Quasi-relativistic description of a quantum particle moving through one-dimensional piecewise constant potentials

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Using a novel wave equation, which is Galileo invariant but can give precise results up to energies ~ mc^2, exact quantum mechanical solutions are found which corresponds to a particle with mass moving through one-dimensional piecewise constant potentials. As expected, at low particle’s speeds, the found solutions coincide with the solutions of the same problems calculated using the Schrödinger equation; however, as it should be, both solutions have a significant difference at quasi-relativistic speeds. Then, it is argued that the Grave de Peralta equation provides a simpler description than a fully relativistic theory or the perturbation approach for a quantum particle moving at quasi-relativistic energies through piecewise constant potentials.

I. INTRODUCTION

Recently, the properties of an intriguing but previously unexplored wave equation describing a free quantum particle with mass m moving at quasi-relativistic speeds, were reported [1]. The so-called Grave de Peralta equation [1]:

\[ i\hbar \frac{\partial}{\partial t} \psi(x, t) = -\frac{\hbar^2}{(\gamma_V+1)m} \frac{\partial^2}{\partial x^2} \psi(x, t). \]  

(1)

Is very similar to the well-known Schrödinger equation [2-5]:

\[ i\hbar \frac{\partial}{\partial t} \psi_{Sch}(x, t) = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \psi_{Sch}(x, t). \]  

(2)

In Eqs. (1) and (2), \( \hbar \) is the Plank constant divided by 2\( \pi \). Formally, Eq. (1) can be obtained from Eq. (2) by substituting the factor 2 which multiples \( m \) in the Schrödinger equation by the relativistic factor \( \gamma_V +1 \) in Eq. (1), where [6-8]:

\[ \gamma_V = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}. \]  

(3)

And \( V \) and \( c \) are the speeds of the particle and the speed of the light in vacuum, respectively. The term “quasi-relativistic” is used in this work as meaning a particle moving at so large speeds that it is necessary to use the correct relativistic relation between the linear momentum of the particle, \( p \), and its kinetical energy, \( K \), i.e. [1, 6]:

\[ K = \frac{p^2}{(\gamma_V+1)m}, \quad p = \gamma_V m V. \]  

(4)

Nevertheless, the speed of the quasi-relativistic particle should not be too much large so that the number of particles remains constant. For a free particle this requires that \( K < mc^2 \) because a new particle could be generated from the kinetic energy of the particle when \( K > mc^2 \) [1, 7-8]. However, when a particle is moving through the one-dimensional (1D) piecewise constant potentials \( U(x) \) studied in this work, the number of particles is constant when \( K + |\Delta U| < mc^2 \). This is because particles can also be generated from a potential that is maintained constant by an external source of energy [9-10]. Limiting the scope of this work to quasi-relativistic
energies excludes the study of relativistic effects like the Klein paradox that occurs for very large potentials (|ΔU| > 2 mc²) [9-10]. Nevertheless, this does not diminish the relevance of problems where a quantum particle moves at quasi-relativistic speeds. These problems include all chemistry and all problems where the number of particles is constant. For instance, for electrons mc² ~ 0.5 MeV; thus, electrons moving at quasi-relativistic speeds were commonly used in large color TV displays based on the now obsolete cathode-ray tube technology, where electron beams with kinetic energies ~ 0.1 mc² were produced by electron guns with voltages of up to tens of kV. The most internal electrons in heavy elements have energies of the same order, while the ionization energy of atoms and the energy per molecular chemical bond are of the order of 1 to 10 eV (~ 10⁻² mc²). This explain why the results obtained using the Schrödinger equation are a good first approximation in chemistry applications [5]. In excellent correspondence with this, Eq. (1) clearly coincides with the Schrödinger equation at low particle’s speeds. Moreover, a positive probability density can be defined for the solutions of Eq. (1) by analogy of how it is defined for the solutions of the Schrödinger equation and, like the Schrödinger equation, Eq. (1) is Galilean invariant for observers traveling at low speeds respect to each other [1]. Despite this, Eq. (1) can be used for obtaining precise solutions of very interesting quantum problems at quasi-relativistic energies [1]. In addition, it has been shown that a plane wave (ψ), which is solution of Eq. (1), is given by [1]:

\[ ψ(x, t) = ψ_{KG}(x, t)e^{iwxmt}, \]

\[ w_m = \frac{mc^2}{\hbar}, \quad ψ_{KG}(x, t) = \frac{i}{\hbar}(px - Et). \] (5)

