GLOBAL EXISTENCE AND LONG TIME BEHAVIOR OF THE
ELLIPSOIDAL-STATISTICAL-FOKKER-PLANCK MODEL FOR
DIATOMIC GASES

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Abstract. We are concerned with the global existence and long time behavior of the solutions to the ES-FP model for diatomic gases proposed in [22]. The global existence of the solutions for this model near the global Maxwellian is established by nonlinear energy method based on the macro-micro decomposition. An algebraic convergence rate in time of the solutions to the equilibrium state is obtained by constructing the compensating function. Since the density distribution function $F(t, x, v, I)$ also contains energy variable $I$, we derive more general Poincaré inequality including variables $v, I$ on $\mathbb{R}^3 \times \mathbb{R}^+$ to establish the coercivity estimate of the linearized operator.

1. Introduction. Recently, the Ellipsoidal-Statistical-Fokker-Planck (ES-FP) model of the Boltzmann equation for polyatomic gases is proposed in [22] to obtain the correct Prandtl number in the Compressible Navier-Stokes asymptotics for polyatomic gases, which can be thought of as an extension of the previous model in [21] from monoatomic gases to polyatomic gases. The ES-FP model takes the form

$$
\frac{\partial_t F + v \cdot \nabla_x F}{\tau} = C(F),
$$

where $F(t, x, v, I) \geq 0$ is the number density function of the particles at time $t > 0$ with the position $x \in \mathbb{R}^3$, the velocity $v \in \mathbb{R}^3$ and a non-translational internal energy parameter $I \in \mathbb{R}^+$. $C(F)$ is the collision operator defined by

$$
C(F) = \frac{1}{\tau} \left( \nabla_v \cdot ((v - U) F + \Pi \nabla_v F) + \partial_I (\delta FI + \frac{\delta^2}{2} RT_{rel} I^2 - \frac{2}{3} \partial_I F) \right).
$$

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where the constant \( \tau > 0 \) is the relaxation time, \( R > 0 \) is the gas constant, and \( \delta > 0 \) is linked to the number of degrees of freedom for the polyatomic gases (\( \delta = 2 \) for diatomic gases). The bulk velocity \( U(t,x) \) is defined by

\[
U(t,x) = \frac{1}{\rho(t,x)} \int_{\mathbb{R} \times \mathbb{R}^+} v F(t,x,v,I) dv dI,
\]

with the macroscopic density defined by \( \rho(t,x) = \int_{\mathbb{R} \times \mathbb{R}^+} F(t,x,v,I) dv dI \). The tensor \( \Pi = \Pi(t,x) \) is the combination among the stress tensor \( \Theta \), the translational temperature tensor \( RT_{tr} I \) and the temperature tensor \( RT I \):

\[
\Pi = (1 - \theta) (\nu \Theta + (1 - \nu) RT_{tr} I) + \theta RT I.
\]

The relaxation temperature \( T_{rel} \) is introduced by the non-translational internal energy temperature \( T_{int} \) and the temperature \( T \) as follows

\[
T_{rel} = (1 - \theta) T_{int} + \theta T,
\]

where the coefficients \( \nu, 0 < \theta < 1 \) are some free parameters, \( I \) is identity matrix, and the macroscopic quantities \( \Theta = \Theta(t,x) \), \( T_{tr} = T_{tr}(t,x) \), \( T_{rel} = T_{rel}(t,x) \) and \( T = T(t,x) \) are defined by

\[
\Theta(t,x) = \frac{1}{\rho(t,x)} \int_{\mathbb{R} \times \mathbb{R}^+} (v - U) \otimes (v - U) F(t,x,v,I) dv dI,
\]

\[
\frac{3}{2} RT_{tr}(t,x) = \frac{1}{\rho(t,x)} \int_{\mathbb{R} \times \mathbb{R}^+} \frac{1}{2} |v - U|^2 F(t,x,v,I) dv dI,
\]

\[
\frac{\delta}{2} RT_{int}(t,x) = \frac{1}{\rho(t,x)} \int_{\mathbb{R} \times \mathbb{R}^+} I \frac{1}{2} F(t,x,v,I) dv dI,
\]

\[
\frac{3 + \delta}{2} RT(t,x) = \frac{1}{\rho(t,x)} \int_{\mathbb{R} \times \mathbb{R}^+} \left( \frac{1}{2} |v - U|^2 + I \frac{1}{2} \right) F(t,x,v,I) dv dI.
\]

Notice that it satisfies from (4) that

\[
T = \frac{3}{3 + \delta} T_{tr} + \frac{\delta}{3 + \delta} T_{int}.
\]

It is proved in [22] that for \( \nu \in (-\frac{1}{2}, 1) \), \( \Pi \) is symmetric positive definite and the collision operator \( C(F) \) satisfies the following conservation laws of the mass, momentum, and energy:

\[
\int_{\mathbb{R} \times \mathbb{R}^+} C(F) dv dI = 0, \quad \int_{\mathbb{R} \times \mathbb{R}^+} v_i C(F) dv dI = 0, \quad i = 1, 2, 3,
\]

\[
\int_{\mathbb{R} \times \mathbb{R}^+} \left( \frac{1}{2} |v|^2 + I \frac{1}{2} \right) C(F) dv dI = 0,
\]

and the dissipation of the entropy

\[
\int_{\mathbb{R} \times \mathbb{R}^+} C(F) \log f dv dI \leq 0.
\]

The inequality above also leads to the equilibrium property

\[
C(F) = 0 \iff F = M_p(v,I),
\]

where the local Maxwellian \( M_p(v,I) \) is defined by

\[
M_p(v,I) := M_p [\rho,v,U,T](v,I) = \frac{\rho \Lambda}{(2\pi)^\frac{3+\delta}{2} (RT)^\frac{3+\delta}{2}} e^{-\frac{|v-U|^2}{2RT} - \frac{I}{RT}},
\]
and \( \Lambda_\delta = \left( \int_0^{\infty} e^{-1/2dI} dI \right)^{-1} \). In the sequel, for simplicity we assume that both \( \tau \) and \( R \) equal to 1.

In this paper, we consider the Cauchy problem for Eq. (1) with initial data

\[
F(0, x, v, I) = F_0(x, v, I),
\]

and the purpose is to study the global existence and long time behavior of the solutions to this Cauchy problem for \( \delta = 2 \) (corresponding to the diatomic gases case), when the initial data (6) is a small perturbation around the global Maxwellian

\[
\mu(v, I) := M_p[v, I] = \frac{\Lambda_2}{(2\pi)^{3/2}} e^{-\|v\|^2/2}.
\]

Before stating the main results, we reformulate the Cauchy problem (1) and (6) around the equilibrium state as follows. Let

\[
\begin{align*}
F(t, x, v, I) &= \mu + \sqrt{\mu}f(t, x, v, I) \\
F_0(x, v, I) &= \mu + \sqrt{\mu}f_0(x, v, I).
\end{align*}
\]

By (8), the Cauchy problem (1) and (6) is changed into

\[
\partial_t f + v \cdot \nabla_x f = L_{\nu, \theta} f + N(f),
\]

\[
f(0, x, v, I) = f_0(x, v, I) = \frac{1}{\sqrt{\mu}}(F_0(x, v, I) - \mu),
\]

where \( L_{\nu, \theta} \) denotes the linearized collision operator

\[
L_{\nu, \theta} f = \frac{1}{\sqrt{\mu}} \left( \nabla_v \cdot \left( \mu \nabla_v \left( \frac{f}{\sqrt{\mu}} \right) - b\mu + \left( a - \frac{(1-\theta)(1-\nu)}{3} \right) r - \frac{2}{5}\theta d \right) v \mu \\
- (1-\nu)\nu \mu \right) + \partial_I \left( 2I \mu \partial_I \left( \frac{f}{\sqrt{\mu}} \right) + 2(a - \frac{\theta}{5} r - (1 - \frac{3}{5}\theta)d) I \mu \right),
\]

and \( N(f) \) denotes the nonlinear part

\[
N(f) = \frac{1}{\sqrt{\mu}} \{ \nabla_v \cdot (\Gamma_1(f)) + \partial_I (\Gamma_2(f)) \}
\]

\[
= \frac{1}{\sqrt{\mu}} \left\{ b \frac{2d}{5(1+a)} (av\mu + \nabla_v(\sqrt{\mu}f)) \right\} + \frac{\theta}{5(1+a)} \left\{ r + b \otimes b \left( \frac{1}{1+a} \left( av\mu + \nabla_v(\sqrt{\mu}f) \right) \right) \}
\]

\[
+ (1-\theta) \left\{ \nu \left( \frac{1}{1+a} \left( av\mu + \nabla_v(\sqrt{\mu}f) \right) \right) - b \otimes b \left( \frac{1}{1+a} \left( av\mu + \nabla_v(\sqrt{\mu}f) \right) \right) \}
\]

\[
+ (1-\nu) \left\{ \frac{r}{3(1+a)} \left( av\mu + \nabla_v(\sqrt{\mu}f) \right) - \frac{\|b\|^2}{3(1+a)^2} \left( av\mu + \nabla_v(\sqrt{\mu}f) \right) \right\} \right\}.
\]
\[
+ \partial_t \left\{ \frac{a^2}{1 + a} \left( -2I\mu + 2I\partial_t(\sqrt{\mu} f) \right) - 2aI\partial_t(\sqrt{\mu} f) \right\} \\
+ \theta \left\{ -\frac{1}{5} \frac{|b|^2}{(1 + a)^2} \left( -2I\mu + 2I\partial_t(\sqrt{\mu} f) \right) \right\} \\
+ (1 - \theta) \left\{ -\frac{ad}{1 + a} \left( -2I\mu + 2I\partial_t(\sqrt{\mu} f) \right) + 2dI\partial_t(\sqrt{\mu} f) \right\} \right\}
\]

with
\[
a = \int_{\mathbb{R}^3 \times \mathbb{R}^+} \sqrt{\mu} f(v, I) dv dI, \quad b = (b_i)_{1 \leq i \leq 3}, \quad b_i = \int_{\mathbb{R}^3 \times \mathbb{R}^+} v_i \sqrt{\mu} f(v, I) dv dI,
\]
\[
r = \int_{\mathbb{R}^3 \times \mathbb{R}^+} |v|^2 \sqrt{\mu} f(v, I) dv dI,
\]
\[
l = (l_{ij})_{1 \leq i, j \leq 3}, \quad l_{ij} = \int_{\mathbb{R}^3 \times \mathbb{R}^+} v_i v_j \sqrt{\mu} f(v, I) dv dI, \quad 1 \leq i \leq j \leq 3,
\]
\[
d = \int_{\mathbb{R}^3 \times \mathbb{R}^+} I \sqrt{\mu} f(v, I) dv dI.
\]

By straightforward calculations, we have
\[
\frac{1}{\sqrt{\mu}} \nabla_v \cdot \left( \mu \nabla_v \left( \frac{f}{\sqrt{\mu}} \right) \right) = \Delta_v f + \frac{3}{2} f - \frac{1}{4} |v|^2 f,
\]
\[
\frac{1}{\sqrt{\mu}} \partial_t \left( 2I\mu \partial_t(\frac{f}{\sqrt{\mu}}) \right) = 2\partial_t(\sqrt{\mu} f) + f - \frac{1}{2} f.
\]

The two equalities above will be used in the estimates of Lemma 2.1 and Proposition 3.

We now state the main results as follows.

**Theorem 1.1.** Let \(-\frac{1}{2} < \nu < 1\), \(0.5 < \theta < 1\), and \(f_0 \in L^2_{v,I}(H^3_{v})\). There exists a positive constant \(\delta_1\) small enough such that if \(\|f_0\|^2_{L^2_{v,I}(H^3_{v})} \leq \delta_1\), then the Cauchy problem (9)-(10) admits a unique global solution \(f = f(t, x, v, I) \in C([0, \infty); L^2_{v,I}(H^3_{v}))\) with \(F(t, x, v, I) = \mu + \sqrt{\mu} f(t, x, v, I) \geq 0\) satisfying
\[
\sup_{t \geq 0} \|f(t)\|_{L^2_{v,I}(H^3_{v})} \leq C \|f_0\|_{L^2_{v,I}(H^3_{v})}
\]
with \(C > 0\) a generic constant.

Moreover, there exists a positive constant \(\delta_2\) small enough, such that if \(\|f_0\|^2_{L^2_{v,I}(H^3_{v})} \leq \delta_2\), it holds that for all \(t > 0\),
\[
\|f(t)\|_{L^2_{v,I}(H^3_{v})} \leq C \delta_2 (1 + t)^{-\frac{1}{2}}
\]
with \(C > 0\) a generic constant.

**Remark 1.** Theorem 1.1 implies the global existence, uniqueness and an algebraic convergence rate in time of the solution to the Cauchy problem (1) and (6).

**Remark 2.** Physically, for diatomic molecular gases, two important parameters Prandtl number \(Pr \simeq 0.7\) and bulk viscosity is in 0.6-0.8. Correspondingly, the value of \(\theta\) should be around 0.5. The condition of Theorem 1.1 conforms this requirement.
Remark 3. In the periodic box case [5, 36], the Poincaré inequality was used to obtain the estimate of $Pf$ in terms of its derivative, then the convergence rate is exponential. In the current whole space case, only the derivative of $Pf$ can be directly estimated from the macroscopic equations (41)-(46), and the Poincaré inequality does not hold to estimate $Pf$ itself, thus the time-decay rate is algebraic here.

