Testing Identity of Multidimensional Histograms

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Abstract

We investigate the problem of identity testing for multidimensional histogram distributions. A distribution $p : D \to \mathbb{R}_+$, where $D \subseteq \mathbb{R}^d$, is called a $k$-histogram if there exists a partition of the domain into $k$ axis-aligned rectangles such that $p$ is constant within each such rectangle. Histograms are one of the most fundamental non-parametric families of distributions and have been extensively studied in computer science and statistics. We give the first identity tester for this problem with sub-learning sample complexity in any fixed dimension and a nearly-matching sample complexity lower bound.

More specifically, let $q$ be an unknown $d$-dimensional $k$-histogram and $p$ be an explicitly given $k$-histogram. We want to correctly distinguish, with probability at least $2/3$, between the case that $p = q$ versus $\|p - q\|_1 \geq \epsilon$. We design a computationally efficient algorithm for this hypothesis testing problem with sample complexity $O((\sqrt{k}/\epsilon^2) \log^{O(d)}(k/\epsilon))$. Our algorithm is robust to model misspecification, i.e., succeeds even if $q$ is only promised to be close to a $k$-histogram. Moreover, for $k = 2^\Omega(d)$, we show a nearly-matching sample complexity lower bound of $\Omega((\sqrt{k}/\epsilon^2)(\log(k/\epsilon)/d)^{\Omega(d)})$ when $d \geq 2$.

Prior to our work, the sample complexity of the $d = 1$ case was well-understood, but no algorithm with sub-learning sample complexity was known, even for $d = 2$. Our new upper and lower bounds have interesting conceptual implications regarding the relation between learning and testing in this setting.

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1 Introduction

1.1 Background

The task of verifying the identity of a statistical model — known as identity testing or goodness of fit — is one of the most fundamental questions in statistical hypothesis testing [Pea00, NP33]. In the past two decades, this question has been extensively studied by the TCS and information-theory communities in the framework of property testing [RS96, GGR98]: Given sample access to an unknown distribution $q$ over a finite domain $[n] := \{1, \ldots, n\}$, an explicit distribution $p$ over $[n]$, and a parameter $\epsilon > 0$, we want to distinguish between the cases that $q$ and $p$ are identical versus $\epsilon$-far from each other in $L_1$ norm (statistical distance). Initial work on this problem focused on characterizing the sample size needed to test the identity of an arbitrary distribution of a given support size $n$. This regime is well-understood: there exists an efficient estimator with sample complexity $O(\sqrt{n}/\epsilon^2)$ [VV14, DKN15b, ADK15] that is worst-case optimal up to constant factors.

The aforementioned sample complexity characterizes worst-case instances and drastically better upper bounds may be possible if we have some a priori qualitative information about the unknown distribution. For example, if $q$ is an arbitrary continuous distribution, no identity tester with finite sample complexity exists. On the other hand, if $q$ is known to have some nice structure, the domain size may not be the right complexity measure for the identity testing problem and one might hope that strong positive results can be obtained even for the continuous setting. This discussion motivates the following natural question: To what extent can we exploit the underlying structure to perform the desired statistical estimation task more efficiently?

A natural formalization of the aforementioned question involves assuming that the unknown distribution belongs to (or is close to) a given family of distributions. Let $\mathcal{D}$ be a family of distributions over $\mathbb{R}^d$. The problem of identity testing for $\mathcal{D}$ is the following: Given sample access to an unknown distribution $q \in \mathcal{D}$, and an explicit distribution $p \in \mathcal{D}$, we want to distinguish between the case that $q = p$ versus $\|q - p\|_1 \geq \epsilon$. (Throughout this paper, $\|p - q\|_1$ denotes the $L_1$-distance between the distributions $p, q$.) We note that the sample complexity of this testing problem depends on the complexity of the underlying class $\mathcal{D}$, and it is of fundamental interest to obtain efficient algorithms that are sample optimal for $\mathcal{D}$. A recent body of work in distribution testing has focused on leveraging such a priori structure to obtain significantly improved sample complexities [BKR04, DDS13, DKN15b, ADK15, CDKS17a, DP17, DDK18, DKN17].

One approach to solve the identity testing problem for a family $\mathcal{D}$ is to learn $q$ up to $L_1$-distance $\epsilon/3$ and then check whether the hypothesis is $\epsilon/3$-close to $p$. Thus, the sample complexity of identity testing for $\mathcal{D}$ is bounded from above by the sample complexity of learning (an arbitrary distribution in) $\mathcal{D}$. It is natural to ask whether a better sample size bound could be achieved for the identity testing problem, since this task is, in some sense, less demanding than the task of learning. In this paper, we provide an affirmative answer to this question for the family of multidimensional histogram distributions.

1.2 Our Results: Identity Testing for Multidimensional Histograms

In this work, we investigate the problem of testing the identity of multidimensional histogram distributions. A $d$-dimensional probability distribution with density $p : D \to \mathbb{R}$, where $D \subset \mathbb{R}^d$ is either $[m]^d$ or $[0,1]^d$, is called a $k$-histogram if there exists a partition of the domain into $k$ axis-aligned rectangles $R_1, \ldots, R_k$ so that $p$ is constant on $R_i$, for all $i = 1, \ldots, k$. We let $\mathcal{H}_k^d$ denote the set of $k$-histograms over $[0,1]^d$. Histograms constitute one of the most basic nonparametric distribution families and have been extensively studied in statistics and computer science.
Specifically, the problem of learning histogram distributions from samples has been extensively studied in the statistics community and many methods have been proposed \cite{Sco79, FDS81, Sco92, LN96, DL04, WN07, Kle09} that unfortunately have a strongly exponential dependence on the dimension. In the database community, histograms \cite{JKM+98, CMN98, TGIK02, GGI+02, GKS06, ILR12, ADH15} constitute the most common tool for the succinct approximation of large datasets. Succinct data representations by multivariate histograms are well-motivated in several data analysis applications in databases, where randomness is used to sub-sample a large dataset \cite{CGHJ12}.

In recent years, histogram distributions have attracted renewed interest from the theoretical computer science community in the context of learning \cite{DDS12, CDSS13, CDSS14a, CDSS14b, DHS15, ADLS16, ADLS17, DKS16a} and testing \cite{ILR12, DDS+13, DKN15b, DKN15a, Can16, CDGR16, DKN17}. The algorithmic difficulty in learning and testing such distributions lies in the fact that the location and size of the rectangle partition is unknown. The majority of the literature has focused on the univariate setting which is by now well-understood. Specifically, it is known that the sample complexity of learning $H_k^1$ is $\Theta(k/\epsilon^2)$ (and this sample bound is achievable with computationally efficient algorithms \cite{CDSS14a, CDSS14b, ADLS17}); while the sample complexity of identity testing $H_k^1$ is $\Theta(\sqrt{k}/\epsilon^2)$ \cite{DKN15a}. That is, in one dimension, the gap between learning and identity testing as a function of the complexity parameter $k$ is known to be quadratic.

