The Chiral Space of Local Operators in $SU(2)$-Invariant Thirring Model

Atsushi Nakayashiki

Abstract

The space of local operators in the $SU(2)$ invariant Thirring model ($SU(2)$ ITM) is studied by the form factor bootstrap method. By constructing sets of form factors explicitly we define a subspace of operators which has the same character as the level one integrable highest weight representation of $\mathfrak{sl}_2$. This makes a correspondence between this subspace and the chiral space of local operators in the underlying conformal field theory, the $su(2)$ Wess-Zumino-Witten model at level one.

1 Introduction

The $SU(2)$ invariant Thirring model ($SU(2)$ ITM) is a massive integrable model in two dimensional quantum field theory which can be considered as a deformation of the level one integrable highest weight representation of $\mathfrak{sl}_2$. In particular there should be a subspace of operators in $SU(2)$ ITM which is isomorphic to the level one integrable highest weight representation of $\hat{\mathfrak{sl}}_2$, the chiral space of the level one $su(2)$ WZW model. In this paper we single out such a space by constructing sets of form factors explicitly.

Let $V = \mathbb{C}v_+ \oplus \mathbb{C}v_-$ be the two dimensional vector space on which $\mathfrak{sl}_2$ acts. The $S$-matrix of $SU(2)$ ITM is the operator $S(\beta) : V^{\otimes 2} \rightarrow V^{\otimes 2}$, the explicit form of which is given in [13]. The set of form factors $(F_{2n})^\infty_{n=0}$, $F_{2n}$ being a $V^{\otimes 2n}$-valued meromorphic function, of a local operator in $SU(2)$ ITM satisfies the following system of equations:

\begin{align}
P_{i,i+1} S_{i,i+1}(\beta_i - \beta_{i+1}) F_{2n}(\beta_1, \cdots, \beta_{2n}) &= F_{2n}(\cdots, \beta_{i+1}, \beta_i, \cdots), \\
P_{2n-1,2n} P_{2n-1,2n-2} \cdots P_{1,2} F_{2n}(\beta_1 - 2\pi i, \cdots, \beta_{2n}) &= (-1)^n F_{2n}(\beta_2, \cdots, \beta_{2n}, \beta_1), \\
2\pi \text{res}_{\beta_{2n} = \beta_{2n-1} + 2\pi} F_{2n}(\beta_1, \cdots, \beta_{2n}) &= \left( I - (-1)^n S_{2n-1,2n-2}(\beta_{2n-1} - \beta_{2n-2}) \cdots S_{2n-1,1}(\beta_{2n-1} - \beta_1) \right) F_{2n-2}(\beta_1, \cdots, \beta_{2n-2}) \otimes e_0.
\end{align}

where $e_0 = v_+ \otimes v_- - v_- \otimes v_+$, $P : V^{\otimes 2} \rightarrow V^{\otimes 2}$ is the permutation operator and $S_{ij}(\beta)$ acts on $i$-th and $j$-th components of $V^{\otimes 2n}$ as $S(\beta)$ etc. Conversely any solution $(F_{2n})^\infty_{n=0}$ of (1), (2), (3) determines a local operator such that its $2n$-particle form factor is $F_{2n}$, [17]. Thus the space of local operators can be identified with the space of solutions of (1), (2), (3).

The complete description of the solution space of (1) and (2) is known (see [11] and references therein). Let $R_n$ be the ring of symmetric polynomials of $x_1, \cdots, x_n$ with $x_j = \exp(\beta_j)$ and $C_n$
the field of symmetric, $2\pi i$-periodic and meromorphic functions of $\beta_1, \cdots, \beta_n$. For $\ell \leq n$ consider a polynomial $P = P(X_1, \cdots, X_l|x_1, \cdots, x_{2n})$ such that it is anti-symmetric in $X_a$'s with the coefficients in $R_{2n}$ satisfying $\deg_{X_a} P \leq 2n - 1$ for any $a$. To each $P$ a solution $\Psi_P$ of (1), (2) is constructed through multi-dimensional $q$-hypergeometric integrals. It takes the value in the space of $sl_2$ highest weight vectors with the weight $2n-2\ell$. Since $S(\beta)$ and $P$ (the permutation operator) commute with $sl_2$, other solutions are created from highest weight solutions by the actions of $sl_2$ and $C_{2n}$.

It is convenient to describe the space of polynomials as above by some exterior space. Let $H^{(2n)} = \bigoplus_{j=0}^{2n-1} R_{2n} X^j$ and $H^{(2n)}_{C_{2n}} = \bigoplus_{j=0}^{2n-1} C_2 X^j$. Then $\wedge^{\ell} H^{(2n)}_{C_{2n}}$ can be identified with the space of polynomials satisfying the conditions above and $\wedge^{\ell} H^{(2n)}$ becomes a subspace of it.

For each $n \geq 0$ we classify solutions of (1)-(3) into the solutions of the form $(0, \cdots, 0, F_{2n}, F_{2n+2}, \cdots)$. Then Equation (3) implies

$$\text{res}_{\beta_{2n} = \beta_{2n-1} + \pi} F_{2n}(\beta_1, \cdots, \beta_{2n}) = 0.$$  

If we write $F_{2n} = \Psi_P$, Equation (4) should be rewritten as equations for a polynomial $P$. In fact the following equations imply (3)

$$P|_{(x_{2n-1}, x_{2n}) = (x, -x), X_l = \pm x^{-1}} = 0.$$  

The converse is conjectured to be true. Let

$$U_{2n, \ell} = \{ P \in \wedge^\ell H^{(2n)} | P \text{ is a solution to (3)} \}.$$

In (11) $U_{2n, \ell}$ is proved to be a free $R_{2n}$-module and its basis has been constructed explicitly.

To obtain the solution space of (1), (2), (4) one has to consider $U_{2n, \ell}$ modulo polynomials $P$ such that $\Psi_P$ vanishes identically. Such polynomials have been determined completely (9). Namely there are polynomials $\Xi^{(2n)}_1 \in U_{2n, 1}$ and $\Xi^{(2n)}_2 \in U_{2n, 2}$ such that $\Psi_P$ is identically zero if and only if $P \in \Xi^{(2n)}_1 \wedge^{\ell-1} H^{(2n)} + \Xi^{(2n)}_2 \wedge^{\ell-2} H^{(2n)}$. We define

$$M_{2n, \ell} = \frac{U_{2n, \ell}}{\Xi^{(2n)}_1 \wedge U_{2n, \ell-1} + \Xi^{(2n)}_2 \wedge U_{2n, \ell-2}},$$

and, for $\lambda \geq 0$, $M^{(0)}_\lambda = \oplus_{2n-2\ell = \lambda} M_{2n, \ell}$. Then $M^{(0)}_\lambda$ becomes isomorphic to the space of $sl_2$ highest weight vectors in the level one integrable highest weight representation $V(\Lambda_0)$ of $sl_2$ (11).

In this paper, to each $P_{2n} \in U_{2n, \ell}$ we construct polynomials $F_{2n}$, $n > m$ such that $(\Psi_{P_{2n}})^{\infty}_{n=0}$ satisfies (3) (Theorem 3, where we set $P_{2n} = 0$ for $n < m$ . This set of functions specifies a local operator which is a highest weight vector of $sl_2$. The operators corresponding to non-highest weight vectors are obtained from the ones above by the action of $sl_2$. Let $V_\lambda$ be the finite dimensional irreducible representation of $sl_2$ with the highest weight $\lambda$. Then $\bigoplus_\lambda M^{(0)}_\lambda \otimes V_\lambda$ specifies the subspace of local operators in $SU(2)$ ITM which is isomorphic to $V(\Lambda_0)$.

Finally we briefly remark on the previous results on the determination of the space of local operators in massive integrable models. For several models with diagonal S-matrices the characters of the space of initial form factors, which are the first non-vanishing form factors, have been calculated (9). They are shown to take fermionic forms (8) of the corresponding CFT characters. For Sine-Gordon model and $SU(2)$ ITM, which are two of simple models with non-diagonal S-matrices, important results toward determining all local operators have been obtained in (8) and (9). However the construction of $2n$-particle form factors for every $n$ to each element of the underlying CFT characters has never been done even for models with diagonal S-matrices other than Ising model (2) and (8) and the model considered in (11). We solve this problem for $SU(2)$ ITM under certain assumptions.
The plan of the present paper is as follows. In section 2 we review the results of [11], in particular, the description of the basis of $U_{2n,t}$. The formulae for polynomials $P_{2n}$, $n \geq m$ and consequently for form factors are given in section 3. In section 4 some theorems on symmetric polynomials which is used in section 3 are proved. They are interesting by itself. The form factors of local operators in the anti-chiral subspace are given in section 5. In appendix A the definitions and properties of the function $\zeta(\beta)$ which appeared in the formulae of form factors is briefly explained for the sake of readers’ convenience.

## 2 Space of minimal form factors

In this section we recall the results of [11].

Let $V = \mathbb{C}v_+ \oplus \mathbb{C}v_-$ be the vector representation of $sl_2 = \mathbb{C}e \oplus \mathbb{C}h \oplus \mathbb{C}f$ with the correspondence $v_+ = \iota(1,0)$, $v_- = \iota(0,1)$ and $P : V \otimes V \to V \otimes V$ the permutation operator $P(v_{n_1} \otimes v_{n_2}) = v_{n_1} \otimes v_{n_2}$.

