Asymmetric particle-antiparticle Dirac equation: first quantization

Gustavo Rigolin

Departamento de Física, Universidade Federal de São Carlos, 13565-905, São Carlos, São Paulo, Brazil

(Dated: November 30, 2023)

We derive a Dirac-like equation, the asymmetric Dirac equation, where particles and antiparticles sharing the same wave number have different energies and momenta. We show that this equation is Lorentz covariant under proper Lorentz transformations (boosts and spatial rotations) and also determine the corresponding transformation law for its wave function. We obtain a formal connection between the asymmetric Dirac equation and the standard Dirac equation and we show that by properly adjusting the free parameters of the present wave equation we can make it reproduce the predictions of the usual Dirac equation. We show that the rest mass of a particle in the theoretical framework of the asymmetric Dirac equation is a function of a set of four parameters, which are relativistic invariants under proper Lorentz transformations. These four parameters are the analog to the mass that appears in the standard Dirac equation. We prove that in order to guarantee the covariance of the asymmetric Dirac equation under parity and time reversal operations (improper Lorentz transformations) as well as under the charge conjugation operation, these four parameters change sign in exactly the same way as the four components of a four-vector. The mass, though, being a function of the square of those parameters remains an invariant. We also extensively study the free particle plane wave solutions to the asymmetric Dirac equation and derive its energy, helicity, and spin projection operators as well as several Gordon’s identities. The hydrogen atom is solved in the present context after applying the minimal coupling prescription to the asymmetric Dirac equation, which also allows us to appropriately obtain its non-relativistic limit.

I. INTRODUCTION

The wave function \( \psi(x) \) that solves the standard non-relativistic Schrödinger equation \([1]\) transforms after a Galilean boost as \([2]\)

\[
\psi(x) = e^{\frac{i}{\hbar} \theta(x')} \psi'(x').
\]

(1)

The phase \( \theta(x') \) is a function of \( x' = (ct', \mathbf{r}') \) and of the relative velocity \( \mathbf{v} \) between the two inertial reference frames \( S \) and \( S' \),

\[
\theta(x') = \frac{mv^2}{2} t' + m \mathbf{v} \cdot \mathbf{r}' + cte.
\]

(2)

Here \( cte \) is a real constant (usually set to zero \([2]\)), \( v^2 = \mathbf{v} \cdot \mathbf{v}, m \) is the mass of the particle, \( t' \) and \( \mathbf{r}' \) are respectively the time and position of the particle in \( S' \), \( h \) is Planck’s constant divided by \( 2\pi \), and \( c \) is the speed of light. We must have \( \psi'(x) \) transforming as given by Eq. \([1]\) if we want the Schrödinger equation to be covariant under a Galilean boost \([2]\).

The transformation law of the Schrödinger wave function shows that we can have a logically consistent theory where a complex ‘scalar’ field obeys a more general transformation law under a symmetry operation. In other words, rather than assuming that the complex field \( \psi(x) \) is a strict scalar, we can relax this assumption and demand only that the bilinear \( \psi(x) \psi^*(x) = |\psi(x)|^2 \) be a scalar under a given symmetry operation \( (|\psi(x)|^2 = |\psi'(x')|^2) \).

Is it possible to extend the transformation rule given by Eqs. \([1]\) and \([2]\) in a consistent way to the relativistic domain? What is then the relativistic wave equation covariant under a proper Lorentz transformation (boosts and spatial rotations) if the wave function transforms now according to this relativistic extension? In Ref. \([3]\) we answered in the affirmative the first question above and derived the most general relativistic wave equation covariant under proper Lorentz transformations compatible with the relativistic extension of the transformation rule given by Eqs. \([1]\) and \([2]\).

It turned out that the relativistic wave equation obtained in Ref. \([3]\), which we called the Lorentz covariant Schrödinger equation, has both first and second order time and space derivatives. The main goal of this work is to obtain a consistent relativistic wave equation that has at most first order time and space derivatives and that is, at the same time, compatible with the dispersion relations for particles and antiparticles that naturally emerge when working with the Lorentz covariant Schrödinger equation \([3]\). We also show how the wave function of this first order differential equation transforms under a proper Lorentz transformation and we prove that the wave equation is covariant under those transformations.

As it will become clear in the following pages, the first order wave equation we obtain is a Dirac-like spinorial equation \([4][7]\) and its connection to the standard Dirac equation will also be given. Throughout our formal developments we will try to build a spinorial wave equation that is as close as possible to the standard Dirac equation, pointing along the way the main differences between those two equations. And since the main difference between them is the fact that the wave equation here derived leads to relativistic energies for particles and antiparticles that are no longer degenerate, we will from
now on call it asymmetric Dirac equation.

We should also mention that the physical motivation underlying the mathematical ideas and techniques of the present work stems from the fact that almost all observables in a quantum field theory are bilinear functions of the fields. Therefore, a spinorial quantum field theory having a more general transformation rule, akin to what we have for the Lorentz covariant Schrödinger fields, should lead to a consistent theory compatible with all known experimental facts. This can be accomplished if the bilinears transform in exactly the same way as the standard Dirac bilinears do after a given symmetry operation. We only need the bilinears, not the fields themselves, to transform as usual in order to recover the predictions of the standard Dirac theory.

In the last part of this work we present some further formal developments, paving the way to the second quantization of the asymmetric Dirac equation that will be presented elsewhere, and we apply the asymmetric Dirac equation in several interesting scenarios. We start by first obtaining the free particle plane wave solutions of the asymmetric Dirac equation, which allows us to build its energy, helicity, and spin projection operators as well as derive several Gordon’s identities. We then introduce electromagnetic interactions via the minimal coupling prescription. This allows us to obtain the non-relativistic limit of the asymmetric Dirac equation and to model the hydrogen atom using the asymmetric Dirac equation. We also study how the asymmetric Dirac equation responds to the parity, time reversal, and charge conjugation symmetry operations. Finally, we show the Lagrangian density that leads to the asymmetric Dirac equation and we derive from it the most important conserved Noether currents.

II. THE LORENTZ COVARIANT SCHRÖDINGER EQUATION

Before we start the derivation of the asymmetric Dirac equation, it is important first to present the Lorentz covariant Schrödinger equation and its main features needed for our subsequent analysis. This will also help us set the notation and most of the terminology that will be used throughout this work.

A careful investigation of the meaning of the first two terms in the right hand side of Eq. 2, carried out in ref. 5, showed that for a free particle $mv^2/2$ and $mv$ are, respectively, the kinetic energy and momentum “gained” by the particle of mass $m$ when we solve the Schrödinger equation in the reference frame $S$ instead of $S'$, with $S'$ moving away from $S$ with velocity $v$. With this understanding, we postulated that in the relativistic regime we have after a boost

$$\psi(x) = e^{i\theta(x')\Psi'(x')}(3)$$

and

$$\theta(x') = (\gamma - 1)mc^2t' + \gamma mv \cdot r' + cte$$

with $\gamma = 1/\sqrt{1 - v^2/c^2}$ the Lorentz factor. Here $(\gamma - 1)mc^2$ and $\gamma mv$ are, respectively, the “gained” relativistic kinetic energy and the relativistic momentum for a particle with rest mass $m$ when we describe the particle in reference frame $S$ instead of $S'$. If in $S'$ the particle is at rest, $(\gamma - 1)mc^2$ and $\gamma mv$ are, respectively, the particle’s relativistic kinetic energy and relativistic momentum from the point of view of $S$.

With this transformation law for $\psi(x)$ we searched for the wave equation whose wave function transforms according to it and that is covariant under proper Lorentz transformations. With the aid of three extra reasonable assumptions that we list below, we obtained the following free particle wave equation,

$$\frac{1}{c^2} \frac{\partial^2 \psi(x)}{\partial t^2} - \nabla^2 \psi - i \frac{2m}{\hbar} \frac{\partial \psi(x)}{\partial t} = 0. (5)$$

The extra three assumptions that together with Eqs. 3 and 4 led uniquely to the wave equation 5 were:

1. The non-relativistic limit of the wave equation we are looking for should be the Schrödinger equation.

2. The wave equation should be isotropic, namely, covariant under three-dimensional spatial rotations in the same sense as the non-relativistic Schrödinger equation is. In other words, after a spatial rotation and assuming $\psi(x) = \psi'(x')$, we must get the same wave equation.

3. The wave equation should be a homogeneous linear partial differential equation of order not greater than two and with constant coefficients multiplying the derivatives.

Looking at Eq. 5 we realize that there is no first order spatial derivatives. This lack of symmetry between the time derivative and the spatial derivatives is a consequence of assumption (2) listed above. In order to remedy that, and get a wave equation fully symmetric in first and second order derivatives, we removed assumption (2) and postulated that for any proper Lorentz transformation (boosts or spatial rotations) the wave function should transform as given in Eq. 4, with $\theta(x')$ being a linear function of the space-time coordinates $x^0 = ct', x^1 = x', x^2 = y', x^3 = z'$.
In particular, for an infinitesimal proper Lorentz transformation,

\[ x^\mu = x'^\mu - \epsilon^{\mu\nu} x'_\nu, \]  

we have

\[ \theta(x') = -i\epsilon_{\mu\nu} \kappa^\mu x'_\nu, \]  

where \( \epsilon^{\mu\nu} \) is the infinitesimal antisymmetric tensor related to the proper Lorentz transformation being implemented. The four real parameters \( \kappa^0, \kappa^1, \kappa^2, \) and \( \kappa^3 \) will be defined in a moment but the important point that we should stress now is the fact that \( \kappa^\mu \) is not a four-vector. They are four relativistic invariants of the present theory which are related to the rest mass of the particle.

Therefore, if the wave function transforms according to Eqs. (3) and (7), the following wave equation is covariant under proper Lorentz transformations,

\[ \partial_\mu \partial^\mu \psi(x) - 2i\kappa^\mu \partial_\mu \psi(x) = 0. \]  

Here the metric is \( g_{\mu\nu} = \text{diag}(1,-1,-1,-1) \), a covariant vector is given by \( x_\mu = g_{\mu\nu} x^\nu \), and the covariant four-gradient is \( \partial_\mu = \left( \frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3} \right) \). It is also implicit in the Einstein summation convention, with Latin indexes running from 1 to 3 and Greek ones from 0 to 3, and \( g^{\mu\nu} = g_{\mu\nu} \) since we are working with a Minkowski spacetime.

If we insert the following ansatz into Eq. (8),

\[ \psi(x) = e^{i\kappa x} \phi(x) = e^{i\kappa x} e^{i\kappa x_0} \phi(x), \]  

we get

\[ \partial_\mu \partial^\mu \phi(x) + \kappa^2 \phi(x) = 0. \]  

Equation (10) can be identified with the Klein-Gordon one if we set

\[ \kappa^2 = \kappa_\mu \kappa^\mu = \mu^2 = m^2 = (mc/\hbar)^2, \]  

with \( m \) being the rest mass of the scalar particle described by the Klein-Gordon equation.

Thus, we identify the rest mass of our particle as

\[ m = \frac{\hbar \sqrt{\kappa^2}}{c}. \]  

With this identification we can show that whenever we have self interactions that respect the Lorentz symmetry or electromagnetic interactions modeled via the minimal coupling prescription, the Lorentz covariant Schrödinger equation and the Klein-Gordon one lead to the same predictions. Note that Eq. (5) is a particular case of (8) since we can get the former from the latter by setting \( \kappa^0 = mc/\hbar \) and \( \kappa^1 = 0 \). Also, if we assume no preferred orientation we must have \( \kappa^1 = \kappa^2 = \kappa^3 \).

Furthermore, Eq. (12) tells us that what we identify as the rest mass of a particle has its origin from essentially two parts. A “time-like” contribution coming from \( \kappa^0 \) and a “space-like” one coming from \( \kappa^1 \). It is the square root of \((\kappa^0)^2 - |\kappa|^2\), where \( \kappa = (\kappa^0, \kappa^1, \kappa^2, \kappa^3) \), that is proportional to the mass of the particle. To avoid an imaginary mass and properly relate the Lorentz covariant Schrödinger equation to the Klein-Gordon one we need \( |\kappa^0| > |\kappa| \). Apart from that, we are free to set any value we wish to \( \kappa^0 \) as long as we guarantee the validity of Eq. (12). The full implications of this freedom to choose \( \kappa^0 \) are not yet completely understood. However, as highlighted in Ref. [3], it may help us model condensed matter systems that are spatially anisotropic or it might shed a different light in our understanding of rest mass and mass renormalization procedures.

We should also highlight that when we second quantize Eq. (3), particles and antiparticles with the same rest mass no longer have degenerate energies. We either have \( E^+_p = -\hbar c \kappa^0 + E_p \) or \( E^-_p = \hbar c \kappa^0 + E_p \), where

\[ E_p = \sqrt{m^2 c^2 + |p|^2 c^2} \]

is the standard relativistic energy. The + and − sign in \( E^\pm_p \) remind us that \( E^+_p \) comes from the positive energy solutions and \( E^-_p \) from the negative energy solutions of the Lorentz covariant Schrödinger equation. The same feature is observed for the momentum, where we either have \( p^+_p = -\hbar k^0 + \hbar k^3 \) or \( p^-_p = \hbar k^0 + \hbar k^3 \), with \( k^0 \) being the particle or antiparticle wave number. The interpretation and physical significance of this “rest momentum” is still an open problem. See also Refs. [20–27] on how to generate an asymmetry between matter and antimatter via Lorentz-violating theories and Refs. [28, 30] on further strategies to build Lorentz-violating theories.

### III. Obtaining the Asymmetric Dirac Equation

We can better appreciate and understand the techniques used to obtain the asymmetric Dirac equation if we first review in a modern notation the path taken by Dirac himself to get to his equation [3]. After that it will become clearer why we have to follow a slightly different route to obtain the asymmetric Dirac equation than the one used by Dirac.

#### A. The Dirac equation

Dirac wanted to get a first order differential equation giving the right energy-momentum relation for a relativistic free particle. As such, each component of the spinor \( \Psi_D(x) \) had to satisfy the Klein-Gordon equation [1, 2]. Note that we are using the subscript “D” to distinguish the standard Dirac spinor from the spinor \( \Psi(x) \) associated with the asymmetric Dirac equation that will be derived in Sec. III B.

In other words, if we write a general homogeneous first
order differential equation as
\[ (i\hbar\gamma^\mu \partial_\mu - mc\tilde{B})\Psi_D(x) = 0, \]  
(13)
where \( \gamma^\mu \) and \( \tilde{B} \) are independent of \( x^\mu \) and yet to be determined, we want
\[ (i\hbar\gamma^\mu \partial_\mu + mc\tilde{B})(i\hbar\gamma^\nu \partial_\nu - mc\tilde{B}) \]  
(14)
to be equivalent to the Klein-Gordon operator
\[ g^{\mu\nu} \partial_\mu \partial_\nu + m^2 c^2 / \hbar^2. \]  
(15)
Note that the inspiration behind Dirac’s approach is the simple operator relation
\[ a^\mu a^{\nu} \partial_\mu \partial_\nu - b^2 = (a^\mu \partial_\mu + b)(a^{\nu} \partial_\nu - b), \]  
with \( a^\mu \) and \( b \) commuting objects independent of \( x^\mu \). The importance of using a plus and a minus sign in the two factors at the right hand side is crucial to obtain a left hand side with no first order derivatives.

Expanding Eq. (14) and dropping the common factor \(-\hbar^2\) we get
\[ \frac{1}{2} \{ \gamma^\mu, \gamma^\nu \} \partial_\mu \partial_\nu - i(mc/\hbar)[\tilde{B}, \gamma^\mu] \partial_\mu + (m^2 c^2 / \hbar^2)\tilde{B}^2, \]  
(16)
where \( \{ X, Y \} = XY + YX \) and \( [X, Y] = XY - YX \) are, respectively, the anticommutator and commutator of the objects \( X \) and \( Y \). Comparing Eq. (16) with (15), they are equal if
\[ \{ \gamma^\mu, \gamma^\nu \} = 2g^{\mu\nu}, \quad [\tilde{B}, \gamma^\mu] = 0, \quad \tilde{B}^2 = 1, \]  
(17)
where it is implicit that \( 2g^{\mu\nu} \) is multiplied by the identity operator \( \mathbb{1} \) (the unity matrix in what follows).

The first condition in Eq. (17) is exactly the Clifford algebra defining Dirac’s gamma matrices. The second condition implies that \( \tilde{B} \) should be proportional to the identity matrix \( \mathbb{1} \) since this is the only class of matrices that commutes with all gamma matrices \( \gamma^\mu \). Finally, the third condition implies that \( \tilde{B} = \pm \mathbb{1} \). Both choices for the sign of \( \tilde{B} \) is equally valid and lead to a consistent description of fermions \([10]\) and historically Dirac chose the plus sign, i.e., \( \tilde{B} = \mathbb{1} \).

**B. The asymmetric Dirac equation**

Similarly to what Dirac did, we want to derive a first order homogeneous differential equation. However, we want this equation to give the energy-momentum relations for free particles and antiparticles associated with the Lorentz covariant Schrödinger equation \([3]\). This is accomplished if each component of the spinor \( \Psi(x) \) that solves the first order differential equation satisfies the Lorentz covariant Schrödinger equation as given by Eq. \([3]\). Contrary to the Klein-Gordon equation, Eq. \([3]\) has both second and first order derivatives. We thus need to start from a first order differential equation and arrive at one with first and second order derivatives. As such, Dirac’s original approach, inspired by the operator relation
\[ a^\mu a^{\nu} \partial_\mu \partial_\nu - b^2 = (a^\mu \partial_\mu + b)(a^{\nu} \partial_\nu - b), \]  
has to be modified. In order to have both first and second order derivatives, we need both signs at the right hand side above equal, for instance, \( (a^\mu \partial_\mu - b)(a^{\nu} \partial_\nu - b) \).

As before, we write our general homogeneous first order differential equation as
\[ (i\hbar\gamma^\mu \partial_\mu - mc\tilde{B})\Psi(x) = 0. \]  
(18)
We now want
\[ (i\hbar\gamma^\mu \partial_\mu - mc\tilde{B})(i\hbar\gamma^\nu \partial_\nu - mc\tilde{B})\Psi(x) = 0 \]  
(19)
to be equivalent to the Lorentz covariant Schrödinger equation \([3]\).
\[ (g^{\mu\nu} \partial_\mu \partial_\nu - i2\kappa \epsilon^{\mu\nu})\Psi(x) = 0. \]  
(20)
In other words, we want that each component of \( \Psi(x) \) be a solution to the Lorentz covariant Schrödinger equation. Expanding Eq. (19) and dividing by \(-\hbar^2\) we get
\[ \left( \frac{1}{2} \{ \gamma^\mu, \gamma^\nu \} \partial_\mu \partial_\nu + i\tilde{m}(B, \gamma^\mu) \partial_\gamma - \tilde{m}^2 \tilde{B}^2 \right) \Psi(x) = 0. \]  
(21)
If we now compare Eqs. (20) and (21) and demand that they should be equal, we obtain the following three relations,
\[ \{ \gamma^\mu, \gamma^\nu \} = 2g^{\mu\nu}, \]  
(22)
\[ (B, \gamma^\mu) = -2\kappa \epsilon^{\mu\nu}, \]  
(23)
\[ B^2 = 0. \]  
(24)

We need to find \( \gamma^\mu \) and \( B \) satisfying Eqs. \((22) - (24)\) to guarantee that the first order differential equation \((18)\) has the same dispersion relations of the Lorentz covariant Schrödinger equation.

