Constraints in two-dimensional Dilaton Gravity with Fermions

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ABSTRACT: Generalized Dilaton Theories in two dimensions coupled to Dirac fermions are subjected to constraint analysis. Three first class secondary constraints are found, corresponding to one local Lorentz symmetry and two diffeomorphisms. Moreover, the system also yields second class constraints from the fermions. The algebra of first class constraints is calculated in some detail, and is found to be related to the classical Virasoro algebra.
1. Introduction

Generalized Dilaton Theories (GDTs) in two dimensions arise from several fields of physics (for an exhaustive review, see [1]). Prominent examples would be spherical reduced Einstein-Hilbert gravity from $D > 2$ dimensions, the “string inspired” dilaton model (CGHS model, [2]) or the models put forward by Jackiw & Teitelboim [3,4] and Katanaev & Volovich [5] in the 1980s. All these models can be subsumed into one action [6–8]

\[ S^{\text{GDT}} = \int d^2x \sqrt{-g} \left[ \frac{R}{2} X - \frac{U(X)}{2} (\nabla X)^2 + V(X) \right] \]  

(1.1)

with $X$ being the dilaton field, $R$ the Ricci scalar and $U(X)$ and $V(X)$ arbitrary potentials specifying the model.

It turns then out that there exists a classically equivalent formulation of (1.1) in terms of Cartan variables, namely the Vielbein $e^a = e^a_\mu dx^\mu$, the spin connection
\( \omega^{ab} = \omega^{ab}_\mu dx^\mu =: \omega \epsilon^{ab} \); the dilaton \( X \) and additional auxiliary fields \( X^\pm \). This First Order Gravity (FOG) action reads [9]

\[
S_{\text{FOG}} = \int_{\mathcal{M}_2} \left[ X^a (De)_a + X d\omega + \epsilon \mathcal{V}(X^+ X^-; X) \right]
\]

Here we allow also for non-vanishing torsion \( T^a = (De)^a \) terms, and the proof of classical equivalence (for potentials \( \mathcal{V}(X^+ X^-; X) = U(X)X^a X_\alpha + V(X) \)), which can be found in ch. 2 of [1], amounts to using the equations of motion for the \( X^a \) to eliminate the torsion dependent part of the spin connection.

The FOG action is our starting point for the analysis presented. We will explain the coupling to Dirac fermions in sec. 2. In Sec. 3 we will analyse the constraints of the theory and obtain the constraint algebra. Sec. 4 relates this algebra with the well known classical Virasoro algebra (or Witt algebra) in two dimensions, and Sec. 5 contains some discussion of the main results. The conventions used in this article can be found in Appendix A. Appendices B and C contain equations necessary for proving the main results of this paper.

It should be noted that a similar analysis has been carried out for scalar fields coupled to FOG in [10] and for massless, not self interacting and minimally coupled fermions [11]. We extend this analysis to the general case of massive, self-interacting Dirac fermions with non-minimal coupling. (What is meant by (non-)minimal coupling will be explained in the next section.) One of the first works considering GDTs (with \( U(X) = 0 \) coupled to fermions was [12]. Even before, specific models were used as toy models for Black Hole evaporation [13, 14] and more recently in a paper by Thorlacius et. al. [15] For another remark on this, see also Sec. 2.

2. Fermions

We add to (1.2) an action for the fermion fields consisting of the well known kinetic term and a general self interaction\(^1\)

\[
S = S_{\text{FOG}} + S_{\text{kin}} + S_{\text{SI}}
\]

\[
S_{\text{kin}} = -\frac{i}{2} \int_{\mathcal{M}_2} f(X) (\ast e^a) \wedge (\overline{\chi} \gamma_a \partial^\mu \chi)
\]

\[
= \frac{i}{2} \int d^2 x (e)(f(X)e_a^\mu (\overline{\chi} \gamma^a \overleftarrow{\partial}_\mu \chi)
\]

\[
S_{\text{SI}} = -\int_{\mathcal{M}_2} e h(X) g(\overline{\chi} \chi)
\]

\[
= \int d^2 x (e) h(X) g(\overline{\chi} \chi)
\]

