq \bar{\theta}^\Delta_{y_j}(x, z_l) + \bar{\theta}^\Delta_{y_j}(z_l, y_k) + \bar{\theta}^\Delta_{y_j}(y_k, y).
\]
Similarly, we have
\[
\pi \leq \theta^\Delta_{y_k}(y, y_j) + \theta^\Delta_{y_k}(y_j, z_l) + \theta^\Delta_{y_k}(z_l, z),
\pi \leq \bar{\theta}^\Delta_{y_k}(y, y_j) + \bar{\theta}^\Delta_{y_k}(y_j, z_l) + \bar{\theta}^\Delta_{y_k}(z_l, z, x).
\]
By stretching the hexagon $\tilde{x}\tilde{y}_j\tilde{y}_k\tilde{z}\tilde{l}$ at the concave vertices $\tilde{y}_j$, $\tilde{y}_k$, and $\tilde{z}_l$, we obtain a comparison $\Delta$-minimizing stretched triangle $\tilde{P}^\Delta(xyz)$ in $M^2_\kappa$ for $P^\Delta(xyz)$. The Alexandrov stretching lemma (see e.g., [11, Lemma I.2.16]) leads to the desired inequalities. □

From Lemma A.3 we also derive the following:

**Lemma A.6.** Let $u_*, u'_* \in R_\kappa$ be distinct points. Assuming $p_1(u) \leq p_1(u')$, we choose a decomposition $\Delta = \{s_i\}_{i=0,1,...,n}$ of $[p_1(u), p_1(u')]$. If $e^\Delta_\sigma(u_*, u'_*) < D_\kappa$, then a $\Delta$-minimizing chain $\{x_i\}_{i=0,1,...,n}$ along $S$ from $\sigma(u)$ to $\sigma(u')$ is uniquely determined.

**Proof.** Let $x := \sigma(u), x' := \sigma(u')$, and suppose that two distinct $\Delta$-minimizing chains $\{x_i\}_{i=0,1,...,n}$ and $\{y_i\}_{i=0,1,...,n}$ along $S$ from $x$ to $x'$ satisfy $x_m \neq y_m$ for $m \in \{1, \ldots, n-1\}$. Then for the $\Delta$-minimizing chain triple $P^\Delta(xyx_mx')$ along $S$ we see that a comparison $\Delta$-minimizing stretched triangle $\tilde{P}^\Delta(xyx_mx')$ degenerates in $M^2_\kappa$. Hence we have $\tilde{x}_m = \tilde{y}_m$. On the other hand, Lemma A.3 implies $|\tilde{x}_m, y_m| \leq |\tilde{x}, \tilde{y}_m|$. This is a contradiction. □

### A.2. Curvature bounds on ruled surfaces

Let $\tilde{\Delta}u_*v_*w_*$ be a geodesic triangle in $(R_\kappa, e_\sigma)$ with distinct vertices and with perimeter $< 2D_\kappa$ determined by $\Delta u_*v_*w_* = \tilde{u}_*\tilde{v}_*\tilde{w}_*$, where $\tilde{u}_*\tilde{v}_*$, $\tilde{v}_*\tilde{w}_*$, and $\tilde{w}_*\tilde{u}_*$ are the edges of $\tilde{\Delta}u_*v_*w_*$. We denote by $\Delta\tilde{u}_*\tilde{v}_*\tilde{w}_*$ a comparison triangle in $M^2_\kappa$ for $\tilde{\Delta}u_*v_*w_*$ with the same side-lengths, and by $\bar{\theta}_{u_*}(v_*, w_*)$ the angle $\angle \tilde{v}_*\tilde{u}_*\tilde{w}_*$ at $\tilde{u}_*$ between $\tilde{u}_*\tilde{v}_*$ and $\tilde{u}_*\tilde{w}_*$.

In order to complete the proof of Theorem 3.17, it suffices to show the following (see e.g., [11, Proposition II.1.7]).

