Fast Algorithms for Computing Eigenvectors of Matrices via Pseudo Annihilating Polynomials

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Abstract

An efficient algorithm for computing eigenvectors of a matrix of integers by exact computation is proposed. The components of calculated eigenvectors are expressed as polynomials in the eigenvalue to which the eigenvector is associated, as a variable. The algorithm, in principle, utilizes the minimal annihilating polynomials for eliminating redundant calculations. Furthermore, in the actual computation, the algorithm computes candidates of eigenvectors by utilizing pseudo annihilating polynomials and verifies their correctness. The experimental results show that our algorithms have better performance compared to conventional methods.

Keywords: Eigenvectors, Pseudo annihilating polynomial, Krylov vector space

2010 MSC: 15A18, 65F15, 68W30

\textsuperscript{*}This work has been partly supported by JSPS KAKENHI Grant Numbers JP15KT0102, JP18K0320, JP16K05035, and by the Research Institute for Mathematical Sciences, a Joint Usage/Research Center located in Kyoto University.

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November 26, 2018
1. Introduction

Exact linear algebra plays important roles in many fields of mathematics and sciences. In recent years, this area has been extensively studied and new algorithms have been proposed for various types of computations, such as computing canonical forms of matrices ([2], [9], [15], [22], [23], [25], [26]), the characteristic or the minimal polynomial of a matrix ([8], [17]), LU and other decompositions and/or solving a system of linear equations ([3], [10], [11], [14], [24]), and several software have been developed ([1], [4], [5], [6], [7]).

We have proposed, in the context of symbolic computation, a series of algorithms on eigenproblems including computation of (generalized) eigendecomposition and spectral decomposition ([19]). In this paper, we propose an effective method for computing eigenvectors of matrices of integers or rational numbers.

Let \( \lambda \) be an eigenvalue of a matrix. In a conventional method of computing eigenvectors, the eigenvector associated to \( \lambda \) is simply computed by solving a system of linear equations. However, the method has a drawback that, if \( \lambda \) is an algebraic number, it uses solving a system of linear equations with algebraic number arithmetic for computing the eigenvector, which is inefficient.

In the proposed method, the components of eigenvectors are expressed as polynomials in eigenvalues to which the eigenvector is associated, as a variable. Furthermore, in the case that \( \lambda \) is an algebraic number and the geometric multiplicity of \( \lambda \) is equal to its algebraic multiplicity, it is sufficient to compute just the algebraic multiplicity of \( \lambda \) of eigenvectors for expressing all the eigenvectors associated to all the conjugates of \( \lambda \). A method for computing eigenvectors in this form has been proposed by Takeshima and Yokoyama ([29]) in the 1990s by using the Frobenius normal form of \( A \), and it has been extended by Moritsugu and Kuriyama ([16]) for the case that the Frobenius normal form has multiple companion blocks and for computing generalized eigenvectors. In contrast, our approach is based on the concept of the minimal annihilating polynomials ([28]) and the Krylov vector spaces. We show that, with the use of minimal annihilating polynomials, eigenvectors are computed in an effective manner without
solving a system of linear equations. Furthermore, the proposed method does not require computation of canonical form of matrices.

We propose algorithms for computing eigenvectors under the assumption that the geometric multiplicity of the eigenvalue is equal to its algebraic multiplicity. The resulting algorithms have following features. First, pseudo minimal annihilating polynomials are used for faster computation of eigenvectors. Second, computation of a candidate of eigenvector is completed almost simultaneously as verification of pseudo annihilating polynomial. Notably, the Horner’s rule for matrices and vectors is used in an effective manner for fast evaluations.

This paper is organized as follows. In Section 2 we recall the notion of minimal annihilating polynomial and other necessary concepts. In Section 3 we describe a main idea of an algorithm for computing eigenvectors just for the case that the algebraic multiplicity of the eigenvalue is equal to 1. In Section 4 we give, by using Krylov vector spaces, an algorithm for computing eigenvectors in the case that the algebraic multiplicity of the eigenvalue is greater than 1. In Section 5 we introduce the concept of pseudo annihilating polynomial and present algorithms for computing eigenvectors using the pseudo annihilating polynomials. In Section 6 experimental results for the proposed algorithms are shown.

2. Preliminaries

Let $A$ be a $n \times n$ matrix over rational numbers, $\chi_A(\lambda)$ the characteristic polynomial of $A$, and $E$ the identity matrix of dimension $n$. Assume that the irreducible factorization

$$\chi_A(\lambda) = f_1(\lambda)^{m_1} f_2(\lambda)^{m_2} \cdots f_q(\lambda)^{m_q}$$

(1)

of $\chi_A(\lambda)$ is given, where $f_p(\lambda) \in \mathbb{Q}[\lambda]$, $p = 1, 2, \ldots, q$. 


2.1. The minimal annihilating polynomial

Let \( \mathbf{v} \) be a non-zero vector in \( \mathbb{Q}^n \). The monic generator of an ideal \( \text{Ann}_{\mathbb{Q}[\lambda]}(A, \mathbf{v}) \) defined to be

\[
\text{Ann}_{\mathbb{Q}[\lambda]}(A, \mathbf{v}) = \{ P(\lambda) \in \mathbb{Q}[\lambda] \mid P(A)\mathbf{v} = \mathbf{0} \},
\]

is called the minimal annihilating polynomial of \( \mathbf{v} \) with respect to \( A \). For \( j \in J := \{1, 2, \ldots, n\} \), let \( \mathbf{e}_j = e_j(0, \ldots, 0, 1, 0, \ldots, 0) \) be the \( n \) dimensional standard unit vector and let \( \pi_{A,j}(\lambda) \) denote the minimal annihilating polynomial of \( \mathbf{e}_j \) with respect to \( A \).

Let \( \pi_{A,j}(\lambda) = f_1(\lambda)^{l_1,j} f_2(\lambda)^{l_2,j} \cdots f_p(\lambda)^{l_p,j} \cdots f_q(\lambda)^{l_q,j} \),

\[
0 \leq l_p,j \leq m_p, \quad j \in J,
\]

be the irreducible factorization of \( \pi_{A,j}(\lambda) \).

Let \( g_{p,j}(\lambda) \) denote the cofactor in \( \pi_{A,j}(\lambda) \) of the eigenfactor \( f_p(\lambda) \) defined to be

\[
g_{p,j}(\lambda) = f_1(\lambda)^{l_1,j} \cdots f_{p-1}(\lambda)^{l_{p-1},j} f_{p+1}(\lambda)^{l_{p+1},j} \cdots f_q(\lambda)^{l_q,j}.
\]

2.2. Horner’s rule for matrix polynomials

Let \( f(\lambda) \) be a polynomial in \( \mathbb{Q}[\lambda] \) of degree \( d \):

\[
f(\lambda) = a_d\lambda^d + a_{d-1}\lambda^{d-1} + \cdots + a_0\lambda^0,
\]

with \( a_d \neq 0 \). Define \( \psi_f(x, y) \) as

\[
\psi_f(x, y) = \frac{f(x) - f(y)}{x - y}.
\]