A recent work \cite{DLS18} obtained a sample near-optimal and computationally efficient algorithm for learning multidimensional $k$-histograms in any fixed dimension. The sample complexity of the \cite{DLS18} algorithm is $O((k/\epsilon^2) \log^{O(d)}(k/\epsilon))$ while the optimal sample complexity of the learning problem (ignoring computational considerations) is $\tilde{\Theta}(dk/\epsilon^2)$. On the other hand, the property testing question in two or more dimensions is poorly understood. Prior to this work, no testing algorithm with sub-learning sample complexity was known, even for $d = 2$. In this paper, we obtain such an algorithm and a nearly-matching sample complexity lower bound for the problem of identity testing. Our main result is the following:

**Theorem 1.1 (Main Result).** Fix $\epsilon > 0$ and $k \in \mathbb{Z}_+$. Let $q \in H_k^d$ be an unknown $k$-histogram over $[m]^d$ or $[0,1]^d$ and $p \in H_k^d$ be explicitly given. There is a computationally efficient algorithm which draws $m = O((\sqrt{k}/\epsilon^2) \log^{O(d)}(dk/\epsilon))$ samples from $q$ and distinguishes, with probability at least $2/3$, between the case that $p = q$ versus $\|p - q\|_1 \geq \epsilon$. Moreover, any algorithm for this hypothesis testing problem requires $(\sqrt{k}/\epsilon^2)\Omega((\log(k)/d)^{d-1})$ samples for $k = 2^{\Omega(d)}$, even for uniformity testing.

In the following paragraphs, we provide a number of remarks regarding Theorem 1.1 and its implications. First, we would like to note that the focus of our work is on the case where the parameter $k$ is much larger than the dimension $d$. For example, this condition is automatically satisfied when $d$ is bounded from above by a fixed constant. This is arguably the most natural setting for several applications of multidimensional histograms. For this parameter regime, our identity tester has sub-learning sample complexity that is near-optimal, within logarithmic factors (as follows from our lower bound).

An important remark is that our identity testing algorithm is robust to model misspecification. Specifically, the algorithm is guaranteed to succeed as long as the unknown distribution $q$ is $\epsilon/10$-close, in $L_1$-norm, to being a $k$-histogram. This robustness property is important in applications and is conceptually interesting for the following reason: In high-dimensions, robust identity testing with sub-learning sample complexity is provably impossible, even for the simplest high-dimensional distributions, including spherical Gaussians \cite{DKS16a} and binary product distributions \cite{DKS18}.

A conceptual implication of Theorem 1.1 concerns the sample complexity gap between learning and identity testing for histograms. It was known prior to this work that the gap between the sample complexity of learning and identity testing for univariate $k$-histograms is quadratic as a function of $k$. Perhaps surprisingly, our results imply that this gap decreases as the dimension $d$
increases. This follows from the sample complexity of our algorithm in Theorem 1.1 and the fact that the sample complexity of learning $H^d_k$ is $\Theta(dk/\epsilon^2)$ (as follows from standard VC-dimension arguments). In particular, even for $d = 3$, the gap between the sample complexities of learning and identity testing is already sub-quadratic and continues to decrease as the dimension increases. (We note that our lower bound applies for $k > 2^d$, so there is no contradiction for large $d$.)

Finally, we note here a qualitative difference between the $d = 1$ and $d \geq 2$ cases. Recall that for $d = 1$ the sample complexity of identity testing $k$-histograms is $\Theta(\sqrt{\epsilon}/\epsilon^2)$. For $d = 2$, the sample complexity of our algorithm is $O((\sqrt{\epsilon}/\epsilon^2)\log(k/\epsilon))$. It would be tempting to conjecture that the multiplicative logarithmic factor is an artifact of our algorithm and/or its analysis. Our lower bound of $O((\sqrt{\epsilon}/\epsilon^2)\log^{1/2}(k/\epsilon))$ shows that some constant power of a logarithm is in fact inherently needed.

1.3 Related Work

The field of distribution property testing [BFR+00] has been extensively investigated in the past couple of decades, see [Rub12, Can15, Gol17] for two recent surveys and a book on the topic. A large body of the literature has focused on characterizing the sample size needed to test properties of arbitrary discrete distributions of a given support size. This regime is fairly well understood: for many properties of interest there exist sample-efficient testers [Pan08, CDVV14, YV14, DKN15b, ADK15, CDGR16, DK16, DGPP16, CDS17, Gol17, DGPP17, DDK18, DKN17]. More recently, an emerging body of work has focused on leveraging a priori structure of the underlying distributions to obtain significantly improved sample complexities [BKR04, DDS+13, DKN15b, DKN15a, CDKS17a, DP17, DDK18, DKN17].

The area of distribution inference under structural assumptions — that is, inference about a distribution under the constraint that its probability density function satisfies certain qualitative properties — is a classical topic in statistics starting with the pioneering work of Grenander [Gre56] on monotone distributions. The reader is referred to [BBBB72] for a summary of the early work and to [GJ14] for a recent book on the subject. This topic is well-motivated in its own right, and has seen a recent surge of research activity in the statistics and econometrics communities, due to the ubiquity of structured distributions in the sciences. The conventional wisdom is that, under such structural constraints, the quality of the resulting estimators may dramatically improve, both in terms of sample size and in terms of computational efficiency.

1.4 Overview of Techniques

In this section, we provide a high-level overview of our algorithmic and lower bounds techniques in tandem with a comparison to prior related work.

Overview of Identity Testing Algorithm We start by describing our uniformity tester for $d$-dimensional $k$-histograms. The first observation is that if the unknown distribution $q \in H^d_k$ and the uniform distribution $p = U$ are $\epsilon$-far in $L_1$-distance, there exists a partition of the domain into $k$ rectangles $R_1, \ldots, R_k$ so that the difference between $q$ and $p$ can be detected based on the reduced distributions on this partition. If we knew the partition $R_1, \ldots, R_k$ ahead of time, the testing problem would be easy: Since the reduced distributions have support $k$, this would yield a uniformity tester with sample complexity $O(\sqrt{\epsilon}/\epsilon^2)$. The main difficulty is that the correct partition is unknown to the testing algorithm (as it depends on the unknown histogram).

A natural approach, employed in [DKN15b] for $d = 1$, is to appropriately “guess” the correct rectangle partition. For the univariate case, a single interval partition already leads to a non-
trivial uniformity tester. Indeed, consider partitioning the domain into $\Theta(k/\epsilon)$ intervals of equal length (hence, of equal mass under the uniform distribution). It is not hard to see that the reduced distributions over these intervals can detect the discrepancy between $q$ and $p$, leading to a uniformity tester with sample complexity $\Theta((k/\epsilon)^{1/2}/\epsilon^2) = \Theta(k^{1/2}/\epsilon^{5/2})$. This very simple scheme gives an identity testing with sub-learning sample complexity when $\epsilon$ is constant — albeit suboptimal for small $\epsilon$. Unfortunately, such an approach can be seen to inherently fail even for two dimensions: Any obliviously chosen partition in two dimensions requires $\Omega(k^2/\epsilon^2)$ rectangles, which leads to an identity tester with sample complexity $\Omega(k^{1}/\epsilon^3)$. Hence, a more sophisticated approach is required in two dimensions to obtain any improvement over learning.

