Diophantine Approximation on varieties IV: Derivated algebraic distance and derivative metric
Bézout Theorem

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1 Introduction

Let \( \mathbb{P}^t = \mathbb{P}(\mathbb{Z}^{t+1}) \) be the \( t \)-dimensional projective space over \( \text{Spec} \mathbb{Z} \), \( \mathbb{P}(W) \subset \mathbb{P}^t \) a projective subspace of codimension \( q \), and \( X \in Z_{\text{eff}}(\mathbb{P}^N) \) an effective cycle. We say that \( X \) is regular with respect to \( \mathbb{P}(W) \) if the irreducible components of codimension \( < t - q \) intersect \( \mathbb{P}(W) \) properly, and the irreducible components of codimension \( \geq t - q \) do not meet \( \mathbb{P}(W) \). For regular cycles the algebraic distance

\[
D(Y, \mathbb{P}(W)) \in \mathbb{R}
\]

is defined in [Ma1], section 4.1. Further for \( p + q \leq t + 1 \), and effective cycles \( X, Y \) of pure codimensions \( p, q \) that intersect properly, the algebraic distance

\[
D(X, Y) \in \mathbb{R}
\]

is defined in [Ma1], Definition 4.1. For \( x, y \) points in \( \mathbb{P}(\mathbb{C}) \) denote by \( |x, y| \) their Fubini-Study distance, i.e. \( \sin(x, y) \). The logarithm of the distance is a nonpositive number.

There are the following Theorems for the algebraic distance.

1.1 Theorem For properly intersecting cycles \( \mathcal{X}, \mathcal{Y} \in Z_{\text{eff}}(\mathbb{P}^N) \) with base extensions \( \mathcal{X}, \mathcal{Y} \) to \( \mathbb{C} \) the equality

\[
D(Y, Z) = h(\mathcal{Y}, \mathcal{Z}) - \deg Z h(\mathcal{Y}) - \deg Y h(\mathcal{Z}) - \sigma_t \log 2 \deg Y \deg Z
\]

holds.

Proof [Ma1], Scholie 4.3.

1.2 Theorem With the previous Definition, let \( \mathcal{X}, \mathcal{Y} \) be effective cycles intersecting properly, and \( \theta \) a point in \( \mathbb{P}(\mathbb{C}) \setminus (\text{supp}(X \cup Y)) \).

1. There are effectively computable constants \( c, c' \) only depending on \( t \) and the codimension of \( X \) such that

\[
\deg(X) \log |\theta, X(\mathbb{C})| \leq D(\theta, X) + c \deg X \leq \log |\theta, X(\mathbb{C})| + c' \deg X,
\]

2. If \( \mathcal{X} = \text{div}(f) \) is an effective cycle of codimension one,

\[
h(\mathcal{X}) \leq \log |f|_{L^2} + D \sigma_t, \quad \text{and}
\]

\[
D(\theta, X) + h(\mathcal{X}) = \log |\langle f | \theta \rangle| + D \sigma_{N-1},
\]

where the \( \sigma_i \)'s are certain constants, and \( |\langle f | \theta \rangle| \) is taken to be the norm of the evaluation of \( f \in \text{Sym}^D(E) = \Gamma(\mathbb{P}^N, O(D)) \) at a vector of length one representing \( \theta \).
3. For \( p + q \leq N + 1 \), assume that \( X \) and \( Y \) have pure codimension \( p \) and \( q \) respectively. There exists an effectively computable positive constant \( d \), only depending on \( t \), and a map

\[
f_{X,Y} : I \rightarrow \deg X \times \deg Y
\]

from the unit interval \( I \) to the set of natural numbers less or equal \( \deg X \) times the set of natural numbers less or equal \( \deg Y \) such that \( f_{X,Y}(0) = (0,0) \), \( f_{X,Y}(1) = (\deg X, \deg Y) \), and the maps \( pr_1 \circ f_{X,Y} : I \rightarrow \deg X \), \( pr_2 \circ f_{X,Y} : I \rightarrow \deg Y \) are monotonously increasing, and surjective, fulfilling: For every \( T \in I \), and \( (\nu, \kappa) = f_{X,Y}(T) \), the inequality

\[
\nu \kappa \log |\theta, X + Y| + D(\theta, X,Y) + h(X,Y) \leq \\
\kappa D(\theta, X) + \nu D(\theta, Y) + \deg Y h(X) + \deg X h(Y) + d \deg X \deg Y
\]

holds.

4. In the situation of 3, if further \( |\theta, X + Y| = |\theta, X| \), then

\[
D(\theta, X,Y) + h(X,Y) \leq D(\theta, Y) + \deg Y h(X) + \deg X h(Y) + d' \deg X \deg Y
\]

with \( d' \) a constant only depending on \( N \).

**Proof**: Let now \( \mathbb{P}(W) \subset \mathbb{P}^t(\mathbb{C}) \) be a subspace of dimension \( q \), and \( \partial^t \) a real differential operator on the Grassmannian \( G_{t+1-q,t+1} \) of \( t + 1 - q \)-dimensional subspaces of \( \mathbb{C}^{t+1} \) with respect to some affine chart, where \( I = (i_1, \ldots, i_{2q(t+1-q)}) \) is a multiindex of order \( |I| = i_1 + \cdots + i_{2q(t+1-q)} \). (More details will be given later on). Defines the derivated algebraic distance of order \( S \) of \( X \) to \( \mathbb{P}(W) \) as

\[
D^S(Y, \mathbb{P}(W)) := \sup_{|I| \leq S} \log |\partial^I(\exp D(X, \mathbb{P}(W)))|.
\]

There are the following Theorems for this derivated algebraic distance.

**1.3 Theorem** For \( s, D \in \mathbb{N} \), and \( f \in \Gamma(\mathbb{P}^t, O(D)) \) let \( F \) be the polynomial of degree at most \( D \) in \( t \) variables that corresponds to \( f \) with respect to affine coordinates of \( \mathbb{P}^t \) centered at \( \theta \). Then, with some positive constant \( c \) only depending on \( t \),

\[
D^S(div f, \theta) \leq \sup_{s \leq S, |I| = s} \log \left( \left| \frac{\partial^s}{(\partial z_1)^{i_1} \cdots (\partial z_t)^{i_t}} f \right| (0) \right) + c(s + D) \log(SD).
\]
in the following Theorems the $O$-notation always signifies that the respective inequalities hold modulo a fixed constant only depending on $t$ and codimensions of cycles times the term inside the $O$-bracket.

1.4 Theorem Let $Z$ be an effective cycle of pure codimension $p$ in $\mathbb{P}^t$, and $\theta$ a point not contained in the support of $Z$, and let $|\cdot,\cdot|$ denote the Fubini-Study distance in $\mathbb{P}^t$.

There is a projective subspace $\mathbb{P}(F) \subset \mathbb{P}^t_{\mathbb{C}}$ of codimension $t - p$ intersecting $Z$ properly such that with $z_1,\ldots,z_\deg Z$ the points in the intersection $\mathbb{P}(F).Z$ counted with multiplicity, $|z_1,\theta| \leq \cdots \leq |z_\deg Z,\theta|$, and for $S < \deg Z / 3$ the equalities

$$ D^S(Z,\theta) = \sum_{i=S+1}^{\deg Z} \log |z_i,\theta| + O(S \log \deg Z), $$

$$ 2 \sum_{i=S+1}^{\deg Z} \log |z_i,\theta| \leq D^{3S}(Z,\theta) + O((\deg Z + S) \log(S \deg Z)) $$

hold.

1.5 Corollary The derived algebraic distance is a negative number modulo $O((\deg Z + S) \log(S \deg Z))$.

Next, for $n \in \mathbb{N}$ denote $\mathbb{N}$ the set of natural numbers less or equal $n$ including 0, and for $Z_0, Z_1$ effective cycles, let $f = (f_0, f_1) : \deg Z_0 + \deg Z_1 \to \deg Z_0 \times \deg Z_1$ be a a path from $(1,1)$ to $(\deg Z_0, \deg Z_1)$ such that in each step exactly one of the coordinates increases. If in the $k$th step the coordinate $i$ increases, set $i_k = i$.

1.6 Theorem For any effective cycles $Z_0, Z_1 \in Z_{\text{eff}}(\mathbb{P}^t)$ that intersect properly, and $\theta \in \mathbb{P}^t_{\mathbb{C}}$ a point not contained in the support of $Z_0.Z_1$, there is a path $f$ such that

$$ 2D(Z_0, Z_1) + 2D(Z_0.z_1, \theta) \leq \sum_{k=1}^n D_{f(k)}(Z_{i_k}, \theta) + O((\deg Z_0 \deg Z_1 + S) \log(S \deg Z_0 \deg Z_1)). $$

1.7 Corollary With any $l \leq \deg Z_0 + \deg Z_1$, and $(\nu_0, \nu_1) = f(l)$,

$$ 2D(Z_0, Z_1) + 2D(Z_0.z_1, \theta) \leq 
\nu_0 D^{3\nu_1}(Z_1, \theta) + \nu_1 D^{3\nu_0}(Z_0, \theta) + O((\deg Z_0 \deg Z_1 + S) \log(S \deg Z_0 \deg Z_1)). $$
This immediately implies

**1.8 Corollary** For every $S \leq \deg Z_1/3$,

$$2D(Z_0, Z_1) + 2D(Z_0, Z_1, \theta) \leq \max(SD(Z_0, \theta), D^{2S-1}(Z_1, \theta)) + O(\deg X \deg Y).$$

**1.9 Theorem** For any $S \leq \deg X \deg Y/9$, there are natural numbers $\nu_0, \nu_1$ with $\nu_0 \nu_1 \leq S$, and a path $f$ such that and a function $h_S : \deg Z_0 + \deg Z_1 \rightarrow \mathbb{N}$ with $h_S(k) = 0$ for $k \geq k_0$ such that

1. $$2D(X, Y) + 2D^S(X, Y, \theta) \leq \sum_{k=1}^n D^3(f_{ik}(k) - h_S(k))(Z_{ik}, \theta).$$

2. For any $k \leq \deg Z_0 + \deg Z_1$ greater or equal $k_0$, and $(\bar{\nu}_0, \bar{\nu}_1) = f(k)$,

$$2(\bar{\nu}_0 - \nu_0)(\bar{\nu}_1 - \nu_1) \log |Z_0 + Z_1, \theta| + 2D^S(Z_0, Z_1, \theta) + D(Z_0, Z_1) \leq$$

$$(\bar{\nu}_0 - \nu_0)D^{3\kappa_0}(Z_1, \theta) + (\bar{\nu}_1 - \nu_1)D^{3\kappa_0}(Z_0, \theta) +$$

$$O((\deg Z_0 \deg Z_1 + S) \log(S \deg Z_0 \deg Z_1)).$$

**1.10 Corollary** For any $S \leq \deg Z_0 \deg Z_1/9$, there are numbers $\nu_0 \nu_1$ with $\nu_0 \nu_1 \leq S$ such that for any $l \leq \deg Z_0 + \deg Z_1$ greater or equal $k_0$ with $(\bar{\nu}_0, \bar{\nu}_1) = f(l)$,

$$2D(Z_0, Z_1) + 2D^S(Z_0, Z_1, \theta) \leq$$

$$(\bar{\nu}_1 - \nu_1)D^{3\kappa_0}(Z_0, \theta) + (\bar{\nu} - \nu_0)D^{3\kappa_0}(Z_1, \theta) + O((\deg Z_0 \deg Z_1 + S) \log(S \deg Z_0 \deg Z_1)).$$

**1.11 Corollary** Let $S_0 \leq \deg Z_0/3$, $S_1 \leq \deg Z_1/3$ be natural numbers, and $S = S_0S_1$. Then,

$$2D(Z_0, Z_1) + 2D^S(Z_0, Z_1, \theta) \leq \max(S_1D^{9S_0}(Z_0, \theta), S_0D^{9S_1}(Z_1, \theta)) +$$

$$O((\deg Z_0 \deg Z_1 + S) \log(S \deg Z_0 \deg Z_1)).$$

**Remarks:** 1. I strongly conjecture that Theorem 1.4 as well as Theorems 1.6 and their corollaries still hold if the factor 2 before $D(\theta, X, Y)$ and $D^S(\theta, X, Y)$ is dropped, and possibly also if the 3 in the exponent on the right hand side is replaced by some smaller number greater or equal 1. In order to obtain this, one would only have to improve Lemma 4.3 in this respect; however, I don’t know right now how to do that. For the applications of the Theorems and Corollaries to
Diophantine Approximation and algebraic independence theory this improvement would be insubstantial.

2. Throughout this paper, constants entailed by the notation \( O(\cdots) \) always depend only on \( t \), and the dimensions of cycles involved in the context. As, there are always only finitely many cycles involved, the constants can also be assumed to be depending only on \( t \).

3. To my knowledge, in the literature, one special case of the above Theorems and Corollaries is known, namely Corollary 1.8 in the case \( t = 1 \), \( \text{codim} Z_0 = \text{codim} Z_1 = 1 \). See [LR], preuve du corollaire 3.

Recall that in [Ma1], section 4 there were given 3 alternative definitions of the algebraic distance. The algebraic distance \( D_\infty(\theta, \cdots) \) is not additive on the cycle group, and has some other deficiencies; therefore it is probably not possible to prove the derivative metric Bézout for \( D_\infty \). Proofs will be given for \( D_{Ch} \) and \( D_1 \).

This paper heavily depends on part one ([Ma1]) of this series on diophantine approximation on varieties and can possibly not be read independently of it. It does not however presuppose any knowledge of part 2 and part 3.

2 Sharp decomposition of the algebraic distance

Recall the following notations from [Ma1]. If \( G = G_{q,t} \) is the Grassmannian of \( q \)-dimensional subspaces of \( \mathbb{C}^{t+1} \), then for a subspace \( \mathbb{P}(W) \subset \mathbb{P}^t \) of dimension \( r \leq q \) the sub Grassmannian of \( G \) consisting of the spaces that contain \( W \) is denoted \( G_W \), and for \( \mathbb{P}(F) \subset \mathbb{P}^t \) a subspace of dimension \( p \geq q \) the sub Grassmannian of spaces being contained in \( F \) is denoted \( G^F \).

Let \( \mathbb{P}(F) \subset \mathbb{P}^t \) be a subspace of codimension \( r \), and \( \pi \) the map

\[
\pi : \mathbb{P}^t \setminus \mathbb{P}(F^\perp) \to \mathbb{P}(F), \quad [v, w] \mapsto [v], \quad v \in F, w \in F^\perp.
\]

For any sub variety \( X \subset \mathbb{P}(F) \) of codimension \( p \), the closure \( X_F := \overline{\pi^{-1}(X)} \) is a subvariety of codimension \( p \) in \( \mathbb{P}^t \) with the same degree as \( X_F \). This induces a map \( \pi^* : Z^p(\mathbb{P}(F)) \to Z^p(\mathbb{P}^t), X \mapsto X_F \) with left inverse \( X \mapsto X \mathbb{P}(F) \). For two effective cycles \( X \in Z^p(\mathbb{P}(F)), Y \in Z^q(\mathbb{P}(F)) \), denote by \( D_{\mathbb{P}(F)}(X_F, Y_F) \) their algebraic distance as cycles in \( \mathbb{P}(F) \).

In [Ma1], Theorem 4.11, and Proposition 4.16, for \( Z \) an effective cycle in \( \mathbb{P}^t \), of codimension \( p \) and \( \mathbb{P}(F) \supset \mathbb{P}(W) \) subspaces of codimensions \( r \geq t - p, q > r \) respectively, regular with respect to \( X \) the relations

\[
D(\mathbb{P}(W), Z) \leq D(\mathbb{P}(W), \mathbb{P}(F).Z) + c_1 \deg Z \leq D(\mathbb{P}(W), Z) - D(\mathbb{P}(F), Z) + c_2 \deg Z
\]

with \( c_1, c_2 \) constants depending only on \( p, q, r, \) and \( t \), were proved, and thereby the algebraic distance of \( Z \) to \( \mathbb{P}(W) \) modulo \( O(\deg Z) \) is reduced to the algebraic distance of \( \mathbb{P}(W) \) to \( Z \mathbb{P}(F) \) and the algebraic distance of \( \mathbb{P}(F) \) ot \( Z \).

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If one wants to consider derivatives of algebraic distances, this decomposition is not good enough, because the derivatives of two functions may have an arbitrarily big difference even if their values don’t differ very much. One needs the following sharper decomposition.

2.1 Proposition With the above notations, there is a positive constant $c_1$ only depending on $p, q, r,$ and $t$ such that

$$D(\mathbb{P}(W), Z) + c_1 \deg Z = D(\mathbb{P}(W), \mathbb{P}(F)) + D(Z, \mathbb{P}(F)).$$

The proof will use two Lemmas.

