On solutions of Rashevskii equation

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Abstract

The solutions of Rashevskii equation for gonometric family of plane curves are considered. Their properties are discussed. The connection with the theory of duality for the second order ODE’s is discussed.

1 Gonometric family of plane curves

Two parametrical family of plane curves is defined by the equation

\[ F(x, y, \xi, \eta) = 0. \]  

From the equation (1) and its differential at fixed \((x, y)\)

\[ F_\xi d\xi + F_\eta d\eta = 0 \]

can be find the coordinates

\[ x = x(\xi, \eta, \frac{d\eta}{d\xi}), \quad y = y(\xi, \eta, \frac{d\eta}{d\xi}). \]  

From the condition

\[ F_x dx + F_y dy = 0 \]

is followed expression for the angle \(\theta\)

\[ \tan \theta = -\frac{F_x}{F_y}. \]

After differentiating this expression one gets the equation for the differential \(d\theta\)

\[ \frac{d\theta}{\cos \theta^2} = \frac{F_x (F_\xi d\xi + F_\eta d\eta) - F_y (F_\xi d\eta + F_\eta d\xi)}{F_y^2}. \]

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Taking in consideration the formulae (2) and the relation
\[
\cos^2 \theta = \frac{F_y^2}{F_x^2 + F_y^2}
\]
we get finally the expression
\[
d\theta^2 = \frac{[(F_x F_y \xi - F_y F_x \xi) \, d\xi - (F_y F_x \eta - F_x F_y \eta) \, d\eta]^2}{(F_x^2 + F_y^2)^2 F_y^2}
\]
which can be considered as the metric between two infinitely located in close proximity curves of a given family.

In general case it has the form
\[
d\theta^2 = \Phi(\xi, \eta, d\xi, d\eta).
\]

Two parametrical family of plane curves with angle metric having the Gauss form
\[
d\theta^2 = g_{11} d\xi^2 + 2g_{12} d\xi d\eta + g_{22} d\eta^2
\]
is called gonometric.

Two parametrical family of curves can be done in form of the second order ODE
\[
y'' = \left(1 + y'^2\right)^{3/2} K(x, y, \phi), \quad \phi = \arctan y'
\]
where the function $K$ is the curvature along a curve from the family.

The problem of description of two parametrical family of curves determined by the equations (4) and having the Gauss angle form was solved by Rashevskii.

As it was shown by Rashevskii the function $K(x, y, \phi)$ in the equation (4) must be solution of the partial differential equation
\[
(4K + 2 \sin \phi \, \partial_x - 2 \cos \phi \, \partial_y + \cos \phi \, \partial_x \partial_\phi + \sin \phi \, \partial_y \partial_\phi + K \partial_\phi \partial_\phi) \, XK = 0,
\]
where
\[
XK = \cos \phi \, \frac{\partial K}{\partial x} + \sin \phi \, \frac{\partial K}{\partial y} + K \, \frac{\partial K}{\partial \phi}.
\]

The aim of our consideration is integration of the equation (5).

## 2 The method of solution

For solutions of partial differential equation
\[
F(x, y, z, f_x, f_y, f_z, f_{xx}, f_{xy}, f_{yy}, f_{zz}, f_{xxy}, f_{xyz}, f_{yyy}, f_{xxx}, \ldots) = 0
\]
we use the method of solution of the p.d.e.’s described first in [3] and developed later in [2].

This method allow us to construct particular solutions of the partial nonlinear differential equation
\[
F(x, y, z, f_x, f_y, f_z, f_{xx}, f_{xy}, f_{yy}, f_{yz}, f_{xxx}, f_{xxy}, f_{xyy}, \ldots) = 0
\]
with the help of transformation of the function and corresponding variables.

