ON SEMICONVEXITY

AND WEAK SEMICONVEXITY

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Abstract. Properties of two classes of generally convex sets in the space \( \mathbb{R}^n \), called \( m \)-semiconvex and weakly \( m \)-semiconvex, \( 1 \leq m < n \), are investigated in the present work. In particular, it is established that an open set with smooth boundary in the plan which is weakly 1-semiconvex but not 1-semiconvex consists minimum of four simply connected components.

Key words. Convex set, smooth boundary, real Euclidean space, open (closed) set, neither open nor closed set.
1 Introduction

The class of $m$-semiconvex sets is one of the classes of generally convex sets. The semiconvexity notion was proposed by Yu. Zeliskii \[4\] and it was used in the formulation of a shadow problem generalization. The shadow problem was proposed by G. Khudaiberganov \[9\] in 1982 and is the following: To find the minimal number of open (closed) balls in the space $\mathbb{R}^n$ that are pairwise disjoint, centered on a sphere $S^{n-1}$ (see \[8\]), do not contain the sphere center, and such that any straight line passing through the sphere center intersects at least one of the balls. To formulate the generalized shadow problem, first, let us give the following definitions which we also use in our investigation.

Any $m$-dimensional affine subspace of Euclidean space $\mathbb{R}^n$, $m < n$, is called an $m$-dimensional plane.

**Definition 1.** One of two parts of an $m$-dimensional plane of the space $\mathbb{R}^n$, $n \geq 2$, into which it is divided by its any $(m - 1)$-dimensional plane, is said to be an $m$-dimensional half-plane.

An $m$-dimensional half-plane can be either open or closed. An open $m$-dimensional half-plane is one of two open sets produced by the subtraction of an $(m - 1)$-dimensional plane from an $m$-dimensional plane. A closed $m$-dimensional half-plane is the union of an open $m$-dimensional half-plane and the $(m - 1)$-dimensional plane that defines it.

For instance, the 1-dimensional half-plane is a ray, the 2-dimensional half-plane is a half-plane, etc.

**Definition 2.** (\[3\]) A set $E \subset \mathbb{R}^n$ is called $m$-semiconvex with respect to a point $x \in \mathbb{R}^n \setminus E$, $1 \leq m < n$, if there exists an $m$-dimensional half-plane $L$ such that $x \in L$ and $L \cap E = \emptyset$.

**Definition 3.** (\[3\]) A set $E \subset \mathbb{R}^n$ is called $m$-semiconvex, $1 \leq m < n$, if it is $m$-semiconvex with respect to every point $x \in \mathbb{R}^n \setminus E$.

One can easily see that both definitions satisfy the axiom of convexity: The intersection of each subfamily of these sets also satisfies the definition. Thus, for any set $E \subset \mathbb{R}^n$ we can consider the minimal $m$-semiconvex set containing $E$. This set is called the $m$-semiconvex hull of set $E$. 

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The generalized shadow problem is *To find the minimum number of pairwise disjoint closed (open) balls in $\mathbb{R}^n$ (centered on the sphere $S^{n-1}$ and whose radii are smaller than the radius of the sphere) such that any ray starting at the center of the sphere necessarily intersects at least one of these balls.*

In the terms of $m$-semiconvexity this problem can be reformulated as follows: *What is the minimum number of pairwise disjoint closed (open) balls in $\mathbb{R}^n$ whose centers are located on a sphere $S^{n-1}$ and the radii are smaller than the radius of this sphere such that the center of the sphere belongs to the 1-semiconvex hull of the family of balls?*

In the paper [4] the problem is solved as $n = 2$. And only the sufficient number of the balls is indicated as $n = 3$.

In the 60’s L. Aizenberg and A. Martineau proposed their notions of a linearly convex set in the multi-dimensional complex space $\mathbb{C}^n$. The first author considered domains and their closures and used boundary points of the domains in his definition [1]-[2]. The second one used all points of the addition to a set of the space $\mathbb{C}^n$ [6]. If one uses these definitions not only for domains and compact sets, then Aizenberg’s definition isolates one connected component of a set linearly convex in the sense of Martineau.

