A practicable guide to the quantum computation architectures

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The primordial model of quantum computation was introduced over thirty years ago and the first quantum algorithms have appeared for over twenty years. Yet the exact architectures for quantum computer seem foreign to an undergraduate student major in computer science or engineering, even though the mass media has helped popularize the terminologies in the past decade. Despite being a cutting-edge technology from both the theoretical and the experimental perspectives, quantum computation is indeed imminent and it would be helpful to give the undergraduate students at least a skeleton understanding of what a quantum computer stands for. Since instruction-set architectures originated from classical computing models are familiar, we propose analogously a set of quantum instructions, which can be composed to implement renowned quantum algorithms. Albeit the similarity one can draw between classical and quantum computer architectures, current quantum instructions are fundamentally incommensurable from their classical counterparts because they lack the innate capability to implement logical deductions and recursions. We discuss this trait in length and illustrate why it is held responsible that current quantum computers not be considered general computers.
I. INTRODUCTION

Though descended from quantum mechanics and computer science, quantum computation remains a difficult subject for both quantum physicists and computer scientists. A major part of the difficulty arises from the underlying counter-intuitive mathematical structure of quantum mechanics, which even physicists find hard to incorporate into the body of computer logic. Existing concepts in classical computation are most likely not extensible to quantum computation, making it too precocious a subject for typical computer science textbooks.

Nevertheless, the counter-intuitiveness is exactly what enables an algorithm, based on quantum mechanical principles, to be radically different from a traditional algorithm. Although the motivation for introducing the "quantum" leap remains the same as every other new algorithm (that is to reduce complexity and save the computation time), its implementation finds no siblings in existing algorithms. The change is fundamental: the data unit is migrated from bit to qubit.

Qubit, short for quantum bit, is entirely a quantum mechanical concept; that is to say it cannot be simulated by a classical program. A student, therefore, equipped with the knowledge of Boolean algebra and discrete mathematics, should experience no problem writing a classical computer program but would find this knowledge insufficient to let them appreciate quantum computation. However, it is not necessary for a student to major in theoretical physics before they can appreciate this new way of computation. This paper aims to distill the difficult concepts into easier ones and make quantum computation accessible to undergraduate students. A programmer ignorant of CPU layout or semiconductor theory has no problem coding a program. Likewise, a quantum computer programmer ignorant of quantum physics should be capable of composing a quantum algorithm.

To achieve this goal, we start with the explanation of qubit, and then proceed to explain how logic operations can be performed on a qubit in Sec. IV. To better understand how quantum algorithms can be carried out on the qubits, we extend the concept of instruction set to quantum computation and break down the algorithms into smaller pieces of instructions in Sec. V. This partitioning into finite series instructions elucidates the inner functioning mechanism of each algorithm and allows one to see more clearly how one algorithm is correlated to another when some of the quantum instructions are shared by both algorithms.

In fact, IBM has released a set of assembly code instructions that are readily executable on their soon-to-be commercially available quantum computers, which is termed "Open Quantum Assembly Language (QASM)" [1]. QASM comprises a set of instructions more fundamental than our approach here: with only a few exceptions, each instruction corresponds conceptually to a quantum logic gate, which is explained in Sec. IV A. The latter, in turn, corresponds directly to a hardware implementation such that QASM can be used to manipulate a qubit on a low level, very much like manipulating the voltage level on a circuit node among the interconnected transistors in a classical computer. Our approach is more pedagogic, aiming to assist the understanding of existing quantum algorithms. In some sense, each instruction covered in Sec. IV C is a higher-level composition of QASM instructions but bears a more intuitive meaning. It is hoped that readers can recombine these instructions to construct their own quantum algorithms.

Before discussing the quantum computation models, we give a brief development history of quantum computation in Sec. II and compare their intuitive difference from classical computation models in Sec. III. After presenting the algorithms, we end the paper with a critique on the current architectures of quantum computation in Sec. VI discussing why they cannot yet be considered architectures for general computation.
II. BRIEF HISTORY

The concept of the "quantum computer" was first introduced when mathematicians tried to blend the tenets of quantum mechanics with prototypical Turing machines in the 1980’s [2–4], when physical systems based on quantum mechanical principles became not merely measurable but, to a large degree, controllable. It was already known by then that certain decision problems based on a probabilistic Turing machine, i.e. allowing a margin of error probability, could reduce algorithmic complexities.

Taking advantage of the probabilistic nature of quantum theory and founded on the primordial concept of quantum Turing machines [4], researchers have devised quantum versions of renowned algorithms in the 1990’s, such as large number factoring by Shor [5] and search by Grover [6, 7]. These algorithms are founded on a set of quantum logic gates, which are inherited from the classical concepts of logic gates, and are proven invariably faster than their classical approaches. The parallelism permitted by quantum states would sometimes cut the execution time on an exponential scale. Deutsch has conceived an heuristic algorithm [8] to illustrate the room of time saving, which is impossible for classical machines.

The new millennium witnessed the substantiation of these ideas on paper. Theoretical physicists proposed the conceptual qubit on experimentally realizable two-level atoms or spin-$\frac{1}{2}$ systems, on which logical operations can be realized as the interaction of the qubit with a quantum field. Such systems constitute the quantum logic computing (QLC) model, which we study below in Sec. IV A. There are many candidate systems existing and the three most prominent ones are trapped ions, nuclear spins in NV centers, and superconducting qubits because of their high degree of manipulability. So far, the superconducting qubit [9, 10] is the most popular candidate among academia and industries because this solid-state system can be controlled and measured by programmable microwave equipment. With the advent of circuit quantum electrodynamics [11], quantum logic gates can be operated on these superconducting qubits and Shor’s algorithm was realized experimentally [12].

