Time-Invariant Feedback Strategies Do Not Increase Capacity of AGN Channels Driven by Stable and Certain Unstable Autoregressive Noise

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Abstract

The capacity of additive Gaussian noise (AGN) channels, \( Y_t = X_t + V_t, t = 1, \ldots, n \), with time-invariant channel input feedback strategies, is characterized and conditions are identified for entropy rates, and limit of average power to exist, when the noise is described by stable and unstable autoregressive models, \( \text{AR}(c) \), \( V_t = cV_{t-1} + W_t, V_0 = v_0, t = 1, \ldots, n \), where \( c \in (-\infty, \infty) \), \( W_t, t = 1, \ldots, n \), is a zero mean, variance \( K_W \), independent Gaussian sequence, independent of \( V_0 \). For stable \( \text{AR}(c), c \in (-1, 1) \) the conditions are necessary and sufficient for asymptotic stationarity of the processes \( (X_t, Y_t), t = 1, 2, \ldots \). New closed form capacity formulas and lower bounds are derived, for the \( \text{AR}(c), c \in (-\infty, \infty) \) noise, which are fundamentally different from existing formulas in the literature, and illustrate multiple regimes of capacity, as a function of the parameters \( (c, K_W, \kappa) \), as follows.

1) feedback increases capacity for the regime, \( c^2 \in (1, \infty) \), for \( \kappa \geq \frac{K_W \left(1 + \sqrt{4c^2 - 3}\right)}{2(c^2-1)^2} \),

2) feedback does not increase capacity for the regime \( c^2 \in (1, \infty) \), for \( \kappa \leq \frac{K_W \left(1 + \sqrt{4c^2 - 3}\right)}{2(c^2-1)^2} \), and

3) feedback does not increase capacity for the regime \( c \in [-1, 1] \), for \( \kappa \in [0, \infty) \).

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When compared to [1], our capacity formulas for AGN channels driven by a stable AR\((c)\), \(c \in (-1,1)\) noise, state that feedback does not increase capacity, contrary to the main results of the characterizations of feedback capacity derived in [1, Theorem 4.1 and Theorem 6.1], for stationary or asymptotically stationary processes. We show that our disagreement with [1] is mainly attributed to the identification of necessary and/or sufficient conditions, to ensure the asymptotic average power and entropy rates exist, expressed in terms of the known detectability and stabilizability conditions of convergence of generalized difference Riccati equations (DREs) to analogous generalized algebraic Riccati equations (AREs), of Gaussian estimation problems, which are not accounted for in [1, Theorem 4.1 and Theorem 6.1].

I. INTRODUCTION, MOTIVATION, AND MAIN RESULTS OF THE PAPER

The feedback capacity of additive Gaussian noise (AGN) channels driven by nonstationary, and stationary limited memory Gaussian noise, is addressed, since the early 1970’s, in an anthology of papers, under various assumptions [1]–[8]. Two of the fundamental problems are related to

(Q1): characterizations and computations of feedback capacity of noiseless feedback codes, when the initial state \(S_0 = s_0\) of the noise is known or not known to the encoder and the decoder, and

(Q2): bound on feedback capacity, based on linear feedback coding schemes of communicating Gaussian random variables (RVs), \(\Theta : \Omega \rightarrow \mathbb{R}\), and coding schemes of communicating digital messages \(W : \Omega \rightarrow \mathcal{H}^{(n)} \triangleq \{1,2,\ldots,\lceil M_n \rceil\}\), when the initial state \(S_0 = s_0\) of the noise is known to the encoder and the decoder.

This paper is mainly concerned with question (Q1), for AGN channels driven by stable and unstable noise, when channel input strategies are time-invariant (not necessarily stationary). The first objective is to identify necessary and/or sufficient conditions for optimal channel input strategies to ensure the limiting average power and the information rates exist, for both stable or unstable channel noise, and then to determine if additional conditions are needed for these limits to be independent of the initial states of the noise. Such conditions are identified. For stable (resp. unstable) noise these conditions also imply the input and output processes (resp. the input process and innovations process of the output) are asymptotically stationary, and the limiting average power and the entropy rates do not depend on the initial states of the channel or their initial distributions. The second main objective is the calculation of closed form capacity formulas for stable and unstable autoregressive unit memory noise. Such formulas are derived. They illustrate multiple regimes of capacity, and that feedback does not increase capacity, when the noise is stable (also for unstable noise for certain values of power \(\kappa\)).

From the main results of this paper follows that several of the answers related to questions (Q1) and (Q2), given in [1], that treats stable, stationary or asymptotically stationary noises, are incorrect due to oversights related to convergence of asymptotic average power and entropy rates. In particular, the oversights affect the validity of feedback capacity described in time-domain [1, Theorem 6.1], by a time-invariant optimization problem, with zero variance of the innovations of the channel input process; when
conditions for convergence are imposed, then the value of feedback capacity stated in [1, Theorem 6.1] is necessarily zero (otherwise the per unit time asymptotic limits of the information theoretic characterization of feedback capacity does not exists). Some of the technical oversights in [1] are also repeated in [9]–[12]. These are discussed at various parts of this paper. For the reader’s convenience these are clarified in Section IV.

To enlarge the scope of the current paper compared to previous literature [1], [8]–[12] that treat feedback capacity for stable noises, we analyze both the feedback and non-feedback capacity, for stable and unstable noises (using a unified approach), although most of our emphasis is on feedback capacity. To avoid extensive notation, this paper is focused on AGN channels, driven by a correlated Gaussian noise, of the simplest form, the autoregressive unit memory, stable and unstable noise. However, the method of this paper apply to more general limited memory noise models [1], [8] (under appropriate assumptions).

A. The Main Problems of Capacity of AGN Channels with Memory

In this section we introduce the precise mathematical formulation, and the underlying assumptions based on which we derive the results of the paper. We consider the following time-varying AGN channel.

\[ Y_t = X_t + V_t, \quad t = 1, \ldots, n, \quad \frac{1}{n} E_0 \left\{ \sum_{t=1}^{n} (X_t)^2 \right\} \leq \kappa, \quad \kappa \in [0, \infty) \]  

(I.1)

\[ X^n = \{X_1, X_2, \ldots, X_n\} \] is the sequence of channel input random variables (RVs) \( X_t : \Omega \rightarrow \mathbb{R}, \)

\[ Y^n = \{Y_1, Y_2, \ldots, Y_n\} \] is the sequence of channel output RVs \( Y_t : \Omega \rightarrow \mathbb{R}, \)

\[ V^n = \{V_1, \ldots, V_n\} \] conditioned on the initial state \( V_0 = v_0, \) is a sequence of jointly Gaussian distributed RVs \( V_t : \Omega \rightarrow \mathbb{R}, \) and \( V^n \in N(0, K_{V^n|V_0}), \)

\( V_0 = v_0, \) is known to the encoder and decoder\(^1\)

\( N(0, K_{V^n|V_0}) \) denotes the distribution of the Gaussian RV \( V^n \) conditional on \( V_0, \) with zero conditional mean, and conditional variance \( K_{V^n|V_0}, \)

\( E_0 \{ \cdot \} \) denotes expectation for fixed initial state \( V_0 = v_0. \)

A time-varying unit memory Gaussian autoregressive noise, with initial state \( V_0 = v_0, \) is defined by

\[ \text{AR}(c_t) : \begin{cases} V_t = c_t V_{t-1} + W_t, & V_0 = v_0, \quad t = 1, \ldots, n, \\ W_t \in N(0, K_{W_t}), & t = 1, \ldots, n, \quad \text{indep. Gaussian, indep. of } V_0 \in N(0, K_{V_0}), \\ K_{V_0} \geq 0, \quad K_{W_t} > 0, \quad c_t \in (-\infty, \infty), \quad t = 1, \ldots, n \quad \text{are non-random}. \end{cases} \]  

(I.2)

We denote by \( \text{AR}(c), c \in (-\infty, \infty) \) the restriction of \( \text{AR}(c_t) \) to the time-invariant autoregressive noise, i.e., \( K_{W_t} = K_W, t = 1, \ldots, n, c_t = c \in (-\infty, \infty), t = 0, \ldots, n. \) For stable noise, \( \text{AR}(c), c \in (-1, 1), \) the variance defined by \( K_{V_t} \triangleq E(V_t)^2, \) satisfies \( K_{V_t} = c^2 K_{V_{t-1}} + K_W, K_{V_0} \geq 0, t = 1, \ldots, n. \) The stable \( \text{AR}(c) \) noise is called asymptotically stationary if \( \lim_{n \rightarrow \infty} K_{V_n} = \frac{K_W}{1-c^2}, \) for all initial values \( K_{V_0} \geq 0, \) i.e., \( |c| < 1. \)

\( \text{AR}(c_t) \) without an initial state is defined by (I.2), for \( t = 2, \ldots, n, \) with \( V_1 \in N(0, K_{V_1}), K_{V_1} \geq 0, \) independent

\(^1\)This assumption is explicit in [8], it is hidden in [1, Theorem 6.1], it is explicit in [1, Lemma 6.1 and comments above it]; see Section IV-B for clarification.
The coding rate is \( r_n \). Similarly, the stable AR(c) noise without an initial state is called asymptotically stationary if \( K_{t-1}^cK_{V_{t-1}} + K_{V_t} \geq 0, t = 2, \ldots, n \), converges, \( \lim_{n \to \infty} K_{V_n} = \frac{K_{V_0}}{1-c} \), for all initial values \( K_{V_1} \geq 0, |c| < 1 \). That is, the invariant distribution of the noise is \( N(0, \frac{K_{V_0}}{1-c^2}, c \in (-1, 1)) \).

At this stage, we introduce the feedback code and non-feedback code of the AGN channel.

**Definition 1.1. Feedback and non-feedback codes**

(a) A noiseless time-varying feedback code\(^2\) for the AGN Channel, is denoted by \( \mathcal{C}_{n+1}^{fb} \triangleq \{(n, [M_n], s_0, \kappa, \varepsilon_n) : \quad n = 1, 2, \ldots, \} \), and consists of the following elements and assumptions.

(i) The set of uniformly distributed messages \( W : \Omega \to \mathcal{M}(n) \triangleq \{1, 2, \ldots, [M_n]\} \).

(ii) The set of codewords of block length \( n \), defined by the set\(^3\)

\[
\mathcal{E}_{[0,n]}(\kappa) \triangleq \left\{ X_1 = e_1(W, V_0), X_2 = e_2(W, V_0, X_1, Y_1), \ldots, X_n = e_n(W, V_0, X^{n-1}, Y^{n-1}) : \right. \\
\left. \frac{1}{n+1} E_{v_0}^e \left( \sum_{i=0}^{n} (X_i)^2 \right) \leq \kappa \right\}. \tag{I.3}
\]

(iii) The decoder functions \( (v_0, y^n) \mapsto d_n(v_0, y^n) \in \mathcal{M}(n) \), with average error probability

\[
P_{\text{error}}^{(n)}(v_0) = P\{d_n(V_0, Y^n) \neq W \mid V_0 = v_0\} = \frac{1}{[M_n]} \sum_{v=1}^{[M_n]} P_{v_0}^{e}(d_n(V_0, Y^n) \neq W) \leq \varepsilon_n. \tag{I.4}
\]

where \( P_{v_0}^{e} \) means the distribution depends on \( e(\cdot) \in \mathcal{E}_{[0,n]}(\kappa) \) and \( V_0 = v_0 \) is fixed.

(iv) “\( X^n \) is causally related to \( V^n \)” \cite[page 39, above Lemma 5]{6}, which is equivalent to the following decomposition of the joint probability distribution of \( (X^n, V^n) \) given \( V_0 \).

\[
P_{X^n,Y^n|V_0} = P_{V^n|V_0} \prod_{t=1}^{n} P_{X_t|X^{t-1}, Y^{t-1}, V_0}.
\tag{I.5}
\]

\[
= P_{V^n|V_0} \prod_{t=1}^{n} P_{X_t|X^{t-1}, Y^{t-1}, V_0}, \quad \text{by } Y_t = X_t + V_t. \tag{I.6}
\]

The coding rate is \( r_n \triangleq \frac{1}{n} \log [M_n] \). Given an initial state \( V_0 = v_0 \), a rate \( R(v_0) \) is called an achievable rate, if there exists a code sequence \( \mathcal{C}_{n+1}^{fb} \), satisfying \( \lim_{n \to \infty} \varepsilon_n = 0 \) and \( \liminf_{n \to \infty} \frac{1}{n} \log [M_n] \geq R(v_0) \).

The operational definition of the feedback capacity of the AGN channel, for fixed \( V_0 = v_0 \), is \( C(K, v_0) \triangleq \sup \{ R(v_0) : R(v_0) \text{ is achievable}\} \).

(b) A time-varying code without feedback for the AGN Channel, denoted by \( \mathcal{C}_{n+1}^{fb} \), is the restriction of the time-varying feedback code \( \mathcal{C}_{n+1}^{fb} \), to the subset \( \mathcal{E}_{[0,n]}^{fb}(\kappa) \subset \mathcal{E}_{[0,n]}(\kappa) \), defined by

\[
\mathcal{E}_{[0,n]}^{fb}(\kappa) \triangleq \left\{ X_1 = e_1^{fb}(W, V_0), X_2 = e_2^{fb}(W, V_0, X_1, Y_1), \ldots, X_n = e_n^{fb}(W, V_0, X^{n-1}) : \right. \\
\left. \frac{1}{n} E_{v_0}^{e} \left( \sum_{i=1}^{n} (X_i)^2 \right) \leq \kappa \right\}. \tag{I.7}
\]

\(^2\) A time-varying feedback code means the channel input distributions \( P_{X_t|X^{t-1}, V_t} \), \( t = 1, \ldots, n \) are time-varying.

\(^3\) The superscript \( e(\cdot) \) on \( E_{v_0}^{e} \) is used to denote that the distribution depends on the strategy \( e(\cdot) \in \mathcal{E}_{[0,n]}(\kappa) \).

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Since the code sequence $\mathcal{C}^{fb}_{Z^+}$ depends on $V_0 = v_0$, then in general, the rate $R(v_0)$, and also $C(\kappa, v_0)$ depend on $v_0$.

**Feedback Capacity of Time-Varying Channel Input Strategies.** Consider the feedback code of Definition I.1(a), i.e., $\mathcal{C}^{fb}_{Z^+}$. Given the elements of the set $\mathcal{E}_{[0,n]}(\kappa)$, by the maximum entropy principle of Gaussian distributions, similar to [6], the upper bound holds$^4$.

$$I^*(W;Y^n|v_0) \leq H(Y^n|v_0) - H(V^n|v_0), \quad \text{if } H(Y^n|v_0) \text{ is evaluated at a Gaussian } P_{Y^n|v_0}$$

(I.8)

where $H(X|S)$ stands for differential entropy of RV $X$ conditioned on the initial state $S = s$. Further, similar to [6], the upper bound in (I.8) is achieved if the input $X^n$ is jointly Gaussian for fixed $V_0 = v_0$, satisfies the average power constraint, and respects (I.5). By the chain rule of mutual information, $I^*(W;Y^n|v_0) = \sum_{t=1}^n I^*(W;Y_t|Y^{t-1}, v_0)$, and the data processing inequality, follows,

$$\sup_{\mathcal{E}_{[0,n]}(\kappa)} I^*(W;Y^n|v_0) \leq \sup_{P_{X_t|X^{t-1},Y^{t-1},v_0}, t=1,...,n} H(Y^n|v_0) - H(V^n|v_0), \quad \text{by (I.1)} \quad \text{(I.9)}$$

where the supremum in the right hand side of (I.9) is taken over conditionally Gaussian time-varying distributions $P_{X_t|X^{t-1},Y^{t-1},v_0}, t = 1, \ldots, n$, such that $(X^n, Y^n)$ are jointly Gaussian for fixed $V_0 = v_0$, and (I.5) is respected.

Define, as in [6], the $n-$finite transmission feedback information (FTFI) capacity of code $\mathcal{C}^{fb}_{Z^+}$, by

$$C_n(\kappa, v_0) \triangleq \sup_{P_{X_t|X^{t-1},Y^{t-1},v_0}, t=1,...,n} H(Y^n|v_0) - H(V^n|s_0) \quad \text{(I.10)}$$

provided the supremum element exists in the set. From the converse and direct coding theorems in [6, Theorem 1], it then follows that the characterization of feedback capacity of code $\mathcal{C}^{fb}_{Z^+}$, is given by

$$C(\kappa, v_0) = \lim_{n \rightarrow \infty} \frac{1}{n} C_n(\kappa, v_0) \quad \text{(I.11)}$$

provided the limit exists in $[0, \infty)$ i.e., it is finite.

**Capacity Without Feedback of Time-Varying Channel Input Strategies.** Let $C^{n_{fb}}(\kappa, v_0)$ be defined as in (I.10), with the time-varying feedback distributions $P_{X_t|X^{t-1},Y^{t-1},v_0}, t = 1, \ldots, n$, replaced by the time-varying non-feedback distributions $P_{X_t|X^{t-1},v_0}, t = 1, \ldots, n$, called $n-$finite transmission without feedback information (FTwFI) capacity. The non-feedback capacity of the code $\mathcal{C}^{n_{fb}}_{Z^+}$ of Definition I.1(b), is characterized by $C^{n_{fb}}(\kappa, v_0) = \lim_{n \rightarrow \infty} \frac{1}{n} C^{n_{fb}}_{Z^+}(\kappa, v_0)$, provided the limit exists.

To the best of our knowledge, no closed form formulas are available in the literature for $C_n(\kappa, v_0)$, $C(\kappa, v_0)$, even for AGN channels driven by a time-invariant AR($c$), $c \in (-1, 1)$ noise. Often, past literature is focused on stationary or stable noise, stationary or asymptotically stationary joint input and output process, and investigates the variant of (I.11) with the limit and supremum operations interchanged [8, Theorem 7 and

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$^4$The superscript $e$ means the underlying distributions are induced by the channel distribution and the elements of the set $e(\cdot) \in \mathcal{E}_{[0,n]}(\kappa)$. 

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Corollary 7.1], [1, Theorem 3.2] and [9]–[12]. However, as it will be apparent in this paper, conditions for existence of the limiting average power and entropy rates are not correctly identified in [1], [8]–[12], and this oversight led to incorrect characterizations of feedback capacity by a time-invariant optimization problem (i.e., [1, Theorem 6.1]).

This brings us to the next definition of capacity, where conditions for existence of the limits of average power and entropy rates are characterized, and they part of our problem formulation.

**Feedback Capacity of Time-Invariant Channel Input Strategies.** We consider (I.10), (I.11) with the per unit time limit and supremum operations interchanged, and time-invariant codes and induced distributions, called strategies. To ensure the feedback capacity (to be defined shortly) is well-posed, we introduce the following condition:

**(C1)** Channel input strategies with feedback are time-invariant, the consistency condition (I.5) holds, and the following limits exist:

(i) \( \lim_{n \to \infty} \frac{1}{n} \mathbb{E}_v \{ \sum_{t=1}^n (X_t)^2 \} \in [0, \infty) \), (ii) \( \lim_{n \to \infty} \frac{1}{n} \left\{ H(Y^n|v_0) - H(V^n|v_0) \right\} \in [0, \infty) \).

We define the operational information feedback capacity under condition (C1), as follows.  
\[
C^{\infty}(\kappa, v_0) \overset{\Delta}{=} \sup_{\lim_{n \to \infty} \frac{1}{n} \mathbb{E}_v \{ \sum_{t=1}^n (X_t)^2 \} \leq \kappa} \lim_{n \to \infty} \frac{1}{n} \left\{ H(Y^n|v_0) - H(V^n|v_0) \right\} 
\]  
where the supremum is taken over all jointly Gaussian channel input processes \( X^n, n = 1, 2, \ldots \) with feedback, or distributions with feedback \( P_{X_t|X^{t-1}, Y^{t-1}, v_0}, t = 1, 2, \ldots \), such that \( (X^n, Y^n), n = 1, 2, \ldots \), is jointly Gaussian, for \( V_0 = v_0 \), and (C1) holds.

In the definition of \( C^{\infty}(\kappa, v_0) \) we do not assume joint stationarity of \( (X^n, Y^n, V^n), n = 1, 2, \ldots \), because this is not required for the limits to exist. Similarly, we do not impose conditions to ensure \( C^{\infty}(\kappa, v_0) = C^{\infty}(\kappa), \forall v_0 \), i.e., is independent of the initial state \( V_0 = v_0 \), and hence \( \int C^{\infty}(\kappa, v_0) P_{y_0}(dv_0) \) is independent of \( V_0 \in N(0, K_{v_0}), \forall K_{v_0} \geq 0 \). Rather, for stable (resp. unstable) noise, we first identify necessary and/or sufficient conditions for (C1) to hold, and then determine if additional conditions are needed to ensure the optimal channel input process is asymptotically stationary, and such that it induces asymptotic stationarity of the channel output process (resp. innovations process of the output), and \( C^{\infty}(\kappa, v_0) = C^{\infty}(\kappa), \forall v_0 \).

For time-invariant stable or unstable noise, it will become apparent, from properties of estimation theory of linear Gaussian systems (introduced at latter parts of the paper), that the well-known detectability and stabilizability conditions of generalized Kalman-filter equations [13], [14] (see Definition III.3), are necessary and sufficient for the limits to exist, and for \( C^{\infty}(\kappa, v_0) \) to be well-posed. It will also be apparent that in prior literature [1], [8]–[12], conditions for existence of the limits in (C1) are not imposed, and this omission leads to incorrect characterizations of feedback rates by a time-invariant optimization problem, which is not the asymptotic limit of (I.12) (i.e., [1, Theorem 6.1]).
Capacity Without Feedback of Time-Invariant Channel Input Strategies. Similar to (I.12), we also analyze the non-feedback capacity analog, under condition (C1), which is defined as follows.

\[
C^\infty_{\text{nf}}(\kappa, v_0) \triangleq \sup_{\lim_{n \to \infty} \frac{1}{n} \mathbb{E}\left\{ \sum_{i=1}^{n} \frac{Y_i^2}{v_0^2} \right\} \leq \kappa} \lim_{n \to \infty} \frac{1}{n} H(Y^n | v_0) - H(V^n | v_0) \tag{I.13}
\]

where the supremum is taken over all jointly Gaussian channel input processes \(X^n, n = 1, 2, \ldots\), without feedback or distributions without feedback, denoted by \(P_{X^n | X^{n-1}, V_0}, t = 1, 2, \ldots\), such that \((X^n, Y^n), n = 1, 2, \ldots\) is jointly Gaussian for \(V_0 = v_0\), (C1) holds (with \(P_{X^n | X^{n-1}, Y^{n-1}, V_0}\) replaced by \(P_{X^n | X^{n-1}, V_0}\)), and (I.5) is respected, for \(n = 1, 2, \ldots\). To our knowledge, for AGN channels driven by an unstable noise \(V\), no closed form expression of non-feedback capacity is ever reported in the literature.

Given the above formulation, in this paper we obtain answers to the questions listed under Problem I.1.

Problem I.1. Main problem

Consider \(C^\infty(\kappa, v_0)\) defined by (I.12), and \(C^\infty_{\text{nf}}(\kappa, v_0)\) defined by (I.13), of the AGN channel driven by a time-invariant stable and unstable, AR(\(c\)) noise, i.e., \(c \in (-\infty, \infty)\):

(a) What are necessary and/or sufficient conditions for (C1) to hold?

(b) What are necessary and/or sufficient conditions for asymptotic stationarity of the process \((X^n, Y^n), n = 1, 2, \ldots\) or of only the marginal process \(X^n\), that achieve \(C^\infty(\kappa, v_0)\), and for \(C^\infty(\kappa, v_0) = C^\infty(\kappa)\), \(\forall v_0\) i.e., to be independent of initial data?

(c) What are the characterizations and closed form formulas of feedback capacity \(C^\infty(\kappa, v_0)\)?

(d) How do we extract simple lower bounds on non-feedback capacity, \(C^\infty_{\text{nf}}(\kappa, v_0)\) from the characterizations of feedback capacity?

