Darboux Transformations for (2+1)-dimensional extensions of the KP hierarchy

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Abstract

Generalizations of the KP hierarchy with self-consistent sources and the corresponding modified hierarchy are proposed. The latter hierarchies contain extensions of N-wave problem and matrix DS-III equation. We also recover a system that contains two types of the KP equation with self-consistent sources as special cases. Darboux and Binary Darboux Transformations are applied to generate solutions of the proposed hierarchies.

Keywords: KP hierarchy; symmetry constraints; Binary Darboux Transformation; Davey-Stewartson equation; KP equation with self-consistent sources

1 Introduction

In the past years, lots of attention have been given to the study of Kadomtsev-Petviashvili hierarchy (KP hierarchy) and its generalizations from both physical and mathematical points of view \cite{1,2,3,4}. KP equation with self-consistent sources and related k-constrained KP (k-cKP) hierarchy also present an interest \cite{5,6,7,8,9,10}. The latter hierarchy contains, in particular, nonlinear Schrödinger equation, Yajima-Oikawa equation, extension of the Boussinesq equation, and KdV equation with self-consistent sources. A modified k-constrained KP (k-cmKP) hierarchy was proposed in \cite{11,12,13}. The k-cKP hierarchy was extended to 2+1 dimensions ((2+1)-dimensional k-cKP hierarchy) in \cite{14,15,16}.

A powerful solution generating method for nonlinear systems from the above mentioned hierarchies is based on the Darboux Transformations (DT) and the Binary Darboux Transformations (BDT) \cite{18}. In \cite{19,20}, the latter transformations were applied to generate solutions of the k-cBP hierarchy. Solutions of (2+1)-dimensional the k-cKP hierarchy were obtained via BDTs in \cite{16,21}. More general (2+1)-dimensional extensions of the k-cKP hierarchy and the corresponding solutions were investigated in \cite{22}. The latter hierarchies cover matrix generalizations of the Davey-Stewartson (DS) and Nizhnik-Novikov-Veselov (NNV) systems, (2+1)-dimensional extensions of the Yajima-Oikawa and modified Korteweg-de Vries equations. The Inverse Spectral Transform Method for (2+1)-dimensional equations, including DS and NNV systems, was presented in \cite{23}.

It should be also pointed out that an interest to matrix and, more generally, noncommutative generalizations of the well-known integrable systems appeared recently in a number of papers (see e.g., \cite{24,25,26,27,28,29,30}).
Hamiltonian analysis for the above mentioned hierarchies, which is based on group-theoretical and Lie-algebraic methods, was elaborated in [31,36]. Analytical scheme of the Hamiltonian analysis is presented in [37].

The aim of this paper is to introduce hierarchies that extend above mentioned cases. It leads to generalizations of the corresponding integrable systems. In particular, we got an equation that contains both types of the KP equation with self-consistent sources as special cases (formula (2.19)). The same holds for the corresponding modified version (4.8).

This work is organized as follows. In Section 2 we present a new (2+1)-dimensional generalizations of the KP hierarchies and enumerate some integrable systems that the latter hierarchy contains. In particular it includes N-wave problem with self-consistent sources, a generalization of the DS-III system and extended KP equation with self-consistent sources as special cases (formula (2.19)). The same holds for the corresponding modified version (4.8).

In Section 3 we present solution generating technique (dressing method) for the obtained hierarchies via DTs and BDTs. In Section 4 we present new (2+1)-dimensional extensions of the modified KP hierarchies and propose solution generating method via BDTs. A short summary of the obtained results and some problems for future investigation are presented in Conclusions.

2 A new (2+1)-dimensional generalization of the k-constrained KP hierarchy

For further purposes we will use the following well-known formulae for integral operator $h_1D^{-1}h_2$ constructed by matrix-valued functions $h_1$ and $h_2$ and the differential operator $A$ with matrix-valued coefficients in the algebra of pseudodifferential operators:

$$Ah_1D^{-1}h_2 = (Ah_1D^{-1}h_2)_{\geq 0} + A\{h_1\}D^{-1}h_2, \quad (2.1)$$

$$h_1D^{-1}h_2A = (h_1D^{-1}h_2A)_{\geq 0} + h_1D^{-1}[A^\tau\{h_2^\tau\}]^{\top}, \quad (2.2)$$

$$D^{-1}h_1h_2D^{-1} = D^{-1}\{h_1h_2\}D^{-1} - D^{-1}D^{-1}\{h_1h_2\}. \quad (2.3)$$

Consider Sato-Zakharov-Shabat dressing operator:

$$W = I + w_1D + w_2D^2 + \ldots \quad (2.4)$$

with $N \times N$-matrix-valued coefficients $w_i$. Introduce two differential operators $J_kD^k$ and $\alpha_n\partial_n - \tilde{J}_nD^n$, $\alpha_n \in \mathbb{C}$, $n, k \in \mathbb{N}$, where $J_k$ and $\tilde{J}_n$ are $N \times N$ commuting matrices (i.e., $[\tilde{J}_n, J_k] = 0$). It is evident that dressed operators have the form:

$$L_k := W J_kD^k W^{-1} = J_kD^k + u_{k-1}D^{k-1} + \ldots + u_0 + u_1D^{-1} + \ldots, \quad (2.5)$$

and

$$M_n := W(\alpha_n\partial_n - \tilde{J}_nD^n)W^{-1} = \alpha_n\partial_n - \tilde{J}_nD^n - v_{n-1}D^{n-1} + \ldots + v_0 + v_1D^{-1} + \ldots, \quad (2.6)$$

Impose the following reduction on the integral part of operators $L_k$ and $M_n$:

$$L_k = \beta_k\partial_{t_k} - B_k - qM_0D^{-1}\mathbf{r}^{\top}, \quad B_k = \sum_{j=0}^{k} u_j D^j, \quad u_j = u_j(x, t_k, t_n), \quad \beta_k \in \mathbb{C},$$

$$M_n = \alpha_n\partial_n - A_n - \tilde{q}\tilde{M}_0D^{-1}\tilde{\mathbf{r}}^{\top}, \quad A_n = \sum_{i=0}^{n} v_i D^i, \quad v_i = v_i(x, t_k, t_n), \quad \alpha_n \in \mathbb{C},$$

where $u_j$ and $v_i$ are matrix-valued functions of dimension $N \times N$; $q$ and $r$ are matrix-valued functions of dimension $N \times m$; $\tilde{q}$ and $\tilde{r}$ are matrix-valued functions with dimension $N \times \tilde{m}$. 

2
$\mathcal{M}_0$ and $\tilde{\mathcal{M}}_0$ are constant matrices with dimensions $m \times m$ and $\tilde{m} \times \tilde{m}$ respectively. We shall also assume that functions $q$, $r$, $\tilde{q}$ and $\tilde{r}$ satisfy equations:

$$L_k\{\tilde{q}\} = \tilde{q}\Lambda_q, \quad L_k^\top\{\tilde{r}\} = \tilde{r}\Lambda_r, \quad M_n\{q\} = q\Lambda_q, \quad M_n^\top\{r\} = r\Lambda_r.$$ 

The following proposition holds for operators in (2.7):

**Proposition 2.1.** *Lax equation* $[L_k, M_n] = 0$ *is satisfied in case the following equations hold:*

$$
\begin{align*}
[L_k, M_n]_{>0} &= 0, \quad L_k\{\tilde{q}\} = \tilde{q}\Lambda_q, \quad L_k^\top\{\tilde{r}\} = \tilde{r}\Lambda_r, \\
M_n\{q\} &= q\Lambda_q, \quad M_n^\top\{r\} = r\Lambda_r,
\end{align*}
\tag{2.8}
$$

where $\Lambda_q$, $\Lambda_r$, $\tilde{\Lambda}_q$, $\tilde{\Lambda}_r$ are constant matrices with dimensions $(m \times m)$ and $(\tilde{m} \times \tilde{m})$ respectively that satisfy equations: $\Lambda_q\tilde{\mathcal{M}}_0 - \tilde{\mathcal{M}}_0\Lambda_r^\top = 0$, $\Lambda_q\mathcal{M}_0 - \mathcal{M}_0\Lambda_r^\top = 0$.

