Non-abelian Extensions of Lie 2-algebras

Shaohan Chen
School of Science, South China University of Technology, Guangzhou 510641, Guangdong, China
e-mail: cshjiayou@126.com

Yunhe Sheng
Department of Mathematics, Jilin University, Changchun 130012, Jilin, China
e-mail: shengyh@jlu.edu.cn

Zhujun Zheng
School of Science, South China University of Technology, Guangzhou 510641, Guangdong, China
e-mail: zhengzj@scut.edu.cn

Abstract

In this paper, we introduce the notion of derivations of Lie 2-algebras and construct the associated derivation Lie 3-algebra. We prove that isomorphism classes of non-abelian extensions of Lie 2-algebras are classified by equivalence classes of morphisms from a Lie 2-algebra to a derivation Lie 3-algebra.

1 Introduction

Eilenberg and MacLane [6] developed a theory of non-abelian extensions of abstract groups in the 1940s, leading to the low dimensional non-abelian group cohomology. Then there are a lot of analogous results for Lie algebras [1, 8, 7, 16]. Nonabelian extensions of Lie algebras can be described by some linear maps regarded as derivations of Lie algebras. This result was generalized to the case of super Lie algebras in [2], and to the case of Lie algebroids in [3, 11, 15].

Lie 2-algebras are the categorification of Lie algebras [3]. In a Lie 2-algebra, the Jacobi identity is replaced by a natural isomorphism, which satisfies its own coherence law, called the Jacobiator identity. The 2-category of Lie 2-algebras is equivalent to the 2-category of 2-term \( L_\infty \)-algebras, so people also view a 2-term \( L_\infty \)-algebra as a Lie 2-algebra. Associated with any Lie algebra \( \mathfrak{g} \), \( \mathfrak{g} \to \text{Der}(\mathfrak{g}) \) is a strict Lie 2-algebra, where \( \text{Der}(\mathfrak{g}) \) is the Lie algebra of the derivations of \( \mathfrak{g} \). Any \( ^{\ast} \text{Keyword: derivations of Lie 2-algebras, derivation Lie 3-algebra, non-abelian extensions} \)

\( ^{\ast} \text{MSC: 17B99, 53D17.} \)

*The second author is supported by NSF of China (11026046,11101179), Doctoral Fund. of MEC (20100061120096) and "the Fundamental Research Funds for the Central Universities" (200903294). The third author is supported by NSF of China (10971071).
non-abelian extension of a Lie algebra $\mathfrak{m}$ by $\mathfrak{k}$ is described by a morphism from $\mathfrak{m}$ (a trivial Lie 2-algebra) to the Lie 2-algebra $\mathfrak{k} \to \text{Der}(\mathfrak{k})$. Semidirect product Lie 2-algebras and the integration of string type Lie 2-algebras were studied in [15].

In this paper, we study the non-abelian extensions of Lie 2-algebras. To do that, first we develop the theory of derivations of Lie 2-algebras. In general, for an $L_\infty$-algebra $L$, degree $p$ derivations of $L$ is defined using coderivations of the coalgebra $\wedge^p L$ [17]. Concentrate on the case of Lie 2-algebras, by truncation, we construct a strict Lie 2-algebra $\text{Der}(g)$ associated with derivations, which plays important role when we consider nonabelian extensions of Lie 2-algebras. Motivated by the nonabelian extension theory of Lie algebras, we construct the associated strict Lie 3-algebra $\text{DER}(g)$, which we call the derivation Lie 3-algebra. Any non-abelian extension of a Lie 2-algebra $g$ by a Lie 2-algebra $\mathfrak{h}$ gives rise to a morphism from $g$ to the derivation Lie 3-algebra $\text{DER}(\mathfrak{h})$. Furthermore, the isomorphism classes of extensions are classified by the equivalence classes of such morphisms.

The paper is organized as follows. In Section 2, we recall some basic definitions regarding Lie 2-algebras and strict Lie 3-algebras. In Section 3, we give the definition of derivations of degree 0 of Lie 2-algebras using explicit formulas. Then by truncation, we obtain the strict Lie 2-algebra $\text{Der}(g)$ associated with derivations. At last, we construct the associated strict Lie 3-algebra $\text{DER}(g)$, which we call the derivation Lie 3-algebra. In Section 4, we prove that by choosing a splitting, any non-abelian extension of the Lie 2-algebra $g$ by $\mathfrak{h}$ gives rise to a morphism from $g$ to the derivation Lie 3-algebra $\text{DER}(\mathfrak{h})$ and different splittings give rise to equivalent morphisms. Moreover, there is a one-to-one correspondence between the isomorphism classes of non-abelian extensions and the equivalence classes of morphisms.

2 Preliminaries

In this section, we recall some basic concepts and facts about Lie 2-algebras and strict Lie 3-algebras, and see [3, 9, 11] for more details. An $L_\infty$-algebra is a graded vector space $L = L_0 \oplus L_1 \oplus \cdots$ equipped with a system $\{l_k | 1 \leq k < \infty\}$ of linear maps $l_k : \wedge^k L \to L$ of degree $\text{deg}(l_k) = k - 2$, where the exterior powers are interpreted in the graded sense and the following relation with Koszul sign “Ksg” is satisfied for all $n \geq 0$:

$$
\sum_{i+j=n+1} (-1)^{(j-1)} \sigma \text{sgn} (\sigma) \text{Ksg} (\sigma) l_j \left( l_i (x_{\sigma(1)}, \cdots, x_{\sigma(i)}), x_{\sigma(i+1)}, \cdots, x_{\sigma(n)} \right) = 0, \quad (1)
$$

where the summation is taken over all $(i, n-i)$-unshuffles with $i \geq 1$. A Lie 2-algebra is a 2-term $L_\infty$-algebra. More precisely, we have

**Definition 2.1.** [3] A Lie 2-algebra $L$ is 2-term complex of vector spaces $L : L_1 \xrightarrow{d} L_0$ with linear maps $\{l_k : \wedge^k L \to L, k = 2, 3\}$ of degree $\text{deg}(l_k) = k - 2$ satisfying the following equalities

- $dl_2(x, a) = l_2(x, da),$
- $l_2(da, b) = l_2(a, db),$
- $l_2(x, l_2(y, z)) + l_2(y, l_2(x, z)) + l_2(z, l_2(x, y)) = dl_3(x, y, z),$
- $l_2(x, l_2(y, a)) + l_2(y, l_2(a, x)) + l_2(a, l_2(x, y)) = l_3(x, y, da),$
- $l_3(l_2(x, y), z, t) + \text{c.p.} = l_2(l_3(x, y, z), t) + \text{c.p.},$

$2$
Theorem 2.2. Let $L = L_0 \oplus L_1 \oplus L_2$ be a Lie 2-algebra. A Lie 2-algebra morphism $f : L \to L'$ consists of:

- two linear maps $f_0 : L_0 \to L'_0$ and $f_1 : L_1 \to L'_1$,
- one skew-symmetric bilinear map $f_2 : L_0 \times L_0 \to L'_1$, such that the following equalities hold for all $x, y, z \in L_0, a \in L_1$,
  
  - $d' \circ f_1 = f_0 \circ d$,
  - $f_0 l_2(x, y) - l'_2(f_0(x), f_0(y)) = d' f_2(x, y)$,
  - $f_1 l_2(x, a) - l'_2(f_0(x), f_1(a)) = f_2(x, da)$,
  - $l'_2(f_0(x), f_2(y, z)) + \text{c.p.} + l'_2(f_0(y), f_0(z)) = f_2(l_2(x, y, z) + \text{c.p.} + f_1(l_3(x, y, z))$, where c.p. means cyclic permutation. If $f_2 = 0$, the morphism $f$ is called a strict morphism.

Definition 2.4. A strict Lie 3-algebra is a graded vector space $L = L_0 \oplus L_1 \oplus L_2$ with linear maps $\{l_i : \wedge^i L \to L, i = 1, 2\}$ of degree $\text{deg}(l_i) = i - 2$, satisfying the following equalities for any $x, y, z \in L$:

(a) $l_1^2 = 0$,

(b) $l_1 l_2(x, y) = l_2(l_1(x), y) + (-1)^{|x||y|} l_2(x, l_1(y))$,

(c) $(-1)^{|x||z|} l_2(l_2(x, y), z) + (-1)^{|x||y|} l_2(l_2(y, z), x) + (-1)^{|y||z|} l_2(l_2(z, x), y) = 0$.

