Lucky 13-th Exercises on Stirling-like numbers and Dobinski-like formulas

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Summary

Extensions of the Stirling numbers of the second kind and Dobinski-like formulas are proposed in a series of exercises for graduates. Some of these new formulas recently discovered by me are to be found in A.K.Kwaśniewski’s source paper [1]. These extensions naturally encompass the well known $q$-extensions. The indicatory references are to point at a part of the vast domain of the foundations of computer science in ArXiv affiliation noted as CO.cs.DM.

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1. In the $q$-extensions realm

Ex.1 Recall and prove it again occasionally noting that The Number 44 is a magic number in Polish Poetry and this had had an implementation in quite recent Polish history in 1968 students revolutionary riots for independence and freedom of thinking under communist regime. Well, then forty four years ago Gian-Carlo Rota [4] proved that the exponential generating function for Bell numbers $B_n$ is of the form

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} B_n = \exp(e^x - 1)$$

using the linear functional $L$ such that

$$L(X^n) = 1, \quad n \geq 0$$

Then Bell numbers (see: formula (4) in [4]) are defined by

$$L(X^n) = B_n, \quad n \geq 0$$

The above formula is exactly the Dobinski formula [5] if $L$ is interpreted as the average value functional for the random variable $X$ with the Poisson distribution with $L(X) = 1$. Recall it and prove it again.

Ex.2 Recall and prove:

The two standard [12], see also [13-17] $q$-extensions Stirling numbers of the second kind are defined by

$$\sum_{k=0}^{n} \binom{n}{k} x_k^q = \sum_{k=0}^{n} \left\{ \binom{n}{k} \right\} x_k^q,$$
where \( x_q = \frac{1}{1-q} \) and \( x_q^k = x_q(x-1)_q\ldots(x-k+1)_q \), which corresponds to the \( \psi \) sequence choice in the \( q \)-Gauss form \( \langle \frac{1}{n!} \rangle_{n \geq 0} \) and \( q \)-Stirling numbers

\[
(5) \quad x^n = \sum_{k=0}^{n} \left\{ \begin{array}{c} n \\ k \end{array} \right\}_q \chi_k(x)
\]

where \( \chi_k(x) = x(x-1)_q\ldots(x-k+1)_q \)

Note that these formulae become the usual unextended Stirling numbers of the second kind formulae when the subscript \( q \) is removed.

**Ex.3 Recall and prove** it again:

For these two classical by now \( q \)-extensions of Stirling numbers of the second kind - the "\( q \)-standard" recurrences hold respectively:

\[
\left\{ \begin{array}{c} n+1 \\ k \end{array} \right\}_q = \sum_{l=0}^{n} \left( \begin{array}{c} n \\ l \end{array} \right)_q q^l \left\{ \begin{array}{c} l \\ k-1 \end{array} \right\}_q ; n \geq 0, k \geq 1,
\]

\[
\left\{ \begin{array}{c} n+1 \\ k \end{array} \right\}^\sim_q = \sum_{l=0}^{n} \left( \begin{array}{c} n \\ l \end{array} \right)_q q^{l-k+1} \left\{ \begin{array}{c} l \\ k-1 \end{array} \right\}_q^\sim ; n \geq 0, k \geq 1.
\]

Show that these formulae become the usual unextended Stirling numbers of the second kind formulae when the subscript \( q \) is removed and the number \( q \) is put equal to one.

**Ex.4 Recall and prove** it again:

From the above it follows immediately that corresponding \( q \)-extensions of \( B_n \) Bell numbers satisfy respective recurrences:

\[
B_q(n+1) = \sum_{l=0}^{n} \left( \begin{array}{c} n \\ l \end{array} \right)_q q^l B_q(l); n \geq 0,
\]

\[
B_q^\sim(n+1) = \sum_{l=0}^{n} \left( \begin{array}{c} n \\ l \end{array} \right)_q q^{l-k+1} \mathcal{B}_q(l); n \geq 0
\]

where

\[
\mathcal{B}_q(l) = \sum_{k=0}^{l} q^k \left\{ \begin{array}{c} l \\ k \end{array} \right\}^\sim_q .
\]

Show that these formulae become the usual unextended Stirling numbers of the second kind formulae when the subscript \( q \) is removed and the number \( q \) is put equal to one.