One of the disadvantages of the QLC model is its reliance on the manipulation of all the qubits individually, making logical operations expensive and scalability a nightmare. Therefore, some scientists took another approach that was coined adiabatic quantum computing (AQC), to evade the problem, which we study below in Sec. IV B. Although the AQC model is still implemented on superconducting qubits [13], the computation method is distinct: the algorithmic result is obtained through the minimization of system energy across an array of qubits. Hence, the success of computation does not rely on the state of a specific qubit. The drawback is that the adiabatic algorithm is heuristic with no guarantee of efficiency improvement.

III. CONCEPTUAL DIFFERENCES

Quantum computers employ the laws of quantum mechanics to provide a vastly different mechanism for computation than that available from classical machines. The first distinguishing trait of a quantum system is known as superposition, or more formally, the superposition principle of quantum mechanics. Rather than existing in one distinct state at a time, a quantum system actually exists in all of its possible states at the same time. With respect to a quantum computer, this means that a quantum register exists in a superposition of all its possible configurations of 0’s and 1’s at the same time, unlike a classical system whose register contains only one value at any given time. It is not until the system is observed that it collapses into an observable, definite classical state. It is still possible to compute using such a seemingly unruly system because probabilities can be assigned
to each of the possible states of the system. Thus, a quantum system is probabilistic: there is a computable probability corresponding to the likelihood that any given state will be observed if the system is measured. Quantum computation is performed by increasing the probability of observing the correct state to a sufficiently high value so that the correct answer may be found with a reasonable amount of certainty. Quantum systems may also exhibit entanglement. A state is considered entangled if it cannot be decomposed into its more fundamental parts. In other words, two distinct elements of a system are entangled if one part cannot be described without taking the other part into consideration. In a quantum computer, it is possible for the probability of observing a given configuration of two qubits to depend on the probability of observing another possible configuration of those qubits, and it is impossible to describe the probability of observing one configuration without considering the other. An especially interesting quality of quantum entanglement is that elements of a quantum system may be entangled even when they are separated by considerable space. The exact physics of quantum entanglement remain elusive even to professionals in the field, but that has not stopped them from applying entanglement to quantum information theory.

IV. QUANTUM COMPUTATIONAL MODELS

A. Quantum logic computing (QLC) model

1. The qubit

A classical bit may be represented as a base-2 number that has either the value 1 or the value 0. Qubits are represented in a similar way: they take on the value 1 or 0, but only when they are measured and hence collapsed to a classical state. Unlike classical bits, a qubit normally assumes a superposition of the measurable values 1 and 0. The most convenient way to mathematically represent the state of a qubit is to use the orthonormal basis vectors $|1\rangle$ and $|0\rangle$, which are also known as eigenvectors or eigenstates, in a so-called Hilbert space $\mathcal{H}$. The state $|\psi\rangle$ of a qubit is represented by a linear combination, i.e. a superposition, of these two basis vectors:

$$|\psi\rangle = a_0 |0\rangle + a_1 |1\rangle$$

(1)

where $a_0$ is the complex scalar amplitude for measuring the $|0\rangle$ state while $a_1$ the amplitude for measuring the state $|1\rangle$. Amplitudes may be thought of as “quantum probabilities” in that they represent the chance that a given quantum state will be observed when the superposition is collapsed. The most fundamental difference between probabilities of states in classical probabilistic algorithms and amplitudes of states in quantum algorithms is that amplitudes are represented by complex numbers, while traditional probabilities are represented by real numbers. Complex numbers are required to fully describe the superposition of states and interference or entanglement inherent in quantum systems. It follows that, as the probabilities of a classical system must sum to 1 in order for the probabilities to form a complete probability distribution, the squares of the absolute values of the amplitudes of states in a quantum system must similarly add up to 1.

Remember that the contents of Dirac bras and kets are labels that describe the underlying vectors. $|0\rangle$ and $|1\rangle$ may be transformed into any two vectors that form an orthonormal basis in $\mathcal{H}$. The most common basis used in quantum computing is referred to as the computational basis:

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$ 

(2)

However, any other orthonormal basis could be used.
2. Quantum register

It is difficult to perform any interesting computation with only a single qubit. Like classical computers, quantum computers use quantum registers made up of multiple qubits. When collapsed, quantum registers are bit strings whose length determines the amount of information they can store. In superposition, each qubit in the register is in a superposition of $|1\rangle$ and $|0\rangle$, and consequently a register of $n$ qubits is in a superposition of all $2^n$ possible bit strings that could be represented using $n$ bits. The state space of a size-$n$ quantum register is a linear combination of $n$ basis vectors, each of length $2^n$:

$$|\psi_n\rangle = \sum_{i=0}^{2^n-1} a_i |i\rangle. \quad (3)$$

As with single qubits, the squared absolute value of the amplitude associated with a given bit string is the probability of observing that bit string upon collapsing the register to a classical state, and the squares of the absolute values of the amplitudes of all $2^n$ possible bit configurations of an $n$-bit register sum to unity:

$$\sum_{i=0}^{2^n-1} |a_i|^2 = 1. \quad (4)$$

Quantum registers are a relatively straightforward extension of quantum bits.

3. Entanglement

Quantum mechanics permits a peculiar feature between two or more physical systems: entanglement. When multiple physical systems are entangled or, equivalently speaking, a system of multiple subsystems is in an entangled state, these subsystems are no longer independent, but inseparable from each other. In other words, if one has two quantum registers entangled, the state assumed by one register will have strong implication on the state obtained by the other register.