**Proof.** From the equality $[L_k, M_n] = [L_k, M_n]_{\geq 0} + [L_k, M_n]_{< 0}$ we obtain that Lax equation $[L_k, M_n] = 0$ is equivalent to the following one:

$$
[L_k, M_n]_{\geq 0} = 0, \quad [L_k, M_n]_{< 0} = 0.
\tag{2.9}
$$

Thus, it is sufficient to prove that equalities $L_k\{\tilde{q}\} = \tilde{q}\Lambda_q$, $L_k^\top\{\tilde{r}\} = \tilde{r}\Lambda_r$, $M_n\{q\} = q\Lambda_q$, $M_n^\top\{r\} = r\Lambda_r$ imply $[L_k, M_n]_{< 0} = 0$. From the form of operators $L_k$, $M_n$ (2.7) we obtain:

$$
[L_k, M_n]_{< 0} = [\tilde{q}\tilde{\mathcal{M}}_0 D^{-1}\tilde{r}^\top, \beta_k\partial_{\tau_k} - B_k]_{< 0} + \\
+ [q\mathcal{M}_0 D^{-1}r^\top, \tilde{q}\tilde{\mathcal{M}}_0 D^{-1}\tilde{r}^\top]_{< 0} + [\alpha_n\partial_{\tau_n} - A_n, q\mathcal{M}_0 D^{-1}r^\top]_{< 0}.
\tag{2.10}
$$

After direct computations for each of the three items at the right-hand side of formula (2.10) we get:

$$
[\tilde{q}\tilde{\mathcal{M}}_0 D^{-1}\tilde{r}^\top, \beta_k\partial_{\tau_k} - B_k]_{< 0} = - (\beta_k \tilde{q}\partial_{\tau_k} - B_k(\tilde{q})) \tilde{\mathcal{M}}_0 D^{-1}\tilde{r}^\top - \\
- \tilde{q}\mathcal{M}_0 D^{-1} (\beta_k \tilde{r}^\top + (B_k^\top(\tilde{r}))^\top),
\tag{2.11}
$$

$$
[\alpha_n\partial_{\tau_n} - A_n, q\mathcal{M}_0 D^{-1}r^\top]_{< 0} = (\alpha_n q_{\tau_n} - A_n(\{q\})) \mathcal{M}_0 D^{-1}r^\top + \\
+ q\mathcal{M}_0 D^{-1} (\alpha_n \tilde{r}^\top + (A_n^\top(\{r\}))^\top),
\tag{2.12}
$$

$$
[q\mathcal{M}_0 D^{-1}r^\top, \tilde{q}\tilde{\mathcal{M}}_0 D^{-1}\tilde{r}^\top]_{< 0} = q\mathcal{M}_0 D^{-1}\{q^\top\tilde{q}\}\tilde{\mathcal{M}}_0 D^{-1}\tilde{r}^\top - \\
- q\mathcal{M}_0 D^{-1}D^{-1}\{q^\top\tilde{r}\}\tilde{\mathcal{M}}_0 D^{-1}\tilde{r}^\top + q\mathcal{M}_0 D^{-1}D^{-1}(\tilde{r}^\top\tilde{q})\mathcal{M}_0 D^{-1}r^\top + \\
+ \tilde{q}\tilde{\mathcal{M}}_0 D^{-1}D^{-1}(\tilde{r}^\top\tilde{q})\tilde{\mathcal{M}}_0 r^\top.
\tag{2.13}
$$

The latter formulae are consequences of (2.1)-(2.3). From formulae (2.10)-(2.13) using (2.8) we have

$$
[L_k, M_n]_{< 0} = M_n\{q\} M_0 D^{-1}r^\top - q\mathcal{M}_0 D^{-1}(M_n^\top\{r\})^\top - L_k\{\tilde{q}\} \tilde{\mathcal{M}}_0 D^{-1}\tilde{r}^\top + \\
+ \tilde{q}\tilde{\mathcal{M}}_0 D^{-1}(L_k^\top\{\tilde{r}\})^\top - q(\Lambda_q\mathcal{M}_0 - \mathcal{M}_0\Lambda_r^\top) D^{-1}r^\top - \\
- \tilde{q}(\Lambda_q\tilde{\mathcal{M}}_0 - \tilde{\mathcal{M}}_0\Lambda_r^\top) D^{-1}\tilde{r}^\top = 0.
\tag{2.14}
$$

From the last formula we obtain that equality $[L_k, M_n] = 0$ is a consequence of (2.8). □
Consider some nonlinear systems that hierarchy given by (2.7)-(2.8) contains. For simplicity we set $\Lambda_q = \Lambda_r = 0$, $\Lambda_\tilde{q} = \Lambda_\tilde{r} = 0$.

1. $k = 1, n = 1$. We shall use the following notation $\beta := \beta_1$, $\alpha := \alpha_1$, $\tau := \tau_1$, $t := t_1$. Then (2.7) reads:

$$L_1 = \beta_1 \partial_t - JD + [J, Q]_r - qM_0D^{-1}r^\top, \quad M_1 = \alpha_1 \partial_\tau - \tilde{J}D + [\tilde{J}, \tilde{Q}]_r - \tilde{q}\tilde{M}_0D^{-1}\tilde{r}^\top, \quad (2.15)$$

with commuting matrices $J$ and $\tilde{J}$. According to Proposition 2.1 the commutator equation $[L_1, M_1] = 0$ is equivalent to the system:

$$\begin{align*}
\beta_1[J, Q] - \alpha_1[J, Q]_r + JD_J - \tilde{J}D_{\tilde{J}}J + [[J, Q], [\tilde{J}, \tilde{Q}]] + \\
+ [J, \tilde{q}\tilde{M}_0\tilde{r}^\top] + [\tilde{q}\tilde{M}_0\tilde{r}^\top, \tilde{J}] = 0, \\
\beta_1q_r - Jq_x + [J, Q]q - \tilde{q}\tilde{M}_0S_1 = 0, \\
-\beta_1\tilde{r}^\top + \tilde{r}^\top J + \tilde{r}^\top [J, Q] + S_2\tilde{M}_0\tilde{r}^\top = 0, \\
a_1q_r - \tilde{J}q_x + [\tilde{J}, \tilde{Q}]q - \tilde{q}\tilde{M}_0S_2 = 0, \\
-\alpha_1\tilde{r}^\top + r^\top \tilde{J} + r^\top [J, \tilde{Q}] + S_1M_0\tilde{r}^\top = 0, \\
S_{1,x} = r^\top \tilde{q}, \quad S_{2,x} = \tilde{r}^\top q.
\end{align*}$$

The latter system is a generalization of the N-wave problem. In case we set $Q = 0$ we obtain a noncommutative generalization of the nonlinear system of four waves \cite{38,39}. Under the Hermitian conjugation reduction $\tilde{r} = \tilde{q}$, $M_0 = M_0^*$, $\tilde{M}_0 = \tilde{M}_0^*$, $r = q$, $Q = -Q^*$, $\alpha, \beta \in \mathbb{R}$, $J = J^*$, $\tilde{J} = \tilde{J}^*$ the latter system reads:

$$\begin{align*}
\beta_1[J, Q] - \alpha_1[J, Q]_r + JD_J - \tilde{J}D_{\tilde{J}}J + [[J, Q], [\tilde{J}, \tilde{Q}]] + \\
+ [J, \tilde{q}\tilde{M}_0\tilde{q}^*] + [\tilde{q}\tilde{M}_0\tilde{q}^*, \tilde{J}] = 0, \quad S_{1,x} = q^* \tilde{q}, \\
\beta_1q_r - Jq_x + [J, Q]q - \tilde{q}\tilde{M}_0S_1 = 0, \quad \alpha_1q_r - \tilde{J}q_x + [\tilde{J}, \tilde{Q}]q - \tilde{q}\tilde{M}_0S_1^* = 0.
\end{align*} \quad (2.16)$$

2. $k = 1, n = 2$.

$$L_1 = \beta_1 \partial_{\tau_1} - qM_0D^{-1}r^\top, \quad M_2 = \alpha_2 \partial_{\tau_2} - cD^2 + v - \tilde{q}\tilde{M}_0D^{-1}\tilde{r}^\top, \quad c \in \mathbb{C} \quad (2.17)$$

Lax equation $[L_1, M_2] = 0$ is equivalent to the following generalization of the DS-III equation:

$$\begin{align*}
\beta_1q_{\tau_1} &= qM_0S_1, \quad \beta_1\tilde{r}_{\tau_1}^\top = S_2M_0r^\top, \quad S_{1,x} = r^\top \tilde{q}, \\
\alpha_2q_{\tau_2} - cqq_{xx} + vq &= \tilde{q}\tilde{M}_0S_2, \quad \alpha_2r_{\tau_2}^\top + cr^\top_{xx} - r^\top v = S_1\tilde{M}_0\tilde{r}^\top, \quad S_{2,x} = \tilde{r}^\top q, \\
\beta_1v_{\tau_1} &= 2(qM_0r^\top)_{xx}.
\end{align*}$$

If we set $\tilde{q} = 0$, $\tilde{r} = 0$ we recover DS-III system:

$$\begin{align*}
\alpha_2q_{\tau_2} - cqq_{xx} + vq &= 0, \quad \alpha_2r_{\tau_2}^\top + cr^\top_{xx} - r^\top v = 0, \quad \beta_1v_{\tau_1} = 2(qM_0r^\top)_{xx}.
\end{align*} \quad (2.18)$$

3. $k = 3, n = 2$.