Remark 4. The Prandtl number $Pr$ satisfies $\frac{6}{5} < Pr < 3$ for $\nu$ and $\theta$ in Theorem 1.1, where $Pr$ is defined by (53) in [22]. Actually, $Pr$ can be chosen arbitrarily smaller than $\frac{6}{5}$ for any $\nu \in (-\infty, -\frac{1}{2}], \theta \in (0, 1)$. By Proposition 2.4 in [22], we can take $\nu \in (-\frac{RT_{tr} + \frac{2}{3}RT}{\lambda_{\max} - RT_{tr}}, 1)$ for the positive definiteness of $\Pi$, where $T_{tr}$ and $T$ are defined by (4), and $\lambda_{\max} > 0$ is the maximum eigenvalue of the tensor $\Theta$ which is defined by (4). By definition $\lambda_{\max} > RT_{tr}$, and both $RT_{tr}$ and $\lambda_{\max}$ get close to $RT$ when $f$ is close to the local Maxwellian $M_p(v, I)$, then the value of $-\frac{RT_{tr} + \frac{2}{3}RT}{\lambda_{\max} - RT_{tr}}$ can be as small as possible. Consequently, this shows that $\nu$ can take any value between $-\infty$ and $\frac{6}{5}$ when $f$ is close to the local Maxwellian equilibrium. In particular, for any $\nu \in (-\infty, -\frac{1}{2}], \theta \in (0, 1)$, the coercivity estimate of the linearized operator $L_{\nu, \theta}$ (22) is still satisfied. We can prove the supremum norm of $f$ is small by the energy estimates method when the initial data is a small perturbation of the Maxwellian equilibrium [12, 13, 14], we omit the details here.

The proof of Theorem 1.1 consists of two parts. On the one part, we construct the global solution motivated by the $L^2$ energy method based on the macro-micro decomposition developed by Guo in the analysis of the well-posedness of the Boltzmann equation [12, 13, 14]. A crucial step in this process is to derive an energy estimate so that the a-priori estimate can be closed. On the other part, to establish the time decay rate of the global solutions, we employ the method of compensating function together with the energy estimates introduced by Kawashima in [17]. This method is first used for the Boltzmann equation [10, 17], and then is extended to other kinetic equations [9, 29, 30, 31, 32, 33].

However, different from the usual situation, because the introduction of internal energy parameter $I$ (cf.[22]) and more complicated structure which the equation have, we face some new difficulties and deal with them as follows. Firstly, the structure of the linearized collision operator $L_{\nu, \theta}$ given by (11) is more complicated than the linearized Fokker-Planck operator. The coercivity estimates of the linearized Fokker-Planck operator cannot be used here, so we have to re-establish the coercivity estimate of $L_{\nu, \theta}$ which is crucial to the theory of well-posedness. To achieve this purpose, the most important step is to derive more general Poincaré inequality (25) including variables $v, I$ on $\mathbb{R}^3 \times \mathbb{R}^+$. For the diatomic case ($\delta = 2$), inspired by [19, 28], using the method of Sturm-Liouville theory, and choosing Laguerre function as an orthonormal basis, we develop the inequality (25) which we need. However, so far as I know, for more general case ($\delta \geq 3$), the approach above establishing the inequality (25) will no longer work. How to obtain the coercivity estimates of the linearized operator for all cases of polyatomic gases and build the existence of solutions, will be our further work to study. Secondly, because of appearance of $I$, we get fourteen moments rather than thirteen moments as usual, so we build new macro-micro system (cf. [12]) to finish energy estimates, and reconstruct the corresponding compensating function to obtain the time convergence rate of the solutions to the equilibrium. Thirdly, we should mention here that the energy estimates obtained for the global existence of the solutions are not sufficient enough for
the time decay rate because of the additional difficulties caused by the second order derivative terms with respect to variable $v, I$ in $N(f)$. We need to establish some new estimates containing higher order derivatives with respect to variable $v, I$ and $x$, which is the main reason why we impose more spatial regularity assumptions on the initial data in Theorem 1.1.

At the end of the introduction, we also review some works related to the Ellipsoidal Statistical model. In [2, 6, 7, 15], the BGK model was extended to the ES-BGK model which was proved to retain the elementary properties of the Boltzmann operator (collisional invariants, Maxwellian equilibrium) and H-theorem. In the past ten years, some works about ES-BGK were studied. Yun et al. proved the existence of global in time of the smooth solutions or the weak solutions to the Cauchy problem and the stationary weak solutions to the boundary value problem of the ES-BGK model for monoatomic gases [3, 24, 34, 35], and the global existence of the classical solutions or mild solutions to the Cauchy problem of the ES-BGK model for polyatomic gases [25, 36]. For the ES-FP model, in a previous paper [26], we have established the global existence of the unique solution of this model for monoatomic gases, and obtained its algebraic time convergence rate to the equilibrium state.

The rest of the paper is organized as follows. The uniform a-priori estimates are established in Section 2. The global existence of the unique solution is constructed in Section 3. The time decay estimate is obtained in Section 4.

**Notation.** Throughout this paper, the constant $C > 0$ denotes a generic constant. We introduce $P$ as the macroscopic projection operator, defined by

$$Pf = a\sqrt{\mu} + b \cdot v\sqrt{\mu} + c \frac{(|v|^2 - 3 + 2I - 2)}{\sqrt{10}} \sqrt{\mu}$$

with

$$a = \int_{\mathbb{R}^3 \times \mathbb{R}^+} \sqrt{\mu} f(v, I) dv dI, \quad b_i = \int_{\mathbb{R}^3 \times \mathbb{R}^+} v_i \sqrt{\mu} f(v, I) dv dI, \quad i = 1, 2, 3,$$

$$c = \int_{\mathbb{R}^3 \times \mathbb{R}^+} \frac{(|v|^2 - 3 + 2I - 2)}{\sqrt{10}} \sqrt{\mu} f(v, I) dv dI.$$  

(17)

Then for any fixed point $(t, x)$, any function $f(t, x, v, I) \in L^2_{x,v,I}(H^3_x)$ can be uniquely decomposed into

$$f = Pf + (I - P)f,$$

(18)

where $Pf$ and $(I - P)f$ are the macroscopic and microscopic components of the function $f$, respectively. Corresponding to the linearized operator $L_{\nu,\theta}$, the dissipation rate is given by the following norm:

$$\|f\|_{L^2_x} = \left( \int_{\mathbb{R}^3 \times \mathbb{R}^+} |\nabla_x f(v, I)|^2 + |I^\frac{3}{2} \partial_I f|^2 + (1 + |v|^2 + I)|f(v, I)|^2 dv dI \right)^{\frac{1}{2}}.$$  

(19)

The inner product $\langle \cdot, \cdot \rangle$ on $L^2(\mathbb{R}^3 \times \mathbb{R}^+)$ is defined by

$$\langle f, g \rangle = \int_{\mathbb{R}^3 \times \mathbb{R}^+} f(v, I)g(v, I) dv dI,$$

and the space $L^2_{x,v,I} = L^2_{v,I}(L^2_x)$. Notice that, by the definitions of (11), (13) and straightforward calculations, we can obtain that the linearized operator $L_{\nu,\theta}$ is self-adjoint: $\langle L_{\nu,\theta} f, g \rangle = \langle L_{\nu,\theta} g, f \rangle$, and $L_{\nu,\theta}(Pf) = 0$. 
The Fourier transform \( \hat{f} = \mathcal{F}f \) of an integrable function \( f \) is defined by
\[
\hat{f} = \mathcal{F}f = \int_{\mathbb{R}^n} e^{-ix \cdot \xi} f(x) dx.
\]
For a multiple index \( \alpha = (\alpha_1, \ldots, \alpha_n) \), we denote \( \partial^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n} \). The length of \( \alpha \) is \( |\alpha| = \alpha_1 + \cdots + \alpha_n \).

2. The uniform a-priori estimates. Our goal of this section is to establish the uniform a-priori estimates for the smooth solution \( f = f(t, x, v, I) \) to the Cauchy problem (9)-(10) in \((0, T) \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^+ \), under the uniform a-priori assumption
\[
\sup_{0 \leq t \leq T} \|f(t)\|_{L^2_{\nu, I}(H^2_x)}^2 \leq N_0
\]
for any given constant \( T > 0 \) and the constant \( N_0 > 0 \) small enough.

In this section, the main result is as follows.

**Proposition 1.** Let \(-\frac{1}{2} < \nu < 1, 0.5 < \theta < 1, \) and \( f(t, x, v, I) \) be a smooth solution to the Cauchy problem (9)-(10) satisfying (20) in \((0, T) \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^+ \). Then it holds that
\[
\sup_{0 \leq t \leq T} \|f(t)\|_{L^2_{\nu, I}(H^2_x)}^2 + \sigma_0 \int_0^T \| (I - P) f(\tau) \|_{L^2_{\nu, I}(H^2_x)}^2 + \| \nabla_x a(\tau) \|_{H^2_x}^2 + \| \nabla_x b(\tau) \|_{H^2_x}^2 + \| \nabla_x c(\tau) \|_{H^2_x}^2 d\tau \leq C \| f_0 \|_{L^2_{\nu, I}(H^2_x)}^2,
\]
where \( a, b \) and \( c \) are defined by (17) and \( C > 0, \sigma_0 > 0 \) are generic positive constants.

The proof of Proposition 1 is mainly based on the properties of the linearized collision operator \( L_{\nu, \theta} \) and the nonlinear energy method. We state the details in the following two subsections.

2.1. Preliminary. In this subsection, we establish the coercivity estimate of the linearized collision operator \( L_{\nu, \theta} \).

**Lemma 2.1.** Let \(-\frac{1}{2} < \nu < 1 \) and \( 0.5 < \theta < 1 \), there exists a positive constant \( \sigma_1 = \sigma_1(\nu, \theta) \), such that for any \( f \in L^2_{\nu, I}(\mathbb{R}^3 \times \mathbb{R}^+) \), the linearized collision operator \( L_{\nu, \theta} \) satisfies:
\[
- \int_{\mathbb{R}^3 \times \mathbb{R}^+} f L_{\nu, \theta} f dv dI \geq \sigma_1 \left( \| \nabla_v (I - P) f \|_{L^2_{\nu, I}}^2 + \| I^{\frac{1}{4}} \partial_t (I - P) f \|_{L^2_{\nu, I}}^2 \right),
\]
where \( P \) is the macroscopic projection operator defined by (16).

**Proof.** By straightforward calculations, we are able to obtain
\[
- \int_{\mathbb{R}^3 \times \mathbb{R}^+} f L_{\nu, \theta} f dv dI,
\]
\[
= \int_{\mathbb{R}^3 \times \mathbb{R}^+} \mu \left| \nabla_v \left( \frac{I - P}{\sqrt{\mu}} \right) f \right|^2 dv dI + 2 \int_{\mathbb{R}^3 \times \mathbb{R}^+} I \mu \left| \partial_t \left( \frac{I - P}{\sqrt{\mu}} \right) f \right|^2 dv dI \]
\[
- 2(1 - \theta) \nu \sum_{1 \leq i < j \leq 3} \left( \int_{\mathbb{R}^3 \times \mathbb{R}^+} \frac{v_i^2 - v_j^2}{\sqrt{6}} \sqrt{\mu} (I - P) f dv dI \right)^2
\]

(23)
By the conclusion of orthogonal projection about \( \{h \} \) and motivated the approach (cf. [4]) which generalized Poincaré inequality from \( \mathbb{R} \) to \( \mathbb{R}^3 \), we are able to derive the following generalized inequality on \( \mathbb{R}^3 \times \mathbb{R}^+ \)

\[
\int_{\mathbb{R}^3} |\nabla h|^2 \mu(v) dv + (\int_{\mathbb{R}^3} h \mu(v) dv)^2 \geq \int_{\mathbb{R}^3} |h|^2 \mu(v) dv, \quad (24)
\]

and the inequality with respect to the measure \( \mu_I(I) \)

\[
2 \int_0^{+\infty} I |\partial_I h|^2 \mu_I(I) dI + (\int_0^{+\infty} h \mu_I(I) dI)^2 \geq \int_0^{+\infty} |h|^2 \mu_I(I) dI, \quad (25)
\]

the proof is given in Appendix) and motivated the approach (cf. [4]) which generalized Poincaré inequality from \( \mathbb{R} \) to \( \mathbb{R}^3 \), we are able to derive the following generalized inequality on \( \mathbb{R}^3 \times \mathbb{R}^+ \)

\[
\int_{\mathbb{R}^3} |\nabla h|^2 \mu(v, I) dv + (\int_{\mathbb{R}^3} h \mu(v, I) dv)^2 \geq \int_{\mathbb{R}^3} |h|^2 \mu(v, I) dv, \quad (26)
\]

for any suitable function \( h(v, I) \) and \( \mu(v, I) = \mu(v) \mu_I(I) \).

