Existence of steady solutions for a general model for micropolar electrorheological fluid flows

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Abstract

In this paper we study the existence of solutions to a steady system that describes the motion of a micropolar electrorheological fluid. The constitutive relations for the stress tensors belong to the class of generalized Newtonian fluids. The analysis of this particular problem leads naturally to weighted Sobolev spaces. By deploying the Lipschitz truncation technique, we establish the existence of solutions without additional assumptions on the electric field.

Keywords: Existence of solutions, Lipschitz truncation, weighted function spaces, variable exponent spaces, micropolar electrorheological fluids.

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1. Introduction

In this paper we establish the existence of solutions of the system

\[ \begin{align*}
- \text{div} S + \text{div}(\mathbf{v} \otimes \mathbf{v}) + \nabla \pi &= \mathbf{f} & \text{in } \Omega, \\
\text{div} \mathbf{v} &= 0 & \text{in } \Omega, \\
- \text{div} \mathbf{N} + \text{div}(\omega \otimes \mathbf{v}) &= \mathbf{t} - \varepsilon : \mathbf{S} & \text{in } \Omega, \\
\mathbf{v} &= \mathbf{0}, \quad \omega &= \mathbf{0} & \text{on } \partial \Omega.
\end{align*} \]

Here, \( \Omega \subseteq \mathbb{R}^d, d \geq 2 \), is a bounded domain. The three equations in (1.1) represent the balance of momentum, mass and angular momentum for an incompressible, micropolar electrorheological fluid. In it, \( \mathbf{v} \) denotes the velocity, \( \omega \) the microrotation, \( \pi \) the pressure, \( S \) the mechanical extra stress tensor, \( \mathbf{N} \) the couple stress tensor, \( \mathbf{f} \) the electromagnetic couple force, \( \mathbf{t} = \mathbf{f} + \chi^E \text{div}(\mathbf{E} \otimes \mathbf{E}) \) the body force, where \( \mathbf{f} \) is the mechanical body force, \( \chi^E \) the dielectric susceptibility and \( \mathbf{E} \) the electric field. The electric field \( \mathbf{E} \) solves the quasi-static Maxwell’s equations

\[ \begin{align*}
\text{div} \mathbf{E} &= 0 & \text{in } \Omega, \\
\text{curl} \mathbf{E} &= \mathbf{0} & \text{in } \Omega, \\
\mathbf{E} \cdot \mathbf{n} &= \mathbf{E}_0 \cdot \mathbf{n} & \text{on } \partial \Omega.
\end{align*} \]
where \( \mathbf{n} \) is the outer normal vector field of \( \partial \Omega \) and \( \mathbf{E}_0 \) is a given electric field. The system (1.1), (1.2) is the steady version of a model derived in [13], which generalizes previous models of electrorheological fluids in [32], [34]. The model in [13] contains a more realistic description of the dependence of the electrorheological effect on the direction of the electric field. Since Maxwell’s equations (1.2) are separated from the balance laws (1.1) and due to the well developed mathematical theory for Maxwell’s equations (cf. Section 3), we can view the electric field \( \mathbf{E} \) with appropriate properties as a given quantity in (1.1). As a consequence, we concentrate in this paper on the investigation of the mechanical properties of the electrorheological fluid governed by (1.1).

A representative example for a constitutive relation for the stress tensors in (1.1) reads, e.g., (cf. [13], [34])

\[
\mathbf{S} = (\alpha_{31} + \alpha_{33})|\mathbf{E}|^2(1 + |\mathbf{D}|)^{p-2}\mathbf{D} + \alpha_{51}(1 + |\mathbf{D}|)^{p-2}(D\mathbf{E} \otimes \mathbf{E} + \mathbf{E} \otimes D\mathbf{E})
\]

\[
+ \alpha_{71}|\mathbf{E}|^2(1 + |\mathbf{R}|)^{p-2}\mathbf{R} + \alpha_{91}(1 + |\mathbf{R}|)^{p-2}(\mathbf{R}\mathbf{E} \otimes \mathbf{E} + \mathbf{E} \otimes \mathbf{R}\mathbf{E}),
\]

(1.3)

\[
\mathbf{N} = (\beta_{31} + \beta_{33})|\mathbf{E}|^2(1 + |\nabla\omega|^2)^{p-2}\nabla\omega
\]

\[
+ \beta_{51}(1 + |\nabla\omega|)^{p-2}((\nabla\omega)\mathbf{E} \otimes \mathbf{E} + \mathbf{E} \otimes (\nabla\omega)\mathbf{E}),
\]

with material constants \( \alpha_{31}, \alpha_{33}, \alpha_{71}, \beta_{31} > 0 \) and \( \beta_{31} \geq 0 \) and a shear exponent \( p = \hat{p} \circ |\mathbf{E}|^2 \), where \( \hat{p} \) is a material function. In (1.3), we employed the common notation\(^2\) \( \mathbf{D} = (\nabla \mathbf{v})^{\text{sym}} \) and \( \mathbf{R} = \mathbf{R}(\mathbf{v}, \omega) := (\nabla \mathbf{v})^{\text{skew}} + : \omega \).

Micropolar fluids have been introduced by Eringen in the sixties (cf. [14]). A model for electrorheological fluids was proposed in [33], [32], [34]. While there exist many investigations of micropolar fluids or electrorheological fluids (cf. [28], [34]), there exist to our knowledge no mathematical investigations of steady motions of micropolar electrorheological fluids except the PhD thesis [15], the diploma thesis [37] and the research paper [16]. Even these investigations only treat the case of constant shear exponents.

For the existence theory of problems of similar type as (1.1), the Lipschitz truncation technique (cf. [20], [10]) has proven to be very powerful. This method is available in the setting of Sobolev spaces (cf. [19], [10], [12]), variable exponent Sobolev spaces (cf. [10], [12]), solenoidal Sobolev spaces (cf. [4]), Sobolev spaces with Muckenhoupt weights (cf. [16]) and functions of bounded variation (cf. [5]). Since, in general, \( |\mathbf{E}|^2 \) does not belong to the correct Muckenhoupt class \( A_p \), the results in [16, Thm. 5.49, Thm. 5.56, Thm. 5.59 & Thm. 6.44] are either sub-optimal with respect to the lower bound for the shear exponent \( p \) or require additional restrictive assumptions on the electric field \( \mathbf{E} \). Apart from that, solely the case of constant shear exponents is treated. As a consequence, there are no results for the general model for micropolar electrorheological fluids (1.1)–(1.3), which is the most realistic from the point of view of modeling and applications.

The present paper improves the previous treatments in two special aspects. First, we show the existence of solutions for constant shear exponents \( p \) larger than the optimal exponent \( \frac{2d}{d+2} \) without the restrictive assumption that \( |\mathbf{E}|^2 \) belongs to the Muckenhoupt class \( A_p \). Second, we extend this result to the general case of shear exponents \( p = \hat{p} \circ |\mathbf{E}|^2 \) satisfying \( \hat{p} > \frac{2d}{d+2} \). In fact, this seems to be the first existence result in weighted variable exponent Sobolev spaces with a weight not satisfying a Muckenhoupt condition.

\(^2\)Here, \( : \mathbf{v} \) denotes the tensor with components \( \varepsilon_{ijk} v_k \), \( i, j = 1, \ldots, d \).
This paper is organized as follows: First, we introduce the functional setting in the constant exponent case, collect auxiliary results and give assumptions for the stress tensors. Section 3 is devoted to the analysis of the electric field and weighted Sobolev spaces, while Section 4 is devoted to the weak stability of the stress tensors. In Section 5, we deploy the Lipschitz truncation technique in order to prove the existence of solutions of (1.1), (1.2) for constant shear exponents. Section 6 contains the generalization of the previous results to the variable exponent case.

2. Preliminaries

2.1. Notation and function spaces

We employ the customary Lebesgue spaces \( L^p(\Omega) \), \( 1 \leq p \leq \infty \), and Sobolev spaces \( W^{1,p}(\Omega) \), \( 1 \leq p \leq \infty \), where \( \Omega \subseteq \mathbb{R}^d \), \( d \in \mathbb{N} \), is a bounded domain. We denote by \( \| \cdot \|_p \) the norm in \( L^p(\Omega) \) and by \( \| \cdot \|_{1,p} \) the norm in \( W^{1,p}(\Omega) \). Moreover, the spaces \( C^k(\Omega) \), \( k \in \mathbb{N}_0 \cup \{\infty\} \), consist of \( k \)-times continuously differentiable functions with compact support in \( \Omega \). The space \( W^{1,p}(\Omega) \), \( 1 \leq p < \infty \), is defined as the completion of \( C^\infty_0(\Omega) \) with respect to the gradient norm \( \| \nabla \cdot \|_p \), while the space \( V_p(\Omega) \), \( 1 \leq p < \infty \), is the closure of \( C^\infty_0(\Omega) := \{ u \in C^\infty_0(\Omega) : \nabla u = 0 \} \) with respect to the gradient norm \( \| \nabla \cdot \|_p \). For a bounded Lipschitz domain \( G \subseteq \mathbb{R}^d \), we define \( W^{1,\infty}(G) \) as the subspace of functions \( u \in W^{1,\infty}(G) \) having a vanishing trace, i.e., \( u_{|\partial G} = 0 \). We use small boldface letters, e.g., \( v \), to denote vector-valued functions and capital boldface letters, e.g., \( S \), to denote tensor-valued functions. However, we do not distinguish between scalar, vector-valued and tensor-valued function spaces in the notation. The standard scalar product between vectors is denoted by \( \mathbf{v} \cdot \mathbf{u} \), while the standard scalar product between tensors is denoted by \( \mathbf{A} : \mathbf{B} \). For a normed linear vector space \( X \), we denote its topological dual space by \( X' \). Moreover, we employ the notation \( \langle u, v \rangle := \int_{\Omega} \mathbf{u} \cdot \mathbf{v} \, dx \), whenever the right-hand side is well-defined. We denote by \( |M| \) the \( d \)-dimensional Lebesgue measure of a measurable set \( M \). The mean value of a locally integrable function \( u \in L^1_{\text{loc}}(\Omega) \) over a measurable set \( M \subseteq \Omega \) is denoted by \( \frac{1}{|M|} \int_M u \, dx \). By \( L^p_{\text{loc}}(\Omega) \) and \( C^\infty_{0,\text{loc}}(\Omega) \), resp., we denote the subspace of \( L^p(\Omega) \) and \( C^\infty_0(\Omega) \), resp., consisting of all functions \( u \) with vanishing mean value, i.e., \( \int_{\Omega} u \, dx = 0 \).

We will also use weighted Lebesgue and Sobolev spaces (cf. [22], [27], [26]). A weight \( \sigma \) on \( \mathbb{R}^d \) is a locally integrable function satisfying \( 0 < \sigma(x) < \infty \) a.e.\(^4\). To each weight \( \sigma \) we associate a Radon measure \( \nu_\sigma \) defined via \( \nu_\sigma(A) := \int_A \sigma \, dx \). The space \( L^p(\Omega; \sigma) \), \( p \in [1, \infty) \), is defined as the set of all Lebesgue measurable functions \( u : \Omega \to \mathbb{R} \) for which \( \int_\Omega |u|^p \sigma \, dx < \infty \). It is a Banach space if equipped with the norm \( \| u \|_{p,\sigma} := \left( \int_\Omega |u|^p \sigma \, dx \right)^{\frac{1}{p}} \). For \( p \in (1, \infty) \), it is separable and reflexive. Note that, in general, the space \( L^p(\Omega; \sigma) \) does not embed into \( L^1_{\text{loc}}(\Omega) \) (cf. [27]). The condition \( \sigma^\frac{1}{p-1} \in \mathcal{L}^1_{\text{loc}}(\Omega) \) is both necessary and sufficient for the embedding \( L^p(\Omega; \sigma) \hookrightarrow L^1_{\text{loc}}(\Omega) \) (cf. [27], [15]). The dual space of \( L^p(\Omega; \sigma) \) can be identified with respect to \( \langle \cdot, \cdot \rangle \) with \( L^{p'}(\Omega; \sigma') \), where \( \sigma' := \frac{\sigma}{\sigma^\frac{1}{p-1}} \). In particular, we have that

\[
|\langle u, v \rangle| \leq \| u \|_{p,\sigma} \| v \|_{p',\sigma'},
\]

\(^4\)The only exception of this is the electric vector field which is denoted as usual by \( \mathbf{E} \).
\(^5\)If not stated otherwise, a.e. is meant with respect to the Lebesgue measure.
if \( u \in L^p(\Omega; \sigma) \) and \( v \in L^p(\Omega; \sigma') \). By \( L^p_0(\Omega; \sigma) \), we denote the subspace of \( L^p(\Omega; \sigma) \) consisting of all functions with vanishing mean value.

In order to define weighted Sobolev spaces, we make the following assumption on the weight \( \sigma \).

**Assumption 2.1.** Let \( \Omega \subseteq \mathbb{R}^d, d \in \mathbb{N} \), be an open set and \( p \in [1, \infty) \). The weight \( \sigma \) is admissible, i.e., if a sequence \( (\varphi_n)_{n \in \mathbb{N}} \subseteq C_\infty(\Omega) \) and \( v \in L^p(\Omega; \sigma) \) satisfy \( \int_{\Omega} |\varphi_n|^p \sigma \, dx \to 0 \) (\( n \to \infty \)) and \( \int_{\Omega} |\nabla \varphi_n - v|^p \sigma \, dx \to 0 \) (\( n \to \infty \)), then it follows that \( v = 0 \) in \( L^p(\Omega; \sigma) \).