Since, \( \psi_f(x, y) \) is the quotient of \( f(x) \) on division by \( x - y \), the coefficients \( c_i \in \mathbb{Q}[x] \), \( i = d - 1, d - 2, \ldots, 0 \), of the expansion

\[
\psi_f(x, y) = c_d y^{d-1} + c_1 y^{d-2} + \cdots + c_1 y + c_0,
\]

of \( \psi(x, y) \) with respect to \( y \) satisfy the following recursion relations:

\[
c_{d-1} = a_d, \quad c_{d-1-j} = c_{d-j}x + a_{d-j} \quad (j = 1, \ldots, d - 1).
\]
Let \( \mathbf{v} \in \mathbb{Q}^n \). Then, the vector \( f(A)\mathbf{v} \) and the coefficient vectors \( \mathbf{c}_i, i = d - 1, d - 2, \ldots, 0 \) are calculated by the Horner’s rule with multiplication of \( \mathbf{v} \) from the right as

\[
f(A)\mathbf{v} = (a_d A^d + a_{d-1} A^{d-1} + \cdots + a_0 E)\mathbf{v} = A(\cdots A(a_n (A\mathbf{v}) + a_{d-1}\mathbf{v}) + a_{d-2}\mathbf{v}) \cdots) + a_0 \mathbf{v},
\]

\[
\psi_f(A, \lambda E)\mathbf{v} = \lambda^{d-1}\mathbf{c}_{d-1} + \lambda^{d-2}\mathbf{c}_{d-2} + \cdots + \lambda \mathbf{c}_1 + \mathbf{c}_0,
\]

where \( \mathbf{c}_{d-1} = a_d \mathbf{v} \), \( \mathbf{c}_{d-1-j} = A\mathbf{c}_{d-j} + a_{d-j}\mathbf{v} \) \((j = 1, \ldots, d - 1)\), respectively. Thus, total cost is bounded by \( O(n^2) \) and \( O(n^2(d - 1)) \), respectively.

Notice that, \( f(A)\mathbf{v} = A\mathbf{c}_0 + a_0 \mathbf{v} \) holds. This relation will be used in Section 5.

**Lemma 1.** Let \( \mathbf{u} \in \mathbb{Q}^n \) be a non-zero vector and let \( f(\lambda) \) be the minimal annihilating polynomial of \( \mathbf{u} \) with respect to \( A \). Let \( \varphi(\lambda) = \psi_f(A, \lambda E)\mathbf{u} \). Let \( \alpha \) be a root of \( f(\lambda) \). Then, \( \varphi(\alpha) \) is an eigenvector of \( A \) associated to the eigenvalue \( \alpha \).

**Proof.** It follows immediately from \( (x - y)\psi_f(x, y) = f(x) - f(y) \) that

\[
(A - \lambda E)\varphi(\lambda) = (f(A) - f(\lambda E))\mathbf{u} = -f(\lambda)\mathbf{u}.
\]

Therefore, \( (A - \alpha E)\varphi(\alpha) = 0 \) holds. Since \( f \) is the minimal annihilating polynomial and \( \deg(\psi_f(x, y)) < \deg(f(\lambda)) \), we have \( \varphi(\alpha) \neq 0 \). This completes the proof. \( \square \)

Since \( f \) is a factor of the characteristic polynomial \( \chi_A(\lambda) \), we call \( \varphi(\lambda) \), an eigenvector associated to the eigenfactor \( f \), for simplicity.

Let us emphasize the fact that the eigenvector \( \varphi(\lambda) \) introduced above represents all the eigenvectors \( \varphi(\alpha_1), \varphi(\alpha_2), \ldots, \varphi(\alpha_d) \) associated to the eigenvalues \( \alpha_1, \alpha_2, \ldots, \alpha_d \) of \( A \) satisfying \( f(\lambda) = 0 \).

**3. Main ideas**

In this section, we show basic ideas of our approach for computing eigenvectors. For this aim, we consider the simplest case where algebraic multiplicity
$m_p$ of an eigenfactor $f_p$ is equal to one. We present a prototype of our method for illustration.

Assume that $m_p = 1$ and all the minimal annihilating polynomials $\pi_{A,1}(\lambda), \pi_{A,2}(\lambda), \ldots, \pi_{A,n}(\lambda)$ are given.

Let

$$J_0 = \{j \in J \mid l_{p,j} = 0\}, \quad J_1 = \{j \in J \mid l_{p,j} = 1\}. \tag{10}$$

Then, for $j \in J_1$, we have $\pi_{A,j}(\lambda) = f_p(\lambda)g_{p,j}(\lambda)$, where $g_{p,j}(\lambda)$ is the cofactor in $\pi_{A,j}$ of the eigenfactor $f_p(\lambda)$. Now consider the vector

$$v_{p,j} = g_{p,j}(A)e_j, \tag{11}$$

for $j \in J_1$. Then, since $f_p(\lambda)$ is the minimal annihilating polynomial of the non-zero vector $v_{p,j}$, $\varphi_j(\lambda)$ defined to be

$$\varphi_j(\lambda) = \psi_p(A, \lambda E)v_{p,j}, \tag{12}$$

is an eigenvector associate with the eigenfactor $f_p(\lambda)$, where $\psi_p(x, y) = \psi_{f_p}(x, y)$.

The argument above leads a prototype for computing eigenvectors as follows.

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**Input:** $A \in \mathbb{Q}^{n \times n}$; $f_p(\lambda) \in \mathbb{Q}[\lambda]$: an eigenfactor of $A$ with $m_p = 1$; $J_1 \subset J$; 

$\{g_{p,j}(\lambda) \mid j \in J_1\}$: a set of cofactors, defined as in eq. (11);

**Output:** $\varphi(\lambda)$: an eigenvector of $A$ associated to the root of $f_p(\lambda) = 0$;

1. Select $j \in J_1$;
2. $v_{p,j} \leftarrow g_{p,j}(A)e_j$ with the Horner’s rule (eq. (11));
3. $\varphi(\lambda) \leftarrow \psi_p(A, \lambda E)v_{p,j}$ with the Horner’s rule (eq. (12));
4. **return** $\varphi(\lambda)$;

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**Example 1.** Let

$$A = \begin{pmatrix}
-3 & -3 & -4 & 2 & 1 \\
-114 & 56 & 12 & 6 & -3 \\
330 & -179 & -50 & -11 & 12 \\
423 & -255 & -88 & -4 & 22 \\
-303 & 3 & -79 & 60 & 5
\end{pmatrix}.$$
The characteristic polynomial $\chi_A(\lambda)$ and the unit minimal annihilating polynomial $\pi_{A,j}(\lambda)$, $j = 1, 2, \ldots, 5$ are

\[
\chi_A(\lambda) = f_1(\lambda)f_2(\lambda),
\]
\[
\pi_{A,3}(\lambda) = f_2(\lambda), \pi_{A,1}(\lambda) = \pi_{A,2}(\lambda) = \pi_{A,4}(\lambda) = \pi_{A,5}(\lambda) = f_1(\lambda)f_2(\lambda),
\]

where $f_1(\lambda) = \lambda^2 + \lambda + 12$, $f_2(\lambda) = \lambda^3 - 5\lambda^2 - 60\lambda - 41$. Let us compute the eigenvector $\varphi(\lambda)$ associated to the eigenfactor $f_2(\lambda)$, by using $\psi_2(x, y) = y^2 + (x - 5)y + x^2 - 5x - 60$. Since $J_1 = \{1, 2, 3, 4, 5\}$, any vector from $v_{2,1}, v_{2,2}, v_{2,3}, v_{2,4}, v_{2,5}$ can be used. Here we take, for instance, two cases:

1. Computing $\varphi(\lambda)$ using $v_{2,3}$: since $g_{2,3}(\lambda) = 1$, $v_{2,3} = e_3$. The eigenvector $\varphi(\lambda)$ is computed as

\[
\varphi(\lambda) = \psi_2(A, \lambda E)e_3 = \{\lambda^2 E + \lambda(A - 5E) + (A^2 - 5A - 60E)\}e_3
\]
\[
= \lambda^2 e_3 + \lambda(\lambda e_3 - 5e_3) + (\lambda^2 e_3 - 5\lambda e_3 - 60e_3)
\]
\[
= \lambda(0, 0, 1, 0, 0)\lambda^2 + \lambda(-4, 12, -55, -88, -79)\lambda
\]
\[
+ \lambda(-59, 177, -758, -1298, -82).
\]

(13)

2. Computing $\varphi(\lambda)$ using $v_{2,1}$: since $g_{2,1}(\lambda) = f_1(\lambda)$,

\[
v_{2,1} = f_1(A)e_1 = (-417, 1251, -5043, -9174, -1941).
\]

(14)

The eigenvector $\varphi(\lambda)$ is computed as

\[
\varphi(\lambda) = \psi_2(A, \lambda E)v_{2,1} = \{\lambda^2 E + \lambda(A - 5E) + (A^2 - 5A - 60E)\}v_{2,1}
\]
\[
= \lambda^2 v_{2,1} + \lambda(Av_{2,1} - 5v_{2,1}) + (Av_{2,1} - 5v_{2,1} - 60v_{2,1})
\]
\[
= \lambda(-417, 1251, -5043, -9174, -1941)\lambda
\]
\[
+ \lambda(-534, 1602, -6552, -11748, -21939)\lambda
\]
\[
+ \lambda(2589, -7767, 33162, 56958, -13899).
\]

(15)

Now consider the Krylov vector space $L_A(v_{2,1}) = \text{Span}_Q\{v_{2,1}, Av_{2,1}, A^2v_{2,1}\}$
generated by \( v_{2,1} \). From
\[
\begin{align*}
v_{2,1} &= t(-417, 1251, -5043, -9174, -1941), \\
Av_{2,1} &= t(-2619, 7857, -31767, -57618, -31644), \\
A^2v_{2,1} &= t(-35526, 106578, -428253, -781572, -288579), \\
\end{align*}
\]
we have
\[
L_A(v_{2,1}) = \text{Span}_Q \left\{ t(1, -3, 0, 22, 0), e_3, e_5 \right\}.
\]

Therefore, the vector
\[
\psi_2(A, \lambda E)e_5 = t(0, 0, 0, 0, 1)\lambda^2 + t(1, -3, 12, 22, 0)\lambda \\
+ t(2, -6, 25, 44, 0), \tag{16}
\]
constructed from the last basis vector \( e_5 = t(0, 0, 0, 0, 1) \) in \( V = L_A(v_{2,1}) \) is also an eigenvector associated to the eigenfactor \( f_2(\lambda) \).

Notice that \( \psi_2(A, \lambda E)e_3, \psi_2(A, \lambda E)e_5 \) have simpler expression than \( \psi_2(A, \lambda E)v_{2,1} \).

Notice also that, in all cases, the leading coefficient vector in \( \phi(\lambda) = \psi_2(A, \lambda E)u \) is equal to \( u \) itself. Note also that, if we consider \( V = \text{Span}_Q \left\{ v_{2,1}, v_{2,2}, \ldots, v_{2,5} \right\} \), we also have
\[
V = \text{Span}_Q \left\{ t(1, -3, 0, 22, 0), e_3, e_5 \right\}.
\]

Let us turn back to the case \( m_p = 1 \). Let
\[
V = \text{Span}_Q \left\{ v_{p,j} \mid v_{p,j} = g_{p,j}(A)e_j, \ j \in J_1 \right\}.
\]
Then,
\[
\phi(\lambda) = \psi_p(A, \lambda E)u,
\]
constructed from any non-zero vector \( u \) in \( V \) gives rise to an eigenvector associated to the eigenfactor \( f_p(\lambda) \). In other words, every non-zero vector \( u \in V \) has the same amount of information on eigenspaces associated to the eigenfactor \( f_p(\lambda) \). In fact, if we consider the Krylov vector space \( L_A(u) \) defined to be
\[
L_A(u) = \text{Span}_Q \left\{ u, Au, A^2u, \ldots, A^{d_p-1}u \right\}, \tag{17}
\]
8
for $\mathbf{u} \in V$, with $d_p = \deg(f_p)$, we have $V = L_A(\mathbf{u})$.

This observation above yields the following two different strategies for computing eigenvectors for the case $m_p = 1$.

(a) The prototype method presented above requires $O(n^2 \deg(\pi_{A,j}))$ operations for computing $\varphi(\lambda)$. This suggests that, if one wants to obtain the eigenvector as quickly as possible, one should select the unit minimal annihilating polynomial $\pi_{A,j}(\lambda)$ of (1) smaller degree, or (2) if there are several ones of the same degree, select one with coefficients of smaller magnitudes, to reduce the amount of computation.

(b) Recall the fact that the leading coefficient vector in $\psi_p(A, \lambda E)\mathbf{u}$ is equal to $\mathbf{u}$ itself. Therefore, if one wants to obtain the eigenvector which has a simple expression, it might be better to select a non-zero vector $\mathbf{u}$ from $V$. We arrive at the following strategy:

(i) select $j$ from $J_1$ as in (a) above;
(ii) compute a basis $B$ of the Krylov vector space $L_A(\mathbf{u})$ by column reductions;
(iii) select a basis vector $\mathbf{u}$ from $B$ that has a simple form, or choose an appropriate one;
(iv) compute the eigenvector $\varphi(\lambda) = \psi_p(A, \lambda E)\mathbf{u}$.

4. Krylov vector space

Let $f_p(\lambda) \in \mathbb{Q}[\lambda]$ be an eigenfactor of $A$, which satisfies the condition $\max_{j \in J}\{l_{p,j}\} = 1$. We give an algorithm for computing the eigenvectors for all the roots $\alpha_1, \alpha_2, \ldots, \alpha_d$ of $f_p(\lambda)$, where $d = d_p$ stands for the degree of $f_p$.

For $i = 1, \ldots, d$, let $F_{p,\alpha_i}$ be the eigenspace associated to the eigenvalue $\alpha_i$, and $F_p$ be the eigenspace associated to the roots of $f_p(\lambda) = 0$. Since the condition $\max_{j \in J}\{l_{p,j}\} = 1$ implies $\dim(F_{p,\alpha_i}) = m_p$, we have $\dim_{\mathbb{C}}(F_p) = d_p m_p = d m_p$. The purpose of this section is therefore to describe a method
for computing \( dm_p \) eigenvectors that constitute a basis of the eigenspace \( F_p \)
associated to the eigenfactor \( f_p(\lambda) \).

Let \( V = \text{Span}_Q\{v_{p,j} | j \in J_1\} \), where \( v_{p,j} \) are defined as in eq. (11). For \( v \in V \), let \( L_A(v) \) be as in eq. (17). We have the following proposition.

**Proposition 2.** Let \( u, w \in V \) with \( u, w \neq 0 \), and \( \alpha_1, \ldots, \alpha_d \) be the roots of \( f_p(\lambda) = 0 \). Then, the followings are equivalent:

(i) \( \text{Span}_C\{\psi_p(A, \alpha_i)u | i = 1, \ldots, d\} = \text{Span}_C\{\psi_p(A, \alpha_i)w | i = 1, \ldots, d\} \),

(ii) \( L_A(u) = L_A(w) \),

(iii) \( w \in L_A(u) \),

(iv) \( u \in L_A(w) \).