In this case we obtain the following pair of operators:

$$L_3 = \beta_3 \partial_{\tau_3} - (c_1D^3 - wD - u) - qM_0D^{-1}r^\top, \quad M_2 = \alpha_2 \partial_{\tau_2} - c_2(D^2 - v) - \tilde{q}\tilde{M}_0D^{-1}\tilde{r}^\top.$$ 

Equation $[L_3, M_2] = 0$ is equivalent to the following system:

$$\begin{align*}
c_1c_2(2w - 3v) &= 0, \quad -\alpha_2c_1w_{\tau_2} - \frac{3}{2}c_1c_2w_{xx} + 3c_1(\tilde{q}\tilde{M}_0\tilde{r}^\top)_{xx} + 2c_1c_2u_{xx} = 0, \\
\beta_3c_2w_{\tau_3} - c_1c_2w_{xxx} + 3c_1(\tilde{q}\tilde{M}_0\tilde{r}^\top)_{xxx} + c_1c_2u_{xxx} - c_1[w, \tilde{q}\tilde{M}_0\tilde{r}^\top] + \\
+ c_1c_2[u, v] + c_1c_2u_{xx} - \alpha_2c_1u_{xx} - 2c_2(qM_0r^\top)_{xx} = 0, \\
\beta_3\tilde{q}_{\tau_3} - c_1\tilde{q}_{xxx} + c_1w_{\tilde{q}_x} + c_1u_{\tilde{q}-qM_0S_1} = 0, \quad S_{1,x} = r^\top \tilde{q}, \\
-\beta_3\tilde{r}_{\tau_3} + c_1\tilde{r}_{xxx} - c_1(\tilde{r}^\top w)_{xx} + c_1\tilde{r}^\top v + S_2\tilde{M}_0\tilde{r}^\top = 0, \quad S_{2,x} = \tilde{r}^\top q, \\
\alpha_2\tilde{q}_{\tau_2} - c_2\tilde{q}_{xx} + c_2v_{\tilde{q}} - \tilde{q}\tilde{M}_0S_2 = 0, \\
\alpha_2\tilde{r}_{\tau_2} + c_2\tilde{r}_{xx} - c_2\tilde{r}^\top v - S_1\tilde{M}_0\tilde{r}^\top = 0.
\end{align*}$$

The latter consists of several special cases:
(a) $c_1 = c_2 = 1$. In this case the latter system can be rewritten in the following way:

\[
\begin{align*}
-\frac{3}{2} \alpha_2 v_t - \frac{3}{2} v_x + 3 (\bar{q} \tilde{M}_0 \bar{r}^T)_x + 2 u_x &= 0, \\
(\beta_3 v_{x_3} - \frac{1}{2} v_{x_3} + \frac{3}{2} \nu v_x)_{x_3} - 3 \alpha_2^2 u_{t_2} + \left( [u, v] - [w, \bar{q} \tilde{M}_0 \bar{r}^T] \right)_{x_3} + \\
+ \frac{3}{2} \left( \bar{q}_{xx} \tilde{M}_0 \bar{r}^T - \bar{q} \tilde{M}_0 \bar{r}^T \right)_{xx} + \alpha (q \tilde{M}_0 \bar{r}^T)_{t_2} - 2 \left( q \tilde{M}_0 r^T \right)_{xx} &= 0, \\
\beta_3 \bar{q}_x - \bar{q}_{xx} + \frac{3}{2} \nu \bar{q}_x + u \bar{q} - q \tilde{M}_0 S_1 &= 0, \\
\beta_3 \tilde{r}_{xx} + \frac{1}{2} \tilde{r}^T v + \bar{r}^T u + S_2 M_0 r^T &= 0, \\
\alpha_2 q_{t_2} - q_{xx} + v q - \bar{q} \tilde{M}_0 S_2 &= 0, \\
\alpha_2 \tilde{r}_{t_2} + r_{xx} - r^T v - S_1 \tilde{M}_0 \tilde{r}^T &= 0.
\end{align*}
\]

In the scalar case ($N = 1$) under the Hermitian conjugation reduction: $\alpha_2 \in i \mathbb{R}$, $r = \bar{q}$, $M_0 = M_0^*$ ($M_2 = M_2^*$) and $\beta_3 \in \mathbb{R}$, $\tilde{M}_0 = -\tilde{M}_0^*$, $\bar{r} = \bar{q}$, $w = w^*$, $u_x^* = u + u^*$, $v = v^*$ ($L_3 = -L_3^*$), the latter equation reads:

\[
\begin{align*}
(\beta_3 v_{x_3} - \frac{1}{2} v_{x_3} + \frac{3}{2} \nu v_x)_{x_3} - 3 \alpha_2^2 u_{t_2} + \\
+ \frac{3}{2} \left( \bar{q}_{xx} \tilde{M}_0 \bar{r}^T - \bar{q} \tilde{M}_0 \bar{r}^T \right)_{xx} + \alpha (q \tilde{M}_0 \bar{r}^T)_{t_2} - 2 \left( q \tilde{M}_0 r^T \right)_{xx} &= 0, \\
\beta_3 \bar{q}_x - \bar{q}_{xx} + \frac{3}{2} \nu \bar{q}_x + u \bar{q} - q \tilde{M}_0 S_1 &= 0, \\
\beta_3 \bar{r}_{xx} + \frac{1}{2} \bar{r}^T v + \bar{r}^T u + S_2 \tilde{M}_0 r^T &= 0, \\
\alpha_2 q_{t_2} - q_{xx} + v q - \bar{q} \tilde{M}_0 S_2 &= 0, \\
\alpha_2 \tilde{r}_{t_2} + r_{xx} - r^T v - S_1 \tilde{M}_0 \tilde{r}^T &= 0.
\end{align*}
\]

This system is a generalization of the KP equation with self-consistent sources (KPSCS). In particular, if we set $M_0 = 0$ we recover KPSCS of the first type. In case $M_0 = 0$ we obtain KPSCS of the second type.

(b) $c_1 = 0, c_2 = 1$. In this case (2.18) becomes the following:

\[
\begin{align*}
\beta_3 v_{x_3} &= 2 (q \tilde{M}_0 r^T)_{x_3}, \\
\beta_3 \bar{q}_x - q \tilde{M}_0 S_1 &= 0, \\
\beta_3 \bar{r}_{xx} + S_2 \tilde{M}_0 r^T &= 0, \\
\alpha_2 q_{t_2} - q_{xx} + v q - \bar{q} \tilde{M}_0 S_2 &= 0, \\
\alpha_2 \tilde{r}_{t_2} + r_{xx} - r^T v - S_2 \tilde{M}_0 \tilde{r}^T &= 0.
\end{align*}
\]

In case $\tilde{M}_0 = 0$ the latter becomes the noncommutative generalization of the DS-III system.

New (2+1)-dimensional k-constrained KP hierarchy connected with the Lax pair (2.7) also are closely related to the bidirectional generalization of (2+1)-dimensional matrix k-constrained KP hierarchy ((2+1)-BDk-cKP hierarchy) that was introduced in [22] and its generalizations. Namely, let us put in formulae (2.7):

\[
\begin{align*}
\bar{q} := (\bar{q}_1, c_1 q[0], c_1 q[1], \ldots, c_1 q[l]), \\
\bar{r} := (\bar{r}_1, r[l], r[l - 1], \ldots, r[0]), \\
\tilde{M}_0 = \text{diag}(M_0, I_{l+1} \otimes \tilde{M}_0),
\end{align*}
\]

where $q[j] = (L_k)^j \{ q \}$, $r[j] = (L_k^l)^i \{ r \}$, $j = 0, l$. I.e., $\tilde{M} = \tilde{M}_1 + m(l + 1)$ and matrices $\bar{q}$ and $\bar{r}$ consist of $N \times \tilde{m}_1$-matrix-valued blocks $q_1$ and $r_1$ and $N \times m$-matrix-valued blocks $q[j]$ and $r[j]$, $j = 0, l$. $\tilde{M}_0$ is a block-diagonal matrix and $I_{l+1} \otimes \tilde{M}_1$ stands for the tensor product of the $l + 1$-dimensional identity matrix $I_{l+1}$ and matrix $\tilde{M}_0$. Then we obtain the following pair of operators in (2.7):

\[
\begin{align*}
L_k &= \beta_k \partial_{t_k} - B_k - q \tilde{M}_0 D^{-1} r^T, \\
B_k &= \sum_{j=0}^{k} u_j D^j, \\
\alpha_n \partial_{t_n} - A_n - \bar{q}_1 \tilde{M}_0 D^{-1} \bar{r}_1^T - c_1 \sum_{j=0}^{l} q[j] M_0 D^{-1} r^T[l - j], \\
A_n &= \sum_{i=0}^{n} v_i D^i, \\
\alpha_n \in \mathbb{C}, \\
\beta_k \in \mathbb{C}.
\end{align*}
\]
If we assume that equations
\[ M_{n,l} \{ q \} = c_l (L_k)^{l+1} \{ q \}, \quad M_{n,l}^r \{ r \} = c_l (L_k^r)^{l+1} \{ r \}, \quad L_k \{ \bar{q}_1 \} = \bar{q}_1 \Lambda_{\bar{q}_1}, \quad L_k^r \{ \bar{r}_1 \} = \bar{r}_1 \Lambda_{\bar{r}_1} \] (2.23)
with constant matrices \( \Lambda_{\bar{q}} \) and \( \Lambda_{\bar{r}} \) are satisfied, then the following proposition holds.