In Eq. (5), \( E = K + mc² \) and \( ψ_{KG} \) is a plane wave which is a solution with positive energy of the relativistic Klein-Gordon equation [1, 7-8]:

\[ \frac{1}{c^2} \frac{\partial^2}{\partial t^2} ψ_{KG}(x, t) = \frac{\partial^2}{\partial x^2} ψ_{KG}(x, t) - \frac{m^2 c^2}{\hbar^2} ψ_{KG}(x, t). \] (6)

Therefore [1]:

\[ ψ(x, t) = e^{i(px - Et)}. \] (7)

The Grave de Peralta equation has been used for obtaining precise quasi-relativistic solutions of the well-known infinite rectangular well and quantum rotor problems [1-5]. The formal similitude between Eqs. (1) and (2) permitted finding exact analytical solutions of Eq. (1) for these problems with no more mathematical difficulty than those found when Eq. (2) is used. This is in contrast with the difficulties and complexities associated with finding solutions of similar relativistic quantum problems [7-11], or with the common theory of perturbations approach for including relativistic corrections to the energy values obtained from the Schrödinger equation [4]. In this work, it is shown that an extension of Eq. (1) can also be used for finding precise solutions of a whole class of interesting problems where a quantum particle with mass \( m \) move at quasi-relativistic speeds through a 1D piecewise constant potential. These problems have real applications and illustrate many important quantum-mechanics effects, such as penetration of a potential barrier, reflection of matter waves by a sharp change in potential, and the energy quantization in bounded states. Due to their importance and simplicity, these problems are often solved using the Schrödinger equation in quantum mechanics textbooks; however, the solutions found are only valid for particle’s speeds much smaller than \( c \). Therefore, the solution of the same problems using Eq. (1) allows extending our knowledge about these quantum-mechanics effects to the
quasi-relativistic domain without a significant increment in the complexity of the theory. In Section II are presented general considerations about the movement of a quantum particle at quasi-relativistic speeds through 1D piecewise constant potentials, while in Sections III, IV, and V the reflection of a quantum particle by a sharp quantum step potential, the transmission through a potential barrier, and the bond states in a rectangular quantum well are discussed, respectively. Finally, the conclusions of this work are given in Section VI. In addition, for completeness, a summary discussion about the Grave de Peralta equation for a free quantum particle is presented in Annex A, where also is discussed the existing relationship between the Klein-Gordon, the Grave de Peralta, and the Schrödinger equation.

II. 1D PIECEWISE CONSTANT POTENTIALS

The wavefunction of a quantum particle slowly traveling through a 1D piecewise constant potential \( U(x) \) can be found solving the following Schrödinger equation [2-5]:

\[
i\hbar \frac{\partial}{\partial t} \psi_{Sch}(x, t) = \frac{-\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \psi_{Sch}(x, t) + U(x) \psi_{Sch}(x, t).
\]

(8)

However, by analogy with the free particle case [1], when the particle is moving at quasi-relativistic speeds, it is necessary to solve the following Grave de Peralta equation:

\[
i\hbar \frac{\partial}{\partial t} \psi(x, t) = \frac{\hbar^2}{(\gamma_\nu + 1)m} \frac{\partial^2}{\partial x^2} \psi(x, t) + U(x) \psi(x, t).
\]

(9)

Due to the formal similarity between Eqs. (8) and (9), one can expect to solve Eq. (9) following similar procedures than the ones used to solve Eq. (8) [2-5]. Looking for solutions of Eq. (9) corresponding to a constant value of the energy \( E = K + U = E + U - mc^2 \). At quasi-relativistic energies, the number of particles is constant; therefore, \( E \) is constant whenever \( E + U \) is constant. For a 1D piecewise constant potential, \( E \) and \( K \) \( V^2 \) are constants in each \( x \)-region where \( U \) is constant; therefore, one can look in each of the regions for a solution of Eq. (9) with the following form [2-5]:

\[
\psi(x, t) = X_K(x)e^{-\frac{i}{\hbar}Et}, \quad \xi = K + U
\]

(10)

In Eq. (10), \( X_K \) is a solution of the following equation [2-5]:

\[
\frac{d^2}{dx^2}X_K(x) + \kappa^2 X_K(x) = 0, \quad \kappa = \frac{p}{\hbar}.
\]

(11)

