On complex surfaces with definite intersection form

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Abstract

A compact complex surface with positive definite intersection lattice is either the projective plane or a false projective plane. If the intersection lattice is negative definite, the surface is either a non-minimal secondary Kodaira surface, a non-minimal elliptic surface with \( b_1 = 1 \), or a class VII surface with \( b_2 > 0 \). In all cases the lattice is odd and diagonalizable over the integers.

1 Introduction

The intersection form of a compact connected orientable 4\(n\)-dimensional manifold is bilinear, symmetric and, by Poincaré duality, unimodular. As is well known (cf. [10, 11]) if such a form is indefinite, its isometry class is uniquely determined by the signature and type of the form. I recall that the type of the form \( b \) can be even or odd. It is even if \( b(x, x) \) is even for all elements \( x \) of the lattice and odd otherwise. Odd forms are diagonalizable over the integers, but unimodular even forms are evidently not diagonalizable\(^\text{1}\).

In the definite situation the situation is dramatically different: the number of isometry classes goes up drastically with the rank. See e.g. [13, Ch. V.2.3]. So one might ask whether all definite forms occur as intersection forms. This is indeed the case for topological manifolds in view of the celebrated result [5] by M. Freedman implying that every form can be realized as the intersection form of a simply connected compact oriented 4-manifold. Moreover, its oriented homeomorphism type is uniquely determined by the intersection form.

Let me next turn to differentiable 4-manifolds. For those Donaldson [3] proved that in the simply connected situation a definite form is diagonalizable. A little later, in [4] he proved this also for the non-simply connected case. Such differentiable 4-manifolds are easily constructed: take a connected sum of projective planes and projective planes with opposite orientation. These cannot all have a complex structure. Indeed, e.g. [11, V. Thm. 1.1] implies that the only possible simply connected complex surface with a positive definite intersection form is the projective plane.

\(^\text{1}\)Consider [10,11] for more precise information.
The main results of this note deal with complex surfaces having definite intersection forms. The first of these covers the (slightly easier) Kähler case. The result reads as follows:

**Theorem 1.1.** Let $X$ be a compact Kähler surface with a definite intersection form, then $X$ is either the projective plane or a fake projective plane, that is a surface of general type with the same Betti numbers as $\mathbb{P}^2$. In these cases the intersection form is isometric to the (trivial) odd positive rank 1 form $(x, y) \mapsto xy$.

This result is probably known to experts but I am not aware of any proof in the literature. It leads to a characterisation of fake planes:

**Corollary 1.2.** The only non-simply connected Kähler surfaces with a definite intersection form are the fake planes.

Let me next consider non-Kähler surfaces. For those the intersection form can also be negative definite. In this case, a distinction has to be made between minimal and non-minimal surfaces, where, I recall, a surface is minimal if it does not contain exceptional curves, i.e., rational curves of self-intersection $(-1)$. Non-minimal surfaces are obtained from minimal surfaces by repeatedly blowing up points. Each blowing up introduces an exceptional curve. The main theorem is as follows:

**Theorem 1.3.** Let $X$ be a compact non-Kähler surface with a definite intersection form. Then either $X$ is a surface of class VII with $b_2 > 0$, a non-minimal secondary Kodaira surface, or a blown up properly elliptic surface whose minimal model has invariants $q = b_1 = 1$ and $b_2 = c_2 = 0$. In all cases the intersection form is negative definite and diagonalizable (and hence odd).

For the (standard) terminology concerning surfaces I refer to [1, Ch. VI].

**Remark 1.4.** 1. The elliptic surfaces in the above theorem have been classified: they are obtained from the product $\mathbb{P}^1 \times E$, $E$ an elliptic curve, by doing logarithmic transformations in (lifts of) torsion points of $E$ with sum zero. See [6, Ch. II.2, Thm. 7.7].

2. Donaldson’s results are not used in the proof in the Kähler case, but instead the Bogomolov–Miyaoka–Yau inequality (cf. [1, §VII.4]) is invoked. For the non-Kähler situation the Donaldson results can likewise be dispensed of provided the Kato conjecture holds, i.e. class VII surfaces have global spherical shells.

