Numerical Study of Order in a Gauge Glass Model

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The XY model with random quenched phase shifts is studied by a $T = 0$ finite size defect energy scaling method in 2d and 3d. The defect energy is defined by a change in the boundary conditions from those compatible with the true ground state configuration for a given realization of disorder. A numerical technique, which is exact in principle, is used to evaluate this energy and to estimate the stiffness exponent $\theta$. This method gives $\theta = -0.36 \pm 0.013$ in 2d and $\theta = +0.31 \pm 0.015$ in 3d, which are considerably larger than previous estimates, strongly suggesting that the lower critical dimension is less than three. Some arguments in favor of these new estimates are given.

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The XY model with random quenched disorder, also known as a gauge glass, as a model for a superconducting glass phase has been intensively investigated over the last decade. Since the mean field theory flux lattice moves in response to a transport current, the existence of disorder, which, however, destroys the flux lattice structure, becomes essential in order to have a superconducting phase in a high $T_c$ superconductor. Since the pinning of flux lines by random disorder can prevent dissipation, a supercurrent can exist in the glass phase.

From numerical and experimental studies, it is believed that the gauge glass has no ordered phase at any finite temperature in two dimensions. In three dimensions, numerical studies indicate that the lower critical dimension seems to be close to three. However the situation is less conclusive, since the simulations are limited to small system sizes. Experimentally there is also some evidence which supports a finite temperature phase transition to a superconducting glass phase.

The Hamiltonian of the gauge glass model is given by

$$ H = \sum_{\langle ij \rangle} V(\theta_i - \theta_j - A_{ij}) $$

where $V(\phi)$ is a $2\pi$ periodic function of $\phi$ with a maximum at $\phi = \pi$, usually taken to be $V(\phi_{ij}) = -J_{ij}\cos(\phi_{ij})$. $\theta_i$ is the phase of the superconducting order parameter at site $i$ of a square lattice in 2d and a simple cubic lattice in 3d. The sum is taken over all nearest neighbor sites. The random bond variables, $A_{ij}$, which are responsible for the random frustration, are uniformly distributed in $(-\pi, \pi)$. The coupling constants, $J_{ij}$, are taken as constant, $J_{ij} = J$. In this paper, we use a domain wall renormalization group (DWRG), or defect energy scaling, method to investigate the possibility of an ordered phase at small but finite temperature $T$. The idea is to find the ground state (GS) energy $E_0(L)$ of a system of linear size $L$, then to change the boundary conditions (BC) in one direction to introduce a defect or domain wall into the system and finally to find the energy $E_D(L)$ of this. The energy difference $\Delta E(L) \equiv E_D(L) - E_0(L)$ is then the energy of a domain wall (defect) of length $L$. This is assumed to scale as

$$ < \Delta E(L) > \sim L^\theta $$

where $< \cdots >$ means an average over disorder and $\theta$ is the stiffness exponent. The sign of $\theta$ determines if a low $T$ ordered phase exists or not. If $\theta < 0$, in the thermodynamic limit the defect energy vanishes and at any $T > 0$ will proliferate and destroy the order. On the other hand, if $\theta > 0$, the defect energy diverges as $L \to \infty$ and the system will be ordered at sufficiently low $T$ and one expects a phase transition at some $T_c > 0$ between two distinct phases. The standard method to implement these ideas is to simply find the energies of the system subject to periodic and antiperiodic BC, despite the fact that neither BC is compatible with the true GS and that the energy difference is not a defect energy but is the energy difference between two randomly chosen defects. Nevertheless, $< \Delta E(L) > \equiv < |E_p(L) - E_0(L)| >$ is assumed to scale as $L^\theta$ and the best fit to this is taken as yielding the exponent $\theta$.