**Ex.5 Recall and prove** it again:

Recursions for both inversion \( q \)-Bell numbers and inversion \( q \)-Stirling numbers of the second kind are not difficult to be derived. Also in a natural way the inversion \( q \)-Stirling numbers of the second kind from [16] satisfy a \( q \)-analogue of the standard recursion for Stirling numbers of the second kind to be written via mnemonic adding "\( q \)" subscript to the binomial and second kind Stirling symbols in the the standard recursion formula i.e.

\[
\left\{ \begin{array}{c} n+1 \\ k \end{array} \right\}^{inv}_q = \sum_{l=0}^{n} \left( \begin{array}{c} n \\ l \end{array} \right)_q \left\{ \begin{array}{c} n-l \\ k-1 \end{array} \right\}^{inv}_q ; n \geq 0, k \geq 1.
\]

Another \( q \)-extended Stirling numbers much different from Carlitz "\( q \)-ones" were introduced in the reference [19], see [1,20,28].

The cigl-\( q \)-Stirling numbers of the second kind are expressed in terms of \( q \)-binomial coefficients and \( q = 1 \) Stirling numbers of the second kind [16,17], (see [1] for more references) as follows
\[
\left\{ \begin{array}{c}
\frac{n+1}{k} \\
\end{array} \right\}^{cigl}_{q} = \sum_{l=0}^{n} \left( \begin{array}{c}
\frac{n}{l} \\
\end{array} \right) q^{(n-l+1)} \left\{ \begin{array}{c}
\frac{n-l}{k-1} \\
(2k-2) \\
\end{array} \right\}^{cigl}_{q}; n \geq 0, k \geq 1.
\]

The corresponding \(cigl-q\)-Bell numbers recently have been equivalently defined via \(cigl-q\)-Dobinski formula [20,28] - which now in more adequate notation reads :

\[
L(X^q) = \overline{\mathbb{N}}_n(q), \quad n \geq 0, X^q \equiv X(X + q - 1) \ldots (X - 1 + q^{n-1}).
\]

The above \(cigl-q\)-Dobinski formula is interpreted as the average of this specific \(n\)-th \(cigl-q\)-power random variable \(X^q\) with the \(q = 1\) Poisson distribution such that \(L(X) = 1\).

For that to see use the identity by Cigler [19]

\[
x(x-1+q) \ldots (x-1+q^{n-1}) = \sum_{k=0}^{n} \left\{ \begin{array}{c}
\frac{n}{k} \\
\end{array} \right\}^{cigl}_{q} x^k.
\]

Note that these formulae become the usual unextended Stirling numbers of the second kind formulae when the subscript \(q\) is removed and the number \(q\) is put equal to one.

### 2. Beyond the \(q\)-extensions realm

The further consecutive umbral extension of Carlitz-Gould \(q\)-Stirling numbers \(\left\{ \begin{array}{c}
\frac{n}{k} \\
\end{array} \right\}_{q}\) and \(\left\{ \begin{array}{c}
\frac{n}{k} \\
\end{array} \right\}_{q}\) is realized two-fold way - one of which leads to a surprise (?) in contrary perhaps to the other way.

**The first** "easy way" consists in almost mnemonic sometimes replacement of \(q\) subscript by \(\psi\) after having realized that via equation (5) we are dealing with the specific case of Comtet numbers [1] i.e. now we have

\[
x^n = \sum_{k=0}^{n} \left\{ \begin{array}{c}
\frac{n}{k} \\
\end{array} \right\}^{\sim}_{\psi} \psi_k(x)
\]

where \(\psi_k(x) = x(x-1) \psi(x-2) \ldots (x-[k-1] \psi).\) Show that these formulae become the usual unextended Stirling numbers of the second kind formulae when the subscript \(\psi\) is removed.

As a consequence we have "for granted" the following:

**Ex.6 Recall standard and prove** its extension:

\[
\left\{ \begin{array}{c}
\frac{n+1}{k} \\
\end{array} \right\}^{\sim}_{\psi} = \left\{ \begin{array}{c}
\frac{n}{k-1} \\
\end{array} \right\}^{\sim}_{\psi} + \psi_k \left\{ \begin{array}{c}
\frac{n}{k} \\
\end{array} \right\}^{\sim}_{\psi}; n \geq 0, k \geq 1;
\]

where \(\left\{ \begin{array}{c}
\frac{n}{0} \\
\end{array} \right\}^{\sim}_{\psi} = \delta_{n,0}, \quad \left\{ \begin{array}{c}
\frac{n}{k} \\
\end{array} \right\}^{\sim}_{\psi} = 0; \quad k > n; \quad \text{and the recurrence for ordinary generating function reads}
\]

\[
G_{k,\psi}^\sim(x) = \frac{x}{1-k\psi} G_{k-1,\psi}^\sim(x), \quad k \geq 1
\]

where naturally

\[
G_{k,\psi}^\sim(x) = \sum_{n \geq 0} \left\{ \begin{array}{c}
\frac{n}{k} \\
\end{array} \right\}^{\sim}_{\psi} x^n, \quad k \geq 1
\]

from where one infers
(9) \[ G_{k\psi}^n(x) = \frac{x^k}{(1 - 1\psi x)(1 - 2\psi x)\ldots(1 - k\psi x)} \quad , \quad k \geq 0 \]

hence we arrive in the standard extended text-book way [21] at the following explicit formula

