Recurrence Relations for Values of the Riemann Zeta Function in Odd Integers

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Abstract
It is commonly known that \( \zeta(2k) = q_k \frac{\zeta(2k+2)}{\pi^{2k}} \) with known rational numbers \( q_k \). In this work we construct recurrence relations of the form \( \sum_{k=1}^\infty r_k \frac{\zeta(2k+1)}{\pi^{2k}} = 0 \) and show that series representations for the coefficients \( r_k \in \mathbb{R} \) can be computed explicitly.

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1 Summary
In the first section we show that \( \frac{\cosh(x)}{\sinh(x)^{2N+1}} \) can be expressed as linear combination of \( \frac{\sinh(x)}{\cosh(x)^{2N+1}} \) and \( \frac{\cosh(2x)}{\sinh(2x)^{2N+1}} \) for some \( k \leq N \). We achieve this by proving four identities between certain rational functions. These in turn are derived from four recurrence relations for binomial coefficients, which are proved first. Then we show that the \( 2n \)-th derivative of \( \coth(x) \) can be expressed as linear combination of \( \frac{\cosh(x)}{\sinh(x)^{2k+1}} \) with \( k \) ranging from 1 to \( n \). We prove some useful recurrence relations between the coefficients of the \( \frac{\cosh(x)}{\sinh(x)^{2k+1}} \)'s and compute explicitly the inverse of the matrix formed by these coefficients.

We derive our main result Theorem 3.3 - the limit identity for \( \lim_{\alpha \to 0} \sum_{n=1}^\infty \frac{\sinh(\alpha n)}{n \cosh(\alpha n)^{2N+1}} \) for a fixed \( N \in \mathbb{N} \) - by applying our previous findings on Ramanujan’s famous identity for the Riemann zeta function values in odd integers. As an application we finally determine recurrence relations of the form \( \sum_{k=1}^\infty r_k \frac{\zeta(2k+1)}{\pi^{2k}} = 0 \).

2 Preliminaries

Remark 2.1. Throughout this work we set \( \binom{n}{k} = 0 \) for \( k < 0 \) and for \( n < k \).
2.1 Identities for \( \frac{\cosh(x)}{\sinh(x)^{N+1}} \)

**Lemma 2.2.** For \( M, j \in \mathbb{N}, 0 \leq j \leq 2M \) we have the identity

\[
\frac{j!}{2M - 2j + 1} \left( \frac{4M + 2}{2j} \right)
= 2 \frac{1}{2M + 1} \sum_{k=0}^{j-1} (-1)^{j-k} \frac{(2M - 2k)(3M - k + 1)}{2M - 2k + 1} k!(j - 1 - k)! \left( \frac{2M - k}{j - 1 - k} \right) \left( \frac{4M + 2}{2k + 1} \right)
- 2(4M + 1) \sum_{k=0}^{j-1} (-1)^{j-k} \frac{2M - 2k}{(2M - 2k - 1)(2M - 2k + 1)} k!(j - 1 - k)! \left( \frac{2M - k}{j - 1 - k} \right) \left( \frac{4M}{2k + 1} \right)
+ (-1)^j \frac{j!}{2M + 1} \left( \frac{2M + 1}{j} \right). \tag{2.1}
\]

**Proof.** Induction over \( j \) using

\[
\left( \frac{2M - k}{j - 1 - k} \right) = \frac{2M - j + 2}{j - 1 - k} \left( \frac{2M - k}{j - 2 - k} \right)
\]

and

\[
\frac{(j - 1)!}{2M - 2j + 3} \left( \frac{2(4M + 1)(2M - 2j + 2)}{2M - 2j + 1} \left( \frac{4M}{2j - 1} \right) - 2(2M - 2j + 2)(3M - j + 2) \left( \frac{4M + 2}{2j - 1} \right) \right)
- (2M - j + 2) \left( \frac{4M + 2}{2j - 2} \right) \right) = \frac{j!}{2M - 2j + 1} \left( \frac{4M + 2}{2j} \right).
\]

\[\square\]

**Lemma 2.3.** For \( M, j \in \mathbb{N}, 0 \leq j \leq 2M \) we have the identity

\[
j! \left( \frac{4M}{2j} \right) = 2M \sum_{k=0}^{j-1} (-1)^{j-1-k} k!(j - 1 - k)! \left( \frac{2M - 1 - k}{2j + 1} \right) + (-1)^j j! \left( \frac{2M}{j} \right). \tag{2.2}
\]

**Proof.** Induction over \( j \) using

\[
\left( \frac{2M - 1 - k}{j - 1 - k} \right) = \frac{2M - j + 1}{j - 1 - k} \left( \frac{2M - 1 - k}{j - 2 - k} \right)
\]

and

\[
- (2M - j + 1)(j - 1)! \left( \frac{4M}{2j - 2} \right) + 2M(j - 1)! \left( \frac{4M}{2j - 1} \right) = j! \left( \frac{4M}{2j} \right).
\]

\[\square\]

**Lemma 2.4.** For \( M, j \in \mathbb{N}, 1 \leq j \leq 2M \) we have the identity

\[
\frac{(4M - 2j + 2)! \left( \frac{2j}{2j + 1} \right)!}{(2M - j + 1)! \left( \frac{2j}{2j + 1} \right)!} \left( \frac{(2j + 1)}{4M + 2} \right) - (4M + 2) \left( \frac{4M + 2}{2j} \right)
= (4M + 2) \sum_{k=1}^{j-1} (-1)^{j-k} \frac{(4M - 2k + 2)! \left( \frac{2k}{2k + 1} \right)!}{(2M - k + 1)! \left( \frac{2k}{2k + 1} \right)!} \left( \frac{4M + 2}{2k} \right) + 2(-1)^j \frac{(4M + 2)!}{(2M)!}. \tag{2.3}
\]

**Proof.** Induction over \( j \) using

\[
- \frac{(4M - 2j + 4)! \left( \frac{2j}{2j + 2} \right)!}{(2M - j + 2)! \left( \frac{2j}{2j + 2} \right)!} \left( \frac{(2j - 1)}{4M + 2} \right)
= \frac{(4M - 2j + 2)! \left( \frac{2j}{2j + 1} \right)!}{(2M - j + 1)! \left( \frac{2j}{2j + 1} \right)!} \left( \frac{(2j + 1)}{4M + 2} \right) - (4M + 2) \left( \frac{4M + 2}{2j} \right).
\]

\[\square\]
Lemma 2.5. For $M, j \in \mathbb{N}$, $1 \leq j \leq 2M - 1$ we have the identity

$$(2M + 1)\binom{4M}{2j + 1} - (2M - 2j)\binom{4M + 2}{2j + 1} = \sum_{k=1}^{j-1} (2M - 2k)\binom{4M + 2}{2k + 1} + 4M(2M + 1). \quad (2.4)$$

Proof. Induction over $j$ using

$$(2M + 1)\binom{4M}{2j - 1} = (2M + 1)\binom{4M}{2j + 1} - (2M - 2j)\binom{4M + 2}{2j + 1}.$$ 

