WAVELET METHODS IN PARTIAL DIFFERENTIAL EQUATIONS ON SPHERES

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Abstract. We propose a method of solving partial differential equations on the n-dimensional unit sphere with methods based on the continuous wavelet transform derived from approximate identities.

1. Introduction

In the last thirty years, many authors developed wavelet methods for solving differential equations. Already in the 1990s numerical solutions of ODEs [33, 11] and PDEs [31, 27] on the Euclidean space were found. Some further examples of wavelet application to ODEs are presented in [29, 20], and to PDEs in [19, 30, 38, 21, 15, 41, 4]. Contrary to the articles listed so far, the paper [16] describes a method for numerical solving of PDEs on the sphere. Recent years brought a number of papers in which wavelet methods were involved to solving fractional differential equations [18, 40, 32, 1]. The list is far from being complete, but it is apparent that wavelets are usually being used to develop algorithms for numerical solving of differential equations. No publication is known to us that presents an analytical solution.

Theories of continuous spherical wavelets have been developed in the last decades, simultaneously to theories of wavelets over Euclidean space. Iglewska-Nowak has shown in [23, Section 5] that there exist only two essentially different continuous wavelet transforms for spherical signals, namely that based on group theory [2, 3] and that derived from approximate identities [10, 12, 22, 23]. In the present paper we show that the latter one can be efficiently applied to solving partial differential equations on the sphere. We give an explicit solution of the Poisson equation, as well as an algorithm for solving the Helmholtz equation.

The present paper seems to be the first attempt to involve wavelet methods to analytical solving of PDEs.

2. Preliminaries

2.1. Functions on the sphere. A square integrable function \( f \) over the n-dimensional unit sphere \( S^n \subseteq \mathbb{R}^{n+1}, n \geq 2 \), with the rotation-invariant measure \( d\sigma \) normalized such
that
\[ \Sigma_n = \int_{S^n} d\sigma(x) = \frac{2\pi^{(n+1)/2}}{\Gamma((n+1)/2)}, \]
can be represented as a Fourier series in terms of the hyperspherical harmonics,
\[ f = \sum_{l=0}^{\infty} \sum_{k \in \mathcal{M}_{n-1}(l)} a^k_l(f) Y^k_l, \]
where \( \mathcal{M}_{n-1}(l) \) denotes the set of sequences \( k = (k_0, k_1, \ldots, k_{n-1}) \) in \( \mathbb{N}_{0}^{n-1} \times \mathbb{Z} \) such that \( l \geq k_0 \geq k_1 \geq \ldots \geq |k_{n-1}| \) and \( a^k_l(f) \) are the Fourier coefficients of \( f \). The hyperspherical harmonics of degree \( l \) nad order \( k \) are given by
\[ Y^k_l(x) = A_l^k \prod_{\tau=1}^{n-1} C_{k_{\tau-1} - k_{\tau}}^{n-\tau+k_{\tau}}(\cos \vartheta_\tau) \sin^{k_\tau} \vartheta_\tau \cdot e^{\pm ik_{n-1}\varphi}, \]
for some constants \( A_l^k \). Here, \((\vartheta_1, \ldots, \vartheta_{n-1}, \varphi)\) are the hyperspherical coordinates of \( x \in S^n \),
\[
\begin{align*}
  x_1 &= \cos \vartheta_1, \\
  x_2 &= \sin \vartheta_1 \cos \vartheta_2, \\
  x_3 &= \sin \vartheta_1 \sin \vartheta_2 \cos \vartheta_3, \\
  \ldots \\
  x_{n-1} &= \sin \vartheta_1 \ldots \sin \vartheta_{n-2} \cos \vartheta_{n-1}, \\
  x_n &= \sin \vartheta_1 \ldots \sin \vartheta_{n-2} \sin \vartheta_{n-1} \cos \varphi, \\
  x_{n+1} &= \sin \vartheta_1 \ldots \sin \vartheta_{n-2} \sin \vartheta_{n-1} \sin \varphi,
\end{align*}
\]
and \( C^K_\kappa \) are the Gegenbauer polynomials of degree \( \kappa \) and order \( K \). The set of the hyperspherical harmonics of degree \( l \) is denoted by \( \mathcal{H}_l \).

The Laplace-Beltrami operator on the sphere \( \Delta^* \) is defined by
\[
\Delta^* f = \sum_{k=1}^{n-1} \left( \prod_{j=1}^{k} \sin \vartheta_j \right)^{-2} (\sin \vartheta_k)^{k+2-n} \frac{\partial}{\partial \vartheta_k} \left( \sin^{n-k} \vartheta_k \frac{\partial f}{\partial \vartheta_k} \right) + \left( \prod_{j=1}^{k} \sin \vartheta_j \right)^{-2} \frac{\partial^2 f}{\partial \varphi^2}.
\]
It is known that the hyperspherical harmonics are the eigenfunctions of \( \Delta^* \), i.e.,
\[ \Delta^* Y^k_l = -l(l+n-1)Y^k_l, \]
see [34 Chap. II, Theorem 4.1]. The relation of \( \Delta^* \) and the Laplace operator \( \Delta \) is given by
\[ \Delta f = R^{-n} \frac{\partial}{\partial R} \left( R^n \frac{\partial f}{\partial R} \right) + \frac{1}{R^2} \Delta^* f, \]
where \( R \geq 0 \) is the radius of \( x \in \mathbb{R}^n \) in the hyperspherical coordinates, see [34 Chap. II, Proposition 3.3].

The Laplace operator is commutative with \( SO(n+1) \)-rotations \( \Upsilon \),
\[ \Delta [f(\Upsilon x)] = (\Delta f)(\Upsilon x), \]
see [39, Chap. IX, § 2, Subsec. 4]. Consequently, from [4] it follows that the same holds for the Laplace-Beltrami operator, see also [34, Chap. II, formula (3.15)]. Since $S^n$ is a manifold without boundary, it follows from the Green’s second surface identity that for $f, g$ of class $C^2$ the following holds:

$$
\int_{S^n} \Delta^* f(x) \cdot g(x) \, d\sigma(x) = \int_{S^n} f(x) \cdot \Delta^* g(x) \, d\sigma(x).
$$

(6)

The scalar product in $L^2(S^n)$ is antilinear in the first variable,

$$
\langle f, g \rangle = \frac{1}{\Sigma_n} \int_{S^n} \overline{f(x)} \, g(x) \, d\sigma(x).
$$

Since $\Delta^*$ is a linear operator, one has

$$
\Delta^* f = \Delta^* f
$$

and (6) can be also written as

$$
\langle \Delta^* f, g \rangle = \langle f, \Delta^* g \rangle.
$$

(7)

Zonal (rotation-invariant) functions are those depending only on the first hyperspherical coordinate $\vartheta = \vartheta_1$. Unless it leads to misunderstandings, we identify them with functions of $\vartheta$ or $t = \cos \vartheta$. A zonal $L^2$-function $f$ has the following Gegenbauer expansion

$$
f(t) = \sum_{l=0}^{\infty} \hat{f}(l) C^{\lambda}_l(t), \quad t = \cos \vartheta,
$$

(8)

where $\hat{f}(l)$ are the Gegenbauer coefficients of $f$ and $\lambda$ is related to the space dimension by

$$
\lambda = \frac{n - 1}{2}.
$$

Consequently, for a zonal $L^2$-function $f$ one has

$$
\hat{f}(l) = A^0_l \cdot a^0_l(f),
$$

compare (1), (2), and (8).

