A DIMENSION DROP PHENOMENON OF FRACTAL CUBES

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ABSTRACT. Let $E$ be a metric space. We introduce a notion of connectedness index of $E$, which is the Hausdorff dimension of the union of non-trivial connected components of $E$. We show that the connectedness index of a fractal cube $E$ is strictly less than the Hausdorff dimension of $E$ provided that $E$ possesses a trivial connected component. Hence the connectedness index is a new Lipschitz invariant. Moreover, we investigate the relation between the connectedness index and topological Hausdorff dimension.

1. Introduction

An iterated function system (IFS) is a family of contractions $\{\varphi_j\}_{j=1}^N$ on $\mathbb{R}^d$, and the attractor of the IFS is the unique nonempty compact set $K$ satisfying $K = \bigcup_{j=1}^N \varphi_j(K)$, and it is called a self-similar set [5]. Let $n \geq 2$ and let $\mathcal{D} = \{d_1, \ldots, d_N\} \subseteq \{0,1,\ldots,n-1\}^d$, which we call a digit set. Denote by $\#\mathcal{D} := N$ the cardinality of $\mathcal{D}$. Then $n$ and $\mathcal{D}$ determine an IFS $\{\varphi_j(z) = \frac{1}{n}(z + d_j)\}_{j=1}^N$, whose attractor $E = E(n, \mathcal{D})$ satisfies the set equation

$$E = \frac{1}{n}(E + \mathcal{D}).$$

We call $E$ a fractal cube [12], especially, when $d = 2$, we call $E$ a fractal square [6].

There are some works on topological and metric properties of fractal cubes. Whyburn [11] studied the homeomorphism classification, Bonk and Merenkov [2] studied the quasi-symmetric classification. Lau, Luo and Rao [6] studied when a fractal square is totally disconnected. Xi and Xiong [12] gave a complete classification of Lipschitz equivalence of fractal cubes which are totally disconnected. Recently, the studies of [10,13] focus on the the Lipschitz equivalence of fractal squares which are not totally disconnected.

Topological Hausdorff dimension is a new fractal dimension introduced by Buczolich and Elekes [1]. It is shown in [1] that for any set $K$ we always have $\dim_t H K \leq \dim_H K$, where $\dim_t H$ and $\dim_H$ denote the topological Hausdorff dimension and Hausdorff dimension respectively. Ma and Zhang [8] calculated topological Hausdorff dimensions of a class of fractal squares.

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Let $K$ be a metric space. A point $x \in K$ is called a trivial point of $K$ if $\{x\}$ is a connected component of $K$. Let $\Lambda(K)$ be the collection of trivial points in $K$. Denote

\[ \mathcal{I}_c(K) := \dim_H K \setminus \Lambda(K), \]

and we call it the connectedness index of $K$. It is obvious that $\mathcal{I}_c(K) \leq \dim_H K$. Clearly, the connectedness index is a Lipschitz invariant. The main results of the present paper are as follows.

**Theorem 1.1.** Let $E = E(n, \mathcal{D})$ be a $d$-dimensional fractal cube. If $E$ has a trivial point, then $\mathcal{I}_c(E) < \dim_H E$.

However, Theorem 1.1 is not valid for general self-similar sets, even if the self-similar sets satisfy the open set condition.

**Example 1.1.** Let $Q = \{0\} \cup \left( \bigcup_{k=0}^{\infty} \left[ \frac{1}{2^{k+2}}, \frac{1}{2^{k}} \right] \right)$. Observe that $Q = \frac{Q}{4} \cup \left[ \frac{1}{2}, 1 \right]$ and $\frac{Q}{2} \cup Q = [0, 1]$. Then $Q$ is a self-similar set satisfying the equation

\[ Q = \frac{Q}{4} \cup \left( \frac{Q}{4} + \frac{1}{2} \right) \cup \left( \frac{Q}{2} + \frac{1}{2} \right). \]

The set $Q$ has only one trivial point, that is 0. Therefore, $\mathcal{I}_c(Q) = \dim_H Q = 1$. Figure 1 illustrates $Q'$, a two dimensional generalization of $Q$. Similarly, $Q'$ is a self-similar set, and the unique trivial point of $Q'$ is 0.

![Figure 1. The self-similar set $Q'$.](image)

Using Theorem 3.7 of [1] we show the following.

**Theorem 1.2.** For a non-empty $\sigma$-compact metric space $K$, we have $\dim_{tH} K \leq \mathcal{I}_c(K)$.

Zhang [14] asked when $\dim_{tH} E = \dim_H E$, where $E$ is a fractal square. According to [14], a digit set $\mathcal{D}$ is called a Latin digit set, if every row and every column has the same number of elements (see Figure 2). For a fractal square $E = E(n, \mathcal{D})$, Zhang showed that if $\dim_{tH} E = \dim_H E$, then either $E = [0, 1] \times C$, or $E = C \times [0, 1]$ for some $C \subset [0, 1]$, or $\mathcal{D}$ is a Latin digit set.

