Certain Periodically Correlated Multi-component Locally Stationary Processes

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Abstract

We study a class of non-stationary processes, say periodically correlated locally stationary (PC-LS). Consider $s_0 < s_1 < \ldots$ to be positive real numbers and $B_j = (s_{j-1}, s_j]$, for $j \in \mathbb{N}$. Let $X^{ls}(t) = \sum_{j=1}^{\infty} X_j^{ls}(t) I_{B_j}(t)$, $t \in \mathbb{R}^+$ where $X_j^{ls}(t)$ is a mixture of two stationary processes with exponentially convex weights. By this, we provide $X^{ls}(t)$ as a multi-component locally stationary process. Also we consider $\{X_j^p\}$ as a sequence of periodically correlated random variables. We define an orthogonally scattered random measure $M_j$ on subsets of $B_j$ by $X_j^p = M_j(B_j)$ and set $X^p(t) = M_j(s_{j-1}, t]$ for $t \in B_j$ with some special correlation. Then $X^p(t) = \sum_{j=1}^{\infty} X_j^p(t) I_{B_j}(t)$ is a continuous time periodically correlated process which we study its spectral representation. Finally we assume that $X^{ls}(t)$ and $X^p(t)$ are independent and define $X(t) = X^{ls}(t) + X^p(t)$ as a certain multi-component PC-LS process which has both periodically correlated and locally stationary properties. The covariance structure and the time dependent spectral representation of such a process are characterized.

Keywords: Periodically correlated; Spectral representation; Multi-component locally stationary processes; exponentially convex covariance.

1 Introduction

Spectral analysis of stochastic processes such as stationary and periodically correlated (PC) processes have a long history with interest in both theory and applications. These processes are interesting for engineers due to their applications

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in signal processing and communication systems, specially in network traffic. The spectral theory of these processes is primarily based on spectral representations.

Properties of stationary processes are well known and have been used in analyzing system performance. Recall that a stationary process is one where all finite dimensional distributions are invariant to shifts in time. Non-stationary processes have extensive applications where random fluctuations change in time or space. If a process is slowly varying and if the interval is short enough, then the process can be approximated in some sense by a stationary one. Recently a large amount of work has been devoted to time series analysis, with the focus placed on locally stationary (LS) and PC processes which give the plausible description of real world. LS processes in the sense of Silverman [15] can be used to model systems where they behave as a function of time. He presented a relation between the covariance and the spectral density of LS processes which constitutes a natural generalization of the Wiener-Khintchine relations.

Berman [2] introduced a class of LS Gaussian processes with index $\alpha$ which allows minor fluctuations of dependence at the global scale and at the same time keeps the stationary structure at the local time. Dehay [6] presented the notion of locally harmonizable process, a large class of non-stationary processes containing LS and harmonizable. A new class of LS processes where their spectral structure varies smoothly over time is introduced by Dahlhaus [4]. Mallat et al. [13] showed that the covariance operator of a LS process has approximate eigenvectors that are local cosine functions. They modeled these processes with pseudo differential operators that are time-varying convolutions. Oxley et al. [14] proposed another definition of LS processes and presented their properties and relationships to the stationary ones. They also showed that the signals that arise in Air Force applications typically has noise that can be modeled as a LS process. Exponentially convex stochastic processes and exponentially convex covariances have been studied by Loeve [12], among others. He found the spectral representation of the exponentially convex process.

Another class of non-stationary processes are PC processes which have periodic structure. Theses processes are a class of processes which are in general non-stationary but exhibit many of the properties of stationary processes. They have been used as models of meteorology, radio physics and communications engineering. This class was introduced by Gladyshev [10], who studied the structure of the covariance function and provided an interesting spectral representation. Hurd, Miamee, Makagon cited in Gardner [8], demonstrated various applications of PC processes in science and engineering.

In this paper we are concerned with a certain non-stationary process called, multi-component periodically correlated locally stationary (PC-LS) process. By
choosing $0 = s_0 < s_1 < \ldots$ and $B_j = (s_{j-1}, s_j]$, $j \in \mathbb{N}$, we partition the positive real line. Let $X(t) = X^{ls}(t) + X^p(t)$, $t \in \mathbb{R}^+$ represents a stochastic process, where $X^{ls}(t) = \sum_{j=1}^{\infty} X_j^{ls}(t) I_{B_j}(t)$ and $X_j^{ls}(t)$ is a multi-component LS process which is an exponentially convex mixture of two stationary processes. $X^p(\cdot)$ is a discrete time PC process that $X^p(t) = \sum_{j=1}^{\infty} X_j^p(t) I_{B_j}(t)$ in which $\{X_j^p(t)\}$ is a sequence of PC processes and $X_j^p(t) = M_j(s_{j-1}, t]$ where $M_j$ for $j \in \mathbb{N}$ is an orthogonally scattered random measure on Borel field of subsets of $B_j$. Then $X(\cdot)$ is a certain multi-component PC-LS process.