Taking \( h(v, I) = \frac{(I-P)f(v, I)}{\sqrt{\mu(v, I)}} \), we get from (26) that

\[
\int_{\mathbb{R}^3} \mu(v, I) |\nabla h|^2 dv + (\int_{\mathbb{R}^3} h \mu(v, I) dv)^2 \geq \int_{\mathbb{R}^3} |h|^2 \mu(v, I) dv, \quad (27)
\]

By the conclusion of orthogonal projection about \( \{v_i, v_j, \sqrt{\mu} \} \) corresponding to Lemma 3.2 of [35], using orthonormal property of \( \{v_i, v_j, \sqrt{\mu}, \sqrt{I-1}, 1 \leq i < j \leq 3 \) and Bessel's inequality, we have

\[
\sum_{1 \leq i < j \leq 3} \left( \int_{\mathbb{R}^3} \frac{v_i^2 - v_j^2}{\sqrt{\mu}} f dv \right)^2 + \left( \int_{\mathbb{R}^3} \frac{|v|^2 - 3}{\sqrt{\mu}} f dv \right)^2 \geq \|f\|^2_{L^2, I}, \quad (28)
\]

By (27) and (28), we have from (24) that

\[
-2(1 - \theta) \nu \int_{\mathbb{R}^3} \left( \sum_{1 \leq i < j \leq 3} \frac{v_i v_j \sqrt{\mu}}{\sqrt{6}} (I - P) f dv + \int_{\mathbb{R}^3} \frac{|v|^2 - 3}{\sqrt{6}} \sqrt{\mu} (I - P) f dv \right)^2 \geq (1 - 2(1 - \theta)) \|f\|^2_{L^2, I}, \quad (29)
\]
We are also able to obtain from (14) and (15) that
\[
- \int_{\mathbb{R}^3 \times \mathbb{R}^+} f L_{\nu, \theta} f dv dI \geq \| \nabla_v (I - P) f \|_{L^2_{\nu, I}}^2 + \frac{1}{4} \| v |(I - P) f \|_{L^2_{\nu, I}}^2 \\
+ 2 \| I \partial_t (I - P) f \|_{L^2_{\nu, I}}^2 + \frac{1}{2} \| I^{\frac{1}{2}} (I - P) f \|_{L^2_{\nu, I}}^2 \\
- \left( \frac{5}{2} + 2 (1 - \theta) \right) \| (I - P) f \|_{L^2_{\nu, I}}^2.
\]
Taking (29) + \varepsilon(30) with \varepsilon > 0 sufficiently small, we conclude that for \(-\frac{1}{2} < \nu < 1, 0.5 < \theta < 1\), there exists a constant \(\sigma_1 = \sigma_1(\nu, \theta) > 0\) such that (22) holds. The proof of the Lemma 2.1 is completed.

\[\square\]

**Corollary 1.** For \(-\frac{1}{2} < \nu < 1\) and \(0.5 < \theta < 1\), the null space of the linearized collision operator \(L_{\nu, \theta}\) is given by
\[
\text{Ker}(L_{\nu, \theta}) = \text{span}\{ \sqrt{\mu}, v_i \sqrt{\mu} \frac{(|v|^2 - 3 + 2I - 2)}{\sqrt{10}} \sqrt{\mu}, i = 1, 2, 3 \}.
\]

It also holds that
\[
\int_{\mathbb{R}^3 \times \mathbb{R}^+} \sqrt{\mu} N(f) dv dI = 0, \int_{\mathbb{R}^3 \times \mathbb{R}^+} v_i \sqrt{\mu} N(f) dv dI = 0, i = 1, 2, 3,
\]
\[
\int_{\mathbb{R}^3 \times \mathbb{R}^+} \frac{(|v|^2 - 3 + 2I - 2)}{\sqrt{10}} \sqrt{\mu} N(f) dv dI = 0.
\]

**2.2. The nonlinear energy estimates.** We cite the lemma 2.1 in [5] which will be used frequently below.

**Lemma 2.2.** ([5]) For any \(f, g \in H^3(\mathbb{R}^3)\) and any multi-index \(\gamma \in \mathbb{N}^3\) verifying \(1 \leq |\gamma| \leq 3\), it holds
\[
\| f \|_{L^\infty(\mathbb{R}^3)} \leq C \| \nabla_x f \|_{L^2(\mathbb{R}^3)} \| \nabla^2_x f \|_{L^2(\mathbb{R}^3)} \| \nabla^3_x f \|_{L^2(\mathbb{R}^3)},
\]
\[
\| f g \|_{H^3(\mathbb{R}^3)} \leq C \| f \|_{H^3(\mathbb{R}^3)} \| \nabla_x g \|_{H^3(\mathbb{R}^3)},
\]
\[
\| \partial^\gamma (f g) \|_{L^2(\mathbb{R}^3)} \leq C \| \nabla_x f \|_{H^2(\mathbb{R}^3)} \| \nabla_x g \|_{H^2(\mathbb{R}^3)},
\]
\[
\| \partial^\gamma (f g h) \|_{L^2(\mathbb{R}^3)} \leq C \| \nabla_x f \|_{H^2(\mathbb{R}^3)} \| \nabla_x g \|_{H^2(\mathbb{R}^3)} \| \nabla_x h \|_{H^2(\mathbb{R}^3)}
\]
with \(C > 0\) a generic constant.

To estimate the terms of \(N(f), L_{\nu, \theta}(f)\) using Lemma 2.2, by straightforward calculations, we can also get
\[
\| \nabla_x \frac{1}{1 + a(t, x)} \|_{H^2_1(\mathbb{R}^3)} \leq C \| \nabla_x a(t, x) \|_{H^2_1(\mathbb{R}^3)} \leq C \| \nabla_x f(t, x, v, I) \|_{L^2_{\nu, I}(H^2_1)}
\]
when \(N_0\) is small enough, where \(a(t, x) = \int_{\mathbb{R}^3 \times \mathbb{R}^+} \sqrt{\mu} f(t, x, v, I) dv dI\) is defined by (13).

Our first result is about the basic energy estimate on \(f\).

**Lemma 2.3.** Let \(-\frac{1}{2} < \nu < 1, 0.5 < \theta < 1, T > 0\), and \(f(t, x, v, I)\) be a smooth solution to the Cauchy problem (9)-(10) satisfying (20) in \((0, T] \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^+\). Then, it holds that
\[
\frac{d}{dt} \| f \|_{L^2_{\nu, I}(H^2_1)}^2 + \sigma_2 \| (I - P) f \|_{L^2_{\nu, I}(H^2_1)}^2 \leq C \sqrt{N_0} (\| \nabla_x a \|_{H^2_1}^2 + \| \nabla_x b \|_{H^2_1}^2 + \| \nabla_x c \|_{H^2_1}^2),
\]
\[
(32)
\]
where \(a, b\) and \(c\) are defined by (17), \(C > 0\) and \(\sigma_2\) are generic constants, and \(N_0\) is small enough.

Proof. The proof is divided into two steps.

Step 1. Estimate on \(\|f(t)\|_{L^2_\varphi(L^2)}^2\).

Multiplying (9) by \(f\) and taking the integrations over \(\mathbb{R}_+^3 \times \mathbb{R}_+^3 \times \mathbb{R}_+^3\), we have

\[
\frac{d}{dt} \int_{\mathbb{R}_+^3 \times \mathbb{R}_+^3 \times \mathbb{R}_+^3} |f(t, x, v, I)|^2 \, dv \, dx - \int_{\mathbb{R}_+^3 \times \mathbb{R}_+^3 \times \mathbb{R}_+^3} f(t, x, v, I) L_{\nu, \theta} f(t, x, v, I) \, dv \, dx = \int_{\mathbb{R}_+^3 \times \mathbb{R}_+^3 \times \mathbb{R}_+^3} N(f) f(t, x, v, I) \, dv \, dx.
\]

(33)

By (22), the second term of (33) is estimated by

\[
- \int_{\mathbb{R}_+^3 \times \mathbb{R}_+^3 \times \mathbb{R}_+^3} f(t, x, v, I) L_{\nu, \theta} f(t, x, v, I) \, dv \, dx \geq C \|f(t)\|_{L^2_\varphi(L^2)}^2.
\]

(34)

By Corollary 1 and Lemma 2.2, and using Hölder’s and Young’s inequalities, one can estimate the last term of (33) as follows

\[
\begin{aligned}
|\int_{\mathbb{R}_+^3 \times \mathbb{R}_+^3 \times \mathbb{R}_+^3} N(f)(t, x, v, I) f(t, x, v, I) \, dv \, dx| & \leq \varepsilon \|f(t)\|_{L^2_\varphi(L^2)}^2 \\
& + C \varepsilon \left\|f(t)\right\|_{L^2_\varphi(L^2)}^2 \|\nabla_x f(t)\|_{L^2_\varphi(L^2)}^2 + \|f(t)\|_{L^2_\varphi(L^2)}^2 \|\nabla_x f(t)\|_{L^2_\varphi(L^2)}^2 + \|f(t)\|_{L^2_\varphi(L^2)}^2 \|\nabla x f(t)\|_{L^2_\varphi(L^2)}^2 + \|f(t)\|_{L^2_\varphi(L^2)}^2 \|\nabla v f(t)\|_{L^2_\varphi(L^2)}^2 + \|f(t)\|_{L^2_\varphi(L^2)}^2 \|\nabla c f(t)\|_{L^2_\varphi(L^2)}^2
\end{aligned}
\]

(35)

where the constant \(\varepsilon > 0\) is to be chosen small enough.

Substituting (34)-(35) into (33), we can obtain

\[
\frac{d}{dt} \|f\|_{L^2_\varphi(L^2)}^2 + \sigma_2 \|(I - P) f\|_{L^2_\varphi(L^2)}^2 \\
\leq CN_0 \|\nabla_x f\|_{L^2_\varphi(L^2)}^2 + \|(I - P) f\|_{L^2_\varphi(L^2)}^2 + \|\nabla x a\|_{H^2_\varphi}^2 + \|\nabla b\|_{H^2_\varphi}^2 + \|\nabla c\|_{H^2_\varphi}^2
\]

(36)

with \(C > 0\) a generic constant, where the inequality holds due to the fact that \(N_0\) is small enough.

Step 2. Estimate on \(\sum_{1 \leq |\alpha| \leq 3} \|\partial^\alpha f(t)\|_{L^2_\varphi(L^2)}^2\).

Taking the spatial derivatives \(\partial^\alpha\) to (9) with \(1 \leq |\alpha| \leq 3\), multiplying by \(\partial^\alpha f\) and taking the integrations over \(\mathbb{R}_+^3 \times \mathbb{R}_+^3 \times \mathbb{R}_+^3\), we get

\[
\begin{aligned}
\frac{d}{dt} \int_{\mathbb{R}_+^3 \times \mathbb{R}_+^3 \times \mathbb{R}_+^3} |\partial^\alpha f(t, x, v, I)|^2 \, dv \, dx \\
- \int_{\mathbb{R}_+^3 \times \mathbb{R}_+^3 \times \mathbb{R}_+^3} \partial^\alpha f(t, x, v, I) L_{\nu, \theta} \partial^\alpha f(t, x, v, I) \, dv \, dx \\
= \int_{\mathbb{R}_+^3 \times \mathbb{R}_+^3 \times \mathbb{R}_+^3} \partial^\alpha N(f)(t, x, v, I) \partial^\alpha f(t, x, v, I) \, dv \, dx.
\end{aligned}
\]

(37)

By (22), the second term of (37) is estimated by

\[
- \int_{\mathbb{R}_+^3 \times \mathbb{R}_+^3 \times \mathbb{R}_+^3} \partial^\alpha f(t, x, v, I) L_{\nu, \theta} \partial^\alpha f(t, x, v, I) \, dv \, dx \geq C \|\partial^\alpha (I - P) f\|_{L^2_\varphi(L^2)}^2,
\]

(38)
By Corollary 1 and Lemma 2.2, and using Hölder’s and Young’s inequalities, one can estimate the last term of (37) as follows

\[
\left| \int_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^+} \partial^\alpha N(f)(t, x, v, I) \partial^\alpha f(t, x, v, I) \mathrm{d}v \mathrm{d}x \right| \leq \varepsilon \| \partial^\alpha (I - P) f(t) \|_{L^2_v(L^2_x)}
\]

\[+ C \left( \| \nabla_x f(t) \|_{L^2_v(L^2_x)}^2 + \| \nabla_x f(t) \|_{L^2_v(L^2_x)}^4 \right) \| \nabla_x f(t) \|_{L^2_v(L^2_x)}^2 \]

\[+ \left( \| \nabla_x f(t) \|_{L^2_v(L^2_x)}^2 + \| \nabla_x f(t) \|_{L^2_v(L^2_x)}^4 \right) \| \nabla_x f(t) \|_{L^2_v(L^2_x)}^2 \]

\[\| \nabla x (I - P) f(t) \|_{L^2_v(L^2_x)}^2 \right),
\]

(39)

where the constant \( \varepsilon > 0 \) is to be chosen small enough.

Substituting (38)-(39) into (37) and taking the summation over \( 1 \leq |\alpha| \leq 3 \), we can obtain the estimate

\[
\frac{d}{dt} \sum_{1 \leq |\alpha| \leq 3} \| \partial^\alpha f \|_{L^2_v(L^2_x)}^2 + \sigma_2 \sum_{1 \leq |\alpha| \leq 3} \| \partial^\alpha (I - P) f \|_{L^2_v(L^2_x)}^2 \leq C \sqrt{N_0} \left( \| (I - P)f \|_{L^2_v(L^2_x)}^2 + \| \nabla_x f \|_{L^2_v(L^2_x)}^2 \right) \]

(40)

\[\leq C \sqrt{N_0} \left( \| (I - P)f \|_{L^2_v(L^2_x)}^2 + \| \nabla_x a \|_{H^2}^2 + \| \nabla_x b \|_{H^2}^2 + \| \nabla_x c \|_{H^2}^2 \right) \]

with \( C > 0 \) a generic constant. The combination of (36) and (40) gives rise to the estimate (32) due to \( N_0 \) small enough. The proof of Lemma 2.3 is completed. \( \square \)

We next estimate the macroscopic quantities \( a, b \) and \( c \) which are defined by (17). Inspired by [12], using \( L_{v, \theta}(P f) = 0 \) and plugging \( f = P f + (I - P)f \) into (9) to express the macroscopic part \( P f \) through the microscopic part \( (I - P)f \), we get

\[
(\partial_t + v \cdot \nabla_x)P f = -(\partial_t + v \cdot \nabla_x - L_{v, \theta})(I - P)f + N(f).
\]

By re-writing the left hand side and right hand side of (9) with respect to the following basis

\[
e_1 = \sqrt{\mu}, \quad e_{j+1} = v_j \sqrt{\mu}, \quad \tilde{e}_{j+4} = \frac{1}{\sqrt{2}} (v_j^2 - 1) \sqrt{\mu},
\]

\[
\tilde{e}_8 = (I - 1) \sqrt{\mu}, \quad e_9 = v_1 v_2 \sqrt{\mu}, \quad e_{10} = v_2 v_3 \sqrt{\mu}, \quad e_{11} = v_3 v_1 \sqrt{\mu},
\]

\[
e_{j+11} = \frac{v_j (|v|^2 - 5 + 2 I - 2)}{\sqrt{14}} \sqrt{\mu}, \quad j = 1, 2, 3,
\]

we obtain the coupled micro-macro system

\[
\partial_t a + \nabla_x \cdot b = 0,
\]

(41)

\[
\partial_t b_j + \partial_x a + \frac{2}{\sqrt{10}} \partial_{x_j} c = - \nabla_x \cdot (v v_j \sqrt{\mu}, (I - P)f), j = 1, 2, 3,
\]

(42)

\[
\frac{1}{\sqrt{5}} \partial_t c + \sqrt{2} \partial_{x_j} b_j = \langle \tilde{e}_{4+j}, \partial_t (I - P)f \rangle - \nabla_x \cdot (v \tilde{e}_{4+j}, (I - P)f) \]

\[+ \langle \tilde{e}_{4+j}, L_{v, \theta} f \rangle + \langle \tilde{e}_{4+j}, N(f) \rangle, j = 1, 2, 3,
\]

(43)

\[
\frac{2}{\sqrt{10}} \partial_{t c} = - \langle \tilde{e}_8, \partial_t (I - P)f \rangle - \nabla_x \cdot (v \tilde{e}_8, (I - P)f) \]

\[+ \langle \tilde{e}_8, L_{v, \theta} f \rangle + \langle \tilde{e}_8, N(f) \rangle,
\]

(44)
The first term on the right hand side of (48) can be estimated as follows. It holds
\[\frac{\sqrt{7}}{\sqrt{5}} \frac{d}{dt} \int |\partial^\alpha \partial_x c|^2 dx = \int \partial^\alpha \partial_x c \partial^\alpha \left( -e_{j+11}, \partial_t (I - P)f \right) \]
\[- \nabla_x \cdot \langle ve_{j+11}, (I - P)f \rangle + \langle e_{j+11}, L_{u, \theta}f \rangle + \langle e_{j+11}, N(f) \rangle dx. \]

Our second result is about the energy estimates of a, b and c.

