The Rigidity and Stability of Gradient Estimates

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Abstract
In this note, we obtain the rigidity of the sharp Cheng–Yau gradient estimate for positive harmonic functions on surfaces with non-negative Gaussian curvature, the rigidity of the sharp Li-Yau gradient estimate for positive solutions to heat equations and the related estimates for Dirichlet Green’s functions on Riemannian manifolds with non-negative Ricci curvature. Moreover, we also obtain the corresponding stability results.

Keywords  Rigidity · Stability · Gradient estimates

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1 Introduction
The connection between harmonic functions and geometry can be traced back to the proof of the uniformization theorem of Riemann surfaces. For manifolds with non-negative Ricci curvature, one important result relying on harmonic functions, is Cheeger–Gromoll’s splitting theorem [3]. Its proof reduces to the fact: the existence of a non-trivial harmonic function with constant norm of gradient results in the splitting of the manifold. This possibly can be viewed as the first geometric application of gradient estimates for harmonic functions.

Dedicated to Professor Peter Li on the occasion of his 70th birthday

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Yau’s Liouville theorem [15] inspired a lot of studies in harmonic functions on Riemannian manifolds with non-negative Ricci curvature. Especially, Cheng-Yau [5] proved the local gradient estimate of harmonic functions, which is not only important as a technical tool, but also provide a universal method to deal with similar analytical objects such as Green’s function, heat kernel [12].

Recall in [6], for the global positive Green’s function $G$ on a complete Riemannian manifold with non-negative Ricci curvature, let $b = G^{\frac{1}{n}}$, we have $|\nabla b| \leq 1$, and $M^n$ is isometric to $\mathbb{R}^n$ if the equality holds at some point in $M^n$. This result also implies: If $G$ and $\overline{G}$ are the global positive Green’s functions on $M$ and $\mathbb{R}^n$ with $n \geq 3$, then $G(p, y) \geq \overline{G}(0, \bar{y})$, where $d(p, y) = d(0, \bar{y})$. Moreover if the equality holds at some point, then $M$ is isometric to $\mathbb{R}^n$. Hence, for the global positive Green’s function, the sharp estimate and the related rigidity are known.

The philosophy of the above rigidity result is: If the sharp bounds of some estimates for certain kinds of functions (gradient estimates of positive harmonic functions, positive solutions to heat equations or Green’s functions) are achieved on some manifold $M^n$, then $M^n$ is isometric to the model space. In the case of non-negative curvature, the model space is usually $\mathbb{R}^n$.

Besides the isometric rigidity, due to Gromov [8], we can discuss the stability of manifolds (in other words, the Gromov–Hausdorff distance between the manifolds and the model spaces) when the estimate is close to be sharp.

In this note, we consider rigidity and stability for the Cheng-Yau gradient estimate of positive harmonic functions on surfaces with non-negative Gaussian curvature, Li-Yau gradient estimates for positive solutions to heat equations and some estimates of Dirichlet Green’s functions on Riemannian manifolds with $Rc \geq 0$. For the stability part, we only focus on local stability, more precisely, the stability of geodesic balls in the manifolds. Our results reveal that the geometry of the domain can be rediscovered by the properties of suitable functions defined on it.

The rest of the paper is organized as follows: We first review some preliminary results in Sect. 2, which will be used to establish the rigidity and stability in later sections. Although the Cheng-Yau gradient estimate for positive harmonic functions is sharp up to decay rate in global sense, it is not sharp pointwisely. The sharp Cheng-Yau gradient on geodesic balls is only known in 2-dim case (see [14]). In this note, we obtain the related rigidity and stability for geodesic balls on surfaces in Sect. 3. It is well known that Li-Yau’s parabolic gradient estimate is sharp as a global inequality on the whole manifold when Ricci curvature is non-negative. We will also establish the rigidity of Li-Yau’s estimate in Sect. 3, which can be viewed as a parabolic analogue of Colding’s rigidity result about global Green’s functions. Finally, inspired by Colding’s rigidity result of global Green’s functions, in Sect. 4, we study the corresponding rigidity result for Dirichlet Green’s functions in geodesic balls, and obtain the stability result for Dirichlet Green’s functions.