Guided by similar reflections, Yu. Zeliskii suggested to distinguish $m$-semiconvex and weakly $m$-semiconvex sets and obtained the following results.

**Definition 4.** ([5]) An open set $G \subset \mathbb{R}^n$ is called *weakly $m$-semiconvex*, $1 \leq m < n$, if it is $m$-semiconvex for any point $x \in \partial G$. A set $E \subset \mathbb{R}^n$ is called *weakly $m$-semiconvex* if it can be approximated from the outside by a family of open weakly $m$-semiconvex sets.

**Theorem 1.** ([5]) Let a set $E \subset \mathbb{R}^2$ be weakly 1-semiconvex and not 1-semiconvex. Then set $E$ is disconnected.

In [5] there was also made the assumption that a weakly 1-semiconvex and not 1-semiconvex set consists of not less than three components. This proposition was proved in [7].

**Theorem 2.** ([7]) Let a set $E \subset \mathbb{R}^2$ be weakly 1-semiconvex and not 1-semiconvex. Then $E$ consists of not less than three components.
The present work proceeds the research of Yu. Zelinskii and contains the investigation of properties of 1-semiconvex and weakly 1-semiconvex sets with smooth boundary of the plane.

Except open and closed sets, a set that contains a subset $Q$ of the set $B$ of its boundary points and does not contain $B \setminus Q$ is also used in this work. The expression "neither open nor closed set" will stand for such a set.

2 Main results

We consider a class of weakly $m$-semiconvex sets by the following definition:

**Definition 5.** A set $E \subset \mathbb{R}^n$ is called weakly $m$-semiconvex, $1 \leq m < n$, if for any point $x \in \partial E$ there exists an open $m$-dimensional half-plane $L$ such that $x \in L$ and $L \cap E = \emptyset$.

Classes of open sets, weakly $m$-semiconvex by Definition 4 and Definition 5, coincide. However, a set containing closed or neither open nor closed connected components which is not weakly $m$-semiconvex by Definition 4 can belong to the class of weakly $m$-semiconvex sets with respect to Definition 5, Figure 4, 6. And conversely, there are sets weakly $m$-semiconvex by Definition 5 but not weakly $m$-semiconvex with respect to Definition 5, as illustrated in Figure 1.

Everywhere further we consider weakly $m$-semiconvex sets with respect to Definition 5. We say that there exists (can be drawn) an $m$-dimensional half-plane starting at a point and satisfying some conditions if there exists an open $m$-dimensional half-plane satisfying the conditions and which boundary contains the point.

Let us provide a number of accessory propositions.

**Proposition 1.** If an open set $E \subset \mathbb{R}^n$ is $m$-semiconvex, then it is weakly $m$-semiconvex.

It fails for closed and neither open nor closed sets. Figure 1 shows examples of sets that are 1-semiconvex but not weakly 1-semiconvex.
Proposition 2. A weakly \((n-1)\)-semiconvex \(((n-1)\)-semiconvex \) set \(E \subset \mathbb{R}^n\) consists of simply connected components.

Proposition 3. There exist weakly 1-semiconvex sets in the plane which are not 1-semiconvex.

Example 1. Figure 2 a) shows the set \(E\) that consists of four open rectangles with some common tangents. For any boundary point of set \(E\) there exists a ray that does not intersect \(E\) but for any point of the interior of rhombus \(ABCD\) such a ray can not be found. In Figure 2 b), the set consisting of three open components is weakly 1-semiconvex, but for any point of the interior of triangle \(ABC\) we can not find a ray that does not intersect the set.
**Theorem 3.** Let a set \( E \subset \mathbb{R}^2 \) be weakly 1-semiconvex and not 1-semiconvex. Then set \( E \) is disconnected.