Equation (11) and the relation between \( \kappa \) and \( p \) are identical to the ones obtained when solving Eq. (8) [2-5]. In addition, the possible values of \( \kappa \) are determined by the boundary conditions [1-5], which for a given problem are the same when resolving Eq. (8) and Eq. (9); therefore, for a given problem, the spatial part of the solution of Eqs. (8) and (9) are equal. Using Eqs. (4) and (10) allows for rewritten \( \kappa \) in the following way:

\[
\kappa = \frac{p}{\hbar} = \frac{1}{\hbar} \sqrt{(\gamma_\nu + 1)mK} = \frac{1}{\hbar} \sqrt{(\gamma_\nu + 1)m(E - U)}.
\]

(12)

Consequently, \( \kappa \) and \( X_K \) are not determined by the values of \( E \) but by the values of \( K = E - U \). Once the allowed values of \( \kappa \) are determined from Eq. (11) and the boundary conditions, the allowed values of \( K = E - U \) are given by:

\[
K = \frac{\hbar^2 \kappa^2}{(\gamma_\nu + 1)m}.
\]

(13)

Equation (13) corresponds to the relativistic kinetic energy of the particle, which is different than the non-relativistic kinetic energy that is obtained when solved Eq. (8). Therefore, for a given value of \( U \), the values of \( E = K + U \) obtained solving Eq. (9) are different than the energy values corresponding to Eq. (8). Nevertheless, as expected, Eq. (13)
gives the non-relativistic values of the particle’s energies at low speeds when $\gamma_V \sim 1$ [2-5]. Moreover, from Eq. (13) and the relativistic equation, $K = (\gamma_V -1) mc^2$, follows that:

$$\gamma_V^2 = 1 + \left(\frac{\lambda C}{\lambda}\right)^2, \quad \lambda_C = \frac{h}{mc}, \quad \lambda = \frac{2\pi}{\kappa}. \quad (14)$$

In Eq. (14), $\lambda_C$ is the Compton wavelength associate to the mass of the particle [6-7], and $\lambda$ is the De Broglie wavelength of the wavefunction given by Eqs. (7) and (10) [2-5]. As expected $\gamma_V^2 \sim 1$ when $p = h/\lambda$ is very small because $\lambda >> \lambda_C$; then $K \sim h^2 \kappa^2/(2m)$, which is the non-relativistic expression of the particle’s kinetic energy [2-5]. Substituting Eq. (14) in Eq. (13) allows obtaining an analytical expression of the precise quasi-relativistic kinetic energy of the particle:

$$K = \frac{\hbar^2 \kappa^2}{1 + \sqrt{1 + \left(\frac{\kappa}{\lambda}\right)^2}} \frac{1}{m}. \quad (15)$$

As expected, Eq. (15) match the non-relativistic expression of the particle’s kinetic energy when $p = h/\lambda$ is very small because $\lambda >> \lambda_C$. However, in each region where the value of $U$ is constant, the values of $K$ and then $E = K + U$ calculated using Eq. (15) are smaller than the ones calculated using the Schrödinger equation. It is worth noting that the wavefunction given by Eq. (10) corresponds to quantum states with well determined values of $E = K + U$. Thus, $E$ is the same everywhere. However, there are different values of $U$ in different regions of the piecewise constant potential; therefore, the values of $K = E - U$ are well determined and constant in each region but different in different regions. In addition, due Eq. (11), the values of $p$ are also well determined and constant in each region but different in different regions. Consequently, due Eqs. (4) and (13), the values of $(\gamma_V +1)$ must be well determined and constant in each region but different in different regions. Also, due the relativistic relation $K = (\gamma_V -1) mc^2$, this must happen for $(\gamma_V +1) too. Consequently, the same must happen for $\gamma_V^2 = (\gamma_V +1)(\gamma_V +1)$ and thus also for $\gamma_V$ and $V^2$. This means that strictly speaking a different Eq. (9) with a different value of $V^2$ should be solved in each region where the potential is constant, or alternatively, the equation that should be solved is the following one:

$$i\hbar \frac{\partial}{\partial t} \psi(x, t) = -\frac{\hbar^2}{(\gamma_V(x) + 1)m} \frac{\partial^2}{\partial x^2} \psi(x, t) + U(x) \psi(x, t). \quad (16)$$

In Eq. (16) $\gamma_V$ is a function of $x$ because, in general, the square of the particle’s speed ($V^2$) depends on the position. The boundary conditions of Eq. (11), at the points in between two regions with constant but different values of $U(x)$, correspond to the continuity of the wavefunction and its first spatial derivative [2-5]. In what follows the general ideas discussed in this Section will be applied to some representative cases of 1D piecewise constant potentials.