**Remark 2.2.**

(i) If \( \sigma \) belongs to the Muckenhoupt class \( A_p \) for some \( p \in [1, \infty) \), then Assumption 2.1 is satisfied for this specific \( p \) (cf. [22, Sec. 1.9]).

(ii) If \( \sigma \in C^0(\Omega) \), then Assumption 2.1 is satisfied for all \( p \in [1, \infty) \). In fact, the set \( \Omega_\sigma := \{ \sigma > 0 \} \) is open and satisfies \( |\Omega \setminus \Omega_\sigma| = 0 \). In addition, for any \( K \subset \subset \Omega_\sigma \), there exists a constant \( c_K > 0 \) such that \( c_K^{-1} \leq \sigma \leq c_K \) in \( K \). Thus, for a sequence \( (\varphi_n)_{n \in \mathbb{N}} \subseteq C_\infty(\Omega) \) from \( \int_{\Omega} |\varphi_n|^p \sigma \, dx \to 0 \) (\( n \to \infty \)) and \( \int_{\Omega} |\nabla \varphi_n - v|^p \sigma \, dx \to 0 \) (\( n \to \infty \)), where \( v \in L^p(\Omega; \sigma) \), it follows that \( \varphi_n \to 0 \) in \( L^p(K) \) (\( n \to \infty \)) and \( \nabla \varphi_n \to v \) in \( L^p(K) \) (\( n \to \infty \)) for all \( K \subset \subset \Omega_\sigma \). Consequently, for every \( \psi \in C^0_\infty(\Omega_\sigma) \), one has that

\[
0 = \lim_{n \to \infty} \int_{\Omega} \varphi_n \text{div} \psi \, dx = \lim_{n \to \infty} \int_{\Omega} \nabla \varphi_n \cdot \psi \, dx = \int_{\Omega} v \cdot \psi \, dx,
\]

i.e., \( v = 0 \) a.e. in \( \Omega_\sigma \), which, in turn, implies that \( v = 0 \) a.e. in \( \Omega \).

(iii) There exist weights \( \sigma \) such that Assumption 2.1 is not satisfied (cf. [17]).

For \( \sigma \) satisfying Assumption 2.1, and \( p \in [1, \infty) \), we introduce the norm

\[
\| u \|_{1,p,\sigma} := \| u \|_{p,\sigma} + \| \nabla u \|_{p,\sigma},
\]

whenever the right-hand side is well-defined. Then, the Sobolev space \( H^{1,p}(\Omega; \sigma) \) is defined to be the completion of

\[
V_{p,\sigma} := \{ u \in C_\infty(\Omega) \mid \| u \|_{1,p,\sigma} < \infty \}
\]

with respect to the norm \( \| \cdot \|_{1,p,\sigma} \). In other words, \( u \in H^{1,p}(\Omega; \sigma) \) if and only if \( u \in L^1(\Omega; \sigma) \) and there exists a function \( v \in L^p(\Omega; \sigma) \) such that for some sequence \( (\varphi_n)_{n \in \mathbb{N}} \subseteq C_\infty(\Omega) \) holds \( \int_{\Omega} |\varphi_n - u|^p \sigma \, dx \to 0 \) (\( n \to \infty \)) and \( \int_{\Omega} |\nabla \varphi_n - v|^p \sigma \, dx \to 0 \) (\( n \to \infty \)). In this case, the function \( v \) is called the gradient of \( u \) in \( H^{1,p}(\Omega; \sigma) \) and denoted by \( \nabla u := v \). Here, Assumption 2.1 implies that \( \nabla u \) is a uniquely defined function in \( L^p(\Omega; \sigma) \). Note that \( W^{1,p}(\Omega) = H^{1,p}(\Omega; 1) \) if \( \sigma = 1 \) a.e. in \( \Omega \) with \( \nabla u = \nabla v \) for all \( u \in W^{1,p}(\Omega) \). However, in general, \( \nabla v \) and the usual weak or distributional gradient \( \nabla u \) do not coincide. The space \( H^{1,p}(\Omega; \sigma) \), \( p \in (1, \infty) \), is a separable and reflexive Banach space. Then, we define the space \( H^{1,p}_0(\Omega; \sigma) \) as the completion of \( C^0_\infty(\Omega) \) with respect to \( \| \cdot \|_{1,p,\sigma} \). We will use the observation that, if \( \sigma \in L^\infty(\Omega) \), then \( W^{1,p}_0(\Omega) \hookrightarrow H^{1,p}_0(\Omega; \sigma) \) and \( \nabla u = \nabla v \) for every \( u \in W^{1,p}(\Omega) \) (cf. [22, Lem. 1.12]), which is a consequence of the inequality \( \| u \|_{p,\sigma} \leq \| \sigma^{1/p} \|_{\infty} \| u \|_{p} \) valid for every \( u \in L^p(\Omega) \) and the density of \( C^\infty(\Omega) \cap W^{1,p}_0(\Omega) \) in \( W^{1,p}_0(\Omega) \).

Another possible approach is to define the weighted Sobolev space \( W^{1,p}(\Omega; \sigma) \) as the set of all functions \( u \in L^p(\Omega; \sigma) \) which posses a distributional gradient \( \nabla u \in L^p(\Omega; \sigma) \). We equip \( W^{1,p}(\Omega; \sigma) \) with the norm \( \| \cdot \|_{1,p,\sigma} \). Note that, in general, the space \( W^{1,p}(\Omega; \sigma) \) need not to be a Banach space (cf. [22]). To make \( W^{1,p}(\Omega; \sigma) \) a Banach space, the condition \( \sigma^{1/p} \in L^{1/p}_{loc}(\Omega) \) is sufficient (cf. [27]). However, this condition is for our purposes too restrictive (cf. Section 3). As a consequence, we will not use \( W^{1,p}(\Omega; \sigma) \), but we will work with the spaces \( H^{1,p}(\Omega; \sigma) \).
2.2. Auxiliary results

The following generalization of a classical result (cf. [21]) is very useful in the identification of limits.

**Theorem 2.3.** Let \( \Omega \subseteq \mathbb{R}^d \), \( d \in \mathbb{N} \), be a bounded domain, \( \sigma \) a weight and \( p \in [1, \infty) \). Then, for a sequence \( (u_n)_{n \in \mathbb{N}} \subseteq L^p(\Omega; \sigma) \) from\(^5\)

\[
\begin{align*}
\text{(i)} & \quad \lim_{n \to \infty} u_n = v \text{ \( \nu_{\sigma} \)-a.e. in } \Omega, \\
\text{(ii)} & \quad u_n \rightharpoonup u \text{ in } L^p(\Omega; \sigma) \ (n \to \infty),
\end{align*}
\]

it follows that \( u = v \) in \( L^p(\Omega; \sigma) \).

**Proof:** See [23, Thm. 13.44].

Our proof relies on the following version of the Lipschitz truncation technique:

**Theorem 2.4.** Let \( G \subseteq \mathbb{R}^d \), \( d \in \mathbb{N} \), be a bounded Lipschitz domain and \( p \in (1, \infty) \). Furthermore, let \( u^n \in W^{1,p}_0(G) \) be such that \( u^n \to 0 \) in \( W^{1,p}_0(G) \) \( (n \to \infty) \). Then, for any \( j, n \in \mathbb{N} \), there exist \( u^{n,j} \in W^{1,\infty}_0(G) \) and \( \lambda_{n,j} \in [2^{2j}, 2^{2j+1}] \) such that

\[
\begin{align*}
\lim_{n \to \infty} (\sup_{j \in \mathbb{N}} \|u^{n,j}\|_{\infty}) &= 0, \\
\|\nabla u^{n,j}\|_{\infty} &\leq c \lambda_{n,j} \leq c 2^{2j+1}, \\
\|\nabla u^{n,j}\|_{p} &\leq c \lambda_{n,j} \|\{u^{n,j} \neq u^n\}\|, \\
\limsup_{n \to \infty} \lambda_{n,j} \|\{u^{n,j} \neq u^n\}\| &\leq c 2^{-j},
\end{align*}
\]

(2.5)

where \( c = c(d, p, G) > 0 \). Moreover, for any \( j \in \mathbb{N} \), \( \nabla u^{n,j} \rightharpoonup 0 \) in \( L^s(G) \) \( (n \to \infty) \), \( s \in [1, \infty) \), and \( \nabla u^{n,j} \rightharpoonup 0 \) in \( L^\infty(G) \) \( (n \to \infty) \).

**Proof:** See [10, Theorem 2.5].

Except classical Korn’s and Poincaré’s inequalities, we also need the following result for the divergence equation.

**Theorem 2.6.** Let \( G \subseteq \mathbb{R}^d \), \( d \geq 2 \), be a bounded Lipschitz domain. Then, there exists a linear operator \( B_G : C_0^\infty(\Omega) \to C_0^\infty(\Omega) \) (which for all \( p \in (1, \infty) \)) extends uniquely to a linear, bounded operator \( B_G : L^p_0(G) \to W^{1,p}_0(G) \) such that \( \|B_G u\|_{1,p} \leq c \|u\|_p \) and \( \text{div} B_G u = u \) for every \( u \in L^p_0(G) \).

**Proof:** See [2], [3].

3. The electric field \( \mathbf{E} \)

We first note that the system (1.2) is separated from (1.1), in the sense that one can first solve the quasi-static Maxwell’s equations yielding an electric field \( \mathbf{E} \), which then, in turn, enters into (1.1) as a parameter through the stress tensors.

It is proved in [30], [31], [34], that for bounded Lipschitz domains, there exists a solution\(^6\) \( \mathbf{E} \in H(\text{curl}) \cap H(\text{div}) \) of the system (1.2) with \( \|\mathbf{E}\|_2 \leq c \|\mathbf{E}_0\|_{H^{-1/2}(\partial \Omega)} \).

---

\(^5\)Recall that \( \nu_{\sigma}(A) = \int_A \sigma \, dx \) for all measurable sets \( A \subseteq \Omega \).

\(^6\)Here, we employ the standard function spaces \( H(\text{curl}) := \{v \in L^2(\Omega) \mid \text{curl } v \in L^2(\Omega)\} \), \( H(\text{div}) := \{v \in L^2(\Omega) \mid \text{div } v \in L^2(\Omega)\} \) and \( H^{-1/2}(\partial \Omega) := (H^{1/2}(\partial \Omega))^* \).
A more detailed analysis of the properties of the electric field \( \mathbf{E} \) can be found in [15]. Let us summarize these results here. First, note that combining (1.2)\(_1\) and (1.2)\(_2\), we obtain that

\[-\Delta \mathbf{E} = \text{curl curl} \mathbf{E} - \nabla \text{div} \mathbf{E} = 0,\]

i.e., the electric field \( \mathbf{E} \) is a harmonic. Moreover, the structure of the stress tensors (cf. Assumption 4.1, Assumption 4.2) yields that the natural functional setting of our problem involves weighted Sobolev spaces, where the weight is given by \( |\mathbf{E}|^2 \).

Using the theory of harmonic functions it is shown in [15, Sec. 3.2] that \( |\mathbf{E}|^2 \) belongs to the Muckenhoupt class \( A_\infty \) and that, in general, \( |\mathbf{E}|^{\frac{2}{p-1}} \) does not belong to \( L^1_{\text{loc}}(\Omega) \). Since for our investigations it is more important to work with a Banach space than that the gradient is a distributional gradient, we, hence, work with the space \( H^{1,p}(\Omega; |\mathbf{E}|^2) \) and not with the space \( W^{1,p}(\Omega; |\mathbf{E}|^2) \).

On the other hand, because any harmonic function is real analytic, one can characterize its zero set as follows:

**Lemma 3.1.** Let \( \Omega \subseteq \mathbb{R}^d, d \in \mathbb{N}, \) be a bounded domain and \( u : \Omega \rightarrow \mathbb{R} \) a non-trivial analytic function. Then, \( u^{-1}(0) \) is a union of \( C^1 \)-manifolds \( (M_i)_{i=1,\cdots,m}, \) \( m \in \mathbb{N}, \) with \( \dim M_i \leq d - 1 \) for every \( i = 1, \cdots, m, \) and \( |u^{-1}(0)| = 0.\)

**Proof:** See [15], [16, Lem. 3.1]. \( \blacksquare \)

Finally, we observe that using the regularity theory for Maxwell’s equations (cf. [35], [34]), one can give conditions on the boundary data \( \mathbf{E}_0 \) ensuring that the electric field \( \mathbf{E} \) is globally bounded, i.e., \( |\mathbf{E}|_\infty \leq c(\mathbf{E}_0). \) Based on these observations, we will make the following assumption on the electric field \( \mathbf{E} \):

**Assumption 3.2.** The electric field \( \mathbf{E} \) satisfies \( \mathbf{E} \in C^\infty(\Omega) \cap L^\infty(\Omega) \) and the closed set \( |\mathbf{E}|^{-1}(0) \) is a null set, i.e., \( \Omega_0 := \{ x \in \Omega \mid |\mathbf{E}(x)| > 0 \} \) has full measure.

In the sequel, we do not use that \( \mathbf{E} \) is the solution of the quasi-static Maxwell’s equations (1.2), but we will only use Assumption 3.2. The following embedding will play a substantial role in our investigation.