To address Problem I.1 we make use of the identities

\[
\lim_{n \to \infty} \frac{1}{n} H(Y^n | v_0) = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} H(Y_t | Y^{t-1}, v_0) = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} H(Y_t - \mathbb{E}(Y_t | Y^{t-1}, v_0) | Y^{t-1}, v_0) \tag{I.14}
\]

\[
= \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} H(I_t), \quad I_t \triangleq Y_t - \mathbb{E}(Y_t | Y^{t-1}, v_0) \quad \text{an indep. innovations process.} \tag{I.15}
\]

Then we identify necessary and/or sufficient conditions for the limits \(\lim_{n \to \infty} \frac{1}{n} \mathbb{E}(H(Y^n | v_0) - H(V^n | v_0))\) and \(\lim_{n \to \infty} \frac{1}{n} \mathbb{E}(\sum_{i=1}^{n} X_t^2)\) to exist, i.e., in \([0, \infty)\).

B. Methodology of the Paper

Our methodology is based on the following main steps.

Step 1. We characterize \(C_n(\kappa, v_0)\) defined by (I.10), i.e., the \(n\)-FTFI capacity, of the AGN channel driven by a time-varying AR(\(c_t\)) noise. We also give a lower bound on the characterization of the \(n\)-FTwFI capacity \(C_n^{\text{ftb}}(\kappa, v_0)\), using a Gaussian channel input process, which is realized by an AR(\(\Lambda_t\)) process,

\[
X_t = \Lambda_t X_{t-1} + Z_t, \quad X_1 = Z_1, \quad \Lambda_t \in (-\infty, \infty), \quad t = 2, \ldots, n \tag{I.16}
\]
where $Z^n$ an independent Gaussian sequence, independent of $(V^n, V_0)$.

**Step 2.** We characterize the feedback capacity $C^n(\kappa, v_0) = C^n(\kappa), \forall v_0$ defined by (I.12), and we give a lower bound on the characterization of $C^{n,fb}(\kappa, v_0)$ defined by (I.13), of the AGN channel driven by a time-invariant stable or unstable noise, AR$(c), c \in (-\infty, \infty)$. Our analysis identifies necessary and/or sufficient conditions for condition (C1) to hold, expressed in terms of the convergence properties of generalized difference Riccati equations (DREs) and algebraic Riccati equations (AREs), of estimating the channel state, that is, the noise $V^n$, from the channel output process $Y^n$, and the initial state $V_0 = v_0$, for $n = 1, 2, \ldots$. This step is analogous to [15, Theorem 4.1], although the models considered in [15] involve a classical control DRE and ARE.

**Step 3.** We derive a closed form formula of feedback capacity $C^n(\kappa, v_0) = C^n(\kappa), \forall v_0$, that shows there are multiple regimes of capacity, and these regimes depend on the parameters $(c, K_W, \kappa)$. Our feedback capacity formulae $C^n(\kappa)$ for AGN channels driven by stable noise AR$(c), c \in (-1, 1)$ is fundamentally different from the one obtained using the characterization of feedback capacity in [1, Theorem 6.1]. We show this difference is mainly attributed to the appended detectability and stabilizability conditions on the characterization of our feedback capacity, to ensure the optimal channel input process $X^n, n = 1, 2, \ldots$ is such that the limits, $\lim_{n \to \infty} \frac{1}{n} E_0 \left\{ \sum_{i=1}^{n} (X_i)^2 \right\} \in [0, \infty)$, $\lim_{n \to \infty} \frac{1}{n} \{ H(Y^n|v_0) - H(V^n|v_0) \} \in [0, \infty)$ exist, and the joint process $(X^n, Y^n), n = 1, 2, \ldots$ is asymptotically stationary, which are not accounted for, in [1, Theorem 6.1].

We also give an achievable lower bound on the non-feedback capacity $C^{n,fb}(\kappa, v_0)$, based on (I.16), with $\bar{X}_i = 0, \forall i$, i.e., $X_i = Z_i, Z^n, n = 1, \ldots$ an independent and identically distributed (IID) sequence, and holds for stable and unstable AR$(c), c \in (-\infty, \infty)$ noise.

**Step 4.** We show the characterization of feedback capacity given in [1, Theorem 6.1, $C_{FB}$] (i.e., the limiting expression of $C^n(\kappa, v_0)$, without the stabilizability condition), and with a zero variance of the innovations part of the channel input process (see [1, Lemma 6.1]), gives an incorrect value of $C_{FB}$. However, when the stabilizability conditions is imposed (which is necessary for asymptotic limits to exists) then it follows the value $C_{FB} = 0$; clearly an incorrect limiting value of (I.12).

We structured the paper as follows.

In Section II, we derive the characterization of the $n$–FTFI capacity, and the lower bound on the characterization of the $n$–FTwFI capacity, for AGN channels driven by the AR$(c_i)$ noise (Section II-A), and present a preliminary elaboration on technical issues that are integral part of capacity definition (I.12).

In Section III, we present the derivations of feedback capacity formulas of $C^n(\kappa, v_0) = C^n(\kappa), \forall v_0$, i.e., (I.12), and the achievable lower bounds on the non-feedback capacity $C^{n,fb}(\kappa, v_0)$, for stable and unstable noise, using the asymptotic analysis of generalized Kalman-filters [13], [14].

In Section IV we collect and discuss some of the oversights in [1], which are also repeated in [9]–[12].

For the rest of this section, we follow a rather unconventional presentation; we present a summary of the main results of the paper, with informative simulations of achievable rates.
C. Summary of the Main Results, Graphical Evaluations, and Discussion

Below, we summarize the main results of the paper, and present graphical evaluations of achievable rates.

(R1) Multiple Regimes of Capacity. There are multiple regimes of feedback capacity $C^\omega(\kappa,v_0) = C^\omega(\kappa), \forall v_0$ defined by (I.12), and these regimes depend on the parameters $(c,K_w,\kappa)$, as follows.

Regime 1: $\mathcal{K}^\omega(c,K_w) \supseteq \left\{ \kappa \in [0,\infty) : 1 < c^2 < \infty, \kappa > \frac{K_w(1 + \sqrt{4c^2 - 3})}{2(c^2 - 1)^2} \right\}$, \hspace{1cm} (I.17)

Regime 2: $\mathcal{K}^\omega_{\text{reg}}(c,K_w) \supseteq \left\{ \kappa \in [0,\infty) : 1 < c^2 < \infty, \kappa \leq \frac{K_w(1 + \sqrt{4c^2 - 3})}{2(c^2 - 1)^2} \right\}$, \hspace{1cm} (I.18)

Regime 3: $\mathcal{K}^\omega_{\text{no fb}}(c) \supseteq \left\{ \kappa \in [0,\infty) : 0 \leq c^2 \leq 1 \right\}$, \hspace{1cm} (I.19)

Regime 1. This corresponds to the unstable AR$(c)$ noise, i.e., $c \in (-\infty,1) \cup (1,\infty)$, there is a threshold effect on power, $\kappa > \kappa_{\text{min}} = \frac{K_w(1 + \sqrt{4c^2 - 3})}{2(c^2 - 1)^2}$, and feedback increases capacity.

Regime 2. This corresponds to the unstable AR$(c)$ noise, i.e., $c \in (-\infty,1) \cup (1,\infty)$, there is a threshold effect on power, $\kappa \leq \kappa_{\text{max}} = \frac{K_w(1 + \sqrt{4c^2 - 3})}{2(c^2 - 1)^2}$, feedback does not increase capacity, $C^\omega(\kappa,v_0) = C^\omega_{\text{reg}}(\kappa), \forall v_0$.

Regime 3. This corresponds to the marginally stable AR$(c)$ noise, i.e., $c \in [-1,1]$, there is no threshold effect on power, feedback does not increase capacity, $C^\omega(\kappa,v_0) = C^\omega_{\text{no fb}}(\kappa), \forall v_0$.

(R2) Feedback capacity $C^\omega(\kappa,v_0) = C^\omega(\kappa), \forall v_0$. For Regime 1 the following hold.

The optimal time-invariant, channel input distribution for $C^\omega(\kappa,v_0)$ defined by (I.12), is induced by a jointly Gaussian channel input process $X^n(t)$ (not necessarily stationary), with a representation

\begin{equation}
X^n_t = \Lambda^n\left(X^n_{t-1} - v_0^n\right) + Z^n_t, \quad t = 2,\ldots,n, \hspace{1cm} (I.20)
\end{equation}

\begin{equation}
= \Lambda^n\left(V_{t-1} - v_0^n\right) + Z^n_t, \quad \Lambda^n = -\Lambda^n, \hspace{1cm} (I.21)
\end{equation}

\begin{equation}
X^n_t = \Lambda^n\left(v_0^n - v_0^n\right) + Z^n_t = Z^n_t, \hspace{1cm} (I.22)
\end{equation}

\begin{equation}
\frac{1}{n}E_n\left\{ \sum_{i=1}^{n} (X^n_i)^2 \right\} = \frac{1}{n} \sum_{i=1}^{n} \left\{ \left(\Lambda^n\right)^2 K^n_{t-1} + K^n_Z \right\} \leq \kappa, \hspace{1cm} (I.23)
\end{equation}

\begin{equation}
Z^n_t \text{ indep. of } (X^n_{t-1},V^t_{t-1},V^t_{v_0^n}), \quad Z^n_t \in N(0,K^n_Z), \quad \text{for } t = 1,\ldots,n, \hspace{1cm} (I.24)
\end{equation}

\begin{equation}
Z^n_{v_0^n} \text{ indep. of } (V^n,v_0^n), \hspace{1cm} (I.25)
\end{equation}

\begin{equation}
K^n_t \triangleq E_n\left(E^n_t\right)^2, \quad E^n_t \triangleq V_t - v_0^n\left(V^n_t\right)^{v_0^n}, \quad t = 1,\ldots,n, \quad K^n_{v_0^n} = 0, \hspace{1cm} (I.26)
\end{equation}

\begin{equation}
V^n_t \triangleq E_n\left(V_t\right)^{v_0^n}, \quad \text{satisfy the generalized Kalman-filter equations (III.123)-(III.130).} \hspace{1cm} (I.27)
\end{equation}

The feedback capacity is characterized by

\begin{equation}
C^\omega(\kappa,v_0) \triangleq \sup_{\left(\Lambda^n,K^n_Z\right)} \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \log \left( \frac{\left(\Lambda^n + c\right)^2 K^n_{t-1} + K^n_Z + K^n_w}{K^n_w} \right), \hspace{1cm} (I.28)
\end{equation}
Here “stabilizing” means \( \kappa^\infty \geq 0 \) is such that the eigenvalue of the linear recursion of the estimation error \( E_n^o \triangleq V_n - \hat{V}_n^o, n = 1, 2, \ldots, \) lies inside \((-1, 1)\), hence it converges in mean-square-sense.

For Regime 1 (see (I.17)) which means \( c^2 \in (1, \infty) \) the closed form capacity formula is

\[
C^\infty(\kappa) = \frac{1}{2} \log \left( \frac{c^2 - 1}{c^2 - 1} \right), \quad \text{for all } \kappa \in \mathcal{K}^\infty(c, K_W),
\]

(33)

\[
= \log |c| + \frac{1}{2} \log \left( \frac{1}{c^2 - 1} + y \right), \quad y \triangleq \frac{\kappa}{K_W}
\]

(34)

\[
\geq \log |c| \in (0, \infty)
\]

(35)

and it is achieved by the unique values \((K^\infty, \Lambda^\infty, K_Z^\infty)\):

\[
K^\infty = \frac{\kappa(c^2 - 1) - K_W}{c^2(c^2 - 1)} \in (0, \infty),
\]

(36)

\[
\Lambda^\infty = \frac{cK_W}{\kappa(c^2 - 1) - K_W} \in (-\infty, \infty),
\]

(37)

\[
K_Z^\infty = \frac{\kappa(c^2 - 1)(\kappa(c^2 - 1) - K_W) - K_W^2}{(c^2 - 1)\left(\kappa(c^2 - 1) - K_W\right)} \in (0, \infty).
\]

(38)

Since for Regime 1, \( \kappa \in \mathcal{K}^\infty(c, K_W) \), and \( c^2 \in (1, \infty) \), inequality (I.35) indicates a threshold effect on the achievable rate for the unstable noise. From (I.34), the behavior of \( C^\infty(\kappa) \) for \( \gamma \) large is

\[
C^\infty(\kappa) \simeq \log |c| + \frac{1}{2} \log \gamma, \quad \text{for large } \gamma.
\]

(39)

---

\footnotesize{For \( \kappa \in \mathcal{K}^\infty(c, K_W) \) the limit in (I.28) converges to a finite number and the supremum exists and it is finite.}
(R3) Capacity without feedback $C^{\infty,nfb}(\kappa,v_0) = C^{\infty,nfb}_o(\kappa), \forall v_0$. For Regime 2 and Regime 3, feedback does not increase capacity. That is, if $\kappa \in \mathcal{H}^{\infty,nfb}(c,K_w)$ or $\kappa \in \mathcal{H}^{\infty,nfb}(c)$, there does not exist channel input process such that the limit in (I.28) converges to a nonzero value. On the other hand, for $\kappa \in \mathcal{H}^{\infty,nfb}(c,K_w)$ or $\kappa \in \mathcal{H}^{\infty,nfb}(c)$, there exists a channel input without feedback, in particular, with $\Lambda^\infty = \Lambda^\infty = 0$ in (I.20) and (I.21), such that the limit in (I.13) converges to a nonzero value, for the AGN channel driven by a stable and unstable AR($c$), $c \in (-\infty,\infty)$ noise.

(R4) Lower bound on capacity without feedback $C^{\infty,nfb}(\kappa,v_0)$. An achievable lower bound on the characterization of nonfeedback capacity, is obtained, for the AGN channel driven by a stable and unstable AR($c$), $c \in (-\infty,\infty)$ noise, which corresponds to a unit memory channel input, $X_t^o = \overline{X} X_{t-1}^o + Z_t^o, X_0^o = Z_0^o, t = 2, \ldots$, and holds for any $\kappa \in [0,\infty)$. Further, another achievable lower bound on the non-feedback capacity for the AGN channel driven by a stable and unstable AR($c$), $c \in (-\infty,\infty)$ noise, is obtained from (R2), by letting $\overline{X}^\infty = \Lambda^\infty = 0$, which holds for any $\kappa \in [0,\infty)$, with corresponding IID channel input, $X_t^o = Z_t^o, t = 1, \ldots, K_Z^\infty = \kappa$. The lower bound based on the IID channel input is given by

$$\begin{align*}
C^{\infty,nfb}(\kappa,v_0) \geq C^{\infty,nfb}_o(\kappa) &= \frac{1}{2} \log \left( \frac{c^2 K_{\infty,s}^{\kappa} + \kappa + K_w}{K_w} \right), \quad \forall \kappa \in [0,\infty),
\end{align*}$$

$$\begin{align*}
= \frac{1}{2} \log \left( \frac{\kappa(1 + c^2) + K_w + \sqrt{\kappa(1 - c^2) + K_w}^2 + 4c^2 K_w \kappa}{2K_w} \right), \quad c \in (-\infty,\infty)
\end{align*}$$

where $K_{\infty,s} \triangleq \lim_{t \to \infty} K_{\infty}^s \in [0,\infty)$ is the unique nonnegative stabilizing solution of (I.32), that corresponds to the optimal strategy $(\Lambda^{\infty,s},K_{Z}^{\infty,s}) = (0,\kappa)$, and given by

$$\begin{align*}
K_{\infty,s} = \left\{ \begin{array}{ll}
-\kappa \left(1-c^2\right) - K_w + \sqrt{\kappa(1-c^2) + K_w}^2 + 4c^2 K_w \kappa & , c \neq 0, \quad \kappa \in [0,\infty), \\
\frac{\kappa K_w}{\kappa + K_w} & , c = 0, \quad \kappa \in [0,\infty),
\end{array} \right.
\end{align*}$$

$$\begin{align*}
\Lambda^{\infty,s} = \Lambda^{\infty,s} = 0, \quad K_{Z}^{\infty,s} = \kappa.
\end{align*}$$

(R5) Numerical evaluations of the feedback capacity $C^\infty(\kappa)$ for $\kappa \in \mathcal{H}^\infty(c,K_w)$, i.e. regime 1, based on (I.33), and the lower bound of non-feedback capacity $C^{\infty,nfb}_o(\kappa)$ for $\kappa \in [0,\infty)$ based on (I.41), are shown in Figure I.1. These illustrate that feedback capacity $C^\infty(\kappa)$ for $\kappa \in \mathcal{H}^\infty(c,K_w)$ is an increasing function of the parameter, $|c| \in (1,\infty)$, that is, the more unstable the AR($c$) noise the higher the value of capacity $C^\infty(\kappa)$. Further, the lower bound on non-feedback capacity $C^{\infty,nfb}_o(\kappa)$ is achievable for all $\kappa \in [0,\infty)$, for stable and unstable AR($c$), $c \in (-\infty,\infty)$ noise. As illustrated in Figure I.1, for values of $|c| > 1$, a discontinuity occurs at $\kappa = \frac{K_w(1 + \sqrt{4c^2 - 3})}{2(c^2 - 1)^2}$. Note, that this occurs since i) for $|c| \in (1,\infty)$ and $\kappa \leq \frac{K_w(1 + \sqrt{4c^2 - 3})}{2(c^2 - 1)^2}$, and ii) for $|c| \in (0,1]$ and $\kappa \in [0,\infty)$, the depicted curves correspond to lower bounds on non-feedback capacity $C^{\infty,nfb}_o(\kappa)$, and not the actual non-feedback capacity, whereas for $|c| \in (1,\infty)$ and $\kappa > \frac{K_w(1 + \sqrt{4c^2 - 3})}{2(c^2 - 1)^2}$, the depicted curves correspond to the feedback capacity. 

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Fig. I.1. Feedback capacity $C^\infty(\kappa)$ for $\kappa \in \mathcal{K}^\infty(c,K_W)$ based on (I.33) and lower bound on non-feedback capacity $C_{LB}^\infty(\kappa)$ for $\kappa \in [0,\infty)$ based on (I.41), of the AGN channel driven by AR($c$) noise, for various values of $c \in (-\infty,\infty)$ and $K_W = 1$.

Fig. I.2. Time-sharing between time-invariant channel input feedback strategy and time-invariant channel input non-feedback strategy (of the lower bound) for an AGN channel driven by AR($c$) noise, with $c = 1.5$ and $K_W = 1$. 
The rate of a time-sharing scheme between feedback capacity of regime $1$, $C^\infty(\kappa)\in \mathcal{K}^\infty(c, K_W)$, and the lower bound on capacity without feedback $C^\infty_{LB}(\kappa), \kappa \in [0, \infty)$, is illustrated in Figure 1.2. This scheme results in higher rates compared to feedback capacity, since it employs a time-varying channel input strategy, that is, two different strategies, one without feedback and one with feedback are applied, whereas the feedback capacity of regime $1$, is only defined over time-invariant channel input strategies.

Now, we digress to recall the closed form non-feedback capacity formulae, that is derived using water-filling in [16, eqn(5.5.14)] (see also (and [5, eqn(6)]), for the AGN channel driven a stable AR($c$) noise.

$$C^{nf}(\kappa) = \frac{1}{2} \log \left( 1 + \kappa + \frac{c^2}{1-c^2} \right), \quad \kappa > \left( \frac{1}{(1-|c|)^2} - \frac{1}{(1-c^2)} \right), \quad c \in (-1, 1), \quad K_W = 1. \quad (I.44)$$

Figure 1.3 compares the non-feedback capacity $C^{nf}(\kappa)$ based on (I.44) to the achievable lower bound on the non-feedback $C^\infty_{LB}(\kappa)$ based on (I.41), which corresponds to transmitting an IID channel input $Z_o \in N(0, \kappa)$. For an AGN channel driven by AR($c$), noise, $c = 0.75$ and $K_W = 1$ (the values correspond to the maximum difference).
\( N(0, \kappa) \), i.e., \( K_Z^{o^*} = \kappa \), for \( c = 0.75 \) and \( K_W = 1 \). Surprisingly, contrary to the non-feedback lower bound formulae (I.41), which holds for all stable or unstable AR\((c)\), \( c \in (-\infty, \infty) \) noise and \( \kappa \in [0, \infty) \), the closed form non-feedback capacity formulae (I.44), based on water-filling, is restricted to \( \kappa > \left( \frac{1}{1-|c|^{2}} - \frac{1}{1-c^{2}} \right) \), and to the stable AR\((c)\), \( c \in (-1, 1) \). The maximum difference \( C^{o^f\text{fb}}(\kappa) - C^{o^f\text{fb}}_{\text{LB}}(\kappa) \), when \( K_W = 1 \) occurs at \( c = 0.75 \), and is less than \( 1.5 \times 10^{-2} \) bits per channel use. This difference is expected to be reduced further if a unit memory optimal input \( X_t^o = \overline{X}_t X_{t-1} + Z_t, X_1^o = Z_1, t = 2, \ldots, \), is used, as stated in (R4).

II. CHARACTERIZATIONS OF \( n - \text{FTFI} \) AND \( n - \text{FTWFI} \) CAPACITY OF AGN CHANNELS

In this section we present the following main results.

1. Theorem II.1 (Section II-A), which gives the characterization of \( n - \text{FTFI} \) capacity for time-varying feedback codes of Definition I.1.(a).
2. high level discussion (Section II-B) on the implications of generalized Kalman-filter equations on the characterizations of \( n - \text{FTFI} \) capacity, and
3. Corollary II.1 (Section II-D), which gives a lower bound on the \( n - \text{FTWFI} \) capacity for time-varying non-feedback codes of Definition I.1.(b), based on a Markov channel input process without feedback, and follows directly from Theorem II.1.

A. Characterization of \( n - \text{FTFI} \) Capacity for AGN Channels Driven by AR\((c)\) Noise

Below, we introduce the characterization of the \( n - \text{FTFI} \) capacity, for an AGN channel, driven by the time-varying AR\((c)\) noise, for the feedback code of Definition I.1.(a). Our presentation, of the next theorem, is based on the degenerate case of the general characterization of the \( n - \text{FTFTI} \) capacity of AGN channels, derived in [17]. We should mention that although, [8], treats AGN channels driven by stable noise, some parts of the representation given below can be extracted from the analysis of [8, Section II-V].

**Theorem II.1. Characterization of \( n - \text{FTFI} \) Capacity for AGN Channels Driven by AR\((c)\) Noise**

Consider the AGN channel (I.1) driven by a time-varying AR\((c)\) noise, i.e., (I.2), and the code of Definition I.1.(a). Then the following hold.

