Non-Integrability of Geodesic dynamics of Chazy-Curzon space-time

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Abstract

We study the integrability of the geodesic equations of the Chazy-Curzon space-time. It was established that for the equilibrium point \( p_p = p_z = z = 0 \) and, \( \rho_0 \in (1, 2) \), there are only periodic solutions, the Hamiltonian system, describing geodesic motion of Chazy-Curzon space-time has no additional analytic first integral. Our approach is based on the following: if the system has a family of periodic solutions around an equilibrium and if the period function is infinitely branched then the system has no additional analytical first integral.

1 Introduction

Let \( H \) be a smooth real-valued function in \( 2n \) real variables \( (p, q) \), \( p, q \in \mathbb{R}^n \). Assume that \( dH(a) = 0 \), where, \( a \) is an equilibrium point for the Hamiltonian system \( X_H \) (with \( n \) degrees of freedom), given by

\[
\frac{dq}{dt} = \frac{\partial H}{\partial p}, \quad \frac{dp}{dt} = -\frac{\partial H}{\partial q}.
\]

We write the Hamiltonian systems in the form

\[
\dot{x} = X_H(x), \quad x \in \mathbb{R}^{2n},
\]

where \( X_H \) is the hamiltonian vector field. The system is called Liouville - Arnold Integrable near \( a \) if there exist \( n \) functions in involution \( f_1 = H, f_2, \ldots f_n \), defined around \( a \), which are functionally independent. The Poisson bracket of \( f \) and \( g \) are

\[
\{f, g\} = X_f(g) = \sum \frac{\partial f}{\partial p} \frac{\partial g}{\partial q} - \frac{\partial f}{\partial q} \frac{\partial g}{\partial p} = -\{g, f\}.
\]

We say that the functions \( f \) and \( g \) are in involution if the Poisson bracket is equal to 0. This means that \( df_1, df_2, \ldots df_n \) are linearly independent around the equilibrium \( a \) and \( f_j = \text{const} \).
for all \( j \) define a smooth submanifold, this level manifold is invariant under \( X_{f_j} \). We have \( X_{f_j} f_j = 0 \) and \([X_{f_j}, X_{f_k}] = X_{\{f_j, f_k\}} = 0\) - the vector fields commute. The compact and connected component of \( M_c := \{f_j = c_j, j = 1, \ldots, n\} \) is diffeomorphic to a torus.

Given a Riemannian manifold \( M \), a geodesic may be defined as curve that results from the application on the principle of least action (we follow[1]). A differential equation describing their shape may be derived using variational principles, by finding extremum of the energy of a curve. Given a smooth curve

\[
\gamma : I \to M
\]

this maps an interval \( I \) in real line to the \( M \), one writes the energy

\[
E(\gamma) = \frac{1}{2} \int_I g(\gamma(t), \gamma'(t)) dt,
\]

where \( \gamma'(t) \) is the tangent vector to the curve \( \gamma \) at the point \( t \in I \). Here \( g(.,.) \) is the metric tensor on the manifold \( M \).

In terms of local coordinates on \( M \), the geodesic equation is

\[
\frac{d^2x^a}{dt^2} + \Gamma^a_{bc} \frac{dx^b}{dt}\frac{dx^c}{dt} = 0,
\]

where the \( x^a(t) \) are coordinates of \( \gamma(t) \), \( \Gamma^a_{bc} \) are the Christoffel symbols and indexes up and below means summation convention.

Geodesics can be understood to be the Hamiltonian flows of a special Hamiltonian vector field defined on the cotangent space of the manifold. The Hamiltonian is constructed from the metric on the manifold, and is thus a quadratic form consisting entirely of the kinetic term. The geodesic equations are second order differential equations, they can be expressed as a first order equations by additional independent variables. Note that \( x_a \) is the coordinates on the neighborhood \( U \) induces local trivialization of

\[
T^* M|_U \simeq U \times \mathbb{R}^n
\]

by the map which sends a point \( \eta \in T^*_x M|_U \) of the form \( \eta = p_a dx^a \) to the point \( (x, p_x) \in U \times \mathbb{R}^n \). The Hamiltonian is

\[
H(x, p_x) = \frac{1}{2} g^{ab} (x) p_a p_b.
\]