Instead of using a single oblivious interval decomposition of the domain, the sample-optimal $\Theta(k^{1/2}/\epsilon^2)$ uniformity tester of [DKN15b] for univariate $k$-histograms partitions the domain into intervals in several different ways, and runs a known $\ell_2$ tester on the reduced distributions (with respect to the intervals in the partition) as a black-box. We appropriate generalize this idea to the multidimensional setting. To achieve this, we proceed by partitioning the domain into approximately $k$ congruent rectangles, distinguishing the different partitions based on the shapes of these rectangles. This requirement to guess the shape is necessary, as for example partitioning the square into rows will not suffice when the true partition is a partition into columns. However, it suffices to consider a poly-logarithmic sized set of partitions, where any desired shape of rectangle can be achieved to within a factor of 2. It is not hard to show that for each of the $k$ rectangles in the true partition that are sufficiently large, at least one of our oblivious partitions will use rectangles of approximately the same size, and thus at least one rectangle in this partition will approximately capture the discrepancy due to this rectangle (note that only considering large rectangles suffices since any rectangle on which the uniform distribution assigns substantially more mass than $q$ must be reasonably large). This means that at least one partition will have an $\epsilon/\text{polylog}(k/\epsilon)$ discrepancy between $p$ and $q$, and by running an identity tester on this partition, we can distinguish them.

One complication that arises here is that for small values of $\epsilon$, the difference between $p$ and $q$ might arise from rectangles with area much less than $1/k$. In order to capture these rectangles, we will need some of our oblivious partitions to be into rectangles with area smaller than $1/k$, for which there will necessarily be more than $k$ rectangles in the partition (in fact as many as $k/\epsilon$ many rectangles). This would appear to be an issue for the following reason: the sample complexity of uniformity testing over a discrete domain of size $n$ if $\Theta(n^{1/2}/\epsilon^2)$. Hence, naively using such a uniformity tester on the reduced distributions obtained by a decomposition into $k/\epsilon$ rectangles would lead to sample complexity of $\Theta((k/\epsilon)^{1/2}/\epsilon^2) = \Theta(k^{1/2}/\epsilon^{5/2})$ We can circumvent this difficulty by using the following insight: Even though the total number of rectangles in the partition might be large, it can be shown that for a well-chosen oblivious partition, a reasonable fraction of this discrepancy is captured by only $k$ of these rectangles. In such a case, the sample complexity of uniformity testing can be notably reduced using an $\ell_1^k$ tester [DKN17] — a uniformity tester under a modified metric that measures the discrepancy of the largest $k$ domain elements. This completes the sketch of our uniformity tester for the multidimensional case.

To generalize our uniformity tester to an identity tester for multidimensional histograms, two significant problems arise. The first is that it is no longer clear what the shape of rectangles in the oblivious partition should be. This is because when the explicit distribution $p$ is not the uniform distribution, equally sized rectangles are not a natural option to consider. Fortunately, this problem can be fixed by breaking the axes into pieces that assign equal mass to the marginals of the known distribution. The more substantial problem is that it is no longer clear that the discrepancy between $p$ and $q$ can be captured by a partition of the square into $k$ rectangles. This is because the two $k$ rectangle partitions corresponding to the $k$-histograms $p$ and $q$ when refined could lead to a partition of the square into as many $k^2$ rectangles. To remedy this, we note that
there is still a partition into \( k \) rectangles so that \( q \) is piecewise constant on that partition. We show (Lemma 2.3) that if we refine this partition slightly — by dividing each region into two regions, the half on which \( p \) is heaviest and the half on which \( p \) is lightest — that this new partition will capture a constant fraction of the difference between \( p \) and \( q \). Given this structural result, our identity testing algorithm becomes similar to our uniformity tester. We obliviously partition our domain into rectangles poly-logarithmically many times, each time we now divide each rectangle further into two regions as described above, and then run identity testers on these partitions. We show that if \( p \) and \( q \) differ by \( \epsilon \) in \( L_1 \)-distance, then at least one such partition will detect at least \( \epsilon / \text{polylog}(k/\epsilon) \) of this discrepancy.

**Overview of Sample Complexity Lower Bound.** Note that \( \Omega(\sqrt{k}/\epsilon^2) \) is a straightforward lower bound on the sample complexity of identity testing \( k \)-histograms, even for \( d = 1 \). (This follows from the fact that a \( k \)-histogram can simulate any discrete distribution over \( k \) elements.)

In order to prove lower bounds of the form \( \omega(\sqrt{k}/\epsilon^2) \), it is important to show that an algorithm must consider many possible shapes of rectangles. This suggests a construction where we have a grid of some unknown dimensions where some squares in the grid is dense and the remainders are sparse in a checkerboard-like pattern. It should be noted that if we have two such grids whose dimensions differ by exactly a factor of 2, it can be arranged so that the distributions are exactly uncorrelated with each other. Using this observation, we can construct \( \log(k) \) such uncorrelated distributions that the tester will need to check for individually. However, this simple construction will not suffice as one could merely run \( \log(k) \) different testers in parallel. To deal with this, we will need a slightly more complicated construction. First, we divide the square into \( \text{polylog}(k) \) equal regions. Each of these regions is turned into one of these randomly-sized checkerboards, but where different regions will have different scales. We claim that this ensemble is hard to distinguish from uniform.

The actual proof of the lower bound is somewhat technical and involves bounding the chi-squared distance of taking \( \text{Poi}(m) \) samples from a random distribution in our ensemble with respect to the distribution obtained by taking \( \text{Poi}(m) \) samples from the uniform distribution. This computation is simplified by noting that since the sets of samples from each of the \( \sqrt{k} \) bins are independent of each other, we can consider each of them independently. For each individual bin we take \( s \sim \text{Poi}(m/\sqrt{k}) \) samples and need to compute \( \chi^2_s(X^s, Y^s) = \chi^2_U(X, Y)^s \), where \( X \) and \( Y \) are random distributions from our ensemble. However, it is not hard to see that if \( X \) and \( Y \) are checkerboards of different scales that the \( \chi^2 \) value is exactly 1. This saves us a factor of \( \log(k) \), as there are \( \log(k) \) many different scales to consider, and leads to the improved lower bound.