For $f, g \in L^1(S^n)$, $g$ zonal, their convolution $f * g$ is defined by [9, Definition 2.1.1]

$$
(f * g)(x) = \frac{1}{\Sigma_n} \int_{S^n} f(y) \, g(x \cdot y) \, d\sigma(y).
$$

With this notation one has

$$
f \in \mathcal{H}_l \implies f = \frac{\lambda + l}{\lambda} \left( f \ast C^\lambda_l \right)
$$

(Funck-Hecke formula), i.e., the function $\frac{\lambda + l}{\lambda} C^\lambda_l$ is the reproducing kernel for $\mathcal{H}_l$.  

2.2. Frames in reproducing kernel Hilbert spaces. One of the wavelet definitions considered in the present paper requires the notion of a continuous frame. Here we present the one given in [28].

**Definition 2.1.** Let $\mathcal{H}$ be a Hilbert space and let $M$ be a measure space with measure $\mu$. A generalized frame in $\mathcal{H}$ indexed by $M$ is a family of vectors $\{h_m \in \mathcal{H} : m \in M\}$ such that

(a) For every $f \in \mathcal{H}$, the function $\tilde{f} : M \to \mathbb{C}$ defined by
\[
\tilde{f}(m) = \langle h_m, f \rangle_{\mathcal{H}}
\]
is measurable.

(b) There exists a pair of constants $0 < A < B < \infty$ such that for every $f \in \mathcal{H}$,
\[
A \|f\|_{\mathcal{H}}^2 \leq \|\tilde{f}\|_{L^2(M)}^2 \leq B \|f\|_{\mathcal{H}}^2.
\] (9)

The mapping
\[
T : \mathcal{H} \to L^2(M)
\]
\[
f \mapsto \tilde{f}
\]
is a linear operator, which is called the frame operator. By the frame condition (9) it is bounded and invertible.

**Remark.** In the theory of discrete frames which are much more popular than continuous ones, see, e.g., [7, 8, 17], operator $T$ is called analysis operator or pre-frame operator, whereas frame operator means $T^*T$.

Theorem 4.4 in [28] states the following.

**Theorem 2.2.** The synthesizing operator $S = (T^*T)^{-1}T^* : L^2(M) \to \mathcal{H}$ is given by
\[
Sg = \int_M h^m g(m) \, d\nu(m),
\]
where
\[
h^m = (T^*T)^{-1}h_m
\]
is the reciprocal frame of $\mathcal{H}_M$. In particular, $f \in \mathcal{H}$ can be reconstructed from $\tilde{f} \in \mathcal{F}$ by
\[
f = S\tilde{f} = \int_M h^m \tilde{f}(m) \, d\nu(m).
\]

On the other hand, signal reconstruction from its frame coefficients can be done iteratively by the so-called frame algorithm, compare [17, Section 5.1].

**Algorithm 2.3.** Given a relaxation parameter $0 < \varrho < \frac{2}{B}$, set $\delta = \max\{|1 - \varrho A|, |1 - \varrho B|\} < 1$. Let $f_0 = 0$ and define recursively
\[
f_{k+1} = f_k + \lambda(T^*T)(f - f_k).
\]
Then, $\lim_{k \to \infty} f_k = f$ with a geometric rate of convergence,
\[
\|f - f_k\| \leq \delta^k \|f\|.
\]
Note that
\[ f_1 = \varrho(T^*T)f = \varrho \int_M h_m \langle h_m, f \rangle \, d\mu(m) \]
contains the frame coefficients as input. This suffices to compute the further approximations \( f_k \) and to reconstruct \( f \).

3. Two wavelet transforms

A wavelet transform based on approximate identities can be performed in a twofold way. In the first case, the constraints on a function to be a wavelet are quite restrictive, but the inverse transform is given directly by an integral. This is the more popular version of the wavelet transform, developed starting from the 1990s by Freeden et al. [12, 13, 14] and Bernstein et al. [5, 10], as well as by Iglewska-Nowak in the recent years [22, 23, 24, 25]. The following definition comes from [22], it is an improved version of the one used in [23], adapted to the case of zonal wavelets and with the most popular weight function \( \alpha(\rho) = \frac{1}{\rho} \).

**Definition 3.1.** A family \( \{ \Psi_\rho \}_{\rho \in \mathbb{R}^+_0} \subseteq L^2(S^n) \) of rotation-invariant functions is called an admissible spherical wavelet if it satisfies the following condition:
\[
\int_0^\infty \left| \hat{\Psi}_\rho(l) \right|^2 \frac{d\rho}{\rho} = \left( \frac{\lambda + l}{\lambda} \right)^2 \quad \text{for } l \in \mathbb{N}_0.
\]

The wavelet transform is defined by
\[
W_\Psi f(\rho, y) = \frac{1}{\sum_n} \int_{S^n} \Psi_\rho(y \cdot x) \cdot f(x) \, d\sigma(x) = \langle \tau_y \Psi_\rho, f \rangle,
\]
where \( \tau_y \) denotes the translation operator of zonal functions,
\[
\tau_y f(x) = f(y \cdot x).
\]

Note that this operator can be represented by
\[
\tau_y f(x) = f(A_y \cdot x),
\]
where \( A_y \) is an \( SO(n+1) \)-matrix, corresponding to the translation \( \tau_y \).

The wavelet transform is invertible by an integral. Theorem 3.2 in [22] reduced to the case of zonal functions states the following.

**Theorem 3.2.** If \( \{ \Psi_\rho \}_{\rho \in \mathbb{R}^+_0} \) is an admissible wavelet, then the wavelet transform is invertible by
\[
f(x) = \frac{1}{4\pi} \int_0^\infty \int_{S^n} \Psi_\rho(x \cdot y) \cdot W_\Psi f(\rho, y) \, d\sigma(y) \frac{d\rho}{\rho}
\]
with the limit in \( L^2 \)-sense.