□

Proposition 2.6. For $M, j \in \mathbb{N}$ we have the following four relations:

$$\sum_{k=0}^{j} (-1)^{j-k}2^{4k} \left(\binom{2(M - k)}{j - k} - \binom{2(M - k)}{j - 1 - k}\right) \binom{M + k}{2k} = \binom{4M + 2}{2j}, \quad j = 0, \ldots, M \quad (2.5)$$

$$\sum_{k=0}^{j} (-1)^{j-k}2^{4k} \frac{M}{M + k} \left(\binom{2(M - k)}{j - k} - \binom{2(M - k)}{j - 1 - k}\right) \binom{M + k}{2k} = \binom{4M}{2j}, \quad j = 0, \ldots, M \quad (2.6)$$

$$\sum_{k=0}^{j} (-1)^{j-k}2^{4k} \frac{4M + 2}{2k + 1} \left(\binom{2(M - k)}{j - k} - \binom{2(M - k)}{j - 1 - k}\right) \binom{M + k}{2k} = \binom{4M + 2}{2j + 1}, \quad j = 0, \ldots, M \quad (2.7)$$

$$\sum_{k=0}^{j} (-1)^{j-k}2^{4k+2} \frac{M - k}{2k + 1} \left(\binom{2(M - 1 - k)}{j - k} - \binom{2(M - 1 - k)}{j - 1 - k}\right) \binom{M + k}{2k} = \binom{4M + 2}{2j + 1}, \quad j = 0, \ldots, M - 1 \quad (2.8)$$

Proof. The proof follows with induction over $j$. First we denote with $S^{4M+2}_{2j}$, $S^{4M}_{2j}$, $S^{4M+2}_{2j+1}$ and $S^{4M}_{2j+1}$ the left hand sides of (2.5), (2.6), (2.7) and (2.8) respectively. We can easily check

$S^{4M+2}_0 = 1, \quad S^{4M}_0 = 1, \quad S^{4M+2}_1 = 4M + 2 \quad \text{and} \quad S^{4M}_1 = 4M.$

In the following we assume $j > 0$.

Using

$$\binom{2(M - k)}{j - 1 - k} = \frac{j - k}{2M - k - j + 1} \binom{2(M - k)}{j - k}$$

we write

$$S^{4M+2}_{2j} = \sum_{k=0}^{j} (-1)^{j-k}2^{4k} \frac{2M - 2j + 1}{2M - k - j + 1} \left(\binom{2(M - k)}{j - k} - \binom{2(M - k)}{j - 1 - k}\right) \binom{M + k}{2k}$$

Then a shift of the index $k$ to $k - 1$ in $S^{4M+2}_{2(j-1)+1}$ and $S^{4M}_{2(j-1)+1}$ and the relation

$$\binom{2(M - k + 1)}{(j - 1) - k + 1} \binom{M + k - 1}{2k - 2} = \frac{4k(2k - 1)(2M - 2k + 1)}{(2M - k - j + 1)(2M - k - j + 2)(M + k)} \binom{2(M - k)}{j - k} \binom{M + k}{2k}$$
altogether give

\[
(2M - 2j + 3)(4M + 2)j S_{2j}^{4M+2} + 4(2M - 2j + 2)(2M - 2j + 1)(3M - j + 2)S_{2(2j-1)+1}^{4M+2} - 2(2M - 2j + 2)(4M + 1)(4M + 2) S_{2(j-1)+1}^{4M+2} \\
= (2M - 2j + 3)(4M + 2)j \sum_{k=0}^{j} (-1)^j - k 2^{4k} \frac{2M - 2j + 1}{2M - k - j + 1} \binom{2(M - k)}{j - k} \binom{M + k}{2k} \\
+ 4(2M - 2j + 2)(2M - 2j + 1)(3M - j + 2) \sum_{k=1}^{j} (-1)^j - k 2^{4k} \frac{k(4M + 2)(2M - 2k + 1)}{(2M - k - j + 2)(2M - k - j + 1)(M + k)} \binom{2(M - k)}{j - k} \binom{M + k}{2k} \\
- 2(2M - 2j + 2)(4M + 1)(4M + 2) \sum_{k=1}^{j} (-1)^j - k 2^{4k} \frac{2k(2M - 2j + 1)}{(2M - k - j + 1)(M + k)} \binom{2(M - k)}{j - k} \binom{M + k}{2k} \\
= - (2M - 2j + 1)(2M - j + 2)(4M + 2) \sum_{k=0}^{j-1} (-1)^{j-1} - k 2^{4k} \frac{2M - 2(j - 1) + 1}{2M - k - (j - 1) + 1} \binom{2(M - k)}{j - 1 - k} \binom{M + k}{2k} \\
= - (2M - 2j + 1)(2M - j + 2)(4M + 2) S_{2(2j-1)+1}^{4M+2}.
\]

Now we apply the last relation on itself for \(j = 2, j = 3, \ldots, 1\) repeatedly which yields

\[
\frac{j!}{(2M - 2j + 1)!!} \frac{(2M + 1)!!}{(4M + 2)} S_{2j}^{4M+2} = 4 \sum_{k=0}^{j-1} (-1)^j - k \frac{(2M + 1)!!(2M - 2k - 1)!!}{(2M - 2k + 1)!!(2M - 2j - 1)!!} k!(j - 1 - k)! (2M - 2k)(3M - k + 1) \binom{2M - k}{j - 1 - k} S_{2k+1}^{4M+2} \\
- 2(4M + 1)(4M + 2) \sum_{k=0}^{j-1} (-1)^j - k \frac{(2M + 1)!!(2M - 2k - 3)!!}{(2M - 2k + 1)!!(2M - 2j - 1)!!} k!(j - 1 - k)! \binom{2M - k}{j - 1 - k} S_{2k+1}^{4M+2} \\
+ (-1)^j \frac{(2M + 1)!!}{(2M - 2j - 1)!!} j! \binom{2M + 1}{j},
\]

where we used \(S_0^{4M+2} = 1\) in the last summand. Now we use \((2k - 1)!! = \frac{(2k)!}{2^k k!}\) to eliminate the double factorials. This results in

\[
\frac{j!}{2M - 2j + 1} S_{2j}^{4M+2} = \frac{1}{2M + 1} \sum_{k=0}^{j-1} (-1)^j - k \frac{(2M - 2k)(3M - k + 1)}{2M - 2k + 1} k!(j - 1 - k)! \binom{2M - k}{j - 1 - k} S_{2k+1}^{4M+2} \\
- 2(4M + 1) \sum_{k=0}^{j-1} (-1)^j - k \frac{2M - 2k}{(2M - 2k - 1)(2M - 2k + 1)} k!(j - 1 - k)! \binom{2M - k}{j - 1 - k} S_{2k+1}^{4M+2} \\
+ (-1)^j \frac{j!}{2M + 1} \frac{2M + 1}{j}.
\]

