Data-Driven Control Design with LMIs and Dynamic Programming

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Abstract—The goal of this paper is to develop data-driven control design and evaluation strategies based on linear matrix inequalities (LMIs) and dynamic programming. We consider deterministic discrete-time LTI systems, where the system model is unknown. We propose efficient data collection schemes from the state-input trajectories together with data-driven LMIs to design state-feedback controllers for stabilization and linear quadratic regulation (LQR) problem. In addition, we investigate theoretically guaranteed exploration schemes to acquire valid data from the trajectories under different scenarios. In particular, we prove that as more and more data is accumulated, the collected data becomes valid for the proposed algorithms with higher probability. Finally, data-driven dynamic programming algorithms with convergence guarantees are then discussed.

Index Terms—Optimal control, LTI system, data-driven design, reinforcement learning, linear matrix inequality, dynamic programming

I. INTRODUCTION

Recently, reinforcement learning (RL) [1] and data-driven control design have captured significant attentions due to its successful demonstrations that outperform humans in several challenging tasks [2], [3]. The goal of this paper is to develop efficient data-driven control design methods for deterministic discrete-time linear-time invariant (LTI) systems. Two different lines of approaches are addressed: linear matrix inequalities (LMIs) [4] and dynamic programming [5], [6]. In particular, we develop simple and efficient data-driven LMIs for stabilization and LQR problems along with data collection algorithms tailored to the proposed LMIs, which allow us to design controllers without the knowledge of the model. We also prove rigorously the optimality of the LMI solutions. Moreover, new data collection algorithms are developed, and we prove that these algorithms guarantee the validity of the data in the probabilistic sense, where the validity implies that it includes sufficient information for the proposed algorithms to successfully solve the given problems. Finally, additional data-driven dynamic programming algorithms are proposed based on the data collection algorithms with their convergence proofs. All these algorithms are sample efficient in the sense that once valid data is collected, then no more data is required to solve the problems completely.

Related works: The previous works can be roughly categorized into two parts: RL (or data-driven dynamic programming) and data-based LMIs. As for RL, the early work [7] proposed a Q-learning algorithm [8] for discrete-time LTI systems, where the approximate Bellman equation is solved using the least-square method and trajectories. More comprehensive least-square reinforcement learning approaches were reported in [9]. A model-based RL has been studied in [10] for discrete-time LTI systems with sample complexity analysis. A policy gradient algorithm for LTI systems and its global convergence were provided in [11]. An efficient online RL with guaranteed finite-time regret bounds has been proposed in [12] based on a novel semidefinite programming relaxation. The paper [13] proposed several model-based and model-free RLs. [14] proposed a policy iteration reinforcement learning based on the Lagrangian duality perspectives of the Bellman equation.

As for the data-based LMIs, several advances have been made recently in deriving numerically tractable data-based LMIs that enable direct data-driven control designs. Data-dependent LMIs were developed in [15] for stabilization of switched systems. The paper [16] introduced a data-dependent controller parameterization, and proposed data-based LMIs for stabilization and optimal control problems. The concept of informative data was introduced in [17], from which necessary and sufficient data-based conditions have been developed for various control problems. The paper [18] proposed LMI conditions for control with guaranteed stability and performance by introducing a notion of noise bounds. Recently, [19] introduced data-driven LMI conditions for stabilization problems based on a matrix version of the classical Finsler’s lemma [20].

Contribution: Compared to the previous works, the proposed data-driven LMIs provide more intuitive conditions with more memory efficient data structures and computational efficiency in terms of the size of LMIs. We additionally provide data-based LMIs for policy evaluations. Moreover, new data generation schemes are developed with different scenarios. We prove that the new data collection approaches is guaranteed to be valid with probability one as more and more trajectories are accumulated. Lastly, data-driven dynamic programming schemes are briefly discussed, whose learning process is off-policy. The algorithms are sample efficient in the sense that once the data is collected, then no more samples are required.

Notation: The adopted notation is as follows: $\mathbb{R}$: set of real numbers; $\mathbb{R}^n$: $n$-dimensional Euclidean space; $\mathbb{R}^{n \times m}$: set of all $n \times m$ real matrices; $A^T$: transpose of matrix $A$; $A^{-T}$: transpose of matrix $A^{-1}$; $A \succ 0$ ($A \preceq 0$, $A \succeq 0$, $A \preceq 0$, respectively): symmetric positive definite (negative definite, positive semi-definite, and negative semi-definite, respectively) matrix $A$; $I$: identity matrix with appropriate dimensions; $S_n^+$: symmetric $n \times n$ matrices; $S_n^{++}$: cone of symmetric $n \times n$ positive semi-definite matrices; $S_n^+$: symmetric $n \times n$ positive definite matrices; $\text{Tr}(A)$: trace of matrix $A$; $\rho(\cdot)$: spectral radius; $\text{diag}(A_1, \ldots, A_n)$: block diagonal matrix with diagonal elements $A_1, \ldots, A_n$.

II. PROBLEM FORMULATIONS AND PRELIMINARIES

Consider the LTI system

$$x(k+1) = Ax(k) + Bu(k), \quad x(0) = z \in \mathbb{R}^n,$$

(1)

where $k \in \mathbb{N}$, $x(k) \in \mathbb{R}^n$ is the state vector, $u(k) \in \mathbb{R}^m$ is the input vector, and $z \in \mathbb{R}^n$ is the initial state.

