The Cavendish Experiment in General Relativity

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Abstract
Solutions of Einstein’s equations are discussed in which the “gravitational force” is balanced by an electrical force, and which can serve as models for the Cavendish experiment.

1 Introduction

One of many useful lessons one can learn from Engelbert is the appreciation of simple situations and examples that nonetheless can teach us valuable physics. For me, such an Engelbert lesson was an introduction to the Bertotti-Robinson universe (which, as Engelbert added with characteristic precision that extends also to the history of physics, was first discovered by Levi-Civita), and its relation to extremal solutions [1]. Below is a bit of physics that we can learn from extremal solutions to Einstein’s equations.

In General Relativity there is a well-defined sense in which the equations of motion for particles follow from the field equations. This is well known but not easily checked out \(^1\) for the necessary manipulations are rather formidable. It has been remarked \(^2\) that when predictions of General Relativity are based on particle equations of motion they appear to lack the transparency and cogency that we appreciate in Newtonian physics and in some alternative theories; that even the outcome of the Cavendish experiment has not been derived in a way that is both simple and rigorous; and that, at least in the case of two-dimensional (“planar”) translational symmetry there exist “anti-Cavendish” solutions of Einstein’s field equations, describing slabs that do not attract each other. (In these solutions there are no other interactions than gravity between the slabs, but the stress-energy of the matter is “exotic.”)

\(^*\)Contribution to Festschrift volume for Engelbert Schucking, to be published by Springer Verlag

\(^1\)I really mean nachvollziehbar, a fashionable German word that seems to have no good English equivalent.
In the present contribution we examine a question suggested by these con-
siderations\textsuperscript{2}, namely whether there are simple models in General Relativity that are relevant to the Cavendish experiment. We will construct one such model that is easily analyzed and whose predictions agree with the expected experimental outcome. These models are not confined to the plane symmetric case for which they were first discussed, and they have no connection with the “exotic” slab solutions. Nevertheless we begin with a few elementary remarks about the special status of planar symmetry in General Relativity as compared to other field theories.

\section{Planar symmetry}

In electrostatics, problems with planar symmetry (such as two parallel charged plates) are among the simplest to treat. The translational and rotational symmetry of the physical setup prevents dependence of physical quantities on the transverse \((y, z)\) directions. The problem therefore becomes one-dimensional. Physically one cannot, of course, realize strict translational symmetry, because the system’s total charge and mass would be infinite. However, one can approximate the one-dimensional situation by systems whose properties are independent of \(y\) and \(z\) out to some large distance \(D\), when one considers only longitudinal distances \(x\) small compared to \(D\). In the limit \(D \to \infty\) the one-dimensional approximation becomes arbitrarily accurate, and reasonable physical quantities have finite limits. These include the electric field, the force per area, and the acceleration of the plates. (However, when the total charge on the plates is non-zero, the electrostatic potential does not have a finite limit, if it is normalized to zero at infinity.)

Another simple feature of translationally symmetric electrostatics is the uniqueness of the relative acceleration between the plates. (The electric field is unique up to an additive constant.) If one has any solution with the appropriate symmetry, it is the correct solution\textsuperscript{3}. This simplicity makes the (approximately) parallel plate geometry so useful in both pedagogy and practice.

In Newtonian gravitation the situation is essentially identical; everything one knows about electrostatics can be taken over (with the appropriate sign of the force), except that there is no arbitrary charge to mass ratio — the (strong) principle of equivalence fixes the ratio of gravitational to inertial mass to be a positive constant. One might expect that the simplicity of the parallel plate

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\textsuperscript{2}I thank Prof. C. Alley (University of Maryland) for numerous discussions which called attention to the status of the Cavendish experiment \textit{vis a vis} General Relativity, and in which he supplied the experimental ideas mentioned in passing below.

\textsuperscript{3}This is true provided the plates are indeed static. In a typical experiment one balances the electric force between plates by an elastic force, and measures how much elastic force is needed to keep the plates static. If plates of finite size and non-negligible charge were allowed to accelerate, they would of course radiate. The radiation reaction would affect the plates’ net acceleration, and this would depend on radiation conditions imposed at infinity.
geometry will also carry over in General Relativity.

