Regularity properties of the solution to a stochastic heat equation driven by a fractional Gaussian noise on $S^2$

Xiaohong Lan ∗
School of Mathematical Sciences
University of Science and Technology of China
E-mail: xhlan@ustc.edu.cn

Yimin Xiao †
Department of Statistics and Probability
Michigan State University
E-mail: xiao@stt.msu.edu

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Abstract

We study the stochastic heat equation driven by an additive infinite dimensional fractional Brownian noise on the unit sphere $S^2$. The existence and uniqueness of its solution in certain Sobolev space is investigated and sample path regularity properties are established. In particular, the exact uniform modulus of continuity of the solution in time/spatial variable is derived.

Key words: Stochastic heat equation; fractional-colored Gaussian noise; spherical random fields; uniform modulus of continuity.

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1 Introduction

For more than three decades, the topic of stochastic partial differential equations (SPDEs, henceforth) has been an active area of research in pure and applied mathematics. The study of SPDEs has recently entered a period of rapid growth.

∗Research of X. Lan is supported by NSFC grant 11501538 and CAS grant QYZDB-SSW-SYS009. E-mail: xhlan@ustc.edu.cn
†Corresponding author. Research of Y. Xiao is partially supported by grants DMS-1612885 and DMS-1607089 from the National Science Foundation.
We refer to [6, 7, 20, 25] for systematic accounts on SPDEs, and to [5, 11, 12] for some recent developments.

In this paper, we consider the following stochastic heat equation driven by an infinite dimensional fractional Brownian noise $W^H = \{W^H(t), t \in \mathbb{R}_+\}$ on the unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$:

$$du(t) = \Delta_{\mathbb{S}^2} u(t) dt + dW^H(t)$$

with initial condition $u(0) = u_0 \in L^2(\Omega \times \mathbb{S}^2, \mathbb{P} \times \nu)$, where $\nu$ is the Lebesgue measure on $\mathbb{S}^2$.

In (1), $\Delta_{\mathbb{S}^2}$ is the Laplace-Beltrami operator on $\mathbb{S}^2$ defined as

$$\Delta_{\mathbb{S}^2} = \frac{\partial^2}{\partial \vartheta^2} + \cot \vartheta \frac{\partial^2}{\partial \varphi^2} + (\sin \vartheta)^{-2} \frac{\partial^2}{\partial \varphi^2},$$

where $\vartheta \in [0, \pi]$ represents the latitude and $\phi \in [0, 2\pi)$ the longitude in spherical coordinates. The Gaussian noise $W^H$ is a specialization of the infinite dimensional fractional Brownian noise in Tindel et al (23) to the sphere $\mathbb{S}^2$. More precisely, it is defined as follows.

**Definition 1.1** The noise $W^H = \{W^H(t), t \in \mathbb{R}_+\}$ is a Gaussian process with the following representation

$$W^H(t, x) = \sum_{\ell \geq 0} \sum_{m = -\ell}^{\ell} \sqrt{C_\ell} \beta_{\ell m}(t) Y_{\ell m}(x),$$

where $\{C_\ell, \ell = 0, 1, 2, \ldots\}$ is a sequence of positive constants, $Y_{\ell m}$, $(\ell = 0, 1, 2, \ldots, m = 0, \pm 1, \ldots, \pm \ell)$ are the spherical harmonic functions on $\mathbb{S}^2$ satisfying

$$\Delta_{\mathbb{S}^2} Y_{\ell m} = -\ell (\ell + 1) Y_{\ell m},$$

and the sequence of complex-valued Gaussian processes $\{\beta_{\ell m}(t)\}_{\ell m}$ satisfies the following two conditions

(a) for every $t \in \mathbb{R}$ and $\ell = 0, 1, 2, \ldots, \ell = 0, 1, 2, \ldots, m = 0, \pm 1, \ldots, \pm \ell$,

$$\overline{\beta_{\ell m}(t)} = (-1)^m \beta_{\ell, -m}(t).$$

(b) $\{\sqrt{2} \text{Re} \beta_{\ell m}(t), \sqrt{2} \text{Im} \beta_{\ell m}(t), \ell = 0, 1, 2, \ldots, m = 0, \ldots, \ell\}$ is a sequence of independent copies of a real-valued fractional Brownian motion $B^H = \{B^H(t), t \in \mathbb{R}_+\} with Hurst index $H \in (0, 1)$.

**Remark 1.2** It is readily seen that the sequence of Gaussian processes $\{\beta_{\ell m}(t)\}_{\ell m}$ satisfies

$$\mathbb{E}[\beta_{\ell m}(t) \overline{\beta_{\ell' m'}(s)}] = \delta_{\ell \ell'} \delta_{m m'} R_H(t, s),$$

for all $\ell = 0, 1, 2, \ldots, m = -\ell, \ldots, \ell$, where $\delta_{\ell}^\ell = 1$ if $\ell = \ell'$ and 0 otherwise, and where

$$R_H(t, s) = \frac{1}{2} \left[t^{2H} + s^{2H} - |t - s|^{2H}\right].$$
For the positive coefficients \( \{c_\ell, \ell = 0,1,2,\ldots\} \) in [24], we assume the following condition:

**Condition (A.1)** There exist constants \( \alpha > 0 \) and \( c_0 > 1 \) such that

\[
C_\ell = \Upsilon(\ell) (\ell + 1/2)^{-\alpha} \quad \text{and} \quad c_0^{-1} \leq \Upsilon(\ell) \leq c_0
\]

for \( \ell = 0,1,2,... \)

Note that for \( \alpha > 2 \), \( W^H = \{W^H(t), t \in \mathbb{R}_+\} \) is an \( L^2(\mathbb{S}^2) \)-valued \( \Lambda \)-fractional Brownian motion with \( \Lambda \) given below in (11). When \( 0 < \alpha \leq 2 \), \( W^H(t) \) can be viewed as a generalized fractional Brownian motion taking values on some Hilbert space \( U \supset L^2(\mathbb{S}^2) \). For instance, \( U \) is a Hilbert space such that for any \( \varphi, \psi \in U \),

\[
\langle \varphi, \psi \rangle_U = \int_{\mathbb{S}^2} \int_{\mathbb{S}^2} \varphi(x) \overline{\Lambda}(x,y) \psi(y) d\nu(x)d\nu(y),
\]

where \( \overline{\Lambda}(x,y) = \sum_{\ell=1}^{\infty} \ell^{-2} P_\ell(\langle x, y \rangle) \) for all \( x, y \in \mathbb{S}^2 \). Here \( P_\ell : [-1,1] \to \mathbb{R}, (\ell = 0,1,2,...) \) are the Legendre polynomials satisfying the normalization condition \( P_\ell(1) = 1 \) for all \( \ell \). So even if \( \sum_{\ell=0}^{\infty} C_\ell = \infty \), \( W^H(t) \) given by (2) is a well-defined \( U \)-valued Gaussian process, see [24] for more discussion about the Gaussian process \( W^H(t) \) taking values on more general Hilbert spaces. In analogy to the Euclidean space setting considered in [4] 26, one may refer to \( W^H \) as a fractional-colored Gaussian noise on the sphere \( \mathbb{S}^2 \).

The present paper is mainly motivated by the recent works of Lang and Schwab [15] who studied (1) driven by a Q-Wiener process which corresponds to the case of \( H = \frac{1}{2} \) and \( \alpha > 2 \) in the setting of the present paper, and by Tindel et al. [23] [24] [19] who studied (1) with \( x \in \mathbb{S}^1 \), which is the unit circle in \( \mathbb{R}^2 \), and the fractional Gaussian noise \( W^H \) on \( \mathbb{S}^1 \) for an arbitrary \( H \in (0,1) \). Our objectives are to establish the existence and uniqueness of the mild solution of (1), and to study the regularity properties of the solution process when it exists. For simplicity, we focus in this paper on the case of \( \frac{1}{2} < H < 1 \). The case of \( 0 < H < \frac{1}{2} \) is more delicate and will be considered in a subsequent paper.

In the stochastic heat equation (1), we make some assumptions on the initial value \( u_0 = \{u_0(x), x \in \mathbb{S}^2\} \). First recall from [17] that, a spherical random field \( Z = \{Z(x), x \in \mathbb{S}^2\} \) is called 2-weakly isotropic if

\[
\mathbb{E}[Z(x)Z(y)] = \mathbb{E}[Z(gx)Z(gy)]
\]

for all \( x, y \in \mathbb{S}^2 \) and \( g \in SO(3) \). If \( u_0 \) is a zero-mean, 2-weakly isotropic random field with finite variance, then by Theorem 5.13 in [17], we have the following spectral representation:

\[
u_0(x) = \sum_{\ell \geq 0} \sum_{m=-\ell}^{\ell} u_{0,\ell m} Y_\ell m(x), \quad \text{a.s.,} \tag{6}
\]

where the random variables \( u_{0,\ell m}, \ell = 0,1,2,..., m = 0,\pm 1,\ldots, \pm \ell, \) satisfy

\[
\mathbb{E}(u_{0,\ell m}u_{0,\ell m'}) = \delta_\ell^{\ell'} \delta_m^{m'} D_\ell
\]
for some nonnegative constants $D_\ell$, $\ell = 0, 1, 2, \ldots$. The sequence $\{D_\ell, \ell \geq 0\}$ is called the angular power spectrum of $u_0$. We will make use of the following assumption.

**Condition (A.2)** Either $u_0 \equiv 0$ or $u_0 = \{u_0(x), x \in S^2\}$ is a zero-mean isotropic Gaussian field which is independent from the Gaussian noise $W^H$. Moreover, there exist finite constants $\beta > 4$ and $D_0 > 0$ such that

$$D_\ell \leq D_0 (\ell + 1/2)^{-\beta}$$

for all $\ell = 0, 1, 2, \ldots$

In order to state our main theorem, we introduce the following notations. Let $I$ be an open interval on $\mathbb{R}$. For a function $u : I \to \mathbb{R}$ and an integer $k > 0$, we say that the $k$th weak derivative of $u$ exists if there exists a locally integrable function $v$ such that for all infinitely differentiable function $\varphi$ with compact support on $I$,

$$\int_I u D^k_t \varphi dt = (-1)^k \int_I v \varphi dt.$$ 

Such kind of function $v$ is uniquely determined up to a zero-measure set on $I$, and we write $v =: D^k_t u$ as usual. Let $\mathbb{H}^k(I)$ be the subspace of $L^2(I)$ such that

$$\mathbb{H}^k(I) = \{ u \in L^2(I) : \forall m = 0, 1, \ldots, k, D^m_t u \text{ exists and belongs to } L^2(I) \}.$$ 

The space $\mathbb{H}^k(I)$ is also called the Sobolev space with $k$th weak derivatives having finite $L^2$-norm, see for instance [8, 20] for more details about this and more general Sobolev spaces.

Throughout this paper, $T$ is a positive and finite constant and $T = [0, T]$. The following theorem is the main result of this paper.