This gives from the induction hypothesis for \(S_{2k+1}^{4M+2}\) and \(S_{2k+1}^{4M+2}\) and \((2.1)\) that \(S_{2j}^{4M+2} = \binom{4M+2}{2j}\) must hold.
The last summand comes from $S^{4M}_{2j-1}$ as before, but now only for $S^{4M}_{2j-1}$. For $S^{4M}_{2j+1}$, we have

\begin{align*}
  jS^{4M}_{2j} - 2MS^{4M}_{2(j-1)+1} &= j \sum_{k=0}^{j} (-1)^{j-k}2^{4k} \frac{M}{M+k} \left( \frac{2(M-k)}{j-k} \right) \left( \frac{M+k}{2k} \right) \\
  &= -2M \sum_{k=1}^{j} (-1)^{j-k}2^{4k-2} \frac{2k(2M-2j+1)}{(2M-k-j+1)(M+k)} \left( \frac{2(M-k)}{j-k} \right) \left( \frac{M+k}{2k} \right) \\
  &= -(2M-j+1) \sum_{k=0}^{j-1} (-1)^{j-1-k}2^{4k} \frac{M}{M+k} \left( \frac{2(M-k)}{j-1-k} \right) \left( \frac{M+k}{2k} \right) \\
  &= -(2M-j+1)S^{4M}_{2(j-1)}. 
\end{align*}

From iterating this last finding we obtain

\begin{align*}
  j!S^{4M}_{2j} &= 2M \sum_{k=0}^{j-1} (-1)^{j-1-k}k!(j-1-k)! \left( \frac{2M-1-k}{j-1-k} \right) S^{4M}_{2k+1} + (-1)^j j! \left( \frac{2M}{j} \right). 
\end{align*}

The last summand comes from $S^{4M}_{0} = 1$. Now we get $S^{4M}_{2j} = \binom{4M}{2j}$ from the induction hypothesis for $S^{4M}_{2k+1}$ and (2.2).

We apply a different approach as before for $S^{4M+2}_{2j+1}$. We do not perform a shift of the index $k$, but involve $S^{4M+2}_{2j}$ instead. Thus we compute

\begin{align*}
  (2j+1)S^{4M+2}_{2j+1} - (4M+2)S^{4M+2}_{2j} &= (2j+1) \sum_{k=0}^{j} (-1)^{j-k}2^{4k} \frac{4M+2}{2k+1} \left( \frac{2(M-k)}{j-k} \right) \left( \frac{M+k}{2k} \right) \\
  &= -(4M+2) \sum_{k=0}^{j} (-1)^{j-k}2^{4k} \frac{2M-2j+1}{2M-k-j+1} \left( \frac{2(M-k)}{j-k} \right) \left( \frac{M+k}{2k} \right) \\
  &= -(4M-2j+3) \sum_{k=0}^{j-1} (-1)^{j-1-k}2^{4k} \frac{4M+2}{2k+1} \left( \frac{2(M-k)}{j-1-k} \right) \left( \frac{M+k}{2k} \right) \\
  &= -(4M-2j+3)S^{4M+2}_{2(j-1)+1}. 
\end{align*}

Iterating this relation yields

\begin{align*}
  (2j+1)!!S^{4M+2}_{2j+1} - (4M+2)(2j-1)!!S^{4M+2}_{2j} &= (4M+2) \sum_{k=1}^{j-1} (-1)^{j-k} \frac{(4M-2k+1)!!(2k-1)!!}{(4M-2j+1)!!} S^{4M+2}_{2k} + (-1)^j (4M+2) \frac{(4M+1)!!}{(4M-2j+1)!!} \\
\end{align*}

where the last summand comes from $S^{4M+2}_{1} = 4M+2$. After eliminating the double factorials we get

\begin{align*}
  \frac{(4M-2j+2)!}{(2M-j+1)!} \frac{(2j)!}{j!} \left( (2j+1)S^{4M+2}_{2j+1} - (4M+2)S^{4M+2}_{2j} \right) &= (4M+2) \sum_{k=1}^{j-1} (-1)^{j-k} \frac{(4M-2k+2)!!(2k)!}{(2M-k+1)!!} k! S^{4M+2}_{2k} + (-1)^j (4M+2) \frac{(4M+2)!}{(2M+1)!} \\
\end{align*}

We already know $S^{4M+2}_{2j} = \binom{4M+2}{2j}$. This, the induction hypothesis for $S^{4M+2}_{2k+2}$ and (2.3) establish $S^{4M+2}_{2j+1} = \binom{4M+2}{2j+1}$.
At last we turn to $S_{2j+1}^{4M}$. We write

$$S_{2j+1}^{4M} = \sum_{k=0}^{i} (-1)^{j-k} 2^{4k+2} \frac{M - k (2M - 2j - 1)(2M - k - j)}{2k + 1} (2M - k) \left( \frac{M + k}{2k} \right)$$

and compute

$$(2M + 1)S_{2j+1}^{4M} - (2M - 2j)S_{2j+1}^{4M+2}$$

$$= (2M + 1) \sum_{k=0}^{j} (-1)^{j-k} 2^{4k+2} \frac{M - k (2M - 2j - 1)(2M - k - j)}{2k + 1} (2M - k) \left( \frac{M + k}{2k} \right)$$

$$- (2M - 2j) \sum_{k=0}^{j} (-1)^{j-k} 2^{4k} \frac{4M + 2}{2k + 1} (2M - k) \left( \frac{M + k}{2k} \right)$$

$$= (2M + 1) \sum_{k=0}^{j-1} (-1)^{j-k} 2^{4k+2} \frac{M - k (2M - 2(j - 1) - 1)(2M - k - (j - 1))}{2k + 1} (2M - k) \left( \frac{M + k}{2k} \right)$$

$$(2M + 1)S_{2(j-1)+1}^{4M}.$$

Iterating the last result gives

$$(2M + 1)S_{2j+1}^{4M} - (2M - 2j)S_{2j+1}^{4M+2} = \sum_{k=1}^{j-1} (2M - 2k)S_{2k+1}^{4M+2} + 4M(2M + 1),$$

where the last summand comes from $S_{1}^{4M} = 4M$. Since we have shown $S_{2j+1}^{4M+2} = \left( \frac{4M + 2}{2j+1} \right)$ before, we get from the induction hypothesis for $S_{2k+1}^{4M+2}$ and (2.4) that $S_{2j+1}^{4M} = \left( \frac{4M}{2j+1} \right)$ must hold. This concludes the proof.