\(^1\)Note that \( \ast e^a \neq e^a \).
Here all possible boundary terms coming from the fermions have been omitted. The functions \( f(X) \) and \( h(X) \) introduce a coupling to the dilaton field. If both are constant, i.e. \( f \propto h = \text{const} \), we call the fermions minimally coupled, and non minimally coupled otherwise. Because spinors are anti commuting Grassmann fields, a Taylor expansion of \( g \) yields at at most a quartic term, \( g(\overline{\chi}) = g_0 + m\overline{\chi} + \lambda(\overline{\chi})^2 \); and the constant term can always be absorbed into \( V(X) \).

Note that in two dimensions the kinetic term is independent of the spin connection: In arbitrary dimension, the action for the kinetic term would be\(^2\) [16]

\[
\frac{i}{2} \int d^n x \det(e^a_\mu) \left[ f(X) e^a_\mu(\overline{\chi}\gamma^a (\partial_\mu + \omega_\mu^{bc} \Sigma_{bc}) \chi) + h.c. \right]
\]

For \( n = 2 \) however, there is only one independent generator of Lorentz transformations \( \Sigma_{01} = \frac{1}{4}[\gamma_0, \gamma_1] = -\frac{\gamma_2}{2} \), and with \( \{\gamma_a, \gamma_s\} = 0 \) the terms in (2.3) containing the spin connection vanish

\[
-\frac{i}{4} f(X)(\ast e^a) \wedge \omega \chi^\dagger (\gamma_0^a \gamma_a - \gamma_s^a \gamma^s_a \gamma^0) \chi
\]

\[
= -\frac{i}{4} f(X)(\ast e^a) \wedge \omega \chi^\dagger (\gamma_0^a \gamma_a - \gamma_s^a \gamma^s_a \gamma^0) \gamma_s \chi = 0
\]

3. Hamiltonian Analysis

For the sake of better memorability, we henceforth denote the canonical coordinates and momenta by

\[
\begin{align*}
\overline{q}^i &= (\omega_0, e_0^-, e_0^+) \\
q^i &= (\omega_i, e_i^-, e_i^+), \quad i = 1, 2, 3 \\
p_i &= (X, X^+, X^-) \\
Q^a &= (\chi_0, \chi_1, \chi_0^\dagger, \chi_1^\dagger), \quad \alpha = 0, 1, 2, 3
\end{align*}
\]

The canonical structure on the phase space is given by Poisson brackets

\[
\{q^i(x), p_j(y)\} = \delta^i_j \delta(x - y) \tag{3.1}
\]

\[
\{Q^a(x), P_\beta(y)\} = -\delta^a_\beta \delta(x - y)
\]

where the \( P_\beta \) are canonical momenta for the spinors, and not present explicitly in (2.2) and (2.3).

\(^2\text{h.c. means hermitian conjugate}\)
3.1 Primary and Secondary Constraints

In our system there occur both primary first and second class constraints. A look at the Lagrangian written in components

\[ L_{\text{FOG}} = \bar{\varepsilon}^{\mu\nu}(X^+(\partial_\mu - \omega_\mu)e^-_\nu + X^-(\partial_\mu + \omega_\mu)e^+_\nu + X\partial_\mu (X^+X^-; X) \]

\[ L_{\text{kin}} = \frac{i}{\sqrt{2}} f(X) \left[ -e^+_0 (\chi^+_0 \partial_1 \chi^+_0) + e^-_0 (\chi^+_1 \partial_0 \chi^+_1) + e^+_1 (\chi^+_0 \partial_0 \chi^+_0) - e^-_1 (\chi^+_1 \partial_0 \chi^+_1) \right] \]

\[ L_{SI} = (e^-_1 e^+_1 - e^-_0 e^+_0) h(X) g(\chi^+_1 \chi^+_0 + \chi^+_0 \chi^+_1) \]

shows that there do not occur any time (i.e. \( x^0 \)) derivatives of the \( \vec{q}^i \) and thus the \( \vec{p}_i \) are constrained to zero, \( \vec{p}_i \approx 0 \), where \( \approx \) means weakly equal to zero.