**Theorem 1.3** Assume $H > 1/2$ and Conditions (A.1) and (A.2) hold. Then eq. (1) has a unique solution $\{u(t, x), t \in T, x \in S^2\}$ in $L^2(\Omega \times T \times S^2)$ which is a mean-zero Gaussian random field that is $2$-weakly isotropic in $x$ (for each fixed $t \in T$). Moreover, the solution has the following regularity properties:

(a) If $\alpha + 4H > 4$, then for every $t \in T$, we have $u(t, \cdot) \in C^1(S^2)$ a.s. Moreover, if $\alpha + 4H > 6$, then for every $t \in T$, $u(t, \cdot) \in C^2(S^2)$ a.s. and, for every $x \in S^2$, $u(\cdot, x) \in H^1(T)$ a.s.

(b) If $u_0 \equiv 0$ or $\beta > 4H + 2$, then for every $x \in S^2$,

$$\lim_{\varepsilon \to 0} \sup_{0 \leq s < t \leq T} \frac{|u(t, x) - u(s, x)|}{(t - s)^{\eta} \sqrt{\log(t - s)}} = K_{1,1}, \text{ a.s.},$$

where $\eta = H - \max \{(2 - \alpha)/4, 0\}$, and where $K_{1,1} > 0$ is a constant depending on $c_0$, $\alpha$ and $H$. 

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(c) If \( \alpha + 4H < 4 \) and \( u_0 \equiv 0 \), then for every \( t \in \mathbb{T} \), there exists a constant \( K_{1,2} > 0 \) depending on \( c_0, \alpha, t \) and \( H \) such that
\[
\lim_{\varepsilon \to 0} \sup_{x, y \in \mathbb{S}^2} \left[ \frac{|u(t, x) - u(t, y)|}{(d_{S^2}(x, y))^\gamma \sqrt{\log d_{S^2}(x, y)}} \right] = K_{1,2}, \text{ a.s.,} \tag{8}
\]
where \( \gamma = \alpha/2 - 1 + 2H \in (0, 1) \) and \( d_{S^2}(x, y) \) denotes the geodesic distance between \( x, y \).

In the following, we give some remarks that compare our results with some existing ones in the literature and raise some unsolved problems.

- Tindel et al. [23] proved the almost-sure Hölder continuity of \( u \) with respect to (w.r.t.) the time variable \( t \) for \( u(t) \) taking values on some general Hilbert space \( V \). Eq. (7) significantly improves their result because it implies the exact uniform modulus of continuity of \( u(t) \) taking values on \( L^2(S^2) \). Moreover, since \( \gamma \) increases with \( \alpha > 0 \), (7) indicates that the Hölder continuity of the solution in the time variable \( t \) can be improved by the smoothness of spatial structure in the fractional-colored noise \( W^H \).

- Parts (a) and (c) show that the smoothness of the solution in the spatial variable \( x \) deteriorates when \( \alpha \) decreases. In particular, when \( 2 < \alpha + 4H < 4 \), \( u \) is not differentiable w.r.t. spatial variable \( x \). Eq. (8) provides the exact uniform modulus of continuity of \( u \) in \( x \). On the other hand, when \( \alpha + 4H > 4 \), we are able to establish more precise information on the smoothness of \( u \) in \( x \) by providing the exact uniform moduli of continuity of higher-order derivatives of \( u \) in \( x \in S^2 \). See Corollary 5.2 below.

- Note that we have left out the cases of \( 4 < \beta \leq 4H + 2 \) and \( \alpha + 4H = 4 \). In the first case, the regularity properties of \( u_0 \) may compete with those of the solution of eq. (1) with \( u_0 \equiv 0 \). In the second case, we have not been able to prove the strong local nondeterminism for the solution \( u \). These cases are more subtle and some new method may be needed.

The plan of this paper is as follows. In Section 2, we recall some basic properties of stochastic integration with respect to fractional Brownian motion and investigate the smoothness of the Gaussian noise \( W^H \). We present the unique existence of the mild solution \( u(t) \) in \( L^2(\Omega \times S^2) \) in Section 3. In particular, we give the uniform convergence of this mild solution when \( \alpha + 4H > 4 \), which leads to that \( u \) also exists in some nice Sobolev spaces. In Section 4, some auxiliary technical tools such as estimation of variogram and strong local nondeterminism of the solution \( \{u(t, x), t \in \mathbb{T}, x \in S^2\} \) are provided. These are not only instrumental for our proofs in this paper but also useful for other purposes (cf. 27). Finally, we prove the exact moduli of continuity (7) and (8) of the solution \( u \) in Section 5.

Throughout this paper, we denote by “\( A \approx B \)” the commensurate case that there exist positive and finite constants \( c_1 < c_2 \) such that \( c_1B \leq A \leq c_2B \).
2 The Fractional-colored Noise

In this section, we provide some preliminaries on the fractional Gaussian noise \( W^H(t) \). Moreover, when \( \alpha > 2 \), we establish some regularity properties of the random field \( \{W^H(t, x), t \in \mathbb{R}_+, x \in S^2\} \).

We first recall briefly some well-known results about stochastic integration with respect to fractional Brownian motion \( B^H = \{B^H(t), t \in \mathbb{R}_+\} \) with \( H > \frac{1}{2} \). For any \( T > 0 \), let \( \mathcal{H}(T) \) be the completion of the space linearly spanned by the indicator functions \( \{1_{[s_1, t_1]}, 0 \leq s < t \leq T\} \) with respect to the inner product

\[
\langle 1_{[s_1, t_1]}, 1_{[s_2, t_2]} \rangle = \int_{s_1}^{t_1} \int_{s_2}^{t_2} |s-t|^{2H-2} dsdt,
\]

where \( 0 \leq s_i < t_i \leq T, i = 1, 2 \). Then, for any \( \varphi \in \mathcal{H}(T) \), the stochastic integral

\[
B^H(\varphi) = \int_0^T \varphi(t) dB^H(t)
\]

is well-defined. Moreover, for any \( \varphi, \psi \in \mathcal{H}(T) \),

\[
E[B^H(\varphi)B^H(\psi)] = \langle \varphi, \psi \rangle = \int_0^T \int_0^T \varphi(t)\psi(s)|t-s|^{2H-2} dt ds.
\]

See for instance [2] or [23] for more details about the stochastic integration with respect to fBm \( B^H \).

In the meantime, it is readily seen by [17] that when \( \alpha > 2 \), the definition of \( W^H \) in Section 1 is equivalent to that the noise \( W^H = \{W^H(t), t \in \mathbb{R}_+\} \) is an \( L^2(S^2) \)-valued Gaussian field such that \( E(W^H(t, x)) = 0 \) for all \( (t, x) \in \mathbb{R}_+ \times S^2 \) and its covariance function is given by

\[
E[W^H(t, x)W^H(s, y)] = R_H(t, s)\Lambda(x, y),
\]

with the spatial covariance function \( \Lambda : S^2 \times S^2 \to \mathbb{R}_+ \) which can be decomposed into

\[
\Lambda(x, y) = \sum_{\ell=0}^{\infty} \frac{2\ell + 1}{4\pi} C_\ell P_\ell(\langle x, y \rangle).
\]

Notice that [10] and (11) imply that

\[
E[W^H(t, x)W^H(s, y)] = E[W^H(t, gx)W^H(s, gy)]
\]

for every pair of \( (t, s) \in \mathbb{R}_+^2 \), all \( g \in SO(3) \), and all \( x, y \in S^2 \). Therefore, for every \( t \in \mathbb{R}_+ \), \( W^H(t) \) is a 2-weakly isotropic random field on \( S^2 \).

The following proposition provides some properties of the fractional-colored noise \( W^H \) which will be exploited later.

**Proposition 2.1** Under Conditions (A.1) and (A.2) with \( \alpha > 2 \), the Gaussian random field \( W^H \) defined in (2) is zero-mean, 2-weakly isotropic and converges in \( L^2(\Omega) \) uniformly for \( (t, x) \in T \times S^2 \). Moreover, the following statements hold:
(i) If $1/2 < H < 1$, then for every $x \in \mathbb{S}^2$, $W^H(\cdot, x) \in \mathbb{H}^1(T)$ a.s.
(ii) If $\alpha > 4$, then for every $t \in T$, $W^H(t, \cdot) \in C^1(\mathbb{S}^2)$ a.s.

**Proof.** Based on the previous discussion, we only need to prove (i) and (ii). First notice that the weak derivatives $\nabla S_2 W^H$ and $D_t W^H$ exist due to the uniform convergence of $W^H$ in $L^2(\Omega)$. For part (i), recall

$$\sum_{m=-\ell}^{\ell} Y_{\ell m}(x)Y_{\ell m}(y) = \frac{2\ell + 1}{4\pi} P_{\ell}((x, y))$$

(c.f. [17], section 3.4.2), then careful calculations show that

$$\mathbb{E} \left| \int_0^T \left| D_t W^H(t, x) \right|^2 dt \right|^2 = \int_0^T \mathbb{E} \left| \sum_{\ell \geq 0} \sum_{m=-\ell}^{\ell} \sqrt{C_\ell} \frac{d}{dt} \beta_{\ell m}(t) Y_{\ell m}(x) \right|^2 dt$$

$$= \sum_{\ell \geq 0} \frac{2\ell + 1}{4\pi} C_\ell \int_0^T \frac{d^2}{dt^2} \mathbb{E} |\beta_{\ell m}(t)|^2 dt$$

$$= \sum_{\ell \geq 0} \frac{2\ell + 1}{4\pi} C_\ell \int_0^T t^{2H-2} dt = c_{2,1} T^{2H-1}$$

for some constant $c_{2,1} > 0$ depending on $H$. The last equality holds if and only if $\alpha > 2$ and $H > 1/2$.

For part (ii), recall the gradient $\nabla S_2$ on $\mathbb{S}^2$ defined as follows:

$$\nabla S_2 = \left( \frac{\partial}{\partial \vartheta}, (\sin \vartheta)^{-1} \frac{\partial}{\partial \varphi} \right).$$

For any $x, y \in \mathbb{S}^2$, denote by $x = (\vartheta_x, \phi_x)$ and $y = (\vartheta_y, \phi_y)$. Then careful calculations show that

$$\frac{\partial^2}{\partial \vartheta_y \partial \varphi_x} P_{\ell}((x, y)) \big|_{x=y} = P'_{\ell}(1) = -\frac{\ell(\ell + 1)}{2},$$

and

$$\frac{\partial^2}{\partial \varphi_y \partial \varphi_x} P_{\ell}((x, y)) \big|_{x=y} = (\sin \vartheta)^2 P'_{\ell}(1) = -\frac{\ell(\ell + 1)}{2}.$$

Therefore, we obtain

$$\mathbb{E} \left| \frac{\partial}{\partial \vartheta} W^H(t, \vartheta, \phi) \right|^2 = \mathbb{E} \left| \frac{\partial}{\partial \varphi_x} \frac{\partial}{\partial \varphi_y} W^H(t, x) \right|_{x=y}$$

$$= \sum_{\ell \geq 0} C_\ell \frac{2\ell + 1}{4\pi} \mathbb{E} \left| \beta_{\ell m}(t) \right|^2 \left\{ \frac{\partial^2}{\partial \vartheta_y \partial \varphi_x} \left| P_{\ell}((x, y)) \right| \big|_{x=y} \right\}$$

$$= t^{2H} \sum_{\ell \geq 0} C_\ell \frac{2\ell + 1}{4\pi} \frac{\ell(\ell + 1)}{2}$$

$$\leq \frac{c_0}{4\pi} t^{2H} \sum_{\ell \geq 0} \sum_{m=-\ell}^{\ell} (\ell + 1)^3 \alpha.$$
The last term above is finite if and only if $\alpha > 4$. Similarly, we have

$$E|\frac{\partial}{\partial \phi} W^H(t, \theta, \phi)|^2 \leq \frac{(\sin \theta)^2}{4\pi} t^{2H} c_0 \sum_{\ell \geq 0} \sum_{m=-\ell}^\ell (\ell + \frac{1}{2})^{3-\alpha} < \infty,$$

and hence

$$E|\nabla S^2 W^H(t, \theta, \phi)|^2 \leq E|\frac{\partial}{\partial \theta} W^H(t, \theta, \phi)|^2 + (\sin \theta)^2 E|\frac{\partial}{\partial \phi} W^H(t, \theta, \phi)|^2 < \infty$$

for $\alpha > 4$. The proof is then completed. $$\blacksquare$$

### 3 Existence of the Solution and Proof of (a) in Theorem 1.3

Recall the form (2) of $W^H$, the following is an immediate consequence of Theorem 1 of Tindel et al. [23].