**Lemma 2.4.** Let \(-\frac{1}{2} < \nu < 1, 0.5 < \theta < 1, T > 0, \) and \(f(t, x, v, I)\) be a smooth solution to the Cauchy problem (9)-(10) satisfying (20) in \((0, T] \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^+\).

Then, there exists some constant \(C > 0\) such that
\[
\|\nabla_x a(t)\|^2_{L^2_t} + \|\nabla_x b(t)\|^2_{L^2_t} + \|\nabla_x c(t)\|^2_{L^2_t} \\
+ k_1 \frac{d}{dt} \sum_{|\alpha| \leq 2} \sum_{1 \leq 1 \leq 3} \int_{\mathbb{R}^3} |\partial^\alpha \partial_x, c(e_{j+11}, \partial^\alpha (I - P)f) dx \\
+ k_2 \frac{d}{dt} \sum_{|\alpha| \leq 2} \sum_{1 \leq 1 \leq 3} \int_{\mathbb{R}^3} |\partial^\alpha (\partial_x b_j + \partial_x b_i)\partial^\alpha (v_j v_j \sqrt{\mu}, (I - P)f) dx \\
+ k_3 \frac{d}{dt} \sum_{|\alpha| \leq 2} \sum_{1 \leq 1 \leq 3} \int_{\mathbb{R}^3} \left( \frac{1}{\sqrt{5}} \partial^\alpha c \partial^\alpha \partial_x b_j + \langle \delta_{4+j}, \partial^\alpha (I - P)f \rangle \partial^\alpha \partial_x b_j \right) dx \\
+ k_4 \frac{d}{dt} \sum_{|\alpha| \leq 2} \sum_{1 \leq 1 \leq 3} \int_{\mathbb{R}^3} \partial^\alpha b_j \partial^\alpha \partial_x a dx \\
\leq C \|\nabla_x (I - P)f\|^2_{L^2_t, (H^2_t)}.
\]

**Proof.** The proof is divided into four steps.

Step 1. Estimate on \(\|\nabla_x c(t)\|_{L^2_t}\).

Taking the spatial derivative \(\partial^\alpha\) to (46) with \(|\alpha| \leq 2\), multiplying by \(\partial^\alpha \partial_x, c\), and taking integration over \(\mathbb{R}^3\), we have
\[
\frac{\sqrt{7}}{\sqrt{5}} \frac{d}{dt} \int |\partial^\alpha \partial_x c|^2 dx = \int \partial^\alpha \partial_x c \partial^\alpha \left( -e_{j+11}, \partial_t (I - P)f \right) \\
- \nabla_x \cdot \langle ve_{j+11}, (I - P)f \rangle + \langle e_{j+11}, L_{u, \theta}f \rangle + \langle e_{j+11}, N(f) \rangle dx.
\]

The first term on the right hand side of (48) can be estimated as follows. It holds that
\[
- \int_{\mathbb{R}^3} \partial^\alpha \partial_x c(e_{j+11}, \partial_t \partial^\alpha (I - P)f) dx \\
= - \frac{d}{dt} \int_{\mathbb{R}^3} \partial^\alpha \partial_x c(e_{j+11}, \partial^\alpha (I - P)f) dx + \int_{\mathbb{R}^3} \partial^\alpha \partial_x c(e_{j+11}, \partial^\alpha (I - P)f) dx.
\]

Multiplying (43) by \(\frac{1}{\sqrt{5}}\), multiplying (44) by \(\frac{2}{\sqrt{10}}\), and combine the two equalities together, we get
\[
\partial_t c + \frac{2}{\sqrt{10}} \nabla_x \cdot b = - \frac{1}{\sqrt{10}} \nabla_x \cdot \langle v|v|^2 - 3 + 2I - 2\sqrt{\mu}, (I - P)f \rangle,
\]
which leads to

\[
\left| \int_{\mathbb{R}^3} \partial^\alpha \partial x_j \partial x_c (\epsilon_{j+11}, \partial^\alpha (I - P) f) \, dx \right|
\]

\[
= \int_{\mathbb{R}^3} \left( - \frac{1}{\sqrt{10}} \partial^\alpha \partial x_j \nabla_x \cdot (v |v|^2 - 3 + 2I - 2) \sqrt{\mu}, (I - P) f - \frac{2}{\sqrt{10}} \partial^\alpha \partial x_j \nabla_x \cdot b \right)
\]

\[
\times (\epsilon_{j+11}, \partial^\alpha (I - P) f) \, dx
\]

\[
= \int_{\mathbb{R}^3} \left( \frac{1}{\sqrt{10}} \partial^\alpha \nabla_x \cdot (v |v|^2 - 3 + 2I - 2) \sqrt{\mu}, (I - P) f + \frac{2}{\sqrt{10}} \partial^\alpha \nabla_x \cdot b \right)
\]

\[
\times (\epsilon_{j+11}, \partial x_j \partial^\alpha (I - P) f) \, dx
\]

\[
\leq \varepsilon \| \nabla_x \cdot b \|_{L^2(H_2)}^2 + C \varepsilon \| (I - P) f \|_{L^2_x(H_2)}^2,
\]

(50)

where the constant \( \varepsilon \) is to be chosen small enough. For the rest terms on the right hand side of (48), applying Lemma 2.2, and using Hölder’s and Young’s inequalities, we have

\[
\left| \int_{\mathbb{R}^3} \partial^\alpha \partial x_j c \partial^\alpha \left( - \nabla_x \cdot (\epsilon_{j+11}, (I - P) f) + (\epsilon_{j+11}, L_{\nu, \theta} f) + (\epsilon_{j+11}, N(f)) \right) \, dx \right|
\]

\[
\leq \varepsilon \| \nabla_x c \|_{L^2(H_2)}^2 + C \varepsilon \left( \| \| f \|_{L^2_x(H_2)}^2 \right) \left( \| \nabla_x f \|_{L^2_x(H_2)}^2 + \| \nabla_x f \|_{L^2_x(H_2)}^2 + \| \nabla_x f \|_{L^2_x(H_2)}^2 \right)
\]

\[
\leq \varepsilon \| (I - P) f \|_{L^2_x(H_2)}^2.
\]

(51)

Substituting (49), (50) and (51) into (48), and taking the summation over \( \alpha, j \) with \( |\alpha| \leq 2 \) and \( 1 \leq j \leq 3 \), we can obtain the estimate

\[
\sum_{|\alpha| \leq 2} \left( \| \partial^\alpha \nabla_x c \|_{L^2(H_2)}^2 + k_1 \frac{d}{dt} \sum_{|\alpha| \leq 2} \sum_{j=1}^3 \int_{\mathbb{R}^3 \times \mathbb{R}^3} e_{j+11} \partial^\alpha (I - P) f \partial x_j \partial^\alpha c \, dx \right)
\]

\[
\leq \varepsilon \| \nabla_x \cdot b \|_{L^2(H_2)}^2 + C \varepsilon \left( N_0 \| \nabla_x f \|_{L^2_x(H_2)}^2 + \| (I - P) f \|_{L^2_x(H_2)}^2 \right).
\]

(52)

Notice that

\[
\| \nabla_x f \|_{L^2_x(H_2)}^2 = \| \nabla_x a(t) \|_{H_2}^2 + \| \nabla_x b(t) \|_{H_2}^2 + \| \nabla_x c(t) \|_{H_2}^2 + \| \nabla_x (I - P) f \|_{L^2_x(H_2)}^2.
\]

Step 2. Estimate on \( \sum_{1 \leq i < j \leq 3} \| \partial x_i b_j + \partial x_j b_i \|_{H_2} \).

Taking the spatial derivatives \( \partial^\alpha \) to (45) with \( |\alpha| \leq 2 \), multiplying by \( \partial^\alpha (\partial x_i b_j + \partial x_j b_i) \), and taking integrations over \( \mathbb{R}^3 \), we have

\[
\int_{\mathbb{R}^3} | \partial^\alpha \partial x_i b_j + \partial^\alpha \partial x_j b_i |^2 \, dx
\]

\[
= \int_{\mathbb{R}^3} \partial^\alpha (\partial x_i b_j + \partial x_j b_i) \partial^\alpha \left( (v_i v_j \sqrt{\mu}, \partial t (I - P) f) - \nabla_x \cdot (v_i v_j \sqrt{\mu}, (I - P) f) \right)
\]

\[
+ \langle v_i v_j \sqrt{\mu}, L_{\nu, \theta} f \rangle + \langle v_i v_j \sqrt{\mu}, N(f) \rangle \, dx.
\]

(53)
Substituting (54), (55) and (56) into (53) and taking the summation over \( \alpha, i, j \), it holds that

\[
- \int_{\mathbb{R}^3} \partial^\alpha (\partial_x b_j + \partial_x b_i) \partial^\alpha (v_i v_j \sqrt{\mu}, \partial_t (I - P)f) dx
\]

\[
= - \frac{d}{dt} \int_{\mathbb{R}^3} \partial^\alpha (\partial_x b_j + \partial_x b_i) \partial^\alpha (v_i v_j \sqrt{\mu}, (I - P)f) dx
\]

\[
+ \int_{\mathbb{R}^3} \partial^\alpha (\partial_x b_j + \partial_x b_i) \partial^\alpha (v_i v_j \sqrt{\mu}, (I - P)f) dx
\]

\[
= - \frac{d}{dt} \int_{\mathbb{R}^3} \partial^\alpha (\partial_x b_j + \partial_x b_i) \partial^\alpha (v_i v_j \sqrt{\mu}, (I - P)f) dx
\]

\[
- \int_{\mathbb{R}^3} \partial^\alpha \partial_t b_j \partial^\alpha (v_i v_j \sqrt{\mu}, \partial_x (I - P)f) + \partial^\alpha \partial_t b_i \partial^\alpha (v_i v_j \sqrt{\mu}, \partial_x (I - P)f) dx. \tag{54}
\]

By (42), and using Hölder’s and Young’s inequalities, one can estimate the last term of (54) as follows

\[
- \int_{\mathbb{R}^3} \partial^\alpha \partial_t b_j \partial^\alpha (v_i v_j \sqrt{\mu}, \partial_x (I - P)f) + \partial^\alpha \partial_t b_i \partial^\alpha (v_i v_j \sqrt{\mu}, \partial_x (I - P)f) dx
\]

\[
= \int_{\mathbb{R}^3} \partial^\alpha \left( \partial_x a + \frac{2}{\sqrt{10}} \partial_x c + \nabla_x \cdot (vv_j \sqrt{\mu}, (I - P)f) \right) \partial^\alpha (v_i v_j \sqrt{\mu}, \partial_x (I - P)f) dx
\]

\[
+ \int_{\mathbb{R}^3} \partial^\alpha \left( \partial_x a + \frac{2}{\sqrt{10}} \partial_x c + \nabla_x \cdot (vv_i \sqrt{\mu}, (I - P)f) \right) \partial^\alpha (v_i v_j \sqrt{\mu}, \partial_x (I - P)f) dx
\]

\[
\leq \varepsilon (\|v_a x d\|^2_{L^2} + \|v_x c\|^2_{L^2}) + C\varepsilon \|(I - P)f\|^2_{L^2_{x, t}(H^2)}, \tag{55}
\]

where the constant \( \varepsilon \) is to be chosen small enough. By Lemma 2.2, and using Hölder’s and Young’s inequalities, the rest terms on the right hand side of (53) can be estimated as follows

\[
\int_{\mathbb{R}^3} \partial^\alpha (\partial_x b_j + \partial_x b_i) \partial^\alpha \left( - \nabla_x \cdot (vv_j \sqrt{\mu}, (I - P)f) + (v_i v_j \sqrt{\mu}, L_{\nu, \theta} f) \right) dx
\]

\[
\leq \varepsilon \|\partial_x b_j + \partial_x b_i\|^2_{L^2} + C\varepsilon \left( \|(I - P)f\|^2_{L^2_{x, t}(H^2)} + \|f\|^2_{L^2_{x, t}(H^2)} \right) \|\nabla_x f\|_{L^2_{x, t}(H^2)}^4 + \|\nabla_x f\|_{L^2_{x, t}(H^2)}^6. \tag{56}
\]

Substituting (54), (55) and (56) into (53) and taking the summation over \( \alpha, i, j \) with \(|\alpha| \leq 2 \) and \( 1 \leq i < j \leq 3 \), we can obtain the estimate

\[
\sum_{|\alpha| \leq 2} \sum_{1 \leq i < j \leq 3} \|\partial^\alpha \partial_x b_j + \partial^\alpha \partial_x b_i\|^2_{L^2} \tag{57}
\]

\[
+ \frac{d}{dt} \sum_{|\alpha| \leq 2} \sum_{1 \leq i < j \leq 3} \int_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^+} \partial^\alpha (I - P) f v_i v_j \sqrt{\mu} (\partial^\alpha \partial_x b_j + \partial^\alpha \partial_x b_i) dvdt dx
\]

\[
\leq \varepsilon (\|v_a x d\|^2_{L^2} + \|v_x c\|^2_{L^2}) + C\varepsilon \left( \|(I - P)f\|^2_{L^2_{x, t}(H^2)} \right).
taking integration over $\mathbb{R}$ term of (59) as follows. Applying (42), using Hölder’s and Young’s inequalities, one can estimate the last term of (58) as follows. It holds that

\[
\sqrt{2} \int_{\mathbb{R}^3} |\partial^\alpha \partial_x b_j|^2 \, dx = \int_{\mathbb{R}^3} \partial^\alpha \partial_x b_j \partial^\alpha \left( - \frac{1}{\sqrt{5}} \partial_t c - \langle \bar{e}_{4+j}, \partial_t (I-P)f \rangle - \langle \bar{e}_{4+j}, (I-P)f \rangle - \langle \bar{e}_{4+j}, (I-P)f \rangle - \langle \bar{e}_{4+j}, (I-P)f \rangle \right) \, dx.
\]  