More precisely this paper is organized as follows. In section 2, we study the general framework and preliminaries of discrete time periodically correlated, exponentially convex function and locally stationary processes. The harmonizable representation of stationary and PC processes are given in this section too. In section 3 we present the covariance structure and spectral representation of such continuous time multi-component PC-LS processes.

2 Preliminaries

We review the spectral theory of unitary operators in subsection 2.1. In subsection 2.2, we study the properties of stationary and periodically correlated processes and their spectral representations. We present some definitions of locally stationary and exponentially convex processes in subsections 2.3 and 2.4 respectively.

2.1 Spectral theory of unitary operators

One of the classical results of operator theory is the spectral theorem. We introduce the notion of spectral measure and briefly discuss their properties. We proceed to define the spectral integral as an operator. At last we represent the unitary operator and the time varying spectral representation of a PC process.

Throughout this subsection we work with a measurable space $(\Omega, \mathcal{F})$ consisting of a set $\Omega$ and a $\sigma$-algebra $\mathcal{F}$ of its subsets. For more details about the following results one could refer to Hurd and Miamee [11].

Definition 2.1 A function $Q$ defined on $\mathcal{F}$ of subsets of $\Omega$ whose values are orthogonal projections in a Hilbert space $\mathcal{H}$ is called a spectral measure, if $Q(\Omega) = I$, the identity operator, and for any sequence $M_n$ of disjoint sets in $\mathcal{F}$ we have $Q(\bigcup_{n=1}^{\infty} M_n) = \sum_{n=1}^{\infty} Q(M_n)$.

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The following proposition lists several useful properties of spectral measure without presenting the proofs.

**Proposition 2.1** If $Q$ is an orthogonally scattered spectral measure on $\mathcal{F}$, where $Q(\emptyset) = 0$, we have the following properties for any two sets $M, N$ in $\mathcal{F}$

(a) **Modular:** $Q(M \cup N) = Q(M) + Q(N) - Q(M \cap N)$,

(b) **Multiplicative:** $Q(M \cap N) = Q(M)Q(N)$.

**Definition 2.2** A unitary operator on a Hilbert space $\mathcal{H}$ is a linear operator $U$ from $\mathcal{H}$ onto $\mathcal{H}$ for which $\langle Ux, Uy \rangle = \langle x, y \rangle$ for every $x, y \in \mathcal{H}$, that is, unitary operators are linear and preserve inner products.

**Definition 2.3** The spectral integral $\int f(\lambda)Q(d\lambda)$ as an operator $A(f)$ such that $A : \mathcal{H} \to \mathcal{H}$ and for any pair $x, y \in \mathcal{H}$

$$\int f(\lambda)d\langle Q(\lambda)x, y \rangle = \langle A(f)x, y \rangle.$$

**Theorem 2.1** For any unitary operator $U$ on a Hilbert space $\mathcal{H}$, there exists a unique spectral measure $Q$ on the Borel subsets of $[0, 2\pi)$ such that $U = \int_{0}^{2\pi} e^{i\lambda}Q(d\lambda)$, and for any integer $t$

$$U^t = \int_{0}^{2\pi} e^{i\lambda t}Q(d\lambda).$$

### 2.2 Spectral representation of PC processes

We present definitions of stationary and PC processes in this subsection and exhibit the harmonizable representation of such processes. For more details see [11].

**Definition 2.4** A random process $X(t)$ taking values in the $L_2$ random variables of a probability space $(\Omega, \mathcal{F}, P)$ and indexed on $\mathbb{Z}$ or $\mathbb{R}$ is called periodically correlated if there exists some $T > 0$ in $\mathbb{Z}$ or $\mathbb{R}$ respectively, such that

$$\mu(t) = E[X(t)] = \mu(t + T), \quad R_X(t, s) = R_X(t + T, s + T)$$

for every $t, s$ in $\mathbb{Z}$ or $\mathbb{R}$, $R_X(t, s) = E[(X(t) - \mu(t))(X(s) - \mu(s))]$. The smallest such $T$ will be called period of $X(t)$. For discrete time, it is required that $T > 1$, otherwise the process is stationary. For continuous time, also the process requires continuity of the correlation function.
The notion of a harmonizable process can be motivated from the fact that every wide sense stationary process has an integral spectral representation of the form

$$X(t) = \int_{-\infty}^{\infty} e^{i\omega t} dZ(\lambda) \quad (2.1)$$

where the frequency indexed random process $Z(\lambda)$ has orthogonal or uncorrelated increments and the equality is in $L_2$ of the probability space $(\Omega, \mathcal{F}, P)$. The notion of harmonizable processes preserves the spectral representation (2.1) but the increments of $Z(\lambda)$ need not be orthogonal.