## 2 Preliminary for local rigidity and stability

In this section, we recall some preliminary results that will be used in latter sections.
The following lemma is well known in Riemannian geometry. We present it here for completeness.

**Lemma 2.1** Let \((M^n, g)\) be a complete Riemannian manifold with \(Rc(g) \geq 0\), \(p \in M\) and \(\Omega\) be a star-shaped domain with respect to \(p\). Let \(r(x) = d(p, x)\). If \(\Delta r(x) = \frac{n - 1}{r(x)}\) for any \(x \in \Omega \setminus \{p\}\) and \(\Omega\) contains no cut-point of \(p\), then \(\Omega\) is isometric to a star-shaped domain \(\hat{\Omega}\) in \(\mathbb{R}^n\).

**Proof** Applying the Bochner formula to \(r\), we get

\[
0 = \frac{1}{2} \Delta |\nabla r|^2 = |\nabla^2 r|^2 + \langle \nabla \Delta r, \nabla r \rangle + Rc(\nabla r, \nabla r) \geq |\nabla^2 r|^2 - \frac{n - 1}{r^2}. \tag{2.1}
\]

On the other hand, by the Cauchy-Schwartz inequality,

\[
|\nabla^2 r|^2 \geq \left(\frac{\Delta r}{n - 1}\right)^2 = \frac{n - 1}{r^2}. \tag{2.2}
\]

Combining (2.2) and (2.1), we know that the equality of (2.2) holds and hence

\[
\nabla^2 r = \frac{1}{r} (g - dr \otimes dr) \tag{2.3}
\]

in \(\Omega\). For any normal minimal geodesic \(\gamma\) in \(\Omega\) starting from \(p\), let \(E_1 = \gamma'\), \(E_2, \ldots, E_n\) be a parallel orthonormal frame along \(\gamma\). Then, by (2.3)

\[
r_{11} = r_{1i} = 0 \quad \text{and} \quad r_{ij} = \frac{1}{r} \delta_{ij} \tag{2.4}
\]

for any \(i, j \geq 2\). Moreover,

\[
0 = \frac{1}{2} \left( |\nabla r|^2 \right)_{ij} = r_{ijk} r_k + r_{kl} r_{kj} + R_{iklj} r_k r_l
\]

\[
= \frac{dr_{ij}}{dr} + r_{ki} r_{kj} + R_{ikjl} r_k r_l = R_{iklj} r_k r_l.
\]

Thus,

\[
R(E_i, \gamma', \gamma', E_j) = 0
\]

for any \(i, j = 1, 2, \ldots, n\). Finally, by Cartan’s isometry theorem (see [9, Theorem 1.12.8]), \(\exp_p^{-1} : \Omega \to \hat{\Omega} \subset T_pM \cong \mathbb{R}^n\) is an isometry.

The following theorem follows from [2], which will be used in the rest of the paper.
Theorem 2.1 On a complete Riemannian manifold \((M^n, g)\) with \(Rc \geq 0\), let \(r(x) = d(p, x)\) and if
\[
\int_{B_1(p)} \left| \nabla^2 \left( r^2 \right) - 2g \right| \leq \delta,
\]
then \(B_1(p)\) is \(\psi(\delta)\)-Gromov–Hausdorff close to \(B_1(0)\), where \(\lim_{\delta \to 0} \psi(\delta) = 0\). Especially, if \(\nabla^2 (r^2) - 2g \equiv 0\), then \(B_1(p)\) is isometric to \(B_1(0)\).