### III. Reflection of a Quantum Particle by a Sharp Potential Step

The simplest example of a pure quantum mechanical effect is the existence of a probability of reflection when a quantum particle with $E > U(x)$ pass by a region where there is a sharp change in the potential, $|\Delta U| = U_o$, such that $E = K + U_o < mc^2$. The one-dimensional piecewise constant potential corresponding to this situation is a potential that undergoes only one sharp discontinuous change and is given by the following expression:

$$U(x) = \begin{cases} 0, & -\infty < x < 0 \\ U_o > 0, & 0 \leq x < +\infty \end{cases}. \quad (17)$$

Due to the formal similitude between Eqs. (8) and (9), one can proceed to solve Eq. (9) as it is done for Eq. (8). The task here is to calculate the reflectivity ($R$) associated with
the sharp potential variation at \( x = 0 \) [2-5].

When \( E_i > U_o \), one can look for a solution as given by Eq. (10) with \( X(x) \) given by [2]:

\[
X(x) = \begin{cases} 
Be^{i\gamma_1 x} + Ce^{-i\gamma_1 x}, & x \leq 0 \\
Ae^{i\gamma_2 x}, & x \geq 0 
\end{cases}
\] (18)

This solution describes a steady flow of particles with mass \( m \) and kinetic energy \( K_i = E_i - mc^2 \), which are traveling with speed \( V_i \) from left to right and then are partially reflected and partially transmitted at \( x = 0 \). Due to Eqs. (4) and (12), in Eq. (18):

\[
p_1 = \gamma_{V_1} mV_1 = \sqrt{(\gamma_{V_1} + 1)mE_i}, \quad p_2 = \gamma_{V_2} mV_2 = \sqrt{(\gamma_{V_2} + 1)m(E_i - U_o)}.
\] (19)

This is in contrast with the following expressions for \( p_1 \) and \( p_2 \) when the Schrödinger equation is solved [2]:

\[
p_1 = mV_1 = \sqrt{2mE_i}, \quad p_2 = mV_2 = \sqrt{2m(E_i - U_o)}.
\] (20)

The movement of the particles is not confined to a finite region; therefore, in Eqs. (19) and (20) the values of \( p_1, p_2 \) and \( E \) are not quantized. Both Eqs. (19) and (20) determine the values of \( V^2 \) everywhere. From Eq. (20) follows that:

\[
V_1^2 = \frac{2E_i}{m}, \quad V_2^2 = \frac{2(E_i - U_o)}{m}.
\] (21)

While from Eq. (19) follows that:

\[
V_1^2 = \frac{E_i (E_i + 2mc^2)}{(E_i + mc^2)^2} c^2, \quad V_2^2 = \frac{(E_i - U_o)(E_i - U_o) + 2mc^2}{[(E_i - U_o) + mc^2]^2} c^2.
\] (22)

For instance, when \( K_i = E_i \sim mc^2 \), then \( V_1^2 \sim \frac{3}{4} c^2, \gamma V_2^2 \sim 4 \), and \( V_1 \sim \pm 0.87 \ c \). The constant \( A, B, C \) must now be determined from the boundary conditions requiring that the wave function and its first derivative are continuous at \( x = 0 \). From this, one can determine that \( R \) is given by the following expressions, which are identical to the ones obtained when solving the Schrödinger equation [2]:

\[
R = \frac{(p_1 - p_2)^2}{(p_1 + p_2)^2}.
\] (23)

\( R(E) \) for particles moving at quasi-relativistic speeds can be obtained from Eqs. (23), (19) and (22), while for particles moving at very low speeds one should use Eqs. (23) and (20). Fig. (1) shows a comparison of \( R(E) \) calculated using the Schrödinger and the Grave de Peralta equations. In both cases, the reflection coefficient becomes large only when \( U_o \) is comparable in size with \( E_i \) (not shown). As expected, both reflection coefficients coincide when \( E_i \ll mc^2 \) (not shown); however, as shown in Fig. 1 for \( m = m_e \) and \( U_o = 0.3 \ m_e c^2 \), at quasi-relativistic energies \( R(E) \) calculated using the Grave de Peralta equation is slightly larger than \( R(E) \) calculated using the Schrödinger equation.