**Proposition 2.7.** Let $M \in \mathbb{N}$ and $z \in \mathbb{C}$. Then we have the following four relations between rational functions:

$$\frac{1}{2} \left( z + \frac{1}{z} \right)^{4M+2} - \frac{1}{2} \left( z - \frac{1}{z} \right)^{4M+2} = \sum_{k=0}^{M} 2^{4k} \frac{4M + 2}{2k + 1} (M + k) \left( \frac{M + k}{2k} \right) \left( z^2 - \frac{1}{z^2} \right)^{2(M-k)}$$

$$= \frac{1}{2} \left( z + \frac{1}{z} \right)^{4M} - \frac{1}{2} \left( z - \frac{1}{z} \right)^{4M} = 4 \left( z^2 + \frac{1}{z^2} \right) \sum_{k=0}^{M-1} 2^{4k} \frac{M - k}{2k + 1} (M + k) \left( z^2 - \frac{1}{z^2} \right)^{2(M-1-k)}$$

$$\frac{1}{2} \left( z + \frac{1}{z} \right)^{4M} + \frac{1}{2} \left( z - \frac{1}{z} \right)^{4M} = \sum_{k=0}^{M} 2^{4k} \frac{M}{M + k} (M + k) \left( \frac{M + k}{2k} \right) \left( z^2 - \frac{1}{z^2} \right)^{2(M-k)}$$

$$\frac{1}{2} \left( z + \frac{1}{z} \right)^{4M+2} + \frac{1}{2} \left( z - \frac{1}{z} \right)^{4M+2} = \left( z^2 + \frac{1}{z^2} \right) \sum_{k=0}^{M} 2^{4k} \left( \frac{M + k}{2k} \right) \left( z^2 - \frac{1}{z^2} \right)^{2(M-k)}$$

**Proof.** We can expand

$$\frac{1}{2} \left( z + \frac{1}{z} \right)^{4M+2} - \frac{1}{2} \left( z - \frac{1}{z} \right)^{4M+2} = \sum_{j=0}^{M-1} \left( \frac{4M + 2}{2j + 1} \right) \left( z^{4(M-j)} + \frac{1}{z^{4(M-j)}} \right) + \left( \frac{4M + 2}{2M + 1} \right)$$
Comparing coefficients and (2.8) give (2.10).

\[
\sum_{k=0}^{M} 2^{4k} \frac{4M + 2}{2k + 1} \left( \frac{M + k}{2k} \right) \left( z^2 - \frac{1}{z^2} \right)^{2(M-k)}
\]

\[
= \sum_{j=0}^{M-1} \sum_{k=0}^{j} (-1)^{j-k} 2^{4k} \frac{4M + 2}{2k + 1} \left( \frac{2(M-j) - (j-k) + 1}{2k} \right) \left( \frac{M + k}{2k} \right) \left( z^{4(M-j)} + \frac{1}{z^{4(M-j)}} \right)
\]

Then comparing coefficients and (2.7) give (2.9).

We compute

\[
\frac{1}{2} \left( z + \frac{1}{z} \right)^{4M} - \frac{1}{2} \left( z - \frac{1}{z} \right)^{4M} = \sum_{j=0}^{M-1} \left( \frac{4M}{2j + 1} \right) \left( z^{4(M-j)} + \frac{1}{z^{4(M-j)}} \right)
\]

and

\[
4 \left( z^2 + \frac{1}{z^2} \right) \sum_{k=0}^{M-1} 2^{4k} \frac{M - k}{2k + 1} \left( \frac{M + k}{2k} \right) \left( z^2 - \frac{1}{z^2} \right)^{2(M-1-k)}
\]

\[
= \sum_{j=0}^{M-1} \sum_{k=0}^{j} (-1)^{j-k} 2^{4k+2} \frac{M - k}{2k + 1} \left( \frac{2(M-j) - (j-k) + 1}{2k} \right) \left( \frac{M + k}{2k} \right) \left( z^{4(M-j)} + \frac{1}{z^{4(M-j)}} \right)
\]

Comparing coefficients and (2.8) give (2.10).

We expand

\[
\frac{1}{2} \left( z + \frac{1}{z} \right)^{4M} + \frac{1}{2} \left( z - \frac{1}{z} \right)^{4M} = \sum_{j=0}^{M-1} \left( \frac{4M}{2j + 1} \right) \left( z^{4(M-j)} + \frac{1}{z^{4(M-j)}} \right) + \left( \frac{4M}{2M} \right)
\]

and

\[
\sum_{k=0}^{M} 2^{4k} \frac{M + k}{M^2 + k} \left( z^2 - \frac{1}{z^2} \right)^{2(M-k)}
\]

\[
= \sum_{j=0}^{M-1} \sum_{k=0}^{j} (-1)^{j-k} 2^{4k} \frac{M}{M^2 + k} \left( \frac{2(M-j) - (j-k) + 1}{2k} \right) \left( \frac{M + k}{2k} \right) \left( z^{4(M-j)} + \frac{1}{z^{4(M-j)}} \right)
\]

Comparing coefficients and (2.8) give (2.11).

At last we expand

\[
\frac{1}{2} \left( z + \frac{1}{z} \right)^{4M+2} + \frac{1}{2} \left( z - \frac{1}{z} \right)^{4M+2} = \sum_{j=0}^{M} \left( \frac{4M + 2}{2j + 1} \right) \left( z^{4(M-j)} + \frac{1}{z^{4(M-j)}} \right)
\]

and

\[
\left( z^2 + \frac{1}{z^2} \right) \sum_{k=0}^{M} 2^{4k} \frac{M + k}{2k} \left( z^2 - \frac{1}{z^2} \right)^{2(M-k)}
\]

\[
= \sum_{j=0}^{M} \sum_{k=0}^{j} (-1)^{j-k} 2^{4k} \left( \frac{2(M-j) - (j-k) + 1}{2k} \right) \left( \frac{M + k}{2k} \right) \left( z^{4(M-j)} + \frac{1}{z^{4(M-j)}} \right)
\]

Comparing coefficients and (2.5) give (2.12).
Corollary 2.8. For \( M \in \mathbb{N} \) and \( x \in \mathbb{C} \) we have

\[
\cosh(x)^{4M} - \sinh(x)^{4M} = \cosh(2x) \sum_{k=0}^{M-1} \frac{1}{2^{2(M-k)-1}} \frac{M-k}{2k+1} \left( \frac{M+k}{2k} \right) \sinh(2x)^{2(M-k)}
\]

\[
\cosh(x)^{4M} + \sinh(x)^{4M} = \sum_{k=0}^{M} \frac{1}{2^{2(M-k)-1}} \frac{M+k}{2k+1} \left( \frac{M+k}{2k} \right) \sinh(2x)^{2(M-k)}
\]

\[
\cosh(x)^{4M+2} - \sinh(x)^{4M+2} = \sum_{k=0}^{M} \frac{1}{2^{2(M-k)}} \frac{2M+1}{2k+1} \sinh(2x)^{2(M-k)}
\]

\[
\cosh(x)^{4M+2} + \sinh(x)^{4M+2} = \cosh(2x) \sum_{k=0}^{M} \frac{1}{2^{2(M-k)}} \left( \frac{M+k}{2k} \right) \sinh(2x)^{2(M-k)}.
\]

Rearranging the latter yields

Corollary 2.9. For \( M \in \mathbb{N} \) and \( x \in \mathbb{C} \) we have

\[
\frac{\cosh(x)}{\sinh(x)^{4M-1}} = -\frac{\sinh(x)}{\cosh(x)^{4M-1}} + \sum_{k=0}^{M} \frac{1}{2^{2(M+k)}} \frac{M}{M+k} \left( \frac{M+k}{2k} \right) \frac{1}{\sinh(2x)^{2(M+k)-1}}
\]

(2.13)

and

\[
\frac{\cosh(x)}{\sinh(x)^{4M+1}} = \frac{\sinh(x)}{\cosh(x)^{4M+1}} + \sum_{k=0}^{M} \frac{1}{2^{2(M+k)+1}} \frac{M}{2k+1} \left( \frac{M+k}{2k} \right) \frac{1}{\sinh(2x)^{2(M+k)+1}}
\]

(2.14)

2.2 The 2n-th Derivative of coth(x) and Some Useful Recurrence and Matrix Relations

The 2n-th derivative of coth(x) will be of special interest in the next section.