**Proof** It follows from [2, Proposition 2.80] and [2, Theorem 3.6]. ⊓⊔

Lemma 2.1 If there is a smooth function \(f\) defined on a complete Riemannian manifold \((M^n, g)\) with \(Rc \geq 0\) such that \(\nabla^2 f = c \cdot g\), where \(c > 0\) is some constant, then \((M^n, g)\) is isometric to \(\mathbb{R}^n\) and \(f(x) = \frac{c}{2} d(p, x)^2 + f(p)\) for some \(p \in M^n\).

**Proof** It is the result of [2, §1]. ⊓⊔

From [2, Theorem 6.33], we have the following existence of cut-off functions.

Lemma 2.2 On a complete Riemannian manifold \((M^n, g)\) with \(Rc \geq 0\), there is a non-negative cut-off function \(\phi\) with \(\phi \leq 1\), such that \(\phi \big|_{B_1^2(p)} \equiv 1\), \(\text{spt}(\phi) \subseteq B_1(p)\), \(|\nabla \phi| \leq C\), and \(|\Delta \phi| \leq C\), where \(C\) is some universal constant.

Finally, by reducing to Theorem 2.1, we have the following local stability for Riemannian surfaces in terms of \(\Delta r\).

Lemma 2.3 On a complete Riemannian surface \((M^2, g)\) with \(Rc \geq 0\), let \(r(x) = d(p, x)\) and if
\[
\int_{B_1(p)} \left| \Delta \left( r^2 \right) - 4 \right| \leq \delta,
\]
then \(B_1(p)\) is \(\psi(\delta)\)-Gromov-Hausdorff close to \(B_1(0)\), where \(\lim_{\delta \to 0} \psi(\delta) = 0\).

**Proof** By Hessian comparison, \(2g - \nabla^2 (r^2)\) is non-negative. So
\[
4 - \Delta \left( r^2 \right) \geq \left| \nabla^2 \left( r^2 \right) - 2g \right|.
\]
Then, the conclusion follows from Theorem 2.1 directly. ⊓⊔

### 3 Rigidity and stability of Cheng-Yau and Li-Yau estimate

In this section, we prove the rigidity and stability result for Cheng-Yau gradient estimates on surfaces, and the rigidity of Li-Yau gradient estimate.

We first prove the rigidity of the sharp Cheng-Yau gradient for complete Riemannian surfaces with non-negative Ricci curvature. In fact, we also provide a simpler proof of [14, Theorem 3.4].
Theorem 3.1 Let \((M^2, g)\) be a complete Riemannian surface with \(Rc \geq 0\) and \(u\) be a positive harmonic function on \(B_1(p)\). Then the follows hold:

(1) Let \(r(x) = d(p, x)\),

\[
|\nabla \ln u|(x) \leq \frac{2}{1 - r^2(x)}.
\]  \(\text{(3.1)}\)

(2) Furthermore, if there is a point \(x_0 \in B_1(p)\) such that

\[
|\nabla \ln u|(x_0) = \frac{2}{1 - r^2(x_0)},
\]

then \(B_1(p)\) is isometric to the unit ball in \(\mathbb{R}^2\) and \(u(x) = P(x, y)\) for some \(y \in \partial B_1(p)\), where \(P(x, y)\) is the Poisson kernel of \(B_p(1)\).

(3) For \(\delta > 0\), if

\[
\int_{B_1(p)} \left( \frac{4}{(1 - r^2)^2} - |\nabla \ln u|^2 \right) + \ln \left( \frac{2}{(1 - r^2)|\nabla \ln u|} \right) \leq \delta,
\]

then \(B_1(p)\) is \(\psi(\delta)\)-Gromov–Hausdorff close to \(B_1(0)\) with \(\lim_{\delta \to 0} \psi(\delta) = 0\).

Remark 3.1 The inequality (3.1) was proved in [14] by a carefully chosen cut-off function. Here, we firstly give a more direct proof of (3.1) without using cut-off functions, and then use the strong maximum principle to get the rigidity result.