**Proposition 2.2** A second order stochastic sequence $X_n$ is PC with period $T$ if and only if for every $n \in \mathbb{Z}$, there exists a unitary operator $V^n$ and a periodic sequence (process) $P_n$ taking values in $\mathcal{H}_X = \overline{\mathcal{L}_X}$ (closure of the $\mathcal{L}_X$) and $\mathcal{L}_X = \text{sp}\{X_n, n \in \mathbb{Z}\}$ for which

$$X_n = V^n P_n$$

where $V = \int_{0}^{2\pi} e^{i\lambda/T} Q(d\lambda)$, $V^T = U$ and $Q$ is the spectral measure defined in Definition 2.1, so [11]

$$X_n = \int_{0}^{2\pi} e^{i\lambda n/T} Q(d\lambda) P_n.$$ 

We turn to spectral integral representation for PC sequences by the following theorem.

**Theorem 2.2** A second order stochastic sequence $X_n$ is PC with period $T$ if and only if there exists a time dependent spectral measure $\xi(\cdot, n) = \xi(\cdot, n + T)$ on the Borel subsets of $[0, 2\pi)$ that is orthogonally scattered in the sense that

$$\langle \xi(A, m), \xi(B, n) \rangle = 0$$

for every $m, n \in \mathbb{Z}$ whenever $A \cap B = \emptyset$ and such that for all $n \in \mathbb{Z}$

$$X_n = \int_{0}^{2\pi} e^{i\lambda n} \xi(d\lambda, n)$$

where the time dependent spectral measure $\xi(\cdot, n)$ is defined through the application of the spectral measure to the vector $P_n$ as for any Borel set $A$, we have $\xi(A, n) = Q(A)P_n$, where $P_n$ is introduced in Proposition 2.2.

### 2.3 Locally stationary processes

We present the concept of locally stationary process, which is a generalization of stationary random process and is introduced by Silverman [15].
Let \( X(t) \) be a random process (in general complex), where the real parameter \( t \) lying in some index set \( I \), which is a closed interval or the infinite real line \( \mathbb{R} \). We assume that, for all \( t \in I \), the second moment of \( X(t) \) exists and the first moment of \( X(t) \) is zero; the latter assumption involves no loss of generality. Before proposing the definition of locally stationary on an interval, we introduce the concept of partitioning of the parameter space \( I \).

**Definition 2.5** Let \( I \subset \mathbb{R} \) be an interval (possibly \( \mathbb{R} \)). A partition of \( I \) is a countable collection of subintervals \( \{B_1, B_2, \ldots\} \) where \( B_k \subset I \) is an interval and \( k \) belongs to some countable index set \( \mathcal{I} \), such that

1. \( B_i \cap B_k = \emptyset \) for all \( i \neq k \) in \( \mathcal{I} \),
2. \( \cup_{k \in \mathcal{I}} B_k = I \).

In the Silverman sense we have the following definition.

**Definition 2.6** The random process \( X(t) \), defined for all real \( t \), is locally stationary in the wide sense, or has a locally stationary covariance, if its covariance can be written as

\[
R_X(t, s) = q\left(\frac{t + s}{2}\right)c(t - s)
\]

where \( R_X(t, s) \) is the covariance function of \( X(t) \) and \( q(t + s) \) is the average power of the process at the point \( t + s \). The quantity \( E[|X(t)|^2] \) is the average instantaneous power of the process \( X(t) \). The symmetrization has been chosen because it makes \( R_X(t, s) \) Hermitian and because of its suggestive meaning as the average or centroid of the points \( t \) and \( s \). We have chosen the correlation function or stationary covariance \( c(\tau) \) to be normalized, by which we mean that \( c(0) = 1 \). The fact that \( R_X(t, s) \) is Hermitian implies that \( c(\tau) = c^*(-\tau) \), the asterisk denotes the complex conjugate.

### 2.4 Exponentially convex function

No we give a brief description of exponentially convex process and its spectral representation [7], [9].

**Definition 2.7** The covariance function of a second order zero mean process \( \{Z(t), t \in \mathbb{R}\} \) with finite variance, \( \text{cov}(Z(t_i), Z(t_j)) = \psi(t_i + t_j) \), is called exponentially convex if and only if it is a complex-valued function in \( \mathcal{C} \) defined on the product space \( \mathbb{R} \times \mathbb{R} \) that satisfies

\[
\sum_{i=1}^{n} \sum_{j=1}^{n} a_i a_j \psi(t_i + t_j) \geq 0
\]
for all finite sets of complex coefficients $a_1, \ldots, a_n$ and points $t_1, \ldots, t_n \in \mathbb{R}$.