**Proof.** Since set \( E \) is weakly 1-semiconvex and not 1-semiconvex, there is a point \( x \in \mathbb{R}^2 \setminus E \) such that any ray starting at point \( x \) intersects set \( E \). We now fix two complementary (lying on the same straight line) rays \( s_1, s_2 \) starting at point \( x \). They intersect \( \partial E \) at some points \( x', x'' \) that are nearest to \( x \) along rays \( s_1, s_2 \) respectively. But, since \( E \) is a weakly 1-semiconvex set, it follows that for points \( x', x'' \) there exist rays that do not cross \( E \). The polygonal chain containing rays \( s_1, s_2 \), with vertexes at points \( x', x'' \), cuts then the plane into two parts if it is simple (without self-intersections) (case 1) or into three parts if the polygonal chain is self-intersecting (case 2).

If in case 1 the whole set \( E \) is contained in one part of the plane, then for point \( x \) we can always draw a ray lying in the other part. Such a ray does not intersect \( E \), which contradicts the lemma conditions.

In case 2 set \( E \) can not be contained in one part of the plane too. Indeed, if the whole set \( E \) is contained in part I (see Figure 3), then ray \( s \) starting at the point \( x \) and passing through the point of intersection of rays \( s_1, s_2 \) is a ray that does not intersect \( E \). If set \( E \) is in one of the other parts, then the ray complementary to \( s \) can be the required ray.

![Fig. 3](image-url)

The idea of the proof of Lemma 1 belongs to Yu. Zelinskii ([5]).

Theorem 2 is true in the sense of Definition 4. But there are sets that are weakly 1-semiconvex but not 1-semiconvex by Definition 5 and consisting of two components, as the following example shows.
Example 2. Let a set \( E \subset \mathbb{R}^2 \) consist of two closed components with common tangent line and suppose \( E \) is not smooth at the points \( A, B \), Figure 4. It is weakly 1-semiconvex but not 1-semiconvex at all points of the open segment \((A, B)\).

Here we propose our own result, similar to Theorem 2, but for sets with smooth boundary.

Theorem 4. Let \( E \subset \mathbb{R}^2 \) be an open (closed), weakly 1-semiconvex and not 1-semiconvex set with smooth boundary. Then set \( E \) consists of more than two simply connected components.

Proof. Set \( E \) is disconnected by Theorem 3 and all its components are simply connected by Proposition 2. Suppose set \( E \) consists of two simply connected components \( E_1, E_2 \). First, we consider the case when \( E \) is open.

Since set \( E \) is not 1-semiconvex and weakly 1-semiconvex, it follows that there is a point \( x \in \mathbb{R}^2 \setminus E \) such that any ray starting at \( x \) intersects set \( E \) and its boundary \( \partial E \) as well. Suppose a ray \( l_1 \) starting at point \( x \) and intersecting set \( E \) crosses \( \partial E \) at some point, nearest to \( x \) along the ray. Since set \( E \) is weakly 1-semiconvex, there is a ray \( l \) starting at this point and not intersecting \( E \).

Among all rays starting at point \( x \) and crossing ray \( l \), there exist such rays that intersect \( E \) before they intersect \( l \). A case when all these rays intersect \( E \) after they intersect \( l \) is possible only when \( E \) is unbounded. But in this case there exists the
ray that is parallel to the ray $l$ and does not intersect $E$, which contradicts the conditions of the theorem.

For definiteness, suppose the rays from the previous paragraph initially intersect component $E_1$. Then from point $x$ we can draw a ray $s_1$ that is tangent to $E_1$ at some point $y_1$ and does not intersect $E_1$. By the way, the extreme positions of ray $s_1$ is ray $l_1$. Suppose $s_1$ intersects $l$ at some point $z_1$.

For point $y_1$ we can draw a ray that does not intersect $E_1$ by the conditions. So we can draw the ray that is tangent to $E_1$ and does not intersect $E$. With that, such a ray is to pass above the straight line $s$ that contains ray $s_1$, because $s$ is tangent to the boundary of $E_1$ and intersects set $E$ (see Figure 5). Here and everywhere below the expression "to pass above the straight line $s$" means to pass in the half-plane with respect to $s$ which does not contain $E_1$ in some neighborhood of point $y_1$. Such neighborhood exists according to the constructions in the previous paragraph.