Assuming the control $u(k)$ is given by a state-feedback control policy $u(k) = Fx(k)$, we denote by $x(k; F, z)$ the solution of (1) starting from $x(0) = z$. Under the state-feedback control policy, the cost function for the classical LQR problem is denoted by

$$J(F, z) := \sum_{k=0}^{\infty} \left[ x(k; F, z) \right]^T A \left[ x(k; F, z) \right],$$

(2)

where $A := \begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix} \succeq 0$ is the weight matrix.
By introducing the augmented state vector \( v(k) := \begin{bmatrix} x(k) \\ u(k) \end{bmatrix} \), we will consider the augmented system
\[
v(k + 1) = A_F v(k), \quad v(0) = v_0 \in \mathbb{R}^{n+m},
\]
where \( A_F := \begin{bmatrix} A & B \\ FA & FB \end{bmatrix} \in \mathbb{R}^{(n+m) \times (n+m)} \), which plays an important role throughout the paper. A useful property of \( A_F \) is that its spectral radius \( \rho(A_F) \) is identical to that of \( A + BF \).

**Lemma 1** ([14]), \( \rho(A + BF) = \rho(A_F) \) holds.

Define \( \mathcal{F} \) as the set of all stabilizing state-feedback gains of system \((A, B)\).

**Definition 1** (Stabilizing set). The set of all stabilizing state-feedback gains of system \((A, B)\) is denoted by
\[
\mathcal{F} := \{ F \in \mathbb{R}^{m \times n} : \rho(A + BF) < 1 \}
\]

Note that \( \mathcal{F} \) is an open set, and not necessarily convex [21, Lemma 2]. However, finding a state feedback gain \( F \in \mathcal{F} \) can be reduced to a simple convex problem. In this paper, we study both the LQR problem and stabilization problem.

**Problem 1** (Stabilization problem). Find a stabilizing feedback gain \( F \in \mathcal{F} \).

**Problem 2** (LQR problem). Solve \( F^* = \arg \min_{F \in \mathbb{R}^{m \times n}} J(F, z) \) if the optimal value of \( \inf_{F \in \mathbb{R}^{m \times n}} J(F, z) \) exists and is attained.

From the standard LQR theory, although \( J^*(F, z) \) has different values for different \( z \in \mathbb{R}^n \), the minimizer \( F^* = \arg \min_{F \in \mathbb{R}^{m \times n}} J(F, z) \) is not dependent on \( z \). Therefore, it follows that \( \arg \min_{F \in \mathbb{R}^{m \times n}} J(F, z) = \arg \min_{F \in \mathbb{R}^{m \times n}} \sum_{i=1}^n J(F, z_i) \) for any \( z, z_i \in \mathbb{R}^n \), \( i \in \{1, 2, \ldots, r\} \). For technical reasons that will become clear later, we solve
\[
F^* := \arg \min_{F \in \mathbb{R}^{m \times n}} \sum_{i=1}^n J(F, e_i)
\]

instead of \( \arg \min_{F \in \mathbb{R}^{m \times n}} J(F, z) \), where \( e_i \in \mathbb{R}^n \) is the \( i \)-th standard basis vector. Therefore, it will be useful to define a standard measure of the cost. In this paper, we will use the following cost index:
\[
J(F) := \sum_{i=1}^n J(F, e_i)
\]

For a given \( z \in \mathbb{R}^n \), if the optimal value of \( \inf_{F \in \mathbb{R}^{m \times n}} J(F, z) \) exists and is attained, then the optimal cost is denoted by \( J^*(z) = J(F^*) \). Assumptions that will be used throughout the paper are summarized below.

**Assumption 1.** Throughout the paper, we assume that

- \( Q \succeq 0, R > 0 \);
- \((A, B)\) is stabilizable, and \( Q \) can be written as \( Q = C^T C \), where \((A, C)\) is detectable.

Under Assumption 1, the optimal value of \( \inf_{F \in \mathbb{R}^{m \times n}} J(F, z) \) exists, is attained, and \( J^*(z) \) is a quadratic function, i.e., \( J^*(z) = z^T X^* z \), where \( X^* \) is the unique solution of the algebraic Riccati equation (ARE) [5, Proposition 4.4.1] for \( X \):
\[
X = A^T X A - A^T X B (R + B^T X B)^{-1} B^T X A + Q, \quad X \succeq 0.
\]

In this case, \( J^*(z) \) as a function of \( z \in \mathbb{R}^n \) is called the optimal value function. The reader can refer to [5] and [22] for more details of the classical LQR results. The corresponding optimal control policy is \( u^*(z) = F^* z \), where
\[
F^* := -(R + B^T X^* B)^{-1} B^T X^* A \in \mathcal{F}
\]
Algorithm 1 On-policy data collection \((S(F), H(F))\) = On − Collect\((F)\) with exploring starts

1: Initialize \(S_0 = 0, H_0 = 0\).
2: for \(i \in \{1, 2, \ldots, n\}\) do
3: Initialize \(x(0) = e_i\).
4: for \(k \in \{0, 1, \ldots, N − 1\}\) do
5: Apply control input \(u(k) = Fx(k)\).
6: Observe \(x(k + 1)\).
7: Update
   \[
   S_{k+1} \leftarrow \frac{k}{k+1} S_k + \frac{1}{k+1} \left[ \begin{array}{c} x(k) \\ u(k) \end{array} \right] \left[ \begin{array}{c} x(k) \\ u(k) \end{array} \right]^T
   \]
   \[
   H_{k+1} \leftarrow \frac{k}{k+1} H_k + \frac{1}{k+1} \left[ \begin{array}{c} x(k+1) \\ u(k+1) \end{array} \right] \left[ \begin{array}{c} x(k+1) \\ u(k+1) \end{array} \right]^T
   \]
8: end for
9: end for
10: Return \((S(F), H(F)) = (S_N, H_N)\)

Algorithm 2 Off-policy data collection \((S, H) = \text{off} − \text{Collect}(z)\) with exploring starts