2.2 Lemma Let $p + r \leq t$, $X \in Z_{\text{eff}}^p(\mathbb{P}(\mathbb{C}))$, and $\mathbb{P}(F) \subset \mathbb{P}(\mathbb{C})$ a subspace of codimension $r$ that intersects $X$ properly. Let further $\mathbb{P}(W) \subset \mathbb{P}(F)$ be a subspace of codimension $q \geq r$ that is regular with respect to $X$. Then

$$D(Z, \mathbb{P}(W)) - D(Z, \mathbb{P}(F)) = D(Z, \mathbb{P}(F)) - D(Z, \mathbb{P}(F)).$$

Proof If $q \leq t + 1 - p$, this is [Ma1], Proposition 5.1.

If $q > t + 1 - p$, the Lemma will be proved for $D_{\text{Ch}}$ and $D_G$ successively, firstly for $D_{\text{Ch}}$: Let $\mathbb{P}(V) \subset \mathbb{P}(\mathbb{C})$ be a subspace of codimension $t + 1 - p$ that does not intersect $Z$, and fulfills $\mathbb{P}(W) \subset \mathbb{P}(V) \subset \mathbb{P}(F)$. By [Ma1], Proposition 5.1, and Proposition 4.14.2,

$$D(Z, \mathbb{P}(W)) - D(Z, \mathbb{P}(F)) = D(Z, \mathbb{P}(F)) - D(Z, \mathbb{P}(F)) = D_{\text{Ch}}(Z, \mathbb{P}(W)) - D_{\text{Ch}}(Z, \mathbb{P}(F)).$$

(1) Next, we repeat the construction of the cycle deformation in [Ma1], section 5. For any effective cycle $X$ in $\mathbb{P}(\mathbb{C})$ that intersects $\mathbb{P}(F)$ properly, define $\Phi$ as the subvariety of $(\mathbb{P}(\mathbb{C}) \times \mathbb{P}(\mathbb{C}))^{t+1-p} \times \mathbb{A}(\mathbb{C})$ given as the Zariski closure of the set

$$\{(\psi_\lambda(x), \ldots, \psi_\lambda(x), \lambda) \in (\mathbb{P}(\mathbb{C}))^{t+1-p} \times \mathbb{C}^* | x \in X(\mathbb{C})\}. \quad (3)$$
Then, $\Phi$ intersects $\mathbb{P}(F) \times \Lambda^1$ properly. Further, for $\lambda \in \mathbb{C}$, and $y$ the coordinate of the affine line, the divisor $\Phi_\lambda$ corresponding to the restriction of the function $y - \lambda$ to $\Phi$ is a proper intersection of $\Phi$ and the zero set of $y - \lambda$ and is of the form $Z_{\lambda}^{t+1-p} \times \{\lambda\}$, for some subvariety $Z_{\lambda}$ of $\mathbb{P}^t_{\mathbb{C}}$; and for $\lambda \neq 0$, we have $Z_{\lambda} = \psi_{\lambda}(X)$. The specialization $\Phi_0$ equals

$$\Phi_0 = \pi^*(X_{\mathbb{P}(F)} = \pi^*(X_{\lambda \mathbb{P}(F)}),$$

for arbitrary $\lambda \in \mathbb{C}^*$. (See [BGS], p.994.)

Recall the correspondence from [Ma1], section 5, for arbitrary $\lambda, \mu, \chi, Z$.

With $(F, \pi, \psi, Z, \lambda)$ for an effective cycle $Z$ of codimension as above, and the corresponding cycle $\Phi \subset \mathbb{P}^t \times \Lambda^1$, the cycle $\Phi := G_* F^* \Delta_\lambda \Phi$ intersects $(\mathbb{P}(W))^{t+1-p} \times \Lambda^1$ properly, and the intersection of $\Phi$ with $(\mathbb{P}^t)^{t+1-p} \times \{y\}$ is likewise proper. For any $\lambda \in \mathbb{C}$ the cycle $G_* F^* \Delta_\lambda (Z_{\lambda} \times \{\lambda\})$ equals $Ch(Z_{\lambda}) \times \{\lambda\}$. Take $g_{\mathbb{P}(W)p}$ the normalized Green form of $\mathbb{P}(W)^{t+1-p}$ in $(\mathbb{P}^t)^{t+1-p}$, and define

$$\varphi_W(\lambda) = \int_{(\mathbb{P}^t)^{t+1-p}} \delta_{Ch(Z_{\lambda})} g_{\mathbb{P}(W)(t+1-p)} \tilde{\mu}^{(q-1)(t+1-p)},$$

$$\varphi_V(\lambda) = \int_{(\mathbb{P}^t)^{t+1-p}} \delta_{Ch(Z_{\lambda})} g_{\mathbb{P}(V)(t+1-p)} \tilde{\mu}^{(t-p)(t+1-p)},$$

where $\tilde{\mu} = c_1(O(1, \ldots, 1))$ (see [Ma1], section 3.3 for details). By definition,

$$D_{Ch}(\mathbb{P}(W), Z_{\lambda}) = \frac{1}{(t+1-p)(t-p)} \varphi_W(\lambda), \quad D_{Ch}(\mathbb{P}(V), Z_{\lambda}) = \frac{1}{(t+1-p)(t-1)} \varphi_V(\lambda),$$

where $\frac{(t+1-p)(t-p)}{t-1, \ldots, t-1}$ is the multinomial coefficient. From the proof of [Ma1], Proposition 4.16.3, it is clear that there are smooth real functions $\chi_W, \chi_F : \mathbb{R} \to \mathbb{R}$ such that $\varphi_W(\lambda) = \chi_W(|\lambda|)$, $\varphi_V(\lambda) = \chi_V(|\lambda|)$, and further that $\chi'_W(0) = \chi'_V = 0$, and $\chi''_W, \chi''_V$ are nonnegative; consequently $\varphi_W(\lambda) \geq \varphi_W(0)$, $\varphi_V(\lambda) \geq \varphi_V(0)$. Finally, with $\mu_i$ being the Fubini-Study form on the $i$th factor of $(\mathbb{P}^t)^{t+1-p},$

$$dd^c([\varphi_W]) = pr_{2*}(\delta_\phi pr_1^*(\mu_1^{t+1-q} \cdots \mu_{t+1-p}^{t+1-q} \mu(q(t+1-p)-1))).$$
\[
\left(\frac{t+1-p}{t-p}\right) \left(\frac{t-p}{t-p}\right) \cdots \left(\frac{t-p}{t-p}\right)\]
\[
\left(\frac{1-p}{1-p}\right) \left(\frac{1-p}{1-p}\right) \cdots \left(\frac{1-p}{1-p}\right)\]

and similarly
\[
\frac{1}{(t+1-p)(t-p)} dd^c ([\varphi_W]) = \frac{1}{(t+1-p)(t-p)} dd^c ([\varphi_V]) = pr_2^* (\delta_1^p (\mu_1 \cdots \mu_{t+1-p})) ,
\]

hence
\[
\frac{1}{(t+1-p)(t-p)} \chi''_W = \frac{1}{(t+1-p)(t-p)} \chi''_V ,
\]

consequently, since \( \chi_W(0) = \chi_V(0)' = 0 \),
\[
\frac{1}{(t+1-p)(t-p)} (\varphi_W(0) - \varphi_W(\lambda)) = \frac{1}{(t+1-p)(t-p)} (\varphi_F(0) - \varphi_F(\lambda)).
\]

Together with (4), this implies
\[
D(P(W), Z) - D(P(W), Z) = D(P(W), Z) - D(P(W), Z) =
\]

which together with (4) entails the Lemma.

**Proof of Proposition 2.1 for \( D_G \):** Again, if \( P(V) \) is a subspace of codimension \( t + 1 - p \) intersecting \( Z \) properly such that \( P(V) \subset P(F) \), then, by the proof of [Ma1], Proposition 4.14,
\[
D(P(F), Z) - D(P(F), Z) = D(P(V), Z) - D(P(V), Z) =
\]

With \( (F, \sigma, X) \) as in (2), and the correspondence
\[
\tilde{C} = C \times A^1
\]

\[
P^t \times A^1 \quad \quad G \times A^1
\]
where again \( F \), and \( G \) are defined by taking the identity on the second factor, and the intersections of \( \Phi \) with \( G \times \{ y \} \) are proper. Thus, we can proceed just in the case \( D_{Ch} \), and define

\[
\varphi_W(\lambda) = \int_G \delta_{V \lambda} g_{GW} \mu_{G}^{(t+1-p)(t+1-q)}, \quad \varphi_V(\lambda) = \int_G \delta_{V \lambda} g_{GV},
\]

where \( \mu_G = c_1(L_G) \) with \( L_G \) the canonical line bundle on \( G \). (see [Ma1], section 3.2.) We have again

\[
[\varphi_W] = (pr_2)_*(\delta_\Phi pr_1*(g_{GW} \mu_G^{(t+1-p)(t+1-q)})), \quad [\varphi_V] = pr_2_*(\delta_\Phi pr_1*(g_{GV})),
\]

and the calculations

\[
\begin{align*}
\ddc(\delta_\Phi pr_1*(g_{GW})) + \delta_\Phi pr_1^*(\omega(g_{GW})) &= \mu_G^{(t+1-p)(t+1-q)}G,
\ddc(\delta_\Phi pr_1^*(g_{GV})) + \delta_\Phi pr_1^*(\omega(g_{GV})) &= \mu_G^{(t+1-p)(t+1-q)}G.
\end{align*}
\]

imply this time

\[
\begin{align*}
\ddc([\varphi_W]) &= pr_2_*(\delta_\Phi pr_1^*(\omega(g_{GW})\mu_G^{(t+1-p)(t+1-q)})),
\ddc([\varphi_V]) &= pr_2_*(\delta_\Phi pr_1^*(\omega(g_{GV})).
\end{align*}
\]

Now, \( G_V \) is the single point \( V \) in \( G \), hence \( \omega(g_V) \) is the canonical generator of the one dimensional space of harmonic forms \( H^p(t+1-p,t+1-p)(G) \). Further, by the intersection theory on \( G \) the form \( \omega(g_{GW})\mu_G^{(t+1-p)(t+1-q)}G \) likewise equals this generator. We get

\[
\ddc([\varphi_W]) = \ddc([\varphi_V]),
\]

and the rest of the proof is in complete analogy to the case \( D_{Ch} \).

2.3 Lemma Let \( X \in Z_{eff}^p (\mathbb{P}_C^t) \), and \( \mathbb{P}(W) \subset \mathbb{P}_C^t \) a subspace of codimension \( q > t - p \) that does not meet the support of \( X \). Finally \( \mathbb{P}(F) \) a subspace of codimension \( r \leq t - p \) containing \( \mathbb{P}(W) \), and intersecting \( X \) properly. Then, for certain constants \( c_3, c_6 \) depending only on \( p, q, r, t \).

1. \( D(\mathbb{P}(W), X_F) = D^{\mathbb{P}(F)}(\mathbb{P}(W), X_F, \mathbb{P}(F)) + c_3 \deg X \).

2. \( D^{\mathbb{P}(F)}(\mathbb{P}(W), X_F, \mathbb{P}(F)) = D(\mathbb{P}(W), X_F, \mathbb{P}(F)) + c_6 \deg X \).
Proof Assume first $r = t - p$

1. In this case the intersection $\mathbb{P}(F).X_F = \mathbb{P}(F).X$ is zero dimensional, hence $X_F$ consists of $t - p$-dimensional subspaces. If $\deg X_F = 1$, the intersection of $X_F$ with $\mathbb{P}(F)$ is a single point, hence by [Ma1], fact 4.8, $D^{\mathbb{P}(F)}(\mathbb{P}(F).X_F, \mathbb{P}(W)) = \log |\mathbb{P}(F).X_F, \mathbb{P}(W)| + c$ where $c$ is a constant only depending on $q, p$ and $t$. Similarly, by the same fact, $D(\mathbb{P}(W), X_F) = \log |\mathbb{P}(W), X_F| + c_1$ with $c_1$ only depending on $r, p$, and $t$. Further, since $X_F$ is orthogonal to $\mathbb{P}(F)$ the equality $|X_F, \mathbb{P}(W)| = |\mathbb{P}(F).X_F, \mathbb{P}(W)|$ holds, and we get

$$D^{\mathbb{P}(F)}(\mathbb{P}(W), \mathbb{P}(F).X_F) = D(\mathbb{P}(W), X_F) + c_3.$$ 

Since the algebraic distance is additive, for $X_F$ of arbitrary degree the equality

$$D(\mathbb{P}(W), \mathbb{P}(F).X_F) = D(\mathbb{P}(W), X_F) + c_3 \deg X$$

follows.

2. Again

$$D(\mathbb{P}(W), X_F.\mathbb{P}(F)) = \log |\mathbb{P}(W), X_F.\mathbb{P}(F)| + c_4 \deg X,$$

and

$$D^{\mathbb{P}(F)}(\mathbb{P}(W), X_F.\mathbb{P}(F)) = \log |\mathbb{P}(W), X_F.\mathbb{P}(F)| + c_5 \deg X,$$

for the same reasons as in 1. The equality

$$D^{\mathbb{P}(F)}(\mathbb{P}(W), X_F.\mathbb{P}(F)) = D(\mathbb{P}(W), X_F.\mathbb{P}(F)) + c_6 \deg X$$

follows.

Let now $r \leq t - p$.

1. Let $\mathbb{P}(V) \subset \mathbb{P}(F)$ be a subspace of codimension $r = t - p$ that contains $\mathbb{P}(W)$, and intersects $X$ properly. Since, $(X_F)_V = X_V$, by Lemma 2.1

$$D(\mathbb{P}(W), X_F) = D(\mathbb{P}(W), X_V) + D(\mathbb{P}(V), X_F) - D(\mathbb{P}(V), X_V).$$

(8)

On the other hand, consider $(X_F.\mathbb{P}(F))_V$ inside $\mathbb{P}(F)$. Again, by Lemma 2.1

$$D^{\mathbb{P}(F)}(\mathbb{P}(W), X_F.\mathbb{P}(F)) =$$

$$D^{\mathbb{P}(F)}(\mathbb{P}(W), (X_F.\mathbb{P}(F))_V) + D^{\mathbb{P}(F)}(\mathbb{P}(V), (X_F.\mathbb{P}(F))) - D^{\mathbb{P}(F)}(\mathbb{P}(W), (X_F.\mathbb{P}(F))_V).$$

(9)

We want to show that the left hand side of (8) equals a constant times $\deg Z$ plus the left hand side of (9). Since, $(X_F, \mathbb{P}(F))_V = X_V.\mathbb{P}(F)$, the first terms on the right hand sides coincide modulo a constant times $\deg Z$ by the Lemma for $r = t - p$. The third terms on the right hand sides are constants times $\deg Z$ by the proof of [BGS], Proposition 5.1.1. Finally the second terms on the right hand sides coincide modulo a constant times $\deg Z$ by [Ma1], Lemma 4.13.1.
2. With \( P(V) \) as in part one, we have by Lemma 2.1

\[
D(P(W), X_F P(F)) =
D(P(W), (X_F P(F))_V) + D(P(V), (X_F P(F))) - D(P(V), (X_F P(F))_V).
\]

The terms on the right hand side of (10), and (8) can be compared completely analogously as in part one, again using the Lemma for \( r = t - p \), the proof of [BGS], Proposition 5.1.1, and [Ma1], Lemma 4.13.1.

Proof of Proposition 2.1: Since \( X_F P(F) = X P(F) \), the Proposition simply follows from the two Lemmata.

3 Affine Differentiation

Let \( Z \) be a Kähler manifold of dimension \( d \), i.e. a complex manifold equipped with a metric on the tangent space \( T_z Z \) for every \( z \in Z \) such that the fundamental form defined by this metric is closed.

A map smooth map \( \mathbb{R}^m \to \mathbb{R}^n \) is called analytic, if its Taylor series locally converges.

Clearly if \( f : \mathbb{C}^m \to \mathbb{C}^n \) is holomorphic, the induced map \( f_* : \mathbb{R}^{2m} \to \mathbb{R}^{2n} \) is analytic, and if \( \mathbb{C}^n \to \mathbb{C} \) is holomorphic the maps \( F = \|f| : \mathbb{R}^{2n} \to \mathbb{R} \) and \( \log F \) are analytic.

Let now \( U_\theta \) be a neighbourhood of \( \theta \in Z \) as above, and \( \varphi, \psi : U \to U_\theta \) holomorphic charts centered at the origin. Further, denote by \( |\cdot, \cdot| \) the distance on \( Z \) as well as the standard distance on \( \Lambda^d(\mathbb{C}) = \mathbb{C}^d \). If \( U_\theta \) is relatively compact, then there are positive constants \( c_1, c_2 \) such that for every \( z_1, z_2 \in U_\theta, \)

\[
|\varphi^{-1} z_1, \varphi^{-1} z_2| \leq c_1 |z_1, z_2|, \quad |z_1, z_2| \leq c_2 |\varphi^{-1} z_1, \varphi^{-1} z_2|,
\]

and the same with \( \psi \).