Proof (1) It was shown in [14] that for any positive harmonic function \(u\) on \(B_1(p)\) in a complete Riemannian surface \((M^2, g)\) with non-negative Ricci curvature, one has

\[
Q \Delta Q \geq 2Q^3 + \|\nabla Q\|^2 \quad \text{(3.2)}
\]

where \(Q = \|\nabla \ln u\|^2\). Let \(v = \ln Q\). Then,

\[
\Delta v \geq 2e^v \quad \text{(3.3)}
\]

on where \(Q > 0\). For \(\epsilon \in (0, 1)\), let

\[
v_\epsilon = 2 \ln \frac{2}{(1 - \epsilon)^2 - r^2}.
\]

By Laplacian comparison,

\[
\Delta v_\epsilon \leq 2e^{v_\epsilon} \quad \text{(3.4)}
\]

We claim that \(v \leq v_\epsilon\) on \(B_{1-\epsilon}(p)\). Then, by letting \(\epsilon \to 0^+\), we will get (1). If the claim is not true, then there is \(x_0 \in B_{1-\epsilon}(p)\) such that

\[
v(x_0) - v_\epsilon(x_0) = \max_{x \in B_{1-\epsilon}(p)} (v - v_\epsilon) > 0.
\]
by noting that $v$ is upper semi-continuous on $B_1(p)$ and $v_{ε}|_{∂B_{1−ε}(p)} = +∞$. It is also clear that $Q(x_0) > 0$ (Otherwise $v(x_0) = −∞$ which is impossible since $x_0$ is the maximum point of $v−v_{ε}$). If $x_0$ is not a cut-point of $p$, then, by (3.3) and (3.4),

$$0 ≥ \Delta(v−v_{ε})(x_0) ≥ 2e^{v(x_0)}−2e^{v_{ε}(x_0)} > 0$$

(3.5)

which is a contradiction. When $x_0$ is a cut-point of $p$, the classical Calabi trick (see [1]) will also give us a contradiction similarly. This complete the proof of (1).

(2) Let $w(x) = 2 \ln \frac{2}{1−r(x)^2}$. If $v = w$ for some point in $B_1(p)$, let

$$Ω = \{x ∈ B_1(p) | v(x) = w(x)\}$$

(3.6)

It is then clear that $Ω$ is non-empty and closed in $B_1(p)$ by continuity. Moreover, for any $x_0 ∈ Ω$, because $v(x_0) = w(x_0)$, there is an open neighborhood $B_{δ}(x_0)$ of $x_0$, such that $Q > 0$ in $B_{δ}(x_0)$. Note that

$$0 ≤ (Δv − 2e^{v}) − (Δw − 2e^{w}) = Δ(v − w) − c(x)(v−w)$$

(3.7)

with

$$c(x) = \begin{cases} 2(e^{v(x)}−e^{w(x)}) & v(x) ≠ w(x) \\ 2e^{v(x)} & v(x) = w(x). \end{cases}$$

in $B_{δ}(x_0)$. Since $c(x) > 0$, by the strong maximum principle (see [1] or [7, Theorem 8.19]), we know that $B_{δ}(x_0) ⊂ Ω$. This implies that $Ω$ is open. Now, $Ω$ as a non-empty open and closed subset of $B_1(p)$ must be $B_1(p)$ since $B_1(p)$ is connected. Hence, $v = w$ all over $B_1(p)$ and moreover the equality of the Laplacian comparison holds. So, $B_1(p)$ is flat and

$$r^2(x) = 1 − 2e^{−\frac{v(x)}{2}}$$

is smooth in $B_1(p)$ which implies that there is no cut-point of $p$ in $B_1(p)$ (If there is a cut-point $q$ of $p$ in $B_1(p)$, then $q$ is not a conjugate point of $p$ since $B_1(p)$ is flat. So, there are at least two different minimal geodesics joining $p$ to $q$ which contradicts that $∇r$ is well defined at $q$ since $r$ is smooth at $q$). Now, by Lemma 2.1, $B_1(p)$ is isometric to the ball $B_1(0)$ in $R^2$. Finally by [14], $u$ is the Poisson kernel.