**Proof.** Since \( f_p(\lambda) \) is the minimal annihilating polynomial of \( u \in V \), \( u, Au, A^2u, \ldots, A^{d-1}u \) are linearly independent. Furthermore, \( A^k u \) satisfies

\[
\psi_p(A, \lambda E)(A^k u) = A^k(\psi_p(A, \lambda E)u) = \alpha_1^k(\psi_p(A, \lambda E)u),
\]

which shows that (i) and (iii) are equivalent. Next, \( w, Aw, A^2w, \ldots, A^{d-1}w \)
satisfy

\[
\psi_p(A, \alpha_i E)(A^k w) = A^k(\psi_p(A, \alpha_i E)w) = \alpha_i^k(\psi_p(A, \alpha E)w),
\]
as in eq. (18). Since \( \psi_p(A, \alpha E)u \) is equal to \( \psi_p(A, \alpha E)w \) up to a scalar, we see that (iii) and (iv) are equivalent, thus we also have (ii) is equivalent to the others, which completes the proof. \( \square \)

Since there exist \( m_p \) vectors \( u_1, u_2, \ldots, u_{m_p} \in V \) that satisfy

\[ V = L_A(u_1) \oplus L_A(u_2) \oplus \cdots \oplus L_A(u_{m_p}), \]

we have

\[ F_p = \text{Span}_C\{\psi_p(A, \lambda)u_k | \lambda = \alpha_1, \alpha_2, \ldots, \alpha_d, k = 1, \ldots, m_p\}, \]

thus \( \psi_p(A, \lambda)u_k \) \( (k = 1, \ldots, m_p) \) are the desired eigenvectors.

Discussions above leads an algorithm for computing eigenvectors; see Algorithm II.
Algorithm 1 Computing eigenvectors in the case that true unit minimal annihilating polynomials are given

**Input:** $A \in \mathbb{Q}^{n \times n}; f_p(\lambda) \in \mathbb{Q}[\lambda]:$ an eigenfactor of $A; J_1 \subset J$ satisfying eq. (10); 
\{g_{p,j}(\lambda) \mid j \in J_1\}: a set of cofactors, defined as in eq. (11); 

**Output:** $\Phi = \{\varphi_1(\lambda), \ldots, \varphi_{m_p}(\lambda)\}:$ the eigenvectors of $A$ associated to the root of $f_p(\lambda) = 0$;

1: $\Phi \leftarrow \{\}; L \leftarrow \{\};$
2: for $j \in J_1$ do $v_j \leftarrow g_{p,j}(A)e_j$ with the Horner’s rule (eq. (9));
3: end for
4: Calculate a basis $B$ of $V = \text{Span}_\mathbb{Q}\{v_j \mid j \in J_1\};$
5: for $k = 1, \ldots, m_p - 1$ do
6: Choose $u \in B$ satisfying $u \notin L$ which has the “simplest” form;
7: Calculate $K_A(u) = \{u, Au, \ldots, A^{d-1}u\}$;
8: $\varphi_k(\lambda) \leftarrow \text{CALCULATEEIGENVECTOR}(f_p(\lambda), K_A(u)); \triangleright$ See Remark 1
9: Calculate a basis of $L_A(u)$ from $K_A(u)$ by the column reduction;
10: $\Phi \leftarrow \Phi \cup \{\varphi_k(\lambda)\};$
11: $L \leftarrow L \oplus L_A(u);$
12: end for
13: Choose $u \in B$ satisfying $u \notin L$ which has the “simplest” form; \triangleright Note that this step does not require computing $L_A(u)$, etc.
14: $\varphi_{m_p}(\lambda) \leftarrow \psi_p(A, \lambda E)u; \quad \triangleright$ Calculated using the Horner’s rule (eq. (9))
15: $\Phi \leftarrow \Phi \cup \{\varphi_{m_p}(\lambda)\};$
16: return $\Phi;$
Remark 1. In Line 8 in Algorithm 1 eigenvectors are computed using the Krylov vectors calculated in the preceding lines, as shown in Procedure \textsc{CalculateEigenvector} below.

\textbf{Input:} $f_p(\lambda) = \lambda^d + a_{p,d-1}\lambda^{d-1} + \cdots + a_{p,0} \in \mathbb{Q}[\lambda]$: an eigenfactor of $A$ with $a_{p,d} = 1$; $K_A$: a list of $d$ vectors of dimension $n$;
\textbf{Output:} $\varphi(\lambda) = \psi_p(A, \lambda E)u$: an eigenvector of $A$ associated to the root of $f_p(\lambda) = 0$;

1: \textbf{procedure} \textsc{CalculateEigenvector}(f_p(\lambda), K_A)
2: \hspace{1em} \textbf{for} $j = 1, \ldots, d$ \textbf{do} $c_{d-j} \leftarrow \sum_{k=0}^{j-1} a_{p,d-k}K_A[j-k]$; $\triangleright K_A[i]$ denotes the $i$-th element in $K_A$
3: \hspace{1em} \textbf{end for}
4: \textbf{return} $\lambda^{d-1}c_{d-1} + \lambda^{d-2}c_{d-2} \cdots + \lambda c_1 + c_0$;
5: \textbf{end procedure}

Remark 2. In several lines in Algorithm 1 we take vectors of “the simplest form” from the certain set of vectors. “The simplest form” may be different according to different criteria, such as bit-length of the components, or the number of zero components in the calculated vectors.

5. Main results

Algorithm 1 uses the minimum annihilating polynomials effectively for computing eigenvectors. However, direct use of the minimum annihilating polynomials often leads to relatively high computational complexity.

In this section, the unit \textit{pseudo} annihilating polynomials (28) are utilized for efficient computation of eigenvectors. Pseudo annihilating polynomials are suitable for computing eigenvectors in place of the minimal annihilating polynomials because they coincide with high possibility. In addition, computation of pseudo annihilating polynomials is more efficient than that of the minimal annihilating polynomials.
First, we recall the notion of unit pseudo annihilating polynomials from our previous paper \([28]\). Let \(r\) be a non-zero row vector over \(\mathbb{Z}\) whose components are randomly given. Let

\[
\begin{align*}
\mathbf{r}^{(0)}_p &= (r^{(0)}_{p,1}, r^{(0)}_{p,2}, \ldots, r^{(0)}_{p,n}) = \mathbf{r}G_p, \\
\mathbf{r}^{(k)}_p &= (r^{(k)}_{p,1}, r^{(k)}_{p,2}, \ldots, r^{(k)}_{p,n}) = \mathbf{r}G_p F^k_p \\
\end{align*}
\]

where \(G_p = g_{p,j}(A)\) and \(F_p = f_p(A)\). Furthermore, for \(j = 1, \ldots, n\), define

\[
l'_{p,j} = \begin{cases} 
0 & \text{if } r^{(0)}_{p,j} = 0, \\
k & \text{if } r^{(k-1)}_{p,j} \neq 0 \text{ and } r^{(k)}_{p,j} = 0.
\end{cases}
\]

Consider the polynomial \(\pi'_{A,j}(\lambda)\) defined by

\[
\pi'_{A,j}(\lambda) = f_1^{l'_{1,j}}(\lambda) f_2^{l'_{2,j}}(\lambda) \cdots f_q^{l'_{q,j}}(\lambda).
\]

We call \(\pi'_{A,j}(\lambda)\) a \(j\)-th unit pseudo annihilating polynomial of \(A\). Notice that \(\pi'_{A,j}(\lambda)\) divides \(\pi_{A,j}(\lambda)\). Therefore, \(\pi'_{A,j}(\lambda) = \pi_{A,j}(\lambda)\) if and only if \(\pi'_{A,j}(\lambda)e_j = \pi_{A,j}(\lambda)e_j\). In the previous paper \([28]\), an effective method for computing \(\pi'_{A,j}(\lambda)\) for \(j \in J\) is given.

