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Phase Retrieval via Randomized Kaczmarz: Theoretical Guarantees

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Abstract

We consider the problem of phase retrieval, i.e. that of solving systems of quadratic equations. A simple variant of randomized Kaczmarz method was recently proposed for phase retrieval, and it was shown numerically to have a computational edge over state-of-the-art Wirtinger flow methods. In this paper, we provide the first theoretical guarantee for the convergence of the randomized Kaczmarz method for phase retrieval. We show that it is sufficient to have as many gaussian measurements as the dimension, up to a constant factor. Along the way, we introduce a sufficient condition on measurement sets for which the randomized Kaczmarz method is guaranteed to work. We show that gaussian measurements satisfy this property with high probability; this is proved using a chaining argument coupled with bounds on VC dimension and metric entropy.

1 Introduction

The phase retrieval problem is that of solving a system of quadratic equations

\[ |\langle a_i, z \rangle|^2 = b_i^2, \quad i = 1, 2, \ldots, m \]

where \( a_i \in \mathbb{R}^n \) (or \( \mathbb{C}^n \)) are known sampling vectors, \( b_i > 0 \) are observed measurements, and \( z \in \mathbb{R}^n \) (or \( \mathbb{C}^n \)) is the decision variable. This problem is well motivated by practical concerns [8] and has been a topic of study from at least the early 1980s. Over the last half a decade, there has been great interest in constructing and analyzing algorithms with provable guarantees given certain classes of sampling vector sets. One line of research involves “lifting” the quadratic system to a linear system, which is then solved using convex relaxation. This is the strategy of the well-known PhaseLift [4] and PhaseMax [9] algorithms. Another popular line of research has been to use methods from non-convex optimization. Recent work on Wirtinger Flow [3], Truncated Wirtinger Flow [5], and other variants fall into this category.

In 2015, Wei [17] proposed adapting a family of randomized Kaczmarz methods for solving the phase retrieval problem. He was able to show using numerical experiments that these methods perform comparably with state-of-the-art Wirtinger flow methods when the sampling vectors are real or complex gaussian, or when they follow the coded diffraction pattern (CDP) model [3]. He also showed that randomized Kaczmarz methods outperform Wirtinger flow when the sampling vectors are the concatenation of a few unitary bases.
Unfortunately, [17] was not able to provide adequate theoretical justification for the convergence of these methods (see Theorem 2.6 in [17]). In this paper, we attempt to bridge this gap by showing that the basic randomized Kaczmarz scheme used in conjunction with truncated spectral initialization achieves \textit{linear convergence} to the solution with high probability, whenever the sampling vectors are drawn uniformly from the sphere\(^1\) \(S^{n-1}\) and the number of measurements \(m\) is larger than a constant times the dimension \(n\).

We have restricted ourselves to real measurements partly for the sake of expositional clarity. It should not be too difficult to generalize the arguments in this paper to the case of complex gaussian measurements, and perhaps also to other classes of sampling vectors.

1.1 \textbf{Randomized Kaczmarz for solving linear systems}

The Kaczmarz method is a fast iterative method for solving systems of overdetermined linear equations that works by iteratively satisfying one equation at a time. In 2009, Strohmer and Vershynin [11] were able to give a provable guarantee on its rate of convergence, provided that the equation to be satisfied at each step is selected using a prescribed randomized scheme.

Suppose our system to be solved is given by

\[ Ax = b, \]  

where \( A \) is an \( m \times n \) matrix. Denoting the rows of \( A \) by \( a_1^T, \ldots, a_m^T \), we can write (1.2) as the system of linear equations

\[ \langle a_i, x \rangle = b_i, \quad i = 1, \ldots, m \]

The solution set of each equation is a hyperplane. The randomized Kacmarz method is a simple iterative algorithm in which we \textit{project the running approximation onto the hyperplane of a randomly chosen equation}. More formally, at each step \( k \) we randomly choose an index \( r(k) \) from \([m]\) such that the probability that \( r(k) = i \) is proportional to \( \|a_i\|^2 \), and update the running approximation as follows:

\[ x_k := x_{k-1} + \frac{b_{r(k)} - \langle a_{r(k)}, x_{k-1} \rangle}{\|a_{r(k)}\|^2} a_{r(k)}. \]  

(1.3)

Strohmer and Vershynin [11] were able to prove the following theorem:

\textbf{Theorem 1.1 (Linear convergence for linear systems).} Let \( \kappa(A) = \|A\|_F/\sigma_{\min}(A) \). Then for any initialization \( x_0 \) to the equation (1.2), the estimates given to us by randomized Kaczmarz satisfy

\[ \mathbb{E}\|x_k - x\|^2_2 \leq (1 - \kappa(A)^{-2})^k \|x_0 - x\|^2_2. \]  

(1.4)

Note that if \( A \) has bounded condition number, then \( \kappa(A) \asymp 1/n \).

1.2 \textbf{Randomized Kaczmarz for phase retrieval}

In the phase retrieval problem (1.1), each equation

\[ |\langle a_i, x \rangle| = b_i \]

\( ^1\)This is essentially equivalent to being real gaussian because of the concentration of norm phenomenon in high dimensions. Also, one may normalize vectors easily.
defines two hyperplanes, one corresponding to each of \( \pm x \). A natural adaptation of the randomized Kaczmarz update for this situation is then to *project the running approximation to the closer hyperplane*. We restrict to the case where each measurement vector \( a_i \) has unit norm, so that in equations, this is given by

\[
x_k := x_{k-1} + \eta_k a_r(k),
\]

(1.5)

where

\[
\eta_k = \text{sign}(\langle a_r(k), x_{k-1} \rangle) b_r(k) - \langle a_r(k), x_{k-1} \rangle).
\]

(1.6)

Unlike the case for linear systems where we could start with an arbitrary initial estimate \( x_0 \in \mathbb{R}^n \), in order to obtain a theoretical guarantee, we need to choose \( x_0 \) so that it is close enough to the ground truth \( x \). One easy method to do this is to use spectral initialization [3]. Here, we set \( x_0 \) to be the leading eigenvector for the matrix \( \frac{1}{m} \sum_{i=1}^m b_i^2 a_i a_i^T \).

It is easy to see why this would work by considering the expectation of this matrix. Nonetheless, the random vectors \( b_i a_i \) have heavy tails, which causes \( \frac{1}{m} \sum_{i=1}^m b_i^2 a_i a_i^T \) to have poor convergence properties. We obtain a better sample complexity by using *truncated spectral initialization* [5], in which we instead take the leading eigenvector of the matrix obtained by only summing terms for which \( b_i \) is beneath some threshold.

We can thus state our algorithm as follows:

**Algorithm 1** Randomized Kaczmarz for Phase Retrieval

**Input:** Measurements \( b_1, \ldots, b_m \), measurement vectors \( a_1, \ldots, a_m \), number of iterations \( K \).

**Output:** An estimate \( x_K \) for \( x \).

1: Initialize \( x_0 := \lambda_0 \tilde{x}_0 \), where \( \lambda_0 = \sqrt{\frac{1}{m} \sum_{i=1}^m b_i^2} \), and \( \tilde{x}_0 \) is the leading eigenvector of

\[
Y = \frac{1}{m} \sum_{i=1}^m b_i^2 a_i a_i^T 1(b_i \leq 3\lambda_0).
\]

(1.7)

2: For \( k = 1, 2, \ldots, K \), update \( x_k \) according to the rule (1.5).

### 1.3 Contributions and main result

The main result of our paper guarantees the linear convergence of randomized Kacmarz algorithm for phase retrieval for random gaussian measurements \( a_i \). Note that there are two sources of randomness here: one is in the creation of the measurements \( a_i \), and the other is in the selection of the equation at every iteration of the algorithm. The following theorem gives a guarantee that holds with high probability over both sources of randomness.