3.1 Lemma Let \( \partial \) denote the vector \( (\partial/\partial x_1, \partial/\partial y_1, \ldots, \partial/\partial x_t, \partial/\partial y_t) \), which will also be denoted \( (\partial_1, \ldots, \partial_{2d}) \) shortly, and by \( I = (i_1, \ldots, i_{2d}) \) a multiindex of order \( S \), i.e. \( |I| = i_1 + \cdots + i_{2d} = S \). Then, with \( \partial^I \) the derivative \( \partial^I/(\partial x_1)^{i_1} \cdots (\partial y_d)^{i_{2d}} \),

\[
\log |\partial^I(\varphi^* f)(0)| \leq \sup_{s \leq S, |J| = s} \log |\partial^J(\psi^* f)(0)| + cS \log S,
\]

and

\[
\log |\partial^I(\psi^* f)(0)| \leq \sup_{s \leq S, |J| = s} \log |\partial^J(\varphi^* f)(0)| + cS \log S,
\]

with \( c \) a constant depending only on \( d \) and the charts \( \varphi, \psi \).
Proof Since \((\varphi^{-1} \circ \psi)^* : \mathbb{C}^d \to \mathbb{C}^d\) is holomorphic, the induced map \((\varphi^{-1} \circ \psi)^*_R : \mathbb{R}^{2d} \to \mathbb{R}^{2d}\) is analytic, hence

\[
\log |\partial^J (\varphi^{-1} \circ \psi)^*_R(0)| \leq C s \log s,
\]

with \(C\) only depending on \(\varphi\), and \(\psi\). Successively applying the chain rule gives the desired result.

Let now \(M\) be a set of functions \(Z \to \mathbb{R}\) closed under sums and differences. A grading on \(M\) is a map \(\deg : M \to \mathbb{Z}\) such that \(\deg(f + g) = \deg f + \deg g\) and \(\deg(-f) = -\deg f\) for every \(f, g \in M\).

3.2 Definition A graded set \(M\) of smooth functions \(f : U_0 \to \mathbb{R}^> 0, i \in I\) is said to have a holomorphic model if there is an analytic function \(g : U_0 \to \mathbb{R}^> 0\), and for every \(f \in M\) a holomorphic function \(F : U_0 \to \mathbb{C}\) such that

\[
f = \log |F| + \deg f \log g.
\]

For multiindices \(I = (i_1, \ldots, i_{2d}), J = (j_1, \ldots, j_{2d})\) write \(J \prec I\) if \(j_k \leq i_k\) for all \(k = 1, \ldots, 2d\). If \(J \prec I\) the multiindex \(I - J\) is defined. Further, for \(I = (i_1, \ldots, i_{2d})\) a multiindex, and \(\partial^I\) the corresponding differential operator, write \(\partial^I_z\) for the differential operator

\[
\partial^I_z / (\partial z_1)^{i_1} \partial^2 / (\partial z_1)^{i_2} \partial^3 / (\partial z^3 z_1)^{i_3} \ldots \partial^{2d-1} / (\partial z_d)^{i_{2d-1}} / (\partial z_d)^{i_{2d}}.
\]

3.3 Lemma If \(\varphi : U \to U_0\) is a holomorphic chart, and \(M\) a set of functions that has a holomorphic model, then for every \(f \in M\) with holomorphic model \(F\), the function \(F\) locally has a square root \(h\), and

\[
\sup_{|I| \leq S} |(\partial^I (\varphi^* f)(0)| \leq \sup_{|J| \leq S} |(\partial^I \varphi^* h)(0)|^2 + O((S + \deg f) \log(S \deg f)),
\]

where the constants implied by the \(O\)-notation depend on the choice of the holomorphic charts only.

Proof For \(s \leq S\), and \(I\) a multiindex of order \(s\),

\[
(\partial^I \varphi^* f)(0) = \sum_{J \prec I} (\partial^I \varphi^* |F|(0) \partial^{I-J} \varphi^* g^{\deg f})(0).
\]

Since \(g\) is an analytic function,

\[
|\partial^{I-J} \varphi^* g^{\deg f})(0)| \leq |(I - J| \deg f)|^c |I - J| + \deg f \leq (s \deg f)^c(s + \deg f)
\]

for some constant \(c\). Hence, the absolute value of \((12)\) is less or equal

\[
(2d)^s \sup_{J \prec I} |(\partial^I \varphi^* |F|(0)|(s \deg f)^c(s + \deg f).
\]

(13)
Next, with \( H = \varphi^* h \),
\[
|\partial^J \varphi^*(F)(0)| = |\partial^J (H \bar{H})(0)|,
\]
and the Cauchy-Riemman differential equations imply
\[
\frac{\partial H}{\partial z_i} = \frac{\partial H}{\partial x_i} = -i \frac{\partial H}{\partial y_i},
\]
consequently,
\[
|\partial^J H| = |(−i)^{i_2 + i_4 + \cdots + i_2j} \partial^j \bar{H}| = |\partial^j \bar{H}|,
\]
and similarly
\[
|\partial^J \bar{H}| = |\partial^j \bar{H}|.
\]
This implies
\[
|\partial^J \varphi^* f| = |\partial^J (H \bar{H})| \leq 2^{|J|} \sup_{K \leq J} \partial^K \bar{H} \partial^{j_{-} K} \bar{H} \leq 2^{|J|} \sup_{|K| \leq s} |\partial^K \bar{H}|^2.\]

Hence, (13) is less or equal
\[
(4d)^s (s \deg f)^c (s + \deg f) \sup_{|K| \leq s} |(\partial^K \bar{H})(0)|^2,
\]
proving the Lemma.

By standard complex analysis, the derivatives of a function \( f \) at \( \theta \) that has a holomorphic model \( F \) can be estimated by the values of \( F \) on \( U_\theta \) in the following way.

**3.4 Proposition** Let \( Z \) be a Kähler manifold of dimension \( d \), and \( \theta \in Z \) a point with neighbourhood \( U_\theta \); further \( \varphi : U \to U_\theta \) an affine chart, and \( f \) a smooth function on \( U_\theta \) that has a holomorphic model \( F \). Then, for every \( S \in \mathbb{N} \), every \( I \) with \( |I| = S \), and every \( R \in \mathbb{R} \) such that the ball of radius \( R \) around \( \theta \) is contained in \( U_\theta \),
\[
\log |(\partial^I \varphi^* f)(0)| \leq \sup_{|\theta, z| \leq R} \log |F(z)| - 2S \log R + O(S \log S).
\]

**Proof** Let \( h \) be a local square root of \( F \) and \( H = \varphi^* h \). Lemma 3.3 implies for any multiindex with \( |I| = S \),
\[
\log |(\partial^I \varphi^* f)(0)| \leq \sup_{|I| \leq S} |(\partial^I \bar{H})(0)|^2 = \sup_{|I| \leq S} \log \frac{\partial^I}{(\partial z_1)^{j_1+j_2} \cdots (\partial z_d)^{j_{2d-1}+j_{2d}}} (0)^2 + (S + \deg f) \log(S \deg f). \quad (14)
\]
By the multidimensional Cauchy formula,

\[
\left| \left( \frac{\partial^s}{(\partial z_1)^{j_1+\cdots+j_d} (\partial z_d)^{j_{d-1}+\cdots+j_{d-1}}} H \right)(0) \right|^2 \leq \frac{1}{(2\pi)^d} \int_{z_1=R'}^{z_2=R'} \cdots \int_{z_d=z_1+k_2 \cdots z_{d-1}+k_{d-1}}^{z_d=0} \frac{H}{z_1^{k_1+k_2} \cdots z_d^{k_{d-1}+k_{d-2}}} d\zeta_1 \cdots d\zeta_d \leq \frac{1}{2\pi d (R')^{-2s}} \sup_{|z,\theta| \leq R'} |H(z)|^2
\]

with \( R' = R/c_2 \), and \( c_2 \) from (11), which in turn equals

\[
\frac{1}{2\pi d (R')^{-2s}} \sup_{|z,\theta| \leq R'} |F(z)|.
\]

Inserting this into (14) finishes the proof.

### 3.1 Projective space

#### 3.5 Lemma Let \( \theta \in \mathbb{P}^t(\mathbb{C}) \).

1. If \( f \in \Gamma(\mathbb{P}^t, O(D)) \) is a global section whose restriction to \( \theta \) is nonzero, then the function
   
   \[ \zeta \mapsto \log |f_\zeta|, \]
   
   hence likewise the function
   
   \[ D(\zeta, \text{div} f) = \log |f_\zeta| - \int_{\mathbb{P}^t} \log |f| \mu^t \]

   have holomorphic models.

2. For \( U_{\theta} \) the circle of radius \( r < 1 \) around \( \theta \), the inequalities (17) hold with \( c_1 = 1/\sqrt{1-r^2}, c_2 = 1 \).

**Proof.** 1. If the global sections \( \Gamma(\mathbb{P}^t, O(D)) \) of the line bundle \( O(D) \) are given by homogeneous polynomials of degree \( D \) in variables \( z_0, \ldots, z_t \) with \( z_t(\theta) = \cdots = z_1(\theta) = 0 \), the map \( f(z_0, \ldots, z_t) \mapsto F(z_1, \ldots, z_t) = f(1, z_1, \ldots, z_t) \) maps \( \Gamma(\mathbb{P}^t, O(D)) \) to the space of polynomials on \( \mathbb{A}^t \) of degree at most \( D \). Further, with \( \varphi : \mathbb{A}^t \to \mathbb{P}^t \) the affine chart with \( \varphi(0) = \theta \) and \( \zeta = (\zeta_0, \ldots, \zeta_t) \) in \( \varphi(\mathbb{A}^t(\mathbb{C})) \),

\[
|f_{\varphi(z)}| = \frac{|F(\zeta_1, \ldots, \zeta_t)|}{|(1, \zeta_1, \ldots, \zeta_t)|^D}.
\]

Since \( g : (z_1, \ldots, z_t) \to |(1, z_1, \ldots, z_t)| \) is analytic, it follows that \( F(\zeta_1, \ldots, \zeta_t) \) is a holomorphic model for \( \zeta \mapsto \log |f_\zeta| \), and \( F(\zeta_1, \ldots, \zeta_t) \exp \left( -\int_{\mathbb{P}^t} \log |f| \mu^t \right) \) is a holomorphic model for \( D(\zeta, \text{div} f) \).
2. With $\zeta = \varphi(z)$ we have $|0, z| = |z|$, and
\[
|\theta, \zeta| = \sqrt{\frac{|(z_1, \ldots, z_t)|^2}{1 + |(z_1, \ldots, z_t)|^2}},
\]
implying the claim.

3.2 Grassmannians

Let now $G(\mathbb{C}) = G_{p,t+1}(\mathbb{C})$ be the Grassmannian of $p$-dimensional subspaces of $\mathbb{C}^{t+1}$. On $G$, there is the line bundle $L$ defined as the determinant of the canonical quotient bundle. Further, there is the canonical harmonic $(1,1)$-form $\mu_G = c_1(L)$. This metric explicitly can be described as follows:

Let $W, W' \in G(\mathbb{C})$, and $S_{W'}$ the unit sphere in $W'$. Then,
\[
|W, W'| = \sup_{w \in S_{W'}} |pr_{W}(w)|,
\]
where $pr_{W}$ is the orthogonal projection to the orthogonal complement of $W$.

Let $W_0$ be any $p$-dimensional subspace of $\mathbb{C}^{t+1}$. There is the following holomorphic chart $\varphi: A^{(t+1-p)p} \to G$: Let $w_1, \ldots, w_p$ be an orthonormal basis of $W_0$, $V = W_0^\perp$, and $U^-$ the unipotent radical of the subgroup of $GL(\mathbb{C}^{t+1})$ that leaves $V$ invariant. Then the big cell in the Bruhat decomposition of $G_{p,t+1}$ centered at $W_0$ consists of the subspaces $uW_0, u \in U^-$. The map
\[
\varphi: A^{(t+1-p)p} \cong U^- \to G, \quad u \mapsto uW_0
\]
is certainly holomorphic.

3.6 Lemma

1. For any hypersurface $Z = \text{div} f$ in $G$ the map
\[
G \to \mathbb{R}, \quad V \mapsto D(V, Z) = \log |f_V| - \int_G \log |f| \mu_G^{p(t+1-p)}
\]
has a holomorphic model.

2. The inequality (11) holds with $c_2 = 1$, and $c_1$ some constant depending on $p$ and $t$.

Proof 1. Let $\tilde{E}$ be the vector bundle on $G_{p,t+1}$ that attaches to each point $W \in G$ the dual vector space $\tilde{W}$ of $W$. The global sections of $\tilde{E}$ are the vectors $\tilde{v} \in (\mathbb{C}^{t+1})^*$,
Since, \( L_G = \Lambda^p \tilde{E} \) the global sections of \( L_G^{D} \) are symmetric products of vectors of the form \( \tilde{v}_1 \wedge \cdots \wedge \tilde{v}_p, \tilde{v}_i \in (\mathbb{C}^{t+1}) \). Now if

\[
f = \prod_{j=1}^D \tilde{v}_{1j} \wedge \cdots \wedge \tilde{v}_{pj}
\]

is such a global section, then for \( \tilde{W} \) not in the support of \( \text{div} f \),

\[
D(\text{div} f, \tilde{W}) = \log |f_{\tilde{W}}| - \int_G \log |f| \mu_G^{p(t+1)}.
\]

Further if \( w_1, \ldots, w_p \) is an orthonormal Basis of \( W_0 \), and \( U^- \) is as defined above, then for \( W = uW \),

\[
|f_{\tilde{W}}| = \left| \frac{\det((\prod_{j=1}^D \tilde{v}_{ij})(uw_k))_{i=1,\ldots,p; k=1,\ldots,p}}{\det(w_1 \wedge \cdots \wedge w_p)^D} \right|
\]

and the function inside the numerator is certainly a holomorphic function of \( u \in U^- \cong \mathbb{A}^{p(t+1)} \).

2. Is clear.

3.7 Lemma Let \( p + q \leq t + 1 \), and \( \mathbb{P}(W) \) a subspace of dimension \( p - 1 \), and \( U_W \) a neighbourhood of \( W \) in \( G_{p,t+1} \). Then, if \( L_{Z^{t+1-q}}(\mathbb{P}^t) \subset Z^{t+1-q}_{eff}(\mathbb{P}^t) \) is the subgroup generated by subspaces of dimension \( p - 1 \) that intersect each \( \mathbb{P}(\tilde{W}) \) with \( \tilde{W} \in U_W \) properly, the set of functions

\[
f_Z : U_W \to \mathbb{R}, \quad \tilde{W} \mapsto D(Z, \mathbb{P}(\tilde{W})), \quad Z \in L_{Z^{t+1-q}}(\mathbb{P}^t)
\]

has a holomorphic model.

Proof For \( \mathbb{P}(V) \subset \mathbb{P}^t \) a subspace of dimension \( p - 1 \) intersecting each \( \mathbb{P}(\tilde{W}) \) with \( \tilde{W} \in U_W \) properly, let \( \tilde{F}_V \) be the vector bundle on \( U_W \) that attaches to each \( \tilde{W} \) the space \( (\tilde{W}/(V \cap \tilde{W})) \), and the line bundle \( L_V = \bigwedge^{t+1-q} \). Further, let

\[
\tilde{v}_1, \ldots, \tilde{v}_{t+1-q} \in (\mathbb{C}^{t+1})^*
\]

be linear forms that are orthonormal and zero on \( V \), and define \( f^V = \tilde{v}_1 \wedge \cdots \wedge \tilde{v}_{t+1-q} \in \Gamma(U_W, L) \). If \( P(V) \) is the parabolic subgroup of \( \text{Gl}(\mathbb{C}^{t+1}) \) that leaves \( V \) invariant, then the group \( U^- \cap P(V) \) operates transitively on \( U_W \), and for \( \tilde{W} = uW \) we have

\[
|f^V_{\tilde{W}}| = \frac{\left| \det(\tilde{v}_i(uw_k))_{i=1,\ldots,t+1-q; k=1,\ldots,t+1-q} \right|}{\det(w_1 \wedge \cdots \wedge w_{t+1-q})}.
\]
The formula inside the absolute value in the numerator is linear from in \( u \). Further, the above expression equals the sine of the angle between \( V \) and \( \tilde{W} \), and by [Ma1, Fact 4.8] \( D(\mathbb{P}(V), \mathbb{P}(W)) \), modula a fixed constant, equals the logarithm of the sine of the angle between \( V \) and \( \tilde{W} \). Now, for \( Z \in L \mathbb{P}^{t+1-q} \) arbitrary, \( Z = \sum_\mathbb{V} n_\mathbb{V} V \), and the function
\[
\prod_\mathbb{V} (f^\mathbb{V})^{n_\mathbb{V}}
\]
models the function \( D(Z, \mathbb{P}(\tilde{W})) \).