For \(j \in J\), let

\[
g'_{p,j}(\lambda) = f_1(\lambda)^{l'_{1,j}} \cdots f_{p-1}(\lambda)^{l'_{p-1,j}} f_{p+1}(\lambda)^{l'_{p+1,j}} \cdots f_q(\lambda)^{l'_{q,j}},
\]

and

\[
J'_j = \{j \in J | l'_{p,j} = 1\}.
\]

Next, we consider for \(j \in J'_j\), two vectors \(v'_j\) and \(\varphi'(\lambda)\) defined to be

\[
v'_j = g'_{p,j}(A)e_j \quad \text{and} \quad \varphi'(\lambda) = \psi_p(A, \lambda E)v'_j,
\]

respectively. Since \(f_p(A) = (A - \lambda E)\psi_p(A, \lambda E)\), we have

\[
\begin{align*}
\pi'_{A,j}(A)e_j &= f_p(A)g'_{p,j}(A)e_j \\
&= f_p(A)v'_{p,j} \\
&= (A - \lambda E)\psi_p(A, \lambda E)v'_{p,j}.
\end{align*}
\]

Therefore, \(\pi'_{A,j}(\lambda) = \pi_{A,j}(\lambda)\) if and only if \(\varphi'(\lambda) = \psi_p(A, \lambda E)v'\) is a true eigenvector associated to the eigenfactor \(f_p(\lambda)\). Furthermore, the formula above
shows that the calculation of $\varphi'(\lambda)$ is contained in the calculation by the Horner’s rule of $\pi'_{A,j}(\lambda)e_j$. More precisely, as we have mentioned in Section 2, $f_p(A)v'_j = \pi'_{A,j}(A)e_j$ is obtained from $\varphi'(\lambda) = \psi_p(A,\lambda E)v'_j$ by just one last step of the Horner’s rule:

$$Ac_0 + a_0v',$$  

(24)

where $\varphi'(\lambda) = \lambda^d c_{d-1} + \lambda^{d-2}c_{d-2} + \cdots + \lambda c_1 + c_0$.

Now, recall a method for computing the minimal annihilating polynomials $\pi_{A,j}(\lambda)$, $j \in J$ proposed in [28]. The method consists of mainly three steps:

Step 1. Compute unit pseudo annihilating polynomials $\pi'_{A,j}(\lambda)$ for $j \in J$.

Step 2. Compute $\pi'_{A,j}(\lambda)e_j$ for $j \in J$ by the Horner’s rule.

Step 3. If $\pi'_{A,j}(\lambda)e_j \neq 0$ for some $j$, then construct the minimal annihilating polynomial $\pi_{A,j}(\lambda)$ by computing the minimal annihilating polynomial of the vector $\pi'_{A,j}(\lambda)e_j$.

The discussion given in the present section shows that Step 2 involves the computation of a lot of eigenvectors. However, all the information on eigenvalues are discarded by the direct use of the Horner’s rule. We conclude, in this regard, that Algorithm 1 which utilizes the true unit minimal annihilating polynomials $\pi_{A,j}(\lambda)$ for $j \in J_1$ has redundancy.

Now we are ready to design an efficient method for computing eigenvectors associated to the eigenfactor $f_p(\lambda)$. Let $l'_p = \max_{j \in J}\{l'_p,j\}$. Assume hereafter that $l'_p = 1$ and set

$$J'_1 = \{j \in J \mid l'_p,j = 1\}, \quad J'_0 = \{j \in J \mid l'_p,j = 0\}.$$

Let

$$G' = \{v'_j = g'_{p,j}(\lambda)e_j \mid j \in J'_1\}, \quad V' = \text{Span}_Q\{v'_j \mid j \in J'_1\},$$

$B'$ denotes a basis of the vector space $V'$. We present two different algorithms. Algorithm 2 uses the set $G'$ and Algorithm 3 uses the set $B'$. Algorithm 2 is designed to compute eigenvectors in an efficient manner. In contrast, Algorithm 3 is designed with an intention of obtaining simpler expression of eigenvectors.
Algorithm 2 Computing eigenvectors with unit pseudo annihilating polynomials (for quick computation of eigenvectors)

**Input:** \( A \in \mathbb{Q}^{n \times n}; f_p(\lambda) \in \mathbb{Q}[\lambda]: \) an eigenfactor of \( A; J'_1 \subset J \) satisfying eq. (23);
\( \{g'_{p,j}(\lambda) \mid j \in J'_1\}: \) a set of cofactors, defined as in eq. (22);

**Output:** \( \Phi = \{\varphi_1(\lambda), \ldots, \varphi_{m_p}(\lambda)\}: \) the eigenvectors of \( A \) associated to the root of \( f_p(\lambda) = 0; \)

1: \( \Phi \leftarrow \{\}; G' \leftarrow \{\}; L \leftarrow \{\}; \\
2: \text{for } j \in J'_1 \text{ do} \\
3: \quad v'_j \leftarrow g'_{p,j}(A)e_j \text{ with the Horner's rule (eq. (9));} \\
4: \quad G' \leftarrow G' \cup \{v'_j\}; \\
5: \text{end for} \\
6: \text{for } k = 1, \ldots, m_p - 1 \text{ do} \\
7: \quad \text{Choose } u' \in G' \text{ satisfying } u' \not\in L \text{ which has the "simplest" form;} \\
8: \quad G' \leftarrow G' \setminus \{u'\}; \\
9: \quad \text{Calculate } K_A(u') = \{u', Au', \ldots, A^{d-1}u'\}; \\
10: \quad \varphi_k(\lambda) \leftarrow \text{CALCULATEEIGENVECTOR}(f_p(\lambda), K_A(u')); \triangleright \text{See Remark[1]} \\
11: \quad \text{if } f_p(A)u' = 0 \text{ then} \triangleright \text{Calculated as in eq. (24)} \\
12: \quad \text{Calculate a basis of } L_A(u') \text{ from } K_A(u') \text{ by the column reduction;} \\
13: \quad \Phi \leftarrow \Phi \cup \{\varphi_k(\lambda)\}; \\
14: \quad L \leftarrow L \oplus L_A(u'); \\
15: \quad \text{else go to Line 7} \\
16: \text{end if} \\
17: \text{end for} \\
18: \text{Choose } u' \in G' \text{ satisfying } u' \not\in L \text{ which has the "simplest" form;} \triangleright \text{Note that this step does not require calculating } L_A(u'), \text{ etc.} \\
19: \quad G' \leftarrow G' \setminus \{u'\}; \\
20: \quad \varphi_{m_p}(\lambda) \leftarrow \psi_p(A, \lambda E)u'; \triangleright \text{Calculated using the Horner's rule (eq. (9))} \\
21: \quad \text{if } f_p(A)u' = 0 \text{ then } \Phi \leftarrow \Phi \cup \{\varphi_{m_p}(\lambda)\}; \triangleright \text{Calculated as in eq. (24)} \\
22: \quad \text{else go to Line 18} \\
23: \text{end if} \\
24: \text{return } \Phi;
Algorithm 3 Computing eigenvectors with unit pseudo annihilating polynomials