**Proposition 3.1** Assume that $H > 1/2$ and Conditions (A.1) and (A.2) hold. Then there exists a unique solution $u \in L^2(\Omega \times T \times S^2)$ for eq.(1) with the following mild form

$$u(t, x) = \sum_{\ell \geq 0} \sum_{m=-\ell}^\ell \left( e^{-\ell(t+1)t} u_{0, \ell m} + \sqrt{C_\ell} \int_0^t e^{-\ell(t+1)(t-s)} d\beta_{\ell m}(s) \right) Y_{\ell m}(x)$$

for $(t, x) \in T \times S^2$.

**Proof.** First we take the spherical harmonics $\{Y_{\ell m}\}_{\ell m}$ as the orthonormal basis of $L^2(S^2)$ and define $\Phi : L^2(S^2) \rightarrow L^2(S^2)$ a positive linear operator such that

$$\Phi(Y_{\ell m}) = \sqrt{C_\ell} Y_{\ell m}.$$ 

It is readily seen that the adjoint operator $\Phi^* = \Phi$. In the meantime, let $B^H$ be the cylindrical fractional Brownian motion on the Hilbert space $L^2(S^2)$ defined as

$$B^H(t) = \sum_{\ell \geq 0} \sum_{m=-\ell}^\ell \beta_{\ell m}(t) Y_{\ell m},$$

where $\{\beta_{\ell m}(t)\}_{\ell, m}$ is the sequence of complex-valued fBms in Definition [23]. Then,

$$\Phi B^H(t) = \sum_{\ell \geq 0} \sum_{m=-\ell}^\ell \beta_{\ell m}(t) \Phi Y_{\ell m} = W^H(t)$$
is well defined in the Hilbert space $U$ with inner product defined in (5). See [23, p.190] for more discussion about the stochastic integration w.r.t. $d(\Phi B^H(t)) = \Phi dB^H(t)$.

Now we are ready to prove Proposition 3.1. It is known from [23] that the mild solution of eq. (1) is unique if it exists and can be written as

$$u(t) = e^{t\Delta_2} u_0 + \int_0^t e^{(t-s)\Delta_2} \Phi dB^H(s), \quad t \in [0,T]. \quad (14)$$

To prove the existence, we only need to establish that the $u(t)$ defined in (14) belongs to $L^2(\Omega \times \mathbb{S}^2)$ for every $t \in [0,T]$. It is readily seen that under Condition (A.2),

$$e^{t\Delta_2} u_0 = \sum_{\ell \geq 0} \sum_{m=-\ell} \sum_{\ell \geq 0} \sum_{m=-\ell} e^{-\ell(t+1)} u_{0,\ell m} Y_{\ell m}(x) \in L^2(\Omega \times \mathbb{S}^2).$$

For the second part on the right hand side of equation (14), we first set the function $G_H(\lambda) = (\max\{\lambda, 1\})^{-2H}$. A simple calculation yields that under the spherical harmonic basis $\{Y_{\ell m}\}_{\ell, m}$, the trace of the operator $\Phi^* G_H(\Delta_2) \Phi$

$$Tr(\Phi^* G_H(\Delta_2) \Phi) = \sum_{\ell \geq 0} \sum_{m=-\ell} \langle \Phi^* G_H(\Delta_2) \Phi Y_{\ell m}, Y_{\ell m} \rangle_{L^2(\mathbb{S}^2)}$$

$$= C_0 + \sum_{\ell \geq 1} \frac{2\ell + 1}{4\pi} C_{\ell} (\ell(\ell+1))^{2H}$$

$$\leq C_0 + c_0 \sum_{\ell \geq 1} \frac{2\ell + 1}{4\pi} (\ell + 1/2)^{-\alpha}(\ell(\ell+1))^{-2H} < \infty$$

under conditions (A.1) and $H > 1/2$. Hence, $\Phi^* G_H(\Delta_2) \Phi$ is in the trace class by the fact that this operator is positive. Therefore, the unique existence of the mild solution (13) to eq. (11) is obtained by Theorem 1 in [23].

We shall study in more details the sample path properties of the solution $u(t, x)$ in (13) in the following sections. At this moment, let us first focus on the special case $H > 1/2$ and $\alpha > 2$. We write $u(t, x)$ in (13) in the following form:

$$u(t) = \sum_{\ell=0}^{\infty} u_{\ell}(t, x) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell} u_{\ell m}(t) Y_{\ell m}(x), \quad (15)$$

where

$$u_{\ell m}(t) = e^{-\ell(t+1)} u_{0,\ell m} + \sqrt{C_{\ell}} \int_0^t e^{-\ell(t+1)(t-s)} d\beta_{\ell m}(s). \quad (16)$$

Note that under Conditions (A.1) and (A.2), $u_0$ and $W^H$ are independent, which implies that the two sequences $\{u_{0,\ell m}\}_{\ell m}$ and $\{\beta_{\ell m}\}_{\ell m}$ are mutually independent. Moreover,

$$E[u_{\ell m}(t) u_{\ell' m'}(s)] = \delta_{\ell'} \delta_{m'} \delta_{\ell m} U_\ell(t,s), \quad (17)$$

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in view of (9). Here $U_t(t, s) = U_t(s, t)$ for any $s, t \in T$.

Now we prove the following lemma which implies that the series in (15) converges uniformly both in the senses of $L^2(\Omega)$ for all $(t, x) \in T \times S^2$ and $L^2(\Omega \times T)$ for all $x \in S^2$.

**Lemma 3.2** Assume $1/2 < H < 1$, Conditions (A.1) and (A.2). Then there exists a constant $K_{3,1} > 0$, depending on $\alpha, \beta$ and $H$, such that for any $L$ large,

$$
\mathbb{E} \left| u(t, x) - \sum_{\ell=0}^{L} u_{\ell}(t, x) \right|^2 \leq K_{3,1} \left\{ t^{\beta/2-1} L^{-\beta} e^{-L(L+1)t} + L^{-(\alpha-2+4H)} \right\}
$$

for any fixed $(t, x) \in T \times S^2$. Moreover,

$$
\mathbb{E} \left\| u(\cdot, x) - \sum_{\ell=0}^{L} u_{\ell}(\cdot, x) \right\|^2_{L^2(\Omega)} \leq K_{3,1} \max \left\{ T^{\beta/2-1}, T \right\} L^{-\min\{\beta-2, \alpha-2+4H\}}
$$

for every $x \in S^2$.

**Proof.** Recall equation (16) for $u_{\ell m}(t)$. We have

$$
\mathbb{E} \left| u(t) - \sum_{\ell=0}^{L} u_{\ell}(t) \right|^2 = \mathbb{E} \left\| \sum_{\ell=L+1}^{\infty} u_{\ell}(t) \right\|^2 = I_1 + I_2,
$$

where

$$
I_1 = \mathbb{E} \left\| \sum_{\ell=L+1}^{\infty} \sum_{m=-\ell}^{\ell} e^{-\ell(t+1)s} u_{0,\ell m} Y_{\ell m} \right\|^2
$$

and

$$
I_2 = \mathbb{E} \left\| \sum_{\ell=L+1}^{\infty} \sum_{m=-\ell}^{\ell} \sqrt{C_{\ell}} \int_{0}^{t} e^{-\ell(t+1)(t-s)} d\beta_{\ell m}(s) Y_{\ell m} \right\|^2,
$$

in view of (17). It is readily seen that under Conditions (A.1) and (A.2),

$$
I_1 = \sum_{\ell=L+1}^{\infty} e^{-\ell(t+1)s} \frac{2\ell+1}{4\pi} D_{\ell} \leq c_{3,1} t^{\beta/2-1} L^{-\beta} e^{-L(L+1)t}
$$

in view of the equality (19). Here $c_{3,1}$ is some positive constant depending on $\beta$. Hence,

$$
\int_{0}^{T} I_1 dt \leq c_{3,2} T^{\beta/2-1} L^{-\beta-2}
$$

where

$$
U_{\ell}(t, s) = e^{-\ell(t+1)(t+s)} D_{\ell} + C_{\ell} \int_{0}^{t} \int_{0}^{s} e^{-\ell(t+1)(t+s-\lambda)} \xi - \lambda)^{2H-2} d\xi d\lambda
$$
for some constant $c_{3,2} > 0$. For the second sum $I_2$, we have

$$I_2 = \sum_{\ell=L+1}^{\infty} \sum_m C_{\ell} \mathbb{E} \left[ \int_0^t e^{-\ell(t+1)(t-s)} d\beta_{\ell m}(s) \right]^2 |Y_{\ell m}|^2$$

in view of the properties (3) and (4) of $\beta_{\ell m}$ and the equality (12). It is readily seen that the process $\int_0^t e^{-\ell(t+1)(t-s)} d\beta_{\ell m}(s)$ is Gaussian with mean zero. Now we focus on its variance, which is

$$\sigma_{\ell}^2(t) =: \mathbb{E} \left[ \int_0^t e^{-\ell(t+1)(t-s)} d\beta_{\ell m}(s) \right]^2$$

$$= \int_0^t \int_0^t e^{-\ell(t+1)(2s-r)} s-r|^{2H-2} dr ds.$$

Let

$$g_{\ell}(s, \lambda) = e^{-\ell(t+1)(s-\lambda)} 1_{[0,s]}(\lambda). \quad (23)$$

It is known that (see for instance [4], Lemma A.1.)

$$\int_0^T \int_0^T g_{\ell}(t, \lambda) g_{\ell}(s, \xi) |\xi - \lambda|^{2H-2} d\xi d\lambda = c_H \int_\mathbb{R} \hat{g}_{\ell}(t, \tau) \hat{g}_{\ell}(s, \tau) \frac{d\tau}{|\tau|^{2H-1}}, \quad (24)$$

where $c_H = \left[ 2^{2(1-H)} \pi^{1/2} \right]^{-1} \Gamma(H-1/2) / \Gamma(1-H)$ and

$$\hat{g}_{\ell}(t, \tau) = \int_0^t e^{-i\tau\lambda} e^{-\ell(t+1)(t-\lambda)} d\lambda = e^{-it\tau} - e^{-\ell(t+1)t} \ell(t+1) - i\tau. \quad (25)$$