In this paper, we take the point of view that, since randomness restricts the accessible system sizes $L$ to rather small values, for the DWRG to be successful it is important to minimize uncontrolled and poorly understood effects which may introduce large corrections to scaling. Also when fitting a continuous function to less than ten data points there is scope for obtaining erroneous results. To attempt to minimize such effects, we first transform the problem described by eq. (1) to a Coulomb gas representation which eliminates the spin wave contribution to the energies and also enables us to estimate numerically the true GS energy by optimizing the BC. In the phase representation the problem would involve applying phase differences across all the bonds between opposite faces of the system on a hypertorus in $d$ dimensions and minimizing the energy with respect to $L^d$ phases and $dL^{d-1}$ phase differences on the bonds connecting opposite faces. In the Coulomb gas representation, one needs only $d$ extra parameters, one for each direction, and minimize with...
respect to \((d + L^d)\) parameters. To introduce a domain wall, we have to change only one of the \(d\) parameters from its value in the GS and minimize the energy with respect to \(L^d\) bulk variables keeping the BC fixed to find the defect energy \(E_D\). We then obtain the true domain wall energy as \(\Delta E(L) = E_D(L) - E_0(L) \geq 0\) for a particular realization of disorder. This procedure is repeated for several realizations of disorder and system sizes \(L\) to obtain \(\Delta E(L)\) which is fitted to eq. (3).

Some support for this point of view comes from a conjecture made by Ney-Nifle and Hilhorst in the context of the 2d \(XY\) spin glass, which is a special case of the model of eq. (1) when the bond variables \(A_{ij} = 0, \pi\) each with probability \(1/2\), based on some analytic work by Ney-Nifle, Hilhorst and Moore on the \(XY\) spin glass on a ladder. The latter authors solve the problem analytically and, by imposing reflective BC which forces a chiral domain wall into the system, find that both the spin and chiral domain wall energies scale with the same exponent \(\theta_s = \theta_c = -1.7972\ldots\). The same conclusion is reached in ref. [14] but with a slightly different value of \(\theta\). In ref. [14], supported by heuristic but plausible arguments, it is conjectured that, for any spin glass system below the lower critical dimension \(d_l\), \(\theta_s = \theta_c < 0\). This result, although lacking rigorous proof, provides an important bench test for a numerical simulation which seems to have been either ignored or overlooked in some recent studies eg. [15]. We then obtain the defect energy \(E\) which is fitted to eq. (2).

By a duality transformation [14][20][21], in 2d the Coulomb gas representation which is more convenient for numerical work from the phase representation of eq. (1) to a Coulomb gas representation which is equivalent to a Villain [19] potential at \(T = 0\)

\[
H = \frac{J}{2} \sum_{<ij>} (\theta_i - \theta_j - A_{ij} - 2\pi n_{ij})^2 \\
\equiv \frac{J}{2} \sum_{<ij>} (\phi_{ij} - A_{ij})^2
\]

(3)

\[
H = 2 \pi^2 J \sum_{r,r'} (q_r - f_r)(q_{r'} - f_{r'}) \\
+ \frac{J}{2L^2} \sum_{\alpha=x,y} \sigma^2_{\alpha}
\]

(4)

where

\[
\sigma_x = -2\pi \left[ L(q_{x1} - f_{x1}) + \sum_r (q_r - f_r)y \right]
\]

\[
\sigma_y = -2\pi \left[ L(q_{y1} - f_{y1}) - \sum_r (q_r - f_r)x \right]
\]

\[
G(r) = \frac{1}{N} \sum_{k \neq 0} e^{ik\cdot r} \left( 4 - 2\cos k_x - 2\cos k_y \right)
\]

(5)

\(r = (x, y)\) represents the coordinates of the sites of the dual lattice and \(G(r)\) is the lattice Green’s function. The topological charge, \(q_r\), is the circulation of the phase round the plaquette at \(r\), \(q_r = \sum_{\phi} \phi_{ij}/2\pi\) and can be any integer subject to the neutrality condition \(\sum_r q_r = 0\). The frustration, \(f_r\), is the circulation of \(A_{ij}/2\pi\) around the plaquette at \(r\). The quantity \(f_{x1}\) is the circulation of \(A_{ij}/2\pi\) round the whole torus on horizontal bonds of the plaquettes at \(y = 1\) and \(q_{x1}\) is the corresponding circulation of the phase. \(q_{y1}\) and \(q_{x1}\) are defined similarly. Periodicity in the phases \(\theta_i\) restricts \(q_{x1}, q_{y1}\) to integers.