**Input:** \( A \in \mathbb{Q}^{n \times n}; f_p(\lambda) \in \mathbb{Q}[\lambda]: \) an eigenfactor of \( A; J'_1 \subset J \) satisfying eq. (23); \( \{g'_{p,j}(\lambda) \mid j \in J'_1\}: \) a set of cofactors, defined as in eq. (22);

**Output:** \( \Phi = \{\varphi_1(\lambda), \ldots, \varphi_{m_p}(\lambda)\}: \) the eigenvectors of \( A \) associated to the root of \( f_p(\lambda) = 0; \)

1: \( \Phi \leftarrow \{\}; L \leftarrow \{\}; \)
2: for \( j \in J'_1 \) do
3: \( v'_j \leftarrow g'_{p,j}(A)e_j \) with the Horner’s rule (eq. (9));
4: \( c_j \leftarrow (\text{a random integer}); \)
5: end for
6: \( v' \leftarrow \sum_{j \in J'_1} c_j v'_j; \)
7: if \( f_p(A)v' \neq 0 \) then exit with an error message: “One or more pseudo annihilating polynomial(s) are wrong”;\)
8: end if
9: Calculate \( B' \) as a basis of \( V' = \text{Span}_\mathbb{Q}\{v'_j \mid j \in J'_1\}; \)
10: for \( k = 1, \ldots, m_p - 1 \) do
11: Choose \( u' \in B' \) satisfying \( u' \notin L \) which has the “simplest” form;
12: \( B' \leftarrow B' \setminus \{u'\}; \)
13: Calculate \( K_A(u') = \{u', Au', \ldots, A^{d-1}u'\}; \)
14: \( \varphi_k(\lambda) \leftarrow \text{CALCULATE Eigenvector}(f_p(\lambda), K_A(u')); \) \( \triangleright \text{See Remark}\)
15: if \( f_p(A)u' = 0 \) then \( \triangleright \text{Calculated as in eq. (24)} \)
16: Calculate a basis of \( L_A(u') \) from \( K_A(u') \) by the column reduction;
17: \( \Phi \leftarrow \Phi \cup \{\varphi_k(\lambda)\}; \)
18: \( L \leftarrow L \oplus L_A(u'); \)
19: else exit with an error message: “One or more pseudo annihilating polynomial(s) are wrong”;
20: end if
21: end for
Algorithm 3 Computing eigenvectors with unit pseudo annihilating polynomials (Continued)

22: Choose \( u' \in B' \) satisfying \( u' \notin L \) which has the “simplest” form; \( \triangleright \) Note that this step does not require calculating \( L_A(u') \), etc.

23: \( B' \leftarrow B' \setminus \{u'\} \);

24: \( \varphi_{mp}(\lambda) \leftarrow \psi_p(A, \lambda E)u' \); \( \triangleright \) Calculated using the Horner’s rule (eq. (9))

25: if \( f_p(A)u' = \mathbf{0} \) then \( \Phi \leftarrow \Phi \cup \{\varphi_{mp}(\lambda)\} \); \( \triangleright \) Calculated as in eq. (24)

26: else exit with an error message: “One or more pseudo annihilating polynomial(s) are wrong”;

27: end if

28: return \( \Phi \);

Remark 3. In Algorithm 2 \( v'_j = g'_p,j(A) e_j \) are directly used for efficient construction of candidates of eigenvectors. Furthermore, in the case that \( \varphi'(\lambda) = \psi_p(A, \lambda E)v'_j \) is not a true eigenvector, another candidate is computed immediately just by picking up \( v'_{j'} \in G' \) with \( j' \neq j \). We continue to pick up new \( v'_j \) until \( m_p \) eigenvectors are computed.

Remark 4. In Algorithm 3 if there is a vector \( u' \in B' \) which does not satisfy the condition \( f_p(A)u' = \mathbf{0} \), there may exist many such vectors, because \( B' \) is calculated from \( G' \) by column reduction. Therefore, in the case when such vector is detected, we recalculate pseudo annihilating polynomials of \( A \) and start over computation of Algorithm 3.

Remark 5. In both algorithms, it is sufficient to verify \( f_p(A)u'_k = \mathbf{0} \) only for vectors \( u'_1, \ldots, u'_m \) in the basis \( V' = \text{Span}_Q \{v'_{j} \mid j \in J'_1\} \). This reduces the cost of computation considerably.

6. Experiments

We have implemented Algorithms 2 and 3 on a computer algebra system Risa/Asir ([18]) and evaluated them. First, for the case of \( m_p = 1 \), we have computed eigenvectors with changing \( \dim(A) \) and \( \deg(\pi'_{A,j}(\lambda)) \). Second, we
have computed eigenvectors for the case $m_p = 2, 3, 4$ with focusing attention on calculation and reduction of “seeds” of eigenvectors. Finally, we have compared performance of our algorithms with an algorithm implemented on Maple (13).

The tests were carried out on the following environment: Intel Xeon E5-2690 at 2.90 GHz, RAM 128GB, Linux 2.6.32 (SMP).

6.1. Computing eigenvectors with $m_p = 1$

In this experiment, test matrices are given as follows. Let $f_1(x), \ldots, f_8(x)$ be monic and pairwise relatively prime polynomials of the same degree. For $\bar{A} = \text{diag}(C(f_1), C(f_2), \ldots, C(f_8))$, where $C(f)$ denotes the companion matrix of $f$, we have calculated dense test matrix $A$ by applying similarity transformations.

Tables 1 and 2 show the results with changing dimension of the matrix. In the amount of memory usage, “$a \times 10^b$” denotes $a \times 10^b$ (bytes). In Table 1, eigenvectors are computed with pseudo annihilating polynomials of degree $\text{deg}(\pi_{A,j}(\lambda)) = \text{dim}(A)$. On the other hand, in Table 2 eigenvectors are computed with pseudo annihilating polynomials of degree $\text{deg}(\pi_{A,j}(\lambda)) = \text{dim}(A)/4$.

Tables 3 and 4 show the results using $\pi_{A,j}(\lambda)$ of different degrees for the same matrix.

From both experiments, we see that eigenvectors are computed more efficiently by using pseudo annihilating polynomials of smaller degrees.

6.2. Computing eigenvectors with $m_p > 1$

In this experiment, test matrices are given in the same way as above. In test matrices, the number of $\pi_{A,j}(\lambda)$ with $l_{p,j} = 1$ is approximately equal to $\text{dim}(A)/4$. Among them, approximately half of them have degree $\text{dim}(A)/4$, the other half have degree $\text{dim}(A)$. For each cases, the same test matrices are used.