Let a ray $s_3$, starting at the point $y_1$, be tangent to the component $E_1$ at some point $y_3$ that is above the straight line $s$, according to the previous explanations. Points $y_1$, $y_3$ belong to the boundary of the same component $E_1$. Ray $s_1$ does not intersect the part of $\partial E_1$ connecting the points, but the ray $s_2$, complement to $s_1$, intersects $\partial E_1$ at some point $y_2$. For $y_2$ we can draw a ray $s_4$ that does not intersect set $E$ but it should intersect the segment $y_1y_3$ at some point $y$.

According to the previous constructions, the polygonal chain $lz_1y_1s_3$ is simple (without self-intersections) and it cuts the plane into two parts, each of which contains a component of set $E$. With that, the domain that is bounded by the part of $\partial E_1$,
connecting points \(y_1, y_3\), on one side and by the segment \(y_1y_3\) on the other one, does not contain points of \(E\). Thus, we can draw the ray, starting at point \(x\) and passing through the point \(y\), which does not intersect \(E\). But this is a contradiction with the theorem conditions.

In the case of closed set \(E\) all statements remain the same as in the case of open set \(E\) with the following small addition. The ray \(s_1\) already intersects \(E\) at the point \(y_1\). But still it should intersect \(E\) after crossing \(l\). In the opposite case, for point \(x\) we can draw a ray that lies between \(s_1\) and the ray tangent to \(E\) behind \(l\) and does not intersect \(E\), which contradicts the theorem conditions.

Thus, \(E\) consists of more than two simply connected components. Theorem 4 is proved.

**Example 3.** The example of weakly 1-semiconvex and not 1-semiconvex set with smooth boundary which consists of two simply connected components is given in Figure 6. In case a), the set \(E\), among all its boundary points, contains only points \(A\) and \(B\). In case b) \(E\) contains all its boundary points except points \(C\) and \(D\). In both cases, for any point of the interval \(AB\) it is not possible to draw a ray that does not intersect the sets. In general, if one cuts a subset of the set of all boundary points of \(E\) which contains points \(C, D\) and does not contain points \(A, B\) from the closure of \(E\), then the obtained set is also weakly 1-semiconvex and not 1-semiconvex.

![Fig. 6](image)
**Example 4.** The system of four open (closed) balls with common tangent lines, illustrated in Figure 7a (7b), is an example of open (closed) set with smooth boundary which is weakly 1-semiconvex, not 1-semiconvex, and consists of four simply connected components. Herewith, it is easy to see that in the case of open balls for any point of the interior of the rhombus $ABCD$ there is no ray that intersects the set (Figure 7a). And in the case of closed balls this holds for the points of the closure of rhombus $ABCD$ (Figure 7b).

![Diagram](image)

**Fig. 7**

The following result is the direct corollary from Theorem \[ \text{8} \]

**Lemma 1.** Let a set $E \subset \mathbb{R}^2$ be simply connected and weakly 1-semiconvex. Then $E$ is 1-semiconvex.

**Definition 6.** A point $x \in \mathbb{R}^n \setminus A$ is called point of $m$-nonsemiconvexity of a set $A \subset \mathbb{R}^n$ if there is no open $m$-dimensional half-plan starting at $x$ and not intersecting $A$.

**Definition 7.** A ray $l$ starting at a point $x \in \mathbb{R}^n \setminus \overline{A}$ is supporting for a set $A \subset \mathbb{R}^n$ at a point $a \in \partial A$ if $a \in l \cap \partial A \neq \emptyset$ and $l \cap \text{Int} A = \emptyset$.