**IV. TUNNELING THROUGH A BARRIER**

Another example of a pure quantum mechanical effect is the tunneling of a quantum
particle through a potential barrier of high \(|\Delta U| = U_o < mc^2\) when \(\mathcal{E}_i < U_o\). The one-dimensional piecewise constant potential corresponding to this situation is given by the following expression \([2]\):

\[
U(x) = \begin{cases} 
0, & x < 0, \ x > L \\
U_o > 0, & 0 \leq x \leq L 
\end{cases}
\] (24)

Due to the formal similitude between Eqs. (8) and (9), one can proceed to solve Eq. (9) as it is done for Eq. (8). Assuming incident particles from the region \(x < 0\) with linear momentum \(p_1\) and quasi-relativistic energy \(\mathcal{E}_i = K_1 < U_o\) such that \(K_1 + U_o < mc^2\), and assuming that the width of the barrier \((L)\) is large enough; i.e., \(p_2L/\hbar >>1\), where \(p_2\) is the particle momentum inside of the barrier; one can then solve the Schrödinger equation for the ratio of the intensity of the wave transmitted to the region \(x > L\) to that of the incident wave, thus obtaining \([2]\):

\[
T = \frac{16e^{-2p_2L/\hbar}}{1 + \left(\frac{p_2}{p_1}\right)^2\left[1 + \left(\frac{p_1}{p_2}\right)^2\right]}, \ p_2 = \sqrt{2m(U_o - \mathcal{E})} .
\] (25)

In Eq. (25), \(p_1\) is given by Eq. (20). Eq. (25) can also be obtained using the Grave de Peralta equation and thus is also valid for quasi-relativistic energies but then \(p_1\) and \(V_1\) are given by Eqs. (19) and (22), respectively, and \(p_2\) and \(V_2\) are given by:

\[
p_2 = \sqrt{(\gamma V_2 + 1)m(V_2 - \mathcal{E})} = \frac{(U_0 - \mathcal{E})[U_0 - \mathcal{E} + 2mc^2]}{(U_0 - \mathcal{E}) + mc^2}]^2 .
\] (26)

Figure (2) shows a comparison of \(T(\mathcal{E})\) calculated using the Schrödinger and the Grave de Peralta equations. As it is well known, there is a small probability that a particles can penetrate a potential barrier which it could not even enter according to classical theory. This probability decreases rapidly as the barrier get thicker and as it gets higher (not shown). As expected, both transmission coefficients coincide when \(\mathcal{E}_i << mc^2\) (not shown); however, as shown in Fig. 2 for \(m = m_e, U_o = 0.5 \ m_e c^2\), and \(L = \lambda_c\), at quasi-relativistic energies \(T(\mathcal{E})\) calculated using the Grave de Peralta equation is slightly smaller than \(T(\mathcal{E})\) calculated using the Schrödinger equation.

**V. BOUND STATES IN THE RECTANGULAR WELL**

Quantization of the energy of a quantum particle trapped in a potential well is one of the most emblematic quantum effects. The one-dimensional piecewise constant potential corresponding to this situation is given by the following expression \([2]\):

\[
U(x) = \begin{cases} 
0, & x < -\frac{L}{2}, \ x > \frac{L}{2} \\
-U_o < 0, \ -\frac{L}{2} \leq x \leq \frac{L}{2} 
\end{cases}
\] (27)

Here again, due to the formal similitude between Eqs. (8) and (9), one can proceed to solve Eq. (9) as it is done for Eq. (8). Consequently, assuming \(\mathcal{E}_i < 0\), it can be obtained in both cases the following transcendental equation \([2]\):
\[
\frac{p_1}{p_2} = \tan\left(\frac{p_2 L}{2\hbar} + \frac{n\pi}{2}\right) = \\
\begin{cases}
\tan\left(\frac{p_2 L}{2\hbar}\right), & n \text{ even} \\
-cot\left(\frac{p_2 L}{2\hbar}\right), & n \text{ odd}
\end{cases}
\] (28)

Where \( n \) is an integer, and \( p_1 \) and \( p_2 \) are the particle’s linear momenta outside and inside of the well, respectively. The allowed values of \( E \) can be obtained from Eq. (28) by expressing \( p_1 \) and \( p_2 \) in terms of \( E \) and \( U_o \). Consequently, a different transcendental equation is obtained when solving Eq. (8) than when solving Eq. (9). For the Schrödinger equation can be obtained the following transcendental equation [2]:

\[
\sqrt{\frac{|E|}{U_o - |E|}} = \\
\begin{cases}
\tan\left[\frac{L}{2\hbar}\sqrt{2m(U_o - |E|)}\right], & n \text{ even} \\
-cot\left[\frac{L}{2\hbar}\sqrt{2m(U_o - |E|)}\right], & n \text{ odd}
\end{cases}
\] (29)