As a corollary of Theorem 1.1 and Theorem 1.2, we obtain a new necessary condition for $\dim_H E = \dim_{tH} E$. 

\[ \]
It is shown in [14] that \( \log 12/\log 6 = \dim H L < \dim tH L = \log 24/\log 6 \). While by Theorem 1.1 and Theorem 1.2, we directly have \( \dim tH L < \dim H L \).

**Corollary 1.1.** Let \( E \) be a \( d \)-dimensional fractal cube. If \( \dim H E = \dim tH E \), then \( E \) has no trivial point.

**Remark 1.1.** Another application of Theorem 1.1 is on the gap sequences of fractal cubes, a Lipschitz equivalent invariant introduced by Rao, Ruan and Yang [9]. For a fractal cube \( K \), let \( \{g_m(K)\}_{m\geq 1} \) be the gap sequence. Using Theorem 3.1 of the present paper, it is proved in [4] that if \( K \) has trivial point, then \( \{g_m(K)\}_{m\geq 1} \) is equivalent to \( \{m^{-1/\gamma}\}_{m\geq 1} \), where \( \gamma = \dim H K \).

Finally, we calculate the connectedness indexes of two fractal squares in Figure 3, and illustrate the application to Lipschitz classification.

**Example 1.2.** Let \( K \) and \( K' \) be two fractal squares indicated by Figure 3. It is seen that \( \dim H K = \dim H K' = \frac{\log 14}{\log 5} \). By Theorem 1.3 of [8], one can obtain that \( \dim tH K = \dim tH K' = 1 + \frac{\log 2}{\log 5} \). We will show in section 5 that

\[
\mathcal{I}_c(K) = \frac{\log(8 + \sqrt{132}/2)}{\log 5} \quad \text{and} \quad \mathcal{I}_c(K') = \frac{\log 13}{\log 5}.
\]

So \( K \) and \( K' \) are not Lipschitz equivalent.

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![Figure 2](image1.png)

**Figure 2.**

![Figure 3](image2.png)

**Figure 3.**

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This article is organized as follows. In section 2, we recall some basic facts of \( r \)-face of the polytope \([0, 1]^d\). In section 3, we prove Theorem 1.1. In section 4, we prove Theorem 1.2. In section 5, we give the details of Example 1.2.

2. Preliminaries on \( r \)-faces of \([0, 1]^d\)

We recall some notions about convex polytopes, see [15]. Let \( C \subset \mathbb{R}^d \) be a convex polytope, let \( F \) be a convex subset of \( C \). The affine hull of \( F \), denoted by \( \text{aff}(F) \), is the smallest affine subspace containing \( F \). We say \( F \) is a face of \( C \), if any closed line segment in \( C \) with a relative interior in \( F \) has both endpoints in \( F \).

The dimension of an affine subspace is defined to be the dimension of the corresponding linear vector space. The dimension of a face \( F \), denoted by \( \dim F \), is the dimension of its affine hull. Moreover, \( F \) is called an \( r \)-face of \( C \), if \( F \) is a face of \( C \) with dimension \( r \). We take it by convention that \( C \) is a \( d \)-face of itself if \( \dim C = d \).

For \( z \in C \), a face \( F \) of \( C \) is called the containing face of \( z \) if \( z \) is a relative interior point of \( F \).

Let \( e_1, \ldots, e_d \) be the canonical basis of \( \mathbb{R}^d \). The following facts about the \( r \)-faces of \([0, 1]^d\) are obvious, see Chapter 2 of [15].

**Lemma 2.1.** (i) Let \( A \cup B = \{1, \ldots, d\} \) be a partition with \( \#A = r \). Then the set

\[
F = \left\{ \sum_{j \in A} c_j e_j; \ c_j \in [0, 1] \right\} + b
\]

is an \( r \)-face of \([0, 1]^d\) if and only if \( b \in T \), where

\[
T := \left\{ \sum_{j \in B} \epsilon_j e_j; \ \epsilon_j \in \{0, 1\} \right\};
\]

(ii) For any \( r \)-face \( F \) of \([0, 1]^d\), there exists a partition \( A \cup B = \{1, \ldots, d\} \) with \( \#A = r \) such that \( F \) can be written as (2.1).