Note that the wavelet transform definition from [22] formula (4)] should be corrected by the factor \( \frac{1}{\sum_n} \) such that it coincides with the one used in the present paper. The article [22] is based on [23], where the wavelet transform is defined in the same way as in formula (11). In the proof of [22] Theorem 3.2] the reproducing property of the kernels \( \frac{\lambda + l}{\lambda} C_l^\lambda \) is used (falsely) without the factor \( \frac{1}{\sum_n} \).
Examples of wavelets satisfying Definition 3.1 are first of all the Gauss-Weierstrass wavelet given by
\[ \widehat{\Psi}^G_{\rho}(l) = \sqrt{2l(l + 2\lambda)\rho} e^{-l(l + 2\lambda)\rho} \cdot \frac{\lambda + l}{\lambda}, \quad l \in \mathbb{N}_0, \]
the Abel-Poisson wavelet,
\[ \widehat{\Psi}^A_{\rho}(l) = \sqrt{2lr\rho} e^{-l\rho} \cdot \frac{\lambda + l}{\lambda}, \quad l \in \mathbb{N}_0, \]
and the Poisson wavelets of order \( d \in \mathbb{N} \),
\[ \widehat{\Psi}^d_{\rho}(l) = \frac{2^d}{\sqrt{\Gamma(2d)}} \cdot (l\rho)^d e^{-l\rho} \cdot \frac{\lambda + l}{\lambda}, \quad l \in \mathbb{N}_0. \]
For more details on the latter family see [26].

Another definition of a spherical wavelet transform based on approximate identities has been introduced in [25] in order to obtain a wider class of functions that can be used as wavelets. The constraints on a wavelets family are weaker as in Definition 3.1 but the price to be paid is the lack of a direct inverse wavelet transform. The analysed signal can be reconstructed by frame methods.

**Definition 3.3.** The family \( \{\Psi_{\rho}\}_{\rho \in \mathbb{R}_+} \subseteq L^2(S^n) \) is called a wavelet (family) of order \( m \) if \( \widehat{\Psi}_{\rho}(l) = 0 \) for \( l = 0, 1, \ldots, m \) and it satisfies
\[
A \cdot \left( \frac{\lambda + l}{\lambda} \right)^2 \leq \int_0^{\infty} |\widehat{\Psi}_{\rho}(l)|^2 \frac{d\rho}{\rho} \leq B \cdot \left( \frac{\lambda + l}{\lambda} \right)^2
\]
for some positive constants \( A \) and \( B \) independent of \( l \in \mathbb{N}_0 \) and \( l > m \). The constants \( A \) and \( B \) are called wavelet family bounds of the wavelet family \( \{\Psi_{\rho}\} \).

The wavelet transform is performed according to (11) and it follows that the wavelet coefficients \( \{W_{\rho} f(\rho, y)\} \) constitute a frame (see [25, Theorem 2.5]).

**Theorem 3.4.** Let \( \{\Psi_{\rho}\} \) be a wavelet family of order \( m \). Then for any \( f \in L^2(S^n) \) with \( m \) vanishing moments (i.e., such that \( a^k_l(f) = 0 \) for \( l = 0, 1, \ldots, m \) and \( k \in \mathcal{M}_{m-1}(l) \)) we have
\[
A \|f\|^2 \leq \|W_{\rho} f\|^2 \leq B \|f\|^2,
\]
i.e., the set \( \{\tau_y \Psi_{\rho}, \rho \in (0, \infty), y \in S^n\} \) is a frame for \( L^2(S^n) \setminus \bigcup_{l=0}^{m} \mathcal{H}_l \).

This means that in this case the wavelet transform is invertible, but with use of frame techniques.

### 4. Differential equations

**4.1. The Poisson equation** \( \Delta^* u = f \). Consider the Poisson equation
\[
\Delta^* u = f, \tag{14}
\]
where both functions \( u \) and \( f \) are given as the Fourier series
\[
u = \sum_{l=0}^{\infty} \sum_{k \in \mathcal{M}_{n-1}(l)} a^k_l(u) Y^k_l, \quad f = \sum_{l=0}^{\infty} \sum_{k \in \mathcal{M}_{n-1}(l)} a^k_l(f) Y^k_l.
\]
Since the hyperspherical harmonics $Y^k_l$ are linearly independent, one obtains by (3) the following relation

$$-l(l + n - 1) a^k_l(u) = a^k_l(f).$$

If $a^0_0(f) = 0$,

$$u = \sum_{l=1}^{\infty} \sum_{k \in M_{n-1}(l)} \frac{-a^k_l(f)}{l(l + n - 1)} \cdot Y^k_l$$

is a solution to the equation (14). This statement is the content of [6, Theorem 2.1], however, convergence of the series is not proved. Wavelet methods yield the same formula for the solution of (14) and, additionally, the proof of convergence.

**Theorem 4.1.** Let $f$ be an $L^1(S^n)$-function such that $a^0_0(f) = 0$. Then, the function

$$u = f * K$$

with

$$K = \sum_{l=1}^{\infty} \frac{-1}{l(n + l - 1)} \cdot \frac{\lambda + l}{\lambda} C^\lambda_l$$

is a solution to the equation

$$\Delta^* u = f.$$ 

**Proof.** Consider the family of functions

$$\psi^d_\rho = \sum_{l=0}^{\infty} \frac{\rho^d l^{d+1}}{l + n - 1} \cdot e^{-\rho l} \cdot \frac{\lambda + l}{\lambda} C^\lambda_l = \frac{\sqrt{\Gamma(2d)}}{2^d} \cdot \sum_{l=0}^{\infty} \frac{l}{l + n - 1} \cdot \frac{\lambda + l}{\lambda} \cdot \widehat{\psi^d_\rho}(l) C^\lambda_l. \quad (17)$$

(Recall that $\Psi^d_\rho$ is the Poisson wavelet of order $d$). Its coefficients satisfy

$$\int_0^\infty |\widehat{\psi^d_\rho}(l)|^2 d\rho = \frac{\Gamma(2d)}{4^d} \cdot \frac{l^2}{(l + n - 1)^2} \cdot \left(\frac{\lambda + l}{\lambda}\right)^2,$$

i.e., $\{\psi^d_\rho\}$ is a wavelet with respect to Definition 3.3. Wavelet family bounds can be chosen to be equal to

$$A := \frac{\Gamma(2d)}{4^d} \cdot \inf_{l \in \mathbb{N}} \frac{l^2}{(l + n - 1)^2} = \frac{\Gamma(2d)}{4^d} \cdot \frac{l^2}{4^d \cdot n^2}, \quad B := \frac{\Gamma(2d)}{4^d} \cdot \sup_{l \in \mathbb{N}} \frac{l^2}{(l + n - 1)^2} = \frac{\Gamma(2d)}{4^d}.$$