Lemma 2.10. let \( n \in \mathbb{N} \) and \( x \in \mathbb{C} \). Then we have

\[
\frac{d^{2n}}{dx^{2n}} \coth(x) = \sum_{k=1}^{n} 2^{k} \sum_{j=1}^{k} (-1)^{k-j} \left( \begin{array}{c} 2k \\ k-j \end{array} \right) (2j)^{2n} \coth(x) \sinh(x)^{2k+1}.
\]

Proof. We make an ansatz of the form

\[
\frac{d^{2n}}{dx^{2n}} \coth(x) = \sum_{k=1}^{n} c_{n,k} \frac{\coth(x)}{\sinh(x)^{2k+1}},
\]

with coefficients \( c_{n,k} \in \mathbb{R} \). Utilizing

\[
\frac{d^{2}}{dx^{2}} \coth(x) = 4k^{2} \frac{\coth(x)}{\sinh(x)^{2k+1}} + (2k+2)(2k+1) \frac{\coth(x)}{\sinh(x)^{2k+3}}
\]
we can derive the recurrence relation for the \( c_{n,k} \)'s

\[
c_{n,1} = \frac{1}{2} 4^n, \quad c_{n,k} = 2k(2k-1)c_{n-1,k-1} + 4k^2 c_{n-1,k} \quad \text{for} \quad 2 \geq k \geq n-1 \quad \text{and} \quad c_{n,n} = (2n)!. \tag{2.15}
\]

Now the choice

\[
c_{n,k} = \frac{2^k}{4^n} \sum_{j=1}^{k} (-1)^{k-j} \binom{2k}{k-j} (2j)^{2n}
\]

fulfills the first two of the latter equations even for \( n = 0 \), which can be readily checked by a small calculation using

\[
\left( \frac{2k-2}{k-1-j}\right) = \frac{k^2-j^2}{2k(2k-1)} \left( \frac{2k}{k-j}\right).
\]

The trivial identity \( 2 \sum_{j=0}^{k-1} (-1)^j \binom{2k}{j} = (-1)^{k+1} \frac{1}{k} \binom{2k}{k} \) directly yields \( c_{0,k} = (-1)^{k+1} \frac{1}{k} \binom{2k}{k} \) for \( k \geq 1 \). This in turn implies \( c_{1,k} = 0 \) for \( k \geq 2 \), which is due to \( c_{1,k} = 2k(2k-1)c_{0,k-1} + 4k^2 c_{0,k} \). Then the same argument gives \( c_{n,k} = 0 \) for \( k > n \). This gives \( c_{n,n} = (2n)! \). Thus the claim follows by induction over \( n \).

\[\square\]

**Lemma 2.11.** The \( c_{n,k} \)’s fulfilling (2.15) also fulfill for \( n \geq k \)

\[
\sum_{i=k}^{n} \binom{k}{i-k} c_{n,i} = 2^{2(n-k)} c_{n,k}.
\tag{2.17}
\]

**Proof.** We set \( a_{n,k} := \sum_{i=k}^{n} \binom{k}{i-k} c_{n,i} \) and \( b_{n,k} := 2^{2(n-k)} c_{n,k} \). The strategy of the proof is to show that both \( a_{n,k} \) and \( b_{n,k} \) suffice

\[
a_{n,k} = 2k(2k-1)a_{n-1,k-1} + 16k^2 a_{n-1,k}.	ag{2.18}
\]

Then since \( a_{n,1} = 2^{4n-3} = b_{n,1} \) and \( a_{n,k} = b_{n,k} = 0 \) for \( k > n \) this gives \( a_{n,k} = b_{n,k} \) for all \( k \geq 1 \) as claimed. Now (2.18) is easily confirmed for the \( b_{n,k} \)’s. For the \( a_{n,k} \)’s we note that for \( i \geq k \)

\[
(2i+2)(2i+1)\binom{k}{i-k+1} - 2k(2k-1)\binom{k-1}{i-k+1} + 4(i^2-4k^2)\binom{k}{i-k} = 0
\]

holds. Therefore we have

\[
\sum_{i=k}^{n-1} (2i+2)(2i+1)\binom{k}{i-k+1} c_{n-1,i} + \sum_{i=k}^{n-1} 4i^2\binom{k}{i-k} c_{n-1,i} = 2k(2k-1)\sum_{i=k}^{n-1} \binom{k-1}{i-k+1} c_{n-1,i} + 16k^2 \sum_{i=k}^{n-1} \binom{k}{i-k} c_{n-1,i}.
\]

An index shift in the first sum of the left hand side and adding \( 2k(2k-1)c_{n-1,k-1} \) on both sides yield

\[
a_{n,k} = \sum_{i=k}^{n} \binom{k}{i-k} (2i(2i-1)c_{n-1,i-1} + 4i^2 c_{n-1,i})
\]

\[
= 2k(2k-1) \sum_{i=k-1}^{n-1} \binom{k-1}{i-k+1} c_{n-1,i} + 16k^2 \sum_{i=k}^{n-1} \binom{k}{i-k} c_{n-1,i}
\]

\[
= 2k(2k-1)a_{n-1,k-1} + 16k^2 a_{n-1,k}.
\]

\[\square\]
Let the matrix $U = (u_{n,k})$ be defined by
\[
u_{n,k} := \frac{(-1)^n}{(2n)!} c_{n,k}, \tag{2.19}\]
with the $c_{n,k}$'s from (2.16), i.e. the $u_{n,k}$'s satisfy
\[
u_{n,k} = \frac{2k(2k - 1)}{2n(2n - 1)} v_{n-1,k-1} - \frac{4k^2}{2n(2n - 1)} u_{n-1,k}. \tag{2.20}\]

Let $V = (v_{n,k})$ denote the inverse of $U$. Then the $v_{n,k}$'s satisfy the recurrence relation
\[
u_{n,k} = \frac{2k(2k - 1)}{2n(2n - 1)} v_{n-1,k-1} - \frac{4(n - 1)^2}{2n(2n - 1)} v_{n-1,k}. \tag{2.21}\]