There are of course important differences between these theories, which can destroy the analogy. One relevant difference that is usually cited is the role of the potential. In electrostatics and in Newtonian gravity the potential has no direct physical meaning, separate from the fields. In general relativity the analogous quantity is the metric, and it measures directly the physically meaningful space-time distances. When the size $D$ of the system increases (with constant mass density), the Newtonian potential, normalized to zero at infinity, typically diverges. This is not a serious problem in electrostatics or in Newtonian gravity, because another normalization can be chosen with impunity. However, a diverging metric offers more serious problems, and is certainly not allowed in an asymptotically flat spacetime. On the other hand, it is not obvious that this divergence cannot be undone in the limit by suitable gauge changes; and in any case one can take the view that if translationally symmetric solutions exist, they should (approximately) describe physically realistic parallel plates, since the General Relativity solutions for finite plates presumably exist.

Static solutions with plane symmetry have in fact been studied in general relativity, for example by Taub\textsuperscript{[3]} for matter with a fluid equation of state. Unfortunately they do not readily lend themselves to physical interpretation of the type sought here. (One can however show on the basis of this work that, as expected, no solution exists with vanishing pressure and positive mass density.) Also, the relation of these solutions to any description of finite parallel plates with proper (asymptotically flat) behavior at infinity is not transparent.

3 Exact, static solutions

Static solutions would not seem to present a very versatile arena for exploring the features of the gravitational interaction. They do however correspond to a possible physical arrangement that would reasonably be used in a sensitive experiment to measure the strength of the gravitational interaction ($G$). In the usual Cavendish experiment\textsuperscript{[4]} even if initially the proof mass is in free fall, the long-time behavior is typically governed by an interplay of gravitational interaction and torsion fiber reaction. The initial acceleration can generally not be measured as accurately as the final displacement, which one can model as masses with constant separation — in other words, a static situation.

\textsuperscript{4}Prof. Alley (private communication) points out that this could be modified to realize the plane-symmetric geometry by replacing the usual masses with parallel plates, one of which is suspended (for example, by means of the traditional torsion fiber) so that the total force on it can be monitored. This geometry has several advantages, for example that the distance between the plates does not have to be known with great accuracy, and that many of the devices used in a parallel plate electrostatic measurement to increase the accuracy, such as “guard rings” to make the field more uniform, could be adapted to the gravitational experiment. However, I do not know of any attempt to obtain a more accurate measurement of $G$ in this way.
The gravitational interaction is then measured by the force necessary to keep the masses apart, and the basic nature of this force is electrical. (One could replace the force of the torsion fiber by the explicitly electrical force obtained, for example in the parallel plate version of the experiment, by putting equal charges on the plates.) We model this force by assuming that each volume also carries a net charge, proportional to the mass of that volume, and all of the same sign. For a suitable choice of the constant charge/mass ratio, the attractive gravitational and repulsive electrical forces will then balance in the Newtonian description.

How do we describe this situation in General Relativity? Because mass and charge is present we must solve the Einstein and the Maxwell equations, as well as the equations of motion of the matter. The source in the Einstein equations is the stress-energy of the electric field and that of the matter; the source in the Maxwell equations is the charge density of the matter; and these equations imply the matter equation of motion, at least for the simplest kind of matter, “charged dust”\textsuperscript{5} If it is indeed possible to balance the forces in detail — an expectation discouraged by the nonlinear nature of Einstein gravitation, but encouraged by the absence of interaction energies in the the corresponding Newtonian situation — then there should be a static solution. It is a remarkable theorem\textsuperscript{6} that such solutions not only exist, but that the fields have a unique form under these conditions, the Majumdar-Papapetrou form\textsuperscript{7} that is well-known when gravity is generated not by matter but only by charged black holes (and the stress-energy of their electric field). In the latter case the geometry and field can represent any static arrangement of a finite number of black holes with an \textit{extremal} charge.