In 3d, the charge Hamiltonian becomes

\[
H = 2 \pi^2 J \sum_{r,r'} (q_r - f_r) \cdot (q_{r'} - f_{r'})G(r - r') \\
+ \frac{J}{2L} \sum_{\alpha=x,y,z} \sigma^2_{\alpha}
\]

(6)

where

\[
\sigma_x = \pi L^{-1} \sum_r \left\{ -z(q_{x}^r - f_{x}^r) + y(q_{y}^r - f_{y}^r) \right\} + Q_x
\]

\[
Q_x = -\pi \sum_r \left\{ -z f_{y1}^r \delta_{y1,1} + y f_{x1}^r \delta_{x1,1} \right\} + 2\pi L(q_{x1} - f_{x1})
\]

\[
G(r) = \frac{1}{N} \sum_{k \neq 0} e^{ik\cdot r} \left( 6 - 2\cos k_x - 2\cos k_y - 2\cos k_z \right)
\]

(7)

with similar expressions for \(\sigma_y, \sigma_z, Q_y, Q_z\) which are obtained by cyclic permutations of \((x, y, z)\). Here, in 2d, \(r = (x, y, z)\) are the coordinates of the dual lattice sites and \(G(r)\) is the Green’s function. The charge \(q_r\) and the frustration \(f_r\) associated with the dual lattice at \(r\) become vector quantities with components \((q_x^r, q_y^r, q_z^r)\) and \((f_x^r, f_y^r, f_z^r)\). Here \(q_{x1}^r, f_{x1}^r\) are the circulations of \(\theta_i\).
and $A_{ij}$ about the plaquette at $r$ normal to the direction $\alpha$. These vector charges satisfy $V \cdot q_{r\alpha} = 0$ at each site $r$ and a neutrality condition as in 2d.

To find the true GS energy of the finite system of $L^d$ sites on a hypertorus, we minimize the energy given by eq.(3) with respect to the variables $q_{r\alpha}$ and $f_{x1}, f_{y1}$ in 2d, and by eq.(4) in 3d. The lowest energy of the system is $2\pi r$ periodic in the twist $\Delta_\mu \equiv 2\pi f_{\mu 1}$, with a minimum at some $\Delta_0^\mu$ which depends on the particular realization of disorder. The lowest energy with $\Delta_\mu \neq \Delta_0^\mu$ includes the excitation energy due to the twist $\Delta_\mu - \Delta_0^\mu$. Adding a twist is equivalent to the gauge transformation $A_{ij} \rightarrow A_{ij} + \Delta_\mu / L$ on all bonds in the direction $\mu$.