**Proposition 2.2.** Lax equation \([L_k, M_{n,l}] = 0\) is equivalent to the system that consists of the operator equation \([L_k, M_{n,l}] \geq 0\) and equations (2.23).

**Proof.** The proof is similar to the proof of the Proposition 2.1 and the proof of the Theorem 1 in [22].

If \( \bar{q}_1 = 0 \) and \( \bar{r}_1 = 0 \) we recover \((2+1)-BDk\)-cKP hierarchy that contains the following subcases:

1. \( \beta_k = 0, c_l = 0 \). Under this assumption we obtain Matrix k-constrained KP hierarchy [19]. We shall point out that the case \( \beta_k = 0 \) and \( c_l \neq 0 \) also leads to Matrix k-constrained KP hierarchy.

2. \( c_l = 0, N = 1, v_n = u_k = 1, v_{n-1} = u_{k-1} = 0 \). In this way we recover \((2+1)\)-dimensional k-cKP hierarchy [16].

3. \( n = 0 \). The differential part of \( M_{0,l} \) (2.22) is equal to zero in this case \((A_0 = 0)\) and we get a new generalization of DS-III hierarchy.

4. \( c_l = 0 \). We obtain \((t_A, \tau_B)\)-Matrix KP Hierarchy that was investigated in [40].

5. If \( l = 0 \) we recover \((\gamma_A, \sigma_B)\)-Matrix KP hierarchy [41].

### 3 Dressing methods for the new \((2+1)\)-dimensional generalization of k-constrained KP hierarchy

#### 3.1 Dressing via Darboux Transformations

In this section we will consider Darboux Transformations (DT) for the pair of operators (2.7) and its reduction (2.22). At first, we shall start with the linear problem associated with the operator \( L_k \) (2.7):

\[
L_k \{ \varphi_1 \} = \beta_k (\varphi_1)_{\tau_k} - \sum_{j=0}^{k} u_j (\varphi_1)^{(j)} - q M_0 D^{-1} \{ r^\top \varphi_1 \} = \varphi_1 \Lambda_1, \tag{3.1}
\]

where \( \varphi_1 \) is \((N \times N)\)-matrix-valued function; \( \Lambda_1 \) is a constant matrix with dimension \( N \times N \). Introduce the DT in the following way:

\[
W_1[\varphi_1] = \varphi_1 D \varphi_1^{-1} = D - \varphi_1 x \varphi_1^{-1}. \tag{3.2}
\]

The following proposition holds.
Proposition 3.1. The operator \( \hat{L}_k[1] := W_1[\varphi_1]L_kW_1^{-1}[\varphi_1] \) obtained from \( L_k \) (3.2) via DT (3.2) has the form

\[
\hat{L}_k[1] := W_1[\varphi_1]L_kW_1^{-1}[\varphi_1] = \beta_k\partial_{\tau_k} - \hat{B}_k - q_1M_0D^{-1}\hat{r}_1^\top, \quad \hat{B}_k[1] = \sum_{j=0}^k \hat{u}_j[1]D^j, \quad (3.3)
\]

where

\[
q_1 = W_1[\varphi_1]\{q\}, \quad \hat{r}_1 = W_1^{-1,\tau}[\varphi_1]\{r\}. \quad (3.4)
\]

\( \hat{u}_j[1] \) are \( N \times N \)-matrix coefficients depending on function \( \varphi_1 \) and coefficients \( u_i, i = 0, k \). In particular, \( \hat{u}_k[1] = u_k \).

Proof. It is evident that the inverse operator to (3.2) has the form \( W_1^{-1}[\varphi_1] = \varphi_1D^{-1}\varphi_1^{-1} \). Thus, we have

\[
\hat{L}_k[1] = W_1[\varphi_1]L_kW_1^{-1}[\varphi_1] = \varphi_1D\varphi_1^{-1}(\beta_k\partial_{\tau_k} - B_k - qM_0D^{-1}r^\top)\varphi_1D^{-1}\varphi_1^{-1} = \beta_k\partial_{\tau_k} + (\hat{L}_k,1)_{\geq 0} + (\hat{L}_k,1)_{<0},
\]

where \((\hat{L}_k[1])_{\geq 0} = -\hat{B}_k[1] = -\sum_{j=0}^k \hat{u}_j[1]D^j \). It remains to find the explicit form of \((\hat{L}_k)_<0 \).

Using formulae (2.1)-(2.3) we have:

\[
(\hat{L}_k[1])_<0 = (\beta_k\varphi_1D\varphi_1^{-1}\varphi_1,\tau_kD^{-1}\varphi_1^{-1})_<0 - (\varphi_1D\varphi_1^{-1}B_k\{\varphi_1\}D^{-1}\varphi_1^{-1})_{<0} - (\varphi_1D\varphi_1^{-1}qM_0D^{-1}\{r^\top\varphi_1\}D^{-1}\varphi_1^{-1} - \varphi_1D\varphi_1^{-1}qM_0D^{-1}(r^\top\varphi_1)\varphi_1^{-1})_{<0} = (\varphi_1D\varphi_1^{-1}\varphi_1\Lambda_1D^{-1}\varphi_1^{-1})_<0 + \varphi_1D\{\varphi_1^{-1}q\}M_0D^{-1}(r^\top\varphi_1)\varphi_1^{-1} = -(\varphi_1D\varphi_1^{-1}\varphi_1\Lambda_1D^{-1}\varphi_1^{-1})_<0 - \varphi_1D\{\varphi_1^{-1}q\}M_0D^{-1}(r^\top\varphi_1)\varphi_1^{-1} = -W_1\{q\}M_0D^{-1}(W_1^{-1,\tau}\{r\})^\top.
\]

It is also possible to generalize the latter theorem to the case of finite number of solutions of linear problems associated with the operator \( L_k \). Namely, let functions \( \varphi_s, s = 1, K \) be solutions of the problems:

\[
L_k\{\varphi_s\} = \beta_k(\varphi_s)\tau_k - \sum_{j=0}^k u_j(\varphi_s)^{(j)} - qM_0D^{-1}\{r^\top\varphi_s\} = \varphi_s\Lambda_s, \quad s = 1, K. \quad (3.7)
\]

For further convenience we shall use the notations \( \varphi_s[1] := \varphi_s, s = 1, K \) and define the following functions:

\[
\varphi_s[2] = W_1[\varphi_1[1]]\{\varphi_s[1]\}, s = 2, K. \quad (3.8)
\]

Now, using functions \( \varphi_1[1], \varphi_2[2], \) we shall define functions \( \varphi_s[3], s = 3, K \):

\[
\varphi_s[3] := W_1[\varphi_2[2]]\{\varphi_s[2]\} = W_1[\varphi_2[2]]W_1[\varphi_1[1]]\{\varphi_s[1]\}, s = 3, K. \quad (3.9)
\]

At the \( p \)-th step we obtain functions: \( \varphi_s[p] := W_1[\varphi_{p-1}[p-1]]\{\varphi_s[p-1]\} = W_1[\varphi_{p-1}[p-1]\ldots W_1[\varphi_2[2]]W_1[\varphi_1[1]]\{\varphi_s[1]\}], s = p, K \). Now we shall construct the following generalization of DT (3.2):

\[
W_K[\varphi_1, \ldots, \varphi_K] = W_1[\varphi_K[K]]\ldots W_1[\varphi_1[1]] = (D - \varphi_{K,x}[K]\varphi_K^{-1}[K])\ldots (D - \varphi_{1,x}[1]\varphi_1^{-1}[1]). \quad (3.10)
\]