Because the kinetic term for the fermions is of first order in the derivatives, the fermion momenta \( P_\alpha := \frac{\partial L}{\partial \dot{Q}_\alpha} \) give rise to primary constraints

\[ \Phi_0 = P_0 + \frac{i}{\sqrt{2}} f(p_1) q^0 q^2 \approx 0 \]

\[ \Phi_1 = P_1 - \frac{i}{\sqrt{2}} f(p_1) q^0 q^3 \approx 0 \]

\[ \Phi_2 = P_2 + \frac{i}{\sqrt{2}} f(p_1) q^0 q^0 \approx 0 \]

\[ \Phi_3 = P_3 - \frac{i}{\sqrt{2}} f(p_1) q^0 q^1 \approx 0 \]

They have a non vanishing Poisson bracket\(^3\) with each other,

\[ C_{\alpha\beta}(x,y) := \{ \Phi_\alpha(x), \Phi_\beta(y) \} \]

\[ = i \sqrt{2} f(X) \begin{pmatrix} 0 & 0 & -e^+_1 & 0 \\ 0 & 0 & 0 & e^-_1 \\ -e^+_1 & 0 & 0 & 0 \\ 0 & e^-_1 & 0 & 0 \end{pmatrix} \delta(x-y) \]

and thus are of second class, according to Dirac’s original classification of constraints [17]. The \( \Phi_\alpha \), however, are independent of the \( \vec{q}^i \) and thus commute with the \( \vec{p}_i \).

\(^3\{\ldots\}\) denotes the graded Poisson bracket. For the definition c.f. App. A
Having computed all the momenta, we obtain the Hamiltonian density

\[ \mathcal{H} = \dot{Q}^a P_a + p_i \dot{q}^i - \mathcal{L} =: \mathcal{H}_{FOG} + \mathcal{H}_{kin} + \mathcal{H}_{SI} \]

\begin{align*}
\mathcal{H}_{FOG} &= X^+ (\partial_1 - \omega_1) e_0^+ + X^- (\partial_1 + \omega_1) e_0^- + X_0 (e^+ - e^-) \omega_0 \\
\mathcal{H}_{kin} &= \frac{i}{\sqrt{2}} f(X) \left[ e_0^+ (\chi_0^+ \partial_1 \chi_0) - e_0^- (\chi_0^+ \partial_1 \chi_0) \right] \\
\mathcal{H}_{SI} &= -(e) h(X) g(\chi \chi)
\end{align*}

To deal with the second class constraints, we pass to the Dirac bracket \[^{[17,18]}\] with \( C^{\alpha \beta}(x, y) \) being the inverse of the matrix-valued distribution \[^{4}\]. Demandng that the primary first class constraints should not change during time evolution, i.e. \[^{5}\] \( G_i := \{ \mathcal{P}_i, \mathcal{H} \} = \{ \mathcal{P}_i, \mathcal{H}' \} \approx 0 \), leads us to secondary constraints

\begin{align*}
G_1 &= G_1^g \\
G_2 &= G_2^g + \frac{i}{\sqrt{2}} f(X) (\chi_0^+ e_0^-) + e_1 \ h(X) g(\chi \chi) \\
G_3 &= G_3^g - \frac{i}{\sqrt{2}} f(X) (\chi_0^+ e_0^-) - e_1 \ h(X) g(\chi \chi)
\end{align*}

with

\begin{align*}
G_1^g &= \partial_1 X + X^- e_1^- - X^+ e_1^+ \\
G_2^g &= \partial_1 X^+ + \omega_1 X^+ - e_1 \ V \\
G_3^g &= \partial_1 X^- - \omega_1 X^- + e_1 \ V
\end{align*}

The Hamiltonian density now turns out to be constrained to zero, as expected for a generally covariant system\[^{6}\] \[^{[18]}\].

\[ \mathcal{H} = -\mathcal{P}^i G_i \]
This already lets us to expect that the $G_i$ are related to the generators of the three gauge symmetries in the system, namely local Lorentz symmetry and the two diffeomorphisms.