Essence of the method consists in a following presentation of the functions and variables
\[
f(x, y, z) \to u(x, t, z), \quad y \to v(x, t, z), \quad f_x \to u_x - \frac{u_t}{v_t} v_x, \quad f_z \to u_z - \frac{u_t}{v_t} v_z, \quad f_y \to \frac{u_t}{v_t}, \quad f_{yy} \to \frac{(u_t v_t)_t}{v_t}, \quad f_{xy} \to \frac{(u_x - \frac{u_t}{v_t} v_x)_t}{v_t}, \quad \ldots
\]
(7)
where variable \( t \) is considered as parameter.

Remark that conditions of the type
\[
f_{xy} = f_{yx}, \quad f_{xz} = f_{zx}, \ldots
\]
are fulfilled at the such type of presentation.

In result instead of equation (6) one get the relation between the new variables \( u(x, t, z), \; v(x, t, z) \) and their partial derivatives
\[
\Psi(u, v, u_x, u_z, u_t, v_x, v_z, v_t, \ldots) = 0.
\]
(8)
This relation coincides with initial p.d.e at the condition \( v(x, t, z) = t \) and lead to the new p.d.e
\[
\Phi(\omega, \omega_x, \omega_t, \omega_{xx}, \omega_{xt}, \omega_{tt}, \ldots) = 0
\]
(9)
when the functions \( u(x, t, s) = F(\omega(x, t, z), \omega_t, \ldots) \) and \( v(x, t, s) = F(\omega(x, t, z), \omega_t, \ldots) \) are expressed through the auxiliary function \( \omega(x, t, s) \).

Remark that there are a various means to reduce the relation (8) into the partial differential equation.

In some cases the solution of equation (9) is a more simple problem than solution of the equation (6).

**Remark 1** As example we consider the system of equations
\[
\frac{\partial}{\partial z} f(x, y, z) + f(x, y, z) \frac{\partial}{\partial x} f(x, y, z) + h(x, y, z) \frac{\partial}{\partial y} f(x, y, z) = 0,
\]
and
\[
\frac{\partial}{\partial z} h(x, y, z) + f(x, y, z) \frac{\partial}{\partial x} h(x, y, z) + h(x, y, z) \frac{\partial}{\partial y} h(x, y, z) = 0.
\]
Such type of equations are meeting in theory of motion of free fluid particles.

After the \((u, v)\)-transformation
\[
u(x, t) = \left( \frac{\partial}{\partial t} \omega(x, t) \right) t - \omega(x, t),
\]
\[
u(x, t) = \left( \frac{\partial}{\partial t} \omega(x, t) \right)
\]
with condition
\[
\omega(x, t, z) = A(t, z) + xt
\]
it takes the form of equation
\[
\left( \frac{\partial}{\partial z} A(t, z) \right)^2 + t^2 \left( \frac{\partial^2}{\partial t \partial z} A(t, z) \right)^2 - 2 t \left( \frac{\partial^2}{\partial t \partial z} A(t, z) \right) \frac{\partial}{\partial z} A(t, z) -
\left( \frac{\partial^2}{\partial t^2} A(t, z) \right) t^2 \frac{\partial^2}{\partial z^2} A(t, z) = 0.
\]

The simplest solution of this equation is
\[
A(t, z) = F_1(t) + F_2(z) + F_2(z) F_1(t),
\]
where
\[
F_1(t) = \left( \frac{t \omega^{-1} - c_2}{(-C_1 - C_2 t) (c_2 - 1)} \right)^{-1} - 1,
\]
and
\[
F_2(z) = \left( (-C_1 z c_2 + c_2 z - C_2 c_2 + C_2 (\omega^{-1} - 1) \right)^{-1} - 1
\]
with parameters $C_1$, $C_2$, $c_2$.

Elimination of the parameter $t$ from corresponding relations give us the functions $f(x, y, z)$ and $h(x, y, z)$ satisfying the above system of equations.