**Theorem 3.3.** Let \( \Omega \subseteq \mathbb{R}^d, d \in \mathbb{N}, \) be open, \( p \in [1, \infty) \) and let Assumption 3.2 be satisfied. Set \( p^* := \frac{dp}{d-1} \) if \( p < d \) and \( p^* := \infty \) if \( p \geq d. \) Then, for any open set \( \Omega' \subset \subset \Omega \) with \( \partial \Omega' \in C^0(\overline{\Omega'}) \) and any \( \alpha \geq 1 + \frac{2}{p^*}, \) it holds

\[ H^{1,p}(\Omega; |\mathbf{E}|^2) \hookrightarrow L^r(\Omega'; |\mathbf{E}|^\alpha) \]

with \( r \in [1, p^*] \) if \( p \neq d \) and \( r \in [1, p^*) \) if \( p = d. \)

**Proof:** The proof of this result is inspired by [1]. First, let \( u \in \mathcal{V}_{p,|\mathbf{E}|^2} \) be arbitrary. Due to \( \Omega' \subset \subset \Omega \) and \( \mathbf{E} \in C^\infty(\Omega), \) it holds \( |\mathbf{E}|^\alpha \in C^1(\overline{\Omega'}) \) for any \( \alpha > 1. \) In fact, \( |\mathbf{E}|^\alpha \in C^1(\overline{\Omega'}) \setminus (|\mathbf{E}|^{-1}(0)) \) holds since we have

\[ \nabla|\mathbf{E}|^\alpha = \alpha|\mathbf{E}|^{\alpha-2}\nabla \mathbf{E} \quad \text{in } \overline{\Omega'} \setminus |\mathbf{E}|^{-1}(0), \]

which can be extended continuously to all of \( \overline{\Omega'} \) for any \( \alpha > 1. \) Apparently, we have \( u|\mathbf{E}|^\alpha \in L^p(\Omega') \) with

\[ \| u|\mathbf{E}|^\alpha \|^p_{L^p(\Omega')} \leq \| \mathbf{E} \|^{p-2}_{L^\infty(\Omega')} \| u \|^p_{L^p(\Omega'; |\mathbf{E}|^2)} \].
since \(ap \geq 2\). Moreover, we have \(u|E|^\alpha \in W^{1,p}(\Omega')\). In fact, due to \(\alpha \geq 1 + \frac{2}{p}\), and \(\nabla(u|E|^\alpha) = \nabla u|E|^\alpha + \alpha u|E|^\alpha - 2 \nabla |E|^\alpha \nabla u\) almost everywhere in \(\Omega'\), we get
\[
\begin{align*}
\|\nabla(u|E|^\alpha)\|_{L^p(\Omega')} & \leq 2^p (\|\nabla u\|_{L^p(\Omega')} + \alpha \|\nabla |E|^\alpha\|_{L^\infty(\Omega')} \|u|E|^\alpha\|_{L^p(\Omega')}) \\
& \leq 2^p (\|E\|_{L^p(\Omega')})^\alpha \|\nabla u\|_{L^p(\Omega'; |E|^{\alpha p})} + \|\nabla |E|^\alpha\|_{L^\infty(\Omega')} \|u|E|^\alpha\|_{L^p(\Omega')} \|u\|_{L^p(\Omega'; |E|^{\alpha p})}.
\end{align*}
\]
Hence, Sobolev’s embedding theorem yields a constant \(c_S > 0\) such that we have for the above specified exponents \(r\)
\[
\|u\|_{L^p(\Omega'; |E|^{\alpha p})} = \|u\|_{W^{1,p}(\Omega')} \leq c_S \|u|E|^\alpha\|_{W^{1,p}(\Omega')} (3.4)
\]
\[
\leq c_S^2 \|\nabla\|_{L^\infty(\Omega')} \|\nabla |E|^\alpha\|_{L^\infty(\Omega')} \|\nabla u\|_{L^p(\Omega'; |E|^{\alpha p})} \|u\|_{L^p(\Omega'; |E|^{\alpha p})}.
\]
Next, let \(u \in H^{1,p}(\Omega; |E|^2)\) be arbitrary. Then, by definition, there is a sequence \((u_n)_{n \in \mathbb{N}} \subseteq V_{p,|E|^2}\) such that \(u_n \to u\) in \(H^{1,p}(\Omega; |E|^2)\) (\(n \to \infty\)). Then, resorting to inequality (3.4), it is readily seen that \((u_n)_{n \in \mathbb{N}} \subseteq V_{p,|E|^2}\) is a Cauchy sequence in \(L^p(\Omega'; |E|^{\alpha p})\). Since \(L^p(\Omega'; |E|^{\alpha p})\) is complete, there exists some \(v \in L^p(\Omega'; |E|^{\alpha p})\) such that \(u_n \to v\) in \(L^p(\Omega'; |E|^{\alpha p})\) (\(n \to \infty\)). To identify \(u\) with \(v\), one usually uses the embeddings \(L^p(\Omega'; |E|^{\alpha p}), L^p(\Omega'; |E|^2) \hookrightarrow L_{loc}^p(\Omega)\). However, in general, we do not have these embeddings available and need to argue differently. We exploit that from \(u_n \to u\) in \(H^{1,p}(\Omega; |E|^2)\) (\(n \to \infty\)) and \(u_n \to v\) in \(L^p(\Omega'; |E|^{\alpha p})\) (\(n \to \infty\)), it follows that, up to a subsequence, it holds \(u_n \to u\) \(\nu_{\Omega'; |E|^{\alpha p}}\)-a.e. in \(\Omega\) (\(n \to \infty\)) and \(u_n \to v\) \(\nu_{\Omega'; |E|^2}\)-a.e. in \(\Omega\) (\(n \to \infty\)). The properties of \(E\) and Tschebyscheff’s inequality imply that the Lebesgue measure is absolutely continuous with respect to the measures \(\nu_{\Omega'; |E|^2}\) and \(\nu_{\Omega'; |E|^{\alpha p}}\). Therefore, we conclude that \(u = v\) a.e. in \(\Omega\). Since \(\nu_{\Omega'; |E|^{\alpha p}}\) is also absolutely continuous with respect to the Lebesgue measure, we just proved \(u = v\) in \(L^p(\Omega'; |E|^{\alpha p})\).

**Lemma 3.5.** Let \(\Omega \subseteq \mathbb{R}^d\), \(d \in \mathbb{N}\), be open, \(p \in [1, \infty)\) and let Assumption 3.2 be satisfied. Then, for any \(\Omega' \subset \subset \Omega_0\), we have that \(W^{1,p}(\Omega') = H^{1,p}(\Omega'; |E|^2)\) with norm equivalence (depending on \(\Omega'\) and \(E\)) and \(\nabla u = \nabla v\) for all \(u \in W^{1,p}(\Omega')\).

**Proof:** Due to \(|E| > 0\) in \(\Omega\) and \(|E| \in C^0(\Omega)\), there is a local constant \(c(\Omega') > 0\) such that \(c(\Omega')^{-1} \leq |E|^2 \leq c(\Omega')\) in \(\Omega\). Thus, we have \(L^p(\Omega') = L^p(\Omega'; |E|^2)\) with
\[
c(\Omega')^{-\frac{1}{2}} \|u\|_{L^p(\Omega')} \leq \|u\|_{L^p(\Omega'; |E|^2)} \leq c(\Omega') \|u\|_{L^p(\Omega')}
\]
for every \(u \in L^p(\Omega') = L^p(\Omega'; |E|^2)\). As a result, we also have \(V_{p,|E|^2} = V_{p,1}\) with
\[
c(\Omega')^{-\frac{1}{2}} \|u\|_{W^{1,p}(\Omega')} \leq \|u\|_{H^{1,p}(\Omega'; |E|^2)} \leq c(\Omega') \|u\|_{W^{1,p}(\Omega')} (3.6)
\]
for every \(u \in V_{p,|E|^2} = V_{p,1}\). Since \(W^{1,p}(\Omega')\), by Meyer–Serrin’s theorem, is the closure of \(V_{p,1}\) and \(H^{1,p}(\Omega'; |E|^2)\), by definition, is the closure of \(V_{p,|E|^2}\), (3.6) implies that \(W^{1,p}(\Omega') = H^{1,p}(\Omega'; |E|^2)\) and \(\nabla u = \nabla v\) for all \(u \in W^{1,p}(\Omega')\).

**4. A weak stability lemma**

The weak stability of problems of \(p\)-Laplace type is well-known (cf. [10]). It also holds for our problem (1.1) if we make appropriate natural assumptions on the extra stress tensor \(S\) and on the couple stress tensor \(N\), which are motivated by the canonical example in (1.3) for constant shear exponents. We denote the symmetric and the skew-symmetric part, resp., of a tensor \(A \in \mathbb{R}^{d \times d}\) by \(A_{\text{sym}} := \frac{1}{2}(A + A^\top)\) and \(A_{\text{skew}} := \frac{1}{2}(A - A^\top)\). Moreover, we define \(\mathbb{R}_{\text{sym}} := \{A \in \mathbb{R}^{d \times d} \mid A = A_{\text{sym}}\}\) and \(\mathbb{R}_{\text{skew}} := \{A \in \mathbb{R}_{\text{sym}} \mid A = A_{\text{skew}}\}\).
Assumption 4.1. For the extra stress tensor $S : \mathbb{R}^{d \times d} \times \mathbb{R}^{d \times d} \times \mathbb{R}^d \to \mathbb{R}^d$ and some $p \in (1, \infty)$, there exist constants $c, C > 0$ such that:

(S.1) $S \in C^0(\mathbb{R}^{d \times d} \times \mathbb{R}^{d \times d} \times \mathbb{R}^d, \mathbb{R}^d)$.

(S.2) For every $D \in \mathbb{R}^{d \times d}$, $R \in \mathbb{R}^{d \times d}$ skew and $E \in \mathbb{R}^d$, it holds

\[ |S^\text{sym}(D, R, E)| \leq c (1 + |E|^2) (1 + |D|^{p-1}), \]
\[ |S^\text{skew}(D, R, E)| \leq c |E|^2 (1 + |R|^{p-1}) \cdot \]

(S.3) For every $D \in \mathbb{R}^{d \times d}$, $R \in \mathbb{R}^{d \times d}$ skew and $E \in \mathbb{R}^d$, it holds

\[ S(D, R, E) : D \geq c (1 + |E|^2) (|D|^p - C), \]
\[ S(D, R, E) : R \geq c |E|^2 (|R|^p - C) \cdot \]

(S.4) For every $D_1, D_2 \in \mathbb{R}^{d \times d}$, $R_1, R_2 \in \mathbb{R}^{d \times d}$ skew and $E \in \mathbb{R}^d$ with $(D_1, |E|R_1) \neq (D_2, |E|R_2)$, it holds

\[ (S(D_1, R_1, E) - S(D_2, R_2, E)) : (D_1 - D_2 + R_1 - R_2) > 0 \cdot \]

Assumption 4.2. For the couple stress tensor $N : \mathbb{R}^{d \times d} \times \mathbb{R}^d \to \mathbb{R}^{d \times d}$ and some $p \in (1, \infty)$, there exist constants $c, C > 0$ such that:

(N.1) $N \in C^0(\mathbb{R}^{d \times d} \times \mathbb{R}^d, \mathbb{R}^{d \times d})$.

(N.2) For every $L \in \mathbb{R}^{d \times d}$ and $E \in \mathbb{R}^d$, it holds

\[ |N(L, E)| \leq c |E|^2 (1 + |L|^{p-1}) \cdot \]

(N.3) For every $L \in \mathbb{R}^{d \times d}$ and $E \in \mathbb{R}^d$, it holds

\[ N(L, E) : L \geq c |E|^2 (|L|^p - C) \cdot \]

(N.4) For every $L_1, L_2 \in \mathbb{R}^{d \times d}$ and $E \in \mathbb{R}^d$ with $|E| > 0$ and $L_1 \neq L_2$, it holds

\[ (N(L_1, E) - N(L_2, E)) : (L_1 - L_2) > 0 \cdot \]

Under these assumptions, the following weak stability of our problem (1.1) is valid.

Lemma 4.3. Let $\Omega \subseteq \mathbb{R}^d$, $d \geq 2$, be a bounded domain, let $p > \frac{2d}{d+2}$ and let Assumption 4.1, Assumption 4.2 and Assumption 3.2 be satisfied. Furthermore, let $(v^n)_{n \in \mathbb{N}} \subseteq V_p(\Omega)$ and $(\omega^n)_{n \in \mathbb{N}} \subseteq H^{-1,p}_0(\Omega; |E|^2)$ be such that

\[ v^n \to v \quad \text{in } V_p(\Omega) \quad (n \to \infty), \]
\[ \omega^n \to \omega \quad \text{in } H^{-1,p}_0(\Omega; |E|^2) \quad (n \to \infty). \quad (4.4) \]

For a ball $B \subseteq \Omega_0$ such that $B' := 2B \subseteq \Omega_0$ and $\tau \in C^\infty_0(B')$ satisfying $\chi_B \leq \tau \leq \chi_B'$ we set $u^n := (v^n - v)\tau \in W^{-1,p}_0(B')$, $\psi^n := (\omega^n - \omega)\tau \in W^{-1,p}_0(B')$, $n \in \mathbb{N}$. Let $u^{n,j} \in W^{1,\infty}_0(B')$, $n, j \in \mathbb{N}$, and $\psi^{n,j} \in W^{1,\infty}_0(B')$, $n, j \in \mathbb{N}$, resp., denote the Lipschitz truncations constructed according to Theorem 2.4. Moreover, assume that for every $j \in \mathbb{N}$, we have that

\[ \limsup_{n \to \infty} \left| \langle S(Dv^n, R(v^n, \omega^n), E) - S(Dv, R(v, \omega), E), Du^{n,j} + R(u^{n,j}, \psi^{n,j}) \rangle \right| + \langle N(\nabla \omega^n, E) - N(\nabla \omega, E), \nabla \psi^{n,j} \rangle \leq \delta_j, \quad (4.5) \]

where $\delta_j \to 0$ ($j \to \infty$). Then, one has that $\nabla v^n \to \nabla v$ a.e. in $B$ ($n \to \infty$), $\nabla \omega^n \to \nabla \omega$ a.e. in $B$ ($n \to \infty$) and $\omega^n \to \omega$ a.e. in $B$ ($n \to \infty$) for suitable subsequences.
Remark 4.6. For each ball $B' \subset \Omega_0$, Lemma 3.5 shows that $W^{1, p}(B') = H^{1, p}(B'; |E|^2)$. Hence, for any $\omega \in H^{1, p}(\Omega; |E|^2)$, it holds $\omega|_{B'} \in W^{1, p}(B')$ with $\nabla(\omega|_{B'}) = (\nabla \omega)|_{B'}$ for each ball $B' \subset \Omega_0$. In precisely that sense, the gradients of $\omega \in H^{1, p}(\Omega; |E|^2)$ and $(\omega^n)_{n \in \mathbb{N}} \subseteq H^{1, p}(\Omega; |E|^2)$ are to be understood in (4.5).