To solve Eq. \((22)\) we just need to assume that \( \gamma^\mu \) are the usual Dirac gamma matrices and all that remains to be done is to find the most general \( B \) compatible with Eqs. \((23)\) and \((24)\).

Since the 16 matrices \( \mathbb{1}, \gamma^5, \gamma^i, \gamma^5 \gamma^i \), and \( \sigma^{\mu\nu} = -\sigma^{\nu\mu} \) are linearly independent \([3]\), an arbitrary matrix \( B \) can be written as
\[ B = b_4 \mathbb{1} + b_5 \gamma^5 + b_\mu \gamma^\mu + b_5 \gamma^5 \gamma^\mu + \frac{b_{\mu\nu}}{2} \sigma^{\mu\nu}, \]  
(25)
where
\[ \gamma^5 = i\gamma^0 \gamma^1 \gamma^2 \gamma^3, \]  
(26)
\[ \sigma^{\mu\nu} = (i/2)[\gamma^\mu, \gamma^\nu], \]  
(27)
and \( b_4, b_5, b_\mu, b_5 \), \( b_{\mu\nu} \) are arbitrary complex numbers that depend on \( B \).
To compute the left hand side of Eq. (23), we need the following anticommutators,
\[
\{1, \gamma^\mu\} = 2\gamma^\mu, \\
\{\gamma^\mu, \gamma^\nu\} = 0, \\
\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}, \\
\{\gamma^\mu, \gamma^\nu\} = -i2\gamma^5\sigma^{\mu\nu} = -\epsilon^{\mu\nu\alpha\beta}g_{\rho\sigma}g_{\lambda\beta}\gamma_{\rho\sigma}, \\
\{\sigma^{\mu\nu}, \gamma^\eta\} = -2\epsilon^{\mu\nu\xi\lambda}\epsilon_{\xi\lambda\eta\rho},
\]
where \(\epsilon^{\alpha\beta\gamma\delta}\) is the completely antisymmetric four-dimensional Levi-Civita symbol, namely, \(\epsilon_{\alpha\beta\gamma\delta} = 1\) for \((\alpha, \beta, \gamma, \delta) = (0, 1, 2, 3)\) and its even permutations while \(\epsilon_{\alpha\beta\gamma\delta} = -1\) for its odd permutations.

Using Eqs. (25)–(32), we can write Eq. (23) as
\[
(2b^\mu + 2b^\mu - 2\eta^{\mu\nu}g_{\alpha\nu}\epsilon^{\mu\lambda\nu\gamma}\gamma^\alpha + i2b^\mu\gamma^5\gamma^\alpha = -\frac{2K^\mu}{m^2}.
\]
The right hand side of Eq. (33) is proportional to the identity matrix. On the other hand, in the left hand side we only have \(2b^\mu\) proportional to it, with the other coefficients multiplying the linearly independent terms \(\gamma^\mu, \gamma^5\gamma^\alpha, \) and \(\gamma^5\sigma^{\mu\nu}.\) Thus, a little algebra making use of the properties of \(\epsilon^{\alpha\beta\gamma\delta}\) necessarily implies that
\[
b^\mu = -\kappa^\mu/\tilde{m}, \\
b_1 = b_\mu = b_{\nu} = 0.
\]

If the coefficients defining \(B\) in Eq. (25) are given by Eqs. (34) and (35), we have relation (23) satisfied and \(B\) can be written as
\[
B = b_5\gamma^5 + b_\mu\gamma^\mu = b_5\gamma^5 - (\kappa^\mu/\tilde{m})\gamma^\mu.
\]

What remains to be done is to check if Eq. (36) also satisfies the last condition, Eq. (24). Inserting Eq. (36) into (24) we get
\[
(b_5)^2 + b_\mu b^\mu = 0,
\]
where we used that \((\gamma^5)^2 = 1, \{\gamma^5, \gamma^\mu\} = 0,\) and Eq. (30) to arrive at Eq. (37). Since Eq. (34) implies that \(b_\mu b^\mu = \kappa^\mu\kappa^\mu/\tilde{m}^2 = \kappa^2/\tilde{m}^2,\) we also have
\[
(b_5)^2 = -\kappa^2/\tilde{m}^2.
\]

Therefore, for \(B\) given by Eq. (36), with \(b_\mu\) and \(b_5\) given by Eqs. (34) and (35), we have all the three relations given by Eqs. (22)–(24) satisfied.

Moreover, using Eq. (11) we see that Eq. (38) implies that
\[
b_5 = \pm i.
\]

And similarly to the choice of the sign of \(\bar{B}\) for the standard Dirac equation, it is not difficult to see that here both choices for the sign of \(b_5\) are equally legitimate, leading to consistent theories. For definiteness, we stick with the positive sign and from now on
\[
b_5 = i.
\]

Using Eqs. (34), (36), and (11), we can finally write the asymmetric Dirac equation as follows,
\[
i\hbar\gamma^\mu\partial_\mu\Psi(x) - mc(i\gamma^5 + b_\mu\gamma^\mu)\Psi(x) = 0.
\]
Or, equivalently, as
\[
i\hbar\gamma^\mu\partial_\mu\Psi(x) - (\imath m\gamma^5 - \hbar\kappa^\mu\gamma^\mu)\Psi(x) = 0.
\]

As we show next, Eq. (44) cannot be satisfied for any \(B.\) To prove that, we need the following commutators,
\[
[1, \gamma^\mu] = 0, \\
[\gamma^5, \gamma^\mu] = 2\gamma^5\gamma^\mu, \\
[\gamma^\mu, \gamma^\nu] = i2\sigma^{\mu\nu}, \\
[\gamma^5\gamma^\mu, \gamma^\nu] = 2g^{\mu\nu}\gamma^5, \\
[\sigma^{\mu\nu}, \gamma^\eta] = i2(g^{\mu\nu}\gamma^\eta - g^{\eta\nu}\gamma^\mu).
\]
Looking at Eqs. (45)–(49) we realize that none of them is proportional to the identity matrix. This implies that we cannot satisfy Eq. (41), no matter how general we choose \(B\) as given by Eq. (25). Indeed, Eqs. (45)–(49) tell us that the left hand side of (44) has no term proportional to \(1\) while its right hand side has a single term proportional to \(\tilde{m}^2.\) Therefore, there is no \(B\) that satisfies (44) for massive particles since in this case we must have at least \(\kappa^0 \neq 0.\)

**IV. RELATIVISTIC PROPERTIES**

So far we have shown that it is possible to have a first order differential equation, which we called asymmetric Dirac equation, whose free particle-antiparticle energy-momentum relations are the ones given by the Lorentz covariant Schrödinger equation, a second order differential equation. But this is just half of the story. We now need to prove that the asymmetric Dirac equation is covariant under proper Lorentz transformations and calculate explicitly how the wave function \(\Psi(x)\) changes under those transformations.
To achieve this goal, we will first obtain the conserved four-current associated with the asymmetric Dirac equation and postulate that it should transform as a four-vector, limiting the choices of how $\Psi(x)$ should transform under a proper Lorentz transformation. The other constraints will appear when we require the asymmetric Dirac equation to be covariant under those transformations.

A. The conserved four-current

Since $b_\nu$ is real, $\gamma^0(\gamma^5)^\dagger \gamma^0 = -\gamma^5$, and $\gamma^0(\gamma^\mu)^\dagger \gamma^0 = \gamma^\mu$, it is not difficult to see that

$$B^\dagger = \gamma^0 B \gamma^0.$$  \hspace{1cm} (50)

Using Eq. (50), the adjoint of the asymmetric Dirac equation (18) becomes

$$i\hbar \partial_\mu \overline{\Psi}(x) \gamma^\mu + m c \overline{\Psi}(x) B = 0,$$  \hspace{1cm} (51)

where

$$\overline{\Psi}(x) = \Psi^\dagger(x) \gamma^0.$$  \hspace{1cm} (52)

Multiplying (18) by $\overline{\Psi}(x)$ at the left, (51) by $\Psi(x)$ at the right, and then summing the two expressions we get

$$\partial_\mu [\overline{\Psi}(x) \gamma^\mu \Psi(x)] = \partial_\mu j^\mu(x) = 0.$$  \hspace{1cm} (53)

Equation (53) tells us that the asymmetric Dirac equation conserve four-current, $j^\mu(x) = \overline{\Psi}(x) \gamma^\mu \Psi(x)$, is formally equal to the one coming from the standard Dirac equation. And since we want to develop the present theory as close as possible to what we have for the standard Dirac equation, we postulate that the four-current $j^\mu(x)$ transforms as a contravariant four-vector after a proper Lorentz transformation. That is, we assume

$$j'^\mu(x') = \Lambda^\mu_\nu j^\nu(x),$$  \hspace{1cm} (54)

where $\Lambda^\mu_\nu$ represents an arbitrary proper Lorentz transformation,

$$x'^\mu = \Lambda^\mu_\nu x^\nu, \hspace{1cm} (55)$$

$$\Lambda^\mu_\nu \Lambda^\nu_\rho = g^\mu_\rho = \delta^\mu_\rho.$$  \hspace{1cm} (56)

Equation (55) is a consequence of the invariance of $x^\mu x'^\mu$ under a proper Lorentz transformation. Rewriting Eq. (54) using the definition of $j'^\mu(x)$ gives

$$\overline{\Psi}'(x') \gamma^\mu \Psi'(x') = \overline{\Psi}(x) \Lambda^\mu_\nu \gamma^\nu \Psi(x).$$  \hspace{1cm} (57)

We also write the transformed spinor $\Psi'(x')$ as

$$\Psi'(x') = M(x) \Psi(x),$$  \hspace{1cm} (58)

where $M(x)$, as implied by the notation, can depend on $x$ and is assumed to be an invertible matrix.

If we insert Eq. (58) into the left hand side of (57) we get

$$\overline{\Psi}'(x') \gamma^\mu \Psi'(x') = \overline{\Psi}(x) \gamma^0 M^\dagger(x) \gamma^0 \gamma^\mu M(x) \Psi(x).$$  \hspace{1cm} (59)

Comparing Eqs. (57) and (59) we immediately obtain

$$N^\mu_{\nu} \gamma^\nu = \gamma^0 M^\dagger(x) \gamma^0 \gamma^\mu M(x),$$  \hspace{1cm} (60)

which, with the help of Eq. (56), becomes

$$\gamma^\nu = [\gamma^0 M^\dagger(x) \gamma^0] \gamma^\mu M(x) \Lambda^\nu_\mu.$$  \hspace{1cm} (61)

Equation (61) is one of the constraints the matrix $M(x)$ has to satisfy.

B. Relativistic covariance

In the inertial reference frame $S'$ the asymmetric Dirac equation should look like

$$i \hbar \gamma^\mu \partial_\mu \Psi'(x') - m c B \Psi'(x') = 0.$$  \hspace{1cm} (62)

If we use Eq. (58) and left multiply Eq. (62) by $M^{-1}(x)$, the inverse of $M(x)$, we get

$$i \hbar M^{-1}(x) \gamma^\mu M(x) \Lambda^\nu_\mu \partial_\nu \Psi(x) + \{i \hbar M^{-1}(x) \gamma^\mu M^\dagger(x) \partial_\nu M(x) \} - m c M^{-1}(x) B M(x) \Psi(x) = 0.$$  \hspace{1cm} (63)

Comparing Eq. (63) with the asymmetric Dirac equation in reference frame $S$, Eq. (18), they look the same (covariance) if

$$M^{-1} \gamma^\mu M \Lambda^\nu_\mu = \gamma^\nu,$$  \hspace{1cm} (64)

$$i \hbar M^{-1}(x) \gamma^\mu \Lambda^\nu_\mu \partial_\nu M - m c M^{-1}(x) B M = -m c B,$$  \hspace{1cm} (65)

where from now on we drop the explicit reference to the dependence of $M$ on $x$. If we left multiply Eq. (65) by $M$ we can rewrite it as

$$i \gamma^\mu \Lambda^\nu_\mu \partial_\nu M = \tilde{m} [B, M].$$  \hspace{1cm} (66)

Furthermore, if we compare Eq. (63) with (66), we get that the constraint (61) is equivalent to

$$M^{-1} = \gamma^0 M^\dagger \gamma^0,$$  \hspace{1cm} (67)

which implies that

$$\overline{\Psi}' = \overline{\Psi} M^{-1}. \hspace{1cm} (68)$$

Putting everything together, the asymmetric Dirac equation is covariant under a proper Lorentz transformation and $j^\mu$ transforms as a four-vector if there exists an invertible matrix $M$ such that it satisfies Eqs. (54), (64), and (67). Our goal in what follows is to explicitly obtain $M$. 
C. Obtaining $M(x)$: infinitesimal transformations

Since we are dealing with a continuous symmetry, we will first obtain $M(x)$ for infinitesimal proper Lorentz transformations and only afterwards the finite ones.

Using the notation already introduced in Eqs. (6) and (55), an infinitesimal proper Lorentz transformation is such that

$$A_{\mu}^\nu = g_{\mu}^\nu + \epsilon_{\mu}^\nu$$

and the respective infinitesimal matrix $M(x)$ and its inverse are

$$M(x) = 1 + \frac{1}{2} \epsilon_{\mu\nu} M^{\mu\nu}(x), \quad (70)$$

$$M^{-1}(x) = 1 - \frac{1}{2} \epsilon_{\mu\nu} M^{\mu\nu}(x). \quad (71)$$

where, due to the fact that $\sigma^{\alpha\beta}$ is antisymmetric, we can work without losing in generality with $h^{\alpha\beta\gamma\delta}$ antisymmetric in the indices $\alpha, \beta$, i.e., $h^{\alpha\beta\gamma\delta} = -h^{\beta\alpha\delta\gamma}$. In addition to that, since $M^{\mu\nu}$ is also antisymmetric, all the coefficients appearing in Eq. (73) are antisymmetric in the indexes $\mu, \nu$.

Inserting Eq. (73) into the left hand side of (72) and using the commutators listed in Eqs. (49)-(50) we get

$$[2f^{\alpha\beta\gamma\delta}(x)\gamma^\gamma + [2,5^{\alpha\beta\gamma\delta}(x)]\gamma^\gamma \gamma + [i2d^{\alpha\beta\gamma\delta}(x)]\sigma^{\alpha\beta\gamma\delta} + ih^{\alpha\beta\gamma\delta}(x)(\gamma^\gamma g^{\delta\mu} - \gamma^\delta g^{\gamma\mu}) = \gamma^\mu g^{\gamma\mu} - \gamma^\gamma g^{\delta\gamma}. \quad (74)$$

Since $\gamma^\gamma, \gamma^\delta, \sigma^{\alpha\beta\gamma\delta}$ and $\gamma^\mu$ are linear independent, it is not difficult to see that Eq. (74) implies that

$$c_5^{\gamma\delta}(x) = d^{\gamma\delta}(x) = f^{\gamma\delta}(x) = 0 \quad (75)$$

and

$$h^{\alpha\beta\gamma\delta}(x) = \frac{i}{2} (\delta^{\alpha\beta\gamma\delta} - \delta^{\beta\gamma\delta\alpha}). \quad (76)$$

If we now insert Eqs. (75) and (76) into (73) we obtain

$$M^{\mu\nu}(x) = c_4^{\mu\nu}(x)\mathbb{1} - \frac{i}{2} \sigma^{\mu\nu}. \quad (77)$$

Moving to the second relation that $M(x)$ has to satisfy, we obtain to first order in $\epsilon_{\mu\nu}$,

$$i\gamma^\nu \partial_{\mu} M^{\alpha\beta} = \tilde{m}[B, M^{\alpha\beta}], \quad (78)$$

after we insert Eqs. (30)-(31) into (66). Using Eqs. (30), (49), (77), and that $[\gamma^\gamma, \sigma^{\mu\nu}] = 0$, we can write Eq. (81) as

$$\partial_{\mu} c_{\alpha}^\beta(x) = i\tilde{m}(b^\beta \gamma^\alpha - b^\alpha \gamma^\beta), \quad (79)$$

whose solution is

$$c_{\alpha}^\beta(x) = i\tilde{m}(b^\alpha x^\beta - b^\beta x^\alpha) + c_{\alpha}^{\mu\nu}. \quad (80)$$

if we remember that $\partial_{\mu} x^\beta = \delta^\mu_\beta$. Here $c_{\alpha}^{\mu\nu}$ is an arbitrary constant.

Using Eqs. (54) and (81), Eq. (73) becomes

$$M^{\mu\nu}(x) = c_0^{\mu\nu} + i(\kappa^\mu x^\nu - \kappa^\nu x^\mu) - \frac{i}{2} \sigma^{\mu\nu}. \quad (81)$$

The last constraint on $M(x)$, Eq. (61), can be written as

$$M^{\mu\nu} = -\gamma^0 M^{\mu\nu} \gamma^0 \quad (82)$$

after Eqs. (70) and (71). Using Eq. (81), noting that $(\sigma^{\mu\nu})^\dagger = \gamma^0 \sigma^{\mu\nu} \gamma^0$, and remembering that $\kappa^\mu$ is real, Eq. (82) gives

$$c_0^{\mu\nu} = -(c_0^{\mu\nu})^\dagger. \quad (83)$$

In other words, $c_0^{\mu\nu}$ is a pure imaginary and we thus write it as

$$c_0^{\mu\nu} = ia^{\mu\nu}, \quad (84)$$

where $a^{\mu\nu}$ is a real constant.

