Kovalevskaya top – an elementary approach

A. M. Perelomov *

Departamento de Física, Facultad de Ciencias, Universidad de Oviedo,
E-33007 Oviedo, Spain

To the memory of Jürgen Moser

Abstract

The goal of this note is to give an elementary and very short solution to equations of motion for the Kovalevskaya top [1]. For this, we use some results from the original papers by Kovalevskaya [1], Kötter [2] and Weber [3] and also the Lax representation from the note [4].

1. The Kovalevskaya top [1] is one of the most beautiful examples of integrable systems. This is the top for which the principal momenta of inertia \( J_1, J_2, J_3 \) satisfy the relation

\[
J_1 = J_2 = 2J_3 = J,
\]

and the center of mass lies in the equatorial plane of the body (for the simplicity, we put further \( J = 1 \)). The dynamical variables are components \( m_1, m_2, m_3 \) of angular momentum and components \( n_1, n_2, n_3 \) of the center mass vector in the system related to the principal axes of the body.

This system is Hamiltonian relative to the Poisson structure for the Lie algebra \( \mathfrak{e}(3) \) of motion of three-dimensional Euclidean space

\[
\{ m_i, m_j \} = \varepsilon_{ijk} m_k, \quad \{ m_i, n_j \} = \varepsilon_{ijk} n_k, \quad \{ n_i, n_j \} = 0,
\]

where \( \varepsilon_{ijk} \) is a standard totally skew-symmetric tensor.

The Hamiltonian has the form

\[
H = \frac{1}{2} \left( m_1^2 + m_2^2 + 2m_3^2 - n_1 \right)
\]
and the equations of motion are (the dot means a derivative in time)
\[
\dot{m}_j = \{H, m_j\}, \quad \dot{n}_j = \{H, n_j\},
\]
(4)
or in the explicit form,
\[
\begin{align*}
\dot{m}_1 &= m_2 m_3, \\
2 \dot{m}_2 &= -(2 m_3 m_1 + n_3), \\
2 \dot{m}_3 &= n_2, \\
\dot{n}_1 &= 2 m_3 n_2 - m_2 n_3, \\
\dot{n}_2 &= m_1 n_3 - 2 m_3 n_1, \\
\dot{n}_3 &= m_2 n_1 - m_1 n_2.
\end{align*}
\]
(5)
Note that the angular velocity vector has the form
\[
(p, q, r) = (m_1, m_2, 2m_3).
\]
(6)

In the celebrated paper [1], Kovalevskaya succeeded in integration of these equations in terms of abelian functions of two variables. The Kovalevskaya approach was simplified later by Kötter [2]. Note also the paper by Kolosov [5], where he reduced this problem to the problem of motion of the point on the plane in a potential field.

One century was gone, and the Kovalevskaya top roused interest again. In the paper [4] the Kovalevskaya top was considered as the projection of the Euler top. This approach gives as the explanation of famous relation (1) as the natural multi-dimensional integrable generalizations of such system.

In papers by Enolsky [6], [7], nontrivial reductions were found which give the elliptic solutions for the Kovalevskaya top. In the paper by Novikov and Veselov [8] (see also [9] and references there), the action-angle variables for this problem were constructed and the Poisson commutativity of variables \(s_1\) and \(s_2\) was discovered. Note that namely these variables are appeared at the consideration of the Kovalevskaya top as the projection of Euler’s top.

Authors of number of papers (see [10], [11], [12], [13] and references therein) used the algebro-geometrical approach to this problem. Unfortunately, this approach is very complicated and till now only some part of original results for the Kovalevskaya top was reproduced in framework of it.

For example, the Lax representation with spectral parameter [10], [12] gives the spectral curve of genus three and correspondingly the abelian functions of three variables, but not abelian functions of two variables as in the original Kovalevskaya paper [1]. Even for the simplest case \((m, n) = 0\), the correspondence between two such approaches is very complicated [13]. So, in author’s opinion, the original Kovalevskaya–Kötter approach being elementary and natural one is more adequate to the problem under consideration.

