Uniformization of $p$-adic curves via Higgs-de Rham flows

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Abstract. Let $k$ be an algebraic closure of a finite field of odd characteristic. We prove that for any rank two graded Higgs bundle with maximal Higgs field over a generic hyperbolic curve $X_1$ defined over $k$, there exists a lifting $X$ of the curve to the ring $W(k)$ of Witt vectors as well as a lifting of the Higgs bundle to a periodic Higgs bundle over $X/W$. As a consequence, it gives rise to a two-dimensional absolutely irreducible representation of the arithmetic fundamental group $\pi_1(X_K)$ of the generic fiber of $X$. This curve $X$ and its associated representation is in close relation with the canonical curve and its associated canonical crystalline representation in the $p$-adic Teichmüller theory for curves due to S. Mochizuki. Our result may be viewed as an analogue of the Hitchin-Simpson’s uniformization theory of hyperbolic Riemann surfaces via Higgs bundles.

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1. Introduction

The work of Deninger-Werner [DW] initiated the problem of associating representations of geometric fundamental groups of $p$-adic curves to vector bundles with suitable conditions. Their result can be viewed as a partial analogue of the classical Narasimhan-Seshadri theory of vector bundles on compact Riemann surfaces. Later, Faltings [Fa], using the theory of almost étale extensions in his work in the $p$-adic Hodge theory, obtained a far-reaching generalization. In particular, as a $p$-adic analogue of Simpson’s theory in the nonabelian Hodge theory, Faltings associates generalized representations to Higgs bundles over $p$-adic curves. However, a fundamental question concerning semistability of Higgs bundles remains: Faltings asked whether semistable Higgs bundles of degree zero over a smooth projective $p$-adic curve corresponds to usual (i.e. continuous $\mathbb{C}_p$-) representations of the geometric fundamental group.

Reading the foundational paper [Hi] of Hitchin in the nonabelian Hodge theory, one finds, on the other hand, that even the question that whether a very basic graded Higgs bundle of the following type corresponds a usual representation is unknown:

$$ (E, \theta) := (L \oplus L^{-1}, \theta), $$

where $L$ is a line bundle over $X$ satisfying $L^2 \cong \Omega^1_X$, and $\theta : L \to L^{-1} \otimes \Omega^1_X$ is the tautological isomorphism.

For $X$ being a compact Riemann surface, this is Example 1.5 loc. cit (Example 1.4 ibid belongs to the theory of Narasimhan-Seshadri on stable vector bundles). Such a Higgs bundle is considered as basic because, as shown to us by Hitchin, it gives a uniformization for a hyperbolic $X$. Simpson [Si90] extended the theory to a noncompact Riemann surface by introducing the log version of the previous Higgs bundles, namely

$$ (E, \theta) := (L \oplus L^{-1}, \theta), $$

where $L$ is a line bundle over $X$ satisfying $L^2 \cong \Omega^1_X(\log D)$, and $\theta : L \to L^{-1} \otimes \Omega^1_X(\log D)$ is the tautological isomorphism, where $X$ is smooth projective and $D \subset X$ a divisor. This paper is devoted to study this type of graded (logarithmic) Higgs bundles in the context of the $p$-adic Hitchin-Simpson correspondence established in [LSZ]. For a rank two graded logarithmic Higgs bundle with trivial determinant, the above type is characterized by the Higgs field being an isomorphism and it is said to be with maximal Higgs field. Note that they are typically Higgs stable and are of degree zero. Recall that in the complex case, a Higgs bundle with maximal Higgs field (1.0.1) is used to recover the uniformization theorem of a compact Riemann surface by solving the Yang-Mills-Higgs equation. Hitchin observed that the unique solution to the Yang-Mills-Higgs equation associates to $(E, \theta)$ defined in (1.0.1) over $X$ a polarized $\mathbb{C}$-variation of Hodge structure $(H, \nabla, Fil, \Psi)$ (in fact it carries also a real structure), where $H$ is the underlying $C^\infty$-bundle of $E$ with a new holomorphic
structure, $\nabla$ is an integrable connection

$$\nabla : H \to H \otimes \Omega^1_X,$$

$Fil$ is a Hodge filtration, i.e., a finite decreasing filtration satisfying Griffiths transversality and $\Psi$ is a horizontal bilinear form satisfying the Hodge-Riemann bilinear relation. By taking the grading of the Hodge filtration, one obtains that the associated graded Higgs bundle $Gr_{Fil}(H, \nabla)$ is isomorphic to $(E, \theta)$. Moreover, the classifying map associated to $(H, \nabla, Fil, \Psi)$ is a holomorphic map

$$\pi : \tilde{X} \to \mathcal{H}$$

from the universal cover $\tilde{X}$ of $X$ to the classifying space of rank two polarized $\mathbb{R}$-Hodge structures of weight one, which turns out to be the upper half plane $\mathcal{H}$. As the derivative of $\pi$ can be identified with $\theta$ via the grading $Gr_{Fil}(H, \nabla)$, one knows that $\pi$ is an isomorphism. This illustrates the approach to the uniformization theorem from the point of view of Higgs bundles.

Turn to the $p$-adic case. Let $k := \overline{\mathbb{F}}_p$ be an algebraic closure of a finite field of odd characteristic $p$, $W := W(k)$ the ring of Witt vectors with coefficients in $k$ and $K$ its fraction field, $\overline{K}$ an algebraic closure of $K$. Based on the fundamental work of Ogus-Vologodsky [OV] on the nonabelian Hodge theory in char $p$, we introduced in [LSZ] the notion of a strongly semistable Higgs bundle, generalizing the notion of a strongly semistable vector bundle which played an important role in the work [DW], and a char $p/p$-adic analogue of Yang-Mills-Higgs flow whose “limit” can be regarded as a char $p/p$-adic analogue of the solution of Yang-Mills-Higgs equation, whose definition is recalled as follows.

Let $X_1$ be a smooth projective variety over $k$ together with a simple normal crossing divisor $D_1 \subset X_1$ such that $(X_1, D_1)$ is $W_2(k)$-liftable. Fix a smooth projective scheme $X_2$ over $W_2 := W_2(k)$ together with a divisor $D_2 \subset X_2$ relative to $W_2(k)$ whose reduction mod $p$ is $(X_1, D_1)$. Let $C^{-1}_1$ be the inverse Cartier transform of Ogus-Vologodsky from the category of nilpotent logarithmic Higgs module of exponent $\leq p - 1$ to the category of nilpotent logarithmic flat module of exponent $\leq p - 1$ with respect to the chosen $W_2$-lifting $(X_2, D_2)$. We refer the reader to [LSZ0] for an elementary approach to the construction of the inverse Cartier/Cartier transform in the case $D_1 = \emptyset$ and the Appendix in the general case. A Higgs-de Rham flow over $X_1$ is a diagram of the following form:

$$\begin{array}{ccc}
(C^{-1}_1(E_0, \theta_0), Fil_0) & \xrightarrow{Gr_{Fil_0}} & (C^{-1}_1(E_1, \theta_1), Fil_1) \\
(E_0, \theta_0) & \xrightarrow{Gr_{Fil_0}} & (E_1, \theta_1) \\
\vdots & & \vdots
\end{array}$$

where the initial term $(E_0, \theta_0)$ is a nilpotent graded Higgs bundle with exponent $\leq p - 1$; for $i \geq 0$, $Fil_i$ is a Hodge filtration on the flat module $C^{-1}_1(E_i, \theta_i)$ of level $\leq p - 1$; for $i \geq 1$, $(E_i, \theta_i)$ is the graded Higgs module associated to the de Rham module $(C^{-1}_1(E_{i-1}, \theta_{i-1}), Fil_{i-1})$. A Higgs-de Rham flow is said to be periodic of period $f \in \mathbb{N}$ if $f$ is the minimal integer satisfying that $\phi : (E_f, \theta_f) \cong (E_0, \theta_0)$ as graded Higgs module. A (logarithmic) Higgs bundle is said to be periodic
if it initiates a periodic Higgs-de Rham flow. There are two points we want to emphasize: first, the Hodge filtrations in the above diagram do not come from the inverse Cartier transform, but rather part of defining datum of a flow; second, since the inverse Cartier transform does depend on the choice of a $W_2$-lifting of $(X_1, D_1)$, a periodic Higgs bundle only makes sense after a $W_2$-lifting is specified, although the Higgs bundle itself is just defined over $k$. To define the notion of periodic Higgs-de Rham flow over a truncated Witt ring, one needs to lift the inverse Cartier transform of Ogus-Vologodsky which has been partially realized in [LSZ]. We refer the reader to Section 5 loc. cit. for details. It is quite straightforward to generalize one of main results [LSZ, Theorem 1.6] to the following logarithmic case, which shall provide us with the basic device of producing representations in this paper.

**Theorem 1.1.** Let $Y$ be a smooth projective scheme over $W$ with a simple normal crossing divisor $D \subset Y$ relative to $W$. Then for each natural number $f \in \mathbb{N}$, there is an equivalence of categories between the category of strict $p^n$-torsion logarithmic Fontaine modules (with pole along $D \times W_n \subset Y \times W_n$) with endomorphism structure $W_n(\mathbb{F}_{p^n})$ whose Hodge-Tate weight $\leq p-2$ and the category of periodic Higgs-de Rham flows over $Y_n$ whose periods are divisors of $f$ and exponents of nilpotency are $\leq p-2$, where $W_n(\mathbb{F}_{p^n})$ (resp. $W_n$) is the truncated Witt ring $W(\mathbb{F}_{p^n})/p^n$ (resp. $W/p^n$) with coefficients in $\mathbb{F}_{p^n}$ (resp. $k$).

After the fundamental theorem of Faltings [Pa88, Theorem 2.6*], to each periodic Higgs-de Rham flows over $Y_n$ whose periods are divisors of $f$ as above, one can now associate a crystalline representation of the arithmetic fundamental group of the generic fiber $Y^n_K := (Y - D) \times K$ with coefficients in $W_n(\mathbb{F}_{p^n})$. According to the theorem, the problem is reduced to show that (1.0.2) over $k$ is periodic and can be lifted to a periodic Higgs bundle over an arbitrary truncated Witt ring $W_n, n \in \mathbb{N}$. Here comes our main theorem.

**Theorem 1.2.** Assume that $2g - 2 + r > 0$ and $r$ is even. Let $X_1$ be a generic curve in the moduli space $\mathcal{M}_{g,r}$ of smooth projective curves over $k$ with $r$ marked points $D$. Let $(L_1 \oplus L_1^{-1}, \theta_1)$ be a logarithmic Higgs bundle with maximal Higgs field (1.0.2) defined over $X_1$. Then there exists a tower of log smooth liftings $(X_n, D_n)\rightarrow W_n, n \in \mathbb{N}$

\[(1.2.1) \quad (X_1, D_1 := D) \hookrightarrow (X_2, D_2) \hookrightarrow \cdots \hookrightarrow (X_n, D_n) \hookrightarrow \cdots ,\]

such that $(L_1 \oplus L_1^{-1}, \theta_1)$ is two-periodic with respect to the $W_2$-lifting $(X_2, D_2)$ and lifts to a two-periodic logarithmic Higgs bundle $(L_n \oplus L_n^{-1}, \theta_n)$ over $X_n$ with the log pole along $D_n$ for each $n \geq 2$.

To be more specific, for all $n \in \mathbb{N}$, there exists a log smooth $W_{n+1}$-lifting $(X_{n+1}, D_{n+1})$ of $(X_n, D_n)$, a logarithmic Higgs bundle $(L_n \oplus L_n^{-1}, \theta_n)$ over $X_n$, a two-torsion line bundle $\mathcal{L}_n$ over $X_n$, a Hodge filtration $Fil_n$ on the inverse Cartier transform $C^{-1}_n(L_n \oplus L_n^{-1}, \theta_n)$ with respect to $X_n \subset X_{n+1}$ and an isomorphism

\[\phi_n : Gr_{Fil_n} \circ C^{-1}_n(L_n \oplus L_n^{-1}, \theta_n) \cong (L_n \oplus L_n^{-1}, \theta_n) \otimes (\mathcal{L}_n, 0),\]
such that for all \( n \geq 2 \),
\[
(L_n \oplus L_n^{-1}, \theta_n) \equiv (L_n^{-1} \oplus L_n^{-1}, \theta_n) \mod p^{-2};
\]
\( \mathcal{L}_n \equiv \mathcal{L}_{n-1} \mod p^{-1}; \quad \text{Fil}_n \equiv \text{Fil}_{n-1} \mod p^{-1}; \quad \phi_n \equiv \phi_{n-1} \mod p^{-1}. \)

Set \( (E_n, \theta_n) := (L_n \oplus L_n^{-1}, \theta_n) \), and denote the trivial filtration by \( \text{Fil}_{tr} \). Then there is a tower of two-periodic Higgs-de Rham flows as below:

\[
\begin{array}{c}
(E_n, \theta_n) \\
\end{array}
\begin{array}{c}
\text{mod } p^* \\
\end{array}
\begin{array}{c}
(C_{n+1}^{-1}(E_{n+1}, \theta_{n+1}), \text{Fil}_{n+1}) \\
\text{mod } p^* \\
\end{array}
\begin{array}{c}
(E_{n+1}, \theta_{n+1}) \\
\end{array}
\begin{array}{c}
\text{mod } p^* \\
\end{array}
\begin{array}{c}
(C_{n+1}^{-1}(E_{n+1}, \theta_{n+1}) \otimes \mathcal{L}_{n+1}, \text{Fil}_{n+1} \otimes \text{Fil}_{n+1}) \\
\text{mod } p^* \\
\end{array}
\begin{array}{c}
(E_{n+1}, \theta_{n+1}) \\
\end{array}
\begin{array}{c}
\text{mod } p^* \\
\end{array}
\begin{array}{c}
(C_{n+1}^{-1}(E_{n+1}, \theta_{n+1}) \otimes \mathcal{L}_n, \text{Fil}_{n+1} \otimes \text{Fil}_{n+1}) \\
\end{array}
\begin{array}{c}
(E_{n, \theta_n}) \\
\end{array}
\end{array}
\]

The following consequence is immediate from the previous results.

**Corollary 1.3.** Notations as above. Then for a logarithmic Higgs bundle with maximal Higgs field over a generic curve \((X_1, D_1)\) in the moduli space \( \mathcal{M}_{g, r} \), one has a log smooth curve \((X_\infty, D_\infty)\) over \( W \) lifting \((X_1, D_1)\) and an irreducible crystalline representation
\[
\rho : \pi_1(X_K) \to GL(2, W(\mathbb{F}_p^2)),
\]
where \( X_K := X \times K \) is the generic fiber of the hyperbolic curve \( X := X_\infty - D_\infty \).

The so-constructed representation \( \rho \) in the above corollary shares the following stronger irreducible property. For simplicity, we give the statement only in the case that the divisor \( D_1 \) in \( X_1 \) is empty.

**Proposition 1.4.** Notations as above. Assume \( D_1 = \emptyset \). Denote by \( \bar{\rho} \) the restriction of \( \rho \) to the geometric fundamental group \( \pi_1(X_K) \). Then for any smooth curve \( Y_K \) over \( K \) with a finite morphism \( f : Y_K \to X_K \), the induced representation of \( \pi_1(Y_K) \) from \( \bar{\rho} \) is absolutely irreducible.