Definition 2.5. Let $(L, d, l_2, l_3)$ be a Lie 2-algebra and $(L', d', l'_2)$ be a strict Lie 3-algebra. A morphism $f$ from $L$ to $L'$ consists of:

- two linear maps $f_0 : L_0 \to L'_0$ and $f_1 : L_1 \to L'_1$,
- two skew-symmetric bilinear maps $f_2 : L_0 \times L_0 \to L'_1$ and $f_3 : L_0 \times L_1 \to L'_2$. 

for any $x, y, z, t \in L_0, a, b \in L_1$. If $l_3 = 0$, $L$ is called a strict Lie 2-algebra.

Sometimes we use $[,]_L$ instead of $l_2$ and we denote a Lie 2-algebra by $(L, d, l_2, l_3)$.

Let $V : V_1 \to V_0$ be a 2-term complex of vector spaces, and we can form a new 2-term complex of vector spaces $\text{End}(V) : \text{End}^1(V) \to \text{End}^0(V)$ by defining $\delta(D) = d \circ D + D \circ d$ for any $D \in \text{End}^1(V)$, where $\text{End}^1(V) = \text{End}(V_0, V_1)$ and

$$\text{End}^0(V) = \{X = (X_0, X_1) \in \text{End}(V_0, V_0) \oplus \text{End}(V_1, V_1) | X_0 \circ d = d \circ X_1 \}.$$

Define $l_2 : \wedge^2 \text{End}(V) \to \text{End}(V)$ by setting:

$$\begin{align*}
l_2(X, Y) &= [X, Y]_C, \\
l_2(X, D) &= [X, D]_C, \\
l_2(D, D') &= 0,
\end{align*}$$

where $[,]_C$ is the graded commutator, for any $X, Y \in \text{End}^0(V)$ and $D, D' \in \text{End}^1(V)$.

Theorem 2.2. With the above notations, $(\text{End}(V), \delta, l_2)$ is a strict Lie 2-algebra.
one skew-symmetric trilinear map $f_3 : L_0 \times L_0 \times L_0 \rightarrow L_2'$, such that for all $x, y, z, t \in L_0$, $a, b \in L_1$, we have

$$d' \circ f_1 = f_0 \circ d,$$

$$f_{0l_2}(x, y) - l_2'(f_0(x), f_0(y)) = d'f_2^0(x, y),$$

$$f_{1l_2}(x, a) - l_2'(f_0(x), f_1(a)) = f_2^0(x, d(a)) + d'f_1^1(x, a),$$

$$l_2'(f_1(a), f_1(b)) = f_2^1(a, d(b)) - f_2^1(d(a), b),$$

$$f_2^1(l_2(x, y), z) + c.p. + f_1(l_3(x, y, z)) = l_2'(f_0(x), f_2^0(y, z)) + c.p. + d'f_3(x, y, z),$$

$$f_2^1(l_2(x, y), a) + c.p. + f_3(x, y, da) = l_2'(f_0(x), f_2^1(y, a)) + l_2'(f_0(y), f_2^1(a, x)) - l_2'(f_1(a), f_2^0(x, y)),$$

and

$$f_2^1(x, l_3(y, z, t)) + l_2'(f_0(x), f_3(y, z, t)) + c.p. = f_3(l_2(x, y), z, t) + c.p. + (l_2'(f_2^0(x, y), f_2^0(z, t)) + c.p.).$$

## 3 Derivations of Lie 2-algebras

For a graded vector space $L$, there is a natural coalgebra structure on $\wedge s(L)$, where $s(L)$ is the graded vector space shifted by 1. Another equivalent definition of an $L_\infty$ structure on $L$ is a coderivation $\partial$ of degree $-1$ satisfying $\partial^2 = 0$ on the coalgebra $\wedge s(L)$. See [3, 10] for more details.

**Definition 3.1.** [17] A derivation of degree $p \geq 1$ of an $L_\infty$-algebra $L$ is a coderivation $f \in \text{Coder}^p(\wedge s(L))$ of degree $p$ of the coalgebra $\wedge s(L)$. A derivation of degree $0$ of an $L_\infty$-algebra $L$ is a coderivation of degree $0$ of the coalgebra $\wedge s(L)$, which is commutative with $\partial$.

Denote by $\overline{\text{Der}^{p\geq r}}(L)$ the set of degree $p$ derivations of $L$ and $\text{Der}^0(L)$ the set of degree $0$ derivations of $L$, then we have a differential graded Lie algebra $[17]$

$$\overline{\text{Der}^0(L)} \rightarrow \overline{\text{Der}^r(L)} \rightarrow \cdots \rightarrow \overline{\text{Der}^0(L)} \rightarrow 0.$$

Concentrate on the case of Lie 2-algebras, we can give the definition of derivations of degree 0 of Lie 2-algebras using explicit formulas as follows.

**Definition 3.2.** Let $(\mathfrak{g} : \mathfrak{g}_1 \xrightarrow{d_3} \mathfrak{g}_0, [\cdot, \cdot]_\mathfrak{g}, l^0_{\mathfrak{g}})$ be a Lie 2-algebra. A derivation of degree 0 of $\mathfrak{g}$ consists of

- an element $X \in \text{End}^0_\mathfrak{g}(\mathfrak{g})$,

- a skew-symmetric bilinear map $l_X : \mathfrak{g}_0 \times \mathfrak{g}_0 \rightarrow \mathfrak{g}_1$, such that for all $x, y, z \in \mathfrak{g}_0$ and $a \in \mathfrak{g}_1$,

  - $X[x, y]_\mathfrak{g} - [Xx, y]_\mathfrak{g} - [x, Xy]_\mathfrak{g} = d_3l_X(x, y),$

  - $X[x, a]_\mathfrak{g} - [Xx, a]_\mathfrak{g} - [x, Xa]_\mathfrak{g} = l_X(x, d_3a),$

  - $l_X(x, [y, z]_\mathfrak{g}) + [x, l_X(y, z)]_\mathfrak{g} + l^0_3(Xx, y, z) + l^0_2(x, Xy, z) + l^0_3(x, y, Xz) = Xl^0_3(x, y, z) + l_X([x, y]_\mathfrak{g}, z) + [Xx, y, z]_\mathfrak{g} + [y, l_X(x, z)]_\mathfrak{g}.$

We denote a derivation of degree 0 of $\mathfrak{g}$ by $(X, l_X)$ and the set of derivations of degree 0 of $\mathfrak{g}$ by $\text{Der}^0(\mathfrak{g})$. 

4
Remark 3.3. In a strict case, derivations of Lie 2-algebras can be realized as normalizers of the corresponding Dirac structures in omni-Lie 2-algebras (see Section 4 in [14] for more details).

Example 3.4. For any $x \in g_0$, define $\text{ad}_x \in \text{End}_1^0(g)$ by $\text{ad}_x(y + a) = [x, y + a]_g$ for any $y \in g_0$ and $a \in g_1$, then $(\text{ad}_x, l_{\text{ad}_x} = l_1^0(x, \cdot, \cdot)) \in \text{Der}^0(g)$, which we call an inner derivation.

