The dressing procedure for the cosmological equations and the indefinite future of the universe

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In this paper we present a new simple method of construction of infinite number of solutions of Freidmann equations from the already known ones, which allows for a startling conclusion of practical impossibility of correct predictions on the universe’s future dynamics which are based solely on astronomical observations on the value of a scale factor. In addition, we present particular examples of newly constructed solutions, such as the ones, describing the smooth dynamical (de-)phantomization, and the models lacking the events horizons (both in classical and brane world cases). The generalization of the method to the simplest anisotropic universes are presented as well.

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I. INTRODUCTION

Imagine for a second an immortal astronomer, restlessly surveying an expansion of the universe, how it goes during $10^{20}$ consequent years. Suppose that she already knows for sure about the flatness of her universe. Then, as time goes, the astronomer will gather an (impressive) amount of observational data, visually presented as a set of points on the graph $\{t_i, a(t_i)\}$. If those points are located with appropriate density, then it should be possible to choose some continuous function $a(t)$, which with high precision will approximate the gained set. Of course, this function must not be unique, but lets assume for now that our astronomer have managed (not without some efforts) to find out the most satisfactory one; when this is done she might conclude that now she posses the actual time dependence of observed scale factor. Moreover, since both matters density and pressure are expressed in terms of scale factor (and its derivatives) explicitly (we remind here, that under the assumption the sign of curvature is already known):

$$\rho = \frac{3 \dot{a}^2}{8\pi G a^2}, \quad p = -\frac{c^2 (2\ddot{a} + \dot{a}^2)}{8\pi G a^2}, \quad (1)$$

then she might as well conclude that the observations have given her the values of all basic characteristics of the universe. Can we say now that the further evolution of her universe will be completely defined? No, we can’t.

This paper contains the discussion of an extremely simple method which allows one to construct new solutions of the cosmological Friedmann equations starting out from the previously known (and more simple) ones. The method itself will be denoted as the method of linearization, and the transformations which allows one to construct new solutions starting out from another ones will be refereed to as "the dressing" or as a "dressing procedure". What is new and interesting about such transformations is that they admit the invariants. More precisely, the special linear combination of the density $\rho$ and pressure $p$ (of a form $U = \alpha \rho + \beta p$) is covariant with respect to these transformations ($\alpha \dot{\rho} + \beta \ddot{p} = \alpha \dot{\rho} + \beta \ddot{p}$). (For example, in one particular case $\dot{p} = p$).

There are in total four aims of this paper. First of all, we will show that for any function $a(t)$ there exist such solutions $a_n$, which, being similar to $a(t)$ at given time interval with any given precision rate, will completely differ from $a(t)$ later (note here, that possibility of such scenario, i.e. of indeterminacy of universe fu-
Our starting point is that the volume function $\psi$ coupled scalar field with Lagrangian $\mathcal{L}$ if the universe is filled with a self-acting and minimally satisfies a simple second-order differential equation (2) one can calculate in independent one as well. Substituting this general solution into the (2) one gets the new potential $\tilde{V}$ with $k = 0$. Then the function $\psi_n = a^n$ is the solution of the Schrödinger equation

$$\frac{\ddot{\psi}_n}{\dot{\psi}_n} = U_n,$$  

with potential

$$U_n = n^2 \rho - \frac{3a}{2} (\rho + p).$$  

For example:

$$U_1 = -\frac{\rho + 3p}{2}, \quad U_2 = \rho - 3p, \quad U_3 = \frac{9}{2}(\rho - p),$$

or

$$U_{1/2} = -\frac{1}{2} \left( \rho + \frac{3p}{2} \right), \quad U_{-1} = \frac{5\rho + 3p}{2}$$

and so on.

Remark 1. If the universe is filled with scalar field $\phi$ whose Lagrangian is then the expression will be

$$U_n = n(n - 3)K + n^2V.$$  

In particular case $n = 3$ $U_3 = 9V(\phi)$ (see (4)). This particular case has been extensively studied in $\textit{[2, 3]}. \quad \text{Remark 2.}$ For small values of $n \ll 1$ one gets $U_n \sim -3n(\rho + p)/2$; for example, if $n = 0.01$ then

$$U_n \sim -0.0149\rho + 0.015p \sim -0.015(\rho + p).$$

Therefore one can use $U_n < 0$ to check whether the weak energy condition is violated $\textit{[17]}. \quad \text{If, on the contrary,} \ n \gg 1 \ \text{then} \ U_n \sim n^2\rho.$

In the case of general position, the solution of the equation has the form

$$\Psi_n = c_1 \psi_n + c_2 \hat{\psi}_n,$$  

where $\hat{\psi}_n$ is linearly independent counterpart of $\psi_n$:

$$\hat{\psi}_n(t) = \psi_n(t) \int^t dt' \frac{d\psi_n(t')}{\psi_n(t')} \equiv \psi_n(t)\xi(t).$$  