**Proof.** Note that $U$ and $V$ are lower triangular matrices. The proof follows by induction over the row index $n$ for $V$. We fix an $s \in \{1, \ldots, n\}$ and compute
\[
\sum_{k=s}^{n} v_{n,k} u_{k,s} = -\sum_{k=s}^{n} \frac{2k(2k - 1)}{2n(2n - 1)} v_{n-1,k-1} u_{k,s} - \sum_{k=s}^{n} \frac{4(n - 1)^2}{2n(2n - 1)} v_{n-1,k} u_{k,s} \\
= -\frac{4(n - 1)^2}{2n(2n - 1)} \sum_{k=s}^{n-1} v_{n-1,k} u_{k,s} + \frac{2s(2s - 1)}{2n(2n - 1)} \sum_{k=s}^{n} v_{n-1,k-1} u_{k-1,s} \\
+ \frac{4s^2}{2n(2n - 1)} \sum_{k=s+1}^{n} v_{n-1,k-1} u_{k-1,s}.
\]
All of the last three sums vanish for $s < n - 1$ by the induction hypothesis. For $s = n - 1$ we get
\[
\sum_{k=n-1}^{n} v_{n,k} u_{k,n-1} = -\frac{4(n - 1)^2}{2n(2n - 1)} + 0 + \frac{4(n - 1)^2}{2n(2n - 1)} = 0
\]
and for $s = n$ we obtain
\[
\sum_{k=n}^{n} v_{n,k} u_{k,n} = 0 + 1 + 0.
\]

**Proposition 2.13.** Let the numbers $h_{k,n}$ be given by
\[
h_{1,n} = 1 \quad \text{and} \quad h_{k,n} = \sum_{j=k-1}^{n-1} \frac{1}{j^2} h_{k-1,j} \quad \text{for} \quad k \geq 2. \tag{2.22}\]

Then the entries $v_{n,k}$ satisfying (2.21) are given by
\[
v_{n,k} = (-1)^n (2k)! \frac{2^{(n-k)}}{n^2 \binom{2n}{n}} h_{k,n}. \tag{2.23}\]

**Proof.** Plugging the $v_{n,k}$’s as given in (2.23) into (2.21) gives a relation equivalent to
\[
h_{k,n} = \frac{1}{(n - 1)^2} h_{k-1,n-1} + h_{k,n-1}.
\]
This in turn is equivalent to the definition (2.22). Furthermore we have
\[
v_{n,n} = (-1)^n (2n) \frac{1}{n^2 \binom{2n}{n}} h_{n,n} = (-1)^n ((n - 1)!)^2 \frac{1}{((n - 1)!)^2} = (-1)^n.
\]

\[\Box\]
Proposition 2.14. We define the lower triangular matrix \( L = (l_{n,k}) \in \mathbb{R}^{N \times N} \) by
\[
l_{n,k} = 2^{2k+1} \binom{k}{n-k}, \quad \text{for } k \leq n \leq \min(2k, N)
l_{n,k} = 0, \quad \text{otherwise}
\]
and the diagonal matrix \( D = (d_{n,i}) \) by \( d_{i,i} = 2^{2i+1} \). Then we have for \( U \) and \( V \) defined in (2.19) and (2.23) respectively the matrix relations
\[
UL = DU \quad \text{and} \quad LV = VD.
\]

Proof. Remembering \( u_{n,k} = (-1)^n \binom{n}{2n} c_{n,k} \) we have
\[
(UL)_{n,k} = 2^{2k+1} \frac{(-1)^n}{(2n)!} \sum_{i=k}^{n} \binom{k}{i-k} c_{n,i} = 2^{2k+1} \frac{(-1)^n}{(2n)!} 2^{2(n-k)} c_{n,k} = 2^{2n+1} u_{n,k}.
\]
Here we used (2.17) in the penultimate step. \( \square \)

Corollary 2.15. The matrix equation \( LV = VD \) can be written out as
\[
\begin{align*}
\sum_{k=0}^{M-1} 2^{2(M+k)+1} \frac{M-k}{2k+1} \binom{M+k}{2k} v_{M+k,j} &= 2^{2j+1} v_{2M-1,j} \\
\sum_{k=0}^{M} 2^{2(M+k)+1} \binom{M+k}{2k} v_{M+k,j} &= 2^{2j+1} v_{2M,j}
\end{align*}
\]

Note that the coefficients occurring above are the same as in (2.13) and (2.14).

3 Main Results

3.1 A Limit Identity for \( \lim_{\alpha \to 0^+} \sum_{n=1}^{\infty} \frac{\sinh(\alpha n)}{n \cosh(\alpha n)^{2n+1}} \)

Our starting point is Ramanujan’s famous formula for \( \zeta(2n+1) \), cf. [1].

Theorem 3.1. Let \( B_r, r \geq 0 \) denote the \( r \)-th Bernoulli number. If \( \alpha \) and \( \beta \) are positive numbers sucht that \( \alpha \beta = \pi^2 \), and if \( n \) is a positive integer, then
\[
\begin{align*}
&\frac{1}{(4\alpha)^n} \left( \frac{1}{2} \zeta(2n+1) + \sum_{m=1}^{\infty} \frac{1}{m^{2n+1}(\exp(2m\alpha) - 1)} \right) \\
&- \frac{1}{(-4\beta)^n} \left( \frac{1}{2} \zeta(2n+1) + \sum_{m=1}^{\infty} \frac{1}{m^{2n+1}(\exp(2m\beta) - 1)} \right) \\
&= \sum_{k=0}^{n+1} (-1)^{k-1} \frac{B_{2k}}{(2k)!} \frac{B_{2n-2k+2}}{(2n-2k+2)!} \alpha^{n-k+1} \beta^k.
\end{align*}
\]

Using Euler’s classical result for \( n \in \mathbb{N} \)
\[
\zeta(2n) = \frac{(-1)^{n-1} B_{2n}}{2(2n)!} (2\pi)^{2n},
\]
we rewrite Ramanujan’s formula in a more convenient form, which is
Corollary 3.2. Let $M \in \mathbb{N}_0$. Then we have for $\alpha > 0$ and for $M \geq 1$

$$
\pi \frac{1}{\alpha^{2M}} \sum_{n=1}^{\infty} \frac{\coth(\alpha \pi n)}{n^{M+1}} - \pi \alpha^{2M} \sum_{n=1}^{\infty} \frac{\coth(\frac{1}{\alpha} \pi n)}{n^{M+1}}
= - \zeta(4M + 2) \left( \frac{\alpha^{2M+1}}{\alpha^{2M+1}} - \sum_{j=1}^{M} (-1)^j \zeta(2j) \zeta(4M + 2 - 2j) \left( \frac{\alpha^{2M+1-2j}}{\alpha^{2M+1-2j}} \right) \right)
= \zeta(4M + 2) \frac{1}{\alpha^{2M+1}} - 2 \sum_{k=1}^{2M} (-1)^k \zeta(2k) \zeta(4M + 2 - 2k) \frac{1}{\alpha^{2(M-k)+1}} - \zeta(4M + 2) \alpha^{2M+1},
$$

and for $M \geq 0$

$$
\pi \frac{1}{\alpha^{2M+1}} \sum_{n=1}^{\infty} \frac{\coth(\alpha \pi n)}{n^{M+3}} + \pi \alpha^{2M+1} \sum_{n=1}^{\infty} \frac{\coth(\frac{1}{\alpha} \pi n)}{n^{M+3}}
= \zeta(4M + 4) \left( \frac{\alpha^{2M+2}}{\alpha^{2M+2}} - \sum_{j=1}^{M} (-1)^j \zeta(2j) \zeta(4M + 4 - 2j) \left( \frac{\alpha^{2(M+1-j)}}{\alpha^{2(M+1-j)}} \right) \right)
+ 2(-1)^M \zeta(2M + 2)^2
= \zeta(4M + 4) \frac{1}{\alpha^{2M+2}} - 2 \sum_{k=1}^{2M+1} (-1)^k \zeta(2k) \zeta(4M + 4 - 2k) \frac{1}{\alpha^{2(M-k)+1}} + \zeta(4M + 4) \alpha^{2M+2}.
$$