While the following transcendental equation can be obtained when solving the Grave de Peralta equation:

\[
\sqrt{\frac{(\gamma_{V_2} + 1)|E|}{(\gamma_{V_2} + 1)(U_o - |E|)}} = \\
\begin{cases}
\tan\left[\frac{L}{2\hbar}\sqrt{(\gamma_{V_2} + 1)m(U_o - |E|)}\right], & n \text{ even} \\
-cot\left[\frac{L}{2\hbar}\sqrt{(\gamma_{V_2} + 1)m(U_o - |E|)}\right], & n \text{ odd}
\end{cases}
\] (30)

In Eq. (30):

\[
V_1^2 = \frac{|E|}{(\frac{|E| + 2mc^2}{(\frac{|E| + mc^2}{2})})^2} c^2, \quad V_2^2 = \frac{(U_o - |E|)(U_o - |E|) + 2mc^2}{((U_o - |E|) + mc^2)^2} c^2.
\] (31)

As expected, Eq. (30) coincide with Eq. (29) at very low particle’s speeds. Using Eq. (31) allows for numerical evaluation of both sizes of Eq. (30). Wherever both sizes match, there is a possible energy level. In contrast, it is well known that one can obtain exact solutions of the Schrödinger equation for the infinite rectangular well problem, which corresponds to the following potential [2-5]:

\[
U(x) = \\
\begin{cases}
U_o \rightarrow +\infty, & x < 0, \quad x > L \\
0, & 0 \leq x \leq L
\end{cases}
\] (32)

Therefore, finding the bound states of the infinite rectangular well problem can be considered as a limit case of the finite problem when \( U_o \rightarrow +\infty \) [2]. This case has a high scholastic value and describes a quantum particle absolutely trapped in a finite region of length \( L \) [2-5]. For the infinite well, the solution of Eq. (11), which gives the spatial dependence of the wave function inside of the infinite well for Eq. (8), is given by the following expression [4]:

\[
X_n(x) = \sqrt{\frac{2}{L}}Sin\left(\frac{n\pi}{L}x\right), \quad \kappa_n = \frac{n\pi}{L}, \quad n = 1, 2, \ldots (33)
\]

Consequently, for the Schrödinger equation, the allowed energies in the infinite rectangular well are given by [2-5]:

\[
K_n = n^2 \frac{\pi^2 h^2}{2mL^2}.
\] (34)

Strictly speaking, the problem corresponding to the potential defined by Eq. (32) is a relativistic problem because \( |\Delta U| = U_o \gg mc^2 \) and thus the number of particles may not be constant [7-12]. Nevertheless, the non-relativistic and quasi-relativistic infinite well problems could be considered approximations to the problem of a quantum particle absolutely trapped in a finite region. This is because for obtaining Eqs. (33) and (34) the infinitude of the potential is only used for arguing that \( X(x) \) should be null everywhere except inside of the well [2-5], thus assigning null boundary conditions to Eq. (11). In this sense, one could resolve Eq. (9) for the infinite rectangular well as it is done for Eq. (8). Proceeding in this way [11], one can demonstrate that Eq. (33) is also valid at quasi-
As expected, Eq. (35) coincides with Eq. (34) when the linear momentum of the particle in the infinite well is very small because \( \lambda = \hbar / p >> 1 \). This happens for small values of \( n \) when \( L >> \lambda_c \). In contrast, when the width of the well is close to \( \lambda_c / 2 \), the minimum particle energy is quasi-relativistic; therefore, Eq. (35) should be used instead of Eq. (34). For instance, \( \gamma v^2 = 2, \gamma V = 0.7 c, \) and \( K = 0.4 m c^2 \) when Eq. (35) is evaluated for \( n = 1 \) and \( L = \lambda c / 2 \). However, \( \gamma v^2 = 5 \) and \( K = 1.2 m c^2 \) when \( n = 1 \) and \( L = \lambda c / 4 \). The number of particles may not be constant at these energies. This result for a 1D infinite rectangular well can easily be extended to the 3D infinite rectangular well as it is done for the Schrödinger equation [5]. Consequently, Eq. (9) establishes a fundamental connection between quantum mechanics and especial theory of relativity: no single particle with mass can be confined in a volume much smaller than \( \lambda c^3 \) because when this occurs, \( K > m c^2 \) and the number of particles may not be constant anymore; therefore, a single point-particle with mass cannot exist. Point-particles with mass can only exist in fully relativistic quantum field theories where the number of particles is not constant. This is true for an electron, a quark, and probably may also be true for a black hole and the whole universe at the beginning of the Big Bang. This is consistent, for instance, with the confinement of an electron in the Hydrogen atom because for an electron \( \lambda_c \sim 2.4 \times 10^{-3} \text{ nm} \), which is \( \sim 20 \) times smaller than the Bohr radius of the Hydrogen atom, \( r_B ~ 5.3 \times 10^{-2} \text{ nm} \) [1-5].