(3) It is straightforward to verify that

$$−Δw + 2e^{w} ≥ \left(1 − r^2\right)(−Δw + 2e^{w}) = 2\left(4 − Δ\left(r^2\right)\right).$$

So, by (3.3),

$$Δ(v−w) + 2(e^{w} − e^{v}) ≥ 2\left(4 − Δ\left(r^2\right)\right).$$
Let $\phi$ be the cut-off function in Lemma 2.2, and note that $4 - \Delta (r^2) \geq 0$ by Laplacian comparison and $w - v \geq 0$ by (1). Then

$$2 \int_{B_1(p)} \left| 4 - \Delta \left( r^2 \right) \right| \leq \int_{B_1(p)} \phi \cdot \left( \Delta (v - w) + 2(e^w - e^v) \right) = 2 \int_{B_1(p)} \left( \frac{4}{(1 - r^2)^2} - |\nabla \ln u|^2 \right) - \int_{B_1(p)} (w - v) \Delta \phi \leq 2 \int_{B_1(p)} \left( \frac{4}{(1 - r^2)^2} - |\nabla \ln u|^2 \right) + 2 \sup_{B_1(p)} |\Delta \phi| \int_{B_1(p)} \ln \left( \frac{2}{(1 - r^2)|\nabla \ln u|} \right).$$

Hence,

$$\int_{B_1(p)} \left| 4 - \Delta \left( r^2 \right) \right| \leq C \int_{B_1(p)} \left( \frac{4}{(1 - r^2)^2} - |\nabla \ln u|^2 \right) + \ln \left( \frac{2}{(1 - r^2)|\nabla \ln u|} \right).$$

Now the conclusion follows from the above inequality and Lemma 2.3. \qed

Next, we come to prove the rigidity for the Li-Yau gradient estimate on complete Riemannian manifolds with non-negative Ricci curvature. We first recall the following Li-Yau’s theorem (see [12]).

**Theorem 3.2** Let $u$ be a positive solution of the heat equation on the complete Riemannian manifold $(M^n, g)$ with $Rc \geq 0$, then

$$(\ln u)_t - |\nabla \ln u|^2 + \frac{n}{2t} \geq 0.$$  

The rigidity result of the Li-Yau gradient estimate on complete Riemannian manifolds with non-negative Ricci curvature was obtained in [13], we present another proof here.

**Theorem 3.3** Let $u$ be a positive solution of the heat equation on the complete Riemannian manifold $(M^n, g)$ with $Rc \geq 0$, and

$$(\ln u)_t - |\nabla \ln u|^2 + \frac{n}{2t} = 0$$

at some point $(x_0, t_0) \in M^n \times \mathbb{R}^+$. Then $(M^n, g)$ is isometric to $\mathbb{R}^n$, and $u(x, t)$ is a multiple of the heat kernel of $\mathbb{R}^n$. 

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Proof  Let \( f = -\ln u \). Then
\[
(2t \Delta f - n)(x_0, t_0) = 0.
\]
Let \( G = t \Delta f - \frac{n}{2} \). It is straightforward to get
\[
\left( \frac{\partial}{\partial t} - \Delta \right) (tG) = -2 \left( tG \right)^2 - 2 \nabla f \cdot \nabla (tG) - 2t \cdot Rc(\nabla f, \nabla f) - \frac{t^2}{n} \left( \sum_{i<j} (f_{ii} - f_{jj})^2 \right),
\]
\[
\leq -2 \nabla f \cdot \nabla (tG).
\]
(3.8)

By Theorem 3.2, we get
\[
t_0 G(x_0, t_0) = t \left( t \Delta f - \frac{n}{2} \right)(x_0, t_0) = \max_{(x,t) \in M^n \times \mathbb{R}^+} tG(x, t).
\]