**Example 2.1** Let $\{Z(t), t \in \mathbb{R}\}$ be a stochastic process with the covariance function

$$\text{cov}(Z(t), Z(s)) = \psi(t + s),$$

$$\psi(u) = (1 + u^2)e^{\frac{u^2}{2}}, \quad u \in \mathbb{R}$$

as $\psi(u)$ is an exponentially convex function [7] and $\{Z(t), t \in \mathbb{R}\}$ is an exponentially convex process.

We conclude the following theorem from the result of Berg et al. [1] (Theorem 5.11, p.12). The corresponding theorem shows that a continuous function is exponentially convex if and only if it is the Laplace transform of a non-negative finite measure.

**Theorem 2.3** Any continuous positive definite function $\psi$ has the representation

$$\psi(t) = \int_{\mathbb{R}} e^{\lambda t} d\mu(\lambda), \quad t \in \mathbb{R}$$

for a uniquely determined positive Radon measure (a Borel measure that is finite on compact sets) $\mu$.

By Theorem 2.3 and the Karhunen-Loeve theorem, it follows that for any symmetric and continuous, in the mean square, process $Z(\cdot)$ there exists a second order stochastic process $F$ with orthogonal increments and spectral measure $\mu$ such that

$$Z(t) = \int_{\mathbb{R}} e^{\lambda t} dF(\lambda), \quad t \in \mathbb{R}$$

### 3 Main results: A continuous time PC-LS model

We introduce a new class of certain non-stationary process which we call multi-component periodically correlated locally stationary (PC-LS) process and is a combination of two locally stationary and periodically correlated processes as

$$X(t) = X^{ls}(t) + X^{p}(t), \quad t > 0$$

(3.1)

where $X^{ls}(\cdot)$ is a multi-component LS process and $X^{p}(\cdot)$ is a sequence of PC process that are independent. We partition the positive real line index space of the process, by disjoint intervals $B_j = (s_{j-1}, s_j]$ where $s_0 = 0$ and $|B_j| = |B_{j+T}|$ for all $j = 1, \ldots, T$ and $T \in \mathbb{N}$. 
3.1 Continuous PC process

We define \( M_j \) as an orthogonally scattered random measure on Borel field of subsets of \( B_j \), by \( X^p_j(t) := M_j(s_{j-1}, t) \) for \( t \in B_j \) and \( X^p_j := M_j(B_j), j \in \mathbb{N} \) where \( \{X^p_j(t)\} \) is a positive second order discrete time PC process with period \( T \) of centered random variables. For \( t \in B_i \), \( \{X^p_i(t)\} \) is a positive second order discrete time PC process with period \( T \) of centered random variables. For \( t \in B_i \), \( \{X^p_i(t)\} \) is a positive second order discrete time PC process with period \( T \) of centered random variables. For \( t \in B_i \), \( \{X^p_i(t)\} \) is a positive second order discrete time PC process with period \( T \) of centered random variables.

\[
X^p(t) = \sum_{j=1}^{\infty} X^p_j(t) I_{B_j}(t), \quad t > 0 \tag{3.2}
\]

is a continuous time PC process. By the following projection, we provide a linear approximation of the process say the traffic flow in the subintervals of each partition.

Let \( A \subset B_j \), \( L_A^j = \mathfrak{p}\{M_j(D), D \subset A\} \) and \( \mathcal{P}_A^j : L^j_{B_j} \to L^j_A \) for \( j \in \mathbb{N} \) be an orthogonal projection which is defined by

\[
\mathcal{P}_A^j X_j^p = M_j(A). \tag{3.3}
\]

with the following property

\[
\mathcal{P}_A^j \mathcal{P}_B^j = \mathcal{P}_{A \cap B}^j \tag{3.4}
\]

Our idea is to provide a proper bilateral correlation of such variables. So we consider covariance function of random measure \( M_j \) on subintervals \( A_1, A_2, B \subset B_j \) where \( |A_1| = |A_2| \), to satisfy

\[
\text{corr}(M_j(A_1), M_j(B)) = \text{corr}(M_j(A_2), M_j(B)).
\]

Thus if \( M_j \) is the flow of traffic, then on each subinterval of \( B_j \) with fixed length, it has the same multivariate distribution with other random measures as the one of any other subintervals of \( B_j \) with the same length. Thus for \( A \subset B_j \) and \( B \subset B_j \) we define the covariance function of \( M_j \) as

\[
\langle M_j(A), M_j(B) \rangle = \frac{2|A||B|}{a_j(|A| + |B|)} \gamma_{jj}^p \tag{3.5}
\]

where \( \langle X, Y \rangle = \text{cov}(X, Y) = E[XY], \gamma_{jk}^p = E[X_j^p X_k^p] \) and \( |B_j| = a_j, j, k \in \mathbb{N} \). The correlation function of this measure is
\[
\text{corr}(M_j(A), M_j(B)) = \frac{2 \sqrt{|A||B|}}{|A| + |B|} = \frac{2}{\sqrt{|A||B|} + |A| + 2}.
\]

As for positive \( x, f(x) = x + 1/x \geq 2 \) and equality holds for \( x = 1 \), so the correlation function is equal to one when \( |A| = |B| \). Also the correlation function decreases when the amounts of \( |A| \) and \( |B| \) differ and increases when they approach to each other.