Using Eq. (84) we can write Eq. (81) as

$$M^{\mu\nu}(x) = ia^{\mu\nu} + i(\kappa^\mu x^\nu - \kappa^\nu x^\mu) - \frac{i}{2} \sigma^{\mu\nu} \quad (85)$$

and, finally, Eq. (70) for an arbitrary infinitesimal proper Lorentz transformation,

$$M(x) = 1 + \frac{i}{2} \epsilon_{\mu\nu} a^{\mu\nu} + \frac{i}{2} \epsilon_{\mu\nu} K^{\mu\nu}(x) - \frac{i}{4} \epsilon_{\mu\nu} \sigma^{\mu\nu}, \quad (86)$$

where

$$K^{\mu\nu}(x) = \kappa^\mu x^\nu - \kappa^\nu x^\mu. \quad (87)$$

Note that if $a^{\mu\nu} = \kappa^\mu = 0$, Eq. (80) reduces to the transformation associated with the standard Dirac equation.
D. Obtaining $M(x)$: finite transformations

1. Finite boosts

For an infinitesimal boost along the $x^1$-axis we have $e_{\mu} = \epsilon (I_x)_{\mu,}$ where $\epsilon > 0$ and the generator is $(I_x)_{0,1} = (I_x)^1_0 = -1$ and zero otherwise [3]. Using this notation, Eq. (88) becomes

$$M(x) = 1 + \frac{i}{2} (I_x)_{\mu, \nu} a^{\mu \nu} + \frac{i}{2} (I_x)_{\mu, \nu} K^{\mu \nu}(x) - \frac{i}{4} (I_x)_{\mu, \sigma} \sigma^{\mu \sigma}.$$  

(88)

To order $\epsilon$ we can write Eq. (88) as

$$M(x) = \left[ 1 + \frac{i}{2} (I_x)_{\mu, \nu} a^{\mu \nu} \right] \left[ 1 + \frac{i}{2} (I_x)_{\mu, \nu} K^{\mu \nu}(x) \right] \times \left[ 1 - \frac{i}{4} (I_x)_{\mu, \sigma} \sigma^{\mu \sigma} \right] + \mathcal{O}(\epsilon).$$  

(89)

To go from the inertial reference frame $S$ to $S'$ moving with speed $v$ in the $x^1$ direction, we apply $N$ infinitesimal boosts with rapidity $\epsilon$ [5], where the finite rapidity (hyperbolic angle) is $\omega = N \epsilon$. Denoting the reference frame $S$ and its variables using the subscript 0, frame $S'$ and its variables using the subscript $N$, and calling the intermediate reference frames $S_n$, we have after $N$ hyperbolic rotations

$$\Psi_N(x_N) = f^N \left( \frac{\omega}{N} \right) \prod_{j=0}^{N-1} g \left( \frac{\omega}{N}, x_j \right) h^N \left( \frac{\omega}{N} \right) \Psi_0(x_0),$$  

(90)

where

$$f(\epsilon) = 1 + \frac{i}{2} (I_x)_{\mu, \nu} a^{\mu \nu},$$  

(91)

$$g(\epsilon, x) = 1 + \frac{i}{2} (I_x)_{\mu, \nu} K^{\mu \nu}(x),$$  

(92)

$$h(\epsilon) = 1 - \frac{i}{4} (I_x)_{\mu, \sigma} \sigma^{\mu \sigma}.$$  

(93)

We have also used that $a^{\mu \nu}, K^{\mu \nu}$, and $\sigma^{\mu \nu}$ commute among each other to obtain Eq. (90).

In the limit where $N \to \infty$ we obtain

$$\Psi'(x') = \lim_{N \to \infty} \left[ f^N \left( \frac{\omega}{N} \right) \right] \lim_{N \to \infty} \left[ \prod_{j=0}^{N-1} g \left( \frac{\omega}{N}, x_j \right) \right] \times \lim_{N \to \infty} \left[ h^N \left( \frac{\omega}{N} \right) \right] \Psi_0(x_0).$$  

(94)

However, it is not difficult to see that using Eqs. (91) and (93) we have [3],

$$\lim_{N \to \infty} \left[ f^N \left( \frac{\omega}{N} \right) \right] = \exp \{ -i/2 \omega a^{\mu \nu}(I_x)^{\mu \nu} \},$$  

(95)

$$\lim_{N \to \infty} \left[ h^N \left( \frac{\omega}{N} \right) \right] = \exp \{ -(i/4) \omega \sigma^{\mu \nu}(I_x)^{\mu \nu} \} = \exp \{ -(i/2) \omega \sigma^{10} \}.$$  

(96)

Equation (94) is related to a global constant phase, $e^{i\theta}$, since $a^{\mu \nu}$ is proportional to the identity matrix that acts on the spinorial space. We can thus drop this term from now on without losing in generality. Equation (96) is exactly the transformation law for the Dirac spinor after a boost in the $x^1$ direction [3]. Henceforth we call it by its usual notation, i.e., $S$ (do not confuse with reference frame $S$).

The mathematical steps needed to compute the remaining limit in Eq. (94) is a little more involved. This comes about since at each step we need to change the variables to the new reference frame. Contrary to Eqs. (91) and (93), Eq. (92) depends on the space-time coordinate $x^\mu$. The details of this long calculation can be found in the appendix C of ref. [3]. The final result is

$$K(x) = \lim_{N \to \infty} \left[ \prod_{j=0}^{N-1} g \left( \frac{\omega}{N}, x_j \right) \right]$$  

$$= \exp \{ i(\gamma - 1)\nu^0 + \gamma \beta^1 x^0 - i[\gamma \beta^0 + (\gamma - 1)\nu^1]x^1 \}.$$

(97)

where $\gamma = 1/\sqrt{1 - \beta^2}$ is the Lorentz factor, $\beta = v/c$, and the rapidity $\omega$ is connected to $\beta$ by the following relation, $\tanh \omega = \beta$. Note that in Ref. [3] the calculation is done from reference frame $S'$ to $S$ while here we are going from $S$ to $S'$. For boosts, this means that we should change $\beta$ to $-\beta$ in the expression in Ref. [3] to obtain Eq. (97) and for spatial rotations we should change the sign of the rotation angle to get the results presented next.

2. Finite spatial rotations

The same steps adopted to obtain the finite boost transformation from $N$ infinitesimal ones apply for finite spatial rotations too. The main difference is that instead of hyperbolic angles we now have euclidean rotation angles about a given spatial axis.

Similarly to finite boosts, we have

$$\Psi'(x') = M(x) \Psi(x) = K(x) S \Psi(x),$$  

(98)

where $S$ again is the corresponding transformation law for a Dirac spinor subjected to a given proper Lorentz transformation, a spatial rotation in the present scenario, and $K(x)$ is the corresponding transformation law of the Lorentz-Schrödinger scalar wave function $\psi(x)$, where $\psi(x)$ satisfies the Lorentz covariant Schrödinger equation [3].

We illustrate the previous point showing the explicit finite transformation law for two cases, namely, rotations about two different orthogonal axes, which are enough to generate any finite rotation.

For a $\varphi$ radian rotation about the $x^3$-axis (z-axis), whose generator is $(I_3)^{1,2} = -(I_3)^{2,1} = 1$ and zero otherwise, we get

$$S = \exp \{ i(\varphi/2)\varphi^{12} \},$$

(99)

$$K(x) = \exp \{ i[\kappa^1 x^1 + \kappa^2 x^2] / (1 - \cos \varphi) \} \times \exp \{ -i[\kappa^1 x^2 - \kappa^2 x^1] \sin \varphi \}.$$  

(100)
For a rotation of $\varphi$ radians about the $x^1$-axis ($x$-axis), whose generator is $(I_1)^2_3 = -(I_1)^3_2 = 1$ and zero otherwise, we get
\[ S = \exp\{i(2\varphi\sigma^{23})\}, \]  
\[ K(x) = \exp\{i[k_1x^2 + k_3x^3](1 - \cos \varphi)\} \times \exp\{-i[k_1^2x^2 - k_3^2x^3] \sin \varphi \}. \]

We have also implemented a consistency check of the validity of the previous calculations that lead to the above three finite proper Lorentz transformations (one boost and two rotations). We have checked that they all satisfy Eqs. (63), (66), and (67), the three relations that any transformation $M(x)$, finite or infinitesimal, should satisfy.

Remark. Since $\Psi(x') = K(x)S\Psi(x)$, with $S$ being exactly the transformation law we get when dealing with a Dirac spinor, and $K(x)$ being proportional to the identity matrix acting on the spinorial space, with $K^\dagger(x) = K^{-1}(x)$, the bilinears of the present theory transform exactly in the same way as the Dirac bilinears do after a proper Lorentz transformation. For improper Lorentz transformations, however, the analysis is more subtle and we will deal with it when we investigate the discrete symmetries of the asymmetric Dirac equation at the end of this work.

V. CONNECTION TO THE DIRAC EQUATION

If we insert the following ansatz,
\[ \Psi(x) = e^{i\kappa_\mu x^\mu} U \Psi_D(x), \]  
where
\[ U = \left( \begin{array}{cc} 1 & i\gamma^5 \sqrt{2} \\ 0 & 1 \end{array} \right), \]
into the asymmetric Dirac equation (12), we get
\[ e^{i\kappa_\mu x^\mu} U^\dagger \{ih\gamma^\mu \partial_\mu \Psi_D(x) - mc\Psi_D(x)\} = 0. \]
To obtain Eq. (105) we used that $\gamma^\mu U = U^\dagger \gamma^\mu$, which follows from $\gamma^\mu \gamma^5 = -\gamma^5 \gamma^\mu$ and $\gamma^5 = \gamma^5\dagger$, and that $\gamma^5 U = -iU^\dagger$, a consequence of $(\gamma^5)^2 = 1$.

If we left multiply Eq. (105) by $e^{-i\kappa_\mu x^\mu} U$ we obtain
\[ ih\gamma^\mu \partial_\mu \Psi_D(x) - mc\Psi_D(x) = 0, \]
which is exactly the Dirac equation. As such, $\Psi_D(x)$ is a solution to the standard Dirac equation and Eq. (103) is the connection between the asymmetric Dirac equation and the standard one.

As anticipated in the remark at the end of the last section, we can now better appreciate that the asymmetric Dirac equation’s behavior under discrete symmetry operations is not straightforward extensions of what we know from the Dirac equation. The reason for this different behavior is the presence of the $\gamma^5$ matrix in the unitary matrix connecting both equations [cf. Eq. (103)]. Under a Lorentz transformation $\Lambda$, it is known that $S^{-1} \gamma^5 S = \gamma^5 \det \Lambda$, where $\det \Lambda$ is the determinant of the matrix $\Lambda$. Therefore, for improper Lorentz transformations a minus sign will show up and thus $\gamma^5 \rightarrow -\gamma^5$, which requires a careful analysis when we build the parity and time reversal operators associated with the asymmetric Dirac equation. We will come back to this issue in Sec. XIII and in Sec. XI we will further explore under what conditions the asymmetric Dirac equation and the Dirac equation lead to the same experimental predictions at the first quantization level.

VI. PLANE WAVE SOLUTIONS

Inserting the ansatz
\[ \Psi(x) = e^{i\kappa_\mu x^\mu} \tilde{\Psi}(x) \] into the asymmetric Dirac equation (12), we obtain after multiplying by $e^{-i\kappa_\mu x^\mu}$,
\[ ih\gamma^\mu \partial_\mu \tilde{\Psi}(x) - imc\gamma^5 \tilde{\Psi}(x) = 0. \]

Two interesting features of Eq. (108) are the following. First, a direct calculation shows that
\[ (i\hbar\gamma^\mu \partial_\mu - imc\gamma^5)(i\hbar\gamma^\mu \partial_\mu - imc\gamma^5)\tilde{\Psi}(x) = 0 \]
is
\[ (\partial_\mu \partial^\mu + \tilde{m}^2)\tilde{\Psi}(x) = 0, \]
i.e., each component of Eq. (108) satisfies the Klein-Gordon equation. As such, the dispersion relation for the plane wave solutions of Eq. (108) are the standard relativistic energy-momentum relation.

Second, as we show in the appendix A the wave function $\tilde{\Psi}(x)$ that solves (108) transforms under a proper Lorentz transformation in exactly the same way as the standard Dirac spinor,
\[ \tilde{\Psi}(x') = S \tilde{\Psi}(x). \]

This implies that Eq. (108) is Lorentz covariant, which can be seen by noting that $S$ commutes with $\gamma^5$ (and this is true because $[\gamma^5, \sigma^{\mu\nu}] = 0$).

Since the free particle solutions to Eq. (108) lead to the standard relativistic dispersion relations, it is not difficult to see that the following ansatz is the most convenient one to represent the positive energy plane wave solutions of the asymmetric Dirac equation,
\[ \Psi(x) = cte \ U_r(p) e^{i\kappa_\mu x^\mu} e^{-ip_\mu x^\mu/\hbar}. \]

Here $cte$ is a normalization constant and $U_r(p)$ is a spinor, with $r = 1, 2$ labeling two linearly independent solutions that we choose to be orthogonal. These solutions are
associated with particles with positive energies given by $E_p^+ = \hbar c k^0 + E_p$, with $E_p = \sqrt{m^2 c^4 + |p|^2 c^2}$.

As we show next, Eq. (112) when inserted into the asymmetric Dirac equation (12) gives a matrix equation for $u_r(p)$ that, as the notation implies, does not depend on $k^0$. Also, $k^\mu = p^\mu / \hbar$ satisfies the same relations of the Dirac’s relativistic four-wave vector and are consistent with the usual interpretations associated with it.

Inserting Eq. (112) into (12) leads to

\[
(\not{p} - imc\gamma^5)u_r(p) = 0 \tag{113}
\]

and, consequently, to its adjoint equation,

\[
\overline{\pi}_r(p)(\not{p} - imc\gamma^5) = 0, \tag{114}
\]

where

\[
\overline{\pi}_r(p) = u_r^\dagger(p)\gamma^0 \tag{115}
\]

and $\not{p} = \gamma^\mu p_\mu$ is Feynman slash notation.

A direct calculation using that $\not{p}\not{p} = p_\mu p^\mu = p^2$ and $\gamma^\mu \gamma_\mu = -\gamma^0$ gives

\[
(\not{p} - imc\gamma^5)^2u_r(p) = (p^2 - m^2 c^2)u_r(p) = 0. \tag{116}
\]

Equation 116 implies that $p^2 = m^2 c^2$, i.e., the standard relativistic energy-momentum relation. Being more explicit, we have $(p^0)^2 = |p|^2 + m^2 c^2$, with $p^0 = E_p/c$. Note, however, that the energy and momentum for particles of mass $m$ described by the asymmetric Dirac equation are, respectively, $E_p^+ = \hbar c k^0 + E_p$ and $p^\mu_+ = -\hbar k^\mu + p^\mu$, with $p^3 = \hbar k^3$.

To describe the plane wave solutions with “negative” energies, i.e., antiparticles with positive energies when we second quantize the asymmetric Dirac equation [8], the following ansatz is the most convenient one,

\[
\Psi(x) = cte \, v_r(p)e^{i\kappa x^\mu x_\mu}e^{ipx}/\hbar. \tag{117}
\]

As before, $cte$ is a normalization constant and $v_r(p)$ is a spinor with $r = 1, 2$ labeling two linearly independent solutions that we choose to be orthogonal. Now, however, these solutions are related to antiparticles with positive energies given by $E_p^+ = \hbar c k^0 + E_p$.

Similarly to the calculations involving $u_r(p)$, it is not difficult to see that we now have

\[
(\not{p} + imc\gamma^5)v_r(p) = 0, \tag{118}
\]

\[
\overline{\pi}_r(p)(\not{p} + imc\gamma^5) = 0, \tag{119}
\]

where

\[
\overline{\pi}_r(p) = v_r^\dagger(p)\gamma^0. \tag{120}
\]

Also,

\[
(\not{p} + imc\gamma^5)^2v_r(p) = (p^2 - m^2 c^2)v_r(p) = 0, \tag{121}
\]

which implies that $p^2 = m^2 c^2$ and $(p^0)^2 = |p|^2 + m^2 c^2$. The energy and momentum for antiparticles of mass $m$ described by the asymmetric Dirac equation are, respectively, $E_p^- = \hbar c k^0 + E_p$ and $p^\mu_- = \hbar k^\mu - p^\mu$, with $p^3 = \hbar k^3$.

We should mention that which vacuum excitation we call particle or antiparticle is rather arbitrary. Following the choice adopted in Ref. [3], we call particles the vacuum excitations with the smallest energy for given a wave number $k^3$ and antiparticles the excitations with the greatest energy for the same wave number.

Following the standard prescription [5, 6], we normalize $u_r(p)$ and $v_r(p)$ as follows,

\[
u_r^\dagger(p)u_r(p) = v_r^\dagger(p)v_r(p) = E_p/mc^2. \tag{122}
\]

Since for a degenerate eigenvalue we can always choose orthogonal eigenvectors, we have

\[
u_r^\dagger(p)u_s(p) = v_r^\dagger(p)v_s(p) = \frac{E_p}{mc^2}\delta_{rs} = \frac{p^0}{mc}\delta_{rs}, \tag{123}
\]

where $\delta_{rs}$ is the Kronecker delta.

As we prove in the appendix B, Eqs. (113)-(131) for all $r$ and $s$ lead to the following orthonormality relations,

\[
u_r^\dagger(p)\nu_s(-p) = \delta_{rs}, \tag{124}
\]

\[
u_r^\dagger(p)u_s(p) = \delta_{rs}, \tag{125}
\]

\[
u_r^\dagger(p)\gamma^5 u_s(p) = \delta_{rs}, \tag{126}
\]

\[
u_r^\dagger(p)i\gamma^5 u_s(p) = \delta_{rs}. \tag{127}
\]

To the following completeness relation (resolution of the identity),

\[
\mathbb{1} = \sum_{r=1}^{2}[u_r(p)\overline{\pi}_r(p)i\gamma^5 - \overline{\pi}_r(p)i\gamma^5 v_r(p)], \tag{128}
\]

and to these identities,

\[
u_s(-p)\nu_r(p) = \frac{-ip^0}{mc}u_s^\dagger(-p)\gamma^5 v_r(p), \tag{129}
\]

\[
u_s(-p)u_r(p) = \frac{ip^0}{mc}u_s^\dagger(-p)\gamma^5 u_r(p). \tag{130}
\]

It is worth noting that several of the previous relations are different from the ones the Dirac spinors satisfy. The standard Dirac spinors $u_r(p)$ and $v_r(p)$ do not satisfy Eqs. (125)-(131) for all $r$ and $s$. For instance, they do not satisfy Eq. (126) for $r = s$ and the equivalent expressions related to Eqs. (128) and (129). However, $\nu_r(p)$ do not have the $\gamma^5$ matrix and the imaginary number $i$.

**VII. ENERGY PROJECTION OPERATORS**

The energy projection operators for the asymmetric Dirac equation are

\[
A^\pm(p) = \frac{\gamma^5(\pm\not{p} + imc\gamma^5)}{2imc} = \frac{(\pm\not{p} - imc\gamma^5)i\gamma^5}{2mc}, \tag{132}
\]
where we used that $\gamma^5 \not{p} = -\not{p} \gamma^5$ to obtain the last term above from the middle one.

Using Eqs. (113)-(114) and (118)-(119) we can prove that

$$\Lambda^+(p) u_r(p) = u_r(p),$$
$$\Lambda^+(p) v_r(p) = 0,$$
$$\Lambda^-(p) u_r(p) = v_r(p),$$
$$\Lambda^-(p) u_r(p) = 0.$$  \tag{136}

Equations (133)-(136) are equal to the relations obtained for the standard Dirac equation.

To prove Eq. (133) we add and subtract $imc\gamma^5$ inside the parenthesis that appear in the definition of $\Lambda^+$ [cf. Eq. (132)]. Then, we proceed as follows,

$$\Lambda^+(p) u_r(p) = \frac{\gamma^5 (\not{p} - imc\gamma^5 + 2imc\gamma^5)}{2imc} u_r(p)$$
$$= \frac{\gamma^5}{2imc} (\not{p} - imc\gamma^5) u_r(p) + (\gamma^5)^2 u_r(p)$$
$$= 0 + i u_r(p) = u_r(p),$$  \tag{137}

where Eq. (113) was used to obtain the last line. To see that Eq. (134) is indeed true, we just need to use Eqs. (114) and (119) in a similar way we prove Eqs. (135) and (136).