We will call \( F_0 = \{ \sum_{j \in A} c_j e_j; \ c_j \in [0, 1] \} \) a basic \( r \)-face related to the partition \( A \cup B \). We give a partition \( B = B_0 \cup B_1 \) according to \( b \) by setting

\[
B_0 = \{ j \in B; \ \text{the } j\text{-th coordinate of } b \text{ is 0} \},
B_1 = \{ j \in B; \ \text{the } j\text{-th coordinate of } b \text{ is 1} \}.
\]

Let \( x = \sum_{j \in A} \alpha_j e_j + \sum_{i \in B} \beta_i e_i \in [0, 1]^d \), we define two projection maps as follows:

\[
\pi_A(x) = \sum_{j \in A} \alpha_j e_j, \quad \pi_B(x) = \sum_{i \in B} \beta_i e_i.
\]

If \( F \) is an \( r \)-face of \([0, 1]^d\), we denote by \( \hat{F} \) the relative interior of \( F \).

**Lemma 2.2** (Chapter 2 of [15]). Let \( C \subset \mathbb{R}^d \) be a polytope.

(i) If \( G \) and \( F \) are faces of \( C \) and \( F \subset G \), then \( \hat{F} \) is a face of \( G \).

(ii) If \( G \) is a face of \( C \), then any face of \( G \) is also a face of \( C \).
The following lemma will be needed in section 3.

**Lemma 2.3.** Let \( F = F_0 + b \) be an \( r \)-face of \([0, 1]^d\) given by (2.1). Let \( u \in \mathbb{Z}^d \). Then \( F \cap (u + [0, 1]^d) \neq \emptyset \) if and only if \( u = b - b' \) for some \( b' \in T \), where \( T \) is defined in (2.2).

**Proof.** 

\( \Rightarrow \) Suppose \( b' \in T \), then \( F - (b - b') = F_0 + b' \), and it is an \( r \)-face of \([0, 1]^d\) by Lemma 2.1 (i). Applying a translation \( b - b' \) we see that \( F \subset (b - b') + [0, 1]^d \), which completes the proof of the sufficiency.

\( \Leftarrow \) Suppose \( F \cap (u + [0, 1]^d) \neq \emptyset \). Let \( z_0 \) be a point in the intersection and let \( F' = [0, 1]^d \cap (u + [0, 1]^d) \). Then \( F' \) is a face of both \([0, 1]^d\) and \( u + [0, 1]^d \). So we have \( F \subset F' \) since \( F' \) contains \( z_0 \), a relative interior point of \( F \). Hence \( F \) is an \( r \)-face of \( F' \) by Lemma 2.2 (i). It follows that \( F - u \) is an \( r \)-face of \( F' - u \).

Notice that \( F' \) is a face of \( u + [0, 1]^d \), then \( F' - u \) is a face of \([0, 1]^d\). By Lemma 2.2 (ii), \( F - u = F_0 + (b - u) \) is an \( r \)-face of \([0, 1]^d\). By Lemma 2.1 (i) we have \( b - u \in T \).

\[ \square \]

3. Trivial points of fractal cubes

Let \( \Sigma = \{1, 2, \ldots, N\} \). Denote by \( \Sigma^\infty \) and \( \Sigma^k \) the sets of infinite words and words of length \( k \) over \( \Sigma \) respectively. Let \( \Sigma^* = \bigcup_{k \geq 0} \Sigma^k \) be the set of all finite words. For any \( \sigma = \sigma_1 \ldots \sigma_k \in \Sigma^k \), let \( \varphi_\sigma = \varphi_{\sigma_1} \circ \cdots \circ \varphi_{\sigma_k} \).

In this section, we always assume that \( E = E(n, D) \) is a \( d \)-dimensional fractal cube defined in (1.1) with IFS \( \{\varphi_j\}_{j \in \Sigma} \). In the following, we always assume that

For a point \( z \in E \), we say \( F \) is the containing face of \( z \) means that \( F \) is a face of the polytope \([0, 1]^d\) and it is the containing face of \( z \).

**Lemma 3.1.** Let \( z_0 \in E \) and \( \sigma \in \Sigma^k \) for some \( k > 0 \). Let \( F \) be the containing face of \( z_0 \), let \( F' \) be the containing face of \( \varphi_\sigma(z_0) \). Then either \( \varphi_\sigma(z_0) \in F \) or \( \dim F' \geq \dim F + 1 \).

**Proof.** Let \( A \cup B \) be the partition in Lemma 2.1 (i) which defines \( F \). By the definition of containing face, we have \( z_0 \in \hat{F} \). Suppose that \( \varphi_\sigma(z_0) \notin F \).