From the strong maximum principle of parabolic PDE (see [4, Theorem 12.40]), \( tG \equiv 0 \) and hence
\[
\Delta f = \frac{n}{2t}
\]
all over \( M^n \times (0, t_0] \). Plugging this into (3.8), we have
\[
Rc(\nabla f, \nabla f) \equiv 0 \quad \text{and} \quad \nabla^2 f = \frac{1}{2t} g.
\]

Combining the above with Lemma 2.1, we get that \( M^n \) is \( \mathbb{R}^n \), and
\[
f(x, t) = \frac{|x - p(t)|^2}{4t} + f(p(t), t),
\]
(3.9)

where \( p(t) \in \mathbb{R}^n \). Substituting (3.9) into the equation of \( f \):
\[
-f_i - |\nabla f|^2 + \frac{n}{2t} = 0,
\]
we get
\[
\frac{n}{2t} - \frac{\partial f}{\partial t}(p(t), t) = -\frac{(x - p(t)) \cdot p'(t)}{2t}.
\]

Hence \( \frac{d}{dt} p(t) = 0 \) and \( \frac{d}{dt} f(p(t), t) = \frac{n}{2t} \). Thus \( p(t) = p \) is a constant point and
\[
f(x, t) = \frac{|x - p|^2}{4t} + \frac{n}{2} \ln t + C
\]
for some constant \( C \in \mathbb{R} \). Therefore \( u = e^{-f} \) is a multiple of the heat kernel. \( \square \)
As a direct consequence of the rigidity for Li-Yau gradient estimate, we have the following rigidity for Li-Yau’s sharp Harnack inequality of positive solutions to the heat equation on complete Riemannian manifolds with non-negative Ricci curvature (see [12]).

**Corollary 3.1** Let $u$ be a positive solution of the heat equation on the complete Riemannian manifold $(M^n, g)$ with $Rc \geq 0$. If there exist $x_1, x_2 \in M^n$ and $0 < t_1 < t_2$, such that

$$u(x, t_1) = u(x_2, t_2) \left( \frac{t_2}{t_1} \right)^{n/2} \exp \left( \frac{d(x_1, x_2)^2}{4(t_2 - t_1)} \right).$$

Then $(M^n, g)$ is isometric to $\mathbb{R}^n$ and $u(x, t)$ is a multiple of the heat kernel of $\mathbb{R}^n$.

**Proof** Let $γ : [t_1, t_2] \to M$ be a minimal geodesic joining $x_1$ to $x_2$. Then, by Theorem 3.2 and noting that $|γ'| = \frac{d(x_1, x_2)}{t_2 - t_1}$,

$$- \frac{n}{2} \ln \frac{t_2}{t_1} - \frac{d^2(x_1, x_2)}{4(t_2 - t_1)} = \ln u(x_2, t_2) - \ln u(x_1, t_1)$$

$$= \int_{t_1}^{t_2} \left( \frac{d}{dt} \ln u(γ(t), t) \right) dt$$

$$= \int_{t_1}^{t_2} \left( (\ln u)_t(γ(t), t) + (\nabla \ln u, γ'(t)) \right) dt$$

$$\geq \int_{t_1}^{t_2} \left( |\nabla \ln u|^2(γ(t), t) - \frac{n}{2t} - \frac{d(x_1, x_2)}{t_2 - t_1} |\nabla \ln u|(γ(t), t) \right) dt$$

$$= \int_{t_1}^{t_2} \left( \left( |\nabla \ln u|^2(γ(t), t) - \frac{d(x_1, x_2)}{2(t_2 - t_1)} \right)^2 - \frac{n}{2t} - \frac{d^2(x_1, x_2)}{4(t_2 - t_1)^2} \right) dt$$

$$\geq - \frac{n}{2} \ln \frac{t_2}{t_1} - \frac{d^2(x_1, x_2)}{4(t_2 - t_1)}.$$

Thus,

$$(\ln u)_t - |\nabla \ln u|^2 + \frac{n}{2t} = 0$$

on $(γ(t), t)$ with $t_1 \leq t \leq t_2$. Then, by Theorem 3.3, we complete the proof of the corollary. \qed

**4 Rigidity and Stability for Dirichlet Green’s Function**

For global Green’s functions and heat kernels, there are sharp estimates (see [6, 11, 12]). In this section, we study the sharp estimate and rigidity property of Dirichlet
Green’s functions, which can be viewed as a local version of Colding’s rigidity theorem for global positive Green’s functions.