## 2 Sample Near-Optimal Identity Testing Algorithm

### 2.1 Algorithm Description

Let \( q \) be the unknown histogram distribution and \( p \) be the explicitly known one. Our algorithm considers several judiciously chosen oblivious decompositions of the domain that will be able to approximate a set on which we can distinguish our distributions. We formalize the properties that we need these decompositions to have with the notion of a good oblivious covering (Definition 2.1 below). The essential idea is that we cover the domain \([0, 1]^d\) with rectangles that don’t overlap too much in such a way that any partition of \([0, 1]^d\) into \( k \) rectangles can be approximated by some union of rectangles in this family.
Definition 2.1 (good oblivious covering). Let $p$ be a probability distribution on $[0, 1]^d$. For $k, j, \ell \in \mathbb{Z}_+$ and $0 < \epsilon \leq 1/2$, an $(k, j, \ell, \epsilon)$-oblivious covering of $p$ is a family $\mathcal{F}$ of subsets of $[0, 1]^d$ satisfying the following:

1. For any partition $\Pi$ of $[0, 1]^d$ into $k$ rectangles, there exists a subfamily $\mathcal{S} \subseteq \mathcal{F}$ such that:
   
   (a) We have that $|\mathcal{S}| \leq k \cdot j$.
   
   (b) The sets in $\mathcal{S}$ are mutually disjoint, i.e., $S \cap T = \emptyset$ for all $S \neq T \in \mathcal{S}$.
   
   (c) The sets in $\mathcal{S}$ together contain all except $\epsilon$ of the probability mass of $[0, 1]^d$ under $p$, i.e., $p(\mathcal{S}) \geq 1 - \epsilon$.
   
   (d) For each $S \in \mathcal{S}$ there is some histogram rectangle $R \in \Pi$ such that $S$ only contains points from $R$, i.e., $S \subseteq R$.

2. For each point $x$ in $[0, 1]^d$, the number of sets in $\mathcal{F}$ containing $x$ is exactly $\ell$.

In Section 2.2, we establish the existence of a $(k, \log^{O(d)}(4kd/\epsilon), \log^{O(d)}(4kd/\epsilon), \epsilon)$-oblivious covering of $p$ for any distribution $p$ on $[0, 1]^d$ and for all $k, d, \epsilon$ with $\epsilon \leq 1/2$.

Our basic plan will be that if $p$ is a distribution with a $(k, j, \ell, \epsilon/2)$-oblivious covering $\mathcal{F}$, and $q$ is a $k$-histogram that differs from $p$ by at least $\epsilon$ in $L_1$-distance, then $q$ defines a partition $\Pi$ of $[0, 1]^d$ into $k$ rectangles. This partition gives rise to a subfamily $\mathcal{S} \subseteq \mathcal{F}$ satisfying the constraints specified in Definition 2.1. We would like to show that a constant fraction of the discrepancy between $p$ and $q$ can be detected by considering their restrictions to $\mathcal{S}$. There are a couple of obstacles to showing this, the first of which is that we do not know what $\mathcal{S}$ is. Fortunately, we do have the guarantee that $|\mathcal{S}|$ is relatively small. We can consider the restrictions of $p$ and $q$ over all sets in $\mathcal{S}$ and try to check if there is a significant discrepancy between the two coming from any small subset. To achieve this, we will make essential use of an identity tester under the $\ell_1$-metric:

Theorem 2.2 (optimal $\ell_1^k$ tester, [DKN17]). Given a known discrete distribution $p$ and sample access to an unknown discrete distribution $q$, each of any finite size, there exists an algorithm that accepts with probability $2/3$ if the distributions are the same and rejects with probability $2/3$ if there exists a set $\mathcal{A}$ of $k$ or fewer domain elements such that $\sum_{s \in \mathcal{A}} |p(s) - q(s)| \geq \epsilon$. That is, these elements alone contribute at least $\epsilon$ to the $\ell_1$-distance between the distributions. The tester requires only knowledge of the known distribution $p$ and $O(\sqrt{k}/\epsilon^2)$ i.i.d. samples from $q$.

The second obstacle is that although $q$ will be constant within each $S \in \mathcal{S}$, it will not necessarily be the case that $p(S)$ and $q(S)$ will differ substantially even if the variation distance between $p$ and $q$ on $S$ is large. To fix this, we show that $S$ can be split into two parts so that at least one of the two parts will necessarily see a large fraction of this difference:

Lemma 2.3. Let $p, q : \mathbb{R}^d \rightarrow \mathbb{R}_{\geq 0}$ and let $S$ be a bounded open subset in $\mathbb{R}^d$ on which $q$ is uniform. Suppose $S$ is partitioned into two subsets $S_1, S_2$ such that $\text{vol}(S_1) = \text{vol}(S_2) = \text{vol}(S)/2$ and $p(s_1) \geq p(s_2)$ for all $s_1 \in S_1, s_2 \in S_2$, where $\text{vol}()$ denotes Euclidean volume. Then,

\[
\max \left( \left| \int_{S_1} p(x) - q(x) dx \right|, \left| \int_{S_2} p(x) - q(x) dx \right| \right) \geq \int_S |p(x) - q(x)| dx / 4.
\]

Proof. First, we note that if $W \subseteq S$ is the set of points $x$ for which $p(x) \geq q(x)$, and $W' = S \setminus W$, then

\[
\int_S |p(x) - q(x)| dx = \int_W p(x) - q(x) dx + \int_{W'} q(x) - p(x) dx.
\]
We will show that
\[
\max \left( \left| \int_{S_1} p(x) - q(x) \, dx \right| , \left| \int_{S_2} p(x) - q(x) \, dx \right| \right) \geq \int_{W} p(x) - q(x) \, dx / 2, \tag{1}
\]
and that by symmetry it will follow that
\[
\max \left( \left| \int_{S_1} p(x) - q(x) \, dx \right| , \left| \int_{S_2} p(x) - q(x) \, dx \right| \right) \geq \int_{W} q(x) - p(x) \, dx / 2.
\]
Combining these will give our final result.

Note that if \( S_1 = S \cap W \), Equation 1 immediately holds. In fact, it holds even without the factor of two on the right hand side. Similarly, if \( S_1 \subseteq S \cap W \), then it also holds (but this time with the factor of two). To see this, note that
\[
\int_{W} p(x) - q(x) \, dx = \int_{S_1 \cap W} p(x) - q(x) \, dx + \int_{S_2 \cap W} p(x) - q(x) \, dx = \int_{S_1} p(x) - q(x) \, dx + \int_{S_2 \cap W} p(x) - q(x) \, dx.
\]
The RHS is a sum of two integrals where the second integral’s integrand is always smaller than the smallest value of the first integrals integrand. Furthermore, the second integral is over a region that is no larger than the first region of the first integral because \( \text{vol}(S_1) = \text{vol}(S) / 2 \), while \( \text{vol}(S_2 \cap W) \leq \text{vol}(S_2) = \text{vol}(S) / 2 \). Thus, we have
\[
\int_{W} p(x) - q(x) \, dx \leq 2 \int_{S_1} p(x) - q(x) \, dx
\]
which implies Equation 1.