1: Initialize \(S_0 = 0, H_0 = 0\).
2: Initialize \(x(0) = z\).
3: Initialize \(\varepsilon > 0\).
4: for \(k \in \{0, 1, \ldots\}\) do
5: Apply control input \(u(k)\) with some excitation input
6: Observe \(x(k + 1)\).
7: Update
   \[
   S_{k+1} \leftarrow \frac{k}{k+1} S_k + \frac{1}{k+1} \left[ \begin{array}{c} x(k) \\ u(k) \end{array} \right] \left[ \begin{array}{c} x(k) \\ u(k) \end{array} \right]^T
   \]
   \[
   H_{k+1} \leftarrow \frac{k}{k+1} H_k + \frac{1}{k+1} \left[ \begin{array}{c} x(k+1) \\ u(k+1) \end{array} \right] \left[ \begin{array}{c} x(k+1) \\ u(k+1) \end{array} \right]^T
   \]
8: if \(\lambda_{\min}(S_{k+1}) > \varepsilon\) then
9: Stop and return \((S, H) = (S_{k+1}, H_{k+1})\)
10: end if
11: end for

so as to collect sufficient information on the model. Note that the data matrices, \((S, H)\), in Algorithm 2 can be expressed as

\[
S := \frac{1}{N} \sum_{k=0}^{N-1} \left[ \begin{array}{c} x(k) \\ u(k) \end{array} \right] \left[ \begin{array}{c} x(k) \\ u(k) \end{array} \right]^T
\]
\[
H := \frac{1}{N} \sum_{k=0}^{N-1} \left[ \begin{array}{c} x(k+1) \\ u(k) \end{array} \right] \left[ \begin{array}{c} x(k+1) \\ u(k) \end{array} \right]^T
\]

Throughout the paper, we call the data generated by the data collection algorithms is valid if the \(S\)-matrix \((S(F)\) or \(S)\) is strictly positive definite. For completeness, the definition is formally stated below.

Definition 2 (Data validity). The data \((S, H)\) and \((S(F), H(F))\) generated by Algorithm 1 and Algorithm 2, respectively, is said to be valid if \(S > 0\) and \(S(F) > 0\), respectively.

The validity of the data ensures that all the proposed methods perform well, and completely solve the desired problems. Due to the exploring starts in Algorithm 1, we can prove that the data from Algorithm 1 is always valid for any \(N > 0\).

Lemma 5 (Data validity of Algorithm 1). With a positive integer \(N > 0\), \(S(F) > 0\) holds.

Proof. We have

\[
S(F) = \frac{1}{nN} \sum_{i=1}^{n} \sum_{k=0}^{N-1} \left[ \begin{array}{c} x(k; F, e_i) \\ u(k) \end{array} \right] \left[ \begin{array}{c} x(k; F, e_i) \\ u(k) \end{array} \right]^T
\]
\[
= \frac{1}{nN} \sum_{i=1}^{n} \sum_{k=0}^{N-1} (A_k)\Gamma e_i e_i^T (A_k)^T
\]
\[
= \frac{1}{nN} \sum_{k=0}^{N-1} (A_k)^T \left( \begin{array}{c} x(k+1; F) \\ u(k+1) \end{array} \right)^T
\]

which completes the proof.

On the other hand, Algorithm 2 cannot theoretically guarantee the validity. Therefore, we adopt the so-called persistent excitation assumption for Algorithm 2, given below.

Assumption 2 (Persistent excitation). There exists a positive integer \(N > 0\) such that \(S > 0\) from Algorithm 2.

We notice that it is typical to apply Assumption 2 in adaptive control and reinforcement learning community \([7, 9, 23]\). Moreover, in the last section, more sophisticated data collection algorithms will be developed, which theoretically guarantee the data validity with different scenarios. Finally, the following lemma will be useful throughout the paper.

Lemma 6 (Data matrix transformation). The following identity holds:

\[
S \left[ \begin{array}{c} A_k^T \\ B_k^T \end{array} \right] = H
\]

Proof.

\[
S \left[ \begin{array}{c} A_k^T \\ B_k^T \end{array} \right] = \frac{1}{N} \sum_{k=0}^{N-1} \left[ \begin{array}{c} x(k) \\ u(k) \end{array} \right] (Ax(k) + Bu(k))^T
\]
\[
= \frac{1}{N} \sum_{k=0}^{N-1} \left[ \begin{array}{c} x(k) \\ u(k) \end{array} \right] x(k+1)^T
\]
\[= H
\]

IV. DATA-DRIVEN LMIS FOR STABILIZATION

In this section, the main focus is on data-driven LMIs for stabilization, where the model \((A, B)\) is unknown. The main breakthrough in this approach lies in augmenting the state and input into a single augmented state as in (3). Then, the model data \((A, B)\) can be eliminated using the data matrices \((S, H)\) or \((S(F), H(F))\) together with Lemma 4 and Lemma 6. We first consider a policy evaluation problem. In this setting, given a potentially unknown state-feedback gain \(F\), we only have an access to the state-input trajectories. Under this situation, the problem is to determine whether or not the unknown feedback gain \(F\) stabilizes the system.

Proposition 1 (Stability evaluation). The system (1) is stable under \(u(k) = Fx(k)\) if and only if there exist \(P \in S^{(n+m)}\) such that the following LMI holds:

\[
H(F)^T PH(F) < S(F)PS(F), \quad P > 0
\]

Proof. From the Lyapunov theory, \(A_F^T\) is stabilizable if and only if there exist \(P \in S^{(n+m)}\) such that \(A_F^T A_F^T < P\). Replacing \(P\) with \(SPS\) leads to the equivalent condition \(A_F S(F)PS(F) A_F^T < S(F)PS(F)\). Using the relation in Lemma 4 yields the conclusion.

□
Proposition 1 will be useful when we want to check if the unknown system is (asymptotically) stable. Its main feature is that it requires the on-policy data from Algorithm 1. Note however that it does not require the knowledge of $F$. Next, a stabilizing state-feedback control design algorithm is proposed using LMIs and data from Algorithm 2.