2. Following [1] and [2], let us remind first that equations (5) have four integrals of motion
\[
\begin{align*}
H_1 &= 2 H = m_1^2 + m_2^2 + 2m_3^2 - n_1 = h_1, \\
\tilde{H}_2 &= \xi_+ \xi_- = k^2, \\
C_3 &= (m, n) = m_1 n_1 + m_2 n_2 + m_3 n_3 = c_3, \\
C_4 &= n_1^2 + n_2^2 + n_3^2 = c_4,
\end{align*}
\]
(7–10)
where
\[
\xi_\pm = m_\pm^2 + n_\pm, \quad m_\pm = (m_1 \pm i m_2), \quad n_\pm = (n_1 \pm i n_2).
\]
(11)
Note that $C_3$ and $C_4$ are Casimir functions and the equations $C_3 = c_3$, $C_4 = c_4$ define the four-dimensional symplectic manifold $\mathcal{M}_c$ – the orbit of coadjoint representation of Lie group $E(3)$ (the group of motion of three-dimensional Euclidean space).

The integration of equations (5) consists from several steps.

3. We start with the Lax representation [4] describing the Kovalevskaya top as the projection of the Euler top

\[\begin{align*}
\dot{L}_2 &= [L_2, M_2], \\
L_2 &= -A (2 \hat{m}^2 + (\gamma \otimes n + n \otimes \gamma)) A, \\
M_2 &= -A \hat{m} A,
\end{align*}\]

where

\[\begin{align*}
\hat{m} &= \begin{pmatrix}
0 & m_3 & -m_2 \\
-m_3 & 0 & m_1 \\
m_2 & -m_1 & 0
\end{pmatrix}, \\
A &= \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{pmatrix}, \\
\gamma &= (1, 0, 0), \\
n &= (n_1, n_2, n_3),
\end{align*}\]

(12)

\[
\text{tr} \left( L_2^2 \right) = 2H_1 = 4H, \quad \text{det} \left( L_2 \right) = H_2^2 - \tilde{H}_2.
\]

(13)

Then we have

\[
\det \left( sI - L_2 \right) = s P_2(s), \quad P_2(s) = s^2 - 2H_1 s + H_2, \\
H_1 = h_1, \quad H_2 = h_2, \quad h_2 = h_1^2 - k^2.
\]

(14)

In the Kovalevskaya case, the equations of motion contain as quadratic as linear terms in dynamical variables $m_1, m_2, m_3; n_1, n_2, n_3$. From this it follows that some of these variables being meromorphic functions of time $t$ have the second order poles in $t$.

Let us try to find the change of variables such that the equations of motion will contain only quadratic terms. This may be achieved by elimination of variables $n_1$ and $n_2$.

From equations (7)–(10) follows two linear equations for variables $n_1$ and $n_2$

\[
m_1 n_1 + m_2 n_2 = c_3 - m_3 n_3, \\
(m_1^2 - m_2^2) n_1 + 2 m_1 m_2 n_2 = \frac{1}{2} \left( n_3^2 - (m_1^2 + m_2^2)^2 + k^2 - c_4 \right).
\]

(16)

Using them we may eliminate $n_1$ and $n_2$ from equations

\[
2H_1 = 2h_1, \quad H_2 = h_2.
\]

(17)

Then we discover that the left hand side of these equations becomes the quadratic form if we introduce new variables $f = (f_1, f_2, f_3)$, $g = (g_1, g_2, g_3)$, where

\[
\begin{align*}
f_1 &= \frac{1}{m_2}, & f_2 &= \frac{m_1}{m_2}, & f_3 &= \frac{m_1^2 + m_2^2}{m_2}; \\
g_1 &= 2\frac{m_3}{m_2}, & g_2 &= \frac{m_3}{m_2}, & g_3 &= -\frac{2}{m_2} \left( (m_1^2 + m_2^2) m_3 + m_1 n_3 \right).
\end{align*}
\]

(18)

(19)