The final case needed to prove Equation 1 holds is when \( S_1 \cap W \subseteq S_1 \), which is equivalent to saying that \( S_1 \) contains points \( x \) for which \( p(x) < q(x) \). Let \( h = -\int_{S_1 \cap W} p(x) - q(x) \, dx \geq 0 \). Then we have
\[
\int_{W} p(x) - q(x) \, dx = \int_{S_1} p(x) - q(x) \, dx - \int_{S_1 \cap W} p(x) - q(x) \, dx = h + \int_{S_1 \cap W} p(x) - q(x) \, dx.
\]
If \( h \leq \int_{S_1 \cap W} p(x) - q(x) \, dx / 2 \), then we can substitute this into the preceding equation and we are done. Otherwise, \( h > \int_{S_1 \cap W} p(x) - q(x) \, dx / 2 \). Note that in this case, \( |\int_{S_2} p(x) - q(x) \, dx| \geq h \).

Putting these together gives
\[
\left| \int_{S_2} p(x) - q(x) \, dx \right| > \int_{S_1 \cap W} p(x) - q(x) \, dx / 2,
\]
completing the proof.

We can now state the general theorem:

**Theorem 2.4.** Let \( p \) be a known distribution on \([0,1]^d\) with an \((k, j, \ell, \epsilon/2)\)-oblivious covering. There exists a tester that given sample access to an unknown \( k \)-histogram \( q \) distinguishes between \( p = q \) and \( d_{TV}(p,q) \geq \epsilon \) with probability at least \( 2/3 \) using \( O(\sqrt{k} \cdot \ell^2 / \epsilon^2) \) samples.

Plugging in the bounds on \( j \) and \( \ell \) of \( \log^{\Theta(d)}(kd/\epsilon) \) for the construction we will prove later yields a sample complexity of \( O(\sqrt{k} \log^{\Theta(d)}(kd/\epsilon^2)) \) for \( \epsilon \leq 1/2 \). This gives the upper bound portion of Theorem 1.4.
Algorithm 1 Identity Tester for $d$-dimensional $k$-histograms (used to prove Theorem 2.4).

1. Let $F$ be a $(k, j, \ell, \epsilon/2)$-oblivious covering of $p$.

2. Obtain a new family of sets $F'$ by taking each $S \in F$ and replacing it with the two sets $S_1$ and $S_2$ as defined in Lemma 2.3.

3. Define discrete distributions $p', q'$ over $F'$ where a random sample, $x$, from $p'$ (resp. $q'$) is obtained by taking a random sample from $p$ (resp. $q$) and then returning a uniform random element of $F'$ containing $x$. We note that the distribution $p'$ can be explicitly computed, and we can take a sample from $q'$ at the cost of taking a sample from $q$.

4. Use the algorithm from Theorem 2.2 to distinguish between $p' = q'$ and the existence of a set $A$ of size at most $2k \cdot j$ with $\sum_{S \in A} |p'(S) - q'(S)| \geq \epsilon/(8\ell)$.

5. Output EQUAL in the former case and NONEQUAL in the latter case.

The basic idea of the algorithm is to take each element of the oblivious cover and divide it in two as in Lemma 2.3 and then use the tester from Theorem 2.2 on the induced distributions of $p$ and $q$ on the resulting sets. The algorithm itself is quite simple.

We can now prove Theorem 2.4.

Proof. We note that the sample complexity of the tester described in Algorithm 1 is $O(\sqrt{k}j\ell^2/\epsilon^2)$, as desired. It remains to prove correctness.

The completeness case is straightforward. If $p = q$, then clearly $p' = q'$ and our tester will accept with probability at least $2/3$.

We now proceed to prove soundness. If $d_{TV}(p, q) \geq \epsilon$, we claim that our tester will reject with probability at least $2/3$. For this we note that $q$ defines some partition $\Pi$ of $[0, 1]^d$ into $k$ rectangles so that $q$ is constant on each part of the partition. By the definition of an oblivious cover, there is a subfamily of disjoint sets $S \subseteq F$ so that:

- $q$ is constant on each element of $S$.
- $|S| \leq k \cdot j$.
- Letting $U = \bigcup_{S \in S} S$, we have that $p(U) \geq 1 - \epsilon/2$.

Since $\epsilon = d_{TV}(p, q) = \int_{[0,1]^d} \max(p - q, 0) dx$, we have that $\int_U \max(p - q, 0) dx \geq \epsilon - \int_{[0,1]^d - U} p dx \geq \epsilon/2$. Therefore, since the elements of $S$ are disjoint, we have that

$$\sum_{S \in S} \int_S |p - q| dx \geq \epsilon/2.$$

We now let $A \subseteq F'$ be the collection of all $S_1$ or $S_2$ corresponding to an $S \in S$. We note that

This is because the integrand on the RHS is always more negative value of the integrand on the RHS and the region the integral on the LHS is over is at least as large as that of the integral on the RHS. This is very similar to the reasoning in the earlier case where $S_1 \cap W = S_1$ above.
\[|A| = 2|S| \leq 2k \cdot j. \] Furthermore, by Lemma 2.3 we have that
\[
\frac{\epsilon}{8} \leq \sum_{S \in \mathcal{S}} \int_S |p - q|dx/4
\leq \sum_{S \in \mathcal{S}} \max(|p(S_1) - q(S_1)|, |p(S_2) - q(S_2)|)
\leq \sum_{A \in \mathcal{A}} |p(A) - q(A)|.
\]

On the other hand, for \(A \in \mathcal{A}\), we have that \(p'(A) = p(A)/\ell\) and \(q'(A) = q(A)/\ell\), so we have that
\[
\sum_{A \in \mathcal{A}} |p'(A) - q'(A)| \geq \epsilon/(8\ell).
\]

Therefore, if \(d_{TV}(p, q) \geq \epsilon\), our tester will reject with probability at least 2/3.

This completes our proof.

\[ \square \]

**Remark 2.5.** Algorithm 1 is robust in the sense that it still works even if \(q\) is only (say) \(\epsilon/10\)-close to some \(k\)-histogram distribution \(\tilde{q}\) instead of actually being one. To show this, one can note that the existing proof applied to \(p\) and \(\tilde{q}\) gives an \(\mathcal{A}\) such that \(\sum_{A \in \mathcal{A}} |p(A) - \tilde{q}(A)|\) is at least \(\epsilon/8\). The triangle inequality then implies \(\sum_{A \in \mathcal{A}} |p(A) - q(A)| \geq \epsilon/40\), which, by the same reasoning given in the proof of the non-robust case, implies the algorithm is still correct.

### 2.2 Construction of Good Oblivious Covering

Here we prove the existence of an oblivious covering:

**Lemma 2.6.** For any continuous distribution \(p\) on \([0, 1]^d\), positive integer \(k\) and \(\epsilon \leq 1\), there exists an \((k, 2^d \log^d(4kd/\epsilon), \log^d(4kd/\epsilon), \epsilon)\)-oblivious covering of \(p\).

**Proof.** The basic idea of our construction will be to let \(\mathcal{F}\) be a union of grids where the number of cells in each direction is a power of 2.

For each coordinate, \(j\), and each non-negative integer \(i\), define the \(i^{th}\) partition of this coordinate to be a partition of \([0, 1]\) into \(2^i\) intervals so that \(j^{th}\) marginal of \(p\) assigns each interval in the partition equal mass, and so that the \(i^{th}\) partition is a refinement of the \((i-1)^{st}\).