Table 5 shows the results for Algorithm 2. “Time ($G'$)" denotes time for computing $G'$, the set of “seeds" of eigenvectors (lines 2–5). “# $G'$” denotes the number of elements in $G'$. Although computing time of $G'$ is long for large $A$, it
Table 1: Computing time and memory usage for the case of $\deg(\pi_{A,j}(\lambda)) = \dim(A)$. See Section 6.1 for details.

| dim(A) | $\deg(\pi_{A,j})$ | Time (sec.) | Memory usage (bytes) |
|-------|-------------------|-------------|----------------------|
| 128   | 128               | 0.205       | 2.37e8               |
| 256   | 256               | 2.037       | 2.11e9               |
| 384   | 384               | 8.971       | 8.76e9               |
| 512   | 512               | 29.57       | 2.61e10              |
| 640   | 640               | 50.48       | 4.37e10              |
| 768   | 768               | 105.58      | 9.22e10              |
| 896   | 896               | 164.75      | 1.50e11              |
| 1024  | 1024              | 289.72      | 2.57e11              |

Table 2: Computing time and memory usage for the case of $\deg(\pi_{A,j}(\lambda)) = \dim(A)/4$. See Section 6.1 for details.

| dim(A) | $\deg(\pi_{A,j})$ | Time (sec.) | Memory usage (bytes) |
|-------|-------------------|-------------|----------------------|
| 128   | 32                | 0.033       | 4.50e7               |
| 256   | 64                | 0.322       | 3.78e8               |
| 384   | 96                | 1.379       | 1.47e9               |
| 512   | 128               | 3.598       | 3.55e9               |
| 640   | 160               | 5.471       | 5.54e9               |
| 768   | 192               | 12.88       | 1.13e10              |
| 896   | 224               | 21.19       | 1.78e10              |
| 1024  | 256               | 34.98       | 3.01e10              |
Table 3: Computing time and memory usage for the case of dim($A$) = 128 with increasing the degree of the minimal annihilating polynomial. See Section 6.1 for details.

| $\deg(\pi_{A,j})$ | Time (sec.) | Memory usage (bytes) |
|-------------------|-------------|----------------------|
| 32                | 0.033       | 4.50e7               |
| 48                | 0.054       | 7.47e7               |
| 64                | 0.085       | 1.05e8               |
| 80                | 0.109       | 1.37e8               |
| 96                | 0.144       | 1.73e8               |
| 112               | 0.170       | 2.04e8               |
| 128               | 0.204       | 2.37e8               |

Table 4: Computing time and memory usage for the case of dim($A$) = 1024 with increasing the degree of the minimal annihilating polynomial. See Section 6.1 for details.

| $\deg(\pi_{A,j})$ | Time (sec.) | Memory usage (bytes) |
|-------------------|-------------|----------------------|
| 256               | 34.98       | 3.01e10              |
| 384               | 61.80       | 5.29e10              |
| 512               | 95.06       | 7.98e10              |
| 640               | 135.71      | 1.16e11              |
| 768               | 172.33      | 1.54e11              |
| 896               | 222.82      | 2.02e11              |
| 1024              | 289.72      | 2.57e11              |
can be reduced by the use of parallel processing (e.g. [12]) since all the vectors in $G'$ can be calculated independently with the Horner’s rule.

Table 6 shows the results for Algorithm 3. "Time ($B'$)" denotes computing time of $B'$ (line 9) from construction of $G'$.

In Table 7, "max$\{\|\varphi_2(\lambda)\|_2\}$" and "max$\{\|\varphi_3(\lambda)\|_2\}$" denote the maximum values of the 2-norms of eigenvectors computed by Algorithms 2 and 3, respectively. Notice that, in Algorithm 3, the norm of computed eigenvectors has remarkably decreased.

6.3. Comparison of performance with Maple

In this experiment, Test matrices are given as $A = (a_{ij})$ with integers $a_{ij}$ satisfying $|a_{ij}| < 10$ and dim($A$) = $8s$ with $s = 1, 2, \ldots, 7$. We have executed “LinearAlgebra:-Eigenvectors” function with “implicit=true” option for expressing eigenvalues as the characteristic polynomial. In each degree, we have measured computing time and memory usage for computing eigenvectors of the same matrix for 5 times and have taken the average.

Table 8 shows the results with computing time in seconds and memory usage in bytes. Furthermore, since Maple calculates the characteristic polynomial of the matrix, we have measured computing time for calculating characteristic polynomial $\chi_A(\lambda)$ of the given matrices independently, which is shown in the rightmost column in the table. We see that, in each dimension of $A$, computing time for the characteristic polynomial accounts only a small portion of computing time for eigenvectors. This result demonstrates efficiency of our method.

7. Concluding remarks

In this paper, we have proposed efficient algorithms for computing eigenvector of matrices of integers under the assumption that the geometric multiplicity of the eigenvalue is equal to the algebraic multiplicity. The resulting algorithms utilize pseudo unit annihilating polynomials, the Horner’s rule for matrix polynomial with vectors and Krylov vector spaces in an efficient manner.
Table 5: Computing time and memory usage of Algorithm 2 for the case of \( m_p > 1 \). See Section 6.2 for details.

| \( \text{dim}(A) \) | \( \deg(f_p) \) | \( m_p \) | Time (sec.) | Memory usage | Time \((G')\) | \#\(G'\) |
|-----------------|-------------|------|----------|-------------|----------|------|
| 128             | 4           | 2    | 4.072    | 4.61e9      | 4.008    | 28   |
| 128             | 4           | 3    | 3.992    | 4.44e9      | 3.860    | 32   |
| 128             | 4           | 4    | 4.304    | 4.48e9      | 3.780    | 32   |
| 256             | 8           | 2    | 77.21    | 8.14e10     | 75.60    | 64   |
| 256             | 8           | 3    | 71.12    | 7.00e10     | 65.33    | 64   |
| 256             | 8           | 4    | 89.37    | 7.86e10     | 76.52    | 64   |
| 512             | 16          | 2    | 1819.1   | 1.79e12     | 1797.61  | 128  |
| 512             | 16          | 3    | 1319.6   | 1.21e12     | 1243.96  | 128  |
| 512             | 16          | 4    | 2302.3   | 1.75e12     | 1780.29  | 128  |

Table 6: Computing time and memory usage of Algorithm 3 for the case of \( m_p > 1 \). See Section 6.2 for details.

| \( \text{dim}(A) \) | \( \deg(f_p) \) | \( m_p \) | Time (sec.) | Memory usage | Time \((B')\) |
|-----------------|-------------|------|----------|-------------|----------|
| 128             | 4           | 2    | 4.156    | 4.74e9      | 0.104    |
| 128             | 4           | 3    | 4.232    | 4.61e9      | 0.204    |
| 128             | 4           | 4    | 4.192    | 4.45e9      | 0.100    |
| 256             | 8           | 2    | 77.81    | 8.12e10     | 0.640    |
| 256             | 8           | 3    | 69.14    | 6.98e10     | 1.308    |
| 256             | 8           | 4    | 85.86    | 7.86e10     | 2.608    |
| 512             | 16          | 2    | 1840.7   | 1.79e12     | 9.376    |
| 512             | 16          | 3    | 1355.7   | 1.21e12     | 27.34    |
| 512             | 16          | 4    | 2253.5   | 1.76e12     | 112.94   |
Table 7: The maximum value of 2-norms of eigenvectors computed by Algorithm $\phi_2(\lambda)$ and Algorithm $\phi_3(\lambda)$. See Section 6.2 for details.

| dim($A$) | deg($f_p$) | $m_p$ | $\max\{\|\phi_2(\lambda)\|_2\}$ | $\max\{\|\phi_3(\lambda)\|_2\}$ |
|----------|------------|-------|----------------------------------|----------------------------------|
| 128      | 4          | 2     | 4.03e24                          | 2.38e3                           |
| 128      | 4          | 3     | 6.64e8                           | 1.03e3                           |
| 128      | 4          | 4     | 2.78e10                          | 1.34e2                           |
| 256      | 8          | 2     | 4.75e15                          | 8.76e1                           |
| 256      | 8          | 3     | 6.90e15                          | 5.66e1                           |
| 256      | 8          | 4     | 2.05e29                          | 1.01e2                           |
| 512      | 16         | 2     | 1.18e27                          | 1.17e2                           |
| 512      | 16         | 3     | 1.40e26                          | 1.06e2                           |
| 512      | 16         | 4     | 6.61e49                          | 1.01e2                           |