**Lemma A.7.** Every geodesic triangle $\tilde{\Delta}u_*v_*w_*$ in $(R_\kappa, e_\sigma)$ as above satisfies the convexity of angle $\kappa$-comparison: Namely for all $w'_* \in \tilde{u}_*\tilde{v}_*$ \ $\{u_*, v_*\}$, $u'_* \in \tilde{v}_*\tilde{w}_*$ \ $\{v_*, w_*\}$, and $v'_* \in \tilde{w}_*\tilde{u}_*$ \ $\{w_*, u_*\}$, we have the following monotonicity:
\[
\bar{\theta}_{u_*}(w'_*, u'_*) \leq \bar{\theta}_{v_*}(v'_*, w'_*), \quad \bar{\theta}_{v_*}(w'_*, u'_*) \leq \bar{\theta}_{w_*}(u'_*, v'_*),
\]
\[
\t\bar{\theta}_{u_*}(w'_*, u'_*) \leq \bar{\theta}_{v_*}(v'_*, w'_*), \quad \bar{\theta}_{v_*}(w'_*, u'_*) \leq \bar{\theta}_{w_*}(u'_*, v'_*).
\]
Before proving Lemma A.7, we show the following sublemma. By Proposition 3.10, for every minimal geodesic \( c_* \) in \( (R_*, e_*) \) there exists a monotone curve \( c \in R \) with \( \pi \circ c = c_* \) up to monotone parametrization.

Sublemma A.8. In the same setting as in Lemma A.7, let \( u_*, u'_* \in (R_*, e_*) \) be distinct points with \( e_\sigma(u_*, u'_*) < D_\kappa \), and let \( c_* \) be a minimal geodesic in \( (R_*, e_*) \) from \( u_* \) to \( u'_* \). Assume \( p_1(u) \leq p_1(u') \) and choose a sequence \( \{\Delta_n\}_{n \in \mathbb{N}} \) of decompositions \( \Delta_n = \{s_i\}_{i=0,1,...,n} \) of \( [p_1(u), p_1(u')] \) with \( \lim_{n \to \infty} |\Delta_n| = 0 \). For \( n \in \mathbb{N} \), let \( \{x_i\}_{i=0,1,...,n} \) be the \( \Delta_n \)-minimizing chain along \( S \) from \( x := \sigma(u) \) to \( x' := \sigma(u') \), and take a sequence \( \{y_i\}_{i=0,1,...,n} \) in the image of \( \gamma := \sigma_* \circ c_* \) in such a way that \( y_0 = x, y_n = x' \), and \( y_i \in \lambda_* \), for all \( i \in \{1, \ldots, n-1\} \). Then we have the following:

1. \( e_\sigma(u_*, u'_*) = \lim_{n \to \infty} \sum_{i=1}^{n} |y_{i-1}, y_i| \);

2. For every \( s \in [p_1(u), p_1(u')] \), and for every sequence \( \{s_i\}_{n \in \mathbb{N}} \) converging to \( s \) with \( s_i \in \Delta_n \), we have
   \( \lim_{n \to \infty} |x_{i_n}, y_{i_n}| = 0 \).

Proof. 1. From Lemma 3.19, we derive \( e_\sigma(u_*, u'_*) = \lim_{n \to \infty} \sum_{i=1}^{n} |x_{i-1}, x_i| \); moreover, \( e_\sigma(u_*, u'_*) = \lim_{n \to \infty} \sum_{i=1}^{n} |y_{i-1}, y_i| \). Indeed, we have

\[
e_\sigma(u_*, u'_*) = \lim_{n \to \infty} \sum_{i=1}^{n} |x_{i-1}, x_i| \leq \lim \inf_{n \to \infty} \sum_{i=1}^{n} |y_{i-1}, y_i|
\]

\[
\leq \lim \sup_{n \to \infty} \sum_{i=1}^{n} |y_{i-1}, y_i| \leq \lim \sup_{n \to \infty} \sum_{i=1}^{n} e_\sigma(y_{i-1}, y_i) = e_\sigma(u_*, u'_*).
\]