**3.8 Lemma** Let \( X \subset \mathbb{P}^t_G \) be a subvariety of dimension \( p \), and \( G = G_{t, t-p} \).

1. There is a positive constant \( c_1 \), only depending on \( t \), and \( p \) such that
   \[
   \mu(V_X) = \int_{V_X} \mu_G^{(t+1-p)p-1} = c_1 \deg X.
   \]

2. Let \( U_\epsilon(V_X) \) be the tubular neighbourhood
   \[
   U_\epsilon(V_X) := \{ V \in G(\mathbb{C})||V, V_X| \leq \epsilon \}
   \]
   of \( V_X \). Then, there is a positive constant, depending only on \( t \), and \( p \) such that
   \[
   \mu_G(U_\epsilon(V_X)) = \int_{U_\epsilon(V_X)} \mu_G^{(t+1-p)p} \leq c_2 \deg X \epsilon.
   \]

**Proof** 1. Since \( \mu_G \), and thereby \( \mu_G(t + 1 - p)p - 1 \) are closed forms, the integral
   \[
   \int_{V_X} \mu_G^{(t+1-p)p-1}
   \]
depends only on the cohomology class of the cycle \( V_X \), and thereby only on the class of \( V_X \) in \( CH^1(G) \). Since in this last group
   \[
   [V_X] = \deg X[V_{\mathbb{P}(W)}],
   \]
where \( \mathbb{P}(W) \subset \mathbb{P}^t \) is any projective subspace of codimension \( p \), with
   \[
   c_1 := \int_{\mathbb{V}_{\mathbb{P}(W)}} \mu_G^{(t+1-p)p-1}
   \]
the equality \( \int_{V_X} \mu_G^{(t+1-p)p-1} = c_1 \deg X \) follows.

2. Since \( G \) has positive curvature, this immediately follows from part one.

**3.9 Lemma** Let \( \mathbb{P}(W) \subset \mathbb{P}(F) \) be subspaces of \( \mathbb{P}^t \) of codimensions \( q \geq r \). Every subspace \( \mathbb{P}^t(F') \) of codimension \( r \) contains a subspace \( \mathbb{P}(W') \) of codimension \( q \) such that
   \[
   |W, W'| \leq |F, F'|
   \]
as points in the corresponding Grassmannians.
The proof is elementary linear algebra. Let $pr_F : \mathbb{C}^{t+1} \to F$ be the orthogonal projection to $F$, and define $W'$ as the intersection $pr_F^{-1}(W) \cap F'$. Then $W'$ has codimension $q$, is contained in $F'$, and every vector $w \in W'$ may be written as $W = w_1 + v_1$ with $w_1 \in W, v_1 \in F' \subset W'$. Hence,

$$pr_{W'}(w) = v_1 = pr_{F'}(w),$$

and since $W' \subset F'$,

$$|W, W'| = \sup_{w \in W'} |pr_{W'}(w)| = \sup_{w \in W'} |pr_{F'}(w)| \leq \sup_{v \in F'} |pr_{F'}(v)| = |F, F'|.$$

Next, there is the following functoriality for differential operators on Grassmannians of subspaces of different dimension. Let $p + q \leq t + 1$, $\mathbb{P}(W) \subset \mathbb{P}(\mathbb{C})$ be a subspace of dimension $q - 1$, and $U_W$ an open subset of $W$ in $G_{q,t}$ that is contained in the big cell in the Bruhat decomposition centered at $W$.

For $\mathbb{P}(F)$ a subspace of dimension $p+q-1$ containing $\mathbb{P}(W)$, let $V$ be the orthogonal complement of $W$ in $F$, and define the map

$$f : U_W \to G_{p+q,t}(\mathbb{C}), \ W \mapsto \bar{W} \oplus V.$$

Clearly, $f$ is a holomorphic map.

3.10 Lemma Let $|\cdot, \cdot|$ be the canonical metric on the Grassmannian. With the above notations,

1. $\forall \bar{W} \in U_W |F, \varphi(\bar{W})| \leq |W, \bar{W}|$.

2. Let $\varphi : A \to U_w, \psi : A \to G$. Then,

$$\sup_{|I| = s} |\partial_z^I (\psi^{-1} \circ f \circ \varphi)(0)| \leq cs \log s,$$

where $c$ is a constant depending only on $p, q$, and $t$ but not on the choice of $W$ and $F$.

Proof 1. Let $\bar{W} \in U_W$ and $\bar{F} = \varphi(\bar{W})$. Since $V$ is contained in $F$, we have

$$|F, \bar{F}| = \sup_{v \in S_F} |pr_{F'}(v)| = \sup_{w \in S_{F \cap W'}} |pr_{F'}(v)|.$$

Now let $v \in S_{F \cap V'}$ be a vector where the last supremum is achieved, and $w \in S_{\bar{W}}$ be a vector such that $|W, \bar{W}| = |pr_{W'}(w)|$. We have to show

$$|pr_{F'}(v)| \leq |pr_{W'}(w)|,$$

which again boils down to an elementary calculation in linear algebra.

2. This is an immediate consequence of the holomorphicity of $f$, and the fact that $\psi^{-1} \circ f \circ \varphi$ is the same for all $W, F$ modulo a transformation by a $g \in SU(t + 1)$. 


4 The derivated algebraic distance

4.1 Hypersurfaces and points

Let $Z$ be a regular projective algebraic variety of dimension $d$ over $\mathbb{C}$, and fix a Kähler structure on $Z$. For $f$ a global section of some line bundle on $Z$, $X = \text{div} f$ an effective cycle of pure codimension 1 on $Z$, and $\theta \in Z$ a point not contained in the support of $X$ the algebraic distance $D(\theta, X)$ equals

$$D(\theta, X) = \log |f_\theta| - \int_Z \log |f| \mu^d = -\frac{1}{2} \int_X g_\theta + \frac{1}{2} \int_Z g_\theta \mu,$$

where $\mu$ is the chosen Kähler form on $Z$, and $g_\theta$ is a green form of log type for $\theta$. (See [Ma1], Definition 4.1).

For $Y$ a cycle in $Z$ of pure codimension 1 the function $D(Y, \theta)$ clearly is smooth in a neighbourhood of $\theta$. This leads to the following definition.

4.1 Definition With the above notations, let $Y \in Z_{\text{eff}}(Z)$ be an effective cycle of pure codimension 1 in $Z$ on $Z$, and $\theta \in Z$ a point not contained in the support of $Z$. Further $\varphi : U \subset \mathbb{A}^d \to U_\theta$ an affine chart. The derivated algebraic distance of $Z$ to $\theta$ is defined as

$$D^S(\theta, Z) := \sup_{|I| \leq S} \log \left| \left( \partial^I \exp(D(z, Z)) \right)(\theta) \right|.$$

4.2 Proposition Let $q \leq t$, and $G = G_{q,t}$ the Grassmannian of $q$-dimensional subspaces of $\mathbb{C}^{t+1}$. Then, for effective cycle $Z$ of codimension $q$ in $G$ and every point $W \in G$ not contained in $Z$, the algebraic distance $D(Z, W)$ has a holomorphic model.

PROOF Since $CH^1(G) \cong \mathbb{Z}$, there is some global section $f$ of the canonical line bundle $L_G$ on $G$ such that $Z = \text{div} f$. The Proposition thus follows from Lemma 3.6.1.

4.3 Corollary With $f$ a global section of a line bundle $L$ on $Z$, $Y = \text{div} f$, $\theta$ a point not contained in the support of $Y$, $\varphi : U \to U_\theta$ an affine chart, and $H$ a local square root of $\varphi^* f$,

$$D^S(Y, \theta) = \sup_{|I| \leq S} \left| \frac{\partial^s}{(\partial z_1)^{\mu_1} \cdots (\partial z_t)^{\mu_t}} H \right|(0)^2 + O(S \log SS).$$
Proof follows from the Proposition and Lemma 3.3.

The derivated algebraic distance of a hypersurface in \( \mathbb{P}^t \) can also be estimated against the values of the derivations of the global sections directly as stated in Theorem 1.3.

**Proof of Theorem 1.3**

By the proof of Lemma 3.5, with \( \varphi : \mathbb{A}^t \to \mathbb{P}^t \) a homogeneous chart centered at \( \theta \), and \( \zeta_1, \ldots, \zeta_t \) the coordinates in \( \mathbb{A}^t \),

\[
D(\text{div} f, \varphi(\zeta)) = \log |F(\zeta)| - \int_{\mathbb{P}^t} |f|\mu^t - D \log |(1, \zeta_1, \ldots, \zeta_t)|,
\]

hence

\[
D^S(\text{div} f, \theta) = \sup_{|I| \leq S} \log |(\partial^I |F|)(0)| - \int_{\mathbb{P}^t} |f|\mu^t + O(S \log S).
\]

By the first equality above the Proposition holds for \( S = 0 \). Assume that for a multiindex \( I \) with \( |I| = S - 1 \),

\[
|(\partial^I |F|)(0)| \leq |(\partial^I_2 F)(0)|.
\]

Then, for any \( j = 1, \ldots, t \),

\[
|(\partial/\partial x_j \partial^I |F|)(0)| \leq |(\partial/\partial x_j |(\partial^I_2 F)(0)|.
\]

With \( G = \partial^I_2 F \), and \( G_r, G_i \) the real and imaginary parts of \( G \) this equals

\[
\left| \frac{G_r \partial G_r/\partial x_j + G_i \partial G_i/\partial x_j}{\sqrt{G_r^2 + G_i^2}}(0) \right| \leq \left| \frac{\partial G_r}{\partial x_j} + \frac{\partial G_i}{\partial x_j}(0) \right| \leq \left| \frac{\partial G_r}{\partial x_j}(0) \right| + \left| \frac{\partial G_i}{\partial x_j}(0) \right|,
\]

which by the Cauchy-Riemman-equations equals

\[
2 \left| \frac{\partial G}{\partial z_j} \right| = 2 \left| (\partial/\partial z_j \partial^I F)(0) \right|.
\]

The Proposition follows by complete induction.

### 4.2 Effective cycles in projective space and projective subspaces

**4.4 Proposition** Let \( p, q \in \mathbb{N}, Z \in Z^p_{\text{eff}}(\mathbb{P}^t_{\mathbb{C}}) \), and \( \mathbb{P}(W) \subset \mathbb{P}^t(\mathbb{C}) \) a subspace of codimension \( q \) that is regular with respect to \( Z \). Let further \( G = G_{t+1-q,t} \) denote the Grassmannian of \( q \)-codimensional subspaces of \( \mathbb{P}^t(\mathbb{C}) \). There is an open neighbourhood \( U_W \) of \( W \) in \( G \) such that every \( \tilde{W} \in U_W \) is regular with respect to \( Z \). Further,
1. If $t \leq p + q \leq t + 1$, or more generally if $p + q \leq t + 1$, and for every $W$ in some neighbourhood $U_W$ of $W$ the intersection $Z, \mathbb{P}(W)$ is a sum of projective subspaces, then the function $D(Z, \mathbb{P}(W))$ has a holomorphic model, in particular is a smooth function.

2. If $p + q > t + 1$, the function $D(Z, \bullet)$ is smooth on $U_W$.

This leads to the following Definition:

4.5 Definition Let $Z \in Z_{eff}(\mathbb{P}^t_{\mathbb{C}})$ be an effective cycle in projective space, and $\mathbb{P}(W) \subset \mathbb{P}^t(\mathbb{C})$ a projective subspace of codimension $q$ that is regular with respect to $Z$. For $W$ the corresponding point in the Grassmannian $G = G_{t+1-q,t+1}$ there is a connected simply connected neighbourhood $U_W$ of $W$ in $G$ such that for every $\tilde{W} \in U_W$ the space $\mathbb{P}(W)$ is likewise regular with respect to $Z$. Let $\varphi^* U \to U_W$ be the affine chart from section 3, and define the derivated algebraic distance of $\mathbb{P}(W)$ to $Z$ as

$$D_S^\bullet(\theta, Z) := \sup_{|I| \leq S} |(\partial^I \exp (\varphi^* (D_\bullet(z, Z))))(0)|,$$

where $D_\bullet(\theta, Z)$ either denotes the algebraic distance $D_G(\theta, Z) = D_1(\theta, Z)$ or the algebraic distance $D_{Ch}(\theta, Z)$ of $\theta$ to $Z$ as defined in [Ma1], section 4.

4.2.1 Points

Proof of Proposition 4.4.2 for $p = t$: For $Z \in Z_{eff}(\mathbb{P}^t_{\mathbb{C}})$, and $\mathbb{P}(W) \subset \mathbb{P}^t$ a subspace of codimension $q \leq t - 1$ that does not meet the support of $Z$, let

$$Z = \sum_{i=1}^{\deg Z} z_i,$$

and $x_i, i = 1, j = 1, \ldots, q$ global section of $O(1)$ of length 1 on $\mathbb{P}^t$ such that

$$\mathbb{P}(W) = \text{div} x_1 \cap \ldots \cap \text{div} x_q.$$

For $U_W$ a neighbourhood of $W$ in $G_{t-q,t+1}$, such that for every $V \in U_W$, no $z_i$ lies in $\mathbb{P}(V)$, and define

$$g_i : U_\theta \to \mathbb{R}, \quad z \mapsto \frac{x_1(z_i)x_1(z_i) + \cdots + x_q(z_i)x_q(z_i)}{1 + x_1(z_i)x_1(z_i) + \cdots + x_q(z_i)x_q(z_i)}.$$

Clearly $g_i$ is smooth and nonzero on $U_W$. Hence, the square root $f_i$ of $g_i$ is also smooth on $U_W$. Since, $|z, z_i| = |g_i(z)|$, and by [Ma1], Fact 4.8, there is constant $c$ such that

$$D(Z, \mathbb{P}(W)) = c \deg Z + \sum_{i=1}^{\deg Z} \log |z, z_i|,$$
the claim follows.

For zero dimensional cycles $X$ the derivated algebraic distance to a point $\theta$ not on $X$ takes a particularly simple form

\section{4.6 Proposition}

1. Let $Z \in Z^t_{\text{eff}}(\mathbb{P}^t)$ be an effective cycle of pure dimension zero, and $\theta \in \mathbb{P}^t$ a point not contained in the support of $Z$. If $Z = \sum_{i=1}^{\deg Z} z_i$ is ordered in such a way that $|z_1, \theta| \leq \cdots \leq |z_{\deg Z}, \theta|$, then for every $S \leq \deg Z$

$$D^S(Z, \theta) \leq \sum_{i=S+1}^{\deg Z} \log |z_i, \theta| + O(S \log \deg Z),$$

and for every $S \leq \deg Z/3$

$$2 \sum_{i=S+1}^{\deg Z} \log |z_i, \theta| \leq D^{3S}(Z, \theta) + O((S + \deg Z) \log \deg Z).$$

2. For $p \leq t$ let $Z \in Z^p_{\text{eff}}(\mathbb{P}^t)$ such that $Z = \sum_{i=1}^{\deg Z} \mathbb{P}(W_i)$ with each $\mathbb{P}(W)$ a projective subspace, and $\theta$ a point not contained in $\mathbb{P}(W_i)$ for all $i = 1, \ldots, \deg Z$. Then, for every $S \leq \deg Z$

$$D^S(\theta, Z) \leq \sum_{i=S+1}^{\deg Z} \log |\theta, \mathbb{P}(W_i)| + O(S \log \deg Z).$$

The next three Lemmata will be proved in the appendix

\section{4.7 Lemma}

Let $x_1, \ldots, x_n \in [-1, 1] \setminus \{0\}$ with

$$|x_1| \leq |x_2| \leq \cdots \leq |x_n|, \quad \text{and} \quad f(x) := \prod_{i=1}^{n} (x - x_i).$$

Then, for $s < n/3$,

$$\frac{1}{(2s + 1)(3n^3)^{s+1}} \prod_{i=s+1}^{n} |x_i|^2 \leq \sup_{0 \leq j \leq 3s} |f^{(j)}(0)|,$$

and for $s \leq n$,

$$\sup_{0 \leq j \leq s} |f^{(j)}(0)| \leq \frac{n!}{(n-s)!} \prod_{i=s+1}^{n} |x_i|. $$

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4.8 Lemma Let $\theta, z_1, \ldots, z_n$ be points in $\mathbb{P}^t(\mathbb{C})$, with $\theta \neq z_i \forall i = 1, \ldots, n$, and $\varphi : \mathbb{A}^t(\mathbb{C}) \to \mathbb{P}^t(\mathbb{C})$ the affine chart centered at $\theta$. With

$$f(z) = \prod_{i=1}^{n} |z, z_i|,$$

and $F = \varphi^* f$, for every multiindex $I = (i_1, \ldots, i_t)$ with $|I| = s \leq n/3$,

$$\frac{1}{(2s + 1)(3n^2)^s + 1} n^n \prod_{s+1}^{n} |\theta, z_i|^2 \leq \sup_{|I| \leq 3s} \left| (\partial^I F)(0) \right|,$$

and for $s \leq n$,

$$\sup_{|I| \leq s} \left| (\partial^I F)(0) \right| \leq n^s \prod_{i=s+1}^{n} |\theta, z_i|.$$

4.9 Lemma Let $\mathbb{P}(W_i) \subset \mathbb{P}^t_{\mathbb{C}}$, $i = 1, \ldots, n$ be subspaces of fixed codimension $p$, and $\theta \in \mathbb{P}^t(\mathbb{C})$ a point contained in none of them. Further, $\varphi : \mathbb{A}^t(\mathbb{C}) \to \mathbb{P}^t(\mathbb{C})$ the affine chart centered at $\theta$ and

$$f(z) = \prod_{i=1}^{n} |z, \mathbb{P}(W_i)|.$$

Then with $F = \varphi^* f$, and $s \leq n$,

$$\sup_{|I| \leq s} \left| (\partial^I F)(0) \right| \leq n^s \prod_{i=1}^{n} |\theta, \mathbb{P}(W_i)|.$$

Proof of Proposition 4.6: 1. The inequality

$$D^S(Z, \theta) \leq O(S \log \deg Z)$$

follows immediately from the second inequality of Lemma 4.8 together with the equality

$$D(\theta, x) = \log |x, \theta| + c,$$

with $c$ a constant only depending on $t$ from [Ma1], Fact 4.10. The second inequality follows from the first equality of Lemma 4.8 and again equality (15).