**Proof.** Let \((E, \theta)\) be the inverse limit of graded Higgs bundle \( \{(E_n, \theta_n)\} \) in Theorem 1.2. By the example in [Fa, Page 861], one sees that the generalized extension corresponding to \((E, \theta)\) is compatible with \( \bar{\rho} \), that is, it is just the scalar extension of \( \bar{\rho} \) by tensoring with \( \mathbb{C}_p \). We can find a finite extension field \( L \) of \( K \), with its integral ring \( \mathcal{O}_L \), such that the curve and the finite morphism descend to \( f_L : Y_L \to X_L \). By taking the semistable reduction (with \( L \) suitably enlarged), there exists a proper smooth scheme \( Y_{\mathcal{O}_L} \) with toroidal singularity over \( \mathcal{O}_L \) such that the map \( f_L \) extends to \( f_{\mathcal{O}_L} : Y_{\mathcal{O}_L} \to X_{\mathcal{O}_L} \). By the theory in [Fa], the twisted pullback of the graded Higgs bundle \( f_{\mathcal{O}_L}^*(E, \theta) \) corresponds to the
restricted representation of \( \hat{\rho} \otimes \mathbb{C}_p \) to \( \pi_1(Y_R) \). By the very construction of the twisted pullback, one has a short exact sequence:

\[
0 \to (f^*_L L^{-1}, 0) \to f^*_L (E, \theta) \to (f^*_L L, 0) \to 0.
\]

So \( f^*_L (E, \theta) \) does not contain any proper Higgs subbundle of degree zero and the irreducibility of the representation follows. \( \square \)

In general, the \( p \)-adic curve appeared in Corollary 1.3 is neither unique nor far from an arbitrary lifting of \((X_1, D_1)\) over \( \mathbb{k} \). In fact, it is in close relation with the notion of a canonical curve due to S. Mochizuki \([Mo]\), Definition 3.1, Ch. III. At this point, it is in demand to make a brief clarification about the relation between the notion of an indigenous bundle, which is central in the \( p \)-adic Teichmüller theory of Mochizuki for curves, and the notion of a logarithmic Higgs bundle with maximal Higgs field (i.e. \((1.0.2)\)). According to Mochizuki loc. cit., an indigenous bundle is a \( \mathbb{P}^1 \)-bundle with connection associated to a rank two flat bundle with a Hodge filtration, and its associated graded Higgs bundle is of the form \((1.0.2)\). He started with indigenous bundles over a Riemann surface admitting an integral structure over \( W(\mathbb{k}) \) for some \( p \), and then studied the moduli space of \( p \)-adic indigenous bundles over the corresponding \( p \)-adic curve. Although related, the approach and setting in \([Mo]\) are very different from ours. From the point of view of nonabelian Hodge theory, one is by no means restricted to the curve case. As an illustration, our approach yields a similar result for an ordinary abelian variety \( A \) as Theorem 1.2 by considering the following Higgs bundle (see \([LSZ]\), Example 5.27):

\[
(\Omega_A \oplus \mathcal{O}_A, \theta), \text{ where } \theta : \Omega_A \to \mathcal{O}_A \otimes \Omega_A \text{ is the tautological isomorphism.}
\]

### 2. Outline of the proof of the main theorem

Let \((E_1 := L_1 \oplus L_1^{-1}, \theta_1)\) be the Higgs bundle \((1.0.2)\). The proof of Theorem 1.2 is divided into two steps. At the first step, we show that there exists a \( W_2 \)-lifting of \( X_1 \) such that \((1.0.2)\) becomes a two-periodic Higgs bundle, see Theorem 2.1; at the second step, we show that under some condition, a periodic Higgs bundle over \( X_n \) is liftable to a periodic Higgs bundle over \( X_{n+1} \), see Theorem 2.5.

**Theorem 2.1.** For a generic curve \( X_1 \in \mathcal{M}_{g,r} \), there exists a \( W_2 \)-lifting \( X_2 \) of \( X_1 \) and a Hodge filtration \( \text{Fil}_1 \) on \( C^{-1}_1(L_1 \oplus L_1^{-1}, \theta_1) \) such that

\[
\text{Gr}_{\text{Fil}_1} \circ C^{-1}_1(L_1 \oplus L_1^{-1}, \theta_1) \cong (L_1 \oplus L_1^{-1}, \theta_1) \otimes (\mathcal{L}, 0),
\]

where \( \mathcal{L} \) is a two-torsion line bundle over \( X_1 \).

**Corollary 2.2.** Notations as above. The Higgs bundle \((L_1 \oplus L_1^{-1}, \theta_1)\) with respect to the \( W_2 \)-lifting \( X_2 \) is two-periodic.

**Proof.** It is to show there is some filtration \( \text{Fil}_2 \) on \( C^{-1}_1((L_1 \oplus L_1^{-1}, \theta_1) \otimes (\mathcal{L}, 0)) \)
such that two graded Higgs bundles are isomorphic:
\[ Gr_{Fil_2} \circ C^{-1}_1 \circ Gr_{Fil_1} \circ C^{-1}_1 (L_1 \oplus L_1^{-1}, \theta_1) \cong (L_1 \oplus L_1^{-1}, \theta_1). \]
Since \( C^{-1}_1((L_1 \oplus L_1^{-1}, \theta_1) \otimes (\mathcal{L}, 0)) \) is naturally isomorphic to
\[ C^{-1}_1(L_1 \oplus L_1^{-1}, \theta_1) \cong (L_1 \oplus L_1^{-1}, \theta_1) \otimes (\mathcal{L}, \nabla_{can}), \]
we can equip it with the filtration \( Fil_2 \) which is the tensor product of the filtration \( Fil_1 \) on \( C^{-1}_1(L_1 \oplus L_1^{-1}, \theta_1) \) and the trivial filtration on \( \mathcal{L} \). Then by Theorem 2.1, it follows that
\[ Gr_{Fil_2} \circ C^{-1}_1 \circ Gr_{Fil_1} \circ C^{-1}_1 (L_1 \oplus L_1^{-1}, \theta_1) \cong (L_1 \oplus L_1^{-1}, \theta_1) \otimes (\mathcal{L} \otimes 2, 0) = (L_1 \oplus L_1^{-1}, \theta_1). \]

To outline the proof of Theorem 2.1, we start with an observation: For every \( W_2 \)-lifting \( X_2 \) of \( X_1 \), there exists a short exact sequence of flat bundles as follows:

\[ (2.2.1) \quad 0 \to (F^* L_1^{-1}, \nabla_{can}) \to C^{-1}_1(E_1, \theta_1) \to (F^* L_1, \nabla_{can}) \to 0, \]
where \( F : X_1 \to X_1 \) is the absolute Frobenius. This comes after applying the exact functor \( C^{-1}_1 \) to the short exact sequence of Higgs bundles:
\[ 0 \to (L_1^{-1}, 0) \to (E_1, \theta_1) \to (L_1, 0) \to 0. \]
Put \( (H, \nabla) = C^{-1}_1(E_1, \theta_1). \) Then forgetting the connection in the exact sequence, we get an extension of vector bundles:

\[ (2.2.2) \quad 0 \to F^* L_1^{-1} \to H \to F^* L_1 \to 0. \]

The cohomology group \( H^1(X_1, F^* L_1^{-2}) \cong \text{Ext}^1(F^* L_1, F^* L_1^{-1}) \) classifies the isomorphism classes of extensions of form (2.2.2). Let \( \{X_1 \subset X_2\} \) be the set of isomorphism classes of \( W_2 \)-liftings of \( X_1 \), which is known to be an \( H^1(X_1, \mathcal{T}_{X_1/k}) \)-torsor, where \( \mathcal{T}_{X_1/k} \) is the log tangent sheaf of \( X_1/k \). To consider the effect of the choices of \( W_2 \)-liftings on the inverse Cartier transform of \( (E_1, \theta_1) \), we shall consider the natural map

\[ (2.2.3) \quad \rho : \{X_1 \subset X_2\} \to H^1(X_1, F^* L_1^{-2}), \]
obtained by sending \( X_1 \subset X_2 \) to the extension class of \( H \) as above. It turns out that the map \( \rho \) behaves well under the torsor action and is injective (Lemma 3.8 and Lemma 3.2). Put \( A = \text{Im}(\rho) \). In order to remember the connection in the inverse Cartier transform, we shall also consider the isomorphism classes \( \text{Ext}^1((F^* L_1, \nabla_{can}), (F^* L_1^{-1}, \nabla_{can})) \) of extensions of flat bundles:

\[ (2.2.4) \quad 0 \to (F^* L_1^{-1}, \nabla_{can}) \to (H, \nabla) \to (F^* L_1, \nabla_{can}) \to 0. \]

There is a natural forgetful map
\[ \text{Ext}^1((F^* L_1, \nabla_{can}), (F^* L_1^{-1}, \nabla_{can})) \to \text{Ext}^1(F^* L_1, F^* L_1^{-1}) \]
sending \( (H, \nabla) \) to \( H \). We denote \( B \) for its image. Thus by the short exact sequence (2.2.3), we see that \( A \subset B \). Later we show that \( A \) is an affine subspace of the ambient space \( H^1(X_1, F^* L_1^{-2}) \), not passing through the origin, and \( B \) is the linear hull of \( A \). On the other hand, if \( C^{-1}_1(X_1 \subset X_2)(E_1, \theta_1) \) satisfies moreover (2.1.1), then it admits a subsheaf \( L_1 \oplus \mathcal{L} \subset C^{-1}_1(X_1 \subset X_2)(E_1, \theta_1) \) with \( \mathcal{L} \in \text{Pic}^0(X_1) \). For a
given element $\xi \in H^1(X_1, F^*L_1^{-2})$, let $H_\xi$ be the corresponding isomorphism class of extension:

\[
(2.2.5) \quad 0 \to F^*L_1^{-1} \to H_\xi \to F^*L_1 \to 0,
\]

Thus this leads us also to considering the subset $K \subset H^1(X_1, F^*L_1^{-2})$, consisting of extensions such that $H_\xi$ admits a subsheaf $L_1 \otimes \mathcal{L} \hookrightarrow H_\xi$ for some $\mathcal{L} \in \text{Pic}^0(X_1)$. $K$ is clearly a cone. We shall call it the periodic cone of $X_1/k$.

**Lemma 2.3.** The periodic cone $K$ can be written in the following form

\[
(2.3.1) \quad K = \bigcup_{s \in \Phi \text{Hom}(L_1 \otimes \mathcal{L}, F^*L_1), \mathcal{L} \in \text{Pic}^0(X_1)} \ker(\phi_s),
\]

where the map on $H^1$

\[
\phi_s : H^1(X_1, F^*L_1^{-1} \otimes F^*L_1^{-1}) \to H^1(X_1, L_1^{-1} \otimes \mathcal{L}^{-1} \otimes F^*L_1^{-1})
\]

is induced from the sheaf morphism $s \otimes \text{id}$ with $s : F^*L_1^{-1} \to L_1^{-1} \otimes \mathcal{L}^{-1}$ the dual of $s$.

**Proof.** If $H_\xi$ admits a subsheaf $L_1 \otimes \mathcal{L}$ with $\mathcal{L} \in \text{Pic}^0(X_1)$, then the composite $L_1 \otimes \mathcal{L} \hookrightarrow H_\xi \to F^*L_1$ cannot be zero for the degree reason. So for a chosen nonzero element $s \in \text{Hom}(L_1 \otimes \mathcal{L}, F^*L_1)$, it is to describe those extensions $H_\xi$ such that $s$ is liftable to a morphism $L_1 \otimes \mathcal{L} \to H_\xi$. Pulling-back the extension $H_\xi$ along $s$, one obtains an extension of $L_1 \otimes \mathcal{L}$ by $F^*L_1^{-1}$ whose class is given by the image of $\xi$ under the map

\[
\phi_s : H^1(X_1, F^*L_1^{-1} \otimes F^*L_1^{-1}) \xrightarrow{s \otimes \text{id}} H^1(X_1, L_1^{-1} \otimes \mathcal{L}^{-1} \otimes F^*L_1^{-1}).
\]

It is clear that the $s$ lifts to a morphism $L_1 \otimes \mathcal{L} \to H_\xi$ if the pull-back extension of $H_\xi$ along $s$ is split, that is, $\phi_s(\xi) = 0$. \qed

Viewing $H^1(X_1, F^*L_1^{-2})$ as the affine space, we shall see that both $A$ and $K$ are actually closed subvarieties, see Propositions 3.1 and 3.5. One of the main points of the paper is to show for a generic curve $X_1 \in M_{g,r}$, $A \cap K \neq \emptyset$. This fact has the consequence that one finds then a $W_2$-lifting $X_2$ and a Hodge filtration $Fil_1$ on $C_{X_1 \subset X_2}^{-1}(E_1, \theta_1)$ such that the isomorphism 2.1.1 holds, see Proposition 3.6. This is the way we prove Theorem 2.1. We use a degeneration argument to show $A \cap K \neq \emptyset$ for a generic curve. In fact, we consider a totally degenerate curve of genus $g$ with $r$-marked points in §4, where we are able to prove the intersection $A \cap K$ consists of a unique element and is particularly nonempty. In §5, we show that $A$s and $K$s glue into families when the base curve $X_1$ deforms and then, by the upper semi-continuity, it follows that $A \cap K \neq \emptyset$ for closed points in an open neighborhood of a totally degenerate curve.

Now we turn to the lifting problem of a periodic Higgs bundle. The key is to resolve the obstruction class of lifting the Hodge filtration. For this purpose, we rediscover the ordinary condition due to S. Mochizuki [Mo] introduced somewhat twenty years ago.
Definition 2.4. Assume that $L \in \text{Pic}^0(X_1)$. We call $s \in \text{Hom}(L_1 \otimes L, F^*L_1)$ ordinary if the composite

\[(2.4.1)\quad H^1(X_1, L_1^{-2}) \xrightarrow{F^*} H^1(X_1, F^*L_1^{-2}) \xrightarrow{s^2} H^1(X_1, L_1^{-2} \otimes L^{-2})\]

is injective, where the second map is induced by the sheaf morphism $s^2$.

This definition is equivalent to the one given by Mochizuki loc. cit., although it was stated in a totally different form. In our context, the ordinarity ensures the existence of lifting Hodge filtrations. See \S 5. Denote by $(E_n, \theta_n, \text{Fil}_n, \phi_n)$ the two-periodic Higgs bundle over $X_n$ in (1.2.2). Here comes Theorem 2.5.

Theorem 2.5. Notations as in Theorem 2.1. Then with respect to a $W_2$-lifting $X_2$ of $X_1$, $(E_1, \theta_1)$ becomes a two-periodic Higgs bundle $(E_1, \theta_1, \text{Fil}_1, \phi_1)$. Let $s$ be the composite $\text{Fil}_1^1 \hookrightarrow C_1^{-1}(E_1, \theta_1) \twoheadrightarrow F^*L_1$, where $\text{Fil}_1^1$ is of the form $L_1 \otimes L$ for a two-torsion line bundle $L$ and the second map is given in (2.2.2). If $s$ is ordinary, then for all $n \geq 1$, inductively there exists a $W_{n+1}$-lifting $X_{n+1}$ of $X_n$ such that the two-periodic Higgs bundle $(E_{n-1}, \theta_{n-1}, \text{Fil}_{n-1}, \phi_{n-1})$ over $X_{n-1}$ can be lifted to a two-periodic Higgs bundle $(E_n, \theta_n, \text{Fil}_n, \phi_n)$ over $X_n$.

Finally, we can give a proof of Theorem 1.2.

Proof of Theorem 1.2. With the ordinary condition ensured by Proposition 4.9, Theorem 1.2 is a direct consequence of Corollary 2.2 and Theorem 2.5. □

3. The smooth case

In this section, we begin to investigate the properties of the subsets $A$ and $K$ in the case where the base curve $X_1/k$ is a smooth projective curve of genus $g$ together with $r$-marked points $D_1 \subset X_1$. It is required that $2g - 2 + r$ be an even positive number. To conform with the notations in later sections, we use the fine logarithmic structures defined in \[KKa\] and notations therein. In particular, we equip $X_1$ with the standard log structure defined by the divisor $D_1$ (Example 1.5 (1) loc. cit.) and the base $\text{Spec } k$ the trivial log structure (so that the structural morphism $X_1 \to \text{Spec } k$ is log smooth Example 3.7 (1) loc. cit.); $\omega_{X_1/k}$ denotes for the invertible sheaf of differential forms with respect to these log structures (1.7 loc. cit.).

3.1. General discussions. In the last section, we introduced the subsets $A$ and $K$ of the affine space $H^1(X_1, F^*L_1^{-2})$. We proceed to study some basic geometric properties of these two subsets and their intersection behavior.