The wavelet transform of $f$ is given by

$$\mathcal{W}_{\psi^d_\rho} f(\rho, y) = \langle \tau_y \psi^d_\rho, f \rangle = \langle \tau_y \psi^d_\rho, \Delta^* u \rangle.$$

From (7) and the differentiability of (17) it follows that

$$\mathcal{W}_{\psi^d_\rho} f(\rho, y) = \langle \Delta^* (\tau_y \psi^d_\rho), u \rangle. \quad (18)$$

Set $\theta^d_\rho := \rho^2 \Delta^* \psi^d_\rho$. According to (3) and (8), the coefficients of this family satisfy

$$\widehat{\theta^d_\rho}(l) = -\rho^2 l(l + n - 1) \widehat{\psi^d_\rho}(l) = -(\rho l)^{d+2} e^{-\rho l} \cdot \frac{\lambda + l}{\lambda}.$$
\[ \rho^2 \cdot \mathcal{W}_{\psi^d} f = \langle \tau_y (\rho^2 \Delta \psi^d), u \rangle = \mathcal{W}_{\psi^d} u. \]  

(19)

Since \( \{-\frac{d+2}{\sqrt{\Gamma(2d+4)}} \cdot \theta^d_\rho \} \) is a wavelet with respect to Definition 3.1, direct inversion is possible, i.e.,

\[ u(x) = \frac{4^{d+2}}{\sum_n \cdot \Gamma(2d + 4)} \cdot \int_{S^n} \theta^d_\rho (x \cdot y) \cdot \mathcal{W}_{\psi^d} u(\rho, y) \, d\sigma(y) \frac{\,d\rho}{\rho}. \]

Hence, by (19),

\[ u(x) = \frac{4^{d+2}}{\sum_n \cdot \Gamma(2d + 4)} \cdot \int_{S^n} \theta^d_\rho (x \cdot y) \cdot \rho^2 \, \mathcal{W}_{\psi^d} f(\rho, y) \, d\sigma(y) \frac{\,d\rho}{\rho}. \]

Substitute the right-hand-side of (11) for \( \mathcal{W}_{\psi^d} f \) to obtain

\[ u(x) = \frac{4^{d+2}}{\sum_n \cdot \Gamma(2d + 4)} \cdot \int_{S^n} \phi^d(x \cdot y) \cdot \rho^2 \, \mathcal{W}_{\psi^d} f(\rho, y) \, d\sigma(y) \frac{\,d\rho}{\rho}. \]

The triple integral is convergent in \( L^2 \)-sense (with \( \int_0^\infty \) understood as \( \lim_{R \to 0} \int_R^{1/R} \)), compare the proof of [22, Theorem 3.2]. Thus, the order of integration may be changed,

\[ u(x) = \frac{4^{d+2}}{\sum_n \cdot \Gamma(2d + 4)} \cdot \int_{S^n} \int_{S^n} \rho \cdot \theta^d_\rho (x \cdot y) \cdot \mathcal{W}_{\psi^d} f(\rho, y) \, d\sigma(y) \, d\sigma(z) \, d\rho. \]

(20)

Now, [23, formula (15)] written for two zonal functions \( f, h \),

\[ f \ast h = \sum_{l=0}^{\infty} \frac{n - 1}{n + 2l - 1} \, \hat{f}(l) \, \hat{h}(l) \, C^\lambda_l. \]

yields

\[ \frac{\rho}{\sum_n} \int_{S^n} \theta^d_\rho (x \cdot y) \cdot \mathcal{W}_{\psi^d}(y \cdot z) \, d\sigma(y) \]

\[ = \rho \cdot \sum_{l=0}^{\infty} \frac{n - 1}{n + 2l - 1} \cdot \left[ - (\rho l)^{d+2} \cdot e^{-\rho l} \cdot \frac{n + 2l - 1}{n - 1} \right] \cdot \frac{\rho^{d+1} \cdot e^{-\rho l} \cdot \frac{n + 2l - 1}{n - 1} \cdot C^\lambda_l(x \cdot z)}{\sum_n \cdot \Gamma(2d + 4)} \frac{\,d\rho}{\rho}. \]

\[ = - \sum_{l=0}^{\infty} \frac{(\rho l)^{2d+3}}{n + l - 1} \cdot e^{-2\rho l} \cdot \frac{\lambda + l}{\lambda} C^\lambda_l(x \cdot z). \]

(22)

Further,

\[ \int_0^\infty (\rho l)^{2d+3} \cdot e^{-2\rho l} \, d\rho = \frac{\Gamma(2d + 4)}{4^{d+1} \cdot l}. \]

(23)

Substitute (22) and (23) into (20) to obtain

\[ u(x) = \frac{1}{\sum_n} \int_{S^n} K(x \cdot z) \cdot f(z) \, d\sigma(z) = f \ast K(x) \]
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with

\[
\mathcal{K} = -\frac{4^{d+2}}{\Gamma(2d+4)} \cdot \int_0^\infty \sum_{l=0}^\infty \left(\frac{pl}{n+l-1}\right)^{2d+3} \cdot e^{-2pl} \cdot \frac{\lambda + l}{\lambda} C_l^\lambda d\rho
\]

(24)

\[
= \sum_{l=1}^\infty \frac{-1}{l(n+l-1)} \cdot \frac{\lambda + l}{\lambda} C_l^\lambda.
\]

(25)

**Remark.** Since \(\frac{\lambda + l}{\lambda} C_l^\lambda\) is the reproducing kernel for \(\mathcal{H}_l\), for \(f\) given as Fourier series this yields (15). Convergence of the series representing both \(\mathcal{K}\) and \(f\) follows from the arguments given in the proof of [22, Theorem 3.2]. Note that \(\mathcal{K}\) is in principle a convolution of two Poisson wavelets, integrated with respect to the scale parameter. Thus, it is twice continuously differentiable. This ensures that \(u \in C^2(S^n)\).