For \( A \subset B_j \) and \( B \subset B_k, j \neq k \)
\[
\langle M_j(A), M_k(B) \rangle = \frac{|A||B|}{a_j a_k} \gamma_{jk}.
\]

If \( B = B_k \) then \( \frac{|B|}{a_k} = 1 \), so for \( j \neq k \) \( \langle M_j(A), M_k(B) \rangle = \frac{|A|}{a_j} \gamma_{jk} \). One can easily verify that, this inner product is well defined.

In this case for \( t, u \in B_j \) and \( t \leq u \) we have
\[
\langle M_j(s_{j-1}, t], M_j(s_{j-1}, u] \rangle = \frac{2a_j a_u}{a_j(a_j + a_u)} \gamma_{jj} \tag{3.6}
\]

and for \( t \in B_j, u \in B_k \)
\[
\langle M_j(s_{j-1}, t], M_k(s_{k-1}, u] \rangle = \frac{a_j a_u}{a_j a_k} \gamma_{jk}
\]

where \( a_j = \frac{|B_j|}{|A|} \) and \( j \in \mathbb{N} \).

### 3.2 Multi-component LS process

Let \( \{Y^s_j(t), t \in B_j \cup B_{j+1}\}_{j=1}^\infty \) be a countable class of independent zero mean stationary processes. In addition we assume that for all \( j \in \mathbb{N}, Y^s_j(s_{j-1}) = 0, B_j = (s_{j-1}, s_j] \) and \( Y^s_j(t) \equiv N_j(s_{j-1}, t] \) for \( t \in B_j \), where \( N_j(A) \) is a random measure, say cumulative traffic on \( A \subset B_j \). We also consider \( X^{ts}_j(t) \) as
\[
X^{ts}_j(t) = U^{j-1}(t)Y^{s}_{j-1}(t) + U^j(t)Y^s_j(t) \tag{3.7}
\]

for \( t \in B_j, j = 2, \ldots, T \) and \( X^{ts}_1(t) = U^1(t)Y^s_1(t) \) for \( t \in B_1 \). Also \( Y^s_0(t) \equiv 0 \) and
$U^j(t)$ is a random weight with exponentially convex covariance which is independent to the processes $Y_j^s(t)$. We call $X_j^{ls}(t)$, a multi-component locally stationary process motivated from its covariance function which is obtained by (3.10). Relation (3.8) is an exponentially convex mixture of two stationary processes. Let

$$X^{ls}(t) = \sum_{j=1}^{\infty} X_j^{ls}(t)I_{B_j}(t), \quad t > 0. \tag{3.8}$$

where $X_j^{ls}(\cdot)$ is defined by (3.7). We show in Theorem 3.1 that $X_j^{ls}(\cdot)$ is a zero mean multi-component LS process indexed by subsets of subintervals of $B_j$. Thus $X^{ls}(t)$ is a simple multi-component LS process.

### 3.3 Covariance function of PC-LS process

By the following theorem, we find the covariance structure of the introduced model in this section.

**Theorem 3.1** Let $\{B_j\}_{j=1}^{\infty}$ be a partition of positive real line defined in subsection 3.2. The covariance function of the multi-component PC-LS process $X(t) = X^{ls}(t) + X^p(t)$, where $X^{ls}(t)$ and $X^p(t)$ are independent and defined by (3.8) and (3.2) respectively, is $\gamma(t,u) = \text{cov}(X(t), X(u)) = \gamma^{ls}(t,u) + \gamma^p(t,u)$ where

$$\gamma^{ls}(t,u) = \text{cov}(X^{ls}(t), X^{ls}(u)) = \sum_{m=1}^{\infty} \sum_{n=m-1}^{m+1} \gamma^{ls}_{mm}(t,u)I_{B_m}(t)I_{B_n}(u) \tag{3.9}$$

where for $t \in B_i$, $u \in B_j$ and $i \neq j$ with $k = \min\{i,j\}$, $\gamma^{ls}_{ij}(t,u) = \psi_k(t+u)\gamma_k(t-u)$ and for $t \in B_i$, $\gamma^{ls}_{ii}(t,u) = \psi_i(t+u)\gamma_i(t-u) + \psi_{i-1}(t+u)\gamma_{i-1}(t-u)$, in which $E[U^m(t)U^m(u)] = \psi_m(t+u)$, $E[Y^{s}_m(t)Y^{s}_m(u)] = \gamma_m(t-u)$. Also by (3.9) we have that for $t \in B_m$ and $u \in B_n$:

$$\gamma^p(t,u) = \text{cov}(X^p(t), X^p(u)) = \begin{cases} \frac{2a^m a^n}{a_m(a_m+a_n)} \gamma^{p}_{mm} & m = n, \\ \frac{a^m a_n}{a_m a_n} \gamma^{p}_{mn} & m \neq n \end{cases} \tag{3.10}$$

in which $\gamma^{p}_{mn} = E[X^{p}_m X^{p}_n]$.