Equation (8) is enough to establish the following theorem:

**Linearization Theorem.** Let $a = a(t)$ be the solution of (2) with $k = 0$ and with $\rho$ and $p$, given by (1). Then the two-parameter function $a_n = a_n(t; c_1, c_2)$:

$$a_n = a \left( c_1 + c_2 \int \frac{dt}{a^{2n}} \right)^{1/n},$$  

II. DRESSING PROCEDURE

Let us write the Friedmann equations as

$$\left( \frac{\dot{a}}{a} \right)^2 = \rho - \frac{k}{a^2}, \quad 2\ddot{a} = -\left( \rho + 3p \right),$$  

(2)

where $k = 0, \pm 1$. Throughout the paper (excluding Sec. V) we’ll stick to the metric units with $8\pi G/3 = c = 1$. If the universe is filled with a self-acting and minimally coupled scalar field with Lagrangian

$$L = \frac{\dot{\phi}^2}{2} - V(\phi) = K - V,$$  

(3)

then the energy density and pressure are

$$\rho = K + V, \quad p = K - V,$$

therefore

$$V = \frac{1}{2}(\rho - p), \quad K = \frac{1}{2}(\rho + p).$$  

(4)

Our starting point is that the volume function $\psi = a^3$ satisfies a simple second-order differential equation (2)

$$\ddot{\psi} = 9V\psi.$$  

(5)

In (5) the potential $V$ is represented as a function of time $t$. For simple forms of the potential one can find the general solution of (5), containing both the solution used for the construction of this potential and a linearly independent one as well. Substituting this general solution into the (2) one can calculate $\rho$ and $p$. Then using one gets the new potential $\tilde{V}$ such that $\tilde{V}(t) = V(t)$ but whose form is different from $V$ if $\tilde{V}$ and $V$ are represented as functions of $\phi$: $\tilde{V}(\phi) \neq V(\phi)$. This simple method allows one to construct cosmological models describing a smooth transition from ordinary dark energy to the phantom one (3, 4).

Therefore the Friedmann equations admits the linearizing substitution and can be studied via different powerful mathematical methods which were developed for the linear differential equations. This is the reason why we call our approach the method of linearization. The crucial point of this paper is connected to the simple generalization of results above. More precisely, the following proposition is hold:

**Proposition.** Let $a = a(t)$ (with $p = p(t)$, $\rho = \rho(t)$) be the solution of (2) with $k = 0$. Then the function $\psi_n = a^n$ is the solution of the Schrödinger equation

$$\frac{\ddot{\psi}_n}{\dot{\psi}_n} = U_n,$$  

(6)

with potential

$$U_n = n^2 \rho - \frac{3a}{2} (\rho + p).$$  

(7)

For example:

$$U_1 = -\frac{\rho + 3p}{2}, \quad U_2 = \rho - 3p, \quad U_3 = \frac{9}{2}(\rho - p),$$

or

$$U_{1/2} = -\frac{1}{2} \left( \rho + \frac{3p}{2} \right), \quad U_{-1} = \frac{5\rho + 3p}{2}$$

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Equation (8) is enough to establish the following theorem:

**Linearization Theorem.** Let $a = a(t)$ be the solution of (2) with $k = 0$ and with $\rho$ and $p$, given by (1). Then the two-parameter function $a_n = a_n(t; c_1, c_2)$:

$$a_n = a \left( c_1 + c_2 \int \frac{dt}{a^{2n}} \right)^{1/n},$$  

(10)
Another way to formulate this theorem is to say that the new energy density 
\[ n^2 \rho_n - \frac{3n}{2} (\rho_n + p_n) = n^2 \rho - \frac{3n}{2} (\rho + p). \]
(11)

Another way to formulate this theorem is to say that the process of transformation of a triple \((a, \rho, p)\), with the resulting triple \((a_n, \rho_n, p_n)\) being referred to as the dressed one.

**Remark 3.** This theorem is valid for the case \( k = 0 \). If \( k = \pm 1 \) then this theorem will hold if and only if \( n = 0, 1 \).

**Remark 4.** If \( n = 3/2 \) then the equation (11) results in \( p_n = p \) (but \( p_n \neq p \)).

Let \( a(0) = 0 \), i.e. suppose that at \( t = 0 \) there exist an initial singularity. Let's assume that \( a(t) \sim t^{\lambda} \) for \( t \to 0 \). One might easily verify that if \( 2n\lambda \leq 1 \) then \( a(0) = 0 \). Now let us choose \(|c_1/c_2| \gg |\xi_{\text{max}}| \) where \( \xi = \xi(t) \) is the quantity from (9); \( \xi \) is a bounded function at the interval \( 0 < t < T \) and \( \xi_{\text{max}} \) is the maximal value of \( \xi(t) \) at this interval. It can be seen that at a given time interval \( a_n(t) \) behaves similar to \( a(t) \) with any given precision rate.