Proof: Since $W^{1, p}(B') = H^{1, p}(B'; |E|^2)$ with norm equivalence (cf. Lemma 3.5), from (4.4), and resorting to Rellich’s compactness theorem, we deduce that

$$
v^n \to v \quad \text{in } L^q(B') \text{ and a.e. in } B' \quad (n \to \infty),$$

$$\omega^n \to \omega \quad \text{in } L^q(B') \text{ and a.e. in } B' \quad (n \to \infty),$$

(4.7)

for any $q \in [1, p^*)$. Throughout the proof, we will employ the particular notation

$$\tilde{S} := S(Dv, R(v, \omega), E), \quad S^n := S(Dv^n, R(v^n, \omega^n), E), \quad \tilde{N} := N(\nabla \omega, E), \quad N^n := N(\nabla \omega^n, E).$$

Using (S.2), (N.2), Assumption 3.2 and (4.4), we see that there exists a constant $K := K(\|E\|_\infty) > 0$ (not depending on $n \in \mathbb{N}$) such that

$$\|v^n\|_{1, p} + \|\omega^n\|_{1, p, |E|^2} + \|\omega\|_{1, p, |E|^2} \leq K,$$

$$\|S^n\|_{p'} + \|\tilde{S}\|_{p'} + \|(S^n)_{\text{skew}}\|_{p', |E|^2}^{\frac{p}{p-1}} + \|(\tilde{S})_{\text{skew}}\|_{p', |E|^2}^{\frac{p}{p-1}} \leq K,$$

$$\|N^n\|_{p', |E|^2}^{\frac{p}{p-1}} + \|N\|_{p', |E|^2}^{\frac{p}{p-1}} \leq K.$$

(4.9)

Recall that $\tau \in C_0^\infty(B')$ with $\chi_B \leq \tau \leq \chi_{B'}$. Hence, using (S.4) and (N.4), we get

$$I^n := \int_B^\tau [(S^n - \tilde{S}):(D(v^n - v) + (v^n - v, \omega^n - \omega))] \theta dx$$

$$\leq \int_{B'} [(S^n - \tilde{S}):(D(v^n - v) + (v^n - v, \omega^n - \omega))] \theta dx$$

$$\leq \int_{B'} [(S^n - \tilde{S}):(D(v^n - v) + (v^n - v, \omega^n - \omega)) \tau] \theta dx$$

$$+ \int_{B'} [(N^n - \tilde{N}) : \nabla(\omega^n - \omega)] \theta dx =: \int_{B'} \alpha^n dx + \int_{B'} \beta^n dx,$$

(4.10)

where we also used that

$$\frac{1}{2} (a^\theta + b^\theta) \leq (a + b)^\theta \leq a^\theta + b^\theta$$

(4.11)

valid for all $a, b \geq 0$ and $\theta \in (0, 1)$. Then, splitting the integral of $\alpha^n$ over $B'$ into an integral over $\{u^n \neq u^{n-1}\}$ and one over $\{u^n = u^{n-1}\}$, also using Hölder’s inequality with exponents $\frac{1}{\theta}$ and $\frac{1}{1-\theta}$, we find that

$$\int_{B'} \alpha^n dx \leq \|\alpha_n\|^\theta_{L^1(B')} \{u^n \neq u^{n-1}\}|^{-\theta} + \|\alpha_n \chi_{\{u^n = u^{n-1}\}}\|^\theta_{L^1(B')} |B'|^{1-\theta}$$

$$= (I^n)^\theta \{u^n \neq u^{n-1}\}|^{-\theta} + \|\alpha_n \chi_{\{u^n = u^{n-1}\}}\|^\theta_{L^1(B')} |B'|^{1-\theta}.$$  

(4.12)
For the first term, we will use (2.5) and, thus, have to show that \((I^n_k)_{n \in \mathbb{N}} \subseteq \mathbb{R}\) is bounded. To this end, we use that for vector fields \(u, w\) and tensor fields \(A\) there holds

\[
A : Du + A : R(u, w) = A : \nabla u + A^{\text{skew}} : (e \cdot w).
\]

(4.13)

Then, combining (4.4), (4.9), (4.13) and that \(\tau \leq 1\) in \(\Omega\), we observe that

\[
I^n_1 \leq \left( \|S^n\|_{p'} + \|\tilde{S}\|_{p'} \right) \|\nabla v^n - \nabla v\|_p
+ \left( \|S^n\|_{p'} \|p'\|_{\frac{2}{p'}} + \|\tilde{S}\|_{p'} \|p'\|_{\frac{2}{p'}} \right) \|\omega^n - \omega\|_{p,|E|^2}
\]

(4.14)

\[\leq 2 K^2.\]

Similarly, we deduce that

\[
\int_{B^r} \beta_n^\theta dx \leq \left\| \beta_n \right\|_{L^1(B^r)} \{ \psi^n \neq \psi^n_j \} \|\nabla \psi^n \|_{\theta} + \left\| \beta_n \chi(\psi^n = \psi^n_j) \right\|_{L^1(B^r)} \|B^r\|^{1 - \theta}
\]

(4.15)

and that

\[
I^n_0 \leq \left( \|N^n\|_{p'} + \|\tilde{N}\|_{p'} \right) \|\nabla \omega^n - \nabla \omega\|_{p,|E|^2} \leq 2 K^2.
\]

(4.16)

Using (4.12), (4.14)–(4.16) and (4.11) we, thus, conclude that

\[
\int_{B^r} \alpha_n^\theta dx + \int_{B^r} \beta_n^\theta dx
\]

(4.17)

\[\leq 2^\theta K^{2\theta} \{(u^n \neq u^{n,j})\}^{1 - \theta} + \{(\psi^n \neq \psi^n_j)\}^{1 - \theta}
+ 2 \|B^r\|^{1 - \theta} \left( \int_{B^r} \alpha_n \chi(u^n = u^{n,j}) dx + \int_{B^r} \beta_n \chi(\psi^n = \psi^n_j) dx \right)^\theta.
\]

Let us now treat the last two integrals, which we denote by \(I^{n,j}_3\) and \(I^{n,j}_4\). We have \(\nabla (v^n - v) = \nabla u^n - (v^n - v) \cdot \nabla \tau\) on \(\{u^n = u^{n,j}\}\), which, using (4.13), implies

\[
I^{n,j}_3 = \langle S^n - \tilde{S}, (\nabla u^n - (v^n - v) \cdot \nabla \tau) \chi(u^n = u^{n,j}) \rangle
\]

(4.18)

\[+ \langle (S^n - \tilde{S})^{\text{skew}}, e \cdot \psi^n_j \chi(\psi^n = \psi^n_j) \rangle
+ \langle (S^n - \tilde{S})^{\text{skew}}, e \cdot (\omega^n - \omega) \chi(u^n = u^{n,j}) \cap (\psi^n = \psi^n_j) \rangle
\]

\[+ \langle (S^n - \tilde{S})^{\text{skew}}, e \cdot (\omega^n - \omega) \chi(u^n \neq u^{n,j}) \cap (\psi^n = \psi^n_j) \rangle.\]

From \(\nabla (\omega^n - \omega) = \nabla \psi^n_j - (\omega^n - \omega) \cdot \nabla \tau\) on \(\{\psi^n = \psi^n_j\}\), it follows that

\[
I^{n,j}_4 = \langle N^n - \tilde{N}, (\nabla \psi^n_j - (\omega^n - \omega) \cdot \nabla \tau) \chi(\psi^n = \psi^n_j) \rangle.
\]

(4.19)

Using (4.13) and adding appropriate terms, we deduce from (4.18) and (4.19) that

\[
I^{n,j}_3 + I^{n,j}_4 \leq \langle S^n - \tilde{S}, (Du^n - R(u^n, \psi^n_j)) \rangle + \langle N^n - \tilde{N}, \nabla \psi^n_j \rangle
+ \langle (S^n - \tilde{S})^{\text{skew}}, e \cdot \psi^n_j \chi(\psi^n = \psi^n_j) \rangle + \langle (S^n - \tilde{S})^{\text{skew}}, \omega^n - \omega \rangle
+ \langle (S^n - \tilde{S})^{\text{skew}}, (v^n - v) \cdot \nabla \tau \rangle + \langle N^n - \tilde{N}, (\omega^n - \omega) \cdot \nabla \tau \rangle
\]

(4.20)

\[= \sum_{k=5}^{11} I^{n,j}_k.\]
The term $I_0^{n,j}$, i.e., the first line on the right-hand side in (4.20), is handled by (4.5). For the other terms we obtain, using Hölder’s inequality and (4.9), that

\[
I_0^{n,j} \leq (|S^n|_{p'} + |\tilde{S}|_{p'}) \|\nabla u^{n,j} \chi_{\{u^n \neq u^{n,j}\}}\|_{L^p(B')}
\leq K \|\nabla u^{n,j} \chi_{\{u^n \neq u^{n,j}\}}\|_{L^p(B')},
\]

(4.21)

\[
I_7^{n,j} \leq (|N^n|_{p',|E|} \frac{1}{\rho^n} + |\tilde{N}|_{p',|E|} \frac{1}{\rho^n}) \|\nabla \psi^{n,j} \chi_{\{\psi^n \neq \psi^{n,j}\}}\|_{L^p(B'|E|^2)}
\leq K \|E\|_\infty^\frac{\theta}{p} \|\nabla \psi^{n,j} \chi_{\{\psi^n \neq \psi^{n,j}\}}\|_{L^p(B')},
\]

(4.22)

\[
I_8^{n,j} \leq (|S'|_{p',|E|} \frac{1}{\rho^n} + |\tilde{S}'|_{p',|E|} \frac{1}{\rho^n}) \|\nabla \tau_{\infty} \|_{L^p(B')} + \|E\|_\infty |\Omega|^{\frac{1}{2}} \|\psi^{n,j}\|_{L_\infty(B')},
\]

(4.23)

\[
I_9^{n,j} \leq (|S'|_{p',|E|} \frac{1}{\rho^n} + |\tilde{S}'|_{p',|E|} \frac{1}{\rho^n}) \|\nabla \omega^n - \omega\|_{L^p(B'|E|^2)}
\leq K \|E\|_\infty \|\omega^n - \omega\|_{L^p(B')},
\]

(4.24)

\[
I_{10}^{n,j} \leq (|S'|_{p',|E|} \frac{1}{\rho^n} + |\tilde{S}'|_{p',|E|} \frac{1}{\rho^n}) \|\nabla \tau_{\infty} \|_{L^p(B')} + \|E\|_\infty |\Omega|^{\frac{1}{2}} \|\psi^{n,j}\|_{L_\infty(B')},
\]

(4.25)

\[
I_{11}^{n,j} \leq (|N^n|_{p',|E|} \frac{1}{\rho^n} + |\tilde{N}|_{p',|E|} \frac{1}{\rho^n}) \|\nabla \tau_{\infty} \|_{L^p(B')} + \|E\|_\infty \|\omega^n - \omega\|_{L^p(B')}.
\]

(4.26)

With (2.5), (4.4)–(4.7) and $1 \leq \lambda_{n,j}$, we get from (4.10), (4.17)–(4.26) for all $j \in \mathbb{N}$

\[
\limsup_{n \to \infty} I^n \leq c \lambda_0^9 + c K^{20} 2^{-j(1-\theta)} + c (1 + \|E\|_\infty^\theta) \theta K^{20} 2\tau^{-\theta}.
\]

Since $\lim_{j \to \infty} \delta_j = 0$, we observe that $I^n \to 0$ ($n \to \infty$), which, owing to $\theta \in (0, 1)$,

(S.4) and (N.4), implies for a suitable subsequence that

\[
(S^n - \tilde{S}) : (D(v^n - v) + R(v^n - v, \omega^n - \omega)) \to 0 \quad \text{a.e. in } B \quad (n \to \infty),
\]

\[
(N^n - \tilde{N}) : (\nabla \omega^n - \nabla \omega) \to 0 \quad \text{a.e. in } B \quad (n \to \infty).
\]

In view of (4.7), we also know that $\omega^n \to \omega$ a.e. in $B$ and, hence, we can conclude the assertion of Lemma 4.3 as in the proof of [7, Lem. 6].

**Corollary 4.27.** Let the assumptions of Lemma 4.3 be satisfied for all balls $B \subset \subset \Omega_0$ with $B' := 2B \subset \subset \Omega_0$. Then, we have for suitable subsequences that $\nabla v^n \to \nabla v$ a.e. in $\Omega$ ($n \to \infty$), $\nabla \omega^n \to \nabla \omega$ a.e. in $\Omega$ ($n \to \infty$) and $\omega^n \to \omega$ a.e. in $\Omega$ ($n \to \infty$).