4.1.1. Direct representations of the kernel \(\mathcal{K}\). Now, in order to obtain a direct expression for \(\mathcal{K}\), note that (25) can be obtained from the Poisson kernel [26, formulae (4) and (5)]

\[
p_r(\cos \vartheta) = \frac{1}{\sum_n} \cdot \frac{1 - r^2}{(1 - 2r \cos \vartheta + r^2)^{(n+1)/2}} = \frac{1}{\sum_n} \cdot \sum_{l=0}^\infty r^l \cdot \frac{\lambda + l}{\lambda} C_l^\lambda(\cos \vartheta).
\]

(26)

**Proposition 4.2.** The kernel \(\mathcal{K}\), given by (16), is equal to

\[
\mathcal{K}(\cos \vartheta) = -\lim_{\epsilon \to 0} \int_0^{1-\epsilon} R^{n-2} \cdot \int_0^R \frac{\sum_n \cdot p_r(\cos \vartheta) - 1}{r} dr dR.
\]

(27)

**Remark.** The upper bound in the outer integral cannot be chosen to be equal to 1, because \(p_r\) is defined only for \(r \in [0, 1)\). However, the integrand \(\frac{\sum_n \cdot p_r(\cos \vartheta) - 1}{r}\) can be continuously extended to \(r = 1\), such that actually, the second integration is performed over the interval \([0, 1]\).

**Proof.** Since the Gegenbauer polynomials \(C_l^\lambda\) over the interval \([-1, 1]\) are bounded by

\[
|C_l^\lambda(\cos \vartheta)| \leq (n + l - 2)^{-n-2}
\]

(28)

uniformly in \(\vartheta \in [0, \pi]\) (compare [35, Theorem 7.33.1]), the series (26) is absolutely convergent for \(r \in [0, 1)\). Note that for \(n \geq 2\) and \(l \geq 1\)

\[
\frac{1}{l(n+l-1)} = \int_0^1 R^{n-2} \cdot \int_0^R r^{l-1} dr dR.
\]

Substitute this expression to (16) to obtain (27), taking into account that \(C_0^\lambda(\cos 0) = 1\).

\(\Box\)

**Example 4.3.** For \(n = 2\) it follows from (26)

\[
\beta_r(t) := \sum_{l=1}^\infty r^{l-1} \cdot (2l + 1) C_l^1(t) = \frac{1 - r^2}{r(1 - 2tr + r^2)^{3/2}} - \frac{1}{r}.
\]
Further, \[ \gamma_r(t) := \int \beta_r(t) \, dr = \frac{2}{\sqrt{1 - 2tr + r^2}} - \ln \left( 1 - rt + \sqrt{1 - 2tr + r^2} \right) + C \]

and \[ \zeta_R(t) := \int [\gamma_R(t) - \gamma_0(t)] \, dR = \ln \left( R - t + \sqrt{1 - 2tR + R^2} \right) - R \left( 1 + \ln \frac{1 - tR + \sqrt{1 - 2tR + R^2}}{2} \right). \]

Consequently, \[ K^{(2)}(t) = \zeta^{(2)}_0(t) - \zeta^{(2)}_1(t) = 1 + \ln \frac{1 - t}{2}. \]

For \( t = \cos \vartheta \) it can be expressed as \[ K^{(2)}(\cos \vartheta) = 1 + \ln \left( \sin \frac{\vartheta}{2} \right)^2. \]

Table 4.1.1 gives the expressions for \( K \) for \( n = 2, 3, 4, 5, 6, 7, 8, 9, 10 \).

### Table 4.1.1. Kernel \( K \) for different values of \( n \), \( t = \cos \vartheta \)

| \( n = 2 \) | \( 1 + 2 \ln \left( \frac{1 - t}{2} \right) \) |
| --- | --- |
| \( n = 3 \) | \( \frac{-2(\pi \vartheta)}{2\sqrt{1-t^2}} + \frac{1}{4} \) |
| \( n = 4 \) | \( \frac{4 - 7t}{9(1-t)} + \frac{1}{3} \ln \left( \frac{1-t}{2} \right) \) |
| \( n = 5 \) | \( \frac{(\pi \vartheta) t (3 - 2t^2)}{2(1-t^2)^{3/2}} - \frac{3 - 5t^2}{16(1-t^2)} \) |
| \( n = 6 \) | \( \frac{23 - 7t + 43t^2}{75(1-t)^2} + \frac{1}{9} \ln \left( \frac{1-t}{2} \right) \) |
| \( n = 7 \) | \( \frac{(\pi \vartheta) (-15 + 20t^2 - 8t^4)}{48(1-t^2)^{5/2}} + \frac{22 - 7t^2 + 49t^4}{144(1-t^2)^2} \) |
| \( n = 8 \) | \( \frac{176 - 75t + 90t^2 - 33t^3}{735(1-t)^3} + \frac{1}{7} \ln \left( \frac{1-t}{2} \right) \) |
| \( n = 9 \) | \( \frac{(\pi \vartheta) (-35 + 70t^2 - 50t^4 + 16t^6)}{128(1-t^2)^{7/2}} + \frac{50 - 237t^2 + 266t^4 - 94t^6}{384(1-t^2)^3} \) |
| \( n = 10 \) | \( \frac{563 - 3089t + 5469t^2 - 4049t^3 + 1091t^4}{2835(1-t)^4} + \frac{1}{9} \ln \left( \frac{1-t}{2} \right) \) |

4.2. The Helmholtz equation \( \Delta^* u + au = f \), \( a \in \mathbb{C} \).

4.2.1. Solution with wavelet methods. Let \{\Psi^d_\lambda\} be the Poisson wavelet family of order \( d \), i.e.,

\[ \Psi^d_\lambda = \sum_{l=0}^{\infty} (\rho l)^d \cdot e^{-\rho l} \cdot \frac{\lambda + l}{\lambda} C^d_\lambda. \]

The wavelet transform of \( f \) is equal to \[ \mathcal{W}_\Psi f(\rho, y) = \langle \tau_y \Psi^d_\rho, f \rangle = \langle \tau_y \Psi^d_\rho, \Delta^* u \rangle + \langle \tau_y \Psi^d_\rho, au \rangle, \]

and it can be written as \[ \mathcal{W}_\Psi f(\rho, y) = \langle \tau_y [(\Delta + \tilde{a})\Psi^d_\rho], u \rangle \]
(by the commutativity of the Laplace-Beltrami operator with rotations). The Gegenbauer coefficients of the family \( \Theta^d_\rho := \rho^2 (\Delta + \bar{a}) \Psi^d_\rho \) are equal to
\[
\hat{\Theta}^d_\rho(l) = \rho^2 \left[ -l(l+n-1) + \bar{a} \right] \hat{\Psi}^d_\rho(l) \\
= -\hat{\mathcal{P}}^{d+2}_\rho(l) - \rho \cdot (n-1) \hat{\Psi}^{d+1}_\rho(l) + \rho^2 \cdot \bar{a} \hat{\mathcal{P}}^d_\rho(l)
\]
and they satisfy
\[
\int_0^\infty \left| \hat{\Theta}^d_\rho(l) \right|^2 \frac{d\rho}{\rho} = \frac{\Gamma(2d+4)}{2^{2d+2} l^4} \cdot \left[ l(l+n-1) - \bar{a} \right]^2 \cdot \left( \frac{\lambda + l}{\lambda} \right)^2.
\]
If \( a \neq l(l+n-1) \) for all \( l \in \mathbb{N} \), the family \( \{\Theta^d_\rho\} \) is a wavelet according to Definition 3.3.