For each vector \(z \in \mathbb{N}^d\), define the \(z\)-grid as the partition of \([0, 1]^d\) into rectangles by taking the product of the \(z^j\)-th partition of the \(j^{th}\) coordinate. We let \(\mathcal{F}\) be the union of the cells in the \(z\)-grid for all \(z \in \mathbb{N}^d \cap [0, m - 1]\) for \(m = \log_2(4kd/\epsilon)\). An illustration is given in Figure [1](#).

We note that each \(x \in [0, 1]^d\) is in exactly one cell in each \(z\)-grid, and therefore is contained in exactly \(m^d\) elements of \(\mathcal{F}\), verifying Property 2.

For Property 1, consider a partition of \([0, 1]^d\) into rectangles \(R_1, \ldots, R_k\). We claim that for each \(R_i\) there is a subfamily \(\mathcal{T}_i \subseteq \mathcal{F}\) of disjoint subsets of \(R_i\) with \(|\mathcal{T}_i| \leq 2^d m^d\), and so that \(p(R_i \setminus \bigcup_{S \in \mathcal{T}_i} S) \leq \epsilon/k\). It is then clear that taking \(S\) to be the union of the \(\mathcal{T}_i\) will suffice. In fact, we will show that for any rectangle \(R_i\), there is a corresponding \(\mathcal{T}_i\) with these properties.

We let \(R_i = \prod_{j=1}^d I_j\) for intervals \(I_j\). We let \(I'_j\) be \(I_j\) minus the intervals of the \((m-1)^{st}\)-partition of the \(j^{th}\) coordinate that contain the endpoints of \(I_j\). We note that \(p(I_j \setminus I'_j) \leq \epsilon/(kd)\) and that \(I'_j\) is a union of consecutive intervals in the \((m-1)^{st}\) partition of this coordinate. We claim that this means that \(I'_j\) is the union of at most \(2m\) intervals of one of the first \(m - 1\) partitions of the \(j^{th}\)
(a) A $z$-grid with $z = [1, 1]$.  
(b) A $z$-grid with $z = [2, 3]$.

Figure 1: $z$-grids for different values of $z$ partition $[0, 1]^2$ into rectangles. In this figure, the axes are scaled such that the marginal distributions of the vertical and horizontal coordinates, respectively, of $p$ are uniform.

Figure 2: How our oblivious covering is used to cover a rectangle $R_i$ in the proof of Lemma 2.6. Each dimension of $R_i$ is separately decomposed into non-overlapping one-dimensional rectangles, with a small amount of area shaded in beige left over on the sides. $\mathcal{T}_i$ is obtained by taking the family of all Cartesian products of the form $I''_1 \times \cdots \times I''_d$ where, for each $j$, $I''_j$ is any subinterval in the decomposition of $I'_j$. In this figure, the axes are scaled such that the marginal distributions of the vertical and horizontal coordinates, respectively, of $p$ are uniform.
coordinate. This is easy to see by induction on \( m \), as \( I_j' \) is a union of consecutive intervals in the \((m - 2)^{nd}\) partition union at most one interval of the \((m - 1)^{st}\) on either end. The one-dimensional intervals on the top and left of Figure 2 show an illustration of this.

In order to produce \( T_i \), we write each \( I_j' \) as a union of at most 2\( m \) intervals from the relevant partitions. We let \( T_i \) be the set of rectangles obtained by taking the product of one rectangle from each of these sets. It is clear then that \( T_i \) partitions \( \prod_{j=1}^{d} I_j' \) into at most \( (2^m)^d \) pieces. Figure 2 shows an illustration of this. However, we note that

\[
p\left( R \setminus \prod_{j=1}^{d} I_j' \right) = p\left( \prod_{j=1}^{d} I_j \setminus \prod_{j=1}^{d} I_j' \right) \leq \sum_{j=1}^{d} p_j(I_j \setminus I_j') \leq \epsilon/k.
\]

Thus, \( T \) satisfies all of the desired properties, and taking the union of the \( T_i \) will yield an appropriate \( S \). This completes our proof.

\[
\square
\]

3 Sample Complexity Lower Bound

In this section, we prove the sample complexity lower bound given in Theorem 3.5. We begin by providing a new proof that \( \Omega(\sqrt{k}/\epsilon^2) \) samples are required to test uniformity of a \( k \)-histogram in one dimension. To start we will introduce some definitions relating to the \( \chi^2 \)-metric.

**Definition 3.1.** For probability distributions \( p, q \) and \( r \) let

\[
\chi_q(p, r) \overset{\text{def}}{=} \int \frac{dpdr}{dq}.
\]

Notice that, for fixed \( q \), \( \chi_q(p, r) \) is an inner product on distributions \( p, r \). Furthermore, by the Cauchy-Schwarz inequality it follows that if \( p \) and \( q \) are probability distributions then

\[
\chi_q(p, p) = \int \frac{dp^2}{dq} = \left( \int \frac{dp^2}{dq} \right) \left( \int dq \right) \geq \left( \int dp \right)^2 = 1.
\]

We also note that this metric is useful for determining whether or not distributions can be distinguished. In particular, if \( p \) and \( q \) can be distinguished from a single sample, it must be the case that \( \chi_q(p, p) \) is much bigger than 1. Formally, we have:

**Lemma 3.2.** Suppose that \( p \) and \( q \) are probability distributions. Suppose furthermore that there is an algorithm that given a random sample from \( p \) accepts with probability at least \( 2/3 \), and given a random sample from \( q \) rejects with probability at least \( 2/3 \). Then, it holds that \( \chi_q(p, p) \geq 4/3. \)

**Proof.** Let \( A \) be the set on which the algorithm accepts. We then have that \( p(A) \geq 2/3 \) and \( q(A) \leq 1/3 \). Therefore, we have that

\[
\chi_q(p, p) \geq \int_A \frac{dp^2}{dq} \geq 3 \left( \int_A \frac{dp^2}{dq} \right) \left( \int_A dq \right) \geq 3 \left( \int_A dp \right)^2 \geq 4/3.
\]

\[
\square
\]

We can now use this simple fact to prove a lower bound on the number of samples required to test uniformity of univariate \( k \)-histograms. The idea is to use a standard adversary argument, using Lemma 3.2 to show that it is impossible to distinguish samples taken from a distribution from a particular ensemble, from those taken from the uniform distribution.
Proposition 3.3. If there exists an algorithm that given $s$ independent samples from a $k$-histogram, $p$, on $[0, 1]$ and accepts with at least $2/3$ probability if $p = U$ and rejects with at least $2/3$ probability if $d_{TV}(p, U) \geq \epsilon$, then $s = \Omega(\sqrt{k}/\epsilon^2)$.