**Theorem 1.2** (Guarantee for Algorithm 1). Let \( \delta_1, \delta_2 > 0 \). There is an absolute constant \( C > 0 \) such that if \( m \geq C(n + \log(1/\delta_1))/\delta_2 \), then with probability at least \( 1 - \delta_1 \), \( m \) measurement vectors selected uniformly and independently from the unit sphere \( S^{n-1} \) form a set such that the following holds: For any \( \epsilon > 0 \), \( K \geq 2(\log(1/\epsilon) + \log(2/\delta_2))n \), and any vector \( x \in \mathbb{R}^n \) such that \( \|x\|_2 \geq f(m) \) where \( f \) is a decreasing function in \( m \), running Algorithm 1 with \( K \) update steps yields an estimate \( x_K \) for \( x \) satisfying \( \|x_K - x\|_2^2 \leq \epsilon \|x\|_2^2 \) with probability at least \( 1 - \delta_1 - \delta_2 \).
The proof of this theorem is given in Section 6. The argument is more nontrivial than Strohmer-Vershynin analysis of randomized Kacmarz algorithm for linear systems [11]. We broke down the argument in smaller steps, each of which may be of independent interest to researchers in this field.

First, we generalize the Kaczmarz update formula (1.5) and define what it means to take a randomized Kaczmarz step with respect to any probability measure on the sphere $S^{n-1}$: we choose a measurement vector at each step according to this measure. Using a simple geometric argument, we then provide a bound for the expected decrement in distance to the solution set in a single step, where the quality of the bound is given in terms of the properties of the measure we are using for the Kaczmarz update (Lemma 2.1).

Performing the generalized Kaczmarz update with respect to the uniform measure on the sphere corresponds to running Algorithm 1 with unlimited measurements. We utilize the symmetry of the uniform measure to compute an explicit formula for the bound on the stepwise expected decrement in distance. This decrement is geometric whenever we make the update from a point making an angle of less than $\pi/8$ with the true solution, so we obtain linear convergence expected decrement conditional on no iterates escaping from the “basin of linear convergence”. We are able to bound the probability of this bad event using a supermartingale inequality (Theorem 3.1).

Next, we abstract out the property of the uniform measure that allows us to obtain local linear convergence. We call this property the anti-concentration on wedges property, calling it ACW for short. Using this convenient definition, we can easily generalize our previous proofs for the uniform measure to show that all ACW measures give rise to randomized Kaczmarz update schemes with local linear convergence (Theorem 4.3).

The usual Kaczmarz update in Algorithm 1 corresponds running the generalized Kaczmarz update with respect to $\mu_A := \frac{1}{m} \sum_{i=1}^{m} \delta_{a_i}$. We are able to prove that when the $a_i$’s are selected uniformly and independently from the sphere, then $\mu_A$ satisfies the ACW condition with high probability, so long as $m$ is a constant times $n$ (Theorem 5.12). The proof of this fact uses an unusual chaining argument together with metric entropy estimates.

Finally, we are able to put everything together to prove a guarantee for the full algorithm.

1.4 Notation

Throughout the paper, $C$ and $c$ are absolute constants that can change from line to line.

2 Computations for a single step

In this section, we will compute what happens in expectation for a single update step of the randomized Kaczmarz. It will be convenient to generalize our sampling scheme slightly as follows. When we work with a fixed matrix $A$, we may view our selection of a random row $a_r(k)$ as drawing a random vector according to the measure $\mu_A := \frac{1}{m} \sum_{i=1}^{m} \delta_{a_i}$. We need not restrict ourselves to sums of Diracs. For any probability measure $\mu$ on the sphere $S^{n-1}$, we define the random map $P = P_\mu$ on vectors $z \in \mathbb{R}^n$ by setting

$$Pz := z + \eta a,$$

where

$$\eta = \text{sign}(\langle a, z \rangle) |\langle a, x \rangle| - \langle a, z \rangle \quad \text{and} \quad a \sim \mu.$$ 

Note that as before, $x$ is a fixed vector in $\mathbb{R}^n$ (think of $x$ as the actual solution of the phase retrieval problem). We call $P_\mu$ the generalized Kaczmarz projection with respect to $\mu$. Using this update rule
over independent realizations of $P$, $P_1, P_2, \ldots$, together with an initial estimate $x_0$, gives rise to a generalized randomized Kaczmarz algorithm for finding $x$: set the $k$-th step estimate to be

$$x_k := P_k P_{k-1} \cdots P_1 x_0.$$  \hfill (2.3)

Fix a vector $z \in \mathbb{R}^n$ that is closer to $x$ than to $-x$, i.e. so that $\langle x, z \rangle > 0$, and suppose that we are trying to find $x$. Examining the formula in (2.2), we see that $P$ projects $z$ onto the right hyperplane (i.e., the one passing through $x$ instead of the one passing through $-x$) if and only if $\langle a, z \rangle$ and $\langle a, x \rangle$ have the same sign. In other words, this occurs if and only if the random vector $a$ does not fall into the region of the sphere defined by

$$W_{x,z} := \{ v \in S^{n-1} \mid \text{sign}(\langle v, x \rangle) \neq \text{sign}(\langle v, z \rangle) \}. \quad (2.4)$$

This is the region lying between the two hemispheres with normal vectors $x$ and $z$. We call such a region a spherical wedge, since in three dimensions it has the shape depicted in Figure 1.

![Figure 1: Geometry of $W_{x,z}$](image)

When $a \notin W_{x,z}$, we can use the Pythagorean theorem to write

$$\|z - x\|^2 = \|Pz - x\|^2 + \langle z - x, a \rangle^2. \quad (2.5)$$

Rearranging gives

$$\|Pz - x\|^2 = \|z - x\|^2 (1 - \langle \tilde{z}, a \rangle^2), \quad (2.6)$$

where $\tilde{z} = (z - x)/\|z - x\|^2$.

In the complement of this event, we get

$$Pz = z + \langle a, (-x) - z \rangle a = z - \langle a, z - x \rangle + \langle a, -2x \rangle, \quad (2.7)$$

and using orthogonality,

$$\|Pz - x\|^2 = \|z - x\|^2 - \langle a, z - x \rangle^2 + \langle a, 2x \rangle^2. \quad (2.8)$$

Since $z$ gets projected to the hyperplane containing $-x$, it may move further away from $x$. However, we can bound how far away it can move. Because $\langle a, x \rangle$ has the opposite sign as $\langle a, z \rangle$, we have

$$|\langle a, z + x \rangle| < |\langle a, z - x \rangle|, \quad (2.9)$$

and so

$$|\langle a, 2x \rangle| = |\langle a, (z - x) - (z + x) \rangle| < 2|\langle a, z - x \rangle|. \quad (2.10)$$
Substituting this into (2.8), we get the bound
\[
\|Pz - x\|_2^2 \leq \|z - x\|_2^2 + 3\langle a, z - x \rangle^2 = \|z - x\|_2^2(1 + 3\langle \tilde{z}, a \rangle^2),
\] (2.11)
where \(\tilde{z}\) is as before.

We can combine (2.6) and (2.11) into a single inequality by writing
\[
\|Pz - x\|_2^2 \leq \|z - x\|_2^2(1 - \langle \tilde{z}, (1 - 4 \cdot 1_{W_{x,z}}(a))aa^T\tilde{z} \rangle) + \|z - x\|_2^2(1 + 3\langle \tilde{z}, a \rangle^2)\](2.12)
\[
\|z - x\|_2^2(1 - \langle \tilde{z}, (1 - 4 \cdot 1_{W_{x,z}}(a))aa^T\tilde{z} \rangle).
\] (2.13)

Taking expectations, we can remove the role that \(\tilde{z}\) plays by bounding this as follows.
\[
E[\|z - x\|_2^2(1 - \langle \tilde{z}, (1 - 4 \cdot 1_{W_{x,z}}(a))aa^T\tilde{z} \rangle)] = \|z - x\|_2^2(1 - \langle \tilde{z}, E[(1 - 4 \cdot 1_{W_{x,z}}(a))aa^T]\tilde{z} \rangle) \leq \|z - x\|_2^2[1 - \lambda_{\text{min}}(Eaa^T - 4Eaa^T1_{W_{x,z}}(a))] \leq \|z - x\|_2^2[1 - \lambda_{\text{min}}(Eaa^T - 4Eaa^T1_{W_{x,z}}(a))] \leq \|z - x\|_2^2[1 - \lambda_{\text{min}}(Eaa^T - 4Eaa^T1_{W_{x,z}}(a))].
\] (2.15)

We may thus summarize what we have obtained in the following lemma.