Figure (3) shows a comparison of the calculated energies of a particle in an infinite rectangular well using the Schrödinger and the Grave de Peralta equations. The values of \( K_n \) calculated using Eq. (35) are smaller than the ones calculated using Eq. (34) (not shown). This in excellent correspondence with more involved numerical results obtained solving the Dirac equation for the 1D infinite rectangular well [12]. Moreover, as shown in Fig. 3 for \( m = m_e \), the values of \( \Delta K_{21} = K_2 - K_1 \) calculated using Eq. (35) are significatively smaller than the ones calculated using Eq. (34) when \( L \sim \lambda_c \). This is important because it is the energy difference between two energy levels what can be experimentally measured. As expected, the difference between the values of \( \Delta K_{21} \) calculated using both approaches coincide at low particle's velocities, i.e., when \( L >> \lambda_c \) (not shown) but are significatively different at quasi-relativistic velocities, i.e., when \( L \sim \lambda_c \).

VI. CONCLUSIONS

It has been shown how to solve the Grave de Peralta equation for a quantum particle
with mass moving at quasi-relativistic energies through one-dimensional piecewise constant potentials. The solutions were found following the same procedures and with no more difficulty than the corresponding to solving the same problems using the Schrödinger equation. As expected, at low particle’s speeds, the solutions found coincide with the solutions of the same problems calculated using the Schrödinger equation; however, as it should be, both solutions have a significative difference at quasi-relativistic energies. This demonstrates the practical scholastic utility of the Grave de Peralta equation, which may impact how relativistic corrections are introduced in future textbooks of Quantum Mechanic. Nevertheless, for reliable comparison with experiments, problems with more realistic potentials should be solved. The author is currently involved in this task.

ANNEX A: THE 1D GRAVE DE PERALTA EQUATION FOR A FREE QUANTUM PARTICLE

Formally, the Schrödinger equation for a free quantum particle can be obtained from the classical relation between $K$ and $p$ for a free particle when $V << c$ [1-5]:

$$K = \frac{p^2}{2m}, \quad p = mV.$$  \hspace{1cm} (A1)

Then, Eq. (2) is obtained by substituting $K$ and $p$ by the following energy and momentum quantum operators [1-4]:

$$\hat{E} = \hat{K} = i\hbar \frac{\partial}{\partial t}, \quad \hat{p} = -i\hbar \frac{\partial}{\partial x}. $$  \hspace{1cm} (A2)

By analogy, Eq. (1) can be simply obtained combining Eqs. 4 and (A2) [1]. Equation (4) can be easily obtained from the following well-known relativistic equations [6-7]:

$$E^2 - m^2c^4 = p^2c^2 \Leftrightarrow (E + mc^2)(E - mc^2) = p^2c^2.$$  \hspace{1cm} (A3)

And:

$$K = E - mc^2, \quad E = \gamma_v mc^2.$$  \hspace{1cm} (A4)

The Klein-Gordon equation (Eq. (6)) can formally be obtained from Eq. (A2) and the first expression of Eq. (A3) by assigning the temporal partial derivative operator to $E$, which is the sum of particle’s kinetic energy plus the energy associated to the mass of the particle [6-7]. However, if one chooses to assign this operator to $K$, as it is done when obtaining the Schrödinger equation, then from Eqs. (4) and (A2) follows Eq. (1) [1]. Alternatively, one can use Eq. (A4) and the second expression of Eq. (A3) for obtaining the following algebraic equation:

$$(E - mc^2) = \frac{\hbar^2}{(\gamma_{V+1})m}$$  \hspace{1cm} (A5)