The following statement holds:
Proposition 3.2. The operator

\[ \hat{L}_k[K] := W_K[\varphi_1[1], \varphi_2[1], \ldots, \varphi_K[1]]L_kW_K^{-1}[\varphi_1[1], \varphi_2[1], \ldots, \varphi_K[1]] = W_KL_kW_K^{-1} \]

obtained from \( L_k \) \(^{(2.22)}\) via DT \(^{(3.10)}\) has the form

\[ \hat{L}_k[K] := W_KL_kW_K^{-1} = \beta_k \partial_{\tau_k} - \hat{B}_{k,K} - \hat{q}_K \mathcal{M}_0 D^{-1} \hat{r}_K^T, \quad \hat{B}_{k,K} = \sum_{j=0}^{k} \hat{u}_j[K]D^j, \quad (3.11) \]

where

\[ \hat{q}_K = W_K[\varphi_1[1], \ldots, \varphi_K[1]]\{q\}, \quad \hat{r}_K = W_K^{-1}[\varphi_1[1], \ldots, \varphi_K[1]]\{r\}. \quad (3.12) \]

\( \hat{u}_j[K] \) are \( N \times N \)-matrix coefficients depending on functions \( \varphi_s, s = 1, K \) and coefficients \( u_i, i = 0, k \). In particular, \( \hat{u}_k[K] = u_k \).

Proof. The proof can be done via induction by \( K \). Namely, assume that the statement holds for \( K - 1 \). I.e.,

\[ \hat{L}_k[K - 1] = W_{K-1}L_kW_{K-1}^{-1} = \beta_k \partial_{\tau_k} - \hat{B}_{k,K} - \hat{q}_{K-1} \mathcal{M}_0 D^{-1} \hat{r}_{K-1}^T, \quad (3.13) \]

with \( \hat{q}_{K-1} = W_{K-1}[\varphi_1[1], \ldots, \varphi_{K-1}[1]]\{q\} \) and \( \hat{r}_{K-1} = W_{K-1}^{-1}[\varphi_1[1], \ldots, \varphi_{K-1}[1]]\{r\} \). The function \( \varphi_K[K] = W_{K-1}\{\varphi_K[1]\} = W_{K-1}[\varphi_1[1], \ldots, \varphi_{K-1}[1]]\{\varphi_K[1]\} \) (see formulae \(^{(3.8)}\)-\(^{(3.10)}\)) satisfies the equation: \( \hat{L}_k[K - 1]\{\varphi_K[K]\} = W_{K-1}L_kW_{K-1}^{-1}\{\varphi_{K-1}[1]\} = W_{K-1}\{\varphi_K[1]\} = \varphi_K[K]\Lambda_K \).

Now, it remains to apply the Proposition 3.1 to operator \( \hat{L}_k[K - 1] \) \(^{(3.13)}\) with the DT \( W_1[\varphi_K[K]] \) (see formula \(^{(3.2)}\)) and use formula \( W_K = W_1[\varphi_K[K]]W_{K-1} \) that immediately follows from \(^{(3.10)}\).

\[ \square \]

Remark 3.3. We shall also point out that in a scalar case \( (N = 1) \) the DT \( W_K \) \(^{(3.10)}\) can be rewritten in the following way:

\[ W_K := \frac{1}{W[\varphi_1, \varphi_2, \ldots, \varphi_K]} \begin{vmatrix} \varphi_1 & \cdots & \varphi_K & 1 \\ \varphi'_1 & \cdots & \varphi'_K & D \\ \vdots & \cdots & \cdots & \cdots \\ \varphi^{(K)}_1 & \cdots & \varphi^{(K)}_K & D^K \end{vmatrix} = D^K + \sum_{i=0}^{K-1} w_i D^i, \quad (3.14) \]

where \( W[\varphi_1, \varphi_2, \ldots, \varphi_K] \) denotes the Wronskian constructed by solutions \( \varphi_j, j = 1, \ldots, K \), of the linear problem \(^{(3.7)}\). It acts on the vector-valued function \( q = (q_1, \ldots, q_m) \) in the following way: \( W_K(q) = (W_K(q_1), \ldots, W_K(q_m)), \) where \( W_K(q_j) = \frac{W[\varphi_1, \varphi_2, \ldots, \varphi_K, q_j]}{W[\varphi_1, \varphi_2, \ldots, \varphi_K]} \).

The Darboux Transformations \(^{(3.2)}, \quad (3.10)\) are widely used for the solution generating technique involving Lax pairs consisting of differential operators \(^{18,42,46}\). Transformations that generalize \(^{(3.2)}, \quad (3.10)\) also arise in the bidualferential calculus approach to integrable systems and their hierarchies \(^{45,46}\).

Dressing methods for integro-differential operator given by the Proposition 3.2 are closely related to the results of papers on constrained KP hierarchies and their \((2+1)\)-dimensional generalizations \(^{15,16,19,21,47}\).

From Proposition 3.2 we obtain the corollary for Lax pairs consisting of operators \( L_k \) and \( M_n \) \(^{(2.7)}\). Namely, let functions \( \varphi_s, s = 1, K \) be solutions of the problems:

\[ L_k\{\varphi_s\} = \beta_k(\varphi_s)\tau_k - \sum_{i=0}^{k} u_j(\varphi_s)^{(j)} - q_M D^{-1}\{r^T \varphi_s\} = \varphi_s \Lambda_s, \]

\[ M_n\{\varphi_s\} = \alpha_n(\varphi_s)\tau_n - \sum_{i=0}^{n} v_i(\varphi_s)^{(i)} - \tilde{q}_M D^{-1}\{r^T \varphi_s\} = \varphi_s \tilde{\Lambda}_s, \quad s = 1, K. \quad (3.15) \]

Then the following statement holds:
Corollary 3.4. Assume that Lax equation with operators $L_k$ and $M_n$ (2.7) holds: $[L_k, M_n] = 0$. Then:

1. Transformed operators $\hat{L}_k[K] := W_K[\varphi_1, \ldots, \varphi_K] L_k W_K^{-1}[\varphi_1, \ldots, \varphi_K]$, $\hat{M}_n[K] := W_K[\varphi_1, \ldots, \varphi_K] M_n W_K^{-1}[\varphi_1, \ldots, \varphi_K]$, where $W_K$ is defined by (3.10), have the form:

$$
\hat{L}_k[K] := W_K L_k W_K^{-1} = \beta_k \partial_{\tau_k} - \hat{B}_k[K] - \hat{q}_K \hat{M}_0 D^{-1} \hat{r}_k^T, \quad \hat{B}_k[K] = \sum_{j=0}^k \hat{u}_j[K] D^j,
\hat{M}_n[K] := W_K M_k W_K^{-1} = \alpha_n \partial_{\tau_n} - \hat{A}_n[K] - \hat{q}_K \hat{M}_0 D^{-1} \hat{r}_k^T, \quad \hat{A}_k[K] = \sum_{i=0}^n \hat{v}_i[K] D^i,
$$

(3.16)

where

$$
\hat{q}_K = W_K[\varphi_1[1], \ldots, \varphi_K[1]]\{q\}, \quad \hat{r}_K = W_K^{-1,\tau}[\varphi_1[1], \ldots, \varphi_K[1]]\{r\},
\hat{q}_K = W_K[\varphi_1[1], \ldots, \varphi_K[1]]\{q\}, \quad \hat{r}_K = W_K^{-1,\tau}[\varphi_1[1], \ldots, \varphi_K[1]]\{r\}.
$$

(3.17)

2. The operators $\hat{L}_k[K]$ and $\hat{M}_n[K]$ (3.16) satisfy Lax equation: $[\hat{L}_k[K], \hat{M}_n[K]] = 0$.

3. In case of reduction (2.21) in Lax pair (2.7) we have: $\hat{q}_K = (\hat{q}_{1,K}, c_0 \hat{q}_K[0], \ldots, c_0 \hat{q}_K[l])$, $\hat{r}_K = (\hat{r}_{1,K}, \hat{r}_K[0], \ldots, \hat{r}_K[l])$, where $\hat{q}_{1,K} = W_K[\hat{q}_1], \hat{r}_{1,K} = W_K^{-1,\tau}[\hat{r}_1], 
\hat{q}_K[j] = (\hat{L}_k[K])^j(\hat{q}_K), \hat{r}_K[j] = (\hat{L}_k[K])^j(\hat{r}_K), \quad j = 0, l$

Proof. 1. Form (3.16) of operators $\hat{L}_k[K], \hat{M}_n[K]$ follows from Proposition 3.2