For any $(X, L_x), (Y, L_y) \in \text{Der}^0(g)$, and $x, y \in g_0$, we have

$$\begin{align*}
[X, Y]c([x, y]_g) - [[X, Y]c(x), y]_g - [x, [X, Y]c(y)]_g \\
= X(Y[x, y]_g) - Y(X[x, y]_g) - [X(Yx) - Y(Xx), y]_g - [x, X(Yy) - Y(Xx)]_g \\
= X([Yx, y]_g + [x, Yy]_g + d_Y^L(x, y)) - Y([Xx, y]_g + [x, Xy]_g + d_X^L(x, y)) \\
- [X(Yx) - Y(Xx), y]_g - [x, X(Yy) - Y(Xx)]_g \\
= [X(Yx), y]_g + [Yx, Yy]_g + d^L_X(x, y) + [x, X(Yy)]_g + d^L_Y(x, y) \\
+ Xd^L_Y(x, y) - Yd^L_X(x, y) - [Y(Xx), y]_g + [x, X(Yy)]_g + [x, X(Yx)]_g \\
= d_y\left(l_X(Yx, y) + l_X(x, Yy) - l_Y(Xx, y) - l_Y(x, Xy) + Xl_Y(x, y) - Yl_X(x, y)\right).
\end{align*}$$

It is straightforward to see that

$$l_{[X,Y]c}(x, y) \triangleq l_X(Yx, y) + l_X(x, Yy) - l_Y(Xx, y) - l_Y(x, Xy) + Xl_Y(x, y) - Yl_X(x, y) \quad (2)$$

satisfies Condition (c) in Definition 3.2. Thus, there is a well-defined bilinear skew-symmetric map $\lbrack \cdot, \cdot \rbrack_{\text{Der}} : \wedge^2\text{Der}^0(g) \rightarrow \text{Der}^0(g) :$

$$\lbrack [X, Y]c, (Y, l_Y) \rbrack_{\text{Der}} \triangleq \lbrack [X, Y]c, l_{[X,Y]c} \rbrack$$

(3)

For any $(X, l_X), (Y, l_Y), (Z, l_Z) \in \text{Der}^0(g)$, it is straightforward to deduce that

$$l_{[X,Y,Z]c} + l_{[Y,Z,X]c} + l_{[Z,X,Y]c} = 0.$$

Thus, we have

Lemma 3.5. With the above notations, $(\text{Der}^0(g), \lbrack \cdot, \cdot \rbrack_{\text{Der}})$ is a Lie algebra.

By Definition 3.1, the degree 1-derivation $\overline{\text{Der}}^1(g) = \text{Coder}^1(\wedge\text{L}(L))$ is given by

$$\overline{\text{Der}}^1(g) = \text{End}^1(g) \oplus \text{End}(g_0, \wedge^2 g_0) \oplus \text{End}(g_1, \wedge^3 g_0).$$

However, we find out that a smaller, thus simpler, sub-Lie 2-algebra of the above (see Theorem 5.7) is enough for the application of non-abelian extensions in our setting. Thus by truncation, we obtain a smaller Lie 2-algebra, which plays essential role when we consider extensions of Lie 2-algebras in Section 4. To do that, first we consider the complex $\text{End}^1(g) \overset{\delta}{\rightarrow} \text{End}^0(g) \oplus \text{Hom}(\wedge^2 g_0, g_1)$, where $\delta$ is given by

$$\overline{\delta}(D) = (\delta(D), l_{\delta(D)}),$$

(4)

in which $l_{\delta(D)} : \wedge^2 g_0 \rightarrow g_1$ is given by

$$l_{\delta(D)}(x, y) = D[x, y]_g - [x, D(y)]_g - [D(x), y]_g.$$
**Theorem 3.7.** Thus, $\delta(D)$ is a derivation, i.e. $\delta(D) \in \text{Der}^0(g)$. Thus, we have a well-defined complex

\[
\text{Der}(g) : \text{Der}^1(g) \cong \text{End}^1(g) \overset{\delta}{\rightarrow} \text{Der}^0(g).
\]

**Proof.** By (8), and the fact that $\delta(D)[x,y]_g = d_g D[x,y]_g$ and $\delta(D)[x,a]_g = D[x,d_g a]_g$, we have the following two equalities obviously:

\[
\delta(D)[x,y]_g = [\delta(D)(x),y]_g + [x,\delta(D)(y)]_g + d_g l(D)(x,y),
\]

\[
\delta(D)[x,a]_g = [\delta(D)(x),a]_g + [x,\delta(D)(a)]_g + l(D)(x,d_g a).
\]

By straightforward computations, we can obtain Condition (c) in Definition 3.2, i.e. the following equality:

\[
\begin{align*}
\langle x,y,z \rangle + l(D)(x,y,z) &+ l(D)(x,y,z) + l(D)(x,y,z) \\
&= \delta(D)[x,y]_g + l(D)(x,y)_g + l(D)(x,y)_g + l(D)(x,y) + [y,l(D)(x,z)]_g.
\end{align*}
\]

Thus, $\delta(D)$ is a derivation. $\blacksquare$

Define a bilinear skew-symmetric map $[,]_{\text{Der}} : \text{Der}^0(g) \wedge \text{Der}^1(g) \rightarrow \text{Der}^1(g)$ by:

\[
[X,l_X],D]_{\text{Der}} \triangleq [X,D]_C.
\]

**Theorem 3.7.** $(\text{Der}(g),\delta,[,]_{\text{Der}})$ is a strict Lie 2-algebra, when the complex $\text{Der}(g)$ is given by (6), the differential $\delta$ is given by (1) and the bracket $[,]_{\text{Der}}$ is given by (8) and (7).

**Proof.** By Theorem 2.2 and Lemma 3.5 we only need to prove that $\delta$ is a graded derivation with respect to the bracket operation $[,]_{\text{Der}}$, i.e.

\[
\begin{align*}
\delta([X,l_X],D]_{\text{Der}} &= [(X,l_X),\delta(D)]_{\text{Der}}, \\
[\delta(D),E]_{\text{Der}} &= [D,\delta(E)]_{\text{Der}},
\end{align*}
\]

for any $(X,l_X) \in \text{Der}^0(g)$ and $D,E \in \text{Der}^1(g)$. The left hand side of (8) is equal to

\[
\delta([X,l_X],D]_{\text{Der}} = \delta([X,D]_C) = (\delta([X,D]_C),\delta([X,D]_C)),
\]

where $l_{\delta([X,D]_C)}$ is given by

\[
\begin{align*}
l_{\delta([X,D]_C)}(x,y) &= [X,D]_C(x,y) - [X,D]_C(x,y) - [X,D]_C(x,y) \\
&= X \circ D[x,y]_g - D \circ X[x,y]_g - [x,X \circ D(y) - D \circ X(y)]_g - [X \circ D(x) - D \circ X(x)]_g \\
&= X([X,D]_C(x,y) + [D(x),y]_g + l(D)(x,y) - D[X(x),y]_g + D(x,X) - D \circ X(x, y)]_g \\
&- [x,X \circ D(y) - D \circ X(y)]_g - [X \circ D(x) - D \circ X(x, y)]_g \\
&= [X(x),D(y)]_g + l(X,D)(D(x),y) + l(X,D)(D(x),y) + l(D)(x,y) + X[l(D)(x),y]_g \\
&- D[X(x),y]_g - D[X(x),y]_g + D \circ X(y)]_g + D \circ X(x, y)]_g.
\end{align*}
\]

By (2), the right hand side of (8) is equal to

\[
[(X,l_X),\delta(D)]_{\text{Der}} = [(X,l_X),\delta(D),l_{\delta([X,D]_C)}]_{\text{Der}} = ([X,\delta(D)]_C,l_{[X,\delta(D)]_C}).
\]
Theorem 3.8. With the above notations, whose degree 0 part $\text{DER}_0$ we call the derivation Lie 3-algebra of $\mathfrak{g}$ for any $(X, \delta \mathfrak{g})$. Motivated by this, if we consider extensions of $\text{Der}(\mathfrak{g})$ to the strict Lie 2-algebra. Thus, we can see that only considering the derivation Lie algebra $\text{Der}(\mathfrak{g})$ can be described by a morphism from the Lie algebra $\mathfrak{g}$ to the strict Lie 2-algebra $\mathfrak{k} \longrightarrow \mathfrak{g} \longrightarrow \mathfrak{g} \longrightarrow 0$.

In the classical case of Lie algebras, a nonabelian extension

$$0 \longrightarrow \mathfrak{k} \longrightarrow \mathfrak{g} \longrightarrow \mathfrak{g} \longrightarrow 0$$

can be described by a morphism from the Lie algebra $\mathfrak{g}$ to the strict Lie 2-algebra $\mathfrak{k} \longrightarrow \text{Der}(\mathfrak{k})$ by choosing a splitting. Thus, we can see that only considering the derivation Lie algebra $\text{Der}(\mathfrak{k})$ is not enough, we have to extend it to a Lie 2-algebra. Motivated by this, if we consider extensions of Lie 2-algebras, we have to extend the strict Lie 2-algebra $\text{Der}(\mathfrak{g})$ given in Theorem 3.7 to a strict Lie 3-algebra.