In this case,
\[
\rho^2 \cdot W \Psi^d f(\rho,y) = \langle \tau_y \Theta^d_\rho, u \rangle = W \Theta^d u(\rho,y)
\]
and the function \( u \) can be recovered by frame methods.

4.2.2. The generalized Green function for the Helmholtz equation. The Helmholtz equation is a subject of research of Szmytkowski [36, 37]. The author studies the case when \( a = L(L+n-1) \), \( L \in \mathbb{Z} \). It turns out that the solution is given by
\[
u = \int_{S^n} \mathcal{G}_L(x \cdot y) \cdot f(y) d\sigma(y).
\]
One has the generalized Green function
\[
\mathcal{G}_L(x \cdot y) = -\sum_{\substack{l \geq 0 \ l \neq L \ k \in \mathcal{M}_{n-1}(l) \ \sum}} Y^k_l(x) \cdot \overline{Y^k_l(y)} \frac{l(l+n-1) - L(L+n-1)}{l(l+n-1) - L(L+n-1)}.
\]
In the papers [36, 37] closed forms of these functions are given. With the method developed in Subsection 4.1.1, we are able to derive an algorithm for finding direct expressions for \( \mathcal{G}_L \).

It is different from the one presented by Szmytkowski. For the set of indices that we have tested, the expressions for \( \mathcal{G}_L \) coincide with those in [36, 37].

**Theorem 4.4.** Let \( n \geq 2 \) be fixed. Then,
\[
\mathcal{G}_L(\cos \vartheta) = -\lim_{\varepsilon \rightarrow 0} \int_0^{\varepsilon} \int_0^R R^{n+2k-1} \left( \sum_{l=0}^L r^l \cdot \frac{\lambda + l}{\lambda} \mathcal{C}_l^\lambda(\cos \vartheta) \right) dr dR - \sum_{l=0}^{L-1} \frac{1}{(l-L)(l+n+L-1)} \cdot \lambda + l \mathcal{C}_l^\lambda(\cos \vartheta).
\]

**Proof.** Note that
\[
\sum_{k \in \mathcal{M}_{n-1}(l)} Y^k_l(x) \cdot \overline{Y^k_l(y)} = \frac{\lambda + l}{\lambda} \mathcal{C}_l^\lambda(x \cdot y)
\]
([9, Theorem 1.2.6] together with [9, Lemma 1.2.3]) and
\[
l(l+n-1) - L(L+n-1) = (l-L)(l+n+L-1).
\]
Further,
\[ \sum_n p_n(\cos \vartheta) - \sum_{l=0}^L r^l \cdot \frac{\lambda + l}{\lambda} C_l^n(\cos \vartheta) = \sum_{l=L+1}^{\infty} r^l \cdot \frac{\lambda + l}{\lambda} C_l^n(\cos \vartheta). \]

The statement (29) follows from
\[ \frac{1}{(l - L)(l + n + L - 1)} = \int_0^1 R^{-(n+2k)} \cdot \int_0^R r^{n+L-2} \cdot r^l \, dr \, dR \]
with the same arguments as in the proof of Proposition 4.2.

Table 4.2.2 gives the expressions for \( G_L(\cos \vartheta) \) for several values of parameters \( n \) and \( L \). They coincide with those derived in \cite{36} and \cite{37}.

**Appendix**

In this section, we collect several hints for computing integrals of irrational functions that arise when one wants to derive a formula for the kernel \( K \) in an even-dimensional space.

**Lemma 5.5.** If \( \lambda \) is a half-integer, \( \lambda \in \mathbb{N}/2 \setminus \mathbb{N} \), then the following holds:
\[
\int \left[ \frac{1 - r^2}{r(1 - 2tr + r^2)^{\lambda+1}} - \frac{1}{r} \right] dr
= \frac{1}{\lambda(t + r^2)^{\lambda}} + \frac{1}{2} \sum_{j=1}^{\lambda+\frac{1}{2}} \frac{1}{(\lambda - j)(t + r^2)^{\lambda-j}} + \sum_{j=1}^{\lambda+\frac{1}{2}} \frac{t Q_{j-1}^{\lambda}(t) \cdot r}{t^j(t + r^2)^{\lambda-j}} - \ln \left( t - tr + \sqrt{t + r^2} \right) + C,
\]
where \( r = r - t \) and \( t = 1 - t^2 \), and polynomials \( Q_j^{\lambda} \), \( j = 0, 1, \ldots, \lambda - \frac{3}{2} \), are given recursively by
\[
1 = 2(\lambda - 1)Q_0^{\lambda}(t),
0 = 2(\lambda - 2)Q_1^{\lambda}(t) - [(2\lambda - 3) + (2\lambda - 1)t] \cdot Q_0^{\lambda}(t),
0 = 2(\lambda - j - 1)Q_j^{\lambda}(t) - [(2\lambda - 2j - 1) + 2(\lambda - j)t] \cdot Q_{j-1}^{\lambda}(t)
+ (2\lambda - 2j + 1)t Q_{j-2}^{\lambda}(t),
\]
where \( j = 2, \ldots, \lambda - \frac{3}{2} \).

**Lemma 5.6.** Suppose \( t \neq 0 \) and consider the integral
\[
I_{k,J+\frac{1}{2}} := \int \frac{R^k}{(t + R^2)^{J+\frac{1}{2}}} \, dR.
\]
If \( k = 2\kappa + 1 \), then
\[
I_{k,J+\frac{1}{2}} = \sum_{\iota=0}^{\kappa} \binom{\kappa}{\iota} \frac{(-1)^{\kappa-\iota+1}}{2(J-\iota) - 1} \cdot \frac{t^{\kappa-\iota}}{(t + R^2)^{J-\iota-\frac{1}{2}}} + C.
\]
\[
\text{Table 2. The Green function } G_L \text{ for different values of } n, L; t = \cos \vartheta
\]