As example, in the case $C_1 = 1$, $C_2 = 0$, $c_2 = 2$ we get the solution
\[
h(x, y, z) = -1/2 \frac{-2 z^2 + \sqrt{4 y z - 4 x z z - 2 y z + 2 x z}}{z^2},
\]
and
\[
f(x, y, z) = -1/2 \frac{4 y z - 2 \sqrt{4 y z - 4 x z z - 4 x z}}{\sqrt{4 y z - 4 x z z}}.
\]

In the case $C_1 = 1$, $C_2 = 0$, $c_2 = 1/2$ we find
\[
h(x, y, z) = - \frac{x z - y z + x^2 - 2 x y + y^2 - z^2}{z^2}
\]
and
\[
f(x, y, z) = - \frac{x^2 - 2 x y + y^2 - z^2}{z^2}.
\]

To construction of more complicated solutions it is necessary to use another type of reduction of $(u, v)$-system.

### 3 Abridged equation

The equation (5) admits solutions satisfying the condition
\[
XK = \cos \phi \frac{\partial K}{\partial x} + \sin \phi \frac{\partial K}{\partial y} + K \frac{\partial K}{\partial \phi} = 0.
\]
3.1 Hodograph-transformation

To integrate the equation (10) we rewrite its in the form

$$\cos(w) \frac{\partial}{\partial u} x(u, v, w) + \sin(w) \frac{\partial}{\partial v} x(u, v, w) + x(u, v, w) \frac{\partial}{\partial w} x(u, v, w) = 0.$$ 

Now transformations of the function and variables

$$u - \lambda(x, v, w) = 0, \quad \frac{\partial}{\partial u} x(u, v, w) = \left(\frac{\partial}{\partial x} \lambda(x, v, w)\right)^{-1},$$

$$\frac{\partial}{\partial v} x(u, v, w) = -\frac{\partial}{\partial x} \lambda(x, v, w), \quad \frac{\partial}{\partial w} x(u, v, w) = -\frac{\partial}{\partial x} \lambda(x, v, w)$$

give us the linear equation on the function $\lambda(x, v, w)$

$$\cos(w) - \sin(w) \frac{\partial}{\partial v} \lambda(x, v, w) - x \frac{\partial}{\partial w} \lambda(x, v, w) = 0$$

with general solution

$$\lambda(x, v, w) = _{F1} \left(x, -\frac{\cos(w) + vx}{x}\right) + \frac{\sin(w)}{x},$$

where $_{F1}$ is arbitrary function of its own arguments.

In result the function $x(u, v, w)$ is determined from the condition

$$u - \lambda(x, v, w) = 0,$$

or

$$u - \frac{\sin(w)}{x} = _{F1} \left(x, -\frac{\cos(w) + vx}{x}\right).$$

As example in the case

$$_{F1} \left(x, -\frac{\cos(w) + vx}{x}\right) = x - \frac{\cos(w) + vx}{x}$$

one get the equation

$$u - x + \frac{\cos(w) + vx}{x} - \frac{\sin(w)}{x} = 0$$

for determination the function $x = x(u, v, w)$.

It is defined by the expression

$$x(u, v, w) = 1/2 v + 1/2 u + 1/2 \sqrt{v^2 + 2uv + u^2 + 4 \cos(w) - 4 \sin(w)}.$$

From the correspondence

$$x(u, v, w) \iff K(x, y, \phi)$$

we find

$$K(x, y, \phi) = 1/2 y + 1/2 x + 1/2 \sqrt{y^2 + 2yx + x^2 + 4 \cos(\phi) - 4 \sin(\phi)}$$

the solution of abridged equation (10).
3.2 \((u,v)\)-transformation

From the sake of convenience we present the equation (10) in the form

\[
\cos(z) \frac{\partial}{\partial x} K(x, y, z) + \sin(z) \frac{\partial}{\partial y} K(x, y, z) + K(x, y, z) \frac{\partial}{\partial z} K(x, y, z) = 0. \tag{11}
\]