**Proof:** Using all rational tuples contained in $\Omega_0$ as centers, we find a countable family $(B_k)_{k \in \mathbb{N}}$ of balls covering $\Omega_0$ such that $B'_k := 2B_k \subset \subset \Omega_0$ for every $k \in \mathbb{N}$. Using the usual diagonalization procedure, we construct suitable subsequences such that $\omega^n \to \omega$ a.e. in $\Omega_0$ ($n \to \infty$), $\nabla \omega^n \to \nabla \omega$ a.e. in $\Omega_0$ ($n \to \infty$) and $\nabla \omega^n \to \nabla \omega$ a.e. in $\Omega_0$ ($n \to \infty$). Since $|\Omega \setminus \Omega_0| = 0$, we proved the assertion. ■

5. Existence theorem for constant shear exponents

Now we are prepared to prove our first main result, namely the existence of solutions to the problem (1.1), (1.2) for $p > \frac{2d}{d+2}$ without imposing the additional assumption that $|E|^2$ belongs to the Muckenhoupt class $A_p$.

---

7Here, we used again that $(\nabla \omega)|_{B'_k} = \nabla(\omega|_{B'_k})$ in $B'_k$ for all $k \in \mathbb{N}$ according to Remark 4.6.
Theorem 5.1. Let \( \Omega \subseteq \mathbb{R}^d \), \( d \geq 2 \), be a bounded domain, let Assumption 4.1 and Assumption 4.2 be satisfied for some \( p > \frac{2d}{d-2} \) and let Assumption 3.2 be satisfied. Then, for every \( f \in (W_0^{1,p}(\Omega))^* \) and \( \ell \in (H_0^{1,p}(\Omega; |E|^2))^* \), there exist \( v \in V_\rho(\Omega) \) and \( \omega \in H_0^{1,p}(\Omega; |E|^2) \) such that for every \( \varphi \in C_0^1(\Omega) \) with \( \nabla \varphi = 0 \) and for every \( \psi \in C_0^1(\Omega) \) with \( \nabla \psi \in L^p(\Omega; |E|^{-\frac{2}{p}}) \) for some \( q \in [1, p^*) \), it holds
\[
\begin{align*}
\langle S(Dv, R(v, \omega), E) - v \otimes v, D\varphi + R(\varphi, \psi) \rangle \\
+ \langle N(\nabla \omega, E) - \omega \otimes v, \nabla \psi \rangle = \langle f, \varphi \rangle + \langle \ell, \psi \rangle. 
\end{align*}
\]
Moreover, there exists a constant \( c > 0 \) such that
\[
\|v\|_{1,p} + \|\omega\|_{1,p,|E|^2} \leq c(1 + + \|f\|_{L^p(\Omega)} + \|\ell\|_{H_0^{1,p}(\Omega; |E|^2))}).
\]

Remark 5.3. (i) The lower bound \( p > \frac{2d}{d+2} \) in [16, Thm. 6.44] is improved in Theorem 5.1 to \( p > \frac{2d}{d-2} \).

(ii) In contrast to [16, Thm. 5.49], [16, Thm. 5.56] and [16, Thm. 5.59], we do not require in Theorem 5.1 that \( |E|^2 \in \mathcal{A}_p \). Even though we know that the real analytic function \( |E|^2 \) belongs to the Muckenhoupt class \( \mathcal{A}_\infty \) (cf. Section 3), this does not imply that \( |E|^2 \) belongs to the Muckenhoupt class \( \mathcal{A}_p \).

(iii) The results [16, Thm. 5.56], [16, Thm. 5.59] and [16, Thm. 6.44] consider test functions \( \psi \in |E|^2 \times C_0^1(\Omega) \) for some \( \beta \geq 2 \) instead of \( \psi \in C_0^1(\Omega) \) with \( \nabla \psi \in L^\frac{2q}{q-2}(\Omega; |E|^{-\frac{2}{q}}) \) for some \( q \in [1, p^*) \). However, any \( \psi \in C_0^1(\Omega)^d \), satisfies both \( \psi \in |E|^2 \times C_0^1(\Omega) \) and \( \nabla \psi \in L^\frac{2q}{q-2}(\Omega; |E|^{-\frac{2}{q}}) \) for all \( q \in [1, p^*) \).

Hence, if we assume that there exist sufficiently smooth solutions \( \psi : \Omega \to \mathbb{R}^d \) and \( \omega : \Omega \to \mathbb{R}^d \) of (5.2) or [16, (6.46)], then testing with \( \psi \in C_0^1(\Omega_0) \), integration by parts, the fundamental theorem of calculus of variations and using that \( \Omega \setminus \Omega_0 = 0 \), we readily deduce that
\[
- \text{div } S + \text{div}(v \otimes v) + \nabla \pi = f \quad \text{in } \Omega,
\]
\[
- \text{div } N + \text{div}(\omega \otimes v) = \ell - \varepsilon : S \quad \text{in } \Omega,
\]
i.e., the weak formulations in [16, Thm. 5.56, Thm. 5.59 & Thm. 6.44] and Theorem 5.1 yield comparable results.

Proof: 1. Non-degenerate approximation and a-priori estimates:

Analogously to [16, Thm. 6.1] or using the standard theory of pseudomonotone operators, one can show that for every \( n \in \mathbb{N} \), there exist functions \( (v^n, \omega^n) \in (V_\rho(\Omega) \cap L^r(\Omega)) \times (W_0^{1,p}(\Omega) \cap L^r(\Omega)) \) satisfying for every \( \varphi \in V_\rho(\Omega) \cap L^r(\Omega) \) and \( \psi \in W_0^{1,p}(\Omega) \cap L^r(\Omega) \)
\[
\begin{align*}
\langle S(Dv^n, R(v^n, \omega^n), E) - v^n \otimes v^n, D\varphi + R(\varphi, \psi) \rangle \\
+ \langle N_{\text{non-deg}}(\omega^n) - \omega^n \otimes v^n, \nabla \psi \rangle + \frac{1}{n}\langle |\omega^n|^{r-2}\omega^n, \psi \rangle = \langle f, \varphi \rangle + \langle \ell, \psi \rangle,
\end{align*}
\]
where \( N_{\text{non-deg}} := N(\nabla \omega, E) + \frac{1}{n}(1 + |\nabla \omega|^n)|n-2|\nabla \omega^n \) and \( r > 2p' \) is fixed. The existence of these solutions is for every \( n \in \mathbb{N} \) based on the a-priori estimate
\[
\begin{align*}
\int_{\Omega} \left( (1 + |E|^2)|Dv^n|^p + |\nabla \omega^n|^p|E|^2 + |R(v^n, \omega^n)|^p|E|^2 \right) dx \\
+ \frac{1}{n}\left( (1 + |E|^2)|v^n|^p + |\nabla \omega^n|^p + |\omega^n|^2 \right) dx \leq c,
\end{align*}
\]}

\footnote{We have chosen the exponent \( r > 2p' \) such that both convective terms \( (v \otimes v, \nabla \varphi) \) and \( (\omega \otimes v, \nabla \varphi) \) define compact operators from \( L^r(\Omega) \times L^r(\Omega) \) to \( (W_0^{1,p}(\Omega))^* \).}
which follows, using (S.3) and (N.3), in a standard way. Using Korn’s inequality in the non-weighted case, the definition of $R(\mathbf{v}, \omega)$ and Poincaré’s inequality in the non-weighted case, we deduce, as in [16, Sec. 4], from (5.5) that there exists a constant $K := \left\{ \|\mathbf{E}\|_{L^2(\Omega)}, \|f\|_{W^{1,p}(\Omega)}, \|\ell\|_{L^p(\Omega; \mathbb{R}^d)} \right\} > 0$ such that for every $n \in \mathbb{N}$

$$
\|\mathbf{v}^n\|_{L^p} + \|\omega^n\|_{L^p|E|^2} + \frac{1}{n}\|\mathbf{v}^n\|_{\tau} + \frac{1}{n}\|\omega^n\|_{L^p} \leq K.
$$

Apart from that, using (S.2), (N.2), (5.6), $E \in L^\infty(\Omega)$ (cf. Assumption 3.2) and the notation introduced in (4.8), we obtain for every $n \in \mathbb{N}$ that

$$
\|\mathbf{v}^n\|_{L^p|E|^2|} + \|\omega^n\|_{L^p|E|^2|} \leq K.
$$

2. Extraction of (weakly) convergent subsequences:

The estimates (5.6), (5.7) and Rellich’s compactness theorem yield no relabeled subsequences as well as functions $v \in V_p(\Omega)$, $\omega \in H^1_p(\Omega; |E|^2)$, $\hat{S} \in L^p(\Omega)$ and $\hat{N} \in L^p(\Omega; |E|^2)$ such that

$$
v^n \rightharpoonup v \quad \text{in} \quad V_p(\Omega),
$$

$$v^n \to v \quad \text{in} \quad L^p(\Omega) \text{ and a.e. in } \Omega \quad \text{as } n \to \infty,
$$

$$\omega^n \to \omega \quad \text{in} \quad H^1_p(\Omega; |E|^2) \quad \text{as } n \to \infty,
$$

$$S^n \rightharpoonup \hat{S} \quad \text{in} \quad L^p(\Omega) \quad \text{as } n \to \infty,
$$

$$(S^n)^{\text{skew}} \rightharpoonup \hat{S}^{\text{skew}} \quad \text{in} \quad L^p(\Omega; |E|^2) \quad \text{as } n \to \infty,
$$

$$N^n \rightharpoonup \hat{N} \quad \text{in} \quad L^p(\Omega; |E|^2) \quad \text{as } n \to \infty,
$$

where $q \in [1, p^*)$.

3. Identification of $\hat{S}$ with $S(Dv, R(\mathbf{v}, \omega), E)$ and $\hat{N}$ with $N(\nabla \omega, E)$:

Recall that $\Omega_0 := \{ x \in \Omega | |E(x)| > 0 \}$. Next, let $B \subset \subset \Omega_0$ be a ball such that $B' := 2B \subset \subset \Omega_0$. Then, due to Lemma 3.5, we have $W^{1,p}(B') = H^1_p(B', |E|^2)$ with norm equivalence (depending on $B'$ and $E$). Therefore, from (5.8) and Rellich’s compactness theorem, we deduce that

$$
\omega^n \rightharpoonup \omega \quad \text{in} \quad W^{1,p}(B') \quad \text{as } n \to \infty,
$$

$$\omega^n \to \omega \quad \text{in} \quad L^p(B') \text{ and a.e. in } B' \quad \text{as } n \to \infty,
$$

where $q \in [1, p^*)$. In particular, this implies that $\omega \in W^{1,p}(B') \cap L^p(B')$. Next, let $\tau \in C_0^\infty(B')$ satisfy $\chi_B \leq \tau \leq \chi_{B'}$. According to (5.8) and (5.10), it follows that

$$
\mathbf{u}^n := (v^n - v) \tau \rightharpoonup 0 \quad \text{in} \quad W^{1,p}(B') \quad \text{as } n \to \infty,
$$

$$\psi^n := (\omega^n - \omega) \tau \rightharpoonup 0 \quad \text{in} \quad W^{1,p}(B') \quad \text{as } n \to \infty.
$$

Denote for $n \in \mathbb{N}$, the Lipschitz truncation of $\mathbf{u}^n \in W^{1,p}(B')$ and $\psi^n \in W^{1,p}(B')$ according to Theorem 2.4 with respect to the ball $B'$ by $(\mathbf{u}^{n,j})_{j \in \mathbb{N}} \subseteq W^{1,\infty}(B')$ and $(\psi^{n,j})_{j \in \mathbb{N}} \subseteq W^{1,\infty}(B')$, resp. In particular, on the basis of (5.11), Theorem 2.4 implies that the Lipschitz truncations satisfy for every $j \in \mathbb{N}$ and $s \in [1, \infty)$

$$
\mathbf{u}^{n,j} \to 0 \quad \text{in} \quad W^{1,s}(B') \quad \text{as } n \to \infty,
$$

$$\psi^{n,j} \to 0 \quad \text{in} \quad L^s(B') \quad \text{as } n \to \infty,
$$

$$\psi^{n,j} \to 0 \quad \text{in} \quad W^{1,s}(B') \quad \text{as } n \to \infty,
$$

$$\psi^{n,j} \to 0 \quad \text{in} \quad H^{1,s}(B'; |E|^2) \quad \text{as } n \to \infty,
$$

$$\psi^{n,j} \to 0 \quad \text{in} \quad L^s(B') \quad \text{as } n \to \infty,
$$

$$\psi^{n,j} \to 0 \quad \text{in} \quad H^{1,s}(B'; |E|^2) \quad \text{as } n \to \infty,
where we used in the last line that \( E \in L^\infty(\Omega) \) holds. Note that \( \psi^{n,j} \in W_0^{1,\infty}(B') \), \( n,j \in \mathbb{N} \), are suitable test-functions in (5.4). However, \( u^{n,j} \in W_0^{1,\infty}(B') \), \( n,j \in \mathbb{N} \), are not admissible in (5.4) because they are not divergence-free. To correct this, we define \( w^{n,j} := B_{B'}(\text{div } u^{n,j}), n,j \in \mathbb{N} \), where \( B_{B'} : L_0^2(B') \to W_0^{1,\infty}(B') \) denotes the Bogovskii operator with respect to \( B' \), ensured by Theorem 2.6. Since \( B_{B'} \) is weakly continuous, (5.12) and Rellich’s compactness theorem imply for every \( j \in \mathbb{N} \) and \( s \in (1,\infty) \) that

\[
\begin{align*}
  w^{n,j} &\to 0 \text{ in } W_0^{1,s}(B') \quad (n \to \infty), \\
  w^{n,j} &\to 0 \text{ in } L^s(B') \quad (n \to \infty).
\end{align*}
\]