The factor $(E + mc^2)$ is always different than 0 for $E > 0$; therefore Eq. (A5) is equivalent to Eq. (A3) for positive energies of $E$. Assigning the temporal partial derivative operator in Eq. (A2) to $E$ in Eq. (A5) results the following differential equation:

$$i\hbar \frac{\partial}{\partial t} \psi_{KG+}(x, t) = -\frac{\hbar^2}{(\gamma_{V+1})m} \frac{\partial^2}{\partial x^2} \psi_{KG+}(x, t) + mc^2 \psi_{KG+}(x, t).$$  \hspace{1cm} (A6)

In Eq. (A6), $\psi_{KG+}$ is a solution of the Klein-Gordon equation given by Eq. (5) for $E > 0$. Thus, the Grave de Peralta equation can be obtained by using Eq. (5) and looking for a solution of Eq. (A6) of the form $\psi_{KG} = \psi e^{-i\hbar \omega_m t}$. Eqs. (5) and (7) suggest that the time-independent equations corresponding to Eqs. (1) and (6) are equal. In fact, looking for solutions of the form $X(x)T(t)$ of Eqs. (1), (2), and (6), where $T(t) = e^{-i\omega_m t}$ for Eqs. (1) and (2) but $T(t) = e^{-i\hbar \omega_m t}$ for Eq. (6), produces the same time-independent equation in the three cases:
\[
\frac{d^2}{dx^2}X(x) + \kappa^2 X(x) = 0, \quad \kappa = \frac{p}{\hbar}. \tag{A7}
\]

Often \(X(x)\) and \(\kappa\) are determined solving Eq. (A7) under adequate boundary conditions [1-5]; then the possible values of \(p\) are determined from the possible values of \(\kappa\). However, the relation between \(K\) and \(p\) are different for non-relativistic and quasi-relativistic speeds; therefore, the solutions of Eqs. (1) and (2) have equal spatial dependences but different values of \(K\). Also, the relation between \(E\) or \(K\) and \(p\) are different for quasi-relativistic speeds; therefore, the solutions of Eqs. (1) and (6) have equal spatial dependences but different values of \(K\) and \(E\). Equation (4) and Eq. (A3) can be obtained from each other using Eq. (A4); however, Eq. (A3) admits solutions with positive and negative energies but \(K\) only can be positive in Eq. (A4). This is in correspondence to the presence of a second-order temporal partial derivative in Eq. (6), which determines that Eq. (6) has solutions with positive and negative energies [7-8]. In contrast, there is a first-order temporal partial derivative in Eqs. (1) and (2). This determines that Eqs. (1) and (2) only have solutions with positive energies. Eq. (5) gives a simple recipe from obtaining a plane wave solution of Eq. (1) from a plane wave solution of Eq. (6) with positive energy and vice versa. The wavefunction of a free particle with mass \(m\) moving at quasi-relativistic speeds, which is given by Eq. (7), have well determined values of \(K\), \(E = K + mc^2\), \(p\), and thus of \(\gamma_V\) and \(V^2\). Consequently, Eq. (1) is well determinate and the same everywhere. Finally, it is worth noting that Eq. (1) is non-linear in the sense that if \(\psi_1\) and \(\psi_2\) are two solutions of Eq. (1) corresponding to two different values of \(V^2\), then strictly they are not solutions of the same Eq. (1) but of slightly different Eqs. (1) with different values of \(\gamma_V\). Moreover, \(\psi = a\psi_1 + b\psi_2\) is not a solution of any Eq. (1) [1]. However, Eq. (6) is linear and the wavefunction \(\psi_{KG+} = (a\psi_1 + a\psi_2)e^{-i\omega mt}\) is indeed a solution of Eq. (6). In this sense, Eqs. (1) and (9) are like Eqs. (11) and (A8) in that solving them requires simultaneously finding eigenvalue-eigenfunction pairs. This can be made evident by using Eq. (A4) for eliminating \(\gamma_V\) from Eq. (8), thus rewriting Eq. (8) as the following eigenvalue equation where \(\psi\) and \(E\) should be found simultaneously:

\[
i\hbar \frac{\partial}{\partial t} \psi(x,t) = -\frac{c^2\hbar^2}{|E-U|+2mc^2} \frac{\partial^2}{\partial x^2} \psi(x,t) + U(x)\psi(x,t). \tag{A8}
\]

While both forms of the same equation are equivalent, Eq. (9) is more suggestive due its striking similarity to the Schrödinger equation. From this point of view, the Grave de Peralta equation provides a useful way to find exact solution of the Klein-Gordon equation with positives energies when \(E - K < mc^2\). The Schrödinger equation then appears as a limit case of the Grave de Peralta equation when \(E << mc^2\).

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