(3.1)

(3.2)

Theorem 3.3. Let $N \in \mathbb{N}$ and $h_{k,n}$ as defined in (2.22). Then we have

$$
\lim_{\alpha \to 0+} \sum_{n=1}^{\infty} \frac{\sinh(\alpha n)}{n \cosh(\alpha \pi n)^{2N+1}} = \frac{1}{N^2 \pi^{2N+1}} \sum_{k=1}^{N} (2k)! \left( 2^{2N+1} - 2^{(N-k)} \right) h_{k,N} \frac{\zeta(2k+1)}{\pi^{2k}}.
$$

(3.3)

Proof. We divide both sides of (3.2) by $\alpha^{2M+1}$ and both sides of (3.1) by $\alpha^{2M}$. Then we apply the operator $\alpha^2 \frac{d}{d\alpha}$ on (3.2) $(4M + 2)$ times and on (3.1) $4M$ times. This results in the linear system

$$
\pi^{4M+3} \sum_{k=1}^{2M+1} c_{2M+1,k} \sum_{n=1}^{\infty} \frac{\cosh(\frac{1}{\alpha} \pi n)}{n \sinh(\frac{1}{\alpha} \pi n)^{2k+1}} = - \pi(4M + 2)! \sum_{n=1}^{\infty} \frac{\coth(\alpha \pi n)}{n^{4M+3}}
+ \{ \text{sums involving } \coth \text{ vanishing for } \alpha \to \infty \}
+ (4M + 3)! \zeta(4M + 4) \frac{1}{\alpha} + (4M + 2)! \zeta(4M + 4) \alpha^{4M+3},
$$

for $M \geq 0$ and

$$
-\pi^{4M+1} \sum_{k=1}^{2M} c_{2M,k} \sum_{n=1}^{\infty} \frac{\cosh(\frac{1}{\alpha} \pi n)}{n \sinh(\frac{1}{\alpha} \pi n)^{2k+1}} = - \pi(4M)! \sum_{n=1}^{\infty} \frac{\coth(\alpha \pi n)}{n^{4M+1}}
+ \{ \text{sums involving } \coth \text{ vanishing for } \alpha \to \infty \}
+ (4M + 1)! \zeta(4M + 2) \frac{1}{\alpha} - (4M)! \zeta(4M + 2) \alpha^{4M+1},
$$

for $M \geq 1$. Here we use the $c_{n,k}$’s from (2.10). Select $N \in \mathbb{N}$. Then solving for $\sum_{n=1}^{\infty} \frac{1}{n \sinh(\frac{1}{\alpha} \pi n)^{2N+1}}$
gives

$$\sum_{n=1}^{\infty} \frac{1}{n} \frac{\cosh(\frac{1}{n} \pi \alpha)}{\sinh(\frac{1}{n} \pi \alpha)} = \sum_{k=1}^{N} v_{N,k} \sum_{n=1}^{\infty} \frac{\coth(\alpha \pi n)}{n^{2k+1}} \sum_{n=1}^{\infty} \frac{1}{\sinh(\frac{1}{n} \pi \alpha)}$$

$$+ \{ \text{sums involving } \coth \text{ vanishing for } \alpha \to \infty \}$$

$$- \sum_{k=1}^{N} v_{N,k}(2k + 1) \frac{\zeta(2k + 2)}{\pi^{2k+1}} \frac{1}{\alpha} + \sum_{k=1}^{N} v_{N,k}(-1)^k \frac{\zeta(2k + 2)}{\pi^{2k+1}} \alpha^{2k+1},$$

with the $v_{n,k}$'s defined in (2.23). Let us first assume that $N$ is odd, i.e. $N = 2M - 1$. Then plugging (2.13) into the left hand side of above equation gives

$$\sum_{n=1}^{\infty} \frac{1}{n} \frac{\sinh(\frac{1}{n} \pi \alpha)}{\cosh(\frac{1}{n} \pi \alpha)} = \sum_{k=0}^{M-1} 2^{2(M+k)+1} \frac{M - k}{2k + 1} \left( \frac{M + k}{2k} \right) \sum_{n=1}^{\infty} \frac{1}{n} \frac{\cosh(\frac{2}{n} \pi \alpha)}{\sinh(\frac{2}{n} \pi \alpha)}$$

$$+ \{ \text{sums involving } \coth \text{ vanishing for } \alpha \to \infty \}$$

$$- \sum_{k=1}^{2M-1} v_{2M-1,k}(2k + 1) \frac{\zeta(2k + 2)}{\pi^{2k+1}} \frac{1}{\alpha} + \sum_{k=1}^{2M-1} v_{2M-1,k}(-1)^k \frac{\zeta(2k + 2)}{\pi^{2k+1}} \alpha^{2k+1}$$

$$= - \sum_{k=0}^{M-1} 2^{2(M+k)+1} \frac{M - k}{2k + 1} \left( \frac{M + k}{2k} \right) \sum_{s=1}^{M+k} v_{M+k,s} \frac{1}{\pi^{2s}} \sum_{n=1}^{\infty} \frac{\coth(\frac{\alpha \pi n}{2s+1})}{n^{2s+1}}$$

$$+ \{ \text{sums involving } \coth \text{ vanishing for } \alpha \to \infty \}$$

$$- \sum_{k=0}^{M-1} 2^{2(M+k)+1} \frac{M - k}{2k + 1} \left( \frac{M + k}{2k} \right) \sum_{s=1}^{M+k} (v_{M+k,s}(2s + 1) - v_{M+k,s}(2s + 1)) \frac{\zeta(2s + 2)}{\pi^{2s+1}} \frac{1}{\alpha}$$

$$+ \sum_{k=1}^{2M-1} v_{2M-1,k}(2k + 1) \frac{\zeta(2k + 2)}{\pi^{2k+1}} \frac{1}{\alpha} + \sum_{k=1}^{2M-1} v_{2M-1,k}(-1)^k \frac{\zeta(2k + 2)}{\pi^{2k+1}} \alpha^{2k+1}$$