**Lemma 2.1** (Expected decrement). Fix vectors \(x, z \in \mathbb{R}^n\), a probability measure \(\mu\) on \(S^{n-1}\), and let \(P = P_\mu\), \(W_{x,z}\) be defined as in (2.1) and (2.4) respectively. Then
\[
E\|Pz - x\|_2^2 \leq [1 - \lambda_{\text{min}}(Eaa^T - 4Eaa^T1_{W_{x,z}}(a))] \|z - x\|_2^2. \quad (2.17)
\]
Let $\phi$ denote the anti-clockwise angle of $Qa$ from $e_2$. We may write
\[
\langle a, e_1 \rangle^2 = \|Qa\|_2^2 (\|Qa\|_2, e_1)^2 = \|Qa\|_2^2 \sin^2 \phi. \tag{2.18}
\]

Note that the magnitude and direction of $Qa$ are independent, and $a \in W_{x,z}$ if either $\phi$ or $\phi - \pi$ lies between $-\theta/2$ and $\theta/2$. We therefore have
\[
M_{11} = \mathbb{E}\langle a, e_1 \rangle^2 1_{W_{x,z}}(a) = \mathbb{E}\|Qa\|_2^2 \mathbb{E}\sin^2 \phi 1_{(-\theta/2, \theta/2)}(\phi \text{ or } \phi - \pi). \tag{2.19}
\]

By a standard calculation using symmetry, we have
\[
\mathbb{E}\|Qa\|_2^2 = \frac{2}{n}. \quad \text{Since } \phi \text{ is distributed uniformly on the circle, we can compute }
\]
\[
\mathbb{E}\sin^2 \phi 1_{(-\theta/2, \theta/2)}(\phi \text{ or } \phi - \pi) = \frac{1}{\pi} \int_{-\theta/2}^{\theta/2} \sin^2 t dt = \frac{1}{\pi} \int_{-\theta/2}^{\theta/2} \frac{1 - \cos(2t)}{2} dt = \frac{\theta - \sin \theta}{2\pi}. \tag{2.20}
\]

As such, we have $M_{11} = (\theta - \sin \theta)/n\pi$, and by a similar calculation, $M_{22} = (\theta + \sin \theta)/n\pi$, and $M_{ii} = \theta/n\pi$ for $i \geq 3$. This implies that
\[
\lambda_{\max}(M_\theta) = \frac{\theta + \sin \theta}{n\pi}. \tag{2.21}
\]

We have now completed proving the following lemma.

**Lemma 2.2 (Expected decrement for uniform measure).** Fix vectors $x, z \in \mathbb{R}^n$ such that $\langle z, x \rangle > 0$, and let $P = P_\sigma$ denote the generalized Kaczmarz projection with respect to $\sigma$, the uniform measure on the sphere. Let $\theta$ be the angle between $z$ and $x$. Then
\[
\mathbb{E}\|Pz - x\|_2^2 \leq \left[1 - \frac{1 - 4(\theta + \sin \theta)/n}{n}\right] \|z - x\|_2^2. \tag{2.22}
\]

**Remark 2.3.** By being more careful, one may compute an exact formula for the expected decrement rather than a bound as is the case in previous lemma. This is not necessary for our purposes and does not give better guarantees in our analysis, so the computation is omitted.

## 3 Local linear convergence using unlimited uniform measurements

In this section, we will show that if we start with an initial estimate that is close enough to the ground truth $x$, then repeatedly applying generalized Kaczmarz projections with respect to the uniform measure $\sigma$ gives linear convergence in expectation. This is exactly the situation we would be in if we were to run randomized Kaczmarz given an unlimited supply of independent measurements $a_1, a_2, \ldots$ drawn uniformly from the sphere.

We would like to imitate the proof for linear convergence of randomized Kaczmarz for linear systems (Theorem 1.1) given in [11]. We denote by $X_k$ the estimate at step $k$, using capital letters to emphasize the fact that it is a random variable. If we know that $X_k$ takes the value $x_k \in \mathbb{R}^n$, and the angle $\theta_k$ that $z$ makes with $x_k$ is smaller than $\pi/8$, then, Lemma 2.2 tells us
\[
\mathbb{E}\|X_{k+1} - x\|_2^2 \mid X_k = x_k \leq (1 - \kappa_\sigma/n)\|x_k - x\|_2^2, \tag{3.1}
\]
where $\kappa_\sigma := 1/2 - 4\sin(\pi/8)/\pi > 0$. 

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The proof for Theorem 1.1 proceeds by unconditioning and iterating the bound for \( \mathbb{E}[\|X_k - x\|_2^2 \mid X_{k-1} = x_{k-1}] \). Unfortunately, our bound depends on \( x_k \) being in a specific region in \( \mathbb{R}^n \) and does not hold arbitrarily. In other words, instead of linear convergence, we have conditional linear convergence.

Let \( B \subset S^{n-1} \) be the region on the sphere comprising all points making an angle less than or equal to \( \pi/8 \) with \( x \). This is our basin of linear convergence. Let us assume a fixed initial estimate \( x_0 \) and condition on the event

\[
\Omega = \Omega(x, x_0) := \{ \text{no iterate } x_k \text{ leaves } B \}.
\]

It is easy to see that \( \|X_{k+1} - x\|_2^2 \) is negatively correlated with \( \Omega \), conditioned on \( X_k \). As such, for each positive integer \( k \), we have

\[
\mathbb{E}[\|X_{k+1} - x\|_2^2 \mid X_k = x_k, \Omega] \leq \mathbb{E}[\|X_{k+1} - x\|_2^2 \mid X_k = x_k] \leq (1 - \kappa_\sigma/n)\|x_k - x\|_2^2.
\]

Since the above bound holds for any possible value \( x_k \) that \( X_k \) can take conditioned on \( \Omega \), we may uncondition with respect to \( X_k \) to get

\[
\mathbb{E}[\|X_{k+1} - x\|_2^2 \mid \Omega] = \mathbb{E}\mathbb{E}[\|X_{k+1} - x\|_2^2 \mid X_k, \Omega] \mid \Omega] \leq (1 - \kappa_\sigma/n)\mathbb{E}[\|X_k - x\|_2^2 \mid \Omega].
\]

By induction, we therefore obtain

\[
\mathbb{E}[\|X_k - x\|_2^2 \mid \Omega] \leq (1 - \kappa_\sigma/n)^k\|x_0 - x\|_2^2.
\]

It remains to show that the good event \( \Omega \) happens with high probability, assuming that \( x_0 \) is close enough to \( x \). One may do so by combining a stopping time argument with a supermartingale maximal inequality. Proceeding in this manner, we get the following theorem.