(a) The optimal time-varying channel input distribution with feedback, for the optimization problem \( C_n(\kappa, v_0) \) defined by (I.10), is conditionally Gaussian, of the form

\[
P_{X_t|V_{t-1}, Y^{t-1}, V_0} = P_{X_t|V_{t-1}, Y^{t-1}, V_0}, \quad t = 1, \ldots, n \tag{II.45}
\]

and it is induced by the time-varying jointly Gaussian channel input process \( X^n \), with a representation\(^6\)

\[
X_t = \overline{\Lambda}_t \left( X_{t-1} - \overline{\Lambda}_{t-1} \right) + Z_t, \quad t = 2, \ldots, n, \tag{II.46}
\]

\[
= \Lambda_t \left( V_{t-1} - \overline{V}_{t-1} \right) + Z_t, \quad \overline{\Lambda}_t = -\Lambda_t, \tag{II.47}
\]

\(^6\)The fact that \( X_1 = Z_1, K_1 = 0, \overline{V}_0 = v_0 \) is due to the code definition, i.e., \( V_0 = v_0 \) is known to the encoder.
\[ X_1 = Z_1, \quad \text{(II.48)} \]

\[ Z_t \in N(0, K_{Z_t}), \quad t = 1, \ldots, n \quad \text{a Gaussian sequence,} \quad \text{(II.49)} \]

\[ Z_t \quad \text{independent of} \quad (V_t^{t-1}, X_t^{t-1}, Y_t^{t-1}, V_0), \quad t = 1, \ldots, n, \quad \text{(II.50)} \]

\[ Z^n \quad \text{independent of} \quad (V^n, V_0), \quad \text{(II.51)} \]

\[ V_t = c_t V_{t-1} + W_t, \quad V_0 = v_0, \quad c_t \in (-\infty, \infty), \quad t = 1, \ldots, n, \quad \text{(II.52)} \]

\[ Y_t = X_t + V_t = \Lambda_t (X_{t-1} - \hat{X}_{t-1}) + c_t (Y_{t-1} - X_{t-1}) + W_t + Z_t, \quad t = 2, \ldots, n \quad \text{(II.53)} \]

\[ Y_1 = Z_1 + c_1 V_0 + W_1, \quad V_0 = v_0, \quad \text{(II.54)} \]

\[ \frac{1}{n} E_{v_0} \left\{ \sum_{t=1}^{n} (X_t)^2 \right\} = \frac{1}{n} \sum_{t=1}^{n} \left\{ (\Lambda_t)^2 K_{t-1} + K_{Z_t} \right\} \leq \kappa, \quad \text{(II.56)} \]

\[ (\Lambda_t, K_{Z_t}) \in (-\infty, \infty) \times [0, \infty) \quad \text{scalar-valued, non-random,} \quad \text{(II.57)} \]

\[ \hat{X}_t \triangleq E_{v_0} \left\{ X_t \mid Y^t \right\}, \quad \hat{V}_t \triangleq E_{v_0} \left\{ V_t \mid Y^t \right\}, \quad \text{(II.58)} \]

\[ K_t \triangleq E_{v_0} \left\{ (X_t - \hat{X}_t)^2 \right\} = E_{v_0} \left\{ (V_t - \hat{V}_t)^2 \right\}, \quad t = 1, \ldots, n. \quad \text{(II.59)} \]

Further, \( H(Y^n | v_0) - H(V^n | v_0) \), \( (\hat{V}_t, K_t), t = 1, \ldots, n \) are determined by the generalized\(^7\) time-varying Kalman-filter and generalized time-varying difference Riccati equation (DRE), of estimating \( V^n \) from \( Y^n \), given below.

**Generalized Kalman-filter Recursion for (II.52)-(II.53)** [13], [14]:

\[ \hat{V}_t = c_t \hat{V}_{t-1} + M_t (K_{t-1}, \Lambda_t, K_{Z_t}) I_t, \quad \hat{V}_0 = v_0, \quad \text{(II.60)} \]

\[ = F_t (K_{t-1}, \Lambda_t, K_{Z_t}) \hat{V}_{t-1} + M_t (K_{t-1}, \Lambda_t, K_{Z_t}) Y_t, \quad \hat{V}_0 = v_0, \quad \text{(II.61)} \]

\[ I_t \triangleq Y_t - E_{v_0} \left\{ Y_t \mid Y^t-1 \right\} = Y_t - c_t \hat{V}_{t-1}, \quad I_1 = Z_1 + W_1, \quad t = 2, \ldots, n, \quad \text{(II.62)} \]

\[ = (\Lambda_t + c_t) (V_{t-1} - \hat{V}_{t-1}) + Z_t + W_t, \quad \text{(II.63)} \]

\[ M_t (K_{t-1}, \Lambda_t, K_{Z_t}) \triangleq \left( K_{W_t} + c_t K_{t-1} (\Lambda_t + c_t) \right) \left( K_{Z_t} + K_{W_t} + (\Lambda_t + c_t)^2 K_{t-1} \right)^{-1}, \quad \text{(II.64)} \]

\[ F_t (K_{t-1}, \Lambda_t, K_{Z_t}) \triangleq c_t - M_t (K_{t-1}, \Lambda_t, K_{Z_t}) (\Lambda_t + c_t) \quad \text{(II.65)} \]

\[ I_t, \quad t = 1, \ldots, n, \quad \text{an orthogonal innovations process.} \quad \text{(II.66)} \]

\(^7\)Unlike [8], we use the term generalized, because, the conditions for the asymptotic analysis to hold, are fundamentally different from those of asymptotic analysis of classical Kalman-filter equations.
Generalized Time-Varying Difference Riccati Equation:

\[
K_t = c_t^2 K_{t-1} + K_W - \left( \frac{K_W + c_t K_{t-1} \left( \Lambda_t + c_t \right)}{K_Z + K_W + \left( \Lambda_t + c_t \right)^2 K_{t-1}} \right)^2, \quad K_t \geq 0, \quad K_0 = 0, \quad t = 1, \ldots, n, \quad (\text{II}.67)
\]

Error Recursion of the Generalized Kalman-filter, \( E_t \triangleq V_t - \hat{V}_t, t = 1, \ldots, n \):

\[
E_t = F_t (K_{t-1}, \Lambda_t, K_Z) E_{t-1} - M_t (K_{t-1}, \Lambda_t, K_Z) \left( Z_t + W_t \right) + W_t, \quad E_0 = v_0 - \hat{V}_0, \quad t = 1, \ldots, n. \quad (\text{II}.68)
\]

Entropy of Channel Output Process:

\[
H(Y^n | v_0) = \sum_{t=1}^{n} H(Y_t | Y_t^{t-1}, v_0) = \sum_{t=1}^{n} H(Y_t - E \{ Y_t | Y_t^{t-1}, v_0 \} | Y_t^{t-1}, v_0) = \sum_{t=1}^{n} H(I_t). \quad (\text{II}.69)
\]

(b) The characterization of the \( n \)-FTFI capacity \( C_n(\kappa, v_0) \) defined by (1.10) is

\[
C_n(\kappa, v_0) \triangleq \sup_{(\Lambda_t, K_Z), t = 1, \ldots, n: \frac{1}{n} \sum_{t=1}^{n} \left( \frac{\Lambda_t + c_t}{K_{t-1}} \right)^2 K_{t-1} + K_Z + K_W} \frac{1}{2} \left( \frac{\Lambda_t + c_t}{K_{t-1}} \right)^2 K_{t-1} + K_Z + K_W \quad (\text{II}.70)
\]

subject to: \( K_t, t = 1, \ldots, n \) satisfies recursion (II.67) and \( K_Z \geq 0, t = 1, \ldots, n. \quad (\text{II}.71) \)

Proof. (a) Representation (II.45) follows, from a degenerate case of [17]. The representation of the jointly Gaussian process \( X^n \), defined by (II.47), such that \( Z^n \) satisfies (II.49) and (II.50), is also a degenerate case of [17], where the channel is more general, of the form \( Y_t = C_{t-1} Y_{t-1} + D_t X_t + D_{t-1} X_{t-1} + V_t, \) \( V_t = F_t V_{t-1} + W_t, \) where \( (C_{t-1}, D_t, D_{t-1}, F_t) \) are nonrandom, i.e., with past dependence on channel inputs and outputs. The representation (II.46) follows from (II.47), by substituting \( V_{t-1} = Y_{t-1} - X_{t-1}. \) Expressions (II.52)-(II.59) follow directly from (II.46) and (II.47), and the channel definition. The generalized Kalman-filter equations follow from standard textbooks, i.e., [13], (II.69) follows from the independent property of the innovations process. (b) Follows from (I.10), (II.69), \( H(V^n | v_0) = \sum_{t=1}^{n} H(W_t) \), and part (a). \( \square \)

Remark II.1. By the definition of the innovations process and entropy, (II.63) and (II.69), it follows that whether the limit exists, \( \lim_{n \to \infty} \frac{1}{n} \left\{ H(Y^n | v_0) - H(V^n | v_0) \right\} = \lim_{n \to \infty} \frac{1}{n} \sum_{t=1}^{n} \left\{ H(I_t) - H(W_t) \right\} \in [0, \infty) \) can be determined from the limiting covariance of the innovations process \( I^n \) and noise \( W^n. \) Similarly, for \( \lim_{n \to \infty} \frac{1}{n} E_{v_0} \left\{ \sum_{t=1}^{n} (X_t)^2 \right\} \in [0, \infty) \) by (II.56).

B. Some Preliminary Facts on Generalized Difference and Algebraic Riccati Equations

At this point we pause to discuss several facts, to provide insight into the questions of asymptotic analysis, of \( C(\kappa, s_0) \), defined by (I.11), and \( C^\infty(\kappa, v_0) \) defined by (I.12), based on the properties of generalized DREs and AREs. These are well-developed in the systems and control theory literature, for control and estimation problems [13], [14]. The reader finds a more detailed presentation in Section III, Section III-B, and Theorem III.1.
Fact 1. The generalized time-varying Kalman-filter with correlated state and observation noise.

The terminology, “generalized DREs” is often used in the literature, because in system (II.72), (II.73), the noises entering \( V^n \) and \( \overline{V}^n \) are correlated. The noise covariance, for system (II.52), (II.54), or system (II.72), (II.73) is thus, defined by

\[
\text{cov}
\begin{bmatrix}
W_t \\
W_t + Z_t
\end{bmatrix}
= \begin{bmatrix}
Q_t & S_t \\
S_t & R_t
\end{bmatrix}
\triangleq K_{W_t} K_{Z_t} \\
K_{W_t} + K_{Z_t} \\
K_{W_t} + K_{Z_t} \\
K_{W_t} + K_{Z_t} > 0.
\]

With the above notation, then the generalized time-varying DRE (II.67), is given by

\[
K_t = A_t^2 K_{t-1} + K_{W_t} - \left( K_{W_t} + A_t K_{t-1} C_t \right) \left( R_t + C_t^2 K_{t-1} \right), \quad K_t \geq 0, \quad K_0 = 0, \quad t = 1, \ldots, n,
\]

By [14], it follows that the generalized time-varying DRE (II.67), equivalently (II.77), is the time-varying version of the one studied in [14, Section 17.7, i.e., eqn(14.7.1), and throughout the book, for \( S \neq 0 \)]. That is, the Kalman-filter (II.60)-(II.67) or equivalently, the Kalman-filter for (II.72), (II.73), is not the classical Kalman-filter. The generalized Kalman-filter (II.60)-(II.67) reduces to the classical Kalman-filter if and only if the noise \( W^n \) does not enter \( Y^n \) or \( \overline{V}^n \), that is, \( S = 0 \) (see the notation in [14, \( S = 0 \)]).

Fact 2. On the convergence of solutions of time-invariant generalized DREs to solutions of AREs.

For the time-invariant AR(e) noise, i.e., \( c_t = c, K_{W_t} = K_W, t = 1, \ldots, n \), consider the time-invariant channel input distributions or strategies, i.e., \( \Lambda_t = \Lambda^\infty, K_{Z_t} = K_Z^\infty, t = 1, \ldots, n \), which do not imply the corresponding generated process \((X^o,Y^o)\) is stationary. Let \( K_t^o, t = 0, 1, \ldots, n \), denote the sequence generated by the strategies \((\Lambda^\infty,K_Z^\infty)\). Then from Theorem II.1, we obtain \( C^o(\kappa,v_0) \) defined by (I.28), (I.30), that is, the time-invariant version of the generalized DRE (II.67), equivalently (II.77):

\[
K_t^o = A_t^2 K_{t-1}^o + K_W - \left( K_{W_t} + A_t K_{t-1}^o C_t^o \right) \left( R_{t} + C_{t}^2 K_{t-1}^o \right), \quad K_t^o \geq 0, \quad K_0^o = 0, \quad t = 1, \ldots, n,
\]

\[
C^o \triangleq C + \Lambda^\infty, \quad R^o \triangleq K_W + K_Z^\infty.
\]

Hence, to determine whether the limit in \( C^o(\kappa,v_0) \) defined by (I.28), exists, it is necessary to understand the convergence properties of \( K_t^o, t = 0, 1, \ldots, n \), as \( n \to \infty \), and these properties depend on the values
of parameters \((c,K_W,\kappa)\). It is well-known, and easily verified from [13], [14], that the convergence properties of the mean-square error \(K^o_t, t = 0, 1, \ldots, n\), as \(n \to \infty\), of the generalized Kalman-filter, are fundamentally different from those of the classical Kalman-filter. In particular, even for the special case of a stable AR\((c)\) noise, i.e., \(c \in (-1, 1)\), by Section III-B (i.e., Theorem III.1.(1)), the conditions, known as detectability and stabilizability, are sufficient and/or necessary conditions, for the convergence of the mean-square error \(K^o_t, t = 0, 1, \ldots, n\), as \(n \to \infty\), to a finite nonnegative, unique limit \(K^\infty \geq 0\), such that \(K^\infty\) satisfies the generalized algebraic Riccati equation (ARE), i.e., the steady state version of (II.78):

\[
K^\infty = A^2 K^\infty + K_W - \left(\frac{K_W + AK^\infty C^\infty}{R^\infty + (C^\infty)^2 K^\infty}\right) K^\infty, \quad K^\infty \geq 0. \quad (II.80)
\]

On the other hand, the following is a well-known property of classical Kalman-filters, which is easily verified from the properties of DREs and AREs presented in [14, with \(S = 0\)] and [13]: for classical time-invariant DREs and classical AREs, i.e., that correspond to a stable state process, to be estimated, driven by a Gaussian noise, and such that, the noise is independent of the Gaussian noise entering the observations, then the detectability and stabilizability conditions are automatically satisfied. This implies the mean-square estimation error of Kalman-filter converges, to a finite nonnegative, unique limit, which satisfies a classical ARE. The unique solution of the classical ARE is also stabilizing.

**Fact 3.** On the zero variance of the innovations process of the channel input.

Suppose \(K^\infty_Z\) in (II.78) and (II.80), is replaced by \(K^\infty_Z = 0\), i.e., \(R^\infty = K_W\). Then the resulting generalized ARE (II.80), with \(K^\infty_Z\) is precisely the Riccati equation in the feedback characterization in [1, Theorem 6.1, \(\Sigma = K^\infty\)] for the AR\((c)\) noise, and this equation is a quadratic polynomial in \(K^\infty\), with two solutions:

\[
K^\infty = 0, \quad K^\infty = K^\infty(\Lambda^\infty) = \frac{K_W \left( (\Lambda^\infty)^2 - 1 \right)}{(\Lambda^\infty + c)^2}, \quad \Lambda^\infty \neq -c. \quad (II.81)
\]

The second solution \(K^\infty(\Lambda^\infty)\) is a functional of \(\Lambda^\infty\), and gives rise to solutions:

\[
K^\infty(\Lambda^\infty) = \frac{K_W \left( (\Lambda^\infty)^2 - 1 \right)}{(\Lambda^\infty + c)^2} \geq 0, \quad \text{if and only if } |\Lambda^\infty| \geq 1, \quad (II.82)
\]

\[
K^\infty(\Lambda^\infty) = \frac{K_W \left( (\Lambda^\infty)^2 - 1 \right)}{(\Lambda^\infty + c)^2} \leq 0, \quad \text{if and only if } |\Lambda^\infty| \leq 1. \quad (II.83)
\]

The main question is then: which one of the solutions \(K^\infty\) is the unique limit of the sequence \(K^o_t, t = 0, 1, \ldots, n\), as \(n \to \infty\), which then defines the unique limit of the entropy rate \(\frac{1}{n} H(Y^n|v_0) = \frac{1}{n} \sum_{t=1}^{n} H(I^n_t)\) in (II.69) and average power \(\frac{1}{n} E_0 \left\{ \sum_{t=1}^{n} (X_t)^2 \right\}\) in (II.56), as \(n \to \infty\)?

The answer to this question is: the solution that corresponds to \(|\Lambda^\infty| < 1\); this follows from the properties of generalized DREs and AREs, stated in Theorem III.1.(1), (see also Lemma III.2.(3)), because \(|\Lambda^\infty| < 1\) is a necessary and sufficient condition for convergence of \(K^o_t, t = 0, 1, \ldots, n\) for all \(K^o_t \geq 0\), as \(n \to \infty\), to a unique limit \(K^\infty \geq 0\) which satisfies the generalized ARE (II.80). Consequently, the unique nonnegative
limit of the $K_t^n, t = 0, 1, \ldots, n \forall K_0^n \geq 0$, as $n \to \infty$, is $K^\infty = 0$, since both solutions (II.82) and (II.83) are ruled out, by the condition $|\Lambda^\infty| < 1$. By the expression of the entropies $H(Y^n|v_0) - H(V^n|v_0) = \sum_{i=1}^n \{H(F^n_i) - H(W_i)\}$, inside the limit in (II.28), if $K_Z^\infty = 0$, then $C^\infty(\kappa, v_0) = 0, \forall \kappa \in [0, \infty), \forall v_0$. This means the feedback capacity characterization in [1, Theorem 6.1, $C_{FB}$] is zero. An alternative illustration is given in Counterexample IV.1.

Remark II.2. Facts 1-3 state that certain fundamental technical issues need to be accounted for in the asymptotic analysis of entropy rates and of the average power in [9]–[12], even if the noise $V^n$ is stable, stationary, etc. to ensure optimal time-invariant channel input strategies induce asymptotic stationarity of the channel input process, and asymptotic stationarity of the output process (for stable noise).

C. Converse Coding Theorem for AGN Channels

By Theorem II.1, the characterization of $n$–FTFI capacity, $C_n(\kappa, v_0)$, is expressed in terms of the mean-square error $K_t, t = 1, \ldots, n$, that satisfies the time-varying generalized RDE (II.67). We recall the error recursion of the generalized Kalman-filter given by (II.68), which satisfies a linear time-varying recursion, and hence its convergence properties, in mean-square sense, i.e., $\lim_{n\to\infty} K_n = \lim_{n\to\infty} E_v \{(E_n)^2\}$ is determined by the properties of $F_t(K_{t-1}, \Lambda_t, K_0^\infty)$ and $M_t(K_{t-1}, \Lambda_t, K_0^\infty), \Lambda_t, K_0^\infty, t = 1, 2, \ldots$. In general, $\lim_{n\to\infty} K_n = \lim_{n\to\infty} E_v \{(E_n)^2\}$ does not exist, for arbitrary $F_t(K_{t-1}, \Lambda_t, K_0^\infty), M_t(K_{t-1}, \Lambda_t, K_0^\infty), \Lambda_t, K_0^\infty, t = 1, 2, \ldots$. In view of the error recursion (II.68), we have the following theorem.

Theorem II.2. Converse coding theorem

Consider the feedback code $C_{FB}^\infty$ of Definition II.1.(a).

Converse Coding Theorem. If there exists a feedback code $C_{FB}^\infty$, i.e., with $\epsilon_n \to 0$, as $n \to \infty$, then the code rate $R(v_0)$ satisfies:

$$R(v_0) \leq C(\kappa, v_0) \triangleq \lim_{n\to\infty} \frac{1}{n} C_n(\kappa, v_0), \quad C_n(\kappa, v_0) \text{ defined in Theorem II.1.(b)}$$

(II.84)

provided the following conditions hold:

(C1) the maximizing element, denoted by $(\Lambda_t^*, K_0^*)$, $t = 1, \ldots, n$ which satisfies the average power constraint exists, and

(C2) the limit exists in $[0, \infty)$.

Proof: (C1) and (C2) follow from the above discussion; the converse coding theorem is similar to [6].

Remark II.3. By the average power (II.56) and optimization problem (II.70), it is necessary to identify sufficient and/or necessary conditions such that the maximizing element, $(\Lambda_t^*, K_0^*)$, $t = 1, \ldots, n$, exists in the set, and to ensure convergence of $K_n = E_v \{(E_n)^2\}$ (that satisfies the time-varying DRE (II.67)), as $n \to \infty$, such that the limit in (II.84) exists in $[0, \infty)$. However, to ensure $C(\kappa, v_0)$ is independent of $v_0$,
it is necessary that the limit is also independent of \( v_0 \). On the other hand, if the limit \( C(\kappa, v_0) \) depends on \( v_0 = v_0 \), then one needs to consider a formulation based on compound capacity, by taking infimum over all initial states \( v_0 = v_0 \), as done, for example, in [18], for finite state feedback channels, otherwise different \( v_0 \) give rise to different rates.

D. Lower Bound on Characterization of \( n - \text{FTwFI Capacity for AGN Channels Driven by AR}(c_t) \) Noise

Next, we give a lower bound on the characterization of \( n - \text{FTwFI Capacity} \), for the non-feedback code of Definition I.1.(b), which follows directly from Theorem II.1.

**Corollary II.1.** Lower bound on characterization of \( n - \text{FTwFI Capacity for AGN Channels Driven by AR}(c_t) \) Noise

Consider the AGN channel (I.1) driven by a time-varying AR\((c_t)\) noise, i.e., (I.2), and the code without feedback, of Definition I.1.(b). Define the information theoretic optimization problem of capacity without feedback, i.e., the analog of (I.10), by

\[
C_n^{\text{FTwFI}}(\kappa, v_0) \equiv \sup_{P_{X_t|X^{t-1}, v_0}} H(Y^n|v_0) - H(\mathbf{V}^n|v_0) = \frac{1}{n} E \{ \sum_t (X_t)^2 \} \leq \kappa
\]  

provided the supremum exists. Then the following hold.

(a) A lower bound on \( C_n^{\text{FTwFI}}(\kappa, v_0) \) is obtained by the conditionally Gaussian, time-varying channel input distribution without feedback, given by

\[
P_{X_t|X^{t-1}, v_0} = P_{X_t|X^{t-1}, v_0}, \quad t = 1, \ldots, n
\]  

which is induced by the time-varying jointly Gaussian channel input process \( X^n \), with a representation

\[
X_t = \bar{X}_t X_{t-1} + Z_t, \quad t = 2, \ldots, n,
\]  

\[
X_1 = Z_1,
\]  

\[
Z_t \in N(0, K_{\bar{Z}}), \quad t = 1, \ldots, n \quad \text{a Gaussian sequence},
\]  

\[
Z_t \text{ independent of } (V^{t-1}, X^{t-1}, Y^{t-1}, V_0), \quad t = 1, \ldots, n,
\]  

\[
Z^n \text{ independent of } (V^n, V_0),
\]  

\[
V_t = c_t V_{t-1} + W_t, \quad V_0 = v_0, \quad c_t \in (-\infty, \infty), \quad t = 1, \ldots, n
\]  

\[
Y_t = X_t + V_t = (\bar{X}_t - c_t) X_{t-1} + c_t Y_{t-1} + W_t + Z_t, \quad t = 2, \ldots, n,
\]  

\[
Y_1 = Z_1 + c V_0 + W_1, \quad V_0 = v_0,
\]  

\[
\frac{1}{n} E_{v_0} \left\{ \sum_t (X_t)^2 \right\} = \frac{1}{n} \sum_t \left\{ (\bar{X}_t)^2 K_{X_{t-1}} + K_{Z_t} \right\} \leq \kappa
\]  

\[
(\bar{X}_t, K_{\bar{Z}}) \in (-\infty, \infty) \times [0, \infty) \quad \text{scalar-valued, non-random},
\]  

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Further, \((\tilde{X}_t, K_t), t = 1, \ldots, n\) are determined by the generalized time-varying Kalman filter and generalized time-varying difference Riccati equation (DRE), of estimating \(X^n\) from \(Y^n\), and \(K_n, t = 1, \ldots, n\) is determined by the time-varying Lyapunov difference equation, given below.