Here \( g^{ab}(x) \) is the inverse of the metric tensor \( g^{ab}(x)g_{bc}(x) = \delta^a_c \). The behavior of the metric tensor under transformation of coordinates implies that \( H \) is invariant under change of variables. The geodesic equations may be written as

\[
\dot{x}^a = \frac{\partial H}{\partial p_a} = g^{ab}(x) p_b, \quad \dot{p}_a = -\frac{\partial H}{\partial x^a} = -\frac{1}{2} \left( \frac{\partial g^{ab}(x)}{\partial x^a} \right) p_b p_c.
\]

The question of the integrability of the geodesic equation has been investigated by several in the authors context of Stationary Axially-symmetric Vacuum (SAV) space-time in [2], [3],
and \[9\]. The properties of stationarity and axial-symmetry suggest that every SAV space-time possesses a pair Killing vector fields \(\partial_t\) and \(\partial_\phi\), and a pair first integrals \(p_t = -E\), and \(p_\phi = L\), corresponding to energy \(E\) and azimuthal angular momentum \(L\). The problems of geodesic motion reduces to two dimensional Hamiltonian system in the meridian plane \((\rho, z)\).

The existence of additional independent integral of motion in involution with others integrals \((E, L, H)\) lead to Liouvillian integrability. The interesting case Kerr space-time - a SAV space-time, describing rotating black hole - admits two-rank irreducible Killing tensor, that is why four constant of motion which is a quadratic in the momenta - the so-called Carter’s constants, \[10\] and \[18\]. The important question is when the Carter’s constant exists in other SAV-models.

Geodesic motion on Zipoy-Voorhees space-times known as a \(\gamma\)-metrics, has been studied via numerical methods, in \[15\] and \[16\] and analytically in \[13\] and \[17\]. The Zipoy-Voorhees metrics with a parameter \(\delta\) have as special cases both Minkowski flat space-time (\(\delta = 0\)) and Schwarzschild (\(\delta = 1\)) space-time. Important for study of Zipoy-Voorhees metrics is that finding obvious regularity in geodesics across large parts of the parameter \(\delta\) and the finding of approximate constants of motion, is not enough for proof of the Liouville integrability, i.e. for the existence of fourth constant of motion.

In \[4\] it was shown numerically that the Chazy-Curzon and Zipoy-Voorhees space-time exhibit obviously-regular dynamics in special parameter \(\delta\) values. After further research in \[7\], Brink suggests that the Zipoy-Voorhees metrics may give rise to integrable systems for \(\delta \neq 0; 1\). In \[5\] and \[6\] Brink show that all SAV metrics admits irreducible second-order Killing tensors are necessarily of Petrov type D. This key result implies that for generic Zipoy-Voorhees metrics without Petrov D property, do not permit in general for any new constant of geodesic motion which is quadratic in momentum variables.

In 2012 in \[15\] using numerical calculations it was shown that the Hamiltonian phase space of the Zipoy-Voorhees system has the features expected of a perturbed non-integrable system such as chaotic layers, Birkhoff chains, and “stickiness” in the rotation number \[15\] and \[16\]. Chaotic layers are mos visible near to tha Lyapunov orbit - near to the unstable periodic orbit at the border between bound and plunging orbits (see \[9\] for details).

2 Formulation of the problem

The Chazy-Curzon space-time model are part of the class of Static Axially-symmetric Vacuum (StAV) metrics - a subclass of SAV metrics. In cylindrical coordinates \((t, \rho, z, \phi)\) the StAV line element can be written in Weyl form as

\[
s^2 = g_{\mu\nu}dx^\mu dx^\nu = -e^{2\psi}dt^2 + e^{-2\psi}(e^{2\gamma}(d\rho^2 + dz^2) + \rho^2 d\phi^2).
\]

The geodesic motion is governed by a hamiltonian \(H_4(x^\mu, p_\nu) = 1/2g^{\mu\nu}p_\mu p_\nu = -1/2\) with \(g^{\mu\nu}\) being the inverse of metric. \(H_4\) is a constant of motion which we set to \(-1/2\) to test the bodies
of unit mass. The momentum variables are given by $p_\mu = g^{\mu\nu} \dot{x}^\nu$, where the dot stands for differentiation with respect to the proper time.