Proof. We assume that $k$ is even. Divide $[0, 1]$ into $k/2$ equally sized bins. Let $P$ be a distribution over $k$ histograms where in each bin either $dp = (1 + \epsilon)dx$ on the first half and $dp = (1 - \epsilon)dx$ on the second half of the bin, or visa versa independently for each bin. Note that a sample from $P$ is always a $k$-histogram $p$ with $d_{TV}(p, U) = \epsilon$. Let $P^s$ be the distribution on $[0, 1]^s$ obtained by randomly picking a distribution $p$ from $P$ and then taking $s$ independent samples from $p$.

Given that an algorithm to distinguish the uniform distribution from $k$-histograms far from it exists, such a distribution can distinguish a single sample from $P^s$ from a sample from $U^s$. Therefore, by Lemma 3.2, we must have that $\chi_{U^s}(P^s, P^s) \geq 4/3$. We will now try to bound this quantity.

Note that $P^s$ is a mixture of the distributions $p^s$ where $p$ is taken from $P$. Therefore, by linearity of the $\chi$-metric, we have that

$$\chi_{U^s}(P^s, P^s) = \mathbb{E}_{p,q \sim P}[\chi_{U^s}(p^s, q^s)] = \mathbb{E}_{p,q \sim P}[(\chi_U(p, q))^s],$$

where the last equality is by noting that the corresponding integral decomposes as a product.

We now need to think about the distribution of $\chi_U(p, q)$ when $p$ and $q$ are taken independently from $P$. We note that for each bin $B$ that $\int_B dp dq dU$ is either $1 + \epsilon^2 k/2$ or $1 - \epsilon^2 k/2$ with equal probability and independently for each bin. Therefore,

$$\chi_{U^s}(P^s, P^s) \leq \mathbb{E}\left[\left(1 + \frac{\epsilon^2}{k/2} \sum_{i=1}^{k/2} X_i\right)^s\right],$$

where $X_i$ are i.i.d. random variables $X_i \in \{\pm 1\}$. Therefore,

$$\chi_{U^s}(P^s, P^s) = \mathbb{E}\left[\left(1 + \frac{\epsilon^2}{k/2} \sum_{i=1}^{k/2} X_i\right)^s\right].$$

To bound this quantity, note that since the $t^{th}$ moment of a Bernoulli random variable is less than or equal to the corresponding moment of the standard Gaussian for each $t$. We thus have that

$$\chi_{U^s}(P^s, P^s) \leq \mathbb{E}\left[\left(1 + \frac{\epsilon^2}{k/2} \sum_{i=1}^{k/2} G_i\right)^s\right],$$

where the $G_i$ are i.i.d. $N(0, 1)$ variables. However, we can bound this as follows:

$$\chi_{U^s}(P^s, P^s) \leq \mathbb{E}\left[\left(1 + \frac{\epsilon}{\sqrt{k/2}} N(0, 1)\right)^s\right] \leq \mathbb{E}\left[\exp\left(\left(\frac{s\epsilon}{\sqrt{k/2}}\right)^2\right) N(0, 1)\right] \leq \mathbb{E}\left[\exp\left(\left(\frac{s\epsilon}{\sqrt{k/2}}\right)^2 / 2\right)\right].$$
Figure 3: An example of a distribution from $\mathcal{P}$. The dark cells have density $1 + \epsilon$, and the light cells have density $1 - \epsilon$. The green lines separate the square into a $4 \times 2$ grid, and each rectangle is filled with a random $2 \times 2$ checkerboard.

Hence, the algorithm can only exist when

$$\left( \frac{st^2}{\sqrt{k}} \right) \geq \sqrt{\log(4/3)},$$

or equivalently when $s = \Omega(\sqrt{k}/\epsilon^2)$. This completes the proof.

We can get a slightly improved bound in $d$-dimensions by slightly modifying our ensemble in order to force our algorithm to guess the dimensions of the rectangles involved in the partition. Specifically, we prove the following:

**Proposition 3.4.** If there exists an algorithm that given $s$ independent samples from a $k$-histogram, $p$, on $[0,1]^d$ with $k > 4^d$ and accepts with at least $2/3$ probability if $p = U$ and rejects with at least $2/3$ probability if $d_{TV}(p,U) \geq \epsilon$, then $s = \Omega(\epsilon^{-2} \sqrt{kd/2^d \log(\log(k-d)/d)})$.

**Proof.** We first assume that $k$ is a power of 2, namely $k = 2^{m+d}$. Since this can always be achieved by decreasing $k$ by a factor of at most 2, this should not affect the final bound. We define an ensemble $\mathcal{P}$ similarly to how we did so in the proof of Proposition 3.3. To define a distribution $p$ in $\mathcal{P}$, first we randomly and uniformly pick a $d$-tuple $(m_1, m_2, \ldots, m_d)$ of non-negative integers summing to $m$. We call this the defining vector of $p$. We next divide $[0,1]^d$ into $k/2$ bins by producing a $2^{m_1} \times 2^{m_2} \times \cdots \times 2^{m_d}$ grid. Each bin we cut into $2^d$ equal sub-bins by diving it in half along each dimension. We divide these sub-bins into two classes based on their parity. We then let $dp = (1+\epsilon)dV$ on a the sub-bins of a random parity and $dp = (1-\epsilon)dV$ on the other sub-bins, where the choices are independent for each bin. We note that a $p$ from $\mathcal{P}$ is always a $k$-histogram that is $\epsilon$-far from $U$. An illustration is given in Figure 3.

We let $\mathcal{P}^s$ be the distribution on $([0,1]^d)^s$ obtained by taking a random $p$ from $\mathcal{P}$ and taking $s$ independent samples from $p$. Once again, it suffices to bound from below $\chi_U(\mathcal{P}^s, \mathcal{P}^s)$. We similarly have that

$$\chi_U(\mathcal{P}^s, \mathcal{P}^s) = \mathbb{E}_{p,q \sim \mathcal{P}}[(\chi_U(p,q))^s].$$

We note that if $p$ and $q$ have the same defining vectors, that the contribution to $\chi_U(p,q)$ from each bin is randomly and independently $2(1 \pm \epsilon^2)/k$. Therefore, by the arguments above, if we condition on $p$ and $q$ having the same defining vectors, the expectation of $(\chi_U(p,q))^s$ is at most

$$\exp \left( \frac{st^2}{\sqrt{k/2^d}} \right).$$

On the other hand, if $p$ and $q$ have different defining vectors, we claim that
\( \chi_U(p, q) = 1 \). In fact, we make the stronger claim that if \( A \) is the intersection of a defining bin of \( p \) and a defining bin of \( q \), then \( \int_A \frac{dp dq}{dU} = q(A) \). This is because without loss of generality we may assume that \( p \)'s associated \( m_1 \) is smaller than \( q \)'s associated \( m_1 \). This in turn means that given any point in \( A \), the entire width of \( A \) along the first axis will be in the same sub-bin for \( q \), but will pass through two sub-bins of opposite parity for \( p \). Thus, the average of \( dp/dU \) over this line will be 1, and thus the integral over \( A \) of \( dp dp/dU \) is the same as the integral of \( dq \).