Therefore, the conjecture of our immortal astronomer was wrong: Her observations might result in \( a(t) \), but she can't be sure that the "real" scale factor is \( a(t) \). It can be \( a_n(t) \) as well.

### III. SMOOTH DYNAMICAL (DE)-PHANTOMIZATION

Let us now impose the equation of state: \( p = w\rho = (\gamma - 1)\rho \), where \( \gamma \) is adiabatic index. In this case the solution of equations (2) with \( k = 0 \) has the form

\[ a = a_t^{2/(3\gamma)}, \quad \rho = \frac{4}{9\gamma^2 t^2}, \]

where \( a_i = \text{const} \). Using "dressing procedure" \( a \to a_n \) one gets a new solution \( a_n = a_n(t; c_1, c_2) \) such that

\[ a_n(t; c_1, 0) = c_1^{1/n} a(t), \quad a_n(t; 0, c_2) = \tilde{a}_t^{2/(3\tilde{\gamma})}, \]

where

\[ \tilde{a}_t = \frac{1}{a_t} \left( \frac{3c_2 \gamma}{3\gamma - 4n} \right)^{1/n}, \quad \tilde{\gamma} = \frac{2\gamma n}{3\gamma - 2n}. \]

or

\[ \tilde{w} = \frac{2n w + 4n - 3(w + 1)}{3(w + 1) - 2n}. \]

Phantom energy always takes place for negative \( \gamma \) (or \( \tilde{\gamma} \)). If, on the other hand, \( \gamma > 0 \) then \( a_n(t; 0, c_2) \) will describe the phantom cosmological model whenever \( n \) lies in the range \( n < 0 \) or \( n > 3\gamma/2 \). For example, if initial \( a(t) \) describes the dust universe with \( w = 0 \) (\( \gamma = 1 \)) then \( a_n(t; 0, c_2) \) will describe the phantom cosmological model for \( n < 0 \) and \( n > 3/2 \); if the initial scale factor describes radiation universe with \( w = 1/3 \) then one gets the phantom model both for negative \( n \) and \( n > 2 \).

If \( c_1 \neq 0 \) and \( c_2 \neq 0 \) then (10) will describe the smooth dynamical (de)-phantomization (see [3] for the case \( n = 3 \)). Phantom energy results in big rip singularity at some time \( t = t_\star \). Controlling the zeroes of the starting solutions \( a(t) \), we can construct exact cosmologies with the presence of big rip. In fact, let \( a(t_\star) = 0 \) and \( a(t) \sim (t - t_\star)^\lambda \) or \( a(t) \sim (t_\star - t)^\lambda \) for \( t \sim t_\star \). Using (10) one can conclude that, for \( n\lambda > 1 \), \( a_n(t; c_1, c_2) \) will describe the big rip as the singularity of \( a(t) \) is approached.

If \( \xi(t) \to 0 \) as \( t \to \pm \infty \) then one has a solution allowing for finite "phantom regions". In other words, there exist two instants \( t_1 \) and \( t_2 \), \( t_1 < t_2 < t_\star \) such that \( (\rho_n + p_n)(t_1) = 0 \) and \( \rho_n + p_n < 0 \) if \( t_1 < t < t_2 \).

All these regions can be cut out, with the edges being newly sewn up in such a way that neither the scale factor (and its first two derivatives) nor density or pressure will experience a jump anymore. To show this let us assume that the universe is born at some time instant \( t = 0 < t_1 \) and suppose there exist a time interval, during which all the energy conditions are satisfied. One can demonstrate that as time goes to \( t_1 \), the strong energy condition is the one bounded to be violated first, the weak one being extant. As a result, the universe would get inflated up to the instant \( t_1 \).

Furthermore, when \( t = t_1 \) we arrive precisely to the De Sitter's phase, but this time **instantaneously**. What follows next is the "phantom region", **which can be cut out** by matching the solutions corresponding to \( t = t_1 \) and \( t = t_2 \). Note that gluing the solutions in such a fashion we shall make sure to use the constraints much stricter than those usually imposed for the similar problems of non-relativistic quantum theory based on the Schrödinger equation. In fact, what the discussed problem calls for is a \( C^2 \) gluing; namely the functions has to be matched up to their second derivatives. The latter requirement serves to eliminate the possibility of discontinuities of pressure and density. A general solution of equation (10) is characterized by two integration constants, and it is not obvious that three matching conditions will be satisfied in general. However, it is easy to verify that the third condition, namely the equality of the second derivatives, is **automatically fulfilled**, provided that the first two, namely the \( C^1 \) matching has been successfully applied. This apparent fluke is due to the fact that the matching is done at an inflection point of \( \log \psi \).