After application of the \((u,v)\)-transformation at the equation (11) we find the relation

\[
\left( -\left( \frac{\partial}{\partial t} u(x, t, z) \right) \frac{\partial}{\partial z} v(x, t, z) + \left( \frac{\partial}{\partial z} u(x, t, z) \right) \frac{\partial}{\partial t} v(x, t, z) \right) u(x, t, z) + \\
+ \left( \left( \frac{\partial}{\partial x} u(x, t, z) \right) \frac{\partial}{\partial t} v(x, t, z) - \left( \frac{\partial}{\partial t} u(x, t, z) \right) \frac{\partial}{\partial x} v(x, t, z) \right) \cos(z) + \sin(z) \frac{\partial}{\partial t} u(x, t, z) = 0. \tag{12}
\]

By means of standard change of variables

\[
v(x, t, z) = t\theta - \theta, \quad u(x, t, z) = \theta \tag{13}
\]

the relation (12) is reduced again to the nonlinear equation

\[
\cos(z) \frac{\partial}{\partial x} \theta(x, t, z) + \sin(z) + \left( \frac{\partial}{\partial t} \theta(x, t, z) \right) \frac{\partial}{\partial z} \theta(x, t, z) = 0.
\]

In order to obtain an integrable reduction let us rewrite relation (12) in new designation as

\[
\left( -\left( \frac{\partial}{\partial v} x(u, v, w) \right) \frac{\partial}{\partial w} y(u, v, w) + \left( \frac{\partial}{\partial w} x(u, v, w) \right) \frac{\partial}{\partial v} y(u, v, w) \right) x(u, v, w) + \\
+ \left( \left( \frac{\partial}{\partial u} x(u, v, w) \right) \frac{\partial}{\partial v} y(u, v, w) - \left( \frac{\partial}{\partial v} x(u, v, w) \right) \frac{\partial}{\partial u} y(u, v, w) \right) \cos(w) + \\
+ \sin(w) \frac{\partial}{\partial v} x(u, v, w) = 0. \tag{13}
\]

In result of application of hodograph-transformation

\[
y - \omega(x, v, w) = 0, \quad u - \lambda(x, v, w) = 0
\]

the relation (12) takes the form

\[
\left( x \frac{\partial}{\partial w} \omega(x, v, w) - \sin(w) \right) \frac{\partial}{\partial v} \lambda(x, v, w) + \left( -x \frac{\partial}{\partial w} \lambda(x, v, w) + \cos(w) \right) \frac{\partial}{\partial v} \omega(x, v, w) = 0. \tag{14}
\]

From here it is easy to obtain the expressions

\[
\lambda(x, v, w) = \frac{\sin(w)}{x} + \mathcal{F}1(x, v)
\]

and

\[
\omega(x, v, w) = -\frac{\cos(w)}{x} + \mathcal{F}2(x, v),
\]

where \(\mathcal{F}2(x, v)\) and \(\mathcal{F}1(x, v)\) are arbitrary.

Now from the conditions (14) can be found the functions \(x(u, v, w)\) and \(y(u, v, w)\) and thereby the solutions of the equation (11).