This special property of entanglement is endowed by the innate mathematical structure that dictates the formulation of quantum mechanics. When two registers are present in a quantum computing system, the way to describe their states simultaneously is to pair up their individual Hilbert-space vectors through a mathematical procedure called tensor product. For example, if the first register assumes the state $|\alpha\rangle$ while the second the state $|\beta\rangle$, then their joint state is written as $|\alpha\rangle \otimes |\beta\rangle$. Since the two registers are part of one computing system, we can equivalently express the state of the system to be $|\alpha, \beta\rangle$. Except for the choice of notation, these two expressions describe the exact same thing. We have seen above that quantum mechanics postulates a linear superposition principle for quantum states. Suppose each of the two registers is one qubit, which individually can assume either the $|0\rangle$ state or the $|1\rangle$ state. Then, it is possible to form the joint state

$$|\psi\rangle = \frac{1}{\sqrt{2}} |0, 1\rangle + \frac{1}{\sqrt{2}} |1, 0\rangle \quad (5)$$

for the two quantum registers combined. This particular state, known as the Bell state, is an entangled state and implies that whenever the first register assumes the state $|0\rangle$, the second register the state $|1\rangle$; whenever the first register assumes the state $|1\rangle$, the second register the state...
TABLE I. Entanglement of one quantum register $|R_1\rangle$ with another register $|R_2\rangle$.

| $|0\rangle$ | $|2\rangle$ | $|7\rangle$ | $|3\rangle$ | $|5\rangle$ | $|6\rangle$ | $|1\rangle$ | $|c\rangle$ | $|a\rangle$ | $|d\rangle$ | $|f\rangle$ | $|b\rangle$ | $|a\rangle$ |

Notice that this state can no longer be separated into a tensor product, meaning we can no longer determine the linear combination of each register alone.

This entanglement property plays a crucial role in developing quantum algorithms. As we shall see in later sections, each of the quantum algorithms as known today rely more or less on the formation of entangled states between two or more registers. The importance and usefulness of entanglement in terms of computation can be understood from an analogy with associative memory. Like the example state given above, once two registers are entangled and thus inseparable, the states of one register would develop a definitive (not necessarily one-to-one) correspondence with the states of the other, i.e. 0 to 1 and 1 to 0. In general, the correspondence can be illustrated by the Table I.

The first row shows the eigenstates that comprise the superposed state for the first register, while the second shows that of the second register. Given the correspondence, we can immediately see that if the first register is in, say, state $|5\rangle$, the second register would definitely be in state $|e\rangle$. Conversely, if the second register is in, say, state $|d\rangle$, the first register would definitely adopt either state $|3\rangle$ or $|7\rangle$. Therefore, the joint states for a quantum computing system will act like an associative memory, in which if the data in one register is known, the data in the other register is instantaneously given. In classical computers, the association between two variables is universally useful since many functions demanded in a computer can be reduced to such association between values stored across multiple registers. For instance, multiplication is nothing but associating a value called “product” with one value called “multiplicand” and with another value called “multiplier.” In other words, given a multiplicand and a multiplier, the goal of multiplication is to find or search for a value that associates with these two numbers. A search in a database is even more straightforward, given the association. In classical computers, the association between two variables is universally useful since many functions demanded in a computer can be reduced to such association between values stored across multiple registers. For instance, multiplication is nothing but associating a value called “product” with one value called “multiplicand” and with another value called “multiplier.”

In a classical computer, the execution of an algorithm is carried out through a series of logical operations: NOT, AND, OR, etc. On the level of assembly code, each operation corresponds to a machine instruction. On the hardware level, each operation corresponds to a logical gate with input and output signal wires such that an algorithm can be conceptually converted into a multi-stage gate diagram. Since the input and output wires naturally correspond to hardware circuit wire nodes, the gate diagram decides on the circuit wiring necessary to carry out an algorithm execution. Analogously, when quantum computation was first devised, researchers had in mind a series of quantum logical gates that would be connected to constitute a complete quantum algorithm.
Such gates include the Controlled-NOT or CNOT gate, Hadamard gate, Pauli gates, Toffoli gate, etc. Though it is tempting to consider these gates as quantum counterparts of classical logic gates, the analogy only remains on a superficial level. Quantum algorithms are indeed decomposable into quantum logic gates and these gates can be graphically represented as diagrams with quantum states as inputs and outputs. Nevertheless, unlike classical logic gates, which pertain to the Boolean algebra as the foundation for carrying out the predicate calculus, quantum logic gates have no such algebraic foundation and cannot be configured into gates that carry logical meanings on a human intentional level. Rather, all quantum logic gates are essentially transformation matrices, which are square unitary matrices applicable on quantum states when quantum states are regarded as vectors.

For example, the NOT gate carries the logical meaning of negation for a given statement or predicate. If the truth value of that statement is stored in a Boolean variable, it flips the value. There is, however, no such negating gate for quantum states. Even though one can find a matrix that transforms the state $|0\rangle$ to $|1\rangle$ (this matrix is called Pauli-$X$ gate), the transformation loses the meaning of negation when the matrix is operated on quantum states such as Eqs. (1) and (5).

The most conspicuous difference between the two types of gates are their reversibility. Classical logic gates are irreversible, meaning the number of outputs are always less than the number of inputs, e.g. AND has one output and two inputs. One can only conduct the logical deduction along the forward direction, but not in reverse. From an information-theoretic perspective, classical logic operations are energy-consuming and entropy-reducing. On the other hand, since quantum logic gates are unitary matrices in essence, they are reversible as there always exists an inverse matrix for a given unitary matrix. No information would be lost in carrying out quantum logic operations and the number of inputs and outputs match, e.g. CNOT gate has two inputs and two outputs. The lack of correspondence between existing quantum logic gates and an algebra for predicate calculus lies at the heart of the difficulty in promoting quantum systems as general computation systems. We will discuss this problem in more detail in Sec. (VI).