Indeed, let us return to (8). It is possible to write \( \Psi_n(\pm) = c_1^{(\pm)} \Psi_n + c_2^{(\pm)} \Psi_0 \) where \( \Psi_n = \Psi_n(\pm) \) at \( t < t_1 \) and
\[ \Psi_n = \Psi_n^{(+)} \text{ at } t > t_2. \] Using the conditions

\[ \Psi_n^{(-)}(t_1) = \Psi_n^{(+)}(t_2), \quad \dot{\Psi}_n^{(-)}(t_1) = \dot{\Psi}_n^{(+)}(t_2), \]

one can express the constants \( c_{1,2}^{(+)} \) via \( c_{1,2}^{(-)}, a(t_{1,2}) \) and \( \dot{a}(t_{1,2}). \) Since \( p_n = n^2 \dot{a}^2/2a^2, \) it follows that \( p_n(t_1) = p_n(t_2). \) But \( p_n(t_{1,2}) = -p_n(t_{1,2}), \) therefore \( p_n(t_1) = p_n(t_2). \) Using (9) one concludes that \( a_n(t_1) = a_n(t_2), \)
\( \dot{a}_n(t_1) = \dot{a}_n(t_2), \) and \( \ddot{a}_n(t_1) = \ddot{a}_n(t_2), \) and hence \( \dot{\rho}(t_1) = \dot{\rho}(t_2). \)

Therefore, the finite phantom region can be cut out. However, an immediate question arises here: is it really \textit{required} to do this? Of course, the smooth phantomization looks extremely suspicious. In the universe filled with scalar field \( \phi \) (\( \phi \) is the dressed field: \( \phi \rightarrow \phi_n \)) it leads to the highly unusual situation: \( \dot{\phi}_n^2 > 0 \) at \( t < t_1, \)
t\( t > t_2 \) while \( \dot{\phi}_n^2 < 0 \) at \( < t_1 < t < t_2. \) Is it possible for kinetic term to change its sign? While absolutely impossible in Minkowski space, this may still be the case in the General Relativity [4] (see also [5, 6]). Our point of view is this: one shall treat the smooth phantomization as one real physical phenomenon. If, however, the finite phantom region is an undesirable one then it can simply be cut out by the method described above.

\section*{IV. SPACE-TIME WITHOUT EVENTS HORIZON}

The equation (10) allows one to construct the space-time without events horizon (see also [7]). In order to show this let \( t_f \) be an instant, such that \( a_n(t_f; c_1, c_2) = 0 \) but \( a(t_f) = a_f \neq 0. \) In other words, suppose that \( c_1 + c_2 \xi_f = 0, \) where \( \xi_f = \xi(t_f). \) Then, for \( t \rightarrow t_f \) (and \( t < t_f \)) one gets

\[ a_n \sim a_f \left( c_1 + c_2 \xi_f - c_2(t_f - t) \xi_f \right)^{1/n} \frac{1}{a_f} \left( t_f - t \right)^{1/n}, \]

where \( c_2 = -\kappa^2 < 0. \) Integrating this equation for future directed radial null geodesics, \( ds^2 = dt^2 - a_n^2 d\chi^2 \) one will get

\[ \Delta \chi \sim \frac{a_f}{\kappa} \int_t^{t_f} \frac{dt'}{(t_f - t')^{1/n}} \sim \frac{na_f}{\kappa(1 - n)} \lim_{t \rightarrow t_f} (t_f - t)^{(n-1)/n}. \]

It is easy to see that for \( 0 < n \leq 1 \) we will have \( \Delta \chi = +\infty \) which shows that radial null geodesics circumnavigate the universe infinite number of times as the future c-boundary at \( t = t_f \) is approached. By homogeneity and isotropy, we can conclude that all future endless timelike curves define the same c-boundary point.

In the case \( n = 1 \) one gets \( \Delta \chi \sim -\log(t_f - t) \rightarrow +\infty \) as the final singularity is approached. This case is extremely interesting because (see Remark 3) when \( n = 1 \) one can use the dressing procedure to construct the exact solutions for the universes with \( k = \pm 1. \) Let’s consider a few examples.

\section*{A. Simple generalization of \( k = +1 \) dust model}

In the simplest dust case with \( p = 0 \) one can solve the system (2) to get

\[ a = a_m \sin^2 \eta, \quad 2\eta - \sin 2\eta = \frac{2t}{a_m}, \] \hspace{1cm} (13)

\[ \rho = \frac{1}{a_m^2 \sin^6 \eta}. \]