(58)

The first two terms on the right hand side of (58) can be estimated as follows. It holds that

\[
\int_{\mathbb{R}^3} \partial^\alpha \partial_x b_j \partial^\alpha \left( - \frac{1}{\sqrt{5}} \partial_t c - \langle \bar{e}_{4+j}, \partial_t (I-P)f \rangle \right) \, dx
\]

\[
= - \frac{d}{dt} \int_{\mathbb{R}^3} \partial^\alpha \partial_x b_j \partial^\alpha \left( \frac{1}{\sqrt{5}} c + \langle \bar{e}_{4+j}, (I-P)f \rangle \right) \, dx
\]

\[
+ \int_{\mathbb{R}^3} \partial^\alpha \partial_x b_j \partial^\alpha \left( \frac{1}{\sqrt{5}} c + \langle \bar{e}_{4+j}, (I-P)f \rangle \right) \, dx
\]

\[
= \frac{d}{dt} \int_{\mathbb{R}^3} \partial^\alpha \partial_x b_j \partial^\alpha \left( \frac{1}{\sqrt{5}} c + \langle \bar{e}_{4+j}, (I-P)f \rangle \right) \, dx
\]

\[
- \int_{\mathbb{R}^3} \partial^\alpha \partial_x b_j \partial^\alpha \left( \frac{1}{\sqrt{5}} c + \langle \bar{e}_{4+j}, \partial_x (I-P)f \rangle \right) \, dx.
\]

Applying (42), using Hölder’s and Young’s inequalities, one can estimate the last term of (59) as follows

\[
\int_{\mathbb{R}^3} \partial^\alpha \partial_x b_j \partial^\alpha \left( \frac{1}{\sqrt{5}} c + \langle \bar{e}_{4+j}, \partial_x (I-P)f \rangle \right) \, dx
\]

\[
= \int_{\mathbb{R}^3} \partial^\alpha \left( \partial_x a + \frac{2}{\sqrt{10}} \partial_x c + \nabla_x \cdot (\alpha V_{e^j} \sqrt{\mu}, (I-P)f) \right) \right) \, dx
\]

\[
\times \partial^\alpha \left( \frac{1}{\sqrt{5}} c + \langle \bar{e}_{4+j}, \partial_x (I-P)f \rangle \right) \, dx
\]

\[
\leq \varepsilon \|\nabla_x a\|_{H^2}^2 + C \left( \|\nabla_x c\|_{H^2}^2 + \|f\|_{L^2_{\alpha,i}(H^2)}^2 \right)
\]

where the constant $\varepsilon$ is small enough to be determined later. By Lemma 2.2, and using Hölder’s and Young’s inequalities, the rest terms on the right hand side of (58) can be estimated as follows

\[
\int_{\mathbb{R}^3} \partial^\alpha \partial_x b_j \partial^\alpha \left( - \nabla_x \cdot (\bar{e}_{4+j}, (I-P)f) + \langle \bar{e}_{4+j}, L_{v,0}f \rangle + \langle \bar{e}_{4+j}, N(f) \rangle \right) \, dx
\]

\[
\leq \varepsilon \|\partial_x b_j\|_{H^2}^2 + C \left( \|f\|_{L^2_{\alpha,i}(H^2)}^2 \right) + \|\nabla_x f\|_{L^2_{\alpha,i}(H^2)}^2
\]

\[\leq \varepsilon \|\partial_x b_j\|_{H^2}^2 + C \left( \|f\|_{L^2_{\alpha,i}(H^2)}^2 \right) + \|\nabla_x f\|_{L^2_{\alpha,i}(H^2)}^2
\]

(60)

(61)
Substituting (59), (60) and (61) into (58) and taking the summation over $\alpha, j$ with $|\alpha| \leq 2$ and $1 \leq j \leq 3$, we can obtain the estimate

\[
\sqrt{2} \sum_{|\alpha| \leq 2} \sum_{2 \leq j \leq 3} \| \partial^\alpha \partial_x_j b_j \|_{L^2}^2 + k_3 \frac{d}{dt} \sum_{|\alpha| \leq 2} \sum_{2 \leq j \leq 3} \int_{\mathbb{R}^3} \left( \frac{1}{\sqrt{5}} \partial^\alpha \partial_x_j b_j \right) \\
+ \langle \hat{e}_{4+j}, \partial^\alpha (I - P) f \rangle \partial^\alpha \partial_x_j b_j \rangle \rangle dx \\
\leq \varepsilon \| \nabla_x a \|_{H^2}^2 + C_\varepsilon \left( \| \nabla_x c \|_{H^2}^2 + \| (I - P) f \|_{L^2_{x,t}(H^2)}^2 \\
+ \| f \|_{L^2_{x,t}(H^2)}^2 \right) \\
\leq \varepsilon \| \nabla_x a \|_{H^2}^2 + C_\varepsilon \left( \| \nabla_x c \|_{H^2}^2 + N_0^2 \| \nabla_x f \|_{L^2_{x,t}(H^2)}^2 + \| (I - P) f \|_{L^2_{x,t}(H^2)}^2 \right).
\]

Since $\| \partial^\alpha \nabla_x \cdot b \|_{L^2}^2 \leq 3 \sum_{|\alpha| \leq 2} \| \partial^\alpha \partial_x_j b_j \|_{L^2}^2$, we also have

\[
\sqrt{2} \sum_{|\alpha| \leq 2} \| \partial^\alpha \nabla_x \cdot b \|_{L^2}^2 + k_3 \frac{d}{dt} \sum_{|\alpha| \leq 2} \sum_{2 \leq j \leq 3} \int_{\mathbb{R}^3} \left( \frac{1}{\sqrt{5}} \partial^\alpha \partial_x_j b_j \right) \\
+ \langle \hat{e}_{4+j}, \partial^\alpha (I - P) f \rangle \partial^\alpha \partial_x_j b_j \rangle \rangle dx \\
\leq \varepsilon \| \nabla_x a \|_{H^2}^2 + C_\varepsilon \left( \| \nabla_x c \|_{H^2}^2 + N_0^2 \| \nabla_x f \|_{L^2_{x,t}(H^2)}^2 + \| (I - P) f \|_{L^2_{x,t}(H^2)}^2 \right).
\]

Step 4. Estimate on $\| \nabla_x a \|_{H^2}$.

Taking the spatial derivatives $\partial^\alpha$ to (42) with $|\alpha| \leq 2$, multiplying by $\partial^\alpha \partial_x a$, and taking integration over $\mathbb{R}^3$, we have

\[
\int_{\mathbb{R}^3} \| \partial^\alpha \partial_x a \|_{L^2}^2 dx = \int_{\mathbb{R}^3} \partial^\alpha \partial_x \partial_x a \partial^\alpha \left( - \partial_t b - \frac{2}{10 \mu} \partial_x c - \nabla_x \cdot \langle v v_j \sqrt{\mu}, (I - P) f \rangle \right) dx.
\]

Applying (41), one can estimate the first term on the right hand side of (66) as follows

\[
\int_{\mathbb{R}^3} \| \partial^\alpha \partial_x a \|_{L^2}^2 dx = \int_{\mathbb{R}^3} \partial^\alpha \partial_x \partial_x a \partial^\alpha \left( - \partial_t b - \frac{2}{10 \mu} \partial_x c - \nabla_x \cdot \langle v v_j \sqrt{\mu}, (I - P) f \rangle \right) dx.
\]

By Lemma 2.2, and using Hölder’s and Young’s inequalities, the remaining terms on the right hand side of (66) can be estimated as follows

\[
\int_{\mathbb{R}^3} \| \partial^\alpha \partial_x a \|_{L^2}^2 dx \leq \varepsilon \| \partial_x a \|_{H^2}^2 + C \left( \| \partial_x c \|_{H^2}^2 + \| (I - P) f \|_{L^2_{x,t}(H^2)}^2 \right),
\]

where the constant $\varepsilon$ is to be chosen small enough.
Substituting (65) and (66) into (64) and taking the summation over $\alpha, j$ with $|\alpha| \leq 2$ and $1 \leq j \leq 3$, we can obtain the estimate

\[ \sum_{|\alpha| \leq 2} \left| \partial^\alpha \nabla_x a \right|^2_{L^2} + k_4 \frac{d}{dt} \sum_{|\alpha| \leq 2} \sum_{1 \leq j \leq 3} \int_{\mathbb{R}^3} \partial^\alpha b_j \partial^\alpha \partial_x a dx \leq C \left( \| \nabla_x b \|_{H^2}^2 + \| \nabla_x c \|_{H^2}^2 + \| (I - P) f \|_{L^2_{0,j}(H^2)}^2 \right), \] (67)

with $C > 0$ a generic constant.

Taking (52) $+ \varepsilon (58) + \varepsilon (62) + 3 \varepsilon (63) + 8 \varepsilon^2 (67)$ with $\varepsilon > 0$ sufficiently small, and choosing $N_0$ small enough, using the fact $N_0$ is small and

\[ \sum_{1 \leq i < j \leq 3} \| \partial_{x_i} b_j + \partial_{x_i} b_i \|_{H^2}^2 + 2 \sum_{1 \leq i \leq 3} \| \partial_{x_i} b_j \|_{H^2}^2 = \| \nabla_x b \|_{H^2}^2 + \| \nabla_x b \|_{H^2}^2 \] (cf. [5]), we obtain the estimate (47). The proof of Lemma 2.4 is completed. $\Box$

**Proof of Proposition 1.** We combine Lemma 2.3 and Lemma 2.4 to prove Proposition 1. Actually, define

\[ \mathcal{E}(t) = \| f(t) \|_{L^2_{0,j}(H^2)}^2 + \varepsilon \left( k_1 \sum_{|\alpha| \leq 2} \sum_{1 \leq j \leq 3} \int_{\mathbb{R}^3} \partial^\alpha \partial_x c(e_{j+1}, \partial^\alpha (I - P) f) dx \right. \]

\[ + k_2 \sum_{|\alpha| \leq 2} \sum_{1 \leq i < j \leq 3} \int_{\mathbb{R}^3} \partial^\alpha (\partial_{x_i} b_j + \partial_{x_i} b_i) \partial^\alpha \langle v_i v_j \sqrt{\mu}, (I - P) f \rangle dx \]

\[ + k_3 \sum_{|\alpha| \leq 2} \sum_{1 \leq i \leq 3} \int_{\mathbb{R}^3} \left( \frac{1}{\sqrt{\nu}} \partial^\alpha c \partial^\alpha \partial_{x_i} b_j + \partial^\alpha (I - P) f \partial^\alpha \partial_{x_i} b_j \right) dx \]

\[ + k_4 \sum_{|\alpha| \leq 2} \sum_{1 \leq i \leq 3} \int_{\mathbb{R}^3} \partial^\alpha b_j \partial^\alpha \partial_{x_i} a dx \right). \]

Obviously, $\mathcal{E}(t) \sim \| f(t) \|_{L^2_{0,j}(H^2)}^2$ for enough small constant $\varepsilon$. Taking (32) $+ \varepsilon (47)$ with $\varepsilon > 0$ sufficiently small, and choosing $N_0$ small enough for each fixed $\varepsilon$, we get

\[ \frac{d}{dt} \mathcal{E}(t) + \sigma_0 \| (I - P) f(t) \|_{L^2_{0,j}(H^2)}^2 + k \left( \| \nabla_x a(t) \|_{H^2}^2 + \| \nabla_x b(t) \|_{H^2}^2 + \| \nabla_x c(t) \|_{H^2}^2 \right) \leq 0, \] (68)

Integrating with respect to the time over $[0,t]$, we conclude the following estimates

\[ \mathcal{E}(t) + \sigma_0 \int_0^t \left( \| (I - P) f(\tau) \|_{L^2_{0,j}(H^2)}^2 + k \left( \| \nabla_x a(\tau) \|_{H^2}^2 + \| \nabla_x b(\tau) \|_{H^2}^2 + \| \nabla_x c(\tau) \|_{H^2}^2 \right) \right) d\tau \leq \mathcal{E}(0). \]

Finally, we get

\[ \| f(t) \|_{L^2_{0,j}(H^2)}^2 + \sigma_0 \int_0^t \left( \| (I - P) f(\tau) \|_{L^2_{0,j}(H^2)}^2 + k \left( \| \nabla_x a(\tau) \|_{H^2}^2 + \| \nabla_x b(\tau) \|_{H^2}^2 + \| \nabla_x c(\tau) \|_{H^2}^2 \right) \right) d\tau \leq C \| f_0 \|_{L^2_{0,j}(H^2)}^2, \]

since $\mathcal{E}(t) \sim \| f(t) \|_{L^2_{0,j}(H^2)}^2$. It gives rise to the estimate (21) by taking supremum on $[0,T]$. The proof of Proposition 1 is completed. $\Box$
3. Global existence. We first construct the local solution to the Cauchy problem \((9)-(10)\) with the initial data \(f_0 \in L^2_{\nu,1}(H^3_{\nu})\).

Considering the following problem
\[
\partial_t f^\varepsilon + v \cdot \nabla_x f^\varepsilon = L_{\nu,\theta} f^\varepsilon + N(f^\varepsilon),
\]
\[
f_0^\varepsilon(x, v, I) = \eta_{\varepsilon} * f_0(x, v, I),
\]
where \(L_{\nu,\theta} f^\varepsilon\) and \(N(f^\varepsilon)\) are defined by \((11)\) and \((12)\), and
\[
\eta_{\varepsilon}(x, v, I) = \frac{1}{\varepsilon^3} \eta\left(\frac{x}{\varepsilon}, \frac{v}{\varepsilon}, \frac{I}{\varepsilon}\right)
\]
with \(\eta\) the standard mollifier.