**Lemma 2.** Let an open (closed) set $E \subset \mathbb{R}^2$ be simply connected, weakly 1-semiconvex, and have smooth boundary. Then there exist two and only two supporting rays of $E$ starting at a point $x \in \mathbb{R}^2 \setminus \overline{E}$.
Proof. The set $E$ is 1-semiconvex by Corollary \[ \square \] So, for any point $x \in \mathbb{R}^2 \setminus \overline{E}$ we can draw a ray that does not intersect $E$. By continuity and since $E$ has smooth boundary, we can draw at least one tangent to $\partial E$ ray starting at $x$ that does not intersect $\text{Int} E$.

Suppose we have a unique supporting ray for a fixed point $x$. This is not possible for closed sets. Let set $E$ be open. Without loss of generality we suppose that there are two possible cases: $\partial E$ and supporting ray have one common point $x_1$; $\partial E$ and supporting ray have two common points $x_1$, $x_2$, with that, $x_1$ is closer to $x$ than $x_2$. The complimentary ray to the supporting one intersects $\partial E$ at some point $x_0$. Since $\partial E$ is smooth, points of both open parts of $\partial E$ between $x_1$, $x_0$ are 1-nonsemiconvexity points in the first case (Figure 8a)), while points of open part of $\partial E$ between $x_1$, $x_0$ are 1-nonsemiconvexity points in the second one (Figure 8b)). In both cases this is a contradiction with 1-semiconvexity of $E$.

\[ \text{Рис. 8} \]

Let us assume that there exist $n$, $n \geq 3$, different supporting rays starting at the point $x \in \mathbb{R}^2 \setminus \overline{E}$. So they cut the plane into $n$ parts and, by the construction, $n - 1$ of them should contain $E$, which contradicts the fact that $E$ is connected.

The lemma is proved.

The open set on Figure 8 a) with boundary that is nonsmooth at the point $x_1$ can have a unique supporting ray starting at the point $x$.

We say that a set $A \subset \mathbb{R}^n$ is projected from a point $x \in \mathbb{R}^n$ on a set $B \subset \mathbb{R}^n$ if any ray, starting at point $x$ and intersecting $A$, intersects $B$ as well.
Lemma 3. Let an open set $E \subset \mathbb{R}^2$ be weakly 1-semiconvex but not 1-semiconvex and consist of three components. Then none of its components is projected on the others from a point of 1-nonsemonic convexity of $E$.

Proof. Let $E$ consist of three components $E_1, E_2, E_3$ and $x \in \mathbb{R}^2 \setminus E$ be a point of 1-nonsemonic convexity of $E$. Without loss of generality, suppose $E_1$ is projected from $x$ on at least one of the other components, Figure 9. Then the set, consisting only of components $E_2, E_3$, is weakly 1-semiconvex and not 1-semiconvex, which contradicts Theorem 2.

![Fig. 9](image-url)

The lemma is proved.

Let sets $A, B \subset \mathbb{R}^n$ be given. Let $l$ be a supporting ray of $A$ starting at a point $x \in \mathbb{R}^n$ and point $a \in l \cap \partial A$. The ray $l$ is called inner supporting ray of the set $A \cup B$ if $l \cap B \neq \emptyset$ and the distance between points $x$ and $a$ is less than a distance between point $x$ and any point $b \in l \cap B$.

Lemma 4. Let a set $E \subset \mathbb{R}^2$ be weakly 1-semiconvex but not 1-semiconvex. Then $E$ has at least one inner supporting ray.

Proof. Set $E$ consists of $n$, $2 < n < \infty$, simply connected components $E_j$, $j = \{1, n\}$, by Theorem 2. Since $E$ is not 1-semiconvex, there exists a 1-nonsemonic convexity point $x \in \mathbb{R}^2 \setminus E$. Let us draw a ray $\xi$ starting at point $x$. It intersects $\partial E$ at some point $y$ that is nearest to $x$ along the ray. Since $E$ is weakly 1-semiconvex, there exists a ray $\gamma$ starting at $y$ and not intersecting $E$. Moreover, $\gamma$ does not lie on the straight line that contains $\xi$. 