B. Adiabatic quantum computing (AQC) model

1. Adiabatic process

Besides the QLC model explained above, there is another computing model called adiabatic quantum computing, which is popular amongst researchers of quantum computation and whose embodiment using superconducting circuits has been commercially available. This quantum computing model is based on a physical process called the quantum adiabatic process.

Usually, under the context of classical thermodynamics, an adiabatic process simply refers to a thermodynamic process that is occurring between two or more thermomechanical systems without exchanging heat, i.e. enthalpy. For such a process to happen, it is often necessary to keep the rates of system variations really slow; so slow that while all the systems experience quantitative changes (e.g. change of volume or pressure) during the process, heat is not transferred from one system into the other.

Quantum adiabatic process, though named after the classical adiabatic process, is not its equivalent quantum process. It retains the feature of slow rate of variation but has nothing to do with the prohibition of heat exchange between systems. The process is determined for one quantum system. By slow rate of variation, we mean that the variation of system energy, which we call Hamiltonian, over time (either increase or decrease) is so slow that it will not affect the linear superposition of the state of the quantum system.
2. Simulated annealing

First, a (potentially complicated) Hamiltonian is found whose ground state describes the solution to the problem of interest. Next, a system with a simple Hamiltonian is prepared and initialized to the ground state. Finally, the simple Hamiltonian is adiabatically evolved to the desired complicated Hamiltonian. By the adiabatic theorem, the system remains in the ground state; therefore at the end, the state of the system describes the solution to the problem. Adiabatic Quantum Computing has been shown to be polynomially equivalent to conventional quantum computing in the circuit model. The time complexity for an adiabatic algorithm is the time taken to complete the adiabatic evolution which is dependent on the gap in the energy eigenvalues (spectral gap) of the Hamiltonian. Specifically, if the system is to be kept in the ground state, the energy gap between the ground state and the first excited state of $H(t)$ provides an upper bound on the rate at which the Hamiltonian can be evolved at time $t$. When the spectral gap is small, the Hamiltonian has to be evolved slowly. AQC is a possible method to circumvent the problem of energy relaxation. Since the quantum system is in the ground state, interference with the outside world cannot make it move to a lower state. If the energy of the outside world (that is, the "temperature of the bath") is kept lower than the energy gap between the ground state and the next higher energy state, the system has a proportionally lower probability of transitioning into a higher energy state. Thus, the system can stay in a single system eigenstate as long as necessary. Quantum annealing is a physical process which attempts to implement such algorithms. Beginning with a set of initial states and a time-dependent Hamiltonian $H(t)$, you find the solutions to your problems encoded in the ground state of $H(t_F)$ at a final time $t_F$. Let the states evolve according to $H(t)$ and anticipate that, in the end, one of the candidate states ends up in the ground state of $H(t_F)$. If you have a qubit that starts in the ground state of $H(t_0)$ and $H(t_0)$ evolves into $H(t_F)$ slowly enough, then, according to the "adiabatic theorem," the qubit always remains in the ground state of $H(t)$ for all time $t$: one is guaranteed to determine the right answer which is encoded in the ground state of $H(t_F)$. This is what the AQC model has in mind. However, because it is experimentally difficult to start in the ground state, we sometimes try to implement AQC algorithms using quantum annealing, which unfortunately might not result in the ground state.

C. Quantum instruction set

To better understand either the QLC model or the AQC model, we introduce in this section a set of seven instructions. These seven instructions form a powerful quantum instruction set that is sufficient to perform significant quantum algorithms. We decompose three famous quantum algorithms in the next section into instructions from this set.

1. **INI $R$**
   
   • Description: initialize $R$ in quantum state $|R⟩$ to the zero state, i.e. let $|R⟩ \rightarrow |0⟩$.

2. **QFT $R$**
   
   • Description: apply a quantum Fourier transform on $R$
Given a state vector $|R\rangle = \sum_{j=0}^{N-1} a_j |j\rangle$ or in the matrix form

$$|R\rangle = \begin{bmatrix} a_0 \\ \vdots \\ a_{N-1} \end{bmatrix}$$

(6)

the resulting vector after the transformation is

$$\text{QFT}(|R\rangle) = \sum_{k=0}^{N-1} b_k |k\rangle \text{ where } b_k = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} a_j \exp \left\{ i \frac{2\pi j k}{N} \right\}$$

(7)

Example: assume we have a two-qubit register $|R\rangle = a_0 |00\rangle + a_1 |01\rangle + a_2 |10\rangle + a_3 |11\rangle$, which can be written as $|R\rangle = \sum_{j=0}^{3} a_j |j\rangle$ if we write the binary numbers in digits; then $b_k = \sum_{j=0}^{3} a_j \exp \{i2\pi jk/4\}$. To be specific,

$$b_0 = \frac{1}{2} [a_0 + a_1 + a_2 + a_3]$$

(8)

$$b_1 = \frac{1}{2} \left[ a_0 + a_1 e^{i\pi/2} + a_2 e^{i\pi} + a_3 e^{3i\pi/2} \right]$$

(9)

$$b_2 = \frac{1}{2} \left[ a_0 + a_1 e^{i\pi} + a_2 e^{2i\pi} + a_3 e^{3i\pi} \right]$$

(10)

$$b_3 = \frac{1}{2} \left[ a_0 + a_1 e^{3i\pi/2} + a_2 e^{3i\pi} + a_3 e^{9i\pi/2} \right]$$

(11)

In the special case where $a_0 = 1$ and $a_1 = a_2 = a_3 = 0$, we have $b_0 = b_1 = b_2 = b_3 = 1/2$.

