SPECTRAL CORRESPONDENCES FOR AFFINE HECKE ALGEBRAS

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Abstract. We introduce the notion of spectral transfer morphisms between normalized affine Hecke algebras, and show that such morphisms induce spectral measure preserving correspondences on the level of the tempered spectra of the affine Hecke algebras involved. We define a partial ordering on the set of isomorphism classes of normalized affine Hecke algebras, which plays an important role for the Langlands parameters of Lusztig’s unipotent representations.

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1. Introduction

In this paper we introduce the notion of "spectral transfer morphisms" between normalized affine Hecke algebras. We prove that such a spectral transfer morphism $\phi : \mathcal{H}_1 \sim \mathcal{H}_2$ induces a finite morphism $\phi_Z$ from the affine variety defined by the center $Z_1$ of $\mathcal{H}_1$ to the affine variety defined by the center $Z_2$ of $\mathcal{H}_2$. Using this map we will construct a correspondence between the tempered spectra of the affine Hecke algebras $\mathcal{H}_1$ and $\mathcal{H}_2$ which respects the connected components in the tempered spectra of $\mathcal{H}_1$ and $\mathcal{H}_2$ and which respects the spectral measures up to rational constants and finite maps. By this we mean the following: If $\mathcal{G}_1$ is a connected component of the tempered spectrum of $\mathcal{H}_1$, then there exists a connected
component $\mathcal{S}_2$ of the tempered spectrum of $\mathcal{H}_2$, a connected space $\mathcal{S}_{12}$ with finite surjective continuous maps $p_1 : \mathcal{S}_{12} \to \mathcal{S}_1$ and $p_2 : \mathcal{S}_{12} \to \mathcal{S}_2$, and a positive measure $\nu_{12}$ on $\mathcal{S}_{12}$ whose push forward to $\mathcal{S}_i$ yields the spectral measure of $\mathcal{H}_i$ restricted to $\mathcal{S}_i$, up to a rational constant factor depending on $\mathcal{S}_1$ and $\mathcal{S}_2$.

In [O4] we apply this notion to unipotent affine Hecke algebras, i.e. to the normalized affine Hecke algebras associated to the unipotent types of inner forms of a given absolutely simple quasi split linear algebraic group $G$ defined over a non-archimedian local field, all normalized in an appropriate way. We will prove in that paper that all unipotent normalized affine Hecke algebras admit an essentially unique spectral transfer morphism to the Iwahori Hecke algebra of $G$, and that these maps are exactly the same as Lusztig's geometric-arithmetic correspondences [Lu2], [Lu3]. One could interpret this remarkable fact as an amplification of Mark Reeder's work on formal degrees of unipotent discrete series representations and L-packets for split exceptional groups [R2] and small rank classical groups [R1].

Thus Lusztig's geometric-arithmetic correspondences can be recovered from the perspective of harmonic analysis alone, and this is in itself a motivation for studying this notion of spectral transfer morphisms. As an application we will give a proof in [O4] of a conjecture of K. Hiraga, A. Ichino and T. Ikeda [HII, Conjecture 1.4] on the formal degree of a discrete series representation $\pi$ of $G(k)$, where $G$ is a connected reductive group over a local field $k$, for unipotent discrete series representations. This conjecture expresses the formal degree in terms of the adjoint gamma factor of the conjectural Langlands parameters of $\pi$.

It is well known [Mo1], [Mo2], [Lu2], [Lu3] that, if $k$ is non-archimedean, the unipotent Bernstein components of the category of smooth representations of $G(k)$ are Morita equivalent to the module category of an affine Hecke algebra, in general with unequal parameters. These Morita equivalences are realized by a type in the sense of [BK], and in general, such Morita equivalences are known to respect the notion of temperedness and are Plancherel measure preserving [BHK]. For more general Bernstein components one still expects variations of such descriptions to hold (see e.g. [Yu]). An alternative approach to such results, suggested by J. Bernstein and established for classical groups by Heiermann [Hei], is to directly establish an isomorphism between the opposite endomorphism ring of a progenerator $\pi_\mathcal{O}$ of a Bernstein component $\mathcal{O}$ and an affine Hecke algebra (or a small variations thereof). In the latter situation there is not yet a complete understanding of the correspondences on the level of harmonic analysis. In both approaches, the corresponding Bernstein variety is equipped with a rational function which has its origin in harmonic analysis: the $\mu$-function. By the work of Heiermann [Hei1], [Hei2], it is known that the central support of the set of tempered representation in a Bernstein component can be completely described in terms of this $\mu$-function in analogy to the results of [O2], [O3] for affine Hecke algebras.

In recent years, from various perspectives, the representation theory and harmonic analysis of unequal parameter affine Hecke algebras has seen progress [HO2], [O2], [OS2], [Sl2], [CK], [CKK], [Lu3], [COT], [CiuOpd1], [CiuOpd2]. The spectral transfer morphisms introduced in the present paper are defined in entirely terms of the $\mu$-functions of the affine Hecke algebras, using the properties of its poles [O3]. It is therefore likely that spectral transfer morphism can also be defined for more general Bernstein components than the unipotent ones, without reference to an affine Hecke algebra. In any case, in view of the natural role of (unequal parameter) affine Hecke
algebras in the representation theory of reductive $p$-adic groups sketched above, it is a natural quest to understand how their irreducible spectra are combined into stable packets. Using the notion of spectral transfer morphisms we will show in this paper (together with [O4] and [FO]) that if such partitioning exists for tempered unipotent representations, then it is essentially uniquely determined by certain natural algebraic relations between the $\mu$-functions of the unipotent affine Hecke algebras, and that this yields the same partitioning as provided by Lusztig [Lu2], [Lu3].

We will introduce a natural partial ordering between the spectral isogeny classes of normalized affine Hecke algebras with respect to spectral transfer maps. For the normalized affine Hecke algebras which appear in the study of Lusztig’s unipotent representations, we will in fact see that the spectral isogeny classes in this sense are the same as the isomorphism classes. This ordering gives a special role to the Iwahori Hecke algebra $H(G)_{IM}$ of an unramified quasi-split semisimple group $G$ (defined over a nonarchimedian local field): Among the isomorphism classes of the unipotent affine Hecke algebras of inner forms of $G$, (the isomorphism class of) $H(G)_{IM}$ is the least element. In view of the remarks above, one may expect that this ordering plays a role more generally for components of the Bernstein variety of $G$.

The aim of the present paper is the study of the Plancherel measure preserving correspondences associated to a spectral transfer morphism, using our knowledge [O2], [OS2], [O3] of the properties of the $\mu$-function and its relations to the Plancherel measure.

2. THE PLANCHEREL MEASURE OF A NORMALIZED AFFINE HECKE ALGEBRA

2.1. Affine Hecke algebras. We shall mainly follow the conventions of [OS2]. However, we will restrict ourselves in the present paper to affine Hecke algebras whose parameters are unequal but which are all integral powers of a fixed invertible indeterminate $v$. Therefore certain notions and definitions of [OS2] require some small adaptations (e.g. the notion of the ”spectral diagram” of the Hecke algebra).

2.1.1. Root data and affine Weyl groups. Recall the notion of a based, semisimple root datum $R = (X, R_0, Y, R_0^\vee, F_0)$, consisting of a semisimple root datum with a given basis $F_0 \subset R_0$ of simple roots. Let $W_0 = W(R_0)$ be the Weyl group of the root system $R_0 \subset X$, acting on the lattice $X$, and let $W = W(R) := X \times W_0$ denote the associated extended affine Weyl group. We equip the real vector space $V^* := X \otimes \mathbb{R}$ with a Euclidean structure which is $W_0$-invariant.

2.1.2. Affine root system and the affine Coxeter group $W^a$. The set $R = R_0^\vee + \mathbb{Z} \subset Y + Z$ is an affine root system whose associated affine Coxeter group $W^a = W(R) \approx \text{Q}(R_0) \rtimes W_0$ acts naturally on $X$. Then $W^a$ is a normal subgroup of $W$. The basis $F_0$ of simple roots of $R_0$ determines a corresponding set of simple affine roots denoted by $F \subset R$. If $S \subset W^a$ is the corresponding set of affine simple reflections then $(W^a, S)$ is an affine Coxeter group with Coxeter generators $S$. The Coxeter group $(W^a, S)$ is determined by the Coxeter graph $X(F)$ of $(W^a, F)$. Since we are assuming that $(W^a, S)$ is an affine Coxeter group (i.e. all connected components of $X(F)$ are irreducible affine Coxeter graphs) it is clear by the classification of irreducible affine Coxeter graphs that $X(F)$ determines a unique untwisted affine Dynkin diagram $D(F)$ (whose nodes are identified with the set $F$).
2.1.3. W as extended Coxeter group. Let \( C \subset V^* \) be the fundamental alcove \( C = \{ \lambda \in V^* \mid \forall a \in F : a(\lambda) > 0 \} \). The closure \( \overline{C} \) of \( C \) is a fundamental domain for the action of \( W^a \) on \( V^* \). Then \( S \) is the set of isometric affine hyperplane reflections in the faces of codimension 1 of \( C \). We introduce the finite subgroup \( \Omega_X = \{ \omega \in W : \omega(C) = C \} \subset W \). Obviously \( \Omega_X \) acts on \( D(F) \) by diagram automorphisms. Hence the natural action of \( \Omega_X \) on \( S \) extends to an action of \( \Omega_X \) on \( W^a \) by automorphisms. It is clear that we have \( W = W^a \rtimes \Omega_X \). It follows easily that \( \Omega_X \approx W/W^a \) is isomorphic to the finitely generated abelian group \( \Omega_X = X/Q(R_0) \), where \( Q(R_0) \) denotes the root lattice of the root system \( R_0 \).

2.1.4. Length function. The canonical length function \( l \) on the Coxeter group \( (W^a, S) \) can be extended uniquely to a length function \( l \) on the extended affine Coxeter group \( W \) such that \( \Omega_X \) is the set of elements of length zero. Thus \( W \) is generated by its distinguished set \( S \) of simple generators (of length 1) and its subgroup group \( \Omega_X \) of elements of length zero.

2.1.5. Generic affine Hecke algebra. We introduce the ring \( \Lambda = \mathbb{C}[v(s)^\pm 1; s \in S] \) of Laurent polynomials in invertible commuting indeterminates \( v(s) \) (with \( s \in S \)) subject to the relations \( v(s) = v(s') \) iff \( s \) and \( s' \) are conjugate in \( W \). The maximal spectrum of \( \Lambda \) is a complex algebraic torus denoted by \( \mathcal{Q}_c \) in \([OS2] \) (we reserve the notation \( \mathcal{Q} \) for the subset \( \mathcal{Q} \subset \mathcal{Q}_c \) of positive (or infinitesimally real) points of \( \mathcal{Q}_c \)).

Let \( \mathcal{L} = \mathbb{C}[v^\pm 1] \) denote be the ring of regular functions on \( \mathbb{C}^\times \), and let \( m : \mathbb{C}^\times \to \mathcal{Q}_c \) be a cocharacter. Then \( m \) is given by a collection of integers \( m_S(s) \in \mathbb{Z} (s \in S) \) defined on the set of \( W \)-conjugacy classes meeting \( S \) by \( m^*(v(s)) = v^{m_S(s)} \) for all \( s \in S \). Thus the collection of such cocharacters \( m \) is in natural bijection with the set of \( W \)-invariant functions \( m_R \) on \( R \) with values in \( \mathbb{Z} \), where \( m \) and \( m_R \) correspond iff \( m_R(a) = m(s_a) \) for all \( a \in F \). We consider \( \mathcal{L} \) as a \( \Lambda \) algebra via the homomorphism \( m^* : \Lambda \to \mathcal{L} \).

In \([OS2] \) the generic affine Hecke algebra \( \mathcal{H}_\Lambda(\mathcal{R}) \) was defined (see below). In the present paper the basic objects of study are Hecke algebras with coefficient ring \( \mathcal{L} \), obtained by specialization from \( \Lambda \) via a cocharacter \( m \).

**Definition 2.1.** Let \( \mathcal{R} \) be a based affine root datum and let \( m : \mathbb{C}^\times \to \mathcal{Q}_c \) be a cocharacter as above. We associated with these data the generic affine Hecke algebra \( \mathcal{H}(\mathcal{R}, m) \) over \( \mathcal{L} \) as follows. Recall that \( \mathcal{H}_\Lambda(\mathcal{R}) \) is the unique unital, associative, \( \Lambda \)-algebra with distinguished \( \Lambda \)-basis \( \{N_w\}_{w \in W} \) parametrized by \( w \in W \), satisfying the relations

(i) \( N_w N_{w'} = N_{ww'} \) for all \( w, w' \in W \) such that \( l(ww') = l(w) + l(w') \).

(ii) \( (N_s - v(s))(N_s + v(s)^{-1}) = 0 \) for all \( s \in S \).

The generic affine Hecke algebra with parameter \( m \) equals \( \mathcal{H}(\mathcal{R}, m) := \mathcal{H}_\Lambda(\mathcal{R}) \otimes_{m^* \mathcal{L}} \mathcal{L} \).

The set of distinguished generators \( \{N_s \otimes 1 \mid s \in S\} \) is considered an intrinsic part of the structure of \( \mathcal{H} = \mathcal{H}(\mathcal{R}, m) \), as well as the set of generators \( \{N_w \otimes 1 \mid \omega \in \Omega_X\} \) of length 0. Finally we consider the subset \( S_0 \subset S \) as part of the structure of \( \mathcal{H} \).

**Remark 2.2.** In particular the \( \mathcal{L} \)-basis \( \{N_w \otimes 1 \mid w \in W\} \) is a distinguished bases of \( \mathcal{H} \) which is an integral part of its structure. We will from now on write \( N_w \) again instead of \( N_w \otimes 1 \).

The following result is well known and its easy proof is left to the reader.

**Proposition 2.3.** The pair of data \( (\mathcal{R}, m) \) is canonically determined by \( \mathcal{H} \).
We include the following simple result on root data, for later reference.

**Proposition 2.4.** Suppose that $\mathcal{R} = (X, R_0, Y, R_0' \setminus Y)$ is a root datum, and that $R_0' \subset R_0$ is an irreducible component of $R_0$. Put $X' := X \cap \mathbb{R}R_0$, and $Y' := Y \cap \mathbb{R}R_0'$. Assume that either (i): $X' = P(R_0')$, or that (ii): There exists an $\alpha \in R_0'$ such that $\alpha^\vee \in 2Y$. Then $\mathcal{R}' := (X', R_0', Y', (R_0')^\vee)$ is a root datum which is a direct summand of $\mathcal{R}$. In the case (ii), $\mathcal{R}'$ has type $C_n^{(1)}$.

**Proof.** If $X' = P(R_0')$ then $X'$ is the complement of the $W_0$-invariant sub lattice $K := \cap_{\alpha \in R_0'} \text{Ker}(\alpha^\vee) \subset X$. If there exists an $\alpha \in R_0'$ such that $\alpha^\vee \in 2Y$ then the same argument applies to $\mathcal{R}_1 := (X, R_1, Y, R_1^\vee)$ (since $X' = Q(R_0') = P(R_1')$ in this case) showing that $\mathcal{R}_1' := (X', R_1', Y', (R_1')^\vee)$ is a direct summand of $\mathcal{R}_1$. But then $\mathcal{R}'$ is also a direct summand of $\mathcal{R}$. Moreover $\alpha^\vee \in 2Y'$. This implies that the irreducible root datum $\mathcal{R}'$ has type $C_n^{(1)}$. □

2.1.6. $\mathcal{H}(\mathcal{R}, m)$ as an extended affine Hecke algebra. As is well known, we can view $\mathcal{H}(\mathcal{R}, m)$ as an extended affine Hecke algebra in the following way. Define $\mathcal{R}^m = (R_0, Q(R_0), R_0^\vee, P(R_0^\vee), F_0)$ and let $m^a$ be the composition of $m$ with the natural map $\mathcal{Q}_c \rightarrow \mathcal{Q}_c^a$. Then we have an algebra isomorphism $\mathcal{H}(\mathcal{R}, m) = \mathcal{H}(\mathcal{R}^m, m^a) \times \Omega_X$.

2.1.7. Homomorphisms of affine Hecke algebras. We introduce some useful classes of $L$-algebra homomorphisms between generic affine Hecke algebras. A strict homomorphism $\lambda: \mathcal{H} \rightarrow \mathcal{H}'$ between two generic extended affine Hecke algebras is an algebra homomorphism given by a homomorphism $l: (W, S, S_0, \Omega_X) \rightarrow (W', S'_0, \Omega_{X'})$ between the underlying (extended affine) Coxeter groups (of course, in order to extend to the Hecke algebra level, the Hecke parameters of $\mathcal{H}$ and $\mathcal{H}'$ should match accordingly). An essentially strict homomorphism is like a strict homomorphism except that $\lambda(N_s) = \epsilon(s)N_{l(s)}$ for some signatures $\epsilon(s) \in \{\pm 1\}$, and $\lambda(N_w) = \delta(\omega)N_{l(\omega)}$ for some character $\delta \in \Omega_X^\ast$. An admissible homomorphism is a homomorphism of algebras respecting the standard bases of $\mathcal{H}$ and $\mathcal{H}'$ in the sense that for all $w \in W$: $\lambda(N_w) = \epsilon(w)N_{l(w)}$, where $\epsilon$ is a linear character of $W$ and $l: W \rightarrow W'$ a group homomorphism. If $\mathcal{H} = \mathcal{H}(\mathcal{R}, m)$ we speak of strict, essentially strict and admissible automorphisms of $\mathcal{H}$ (although this is an abuse of language, since a non-strict automorphism of $\mathcal{H}$ does not respect all the intrinsic structures of $\mathcal{H}$).

2.1.8. Notational conventions. We use the following notational conventions for change of coefficients. If $\mathcal{A}$ is an $L$-algebra and $\mathcal{M}$ is a unital, commutative $L$-algebra, then $\mathcal{A}M := \mathcal{A} \otimes_L \mathcal{M}$ denotes the extension of scalars from $L$ to $M$. Given $v \in \mathbb{C}^X$ we use the notation $\mathcal{A}_v$ as a shorthand for the $\mathbb{C}$-algebra $\mathcal{A}_C^X$, where $\mathbb{C}_v$ denotes the residue field of $L$ at $v$.

Now let $\mathcal{A}$ be a commutative $L$-algebra, and let $X = \text{Spec}(\mathcal{A})$ be its spectrum viewed as an affine scheme over $\text{Spec}(L)$. If $v \in \text{Spec}(L)$ we denote by $X_v = \text{Spec}(\mathcal{A}_v)$ its fiber at $v$. We denote by $X(L)$ the set of $L$-valued points of $X$.

**Remark 2.5.** We will often use the indeterminate $q = v^2$ (since various naturally given functions turn out to be functions of $v^2$). If we specialize $q$ at $q > 0$ we will tacitly assume that $v$ is the positive square root of $q > 0$. 

2.1.9. The Bernstein basis and the center. Let $\mathcal{H} = \mathcal{H}(R, m)$ be a generic affine Hecke algebra, and let $X^+ \subset X$ denote the cone of dominant elements in the lattice $X$. As is well known, the map $X^+ \to \mathcal{H}^\times$ defined by $x \to N_x := \theta_x$ is a monomorphism of monoids, and the commutativity of the monoid $X^+$ implies that this can be uniquely extended to a group monomorphism $X \ni x \to \theta_x \in \mathcal{H}^\times$. We denote by $A \subset \mathcal{H}$ the commutative $L$-subalgebra of $\mathcal{H}$ generated by the elements $\theta_x$ with $x \in X$. Observe that $A \approx L[X]$. Let $H_0 = \mathcal{H}(R_0, m)$ be the Hecke subalgebra (of finite rank over the algebra $L$) corresponding to the Coxeter subgroup $(W_0, S_0) \subset (W, S)$. We have the following well known result due to Bernstein-Zelevinski (unpublished) and Lusztig ([Lu1]):

**Theorem 2.6.** Let $\mathcal{H} = \mathcal{H}(R, m)$ be a generic affine Hecke algebra. The multiplication map defines an isomorphism of $A = \mathcal{H}_0$-modules $A \otimes \mathcal{H}_0 \to \mathcal{H}$ and an isomorphism of $\mathcal{H}_0 - A$-modules $\mathcal{H}_0 \otimes A \to \mathcal{H}$. The algebra structure on $\mathcal{H}$ is determined by the following cross relation (with $x \in X$, $\alpha \in F_0$, $s = r_\alpha^\vee$, and $s' \in S$ is a simple reflection such that $s' \not\sim_W r_{\alpha^\vee + 1}$):

$$\theta_x N_s - N_s \theta_s(x) = \left((v^{m_\alpha(s)} - v^{-m_\alpha(s)}) + (v^{m_\alpha(s')} - v^{-m_\alpha(s')})\theta_{-\alpha}\right) \frac{\theta_x - \theta_s(x)}{1 - \theta_{-2\alpha}}$$

(Note that if $s' \not\sim_W s$ then $\alpha^\vee \in 2Y$, which implies $x - s(x) \in 2\mathbb{Z}a$ for all $x \in X$. This guarantees that the right hand side of [7] is always an element of $A$).