(10) \[ \left\{ \begin{array}{l} n \\ k \end{array} \right\}_\psi = \frac{r^n}{k!} \sum_{r=1}^{k} (-1)^{k-r} \binom{k}{r} \psi^r; \quad n, k \geq 0. \]

Show that these formulae become the usual unextended Stirling numbers of the second kind formulae when the subscript \(\psi\) is removed.

Expanding the right hand side of the corresponding equation above results in another explicit formula for these \(\psi\)-case Comtet numbers [1] i.e. we have

\[ \text{Ex.7 Recall standard and prove its extension:} \]

(11) \[ \left\{ \begin{array}{l} n \\ k \end{array} \right\}_\psi = \sum_{1 \leq i_1 \leq i_2 \leq \ldots \leq i_{n-k} \leq k} (i_1)_\psi (i_2)_\psi \ldots (i_{n-k})_\psi; \quad n, k \geq 0. \]

or equivalently (compare with [13], see [12,14,15])

(12) \[ \left\{ \begin{array}{l} n \\ k \end{array} \right\}_\psi = \sum_{d_1 + d_2 + \ldots + d_k = n-k, \quad d_i \geq 0} 1^{d_1} 2^{d_2} \ldots k^{d_k}; \quad n, k \geq 0. \]

\(\psi\)-Stirling numbers of the second kind being defined equivalently by (10) , (14), (15) or (16) yield \(\psi\)-Bell numbers

\[ B_n\psi(\psi) = \sum_{k=0}^{n} \left\{ \begin{array}{l} n \\ k \end{array} \right\}_\psi, \quad n \geq 0. \]

Show that these formulae become the usual unextended Stirling numbers of the second kind formulae when the subscript \(\psi\) is removed.

\[ \text{Ex.8 Recall standard and prove its extension:} \]

Adapting the standard text-book method [21] we have for two variable ordinary generating function for \(\left\{ \begin{array}{l} n \\ k \end{array} \right\}_\psi\) Stirling numbers of the second kind and the \(\psi\)-exponential generating function for \(B_n\psi(\psi)\) Bell numbers the following formulæ

(13) \[ C_{n\psi}(x, y) = \sum_{n \geq 0} A_n\psi(\psi, y)x^n, \]

where the \(\psi\)-exponential-like polynomials \(A_n\psi(\psi, y)\)

\[ A_n\psi(\psi, y) = \sum_{k=0}^{n} \left\{ \begin{array}{l} n \\ k \end{array} \right\}_\psi y^k \]

do satisfy the recurrence

\[ A_n\psi(\psi, y) = [y(1 + \psi)A_{n-1}\psi(\psi, y) \quad n \geq 1, \]

hence \[ A_n\psi(\psi, y) = [y(1 + \psi)^n 1, \quad n \geq 0, \]

where the linear operator \(\psi\) acting on the algebra of formal power series is being called (see: [1,2,3,24,25,31]) the "\(\psi\)-derivative" and \(\psi y^n = n\psi y^{n-1}. \)

Show that these formulæ become the usual unextended Stirling numbers of the second kind formulæ when the subscript \(\psi\) is removed.

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Ex.9 Recall standard and prove its extension:
The ψ-exponential generating function $B_\sim^n(x) = \sum_{n \geq 0} B_\sim^n(\psi) \frac{x^n}{n!}$ for $B_\sim^n(\psi)$ Bell numbers - after cautious adaptation of the method from the Wilf’s generatingfunctionology book [21] can be seen to be given by the following formula

\begin{equation}
B_\sim^n(\psi) = \sum_{r \geq 0} \frac{1}{\epsilon(\psi, r)} \frac{e_\psi[r_\psi x]}{r_\psi!}
\end{equation}

where (see: [1,2,3,24,25,31])

\begin{equation}
e_\psi(x) = \sum_{n \geq 0} x^n \frac{\psi^n}{n!}
\end{equation}

while

\begin{equation}
\epsilon(\psi, r) = \sum_{k=r}^{\infty} \frac{(k-r)!}{(k_\psi - r)!}
\end{equation}

and for the ψ-extensions the Dobinski like formula here now reads

\begin{equation}
B_\sim^n(\psi) = \sum_{r \geq 0} \frac{1}{\epsilon(\psi, r)} \frac{r_\psi^n}{r_\psi!}.
\end{equation}

Show that these formulae become the usual unextended Stirling numbers of the second kind formulae when the subscript ψ is removed.