2. For \( n \in \mathbb{N} \), let \( P_n = (\bigcup_{i=1}^{n} x_{i-1}, x_i) \cup (\bigcup_{i=1}^{n} y_{i-1}, y_i) \) be the polygon in \( X \). In the model surface \( M_\kappa^2 \), we construct a comparison \((n+1)-gon \) \( \tilde{P}_n \) for \( P_n \) as follows: Let \( \triangle \hat{x}_i \hat{y}_1 \) and \( \triangle \hat{x}_{n-1} \hat{y}_n \) be comparison triangles in \( M_\kappa^2 \) for \( \triangle x_{i-1} x_i y_1 \) and \( \triangle x_{n-1} y_n \), respectively. For each \( i \in \{1, \ldots, n-1\} \), take comparison triangles \( \triangle \hat{x}_i \hat{x}_{i+1} \hat{y}_i \) and \( \triangle \hat{x}_{i-1} \hat{x}_i \hat{y}_{i+1} \) in \( M_\kappa^2 \) for \( \triangle x_{i-1} x_i y_{i+1} \) and \( \triangle x_{i-1} x_i y_{i+1} \), respectively, and then glue all the comparison triangles in \( M_\kappa^2 \) along \( \hat{x}_i \hat{y}_i \), and along \( \hat{x}_{i+1} \hat{y}_{i+1} \), for all \( i \in \{1, \ldots, n-1\} \). Then we put \( \hat{P}_n := (\bigcup_{i=1}^{n} \hat{x}_{i-1}, \hat{x}_i) \cup (\bigcup_{i=1}^{n} \hat{y}_{i-1}, \hat{y}_i) \). From Lemma A.3 it follows that for each \( i \in \{1, \ldots, n-1\} \) the inner angle at \( \hat{x}_i \) in \( \hat{P}_n \) is at least \( \pi \). By stretching the polygon \( \hat{P}_n \) at the concave vertices, we obtain an \((n+1)-gon \) \( \hat{P}_n = \hat{x} \hat{x}' \cup (\bigcup_{i=1}^{n} \hat{y}_{i-1}, \hat{y}_i) \) in \( M_\kappa^2 \) whose side-lengths satisfy \( |\hat{x}, \hat{x}'| = e_\sigma^{\kappa}(u_*, u'_*) \) and \( |\hat{y}_{i-1}, \hat{y}_i| = |y_{i-1}, y_i| \) for all \( i \in \{1, \ldots, n\} \). Let \( \hat{x}_i \in \hat{x} \hat{x}' \), \( i \in \{1, \ldots, n-1\} \), be the points corresponding to \( \hat{x}_i \). The Alexandrov stretching lemma (see e.g., [11], Lemma I.2.16) leads to that \( |x_i, y_i| \leq |\hat{x}_{i}, \hat{y}_i| \) for all \( i \in \{1, \ldots, n-1\} \).

Suppose that the second half of the sublemma is false. Then we find \( s \in (p_1(u), p_1(u')) \), and a sequence \( \{s_{i_n}\}_{n \in \mathbb{N}} \) converging to \( s \) such that
for all \( n \in \mathbb{N} \) we have \( s_{i_n} \in \Delta_n \), and we have \( |x_{i_n}, y_{i_n}| \geq C \) for some \( C > 0 \). Then for the points \( \bar{x}_{i_n}, \bar{y}_{i_n} \) on the comparison \((n + 1)\)-gon \( \bar{P}_n \) for \( P_n \) we have
\[
C \leq \liminf_{n \to \infty} |x_{i_n}, y_{i_n}| \leq \liminf_{n \to \infty} |\bar{x}_{i_n}, \bar{y}_{i_n}|.
\]
On the other hand, since \( e_\sigma(u_* u') = \lim_{n \to \infty} \sum_{i=1}^{n} |x_{i-1}, x_i| \), and since \( e_\sigma(u_* u') = \lim_{n \to \infty} \sum_{i=1}^{n} |y_{i-1}, y_i| \), the comparison \((n + 1)\)-gon \( \bar{P}_n \) degenerates in \( M^2 \) as \( n \to \infty \). This yields a contradiction. \( \square \)