**Corollary 2.7.** The center $Z$ of $\mathcal{H}$ is the algebra $Z = A^{W_0}$. For any $v \in Q_c$ the center of $\mathcal{H}_v$ is equal to the subalgebra $Z_v = A^{W_0}_v \subset \mathcal{H}_v$.

In particular $\mathcal{H}$ is a finite type algebra over its center $Z \approx L[X]^{W_0}$, and similarly $\mathcal{H}_v$ is a finite type algebra over its center $Z_v$.

2.1.10. Arithmetic and spectral diagrams for semisimple affine Hecke algebras. Let $\mathcal{H} = \mathcal{H}(R, m)$ be a generic affine Hecke algebra for a *semisimple* root datum $R$. Then the affine Hecke algebra $\mathcal{H}(R, m)$ is determined by the “arithmetic diagram” $\Sigma_a = \Sigma_a(R, m)$ derived from the data $(R, m)$ as follows. The arithmetic diagram consists of the affine Dynkin diagram $(R_0^\vee)^{(1)}$ of the based affine root system $(R, F)$, with one marked special vertex in each connected component (corresponding to the simple affine roots which are not in $F_0^\vee$), the action of $\Omega_X = X/Q(R_0)$ on this diagram, and a labelling of the vertices by the values of the integer valued function $m_R$. Clearly $\Sigma_a$ determines the tuple of data $(W, S, S_0, \Omega_X, m_S)$ and therefore it determines $\mathcal{H}$.

**Definition 2.8.** We call the data $(R, m)$ standard if (i) the root datum $R$ is semisimple, and (ii) $\forall s \in S$: $m_S(s) \neq 0$. If (ii) is replaced by (ii)”$\forall s \in S$: either $m_S(s) \neq 0$ or $m_S(s') \neq 0$ (using the notation of Theorem 2.6), then we call $(R, m)$ semi-standard.

We call an arithmetic diagram $\Sigma_a(R, m)$ of standard data $(R, m)$ a standard arithmetic diagram.

**Lemma 2.9.** Let $(R, m)$ be semi-standard and let $\mathcal{H} = \mathcal{H}(R, m)$. There exists an admissible isomorphism $\mathcal{H} \to \mathcal{H}(R', m')$ with $(R', m')$ standard. Hence up to an admissible isomorphism, $\mathcal{H}$ is represented by a standard arithmetic diagram.
Proof. Suppose that \(m_S(s) = 0\). Then \(m_S(s') \neq 0\) by assumption, and either \(s\) or \(s'\) belongs to a simple root \(\alpha \in F_0\) such that \(\alpha^\vee \in 2Y\). By Proposition \[2.4\] we see that \(s\) and \(s'\) belong to a component \(C\) of \(\Sigma_d\) of type \(C_n^{(1)}\), and \(\Omega_X\) fixes all the vertices of \(C\) point wise. We denote the corresponding direct summand of \(R\) by \(R_C = (R_{0C}, X_C, (R_{0C}^\vee), Y_C, F_C)\) (hence \(R_0\) has type \(B_n\) and \(X\) is its root lattice).

By applying an admissible isomorphism to \(H\) we may assume that \(m_R(\alpha_0^\vee) = 0\) (where \(\alpha_0^\vee\) is the unique simple affine root of \(R_C\) which is not in \(F_0^\vee\)). Applying another admissible isomorphism we may replace \((R_C, m_C)\) by standard data \((R_{1C}, m_{1C})\) with \(R_{1C} = (R_{1C}, X_{1C}, (R_{1C}^\vee), Y_{1C}, F_{1C})\) (where \(R_1\) is of type \(C_n\) such that its long roots are twice the short roots of \(R_0\), and \(m_{1C,R}(\alpha^\vee) = m_{R_0}(2\alpha^\vee)\) for all long roots of \(R_1\).

We repeat this procedure for all components of type \(C_n^{(1)}\) if necessary. \(\square\)

For the purpose of this paper it is also convenient to describe \(H\) in a “dual” way, by means of the “spectral diagram” \(\Sigma(a, m)\). A similar (but different) notion of the spectral diagram for a pair \((R, q)\) was defined in [OS2, Definition 8.1]. We will use a slight variation of this definition (see below) adapted to generic affine Hecke algebras in the present context of Definition \[2.1\]

**Definition 2.10.** Let \(n_m : R_0 \to \{1, 2\}\) be the \(W_0\)-invariant function on \(R_0\) defined by \(n_m(\alpha) = 2\) iff \(m_R(1-\alpha^\vee) \neq m_R(\alpha^\vee)\) (notice that this inequality in particular implies that \(\alpha^\vee \in 2Y\)).

We introduce a set \(R_m \subset X\) by \(R_m = \{n_m(\alpha) \alpha \mid \alpha \in R_0\}\). Then \(R_m\) is again a root system, and \(W_0 = W_0(R_m)\). By construction we have \(Q(R_m) \subset Q(R_0) \subset X \subset P(R_m) \subset P(R_0)\). Let \(\Omega_Y\) be the abelian group \(\Omega_Y = Y/Q(R_{m0})\). Let \(W^\vee = Y \times W_0\) be the affine Weyl group associated with the dual \(R^\vee\) of the root datum \(R\). Observe that

\[
\mathcal{H}(R_0, m) \subset \mathcal{H}(R, m) = (\mathcal{H}(R^m, m))^{\Omega_Y^\vee} \subset \mathcal{H}(R^m, m)
\]

with \(R_0 = (Q(R_0), R_0, P(R_0), R_0^\vee, F_0)\) and \(R^m = (P(R_m), R_0, Q(R_{m0}), R_0^\vee, F_{m0})\).

Both inclusions have finite index (here we use that \(R\) is semisimple). We have

\[
W^\vee = W((R^m)^\vee) \rtimes \Omega_Y^\vee
\]

where \(W((R^m)^\vee) = W(R_{m0}^{(1)})\) is the affine Coxeter group associated with the affine extension of the root system \(R_m\). Since \(R\) is semi-simple then \(X_m := P(R_m)\) is the largest sublattice of \(P(R_0)\) such that the function \(m_R\) on \(R = R_0 \rtimes \mathbb{Z}\) is invariant for the natural action of \(W(R^m) = X_m \rtimes W_0\) on \(R\). Thus we see that \(\mathcal{H}(R^m, m)\) is the largest affine Hecke algebra containing \(\mathcal{H}(R, m)\) as a subalgebra of finite index.

Observe that both \(R_0\) and \(R^m\) are direct products of their irreducible components.

The diagram underlying the spectral diagram \(\Sigma(a, m)\) is the diagram of the affine extension \(R_m^{(1)}\) of \((R_m, F_m)\). We identify its vertices with the set \(F_m^{(1)}\) of affine simple roots. We have a natural action of \(\Omega_Y^\vee\) on this diagram, and this information is included in the spectral diagram. The last piece of data of the spectral diagram \(\Sigma\) is a \(W^\vee\)-invariant labeling of its nodes by integers. We will define this labeling by restriction to \(F^{(1)}\) of a \(W^\vee\)-invariant function \(m_R^{(1)}\) on \(R_m^{(1)}\) with integer values.

Recall that \(m\) determines a \(W\)-invariant integer valued function \(m_R\) on \(R\). From this function we construct two half-integral \(W_0\)-invariant functions \(m_\pm\) on \(R_0\) as
follows: for $\alpha \in R_0$ we define
\begin{align}
\begin{cases}
m_+(\alpha) &= \frac{1}{2}(m_R(\alpha^\vee) + m_R(1 - \alpha^\vee)) \\
m_-(\alpha) &= \frac{1}{2}(m_R(\alpha^\vee) - m_R(1 - \alpha^\vee))
\end{cases}
\end{align}

The nodes of the Dynkin diagram of $R_m^{(1)}$ are labelled by the values of a $W^\vee$-invariant function $m^\vee_R$ on the affine root system $R_m^{(1)}$ defined as follows. Let $a^\vee = n_m(\alpha)\alpha + k \in R_m^{(1)}$. We introduce a $W^\vee$-invariant signature function $\epsilon : R_m^{(1)} \to \{\pm\}$ by defining $\epsilon(a^\vee)1 = (-1)^{k(n_m(\alpha)-1)}$ (one easily checks that this is indeed $W^\vee$-invariant). Finally we define an integer valued function $m^\vee_R$ by:
\begin{equation}
m^\vee_R(a^\vee) := n_m(\alpha)m_\epsilon(a^\vee)(\alpha)
\end{equation}

**Definition 2.11.** The spectral diagram $\Sigma_s(R, m)$ associated with $(R, m)$ consists of the extended Dynkin diagram of $(R_m^{(1)}, F_m^{(1)})$, the finite abelian group $\Omega^\vee_Y = Y/Q(R_m^\vee)$ of automorphisms of this diagram, and the function $m^\vee_R$ defined above.

The following is clear:

**Proposition 2.12.** If $R$ is semisimple then the algebra $\mathcal{H}(R, m)$ is completely determined by its spectral diagram $\Sigma_s = \Sigma_s(R, m)$. Note that $(R, m)$ is semi-standard iff $\Sigma_s$ is semi-standard in the sense that for all $a^\vee = n_m(\alpha)\alpha + k \in R_m^\vee$, either $m_-(\alpha) \neq 0$ or $m_+(\alpha) \neq 0$.

### 2.2 Normalized affine Hecke algebras and the $\mu$-function.

#### 2.2.1. The standard trace of an affine Hecke algebra.

We equip a generic affine Hecke algebra $\mathcal{H} = \mathcal{H}(R, m)$ with a $L$-valued trace $\tau^1 : \mathcal{H} \to L$ defined by
\begin{equation}
\tau^1(N_w) = \delta_{w, e}
\end{equation}

Then $\tau^1$ defines a family of $C$-valued traces $R_{>1} \ni v \to \tau^1_v$ on the $R_{>1}$-family of algebras of complex algebras $\mathcal{H}_v$. It is an elementary but fundamental fact that the traces $\tau^1_v$ are positive with respect to the conjugate linear anti-involution $\ast$ on $\mathcal{H}_v$ defined by $N_{w^*} = N_w^{-1}$. For all $v \in R_{>1}$ the trace $\tau^1_v$ and the $\ast$ operator define the structure of a type I Hilbert algebra on $\mathcal{H}_v$ (see [O2]). It will be of crucial importance to normalize the family of traces $\tau^1_v$ in a suitable way.

#### 2.2.2. Definition of normalized affine Hecke algebras.

Let $K$ denote the field of fractions of $L$, and let $M' \subset K^\times$ be the subgroup of the multiplicative group $K^\times$ generated by $(v - v^{-1})$, by $Q^\times$, and by the $q$-integers $[n]_q := \frac{q^n - q^{-n}}{q - q^{-1}}$. Observe that $d(v^{-1}) = \pm d(v)$ for all $d \in M'$. We put
\begin{equation}
M = \{d \in M' \mid d(v) > 0 \text{ if } v > 1\}
\end{equation}

We write $M'_{(n)}$ and $M_{(n)}$ for the subsets of elements with vanishing order $n$ at $v = 1$. Observe that $M = \cup_{n \in \mathbb{Z}} M_{(n)}$.

**Definition 2.13.** A normalized affine Hecke algebra is a pair $(\mathcal{H}, \tau^d)$ where $\mathcal{H}$ is a generic affine Hecke algebra and where $\tau^d$ is a $K$-valued trace on $\mathcal{H}$ of the form $\tau^d = d\tau^1$ with $d \in M$ as above.

In a normalized affine Hecke algebra $(\mathcal{H}, \tau^d)$ we have a family of positive traces $\{\tau^d_v = d(v)\tau^1_v\}_{v > 1}$ on the family of specializations $\{\mathcal{H}_v\}_{v > 1}$. Before we go into the harmonic analytic aspects of the spectral decomposition of these traces we discuss the so-called $\mu$-function of a normalized affine Hecke algebra.
2.2.3. The $\mu$ function of $(\mathcal{H}, \tau^d)$. The $\mu$ function captures the structure of a normalized affine Hecke algebra and plays a key role in the harmonic analysis related to the spectral decomposition of the traces $\tau^d$.

Let $(\mathcal{H}, \tau^d)$ be a normalized affine Hecke algebra, and let $T$ be the diagonalizable group scheme with character lattice $\mathbb{Z} \times X$, viewed as diagonalizable group scheme over $\text{Spec}(\mathbb{L})$ via the $\mathbb{C}$-algebra homomorphism $L \to \mathbb{C}[\mathbb{Z} \times X]$ given by $v^n \to (n,0)$. We denote by $T(L)$ its group of $L$-points, and by $T_\mathfrak{v}$ the fibre at $\mathfrak{v} \in \mathbb{C}^\times$.

The $\mu$-function of $(\mathcal{H}, \tau^d)$ is a rational function on $T$ which we define in terms of $d$ and the spectral diagram of $\mathcal{H}$. We first assign the so-called Macdonald $c$-function $c_{m,\alpha}$ to the roots $\alpha \in R_0$ by

$$c_{m,\alpha} := \frac{(1 + v^{-2m_-(\alpha)}\alpha^{-1})(1 - v^{-2m_+(\alpha)}\alpha^{-1})}{1 - \alpha^{-2}}$$

The $c$-function $c_m$ of $\mathcal{H}$ is defined by $c_m := \prod_{\alpha \in R_{0, +}} c_\alpha$. Finally we define

**Definition 2.14.** The $\mu$-function of $(\mathcal{H}, \tau^d)$ by

$$\mu = v^{-2m_W(w_0)} \frac{d}{c_m c_{m_0}}$$

where $w_0 \in W_0$ is the longest Weyl group element, and where $m_W : W \to \mathbb{Z}$ is defined by

$$m_W(w) = \sum_{a \in R_+ \cap w^{-1}(R_-)} m_R(a)$$

From Proposition 2.23 it is clear that:

**Proposition 2.15.** The $\mu$-function is canonically determined by $\mathcal{H}$.

Let us comment on the meaning of the spectral diagram in terms of the $\mu$-function. We are going to consider below the group of affine symmetries of the rational function $\mu$ on $T$ (with $T$ viewed as $\mathbb{L}$-scheme). This symmetry group is determined by its action on the group of complex points $T_\mathfrak{v}(\mathbb{C})$ of a generic fibre $T_\mathfrak{v}$. We fix a fibre at $\mathfrak{v} > 1$. If $\mathcal{R}$ is semisimple then it is also easy to see that this group acts faithfully on the compact form $T_u := (T_\mathfrak{v}(\mathbb{C}))_u$. We lift the $\mu$-function to the vector space $V = \mathbb{R} \otimes_{\mathbb{Z}} Y$ via the exponential map $x \to \exp(2\pi i x) \in T_u$ of the compact torus $T_u$. The zero set of $\mu$, considered as periodic function on $V$, is a union of affine hyperplanes in $V$. Let us call these hyperplanes the $\mu$-mirrors.

**Proposition 2.16.** Assume that $(\mathcal{R}, m)$ is semi-standard, with spectral diagram $\Sigma$.

(i) The group of affine symmetries of $\mu$ on $V$ generated by $Y$ and by the reflections in the $\mu$-mirrors is equal to the group $W^\vee = W(R_m^{(1)} \times \Omega^Y_X)$ (the affine Weyl group of the affine Dynkin diagram (equipped with $\Omega^Y_X$-action) underlying $\Sigma$).

(ii) The group $\text{Aut}_T(\mu)$ of affine symmetries of $\mu$ on $T$ is a semidirect product of the form $\text{Aut}_T(\mu) = W_0 \rtimes \text{Out}_T(\mu)$ for some subgroup $\text{Out}_T(\mu)$.

(iii) $(\mathcal{R}, m)$ can be replaced by a standard pair $(\mathcal{R}_1, m_1)$ without changing $\mu$.

(iv) If $(\mathcal{R}, m)$ is standard then $\text{Out}_T(\mu) = \Omega^*_X \rtimes \Omega^Y_0$, with $\Omega^Y_0 = \text{Out}_T(R_0^Y)$, the group of diagram automorphisms of $(R_0^Y, F_0^Y)$ which normalize the lattice $Y$, and $\Omega^*_X = P(R_0^Y)/Y < T$ (the dual of $\Omega^Y_X$), acting trivially on $W_0$.

(v) There is a canonical isomorphism between the group $\text{Aut}_{es}(\mathcal{H})$, the opposite of the group of essentially strict automorphisms of $\mathcal{H}$, and $\text{Out}_T(\mu)$. 
For (i) we only need to consider the case where there exists $\alpha \in R_0$ such that $m_+(\alpha) = 0$ (and thus $m_-(\alpha) \neq 0$ by assumption). But then $\alpha$ belongs to an indecomposable summand $\mathcal{R}'$ of $\mathcal{R}$ of type $C_n^{(1)}$ (see the proof of Proposition 2.4). In particular, the translation lattice $Y'$ of this summand is equal to $Q((R_0')^\vee) = P((R_0')^\vee)$. This shows that the reflections of $W(R_0)$ (acting on $V$) are contained in the group generated by $Y$ and the reflections in the $\mu$-mirrors.

Assertion (ii) will follow from the proof of (iv) using (iii).

For (iii), consider the $\mu$-function of a semi-standard but not standard pair of data $(\mathcal{R}, m)$. Hence there exists a component in $\mathcal{R}$ of type $C_n^{(1)}$ such that $m_S$ has a zero at precisely one of the extreme ends of $\Sigma_\alpha$. We may (and will) assume without loss of generality that $\mathcal{R}$ equals this component. The assumption implies that $m_-(\alpha) = m_+(\alpha)$ or that $m_-(\alpha) = -m_+(\alpha)$ for a root $\alpha \in R_0$ such that $\alpha^\vee \in 2Y$. It is easy to see that the $\mu$ function is unchanged if we change $m_+$ or $m_-$ to its negative on any of the $W_0$-orbits of roots in $R_0$. So after such a symmetry transformation of $\mu$ we may assume that $m_+(\alpha) = m_-(\alpha)$. Now we can simplify the denominator of $\mu$ to $(1 - v^{-4m_+(\alpha)}\alpha^{-2})$, or (which amounts to the same thing) multiply all roots of $R_0$ in the orbit of $\alpha$ by 2. Hence $R_0$ is now changed to a new root system $R_1$ of type $C_n$, and the corresponding pair $(\mathcal{R}_1, m_1)$ is obviously standard.

Next we consider (ii) and (iv). By elementary covering theory we have $\text{Aut}_T(\mu) = \text{Aut}_V(\mu, Y)/Y$, where $\text{Aut}_V(\mu, Y)$ denotes the group of affine linear transformation of $V$ which fix $\mu$ and normalize $Y$. Using that we are in the standard case, this last group is easily seen to be equal to $(\Omega_X^X \times W_0) \times \Omega_Y^Y$. But $W_0$ acts trivially on $P(R_0^\vee)/Q(R_0^\vee)$, and in particular on $\Omega_X^X = P(R_0^\vee)/Y$. Hence $W_0$ is a normal subgroup, and we can rewrite the answer in the stated form.

Finally we prove (v). By Lemma 2.9 and parts (ii),(iv) we may assume that $(\mathcal{R}, m)$ is standard. This implies easily that an essentially strict automorphisms $\lambda$ acts on the distinguished basis elements $N_s$ with $s \in S$ by $\lambda(N_s) = N_{l(s)}$ for an element $l \in \Omega_0^X$, the group of diagram automorphisms of $(R_0, F_0)$ stabilizing $X$ (in fact, these automorphisms are automatically strict in the standard case). In addition we have that action $N_\omega \to \lambda(\omega)N_\omega$ of $\lambda \in \Omega_X^X$ on the elements of the form $N_\omega$ with $\omega \in \Omega_X$. It is clear that $\Omega_X^X \times \Omega_0^Y$ exhausts $\text{Aut}_{ex}(\mathcal{H})$, and that its action on $\mathcal{H}$ induces by duality a faithful right action on $T$ by elements of $\text{Aut}_T(\mu)$. Now (iv) completes the proof $(\Omega_0^X$ and $\Omega_Y^Y$ correspond, as well as the actions of $\Omega_X^X$).

**Proposition 2.17.** Suppose that $(\mathcal{R}, m)$ is standard and put $\Omega_m^* = P(R_0^\vee)/Q(R_m^\vee)$. The group $\Omega_m^* \times \Omega_0^Y$ acts faithfully on the labelled Dynkin diagram underlying $\Sigma_s$, and we have an exact sequence

$$1 \to \Omega_m^{\vee} \to \Omega_m^* \times \Omega_0^{\vee} \to \text{Out}_T(\mu) \to 1$$

In particular, $\Omega_m^* \times \Omega_0^{\vee}$ acts faithfully on $\Sigma_s(\mathcal{R}^m, m)$ (since $\Omega_0^{\vee} = 1$ in this case).

**2.3. Poles and zeroes of $\mu$-functions.** The poles and zeroes of the $\mu$-function $\mu = \mu_{\mathcal{R}, m, d}$ have certain remarkable properties which reflect the role of $\mu$ in the spectral decomposition of the positive trace $\tau^d$ (and can in fact be deduced from this, see [O3]).

**2.3.1. Cosets of tori.** Recall that the category of diagonalizable group schemes of finite type over $\text{Spec}(L)$ is anti-equivalent with the category of finitely generated...
abelian groups via the anti-equivalence \( S \rightarrow X_S := \text{Hom}_{\text{Spec}(\mathbb{L})}(S, \mathbb{G}_{m, \text{Spec}(\mathbb{L})}) \). Below we will use the phrase “torus over \( \mathbb{L} \)” as shorthand for a “connected diagonalizable group scheme of finite type over \( \text{Spec}(\mathbb{L}) \)”.