Let $\sigma$ be the Frobenius automorphism of the field $k$. Then for any vector bundle $V$ over $X_1$, the absolute Frobenius of $X_1$ induces a $\sigma$-linear morphism
There are natural morphisms of complexes

\[ F^*: H^i(X_1, V) \to H^i(X_1, F^*V) \] for \( i \geq 0 \). Indeed, it is the composite of natural morphisms:

\[ H^i(X_1, V) \cong H^i(X_1, V \otimes_{\mathcal{O}_{X_1}} \mathcal{O}_{X_1}) \to H^i(X_1, V \otimes_{\mathcal{O}_{X_1}} F, \mathcal{O}_{X_1}) \cong H^i(X_1, F^*V). \]

In our case, we denote \( W_F \) for the image of the natural map \( F^*: H^1(X_1, L_1^{-2}) \to H^1(X_1, F^*L_1^{-2}) \). Then we have

**Proposition 3.1.** The subset \( A \subset H^1(X_1, F^*L_1^{-2}) \) is a nontrivial translation of the linear subspace \( W_F \). Moreover, \( \dim A = 3(g - 1) + r \).

We postpone the proof. We point out first that \( W_F \) is actually a linear subspace of \( B \). Indeed, one can also interpret the natural map \( F^*: H^1(X_1, L_1^{-2}) \to H^1(X_1, F^*L_1^{-2}) \) in terms of extensions. That is, it is the map obtained by pulling-back an extension of form

(3.1.1) \[ 0 \to L_1^{-1} \to E \to L_1 \to 0 \]

via the absolute Frobenius to an extension of form

(3.1.2) \[ 0 \to F^*L_1^{-1} \to F^*E \to F^*L_1 \to 0. \]

Note the latter extension is tautological to the extension equipped with the canonical connections:

(3.1.3) \[ 0 \to (F^*L_1^{-1}, \nabla_{can}) \to (F^*E, \nabla_{can}) \to (F^*L_1, \nabla_{can}) \to 0. \]

Thus, we see that \( W_F \) lies in \( B \). In fact, there is also a natural map \( H^1(X_1, L_1^{-2}) \) to \( \text{Ext}^1((F^*L_1, \nabla_{can}), (F^*L_1^{-1}, \nabla_{can})) \). To see this, we let \( H^1_{dR} := H^1_{dR}(F^*L_1^{-2}, \nabla_{can}) \) be the first hypercohomology of the de Rham complex

\[ \Omega^*_{dR}(F^*L_1^{-2}, \nabla_{can}) := F^*L_1^{-2} \nabla_{can} \to F^*L_1^{-2} \otimes \omega_{X_1/k}. \]

There are natural morphisms of complexes

\[ L_1^{-2} \to \Omega^*_{dR}(F^*L_1^{-2}, \nabla) \]

and

\[ \Omega^*_{dR}(F^*L_1^{-2}, \nabla) \to F^*L_1^{-2}. \]

Identifying \( \text{Ext}^1((F^*L_1, \nabla_{can}), (F^*L_1^{-1}, \nabla_{can})) \) with \( H^1_{dR} \), one sees that the former morphism induces the map on hypercohomologies

\[ \beta : H^1(X_1, L_1^{-2}) \to \text{Ext}^1((F^*L_1, \nabla_{can}), (F^*L_1^{-1}, \nabla_{can})), \]

while the latter induces

\[ \alpha : \text{Ext}^1((F^*L_1, \nabla_{can}), (F^*L_1^{-1}, \nabla_{can})) \to H^1(X_1, F^*L_1^{-2}), \]

which is just the forgetful map defining \( B \). Moreover, the composite map

\[ \alpha \circ \beta : H^1(X_1, L_1^{-2}) \to H^1(X_1, F^*L_1^{-2}) \]

is the map \( F^* \) defining \( W_F \). An injectivity result is in demand.

**Lemma 3.2.** Both the map \( \alpha : H^1_{dR}(F^*L_1^{-2}, \nabla_{can}) \to H^1(X_1, F^*L_1^{-2}) \) and the map \( \alpha \circ \beta = F^* : H^1(X_1, L_1^{-2}) \to H^1(X_1, F^*L_1^{-2}) \) are injective.
Proof. Consider first the map $\alpha$. Suppose the exact sequence (2.2.4) is split as an exact sequence of vector bundles, i.e. there is a splitting $F^*L_1 \to H$, but not split as an exact sequence of flat bundles, i.e. the image of the splitting is not preserved by the connection $\nabla$. Then the associated Higgs bundle by taking grading with respect to the image of the splitting has nonzero Higgs field. That is,

$$\bar{\nabla} : F^*L_1 \to (H/F^*L_1 \cong F^*L_1^{-1}) \otimes \omega_{X_1/k}$$

is nonzero. But since

$$\deg F^*L_1 = \frac{p}{2}(2g - 2 + r) > \frac{1-p}{2}(2g - 2 + r) = \deg(F^*L_1^{-1} \otimes \omega_{X_1/k}),$$

a contradiction! The injectivity for the map $\bar{\nabla}$ also follows. Indeed, the previous argument shows if the exact sequence (3.1.2) is split then the exact sequence (3.1.3) with connections is also split. Then one applies the Cartier descent theorem to conclude the splitting of (3.1.1).

Remark 3.3. One can extend the above argument for $F^*$ to show a general injectivity statement: for a smooth projective variety $X/k$ and an ample line bundle $L$ over $X$, the natural $\sigma$-linear map $H^1(X, L^{-1}) \to H^1(X, F^*L^{-1})$ is injective. Of course, when $X$ is of dimension greater than one, the statement is only meaningful for a non $W_2$-liftable $X$, since otherwise $H^1(X, L^{-1})$ vanishes by the Kodaira vanishing theorem of Raynaud, see Corollaire 2.8 [DI].

By the above lemma, one computes the dimension of $W_F$ (and hence of $A$) by Riemann-Roch. The dimension for $H^1_{dR}$ is computed via its natural isomorphism to its corresponding Higgs cohomology [DI], [OV]. Define $H^1_{Hig} := H^1_{Hig}(L_1^{-2}, 0)$ to be the hypercohomology of the Higgs complex

$$\Omega^*_{Hig}(L_1^{-2}, 0) := L_1^{-2} \to L_1^{-2} \otimes \omega_{X_1/k}.$$  

Clearly,

$$H^1_{Hig} \cong H^1(X_1, L_1^{-2}) \oplus H^0(X_1, L_1^{-2} \otimes \omega_{X_1/k} \cong \mathcal{O}_{X_1}).$$

Lemma 3.4. There is a natural isomorphism

$$H^1_{dR}(F^*L_1^{-2}, \nabla_{can}) \cong H^1_{Hig}(L_1^{-2}, 0).$$

Therefore $\dim B = \dim A + 1$.

Proof. See Corollary 2.27 [OV] for the case $D_1 = \emptyset$. However in the zero Higgs field case, one extends directly the splitting formula due to Deligne-Illusie [DI], to construct an explicit quasi-isomorphism from the Higgs complex $\Omega^*_{Hig}(L_1^{-2}, 0)$ to the simple complex associated to the Čech double complex of $F_*(\Omega^*_{dR}(F^*L_1^{-2}, \nabla_{can}))$. See e.g. the proof of Theorem 10.7 [EV]. In particular, the argument works also in this special log case.

Proposition 3.5. The periodic cone $K$ is a closed subvariety of $H^1(X_1, F^*L_1^{-2})$. Moreover $\dim K = (p-1)(2g-2+r)$. 

Thus one obtains relations:

\[(3.5.1) \quad \dim A + \dim K = \dim H^1(X_1, F^*L_1^{-2}).\]

We shall projectivize everything: Let \( \mathbb{P} := \mathbb{P}(H^1(X_1, F^*L_1^{-2})) \) be the associated projective space and \( p : H^1(X_1, F^*L_1^{-2}) \setminus \{0\} \to \mathbb{P} \) the natural map. By Proposition 3.1, \( p(A) \subset \mathbb{P} \) is an affine space of the same dimension as \( A \). Denote \( \mathbb{P}(K) \) for the projectivized periodic cone, and similarly for \( \mathbb{P}(W_F) \) and \( \mathbb{P}(B) \). Now, by the above discussions, we get

\[(3.5.2) \quad \dim \mathbb{P}(B) + \dim \mathbb{P}(K) = \dim \mathbb{P}.\]

Hence \( \mathbb{P}(B) \) and \( \mathbb{P}(K) \) always intersects. For each \( \xi \in \mathbb{P}(B) \cap \mathbb{P}(K) \), the corresponding extension \( (H, \nabla) \) admits an invertible subsheaf \( L_1 \otimes L \) for some \( L \in \text{Pic}^0(X_1) \) and \( \xi \in \text{Ker}(\phi_s) \) with \( s \) the composite map \( L_1 \otimes L \hookrightarrow H \to F^*L_1 \).

This is near to what we need for the periodicity but not exactly: the next proposition shows that those \( \xi \)'s in the subset \( p(A) \cap \mathbb{P}(K) \), if nonempty, give rise to periodicity. But clearly

\[ p(A) \cap \mathbb{P}(K) = \mathbb{P}(B) \cap \mathbb{P}(K) - \mathbb{P}(W_F) \cap \mathbb{P}(K). \]

**Proposition 3.6.** If \( p(A) \cap \mathbb{P}(K) \neq \emptyset \) or equivalently \( A \cap K \neq \emptyset \), then \( (E_1, \theta_1) \) is a two-periodic Higgs bundle, that is, there exists a \( W_2 \)-lifting \( X_2 \) of \( X_1 \) and a Hodge filtration \( \text{Fil}_1 \) such that

\[(3.6.1) \quad \text{Gr}_{\text{Fil}_1} \circ C^{-1}_{X_1 \subset X_2}(E_1, \theta_1) \cong (E_1, \theta_1) \otimes (L, 0),\]

with \( L \) a certain two-torsion line bundle.

**Proof.** Take \( \xi \in A \cap K \). By definition, the corresponding extension is of form \( H_\xi = C^{-1}_{X_1 \subset X_2}(E_1, \theta_1) \) for some \( W_2 \)-lifting \( X_2 \) of \( X_1 \) and \( H_\xi \) has a subsheaf \( L_1 \otimes L \hookrightarrow H_\xi \) for \( L \in \text{Pic}^0(X_1) \).

Next, we show that the subsheaf \( L_1 \otimes L \hookrightarrow H_\xi \) is saturated and not \( \nabla \)-invariant. Let \( \text{Fil}_1 \) be the saturated subbundle of \( L_1 \otimes L \hookrightarrow H_\xi \). We claim that \( \text{Fil}_1 \) is not \( \nabla \)-invariant. Otherwise, since the \( p \)-curvature of \( \nabla \) on \( H_\xi \) is nilpotent, therefore, the \( p \)-curvature of \( \nabla \) on \( \text{Fil}_1 \) is zero; as \( H_\xi \) has another subbundle \( F^*L_1^{-1} \) with trivial \( p \)-curvature, the \( p \)-curvature of \( H_\xi \) has to be zero. But since \( H_\xi \) is the inverse Cartier of a Higgs bundle with maximal Higgs field, its \( p \)-curvature is nonzero (actually nowhere zero). A contradiction. Since \( \text{Fil}_1 \) is not \( \nabla \)-invariant, one gets a nonzero Higgs field

\[ \nabla: (\text{Fil}_1) \to (\text{Fil}_1)^{-1} \otimes \omega_{X_1/k}. \]

Thus one obtains relations:

\[ \frac{2g - 2 + r}{2} = \deg(L_1 \otimes L) \leq \deg \text{Fil}_1 \leq \frac{1}{2} \deg \omega_{X_1/k} = \frac{2g - 2 + r}{2}. \]

It follows immediately that \( \text{Fil}_1 \cong L_1 \otimes L \), the Higgs field \( \nabla \) is an isomorphism and \( L \) is a two-torsion. To summarize, one gets

\[ \text{Gr}_{\text{Fil}_1} \circ C^{-1}_{X_1 \subset X_2}(E_1, \theta_1) \cong (E_1, \theta_1) \otimes (L, 0) \]
The last proposition reveals the intimate relation of the intersection $A \cap K$ and the periodicity of the Higgs bundle $(E_1, \theta_1)$ with maximal Higgs field. In the case of $\mathbb{P}^1$ with 4 marked points, one can directly show $A \cap K \neq \emptyset$. In general, we can only show the nonempty intersection for a generic $X_1$ in the moduli using a degeneration argument.

**Proposition 3.7.** Let $X_1$ be a $\mathbb{P}^1$ with 4 marked points. Then the Higgs bundle $(E_1 = \mathcal{O}(1) \oplus \mathcal{O}(-1), \theta_1)$ of type $\{1.0.2\}$ is one-periodic.

**Proof.** By the above discussions, it suffices to show that $\mathbb{P}(W_F) \cap \mathbb{P}(K) = \emptyset$, or equivalently $W_F \cap K = 0$. First notice that the only nontrivial extension of $\mathcal{O}(-1)$ by $\mathcal{O}(1)$ over $\mathbb{P}^1$ is isomorphic to $\mathcal{O}^{\oplus 2}$. Therefore, $H_\xi$ corresponding to a nonzero $\xi \in W_F$ is also isomorphic to $\mathcal{O}^{\oplus 2}$. It follows that $H_\xi$ does not admit any invertible subsheaf of positive degree, which implies $\xi \notin K$ and then $W_F \cap K = 0$. $\square$

**3.2. Proof of Proposition 3.1.** Let $\tilde{\theta}_1 : \mathcal{T}_{X_1/k} \xrightarrow{\cong} L_1^{-2}$ be the isomorphism induced by the Higgs field. By abuse of notation, it denotes also for the induced isomorphism on cohomologies. The following diagram commutes:

$$
\begin{array}{ccc}
H^1(X_1, \mathcal{T}_{X_1/k}) & \xrightarrow{\tilde{\theta}_1} & H^1(X_1, L_1^{-2}) \\
\downarrow \mathcal{F}^* & & \downarrow \mathcal{F}^* \\
H^1(X_1, \mathcal{F}^* \mathcal{T}_{X_1/k}) & \xrightarrow{\mathcal{F}^*(\tilde{\theta}_1)} & H^1(X_1, \mathcal{F}^* L_1^{-2}).
\end{array}
$$

Proposition 3.1 follows from the following observations.

**Lemma 3.8.** (1) Consider $\rho : \{X_1 \subset X_2\} \to H^1(X_1, \mathcal{F}^* L_1^{-2})$. Then for $\tau \in \{X_1 \subset X_2\}$ and $\nu \in H^1(X_1, \mathcal{T}_{X_1/k})$, it holds that

$$
\rho(\tau + \nu) = \rho(\tau) + \mathcal{F}^* \circ \tilde{\theta}_1(\nu).
$$

(2) The image of $\rho$ does not pass through the origin of $H^1(X_1, \mathcal{F}^* L_1^{-2})$.

**Proof.** (1) Given a $W_2$-lifting $X_2$ of $X_1$, the obstruction to lifting the absolute Frobenius over $X_2$ lies in $H^1(X_1, \mathcal{F}^* \mathcal{T}_{X_1/k})$. It defines the following map

$$
\mu : \{X_1 \subset X_2\} \to H^1(X_1, \mathcal{F}^* \mathcal{T}_{X_1/k}).
$$

By the construction of the inverse Cartier transform [LSZ0] (see also Appendix), one sees that $\rho$ is nothing but the composite $\mathcal{F}^*(\tilde{\theta}_1) \circ \mu$. So it suffices to show that $\mu$ is a torsor map under the map

$$
\mathcal{F}^* : H^1(X_1, \mathcal{T}_{X_1/k}) \to H^1(X_1, \mathcal{F}^* \mathcal{T}_{X_1/k}),
$$

which is injective after Lemma 3.2 and the commutative diagram (3.7.1).