We can prove the following local existence results to the Cauchy problem \((69)-(70)\) (cf. \([13], [14]\)). There exists a constant \(C > 0\) for the equation \((9)\), if \(f_0 \in L^2_{\nu,1}(H^3_{\nu})\) with the initial data \(\eta_{\varepsilon}\) introduced by Kawashima \([17]\). Let us recall the definition of compensating function with the energy estimates to obtain the time decay rate of the global solutions to the Cauchy problem \((9)-(10)\).

4. The time decay rate. In this section, we will combine the compensating function with the energy estimates to obtain the time decay rate of the global solutions to the Cauchy problem \((9)-(10)\). Let us recall the definition of compensating function introduced by Kawashima \([17]\).

**Definition 4.1.** Let \(S(\omega), \omega \in S^2\) be a bounded linear operator on \(L^2_{\nu,I}\). \(S(\omega)\) is called a compensating function for the equation \((9)\), if

1. \(S(\cdot)\) is \(C^\infty\) on \(S^2\) with values in the space of bounded linear operators on \(L^2_{\nu,I}\),
2. \(iS(\omega)\) is self-adjoint on \(L^2_{\nu,I}\) for all \(\omega \in S^2\).
3. There is a constant \(c_1 > 0\) such that
\[
\Re \langle S(\omega)(\omega \cdot v)f, f \rangle + \langle -L_{\nu,\theta} f, f \rangle \geq c_1 \|f\|_{L^2_{\nu,I}}^2
\]
for all \(f \in L^2_{\nu,I}, \omega \in S^2\).

The main idea to construct \(S(\omega)\) owns to \([17]\). Indeed, let \(X\) be the closed subspace spanned by fourteen moments consisting of the kernel of \(L_{\nu,\theta}\) and the images of linear operator \(v_j(j = 1, 2, 3)\) from \(\text{Ker}(L_{\nu,\theta}) \to L^2_{\nu,I}\) denoted by
\[
X = \text{span}\{e_j | j = 1, 2, \ldots, 14\},
\]
where the orthonormal basis \( \{e_i\} \) for \( X \) is given by

\[
\begin{align*}
e_1 &= \sqrt{\mu}, \\
e_{j + 1} &= v_j \sqrt{\mu}, j = 1, 2, 3, \\
e_5 &= \frac{(|v|^2 - 3 + 2I - 2)}{\sqrt{10}} \sqrt{\mu}, \\
e_6 &= \frac{1}{2}(v_1^2 - v_2^2) \sqrt{\mu}, \\
e_7 &= \frac{|v|^2 - 3v_3^2}{\sqrt{12}} \sqrt{\mu}, \\
e_8 &= \frac{(|v|^2 - 3 - 3I + 3)}{\sqrt{15}} \sqrt{\mu}, \\
e_9 &= v_1 v_2 \sqrt{\mu}, \\
e_{10} &= v_2 v_3 \sqrt{\mu}, \\
e_{11} &= v_3 v_1 \sqrt{\mu}, \\
e_{j + 11} &= v_j (|v|^2 - 5 + 2I - 2) \sqrt{\mu}, j = 1, 2, 3.
\end{align*}
\]

Set \( V = \{\langle v_k e_l, e_j \rangle\}^{14}_{k,l=1} \), by a straightforward calculation, one has

\[
V(\xi) = \sum_{j=1}^{3} V_j \xi_j = \begin{pmatrix}
V_{11}(\xi) & V_{12}(\xi) & V_{13}(\xi) \\
V_{21}(\xi) & V_{22}(\xi) & V_{23}(\xi)
\end{pmatrix}^{5 \times 9}
\]

with

\[
V_{11} = \left(\begin{array}{ccccc}
0 & \xi_1 & \xi_2 & \xi_3 & 0 \\
\xi_1 & 0 & 0 & 0 & \frac{2}{\sqrt{10}} \xi_1 \\
\xi_2 & 0 & 0 & 0 & \frac{2}{\sqrt{10}} \xi_2 \\
\xi_3 & 0 & 0 & 0 & \frac{2}{\sqrt{10}} \xi_3 \\
0 & \frac{2}{\sqrt{10}} \xi_1 & \frac{2}{\sqrt{10}} \xi_2 & \frac{2}{\sqrt{10}} \xi_3 & 0
\end{array}\right)
\]

and

\[
V_{21} = \left(\begin{array}{ccccc}
0 & \xi_1 & \xi_2 & \xi_3 & 0 \\
\frac{2}{\sqrt{15}} \xi_1 & 0 & 0 & 0 & \frac{2}{\sqrt{15}} \xi_1 \\
\frac{2}{\sqrt{15}} \xi_2 & \xi_1 & 0 & 0 & \frac{2}{\sqrt{15}} \xi_2 \\
\frac{2}{\sqrt{15}} \xi_3 & 0 & \xi_1 & 0 & \frac{2}{\sqrt{15}} \xi_3 \\
0 & 0 & 0 & 0 & \frac{2}{\sqrt{3}} \xi_1 \\
0 & 0 & 0 & 0 & \frac{2}{\sqrt{3}} \xi_2 \\
0 & 0 & 0 & 0 & \frac{2}{\sqrt{3}} \xi_3
\end{array}\right).
\]

Denote \( R(\omega) \) by

\[
R(\omega) = \begin{pmatrix}
\alpha \bar{R}_{11}(\omega) & V_{12}(\omega) \\
-V_{21}(\omega) & 0
\end{pmatrix}^{5 \times 9}
\]

with the constant \( \alpha > 0 \) to be determined later, \( \omega = (\omega_1, \omega_2, \omega_3) = \frac{\xi}{|\xi|} \), and

\[
\bar{R}_{11} = \left(\begin{array}{ccc}
0 & \omega_1 & \omega_2 & \omega_3 & 0 \\
-\omega_1 & 0 & 0 & 0 & 0 \\
-\omega_2 & 0 & 0 & 0 & 0 \\
-\omega_3 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{array}\right).
\]

Define

\[
S(\omega)f = \sum_{k,l=1}^{14} \beta r_{kl}(\omega) (f, e_l) e_k
\]

with \( \beta > 0 \) some constant to be determined later. It is easy to verify (cf. [17]) that there exist a constant \( \alpha > 0 \) and a constant \( \beta > 0 \) small enough, such that \( S(\omega) \) defined by (73) is a compensating function to (9).
Taking the Fourier transform in $x$ for equation (9), we have
\[
\partial_t \hat{f} + i|\xi|v \cdot \hat{f} - L_{\nu,\theta} \hat{f} = \hat{N}(f) = \frac{1}{\sqrt{\mu}}(\nabla_v \cdot \Gamma_1(f) + \partial_t \Gamma_2(f)). \tag{74}
\]

Taking the inner product of (74) with $(1 + |\xi|^2 - i\kappa S(\omega))\hat{f}$ and using the property (72) of the compensating function, we obtain
\[
\frac{d}{dt}\left\{ \frac{1 + |\xi|^2}{2} E(\hat{f}) \right\} + (1 + (1 - \kappa)|\xi|^2)(-L_{\nu,\theta} \hat{f}, f) + \kappa|\xi|^2\{Re\langle S(\omega)(v \cdot \omega)\hat{f}, \hat{f} \rangle + \langle -L_{\nu,\theta} \hat{f}, f \rangle \}
= (1 + |\xi|^2)Re\langle \hat{N}(f), \hat{f} \rangle + \kappa|\xi|^2Re\{\langle iS(\omega)L_{\nu,\theta}\hat{f}, \hat{f} \rangle - \langle iS(\omega)\hat{N}(f), \hat{f} \rangle \},
\tag{75}
\]
where
\[
E(\hat{f})(\xi, t) = \left\| \hat{f}(\xi, t) \right\|^2_{L^2_v} - \kappa \left| \frac{\xi}{1 + |\xi|^2} \langle iS(\omega)\hat{f}(\xi, t), \hat{f}(\xi, t) \rangle \right| \sim \left\| \hat{f} \right\|^2_{L^2_v},
\]
with $\kappa > 0$ a constant small enough. The left of (75) is bounded below by
\[
\frac{d}{dt}\left\{ \frac{1 + |\xi|^2}{2} E(\hat{f}) \right\} + (1 - \kappa)\sigma_1(1 + |\xi|^2)(\|I - P\|\hat{f}\|^2_{L^2_v} + \kappa|\xi|^2c_1\|\hat{f}\|^2_{L^2_v}),
\]
because of the coercivity estimate (22) of $L_{\nu,\theta}$ and inequality (72).

On the other hand, by using Corollary 1 and integration by parts, the absolute value of the first term on the right of (75) satisfies
\[
Re\langle \hat{N}(f), \hat{f} \rangle \leq \left| \int_{\mathbb{R}^3 \times \mathbb{R}^+} \hat{f} \hat{N}(f) dv dI \right| = \left| \int_{\mathbb{R}^3 \times \mathbb{R}^+} (I - P) \hat{f} \frac{1}{\sqrt{\mu}}(\nabla_v \cdot \Gamma_1(f)) \hat{f} + \partial_t \Gamma_2(f) dv dI \right|
= \left| \int_{\mathbb{R}^3 \times \mathbb{R}^+} \nabla_v (I - P) \hat{f} \frac{1}{\sqrt{\mu}} \Gamma_1(f) + (I - P) \hat{f} \left( -\frac{1}{2} v \right) \nabla_v \cdot \Gamma_1(f) \left( I^{-\frac{1}{2}} \right) \hat{f} + \partial_t (I - P) \hat{f} \frac{1}{\sqrt{\mu}} \left( I^{-\frac{1}{2}} \right) \hat{f} \hat{f} dv dI \right|
\leq \varepsilon \left\| (I - P) \hat{f} \right\|^2_{L^2_v} + C_\varepsilon (\|G_1(f)\|^2 + \|G_2(f)\|^2),
\tag{76}
\]
where
\[
G_1(f) =: \frac{1}{\sqrt{\mu}} \Gamma_1(f), \quad G_2(f) =: \frac{1}{\sqrt{\mu}} \Gamma_2(f).
\]

By the definition (73) and integration by part, the absolute value of the third term on the right of (75) equals
\[
|Re\langle iS(\omega)\hat{N}(f), \hat{f} \rangle| \leq C\|\langle \hat{f}, e_i \rangle \| \|\hat{N}(f), e_i \| = C\|\hat{f}\| \left| \int_{\mathbb{R}^3 \times \mathbb{R}^+} e_i \hat{N}(f) dv dI \right|
= C\|\hat{f}\| \left| \int_{\mathbb{R}^3 \times \mathbb{R}^+} e_i \hat{f} \frac{1}{\sqrt{\mu}}(\nabla_v \cdot \Gamma_1(f)) \hat{f} + \partial_t \Gamma_2(f) dv dI \right|
\leq \varepsilon \left\| \hat{f} \right\|^2 + C_\varepsilon (\|G_1(f)\|^2 + \|G_2(f)\|^2).
Notice that
\[
S(\omega)f = \sum_{k,l=1}^{14} \beta r_{kl}(\omega) \langle L_{\nu,\theta} f, e_l \rangle e_k = \sum_{k,l=1}^{14} \beta r_{kl}(\omega) \langle L_{\nu,\theta}(I-P) f, e_l \rangle e_k \tag{77}
\]
and
\[
\langle L_{\nu,\theta}(I-P) f, e_l \rangle = \langle (I-P) f, L_{\nu,\theta}(e_l) \rangle \leq C \| (I-P) f \|,
\tag{78}
\]
where we used the self-adjoint property of \( L_{\nu,\theta} \) and the exponential decay of \( e_l(v,I) \) in \( v,I \). For the absolute value of the second term on the right of (75), we have
\[
|\text{Re}(iS(\omega)L\hat{f}, \hat{f}^*)| \leq | \sum_{k,l=1}^{14} \beta r_{kl}(\omega) \langle L\hat{f}, e_l \rangle \langle \hat{f}, e_k \rangle | \leq C \| (I-P) \hat{f} \| || \hat{f} || \tag{79}
\]
\[
\leq \varepsilon \| (I-P) \hat{f} \|^2_{L^2} + C \varepsilon \| \hat{f} \|^2.
\]
Therefore for any \( \varepsilon > 0 \), the right of (75) is not more than
\[
\varepsilon (1 + |\xi|^2) \| (I-P) \hat{f} \|^2_{L^2} + C \varepsilon (1 + |\xi|^2)(||G_1(\hat{f})||^2 + ||G_2(\hat{f})||^2) + \frac{\kappa \varepsilon}{2} |\xi|^2 \| \hat{f} \|^2
\]
\[
+ \kappa \varepsilon \| (I-P) \hat{f} \|^2_{L^2} + \frac{\kappa \varepsilon}{2} |\xi|^2 \| \hat{f} \|^2 + C \varepsilon (||G_1(\hat{f})||^2 + ||G_2(\hat{f})||^2)
\]
\[
\leq (\varepsilon + \kappa \varepsilon)(1 + |\xi|^2)(||G_1(\hat{f})||^2 + ||G_2(\hat{f})||^2)
\]
\[
+ C \varepsilon (1 + |\xi|^2)(||G_1(\hat{f})||^2 + ||G_2(\hat{f})||^2).
\]
Now take \( \varepsilon, \kappa \) small enough such that \( \varepsilon = \min \{ \frac{\sigma_1}{6}, \frac{\sigma_2}{\kappa} \} \), and then \( 0 < \kappa \leq \kappa_2 \), where
\[
\kappa_2 = \min \{ \frac{1}{6}, \frac{\sigma_1}{6C \varepsilon} \}.
\]
Finally, we get
\[
\partial_t E(\hat{f}) + \frac{|\xi|^2}{1 + |\xi|^2} E(\hat{f}) \leq C(||G_1(\hat{f})||^2 + ||G_2(\hat{f})||^2).
\tag{80}
\]
With the help of (80), we are able to establish the following estimates.

**Proposition 2** (cf. [9]). Let \( l \geq 0 \). Assume that

1. \( f_0 \in L^2_{v,I}(H^l_2) \cap L^2_{v,I}(L^l_2) \),
2. \( G_1(f), G_2(f) \in C^0([0, \infty); L^2_{v,I}(H^l_2)) \cap L^2_{v,I}(L^l_2) \).