Let \( \mathbb{L} \) be a commutative unital \( \mathbb{L} \)-algebra. A subtorus \( \iota_S : S_A \rightarrow T_A \) of \( T_A \) over \( A \) corresponds to an epimorphism \( \iota_S^* : X \rightarrow X_S \) of lattices. By a coset \( L_A \subset T_A \) of \( S_A \) we mean a closed subscheme of \( T_A \) defined by a closed immersion \( i : S_A \rightarrow T_A \) of \( S_A \) in \( T_A \) that is given by a group homomorphism \( i^* : X \rightarrow A[X_S]^\times \) of the form \( i^*(x) = r(x)i_S^*(x) \), where \( r : X \rightarrow A^\times \) denotes an \( A \)-point \( r \in T(A) \). In this situation we write \( L_A = rS_A \). We define \( \text{codim}(L_A) := \text{rank}(X) - \text{rank}(X_S) \).

### 2.3.2. Residual cosets.

Given a coset \( L_A \subset T_A \) in \( T_A \) over \( A \) we define \( p_\epsilon(L_A) = \{ \alpha \in R_0 \mid \epsilon \alpha|_{L_A} = v^{-m_\epsilon(\alpha)}1_A \} \) and \( z_\epsilon(L_A) = \{ \alpha \in R_0 \mid \epsilon \alpha|_{L_A} = 1_A \} \) (with \( \epsilon = + \) or \( \epsilon = - \)). Then \( L_A \subset T_A \) is called a residual coset over \( A \) if

\[
\#p_+(L_A) + \#p_-(L_A) - \#z_+(L_A) - \#z_-(L_A) \geq \text{codim}(L_A) \tag{12}
\]

We denote by \( \mathcal{L}_A \) the set of residual cosets in \( T_A \) over \( A \). If \( \nu \in \mathbb{R}_+ \subset \mathbb{C}^\times \) we use the shorthand notation \( \mathcal{L}_\nu \) for the set of residual cosets in \( T_\nu \) over \( \mathbb{C}_\nu \). We write \( \mathcal{L} \) for the set of residual cosets over \( \mathcal{L} \). Of special interest are the residual points:

**Definition 2.18.** Let \( \text{Res}(\mathcal{R}, m) \subset \mathcal{L}(\mathcal{R}, m) \) denote the set of \( \mathcal{L} \)-valued points of \( T \) which belong to \( \mathcal{L}(\mathcal{R}, m) \). This set is called the set of generic residual points of \( \mu = \mu_{\mathcal{R}, m, d} \).

Let \( L \subset T \) be a coset of a subtorus \( T^L \subset T \) and suppose that \( L \) is residual. Let \( q : T \rightarrow T^L := T/T^L \) be the quotient map, and put \( LX \subset X \) for the corresponding sublattice of \( X \). Then \( R_L = R_0 \cap LX \) is a parabolic root subsystem. Putting \( LY = Y/(LX) \) we obtain a root datum \( \mathcal{R}_L := (R_L, LX, R^L_L, LY) \). If \( m_{L, \pm} = m_{\pm | R_L} \) then there exists a unique cocharacter \( m_L : \mathbb{C}^\times \rightarrow Q_\epsilon(\mathcal{R}_L) \) such that \( m_{L, \pm} \) is associated with a spectral diagram \( \Sigma_\epsilon(\mathcal{R}_L, m_L) \) as described in \([2.1.10]\).

**Theorem 2.19.**

(i) If \( L_\nu \subset T_\nu \) is a residual coset over \( A = \mathbb{C}_\nu \) with \( \nu \in \mathbb{R}_+ \) then the inequality \((12)\) is an equality. Similarly for all residual cosets \( L \in \mathcal{L} \) and \( L_K \in \mathcal{L}_K \) the inequality \((12)\) is an equality.

(ii) \( L \in \mathcal{L} \) if and only if \( L = q^{-1}(r) \) with \( r \in T^L_\nu(L) \) a generic \( (\mathcal{R}_L, m_L) \)-residual \( \mathcal{L} \)-valued point. Analogous statements hold for \( \mathcal{L}_K \) and for \( \mathcal{L}_\nu \), provided \( \nu \in \mathbb{R}_+ \).

(iii) The sets \( \mathcal{L}_K, \mathcal{L}_\nu \) and \( \mathcal{L} \) are finite, \( W_0 \)-invariant and consist of cosets of subtori of \( T_K, T_\nu \) and \( T \) respectively.

(iv) The map \( \mathcal{L} \rightarrow \mathcal{L}_K \) given by \( L \rightarrow L_K \) is bijective.

(v) If \( \nu \in \mathbb{R}_+ \) and \( \nu \neq 1 \) then the map \( \mathcal{L} \rightarrow \mathcal{L}_\nu \) given by \( L \rightarrow L_\nu \) is bijective.

**Proof.** The assertion (i) for \( \mathcal{L}_\nu \) (with \( \nu \in \mathbb{R}_+ \)) is \([O3 \ Theorem 6.1(B)]\). Let \( L \in \mathcal{L} \). Since \( \mathbb{R}_+ \subset \mathbb{C}^\times \) is Zariski dense there exists \( \nu \in \mathbb{R}_+ \) such that \( p_\nu(L) = p_\nu(L_\nu) \) and \( z_\nu(L) = z_\nu(L_\nu) \), proving the assertion of (i) for the generic case as well. A similar argument works for \( \mathcal{L}_K = K \).

To prove (ii) first observe that \( p_\nu(L), z_\nu(L) \subset R_L \). A glance at \((12)\) makes it clear that \( q(L) \subset T_L \) is an \( \mathcal{L} \)-valued point \( r_L \) of \( T_L \) which is an \( (\mathcal{R}_L, m_L) \)-residual point over \( \mathcal{L} \). The converse is clear from the definitions. Analogous arguments can be given for \( \mathcal{L}_K \) and for \( \mathcal{L}_\nu \). This proves (ii).

Now we are in the position to reduce the remaining statements to the case of (generic) residual points and use the results of \([O2]\).
By [OS2] Theorem 2.44, Proposition 2.52, Proposition 2.56, Proposition 2.63] the assertions of (iii) and (iv) can be easily checked.

Finally observe that [OS2] Proposition 2.56] implies that if \( r \) is as above and \( \alpha \in R_L \) then \( \alpha(r) = \zeta v^n \) with \( \zeta \) a root of unity, and \( n \in \mathbb{Z} \). This implies that \( r(v) \) is residual for all \( v \in \mathbb{R}_+ \setminus \{1\} \), proving (v).

### 2.3.3. Regularization of \( \mu \) along a residual coset

Let \( L \in \mathcal{L}(R, m) \) be a residual coset for \( \mu = \mu_{R, m,d} \). We define the regularization \( \mu^L \) of \( \mu \) along \( L \) as follows:

\[
\mu^L = \frac{\mu(v) \prod_{\alpha \in R_0 \setminus z_v(L)}(1 + \alpha^{-1}) \prod_{\alpha \in R_0 \setminus z_v(L)}(1 - \alpha^{-1})}{\prod_{\alpha \in R_0 \setminus z_v(L)}(1 + v^{-2m_+(\alpha)} \alpha^{-1}) \prod_{\alpha \in R_0 \setminus z_v(L)}(1 - v^{-2m_+(\alpha)} \alpha^{-1})}
\]

where \( p_v(L) \) and \( z_v(L) \) are as defined in 2.3.

**Proposition 2.20.** The function \( \mu^L \) restricts to a nonzero rational function \( \mu^L = \mu^L_{R, m,d} \) on \( L \). For all \( v \in \mathbb{R}_+ \setminus \{1\} \) the rational function \( \mu^L \) on \( L \) is regular at the generic point of the fiber \( L_v \), and restricts to a nonzero rational function on \( L_v \). The function \( \mu^L_{R, m,1} \) has vanishing order \( \text{codim}(L) \) at the generic point of the fiber \( L_v \) for \( v = 1 \).

**Proof.** We need to check that \( \mu^L \) maps to an element of the function field \( K(LK) \) of \( L \) if we restrict to \( L \). This is clear by the definition of the sets \( p_v(L) \) and \( z_v(L) \).

Let \( v \in \mathbb{R}_+ \). Let \( r \in T_L \) be as in Theorem 2.19. As was mentioned at the end of the proof of Theorem 2.19 for all \( \alpha \in R_L \) we have that \( \alpha(r) = \zeta v^n \) for some \( n \in \mathbb{Z} \) and \( \zeta \) a root of 1. Hence \( p_v(L_v) = p_v(L) \) and \( z_v(L_v) = z_v(L) \) if \( v \neq 1 \) (recall that these sets are subsets of \( R_L \), see the proof of Theorem 2.19). On the other hand it follows that \( p_v(L_v) = z_v(L_v) \) if \( v = 1 \). Using (12) for \( A = L \) we are done.

Observe that \( \mu^L_{R, m,d} \in K^* \) if \( r \in \text{Res}(R, m) \) (see Theorem 2.28 for more precise statements). We decompose \( \mu^L_{R, m,d} \) as follows. Assume (by replacing \( L \) by a \( W_0 \)-translate \( w(L) \) if necessary) that \( R_L = R_P \) for a standard parabolic root subsystem \( R_P \subset R_0 \) defined by \( P \subset F_0 \). Using the notation of Theorem 2.19(ii) we write

\[
\mu^L_{R, m,d} = \frac{\mu^L_{P, m, d}}{\prod_{\alpha \in R_0 \setminus R_P} \prod_{\alpha \in R_0 \setminus R_P}(1 + v^{2m_+(\alpha)} \alpha^{-1}) \prod_{\alpha \in R_0 \setminus R_P}(1 - v^{2m_+(\alpha)} \alpha^{-1})}
\]

where \( W^P = w_0 w_0^{-1} \in W^P \) is the longest element.

### 2.4. The tempered spectrum of generic affine Hecke algebras

Suppose we are given a normalized affine Hecke algebra \( (H, \tau^d) \) (with \( H = H(R, m) \) and \( d \in M \)) Recall that \( \tau^d \) defines a family of traces \( \mathbb{R}_{>1} \ni v \to \tau^d_v = d(v)\tau^d_1 \) on the family of \( \mathbb{C} \)-algebras \( H_v \). The trace \( \tau^d \) is positive with respect to the conjugate linear anti-involution \( \ast \) on \( H_v \) defined by \( N^*_v = N_v^{-1} \), and in fact defines the structure of a type I Hilbert algebra on \( H_v \) (see [O2]).

**Definition 2.21.** We denote by \( \mathcal{G}_v = \mathcal{G}_v(R, m) \) the irreducible spectrum of the \( C^* \)-algebra completion \( C_v = C_v(R, m) \) of \( H_v \). In particular \( \mathcal{G}_v \) is equal to the support of the spectral measure \( \tau^d_v \).

It is known that the underlying set of \( \mathcal{G}_v \) can also be characterized as the set of equivalence classes of irreducible tempered representations of \( H_v \) (see [DeOp1]), and for this reason we refer to \( \mathcal{G}_v \) as the tempered spectrum of \( H_v \).
It is an important fact for our purpose that the tempered spectra $\mathcal{S}_\nu$ of the algebras $\mathcal{H}_\nu$ are the fibers of the spectrum $\mathcal{S} \to \mathbb{R}_{>1}$ of a trivial bundle $\mathcal{C}$ of $C^*$-algebras. We will discuss this in some detail this section.

2.4.1. **Generic families of discrete series characters.** The irreducible discrete series characters of $\mathcal{H}_\nu(\mathbb{R},m)$ (with $\nu \in \mathbb{R}_{>1}$) arise in finitely many families depending continuously on the parameters of the Hecke algebra ([OS2], Theorem 5.7). In our present setup we only consider a very special instance of this general phenomenon called “scaling” of the parameters (i.e. varying $\nu > 1$ while keeping $m$ fixed):

**Theorem 2.22** ([OS1], Theorem 1.7). There exists a finite set $[\Delta] = [\Delta(\mathbb{R},m)]$ of weakly continuous families $\mathbb{R}_{>1} \ni \nu \to \delta_\nu$ of irreducible characters of $\mathcal{H} = \mathcal{H}(\mathbb{R},m)$ such that

1. For all $\nu \in \mathbb{R}_{>1}$, the character $\delta_\nu$ is an irreducible discrete series character of $\mathcal{H}_\nu$.
2. Given $\nu \in \mathbb{R}_{>1}$ we denote by $[\Delta_\nu]$ the set of discrete series characters of $\mathcal{H}_\nu$. Then the evaluation map $[\Delta] \ni \delta \to \delta_\nu \in [\Delta_\nu]$ is a bijection.

2.4.2. **Scaling of the parameters.** The continuous families of discrete series characters described in the previous paragraph arise from underlying families of “scaling isomorphisms” which can be defined on the level of formal completions of affine Hecke algebras at a fixed central character. This construction yields (see [OS1], Theorem 1.7) a family bijections for $\epsilon \in (0,1]$

\[
\sigma_\epsilon : \text{Irr}(\mathcal{H}_\nu) \to \text{Irr}(\mathcal{H}_{\nu^\epsilon})
\]

\[
(\pi, V) \to (\pi_\epsilon, V)
\]

preserving unitarity, temperedness, and the discrete series, and having the property that for all $h \in \mathcal{H}_\nu$ the family $0 \leq \epsilon \leq 1$ is real analytic in $\epsilon$. These results enable us to choose a set $\Delta$ of real analytic families of irreducible representations $\nu \to (\tilde{\delta}_\nu, V_\delta)$ for $\mathcal{H}$ such that the character map $\delta \to \chi_\delta$ yields a bijection $\Delta \to [\Delta]$. In other words, we can and will choose a complete set $\Delta$ of real analytic representatives for the set of continuous families of irreducible discrete series characters $[\Delta]$.

**Definition 2.23.** We call $\Delta = \Delta(\mathbb{R},m)$ a complete set of generic irreducible discrete series characters. For all subsets $P \subset F_0$ we fix a complete set of generic irreducible discrete series representations $\Delta_P = \Delta(\mathbb{R}_P, m_P)$ of the semisimple quotient algebra $\mathcal{H}_P := \mathcal{H}(\mathbb{R}_P, m_P)$ of the “Levi-subalgebra” $\mathcal{H}_P^P := \mathcal{H}(\mathbb{R}_P^P, m_P^P) \subset \mathcal{H}(\mathbb{R}, m)$ (see [DeOp1] Section 3.5, Section 4.1] for the precise definitions of these subquotient algebras associated with $P$).

2.4.3. **Scaling transformations on the tempered spectrum.** Let $\mathcal{S}_\nu$ denote the Schwartz algebra of $\mathcal{H}_\nu$. As subspace of the $L^2$-completion $L^2(\mathcal{H}_\nu)$ we have the following description of $\mathcal{S}_\nu$:

\[
\mathcal{S}_\nu := \{s = \sum_{w \in W} c_w N_w \mid \forall k \in \mathbb{N} \exists M_k \in \mathbb{R}_{\geq 0} \forall w \in W : (1 + N(w))^k |c_w| \leq M_k\}
\]

(where $N(w)$ is a norm function on $W$, see [DeOp1]). This is a Fréchet completion of $\mathcal{H}_\nu$ whose irreducible spectrum, the space of tempered irreducible representation, is known to be equal (see [DeOp1] Corollary 4.4]) to the support $\mathcal{S}_\nu$ of the spectral measure $\mu_{\mathcal{P}1,\nu}$ of $\tau_\nu^d$. (We remark in passing that the (Jacobson) topologies of $\mathcal{S}_\nu$ when viewed as the irreducible spectrum of $\mathcal{C}_\nu$ or when viewed as the irreducible...
Proof. In [Sol1, Theorem 5.21] the isomorphisms of pre-$C^*$-algebras this implies that the maps $\pi_i, \tau_i, \delta_i$ are independent of $v$ when viewed as Fréchet spaces (forgetting the algebra structure). We sometimes write $S(R)$ to denote this underlying Fréchet space.

The scaling maps $\tilde{\sigma}_\epsilon$ (with $\epsilon > 0$) of (15) define homeomorphisms on the tempered spectrum:

**Theorem 2.24** ([Sol1], Lemma 5.19 and Theorem 5.21). There exists a family of isomorphisms of pre-$C^*$-algebras

\begin{equation}
\phi_{\epsilon,v} : S_v \to S_v
\end{equation}

(with $v > 1$ and $\epsilon > 0$) such that $\phi_{1,v} = \text{Id}$ and such that the following holds:

(i) For all $s \in S(R)$ fixed the map $\mathbb{R}_+ \times \mathbb{R}_{>1} \ni (\epsilon, v) \to \phi_{\epsilon,v}(s) \in S_v$ is continuous if we equip $S_v$ with its canonical pre-$C^*$-algebra topology (recall that $S(R)$ denotes the underlying Fréchet space of $S_v$) and as such is independent of $v$.

(ii) For all $\pi \in S_v$ we have $\tilde{\sigma}_\epsilon, \pi \simeq \pi \circ \phi_{\epsilon,v}$ (with $\tilde{\sigma}_\epsilon$ as in (15), but extended to all $\epsilon > 0$ by defining $\tilde{\sigma}_\epsilon, \pi := \tilde{\sigma}_{\epsilon^{-1}, \epsilon \pi}$ if $\epsilon \geq 1$).

(iii) The map $\tilde{\sigma}_\epsilon$ (with $\epsilon > 0$) induces a homeomorphism from $S_v$ onto $S_v$.

**Proof.** In [Sol1, Theorem 5.21] the isomorphisms of pre-$C^*$-algebras $\phi_{\epsilon,v}$ were constructed for $\epsilon \in (0,1]$ in such a way that $\phi_{1,v}$ is the identity, and such that the continuity (i) holds (this is [Sol1, Lemma 5.19]). For $\epsilon \geq 1$ we define $\phi_{\epsilon,v} := \phi_{\epsilon^{-1}, v}$.

It is easy to see that the required continuity holds in this case as well, by the same arguments as used in [Sol1, Lemma 5.19]. According to [Sol1, Theorem 5.19(2)], for all triples $(P, \delta, t)$ with $P \subset F_0$, $\delta \in \Delta_P$ and $t \in T^P_u$ one has $\pi(P, \delta, t) \circ \phi_{\epsilon, v} \simeq \pi(P, \delta, t)$.

Since these induced characters are finite sums of irreducible tempered characters this implies that $\pi_\epsilon \circ \phi_{\epsilon, v} \simeq \tilde{\sigma}_\epsilon(\pi_\epsilon)$ for all irreducible summands $\pi_\epsilon$ of $\pi(P, \delta, t)$. But all irreducible tempered representations are direct summands of unitarily induced standard representations of the form $\pi(P, \delta, t)$ [DeOp1, Theorem 3.22], hence we deduce the second assertion (ii). Since the maps $\phi_{\epsilon}$ constructed by Solleveld are isomorphisms of pre-$C^*$-algebras this implies that the maps $\tilde{\sigma}_\epsilon$ define homeomorphisms between the spectra of the $C^*$-algebras $C_v$ and $C_v$, proving (iii). \hfill \Box

2.4.4. The generic $C^*$-algebra. It is natural at this point to construct the “generic $C^*$-algebra $C = C(R, m)$”, a continuous bundle of $C^*$-algebras over $\mathbb{R}_{>1}$ which is the analytic counterpart of the generic Hecke algebra $H = H(R, m)$ (this construction was briefly discussed in [Sol1, Section 5.2]).

Consider the bundle of Banach spaces $\prod_{v > 1} C_v$. Recall that the underlying Fréchet space $S(R)$ of the Schwartz algebras $S_v$ is independent of $v$ and is contained in $C_v$ for all $v > 1$. Using this fact we construct a linear subspace (isomorphic to $C_c(\mathbb{R}_{>1}) \otimes S(R)$) of sections $\sigma_{f,s}$ of the above bundle by the linear map defined by $f \otimes s \to \sigma_{f,s}$, where $\sigma_{f,s}$ is the section defined by $\sigma_{f,s}(v) := f(v)s \in C_v$. By [Sol1, Proposition 5.6] the map $v \to \|\sigma_{f,s}(v)\|_o$ is continuous. Hence by [Dix, Section 10.3] this collection of sections generates a continuous bundle of $C^*$-algebras $C$. By construction $C$ contains all compactly supported continuous maps $\sigma : \mathbb{R}_{>1} \to S(R)$.
**Theorem 2.25.** The continuous bundle of $C^*$-algebras $\mathcal{C}$ is trivial. In particular, for any $v_0 > 1$ we have a canonical homeomorphism

$$R_{>1} \times \mathcal{S}_{v_0} \to \mathcal{S}$$

(18)

$$\nu \mapsto \tilde{\sigma}_{\log v_0}(v, \nu_0(\nu))$$

**Proof.** If we fix $v_0 > 1$ then Theorem 2.24(i) implies that for all $s \in \mathcal{S}(R)$ and $f \in C_c(R_{>1})$ the function $\sigma_{f,s}$ defined by $v \mapsto \sigma_{f,s}(v) := f(v)\phi_{\log v_0}(v, \nu_0(s))$ defines a section of $C_0(R_{>1}) \otimes \mathcal{C}_{v_0}$. It is clear that these sections are dense in $C_0(R_{>1}) \otimes \mathcal{C}_{v_0}$, and using Theorem 2.24 we easily see that the linear map defined by $\sigma_{f,s} \mapsto \sigma_{f,s}$ induces a trivialization of $\mathcal{C}$. The assertion on the spectrum of $\mathcal{C}$ follows immediately from this fact. \qed

2.5. **The Plancherel measure.** Given a normalized affine Hecke algebra $(\mathcal{H}, \tau^d)$ we define a family of positive traces $\{\tau^d_v\}_{v>1}$ on $\mathcal{C}$ by $\tau^d_v(h) = d(v)h_v$ if $h = \sum_{w \in W} h_w N_w \in \mathcal{C}$.