3. REA $R$

- Description: read the state of the register $R$.
- The chance of finding the value $x$ stored in $R$ follows the probability distribution of each state $|x\rangle$ of $|R\rangle$.
- After reading, the state of $|R\rangle$ collapses into $|x\rangle$, i.e. $|R\rangle \rightarrow |x\rangle$ where all other possible states vanish.

4. ENT $R_1, R_2, M$

- Description: entangle the register $R_1$ to $R_2$ using transform mapping $M$.
- After the operation, each state $|j\rangle$ of $R_1$ is entangled with state $|M(j)\rangle$ of $R_2$. The state distribution of $R_1$ is not changed and the state distribution of $R_2$ is calculated based upon its association with $R_1$.
- Example: consider the mapping of modular exponentiation $M(j) = a^j \mod N$. If $N = 39$ while $a = 7$, then values of $M(j)$ are shown in table II.

i.e. $|R_1, R_2\rangle = a_0 |0, 1\rangle + a_1 |1, 7\rangle + a_2 |2, 10\rangle + \cdots$. 

i.e. $|R_1, R_2\rangle = a_0 |0, 1\rangle + a_1 |1, 7\rangle + a_2 |2, 10\rangle + \cdots$. 


| $j$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|-----|---|---|---|---|---|---|---|---|---|---|----|----|----|
| $M(j)$ | 1 | 7 | 10 | 31 | 22 | 37 | 25 | 19 | 16 | 34 | 4 | 28 | 1 |

TABLE II. Mapping of the modular exponentiation from $j$ to $M(j)$.

5. **DIF $R, N$**

- Description: given an integer $N$, diffuse the states with higher (concentrated) probabilities in the register $R$ into the states with lower (diluted) probabilities
- The operation is equivalent to the transformation under the symmetric matrix

\[
\begin{pmatrix}
\frac{2}{N} - 1 & \frac{2}{N} & \cdots & \frac{2}{N} \\
\frac{2}{N} & \frac{2}{N} - 1 & \cdots & \frac{2}{N} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{2}{N} & \frac{2}{N} & \cdots & \frac{2}{N} - 1
\end{pmatrix}
\]  

(12)

6. **PHA $R, \phi, n$**

- Description: perform a phase rotation of angle $\phi$ on $n$-th state of the register $R$
- The operation is equivalent to the transformation under the symmetric matrix

\[
\begin{pmatrix}
1 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 & 0 \\
0 & \cdots & 0 & 1 & 0 \\
0 & 0 & \cdots & 0 & 1
\end{pmatrix}
\]  

(13)

7. **ANN $h, J$**

- Description: use quantum annealing to minimize the dimensionless energy of the Ising model with parameter vector $h$ and parameter matrix $J$

\[
\mathcal{E}(s|h, J) = \sum_{i \in V(G)} h_i s_i + \sum_{(i,j) \in E(G)} J_{i,j} s_i s_j
\]  

(14)

where the quantum spin number $s_i$ is either 1 or -1. $i$ indexes the vertices $V(G)$ of the associated graph $G$ fixed by the device whereas $(i,j)$ indexes allowed pairwise interactions given by edges $E(G)$ of this graph. Both $h_i$ and $J_{i,j}$ are real-valued dimensionless coefficients.
V. QUANTUM ALGORITHMS

A. Shor’s integer factoring algorithm

1. Purpose of the algorithm

Shor’s approach to large integer factoring relies on the periodicity (a.k.a. multiplicative order) of a random-base integer modulus of the integer to be factored, i.e. if we denote the integer to be factored by \( N \) and pick a random base \( x \), then we want to find the period \( r \) such that

\[
x^r = 1 \mod N.
\] (15)

Once \( r \) is determined, \( \gcd(x^{r/2} - 1, N) \) will be a factor of \( N \) where the gcd can be found using an algorithm, such as Euclid’s method, that has much less complexity than those of the factoring algorithms. To see why, consider that the congruence relation above allows one to write \((x^{r/2} - 1)(x^{r/2} + 1) = mN\), where \( m \) denotes the quotient. Since \( x^{r/2} - 1 \) and \( x^{r/2} + 1 \) differ by 2 and cannot be both factors of \( m \), \( x^{r/2} - 1 \) must contain at least one factor of \( N \) and this factor can be determined by the gcd. Hence, the factoring problem is reduced to the problem of order finding.

Given \( x \) and \( N \), to find the order \( r \), one needs to perform the modular exponentiation \( x^j \mod N \) until \( j = r \) is found, whose time complexity Shor showed can be reduced exponentially at the expense of a logarithmic space complexity when the computation is carried out on a quantum state. Shor showed that the maximum value of \( j - 1 \) needed to cover the testing range is \( 2^n \) where \( N^2 < 2^n < 2N^2 \). Therefore, two quantum registers \( |R1\rangle \) and \( |R2\rangle \) are required, where the first one storing the values of \( j \) is entangled with the second one carrying the modular exponential value \( x^j \mod N \). When this is done, the job of finding the correct \( j \) in \( |R1\rangle \) for the period \( r \) is accomplished by applying a quantum Fourier transform on the joint state \( |R1, R2\rangle \).

In other words, a quantum computer executing Shor’s algorithm is actually carrying out two operations: i) entanglement with modular exponentiation and ii) quantum Fourier transform. We show the detailed steps below, assuming \( N = 9, x = 4 \), and \( j \) ranges from 0 to \( 2^7 - 1 \) as an example.