Suppose $X_2$ and $X'_2$ are two $W_2$-liftings of $X_1$. Let $X_1 = \bigcup U_i$ be an open affine covering of $X_1$. There is a unique $W_2$-lifting $V_i$ of $U_i$ up to isomorphism. Let $V_{ij}$
be a $W_2$-lifting of $U_{ij} = U_i \cap U_j$. Fixing embeddings $\{V_{ij} \to V_j\}$. Corresponding to $X_2$ and $X'_2$, there are two embeddings $g_{ij}: V_{ij} \to V_i$, and $g'_{ij}: V_{ij} \to V_i$. Note that $g_{ij}$ and $g'_{ij}$ have the same reduction $\overline{U_{ij} \to U_i}$ modulo $p$. Then the image of $g_{ij}^* - (g'_{ij})^*: \mathcal{O}_{V_i} \to \mathcal{O}_{V_{ij}}$ is in $p\mathcal{O}_{V_{ij}}$. Then $\frac{g_{ij}^* - (g'_{ij})^*}{p}: \mathcal{O}_{U_i} \to \mathcal{O}_{U_{ij}}$ is a derivation. And it defines a class $\{\tau_{ij}\}$ in $H^1(X_1, \mathcal{T}_{X_1})$ representing $[X_2] - [X'_2]$.

Now we choose a Frobenius lifting $\{F_i: V_i \to V_i\}_{i \in I}$. For simplicity, we may assume that $U_j \subset U_i$ and $V_j \subset V_i$. We have the Cartesian diagrams

$$
\begin{array}{ccc}
V_j & \xrightarrow{F_j} & V_i \\
g_{ij} \downarrow & & \downarrow g_{ij} \\
V_i & \xrightarrow{F_i} & V_i.
\end{array}
$$

Then we have two Frobenius liftings $g_{ij} \circ F_j$ and $F_i \circ g_{ij}$. Set

$$
h_{ij} = \frac{(g_{ij} \circ F_j)^* - (F_i \circ g_{ij})^*}{p}: \mathcal{O}_{U_i} \to \mathcal{O}_{U_{ij}},
$$

and

$$
h'_{ij} = \frac{(g'_{ij} \circ F_j)^* - (F_i \circ g'_{ij})^*}{p}: \mathcal{O}_{U_i} \to \mathcal{O}_{U_{ij}}.
$$

They induce the class $\rho(X_1 \subset X_2)$ and $\rho(X_1 \subset X'_2)$ respectively. Thus it is reduced to show the following equality:

$$
h_{ij} - h'_{ij} = \frac{F^* \circ g_{ij}^* - F^* \circ (g'_{ij})^*}{p}.
$$

Notice that

$$
h_{ij} - h'_{ij} = \frac{F_j^* \circ (g_{ij}^* - (g'_{ij})^*) - (g_{ij}^* - (g'_{ij})^*) \circ F_i^*}{p}.
$$

Since $\frac{g_{ij}^* - (g'_{ij})^*}{p}: \mathcal{O}_{U_i} \to \mathcal{O}_{U_{ij}}$ is a derivation, and the image of $dF_i^*: \Omega_{V_i} \to \Omega_{V_i}$ lies in $p\Omega_{V_i}$, it follows that

$$
g_{ij}^* - (g'_{ij})^* \circ F_i^* = 0,
$$

that is,

$$
h_{ij} - h'_{ij} = \frac{F_j^* \circ (g_{ij}^* - (g'_{ij})^*)}{p} = \frac{F^* \circ g_{ij}^* - F^* \circ (g'_{ij})^*}{p}.
$$

This finishes the proof.

(2) Assume the contrary, namely, for some $W_2$-lifting, the extension $\text{(2.2.2)}$ given by the bundle part of the inverse Cartier transform of $(E_1', \theta_1)$ splits. By Lemma 3.2, the extension as flat bundles $\text{(2.2.1)}$ splits as well. It follows that its $p$-curvature is zero, which is however impossible. \qed
3.3. Proof of Proposition 3.5. To exhibit the algebraic structure of the periodic cone $K$, we proceed as follows. Note that the notions of locally free sheaves of finite rank and vector bundles will be used interchangeably in our description.

**Step 1:** Denote by $p_1 : \text{Pic}^0(X_1) \times X_1 \to \text{Pic}^0(X_1)$ and $p_2 : \text{Pic}^0(X_1) \times X_1 \to X_1$ the projections. Write $\mathcal{O} = \mathcal{O}_{\text{Pic}^0(X_1) \times X_1}$. Let $\mathcal{L}$ be the universal line bundle over $\text{Pic}^0(X_1) \times X_1$. Consider two invertible sheaves $p_2^* F^* L_1^{-1}$ and $p_2^* F^* L_1^{-1} \otimes \mathcal{L}^{-1}$ which form the sheaf

$$\mathcal{H}om_\mathcal{O}(p_2^* F^* L_1^{-1}, p_2^* F^* L_1^{-1} \otimes \mathcal{L}^{-1}) \cong \mathcal{H}om_\mathcal{O}(p_2^* F^* L_1 \otimes \mathcal{L}, p_2^* F^* L_1).$$

Its direct image $p_1_* \mathcal{H}om_\mathcal{O}(p_2^* F^* L_1^{-1}, p_2^* F^* L_1^{-1} \otimes \mathcal{L}^{-1})$ is a locally free $\mathcal{O}_{\text{Pic}^0(X_1)}$-modules of rank $(2g - 2 + r) - (g - 1)$ by Riemann-Roch. Denote $\mathcal{V}$ for the vector bundle $p_1_* \mathcal{H}om(p_2^* F^* L_1^{-1}, p_2^* F^* L_1^{-1} \otimes \mathcal{L}^{-1})$ and $\mathbb{P}$ its associated projective bundle. Let $\pi_0 : \mathcal{V} \to \text{Pic}^0(X_1)$ and $q_0 : \mathcal{V} - \{0\} \to \mathbb{P}$ be the natural projections, where $\{0\}$ is the zero section of $\pi_0$. Put $\pi = \pi_0 \times id : \mathcal{V} \times X_1 \to \text{Pic}^0(X_1) \times X_1$ and $q : \mathcal{V} - \{0\} \times X_1 \xrightarrow{q_0 \times id} \mathbb{P} \times X_1 \to \mathbb{P}$, where the second map is the natural projection.

**Step 2:** Consider the invertible sheaves $\pi^* p_2^* F^* L_1^{-1}$ and $\pi^* p_2^* F^* L_1^{-1} \otimes \pi^* \mathcal{L}^{-1}$. By construction, there is the tautological morphism

$$t : \pi^* p_2^* F^* L_1^{-1} \to \pi^* p_2^* F^* L_1^{-1} \otimes \pi^* \mathcal{L}^{-1}.$$  

Consider further

$$id \otimes t : \pi^* p_2^* F^* L_1^{-1} \otimes \pi^* p_2^* F^* L_1^{-1} \to \pi^* p_2^* F^* L_1^{-1} \otimes \pi^* p_2^* F^* L_1^{-1} \otimes \pi^* \mathcal{L}^{-1},$$

and its restriction to the open subset $\mathcal{V} - \{0\} \times X_1$. For simplicity, the same notation will be used for this restriction. It induces then the following morphism on the higher direct images:

$$\phi : R^1 q_* \pi^* p_2^* F^* L_1^{-2} \to R^1 q_* \pi^* (p_2^* F^* L_1^{-1} \otimes p_2^* L_1^{-1} \otimes \mathcal{L}^{-1}).$$

Call $G$ for $R^1 q_* \pi^* p_2^* F^* L_1^{-2}$ and $H$ for $R^1 q_* \pi^* (p_2^* F^* L_1^{-1} \otimes p_2^* L_1^{-1} \otimes \mathcal{L}^{-1})$ temporarily. Consider $\phi : G \to H$ as a bundle homomorphism over $\mathbb{P}$. Then define $\text{Ker}(\phi)$ to be the inverse image of the zero section of the bundle $H$. It is a closed subvariety of $G$. In fact, let $0 : \mathbb{P} \to H$ be the closed immersion whose image is the zero section, then $\text{Ker}(\phi)$ is the image of the closed immersion $G \times_H \mathbb{P} \to G$ given by the fiber product.

**Step 3:** Note that $G = R^1 q_* \pi^* p_2^* F^* L_1^{-2} \cong H^1(X_1, F^* L_1^{-2}) \times \mathbb{P}$ as vector bundle. Thus consider the projections

$$\pi_1 : \text{Ker}(\phi) \to H^1(X_1, F^* L_1^{-2}); \quad \pi_2 : \text{Ker}(\phi) \to \mathbb{P}.$$  

Then since $\pi_1$ is proper, the image of $\text{Ker}(\phi)$ under $\pi_1$, for which we consider only its reduced structure, is a closed subset of $H^1(X_1, F^* L_1^{-2})$. By our construction, $\pi_1(\text{Ker}(\phi))$ is nothing but the periodic cone $K$. 

Next, we determine the dimension of $K$. It is easy to calculate the dimension of $\ker(\phi)$ using the second projection $\overline{\pi}_2$. Take a closed point $x = ([s], [\mathcal{L}]) \in \mathbb{P}$ with $[\mathcal{L}] = \pi_0(x) \in \text{Pic}^0(X_1)$ and $s \in \text{Hom}(L_1 \otimes \mathcal{L}, F^*L_1)$. The fiber $\overline{\pi}_2^{-1}(x) \subset \ker(\phi)$ is naturally identified with $\ker(\phi_s) \subset H^1(X_1, F^*L^{-1})$ defined before, where $\phi_s : H^1(X_1, F^*L^{-1}) \to H^1(X_1, L^{-1} \otimes \mathcal{L}^{-1} \otimes F^*L^{-1})$ is the induced morphism by $s$. Let $\text{Div}(s)$ be the zero divisor of the morphism $L_1 \otimes \mathcal{L} \hookrightarrow F^*L_1$. Then one computes that

$$\text{dim} \ker(\phi_s) = \deg(\text{Div}(s)) = \frac{p - 1}{2}(2g - 2 + r).$$

Therefore,

$$\text{dim} \ker(\phi) = \text{dim} \mathbb{P} + \text{dim} \ker(\phi_s) = (p - 1)(2g - 2 + r).$$

Note also that $\ker(\phi)$ is actually a vector bundle over $\mathbb{P}$ and therefore irreducible. The following lemma asserts that $\pi_1 : \ker(\phi) \to K$ is injective over a nonempty open subset and therefore $K$ is irreducible of the same dimension as $\ker(\phi)$.

**Lemma 3.9.** For every closed point $x = ([s], [\mathcal{L}]) \in \mathbb{P}(p_1s(L_1^{-1} \otimes \mathcal{L}^{-1} \otimes F^*L_1))$, the intersection

$$\ker(\phi_s) \cap (\bigcup_{([s'], [\mathcal{L}']) \neq ([s], [\mathcal{L}]) \in \mathbb{P}(p_1s, (L_1^{-1} \otimes \mathcal{L}^{-1} \otimes F^*L_1)))$$

in the ambient space $H^1(X_1, F^*L_1^{-2})$ is contained in a proper closed subset of $\ker(\phi_s)$. Therefore, the map $\pi_1$ is injective over any closed point of $\ker(\phi_s)$ away from the above intersection.

**Proof.** Assume that $\ker(\phi_s) \cap \ker(\phi_{s'}) \neq 0$ for some $([s'], [\mathcal{L}']) \in \mathbb{P}$. Note that $\ker(\phi_s) \cap \ker(\phi_{s'})$ is the kernel of the map

$$\text{Ext}^1(F^*L_1, F^*L_1^{-1}) \xrightarrow{(\phi_s, \phi_{s'})} \text{Ext}^1(L_1 \otimes \mathcal{L}, F^*L_1^{-1}) \oplus \text{Ext}^1(L_1 \otimes \mathcal{L}', F^*L_1^{-1}),$$

which is induced by the sheaf morphism

$$F^*L_1^{-2} \xrightarrow{(s, s')} L_1^{-1} \otimes \mathcal{L}^{-1} \otimes F^*L_1^{-1} \oplus L_1^{-1} \otimes \mathcal{L}'^{-1} \otimes F^*L_1^{-1}.$$

A moment of thought gives you the saturation of the image of (3.9.1), which is $F^*L_1^{-2} \otimes \mathcal{O}(D_{s, s'})$ with $D_{s, s'} = \text{Div}(s) \cap \text{Div}(s')$. Therefore the morphism (3.9.1) factors as

$$F^*L_1^{-2} \hookrightarrow F^*L_1^{-2} \otimes \mathcal{O}(D_{s, s'}) \hookrightarrow L_1^{-1} \otimes \mathcal{L}^{-1} \otimes F^*L_1^{-1} \oplus L_1^{-1} \otimes \mathcal{L}'^{-1} \otimes F^*L_1^{-1}.$$

The long exact sequence of cohomologies of the following short exact sequence of vector bundles

$$0 \to F^*L_1^{-2} \otimes \mathcal{O}(D_{s, s'}) \to L_1^{-1} \otimes \mathcal{L}^{-1} \otimes F^*L_1^{-1} \oplus L_1^{-1} \otimes \mathcal{L}'^{-1} \otimes F^*L_1^{-1} \to 0$$

gives the injective map

$$H^1(X_1, F^*L_1^{-2} \otimes \mathcal{O}(D_{s, s'})) \to H^1(X_1, L_1^{-1} \otimes \mathcal{L}^{-1} \otimes F^*L_1^{-1}) \oplus H^1(X_1, L_1^{-1} \otimes \mathcal{L}'^{-1} \otimes F^*L_1^{-1}),$$
as $H^0(X_1, L_1^{-2} \otimes \mathcal{L}^{-1} \otimes \mathcal{L}'^{-1} \otimes \mathcal{O}(-D_{s,s'})) = 0$ for degree reason. Therefore, Ker$(\phi_s) \cap$ Ker$(\phi_{s'})$ is the kernel of the map

$$H^1(X_1, F^*L_1^{-2}) \rightarrow H^1(X_1, F^*L_1^{-2} \otimes \mathcal{O}(D_{s,s'})),$$

which turns to be the image of the injection

$$H^0(X_1, \mathcal{O}_{D_{s,s'}}) \hookrightarrow H^1(X_1, F^*L_1^{-2}).$$

Here $H^0(X_1, F^*L_1^{-2} \otimes \mathcal{O}(D_{s,s'})) = 0$ again for the degree reason. Clearly,

$$\dim(\text{Ker}(\phi_s) \cap \text{Ker}(\phi_{s'})) = \deg(D_{s,s'}) < \deg(\text{Div}(s)) = \dim\text{Ker}(\phi_s).$$

The above argument actually proves that

$$\text{Ker}(\phi_s) \cap \bigcup_{\{s',[\mathcal{L}']\neq[s],[\mathcal{L}]\in \mathcal{F}(p_1,(L_1^{-1} \otimes \mathcal{L}'^{-1} \otimes F^*L_1))} \text{Ker}(\phi_{s'}) \subseteq \bigcup_{D' \subset \text{Div}(s)} \text{Im}(H^0(X_1, \mathcal{O}_{D'}) \hookrightarrow H^1(X_1, F^*L_1^{-2})),$$

while the latter is a finite union of closed subsets of dimension strictly less than $\dim\text{Ker}(\phi_s)$. This completes the proof. \hfill \Box

We conclude this section with the following

**Remark 3.10.** To the study of $A$ and $K$, we could replace $(E_1, \theta_1)$ of (1.0.2) with one of its twists $(E_1, \theta_1) \otimes (M,0)$ by a line bundle $M$ equipped with the zero Higgs field. Indeed, first one gets the same ambient space

$$\text{Ext}^1(F^*L_1 \otimes F^*M, F^*L_1^{-1} \otimes F^*M) = \text{Ext}^1(F^*L_1, F^*L_1^{-1}),$$

and then the same affine subspace $A$, which is just because of the canonical isomorphism

$$C^{-1}_{X_1 \subset X_2}((E_1, \theta_1) \otimes (M,0)) \cong C^{-1}_{X_1 \subset X_2}((E_1, \theta_1) \otimes (F^*M, \nabla_{\text{can}})),$$

and also the same periodic cone $K$ where one considers those extensions of $F^*L_1^{-1} \otimes F^*M$ by $F^*L_1 \otimes F^*M$ containing the invertible subsheaf $L_1 \otimes F^*M$.