**Definition 2.26.** Let $v > 1$. The spectral measure of the positive trace $\tau^d_v$ (supported on the fibre $\mathcal{S}_v$ of $\mathcal{S}$) is called the Plancherel measure of $(\mathcal{H}_v, \tau^d_v)$ and is denoted by $\nu_{Pl,v}$. We denote the family of spectral measures $\{\nu_{Pl,v}\}_{v>1}$ by $\nu_{Pl}$ and call this the generic Plancherel measure.

2.5.1. **The central character of generic discrete series.** The Plancherel measure is given by density functions which are rational functions with rational coefficients (this will be discussed in more detail below). A crucial step towards this result is the following:

**Theorem 2.27.** The central character $gcc(\delta)$ of a generic irreducible discrete series representation $\delta \in \Delta := \Delta(\mathcal{R}, m)$ is a $W_0$-orbit of generic residual points $gcc(\delta) \in W_0 \setminus \text{Res}(\mathcal{R}, m)$.

2.5.2. **The generic formal degree.** Given $\delta \in \Delta$ it is known [OS2 Theorem 5.12] that the Plancherel measure $\mathbb{R}_{>1} \ni v \mapsto \nu_{Pl,v}(\{\delta_v\})$ is a function on $\mathbb{R}_{>1}$ whose value at $v \in \mathbb{R}_{>1}$ is the specialization at $v$ of a rational function $fdeg(\delta)$ with rational coefficients. The rational function $fdeg(\delta)$ is called the generic formal degree of $\delta$.

**Theorem 2.28.** Let $(\mathcal{H}, \tau^d)$ be a normalized affine Hecke algebra of rank $l$ and with $d \in M_{(k)}$. Let $\delta \in \Delta$ be a generic discrete series character of $\mathcal{H}$, and let $gcc(\delta) = W_0^r$ with $r \in \text{Res}(\mathcal{R}, m)$.

(i) For all $r \in \text{Res}(\mathcal{R}, m)$ the set $\Delta_{W_0^r}(\mathcal{R}, m) := \{\delta \in \Delta(\mathcal{R}, m) \mid gcc(\delta) = W_0^r\}$ is a nonempty, finite set.

(ii) There exists a nonzero rational constant $D_\delta \in \mathbb{Q}^*$ such that for all $v > 1$

$$fdeg(\delta)(v) = D_\delta h^k(r)(v, r(v))$$

(iii) One has $fdeg(\delta) \in M_{(k+l)}$.

**Proof.** The assertion (i) is [OS2 Theorem 5.7].

Claim (ii) follows from [O2 Proposition 3.27(v)] and [OS2 Theorem 5.12]. For (iii) we will first show that $fdeg(\delta) \in M$, i.e. that $fdeg(\delta)(v) = fdeg(\delta) (-v) = \pm fdeg(\delta)(v^{-1})$. The first equality follows from the Euler-Poincaré formula [OS2 Theorem 4.3] for the generic formal degree since the corresponding identity for the generic formal degrees of irreducible representations of finite type Hecke algebras is
true by the explicit results in [Car] Section 13.5). (Here we rely on the technique due to [G] and discussed in the proof of [OS2] Lemma 4.1 to reduce the computation of formal generic degrees of the cross product of a finite type Hecke algebra by a finite group of diagram automorphisms to the case without the cross product).

Let us now look at the second identity. It is easy to see that there exists a unique ring automorphism (C-linear, but obviously not L-linear) $\phi : \mathcal{H} \to \mathcal{H}$ defined on the generators by sending $v \to v^{-1}$, $\omega \to \omega$ (for all $\omega \in \Omega_X$), and $N_s \to -N_s$ (for all $s \in S$). Given $\nu \in \mathbb{R}_+ \subset \mathbb{C}^\times$ we obtain an isomorphism

$$\phi : \mathcal{H}_\nu \to \mathcal{H}_{\nu^{-1}}$$

This algebra isomorphism is an isomorphism of Hilbert algebras as one easily checks.

Recall from [O2, Theorem 3.29, Lemma 3.31] \((20)\)

$$\phi_{|_S} : \mathcal{H} \to \mathcal{H}$$

Recall that the translations of residual points. On the other hand, by [O2, Corollary 3.25] we have, for all $\nu \in \mathbb{R}_+ \setminus \{1\}$, that

$$\nu_{\nu}(|\{W_0\}|) = \nu_{\nu^{-1}}(|\{W_0\}|)$$

for all $W_0$-orbits $W_0\nu$ of residual points. In addition [O2 Lemma 3.31] and [O2 Theorem A.14(ii)] (also see [O3] for classification free proofs) imply that

$$gcc(\delta)(\nu) = gcc(\delta)(\nu^{-1})$$

where the central character $gcc(\delta)(\nu)$ is now viewed as a discrete series character of $\mathcal{Z}_\nu$, and where $\kappa_{\nu} \in \mathbb{Q}^\times$ is a rational number depending only on $\epsilon := \text{sign}(\log(r))$. In addition [O2 Lemma 3.31] and [O2 Theorem A.14(ii)] (also see [O3] for classification free proofs) imply that

$$gcc(\delta)(\nu) = gcc(\delta)(\nu^{-1})$$

for all $\nu \in \mathbb{R}_+ \setminus \{1\}$, that

$$\mu_{(r)}(\nu, r(\nu)) = \kappa_{(r)}(\nu^{-1}, r(\nu^{-1}))$$

for some nonzero rational number $\kappa$. Combining with \((13)\) we see that $\kappa \in \{\pm 1\}$, and by (i) this implies that $fdeg(\delta)(\nu) = \pm fdeg(\delta)(\nu^{-1})$ as required.

Finally we need to check that the vanishing order of $fdeg(\delta)$ at $\nu = 1$ equals $k + l$. Using (i) this follows directly from Proposition \(2.20\) \(\Box\)

2.5.3. The tempered central character map. A residual coset $L$ (over $\mathbb{L}$) determines a parabolic subsystem $R_L := \{\alpha \in R_0 \mid \alpha|_L \text{ is constant}\}$. Let $T_L \subset T$ be the subtorus whose cocharacter lattice $Y_L = Y \cap \mathbb{Q}R_L^\vee$. Let $T^L \subset T$ be the subtorus of which $L$ is a coset (i.e. $T_L$ is the subtorus whose cocharacter lattice is $Y_L := Y \cap (R_L)^\perp$. It is clear from the results stated in paragraph \(2.3.2\) that we have $L = rT^L$ for some residual point $r \in T_L$ of the affine Hecke algebra $\mathcal{H}(R_L, m_L)$, where $R_L$ is the based root datum $R_L := (R_L, X_L, R_L^\vee, Y_L, F_L)$ where $X_L$ is the dual lattice to $Y_L$ (this is the quotient of the lattice $X$ by the sublattice $X_L$ of characters of $T^L$), $m_L$ is defined as in \(2.3.2\) and $F_L$ the base of $R_L$ such that $R_{L,+} = R_L \cap R_{0,+}$. We denote by $K_L$ the finite subgroup $T_L \cap T^L$ (this is the group of characters of the finite abelian group $X_L/(X \cap \mathbb{Q}R_L)$).

A crucial role in the Fourier transform of the Schwartz algebra completion $\mathcal{S}_\nu$ of $\mathcal{H}_\nu$ is played by the groupoid $\mathcal{W}_{\Xi,\nu}$ of tempered standard induction data via the
Fourier transform (see [DeOp1]). In the present context it is natural to consider \( \mathcal{W}_{\Xi, v} \) as the fiber at \( v \) of a smooth, étale groupoid \( \mathcal{W}_{\Xi} \) trivially fibred over \( \mathbb{R}_{> 1} \) (equipped with the trivial groupoid structure). We will first describe \( \mathcal{W}_{\Xi} \) now.

Let \( P \subset F_0 \) be a subset and let \( T^P \subset T \) be the subtorus over \( L \) whose character lattice is \( X \cap Q R P \). There exists a unique real structure on \( T^P \) given by the \( \mathbb{C} \)-anti-linear \( \mathbb{R} \)-algebra automorphism \( L[X^P] \to L[X^P] \) defined by \( v \to v \) and \( x \to -x \) for all \( x \in X^P \). We denote by \( T^P(\mathbb{R}) \to \mathbb{R}^\times \) the corresponding set of \( \mathbb{R} \)-points of \( T^P \) (thus \( T^P(\mathbb{R}) \approx \mathbb{R}^\times \times (S^1)^{|F_0| - |P|} \)). Let \( T^{P,v} \subset T^P(\mathbb{R}) \) be the subset of points that lie above \( \mathbb{R}_{> 1} \subset \mathbb{R}^\times \). We define \( T^P \to \mathbb{R}_{> 1} \) similarly. We equip \( T^{P,v} \to \mathbb{R}_{> 1} \) and \( T^P \to \mathbb{R}_{> 1} \) with the analytic topology.

The set of objects \( \Xi \) is a disjoint union of components \( \Xi_{(P, \delta)} \), where \( P \) runs over all subsets \( P \subset F_0 \) and \( \delta \in \Delta_{P,m} \). The elements of the component \( \Xi_{(P, \delta)} \) are triples \((P, \delta, t)\) with \( t \in T^{P,v} \). We denote by \( \Xi_v \) the subset of objects \((P, \delta, t)\) with \( t \in T^P \). Thus \( \Xi_{(P, \delta, v)} \) is a copy of the compact torus \( T^{P,v} \) for each \( v > 1 \).

The arrows of \( \mathcal{W}_{\Xi} \) are defined as follows. Let \( K_Q = T^P \cap T^{P,v} \to \mathbb{R}_{> 1} \) (thus \( K_{P,v} \) is a finite abelian group for all \( v > 1 \)). Given \( P, Q \subset F_0 \), let \( W(P, Q) := \{ w \in W_0 \mid w(P) = Q \} \). Given objects \( \xi = (P, \delta, t) \in \Xi_v \) and \( \eta = (Q, \sigma, s) \in \Xi_v \), we put

\[
\text{Hom}_W(\xi, \eta) = \{ (k, w) \in K_Q \times W(P, Q) \mid w(P) = Q, (\delta^w)_{k^{-1}} \simeq \sigma, kw(t) = s \}
\]

The composition of arrows is defined by \((k, u) \circ (l, v) = (u(l)k, lv) \). Observe that the arrows respect the fibers \( \Xi_v \) of the natural map \( \Xi \to \mathbb{R}_{> 1} \). Hence if we equip \( \mathbb{R}_{> 1} \) with the trivial groupoid structure then the above map extends in the obvious way to a homomorphism \( \mathcal{W}_{\Xi} \to \mathbb{R}_{> 1} \) of groupoids. We introduce “scaling isomorphisms” between the fibers of the groupoid \( \mathcal{W}_{\Xi} \)

\[
\phi_{\epsilon, v} : \mathcal{W}_{\Xi, v} \to \mathcal{W}_{\Xi, v^\epsilon}
\]

for \( \epsilon > 0 \) which are identical on the sets of morphisms (in the obvious sense, viewing a morphism as an element of some \( K_Q \times W(P, Q) \)) and which act on objects by \( \phi_{\epsilon, v}(P, \delta, t) := (P, \delta, t^\epsilon) \), where \( t^\epsilon \in T^{P,v} \) is defined by \( v(t^\epsilon) = v^\epsilon \) and for all \( x \in X^P \), \( x(t^\epsilon) = x(t) \). In particular the fibration \( \mathcal{W}_{\Xi} \to \mathbb{R}_{> 1} \) is trivial.

The restriction \( \mathcal{F}_Z \) of the Fourier transform of \( \mathcal{S}_v \) to the center \( Z_{S, v} \) of \( \mathcal{S}_v \) defines an isomorphism [DeOp1] Corollary 5.5]

\[
\mathcal{F}_Z : Z_{S, v} \to C^\infty(\Xi_v)^W
\]

between \( Z_{S, v} \) and the ring of smooth, \( W \)-invariant functions on \( \Xi_v \). This isomorphism extends to an isomorphism of \( C^* \)-algebras, and shows that we may define a continuous map

\[
cc^{\text{temp}}_v : \mathcal{S}_v \to W/\Xi_v
\]

which sends a subrepresentation \( \pi < \pi(\xi) \) of the standard tempered representation \( \pi(\xi) \) to the orbit \( W\xi \in W/\Xi_v \) (“disjointness”, see [DeOp1] Corollary 5.6)). This map is clearly compatible with the scaling maps (17) and (25). In particular these maps are the fibers of a continuous map

\[
cc^{\text{temp}} : \mathcal{S} \to W/\Xi
\]

Another consequence of (26) is the fact that the components \( \mathcal{S}_{(P, \delta, v)} \) of the tempered spectrum [DeOp1] Corollary 5.8] of \( \mathcal{H}_v \) are parametrized by association classes of
pairs \((P, \delta)\) with \(P \subset F_0\) a standard parabolic subset and \(\delta \in \Delta_P\). Thus the same holds for the spectrum of \(C\):

\[
\mathcal{G} = \coprod_{(P, \delta) \sim} \mathcal{G}(P, \delta)
\]

In addition, for each \(v > 1\) there exist dense open subsets \(\mathcal{G}'(P, \delta, v) \subset \mathcal{G}(P, \delta, v)\) and a dense open subsets \(\Xi'(P, \delta, v) \subset \Xi(P, \delta, v)\) such that

\[
\text{cc} temp \mid_{\mathcal{G}'(P, \delta)} : \mathcal{G}'(P, \delta) \to W(P, \delta) \setminus \Xi'(P, \delta, v)
\]

is a homeomorphism (see [O2, Theorem 4.39]; the map \([\pi]\) (loc. cit.) yields the inverse homeomorphism). It is obvious from the definitions of [O2, paragraph 4.5.2] that the sets \(\mathcal{G}'(P, \delta, v)\) and \(\Xi'(v)\) are compatible with the scaling isomorphisms \((17), (25)\). Hence we obtain open dense subsets \(\mathcal{G}'(P, \delta) \subset \mathcal{G}(P, \delta)\), and open dense subsets \(\Xi(P, \delta, v) \subset \Xi(P, \delta)\), and a homeomorphism

\[
\text{cc} temp \mid_{\mathcal{G}(P, \delta)} : \mathcal{G}'(P, \delta) \to W(P, \delta) \setminus \Xi'(P, \delta, v)
\]

### 2.5.4. The algebraic central character map.

The restriction \(p^\text{temp}_Z\) of the algebraic central character map

\[
p_Z : \text{Irr}(\mathcal{H}) \to W_0 \setminus T(C)
\]

(where \(T(C)\) denotes (by abuse of notation) the group of points of the restriction of scalars \(R_{L/C}(T)\) of the algebraic torus \(T\) defined over \(L\)) to \(\mathcal{G} \subset \text{Irr}(\mathcal{H})\) factors via \(\text{cc} temp\). Indeed, the map

\[
p_Z \mid \Xi \to W_0 \setminus T(C)
\]

is well defined and it is obvious that we have a commuting diagram as follows:

\[
\begin{array}{ccc}
\mathcal{G} & \xrightarrow{\text{cc} temp} & W \setminus \Xi \\
\downarrow p^\text{temp}_Z & & \downarrow p_Z \\
W_0 \setminus T(C) & = & W_0 \setminus T(C)
\end{array}
\]

Now we describe the image \(S := p^\text{temp}_Z(\mathcal{G})\) of \(p^\text{temp}_Z\), and its components. If \(L\) is a residual coset over \(L\) (see section 2.3) we can write it in the form \(L = rT^P\) where \(r\) is a \((R_L, m_L)\)-residual point over \(L\) by Theorem 2.19(ii). We define the corresponding tempered residual coset \(L^\text{temp}\) to be the subset

\[
L^\text{temp} := \bigsqcup_{v > 1} r(v)T^P_{V, n} \subset T(C)
\]

of its \(C\)-points. We equip \(L^\text{temp}\) with the relative topology with respect to the analytic topology on the complex variety \(T(C)\). The set of points of \(L^\text{temp}\) that lie over \(v > 1\) is denoted by \(L^\text{temp}_v\). This is a tempered residual coset in the sense of [O2]. We remark that \(L^\text{temp}\) is independent of the chosen decomposition \(L = rT^P\).

**Theorem 2.29.** The image \(S = p^\text{temp}_Z(\mathcal{G})\) of \(p^\text{temp}_Z\) is equal the union

\[
S := \bigsqcup_{L \text{ residual over } L} L^\text{temp} \subset W_0 \setminus T(C)
\]
For all \( L \in \mathcal{L} \) the subset \( W_0 L^{\text{temp}} \subset W_0 \setminus T(\mathbb{C}) \subset S \) is a component of \( S \).

Proof. Consider \( S_{(P, \delta)} := p_{\Sigma}^{\text{temp}}(\mathfrak{S}(P, \delta)) \). By the above commuting diagram we can rewrite this as \( S_{(P, \delta)} = p_{\Sigma}(W_0(\mathfrak{P}(P, \delta)) \setminus \Xi_{(P, \delta)}) \). Using the definition of \( p_{\Sigma} \) and of \( \Xi_{(P, \delta)} \), Theorem \ref{thm:lemma1} and Theorem \ref{thm:lemma2} it is clear that

\[
S_{(P, \delta)} = \bigsqcup_{\nu > 1} W_0(\nu(r(T)_{P_{(\nu)}}) \subset W_0 \setminus T(\mathbb{C})
\]

where \( r \) is an \((P, m_P)\)-residual point of \( T_P \) such that \( \text{gcc}(\delta) = W_P r \). By Theorem \ref{thm:lemma1}(ii) we see that \( L_{(P, r)} = rT_P \) is a residual coset, and clearly \( S_{(P, \delta)} = W_0 L_{(P, r)}^{\text{temp}} \).

This proves that \( S = \bigcup_{(P, \delta) / \sim} S_{(P, \delta)} \). The subsets \( S_{(P, \delta)} \) are clearly connected. We will now show that these sets are in fact the components of \( S \) (thus proving the remaining assertions of the Theorem). It is enough to show that if \( L_1 = r_1 T_{P_1} \) and \( L_2 = r_2 T_{P_2} \) are residual cosets then their tempered parts do not intersect each other unless possibly if \( L_1 \) and \( L_2 \) are in the same \( W_0 \)-orbit. It is enough to check this for the fibers \( L_{i, \nu} \) over \( \nu > 1 \), and this is the content of Lemma \ref{lem:lemma2} below. \( \Box \)

Lemma 2.30. Let \( L_1, L_2 \in \mathcal{L}_v \) for some \( \nu > 1 \). Then

\[
L_1^{\text{temp}} \cap L_2^{\text{temp}} \neq \emptyset \implies L_1 = w(L_2) \text{ for some } w \in W_0
\]

Proof. This is \cite{O2} Theorem A.18, but since this fact did not play a role in \cite{O2} the proof was indicated only very briefly there. We will fill in the details now.

Write \( L_1 = r_1 T_{P_1} \) and \( L_2 = r_2 T_{P_2} \) with \( r_i \in T_{P_i, v} \) residual points for \((P_i, m_{P_i})\), where \( R_i \subset R_0 \) are parabolic subsets of roots. Write \( c_i = |r_i| \). Suppose that \( t \in L_1^{\text{temp}} \cap L_2^{\text{temp}} \). Then clearly \( |t| = c_1 = c_2 \). Unfortunately this real point is not necessarily the center of a real residual affine subspace (in the sense of \cite{HO1}, also see \cite{O2} Appendix A and \cite{O3}) of the vector group \( V := \text{Hom}(X, \mathbb{R}_+) \), and we need to manipulate a bit (by shrinking \( R_0 \) in a suitable way) to arrive at such a favorable situation.

Consider the subset \( R_{12} \subset R_0 \) consisting of the roots \( \alpha \in R_0 \cap R_m \) such that \( \alpha(L_1 \cap L_2) \subset \mathbb{R}_+ \) together with the roots \( \beta \in R_0 \setminus R_m \) such that \( \pm \alpha(L_1 \cap L_2) \subset \mathbb{R}_+ \). It is clear that \( R_{12} \subset R_0 \) is a subsystem of roots. We form the affine root datum \( \mathcal{R}_{12} = (X, R_{12}, Y, R_{12}^+) \). We equip \( R_{12} \) with the restriction to \( R_{12} \) of the half integral functions \( m_{12} \) on \( R_0 \). Then this corresponds to a multiplicity function \( m_{12} \) for \( \mathcal{R}_{12} \).