On the other hand, the action of $\Lambda^\pm(p)$ on the adjoint spinors are not the same we obtain for the standard Dirac equation. Here we have

$$\Pi^+(p) \Lambda^+(p) = 0,$$
$$\Pi^+(p) \Lambda^-(p) = \Pi_\uparrow(p),$$
$$\Pi^- \Lambda^+(p) = 0,$$
$$\Pi^- \Lambda^-(p) = \Pi_\uparrow(p).$$  \tag{141}

The same techniques used to prove Eqs. (138)-(141) apply here. We just need to use Eqs. (114) and (119) instead of (113) and (118) to complete the proofs. The corresponding expressions for the Dirac spinors are [5, 6], $\Pi_\uparrow = \Pi_\uparrow$, $\Pi_\downarrow = 0$, and $\Pi_\uparrow = \Pi_\uparrow$, where $\Lambda^\pm = (\pm \not{p} + imc)/(2mc)$.

Note that if we look at Eq. (126), we can also understand why we must get Eqs. (138) and (140). For instance, Eq. (126) tells us that $\Pi_r(p) u_s(p)$ is orthogonal to $u_s(p)$, for any $r, s$. Therefore, $u_s(p)$ and $\Pi_r(p)$ must be associated with states having different eigenvalues. As such, if $\Lambda^+(p) u_r(p) \neq 0$ it is expected that $\Pi_r(p) \Lambda^+(p) = 0$. A similar reasoning can be made to explain Eq. (140).

Using that $p p = p^2 = mc^2$, $\not{p} \gamma^5 = -\gamma^5 \not{p}$, and $(\gamma^5)^2 = 1$, we can also show that $\Lambda^\pm(p)$ satisfies the usual properties of projection operators,

$$[\Lambda^\pm(p)]^2 = \Lambda^\pm(p),$$
$$\Lambda^\pm(p) \Lambda^\mp(p) = 0,$$
$$\Lambda^+(p) + \Lambda^-(p) = 1.$$  \tag{144}

Finally, if we apply the completeness relation (128) at the left and right of $\Lambda^\pm(p)$ we get

$$\Lambda^+(p) = \sum_{r=1}^2 u_r(p)\Pi_r(p)i\gamma^5,$$  \tag{145}
$$\Lambda^-(p) = -\sum_{r=1}^2 v_r(p)\Pi_r(p)i\gamma^5.$$  \tag{146}

VIII. HELICITY AND SPIN PROJECTION OPERATORS

A. Helicity projection operators

Similarly to the standard Dirac equation, helicity is another quantum number that can be used to classify the free particle solutions of the asymmetric Dirac equation. We define it here in exactly the same way we do for the standard Dirac equation. The helicity projection operators are [5, 6],

$$\Pi^\pm(p) = \frac{1}{2}(1 \pm \sigma_p),$$  \tag{147}

where

$$\sigma_p = \frac{\sigma \cdot p}{|p|}$$  \tag{148}

and

$$\sigma = (\sigma^{23}, \sigma^{31}, \sigma^{12}).$$  \tag{149}

with $\sigma^{\mu\nu}$ given by Eq. (21).

As expected, the helicity projector operators satisfy the following identities [5, 6],

$$[\Pi^\pm(p)]^2 = \Pi^\pm(p),$$  \tag{150}
$$\Pi^+(p)\Pi^+(p) = 0,$$  \tag{151}
$$\Pi^+(p) + \Pi^- = 1.$$  \tag{152}

Also, since the eigenvalues of $\sigma_p$ are $\pm 1$, the eigenvalues of $\Pi^\pm(p)$ are 0 or 1.

To carry over the remaining properties associated with $\Pi^\pm(p)$ that are true for the standard Dirac equation to the present one (those properties based on Eqs. (153) and (155)), we just need to prove that

$$[\Lambda^+(p), \Pi^\pm(p)] = [\Lambda^-(p), \Pi^\pm(p)] = 0.$$  \tag{153}

This is accomplished in appendix C.

Equation (163) tells us that $\Lambda^\pm(p)$, $\Pi^\pm(p)$, $\sigma_p$, and $p = p_1$ can all be diagonalized together. Thus, we can choose the spinors $u_r(p)$ and $v_r(p)$, which are eigenstates of $\Lambda^\pm(p)$, as follows,

$$\sigma_p u_r(p) = (-1)^{r+1} u_r(p),$$  \tag{154}
$$\sigma_p v_r(p) = (-1)^r v_r(p).$$  \tag{155}
The choices given by Eqs. (154) and (155) are very useful when dealing with the second quantization of the asymmetric Dirac equation $\hat{A}$, allowing a similar interpretation to the meaning of $u_r(p)$ and $v_r(p)$ which is obtained when we second quantize the standard Dirac equation $\hat{A}$. For instance, $u_1(p)$ represents a particle with positive energy $E_p^+$ and spin parallel to the direction of its motion given by $p$, which we call a positive helicity state. For $u_2(p)$ we have a negative helicity state, with its spin antiparallel to its momentum $p$. On the other hand, $v_r(p)$ will be associated with antiparticles having positive energies given by $E_p^-$, with the same momentum and helicity of the corresponding particle represented by $u_r(p)$.

Finally, using Eqs. (147), (154) and (155) it can be shown that

$$\Pi^+(p)u_r(p) = \delta_{1r}u_r(p), \quad (156)$$
$$\Pi^+(p)v_r(p) = \delta_{2r}v_r(p), \quad (157)$$
$$\Pi^-(p)u_r(p) = \delta_{2r}u_r(p), \quad (158)$$
$$\Pi^-(p)v_r(p) = \delta_{1r}v_r(p). \quad (159)$$

We should also mention that since Eqs. (113) and (118) are equal to the corresponding ones related to the standard Dirac equation whenever $m = 0$, we can express the helicity projection operators for a massless particle as

$$\Pi^+(p) = \frac{1}{2}(1 \mp \gamma^5). \quad (160)$$

B. Spin projection operators

Similarly to a standard Dirac particle, we have that only in the rest frame of the particle the spin component in an arbitrary direction is a good quantum number. However, it is possible to define in the rest frame and in a covariant way spin projection operators for an arbitrary quantization axis. From this definition, we can go to any frame by implementing the appropriate Lorentz transformation.

For the asymmetric Dirac equation, the covariant spin projection operators in the rest frame along the direction $\hat{n}$ are

$$\Pi^\pm(n) = \frac{1}{2}(1 \mp i\gamma^5), \quad (161)$$

where

$$n^\mu = (0, \hat{n}). \quad (162)$$

Note that Eq. (162) implies that

$$n^2 = n^{\mu}n_\mu = -1 \quad \text{and} \quad np = n^\mu p_\mu = 0 \quad (163)$$

in all frames due to the invariance of scalar products. Indeed, in the rest frame Eq. (162) gives $n^2 = -|\hat{n}|^2 = -1$ and using that in the rest frame $p = (mc, 0)$ we get $np = 0$.

Equation (161) is different from the standard Dirac particle spin projection operators, namely, $\Pi^\pm_D(n) = (1/2)(1 \pm \gamma^5)\hat{n}$. The reason for this difference stems from Eq. (118) and the presence of the matrix $\gamma^5$ in the inertia term of that equation.

In the appendix E we prove that Eq. (161) satisfies all the required properties of good spin projector operators $\hat{A}$, in particular the fact that they commute with the energy projection operators for all $p$ satisfying Eq. (163).

IX. GORDON’S IDENTITIES

As we prove in the appendix E if we use that

$$(p' + p)^\mu + i\sigma_{\mu\nu}(p' - p)_\nu = \not{p}'\gamma^\mu + \gamma^\mu\not{p} \quad (164)$$

and Eqs. (113), (114), (118), and (119), we obtain the following four Gordon’s identities,

$$\overline{\not{p}}(p')[(p' + p)^\mu + i\sigma_{\mu\nu}(p' - p)_\nu]u_r(p) = 0, \quad (165)$$
$$\overline{\not{p}}(p')[(p' + p)^\mu + i\sigma_{\mu\nu}(p' - p)_\nu]v_r(p) = 0, \quad (166)$$
$$\overline{\not{p}}(p')[(p' + p)^\mu + i\sigma_{\mu\nu}(p' - p)_\nu]u_r(p) = 2imc\overline{\not{p}}(p')(\gamma^5)\gamma^\mu\gamma^5u_r(p), \quad (167)$$
$$\overline{\not{p}}(p')[(p' + p)^\mu + i\sigma_{\mu\nu}(p' - p)_\nu]v_r(p) = -2imc\overline{\not{p}}(p')(\gamma^5)\gamma^\mu\gamma^5v_r(p). \quad (168)$$

Furthermore, if repeat the steps leading to the previous Gordon’s identities using Eq. (164) with $p \to -p$, we get these other four Gordon’s identities,$n^\mu = (0, \hat{n}). \quad (162)$

$$\overline{\not{p}}(p')[(p' + p)^\mu + i\sigma_{\mu\nu}(p' + p)_\nu]u_r(p) = -2imc\overline{\not{p}}(p')(\gamma^5)\gamma^\mu\gamma^5u_r(p), \quad (169)$$
$$\overline{\not{p}}(p')[(p' + p)^\mu + i\sigma_{\mu\nu}(p' + p)_\nu]v_r(p) = 2imc\overline{\not{p}}(p')(\gamma^5)\gamma^\mu\gamma^5v_r(p), \quad (170)$$
$$\overline{\not{p}}(p')[(p' - p)^\mu + i\sigma_{\mu\nu}(p' - p)_\nu]u_r(p) = 0, \quad (171)$$
$$\overline{\not{p}}(p')[(p' - p)^\mu + i\sigma_{\mu\nu}(p' - p)_\nu]v_r(p) = 0. \quad (172)$$

X. PLANE WAVE SOLUTIONS IN THE DIRAC-PAULI REPRESENTATION

Remembering that $\not{p}\not{p} = p^2 = m^2c^2$, it is not difficult to see that

$$(\not{p} + imc\gamma^5)(\not{p} + imc\gamma^5) = 0. \quad (173)$$

Using Eq. (173) and Eqs. (113) and (118), we have the following free particle solutions to the asymmetric Dirac equation in momentum space,

$$u_r(p) = N(p)(\not{p} - imc\gamma^5)v_r(0), \quad (174)$$
$$v_r(p) = N(p)(\not{p} + imc\gamma^5)u_r(0). \quad (175)$$
where
\[ \mathbb{T} = \begin{cases} 1, & \text{if } r = 2, \\ 2, & \text{if } r = 1 \end{cases} \]  
(176)
and
\[ N(p) = \frac{1}{\sqrt{2mc(p^0 + mc)}} = \frac{1}{\sqrt{2mE_p + m^2c^2}} \]  
(177)
to guarantee that the normalization given by Eq. (123) is satisfied.

The zero momentum spinors \( u_\tau(0) \) and \( v_\tau(0) \) in the Dirac-Pauli representation (see appendix [1]) are, respectively, given by Eqs. (173) and (177). And in Eqs. (174) and (175), we have \( v_\tau(0) \) and \( u_\tau(0) \) because, as a direct calculation in the present representation shows, we have
\[ -i\gamma^5 u_\tau(0) = \gamma^0 v_\tau(0) = u_r(0), \]  
(178)
\[ i\gamma^5 u_\tau(0) = \gamma^0 u_r(0) = v_r(0). \]  
(179)
This ensures the correct “initial condition”, namely, \( u_r(p) \) and \( v_r(p) \) in Eqs. (173) and (175) become \( u_r(0) \) and \( v_r(0) \) when \( p = 0 \).

The corresponding plane wave solutions are, according to Eqs. (172) and (177),
\[ \Psi(x) = \begin{cases} \text{cte } u_r(p)e^{i\kappa_n x^n}e^{-ip_0x^0}/\hbar, \\ \text{cte } v_r(p)e^{i\kappa_n x^n}e^{ip_0x^0}/\hbar, \end{cases} \]  
(180)
and, if we explicitly compute Eqs. (174) and (175), we get
\[ u_1(p) = \frac{N(p)}{\sqrt{2}} \left( \begin{array}{c} p^0 + mc - ip^3 \\ p^2 - ip^1 \\ p^3 - i(p^0 + mc) \end{array} \right), \]  
(181)
\[ u_2(p) = \frac{N(p)}{\sqrt{2}} \left( \begin{array}{c} -p^2 - ip^1 \\ p^0 + mc + ip^3 \\ -p^3 - i(p^0 + mc) \end{array} \right), \]  
(182)
\[ v_1(p) = \frac{N(p)}{\sqrt{2}} \left( \begin{array}{c} p^2 + ip^1 \\ p^0 + mc - ip^3 \\ -p^3 + i(p^0 + mc) \end{array} \right), \]  
(183)
\[ v_2(p) = \frac{N(p)}{\sqrt{2}} \left( \begin{array}{c} -p^2 + ip^1 \\ p^0 + mc + ip^3 \\ p^3 + i(p^0 + mc) \end{array} \right). \]  
(184)

There is a feature characteristic of Eqs. (181)-(184) that sets them apart from the corresponding solutions of the standard Dirac equation [3, 4]. This is related to the fact that we cannot split them in an upper two-dimensional spinor and a lower two-dimensional one, where only the upper spinor or the lower one depends on \( p \). Also, looking at Eqs. (181)-(184) when \( p = 0 \) [cf. Eqs. (D15)-(D16)], we realize that, in contradistinction to the plane wave solutions of the standard Dirac equation in the rest frame, both the upper and lower parts of \( u_r(0) \) and \( v_r(0) \) are not zero.

Before we finish this section, we will prove the following identities that will be helpful when we deal with the discrete symmetries associated with the asymmetric Dirac equation,
\[ u_r(p) = i\gamma^0\gamma^5 u_r(-p), \]  
(185)
\[ v_r(p) = -i\gamma^0\gamma^5 v_r(-p). \]  
(186)
To prove Eq. (185), we first realize that for \( p = 0 \) Eq. (175) can be written as
\[ v_\tau(0) = \frac{1}{2}(\gamma^0 + i\gamma^5)u_r(0). \]  
(187)
Inserting Eq. (187) into (174) we get after a little algebra and using Eq. (175),
\[ u_r(p) = \frac{1}{2}\gamma^0 v_\tau(-p) - \frac{i}{2}\gamma^5 v_\tau(p). \]  
(188)
If we now change \( p \) to \( -p \) in Eq. (188) and left multiply it by \( i\gamma^5\gamma^0 \), we obtain
\[ i\gamma^5\gamma^0 u_r(-p) = -\left[ \frac{1}{2}\gamma^0 v_\tau(-p) - \frac{i}{2}\gamma^5 v_\tau(p) \right] = -u_r(p). \]  
(189)
And since \( \gamma^5\gamma^0 = -\gamma^0\gamma^5 \), Eq. (189) is equal to (185), completing the proof. The same reasoning can be used to prove Eq. (186), exchanging the roles of Eqs. (175) and (174), while repeating the above logical steps.

**XI. MINIMAL COUPLING PRESCRIPTION**

Using SI units and the metric signature of the present work, the electromagnetic minimal coupling prescription [3, 4] is implemented in the asymmetric Dirac equation by changing all derivatives \( \partial_\mu \) to
\[ \partial_\mu \rightarrow D_\mu = \partial_\mu + \frac{iq}{\hbar} A_\mu. \]  
(190)
The covariant four-vector potential is given by
\[ A_\mu = \left( \begin{array}{c} \varphi \\ c \mu \end{array} \right), \]  
(191)
where \( \varphi \) and \( A = (A^1, A^2, A^3) \) are, respectively, the electric and vector potentials describing an electromagnetic field.
Therefore, using Eq. (190) in Eq. (12) we obtain
\[ i\hbar \gamma^\mu D_\mu \Psi(x) - (imc\gamma^5 - \hbar\kappa_\mu \gamma^\mu) \Psi(x) = 0. \] (192)

The first term above, i.e., \( i\hbar \gamma^\mu D_\mu \Psi(x) \), is formally equal to the one coming from the standard Dirac equation when it is minimally coupled to the electromagnetic field. As such, many techniques used to solve the Dirac equation in the presence of an external electromagnetic field can be carried over to solve the equivalent problem using the asymmetric Dirac equation.

A direct calculation shows that Eq. (192) is covariant under proper Lorentz transformations if \( \Psi \) transforms as given by Eq. (195). To arrive at that conclusion we should remember that after a proper Lorentz transformation \( D_\mu \) transforms as a covariant four-vector since \( \partial_\mu \) transforms as a covariant four-vector. This means that we need to solve the standard Dirac equation to obtain the solutions to the asymmetric Dirac one.

Also, the gauge transformation
\[ A_\mu(x) = \tilde{A}_\mu(x) - \partial_\mu \chi(x) \] (193)
leads to
\[ i\hbar \gamma^\mu \tilde{D}_\mu \tilde{\Psi}(x) - (imc\gamma^5 - \hbar\kappa_\mu \gamma^\mu) \tilde{\Psi}(x) = 0. \] (194)
In Eq. (194) we have
\[ \tilde{D}_\mu = \partial_\mu + \frac{iq}{\hbar} \tilde{A}_\mu \]
and
\[ \tilde{\Psi}(x) = e^{i(q/\hbar)\chi(x)} \tilde{\Psi}(x). \] (195)

The above argument proves that the asymmetric Dirac equation minimally coupled to the electromagnetic field is covariant after a local gauge transformation, as given by Eq. (196), if the wave function changes as prescribed by Eq. (197). Moreover, this covariance also implies that a local gauge transformation cannot be used to get rid of the constants \( \kappa^\mu \).

### A. The Hydrogen atom

Computing explicitly Eq. (192) we get
\[ i\hbar \gamma^\mu \partial_\mu \Psi(x) - mcB\Psi(x) - qA_\mu \gamma^\mu \Psi(x) = 0, \] (196)
where \( B \) is given by Eq. (36). Using Eq. (103), carrying out the derivatives, and left multiplying by \( Ue^{-i\kappa_\mu x^\mu} \), we obtain
\[ i\hbar \gamma^\mu \partial_\mu \Psi_D(x) - mc\Psi_D(x) - qA_\mu \gamma^\mu \Psi_D(x) = 0. \] (197)
Equation (197) is the standard Dirac equation after we apply to it the minimal coupling prescription. This means that we need to solve the standard Dirac equation to obtain the solutions to the asymmetric Dirac one.

Being more specific, any bound state problem involving regular matter associated with the asymmetric Dirac equation will have the eigenvalues of the equivalent problem related to the standard Dirac equation displaced by \(-\hbar\kappa^0\) [cf. the phase appearing in Eq. (103) and also Ref. [3]]. The corresponding wave function (eigenvector) will be given by Eq. (103), where \( \Psi_D(x) \) is the respective solution to the standard Dirac equation.