Using (10) for the case \( n = 1 \) one will obtain the general solution

\[ a(t)_{\text{gen}} = c_1 a(t) + c_2 \dot{a}(t), \] \hspace{1cm} (14)

where \( c_1, c_2 \) are the arbitrary constants. It is possible to rewrite (13) in the form

\[ a(t) \equiv a(t)_{\text{gen}} = A \sin \eta \sin(\delta - \eta), \] \hspace{1cm} (15)

with two arbitrary constants \( A \) and \( \delta. \) This solution describes the universe being born from the initial singularity (\( \eta_i = 0, t_i = 0 \)) and collapsing thereafter into the final singularity at \( \eta_f = \delta \) or

\[ t_f = \frac{1}{2 \alpha} (2\delta - \sin 2\delta) \]

where we have introduced a new parameter \( \alpha \) such that\n
\[ 2\eta - \sin 2\eta = 4\alpha t, \quad \alpha = 1/2 a_m \sin^2 \delta/2 \]

hence,

\[ a = \frac{a_m}{\sin^2(\delta/2)} \sin \eta \sin(\delta - \eta), \] \hspace{1cm} (16)

and the maximum value of \( a = a_m \) will occur at \( \eta = \delta/2. \)

It is easy to see that upon the choice \( \delta = \pi \) one gets the well known \textquotedblleft dust solution" \hspace{1cm} (14). In case of general position one shall choose \( 0 < \delta < \pi. \) It can be seen that for \( t \ll t_f, \delta = \pi - \epsilon \) and \( \epsilon \ll 1, \hspace{1cm} (16) \) will behave similar to (13). But if \( t \sim t_f \) then one gets something really different: a universe without events horizon. To show this lets consider\n
\[ ds^2 = dt^2 - \frac{1}{4 \alpha^2 \sin^2 \eta \sin^2(\delta - \eta)} [d\chi^2 + \sin^2 \chi d\Omega^2_3]. \]

Upon integration of the equation \( ds^2 = 0 \) (describing the future directed radial null geodesics) one gets:

\[ \Delta \chi = 2 \int_{\eta}^{\delta} \frac{\sin \eta}{\sin(\delta - \eta)} d\eta = +\infty. \] \hspace{1cm} (17)

We note that if \( \delta = \pi \) then

\[ \Delta \chi = 2(\pi - \eta) < \infty. \]

The result (17) shows that radial null geodesics circumnavigate the universe an infinite number of times as
t \to t_f$. This fact and the homogeneity-isotropy results in conclusion that (i) this universe has no event horizons and (ii) all future endless timelike curves define the same c-boundary point. At last,

$$\Delta \chi = 2 \int_{0}^{\pi} \frac{\sin \eta}{\sin(\delta - \eta)} d\eta < +\infty.$$ 

One can show that all energy conditions are valid. For this let us point out that sum $\rho + 3p$ in general model is equal to the sum $\rho + 3\rho$ in starting model (see (11) for the case $n = 1$). This fact results in validity of strong energy condition for our model. Finally, from Friedmann equations one can see that density of energy is always positive at $k = 1$. By this property and the validity of the strong energy condition, the weak energy one will be satisfied automatically.

### B. Generalization of a Lambda-radiation model in flat space

Let's consider the flat universe which has a positive vacuum energy $\Lambda$. Let's also assume that the universe is filled with the radiation. Solving system (2) we will obtain the initial solution for the scale factor

$$a = a_0 \sinh^{1/2} \theta, \quad \theta = 2\sqrt{\Lambda} t,$$

$$\rho = \Lambda(1 + \sinh^{-2} \theta),$$

where $a_0$ is a positive constant. If $t > t_v = \sqrt{\frac{1}{\Lambda}} \text{arsinh}(1)$, the strong energy condition will necessarily be violated. Using (11) for the case $n = 1$ one can see that general solution can be written in the following form

$$a_{gen} \equiv a = a_0 \sinh^{1/2} \theta \ln \frac{\coth^{\delta/2}}{\delta},$$

where $\delta$ is a positive constant. The parameter $\epsilon$, introduced here, plays an important role in our reasonings. If $\epsilon = 0$, then (11) will be equivalent to the initial solution. In the remaining cases one can without any loss of generality assume $\epsilon = 1$. There will be three types of solutions. If $\delta < 1$ the universe will be open. This type of solutions has the following asymptotic behavior

$$a \to -0.5a_0 \ln \delta \exp(\sqrt{\Lambda} t), \quad t \to \infty.$$ 

It is easy to see that this universe is plagued by the events horizon.

The case $\delta \geq 1$ is a more interesting one. If $\delta < 1$, then solution will describe the universe, starting from an initial singularity ($\theta = 0, t = 0$) and ending up in the final singularity at $t_f = \frac{1}{\sqrt{\Lambda}} \text{arcoth} \delta$. One shall note that, if $\delta \geq 1 + \sqrt{2}$ the strong energy condition will always be satisfied (in fact, universe will end up in singularity long before the time $t_v$).

When $\delta = 1$ the resulting solution might be denoted as "quasisingular" because

$$\lim_{t \to \infty} a \sim \lim_{\theta \to \infty} \sinh^{1/2} \theta \ln \frac{\theta}{2} \sim \exp(-\theta/2).$$

From this relation one can see that scale factor tends to singularity but never achieves it.