### 4. The degeneration argument

For a given curve $X_1 \in \mathcal{M}_{g,r}$, it seems hard in general to determine whether $A$ intersects with $K$ or not. The dimension counting 3.5.1 makes the problem more interesting. The principal aim of this section is to show $A \cap K \neq \emptyset$ for a generic curve via a degeneration argument. So we shall consider the coarse moduli space of stable curves $\overline{\mathcal{M}}_{g,r}$ and investigate the intersection behavior of $A$ and $K$ in a neighborhood of a so-called *totally degenerate* $(g,r)$-curve (abbreviated as t.d. $(g,r)$-curve). The advantage of considering a t.d curve is that one can exhibit explicitly the nonemptiness of $A \cap K$. In fact, Proposition 4.5 asserts that it consists of exactly one point (multiplicity not counted). As another output of this method, we also show the existence of ordinary sections for a generic curve, which is the crucial input for lifting the periodic Higgs bundle to an arbitrary truncated Witt ring.
4.1. The relative $A$ and $K$ over a local base. Recall that a t.d. $(g, r)$-curve is a nodal curve with $r$-marked points whose arithmetic genus is $g$ and normalization consists of disjoint copies of $\mathbb{P}^1$ with 3 marked points, say $\{0, 1, \infty\}$ (the inverse image of nodes are also considered as marked points on the each component of the normalization). In the moduli $\overline{M}_{g,r}$ of stable $(g, r)$-curves, there always exists such a t.d. $(g, r)$-curve.

Let $\chi$ be a t.d. $(g, r)$-curve. We shall restrict ourself to a local smooth deformation of $\chi$. Let $f: \mathcal{X}_1 \to B_1$ over $k$ be a local log smooth deformation of $\chi$ (see [FKa00]). That is, $f$ is a log smooth projective morphism from smooth quasi-projective $\mathcal{X}_1$ to a log smooth affine curve $B_1$ whose central fiber over $0 \in B_1$ is isomorphic to $\chi$ and any other closed fiber is a smooth $(g, r)$-curve. The family $f$ is subject to an étale base change $B_1' \to U_1$ with $U_1 \subset B_1$ an open affine neighborhood of 0, always equipped with the induced log structure. The relative sheaf of log differential forms $\omega_{\mathcal{X}_1/B_1}$ (w.r.t the given log structures on $\mathcal{X}_1$ and $B_1$) is an invertible sheaf by Proposition 3.10 [KKa]. Let $\mathcal{T}_{\mathcal{X}_1/B_1}$ be its $\mathcal{O}_{\mathcal{X}_1}$-dual. We shall consider the following relative Higgs bundle

\begin{equation}
(\mathcal{E}, \theta) = (\omega_{\mathcal{X}_1/B_1} \oplus \mathcal{O}_{\mathcal{X}_1}, \theta),
\end{equation}

where $\theta: \omega_{\mathcal{X}_1/B_1} \to \mathcal{O}_{\mathcal{X}_1} \otimes \omega_{\mathcal{X}_1/B_1}$ is the natural isomorphism. For a closed point $b \neq 0$, it restricts to a Higgs bundle (1.0.2) over $\mathcal{X}_{1,b} := f^{-1}(b)$ twisted by $L_1$, i.e,

\begin{equation}
(L_1^2 \oplus \mathcal{O}_{\mathcal{X}_1}, \theta_1), \quad \theta_1 : L_1^2 \xrightarrow{=} \mathcal{O}_{\mathcal{X}_1} \otimes \omega_{\mathcal{X}_1/k}.
\end{equation}

**Lemma 4.1.** The coherent sheaf of $\mathcal{O}_{B_1}$-modules $R^1 f_* F_{\mathcal{X}_1, B_1}^* \mathcal{T}_{\mathcal{X}_1/B_1}$ is locally free of rank $(2p + 1)(g - 1) + pr$.

**Proof.** For each closed point $b \in B_1$, the fiber $(F_{\mathcal{X}_1, B_1}^* \mathcal{T}_{\mathcal{X}_1/B_1})_b$ is naturally isomorphic to $F^* \mathcal{T}_{\mathcal{X}_{1,b}/k}$. We have shown that for $b \neq 0$, the dimension $h^1(F^* \mathcal{T}_{\mathcal{X}_{1,b}/k})$ is equal to $(2p + 1)(g - 1) + pr$. By the theorem of Grauert, it is reduced to show that it is so for the fiber over 0, which is Proposition 4.4 (1) below. \hfill \Box

Now we shall construct two closed subsets $A$ and $K$ inside the vector bundle $R^1 f_* F_{\mathcal{X}_1, B_1}^* \mathcal{T}_{\mathcal{X}_1/B_1}$ over $B_1$, such that over a closed point $0 \neq b \in B_1$, they specialize to $A$ and $K$ given before.

**Construction of $A$:** Let

\[ \nabla_{can} : F_{\mathcal{X}_1}^* \mathcal{T}_{\mathcal{X}_1/B_1} \to F_{\mathcal{X}_1}^* \mathcal{T}_{\mathcal{X}_1/B_1} \otimes \omega_{\mathcal{X}_1/B_1} \]

be the relative flat bundle. From it, one gets the relative de-Rham complex

\[ \Omega_{dR}^* := \Omega_{dR}^*(F_{\mathcal{X}_1}^* \mathcal{T}_{\mathcal{X}_1/B_1}, \nabla_{can}). \]

There are natural morphisms of complexes $\Omega_{dR}^* \to F_{\mathcal{X}_1}^* \mathcal{T}_{\mathcal{X}_1/B_1}$ and $\mathcal{T}_{\mathcal{X}_1/B_1} \to \Omega_{dR}^*$ which induce the corresponding morphism on higher direct images:

\[ \Gamma : R^1 f_* dR(F_{\mathcal{X}_1}^* \mathcal{T}_{\mathcal{X}_1/B_1}, \nabla_{can}) \to R^1 f_* F_{\mathcal{X}_1}^* \mathcal{T}_{\mathcal{X}_1/B_1}, \]

and

\[ \Lambda : R^1 f_* \mathcal{T}_{\mathcal{X}_1/B_1} \to R^1 f_* dR(F_{\mathcal{X}_1}^* \mathcal{T}_{\mathcal{X}_1/B_1}, \nabla_{can}). \]
The coherent sheaves $R^1f_*dR(F_{\mathcal{X}_1/B_1}^*, T_{\mathcal{X}_1/B_1}, \nabla_{\text{can}})$ and $R^1f_*T_{\mathcal{X}_1/B_1}$ are locally free of rank \(3g - 2 + r\) and \(3g - 3 + r\) respectively by Proposition \[1\] (2)-(3). Also, the sheaf morphisms $\Gamma$ and $\Gamma \circ \Lambda$ are injective as they are so at each fiber over a closed point by ibid. Define $\mathcal{B}$ (resp. $\mathcal{W}_F$) to be subsheaf $\Gamma(R^1f_*dR(F_{\mathcal{X}_1/B_1}^*, T_{\mathcal{X}_1/B_1}, \nabla_{\text{can}}))$ (resp. $\Gamma \circ \Lambda(R^1f_*T_{\mathcal{X}_1/B_1})$). Clearly, $\mathcal{W}_F \subseteq \mathcal{B}$. Take and then fix a nowhere vanishing section $\xi_0$ of $B - \mathcal{W}_F$ (it always exists after shrinking $B_1$ if necessary). We define $\mathcal{A} \subset R^1f_*F^*_{\mathcal{X}_1/B_1}$ to be the translation of $\mathcal{W}_F$ by $\xi_0$.

**Construction of $\mathcal{K}$:** We shall follow the construction of the periodic cone in the smooth case, see Section \[3,3\]. In Step 1, one replaces $p_i$ with the natural maps in the following Cartesian diagram of the fiber product:

\[
\begin{array}{c}
\text{Jac}(\mathcal{X}_1/B_1) \times_{B_1} \mathcal{X}_1 & \xrightarrow{p_2} & \mathcal{X}_1 \\
\downarrow p_1 & \downarrow f_1 & \\
\text{Jac}(\mathcal{X}_1/B_1) & \longrightarrow & B_1,
\end{array}
\]

where $\text{Jac}(\mathcal{X}_1/B_1)$ is the relative Jacobian. Let $\mathcal{L}$ be the universal line bundle over $\text{Jac}(\mathcal{X}_1/B_1) \times_{B_1} \mathcal{X}_1$. Consider the vector bundle

\[
\mathbb{V} := p_1^* \text{Hom}(p_2^*F^*_{\mathcal{X}_1/B_1}, p_2^*T_{\mathcal{X}_1/B_1} \otimes \mathcal{L}^{-1}),
\]

and its associated projective bundle $\mathbb{P}$. Let

\[
\pi : \mathbb{V} \times_{B_1} \mathcal{X}_1 \to \text{Jac}(\mathcal{X}_1/B_1) \times_{B_1} \mathcal{X}_1, \quad q : \mathbb{V} - \{0\} \times_{B_1} \mathcal{X}_1 \to \mathbb{P}
\]

be the maps defined similarly in the smooth case. Then Step 2 proceeds as in the smooth case. So one obtains a closed subvariety $\text{Ker}(\phi) \subset R^1q_*\pi^*p_2^*F^*_{\mathcal{X}_1/B_1}$. To carry out Step 3, we need to show

\[
R^1q_*\pi^*p_2^*F^*_{\mathcal{X}_1/B_1} \cong R^1f_*F^*_{\mathcal{X}_1/B_1} \times_{B_1} \mathbb{P}
\]

as vector bundle. Here we consider $R^1f_*F^*_{\mathcal{X}_1/B_1}$ as a vector bundle over $B_1$. To see that, one considers the fiber product

\[
\mathbb{P} \times_{B_1} \mathcal{X}_1 \xrightarrow{u} \mathcal{X}_1, \quad \downarrow g_1 \quad \downarrow f_1 \\
\mathbb{P} \xrightarrow{v} B_1.
\]

Note first that the map $q : \mathbb{V} - \{0\} \times_{B_1} \mathcal{X}_1 \to \mathbb{P}$ is just the composite

\[
\mathbb{V} - \{0\} \times_{B_1} \mathcal{X}_1 \xrightarrow{q_0 \times \text{id}} \mathbb{P} \times_{B_1} \mathcal{X}_1 \xrightarrow{g_1} \mathbb{P}.
\]

Second, one notices that there is a sheaf isomorphism

\[
\pi^*p_2^*F^*_{\mathcal{X}_1/B_1} \cong (q_0 \times \text{id})^*u^*F^*_{\mathcal{X}_1/B_1},
\]

so that $R^1q_*\pi^*p_2^*F^*_{\mathcal{X}_1/B_1} \cong R^1g_1^*u^*F^*_{\mathcal{X}_1/B_1}$. Finally, by the flat base change theorem, one gets a sheaf isomorphism

\[
R^1g_1^*u^*F^*_{\mathcal{X}_1/B_1} \cong v^*R^1f_*F^*_{\mathcal{X}_1/B_1},
\]
while the latter as vector bundle is isomorphic to $R^1f_*F^*\mathcal{T}_{X_1/B_1} \times_{B_1} \mathbb{P}$. So one gets the proper map

$$\pi_1 : \text{Ker}(\phi) \rightarrow R^1f_*F^*\mathcal{T}_{X_1/B_1}$$

obtained from the composite

$$\text{Ker}(\phi) \hookrightarrow R^1q_*\pi^*p^*_2F^*\mathcal{T}_{X_1/B_1} \cong R^1f_*F^*\mathcal{T}_{X_1/B_1} \times_{B_1} \mathbb{P} \rightarrow R^1f_*F^*\mathcal{T}_{X_1/B_1}.$$ 

We define $K$ to be the closed subvariety $\pi_1(\text{Ker}(\phi))$.

By specializing to a closed point $b \neq 0 \in B_1$, one finds that $\mathcal{A}_b$ is a nonzero translation of $\mathcal{W}_{F,b}$. Of course, this may differ from the original definition of $A$ in the smooth case by a translation. See Proposition 3.1. But it is clear that $\mathcal{A}_b \cap \mathcal{K}_b \neq \emptyset$ iff $A \cap K \neq \emptyset$ by Remark 3.10. The following paragraphs are devoted to $\mathcal{A}_0 \cap \mathcal{K}_0$. For simplicity of notation, we shall again use $A$ for $\mathcal{A}_0$, $B$ for $\mathcal{B}_0$, $K$ for $\mathcal{K}_0$, $\alpha$ for the induced morphism of $\Gamma$ on fibers over $0$ and similarly $\beta$ for $\Lambda$ over $0$.

4.2. The $(0,3)$-curve. We study first the unique $(0,3)$-curve up to isomorphism. In this case, $\omega_{\chi/k} = \mathcal{O}_{\mathbb{P}^1}(1)$ and we are considering the following logarithmic Higgs bundle

$$(\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}, \theta) = id : \mathcal{O}_{\mathbb{P}^1}(1) \rightarrow \mathcal{O}_{\mathbb{P}^1} \otimes \omega_{\chi/k}.$$ 

Thus the ambient vector space $H^1(\chi, F^*\mathcal{T}_{\chi/k}) = H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-p))$ is of dimension $p - 1$; The linear subspace

$$W_F = \text{Im}(F^* : H^1(\chi, \mathcal{T}_{\chi/k}) \rightarrow H^1(\chi, F^*\mathcal{T}_{\chi/k}))$$

is of dimension zero. Hence $A$ is just a nonzero vector inside the line $B : \text{Im}(H^1_{\text{dr}}(F^*\mathcal{T}_{\chi/k}, \nabla_{\text{can}}) \rightarrow H^1(\chi, F^*\mathcal{T}_{\chi/k}))$. Here the dimension and the injectivity are proved in the same way as Lemma 3.4 and Lemma 3.2.

**Lemma 4.2.** The periodic cone $K \subset H^1(\chi, F^*\mathcal{T}_{\chi/k})$ is the whole space. Therefore, $A \cap K$ consists of one point $\xi_0$.

**Proof.** Take any $\xi \in H^1(\chi, F^*\mathcal{T}_{\chi/k})$ which is the extension class of the following short exact sequence:

$$(4.2.1) \quad 0 \rightarrow \mathcal{O}_{\mathbb{P}^1} \rightarrow H_\xi \rightarrow F^*\omega_{\chi/k} = \mathcal{O}_{\mathbb{P}^1}(p) \rightarrow 0.$$ 

Note that $H_\xi \cong \mathcal{O}_{\mathbb{P}^1}(d_1) \oplus \mathcal{O}_{\mathbb{P}^1}(d_2)$ for two integers $d_1, d_2$ with $d_1 + d_2 = p$. Thus the larger number between $d_1$ and $d_2$ must be $\geq \frac{p+1}{2}$. It implies that $H_\xi$ has an invertible subsheaf of degree $\frac{p+1}{2}$. Therefore, $K$ is just the whole space $H^1(\chi, F^*\mathcal{T}_{\chi/k})$. The unique element of $\xi_0 = A \cap K$ corresponds to the following nonsplit extension (unique up to isomorphism):

$$(4.2.2) \quad 0 \rightarrow (\mathcal{O}_{\mathbb{P}^1}, \nabla_{\text{can}}) \rightarrow (H_0, \nabla) \rightarrow (F^*\omega_{\chi/k}, \nabla_{\text{can}}) \rightarrow 0.$$ 

□
Consider further

\[ 0 \longrightarrow \mathcal{O}_X \longrightarrow H_0 \longrightarrow F^*\omega_{\chi/k} \longrightarrow 0 \]

Here \( s : \omega_{\chi/k}^{p-1} \to F^*\omega_{\chi/k} \) is the composite \( \omega_{\chi/k}^{p-1} \to H_0 \to F^*\omega_{\chi/k} \). Using the same argument involving the nonvanishing \( p \)-curvature and the degree comparison as in the proof of Proposition 3.6, one concludes that

\begin{equation}
\nabla : \omega_{\chi/k}^{p-1} \cong (H_0/\omega_{\chi/k}^{p-1}) \otimes \omega_{\chi/k}
\end{equation}

(One can also deduce that \( H_0 \cong \mathcal{O}_{\mathbb{P}^1}(p-1) \oplus \mathcal{O}_{\mathbb{P}^1}(\frac{p-1}{2}) \)).