The Majumdar-Papapetrou ansatz for the metric and field can be written as

\[ ds^2 = -V^{-2}dt^2 + V^2(dx^2 + dy^2 + dz^2) \]
\[ A^\mu = V \delta^\mu_t \]  \hspace{1cm} (1)

By explicit computation\textsuperscript{8} of the Einstein tensor \( G_{\alpha\beta} \) and the Maxwell stress-energy tensor \( T^\text{EM}_{\alpha\beta} \) one finds agreement of most of the components of the two, for example

\[ G_{xx} = V^{-2} \left( -V_x^2 + V_y^2 + V_z^2 \right) = T^\text{EM}_{xx} \]
\[ G_{xy} = -2V_x V_y = T^\text{EM}_{xy} \]

\[ \text{etc.} \]

\textsuperscript{5}The static sources in these solutions are unstressed, due to the detailed balance between electric and gravitational forces. So we can imagine that these are elastic bodies rather than dust, but with vanishing stress and strain. The stress-energy tensor is then the same as that of dust, and the solution still applies. It is clear that in this model the stress-energy tensor satisfies all energy conditions one might reasonably want to impose.

\textsuperscript{6}I am grateful to the group of Prof. C. Alley for providing me with many of the results cited below, as obtained by their computer calculations. The conclusions drawn from these calculations are my own and have not been fully discussed with Prof. Alley.
The exception is \((\alpha, \beta) = (t, t)\). Similarly one finds that the \(A^\mu\) of Eq (1.1) satisfy most of the components of the vacuum Maxwell equations,

\[
F_{\alpha,\beta}^\beta = 0 \quad \text{except for} \quad \alpha = t.
\]

This structure of the field equations is appropriate for a static dust source, since that type of source contributes only to the components that are excepted above. For these one has the condition (in units where the gravitational and electromagnetic coupling is unity)

\[
\begin{align*}
G_{tt} - T_{tt}^{\text{EM}} &= -2V^{-5}\nabla^2 V = T_{tt}^{\text{matter}} \\
F_{t}^\beta &_{;\beta} = -V^{-4}\nabla^2 V = J_{t}^{\text{matter}}.
\end{align*}
\] (2)

Here the Laplacian \(\nabla^2\) is to be evaluated in the flat three-dimensional background metric \(dx^2 + dy^2 + dz^2\).

All the equations are satisfied if charged dust can supply both sources in the equations (1.2). For this type of matter, with mass density \(\rho\) and charge density \(\sigma\), we have

\[
\begin{align*}
T_{\alpha\beta}^{\text{matter}} &= \rho u_\alpha u_\beta \\
J_{\alpha}^{\text{matter}} &= \sigma u_\alpha.
\end{align*}
\]

From the metric (1.1) we find that the unit four-velocity for the static matter has the form \(u_\alpha = V^{-1}\delta^t_\alpha\). Thus we see that equations (1.2) are satisfied if we choose

\[
\rho = \sigma = V^{-3}\nabla^2 V.
\] (3)

The equation relating the “potential” \(V\) and the source \(\rho\) is very similar to the Newtonian equation; in the vacuum region they are identical. One way to make a correspondence between the two is the following: Given any Newtonian potential \(V_N\) (vanishing at infinity) and source \(\rho_N\) one finds a solution of equation (1.3) by

\[
V = 1 + V_N \quad \rho = \rho_N/V^3.
\]

Thus \(\rho\) has the same support as \(\rho_N\), and the two differ only slightly if the gravitational fields are weak, \(|V_N| \ll 1\).

This solution to the equations of general relativity has all the physically reasonable properties one expects; but could there be other solutions to the same problem with different properties? Suppose any static solution to the Einstein-Maxwell equations is given. For simplicity, confine attention to the region outside the matter. Let the electrostatic potential \(A^t\) of the solution be \(V\). Let \(g_{ij}\) be the spacelike metric on the three-dimensional hypersurfaces that are orthogonal to the timelike Killing vector. One can then show that when one modifies the metric by a conformal factor \(V^{-2}\), its Ricci tensor vanishes, \(R_{ij}[V^{-2}g_{ij}] = 0\). The three-dimensional modified metric must therefore be flat, and hence the original metric and field must have the form of Eq (1.1). In this sense, then, the solutions given above are unique.
4 Test particle motion

It is not necessary to verify separately that the equations of motion for the matter are satisfied by the solution given by equations (1.1, 1.3), because the matter motion is a consequence of the field equations. However, as a further check that this solution is reasonable, we derive the equation of motion for a test particle in the fields of this solution.

As in Newtonian physics, the general relativistic motion of test particles in the general metric (1.1) (even with $V$ harmonic) is not integrable, but there is always an energy integral. The energy integral is enough to find the motion if we know that it is confined to one coordinate line. This is the case for example when there is planar ($x, y$) or axial (about the $z$-axis) symmetry. We therefore confine attention to these cases where the energy integral yields the essential information about the motion.