Both singular and quasisingular models contains no events horizons. To show this for the quasisingular case let us integrate the equation for future directed radial null geodesics ($ds^2 = 0$) just like it has been done in the previous subsection:

$$\Delta \chi \sim \int_{0}^{\infty} \frac{d\theta}{\sinh^{1/2} \theta \ln \coth \frac{\theta}{2}}.$$ 

This integral diverges because subintegral expression has an exponential asymptot at large $\theta$. Therefore radial null geodesics circumnavigate the universe an infinite number of times as $t \to \infty$. This fact and the homogeneity-isotropy result in the conclusion similar to the one from the subsection B, namely that such universe possess no events horizon. Absence of events horizon for singularity model can be proved by analogy.

In conclusion let us analyze the equation of state for the generalization of a lambda-radiation model. One can show that the value $w = p/\rho$ is equal to

$$w = -\frac{1}{3} + \frac{2}{3} \frac{(1 - \sinh^2 \theta) \ln^2 \coth \frac{\theta}{2}}{(\cosh \theta \ln \coth \frac{\theta}{2})^2}.$$ 

(20)

From this relation it follows immediately that for both open and quasisingular models $w \to -1$ at large $t$ (large $\theta$). For the closed model $w = -1/3$ whenever we approach the final singularity. In the initial singularity $w = 1/3$ for all models. Close examination of equation (20) shows that $w$ is always greater than -1 for all cases, i.e. weak energy condition will always be satisfied.

### V. POSSIBLE GENERALIZATIONS OF THE METHOD

In the previous sections we have shown that no finite-time observations made by our astronomer will yield an exact picture of the future dynamics of her universe. One can claim, however, that the cited "nonuniqueness" is due to the special properties of the Friedmann equations and will probably disappear for the cosmological models, distinct from this one. In current section we are going to show that, in fact, the above-mentioned ideology can be applied absolutely similarly in at least two particular cases: for anisotropic universes and for a certain brane world models.
A. Anisotropic universes

In this subsection we are going to show that the described method can be generalized to some cases of anisotropic universes. Consider a simplest possible case of anisotropic model:

\[ ds^2 = dt^2 - a^2(t)dx_1^2 - b^2(t)dx_2^2 - c^2(t)dx_3^2, \]

with the perfect fluid energy-momentum tensor. The corresponding system of equation has the form

\[
\begin{align*}
\frac{\ddot{a}}{a} + \frac{\dot{b}}{b} + \frac{\dot{c}}{c} &= \frac{3}{2}(\rho - p), \\
\frac{\ddot{b}}{b} + \frac{\dot{a}}{a} + \frac{\dot{c}}{c} &= \frac{3}{2}(\rho - p), \\
\frac{\ddot{c}}{c} + \frac{\dot{a}}{a} + \frac{\dot{b}}{b} &= \frac{3}{2}(\rho - p),
\end{align*}
\]

and

\[
\begin{align*}
\frac{\ddot{a}}{a} + \frac{\ddot{b}}{b} + \frac{\ddot{c}}{c} &= -\frac{3}{2}(\rho + 3p). \tag{22}
\end{align*}
\]

Using (21) one can rewrite (22) as

\[ \dot{abc} + \dot{a}bc + \dot{ab}c = 3\rho abc. \tag{23} \]

There are known quite a few exact solutions of the system (21), (22), such as the vacuum solution of Kasner \[8\], or dust solution of Saffucing, Heckmann \[9\]. Our aim, however, is to develop the dressing procedure for the (21), (22).

In order to do this lets first define a new function \( \psi = abc \). Using (21) one can check this function to be a solution of the Schrödinger equation:

\[ \dot{\psi} = V\psi, \tag{24} \]

with potential \( V = 9(\rho - p)/2 \). In fact, this is just a replica of the case \( n = 3 \) of the Friedmann equation.

The general solution of the equation (24) is

\[ \Psi = c_1\psi + c_2\dot{\psi}, \tag{25} \]

where \( \dot{\psi} \) is linearly independent counterpart of \( \psi \). Since \( \Psi \) is the solution of the (24) with initial potential \( V \) (with the new density and pressure \( \hat{\rho} \) and \( \hat{p} \), but same difference: \( \hat{\rho} - \hat{p} = \rho - p \)), we can define the new solutions of the system (21) \( A, B, C \) with similar right hand side:

\[
\begin{align*}
\frac{\ddot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C} &= \frac{3}{2}(\rho - p), \\
\frac{\ddot{B}}{B} + \frac{\dot{A}}{A} + \frac{\dot{C}}{C} &= \frac{3}{2}(\rho - p), \\
\frac{\ddot{C}}{C} + \frac{\dot{B}}{B} + \frac{\dot{A}}{A} &= \frac{3}{2}(\rho - p), \tag{26}
\end{align*}
\]

and

\[ \dot{A}B + \dot{A}CB + \dot{ABC} = 3\hat{\rho}\Psi. \tag{29} \]