We thus obtain a pair \((\mathcal{R}_{12}, m_{12})\) consisting of an affine root datum and a multiplicity function. The first important property of \((\mathcal{R}_{12}, m_{12})\) is the fact that \( L_1 \) and \( L_2 \) are still residual cosets with respect to \((\mathcal{R}_{12}, m_{12})\). Indeed, the pole sets and the zero sets of roots for either \( L_1 \) or \( L_2 \) are clearly subsets of \( R_{12} \) (see \ref{thm:lemma2}.

Put \( s_i = t c_i^{-1} \) (recall that \( t \in L_1^{\text{temp}} \cap L_2^{\text{temp}} \)). By definition of \( R_{12} \) the unitary points \( s_i \) satisfy \( \alpha(s_i) = 1 \) for all \( \alpha \in R_{12, m} \). Hence the points \( s_i \) are fixed by \( W(R_{12}) \), and \( L'_i := s_i^{-1}L_i = c_i T_{P_i} \) is also a \((\mathcal{R}_{12}, m_{12})\)-residual coset. We define \( L'_i := L'_i \cap V \) where \( V := \text{Hom}(X, \mathbb{R}_+) \) denotes the vector group of “infinitesimally real” points of \( T \). We view \( V \) as a real vector space via the exponential homeomorphism. We have obtained two residual real affine subspaces \( L'_i \) of \( V \) with respect to the root system \( R_{12} \subset R_0 \) and a multiplicity function \( m_{12}^h \) for a suitable \( h \) (see \cite{O2} Theorem A.7). Moreover, the center of \( L'_i \) equals \( c_i \) and, as we have seen above, these centers coincide \( c_1 = c_2 \). Now we are reduced to showing that the set of \( W(R_{12}) \)-orbits of \((R_{12}, m_{12}^h)\) residual real affine subspaces is in bijection with the set of \( W(R_{12}) \)-orbits of the centers of residual real affine subspaces.
If the parameters function $m_{12}^L$ is constant on $R_{12}$ then this assertion follows from the Bala-Carter theorem on the classification of nilpotent orbits of $\mathfrak{g}(R_{12}, \mathbb{C})$, see [O3, Remark 7.5]. Observe that this problem for affine residual subspaces is completely reducible, and thus we may assume that $R_{12}$ is irreducible without loss of generality.

If $R_{12}$ is simply laced we are done by the above application of the Bala-Carter theorem. If $R_{12}$ is of type $B_n$ or $C_n$ then this was proved by Slooten [Slo1, Theorem 1.5.3, Theorem 1.5.5]. For $R_{12}$ of type $G_2$ it is a trivial verification, and for type $F_4$ is a more involved case by case verification which was done by Slooten (private communication, 2001).

**Proposition 2.31.**

(i) The finite map $p_{Z}^{\text{temp}} : \mathfrak{g} \to S$ maps components to components. The components of $S$ are of the form $p_{Z}^{\text{temp}}(\mathfrak{g}(P,\delta)) = S(P,\delta)$.

(ii) For each component $S(P,\delta)$ of $S$ there exists a residual coset $L \in L$, unique up to the action of $W_0$ on $L$, such that $S(P,\delta) = W_0 \backslash W_0 L^{\text{temp}}$.

(iii) Let $L \in L$ be as in (ii). The map $q_L : L^{\text{temp}} \to S(P,\delta)$ given by $q_L(t) = W_0 t$ is an identification map whose fibers are the $N_{W_0}(L)$-orbits in $L^{\text{temp}}$. Hence $S(P,\delta) \approx N_{W_0}(L) \backslash L^{\text{temp}}$.

(iv) The restriction of $p_{Z}^{\text{temp}}$ to $\mathfrak{g}(P,\delta)$ is a finite covering map onto an open, dense subset $S'(P,\delta) \subset S(P,\delta)$ of $S(P,\delta)$.

(v) Let $L \in L$ be as in (ii), and let $(L^{\text{temp}})' \subset L^{\text{temp}}$ be the inverse image of $\mathfrak{g}(P,\delta)$ in $S(P,\delta)$ with respect of the quotient map $q_L$ of (iii). Then $N_{W_0}(L)$ acts freely on $(L^{\text{temp}})'$. This gives $S'(P,\delta)$ the structure of a smooth manifold.

**Proof.** The first assertion (i) was shown in the first part of the proof of Theorem 2.29. In this argument it was also explained that $S(P,\delta)$ has the form $S(P,\delta) = W_0 \backslash W_0 L^{\text{temp}}$ for a residual coset $L$ of the form $L = L_{(P,\gamma)}$, implying (ii). Assertion (iii) is obvious, and (iv) follows directly from [O2, Corollary 4.40] and Theorem 2.25. Finally (v) follows from [O2, Lemma 4.35, Lemma 4.36].

**2.5.5.** The Plancherel measure $\nu_{P_1}$.

**Lemma 2.32.** Let $L \in L$. There exists a $\epsilon_L \in \{\pm 1\}$ such that the function $m_L := \epsilon_L \mu^{(L)}|_{L^{\text{temp}}}$ is a smooth, $N_{W_0}(L)$-invariant positive function on $L^{\text{temp}}$.

Consider a component $\mathfrak{g}(P,\delta)$ of $\mathfrak{g}$. As we have seen in Theorem 2.29 its image $p_{Z}^{\text{temp}}(\mathfrak{g}(P,\delta)) = S(P,\delta) \subset S$ is a component of $S$. From Corollary 2.31 we see that there exists a residual coset $L$ such that $S(P,\delta) = W_0 \backslash W_0 L^{\text{temp}}$. Consider the smooth $\mathbb{R}_{>1}$-family of volume forms $\nu_L$ on $L^{\text{temp}}$ which has density function $m_L$ with respect to the family $dL(t)$ of normalized $T_{\nu_L}^{W_0 \gamma}$-invariant volume forms on $L^{\text{temp}}$. Let $\nu'_L$ denote the restriction of $\nu_L$ to $(L^{\text{temp}})'$. By Proposition 2.31(v) there exists a unique smooth $\mathbb{R}_{>1}$-family of volume forms $\nu'_S$ on $S'(P,\delta)$ such that $\nu'_L = q_L^{*}(\nu'_S)$. Using Proposition 2.31(iv) we define a smooth $\mathbb{R}_{>1}$-family of volume forms on $\mathfrak{g}(P,\delta)$ by $\nu'_\mathfrak{g} := (p_{Z}^{\text{temp}})^{*}(\nu'_S)$.

**Theorem 2.33.** Let $P \subset F_0$, $\delta \in \Delta_P$ and let $L \in L$ such that $S(P,\delta) = W_0 \backslash W_0 L^{\text{temp}}$. There exists a constant $a_{(P,\delta)} \in \mathbb{Q}_{+}$ such that the restriction of $\nu_{P_1}$ to $\mathfrak{g}(P,\delta)$ is given by $\nu'_{P_1} = a_{(P,\delta)} \cdot (\nu'_\mathfrak{g})$, where $i$ denotes the open embedding $i : \mathfrak{g}(P,\delta) \to \mathfrak{g}(P,\delta)$.

**Proof.** This follows directly from [O2, Theorem 4.43] and [OS2, Theorem 5.12].
3. The spectral transfer category

3.1. Spectral transfer maps.

3.1.1. Definition of spectral transfer maps. In view of Proposition 2.15 the following definition makes sense. Let \( \mathcal{H}_i = \mathcal{H}(\mathcal{R}_i, m_i) \) (\( i = 1, 2 \)) be two (possibly extended, not necessarily semisimple) affine Hecke algebras, with normalizing elements \( d_i \in M \). Recall from paragraph 2.5.3 that given a residual coset \( L \) of \( T_2 \) we can write \( L = rT_L \) with \( r \in T_L \) a residual point. Let \( K_L := T_L \cap T_L^e \), a \( N_{W_2,0}(L) \)-stable finite abelian group acting faithfully on \( L \), where \( N_{W_2,0}(L) \) is the stabilizer of \( L \) in \( W_2,0 \). Then \( L \cap T_L = K_{Lr} \). Let \( K_L^n := K_L \cap N_{W_2,0}(L)/Z_{W_2,0}(L) \) (with \( Z_{W_2,0}(L) \) the pointwise stabilizer of \( L \)), and \( K_L^0 \subset N_{W_2,0}(L) \) its inverse image in \( N_{W_2,0}(L) \). Then \( K_L^n = W(R_L) \cap N_{W_2,0}(L) = N_{W(L)}(L) \), and \( N_{W,0}(L) \) acts on \( L := L/K_L^n \). Clearly \( \mu_{\mathcal{R}, m, d} \) is \( K_L^n \)-invariant, and thus descends to a rational function \( \mu_{\mathcal{R}, m, d} \) on \( L \).

**Definition 3.1.** We first assume that \((\mathcal{R}_1, m_1)\) is (semi)-standard. By a (semi)-standard spectral transfer map from \( \mathcal{H}_1 \) to \( \mathcal{H}_2 \) we mean a morphism \( \phi_T : T_1 \to L_n := L/K_L^n \) over \( L \), where \( L \subset T_2 \) denotes a residual coset, such that:

- (T1) \( \phi_T \) is finite.
- (T2) \( \phi_T(e) \in (T_L \cap L)/K_L^n \), and if we declare \( \phi_T(e) \) to be the unit of the \( T_L^e := T_L/K_L^n \)-torsor \( L_n \), then \( \phi_T \) is a homomorphism of algebraic tori over \( L \).
- (T3) There exists an \( a \in \mathbb{C}^\times \) such that \( \phi_T^*(\mu_{\mathcal{R}_2, m_2, d_2}) = a\mu_{\mathcal{R}_1, m_1, d_1} \).

If \((\mathcal{R}_1, m_1)\) is not semi-standard we impose the additional condition:

- (T4) The \( \phi_T \)-image of a \( W_{1,0} \)-orbit is contained in a single \( N_{W_2,0}(L) \)-orbit.

**Proposition 3.2.** A spectral transfer map \( \phi_T \) has the following property: if \( L_1 \subset T_1 \) is a residual coset with respect to \((\mathcal{R}_1, m_1)\) then there exists a residual coset \( L_2 \subset L \) such that \( \phi_T(L_1) = K_L^nL_2/K_L^n \). In this case we also have \( \phi_T(L_1^{temp}) = L_2^{temp} := K_L^nL_2^{temp}/K_L^n \). Conversely, if \( L_2 \subset L \) is a residual coset then \( \phi_T^{-1}(L_2^n) \) (with \( L_2^n := K_L^nL_2/K_L^n \)) is a finite union of residual cosets of \( T_1 \) (and analogously for tempered forms of residual cosets of \( T_2 \)).

**Proof.** Clear by the fundamental property Theorem 2.19(i) of residual cosets. The assertion about the tempered forms of the residual cosets follows easily from (T1) and (T2). \( \square \)

**Proposition 3.3.** The constant \( a \) of (T3) is always rational.

**Proof.** Let \( \phi_T(T_1) = L_n \) with \( L \subset T_2 \) a residual coset. By Theorem 2.19(ii), equation (T3) is equivalent to

\[
\phi_T^*(\mu_{\mathcal{R}_2, m_2, d_2}) = a\mu_{\mathcal{R}_1, m_1, d_1}
\]

Now consider formula (14) for \( \mu_{\mathcal{R}_2, m_2, d_2} \). By [O2] Theorem 3.27(v)] and Proposition 2.20 we have, if \( d_2 \in M_k \) and \( l = \text{codim}(L) \), that \( \mu_{\mathcal{R}_2, m_2, d_2} \in M_{k+l} \). Obviously (38) implies that \( d_1 \in M_{k+l} \) as well, and thus after dividing by \( d_1 \) we can specialize the resulting identity

\[
\phi_T^*(\mu_{\mathcal{R}_2, m_2, d_2/d_1}) = a\mu_{\mathcal{R}_1, m_1, 1}
\]

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at the generic point of the fiber \( T_{1,v} \) of \( T_1 \) at \( v = 1 \). But from the definition of \( \mu_{R_2, m_1, 1} \) and from equation (14) for \( \mu_{R_2, m_2, d_1} \), we see that this yields an equation of constants of the form \( c_i = ac_2 \) with \( c_i \in \mathbb{Q}^\times \), hence the conclusion. \( \square \)

**Proposition 3.4.** If \( \phi_T \) is a spectral transfer map from \((\mathcal{H}_1, \tau_{d_1})\) to \((\mathcal{H}_2, \tau_{d_2})\) such that \( \phi_T(T_1) = L_n \) with \( L \subset T_2 \) a residual coset. Then \( \forall w_1 \in W_{1,0} \exists w_2 \in N_{W_2,0}(L) \) such that \( \phi_T \circ w_1 = w_2 \circ \phi_T \).

**Proof.** In the non-semi-standard case there is nothing to prove (by (T4)), so let us assume that we are in the semi-standard case. We may assume that \( T^L = T^P \) for some standard parabolic subset \( P \subset F_{2,0} \). Choose \( \delta \in \Delta_P \) such that \( S_{(P,\delta)} = N_{W,0,L} \setminus L^{temp} \). Recall from Theorem 2.33 that the Plancherel measure on \( S_{(P,\delta)} \) is given by the invariant smooth density function (with respect to the normalized \( T_u^P \)-invariant measures \( d^L(t) \)) \( m_L \) on \( L^{temp} \). Let us denote by \( m_\Xi \) the lift of \( m_L \) to \( \Xi_{(P,\delta)}^{temp} \) via the isomorphism \( j_L : \Xi_{(P,\delta)} \to L \) given by \( j_L((P,\delta,t)) = rt \), where we have chosen a generic residual point \( r \in T_P \) such that \( W_P r \) is the generic central character of \( \delta \). Then the rational function \( c(\xi) \) on \( \Xi_{(P,\delta)} \) defined by

\[
(40) \quad c(\xi) := \prod_{\alpha \in R_0, + \setminus R_P, +} c_\alpha(j_L(\xi))
\]

is independent of the choice of \( r \) as above. We have (see [DeOp1, Theorem 4.3, Proposition 9.8]) (compare to (14) and Theorem 2.28):

\[
(41) \quad m_\Xi(\xi) = v^{-2m_{w_0}(w^P)}|W_P/K_P|^{-1}d\xi(c(\xi)c(w^P(\xi)))^{-1}
\]

where \( w^P = w_0 w_{-1} \in W^P \) is the longest element.

Recall that \( c(\xi)^{-1} \) is smooth on \( \Xi_{(P,\delta)}^{temp} \) and \( c(w^P(\xi)) = c(\xi) \) on \( \Xi_{(P,\delta)}^{temp} \) (by [DeOp1, Proposition 9.8]). Hence the zero set of \( m_\Xi \) on \( \Xi_{(P,\delta)}^{temp} \) is the same as the zero set of \( c(\xi)^{-1} \) on \( \Xi_{(P,\delta)}^{temp} \). This zero set is a union of connected components of certain hypersurfaces of the form \( (P,\alpha)(\xi) = \text{constant} \), where \( \alpha \in R_2, \Omega \setminus R_2, \text{root system} \) \( \mathfrak{g_\xi} \), and all elements of \( W(\mathfrak{g_\xi}) \) belong to the isotropy group \( W_\xi \) of \( \xi \) for the action of \( W \).

Let \( L_1 = T_1 \) (which is a residual coset of \( T_1 \)) and let \( (P_1, \delta_1) \) be the unique pair \( P_1 = F_{1,0}, \delta_1 = 1 \) associated to \( L_1 \). We will identify \( \Xi_{(P_1,\delta_1)} \) and \( T_1 \) via \( j_{L_1} \). The assumption on \( m_+^{(\alpha)} \) implies that there exists a point \( o_1 \in T_{1,u} = L_1^{temp} \) such that for all \( \alpha \in R_1,0 \) there exists a \( (P_1,\alpha) \)-mirror containing \( o_1 \). In particular, \( o_1 \) is \( W_1 \)-invariant, and the action of \( W_1 \) on \( T_1^{temp} \) is generated by the mirror reflections in the mirrors through \( o_1 \) (see Proposition 2.16). Choose \( \xi_o \in \Xi_{(P_1,\delta_1)}^{temp} \) such that \( K_1^{\xi_o} j_{L_1}(\xi_o) = \phi_T(o_1) \). Observe that (T3) and (40) imply that for each mirror \( M(L_1,\alpha) \subset L_1^{temp} \) \( \Xi_{(P_1,\delta_1)}^{temp} \) there exists a mirror \( M(P,\beta) \subset \Xi_{(P,\delta)}^{temp} \) through \( \xi_o \) such that \( \phi_T(M(P,\alpha)) = K_1^{\xi_o} j_{L_1}(M(P,\beta)) \). Let \( r_{(P,\beta)} \in W(\mathfrak{g_\xi}) \) be the reflection in \( M(P,\beta) \). Then (T1) and (T2) imply that we can choose an open \( r_{\alpha} \)-invariant neighborhood \( V \subset T_1 \) of \( o_1 \) such that \( \phi_T \) restricts to an isomorphism on \( V \), and a \( r_{(P,\beta)} \)-invariant open neighborhood \( U \supset \xi_o \) such that the covering \( \pi : L \to L_n \) restricts to an isomorphism on \( j_L(U) \) and such that \( \pi(j_L(U)) = \phi_T(V) \), and such that \( r_{(P,\beta)}(\xi_o) = (\phi_T^{-1} \circ \pi \circ j_L)^{-1} \circ r_{\alpha} \circ (\phi_T^{-1} \circ \pi \circ j_L)(\xi_o) \).
By the previous paragraph $\tau_{(P,\beta)} \in W$. This means by definition that there exists a $k \in K_P$ and a $n \in W_{2,0}$ with the following properties: $n(P) = P$, $(\delta^n)_{k-1} \simeq \delta$ and $\tau_{(P,\beta)}(\xi) = (k \times n)(\xi)$ for all $\xi = (P,\delta,t) \in \Xi_{(P,\beta)}$. In other words, we have for all $t \in T_P$ that $j_L(\tau_{(P,\beta)}(\xi)) = rkn(t)$. Since $(\delta^n)_{k-1} \simeq \delta$ we have $w = k^{-1}n(r)$ for some $w \in W(R_{2,P})$. Consequently $w(j_L(\tau_{(P,\beta)}(\xi))) = n(rt) = n(j_L(\xi))$, and we see that the action of $\tau_{(P,\beta)}$ on $\Xi_{(P,\beta)}$ is given by the Weyl group element $w^{-1}n \in N_{W_{2,0}}(L)$ via the isomorphism $j_L$. In view of the above we obtain that $\phi_T \circ r_\alpha = w^{-1}n \circ \phi_T$ with $w^{-1}n \in N_{W_{2,0}}(L)$. Since we can do this construction for any $\alpha \in R_{1,0}$ the claim is proved.

**Corollary 3.5.** A spectral transfer map $\phi_T$ from $\mathcal{H}_1$ to $\mathcal{H}_2$ induces a morphism

$$
\phi_Z : W_{1,0}\setminus T_1(\mathbb{C}) \to W_{2,0}\setminus T_2(\mathbb{C})
W_{1,0}(t) \to W_{2,0}(\phi_T(t))
$$

We have $\phi_Z(S_1) \subset S_2$, and the restriction $\phi_T^{\text{temp}}$ of $\phi_Z$ to $S_1$ is a finite map sending components to components.

**Proof.** Let $L \subset T_2$ be a residual coset such that $\phi_T(T_1) = L^n = L/K^n_L$. The existence of $\phi_Z$ follows from Proposition 3.4 and the fact that $K^n_L \subset W_{2,0}(L) := N_{W_{2,0}}(L)/Z_{W_{2,0}}(L)$. This map is finite by (T1). By Proposition 2.31 and Proposition 3.2 we see that $\phi_Z$ maps components of $S_1$ onto components of $S_2$. □

### 3.1.2. Reduction to irreducible types in the general semisimple case

Suppose that $\phi_T$ is a spectral transfer map from $\mathcal{H}_1$ to $\mathcal{H}$, with $\mathcal{H} = \mathcal{H}(\mathcal{R}, m)$ and $\mathcal{H}_1 = \mathcal{H}(\mathcal{R}_1, m_1)$ both semi-simple affine Hecke algebras. We will show in this paragraph that we can essentially reduce to a product of situations where $\mathcal{R}$ is irreducible. Moreover we will see below (cf. Proposition 3.6(3)), that if $\mathcal{H}(\mathcal{R}, m)$ is semi-standard and irreducible, then $\mathcal{R}_1$ is forced to be irreducible too. Hence if $\mathcal{R}$ is semi standard it will follow that modulo covering maps, an STM from $\mathcal{H}_1$ to $\mathcal{H}$ is always a fibered product (over $\text{Spec}(L)$) of STMs between the simple factors, provided we normalize the simple factors of $\mathcal{H}_1$ and of $\mathcal{H}$ in a coherent way.