**Proof:** According to (3.7) and the fact that $U^j(\cdot)$ and $Y^s_j(\cdot)$, $j \in \mathbb{N}$ are independent processes, for $t,u \in B_m$ and $t \leq u$, we have

$$\gamma^{ls}_{mm}(t,u) = E[X^{ls}_m(t)X^{ls}_m(u)] = \psi_{m-1}(t+u)\gamma_{m-1}(t-u) + \psi_m(t+u)\gamma_m(t-u)$$

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For \( t \in B_m \) and \( u \in B_{m+1} \)
\[
\gamma_{m,m+1}^{ls}(t, u) = E[X_m^{ls}(t)X_{m+1}^{ls}(u)] = \psi_m(t + u)\gamma_m(t - u).
\]

For \( t \in B_m \) and \( u \in B_{m-1} \)
\[
\gamma_{m,m-1}^{ls}(t, u) = E[X_m^{ls}(t)X_{m-1}^{ls}(u)] = \psi_{m-1}(t + u)\gamma_{m-1}(t - u)
\]
and for the other cases the process is uncorrelated. This covariance function confirms that, \( X_t^{ls}(\cdot) \) is multi-component PC-LS process in the Silverman sense. Therefore \( \gamma^{ls}(t, u) = E[X^{ls}(t)X^{ls}(u)] \) is
\[
\gamma^{ls}(t, u) = E[\sum_{m=1}^{\infty} X_m^{ls}(t)I_{B_m}(t)X_m^{ls}(u)] = \sum_{m=1}^{\infty} E[X_m^{ls}(t)X_m^{ls}(u)]I_{B_m}(u)
\]
\[
= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} E[X_m^{ls}(t)X_n^{ls}(u)]I_{B_m}(t)I_{B_n}(u) = \sum_{m=1}^{\infty} \sum_{n=m+1}^{\infty} \gamma_{mn}^{ls}(t, u)I_{B_m}(t)I_{B_n}(u).
\]
By (3.2) the covariance function of \( X^p(\cdot) \) for \( t \in B_m, u \in B_n \) is \( \gamma^p(t, u) = E[X_m^p(t)X_n^p(u)] \) and by (3.6) we have the result.

### 3.4 Spectral representation

By the integral representation of stationary and exponentially convex processes as mentioned in preliminaries section, the spectral representation of the process \( X_t^{ls}(t) \) is provided. Using the orthogonal projection \( P_j \) defined by (3.3) and the spectral representation of discrete time PC process \( \{X_j^p\}_{j=1}^{\infty} \), spectral representation of the continuous time PC process \( X^p(t) \) is obtained. Finally the harmonizable representation of the multi-component PC-LS process \( X(\cdot) \) and its spectral measure is characterized.

**Lemma 3.1** Let \( \{X_j^p\}_{j=1}^{\infty} \) be a sequence of PC process with period \( T \), that by proposition 2.2 has the spectral representation \( X_j^p = \int_0^{2\pi} e^{i\lambda j/T} \xi(d\lambda, j) \) where \( \xi(d\lambda, j) = Q(d\lambda)\hat{P}_j \) and \( \hat{P}_j \) is a periodic sequence with period \( T \). Then \( M_j(s_{j-1}, t) = P_{\{s_{j-1}, j\}}X_j^p \), \( t \in B_j \) can be represented as \( X_j^p(t) = M_j(s_{j-1}, t) = \int_0^{2\pi} e^{i\omega j/T} \xi_m(d\lambda, t) \) where \( \xi_m(d\lambda, t) = Q(d\lambda)\hat{P}_{j,t} \) and \( \hat{P}_{j,t} \) is the periodic sequence corresponding to \( M_j(s_{j-1}, t) = V_j\hat{P}_{j,t} \), where \( P_{j,s_j} \equiv \hat{P}_j \). Also
\[
\Theta_{j,k}(d\lambda, d\lambda, t, u) = \langle \xi_j(d\lambda, t), \xi_k(d\lambda, u) \rangle = \begin{cases} 
\frac{2a_j^*a_k}{a_j^2 + a_k^2} \hat{\theta}_{j, j}(d\lambda) & j = k \\
\frac{a_j^*a_k}{a_j a_k} \hat{\theta}_{j, k}(d\lambda) & j \neq k
\end{cases}
\]
and if \( \lambda \neq \omega \), \( \Theta_{j,k}(d\lambda, d\omega, t, u) = 0 \) where \( \tilde{\theta}_{j,k}(d\lambda) = \langle \xi(d\lambda, j), \xi(d\lambda, k) \rangle \) where \( a_j^t = t - s_{j-1} \) for \( t \in B_j \), \( a_j = |B_j| \).