Associated with the 2-term complex $\text{Der}(\mathfrak{g})$, we can form a 3-term complex of vector spaces

$$\text{DER}(\mathfrak{g}) = \mathfrak{g}_1 \longrightarrow \text{Der}^1(\mathfrak{g}) \oplus \mathfrak{g}_0 \longrightarrow \text{Der}^0(\mathfrak{g}),$$

whose degree 0 part $\text{DER}_0(\mathfrak{g})$ is $\text{Der}_0(\mathfrak{g})$, degree 1 part $\text{DER}^1(\mathfrak{g})$ is $\text{Der}^1(\mathfrak{g}) \oplus \mathfrak{g}_0$, degree 2 part $\text{DER}^2(\mathfrak{g})$ is $\mathfrak{g}_1$ and for any $a \in \mathfrak{g}_1$, $(D, x) \in \text{Der}^1(\mathfrak{g}) \oplus \mathfrak{g}_0$, $d_D$ is given by

$$d_D(a) = (ad - \delta_D)(a) = (ad_a, -\delta_D(a)),
\quad d_D(D, x) = (\delta + ad)(D, x) = \delta D + (ad_x, l_{ad_x}).$$

$d_D^2 = 0$ follows from

$$\delta(ad_a) = ad_{ad_a}. \tag{11}$$

Define a bilinear degree 0 bracket $\cdot, \cdot : \text{DER}(\mathfrak{g}) \wedge \text{DER}(\mathfrak{g}) \longrightarrow \text{DER}(\mathfrak{g})$ by

$$\begin{align*}
    [(X, l_X), (Y, l_Y)]_{\text{DER}} &= [(X, l_X), (Y, l_Y)]_{\text{Der}}, \\
    [(X, l_X), (D, x)]_{\text{DER}} &= \left( [(X, l_X), D]_{\text{Der}} + l_X(x, \cdot) + \delta_X(x), X(x) \right), \\
    [(D, x), (D', x')]_{\text{DER}} &= -Dx' - D'x, \\
    [(X, l_X), a]_{\text{DER}} &= X(a),
\end{align*} \tag{12}$$

for any $(X, l_X), (Y, l_Y) \in \text{DER}_0(\mathfrak{g})$, $(D, x), (D', x') \in \text{Der}^1(\mathfrak{g})$ and $a \in \text{Der}^2(\mathfrak{g})$.

**Theorem 3.8.** With the above notations, $(\text{DER}(\mathfrak{g}), d_D, [\cdot, \cdot]_{\text{DER}})$ is a strict Lie 3-algebra, which we call the derivation Lie 3-algebra of $\mathfrak{g}$.

---

1 For an $L_\infty$-algebra $L$, $\text{Der}(L)$ has been already considered by Danny Stevenson, see [17] for more details.
Proof. We only need to show that $d_D$ is a graded derivation with respect to the bracket operation $[\cdot, \cdot]_{DER}$, and $[\cdot, \cdot]_{DER}$ satisfies the graded Jacobi identity. The condition that $d_D$ is a graded derivation is equivalent to

$$d_D([X, l_X], a)_{DER} = [(X, l_X), d_D(a)]_{DER},$$  \hspace{1cm} (13)

$$d_D([X, l_X], (D, x)]_{DER} = [(X, l_X), d_D(D, x)]_{DER},$$  \hspace{1cm} (14)

$$d_D([D, x], (D', x')]_{DER} = [d_D(D, x), (D', x')]_{DER} - [(D, D), d_D([D', x'])_{DER},$$  \hspace{1cm} (15)

$$[d_D(D, x), a]_{DER} = [(D, x), d_D(a)]_{DER}. \hspace{1cm} (16)$$

The left hand side of (13) is equal to $(ad_{X(a)}, -d_x X(a))$, and the right hand side is equal to $[(X, ad_a)_C - l_X(d_D a, \cdot) - X(d_D a))$.

By the fact that $[X, ad_a]_C = ad_{X(a)} + l_X(d_D a, \cdot)$, we obtain (13).

The left hand side of (14) is equal to

$$d_D([X, l_X], (D, x)]_{DER}$$

$$= d_D\left([(X, l_X), D]_{DER} + l_X(x, \cdot), X(x)\right)$$

$$= d_D\left([X, D]_{C} + l_X(x, \cdot), X(x)\right)$$

$$= \left(\delta([X, D]_{C} + l_X(x, \cdot)) + ad_{X(x)}l_{[X, D]_{C}} + l_{\delta(l_X(x, \cdot))} + l_{ad_{X(x)}}\right).$$

The right hand side of (14) is equal to

$$[X, l_X], d_D(D, x)]_{DER}$$

$$= [X, l_X), (\delta(D), l_{[X, D]_{C}}) + (ad_x, l_{ad_x})]_{DER}$$

$$= ([X, \delta(D) + ad_x]_{C}, l_{[X, \delta(D) + ad_x]_{C}})$$

$$= ([X, \delta(D)]_{C} + ad_{X(x)} + \delta(l_X(x, \cdot)), l_{[X, \delta(D)]_{C} + ad_{X(x)} + \delta(l_X(x, \cdot))}).$$

The last equality holds since $(X, l_X)$ is a derivation. Therefore, by the fact that $\delta$ is a graded derivation with respect to the bracket operation $[\cdot, \cdot]_{C}$, we deduce that

$$d_D([X, l_X], (D, x)]_{DER} = [(X, l_X), d_D(D, x)]_{DER}. \hspace{1cm} (15)$$

The left hand side of (15) is equal to

$$d_D(-Dx' - D'x) = (-ad_{D(x')} - ad_{D'(x)}, d_D(D(x')) + d_D(D'(x))).$$

The right hand side of (15) is equal to

$$[\delta(D) + ad_x, (D', x')]_{DER} - [(D, x), \delta(D') + ad_{x'}]_{DER}$$

$$= \left(\delta(D), D']_{C} + [ad_x, D']_{C} + l_{[X, D]}(x', \cdot) + l_{ad_x}(x', \cdot), \delta(D)(x') + [x, x']\right)$$

$$- \left([D, \delta(D')]_{C} + [D, ad_{x'}]_{C} - l_{[\delta(D'), x]}(x, \cdot) - l_{ad_{x'}}(x, \cdot), -\delta(D'(x) - [x', x])\right)$$

$$= \left([ad_x, D']_{C} + l_{[\delta(D'), x]}(x, \cdot) - [D, ad_{x'}]_{C} + l_{[\delta(D)](x', \cdot)}, d_D(D(x')) + d_D(D'(x))\right).$$
By (5), we deduce that (15) holds. It is straightforward to deduce that (16) holds.

The bracket operation \([\cdot,\cdot]_{\text{DER}}\) satisfies graded Jacobi identity, and it is equivalent to

\[
[[[(X,l_X),(Y,l_Y)]_{\text{DER}},a]_{\text{DER}} + c.p. = 0, \quad (17)
\]

and

\[
[[[(X,l_X),(D,x)]_{\text{DER}},(D',x')]_{\text{DER}} + [[[D,x],(D',x')]_{\text{DER}},(X,l_X)]_{\text{DER}}
- [[[D',x']),(X,l_X)]_{\text{DER}},(D,x)]_{\text{DER}} = 0. \quad (19)
\]

It is obvious that (17) holds. By straightforward computations, the left hand side of (18) is equal to

\[
[[[X,Y]_{c},l_{[X,Y]c}), (D,x)]_{\text{DER}} + [[Y,D|c + l_Y(x,\cdot),Yx),(X,l_X)]_{\text{DER}}
+[[[D,X]_{c} - l_X(x,\cdot),-Xx), (Y,l_Y)]_{\text{DER}}
= ([[[X,Y]_{c},D|c + l_{[X,Y]c}(x,\cdot),[X,Y]_{c}(x))]
+ ([[Y,D|c + l_Y(x,\cdot),X]_{c} - l_X(Yx,\cdot),-X(Yx))
+ ([[D,X]_{c} - l_X(x,\cdot),Y]_{c} + l_Y(Xx,\cdot),Y(Xx)).
\]

Since \([\cdot,\cdot]_{c}\) satisfies the Jacobi identity and by the definition of \(l_{[X,Y]c}\) (see (2)), we get (18), (19) can be deduced similarly.