2. Follows in the same way using Lemma 4.9 and the equality

$$D(\theta, Z) = \sum_{i=1}^{\deg Z} \log |\theta, \mathbb{P}(W_i)| + c \deg Z,$$

which follows again from [Ma1], Fact 4.10 together with the additivity of the algebraic distance.
4.2.2 The general case

**Proof of Proposition 4.14.** 1: Assume first that \( p + q = t + 1 \). Then, by the proof of [Ma1], Proposition 4.14,

\[
D(Z, \mathbb{P}(W)) = D(V_Z, W),
\]

with \( V\mathcal{Z} \) \( G^*F_*Z \in \mathcal{Z}_{eff}(G_{t+1-p,t}) \) from the Correspondence (7). Thus, the claim in this case is Proposition 4.2.

For \( p + q \leq t \), and \( Z, \mathbb{P}(\tilde{W}) \) equal to a sum of projective subspaces for every \( \tilde{W} \) in some neighbourhood of \( W \) assume first \( p + q = t \), and let \( \mathbb{P}(V) \subset \mathbb{P}^t(\mathbb{C}) \) be a subspace of codimension one that does not meet the support \( \mathbb{P}(\tilde{W}).Z \) for every \( \tilde{W} \in U_W \). By [Ma1], Proposition 4.12,

\[
D(Z, \mathbb{P}(\tilde{W})) = D(\mathbb{P}(V), \mathbb{P}(\tilde{W})) + D(Z, \mathbb{P}(V) \cap \mathbb{P}(\tilde{W})) - D(Z, \mathbb{P}(\tilde{W}), \mathbb{P}(V)).
\] (16)

Hence, the claim of the Proposition holds for \( D(Z, \mathbb{P}(\tilde{W})) \) if it holds for every term on the left hand side of (16). For the second term, the claim follows from Case A. For the first term, it follows from Lemma 3.7.

For the third term, let \( U^0 \supset U_W \) be the subset such that for every \( \tilde{W} \in U_W \) the space \( \mathbb{P}(\tilde{W}) \) does not meet the singular locus of \( Z \), and \( Z, \mathbb{P}(\tilde{W}) \) has no double points. Clearly \( U_W \setminus U^0_W \) is contained in the intersection of \( U_W \) with the zero set of a holomorphic function. Further, let \( X \) be the global section of \( O(1) \) such that \( \mathbb{P}(V) = \text{div}X \). Then,

\[
D(Z, \mathbb{P}(\tilde{W}), \mathbb{P}(V)) = \log \prod_{i=1}^{\deg Z} \frac{|X(z_i)|}{\sqrt{1 + X(z_i)X(z_i)}} + c \deg Z.
\]

On \( U^0_W \) the coordinates of \( z_1, \ldots, z_{\deg Z} \) are holomorphic functions of \( \tilde{W} \). Hence \( X(z_i) \) is holomorphic on \( U^0_W \). Since \( X(z_i) \) is continuous on \( U_W \), and \( U_W \setminus U^0_W \) is contained in the intersection of \( U_W \) with the zero set of a holomorphic function, it follows that \( X(z_i) \) is holomorphic on all of \( U_W \), finishing the proof.

If \( p + q < t \), let \( \mathbb{P}(V) \subset \mathbb{P}^t(\mathbb{C}) \) be a subspace of codimension \( t + 1 - p - q \) that does not meet the support of \( \mathbb{P}(\tilde{W}).Z \) for every \( \tilde{W} \in U_W \). Then (16) still holds and the first and second term on the right hand side by the same arguments have holomorphic models. Further, by assumption \( Z, \tilde{W} = \sum_{i=1}^{\deg Z} \mathbb{P}(\tilde{W}_i) \), hence by [Ma1], p. 28,

\[
D(Z, \mathbb{P}(\tilde{W}, \mathbb{P}(V)) = \sum_{i=1}^{n} D(\mathbb{P}(\tilde{W}_i), \mathbb{P}(V)) = \sum_{i=1}^{n} D(\tilde{W}_i, V_{\mathbb{P}(V)}),
\]

where the \( \tilde{W}_i \) are points and \( V_{\mathbb{P}(V)} \) is a hypersurface in \( G_{t-p-q,t+1} \), and algebraic...
distance of effective cycles in $\mathbb{P}(F)$,

$$D(Z, \mathbb{P}(\tilde{W}), \mathbb{P}(V)) = \log \prod_{i=1}^{\infty} \frac{X(\tilde{W}_i)}{\sqrt{1 + X(\tilde{W}_i)X(\tilde{W}_i)}};$$

with $X$ a global section of the canonical line bundle of the Grassmannian such that $V_{\mathbb{P}(V)} = \text{div} X$. Taking again $U^0_\tilde{W} \subset U_\tilde{W}$ as the subset such that for every $\tilde{W} \in U^0_\tilde{W}$ the space does not meet the singular locus of $Z$, and $Z.\mathbb{P}(\tilde{W})$ has no double points, the coordinates of each $\tilde{W}_i$ depend holomorphically on $\mathbb{P}(\tilde{W})$, and one can repeat the argument above.

**Proof of Proposition 4.4 for $p + q > t + 1$:** Let $\mathbb{P}(F)$ be a subspace of codimension $t - p$ containing $\mathbb{P}(W)$, and intersecting $Z$ properly. By Proposition 2.1,

$$D(Z, \mathbb{P}(W)) - D(Z_F, \mathbb{P}(W)) = D(Z, \mathbb{P}(F)) - D(Z_F, \mathbb{P}(F)).$$

By part one of the Proposition, the left hand side is smooth. Further, since $D(Z_F, \mathbb{P}(W)) = D(Z_F, \mathbb{P}(F), \mathbb{P}(W)) + c \deg X$, and $Z_F, \mathbb{P}(F)$ consist of points, the second term on the left hand side is smooth by part two of the proposition for $p = t$ proved above. Hence, $D(Z, \mathbb{P}(W))$ is likewise smooth.

5 **Reduction of the derivated algebraic distance to derivated algebraic distances to points**

Let $Z \subset \mathbb{P}^t_C$ be an algebraic subvariety of codimension $p$, and $\theta \in \mathbb{P}^t(\mathbb{C})$ a point not contained in $Z$. $[Ma1]$, Proposition 4.16 implies the existence of a projective subspace $\mathbb{P}(F) \subset \mathbb{P}^t$ of codimension $t - p$ that intersects $Z$ properly and contains $\theta$, such that

$$c_1 \deg Z \geq D(\mathbb{P}(F), Z) \geq -c_2 \deg Z,$$

with positive constants $c_1, c_2$ only depending on $p$, and $t$, and, if $D^{\mathbb{P}(F)}(\bullet, \bullet)$ denotes the

$$D(Z, \theta) = D^{\mathbb{P}(F)}(Z.\mathbb{P}(F), \theta) + O(\deg Z).$$

The existence of such a space thus allowed to reduce the algebraic distance of a point $\theta$ to an effective cycle $Z$ to the algebraic cycle to the algebraic distance of $\theta$ to $Z.\mathbb{P}(F)$ which is zero dimensional. For decomposing the derivated algebraic distance however, the condition $D(Z, \mathbb{P}(F)) \geq -c_2 \deg Z$ is not good enough, because derivatives of a function may be very small or big even if the values of the function are not; to assure that there is a space $\mathbb{P}(F)$ such that the derivations of $\exp D(\mathbb{P}(F), Z)$ and $\exp(-D(\mathbb{P}(F), Z))$ are also bounded in terms of $\det Z$, one has to look for a space that contains a smaller subspace that has not too small distance to $Z$.
5.1 Theorem Let $p \leq t$, and $Z \subset \mathbb{P}^t(\mathbb{C})$ be an effective cycle of pure codimension $p$; further $\theta \in \mathbb{P}^t(\mathbb{C})$ a point not contained in the support of $Z$, and $S$ a natural number at most $\deg Z/3$.

1. There are fixed constants $c_1, c_2, c_3$ only depending on $p$, and $t$, and a subspace $\mathbb{P}(F) \subset \mathbb{P}^N$ of codimension $t - p$, containing $\theta$, intersecting $Z$ properly, and fullfilling

$$c_1 \deg Z \geq D(\mathbb{P}(F), Z) \geq -c_2 \deg Z \log \deg Z,$$

and

$$D^S(Z, \theta) = (D^{\mathbb{P}(F)})^S(Z, \mathbb{P}(F), \theta) + D(\mathbb{P}(F), Z) + O((S + \deg Z) \log(S \deg Z)).$$

2. If $\mathbb{P}(F)$ is any subspace of codimension $q \geq t - p$ containing $\theta$, then

$$(D^{\mathbb{P}(F)})^S(Z, \mathbb{P}(F), \theta) \leq D^S(\theta, Z) + D(\mathbb{P}(F), Z) + O((S + \deg Z) \log(S \deg Z)).$$

The prove will be given for the $D_G$, and $D_{Ch}$ seperately. Consider first $D_{Ch}$, and recall that the Chow divisor of an algebraic cycle $X \in Z^p_{eff}(\mathbb{P}^t)$ is defined in the following way. Let $\delta : \mathbb{P}^t \to (\mathbb{P}^t)^p$ be the diagonal, and define the correspondence

$$
\begin{array}{c}
\mathcal{C} \\
(\mathbb{P}^t)^p \\
\end{array}
\xleftarrow{f} 
\xrightarrow{g}
\begin{array}{c}
(\mathbb{P}^t)^p \\
\end{array}
$$

(17)

where $\mathcal{C}$ is the subscheme of $(\mathbb{P}^t)^p \times (\mathbb{P}^t)^p$ assigning to each $t + 1$ dimensional vector space $V$ over a field $k$ the set

$$\{(v_1, \ldots, v_p, \overline{v}_1, \ldots, \overline{v}_p| v_i \in V, \overline{v}_i \in \overline{V}, \overline{v}_i(v_i) = 0, \forall i = 1, \ldots p\}.$$

The maps $f : \mathcal{C} \to (\mathbb{P}^t)^p, g : \mathcal{C} \to (\mathbb{P}^t)^p$ are just the restrictions of the projections. They are flat, projective, surjective, and smooth. For $Z \in Z^p_{eff}(\mathbb{P}^t)$, the Chow divisor $Ch(Z) \subset (\mathbb{P}^t)^p$ is defined as $Ch(Z) := g_* \circ f^* \circ \delta_*(Z)$.

5.2 Lemma Let $Z \in Z^p_{eff}(\mathbb{P}^t)$.

1. For every $l \geq t + 1 - p$, there is a subspace $\mathbb{P}(V) \subset \mathbb{P}^t(\mathbb{C})$ of codimension $l$ such that with $V$ the corresponding point in $G_{t+1-t, t+1}$

$$\log |V, V^*_Z| \geq -c_1 - \log \deg Z,$$

where $c_1$ is a positive constant only depening on $t$ and $p$. 27
2. Let $\mathbb{P}(F) \subset \mathbb{P}^t(\mathbb{C})$ be a subspace of codimension $r \leq t + 1 - p$ that contains a subspace $\mathbb{P}(V)$ of codimension $l \geq t + 1 - p$ with

$$\log |V, V_Z| \geq -c_1 - \log \deg Z.$$ 

Then,

$$D(\mathbb{P}(F), Z) \geq -c_3 \deg Z \log \deg Z$$

with $c_3$ a positive constant depending only on $p, q$ and $t$.

**Proof** 1. Let $c_2$ be the constant from Lemma 3.8.2, and $U(V_Z)$ the tubular neighbourhood of diameter $\frac{1}{2c_2 \deg Z}$ of the support of $V_Z$ in $G$. By Lemma 3.8.2,

$$\mu(U(V_Z)) \leq \frac{c_2 \deg Z}{2c_2 \deg Z} = 1/2 < 1 = \mu(G).$$

Hence, there is a subspace $\mathbb{P}(V)$ of dimension $t - q$ such that the point $V \in G$ corresponding to $\mathbb{P}(V)$ does not lie in $U(V_Z)$, i.e. $\log |V, V_Z| \geq -\log c_2 - \log(2 \deg Z)$. Take $c_1 = 2c_2$.

2. Follows in the same way as [Ma1], Proposition 4.17.

### 5.3 Proposition

For $p + r = t$, let $Z \in Z_{\text{eff}}^p(\mathbb{P}^t(\mathbb{C}))$, and $\mathbb{P}(F) \subset \mathbb{P}^t(\mathbb{C})$ be a subspace of codimension $r$, regular with respect to $Z$ that contains a subspace $\mathbb{P}(V)$ of codimension $l \geq t + 1 - p$ such that in the Grassmanian $G_{p,t+1}$ one has $\log |V, V_Z| \geq -\log c_1 - \log \deg Z$ with $c_1 > 0$ a constant only depending on $p, q, t$. Then, for every $S \leq \deg Z$

$$D^S(\mathbb{P}(F), Z) \leq c_3((\deg Z + S)(\log \deg Z + \log S)),$$

and with $U_V$ a neighbourhood of $V$ in $G$, and $\varphi : U \rightarrow U_V$ an affine chart, the function

$$D^S_*(\mathbb{P}(F), X) := \log \sup_{s \leq S, |I| = s} \partial I(\exp -D(\mathbb{P}(F), Z)),$$

also fulfills

$$D^S_*(\mathbb{P}(F), X) \leq c_3((\deg Z + S)(\log \deg Z + \log S)).$$

with $c_3$ a constant depending only on $p, q, r$, and $t$.

**Proof** Let $U_F$ be the ball with logarithmic radius $-\log 2c_1 - \log \deg Z$ around $F$ in $G_{t+1-r,t+1}$. By Lemma 3.9, for every $\tilde{F} \in U_F$ there is a $\tilde{V} \subset \tilde{F}$ of dimension $t - r$ such that $|V, \tilde{V}| \leq |F, \tilde{F}| \leq -2c_1 - \log \deg Z$. The assumption on $\mathbb{P}(V)$, together
with the triangle inequality, implies $|\tilde{V}, V_Z| \geq -\log 2c_1 - \log \deg Z$, which in turn by Lemma 5.2 implies
\[
D(\mathbb{P}(\tilde{F}), Z) \geq -c_2 \deg Z \log \deg Z,
\]
or
\[
D_\ast(\mathbb{P}(\tilde{F}), Z) \leq 2c_2 \deg Z \log \deg Z.
\]
Further by Proposition 4.4.1 the function $D(\mathbb{P}(\tilde{F}), Z)$ and thereby the function $D_\ast(\mathbb{P}(\tilde{F}, Z))$ has a holomorphic model $g$ on $U_F$. Hence Proposition 3.4 together with the above inequalities implies
\[
D_\ast^\circ(\mathbb{P}(F), Z) \leq \sup_{\tilde{F} \in U_F} \log |g(\tilde{F})| + 2c_1 S \log \deg Z + O((S + \deg Z) \log(S \deg Z)) \leq 2c_2 \deg Z \log \deg Z + 2c_1 S \log \deg Z + O(S \log S) \leq c_3(S + \deg Z)(\log S + \log \deg Z),
\]
with a suitable constant $c_3$.
The inequality
\[
D_\ast^\circ(\mathbb{P}(F), X) \leq c_3((\deg X + S)(\log \deg X + \log S))
\]
follows in the same way, this time using $D(\mathbb{P}(\tilde{F}), Z) \leq c_4 \deg Z$ for every $\tilde{F}$ regular with respect to $Z$ which is just a reformulation of [BGS], Propostions 5.1, and the holomorphic model for $D(\tilde{F}, Z)$.

