ON THE GRAVITATIONAL FIELDS
CREATED BY THE ELECTROMAGNETIC WAVES

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Abstract. We show that the Maxwell equations describing an electromagnetic wave are a mathematical consequence of the Einstein equations for the same wave. This fact is significant for the problem of the Einsteinian metrics corresponding to the electromagnetic waves.

Summary – Introduction – 1. On a consequence of the fact that the light-rays are null geodesics in any spacetime manifold. – 2. The Maxwell equations of an electromagnetic wave are a consequence of the Einstein equations for the same wave. – 2bis. An example. – 3. A result analogous to that of sect. 2 holds in the linear version of GR. – 3bis, 3ter. An example. – 4. A final remark. – Appendix.

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Introduction. – Very rarely this subject has been approached from a really physical standpoint. On the contrary, there are in the literature many interesting papers of a mathematical character about metrics which are linked in some way to the propagation of electromagnetic (e.m.) waves. We recall sect. 8 of a famous paper (1923) by Eddington [1], the paper of 1926 by Baldwin and Jeffery [2], the paper by Bonnor of 1969 [3], and the overflowing of geometrical articles on the plane-fronted waves with parallel propagation (briefly, p-p waves) [4].

The classic treatment by Tolman [5], which is limited to the linear version of GR and to very particular models, is not fully satisfying, because it neglects the important role of the equation of the characteristic surfaces of Maxwell theory.

A new approach to the topic is given in the present paper.

1. – In any spacetime the e.m. rays are null geodesics, as it is well known. Consequently, no undulatory, purely gravitational, and autonomous field is created by the propagation of any e.m. wave in any spacetime manifold. (This propagation is a “natural” one, like that of any e.m. wave in a Minkowski spacetime). And clearly, no gravitational interaction exists among the various portions of an e.m. wave.
Einstein field equations in a space devoid of bodies and of charges are 

c = G = 1

\[ R_{jk} = -8\pi E_{jk} \quad (E_{j} = 0) \quad (j, k = 0, 1, 2, 3) \]

if \( E_{jk} \) is the e.m.

\[
4\pi E^{jk} = -F_{j}^{m} F^{km} + \frac{1}{4} g^{jk} F_{mn} F^{mn} ,
\]

where \( F_{mn}, (m, n = 0, 1, 2, 3) \), is the e.m. field of the considered e.m.

\[ E_{jk} = 0 \quad (j, k = 0, 1, 2, 3) \]

\[ F_{mn} = \Phi_{m:n} - \Phi_{n:m} = \Phi_{m,n} - \Phi_{n,m} \]

\[ F_{mn} = \Phi_{m:n} - \Phi_{n:m} \quad \text{if} \quad \Phi_{m} \]

\[ G^{jk} \left( F_{jk} \sqrt{-g} \right)_{,k} = 0 \]

Eqs. (1) tell us that

\[ E^{jk} = 0 \quad \text{from which, taking into account} \quad (1') \quad \text{and} \quad (1''), \]

we have Maxwell equations

\[ (F_{j}^{k} \sqrt{-g})_{,k} = 0 \]

\[ (F_{j}^{k} \sqrt{-g})_{,k} = 0 \]

which coincide with eqs. (2).

This means that if we express \( E_{jk} \), as a function of \( \Phi_{m} \), Einstein eqs. (1) have Maxwell eqs. (2) as a mathematical consequence. Gravitation has “absorbed” the e.m. properties of the e.m. wave. This result has as a necessary condition that the differential equation of the characteristic

\[ g^{jk} \frac{\partial z(x)}{\partial x^{j}} \frac{\partial z(x)}{\partial x^{k}} = 0 \]

is the same for both Maxwell and Einstein fields (Whittaker and Levi-Civita).

At this point, it is very natural to specify the reference frame in such a way that four components of the metric tensor \( g_{jk} \) are functionally identical to the four components of the e.m. potential \( \Phi_{m} \), which describes our e.m.