2. We obtain the proof of this item from the following formulae

$$
[\hat{L}_k[K], \hat{M}_n[K]] = [W_K L_k W_K^{-1}, W_K M_n W_K^{-1}] = W_K[L_k, M_n] W_K^{-1} = 0.
$$

3. From formulae:

$$
\hat{q}_K = W_K\{(\hat{q}_{1,K}, c_0 \hat{q}_K[0], \ldots, c_0 \hat{q}_K[l])\} = (W_K\{\hat{q}_{1,K}\}, c_0 W_K\{q[0]\}, \ldots, c_0 W_K\{q[l]\}),
W_K\{q[j]\} = W_K\{(L_k)^j(q)\} = (W_K L_k W_K^{-1})^j(W_K\{q\}) = \hat{L}_k[K]\{\hat{q}_K\}
$$

we get the form of $\hat{q}_K$ mentioned in item 3. The form of $\hat{r}_K$ can be obtained in a similar way.

\[\square\]

### 3.2 Dressing via Binary Darboux Transformations

Let $N \times K$-matrix functions $\varphi$ and $\psi$ be solutions of linear problems:

$$
L_k\{\varphi\} = \varphi \Lambda_k, \quad L_k^T\{\psi\} = \psi \bar{\Lambda}_k, \quad \Lambda_k, \bar{\Lambda}_k \in \text{Mat}_{K \times K}(\mathbb{C}).
$$

(3.18)

Introduce binary Darboux transformation (BDT) in the following way:

$$
W = I - \varphi (C + D^{-1}\psi^T\varphi)^{-1} D^{-1}\psi^T,
$$

(3.19)

where $C$ is a $K \times K$-constant nondegenerate matrix. The inverse operator $W^{-1}$ has the form:

$$
W^{-1} = I + \varphi D^{-1} (C + D^{-1}\psi^T\varphi)^{-1} \psi^T.
$$

(3.20)

The following theorem is proven in [48].
Theorem 3.5. The operator \( \hat{L}_k := WL_k W^{-1} \) obtained from \( L_k \) in (2.7) via BDT (3.19) has the form

\[
\hat{L}_k := WL_k W^{-1} = \beta_k \partial_{\tau_k} - \hat{B}_k - \hat{q} M_0 D^{-1} r^\top + \Phi M_k D^{-1} \Psi^\top, \quad \hat{B}_k = \sum_{j=0}^k \hat{u}_j D^j, 
\]

where

\[
M_k = C \Lambda_k - \hat{\Lambda}_k^\top C, \quad \Phi = \varphi \Delta^{-1}, \quad \Psi = \psi \Delta^{-1,\top}, \quad \Delta = C + D^{-1} \{\psi^\top \varphi\}, \quad \hat{q} = W \{ q \}, \quad \hat{r} = W^{-1,\tau} \{ r \}.
\]

\( \hat{u}_j \) are \( N \times N \)-matrix coefficients depending on functions \( \varphi, \psi \) and \( u_j \). In particular,

\[
\hat{u}_k = u_k, \quad \hat{u}_{k-1} = u_{k-1} + \left[ u_k, \varphi \left( C + D^{-1} \{\psi^\top \varphi\}\right)^{-1} \psi^\top \right].
\]

Solution generating method for the hierarchy (2.7)-(2.8) is given by the corollary, which follows from the previous theorem.

Corollary 3.6. Let \( N \times K \)-matrix functions \( \varphi \) and \( \psi \) satisfy equations:

\[
L_k \{ \varphi \} = \varphi \Lambda_k, \quad L_k \{ \psi \} = \psi \hat{\Lambda}_k, \quad \Lambda_k, \hat{\Lambda}_k \in \text{Mat}_{K \times K}(\mathbb{C}), \\
M_n \{ \varphi \} = \varphi \Lambda_n, \quad M_k \{ \psi \} = \psi \hat{\Lambda}_n, \quad \Lambda_n, \hat{\Lambda}_n \in \text{Mat}_{K \times K}(\mathbb{C})
\]

with operators \( L_k \) and \( M_n \) (2.7) satisfying \([L_k, M_n] = 0\). Then transformed operators \( \hat{L}_k \) and \( \hat{M}_n \) satisfy Lax equation \([\hat{L}_k, \hat{M}_n] = 0\) and have the form:

\[
\hat{L}_k := WL_k W^{-1} = \beta_k \partial_{\tau_k} - \hat{B}_k - \hat{q} M_0 D^{-1} r^\top + \Phi M_k D^{-1} \Psi^\top, \quad \hat{B}_k = \sum_{j=0}^k \hat{u}_j D^j, \\
\hat{M}_n := WM_n W^{-1} = \alpha_n \partial_{\tau_n} - \hat{\Lambda}_n - \hat{q} M_0 D^{-1} r^\top + \Phi M_n D^{-1} \Psi^\top, \quad \hat{\Lambda}_n = \sum_{i=0}^n \hat{v}_i D^i.
\]

where

\[
\hat{M}_n = C \Lambda_n - \hat{\Lambda}_n^\top C, \quad \hat{M}_n = C \Lambda_n - \hat{\Lambda}_n^\top C, \quad \Phi = \varphi \Delta^{-1}, \quad \Psi = \psi \Delta^{-1,\top}, \\
\Delta = C + D^{-1} \{\psi^\top \varphi\}, \quad \hat{q} = W \{ q \}, \quad \hat{r} = W^{-1,\tau} \{ r \}, \quad \hat{q} = W \{ q \}, \quad \hat{r} = W^{-1,\tau} \{ r \}
\]

\( \hat{u}_j \) are \( N \times N \)-matrix coefficients depending on functions \( \varphi, \psi \) and \( u_j, v_i \). In particular,

\[
\hat{u}_k = u_k, \quad \hat{u}_{k-1} = u_{k-1} + \left[ u_k, \varphi \left( C + D^{-1} \{\psi^\top \varphi\}\right)^{-1} \psi^\top \right], \\
\hat{v}_n = v_n, \quad \hat{v}_{n-1} = v_{n-1} + \left[ v_n, \varphi \left( C + D^{-1} \{\psi^\top \varphi\}\right)^{-1} \psi^\top \right].
\]

Proof. From formulae \( W[L_k, M_n] W^{-1} = [\hat{L}_k, \hat{M}_n] = 0 \) we obtain that Lax equation with transformed operators is satisfied. Form (3.25) of the transformed operators \( \hat{L}_k \) and \( \hat{M}_n \) follows from Theorem 3.5.

4 New (2+1)-dimensional generalizations of the modified k-constrained KP hierarchy

Consider the following pair of integro-differential operators:

\[
L_k = \beta_k \partial_{\tau_k} - B_k - q M_0 D^{-1} r^\top D, \quad B_k = \sum_{j=1}^k u_j D^j, \quad u_j = u_j(x, \tau_k, t_n), \quad \beta_k \in \mathbb{C}, \\
M_n = \alpha_n \partial_{\tau_n} - A_n - \hat{q} M_0 D^{-1} r^\top D, \quad A_n = \sum_{i=1}^n v_i D^i, \quad v_i = v_i(x, \tau_k, t_n), \quad \alpha_n \in \mathbb{C},
\]
where \( u_j \) and \( v_i \) are matrix-valued functions of dimension \( N \times N \); \( q \) and \( r \) are matrix-valued functions of dimension \( N \times m \); \( \bar{q} \) and \( \bar{r} \) are matrix-valued functions with dimension \( N \times \bar{m} \). \( M_0 \) and \( \bar{M}_0 \) are constant matrices with dimensions \( m \times m \) and \( \bar{m} \times \bar{m} \) respectively. The following proposition is a consequence of Proposition 2.1.

**Proposition 4.1.** Lax equation \([L_k, M_n] = 0\) is satisfied in case the following equations hold:

\[
[L_k, M_n]_{\geq 0} = 0, \quad L_k \{ \bar{q} \} = \bar{q} \Lambda q, \quad (D^{-1} L_k^T D) \{ \bar{r} \} = \bar{r} \Lambda r, \\
M_n \{ q \} = q \Lambda q, \quad (D^{-1} M_n^T D) \{ r \} = r \Lambda r,
\]

(4.2)

where \( \Lambda q, \Lambda r, \bar{\Lambda} q, \bar{\Lambda} r \) are constant matrices with dimensions \((m\times m)\) and \((\bar{m}\times \bar{m})\) respectively that satisfy equations: \( \Lambda q M_0 - \bar{M}_0 \Lambda r = 0, \bar{\Lambda} q M_0 - M_0 \bar{\Lambda} r = 0 \).