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Let us rotate ray $\xi$ round $x$ in the half-plane where $\gamma$ lies. In other words, we choose the polar coordinate system where point $x$ is the pole, ray $\xi$ is the polar axis and a positive angular coordinate is determined by the angle $\varphi$ between the ray $\xi = \xi(0)$ and a ray $\xi(\varphi)$, $0 < \varphi < \pi$, that intersects ray $\gamma$.

So, among all rays starting at $x$ and intersecting $\gamma$, there are those that intersect component $E_1$, for definiteness, before they intersect $\gamma$. A case when all these rays intersect $E$ after they intersect $\gamma$ is possible only when $E$ is unbounded. But in this case there exists the ray that is parallel to ray $\gamma$ and does not intersect $E$, which contradicts the lemma conditions. Thus, by continuity, there exists the ray $\xi(\varphi_0)$, $0 \leq \varphi_0 < \pi$, such that $\xi(\varphi_0) \cap \partial E_1 \neq \emptyset$, $\xi(\varphi_0) \cap \text{Int} E_1 = \emptyset$, and for $\varepsilon > 0$ small enough rays $\xi(\varphi)$, $\varphi_0 < \varphi < \varphi_0 + \varepsilon$, intersect $E_1$ before they intersect $\gamma_1$. Ray $\xi(\varphi_0)$ does not intersect component $E_1$ after intersecting $\gamma$, otherwise, the point of $\partial E_1$ which is nearest to point $x$ along a ray $\xi(\varphi)$, $\varphi_0 < \varphi \leq \varphi_0 + \pi$, is a point of 1-nonsenseconvexity of $E$. Thus, $\xi(\varphi_0) \cap E_2 \neq \emptyset$, for definiteness, and $\xi(\varphi_0)$ is an inner supporting ray of $E$.

Further in this work we will often use the constructions of the proof of Lemma 4.

The main result of the paper is the following

**Theorem 5.** Let $E \subset \mathbb{R}^2$ be an open, weakly 1-semiconvex, and not 1-semiconvex set with smooth boundary. Then $E$ consists minimum of four simply connected components.

**Proof.** The set does not consist of one or two simply connected components by Theorem 4. Suppose $E$ consists of three simply connected components $E_i$, $i = 1, 2, 3$.

Since $E$ is not 1-semiconvex, it follows that there exists a 1-nonsenseconvexity point $x \in \mathbb{R}^2 \setminus \overline{E}$ of $E$. Since $E$ is a weakly 1-semiconvex set, we do not consider points of $\partial E$ as points of 1-nonsenseconvexity of $E$.

By Lemma 3, none of the components of $E$ is projected on the others. Then there are three rays $\tau_1$, $\tau_2$, $\tau_3$ starting at $x$ and intersecting a unique component $E_1$, $E_2$, $E_3$ respectively. That is to say, $\tau_1 \cap E \equiv \tau_1 \cap E_1$, $\tau_2 \cap E \equiv \tau_2 \cap E_2$, $\tau_3 \cap E \equiv \tau_3 \cap E_3$. So, rays $\tau_1$, $\tau_2$, $\tau_3$ cut the plane by three nonempty parts $G_1$, $G_2$, $G_3$.

Without loss of generality, let us consider the closure of domain $G_1$ between the rays $\tau_1$, $\tau_2$. Since $\tau_3$ intersects component $E_3$ in $\mathbb{R}^2 \setminus \overline{G_1}$, set $\overline{G_1}$ does not contain points of $E_3$. Then $\overline{G_1}$ consists of rays that intersect only $E_1$, only $E_2$, and of those
that intersect both $E_1$, $E_2$. With that, the set of rays, intersecting both $E_1$, $E_2$, is open in $\overline{G_1}$ and its boundary consists of one supporting ray of $E_1$ and one of $E_2$. If one assumes that there is another supporting ray, then it should be supporting for one of the components $E_1$, $E_2$. Without loss of generality, we assume that it is supporting for $E_1$. Since each component of $E$ has exactly two supporting rays, by Lemma 2 then $E_1$ should be completely contained in $\overline{G_1}$. But this contradicts the fact that $E_1$ is open and $\tau_1 \cap E_1 \neq \emptyset$.