Further on, one can represent equations (20), (27) as the particular system

\[
\begin{align*}
2A\dot{A}\Psi + 2A\dot{A}\Psi - 2\Psi\dot{A}^2 + 3(\rho - p)A^2\Psi &= 0, \\
2B\dot{B}\Psi + 2B\dot{B}\Psi - 2\Psi B^2 + 3(\rho - p)B^2\Psi &= 0.
\end{align*}
\]

One can show that if \( A, B \) and \( \Psi \) are solutions of respectively (20) and (24) then the function

\[ C = \frac{\Psi}{AB}, \tag{31} \]

would be a solution of (28). Thus, assuming \( \rho, p \) and \( \Psi \) to have certain values, and solving (30) for \( A \) and \( B \), one can automatically calculate \( C \) (via (31)). The new density \( \hat{\rho} \) will then be defined from the eq. (29) and the new pressure from

\[ \hat{p} = \hat{\rho} - \rho + p. \]

Equations (30) can be solved by simple substitution \( A = \exp(\alpha), B = \exp(\beta) \) which results in system

\[
\begin{align*}
\dot{\alpha} + \frac{\dot{\Psi}}{\Psi}\alpha &= \frac{3}{2}(\rho - p), \\
\dot{\beta} + \frac{\dot{\Psi}}{\Psi}\beta &= \frac{3}{2}(\rho - p),
\end{align*}
\]

therefore

\[ A = F(t)\exp\left(c_1\int^t dt' \frac{\dot{\Psi}(t')}{\Psi(t')}\right), \]

and

\[ B = F(t)\exp\left(c_{11}\int^t dt' \frac{\dot{\Psi}(t')}{\Psi(t')}\right), \]

where

\[
F(t) = \exp\left[\frac{3}{2}\int^t dt' \frac{\dot{\Psi}(t')}{\Psi(t')} \int^t dt'' (\rho(t'') - p(t''))\right],
\]

\( c_i \) and \( c_{11} \) are arbitrary constants.

B. A Few notes about the brane universe

Let’s start our considerations from the following system:

\[
\begin{align*}
\frac{\dot{a}}{a} &= \rho \left(1 + \frac{\rho}{2\lambda}\right), \\
-2\frac{\ddot{a}}{a} &= \rho + 3p + \frac{\rho}{\lambda}(2\rho + 3p), \tag{32}
\end{align*}
\]
where $\rho$ is the density, $p$ - pressure, and $\lambda$ is the tension on the brane.

Before we actually start applying our method, let us make one simple but interesting observation. If we’ll try to solve the system (32) for the $\rho$ as a function of $a$ (for finite values of $\lambda \neq 0$), we will unexpectedly end up with not just one, but two distinct solutions:

$$\rho_{1,2} = -\lambda \pm \sqrt{\lambda^2 + 2\lambda \left(\frac{\ddot{a}}{a}\right)^2}$$  \hspace{1cm} (33)

For the negative values of $\lambda$ this (interestingly enough) gives an upper bound on the absolute value of the Hubble constant: $H^2 = \left(\frac{\ddot{a}}{a}\right)^2 \leq -\lambda/2$ and presents two positive values of $\rho$. In the case of positive $\lambda$ (which in theory might correspond to our universe) one of the values of $\rho$ will be negative (it might seem strange on a first glance, but the study of the models with the negative $\rho$ has experienced quite a surge in a past few years. See, for example [10]). In other words, even the exact knowledge of the very functional representation of scale factor would not allow the astronomer (living in such universe) to uniquely determine the density function!

With this being said let us finally state the following version of our linearization theorem:

**Linearization Theorem for Branes.** Let $a = a(t)$ be the solution of (32) with $k = 0$ and with $\rho$ and $p$ being the density and pressure functions correspondingly. Then

1. Function $\psi_n \equiv a^n$ will be solution of the Schrödinger equation

$$\frac{\psi''}{\psi} = W_n,$$ \hspace{1cm} (34)

with the potential

$$W_n = \frac{n}{2} \left(2n\rho - 3(\rho + p) + \frac{\rho}{\lambda} (n\rho - 3(\rho + p))\right) = \frac{n}{2} \left[2n(K + V) - 6K + \frac{1}{\lambda}(K + V)(n(K + V) - 6K)\right];$$ \hspace{1cm} (35)

(Note, that despite the discussed non-uniqueness of the density function, potential $W_n$ is unique!)

2. The two-parameter function $a_n = a_n(t; c_1, c_2)$, satisfying (10) will be solution of (32) with new energy density $\rho_n$ and pressure $p_n$ satisfying:

$$W_n(\rho, p) = W_n(\rho_n, p_n).$$ \hspace{1cm} (36)

As before, the proof is nothing more but a technical exercise that will be left out for our readers.