Table 8: Computing time and memory usage by Maple. See Section 6.3 for details.

| dim($A$) | Time (sec.) | Memory usage | Time for $\chi_A(\lambda)$ |
|----------|-------------|--------------|------------------------------|
| 8        | 0.24        | 8.40e6       | 4.8e−3                       |
| 16       | 9.40        | 7.68e7       | 5.8e−3                       |
| 24       | 146.80      | 1.26e8       | 7.2e−3                       |
| 32       | 2128.74     | 3.14e8       | 7.4e−3                       |
| 40       | 21584.16    | 2.08e9       | 1.6e−3                       |
| 48       | 41478.60    | 1.64e11      | 1.28e−2                      |
| 56       | 159304.81   | 2.89e11      | 3.12e−2                      |
The results of experiments show high performance of the resulting algorithms.

Based on the concept of (pseudo) annihilating polynomials, the first and the second authors of the present paper studied a method for computing generalized eigenvectors and reported basic ideas ([20], [21], [27]). Algorithms for computing generalized eigenvectors will be described in forthcoming papers.

References

[1] M. R. Albrecht. The M4RIE library for dense linear algebra over small fields with even characteristic. *Proc. ISSAC ’12*, 28–34, ACM Press, 2012.

[2] D. Augot, P. Camion. On the computation of minimal polynomials, cyclic vectors, and Frobenius forms. *Linear Algebra and its Applications*, 260, 61–94, 1997.

[3] A. Bostan, C.-P. Jeannerod, É. Schost. Solving structured linear systems with large displacement rank. *Theoretical Computer Science*, 407, 155–181, 2008.

[4] Z. Chen, A. Storjohann. A BLAS based C library for exact linear algebra on integer matrices. *Proc. ISSAC ’05*, 92–99, ACM Press, 2005.

[5] J.-G. Dumas, T. Gautier, M. Giesbrecht, P. Giorgi, B. Hovinen, E. Kaltofen, B. D. Saunders, W. J. Turner, G. Villard. LinBox: A generic library for exact linear algebra. *Proc. ICMS 2002*, 40–50, World Scientific, 2002.

[6] J.-G. Dumas, T. Gautier, C. Pernet. Finite field linear algebra subroutines. *Proc. ISSAC ’02*, 63–74, ACM Press, 2002.

[7] J.-G. Dumas, P. Giorgi, C. Pernet. FFPACK: Finite field linear algebra package. *Proc. ISSAC ’04*, 119–126, ACM Press, 2004.

[8] J.-G. Dumas, C. Pernet, Z. Wan. Efficient computation of the characteristic polynomial. *Proc. ISSAC ’05*, 140–147, ACM Press, 2005.
[9] J.-G. Dumas, B. D. Saunders, G. Villard. On efficient sparse integer matrix Smith normal form computations. Journal of Symbolic Computation, 32, 71–99, 2001.

[10] W. Eberly, M. Giesbrecht, P. Giorgi, A. Storjohann, G. Villard. Solving sparse rational linear systems. Proc. ISSAC ’06, 63–70, ACM Press, 2006.

[11] C.-P. Jeannerod, C. Pernet, A. Storjohann. Rank-profile revealing Gaussian elimination and the CUP matrix decomposition. Journal of Symbolic Computation, 56, 46–68, 2013.

[12] M. Maekawa, M. Noro, K. Ohara, N. Takayama, K. Tamura. The Design and Implementation of OpenXM-RFC 100 and 101. Computer Mathematics, Proc. the Fifth Asian Symposium: ASCM 2001, World Scientific, 102–111, 2001.

[13] Maplesoft, a division of Waterloo Maple Inc. Maple 2016 [computer software], 2016.

[14] J. P. May, D. Saunders, Z. Wan. Efficient matrix rank computation with application to the study of strongly regular graphs. Proc. ISSAC ’07, 277–284, ACM Press, 2007.

[15] S. Moritsugu. A practical implementation of modular algorithms for Frobenius normal forms of rational matrices. IPSJ Journal, 45, 1630–1641, 2004.

[16] S. Moritsugu, K. Kuriyama. Symbolic computation of eigenvalues, eigenvectors and generalized eigenvectors of matrices by computer algebra (in Japanese). Transactions of the Japan Society for Industrial and Applied Mathematics, 11, 103–120, 2001.

[17] M. Neunhöffer, C. E. Praeger. Computing minimal polynomials of matrices. LMS Journal of Computation and Mathematics, 11, 252–279, 2008.

[18] M. Noro. A computer algebra system: Risa/Asir. Algebra, Geometry and Software Systems (Michael Joswig and Nobuki Takayama, editors), 147–162, Springer, 2003.
[19] K. Ohara, S. Tajima. Spectral decomposition and eigenvectors of matrices by residue calculus. *The Joint Conference of ASCM 2009 and MACIS 2009, COE Lecture Note*, 22, 137–140, Faculty of Mathematics, Kyushu University, 2009.

[20] K. Ohara, S. Tajima. Algorithms for calculating generalized eigenspaces using pseudo annihilating polynomials (in Japanese). *Computer Algebra and Related Topics, RIMS Kôkyûroku*, 1907, 62–70, Research Institute for Mathematical Sciences, Kyoto University, 2014.

[21] K. Ohara, S. Tajima. Algorithms for calculating generalized eigenspaces using the minimum annihilating polynomials (in Japanese). *Computer Algebra and Related Topics, RIMS Kôkyûroku*, 1955, 198–205, Research Institute for Mathematical Sciences, Kyoto University, 2015.

[22] C. Pernet, W. Stein. Fast computation of Hermite normal forms of random integer matrices. *Journal of Number Theory*, 130, 1675–1683, 2010.

[23] D. Saunders, Z. Wan. Smith normal form of dense integer matrices fast algorithms into practice. *Proc. ISSAC ’04*, 274–281, ACM Press, 2004.

[24] B. D. Saunders, D. H. Wood, B. S. Youse. Numeric-symbolic exact rational linear system solver. *Proc. ISSAC ’11*, 305–312, ACM Press, 2011.

[25] A. Storjohann. Deterministic computation of the Frobenius form. *Proc. 2001 IEEE International Conference on Cluster Computing*, 368–377, IEEE Computer Society, 2001.

[26] A. Storjohann, G. Labahn. Asymptotically fast computation of Hermite normal forms of integer matrices. *Proc. ISSAC ’96*, 259–266, ACM Press, 1996.

[27] S. Tajima. Calculating generalized eigenspace of matrices (in Japanese). *Computer Algebra: The Algorithms, Implementations and the Next Generation, RIMS Kôkyûroku*, 1843, 146–154, Research Institute for Mathematical Sciences, Kyoto University, 2013.

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[28] S. Tajima, K. Ohara, A. Terui. Fast algorithm for calculating the minimal annihilating polynomials of matrices via pseudo annihilating polynomials, preprint, arXiv:1801.08437 [math.AC].

[29] T. Takeshima, K. Yokoyama. A method for solving systems of algebraic equations — using eigenvectors of linear maps on residue class rings (in Japanese). Communications for Symbolic and Algebraic Manipulation, 6, 27–36, 1990.