2. Quantum assembly code

1. INIT \( R1 \)
   - Effect: \( |R1\rangle = |0\rangle \).

2. INIT \( R2 \)
   - Effect: \( |R1\rangle = |0\rangle \).

3. QFT \( R1 \)
   - Effect: \( |R1\rangle = \frac{1}{\sqrt{2^7}} \sum_{j=0}^{2^7-1} |j\rangle \).

4. ENT \( R1, R2, x^j \mod N \)
   - Effect: \( R1 \) stays unchanged.
TABLE III. Transformation mapping from $j$ to $f(j)$ for entangling modular exponentiated states.

| $j$  | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | ... | 127 |
|------|---|---|---|---|---|---|---|---|---|-----|------|
| $f(j)$ | 1 | 4 | 7 | 4 | 7 | 1 | 4 | 7 | 1 | ... | 4 |

- $R2$ is entangled to each state of $R1$ given by the map $f(j) = x^j \mod N$, i.e.

$$|R1, R2\rangle = \frac{1}{\sqrt{2^7}} \sum_{j=0}^{2^7-1} |j, f(j)\rangle$$

(16)

where the pair $j$ and $f(j)$ are given by the table III.

5. REA $R2$

- Effect: $R2$ collapses into one $|f_0(j)\rangle$ state among all $\{|f(j)\rangle\}$ states
- $R1$ is reduced to a superposition state of all $|j\rangle$ eigenstates associated with $|f_0(j)\rangle$. Since in the example assumed, $r = 3$, we can write the superposition as

$$|R1\rangle = \frac{1}{\sqrt{43}} \sum_{m=0}^{42} |3m + c\rangle$$

(17)

given $c \in \{0, 1, 2\}$ as the residue in congruence with the period.

6. QFT $R1$

- Effect: the state $|R1\rangle$ of the first register becomes

$$|R1\rangle = \frac{1}{\sqrt{43 \times 2^7}} \sum_{m=0}^{42} \sum_{k=0}^{2^7-1} \exp \left\{ i2\pi \frac{k(3m + c)}{2^7} \right\} |k\rangle .$$

(18)

- The probability of finding $|R1\rangle$ in one particular $|k\rangle$ state among all $2^7$ states is

$$\mathcal{P}(k) = \frac{1}{43 \times 2^7} \left| \sum_{m=0}^{42} \exp \left\{ i2\pi \frac{k(3m + c)}{2^7} \right\} \right|^2$$

$$= \frac{1}{43 \times 2^7} \left| \sum_{m=0}^{42} \exp \left\{ i2\pi \frac{3mk}{2^7} \right\} \right|^2$$

$$= \frac{1}{43 \times 2^7} \left| \exp \left\{ i2\pi \frac{3 \cdot 43k}{2^7} \right\} \right|^2$$

(19)

where the second line is obtained because the imaginary $\exp \{i2\pi kc/2^7\}$ vanishes when the norm is taken and the third line completes the summation over the geometric series.

7. REA $R1$
• Effect: $|R1\rangle$ collapses into one of the $|k\rangle$ state out of $2^7$ states

• The likelihood of observing a particular $|k\rangle$ state is exactly the probability $P(k)$, which maximizes when $3k/2^7$ approaches unity.

The last readout yields the desired result of $k$ where

$$k \approx \frac{2^7}{3} \quad (20)$$

is about 43. Then the correct order $r = 3$ can be inferred from the $k$ value, i.e. $2^7/k \approx 3$.

B. Grover’s search algorithm

1. Purpose of the algorithm

Grover’s algorithm is a search algorithm, which means it gives the address in terms of a quantum “index” state from a given quantum “data” state. The address is stored in the first quantum register $|R1\rangle$ and the data in the second quantum register $|R2\rangle$. The key is to correlate the address with the data in one-one correspondence through entanglement. Then when the data is fed, the system collapses onto the joint $|R1, R2\rangle$ state that corresponds to the data, giving the correct address in $|R1\rangle$. The detailed process is the following.

2. Quantum assembly code

1. INI $R1$

2. INI $R2$

• Effect: to clear both registers to zero and prepare them for further operations, similar to what occurred in Shor’s algorithm

3. QFT $R1$

• Effect: to uniformly distribute probability in all possible indexing values, from 1 to the maximal $N$, for the first register as an indexing register.

4. ENT $R1, R2, M$

• Effect: to associate each indexing value in $R1$ to a unique value to be searched in $R2$. In other words, states of $R1$ are analogous to addresses while states of $R2$ are analogous to data.

• The elements $M_{mn}$ of the transformation matrix $M$ is used to specify the association.

5. for $i = 1 : \sqrt{N}$

(a) PHA $R2$, $\pi$, $x$

(b) DIF $R2$, $N$
(c) Effect: through the iteration of this loop, the quantum state for the value \(x\) to be searched will have an iteratively increasing probability weight while all other states will exhibit iteratively decreasing probability weights. After \(\sqrt{N}\) steps of iteration, the correct quantum state will have a probability asymptotically close to unity.

6. **REA R1**

- Effect: one can read out the corresponding index state in \(|R1\rangle\) that associates with the data state \(|x\rangle\).

### C. Deutsch-Jozsa’s heuristic algorithm

#### 1. Purpose of the algorithm

Deutsch-Jozsa (DZ) algorithm is arguably the first quantum algorithm, which predates Shor’s algorithm and Grover’s algorithm. Though the algorithm does not carry an imminent practicality for its function, it purposely illustrates the complexity advantage of a quantum algorithm over its classical counterparts.