From Hamiltonian equations we have $p_t = p_\phi = 0$ and we obtain two first integrals $E = -p_t$ and $L = p_\phi$. Motion on the meridian plane is governed by the reduced Hamiltonian $H_2$, with the restriction, $H_2 = 0$ given by

$$H_2 = \Omega^2(\rho, z). \left( \frac{1}{2}(p_\rho^2 + p_z^2) + V_{eff}(\rho, z, E^2, L^2) \right),$$

(2.1)

where $\Omega = e^{(\psi - \gamma)}$, $V_{eff} = -\frac{1}{2} \Phi$ with $\Phi = e^{-2\gamma}(E^2 - e^{2\psi} - \rho^{-2}e^{4\psi}L^2)$. The contour $V_{eff} = 0$ defines a curve of zero velocity. The geodesics are found by solving Hamiltonian equations

$$\dot{\rho} = \frac{\partial H_2}{\partial p_\rho}, \dot{z} = \frac{\partial H_2}{\partial p_z}, \dot{p}_\rho = -\frac{\partial H_2}{\partial \rho}, \dot{p}_z = -\frac{\partial H_2}{\partial z},$$

(2.2)

with a choice of $E$ and $L$ and initial conditions for $p_\rho, p_z, \rho, z$ such that $H_2 = 0$.

Einstein’s vacuum field equations for the line part reduce to Laplace equation

$$\partial^2 \rho\rho \psi + \rho^{-1}\partial_\rho \psi + \partial_\rho \psi = 0$$

with $\gamma_\rho = \rho(\psi_\rho^2 + \psi_z^2)$ and $\gamma_{\rho z} = 2\rho \psi_\rho \psi_z$.

The solution of the Laplace equation is

$$\psi = -\frac{m}{\sqrt{\rho^2 + z^2}}, \gamma = -\frac{m^2 \rho^2}{2(\rho^2 + z^2)^4}.$$ 

This is a single-particle Chazy-Curzon solution. In this paper, we establish, that the Hamiltonian system, describing this motion is not Liouville-integrable.

Next we will find the points of equilibrium of the system, describing geodesic motion of Chazy-Curzon space-time. We take $m = 1$, by scaling, and we have

$$H_2(p_\rho, p_z, \rho, z) = \Omega^2(\rho, z). \left( \frac{1}{2}(p_\rho^2 + p_z^2) + V_{eff}(\rho, z, E^2, L^2) \right),$$

where $\Omega(\rho, z) = e^{\left(-\frac{1}{4\sqrt{\rho^2 + z^2}} + \frac{1}{2(\rho^2 + z^2)^2}\right)}$ and change the time

$$d\tau = \Omega^2(\rho, z) dt$$

(2.3)

for the Hamiltonian and the equations (2.2).

We found the new “Hamiltonian” - change of the time variable is not a canonical transformation, but preserves the integrability properties (see [19] for details)

$$F(p_\rho, p_z, \rho, z) = \frac{1}{2}(p_\rho^2 + p_z^2) + \frac{1}{2} - \frac{E^2}{2} e^{\frac{2}{\sqrt{\rho^2 + z^2}}} + \frac{L^2}{2\rho^2} e^{\frac{2}{\sqrt{\rho^2 + z^2}}}$$

(2.4)

and we obtain

$$\frac{\partial F(p_\rho, p_z, \rho, z)}{\partial p_\rho} = p_\rho,$$
\[ \frac{\partial F(p_\rho, p_z, \rho, z)}{\partial p_z} = p_z, \]
\[ -\frac{\partial F(p_\rho, p_z, \rho, z)}{\partial \rho} = -\frac{E^2 \rho^2 e^{\frac{\rho}{\sqrt{\rho^2 + z^2}}}}{(\rho^2 + z^2)^{3/2}} + \frac{L^2 e^{-\frac{\rho}{\sqrt{\rho^2 + z^2}}}}{\rho^3} + \frac{L^2 e^{-\frac{\rho}{\sqrt{\rho^2 + z^2}}}}{\rho^3}, \]
and
\[ -\frac{\partial F(p_\rho, p_z, \rho, z)}{\partial z} = -z. \]