Moreover, owing to the boundedness of \( B_{B'} \), one has for any \( n,j \in \mathbb{N} \) and \( s \in (1,\infty) \) that

\[
\|w^{n,j}\|_{W_0^{1,s}(B')} \leq c \|\text{div } u^{n,j}\|_{L^s(B')}.
\]

On the basis of \( \nabla u^n = \nabla \omega^{n,j} \) on the set \( \{u^n = u^{n,j}\} \) (cf. [29, Cor. 1.43]) and \( \text{div } u^n = \nabla \tau \cdot (v^n - v) \) for every \( n,j \in \mathbb{N} \), we further get for every \( n,j \in \mathbb{N} \) that

\[
\text{div } u^{n,j} = \chi_{\{u^n \neq u^{n,j}\}} \text{div } u^{n,j} + \chi_{\{u^n = u^{n,j}\}} \nabla \tau \cdot (v^n - v) \quad \text{a.e. in } B'.
\]

Then, (5.14) with \( s = p \) and (5.15) together imply for every \( n,j \in \mathbb{N} \) that

\[
\|w^{n,j}\|_{W_0^{1,p}(B')} \leq c \|\nabla u^{n,j}\|_{L^p(B')} + c (\|\nabla \tau\|_{\infty}) \|v^n - v\|_{L^p(B')},
\]

which in conjunction with (2.5) and (5.8) yields for every \( j \in \mathbb{N} \) that

\[
\limsup_{n \to \infty} \|w^{n,j}\|_{W_0^{1,p}(B')} \leq c 2^{\frac{d}{p}}.
\]
From (5.6), (5.12) and (5.13), we obtain for every \(j \in \mathbb{N}\) that

\[
\lim_{n \to \infty} J_{n}^{n,j} + J_{s}^{n,j} + J_{t}^{n,j} = 0. \tag{5.19}
\]

Using the notation (4.8), the estimates (5.7) and (5.16), we get for every \(j \in \mathbb{N}\) that

\[
\limsup_{n \to \infty} J_{s}^{n,j} \leq \limsup_{n \to \infty} \|S^{n} \| \|\nabla w^{n,j} \|_{L^{p}(B')} \leq c K \frac{1}{2^{j}} =: \delta_{j}. \tag{5.20}
\]

From (5.8) and (5.10), it further follows that

\[
\begin{align*}
\mathbf{v}^{n} \otimes \mathbf{v}^{n} & \to \mathbf{v} \otimes \mathbf{v} \quad \text{in } L^{s'}(\Omega) \quad (n \to \infty), \\
\omega^{n} \otimes \omega^{n} & \to \omega \otimes \omega \quad \text{in } L^{s'}(B') \quad (n \to \infty), \quad s' \in \left[1, \frac{p}{2}\right].
\end{align*} \tag{5.21}
\]

Thus, combining (5.12), (5.13) and (5.21), we find that for every \(j \in \mathbb{N}\) that

\[
\lim_{n \to \infty} J_{t}^{n,j} = 0. \tag{5.22}
\]

From (5.17)–(5.22), it follows (4.5). Thus, Corollary 4.27 yields subsequences with

\[
\begin{align*}
\nabla \mathbf{v}^{n} & \to \nabla \mathbf{v} \quad \text{a.e. in } \Omega, \\
\nabla \omega^{n} & \to \nabla \omega \quad \text{a.e. in } \Omega, \\
\omega^{n} & \to \omega \quad \text{a.e. in } \Omega. \tag{5.23}
\end{align*}
\]

Since \(S \in C^{0}(\mathbb{R}^{d \times d} \times \mathbb{R}_{\text{skew}}^{d \times d} \times \mathbb{R}^{d} ; \mathbb{R}^{d \times d})\) (cf. (S.1)) and \(N \in C^{0}(\mathbb{R}^{d \times d} \times \mathbb{R}^{d} ; \mathbb{R}^{d \times d})\) (cf. (N.1)), we deduce from (5.23) that

\[
\begin{align*}
S^{n} & \to S(\nabla \mathbf{v}, \mathbf{R}(\mathbf{v}, \omega), E) \quad \text{a.e. in } \Omega \quad (n \to \infty), \\
N^{n} & \to N(\nabla \omega, E) \quad \text{a.e. in } \Omega \quad (n \to \infty). \tag{5.24}
\end{align*}
\]

To identify \(\widehat{S}\), we now argue as in the proof of [16, Thm. 4.6 (cf. (4.21)–(4.23))], while Theorem 2.3 (with \(G = \Omega \) and \(\sigma = |E|^{2}\), (5.9), (5.24) and the absolute continuity of Lebesgue measure with respect to \(\nu_{|E|^{2}}\) is used to identify \(\widehat{N}\). Thus, we just proved

\[
\widehat{S} = S(\nabla \mathbf{v}, \mathbf{R}(\mathbf{v}, \omega), E) \quad \text{and} \quad \widehat{N} = N(\nabla \omega, E). \tag{5.25}
\]

4. Limiting process \(n \to \infty\):

Now we have at our disposal everything to identify the limits of all but one term in (5.4). Using (5.6), (5.8), (5.9), (5.21), (5.25) as well as \(p > \frac{2d}{d-2}\), we obtain from (5.4) that for every \(\varphi \in C^{0}_{0}(\Omega)\) with div \(\varphi = 0\) and for every \(\psi \in C^{1}_{0}(\Omega)\), it holds

\[
\begin{align*}
\langle S(\nabla \mathbf{v}, \mathbf{R}(\mathbf{v}, \omega), E) - \mathbf{v} \otimes \mathbf{v}, D\varphi \rangle + \langle N(\nabla \omega, E), \nabla \psi \rangle & - \lim_{n \to \infty} \langle \omega^{n} \otimes \mathbf{v}^{n}, \nabla \psi \rangle = \langle f, \varphi \rangle + \langle \ell, \psi \rangle. \tag{5.26}
\end{align*}
\]

Finally, we have to identify the remaining limit in (5.26). To this end, we fix an arbitrary \(\psi \in C^{1}_{0}(\Omega)\) with \(\nabla \psi \in L^{\frac{2d}{d-2}}(\Omega; |E|^{\frac{2}{d-2}})\) and choose \(\Omega'\) with Lipschitz boundary such that int(supp(\(\psi\))) \subset \Omega' \subset \subset \Omega\) holds. Due to Theorem 3.3 and (5.8), for all \(q \in [1, p')\), it holds

\[
\omega^{n} \to \omega \quad \text{in } L^{q}(\Omega'; |E|^{aq}) \quad (n \to \infty).
\]
for every $\alpha \geq 1 + \frac{2}{p}$. On the other hand, due to $\nabla \psi \in L^{\frac{n}{n-2}}(\Omega; |E|^{-\frac{2}{n-2}})$ and (5.8), using Hölder’s inequality, for any $q \in [1, p^*)$, we also see that

$$\nabla \psi \nu^n \to \nabla \psi \nu \quad \text{in} \quad L^{q'}(\Omega'; |E|^{-\frac{2}{n-2}}) \quad (n \to \infty).$$

Since $(L^q(\Omega'; |E|^{\alpha q}))^* = L^{q'}(\Omega'; |E|^{-\frac{2}{n-2}})$, we infer that

$$\lim_{n \to \infty} \langle \omega^\alpha \otimes \nu^n, \nabla \psi \rangle = \langle \omega \otimes \nu, \nabla \psi \rangle,$$

which, looking back to (5.26), concludes the proof of Theorem 5.1.

6. Variable shear exponent

In this section, we extend the existence result in Theorem 5.1 to the case of variable exponents. Before we do so, we first give a brief introduction into weighted variable exponent Lebesgue and Sobolev spaces. Then, we explain the changes in the arguments in the previous sections due to the variable exponent setting.

6.1. Weighted variable exponent Lebesgue and Sobolev spaces

Let $\Omega \subseteq \mathbb{R}^d$, $d \in \mathbb{N}$, be an open set and $p : \Omega \to [1, \infty)$ be a measurable function, called variable exponent. By $\mathcal{P}(\Omega)$, we denote the set of all variable exponent. For $p \in \mathcal{P}(\Omega)$, we denote by $p^+ := \text{ess sup}_{x \in \Omega} p(x)$ and $p^- := \text{ess inf}_{x \in \Omega} p(x)$ its constant limit exponents. By $\mathcal{P}^\infty(\Omega) := \{ p \in \mathcal{P}(\Omega) \mid p^+ < \infty \}$, we denote the set of all bounded variable exponents. For $p \in \mathcal{P}(\Omega)$, we use the, by now standard, variable exponent Lebesgue spaces $L^{p(\cdot)}(\Omega)$ equipped with the Luxembourg norm $\| \cdot \|_{p(\cdot)}$ and Sobolev spaces $W^{1,p(\cdot)}(\Omega)$ equipped with the Luxembourg norm $\| \cdot \|_{1,p(\cdot)}$. These spaces are separable Banach spaces. The space $W^{1,p(\cdot)}_0(\Omega)$ is defined as the completion of $C_0^\infty(\Omega)$ with respect to the gradient norm $\| \nabla \cdot \|_{p(\cdot)}$. By $L^{p(\cdot)}_0(\Omega)$, we denote the subspace of $L^{p(\cdot)}(\Omega)$ consisting of all functions with vanishing mean value. If $p \in \mathcal{P}^\infty(\Omega)$, in addition, satisfies $p^+ > 1$, then the spaces $L^{p(\cdot)}(\Omega)$, $L^{p(\cdot)}_0(\Omega)$, $W^{1,p(\cdot)}(\Omega)$, $W^{1,p(\cdot)}_0(\Omega)$ and $V_{p(\cdot)}(\Omega)$ are reflexive. For a more in-depth analysis of these spaces, we refer to [25], [18], [9] and [6].

For a variable exponent $p \in \mathcal{P}^\infty(\Omega)$ and a weight $\sigma \in L^1_{\text{loc}}(\mathbb{R}^d)$, the weighted variable exponent Lebesgue space $L^{p(\cdot)}(\Omega; \sigma)$ consists of all measurable functions $u : \Omega \to \mathbb{R}$, i.e., $u \in \mathcal{M}(\Omega)$, for which the modular

$$\rho_{p(\cdot), \sigma}(u) := \int_{\Omega} |u(x)|^{p(x)} \, d\nu_{\sigma}(x) := \int_{\Omega} |u(x)|^{p(x)} \sigma(x) \, dx$$

is finite, i.e., we have that $L^{p(\cdot)}(\Omega; \sigma) := \{ u \in \mathcal{M}(\Omega) \mid \sigma^{1/p(\cdot)} u \in L^{p(\cdot)}(\Omega) \}$. Then, we equip $L^{p(\cdot)}(\Omega; \sigma)$ with the Luxembourg norm

$$\| u \|_{p(\cdot), \sigma} := \inf \{ \lambda > 0 \mid \rho_{p(\cdot), \sigma}(u/\lambda) \leq 1 \},$$

which turns $L^{p(\cdot)}(\Omega; \sigma)$ into a separable Banach space. If $p \in \mathcal{P}^\infty(\Omega)$, in addition, satisfies $p^- > 1$, then $L^{p(\cdot)}_0(\Omega; \sigma)$ is reflexive. The dual space $L^{p(\cdot)}(\Omega; \sigma)^*$ can be identified with respect to $(\cdot, \cdot)$ with $L^{p'/(\cdot)}(\Omega; \sigma')$, where $\sigma' := \sigma^{p'/p^*}$. These properties, as many other basic properties of weighted variable Lebesgue spaces,
can be proved in the same way as for variable Lebesgue spaces. This observation works for all results for which no particular property of the Lebesgue measure is used that is not shared by a Radon measure \( \nu_\sigma \) (cf. [34], [9]).

The identity \( \rho_{p(\cdot),\sigma}(u) = \rho_{p(\cdot)}(u^{1/p(\cdot)}) \) implies that

\[
\|u\|_{p(\cdot),\sigma} = \|u^{1/p(\cdot)}\|_{p(\cdot)}
\]

for all \( u \in L^{p(\cdot)}(\Omega; \sigma) \). This and Hölder’s inequality in variable Lebesgue spaces, for every \( u \in L^{p(\cdot)}(\Omega; \sigma) \) and \( v \in L^{p'(\cdot)}(\Omega; \sigma') \), where \( \sigma' = \sigma^{1/p' - 1} \), yields that

\[
\langle u, v \rangle \leq 2 \|u\|_{p(\cdot),\sigma} \|v\|_{p'(\cdot),\sigma'}.
\]

The relation between the modular and the norm is clarified by the following lemma, which is called norm-modular unit ball property.

**Lemma 6.1.** Let \( \Omega \subseteq \mathbb{R}^d \), \( d \in \mathbb{N} \), be open and let \( p \in \mathcal{P}^\infty(\Omega) \). Then, we have for any \( u \in L^{p(\cdot)}(\Omega; \sigma) \):

(i) \( \|u\|_{p(\cdot),\sigma} \leq 1 \) if and only if \( \rho_{p(\cdot),\sigma}(u) \leq 1 \).

(ii) If \( \|u\|_{p(\cdot),\sigma} \leq 1 \), then \( \rho_{p(\cdot),\sigma}(u) \leq \|u\|_{p(\cdot),\sigma} \).

(iii) If \( 1 < \|u\|_{p(\cdot),\sigma} \), then \( \|u\|_{p(\cdot),\sigma} \leq \rho_{p(\cdot),\sigma}(u) \).