The $S$-matrix of the $SU(2)$ invariant Thirring model is the linear operator in $\text{End}(V \otimes V)$ given as

$$S(\beta) = S_0(\beta) \tilde{S}(\beta), \quad \tilde{S}(\beta) = \frac{\beta - \pi i P}{\beta - \pi i}, \quad S_0(\beta) = \frac{\Gamma(\frac{\beta + \pi i}{2\pi i})\Gamma(-\frac{\beta}{2\pi i})}{\Gamma(\frac{\beta - \pi i}{2\pi i})\Gamma(\frac{-\beta}{2\pi i})} \tag{6}$$

For a linear operator $A$ in $\text{End}(V \otimes V)$ we define $A_{ij} \in \text{End}(V \otimes ^n)$ as follows. Write

$$A = \sum_a B_a \otimes C_a, \quad B_a, C_a \in \text{End}(V).$$

Then

$$A_{ij} = \sum_a 1 \otimes \cdots \otimes B_a \otimes \cdots \otimes C_a \otimes \cdots \otimes 1,$$

where $B_a$ and $C_a$ are in $i$-th and $j$-th components respectively.

Consider the following system of equations for a $V \otimes ^n$-valued function $F$:

$$P_{n-1,n}P_{n-2} \cdots P_{i+1,i} P_i(\beta_i - \iota P)(\beta_i - \iota P)^{-1} F(\beta_1, \cdots, \beta_n) = F(\beta_1, \cdots, \beta_i, \beta_i, \cdots), \tag{7}$$

$$P_n(\beta_i - 2\pi i) F(\beta_1, \cdots, \beta_n) = (-1)^{\frac{n}{2}} F(\beta_i, \cdots, \beta_n, \beta_1). \tag{8}$$

All meromorphic solutions of those equations are given by the multi-dimensional integrals of $q$-hypergeometric type in the following manner.

Notice that $S(\beta)$ commutes with the action of $sl_2$. Therefore any solution can be obtained, by the action of $sl_2$, from the solutions taking values in the space of highest weight vectors $(V \otimes ^n)_{\lambda}^{\text{sing}}$:

$$(V \otimes ^n)_{\lambda}^{\text{sing}} = \{v \in V \otimes ^n \mid \text{ev} = 0, \text{hv} = \lambda v\}.$$ 

We remark that $(V \otimes ^n)_{\lambda}^{\text{sing}} \neq \{0\}$ if and only if $\lambda$ is written as $\lambda = n - 2\ell$ for some $0 \leq \ell \leq n/2$.

For a subset $M \subset \{1, 2, \ldots, n\}$ we set

$$v_M = v_{\epsilon_1} \otimes \cdots \otimes v_{\epsilon_n} \in V \otimes ^n, \quad M = \{j \mid \epsilon_j = -\}.$$ 

Let $C_n$ be the field of symmetric, $2\pi i$-periodic and meromorphic functions of $\beta_1, \ldots, \beta_n$ and

$$H^{(n)} := \oplus_{k=0}^{n-1} R_n X^k, \quad H_{C_n}^{(n)} := \oplus_{k=0}^{n-1} C_n X^k$$

3
We use the variables $X_a$, $x_j$ and $\alpha_j$, $\beta_j$ which are related by $X_a = \exp(-\alpha_a)$, $x_j = \exp(\beta_j)$.

Now for each $P \in \wedge^\ell H_{C_n}^{(n)}$ define the $V^{\otimes n}$-valued function $\psi_P$ by

$$\psi_P = \sum_{\ell = \ell} I_M(P)v_M,$$

where

$$I_M(P) := \int_{C_\ell} \prod_{a=1}^{\ell} d\alpha_a \prod_{a=1}^{\ell} \phi_n(\alpha_a) w_M \frac{P(X_1, \ldots, X_{\ell}|x_1, \ldots, x_n)}{\Pi_{a=1}^{\ell} \Pi_{j=1}^{a} (1 - X_a x_j)},$$

(9)

where, $w_M = \text{Asym}(g_M)$ and, for $M = (m_1, \ldots, m_\ell)$ with $m_1 < \cdots < m_\ell$,

$$\phi_n(\alpha) = \prod_{j=1}^{n} \frac{\Gamma\left(\frac{\alpha_j + \pi i}{2\pi i}\right)}{\Gamma\left(\frac{\alpha_j}{2\pi}\right)}, \quad g_M = \prod_{a=1}^{\ell} \left(\frac{1}{\alpha_a - \beta_j} \prod_{j=1}^{m_a} \frac{\alpha_a - \beta_j + \pi i}{\alpha_a - \beta_j} \prod_{1 \leq a < b \leq \ell} (\alpha_a - \alpha_b + \pi i)\right).$$

The integration contour $C$ is defined as follows. It goes from $-\infty$ to $+\infty$, separating two sets $\{\beta_j - 2\pi ik \mid 1 \leq j \leq n, k \geq 0\}$ and $\{\beta_j + (2k - 1)\pi i \mid 1 \leq j \leq n, k \geq 0\}$.

We set

$$\Psi_P = \exp\left(\frac{1}{n} \sum_{j=1}^{n} \beta_j\right) \prod_{j<k} \zeta(\beta_j - \beta_k) \psi_P,$$

(10)

where $\zeta(\beta)$ is a certain meromorphic function described in Appendix A.

Let $\text{Sol}_{n}^{n}$ be the space of meromorphic solutions of (7) and (8) taking values in $(V^{\otimes n})^{\text{sing}}$. Then $\Psi_P \in \text{Sol}_{n-2\ell}^{n}$.  

**Remark.** The integral $I_M(P)$ is well defined for any polynomial $P(X_1, \ldots, X_{\ell}|x_1, \ldots, x_n)$ such that $\deg X_a, P \leq n$ for any $a$. If $P$ is symmetric in $x_j$'s, the function $\Psi_P$ becomes a solution of (7) and (8).

Due to the anti-symmetry of $w_M$ the relation $\ell \Psi_P = \Psi_{\text{Asym}(P)}$ holds. The highest degree $X_a^n$ can be eliminated by $\text{Ker } \Psi$ below. Thus for any $P$ we have $\Psi_P = \Psi_{P'}$ for some $P' \in \wedge^\ell H_{C_n}^{(n)}$. We later use this property in the description of polynomials corresponding to form factors.

We have the linear map:

$$\Psi : \wedge^\ell H_{C_n}^{(n)} \longrightarrow \text{Sol}_{n-2\ell}^{n}, \quad P \mapsto \Psi_P.$$

This map is surjective. The kernel of $\Psi$ is described as follows. Let $\Theta^{(n)}_{\pm}(X) = \prod_{j=1}^{n} (1 \pm X x_j)$ and

$$2\Theta^{(n)}_{1}(X) = \Theta^{(n)}_{+}(X) + (-1)^{n-1} \Theta^{(n)}_{-}(X),$$

$$2\Theta^{(n)}_{2}(X_1, X_2) = \left(\Theta^{(n)}_{+}(X_1) \Theta^{(n)}_{+}(X_2) - \Theta^{(n)}_{-}(X_1) \Theta^{(n)}_{-}(X_2)\right) \frac{X_1 - X_2}{X_1 + X_2} + (-1)^{n} \left(\Theta^{(n)}_{+}(X_1) \Theta^{(n)}_{-}(X_2) - \Theta^{(n)}_{+}(X_2) \Theta^{(n)}_{-}(X_1)\right).$$
Then
\[ \text{Ker } \Psi = \wedge^{\ell-1} H_{C_n}^{(n)} \wedge \Xi_1^{(n)} + \wedge^{\ell-2} H_{C_n}^{(n)} \wedge \Xi_2^{(n)}. \]

Consider the restriction of \( \Psi \) to \( \wedge^\ell H^{(n)} \). Then the function \( \Psi_P \) has at most a simple pole at \( \beta_n = \beta_{n-1} + \pi i \) as a function of \( \beta_n \). In \([11]\) the subspace of \( \wedge^\ell H^{(n)} \) such that the function \( \Psi_P \) is holomorphic at \( \beta_n = \beta_{n-1} + \pi i \) has been studied. Let us recall the results.

For a polynomial \( P(X_1, \ldots, X_\ell|x_1, \cdots, x_n, x, -x) \) in \( X_\ell \)'s with the coefficients in Laurent polynomials of \( x_j \)'s define
\[ \rho_\pm(P) = P(X_1, \ldots, X_{\ell-1}, \pm x^{-1} | x_1, \cdots, x_{n-2}, x, -x), \]
and set
\[ U_{n, \ell} = \{ P \in \wedge^\ell H^{(n)} | \rho_+(P) = \rho_-(P) = 0 \}. \]
Then \( \Xi_1^{(n)} \in U_{n,1}, \Xi_2^{(n)} \in U_{n,2} \) and if \( P \) is in \( U_{n, \ell} \) then \( \Psi_P \) becomes regular at \( \beta_n = \beta_{n-1} + \pi i \). Let
\[ M_{n, \ell} = \frac{U_{n, \ell}}{U_{n, \ell-1} \wedge \Xi_1^{(n)} + U_{n, \ell-2} \wedge \Xi_2^{(n)}}, \]
and, for \( i = 0, 1, \lambda \in \mathbb{Z}_{\geq 0}, \)
\[ M_\lambda^{(i)} = \oplus_{n \equiv i \mod. 2, n-2\ell = \lambda} M_{n, \ell}. \]

**Remark.** We conjecture the following equation:
\[ U_{n, \ell} \cap \left( \wedge^{\ell-1} H_{C_n}^{(n)} \wedge \Xi_1^{(n)} + \wedge^{\ell-2} H_{C_n}^{(n)} \wedge \Xi_2^{(n)} \right) = U_{n, \ell-1} \wedge \Xi_1^{(n)} + U_{n, \ell-2} \wedge \Xi_2^{(n)}. \] (11)

For \( n = 1, 2 \) \([11]\) holds. If \([11]\) holds, then the space \( M_{n, \ell} \) becomes isomorphic to the subspace of \( \text{Sol}_{n-2\ell} \).