The secondary constraints commute with the $p_i$ because both the $G_i$ and $\Phi_\alpha$ are independent of the $q_i$. They also trivially commute with the primary second class constraints, $\{\Phi_\alpha, G'_j\}^* = 0$, because of the definition of the Dirac bracket. For the same reason the $\Phi_\alpha$ do not give rise to new secondary constraints.

3.2 Algebra of the secondary constraints

Dirac conjectured [17] that every first class constraint generates a gauge symmetry. The proof of this conjecture is possible in a very general setting [19], but some additional assumptions (see paragraph 3.3.2 of [18]) to rule out “pathological” examples make it easier. These assumptions are fulfilled in our case, because 1. every constraint belongs to a well defined generation; 2. the Dirac bracket ensures that the primary second class constraints do not generate new ones and, as will be seen below, the secondary constraints are first class and there are no ternary constraints; and 3. every primary first class constraint $p_i = 0$ generates one $G_i$.

To show that the system doesn’t admit any ternary constraints, it is sufficient to show that the algebra of secondary constraints closes, i.e. $\{G_i, G'_j\}^* = 0$ and thus the secondary constraints are preserved under the time evolution generated through the Dirac bracket,

$$\dot{G}_i = \{G_i, H'\}^* = -\overline{\pi}' \{G_i, G'_j\}^* \approx 0$$

To calculate all the Dirac brackets, one first needs the Poisson brackets $\{\Phi_\alpha, G_j\}$. They are rather lengthy and thus listed in Appendix B. The resulting algebra then reads

$$\{G_i, G'_i\}^* = 0 \quad i = 1, 2, 3 \quad (3.13)$$
$$\{G_1, G'_2\}^* = -G_2 \delta \quad (3.14)$$
$$\{G_1, G'_3\}^* = G_3 \delta \quad (3.15)$$
$$\{G_2, G'_3\}^* = \left[ -\sum_{i=1}^{3} \frac{dV}{dp_i} G_i + \left( gh' - \frac{h}{f} f' g' \cdot \overline{\chi} \chi \right) G_1 \right] \delta \quad (3.16)$$

We’d like to comment on some technical points. Only obtaining (3.16) needs some care, the others brackets are rather straightforward, using the Poisson structure of our phase space (3.1). However, one should keep in mind that the $Q^\alpha$ are anticommuting. The tricky part of (3.16) is actually not the $C^{\alpha\beta}$-term in the Dirac bracket, but the bracket $\{G_2[\varphi], G_3[\psi]\}$ and therein the integrations by part during calculation, which have to be performed using smeared constraints, i.e.

$$G_i[\varphi] = \int dx \varphi(x)G_i(x)$$
The bracket itself reads with (3.10), (3.11)

\[
\{ G_2[\varphi], G_3[\psi] \} = \iint dx dz \varphi(x) \psi(z) \left\{ \{ G_2^{old}(x), G_3^{old}(z) \} + \{ q_2^3(x) h(x) g(x), G_3^{old}(z) \} - \{ G_2^{old}(x), q_2^3(z) h(z) g(z) \} \right\}
\]

(3.17)

Here we denote with \( G^{old} \) the constraints with \( h = 0 \),

\[
\begin{align*}
G_1^{old} &= G_1^g = G_1 \\
G_2^{old} &= G_2^g + \frac{i}{\sqrt{2}} f(X)(\chi_1^x \partial_x \chi_1) \\
G_3^{old} &= G_3^g - \frac{i}{\sqrt{2}} f(X)(\chi_0 \partial_x \chi_0)
\end{align*}
\]

and

\[
\{ G_2^{old}(x), G_3^{old}(z) \} = - \sum_{i=1}^3 \frac{dV}{dp_i} G_i^g
\]