Let us consider an example.
4 Simplest solutions of complete equation

In open form the Rashevskii equation has the form

\[
\frac{1}{2} \frac{\partial^3}{\partial y^2 \partial z} K(x, y, z) + 2 \sin(z) \left( \frac{\partial}{\partial x} K(x, y, z) \right) \frac{\partial}{\partial z} K(x, y, z) - 1/2 \left( \frac{\partial^2}{\partial y^2} K(x, y, z) \right) \sin(2z) - \\
-2 \cos(z) \left( \frac{\partial}{\partial y} K(x, y, z) \right) \frac{\partial}{\partial z} K(x, y, z) + \left( \frac{\partial^3}{\partial y \partial x \partial z} K(x, y, z) \right) \frac{\partial^2}{\partial z^2} K(x, y, z) + \\
+2 \cos(z) \left( \frac{\partial^2}{\partial x \partial z} K(x, y, z) \right) \frac{\partial}{\partial z} K(x, y, z) + \cos(z) \left( \frac{\partial}{\partial x} K(x, y, z) \right) \frac{\partial^2}{\partial z^2} K(x, y, z) + \\
+2 \cos(z) K(x, y, z) \frac{\partial}{\partial z} K(x, y, z) + 2 \sin(z) \left( \frac{\partial^2}{\partial y \partial z} K(x, y, z) \right) \frac{\partial}{\partial z} K(x, y, z) + \\
+ \sin(z) \left( \frac{\partial}{\partial y} K(x, y, z) \right) \frac{\partial^2}{\partial z^2} K(x, y, z) + 2 \sin(z) K(x, y, z) \frac{\partial^3}{\partial z \partial y \partial z} K(x, y, z) + \\
+3 K(x, y, z) \left( \frac{\partial}{\partial y} K(x, y, z) \right) \frac{\partial^2}{\partial z^2} K(x, y, z) - 1/2 \left( \frac{\partial^3}{\partial y^2 \partial z} K(x, y, z) \right) \cos(2z) + \\
+ (K(x, y, z))^2 \frac{\partial^3}{\partial z^2} K(x, y, z) + 4 (K(x, y, z))^2 \frac{\partial}{\partial z} K(x, y, z) - \\
- \left( \frac{\partial^2}{\partial x \partial y} K(x, y, z) \right) \cos(2z) + 1/2 \left( \frac{\partial^3}{\partial x^2 \partial z} K(x, y, z) \right) \cos(2z) + 1/2 \frac{\partial^3}{\partial x^2 \partial z} K(x, y, z) + \\
+3 K(x, y, z) \cos(z) \frac{\partial}{\partial x} K(x, y, z) + 3 K(x, y, z) \sin(z) \frac{\partial}{\partial y} K(x, y, z) + \\
+1/2 \left( \frac{\partial^2}{\partial x^2} K(x, y, z) \right) \sin(2z) = 0. \tag{15}
\]

Let us consider some simplest solutions (15).
The substitution of the form

\[ K(x, y, z) = \frac{\sin(z) + \frac{1}{U(x)}}{U(x)}, \]

give us the equation for the function \( U(x) \)

\[ -5 \frac{d}{dx} U(x) - \left( \frac{d^2}{dx^2} U(x) \right) U(x) + 3 + 2 \left( \frac{d}{dx} U(x) \right)^2 = 0. \]

Its particular solutions are

\[ U(x) = x, \quad U(x) = \frac{3}{2} x. \]

Remark that solution \( U(x) = x \) corresponds the function

\[ K(x, y, z) = \frac{\sin(z) + \frac{1}{x}}{x}, \]

which is also the solution of abridged equation, but the second \( U(x) = \frac{3}{2x} \) corresponds the solution of the complete equation.
The substitution of the form

\[ K(x, y, z) = U(x) \sin(z) + V(x) \]

lead to the system of equations

\[
3 \left( U(x) \right)^2 V(x) + 4 U(x) \frac{d}{dx} V(x) + \left( \frac{d}{dx} U(x) \right) V(x) + \frac{d^2}{dx^2} V(x) = 0, \\
2 \frac{d^2}{dx^2} U(x) + 6 U(x) \left( V(x) \right)^2 + 4 U(x) \frac{d}{dx} U(x) + 6 V(x) \frac{d}{dx} V(x) = 0
\]

having the particular solutions

\[ U(x) = \tanh\left( \frac{x + C_2}{C_1} \right) C_1^{-1}, \quad V(x) = 0, \]

and

\[ U(x) = \tanh\left( \frac{x + C_2}{C_1} \right) C_1^{-1}, \quad V(x) = C_1 e^{\int -U(x) \, dx}. \]

The substitution of the form

\[ K(x, y, z) = \cos(z) \frac{\partial}{\partial y} U(x, y) - \sin(z) \frac{\partial}{\partial x} U(x, y) \]

give us the Liouville equation for the function \( U(x, y) \)

\[
\frac{\partial^2}{\partial y^2} U(x, y) + \frac{\partial^2}{\partial x^2} U(x, y) = M e^{2U(x,y)}. 
\]