## 2 Basic facts from surface theory

It is well known that the Chern numbers $c_1^2(X)$ and $c_2(X)$ are topological invariants. This is obvious for $c_2$ since it is the Euler number. For $c_1^2$ this is a consequence of a special case of the index theorem [7, Thm. 8.2.2] which in this case reads

$$\tau(X) = \text{index of } X = \frac{1}{3} (c_1^2(X) - 2c_2(X)).$$

\(\text{Surfaces of the latter sort have been classified in [12] and turn out to have large fundamental groups.}\)
Here the index is the index of the intersection form of \( X \). Also Noether’s formula (cf. [1, p. 26]) is used below. It is a special case of the Riemann–Roch formula and reads:

\[
1 - q(X) + p_g(X) = \frac{1}{12}(c_1^2(X) + c_2(X)),
\]

where \( q(X) = \dim H^1(X, \mathcal{O}_X) \) and \( p_g = \dim H^2(X, \mathcal{O}_X) \). Furthermore, I shall need an expression for the signature of the intersection form in terms of these invariants (cf. [1, Ch. IV.2–3]):

**Proposition 2.1.** Let \( X \) be a compact complex surface. Then

1. \( b_1(X) \) is even and equal to \( 2q(X) \) if and only if \( X \) is Kähler. Otherwise \( b_1(X) = 2q(X) - 1 \).

2. In the Kähler case the signature of the intersection form equals \( (2p_g(X)+1, b_2(X) - 2p_g(X) - 1) \) and \( (2p_g(X), b_2(X) - 2p_g(X)) \) otherwise.

As a consequence, firstly, \( q(X) \) and \( p_g(X) \) are topological invariants. Secondly, for a Kähler surface the intersection form \( S_X \) can only be indefinite or positive definite while for a non-Kähler surface it can a priori be indefinite, positive definite or negative definite. It is positive definite if and only if \( b_2 = 2p_g \neq 0 \) and negative definite if and only if \( p_g = 0 \) and \( b_2 \neq 0 \).

The proof of the main results uses the Enriques–Kodaira classification. I recall it in the form in which it is needed (cf. [1, Ch. VI]):

**Theorem 2.2** (Enriques–Kodaira classification). Every compact complex surface belongs to exactly one of the following classes according to their Kodaira dimension \( \kappa \). The invariants \((c_1^2, c_2)\) are given for their minimal models:

| \( \kappa \) | Class | \( q \) | \( c_1^2 \) | \( c_2 \) |
|---|---|---|---|---|
| \(-\infty\) | rational surfaces | algebraic | 0 | 8 or 9 |
| | ruled surfaces of genus \( > 0 \) | algebraic | \( g \) | \( 8(1-g) \) |
| | class VII surfaces | non-Kähler | 1 | \( -c_2 \) |
| 0 | Two-dimensional tori | Kähler | 4 | 0 |
| | K3 surfaces | Kähler | 0 | 24 |
| | primary Kodaira surfaces | non-Kähler | 2 | 0 |
| | secondary Kodaira surfaces | non-Kähler | 1 | 0 |
| | Enriques surfaces | algebraic | 0 | 12 |
| | bielliptic surfaces | algebraic | 2 | 0 |
| 1 | proper elliptic surfaces | algebraic | 0 | \( \geq 0 \) |
| 2 | surfaces of general type | algebraic | > 0 | > 0 |

### 3 Proofs of Theorems 1.1 and 1.3

Let \( X \) be a compact complex surface, \( S_X \) the intersection form on the free \( \mathbb{Z} \)-module \( H_X = H^2(X, \mathbb{Z})/\text{torsion} \). So \((H_X, S_X)\) is the intersection lattice of \( X \).
Let me introduce some further useful notation. The rank 1 unimodular positive, respectively negative definite lattices are denoted \(\langle 1 \rangle\) and \(\langle -1 \rangle\) respectively. The hyperbolic plane \(\mathbb{H}\) is the rank 2 lattice with basis \(\{e, f\}\) and form (denoted by a dot) given by \(e \cdot e = f \cdot f = 0, e \cdot f = 1\) For rational and ruled surfaces the intersection forms are well known: for \(\mathbb{P}^2\) it is \(\langle 1 \rangle\), for the other minimal rational or ruled surfaces it is either \(\langle 1 \rangle \oplus \langle -1 \rangle\) or \(\mathbb{H}\). See for example [2, Prop. II.18, Prop. V.1.]. So only \(\mathbb{P}^2\) gives a definite intersection form and the other surfaces can be discarded for the proof of Theorem 1.1.

As to minimality, observe the following result:

**Lemma 3.1.** If \(X\) is not minimal, then \(\mathcal{H}_X\) is odd and splits off as many copies of \(\langle -1 \rangle\) as blow-ups from a minimal model are needed to obtain \(X\). If, moreover \(X\) is Kähler, \(\mathcal{H}_X\) is indefinite.