**Definition 3.9.** Let \((g,d_g,[\cdot,\cdot]_g, l^0_3)\), \((h,d_h,[\cdot,\cdot]_h, l^0_3)\) be two Lie 2-algebras. Assume that \(f = (f_0, f_1, f_2^0, f_2^1, f_3)\) and \(f' = (f'_0, f'_1, f'_2^0, f'_2^1, f'_3)\) are two morphisms from \(g\) to \(\text{DER}(h)\). We say that \(f'\) is equivalent to \(f\) if there exist:

- linear maps \(b_0 : g_0 \rightarrow h_0\) and \(b_1 : g_1 \rightarrow h_1\),
- a bilinear map \(b_2 : \wedge^2 g_0 \rightarrow h_1\),

such that \((b_0, b_1)\) is a chain homotopy from \((f'_0, f'_1)\) to \((f_0, f_1)\):

\[
\begin{align*}
(f_0 - f'_0) &= d_D \circ b_0, \\
(f_1 - f'_1) &= b_0 \circ d_g + d_D \circ b_1,
\end{align*}
\]

and the following equalities hold for all \(x, y, z \in g_0\) and \(a \in g_1\),

\[
\begin{align*}
(f'_2 - f_2^0)(x,y) &= [f'_0(x),b_0(y)]_{\text{DER}} - [f'_0(y),b_0(x)]_{\text{DER}} - b_0([x,y]_g) \\
&\quad + [d_Db_0(x), b_0(y)]_{\text{DER}} - d_Db_2(x,y), \\
(f'_2 - f_2^1)(x,a) &= [f'_0(x),b_1(a)]_{\text{DER}} + [f'_1(a),b_0(x)]_{\text{DER}} - b_1([x,a]_g) \\
&\quad + [d_Db_0(x), b_1(a)]_{\text{DER}} + b_2(x,d_ga), \\
(f'_3 - f_3)(x,y,z) &= [f'_0(x),b_2(y,z)]_{\text{DER}} - b_2([x,y]_g, z) + c.p. \\
&\quad + [f'_2(x,y),b_0(z)]_{\text{DER}} + [d_Db_0(x), b_2(y,z)]_{\text{DER}} + l^3_{D(x)}(b_0(y), b_0(z)) + c.p. \\
&\quad - b_1(t^0_3(x,y,z)) + t^0_3(b_0(x), b_0(y), b_0(z)).
\end{align*}
\]
Remark 3.10. Let $h = (0 \xrightarrow{\delta} h_0, [\cdot, \cdot]_{h_0}, l_3 = 0)$ be the trivial Lie 2-algebra determined by a Lie algebra $h_0$, then the Lie 3-algebra $\text{DER}(h)$ reduces to the well-known Lie 2-algebra $h_0 \xrightarrow{ad} \text{Der}(h_0)$.

Two morphisms $f = (f_0, f_1, f_2)$ and $f' = (f'_0, f'_1, f'_2)$ from $g$ to $h_0 \xrightarrow{ad} \text{Der}(h_0)$ are equivalent if and only if there is a linear map $b_0 : g_0 \rightarrow h_1$ such that

$$f_0 - f'_0 = \text{ad} b_0,$$
$$f_1 - f'_1 = b_0 \circ d_g,$$
$$(f'_2 - f'_2)(x, y) = f'_0(x)(b_0(y)) - f'_1(y)(b_0(x)) - b_0([x, y]_g) + [b_0(x), b_0(y)]_h,$$

i.e. $b_0$ is a 2-morphism from $f'$ to $f$ in the sense of Baez-Crans.

4 Non-abelian Extensions of Lie 2-algebras

Definition 4.1. (i) Let $g : g_1 \rightarrow h_0$, $h : h_1 \rightarrow h_0$, $\hat{g} : \hat{g}_1 \rightarrow \hat{h}_0$ be Lie 2-algebras and $i = (i_1, i_0) : h \rightarrow \hat{h}$, $p = (p_1, p_0) : \hat{g} \rightarrow g$ be strict morphisms. The following sequence of Lie 2-algebras is a short exact sequence if $\text{Im}(i) = \ker(p)$, $\ker(i) = 0$ and $\text{Im}(p) = g$.

$$0 \xrightarrow{0} h_1 \xrightarrow{i_1} \hat{g}_1 \xrightarrow{p_1} g_1 \xrightarrow{0} 0$$
$$0 \xrightarrow{0} h_0 \xrightarrow{i_0} \hat{g}_0 \xrightarrow{p_0} g_0 \xrightarrow{0} 0$$

We call $\hat{g}$ an extension of $g$ by $h$, and denote it by $E_{\hat{g}}$.

(ii) A splitting $\sigma : g \rightarrow \hat{g}$ of $p : \hat{g} \rightarrow g$ consists of linear maps $\sigma_0 : g_0 \rightarrow \hat{g}_0$ and $\sigma_1 : g_1 \rightarrow \hat{g}_1$ such that $p_0 \circ \sigma_0 = \text{id}_{\hat{g}_0}$ and $p_1 \circ \sigma_1 = \text{id}_{\hat{g}_1}$.

(iii) We say that two extensions of Lie 2-algebras $E_{\hat{g}} : h \rightarrow \hat{g}$ and $E_{\hat{g}}' : h \rightarrow \hat{g}'$ are isomorphic if there exists a Lie 2-algebra morphism $F : \hat{g} \rightarrow \hat{g}'$ such that $F \circ i = j$, $q \circ F = p$ and $F_2(i(u, \alpha)) = 0$, for any $u \in h_0$, $\alpha \in \hat{g}_0$.

In the sequel, we will write an element $(X, l_X) \in \text{Der}^0(h)$, by $X$ to simplify the computation.

Given a splitting $\sigma$, we have $\hat{g}_0 \cong g_0 \oplus h_0$ and $\hat{g}_1 \cong g_1 \oplus h_1$ as vector spaces. Furthermore, $(i_0, i_1)$ are inclusions and $(p_0, p_1)$ are projections. $\sigma$ induces linear maps:

$$\varphi : g_0 \rightarrow h_0, \quad \varphi(a) := \hat{d}\sigma(a) - \sigma(d_g a),$$
$$\mu_0 : g_0 \rightarrow \text{Der}^0(h), \quad \mu_0(x)(u + m) := [\sigma(x), u + m]_\hat{g},$$
$$\mu_1 : g_1 \rightarrow \text{Der}^1(h), \quad \mu_1(u)(a) := [\sigma(a), u]_\hat{g},$$
$$\omega : \Lambda^2 g_0 \rightarrow h_0, \quad \omega(x, y) := \sigma(x, y) - \sigma(x) - \sigma(y),$$
$$\nu : g_0 \wedge g_1 \rightarrow h_1, \quad \nu(x, a) := \sigma(x, a) - \sigma(x) - \sigma(a),$$
$$\theta : \Lambda^3 g_0 \rightarrow h_1, \quad \theta(x, y, z) := \sigma(x, y, z) - \sigma(x, y) - \sigma(y, z).$$

for any $x, y, z \in g_0$, $a \in g_1$, $u \in h_0$ and $m \in h_1$. 

10
Proposition 4.2. The splitting \( \sigma \) induces a morphism

\[
f = (f_0, f_1, f_2, f_3) = (\mu_0, \mu_1 - \varphi, -\mu_2 + \omega, \nu, \theta)
\]  

(21)

from the Lie 2-algebra \( \mathfrak{g} \) to the derivation Lie 3-algebra \( \text{DER}(\mathfrak{h}) \). Moreover, different splittings give equivalent morphisms.