**Theorem 3.1** (Linear convergence from unlimited measurements). Let \( x \) be a vector in \( \mathbb{R}^n \), let \( \delta > 0 \), and let \( x_0 \) be an initial estimate to \( x \) such that \( \|x_0 - x\|_2 \leq \delta \|x\|_2 \). Suppose that our measurements \( a_1, a_2, \ldots \) are fully independent random vectors distributed uniformly on the sphere \( S^{n-1} \). Let \( \Omega \) be the event that for every \( k \in \mathbb{Z}_+ \), \( X_k \), the \( k \)-th step of the randomized Kaczmarz method using the update rule (2.1) makes an angle less than \( \pi/8 \) with \( x \). Then for every \( k \in \mathbb{Z}_+ \),

\[
\mathbb{E}[\|X_k - x\|_2^2 \mid \Omega] \leq (1 - \kappa_\sigma/n)^k\|x_0 - x\|_2^2,
\]

where \( \kappa_\sigma = 1/2 - 4\sin(\pi/8)/\pi > 0 \). Furthermore, \( \mathbb{P}(\Omega^c) \leq (\delta/\sin(\pi/8))^2 \).

**Proof.** Let \( \{\mathcal{F}_k\} \) denote the filtration defined by the \( a_k \)'s. We let \( \theta_k \) denote the angle that \( X_k \) makes with \( x \). We define a stopping time \( \tau \), setting it to be the earliest time \( k \) for which \( \theta_k \geq \pi/8 \). Now set \( Y_k := \|X_{\tau \wedge k} - x\|_2^2 \). We claim that \( Y_k \) is a supermartingale. To see this, we break up its conditional expectation as follows:

\[
\mathbb{E}[Y_{k+1} \mid \mathcal{F}_k] = \mathbb{E}[\|X_{\tau \wedge (k+1)} - x\|_2^2 \mid \mathcal{F}_k] + \mathbb{E}[\|X_{\tau \wedge (k+1)} - x\|_2^2 \mid \mathcal{F}_k] \mathcal{F}_k] = \mathbb{E}[\|X_{\tau \wedge k} - x\|_2^2 \mid \mathcal{F}_k] + \mathbb{E}[\|X_{k+1} - x\|_2^2 \mathcal{F}_k] \mathcal{F}_k].
\]

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Since $\|X_{\tau \wedge k} - x\|^2_2$ is measurable with respect to $F_k$, we get
\[
\mathbb{E}[\|X_{\tau \wedge k} - x\|^2_1 \mid F_k] = \|X_{\tau \wedge k} - x\|^2_1 \mathbb{1}_{\tau \leq k} = Y_k \mathbb{1}_{\tau \leq k}.
\] (3.10)

Meanwhile, on the event $\tau > k$, we have $X_k \in B$, so we may use (3.1) to obtain
\[
\mathbb{E}[\|X_{k+1} - x\|^2_1 \mid F_k] = \mathbb{E}[\|X_{k+1} - x\|^2_1 \mid F_k] \mathbb{1}_{\tau > k} \leq (1 - \kappa_\sigma/n)\|X_k - x\|^2_1 \mathbb{1}_{\tau > k}.
\] (3.11)

Next, notice that
\[
\|X_k - x\|^2_1 \mathbb{1}_{\tau > k} = \|X_{\tau \wedge k} - x\|^2_1 \mathbb{1}_{\tau > k} = Y_k \mathbb{1}_{\tau > k}.
\] (3.12)

Combining these calculations gives
\[
\mathbb{E}[Y_{k+1} \mid F_k] \leq Y_k \mathbb{1}_{\tau \leq k} + (1 - \kappa_\sigma/n)Y_k \mathbb{1}_{\tau > k} \leq Y_k.
\] (3.13)

The bad event $\Omega^c$ corresponds to $\{\tau < \infty\}$. We define a second stopping time $T$ to be the earliest time $k$ such that $\|X_k - x\|^2_2 \geq \sin(\pi/8) \cdot \|x\|_2$. A simple geometric argument tells us that $T \leq \tau$, and that $T$ also satisfies
\[
T = \inf\{k \mid Y_k \geq \sin^2(\pi/8)\|x\|^2_2\}.
\] (3.14)

As such, we have
\[
\mathbb{P}(\tau < \infty) \leq \mathbb{P}(T < \infty) = \mathbb{P}\left(\sup_{1 \leq k < \infty} Y_k \geq \sin^2(\pi/8)\|x\|^2_2\right).
\] (3.15)

Since $(Y_k)$ is a non-negative supermartingale, we may apply the supermartingale maximal inequality to obtain a bound on the right hand side:
\[
\mathbb{P}\left(\sup_{1 \leq k < \infty} Y_k \geq \sin^2(\pi/8)\|x\|^2_2\right) \leq \frac{\mathbb{E} Y_0}{\sin^2(\pi/8)\|x\|^2_2} \leq \frac{(\delta/\sin(\pi/8))^2}{(\delta/\sin(\pi/8))^2}.
\] (3.16)

This completes the proof of the theorem. \qed

**Corollary 3.2.** Fix $\epsilon, \delta > 0$. In the setting of Theorem 3.1, suppose that $\|x_0 - x\|_2 \leq \sqrt{\delta/2} \sin(\pi/8)\|x\|_2$. Then with probability at least $1 - \delta$, if $k \geq (\log(1/\epsilon) + \log(2/\delta)) n/\kappa_\sigma$ then $\|X_k - x\|^2_2 \leq \epsilon\|x_0 - x\|^2_2$.

**Proof.** Use the previous theorem and Markov’s inequality. \qed

### 4 Local linear convergence for ACW($\theta, \kappa$) measures

We would like to extend the analysis in the previous section to the setting where we only have access to finitely many uniform measurements, i.e. when we are back in the situation of (1.1). When we sample uniformly from the rows of $A$, this can be seen as running the generalized randomized Kaczmarz algorithm using the measure $\mu_A = \frac{1}{m} \sum_{i=1}^m \delta_{a_i}$ as opposed to $\mu = \sigma$.

If we retrace our steps, we will see that the key property of the uniform measure $\sigma$ that we used was that if $W \subset S^{n-1}$ is a wedge$^2$ of angle $\theta$, then we could make $\lambda_{\text{max}}(\mathbb{E}_\sigma a a^T 1_W(a))$ arbitrarily

---

$^2$Recall that a wedge of angle $\theta$ is the region of the sphere between two hemispheres with normal vectors making an angle of $\theta$. 

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small by taking $\theta$ small enough (see equation (2.21)). We do not actually need such a strong statement. It suffices for there to be an absolute constant $\kappa$ such that

$$\lambda_{\min}(Eaa^T - 4Eaa^T 1_W(a)) \geq \frac{\kappa}{n}$$

(4.1)

holds for $\theta$ small enough.

**Definition 4.1** (Anti-concentration). If a probability measure $\mu$ on $S^{n-1}$ satisfies (4.1) for all wedges $W$ of angle less than $\theta$, we say that it is anti-concentrated on wedges of angle $\theta$ at level $\kappa$, or for short, that it satisfies the $\text{ACW}(\theta, \kappa)$ condition.

Abusing notation, we say that a measurement matrix $A$ is $\text{ACW}(\theta, \kappa)$ if the uniform measure on its rows is $\text{ACW}(\theta, \kappa)$. Plugging in this definition into Lemma 2.1, we immediately get the following statement.

**Lemma 4.2** (Expected decrement for ACW measure). Let $\mu$ be a probability measure on the sphere $S^{n-1}$ satisfying the $\text{ACS}(\theta, \kappa)$ condition for some $\kappa > 0$ and some acute angle $\theta > 0$. Let $\Omega$ be the event that for every $k \in \mathbb{Z}_+$, $X_k$ makes an angle less than $\theta$ with $x$. Then for any $x, z \in \mathbb{R}^n$ such that the angle between them is less than $\theta$, we have

$$E\|Pz - x\|^2 \leq (1 - \kappa/n)\|z - x\|^2.$$  

(4.2)

We may now imitate the arguments in the previous section to obtain a guarantee for local linear convergence for the generalized randomized Kaczmarz algorithm using such a measure $\mu$.

