Twisted submersions in nonnegative sectional curvature

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Abstract. In [16], Wilking introduced the dual foliation associated to a metric foliation in a Riemannian manifold with nonnegative sectional curvature and proved that when the curvature is strictly positive, the dual foliation contains a single leaf, so that any two points in the ambient space can be joined by a horizontal curve. We show that the same phenomenon often occurs for Riemannian submersions from nonnegatively curved spaces even without the strict positive curvature assumption and irrespective of the particular metric.

Mathematics Subject Classification (1991). 53C20.

1. Introduction and statements of results. Recall that in a Riemannian manifold $M$ with a metric foliation $\mathcal{F}$, the horizontal curves are those orthogonal to the leaves of $\mathcal{F}$ at every point. Given any point $p \in M$, consider the subset of $M$ that can be reached by horizontal curves emanating from $p$. In general, these subsets have no particularly interesting structure. However, in the presence of nonnegative sectional curvature (and assuming completeness of leaves), Wilking showed in [16] that they form a singular metric foliation, called the dual foliation to $\mathcal{F}$, which we hereafter denote by $\mathcal{F}^\#$. He also showed that, when the metric is positively curved, the dual foliation contains a single leaf, and hence any two points are connected by a horizontal curve.

The simplest metric foliations appear as the collection of fibers of Riemannian submersions; it is therefore natural to study the properties of the dual foliation in this setting. If the total space of the submersion is compact, Theorem 3 in [16] guarantees intrinsic completeness of the dual leaves, and hence $\mathcal{F}^\#$ is a legitimate metric foliation. Moreover, Riemannian submersions also preserve nonnegative sectional curvature and provide most of the known examples of such spaces.

P. Angulo-Ardoy and L. Guijarro were supported by research grants MTM2011-22612 from the Ministerio de Ciencia e Innovación (MCINN) and MINECO: ICMAT Severo Ochoa project SEV-2011-0087.
The aim of this note is to show that relatively mild restrictions on a submersion often results in triviality of the dual foliation. Before describing this in more detail, it is convenient to introduce some terminology:

**Definition 1.** A Riemannian submersion $\pi : M \to B$ is said to be twisted if the foliation dual to the fibers of $\pi$ contains only one leaf.

The term twisted is meant to contrast with the case of Riemannian products $F \times B \to B$, where the dual foliations correspond to the submanifolds $\{p\} \times B$, $p \in F$. For an arbitrary Riemannian submersion $M \to B$, any closed curve in $B$ beginning and ending at $b \in B$ induces a diffeomorphism of the fiber $F = \pi^{-1}(b)$ over $b$ by lifting the curve horizontally to points of $F$. The collection of all these diffeomorphisms forms a group called the **holonomy group** of $\pi$ at $b$. In the case of a Riemannian product $F \times B \to B$, that group is trivial. In general, the holonomy group is not a Lie group. However, one large class of Riemannian submersions are the so-called **homogeneous** ones: if $M$ is a Riemannian manifold and $G$ a compact Lie group acting by isometries on $M$ with principal orbits, then the space $B = G \setminus M$ of orbits inherits in a natural way a Riemannian metric for which the quotient map $\pi : M \to B$ is a Riemannian submersion. In this case, $\pi$ is actually a fiber bundle, the holonomy group at any point is a Lie group, and is, in fact, the structure group of the bundle (see [8]). Thus, if the dual foliation consists of a single leaf, then the structure group acts transitively on the fiber, and the bundle may be thought of as being in essence twisted.

Among the several further reasons why twisted Riemannian submersions are interesting, two in particular stand out: first of all, as mentioned earlier, they automatically occur in positive curvature and could help us understand topological similarities between positive and nonegative sectional curvature; second, they provide a rich extra structure on the space, such as the subriemannian distance on $M$ obtained by considering the infimum of the length of horizontal curves between two points.

In this paper, we provide several sources of twisting for Riemannian submersions. Recall that even non homogenous Riemannian submersions are topological fibrations, and twisting can already be built into the structure of the fibration; this is studied here in terms of homotopy, using the boundary operator of the long exact homotopy sequence of the fibration, and in cohomological terms through the use of transgressive elements. Both are described in Section 2. Next, we consider cases where a submersion is either twisted or else the metric is rigidly constrained; this is illustrated for submersions of the form $\pi : M \times S^2 \to M$ for arbitrary metrics on the product. Section 4 deals with the special case of principal torus bundles, showing that the corresponding quotient map will be twisted whenever the total space is simply connected. Finally, we conclude with some local conditions on the metric that guarantee that the submersion is twisted.

The authors are indebted to the referee for pointing out a mistake in the first version of Theorem 3 and to Kris Tapp for pointing out a mistake in Section 2.2 of the first version of this paper and for telling us about reference [10].

All the manifolds appearing in this paper are considered connected.