We define a degree of an element in \( U_{n, \ell} \), denoted by \( \deg_1 \), assigning \( \deg_1 X_a = -1, \deg_1 x_j = 1 \). Then we set \( \deg_2 P = \deg_1 P + n^2/4 \) for a homogeneous element \( P \) in \( U_{n, \ell} \). Then \( \deg_1 \Xi_1^{(n)} = \deg_1 \Xi_2^{(n)} = 0 \). We introduce a grading on \( M_{n, \ell} \) by \( \deg_2 \). In general for a graded vector space \( V = \oplus_n V_n \) we define its character by
\[ \text{ch } V = \sum_n q^n \dim V_n. \]

Let \( \Lambda_i \) be the fundamental weights of \( \widehat{sl}_2 \), \( V(\Lambda_i) \) the integrable highest weight representation of \( \widehat{sl}_2 \) with the highest weight \( \Lambda_i, V(\Lambda_i|\lambda), \lambda \in \mathbb{Z}_{\geq 0}, \) the subspace of \( V(\Lambda_i) \) consisting of \( sl_2 \)-highest weight vectors with the weight \( \lambda \) and \( d \) the scaling element of \( sl_2 \) \([7]\).

**Theorem 1** \([11]\)

\[
\text{ch } M_\lambda^{(i)} = \text{tr}_{V(\Lambda_i|\lambda)}(q^{-d+i/4}) = \sum_{n-2\ell = \lambda, n \equiv i \mod. 2} \frac{q^\frac{n^2}{4}}{[n]_q!} \left( \begin{bmatrix} n \\ \ell \end{bmatrix}_q - \begin{bmatrix} n \\ \ell - 1 \end{bmatrix}_q \right), \quad (12)
\]

where
\[
[n]_q = 1 - q^n, \quad [n]_q! = \prod_{j=1}^{n} [j]_q! = \begin{bmatrix} n \\ \ell \end{bmatrix}_q! / [\ell]_q! [n-\ell]_q!.
\]
The $R_n$-module $U_{n,\ell}$ becomes a free module and a basis is given explicitly. Since we consider the case of $n$ even in this paper, we recall the basis only in the case of $n$ even here.

Let $e^{(n)}_k$ be the elementary symmetric polynomial defined by

$$
\prod_{j=1}^{n}(1 + x_j t) = \sum_{k=0}^{n} e^{(n)}_k t^k.
$$

The symmetric polynomial $P^{(2n)}_{r,s}$, $1 \leq r \leq n$, $s \in \mathbb{Z}$ is defined by the following recursion relations:

$$
P^{(2n)}_{1,s} = e^{(2n)}_{2s},
$$

$$
P^{(2n)}_{r,s} = P^{(2n)}_{r-1,s+1} - e^{(2n)}_{2s} P^{(2n)}_{r-1,1} \quad \text{for } r \geq 2.
$$

For a function $f(x_1, \cdots, x_n)$ of $x_j$'s we denote $\bar{f}$ the function obtained from $f$ by specializing the last two variables to $x, -x$:

$$
\bar{f} = f(x_1, \cdots, x_{n-2}, x, -x), \quad x = x_{n-1},
$$

if it has a sense. Then $P^{(2n)}_{r,s}$ satisfy

$$
P^{(2n)}_{r,s} = P^{(2n-2)}_{r,s} - x^2 P^{(2n-2)}_{r,s-1}.
$$

(13)

We set

$$
v_0^{(2n)} = \sum_{j=0}^{n} e^{(2n)}_j X^{2j}, \quad v_r^{(2n)} = \sum_{s=1}^{n} P^{(2n)}_{r,s} X^{2(s-1)}, \quad w_r^{(2n)} = X v_r^{(2n)}, \quad 1 \leq r \leq n,
$$

(14)

and, for $1 \leq j \leq n$,

$$
2 \xi_j^{(2n)}(X_1, X_2) = \frac{X_1 - X_2}{X_1 + X_2} \left( v_0^{(2n)}(X_1)w_j^{(2n)}(X_2) + v_0^{(2n)}(X_2)w_j^{(2n)}(X_1) \right)
$$

$$
\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad - v_0^{(2n)}(X_1)w_j^{(2n)}(X_2) + v_0^{(2n)}(X_2)w_j^{(2n)}(X_1).
$$

(15)

Equation (13) implies that $v_r^{(2n)}, w_r^{(2n)} \in U_{2n,1}$, $1 \leq r \leq n$ and $\xi_j^{(2n)} \in U_{2n,2}$.

We use the multi-index notations like $v_I = v_{i_1} \wedge \cdots \wedge v_{i_{\ell}}$ for $I = (i_1, \cdots, i_{\ell})$. Then, in general, we have the following theorem.

Theorem 2 As an $R_{2n}$-module, $U_{2n,\ell}$ is a free module with the following set of elements as a basis:

$$
v_j^{(2n)} \wedge w_j^{(2n)} \wedge \xi_k^{(2n)},
$$

$I = (i_1, \cdots, i_{\ell_1})$, $1 \leq i_1 < \cdots < i_{\ell_1} \leq n$,

$J = (j_1, \cdots, j_{\ell_2})$, $1 \leq j_1 < \cdots < j_{\ell_2} \leq n - \ell_1 - \ell_3$,

$K = (k_1, \cdots, k_{\ell_3})$, $1 \leq k_1 \leq \cdots \leq k_{\ell_3} \leq n - \ell_1 - \ell_3 + 1$,

$\ell_1 + \ell_2 + 2\ell_3 = \ell$. 

3 Chiral subspace

Let $m$ and $r$ be non-negative integers such that $0 \leq r \leq m$. We fix them once for all in this section. We set $\lambda = 2m - 2r$ and $\ell_2 = r + n - m$. Define $P_{2n} = 0$ for $n < m$. To each $P_{2n} \in U_{2m, r}$ we shall construct polynomials $P_{2n}(X_1, ..., X_{2n}, |x_1, ..., x_{2n})$, $n \geq m$ such that $(\Psi_{P_{2n}})_{n=0}^\infty$ satisfy $\text{[11], [22], [33]}$. Let us first discuss the symmetry of equations (1), (2), (3). Let $p = 1$ and $\lambda = 0$. Suppose that $(\Psi_{p_{2n-1}})_{n=0}^\infty$ is a solution of the same equations, since $p_{2n-1}^{(2n)} = p_{2n-1}^{(2n-2)}$. Thus the space of solutions of $\text{[11], [22], [33]}$ becomes a module over the ring $Z[p_{\pm 1}, p_{\pm 3}, \cdots]$. The action of $p_{2n-1}$ is that of the local integral of motion with spin $2s - 1$. We introduce the generating function of $p_{2n-1}^{(n)}$:

$$E_{\text{odd}}^{(n)}(t) = \exp(\sum_{j \in \mathbb{Z}} t_{2j-1}p_{2j-1}^{(n)}),$$

which satisfies the equation

$$E_{\text{odd}}^{(n)}(t) = E_{\text{odd}}^{(n-2)}(t).$$

Next we consider the functions

$$Q_{2n}^{(\pm)}(z) = \prod_{j=1}^{2n} (1 - x_j^{\pm 1}z),$$

for $n \geq m$. For $n = m$ the empty product in the numerator is understood as 1, that is,

$$Q_{2m}^{(\pm)}(z) = \prod_{j=1}^{2m} (1 - x_j^{\pm 1}z) = \sum_{j=0}^{\infty} h_j^{(2m)} z^j,$$

where $h_j^{(2m)} := h_j^{(2m)}$ is the complete symmetric function $\text{[10]}$. They satisfy the equation

$$\rho_\epsilon(Q_{2n}^{(\epsilon)}(z)) = Q_{2n-2}^{(\epsilon)}(z),$$

for $\epsilon = \pm$. These functions were extracted from the integral formulae for the form factors of local operators defined by Lukyanov $\text{[13], [14], [40]}$.

Take $I, J, K$ such that they satisfy conditions in Theorem 2 with $\ell_1 + \ell_2 + 2\ell_3 = r$. We set

$$P_{2m} = E_{\text{odd}}^{(2m)}(t) \prod_{i=1}^{m} Q_{2n}^{(\pm)}(z_i) v_{\ell_1}^{(2m)} w_{\ell_2}^{(2m)} \xi_{K}^{(2m)},$$

$$P_{2n} = c_{2n} E_{\text{odd}}^{(2n)}(t) \prod_{i=1}^{m} Q_{2n}^{(\pm)}(z_i) v_{\ell_1}^{(2n)} w_{\ell_2}^{(2n)} \xi_{K}^{(2n)} \prod_{a=r+1}^{\ell_2} \prod_{n} x_n^{2n+1+2r-2a}, \text{ for } n \geq m,$$

where the constant $c_{2n}$ is given by

$$c_{2n} = (-1)^{(n-m)(n+m-2r-1)(i\xi(-\pi i))^{m-n}(2\pi i)^{(m-n)(n+r)}}.$$ 

In $\text{[17], [18]}$, $v_{\ell_1}^{(2n)} \land w_{\ell_2}^{(2n)} \land \xi_{K}^{(2n)}$ is understood as the polynomial of $X_1, \cdots, X_r$. We expand $P_{2n}$ as

$$P_{2n} = \sum_{\alpha, \gamma} P_{2n, \alpha, \gamma} t_{\alpha} z_{\gamma},$$

where $\alpha = (\ldots, \alpha_1, \alpha_3, \ldots)$, $\gamma = (\gamma_1, \ldots, \gamma_m)$ and $t_{\alpha} = \prod t_{\alpha_i}^2$, $z_{\gamma} = \prod z_{\gamma_i}^2$. 