Take any point \( x \in F \setminus \{z_0\} \) and let \( I \) be the closed line segment in \( F \) such that \( x \) is an endpoint of \( I \) and \( z_0 \) is a relative interior point of \( I \). It is clear that \( \varphi_\sigma(I) \subset \varphi_\sigma([0, 1]^d) \subset [0, 1]^d \). Since \( \varphi_\sigma(z_0) \in F' \), we have \( \varphi_\sigma(I) \subset F' \). By the arbitrary of \( x \) we deduce that \( \varphi_\sigma(F) \subset F' \), hence

\[ \dim F' \geq \dim \varphi_\sigma(F) = \dim F. \]

We claim that \( F' \) is not an \( r \)-face of \([0, 1]^d\). This claim together with (3.1) imply \( \dim F' \geq \dim F + 1 \).

Suppose on the contrary that \( F' \) is an \( r \)-face of \([0, 1]^d\). Then there exists a partition \( A' \cup B' = \{1, \ldots, d\} \) such that \( F' = F_0' + b' \), where \( F_0' = \bigcup_{j \in A'} c_j e_j; \ c_j \in [0, 1] \) and \( b' \in \{ \sum_{j \in B'} \varepsilon_j e_j; \ \varepsilon_j \in \{0, 1\} \} \). Since

\[ \frac{F_0}{n^k} + \frac{b}{n^k} + \varphi_\sigma(0) = \varphi_\sigma(F) \subset F' = F_0' + b', \]
we have $F'_0 = F_0$. Hence $A' = A$ and $B' = B$. It follows that
\[(3.2) \quad b' = \pi_B(\varphi_\sigma(z_0)) = \frac{b}{n^k} + \pi_B(\varphi_\sigma(0)) \in T.\]

Notice that
\[(3.3) \quad \pi_B(\varphi_\omega(0)) \in \left\{ \sum_{j \in B} c_j e_j; \ c_j \in [0, \frac{n^k - 1}{n^k}] \right\}\]
for any $\omega \in \Sigma^k$, which together with (3.2) imply that $\pi_B(\varphi_\sigma(0)) = \frac{(n^k - 1)}{n^k} b$. Hence $b' = b$ and it follows that $\varphi_\sigma(z_0) \in F$, a contradiction. The claim is confirmed and the lemma is proven. $\square$

For each $\sigma = \sigma_1 \ldots \sigma_k \in \Sigma^k$, we call $\varphi_\sigma([0,1]^d) \subset E_k$ a $k$-th cell of $E_k$. Denote
\[(3.4) \quad \Sigma_\sigma = \{ \omega \in \Sigma^k; \pi_A(\varphi_\omega(0)) = \pi_A(\varphi_\sigma(0)) \}\]
and set
\[(3.5) \quad H_\sigma = \bigcup_{\omega \in \Sigma_\sigma} \varphi_\omega([0,1]^d).\]

Indeed, $H_\sigma$ is the union of all $k$-th cells having the same projection with $\varphi_\sigma([0,1]^d)$ under $\pi_A$. From now on, we always assume that
\[(3.6) \quad z_0 \text{ is a trivial point of } E \text{ and } F \text{ is the containing face of } z_0.\]

**Lemma 3.2.** Let $k > 0$, fix $\sigma \in \Sigma^k$. If $H_\sigma$ is not connected or $H_\sigma \cap F = \emptyset$, then there exists $\omega^* \in \Sigma_\sigma$ such that $\varphi_{\omega^*}(z_0) \notin F$ and it is a trivial point of $E$.

**Proof.** Let $\dim F = r$ and let $A \cup B$ be the partition in Lemma 2.1 (i) which defines $F$.

We claim that if $H_\sigma \cap F \neq \emptyset$, then there is only one $k$-th cell in $H_\sigma$ which intersects $F$. Actually, since $\varphi_\omega(0) \in [0, \frac{n^k - 1}{n^k}]^d$ for any $\omega \in \Sigma^k$, if $\varphi_\omega([0,1]^d) \cap F \neq \emptyset$ for some $\omega \in \Sigma_\sigma$, then similar to the proof of Lemma 3.1 we must have $\pi_B(\varphi_\omega(0)) = \frac{(n^k - 1)}{n^k} b$. On the other hand, $\pi_A(\varphi_\omega(0)) = \pi_A(\varphi_\sigma(0))$, so $\omega$ is unique in $\Sigma_\sigma$. Furthermore, $\varphi_\omega(z_0) \in F$ in this scenario.