Moreover,

$$\int_\mathbb{R} \frac{|\tau|^{-2H+1}}{[\ell(t+1)]^2 + \tau^2} d\tau = c_{3,3} \left[ \ell(t+1) \right]^{-2H} \quad (26)$$

with $c_{3,3} = \int_\mathbb{R} \frac{|\tau|^{-2H+1}}{1+\tau^2} d\tau$. Thus,

$$I_2 = \sum_{\ell=L+1}^{\infty} \frac{2\ell + 1}{4\pi} C_{\ell} \sigma_{\ell}^2(t) \leq \frac{c_{3,3}}{2\pi} \sum_{\ell=L+1}^{\infty} (\ell + \frac{1}{2})^{1-\alpha-4H}$$

$$\leq c_{3,4} L^{-(\alpha-2+4H)}, \quad (27)$$

with some positive constant $c_{3,4}$ depending on $\alpha$ and $H$, which leads to

$$\int_0^T I_2 dt \leq c_{3,4} L^{-(\alpha-2+4H)} T. \quad (28)$$

Combining inequalities (21), (22), (24) and (28), we obtain the approximations in Lemma 3.2. \[\blacksquare\]

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Based on the uniform convergence (19) in Lemma 3.2 and (17), we have for every pair of $(t, s) \in T \times T$,

\[
\mathbb{E}[u(t, x)u(s, y)] = \sum_{\ell=0}^{\infty} U_\ell(t, s) \sum_{m=-\ell}^{\ell} Y_{\ell m}(x)Y_{\ell m}(y) = \sum_{\ell=0}^{\infty} \frac{2\ell + 1}{4\pi} U_\ell(t, s)P_\ell((x, y)) = \mathbb{E}[u(t, g x)u(s, g y)]
\]

for all $g \in SO(3)$ and all $x, y \in \mathbb{S}^2$. In particular, for every $t \in T$, the random field $u(t) = \{u(t, x), x \in \mathbb{S}^2\}$ is 2-weakly isotropic.

Now we are ready to give the following result, which is also part (a) of Theorem 1.3. More precise information on differentiability of $u(t, x)$ in variable $x$ can be found in Corollary 5.2 below.

**Theorem 3.3** Assume $1/2 < H < 1$ and Conditions (A.1) and (A.2) hold with $\alpha + 4H > 4$. Then the following properties hold:

(i) for every $t \in T$, we have $u(t, \cdot) \in C^1(\mathbb{S}^2)$ a.s.

(ii) if $\alpha + 4H > 6$, then for every $t \in T$, $u(t, \cdot) \in C^2(\mathbb{S}^2)$ a.s. and, for every $x \in \mathbb{S}^2$, $u(\cdot, x) \in H^1(T)$ a.s.

**Proof.** For part (i), we only need to prove that $\mathbb{E} |\nabla_{\mathbb{S}^2} u(t)|^2 < \infty$. Similar to the argument for obtaining $\mathbb{E} |\nabla_{\mathbb{S}^2} W^H(t, x)|^2$ in the proof of Proposition 2.1, we have

\[
\mathbb{E} |\nabla_{\mathbb{S}^2} u|^2 = \sum_{\ell \geq 0} \ell(\ell + 1) U_\ell(t, t)
\]

in view of Lemma 3.2. Now recall (18) and (27), we have

\[
\mathbb{E} |\nabla_{\mathbb{S}^2} u|^2 = \sum_{\ell \geq 0} \ell(\ell + 1) \frac{2\ell + 1}{4\pi} U_\ell(t, t)
\]

\[
+ \sum_{\ell \geq 0} \ell(\ell + 1) \frac{2\ell + 1}{4\pi} C_{\ell m} \left| \int_0^t e^{-\ell(\ell+1)(t-s)} d\beta_{\ell m}(s) \right|^2
\]

\[
\leq c_{3,5} \left[ t^{\beta/2-1} + \sum_{\ell \geq 0} (\ell + 1)^{3-\alpha - 4H} \right] ^2
\]

\[
\leq c_{3,6},
\]

where the last inequality is true because $\alpha + 4H > 4$. The constants $c_{3,5}$ and $c_{3,6}$ are finite and positive depending only on $T, \alpha, \beta$ and $H$.

For part (ii), notice that by eq. (11) we have

\[
\int_0^T |D_{t} u|^2 dt \leq \int_0^T |\Delta_{\mathbb{S}^2} u|^2 dt + \int_0^T |D_{t} W^H|^2 dt.
\]
Since
\[ \Delta_{\mathcal{S}^2} u = - \sum_{\ell=0}^{\infty} \ell(\ell+1)u_\ell \]
in view of Lemma 3.2 and the fact that \( \alpha + 4H > 6 \), we have
\[ E \int_0^T |\Delta_{\mathcal{S}^2} u|^2 dt = \sum_{\ell=0}^{\infty} \ell^2(\ell+1)^2 \frac{2\ell + 1}{4\pi} e^{-\ell(t+1)(t+s)} D_\ell \]
\[ + \sum_{\ell \geq 0} \ell^2(\ell+1)^2 \frac{2\ell + 1}{4\pi} C_\ell E \int_0^t e^{-\ell(t+1)(t-s)} d\beta_{\ell m}(s) \]
\[ \leq c_{3.7} \left[ t^{3/2-1} + \sum_{\ell=0}^{\infty} \left( \ell + \frac{1}{2} \right)^{5-\alpha-4H} \right] (29) \]
\[ \leq c_{3.8}, \]
for some positive constants \( c_{3.7} \) and \( c_{3.8} \) depending only on \( T, \alpha, \beta \) and \( H \). Thus, we have proved the first conclusion in part (ii) as well as the second one since
\[ E \int_0^T |D_t u|^2 dt < \infty, \]
in view of (29) and Proposition 2.1. Hence the proof is completed. \( \blacksquare \)

4 Some Technical Tools

In this section, we study the variogram and strong local nondeterminism of the solution (13). These properties are the key for investigating the exact modulus of continuity for the solution \( u(t, x) \).

4.1 Estimation of the Variogram

**Proposition 4.1** Assume that \( 1/2 < H < 1 \) and Conditions (A.1) and (A.2) hold. Then the solution \( u(t, x) \) defined in (13) satisfies the following condition: There exist constants \( K_{4.1} > 0 \) and \( 0 < \delta < 1 \), such that for any \( |t - s| < \delta \), \( \theta = d_{\mathcal{S}^2}(x, y) < \delta \),
\[ E|u(t, x) - u(s, y)|^2 \leq K_{4.1} [t - s]^{2\eta} + \rho_\gamma^2(\theta), \] (30)
where \( \eta = H - \max\{(2 - \alpha)/4, 0\} \), \( \gamma = \alpha/2 - 1 + 2H \), and the function \( \rho_\gamma : \mathbb{R}^+ / \{0\} \rightarrow \mathbb{R}^+ \) is defined as follows:
\[ \rho_\gamma(s) = \begin{cases} 
    s^{\gamma}, & \text{if } \gamma < 1, \\
    s^{\gamma} & \text{if } \gamma = 1, \\
    s, & \text{if } \gamma > 1.
\end{cases} \] (31)
Proof. For any \((t, x), (s, y) \in T \times S^2\), we have
\[
E|u(t, x) - u(s, y)|^2 = E \left| \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} u_{\ell m}(t) Y_{\ell m} - u_{\ell m}(s) Y_{\ell m} \right|^2
\]
(32)
\[
= \sum_{\ell=1}^{\infty} \frac{2\ell + 1}{4\pi} \left[ U_\ell(t, t) + U_\ell(s, s) - 2U_\ell(t, s) P_\ell(\langle x, y \rangle) \right]
\]
in view of (17) and (12). Recall (18), we obtain
\[
U_\ell(t, t) + U_\ell(s, s) - 2U_\ell(t, s) P_\ell(\langle x, y \rangle) = D_\ell A_{\ell,1} + C_\ell A_{\ell,2},
\]
(33)
where
\[
A_{\ell,1} := e^{-2\ell(t+1)t} + e^{-2\ell(t+1)s} - 2e^{-\ell(t+1)(t+s)} P_\ell(\langle x, y \rangle),
\]
and
\[
A_{\ell,2} := \sum_{\mu=t,s} \int_0^\mu \int_0^{\mu} e^{-\ell(t+1)(2\mu-\lambda-\xi)} |\lambda - \xi|^{2-2H} d\xi d\lambda
\]
\[
- 2P_\ell(\langle x, y \rangle) \int_0^t \int_0^s e^{-\ell(t+1)(t+s-\lambda-\xi)} |\lambda - \xi|^{2H-2} d\xi d\lambda.
\]
Hence (30) follows from Lemmas 4.2 and 4.3 below.

Lemma 4.2 There exists a constant \(K_{1,2} > 1\) such that for any \(0 < t-s, \theta < \delta\), we have
\[
\sum_{\ell=0}^{\infty} \frac{2\ell + 1}{4\pi} D_\ell A_{\ell,1} \leq K_{1,2} \left\{ (t-s)^{2(\beta/2-1)} + [1 + (t+s)^{\beta/2-2}] \theta^2 \right\}.
\]
Proof. Let \(\langle x, y \rangle = \cos \theta\) where \(\theta \in [0, \pi]\) is the geodesic distance between \(x\) and \(y\) on \(S^2\), then we have
\[
A_{\ell,1} = \left[ e^{-\ell(t+1)t} - e^{-\ell(t+1)s} \right]^2 + 2e^{-\ell(t+1)(t+s)} \left[ 1 - P_\ell(\cos \theta) \right]
\]
(34)
\[
= \tilde{A}_{\ell,1}^0 + \tilde{A}_{\ell,1}^1.
\]
Notice that
\[
\sum_{\ell=0}^{\infty} \frac{2\ell + 1}{4\pi} D_\ell \tilde{A}_{\ell,1}^0 \leq \int_s^t \int_s^t \left[ \int_1^\infty x^{5-\beta} e^{-x^2(w+v)} dx \right] dw dv
\]
\[
\leq \int_s^t \int_s^t (w + v)^{\beta/2-3} dw dv
\]
(35)
\[
\leq \begin{cases} (2t)^{\beta/2-3}, & \text{if } \beta \geq 6, \\
\frac{2}{\beta-4} (t-s)^{\beta/2-1}, & \text{if } 4 < \beta < 6,
\end{cases}
\]
In deriving the last inequality, we have used the elementary fact that, if $4 < \beta < 6$, then $0 < \frac{\beta}{2} - 2 < 1$ and, consequently, $b^{\beta/2 - 2} - a^{\beta/2 - 2} \leq (b - a)^{\beta/2 - 2}$ for all $0 < a < b < \infty$.