$$= - \sum_{s=1}^{M-1} \sum_{k=0}^{M-1} 2^{2(M+k)+1} \frac{M - k}{2k + 1} \left( \frac{M + k}{2k} \right) v_{M+k,s} \frac{1}{\pi^{2s}} \sum_{n=1}^{\infty} \frac{\coth(\frac{\alpha \pi n}{2s+1})}{n^{2s+1}}$$

$$+ \{ \text{sums involving } \coth \text{ vanishing for } \alpha \to \infty \}$$

$$+ v_{2M-1,k} \frac{1}{\alpha} + P_{2M-1}(\alpha),$$

where $c_{2M-1}$ denotes the coefficient of $\frac{1}{\alpha}$ and $P_{2M-1}(\alpha)$ the polynomial of degree $4M - 1$ occurring
right after $c_{2M-1,\frac{1}{\alpha}}$. Now (2.24) and $v_{n,k} = 0$ for $k > n$ yield

$$\sum_{n=1}^{\infty} \frac{1}{n} \frac{\sinh(\frac{1}{\alpha} \pi n)}{\cosh(\frac{1}{\alpha} \pi n)^{M-1}} = \sum_{k=1}^{2M-1} \frac{1}{\pi 2k} \left( \sum_{n=1}^{\infty} \frac{\coth(\alpha \pi n)}{n^{2k+1}} \right) - \sum_{n=1}^{\infty} \frac{\coth(\frac{1}{\alpha} \pi n)}{n^{2k+1}}$$

$$+ \{\text{sums involving coth vanishing for } \alpha \to \infty \}$$

$$+ c_{2M-1,\frac{1}{\alpha}} + P_{2M-1}(\alpha).$$

For $M = 1$ we can with a little effort directly compute

$$\lim_{\alpha \to \infty} \sum_{n=1}^{\infty} \frac{1}{n} \frac{\sinh(\frac{1}{\alpha} \pi n)}{\cosh(\frac{1}{\alpha} \pi n)^{3}} = \frac{7}{\pi} \zeta(3).$$

So

$$0 < \sum_{n=1}^{\infty} \frac{1}{n} \frac{\sinh(\frac{1}{\alpha} \pi n)}{\cosh(\frac{1}{\alpha} \pi n)^{2M+1}} < \frac{7}{\pi} \zeta(3) + \epsilon$$

for a given $\epsilon$ and $\alpha$ sufficiently large. All other terms than $P_{2M-1}(\alpha)$ in the right hand side of (3.4) are bounded. Since $P_{2M-1}(\alpha)$ does not have a constant term it follows with above result $P_{2M-1}(\alpha) \equiv 0$. Inserting (2.23) into (3.4) and the fact that $\lim_{\alpha \to \infty} \coth(\alpha \pi n) = 1$ holds for all $n \in \mathbb{N}$ proves the claim for $N = 2M - 1$. The proof for $N = 2M$ follows completely analogously using (2.14) and (2.25).

\[\Box\]

3.2 Applications

**Corollary 3.4.** We have the identity

$$14 \sum_{n=1}^{\infty} \frac{n^{2n}}{(n+1)^{(2n+2)n+1}} \frac{\zeta(3)}{\pi^{2}} + \sum_{j=2}^{\infty} \left( \sum_{n=j-1}^{\infty} \frac{n^{2n}}{(n+1)^{(2n+2)n+1}} h_{j,n+1} \right) 4(2j)! \left( 2 - \frac{1}{2j} \right) \frac{\zeta(2j+1)}{\pi^{2j}} = 28 \frac{\zeta(3)}{\pi^{2}}.$$  

(3.5)

**Proof.** Differentiating

$$\tanh(x) \sum_{k=0}^{\infty} \left( \frac{1}{2 \cosh(x)^{2}} \right)^{k} = \tanh(2x)$$

(3.6)

twice gives

$$\sum_{k=1}^{\infty} \frac{k(k+1)}{2^{k}} \frac{\sinh(x)}{\cosh(x)^{2k+3}} = 4 \frac{\sinh(2x)}{\cosh(2x)^{3}}.$$  

(3.7)

Inserting (3.3), taking the limit $\alpha \to 0_{+}$ and rearranging give the result. \[\Box\]

**Remark 3.5.** More identities like (3.5) can be obtained from differentiating (3.7) $2N$ times.

**Corollary 3.6.** Choose $K \in \mathbb{N}_{0}$ and $N \in \mathbb{N}$. Then the limits

$$\lim_{\alpha \to 0_{+}} \frac{1}{\alpha^{K}} \sum_{n=1}^{\infty} \frac{\sinh(\alpha n)^{1+K}}{\cosh(\alpha n)^{2M+1+K}}$$

(3.8)

exist and are a finite linear combination of $\frac{\zeta(3)}{\pi^{2}}, \ldots, \frac{\zeta(2N+2K+1)}{\pi^{2N+2K}}$ with rational coefficients.
Proof. For $K = 0$ we the claim becomes (3.3). For $K > 0$ l’Hôpital’s rule yields

\[
\lim_{\alpha \to 0_+} \frac{1}{\alpha K} \sum_{n=1}^{\infty} \frac{1}{n^{K+1}} \frac{\sinh(\alpha n)^{1+K}}{\cosh(\alpha n)^{2M+1+K}} = -\frac{2N}{K} \lim_{\alpha \to 0_+} \frac{1}{\alpha K^{-1}} \sum_{n=1}^{\infty} \frac{1}{n^K} \frac{\sinh(\alpha n)^{1+K-1}}{\cosh(\alpha n)^{2M+1+K-1}} \\
+ \frac{K + 2N + 1}{K} \lim_{\alpha \to 0_+} \frac{1}{\alpha K^{-1}} \sum_{n=1}^{\infty} \frac{1}{n^K} \frac{\sinh(\alpha n)^{1+K-1}}{\cosh(\alpha n)^{2M+3+K-1}}.
\]

Iterating this finding $K$ times and (3.3) give the result.

Remark 3.7. Using (3.8) one can obtain for $K \in \mathbb{N}$ more identities like (3.5) from (3.6) and (3.7) by sending $\alpha \to 0_+$ in

\[
\frac{1}{\alpha K} \sum_{n=1}^{\infty} \frac{1}{n^{1+K}} \left( \tanh(\alpha n) \sum_{k=0}^{\infty} \left( \frac{1}{2 \cosh(\alpha n)^2} \right)^k \right)^K \sum_{k=1}^{\infty} \frac{k(k+1)}{2^k} \frac{\sinh(\alpha n)}{\cosh(\alpha n)^{2k+3}} \\
= 4 \frac{1}{\alpha K} \sum_{n=1}^{\infty} \frac{1}{n^{1+K}} \tanh(2\alpha n)^K \frac{\sinh(2\alpha n)}{\cosh(2\alpha n)^3}.
\]

References

[1] B. C. Berndt, “Ramunujans formula for $\zeta(2n+1)$,” Professor Srinivasa Ramanujan Commemoration Volume, pp. 1–7, 1974.