We first recall the definition of Dirichlet Green’s functions.

**Definition 4.1** On a compact Riemannian manifold \((M^n, g)\) with boundary \(\partial M^n \neq \emptyset\), the **Dirichlet Green’s function** \(G(x, y)\) is a symmetric function defined on \((M^n \times M^n) \setminus \mathcal{D}(M^n)\) with the properties:

(i) for any \(f \in C^\infty(M^n)\) satisfying \(f|_{\partial M} = 0\),

\[
\int_M G(x, y) \Delta f(y) dy = -f(x) \tag{4.1}
\]

(ii) \(G(x, y) > 0\) on \((M \setminus \partial M) \times (M \setminus \partial M)\).

(iii) \(G(x, y) = 0\) for any \(y \in \partial M^n, x \in M^n \setminus \partial M^n\).

Here \(\mathcal{D}(M^n) = \{(x, x) : x \in M^n\} \subset M^n \times M^n\).

For Dirichlet Green’s functions, we have the following rigidity result related to the gradient of the Dirichlet Green functions, which only applies to higher dimensional case (the dimension is \(n \geq 3\)).

**Lemma 4.1** On a complete Riemannian manifold \((M^n, g)\) with \(n \geq 3\) and \(Rc \geq 0\), let \(G(x, y)\) be the Dirichlet Green’s function of \(B_1(p)\). Define

\[
b(y) = (C(n)G(p, y) + 1)^{\frac{1}{n-1}},
\]

where \(C(n) = (2 - n)n\omega_n\). Then,

\[
\sup_{y \in \partial B_1(p)} |\nabla b(y)| \geq \frac{n\omega_n}{V(\partial B_1(p))}.
\]

In particular, if \(\sup_{y \in \partial B_1(p)} |\nabla b(y)| \leq 1\), then \(B_1(p)\) is isometric to \(B_1(0) \subseteq \mathbb{R}^n\).

**Proof** Note for \(y \in \partial B_1(p)\), we have

\[
\frac{\partial b(y)}{\partial n_y} = n\omega_n \frac{\partial G(p, y)}{\partial n_y},
\]

where \(\partial n_y\) is unit inward normal vector at \(y\). Recall that

\[
\int_{\partial B_1(p)} \frac{\partial G(p, y)}{\partial n_y} = 1.
\]

So

\[
\sup_{y \in \partial B_1(p)} |\nabla b(y)| \cdot V(\partial B_1(p)) \geq \int_{\partial B_1(p)} |\nabla b| = n\omega_n \int_{\partial B_1(p)} \left| \frac{\partial G(p, y)}{\partial n_y} \right| \geq n\omega_n.
\]
The first conclusion then follows. From the rigidity part of the Bishop–Gromov’s volume comparison theorem, we get the second conclusion. □

Note that we do not have the global rigidity result as [6] for the case $n = 2$ because of the following example.

**Example 4.1** Let $M$ be a rotationally symmetric Riemannian surface with center $p$. The metric of $M$ in polar coordinate is $g = dr^2 + f(r)^2 d\theta^2$. Then by direct computation, we get the Green function of $M$ at point $p$ has the form $G(p, y) = -\int_1^{r(y)} \frac{1}{f(r)} \, dr$. If $M$ is Euclidean near $p$, then the Green function of $M$ equals to the Green function of $\mathbb{R}^2$ near $p$, but $M$ may be not isometric to $\mathbb{R}^2$.