**Theorem 4.3** (Linear convergence for ACW measure). Suppose $\mu$ is an $\text{ACW}(\theta, \kappa)$ measure. Let $x$ be a vector in $\mathbb{R}^n$, let $\delta > 0$, and let $x_0$ be an initial estimate to $x$ such that $\|x_0 - x\| \leq \delta \|x\|$. Let $X_k\kappa$ denote the k-th step of the generalized randomized Kaczmarz method with respect to the measure $\mu$, defined as in (2.3). Let $\Omega$ be the event that for every $k \in \mathbb{Z}_+$, $X_k$ makes an angle less than $\theta$ with $x$. Then for every $k \in \mathbb{Z}_+$,

$$E[\|X_k - x\|^2 \mid \Omega] \leq (1 - \kappa/n)^k\|x_0 - x\|^2.$$  

(4.3)

Furthermore, $P(\Omega^c) \leq (\delta / \sin \theta)^2$.

**Proof.** We repeat the proof of Theorem 3.1, defining stopping times $\tau_x$ and $T_x$ as the earliest times that $\theta_k \geq \theta$ and $\|X_k - x_0\|_2 \geq \sin(\theta)\|x\|_2$ respectively. Again, $Y_k := X_{k \wedge \tau_x}$ is a supermartingale, so we may use the supermartingale inequality to bound the probability of $\Omega^c$. Conditioned on the event $\Omega$, we may iterate the bound given by Lemma 4.2 to obtain (4.3).

**Corollary 4.4.** Fix $\epsilon, \delta > 0$. In the setting of Theorem 4.3, suppose that $\|x_0-x\|_2 \leq \sqrt{\delta/2 \sin(\theta)}\|x\|_2$. Then with probability at least $1 - \delta$, if $k \geq (\log(1/\epsilon) + \log(2/\delta))n/\kappa$ then $\|X_k - x\|^2 \leq \epsilon \|x_0 - x\|^2$.

### 5 ACW($\theta, \kappa$) condition for finitely many uniform measurements

Following the theory in the previous section, we see that to prove linear convergence from finitely many uniform measurements, it suffices to show that the measurement matrix $A$ is $\text{ACW}(\theta, \kappa)$ for some $\theta$ and $\kappa$. 


For a fixed wedge $W$, we can easily achieve (4.1) by using a standard matrix concentration theorem. By taking a union bound, we can guarantee that it holds over exponentially many wedges with high probability. However, the function $W \mapsto \lambda_{\text{max}}(Eaa^T1_W(a))$ is not Lipschitz with respect to any natural parametization of wedges in $S^{n-1}$, so a naive net argument fails. To get around this, we use VC theory, metric entropy, and a chaining theorem from [7].

First, we will use the theory of VC dimension and growth functions to argue that all wedges contain approximately the right fraction of points. For an introduction to these topics, we will refer the reader to Chapter 3 in [10].

Claim 5.1. Let $C$ be the collection of all hemispheres in $S^{n-1}$. Then the VC dimension of $C$ is bounded from above by $n + 1$.

Proof. It is a standard fact from statistical learning theory [10] that the VC dimension of half-spaces in $\mathbb{R}^n$ is $n + 1$. Since $S^{n-1}$ is a subset of $\mathbb{R}^n$, the claim follows by the definition of VC dimension.

Claim 5.2. Let $C$ and $D$ be two collections of functions from a set $X$ to $\{0, 1\}$. Using $\triangle$ to denote symmetric difference, we define

$$C \triangle D := \{C \triangle D \mid C \in C, D \in D\}.$$ (5.1)

Then the growth function $\Pi_{C \triangle D}$ of $C \triangle D$ satisfies $\Pi_{C \triangle D}(m) \leq \Pi_C(m) \cdot \Pi_D(m)$ for all $m \in \mathbb{Z}_+$.

Proof. Fix $m$, and points $x_1, \ldots, x_m \in X$. Then every possible configuration $(f(x_1), f(x_2), \ldots, f(x_m))$ arising from some $f \in C \triangle D$ is the point-wise symmetric difference

$$(f(x_1), f(x_2), \ldots, f(x_m)) = (C(x_1), C(x_2), \ldots, C(x_m)) \triangle (D(x_1), D(x_2), \ldots, D(x_m))$$ (5.2)

of configurations arising from some $C \in C$ and $D \in D$. By the definition of growth functions, there are at most $\Pi_C(m) \cdot \Pi_D(m)$ pairs of these configurations, from which the bound follows.

Remark 5.3. There is an extensive literature on how to bound the VC dimension of concept classes that arise from finite intersections or unions of those from a known collection of concept classes, each of which has bounded VC dimension. We won’t require this much sophistication here, and refer the reader to [2] for more details.

Lemma 5.4 (Uniform concentration of empirical measure over wedges). Fix an acute angle $\theta > 0$. Let $W_\theta$ denote the collection of all wedges of $S^{n-1}$ of angle less than $\theta$. Suppose $A$ is an $m$ by $n$ matrix with rows $a_i$ that are independent uniform random vectors on $S^{n-1}$, and let $\mu_A = \frac{1}{m} \sum_{i=1}^m \delta_{a_i}$. Then if $m \geq (4\pi/\theta)^2(2n \log(2m/n) + \log(2/\delta))$, with probability at least $1 - \delta$, we have

$$\sup_{W \in W_\theta} \mu_A(W) \leq 2\theta/\pi.$$ (5.3)

Proof. Using VC theory [14], we have

$$\mathbb{P}(\sup_{W \in W_\theta} |\mu_A(W) - \sigma(W)| \geq u) \leq 2\Pi_{W_\theta}(2m) \exp(-mu^2/16)$$ (5.4)

whenever $m \geq 2/u^2$. Let $S$ be the collection of all sectors of any angle, and let $H$ denote the collection of all hemispheres. By Claim 5.1 and the Sauer-Shelah lemma [10] relating VC dimension to growth functions, we have $\Pi_H(2m) \leq (2em/n)^n$. 

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Next, notice that using the notation in (5.1), we have $W = \mathcal{H} \triangle \mathcal{H}$. As such, we may apply Claim 5.2 to get

$$\Pi_W(2m) \leq (2em/n)^{2n}. \tag{5.5}$$

We now plug this bound into the right hand side of (5.4), set $u = \theta/\pi$, and simplify to get

$$P( \sup_{W \in W} |\mu_A(W) - \sigma(W)| \geq \theta/\pi) \leq 2 \exp(2n \log(2em/n) - m(\theta/\pi)^2/16). \tag{5.6}$$

Our assumption implies that $m \geq 2/(\theta/\pi)^2$ so the bound holds, and also that the bound is less than $\delta$. Finally, since $W_\theta \subset W$, on the complement of this event, any $W \in W_\theta$ satisfies

$$\mu_A(W) \leq \sigma(W) + \theta/\pi \leq 2\theta/\pi \tag{5.7}$$

as we wanted.

For every wedge $W \in W_\theta$, we may associate the configuration vector

$$s_{W,A} := (1W(a_1), 1W(a_2), \ldots, 1W(a_m)). \tag{5.8}$$

We can write

$$\lambda_{\max}(\mathbb{E}_{\mu_A} a a^T 1_W(a)) = \frac{1}{m} \lambda_{\max}(A S_{W,A} A^T), \tag{5.9}$$

where $S_{W,A} = \text{diag}(s_{W,A})$. $S_{W,A}$ is thus a selector matrix, and if we condition on the good event given to us by the previous theorem, it selects at most a $2\theta/\pi$ fraction of the rows of $A$. This means that $s_{W,A} \in S_{2\theta/\pi}$, where we define

$$S_\tau := \{d \in \{0,1\}^m \mid \langle d, 1 \rangle \leq \tau \cdot m\}. \tag{5.10}$$

It is helpful to remember here that there are now two sources of randomness: one source is in the creation of the measurement matrix $A$, and the second is in the selection of the row index at every iteration of the randomized Kaczmarz algorithm. With high probability over the first source of randomness, we would like to majorize the quantity in (5.9) uniformly over all wedges $W$ by the quantity $\frac{1}{2} \lambda_{\min}(\mathbb{E}_{\mu_A} a a^T)$.