Proposition 2 (Stabilization). The system (1) is stabilizable if and only if there exist $G \in \mathbb{R}^{n \times n}$, $P \in \mathbb{S}^{n+m}$, and $X \in \mathbb{R}^{n \times m}$, such that the following LMI holds:

$$
\begin{bmatrix}
-SPS & G & X
\end{bmatrix}^T \begin{bmatrix}
0 \\
G^T \\
X^T
\end{bmatrix} < 0
$$

(7)

If a solution, $(P, G, X)$, exists, then a stabilizing state-feedback gain is given by $F = X^T(G^T)^{-1}$, and $V(x) = x^TSPSx$ is the corresponding Lyapunov function of $A_F$.

Proof. Lemma 1 tells us that the original system (1) is stabilizable if and only if the augmented system (3) is stabilizable, or equivalently $A_F$ is Schur. Moreover, from a standard result of the linear system theory, we know that $A_F$ is Schur if and only if the corresponding dual system $A_F^T$ is Schur. From the Lyapunov theory, the dual system $A_F^T$ is Schur if and only if there exists a Lyapunov matrix $P \in \mathbb{R}^{n+m}$ such that $A_FPA_F^T < P$. The Lyapunov inequality can be expressed as

$$
\begin{bmatrix}
[I & F^T]
\end{bmatrix}
\begin{bmatrix}
-P & 0 \\
0 & P
\end{bmatrix}
\begin{bmatrix}
A^T \\
B^T
\end{bmatrix}
\begin{bmatrix}
A^T \\
B^T
\end{bmatrix}
\begin{bmatrix}
-I & F^T
\end{bmatrix}
< 0
$$

From Lemma 2 (Finsler lemma), we have that dual system $A_F^T$ is Schur if and only if there exist $P, F, G$ such that

$$
\begin{bmatrix}
-P & 0 \\
G & I & F^T
\end{bmatrix}
\begin{bmatrix}
A^T \\
B^T
\end{bmatrix}
\begin{bmatrix}
A^T \\
B^T
\end{bmatrix}
\begin{bmatrix}
G & I & F^T
\end{bmatrix}
< 0
$$

which is a non-convex bilinear matrix inequality. The first block diagonal matrix ensures $P > 0$, and the second block diagonal matrix implies $G + G^T > 0$. This guarantees that $G$ is nonsingular. With the change of variables, $X = GF^T$, the last matrix inequality becomes

$$
\begin{bmatrix}
-P & 0 \\
G & I & F^T
\end{bmatrix}
\begin{bmatrix}
A^T \\
B^T
\end{bmatrix}
\begin{bmatrix}
A^T \\
B^T
\end{bmatrix}
\begin{bmatrix}
G & I & F^T
\end{bmatrix}
< 0
$$

(8)

Clearly, the above linear matrix inequality (8) holds if and only if the previous bilinear matrix inequality is satisfied from the bijective mapping $F^T = G^{-1}X$. Next, we replace $P$ with $SPS$ and use the identity Lemma 6 to obtain the LMI (7) in the statement. Note that (8) holds if and only if (7) because $S \in \mathbb{S}^{n+m}$ is nonsingular. This completes the proof.

Using the LMI condition in Proposition 2, a stabilizing state-feedback controller can be found only using the trajectories. Note that the data used in Proposition 2 is generated from the off-policy method Algorithm 1.

V. DATA-DRIVEN LMIS FOR LQR DESIGN

Beyond the stabilization problem, the idea in the previous section can be also applied to LQR design problems. We first consider a policy evaluation problem again. Given a potentially unknown state-feedback gain $F$, suppose that we only have an access to the state-input trajectories. Under this situation, the problem is to determine the LQR performance of the unknown $F$.

Proposition 3 (Performance evaluation). Consider the optimization problem

$$
\begin{align*}
\min_{P \in \mathbb{S}^{n+m}} \text{Tr}(ASF)PSF \\
\text{subject to } H(F)^TPH(F) + I \preceq S(F)PS(F)
\end{align*}
$$

and $P \in \mathbb{S}^{n+m}$ is the corresponding optimal point. Then, the optimal objective function value (9) is the cost corresponding to $F$, i.e., $\text{Tr}(ASF)PSF = J(F)$.

Proof. The optimal solution $P$ satisfies

$$
H(F)^TPH(F) + I \preceq S(F)PS(F), \quad P > 0.
$$

Using Lemma 4 and letting $\hat{P} = S(F)PS(F)$, it follows that

$$
A_F\hat{P}A_F^T + I \preceq \hat{P}, \quad \hat{P} > 0.
$$

(10)

Since the above inequality is a Lyapunov inequality, $A_F$ is Schur. Therefore, there exists $\hat{P} \in \mathbb{S}^{n+m}$ such that $A_F\hat{P}A_F^T + I = \hat{P}$, where $\hat{P} := \sum_{k=0}^{\infty} A_F^k \hat{P}$. Replacing $P$ with $S(F)MS(F)$, where $M = (S(F)^{-1}PS(F))^{-1}$, we can see that $M$ satisfies $H(F)^TMMH(F) + I = S(F)MS(F)$. This implies that $M$ is a feasible point for (9). Therefore,

$$
\text{Tr}(ASF)PSF \leq \text{Tr}(ASF)MS(F) = \text{Tr}(\hat{P})
$$

On the other hand, repeatedly applying the inequality (10) yields

$$
\sum_{k=0}^{\infty} A_F^k \hat{P} A_F^k \leq \hat{P} \leq S(F)PS(F)
$$

by which we have $\text{Tr}(ASF)PSF) \geq \text{Tr}(\hat{P})$, implying $\text{Tr}(ASF)PSF) = \text{Tr}(\hat{P})$. Then, we can conclude

$$
\text{Tr}(ASF)PSF) = \text{Tr}(\hat{P})
$$

$$
= \text{Tr} \left( \sum_{k=0}^{\infty} (A_F^k)^{\Lambda} A_F^k \right)
$$

$$
= \sum_{i=1}^{\infty} \sum_{k=0}^{\infty} \epsilon_i^T (A_F^k)^{\Lambda} A_F^k \epsilon_i
$$

$$
= J(F)
$$

This completes the proof.