Now let us focus on $\tilde{A}_{t,1}$. Recall the following Hilb’s asymptotics (see [21], page 195, Theorem 8.21.6 or [9] 8.722): there exists a constant $c_{4,1} > 0$, such that, uniformly for all $\theta \in (0, \pi)$,

$$P_\ell(\cos \theta) = \left\{ \theta \sin \theta \right\}^{1/2} J_0\left((\ell + \frac{1}{2})\theta\right) + \delta_\ell(\theta),$$

where

$$\delta_\ell(\theta) \lesssim \begin{cases} \theta^2O(1) & \text{for } 0 < \theta < c_{4,1}\ell^{-1}, \\ \theta^{3/2}O(\ell^{-3/2}) & \text{for } \theta > c_{4,1}\ell^{-1} \end{cases},$$

and $J_0$ is the Bessel function defined as

$$J_0(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k!)^2} \left( \frac{x}{2} \right)^{2k}$$

(c.f. [9], 8.402), which yields that

$$\lim_{u \to 0} \frac{1 - J_0(c_{4,1}u)}{c_{4,1}u^2} = \frac{1}{2}.$$

Thus, by using the fact that

$$\frac{\theta}{\sin \theta} - 1 = \frac{\theta^2}{6} + O(\theta^3), \quad \text{as } \theta \to 0,$$

we obtain that for any positive integer $L < c_{4,1}\theta^{-1}$,

$$\sum_{\ell=1}^{L} \frac{2\ell + 1}{4\pi} D_\ell e^{-\ell(t+1)(t+s)} \left\{ 1 - P_\ell(\cos \theta) \right\} \leq D_0 \sum_{\ell=1}^{L} (\ell + \frac{1}{2})^{1-\beta} e^{-\ell(t+1)(t+s)} \left( \frac{\ell^2}{c_{4,1}^2} - \frac{1}{6} \right) \theta^2 \leq c_{4,2} e^{(t+s)/4(t+s)^{\beta/2-2}} \cdot C(t + s, \theta) \theta^2,$$

where $c_{4,2}$ is a positive constant depending on $D_0$, $c_{4,1}$ and $\beta$, and

$$C(t + s, \theta) = \left[ \Gamma(2 - \beta/2, t + s) - \Gamma(2 - \beta/2, c_{4,1}^2(t + s)\theta^{-2}) \right] \leq C(1 + (t + s)^{2-\beta/2})e^{-(t+s)},$$

with $\Gamma(a, x)$ the incomplete Gamma function defined as

$$\Gamma(a, x) = \int_{x}^{\infty} e^{-u}u^{a-1}du.$$
(c.f. [9] 3.381 and 8.350). On the other hand, recall that \( 1 - P_t(\cos \theta) \leq 2 \) uniformly for all \( \theta \), whence we have, for any \( U > \frac{c_1}{\theta^2} \),

\[
\sum_{\ell=U}^{\infty} \frac{2\ell + 1}{4\pi} D'e^{-\ell(t+1)(t+s)} \{ 1 - P_t(\cos \theta) \} 
\leq D_0 \int_U^{+\infty} (\ell + \frac{1}{2})^{1-\beta} e^{-\ell(t+1)(t+s)} d\ell 
\leq D_0 e^{(t+s)/4} (t+s)^{\beta/2-1} \int_{c_1(t+s)/\theta^2}^{\infty} e^{-u-u^{-\beta/2}} du 
\leq D_0 e^{(t+s)/4} (t+s)^{\beta/2-2} T(2 - \beta/2, c_2(t+s)^{-\theta^2}) \cdot \theta^2.
\]

Thus,

\[
\sum_{\ell=0}^{\infty} \frac{2\ell + 1}{4\pi} D'e^{-\ell(t+1)(t+s)} \{ 1 - P_t(\cos \theta) \} \leq c_{4,3} \left[ 1 + (t+s)^{\beta/2-2} \right] \theta^2 \tag{37}
\]

for some constant \( c_{4,3} > 0 \) depending on \( D_0 \). Lemma (4.2) is then obtained by (34), (35) and (37).

By now let us focus on \( A_{\ell,2} \).

**Lemma 4.3** There exists a constant \( K_{4,3} > 1 \) such that for any \( 0 < t-s, \theta < \delta \), we have

\[
\sum_{\ell=0}^{\infty} 2\ell + 1 \frac{1}{4\pi} C_\ell A_{\ell,2} \leq K_{4,3} \{(t-s)^{2\eta + \rho_\eta^2(\theta)} \},
\]

with \( \eta \) and \( \rho_\eta \) defined in Proposition (4.7).

**Proof.** Recall (23), we decompose \( A_{\ell,2} \) into the following form

\[
A_{\ell,2} = \int_0^{T} \int_0^{T} \left[ g(t, \lambda) - g(t, \xi) \right] \left[ \xi - \lambda \right]^{2H-2} d\xi d\lambda 
+ 2 \left[ 1 - P_t((x, y)) \right] \int_0^{t} \int_0^{t} e^{-\ell(t+1)(t+s-\lambda-\xi)} |\lambda - \xi|^{2H-2} d\xi d\lambda \tag{38}
\]

and we divide the proof into two steps.

Step 1: Approximation for \( \tilde{A}_{\ell,2}^0 \). Recall (24) and (25),

\[
\tilde{A}_{\ell,2}^0 = c_H \int_\mathbb{R} \left| \tilde{g}_t(t, \tau) - \tilde{g}_t(s, \tau) \right|^2 |\tau|^{-(2H-1)} d\tau 
\leq c_H \int_\mathbb{R} \left[ e^{-\ell(t+1)s} - e^{-\ell(t+1)t} \right]^2 |\tau|^{-(2H-1)} d\tau 
+ c_H \int_\mathbb{R} \left[ e^{-it\tau} - e^{-is\tau} \right]^2 |\tau|^{-(2H-1)} d\tau 
:= c_H \left\{ \tilde{\bar{A}}_{\ell,2}^0 + 2\tilde{A}_{\ell,2}^0 \right\}.
\]
Recall (26) and use the fact
\[
\left| e^{-\ell(t+1)s} - e^{-\ell(t+1)t} \right|^2 = \int_s^t \int_s^t e^{-\ell(t+1)(w+v)} dwdv,
\]
we have
\[
\sum_{\ell=1}^{\infty} \frac{2\ell + 1}{4\pi} C_\ell \cdot \tilde{A}_{\ell,2}^0 \leq c_{3,3} \sum_{\ell=1}^{\infty} \frac{2\ell + 1}{4\pi} C_\ell \int_s^t \int_s^t e^{-\ell(t+1)(w+v)} dwdv \left| \ell(t+1) \right|^{2-2H}
\]
\[
\leq c_{4,4} \int_s^t \int_s^t (w+v)^{\alpha/2-3+2H} dwdv
\]
\[
\leq c_{4,5} |t-s|^{\min\{\alpha/2-1+2H,2\}}.
\]
(39)

In the meantime,
\[
\sum_{\ell=1}^{\infty} \frac{2\ell + 1}{4\pi} C_\ell \cdot \tilde{A}_{\ell,2}^0 \leq \sum_{\ell=1}^{\infty} \frac{(2\ell + 1)C_\ell}{4\pi \left| \ell(t+1) \right|^2} \int_{\mathbb{R}} \left| e^{-it\tau} - e^{-ist\tau} \right|^2 \left| \tau \right|^{-(2H-1)} d\tau
\]
\[
\leq c_{4,6} c_H \int_s^t \int_s^t |\lambda - \xi|^{2H-2} d\xi d\lambda = c_{4,7} |t-s|^{2H}.
\]
(40)

Here and above, \(c_{4,4}, \ldots, c_{4,7}\) are positive constants depending on \(c_0, \alpha\) and \(H\).

Therefore, combining inequalities (39) and (40), we have that for \(|t-s|\) small enough,
\[
\sum_{\ell=1}^{\infty} \frac{2\ell + 1}{4\pi} C_\ell \tilde{A}_{\ell,2}^0 \leq 2c_{4,6} |t-s|^{2H-\max\{1-\alpha/2,0\}}.
\]
(41)

Step 2: Approximation for \(\tilde{A}_{\ell,2}^1\). Recall (24), (25) and (26), we have
\[
\left| \int_0^t \int_0^s e^{-\ell(t+1)(t+s-\lambda-\xi)} |\lambda - \xi|^{2H-2} d\xi d\lambda \right|
\]
\[
\leq c_H \int_{\mathbb{R}} \overline{g(t,\tau)} \overline{g(s,\tau)} \left| \tau \right|^{-(2H-1)} d\tau = c_{4,3} \left| \ell(t+1) \right|^{-2H}.
\]
Therefore, by Lemma 10, we have
\[
\sum_{\ell=0}^{\infty} \frac{2\ell + 1}{4\pi} C_\ell \tilde{A}_{\ell,2}^1 \leq c_0 c_H c_{4,4} \sum_{\ell=0}^{\infty} \left( \ell + \frac{1}{2} \right)^{1-\alpha-4H} [1 - P_\ell(\langle x, y \rangle)]
\]
\[
\leq c_{4,8} \rho_{\gamma}^2(\theta)
\]
for some positive constant \(c_{4,8}\) depending on \(c_0, \alpha\) and \(H\). Hence, combining inequalities (35) and (41) together with (42), we obtain Lemma (4.3).
4.2 Strong Local Nondeterminism

In this section we prove the properties of strong local nondeterminism of the solution \( \{u(t, x) : t \in \mathbb{T}, x \in \mathbb{S}^2\} \) in time variable \( t \in \mathbb{T} \) and spatial variable \( x \in \mathbb{S}^2 \), respectively.

First, let \( x \in \mathbb{S}^2 \) be fixed and we consider the Gaussian process \( \{u(t, x), t \in \mathbb{T}\} \). Without loss of generality, we write \( u(t) = u(t, x) \) for brevity.

**Proposition 4.4** Assume \( 1/2 < H < 1 \) and Conditions (A.1) and (A.2). Then there exist constants \( K_{4.4} > 0 \) and \( 0 < \varepsilon < \delta \) (which do not depend on \( x \in \mathbb{S}^2 \)) such that for all \( t \in (0, T] \) and \( r \in (0, \varepsilon) \),

\[
\text{Var}(u(t)|u(s) : s \in \mathbb{T}, t - s \geq r) \geq K_{4.4} r^{2\eta},
\]

where \( \eta = H - \max\{(2 - \alpha)/4, 0\} \).

**Proof.** The proof is inspired by the proof of [27, Theorem 2.1], but with a modification. It is sufficient to prove that, there exists some positive constant \( c_{4.9} \) such that

\[
V_i := \mathbb{E}\left| u(t) - \sum_{j=1}^{n} a_j u(t_j) \right|^2 \geq c_{4.9} r^{2H}
\]

for all integers \( n \geq 1 \) and all \( t_1, ..., t_n \in \mathbb{T} \) satisfying \( |t - t_j| \geq r \).