In order to do this, we define a stochastic process $(Y_{s,v})$ indexed by $s \in S_{2\theta/\pi}$ and $v \in B_2^n$, setting

$$Y_{s,v} := n v^T A \text{diag}(s) A^T = \sum_{i=1}^m s_i (\sqrt{n} a_i, v)^2. \tag{5.11}$$

If we condition on the good set in Lemma 5.4, it is clear that

$$\sup_{W \in W_\theta} \frac{1}{m} \lambda_{\max}(A S_{W,A} A^T) \leq \frac{1}{nm} \sup_{s \in S_{2\theta/\pi}, v \in B_2^n} Y_{s,v}, \tag{5.12}$$

so it suffices to bound the quantity on the right. We will do this using a slightly sophisticated form of chaining, which requires us to make a few definitions.

Let $(T, d)$ be a metric space. A sequence $\mathcal{T} = (T_k)_{k \in \mathbb{Z}^+}$ of subsets of $T$ is called admissible if $|T_0| = 1$, and $|T_k| \leq 2^{2^k}$ for all $k \geq 1$. For any $0 < \alpha < \infty$, we define the $\gamma_\alpha$ functional of $(T, d)$ to be

$$\gamma_\alpha(T, d) := \inf_{T \in \mathcal{T}} \sup_{t \in T} \sum_{k=0}^{\infty} 2^{k/\alpha} d(t, T_k). \tag{5.13}$$
Let $d_1$ and $d_2$ be two metrics on $T$. We say that a process $(Y_t)$ has mixed tail increments with respect to $(d_1,d_2)$ if there are constants $c$ and $C$ such that for all $s,t \in T$, we have the bound

$$
\mathbb{P}(|Y_s - Y_t| \geq c(\sqrt{ud_2(s,t)} + ud_1(s,t))) \leq Ce^{-u}.
$$

(5.14)

Remark 5.5. In [7], processes with mixed tail increments are defined as above but with the further restriction that $c = 1$ and $C = 2$. This is not necessary for the result that we need (Lemma 5.6) to hold. The indeterminacy of $c$ and $C$ gets absorbed into the final constant in the bound.

Lemma 5.6 (Theorem 5, [7]). If $(Y_t)_{t \in T}$ has mixed tail increments, then there is a constant $C$ such that for any $u \geq 1$, with probability at least $1 - e^{-u}$,

$$
\sup_{t \in T}|Y_t - Y_{t_0}| \leq C(\gamma_2(T,d_2) + \gamma_1(T,d_1) + \sqrt{udiam(T,d_2)} + udiam(T,d_1)).
$$

(5.15)

At first glance, the $\gamma_2$ and $\gamma_1$ quantities seem mysterious and intractable. We will show however, that they can be bounded by more familiar quantities that are easily computable in our situation. Let us postpone this for the moment, and first show that our process $(Y_{s,v})$ has mixed tail increments.

Lemma 5.7 (($Y_{s,v}$) has mixed tail increments). Let $(Y_{s,v})$ be the process defined in (5.11). Define the metrics $d_1$ and $d_2$ on $S_{2\theta/\pi} \times B_{\theta}^2$ using the norms \( \| (w,v) \|_1 = \max\{\|w\|_\infty,\|v\|_2 \} \) and \( \| (w,v) \|_2 = \max\{\|w\|_2,\sqrt{2m\theta/\pi}\|v\|_2 \} \). Then the process has mixed tail increments with respect to $(d_1,d_2)$.

Proof. The main tool that we use is Bernstein’s inequality [16] for sums of subexponential random variables. Observe that each $\sqrt{n}a_i$ is a subgaussian random vector with bounded subgaussian norm \( \|\sqrt{n}a_i\|_{\psi_2} \leq C \), where $C$ by an absolute constant. As such, for any $v \in B_{\theta}^2$, $(\sqrt{n}a_i,v)^2$ is a subexponential random variable with bounded subexponential norm \( \|(\sqrt{n}a_i,v)^2\|_{\psi_1} \leq C^2 [16] \).

Now fix $v$ and let $s,s' \in S_{2\theta/\pi}$. Then

$$
Y_{s,v} - Y_{s',v} = \sum_{i=1}^m (s_i - s'_i)(\sqrt{n}a_i,v)^2.
$$

(5.16)

Using Bernstein, we have

$$
\mathbb{P}(|Y_{s,v} - Y_{s',v}| \geq u) \leq 2 \exp(-c \min\{u^2/\|s-s'\|_2^2, u/\|s-s'\|_\infty \}).
$$

(5.17)

Similarly, if we fix $s \in S_{2\theta/\pi}$ and let $v,v' \in B_{\theta}^2$, then

$$
Y_{s,v} - Y_{s,v'} = \sum_{i=1}^m s_i(\langle \sqrt{n}a_i, v\rangle^2 - \langle \sqrt{n}a_i, v'\rangle^2)
$$

(5.18)

$$
= \sum_{i=1}^m s_i \langle \sqrt{n}a_i, v-v'\rangle \langle \sqrt{n}a_i, v+v'\rangle.
$$

(5.19)

We can bound the subexponential norm of each summand via

$$
\|s_i(\sqrt{n}a_i, v-v')\langle \sqrt{n}a_i, v+v'\rangle\|_{\psi_1} \leq s_i \|\langle \sqrt{n}a_i, v-v'\rangle\|_{\psi_2} \cdot \|\langle \sqrt{n}a_i, v-v'\rangle\|_{\psi_1} 
\leq Cs_i \|v-v'\|_2.
$$

(5.20)

(5.21)
As such,

\[
\sum_{i=1}^{m} |s_i (\sqrt{n}a, v - v') (\sqrt{n}a, v + v')|^2 \leq C \|v - v'\|_2^2 \sum_{i=1}^{m} s_i^2 \leq C (2\theta/\pi) m \|v - v'\|_2^2.
\] (5.22)

Applying Bernstein as before, we get

\[ \mathbb{P}(|Y_{s,v} - Y_{s',v'}| \geq u) \leq 2 \exp(-c \min\{u^2/(2\theta/\pi)m \|v - v'\|^2_2, u/\|v - v'\|_2\}). \] (5.23)

Now, recall the simple observation that for any numbers \(a, b \in \mathbb{R}\), we have

\[ \max\{|a|, |b|\} \leq |a + b| \leq 2 \max\{|a|, |b|\}. \] (5.24)

As such, for any \(u > 0\), given \(s, s' \in S_{2\theta/\pi}, v, v' \in B^n_2\), we have

\[
\sqrt{u} \|(s, v) - (s', v')\|_2 + u\|(s, v) - (s', v')\|_1 \geq \frac{1}{2} (\sqrt{u} \|s - s'\|_2 + \sqrt{u} \|v - v'\|_2 + u \|s - s'\|_{\infty}
\]
\[
+ u \|v - v'\|_2) \geq \frac{1}{2} \max\{\sqrt{u} \|s - s'\|_2 + u \|s - s'\|_{\infty}, \sqrt{u} \|v - v'\|_2\}. \] (5.25)

Since

\[ |Y_{s,v} - Y_{s',v'}| \leq |Y_{s,v} - Y_{s',v}| + |Y_{s',v} - Y_{s',v'}|, \] (5.27)

we have that if

\[ |Y_{s,v} - Y_{s',v'}| \geq c (\sqrt{u} \|(s, v) - (s', v')\|_2 + u\|(s, v) - (s', v')\|_1), \] (5.28)

then either

\[ |Y_{s,v} - Y_{s',v'}| \geq \frac{c}{4} (\sqrt{u} \|s - s'\|_2 + u \|s - s'\|_{\infty}) \] (5.29)

or

\[ |Y_{s',v} - Y_{s',v'}| \geq \frac{c}{4} (\sqrt{u} \|v - v'\|_2 + u \|v - v'\|_2). \] (5.30)

We can then combine the bounds (5.23) and (5.17) to get

\[ \mathbb{P}(|Y_{s,v} - Y_{s',v'}| \geq c (\sqrt{u} \|(w, v) - (w', v')\|_2 + u\|(w, v) - (w', v')\|_1)) \leq 4e^{-u}. \] (5.31)

Hence, we see that the process \((Y_{s,v})\) satisfies the definition (5.14) for having mixed tail increments.