Proof. By computations, we have

\[
(d_D \circ (\mu_1 - \varphi)(a))(u + m) = (\delta + \text{ad})(\mu_1(a), -\varphi(a))(u + m)
\]

\[
= \delta(\mu_1(a))(u + m) - \text{ad}_\varphi(a)(u + m)
\]

\[
= d_\mathfrak{h}[\sigma(a), u]_\mathfrak{h} + [\sigma(a), d_\mathfrak{h}m]_\mathfrak{h} - [\varphi(a), u + m]_\mathfrak{h}
\]

\[
= [\sigma(d_\mathfrak{h}a) + \varphi(a), u + m]_\mathfrak{h} - [\varphi(a), u + m]_\mathfrak{h}
\]

\[
= \mu_0(d_\mathfrak{h}a)(u + m),
\]

which implies that

\[
d_D \circ f_1 = f_0 \circ d_\mathfrak{g}.
\]  

(22)

We have the equalities

\[
[\sigma x, [\sigma y, u]_\mathfrak{h}]_\mathfrak{h} + c.p. = \hat{\delta}_3(\sigma x, \sigma y, u),
\]

(23)

\[
[\sigma x, [\sigma y, m]_\mathfrak{h}]_\mathfrak{h} + c.p. = \hat{\ell}_3(\sigma x, \sigma y, d_\mathfrak{h}m).
\]  

(24)

The left hand side of (23) is equal to

\[
[u, \sigma [x, y]_\mathfrak{h} - \omega(x, y)]_\mathfrak{h} + [\sigma(x), \mu_0(y)u]_\mathfrak{h} - [\sigma(y), \mu_0(x)u]_\mathfrak{h}
\]

\[
= [\mu_0(x), \mu_0(y)]_{\text{DER}}(u) - \mu_0([x, y]_\mathfrak{h})(u) + \text{ad}_\omega(x, y)(u),
\]

and the right hand side is equal to \( d_\mathfrak{h}(\mu_2(x, y)u) \), which implies that

\[
[\mu_0(x), \mu_0(y)]_{\text{DER}}(u) - \mu_0([x, y]_\mathfrak{h})(u) = d_\mathfrak{h}(\mu_2(x, y)(u)) - \text{ad}_\omega(x, y)u.
\]

Similarly, by (24), we get

\[
[\mu_0(x), \mu_0(y)]_{\text{DER}}(m) - \mu_0([x, y]_\mathfrak{h})(m) = \mu_2(x, y)(d_\mathfrak{h}m) - \text{ad}_\omega(x, y)m.
\]

Therefore, we have

\[
f_0([x, y]_\mathfrak{h}) - [f_0(x), f_0(y)]_{\text{DER}} = \mu_0([x, y]_\mathfrak{h}) - [\mu_0(x), \mu_0(y)]_{\text{DER}}
\]

\[
= -\delta(\mu_2(x, y)) + \text{ad}_\omega(x, y) = d_D \circ f_2^0(x, y).
\]  

(25)

We have the equality

\[
[\sigma x, [\sigma a, u]_\mathfrak{h}]_\mathfrak{h} + c.p. = \hat{\delta}_3(\sigma x, \hat{d}\sigma a, u).
\]

Thus, we have

\[
[\mu_0(x), \mu_1(a)]_{\text{DER}}(u) - \mu_1([x, a]_\mathfrak{h})(u) + \text{ad}_\nu(x, a)u
\]

\[
= [\sigma x, \mu_1(a)(u)]_\mathfrak{h} + [\sigma(a), -\mu_0(x)(u)]_\mathfrak{h} + [u, \sigma[x, a]_\mathfrak{h} - \nu(x, a)]_\mathfrak{h}
\]

\[
= \hat{\delta}_3(\sigma x, \hat{d}\sigma a(a) + \varphi(a), u)
\]

\[
= \mu_2(x, d_\mathfrak{h}(a))(u) + \hat{\delta}_3(\sigma x, \varphi(a), u).
\]
By the equality \( \hat{d}[\sigma x, \sigma a]_\% \), we obtain that
\[
\mu_0(x)(\varphi(a)) - \varphi([x, a]_\%) = \omega(x, d_\% a) - d_\% \nu(x, a).
\] (26)
Therefore, we have
\[
f_1([x, a]_\%) = [f_0(x), f_1(a)]_{\text{DER}} = \mu_1([x, a]_\%) - \varphi([x, a]_\%) - [\mu_0(x), \mu_1(a) - \varphi(a)]_{\text{DER}} = \mu_1([x, a]_\%) - [\mu_0(x), \mu_1(a)]_{\text{DER}} + l_{\mu_0(x)}(\varphi(a), \cdot) + \mu_0(x)(\varphi(a)) - \varphi([x, a]_\%)
\]
\[
= -\mu_2(x, d_\% a) - \hat{\ell}_3(\sigma x, \varphi(a), \cdot) + \text{ad}_{\nu(x, a)} + l_{\mu_0(x)}(\varphi(a), \cdot) + \omega(x, d_\% a) - d_\% \nu(x, a)
\]
\[
= f_2^0(x, d_\% a) + d_\% f_2^1(x, a).
\] (27)
By the equality \( \hat{[\sigma a, \sigma b]_\%} = [\sigma a, \hat{d}\sigma b]_\% \), we obtain that
\[
[f_1(a), f_1(b)]_{\text{DER}} = [\mu_1(a) - \varphi(a), \mu_1(b) - \varphi(b)]_{\text{DER}} = \mu_1(a)\varphi(b) + \mu_1(b)\varphi(a)
\]
\[
= \nu(a, d_\% b) - \nu(d_\% a, b) = f_2^1(a, d_\% b) - f_2^0(d_\% a, b).
\] (28)
By the equality
\[
[\sigma x, [\sigma y, \sigma z]_\%]_{\text{DER}} + c.p. = \hat{d}\ell_3(\sigma x, \sigma y, \sigma z),
\]
we get
\[
-\mu_0(x)\omega(y, z) - \omega(x, [y, z]_\%) + c.p. = -d_\% \theta(x, y, z) + \varphi(l_3^0(x, y, z))
\] (29)
By the Jacobiator identity:
\[
\ell_3([\sigma x, \sigma y]_\%, \sigma z, u) + c.p. = [\sigma x, \ell_3(\sigma y, \sigma z, u)]_\% + c.p.,
\]
we have
\[
[\mu_0(x), \mu_2(y, z)]_{\text{DER}} - l_{\mu_0(x)}(\omega(y, z), \cdot) + c.p. = \mu_2([x, y]_\%, z) + c.p. + \text{ad}_{\theta(x, y, z)} - \mu_1 l_3^0(x, y, z).
\] (30)
By (29) and (30), we have
\[
[f_0(x), f_2^0(y, z)]_{\text{DER}} + c.p. + d_\% f_3(x, y, z)
\]
\[
= [\mu_0(x), (-\mu_2 + \omega)(y, z)]_{\text{DER}} + c.p. + d_\% \theta(x, y, z)
\]
\[
= (-\mu_0(x), \mu_2(y, z)]_{\text{DER}} + l_{\mu_0(x)}(\omega(y, z), \cdot) + \mu_0(x)\omega(y, z) + c.p.) + d_\% \theta(x, y, z)
\]
\[
= (-\mu_2([x, y]_\%), z + \omega([x, y]_\%, z) + c.p.) + \mu_1 l_3^0(x, y, z) - \varphi(l_3^0(x, y, z))
\]
\[
= f_2^0([x, y]_\%), z + c.p. + f_1 l_3^0(x, y, z).
\] (31)
By the equality
\[
[\sigma x, [\sigma y, \sigma a]_\%]_{\text{DER}} + c.p. = \hat{\ell}_3(\sigma x, \sigma y, \hat{d}\sigma a),
\] (32)
we have
\[
[\sigma x, [\sigma y, \sigma a]_\%]_\% + [\sigma y, [\sigma a, x]_\% - \nu(a, x)]_\% + [\sigma a, [\sigma x, y]_\% - \omega(x, y)]_\%
\]
\[
= \sigma[x, [y, a]_\%] - \nu([x, [y, a]_\%] - \mu_0(x)\nu(y, a) + \sigma[y, [a, x]_\%] - \nu(y, [a, x]_\%) - \nu(y, a)\nu(a, x)
\]
\[
\quad + \sigma[a, [x, y]_\%] - \nu(a, [x, y]_\%) - \mu_1(a)\omega(x, y)
\]
\[
= \sigma[l_3^0(x, y, d_\% a) - \mu_0(x)\nu(y, a) - \mu_0(y)\nu(a, x) - \mu_1(a)\omega(x, y)
\]
\[
- \nu([x, [y, a]_\%] - \nu(y, [a, x]_\%) - \nu(a, [x, y]_\%)
\]
\[
= \hat{\ell}_3(\sigma x, \sigma y, \sigma(d_\% a + \varphi(a))
\]
\[
= \sigma[l_3^0(x, y, d_\% a) - \theta(x, y, d_\% a) + \mu_2(x, y)\varphi(a),
\]
which implies that
\[ [f_0(x), f_1^2(y, a)]_{\text{DER}} + [f_0(y), f_1^2(a, x)]_{\text{DER}} - [f_1(a), f_1^2(x, y)]_{\text{DER}} \]
\[ = [\mu_0(x), \nu(y, a)]_{\text{DER}} + [\mu_0(y), \nu(a, x)]_{\text{DER}} - [(\mu_1 - \varphi)(a), (-\mu_2 + \omega)(x, y)]_{\text{DER}} \]
\[ = \nu([x, y]_g, a) + \nu([y, a]_g, x) + \nu([a, x]_g, y) + \theta(x, y, d_g a) \]
\[ = f_1^2([x, y]_g, a) + f_1^2([y, a]_g, x) + f_1^2([a, x]_g, y) + f_3(x, y, d_g a). \quad (33) \]