Ex.10 Recall standard and prove its extension:
In the case of Gauss $q$-extended choice of $\langle n_q^{k}\rangle_{n \geq 0}$ admissible sequence of extended umbral operator calculus equations (19) and (20) take the form

\begin{equation}
\epsilon(q, r) = \sum_{k=r}^{\infty} \frac{(-1)^{k-r}}{(k-r)_q!} q^{-\binom{k}{2}}
\end{equation}

and the $q$-Dobinski formula is given by

\begin{equation}
B_\sim^n(q) = \sum_{r \geq 0} \frac{1}{\epsilon(q, r)} \frac{r_q^n}{r_q!},
\end{equation}

which for $q = 1$ becomes the Dobinski formula from 1887 [5].

Ex.11 Recall standard and prove its extension:
In a dual inverse way we define the $\psi$- Stirling numbers of the first kind as coefficients in the following expansion

\begin{equation}
\psi_\sim^k(x) = \sum_{r=0}^{k} \left[ \begin{array}{c} k \\ r \end{array} \right]_\sim x^r
\end{equation}

where - recall $\psi_\sim^k(x) = x(x-1_\psi)(x-2_\psi)...(x-[k-1]_\psi)$; (attention: see equations (10)-(16) in [7,8] and note the difference with the present definition).

Therefore from the above we infer that

\begin{equation}
\sum_{r=0}^{k} \left[ \begin{array}{c} k \\ r \end{array} \right]_\sim \left[ \begin{array}{c} r \\ l \end{array} \right]_\sim = \delta_{k,l}.
\end{equation}

Show that these formulae become the usual unextended Stirling numbers of the second kind formulae when the subscript ψ is removed.

Ex.12 Recall standard and prove its extension:
Consider now another Stirling-like numbers (as expected Whitney numbers [31,1,32]) which are natural counterpart to $\psi$- Stirling numbers of the second kind. These are $\psi$- Stirling numbers of the first kind defined here down as coefficients in the following expansion (upperscript "e" is used because of cycles in non-extended case).
\( \psi(x) = \sum_{r=0}^{k} \binom{k}{r} x^r \)

where - now \( \psi(x) = x(x+1)\psi(x+2)\psi(x+3)\psi(...(x+[k-1]\psi); 

Show that these formulae become the usual unextended Stirling numbers of the second kind formulae when the subscript \( \psi \) is removed.

**Ex.13** Recall standard and prove its extension:

Show that the definition here upstairs above of the \( q \)-Stirling numbers of the second kind \( \binom{n}{k}_q \) is equivalent with the definition by recursion

\[
\binom{n+1}{k}_q = q^{k-1} \binom{n}{k-1}_q + k q^{n-1}_k q_k \quad ; \quad n \geq 0, k \geq 1;
\]

where \( \binom{a}{0}_q = \delta_{a,0}, \binom{a}{k}_q = 0, \quad k > n \)

Show that these in turn (just use the \( Q \)-Leibnitz rule \([2,3,24,25,31]\) for Jackson derivative \( \partial_q \)) are equivalent to

\[
(\hat{x} \partial_q)^n = \sum_{k=0}^{n} \binom{n}{k}_q \hat{x}^k \partial_q^k
\]

where \( \binom{a}{0}_q = \delta_{a,0}, \binom{a}{k}_q = 0, \quad k > n \).

Here \( \hat{x} \) denotes the multiplication by the argument of a function.

Show that these formulae become the usual unextended Stirling numbers of the second kind formulae when the subscript \( q \) is removed.

Consult \([33-35]\) for some new open problems arising in related the domain of the so called cobweb posets and their acyclic digraphs representatives which had served the present author to discover a joint combinatorial interpretation for all \( F \)-nomial coefficients. These family encompasses binomial and \( q \)-Gaussian coefficient, Fibonomial coefficient, Stirling numbers of both kinds and all classical \( F \)-nomial coefficients hence specifically incidence coefficients of reduced incidence algebras of full binomial type and Whitney numbers are given the joint cobweb combinatorial interpretation also \([36,37]\).

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