The tricky parts are the second and third bracket in (3.17) 7:

\[
\begin{align*}
\{ (q^3 h(p_1) g(\chi\chi)) [\varphi], G_3^{old}[\psi] \} &= \iint dx dz \varphi(x) \psi(z) g_x(\chi\chi) \{ q_2^3 h_x(p_1), G_3^{old}[\psi] \} \\
&= \iint dx dz \varphi(x) \psi(z) g_x(\chi\chi) [(\partial_x \delta(x - z)) h_x(p_1) \\
&\quad - (q^1 h - q^3 p_3 h' - q^2 p_2 U h)_x \delta(x - z)] \\
\{ G_2^{old}[\varphi], (q^2 h(p_1) g(\chi\chi))[\psi] \} &= \iint dx dz \varphi(x) \psi(z) g_x(\chi\chi) \{ G_2^{old}[\varphi], q_2^2 h_z(p_1) \} \\
&= \iint dx dz \varphi(x) \psi(z) g_x(\chi\chi) [(-\partial_x \delta(x - z)) h_z(p_1) \\
&\quad - (q^1 h - q^3 p_2 h' - q^2 p_3 U h)_x \delta(x - z)]
\end{align*}
\]

\[
\Rightarrow \quad \{ (q^3 h(p_1) g(\chi\chi)) [\varphi], G_3^{old}[\psi] \} - \{ G_2^{old}[\varphi], (q^2 h(p_1) g(\chi\chi))[\psi] \} = \iint dx dz \varphi(x) \psi(z) \left[ (\partial_x \delta(x - z)) g_x(\chi\chi) h_x(p_1) + (\partial_x \delta(x - z)) g_x(\chi\chi) h_z(p_1) \right. \\
&\quad - g(q^2 p_2 - q^3 p_3)(h'(p_1) - U(p_1) h(p_1)) \delta(x - z) \\
&\quad \left. \right|_{int.p.p.} \iint dx dz \varphi(x) \psi(z) \delta(x - z) [\partial_x(hg) - g(q^2 p_2 - q^3 p_3)(h' - U h)]
\]

Thus we get for the graded Poisson bracket of \( G_2 \) and \( G_3 \)

\[
\{ G_2, G_3' \} = \left[ -\frac{dV}{dp_i} G_i^g + \frac{i}{\sqrt{2}} f'[p_3(Q_3^3 \partial_x Q_1) - p_2(Q_2^3 \partial_x Q_0)] \\
+ \partial_x(hg) - g(q^2 p_2 - q^3 p_3)(h' - U h) \right] \delta
\]

7The points in space where the functions are evaluated are denoted by subscript here, e.g. \( h_x(p_1) := h(p_1(x)) \)
The $C^{\alpha\beta}$-terms of the Dirac bracket are (with $\partial_x g = g' \partial_x (\bar{\chi} \chi)$, $\partial_x f = f' \partial_x p_1$ and $p_2 q^2 - p_3 q^3 - \partial_x p_1 = -G_1$)
\[
-\frac{i}{\sqrt{2}} [f' + U f][p_3 (\bar{Q}_3 Q) - p_2 (\bar{Q}_2 Q^0)] - \frac{h}{f} f' g' \cdot (\bar{\chi} \chi) G_1 - h(\partial_x g)
\]
With these results, we obtain (omitting the $\delta(x - x')$)
\[
\{G_2, G'_3\}^* = -\frac{dV}{dp_1} G_1 - \frac{dV}{dp_2} \left( G'_2 + \frac{i}{\sqrt{2}} f (\bar{Q}^3 \partial_x Q^1) + q^3 h g \right) - \frac{dV}{dp_3} \left( G'_3 - \frac{i}{\sqrt{2}} f (\bar{Q}^3 \partial_x Q^1) - q^2 h g \right) + \partial_x (hg) - h(\partial_x g) - gh' (\partial_x p_1) + gh' G_1
\]
\[
= -\frac{dV}{dp_1} G_1 + (gh' - \frac{h}{f} f' g' \cdot (\bar{\chi} \chi)) G_1
\]