2bis. – Let us consider, e.g., a continuous flow of e.m. waves described by the following four-potential \( \Phi_{m} \):
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\(\Phi_0 = \Phi_1 = \Phi_3 = 0\); \(\Phi_2(t+x) = A\sin[\omega(t+x)]\); \(t \equiv x^0; x \equiv x^1\); in a Minkowskian spacetime, eqs. (6) represent an ordinary plane wave. We have:

\[
\begin{align*}
F_{21} &= \Phi_{2,1} = A\omega \cos[\omega(t+x)] \\
F_{20} &= \Phi_{2,0} = A\omega \cos[\omega(t+x)]
\end{align*}
\]

The e.m. energy tensor \(E_{jk}\) reduces to (2):

\[
E_{jk} = -g^{rs}F_{jr}F_{ks}
\]

and eqs. (11) give:

\[
R_{jk} = 8\pi g^{rs}F_{jr}F_{ks}
\]

The e.m. field of the e.m. wave is thus fully described by its own gravitational field.

3. In the linear version (LV) of GR we have approximately:

\[
g_{jk} = \eta_{jk} + h_{jk}
\]

where \(\eta_{jk}\) is the Minkowski tensor \((1, -1, -1, -1)\), and the \(h_{jk}\)'s are small deviations from it. LV is a Lorentz-invariant theory. Its equations are also invariant under the following gauge transformation of the symmetric tensor \(h_{jk}\):

\[
h_{jk} \rightarrow h_{jk} + \xi_{j,k} + \xi_{k,j}
\]

where \(\xi_j(x)\) is an infinitesimal vector function of \((x^0, x^1, x^2, x^3)\). Equivalently, formula (12) can be viewed as the result of a transformation of metric (11) under an infinitesimal change of the Lorentzian coordinates \(x^j\):

\[
x^j \rightarrow x^j + \xi^j(x)
\]

In lieu of the exact eqs. (11), we have, as it is known:

\[
\frac{1}{2} \Box h_{jk} = -8\pi E_{jk}; \quad h_{jk} = 0
\]
in the following equations of sect. 2, we must now substitute $g_{jk}$ with $\eta_{jk}$, and the covariant derivatives with the ordinary ones. Eq. (15) becomes:

$\eta_{jk} \frac{\partial z(x)}{\partial x^j} \frac{\partial z(x)}{\partial x^k} = 0$ ;

the light-rays are rectilinear null-geodesics; the e.m. waves and the field $h_{jk}$ are propagated in Minkowski spacetime.

Four components of tensor $h_{jk}$ can be identified with the four components of the e.m. potential $\Phi_m$.  

3bis. – Let us consider in the LV a continuous flow of plane e.m. waves, described by the four-potential $\Phi_m$ of eqs. (6). We have (1):

$E_{00} = E_{01}(= E_{10}) = E_{11} = A^2 \omega^2 \cos[\omega(t + x)] ;$

the other components of $E_{jk}$ are equal to zero.

Accordingly,

$\Box h_{00}(t, x) = \Box h_{01}(t, x) = \Box h_{11}(t, x) = -16\pi A^2 \omega^2 \cos^2[\omega(t + x)] ;$

$\Box h_{jk} = 0$ , for $(j, k) \neq [(00), (01), (11)]$ .

The e.m. field does not appear in eqs. (17'), and therefore these $h_{jk}$’s can be put equal to zero: they do not “feel” the action of the e.m. waves. However, if we prefer to follow the procedure of sect. 2bis, we can put, e.g., (with gauge transformed $h_{jk}$’s ):

$h_{02}(t + x) = \Phi_2(t + x) = A \sin[\omega(t + x)] ;$

$h_{03} = \Phi_3 = 0 ; h_{12} = \Phi_1 = 0 ; h_{13} = \Phi_0 = 0 ;$

$h_{22} = h_{23} = h_{33} = 0$ .

Then, eqs. (17) can be re-written as follows:

$\Box h_{00} = \Box h_{01} = \Box h_{11} = \left(\frac{\partial h_{02}}{\partial \xi}\right)^2 ,$

if $\xi := t + x$; the e.m. field is thus fully described by its own gravitational field.