If $\mathcal{H}_1$ has positive rank, consider the covering maps $\text{2}$. If we precompose $\phi_T$ with the covering map $\text{Hom}(P((\mathcal{R}_1)_m), C^\times) \to T_1$, it is obvious from the definition that we obtain a spectral transfer map from $\mathcal{H}(\mathcal{R}_1^m, m)$ to $\mathcal{H}$, with the same image. Hence we may and will assume from now on that the $\mathcal{H}_1$ is a tensor product (over $\mathcal{L}$) of affine Hecke algebras $\mathcal{H}_1^i$, each of irreducible type and of positive rank, or $\mathcal{H}_1$ has rank 0, in which case $\mathcal{H}_1 \simeq \mathcal{L}$. In either case, let $T_1 = T_1^1 \times_{\text{Spec}(\mathcal{L})} T_1^2 \times_{\text{Spec}(\mathcal{L})} \cdots \times_{\text{Spec}(\mathcal{L})} T_1^l$ be the corresponding fibered product decomposition of $T_1$. From now on we will suppress $\text{Spec}(\mathcal{L})$, but all products are tacitly assumed to be fibered products over $\text{Spec}(\mathcal{L})$. We normalize the factors $\mathcal{H}_1^i$ such that the $\mu$-function $\mu_1$ of $\mathcal{H}_1$ is the product of the $\mu$-functions $\mu_1^i$ of $\mathcal{H}_1^i$. We embed the factors $T_1^i$ in $T_1$ in the canonical way, and identify the image with $T_1^i$.

Assume first that the rank of $\mathcal{H}_1$ (hence, a fortiori, of $\mathcal{H}$) is positive, and let $L \subset T$ be the residual coset such that $L_n = \text{Im}(\phi_T)$, and write $L = rTP$ with $T_P \subset T$. Using the action of $W_0$ we may and will assume that there exists a subset $P \subset F_0$ such that $T^P$ is the maximal subtorus on which the roots in $P$ all vanish. Let $R_P \subset R_m$ denote the corresponding ”standard parabolic subsystem” of $R_m$. Assume that $R_0$ is a disjoint union of irreducible parabolic subsystems $R_0 = R_0^1 \cup R_0^2 \cup \cdots \cup R_0^l$. Then, with $K := \text{Hom}(X/Q(R_0), C^\times)$, we have $\overline{T} := T/K = \overline{T}^1 \times \cdots \times \overline{T}^l$, with each
factor a torus over \( L \) of positive rank. Moreover \( L = \langle L^1 \times L^2 \times \cdots \times L^l \rangle / (K \cap T^P) \), where \( L^i \) is a connected component in the inverse image in \( T \) of a residual coset \( \overline{L}^i = \overline{r}T^P \subset \overline{T}^i \) (of positive dimension) with respect to the irreducible root datum \( \mathcal{R}_0^1 := (Q(R_0^1), R_0^1, P((R_0^1)^\lor), R_0^1, F_0^1) \). It follows directly from the definition that \( L_n = \prod_i(L_n^i) / (K \cap T^P), \mathcal{T}_n = \prod_i(L_n^i) \) and that \( (\overline{L})_n \) is a quotient of \( L_n^i \). Let \( \pi_i \) denote the projection from \( L_n \) to \( (\overline{L})_n \). Hence we can compose \( \phi_T \) with the quotient map \( L_n \to \mathcal{T}_n, \) and it is obvious from the definitions that the result is a spectral transfer map \( \overline{\psi}_T \) from \( \mathcal{H}_1 = \mathcal{H}(\mathcal{R}_0^1, m_1) \) to \( \mathcal{H}(\mathcal{R}_0^1, m) \) (cf. (2)). Proposition 3.4 implies that for each \( j \in \{1, \ldots, 1\} \) there exists a unique \( i \in \{1, \ldots, l\} \) such that \( \pi_i^i \circ \overline{\psi}_T|_{\mathcal{T}_n^i} \) is nonconstant. This implies that we can partition the set \( \{1, ..., l\} \) into disjoint nonempty subsets \( \Pi(i) \) such that for each \( i \), with \( T_1(i) := \prod_{j \in \Pi(i)} T_j^1 \subset T_1 \), the map \( \phi(i) := \pi_i^\circ \overline{\psi}_T|_{T_1(i)} \) is an STM from \( \otimes_{j \in \Pi(i)} \mathcal{H}_1^j \) (tensor products over \( L \)) to \( \overline{\mathcal{T}}^i := \mathcal{H}(\mathcal{R}_0^1, m_i) \).

If \( \mathcal{H}_1 \) has rank 0 then \( \text{Im}(\phi_T) = r \in T \) is a residual point, and by definition we have \( \mu(\overline{r}) = \tau_1(1) \) up to a constant. The image \( \overline{r} \in \overline{T} \) of \( r \) is also a residual point, and \( \ast \) is equivalent to \( \mu(\overline{r}) = \tau_1(1) \). In the latter equation, the left hand side is a product over the irreducible factors \( \mu(\overline{r}_j) = \tau_1(1) \) (with \( j = 1, \ldots, l \)). Hence we can formally write \( \mathcal{H}_1 = \mathcal{H}_1^1 \otimes \mathcal{L} \cdots \otimes \mathcal{H}_1^l, \) with traces \( \tau_i \) normalized appropriately such that the canonical maps \( \overline{\psi}_T : \text{Spec}(L) \to \overline{T}^i \) associated to \( \tau_i \) are STMs. This achieves our goal in the rank 0 case, \( \overline{\psi}_T \) being the product of the STMs \( \overline{\psi}_T \).

3.1.3. Further properties of spectral transfer maps. The proof of Proposition 3.4 can be refined to yield important additional information about the image of a spectral transfer map. Suppose that \( \phi_T \) is a spectral transfer map from \( \mathcal{H}_1 \) to \( \mathcal{H}, \) with \( \mathcal{H} = \mathcal{H}(\mathcal{R}_0, m) \) and \( \mathcal{H}_1 = \mathcal{H}(\mathcal{R}_1, m_1) \) both semi-simple affine Hecke algebras.

By paragraph 3.1.2 we may assume that \( \mathcal{H} \) is of irreducible type. The dual affine Weyl group \( \mathcal{W} = W(R_{m}) \times Y \) (cf. (3)) naturally acts on \( t \) via translations over \( Y \) (a lattice which contains the lattice \( Q(t) \)) and the Weyl group \( W(R_{m}) \) (or equivalently \( W(R_{m}^0) \) or \( W_0 \)). In particular, the affine Weyl group \( W(R_{m}^{(1)}) \) is a subgroup of \( \mathcal{W} \) of finite index. Let us consider the \( W(R_{m}^0) \)-equivariant covering \( t \to T_u \) whose group of deck transformations is \( Q(R_{m}) \). We choose a fundamental dual alcove \( C^0 \subset t \). It is a fundamental domain for the action of the normal subgroup \( W(R_{m}^{(1)}) \) of \( \mathcal{W} \). Recall that \( \mathcal{R}_m^m \) is a based root datum, with base \( F_{m,0} \). Let \( \mathcal{F}^m \) be the corresponding base for the affine root system \( R^{(1)}_m \). It consists of the base of \( F_{m,0} \) together with the affine root \( a_0^\lor := 1 - \psi, \) where \( \psi \) denotes the highest root of \( R_{m} \).

We will assume until Corollary 3.7 that \( P \subset F_0 \) is a proper subset. Then \( r \in T_P \cap L \) is a residual point in \( T_P \) for the semisimple quotient affine Hecke algebra \( \mathcal{H}_P \) of the “Levi subalgebra” \( \mathcal{H}_P \subset \mathcal{H} \) of positive rank. Let \( r = sc \) be the polar decomposition of \( r \in T_P \). Then \( s \in T_{P,u} \) and \( c(v) \in T_{P,v} \) for all \( v > 0, \) and \( c \) is itself a residual point for the subsystem \( R_{P,s} = \{ \alpha \in R_P \mid \alpha(s) = 1 \} \) such that \( c(1) = e \).

In particular it follows that \( R_{P,u} \subset R_P \) is a maximal rank sub root system. It follows easily that a lift \( l(1) \subset t \) of \( L(1)^{temp} = sT_u^P \subset T_u \) in \( t \) with respect to the covering \( t \to T_u \) is conjugate under \( W(R_{m}^{(1)}) \) to a unique affine subspace \( t^{J} \) which is generated by a facet \( C^{0,J} \) of \( C^0 \). Here \( J \subset \mathcal{F}^m \) is a subset whose complement contains at least two elements, and is uniquely determined by \( L \). Let \( R_J \subset R_m^{(1)} \) be the subset of dual
affine roots which vanish on $t^J$. We may and will assume from now that we have moved $L$ by an appropriate element of $W(R_m)$ so that there exists a lifting $l(1) \subset t$ of $L(1)^\text{temp}$ of the form $l(1) = t^J$ for a (proper) subset $J \subset F^m$. Notice that $J \subset F_{m,0}$ if and only if $s = 1$. In this case we call $0 \in t^J$ the origin of $t^J$. Observe that in this situation $J = P$ is a standard parabolic subset, and therefore $R_J = R_P$ is a parallel parabolic subsystem.

Let us now assume that $s \neq 1$. Then $a_0^\lor \in J$. Let $R_P \subset R_m$ be the parabolic subset of roots which are constant on $t^J$, and let $R_{P,+} = R_P \cap R_{m,+}$. Obviously $J_0 := J \backslash \{a_0^\lor\} \subset F_{m,0}$. Let $\beta \in R_{P,+}$ be the unique positive root such that $J_0 \cup \{\beta\}$ forms a basis of $R_{P,+}$. Observe that the gradient projection $D$ maps $R_J$ isomorphically to a root subsystem of maximal rank obtained from omitting the simple $R_{P,+}$-basis element $\beta$ from the basis of the affine extension $R_P^{(1)}$ with affine basis $J \cup \{\beta\}$.

Consider the affine isomorphism between $L_m^{\text{temp}} = sT_u^P/K_L^P$ and $L_n(1)^{\text{temp}} := sT_u^P/K_L^P$ given by multiplication with $e^{-1}$. Using this we can choose an affine linear isomorphism $D^aT_1$ from $t_1$ to $t^J$ which lifts the finite affine morphism $\phi_T : T_{1,u} = L_1^{\text{temp}} \rightarrow L_n^{\text{temp}}$. Consider inside the Lie algebra $t_1$ of $L_1^{\text{temp}} = T_{1,u}$ the affine hyperplanes with respect the action of dual affine Weyl group $W_1^{\lor,a} := W(R_{m,1}) \times Q(R_{m,1})$ on $t_1$. Analogous to what was said in in the proof of Proposition 3.4, the images under $D^aT_1$ of the affine reflection hyperplanes in $t_1$ are the intersections of $t^J$ with the affine hyperplanes of $R^m$ on $t$ which are not in $\mathbb{Z}J$. In the semi-standard case this is a direct consequence of the way we choose $L$ and the fact that $\phi_T$ is a spectral transfer map with image $L$. In the remaining case this statement follows easily from (T4). As a consequence we may and will choose $D^aT_1$ such that the fundamental alcove $C_1^{\lor}$ of $t_1$ with respect to $F_1^{\lor}$ is mapped to $C^{\lor,J}$ and this fixes $D^aT_1$ uniquely. Let $D_1^aT_1 : t_1 \rightarrow t_1'^J$ be the gradient of $D^aT_1$, with $t_1'^J$ is the linear subspace in $t$ parallel to $t^J$.

**Proposition 3.6.** Assume $H$ is irreducible and semi standard. Recall the proof of Proposition 3.4. Consider a spectral transfer map $\phi_T$ from $H_1$ to $H$ with image $L_n = rT_n^P$, with $L$ a residual coset of positive dimension in the position as described in the text above. Let $r = sc$ and let $t^J$ be a lift of $L(1)^{\text{temp}}$. Then $J \subset F^m$ is a subset whose complement has at least two elements.

1. In the above situation, the subset $J \subset F^m$ is excellent in the sense of [Lu1, paragraph 2.28].
2. For each affine root $a^{\lor} \in R_{m,1}$, the affine linear automorphism $r_{a^{\lor}} := D_1^aT_1 \circ r_{a^{\lor}} \circ D_1^{-1}$ of $t^J$ belongs to $N_{W_1^{\lor}}(W_J)$.
3. The elements $r_{a^{\lor}}$, where $a^{\lor}$ runs over a basis of simple reflections of $W_1^{\lor,a}$ with respect to $C_1^{\lor}$, generate a subgroup $W^*$ of $N_{W_1^{\lor}}(W_J)$ isomorphic to $W_1^{\lor,a}$ which acts as an affine reflection group on $t^J$. Hence $D_1^aT_1$ defines an isomorphism of affine reflection groups. In particular $R_1$ is irreducible. We have $N_{W_1^{\lor}}(W_J) = W_*J$.
4. The element $D_1^aT_1(0) \in C^{\lor,J} \subset t^J$ is a special vertex of the alcove $C^{\lor,J}$ for the action of $W^*$. Since $C^{\lor,J}$ is a face of $C^{\lor}$, we have $D_1^aT_1(0) = \omega_i \in C^{\lor}$, a vertex of $C^{\lor}$ corresponding to an affine root $a_{\omega_0} \in F^m \backslash J$. Observe that $s = 1$ if and only if $\omega_0$ is a (special) vertex in $C^{\lor}$ which is mapped to $1 \in T$ by the covering map $t \rightarrow T$. 

Proof. Observe that the translation lattice \( Q(R_{m,1}^*) \) is mapped injectively to a sublattice of the lattice \( Y_J = t_0' \cap Y \) in \( t_1' \). As in the proof of Proposition 3.4, for any mirror \( M_{P,\beta} \) of \( \Xi_{(P,\beta)} \) a reflection \( r_{P,\beta} \) exists in \( W_Z \) which is an involutive automorphism on the space of induction data leaving \( M_{P,\beta} \) pointwise fixed, and mapping \((P,\beta)\) to \((P,-\beta)\). It also follows from the proof of Proposition 3.4 that \( r_{P,\beta} \) can be represented by an element \( \sigma = n w \in W_0 \) for some \( n \in W_0 \) with \( w(P) = P \) and \( w \in W_P \), such that \( \sigma(L) = L \). In particular \( \sigma(t_0') = t_0' \). Suppose that \( \xi_1 \in M_{P,\beta} \), implying that \( \xi_1 \) is fixed by \( r_{P,\beta} \). Then \( \sigma(t^J) = t^J + y \) for \( y = \sigma(e) - e \in Y \). Let \( r^* = t^J - y \circ \sigma \), then \( r^* \in W^\vee \) is the unique involutive affine isomorphism of \( t^J \) whose image in \( W(R_{m,1}^*) \) is equal to \( \sigma \), normalizing \( t^J \), and fixing \( e \in C^\vee \). Clearly \( r^* \) acts on \( t^J \) as the affine reflection in the affine hyperplane of \( t^J \) through \( e \) corresponding to \( M_{P,\beta} \). Now let \( M_{(P,\beta)} \) be defined by a dual affine simple root \( b^\vee \in F^m \setminus J \). It follows in a standard way that \( r^* \) must be equal to \( w_{J,J(b^\vee)} w_J \) (where \( w_J \) and \( w_{J,J(b^\vee)} \) denote the longest elements in the finite Weyl groups \( W_J \) and \( W_{J,J(b^\vee)} \) respectively) and that \( r^*(J) = J \). It follows that \( J \) is excellent as claimed. This proves (1), and along the way we also proved (2) and (3) (the remark on \( N_{W^\vee}(W_J) \) is [Lu2, Proposition 2.26], and the irreducibility of \( W_{1,a}^{\vee} \) follows from [Lu2 2.28(a)]), and (4) is obvious.

\[ \square \]

**Corollary 3.7.** The map \( D^a \phi_T \) induces a bijective correspondence between \( \{ I \subset F^m \} \) and \( \{ I' \subset F^m \mid J \subset I' \} \) such that \( D^a \phi_T(C_1 \cap J) = C_1 \cap I' \). The map \( D \phi_T \) induces a bijective correspondence between the set of parabolic subsystems \( R_Q \subset R_m \) and the set of parabolic subsystems \( R_{Q'} \subset R_m \) which contain \( R_P \), such that \( D \phi_T(t_Q^J) = t_{Q'} \).

**Proof.** Both claims is are simple consequences of Proposition 3.6 (first note that the result is trivial if \( P = F_0 \)). For the second claim, observe that by definition of \( P \), we have \( t^P = t_0' \). Since \( \omega_i \) is special with respect to the action of \( W^* \), every intersection affine root hyperplanes \( V' \) of a affine root of \( R_{m,1}^* \) with \( t^J \) is parallel to a unique affine subspace of \( t^J \) through \( \omega_i \) which is an intersection of affine hyperplanes. This corresponds via \( D \phi_T^a \) to a linear subspace \( V \) of \( t_1 \) which is an intersection of linear root hyperplanes in \( t_1 \). Thus \( D^a \phi_T \) sets up a bijection between the collection of linear subspaces of \( t_1 \) which are an intersection of roots of \( R_{1,m} \) and the collection of classes of affine subspaces \( V' \) of \( t^J \) which are intersections of affine root hyperplanes for \( R_{m,1}^* \) and which are parallel to each other. To each such class of affine subspaces parallel to \( V' \) we attach the set of roots \( R(V') \subset R_m \) which are constant on \( V' \). Then \( R(V') = R_{Q'}, \) a parabolic subsystem containing \( P \), and it is clear that \( D \phi_T \) yields the desired bijection.

\[ \square \]

**Corollary 3.8.** In Corollary 3.7 we have \( \phi_T(T_{1,Q}) = K^n{L}(T_{Q'} \cap L)^0/K^n{L} \).

**Proof.** It is enough to prove that \( D \phi_T(t_1,Q) = t_{Q'} \cap t^P \). The dimensions of these subspaces are equal since \( t_1^Q \subset t_1 \) and \( t_{Q'} \subset t^P \) have the same codimension, by definition of the correspondence. Hence it is enough to prove that \( D \phi_T(t_1,Q) \subset t_{Q'} \cap t^P \). This follows by the remark that \( D \phi_T \) must map \( \alpha^\vee \in R_{Q'} \) to a multiple of the projection of a coroot in \( R_{Q'} \) onto \( t^P \) along \( t_P \). Since \( t_P \subset t_{Q'} \) this implies that the image is in \( t_{Q'} \cap t^P \).

\[ \square \]

3.1.4. Composition of spectral transfer maps and of spectral transfer morphisms.

Let \( \phi_T \) be a spectral transfer map from \( (H_1,\tau^{d_1}) \) to \( (H_2,\tau^{d_2}) \) and let \( \psi_T \) be a
spectral transfer map from \((H_2, \tau^{d_2})\) to \((H_3, \tau^{d_3})\). We would like to define the the composition \(\rho_T := \psi_T \circ \phi_T\), but in order to do so we need to come to grips with the fact that the image of \(\phi_T\) is the quotient \(L_n\) of a residual coset \(L \subset T_2\). We are saved here by Proposition 3.4. Indeed, we have \(L_n := L/K^n_{L}\) and \(K^n_{L} \subset N_{W_{3,0}(L)}\) by definition. By Proposition 3.4 for all \(k \in K^n_{\rho} \subset N_{W_{3,0}(L)}\) there exists a \(w \in N_{W_{3,0}(M)}\) such that \(\psi_T \circ k = w \circ \psi_T\), where \(M \subset T_3\) is a residual coset such that \(\psi_T(T_2) = M_n\). By Proposition 3.2 there exists a residual coset \(N \subset M\) such that \(\psi_T(L) = N := N/K^n_{M}(N)\). Observe that \(K^n_{N}(N) \subset K^n_{M}\) since the latter is the subgroup of elements of \(W_{3,0}\) which stabilize \(N\) and restrict to a translation on \(N\) (i.e. a multiplication by some \(k' \in T^N\)). Hence \(N_n := N/K^n_{N}\) is a quotient of \(N\). By the above and property (T2) of \(\psi_T\) (which implies that \(D\psi_T|_L : L^\infty \to t^N\) is a linear isomorphism), we see that \(w \in N_{W_{3,0}(N)}\) and that \(w\) restricts to a translation on \(N\). Hence \(w \in K^n_{N}\). In other words, \(\psi_T|_L : L \to N\) maps \(K^n_{L}\) orbits on \(L\) to \(K^n_{N}/K^n_{M}(N)\) orbits on \(N\), and thus defines a map \(\psi_T : L_n \to N_n\). Now we can finally define the composition of \(\phi_T\) and \(\psi_T\) to be the map \(\rho_T := \psi_T \circ \phi_T\).

It is clear that \(\rho_T\) satisfies (T1) and that \(\rho_T\) is an affine homomorphism from \(T_1\) onto \(N_n\) where \(N \subset T_3\) is a residual coset. In order to verify (T2) we need to check in addition that \(\rho_T(e) \in (T_N \cap N)/K^n_{N}\). Let \(T_L \subset T_2\) be the subtorus whose Lie algebra is spanned by the coroots of the roots which restrict to constants on \(L\) (and similarly we define \(T_M\) and \(T_N\)). Since \(N \subset M\) we see that \(T_M \subset T_N\). Observe that \(R_{2,m,L} \subset R_{3,m,N}\) correspond to each other under \(\psi_T\) in the sense of Corollary 3.7 and therefore we have \(\psi_T(T_L) = K^0_T(T_N \cap M)/K^0_M \subset T_N := K^0_TN/K^0_M\) by Corollary 3.8. In particular, if \(\tau_L \in K^0_T\phi_T(e) \subset L \cap T_L\) (using (T2) for \(\phi_T\)) then \(\psi_T(\tau_L) \in N\) clearly \(\psi_T(\tau_L) \in N\) by definition of \(N\), so that \(\rho_T(\tau_L) \in N \cap T_N\). It follows that \(\rho_T(\tau) = \psi_T(\tau_L K^n_{L}/K^n_{M}) \in (N \cap T_N)/K^n_{N}\). Hence \(\rho_T\) also satisfies (T2).