### 5 Connection with dual equation

If the equation

\[ y'' = \phi(x, y, y') \]

is dual the second order ODE cubic on the first derivative

\[ b'' = A(a, b)b'^3 + B(a, b)b'^2 + C(a, b)b' + E(a, b) \]

then its function \( \phi(x, y, y') \) satisfies the p.d.e

\[
\frac{d^2}{dx^2} \phi_{uu} - \phi_u \frac{d}{dx} \phi_{uu} - 4 \frac{d}{dx} \phi_{yu} + 4 \phi_u \phi_{yu} - 3 \phi_y \phi_{uu} + 6 \phi_{yy} = 0, \quad (16)
\]

here

\[
\frac{d}{dx} = \frac{\partial}{\partial x} + u \frac{\partial}{\partial y} + \phi \frac{\partial}{\partial u}
\]

and \( u = y' \).

It can be presented in form of the system [3], [5]

\[ \phi_{xx} + \phi_{yy} - \frac{1}{2} \phi_u^2 + u \phi_{yu} - 2 \phi_y = h(x, y, u) \]
\[ h_{xu} + \phi h_{uu} - \phi_u h_u + uh_{yu} - 3h_y = 0. \]

Such type of couple of equations has common General Integral

\[ F(x, y, a, b) = 0 \]

\[ y'' = \phi(x, y, y') \]

\[ b'' = A(a, b)b^3 + B(a, b)b^2 + C(a, b)b' + E(a, b). \]

In theory of the second order ODE having goniometric property its General Integrals

\[ F(x, y, a, b) = 0 \]

is also special - angle metric in the family of its curves has the Gauss form

\[ d\theta^2 = g_{11}(a, b)da^2 + 2g_{12}(a, b)dadb + g_{22}(a, b)db^2. \]

Thereby the geodesic equations of such type metric are in the form

\[ b'' = A(a, b)b^3 + B(a, b)b^2 + C(a, b)b' + E(a, b). \]

The relation between the equations

\[ y'' = \phi(x, y, y') \]

with condition (16) and the equations

\[ y'' = (1 + y'^2)(3/2)K(x, y, z) \]

with the function \( K(x, y, z) \) satisfying the Rashevskii equation (15) is the subject of our consideration.

The equation (16) can be written as

\[ A(x, y, z) \cos(2 z) + B(x, y, z) \sin(2 z) + C(x, y, z) \cos(z) + E(x, y, z) \sin(z) + F(x, y, z) = 0, \quad (17) \]

where

\[ A(x, y, z) = \]

\[ B(x, y, z) = -4 \frac{\partial^3}{\partial y^2 \partial z} K(x, y, z) + 2 \frac{\partial^4}{\partial z \partial y \partial x \partial z} K(x, y, z) + 4 \frac{\partial^3}{\partial x^2 \partial z} K(x, y, z) - 6 \frac{\partial^2}{\partial x \partial y} K(x, y, z), \]

\[ C(x, y, z) = 4 K(x, y, z) \frac{\partial^4}{\partial z^2 \partial x^2} K(x, y, z) + 28 K(x, y, z) \frac{\partial^2}{\partial x \partial z} K(x, y, z) + 
\]

\[ + 2 \left( \frac{\partial}{\partial x} K(x, y, z) \right) \frac{\partial^3}{\partial z^3} K(x, y, z) + 8 \left( \frac{\partial}{\partial x} K(x, y, z) \right) \frac{\partial^2}{\partial z} K(x, y, z) - 36 K(x, y, z) \frac{\partial}{\partial y} K(x, y, z) - 
\]

\[ - 6 \left( \frac{\partial}{\partial y} K(x, y, z) \right) \frac{\partial^2}{\partial z^2} K(x, y, z) + 8 \left( \frac{\partial}{\partial z} K(x, y, z) \right) \frac{\partial^2}{\partial y} K(x, y, z) - 
\]