Now since there are \( \binom{m+d-1}{d-1} \) different possible defining vectors, we have that

\[
\chi_U(P^s, P^s) \leq 1 + \left( \frac{m + d - 1}{d - 1} \right)^{-1} \exp \left( \frac{s \epsilon^2}{\sqrt{k/2^d}} \right)^2.
\]

In order for this to be at least 4/3, it must be the case that

\[
\left( \frac{s \epsilon^2}{\sqrt{k/2^d}} \right) \geq \sqrt{\log \left( \frac{m + d - 1}{d - 1} \right)},
\]

or

\[
s = \Omega(\epsilon^{-2} \sqrt{kd/2^d \log(k-d)/d}).
\]

This completes the proof. \( \square \)

Unfortunately, the above bound only saves us a \( \log \log(k) \) factor. This is essentially because the algorithm only needs to correctly guess one of poly-logarithmically many defining vectors, and once it has guessed the correct one, it only needs to see a signal large enough that the probability of error is only inverse poly-logarithmic. This can be done by increasing the number of samples by only a doubly logarithmic factor. In order to do better we will need a slightly more complicated construction, where we chop our domain into pieces and fill each piece with rectangles, but where different pieces might have rectangles of different sizes.

**Theorem 3.5.** If there exists an algorithm that given \( s \) independent samples from a \( k \)-histogram, \( p \), on \([0, 1]^d\) with \( k > 2^{100d} \) and accepts with at least 2/3 probability if \( p = U \) and rejects with at least 2/3 probability if \( d_{TV}(p, U) \geq \epsilon \), then \( s = \Omega(\log(k)/d)^{d-1} \sqrt{k}/\epsilon^2 \).

**Proof.** We first assume that \( k \) can be written in the form \( k = n2^{m+d} \), where \( n \leq \binom{m+d-1}{d-1}/4 \). We note that (perhaps decreasing \( k \) by a constant factor) we can achieve this with \( n = \Omega(\log(k)/d)^d \), and therefore we can assume this throughout the rest of the argument. We describe a new ensemble \( Q \) over \( k \)-histograms on \([0, 1]^d\) in the following way: First divide \([0, 1]^d\) into \( n \) equal volume boxes in some arbitrary way. For each box \( B_i \), pick a member \( p_i \) from \( P \), the ensemble from the proof of Proposition 3.4 independently for different \( i \). We let the restriction of \( q \) to \( B_i \) be \( p_i \) rescaled so that it assigns \( B_i \) total mass \( 1/n \), and so that the domain of definition is \( B_i \) rather than \([0, 1]^d\). An example element of \( Q \) is illustrated in Figure 4.

Once again, it suffices to show that if \( s \) is below our bound that

\[
\chi_U(Q^s, Q^s) = \mathbb{E}_{p,q \sim Q}[\chi_U(p,q)^s]
\]

is less than 4/3.

We note that for \( p \) and \( q \) taken from \( Q \) that \( \int_{B_i} \frac{dp dq}{dU} \) is distributed as \( \chi_U(p', q')/n \) with \( p' \) and \( q' \) taken from \( P \). This is \( 1/n \) except with probability \( \alpha := \binom{m+d-1}{d-1}^{-1} \) and otherwise is distributed
Figure 4: An example of a probability distribution from ensemble $Q$. The square is divided into $n = 4$ regions by the black lines. Each sub-square is divided into a randomly sized grid of $2^m = 8$ equal rectangles by the green lines. To get the final distribution, each of those rectangles should be filled with a random checkerboard as in Figure 3.

as $1/n + \epsilon^2/n^{2m} \sum_{j=1}^{2^m} X_{ij}$, where the $X_{ij}$ are i.i.d. $\pm 1$ random variables. Notice that these are independent for different $i$ and sum to $\chi_U(p,q)$. Therefore,

$$\chi_U(p,q) \sim 1 + \sum_{i=1}^{n} Y_i \left( \frac{\epsilon^2}{n^{2m}} \sum_{j=1}^{2^m} X_{ij} \right),$$

where the $Y_i$ are i.i.d. 1 with probability $\alpha$ and 0 otherwise. Therefore, we have that

$$\chi_{U^s}(Q^s, Q^s) = \mathbb{E} \left[ \left( 1 + \sum_{i=1}^{n} Y_i \left( \frac{\epsilon^2}{n^{2m}} \sum_{j=1}^{2^m} X_{ij} \right) \right)^s \right].$$

Once again, this expectation is only increased if the $X_{ij}$ are replaced by standard Gaussians, and so this is at most

$$\mathbb{E} \left[ \left( 1 + \sum_{i=1}^{n} Y_i \left( \frac{\epsilon^2}{n^{2m/2}} G_i \right) \right)^s \right]$$

with $G_i$ i.i.d. standard normals. Noting that we still have a sum of $\sum_{i=1}^{n} Y_i \sim B(n, \alpha)$ independent
Gaussians, this simplifies to

\[
\chi_{U^s}(Q^s, Q^n) \leq E \left[ \left( 1 + \left( \frac{\epsilon \sqrt{B(n, \alpha)}}{n^{2m/2}} N(0, 1) \right) \right)^s \right]
\]

\[
\leq E \left[ \exp \left( \left( \frac{s \epsilon \sqrt{B(n, \alpha)}}{n^{2m/2}} \right)^2 \right) \right]
\]

\[
= E \left[ \exp \left( B(n, \alpha) \left( \frac{s^2 \epsilon^4}{2n^2 2^m} \right) \right) \right]
\]

\[
\leq \left( 1 + \alpha \exp \left( \frac{s^2 \epsilon^4}{2n^2 2^m} \right) \right)^n
\]

\[
\leq \exp \left( n \alpha \exp \left( \frac{s^2 \epsilon^4}{2n^2 2^m} \right) \right)
\]

\[
\leq \exp \left( \exp \left( \frac{s^2 \epsilon^4}{2n^2 2^m} \right) / 4 \right).
\]

In order for this to be at least 4/3, it must be the case that

\[
\frac{s^2 \epsilon^4}{2n^2 2^m} \gg 1,
\]

or equivalently that

\[
s = \Omega(2^{m/2}n/\epsilon^2) = \Omega(\sqrt{kn}/2^d/\epsilon^2) = \Omega((\log(k)/d)^d \sqrt{k}/\epsilon^2).
\]

This completes our proof.

\[\Box\]

4 Conclusions and Future Directions

In this work, we gave a computationally efficient and sample near-optimal algorithm for the problem of testing the identity of multidimensional histogram distributions. Our nearly matching upper and lower bounds have interesting consequences regarding the relation of learning and identity testing for this important nonparametric family of distributions. A natural direction for future work is to generalize our results to the problem of testing equivalence between two unknown multidimensional histograms. The one-dimensional version of this problem was essentially resolved in [DKN15a, DKN17]. Additional ideas are required for this setting, as the algorithm and analysis in this work exploit the a priori knowledge of the explicit distribution.

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