Write \(\mu_i = \mu_{R_i,m_i,d_i}\) for the \(\mu\)-functions of the \((H_i, \tau^{d_i})\). Then \(\phi_T(\mu_{2}^{(L)}) = a\mu_1\) and \(\psi_T(\mu_{3}^{(M)}) = b\mu_2\). We see that this last identity implies that \(\psi_T(\mu_{3}^{(N)}) = b'\mu_{2}^{(L)}\) for some nonzero complex number \(b'\). For this it suffices to prove that \(\psi_T(\mu_{3}^{(N)}) = b'\mu_{2}^{(L)}\), where \(\mu_{3}^{(N)}\) is the restriction to \(N\) of the regularization of \(\mu_{3}^{(M)}\) along \(N\). To prove this statement, note that the irreducible factors of \(\mu_{3}^{(M)}\) are of the form \((1 - v^i\zeta \alpha/n)\) with \(\alpha \in R_{3,m}\) not constant on \(t^M_{3,i}\), \(i\) and \(n\) certain integers (depending on \(\alpha\)), and \(\zeta\) an \(n\)-th root of 1. Therefore we have (using (T1) for \(\psi_T\)) \(\psi_T(1 - v^i\zeta \alpha/n) = (1 - v^i\zeta' \beta)\) where \(\beta\) is a character of \(T_2\), \(\zeta'\) a complex root of 1, and \(i\) an integer. By construction \((1 - v^i\zeta' \beta)\) is identically zero on \(L\) if \((1 - v^i\zeta \alpha/n)\) is identically zero on \(N\), and in this case all irreducible factors of \((1 - v^i\zeta' \beta)\) restrict to constants on \(L\). This easily implies the claim. We conclude that \(\rho_T = \psi_T \circ \phi_T\) is a spectral transfer map. From now on we will denote the composition \(\rho_T\) simply by \(\psi_T \circ \phi_T\) instead of \(\psi_T \circ \phi_T\).

The most trivial examples of spectral transfer maps are Weyl group elements. If \((H, \tau^d)\) is a normalized affine Hecke algebra with \(H = H(R, m)\) then for any \(w \in W_0\) we have an invertible spectral transfer map \(w : T \to T\). Given a spectral transfer map \(\phi_T\) from \((H_1, \tau^{d_1})\) to \((H_2, \tau^{d_2})\) with image \(L_n\) where \(L \subset v^dT^P \subset T\) (with \(R_P \subset R_0\) a parabolic subsystem) it follows from Proposition 3.4 that \(W_{2,0} \circ \phi_T \circ W_{1,0} = W_{2,0} \circ \phi_T\). Therefore we can define an equivalence relation as follows. We call two spectral transfer maps \(\phi_1, \phi_2 : T_1 \to T_2\) from \((H_1, \tau^{d_1})\) to \((H_2, \tau^{d_2})\)
equivalent if $φ_{2,T} ∈ W_{2,0} ∘ φ_{1,T}$. As a special case, we may post-compose $φ_T$ with an affine map $M_k : L → L$ defined by $M_k(t) = kt$ where $k ∈ K_t^n$ within the same equivalence class.

By the above remarks it is not difficult to check that this notion of equivalence of spectral transfer maps is compatible with the operation of composition of spectral transfer maps (this follows from Proposition 3.4). This shows that the following definitions makes sense:

**Definition 3.9.** The spectral transfer category $C$ is the category formed by the normalized affine Hecke algebras as objects, and equivalence classes of spectral transfer maps as morphisms. A spectral transfer morphism (abbreviated to STM) from $H_1$ to $H_2$ is denoted by $φ : H_1 ↣ H_2$. If $φ_T : T_1 → T_2$ is an affine morphism in the equivalence class of the STM $φ$ we say that $φ$ is represented by $φ_T$. The composition $ψ ∘ φ$ of STMs is defined by composing representing STMs: $(ψ ∘ φ)_T = ψ_T ∘ φ_T$.

It is obvious that two spectral transfer maps $φ_{1,T}, φ_{2,T} : T_1 → T_2$ are equivalent iff $φ_{1,Z} = φ_{2,Z}$. Hence a STM $φ : (H_1, τ_{d_1}) ↣ (H_2, τ_{d_2})$ defines a morphism $φ_Z$ from the spectrum of the center of $H_1$ to the spectrum of the center of $H_2$. Similarly we obtain a smooth finite map $φ_Z^{temp}$.

**Definition 3.10.** Let $φ : (H_1, τ_{d_1}) ↣ (H_2, τ_{d_2})$ be a STM and let $Φ = W_{2,0} ∘ φ_T$ be the associated class of morphisms. We attach various invariants to $Φ$. The morphism $φ_Z$ (see Corollary 3.5) clearly only depends on $Φ$, which we express by writing $Φ_Z$ and $Φ_Z^{temp}$. By the image $Im(Φ)$ of $Φ$ we mean the $W_{2,0}$-orbit of residual cosets $W_{2,0}φ_T(T_1) ⊂ W_{2,0}T_2$. Similarly, the tempered image of $Φ$ is the image $φ_Z^{temp}(S_1) ⊂ S_2$. We define two nonnegative integers associated with $Φ$, the rank $rk(Φ) = dim(T_1) − 1$ and the co-rank $cork(Φ) = dim(T_2) − dim(T_1)$.

**3.2. The tempered correspondence of a STM.** The following result is the main theorem of the paper:

**Theorem 3.11.** Let $φ : (H_1, τ_{d_1}) ↣ (H_2, τ_{d_2})$ be a STM. Consider the correspondence between $G_1$ and $G_2$ given by the fibered product $G_{12}$ of $G_1$ and $G_2$ with respect to the diagram:

\[
\begin{array}{ccc}
G_{12} & \xrightarrow{p_1} & G_1 \\
\downarrow p_2 & & \downarrow φ_Z^{temp} p_{Z,1}^{temp} \\
G_2 & \xrightarrow{p_{Z,2}^{temp}} & S_2
\end{array}
\]

(43)

If $C ⊂ G_{12}$ is a component then $C_i := p_i(C) ⊂ G_i$ is a component ($i = 1, 2$). Moreover there exists a positive measure $ν$ on $C$ and $r_i ∈ Q_+$ such that

\[
(p_i)_*(ν) = r_i ν_{P_{i,1}|C_i}
\]

for $i = 1, 2$.

**Proof.** Let $L ⊂ T_2$ be a residual coset which belongs to the image of $Φ$. Let $B_2 ⊂ S_2$ be a component. Then $B_2 ⊂ φ_Z^{temp}(S_1)$ iff $B_2$ is associated to a residual coset $L_2 ⊂ T_2$ (in the sense of Proposition 2.31(i)) such that $L_2 ⊂ W_0L$. By Proposition 3.2 and Corollary 3.5 we see that $φ_Z^{temp}(L_2)$ consists of a union of finitely many $W_{i,0}$-orbits of residual cosets in $T_1$. Using Proposition 2.31 we see that $(φ_Z^{temp})^{-1}(B_2)$ consists
of a union of finitely many components $B_1 \subset S_1$, each of which has the property that $\phi_{Z}^{\text{temp}}(B_1) = B_2$. It follows in particular that for any component $B_1 \subset S_1$ its image $\phi_{Z}^{\text{temp}}(B_1) \subset S_2$ is a component of $S_2$. By Proposition 2.31 we see that the correspondence associated to $S_{12}$ indeed yields a finite correspondence between the components of $S_{1}$ and $S_{2}$. This proves the first assertion of the Theorem.

Let $C_1 \subset S_{1}$ and $C_2 \subset S_{2}$ be corresponding components. Put $B_i = p_{Z_i}^{\text{temp}}(C_i)$ for the corresponding components of $S_i$. Let $L_i \subset T_i$ be corresponding residual cosets in the sense of Proposition 2.31. We may assume without loss of generality that $L_2 \subset L$ and that $L_1$ is a component of $\phi_{Z}^{-1}(K_n^{L}L_2/K_n^{L})$. By Proposition 2.31 and Lemma 4.35 we see that $B_1$ corresponds to $N_{W_i,0}(L_i)$-orbits of $K_{L_i}$-generic points of $L_{i}^{\text{temp}}$. By Proposition 3.4 we see that $B_1 \subset B_2' := (\phi_{Z}^{\text{temp}}|_{B_1})^{-1}(B_2')$. Using Lemma 4.35 and (T1) we see that $\phi_{Z}^{\text{temp}}|_{B_1} : B_1' \to B_2'$ is a finite covering map. By (T1) and Theorem 2.32 the subset $C_1'' = (p_{Z_1}^{\text{temp}}|_{C_1})^{-1}(B_2'') \subset C_1$ is dense and its complement has measure zero with respect to $\nu_{P_1,1}|_{C_1}$. Consider the commuting diagram

$$
\begin{array}{ccc}
C'' & \xrightarrow{p_1|_{C''}} & C_1'' \\
p_2|_{C''} \downarrow & & \downarrow \phi_{Z}^{\text{temp}} \circ \phi_{Z_1}^{\text{temp}}|_{C_1''} \\
C_2' & \xrightarrow{r_{Z_2}^{\text{temp}}|_{C_2'}} & S_2' \\
\end{array}
$$

By the above this is a diagram of finite covering maps of smooth manifolds. We use the notation of (the proof of) Lemma 2.32 and Theorem 2.33. Let $\nu$ be the measure on $C$ obtained by the extension by zero of the pull back of the smooth volume form $\nu_{Z_2}'|_{B_2'}$. Since $p_{2}|_{C''}$ is a finite covering map, the push forward of the measure $\nu|_{C''}$ to $C_2$ is a nonzero integer constant multiple of the pull back of $\nu_{Z_2}'$ to $C_2$. By Theorem 2.33 the Plancherel measure $\nu_{P_1,2}$ is a nonzero rational multiple of the extension by zero of this measure. Hence $(p_2)_*(\nu) = r_2 \nu_{P_1,2}|_{C_2}$ for a certain $r_2 \in \mathbb{Q}^\times$, as desired.

Finally we need to show that

$$
(p_1)_*(\nu) = r_1 \nu_{P_1,1}|_{C_1}
$$

for some nonzero rational constant $r_1 \in \mathbb{Q}^\times$. By Theorem 2.33 it suffices to showing

$$
(\phi_{Z}^{\text{temp}}|_{B_1'})(\nu_{S_2}') = r_1'' \nu_{S_1}'
$$

where $\nu_{S_2}'$ is the restriction of $\nu_{S_2}'$ to $S_2''$, and $r_1'' \in \mathbb{Q}^\times$. By the text above Theorem 2.33 this boils down to proving that

$$
(\phi_{T}|_{L_1})^*(\mu_2(K_n^{L}L_2/K_n^{L})) = r_1'' \mu_1(L_1)
$$

for some constant $r_1'' \in \mathbb{Q}^\times$. This is true by definition, and by Proposition 3.3.

**Remark 3.12.** All results discussed so far are true, mutatis mutandis, in the slightly more general setting where we lift the condition on $\phi_T$ that it has to satisfy $\phi_{T}(e) \in (T_{L} \cap L)/K_n^{L}$ (if $L/K_n^{L}$ denotes the image of $\phi_{T}$), and keep all other requirements. We call such a map an essentially strict STM if (in the notation of paragraph 3.1.3) $t_{P} \cap t'$ is a special vertex of $C^{\nu,J}$ for $W^{*}$. We denote the category of essentially strict STMs by $\mathcal{C}_{es}$.

3.3. Examples of spectral transfermorphisms.
3.3.1. Semi-standard spectral automorphisms. A semi-standard spectral endomorphism \( \Phi : (\mathcal{H}, \tau^d) \rightrightarrows (\mathcal{H}, \tau^d) \) is represented by a homomorphism \( \phi_T : T \to T \) of algebraic tori with finite kernel, such that \( \phi^*_T(\mu) = a\mu \) for some nonzero constant \( a \). Thus \( \phi_T \) lifts via the exponential mapping of \( T \) to a linear isomorphism \( D^a\phi_T : V \to V \) with the property that \( D^a\phi_T \) induces a bijection on the \( \mu \)-mirrors such that the functions \( m_+ \) and \( m_- \) associated with the mirrors are preserved (see Proposition 2.16 for the notion of \( \mu \)-mirror), and in addition normalizes \( Y \). In the language of the proof of Proposition 2.16(ii),(iv) this means that \( D^a\phi_T \in \text{Aut}_V(\mu, Y) \). If we do not require \( \phi_T \) to be strict then we similarly obtain \( D^a\phi_T \in \text{Aut}_V(\mu, Y) \). It follows that:

**Proposition 3.13.** Assume that \( \mathcal{H} = \mathcal{H}(\mathcal{R}, m) \) is semi-simple with \( (\mathcal{R}, m) \) semi-standard. The spectral endomorphisms of \( (\mathcal{H}, \tau^d) \) are invertible, and there is a canonical isomorphism of \( \text{Aut}_\mathcal{E}(\mathcal{H}, \tau^d) \) and the group \( \text{Out}_\mathcal{T}(\mu)_c := \{ \psi \in \text{Out}_\mathcal{T}(\mu) \mid \psi(e) = e \} \). If \( (\mathcal{H}, \tau^d) \) is standard then this equals \( \Omega^V_0(R^\vee_0) \) (see Proposition 2.16). Similarly non-strict spectral transfer endomorphisms are in fact essentially strict spectral transfer automorphisms. These form a group \( \text{Aut}_{\mathcal{E}_es}(\mathcal{H}, \tau^d) \) which is canonically isomorphic to \( \text{Out}_\mathcal{T}(\mu) \). In particular the constant \( a \) of (T3) equals \( a = 1 \) for all (not necessarily strict) spectral endomorphism in this situation.

Similarly we have:

**Proposition 3.14.** The group \( \text{Aut}_\mathcal{E}(\mathcal{H}, \tau^d) \) of spectral automorphisms of \( \mathcal{H} = \mathcal{H}(\mathcal{R}, m) \) (with \( (\mathcal{R}, m) \) semi-standard) is canonically anti-isomorphic to the group \( \text{Aut}_\mathcal{E}(\mathcal{H}) \) of strict algebra automorphisms, via the right action of this last group on \( T \). Similarly, the group \( \text{Aut}_{\mathcal{E}_es}(\mathcal{H}, \tau^d) \) of essentially strict STMs is anti-isomorphic to \( \text{Aut}_{\mathcal{E}_es}(\mathcal{H}) \).

**Proof.** Apply Proposition 2.16(v). \( \Box \)

These results show that the spectral correspondences of semi-standard spectral automorphism can be refined to a Plancherel measure preserving automorphism in view of the following result which follows trivially from the definitions.

**Proposition 3.15.** Admissible algebra automorphisms \( \sigma \in \text{Aut}_{\text{adm}}(\mathcal{H}) \) preserve both the \(*\)-operator on \( \mathcal{H} \) and the trace \( \tau^d \) of \( \mathcal{H} \). In particular such an automorphism induces a Plancherel measure preserving homeomorphism of \( \mathcal{S} \).

**Corollary 3.16.** If \( \phi \in \text{Aut}_{\mathcal{E}_es}(\mathcal{H}) \) corresponds to \( \sigma \in \text{Aut}_{\mathcal{E}_es}(\mathcal{H}) \) (i.e. the action of \( \sigma \) on \( T \) is equal to \( \phi_T \)) then \( \sigma : \mathcal{H} \rightrightarrows \mathcal{H} \) induces a Plancherel measure preserving automorphism on \( \mathcal{S} \) refining the spectral correspondence of \( \phi^\sigma \).

3.3.2. Rigidity in \( \mathcal{C} \) and \( \mathcal{E}_{es} \). The following result states a fundamental rigidity property of semi-standard morphisms of semi-simple objects of category \( \mathcal{C} \), and is a strengthening of Proposition 3.13 Its elementary proof is similar to the above, and is left to the reader.

**Proposition 3.17.** A spectral transfer morphism \( \Phi : (\mathcal{H}_1, \tau^{d_1}) \rightrightarrows (\mathcal{H}_2, \tau^{d_2}) \) with \( (\mathcal{H}_1, \tau^{d_1}) \) semi-simple and semi-standard is essentially determined by its image: If \( \Phi_1, \Phi_2 \in \text{Hom}_{\mathcal{E}}((\mathcal{H}_1, \tau^{d_1}), (\mathcal{H}_2, \tau^{d_2})) \) and \( \text{Im}(\Phi_1) = \text{Im}(\Phi_2) \) then there exists a \( \gamma \in \text{Aut}_{\mathcal{E}}(\mathcal{H}_1, \tau^{d_1}) \) such that \( \Phi_2 = \Phi_1 \circ \gamma \). Similarly for essentially strict STMs.
3.3.3. Spectral isomorphisms. Let $\mathcal{R}$ be a semisimple root datum. The $\mu$ function has certain important symmetries with respect to the cocharacter $m$. Consider the group $\text{Iso} = \text{Iso}(\mathcal{R})$ of automorphism of $\mathcal{Q}_c = \mathcal{Q}_c(\mathcal{R})$ generated by the following involutions $\eta^c$ associated with the $W$-conjugation classes $c \subset S$ of affine simple reflections. If $c \cap S_0 \neq \emptyset$ we define (using notations as in Theorem 2.6) $\eta^c(v(s)) = v(s')^{-1}$ if $s \in c$ or $s' \in c$, and $\eta^c(v(s)) = v(s)$ else. For conjugacy classes $c$ such that $S_0 \cap c = \emptyset$ we define $\eta^c$ by $\eta^c(v(s)) = v(s)^{-1}$ if $s \in c$ and $\eta^c(v(s)) = v(s)$ else. We denote the action of $\eta \in \text{Iso}$ on the set of cocharacters of $\mathcal{Q}_c$ by $m \rightarrow \eta(m)$.

Proposition 3.18. For all $W$-conjugacy classes $c$ of affine simple reflections we can define an essentially strict spectral transfer isomorphism $\phi^c : (\mathcal{H}(\mathcal{R}, m), \tau^d) \sim (\mathcal{H}(\mathcal{R}, \eta^c(m)), \tau^d)$ as follows. If $c \cap S_0 \neq \emptyset$ we put $\phi^c_T = \text{id}_T$. If $c \cap S_0 = \emptyset$ we put $\phi^c_T(t) = \text{set}$, where $s_c \in T$ is the $W_0$-invariant element defined by $\alpha(s_c) = -1$ if $\alpha \in R_0$ is such that $s'_\alpha \in c$, and $\alpha(s_c) = 1$ else.

Proof. We need to verify (T3) (the other conditions being trivially satisfied) and this is an easy computation. 

These spectral isomorphisms too are given by admissible algebra isomorphisms. If $c \cap S_0 \neq \emptyset$ we define $\sigma^c \in \text{Hom}_{adm}(\mathcal{H}(\mathcal{R}, m), \mathcal{H}(\mathcal{R}, \eta^c(m)))$ by $\sigma^c(N_s) = -N_s$ for $s \in c$ or $s' \in c$ and $\sigma^c(N_s) = N_s$ else. For conjugacy classes $c$ such that $c \cap S_0 = \emptyset$ we define $\sigma^c \in \text{Hom}_{adm}(\mathcal{H}(\mathcal{R}, m), \mathcal{H}(\mathcal{R}, \eta^c(m)))$ by $\sigma^c(N_s) = -N_s$ for $s \in c$ and $\sigma^c(N_s) = N_s$ else. We remark that if $c$ is such that $c \cap S_0 \neq \emptyset$ and $c \neq c'$ then $\sigma^c$ is not essentially strict and does not send $\mathcal{A}$ to $\mathcal{A}$. However, the induced morphism $\sigma^c_\mathcal{Z} = \sigma^c|_{\mathcal{Z}}$ on the center $\mathcal{Z}$ of $\mathcal{H}(\mathcal{R}, m)$ is equal to $\phi^c_\mathcal{Z}$ in all cases. Hence we obtain:

Proposition 3.19. The algebra isomorphism $\sigma^c$ defines a Plancherel measure preserving homeomorphism

$$\sigma^c_{\mathcal{Z}, \text{temp}} : \mathcal{G}(\mathcal{R}, \eta^c(m)) \rightarrow \mathcal{G}(\mathcal{R}, m)$$

refining the spectral correspondence of the spectral transfer morphism $\phi^c$.

Remark 3.20. In terms of $\Sigma_s$ the action $\text{Iso}(\mathcal{R}, m)$ is given as follows. If $c \cap S_0 \neq \emptyset$ then $\eta^c$ replaces the parameter $m_+(c)$ by $-m_+(c)$ and leaves the rest of the spectral diagram unchanged. If $c \cap S_0 = \emptyset$ then $\eta^c$ interchanges the role of $m_+(c)$ and $m_-(c)$, and leaves the rest of the spectral diagram unchanged. In particular, if $c \neq c'$ then the two involutions $\eta^c$ and $\eta^{c'}$ acting on the corresponding component of type $C^{(1)}_n$ of the spectral diagram generate a dihedral group of order 8.

3.3.4. Spectral covering morphisms. We call a STM $\phi$ a spectral covering morphism if $\phi_T$ is surjective, or otherwise said, if cork($\phi$) = 0.