\footnote{Note that these variables were used already in papers [1] and [2].}
Namely, we get

\begin{align}
2H_1 &= S_1(f) + T_1(g), \\
H_2 &= S_2(f) + T_2(g),
\end{align}

(20)

where

\begin{align}
S_1 &= \frac{1}{2} \left( (f_3 + h_1 f_1)^2 - 4 c_3 f_1 f_2 - (c_4 + h_2) f_1^2 - 4 h_1 f_2^2 \right), \\
S_2 &= -2 c_3 (f_3 + h_1 f_1) f_2 - (c_4 + h_2) f_2^2 - c_3^2 f_1^2, \\
T_1 &= \frac{1}{2} (-g_1 g_3 + g_2^2), \\
T_2 &= \frac{1}{4} ((h_1 g_1 - g_3)^2 - (c_4 + h_2) g_1^2 + 4 c_3 g_1 g_2).
\end{align}

(21-24)

From (20) it follows that the quantities \( S_1(f) \) and \( S_2(f) \) may be considered as a natural projection of two nontrivial integrals of motion \( 2H_1 \) and \( H_2 \)

\[ \pi : 2H_1 \to S_1(f); \quad \pi : H_2 \to S_2(f). \]

(25)

So we take them as the new dynamical variables. It is natural also to unify them (as in (15)) to

\[ \mathcal{F}(s, f) = s^2 - S_1(f) s + S_2(f) \]

(26)

and consider the equation

\[ \mathcal{F}(s, f) = 0. \]

(27)

The roots \( s_1 \) and \( s_2 \) of equation (27) are the famous Kovalevskaya variables

\[ s_1 + s_2 = S_1, \quad s_1 s_2 = S_2. \]

(28)

Note that as it was shown by Novikov and Veselov [9], the variables \( s_1 \) and \( s_2 \) are Poisson commuting,

\[ \{ s_1, s_2 \} = 0, \quad \{ S_1, S_2 \} = 0. \]

(29)

Note also another property of these variables

\[ \{ T_1, T_2 \} = 0, \quad 2\{ H_1, S_2 \} = \{ H_2, S_1 \}. \]

(30)

So we have also one-parametric family of Poisson commuting variables \( S_1(\lambda) = S_1 + 2\lambda H_1 \) and \( S_2(\lambda) = S_2 + \lambda H_2 \)

\[ \{ S_1(\lambda), S_2(\lambda) \} = 0. \]

(31)

Note that functions \( f_j, g_k \) are not independent but they satisfy the relations

\begin{align}
f_1 f_3 - f_2^2 &= 1, \\
f_1 g_3 + 2 f_2 g_2 + f_3 g_1 &= 0.
\end{align}

(32)

These relations are standard for the cotangent bundle of two-dimensional two-sheet hyperboloid. So after change of variables we come to the dynamical system on two-dimensional two-sheet hyperboloid.
Namely in terms of new variables, equations of motion (5) have the form

\[
\begin{align*}
\dot{f}_1 &= \frac{1}{2} (f_1 g_2 + f_2 g_1), \\
\dot{f}_2 &= -\frac{1}{2} (f_1 g_3 + f_2 g_2) = \frac{1}{4} (-f_1 g_3 + f_3 g_1), \\
\dot{f}_3 &= -\frac{1}{2} (f_2 g_3 + f_3 g_2)
\end{align*}
\]  

(33)

and

\[
\begin{align*}
\dot{g}_1 &= c_3 f_1^2 + h_1 f_1 f_2 - f_2 f_3, \\
\dot{g}_2 &= \frac{1}{2} \gamma f_1^2 + c_3 f_1 f_2 + \frac{1}{2} f_3^2, \\
\dot{g}_3 &= -c_3 \left( f_1 f_3 + 2 f_2^2 \right) - h_1 f_2 f_3 - \gamma_4 f_1 f_2,
\end{align*}
\]  

(34)

where \(\gamma_4 = c_4 - k^2 = c_4 + h_2 - h_1^2\).