**Proposition 4.3.** Let \( P \) be one of three marked points of \( \chi \). Then the value \( s(P) \) is nonzero.

We take a digression into discussion on the local behaviour of the connection around a marked point. Assume the three marked points of \( \chi \) are given by \( \{0, 1, \infty\} \). Then there is an obvious \( \mathbb{W}_2 \)-lifting, say \( \tilde{\chi} \), of the log curve \( \chi \), which is given by \( (\mathbb{P}^1, 0, 1, \infty) \) over \( \mathbb{W}_2 \). Let \( C = C_{\chi \subset \tilde{\chi}} \) (resp. \( C^{-1} = C_{\chi \subset \tilde{\chi}}^{-1} \)) be the Cartier transform (resp. inverse Cartier transform), see Appendix. It is an equivalence of categories. From an extension of flat bundles,

\begin{equation}
0 \to (\mathcal{O}_X, \nabla_{\text{can}}) \to (H, \nabla) \to (F^*\omega_{\chi/k}, \nabla_{\text{can}}) \to 0,
\end{equation}

one obtains an extension of Higgs bundles

\begin{equation}
0 \to (\mathcal{O}_X, 0) \to (E, \theta) = C(H, \nabla) \to (\omega_{\chi/k}, 0) \to 0,
\end{equation}

and vice versa. Forgetting the Higgs field in the above extension, one sees that \( E \) is the direct sum \( \mathcal{O}_X \oplus \omega_{\chi/k} \), and therefore \( (E, \theta) \) is of form

\[ (E, \theta_a) := (\omega_{\mathbb{P}^1_{\log}} \oplus \mathcal{O}_{\mathbb{P}^1}, \theta_a), \theta = a : \omega_{\mathbb{P}^1_{\log}} \to \mathcal{O}_X \otimes \omega_{\mathbb{P}^1_{\log}} = \omega_{\mathbb{P}^1_{\log}} \text{ for some } a \in k. \]

Here \( \omega_{\mathbb{P}^1_{\log}} \) is the sheaf of log differentials on \( \mathbb{P}^1 \) with poles along \( \{0, 1, \infty\} \). We look at the connection of \( (H, \nabla_a) := C_{\chi \subset \tilde{\chi}}^{-1}(E, \theta_a) \) around the node 0. Let \([x : y]\) be the homogenous coordinate of \( \mathbb{P}^1 \) with \( t = x/y \) an affine coordinate of \( \{y \neq 0\} \subset \mathbb{P}^1 \). Then \( U = \text{Spec} \ k[t, (t-1)^{-1}] \) is an open affine neighborhood of 0. Take \( \bar{U} = \text{Spec} \ W_2[t, (t-1)^{-1}] \subset \tilde{\chi} \), together with the standard log Frobenius lifting \( \bar{F} \) determined by \( t \mapsto t^p \). Take the global basis \( e_1 = 1 \) for \( \mathcal{O}_{\mathbb{P}^1} \) and the local basis \( e_2 = \frac{dt}{t} \) for \( \omega_{\mathbb{P}^1_{\log}} \) over \( U \). Then by the construction of the inverse Cartier transform in the Appendix, a local expression of the connection \( \nabla_a \) over \( U \) is given by:

\[ \nabla_a \{e_1 \otimes 1, e_2 \otimes 1\} = \{e_1 \otimes 1, e_2 \otimes 1\} \begin{pmatrix} 0 & a^p \frac{dt}{t} \\ 0 & 0 \end{pmatrix}. \]
where \([e_i] := e_i \otimes 1, i = 1, 2\) is the natural basis of \(H\) over \(U\). It follows immediately that the residue of \(\nabla_a\) at the origin is of expression

\[
\text{Res}_0(\nabla_a)\{[e_1](0), [e_2](0)\} = \{[e_1](0), [e_2](0)\} \left( \begin{array}{c} 0 & a^p \\ 0 & 0 \end{array} \right).
\]

Using the transformations \(t \mapsto t - 1\) resp. \(t \mapsto t^{-1}\), one obtains the analogue for the marked points \(1\) resp. \(\infty\).

**Proof of Proposition 4.3.** The logarithmic flat bundle \((H_0, \nabla)\) is given by \(C^{-1}(E, \theta_1)\) up to isomorphism. Let \(P\) be any marked point. By the previous discussion, the \(k\)-linear map \(\text{Res}_P(\nabla) : H_0(P) \to H_0(P)\) has the property that \(\text{Res}_P(\nabla)([e_1](P)) = 0\). On the other hand, the isomorphism \(\text{(4.2.3)}\), which follows from a global reason i.e. the degree comparison, induces the isomorphism at the point \(P\):

\[
\text{Res}_P(\nabla) : \omega^{\frac{p+1}{\chi}}_{\chi/k}(P) \xrightarrow{\cong} \omega^{\frac{p-1}{\chi}}_{\chi/k}(P).
\]

Now if \(s(P)\) was zero, then \([e_1](P)\) is a basis for the image of \(\omega^{\frac{p+1}{\chi}}_{\chi/k}(P) \to H_0(P)\). But this is impossible. \(\square\)

**4.3. A totally degenerate \((g, r)\)-curve.** Let \(\chi\) be a t.d \((g, r)\)-curve. A direct calculation shows that \(\chi\) has \(\nu = 2g - 2 + r\) irreducible components and \(\delta = 3g - 3 + r\) nodes. Let \(P := \{P_i \in \chi\mid 1 \leq i \leq \delta\}\) and \(\{\chi_i \mid 1 \leq i \leq \nu\}\) respectively be the sets of nodes and irreducible components. Let \(\gamma : \tilde{\chi} \to \chi\) be the normalization. Considering the inverse images of nodes as marked points on \(\tilde{\chi}\), \(\tilde{\chi}\) is a disjoint union of \((0, 3)\)-curves. In other words, one obtains \(\chi\) by gluing \(\nu\)-copies of \((0, 3)\)-curves along marked points in a suitable way. We can assume each copy to be \((\mathbb{P}^1, 0, 1, \infty)\). For good reason, we shall label the set \(P\) of nodes and the set of marked points \(Q := \{Q_j, 1 \leq j \leq \nu\}\), that is, give a map \(P \cup Q \to \{0, 1, \infty\}\). It is not difficult to see that there is an excellent labeling of \(P \cup Q\) such that to each component \(\chi_i\) it restricts to a bijection to \(\{0, 1, \infty\}\). It is more convenient to discuss the gluing of objects over two irreducible components along a node via an excellent labeling, as we can use the same local coordinate of \(\mathbb{P}^1\) around one of three marked points as well as the same local basis of a vector bundle and etc.

**Proposition 4.4.**

1. \(\dim H^1(\chi, F^*\mathcal{T}_{\chi/k}) = (2p + 1)(g - 1) + pr\);
2. the map \(F^* = \alpha \circ \beta : H^1(\chi, \mathcal{T}_{\chi/k}) \to H^1(\chi, F^*\mathcal{T}_{\chi/k})\) is injective and \(\dim W_F = 3g - 3 + r\);
3. the map \(\alpha : H^1_{dR}(F^*\mathcal{T}_{\chi/k}, \nabla_{can}) \to H^1(\chi, F^*\mathcal{T}_{\chi/k})\) is injective and \(\dim B = 3g - 2 + r\).

**Proof.** (1) One has the following short exact sequence of \(\mathcal{O}_\chi\)-modules:

\[
0 \to F^*\mathcal{T}_{\chi/k} \to \gamma^*F^*\mathcal{T}_{\chi/k} \to \bigoplus_{i=1}^{\nu} k_{P_i} \to 0,
\]

where \(k_{P_i}\) denotes the skyscraper sheaf with stalk \(k\) at \(P_i\) and 0 elsewhere. Over each component \(\chi_i\), one has

\[
\gamma^*(F^*\mathcal{T}_{\chi})|_{\chi_i} \cong F^*\mathcal{T}_{\chi_i} \cong \mathcal{O}_{\mathbb{P}^1}(-p),
\]
and since \( H^1(\chi, \gamma^* (F^* T_\chi)) = H^1(\tilde{\chi}, \gamma^* (F^* T_\chi)) \) as \( \gamma \) is finite, it follows that that

\[
h^1(\chi, F^* T_\chi/k) = \sum_{i=1}^\nu h^1(\chi_i, \mathcal{O}_{\tilde{p}^i}(-p)) + \delta = (2p + 1)(g - 1) + pr.
\]

(2) Using the long exact sequence of cohomologies associated to the short exact sequence

\[
(4.4.2) \quad 0 \to T_\chi/k \to \gamma^* T_\chi/k \to \bigoplus_{i=1}^\delta kP_i \to 0,
\]

one calculates the dimension:

\[
h^1(\chi, T_\chi/k) = h^0(\chi, \bigoplus_{i=1}^\delta kP_i) = \delta.
\]

It remains to show the injectivity of \( F^* \). Like in the smooth case, we shall show that, if an extension of the form

\[
(4.4.3) \quad 0 \to \mathcal{O}_\chi \to E \to \omega_{\chi/k} \to 0
\]

is nonsplit, then its Frobenius pullback is nonsplit.

The restriction of \( E \) to each component \( \chi_i \) is split. In other words, \( E \) is obtained by gluing copies of \( \mathcal{O}_{p^1} \oplus \omega_{p^1}^{\log} \) along the nodes. Let \( P_i \) be an arbitrary node where the components \( \chi_1 \) and \( \chi_2 \) intersect and which goes to 0 (it could be any other marked point) under an excellent labeling. Take the basis \( \{ e_1 \} \) (resp. \( \{ e_2 \} \)) of \( \mathcal{O}_{p^1} \) (resp. \( \omega_{p^1}^{\log} \)) at the neighborhood \( U \) of 0, as discussed in the \((0, 3)\)-curve case. Let \( \{ e_{1,i}, e_{2,i} \}_{i=1,2} \) be two copies of such. We shall see that the set of isomorphism classes of extensions \((4.4.3)\) is in one-to-one correspondence of the set \( \{ A_i \}_{1 \leq i \leq \delta} \) with \( A_i \) the following upper triangular matrix

\[
A_i = \begin{pmatrix}
1 & \lambda_i \\
0 & -1
\end{pmatrix}.
\]

Indeed, the formula

\[
(4.4.4) \quad \{ e_{1,2}(0), e_{2,2}(0) \} = \{ e_{1,1}(0), e_{2,1}(0) \} A_i
\]

gives the gluing data of the fibers of two copies of \( \mathcal{O}_{p^1} \oplus \omega_{p^1}^{\log} \) at 0 and different \( \lambda_i \) gives locally non-isomorphic \( E \) over an open neighborhood of \( P_i \). The second diagonal element in \( A_i \) is \(-1\) for the following reason: Let \( \{ t_i \}_{i=1,2} \) be the local coordinate of an open affine neighborhood \( U_i \subset \chi_i, i = 1, 2 \) of the node \( P_i \). Set \( U := U_1 \cup U_2 \). Then \( \omega_{\chi/k}(U) \) is expressed by

\[
\mathcal{O}_U \left\{ \frac{dt_1}{t_1} \right\} \oplus \mathcal{O}_U \left\{ \frac{dt_2}{t_2} \right\} \frac{\mathcal{O}_U \left\{ \frac{dt_1}{t_1} + \frac{dt_2}{t_2} \right\}}{\mathcal{O}_U \left\{ \frac{dt_1}{t_1} \right\} \oplus \mathcal{O}_U \left\{ \frac{dt_2}{t_2} \right\}}.
\]

Therefore, \( \frac{dt_2}{t_2} = -\frac{dt_1}{t_1} \) is the transition relation for the gluing of \( \omega_{\chi/k} \) at the node. Thus, we have obtained an explicit \( k \)-linear isomorphism

\[
H^1(\chi, T_{\chi/k}) \cong k^\delta, \quad [E] \mapsto (\lambda_1, \cdots, \lambda_\delta).
\]
Now let $\{[e_{1,i}], [e_{2,i}]\}_{i=1,2}$ be the natural basis of the restrictions of $F^*E$ to $\chi_1$ and $\chi_2$. It is clear that at $P_i$, the following formula holds:

$$
\{[e_{1,2}](0), [e_{2,2}](0)\} = \{[e_{1,1}](0), [e_{2,1}](0)\} \begin{pmatrix}
1 & \lambda_{i}^p \\
0 & -1
\end{pmatrix}.
$$

Thus, if $F^*E$, as an extension of $F^*\omega_{\chi/k}$ by $O_\chi$, is nonsplit, then there is a nonzero $\lambda_{i_0}$ for some $i_0$. As $\lambda_{i_0}$ is nonzero, $E$ is nonsplit as well.

(3) Note that one still has the isomorphism

$$
H^1_{dR}(F^*T_{\chi/k}, \nabla_{can}) \cong \text{Ext}^1((F^*\omega_{\chi/k}, \nabla_{can}); (O_\chi, \nabla_{can})).
$$

The latter space classifies the isomorphism classes of extensions of the following form:

$$
(4.4.5) \quad 0 \to (O_\chi, \nabla_{can}) \to (H, \nabla) \to (F^*\omega_{\chi/k}, \nabla_{can}) \to 0.
$$

Like in the smooth case, one interprets $\alpha$ as the forgetful map

$$
\text{Ext}^1((F^*\omega_{\chi/k}, \nabla_{can}); (O_\chi, \nabla_{can})) \to \text{Ext}^1(F^*\omega_{\chi/k}, O_\chi) \cong H^1(\chi, F^*T_{\chi/k}).
$$

Then we shall show that if the bundle $H$ of the extension $(H, \nabla)$ of form $(4.4.5)$ is split, then the extension is split. The proof is similar to (2) but easier. First of all, the splitting of $H$ restricts to a splitting of $H|_{\chi_i}$ on the irreducible component $\chi_i$ for each $i$. As $\chi_i$ is the $(0,3)$ curve, it follows that $(H, \nabla)|_{\chi_i}$ is also split. Thus $\nabla|_{\chi_i}$ is nothing but the canonical connection $\nabla_{can}$. Therefore, $\nabla$ is just the canonical connection on $H = F^*\omega_{\chi/k} \oplus O_\chi$, and therefore $(H, \nabla)$ is split.

As $H^1_{dR}(F^*T_{\chi/k}, \nabla_{can})$ is the fiber of the coherent sheaf $R^1 f_*dR(F^*_{\chi_1} T_{\chi_1/B_1}, \nabla_{can})$ at $0 \in B_1$, its dimension is $\geq 3g - 2 + r$ by the upper semi-continuity. Thus it suffices to show $\dim H^1_{dR}(F^*T_{\chi/k}, \nabla_{can}) \leq 3g - 2 + r$, and it suffices to consider only nonsplit extensions for the sake of counting the dimension. First of all, one notices that, if two nonsplit extensions of form $(4.4.5)$ have the isomorphic middle terms, then their corresponding extension classes differ by a scalar. This is because any isomorphism between the middle terms of two extensions must preserve the first terms i.e. $(O_\chi, \nabla_{can})$, since it is the unique flat subbundle of each middle term. Next, for a nonsplit extension $(H, \nabla)$, the restrictions of $H$ to any two components are isomorphic. Thus the isomorphism classes of $H$ are uniquely determined by the gluing datum at the nodes. As shown in the proof of (2), the gluing data under the natural basis at a node $P_i$ is given by a upper triangular matrix $B_i = \begin{pmatrix} 1 & \mu_i \\ 0 & -1 \end{pmatrix}$ with the transition relation

$$
(4.4.6) \quad \{[e_{1,2}](P_i), [e_{2,2}](P_i)\} = \{[e_{1,1}](P_i), [e_{2,1}](P_i)\} B_i.
$$

Finally, we need to count how many connections there are on a given nonsplit $H$. We claim that it is unique up to isomorphism. To see this, we consider a node $P$ where components $\chi_1$ and $\chi_2$ intersect. Let $(H_i, \nabla_i)$ be the restriction of $(H, \nabla)$ to $\chi_i$. As discussed in the $(0,3)$-curve case, $(H_i, \nabla_i) = C^{-1}(\omega_{\chi_1}^{2g-1} \oplus O_{\chi_1}, \theta_{a_i})$ (and we can assume $a_1 \neq 0$). Assume the gluing matrix under the natural basis at the
node is given by \( \begin{pmatrix} 1 & \mu \\ 0 & -1 \end{pmatrix} \). By the discussions on the residues of connections at marked points in the \((0, 3)\)-curve case, and since \( \nabla_1 \) and \( \nabla_2 \) glue at the node \( P \), we obtain the equality
\[
\begin{pmatrix} 1 & \mu \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -a^p_1 \\ 0 & 0 \end{pmatrix}.
\]
It follows that \( a_1 = a_2 \) (and particularly \( a_2 \neq 0 \)!) As \( \chi \) is connected, it follows that for each \( i \),
\[
(H_i, \nabla_i) = C^{-1}((\omega_{\varphi_1}^1 \oplus O_{\varphi_1}, \theta_{a_1}).
\]
Thus \((H, \nabla)\) is unique up to isomorphism. Summarizing the discussions, one finds that the isomorphism classes of extensions are bounded from above by \( 3g - 3 + r \) plus 1, where \( 3g - 3 + r \) denotes for the dimension of the tuples \( \{ (\mu_1, \cdots, \mu_g) \} \) while 1 denotes for the scaling of the extension classes. □

After the above preparation, we can now prove \( A \cap K \neq \emptyset \).