Since for any \( x, y, z, t \in g_0, \)
\[ \hat{\iota}_3([\sigma x, \sigma y]_g, \sigma z, \sigma t) + c.p. = [\sigma x, \hat{\iota}_3(\sigma y, \sigma z, \sigma t)]_g + c.p. \]
The left hand side is equal to
\[ \hat{\iota}_3(\sigma x, \sigma y)_g - \omega(x, y), \sigma z, \sigma t) + c.p. \]
\[ = \sigma \hat{\iota}_3^0([x, y]_g, z, t) - \theta([x, y]_g, z, t) = \mu_2(z, t) \omega(x, y) + c.p. \]
and the right hand side is equal to
\[ [\sigma x, \sigma \hat{\iota}_3^0(y, z, t)]_g - \mu_0(x) \theta(y, z, t) + c.p. \]

Thus, we have
\[ [f_0(x), f_3(y, z, t)]_{\text{DER}} + f_1^2(x, \sigma \hat{\iota}_3^0(y, z, t)) + c.p. \]
\[ = [\mu_0(x), \theta(y, z, t)]_{\text{DER}} + \nu(x, \sigma \hat{\iota}_3^0(y, z, t)) + c.p. \]
\[ = \theta([x, y]_g, z, t) + \mu_2(z, t) \omega(x, y) + c.p. \]
\[ = f_3([x, y]_g, z, t) + [f_1^2(x, y), f_1^2(z, t)]_{\text{DER}} + c.p. \quad (34) \]

By (22), (25), (27), (28), (31), (33), (34), we obtain that \( f \) is a morphism from \( g \) to \( \text{DER}(h) \).

Given another splitting \( \sigma' \) of the extension, there are the induced linear maps \((\varphi', \mu_0', \mu_1', \mu_2', \omega', \nu', \theta')\) such that
\[ f' = (f_0', f_1'^0, f_1'^1, f_3') = (\mu_0', \mu_1' - \varphi', -\mu_2' + \omega', \nu', \theta') \]
is a morphism from \( g \) to \( \text{DER}(h) \).

Assume that
\[ \sigma(x) = \sigma'(x) + b_0(x), \quad \sigma(a) = \sigma'(a) + b_1(a), \]
where \( b_0 : g_0 \rightarrow h_0 \) and \( b_1 : g_1 \rightarrow h_1 \) are linear maps. Then it is straightforward to deduce that
\[ \mu_0(x) - \mu_0'(x) = \text{ad}(b_0(x)), \]
\[ \varphi'(a) - \varphi(a) = b_0(d_g a) - d_g b_1(a), \]
\[ \mu_1(a) - \mu_1'(a) = \text{ad}_{b_1(a)}, \]
\[ (\mu_2 - \mu_2')(x, y) = l_{\mu_0'}(b_0(x), -) - l_{\mu_0'}(b_0(y), -) + l_{\mu_0}^0(b_0(x), b_0(y), -), \]
\[ (\omega' - \omega)(x, y) = \mu_0'(b_0(x)) - \mu_0'(b_0(y)) + [b_0(x), b_0(y)]_g - b_0[x, y]_g + d_g \circ b_2(x, y), \]
\[ (\nu' - \nu)(x, a) = \mu_0'(b_1(a)) - \mu_1'(b_0(a)) - b_1([x, a]_g) + [b_0(x), b_1(a)]_h, \]
\[ (\theta' - \theta)(x, y, z) = \mu_2'(x, y)b_0(z) + l_{\mu_0'}(b_0(y), b_0(z)) + c.p. - b_1(l_3^0(x, y, z)) + l_3^0(b_0(x), b_0(y), b_0(z)). \]
Then it is straightforward to see that \( f' \) is equivalent to \( f \) via \((b_0, b_1, b_2 = 0)\). \( \Box \)

Thus by choosing a splitting, we can transfer the Lie 2-algebra structure on \( \hat{\mathfrak{g}} \) to \( \mathfrak{g} \oplus \mathfrak{h} \), which we denote by \((\hat{\mathfrak{g}} \oplus \mathfrak{h}, \hat{\varphi}, [\cdot, \cdot], \hat{l}_3)\):

\[
\begin{align*}
\hat{d}(a + m) &\triangleq d_\mathfrak{g}(a) + \varphi(a) + d_\mathfrak{h}(m), \\
[x + u, y + v] &\triangleq [x, y]_\mathfrak{g} - \omega(x, y) + \mu_0(x)v - \mu_0(y)u + [u, v]_\mathfrak{h}, \\
[x + u, a + m] &\triangleq [x, a]_\mathfrak{g} - \nu(x, a) + \mu_0(x)m - \mu_1(a)u + [u, m]_\mathfrak{h}, \\
\hat{l}_3(x + u, y + v, z + w) &\triangleq l^g_3(x, y, z) - \theta(x, y, z) + l^h_3(u, v, w) + \mu_1(x)(w) + \mu_2(x, y)(v) + \mu_2(y, z)(u)
\end{align*}
\]

for any \( x, y, z \in \mathfrak{g}_0, u, v, w \in \mathfrak{h}_0, a \in \mathfrak{g}_1 \) and \( m \in \mathfrak{h}_1 \).

Thus any extension \( E_\mathfrak{g} \) given by (36) is isomorphic to

\[
\begin{array}{ccc}
0 & \xrightarrow{0} & \mathfrak{h}_1 \\
\downarrow & d_\mathfrak{h} & \downarrow d_\mathfrak{h} \\
0 & \xrightarrow{0} & \mathfrak{g}_0 \oplus \mathfrak{h}_0 \\
\downarrow & p_0 & \downarrow p_0 \\
0 & \xrightarrow{0} & \mathfrak{g}_0 \\
\end{array}
\]

where the Lie 2-algebra structure on \( \mathfrak{g} \oplus \mathfrak{h} \) is given by (35) for some morphism (21), \((i_0, i_1)\) is the inclusion and \((p_0, p_1)\) is the projection. We denote the extension (36) by \( \hat{E}_\mathfrak{g} \oplus \mathfrak{h} \).

**Theorem 4.3.** There is a 1-1 correspondence between isomorphism classes of extensions of Lie 2-algebras given by (35) and equivalence classes of morphisms (21) from the Lie 2-algebra \( \mathfrak{g} \) to the derivation Lie 2-algebra \( \text{DER}(\mathfrak{h}) \).

**Proof.** Given two isomorphic extensions \( \hat{E}_\mathfrak{g} \oplus \mathfrak{h} \) and \( E'_\mathfrak{g} \oplus \mathfrak{h} \). Let \( F = (F_0, F_1, F_2) : \hat{E}_\mathfrak{g} \oplus \mathfrak{h} \rightarrow E'_\mathfrak{g} \oplus \mathfrak{h} \) be the corresponding isomorphism. By choosing two splittings \( \sigma \) and \( \sigma' \) respectively, we get two morphisms \( f \) and \( f' \) from \( \mathfrak{g} \) to \( \text{DER}(\mathfrak{h}) \). In the following, we prove that \( f' \) is equivalent to \( f \).

Since \( F \) is an isomorphism of extensions, we have

\[
F_2(u, v) = 0, \quad F_2(x, u) = 0, \quad F_2(x, y) \in \mathfrak{h}_1,
\]

and there exist two linear maps \( \psi_0 : \mathfrak{g}_0 \rightarrow \mathfrak{h}_0 \) and \( \psi_1 : \mathfrak{g}_1 \rightarrow \mathfrak{h}_1 \) such that

\[
F_0(x + u) = x + \psi_0(x) + u, \quad F_1(a + m) = a + \psi_1(a) + m.
\]

Set \( b_0 = \psi_0, b_1 = \psi_1 \) and \( b_2 = F_2 \).