We next bound the \(\gamma_1\) and \(\gamma_2\) functions for \(S_{2\theta/\pi} \times B^n_2\).

**Lemma 5.8.** We may bound the \(\gamma_1\) functional of \(S_{2\theta/\pi} \times B^n_2\) by

\[ \gamma_1(S_{2\theta/\pi} \times B^n_2, \|\cdot\|_1) \leq C((2\theta/\pi) \log(\pi/2\theta)m + n). \] (5.32)
Proof. The proof of the bound uses metric entropy and a version of Dudley’s inequality. Let \((T,d)\) be a metric space, and for any \(u > 0\), let \(N(T,d,u)\) denote the covering number of \(T\) at scale \(u\), i.e. the smallest number of radius \(u\) balls needed to cover \(T\). Dudley’s inequality states that there is an absolute constant \(C\) for which

\[
\gamma_1(T,d) \leq C \int_0^\infty \log N(T,d,u) du.
\] (5.33)

Recall that \(S_{2\theta/\pi}\) is the set of all \([0,1]\) vectors with fewer than \(2\theta/\pi\) ones. For convenience, let us assume that \(2m\theta/\pi\) is an integer. We then have the inclusion

\[
S_{2\theta/\pi} \subset \bigcup_{I \in \mathcal{I}} [0,1]^I,
\] (5.34)

where \(\mathcal{I}\) is the collection of all subsets of \([m]\) of size \(2m\theta/\pi\), and \([0,1]^I\) denotes the unit cube in the coordinate set \(I\). We may then also write

\[
S_{2\theta/\pi} \times B^n_2 \subset \bigcup_{I \in \mathcal{I}} ([0,1]^I \times B^n_2).
\] (5.35)

Note that a union of covers for each \([0,1]^I \times B^n_2\) gives a cover for \(S_{2\theta/\pi} \times B^n_2\). This, together with the symmetry of \(\|\cdot\|_\infty\) with respect to permutation of the coordinates gives

\[
N(S_{2\theta/\pi} \times B^n_2, \|\cdot\|_1, u) \leq |\mathcal{I}| \cdot N([0,1]^I \times B^n_2, \|\cdot\|_1, u)
\] (5.36)

for some fixed index set \(I\).

We next generalize the notion of covering numbers slightly. Given two sets \(T\) and \(K\), we let \(N(T,K)\) denote the number of translates of \(K\) needed to cover the set \(T\). It is easy to see that we have \(N(T,d,u) = N(T,uB_d)\), where \(B_d\) is the unit ball with respect to the metric \(d\). Since the unit ball for \(\|\cdot\|_1\) is \(B^m_\infty \times B^n_2\), we therefore have

\[
N([0,1]^I \times B^n_2, \|\cdot\|_1, u) = N([0,1]^I \times B^n_2, u(B^m_\infty \times B^n_2))
\] (5.37)

\[
\leq N(B^{(2\theta/\pi)m}_\infty \times B^n_2, u(B^{(2\theta/\pi)m}_\infty \times B^n_2)).
\] (5.38)

Such a quantity can be bounded using a volumetric argument. Generally, for any centrally symmetric convex body \(K\) in \(\mathbb{R}^n\), we have (see Corollary 4.1.15 in [1])

\[
N(K,uK) \leq (3/u)^n.
\] (5.39)

This implies that

\[
\log N([0,1]^I \times S^{n-1}, \|\cdot\|_1, u) \leq \log(3/u)((2\theta/\pi)m + n).
\] (5.40)

Finally, observe that

\[
\log |\mathcal{I}| = \log \binom{m}{(2\theta/\pi)m} \leq (2\theta/\pi)m \log(e\pi/2\theta).
\] (5.41)

We can thus plug these last two bounds into (5.33), noting that the integrand is zero for \(u \geq 1\) to get

\[
\gamma_1(S_{2\theta/\pi} \times B^n_2, \|\cdot\|_1) \leq C \int_0^1 (2\theta/\pi)m \log(e\pi/2\theta) + \log(3/u)((2\theta/\pi)m + n) du
\] (5.42)

\[
\leq C((2\theta/\pi) \log(\pi/2\theta)m + n)
\] (5.43)

as was to be shown.

\(\square\)
Lemma 5.9. We may bound the $\gamma_2$ functional of $S_{2\theta/\pi} \times B_2^n$ by
\[
\gamma_2(S_{2\theta/\pi} \times B_2^n, \|\cdot\|_2) \leq C \sqrt{2\theta/\pi} (m + \sqrt{mn}).
\] (5.44)

Proof. Since $\alpha = 2$, we may appeal directly to the theory of gaussian complexity [15]. However, since we have already introduced some of the theory of metric entropy in the previous lemma, we might as well continue down this path. In this case, we have the Dudley bound
\[
\gamma_2(T, d) \leq C \int_0^\infty \sqrt{\log N(T, d, u)} du
\] (5.45)
for any metric space $(T, d)$.

Observe that the unit ball for $\|\cdot\|_2$ is $B_2^m \times (2m\theta/\pi)^{-1/2}B_2^n$. On the other hand, we conveniently have
\[
S_{2\theta/\pi} \times B_2^n \subset \sqrt{2m\theta/\pi} B_2^m \times B_2^n.
\] (5.46)
As such, we have
\[
N(S_{2\theta/\pi} \times B_2^n, \|\cdot\|_2,u) \leq N(- \sqrt{2m\theta/\pi} B_2^m \times B_2^n, \|\cdot\|_2, u)
\] (5.47)
\[
= N(- \sqrt{2m\theta/\pi} B_2^m \times B_2^n, u(2m\theta/\pi)^{-1/2}B_2^n))
\] (5.48)
\[
= N(T, (2m\theta/\pi)^{-1/2}uT),
\] (5.49)
where $T = \sqrt{2m\theta/\pi} B_2^m \times B_2^n$.

Plugging this into (5.45) and subsequently using the volumetric bound (5.39), we get
\[
\gamma_2(S_{2\theta/\pi} \times B_2^n, \|\cdot\|_2) \leq C \int_0^\infty \sqrt{\log N(T, (2m\theta/\pi)^{-1/2}uT) du}
\] (5.50)
\[
= C \sqrt{2m\theta/\pi} \int_0^\infty \sqrt{\log N(T, uT)} du
\] (5.51)
\[
\leq C \sqrt{2m\theta/\pi} \sqrt{m + n},
\] (5.52)
which is clearly equivalent to the bound that we want. 

At this stage, we can put everything together to bound the supremum of our stochastic process.