Similar to (32), we have

\[
V_i = \mathbb{E}\left[ \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} u_{\ell m}(t) Y_{\ell m}(x) - \sum_{j=1}^{n} a_j u_{\ell m}(t_j) Y_{\ell m}(x) \right]^2
\]

\[
= \sum_{\ell=0}^{\infty} \frac{2\ell + 1}{4\pi} \left[ U_{\ell}(t, t) + \sum_{i,j=1}^{n} a_i a_j U_{\ell}(t_i, t_j) - \sum_{j=1}^{n} a_j U_{\ell}(t, t_j) \right].
\]

Recall (43), we have

\[
V_i = \sum_{i,j=0}^{n} a_i a_j \sum_{\ell=0}^{\infty} \frac{2\ell + 1}{4\pi} C_{\ell} \int_{t_i}^{t_j} \int_{0}^{\infty} e^{-\ell(t+1)(t_i+t_j-\xi-\lambda)} \left| \xi - \lambda \right|^{2H-2} d\xi d\lambda
\]

\[
+ \sum_{\ell=0}^{\infty} \frac{2\ell + 1}{4\pi} D_{\ell} \left| \sum_{j=1}^{n} a_j e^{-\ell(t+1) t_j} \right|^2
\]

\[
\geq \sum_{i,j=0}^{n} a_i a_j \sum_{\ell=0}^{\infty} \frac{2\ell + 1}{4\pi} C_{\ell} \int_{t_i}^{t_j} \int_{0}^{\infty} e^{-\ell(t+1)(t_i+t_j-\xi-\lambda)} \left| \xi - \lambda \right|^{2H-2} d\xi d\lambda
\]

\[
:= M(t),
\]

where the coefficient \( a_0 = -1 \).

Now we construct a bump function \( \delta_{t, r}(\cdot) \in \mathcal{S}(\mathbb{R}) \) (the Schwartz space on \( \mathbb{R} \)) for any \( t > 0 \) such that its Fourier transform \( \hat{\delta}_{t, r} \) vanishes outside the open
interval \((t - r, t + r)\). Let
\[
\hat{\delta}_{t,r}(\lambda) = \begin{cases} 
\exp\left\{-\frac{r^2}{r^2 - (\lambda - t)^2}\right\}, & \text{if } |\lambda - t| < r, \\
0, & \text{otherwise.}
\end{cases}
\]
Then \(\text{supp } \hat{\delta}_{t,r} \subseteq (t - r, t + r)\). Note that
\[
\delta_{t,r}(\tau) = \int_{\infty}^{-\infty} e^{i\tau\lambda} \hat{\delta}_{t,r}(\lambda) d\lambda = \int_{t - r}^{t + r} \exp \left\{ i\tau\lambda - \frac{r^2}{(\lambda - t)^2} \right\} d\lambda
= r e^{i\tau t} \int_{-1}^{1} \exp \left\{ i\tau y - \frac{1}{1 - y^2} \right\} dy.
\]
Simple calculation yields that
\[
|\delta_{t,r}(\tau)| \leq 2r, \text{ for any } \tau \in \mathbb{R}
\]
and
\[
|\delta_{t,r}(\tau)| = \left| \int_{-\infty}^{\infty} e^{i\tau\lambda} \hat{\delta}_{t,r}(\lambda) d\lambda \right| \approx r\tau^{-3/4} e^{-\sqrt{\tau}/2}, \text{ as } \tau \to \infty.
\]
See for instance [11] as well. Hence \(\delta_{t,r}(\tau) \in S(\mathbb{R})\); that is, there exists \(\hat{\delta}_{t,r} \in S(\mathbb{R})\), such that \(\delta_{t,r}(\tau) = \int_{\mathbb{R}} \delta_{t,r}(s) e^{-i\tau s} ds\).

We are ready to prove (47). Recall (24) and (25), we can rewrite \(M(t)\) as
\[
M(t) = c_H \sum_{\ell=0}^{\infty} \frac{2\ell + 1}{4\pi} C_{\ell} \int_{-\infty}^{\infty} |\hat{g}_\ell(t,\tau) - \sum_{j=1}^{n} a_j \hat{g}_\ell(t_j,\tau)|^2 \frac{d\tau}{|\tau|^{2H-1}}. (45)
\]

We distinguish the two cases \(\alpha > 2\) and \(0 < \alpha \leq 2\).

(i). If \(\alpha > 2\), let
\[
G_s(\lambda) = \sum_{\ell=0}^{\infty} \frac{2\ell + 1}{4\pi} C_{\ell} g_\ell(s,\lambda), \quad s \in \mathbb{T}, \lambda \in \mathbb{R},
\]
then
\[
Q(s,t) =: \int_{-\infty}^{\infty} \hat{G}_s(\tau) \hat{\delta}_{t,r}(\tau) d\tau
= \int_{0}^{t} G_s(\lambda) \left[ \int_{-\infty}^{\infty} \hat{\delta}_{t,r}(\tau) e^{-i\tau\lambda} d\tau \right] d\lambda = \int_{0}^{t} G_s(\lambda) \hat{\delta}_{t,r}(\lambda) d\lambda.
\]
It is readily seen that \(\text{supp } Q \subseteq \{(t, s) : s > t - r\}\). Hence \(Q(t_j, t) = 0\) for any
0 < t_j < t - r, \ j = 1, \ldots, n. Moreover, let 0 < r < \varepsilon < 1, then

\[ Q(t, t) = \int_0^t G_t(\lambda) \delta_{t,r}(\lambda) d\lambda = \int_{t-r}^t G_t(\lambda) \delta_{t,r}(\lambda) d\lambda \]

\[ \approx \int_{t-r}^t \left[ \sum_{\ell=0}^{\infty} \frac{2\ell + 1}{4\pi} (\ell + \frac{1}{2})^{-\alpha}(t - \lambda)^{(1-\alpha)/2} e^{-\ell(t+1)(t-\lambda)/2} \right] \]

\[ \cdot \frac{1}{(t-\lambda)^{1-\alpha/2}} \delta_{t,r}(\lambda) d\lambda \]

\[ \approx \int_{t-r}^t (1 + (t - \lambda)^{\alpha/2 - 1}) \delta_{t,r}(\lambda) d\lambda \]

\[ = r \int_0^r (1 + (ty)^{\alpha/2 - 1}) \exp \left( -\frac{1}{1 - y^2} \right) dy \geq r, \]

which leads to

\[ \left| Q(t, t) - \sum_{j=1}^n a_j Q(t_j, t) \right| \geq r \]

for \( \max_{j=1,\ldots,n} t_j \leq t - r \). Meanwhile, recall the representation (45) of \( M(t) \), then by the Cauchy-Schwartz inequality, we have

\[ \left| Q(t, t) - \sum_{j=1}^n a_j Q(t_j, t) \right|^2 \]

\[ = \left| \sum_{\ell=0}^{\infty} \sum_{m} \frac{2\ell + 1}{4\pi} C_\ell \int_{-\infty}^{\infty} \left( \hat{g}_\ell(t, \tau) - \sum_{j=1}^n a_j \hat{g}_\ell(t_j, \tau) \right) \delta_\ell(\tau) d\tau \right|^2 \]

\[ \leq \left\{ \sum_{\ell=0}^{\infty} \frac{2\ell + 1}{4\pi} C_\ell \sum_{m} \left| \int_{-\infty}^{\infty} \left( \hat{g}_\ell(t, \tau) - \sum_{j=1}^n a_j \hat{g}_\ell(t_j, \tau) \right) \delta_\ell(\tau) d\tau \right|^2 \right\} \]

\[ \leq c_{4,10} \sum_{\ell=0}^{\infty} \frac{2\ell + 1}{4\pi} C_\ell \int_{-\infty}^{\infty} \left| \hat{g}_\ell(t, \tau) - \sum_{j=1}^n a_j \hat{g}_\ell(t_j, \tau) \right|^2 |\tau|^{-(2H-1)} d\tau \]

\[ \cdot \left\{ \int_{-\infty}^{\infty} |\delta_\ell(\tau)|^2 |\tau|^{2H-1} d\tau \right\} \]

\[ = c_{4,10} M(t) \int_{-\infty}^{\infty} |\delta_\ell(\tau)|^2 |\tau|^{2H-1} d\tau, \]

for some positive constant \( c_{4,10} \) depending on \( \alpha \). Now recall formula (41) again,
we have

\[ c_H \int_{-\infty}^{\infty} |\delta_t(\tau)|^2 |\tau|^{2H} d\tau = c_H \int_{-\infty}^{\infty} |\delta_t(\tau)|^2 |\tau|^{1-2(1-H)} d\tau \]

\[ = \int_{t-\tau}^{t+r} \int_{t-\tau}^{t+r} \hat{\delta}_t(\lambda) \delta_t(\xi) (\xi - \lambda)^{2(1-H)-2} d\xi d\lambda \]

\[ = r^{2-2H} \int_{-1}^{1} \int_{-1}^{1} \exp \left\{ \frac{1}{1 - \xi^2} - \frac{1}{1 - \lambda^2} \right\} |\xi' - \lambda|^1 d\xi' d\lambda \]

\leq c_{4,13} r^{2(1-H)} \quad \text{(46)}

for some positive constant \( c_{4,11} \) depending on \( H \). Hence,

\[ M(t) \geq \frac{r^2}{c_{4,10} c_{4,11} r^{2-2H}} = \frac{r^{2H}}{c_{4,10} c_{4,11}}. \]

(ii). If \( 0 < \alpha \leq 2 \), let

\[ G_s(\lambda) =: \sum_{\ell=0}^{\infty} (\ln \ell)^{-1} \sqrt{C_\ell} g_\ell(s, \lambda), \quad s \in \mathbb{T}, \lambda \in \mathbb{R}. \]

Consider the function

\[ Q(s, t) =: \int_{-\infty}^{\infty} \hat{G}_s(\tau) \delta_{t, r}(\tau) d\tau \]

\[ = \int_0^s G_s(\lambda) \left[ \int_{-\infty}^{\infty} \delta_{t, r}(\tau) e^{-i\tau \lambda} d\tau \right] d\lambda = \int_0^s G_s(\lambda) \hat{\delta}_{t, r}(\lambda) d\lambda. \]

Then \( \text{supp} \, Q \subseteq \{(t, s) : s > t - r\} \), which implies \( Q(t_j, t) = 0 \) for any \( 0 < t_j < t - r, \; j = 1, \ldots, n \). Moreover, let \( 0 < r < \varepsilon < 1 \), then

\[ Q(t, t) = \int_0^t G_t(\lambda) \hat{\delta}_{t, r}(\lambda) d\lambda = \int_0^t G_t(\lambda) \hat{\delta}_{t, r}(\lambda) d\lambda \]

\[ \approx \int_{t-r}^t \sum_{\ell=0}^{\infty} (\ln \ell)^{-1} \left( 1 + \frac{1}{2} \right)^{-\alpha/2} (t - \lambda)^{-\alpha/4} e^{-\ell(\ell+1)(t-\lambda) \sqrt{t - \lambda}} \]

\[ \cdot \frac{e^{(t-\lambda)/4}}{(t-\lambda)^{(2-\alpha)/4}} \hat{\delta}_{t, r}(\lambda) d\lambda \]

\[ \geq \frac{1}{\ln^2} \int_{t-r}^t \left( 1 + \frac{1}{(t-\lambda)^{(2-\alpha)/4}} \right) \hat{\delta}_t(\lambda) d\lambda \]

\[ = r \int_0^1 \left( 1 + \frac{1}{(ry)^{(2-\alpha)/4}} \right) \exp \left\{ -\frac{1}{1 - y^2} \right\} dy \geq r^{1 - \max\{(2-\alpha)/4, 0\}}. \]