Proposition 3.21. Assume that $(\mathcal{R}_i, m_i)$ ($i = 1, 2$) both are semi-standard and let $\mathcal{H}_i = \mathcal{H}(\mathcal{R}_i, m_i)$. Let $\phi : (\mathcal{H}_1, \tau^{d_1}) \sim (\mathcal{H}_2, \tau^{d_2})$ be a (not necessarily strict) spectral covering morphism. After replacing $(\mathcal{H}_1, \tau^{d_1})$ by an isomorphic object in $\mathcal{C}_es$ if necessary, and up to the action of Out$_T(m_1)$ (see Proposition 3.13), $\phi$ is represented by a finite morphism $\phi_T : T_1 \rightarrow T_2$ of tori associated to a sublattice $X_2 \subset X_1$ of maximal rank such that $\mathcal{R}_1 = (X_1, R_{1,1}, Y_1, R_{1,0}^\vee)$ and $\mathcal{R}_2 = (X_2, R_{2,0}, Y_2, R_{2,0}^\vee)$ where $R_{1,m_1} = R_{2,m_2}$ and $m_{1,R_1} = m_{2,R_2}$. In this situation the labelled Dynkin diagram of $\Sigma_s(\mathcal{R}_1, m_1)$ can be identified with the underlying labelled Dynkin diagram of $\Sigma_s(\mathcal{R}_2, m_2)$ while $\Omega_{Y_1} \subset \Omega_{Y_2}$. In addition, $\phi$ is essentially strict.
Proof. Let \( \mathcal{H}_i = \mathcal{H}(R_i, m_i) \) and let \( R_i = (X_i, R_{i,0}, Y_i, R_{i,0}') \). Clearly \( X_2 \subset X_1 \) via the morphism of algebraic tori \( T_1 \to T_2 \) underlying the affine morphism \( \phi_T \). By (T3) we have an isomorphism of reflection groups \( W(R_{1,0}) \cong W(R_{2,0}) \) via the map \( (D\phi_T)_p \) where \( p \in T_1 \) is such that \( \phi_T(p) = e \) (using the argument given in the proof of Proposition 3.24). Hence \( p \in T_1 \) is a \( W(R_{1,0}) \)-invariant point, and there exists an essentially strict spectral transfer automorphism of \( \mathcal{H}_1 \) mapping \( e \in T_1 \) to \( p \). Therefore we may assume without loss of generality that \( \phi_T \) is a morphism of algebraic tori. From (T3) it also follows that Proposition 2.16 applies both to \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \). We conclude that \( d(\phi_T)_e \) induces a monomorphism \( W(R_{m_i}(1)) \times \Omega_{Y_1} \to W(R_{m_2}(1)) \times \Omega_{Y_2} \). We have \( R_{2,0} \subset X_2 \subset X_1 \) and there exists a \( W(R_{1,0}) \)-invariant function \( n^\phi : R_{1,0} \to \{1/2, 1, 2\} \) such that \( R_{2,0} = \{n^\phi(\alpha)\alpha \mid \alpha \in R_{1,0}\} \). If \( n^\phi(\alpha) = 1/2 \) then \( \alpha \) belongs to an irreducible component of \( R_{1,0} \) of type \( C_n \) (with \( n \geq 1 \)) whose weight lattice is contained in \( X_2 \). Hence by Proposition 2.16 there is a corresponding direct summand \( C^1 \) of \( R_1 \) with root system \( R_1^0 \) of type \( C_n \) and \( X^1 = P(R_1^0) \), and a direct summand \( C^2 \) of \( R_2 \) with root system \( R_2^0 \) of type \( B_n \) and lattice \( X^2 = X^1 \). By (T3) we see that \( m_{C^1,-} = m_{C^2,+} \), and the tensor factor \( \mathcal{H}(C^1, m_{C^1}) \) of \( \mathcal{H}_1 \) is isomorphic (as algebras) to the tensor factor \( \mathcal{H}(C^2, m_{C^2}) \) of \( \mathcal{H}_2 \) in a sense compatible with the STM. Observe in particular that \( R_1^0 = R_2^0 \) in this situation. If on the other hand \( n^\phi(\alpha) = 2 \) for some \( \alpha \in R_{1,0} \) then \( \alpha \) belongs to an irreducible component \( R^0_1 \) of \( R_{1,0} \) of type \( B_n \). Using (T3) one checks that the only possibility is in this case that \( X^2 = X^1 = X_1 \cap \mathbb{R} R_1^0 = Q(R_1^0) \), and again we have corresponding isomorphic tensor factors in \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \) with \( R_1^0 = R_2^0 \).

Hence we may assume that \( R_{1,0} = R_{2,0} \). By (T3) it is clear that we can replace \((\mathcal{H}_1, \tau^{d_1})\) by an isomorphic object of \( \mathcal{C} \) such that \( m_{1,+} = m_{2,+} \) and \( m_{1,-} = m_{2,-} \), and the remaining assertions easily follow. \( \square \)

Corollary 3.22. If \( \phi : (\mathcal{H}_1, \tau^{d_1}) \sim (\mathcal{H}_2, \tau^{d_2}) \) is a not necessarily strict spectral covering morphism with both \((\mathcal{R}_i, m_i) \ (i = 1, 2) \) are semi-standard, then \( \phi \) is, after replacing \((\mathcal{H}_1, \tau^{d_1})\) by an isomorphic object of \( \mathcal{C} \) if necessary, given by a (strict) algebra embedding \( \sigma : \mathcal{H}_2 \hookrightarrow \mathcal{H}_1 \) of \( \mathcal{H}_2 \) corresponding to an isogenous sub root datum of \( \mathcal{R}_1 \) and \( d_1 d_2^{-1} \in \mathbb{Q}^\times \).

Proof. This follows from the two results above in combination with \[2\]. \( \square \)

Corollary 3.23. Given a spectral covering morphism \( \phi : (\mathcal{H}_1, \tau^{d_1}) \sim (\mathcal{H}_2, \tau^{d_2}) \) where both \((\mathcal{R}_i, m_i) \ (i = 1, 2) \) are semi-standard, there exists an admissible algebra isomorphism \( \sigma : \mathcal{H}_2 \sim \mathcal{H} \subset \mathcal{H}_1 \) of \( \mathcal{H}_2 \) to a subalgebra of \( \mathcal{H}_1 \) whose embedding is a strict algebra morphism. The branching of tempered representations of \( \mathcal{H}_1 \) via \( \sigma \) is a refinement of the spectral correspondence of \( \phi \).

3.3.5. Spectral covering morphisms to non-semi-standard affine Hecke algebras. In the previous paragraph we looked at covering morphisms \( \phi : (\mathcal{H}_1, \tau^{d_1}) \sim (\mathcal{H}_2, \tau^{d_2}) \) with \((\mathcal{H}_i, \tau^{d_i}) \) both semi-standard. We found that essentially such covering STM s correspond to admissible algebra embeddings \( \mathcal{H}_2 \subset \mathcal{H}_1 \). And by Proposition 3.13 the endomorphisms in \( \mathcal{C}_{es} \) of a semisimple, semi-standard normalized affine Hecke algebra are automorphisms.

In general things are a bit more complicated. For instance, if \( \mathcal{H} \) is the group algebra of a lattice \( X \) (normalized by \( d = 1 \) say), then there obviously exist spectral transfer endomorphisms \( \mathcal{H} \sim \mathcal{H} \) which are not invertible. And also, if \((X, R_0, Y, R_0') \)
is a root datum with \( R_0 \) nonempty, and \( T = \text{Spec}(L[X]) \), then the identity map \( T \to T \) defines an STM from \( \mathcal{H} = L[X] \) to \( \mathcal{H}' := L[X \rtimes \omega_0] \) which is a spectral covering map. Observe that the target affine Hecke algebra \( \mathcal{H}' \) is not semi-standard here, and that the embedding \( \mathcal{H} \subset \mathcal{H}' \) goes "the wrong way" compared to the case of semi-standard spectral covering maps.

The more serious examples of non-semistandard spectral covering morphisms discussed below are important for later applications in [O4]. Consider

\[
\phi : \mathcal{H}(\mathcal{R}^D_{sc}, m^D) \to \mathcal{H}(\mathcal{R}^C_{sc}, m^C)
\]

where \( \mathcal{R}^D_{sc} \) denotes the root datum of the simply connected cover of \( SO_{2n} \), i.e. the root datum with \( R_0 = R_0^D \) of type \( D_n \) (\( n \geq 3 \)), and \( X \) the root lattice of type \( D_n \), and \( \mathcal{R}^C_{sc} \) is the root datum with root system \( R_0^C \supset R_0^D \) of type \( C_n \) containing \( R_0^D \) as its set of short roots, with the same \( X \), now viewed as the root lattice of \( C_n \).

Here we have defined the co-character \( m^D \) by \( m^D_S(s) = 1 \) and \( m^C \) by \( m^C_S(s) = 1 \) is \( s \) is a simple reflection of a simple root in \( F_0^D \cap F_0^C \), and \( m^C_S(s) = 0 \) else. We normalized the traces of these algebras by 1.

Then \( \mathcal{H}(\mathcal{R}^C_{sc}, m^C) \) is represented by the non-standard spectral diagram of Figure 1. We define \( \phi \) by \( \phi_T = \text{id}_T \), where \( T \) is the torus with character lattice \( X \). This is obviously a STM. Notice however that although \( \phi_T \) is an isomorphism, \( \phi \) itself is not.

On the other hand there exists an admissible algebra embedding

\[
\sigma : \mathcal{H}(\mathcal{R}^D_{sc}, m^D) \to \mathcal{H}(\mathcal{R}^C_{sc}, m^C)
\]

mapping \( \mathcal{H}(\mathcal{R}^D_{sc}, m^D) \) onto the subalgebra of \( \mathcal{H}(\mathcal{R}^C_{sc}, m^C) \) which is invariant for the algebra involution \( \sigma^c \) where \( c \) is the class of reflections of the short roots of \( B_n = C_n \).

In fact, one can define \( \sigma \) or \( c \) as follows. Assume that

\[
F_0^D = (\alpha_1 = e_1 - e_2, \ldots, \alpha_{n-1} = e_{n-1} - e_n, \alpha_n = e_{n-1} + e_n)
\]

and

\[
F_0^C = (\alpha'_1 = e_1 - e_2, \ldots, \alpha'_{n-1} = e_{n-1} - e_n, \alpha'_n = 2e_n)
\]

Then we may define \( \sigma(N_i) = N_i' \) for \( i = 0, \ldots, n - 1 \) and \( \sigma(N_n) = N_n\prime N_{n-1}' \).

Using that \( (N_n')^2 = 1 \) one checks easily that this defines an admissible algebra homomorphism whose image is the invariant algebra of the algebra involution \( \sigma^c \) of \( \mathcal{H}(\mathcal{R}^C_{sc}, m^C) \) (given by \( \sigma^c(N_i') = N_i' \) for \( i = 0, \ldots, n - 1 \) and \( \sigma^c(N_n') = -N_n' \)).

Then \( \sigma \) is compatible with \( \phi \) in the sense that \( \sigma \) induces the identity on the lattice \( X \). Hence, as before, the branching correspondence of this embedding does refine the spectral correspondence of \( \phi \). Observe however that this admissible algebra embedding goes in the “wrong way” compared to the situation of spectral coverings of (semi)-standard Hecke algebras.
We can combine the algebra embedding $\sigma$ with the strict algebra embedding of $\mathcal{H}(\mathcal{R}_{sc}^C, m^C)$ as a subalgebra of the algebra $\mathcal{H}(\mathcal{R}_{sc}^{(1)}, m^0)$, corresponding to the embedding of the root lattice the type $D_n$ root system as a sub lattice of index two in $\mathbb{Z}^n$, and where $m^0$ has $m^0_-=m^0_+=0$ (and on the long roots $\alpha$ of $B_n$ we have $m^0(\alpha)=1$). This amounts to removing the arrow in the spectral diagram of Figure 1 and corresponds to a $2:1$ covering $\mathcal{H}(\mathcal{R}_{sc}^{(1)}, m^0) \simeq \mathcal{H}(\mathcal{R}_{sc}^C, m^C)$. Altogether we have an admissible algebra embedding of $\mathcal{H}(\mathcal{R}_{sc}^D, m^D)$ into $\mathcal{H}(\mathcal{R}_{sc}^{(1)}, m^0)$ as subalgebra of index 4. Notice that the underlying morphism of tori does not represent a spectral covering morphism.

On the other hand, $\mathcal{H}(\mathcal{R}_{sc}^D, m^D)$ is also embedded as subalgebra of index 2 into the algebra $\mathcal{H}(\mathcal{R}_{ad}^D, m^B)$, by the same embedding of the root lattice of type $D_n$ as index two sub lattice of $\mathbb{Z}^n$, corresponding to the standard covering $\mathcal{H}(\mathcal{R}_{ad}^D, m^B) \simeq \mathcal{H}(\mathcal{R}_{ad}^{(1)}, m^0)$ represented by $id_R$. In turn $\mathcal{H}(\mathcal{R}_{ad}^D, m^B)$ is contained in the maximally extended affine Hecke algebra $\mathcal{H}(\mathcal{R}_{ad}^B, m^B)$ with $R_0$ of type $B_n$, and $X$ equal to the weight lattice $R_0$ (which equals the weight lattice of type $D_n$), and where $m^B(s)=0$ if $s$ is the simple reflection corresponding to the short simple root of $R_0$. We have admissible algebra embeddings of index two of the form $\mathcal{H}(\mathcal{R}_{ad}^{(1)}, m^0) \subset \mathcal{H}(\mathcal{R}_{ad}^B, m^B)$ and $\mathcal{H}(\mathcal{R}_{ad}^D, m^B) \subset \mathcal{H}(\mathcal{R}_{ad}^B, m^B)$ corresponding to STMs $\mathcal{H}(\mathcal{R}_{ad}^B, m^B) \simeq \mathcal{H}(\mathcal{R}_{ad}^{(1)}, m^0)$ and $\mathcal{H}(\mathcal{R}_{ad}^D, m^B) \simeq \mathcal{H}(\mathcal{R}_{ad}^B, m^B)$, which completes the diagram as follows:

$$
\begin{array}{ccc}
\mathcal{H}(\mathcal{R}_{ad}^D, m^D) & \xrightarrow{\simeq} & \mathcal{H}(\mathcal{R}_{ad}^B, m^B) \\
\mathcal{H}(\mathcal{R}_{ad}^B, m^B) & \xrightarrow{\simeq} & \mathcal{H}(\mathcal{R}_{ad}^{(1)}, m^0) \\
\mathcal{H}(\mathcal{R}_{ad}^D, m^D) & \xleftarrow{\simeq} & \mathcal{H}(\mathcal{R}_{ad}^B, m^B) \\
\mathcal{H}(\mathcal{R}_{ad}^B, m^B) & \xleftarrow{\simeq} & \mathcal{H}(\mathcal{R}_{ad}^{(1)}, m^0) \\
\end{array}
$$

3.3.6. Spectral transfer morphisms of rank 0. So far we have only considered STMs $\phi$ of co-rank $\text{cork}(\phi)=0$. In the examples of this kind we have seen only spectral correspondences arising from the branching of tempered representations of $\mathcal{H}$ to admissibly embedded subalgebras of $\mathcal{H}' \subset \mathcal{H}$.

At the opposite other end we have STMs $\phi$ with $\text{rk}(\phi)=0$, and these have a very different behaviour. Let $\mathcal{H}^0=\mathcal{L}$ denote the rank 0 affine Hecke algebra.

**Proposition 3.24.** Let $(\mathcal{H}, \tau^d)$ be an arbitrary normalized affine Hecke algebra with $\mathcal{H}=\mathcal{H}(\mathcal{R}, m)$. Let $r \in \text{Res}(\mathcal{R}, m)$ be a generic residual point for $(\mathcal{R}, m)$. Define $d^0 \in \mathbb{K}^\times$ by

$$
d^0(v) = \lambda \mu(v, r(v))
$$

where $\lambda \in \mathbb{Q}^\times$. Then $d^0 \in \mathcal{M}$, and $\phi_T=r$ defines a STM $\phi:(\mathcal{H}^0, \tau^d) \simeq (\mathcal{H}, \tau^d)$ with $\text{rk}(\phi)=0$. Conversely, all STMs $\phi$ to $\mathcal{H}$ with $\text{rk}(\phi)=0$ are of this form.

**Proof.** The fact that $d^0 \in \mathcal{M}$ is clear from Theorem 2.28. The fact that $\phi_T=r$ defines a STM is obvious from the definitions. \hfill $\square$
Hence for any $d^0 \in M$ we have (with $\text{Res} = \text{Res}(R, m))$:

$$\text{Hom}_c((H^0, \tau^{d^0}),(H, \tau^d)) = \{W_0r \in W_0 \text{Res} \mid \exists \lambda \in \mathbb{Q}^\times : d^0 = \lambda \mu^{(r)}\}$$

and given $\Phi \in \text{Hom}_c((H^0, \tau^{d^0}),(H, \tau^d))$ defined by $r \in \text{Res}(R, m)$, the spectral correspondence $S_{12}(\Phi)$ of $\Phi$ is given by (with $\Delta = \Delta(R, m))$:

$$S_{12}(\Phi) = \{(d^0, \delta) \in \Delta^0 \times \Delta \mid gcc(\delta) = W_0r\}$$

where $\delta^0 \in \Delta^0$ denotes the canonical $\mathbb{R}_{>1}$-family of characters of $H^0 = L$ defined by $\delta^0(1) = 1$. In particular, Theorem 3.11 reduces in this special case to Theorem 2.28(ii).

**Remark 3.25.** If the rank of $H$ is positive and $\Phi \in \text{Hom}_c((H^0, \tau^{d^0}),(H, \tau^d))$ then the spectral correspondence $S_{12}(\Phi)$ is obviously not associated to a branching law of a morphism of $L$-algebras.

3.3.7. General spectral transfer morphisms. For a study of general STMs (including STMs of positive rank and positive co-rank) we refer the reader to [O4], where we describe the structure of the spectral transfer category of unipotent affine Hecke algebras in detail.

3.4. A partial ordering of normalized affine Hecke algebras. The following notions will play an important role in the study of STMs of unipotent affine Hecke algebras [O4].

**Definition 3.26.** We say that $(H_1, d_1)$ and $(H_2, d_2)$ are spectrally isogenous if there exist essentially strict STMs $\phi : (H_1, d_1) \sim (H_2, d_2)$ and $\psi : (H_2, d_2) \sim (H_1, d_1)$.

Clearly this is an equivalence relation. Let us denote by $[(H, d)]$ the spectral isogeny class of a normalized affine Hecke algebra. There exists a canonical partial ordering on the set of spectral isogeny classes:

**Definition 3.27.** We say that $[(H_2, \tau_2)]$ is lower than $[(H_1, \tau_1)]$, notation $[(H_2, \tau_2)] \lesssim [(H_1, \tau_1)]$ (or equivalently, $[(H_1, \tau_1)]$ is higher than $[(H_2, \tau_2)]$, denoted by $[(H_1, \tau_1)] \gtrsim [(H_2, \tau_2)]$) if there exists a spectral transfer morphism $\phi : (H_1, \tau_1) \sim (H_2, \tau_2)$. This defines a partial ordering on the set of spectral isogeny classes of normalized affine Hecke algebras.

In most cases, the underlying affine Hecke algebras of the normalized affine Hecke algebras in a spectral isogeny class are isomorphic as affine Hecke algebras. Exceptions to this are of a trivial nature, caused by the fact that we do not assume that normalized affine Hecke algebras under consideration are semisimple and semi-standard. The spectral isogeny classes of Lusztig’s unipotent affine Hecke algebras as they arise in the theory of unramified almost simple algebraic groups defined over a local field are always simply equal to their isomorphism classes (see Proposition 3.28 below). We will just write $H_1 \lesssim H_2$ instead of $[(H_1, \tau_1)] \lesssim [(H_2, \tau_2)]$.

**Proposition 3.28.** Let $(H_1, d_1)$ and $(H_2, d_2)$ be spectrally isogenous normalized affine Hecke algebras with essentially strict STMs $\phi_1 : (H_1, d_1) \sim (H_2, d_2)$ and $\phi_2 : (H_2, d_2) \sim (H_1, d_1)$. Then $\phi_1$ and $\phi_2$ are spectral covering maps, and if $H_1$ or $H_2$ is semisimple and semi-standard then $\phi_i$ ($i = 1, 2$) both are essentially strict spectral isomorphisms. Hence the spectral isogeny classes of semisimple, semi-standard normalized affine Hecke algebras are simply their isomorphisms classes in $C_{es}$. The same holds true for $H(R_{ad}^B, m^B)$, $H(C_{n}^{(1)}, m^0)$, and $H(R_{sc}^C, m^C)$.
Proof. This is an easy consequence of the fact that for semisimple, semi-standard affine Hecke algebras, spectral transfer endomorphisms are in fact automorphisms, according to Proposition 3.13. For the listed non-semi-standard cases the spectral endomorphisms are also automorphisms, as follow easily from (54). □

Nota bene: The notion of spectral isogeny of normalized affine Hecke algebras is obviously very different from the notion of isogeny of the underlying root data, as we saw in Proposition 3.28. The main reason for introducing this notion is the importance for unipotent affine Hecke algebras (in [O4]) of the partial ordering on the spectral isogeny classes. Also observe from (54) that this partial ordering is not related to embeddings of algebras.

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