4. Relation to the Conformal algebra

As first noted in [20], certain linear combinations of the $G_i$ fulfil the Witt algebra. In that work FOG coupled to scalar matter was considered. However, the same result holds in our case. New generators $G = G_1$, $H_0/1 = q^1 G_1 \mp q^2 G_2 + q^3 G_3$ fulfil an algebra (with $\delta' = \frac{\partial \delta(x-x')}{\partial x'}$)
\[
\{G, G'\}^* = 0 \quad \{H_1, H'_1\}^* = (H_1 + H'_1) \delta'
\]
\[
\{G, H_i\}^* = -G \delta' \quad \{H_0, H'_1\}^* = (H_0 + H'_0) \delta'
\]
(4.1)

Some Dirac brackets needed for calculating this algebra are listed in App. [C]. Fourier transforming the light cone combinations $H^\pm = H_0 \pm H_1$ according to $H^+(x) = \int \frac{dk}{2\pi} L_k e^{ikx}$; $H^-(x) = \int \frac{dk}{2\pi} \overline{T}_k e^{ikx}$ shows that the $L_k$ (and equally $\overline{T}_k$) fulfil the classical Virasoro algebra
\[
\{L_k, L_m\}^* = i(k - m) L_{k+m}
\]
(4.2)

5. Discussion & Outlook

From (3.13) - (3.16) it’s clear that the algebra of secondary constraints closes with delta functions. This implies the absence of ternary constraints. The $G_i$ on-shell generate three gauge symmetries, namely one local Lorentz symmetry ($G_1$) and two diffeomorphisms ($G_2$, $G_3$), which can be seen from
\[
\{G_1, X^\pm\}^* = \mp X^\pm \delta
\]
(5.1)
\[
\{G_{2/3}, X\}^* = \pm X^\pm \delta
\]
(5.2)
by comparing with the transformation property of $X^\pm$ under Lorentz transformations and the Lorentz scalar $X$ under diffeomorphisms.

The nontrivial part of the algebra of first class constraints still closes like in the case of a compact Lie groups, $[\delta_A, \delta_B] = f^C_{\ AB}(x)\delta_C$, but rather with structure functions than with constants. This is especially important when considering BRST symmetry, because the homologic perturbation series for the BRST charge then terminates at Yang-Mills level [21].

The second term in (3.16) deserves some remarks: First, it vanishes for minimal coupling, i.e. for $h = f = \text{const}$. Second, if $h \propto f$, it becomes proportional to $f'(g - g' \cdot \chi)G_1 \delta = -f'\lambda(\chi)G_1 \delta$. Thus a mass term $m\chi$ doesn’t change the constraint algebra at all. Third, it doesn’t contain derivatives of $\chi$. This is different from the case of scalar matter (see eq. E.31 in [10]), where the additional contribution to $\{G_2, G_3\}$ is proportional to $\frac{f'}{f}\mathcal{L}_\text{scalar}$.

Boundary contributions both to the dilaton and the fermionic action have been omitted this work. Dilaton theories with boundaries (but without matter) have been considered recently in detail [22], with the result that a consistent variational principle can be defined. We don’t expect problems from the interplay of fermion boundary terms and gravitational ones. Nevertheless this point still has to be worked out.

As noted in Sec. [1], another motivation for this work stems from the recent paper by Frolov, Kristjánsson and Thorlacius [15], who considered a two-dimensional Schwinger model on a curved background manifold to investigate the effect of pair-production on the global structure of black hole space times. To this end they used the quantum equivalence of the Schwinger model in 1+1 dimensions and the Sine-Gordon model found by Coleman, Jackiw & Susskind [23] to do calculations on the Sine-Gordon side. It is an interesting question whether Bosonisation still shows up in a quantum theory with dynamical gravitational background. For matter less generalised dilaton theories [11] and for ones coupled to scalar fields an exact path integral quantisation of the geometric sector is known [24] and gives rise to a nonlocal vertices effective theory which turn out to be interesting intermediary states like Virtual Black Holes [25]. A similar analysis for our case is in preparation [21] and will shed some light on the question posted above.