(iv) \( \|u\|_{p(\cdot),\sigma} - 1 \leq \rho_{p(\cdot),\sigma}(u) \leq \|u\|_{p(\cdot),\sigma} + 1 \).

**Proof:** See [9, Lem. 3.2.4 & Lem. 3.2.5].

In order to define weighted variable exponent Sobolev spaces, in analogy with Assumption 2.1, we make the following assumption.

**Assumption 6.2.** Let \( \Omega \subseteq \mathbb{R}^d \), \( d \in \mathbb{N} \), be an open set and \( p \in \mathcal{P}^\infty(\Omega) \). The weight \( \sigma \) is admissible, i.e., if a sequence \( (\varphi_n)_{n \in \mathbb{N}} \subseteq C^\infty(\Omega) \) and \( v \in L^{p(\cdot)}(\Omega; \sigma) \) satisfy \( \int_\Omega |\varphi_n(x)|^{p(\cdot)} \sigma(x) \, dx \to 0 \) \( (n \to \infty) \) and \( \int_\Omega |\nabla \varphi_n(x) - v(x)|^{p(\cdot)} \sigma(x) \, dx \to 0 \) \( (n \to \infty) \), then it follows that \( v = 0 \) in \( L^{p(\cdot)}(\Omega; \sigma) \).

**Remark 6.3.** If \( \sigma \in C^0(\Omega) \), then the same argumentation as in Remark 2.2 (ii) shows that Assumption 6.2 is satisfied for every \( p \in \mathcal{P}^\infty(\Omega) \).

For \( \sigma \) satisfying Assumption 6.2 and \( p \in \mathcal{P}^\infty(\Omega) \), we introduce the norm

\[
\|u\|_{1,p(\cdot),\sigma} := \|u\|_{p(\cdot),\sigma} + \|\nabla u\|_{p(\cdot),\sigma},
\]

whenever the right-hand side is well-defined.

**Definition 6.4.** Let \( \Omega \subseteq \mathbb{R}^d \), \( d \in \mathbb{N} \), be open and let Assumption 6.2 be satisfied. Then, the weighted variable exponent Sobolev space \( H^{1,p(\cdot)}(\Omega; \sigma) \) is defined as the completion of \( V_{p(\cdot),\sigma} \) := \( \{ u \in C^\infty(\Omega) \mid \|u\|_{1,p(\cdot),\sigma} < \infty \} \) with respect to the \( \|\cdot\|_{1,p(\cdot),\sigma} \)-norm. In other words, \( u \in H^{1,p(\cdot)}(\Omega; \sigma) \) if and only if \( u \in L^{p(\cdot)}(\Omega; \sigma) \) and there exists a function \( v \in L^{p(\cdot)}(\Omega; \sigma) \) such that for some sequence \( (\varphi_n)_{n \in \mathbb{N}} \subseteq C^\infty(\Omega) \) holds both \( \int_\Omega |\varphi_n - u|^{p(\cdot)} \sigma \, dx \to 0 \) \( (n \to \infty) \) and \( \int_\Omega |\nabla \varphi_n - v|^{p(\cdot)} \sigma \, dx \to 0 \) \( (n \to \infty) \). Assumption 6.2 implies that \( v \) is a uniquely defined function in \( L^{p(\cdot)}(\Omega; \sigma) \) and we, thus, define \( \nabla u := v \). Note that \( W^{1,p(\cdot)}(\Omega) = H^{1,p(\cdot)}(\Omega; 1) \) if \( \sigma = 1 \) a.e. in \( \Omega \) with \( \nabla u = \nabla u \) for all \( u \in W^{1,p(\cdot)}(\Omega) \). However, in general, \( \nabla u \) and the usual weak or distributional gradient \( \nabla u \) do not coincide. Then, the space \( H^{1,p(\cdot)}_0(\Omega; \sigma) \) is defined as the closure of \( C_c^\infty(\Omega) \) with respect to the \( \|\cdot\|_{1,p(\cdot),\sigma} \)-norm. If \( \sigma \in L^\infty(\Omega) \),
We say that \( p \in P \subseteq \mathbb{R} \) and \( \nabla u = \hat{\nabla} u \) for all \( u \in W^{1,p}(\Omega) \), which is a consequence of
\[
\|v\|_{p,\sigma} = \|v\sigma^{1/p}\|_{p} \leq 2 \|\sigma\|_{\infty}^{1/p^-} \|v\|_{p}
\]
valid for every \( v \in L^{p}(\Omega) \).

Another possible approach is to define the weighted variable Sobolev space \( W^{1,p}(\Omega; \sigma) \) as the set of all functions \( u \in L^{p}(\Omega; \sigma) \) which possess a distributional gradient \( \nabla u \in L^{p}(\Omega; \sigma) \). We equip \( W^{1,p}(\Omega; \sigma) \) with the norm \( \|v\|_{1,p,\sigma} + \|\hat{\nabla} u\|_{p,\sigma} \). As constant exponents are a particular case we have that, in general, \( \|\cdot\|_{1,p,\sigma} \) is an isometry from \( L^{1}(\Omega; \sigma) \) onto its range \( \tilde{\Pi} \). However, this condition is again for our purposes too restrictive (cf. Section 3). Thus, we will not use \( W^{1,p}(\Omega; \sigma) \), but we will work with the spaces \( H^{1,p}(\Omega; \sigma) \). Since the space \( H^{1,p}(\Omega; \sigma) \) is even less studied (we are only aware of the study in [36]), we prove its basic properties.

**Theorem 6.5.** Let \( \Omega \subseteq \mathbb{R}^{d} \), \( d \in \mathbb{N} \), be an open set and let \( p \in P^{\infty}(\Omega) \) satisfy \( p^- > 1 \). Then, the space \( H^{1,p}(\Omega; \sigma) \) is a separable and reflexive Banach space.

**Proof:** The space \( H^{1,p}(\Omega; \sigma) \), by definition, is a Banach space. So, it is left to check that it is separable and reflexive. For this, we first note that
\[
\|u\|_{1,p(\cdot),\sigma} = \|u\|_{p(\cdot),\sigma} + \|\hat{\nabla} u\|_{p(\cdot),\sigma}
\]
for all \( u \in H^{1,p}(\Omega; \sigma) \). In fact, for any \( u \in H^{1,p}(\Omega; \sigma) \), by definition, there exists a sequence \( \{\varphi_{n}\}_{n \in \mathbb{N}} \subseteq V_{p(\cdot),\sigma} \) such that \( \varphi_{n} \to u \) in \( L^{p}(\Omega; \sigma) \) \((n \to \infty)\), \( \nabla \varphi_{n} \to \nabla u \) in \( L^{p}(\Omega; \sigma) \) \((n \to \infty)\) and \( \|\varphi_{n}\|_{1,p(\cdot),\sigma} = \|\varphi_{n}\|_{p(\cdot),\sigma} + \|\nabla \varphi_{n}\|_{p(\cdot),\sigma} \) for all \( n \in \mathbb{N} \). Thus, by passing \( n \to \infty \), we obtain (6.6) for all \( u \in H^{1,p}(\Omega; \sigma) \). The equality (6.6) in turn implies that \( \Pi : H^{1,p}(\Omega; \sigma) \to L^{p}(\Omega; \sigma)^{d+1} \), defined via \( \Pi u := (u,\nabla u)^{\top} \) for every \( u \in H^{1,p}(\Omega; \sigma) \), is an isometry. In particular, \( \Pi \) is an isometric isomorphism from \( H^{1,p}(\Omega; \sigma) \) onto its range \( R(\Pi) \). Thus, \( R(\Pi) \) inherits the separability and reflexivity of \( L^{p}(\Omega; \sigma)^{d+1} \) and, by virtue of the isometric isomorphism, \( H^{1,p}(\Omega; \sigma) \) as well.

**6.2. log–Hölder continuity and related results**

We say that a bounded exponent \( p \in P^{\infty}(G) \) is locally log–Hölder continuous, if there is a constant \( c_{1} > 0 \) such that for all \( x, y \in G \)
\[
|p(x) - p(y)| \leq \frac{c_{1}}{\log(e + 1/|x - y|)}.
\]
We say that \( p \in P^{\infty}(G) \) satisfies the log–Hölder decay condition, if there exist constants \( c_{2} > 0 \) and \( p_{\infty} \in \mathbb{R} \) such that for all \( x \in G \)
\[
|p(x) - p_{\infty}| \leq \frac{c_{2}}{\log(e + 1/|x|)}.
\]
The exponent \( p \) is called globally log–Hölder continuous on \( G \), if it is locally log–Hölder continuous and satisfies the log–Hölder decay condition. The maximum \( \epsilon_{p} := \max\{c_{1}, c_{2}\} \) is just called the log–Hölder constant of \( p \). Furthermore, we denote by \( P^{p_{\infty}}(G) \) the set of globally log–Hölder continuous functions on \( G \).

log–Hölder continuity is a special modulus of continuity for variable exponents that is sufficient for the validity of the following results.
Theorem 6.7. Let $G \subseteq \mathbb{R}^d$, $d \geq 2$, be a bounded Lipschitz domain. Then, there exists a linear operator $\mathcal{B}_G : C^{0,0}_0(G) \to C^\infty_0(G)$ which for all exponents $p \in \mathcal{P}^0(G)$ satisfying $p^- > 1$ extends uniquely to a linear, bounded operator $\mathcal{B}_G : L^{p^1}(G) \to W^{1,p^1}_0(G)$ such that $\|\mathcal{B}_G u\|_{1,p^1} \leq c\|u\|_{p^1}$ and $\text{div} \mathcal{B}_G u = u$ for every $u \in L^{p^1}_0(G)$.

Proof: See [11, Thm. 2.2], [8, Thm. 6.4], [9, Thm. 14.3.15].

Theorem 6.8. Let $G \subseteq \mathbb{R}^d$, $d \in \mathbb{N}$, be a bounded Lipschitz domain and let $p \in \mathcal{P}^0(G)$ satisfy $p^- > 1$. Then, there exists a constant $c > 0$ such that $\|u\|_{p^1} \leq c\|\nabla u\|_{p^1}$ for every $u \in W^{1,p^1}_0(G)$.

Proof: See [9, Thm. 8.2.4].

Theorem 6.9. Let $G \subseteq \mathbb{R}^d$, $d \in \mathbb{N}$, be a bounded Lipschitz domain and let $p \in \mathcal{P}^0(G)$ satisfy $p^- > 1$. Then, there exists a constant $c > 0$ such that $\|\nabla u\|_{p^1} \leq c\|\text{div} u\|_{p^1}$ for every $u \in W^{1,p^1}_0(G)$.

Proof: See [8, Thm. 5.5], [9, Thm. 14.3.21].

Theorem 6.10. Let $G \subseteq \mathbb{R}^d$, $d \in \mathbb{N}$, be a bounded Lipschitz domain, $p \in \mathcal{P}^0(G)$ with $p^- > 1$ and let $u^n \in W^{1,p^1}_0(G)$ be such that $u^n \to 0$ in $W^{1,p^1}_0(G)$ ($n \to \infty$). Then, for any $j, n, \lambda \in \mathbb{N}$, there exist $u^{n,j} \in W^{1,\infty}_0(G)$ and $\lambda_{n,j} \in [2^j, 2^{j+1}]$ such that

$$
\lim_{n \to \infty} \sup_{j \in \mathbb{N}} \|u^{n,j}\|_{\infty} = 0,
$$

$$
\|\nabla u^{n,j}\|_{\infty} \leq c \lambda_{n,j} \leq c 2^{2j+1},
$$

$$
\|\nabla u^{n,j} - \lambda_{n,j} x(u^{n,j} - u^n)\|_{p^1} \leq c \|\lambda_{n,j} x(u^{n,j} - u^n)\|_{p^1},
$$

$$
\limsup_{n \to \infty} \|\lambda_{n,j} x(u^{n,j} - u^n)\|_{p^1} \leq c 2^{-j/p^+},
$$

where $c = c(d, p, G) > 0$. Moreover, for any $j \in \mathbb{N}$, $\nabla u^{n,j} \to 0$ in $L^s(G)$ ($n \to \infty$), $s \in [1, \infty)$, and $\nabla u^{n,j} \to 0$ in $L^\infty(G)$ ($n \to \infty$).

Proof: See [10, Thm. 4.4], [9, Cor. 9.5.2].

log–Hölder continuity is also sufficient to prove the analogue of Lemma 3.5 in the variable exponent case.

Lemma 6.11. Let $\Omega \subseteq \mathbb{R}^d$, $d \in \mathbb{N}$, be open, $p \in \mathcal{P}^0(\Omega)$ and let Assumption 3.2 be satisfied. Then, for any $\Omega' \subset\subset \Omega_0$, we have that $W^{1,p(\cdot)}(\Omega') = H^{1,p(\cdot)}(\Omega'; |E|^2)$ with norm equivalence (depending on $\Omega'$ and $E$) and $\nabla u = \nabla u$ for all $u \in W^{1,p(\cdot)}(\Omega')$.