Next, the LQR design problem is addressed using a data-driven LMI. The following LMI condition allows us to design an LQR control of unknown system in a simple and efficient way.

Proposition 4 (LQR design). Consider the optimization problem

$$
\begin{align*}
\min_{P \in \mathbb{S}^{n+m}, G \in \mathbb{R}^{n \times n}, X \in \mathbb{R}^{n \times m}} \text{Tr}(ASPS) \\
\text{subject to } -SPS + I \\
\begin{bmatrix}
X & 0
\end{bmatrix}^T \begin{bmatrix}
G & X
\end{bmatrix} \begin{bmatrix}
0 & 0 \\
G & X
\end{bmatrix} < 0
\end{align*}
$$

(11)

and $G \in \mathbb{R}^{n \times n}$, $P \in \mathbb{S}^{n+m}$, and $Y \in \mathbb{R}^{m \times m}$ are the corresponding optimal points. Then, the optimal objective function value upper bounds the optimal cost, $J(F^*)$, and the corresponding state-feedback gain is given by $F = \hat{F}$.

Proof. The LMI constraint is identical to the stabilization case. Therefore, we can follow the same procedure to arrive that the conclusion that the optimization is equivalent to the following optimization:

$$
\begin{align*}
\min_{P \in \mathbb{S}^{n+m}, F \in \mathbb{R}^{m \times n}} \text{Tr}(AP) \\
\text{subject to } \begin{bmatrix}
I & F
\end{bmatrix} \left( \begin{bmatrix}
A^T & F
\end{bmatrix}^T \begin{bmatrix}
A^T & F
\end{bmatrix} \right) \begin{bmatrix}
I & F
\end{bmatrix}^T + I \preceq P
\end{align*}
$$

(12)
Therefore, \((SPS, \bar{F})\) is an optimal solution of the above problem. Applying the inequality recursively leads to

\[
\sum_{k=0}^{N-1} A^k_F (A^k_F)^T \preceq A^k_F^{N-1} SPS (A^k_F)^T + \sum_{k=0}^{N-1} A^k_F (A^k_F)^T \prec SPS
\]

Multiplying with \(\Lambda\) and taking the trace on the last inequality, one gets

\[
\text{Tr}(\Lambda SPS) \geq \text{Tr} \left( \sum_{k=0}^{N-1} A^k_F (A^k_F)^T \right)
\]

\[
= \sum_{k=1}^{N-1} \sum_{i=0}^{k-1} x(k; \bar{F}, e_i)^T \Lambda x(k; \bar{F}, e_i)
\]

\[
= J(\bar{F}).
\]

Taking the limit \(N \to \infty\), we obtain the desired conclusion. \(\square\)

**Proposition 4** allows us to design a controller with a guaranteed upper bound on the LQR performance. A natural question arising here is whether or not the obtained controller from **Proposition 4** is optimal. If not, then how far is it away from the optimal gain \(F^*\)? A potential answer is given in the following result. In particular, to answer this question, one needs to make it clear that the LMI in **Proposition 4** is strict. Note that the LMI needs the strictness to use the Finsler’s lemma. Therefore, the feasible set satisfying the LMI constraint is open and set, and therefore, there would be no solution to the optimization in **Proposition 4**. In practice, to find an approximate solution to **Proposition 4**, most LMI solvers try to solve the semi-definite problem with a small margin. For simplicity and convenience, let us start with (12), and consider the modified problem

\[
\min_{P \in \mathbb{R}^{n \times m}, F \in \mathbb{R}^{m \times n}} \text{Tr} (\Lambda SPS)
\]

subject to

\[
\begin{bmatrix} I \\ F \end{bmatrix} \begin{bmatrix} A^T & B^T \end{bmatrix}^T SPS \begin{bmatrix} A^T & B^T \end{bmatrix} \begin{bmatrix} I \\ F \end{bmatrix}^T + I
\preceq -\varepsilon I + SPS
\]

with a sufficiently small \(\varepsilon > 0\). Define solution \((G, P, Y) = (\bar{G}_c, \bar{P}_c, \bar{Y}_c)\) to (13). We characterize the solution to (11) and equivalently (13) as \((G, P, Y) = (\bar{G}_c, \bar{P}_c, \bar{Y}_c)\) in the limit \(\varepsilon \to 0\). Based on this definition, we can obtain an optimality of the solution to **Proposition 4**. Indeed, we prove that the feedback gain \(\bar{F}_c\) is optimal for any \(\varepsilon > 0\).

**Proposition 5.** Suppose that \((G, P, Y) = (\bar{G}_c, \bar{P}_c, \bar{Y}_c)\) is a solution of (13), and let \(\bar{F}_c = X^T_c (G^T_c)^{-1}\). The feedback gain \(\bar{F}_c\) is optimal for any \(\varepsilon > 0\).