By

\[
F_0([x, u]) - [F_0(x), F_0(u)]' = 0,
\]

\[
F_1([x, m]) - [F_0(x), F_1(m)]' = 0,
\]

we get

\[
(\mu_0(x) - \mu'_0(x))(u) = ad_{\psi_0(x)}(u),
\]

\[
(\mu_0(x) - \mu'_0(x))(m) = ad_{\psi_0(x)}(m),
\]

14
which implies that

\[(f_0 - f'_0)(x) = \mu_0(x) - \mu'_0(x) = d_D(\psi_0(x)) = d_D(b_0(x)).\]  

(37)

We also have

\[d'F_1(a) = F_0\hat{d}(a),\]

\[F_1([u, a]) - [F_0(u), F_1(a)] = 0,\]

which implies that

\[\varphi'(a) - \varphi(a) = \psi_0(d_g a) - d_b\psi_1(a),\]

\[\mu_1(a) - \mu'_1(a) = \mu_0(a) - \mu'_0(a) = d_D(b_1(a)) + b_0(d_g a).\]

Therefore, we have

\[(f_1 - f'_1)(a) = (\mu_1 - \varphi)(a) - (\mu'_1 - \varphi')(a) = d_D(\psi_1(a)) + \psi_0(d_g a)\]

\[= d_D(b_1(a)) + b_0(d_g a).\]

(38)

Furthermore, we have

\[F_0[x, y] - [F_0(x), F_0(y)]' = d'F_2(x, y),\]

which implies that

\[\omega'(x, y) - \omega(x, y) = \mu'_0(x)b_0(y) - \mu'_0(y)b_0(x) + [b_0(x), b_0(y)]_b - b_0[x, y]_b + d_b \circ b_2(x, y).\]

Since \(F\) is a Lie 2-algebra morphism, we have the equality:

\[[F_0(x), F_2(y, u)]' + c.p. + l_3'(F_0(x), F_0(y), F_0(u)) = F_2([x, y], u) + c.p. + F_1l_3(x, y, u).\]

The left hand side is equal to

\[-\text{ad}_{F_2(x, y)}(u) + l'_2(\psi_0(x), \psi_0(y), u) + \psi_0(d_g x) + \psi_0(d_g y) + \mu'_0(x) + \mu'_0(y) + \mu'_0(x) + \mu'_0(y),\]

and the right hand side is equal to \(\mu_2(x, y)(u)\), which implies that

\[\mu_2(x, y) - \mu'_2(x, y) = -\text{ad}_{b_2(x, y)} + l'_2(b_0(x), b_0(y), u) + \mu'_0(x) + \mu'_0(y),\]

Thus, we have

\[(f''_0 - f'_0)(x, y) = ((\mu_2 - \mu'_2)(x, y), (\omega' - \omega)(x, y))\]

\[= [\mu'_0(x), b_0(y)]_{\text{DER}} - [\mu'_0(y), b_0(x)]_{\text{DER}} - b_0([x, y]_b)\]

\[= d_D(b_2(x, y)) + b_0(d_g x) + b_0(d_g y)\]

\[= [f'_0(x, b_0(y)) + f'_0(y, b_0(x)]_{\text{DER}} - b_0([x, y]_b)\]

\[= d_D(b_2(x, y)) + d_D(b_0(x)) + d_D(b_0(y))\]

(39)

Similarly, by \(F_1[x, a] = [F_0(x), F_1(a)] = F_2(x, \hat{a} a)\), we get

\[\nu'(x, a) - \nu(x, a) = [\mu'_0(x, b_1(a)]_{\text{DER}} + [\mu'_1(a) - \varphi'(a), b_0(x)]_{\text{DER}} - b_1([x, a]_b)\]

\[+ b_2(x, d_g a) + [d_D(b_0(x)), b_1(a)]_{\text{DER}}\]

\[= [f'_0(x, b_1(a)]_{\text{DER}} + [f'_1(a), b_0(x)]_{\text{DER}} - b_1([x, a]_b)\]

\[+ b_2(x, d_g a) + [d_D(b_0(x)), b_1(a)]_{\text{DER}}.\]

(40)
At last, by the equality

\[ [F_0(x), F_2(y, z)]' + c.p. + l_3'(F_0(x), F_0(y), F_0(z)) = F_2([x, y], z) + c.p. + F_1 l_3(x, y, z), \]

we have

\[ (\theta' - \theta)(x, y, z) = \mu_0'(x)b_2(y, z) - b_2([x, y], z) + c.p. \]
\[ + \mu_2'(x, y)b_0(z) + [b_0(x), b_2(y, z)]_b + l_{\nu'(x)}(b_0(y), b_0(z)) + c.p. \]
\[ - b_1(l_{\theta'}(x, y, z)) + l_3'(b_0(x), b_0(y), b_0(z)). \]

(41)

By (37), (38), (39), (40), (41), we deduce that \( f' \) and \( f \) are equivalent.

Conversely, assume that \( f' = (\mu'_0, \mu'_1 - \varphi', -\mu'_2 + \omega', \theta') \) is equivalent to \( f = (\mu_0, \mu_1 - \varphi, -\mu_2 + \omega, \nu, \theta) \) in the sense of Definition 3.9. For any \( u, v \in h_0 \), \( x, y \in g_0 \), \( m \in h_1 \) and \( a \in g_1 \), set

\[ F_0(x + u) = x + b_0(x) + u, \]
\[ F_1(a + m) = a + b_1(a) + m, \]
\[ F_2(x + u, y + v) = b_2(x, y). \]

By similar computations to the first part of the proof, we can deduce that \( F = (F_0, F_1, F_2) \) is an isomorphism from the extension \( \hat{E}_{g \oplus h} \) to \( E'_{g \oplus h} \). This completes the proof.

References

[1] Alekseevsky D, Michor P W, Ruppert W. Extensions of Lie algebras. arXiv:math.DG/0005042.
[2] Alekseevsky D, Michor P W, Ruppert W. Extensions of Super Lie algebras. J. Lie Theory 15(1):125-134 (2005).
[3] Baez J C, Crans A S. Higher-dimensional algebra. VI. Lie 2-algebras. Theory Appl.Categ., 12:492-538 (electronic) (2004).
[4] Brahic O. Extensions of Lie brackets. J. Geom. Phys. 60, no. 2, 352-374 (2010).
[5] Dehling M. Shifted \( L_\infty \)-bialgebras. master thesis, Göttingen University, 2011.
[6] Eilenberg S, MacLane S.:Cohomology theory in abstract groups. II. Group extensions with non-abelian kernel. Ann. Math., 48:326-341 (1947).
[7] Hochschild G. Cohomology classes of finite type and finite dimensional kernels for Lie algebras. Am. J. Math. 76, 763-778 (1954).
[8] Inassaridze N, Khmaladze E, Ladra M. Non-abelian cohomology and extensions of Lie algebras. Journal of Lie Theory, 8:413-432 (2008).
[9] Lada T, Stasheff J. Introduction to sh Lie algebras for physicists. Int. J. Theo. Phys., Vol. 32(7):1087-1103 (1993).
[10] Lada T, Markl M. Strongly homotopy Lie algebras. Comm. Alg., 23(6):2147-2161 (1995).
[11] Mackenzie K. Lie groupoids and Lie algebroids in differential geometry. London Mathematical Society Lecture Note Series, 124, Cambridge University Press, 1987.
[12] Schreiber U, Stasheff J. Structure of Lie \( n \)-Algebras. unpublished work.

[13] Sheng Y, Zhu C. Semidirect products of representations up to homotopy. *Pacific J. Math.*, 249 (1), 211-236 (2011).

[14] Sheng Y, Liu Z-J, Zhu C. Omni-Lie 2-algebras and their Dirac structures. *J. Geom. Phys.*, 61:560-575, (2011).

[15] Sheng Y, Zhu C. Higher Extensions of Lie Algebroids and Application to Courant Algebroids. [arXiv:1103.5920v1](http://arxiv.org/abs/1103.5920).

[16] Shukla U. A cohomology for Lie algebras. *J. Math. Soc. Japan* 18, 275-289 (1966).

[17] Stevenson D. Schreier Theory for Lie 2-algebras. unpublished work.