Thus, each closed set $\overline{G_1}$, $\overline{G_2}$, $\overline{G_3}$ contains one and only one inner supporting ray $\xi_i$, $i = 1, 2, 3$ of $E$. Since $E$ has the smooth boundary, each ray $\xi_i$ is also tangent to the boundary of corresponding component at some point $y_i$, $i = 1, 2, 3$. The rays complementary to $\xi_i$ also intersect $E$ at some points that we denote as $z_i$ respectively.

Since sets $G_1$, $G_2$, $G_3$ have common boundary rays $\tau_j$, $j = 1, 2, 3$, and each inner supporting ray $\xi_i$, $i = 1, 2, 3$, belongs to the respective $\overline{G_i}$, without loss of generality, let us consider the case when two supporting rays coincide with the boundary ray $\tau_3$. Since $\tau_3$ intersects component $E_3$, this ray is supporting for components $E_1$, $E_2$ and is also tangent to their boundaries at points $y_1$, $y_2$ respectively. Points $y_1$, $y_2$ do not coincide; otherwise one can not draw a ray that starts at such a point and does not intersect $E$, which contradicts weakly 1-semiconvexity of $E$.

Let point $y_1$ be nearer to point $x$ than to point $y_2$, for definiteness. Then, let us draw a ray $\gamma_1$ starting at $y_1$ and not intersecting $E$. It lies above the straight that contains ray $\tau_3$. The ray complementary to $\tau_3$ intersects $\partial E$ at a point $z_3$. Let us draw a ray $\zeta_3$ starting at $z_3$ that also does not intersect $E$. Among all rays starting at point $x$ and crossing rays $\tau_3$ and $\zeta_3$ there exist two distinct inner supporting rays different from $\tau_3$, which can be shown as in prove of Lemma 4. This contradicts the fact that $E$ has only two inner supporting rays by our assumption.

Thus, further we consider sets $E$ that have three and only three inner supporting rays $\xi_i$, $i = 1, 2, 3$, starting at the point $x$. Each point $y_i$ belongs to the boundary of one of two neighboring components. Depending on which boundary of two neighboring components each point $y_i$ belongs, we consider two possible cases: 1) the boundary of each component contains only one point $y_i$ ($y_i \in \partial E_i$, $i = 1, 2, 3$), Figure 10 a); two points $y_i$ belong to the boundary of the same component ($y_1, y_2 \in \partial E_1$, $y_3 \in \partial E_3$), Figure 10 b).
1) Without loss of generality, let us consider the polar coordinate system where point $x$ is the pole and inner supporting ray $\xi_1$ is the polar axis. Since $E$ is weakly 1-semiconvex, there exists a ray $\gamma_1$ starting at the point $y_1$ and not intersecting $E$, Figure 11. The ray $\gamma_1$ should lie above the straight that contains ray $\xi_1$. Let us chose a positive angular coordinate determined by the angle $\varphi$ between the ray $\xi_1 = \xi_1(0)$ and a ray $\xi_1(\varphi), 0 < \varphi < \pi$, that intersects ray $\gamma_1$. By the proof of Lemma 4, there exists an angle $0 < \varphi' < \pi$ such that the ray $\xi_1(\varphi')$ is an inner supporting ray and for $\varepsilon > 0$, small enough, rays $\xi_1(\varphi), \varphi < \varphi < \varphi' + \varepsilon$, intersect $E$ before they intersect $\gamma_1$. Suppose ray $\xi_1(\varphi')$ is tangent to the boundary of one of the components $E_i$, $i = 1, 2, 3$, at a point $y'$. Let the third inner supporting ray $\xi_1(\varphi'')$ touch $E$ at a point $y''$ and $0 < \varphi'' < \varphi'$, then points $y_1, y'$ belong to the boundary of the component $E_1$. If $\varphi'' > \varphi'$, then one of the points $y_1, y'$ belongs to the boundary of the same component that point $y''$ does. Both cases contradict the fact that the boundary of each component contains only one point.
2) Let $\alpha$ be the angle of the sector between rays $\xi_1, \xi_2$ which contains component $E_1$. Then two cases are possible.