The substitution (2.3), transforms the Hamiltonian system to the system \( \dot{\mathbf{r}} = \mathbf{F}(\mathbf{r}, \dot{\mathbf{r}}) \)

\[ \rho' = p_\rho, \]
\[ z' = p_z, \]
\[ p_\rho' = \frac{-E^2 \rho^2 e^{\frac{\rho}{\sqrt{\rho^2 + z^2}}}}{(\rho^2 + z^2)^{3/2}} + \frac{L^2 e^{-\frac{\rho}{\sqrt{\rho^2 + z^2}}}}{(\rho^2 + z^2)^{3/2} \rho} + \frac{L^2 e^{-\frac{\rho}{\sqrt{\rho^2 + z^2}}}}{(\rho^2 + z^2)^{3/2} \rho}, \]
\[ p_z' = -z. \]

Then the conditions for the equilibrium are \( p_\rho = p_z = z = 0 \) and
\[ E^2 \rho^2 e^\rho - L^2 (\rho - 1) = 0, \]
from \( F = 0 \) we get
\[ F(0, 0, \rho, 0) = -\frac{E^2}{2} e^{\rho} + \frac{L^2}{2 \rho^2} e^{-\rho} + \frac{1}{2} = 0. \]

Solving this linear system for \( L^2 \) and \( E^2 \) we obtain for the equilibrium
\[ L^2 = -\frac{\rho^2 e^\rho}{\rho - 2}, \quad E^2 = -\frac{\rho (\rho - 1) e^{-\rho}}{\rho - 2}. \]

Finally, if the point \((p_\rho, p_z, z, \rho)\) is an equilibrium for Chazy- Curzon space-time Hamiltonian system, then \( p_\rho = p_z = z = 0 \) and \( \rho \) satisfies \( L^2 = -\frac{e^\rho}{\rho - 2} \) and \( E^2 = -\frac{(\rho - 1) e^{-\rho}}{\rho - 2} \). For real \( E^2 \) and \( L^2 \) we have \( 1 < \rho < 2 \).

### 3 Main result

**Proposition 1.** On the manifold \( P := \{ p_\rho = p_z = z = 0, \rho \} \), invariant under \( X_F \) the system (2.5) has only periodical solutions around equilibrium \( p_\rho = p_z = z = 0, \rho_0 \in (1, 2) \) with restrictions \( L^2 = -\frac{\rho_0^2 e^{\rho_0}}{\rho_0 - 2}, \quad E^2 = -\frac{(\rho_0 - 1) e^{-\rho_0}}{\rho_0 - 2} \).
Proof: For the potential $v(\rho, z, E^2, L^2)$ we have

$$v(\rho, z, E^2, L^2) = \frac{1}{2} - \frac{E^2}{2} \frac{\rho^2}{e^{\rho^2 + z^2}} + \frac{L^2}{2\rho^2} \frac{\rho^2}{e^{\rho^2 + z^2}}$$

around the points of equilibrium we have $v(\rho, 0, -\frac{(\rho-1)e^{\frac{z^2}{\rho-2}}}{\rho-2}, -\frac{e^{\frac{z^2}{\rho-2}}}{\rho-2}) = 0$. We compute the second derivative near the equilibrium,

$$\frac{\partial^2 v(\rho, 0, -\frac{(\rho-1)e^{\frac{z^2}{\rho-2}}}{\rho-2}, -\frac{e^{\frac{z^2}{\rho-2}}}{\rho-2})}{\partial \rho^2} = -\frac{\rho^2 - 6\rho + 4}{\rho^4(\rho - 2)}.$$

We have only periodic solutions around the equilibrium if

$$v(\rho) = v(\rho, 0, -\frac{(\rho-1)e^{\frac{z^2}{\rho-2}}}{\rho-2}, -\frac{e^{\frac{z^2}{\rho-2}}}{\rho-2}) = 0, v'(\rho) = \frac{\partial v(\rho, 0, -\frac{(\rho-1)e^{\frac{z^2}{\rho-2}}}{\rho-2}, -\frac{e^{\frac{z^2}{\rho-2}}}{\rho-2})}{\partial \rho} = 0$$

and

$$v''(\rho) = -\frac{\partial^2 v(\rho, 0, -\frac{(\rho-1)e^{\frac{z^2}{\rho-2}}}{\rho-2}, -\frac{e^{\frac{z^2}{\rho-2}}}{\rho-2})}{\partial \rho^2} = \frac{\rho^2 - 6\rho + 4}{\rho^4(\rho - 2)} > 0,$$

which is true for $\rho \in (1, 2)$.