By the assumption of the lemma and the claim above, there is a connected component $U$ of $H_\sigma$ such that $U \cap F = \emptyset$. Let $W = \{ \omega \in \Sigma_\sigma; \varphi_\omega([0,1]^d) \subset U \}$. For each $\omega \in W$, write
\[
\pi_B(\varphi_\omega(0)) = \sum_{j \in B_0} \alpha_j(\omega)e_j + \sum_{j \in B_1} \beta_j(\omega)e_j
\]
First, we take the subset $W' \subset W$ by
\[
W' = \left\{ \omega \in W; \sum_{j \in B_0} \alpha_j(\omega) \text{ attains the minimum} \right\}.
\]
Then we take $\omega^* \in W'$ such that
\[
\sum_{j \in B_1} \beta_j(\omega^*) = \max_{\omega \in W'} \left\{ \sum_{j \in B_1} \beta_j(\omega); \ \omega \in W' \right\}.
\]
Since $U \cap F = \emptyset$, we have $\varphi_\sigma(z_0) \notin F$.

Let us check that $\varphi_\sigma(z_0)$ is a trivial point of $E$. To this end, we only need to show that

\begin{equation}
\varphi_\sigma(z_0) \notin \varphi_\sigma([0,1]^d),
\end{equation}

where $\omega \in \Sigma^k \setminus \{\omega^*\}$. Notice that $\varphi_\sigma(z_0) \in \varphi_\sigma(\bar{F})$, it is clear that (3.7) holds for any $\omega \notin \Sigma_\sigma$. Since $U$ is a connected component of $H_\sigma$, we see that (3.7) holds for any $\omega \notin W$.

Now suppose $\varphi_\sigma(z_0) \in \varphi_\sigma([0,1]^d)$ for some $\omega \in W$, then $\varphi_\sigma(z_0) \in \varphi_\sigma([0,1]^d) \cap \varphi_\sigma([0,1]^d)$. By Lemma 2.3 we have

\[ \pi_B(\varphi_\sigma(z_0)) - \pi_B(\varphi_\sigma(z_0)) = \pi_B(\varphi_\sigma(0)) - \pi_B(\varphi_\sigma(0)) \leq \frac{b - b'}{n^k}, \]

where $b' \in T$. By the definition of $B_0$ and $B_1$ in (2.3), we know that the $j$-th coordinate of $b - b'$ is 0 or -1 if $j \in B_0$ and is 0 or 1 if $j \in B_1$. According to the choosing process of $\omega^*$, on one hand, we have $\sum_{j \in B_0} (\alpha_j(\omega) - \alpha_j(\omega^*)) \geq 0$. So $\alpha_j(\omega) = \alpha_j(\omega^*)$ for $j \in B_0$, that is to say, $\omega \in W'$. On the other hand, since $\omega \in W'$, we have $\sum_{j \in B_1} (\beta_j(\omega) - \beta_j(\omega^*)) \leq 0$, which forces that $\beta_j(\omega) = \beta_j(\omega^*)$ for $j \in B_1$. Therefore, $b = b'$ and hence $\omega = \omega^*$. This finishes the proof. \hfill \Box

For $k > 0$, denote $D_k = D + nD + \cdots + n^{k-1}D$. We call $E_k = ([0,1]^d + D_k)/n^k$ the $k$-th approximation of $E$. Clearly, $E_k \subset E_{k-1}$ for all $k \geq 1$ and $E = \bigcap_{k=0}^{\infty} E_k$. For $\sigma = (\sigma_i)_{i \geq 1} \in \Sigma^\infty$, we denote $\sigma|_k = \sigma_1 \cdots \sigma_k$ for $k > 0$. We say $\sigma$ is a coding of a point $x \in E$ if $\{x\} = \bigcap_{k \geq 1} \varphi_{\sigma_1 \cdots \sigma_k}(E)$.

**Definition 3.1.** Let $U$ be a connected component of $E_k$, we call $U$ a $k$-th island if $U \cap \partial[0,1]^d = \emptyset$.

**Lemma 3.3.** If $E_k$ contains a $k$-th island for some $k > 0$, then $E$ has a trivial point.

**Proof.** Since we can regard $E(n,D)$ as $E(n^k,D_k)$, without loss of generality, we assume that $E_1$ has an island and denote it by $U$. Write $U = \bigcup_{j \in J} \varphi_j([0,1]^d)$, where $J \subset \Sigma$. We call a letter $j \in J$ a special letter. A sequence $\sigma = (\sigma_i)_{i \geq 1} \in \Sigma^\infty$ is called a special sequence, if special letters occur infinitely many times in $\sigma$.