**Generalized Kalman filter Recursion for (II.87)-(II.94):**

\[
\tilde{X}_t = \tilde{X}_{t-1} + M_{t}^{n fb}(K_{t-1}, \tilde{X}_{t-1}, Z_{t}) I_t, \quad \tilde{X}_1 = \tilde{x}_1, \quad t = 2, \ldots, n
\]  
\(= F_t^{n fb}(K_{t-1}, \tilde{X}_{t-1}, Z_{t}) \tilde{X}_{t-1} + M_t^{n fb}(K_{t-1}, \tilde{X}_{t-1}, Z_{t}) (Y_t - c_t Y_{t-1}), \)  
\(I_t \triangleq Y_t - (\tilde{X}_{t-1} - c_t Y_{t-1}), \quad I_t = Z_t + W_t, \quad t = 2, \ldots, n,
\]
\(= (\tilde{X}_{t-1} - c_t) (X_{t-1} - \tilde{X}_{t-1}) + Z_t + W_t; \)
\(M_{t}^{n fb}(K_{t-1}, \tilde{X}_{t-1}, Z_{t}) \triangleq (K_{t-1} + \tilde{X}_{t-1} (\tilde{X}_{t-1} - c_t) (K_{t-1} + K_{W_t} + (\tilde{X}_{t-1} - c_t)^2 K_{Z_t})^{-1}, \)
\(F_t^{n fb}(K_{t-1}, \tilde{X}_{t-1}, Z_{t}) \triangleq \tilde{X}_{t-1} - M_t^{n fb}(K_{t-1}, \tilde{X}_{t-1}, Z_{t})(\tilde{X}_{t-1} - c_t)
\)
\(I_t, \quad t = 1, \ldots, n, \quad \text{an orthogonal innovations process.} \)

**Generalized Time-Varying Difference Riccati Equation:**
\[
K_t = \tilde{X}_t^2 K_{t-1} + Z_{t} - \left( \frac{(K_{t-1} + \tilde{X}_{t-1} (\tilde{X}_{t-1} - c_t))}{(K_{t-1} + K_{W_t} + (\tilde{X}_{t-1} - c_t)^2 K_{Z_t})} \right)^2, \quad K_t \geq 0, \quad K_0 = 0, \quad t = 1, \ldots, n, \quad (II.107)
\]

**Time-Varying Difference Lyapunov Equation:**
\[
K_n = \tilde{X}_n^2 K_{X_t} + Z_{X_t}, \quad K_n \geq 0, \quad K_{X_0} = 0, \quad t = 1, \ldots, n, \quad (II.108)
\]

**Error Recursion of the Generalized Kalman filter:** \(E_t^{n fb} \triangleq X_t - \tilde{X}_t, t = 1, \ldots, n\):
\[
E_t^{n fb} = F_t^{n fb}(K_{t-1}, \tilde{X}_{t-1}, Z_{t}) E_{t-1} - M_t^{n fb}(K_{t-1}, \tilde{X}_{t-1}, Z_{t}) (Z_t + W_t) + Z_t, \quad E_0^{n fb} = \text{given,} \quad t = 1, \ldots, n. \quad (II.109)
\]

(b) The lower bound characterization of the \(n\)--FTwFI capacity \(C_n^{n fb}(\mathbf{v}_0), \) defined by (II.85), is
\[
C_n^{n fb}(\mathbf{v}_0) \geq C_{n, LB}^{n fb}(\mathbf{v}_0) \triangleq \sup_{(\tilde{X}_t, Z_t)} \frac{1}{2} \sum_{t=1}^{n} \log \left( \frac{(\tilde{X}_t - c_t)^2 K_{t-1} + K_{Z_t} + K_{W_t}}{K_{W_t}} \right) \quad \sup_{(\tilde{X}_t, Z_t)} \left\{ \left( \tilde{X}_t \right)^2 K_{X_t} + K_{Z_t} \right\} \leq \mathbf{v}_0 \quad (II.110)
\]
\(\text{subject to: } K_t, K_{X_t}, t = 1, \ldots, n \text{ satisfy recursions (II.107), (II.108), and } K_{Z_t} \geq 0, \ t = 1, \ldots, n. \quad (II.111)\)
Proof: (a) Similar to the feedback capacity of Theorem II.1, by the maximum entropy of Gaussian distributions, the maximizing distributions $\mathbf{P}_{X_n|Y^{n-1},V_0}, t = 1,\ldots,n$ for the optimization problem (II.85) are conditionally Gaussian, such that $(X^n,Y^n)$ for $V_0 = v_0$, is jointly Gaussian, the average power constraint is satisfied, and condition (I.5) is respected. Clearly, the restriction to distributions that satisfy (II.86) result in a lower bound on $C_n^{c_f b}(\kappa,v_0)$ defined by (II.85). Note that the restriction to (II.86) is precisely the restriction of feedback distributions (II.45) to non-feedback distributions. The rest of the equations follow, similarly to Theorem II.1.(a), and in particular, (II.46), if the channel is used without feedback, i.e., $X_t = \overline{K}_tX_{t-1} + Z_t$. The rest of the expression of part (a) are obtained as in Theorem II.1.(a), and the generalized Kalman-filter recursions follow from [13], [14]. (b) Due to the expressions of part (a).

Remark II.4. Corollary II.1 is useful, because the lower bound is much easier to compute, compared to $C_n^{c_f b}(\kappa,v_0)$, defined by (II.85), where the supremum is taken over all jointly Gaussian channel input processes $X^n,n = 1,2,\ldots$, without feedback or distributions without feedback, $\mathbf{P}_{X_n|Y^{n-1},V_0}, t = 1,2,\ldots$.

III. NEW FORMULAS OF CAPACITY OF AGN CHANNELS DRIVEN BY STABLE AND UNSTABLE AR(c) NOISE AND GENERALIZED RICCATI EQUATIONS

In this section we derive a closed form formula for feedback capacity $C_0(\kappa,v_0)$, defined by (I.12), and lower bounds on capacity without feedback $C_0^{c_f b}(\kappa,v_0)$, defined by (I.13), of AGN channels driven by AR(c), stable and unstable noise, when channel input strategies or distributions are time-invariant. This section includes material on basic properties of generalized DREs, AREs, and definitions and implications of the notions of detectability and stabilizability, which are discussed in Section II-B.

A. Characterization of Feedback Capacity for Time-Invariant Channel Input Distributions

We restrict the class of channel input distributions of Theorem II.1 to the class of time-invariant distributions. We note that our restriction is weaker than the analysis in [1], which presupposes stationarity or asymptotic joint stationarity of the joint Gaussian process $(X^n,Y^n), n = 1,2,\ldots$ (the author also considers a double sided joint process). However, unlike [1], [8], we do not assume the AR(c) noise is stable.

By Theorem II.1, and restricting the channel input strategies to the time-invariant channel input strategies, $(\Lambda_t,K_Z) = (\Lambda^\infty,K_Z^\infty), t = 1,\ldots,n$, (not necessarily stationary) then we have the following representation:

$$X_i^o = \Lambda^\infty(V_{t-1} - \hat{V}_{t-1}^o) + Z_t^o, \quad X_1^o = Z_1^o, \quad t = 2,\ldots,n,$$

$$V_t = cV_{t-1} + W_t, \quad V_0 = v_0, \quad t = 1,\ldots,n, \quad (III.112)$$

$$Y_t^o = X_t^o + V_t = \Lambda^\infty(V_{t-1} - \hat{V}_{t-1}^o) + cV_{t-1} + W_t + Z_t^o, \quad t = 2,\ldots,n,$$

$$V_t = cV_{t-1} + W_t, \quad V_0 = v_0, \quad t = 1,\ldots,n, \quad (III.113)$$

$$Y_t^o = X_t^o + V_t = \Lambda^\infty(V_{t-1} - \hat{V}_{t-1}^o) + cV_{t-1} + W_t + Z_t^o, \quad t = 2,\ldots,n, \quad (III.114)$$

The variation of notation is judged necessary to distinguish it from the time-varying channel input strategies $(\Lambda_t,K_Z)$ and corresponding distributions $\mathbf{P}_{X_n|V_n,Y^{n-1}} = \mathbf{P}_t(dx|v_{t-1},y_{t-1}), t = 1,\ldots,n$. 

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\[ Y_t^o = Z_t^o + cV_0 + W_1, \quad V_0 = v_0, \]  
\[ Z_t^o \sim N(0, K_Z^o), \quad t = 1, \ldots, n \quad \text{is a Gaussian sequence,} \]  
\[ Z_t^o \text{ is independent of } (V_t^{t-1}, X^{o,t-1}, Y^{o,t-1}, V_0), \quad t = 1, \ldots, n, \]  
\[ Z^{o,n} \text{ is independent of } (V^n, V_0), \]  
\[ \text{cov} \begin{bmatrix} W_t \\ W_t + Z_t^o \end{bmatrix} \begin{bmatrix} W_t \\ W_t + Z_t^o \end{bmatrix}^T = \begin{bmatrix} K_W & K_Z \\ K_W & K_W + K_Z^o \end{bmatrix}, \]  
\[ \frac{1}{n} \mathbb{E}_v \left\{ \sum_{t=1}^{n} (X_t^o)^2 \right\} = \frac{1}{n} \sum_{t=1}^{n} (\Lambda^o)^2 K_{t-1}^o + K_Z^o \leq \kappa, \]  
\[ (\Lambda^o, K_Z^o) \in (]-\infty, \infty[ \times [0, \infty[ \quad \text{are non-random,} \]  
\[ P_{X_t^o|V_{t-1}, Y^{o,t-1}} = P^o(dx_t|V_{t-1}, Y^{t-1}), \quad t = 1, \ldots, n, \quad \text{that is, the distribution is time-invariant} \]  
\[ \text{where } (\hat{V}_t^o, K_t^o), t = 1, \ldots, n \text{ satisfy the generalized Kalman-filter and time-invariant DRE, given below.} \]

**Generalized Kalman-filter Recursion:**

\[ \hat{V}_t^o = c\hat{V}_{t-1}^o + M(K_{t-1}^o, \Lambda^o, K_Z^o)I_t^o, \quad \hat{V}_0^o = v_0, \]  
\[ = F(K_{t-1}^o, \Lambda^o, K_Z^o)\hat{V}_{t-1}^o + M(K_{t-1}^o, \Lambda^o, K_Z^o)Y_t^o, \quad \hat{V}_0^o = v_0, \]  
\[ I_t^o \triangleq Y_t^o - c\hat{V}_{t-1}^o, \quad I_t^o = Z_t^o + W_1, \quad t = 2, \ldots, n, \]  
\[ = (\Lambda^o + c)(V_{t-1} - \hat{V}_{t-1}^o) + Z_t^o + W_t, \]  
\[ M(K_{t-1}^o, \Lambda^o, K_Z^o) \triangleq \left( K_W + cK_{t-1}^o(\Lambda^o + c) \right) \left( K_Z^o + K_W + (\Lambda^o + c)K_{t-1}^o \right)^{-1}, \]  
\[ F(K_{t-1}^o, \Lambda^o, K_Z^o) \triangleq c - M(K_{t-1}^o, \Lambda^o, K_Z^o) \left( \Lambda^o + c \right), \]  
\[ I_t^o, \quad t = 1, \ldots, n, \quad \text{an orthogonal innovations process.} \]

**Generalized Time-Invariant Difference Riccati Equation:**

\[ K_t^o = c^2 K_{t-1}^o + K_W - \frac{(K_W + cK_{t-1}^o(\Lambda^o + c))^2}{(K_Z^o + K_W + (\Lambda^o + c)K_{t-1}^o)}, \quad K_0^o = 0, \quad t = 1, \ldots, n, \]  
\[ \text{We note that the Kalman-filter recursion (III.123) is time-varying, but the DRE (III.130) is time-invariant.} \]

The analog of the error recursion (II.68), for time-invariant strategies, is the following.

**Error Recursion of the Generalized Kalman-filter, \( E_t^o \triangleq V_t - \hat{V}_t^o, t = 1, \ldots, n \):**

\[ E_t^o = F(K_{t-1}^o, \Lambda^o, K_Z^o)E_{t-1}^o - M(K_{t-1}^o, \Lambda^o, K_Z^o)\left( Z_t^o + W_t \right) + W_t, \quad E_0^o = 0, \quad t = 1, \ldots, n, \]  
\[ Z_t^o \in N(0, K_Z^o), \quad t = 1, 2, \ldots, n. \]
We note that recursion (III.131) is linear time-varying. Hence, \( \lim_{n \to \infty} K_n^\omega = \lim_{n \to \infty} E_{01} \{ (E_n^\omega)^2 \} \) is not expected to exist, for arbitrary \( F(K_{t-1}^\omega, \Lambda^\infty, K_Z^\infty), M(K_{t-1}^\omega, \Lambda^\infty, K_Z^\infty), t = 1, 2, \ldots \). Indeed, the convergence properties of the sequence \( K_0^\omega, K_1^\omega, \ldots, K_n^\omega \) generated by (III.130), as \( n \to \infty \), are characterized by the detectability and stabilizability conditions [13], [14] (which we introduce shortly). These conditions ensure existence of a unique nonnegative limit, \( \lim_{n \to \infty} K_n^\omega = K^\infty \), such that \( K^\infty \geq 0 \) is the unique solution of a generalized ARE and satisfies the stability property: \( \lim_{n \to \infty} F(K_{n-1}^\omega, \Lambda^\infty, K_Z^\infty) = F(K^\infty, \Lambda^\infty, K_Z^\infty) \in (-1, 1) \).

Next, we define the characterization of the \( n \)--FTFI capacity, its per unit time limit, and the alternative definition, with the per unit time limit and maximization interchanged.

**Definition III.1. Characterizations of asymptotic limits**

Consider the characterization of the \( n \)--FTFI capacity of Theorem II.1, restricted to the time-invariant strategies \( (\Lambda_t = \Lambda^\infty, K_Z = K^\infty), t = 1, \ldots, n \), as defined by (III.112)-(III.130).

(a) The characterization of the \( n \)--FTFI capacity for time-invariant strategies is defined by

\[
C_n^\omega(\kappa, v_0) \triangleq \sup_{(\Lambda^\omega, K_Z^\omega)} \frac{1}{n} \sum_{t=1}^{n} \frac{1}{2} \log \left( \frac{(\Lambda^\omega + c)^2 K_{t-1}^\omega + K_Z^\omega + K_W}{K_W} \right)
\]

subject to: \( K_t^\omega, t = 1, \ldots, n \) satisfies recursion (III.130) and \( K_Z^\omega \geq 0, t = 1, \ldots, n \) (III.134) provided the supremum exists in the set. The per unit time-limit is then defined by

\[
C^\omega(\kappa, v_0) \triangleq \lim_{n \to \infty} \frac{1}{n} C_n^\omega(\kappa, v_0).
\]

provided the supremum exists in \([0, \infty)\).

(b) The characterization of the \( n \)--FTFI capacity for time-invariant strategies, with limit and maximization interchanged is defined by

\[
\frac{1}{n} C^\omega(\kappa, v_0) \triangleq \sup_{(\Lambda^\omega, K_Z^\omega)} \frac{1}{n} \sum_{t=1}^{n} \frac{1}{2} \log \left( \frac{(\Lambda^\omega + c)^2 K_{t-1}^\omega + K_Z^\omega + K_W}{K_W} \right)
\]

provide the limit exists in \([0, \infty)\) and the supremum also exists in the set.

To ensure \( C^\omega(\kappa, v_0) \) defined by (III.136) is well defined, i.e., that the optimal time-invariant channel input strategy or distribution ensures the limit exits and \( C^\omega(\kappa, v_0) \) is independent of \( v_0 \), we shall impose condition (C1). We shall express condition (C1) in terms of properties of generalized time-invariant DREs and AREs, introduced in the next section, from which answers to questions of Problem I.1 are obtained.

**B. Convergence Properties of Time-Invariant Generalized RDEs**

We recall that in the study of mean-square estimation, and in particular, the filtering theory, of time-invariant jointly Gaussian processes described by linear recursions, driven by Gaussian noise processes,
and of jointly stationary Gaussian processes, the concepts of detectability and stabilizability, have been very effective \cite{13, 14}. In this section, we summarize these concepts in relation to the properties of generalized DREs and AREs.

Let \( \{K_t, t = 1, 2, \ldots, n\} \) denote a sequence that satisfies the time-invariant generalized DRE with arbitrary initial condition

\[
K_t = c^2 K_{t-1} + K_W - \frac{(K_W + cK_{t-1}(\Lambda + c))}{(K_Z + K_W + (\Lambda + c)^2 K_{t-1})}^2, \quad K_0 = \text{given,} \quad t = 1, \ldots, n. \tag{III.137}
\]

We note that a solution of (III.137) is a functional of the parameters of the right hand side, that is, \( K_t \equiv K_t(c, K_W, \Lambda, K_Z, K_0), t = 1, \ldots, n \). To discuss the properties of the generalized DRE (III.137), we introduce, as often done in the analysis of generalized DREs \cite{13} and \cite{14, Section 14.7, page 540}, the following definitions.

\[
A \overset{\triangle}{=} c, \quad C \overset{\triangle}{=} \Lambda + c, \quad A^* \overset{\triangle}{=} c - K_W R^{-1} C, \quad B^* \overset{\triangle}{=} K_W^2 B^2 \tag{III.138}
\]

\[
R \overset{\triangle}{=} K_Z + K_W, \quad B \overset{\triangle}{=} 1 - K_W (K_Z + K_W)^{-1}. \tag{III.139}
\]

By (III.127) and (III.128), we also have

\[
M(K, \Lambda, K_Z) \overset{\triangle}{=} (K_W + AKC)(R + (C)^2 K)^{-1}, \tag{III.140}
\]

\[
F(K, \Lambda, K_Z) = A - M(K, \Lambda, K_Z)C. \tag{III.141}
\]

The generalized algebraic Riccati equation (ARE) corresponding to (III.137) is

\[
K = c^2 K + K_W - \frac{(K_W + cK(\Lambda + c))}{(K_Z + K_W + (\Lambda + c)^2 K)^2}, \quad K \geq 0. \tag{III.142}
\]

Next, we introduce the definition of asymptotic stability of the error recursion (III.131).

**Definition III.2. Asymptotic stability**

A solution \( K \geq 0 \) to the generalized ARE (III.142), assuming it exists, is called stabilizing if \( |F(K, \Lambda, K_Z)| < 1 \). In this case, we say \( F(K, \Lambda, K_Z) \) is asymptotically stable, that is, \( |F(K, \Lambda, K_Z)| < 1 \).

With respect to any of the above generalized DRE and ARE, we define the important notions of detectability, unit circle controllability, and stabilizability.

**Definition III.3. Detectability, Stabilizability, Unit Circle controllability**

(a) The pair \( \{A, C\} \) is called detectable if there exists a \( G \in \mathbb{R} \) such that \( |A - GC| < 1 \) (stable).

(b) The pair \( \{A^*, B^*, \frac{1}{2}\} \) is called unit circle controllable if there exists a \( G \in \mathbb{R} \) such that \( |A^* - B^* \frac{1}{2} G| \neq 1 \).

(c) The pair \( \{A^*, B^*, \frac{1}{2}\} \) is called stabilizable if there exists a \( G \in \mathbb{R} \) such that \( |A^* - B^* \frac{1}{2} G| < 1 \).
The next theorem characterizes, detectability, unit circle controllability, and stabilizability [14], [19].

**Lemma III.1.** [14], [19] Necessary and sufficient conditions for detectability, unit circle controllability, stabilizability

(a) The pair \(\{A,C\}\) is detectable if and only if there exists no eigenvalue, eigenvector \(\{\lambda,x\}\), of \(A\), i.e., \(Ax = \lambda x\) such that \(|\lambda| \geq 1\), and such that \(Cx = 0\)

(b) The pair \(\{A^*,B^{\frac{1}{2}}\}\) is unit circle controllable if and only if there exists no eigenvalue, eigenvector \(\{\lambda,x\}\), \(xA^* = x\lambda\), such that \(|\lambda| = 1\), and such that \(xB^{\frac{1}{2}} = 0\).

(c) The pair \(\{A^*,B^{\frac{1}{2}}\}\) is stabilizable if and only if there exists no eigenvalue, eigenvector \(\{\lambda,x\}\), \(xA^* = x\lambda\) such that \(|\lambda| \geq 1\), and such that \(xB^{\frac{1}{2}} = 0\).

In the next theorem we summarize known results on sufficient and/or necessary conditions for convergence of solutions \(\{K_t, t = 1,2,\ldots,n\}\) of the generalized time-invariant DRE, as \(n \rightarrow \infty\), to a nonnegative \(K\), which is the unique stabilizing solution of a corresponding generalized ARE.

**Theorem III.1.** [13], [14] Convergence of time-invariant generalized DRE

Let \(\{K_t, t = 1,2,\ldots,n\}\) denote a sequence that satisfies the time-invariant generalized DRE (III.137) with arbitrary initial condition, and \((A,C,A^*,B^{\frac{1}{2}})\) defined by (III.138), (III.139). Then the following hold.

1. Consider the generalized RDE (III.137) with zero initial condition, i.e., \(K_0 = 0\), and assume, the pair \(\{A,C\}\) is detectable, and the pair \(\{A^*,B^{\frac{1}{2}}\}\) is unit circle controllable.

Then the sequence \(\{K_t : t = 1,2,\ldots,n\}\) that satisfies the generalized DRE (III.137), with zero initial condition \(K_0 = 0\), converges to \(K\), i.e., \(\lim_{n \rightarrow \infty} K_n = K\), where \(K\) satisfies the ARE

\[
K = c^2K + K_W = \frac{(K_W + cK(\Lambda + c))^2}{(K_Z + K_W + (\Lambda + c)^2K)} \quad (III.143)
\]

if and only if the pair \(\{A^*,B^{\frac{1}{2}}\}\) is stabilizable.

2. Assume, the pair \(\{A,C\}\) is detectable, and the pair \(\{A^*,B^{\frac{1}{2}}\}\) is unit circle controllable. Then there exists a unique stabilizing solution \(K \geq 0\) to the generalized ARE (III.137), i.e., such that, \(|F(K,\Lambda,K_Z)| < 1\), if and only if \(\{A^*,B^{\frac{1}{2}}\}\) is stabilizable.

3. If \(\{A,C\}\) is detectable and \(\{A^*,B^{\frac{1}{2}}\}\) is stabilizable, then any solution \(K_t, t = 1,2,\ldots,n\) to the generalized RDE (III.137) with arbitrary initial condition, \(K_0\) is such that \(\lim_{n \rightarrow \infty} K_n = K\), where \(K \geq 0\) is the unique solution of the generalized ARE (III.137) with \(|F(K,\Lambda,K_Z)| < 1\), i.e., it is stabilizing.

Theorem III.1.(1) follows by combining [14, Lemma 14.2.1, page 507] of classical DREs and AREs with [14, Section 14.7] of generalized DREs and AREs. Theorem III.1.(2) is given in [14, Theorem E.6.1, page 784]. Theorem III.1.(3) is obtained from [13, Theorem 4.2, page 164], and also [14].
From Theorem III.1, we can easily re-confirm Facts 2, 3 of Section II-B, as shown in the next lemma.

**Lemma III.2. Properties of Solutions of DREs and AREs for different cases**

Let \((A, C, A^*, B^*, 1)\) be defined by (III.138), (III.139).

1. Suppose \(c \in (-1, 1)\). Then the pair \((A, C)\) is detectable.
2. Suppose \(K_Z = 0\). Then the pair \((A^*, B^*, 1)\) is unit circle controllable if and only if \(|\Lambda| \neq 1\).
3. Suppose \(K_Z = 0\). Then the pair \((A^*, B^*, 1)\) is stabilizable if and only if \(|\Lambda| < 1\).
4. Suppose \(c \in (-1, 1), K_Z = 0\). The sequence \(\{K_t, t = 1, 2, \ldots, n\}\) that satisfies the generalized DRE with zero initial condition, i.e.,

\[
K_t = c^2 K_{t-1} + K_W - \frac{\left(K_W + c K_{t-1}(\Lambda + c)\right)^2}{\left(K_W + (\Lambda + c)^2 K_{t-1}\right)}, \quad K_0 = 0, \quad t = 1, \ldots, n \quad (III.144)
\]

converges to \(K \geq 0\), i.e., \(\lim_{n \to \infty} K_n = K\), where \(K\) satisfies the generalized ARE (III.142) if and only if the \((A^*, B^*, 1)\) is stabilizable, equivalently, \(|\Lambda| < 1\).
5. Suppose \(K_Z = 0\), and \(|\Lambda| \neq 1\), with the corresponding ARE,

\[
K = c^2 K + K_W - \frac{\left(K_W + c K(\Lambda + c)\right)^2}{\left(K_W + (\Lambda + c)^2 K\right)} \quad \text{(III.145)}
\]

Then the two solution, without the restriction \(K \geq 0\), are given by

\[
K = 0, \quad K = \frac{K_W \left(\Lambda^2 - 1\right)}{(\Lambda + c)^2}, \quad c \neq -\Lambda \quad \text{(III.146)}
\]

Moreover, \(K = 0\) is the unique and stabilizing solution \(K \geq 0\) to (III.145), i.e., such that \(|F(K, \Lambda, K_Z)| < 1\), if and only if \(|\Lambda| < 1\).