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A. Conventions

For the Levi-Civitá symbols both in tangent space $\varepsilon^{ab}$ and on the manifold $\tilde{\varepsilon}^{\mu\nu}$, we fix $\varepsilon^{01} := +1$. This is necessary to retain $\varepsilon_{\mu\nu} = e_{\mu}^a e_{\nu}^b \varepsilon^{ab}$, with $\varepsilon_{\mu\nu}$ now being the Levi-Civitá tensor. In the tangent space we use light cone coordinates $X^\pm = \frac{1}{\sqrt{2}}(X^0 \pm X^1)$, and thus $\varepsilon^{LC}_{ab} = -\varepsilon^{ab}$. For the square root of the determinant of the metric we sometimes write $(e_{\mu}) := e^{-\epsilon_{0} e^1_- e^1_+} = \sqrt{-\det(g_{\mu\nu})}$, whereas the volume 2-form is $\varepsilon = -(e_{\mu}) dx^0 \wedge dx^1$. The Hodge star is defined as $[1]$. The two-sided derivative is $a \dd b := a(db) - (da)b$

Our Dirac matrices

\[
\begin{align*}
\gamma^0 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
\gamma^1 &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\
\gamma^+ &= \begin{pmatrix} 0 & \sqrt{2} \\ 0 & 0 \end{pmatrix} \\
\gamma^- &= \begin{pmatrix} 0 & 0 \\ \sqrt{2} & 0 \end{pmatrix}
\end{align*}
\]

The analogue of $\gamma_5$ is defined as $\gamma_* := \gamma^0 \gamma^1 = \frac{1}{2}[\gamma^0, \gamma^1]$. Because our fermion fields are Grassmann variables, throughout this article we use the graded Poisson bracket defined as $[18]$ ($\partial^L$ is the usual left derivative)

\[
\{F, G\} = \int dz \left[ \frac{\partial F}{\partial q^i(z)} \frac{\partial G}{\partial p_i(z)} - \frac{\partial F}{\partial p_i(z)} \frac{\partial G}{\partial q^i(z)} \right] + (-)^{\varepsilon(F)} \left[ \frac{\partial^L F}{\partial Q^\alpha(z)} \frac{\partial^L G}{\partial P_\alpha(z)} - \frac{\partial^L F}{\partial P_\alpha(z)} \frac{\partial^L G}{\partial Q^\alpha(z)} \right]
\]

with $(q, p)$ and $(Q, P)$ being a set of bosonic ($\varepsilon(q) = \varepsilon(p) = 0$) and fermionic ($\varepsilon(Q) = \varepsilon(P) = 1$), and $\varepsilon(F)$ the Grassmann parity of $F$. Its main properties used here are

\[
\begin{align*}
\{F, G\} &= (-)^{\varepsilon(F)\varepsilon(G)+1}\{G, F\} \\
\{F, G_1 G_2\} &= \{F, G_1\} G_2 + (-)^{\varepsilon(F)\varepsilon(G_1)} G_1 \{F, G_2\}
\end{align*}
\]

All these properties carry over to the corresponding Dirac bracket defined by $[18]$.

B. Brackets of the secondary with the second class constraints

To calculate the Dirac brackets, we need all the graded Poisson brackets of the $G_i$ with the $\Phi_\alpha$. They are easily obtained by using the algebraic properties of the graded
Poisson bracket (see App. A).