Note also useful equations.

\[
\begin{align*}
\ddot{f}_1 &= \nu f_1 + \frac{1}{2} (h_1 f_1 + f_3), \\
\ddot{f}_2 &= \nu f_2 + (h_1 f_2 + \frac{1}{2} c_3 f_1), \\
\ddot{f}_3 &= \nu f_3 + \frac{1}{2} \left( h_1 f_3 - 2 c_3 f_2 - \gamma_4 f_1 \right), \\
\nu &= h_1 - S_1.
\end{align*}
\]  

(35 - 38)

One can show that the equations of motion for \(f_1, f_2, f_3\) have the Lax form

\[
\hat{L} = [L, M],
\]  

(39)

where

\[
L = \begin{pmatrix} f_2 & f_1 \\ -f_3 & -f_2 \end{pmatrix}, \quad M = \frac{1}{4} \begin{pmatrix} -g_2 & g_1 \\ -g_3 & g_2 \end{pmatrix}.
\]  

(40)

The equations for \(g_1, g_2, g_3\) have the form

\[
\hat{M} = [L, N],
\]  

(41)

where

\[
N = -\frac{1}{8} \begin{pmatrix} c_3 f_1 + 2 h_1 f_2, & f_3 + h_1 f_1 \\ \gamma_4 f_1 + 2 c_3 f_2 - h_1 f_3, & -c_3 f_1 + 2 h_1 f_2 \end{pmatrix}.
\]  

(42)

Let us consider now the Clebsch problem [14] (see also [15]–[18]), i.e. the problem of motion of rigid body in ideal fluid. The dynamical variables here are the components of momenta \(p_1, p_2, p_3\) and angular momenta \(l_1, l_2, l_3\) and for special case they satisfy also the additional constraint \((l, p) = 0\).
This system is Hamiltonian relative to the Poisson structure for the Lie algebra $\mathfrak{e}(3)$ of motion of three-dimensional Euclidean space

$$\{l_i, l_j\} = \varepsilon_{ijk} l_k, \quad \{l_i, p_j\} = \varepsilon_{ijk} p_k, \quad \{p_i, p_j\} = 0,$$

where $\varepsilon_{ijk}$ is a standard totally skew-symmetric tensor.

The Hamiltonian has the form

$$H = \frac{1}{2} \left( \sum_{j=1}^{3} l_j^2 + \sum_{j,k=1}^{3} B_{jk} p_j p_k \right)$$

where the quantities $B_{jk}$ are constants.

One can check that the equations of motion for this case have the same form as the equations (33), (34) and that the second equation in (32) is equivalent to the condition $(l, p) = 0$.

Note that for diagonal matrix $B$ this problem was solved by Weber [3] in terms of abelian functions of two variables.

4. The last step is to reduce our problem to the case of diagonal matrix $B$. For this it is convenient to use the important identity discovered by Kötter [2]:

$$-2s \mathcal{F}(s) = Q_2^2(s) - P_3(s) Q_1(s),$$

where

$$P_3(s) = sp_2(s) + c_4 s - 2 c_3^2,$$

$$P_2(s) = s^2 - 2 h_1 s + h_2,$$

$$P_5(s) = P_3(s) P_2(s),$$

$$Q_1(s) = f_1^2 s - 2 f_2^2,$$

$$Q_2(s) = s^2 f_1 - (f_3 + h_1 f_1) s - 2 c_3 f_2.$$

Let us introduce instead variables $f_j, g_k$ the variables $x_j, y_k$ by the formulae\footnote{Such kind formulae were introduced by Weierstrass and they are very useful in the theory of abelian functions.}

$$x_j = \sqrt{(s_1 - a_j)(s_2 - a_j)},$$

$$y_j = \frac{x_k x_l}{s_1 - s_2} \left( \frac{\sqrt{P_5(s_1)}}{(s_1 - a_k)(s_1 - a_l)} - \frac{\sqrt{P_5(s_2)}}{(s_2 - a_k)(s_2 - a_l)} \right),$$

where $a_j$ is the root of the equation $P_3(s) = 0$ and $\{j,k,l\}$ is the cyclic permutation of $\{1, 2, 3\}$.