**Proof.** See Appendix VI-A.

In the next remark we make some comments on [1, Theorem 6.1, see also Lemma 6.1], i.e., that a zero variance of the innovations process of the channel input process is not optimal.

**Remark III.1. Asymptotic stationarity of optimal process of [1]**

Consider the characterization of feedback capacity given in [1, Theorem 6.1, \(\Sigma\) satisfying eqn(61)], in which the variance of the innovations process is replaced by a zero value (see comment below [1, Theorem 6.1]). Then \(\Sigma = 0\) is one solution of [1, \(\Sigma\) satisfying eqn(61)].

We ask: what are necessary and/or sufficient conditions for convergence \(\lim_{n \to \infty} \Sigma_n = \Sigma\), where \(\Sigma \geq 0\) is the unique limit that stabilizes the estimation error of the noise?

By the multidimensional version of Theorem III.1.(1), and Lemma III.2.(3), then the limit \(\lim_{n \to \infty} \Sigma_n \)
converges if and only if the stabilizability condition holds. For the AR(c) noise model, since the characterization of feedback capacity given [1, Theorem 6.1], presupposes a zero variance of the innovations process, i.e., \( K_0^x = 0 \), then the value of feedback capacity [1, Theorem 6.1, \( C_{FB} = 0, \forall \kappa \in [0, \infty) \)] (see also (IV.232)-(IV.236) with \( K^\infty = 0 \), which implies \( C^K(\kappa) = 0, \forall \kappa \in [0, \infty) \)).

C. Feedback Capacity of AGN Channels Driven by Time-Invariant Stable/Unstable AR(c) Noise

In this section we analyze the asymptotic per unit time limit of the \( n \)--FTFI capacity of Definition III.1, by making use of the properties of generalized DREs and AREs of Section III-B to identify sufficient and necessary conditions, such that condition (C1) holds. Then we derive closed form expressions for \( C^\infty(\kappa, v_0) = C^\infty(\kappa), \forall v_0 \) defined by (III.136), for Regime 1 given by (I.17), and we show that feedback does not increase \( C^\infty(\kappa) \), for Regimes 2 and 3 given by (I.19) and (I.18).

First, we define the main problem of asymptotic analysis.

**Problem III.1.** Problem of feedback capacity \( C^\infty(\kappa, s_0) \) for stable/unstable time-invariant AR(c) noise

Consider the characterization of the \( n \)--FTFI capacity of Theorem II.1, and restrict the admissible strategies or distributions to the time-invariant strategies or distributions, defined by (III.112)-(III.121), which generate \((X^{o,n}, Y^{o,n})\).

Define the per unit time limit and maximum by

\[
C^\infty(\kappa, v_0) \triangleq \max_{\mathcal{P}^\infty_{[0,\infty]}(\kappa)} \lim_{n \to \infty} \frac{1}{2n} \sum_{t=1}^{n} \log \left( \frac{(\Lambda^\infty + c)^2 K_{t-1} + K^x_t + K_W}{K_W} \right) \tag{III.147}
\]

where the average power constraint is defined by

\[
\mathcal{P}^\infty_{[0,\infty]}(\kappa) \triangleq \left\{ (\Lambda^\infty, K^x_Z) : X^o_t = \Lambda^\infty (V_{t-1} - \tilde{V}_{t-1}) + Z^o_t, X_1 = Z^o_1, t = 2, \ldots, n, Z^o_t \in N(0, K^x_Z), K^x_Z \geq 0, \lim_{n \to \infty} \frac{1}{n} \mathbb{E}_{v_0} \left( \sum_{t=1}^{n} (X^o_t)^2 \right) = \lim_{n \to \infty} \frac{1}{n} \sum_{t=1}^{n} (\Lambda^\infty)^2 K_{t-1} + K^x_Z \leq \kappa \right\}. \tag{III.148}
\]

Determine sufficient and/or necessary conditions such that

(a) the per unit time limit exists, i.e., condition (C1) holds, and
(b) the maximum over \((\Lambda^\infty, K^x_Z)\) exists, and the optimal strategy is such that \( C^\infty(\kappa, v_0) = C^\infty(\kappa) \) is independent of the initial state \( v_0 \).

In the next theorem we provide the answer to Problem III.1, by invoking Theorem III.1.

**Theorem III.2.** Feedback capacity \( C^\infty(\kappa, s_0) \)

Consider the Problem III.1, defined by (III.147), (III.148).

Define the set

\[
\mathcal{P}^\infty \triangleq \left\{ (\Lambda^\infty, K^x_Z) \in (-\infty, \infty) \times [0, \infty) : \right\}
\]
(i) the pair \( \{A,C\} \equiv \{A,C(\Lambda^\infty)\} \) is detectable,

(ii) the pair \( \{A^+,B^{+\frac{1}{2}}\} \equiv \{A^+(K_Z^\infty),B^{+\frac{1}{2}}(K_Z^\infty)\} \) is stabilizable. \( \text{ (III.149)} \)

Then

\[
C^\infty(\kappa,v_0) = C^\infty(\kappa) \triangleq \max_{(\Lambda^\infty,K_Z^\infty) \in \mathcal{P}^\infty(\kappa)} \frac{1}{2} \log \left( \frac{(\Lambda^\infty + c)^2 K^\infty + K_Z^\infty + K_W}{K_W} \right)
\]

(III.150)

that is, \( C^\infty(\kappa,v_0) \) is independent of \( v_0 \), where

\[
\mathcal{P}^\infty(\kappa) \triangleq \left\{ (\Lambda^\infty,K_Z^\infty) \in \mathcal{P}^\infty : K_Z^\infty \geq 0, \ (\Lambda^\infty)^2 K^\infty + K_Z^\infty \leq \kappa, \right\}
\]

Moreover, the maximum element \( (\Lambda^\infty,K_Z^\infty) \in \mathcal{P}^\infty(\kappa) \), is such that,

(i) if the noise is stable, i.e., \( c \in (-1,1) \) then the input and the output processes \( (X^\circ_t,Y^\circ_t),t=1,\ldots \) are asymptotic stationary, and

(ii) if the noise is unstable i.e., \( c \notin (-1,1) \) then the input and the innovations processes \( (X^\circ_t,I^\circ_t),t=1,\ldots \) are asymptotic stationary.

Proof. The sequence \( \{K_n^\circ : t=1,2,\ldots,n\} \) satisfies the time-invariant generalized DRE (III.130), with zero initial condition, \( K_0^\circ = 0 \). Then for elements in the set \( \mathcal{P}^\infty \), an application of Theorem III.1.1(1), (2), states that the sequence generated by (II.130) converges, i.e., \( \lim_{n \to \infty} K_n^\circ = K^\infty \), where \( K^\infty = K^\infty(\Lambda^\infty,K_Z^\infty) \geq 0 \) is the unique stabilizing solution of the generalized ARE given in (III.151). Hence, the following summands converge, and so the limits exist in \( [0,\infty) \).

(III.154)

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{t=1}^{n} \left( (\Lambda^\infty)^2 K_{t-1}^\circ + K_Z^\infty \right) = (\Lambda^\infty)^2 K^\infty + K_Z^\infty
\]

This establishes the characterization of the right hand side of (III.150), and its independence on \( v_0 \). The last part of the theorem follows from the asymptotic properties of the Kalman-filter, as follows. For (i), \( E^\circ_t,t=1,\ldots \) is asymptotically stationary, which implies \( X^\circ_t = \Lambda^\infty E^\circ_{t-1} + Z^\circ_t,t=1,\ldots \), the innovations process \( I^\circ_t,t=1,\ldots \) and \( Y^\circ_t = X^\circ_t + V_t,t=1,\ldots \) are asymptotically stationary. Similarly for (ii), with the exception that \( Y^\circ_t = X^\circ_t + V_t,t=1,\ldots \) is not asymptotically stationary, because \( V_t,t=1,\ldots \) is unstable. \( \square \)
Clearly, the set $\mathcal{P}^\infty$, defined in Theorem III.2 characterizes condition (C1), and (III.150) characterizes the asymptotic limit of feedback capacity defined by (1.12).

In the next remark, we discuss some aspects of Theorem III.2, and we show that $\mathcal{P}^\infty(\kappa) \subseteq \mathcal{P}^\infty$ is non-empty for some values of $\kappa \in [0, \infty)$.

**Remark III.2. Comments on Theorem III.2**

1. Theorem III.2 characterizes the feedback capacity $C^\infty(\kappa, s_0) = C^\infty(\kappa)$, independently of $v_0$, for AGN channels, driven by stable or unstable AR(c) noise, i.e., $c \in (-\infty, \infty)$.

2. Let $(\Lambda^\infty, K_Z^\infty) \in \mathcal{P}^\infty(\kappa)$ denote the optimal pair for the optimization problem $C^\infty(\kappa)$. Then we need to characterize the set of all $\kappa \in [0, \infty)$ such that $(\Lambda^\infty, K_Z^\infty) \in \mathcal{P}^\infty(\kappa)$.

**Case 1-Stable.** Suppose $c \in (-1, 1)$.

1. Suppose $K_Z^\infty = 0$. By Lemma III.2, $\{A, C\}$ is detectable and $\{A^*, B^{1/2}\}$ is stabilizable if and only if $|\Lambda^\infty| < 1$. For such a choice of $(\Lambda^\infty, K_Z^\infty) \in \mathcal{P}^\infty$, defined by (III.149) then $K^\infty = 0$, and $\mathcal{P}^\infty(\kappa)$ in non-empty for all $\kappa \in [0, \infty)$.

2. Suppose $\Lambda^\infty = 0$. Then $K_Z^\infty = \kappa$, and similar to Lemma III.2, $\{A, C\}$ is detectable and $\{A^*, B^{1/2}\}$ is stabilizable for all $\kappa \in (0, \infty)$, and the non-feedback channel input $X^\infty_t = Z^\infty_t, t = 1, 2, \ldots$ induces a strictly positive achievable rate.

**Case 2-Unstable.** Suppose $|c| \geq 1$.

1. Suppose $K_Z^\infty = 0$. By Lemma III.2, $\{A^*, B^{1/2}\}$ is stabilizable if and only if $|\Lambda^\infty| < 1$, and there exists a $G \in (-\infty, \infty)$ such that $|A - GC| = |c - G(\Lambda^\infty + c)| < 1$, i.e., for $G = 1$, then $\{A, C\}$ detectable if $|\Lambda^\infty| < 1$. For such a choice of $(\Lambda^\infty, K_Z^\infty) \in \mathcal{P}^\infty$, then $K^\infty = 0$, and hence $\mathcal{P}^\infty(\kappa)$ in non-empty for all $\kappa \in [0, \infty)$.

2. Suppose $\Lambda^\infty = 0$. Then $K_Z^\infty = \kappa$, and $\{A, C\}$ is detectable and $\{A^*, B^{1/2}\}$ is stabilizable for all $\kappa \in (0, \infty)$, and the non-feedback channel input $X^\infty_t = Z^\infty_t, t = 1, 2, \ldots$ induces a strictly positive achievable rate. However, from Case 1.(i) and Case 2.(i), since $K_Z^\infty = 0$ then $C^\infty(\kappa) = 0, \forall \kappa \in [0, \infty)$. On the other hand, Case 1.(ii) and Case 2.(ii), with $\Lambda^\infty = 0$, $K_Z^\infty = \kappa$ in $(0, \infty)$ gives a strictly positive non-feedback achievable rate. This is re-visited in Theorem III.3.

In the next lemma we give necessary conditions for the optimization problem $C^\infty(\kappa)$ defined by (III.150).

**Lemma III.3. Necessary conditions for the optimization problem of Theorem III.2**

Suppose there exists a policy $(\Lambda^\infty, K_Z^\infty) \in \mathcal{P}^\infty(\kappa)$ for the optimization problem $C^\infty(\kappa)$ in (III.150). Define the Lagrangian by

$$
\mathcal{L}(\Lambda^\infty, K_Z^\infty, K^\infty, \lambda) \triangleq (\Lambda^\infty + c)^2 K^\infty + K_Z^\infty + K_W
$$

- $-\lambda_1 \left\{ \left( K^\infty - c^2 K^\infty - K_W \right) \left( K_Z^\infty + K_W + (\Lambda^\infty + c)^2 K^\infty \right) + \left( K_W + c K^\infty (\Lambda^\infty + c) \right)^2 \right\}$

- $-\lambda_2 \left( (\Lambda^\infty)^2 K^\infty + K_Z^\infty - \kappa \right) - \lambda_3 \left( -K^\infty - \lambda_4 \right) - K_Z^\infty, \tag{III.156}$

$$
\lambda \triangleq (\lambda_1, \lambda_2, \lambda_3, \lambda_4) \in \mathbb{R}^4. \tag{III.157}
$$
Then the following hold.

(i) Stationarity:

\[ \frac{\partial}{\partial \lambda^0} \mathcal{L}(\Lambda^\infty, K_Z^\infty, K_W^\infty, \lambda) \Big|_{\lambda^0 = \Lambda^\infty, K_Z^\infty = K_W^\infty, \lambda = \lambda^*} = 0, \]  
\[ \frac{\partial}{\partial \Lambda^\infty} \mathcal{L}(\Lambda^\infty, K_Z^\infty, K_W^\infty, \lambda) \Big|_{\lambda^0 = \Lambda^\infty, K_Z^\infty = K_W^\infty, \lambda = \lambda^*} = 0, \]  
\[ \frac{\partial}{\partial K_Z^\infty} \mathcal{L}(\Lambda^\infty, K_Z^\infty, K_W^\infty, \lambda) \Big|_{\lambda^0 = \Lambda^\infty, K_Z^\infty = K_W^\infty, \lambda = \lambda^*} = 0. \]  

Complementary Slackness:

\[ \lambda_2^* (\Lambda^\infty)^2 K_W^\infty + K_Z^\infty - \kappa = 0, \quad \lambda_3^* K_W^\infty = 0, \quad \lambda_4^* K_Z^\infty = 0, \]  
\[ \lambda_1^* \left\{ (K_W^\infty - c^2 K_Z^\infty - K_W) \left( K_W^\infty + K_W + (\Lambda^\infty + c)^2 K_W^\infty \right) + (K_W + c K_W^\infty (\Lambda^\infty + c))^2 \right\} = 0. \]  

Primal Feasibility:

\[ (\Lambda^\infty)^2 K_W^\infty + K_Z^\infty \leq \kappa, \quad K_Z^\infty \geq 0, \quad K_W^\infty \geq 0, \]  
\[ (K_W^\infty - c^2 K_Z^\infty - K_W) \left( K_Z^\infty + K_W + (\Lambda^\infty + c)^2 K_W^\infty \right) + (K_W + c K_W^\infty (\Lambda^\infty + c))^2 \leq 0. \]  

Dual Feasibility:

\[ \lambda_1^* \geq 0, \quad \lambda_2^* \geq 0, \quad \lambda_3^* \geq 0, \quad \lambda_4^* \geq 0. \]  

(ii) If \( K_Z^\infty = 0 \) then \( K_W^\infty = 0 \), if \( K_W^\infty = 0 \) then \( K_Z^\infty = 0 \), and if either \( K_Z^\infty = 0 \) or \( K_W^\infty = 0 \) then \( C^\infty(\kappa) = 0, \forall \kappa \in [0, \infty) \).

(iii) A necessary condition for existence of \( \kappa \in (0, \infty) \) such that \( C^\infty(\kappa) > 0 \) is \( \lambda_1^* > 0, \lambda_2^* > 0, \lambda_3^* = 0, \lambda_4^* = 0 \), and \( (\text{III.164}) \) holds with equality.

\[ \square \]

\textbf{Proof.} See Section VI-B.

We shall derive the main Theorem III.4, after we introduce the lower bound on feedback and non-feedback capacity, of Section III-C1 (below).

1) Achievable Rates Without Feedback for Stable and Unstable AR(c) Noise: By Remark III.2, for \( \Lambda^\infty = 0 \), the optimization problem of Theorem III.2 reduces to \( C^\infty(\kappa) \Big|_{\Lambda^\infty = 0} \), which is an achievable rate, because the pair \( \{A, C\} \) is detectable and the pair \( \{A^*, B^+\} \) is stabilizable, when \( \Lambda^\infty = 0 \). For \( \Lambda^\infty = 0 \), by (III.112), the channel input is an independent innovations process \( X_t^\infty = Z_t^0, t = 1, \ldots, n \), and hence the code does not use feedback. In the next theorem we calculate \( C^\infty(\kappa) \Big|_{\Lambda^\infty = 0} \).

\textbf{Theorem III.3.} Achievable rates without feedback for stable and unstable AR(c) noise

For \( \Lambda^\infty = 0 \), define the set

\[ \mathcal{P}^\infty_{0, \text{fb}} \triangleq \left\{ K_Z^\infty \in [0, \infty) : \right\} . \]
(i) the pair \{A, C\} \bigg|_{n=0} \equiv \{A, C(A^{\infty})\} \bigg|_{n=0} is detectable, \hspace{1cm} (III.166)

(ii) the pair \{A^{*}, B^{*} + 1\} \bigg|_{n=0} \equiv \{A^{*}(K_{Z}^{\infty}), B^{*} + 1(K_{Z}^{\infty})\} \bigg|_{n=0} is stabilizable. \hspace{1cm} (III.167)

For \(\Lambda^{\infty} = 0\), define the channel input and output processes by

\[
\begin{align*}
X_{t}^{o} &= Z_{t}^{o}, \quad t = 1, \ldots, n, \hspace{1cm} (III.168) \\
V_{t} &= cV_{t-1} + W_{t}, \quad V_{0} = v_{0}, \hspace{1cm} (III.169) \\
Y_{t}^{o} &= X_{t}^{o} + V_{t} = cV_{t-1} + W_{t} + Z_{t}^{o}, \hspace{1cm} (III.170) \\
Y_{1}^{o} &= Z_{1}^{o} + cV_{0} + W_{1}. \hspace{1cm} (III.171)
\end{align*}
\]

(1) A lower bound on non-feedback capacity \(C_{\infty,nfb}^{\infty}(\kappa, v_{0})\) is \(C_{LB}^{\infty,nfb}(\kappa)\) given by

\[
C_{\infty,nfb}^{\infty}(\kappa, v_{0}) \geq C_{LB}^{\infty,nfb}(\kappa) \triangleq \max_{K_{Z}^{\infty} \in \mathcal{P}_{0}^{\infty,nfb}(\kappa)} \frac{1}{2} \log \left( \frac{c^{2}K_{\infty}^{\infty} + K_{Z}^{\infty} + K_{W}}{K_{W}} \right) \hspace{1cm} (III.172)
\]

where

\[
\mathcal{P}_{0}^{\infty,nfb}(\kappa) \triangleq \left\{ K_{Z}^{\infty} \in \mathcal{P}_{0}^{\infty,nfb} : K_{Z}^{\infty} \geq 0, \ K_{Z}^{\infty} \leq \kappa, \ K_{\infty}^{\infty} = c^{2}K_{\infty}^{\infty} + K_{W} - \frac{(K_{W} + c^{2}K_{\infty}^{\infty})^{2}}{(K_{Z}^{\infty} + K_{\infty}^{\infty} + c^{2}K_{\infty}^{\infty})} \right\} \hspace{1cm} (III.173)
\]

\[
F_{nfb}^{\infty}(K_{\infty}, K_{Z}^{\infty}) \triangleq c - M_{nfb}^{\infty}(K_{\infty}, K_{Z}^{\infty})c, \hspace{1cm} (III.174)
\]

\[
M_{nfb}^{\infty}(K_{\infty}, K_{Z}^{\infty}) \triangleq (K_{W} + c^{2}K_{\infty}^{\infty})(K_{Z}^{\infty} + K_{\infty}^{\infty} + c^{2}K_{\infty}^{\infty})^{-1} \hspace{1cm} (III.175)
\]

provided there exists \(\kappa \in [0, \infty)\) such that the set \(\mathcal{P}_{0}^{\infty,nfb}(\kappa)\) is non-empty.

Moreover, \(C_{LB}^{\infty,nfb}(\kappa)\) is an achievable rate without feedback, independent of the initial state \(V_{0} = v_{0}\), and

(i) if the noise is stable, i.e., \(c \in (-1, 1)\) then the input and the output processes \((X_{t}^{o}, Y_{t}^{o}), t = 1, \ldots\) are asymptotically stationary, and

(ii) if the noise is unstable i.e., \(c \notin (-1, 1)\) then the input and the innovations processes \((X_{t}^{o}, I_{t}^{o}), t = 1, \ldots\) are asymptotic stationary.

(2) The lower bound on non-feedback capacity of (1) is given by

\[
C_{LB}^{\infty,nfb}(\kappa) = \frac{1}{2} \log \left( \frac{c^{2}K_{\infty}^{\infty} + \kappa + K_{W}}{K_{W}} \right), \quad \forall \kappa \in \mathcal{N}_{\infty,nfb}(c, K_{W}) \hspace{1cm} (III.176)
\]

where \(K_{\infty}^{\infty} \geq 0\) is unique and stabilizing, and \(K_{Z}^{\infty,*}\), are given by

\[
K_{\infty}^{\infty,*} = \left\{ \begin{array}{ll}
\frac{-\kappa(1-c^{2}) - K_{W} + \sqrt{(\kappa(1-c^{2}) + K_{W})^{2} + 4c^{2}K_{W}\kappa}}{2c^{2}}, & \forall c \neq 0, \quad \forall \kappa \in \mathcal{N}_{\infty,nfb}(c, K_{W}), \\
\frac{K_{W}}{\kappa + K_{W}}, & \kappa = 0, \quad \forall \kappa \in [0, \infty), \end{array} \right. \hspace{1cm} (III.177)
\]
\[ K_{Z}^{\infty,*} = \kappa, \quad (III.178) \]

\[ \mathcal{H}^{\infty,nfb}(c, K_{W}) \doteq \{ \kappa \in [0, \infty) : \ K_{Z}^{\infty} \geq 0 \} = [0, \infty). \quad (III.179) \]

**Proof.** (1) Note that by setting \( \Lambda^{\infty} = 0 \), then the representation of channel input \( X^{o,n} \) defined by (III.112)-(III.130) is used without feedback, and this is a lower bound on the non-feedback capacity. Hence, the statements follow from Theorem III.2, as a special case.

(2) This follows from (1), since the optimal \( K_{Z}^{\infty} \) occurs on the boundary, i.e., \( K_{Z}^{\infty,*} = \kappa \). Then by substituting \( (\Lambda^{\infty}, K_{Z}^{\infty,*}) = (0, \kappa) \) into the generalized ARE of the constraint (III.173), we obtain

\[ c^2 \left( K_{Z}^{\infty,*} \right)^2 + K_{W} \{ \kappa \left( 1 - c^2 \right) + K_{W} \} - K_{W} \kappa = 0, \quad c \neq 0. \quad (III.180) \]

Hence, the unique and non-negative stabilizing solution of the generalized ARE (III.180) is given by (III.177), for \( c \neq 0 \). For \( c = 0 \), then the generalized ARE reduces to \( K_{Z}^{\infty,*} \{ \kappa + K_{W} \} - K_{W} \kappa = 0 \), and hence all equations under (2) are obtained. The validity of the stability condition, i.e., \( |F_{n fb}(K_{Z}^{\infty,*}, K_{Z}^{\infty})| < 1 \), although it is ensured by the conditions, it is shown in Appendix VI-C. \( \square \)

**Remark III.3.** On the achievable rate without feedback of Theorem III.3

By Theorem III.3.(2) we deduce that the lower bound \( C_{LB}^{\infty,nfb}(\kappa), \kappa \in [0, \infty), \) holds for stable and unstable AR(1) noise. We should explicitly mention the two special cases \( c = 0 \) and \( |c| = 1 \).