For the Hydrogen atom (static Coulomb problem), we have the eigenvalues
\[ E_{n,j+1/2} = E_{n,j+1/2}^D - \hbar\kappa^0, \] (198)
with \( E_{n,j+1/2}^D \) being the eigenvalues we obtain by solving the standard Dirac equation [3],
\[ E_{n,j+1/2}^D = mc^2 \left[ 1 + \left( \frac{\alpha}{n - (j + \frac{1}{2}) + \sqrt{(j + \frac{1}{2})^2 - \alpha^2}} \right)^2 \right]^{1/2}. \] (199)
Here \( n \geq 1 \) is a positive integer and \( j + 1/2 = 1, 2, \ldots, n \). The corresponding eigenvectors are according to Eq. (103),
\[ \Psi_{n,j+1/2}(x) = e^{i\kappa_\mu x^\mu} \left( 1 - i\gamma^5 \right) \Psi_{D,n,j+1/2}(x), \] (200)
with \( \Psi_{D,n,j+1/2}(x) \) being the respective solution to the standard Dirac equation. Note that in the particular case where \( \kappa^0 = mc/\hbar \) and \( \kappa = 0 \), the eigenvalues (199) are simply the ones coming from the usual Dirac equation with the rest energy \( mc^2 \) subtracted from them.

Moreover, if we define the unitary operator
\[ V(x) = e^{i\kappa_\mu x^\mu} \left( \frac{1 - i\gamma^5}{\sqrt{2}} \right) \] (201)
the expectation value of an observable \( \hat{O} \) according to Eq. (200), using the Dirac bra and ket notation, is
\[ \langle \Psi_\lambda | \hat{O} | \Psi_\lambda \rangle = \langle \Psi_{D,\lambda} | V^\dagger \hat{O} V | \Psi_{D,\lambda} \rangle, \] (202)
where \( \lambda \) denotes all relevant quantum numbers. Now, if
\[ V^\dagger \hat{O} V = \hat{O}, \] (203)
i.e., if \( V \) and \( \hat{O} \) commute, the predictions for the expectation value of the observable \( \hat{O} \) are the same as the ones coming from the standard Dirac equation. And since the non-trivial part of \( V \) (the part not proportional to the identity matrix) is \( \gamma^5 \), whenever \( [\hat{O}, \gamma^5] = 0 \) the asymmetric and standard Dirac equations lead to the same predictions.

We can also build the asymmetric Dirac equation’s observables in such a way that we enforce the two theories to give the same predictions. This can be done by postulating that the any observable \( \hat{O}_D \) associated with the standard Dirac equation is mapped to an observable related to the asymmetric Dirac equation as follows,
\[ \hat{O} = V \hat{O}_D V^\dagger. \] (204)
Inserting Eq. (204) into the right hand side of (202) we get
\[ \langle \Psi_\lambda | \hat{O} | \Psi_\lambda \rangle = \langle \Psi_{D,\lambda} | \hat{O}_D | \Psi_{D,\lambda} \rangle, \] (205)
which tells us that both theories yield the same predictions. The previous analysis is valid, as we just showed, in the first quantization level. This is also true for the second quantized theory as we show in Ref. [8].

XII. NON-RELATIVISTIC LIMIT

The non-relativistic limit of the asymmetric Dirac equation is most easily obtained by using the transformation (103) that connects it with the standard Dirac equation and working in the Dirac-Pauli representation for the gamma matrices. As such, by using the non-relativistic limit of the latter equation we can obtain the non-relativistic limit of the former via Eq. (103).

According to Ref. [3], if we insert the ansatz
\[ \Psi_D(x) = e^{-imc/\hbar}\tilde{\Psi}_D(x) \] (206)
into the minimally coupled standard Dirac equation we get
\[ i\hbar \partial_t \left( \frac{\bar{\chi}}{x} \right) = \left( \frac{c\hat{\sigma} \cdot \hat{\pi} \bar{\chi}}{x} \right) + qcA^0 \left( \bar{\chi} \bar{\chi}_\lambda \right) - 2mc^2 \left( \frac{0}{\chi} \right), \] (207)
where
\[ \hat{\pi} = \hat{p} - qA, \] (208)
\[ \hat{p} = -i\hbar \nabla, \] (209)
\[ \hat{\sigma} = (\sigma_1, \sigma_2, \sigma_3), \] (210)
\[ \tilde{\Psi}_D(x) = \left( \bar{\chi}_\lambda \right)(x). \] (211)

In Eq. (211) we should understand \( \bar{\chi}_\lambda \) and \( \bar{\chi}_\lambda \) as two dimensional spinors and in Eq. (210) \( \hat{\sigma}_j \) as the standard \( 2 \times 2 \) Pauli matrices.

In the non-relativistic limit the particle’s kinetic energy and its potential energy are small when compared to its rest energy. This means that
\[ |i\hbar \partial_t \bar{\chi}_\lambda| \ll |mc^2\bar{\chi}_\lambda|, \] (212)
\[ |qcA^0\bar{\chi}_\lambda| \ll |mc^2\bar{\chi}_\lambda|. \] (213)

Within this level of approximation, Eqs. (212) and (213) imply that the lower part of Eq. (207) is solved if
\[ \bar{\chi}_\lambda = \tilde{\chi}_\lambda = \frac{\hat{\sigma} \cdot \hat{\pi}}{2mc} \bar{\chi}_\lambda, \] (214)
which, when inserted into the upper part of Eq. (207) gives
\[ i\hbar \partial_t \bar{\chi}_\lambda = \left[ \frac{\hat{\sigma} \cdot \hat{\pi}}{2m} + qcA^0 \right] \bar{\chi}_\lambda, \] (215)
Equation (215) is the Pauli equation written in SI units, the non-relativistic limit of the standard Dirac equation. Here \( \mathbf{B} = \nabla \times \mathbf{A} \) and to obtain the last line of (215) we used the identity \( (\hat{\sigma} \cdot \hat{\pi})^2 = \pi^2 - q\hbar\hat{\sigma} \cdot \mathbf{B} \), a consequence of the algebra of the Pauli matrices [3].

Inserting Eq. (206) into (103) we get
\[ \Psi(x) = e^{in\mu x^n - i\bar{m}x^0} \frac{1}{\sqrt{2}} \left( \bar{\phi}_\lambda - i\bar{\chi}_\lambda \right). \] (216)
But in the non-relativistic limit \( |\chi_\lambda| \ll |\bar{\chi}_\lambda| \) because \( \chi_\lambda \approx (v/2c)\bar{\chi}_\lambda \) [cf. Eq. (214) and Ref. [3]]. Thus, in the non-relativistic approximation, we can neglect the \( \bar{\chi}_\lambda \) terms in Eq. (216),
\[ \Psi(x) = e^{in\mu x^n - i\bar{m}x^0} \frac{1}{\sqrt{2}} \left( -i\bar{\phi}_\lambda(x) \right). \] (217)
If we now compute the time derivative of Eq. (217) and use (215) we get
\[ i\hbar \partial_t \Psi(x) = (\bar{m} - \kappa^0)\hbar c\Psi(x) + e^{in\mu x^n - i\bar{m}x^0} \times \left[ \frac{\hat{\pi}^2}{2m} - \frac{q\hbar}{2m} \hat{\sigma} \cdot \mathbf{B} + qcA^0 \right] \frac{1}{\sqrt{2}} \left( -i\bar{\phi}_\lambda(x) \right) \]
\[ \left[ \bar{m} - \kappa^0 \right] \hbar c\Psi(x) + e^{in\mu x^n - i\bar{m}x^0} \times \left[ \frac{\hat{\pi}^2}{2m} - \frac{q\hbar}{2m} \hat{\sigma} \cdot \mathbf{B} + qcA^0 \right] e^{-in\mu x^n + i\bar{m}x^0} \Psi(x), \] (218)
where the last line is obtained using Eq. (217). With the aid of the identity,
\[ \frac{\hat{\pi}^2}{2m}[e^{i\kappa \tau} \Psi(x)] = e^{i\kappa \tau} \left( \frac{\hat{\pi} + \hbar \kappa}{2m} \right)^2 \Psi(x), \] (219)
we can finally rewrite Eq. (218) as
\[ i\hbar \partial_t \Psi(x) = (mc^2 - ch\kappa^0)\Psi(x) \]
\[ + \left[ \frac{(\hat{\pi} + \hbar \kappa)^2}{2m} - \frac{q\hbar}{2m} \hat{\sigma} \cdot \mathbf{B} + qcA^0 \right] \Psi(x). \] (220)
Equation (220) is the non-relativistic approximation of the asymmetric Dirac equation. It is the analog of the Pauli equation in the present context.

In the particular case where \( \kappa^0 = mc/\hbar \) and \( \kappa = 0 \), Eq. (220) becomes
\[ i\hbar \partial_t \Psi(x) = \left[ \frac{\hat{\pi}^2}{2m} - \frac{q\hbar}{2m} \hat{\sigma} \cdot \mathbf{B} + qcA^0 \right] \Psi(x), \] (221)
which is formally equivalent to the Pauli equation (215).

On the other hand, in the general case in which \( \kappa \neq 0 \), the following transformation,
\[ \Psi(x) = e^{in\mu x^n - i\bar{m}x^0} \Psi_P(x), \] (222)
When inserted into Eq. (220) leads to
\[ i\hbar \partial_t \Psi_p(x) = \left[ \frac{\hat{\pi}^2}{2m} - \frac{q\hbar}{2m} \cdot \mathbf{B} + qeA^0 \right] \Psi_p(x), \tag{223} \]
an equation formally equal to the Pauli equation. In this scenario, it is more appropriate to consider \( \Psi_p(x) \) as the non-relativistic approximation for the asymmetric Dirac equation wave function. Combining Eqs. (217) and (222) we immediately see that
\[ \Psi_p(x) = \frac{1}{\sqrt{2}} \left( \tilde{\varphi}_p(x) - i\tilde{\varphi}_p(x) \right). \tag{224} \]

### XIII. DISCRETE SYMMETRIES

Our goal here is to build the parity, time reversal, and charge conjugation operators associated with the asymmetric Dirac equation. We will try to be as close as possible to the way they are built for the standard Dirac equation. In addition to that, we want the asymmetric Dirac equation to behave in the same way the usual Dirac equation behaves under the action of those discrete symmetry operations.

#### A. Parity

We want the asymmetric Dirac equation to be covariant under a parity operation. The parity operation, or space inversion operation, changes the sign of the space coordinates describing a given physical system,
\[ x = (x^0, \mathbf{r}) \rightarrow x' = (x^0, \mathbf{r}') = (x^0, -\mathbf{r}). \tag{225} \]

The parity transformation (225) can be studied along the same lines used to investigate proper Lorentz transformations. For the space inversion operation we have
\[ x'_\mu = \Lambda^\nu_\mu x_\nu, \tag{226} \]
where
\[ \Lambda_0^0 = 1, \quad \Lambda_1^1 = \Lambda_2^2 = \Lambda_3^3 = -1, \quad \Lambda_\mu^\nu = 0, \text{ if } \mu \neq \nu. \tag{227} \]
Note that \( \text{det}(\Lambda) = -1 \), which characterizes it as an improper Lorentz transformation.

The wave function changes according to
\[ \Psi'(x') = M\Psi(x), \tag{228} \]
where we assume that \( M \) does not depend on the space-time coordinates,
\[ \partial_\mu M = 0. \tag{229} \]
Repeating the steps given in Sec. 14X and using Eq. (229), the asymmetric Dirac equation is covariant under the parity operation if
\[ M^{-1} \gamma^\mu \Lambda^\nu_\mu M = e^{i\theta} \gamma^\nu, \tag{230} \]
\[ M^{-1} BM = e^{i\theta} B. \tag{231} \]

Note that we are including an arbitrary global phase \( \theta \) above. In this way, Eqs. (230) and (231) are the most general conditions guaranteeing the covariance of the asymmetric Dirac equation. If we set \( \theta = 0 \), as we did when we studied its covariance under proper Lorentz transformations, we cannot obtain \( M \) such that Eqs. (230) and (231) are simultaneously satisfied.

Inspired by the parity operator of the Lorentz covariant Schrödinger equation \[3\] and by the fact that \( B \) is a function of \( \gamma^5, \gamma^\mu \), and \( \kappa_\mu \), we define the parity operator for the asymmetric Dirac equation as
\[ M = P_\kappa = \gamma^5 K_1 P. \tag{232} \]

Here \( P \) is the parity operator of the standard Dirac equation \[3\],
\[ P = e^{i\varphi_\nu \gamma^\nu}, \tag{233} \]
with \( \varphi_\nu = 0, \pm \pi, \) or \( \pm \pi/2 \). Note that these values for \( \varphi_\nu \) are obtained by postulating that four successive space inversions bring us back to the original spinor \[3\]. We also have
\[ K_1 f(\kappa^\mu)K_1^\dagger = f(\kappa^\mu), \tag{234} \]
where \( f(\kappa^\mu) \) is an arbitrary function of \( \kappa^\mu \), \( K_1 = K_1^\dagger \), and \( (K_1)^2 = 1 \). Remembering the metric signature we are using in this work, the operator \( K_1 \) changes the sign of \( \kappa^3 \) while leaving \( \kappa^0 \) unaltered. It is worth noting that \( m \) is not changed by the action of \( K_1 \) since the dependence of \( m \) on \( \kappa^0 \) is quadratic.

As we show in the appendix \[15\] the parity operator defined in Eq. (232) satisfies Eqs. (230) and (231) for \( \theta = \pi \), which guarantees the covariance of the asymmetric Dirac equation under the parity operation.

#### B. Time reversal

Similarly to what we did for the parity operation, we want the asymmetric Dirac equation to be covariant after the time reversal operation. The time reversal operation is an improper Lorentz transformation that changes the sign of the time coordinate associated with a given physical system,
\[ x = (x^0, \mathbf{r}) \rightarrow x' = (x^0, \mathbf{r}') = (-x^0, \mathbf{r}). \tag{235} \]

In other words,
\[ x'_\mu = \Lambda^\nu_\mu x_\nu, \tag{236} \]
where
\[ \Lambda_0^0 = -1, \quad \Lambda_1^1 = \Lambda_2^2 = \Lambda_3^3 = 1, \quad \Lambda_\mu^\nu = 0, \text{ if } \mu \neq \nu. \tag{237} \]

The wave function changes as
\[ \Psi'(x') = T_\kappa \Psi(x). \tag{238} \]
after the time reversal operation, with \( T_\kappa \) assumed to not depend on \( x^\mu \).

Repeating the steps given in Sec. \( \text{[5]} \) and now taking into account that \( T_\kappa \) should contain the complex conjugation operation (\( T_\kappa \) does not commute with a complex number), the asymmetric Dirac equation is covariant under the time reversal operation if

\[
\begin{align*}
T_\kappa^{-1} i\gamma^\mu A_\mu T_\kappa &= e^{i\theta} i\gamma^\nu, \\
T_\kappa^{-1} BT_\kappa &= e^{i\phi} B.
\end{align*}
\]

As we prove in the appendix \( \text{[5]} \) the operator

\[
T_\kappa = \gamma^5 K_1 T,
\]

where \( T \) is the time reversal operator of the standard Dirac equation \( \text{[3]} \), satisfies Eqs. (239) and (240) if \( \theta = \pi \), guaranteeing the covariance of the asymmetric Dirac equation under the time reversal operation.

Note that in the Dirac-Pauli representation for the gamma matrices,

\[
T = e^{i\varphi_\tau} i\gamma^1\gamma^3 K,
\]

with the complex conjugation operation \( K = K^\dagger = K^{-1} \) defined by

\[
K z K = z^*,
\]

where \( z \) is an arbitrary complex number and \( \varphi_\tau \) a real number.

C. Charge conjugation

The asymmetric Dirac equation after we apply the minimal coupling prescription, Eq. \( \text{[196]} \), can be written as

\[
\begin{align*}
\frac{\hbar}{i} \partial_t \Psi_c(x) - i\hbar \gamma^5 \Psi_c(x) + \hbar \Psi_c(x) - q A(x) \Psi_c(x) &= 0. \\
\end{align*}
\]

The positive energy solutions to Eq. \( \text{[241]} \) are identified with particles of charge \( q \) while the negative energy solutions are identified with antiparticles.

Since it is expected that antiparticles behave similarly to particles but with an opposite sign for their charge, we define the following asymmetric Dirac equation whose positive energy solutions have charge \(-q\),

\[
\begin{align*}
\frac{\hbar}{i} \partial_t \Psi_c(x) - i\hbar \gamma^5 \Psi_c(x) + \hbar \Psi_c(x) + q A(x) \Psi_c(x) &= 0. \\
\end{align*}
\]

As we will see, the charge conjugation operator is the operator that connects the solutions of Eq. \( \text{[241]} \) to those of Eq. \( \text{[244]} \). In other words, it is the operator that transforms one equation into the other one \( \text{[5]} \). To obtain that operator, we first rewrite Eq. \( \text{[241]} \) by taking its complex conjugate, where we have taken into account that \( \bar{m}, \kappa^\mu \), and \( A^\mu \) are all real quantities,

\[
[(i\hbar \partial_t - \hbar \kappa_\mu + q A_\mu) \gamma^\mu - i\hbar \gamma^5] \Psi_c^*(x) = 0.
\]

If we now introduce an invertible matrix such that

\[
\Psi_c(x) = U_\kappa \Psi_c(x),
\]

we can rewrite Eq. \( \text{[246]} \) after left multiplying it by \( U_\kappa \) as

\[
\begin{align*}
[i\hbar U_\kappa \gamma^\mu U_\kappa^{-1} \partial_t - \hbar U_\kappa \kappa_\mu \gamma^\mu U_\kappa^{-1} + q A_\mu U_\kappa \gamma^\mu U_\kappa^{-1} - i\hbar \bar{m} U_\kappa \gamma^5 U_\kappa^{-1}] \Psi_c(x) &= 0. \\
\end{align*}
\]

Note that we are already assuming that \( U_\kappa \) commutes with complex numbers and with \( \bar{m} \) to write Eq. \( \text{[248]} \) as given above. The justification for this rests in the fact that \( U_\kappa \) does not depend on the complex conjugation operator \( K \) and that \( \bar{m} \) depends quadratically on \( \kappa^\mu \), being unaffected by \( K_2 \) as given by Eq. \( \text{[245]} \).

Comparing Eqs. \( \text{[241]} \) and \( \text{[248]} \), they are the same up to an overall global phase if

\[
\begin{align*}
U_\kappa \gamma^\mu U_\kappa^{-1} &= e^{i\theta} \gamma^\mu, \\
U_\kappa \gamma^\mu \kappa_\mu U_\kappa^{-1} &= -e^{i\theta} \gamma^\mu \kappa_\mu, \\
U_\kappa \gamma^5 U_\kappa^{-1} &= e^{i\phi} \gamma^5.
\end{align*}
\]

As we prove in the appendix \( \text{[5]} \) in the Dirac-Pauli representation for the gamma matrices the following operator satisfies Eqs. \( \text{[249]} \) and \( \text{[250]} \),

\[
U_\kappa = C_\kappa \gamma^0,
\]

where

\[
\begin{align*}
C_\kappa &= K_2 C, \\
C &= e^{i\phi} i\gamma^2 \gamma^0.
\end{align*}
\]

Here \( C_\kappa \) is an invertible operator, \( C \) is the charge conjugation operator of the standard Dirac equation \( \text{[5]} \), with \( \phi \) being a real number, and

\[
K_2 f(\kappa^\mu) K_2^\dagger = f(-\kappa^\mu).
\]

Note that the operator \( K_2 \) changes the sign of \( \kappa^\mu \) and, as already anticipated, \( \bar{m} \) is not changed by its action since it depends quadratically on \( \kappa^\mu \).