Then if \( f \in C^0([0, \infty); L^2_{v,I}(H^l_2)) \cap C^1([0, \infty); L^2_{v,I}(H^{l-1}_2)) \) is a solution of (9), we have
\[
\| f(t) \|^2_{L^2_{v,I}(H^l_2)} \leq C \{ (1 + t)^{-\frac{3}{2}} (\| f_0 \|_{L^2_{v,I}(H^l_2)} + \| f_0 \|_{L^2_{v,I}(L^l_2)})^2
\]
\[
+ \int_0^t (1 + t - \tau)^{-\frac{3}{2}} (\| G_1(f)(\tau) \|_{L^2_{v,I}(H^l_2)} + \| G_2(f)(\tau) \|_{L^2_{v,I}(H^l_2)})^2 d\tau \}
\]
\[
+ \| G_1(f)(\tau) \|_{L^2_{v,I}(L^l_2)} + \| G_2(f)(\tau) \|_{L^2_{v,I}(L^l_2)})^2 d\tau \}
\tag{81}
\]

Proceeding with the estimates similar to Lemma 2.3-Lemma 2.4, we can obtain the following the fourth-order energy estimate similar to (68).
Proposition 3. Let $-\frac{1}{2} < \nu < 1$, $0.5 < \theta < 1$, and $f(t, x, v, I)$ be a smooth solution to the Cauchy problem (9)-(10) satisfying $\sup_{0 \leq t \leq T} \|f(t)\|_{L^2_t, (H^2)} \leq N_1$ in $(0, T] \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^+$ with $N_1$ small enough. Then it holds that

$$\frac{d}{dt} \mathcal{E}(t) + \sigma_0 (\|I - P\| f(t))_0^2 + k(\|\nabla_x a(t)\|_{H^2}^2 + \|\nabla_x b(t)\|_{H^2}^2 + \|\nabla_x c(t)\|_{H^2}^2) \leq 0. \tag{82}$$

where $a$, $b$, and $c$ are defined by (17), $k > 0$, $\sigma_0 > 0$ are generic constants and $\mathcal{E}(t) \sim \|f(t)\|_{L^2_t, (H^2)}^2$.

In this section, the main result is as follows.

Proposition 4. Let $-\frac{1}{2} < \nu < 1$ and $0.5 < \theta < 1$. Then there exists a constant $\delta_2 > 0$ small enough, such that if $\|f_0\|_{L^2_{x,v}(H^2)}^2 + \|\nabla_v f_0\|_{L^2_{x,v}(H^2)}^2 + \|v f_0\|_{L^2_{x,v}(H^2)}^2 + \|I \partial_t f_0\|_{L^2_{x,v}(H^2)}^2 + \|I \partial_v f_0\|_{L^2_{x,v}(H^2)}^2 + \|f_0\|_{L^2_{x,v}(H^2)}^2 \leq \delta_2$, then the solution $f$ to the Cauchy problem (9)-(10) satisfies

$$\|f(t)\|_{L^2_{x,v}(H^2)} \leq C \delta_2 (1 + t)^{-\frac{\nu}{2}}. \tag{83}$$

Proof. Taking $\partial_{v_i}$ to (9) with $1 \leq i \leq 3$, we get

$$\begin{align*}
\partial_{v_i} f &= v_i \cdot \nabla_v (\partial_{v_i} f) + \partial_{v_i} f \\
&= \partial_{v_i} (L_{v,v} f) + \partial_{v_i} N(f) \\
&= \Delta_v \partial_{v_i} f - \frac{1}{4} \|v\|^2 \partial_{v_i} f - \frac{1}{2} v_i f + 3 \partial_{v_i} f + 2 \partial_{v_i} (\partial_{\alpha} (I \partial_t f)) - \frac{1}{2} \partial_{v_i} (I f) + \partial_{v_i} f \tag{84}
\end{align*}$$

where $\tilde{L}f$ is defined by

$$\begin{align*}
\tilde{L}f &= \frac{1}{\sqrt{\mu}} \left\{ \nabla_v \cdot \left( b \mu + (a - \left( \frac{(1 - \theta)(1 - \nu)}{3} + \frac{\theta}{3} \right) r - \frac{2}{5} \theta d \right) v \mu - (1 - \theta) v \mu \right\} \\
&\quad + 2 \partial_t \left( \left( a - \frac{\theta}{3} r - (1 - \frac{3}{5} \theta) d \right) I \mu \right)
\end{align*}$$

with $a$, $b = (b_1, b_2, b_3)$, $r$, $l = (l_{ij})_{1 \leq i, j \leq 3}$, $d$ defined by (13).

Taking $\partial_{\alpha}^p$ to (84) with $1 \leq |\alpha| \leq 3$, multiplying by $\partial_{\alpha}^p \partial_{v_i} f$, taking integrations over $\mathbb{R}^3_x \times \mathbb{R}_v^3 \times \mathbb{R}_I^3$, and taking the summation over $\alpha$ and $1 \leq i \leq 3$, we obtain

$$\begin{align*}
\sum_{1 \leq |\alpha| \leq 3} \left( \frac{d}{dt} \|\nabla_v \partial_{\alpha}^p f(t)\|_{L^2_{x,v}(L^2_x)}^2 + c_1 \|\nabla_v \partial_{\alpha}^p f(t)\|_{L^2_{x,v}(L^2_x)}^2 + c_2 \|\nabla_v \partial_{\alpha}^p f(t)\|_{L^2_{x,v}(L^2_x)}^2 \right) \\
&\quad + c_3 \|\nabla_v (I \partial_t \partial_{\alpha}^p f(t))\|_{L^2_{x,v}(L^2_x)}^2 + c_4 \|\nabla_v (I \partial_t \partial_{\alpha}^p f(t))\|_{L^2_{x,v}(L^2_x)}^2 \\
&\quad \leq \varepsilon \left( \|\nabla_x a(t)\|_{L^2_x}^2 + \|\nabla_x b(t)\|_{L^2_x}^2 + \|\nabla_x c(t)\|_{L^2_x}^2 + \|v \nabla (I - P)\|_{L^2_{x,v}(H^2)}^2 + N_1 A \right), \tag{85}
\end{align*}$$

where $A = \sum_{1 \leq |\alpha| \leq 3} (\|\nabla_v \partial_{\alpha}^p f(t)\|_{L^2_{x,v}(L^2_x)}^2 + \|v \nabla \partial_{\alpha}^p f(t)\|_{L^2_{x,v}(L^2_x)}^2 + \|\nabla_x \partial_{\alpha}^p f(t)\|_{L^2_{x,v}(L^2_x)}^2 + \|I \partial_t \partial_{\alpha}^p f(t)\|_{L^2_{x,v}(L^2_x)}^2 + \|\partial_{\alpha} (I \partial_t \partial_{\alpha}^p f(t))\|_{L^2_{x,v}(L^2_x)}^2)$, and the constant $\varepsilon > 0$ is to be chosen small enough.
Similarly, taking \( \partial_x^\alpha \) to (9) with \( 1 \leq |\alpha| \leq 3 \), multiplying by \(|v|^2 \partial_x^\alpha f\), taking integrations over \( \mathbb{R}_x^3 \times \mathbb{R}_v^3 \times \mathbb{R}_t^+ \), and taking the summation over \( \alpha \), we obtain

\[
\sum_{1 \leq |\alpha| \leq 3} \left( \frac{d}{dt} \|v \partial_x^\alpha f(t)\|_{L_{v,I}^2(\mathbb{R}^3_v)}^2 + c_1 \| v \nabla_v \partial_x^\alpha f(t)\|_{L_{v,I}^2(\mathbb{R}^3_v)}^2 + c_2 \| |v|^2 \partial_x^\alpha f(t)\|_{L_{v,I}^2(\mathbb{R}^3_v)}^2 \right) + c_3 \| I \partial_t \partial_x^\alpha f(t)\|_{L_{v,I}^2(\mathbb{R}^3_v)}^2 + c_4 \| I^2 \partial_x^\alpha f(t)\|_{L_{v,I}^2(\mathbb{R}^3_v)}^2 \right) \\
\leq C \left( \| f(t)\|_{L_{v,I}^2(\mathbb{R}^3_v)}^2 + \| (I - P) f(t)\|_{L_{v,I}^2(\mathbb{R}^3_v)}^2 + N_1 A \right). \tag{86}
\]

Taking \( \partial_x^\alpha \partial_t \) to (9) with \( 1 \leq |\alpha| \leq 3 \), multiplying by \( I^2 \partial_x^\alpha f\), taking integrations over \( \mathbb{R}_x^3 \times \mathbb{R}_v^3 \times \mathbb{R}_t^+ \), and taking the summation over \( \alpha \), we obtain

\[
\sum_{1 \leq |\alpha| \leq 3} \left\{ \frac{d}{dt} \| I^2 \partial_t \partial_x^\alpha f(t)\|_{L_{v,I}^2(\mathbb{R}^3_v)}^2 + c_1 \| \partial_t (I \partial_x^\alpha f(t))\|_{L_{v,I}^2(\mathbb{R}^3_v)}^2 \right\} + c_2 \| I \partial_t \partial_x^\alpha f(t)\|_{L_{v,I}^2(\mathbb{R}^3_v)}^2 + c_3 \| I^2 \nabla_v \partial_t \partial_x^\alpha f(t)\|_{L_{v,I}^2(\mathbb{R}^3_v)}^2 + c_4 \| I^2 \partial_t \partial_x^\alpha f(t)\|_{L_{v,I}^2(\mathbb{R}^3_v)}^2 \right) \\
\leq C \left( \| f(t)\|_{L_{v,I}^2(\mathbb{R}^3_v)}^2 + \| (I - P) f(t)\|_{L_{v,I}^2(\mathbb{R}^3_v)}^2 + N_1 A \right). \tag{87}
\]

Taking \( \partial_x^\alpha \) to (9) with \( 1 \leq |\alpha| \leq 3 \), multiplying by \( I \partial_x^\alpha f\), taking integrations over \( \mathbb{R}_x^3 \times \mathbb{R}_v^3 \times \mathbb{R}_t^+ \), and taking the summation over \( \alpha \), we obtain

\[
\sum_{1 \leq |\alpha| \leq 3} \left\{ \frac{d}{dt} \| I^2 \partial_x^\alpha f(t)\|_{L_{v,I}^2(\mathbb{R}^3_v)}^2 + c_1 \| \partial_t I \partial_x^\alpha f(t)\|_{L_{v,I}^2(\mathbb{R}^3_v)}^2 + c_2 \| I^2 \partial_x^\alpha f(t)\|_{L_{v,I}^2(\mathbb{R}^3_v)}^2 \right\} + c_3 \| \nabla_v (I^2 \partial_t \partial_x^\alpha f(t))\|_{L_{v,I}^2(\mathbb{R}^3_v)}^2 + c_4 \| I^2 \partial_x^\alpha f(t)\|_{L_{v,I}^2(\mathbb{R}^3_v)}^2 \right) \\
\leq C \left( \| f(t)\|_{L_{v,I}^2(\mathbb{R}^3_v)}^2 + \| (I - P) f(t)\|_{L_{v,I}^2(\mathbb{R}^3_v)}^2 + N_1 A \right). \tag{88}
\]

When \( |\alpha| = 0 \), we are able to get inequalities similar completely to (85)-(88) except that there are some subtle differences produced in the process of enlarging the inequalities, which are as the case of the proof of Lemma 2.3.

By Proposition 3, multiplying (85)–(88) and the case of \( |\alpha| = 0 \) by a small constant \( \kappa > 0 \), adding them to (82), and using the assumption that \( N_1 \) is small enough, one can remove the terms that are not needed, and get

\[
\frac{d}{dt} \left( \| f(t)\|_{L_{v,I}^2(\mathbb{R}^3_v)}^2 + \| v \nabla_v f\|_{L_{v,I}^2(\mathbb{R}^3_v)}^2 + \| I^2 \partial_t f\|_{L_{v,I}^2(\mathbb{R}^3_v)}^2 \right) + \kappa (\| v \nabla_v f\|_{L_{v,I}^2(\mathbb{R}^3_v)}^2 + \| v f\|_{L_{v,I}^2(\mathbb{R}^3_v)}^2 + \| I^2 \partial_t f\|_{L_{v,I}^2(\mathbb{R}^3_v)}^2 ) \leq C \| f(t)\|_{L_{v,I}^2(\mathbb{R}^3_v)}^2, \tag{89}
\]

because \( \mathcal{E}(t) \sim \| f\|_{L_{v,I}^2(\mathbb{R}^3_v)}^2 \) in Proposition 3.