(1) For \( c = 0 \), we recover, as expected, the capacity of the AGN channel with memoryless noise.

(2) For \( |c| = 1 \), we obtain the lower bound \( C_{LB}^{\infty,nfb}(\kappa) \) on the non-feedback capacity \( C^{\infty,nfb}(\kappa) \), given by

\[ C_{LB}^{\infty,nfb}(\kappa) = \frac{1}{2} \log \left( \frac{c^2 K_{Z}^{\infty,*} + \kappa + K_{W}}{K_{W}} \right), \quad (III.181) \]

\[ K_{Z}^{\infty,*} = \frac{-K_{W} + \sqrt{K_{W}^2 + 4K_{W} \kappa}}{2}, \quad K_{Z}^{\infty,*} = \kappa \in [0, \infty). \quad (III.182) \]

The above choice of a channel input strategy ensures the pair \( \{A, C\} |_{\Lambda^{\infty,c}=0} = \{c, c\} \) is detectable, and the pair \( \{A^*, B^* \} \}_{K_{Z}^{\infty,*} = \kappa} = \{c - \frac{K_{W}}{\kappa + K_{W}}, \frac{K_{W}}{2} \sqrt{1 - K_{W} (\kappa + K_{W})^{-1}} \} \) is stabilizable, for any \( \kappa \in (0, \infty) \).

2) Feedback Capacity for Stable and Unstable AR(c) Noise: Next we derive closed form expressions for the feedback capacity, by solving the optimization problem of Theorem III.2, i.e., (III.150).

First, by the definition of the sets \( \mathcal{P}^{\infty} \) and \( \mathcal{P}_{0}^{\infty,nfb} \) of Theorem III.2 and Theorem III.3, we have

\[ \mathcal{P}^{\infty} = \mathcal{P}_{0}^{\infty,nfb} \cup \mathcal{P}^{\infty,fb}, \quad \mathcal{P}^{\infty,fb} \triangleq \{ (\Lambda^{\infty}, K_{Z}^{\infty}) \in \mathcal{P}^{\infty} : \Lambda^{\infty} \neq 0 \}. \quad (III.183) \]

Thus, if \( (\Lambda^{\infty}, K_{Z}^{\infty}) \in \mathcal{P}^{\infty,fb} \) then the channel input process applies feedback, and if \( K_{Z}^{\infty} \in \mathcal{P}_{0}^{\infty,nfb} \) then the channel input process does not apply feedback.
Theorem III.4. Feedback capacity-solution of optimization problem of Theorem III.2

(1) The non-zero feedback capacity $C^\infty(\kappa)$ defined by (III.150), for a stable and unstable AR(c) noise, i.e., $c \in (-\infty, \infty)$, with $c \neq 0, c \neq 1$, occurs in the set $\mathcal{E}^{\infty, fb}$, and is given, as follows.

$$C^\infty(\kappa) = \frac{1}{2} \log \left( \frac{(\Lambda^\infty,^* + c)^2 K^\infty,^* + K^\infty_z + K_W}{K_W} \right), \quad \forall \kappa \in \mathcal{E}^{\infty}(c, K_W)$$  (III.184)

$$= \frac{1}{2} \log \left( \frac{c^2 (c^2 - 1) \kappa + K_W}{(c^2 - 1)K_W} \right),$$  (III.185)

$$\Lambda^\infty,^* = \frac{\kappa (1 - c^2) + K_W + c^2 K^\infty,^*}{c (c^2 - 2) K^\infty,^*} \in (-\infty, \infty),$$  (III.186)

$$K^\infty_z + (\Lambda^\infty,^*)^2 K^\infty,^* = \kappa,$$  (III.187)

$$\mathcal{E}^{\infty}(c, K_W) \triangleq \{ \kappa \in [0, \infty) : K^\infty,^* > 0, \ K^\infty_z,^* > 0 \}, \quad c \in (-\infty, \infty), \ c \neq 0, c \neq 1.$$  (III.188)

where $K^\infty,^*$ is the unique positive and stabilizing solution, i.e., $|F(K^\infty,^*, \Lambda^\infty,^*, K^\infty_z)| < 1$, of the quadratic equation

$$c^4 \left( c^2 - 1 \right) \left( K^\infty,^* \right)^2 + c^4 \left( (1 - c^2) \kappa + K_W \right) K^\infty,^* + \left( (1 - c^2) \kappa + K_W \right)^2 + 4 \left( c^2 - 1 \right) K_W \kappa - c^4 \kappa K_W = 0.$$  (III.189)

Further, for any $\kappa \in \mathcal{E}^{\infty}(c, K_W)$, then

$$K^\infty,^* = \frac{\kappa (c^2 - 1)^2 - K_W}{c^2 (c^2 - 1)} \in (0, \infty),$$  (III.190)

$$\Lambda^\infty,^* = \frac{c K_W}{\kappa (c^2 - 1)^2 - K_W} \in (-\infty, \infty),$$  (III.191)

$$K^\infty_z = \frac{\kappa (c^2 - 1) \left( \kappa (c^2 - 1)^2 - K_W \right) - K_W^2}{(c^2 - 1) \left( \kappa (c^2 - 1)^2 - K_W \right)} \in (0, \infty).$$  (III.192)

(2) The non-zero feedback capacity $C^\infty(\kappa)$, $\kappa \in \mathcal{E}^{\infty}(c, K_W)$ of part (1), is restricted to the region:

(a) $\mathcal{E}^{\infty}(c, K_W) \triangleq \{ \kappa \in [0, \infty) : 1 < c^2 < \infty, \kappa > \frac{K_W + K_W \sqrt{4c^2 - 3}}{2(c^2 - 1)^2} \}$.

(3) A non-zero feedback capacity $C^\infty(\kappa)$, i.e., with $\Lambda^\infty \neq 0$, does not exist for the two regions:

(a) $\mathcal{E}^{\infty, fb}(c) \triangleq \{ \kappa \in [0, \infty) : 0 \leq c^2 \leq 1 \}$;

(b) $\mathcal{E}^{\infty, fb}(c, K_W) \triangleq \{ \kappa \in [0, \infty) : 1 < c^2 < \infty, \kappa \leq \frac{K_W + K_W \sqrt{4c^2 - 3}}{2(c^2 - 1)^2} \}$.

Proof. See Appendix VI-D.
From the previous theorem it then follows the next theorem, that states feedback does not increase capacity $C^\infty(\kappa)$ defined by (III.150), for the two regions $\mathcal{X}^{\infty,fb}(c)$ and $\mathcal{X}^{\infty,nfb}(c, K_W)$.

**Theorem III.5.** Feedback does not increase capacity for certain regions

Feedback does not increase capacity $C^\infty(\kappa)$ defined by (III.150), for the two regions:

(a) $\kappa \in \mathcal{X}^{\infty,nfb}(c) \triangleq \left\{ \kappa \in [0, \infty) : 0 \leq c^2 \leq 1 \right\}$;

(b) $\kappa \in \mathcal{X}^{\infty,nfb}(c, K_W) \triangleq \left\{ \kappa \in [0, \infty) : 1 < c^2 < \infty, \kappa \leq \frac{K_W + K_W \sqrt{4c^2 - 1}}{2(c^2 - 1)} \right\}$.

**Proof.** By Theorem III.4.(3) we deduce that, if $\Lambda^\infty \neq 0$, i.e., if feedback is used, then there does not exists a non-zero value of $C^\infty(\kappa)$, for $\kappa \in \mathcal{X}^{\infty,nfb}(c) \cup \mathcal{X}^{\infty,nfb}(c, K_W)$. On the other hand, by Theorem III.3, an achievable rate without feedback exists for all $\kappa \in \mathcal{X}^{\infty,nfb}(c) \cup \mathcal{X}^{\infty,nfb}(c, K_W)$, by letting $\Lambda^\infty = 0$, and $C_{\infty,nfb}^\infty(\kappa)$ is a lower bound on capacity without feedback. \hfill \blacksquare

**Remark III.4.** Implications of Theorem III.4 on operational non-feedback capacity for unstable noise.

Theorem III.4.(3), when combined with Theorem III.3, states that, one does not need to presuppose, as usually done in information theory literature [20]–[22], that the joint channel input and output process, and noise, are stationary to show existence of achievable non-feedback rates, as done, in the water-filling solution. We shall give a tighter lower bound on the non-feedback capacity in Corollary III.1.

Another lower bound on feedback capacity is presented in Section III-C3 (below).

3) Achievable Feedback Rates with Noise Cancellation for Stable AR(c) Noise: Let us recall the representation of channel input described by (III.112)-(III.121). If we let $\Lambda^\infty = -c$, then (III.114) and (III.115), reduce to the equations $Y_t^o = cV_{t-1} + W_t + Z_t^o, t = 2, \ldots, n, Y_0^o = Z_0^o + cV_0 + W_1$. This means, for the choice of strategy $\Lambda^\infty = -c$, the channel output process at time $t$, $Y_t^o$ is driven by past observations, and $(Z_t^o, W_t)$, which mean $V_t$ and $Y_t^o$ are correlated. Hence, if the resulting pair $\{A, C\} \bigg|_{\Lambda^\infty = -c} \equiv \{A, C(\Lambda^\infty)\} \bigg|_{\Lambda^\infty = -c}$ is detectable, and the pair $\{A^*, B^\infty, \tau\} \bigg|_{\Lambda^\infty = -c} \equiv \{A^*(K_Z^\infty), B^\infty, \tau(K_Z^\infty)\} \bigg|_{\Lambda^\infty = -c}$ is stabilizable, then we can have a corresponding rate which is achievable.

In the next theorem, we give an achievable rate with noise cancellation, that corresponds to $C^\infty(\kappa) \bigg|_{\Lambda^\infty = -c}$.

**Theorem III.6.** Achievable rates with feedback and noise cancellation for stable AR(c) noise

For $\Lambda^\infty = -c$, define the set

$$\mathcal{P}^{\infty,nc} \triangleq \left\{ K_Z^\infty \in [0, \infty) : \right. \quad \left. \begin{array}{l} (i) \text{ the pair } \{A, C\} \bigg|_{\Lambda^\infty = -c} \equiv \{A, C(\Lambda^\infty)\} \bigg|_{\Lambda^\infty = -c} \text{ is detectable,} \\ (III.193) \end{array} \right.$$
Moreover, consider the channel input and output processes, given by (III.112)-(III.130), with \( \Lambda^\infty = -c \):

\[
X^o_t = -c(V_{t-1} - \tilde{V}^o_{t-1}) + Z^o_t, \quad X^o_1 = Z^o_1, \quad t = 2, \ldots, n,
\]

\[
V_t = cV_{t-1} + W_t, \quad V_0 = v_0,
\]

\[
Y^o_t = X^o_t + V_t = c\tilde{V}^o_{t-1} + W_t + Z^o_t,
\]

(1) A lower bound on the feedback capacity is \( C_{LB}^{\infty,nc}(\kappa) \) given by

\[
C^\infty(\kappa) \geq C^\infty(\kappa)\bigg|_{\Lambda^\infty = -c} = C_{LB}^{\infty,nc}(\kappa) \triangleq \max_{K_Z^\infty \in \mathcal{P}^{\infty,nc}(\kappa)} \frac{1}{2} \log \left( \frac{K_Z^\infty + K_W}{K_W} \right)
\]

where

\[
\mathcal{P}^{\infty,nc}(\kappa) \triangleq \left\{ K_Z^\infty \in \mathcal{P}^{\infty,nc} : K_Z^\infty \geq 0, \ c^2 K^\infty + K_Z^\infty \leq \kappa, \ K^\infty = c^2 K^\infty + K_W - \frac{K^\infty}{K_Z^\infty + K_W} \right\}
\]

\[
F^{nc}(K^\infty, K^\infty^*) \triangleq c
\]

provided there exists \( \kappa \in [0, \infty) \) such that the set \( \mathcal{P}^{\infty,nc}(\kappa) \) is nonempty.

Moreover, \( C_{LB}^{\infty,nc}(\kappa) \) is achievable and is independent of the initial state \( V_0 = v_0 \).

(2) The lower bound of (1) is given by

\[
C_{LB}^{\infty,nc}(\kappa) = \frac{1}{2} \log \left( \frac{K^\infty + K^\infty_*}{K_W} \right), \quad c \in (-1, 1), \quad \forall \kappa \in [0, \infty)
\]

where \( K^\infty_* \) and the unique and stabilizing \( K^\infty_* \geq 0 \), are given by

\[
K^\infty_* = \frac{K^\infty_* K_W}{(K^\infty_* + K_W)(1 - c^2)}, \quad c \in (-1, 1), \quad \kappa \in [0, \infty),
\]

\[
K^\infty_* = \frac{-(K_W - \kappa(1 - c^2)) + \sqrt{(K_W - \kappa(1 - c^2))^2 + 4\kappa K_W(1 - c^2)^2}}{2(1 - c^2)}, \quad c \in (-1, 1), \quad \kappa \in [0, \infty).
\]

**Proof.** (1) By setting \( \Lambda^\infty = -c \), in (III.112)-(III.130), then the representation of channel input \( X^o_t \) cancels the noise \( V_{t-1} \) in the output \( Y^o_t \), and (III.195)-(III.197) are obtained. Clearly, the set \( \mathcal{P}^{\infty,nc} \) is non-empty, since the pair \( \{A, C\} \bigg|_{\Lambda^\infty = -c} = \{c, 0\} \) is detectable, for \( c \in (-1, 1), c \neq 0 \). Hence, the statements follow from Theorem III.2, as a special case.

(2) This follows from (1), by simple computations. In particular, from the equation of \( K^\infty \) that appears in constraint (III.199), we obtain

\[
K^\infty = \frac{K^\infty K_W}{(K^\infty + K_W)(1 - c^2)} \geq 0, \quad \text{if and only if} \quad |c| < 1.
\]
Since the optimal $K_Z^{\infty} = K_Z^{\infty,*}$ is such that the average constraint holds with equality, then $c^2 K_Z^{\infty,*} + K_Z^{\infty,*} = \kappa$. Substituting $K_Z^{\infty,*}$ into the average constraint we arrive at the quadratic equation

$$
(1 - c^2) (K_Z^{\infty,*})^2 + \left( K_W - \kappa (1 - c^2) \right) K_Z^{\infty,*} - K_W \kappa \left( 1 - c^2 \right) = 0, \quad |c| < 1. \tag{III.205}
$$

The rest follows from the above equation and the fact that $K_Z^{\infty,*} \geq 0$ belongs to the set $\mathcal{P}_{\kappa, nc}$.

In the lemma we show that the achievable rate without feedback of Theorem III.3, is higher than the achievable rate with feedback and noise cancellation of Theorem III.6, when the noise is stable AR(c).

**Lemma III.4.** On the comparison of lower bounds of Theorem III.3 and Theorem III.6

For $c \in (-1, 1), c \neq 0$ then the following inequality holds.

$$
C_{LB}^{\infty, nc}(\kappa) < C_{LB}^{\infty, nf}(\kappa), \quad \forall \kappa \in (0, \infty). \tag{III.206}
$$

**Proof.** See Appendix VI-E.

D. Lower Bound on Characterization of Capacity without Feedback for Time-Invariant Channel Input Distributions

By recalling the lower bound of Corollary II.1, and by restricting the non-feedback strategies to the time-invariant channel input strategies, $(\overline{\Lambda}_t, K_Z) = (\overline{\Lambda}_0, K_Z^*), t = 1, \ldots, n$, then we have a representation of $(X^{o,n}, Y^{o,n})$, as in Section III-A. The next corollary gives an achievable lower bound on the characterization of non-feedback capacity, as a time-invariant optimization problem, which is analogous to Theorem III.2.

**Corollary III.1.** Achievable lower bound on non-feedback capacity

Consider the lower bound on the characterization of the $n$-FTFI capacity of Corollary II.1.(b), and restrict the admissible strategies to the time-invariant strategies, $(\overline{\Lambda}_t, K_Z) = (\overline{\Lambda}_0, K_Z^*), t = 1, \ldots, n$, which generate $(X^{o,n}, Y^{o,n})$.

Define the lower bound on non-feedback capacity by

$$
C^{\infty,nf}(\kappa, v_0) \geq C_{LB}^{\infty,nf}(\kappa, v_0) \triangleq \sup \left\{ \left( \overline{\Lambda}_t, K_Z^* \right) : \lim_{n \to \infty} \frac{1}{n} \sum_{t=1}^{n} \left( \overline{\Lambda}_t \right)^2 K_{X_{t-1}}^{\infty} + K_Z^{\infty} \leq \kappa \right\}
$$

subject to: $K_Z^{\infty} \geq 0$, $K_{X_t}^{\infty}$ and $K_{X_t}^{\infty}$, $t = 1, \ldots, n$ satisfy the generalized DRE and Lyapunov equation

$$
K_t^{\infty} = \left( \overline{\Lambda}_t \right)^2 K_{t-1}^{\infty} + K_Z^{\infty} - \frac{\left( K_Z^{\infty} + \overline{\Lambda}_t K_{t-1}^{\infty} \left( \overline{\Lambda}_t - c \right) \right)^2}{K_Z^{\infty} + K_W + \left( \overline{\Lambda}_t - c \right)^2 K_{t-1}^{\infty}}, \quad K_t^{\infty} \geq 0, \quad K_0^{\infty} = 0, \quad t = 1, \ldots, n. \tag{III.208}
$$
Define the set\(^9\)

\[ \mathcal{S}^{\infty,nfb} \triangleq \left\{ (\overline{\Lambda}^\infty, K_Z^\infty) \in (-1,1) \times [0,\infty) : \right. \]

(i) the pair \( \{A,C\} \equiv A(\overline{\Lambda}^\infty), C(\overline{\Lambda}^\infty) \) is detectable,

(ii) the pair \( \{A^*,B^{*,\frac{1}{2}}\} \equiv \{A^*(\overline{\Lambda}^\infty, K_Z^\infty),B^{*,\frac{1}{2}}(K_Z^\infty)\} \) is stabilizable. \hspace{1cm} (III.210)

where

\[ A \triangleq \overline{\Lambda}^\infty, \quad C \triangleq \overline{\Lambda}^\infty - c, \quad A^* = \overline{\Lambda}^\infty - K_Z^\infty R^{-1} C, \quad B^{*,\frac{1}{2}} = K_Z^\infty \frac{1}{2} B^{\frac{1}{2}}, \]

\[ R \triangleq K_Z^\infty + K_W, \quad B \triangleq 1 - K_Z^\infty (K_Z^\infty + K_W)^{-1}. \hspace{1cm} (III.211) \]

Then the lower bound on non-feedback capacity is given by the time-invariant optimization problem:

\[ C_{LB}(\kappa, y_0) = C_{LB}(\kappa) \triangleq \max_{(\overline{\Lambda}^\infty,K_Z^\infty)} \frac{1}{2} \log \left( \frac{(\overline{\Lambda}^\infty - c)^2 K_Z^\infty + K_Z^\infty + K_W}{K_W} \right) \hspace{1cm} (III.213) \]

where

\[ \mathcal{S}^{\infty,nfb}(\kappa) \triangleq \left\{ (\overline{\Lambda}^\infty, K_Z^\infty) \in \mathcal{S}^{\infty,nfb} : K_Z^\infty \geq 0, \quad \frac{K_Z^\infty}{1 - (\overline{\Lambda}^\infty)^2} \leq \kappa, \right. \]

\[ K_Z^\infty = (\Lambda)^2 K^\infty + K_Z^\infty - \frac{(K_Z^\infty + \overline{\Lambda}^\infty K^\infty (\overline{\Lambda}^\infty - c))^2}{(K_Z^\infty + K_W + (\overline{\Lambda}^\infty - c)^2 K^\infty)} \]

\[ K^\infty \geq 0 \text{ is unique and stabilizable, i.e., } |F^{nfb}(K^\infty, \overline{\Lambda}^\infty, K_Z^\infty)| < 1 \}, \hspace{1cm} (III.214) \]

\[ F^{nfb}(K^\infty, \overline{\Lambda}^\infty, K_Z^\infty) \triangleq \overline{\Lambda}^\infty - M^{nfb}(K^\infty, \overline{\Lambda}^\infty, K_Z^\infty)(\overline{\Lambda}^\infty - c), \hspace{1cm} (III.215) \]

\[ M^{nfb}(K^\infty, \overline{\Lambda}^\infty, K_Z^\infty) \triangleq K_Z^\infty + \overline{\Lambda}^\infty K^\infty (\overline{\Lambda}^\infty - c) \left( K_Z^\infty + K_W + (\overline{\Lambda}^\infty - c)^2 K^\infty \right)^{-1} \hspace{1cm} (III.216) \]

and the set \( \mathcal{S}^{\infty,nfb}(\kappa) \) is nonempty for all \( \kappa \in [0,\infty) \).

Moreover, there exist maximum element \( (\overline{\Lambda}^\infty, K_Z^\infty) \in \mathcal{S}^{\infty,nfb}(\kappa) \), such that for all \( \kappa \in [0,\infty) \),

(i) if the noise is stable, i.e., \( c \in (-1,1) \) then the input and the output processes \( (X_t^n,Y_t^n), t = 1,\ldots \) are asymptotically stationary, and

(ii) if the noise is unstable i.e., \( c \notin (-1,1) \) then input and the innovations processes \( (X_t^n,I_t^n), t = 1,\ldots \) are asymptotically stationary.

Proof. First, note that definition (III.207)-(III.212), follows from the restriction of time-varying strategies of Corollary II.1 to time-invariant, and by imposing the detectability and stabilizability conditions of

\(^9\)The definition of detectability and stabilizability follow from the DRE (III.208), while \( \overline{\Lambda}^\infty \in (-1,1) \) is necessary for \( K_Z^\infty \) to converge as \( n \longrightarrow \infty \).
Remark III.5. On the lower bound on achievable rates without feedback

(1) We should note that the achievable lower bound on non-feedback capacity, given in Corollary III.1, i.e., $C_{LB}^{\infty}(\kappa, v_0) = C_{LB}^{\infty}(\kappa)$ defined by (III.213), on the non-feedback capacity $C_{nf}^{\infty}(\kappa, v_0)$, defined by (I.13), holds for stable and unstable noise $AR(c), c \in (-\infty, \infty)$. This is contrary to the well-known water-filling solution, given by (I.44), which presupposes the noise is stable, i.e., $AR(c), c \in (-1, 1)$.

(2) We expect that the closed form expression of the lower bound $C_{nf}^{\infty}(\kappa, v_0) = C_{LB}^{\infty}(\kappa)$ defined by (III.213), can be found by carrying out the optimization problem, as in Theorem III.4.

(3) By Figure I.3, the difference between the non-feedback capacity $C_{nf}^{\infty}(\kappa)$ based on water-filling formulae (I.44) (see [16, eqn(5.5.14)]) and [5, eqn(6)]), and the lower bound $C_{LB}^{\infty, nfb}(\kappa)$ on achievable rate without feedback based on the formulae (I.41), of transmitting an IID channel input $Z^n \in N(0, \kappa)$, for an AGN channel driven by $AR(c)$, noise, $c = 0.75$ and $K_w = 1$, is less than $1.5 \times 10^{-2}$ bits per channel use. This difference is expected to be reduced further, if the lower bound $C_{LB}^{\infty}(\kappa, v_0) = C_{LB}^{\infty}(\kappa)$ defined by (III.213) is used, because it employs a Markov channel input, instead of an IID channel input.

IV. Discussion of Related Literature and Comparison with Main Results of the Paper

In this section we recall the formulation and some of the results of [6] and compare them with [1], [8] and with recent results in [9]–[12], with emphasis on the assumptions based on which these are derived. Then we specialize some of the results to AGN channels driven by the $AR(c)$ stable noise, to point out some of the oversights, which are overlooked in [1], and are repeated in [9]–[12].