\[ - 6 K(x, y, z) \frac{\partial^3}{\partial z \partial y \partial z} K(x, y, z) - 2 \left( \frac{\partial}{\partial z} K(x, y, z) \right) \frac{\partial^3}{\partial z \partial x \partial z} K(x, y, z), \]
\[
E(x, y, z) = 36 \left( \frac{\partial}{\partial x} K(x, y, z) \right) K(x, y, z) - 8 \left( \frac{\partial}{\partial x} K(x, y, z) \right) \frac{\partial^2}{\partial x \partial z} K(x, y, z) + \\
+6 \left( \frac{\partial}{\partial x} K(x, y, z) \right) \frac{\partial^2}{\partial z^2} K(x, y, z) + 4 K(x, y, z) \frac{\partial^4}{\partial z^2 \partial y \partial z} K(x, y, z) + \\
+6 K(x, y, z) \frac{\partial^3}{\partial z \partial x \partial z} K(x, y, z) + 28 K(x, y, z) \frac{\partial^2}{\partial y \partial z} K(x, y, z) - \\
-2 \left( \frac{\partial}{\partial z} K(x, y, z) \right) \frac{\partial^3}{\partial z \partial y \partial z} K(x, y, z) + 8 \left( \frac{\partial}{\partial z} K(x, y, z) \right) \frac{\partial}{\partial y} K(x, y, z) + \\
+2 \left( \frac{\partial}{\partial y} K(x, y, z) \right) \frac{\partial^3}{\partial z^3} K(x, y, z),
\]

\[
F(x, y, z) = 18 (K(x, y, z))^3 + 20 (K(x, y, z))^2 \frac{\partial^2}{\partial z^2} K(x, y, z) + 2 (K(x, y, z))^2 \frac{\partial^4}{\partial z^4} K(x, y, z) + \\
+9 \frac{\partial^2}{\partial y^2} K(x, y, z) + \frac{\partial^4}{\partial z \partial y^2 \partial z} K(x, y, z) + 9 \frac{\partial^2}{\partial x^2} K(x, y, z) + \frac{\partial^4}{\partial z \partial x^2 \partial z} K(x, y, z).
\]

So the problem is to find a common solutions of the equations (17) and (15).
An existence of such solutions show the example.
The function
\[
K(x, y, z) = \frac{A \sin(z) + 1}{x}
\]
is the solutions of both equations (17) and (15) at the conditions
\[
A = 0, \quad A = 1, \quad A = \frac{2}{3}.
\]

In the case \( A = 1 \) a corresponding pair of ODE’s looks as [5]
\[
\frac{d^2}{dx^2} y(x) = \frac{\left( \frac{d}{dx} y(x) \right)^3 + \frac{d}{dx} y(x) + \left( \frac{d}{dx} y(x) \right)^2 + 1}{x}^{3/2}
\]

and
\[
\frac{d^2}{da^2} b(a) = -1/2 \frac{\left( \frac{d}{da} b(a) \right) \left( \frac{d}{da} b(a) \right)^2 + 1}{a}.
\]

Both equations have gonometric angle metric in space of their integral curves.
The quantity of such type of examples my be increased.
In the case
\[
K(x, y, z) = A(x) \sin(z) + B(x)
\]
we get the system of equations for determination of \( A(x) \) and \( B(x) \)
\[
\frac{d}{dx} B(x) = \frac{B(x) \left( \frac{d}{dx} A(x) - (A(x))^2 + (B(x))^2 \right)}{A(x)}
\]
\[
\frac{d^2}{dx^2} A(x) = -3 \frac{B(x)^2}{A(x)} \frac{d}{dx} A(x) + 3 \left( B(x) \right)^4 + 2 \left( A(x) \right)^2 \frac{d}{dx} A(x).
\]
In the case

\[ K(x, y, z) = \left( \frac{\partial}{\partial y} U(x, y) \right) \cos(z) - \left( \frac{\partial}{\partial x} U(x, y) \right) \sin(z) \]

which corresponds the equations cubic on the first derivative the function \( U(x, y) \) is solution of the Liouville equation.

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