   a) $\angle \alpha > \pi$, Figure 12. Then ray $\xi_3$ is contained in the complementary of $\angle \alpha$ to $2\pi$ and point $z_3 \in \partial E_1$. Let us draw a ray $\zeta$ that starts at point $z_3$ and does not intersect $E$. We consider the polar coordinate system $(\xi(\varphi), \rho)$ where point $x$ is the pole, the ray complementary to the inner supporting ray $\xi_3$ is the polar axis $\xi(0)$, and a positive angular coordinate is determined by the angle $\varphi$ between the ray $\xi(0)$ and a ray $\xi(\varphi)$, $0 < \varphi < \pi$, that intersects ray $\zeta$. Then there exists an angle $0 < \varphi' < \pi$ such that the ray $\xi_4 = \xi(\varphi')$ is an inner supporting ray and for $\varepsilon > 0$ small enough rays $\xi(\varphi)$, $\varphi < \varphi < \varphi' + \varepsilon$, intersect $E$ before they intersect ray $\zeta$. Since ray $\zeta$ does not coincide with inner supporting ray $\xi_3$, ray $\xi_4$ does not coincide with ray $\xi_3$ too. Since inner supporting rays $\xi_1, \xi_2$ lie in distinct half-planes with respect to the straight that contains ray $\xi_3$, ray $\xi_4$ does not coincide with the ray $\xi_i$, $i \in \{1, 2\}$, that lies in the other half-plane. If ray $\xi_4$ coincides with third inner supporting ray, then we have the case when two inner supporting rays coincide with a ray that intersects a unique component, which is not possible too. So, we have four distinct inner supporting rays, which contradicts the condition that there are three of them.
b) \( \angle \alpha \leq \pi \). If the ray complementary to the inner supporting ray \( \xi_3 \) is contained in the sector \( S \) between the rays complementary to \( \xi_1, \xi_2 \), then the statements should be as in case 2 b).

Let \( \alpha_1, \alpha_2 \) be angles between nearest-neighbor rays \( \xi_3, \xi_1 \) and \( \xi_3, \xi_2 \), respectively, Figure 13. Suppose ray \( \xi_3 \) does not lie in sector \( S \), then \( \alpha_1 \neq \alpha_2 \). Let \( \alpha_1 > \alpha_2 \), for definiteness, with that \( \alpha_1 > \pi \). A ray that starts at point \( y_1 \) and does not intersect \( E \) should lie above the straight that contains \( \xi_1 \), by smoothness of \( \partial E \) and 1-nonsemiconvexity of \( E \) at the point \( x \). But any such a ray intersects a part of the component which lies in the sector between rays \( \xi_3, \xi_1 \). Then \( y_1 \) is a point of 1-nonsemiconvexity, which contradicts the theorem conditions.
Thus, set $E$ does not consist of three components. Example 4 completes the proof.

**Corollary 1.** Let an open set $E \subset \mathbb{R}^2$ be weakly 1-semiconvex but not 1-semiconvex and consist of four simply connected components with smooth boundary. Then none of its components is projected on the others from a point of 1-nonsemiconvexity of $E$.

**Proof.** Let $E = \bigcup_{i=1}^{4} E_i$ consist of four simply connected components with smooth boundary and $x \in \mathbb{R}^2 \setminus \overline{E}$ be a point of 1-nonsemiconvexity. Without loss of generality, suppose $E_1$ is projected from $x$ on at least one of the other components. Then the set, consisting only of components $E_i, i = 2, 3, 4$, is weakly 1-semiconvex and not 1-semiconvex, which contradicts Theorem 5.
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