A. Cover and Pombra Characterization of Feedback Capacity

Cover and Pombra [6] considered the AGN channel (I.1), when the noise is time-varying, i.e., nonstationary Gaussian, with distribution $P_{V_t|V_{t-1}, \ldots, V_1, V_0} = P_{V_t}$, and derived converse and direct coding theorems, through the characterization of the $n$–finite (block length) transmission of feedback information ($n$–FTFI) capacity. The code in [6], is analogous to Definition I.1, with the fundamental difference that the encoder and decoder are replaced by $X_1 = e_1(W), X_2 = e_2(W, X_1, Y_1), \ldots, X_n = e_n(W, X_{n-1}, Y_{n-1}), y^n \rightarrow d_n(y^n) \in \mathcal{M}(n)$. The average error probability is defined by (I.4), without knowledge of the initial state $v_0$, and depends on the initial distribution $P_{V_1}$, induced by $Y_1 = X_1 + V_1$, where $P_{V_1}$ is fixed.
Cover and Pombra derived the characterization of $n$–FTFI capacity, by recognizing that entropy $H(Y^n)$ is maximized, if the input $X^n$ is jointly Gaussian, driven by a jointly Gaussian process $\mathcal{Z}^n$, of the form:

$$X_t = \sum_{j=1}^{t-1} \Gamma_{t,j} V_j + Z_t, \quad X_1 = Z_1, \quad t = 2, \ldots, n,$$

(IV.217)

equivalently $X^n = \Gamma^n V^n + \mathcal{Z}^n$, $\Gamma^n$ is lower diagonal non-random matrix,

$\mathcal{Z}^n$ is jointly Gaussian, $N(0, K_{\mathcal{Z}^n})$,

$\mathcal{Z}^n$ is independent of $V^n$,

$$\frac{1}{n} \mathbb{E}\left\{ \sum_{t=1}^{n} (X_t)^2 \right\} = \frac{1}{n} \text{Trace} \left\{ \mathbb{E}\left( X^n (X^n)^T \right) \right\} \leq \kappa.$$

(IV.221)

The $n$–FTFI capacity is defined by maximizing $H(Y^n) - H(V^n)$ over all time-varying channel input strategies $(\Gamma^n, K_{\mathcal{Z}^n})$, which induce time-varying feedback distributions, as follows.

$$C_n(\kappa) \triangleq \max_{(\Gamma^n, K_{\mathcal{Z}^n})} \frac{1}{n} \text{Trace} \left\{ \mathbb{E}\left( X^n (X^n)^T \right) \right\} \leq \kappa \quad H(Y^n) - H(V^n)$$

(IV.222)

$$= \max_{(\Gamma^n, K_{\mathcal{Z}^n})} \frac{1}{n} \text{Trace} \left\{ \Gamma^n (1 + I) K_{V^n} (1 + I)^T + K_{\mathcal{Z}^n} \right\} \leq \kappa \quad \frac{1}{2} \log \frac{|(1 + I) K_{V^n} (1 + I)^T + K_{\mathcal{Z}^n}|}{|K_{V^n}|}.$$

(IV.223)

Coding theorems, are stated in [6, Theorem 1], based on $\frac{1}{n} C_n(\kappa)$ for sufficiently large “n”.

The $n$–finite transmission without feedback information (FTwFI) capacity is characterized by the input process (IV.217), with $\Gamma_{t,j} = 0, \forall (t, j)$, i.e., $X_t = Z_t, t = 1, \ldots n$, or $\Gamma^n = 0$, which imply $K_{X^n} = K_{\mathcal{Z}^n}$, as follows [6, eqn(14)].

$$C_n^{\text{fb}}(\kappa) \triangleq \max_{K_{X^n}} \frac{1}{2} \log \frac{|K_{X^n} + K_{V^n}|}{|K_{V^n}|}.$$

(IV.224)

In [6, Theorem 2, Eqn(15), Eqn(16)], it is proved that $C_n(\kappa)$ and $C_n^{\text{fb}}(\kappa)$, satisfy

$$\frac{1}{n} C_n^{\text{fb}}(\kappa) \leq \frac{1}{n} C_n^{\text{fb}}(\kappa) \leq \frac{1}{n} C_n^{\text{fb}}(\kappa) + \frac{1}{2}.$$

(IV.225)

To this date no closed formulas are available in the literature for $C_n(\kappa)$ and $C_n^{\text{fb}}(\kappa)$ for unstable or unstable noise. Bounds (IV.225) are analyzed extensively in the literature [7], [23]–[28] (and references therein), under stationarity or asymptotic stationarity.

**B. Observations on the Feedback Codes and Optimal Channel Inputs of Past Literature**

In this section, we make some observations on the optimal channel inputs, that are used in [8], and applied in [1] to derive the main theorem [1, Theorem 6.1]. From these observations then follows that neither [1, Theorem 6.1] (and also [1, Theorem 4.1]) nor [9]–[12], characterize the feedback capacity for the Cover and Pombra [6] formulation of feedback code (with limit and supremum operations interchanged) of stationary channels.
Optimal channel inputs in [1], [8]–[12]

1. In [8] the information structure of optimal channel inputs with feedback are derived for AGN channels driven by stationary autoregressive noise, described by a power spectral density (PSD) function $S_V(e^{j\omega}),\omega \in [-\pi,\pi]$ [8, eqn(7)]:

$$S_V(e^{j\omega}) \triangleq K_W \frac{(1 - \sum_{k=1}^{L} a(k)e^{j\omega}) (1 - \sum_{k=1}^{L} c(k)e^{-j\omega})}{(1 - \sum_{k=1}^{L} c(k)e^{j\omega}) (1 - \sum_{k=1}^{L} c(k)e^{-j\omega})}, \quad |c(k)| < 1, \quad |a(k)| \leq 1. \quad (IV.226)$$

In [8, Section II.C, eqn(19), eqn(20)], the noise is represented by a particular state space realization, with state variables $S^n \triangleq (S_1,\ldots,S_n)$, under the following crucial assumption.

(A1) [8, page 933, I)-III]: given the initial state of the noise $S_0 = s_0$, which is known to the encoder and the decoder, the channel input $X^n \triangleq (X_1,\ldots,X_n)$ uniquely defines the state variables $S^n$ and vice-versa.

Then, under Assumption (A1), in [8, Section II-V], the optimal channel input $X^n$ is expressed in terms of the state $S^n$, its initial state $S_0 = s_0$, the outputs $Y^n$, and a Gaussian innovations process (see [8, Theorem 5, eqn(141)]). The $n$–FTFI capacity is characterized by [8, eqn(131)].

2. In [1, Theorem 6.1] the characterization of feedback capacity is derived by invoking an equivalent state space representation of the noise (IV.226), given by [1, eqn(58)], i.e.,

$$S_{t+1} = FS_t + GW_t, \quad S_0 = 0, W_0 = 0, \quad t = 0,\ldots, \quad (IV.227)$$

$$V_t = HS_t + GW_t, \quad t = 1,\ldots, \quad (IV.228)$$

where $W_t, t = 1,\ldots,$ are independent, Gaussian, independent of the Gaussian RV $S_0$, and $F,G,H$ are matrices of dimensions $L \times L, L \times 1, 1 \times L$, respectively. $S_t$ is called the state of the noise.

Although, not explicitly stated in [1], from [1, Lemma 6.1] (which is used to derive [1, Theorem 6.1]), it follows a crucial assumption, which is analogous to (A1) (see also [1, page 78, left column, first paragraph]):

(A2) [1, Theorem 6.1 and Lemma 6.1]: Given the initial state of the noise $S_0 = s_0, W_0 = w_0$, which is known to the encoder and the decoder, the noise $V^n$ uniquely defines the state variables $S^n$ and vice-versa.

3. In Cover and Pombra [6] the code definition and optimal channel input process $X^n$, i.e., (IV.217), are not allowed to depend on the state variables $S^n$ and the initial states, i.e., assumptions (A1) or (A2) are not imposed, and hence $S^n$ needs to be estimated at the encoder. Because assumptions (A1) and (A2) are imposed in the feedback capacity problem analyzed in [8] and characterization of feedback capacity in [1, Theorem 6.1] (see also [1, Lemma 6.1]), respectively, then [8] and [1, Theorem 6.1] treat a fundamentally different and much easier problem than the Cover and Pombra [6].

Papers [9]–[12] should be read with caution, because the misinterpretations discussed above are repeated. For example, [9], [10], [12] developed their results based on [1, Theorem 4.1] which is equivalent to

If (A1) does not hold then the characterization of the $n$–FTFI capacity [8, Sections II-V] is not valid.
C. Preliminary Observations on Convergence of Feedback Rates

In this section we make preliminary observation on conditions for existence of the limits of the feedback capacity definition (I.12), which are not accounted for in [1, Theorem 6.1] (equivalently in [1, Theorem 4.1]). These are also overlooked in [9]–[12], when use is made of the results of [1]. For the stationary AR(c) noise, the feedback capacity characterizations given in [1, Theorem 6.1], and similarly in [1, Theorem 4.1], correspond to $\mathcal{K}_\infty = 0$, (see [1, Lemma 6.1] and comments above it), which means the stabilizability condition (see Definition III.3) is not satisfied.

Observation IV.2. On the feedback capacity characterizations given in [1]

(1) Observe that the characterization of feedback capacity given in [1, Theorem 6.1] for the state space representation of the noise (IV.227), (IV.228), with zero variance of the innovations of the channel input process, implies one solution to the symmetric matrix algebraic Riccati equation (ARE) [1, Eqn(61)] satisfied by $\Sigma$, is always the zero matrix solution $\Sigma = 0$.

An important technical issue, which is overlooked in the treatment of the asymptotic analysis in [1], is the fact that, the estimation error covariance$^{11}$ of the noise $V^n$ satisfies a generalized difference Riccati equation$^{12}$ (DRE) [14], and not a classical DRE (see Section III). It then follows from the theory of generalized DREs and AREs [13], [14], (see properties of Theorem III.1), that even for the stable and stationary channel noise considered in [1], to ensure asymptotic stationarity of the optimal joint channel input and output process, it is necessary that the conditions, known as detectability and stabilizability conditions should be appended to the definition of characterization of feedback capacity$^{13}$ in [1, Theorem 6.1]. When these conditions are appended, then the feedback capacity is fundamentally different from the one given in [1, Theorem 6.1].

Two special cases are discussed below.

$^{11}$\Sigma in the notation of [1].

$^{12}$Generalized DREs and algebraic Riccati equations (ARE) are well-known in filtering problems of Gaussian systems, where the state to be estimated is driven by a Gaussian noise, which is correlated with the Gaussian noise that drives the observed process, i.e., the process that is used to estimate the state process [13], [14].

$^{13}$It appears [1] did not recognize that stability or stationarity of the noise is not sufficient for detectability and stabilizability to hold, as in classical difference and algebraic equations. This is also not accounted for in [11].
Case 1. Consider the asymptotically stationary AR(c) noise. From the generalized ARE (I.32) of $K^\infty_\infty$, with a zero variance of the innovations of the channel input, $K^\infty_Z = 0$, then it follows that $(K^\infty_\infty, \Lambda^\infty_\infty)$ is the analog of $(\Sigma, X)$ in [1, Theorem 6.1]. The equation of $K^\infty_\infty$, with $K^\infty_Z = 0$, is a quadratic polynomial, with two roots, one at $K^\infty_\infty = 0$, and another at $K^\infty_\infty = K^\infty_\infty(\Lambda^\infty_\infty)$, given in (II.81)-(II.83) (by simple algebra).

It then follows from the general properties of generalized matrix difference Riccati equations [14] (see Theorem III.1, Lemma III.2 for its application to the AR(c) noise), that the asymptotic limit of the mean-square estimation error converges to the zero solution $\Sigma \equiv K^\infty_\infty = 0$, and not the maximal solution. This then implies, the feedback capacity based on [1, Theorem 6.1, $C_{FB}$] is zero.

Case 2. Consider the PSD (IV.226), with $L = 1$, denoted by AR(a,c), given in [1, eqn(43)]:

$$S_V(e^{j\omega}) \triangleq K_W \frac{(1 - ae^{j\omega})(1 - ae^{-j\omega})}{(1 - ce^{j\omega})(1 - ce^{-j\omega})}, \quad |c| < 1, \quad |a| \leq 1, \quad K_W = 1.$$ (IV.229)

In [1, above and below eqn(43)], the following state space representation is introduced.

$$S_{t+1} = cS_t + W_t, \quad S_0 = W_0 = 0, \quad t = 0, \ldots,$$ (IV.230)

$$V_t = (c - a)S_t + W_t, \quad t = 1, \ldots,$$ (IV.231)

Similarly to Case 1, the ARE [1, Theorem 6.1, eqn(61)] satisfied by $\Sigma$, is a quadratic polynomial in $\Sigma$, with two solutions, the zero solution $\Sigma = 0$, and a nonzero solution $\Sigma \neq 0$ (one root is always zero because $K^\infty_Z = 0$). It is easy to verify that the choice of the nonzero solution gives directly the statement of [1, Theorem 5.3] (which is derived using a lengthy derivation from the frequency domain characterization of [1, Theorem 4.1]).

However, similarly to (1) it follows from the general properties of generalized matrix difference Riccati equations [14], that the asymptotic limit of the mean-square estimation error converges to the zero solution $\Sigma \equiv K^\infty_\infty = 0$, and not the nonzero solution, which then implies the feedback capacity based on [1, Theorem 6.1, $C_{FB}$] is zero.

The above technical oversights are also discussed in Remark III.1, and follow from Section II-A, Facts 1-3. We also give an alternative derivation in Section IV-D (see Counterexample IV.1).

Papers [9]–[12] should be read with caution, because the technical issues discussed above are repeated.

D. The Kim Characterization of Feedback Capacity

Let us now recall the characterization of feedback capacity given in [1, Theorem 6.1] for the stable AR(c) noise, with $V_0 = v_0$, known to the encoder and the decoder, which presupposes the variance of the innovations of the channel input process is zero, i.e., $K^\infty_Z = 0$ in the generalized ARE of Theorem III.2.

The Feedback Capacity for the stable AR(c), $c \in (-1, 1)$ Noise [1]. From [1, Theorem 6.1], under the stationarity or asymptotic stationarity of the input and output process, the following is obtained.
The characterization of feedback capacity for the stable AR$(c)$, $c \in (-1,1)$ noise, is given by\footnote{The reader may verify that characterization (IV.232)-(IV.236), with $K_W = 1$, is a degenerate version of [1, Theorem 6].} [1, Theorem 6.1, eqn(61)]

$$C^K(\kappa) \triangleq \max_{\Lambda^\infty} \log \left( \frac{(\Lambda^\infty + c)^2 K^\infty + K_W}{K_W} \right)$$ \hspace{1cm} (IV.232)

$$\Lambda^\infty \in \mathbb{R} \text{ such that } |\Lambda^\infty| \neq 1,$$ \hspace{1cm} (IV.233)

$$c^2 K^\infty \leq \kappa,$$ \hspace{1cm} (IV.234)

$K^\infty$ is the maximal solution of the ARE,

$$K^\infty = c^2 K^\infty + K_W - \left( \frac{K_W + cK^\infty(\Lambda^\infty + c)}{K^\infty + (\Lambda^\infty + c)^2 K^\infty} \right)^2, \quad |c| < 1.$$ \hspace{1cm} (IV.235)

Note that (IV.236) corresponds to the generalized ARE of Theorem III.2 with $K_Z^\infty = 0$. Condition (IV.233) is known as unit circle controllability [14], (see Definition III.3.(b)). Fact 3, (II.81)-(II.83) give all possible solutions of (IV.236). By solving (IV.236) we obtain two solutions, $K^\infty = 0$ and $K^\infty = \frac{K_W(\Lambda^\infty)^2 - 1}{(\Lambda^\infty + c)^2}, |\Lambda^\infty| \neq 1$. Note, that by Lemma III.2.(3), the stabilizability condition is satisfied if and only if $|\Lambda^\infty| < 1$, and hence stabilizability implies the only nonnegative solution is $K^\infty = 0$. Hence, the solution that correspond to the per unit time limit (III.147) is $K^\infty = 0$, i.e., $|\Lambda^\infty| < 1$, which then implies $C^K(\kappa) = 0, \forall \kappa \in [0,\infty)$.}

(K2) In [1, page 58, second column, last paragraph], the solution of the optimization Problem (IV.232)-(IV.236), for the AR$(c)$ noise, described by the power spectral density, $S_V(e^{j\omega}) = \frac{1}{|1+ce^{j\omega}|^2}, -1 < c < 1$, i.e., for the an AR$(c)$ stationary noise with $K_W = 1$, is given as follows.

$$C^K(\kappa) = -\log x_0,$$ \hspace{1cm} (IV.237)

where $x_0$ is the unique positive root of $\kappa x^2 = \frac{(1-x^2)}{(1+|x|)}$. \hspace{1cm} (IV.238)

The (incorrect) simplified characterizations of the feedback capacity i.e., with zero power spectral density and zero variance of the innovations process of the channel input, of [1, Theorem 4.1] and [1, Theorem 6.1], respectively, are utilized extensively in the literature, and stimulated the interest into additional investigations, by many authors, for example, [9]–[12].

The next proposition summarizes the implications of the properties of generalized DREs and AREs of Theorem III.1, and Lemma III.2, on the characterization of feedback capacity given in [1, Theorem 4.1 and Theorem 6.1].

**Proposition IV.1.** On [1, Theorem 6.1 and Lemma 6.1] for stable AR$(c)$ noise

Consider the characterization of [1, Theorem 6] for stable AR$(c)$ noise, which is equivalent to the
optimization problem (IV.232)-(IV.236).

(a) There are two solutions to the optimization problem (IV.232)-(IV.236), listed under Solutions # 1, #2. Solution #1:

\[
K^\infty = \frac{K_W \left( (\Lambda^\infty)^2 - 1 \right)}{\left( \Lambda^\infty + c \right)^2}, \quad |\Lambda^\infty| > 1, \quad \Lambda^\infty \neq -c,
\]

(IV.239)

\[
(\Lambda^\infty)^2 K^\infty = \kappa \implies K_W (\Lambda^\infty)^4 - (K_W + \kappa) (\Lambda^\infty)^2 - 2c \kappa \Lambda^\infty - c^2 \kappa = 0, \quad |\Lambda^\infty| > 1
\]

(IV.240)

\[
C(\kappa) = \log |\Lambda^\infty|,
\]

(IV.241)

the pair \(\{A,C\}\) is detectable,

(IV.242)

the pair \(\{A^*,B^*\}\) is unit circle controllable, which is equivalent to (IV.233),

(IV.243)

the pair \(\{A^*,B^*\}\) is not stabilizable.

(IV.244)

In particular, for \(K_W = 1\) then \(x = \frac{1}{\kappa^2}\) satisfies (IV.238), and (IV.237) and (IV.238) are obtained from (IV.239)-(IV.241), i.e., letting \(K_W = 1\).

Solution #2:

\[
K^\infty = 0, \quad |\Lambda^\infty| < 1,
\]

(IV.245)

\[
(\Lambda^\infty)^2 K^\infty = 0 \leq \kappa, \quad C^K(\kappa) = 0, \quad \forall \kappa \in [0,\infty),
\]

(IV.246)

the pair \(\{A,C\}\) is detectable,

(IV.247)

the pair \(\{A^*,B^*\}\) is stabilizable.

(IV.248)

(b) The asymptotic per unit time limit, as defined by (III.136), converges to the characterization (IV.232)-(IV.236) if and only if \(|\Lambda^\infty| < 1\), and Solution #2 is the unique limit.

Proof. The statements follow from Theorem III.1 and Lemma III.2.

To gain additional insight, we construct a counterexample to [1, Lemma 6.1], which states that the variance of the innovations of the channel input process is optimal, i.e., \(K^\infty_Z = 0\), that led to the incorrect characterization of feedback capacity [1, Theorem 6.1] (and also to [1, Theorem 4.1]), by calculating the transition map of the error recursion of the generalized Kalman-filter. An alternative approach to construct a counterexample, is to follow the statements prior to [14, Lemma 14.2.1, page 507], for the generalized DRE and ARE, using its transform version, as discussed in [14, Section 14.7, page 540].

Counterexample IV.1. On [1, Theorem 6.1 and Lemma 6.1]

Our objective is to evaluate \(C^K(\kappa)\), defined by (IV.232)-(IV.236), from (III.136), by first calculating the transition map of the error recursion of the generalized Kalman-filter.
Let $K_0^o, K_1^o, \ldots, K_n^o$ denote the sequence generated from the generalized DRE (III.130), with $K_Z^\infty = 0$, as stated in [1, Theorem 6], i.e.,

$$K_t^o = c^2 K_{t-1}^o + K_W - \frac{(K_W + c K_{t-1}^o (\Lambda^\infty + c))^2}{(K_W + (\Lambda^\infty + c) K_{t-1}^o)^2}, \quad K_t^o \geq 0, \quad K_0^o = 0, \quad t = 1, \ldots, n, \quad (IV.249)$$

We shall compute the solution of (IV.249) recursively, to determine whether the transition map of the error recursion (III.131) converges, i.e., whether it is stabilizing, by checking:

$$\lim_{n \to \infty} |F(K_{t-1}^o, \Lambda^\infty, K_Z^\infty = 0)| = \lim_{n \to \infty} |c - M(K_{t-1}^o, \Lambda^\infty, K_Z^\infty = 0)(\Lambda^\infty + c)| < 1. \quad (IV.250)$$

(i) By the generalized DRE (IV.249), at time $t = 1$, we have

$$K_1^o = 0 \implies F(K_0^o, \Lambda^\infty, K_Z^\infty = 0) = c - \Lambda^\infty - c = -\Lambda^\infty. \quad (IV.251)$$

At time $t = 2$, we have

$$K_2^o = 0 \implies F(K_1^o, \Lambda^\infty, K_Z^\infty = 0) = -\Lambda^\infty. \quad (IV.252)$$

For all $t \in \{3, 4, \ldots, n\}$, by recursive calculations, we have

$$K_t^o = 0 \implies F(K_{t-1}^o, \Lambda^\infty, K_Z^\infty = 0) = -\Lambda^\infty, \quad t = 2, 3, \ldots, n. \quad (IV.253)$$

Clearly, a necessary condition for the transition map of the error recursion to convergence is

$$|F(K_{n-1}^o, \Lambda^\infty, K_Z^\infty)| = | -\Lambda^\infty | < 1, \quad n = 0, 1, \ldots, \quad (IV.254)$$

Thus, the necessity of $|\Lambda^\infty | < 1$ implies Proposition IV.1, Solution #2 is the valid solution, re-confirming our earlier discussions. Hence, the unique, nonnegative solution of (IV.249) is $K_t^o = 0, n = 0, 1, \ldots$. It then follows that $\Lambda^\infty K_n^o = 0, n = 0, 1, \ldots$. Then for any finite $n$, by (III.136) with $K_Z^\infty = 0$, we have

$$\frac{1}{n} \mathbf{E}_0^o \left\{ \sum_{t=1}^{n} (X_t^o)^2 \right\} = \frac{1}{n} \sum_{t=1}^{n} (\Lambda^\infty)^2 K_{t-1}^o = 0, \quad (IV.255)$$

$$\frac{1}{2n} \sum_{t=1}^{n} \log \left( \frac{(\Lambda^\infty + c)^2 K_{t-1}^o + K_W}{K_W} \right) = 0. \quad (IV.256)$$

The last two equations imply that the valid solution of (IV.232)-(IV.236), i.e., that corresponds to (III.136), when $K_Z^\infty = 0$, is given by $C^K(\kappa) = 0, \forall \kappa \in [0, \infty)$.

(ii) It is easy to verify that the calculations in (i) are also predicted from the properties of the generalized DREs and AREs.