**Proof.** Plugging \((\bar{G}_c, \bar{P}_c, \bar{Y}_c)\) into \((G, P, Y)\) in the constraint (13), we have

\[
\begin{bmatrix} I \\ \bar{F}_c \end{bmatrix} \begin{bmatrix} A^T & B^T \end{bmatrix}^T S \bar{P}_c S \begin{bmatrix} A^T & B^T \end{bmatrix} \begin{bmatrix} I \\ \bar{F}_c \end{bmatrix}^T + (1 + \varepsilon) I
\preceq S \bar{P}_c S,
\]

which is a Lyapunov inequality. Therefore, \(\bar{A}_F\) is Schur, and by the Lyapunov theorem, there exists a Lyapunov matrix

\[
S \bar{P}_c S := (1 + \varepsilon) \sum_{k=0}^{\infty} A^k_F (A^k_F)^T
\]

such that \(A_F S \bar{P}_c S A^T_F + (1 + \varepsilon) I = S \bar{P}_c S\). Obviously, \(\bar{F}_c\) is a feasible solution to (13), and hence, from the optimality of \(\bar{F}_c\), it holds that \(\text{Tr}(A_S P_S) \leq \text{Tr}(A_S \bar{P}_c S) = (1 + \varepsilon) J(\bar{F}_c)\). On the other hand, recursively applying (14) leads to

\[
S \bar{P}_c S \geq (1 + \varepsilon) \sum_{k=0}^{\infty} A^k_F (A^k_F)^T
\]

and hence, \(\text{Tr}(A_S \bar{P}_c S) \geq \text{Tr}(A_S S) = (1 + \varepsilon) J(\bar{F}_c)\). Combining the last two inequalities, we have \(\text{Tr}(A_S S) = (1 + \varepsilon) J(\bar{F}_c)\). By contradiction, assume that there exists an optimal feedback gain \(F^*\) such that

\[
(1 + \varepsilon) J(\bar{F}_c) > (1 + \varepsilon) J(F^*)
\]

Then, there exists

\[
SP^* S := (1 + \varepsilon) \sum_{k=0}^{\infty} A^k_F (A^k_F)^T
\]

such that \(A_F SP^* S A^T_F + (1 + \varepsilon) I = SP^* S\). Since \((P, F) = (P^*, F^*)\) is feasible solution to (13), we have \(\text{Tr}(A_S \bar{P}_c S) \leq \text{Tr}(A_S S) = (1 + \varepsilon) J(F^*)\). Combining the last inequality with (15), we arrive at a contradiction. Therefore, \(\bar{F}_c\) is the optimal feedback gain for any \(\varepsilon\). This completes the proof. \(\square\)

**VI. DATA-DRIVEN DYNAMIC PROGRAMMING**

Although the data-driven LMIs in the previous sections are efficient, it is still meaningful to briefly discuss and summarize dynamic programming methods [6], which does not depend on LMI solvers. The previous ideas can be extended to dynamic programming summarized in Algorithm 3 and Algorithm 4. Algorithm 3 summarizes

**Algorithm 3** Data-Driven Policy Iteration

1: Initialize \(F_0 = 0\).
2: for \(k \in \{0, 1, \ldots\}\) do
3: Collect data \((S(F_k), H(F_k)) = \text{On} - \text{Collect}(F_k)\)
4: Solve for \(P_{k+1}\) the linear equation

\[
H(F_k) P_{k+1} H(F_k) + S(F_k) A S(F_k) = S(F_k) P_{k+1} S(F_k)
\]

5: Update \(F_{k+1} = -P_{k+1}^{-1} P_{k+1}^{11} P_{k+1}^{12}\)
6: if \(|P_k - P_{k+1}| \leq \varepsilon\) then
7: Stop and return \(P_{k+1}\) and \(F_{k+1} = -P_{k+1}^{-1} P_{k+1}^{12}\)
8: end if
9: end for

a policy iteration algorithm proposed in [14] for completeness. Its convergence was also proved in [14].

**Proposition 6 (Convergence of Algorithm 3, [14]).** The iteration \(P_k\) in **Algorithm 3** converges to \(P^*\) defined in (6).

The main feature of **Algorithm 3** is that it uses on-policy data generated by **Algorithm 1**. Therefore, it needs to collect new data at every iteration, and each data collection should apply the exploring starts scheme. The newly proposed value iteration algorithm presented in **Algorithm 4** suggests an off-policy algorithm in the sense that the policy used to generate the data is independent of the policy we want to learn or the intermediate policies while learning. Therefore, it collects data once at the beginning. Moreover, it does not need to stick to the exploring starts scheme because the exploratory inputs can be used during the data collection. In this sense, the new **Algorithm 4** is more sample efficient than **Algorithm 3**. Multiplying both sides of the linear matrix equation in **Algorithm 4** by \(S\), it is reduced to

\[
P_{k+1} = \Lambda + S^{-1} H(P_{k, 11} - P_{k, 12} P_{k, 12}^{-1} P_{k, 12}^T H^T S^{-1} \)
Lemma 6

Algorithm 4 Data-Driven Value Iteration

1: Initialize $P_0 = 0$.
2: Given fixed initial state $x(0) = z$, collect data $(S, H) = \text{Off} - \text{Collect}(z)$
3: for $k \in \{0, 1, \ldots \}$ do
4: solve for $P_{k+1}$ the linear matrix equation
5: $SP_{k+1}S = SAS + H(P_{k, 11} - P_{k, 12}P_{k, 22}^{-1}P_{k, 12}^T)H^T$
6: if $\|P_k - P_{k+1}\| \leq \varepsilon$ then
7: stop and return $P_{k+1}$ and $F_{k+1} = -P_{k+1, 12}^{-1}P_{k+1, 12}^T$
8: end if
9: end

which can be interpreted as a model-based value iteration because $H^T S^{-1} = \begin{bmatrix} A & B \end{bmatrix}$ from Lemma 6. Lastly, we establish the convergence of Algorithm 4.

Proposition 7 (Convergence of Algorithm 4). The iteration $P_k$ in Algorithm 4 converges to $P^\star$.