It follows that

\[ \left| Q(t, t) - \sum_{j=1}^{n} a_j Q(t_j, t) \right|^2 \geq r^{2 - \max\{(2-\alpha)/2, 0\}} \]

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for $\max_{j=1,\ldots,n} t_j \leq t - r$. Meanwhile, recall the representation $[15]$ of $M(t)$, then by the Cauchy-Schwartz inequality, we have

$$|Q(t,t) - \sum_{j=1}^{n} a_j Q(t_j,t)|^2$$

$$= \left| \sum_{\ell=0}^{\infty} \sum_{m=0}^{2\ell+1} \frac{C_{\ell}}{4\pi} C_{\ell} \int_{-\infty}^{\infty} \left| \tilde{g}_{\ell}(t,\tau) - \sum_{j=1}^{n} a_j \tilde{g}_{\ell}(t_j,\tau) \right| \delta_{\ell}(\tau) d\tau \right|^2$$

$$\leq \left\{ \sum_{\ell=0}^{\infty} \frac{2\ell+1}{4\pi} C_{\ell} \int_{-\infty}^{\infty} \left| \tilde{g}_{\ell}(t,\tau) - \sum_{j=1}^{n} a_j \tilde{g}_{\ell}(t_j,\tau) \right|^2 |\tau|^{-2H-1} d\tau \right\}$$

$$= c_{4,12} M(t) \int_{-\infty}^{\infty} |\delta_{\ell}(\tau)|^2 |\tau|^{-2H-1} d\tau.$$

for some universal constant $c_{4,12} > 0$. The last inequality is obtained similar to the argument in (i) and hence by [16] we have

$$M(t) \geq \frac{r^{2-\max\{(2-\alpha)/2,0\}}}{c_{4,10}c_{4,11}r^{2-2H}} = \frac{r^{2\gamma}}{c_{4,10}c_{4,11}}.$$

Thus, the proof is completed.

The next proposition is concerned with the strong local nondeterminism of $u(t,x)$ in space variable $x \in S^2$, when $t \in T$ fixed. Again, with a slight abuse of notation, we write $u(x) = u(t,x)$ for brevity.

**Proposition 4.5** Assume that $1/2 < H < 1$, Condition (A.1) with $0 < \alpha < 2$, and $u_0 \equiv 0$. If $\gamma = \alpha/2 - 1 + 2H \in (0,1)$, then, for every $t \in (0,T]$ fixed, there exist constants $K_{4.5} > 0$, depending on $t$ and $0 < \varepsilon < \delta$, such that we have for all $x_0, x_1, \ldots, x_n \in S^2$ with min $\{d_{S^2}(x_0, x_j), j = 1, \ldots, n\} = r \in (0,\varepsilon)$,

$$\text{Var}(u(t,x_0)|u(t,x_1), \ldots, u(t,x_n)) \geq K_{4.5} r^{2\gamma}. \quad (47)$$

**Proof.** Since the Gaussian random field $u(t) = \{u(t,x), x \in S^2\}$ is 2-weakly isotropic, the results in [13] is applicable. Hence, in order to prove [17], we only need to derive the asymptotic property of the angular power spectrum of $\{u(x), x \in S^2\}$. Recall [17], under the condition of $u_0 \equiv 0$, the angular power spectrum of $u(t,x)$

$$\mathbb{E}|u_{\ell m}(t)|^2 = C_{\ell} \int_{0}^{T} \int_{0}^{T} g_{\ell}(t,\lambda)g_{\ell}(t,\xi)|\xi - \lambda|^{2H-2} d\xi d\lambda$$
where \( g_\ell \) is the function defined in (23). Recall formulae (24), (25) and (26), we have

\[
\mathbb{E}|u_{\ell m}(t)|^2 = c_H C_{\ell} \int_\mathbb{R} \left| e^{-it\tau} - e^{-i(t+1)\tau} \right|^2 \frac{|\tau|^{-(2H-1)}}{|\ell(\ell+1)|^2 + \tau^2} d\tau
\]

and there exists a constant \( c_{4,13} > 1 \) such that

\[
c_{4,13}^{-1}(\ell + \frac{1}{2})^{-\alpha H} \leq \mathbb{E}|u_{\ell m}(t)|^2 \leq c_{4,13}(\ell + \frac{1}{2})^{-(\alpha+4H)}
\]  

(48)

The inequality (47) is then obtained by Theorem 1 in [13] for \( 2 < \alpha + 4H < 4 \).

As an immediate consequence of Propositions 4.1, 4.4 and Proposition 4.5, we have the following corollary.

**Corollary 4.6** Assume \( 1/2 < H < 1 \) and Conditions (A.1) and (A.2) hold.

(i) if \( \beta > 4H + 2 \) or \( u_0 \equiv 0 \), then for any \( x \in S^2 \) fixed, there exists a constant \( K_{4,6} > 1 \) such that

\[
K_{4,6}^{-1} \|t - s\|^{2\eta} \leq \mathbb{E}|u(t, x) - u(s, x)|^2 \leq K_{4,6} \|t - s\|^{2\eta},
\]

where \( \eta = H - \max\{(2 - \alpha)/4, 0\} \).

(ii) if \( u_0 \equiv 0 \) and \( \gamma = \alpha/2 - 1 + 2H \in (0, 1) \), then for any \( t \in T \) fixed, there exists a constant \( K_{4,7} > 1 \) such that

\[
K_{4,7}^{-1} (d_{S^2}(x, y))^{2\gamma} \leq \mathbb{E}|u(t, x) - u(t, y)|^2 \leq K_{4,7} (d_{S^2}(x, y))^{2\gamma}.
\]

5 Exact Uniform Modulus of Continuity

Now we are ready to prove (7) and (8) in Theorem 1.3. We start by stating a Kolmogorov’s 0-1 law regarding the uniform moduli of continuity for \( u \). It is a consequence of Lemma 7.1.1 in Marcus and Rosen [16].

**Lemma 5.1** Let \( \{u(t), t \in T\} \) be a centered Gaussian random process on \( T \), and \( \varphi : \mathbb{R}_+ \to \mathbb{R}_+ \) be a function with \( \varphi(0^+) = 0 \). If

\[
\lim_{\varepsilon \to 0} \sup_{0 \leq s < t \leq T, t-s \leq \varepsilon} \frac{|u(t) - u(s)|}{\varphi(t-s)} = K_{5,1}, \text{ a.s.}
\]

for some constant \( K_{5,1} < \infty \), then

\[
\lim_{\varepsilon \to 0} \sup_{0 \leq s < t \leq T, t-s \leq \varepsilon} \frac{|u(t) - u(s)|}{\varphi(t-s)} = K'_{5,1}, \text{ a.s. for some constant } K'_{5,1} < \infty.
\]

We remark that Lemma 5.1 does not exclude the possibility of \( K'_{5,1} = 0 \). One of the main difficulties in establishing an exact uniform modulus of continuity is to find conditions under which \( K'_{5,1} > 0 \).
Proof of (7) in Theorem 1.3. The argument is similar to that in the proof of Theorem 4.1 in [18]. Because of Lemma 5.1, we see that (7) in Theorem 1.3 will be proved after we establish upper and lower bounds of the following form: there exist positive and finite constants $K_{5,2}$ and $K_{5,3}$ such that

$$\lim_{\varepsilon \to 0} \sup_{0 \leq s < t \leq T, t-s \leq \varepsilon} \frac{|u(t) - u(s)|}{(t-s)^\eta \sqrt{\log (t-s)}} \leq K_{5,2}, \quad \text{a.s.} \quad (49)$$

and

$$\lim_{\varepsilon \to 0} \sup_{0 \leq s < t \leq T, t-s \leq \varepsilon} \frac{|u(t) - u(s)|}{(t-s)^\eta \sqrt{\log (t-s)}} \geq K_{5,3}, \quad \text{a.s.} \quad (50)$$

These and Lemma 5.1 with $\varphi(r) = r^n \sqrt{\log r}$ imply (7) with $K_{1,1} \in [K_{5,3}, K_{5,2}]$.

We divide the rest of the proof into three parts.

Step 1: Proof of (49). We introduce an auxiliary Gaussian field:

$$Y = \{Y(t,s) : 0 \leq s < t \leq T, t-s \leq \varepsilon\}$$

defined by $Y(t,s) = u(t) - u(s)$, where $0 < \varepsilon \leq \delta$ so that Lemma 4.1 holds. By the triangle inequality, we see that the canonical metric $d_Y$ on $\Gamma := \{(t,s) \in T \times T : |t-s| \leq \delta\}$ associated with $Y$ satisfies the following inequality:

$$d_Y((t,s), (t',s')) \leq \min\{d_T(t,t') + d_T(s,s'), d_T(t,s) + d_T(t',s')\}, \quad (51)$$

where $d_T(t,s) = |t-s|^\eta$. Denote the diameter of $\Gamma$ in the metric $d_Y$ by $D$. Then, by Lemma 5.1 we have

$$D \leq \sup_{(s,t),(s',t') \in \Gamma} (d_T(s,t) + d_T(s',t')) \leq 2K_{3,1} \varepsilon^n.$$

For any $\tau > 0$, let $N_Y(\Gamma, \tau)$ be the smallest number of open $d_Y$-balls of radius $\tau$ needed to cover $\Gamma$. It follows from Lemma 5.1 that,

$$N_Y(\Gamma, \tau) \leq c_{5,1} \tau^{-\frac{\eta}{\eta-1}},$$

one can verify that

$$\int_0^D \sqrt{\ln N_Y(T, \tau)} d\tau \leq c_{5,2} \varepsilon^n \sqrt{\ln(1+\varepsilon^{-1})}.$$

Hence, by Theorem 1.3.5 in [11], we have

$$\limsup_{\varepsilon \to 0} \sup_{0 \leq s < t \leq T, t-s \leq \varepsilon} \frac{|u(t) - u(s)|}{(t-s)^\eta \sqrt{\log (t-s)}} \leq c_{5,3}, \quad \text{a.s.},$$

which implies (49). Here, $c_{5,1}$, $c_{5,2}$ and $c_{5,3}$ are positive constants depending only on $c_0$, $\alpha$ and $H$.

Step 2: Proof of (50). For any $n \geq \lfloor \log_2 \delta \rfloor + 1$, where $\delta$ is the constant same as in Proposition 4.1, we chose a sequence of $2^n$ points $\{t_{n,i}, 1 \leq i \leq 2^n\}$
on \( \mathbb{T} \) that are equally separated in the following sense: For every \( 2 \leq k \leq 2^n \), we have
\[
t_{n,k} - t_{n,k-1} = 2^{-n}.
\] (52)

Notice that
\[
\lim_{\varepsilon \to 0} \sup_{0 \leq s < t \leq T, t-s \leq \varepsilon} \frac{|u(t) - u(s)|}{(t-s)^\eta \sqrt{\log(t-s)}} \geq \lim_{n \to \infty} \max_{2 \leq k \leq 2^n} \frac{|u(t_{n,k}) - u(t_{n,k-1})|}{2^{-n} \eta \sqrt{n}}. 
\] (53)

It is sufficient to prove that, almost surely, the last limit in (53) is bounded below by a positive constant. This is done by applying the property of strong local nondeterminism in Proposition 4.4 and a standard Borel-Cantelli argument.