**Proof of Lemma A.7** Without loss of generality, we may assume that
\[
p_1(u) \leq p_1(v) \leq p_1(w).
\]
For each \( n \in \mathbb{N} \), let us choose a decomposition \( \Delta_n = \{s_i\}_{i=0,1,\ldots,n} \) of \([p_{1}(u), p_{1}(v)]\) with \( \lim_{n \to \infty} |\Delta_n| = 0 \) such that \( p_{1}(v) = s_{m} \) for some \( m \in \{1, \ldots, n-1\} \). Let \( \Delta'_n := \{s_i\}_{i=0,1,\ldots,m} \) be the decomposition of \([p_{1}(u), p_{1}(v)]\), and let \( \Delta''_n := \{s_{m+i}\}_{i=0,1,\ldots,n-m} \) be the decomposition of \([p_{1}(v), p_{1}(w)]\). Set \( x := \sigma(u), y := \sigma(v), z := \sigma(w) \) and take the (unique) \( \Delta'_n \)-minimizing chain \( \{y_i\}_{i=0,1,\ldots,m} \) along \( S \) from \( x \) to \( y \), and the \( \Delta''_n \)-minimizing chain \( \{y_{m+i}\}_{i=0,1,\ldots,n-m} \) along \( S \) from \( y \) to \( z \), and the \( \Delta_n \)-minimizing chain \( \{z_i\}_{i=n-1,\ldots,1} \) along \( S \) from \( z \) to \( x \).

Let \( P^{\Delta_n}(xyz) \) be the \( \Delta_n \)-minimizing chain triple along \( S \) defined by
\[
P^{\Delta_n}(xyz) := B^{\Delta_n}(xy) \cup B^{\Delta_n}(yz) \cup B^{\Delta_n}(zx),
\]
where \( B^{\Delta_n}(xy) \) is the broken geodesic \( \bigcup_{i=1}^{m} y_{i-1}y_i \) in \( X \) joining \( x \) and \( y \), \( B^{\Delta_n}(yz) \) is the broken geodesic \( \bigcup_{i=1}^{m} y_{m+i-1}y_{m+i} \) in \( X \) joining \( y \) and \( z \), and \( B^{\Delta_n}(zx) \) is the broken geodesic \( \bigcup_{i=1}^{n-m} z_{i-1}z_i \) in \( X \) joining \( z \) and \( x \). Set \( x' := \sigma(u'), y' := \sigma(v') \) and \( z' := \sigma(w') \). By Sublemma A.8, we can take sequences \( \{y_{j_n}\}_{n \in \mathbb{N}}, \{y_{k_n}\}_{n \in \mathbb{N}}, \{z_{l_n}\}_{n \in \mathbb{N}} \) of broken points on \( P^{\Delta_n}(xyz) \setminus \{x, y, z\} \) satisfying
\[
\lim_{n \to \infty} |y_{j_n}, z'| = 0, \quad \lim_{n \to \infty} |y_{k_n}, x'| = 0, \quad \lim_{n \to \infty} |z_{l_n}, y'| = 0,
\]
where \( j_n \in \{1, \ldots, m-1\}, k_n \in \{m+1, \ldots, n-1\}, l_n \in \{1, \ldots, n-1\} \).