5.4 Proposition Let $p, q, r$ be numbers fullfilling $q \geq t + 1 - p$, $r = t - p$, $t - p - q + 1 \leq 0$ and $Z$ be an effective cycle of codimension $p$ in $\mathbb{P}^t_{\mathbb{C}}$. Further, $\mathbb{P}(W) \subset \mathbb{P}^t(\mathbb{C})$ a subspace of codimension $q$ that does not meet the support of $Z$. There is a subspace $\mathbb{P}(F) \subset \mathbb{P}^t$ of codimension $r$ that contains $\mathbb{P}(W)$, hence intersects $Z$ properly such that with $Z_F$ as defined in section 2,
\[
D_\ast^\circ(Z, \mathbb{P}(W)) \leq D_\ast^\circ(Z_F, \mathbb{P}(W)) + O((\deg Z + S) \log(S \deg Z)),
\]
and
\[
D_\ast^\circ(Z_F, \mathbb{P}(W)) \leq D_\ast^\circ(Z, \mathbb{P}(F)) + O((\deg Z + S) \log(S \deg Z)),
\]
where $D_\ast$ may be chosen to mean either $D_{Ch}$ or $D_G$. Further, if $\mathbb{P}(F)$ is a subspace of codimension $l \geq q$, contains $\theta$ as well as a subspace $\mathbb{P}(V)$ of codimension $t + 1 - p$ such that $\log |\mathbb{P}(V), Z| \geq -c - \log \deg Z$, and $Z, \mathbb{P}(\tilde{F})$ for every $\tilde{F}$ in some neighbourhood of $F$ is a sum of projective subspaces, then the above inequalities still hold.

Proof Let $\mathbb{P}(V) \subset \mathbb{P}(F)$ be a subspace of codimension $l = 2t + 1 - p - q \geq t + 1 - p$ such that $V$ has maximal distance to the support of $V_Z$ with this property. By Lemma 5.2, $\log |V, V_Z| \geq -c_1 \log \deg Z$. Let further $\mathbb{P}(F)$ be the unique subspace of
\[ \mathbb{P}^t \text{ that contains } \mathbb{P}(W) \text{ as well as } \mathbb{P}(V), \text{ and } U_W \text{ a neighbourhood of } W \text{ in } G_{t+1-q,t} \text{ such that for each } \tilde{W} \in U_W \text{ the intersection of } \mathbb{P}(\tilde{W}) \text{ with } Z \text{ is proper. Finally, let } f : U_W \to G_{t+1-q,t} \text{ be the map from Lemma 3.10. By Proposition 2.1 for every } \tilde{W} \in U_W \text{ and } \tilde{F} = f(\tilde{W}), \]
\[
D(Z_{\tilde{F}}, \mathbb{P}(\tilde{W})) - D(Z, \mathbb{P}(\tilde{W})) = D(Z_{\tilde{F}}, \mathbb{P}(\tilde{F})) - D(Z, \mathbb{P}(\tilde{F})),
\]
where \( D(Z_{\tilde{F}}, \mathbb{P}(\tilde{F})) = c \deg Z \) by \([Ma1]\). Let \( \varphi : A^{(t+1-r)r} \to G_{t+1-r,t}, A^{(t+1-r)r} \to G_{t+1-q,t} \)
be the canonical affine charts from chapter 3 centered at \( W \) and \( F \) respectively, and define
\[
G(\tilde{W}, \tilde{F}) = \exp(D(Z_{\tilde{F}}, \mathbb{P}(\tilde{W}))) = \exp(D(Z_{f(\tilde{W})}, \mathbb{P}(\tilde{W}))),
\]
\[
F(\tilde{W}, \cdot) = \exp(D(Z, \mathbb{P}(\tilde{W}))), \quad \text{and } H(\tilde{F}) = \exp(D(Z, \mathbb{P}(\tilde{F}))).
\]
Then, \([18]\) reads
\[
F(\tilde{W}) = \exp(-c \deg Z) H \circ f(\tilde{W}) G(\tilde{W}, f\tilde{W}),
\]
hence for every \( I \) with \( |I| = s \leq S \),
\[
\partial^I((\varphi^*F) = \exp(-c \deg Z) \partial^I((H \circ f \circ \varphi)(G \circ (\varphi, f \circ \varphi)))
\]
\[
= \exp(-c \deg Z) \partial^I((H \circ \psi \circ \psi^{-1} \circ f \circ \varphi)(G \circ \psi \circ \psi^{-1} \circ (\varphi, f \circ \varphi))).
\]
By Lemma \([52]\) \((\partial^J(\psi^{-1} \circ f \circ \varphi))(0) \leq c S \log S \) for every \( J < I \), and by the previous Lemma
\[
|J(\partial^I(H \circ \psi))(0)| \leq c_1(\deg Z + S) \log(S \deg Z),
\]
which for every \( I \) with \(|I| \leq S \), which by elementary differentiation techniques implies
\[
|\partial^I(\varphi^*F)))(0)| \leq \sup_{|J| \leq S} |\partial^I_{\tilde{W}}(\psi^*G))(0)| + c S \log S + c_1(S + \deg Z) \log(S \deg Z) + c_2 S,
\]
where \( \partial^I_{\tilde{W}} \) denotes partial derivatives by the first component; hence
\[
D^S(Z, \mathbb{P}(W)) \leq D^S(Z_{\tilde{F}}, \mathbb{P}(W)) - c \deg Z + c S \log S + c_1(S + \deg Z) \log(S \deg Z) + c_2 S.
\]
The inequality in the other direction is proved analogously.

\[ \textbf{5.5 Lemma} \] Let \( Z \subseteq \mathbb{P}^t_{\mathbb{C}} \) be a subvariety of codimension \( p \), and \( \mathbb{P}(W) \subseteq \mathbb{P}^t_{\mathbb{C}} \) a subspace of codimension \( q \geq t - p \) that does not meet \( Z \). Finally \( \mathbb{P}(F) \) a subspace of codimension \( r = t - p \) containing \( \mathbb{P}(W) \), and intersecting \( Z \) properly,
\[
D^S(\mathbb{P}(W), Z_{\tilde{F}}) = (D^{\mathbb{P}(F)})^S(\mathbb{P}(W), Z_{\tilde{F}}, \mathbb{P}(F)) + c \cdot \deg Z.
\]
By Lemma 2.3, 

Since 

coordinates, and for 

\( t \) to the first \((t + 1 - q)\) \((p + q - t)\) coordinates. The restriction of \( \varphi \) to \( \mathbb{A}^{(t+1-q)(p+q-t)} \) is an affine chart for \( G^F \). Further, let \((x_1, y_1, \ldots, x_{(t+1-q)(p+q-t)}, y_{(t+1-q)(p+q-t)})\) be the real coordinates of \( \mathbb{A}^{(t+1-q)(p+q-t)} \), and denote by \( \partial_{x_1}, \ldots, \partial_{y_{(t+1-q)(p+q-t)}} \) or simply \( \partial_1, \ldots, \partial_{2(t+1-q)(p+q-t)} \) the partial derivatives with respect to these coordinates, and for \( I_F = (i_1, \ldots, i_{2(t+1-q)(p+q-t)}) \) let \( \partial^{I_F}_F \) be the differential operator 

By Lemma 2.3, 

\[
D(\mathbb{P}(W), Z_F) = D^{\mathbb{P}(F)}(\mathbb{P}(W), Z_F, \mathbb{P}(F)) + c \deg Z.
\]

Since, with these notations, for \( i = 2(t+1-q)(p+q-t) + 1, \ldots, 2(t+1-q)(t+1) \) we have \( \partial_i D(Z_F, \mathbb{P}(W)) = 0 \), it follows

\[
D^{\mathbb{P}(F)}(\mathbb{P}(W), Z_F, \mathbb{P}(F)) = \log \left| \sup_{s \leq S, |I_F| = s} \partial^{I_F}_F \exp D(\mathbb{P}(W), Z_F, \mathbb{P}(F)) \right| = \left| \sup_{s \leq S, |I| = s} \partial^I F \exp(D(\mathbb{P}(W), Z_F) - c \deg Z) \right| = D^{\mathbb{P}(F)}(\mathbb{P}(W), Z_F) - c \deg Z.
\]

5.6 Corollary In the situation of Proposition 5.3,

\[
D^S(Z, \mathbb{P}(W)) \leq (D^{\mathbb{P}(F)}_*)^S(Z, \mathbb{P}(F), \mathbb{P}(W)) + O((\deg Z + S) \log(S \deg Z)),
\]

and

\[
(D^{\mathbb{P}(F)}_*)^S(Z, \mathbb{P}(F), \mathbb{P}(W)) \leq D^S(Z, \theta) + O((\deg Z + S) \log(S \deg Z)),
\]

Proof Follows immediately from proposition 5.3 and the previous Lemma.

Proof of Theorem 5.1 In the Corollary, take \( q = t \).

Proof of Theorem 5.4: Let \( \mathbb{P}(F) \) be as in Theorem 5.1 Then,

\[
D^S(\theta, Z) \leq (D^{\mathbb{P}(F)}_*)^S(\theta, Z, \mathbb{P}(F)) + O((S \deg Z) \log(S + \deg Z)).
\]

Since

\[
Z, \mathbb{P}(F) = \sum_{i=1}^{\deg Z} z_i
\]

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with \( |z_1, \theta| \leq \cdots \leq |z_{\deg Z}, \theta| \) is zero-dimensional, by Proposition 4.6

\[
(D^p(F))^S(\theta, Z, \mathbb{P}(F)) \leq \sum_{i=S+1}^{\deg Z} \log |z_i, \theta| + O(S \log \deg Z).
\]

The two inequalities together imply

\[
D^S(\theta, Z) \leq \sum_{i=S+1}^{\deg Z} \log |z_i, \theta| + O((S + \deg Z) \log(S \deg Z)).
\]

Similarly,

\[
(D^p(F))^{3S}(\theta, Z, \mathbb{P}(F)) \leq D^{3S}(\theta, Z) + O((S + \deg Z) \log(S \deg Z)),
\]

and

\[
2 \sum_{S+1}^{\deg Z} \log |z_i, \theta| \leq (D^{3S})^p(F)(\theta, Z, \mathbb{P}(F)) + O((S + \deg Z) \log(S \deg Z)),
\]

imply

\[
\sum_{S+1}^{\deg Z} 2 \log |z_i, \theta| \leq D^{3S}(\theta, Z) + O(\deg Z \log \deg Z).
\]

### 6 Proof of the main Theorems

#### 6.1 Combinatorics of the intersection points

Let \( Z_0, Z_1 \) be properly intersecting effective cycles of pure codimensions \( p \) and \( q \) in \( \mathbb{P}^t(\mathbb{C}) \), and \( Z_0 \# Z_1 \) their join. (See [Ma1], section 6 for details.) Let further \( \theta \in \mathbb{P}^t(\mathbb{C}) \) be a point not contained in the support of \( Z_0, Z_1 \), and \( \mathbb{P}(F_0), \mathbb{P}(F_1) \subset \mathbb{P}^t \) projective subspaces of dimensions \( p, q \) such that \( \mathbb{P}(F_i) \) intersects \( Z_i \) properly for \( i = 0, 1 \). Denote

\[
Z_i, \mathbb{P}(F_i) = \sum_{j=1}^{\deg Z_i} z_j^i \quad \text{with} \quad |z_1^i, \theta| \leq \cdots \leq |z_{\deg Z_j}^i, \theta|, \quad j = 0, 1,
\]

and assume that the numbers

\[
|z_1^0, \theta|, \ldots, |z_{\deg Z_0}^0, \theta|, |z_1^1, \theta|, \ldots, |z_{\deg Z_1}^1, \theta|
\]

together with the numbers

\[
|z_i^0 \# z_j^1, (\theta, \theta)|, \quad i = 1, \ldots \deg Z_0, \quad j = 1, \ldots, Z_1
\]
are pairwise distinct.
Let next \( n \) for \( n \in \mathbb{N} \) denote the set \( \{1, \ldots, n\} \) and define a path
\[
f = f_\theta = (f_0, f_1) : \text{deg} Z_0 + \text{deg} Z_1 \to \text{deg} Z_0 \times \text{deg} Z_1
\]
as in the introduction in the following way: \( f(k) = (f_0(k), f_1(k)) = (\nu_0, \nu_1) \), iff \( k = \nu_0 + \nu_1 \) and there is a number \( t \in [0, 1] \) such that
\[
|z_0^0, \theta| < t < |z_0^0 + 1, \theta| \quad \text{and} \quad |z_1^1, \theta| < t < |z_{n+1}^1, \theta|.
\]
The maps \( f_0 \) and \( f_1 \) are surjective. Further for \( k \in \text{deg} Z_0 + \text{deg} Z_1 \), set \( i_k = 0 \) if \( f_1(k) > f_1(k-1) \), hence \( |z_{f_1(k)}^1, \theta| < |z_{f_0(k-1)}^0, \theta| \), and \( i_k = 1 \) otherwise.
Recall that for \( x, y, \theta \in \mathbb{P}^1(\mathbb{C}) \) the inequalities
\[
\min(|x, \theta|, |y, \theta|) \leq |x \# y, (\theta, \theta)| \leq \max(|x, \theta|, |y, \theta|) \tag{19}
\]
hold. ([Ma1], Lemma 6.4)

6.1 Lemma

1. With the above notations, let \( K \) be a number such that either \( f_0(K-1) < \text{deg} Z_0 \) or \( f_1(K_1) < \text{deg} Z_1 \). Then,
\[
\sum_{i=1}^{\text{deg} Z_0} \sum_{j=1}^{\text{deg} Z_1} \log |(\theta, \theta), z_i^0 \# z_j^1| \leq \sum_{k=0}^{K} \sum_{l=f_{ik}(k)+1}^{\text{deg} Z_{ik}} \log |\theta, z_{ik}^l|.
\]

2. For every \( K \leq \text{deg} Z_0 + \text{deg} Z_1 \), and \( (\nu_1, \nu_0) = f(K) \),
\[
\sum_{i=1}^{\text{deg} Z_0} \sum_{j=1}^{\text{deg} Z_1} \log |(\theta, \theta), z_i^0 \# z_j^1| \leq \nu_1 \sum_{k=\nu_0+1}^{\text{deg} Z_0} \log |\theta, z_k^0| + \nu_0 \sum_{k=\nu_1}^{\text{deg} Z_1} \log |\theta, z_k^1|.
\]

Proof 1. The equation
\[
\sum_{i=1}^{\text{deg} Z_0} \sum_{j=1}^{\text{deg} Z_1} \log |(\theta, \theta), z_i^0 \# z_j^1| = \sum_{k=1}^{K} \sum_{l=f_{ik}(k)+1}^{\text{deg} Z_{ik}} \log |(\theta, \theta), z_{ik}^l \# z_{ik}^l|
\]
is just a reordering of the sum. By (19), and the fact \( |\theta, z_{ik}^l| \leq |\theta, z_{ik}^l| \) following immediately from the definition of \( f \), and \( i_k \), the left hand side is less or equal
\[
\sum_{k=1}^{K} \sum_{l=f_{ik}(k)+1}^{\text{deg} Z_{ik}} \log |\theta, z_{ik}^l|,
\]
as was to be proved.

2. Follows from 1, by taking on the right hand side only the first \( \nu_0 + \nu_1 \leq K \) summands in the first summation, and the last \( \deg Z_{ik} - \nu_{ik} \) summands in the second summation, which is possible since \( \nu_{ik} \geq f_{ik}(k) \).

Let now \( S \leq \deg Z_0 \deg Z_1 \), and \( M \subset \{ z_i^{0 \#} z_j^1 | i = 1, \ldots, \deg Z_0, j = 1, \ldots, \deg Z_1 \} \) be the subset with \( |M| = S \), and \( |z, (\theta, \theta)| < |z', (\theta, \theta)| \) for every \( z \in M, z' \notin M \). Because by (19),

\[
|x, \theta| \leq |x', \theta| \quad \text{and} \quad |y, \theta| \leq |y', \theta|, \quad \Rightarrow \quad |x\#y, (\theta, \theta)| \leq |x'\#y', (\theta, \theta)|
\]

the set \( M \) fullfills the condition

\[
\forall i, i' \in \deg Z_0, \quad \forall j, j' \in Z_1 \quad i \leq i' \land j \leq j' \land (i, j) \notin M \Rightarrow (i', j') \notin M; \quad (20)
\]

consequently, there is a number \( k_0 \leq K \) such that \( f(k_0) = (\nu_0, \nu_1) \notin M \), and \( \nu_0 \nu_1 \leq S \). For any \( k \geq k_0 \) also \( f_0(k) \geq \nu_0, f_1(k) \geq \nu_1 \), and \( f(k) \notin M \). Hence, if \( h_S(k) \) denotes the number \( \min \{ |l| z_{f_0(k)}^{1-i_k} z_{f_1(k)}^{i_k} \notin M \} - f_{ik} \), we have \( h_S(k) = 0 \) for \( k \geq k_0 \).