Theorem 5.10 (Bound on supremum of $(Y_{s,v})$). Let $(Y_{s,v})$ be the process defined in (5.11). Let $0 < \delta < 1/e$, let $\theta$ be an acute angle, and suppose $m \geq \max\{n, \log(1/\delta)\pi/2\theta\}$. Then with probability at least $1 - \delta$, the supremum of the process satisfies
\[
\sup_{s \in S_{2\theta/\pi}, v \in B_2^n} Y_{s,v} \leq C \sqrt{2\theta/\pi} \cdot m
\] (5.53)

Proof. It is easy to see that we have
\[
\text{diam}(S_{2\theta/\pi} \times B_2^n, \|\cdot\|_1) = 2,
\] (5.54)
and
\[
\text{diam}(S_{2\theta/\pi} \times B_2^n, \|\cdot\|_2) = 2 \sqrt{2m\theta/\pi}.
\] (5.55)
Also observe that we have $Y_{s,0} = 0$ for any $s \in S_{2\theta/\pi}$.
Using these, together with the previous two lemmas bounding the $\gamma_1$ and $\gamma_2$ functionals, we may apply Lemma 5.6 to see that

$$\sup_{s \in S_{2\theta/\pi}, v \in B_n^2} Y_{s,v} \leq C\left( ((2\theta/\pi) \log(\pi/2\theta)m + n) + \sqrt{2\theta/\pi}(m + \sqrt{mn}) + u + \sqrt{u}\sqrt{2m\theta/\pi} \right).$$  (5.56)

with probability at least $1 - e^{-u}$.

Using our assumptions on $m$, we may simplify this bound to obtain (5.53).

Finally, we show that $\frac{1}{m} \sum_{i=1}^{m} a_i a_i^T$ is well-behaved.

**Lemma 5.11.** Let $\delta > 0$. Then if $m \geq C(n + \sqrt{\log(1/\delta)})$, with probability at least $1 - \delta$, we have

$$\left\| \frac{1}{m} \sum_{i=1}^{m} a_i a_i^T - I_n \right\| \leq 0.1$$ (5.57)

**Proof.** Note, as before, that the $\sqrt{n} a_i$'s are isotropic subgaussian random variables with subgaussian norm bounded by an absolute constant. The claim then follows immediately from Theorem 5.39 in [16], which itself is proved using Bernstein and a simple net argument.

**Theorem 5.12** (Finite measurement sets satisfy ACW condition). There is some $\theta_0 > 0$ and an absolute constant $C$ such that for all angles $0 < \theta \leq \theta_0$, for all dimensions $n$, and any $\delta > 0$, if $m$ satisfies

$$m \geq C(\pi/2\theta)^2(\log(m/n) + \log(1/\delta)),$$ (5.58)

then with probability at least $1 - \delta$, the measurement set $A$ comprising $m$ independent random vectors drawn uniformly from $S^{n-1}$ satisfies the ACW$(\theta, \kappa)$ condition with $\kappa = 1/2$.

**Proof.** Fix $n, \delta > 0$. Choose $\theta_0$ such that the constant $C$ in the statement in Theorem 5.10 satisfies $C\sqrt{2\theta_0/\pi} \leq 0.1$. Fix $0 < \theta \leq \theta_0$, and let $\Omega_1$, $\Omega_2$, and $\Omega_3$ denote the good events in Lemma 5.4, Theorem 5.10, and Lemma 5.11 with this choice of $\theta$. Whenever $m$ satisfies our assumption (5.58), the intersection of these events occurs with probability at least $1 - 3\delta$ by the union bound.

Let us condition on being in the intersection of these events. For any wedge $W \in \mathcal{W}_\theta$ (i.e. of angle less than $\theta$), Lemma 5.4 tells us that its associated selector vector satisfies $s_{W,A} \in S_{2\theta/\pi}$ (i.e. that it has at most $2m\theta/\pi$ ones,). By Theorem 5.10 and our assumption on $\theta_0$, we then have

$$\lambda_{\max} \left( \frac{1}{m} \sum_{i=1}^{m} a_i a_i^T 1_W(a_i) \right) \leq \frac{1}{nm} \sup_{s \in S_{2\theta/\pi}, v \in B_n^2} Y_{s,v} \leq \frac{0.1}{n}. \quad (5.59)$$

On the other hand, Lemma 5.11 guarantees that

$$\lambda_{\min} \left( \frac{1}{m} \sum_{i=1}^{m} a_i a_i^T \right) \geq \frac{0.9}{n}. \quad (5.60)$$

Combining these, we get

$$\lambda_{\min} \left( \frac{1}{m} \sum_{i=1}^{m} a_i a_i^T - \frac{4}{m} \sum_{i=1}^{m} a_i a_i^T 1_W(a_i) \right) \geq \frac{0.5}{2n}, \quad (5.61)$$

which was to be shown.
6 Proof of Theorem 1.2

We restate the theorem here for convenience.

**Theorem 6.1.** Let $\delta_1, \delta_2 > 0$. There is an absolute constant $C > 0$ such that if $m \geq C(n + \log(1/\delta_1))/\delta_2$, then with probability at least $1 - \delta_1$, $m$ measurement vectors selected uniformly and independently from the unit sphere $S^{n-1}$ form a set such that the following holds: For any $\epsilon > 0$, $K \geq 2(\log(1/\epsilon) + \log(2/\delta_2))n$, and any vector $x \in \mathbb{R}^n$ such that $\|x\|_2 \geq f(m)$ where $f$ is a decreasing function in $m$, running Algorithm 1 with $K$ update steps yields an estimate $x_K$ for $x$ satisfying

$$
\|x_K - x\|_2^2 \leq \epsilon \|x\|_2^2 \text{ with probability at least } 1 - \delta_1 - \delta_2.
$$

**Proof.** Let $A$ be our $m$ by $n$ measurement matrix. By Theorem 5.12, there is an angle $\theta_0$, and a constant $C$ such that for $m \geq C(n + \log(1/\delta_1)), A$ is ACW($\theta_0, 1/2$) with probability at least $1 - \delta_1$.

Let $x \in \mathbb{R}^n$ be any vector with $\|x\|_2 \geq f(m)$, i.e. not too close to 0.\(^3\) We now run Algorithm 1 with the rows of $A$, $a_1, \ldots, a_m$, and the measurements $b_i = |\langle a_i, x \rangle|$ for $i = 1, \ldots, m$. By Proposition 3 in [5], with probability at least $1 - \delta_1$, the truncated spectral initialization step gives us an initial estimate $x_0$ satisfying

$$
\|x_0 - x\|_2 \leq \sqrt{2} \sin(\theta_0) \|x\|_2. \quad (6.1)
$$

We can then use Corollary 4.4 to guarantee that with probability at least $1 - \delta_2$, running the randomized Kaczmarz update $K$ times gives an estimate $x_K$ satisfying

$$
\|x_K - x\|_2^2 \leq \epsilon \|x\|_2^2. \quad (6.2)
$$

This completes the proof of the theorem. \qed

7 Extensions

7.1 Arbitrary initialization

In order to obtain a theoretical guarantee for Algorithm 1, we used a truncated spectral initialization to obtain an initial estimate before running randomized Kaczmarz updates. Since the number of steps that we require is only linear in the dimension, and each step requires only linear time, the iteration phase of the algorithm only requires $O(n^2)$ time, and furthermore does not need to see all the data in order to start running.

The spectral initialization on the other hand requires one to see all the data. Forming the matrix from which we obtain the estimate involves adding $m$ rank 1 matrices, and hence naively requires $O(n^3)$ time. There is hence an incentive to do away with this step altogether, and ask whether the randomized Kaczmarz algorithm works well even if we start from an arbitrary initialization.

We have some numerical evidence that this is indeed true, at least for real gaussian measurements. Unfortunately, we do not have any theoretical justification for this phenomenon, and it will be interesting to see if any results can be obtained in this direction.

\(^3\)The lower bound assumption on $\|x\|_2$ is required in order for the spectral initialization step to work properly.
7.2 Deterministic constructions of measurement sets

We have proved a guarantee for a randomized Kaczmarz algorithm for the case where our measurement vectors are real gaussian. These are not the measurement sets that are used in practical applications. Fortunately, the theory that we have developed in this paper does not apply solely to gaussian measurements, and generalizes to any measurement sets that satisfy the ACW condition that we introduced in Section 5.

It will be interesting to investigate what natural classes of measurement sets satisfy this condition.

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