Proof. We only need to prove that Algorithm 4 is equivalent to the Q-value iteration, which is known to converge to the optimal $P^\star$ [6]. Applying Lemma 6 and multiplying both sides of the $P$-update equation in Algorithm 4 by $S^{-1}$, we obtain

$$P_{k+1} = \Lambda + \begin{bmatrix} A & B \\ -P_{k, 22}^{-1}P_{k, 12}A & -P_{k, 22}^{-1}P_{k, 12}B \end{bmatrix}^T \times P_k \begin{bmatrix} A & B \\ -P_{k, 22}^{-1}P_{k, 12}A & -P_{k, 22}^{-1}P_{k, 12}B \end{bmatrix}$$

Multiplying both sides by $\begin{bmatrix} x \\ u \end{bmatrix}$ from the right and its transpose from the left, we have

$$Q_{k+1}(x, u) = \begin{bmatrix} x \\ u \end{bmatrix}^T \Lambda \begin{bmatrix} x \\ u \end{bmatrix} + \min_{v \in \mathbb{R}^n} Q_k(x, v)$$

with $Q_k(x, u) = \begin{bmatrix} x \\ u \end{bmatrix}^T \times P_k \begin{bmatrix} x \\ u \end{bmatrix}$. It is equivalent to the Q-value iteration [5], which is known to converge to $P^\star$, where $Q^\star(x, u) = \begin{bmatrix} x \\ u \end{bmatrix}^T \times P^\star \begin{bmatrix} x \\ u \end{bmatrix}$. This completes the proof.

Algorithm 5 Off-policy data collection $(S, H) = \text{Off - Collect2}(z)$ with restarting

1: Initialize $S_0 = 0, H_0 = 0$.
2: for $i \in \{1, 2, \ldots, N\}$ do
3: initialize $x(0; i) = z$.
4: Initialize $S_{0,i} = 0, H_{0,i} = 0$.
5: for $k \in \{0, 1, \ldots, n - 1\}$ do
6: apply control input $u(k; i) = \zeta(k; i), \zeta(k; i) \sim \mathcal{N}(0, U)$
7: observe $x(k + 1; i)$
8: update
9: $\tilde{S}_{k+1,i} \leftarrow \tilde{S}_{k,i} + \begin{bmatrix} x(k; i) \\ u(k; i) \end{bmatrix} \begin{bmatrix} x(k; i) \\ u(k; i) \end{bmatrix}^T$
10: $\tilde{H}_{k+1,i} \leftarrow \tilde{H}_{k,i} + \begin{bmatrix} x(k; i) \\ u(k; i) \end{bmatrix} x(k + 1; i)^T$
11: end for
12: return $(S, H) = (S_N, H_N)$

then the data collection strategy guarantees that $S_N$ converges to a strictly positive definite matrix with probability one as $N \to \infty$.

Theorem 1. Suppose that $(A, B)$ is controllable, and consider Algorithm 5, whose output is

$$S_N = \frac{1}{N} \sum_{i=1}^{N} \left( \sum_{k=0}^{n} \begin{bmatrix} x(k; i) \\ u(k; i) \end{bmatrix} \begin{bmatrix} x(k; i) \\ u(k; i) \end{bmatrix}^T \right)$$

where $x(k; i)$ and $u(k; i)$ stand for the state and input at time $k$ at the $i$th outer iteration. Then, we have

$$\mathbb{P} \left[ \lim_{N \to \infty} S_N \succ 0 \right] = 1$$

Proof. Define

$$Q_k := \begin{bmatrix} B & AB & \cdots & A^{k-1}B \end{bmatrix}$$

and

$$U_k := \text{diag}(U, \ldots, U)_{k\text{-times}}$$

and

$$\zeta(k; i) = \begin{bmatrix} \zeta(k; i) \\ \zeta(1; i) \\ \zeta(0; i) \end{bmatrix}$$

Then, $x(k; i)$ is expressed as $x(k; i) = A^k z + O_k u_{k,i}$, and thus

$$x(k; i)^T x(k; i)^T = A^k z^T z (A^T)^k + 2A^k z u_{k,i}^T O_k^T + O_k u_{k,i} u_{k,i}^T O_k^T$$

Taking the expectation leads to

$$\mathbb{E}[x(k; i)^T x(k; i)^T] = A^k z z^T (A^T)^k + O_k H_k O_k^T$$

At $k = n$, $O_n$ is the controllability matrix, and it is full row rank due to the controllability in Assumption 1. Since $U_k \succ 0$, one concludes $\mathbb{E}[x(n; i)^T x(n; i)^T] > 0$. Since $u(k; i)$ is i.i.d. and the initial state is reset periodically after $n$ steps, $S_N$ is written as

$$S_N = \frac{1}{N} \sum_{i=1}^{N} \sum_{k=0}^{n} \begin{bmatrix} x(k; i) \\ u(k; i) \end{bmatrix} \begin{bmatrix} x(k; i) \\ u(k; i) \end{bmatrix}^T = \frac{1}{N} \sum_{i=1}^{N} M_i$$

VII. Exploration schemes

For the off-policy data collection, Algorithm 1, the exploring starts always guarantee $S(F) \succ 0$. However, collecting the trajectories with different initial points which span $\mathbb{R}^n$ may not be tractable in practice. The off-policy data collection, Algorithm 2, is relatively more promising in this respect, because it can use the exploratory inputs while generating the trajectories, and can be used in the case that the initial state is given and fixed. We can apply an arbitrary inputs, $u(k)$, and expect that $S \succ 0$ eventually under the persistent excitation assumption. A standard exploration strategy is to inject the i.i.d. Gaussian noises, $u(k) \sim \mathcal{N}(0, U)$, where $U \in \mathbb{R}^{n \times n}$ is the covariance matrix. If trajectories starting from the fixed $x(0) = z$ can be collected as many as possible, then we can develop a new version of the off-policy exploration strategy given in Algorithm 5, which offers theoretical guarantees of the data validity under a mild assumption, i.e., the controllability.