\[
\begin{array}{c|c}
\hline
n = 2, L = 1 & 1 + \frac{2}{3}t + t \cdot \ln \left(\frac{1-t}{2}\right) \\
\hline
n = 2, L = 2 & -\frac{7+30t+41t^2}{20} - \frac{1-3t^2}{2} \cdot \ln \left(\frac{1-t}{2}\right) \\
\hline
n = 2, L = 3 & -\frac{56-123t+210t^2+289t^3}{84} - \frac{t(3-5t^2)}{2} \cdot \ln \left(\frac{1-t}{2}\right) \\
\hline
n = 2, L = 4 & \frac{75-660t-1189t^2+1260t^3+1739t^4}{288} + \frac{3-30t^2+35t^4}{8} \cdot \ln \left(\frac{1-t}{2}\right) \\
\hline
n = 3, L = 1 & \frac{(\pi-\vartheta)(1-2t^2)}{2\sqrt{1-t^2}} + \frac{t}{4} \\
\hline
n = 3, L = 2 & \frac{(\pi-\vartheta)(3-4t^2)}{2\sqrt{1-t^2}} - \frac{1-4t^2}{12} \\
\hline
n = 3, L = 3 & \frac{(\pi-\vartheta)(-1+8t^2-8t^4)}{2\sqrt{1-t^2}} - \frac{t(1-2t^2)}{4} \\
\hline
n = 3, L = 4 & \frac{(\pi-\vartheta)(5+20t^2-16t^4)}{2\sqrt{1-t^2}} + \frac{1-12t^2+16t^4}{20} \\
\hline
n = 4, L = 1 & \frac{10+13t-28t^2}{15(1-t)} + t \cdot \ln \left(\frac{1-t}{2}\right) \\
\hline
n = 4, L = 2 & \frac{-41+223t+149t^2-359t^3}{84(1-t)} - \frac{1-5t^2}{2} \cdot \ln \left(\frac{1-t}{2}\right) \\
\hline
n = 4, L = 3 & \frac{-96-213t+90t^2+397t^3-1027t^4}{105(1-t)} - \frac{5t(3-7t^2)}{6} \cdot \ln \left(\frac{1-t}{2}\right) \\
\hline
n = 4, L = 4 & \frac{577-5549t-6406t^2+24886t^3+8069t^4-21929t^5}{1056(1-t)} + \frac{5(1-14t^2+21t^4)}{8} \cdot \ln \left(\frac{1-t}{2}\right) \\
\hline
n = 5, L = 1 & \frac{(\pi-\vartheta)(-12t^2+8t^4)}{8(1-t^2)^{3/2}} + \frac{t(13-16t^2)}{24(1-t^2)} \\
\hline
n = 5, L = 2 & \frac{(\pi-\vartheta)(15-40t^2+24t^4)}{8(1-t^2)^{3/2}} - \frac{3-23t^2+22t^4}{16(1-t^2)} \\
\hline
n = 5, L = 3 & \frac{(\pi-\vartheta)(-5-60t^2-120t^4+64t^6)}{8(1-t^2)^{3/2}} - \frac{t(-37+144t^2+112t^4)}{40(1-t^2)} \\
\hline
n = 5, L = 4 & \frac{(\pi-\vartheta)(-35+210t^2-336t^4+160t^6)}{8(1-t^2)^{3/2}} + \frac{9-159t^2+161t^4-272t^6}{48(1-t^2)} \\
\hline
n = 6, L = 1 & \frac{56+64t^2-359t^2+232t^3}{105(1-t)} + t \cdot \ln \left(\frac{1-t}{2}\right) \\
\hline
n = 6, L = 2 & \frac{-103+692t+46t^2-1844t^3+1237t^4}{180(1-t)^2} - 2(1 - 7t^2) \cdot \ln \left(\frac{1-t}{2}\right) \\
\hline
n = 6, L = 3 & \frac{-704+1519t+11288t^2-33432t^3-18464t^4+12697t^5}{660(1-t)^3} - \frac{7t(1-3t^2)}{2} \cdot \ln \left(\frac{1-t}{2}\right) \\
\hline
n = 6, L = 4 & \frac{5477-58742t-30293t^2+384684t^3-166405t^4-45045t^5+315317t^6}{6420(1-t)^4} + \frac{7(1-18t^2+33t^4)}{8} \cdot \ln \left(\frac{1-t}{2}\right) \\
\hline
n = 7, L = 1 & \frac{(\pi-\vartheta)(-5-30t^2+40t^4-16t^6)}{16(1-t^2)^{3/2}} + \frac{t(35-84t^2+46t^4)}{48(1-t^2)^3} \\
\hline
n = 7, L = 2 & \frac{(\pi-\vartheta)(-35+140t^2+68t^4-64t^6)}{16(1-t^2)^{5/2}} - \frac{62-695t^2+1304t^4-650t^6}{240(1-t^2)^2} \\
\hline
n = 7, L = 3 & \frac{(\pi-\vartheta)(-35+660t^2-1680t^4+1792t^6-640t^8)}{48(1-t^2)^{5/2}} - \frac{t(255-1462t^2+2240t^4-1024t^6)}{144(1-t^2)^2} \\
\hline
n = 7, L = 4 & \frac{(\pi-\vartheta)(-105+840t^2-2016t^4+1920t^6-640t^8)}{16(1-t^2)^{5/2}} + \frac{122-2832t^2+10960t^4-14160t^6+5888t^8}{336(1-t^2)^3} \\
\hline
n = 8, L = 1 & \frac{144+131t-1518t^2+2013t^3-777t^4}{315(1-t)^3} + t \cdot \ln \left(\frac{1-t}{2}\right) \\
\hline
n = 8, L = 2 & \frac{-2927+23367t-14646t^2-75134t^3+114273t^4-45021t^5}{4620(1-t)^3} - \frac{1-9t^2}{2} \cdot \ln \left(\frac{1-t}{2}\right) \\
\hline
n = 8, L = 3 & \frac{-6656-13551+155859t^2-155859t^3-250170t^4+45045t^5-180181t^6}{5460(1-t)^3} - \frac{3(3-11t^2)^2}{8} \cdot \ln \left(\frac{1-t}{2}\right) \\
\hline
n = 8, L = 4 & \frac{4173-49699t+5793t^2+402945t^3-485545t^4-379929t^5+843387t^6-341189t^7}{3360(1-t)^3} + \frac{3(3-66t^2-143t^4)}{8} \cdot \ln \left(\frac{1-t}{2}\right) \\
\hline
\end{array}
\]
If $k = 2\kappa$ and $\kappa < J$, then

$$I_{k,J+\frac{1}{2}} = t^{\kappa-J} \sum_{\iota=0}^{J-\kappa-1} \binom{J-\kappa-1}{\iota} (-1)^{J-\kappa-\iota-1} \left( \frac{R^2}{t + R^2} \right)^{J-\iota-\frac{3}{2}} + C.$$ 

If $k = 2\kappa$ and $\kappa \geq J$, then

$$I_{k,J+\frac{1}{2}} = \frac{R}{(t + R^2)^{J+\frac{1}{2}}} \cdot P_{k,J+\frac{1}{2}}(t, R^2) + \alpha t^{\kappa-J} \ln \left( R + \sqrt{t + R^2} \right) + C,$$

where

$$\alpha^{\kappa,J+\frac{1}{2}} = \frac{(-1)^{\kappa-J}(2\kappa - 1)!!}{2^{\kappa-J} \cdot (\kappa - J)! \cdot (2J - 1)!!}.$$ 