The DZ algorithm accomplishes this illustration through a deliberately imposed mathematical function that transverses its entire input space or domain. Its output space or codomain is the rather simple \(\{0, 1\}\) space. That is, for a natural number \(N\), we assume the discrete valued function \(f(n) : \mathbb{Z}_{2N} \to \mathbb{Z}_2\). The goal is to determine which of the following two statement about \(f\) is true:

- either (i) \(f(n) \neq 0\) and \(f(n) \neq 1\);
- or (ii) \(\{f(0), f(1), \ldots, f(2N - 1)\}\) not contain exactly \(N\) zeros.

In short, the problem is to iterate through all \(2N\) input values and count the number of zeros obtained. No zeros, \(N\) zeros, or \(2N\) zeros constitute one case. All other scenarios constitute the complementary case. Using two registers \(|R1\rangle\) and \(|R2\rangle\), the quantum algorithm follows.

#### 2. Quantum assembly code

1. **INI R1**
2. **INI R2**
3. **QFT R1**

- Effect: the three steps above are identical in purpose to those of Shor’s and Grover’s algorithms.

4. **ENT R1, R2, \(M_f\)**

- Effect: the entanglement through transformation matrix \(M_f\) fulfills the mapping of \(f\) in \(|R2\rangle\) for each \(i\), i.e. \(|i, j\rangle \to |i, j + f(i)\rangle\).

- Since the second register \(|R2\rangle = |0\rangle\) (i.e. \(j = 0\)), the joint state of the two registers becomes

\[
|R1, R2\rangle = \frac{1}{\sqrt{2N}} \sum_{i=0}^{2N-1} |i, f(i)\rangle.
\]  

(21)
5. for $i = 0 : 2N - 1$
   
   (a) REA $R2$
   
   (b) PHA $R2, \pi f(i), f(i)$
   
   • Effect: read out the value of $f(i)$ for each $i$-th state in $R2$ and use the value to perform
     add a phase $e^{i\pi f(i)}$ on the state $|f(i)\rangle$.
   
   • Since $e^{i\pi} = -1$, this loop essentially inverts the sign for each state $|f(i)\rangle$ according to
     the value $f(i)$, i.e.
     \[
     \frac{1}{\sqrt{2N}} \sum_{i=0}^{2N-1} |i, f(i)\rangle \rightarrow \frac{1}{\sqrt{2N}} \sum_{i=0}^{2N-1} (-1)^{f(i)} |i, f(i)\rangle
     \] (22)

6. ENT $R1, R2, M_f$
   
   • Effect: Since $f(n)$ is a binary value function, one always has $f(i) + f(i) = 0$. Performing
     the entanglement operation again leads $|R2\rangle$ back to $|0\rangle$.
   
   • Essentially, we have
     \[
     |R1, R2\rangle = \frac{1}{\sqrt{2N}} \sum_{i=0}^{2N-1} (-1)^{f(i)} |i, f(i) + f(i)\rangle = \frac{1}{\sqrt{2N}} \sum_{i=0}^{2N-1} (-1)^{f(i)} |i\rangle \otimes |0\rangle.
     \] (23)

After executing the instructions above, the first register contains all the information one needs to
determine the truth value of statements (i) and (ii). To see this, consider the inner product of $|R1\rangle$
with itself,
\[
|\langle R1|R1 \rangle | = \frac{1}{2N} \sum_{i=0}^{2N-1} (-1)^{f(i)}
\] (24)
which is 0 when statement (i) is true and is 1 when statement (ii) is true.

VI. CRITIQUE OF CURRENT ARCHITECTURES

In the above sections, we have formally dissembled Shor’s algorithm into a set of high-level
instructions, each of which can be further reduced to a set of atomic instructions implementable
on a superconducting-circuit based quantum computer. It is not hard to see that even equipped
with these instructions, one cannot implement an arbitrary quantum algorithm other than integer
factoring at ease. This is because the given instruction set lacks a sense of propositional logic
implied in every classical computer architecture.

Specifically, the sense of truth or falsity, which has always been epitomically represented by 1 and
0 in Boolean logic, is not supplied by quantum operations. The latter are really transformations
of the state vector representing a qubit and have no intrinsic logical implications. Without ascertaining
true or false as a terminal value in a conditional statement, recursions cannot be implemented. Since
recursive programming is pivotal to classical computer programs, the lack of it deprives the sense
of automation when one talks about quantum computing. In fact, many people have suspected
that the quantum computer is at most an ASIC-type device suitable only for a specific task, e.g.
integer factoring, and cannot be regarded as a general-purpose computer. The reason behind can be attributed to this privation of logic and recursion. Without the support of logic as the skeleton of quantum algorithms, the specific algorithms explained in the sections above cannot be dissembled into subroutines and then reassembled to become new algorithms of the same class. The critiques here can be illustrated through a comparison of the existing hardware for classical computers and quantum computers.

A. The classical computer

All modern computers are based on devices fabricated on silicon. The most basic device that can be fabricated is the PN-junction also known as the diode (P and N, respectively, stands for the positively and negatively doped silicon). It is, circuit-wise, a voltage-dependent current source. Putting two such junctions back-to-back, one obtains a sandwiched Source-Bulk-Drain structure. When a Gate-terminal over an insulating oxide layer is deposited on top of this sandwiched structure, one can control arbitrarily whether the current runs from Source to Drain or not, thus a field-effect transistor is essentially accomplished. The transistor is, thus, a voltage controlled switch and is the most fundamental device of a classical computer.

When several such transistors are so wired that their Drains are commonly connected to a suspended node, a binary memory device is essentially accomplished. The Boolean logical 1 is obtained when an electric current fills the suspended Drain node up to certain voltage prescribed by the voltage-current relationship of the PN-junction between Bulk and Drain. Adversely, the Boolean logical 0 is obtained when the current reverses its direction to deplete the charges at Drain and sets its voltage equal to ground.