**Proof.** Operators (4.1) can be rewritten as:

\[
L_k = \beta_k \partial_{r_k} - B_k - q M_0 r^T + q M_0 D^{-1} r^T x, \\
M_n = \alpha_n \partial_{t_n} - A_n - \bar{q} \bar{M}_0 r^T + \bar{q} \bar{M}_0 D^{-1} \bar{r}^T x.
\]

(4.3)

It remains to use the Proposition 2.1 to complete the proof. \(\square\)

Consider some examples of the hierarchy given by (4.1)-(4.2).

1. \( k = 1, n = 2 \).

\[
L_1 = \beta_1 \partial_{\tau_1} - D - q M_0 D^{-1} r^T D, \\
M_2 = \alpha_2 \partial_{\tau_2} - D^2 - v D - \bar{q} \bar{M}_0 D^{-1} \bar{r}^T D.
\]

(4.4)

Lax representation \([L_1, M_2] = 0\) is equivalent to the following system:

\[
\alpha_2 q_{t_2} - q_{xx} - v q_x - \bar{q} \bar{M}_0 D^{-1} \{ \bar{r}^T q_x \} = q \Lambda q, \\
-\alpha_2 r_{t_2} - r_{xx} + v^T r_x - \bar{r} \bar{M}_0 D^{-1} \{ q^T r_x \} = r \Lambda r, \\
\beta_1 q_{t_1} - \bar{q} x - q M_0 D^{-1} \{ \bar{r}^T q_x \} = q \bar{\Lambda} q, \\
v_x - \beta_1 v_{t_1} = 2 (q M_0 r^T) x.
\]

In case of the Hermitian conjugation reduction \( \beta \in \mathbb{R}, \alpha \in i \mathbb{R}, \bar{M}_0^* = -M_0, \bar{\Lambda}_0 = \bar{M}_0, \bar{\bar{r}} = \bar{\bar{q}}, \bar{r} = q, v = -v^* (L_1^* = -D L_1 D^{-1}, M_2^* = D M_2 D^{-1})\) the latter equation reduces to the following:

\[
\alpha_2 q_{t_2} - q_{xx} - v q_x - \bar{q} \bar{M}_0 D^{-1} \{ \bar{q}^* q_x \} = q \Lambda q, \\
\beta_1 q_{t_1} - \bar{q} x - q M_0 D^{-1} \{ q^* \bar{q}_x \} = q \bar{\Lambda} q, \\
v_x - \beta_1 v_{t_1} = 2 (q M_0 q^*) x.
\]

In case we set \( \bar{q} = 0, \Lambda q = 0 \) we get a matrix (2+1)-dimensional generalization of the Chen-Lee-Liu equation.

2. \( k = 3, n = 2 \).

\[
L_3 = \beta_3 \partial_{\tau_3} - c_1 (D^3 + w D^2 + v D) - q M_0 D^{-1} r^T D, \\
M_2 = \alpha_2 \partial_{\tau_2} - c_2 D^2 - u D - \bar{q} \bar{M}_0 D^{-1} \bar{r}^T D.
\]

(4.5)
Using (4.2) we get that Lax representation \([L_3, M_2] = 0\) is equivalent to the following system

\[
3u_x - 2c_2w_x - [u, w] = 0,
\]

\[
\alpha_2 c_1 u_{t_2} - c_2 w_{x} + 2c_1 w u_x - c_1 w w_x + 3c_1 u_{x x} +
+ 3c_1 (q \mathcal{M}_0 \mathcal{R})_x - 2c_2 c_1 v_x - c_2 [q \mathcal{M}_0 \mathcal{R}, w] + c_1 [v, u] = 0,
\]

\[
- \beta_3 u_{x x} + c_1 u_{x x} + c_1 w u_{x x} + 3c_1 (q \mathcal{M}_0 \mathcal{R})_x + 2c_1 w q \mathcal{M}_0 \mathcal{R} +
+ c_1 q \mathcal{M}_0 \mathcal{R} w + \alpha_2 c_1 u_{t_2} - c_2 c_1 v_x + c_1 u_{x x} - c_1 w w_x + c_1 [v, q \mathcal{M}_0 \mathcal{R}] -
- 2c_2 (q \mathcal{M}_0 \mathcal{R})_x + c_1 (q \mathcal{M}_0 \mathcal{R}, u) = 0,
\]

\[
\beta_3 q \mathcal{R}_3 - q \mathcal{R}_{xx} - c_1 q \mathcal{R}_x - c_3 q \mathcal{R}_x - q \mathcal{M}_0 D^{-1} \{q ^{\top} \mathcal{R}_x\} = \tilde{q} \Lambda \mathcal{q},
\]

\[
\alpha_2 q_{t_2} - c_2 q_{x x} - u q_x - q \mathcal{M}_0 D^{-1} \{q ^{\top} \mathcal{R}_x\} = \mathcal{q} \Lambda \mathcal{q}.
\]

Set \(c_1 = c_2 = 1\) in the scalar case \((N = 1)\). Eliminating variables \(w\) and \(v\) from the first and second equation respectively, we get

\[
- \beta_3 u_{x x} - \frac{1}{4} u_{x x} - \frac{3}{4} u_x^2 + \frac{1}{4} u_{x x} + \frac{3}{4} u_{x x} D^{-1} \{u_{x x}\} + \frac{3}{4} \alpha_2 u_{x x} D^{-1} \{u_{x x}\} +
+ \frac{1}{3} (u q \mathcal{M}_0 \mathcal{R})_x + \frac{3}{4} \alpha_2 (q \mathcal{M}_0 \mathcal{R})_x - \frac{3}{4} (q \mathcal{M}_0 \mathcal{R})_x = 0,
\]

\[
- \beta_3 q \mathcal{R}_3 - q \mathcal{R}_{xx} - \frac{3}{4} u q \mathcal{R}_x + \frac{1}{3} u q \mathcal{R}_x - q \mathcal{M}_0 D^{-1} \{q ^{\top} \mathcal{R}_x\} = \mathcal{q} \Lambda \mathcal{q},
\]

\[
\alpha_2 q_{t_2} - q_{x x} - u q_x - q \mathcal{M}_0 D^{-1} \{q ^{\top} \mathcal{R}_x\} = \mathcal{q} \Lambda \mathcal{q}.
\]

The latter under the Hermitian conjugation reduction \(\alpha_2 \in i \mathbb{R}, \beta_3 \in \mathbb{R}, \mathcal{M}_0 = -\mathcal{M}_0, \mathcal{M}_0 = \mathcal{M}_0^\ast\), \(u = -u, \mathcal{q} = \tilde{q}, \mathcal{q} = q, (L^3 = -DL_3 D^{-1}, M^2 = DM_2 D^{-1})\) reads:

\[
- \beta_3 u_{x x} - \frac{1}{4} u_{x x} - \frac{3}{4} u_x^2 + \frac{1}{4} u_{x x} + \frac{3}{4} u_{x x} D^{-1} \{u_{x x}\} + \frac{3}{4} \alpha_2 u_{x x} D^{-1} \{u_{x x}\} +
+ \frac{3}{4} (u q \mathcal{M}_0 q^\ast)_x + \frac{3}{4} \alpha_2 (q \mathcal{M}_0 q^\ast)_x - \frac{3}{4} (q \mathcal{M}_0 q^\ast)_x = 0,
\]

\[
- \beta_3 q \mathcal{R}_3 - q \mathcal{R}_{xx} - \frac{3}{4} u q \mathcal{R}_x + \frac{3}{4} u q \mathcal{R}_x + \frac{3}{4} (q \mathcal{M}_0 q^\ast) + \frac{3}{4} \alpha_2 D^{-1} \{u_{x x}\} \mathcal{q} -
- q \mathcal{M}_0 D^{-1} \{q ^{\ast} \mathcal{q}_x\} = \tilde{q} \Lambda \mathcal{q}, \quad \alpha_2 q_{t_2} - q_{x x} - u q_x - q \mathcal{M}_0 D^{-1} \{q ^{\top} \mathcal{R}_x\} = \mathcal{q} \Lambda \mathcal{q}.
\]

\(\tilde{q} = 0\) leads to the modified KPSCS of the first type. If \(q = 0\) we recover the second type of the modified KPSCS.

### 4.1 Dressing via Binary Darboux Transformations

In this subsection we consider dressing methods for \((2+1)\)-dimensional extensions of the modified k-constrained KP hierarchy given by the family of Lax pairs (4.1). First of all, we start with the matrix version of the theorem that was proven in [50].