In the case of uncharged particles we can apply the usual theorems about geodesics, that any Killing vector like $\partial/\partial t$ yields a conservation law of the corresponding covariant component of the 4-velocity $u$, $u_t = -E$. From the metric (1.1) we therefore have, with $\tau =$ proper time

$$ E = -g_{tt} \frac{dt}{d\tau} = \frac{1}{V^2} \frac{dt}{d\tau}. $$

We also know that setting the length of $u$ to unity is always an integral of the equation of motion,

$$ u \cdot u = -1 = -\frac{1}{V^2} \left( \frac{dt}{d\tau} \right)^2 + V^2 \left( \frac{dx}{d\tau} \right)^2. \quad (4) $$

Elimination of $dt/d\tau$ yields

$$ \left( \frac{dx}{d\tau} \right)^2 + \frac{1}{V^2} = E^2. \quad (5) $$

So for geodesic motion the quantity $1/V^2$ acts as an effective potential, and the particle will be deflected from the Killing orbit $(x, y, z) = \text{const}$ by an amount proportional to $\nabla V$, as in the Newtonian description.

For weak fields we have $V \approx 1 - \Phi$ where $\Phi$ is the Newtonian potential (for a spherically symmetric mass $M$, $\Phi = -M/r$), hence

$$ \frac{1}{2} \left( \frac{dx}{d\tau} \right)^2 + \Phi \approx \frac{1}{2}(E^2 - 1), $$

which is (essentially) the usual energy integral for potential motion, yielding the usual motion corresponding to attractive gravity.
When the particle is charged (with charge/mass $q$) the corresponding conservation laws are derived from the variational principle

$$\delta \int (u \cdot u + 2qu \cdot A) d\tau = 0.$$  

If there is a Killing vector that leaves the metric and the potential $A$ invariant, there is a conserved quantity (Noether’s theorem); for the Killing vector $\partial/\partial t$ the conserved quantity is the momentum conjugate to $t$, $-E = ut + qAt$, or

$$E = \frac{1}{V^2} \frac{dt}{d\tau} - \frac{q}{V}.$$  

We substitute this in (1.4), eliminate $dt/d\tau$, and find

$$\left(\frac{dx}{d\tau}\right)^2 + 1 - \frac{q^2}{V^2} + \frac{2Eq}{V} = E^2.$$  

We see that the attractive gravitational potential of the comparable equation (1.5) is reduced, and becomes repulsive for $q^2 > 1$. There is also a contribution to the potential that is proportional to $E$.

In the special (“extremal”) case $q^2 = 1$ and $E = 0$, an initially static test particle remains at rest at any position (as does the matter that produces these fields). This is the case where attractive gravity and repulsive electrostatics are in perfect balance. If the particle is initially not quite at rest, then $Eq$ must be slightly negative (since $V > 0$), hence the term $2Eq/V$ represents an attractive potential. It may be interpreted as a response of the particle’s increased “relativistic mass” to gravity, with no compensating increase in the particle’s charge.

## 5 Conclusion

We have exhibited the unique static solutions to the Einstein-Maxwell-matter field equations that represent an arbitrary distribution of extremally charged matter in the form of dust. In particular, these solutions can be a good approximation to the geometry of a Cavendish experiment. Because all the charges have the same sign, the electric interaction is repulsive between two volumes of matter. The constancy in time of the physical distance between the masses implies, in ordinary language, a balancing attractive gravitational interaction. In this sense we have shown that in general relativity, as in Newtonian gravity, the gravitational interaction between the bodies is nonzero and attractive. Because the solution is valid only when the charge has the extremal value, such

$^7$In this context the term “extremal” is somewhat misleading: it is the maximum charge that a black hole of the given mass could have, but is not a large charge at all for that mass of ordinary matter to carry.
a balancing Cavendish experiment could be used to find the extremal charge value for a given mass.\footnote{Results from an electrically balancing Cavendish experiment have recently been reported \cite{6}; however, that experiment did not measure the extremal charge value because (for good reasons) the attracting “large mass” was not the same as the repelling electrode.} Measurement of this extremal charge to mass ratio is equivalent to measuring the gravitational constant $G$.

References

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