From (45) – (48) we get the expression for $f_j$ in terms of $x_k$

$$f_1 = -i \sum_{j=1}^{3} \frac{\sqrt{2a_j}}{P_3'(a_j)} x_j,$$

$$f_2 = i \sum_{j=1}^{3} \frac{\sqrt{a_k a_l}}{P_3'(a_j)} x_j,$$

$$f_3 = -h_1 f_1 - 2i \sum_{j=1}^{3} \frac{\sqrt{2a_j}}{P_3'(a_j)} (a_1 + a_2 + a_3 - a_j) x_j.$$
and
\begin{align*}
g_1 &= -i \sum_{j=1}^{3} \frac{\sqrt{2a_j}}{P_3'(a_j)} y_j, \\
g_2 &= i \sum_{j=1}^{3} \frac{\sqrt{a_k a_l}}{P_3'(a_j)} y_j, \\
g_3 &= -h_1 g_1 - 2i \sum_{j=1}^{3} \frac{\sqrt{2a_j}}{P_3'(a_j)} (a_1 + a_2 + a_3 - a_j) y_j.
\end{align*}

Note also that
\begin{equation}
Q_2(a_j) = i \sqrt{2a_j} x_j, \quad j = 1, 2, 3.
\end{equation}

One can check that after substitution of expressions (49)–(54) for \( f_j \) and \( g_k \) into the equations of motion (33), (34) the equations for variables \( x_j, y_k \) become the equations for the special Clebsch case \(((l, p) = 0)\) with diagonal \( B \) matrix. So we may use the Weber solution \([3]\) given by the formulae
\begin{equation}
x_j = x_{j0} \frac{\theta_{j4}(u_1, u_2)}{\theta_0(u_1, u_2)}, \quad y_j = y_{j0} \frac{\theta_j(u_1, u_2)}{\theta_0(u_1, u_2)}, \quad j = 1, 2, 3,
\end{equation}
where \( x_{j0} \) and \( y_{j0} \) are constants, \( u_1 \) and \( u_2 \) are linear functions of \( t \) and \( \theta_0(u_1, u_2), \theta_j(u_1, u_2), \theta_{j4}(u_1, u_2) \) are standard theta functions with half-integer theta characteristics:
\begin{align*}
\theta_1 &= \theta \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} (u_1, u_2), \quad \theta_2 = \theta \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} (u_1, u_2), \quad \theta_3 = \theta \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} (u_1, u_2), \\
\theta_{14} &= \theta \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} (u_1, u_2), \quad \theta_{24} = \theta \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} (u_1, u_2), \quad \theta_{34} = \theta \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} (u_1, u_2), \\
\theta_0 &= \theta \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (u_1, u_2).
\end{align*}

These functions are defined by the standard formulae
\begin{equation}
\theta \begin{bmatrix} \varepsilon_1 & \varepsilon_2 \\ \delta_1 & \delta_2 \end{bmatrix} (u_1, u_2) = \sum_{n_j, n_k = -\infty}^{\infty} \exp \left\{ i \pi \left( \tau_{jk} \left( n_j + \frac{\varepsilon_j}{2} \right) \left( n_k + \frac{\varepsilon_k}{2} \right) + \left( n_j + \frac{\varepsilon_j}{2} \right) \left( 2u_j + \delta_j \right) \right) \right\},
\end{equation}
where \( \tau_{jk} \) is the period matrix related to the algebraic curve
\begin{equation}
y^2 = P_5(x).
\end{equation}

So, the formulae (18), (19), (49)–(54) and (56)–(59) give the explicit solution for the Kovalevskaya top. (For more details see \([1],[2]\) and \([3]\)).

5. Similarly to the Weierstrass approach for geodesics on an ellipsoid \([19]\), we may obtain once more important identity
\begin{equation}
\mathcal{F}(s) \mathcal{H}(s) - \mathcal{G}^2(s) = 2P_5(s)
\end{equation}
where

\[
\mathcal{F}(s) = s^2 - S_1 s + S_2, \quad (61)
\]
\[
\mathcal{G}(s) = \dot{S}_1 s - \dot{S}_2, \quad (62)
\]
\[
\mathcal{H}(s) = 2(s^3 - b_1 s^2 + b_2 s - b_3), \quad (63)
\]
\[
P_5 = (s^3 - 2h_1 s^2 + (c_4 + h_2) s - 2c_3^2)(s^2 - 2h_1 s + h_2) \quad (64)
\]

and

\[
b_1 = -S_1 + 4h_1, \\
b_2 = S_1^2 - S_2 - 4h_1 S_1 + 2h_2 + 4h_1^2 + c_4, \\
b_3 = -S_1^3 + 4h_1 S_1^2 + 2S_1 S_2 - 4h_1 S_2 - (2h_2 + 4h_1^2 + c_4) S_1 - \frac{1}{2} \dot{S}_1^2 + 4h_1 h_2 + 2h_1 c_4 + 2c_3^2. \quad (65)
\]