Let

\begin{equation}
P = \{x \in E; \text{ at least one coding of } x \text{ is a special sequence}\}.
\end{equation}

We claim that every point in $P$ is a trivial point. Let $z \in P$ and let $\sigma = (\sigma_i)_{i \geq 1}$ be a coding of $z$ such that $\sigma$ is a special sequence. Suppose $\sigma_k$ is a special letter, it is easy to see that $z \in \varphi_{\sigma_1 \cdots \sigma_k}([0,1]^d) \subset \varphi_{\sigma_1 \cdots \sigma_{k-1}}(U)$ and $\varphi_{\sigma_1 \cdots \sigma_{k-1}}(U)$ is a connected component of $E_k$ with $\text{diam}(\varphi_{\sigma_1 \cdots \sigma_{k-1}}(U)) \leq \sqrt{d}/n^{k-2}$. Notice that special letters occur infinitely often in $\sigma$, we conclude that $z$ is a trivial point. \hfill \Box

**Theorem 3.1.** Let $E$ be a fractal cube with $\text{dim aff}(E) = d$. Then $E$ has a trivial point if and only if $E_k$ contains a $k$-th island for some $k \geq 1$. 

Proof. Let $z_0 \in E$ be a trivial point. We claim that there exists another trivial point $z^* \in E \cap (0,1)^d$, that is, the dimension of the containing face of $z^*$ is $d$.

Suppose $F$ is the containing face of $z_0$ with $\dim F = r$, where $0 \leq r \leq d - 1$. Let $A \cup B$ be the partition in Lemma 2.1 (i) which defines $F$. Let $\sigma = (\sigma_k)_{k \geq 1} \in \Sigma^\infty$ be a coding of $z_0$. Then for each $k > 0$, $z_0 \in H_{\sigma_k} \cap F$, where $H_{\sigma_k}$ is defined in (3.5). We will show by two cases that $E$ contains another trivial point of the form $\varphi_\omega(z_0), \omega \in \Sigma^*$, and it is not in $F$.

Case 1. $H_{\sigma_k}$ is not connected for some $k > 0$.

By Lemma 3.2, there exists $\omega^* \in \Sigma_{\sigma_k}$ such that $z_1 = \varphi_{\omega^*}(z_0) \notin F$ is a trivial point of $E$.

Case 2. $H_{\sigma_k}$ is connected for all $k > 0$.

Let $p > 0$ be an integer such that $C_p$ is the connected component of $E_p$ containing $z_0$ and $\text{diam}(C_p) < \frac{1}{3}$. It is clear that $H_{\sigma_k} \subset C_p$, so we have $\text{diam}(H_{\sigma_k}) < \frac{1}{3}$. Since $\dim \text{aff}(E) = d$, there exist $j \in \Sigma$ such that

$$\varphi_j([0,1]^d) \cap F = \emptyset. \quad (3.9)$$

We consider the set $H_{j_{\sigma_1 \ldots \sigma_p}}$. Let $\Sigma_j = \{i \in \Sigma; \pi_A(\varphi_i(0)) = \pi_A(\varphi_j(0))\}$. It is easy to see that $H_{j_{\sigma_1 \ldots \sigma_p}} = \bigcup_{i \in \Sigma_j} \varphi_i(H_{\sigma_i})$.

If $\#\Sigma_j = 1$, then $H_{j_{\sigma_1 \ldots \sigma_p}} \cap F = \varphi_j(H_{\sigma_1}) \cap F = \emptyset$. If $\#\Sigma_j > 1$, we have $\varphi_i(H_{\sigma_1}) \cap \varphi_i'(H_{\sigma_1}) = \emptyset$ for any $i, i' \in \Sigma_j$ since $\text{diam}(H_{\sigma_1}) < \frac{1}{3}$. Hence $H_{j_{\sigma_1 \ldots \sigma_p}}$ is not connected. So by Lemma 3.2, there exists $\omega^* \in \Sigma_{j_{\sigma_1 \ldots \sigma_p}}$ such that $\varphi_{\omega^*}(z_0) \notin F$ and it is a trivial point of $E$.

Then by Lemma 3.1, the containing face of this trivial point has dimension no less than $r + 1$. Inductively, we can finally obtain a trivial point $z^*$ whose containing face is $[0,1]^d$. The claim is proved.

Now suppose on the contrary that $E_k$ contains no $k$-th island for all $k \geq 1$. We will derive a contradiction. Let $z^* \in E \cap (0,1)^d$ be a trivial point. Let $U_k$ be the connected component of $E_k$ containing $z^*$, then we have $U_k \cap \partial[z,0,1]^d \neq \emptyset$. By the Weierstrass-Balzano property of the Hausdorff metric, there exists a subsequence $k_j$ such that $U_{k_j}$ converge. We denote $U^*$ to be the limit. On one hand, $U^*$ is connected since $U_{k_j}$ is connected for each $k_j$. On the other hand, $z^* \in U^*$ and $U^* \cap \partial[z,0,1]^d \neq \emptyset$. So $U^*$ is a non-trivial connected component of $E$ containing $z^*$, a contradiction. This together with Lemma 3.3 finish the proof of the theorem. \hfill \Box

Proof of Theorem 1.1. First, let us assume $\dim \text{aff}(E) = d$. Since $E$ contains a trivial point, by Theorem 3.1, there exists $k > 0$ such that $E_k$ contains a $k$-th island. Without lose of generality, suppose $E_1$ has an island $C$. Write $C = \bigcup_{j \in J} \varphi_j([0,1]^d)$, where $J \subset \Sigma$. Let $P$ be defined as (3.8). It has been proved in Lemma 3.3 that every point in $P$ is a trivial point.