$P_{k,J+\frac{1}{2}}$ is a homogeneous polynomial of degree $\kappa - 1$ given by

$$P_{k,J+\frac{1}{2}}(t, R^2) = \sum_{\iota=0}^{\kappa-1} a^{\kappa,J+\frac{1}{2}}_{\iota} t^{\kappa-\iota-1} R^{2\iota},$$

with coefficients obtained recurrently

$$a^{\kappa,J+\frac{1}{2}}_{0} = -a^{\kappa,J+\frac{1}{2}},$$

$$a^{\kappa,J+\frac{1}{2}}_{1} = \frac{(3J - 2) a^{\kappa,J+\frac{1}{2}}_{0}}{3},$$

$$a^{\kappa,J+\frac{1}{2}}_{\iota} = \frac{2(J-\iota) a^{\kappa,J+\frac{1}{2}}_{\iota-1} - a^{\kappa,J+\frac{1}{2}}_{\iota}(J)}{2t + 1}, \quad \iota = 2, 3, \ldots, \kappa - 1,$$

and $P_{0,J+\frac{1}{2}} \equiv 0$ (with the conventions $(-1)!! = 1$ and $\binom{J}{\iota} = 0$ for $\iota > J$).

**Lemma 5.7.** The integral

$$\mathcal{I}_{L,J+\frac{1}{2}}(t, R) := \int \frac{R^L}{(1 - 2tR + R^2)^{J+\frac{1}{2}}} dR,$$

$L, J \in \mathbb{N}_0$, is equal to

$$\mathcal{I}_{L,J+\frac{1}{2}} = \frac{A_{L,J+\frac{1}{2}}(t, R)}{(1 - 2tR + R^2)^{J+\frac{1}{2}} \cdot (1 - t^2)^J} + B_{L,J+\frac{1}{2}}(t) \cdot \ln \left( R - t + \sqrt{1 - 2tR + R^2} \right) + C,$$
where the polynomials $A$ and $B$ are given by

\[
A_{L,J^{\frac{1}{2}}}(t, R) = \sum_{\kappa=0}^{J-1} \sum_{\ell=0}^{J-\kappa-1} \alpha_{\kappa,\ell} \frac{L}{2\kappa} t^{L-2\kappa} \left( t + R^2 \right)^{\ell} t^{R} R^{2(J-\frac{1}{2})} (t + R^2)^{\ell} \\
+ \sum_{\kappa=J}^{J} \sum_{\ell=0}^{\kappa-1} \beta_{\kappa,\ell} \frac{L}{2\kappa} t^{J+\kappa-\ell-1} R^{2\ell+1} + \sum_{\kappa=0}^{J} \sum_{\ell=0}^{\kappa} \gamma_{\kappa,\ell} \frac{L}{2\kappa} t^{L-2\kappa-1} t^{J+\kappa-\ell} (t + R^2)^{\ell}
\]

\[
B_{L,J^{\frac{1}{2}}}(t) = \sum_{\kappa=J}^{J} \mu_{\kappa} \frac{L}{2\kappa} t^{L-2\kappa} t^{\kappa-J},
\]

with $R = R - t$, $t = 1 - t^2$, and

\[
\alpha_{\kappa,\ell} \frac{L}{2\kappa} = \binom{L}{2\kappa} \left( J - \kappa - 1 \right) \left( t \right) \left( -1 \right)^{J-\kappa-\ell-1} \frac{2(J-\ell-1)}{2(J-\ell-1)},
\]

\[
\beta_{\kappa,\ell} \frac{L}{2\kappa} = \binom{L}{2\kappa} \left( \kappa, J^{\frac{1}{2}} \right),
\]

\[
\gamma_{\kappa,\ell} \frac{L}{2\kappa} = \binom{L}{2\kappa} \left( \kappa, J^{\frac{1}{2}} \right) \left( t \right) \left( -1 \right)^{J-\kappa-1} \frac{2(J-\ell-1)}{2(J-\ell-1)}.
\]

**Proof.** Follows from

\[
\mathcal{I}_{L,J^{\frac{1}{2}}} = \int \frac{(R - t)(R + t) \ln \left( R - t + R^2 + R^2 + R^2 \right) (t + R^2)^{\ell} dR}{} = \sum_{k=0}^{L} \binom{L}{2\kappa} t^{L-k} I_{k,J^{\frac{1}{2}}}
\]

\[
+ \min \left\{ \left[ \frac{L}{2\kappa} \right]^{-1} \right\} \sum_{\kappa=J}^{J} \binom{L}{2\kappa} t^{L-2\kappa} I_{2\kappa,J^{\frac{1}{2}}} + \sum_{\kappa=J}^{J} \binom{L}{2\kappa} t^{L-2\kappa} I_{2\kappa,J^{\frac{1}{2}}}
\]

\[
+ \sum_{\kappa=J}^{J} \binom{L}{2\kappa} t^{L-2\kappa-1} I_{2\kappa+1,J^{\frac{1}{2}}}.
\]

**Lemma 5.8.** Let $k$ be a positive integer. The integral

\[
\int R^k \ln \left( 1 - tR + \sqrt{1 - 2tR + R^2} \right) dR
\]

equals

\[
p_k(t, R) \cdot \sqrt{1 - 2tR + R^2} + q_k(t) \cdot \ln \left( R - t + \sqrt{1 - 2tR + R^2} \right) \\
+ \frac{R^{k+1}}{k + 1} \left[ \ln \left( 1 - tR + \sqrt{1 - 2tR + R^2} \right) - \frac{1}{k + 1} \right] + C,
\]
where

\[ p_k(t, R) = \sum_{j=0}^{k-1} R^j \cdot \pi_j^k(t) \]

with \( \pi_j^k \) defined recursively by

\[ \pi_{k-1}^k(t) = \frac{1}{k(k+1)}, \]

\[ \pi_{k-2}^k(t) = \frac{2k-1}{k-2} t \pi_{k-1}^k(t), \]

\[ \pi_j^k(t) = \frac{1}{j+1} \left[ (2j+3) t \pi_{j+1}^k(t) - (j+2) \pi_{j+2}^k(t) \right], \quad j = k-3, k-4, \ldots, 1, \]

\[ \pi_0^k(t) = 3t \pi_1^k(t) - 2\pi_2^k(t), \]

and

\[ q_k(t) = t \pi_0^k(t) - \pi_1^k(t). \]

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