Also, in the Dirac-Pauli representation for the gamma matrices we have \( \gamma^0 = \gamma^5 \) and, thus, Eqs. \( \text{[247]} \) and \( \text{[252]} \) imply that

\[
\Psi_c(x) = C_\kappa \overline{\Psi}(T),
\]

where \( T \) means transposition. The operator \( C_\kappa \) is the charge conjugation operator for the asymmetric Dirac equation.

D. \( \kappa^\mu \) and the discrete symmetries

In the asymmetric Dirac equation the four parameters \( \kappa^0 \) and \( \kappa = (\kappa^1, \kappa^2, \kappa^3) \) are ubiquitous. What we usually
call the mass \( m \) of a particle is a function of them [cf. Eqs. (111) and (122)],

\[
m = \frac{\hbar}{c} \sqrt{(\kappa^0)^2 - |\kappa|^2}.
\] (257)

We can understand the mass of a particle as originating from two different yet complementary aspects. The first one is a “time-like” inertial contribution to the mass, given by \( \kappa^0 \), and the second one is a “space-like” inertial contribution, given by \( |\kappa| \). The mass \( m \) is proportional to \( \sqrt{\kappa^0} \). Also, \( \kappa^0 \) is responsible for breaking the degeneracy of the energies associated with particles and antiparticles sharing the same wave number \( \kappa \), while \( \kappa \) is responsible for breaking the degeneracy of their linear momenta [cf. Eqs. (273) and (274)].

The parameters \( \kappa^0, \kappa^1, \kappa^2, \) and \( \kappa^3 \) are invariant under proper Lorentz transformations (boosts or spatial rotations). \[\kappa^\mu \rightarrow \kappa^\mu . \] (264)

In other words, if \( \kappa^\mu \) behaves as a strict scalar under the \( C \) and \( P \) operations, the asymmetric Dirac equation will not be covariant under those symmetry operations or combinations thereof. This can be seen by noting that the assumptions given by Eqs. (261) and (263) are crucial to the proof that the asymmetric Dirac equation is covariant under the parity and charge conjugation operations (see appendix \[\kappa^\mu \rightarrow \kappa^\mu . \] (264)

In particular, we can build a theory violating the \( P \)-symmetry, the \( C \)-symmetry, or the \( CP \)-symmetry if we postulate a different transformation rule for \( \kappa^\mu \) after the parity and charge conjugation operations. To achieve that, instead of Eqs. (261) and (263) we postulate that \( \kappa^\mu \) transforms under those symmetry operations as follows,

\[
\kappa^\mu \rightarrow \kappa^\mu .
\] (264)

On the other hand, \( \kappa^\mu \) behaves as given by Eqs. (261) and (262) under improper Lorentz transformations because we imposed that the asymmetric Dirac equation should be covariant after the parity or the time reversal operations. Similarly, by imposing that the asymmetric Dirac equation is covariant under the charge conjugation operation, we get Eq. (263). These latter assumptions were brought to the present theory such that the asymmetric Dirac equation behaves in exactly the same way as the standard Dirac equation does when subjected to those discrete symmetry operations. If we change the assumptions given by Eqs. (261)–(263), we can build a Lorentz covariant wave equation under proper Lorentz transformations that responds differently under improper Lorentz transformations. We can build, for instance, a QED-like theory where we break the \( CP \)-symmetry from the start. See also Refs. [37, 40] for other interesting approaches along this line.

In particular, looking carefully at the behavior of \( \kappa^\mu \) under the discrete symmetry operations studied in Sec. XIV and in the appendix \[\kappa^\mu \rightarrow \kappa^\mu . \] (264)

The above analysis also expands and clarifies the discussion initiated in Ref. [3] on the possibility of ascribing to particles and antiparticles masses with opposite signs. What the discussion above tells us is that the mass \( m \), as defined by Eq. (257), is positive for both particles and antiparticles, being invariant under proper and improper Lorentz transformations as well as under the charge conjugation operation. Under the charge conjugation operation, though, we see that particles are associated with \( \kappa^\mu \) while antiparticles are associated with \( -\kappa^\mu \). In the particular and important case where \( \kappa^3 = 0 \), we have \( \kappa^0 = mc/\hbar \) for particles and \( \kappa^0 = -mc/\hbar \) for antiparticles, with both particles and antiparticles having a positive mass \( m \) given by Eq. (257). The implications of \( \kappa^\mu \) having different signs for particles and antiparticles may open the possibility to build a logical and coherent quantum theory of gravity in which particles and antiparticles repel each other gravitationally, while particles attract particles and antiparticles attract antiparticles, with both particles and antiparticles having positive masses \[\kappa^\mu \rightarrow \kappa^\mu . \] (264)

This comes about since the gravitational interaction coupling constant is proportional to \( \kappa^0 \) and not to \( m \). However, further work is needed to complete this program (see also Sec. XIV).
Finally, we can understand the term

\[-\Psi (i mc^2\gamma^\beta - \hbar c k_\mu \gamma^\mu) \Psi\]  

(265)
in the Lagrangian density \(271\), where \(mc/\hbar = \sqrt{\kappa_\mu k^\mu}\), as an alternative to the standard mass term of the usual Dirac equation \[3, 7\],

\[-mc\nabla \Psi.\]  

(266)

These two terms, when added to the Lagrangian density \(i mc\nabla \gamma^\mu \partial_\mu + c\), are different ways of building a nonzero mass-spin-1/2 theory from a massless one. The fundamental differences between the two approaches are: (1) the spinor \(\Psi\) will have different transformation laws to guarantee the Lorentz covariance of the theory; (2) four parameters, \(k^0, k^1, k^2, k^3\), instead of just one, \(m\), to describe the inertial aspects of the particle; and (3) different energy-momentum expressions for particles and antiparticles. And despite these differences, for interactions that respect the Lorentz symmetry we can build both theories to yield the same predictions \[3, 8\].

**XIV. LAGRANGIAN FORMULATION**

We now consider \(\Psi(x)\) and \(\overline{\Psi}(x)\) as two independent fields and

\[L = \int d^4x L[\Psi(x), \overline{\Psi}(x), \partial_\mu \Psi(x), \partial_\mu \overline{\Psi}(x)]\]  

(267)

the Lagrangian that completely characterizes the dynamics of those fields. The Lagrangian density \(L\) depends on the fields and on their first derivatives. Here \(d^4x = dx^0 dx^1 dx^2 dx^3\) is the infinitesimal spatial volume and the integrals above cover the entire space. As usual, the fields and their derivatives are supposed to vanish at the boundaries of integration and the dimension of \(L\) is compatible with \(L\) having the dimension of energy.

Noting that \(d^4x = dx^0 dx^1 dx^2 dx^3\) is the infinitesimal four-volume, the action is defined as \[3, 7\],

\[S = \int dt L = \frac{1}{c} \int d^3x L(\Psi, \overline{\Psi}, \partial_\mu \Psi, \partial_\mu \overline{\Psi}).\]  

(268)

If the infinitesimal variation of the action vanishes, \(\delta S = 0\), we obtain the following Euler-Lagrange equations \[1, 2\],

\[\frac{\partial L}{\partial \Psi} = \partial_\mu \left( \frac{\partial L}{\partial (\partial_\mu \Psi)} \right), \quad \frac{\partial L}{\partial \overline{\Psi}} = \partial_\mu \left( \frac{\partial L}{\partial (\partial_\mu \overline{\Psi})} \right).\]  

(269)

It is not difficult to see that the following Lagrangian density gives the asymmetric Dirac equation and its adjoint when inserted into \(269\).

\[L = i \hbar c \overline{\Psi}(x)\gamma^\mu \partial_\mu \Psi(x) - mc^2 \overline{\Psi}(x)B\Psi(x),\]  

(270)

with \(B\) given by Eq. \[36\]. Substituting \(B\), and omitting the explicit dependence of the fields on \(x\), we have

\[L = i \hbar c \overline{\Psi} \gamma^\mu \partial_\mu \Psi - \Psi (i mc^2\gamma^\beta - \hbar c k_\mu \gamma^\mu)\]  

(271)

where we should not forget that \(mc/\hbar = \sqrt{\kappa_\mu k^\mu}\). A direct calculation shows that the Lagrangian density \(271\) is invariant under proper Lorentz transformations if \(\Psi\) transforms according to Eq. \[68\] with \(M(x)\) satisfying Eqs. \[64, 66, 67\].

Since the time derivative appearing in Eq. \(271\) are formally equal to the one in the Lagrangian density associated with the standard Dirac equation, we have similar expressions for the conjugate momenta of the fields \(\Psi\) and \(\overline{\Psi}\),

\[\Pi(x) = \frac{\partial L}{\partial (\partial_0 \Psi)} = i \hbar \Psi^\dagger(x), \quad \overline{\Pi}(x) = \frac{\partial L}{\partial (\partial_0 \overline{\Psi})} = 0,\]  

(272)

The canonical energy-momentum tensor for the asymmetric Dirac equation can be written as \[3, 7\],

\[^T_{\mu\nu} = \frac{\partial L}{\partial (\partial_\mu \Psi)} \partial^\nu \Psi + \frac{\partial L}{\partial (\partial_\mu \overline{\Psi})} \partial^\nu \overline{\Psi} - g^{\mu\nu} L,\]  

(273)

leading to the following conserved Noether “charges”,

\[c P^\mu = \int d^3x [T^0\mu](x).\]  

(274)

In Eq. \(273\) \(c P^0\) is the Hamiltonian \(H\) associated with the asymmetric Dirac Lagrangian and \(P^j\) is its linear momentum.

A direct calculation gives

\[^T_{\mu\nu} = i \hbar \overline{\Psi} \gamma^\mu \partial^\nu \Psi - \gamma^\mu c \Psi \{i \hbar \partial_0 - mcB\} \Psi = i \hbar \overline{\Psi} \gamma^\mu \partial^\nu \Psi,\]  

(275)

where we used that \(\Psi\) is a solution to the asymmetric Dirac equation \(42\) to arrive at the last equality.

In the metric signature we have been using in this work, Eqs. \(274\) and \(275\) imply that

\[H = i \hbar \int d^3x \Psi^\dagger(x) \partial_0 \Psi(x),\]  

(276)

\[P = -i \hbar \int d^3x \Psi^\dagger(x) \nabla \Psi(x).\]  

(277)

2 The results of Secs. \[14, 11\] and \[13, 14\] imply that the present theory can be an alternative description of spin-1/2 particles, for instance, electrons and positrons. By properly adjusting the free parameters of the present theory, the standard Dirac equation and the asymmetric Dirac equation are equivalent descriptions of spin-1/2 particles.

3 Note that \(274\) also leads to an action that is invariant under space-time translations. This, together with the fact that \(271\) is invariant under Lorentz transformations, implies that we have a theory respecting the Poincaré symmetry \[1, 2\].
When we second quantize this theory, we will have \[ H = \sum_{r=1}^{2} \int d^3 p \left[ (E_p - \hbar c \kappa^0) c_r^\dagger(p) c_r(p) 
+ (E_p + \hbar c \kappa^0) d_r^\dagger(p) d_r(p) \right], \tag{278} \]

\[ P = \sum_{r=1}^{2} \int d^3 p \left[ (p - \hbar \kappa) c_r^\dagger(p) c_r(p) 
+ (p + \hbar \kappa) d_r^\dagger(p) d_r(p) \right], \tag{279} \]

with \( c_r(p) \) and \( d_r(p) \) being annihilation operators and \( c_r^\dagger(p) \) and \( d_r^\dagger(p) \) creation operators associated with fermionic particles and antiparticles.

The conserved total angular momentum is computed in exactly the same way that is done for the standard Dirac Lagrangian if we use \( M^{ij}(x) \) instead of \( S^{ij} = -(i/2)\sigma^{ij} \) if we use \( M^{ij}(x) \). Eq. (85) instead of \( \sigma^{ij} \).

The final result is
\[ J^k = \frac{1}{c} \int d^3 x x^i \Gamma^{0j}(x) - x^j \Gamma^{0i}(x) + c \Pi(x) M^{ij}(x) \Psi(x) \], \tag{280} \]

with \( i, j, k = 1, 2, 3 \) in cyclic order. Inserting Eqs. (83) and (277) into (280), we get in vector notation,
\[ J = \int d^3 x \Psi^\dagger(x) \left[ \mathbf{r} \times (-i \hbar \nabla + \hbar \kappa) \right] \Psi(x) 
+ \int d^3 x \Psi^\dagger(x) \left[ \frac{\hbar}{2} \mathbf{\sigma} \right] \Psi(x). \tag{281} \]

Since \( \Psi(x) \) is a solution to the asymmetric Dirac equation, we can insert the ansatz (107) into (281). This gives
\[ J = \int d^3 x \bar{\Psi}^\dagger(x) \left[ \mathbf{r} \times \hat{\mathbf{p}} + \frac{\hbar}{2} \mathbf{\sigma} \right] \bar{\Psi}(x), \tag{282} \]

where \( \bar{\Psi}(x) \) does not depend on \( \kappa \) and \( \hat{\mathbf{p}} = -i \hbar \nabla \).

Equation (282) tells us that the total angular momentum does not depend on \( \kappa \) and that the first term in its right hand side is the orbital angular momentum while the second term is the intrinsic (spin) angular momentum associated with the field \( \Psi(x) \). Note that if we insert the ansatz (107) into Eq. (277), we will get a term depending on \( \kappa \), a feature that is reflected in the second quantized expression for the linear momentum as given by Eq. (279). In other words, the asymmetry between particles and antiparticles manifests itself in different values for their energy and linear momentum at a given wave number \( \mathbf{k} = \mathbf{p}/\hbar \), while for the angular momentum the symmetry is still preserved (no \( \kappa \) dependence).

In addition to being invariant under space-time translations and spatial rotations, the asymmetric Dirac Lagrangian density is also invariant under a global gauge transformation, \( \Psi \rightarrow e^{i\xi} \Psi \), with \( \xi \) an arbitrary real number. The Noether theorem then leads to the following conserved charge \[ Q = q \int d^3 x \Psi^\dagger(x) \Psi(x), \tag{283} \]

the following current,
\[ \mathbf{j}(x) = cq \bar{\Psi}(x) \gamma^\mu \Psi(x), \tag{284} \]

with \( \gamma = (\gamma^1, \gamma^2, \gamma^3) \), and the continuity equation
\[ \partial_\mu s^\mu(x) = 0, \tag{285} \]

where the four-current is
\[ s^\mu(x) = (cp(x), \mathbf{j}(x)) = cq \bar{\Psi}(x) \gamma^\mu \Psi(x). \tag{286} \]

The constant \( q \) above is interpreted in the second quantization framework as the electric charge associated with the vacuum excitation created by \( c \mathbf{j}^\dagger(p) \) and \( -q \) the corresponding charge of the antiparticle created by \( d \mathbf{j}^\dagger(p) \).

Note that these results are consistent with the four-current obtained in Sec. \[ \nabla A \] by more elementary methods.

We can also apply the minimal coupling prescription to the Lagrangian density (271). This gives the following interaction Lagrangian density,
\[ \mathcal{L}_{int} = -cq \bar{\Psi}(x) \gamma^\mu \Psi(x) A_\mu(x), \tag{287} \]

which is formally equal to the one we obtain applying the minimal coupling prescription to the standard Dirac equation.

On the other hand, we can phenomenologically model a non-electromagnetic interaction using an external scalar potential, which is included in the Lagrangian density similarly to the way we add the standard mass term \( \bar{\kappa} \). In this case the interaction Lagrangian density becomes
\[ \mathcal{L}_{ext} = -\mathcal{V}(x) \bar{\Psi}(x) \Psi(x), \tag{288} \]

where \( \mathcal{V}(x) \) is the potential energy associated to the interaction of the fermion field with the external field.

In the theoretical framework of the asymmetric Dirac equation, in particular in the scenario where \( \kappa = 0 \) and \( m = \hbar c k^0 / c \) [see Eqs. (11) and (12)], we can model the interaction of a fermion with a static gravitational field by setting
\[ \mathcal{V}(x) = \frac{\hbar c k^0}{c} \varphi(x), \tag{289} \]

with \( \varphi(x) \) being the gravitational potential related to the external field acting on the fermion. Note that \( \hbar c k^0 / c \) is the mass of the particle if \( k^0 > 0 \). If we insert Eq. (283) into (288) we get
\[ \mathcal{L}_{ext} = -\frac{\hbar c k^0}{c} \varphi(x) \bar{\Psi}(x) \Psi(x). \tag{290} \]

Now, looking at Eq. (289), we realize that the charge conjugation operation changes \( k^0 \) to \( -k^0 \) while \( \bar{\Psi}(x) \Psi(x) \) is a scalar under this symmetry operation \( \bar{k} \). In this way, the interaction Lagrangian density for an antiparticle interacting with this very same external gravitational field is
\[ \mathcal{L}_{ext} = \frac{\hbar c k^0}{c} \varphi(x) \bar{\Psi}(x) \Psi(x). \tag{291} \]
Looking at Eqs. (290) and (291), we notice that they differ by a minus sign. This means that if a particle is attracted by the gravitational field an antiparticle will be repelled by it. Particles and antiparticles can be modeled to respond differently to a gravitational field within the framework of the asymmetric Dirac equation if we model the interaction with the gravitational field using Eqs. (289) and (290).

We can also make particles and antiparticles respond the same way to a gravitational field if instead of Eq. (290) we model their interaction as follows,

\[ \tilde{L}_{\text{ext}} = -\frac{\hbar|\kappa|^0}{c} \varphi(x)\overline{\Psi}(x)y(x) = -m\varphi(x)\overline{\Psi}(x)\Psi(x). \]  

(292)

In other words, we now couple the fermion field with the gravitational field using the magnitude of \( \kappa^0 \). The bottom line here is that in the theoretical framework of the asymmetric Dirac equation, we can either assume that particles and antiparticles attract each other gravitationally or that they repel each other without facing any logical contradiction. This comes about since in the two cases we do not need to make \( \kappa \) negative. In the context of the asymmetric fields \[ \kappa \], the mass \( m \) is always positive by construction [see Sec. 11] and the way particles and antiparticles couple to gravity is determined by \( \kappa^0 \) and not by \( m \).

We finish this section by noting that the Lagrangian density related to the wave equation \[ 105 \] is

\[ \tilde{L} = i\hbar\overline{\Psi}\gamma^\mu\partial_\mu\Psi - imc\overline{\Psi}\gamma^0\Psi. \]  

(293)

The corresponding conserved Noether “charges” related to its space-time invariance and the conserved electric charge due to its global gauge invariance are exactly the ones given by Eqs. \[ 276 \], \[ 277 \], and \[ 283 \], with \( \Psi \) replaced by \( \overline{\Psi} \), while its conserved total angular momentum is given by Eq. \[ 282 \].

XV. CONCLUSION

We derived a first order spinorial wave equation whose free particle dispersion relations are equal to the dispersion relations associated with the Lorentz covariant Schrödinger equation \[ 8 \]. This latter equation is the relativistic analog of the standard non-relativistic Schrödinger equation, obtained by demanding Lorentz covariance and that its wave function transforms under a proper Lorentz transformation according to the relativistic extension of the Schrödinger wave function’s transformation law after a Galilean boost.