We denote $P^c = E \setminus P$. Let $\mathcal{D}' = \mathcal{D} \setminus \{d_j; j \in J\}$ and let $E'$ be the fractal cube determined by $n$ and $\mathcal{D}'$. It is easy to see that

$$P^c = \bigcup_{k=0}^{\infty} \bigcup_{\sigma_1 \ldots \sigma_k \in \Sigma^k, \sigma_k \in J} \varphi_{\sigma_1 \ldots \sigma_k}(E') \subset \bigcup_{\sigma \in \Sigma^*} \varphi_\sigma(E').$$
Consequently, $\dim H P^c \leq \dim H E^\prime = \frac{\log \# D'}{\log n} < \dim H E$. Notice that $E \setminus \Lambda(E) \subset P^c$, we have $\mathcal{I}_c(E) = \dim H E \setminus \Lambda(E) < \dim H E$.

Next, assume that $\dim \text{aff}(E) < d$. Then there exist $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{R}^d \setminus \{0\}$ and $c \in \mathbb{R}$ such that
\begin{equation}
(3.10) \quad \langle x, \alpha \rangle = c, \quad \forall x \in E.
\end{equation}
Without loss of generality, we may assume that $\alpha_1 \neq 0$. Since $x + h n \in E$ for any $x \in E$ and any $h \in D$, we deduce that
\begin{equation}
(3.11) \quad \langle h, \alpha \rangle = (n - 1)c.
\end{equation}
Let $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$, we define a map by $\pi(x) = (x_2, \ldots, x_d)$. Denote $\tilde{D} = \{\pi(h); h \in D\}$ and let $\tilde{E}$ be the fractal cube determined by $n$ and $\tilde{D}$. Define $g : \mathbb{R}^{d-1} \to \mathbb{R}^d$ by
$$g(x_2, \ldots, x_d) = (c - \langle \pi(x), \pi(\alpha) \rangle, \pi(x)).$$
According to (3.10) and (3.11), one can show that $E = g(\tilde{E})$. So we have $\mathcal{I}_c(E) = \mathcal{I}_c(\tilde{E})$ and $\dim H E = \dim H \tilde{E}$.

Therefore, by the first part of the proof and induction we have $\mathcal{I}_c(E) < \dim H E$. This finish the proof. □

4. Application to topological Hausdorff dimension

The topological Hausdorff dimension is defined as follows:

**Definition 4.1** ([1]). Let $X$ be a metric space. The topological Hausdorff dimension of $X$ is defined as
\begin{equation}
(4.1) \quad \dim_{tH} X = \inf_{\mathcal{U} \text{ is a basis of } X} \left( 1 + \sup_{U \in \mathcal{U}} \dim H \partial U \right),
\end{equation}
where $\dim H \partial U$ denotes the Hausdorff dimension of the boundary of $U$ and we adopt the convention that $\dim H \emptyset = \dim H \emptyset = -1$.

The following theorem gives an alternative definition of the topological Hausdorff dimension.

**Theorem 4.1** (Theorem 3.7 of [1]). For a non-empty $\sigma$-compact metric space $X$, it holds that
$$\dim_{tH} X = \min\{h; \exists S \subset X \text{ such that } \dim H S \leq h - 1 \text{ and } X \setminus S \text{ is totally disconnected}\}.$$ 

**Proof of Theorem 1.2.** Let $G = X \setminus \Lambda(X)$. Clearly $X \setminus G = \Lambda(X)$ is totally disconnected. Let $t = \dim_{tH} G$. By Theorem 4.1, for any $\delta > 0$, there exists $S \subset G$ such that $G \setminus S$ is totally disconnected, and
$$\dim H S + 1 < t + \delta.$$ 
We can see that $X \setminus S = \Lambda(X) \cup (G \setminus S)$ is also totally disconnected; for otherwise there is a connected component of $E$ connecting a point $x \in \Lambda(X)$ and a point