Proof: Due to $|E| > 0$ in $\overline{\Omega'}$ and $|E| \in C^0(\overline{\Omega'})$, there is a local constant $c(\Omega') > 0$ such that $c(\Omega')^{-1} \leq |E|^2 \leq c(\Omega')$ in $\overline{\Omega'}$. Thus, $L^{p(\cdot)}(\Omega') = L^{p(\cdot)}(\Omega'; |E|^2)$ with

$$
c(\Omega')^{-1/2} \|u\|_{L^{p(\cdot)}(\Omega')} \leq \|u\|_{L^{p(\cdot)}(\Omega'; |E|^2)} \leq c(\Omega')^{1/2} \|u\|_{L^{p(\cdot)}(\Omega')}
$$

for all $u \in L^{p(\cdot)}(\Omega') = L^{p(\cdot)}(\Omega'; |E|^2)$. As a result, it holds $V_{p(\cdot)}(\Omega' |E|^2) = V_{p(\cdot)}(\Omega' |E|^2)$ with

$$
c(\Omega')^{-1/2} \|u\|_{W^{1,p(\cdot)}(\Omega')} \leq \|u\|_{H^{1,p(\cdot)}(\Omega'; |E|^2)} \leq c(\Omega')^{1/2} \|u\|_{W^{1,p(\cdot)}(\Omega')}
$$

(6.12) for all $u \in V_{p(\cdot)}(\Omega' |E|^2) = V_{p(\cdot)}(\Omega' |E|^2)$. Since $W^{1,p(\cdot)}(\Omega')$, by [9, Thm. 9.1.8.], is the closure of $V_{p(\cdot)}(\Omega' |E|^2)$ and $H^{1,p(\cdot)}(\Omega'; |E|^2)$, by definition, is the closure of $V_{p(\cdot)}(\Omega' |E|^2)$, (6.12) implies that $W^{1,p(\cdot)}(\Omega') = H^{1,p(\cdot)}(\Omega'; |E|^2)$ and $\nabla u = \nabla u$ for all $u \in W^{1,p(\cdot)}(\Omega')$. ■
6.3. A weak stability lemma for variable exponents

Also the weak stability of problems of $p(\cdot)$–Laplace type is well known (cf. [10]). It also holds for our problem (1.1) if we make appropriate natural assumptions on the extra stress tensor $\Sigma$ and on the couple stress tensor $\Pi$, which are motivated by the canonical example in (1.3).

Assumption 6.13. For the extra stress tensor $\Sigma: \mathbb{R}^{d \times d}_{\text{sym}} \times \mathbb{R}^{d \times d}_{\text{skew}} \times \mathbb{R}^{d} \to \mathbb{R}^{d}$ and some $\hat{p} \in \mathcal{P}^{\log}(\mathbb{R})$ with $\hat{p}^{-1} > 1$, there exist constants $c, C > 0$ such that: 

(S.1) $\Sigma \in C^{0}(\mathbb{R}^{d \times d}_{\text{sym}} \times \mathbb{R}^{d \times d}_{\text{skew}} \times \mathbb{R}^{d}, \mathbb{R}^{d})$.
(S.2) For every $D \in \mathbb{R}^{d \times d}_{\text{sym}}, R \in \mathbb{R}^{d \times d}_{\text{skew}}$ and $E \in \mathbb{R}^{d}$, it holds
\[
|\Sigma^{\text{sym}}(D, R, E)| \leq c (1 + |E|^2) (1 + |D|^\hat{p}(|E|^2)^{-1}) ,
\]
\[
|\Sigma^{\text{skew}}(D, R, E)| \leq c |E|^2 (1 + |R|^\hat{p}(|E|^2)^{-1}) .
\]
(S.3) For every $D \in \mathbb{R}^{d \times d}_{\text{sym}}, R \in \mathbb{R}^{d \times d}_{\text{skew}}$ and $E \in \mathbb{R}^{d}$, it holds
\[
\Sigma(D, R, E) : D \geq c (1 + |E|^2) (|D|^\hat{p}(|E|^2) - C) ,
\]
\[
\Sigma(D, R, E) : R \geq c |E|^2 (|R|^\hat{p}(|E|^2) - C) .
\]
(S.4) For every $D_{1}, D_{2} \in \mathbb{R}^{d \times d}_{\text{sym}}, R_{1}, R_{2} \in \mathbb{R}^{d \times d}_{\text{skew}}$ and $E \in \mathbb{R}^{d}$ with $(D_{1}, |E| R_{1}) \neq (D_{2}, |E| R_{2})$, it holds
\[
\langle \Sigma(D_{1}, R_{1}, E) - \Sigma(D_{2}, R_{2}, E) \rangle : (D_{1} - D_{2} + R_{1} - R_{2}) > 0 .
\]

Assumption 6.14. For the couple stress tensor $\Pi: \mathbb{R}^{d \times d} \times \mathbb{R}^{d} \to \mathbb{R}^{d \times d}$ and some $\hat{p} \in \mathcal{P}^{\log}(\mathbb{R})$ with $\hat{p}^{-1} > 1$, there exist constants $c, C > 0$ such that:

(N.1) $\Pi \in C^{0}(\mathbb{R}^{d \times d} \times \mathbb{R}^{d}, \mathbb{R}^{d \times d})$.
(N.2) For every $L \in \mathbb{R}^{d \times d}$ and $E \in \mathbb{R}^{d}$, it holds
\[
|\Pi(L, E)| \leq c |E|^2 (1 + |L|^\hat{p}(|E|^2)^{-1}) .
\]
(N.3) For every $L \in \mathbb{R}^{d \times d}$ and $E \in \mathbb{R}^{d}$, it holds
\[
\Pi(L, E) : L \geq c |E|^2 (|L|^\hat{p}(|E|^2) - C) .
\]
(N.4) For every $L_{1}, L_{2} \in \mathbb{R}^{d \times d}$ and $E \in \mathbb{R}^{d}$ with $|E| > 0$ and $L_{1} \neq L_{2}$, it holds
\[
\langle \Pi(L_{1}, E) - \Pi(L_{2}, E) \rangle : (L_{1} - L_{2}) > 0 .
\]

Concerning the material function $\hat{p}$ in Assumption 6.13 and Assumption 6.14, we assume the following:

Assumption 6.15. Let Assumption 3.2 be satisfied and let $\hat{p} \in \mathcal{P}^{\log}(\mathbb{R})$. Then, the exponent $p: \Omega \to [1, \infty)$, defined via
\[
p(x) := \hat{p}(|E(x)|^2)
\]
for every $x \in \Omega$, satisfies $p \in \mathcal{P}^{\log}(\Omega)$.

Remark 6.16. Assumption 6.15 can be verified under certain conditions on the boundary data $E_{0}$. In fact, the regularity theory of Maxwell’s equations (cf. [35], [34]) implies $E \in C^{0,\alpha}(\overline{\Omega}), \alpha \in (0, 1)$, if $E_{0}$ is sufficiently smooth. This yields that $p = \hat{p} \circ |E|^2$ satisfies Assumption 6.15, as easy calculations show.
Under these assumptions, we have the following weak stability for problem (1.1).

**Lemma 6.17.** Let \( \Omega \subseteq \mathbb{R}^d, d \geq 2, \) be a bounded domain and let Assumption 4.1, Assumption 6.12 and Assumption 6.15 be satisfied. Moreover, let \((v^n)_{n \in \mathbb{N}} \subseteq V_{p(\cdot)}(\Omega)\) and \((\omega^n)_{n \in \mathbb{N}} \subseteq H^{1,p(\cdot)}_0(\Omega; |E|^2)\) be such that

\[
\begin{align*}
v^n & \rightharpoonup v & \text{ in } V_{p(\cdot)}(\Omega) \quad (n \to \infty), \\
\omega^n & \rightharpoonup \omega & \text{ in } H^{1,p(\cdot)}_0(\Omega; |E|^2) \quad (n \to \infty).
\end{align*}
\]

For every ball \( B \subset \subset \Omega_0 \) such that \( B' := 2B \subset \subset \Omega_0 \) and \( \tau \in C_0^\infty(B') \) satisfying \( \tau \leq \chi_{B'} \), we set \( u^n := (v^n - v)\tau, \psi^n := (\omega^n - \omega)\tau \in W^{1,p(\cdot)}_0(B'), n \in \mathbb{N}. \)

Let \( u^{n,j} \in W^{1,\infty}_0(B'), n, j \in \mathbb{N}, \) and \( \psi^{n,j} \in W^{1,\infty}_0(B'), n, j \in \mathbb{N}, \) resp., denote the Lipschitz truncations constructed according to Theorem 6.10. Furthermore, assume that for every \( j \in \mathbb{N}, \) we have that

\[
\limsup_{n \to \infty} \left( |S(Dv^n, R(v^n, \omega^n), E) - S(Dv, R(v, \omega), E)| + \langle N(\nabla \omega^n, E) - N(\nabla \omega, E), \nabla \psi^{n,j} \rangle \right) \leq \delta_j,
\]

where \( \delta_j \to 0 (j \to 0). \) Then, one has \( \nabla v^n \rightharpoonup \nabla v \) a.e. in \( B (n \to \infty), \nabla \omega^n \rightharpoonup \nabla \omega \) a.e. in \( B (n \to \infty) \) and \( \omega^n \rightharpoonup \omega \) a.e. in \( B (n \to \infty) \) for suitable subsequences.

**Proof:** We follow, word by word, the procedure as in the proof of Lemma 4.3. In doing so, we employ Lemma 6.11 instead of Lemma 3.5, which results in \( H^{1,p(\cdot)}_0(B'; |E|^2) = W^{1,p(\cdot)}_0(B'). \) The trivial embedding \( W^{1,p(\cdot)}_0(B') \hookrightarrow W^{1,\infty}_0(B') \) together with the classical Rellich’s compactness theorem yields that we have to replace \( q \in [1, p^+) \) by \( q \in [1, (p^-)^+] \). Moreover, we have to replace the constant exponent \( p \in (1, \infty) \) by the variable exponent \( p \in \mathcal{P}(\log(\Omega)) \), wherever it occurs. This applies, in particular, to all Lebesgue, weighted Lebesgue, Sobolev and weighted Sobolev norms containing \( p \) or \( p' \). Whenever we use Hölder’s inequality, we get an additional multiplicative factor \( 2. \) Finally, we replace \( |E|^2/p \) by \( |E|^{2/p} \), \( |\Omega|^{1/p} \) by \( \max\{|\Omega|^{1/p}, |\Omega|^{1/p^+}\} \) (cf. [9, Lem. 3.2.12]) and \( 2^{-j/p} \) by \( 2^{-j/p^+}. \)

**Corollary 6.18.** Let the assumptions of Lemma 6.17 be satisfied for all balls \( B \subset \subset \Omega_0 \) with \( B' := 2B \subset \subset \Omega_0. \) Then, one has that \( \nabla v^n \rightharpoonup \nabla v \) a.e. in \( \Omega (n \to \infty), \nabla \omega^n \rightharpoonup \nabla \omega \) a.e. in \( \Omega (n \to \infty) \) and \( \omega^n \rightharpoonup \omega \) a.e. in \( \Omega (n \to \infty) \) for suitable subsequences.

**Proof:** The proof coincides with that of Corollary 4.27.

### 6.4. Existence theorem for variable exponents

Now we have all tools at our disposal to formulate and prove our existence result in the case of variable exponents.

**Theorem 6.19.** Let \( \Omega \subseteq \mathbb{R}^d, d \geq 2, \) be a bounded domain, let Assumption 6.13, Assumption 6.14 and Assumption 6.15 be satisfied, and let \( p^- > \frac{2d}{d+2} \). Then, for every \( f \in (W^{1,p(\cdot)}_0(\Omega))^* \) and \( \ell \in (H^{1,p(\cdot)}_0(\Omega; |E|^2))^* \), there exist functions \( v \in V_{p(\cdot)}(\Omega) \) and \( \omega \in H^{1,p(\cdot)}_0(\Omega; |E|^2) \) such that for every \( \varphi \in C_0^\infty(\Omega) \) with \( \text{div} \varphi = 0 \) and \( \psi \in C_0^\infty(\Omega) \) with \( \nabla \psi \in L^{\frac{d}{d-2}}(\Omega; |E|^{\frac{d}{d-2}}) \) for some \( q \in [1, (p^-)^+] \), it holds

\[
\begin{align*}
\langle S(Dv, R(v, \omega), E) - v \otimes v, D\varphi + R(\varphi, \psi) \rangle \\
+ \langle N(\nabla \omega, E) - \omega \otimes v, \nabla \psi \rangle = \langle f, \varphi \rangle + \langle \ell, \psi \rangle.
\end{align*}
\]

Moreover, we have the following a-priori estimate

\[
\|v\|_{V_{p(\cdot)}} + \|\omega\|_{H^{1,p(\cdot)}_0(\Omega; |E|^2)} \leq c \left( \|f\|_{(W^{1,p(\cdot)}_0(\Omega))^*}, \|\ell\|_{(H^{1,p(\cdot)}_0(\Omega; |E|^2))^*} \right).
\]
Proof: We follow, word by word, the procedure as in the proof of Theorem 5.1. In doing so, we again have to replace the constant exponent $p \in (1, \infty)$ by the variable exponent $p \in P^{\log}(\Omega)$, classical Lebesgue, weighted Lebesgue, Sobolev and weighted Sobolev norms containing $p$ or $p'$ by their variable exponent counterparts. Moreover, we replace $r > 2p'$ by $r > 2(p^*)'$ in the definition of the approximate problem. To show that (5.5) implies (5.6) in the variable exponent case, the constant exponent Korn’s and Poincaré’s inequalities is replaced by their variable exponent counterparts in Theorem 6.9 and Theorem 6.8, and [9, Lem. 3.2.5.] is used to pass from the modular estimate to the norm estimate. Concerning the usage of Rellich’s compactness theorem, we proceed as in the proof of Lemma 6.17 and, thus, replace $q \in [1, p^*)$ by $q \in [1, (p^*)^*)$. Moreover, we replace Lemma 3.5 by Lemma 6.11, Theorem 2.6 by Theorem 6.7, Theorem 2.4 by Theorem 6.10 and Corollary 4.27 by Corollary 6.18.

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