Once the logical values 1 and 0 are set equivalent to physical high and low voltages, elementary logical operations can be translated into serial and parallel connections of multiple transistors together. That is, logical gates are formed by feeding input signals (high or low voltages) to the Gate terminals of these transistors to determine whether currents would replete or deplete a drain node such that a logical 1 (high voltage) or 0 (low voltage) would be obtained as the output. For example, the classical logic NOT is translated into a serial connection of two transistors where the input is connected to the gates of both transistors whose sources are each in contact with a high-voltage reservoir and the ground. The output is the terminal of the two drains wired together.

If four transistors are given with two of the gates connected to one input and the other two to another input, a NAND gate is made. Connecting the NAND gate with the NOT above (4 + 2 = 6 transistors), one finds the logical AND; connecting three NAND gates to two NOT gates (12+4 = 16 transistors), one finds the logical XOR (exclusive OR). In other words, with a few transistors, we have implemented an adder with XOR giving the sum bit and AND giving the carry-over bit of the result. Throughout the computation, the logical values and, hence, the numeric values at both the input and the output ends are designated by a set of high and low voltages. The media that sustain the numeric information are therefore the suspended circuit nodes that carry these electric potentials and are not the transistors comprising the PN-junctions.

The wired circuits of the transistors make the instructions of a classical computer well-defined, in the sense that each part of an instruction corresponds uniquely to a specific part of a specifically wired circuit. Consider, for example, a typical instruction: ADD InA, InB, Out. The three letters ADD, signifying an addition, recall the specific adder circuit consisting of 22 transistors, as explained above, for each bit of InA, InB, and Out. Correspondingly, each bit of InA or InB (Out) would become a specific node at the input (output) end of the 22-transistor circuit. The correspondence between the software and the hardware is clear. But, as we will discuss below, the quantum
computer has not yet enjoyed this clear distinction, at least for the implemented prototypes thus far.

The unique correspondence between a logical circuit and an instruction not only facilitates the implementation of an extensive set of algebraic instructions, but also makes logical determinations and, hence, branching instructions for recursive programs possible. Consider, for instance, a typical recursive function that ends its iteration when a variable reaches a given integer value. If that variable is kept count in a register $R_1$, then the recursion loop always contains a statement that calls the adder to increment $R_1$ by one while keeping track of its value to determine if the loop breaking condition is satisfied during every iteration.

Therefore, since Kleene constructed the natural number system on top of the first-order logic, every computer relies on integers not only for direct calculation, but also for counting the steps in executing algorithms. Offloading the job of counting to a register is what endows the classical computer with the sense of automation. In the picture of von-Neumann architecture, a classical computer is a finite-state automaton that executes logical operations and is equipped with a dedicated “address” register for keeping track of the machine state.

### B. The quantum computer

In contrast, the quantum computer, at its current state of development, lacks such a device for keeping track of the machine state. As we have dissembled the existing quantum algorithms in the sections above, they are all equivalent to one-directional sequences of quantum state transformations without any branching statements or recursions.

Therefore, they do not rely on an address register to keep track of the quantum machine state and hence they do not yet possess the classical sense of automata. One can sense that quantum devices which execute these algorithms resemble a desktop calculator more than a programmable computer, albeit one that performs state transformation rather than arithmetic operations. It is for this reason that many computer scientists do not regard quantum computers, architecturally speaking, as general-purpose computers.

In terms of hardware, the current quantum computer design does not correspond one-to-one to a classical computer. The most basic unit of a quantum computer is, as already explained, a qubit. Unlike a transistor, which is a gating or switching device, a qubit is rather a storage or memory device. In other words, while we need several transistors wired in a particular fashion to form a one-bit device that can store one classical bit of information, the qubit is an indivisible quantum device that can function as the memory unit for one qubit of information. So far, memory is the only function a qubit can perform and, further unlike transistors, qubits and even several qubits wired together cannot function as quantum logical gates.

To carry out instructions or logical operations on the qubits, one relies on external signals such as laser pulses or microwave pulses to perform the relevant transformations. The sources of these external signals, such as a laser diode or a microwave generator, can be regarded as the controllers to the qubits. They are classical devices and serve as the human-machine interfaces to current quantum computers. It is these external sources which convert the quantum instructions depicted in previous sections into specific sequences of pulses, which in turn transform the qubit to a desired state.

We can therefore notice that:

1. programming is conducted on classical devices instead of the qubits;

...
2. the qubits do not actively perform the quantum instructions but are passively operated by the instructions fed externally; and
3. though each qubit is described by a quantum state (i.e. the linear superposition of $|0\rangle$ and $|1\rangle$), the combined state of all the qubits in a quantum computer does not constitute a machine state of an automaton.

The reason for (iii) is that during the execution of an algorithm, the states of the qubits are not read by the external sources to influence the subsequent instructions to be fed. If the quantum computer possessed a pointer similar to a classical computer, this quantum pointer would only continue to the next instruction and never skip ahead or jump backward.

Although we discussed quantum logic gates, such as Hadamard gates and Toffoli gates, they cannot be seen as extensions of classical logic gates since their compositions do not exhaust all the possible logical combinations in the quantum state space while classical logic gates (NAND or NOR, as explained above) do. More importantly, when they are performed on qubits, the results are not true or false logic termination values, but still states of qubits. Thus, programs, at least in the classical sense, cannot rely on them to determine a branching condition or the exiting condition of a recursion. The lack of a direct mapping from superposition states in a Hilbert space into the ring of integers or the Boolean algebra makes programmers who are accustomed to coding classical algorithms unable to conceive of quantum algorithms in a natural way.

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