**Proposition 4.5.** For every totally degenerate \((g, r)\)-curve, the set \( A \cap K \) consists of one unique element.

**Proof.** Recall that \( A \) is a translation of \( W_F \) by an element in \( B - W_F \). By the previous discussions, one knows that \( B = W_F \oplus k\{\xi_0\} \). Let \((H_0, \nabla)\) be the corresponding extension to \( \xi_0 \). Then the elements \((H, \nabla)\) of \( A \) is in one-to-one correspondence to tuples \( \mu := \{ \mu_1, \cdots, \mu_g \} \in k^\delta \), so that \( \mu = 0 \) corresponds to \((H_0, \nabla)\). We claim that there exists a unique tuple \( \mu \) such that the corresponding \( H \) admits an invertible subsheaf \((\omega_{\chi/k}^{\mu_1}) \oplus \mathcal{L} \) for some degree zero invertible sheaf \( \mathcal{L} \). This claim is equivalent to saying that \( A \) intersects with \( K \) at one unique point (multiplicity not counted). By Lemma 4.2 we know that the restriction \( H_i \) of \( H \) to \( \chi_i \) (i arbitrary) admits a unique \( \omega_{\chi/k}^{\mu_1} \mapsto H_i \). Assume that \( P_j \) is a node lying in \( H_i \). By Lemma 4.3 the image of the fiber \( \omega_{\chi/k}^{\mu_1}(P_j) \) inside \( H_i(P_j) \) admits a basis of the form
\[
h_j = [e_{1,j}](P_j) + b_j[e_{2,j}](P_j), \quad b_j \neq 0.
\]
The nonzeroness of \( b_j \) for all \( j \) follows from the proof of Proposition 4.4 (3) for \( \xi_0 \notin W_F \). Recall that the transition relation of \( H_i(P_j) \) in terms of a natural basis is given by (4.4.6). Thus, in order to glue these two invertible subsheaves \( \omega_{\chi/k}^{\mu_1}, i = 1, 2 \) together, the necessary and sufficient condition is the solvability of the following equation:
\[
\begin{pmatrix} 1 \\ b_j \end{pmatrix} = u \begin{pmatrix} 1 & \mu_j \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ b_j \end{pmatrix}, \quad \text{for some } u \in k^*.
\]
One obtains the unique solution \( u = -1; \mu_j = \frac{\mu_1}{b_1} \). It follows that there is a unique \( \mu \in k^\delta \) such that the invertible subsheaves \( \{ \omega_{\chi/k}^{\mu_1} \} \) glue into a global invertible subsheaf of the corresponding \( H \). It is also easy to see that, the so-glued invertible
subsheaf is the same as \( \omega_{X/k}^{\frac{p+1}{2}} \) for \( \frac{p+1}{2} \) odd and differs from it by a two-torsion for \( \frac{p+1}{2} \) even.

Let \( \xi_1 \) be the unique point \( A \cap K \). Assume that \( \xi_1 \in \text{Ker}(\phi_s) \) for some section \( s \in \text{Hom}(\omega_{X/k}^{\frac{p+1}{2}} \otimes L, F^*(\omega_{X/k})) \), where \( L \) is a certain two-torsion line bundle. Its dual \( \check{s} \) induces the composite

\[
F^*T_{X/k} = T_{X/k}^{\frac{p+1}{2}} \otimes L^{-1} \rightarrow T_{X/k} \otimes L^{-2} = T_{X/k},
\]
and the map on \( H^1 \).

**Proposition 4.6.** Notation as above. The composite map

\[
H^1(\chi, \mathcal{T}_{X/k}) F^* \rightarrow H^1(\chi, F^*T_{X/k}) \rightarrow H^1(\chi, \mathcal{T}_{X/k})
\]

is injective.

**Proof.** First of all, Proposition 4.5 shows that the value of \( s \) at each node of \( \chi \) is nonzero. So is its dual \( \check{s} \). This fact gives rise to the isomorphism in the right-down corner of the following commutative diagram:

\[
\begin{array}{ccccccccc}
0 & \rightarrow & \mathcal{T}_{X/k} & \rightarrow & \gamma_* \gamma^* \mathcal{T}_{X/k} & \rightarrow & \oplus_{i=1}^\delta k_{P_i} & \rightarrow & 0 \\
& & F^* & \downarrow & F^* & \downarrow & (\cdot)^p & & \\
0 & \rightarrow & F^* \mathcal{T}_{X/k} & \rightarrow & \gamma_* \gamma^* F^* \mathcal{T}_{X/k} & \rightarrow & \oplus_{i=1}^\delta k_{P_i} & \rightarrow & 0 \\
& & s^2 & \downarrow & s^2 & \downarrow & \cong & & \\
0 & \rightarrow & \mathcal{T}_{X/k} & \rightarrow & \gamma_* \gamma^* \mathcal{T}_{X/k} & \rightarrow & \oplus_{i=1}^\delta k_{P_i} & \rightarrow & 0.
\end{array}
\]

After taking cohomologies, we get

\[
\begin{array}{ccccccccc}
\oplus_{i=1}^\delta k_{P_i} & \cong & H^1(\mathcal{T}_{X/k}) \\
& \downarrow & (\cdot)^p & \downarrow F^* & \\
\oplus_{i=1}^\delta k_{P_i} & \cong & H^1(F^* \mathcal{T}_{X/k}) \\
& \cong & s^2 & \downarrow s^2 & \\
\oplus_{i=1}^\delta k_{P_i} & \cong & H^1(\mathcal{T}_{X/k}).
\end{array}
\]

Clearly \((4.6.1)\) is injective. This completes the proof.

**4.4. Nonemptiness of \( A \cap K \) and ordinariness for a generic smooth \((g, r)\)-curve.** Building upon Proposition 4.5 and Proposition 4.6, we show the nonemptiness of \( A \cap K \) and the existence of ordinary sections for a generic smooth \((g, r)\)-curve, by applying a continuity argument.

**Proposition 4.7.** For a generic curve \( X_1 \) in \( \mathcal{M}_{g,r} \), \( A \) and \( K \) intersects.
PROOF. Take an arbitrary t.d. curve $\chi \in \overline{M}_{g,r}$ and a log smooth deformation $f : X_1 \to B_1$ of $\chi$ as given in \textsection4.1. Write $\mathcal{V}$ for the vector bundle $R^1f_*(F^*\omega_{X_1/B_1}^{-1})$ over $B_1$. Consider the irreducible closed subsets $\mathcal{A}$ and $\mathcal{K}$ and the composite

$$
\pi : \mathcal{A} \cap \mathcal{K} \subset \mathcal{V} \to B_1.
$$

We know that: (i). $\dim \mathcal{A} + \dim \mathcal{K} = \dim \mathcal{V} + \dim B_1$ which can be seen from the formula \textsection3.3.1. It implies that any irreducible component of $\mathcal{A} \cap \mathcal{K}$ is of dimension $\geq \dim \mathcal{A} + \dim \mathcal{K} - \dim \mathcal{V} = \dim B_1 = 1$; (ii). by Proposition \textsection4.5, $\xi_1 = \pi^{-1}(0)$. Then take any irreducible component $Z$ of $\mathcal{A} \cap \mathcal{K}$ passing through $\xi_1$. Thus, the restricted map $\pi|_Z : Z \to B_1$ has to be dominant for dimension reason. Then, one finds a closed point $0 \neq b_1 \in B_1$ so that $A \cap K \neq \emptyset$ over $b_1$. Now consider the analogous set-up as \textsection4.1 for a versal deformation of $[X_1] = b_1 \in M_{g,r}$, then repeating the previous argument yields that there is a Zariski open subset $U$ of $M_{g,r}$ such that $A \cap K \neq \emptyset$ over any closed point in $U$. \hfill \Box

PROOF OF THEOREM \textsection2.1 It follows from Proposition \textsection3.6 and the last proposition. \hfill \Box

DEFINITION 4.8. A smooth $(g,r)$-curve $X_1$ is called ordinary if $A \cap K \neq \emptyset$ and moreover if there exists an ordinary section $s \in \text{Hom}(L_1 \otimes L, F^*L_1)$ such that $A \cap K \cap \text{Ker}(\phi_s) \neq \emptyset$.

PROPOSITION 4.9. A generic curve $X_1$ in $M_{g,r}$ is ordinary.

PROOF. By definition of ordinarity, the set of ordinary sections makes an open subset of $\text{Hom}(L_1 \otimes L, F^*L_1)$. Thus for every smooth $(g,r)$-curve, the subset $K^0 := \bigcup_{s \text{ ordinary}} \text{Ker}(\phi_s)$ is an open subset of $K$ in view of \textsection2.3.1. By extending to the relative case, we obtain an open subscheme $K^0 \subset \mathcal{K}$. By Proposition \textsection2.4.1 $\mathcal{A} \cap K^0$ is a nonempty open subset of $\mathcal{A} \cap \mathcal{K}$. Thus replacing $\mathcal{A} \cap \mathcal{K}$ with $\mathcal{A} \cap K^0$ in the argument of Proposition \textsection4.7 gives the desired result. \hfill \Box

5. Lifting periodic Higgs bundles to a truncated Witt ring

In this section, we consider the lifting question of periodic Higgs bundles. For simplicity, the proof in the following shall be given only for the case that the divisor $D_1$ is absent. The general case follows by replacing Frobenius liftings with logarithmic Frobenius liftings in the argument. Assume that $(E_{n-1}, \theta_{n-1}, Fil_{n-1}, \phi_{n-1})$ is a two-periodic Higgs bundle over $X_{n-1}$ of the same form as in \textsection1.2.2, where

$$
E_{n-1} = L_{n-1} \oplus L_{n-1}^{-1}, \quad \theta_{n-1} : L_{n-1} \cong L_{n-1}^{-1} \otimes \omega_{X_{n-1}/W_{n-1}};
$$

for some $W_n$-lifting $X_n$ of $X_{n-1}$, the inverse Cartier transform

$$(H_{n-1}, \nabla_{n-1}) := C_{n-1}^{-1}(E_{n-1}, \theta_{n-1}).$$

has a Hodge filtration $Fil_{n-1}$, which induces an isomorphism

$$
\psi_{n-1} : (Gr \circ C_{n-1}^{-1})^2(E_{n-1}, \theta_{n-1}) \cong (E_{n-1}, \theta_{n-1}).
$$

Given a Higgs bundle $(E_n, \theta_n)$ on $X_n$, which is a lifting of $(E_{n-1}, \theta_{n-1})$, we recall its inverse Cartier transform $C_n^{-1}(E_n, \theta_n)$ from [LSZ].
Fix a choice of $W_{n+1}$-lifting $X_{n+1}$ of $X_n$. Choose an open affine covering $X_{n+1} = \bigcup_{i \in I} U''_i$ (resp. $X_n = \bigcup_{i \in I} U'_i$) and Frobenius liftings $\{F''_i : U''_i \to U''_i\}$ (resp. $\{F'_i : U'_i \to U'_i\}$). The data $(H_{n-1}, \nabla_{n-1}, \psi_{n-1}, E_n, \theta_n)$ forms an object in $\mathcal{H}(X_n)$. First, apply the functor $T_n$ to it (see [LSZ, §5] for details) to obtain a twisted flat bundle $(\tilde{H}_{-1}, \tilde{\nabla}_{-1})$ whose local model reads:

$$\{H_i := F'_i \tilde{H}_{-1}|_{U_i}, \nabla_i = \nabla_{\text{can}} + \frac{dF''_i}{p}((F'_i)^*\tilde{\nabla}_{-1})\}_{i \in I}.$$  

Second, apply the Taylor formula $\{G_{ij}\}$ to glue the local models into a global de Rham bundle $(H_n, \nabla_n)$, which is defined to be $C^{-1}_n(E_n, \theta_n)$. Choose a coordinate $\{t''\}$ for $U''_i := U''_i \cap U''_j$ (resp. $\{t'\}$ for $U'_i$ and $\{t\}$ for $U_{ij}$), assume $U''_i \subset U''_j$ then for $s \in \tilde{H}_{-1}$,

$$G_{ij}(s \otimes 1) = \sum_{k \geq 0} \tilde{\nabla}_{-1}(\partial t')^k(s) \otimes \frac{z_{ij}}{k!},$$

with

$$z_{ij} := \frac{F''_i}{p}(t''_i) - \frac{F''_j}{p}(t''_j).$$

Consider the obstruction class to lifting the filtration $\text{Fil}_{n-1}^1$ from $(H_{n-1}, \nabla_{n-1})$ to $(H_n, \nabla_n)$. Clearly the obstruction lies in $H^1(X_1, \text{Hom}(\text{Fil}_1^1, H_1/\text{Fil}_1^1))$, and depends on $(H_n, \nabla_n)$ and thus on the choice of $X_{n+1}$. Recall that the deformation space $\{X_n \subset X_{n+1}\}$ of $X_n$ is a torsor of $H^1(X_1, T_{X_1/k})$. Therefore, we obtain a map

$$\varrho : \{X_n \subset X_{n+1}\} \to H^1(X_1, \text{Hom}(\text{Fil}_1^1, H_1/\text{Fil}_1^1)).$$

This map is a torsor map, and its derivative

$$d\varrho : H^1(X_1, T_{X_1/k}) \to H^1(X_1, \text{Hom}(\text{Fil}_1^1, H_1/\text{Fil}_1^1))$$

is a semi-linear map.

Let $X_{n+1}$ and $\tilde{X}_{n+1}$ be two $W_{n+1}$-liftings of $X_n$. Take an affine covering $\tilde{X}_{n+1} = \bigcup_{i \in I} U''_i$ (resp. $X_{n+1} = \bigcup_{i \in I} U''_i$). Assume that the reduction by modulo $p^n$ of $\{U''_i\}_{i \in I}$ coincide with reduction of $\{U'_i\}_{i \in I}$, denoted by $X_n = \bigcup_{i \in I} U'_i$. Choose Frobenius liftings $\{\tilde{F}''_i : \tilde{U}''_i \to \tilde{U}''_i\}$ (resp. $\{F''_i : U''_i \to U''_i\}$) and assume that their reductions by modulo $p^n$ are the same, denote by $\{F'_i : U'_i \to U'_i\}$.

Choose a coordinate $\tilde{t}''_i$ for $\tilde{U}''_i$ (resp. $t''_i$ for $U''_i$ and $t_i$ for $U_{ij}$) and assume that their reduction by modulo $p^n$ are the same, denote by $\tilde{t}'_i$; denote by $t'_{ij}$ the reduction by modulo $p$ of $t''_{ij}$. Set $\hat{z}_{ij} := \frac{F''_j}{p}(\tilde{t}''_j) - \frac{F''_i}{p}(\tilde{t}''_i)$, $z_{ij} := \frac{F''_j}{p}(t''_j) - \frac{F''_i}{p}(t''_i)$, and $\nu_{ij} := z_{ij} - \hat{z}_{ij}$.