V. Conclusion

The $n$–finite transmission feedback information (FTFI) capacity for additive Gaussian noise (AGN) channels with feedback, is characterized, and lower bounds on the characterization of the $n$–finite transmission without feedback information (FTwFI) capacity are derived, when the noise is described by stable and unstable autoregressive models. Closed form feedback capacity formulas are derived,
when channel input strategies or distributions are time-invariant, for autoregressive memory one, stable and unstable noises, It is shown that feedback does not increase capacity, when the noise is stable and for certain unstable noise. Lower bounds on the non-feedback capacity are also derived, based on Markov channel input distributions, i.e., induced by a Gaussian Markov channel input process, and also by an independent and identically distributed channel input process. These achievable lower bounds on non-feedback capacity hold for any autoregressive noise model, irrespective of whether it is stable or unstable. An interesting long standing unanswered question, is to derive a closed form expression for the $n-$FTFI capacity, to gain insight into the performance of achievable time-varying rates.

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VI. Appendix

A. proof of Lemma III.2

Recall (III.138) and (III.139). (1) By definition $A = c, C = \Lambda + c$. By $c \in (-1, 1)$, then there always exists a $G \in \mathbb{R}$ such that $|A - GC| = |c - G(\Lambda + c)| < 1$, i.e., take $G = 0$. This shows (1).

(2) Since $K_Z = 0$, then $B = 1 - K_W(K_Z + K_W)^{-1} = 0, B^* = 0$, $A^* = c - K_W(K_Z + K_W)^{-1}C = c - \Lambda - c = -\Lambda$, and hence, there exists a $G \in \mathbb{R}$ such that $|A^* - B^*G| = |\Lambda| \neq 1$ if and only if $|\Lambda| \neq 1$. This shows (2).

(3) Since $K_Z = 0$, similar to the prove in (2), there exists a $G \in \mathbb{R}$ such that $|A^* - B^*G| = |\Lambda| < 1$ if and only if $|\Lambda| < 1$. This shows (3).

(4) Since $c \in (-1, 1), K_Z = 0$, then, by (1), the pair $\{A, C\}$ is detectable, by (2) the pair $\{A^*, B^*\}$ is unit circle controllable if and only if $|\Lambda| \neq 1$, and by (3) the pair $\{A^*, B^*\}$ is stabilizable if and only if $|\Lambda| < 1$. By Theorem III.1.(1) we deduce the claim. This shows (4).

(5) Clearly, (III.145) is equivalent to the quadratic equation

$$K^2(\Lambda + c)^2 + K(K_W - K_W^2A^2) = 0. \quad (VI.257)$$

Hence, the two solutions are (III.146). The last statement is also obtained by applying Theorem III.1.(2), as follows. By (1), $\{A, C\}$ is detectable, by (2), $\{A^*, B^*\}$ is unit circle controllable if and only if $|\Lambda| \neq 1$, and by (3), $\{A^*, B^*\}$ is stabilizable if and only if $|\Lambda| < 1$. By invoking Theorem III.1.(2), then the last statement follows. On the other hand, it is easily verified from (III.146), that the uniqueness of solutions $K \geq 0$ holds if and only if $|\Lambda| < 1$, because for $|\Lambda| > 1$ there are two non-negative solutions. By (III.140) and (III.141), evaluated at $K_Z = 0, K = 0$, we have $F(K, \Lambda, K_Z)\big|_{K_Z=0, K=0} = -\Lambda$. Hence, the non-negative solution $K = 0$ is unique and stabilizable if and only if $|\Lambda| < 1$.

B. Proof of Lemma III.3

(i) The conditions are a consequence of the optimization problem, while (III.164) is due to the following standard relaxation. For any $(\Lambda^\infty, K_Z^\infty) \in \mathcal{P}(\kappa)$, then detectability and stabilizability hold, and by prop-
erties of time-invariant generalized DRE \cite{13,14}, the corresponding sequence $K^n_m, n = 1, 2, \ldots, K^0_0 = 0$
generated by the generalized DRE (III.130) is non-decreasing, i.e., $K^n_m \leq K^m_n, m \geq n$ and also bounded.

Hence, $K^n_{n+1} = c^2 K^n_m + K_W - \left( \frac{W + c n}{l_K + W + (\Lambda_{\infty} + c)^2} \right)^2$, and by taking the limit, as $n \to \infty$, the inequality (III.164) is obtained.

(ii) Suppose $K_{Z}^{\infty, *} = 0$. By (III.164), with $K_{Z}^{\infty, *} = 0$, then

$$
\left( K_{\infty}^{\infty, *} - c^2 K_{\infty}^{\infty, *} - K_W \right) \left( K_W + (\Lambda_{\infty} + c)^2 K_{\infty}^{\infty, *} \right) + \left( K_W + c K_{\infty}^{\infty, *} (\Lambda_{\infty} + c) \right)^2 \leq 0, \quad (VI.258)
$$

$$
K_{\infty}^{\infty, *} \left( (\Lambda_{\infty} + c)^2 + K_W \left( 1 - (\Lambda_{\infty} + \pi)^2 \right) \right) \leq 0. \quad (VI.259)
$$

By Lemma III.2, a necessary and sufficient condition for stabilizability of the pair $\{A, B^\frac{1}{\lambda}\}$, when $K_{Z}^{\infty} = 0$, is $|\Lambda_{\infty} + c| < 1$, and therefore (VI.259) is satisfied if and only if $K_{Z}^{\infty, *}$ lies in the region

$$
K_W \left( (\Lambda_{\infty} + c)^2 - 1 \right) \leq K_{\infty}^{\infty, *} \leq 0, \quad \text{for} \quad |\Lambda_{\infty} + c| < 1. \quad (VI.260)
$$

Since it must be that $K_{Z}^{\infty, *} \geq 0$ then necessarily $K_{Z}^{\infty, *} = 0$, which implies $C^{\infty}(\pi) = 0, \forall \pi \in [0, \infty)$. Similarly, if $K_{Z}^{\infty, *} = 0$ then necessarily $K_{Z}^{\infty, *} = 0$, and hence $C^{\infty}(\pi) = 0, \forall \pi \in [0, \infty)$.

(iii) By the stationarity conditions (III.158)-(III.160), with $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$:

$$
\frac{\partial}{\partial K_{Z}^{\infty, *}} \mathcal{L}(A^{\infty, *}, K_{Z}^{\infty, *}, K_{\infty}^{\infty, *}, \lambda) = 1 - \lambda_1^* \left( K_{\infty}^{\infty, *} - c^2 K_{\infty}^{\infty, *} - K_W \right) - \lambda_2^* + \lambda_4^* = 0, \quad (VI.261)
$$

$$
\frac{\partial}{\partial \Lambda_{\infty}^{\infty, *}} \mathcal{L}(A^{\infty, *}, K_{Z}^{\infty, *}, K_{\infty}^{\infty, *}, \lambda) = \left( \Lambda_{\infty}^{\infty, *} + c \right) K_{\infty}^{\infty, *} - \lambda_1^* \left( \left( K_{\infty}^{\infty, *} - c^2 K_{\infty}^{\infty, *} - K_W \right) (\Lambda_{\infty}^{\infty, *} + c) \right) K_{\infty}^{\infty, *}
+ \left( K_W + c K_{\infty}^{\infty, *} (\Lambda_{\infty}^{\infty, *} + c) \right) c K_{\infty}^{\infty, *} - \lambda_2^* \Lambda_{\infty}^{\infty, *} K_{\infty}^{\infty, *} = 0, \quad (VI.262)
$$

$$
\frac{\partial}{\partial K_{\infty}^{\infty, *}} \mathcal{L}(A^{\infty, *}, K_{Z}^{\infty, *}, K_{\infty}^{\infty, *}, \lambda) = \left( \Lambda_{\infty}^{\infty, *} + c \right)^2 - \lambda_1^* \left( 1 - c^2 \right) \left( K_{Z}^{\infty} + K_W + (\Lambda_{\infty}^{\infty, *} + c)^2 K_{\infty}^{\infty, *} \right)
+ \left( K_{\infty}^{\infty, *} - c^2 K_{\infty}^{\infty, *} - K_W \right) (\Lambda_{\infty}^{\infty, *} + c)^2 + 2c \left( K_W + c K_{\infty}^{\infty, *} (\Lambda_{\infty}^{\infty, *} + c) \right) \left( \Lambda_{\infty}^{\infty, *} + c \right)
- \lambda_2^* (\Lambda_{\infty}^{\infty, *})^2 + \lambda_3^* = 0, \quad (VI.263)
$$

Suppose $\lambda_1^* \neq 0$. Then, by complementary slackness (III.161), we have $\lambda_4^* K_{Z}^{\infty, *} = 0$, which implies $K_{Z}^{\infty, *} = 0$, and hence by (ii), $K_{\infty}^{\infty, *} = 0$. By complementary slackness (III.161), we also have $\lambda_2^* ((\Lambda_{\infty}^{\infty, *})^2 + K_{Z}^{\infty, *}) = \lambda_2^*(0-\pi) = 0$, hence $\lambda_2^* = 0$. By (ii) it follows that $C^{\infty}(\pi) = 0, \forall \pi \in [0, \infty)$, hence the rate is zero. Similarly, if $\lambda_4^* \neq 0$ then $K_{Z}^{\infty, *} = 0$ and $K_{\infty}^{\infty, *}$ = 0, which lead to a zero rate. However, it can be verified (see Theorem III.3), that for $\Lambda_{\infty}^{\infty, *} = 0, K_{Z}^{\infty, *} \neq 0$, we exhibit a non-zero rate, which is a lower bound on the non-feedback rate. Next, consider the case $\lambda_1^* = 0, \lambda_4^* = 0$ and $\lambda_3^* = 0$. Then, by (VI.261) $\lambda_2^* = 1$. However, when $\lambda_1^* = 1$ equalities (VI.262) and (VI.263) hold only if $c = 0$. Since, $c \neq 0$, otherwise the channel is memoryless, then the only choice is $\lambda_1^* > 0$. Moreover, since $\lambda_1^* > 0$, then in order to satisfy the complementary slackness condition (III.164), then inequality must hold with equality, i.e.,

$$
\left( K_{\infty}^{\infty, *} - c^2 K_{\infty}^{\infty, *} - K_W \right) (K_{Z}^{\infty, *} + K_W + (\Lambda_{\infty}^{\infty, *} + c)^2 K_{\infty}^{\infty, *}) + \left( K_W + c K_{\infty}^{\infty, *} (\Lambda_{\infty}^{\infty, *} + c) \right)^2 = 0. \quad (VI.264)
$$
Finally we consider the case \( \lambda_4^* = 0, \lambda_4^* = 0 \) and \( \lambda_2^* = 0 \). Solving the system of equations (III.161), (VI.261)-(VI.264), the following sets of solutions are obtained.

The first solution is

\[
K_{\infty,*} = \frac{K_W + 1}{1 - c^2}, \quad \Lambda_{\infty,*} = -\frac{c^2 + K_W}{c(K_W + 1)}, \quad K_Z^{\infty,*} = -\frac{K_W(c^2 + K_W)}{c^2(K_W + 1)}, \quad \Lambda_1^* = 1.
\]  

(VI.265)

The second solution is

\[
K_{\infty,*} = 0, \quad \Lambda_{\infty,*} = -\frac{c^2 + 1}{2c}, \quad K_Z^{\infty,*} = 0, \quad \Lambda_1^* = -\frac{1}{K_W}.
\]  

(VI.266)

The first solution is discarded since for \( K_W > 0 \), then, \( K_Z^{\infty,*} < 0 \), while the second solution is discarded due to (ii). Therefore \( \lambda_2^* \neq 0 \), which by the complementary slackness condition (III.161) implies that

\[
(\Lambda_{\infty,*})^2 K_{\infty,*} + K_Z^{\infty,*} = \kappa
\]  

(VI.267)

Thus, we have shown that a necessary condition for existence of \( \kappa \in (0, \infty) \) such that \( C_0(\kappa) > 0 \) is \( \lambda_1^* > 0 \), \( \lambda_2^* > 0 \), \( \lambda_3^* = 0 \) and \( \lambda_4^* = 0 \). This completes the proof.

C. proof of the stability condition of Theorem III.3

From (III.174) and (III.175), we have

\[
F^{nf}b(K_\infty, K_Z^{\infty,*}) \triangleq c - M^{nf}b(K_\infty, K_Z^{\infty,*})c = c\left(1 - (K_W + c^2 K_\infty)\left(\kappa + K_W + c^2 K_\infty\right)^{-1}\right).
\]  

(VI.268)

We begin the evaluation for the stable case, i.e., \( |c| < 1 \). Then,

\[
|F^{nf}b(K_\infty, K_Z^{\infty,*})| = |c| \left| \left(\kappa + K_W + c^2 K_\infty\right)^{-1}\right|^{(a)} < |c| < 1,
\]  

(VI.269)

where (a) holds since \( \kappa \geq 0 \), \( K_W > 0 \) and \( K_Z^{\infty,*} > 0 \). Thus, the stability condition holds for \( |c| < 1 \). For the unstable case, i.e., \( |c| > 1 \), by substituting \( K_Z^{\infty,*} \), we have the following

\[
|F^{nf}b(K_Z^{\infty,*})| = \left| \frac{2c \kappa}{\kappa + K_W + c^2 \kappa + \sqrt{(K_W + \kappa(c^2 - 1))^2 + 4c^2 \kappa K_W}} \right|
\]  

(VI.270)

\[
\leq \left| \frac{2c \kappa}{\kappa + K_W + c^2 \kappa + \sqrt{(K_W + \kappa(c^2 - 1))^2}} \right|
\]  

(VI.271)

\[
\leq \left| \frac{2c \kappa}{\kappa + K_W + c^2 \kappa + \sqrt{(K_W + \kappa(c^2 - 1))^2}} \right| = \left| \frac{2c \kappa}{K_W + 2c^2 \kappa} \right| \leq \frac{|2c \kappa|}{|2c^2 \kappa|} = \frac{1}{|c|} < 1,
\]  

(VI.272)

where (b)-(d) hold since \( \kappa \geq 0 \) and \( K_W > 0 \). Thus, the stability condition also holds for \( |c| > 1 \). The case \( |c| = 1 \) follows from the equality in (VI.269).

D. proof of Theorem III.4.

We prove the statements in several steps.

(i) First, we recall Lemma III.3.(iv) that states if \( K_Z^{\infty,*} = 0 \) or \( K_Z^{\infty,*} = 0 \) the \( C_0(\kappa) = 0, \forall \kappa \in [0, \infty) \). However, by Theorem III.3, if we restrict our optimization to \( \Lambda_{\infty,*} = 0 \), and \( K_Z^{\infty,*} \neq 0 \), then we exhibit a non-zero rate without feedback, which is a lower bound on the non-feedback rate. This shows that the
only choice for non-zero rate with feedback is to exists a \( \kappa \in (0, \infty) \) such that \( K_1^{\infty,*} > 0, K_2^{\infty,*} > 0 \). If such \( \kappa \) does not exists, then, we can exhibit a non-zero rate without feedback by considering time-invariant channel inputs strategies without feedback, \( \mathbf{P}_{X_t^{n}|X_{t-1},Y_0} = \mathbf{P}^{x_n}(dx_t|x_{t-1},v_0), t = 1, \ldots, n \).

(ii) Second, we recall Lemma III.3.(v), which states a necessary condition for existence of a non-zero feedback rate for some \( \kappa \in (0, \infty) \) is \( \lambda_1^* > 0, \lambda_2^* > 0, \lambda_3^* = 0 \) and \( \lambda_4^* = 0 \). For the rest of the derivation we characterize the set of all values \( \kappa \in \mathcal{K}^\infty(c,K_W) \) if such exist, and treat the case when \( \mathcal{K}^\infty(c,K_W) \) is empty separately, using non-feedback channel inputs.

(iii) Consider \( \lambda_1^* > 0, \lambda_2^* > 0, \lambda_3^* = 0 \) and \( \lambda_4^* = 0 \). We solve the system of equations (VI.261)-(VI.263), (VI.264) and (VI.267). First we solve the system of equations (VI.261) and (VI.262) to obtain \( \lambda_1^* \) and \( \lambda_2^* \) as a function of \( \{K_1^{\infty,*}, K_2^{\infty,*}, \Lambda^{\infty,*}\} \). By substituting \( \lambda_1^*, \lambda_2^* \) and \( K_2^{\infty,*} \) from (VI.267) in (VI.263), we obtain (III.186). Finally, by substituting \( K_2^{\infty,*} \) and \( \Lambda^{\infty,*} \) in (VI.264) we deduce the quadratic equation (III.189). The two solutions of the the quadratic equation (III.189) give rise to the following two solutions.

The first solution is

\[
K_1^{\infty,*} = K_1^{\infty,*} = \frac{\kappa \left( c^2 - 1 \right)^2 - K_W}{c^2 (c^2 - 1)}, \tag{VI.273}
\]

\[
\Lambda_1^{\infty,*} = \Lambda_1^{\infty,*} = \frac{c K_W}{\kappa (c^2 - 1)^2 - K_W}, \tag{VI.274}
\]

\[
K_2^{\infty,*} = K_2^{\infty,*} = \frac{\kappa \left( (\kappa (c^2 - 1)^2 - K_W) (c^2 - 1) \right) - K_W}{(\kappa (c^2 - 1)^2 - K_W) (c^2 - 1)}, \tag{VI.275}
\]

\[
\lambda_1^* = \frac{c^2}{K_W - \kappa (1 - c^2)}, \quad \lambda_2^* = c^2. \tag{VI.276}
\]

The second solution is

\[
K_2^{\infty,*} = K_2^{\infty,*} = \frac{\kappa - K_W}{c^2}, \tag{VI.277}
\]

\[
\Lambda_2^{\infty,*} = \Lambda_2^{\infty,*} = \frac{c \kappa}{K_W - \kappa}, \tag{VI.278}
\]

\[
K_2^{\infty,*} = K_2^{\infty,*} = \frac{\kappa K_W}{K_W - \kappa}, \tag{VI.279}
\]

\[
\lambda_1^* = \frac{c^2}{\kappa - K_W (1 - c^2)}, \quad \lambda_2^* = 0. \tag{VI.280}
\]

Solution \( K_1^{\infty,*} = K_2^{\infty,*}, \Lambda_1^{\infty,*} = \Lambda_2^{\infty,*}, K_2^{\infty,*} = K_2^{\infty,*}, \) \( (\lambda_1^*, \lambda_2^*) \) given by (VI.280), are not valid solutions, because if \( K_1^{\infty,*} = K_2^{\infty,*} > 0 \), then \( K_2^{\infty,*} = K_2^{\infty,*} < 0 \), and vice-versa. Thus, the only valid solution is Solution 1, from which all statements of (1) are obtained. It remains to show the statements under (2) and (3).

(iv) Consider \( c^2 < 1 \) and define the set

\[
\mathcal{A}(c,K_W) \Delta \{ \kappa \in [0, \infty) : K_1^{\infty,*} = K_1^{\infty,*} > 0, K_2^{\infty,*} = K_2^{\infty,*} > 0, c^2 < 1, c \neq 0, \lambda_1^* > 0, \lambda_2^* > 0 \}. \tag{VI.281}
\]
Similarly, define the sets for $c^2 > 1$

$$\mathcal{A}_2(c, K_w) \triangleq \left\{ \kappa \in [0, \infty) : K_{11}^{\infty,*} = K_{11}^{\infty,*} > 0, K_{22}^{\infty,*} = K_{22}^{\infty,*} > 0, c^2 > 1, c \neq 0, \lambda_1^* > 0, \lambda_2^* > 0 \right\}$$ (VI.282)

and the set for $c^2 = 1$

$$\mathcal{A}_3(c, K_w) \triangleq \left\{ \kappa \in [0, \infty) : K_{11}^{\infty,*} = K_{11}^{\infty,*} > 0, K_{22}^{\infty,*} = K_{22}^{\infty,*} > 0, c^2 = 1, c \neq 0, \lambda_1^* > 0, \lambda_2^* > 0 \right\}$$ (VI.283)

The proof is then completed by determining the values of $\kappa \in \mathcal{A}_1(c, K_w)$ and $\kappa \in \mathcal{A}_2(c, K_w)$, if such exist. If the set is empty then, we need to consider time-invariant channel input strategies without feedback; such strategies always exists, in view of the lower bound of Theorem III.3.

(iv.1) For $c^2 < 1$, we have the following

$$K_{11}^{\infty,*} = K_{11}^{\infty,*} > 0 \implies \kappa < \frac{K_w}{(c^2 - 1)^2},$$ (VI.284)

$$\lambda_1^* > 0 \implies \kappa < \frac{K_w}{1 - c^2},$$ (VI.285)

$$K_{22}^{\infty,*} = K_{22}^{\infty,*} > 0 \implies \kappa^2 \left( c^2 - 1 \right)^3 - \kappa K_w \left( c^2 - 1 \right) - K_w^2 > 0,$$ (VI.286)

$$\implies \frac{K_w - K_w \sqrt{4c^2 - 3}}{2(1-c^2)} < \kappa < \frac{K_w + K_w \sqrt{4c^2 - 3}}{2(1-c^2)}, \quad 4c^2 \geq 3.$$ (VI.287)

Next, we show the set $\mathcal{A}_1(c, K_w)$ is empty. For $4c^2 < 3$, $\kappa^2 \left( c^2 - 1 \right)^3 - \kappa K_w \left( c^2 - 1 \right) - K_w^2 < 0$, thus $K_{22}^{\infty,*} < 0$. For $4c^2 \geq 3$, and from (VI.285) and (VI.287), it suffices to show that

$$\frac{K_w - K_w \sqrt{4c^2 - 3}}{2(1-c^2)} > \frac{K_w}{1-c^2}, \quad \text{for} \quad 4c^2 \geq 3.$$ (VI.288)

Clearly, after simple algebra, we can show that (VI.288) holds iff $\left( c^2 - 1 \right)^2 > 0$, for $4c^2 \geq 3$, thus it hold for all $c \neq 1$. Therefore, the set defined by $\mathcal{A}_1(c, K_w)$ is empty.

(iv.2) For $c^2 > 1$, we have the following

$$K_{11}^{\infty,*} = K_{11}^{\infty,*} > \implies \kappa > \frac{K_w}{(c^2 - 1)^2},$$ (VI.289)

$$\lambda_1^* > 0 \implies \kappa > \frac{K_w}{(1-c^2)},$$ (VI.290)

$$K_{22}^{\infty,*} = K_{22}^{\infty,*} > 0 \implies \kappa^2 \left( c^2 - 1 \right)^3 - \kappa K_w \left( c^2 - 1 \right) - K_w^2 > 0,$$ (VI.291)

$$\implies \kappa < \frac{K_w - K_w \sqrt{4c^2 - 3}}{2(1-c^2)} \quad \text{or} \quad \kappa > \frac{K_w + K_w \sqrt{4c^2 - 3}}{2(1-c^2)}.$$ (VI.292)

Next, note that for $c^2 > 1$ the following inequalities hold

$$\frac{K_w}{(1-c^2)} < \frac{K_w - K_w \sqrt{4c^2 - 3}}{2(1-c^2)} < 0 < \frac{K_w}{(c^2 - 1)^2} < \frac{K_w + K_w \sqrt{4c^2 - 3}}{2(1-c^2)}.$$ (VI.293)
Then, from (VI.289)-(VI.293), we deduce that the set \( \mathcal{A}_2(c, K_W) \) is non empty, only if the power \( \kappa \) satisfies 
\[
\kappa > \frac{K_W \left( 1 + \sqrt{4c^2-3} \right)}{2 \left( 1-c^2 \right)}.
\]
For values of \( \kappa \leq \frac{K_W \left( 1 + \sqrt{4c^2-3} \right)}{2 \left( 1-c^2 \right)} \), since they do not belong in the set \( \mathcal{A}_2(c, K_W) \), we need to consider time-invariant channel input strategies without feedback.

(iv.3) For \( c^2 = 1 \) clearly the set \( \mathcal{A}_3(c, K_W) \) is empty, thus we need to consider time-invariant channel input strategies without feedback.

Putting all the above together we obtain the statements under (2) and (3). The proof is completed.

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