**Proof:** By using the result of Proposition 2.2 and the assumptions of model we can construct a new representation for the random measure \( M_j \) as \( M_j(s_{j-1}, t] = V^j \tilde{P}_{j,t} \) where \( V^j \) is an operator and \( \tilde{P}_{j,t} \) is a periodic function in \( j \) with period \( T \) for fixed \( t \in B_j \). Also by (3.3)

\[
X_j^p(t) = M_j(s_{j-1}, t] = P_{(s_{j-1},t]}^j X_j^p
\]

and by the representation of \( X_j^p \) we have

\[
X_j^p(t) = \int_0^{2\pi} e^{i\omega_j/T} Q(d\lambda) P_{(s_{j-1},t]}^j \tilde{P}_j = \int_0^{2\pi} e^{i\omega_j/T} \zeta_j(d\lambda, t)
\]

where \( \zeta_j(d\lambda, t) = Q(d\lambda) \tilde{P}_{j,t} \) such that \( \tilde{P}_{j,s_j} \equiv \tilde{P}_j \) is a time varying spectral measure.

Using the covariance structure of random measures \( M_j \), \( j \in \mathbb{N} \) we find the covariance function of \( \zeta_j(d\lambda, t) \). By the above representation

\[
\langle M_j(s_{j-1}, t], M_k(s_{k-1}, u] \rangle = \langle \int_0^{2\pi} e^{i\omega_j/T} \zeta_j(d\omega, t), \int_0^{2\pi} e^{i\lambda_k/T} \zeta_k(d\lambda, u) \rangle
\]

\[
= \int_0^{2\pi} \int_0^{2\pi} e^{i(\omega_j - \lambda_k)} \Theta_{j,k}(d\lambda, d\omega, t, u) \rangle
\]

where \( \Theta_{m,n}(d\lambda, d\omega, t, u) = \langle \zeta_m(d\lambda, t), \zeta_n(d\omega, u) \rangle \). On the other hand as \( M_j \) is orthogonally scattered on subsets of \( B_j \), by (3.5) and (3.6) we have

\[
\langle M_j(s_{j-1}, t], M_j(s_{j-1}, u] \rangle = \frac{2a_j^t a_j^u}{a_j(a_j^t + a_j^u)} \int_0^{2\pi} \tilde{\theta}_{j,j}(d\lambda)
\]

and in the same way

\[
\langle M_j(s_{j-1}, t], M_k(s_{k-1}, u] \rangle = \frac{a_j^t a_k^u}{a_j a_k} \int_0^{2\pi} e^{i\omega_j a_j a_k} \tilde{\theta}_{j,k}(d\lambda).
\]

So we arrive at the last assertion of the theorem.

For finding the spectral representation of \( X(t) \), we find the spectral representation of \( X_j^{ls}(t) \) in the following lemma too.
Lemma 3.2  Spectral representation of the process $X_j^{ls}(t) = U_j^{-1}(t)Y_j^s(t) + U_j^s(t)Y_j^s(t)$, $j \in \mathbb{N}$ defined by (3.7) is

$$X_j^{ls}(t) = \int_{-\infty}^{\infty} e^{i\lambda t} \Phi_j(d\lambda, t), \quad t \in B_j$$ (3.11)

where $\Phi_j(d\lambda, t) = \phi_{j-1}(d\lambda, t) + \phi_j(d\lambda, t)$ in which $\phi_j(d\lambda, t) = U_j^s(t)\eta_j(d\lambda)$ that $\eta_j$ is the orthogonally scattered random measure in the harmonizable representation of the stationary process $Y_j^s(\cdot)$. Also

$$F_{j,k}(d\lambda, d\omega, t, u) = \langle \Phi_j(d\lambda, t), \Phi_k(d\omega, u) \rangle$$

where $\psi_j(t + u) = \langle U_j^s(t), U_j^s(u) \rangle$ and $G_j(d\lambda) = E|\eta_j(d\lambda)|^2$.