6.2 Lemma With the above notations,

1.

\[
\sum_{z \notin M} \log |(\theta, \theta), z| \leq \sum_{k=k_0}^{\deg Z_{ik}} \sum_{l=f_{ik}(k)-h_S(k)+1} \log |\theta, z_{ik}^l|.
\]

2. For any \( k \geq k_0 \), and \( (\tilde{\nu}_0, \tilde{\nu}_1) = f(k) \),

\[
\sum_{z \notin M} \log |(\theta, \theta), z| \leq (\tilde{\nu}_1 - \nu_1) \sum_{l=\tilde{\nu}_0+1}^{\deg Z_0} \log |\theta, z_0^l| + (\tilde{\nu}_0 - \nu_0) \sum_{l=\tilde{\nu}_1+1}^{\deg Z_1} \log |\theta, z_1^l|.
\]

3. With \( k, \tilde{\nu}_0, \tilde{\nu}_1 \) as in 2,

\[
(\tilde{\nu}_0 - \nu_0)(\tilde{\nu}_1 - \nu_1) \log |Z_0 + Z_1, \theta| + \sum_{z \notin M} \log |(\theta, \theta), z| \leq
\]

\[
(\tilde{\nu}_1 - \nu_1) \sum_{l=\tilde{\nu}_0+1}^{\deg Z_0} \log |\theta, z_0^l| + (\tilde{\nu}_0 - \nu_0) \sum_{l=\tilde{\nu}_1+1}^{\deg Z_1} \log |\theta, z_1^l|.
\]

Proof 1. Since \( z_i^0 \# z_j^1 \notin M \) implies \( i \geq \nu_0 \) or \( j \geq \nu_1 \), the inequality

\[
\sum_{z \notin M} \log |(\theta, \theta), z| \leq \sum_{k=k_0}^{K} \sum_{l=f_{ik}(k)-h_S(k)+1}^{\deg Z_{ik}} \log |(\theta, \theta), z_{ik}^{1-i_k} \# z_{ik}^{i_k}|
\]
again follows from a renumbering of the sum. The inequality
\[
\sum_{k=k_0}^{K} \sum_{l=f_{ik}(k)}^{\deg Z_{ik}} \log |(\theta, \theta), z_{ik}^{l-i_k}| \leq \sum_{k=k_0}^{K} \sum_{l=f_{ik}+1}^{\deg Z_{ik}} \log |\theta, z_{ik}^l|,
\]
follows from (19) as in the previous Lemma.

2. Follows again from 1 by leaving out the first \(\nu_1 - \nu_0 - \nu_1 - \nu_0\) in the first summation, and taking only the last \(\deg Z_{ik} - \nu\) summands in the second summation.

3. Define the sets
\[
N_0 := \{(i, j) \in \deg Z_0 \times \deg Z_1 | \nu_0 \leq i \leq \nu_0\}, \quad N_1 := \{(i, j) \in \deg Z_0 \times \deg Z_1 | \nu_1 \leq j \leq \nu_1\}.
\]
The set \(N_0 \cap N_1\) has cardinality \((\nu_0 - \nu_0)(\nu_1 - \nu_1)\). Thus the first inequality of (19) implies
\[
(\nu_0 - \nu_0)(\nu_1 - \nu_1) \log |Z_0 + Z_1, \theta| \leq \sum_{z \in N_0 \cap N_1} \log |(\theta, \theta), z|
\]
for each \(z \in N_0 \cap N_1\). Further, \(N_0 \cap N_1\) is contained in the complement of \(M\). Hence, by the second inequality of (19),
\[
(\nu_0 - \nu_0)(\nu_1 - \nu_1) \log |Z_0 + Z_1, \theta| + \sum_{z \in N_0 \cap N_1} \log |(\theta, \theta), z| \leq
\]
\[
\nu_0 \sum_{l=\nu_0+1}^{\nu_0} |(\theta, \theta), z_{i_k}^0| + \nu_1 \sum_{l=\nu_1+1}^{\nu_1} |(\theta, \theta), z_{i_k}^l|.
\]
Adding the equality from part 2 of the Lemma proves the claim.

6.2 Finish of proofs
By Lemma 5.2 and Proposition 5.3, the \(\mathbb{P}(F_i)\) \(i = 0, 1\) from the previous section may be chosen in such a way that they contain subspaces \(\mathbb{P}(V_i)\) of codimension \(t + 1 - \text{codim} Z_i\) such that
\[
\log |\mathbb{P}(V_i), Z_i| \geq -c - \deg Z_i,
\]
hence
\[
D^S(\mathbb{P}(F_i), Z_i) \geq -O((\deg Z_i + S)(\log \deg Z_i + \log S)).
\]
for every \(S\), and by Theorem 5.1
\[
2 \sum_{j=S+1}^{\deg Z_0} \log |z_{ij}^0, \theta| \leq D^{3S}(\theta, Z_0) + O((S + \deg Z_0) \log (S \deg Z_0))
\]
Further, we have
\[
(\mathbb{P}(F_0)\#\mathbb{P}(F_1)).(Z_0\#Z_1) = \sum_{i=1}^{\deg Z_0} \sum_{j=1}^{\deg Z_1} z_i^0 \# z_j^1,
\]
and each of the \(z_i^0 \# z_j^1\) is a one dimensional projective subspace of \(\mathbb{P}^{2t+1}\). We denote
\[
(\mathbb{P}(F_0)\#\mathbb{P}(F_1)).(Z_0\#Z_1) = \sum_{i=1}^{\deg Z_0} z_i,
\]
such that \(|(\theta,\theta), z_1| < \ldots < |(\theta,\theta), z_{\deg Z_0 \deg Z_1}|\).

6.3 Proposition With the above notations, and \(K\) as in Lemma 6.1

1. \[
2 \sum_{i=1}^{\deg Z_0} \sum_{j=1}^{\deg Z_1} \log |(\theta,\theta), z_i^0 \# z_j^1| \leq \sum_{k=0}^{K} D^{3(f_{ik}(k)-h_S(k))}(Z_{ik},\theta) + O(\deg Z_0 \deg Z_1 \log(\deg Z_0 \deg Z_1)).
\]

2. With \(S, M, \nu_0, \nu_1, k_0\) as in Lemma 6.2

\[
2 \sum_{z \notin M} \log |(\theta,\theta), z| \leq \sum_{k=k_0}^{K} D^{f_{ik}(Z_{kK},\theta)} + O(\deg Z_0 \deg Z_1 \log(\deg Z_0 \deg Z_1)).
\]

3. For every \(k \geq k_0\), and \((\breve{\nu}_0, \breve{\nu}_1) = f(k)\),

\[
2(\nu - \nu_0)(\kappa - \kappa_0) \log |Z_0 + Z_2,\theta| + 2 \sum_{z \notin M} \log |(\theta,\theta), z| \leq (\breve{\nu}_1 - \nu_1)D^{\nu_0}(\theta, Z_0) + (\breve{\nu}_1 - \nu_1)D^{\nu_1}(\theta, Z_1) + O(\deg Z_0 \deg Z_1 \log(\deg Z_0 \deg Z_1)).
\]

Proof 1. By Lemma 6.11,
\[
2 \sum_{i=1}^{\deg Z_0} \sum_{j=1}^{\deg Z_1} \log |(\theta,\theta), z_i^0 \# z_j^1| \leq \sum_{k=0}^{K} \sum_{l=f_{ik}(k)-h_S(k)+1}^{\deg Z_{ik}} \log |\theta, z^k_l|,
\]
Now for each $k$,

$$2 \sum_{l=f_{ik}(k)-hS(k)+1}^{\deg Z_{ik}} \log |\theta, z_{ik}^l| \leq D^3(f_{ik}(k)-hS(k))(Z_{ik}, \theta) + c \deg Z_{ik} \log \deg Z_{ik},$$

by (21). Since $i_k = 0$ for at most $\deg Z_1$ values of $k$ and $i_k = 1$ for at most $\deg Z_0$ values of $k$, we have

$$\sum_{k=0}^{K} \deg Z_{ik} \log \deg Z_{ik} \leq 2c \deg Z_0 \deg Z_1 \log(\deg Z_0 \deg Z_1).$$

Hence, the left hand side of (23) is less or equal

$$\sum_{k=1}^{K} D_{f_{ik}(k)}(Z_{ik}, \theta) + 2 \deg Z_0 \deg Z_1 \log(\deg Z_0 \deg Z_1),$$

and the claim follows.

2. Follows in exactly the same way as 1, this time using Lemma 6.2.1.

3. Follows from Lemma 6.2.3.

**6.4 Lemma** Let $\mathbb{P}(V) \subset \mathbb{P}(F)$ be a subspace of codimension $t-p+1$ such that $\log |\mathbb{P}(V), X| \geq -c \log \deg X$, and $\mathbb{P}(V') \subset \mathbb{P}(F')$ a subspace of codimension $t-q+1$ such that $\log |\mathbb{P}(V'), Y| \geq -c \log \deg Y$. Then,

$$\log |\mathbb{P}(V)\#\mathbb{P}(V'), X\#Y| \geq -c \log(\deg X \deg Y).$$

**Proof** Follows from the inequality

$$|x\#y, v\#w| \geq \min(|x, v|, |y, w|)$$

from [Ma1], Lemma 6.4.

By this Lemma the pair $\mathbb{P}(F_0)\#\mathbb{P}(F_1), Z_0\#Z_1$ fullfills the condition of Proposition 5.3.

**6.5 Proposition** With the above notaions from (22),

$$D^S(\theta, Z_0, Z_1) + D(Z_0, Z_1) \leq \sum_{i=S+1}^{\deg Z_0 \deg Z_1} \log |(\theta, \theta), z_i| + O((S + \deg Z_0 \deg Z_1) \log(s \deg Z_0 \deg Z_1)).$$
Proof Firstly, by Proposition 4.6
\[ D^S((\theta, \theta), X \# Y) \leq \sum_{i=S+1}^{\deg Z_0 \deg Z_1} \log |(\theta, \theta), z_i| + O(S \log(\deg Z_0 \deg Z_1)). \]

Next, together with the preceding Lemma, the Propositions 5.4, and 5.5 just as in the proof of Corollary 5.6 imply
\[ (D^S/C8(\Delta)) S((\theta, \theta), (Z_0 \# Z_1)/C8(\Delta)) \leq D^S((\theta, \theta), Z_0 \# Z_1) - D(\mathbb{P}(\Delta), (X \# Y)) + O((S + \deg Z_0 \deg Z_1) \log(S \deg Z_0 \deg Z_1)) \]

Since \((X \# Y)/C8(\Delta)) = \delta(X \cdot Y)\) with \(\delta: \mathbb{P}^t \rightarrow \mathbb{P}(\Delta) \subset \mathbb{P}^{2t+1}\) the diagonal map,
\[ D^S((\theta, \theta), (Z_0 \# Z_1)/C8(\Delta)) = D(\theta, X \cdot Y) + \log 2 \deg Z_0 \deg Z_1, \]
and consequently
\[ (D^S/C8(\Delta)) S((\theta, \theta), (X \# Y)/C8(\Delta)) = D^S(\theta, X \cdot Y) + \log 2 \deg Z_0 \deg Z_1. \]

Similarly,
\[ D(\mathbb{P}(\Delta), X \# Y) = D(X, Y) + \log 2 \deg Z_0 \deg Z_1. \]

The claim follows.

Proof of Theorem 1.6: Follows from Proposition 6.3 together with Proposition 6.5 for \(S = 0\).

Proof of Corollary 1.7: Since \(f_{ik}(k) \leq f_{ik}(l)\) for \(k \leq l\), we get
\[ D^{f_{ik}(k)}(Z_{ik}, \theta) \leq D^{f_{ik}(l)}(Z_{ik}, \theta) = D^{\nu_1}(Z_{ik}, \theta), \]
and the claim follows from Theorem 1.6 by cutting the sum on the left hand side at \(l\).

Proof of Corollary 1.8: Since in the path \(f\) in each step exactly one coordinate increase, there is a \(k \leq S\) such that either \(f(k) = (1, \nu_1)\) with \(\nu_1 \leq S\) or \(f(k) = (0, S)\). In the first case, by Corollary 1.7
\[ 2D(Z_0, Z_1) + 2D(Z_0, Z_1, \theta) \leq D^{3\nu_1}(Z_1, \theta) + O((S + \deg Z_0 \deg Z_1) \log(S \deg Z_0 \deg Z_1)), \]
which trivially is less or equal
\[ D^{3S}(Z_0, \theta) + O((S + \deg Z_0 \deg Z_1) \log(S \deg Z_0 \deg Z_1)). \]
In the second case, by the same Corollary,

\[ 2D(Z_0, Z_1) + 2D(Z_0, Z_1, \theta) \leq SD(Z_0, \theta) + O((S + \deg Z_0 \deg Z_1) \log(S \deg Z_0 \deg Z_1)). \]

**Proof of Theorem 1.9**: Follows from the Propositions 6.3 and 6.5.

**Proof of Corollary 1.10**: Follows in the same way as Corollary 1.7.

**Proof of Corollary 1.11**: Let \( k_0, \nu_1, \nu_1 \) be as above. We have \( \nu_0 \nu_1 \leq S = S_0 S_1 \), and without loss of generality one may assume \( S_0 \geq \nu_0 = f_0(k_0) \). Let \( l \geq k_0 \) be the smallest number such that \( f_0(l) = 2S_0 \), and \((\bar{\nu}_0, \bar{\nu}_1) = f(l)\). If \( \bar{\nu}_1 - \nu_1 \geq S_1 \), then by Corollary 1.10,

\[ 2D(Z_0, Z_1) + 2D^S(Z_0, Z_1, \theta) \leq (\bar{\nu}_1 - \nu_1)D^{6S_0}(Z_0, \theta) + O((S + \deg Z_0 \deg Z_1) \log(S \deg Z_0 \deg Z_1)) \]

\[ \leq S_1D^{6S_0}(Z_0, \theta) + O((S + \deg Z_0 \deg Z_1) \log(S \deg Z_0 \deg Z_1)). \]

If \( \bar{\nu}_1 - \nu_1 \leq S_1 \), and \( \nu_1 \leq 2S_1 \), then \( \nu_1 \leq 3S_1 \), hence

\[ 2D(Z_0, Z_1) + 2D^S(Z_0, Z_1, \theta) \leq (\bar{\nu}_0 - \nu_0)D^{3S_1}(Z_0, \theta) + O((S + \deg Z_0 \deg Z_1) \log(S \deg Z_0 \deg Z_1)) \]

\[ \leq S_0D^{9S_1}(Z_0, \theta) + O((S + \deg Z_0 \deg Z_1) \log(S \deg Z_0 \deg Z_1)), \]

since \( \bar{\nu}_0 - \nu_0 \geq 2S_0 - S_0 = S_0 \), also by Corollary 1.10.

Finally if \( \bar{\nu}_1 - \nu_1 \leq S_1 \), and \( \nu_1 \geq 2S_1 \), then \( \nu_1/2S_0 \geq S \), hence the complement of the set \( M \) from Lemma 6.2 is contained in the set

\[ \{ z_0^i z_j^j | i \geq \nu_1/2 \land j \geq S_0 \}. \]

This means that for \( \nu_1/2 \leq k \leq \nu_1 \) and \( i_k = 0 \) the value \( h_S(k) \) is less or equal \( f_0(k) - S_0 \). Hence, by Theorem 1.9, with \( l \leq k_0 \) the smallest number such that
\[ f_1(l) = \nu_1/2 \] and \[ \nu_1 \] we get
\[ 2D(Z_0, Z_1) + D_S(Z_0, Z_1, \theta) \leq \sum_{k=l+1}^{k_0} D^3(f_k - h_S(k))(Z_k, \theta) + O((S + \deg Z_0 \deg Z_1) \log(S \deg Z_0 \deg Z_1)) \]
\[ \leq \sum_{k=\nu_1/2+1}^{\nu_1} D^{3\nu_0}(Z_0, \theta) + O((S + \deg Z_0 \deg Z_1) \log(S \deg Z_0 \deg Z_1)) \]
\[ \leq S_1 D^{3\nu_0}(Z_0, \theta) + O((S + \deg Z_0 \deg Z_1) \log(S \deg Z_0 \deg Z_1)). \]

**A Proof of Lemmas 4.7, 4.8 and 4.9**

**Proof of Lemma 4.7** Firstly,
\[ f^{(s)}(0) = s! \sum_{i_1=1}^{n} \sum_{i_2=i_1+1}^{n} \cdots \sum_{i_s=i_{s-1}+1}^{n} \frac{f(0)}{\prod_{k=1}^{s} x_{i_k}}, \]
for every \( s \leq n \). Since \( \prod_{k=1}^{s} |x_{i_k}| \geq \prod_{k=1}^{s} |x_k| \) for every \( s \)-tupel \((x_{i_1}, \ldots, x_{i_k})\) consequently,
\[ |f^{(s)}(0)| \leq \frac{n!}{(n-s)!} \prod_{k=1}^{s} |x_{i_k}| = \frac{n!}{(n-s)!} \prod_{i=s+1}^{n} |x_i|, \]
proving the second inequality.