7
Theorem 3 (1). The set of functions \((\Psi_{P_{2n}})_{n=0}^{\infty}\) satisfies \([11], [12], [13]\) and each \(\Psi_{P_{2n}}\) takes the value in \((V \otimes 2n)^{\text{sing}}\), \(\lambda = 2m - 2r\).

(2). The following property holds for all \(n\):

\[
\Psi_{P_{2n, \alpha, \gamma}}(\beta_1 + \theta, \cdots, \beta_{2n} + \theta) = \exp(\theta \text{deg}_2 P_{2m, \alpha, \gamma}) \Psi_{P_{2n, \alpha, \gamma}}(\beta_1, \cdots, \beta_{2n}).
\]

The property (2) in Theorem 3 was proved in \([11], [13]\). It means that the Lorentz spin of the local operator specified by the set of form factors \((\Psi_{P_{2n}})_{n=0}^{\infty}\) is \(\text{deg}_2 P_{2m, \alpha, \gamma}\). Explicitly \(\text{deg}_2 P_{2m, \alpha, \gamma}\) is given by

\[
\text{deg}_2 P_{2m, \alpha, \gamma} = m^2 + \sum_i \alpha_i + \sum \gamma_i + \text{deg}_1(v_i^{(2m)} \wedge w_j^{(2m)} \wedge \xi_K^{(2m)}).
\]

As shown in Theorem 3 in the next section \(R_{2m}\) is a free \(\mathbb{C}[p_1^{(2m)}, p_2^{(2m)}, \cdots, p_{2m-1}]\) module with the \(n\)-th ordered products of \(h_{2j}\)'s as a basis, that is,

\[
R_{2m} = \oplus_{0 \leq r_1 \leq \cdots \leq r_m} \mathbb{C}[p_1^{(2m)}, p_2^{(2m)}, \cdots, p_{2m-1}] h_{2r_1} \cdots h_{2r_m}.
\]

Thus any element of \(U_{2m,r}\) can be expressed as a linear combination of \(P_{2m, \alpha, \gamma}\)'s, that is,

\[
U_{2m,r} = \sum_{\alpha, \gamma} \mathbb{C} P_{2m, \alpha, \gamma},
\]

where the summation is taken for all \(\gamma\) and \(\alpha\) with \(\alpha_i = 0, i < 0\) or \(i > 2m\).

Let \(\mathcal{O}_\lambda^{\text{sing}}(2m)\) be the space of meromorphic solutions to \([11], [12], [13]\) such that \(F_{2n} = 0, n < m\) and \(F_{2n} \in \text{Sol}_\lambda^n\) for all \(n\). We set \(\mathcal{O}_\lambda^{\text{sing}} = \mathcal{O}_\lambda^{\text{sing}}(0)\). Then we have

\[
\mathcal{O}_\lambda^{\text{sing}} = \mathcal{O}_\lambda^{\text{sing}}(0) \supset \mathcal{O}_\lambda^{\text{sing}}(2) \supset \mathcal{O}_\lambda^{\text{sing}}(4) \supset \cdots.
\]

For each \(m\) there is a map

\[
\Psi : \mathcal{O}^{\text{sing}}_{2m-2r}(2m) \to \mathcal{O}^{\text{sing}}_{2m-2r}(2m), \quad P_{2m, \alpha, \gamma} \mapsto (\Psi_{P_{2n, \alpha, \gamma}})_{n=0}^{\infty},
\]

where \(\alpha\) is such that \(\alpha_i = 0\) for \(i < 0\) or \(i > 2m\). It induces the map

\[
\Psi : \mathcal{O}^{\text{sing}}_{2m,r} \to \mathcal{O}^{\text{sing}}_{2m-2r}(2m).
\]

Finally we obtain the map

\[
\Psi : \mathcal{M}^{(0)}_{\lambda} = \oplus_{\lambda = 2m-2r} \mathcal{M}_{2m,r} \to \mathcal{O}_{\lambda}^{\text{sing}}. \quad (20)
\]

Let \(\mathcal{O}\) be the space of meromorphic solutions to \([11], [12], [13]\) and \(V_\lambda \simeq \mathbb{C}^{\lambda+1}\) the irreducible representation of \(sl_2\) with the highest weight \(\lambda\). We consider \(\mathcal{M}^{(0)}_{\lambda}\) as the trivial \(sl_2\)-module. Then \(20\) induces the map of \(sl_2\)-modules

\[
\oplus_{\lambda} \mathcal{M}^{(0)}_{\lambda} \otimes V_\lambda \to \mathcal{O}. \quad (21)
\]

We call the image of \((21)\) the chiral subspace of the space of local operators in \(SU(2)\) ITM. If we assume the conjecture \([11]\), the map \((21)\) is injective and the chiral subspace is isomorphic to \(V(\Lambda_0)\) due to Theorem 3

Now let us prove Theorem 3. The following theorem has been proved in \([13]\).
Theorem 4 \cite{13} Let \( P_{2n}(X_1, \ldots, X_{\ell_{2n}}, |x_1, \ldots, x_{2n}) \), \( n \geq m \) be polynomials in \( X_1, \ldots, X_{\ell_{2n}} \) with the coefficients in the ring of symmetric Laurent polynomials of \( x_j \)'s such that \( \deg_{X_a} P \leq 2n \) for \( 1 \leq a \leq \ell_{2n} \). Set \( P_{2n} = 0 \) for \( n < m \). Suppose that \( (P_{2n})_{n=0}^{\infty} \) satisfy the following conditions: there exists a set of polynomials \( (P'_{2n})_{n=0}^{\infty} \) such that

\[
\text{Asym}\left( P_{2n} \right) = \text{Asym}\left( \prod_{a=1}^{\ell_{2n}-1} (1 - x^2 X_a^2) P'_{2n} \right),
\]

\[
P'_{2n}|_{X_{\ell_{2n}} = \pm x^{-1}} = \pm x^{-(2n-1)} d_{2n} P_{2n-2},
\]

\[
d_{2n} = \frac{2\pi}{\zeta(-\pi i)} (-2\pi i)^{-r+m-2n},
\]

where Asym is the anti-symmetrization with respect to \( X_{r+1}, \ldots, X_{\ell_{2n}} \). Then \( (\Psi P_{2n})_{n=0}^{\infty} \) satisfies \( \chi \), \( \Re \), \( \Re \).

Let us show that \( \chi \), \( \Re \) satisfy \( \mu \) and \( \mu \).

Lemma 1 For \( 0 \leq i \leq n \), \( 1 \leq j \leq n \) we have

\[
v_i^{(2n)}(X) = (1 - x^2 X^2) v_i^{(2n-2)}(X), \quad w_j^{(2n)}(X) = (1 - x^2 X^2) w_j^{(2n-2)}(X),
\]

\[
\xi_j^{(2n)}(X_1, X_2) = \prod_{a=1}^{\ell_{2n}-1} (1 - x^2 X_a^2) \xi_j^{(2n-2)}(X_1, X_2).
\]

Proof. Using \( \Re \) we easily have the results for \( v_i^{(2n)} \) and \( w_i^{(2n)} \) for \( i \neq 0 \). The equation for \( v_0^{(2n)} \) is similarly verified using

\[
e_i^{(2n)} = e_i^{(2n-2)} - x^2 e_i^{(2n-2)}. \]

The equation for \( \xi_j^{(2n)} \) follows from those of \( v_i^{(2n)} \) and \( w_i^{(2n)} \).

By this lemma we have

\[
v_j^{(2n)} \wedge w_j^{(2n)} \wedge \xi_j^{(2n)} = \prod_{a=1}^{\ell_{2n}-1} (1 - x^2 X_a^2) v_j^{(2n-2)} \wedge w_j^{(2n-2)} \wedge \xi_j^{(2n-2)}. \]  

The following lemma is proved in \( \Re \).

Lemma 2 Let Asym denote the anti-symmetrization with respect to \( X_{r+1}, \ldots, X_{\ell_{2n}} \). We have

\[
\text{Asym}\left( \prod_{a=r+1}^{\ell_{2n}} X_a^{n+1+2r-2a} \right)
\]

\[
= (-1)^{\ell_{2n}-r-1} \text{Asym}(X_{\ell_{2n}}^{2n-1} \prod_{a=r+1}^{\ell_{2n}-1} (1 - x^2 X_a^2) X_a^{2n-1+2r-2a}).
\]

It follows from \( \mu \) that

\[
P_{2n} = \prod_{a=1}^{r} (1 - x^2 X_a^2) P_{2n},
\]

\[
P_{2n}'' = c_{2n} E_{odd}^{(2n-2)}(t) \prod_{i=1}^{n} Q^{(1)}_{2n}(z_i) v_j^{(2n-2)} \wedge w_j^{(2n-2)} \wedge \xi_j^{(2n-2)} \prod_{a=r+1}^{\ell_{2n}} X_a^{2n+1+2r-2a}.
\]

9
As a basis:

Theorem 5

The module \( R_n \) is a free \( R_{n'}^{add} \)-module with the elements \( \{ h_{2i_1}, \ldots, h_{2i_{n'}} \} \) as a basis:

\[
R_n = \bigoplus_{0 \leq i_1 \leq \cdots \leq i_{n'}} R_{n'}^{odd} h_{2i_1} \cdots h_{2i_{n'}}.
\]

For the proof of this theorem we need to prepare some propositions and lemmas. We denote \( R'_n \) the subring of \( R_n \) generated by \( e_j \) with \( j \) odd, \( j \leq n' \):

\[
R'_n = \mathbb{C}[e_{2j+1} | 0 \leq j \leq \frac{n-1}{2}].
\]
**Proposition 1** If $R_n$ is generated by $\{h_{i_1} \cdots h_{i_n}, |i_1, \cdots, i_n| \geq 0\}$ over $R_n'$, it is generated by the same set over $R_n^{\text{odd}}$.