We compute the domain wall energy $\Delta E(L)$ in two different ways. The first is with the usual periodic and antiperiodic BC, which correspond to twists $\Delta \neq \Delta^0$ and $\Delta + \pi \hat{x}$ where $\Delta$ is determined by the particular sample which we call a random twist (RT) measurement. The second is by computing the true domain wall energy $\Delta E^{RT}(L) = E_L(\Delta^0 + \pi \hat{x}) - E_L(\Delta^0)$ where $E_L(\Delta^0)$ is obtained by minimizing the energy with respect to the bulk charges $q_r^\alpha$ and the twists $f_{\mu 0}$. The domain wall is induced by changing one of these $f_{\mu 0}$ round loops enclosing the hypertorus. Once these are computed and is found to scale rather accurately as $L^\theta$ for $L = 6$ and 60 for $L = 7$. The error in $\Delta E(L = 7)$ is very large, but this point was included to check that it is consistent with the behavior deduced from the smaller systems. The results are shown in fig.(2). For the RT measurement, there seems to be a crossover around $L = 5$ from a very small value of $\theta^{RT}$ to a larger positive value as is also seen in [3], but our sizes do not allow any estimate of $\theta^{RT}$. On the other hand, the BT measurement for these sizes is consistent with a power law scaling with a stiffness exponent $\theta^{BT} = +0.31 \pm 0.010$ using sizes $L \leq 6$, which is strong evidence in favor of a superconducting glass phase at finite $T$ and of $d_1 < 3$. This is also consistent with finite $T$ Monte Carlo results on the 3d gauge glass [3,4,22] which indicate $T_c \sim O(J)$, which is difficult to reconcile with the very small value of $\theta^{RT}$ which, if it were the stiffness exponent for the 3d gauge glass, would imply $d_1 \approx 3$ and a small value of $T_c / J$. If we include the $L = 7$ point, the best fit gives $\theta^{BT} = +0.30 \pm 0.015$. Note that the errors quoted here in $\theta^{BT,RT}$ come from a naive least squares fit to the data, and should not be taken too seriously. The $L = 7$ data is suspect because, in $10^3$ CPU hours on a Cray J90, 4 samples of a batch of 64 violated the BT condition $\Delta E^{RT}(L) \geq 0$ implying insufficient annealing to reach the true energy minima. What data we have is entirely consistent with the scaling form of eq.(4) with $\theta \approx +0.3$ with no sign of any deviation from this.

We also studied the effects of screening on the domain wall energy using the BT measurement in 3d. Screening of the interaction of charges is implemented by adding a term $\lambda^{-2}$, where $\lambda$ is the screening length, to the denominators of the Green’s function of eq.(6) [25]. The results are also shown in fig.(2). We averaged over $10^3$ samples for $L = 2, 3, 4$ and 250 for $L = 5$. For the shorter screening lengths, screening is clearly a relevant perturbation and destroys the ordered phase while for longer screening lengths $\Delta E(L)$ seems to scale the same way as the unscreened case but we expect there is a crossover to a negative stiffness exponent at length scales $L$ which are inaccessible with our computing power. Our results are consistent with those of Bokil and Young [26] who studied the screening question using the RT measurement.

The major result of this study is that one can, in principle, find the exact GS energy of a random $XY$ system by a suitable choice of boundary conditions which are consistent with the unknown GS. This is implemented for the gauge glass model in both 2d and 3d and it is also argued that a domain wall is created from the GS by an appropriate change of the BC. In the Coulomb gas representation of the system on a hypertorus, the BC are parametrized by $d$ numbers which are the circulations of $A_{ij}$ round loops enclosing the hypertorus. Once these are known, a domain wall is induced by changing one of these by $\pi$. The size dependence of the domain wall energy is computed and is found to scale rather accurately as $L^\theta$. The values of the stiffness exponent $\theta$ are much larger
than previous estimates and is in accord with \( d_t < 3 \). This also reconciles the finite \( T \) Monte Carlo results with one’s physical intuition. We stress that the disagreement between the stiffness exponent \( \theta_{BT} \) in this work and all previous estimates is because these measure \( \theta_{RT} \) which is a quantity whose meaning is unclear and is more likely to suffer from large corrections to scaling, especially for the small \( L \) values which can be simulated (see fig. (2) and ref. [3]). We conjecture that they would coincide if \( \text{much larger} \) values of \( L \) could be reached. The stiffness exponent \( \theta_{BT} \) is larger than the \( \theta_{ISG} \approx +0.2 \) for the 3d Ising spin glass [26] but we see no contradiction here as continuous variables can adjust to frustration more easily than Ising spins and one expects gauge glass order to resist distortions better than Ising spin glass order.

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