In Algorithm 5, $N$ trajectories are collected and then averaged, i.e., $S_N = \frac{1}{N} \sum_{i=1}^{N} \tilde{S}_{n,i}, H_N = \frac{1}{N} \sum_{i=1}^{N} \tilde{H}_{n,i}$. Each trajectory starts from $x(0) = z$ which is fixed. We can easily prove that the data matrices from Algorithm 5 also satisfies the data transformation property Lemma 6. We can also prove that if $(A, B)$ is controllable,
where
\[ M_i = \sum_{k=0}^{n} \begin{bmatrix} x(k;i) \\ u(k;i) \end{bmatrix}^T \]
is an i.i.d. random variables with mean
\[ \mathbb{E}[M_i] = M := \sum_{k=0}^{n} \begin{bmatrix} A^k z \gamma^T (A^T)^k + \mathcal{O}_k u_k \mathcal{O}_k^T & 0 \\ 0 & U \end{bmatrix} > 0 \]
By the strong law of large numbers, we get \( \lim_{N \to \infty} S_N = M = 1 \), which leads to the desired conclusion.

Algorithm 5 provides a data collection scheme with theoretical guarantees of the validity of the data. It is useful especially when the exploring starts scheme (starting with arbitrary initial states) is not available. However, it still requires the ability to generate \( N \) trajectories from the given initial state \( z \). In practice, if only a single trajectory starting from a fixed \( z \) is available, we can develop another data acquisition method given in Algorithm 6. The benefit comes from some cost to pay. In particular, it initially needs a stabilizing state-feedback gain \( K \) or at least, the system \( A \) itself needs to be stable. In such cases, we can approximately mimic the restarting strategy in Algorithm 5 using the stability of the closed-loop system \( A + BK \). Algorithm 6 will be called the off-policy data collection with periodic excitation. The main feature of Algorithm 6 lies in that injecting Gaussian noises in the input. As in Theorem 1, we can prove that Algorithm 6 theoretically ensures the validity of the data output provided that \( (A, B) \) is controllable.

**Theorem 2.** Suppose that \( (A, B) \) is controllable, and consider Algorithm 5, whose output is
\[ S_N = \frac{1}{N} \sum_{i=1}^{N} \sum_{k=0}^{n} \begin{bmatrix} x(k;i) \\ u(k;i) \end{bmatrix}^T \]
where \( x(k;i) \) and \( u(k;i) \) stand for the state and input, respectively, at time \( k \) at the \( i \)th outer iteration. Then, there exists a sufficient \( \varepsilon > 0 \) such that
\[ \mathbb{P} \left[ \lim_{N \to \infty} S_N > 1 \right] = 1 \]
**Proof.** Define
\[ \mathcal{O}_k := \left[ B 
( A + BK)B 
( A + BK)^2 
\cdots 
( A + BK)^{k-1}B \right] \]
Then, the state at time \( k \) is \( x(k;i) = (A + BK)^k z_i + \mathcal{O}_k u_{k;i} \), where \( z_i \) is the initial state, \( x(0;i) = z_i \) at the \( i \)th period such that \( \|z_i\| \leq \varepsilon \), and \( u_{k;i} \) is defined in (17). Then, one gets
\[ x(k;i)x(k;i)^T = (A + BK)^k z_i z_i^T (A + BK)^k + 2(A + BK)^k z_i u_{k;i}\mathcal{O}_k^T + \mathcal{O}_k u_{k;i}^T \mathcal{O}_k^T \]
which is lower bounded by
\[ x(k;i)x(k;i)^T \geq (A + BK)^k z_i z_i^T (A + BK)^k + 2(A + BK)^k z_i u_{k;i}\mathcal{O}_k^T + \mathcal{O}_k u_{k;i}^T \mathcal{O}_k^T \]
where Lemma 3 was applied to (18). Again, the last bound is further bounded from below as
\[ x(k;i)x(k;i)^T \]
\[ \geq (A + BK)^k z_i z_i^T (A + BK)^k + 2(A + BK)^k z_i u_{k;i}\mathcal{O}_k^T + \mathcal{O}_k u_{k;i}^T \mathcal{O}_k^T \]
where \( \lambda_{\text{max}}(\cdot) \) denotes the maximum eigenvalue of a symmetric matrix, and the last inequality uses the fact that \( \|z_i\| \leq \varepsilon \).

On the other hand, noting \( x(k;i)u(k;i)^T = (A + BK)^k z_i u_{k;i}^T + \mathcal{O}_k u_{k;i}^T \mathcal{O}_k^T \), we have
\[ \begin{bmatrix} 0 \\ u(k;i)x(k;i)^T \end{bmatrix} \]
\[ \begin{bmatrix} 0 \\ u(k;i)z_i^T (A + BK)^k \end{bmatrix} \]
\[ \begin{bmatrix} 0 \\ u(k;i)u_{k;i}^T \end{bmatrix} \]
\[ \begin{bmatrix} 0 \\ \mathcal{O}_k u_{k;i}^T \mathcal{O}_k^T \end{bmatrix} \]
Lemma 3

...schemes to acquire valid data from the trajectories under different positive definite matrix with probability one. Where $U(\varepsilon)$ numbers, with $\|x\| \geq \varepsilon$. Therefore, for a sufficiently small $\varepsilon$, $S_N$ converges to a positive definite matrix with probability one.

VIII. Conclusion

We have developed data-driven control evaluation and design strategies based on LMI and dynamic programming, where stabilization and LQR problems are addressed. Efficient data collection schemes have been investigated. Finally, we investigate exploration schemes to acquire valid data from the trajectories under different scenarios with theoretical guarantees of convergence. In particular, we prove that as more data is accumulated, the collected data becomes valid for the proposed algorithms with higher probability.

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