**Lemma 5.1.** With the above notations. Denote by $\alpha \in H^1(X_1, T_{X_1/k})$ the class $[X_{n+1}] - [\tilde{X}_{n+1}]$. Then the image of $\alpha$ under the map $H^1(X_1, T_{X_1/k}) \to H^1(X_1, F^*T_{X_1/k})$ is represented by the class $\{\frac{\nu_{ij}}{p} \partial t_{ij}\}$. 
Fix embeddings \( \{ V''_i \to V'_i \} \). Corresponding \( X_{n+1} \) (resp. \( X_n \)), there are embeddings \( \{ g_{ij} : V''_i \to V''_j \} \) (resp. \( \{ g_{ij} : V''_i \to V''_j \} \)). Note that \( g_{ij} \) and \( \hat{g}_{ij} \) have the same reduction by modulo \( p^n \). Thus the image of \( g_{ij} - \hat{g}_{ij} : \mathcal{O}_{V''_i} \to \mathcal{O}_{V''_j} \) lies in \( p^n \mathcal{O}_{V''_j} \). Then \( g_{ij} - \hat{g}_{ij} : \mathcal{O}_{U_i} \to \mathcal{O}_{U_{ij}} \) is a derivation and factors through \( \mathcal{O}_{U_{ij}} \), i.e. \( \frac{g_{ij} - \hat{g}_{ij}}{p^n} : \mathcal{O}_{U_{ij}} \to \mathcal{O}_{U_{ij}} \). This defines a class \( \{ \alpha_{ij} \} \) in \( H^1(X_1, \mathcal{T}_{X_1/k}) \) representing \( [X_{n+1}] - [X_{n+1}] \).

Choose Frobenius liftings \( \{ F''_i : V''_i \to V''_i \} \). For convenience, we may assume \( U'_j \subset U'_i \) and \( V''_j \subset V''_i \). There are Cartesian diagrams as below

\[
\begin{array}{ccc}
V''_i & \xrightarrow{F''_i} & V''_i \\
g_{ij} \downarrow & & \downarrow g_{ij} \\
V''_j & \xrightarrow{F''_j} & V''_j
\end{array}
\]

Then we have two Frobenius liftings \( g_{ij} \circ F_j \) and \( F_i \circ g_{ij} \). Pick a local coordinate \( t \) over \( V''_j \), then \( z_{ij} = \frac{F_j \circ g_{ij}(t) - g_{ij}(t) \circ F_i(t)}{p} \).

Similarly, we obtain \( \hat{z}_{ij} = \frac{F_j \circ g_{ij}(t) - \hat{g}_{ij}(t)}{p} \).

Then

\[
\frac{\nu_{ij}}{p^{n-1}} = \frac{z_{ij} - \hat{z}_{ij}}{p^{n-1}} = \frac{1}{p^n} \left( F_j^* \circ (g_{ij}^*- \hat{g}_{ij}^*(t)) - (g_{ij} - \hat{g}_{ij}) \circ F_i(t) \right),
\]

Note that \( \frac{g_{ij} - \hat{g}_{ij}}{p^n} : \mathcal{O}_{U_i} \to \mathcal{O}_{U_{ij}} \) is a derivative, and \( p|dF_i^*(t) \), so \( \frac{g_{ij} - \hat{g}_{ij}}{p^n} \) = 0. Then one can immediately that

\[
\frac{\nu_{ij}}{p^{n-1}} = \frac{1}{p^n} F_j^* \circ (g_{ij}^*(t) - \hat{g}_{ij}^*) = F^* \circ \frac{g_{ij}^*(t) - \hat{g}_{ij}^*}{p^n}.
\]

So \( \{ \frac{\nu_{ij}}{p^{n-1}} \partial t \} \) represents the class \( \{ F^*(\alpha) \} \).

**Proposition 5.2.** \( d\varrho \) is the cohomology map induced by the following composite morphism of sheaves

\[
\mathcal{T}_{X_1/k} \to F^*(\mathcal{T}_{X_1/k}) = C_{-1}^1(\mathcal{T}_{X_1/k}, 0) \xrightarrow{C_{-1}^1(\theta_1)} C_{1}^{-1}(\text{End}(E_1), \text{End}(\theta_1)) \xrightarrow{Pr} \text{Hom}(\text{Fil}_1^1, H_1/\text{Fil}_1^1).
\]

**Proof.** Assume that \( \hat{X}_{n+1} \) is another \( W_{n+1} \)-lifting of \( X_n \). Take an open affine covering \( \hat{U} := \{ \hat{U}_i'' \}_{i \in I} \). For simplicity, we may assume that the reduction by modulo \( p^n \) of \( \hat{U}_i'' \) is \( U_i' \). Choose Frobenius liftings \( \{ F''_i : U_i'' \to 

\hat{U}_i'' \} \), and assume that the reduction by modulo \( p^n \) of \( F''_i \) is \( F_i' \). Choose a coordinate \( \{ t'' \} \) for \( \hat{U}_i'' \), and assume that \( \{ t' \} \) (resp. \( \{ t \} \)) is the reduction by modulo \( p^n \) (resp. \( p \)).
Let \( \{ \hat{G}_{ij} \} \) denote the gluing map with respect to \( \hat{X}_2 \), and
\[
\hat{z}_{ij} = \frac{\hat{F}''}{p}(\hat{t}'') - \frac{\hat{F}''}{p}(\hat{t}'').
\]
Set
\[
\nu_{ij} = z_{ij} - \hat{z}_{ij}.
\]
As \( \hat{F}'' = F'' \mod p^n \), so \( p^n - 1 \mid \nu_{ij} \). Let \( \alpha \in H^1(X_1, T_{X_1/k}) \) denote the class of \( [\hat{X}_{n+1}] - [X_{n+1}] \). By Lemma 5.1 we see that \( \{ \frac{\nu_{ij}}{p^n - 1} \partial t \} \) represents the class of \( F^* \alpha \). By a direct calculation, we have
\[
\hat{G}_{ij} = W_{ij} \cdot G_{ij},
\]
where for \( \tilde{s} \in \tilde{H}_1 \),
\[
W_{ij}(\tilde{s} \otimes 1) = \tilde{s} \otimes 1 + \tilde{\nabla}_{-1}(\partial t')(\tilde{s}) \otimes \nu_{ij}.
\]
Consider the obstruction to lifting \( Fil_{n-1} \) to \( H_n \). We see that the obstruction class with respect to \( X_{n+1} \) is represented by
\[
\{ Pr \circ (\frac{G_{ij} - 1}{p^n - 1}) \}.
\]
So the difference between two obstruction classes with respect to \( X_{n+1} \) and \( \hat{X}_{n+1} \) is represented by
\[
\{ Pr \circ (\frac{\tilde{G}_{ij} - 1}{p^n - 1}) - Pr \circ (\frac{G_{ij} - 1}{p^n - 1}) = Pr \circ (\frac{W_{ij} - 1}{p^n - 1}) \circ \hat{G}_{ij} \},
\]
where \( \tilde{G}_{ij} \) denotes the reduction by modulo \( p \) of \( G_{ij} \). As
\[
Pr \circ \frac{W_{ij} - 1}{p^n - 1} = F^*(\theta_1(\partial t)) \frac{\nu_{ij}}{p^n - 1},
\]
thus \( \{ Pr \circ \frac{W_{ij} - 1}{p^n - 1} \circ \tilde{G}_{ij} \} \) represents the class \( \tilde{C}_{-1}(\theta_1)(F^* \alpha) \). We finish the proof. □

Now we complete the proof of Theorem 2.5.

**Proof of Theorem 2.5.** As we have discussed before, the map \( \varrho \) in (5.0.1) describes how the obstruction class to the lifting of the Hodge filtration \( Fil_{n-1} \) depends on the \( W_{n+1} \)-deformation space of \( X_n/W_n \). By Proposition 5.2, the ordinary condition is equivalent to that \( d\varrho \) is an isomorphism, thus there exists at least one \( W_{n+1} \)-lifting \( X_{n+1} \) such that the obstruction class vanishes. The rest of the proof reads similarly as in Corollary 2.2, provided one notices also that there is a unique two-torsion line bundle \( L_n \) on \( X_n \) which lifts \( L_{n-1} \) and, by the very construction of the inverse Cartier transform, \( C_{n-1}(L_n, 0) = L_n^* = L_n \) holds. □
Let $k$ be a perfect field of positive characteristic and $X$ a smooth algebraic variety over $k$. Let $D \subset X$ be a simple normal crossing divisor. This gives rise to one of standard examples of the logarithmic structure on $X$ (Example 1.5 (1) [KKa]). The aim of this appendix is to provide the Cartier/inverse Cartier transform of Ogus-Vologodsky [OV] in this special log case. This result generalizes our previous result [LSZ0] for the case $D = \emptyset$. The construction of the inverse Cartier transform also serves as the basis for its lifting to $W_n, n \geq 2$.

Let $\omega_{X,log}/k = \Omega_X(\log D)$ be the sheaf of log differential forms which is locally free in this case. Let $T_{X,log}/k$ be its $O_X$-dual. First we formulate two categories:

Let $HIG^{\leq n}(X_{log}/k)$ be the category of nilpotent logarithmic Higgs sheaves over $X/k$ of exponent $\leq n$. An object $(E, \theta) \in HIG^{\leq n}(X_{log}/k)$ is of form

$$\theta : E \to E \otimes \omega_{X,log}/k$$

satisfying

$$\theta_{\partial_1} \cdots \theta_{\partial_{n+1}} = 0$$

for any local sections $\partial_1, \cdots, \partial_{n+1}$ of $T_{X,log}/k$. Let $MIC^{\leq n}(X_{log}/k)$ be the category of nilpotent logarithmic flat sheaves of exponent $\leq n$ (its object is similarly defined with the nilpotent condition referring to its $p$-curvature) and $MIC^{0,\leq n}(X_{log}/k)$ the full subcategory of $MIC^{\leq n}(X_{log}/k)$ with nilpotent residue of exponent $\leq n$. For a logarithmic flat (resp. Higgs) sheaf $(H, \nabla)$ (resp. $(E, \theta)$), the residue is an $O_D$-linear morphism $\text{Res} \nabla : H|_D := H \otimes O_D \to H|_D$ (resp. $\text{Res} \theta : E|_D \to E|_D$) obtained from the composite

$$H \xrightarrow{\nabla} H \otimes \omega_{X,log}/k \xrightarrow{id \otimes \text{res}} H \otimes O_D.$$ 

In the above, $\text{Res} \nabla$ is said to be nilpotent of exponent $\leq n$ if $(\text{Res} \nabla)^{n+1} = 0$. Note that, for $(E, \theta) \in HIG^{\leq n}(X_{log}/k)$, $(\text{Res} \theta)^{n+1} = 0$ holds automatically; but it is not the case for $MIC^{\leq n}(X_{log}/k)$, that is $MIC^{0,\leq n}(X_{log}/k) \subsetneq MIC^{\leq n}(X_{log}/k)$.

**Theorem 6.1** (Theorem 1.2 [LSZ0] for $D = \emptyset$). Assume $(X, D)$ is $W_2(k)$-liftable. Then there is an equivalence of categories

$$HIG^{p-1}(X_{log}/k) \xrightarrow{C^{-1}} MIC^{0, p-1}(X_{log}/k).$$

Note this result strengthens Corollary 4.11 [S] in this special log case. Our argument follows the line of [LSZ0] which is completely elementary.

**Proof.** Given the explicit exposition of the constructions in [LSZ0] for the case where $D$ is absent, we shall not repeat the whole argument but rather emphasize the new ingredients in the new situation.

\footnote{The convention of exponent adopted here differs from the one used in [LSZ0] which originated from N. Katz [KA], but conforms with the one used in Ogus-Vologodsky [OV].}
We start with the trivial observation: The category $\text{HIG}_{\leq 0}(X/k)$ is just $\text{HIG}_{\leq 0}(X/k)$ while the category $\text{MIC}_{\leq 0}^0(X/k)$ is just $\text{MIC}_{\leq 0}^0(X/k)$. Thus the classical Cartier theorem gives the equivalence of categories. The general case is reduced to this theorem by an exponential twisting $[\text{LSZ0}].$

Fix a $W_2$-lifting $(\tilde{X}, \tilde{D})$ of $(X, D)$, see Definition 8.11 $[\text{EV}].$ For an open affine subset $\tilde{U} \subset \tilde{X}$, one takes the log Frobenius lifting respecting the divisor $\tilde{D}_U := \tilde{D} \cap \tilde{U}$, that is, a morphism $\tilde{F}_{(\tilde{U}, \tilde{D}_U)} : \mathcal{O}_U \to \mathcal{O}_{\tilde{U}}$ lifting the absolute Frobenius morphism on $U := \tilde{U} \times k$ and satisfying $\tilde{F}_{(\tilde{U}, \tilde{D}_U)}^* \mathcal{O}_{\tilde{U}}(\tilde{D}_U) = \mathcal{O}_U(-p\tilde{D}_U)$.

Such a lifting exists and two such differ by an element in $T_{X_{\log}/k}(U)$ over $U$. See Propositions 9.7, 9.9 ibid. Then it proceeds the same way as Section 2.2 for the inverse Cartier transform $C^{-1}$, and one obtains $(H, \nabla) = C^{-1}(E, \theta)$. It belongs clearly to $\text{MIC}_{\leq p-1}^0(X_{\log}/k)$. Writing $\zeta$ for the map $\frac{d}{p}\tilde{F}_{(\tilde{U}, \tilde{D}_U)} : F^*\omega_{X_{\log}/k}(U) \to \omega_{X_{\log}/k}(U)$.

Then locally over $U$, the connection is given by

$$\nabla|_U = \nabla_{\text{can}} + \zeta(F^*\theta|_U).$$

It follows that $\text{Res} \nabla|_U = F^*\text{Res} \theta|_U$ and therefore $(H, \nabla) \in \text{MIC}_{\leq p-1}^0(X_{\log}/k)$. Conversely, given an object $(H, \nabla) \in \text{MIC}_{\leq p-1}^0(X_{\log}/k)$, one proceeds as Section 2.3 $[\text{LSZ0}].$ In the same way, one shows the new connection

$$\nabla'|_U = \nabla|_U + \zeta(\psi|_U),$$

where $\psi = \psi_{\nabla} : H \to H \otimes F^*\omega_{X_{\log}/k}$ is the $p$-curvature of $\nabla$, has the vanishing $p$-curvature. However, there is one new ingredient here, namely, the following

**Claim 6.2.** The following diagram commutes:

\[
\begin{array}{ccc}
H & \xrightarrow{\psi_{\nabla}} & H \otimes F^*\omega_{X_{\log}/k} \\
\downarrow & & \downarrow \\
H \otimes \mathcal{O}_D & \xrightarrow{-\text{Res} \nabla} & H \otimes \mathcal{O}_D.
\end{array}
\]

Granted the truth of the claim, the residue of $\nabla'|_U$ vanishes. In this way, we reduce it to the Cartier descent. \hfill \Box

Here is the proof of the above claim:

**Proof.** One reduces immediately the proof of Claim 6.2 to the curve case. So we consider the $\tilde{U} = \text{Spec} W_2[t]$ with $\tilde{D}_U$ the closed subscheme defined by the ideal $(t)$, and the standard log Frobenius lifting $\tilde{F} : t \mapsto t^p$. Since two log Frobenius liftings differ by an element in $T_{X_{\log}/k}(U)$, the residue of $(id \otimes \zeta) \circ \psi_{\nabla}$ is independent from such a choice.
Now $t\partial_t$ is a local basis for $\mathcal{T}_{X_{\log}/k}(U)$. Then by an elementary calculation, one finds $(t\partial_t)^p = t\partial_t$. Thus,

$$
\psi_{t\partial_t} = \nabla_{t\partial_t}^p - \nabla_{t\partial_t},
$$

so that its residue at $D = \{t = 0\}$ equals

$$
\nabla_{t\partial_t}(0)^p - \nabla_{t\partial_t}(0) = (\nabla_{t\partial_t}(0))^p - \nabla_{t\partial_t}(0) = -\nabla_{t\partial_t}(0)
$$

which is just $-\text{Res} \nabla$ at $D$. The second equality in above follows from the assumption $(\text{Res} \nabla)^p = 0$. Finally, one computes directly that

$$
[(id \otimes \zeta) \circ \psi]_{t\partial_t} = \psi_{t\partial_t}.
$$

Combining these discussions, the claim follows.

\[ \square \]

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