Proof: By the result of (2.1) and the definition of the process, we have

$$X_j^{ls}(t) = U_j^{-1}(t) \int_{-\infty}^{\infty} e^{i\lambda t} d\eta_{j-1}(\lambda) + U_j^s(t) \int_{-\infty}^{\infty} e^{i\lambda t} d\eta_j(\lambda).$$

So $X_j^{ls}(\cdot)$ is a mixture of two processes with exponentially convex weights. Also

$$X_j^{ls}(t) = \int_{-\infty}^{\infty} e^{i\lambda t} \phi_{j-1}(d\lambda, t) + \int_{-\infty}^{\infty} e^{i\lambda t} \phi_j(d\lambda, t) = \int_{-\infty}^{\infty} e^{i\lambda t} \left[ \phi_{j-1}(d\lambda, t) + \phi_j(d\lambda, t) \right]$$

where $\phi_j(d\lambda, t) = U_j^s(t)\eta_j(d\lambda)$, thus we have the result. The correlation of the spectral measure $\Phi_j(d\lambda, \cdot)$ is defined as

$$F_{j,k}(d\lambda, d\omega, t, u) = E[\Phi_j(d\lambda, t)\Phi_k(d\omega, u)]$$

$$= E\left[ (U_j^{-1}(t)\eta_{j-1}(d\lambda) + U_j^s(t)\eta_j(d\lambda)) (U_k^{-1}(u)\eta_{k-1}(d\omega) + U_k^s(u)\eta_k(d\omega)) \right]$$

where $U_j^s(t)$ has the representation of Theorem 2.3. Finally by (3.7), independence of the processes $\{U_j^s(\cdot)\}$, $\{Y_j^s(\cdot)\}$ and assumptions of lemma, one can easily obtain $F_{j,k}(d\lambda, d\omega, t, u)$ as expressed in the lemma.
Theorem 3.2 The spectral representation of the multi-component PC-LS process $X(t) = X^{ls}(t) + X^{p}(t)$, where $X^{p}(t)$ and $X^{ls}(t)$ are defined by (3.2) and (3.8) is

$$X(t) = \int_{-\infty}^{\infty} e^{i\lambda t} Z_m(d\lambda, t), \quad t \in B_m \quad (3.12)$$

where the time varying spectral measure $Z_m(\cdot, t)$ is defined as

$$Z_m(d\lambda, t) = \Phi_m(d\lambda, t) + e^{i\lambda (m/T - t)} I_{[0, 2\pi)}(d\lambda) \zeta_m(d\lambda, t) \quad (3.13)$$

in which $\Phi_m(d\lambda, t)$ is characterized by (3.11). Also

$$\langle Z_m(d\lambda, t), Z_n(d\omega, u) \rangle = F_{m,n}(d\lambda, d\omega, t, u) + K \Theta_{m,n}(d\lambda, d\omega, t, u)$$

where $F_{m,n}(d\lambda, d\omega, t, u)$ and $\Theta_{m,n}(d\lambda, d\omega, t, u)$ are as in Lemma 3.2 and Lemma 3.1 respectively, and $K = e^{im/T}(\lambda + \omega - i(\lambda_2 + \omega)) I_{[0, 2\pi)}(d\lambda) I_{[0, 2\pi)}(d\omega)$.

Proof: By Lemma 3.2

$$X^{ls}_m(t) = \int_{-\infty}^{\infty} e^{i\lambda t} \Phi_m(d\lambda, t), \quad t \in B_m$$

where $\Phi_m(d\lambda, t)$ has orthogonally scattered property for $m \in \mathbb{N}$.

For finding the spectral representation of $X^{p}(t)$, $t \in \mathbb{R}$ we have

$$X^{p}(t) = \sum_{m=1}^{\infty} X^{p}_m(t) I_{B_m}(t) = \sum_{m=1}^{\infty} M_m(s_{m-1}, t) I_{B_m}(t)$$

where $M_m$ is the random measure corresponding to partition $B_m$. By the result of Lemmas 3.2, 3.1 and relation (3.2) we have

$$X(t) = \sum_{m=1}^{\infty} \left( \int_{-\infty}^{\infty} e^{i\lambda t} \Phi_m(d\lambda, t) + \int_{0}^{2\pi} e^{i\lambda m/T} \zeta_m(d\lambda, t) \right) I_{B_m}(t)$$

$$= \sum_{m=1}^{\infty} \left( \int_{-\infty}^{\infty} e^{i\lambda t} \Phi_m(d\lambda, t) + \int_{0}^{2\pi} e^{i\lambda t} e^{i\lambda (m/T - t)} \zeta_m(d\lambda, t) \right) I_{B_m}(t)$$

$$= \sum_{m=1}^{\infty} \int_{-\infty}^{\infty} e^{i\lambda t} \left( \Phi_m(d\lambda, t) + e^{i\lambda (m/T - t)} I_{[0, 2\pi)}(d\lambda) \zeta_m(d\lambda, t) \right) I_{B_m}(t).$$

Thus the aggregated process has a time varying spectral representation by (3.12). Also by Lemmas 3.1 and 3.2, the last assertion of the theorem as the covariance function of the random measure is obtained.
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