3ter. – One finds easily the solution of eqs. (17). Indeed, the solution of d’Alembert inhomogeneous equation

\[
(15) \quad \eta^{jk} \frac{\partial z(x)}{\partial x^j} \frac{\partial z(x)}{\partial x^k} = 0 ;
\]

\[
(16) \quad E_{00} = E_{01}(= E_{10}) = E_{11} = A^2 \omega^2 \cos[\omega(t + x)] ;
\]

\[
(17) \quad \Box h_{00}(t, x) = \Box h_{01}(t, x) = \Box h_{11}(t, x) = -16\pi A^2 \omega^2 \cos^2[\omega(t + x)] ;
\]

\[
(17') \quad \Box h_{jk} = 0 , \text{ for } (j, k) \neq [(00), (01), (11)] .
\]

\[
(18) \quad h_{02}(t + x) = \Phi_2(t + x) = A \sin[\omega(t + x)] ;
\]

\[
(18') \quad h_{03} = \Phi_3 = 0 ; h_{12} = \Phi_1 = 0 ; h_{13} = \Phi_0 = 0 ;
\]

\[
(18'') \quad h_{22} = h_{23} = h_{33} = 0 .
\]
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\[
\frac{\partial^2 F}{\partial t^2} - \frac{\partial^2 F}{\partial x^2} = -16\pi A^2 \omega^2 \cos^2[\omega(t + x)]
\]

is given by \( F(t, x) = F_0(t, x) + F_1(t, x) \), where

\[
F_0(t, x) = \varphi(t + x) + \chi(t - x)
\]

is the general solution of the homogeneous equation, with \( \varphi \) and \( \chi \) any functions of their arguments, and \( F_1(t, x) \) is given by

\[
F_1(t, x) = -16\pi A^2 \omega^2 \cdot (t - x) \left\{ \frac{1}{2} (t + x) + \frac{1}{4\omega} \sin[2\pi(t + x)] \right\}
\]

Of course, only \( F_1(t, x) \) concerns the gravitational field generated by our e.m. waves.

4. – A final remark. Hilbert [6] considered the coupled equations of Einstein and Mie (with the gravitational and e.m. potentials, \( g_{jk} \) and \( q_j \), as unique dynamical variables) in lieu of the coupled equations of Einstein and Maxwell. According to Mie’s theory [7] (which has revealed itself as impractical), the electric charges would emerge as solutions of the field equations for the potential \( q_j \), while in Einstein’s theory the point-masses emerge as singularities of the metric tensor \( g_{jk} \). Hilbert remarked that Mie’s equations, referred to the general-relativistic metric, are an analytical consequence of Einstein’s equations with Mie’s e.m. energy tensor. Therefore, for the 14 components of the potentials \( g_{jk} \) and \( q_j \), we have only the 10 functionally independent Einstein’s equations.

In our previous treatment of the gravitational field created by an e.m. wave we had a formalism which is quite analogous, from the mathematical standpoint, to the above Hilbertian formalism, with Maxwell’s e.m. potential \( \Phi \); instead of \( q_j \). We have applied and developed Hilbert’s remark.

APPENDIX

Our approach is a direct one (like that of Tolman for the linear version of GR [5]): we start from a given e.m.–wave potential and give a prescription to compute the generated gravitational field according to Einstein field equations. On the contrary, we find in the literature a clear prevalence of an indirect method: one postulates intuitively the more or less detailed structure of a \( ds^2 \) – or of a gravitational potential \( g_{jk} \); then, from the corresponding expression for \( R_{jk} - (1/2)g_{jk}R \), one derives the tensor \( T_{jk} \) and one verifies whether it can be interpreted as the energy tensor \( E_{jk} \) of an e.m. wave. This method has a weak point: it depends on an interpretation, and thus its physical meaning can be dubious. This adjective is appropriate
also for those mixed procedures that make a partial use of both the above mentioned methods.

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