In order to highlight the non-degenerate aspect of the energy-momentum relations for particles and antiparticles, we called it asymmetric Dirac equation. We also determined how its solutions transform under proper Lorentz transformations by imposing that the asymmetric Dirac equation should have the same form after those transformations (Lorentz covariance).

We then investigated the main similarities and differences between the present equation and the standard Dirac equation, providing a formal connection between the two equations. Throughout the development of the present theory, in particular when dealing with improper Lorentz transformations, we chose the path that led the asymmetric Dirac equation to behave as close as possible to the standard Dirac equation under the same circumstances (physical conditions). It turned out that we can build the present theory to either reproduce almost all predictions of the standard Dirac equation or we can follow a different yet logically consistent path, in which different predictions arise, such as a QED-like theory that violates the CP-symmetry from the start (see Sec. XIII and appendix \[ E \]).

We studied in details the plane wave solutions of the asymmetric Dirac equation as well as its energy, helicity, and spin projection operators. We then obtained several Gordon’s identities related to the asymmetric Dirac equation and we introduced electromagnetic interactions via the minimal coupling prescription, solving the respective Coulomb problem (hydrogen atom). We then determined the asymmetric Dirac equation non-relativistic limit, investigated its behavior under several discrete symmetry operations, and laid down the foundations of its classical field theory (Lagrangian formulation), preparing the ground to its second quantization that will be presented in Ref. \[ 8 \].

Finally, the present work shows that it is theoretically possible to construct a consistent Lorentz covariant spinorial wave equation using a more general transformation law for its wave function under a proper Lorentz transformation. Moreover, the free parameters of this wave equation can be adjusted to reproduce the predictions stemming from the standard Dirac equation. This latter fact is important since we must be able to predict all the experimentally validated results coming from the standard Dirac equation when applying the present theory in the domain of validity of the standard Dirac equation. We showed that when working in this domain, the present theory can be adjusted to give exactly the same experimental predictions of the original Dirac equation. On the other hand, as a bonus, the present theory breaks the degeneracy between the energies of particles and antiparticles in its simplest version \( (\kappa^0 = mc/\hbar, \kappa = 0) \) and also the degeneracy of their energies and momenta for a given wave number in its general version \( (\kappa^0 \neq |\kappa| \neq 0) \). The implications of the non-degenerate energy-momentum relations for particles and antiparticles at the second quantization level is more subtle and will be discussed elsewhere \[ 8 \].

Acknowledgments

The author thanks the Brazilian agency CNPq (National Council for Scientific and Technological Development) for partially funding this research.
Appendix A: Proof of Eq. (111)

First, let us assume that
\[ \Psi'(x') = S \bar{\Psi}(x). \]  
(A1)

Our goal is to prove that \( S = S \), with \( S \) being the usual matrix that transforms a standard Dirac spinor after a proper Lorentz transformation.

Expressing the ansatz Eq. (107) in the primed reference frame we have
\[ \Psi'(x') = e^{i\kappa_\nu x'\nu'} \bar{\Psi}'(x'). \]  
(A2)

Also, using Eqs. (B3) and (107) we get
\[ \Psi'(x') = e^{i\kappa_\nu x'\nu} M(x) \bar{\Psi}(x). \]  
(A3)

Noting that the left hand sides of Eqs. (A2) and (A3) are equal we obtain
\[ \Psi'(x') = e^{i\kappa_\nu (x'\nu - x'\nu')} M(x) \bar{\Psi}(x). \]  
(A4)

Comparing Eqs. (A1) and (A4) we finally arrive at
\[ S = e^{i\kappa_\nu (x'\nu - x'\nu')} M(x). \]  
(A5)

For an infinitesimal proper Lorentz transformation Eqs. (10) and (80) imply, to first order in \( \epsilon_{\mu\nu} \), that Eq. (A5) becomes
\[ S = 1 - i \frac{1}{4} \epsilon_{\mu\nu} \sigma^{\mu\nu} + O(\epsilon^2_{\mu\nu}) = S. \]  
(A6)

And since for any continuous symmetry operation we can build finite transformations from the infinitesimal ones, Eq. (A6) proves that \( S = \bar{S} \) whether or not we have infinitesimal transformations. Finally, note that to arrive at Eq. (A6), the term \( i\epsilon_{\mu\nu} \sigma^{\mu\nu} x'\nu \) coming from \( M(x) \) was exactly canceled by the term \( -i\epsilon_{\mu\nu} \sigma^{\mu\nu} x'\nu' \) coming from \( e^{i\kappa_\nu (x'\nu - x'\nu')} \).

Appendix B: The orthonormality relations and other identities

Let us start proving that the left hand side of Eq. (121) is zero. If we left multiply Eq. (118) by \( \tau_r(-\mathbf{p}) \) we get
\[ \tau_r(-\mathbf{p})(\gamma^0 p_0 + \gamma^j p_j + imc\gamma^5) u_s(\mathbf{p}) = 0. \]  
(B1)

On the other hand, changing \( \mathbf{p} \) to \( -\mathbf{p} \) in Eq. (118) and right multiplying it by \( \tau_r(\mathbf{p}) \) we obtain
\[ \tau_r(-\mathbf{p})(\gamma^0 p_0 - \gamma^j p_j - imc\gamma^5) u_s(\mathbf{p}) = 0. \]  
(B2)

Summing Eqs. (B1) and (B2) we get
\[ p_0 \tau_r(-\mathbf{p}) \gamma^0 u_s(\mathbf{p}) = 0. \]  
(B3)

Since by definition \( p^0 \neq 0 \) we arrive at the desired expression,
\[ u^1_s(-\mathbf{p}) u_s(\mathbf{p}) = 0 \iff u^1_s(\mathbf{p}) u_s(-\mathbf{p}) = 0, \]  
(B4)

after using Eq. (115). To prove that the middle term of Eq. (124) is zero we just need to take the adjoint of (B4). Let us now prove that the left hand side of Eq. (126) is zero. Left multiplying Eq. (119) by \( \tau_r(\mathbf{p}) \gamma^0 \) we have
\[ \tau_r(\mathbf{p}) \gamma^0(\gamma^0 p_0 + \gamma^j p_j - imc\gamma^5) u_s(\mathbf{p}) = 0. \]  
(B5)

On the other hand, right multiplying Eq. (114) by \( \gamma^0 u_s(\mathbf{p}) \) and using the anticommutation relations of \( \gamma^\mu \) to bring \( \gamma^0 \) to the left we get
\[ \tau_r(\mathbf{p}) \gamma^0(\gamma^0 p_0 - \gamma^j p_j + imc\gamma^5) u_s(\mathbf{p}) = 0. \]  
(B6)

Summing Eqs. (B5) and (B6) and remembering that \( (\gamma^0)^2 = 1 \) and that \( p_0 \neq 0 \) we obtain
\[ \tau_r(\mathbf{p}) u_s(\mathbf{p}) = 0, \]  
(B7)

which is what we wanted to prove.

Similarly we prove that the middle term of Eq. (125) is zero. We just need to repeat the above proof using Eqs. (119) and (118) instead of (113) and (114) and change appropriately the objects we left and right multiply them (use \( v_r \) instead of \( u_r \)).

To prove that the left hand side of Eq. (126) is zero, we left multiply Eq. (118) by \( \tau_r(\mathbf{p}) \) and subtract from it Eq. (114) right multiplied by \( v_s(\mathbf{p}) \). The middle term of Eq. (126) is zero since it is proportional to the adjoint of the left hand side term.

If we now left multiply Eq. (113) by \( \tau_r(\mathbf{p}) \gamma^0 \gamma^5 \) we get
\[ \tau_r(\mathbf{p}) \gamma^0 \gamma^5(\gamma^0 p_0 + \gamma^j p_j - imc\gamma^5) u_s(\mathbf{p}) = 0. \]  
(B8)

Right multiplying Eq. (114) by \( \gamma^0 \gamma^5 u_s(\mathbf{p}) \) and using the anticommutation rules involving \( \gamma^5 \) and \( \gamma^\mu \) we obtain
\[ \tau_r(\mathbf{p}) \gamma^0 \gamma^5(\gamma^0 p_0 + \gamma^j p_j + imc\gamma^5) u_s(\mathbf{p}) = 0. \]  
(B9)

Subtracting Eq. (B8) from (B9) and using that \( \gamma^0 \gamma^5 = -\gamma^5 \gamma^0 \) and \( (\gamma^0)^2 = (\gamma^5)^2 = 1 \), we arrive at
\[ \tau_r(\mathbf{p}) \gamma^5 u_s(\mathbf{p}) = \frac{mc}{p_0} u^1_s(\mathbf{p}) u_s(\mathbf{p}). \]  
(B10)

after using Eq. (115). And with the help of Eq. (128) we prove that the left hand side of Eq. (127) is \( \delta_{\tau_s} \). To prove that the middle term of (127) is also \( \delta_{\tau_s} \), we proceed similarly, using Eq. (118) left multiplied by \( \tau_r(\mathbf{p}) \gamma^0 \gamma^5 \) and Eq. (119) right multiplied by \( \gamma^0 \gamma^5 \tau_r(\mathbf{p}) \).

The first resolution of the identity, Eq. (128), can be proven by acting on the four base vectors \( u_s(\mathbf{p}) \) and \( v_s(\mathbf{p}) \), with \( s = 1, 2 \), and verifying that we obtain the expected results, namely, \( \Pi u_s(\mathbf{p}) = u_s(\mathbf{p}) \) and \( \Pi v_s(\mathbf{p}) = v_s(\mathbf{p}) \).
For instance, if we use Eqs. (120) and (127) we get
\[ 1 u_s(p) = \sum_{r=1}^{2} [u_r(p) \Pi_r(p) i \gamma^5 u_s(p)] - v_r(p) \Pi_r(p) i \gamma^5 u_s(p) \]
\[ = \sum_{r=1}^{2} [u_r(p) \delta_{rs} - 0] = u_s(p). \] (B11)

In an analogous way we show that \( 1 v_s(p) = v_s(p) \).

To prove the second resolution of the identity, Eq. (129), we take the adjoint of
\[ \pi_s(p) = \pi_s\left(-p\right) \gamma^5 \left(\gamma^0 p_0 + \gamma^j p_j - i m c \gamma^5\right) v_r(p) = 0. \] (B12)

Similarly, multiplying Eq. (113) by \( \pi_s(-p) \gamma^5 \) at the left we get
\[ \pi_s(-p) \gamma^5 \left(\gamma^0 p_0 + \gamma^j p_j - i m c \gamma^5\right) v_r(p) = 0. \] (B13)

Subtracting Eq. (B12) from (B13), using Eq. (115), employing that \( (\gamma^5)^2 = 1 \), and that \( p_0 = p^0 \), we arrive at the desired expression,
\[ \pi_s(-p) v_r(p) = \frac{i p^0}{m c} u_s(-p) \gamma^5 v_r(p). \] (B14)

**Appendix C: Proof of Eq. (153)**

Using Eqs. (132) and (147) we have
\[ [\Lambda^+(p), \Pi^\pm(n)] = \pm \frac{1}{4 m c} \gamma^5 \gamma^0 \gamma^5 \gamma^5 \gamma^5 [\sigma_p] \]
\[ = \pm \frac{1}{4 m c} \left[ (\gamma^5 \gamma^0 \gamma^5 \gamma^5 \gamma^5) [\sigma_p] + [\gamma^5 \gamma^0 \gamma^5 \gamma^5 \gamma^5] [\sigma_p] \right] \]
\[ = \pm \frac{1}{4 m c} \left[ (\gamma^5 \gamma^0 \gamma^5 \gamma^5 \gamma^5) [\sigma_p] \right], \] (C1)
where we used that \( [\sigma_p] = 0 \).

If we now use Eq. (148) and that
\[ \sigma^{ij} = -\delta^{ij} \gamma^5, \] (C2)

with \( i, j, k = 1, 2, 3 \) in cyclic order, we obtain
\[ [\gamma^5, \sigma_p] = \frac{1}{|p|} [\gamma^5, \gamma^5 \gamma^0 \gamma^5 \gamma^5 \gamma^5 p_k] \]
\[ = \frac{1}{|p|} (\gamma^5 \gamma^0 \gamma^5 \gamma^5 \gamma^5 p_k - \gamma^0 \gamma^5 \gamma^5 \gamma^5 \gamma^5 p_k) \]
\[ = \frac{1}{|p|} (\gamma^5 \gamma^5 \gamma^5 \gamma^5 \gamma^5 p_k + \gamma^0 \gamma^5 \gamma^5 \gamma^5 \gamma^5 p_k) = 0, \] (C3)
where we used that \( \gamma^5 \gamma^5 = -\gamma^5 \) and \( \gamma^5 \gamma^5 = 1 \) to obtain the last line.

Using Eqs. (C1) and (C3) we get \( [\Lambda^+(p), \Pi^\pm(n)] = 0 \).

In an analogous way we prove that \( [\Lambda^-(p), \Pi^\pm(n)] = 0 \).

**Appendix D: The spin projector operators**

Let us start proving that the spin projector operators commute with the energy projection operators. Using Eqs. (132) and (161) we get
\[ [\Lambda^+(p), \Pi^\pm(n)] = \frac{\mp}{4 m c} [\gamma^5 \gamma^5 \gamma^5 \gamma^5 \gamma^5 p_k]. \] (D1)

But
\[ [\gamma^5 \gamma^5 \gamma^5 \gamma^5 \gamma^5 p_k] = \gamma^5 \gamma^5 \gamma^5 p_k - \gamma^5 \gamma^5 \gamma^5 p_k = \gamma^5 p_k + \gamma^5 p_k = 0. \] (D2)

However, since \( pm = 0 \) [cf. Eq. (163)] we have \( p p = -p p \) and thus \( [\gamma^5 \gamma^5 \gamma^5 \gamma^5 \gamma^5 p_k] = 0 \) in (D3).

If we now use that \( p p = -p p = 1 \), we obtain
\[ [\Pi^\pm(n)]^2 = \Pi^\pm(n), \] (D4)
\[ \Pi^\pm(n) \Pi^\mp(n) = 0, \] (D5)
\[ \Pi^\pm(n) + \Pi^\mp(n) = 1, \] (D6)

the expected properties of a complete set of projector operators.

What remains to be done is to check that \( \Pi^\pm(n) \) has the expected properties in the particle’s rest frame. To simplify the following calculations, we need to choose a particular representation for the gamma matrices. From now on we will be working with the Dirac-Pauli representation of the gamma matrices, which is defined as follows
\[ \gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^j = \begin{pmatrix} 0 & \sigma_j \\ -\sigma_j & 0 \end{pmatrix}, \] (D7)
where \( 1 \) and \( 0 \) are \( 2 \times 2 \) matrices,
\[ 1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad 0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \] (D8)
and the $2 \times 2$ Pauli matrices are
\[
\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (D9)
\]

Using Eqs. (D7)-(D9) and Eqs. (26)-(27) we obtain
\[
\gamma^5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^{ij} = \begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix}, \quad \quad (D10)
\]
with $i,j,k = 1,2,3$ in cyclic order.

The operators of Eq. (123). In the rest frame, Eqs. (113) and (118) give
\[
u_r(0) = i \gamma^0 i \gamma^5 \gamma^0 v_r(0). \quad (D12)
\]
Using Eqs. (D11) and (D12) we obtain
\[
\Pi^\pm(\hat{n})u_r(0) = \frac{1}{2}(1 \pm \sigma \cdot \hat{n}) u_r(0), \quad (D13)
\]
\[
\Pi^\pm(\hat{n})v_r(0) = \frac{1}{2}(1 \mp \sigma \cdot \hat{n}) v_r(0). \quad (D14)
\]
The operators $\frac{1}{2}(1 \pm \sigma \cdot \hat{n})$ are exactly the spin projection operators along the direction $\hat{n}$ of the non-relativistic quantum mechanics ($v/c \ll 1$).

In the Dirac-Pauli representation of the gamma matrices, Eqs. (119) and (118) imply that when $p = 0$ (particle's rest frame) we have,
\[
u_1(0) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}, \quad \nu_2(0) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad (D15)
\]
\[
u_1(0) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \nu_2(0) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ i \end{pmatrix}, \quad (D16)
\]
The normalization chosen above guarantees the validity of Eq. (123).

We now orient, for simplicity of calculation and without loss of generality, $\hat{n}$ along the $z$-direction ($x^3$-direction), i.e., we set $\hat{n} = \hat{z} = (0,0,1)$. In this scenario Eqs. (D15) and (D16) give
\[
(\sigma \cdot \hat{n}) u_r(0) = (-1)^{r+1} u_r(0), \quad (D17)
\]
\[
(\sigma \cdot \hat{n}) v_r(0) = (-1)^r v_r(0), \quad (D18)
\]
the analogs of Eqs. (124) and (155).

Finally, using Eqs. (D17) and (D18) in Eqs. (D13) and (D14) we get
\[
\Pi^+(\hat{n}) u_r(0) = \delta_{1r} u_r(0), \quad (D19)
\]
\[
\Pi^-(\hat{n}) u_r(0) = \delta_{1r} u_r(0), \quad (D20)
\]
\[
\Pi^+(\hat{n}) v_r(0) = \delta_{2r} v_r(0), \quad (D21)
\]
\[
\Pi^-(\hat{n}) v_r(0) = \delta_{2r} v_r(0). \quad (D22)
\]

The above relations tell us that $\nu_1(0)$ and $\nu_1(0)$ have spins pointing along the direction of $\hat{n}$ while $\nu_2(0)$ and $\nu_2(0)$ have spins oriented in the opposite direction.

Note that due to the different ordering in sign that we see at the right hand side of Eqs. (D13) and (D14), namely, $\pm$ and $\mp$, Eqs. (120) and (121) are formally different from Eqs. (157) and (158).

Appendix E: Proof of the Gordon's identities

Let us start proving Eq. (164). If we use Eq. (27), the left hand side (lhs) of Eq. (164) can be written after a little algebra as
\[
\text{lhs} = p^{\mu'} + p^{\mu} + \frac{1}{2}(-\gamma^{\mu'} p_{\nu'} + \gamma_{\mu'}) + \frac{1}{2}(-\gamma^{\nu'} p_{\mu'} + \gamma^{\mu'}). \quad (E1)
\]
If we now use Eq. (22) to anticommute $\gamma^{\mu'} \gamma^\nu$ and $\gamma^\nu \gamma^\mu$ above we get
\[
\text{lhs} = p^{\mu'} + \gamma^{\mu'}, \quad (E2)
\]
which is the right hand side (rhs) of Eq. (164).

To prove Eq. (165) we insert Eq. (164) into the lhs of (165) and then use Eqs. (113) and (114). Proceeding as described, we arrive at
\[
\text{lhs} = \imath \epsilon_{\sigma\mu\nu}(p') \gamma^\nu \gamma^\mu \gamma^{\sigma'}) \nu_r(p) = 0, \quad (E3)
\]
which is equal to the rhs of Eq. (167). We obtain Eq. (168) in the same way, using, at the end of the proof, Eqs. (114) and (118) instead of Eqs. (113) and (114).