Important for the integrability of the system is the behavior of the period function,

$$T = 2 \int_{\rho-}^{\rho+} \frac{d\rho}{\sqrt{0 - v(\rho, 0, L^2, E^2)}}$$

where $\rho-$ and $\rho+$ are the roots of $0 - v(\rho, 0, L^2, E^2) = 0, v(\rho, 0, L^2, E^2)$ is the potential function, and $L^2 = -\frac{\rho^2 e^{\frac{z^2}{\rho-2}}}{\rho-2}, E^2 = -\frac{(\rho-1)e^{\frac{z^2}{\rho-2}}}{\rho-2}$ are conditions for equilibrium.

Proposition 2. $T = 2g(\rho_0) \log \eta + \Phi(\epsilon, \delta)$, where $\Phi(\epsilon, \delta)$ is analytical function, $\eta = \frac{z}{\delta}$, and $g(\rho_0) = \frac{\sqrt{\eta}}{-v''(\rho_0)}$.

Proof: Around the point of equilibrium $(p_0, p_0, 0, z) = (0, 0, \rho_0, 0)$, the period is

$$T = 2 \int_{\rho-}^{\rho+} \frac{d\rho}{\sqrt{0 - v(\rho, 0, \frac{\rho^2 e^{\frac{z^2}{\rho-2}}}{\rho-2}, \frac{(\rho-1)e^{\frac{z^2}{\rho-2}}}{\rho-2})}}$$

$$= 2 \int_{\rho_0-\delta}^{\rho_0+\epsilon} \frac{d\rho}{\sqrt{0 - v(\rho, 0, \frac{\rho^2 e^{\frac{z^2}{\rho-2}}}{\rho-2}, \frac{(\rho-1)e^{\frac{z^2}{\rho-2}}}{\rho-2})}}$$

$$= 2g(\rho_0) \int_{\rho_0-\delta}^{\rho_0+\epsilon} \left( \frac{1}{\rho - \rho_0} \right) d\rho + \Phi(\epsilon, \delta)$$

$$= 2g(\rho_0)(\log(\epsilon) - \log(\delta)) + \Phi(\epsilon, \delta)$$

$$= 2g(\rho_0) \log(\epsilon/\delta) + \Phi(\epsilon, \delta)$$

$$= 2g(\rho_0) \log \eta + \Phi(\epsilon, \delta),$$
where $\Phi(\epsilon, \delta)$ is analytical function, $\eta = \frac{\epsilon}{\delta}$, $g(\rho_0) = \frac{\sqrt{2}}{\sqrt{-\nu''(\rho_0)}}$ and $-\nu''(\rho_0) = -\frac{\partial^2 v(\rho_0, 0, \frac{(\rho_0 - 1)e^{\frac{\rho_0}{\rho_0 - 2}}}{\rho_0 - 2}, \frac{e^{\frac{\rho_0}{\rho_0 - 2}}}{\rho_0 - 2})}{\partial \rho^2} > 0$ for $\rho_0 \in (1, 2)$.

The last condition follows from Taylor’s formula applied to $v(\rho)$ around $\rho_0$. We have

$$v(\rho) = v(\rho_0) + v'(\rho_0)(\rho - \rho_0) + \frac{1}{2}v''(\rho_0)(\rho - \rho_0)^2 + O((\rho - \rho_0)^3)$$

because it $v(\rho_0) = v'(\rho_0) = 0$ for the points of equilibrium. Then,

$$T = 2g(\rho_0) \log \eta + \Phi(\epsilon, \delta),$$

around equilibrium point.

Our goal is the following:

**Theorem 1.** The system with Hamiltonian (2.1) is not integrable by means of analytical first integral.

We need an equivalent formulation to prove Theorem 1.

**Proposition 3.** The system (2.7) has no additional analytical first integral.

**Proof:** For the proof we know that there are only a periodical of solutions on $P$, of the Hamiltonian system (2.5) on the hypersurface $F = 0$. If $T$ is the period function then complex continuation of the manifolds $T = \text{const}$ turns out to be infinitely branched - this excludes the existence of a nontrivial analytic integral on any open subset of the complex domain where this infinite branching is true. We need to show that if $G$ is a smooth function on open set $U$ such that $V = U \cap (F = 0)$ is $X_F$ invariant, $\{F, G\} = 0$ and derivative $d\{F, G\} = 0$ on $V$, then $G$ is a function of $F$ and $T$ on $V$.

The proof is similar like in the remarkable paper of J. J. Duistermaat [14].

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7
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