The first inequality will be proved for \(|x_1| < |x_2| < \cdots < |x_n|\), and follows for \(|x_1| \leq \cdots \leq |x_n|\) by continuity.

**Step 1:** There are points
\[ x_{11}, \ldots, x_{1n}, x_{22}, \ldots, x_{2n}, \ldots, x_{n-1n-1}, x_{n-1n}, x_{nn} \in [-1, 1], \]
such that
\[ 0 < |x_{ii}| < |x_{ii+1}| < \cdots < |x_{in}| < 1, \quad \sgn(x_{ij}) = \sgn(x_{i-1j}), \quad \forall i = 1, \ldots, n, \]
\[ 0 < |x_{ij}| < |x_{i-1j}| \leq 1, \quad i = 2, \ldots, n, j = i, \ldots, n, \]
\[ f^{(s-1)}(x_{sj}) = 0, \quad s = 1, \ldots, n, j = s, \ldots, n. \]
PROOF The points are defined recursively. With \( x_{1j} = x_j \) the claims are fullfilled or empty. Assume the points \( x_{ss}, \ldots, x_{sn} \) are defined. If \( \text{sgn}x_{sj} = \text{sgn}x_{sj+1} \), since \( f^{(s-1)}(x_{sj}) = f^{(s-1)}(x_{sj+1}) = 0 \), there is a point \( x_{sj+1} \in (x_{sj}, x_{sj+1}) \), if \( \text{sgn}x_{sj} = 1 \), and \( x_{sj+1} \in (x_{sj}, x_{sj+1}) \) if \( \text{sgn}x_{sj} = -1 \) such that \( f^{(s)}(x_{sj+1}) = 0 \). If \( \text{sgn}x_{sj} \neq x_{sj+1} \), let \( k \) be the biggest number less than \( j \) such that \( \text{sgn}x_{sk} = \text{sgn}x_{sj+1} \); if there is no such number take 0 instead of \( x_{sk} \) in the following step. Then there is a \( x_{s+1} \in (x_{sk}, x_{sj+1}) \) if \( \text{sgn}x_{sk} = 1 \), and \( x_{s+1} \in (x_{sj+1}, x_{sk}) \) if \( \text{sgn}x_{sk} = -1 \) such that \( f^{(s)}(x_{s+1}) = 0 \). The so constructed points obviously fulfill the required conditions.

STEP 2: For every \( s \) with \( 0 \leq s < n \) there is a point \( \bar{x}_s \in [-1, 1] \) with \( |\bar{x}_s| < |x_s| \), and

\[
|f^{(s)}(\bar{x}_s)| \geq \frac{1}{2^s} \prod_{i=s+1}^{n} |x_i| > 0. \tag{25}
\]

PROOF The claim obviously holds for \( s = 0 \) with \( \bar{x}_0 = 0 \). Let \( y_i = x_{i+1} + 1 \) for \( i = 1, \ldots, n \) with \( x_{i+1} \) the points from step 1, and assume the claim holds for \( s \). Then, by (25), and step 1, \( f^{(s)}(\bar{x}_s) \neq 0 \), and \( f^{(s)}(y_s) = 0 \), we have \( \bar{x}_s \neq y_s \). By the mean value Theorem, there is a \( \bar{x}_{s+1} \) inside the intervall between \( \bar{x}_s \) and \( y_s \), hence of absolute value at most \( \max(|x_s|, |y_s|) \leq \max(|x_s|, |x_{s+1}|) = |x_{s+1}| \) such that

\[
|f^{(s+1)}(\bar{x}_{s+1})| = \frac{|f^{(s)}(\bar{x}_s) - f^{(s)}(y_s)|}{|\bar{x}_s - y_s|} = \frac{|f^{(s)}(\bar{x}_s)|}{|\bar{x}_s - y_s|} \geq \frac{1}{2^s} \prod_{i=s+1}^{n} |x_i|.
\]

Since also \( |y_s| \leq |x_{s+1}| \), the inequality \( |\bar{x}_s - y_s| \leq 2|x_{s+1}| \) holds, and consequently,

\[
|f^{(s+1)}(\bar{x}_{s+1})| \geq \frac{1}{2^s} \prod_{i=s+1}^{n} |x_i| = \frac{1}{2^{s+1}} \prod_{i=s+2}^{n} |x_i|,
\]

as was to be proved.

STEP 3: If \( 3n^3|x_{s-1}| \leq |x_{2s-1}| \), with \( \bar{x}_s \) from step 2, then

\[
\prod_{i=s+1}^{n} |x_i| \leq 2^{s+1}(2s + 1) \max_{s \leq j \leq 3s} |f^{(j)}(0)|.
\]

PROOF By Taylor’s formula,

\[
f^{(s)}(\bar{x}_s) = \sum_{i=s}^{3s} \frac{f^{(i)}(0)}{(i - s)!} \bar{x}_s^{i-s} + \sum_{i=3s+1}^{n} \frac{f^{(i)}(0)}{(i - s)!} \bar{x}_s^{i-s},
\]

hence

\[
|f^{(s)}(\bar{x}_s)| \leq (2s + 1) \max_{s \leq j \leq 3s} |f^{(j)}(0)| + \sum_{i=3s+1}^{n} \frac{|f^{(i)}(0)|}{(i - s)!} |\bar{x}_s|^{i-s}. \tag{26}
\]
Next, for $i \geq 3s + 1$, (24) implies

$$|f^{(i)}(0)||\bar{x}_s|^{i-s} \leq \frac{n!}{(n-i)!}|\bar{x}_s|^{i-s} \prod_{j=s+1}^{n} |x_j| = \frac{n!}{(n-i)!} \prod_{j=s+1}^{n} |x_j| \prod_{j=2s+1}^{n} x_j |x_j| \prod_{j=2s+1}^{n} |x_j|.$$  

Since $\bar{x}_s \leq |x_s| \leq |x_j|$ for $j \geq s + 1$, and $\bar{x}_s \leq \frac{1}{3n^s}|x_{2s+1}| \leq \frac{1}{3n^s}|x_j|$ for $j \geq 2s + 1$, the above is less or equal

$$\frac{n!}{(n-i)!} \left( \frac{1}{3n^3} \right)^{i-2s} \prod_{j=s+1}^{n} |x_j| \leq \frac{n!}{(n-i)!} \left( \frac{1}{3n^3} \right)^{i-2s} \prod_{j=s+1}^{n} |x_j| \leq \left( \frac{1}{3} \right)^{i-2s} \prod_{j=s+1}^{n} |x_j|.$$  

Consequently,

$$\sum_{i=3s+1}^{n} |f^{(i)}(0)||\bar{x}_s|^{i-s} \leq \prod_{j=s+1}^{n} |x_j| \sum_{i=3s+1}^{n} \frac{1}{(i-s)!} \left( \frac{1}{3} \right)^{i-2s} \prod_{j=s+1}^{n} |x_j| \leq \frac{1}{2 \cdot 3^s} \prod_{j=s+1}^{n} |x_j|.$$  

Together with (25), and (26), this implies

$$\frac{1}{2^s} \prod_{s=1}^{n} |x_i| \leq (2s + 1) \max_{s \leq j \leq 3s} |f^{(j)}(0)| + \frac{1}{2 \cdot 3^s} \prod_{j=s+1}^{n} |x_j|,$$

hence

$$\frac{1}{2s+1} \prod_{i=s+1}^{n} |x_i| \leq (2s + 1) \max_{s \leq j \leq 3s} |f^{(j)}(0)|,$$

and the claim follows.  

STEP 4: For every $\bar{s} \leq s$,

$$\prod_{i=2s-\bar{s}+1}^{2s} |x_i| \prod_{i=s+1}^{n} |x_i| \leq \max \left( 4(s+1)(3n^3)^{s+1} \max_{0 \leq j \leq 3s} f^{(j)}(0), (3n^3)^s \prod_{i=s-\bar{s}+1}^{n} |x_i| \right). \quad (27)$$  

PROOF The claim obviously holds for $\bar{s} = 0$. Assume (27) holds for $\bar{s} \leq s - 1$. If in (27),

$$\prod_{i=2s-\bar{s}+1}^{2s} |x_i| \prod_{i=s+1}^{n} |x_i| \leq 4(s+1)(3n^3)^{s+1} \max_{0 \leq j \leq 3s} f^{(j)}(0),$$

then

$$\prod_{i=2s-\bar{s}}^{2s} |x_i| \prod_{i=s+1}^{n} |x_i| = |x_{2s-\bar{s}}| \prod_{i=2s-\bar{s}+1}^{2s} |x_i| \prod_{i=s+1}^{n} |x_i| \leq$$
\[|x_{2s-\bar{s}}|4(s+1)(3n^3)^{s+1} \max_{0 \leq j \leq 3s} f^{(j)}(0) \leq 4(s+1)(3n^3)^{s+1} \max_{0 \leq j \leq 3s} f^{(j)}(0).\]

If in (27),
\[
\prod_{i=2s-\bar{s}+1}^{2s} |x_i| \prod_{i=s+1}^{n} |x_i| \leq (3n^3)^{\bar{s}} \prod_{i=s-\bar{s}+1}^{n} |x_i|,
\]
then if \(3n^3|\bar{x}_{s-\bar{s}}| \leq |x_{2(s-\bar{s})-1}|\), step 3, with \(s - \bar{s}\) instead of \(s\) implies
\[
\prod_{i=s-\bar{s}+1}^{n} |x_i| \leq 2^{s-\bar{s}+1}(2(s - \bar{s}) + 1)\max_{s-\bar{s} \leq j \leq 3(s-\bar{s})} |f^{(j)}(0)|,
\]
hence,
\[
\prod_{i=2s-\bar{s}}^{2s} |x_i| \prod_{i=s+1}^{n} |x_i| \leq |x_{2s-\bar{s}}|(3n^3)^{\bar{s}} \prod_{i=s-\bar{s}+1}^{n} |x_i| \leq (3n^3)^{\bar{s}}2^{s-\bar{s}+1}(2(s - \bar{s}) + 1)\max_{s-\bar{s} \leq j \leq 3(s-\bar{s})} |f^{(j)}(0)| \leq (3n^3)^{s+1}(2(s - \bar{s}) + 1)\max_{s-\bar{s}-1 \leq j \leq 3(s-\bar{s}-1)} |f^{(j)}(0)| \leq (3n^3)^{s+1}(2s + 1)\max_{0 \leq j \leq 3s} |f^{(j)}(0)|,
\]
hence (27) for \(\bar{s} + 1\).

If on the other hand \(3n^3|\bar{x}_{s-\bar{s}}| \geq |x_{2s-\bar{s}}|\), by (27), either
\[
\prod_{i=2s-\bar{s}}^{2s} |x_i| \prod_{i=s+1}^{n} |x_i| \leq (3n^3)^{\bar{s}} \prod_{i=s-\bar{s}+1}^{n} |x_i|,
\]
or
\[
\prod_{i=2s-\bar{s}}^{2s} |x_i| \prod_{i=s+1}^{n} |x_i| \leq \prod_{i=2s-\bar{s}+1}^{2s} |x_i| \prod_{i=s+1}^{n} |x_i| \leq 2(s + 1)(3n^3)^{s+1} \max_{0 \leq j \leq 3s} |f^{(j)}(0)|,
\]
hence (27) for \(\bar{s} + 1\).

**Step 5:** Now, (27) for \(\bar{s} = s\) reads
\[
\prod_{i=s+1}^{2s} |x_i| \prod_{i=s+1}^{n} |x_i| \leq \max \left( 2(s + 1)(3n^3)^{s+1} \max_{0 \leq j \leq 3s} |f^{(j)}(0)|, (3n^3)^{s} \prod_{i=1}^{n} |x_i| \right) = 2(s + 1)(3n^3)^{s+1} \max_{0 \leq j \leq 3s} |f^{(j)}(0)|,
\]
since \(\prod_{i=1}^{n} |x_i| = f^{(0)}(0)\). Hence,
\[
\prod_{i=s+1}^{n} |x_i|^2 \leq \prod_{s+1}^{2s} |x_i| \prod_{i=s+1}^{n} |x_i| \leq 2(s + 1)(3n^3)^{s+1} \max_{0 \leq j \leq 3s} |f^{(j)}(0)|,
\]
43
that is the first inequality of the Lemma.

**Proof of Lemma 4.8** Define $x_i := \varphi^{-1}(z_i), i = 1, \ldots, n$. Using

$$f(z) = \prod_{i=1}^{n} |\theta, z_i| = \frac{|x_i - x|}{\sqrt{1 + |x_i - x|^2}} = F(x),$$

the inequality

$$||(\partial^j F)(0)|| \leq n^s \prod_{s+1}^{n} |\theta, z_i|$$

is a straightforward calculation.

Again, we may assume that $|\theta, z_1| < \cdots < |\theta, z_n|$, and also that $|\theta, z_i| < 1$, for $i = 1, \ldots, n$ i.e. the $z_i$ are not at infinity with respect to $\theta$. Hence, $|x_1| < \cdots < |x_n|$, and $|x_n| < \infty$.

Since there are only $n$ points $x_i$, there exists a real line through the origin $L \subset \mathbb{C}$ and a permutation $\pi \in \Sigma_n$, such that with $pr_L$ the projection of $\mathbb{C}$ to $L$, and $y_i = pr_L(x_{\pi i})$,

$$|y_1| < \cdots < |y_n|, \quad |x_i| \geq |y_{\pi i}| \geq \frac{|x_i|}{n},$$

and consequently,

$$|y_i| \leq n |y_{\pi i}| \leq n^2 |y_i|, \quad |x_i| \leq n |x_{\pi i}| \leq n^2 |x_i|.$$  

(29)

Let $\partial = \partial_L$ be the directional derivative in the direction of $L$, and $\bar{z}_i$ the point in $\mathbb{P}(\mathbb{C})$ corresponding to $y_i$. Then, with $g(z) = \prod_{i=1}^{n} |\theta, \bar{z}_i|, G = \varphi^* g$, and $s \leq n$,

$$||(\partial^L G)(0)|| \leq n^s ||(\partial^L F)(0)||.$$  

(30)

To prove for $s \leq n/3$ that

$$\prod_{s+1}^{n} |z_i, \theta| \leq 2(s + 1)(3n^3)^{s+1}n^{n-s} \sup_{i \leq 3s} |\partial^i f^s(0)|,$$

assume first that $|\theta, z_s| \geq 1/\sqrt{2}$. Since $\prod_{i=1}^{n} |\theta, z_i| = F(0)$, one may assume

$$\prod_{i=s}^{n} |\theta, z_i| \leq 2s(3n^3)^s n^{n-s+1} \sup_{|I| \leq 3(s-1)} ||(\partial^I F)(0)||.$$  

and then

$$\prod_{i=s+1}^{n} |\theta, z_i| \leq \sqrt{2} \prod_{i=s}^{n} |\theta, z_i| \leq \sqrt{2} 2s(3n^3)^s n^{n-s+1} \sup_{|I| \leq 3(s-1)} ||(\partial^I F)(0)|| \leq$$
Thus, we may from now on assume that $|\theta, z_s| < 1/\sqrt{2}$, hence $|y_s| \leq |x_s| < 1$. Then, with $\bar{x}_i := x_i$ if $|\theta, z_i| < 1/\sqrt{2}$, and $\bar{x}_i := x_i/|x_i| \geq |\theta, z_i|$ Lemma 4.7 implies

$$\prod_{i=s+1}^{n} |\bar{x}_i| \leq (2s + 1)(3n^3)^{s+1} \sup_{0 \leq j \leq 3s} |(\partial^j L F)(0)|,$$

where $G(x) = \prod_{i=1}^{n} |\bar{x}_i|$. Since $|\bar{x}_i| = c_i |\theta, \bar{z}_i|$, with $c_i \leq 1$ and $|\theta, \bar{z}_i| \leq n|\theta, z_i|$ for $i = 1, \ldots, n$, we have $|(\partial^j_L G)(0)| \leq |(\partial^j_L F)(0)|$. Further, $\prod_{i=s+1}^{n} |\theta, z_i| \leq n^n \prod_{i=1}^{n} |\bar{x}_i|$, hence

$$\prod_{i=s+1}^{n} |\theta, z_i| \leq (2s + 1)(3n^3)^{s+1} n^n \sup_{0 \leq j \leq 3s} |(\partial^j L F)(0)|.$$

Since clearly $\sup_{0 \leq j \leq 3s} |(\partial^j_L F)(0)| \leq \sup_{0 \leq j \leq 3s} |(\partial^j F)(0)|$, the Lemma follows.

**Proof of Lemma 4.9:** By the definition of the Fubini-Study metric for any multiindex, and any $i = 1, \ldots, n$

$$|\partial^j \varphi^* (|\theta, \mathbb{P}(W), i)| | \leq 1.$$

The Lemma hence follows from the Leibniz rule.

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