Let \( \tau > 0 \) be a constant whose value will be chosen later. We consider the events
\[
A_m = \left\{ \max_{2 \leq k \leq m} \left| u(t_{n,k}) - u(t_{n,k-1}) \right| \leq \tau 2^{-nH} \sqrt{n} \right\}
\]
for \( m = 2, \ldots, 2^n \). By conditioning on \( A_{2^n-1} \) first, we can write
\[
P(A_{2^n}) = P\left( \left| u(t_{n,2^n}) - u(t_{n,2^n-1}) \right| \leq \tau 2^{-nH} \sqrt{n} \left| A_{2^n-1} \right\} \right) 
\]
\[
\times P\left( A_{2^n-1} \right). 
\] (54)

Recall that, given the random variables in \( A_{2^n-1} \), the conditional distribution of the Gaussian random variable \( u(t_{n,2^n}) - u(t_{n,2^n-1}) \) is still Gaussian, with the corresponding conditional mean and variance as its mean and variance. By Proposition 4.4 the aforementioned conditional variance for \( k = 2, \ldots, 2^n \) satisfies
\[
\text{Var}(u(t_{n,k}) - u(t_{n,k-1}) | A_{k-1}) \geq K_{4,4} 2^{-2nH}.
\]

This and Anderson’s inequality (see [3]) imply
\[
P\left( \left| u(t_{n,k}, x_{n,k}) - u(t_{n,k-1}, x_{n,k-1}) \right| \leq \tau 2^{-n\eta} \sqrt{n} | A_{k-1} \right) 
\]
\[
\leq P\left( N(0,1) \leq \frac{2^{-n\eta} \tau \sqrt{n}}{K_{4,4}^{1/2} 2^{-n\eta}} \right) 
\]
\[
\leq 1 - \frac{K_{4,4}^{1/2} 2^{-n\eta}}{\tau 2^{-n\eta} \sqrt{n}} \exp \left( - \frac{2^{-2n\eta} C_2 n}{K_{4,4} 2^{1-2n\eta}} \right) 
\]
\[
\leq \exp \left( - \frac{K_{4,4}^{1/2}}{\tau \sqrt{n}} \exp \left( - \frac{\tau^2 n}{2K_{4,4}} \right) \right). 
\] (55)

In deriving the last two inequalities, we have applied Mill’s ratio and the elementary inequality \( 1 - x \leq e^{-x} \) for \( x > 0 \). Iterating this procedure in (54) and
For $2^n - 1$ more times, we obtain

$$P(A_{2^n}) \leq \exp \left\{ - \sum_{k=1}^{2^n} \frac{K_{4,4}^{1/2} \tau^{2n}}{2K_{4,4}^{1/2}} \exp \left( - \frac{\tau^{2n}}{2K_{4,4}^{1/2}} \right) \right\} \leq \exp \left\{ - \frac{K_{4,4}^{1/2}}{2\tau^{2n}} \left( \frac{2}{e^{\tau^2/(2K_{4,4}^{1/2})}} \right)^n \right\},$$

which yields that $\sum_{n=1}^{\infty} P(A_{2^n}) < \infty$. Hence the Borel-Cantelli lemma implies that almost surely,

$$\max_{2 \leq k \leq 2^n} |u(t_{n,k}) - u(t_{n,k-1})| \geq \tau^{2-n\eta} \sqrt{n}$$

for all $n$ large enough. This implies that the right-hand side of (53) is bounded from below almost surely by some $C > 0$. Hence (50) follows from this and Lemma 5.1. This finishes the proof of (7) in Theorem 1.3. 

Proof of (8) in Theorem 1.3. The proof is similar to the argument above (see also proof of Theorem 2 in [13]), and we omit it here. 

As an ending of this section, we further study the regularity properties of higher-order derivatives of $u$ w.r.t. the spatial variable $x \in S^2$ based on pseudo-differential operators, defined as follows: for a real $k \in \mathbb{R}$,

$$\nabla^{(k)} u(t) := (1 - \Delta_{S^2})^{k/2} u(t) = \sum_{\ell m} u_{\ell m}(t)(1 + \ell(\ell + 1))^{k/2} Y_{\ell m}$$

provided the right-hand side converges in $L^2(\Omega \times S^2)$.

It is shown in [22, Chapter XI] that the Sobolev space $H^k(S^2)$ of functions with square-integrable $k$-th weak derivatives can be viewed as the image of $L^2(S^2)$ under the operator $(1 - \Delta_{S^2})^{-k/2}$; this and related property are exploited by Lang et al. [15] and Lan et al. [13] to prove their results on regularity of higher-order derivatives.

Again, with a slight abuse of notation, we write $u(x) = u(t, x)$ for brevity and recall the estimation of the angular power spectrum (48) of $u$. An immediate consequence of Theorem 3 in [13] is as follows, which derives the exact uniform modulus of continuity for $\nabla^{(k)} u(x)$.

**Corollary 5.2** Assume $1/2 < H < 1$ and Conditions (A.1) and (A.2) hold. If $u_0 \equiv 0$ and $2k < \alpha + 4H < 2 + 2k$ for some integer $k \geq 2$, then $\nabla^{(k-1)} u$ satisfies the following exact uniform modulus of continuity:

$$\lim_{\epsilon \to 0} \sup_{x, y \in S^2 \atop d_{S^2}(x, y) \leq \epsilon} \frac{\left| \nabla^{(k-1)} u(x) - \nabla^{(k-1)} u(y) \right|}{(d_{S^2}(x, y))^{\alpha/2 + 2H - k} \sqrt{\log d_{S^2}(x, y)}} = K_{5,4}, \quad \text{a.s.}$$
References

[1] Adler, R. J. and Taylor, J. E. (2007), Random Fields and Geometry. Springer, New York.
[2] Alós, E., Mazet, O. and Nualart, D. (1999), Stochastic calculus with respect to Gaussian processes. Ann. Probab. 29, 766–801.
[3] Anderson, T. W. (1955), The integral of a symmetric unimodal function over a symmetric convex set and some probability inequalities. Proc. Amer. Math. Soc. 6, 170–176.
[4] Balan, R. M. and Tudor, C. A. (2010), The stochastic heat equation with fractional-colored noise: Existence of the solution. J. Theor. Probab. 23, 834–870.
[5] Dalang, R., Khoshnevisan, D., Mueller, C., Nualart, D. and Xiao, Y. (2009), A Minicourse on Stochastic Partial Differential Equations. (D. Khoshnevisan and F. Rassoul-Agha, editors). Lecture Notes in Math., 1962. Springer-Verlag, Berlin.
[6] Da Prato, G. and Zabczyk, J. (1992), Stochastic Equations in Infinite Dimensions. Cambridge Univ. Press, Cambridge.
[7] Gawarecki, L. and Mandrekar, V. (2011), Stochastic Differential Equations in Infinite Dimensions with Applications to Stochastic Partial Differential Equations. Springer, Heidelberg.
[8] Gilbarg, D. and Trudinger, N.S. (1998), Elliptic Partial Differential Equations of Second Order. Springer.
[9] Gradshteyn, I. S. and Ryzhik, I. M. (1980), Tables of Integrals, Series, and Products. (4th ed.), Academic Press, New York.
[10] Johnson, S. G. (2015), Saddle-point integration of $C^\infty$ “bump” functions. arXiv:1508.04376v1.
[11] Khoshnevisan, D. (2016), Invariance and comparison principles for parabolic stochastic partial differential equations. In: From Lévy-type Processes to Parabolic SPDEs, pp. 127-216, Birkhäuser/Springer, Cham.
[12] Khoshnevisan, D. (2014), Analysis of Stochastic Partial Differential Equations. CBMS Regional Conference Series in Mathematics, 119. The American Mathematical Society, Providence, RI, 2014.
[13] Lan, X., Marinucci, D. and Xiao, Y. (2018), Strong local nondeterminism and exact modulus of continuity for spherical Gaussian fields. Stoch. Process. Appl. 128, 1294–1315.
[14] Lang, A., Larsson, S. and Schwab, C. (2013), Covariance structure of parabolic stochastic partial differential equations. Stoch. PDE: Anal. Comp. 1, 351–364.
[15] Lang, A. and Schwab, C. (2015), Isotropic Gaussian random fields on the sphere: regularity, fast simulation and stochastic partial differential equations. Ann. Appl. Probab. 25, 3047–3094.
[16] Marcus, M. B. and Rosen J. (2006), *Markov Processes, Gaussian Processes, and Local Times*. Cambridge University Press, Cambridge.

[17] Marinucci, D. and Peccati, G. (2011), *Random Fields on the Sphere. Representation, Limit Theorem and Cosmological Applications*. Cambridge University Press, Cambridge.

[18] Meerschaert, M. M., Wang, W. and Xiao, Y. (2013), Fernique-type inequalities and moduli of continuity for anisotropic Gaussian random fields. *Trans. Amer. Math. Soc.* 365, 1081–1107.

[19] Nualart, E. and Viens, F. (2009), The fractional stochastic heat equation on the circle: time regularity and potential theory. *Stoch. Process. Appl.* 119, 1505–1540.

[20] Prévôt, C. and Röckner, M. (2007), *A Concise Course on Stochastic Partial Differential Equations*. Lecture Notes in Math., 1905, Springer, Berlin.

[21] Szego, G. (1975), *Orthogonal Polynomials*. American Mathematical Society Colloquium Publications, 4th Edition, Volume XXIII, Providence, RI.

[22] Taylor, M. E. (1981), *Pseudodifferential Operators*. Princeton University Press, Princeton, NJ.

[23] Tindel, S., Tudor, C. A. and Viens, F. (2003), Stochastic evolution equations with fractional Brownian motion. *Probab. Th. Relat. Fields* 127, 186–204.

[24] Tindel, S., Tudor, C. A. and Viens, F. (2004), Sharp Gaussian regularity on the circle, and applications to the fractional stochastic heat equation. *J. Funct. Anal.* 217, 280–313.

[25] Walsh, J. B. (1986), An introduction to stochastic partial differential equations. *École d’été de probabilités de Saint-Flour, XIV—1984*, 265–439, Lecture Notes in Math., 1180, Springer, Berlin.

[26] Tudor, C. A. and Xiao, Y. (2017), Sample paths of the solution to the fractional-colored stochastic heat equation. *Stoch. Dyn.* 17, no. 1, 1750004 (20 pages).

[27] Xiao, Y. (2007), Strong local nondeterminism and sample path properties of Gaussian random fields. In *Asymptotic Theory in Probability and Statistics with Applications*, eds. T. Lai, Q. Shao and L. Qian (Higher Education Press), pp. 136–176.