To prove this proposition we notice the following lemma.

**Lemma 3** In $R_n$, the following conditions are equivalent.

1. $e_{2j+1} = 0$ for $0 \leq j \leq (n-1)/2$.
2. $p_{2j+1} = 0$ for $0 \leq j \leq (n-1)/2$.

**Proof.** By the formula (cf. [10])

$$ke_k = \sum_{r=1}^{k} (-1)^{r-1} p_r e_{k-r},$$

(27)

each $e_k$ is expressed as a homogeneous polynomial of $p_1, \ldots, p_k$ and vice versa where the degrees are defined by $\deg p_k = k, \deg e_k = k$. Suppose that $e_j = 0$ for all odd $j$. Then, for odd $k$, if $p_k$ is written as a polynomial of $e_i$'s, each monomial contains at least one $e_l$ with $l$ being odd and $p_k = 0$. Thus (1) $\Rightarrow$ (2) is proved. The converse is similarly proved. ■

**Proof of Proposition [4]**

To avoid the notational complexity we give a proof in the case of $n$ even and use $2n$ instead of $n$. The case of $n$ odd is similarly proved.

Let $E_{r_1 \cdots r_n} = e_{r_1}^1 e_{r_2}^2 \cdots e_{r_n}^n$. Since $R_{2n}' = \mathbb{C}[e_1, e_3, \cdots, e_{2n-1}]$, $R_{2n}$ is a free $R_{2n}'$-module with the elements $\{E_{r_1 \cdots r_n} | r_j \in \mathbb{Z}_{\geq 0}, \forall j\}$ as a basis. By the assumption one can write

$$E_{r_1 \cdots r_n} = \sum_{s_1, \ldots, s_n \geq 0} f_{1 \cdots n}^{s_1 \cdots s_n} (e_1, e_3, \cdots, e_{2n-1}) h_{s_1} \cdots h_{s_n},$$

for some $f_{1 \cdots n}^{s_1 \cdots s_n} \in R_{2n}'$. Set

$$F_{r_1 \cdots r_n} = \sum_{s_1, \ldots, s_n \geq 0} f_{1 \cdots n}^{s_1 \cdots s_n} (p_1, p_3, \cdots, p_{2n-1}) h_{s_1} \cdots h_{s_n}.$$  

**Lemma 4** The set of polynomials $\{F_{r_1 \cdots r_n} | r_j \in \mathbb{Z}_{\geq 0} \forall j\}$ is linearly independent over $R_{2n}^{\text{odd}}$.

**Proof.** It is sufficient to prove that $\{F_{r_1 \cdots r_n}\}$ is linearly independent over $\mathbb{C}$ at $p_1 = p_3 = \cdots = p_{2n-1} = 0$. By Lemma [3]

$$F_{r_1 \cdots r_n} |_{p_{2j-1} = 0, j \leq n} = \sum_{s_1, \ldots, s_n \geq 0} f_{1 \cdots n}^{s_1 \cdots s_n} (0, \cdots, 0) (h_{s_1} \cdots h_{s_n}) |_{p_{2j-1} = 0, j \leq n} = E_{r_1 \cdots r_n} |_{p_{2j-1} = 0, j \leq n}.$$  

Thus we have to prove that $E_{r_1 \cdots r_n} := E_{r_1 \cdots r_n} |_{p_{2j-1} = 0, j \leq n}$ are linearly independent.

**Lemma 5** At $e_{2j-1} = 0, 1 \leq j \leq n$ we have

$$e_{2k} = -\frac{1}{2k} p_{2k} + \cdots, \quad 1 \leq k \leq n,$$

where $\cdots$ part is a polynomial of $p_2, p_4, \ldots, p_{2k-2}$.  

11
Proof. The lemma easily follows from \([\ref{eq:27}]\). On the set \(\{(r_1, \ldots, r_n)\}\) of non-negative integers we introduce the lexicographical order from the right. Suppose that

\[
\sum c_{r_1, \ldots, r_n} \tilde{E}_{r_1, \ldots, r_n} = 0,
\]

and \((r_1^0, \ldots, r_n^0)\) is the largest index such that \(c_{r_1, \ldots, r_n} \neq 0\). If one expresses each \(\tilde{E}_{r_1, \ldots, r_n}\) as a polynomial of \(p_2, p_4, \ldots, p_{2n}\), one has, by Lemma \([\ref{lem:2}]\)

\[
c_{r_1, \ldots, r_n} p_2^{r_1} \cdots p_{2n}^{r_n} + \sum_{(r_1, \ldots, r_n) < (r_1^0, \ldots, r_n^0)} c'_{r_1, \ldots, r_n} p_2^{r_1} \cdots p_{2n}^{r_n} = 0,
\]

for some constants \(c'_{r_1, \ldots, r_n}\). Thus \(c_{r_1, \ldots, r_n} \neq 0\) which contradicts the assumption. Therefore Lemma \([\ref{lem:4}]\) is proved.

Since \(R_{2n} = \oplus R_{2n}^- E_{r_1, \ldots, r_n}\), \(\deg F_{r_1, \ldots, r_n} = \deg E_{r_1, \ldots, r_n}\) and \(R_{2n}^2 = \deg R_{2n}^2\), we have

\[
\operatorname{ch} \left( \oplus R_{2n}^{odd} F_{r_1, \ldots, r_n} \right) = \operatorname{ch} R_{2n}.
\]

Thus \(R_{2n} = \oplus R_{2n}^{odd} F_{r_1, \ldots, r_n}\) which proves Proposition \([\ref{prop:2}]\).

Proposition 2 The module \(R_n\) is generated by \(\{h_{i_1} \cdots h_{i_n} \mid i_j \in \mathbb{Z}_{\geq 0} \forall j\}\) over \(R'_n\).

Proof. As in the previous proposition we give a proof for \(n\) being even. The proof of the case \(n\) odd is similar. In this proof we again use \(2n\) instead of \(n\). Then \((2n)^! = n\).

For a partition \(\lambda = (\lambda_1, \ldots, \lambda_m)\) we denote \(s_\lambda\) the Schur function and \(\ell(\lambda)\) the length of \(\lambda\) \([\ref{eq:10}]\).

Let \(L_{2n}\) be the submodule of \(R_{2n}\) generated by \(\{s_\lambda \mid \ell(\lambda) \leq n\}\) over \(R_{2n}\). Due to the formula

\[
s_\lambda = \det(h_{\lambda_1-i+j})_{1 \leq i, j \leq \ell(\lambda)},
\]

it is sufficient to prove \(R_{2n} = L_{2n}\). Since \(\{s_\lambda \mid \ell(\lambda) \leq n\}\) is a \(\mathbb{C}\)-linear basis of \(R_{2n}\), we have to prove \(s_\lambda \in L_{2n}\) for any partition \(\lambda = (\lambda_1, \ldots, \lambda_{2n})\). We prove this by induction on the degree \(|\lambda| = \lambda_1 + \cdots + \lambda_{2n}\) of \(s_\lambda\).

If \(|\lambda| \leq n\) for \(\lambda = (\lambda_1, \ldots, \lambda_{2n})\), \(\lambda_j = 0\) for \(j \geq n + 1\) and \(s_\lambda \in L_{2n}\).

Let \(d > n\). Suppose that \(s_{\mu} \in L_{2n}\) for any partition \(\mu = (\mu_1, \ldots, \mu_{2n})\) with \(|\mu| < d\). We prove \(s_\lambda \in L_{2n}\) for any \(\lambda = (\lambda_1, \ldots, \lambda_{2n})\) with \(|\lambda| = d\). To this end we introduce the lexicographical order from the left on the set of partitions \(\{(\mu_1, \ldots, \mu_{2n})\}\). We prove \(s_\lambda \in L_{2n}\) by descending induction on this order of \(\lambda\).

For \(\lambda = (d)\) we have \(s_\lambda \in L_{2n}\) since \(\ell(\lambda) = 1 \leq n\).

Let \(\lambda\) be such that \(\lambda < (d)\). Suppose that \(s_\mu \in L_{2n}\) for any \(\mu > \lambda\) with \(|\mu| = d\). We assume that \(\lambda\) is of the form \(\lambda = (\lambda_1, \ldots, \lambda_m)\), \(\lambda_m \neq 0\). Since \(s_\lambda \in L_{2n}\) for \(m \leq n\), we assume that \(n + 1 \leq m \leq 2n\).

First consider the case \(\lambda_m = 1\). Let \(\lambda = (\lambda_1, \ldots, \lambda_{m-1})\). Then \(|\lambda| = |\lambda| - 1\) and \(s_\lambda \in L_{2n}\) by the assumption of induction on \(|\lambda|\). Recall that \(s_{(1^{k})} = e_k\). Then \(s_\lambda s_{(1)} \in L_{2n}\). By Pieri’s formula (cf. \([\ref{eq:10}]\))

\[
s_\lambda s_{(1)} = s_\lambda + \sum_{\mu > \lambda} c_\mu s_\mu,
\]

for some constants \(c_\mu\). By the hypothesis of induction on the order of \(\lambda\) the second term of the right hand side of \([\ref{eq:28}]\) is in \(L_{2n}\). Thus \(s_\lambda \in L_{2n}\).