\[
\{G_1, \Phi_0'\} = -\frac{i}{\sqrt{2}} f e_1^+ \chi_0^\dagger \delta(x - x') \\
\{G_1, \Phi_2'\} = -\frac{i}{\sqrt{2}} f e_1^+ \chi_0 \delta(x - x') \\
\{G_1, \Phi_1'\} = -\frac{i}{\sqrt{2}} f e_1^- \chi_1^\dagger \delta(x - x') \\
\{G_1, \Phi_3'\} = -\frac{i}{\sqrt{2}} f e_1^- \chi_1 \delta(x - x') \\
\{G_2, \Phi_0'\} = \frac{i}{\sqrt{2}} [f' + U f] X^+ e_1^+ \chi_0^\dagger \delta(x - x') 
- e_1^+ h g' \chi_0^\dagger \delta(x - x') \\
\{G_2, \Phi_2'\} = \frac{i}{\sqrt{2}} [f' + U f] X^+ e_1^+ \chi_0 \delta(x - x') 
+ e_1^+ h g' \chi_1 \delta(x - x') \\
\{G_3, \Phi_0'\} = \frac{i}{\sqrt{2}} [\chi_1 (\omega f - X^+ e_1^- f') - X^- e_1^+ U f] + 2(\partial_x \chi_1^\dagger)f + (\partial_x f)\chi_1^\dagger \delta(x - x') 
+ e_1^+ h g' \chi_0 \delta(x - x') \\
\{G_3, \Phi_2'\} = \frac{i}{\sqrt{2}} [\chi_0 (\omega f - X^- e_1^- f') - X^+ e_1^- U f] - 2(\partial_x \chi_0^\dagger)f - (\partial_x f)\chi_0 \delta(x - x') 
- e_1^- e_1^- h g' \chi_1 \delta(x - x') \\
\{G_3, \Phi_1'\} = \frac{i}{\sqrt{2}} [f' + U f] X^- e_1^- \chi_1^\dagger \delta(x - x') 
+ e_1^- h g' \chi_0 \delta(x - x') \\
\{G_3, \Phi_3'\} = \frac{i}{\sqrt{2}} [f' + U f] X^- e_1^- \chi_1 \delta(x - x') 
- e_1^- h g' \chi_0 \delta(x - x')
\]

However one must be careful with integrating by parts the derivatives of the delta distributions. This is most easily done by smearing the fields with appropriate test functions.

\[
\int dx \varphi(x) [f(x) - f(y)] \partial_x \delta(x - y) = -\int dx \varphi(x)(\partial_x f(x))\delta(x - y)
\]

\*After integrating by parts one obtains
C. Dirac Brackets needed for eq. (4.2)

\[ \{G_1, q^1\}^* = -\partial_1 \delta \]
\[ \{G_1, q^2\}^* = q^2 \delta \]
\[ \{G_1, q^3\}^* = -q^3 \delta \]
\[ \{G_2, q^1\}^* = q^3 \left[ \frac{\partial V}{\partial p_1} - \left( h'g - \frac{f'}{f} hg' \cdot (\chi \chi) \right) \right] \delta \]
\[ \{G_2, q^2\}^* = -\left[ \partial_1 + q^1 - q^3 \frac{\partial V}{\partial p_2} \right] \delta \]
\[ \{G_2, q^3\}^* = q^3 \frac{\partial V}{\partial p_3} \delta \]
\[ \{G_3, q^1\}^* = -q^2 \left[ \frac{\partial V}{\partial p_1} - \left( h'g - \frac{f'}{f} hg' \cdot (\chi \chi) \right) \right] \delta \]
\[ \{G_3, q^2\}^* = -q^2 \frac{\partial V}{\partial p_2} \delta \]
\[ \{G_3, q^3\}^* = -\left[ \partial_1 - q^1 + q^2 \frac{\partial V}{\partial p_3} \right] \delta \]

\[ \{q^i G_i, q^i G_i\}^* = -(\partial_1 \delta) q^i G_i \text{ (no summation)} \]
\[ \{q^1 G_1, q^2 G_2\}^* = -q^2 q^3 \left[ \frac{\partial V}{\partial p_1} - \left( h'g - \frac{f'}{f} hg' \cdot (\chi \chi) \right) \right] G_1 \delta = -\{q^1 G_1, q^3 G_3\}^* = \{q^2 G_2, q^3 G_3\}^* \]

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