**Example.** System (32) can easily be solved when $p/\rho = w = \text{const}$. Let $\lambda > 0$, $w > -1$ ($\gamma > 0$) and $a(0) = 0$ then

$$a(t) = a(t_0) \left(\frac{t(3\lambda\gamma t + 2\sqrt{2\lambda})}{t_0(3\lambda\gamma t_0 + 2\sqrt{2\lambda})}\right)^{1/3\gamma},$$

$$H = \frac{\dot{a}}{a} = \frac{2(3\sqrt{\lambda}\gamma t + \sqrt{2})}{3\gamma(3\sqrt{\lambda}\gamma t + 2\sqrt{2})},$$ \hspace{1cm} (37)

$$\rho = \frac{4\lambda}{3\gamma t(3\sqrt{\lambda}\gamma t + 2\sqrt{2})}.$$ 

For the dust case $w = 0$ ($\gamma = 1$) and $n = 3$ one gets the following two-parameter function $a_3 = a_3(t; c_1, c_2)$:

$$a_3 = \left[\left(c_1 - \frac{3c_2\sqrt{2}}{\sqrt{\lambda}} \log \left(\frac{\zeta(t)}{t}\right)\right) \zeta(t) + 2c_2 \left(\frac{\zeta(t)}{\lambda} + 3t\right)^{1/3}\right],$$ \hspace{1cm} (38)

where $\zeta(t) = 3\lambda t + 2\sqrt{2}\lambda$. One can show that in this case the solution might above all describe the space-times without events horizon (for a special choice of values of $c_1, c_2$).

**Remark 5.** If $\gamma < 0$ then one get the phantom model with two big rips [10]:

$$a(t) = \left(\frac{\mu^2}{1 - 4.5\lambda\gamma^2(t - T)^2}\right)^{1/3|\gamma|},$$

where $\mu^2$ and $T$ are constants. Using our method one can obtain two-parameter (or 2N-parameter) function $a_n = a_n(t; c_1, c_2)$ such that $a_n(t; c_1 = 1, c_2 = 0) = a(t)$.

**VI. CONCLUSION**

In this paper we have discussed a simple (and easily automatizable) method of construction of exact solutions of Friedmann equations. Despite simplicity, the method allows for acquirement of solutions characterized by extremely interesting properties. What is more, it appears (for $k=0$) that the very abundance of the set of solutions that are to be obtained this way leads us to a stunning conclusion: no matter how accurate our astronomical observations are, there exist not just one, but a whole set of solutions that will satisfy the observational data while leading to essentially different dynamics in future. This sudden twist leads us to seemingly unavoidable conclusion about the principle indefiniteness of the future, hidden in the Friedmann equations. For a first glance such conclusion looks really disappointing, rendering useless all our efforts to build a suitable cosmological model describing our universe.

However, everything above-said doesn’t mean the impossibility to determine the actual dynamics of the universe in principle. Even though the usual observational methods might not give us the final answer, we can use the statistical methods to get it - that is, anthropic principle.
For example, in the article we have studied the particular models possessing the final singularity and no event horizons. What is the practical significance of such models? As follows from the recent observations (see [11], [12]) our universe suffers the accelerated expansion [13], [14]. As for now, the most probable cause of such expansion lies in nonzero cosmological constant. If this is really the case, then the future dynamics of observable universe is confined within the particles horizon and, as such, leads to problems with formulation of a fundamental physical theory (like the string theory or hypothetical M-theory) in a finite volume [15]. If, on the other hand, the lifetime of the universe (with an observable expansion rate) will exceed the limit of $10^{100}$ years, the dominating observers will, as follows from [16], be the ones of a quantum fluctuations origin, which, of course, could hardly be called compatible with our observations. Therefore, as follows from the anthropic reasons of [16], the most probable scenarios would be those describing the contemporary expansion, being traced by the consequent contraction phase and the “horizonless” collapse [19]. In other words, exactly that type of scenarios, that has been naturally constructed with the aid of “dressed” initial solutions.

Finally, let us emphasize that the purpose of this paper was a mere study of Friedmann equation’s solutions that has been constructed in this article by the linearization procedure. The practical aspects of application of the described technique to the observed universe lied out of scope of this work. However, we are hoping to get back to it (as well, as to applications of the method to more interesting anisotropic and brane models) in one of our future articles.

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[17] At the same time, one shall keep in mind that this equation is nothing but approximate. To ensure (with the help of $\psi_n$) that the weak energy condition will indeed be violated, one can use exact equation $\sigma_n = -3n(\rho + p)/2$, where $\sigma_n = \psi_n/\psi_n$.
[18] But this would not be true for pressure: $\dot{p}_n(t_1) \neq \dot{p}_n(t_2)$!
[19] Another way to avoid this conclusion is to suppose that our universe has a certain decay rate for tunneling into oblivion: Don N. Page, [hep-th/0612137]; Don N. Page, [hep-th/0610079].