Next consider the case \(m = 2n\). We set \(\tilde{\lambda} = (\lambda_1, \ldots, \lambda_{2n-1}, \lambda_{2n} - 1)\). Then \(\tilde{\lambda}\) is a partition and \(s_\tilde{\lambda} \in L_{2n}\) by the induction hypothesis on \(|\lambda|\). Then \(s_\tilde{\lambda} s_{(1)} \in L_{2n}\) and the equation \([\ref{eq:28}]\) holds. Thus \(s_\lambda \in L_{2n}\).

Let us assume \(\lambda_m \geq 2\) and \(m < 2n\).
Lemma 6 Let $m \leq m' \leq 2n$ and $i_j \in \mathbb{Z}_{\geq 0}$, $m + m' - 2n + 1 \leq j \leq m$ be such that $\sum i_j = 2n - m'$ and

$$\lambda^{(m')} := (\lambda_1, \cdots, \lambda_{m+m'-2n}, \lambda_{m+m'-2n+1} - i_{m+m'-2n+1}, \cdots, \lambda_m - i_m, 1^{2n-m'})$$

is a partition. Then $s_{\lambda^{(m')}} \in L_{2n}$.

Proof. In this proof we use the following notations. Let $e_i = (0, \cdots, 1, \cdots, 0)$ where 1 is on the $i$-th position. For $N \geq 1$ and $a = (a_i), b = (b_i) \in \mathbb{Z}^N$ we define $a + b = (a_i + b_i)$. For $N_1 \leq N_2$, $a = (a_i) \in \mathbb{Z}^{N_1}$ is considered as an element of $\mathbb{Z}^{N_2}$ by setting $a = (a_i)$, $a_i = 0$ for $N_1 < i$.

We prove the lemma by induction on $m'$.

Suppose that $m' = m$. Set

$$\tilde{\lambda}^{(m)} := (\lambda_1, \cdots, \lambda_{2m-2n-1}, \lambda_{2m-2n} - 1, \lambda_{2m-2n+1} - i_{2m-2n+1} - 1, \cdots, \lambda_m - i_m - 1).$$

Since $m < 2n$ and $\lambda^{(m)}$ is a partition, $\lambda_m - i_m \geq 1$. Then $\tilde{\lambda}^{(m)}$ becomes a partition and $|\tilde{\lambda}^{(m)}| = |\lambda| - (2(2n - m) + 1) < |\lambda|$. Thus $s_{\lambda^{(m)}} \in L_{2n}$ by the assumption of induction on $|\lambda|$. Then

$$s_{\tilde{\lambda}^{(m)}} s_{\lambda^{(2n-m')}} = s_{\lambda^{(m)}} + \sum_{\mu > \lambda} c_{\mu} s_{\mu},$$

for some constants $c_{\mu}$'s. Thus $s_{\lambda^{(m)}} \in L_{2n}$ by induction on the order of $\lambda$.

Assume that $m < m' \leq 2n$ and that the lemma holds for any $m''$ satisfying $m \leq m'' < m'$. Let

$$\tilde{\lambda}^{(m')} = (\lambda_1, \cdots, \lambda_{m+m'-2n-1}, \lambda_{m+m'-2n} - 1, \lambda_{m+m'-2n+1} - i_{m+m'-2n+1} - 1, \cdots, \lambda_m - i_m - 1).$$

Here $\lambda_m - i_m \geq 1$. In fact if $m' < 2n$ then it is obvious that $\lambda_m - i_m \geq 1$. If $m' = 2n$ then $i_j = 0$ for all $j$ and $\lambda_m - i_m = \lambda_m \geq 2$ by assumption. Then $\tilde{\lambda}^{(m')}$ is a partition. Since $|\tilde{\lambda}^{(m')}| < |\lambda|$, $s_{\tilde{\lambda}^{(m')}} \in L_{2n}$ by the hypothesis of induction on $|\lambda|$. Then

$$s_{\tilde{\lambda}^{(m')}} s_{\lambda^{(2n-m'-1)}} = s_{\lambda^{(m')}} + \sum_{k \geq 1} \sum_{\{j\}} s_{\lambda^{(k)}}(\{j\}) + \sum_{\mu > \lambda} c_{\mu} s_{\mu}.$$

Here $\{j\}$ and $\lambda^{(k)}(\{j\})$ are as follows. For a given $k$, $\{j\}$ is a set of numbers satisfying

$$m + m' - 2n \leq j_1 < \cdots < j_{m+m'-k+1} \leq m.$$

Then $\lambda^{(k)}(\{j\})$ is defined by

$$\lambda^{(k)}(\{j\}) = \tilde{\lambda}^{(m')} + \sum_{i} e_{j_i} + (0^m, 1^{2n-m'+k}).$$

If $\lambda^{(k)}(\{j\})$ is not a partition we define $s_{\lambda^{(k)}}(\{j\}) = 0$. If $\lambda^{(k)}(\{j\})$ is a partition and $\ell(\lambda^{(k)}(\{j\})) > 2n$ then $s_{\lambda^{(k)}}(\{j\}) = 0$. Consider $k$ such that $\lambda^{(k)}(\{j\})$ is a partition and $\ell(\lambda^{(k)}(\{j\})) \leq 2n$. Since

$$\lambda = \tilde{\lambda}^{(m')} + \sum_{p=m+m'-2n} \epsilon_p + \sum_{j=m+m'-2n+1} i_j \epsilon_j,$$

we have

$$\lambda^{(k)}(\{j\}) - \lambda = - \sum_{p=m+m'-2n} \epsilon_p + \sum_{i=1}^m \epsilon_i + \sum_{j=m+m'-2n+1} \sum_{i} i_j \epsilon_j + (0^m, 1^{2n-m'+k}).$$

(29)
Let us write
\[ \sum_{p=m+m'-2n}^{m} \epsilon_{p} - \sum_{i=1}^{2n-m'-k+1} \epsilon_{j_{i}} + \sum_{j=m+m'-2n+1}^{m} \epsilon_{j} \epsilon_{j} = (0^{m+m'-2n-k}, i'_{m+m'-k-2n+1}, \ldots, i'_{m}) =: \lambda'. \]

Then \( i'_{j} \geq 0 \) for any \( j \), \( \sum i'_{j} = 2n - (m' - k) \) and
\[ \lambda - \lambda' + (0^{m}, 1^{2n-m'+k}) = \lambda^{[k]}(\{j_{i}\}) \]
is a partition. Thus \( (i'_{j}) \) satisfies the condition of \( (i_{j}) \) for \( m' - k \). Since \( k \geq 1 \), \( s_{\lambda^{[k]}}(\{j_{i}\}) \in L_{2n} \) by the assumption of induction on \( m' \). Thus \( s_{\lambda^{[m']}} \in L_{2n} \). 

Consider the case \( m' = 2n \) in Lemma 6 Then \( \lambda^{(2m)} = \lambda \). Thus \( s_{\lambda} \in L_{2n} \). This completes the proof of Proposition 2. 

**Lemma 7** If \( e_{2j+1} = 0 \) for all \( j \leq (n - 1)/2 \), \( h_{2r-1} = 0 \) for any \( r \geq 1 \).

**Proof.** The lemma is easily proved by induction on \( r \) using the relation
\[ \sum_{i=0}^{m} (-1)^{i} e_{i} h_{m-i} = 0, \quad m \geq 1. \]

**Proof of Theorem 3**
Consider the specialization \( p_{2j+1} = 0 \) for all \( j \leq (n - 1)/2 \). Then \( R_{n} = \mathbb{C}[p_{2}, p_{4}, \ldots, p_{2m}] \) and
\[ \chi_{R_{n}} = \prod_{j=1}^{n'} \frac{1}{1 - q^{2j}}. \tag{30} \]

By Lemma 3 we have \( e_{2j+1} = 0 \) for \( j \leq (n - 1)/2 \). Then \( h_{2r-1} = 0 \) for any \( r \geq 1 \) by Lemma 7. Thus
\[ R_{n} = \sum_{0 \leq r_{1} \leq \cdots \leq r_{n'}} \mathbb{C} h_{2r_{1}} \cdots h_{2r_{n'}}, \tag{31} \]
by Proposition 1 and Proposition 2. For two formal power series \( f(q), g(q) \) of \( q \) with the coefficients in \( \mathbb{R} \) we denote \( g(q) \leq f(q) \) if all coefficients of \( f(q) - g(q) \) is non-negative. Then
\[ \chi(\text{RHS of } \text{31}) \leq \sum_{0 \leq r_{1} \leq \cdots \leq r_{n'}} q^{2(r_{1} + \cdots + r_{n'})} = \prod_{j=1}^{n'} \frac{1}{1 - q^{2j}} = \chi_{R_{n}}. \tag{32} \]

Thus the inequality in 32 is in fact the equality. This means that \( \{h_{2r_{1}} \cdots h_{2r_{n'}} | 0 \leq r_{1} \leq \cdots \leq r_{n'} \} \) are linearly independent over \( \mathbb{C} \) at \( p_{2j+1} = 0, \ j \leq (n - 1)/2 \). Thus \( \{h_{2r_{1}} \cdots h_{2r_{n'}} | 0 \leq r_{1} \leq \cdots \leq r_{n'} \} \) are linearly independent over \( R_{n}^{\text{odd}} \). Then
\[ \chi(\oplus_{0 \leq r_{1} \leq \cdots \leq r_{n'}} R_{n}^{\text{odd}} h_{2r_{1}} \cdots h_{2r_{n'}}) = \prod_{j=1}^{n} \frac{1}{1 - q^{2j}} = \chi_{R_{n}}. \]

This completes the proof. 