**Theorem 4.2.** Let \((N \times K)\)-matrix functions \(\varphi\) and \(\psi\) satisfy linear problems:

\[
L_k \{\varphi\} = \varphi \Lambda_k, \quad L_k \{\psi\} = \psi \tilde{\Lambda}_k, \quad \Lambda_k, \tilde{\Lambda}_k \in \text{Mat}_{K \times K}(\mathbb{C}),
\]

\[
L_k = \beta_k \partial_{t_k} + B_k - q \mathcal{M}_0 D^{-1} \mathcal{R} \mathcal{T} D, \quad B_k = \sum_{i=1}^k u_i D^i.
\]

Then the operator \(L_k\) transformed via

\[
W_m := w_0^{-1} W = w_0^{-1} \left( I - \varphi \Delta^{-1} D^{-1} \psi^\top \right) = I - \varphi \tilde{\Delta}^{-1} D^{-1} \{\psi\} D^\top D,
\]

\(w_0\) is an initial solution of the modified k-constrained KP hierarchy.
where
\[
\begin{align*}
w_0 &= I_N - \varphi \Delta^{-1} D^{-1}\{\psi^\top\}, \quad \Delta = C + D^{-1}\{D^{-1}\{\psi^\top\}\varphi_x\}, \\
\end{align*}
\] (4.11)
has the form:
\[
\begin{align*}
\dot{L}_k &:= W_m L_k W_m^{-1} = \beta_k \partial_{t_k} + \dot{B}_k - \tilde{q} M_0 D^{-1} \tilde{\Phi}^\top D + \Phi M_k D^{-1} \tilde{\Psi}^\top D, \\
\dot{B}_k &= \sum_{j=1}^k \dot{u}_j D^j, \quad \dot{u}_k = u_k, \quad \dot{u}_{k-1} = u_{k-1} + ku_0 w_0 w_{0,x}, \ldots,
\end{align*}
\] (4.12)
where
\[
\begin{align*}
\mathcal{M}_k &= CA_k - \tilde{\Lambda}_k^\top C, \quad \tilde{\Phi} = -W_m \{\varphi\} C^{-1} = \varphi \tilde{\Delta}^{-1}, \\
\tilde{\Psi} &= D^{-1}\{W_m^{-1}\{\psi\}\} C^{-1,\top} = D^{-1}\{\psi\} \Delta^{-1,\top}, \quad \tilde{q} = W_m \{q\}, \quad \tilde{r} = D^{-1} W_m^{-1,\tau} D \{r\}.
\end{align*}
\] (4.13)

Proof. Proof is similar to the proof of Theorem 2 in [50].

The following consequence of the latter theorem provides a solution generating method for hierarchy (4.1)-(4.2):

**Corollary 4.3.** Let \((N \times K)\)-matrix functions \(\varphi\) and \(\psi\) satisfy linear problems:
\[
\begin{align*}
L_k \{\varphi\} &= \varphi \Lambda_k, \quad L_k \{\psi\} = \psi \tilde{\Lambda}_k, \quad \Lambda_k, \tilde{\Lambda}_k \in Mat_{K \times K}(\mathbb{C}), \\
M_n \{\varphi\} &= \varphi \Lambda_n, \quad M_n \{\psi\} = \psi \Lambda_n, \quad \Lambda_n, \tilde{\Lambda}_n \in Mat_{K \times K}(\mathbb{C})
\end{align*}
\]
with operators \(L_k\) and \(M_n\) given by (4.1). Then operators \(\dot{L}_k = W_m L_k W_m^{-1}\) and \(\dot{M}_n = W_m M_n W_m^{-1}\) transformed via \(W_m\) (4.10)-(4.11) have the form:
\[
\begin{align*}
\dot{L}_k &:= \beta_k \partial_{t_k} + \dot{B}_k - \tilde{q} M_0 D^{-1} \tilde{\Phi}^\top D + \Phi M_k D^{-1} \tilde{\Psi}^\top D, \\
\dot{B}_k &= \sum_{j=1}^k \dot{u}_j D^j, \quad \dot{u}_k = u_k, \quad \dot{u}_{k-1} = u_{k-1} + ku_0 w_0 w_{0,x}, \ldots,
\end{align*}
\] (4.14)
where
\[
\begin{align*}
\mathcal{M}_k &= CA_k - \tilde{\Lambda}_k^\top C, \quad \mathcal{M}_n = CA_n - \tilde{\Lambda}_n^\top C, \quad \tilde{\Phi} = -W_m \{\varphi\} C^{-1} = \varphi \tilde{\Delta}^{-1}, \\
\tilde{\Psi} &= D^{-1}\{W_m^{-1}\{\psi\}\} C^{-1,\top} = D^{-1}\{\psi\} \Delta^{-1,\top}, \quad \tilde{q} = W_m \{q\}, \quad \tilde{r} = D^{-1} W_m^{-1,\tau} D \{r\}.
\end{align*}
\] (4.15)

5 Conclusions

In this work we proposed new integrable generalizations of the KP hierarchy with self-consistent sources. The obtained hierarchies of nonlinear equations include, in particular, matrix integrable system that contains as special cases two types of the matrix KP equation with self-consistent sources (KPSCS) and its modified version. They also cover new generalizations of the N-wave problem and the DS-III system. Under reductions (2.21) imposed on the obtained hierarchies one recovers (2+1)-BDk-cKP hierarchy. The latter contains \((t_A, t_B)-\) and \((\gamma_A, \gamma_B)\)-Matrix KP hierarchies [40][41] (see [22] for details).

**Remark 5.1.** It should be pointed out that extended mKP hierarchy (4.1) admits the following reduction:
\[
\begin{align*}
q &= (q_1, q_2, -q_2 M_0 r_2^\top - D^{-1}\{u\}, 1), \quad r = (r_1, D^{-1}\{r_2\}, 1, D^{-1}\{u\}), \\
\tilde{q} &= (q_1, q_2, -q_2 M_0 \tilde{r}_2^\top - D^{-1}\{\tilde{u}\}, 1, q_1[0], q_1[1], \ldots, q_1[l]), \\
\tilde{r} &= (\tilde{r}_1, \tilde{r}_2, 1, D^{-1}\{\tilde{u}\}, r_1[l], r_1[l - 1], \ldots, r_1[0]).
\end{align*}
\]
It leads to the following family of integro-differential operators in (4.1)

\[ L_k = \beta_k \partial \tau_k - B_k - q_1 M_0 D^{-1} r_1^T D + q_2 M_0 D^{-1} r_2^T D^{-1} u, \]
\[ M_n = \alpha_n \partial t_n - A_n - \tilde{q}_1 \tilde{M}_0 D^{-1} \tilde{r}_1^T D + \tilde{q}_2 \tilde{M}_0 D^{-1} \tilde{r}_2^T D^{-1} \tilde{u} - \alpha \sum_{j=0}^{l} q_l[j] M_0 D^{-1} r_1^T [l - j] D. \]

Lax equation \([L_k, M_n] = 0\) involving the latter operators should lead (under additional reductions) to (2+1)-dimensional generalizations of the corresponding integrable systems that were obtained in [50]. In particular it concerns systems that extend KdV, mKdV and Kaup-Broer equations.

In this paper we also elaborated solution generating methods for the proposed hierarchies (2.7)-(2.8) and (4.1)-(4.2) respectively via DTs and BDTs.

The latter involve fixed solutions of linear problems and an arbitrary seed (initial) solution of the corresponding integrable system. Exact solutions of equations with self-consistent sources (complexitons, negatons, positons) and the underlying hierarchies were studied in [40,51,52]. One of the problems for future interest consists in looking for the corresponding analogues of these solutions in the obtained generalizations. The same question concerns lumps and rogue wave solutions that were investigated in several integrable systems recently [53–60].

It is also known that Inverse Scattering and Spectral methods were applied to generate solutions of equations with self-consistent sources [61–63]. An extension of these methods to the obtained hierarchies and comparison with results that can be provided by BDTs (e.g., following [64]) presents an interest for us.

The search for the corresponding discrete counterparts of the constructed hierarchies is another problem for future investigation. The latter is expected to contain the discrete KP equation with self-consistent sources [65,66]. One of the possible ways to solve the problem consists in looking for the formulation of the corresponding continuous hierarchy within a framework of bidifferential calculus. The latter framework provides better possibilities to search for the discrete counterparts of the corresponding continuous systems (see, e.g., [67]).

**Acknowledgements**

The authors are grateful to Professor Müller-Hoissen for fruitful discussions and useful advice in preparation of this paper. Yu.M. Sydorenko (J. Sidorenko till 1998 and Yu.M. Sidorenko till 2002 in earlier transliteration) is grateful to the Ministry of Education, Science, Youth and Sports of Ukraine for partial financial support (Research Grant MA-107F).

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