The reason is that if \(X\) is not minimal, the class of an exceptional curve splits off orthogonally whereas a Kähler class has positive self-intersection. This makes the latter somewhat easier to handle. Hence, I first consider the Kähler case where one only has to consider positive definite forms. So let me assume this. Then, by Proposition (2.1) one has \(\tau = 2p_g + 1\). The index theorem (1) combined with the Noether formula (2) then yields the following expressions for \(c_1^2\) and \(c_2\):

\[
\begin{align*}
c_1^2 &= 10p_g - 8q + 9 \\
c_2 &= 2p_g - 4q + 3
\end{align*}
\]

so that \(c_1^2 - 3c_2 = 4(p_g + q)\). The class of surfaces with Kodaira dimension \(-\infty\) has already be dealt with. From the table of the classification theorem[2,2] one sees that for surfaces with Kodaira dimension 0, 1 one has \(c_1^2 - 3c_2 \leq 0\). For surfaces of general type this is the Bogomolov–Miyaoka–Yau inequality (cf. [11, §VII.4]). Consequently, \(p_g = q = 0\) and then necessarily \(S_X \cong \langle 1 \rangle\).

Next, consider the non-Kähler surfaces. The intersection form can either be positive definite or negative definite. In the former case, the index equals \(\tau = 2p_g\) and in the latter \(\tau = -b_2\) and \(p_g = 0\). From the list of Theorem 2.2 the surfaces concerned are the class VII surfaces, the Kodaira surfaces and the properly elliptic surfaces.

- For minimal class VII surfaces the list shows that \(\tau = \frac{1}{2}(c_1^2 - 2c_2) = -c_2 \leq 0\) and so one only the negative definite case has to be investigated. Since \(p_g = 0, b_2 = c_2\) and so the intersection form is negative definite if and only if \(b_2 > 0\). Minimal such surfaces have been constructed by Inoue in [3]. M. Kato has shown in [9] that these admit a holomorphically embedded copy of \(\{z \in \mathbb{C}^2 \mid 1 - \varepsilon < |z| < 1 + \varepsilon\}\) for some \(\varepsilon > 0\), and for which, moreover, the complement in the surface is connected. Conversely, any such Kato surface, by definition a compact complex surface containing such a so-called “global spherical shell” must be of class VII and is a deformation of a blown up primary Hopf surface (a complex surface diffeomorphic to \(S^3 \times S^1\)). This implies that the intersection form is diagonalizable and negative definite. By
Donaldson’s result \cite{4} this is true for any class VII surface with $b_2 > 0$.

- By \cite[Ch V.5]{1} minimal Kodaira surfaces either have $b_2 = 4$ and $p_g = 1$ (primary Kodaira surfaces). These have signature $(2, 2)$ and since the form is even, it is isometric to $U \oplus U$. In particular, these need not be considered, Else $b_2 = 0$, $p_g = 0$ (secondary Kodaira surfaces) with zero intersection form and so only non-minimal such surfaces have negative definite intersection form.

- Minimal non-Kähler elliptic surfaces. Since $c_1^2 = 0$ and $c_2 \geq 0$, the index theorem (11) shows that $\tau \leq 0$ and so we only need to consider the negative definite case. Then $p_g = 0$, and thus $p_g - q + 1 = -q + 1 = \frac{1}{12}c_2 \geq 0$ implying $q = 1$, $b_1 = 1$, $c_2 = b_2 = 0$. Again only non-minimal such surfaces have negative definite diagonalizable intersection form.

**Remark 3.2.** As a consequence of this result, in the case of compact complex surfaces the intersection form is completely determinable from the Stiefel–Whitney class class $w_2 = c_1 \mod 2$ (this determines whether the form is odd or even), $c_1$, and the Euler number. So the intersection form does not give supplementary topological information unlike for topological manifolds. It then follows from \cite{5} that the oriented homeomorphism type of a simply connected surface is uniquely determined by the invariants $w_2, c_1^2$ together with $c_2$. It is an open question whether this remain true for any compact complex surface if one adds the fundamental group to the list of invariants. One can at least say that the latter determines whether the surface is Kähler or not so that the two classes (Kähler or not) can be dealt with separately.

**References**

\[\begin{align*}
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\end{align*}\]

\footnote{It is conjectured that all class VII surfaces with $b_2 > 0$ are Kato surfaces, which would prove this directly.}
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