Now we discuss the sharp $C^0$ estimate of Dirichlet Green’s functions and its rigidity and stability result. One difference is that we have the rigidity for all dimension $n \geq 2$ and stability only for $n = 2$.

**Theorem 4.1** On a complete Riemannian manifold $(M^n, g)$ with $Rc \geq 0$, let $G(x, y)$ be the Dirichlet Green’s function of $B_1(p)$, and

$$G(p, y) = \begin{cases} \frac{d(p, y)^2-n-1}{n(n-2)\omega_n}, & n \geq 3, \\ -\frac{\ln d(p, y)}{2\pi}, & n = 2. \end{cases}$$

Then

(1) $G(p, y) \geq G(p, y)$;

(2) if $G(p, y) = G(p, y)$ holds at some point $y \in B_1(p)$, then $B_1(p)$ is isometric to $B_1(0) \subseteq \mathbb{R}^n$;

(3) If $n = 2$ and $\int_{B_1(p)} |G(p, y) - G(p, y)| \leq \delta$, then $B_{\frac{1}{2}}(p)$ is $\psi(\delta)$-Gromov–Hausdorff close to $B_{\frac{1}{2}}(0)$ with $\lim_{\delta \to 0} \psi(\delta) = 0$.

**Remark 4.1** Because $G(x, y) = \int_1^\infty H(x, y, t) \, dt$, the inequality and the rigidity result for Dirichlet Green’s functions can be derived from [10, Lemma 13.6]. Here we give an elliptic proof for it.

**Proof** (1) By Laplacian comparison, we have $\Delta G \geq 0$. It is well-known that

$$\lim_{x \to p} \frac{G(p, x)}{G(p, y)} = 1.$$ 

Hence for any $\alpha > 0$, there exist $\delta > 0$ sufficiently small, such that

$$(1 + \alpha)G(p, y) - G(y) > 0$$

for $y \in \partial B_\delta(p)$. Moreover,

$$(1 + \alpha)G(p, y) - G(y) = 0$$
for \( y \in \partial B_1(p) \). So by maximum principle we have

\[
(1 + \alpha)G(p, y) \geq \bar{G}(y)
\]
on \( B_1(p) \setminus B_\delta(p) \). Letting \( \alpha \to 0, \delta \to 0 \), we get \( G(p, y) \geq \bar{G}(y) \).

(2) If the equality of (1) holds at some point, then

\[
G(p, y) = \bar{G}(y)
\]
for \( y \in B_1(p) \setminus \{p\} \) by the strong maximum principle. Thus \( B_1(p) \) is isometric to \( B_1(0) \subset \mathbb{R}^n \) by Lemma 2.1.

(3) Let \( r(x) = d(p, x) \). For simplicity, we write \( G(p, \cdot) \) and \( \bar{G}(p, \cdot) \) as \( G \) and \( \bar{G} \).

By direct computation,

\[
2n\omega_n r^n \Delta G = 2n - \Delta \left( r^2 \right).
\]

So,

\[
2n - \Delta \left( r^2 \right) = 2n\omega_n r^n \Delta \left( \bar{G} - G \right).
\]

Now let \( \phi \) be the cut-off function as in Lemma 2.2. Then, by the Laplacian comparison, and noting that \( \Delta \bar{G} \geq 0 \),

\[
\begin{align*}
\int_{B_1(p)} \left| 2n - \Delta \left( r^2 \right) \right| \\
\leq 2n\omega_n \int_{B_1(p)} r^n \Delta \left( \bar{G} - G \right) \cdot \phi \\
\leq 2n\omega_n \int_{B_1(p)} \Delta \left( \bar{G} - G \right) \cdot \phi = 2n\omega_n \int_{B_1(p)} \left( \bar{G} - G \right) \Delta \phi \\
\leq 2n\omega_n \sup_{B_1(p)} |\Delta \phi| \int_{B_1(p)} |\bar{G} - G|.
\end{align*}
\]

Then, the conclusion follows from Lemma 2.3 and the above estimate. \( \square \)

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