We define

\[
Q(t) = \sup_{0 \leq s \leq t} (1 + s)^{\frac{1}{2}} \| f(s)\|_{L_{v,I}^2(\mathbb{R}^3_v)}^2. \tag{90}
\]

By (90), we are able to obtain from (89) that

\[
\| v \nabla_v f\|_{L_{v,I}^2(\mathbb{R}^3_v)}^2 + \| v f\|_{L_{v,I}^2(\mathbb{R}^3_v)}^2 + \| I^2 \partial_t f\|_{L_{v,I}^2(\mathbb{R}^3_v)}^2 + \| I^2 \partial_x f\|_{L_{v,I}^2(\mathbb{R}^3_v)}^2 \\
\leq C(Q(t) + \| f_0\|_{L_{v,I}^2(\mathbb{R}^3_v)}^2 + \| v \nabla_v f_0\|_{L_{v,I}^2(\mathbb{R}^3_v)}^2 + \| v f_0\|_{L_{v,I}^2(\mathbb{R}^3_v)}^2 + \| I^2 \partial_t f_0\|_{L_{v,I}^2(\mathbb{R}^3_v)}^2 + \| I^2 \partial_x f_0\|_{L_{v,I}^2(\mathbb{R}^3_v)}^2 ). \tag{91}
\]
We have proved in Proposition 2 that
\[
\|f(t)\|_{L^2_{v,I}(H^+)}^2 \leq C \left( (1 + t)^{-\frac{1}{2}} \left( \|f_0\|_{L^2_{v,I}(H^+)} + \|f_0\|_{L^2_{v,I}(L^+)} \right) \right)^2 \\
+ \int_0^t (1 + t - \tau)^{-\frac{1}{2}} \left( \|G_1(t)\|_{L^2_{v,I}(H^+)} + \|G_2(t)\|_{L^2_{v,I}(H^+)} \right) \|G_1(\tau)\|_{L^2_{v,I}(L^+)} + \|G_2(\tau)\|_{L^2_{v,I}(L^+)}^2 d\tau \right). 
\]
(92)

By straightforward calculations from the definition of (12) and (76), we have
\[
\|G_1(t)\|_{L^2_{v,I}(H^+)}^2 + \|G_2(t)\|_{L^2_{v,I}(H^+)}^2 + \|G_1(t)\|_{L^2_{v,I}(L^+)}^2 + \|G_2(t)\|_{L^2_{v,I}(L^+)}^2 \\
\leq C \left( \|f(t)\|_{L^2_{v,I}(H^+)}^2 + \|f(t)\|_{L^2_{v,I}(H^+)}^2 \right) \|f(t)\|_{L^2_{v,I}(H^+)}^2 \\
+ \|\nabla_v f(t)\|_{L^2_{v,I}(H^+)}^2 + \|v f(t)\|_{L^2_{v,I}(H^+)}^2 + \|I^\frac{1}{2} \partial_t f\|_{L^2_{v,I}(H^+)}^2 + \|I^\frac{1}{2} f\|_{L^2_{v,I}(H^+)}^2, 
\]
(93)

Furthermore, by (90), (91), (93), we get
\[
\int_0^t (1 + t - \tau)^{-\frac{1}{2}} \left( \|G_1(t)\|_{L^2_{v,I}(H^+)} + \|G_2(t)\|_{L^2_{v,I}(H^+)} \right) \|G_1(\tau)\|_{L^2_{v,I}(L^+)} + \|G_2(\tau)\|_{L^2_{v,I}(L^+)}^2 d\tau \\
\leq CQ(t) (1 + t)^{-\frac{1}{2}} \left( \|f_0\|_{L^2_{v,I}(H^+)}^2 + \|\nabla_v f_0\|_{L^2_{v,I}(H^+)}^2 + \|v f_0\|_{L^2_{v,I}(H^+)}^2 + \|I^\frac{1}{2} \partial_t f_0\|_{L^2_{v,I}(H^+)}^2 \\
+ \|I^\frac{1}{2} f_0\|_{L^2_{v,I}(H^+)}^2 + \|f_0\|_{L^2_{v,I}(L^+)}^2 \right), 
\]

Finally, we obtain from (92) that
\[
(1 + t)^\frac{1}{2} \|f(t)\|_{L^2_{v,I}(H^+)}^2 \\
\leq C \left( Q(t)^2 + \|f_0\|_{L^2_{v,I}(H^+)}^2 + \|\nabla_v f_0\|_{L^2_{v,I}(H^+)}^2 + \|v f_0\|_{L^2_{v,I}(H^+)}^2 + \|I^\frac{1}{2} \partial_t f_0\|_{L^2_{v,I}(H^+)}^2 \\
+ \|I^\frac{1}{2} f_0\|_{L^2_{v,I}(H^+)}^2 + \|f_0\|_{L^2_{v,I}(L^+)}^2 \right), 
\]
which leads to
\[
Q(t) \leq C \left( Q(t)^2 + \|f_0\|_{L^2_{v,I}(H^+)}^2 + \|\nabla_v f_0\|_{L^2_{v,I}(H^+)}^2 + \|v f_0\|_{L^2_{v,I}(H^+)}^2 + \|I^\frac{1}{2} \partial_t f_0\|_{L^2_{v,I}(H^+)}^2 \\
+ \|I^\frac{1}{2} f_0\|_{L^2_{v,I}(H^+)}^2 + \|f_0\|_{L^2_{v,I}(L^+)}^2 \right). 
\]
(94)

Thus, if
\[
\|f_0\|_{L^2_{v,I}(H^+)}^2 + \|\nabla_v f_0\|_{L^2_{v,I}(H^+)}^2 + \|v f_0\|_{L^2_{v,I}(H^+)}^2 + \|I^\frac{1}{2} \partial_t f_0\|_{L^2_{v,I}(H^+)}^2 + \|I^\frac{1}{2} f_0\|_{L^2_{v,I}(H^+)}^2 \\
+ \|f_0\|_{L^2_{v,I}(L^+)}^2
\]
is small enough, we are able to obtain (83) from (94). The proof of the Proposition 4 is completed.

---

**Appendix A. Proof of (25).**

**Proof.** We now prove the inequality (25):
\[
\int_0^{+\infty} 2|\partial_t h|^2 \mu_I(I) dI + \left( \int_0^{+\infty} h \mu_I(I) dI \right)^2 \geq \int_0^{+\infty} |h|^2 \mu_I(I) dI.
\]

We divide the proof into two steps.
Firstly, we construct an orthonormal basis in $L^2_I(0, +\infty)$. Consider the following eigenvalue problem

$$\frac{1}{\sqrt{\mu_I}} \partial_I \left( 2I \mu_I \partial_I \left( \frac{f}{\sqrt{\mu_I}} \right) \right) = 2 \partial_I (I \partial_I f) + f - \frac{1}{2} If = \lambda f, \quad (95)$$

where $\mu_I(I) = \Lambda_I e^{-I}$.

Letting $f = h \sqrt{\mu_I}$, and substituting it into (95), we get

$$2I \partial^2_I h - 2I \partial_I h + \partial_I h = \lambda h. \quad (96)$$

In fact, the equation above is Laguerre equation.

Letting $h_n = a_0 + a_1 I + \ldots + a_n I^n (a_n \neq 0)$, and substituting it into (95), we have

$$-2na_n I^n = \lambda a_n I^n,$$

which gives rise to

$$\lambda = -2n, \quad (97)$$

and

$$2 \sum_{k=1}^{n-1} (k+1)ka_{k+1}I^k - 2 \sum_{k=1}^{n-1} ka_k I^k + 2 \sum_{k=1}^{n-1} (k+1)a_{k+1}I^k = -2n \sum_{k=1}^{n-1} ka_k I^k, \quad (98)$$

which gives rise to $a_{k+1} = -\frac{n-k}{(k+1)} a_k$, $k = 1, 2, \ldots, n-1$, and $2a_1 = -2na_0$.

From (98), we have the following Laguerre polynomials

$$h_n = a_n(I^n - \frac{n^2}{1} I^{n-1} + \frac{n^2}{2} (n-1)^2 I^{n-2} + \ldots + \frac{n^2(n-1)^2 \ldots (k)^2}{1 \cdot 2 \ldots (n-k)} I^k + \ldots), \quad (99)$$

such as

$$h_0(I) = 1, \quad h_1(I) = -I + 1, \quad h_2(I) = I^2 - 4I + 2, \quad h_3(I) = -I^3 + 9I^2 - 18I + 6, \quad h_4(I) = I^4 - 16I^3 + 72I^2 - 96I + 24, \ldots,$$

here, $a_n \neq 0$ is a constant to be determined. Letting Laguerre function $\phi_n = h_n \sqrt{\mu_I}$, we can choose suitably the value of $a_n$ such that

$$\int_0^{+\infty} \phi_n^2 dI = \int_0^{+\infty} h_n^2 \mu_I dI = 1.$$

Specially, we have $\phi_0 = \sqrt{\mu_I}$. Obviously, the set of Laguerre functions $\{\phi_n\}$ is orthonormal and complete in $L^2_I(0, +\infty)$ (cf. [1]).

Secondly, we prove (25). Taking any smooth enough function $f \in L^2_I(0, +\infty)$, and by definition of Fourier series and Parseval equality, we have

$$f = \sum_{n=0}^{+\infty} c_n \phi_n, \quad c_n = \int_0^{+\infty} f \phi_n dI, \quad (100)$$

and

$$\int_0^{+\infty} f^2 dI = \sum_{n=0}^{+\infty} c_n^2. \quad (101)$$

Obviously, we have

$$c_0 = \int_0^{+\infty} f \sqrt{\mu_I} dI \quad \text{(102).}$$
We next prove \( \{ I^\frac{1}{2} \sqrt{\mu_I} \partial_I h_n \} \) are orthogonal each other. Let \( \lambda_n, \phi_n \) are eigenvalue and corresponding eigenvector of (95), where \( \phi_n = h_n \sqrt{\mu_I} \). Substituting them into (95), we get

\[
\frac{1}{\sqrt{\mu_I}} \partial_I (2I \mu_I \partial_I h_n) = \lambda_n h_n \sqrt{\mu_I},
\]

hence

\[
2I^\frac{1}{2} \sqrt{\mu_I} \partial_I \left( \frac{1}{\mu_I} \partial_I (I \mu_I \partial_I h_n) \right) = \lambda_n I^\frac{1}{2} \sqrt{\mu_I} \partial_I h_n.
\]  \hspace{1cm} (103)

Taking \( n = n_1, n_2, (n_1 \neq n_2) \) in (103), we obtain

\[
2I^\frac{1}{2} \sqrt{\mu_I} \partial_I \left( \frac{1}{\mu_I} \partial_I (I \mu_I \partial_I h_{n_1}) \right) = \lambda_{n_1} I^\frac{1}{2} \sqrt{\mu_I} \partial_I h_{n_1} \hspace{1cm} (104)
\]

and

\[
2I^\frac{1}{2} \sqrt{\mu_I} \partial_I \left( \frac{1}{\mu_I} \partial_I (I \mu_I \partial_I h_{n_2}) \right) = \lambda_{n_2} I^\frac{1}{2} \sqrt{\mu_I} \partial_I h_{n_2} \hspace{1cm} (105)
\]

Multiplying (104) by \( I^\frac{1}{2} \sqrt{\mu_I} \partial_I h_{n_2} \), taking integration over \((0, +\infty)\) and integrating by parts, we have

\[
\lambda_{n_1} \int_0^{+\infty} I^\frac{1}{2} \sqrt{\mu_I} \partial_I h_{n_2} I^\frac{1}{2} \sqrt{\mu_I} \partial_I h_{n_1} dI = 2 \int_0^{+\infty} I^\frac{1}{2} \sqrt{\mu_I} \partial_I h_{n_2} \left( \frac{1}{\mu_I} \partial_I (I \mu_I \partial_I h_{n_1}) \right) dI
\]

\[
= -2 \int_0^{+\infty} \frac{1}{\mu_I} \partial_I (I \mu_I \partial_I h_{n_1}) \partial_I (I \mu_I \partial_I h_{n_2}) dI,
\]  \hspace{1cm} (106)

due to the fact that \( I \mu_I \partial_I h_{n_2} \) vanishes at \( I = 0, +\infty \). Similarly, we have also

\[
\lambda_{n_2} \int_0^{+\infty} I^\frac{1}{2} \sqrt{\mu_I} \partial_I h_{n_1} I^\frac{1}{2} \sqrt{\mu_I} \partial_I h_{n_2} dI
\]

\[
= -2 \int_0^{+\infty} \frac{1}{\mu_I} \partial_I (I \mu_I \partial_I h_{n_2}) \partial_I (I \mu_I \partial_I h_{n_1}) dI.
\]  \hspace{1cm} (107)

Then

\[
(\lambda_{n_1} - \lambda_{n_2}) \int_0^{+\infty} I^\frac{1}{2} \sqrt{\mu_I} \partial_I h_{n_2} I^\frac{1}{2} \sqrt{\mu_I} \partial_I h_{n_1} dI = 0,
\]

which gives rise to

\[
\int_0^{+\infty} I^\frac{1}{2} \sqrt{\mu_I} \partial_I h_{n_2} I^\frac{1}{2} \sqrt{\mu_I} \partial_I h_{n_1} dI = 0.
\]  \hspace{1cm} (108)

From (100), [28](cf. p54) and [23], we have

\[
\sqrt{2} I^\frac{1}{2} \sqrt{\mu_I} \partial_I \left( \frac{f}{\sqrt{\mu_I}} \right) = \sum_{n=1}^{+\infty} c_n \sqrt{2} I^\frac{1}{2} \sqrt{\mu_I} \partial_I h_n.
\]  \hspace{1cm} (109)
If \( \int_0^{+\infty} 2I_\mu \left( \partial_1 \left( \frac{f}{\sqrt{\mu I}} \right) \right)^2 dI = +\infty \), (25) holds. Otherwise using the orthogonal property of \( \{ I_1^2 \sqrt{\mu I} \partial_1 h_n \} \) and (109), we get

\[
\int_0^{+\infty} 2I_\mu \left( \partial_1 \left( \frac{f}{\sqrt{\mu I}} \right) \right)^2 dI = \left( \sum_{n=1}^{+\infty} c_n \sqrt{2I_1^2 \sqrt{\mu I}} \partial_1 h_n, \sum_{n=1}^{+\infty} c_n \sqrt{2I_1^2 \sqrt{\mu I}} \partial_1 h_n \right)_{L^2_2(0, +\infty)}
\]

\[
= \lim_{N \to +\infty} \left( \sum_{n=1}^{N} c_n \sqrt{2I_1^2 \sqrt{\mu I}} \partial_1 h_n, \sum_{n=1}^{N} c_n \sqrt{2I_1^2 \sqrt{\mu I}} \partial_1 h_n \right)_{L^2_2(0, +\infty)}
\]

\[
= \sum_{n=1}^{+\infty} c_n^2 2 \int_0^{+\infty} I_\mu \left( \partial_1 h_n \right)^2 dI.
\]

Multiplying (103) by \( -h_n \sqrt{\mu I} \), taking integration over \((0, +\infty)\) and using (97), we obtain

\[
2 \int_0^{+\infty} I_\mu \left( \partial_1 h_n \right)^2 dI = -\lambda_n = 2n \geq 1.
\]

(110), (111) implies

\[
\int_0^{+\infty} 2I_\mu \left( \partial_1 \left( \frac{f}{\sqrt{\mu I}} \right) \right)^2 \geq \sum_{n=1}^{+\infty} c_n^2.
\]

Combining (101), (102), (112) and noticing \( h = \frac{f}{\sqrt{\mu I}} \), (25) is proved.

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