14
5 Anti-chiral subspace

In this section we construct local operators whose values of spins are minus of those of the operators in the chiral subspace in section 3. We call the subspace of such operators the anti-chiral subspace.

Let $R_n = \mathbb{C}[x_1^{-1}, \ldots, x_n^{-1}]$, $H^{(n)} = \bigotimes_{j=0}^{n-1} X_j^{-1}$, $\prod_{j=1}^{n} (1 + x_j^{-1}t) = \sum_{j=0}^{n} \epsilon_j^{(n)} t^j$ and $U_{n, \ell}^{-} = \{ P \in \wedge^\ell H^{(n)} | \rho_{\pm}(P) = 0 \}$.

Define the map

$$- : \wedge^\ell H^{(n)} \to \wedge^\ell H^{(n)}$$

by

$$P(X_1, \cdots, X_\ell | x_1, \cdots, x_n) \mapsto P^- := P(X_1^{-1}, \cdots, X_\ell^{-1} | x_1^{-1}, \cdots, x_n^{-1}),$$

where we set $H^{(n)+} = H^{(n)}$. Obviously $P^{-} = P$ and the map $-$ gives an isomorphism of $\mathbb{C}$-vector spaces. This map induces an isomorphism between $U_{n, \ell}$ and $U_{n, \ell}^{-}$ as a vector space. In fact for $P \in \wedge^\ell H^{(n)}$ we have

$$P|_{X_{\ell}=\pm x^{-1}, x_n=-x_{n-1}} = 0 \iff P^-|_{X_{\ell}=\pm x,x_{n}=-x_{n-1}} = 0.$$

The last condition is equivalent to $\rho_{\pm}(P^-) = 0$. Moreover the map satisfies, for $f \in R_n$ and $P \in \wedge^\ell H^{(n)}$,

$$(fP)^- = f^- P^-.$$

Thus we have

Corollary 1 The module $U_{n, \ell}^{-}$ is a free $R_n$-module with the elements $\{e_{I}^{(2n)} \wedge w_{j}^{(2n)} \wedge \xi_{K}^{(2n)} \}$ as a basis, where $I, J, K$ satisfy the same conditions as in Theorem 3.

We set

$$M_{n, \ell}^{-} = \frac{U_{n, \ell}^{-}}{U_{n, \ell-1}^{-} \wedge \Xi_1^{(n)} + U_{n, \ell-2}^{-} \wedge \Xi_2^{(n)}},$$

and, for $i = 0, 1, \lambda \in \mathbb{Z}_{\geq 0}$,

$$M_{\lambda}^{(i)-} = \oplus_{n \equiv i \mod. 2, n-2\ell=\lambda} M_{n, \ell}^{-}.$$

We introduce the degree on $M_{n, \ell}^{-}$ and $M_{\lambda}^{(i)-}$ by $\deg_{2\ell} P = -n^2/4 + \deg_1 P$. It is obvious that $\deg_1 P^- = -\deg_1 P$ for a homogeneous element $P \in H^{(n)\pm}$. Then

$$\text{ch} M_{\lambda}^{(i)-} = \sum_{n-2\ell=\lambda, n \equiv i \mod. 2} q^{\frac{n^2}{8}} \left[ \begin{array}{c} n \\ \ell \end{array} \right]_{q^{-1}} - \left[ \begin{array}{c} n \\ \ell - 1 \end{array} \right]_{q^{-1}}.$$

In the remaining part of this section we consider the case of $n$ even and use $2n$ instead of $n$. Let us consider the part of $\Psi_P$ which is relevant to determining the homogeneous degree of $\Psi_P$. It is

$$\exp \left( \frac{1}{2} \sum_{j=1}^{2n} \beta_j \right) \prod_{a=1}^{\ell} \prod_{j=1}^{2n} (1 - X_a x_j).$$
We rewrite it as
\[
\exp \left( -\frac{n}{2} \sum_{j=1}^{2n} \beta_j \right) \frac{(e^{2n})_{n-t}(\prod_{a=1}^{\ell} X_a^{-2n}) P}{\prod_{a=1}^{\ell} X_a^{-2n}(1 - X_a^{-1} x_j)}.
\]

For a polynomial \( P = P(X_1, \cdots, X_\ell | x_1, \cdots, x_{2n}) \) of \( X_a \)'s such that \( \deg_{X_a} P \leq 2n \) set
\[
\hat{P} = (e^{2n})_{n-t}(\prod_{a=1}^{\ell} X_a^{-2n}) P.
\]

It is a polynomial of \( X_a^{-1} \)'s such that \( \deg_{X_a} P \leq 2n \). By this notation it is obvious that \( \Psi_P \) is homogeneous of degree \(-n^2 + \deg_1 \hat{P} = \deg_1 \hat{P}\). Rewriting Theorem 4 in terms of \( \hat{P} \) we have

**Theorem 6** Let \( \hat{P}_{2n}(X_1, \ldots, X_{\ell_2}, x_1, \ldots, x_{2n}) \), \( n \geq m \) be polynomials of \( X_a^{-1} \) with the coefficients in the ring of symmetric Laurent polynomials of \( x_j \)'s such that \( \deg_{X_a} \hat{P}_{2n} \leq 2n \) for \( 1 \leq a \leq \ell_{2n} \). Suppose that \((\hat{P}_{2n})_{n=0}^\infty\) satisfy the following conditions: there exists a set of polynomials \((\hat{P}_{2n}')\) such that

\[
\text{Asym} \left( \hat{P}_{2n} \right) = \text{Asym} \left( \prod_{a=1}^{\ell_{2n}-1} (1 - x^{-2} X_a^{-2}) \hat{P}_{2n}', \right),
\]

\[
\hat{P}_{2n}'|_{X_{2n}=\pm x^{-1}} = \pm (-1)^{n-1} x^{2n-1} d_{2n} \hat{P}_{2n-2},
\]

where the notations are same as those in Theorem 4. Set \( P_{2n} = (e_{2n})^{r-m}((\prod_{a=1}^{\ell_2} X_a^{2n}) \hat{P}_{2n} \). Then \((\Psi_{P_{2n}})_{n=0}^\infty\) satisfies \( \Psi \), \( \Theta \), \( \Xi \).

**Proposition 3** The following \((\hat{P}_{2n})_{n=0}^\infty\) satisfies the conditions of Theorem 6

\[
\hat{P}_{2m} = v_{j}^{(2m)^{-}} \wedge w_{j}^{(2m)^{-}} \wedge \xi_{K}^{(2m)^{-}},
\]

\[
\hat{P}_{2n} = c_{2n} v_{j}^{(2n)^{-}} \wedge w_{j}^{(2n)^{-}} \wedge \xi_{K}^{(2n)^{-}} \prod_{a=r+1}^{\ell_{2n}} X_a^{-(2n+1+2r-2a)},
\]

\[
c_{2n} = (i\zeta(-\pi i))^{m-n}(2\pi i)^{(m-n)(n+r)},
\]

where \( \ell_1 + \ell_2 + 2\ell_3 = r \) and \( P_{2n} = 0 \), \( n < m \).

It is possible to multiply \( Q_{2n}^{(-)}(z) \) to \( \hat{P}_{2n} \) simultaneously without destroying the conditions in Theorem 4 if the degrees in \( X_a^{-1} \) remains the permitted range. Let us write the final formula of \( P \) corresponding to the operators in the anti-chiral subspace. Define

\[
P_{2n}^{(-)} = c_{2n} (e_{2n})^{r-m} E_{odd}(t) \prod_{i=1}^{m} Q_{2n}^{(-)}(z) v_{j}^{(2n)^{-}} \wedge w_{j}^{(2n)^{-}} \wedge \xi_{K}^{(2n)^{-}}
\]

\[
\times \prod_{a=1}^{r} X_a^{2n} \prod_{a=r+1}^{\ell_{2n}} X_a^{2(a-r)-1},
\]

and, as in the chiral case,

\[
P_{2n}^{(-)} = \sum I_{2n,a,\alpha}^{(-)} f^{a} z^{\alpha}.
\]
Then
\[ \text{deg}_3 P_{2n,\alpha,\gamma}^{(-)} = -m^2 + \sum i \alpha_i - \sum \gamma_i - \text{deg}_3 (v_i^{(2m)} \wedge w_j^{(2m)} \wedge \xi_k^{(2m)}) , \]
for all \( n \geq m \).

**Corollary 2** (1). The set of functions \( (\Psi_{P_{2n}^{(-)}})_{n=0}^{\infty} \) satisfies (1), (2), (3) and each \( \Psi_{P_{2n}^{(-)}} \) takes the value in \((V \otimes 2^m)^{sing}, \lambda = 2m - 2r\).

(2). The following property holds for all \( n \):
\[ \Psi_{P_{2n,\alpha,\gamma}^{(-)}}(\beta_1 + \theta, \ldots, \beta_{2n} + \theta) = \exp(\theta \text{deg}_3 P_{2m,\alpha,\gamma}^{(-)}) \Psi_{P_{2n,\alpha,\gamma}^{(-)}}(\beta_1, \ldots, \beta_{2n}). \]

Obviously the following equation holds:
\[ \sum_{\alpha,\gamma} C \Psi_{P_{2m,\alpha,\gamma}^{(-)}} = U_{2m,r}^{(-)}, \]
where the summation is taken for all \( \gamma = (\gamma_1, \gamma_2, \ldots) \) and \( \alpha = (\ldots, \alpha_{-3}, \alpha_{-1}, \alpha_1, \ldots) \) with \( \alpha_i = 0 \), \( i \geq 1 \).