Finally, the proofs leading to Eqs. (169)-(172) can be readily obtained by slightly changing the four proofs given above if we insert Eq. (164) with $p \to -p$ at the lhs of Eqs. (169)-(172) and then use the appropriate pair of equations among Eqs. (113), (114), (118), and (119).

Appendix F: More on discrete symmetries

1. Parity

We want to prove that the parity operator defined in Eq. (252) satisfies the conditions given by Eqs. (251) and (253).

Inserting Eq. (252) and its inverse,
\[
P^{-1}_{\kappa} = e^{i\varphi_r \gamma^0 \gamma^5 K_1},
\]
into the left hand side of (230), and using that \((K_1)^2 = 1\), 
\(\gamma^0 \gamma^5 = -\gamma^5 \gamma^0\), \((\gamma^5)^2 = 1\), and \(\gamma^0 \gamma^\mu \gamma^0 = \gamma^\mu\) we get

\[
P^{-1}_\kappa \gamma^\mu \Lambda^\mu_\nu P_\kappa = -\gamma^\nu \Lambda^\mu_\nu. \tag{F1}
\]

If we now use Eq. (224) and that \(\gamma^0 = \gamma^0\) and \(\gamma^5 = -\gamma^5\), we realize that \(-\gamma^\mu \Lambda^\mu_\nu = -\gamma^\nu\). Thus

\[
P^{-1}_\kappa \gamma^\mu \Lambda^\mu_\nu P_\kappa = -\gamma^\nu, \tag{F2}
\]

which is equal to the right hand side of (240) if \(\theta = \pi\).

Using that \(\theta = \pi\) and multiplying the left and right hand sides of Eq. (240) by \(-mc\), we can recast it as follows,

\[
P^{-1}_\kappa \overline{\Gamma} P_\kappa = -\overline{\Gamma}, \tag{F3}
\]

This is exactly the right hand side of (283), proving consequently the validity of Eq. (240). Note that the property (244) of the operator \(K_1\) is crucial to arrive at Eq. (244).

Equation (242) also implies that under the parity operation the four-current \(j^\mu\) transforms according to what one expects of a contravariant four-vector. To see this, we repeat the argument given in Sec. [IV.A]. The assumption that \(j^\mu\) is a four-vector leads to

\[
\gamma^\nu = [\gamma^0 P^\dagger_\kappa \gamma^0] \gamma^\mu \Lambda^\mu_\nu P_\kappa. \tag{F5}
\]

Comparing with Eq. (F2) we see that we must have

\[
\gamma^0 P^\dagger_\kappa \gamma^0 = -P^{-1}_\kappa \tag{F6}
\]

if \(j^\mu\) transforms as a contravariant four-vector. And a direct calculation, using the definition of \(P_\kappa\), shows that Eq. (F6) is true.

Working in the Dirac-Pauli representation of the gamma matrices, we have that the non-relativistic limit of the wave function of the asymmetric Dirac equation is given by Eq. (241). A simple calculation using \(P_\kappa\) in this representation gives

\[
P_\kappa \Psi_P(x) = i e^{i \hat{\tau}_c} K_1 \Psi_P(x), \tag{F7}
\]

which tells us that the wave function for the asymmetric Dirac equation in the non-relativistic limit is an eigenvector of the parity operator. Moreover, using Eqs. (D15) and (D16) we have

\[
P_\kappa u_r(0) = i e^{i \hat{\tau}_c} K_1 u_r(0), \tag{F8}
\]

\[
P_\kappa v_r(0) = -i e^{i \hat{\tau}_c} K_1 v_r(0). \tag{F9}
\]

Equations (283) and (294) show that in the particle’s rest frame the spinors \(u_r(0)\) and \(v_r(0)\) have opposite parity, similarly to the behavior of the standard Dirac spinors \(u_{\nu r}(0)\) and \(v_{\nu r}(0)\).

Using that \(\Psi'(x') = P_\kappa \Psi(x),\overline{\Psi}'(x') = -\overline{\Psi}(x) P^\dagger_\kappa, P_\kappa = P^{-1}_\kappa\), Eqs. (224), (225), (226), (227), and that \([P_\kappa, \sigma] = 0\), we can show that

\[
\overline{\mathcal{L}}'(x') = \mathcal{L}(x), \tag{F10}
\]

\[
H' = H, \tag{F11}
\]

\[
P' = -P, \tag{F12}
\]

\[
J' = J, \tag{F13}
\]

where \(\mathcal{L}, H, P\), and \(J\) are the Lagrangian density, the Hamiltonian, the linear momentum vector, and the total angular momentum vector related to the asymmetric Dirac equation [cf. Eqs. (240), (245), (246), and (247)].

Equations (F10)-(F13) are the expected behavior of how \(\mathcal{L}, H, P\), and \(J\) transform after the space inversion operation, the same behavior one obtains for the corresponding quantities related to the standard Dirac equation.

2. Time reversal

Our goal here is to prove that the time reversal operator (241) is compatible with Eqs. (249) and (2410).

Inserting Eq. (2411) and its inverse, \(T^{-1}_\kappa = T^{-1}K_1\gamma^5\), into the left hand side of Eq. (249) leads to

\[
T^{-1}_\kappa i \gamma^\mu \Lambda^\mu_\nu T_\kappa = i \gamma^\mu K \gamma^\nu K \Lambda^\mu_\nu (\gamma^\alpha \gamma^\beta J^\mu J^\nu), \tag{F14}
\]

where, in addition to the identities highlighted before in the Sec. (F1) we used that in the Dirac-Pauli representation \(\gamma^2\) is an imaginary matrix, i.e., \(K \gamma^2 K = -\gamma^2\), while for the real ones \(K \gamma^\mu K = \gamma^\mu\), where \(\mu \neq 2\). Using these latter identities, the anticommutation properties of \(\gamma^\mu\), and Eq. (237) we get

\[
i \gamma^\mu \gamma^\nu (K \gamma^\mu K \Lambda^\mu_\nu) (\gamma^\alpha \gamma^\beta J^\mu J^\nu) = -i \gamma^\nu \tag{F15}
\]

and thus

\[
T^{-1}_\kappa i \gamma^\mu \Lambda^\mu_\nu T_\kappa = -i \gamma^\nu, \tag{F16}
\]

which proves the validity of Eq. (249) if \(\theta = \pi\).

Setting \(\theta = \pi\) and multiplying the left and right hand sides of Eq. (2410) by \(-mc\) we obtain the equivalent condition

\[
T^{-1}_\kappa \overline{\mathcal{B}} T_\kappa = -\overline{\mathcal{B}}, \tag{F17}
\]

with \(\overline{\mathcal{B}}\) defined in (F1). Using Eq. (2411), its inverse, the identities involving the gamma matrices highlighted in (F1), and remembering that in the Dirac-Pauli representation \(\gamma^5\) is real, the left hand side of (F17) becomes

\[
T^{-1}_\kappa \overline{\mathcal{B}} T_\kappa = imc \gamma^5 - h \gamma^3 \gamma^1 \left( \sum_{\mu} \kappa^\mu K \gamma^\mu K \right) \gamma^1 \gamma^3. \tag{F18}
\]
However, a direct calculation gives
\[ \gamma^3 \gamma^1 \left( \sum_{\mu} \kappa^\mu K \gamma^\mu K \right) \gamma^1 \gamma^3 = \kappa^0 \gamma^0 - \sum_{j=1}^{3} \kappa^j \gamma^j = \kappa_\mu \gamma^\mu, \] and therefore Eq. (18) becomes
\[ T_{\kappa}^{-1} B T_{\kappa} = i m c \gamma^5 - h \kappa_\mu \gamma^\mu = -B. \] (F20)
This is equivalent to Eq. (240) with \( \theta = \pi \) that we wanted to prove [cf. Eq. (17)].
Equation (241) also gives that under the time reversal operation the four-current \( j^\mu \) transforms according to a contravariant four-vector, namely,
\[ j^\mu = \kappa^\nu j^\nu, \] and therefore Eq. (F18) becomes
\[ T_{\kappa}^{-1} B T_{\kappa} = i m c \gamma^5 - h \kappa_\mu \gamma^\mu = -B. \] (F20)

3. Charge conjugation

To prove Eqs. (249)-(251), we first note that if, without losing in generality, we set \( \varphi_c = 0 \), Eqs. (252) give
\[ U_{\kappa} = U_{\kappa}^{-1} = i \gamma^2 K_2. \] (F28)
Inserting Eq. (F28) into the left hand side of (249) we get
\[ U_{\kappa} \gamma^\mu U_{\kappa}^{-1} = -\gamma^2 \gamma^\mu \gamma^2 = -\gamma^\mu, \] (F29)
where the latter equality is a consequence of the algebra of the gamma matrices and that we are working in the Dirac-Pauli representation, where \( \gamma^2 \) is a pure imaginary matrix while the other gamma matrices are real matrices. Equation (F29) is exactly Eq. (249) when \( \theta = \pi \).

Similarly, if we use Eqs. (28), (255), and remember that in the Dirac-Pauli representation for the gamma matrices \( \gamma^2 \gamma^\mu \gamma^2 = \gamma^\mu \), the left hand side of Eq. (250) becomes
\[ U_{\kappa} \gamma^5 U_{\kappa}^{-1} = \gamma^\mu \kappa_\mu, \] (F30)
which is exactly the right hand side of Eq. (250) if \( \theta = \pi \).

Finally, the proof of Eq. (251) is obtained by using the identity \( \gamma^2 \gamma^\nu \gamma^2 = \gamma^5 \) and that in the Dirac-Pauli representation of the gamma matrices \( \gamma^5 = \gamma^5 \). These facts, together with Eq. (F28), allow us to write the left hand side of Eq. (251) as
\[ U_{\kappa} \gamma^5 U_{\kappa}^{-1} = -\gamma^5, \] (F31)
which equals the right hand side of Eq. (251) when \( \theta = \pi \).

To obtain the equivalent expressions of Eqs. (250)-(251), and (224)-(227), where \( \Psi_\omega(x) \) is interpreted as the field describing antiparticles, we have to go to the second quantization formalism and use the fermion normal ordering prescription in the calculations below \( \Omega \). As such, if \( \Omega \) is any function of the gamma matrices not depending on fermion fields, we will have identities of the following type,
\[ \Psi^T \Omega \Psi^T = -\left[ \Psi^T \Omega^T \Psi \right]^T = -\Psi^T \Omega^T \Psi. \]
The minus sign comes from the normal ordering prescription, i.e., whenever we exchange the order of two fermion operators we gain a minus sign. And the last equality above follows from the fact that the transpose of a c-number is the c-number itself.

With the above provisos, using that \( \Psi_{\omega}(x) = U_{\kappa} \Psi_{\mu}(x), \) \( \Psi_{\mu}(x') = -\Psi^T(x) \gamma^\mu U_{\kappa}^\dagger, U_{\kappa}^\dagger = U_{\kappa}^{-1}, \) Eqs. (252), (259), (258), (230), (231), \( \gamma^0 \gamma^\mu \gamma^0 = \gamma^\mu, \) [256], \( \gamma_{\kappa}^0 \gamma^0 = 0, \) and that \( U_{\kappa}^T \gamma^0 \sigma U_{\kappa} = \sigma^\ast \gamma^0, \) we can show that
\[ L_{\omega}(x) = L(x), \] (F32)
\[ L_{\omega\mu}(x) = -L_{\mu\omega}(x), \] (F33)
\[ H_{\omega} = H, \] (F34)
\[ P_{\omega} = P, \] (F35)
\[ J_{\omega} = J. \] (F36)
where \( L, L_{\mu\nu}, H, P, \) and \( J \) are given respectively by Eqs. (270), (277), (276), (277), and (281).

It is worth noting that during the calculations leading to Eq. (F32) and to Eqs. (F34)-(F36), we obtain that the
left hand sides and the right hand sides are equal up to a four-divergence [Eq. (1)] or to spatial volume integrals that can all be converted to surface integrals that vanish since we are assuming the fields go to zero sufficiently fast as we tend to infinity.

[1] E. Schrödinger, An Undulatory Theory of The Mechanics of Atoms and Molecules, Phys. Rev. 28, 1049 (1926).
[2] L. E. Ballentine, Quantum Mechanics: A Modern Development (World Scientific, Singapore, 1998).
[3] G. Rigolin, On Lorentz Invariant Complex Scalar Fields, Adv. High Energy Phys. 2022, 5511428 (2022).
[4] P. A. M. Dirac, The Principles of Quantum Mechanics (Oxford University Press, London, 1967).
[5] W. Greiner, Relativistic Quantum Mechanics: Wave Equations (Springer-Verlag, Berlin, 2000).
[6] F. Mandl and G. Shaw, Quantum Field Theory (John Wiley & Sons, Chichester, 1986).
[7] W. Greiner and J. Reinhardt, Field Quantization (Springer-Verlag, Berlin, 1996).
[8] G. Rigolin, Asymmetric particle-antiparticle Dirac equation: second quantization, arXiv:2208.12239.
[9] F. Schwabl, Advanced Quantum Mechanics (Springer-Verlag, Berlin, 2005).
[10] N. N. Bogoliubov and D. V. Shirkov, Introduction to the Theory of Quantized Fields (John Wiley & Sons, New York, 1980).
[11] V. I. Safronov, Conduction Electron in the Anisotropic Medium, Int. J. Mod. Phys. B 7, 3899 (1993).
[12] A. Zhao, J. Zhang, Q. Gu, and R. A. Klemm, A relativistic electron in an anisotropic conduction band, arXiv:1905.03127.
[13] E. C. G. Stueckelberg, Remarque à propos de la création de paires de particules en théorie de relativité, Helv. Phys. Acta 14, 588 (1941).
[14] L. L. Foldy and S. A. Wouthuysen, On the Dirac Theory of Spin 1/2 Particles and Its Non-Relativistic Limit, Phys. Rev. 78, 29 (1950).
[15] M. A. H. Tucker and A. F. G. Wyatt, Direct Evidence for R-Rotons Having Antiparallel Momentum and Velocity, Science 283, 1150 (1999).
[16] B. Juulgaard, A. Kozhokin, and E. S. Polzik, Experimental long-lived entanglement of two macroscopic objects, Nature (London) 413, 400 (2001).
[17] M. Tsang and C. M. Caves, Evading Quantum Mechanics: Engineering a Classical Subsystem within a Quantum Environment, Phys. Rev. X 2, 031016 (2012).
[18] M. Conforti, S. Trillo, A. Mussot, and A. Kudlinski, Parametric excitation of multiple resonant radiations from localized wavepackets, Sci. Rep. 5, 9433 (2015).
[19] M. A. Khamelchi, K. Hossain, M. E. Mossman, Y. Zhang, Th. Busch, M. McNeil Forbes, and P. Engels, Negative-Mass Hydrodynamics in a Spin-Orbit-Coupled Bose-Einstein Condensate, Phys. Rev. Lett. 118, 155301 (2017).
[20] M. Dine and A. Kusenko, Origin of the matter-antimatter asymmetry, Rev. Mod. Phys. 76, 1 (2004).
[21] O. Bertolami, D. Colladay, V. A. Kostelecký, and R. Potting, CPT violation and baryogenesis, Phys. Lett. B 395, 178 (1997).
[22] D. Colladay and V. A. Kostelecký, Lorentz-violating extension of the standard model, Phys. Rev. D 58, 116002 (1998).
[23] S. M. Carroll and J. Shu, Models of baryogenesis via spontaneous Lorentz violation, Phys. Rev. D 73, 103515 (2006).
[24] M. de Cesare, N. E. Mavromatos, and S. Sarkar, On the possibility of tree-level leptogenesis from Kalb–Ramond torsion background, Eur. Phys. J. C 75, 514 (2015).
[25] J. Sakstein and A. R. Solomon, Baryogenesis in Lorentz-violating gravity theories, Phys. Lett. B 773, 186 (2017).
[26] B. R. Edwards and V. A. Kostelecký, Riemann-Finsler geometry and Lorentz-violating scalar fields, Phys. Lett. B 786, 319 (2018).
[27] D. L. Anderson, M. Sher, and I. Turan, Lorentz and CPT violation in the Higgs sector, Phys. Rev. D 70, 016001 (2004).
[28] V. A. Kostelecký and S. Samuel, Spontaneous breaking of Lorentz symmetry in string theory, Phys. Rev. D 39, 683 (1989).
[29] V. A. Kostelecký and R. Potting, CPT, strings, and meson factories, Phys. Rev. D 51, 3923 (1995).
[30] A. E. Bernardini and R. da Rocha, Lorentz-violating dilatations in momentum space and some extensions on nonlinear actions of Lorentz-algebra-preserving systems, Phys. Rev. D 75, 065014 (2007).
[31] A. E. Bernardini, Dirac neutrino mass from the beta-decay end point modified by the dynamics of a Lorentz-violating equation of motion, Phys. Rev. D 75, 097901 (2007).
[32] A. E. Bernardini and O. Bertolami, Lorentz violating extension of the standard model and the β-decay endpoint, Phys. Rev. D 77, 085032 (2008).
[33] A. E. Bernardini and R. da Rocha, Obtaining the equation of motion for a fermionic particle in a generalized Lorentz-violating system framework, EPL 81, 40010 (2008).
[34] J. M. Hoff da Silva and R. da Rocha, Unfolding physics from the algebraic classification of spinor fields, Phys. Lett. B 718, 1519 (2013).
[35] R. A. C. Correa, R. da Rocha, and A. de Souza Dutra, Entropic information for travelling solitons in Lorentz and CPT breaking systems, Ann. Phys. (N.Y.) 359, 198 (2015).
[36] A. Ferrari, J. A. S. Neto, and R. da Rocha, The role of singular spinor fields in a torsional gravity, Lorentz-violating, framework, Gen. Relativ. Gravit. 49, 70 (2017).
[37] C. C. Nishi, CP violation conditions in N-Higgs-doublet potentials, Phys. Rev. D 74, 036003 (2006).
[38] R. N. Mohapatra and C. C. Nishi, S4 flavored CP symmetry for neutrinos, Phys. Rev. D 86, 073007 (2012).
[39] C. C. Nishi, Generalized CP symmetries in Δ(27) flavor models, Phys. Rev. D 88, 033010 (2013).
[40] R. N. Mohapatra and C. C. Nishi, Implications of μ − τ flavored CP symmetry of leptons, J. High Energ. Phys. 2015, 92 (2015).
[41] M. S. Safronova, D. Budker, D. DeMille, D. F. J. Kimball,
A. Derevianko, and C. W. Clark, Search for new physics with atoms and molecules, Rev. Mod. Phys. 90, 025008 (2018).

[42] H. Bondi, Negative Mass in General Relativity, Rev. Mod. Phys. 29, 423 (1957).

[43] M. Kowitt, Gravitational Repulsion and Dirac Antimatter, Int. J. Theor. Phys. 35, 605 (1996).

[44] M. Villata, CPT symmetry and antimatter gravity in general relativity, EPL 94, 20001 (2011).

[45] J. S. Farnes, A unifying theory of dark energy and dark matter: Negative masses and matter creation within a modified $\Lambda CDM$ framework, A&A 620, A92 (2018).