From (60) it is easy to get the equations of motion in standard Abel-Jacobi form

\[
\dot{s}_1 = i(s_1 - s_2)^{-1} \sqrt{2P_5(s_1)}, \quad \dot{s}_2 = i(s_2 - s_1)^{-1} \sqrt{2P_5(s_2)} \quad (66)
\]

and also the Lax representation with spectral parameter \(s\) in terms of 2 by 2 matrices

\[
\dot{\mathcal{L}}(s) = [\mathcal{L}(s), \mathcal{M}(s)], \quad (67)
\]
\[
\mathcal{L}(s) = \begin{pmatrix} \mathcal{G} & \mathcal{F} \\ -\mathcal{H} & -\mathcal{G} \end{pmatrix}, \quad (68)
\]
\[
\mathcal{M}(s) = \mathcal{F}^{-1}(s) \begin{pmatrix} 0 & 0 \\ \mathcal{C} & \mathcal{D} \end{pmatrix}, \quad (69)
\]

where

\[
\mathcal{C} = (s - 2T_1) \mathcal{F}(s) - (1/2) \mathcal{H}(s), \quad \mathcal{D} = -\mathcal{G}(s). \quad (70)
\]

6. I would like to conclude this paper by the conjecture that similar results are valid also for the \(n\)-dimensional generalization of the Kovalevskaya top given in [4].

Acknowledgments

The main result of this note on the equivalence of the Kovalevskaya system to the special case of the Clebsch system has been obtained in 1983 during the preparation of the review on the motion of the rigid body around the fixed point [17]. Later I had the possibility to discuss with Prof. Moser the Kovalevskaya problem and other problems of classical mechanics. These discussions had great influence on my point of view on the problems of classical mechanics in general. In particular, Prof. Moser emphasized the important role of factorization of the Kovalevskaya polynomial \(P_5(x)\) into polynomials \(P_3(x)\) and \(P_2(x)\). The simple explanation of this fact is absent unfortunately till now.

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References

[1] Kovalevskaya S 1889 *Acta Math.* **12** 177–232

[2] Kötter F 1893 *Acta Math.* **17** 209–263

[3] Weber H 1879 *Math. Ann.* **14** 173–206

[4] Perelomov A M 1981 *Commun. Math. Phys.* **81** 239-241 ; [math-ph/0111024](http://arxiv.org/abs/math-ph/0111024)

[5] Kolosov G V 1902 *Math. Ann.* **56** 265–272

[6] Enolsky V Z 1984 *Phys. Lett.* **A100** 463–466

[7] Enolsky V Z 1984 *Sov. Math. Dokl.* **30** 394-397

[8] Novikov S P and Veselov A P 1985 Poisson brackets and complex tori in: *Proc. Steklov Inst. of Math.* **165** 53-65

[9] Dullin H, Richter P and Veselov A P 1998 *Regular and Chaot. Dynam.* **3** 18–31

[10] Reyman A G and Semenov-Tian-Shansky M A 1987 *Lett. Math. Phys.* **14** 55–61

[11] Adler M and van Moerbeke P 1988 *Commun. Math. Phys.* **113** 659–700

[12] Bobenko A I, Reyman A G and Semenov-Tian-Shansky M A 1989 *Commun. Math. Phys.* **122** 321–354

[13] Markushevich D G 2001 *J. Phys.* **A34** 2125–2135

[14] Clebsch A 1871 *Math. Ann.* **3** 238-262

[15] Moser Ju 1980 in: *The Chern Symposium 1979* 147–188

[16] Perelomov A M 1981 *Funct. Anal. Appl.* **15** 144–146

[17] Perelomov A M 1983 Integrable systems of classical mechanics and Lie algebras. Motion of rigid body around fixed point *Preprint ITEP-147*

[18] Perelomov A M 2000 *Regular and Chaot. Dynam.* **5** 89–91

[19] Weierstrass K 1861 *Monatsberichte Akad. Wiss. zu Berlin* 986