### 6 Concluding remarks

In [18] Smirnov has given a systematic construction of form factors of a large number of chargeless local operators in Sine-Gordon model. This construction is extended to the charged operators in SU(2) ITM in [13]. In these descriptions of local operators, however, finding the subspace which has the same character, with respect to spins, as the chiral subspace of local operators in the corresponding conformal field theory has not been successful up to now.

In this paper we have given an alternative construction of local operators in SU(2) ITM based on the results of our previous paper [11] and the ideas of [18, 13]. With this description of local operators we have specified the subspace of operators isomorphic to the level one integrable highest weight representation of \( \hat{sl}_2 \) which is the chiral space of the level one \( su(2) \) WZW model.

Let us give a brief comment here on the differences of the formulae (17), (18) and those of Smirnov in [18]. In [18] the initial condition is taken as
\[ P_{2m} = (e^{(2m)})^{-N} f \Delta_{2m}^+ \prod_{a=1}^{r} X_a^{r_a}, \quad 0 \leq r_a \leq 2m - 1, \quad \Delta_{2m}^+ = \prod_{i<j} (x_i + x_j), \]
for some \( N \geq 0 \) and \( f \in R_{2m} \). Since \( \Delta_{2m}^+ = 0, \text{Asym} X_1, \ldots, X_r \left( (e^{(2m)})^N P_{2m} \right) \in U_{2m,t} \). The important point is that, in general, \( U_{2m,r} \) is not generated by such functions only. That is why we have introduced the polynomials \( v_i^{(n)}, w_j^{(n)} \) and \( \xi_k^{(n)} \). This difference is crucial in the calculation of the character of the chiral subspace of local operators.

Due to this difference of the initial condition the degrees of freedom corresponding to the parameter \( t_2, s \in \mathbb{Z} \) in [13] could not be introduced in our formula. This drawback is compensated by the introduction of the functions \( Q_{2n}^{(\pm)}(z) \). With this respect the structure of symmetric polynomials given in Theorem 5 is important.
A Function $\zeta(\beta)$

The function $\zeta(\beta)$ was introduced in [8, 15, 17]. It is conveniently described using the double gamma function of Barnes [13, 4]. We review it here. We follow [14] about the notations of double gamma function.

For a set of positive real numbers $\omega = (\omega_1, \omega_2)$ define

$$\Gamma_2(z|\omega) = z \exp \left( \gamma_{22}(\omega) + \frac{z^2}{2} \gamma_{21}(\omega) \right) \prod' \left( 1 + \frac{z}{m \omega_1 + n \omega_2} \right) \exp \left( - \frac{z}{m \omega_1 + n \omega_2} + \frac{z^2}{2(m \omega_1 + n \omega_2)^2} \right),$$

where the product $\prod'$ is over all sets of non-negative integers except $(m, n) = (0, 0)$ and $\gamma_{22}(\omega), \gamma_{21}(\omega)$ are given by the formulae

$$-\gamma_{22}(\omega) = 1 - \frac{1}{\omega_1} \sum_{n=1}^\infty \left( \psi\left( \frac{n \omega_2}{\omega_1} \right) - \log \left( \frac{n \omega_2}{\omega_1} \right) + \frac{1}{2} \left( \frac{1}{\omega_1} + \frac{1}{\omega_2} \right) \log \omega_1 \right) \right),$$

$$-\frac{1}{2 \omega_1} (\gamma - \log 2 \pi) + \frac{\omega_1 - \omega_2}{2 \omega_1 \omega_2} \log \frac{\omega_2}{\omega_1} - \frac{\gamma}{2} \left( \frac{1}{\omega_1} + \frac{1}{\omega_2} \right), \quad (36)$$

$$\gamma_{21} = \frac{1}{\omega_1^2} \left( \psi'(\frac{n \omega_2}{\omega_1}) - \frac{\omega_1}{\omega_2} \right) + \frac{\pi^2}{6 \omega_1^2} - \frac{1}{\omega_1 \omega_2} \log \omega_2 + \frac{\gamma}{\omega_1 \omega_2}. \quad (37)$$

Here $\psi$ is the logarithmic derivative of the gamma function and $\gamma$ is the Euler’s constant. Then $\Gamma_2(z|\omega)^{-1}$ becomes an entire function of $z$, is symmetric with respect to $\omega_1$ and $\omega_2$ and satisfies the following difference equation

$$\frac{\Gamma_2(z + \omega_1|\omega)}{\Gamma_2(z|\omega)} = \sqrt{2\pi} \exp \left( \frac{-z}{\omega_2} + \frac{1}{2} \log \omega_2 \right) \frac{1}{\Gamma(\frac{z}{\omega_2})}. \quad (38)$$

Set $\Gamma_2(z) = \Gamma_2(z|2\pi, 2\pi)$ and define

$$\zeta(\beta) = \frac{\Gamma_2(-i\beta + \pi) \Gamma_2(i\beta + 3\pi)}{\Gamma_2(-i\beta) \Gamma_2(i\beta + 2\pi)}. \quad (39)$$

Then it satisfies the following equations

$$\zeta(\beta - 2\pi i) = \zeta(-\beta), \quad (40)$$

$$\zeta(\beta) \zeta(\beta - \pi i) = \frac{(2\pi)^{3/2}}{\Gamma(-i\beta + \pi) \Gamma(\frac{3\beta}{2\pi})}, \quad (41)$$

$$\frac{\zeta(-\beta)}{\zeta(\beta)} = \frac{\Gamma(\frac{\pi + i\beta}{2\pi}) \Gamma(\frac{i\beta}{2\pi})}{\Gamma(\frac{\pi - i\beta}{2\pi}) \Gamma(\frac{\beta}{2\pi})}. \quad (42)$$

The right hand side of (42) is $S_0(\beta)$ in the $S$-matrix.

References

[1] Babelon, O., Bernard, D. and Smirnov, F., Null-vectors in integrable field theory, *Comm. Math. Phys.* **186** (1997), 601-648.

[2] Cardy, J. and Mussardo, G., Form factors of descendent operators in perturbed conformal field theories, *Nucl. Phys.* **B340** (1990), 387-402.
[3] Christe, P., Factorized characters and form factors of descendant operators in perturbed conformal systems, *Int. J. Mod. Phys.* **A6** (1991), 5271-5286.

[4] Jimbo, M. and Miwa, T., Quantum KZ equation with $|q| = 1$ and correlation functions of the XXZ model in the gapless regime, *J. Phys. A: Math. Gen.* **29** (1996), 2923-2958.

[5] Kedem, R., McCoy, B. and Melzer, E., The sums of Rogers, Schur and Ramanujan and the Bose-Fermi correspondence in 1 + 1-dimensional quantum field theory, in *Recent Progress in Statistical Mechanics and Quantum Field Theory*, Ed. P. Bouwknegt et al. World Scientific, Singapore (1995), 195-219.

[6] Koubek, A., The space of local operators in perturbed conformal field theories, *Nucl.Phys. B.* **435** (1995), 703–734.

[7] Kac, V., Infinite dimensional Lie algebras, third edition, Cambridge University Press, 1992.

[8] Kirillov, A.N. and Smirnov, F., A representation of the current algebra connected with the $su(2)$-invariant Thirring model, *Phys. Lett. B* **198** (1987), 506-510.

[9] Lukyanov, S., Free field representation for massive integrable models, *Comm. Math. Phys.* **167** (1995), 183–226.

[10] Macdonald, I.G., Symmetric functions and Hall polynomials, second edition, Oxford University Press, 1995.

[11] Nakayashiki, A., Residues of $q$-hypergeometric integrals and characters of affine Lie algebras, *math.QA/0210168*.

[12] Nakayashiki, A., Pakuliak, S. and Tarasov, V., On solutions of the KZ and qKZ equations at level zero, *Ann. Inst. Henri Poincaré* **71** (1999), 459–496.

[13] Nakayashiki, A., Takeyama, Y., On form factors of SU(2) invariant Thirring model, in Math-Phys Odyssey 2001, Integrable Models and Beyond- in honor of Barry M. McCoy, ed. Kashiwara, M. and Miwa, T., Progr. in Math. Phys., Birkhäuser, 2002, 357–390.

[14] Shintani, T., On a Kronecker limit formula for real quadratic fields, *J. Fac. Sci. Univ. Tokyo* **24** (1977), 167-199.

[15] Smirnov, F., Lectures on integrable massive models of quantum field theory, *Nankai Lectures on Mathem. Physics* (Ge, M-L. and Zhao, B-H. eds.), World Scientific, Singapore, 1990, 1–68.

[16] Smirnov, F., Dynamical symmetries of massive integrable models, *Int. J. Mod. Phys. A* **7** Suppl. **1B** (1992), 813–837.

[17] Smirnov, F., Form factors in completely integrable models of quantum field theories, World Scientific, Singapore, 1992.

[18] Smirnov, F., Counting the local fields in SG theory, *Nucl. Phys. B* **453** (1995), 807–824.

[19] Tarasov, V., Completeness of the hypergeometric solutions of the qKZ equations at level zero, *Amer. Math. Soc. Translations* Ser.2 **201** (2000), 309–321.

[20] Zamolodchikov, A. B., Integrable field theory from conformal field theory, *Adv. Stud. in Pure Math.* **19** (1989), 641–674.