Let \( \tilde{P}^{\Delta_n}(xyz) = \Delta\tilde{x}\tilde{y}\tilde{z} \) be a comparison \( \Delta_n \)-minimizing stretched triangle in \( M^2 \) for \( P^{\Delta_n}(xyz) \) whose side-lengths satisfy
\[
|\tilde{x}, \tilde{y}| = e^{\Delta_n}_\sigma(u_*, v_*) = e^{\Delta''_n}(v_*, w_*), \quad |\tilde{y}, \tilde{z}| = e^{\Delta'_n}(u_*, w_*), \quad |\tilde{z}, \tilde{x}| = e^{\Delta_n}(w_*, u_*).
\]
Set \( \tilde{d}_n^\Delta(y, z) := \angle \tilde{x}\tilde{y}\tilde{z} \), \( \tilde{d}_y^\Delta(z, x) := \angle \tilde{z}\tilde{y}\tilde{x} \), and \( \tilde{d}_z^\Delta(x, y) := \angle \tilde{x}\tilde{z}\tilde{y} \). Choose the three \( \Delta_n \)-minimizing chain triples \( P^{\Delta_n}(xy_1z_1), P^{\Delta_n}(y_1y_2y_3), \) and \( P^{\Delta_n}(z_1y_1z_2) \) along \( S \), and take comparison \( \Delta_n \)-minimizing chain triangles \( \tilde{P}^{\Delta_n}(xy_1z_1), \tilde{P}^{\Delta_n}(y_1y_2y_3), \) and \( \tilde{P}^{\Delta_n}(z_1y_1z_2) \) in \( M^2 \) for \( P^{\Delta_n}(xy_1z_1), P^{\Delta_n}(y_1y_2y_3), \) and \( P^{\Delta_n}(z_1y_1z_2) \), respectively. As shown in Lemma A.5, we have the monotonicity \( \tilde{d}_n^\Delta(y_1, z_1) \leq \tilde{d}_n^\Delta(x, y), \tilde{d}_n^\Delta(x, y) \leq \tilde{d}_n^\Delta(z, x), \) and \( \tilde{d}_n^\Delta(z_1, y_1) \leq \tilde{d}_n^\Delta(x, y) \). From the choices of the sequences \( \{y_{j_n}\}_{n \in \mathbb{N}}, \{y_{k_n}\}_{n \in \mathbb{N}}, \) and \( \{z_{l_n}\}_{n \in \mathbb{N}} \), it follows that \( \tilde{P}^{\Delta_n}(xy_{j_n}z_{l_n}), \tilde{P}^{\Delta_n}(y_{j_n}y_{k_n}z_{l_n}), \) and \( \tilde{P}^{\Delta_n}(z_{l_n}y_{j_n}z_{l_n}) \) converge to comparison triangles in \( M^2 \) for triangles \( \Delta u_* u'_* v'_*, \Delta u'_* v_* u'_* \), and \( \Delta v'_* u'_* w_* \) in \( (R_*, e_\sigma) \), respectively.
\( \tilde{P}^\Delta_n(xyz) \) converges to a comparison triangle in \( M_\kappa^2 \) for the triangle \( \hat{\triangle}u_*,v_*,w_* \). Therefore we obtain

\[
\tilde{\theta}_{u_*}(w'_*,v'_*) = \lim_{n \to \infty} \tilde{\theta}_x^\Delta_n(y_{jn},z_{ln}) \leq \lim_{n \to \infty} \tilde{\theta}_x^\Delta_n(y,z) = \tilde{\theta}_{u_*}(v_*,w_*).
\]

Similarly, we see \( \tilde{\theta}_{v_*}(u'_*,w'_*) \leq \tilde{\theta}_{v_*}(w_*,u_*) \), and \( \tilde{\theta}_{w_*}(v'_*,u'_*) \leq \tilde{\theta}_{w_*}(u_*,v_*) \).

Thus \( \hat{\triangle}u_*,v_*,w_* \) satisfies the convexity of angle \( \kappa \)-comparison. \( \square \)

From Lemma A.7, we conclude that \((R_*,e_\sigma)\) is a CAT(\( \kappa \))-space. This completes the proof of Theorem 3.17. \( \square \)

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