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Again by Theorem 4.1, \( \dim_{tH} X \leq \dim_{tH} S + 1 < t + \delta \). Since \( \delta \) is arbitrary, we have \( \dim_{tH} X \leq \dim_{tH} G \). Therefore,
\[
\dim_{tH} X \leq \dim_{tH} G \leq \dim H G = \Ic(X).
\]

5. Calculation of \( \Ic(K) \) in Example 1.2

We identify \( \mathbb{R}^2 \) with \( \mathbb{C} \). Let \( n = 5 \). Let \( \mathcal{D} = \{d_1, \ldots, d_{14}\} \) be the digit set illustrated in Figure 3 (a), denote \( \Sigma = \{1, \ldots, 14\} \). Let \( K \) be the fractal square determined by \( n \) and \( \mathcal{D} \), and let \( \{\varphi_j = \frac{z + d_j}{5}\}_{j \in \Sigma} \) be the IFS of \( K \). Denote

\[
\begin{align*}
J_{XX} &= \{j \in \Sigma; \ d_j \in \mathcal{D} \setminus \{i, 2i, 3i\}\}; \\
J_{XY} &= \{j \in \Sigma; \ d_j \in \{i, 2i, 3i\}\}; \\
J_{YX} &= \{j \in \Sigma; \ d_j \in \mathcal{D} \setminus \{i, 2i, 3i, 4i + 4i\}\}; \\
J_{YY} &= \{j \in \Sigma; \ d_j \in \{i, 2i, 3i, 4i + 4i\}\},
\end{align*}
\]

(5.1)

see Figure 4. Let
\[
X = \left( \bigcup_{j \in J_{XX}} \varphi_j(X) \right) \cup \left( \bigcup_{j \in J_{XY}} \varphi_j(Y) \right), \quad Y = \left( \bigcup_{j \in J_{YX}} \varphi_j(X) \right) \cup \left( \bigcup_{j \in J_{YY}} \varphi_j(Y) \right).
\]

Then \( X \) and \( Y \) are graph-directed sets (see [7]). The directed graph \( G \) is given in Figure 5.

\( \begin{aligned} 
\text{(a) The first iteration of } &X. \\
\text{(b) The first iteration of } &Y.
\end{aligned} \)

**Figure 4**.

\( \begin{aligned} 
\text{Figure 5.} \\
&\text{The directed graph } G. \text{ Each } d \in J_{XY} \text{ defined an edge from } X \text{ to } Y, \text{ and the corresponding map of this edge is } (z + d)/5. \text{ The same hold for } J_{XX}, J_{YX} \text{ and } J_{YY}. 
\end{aligned} \)
For each $\ell > 0$, let $J_{Y_X}^{(\ell)}$ be the collection of paths with length $\ell$ which start from $Y$ and end at $X$ in the graph $G$. Similarly, we can define $J_{XX}^{(\ell)}$, $J_{XY}^{(\ell)}$ and $J_{YY}^{(\ell)}$. Let $K_\ell = \bigcup_{\sigma \in \Sigma'} \varphi_\sigma([0,1]^2)$ and $Y_\ell = \bigcup_{\sigma \in J_{Y_X}^{(\ell)} \cup J_{YY}^{(\ell)}} \varphi_\sigma([0,1]^2)$ be the $\ell$-th approximations of $K$ and $Y$ respectively. Then $K = \bigcap_{\ell > 0} K_\ell$ and $Y = \bigcap_{\ell > 0} Y_\ell$.

**Lemma 5.1.** Let $C$ be the connected component of $K$ containing $0$. Then

(i) $C = Y$;
(ii) for any non-trivial connected component $C'' \neq C$ of $K$, there exists $\omega \in \Sigma^*$ such that $C'' = \varphi_\omega(C)$.

**Proof.** (i) Let $C_\ell$ be the connected component of $K_\ell$ containing $0$. We only need to show that $C_\ell = Y_\ell$ for all $\ell > 0$. Now we define a label map $h$ on the cells in $C_\ell$ as follows. We set $h(\sigma_1 \ldots \sigma_\ell) = X$ if there exists $\omega_1 \ldots \omega_\ell \in \Sigma^*$ such that

$$
\varphi_{\omega_1 \ldots \omega_\ell}([0,1]^2) = \varphi_{\sigma_1 \ldots \sigma_\ell}([0,1]^2) + \frac{1}{n_\ell} \in C_\ell,
$$

otherwise set $h(\sigma_1 \ldots \sigma_\ell) = Y$. We will prove by induction that

$$
\sigma_1 \ldots \sigma_\ell \in \begin{cases} J_{Y_X}^{(\ell)}, & \text{if } h(\sigma_1 \ldots \sigma_\ell) = X, \\ J_{YY}^{(\ell)}, & \text{if } h(\sigma_1 \ldots \sigma_\ell) = Y. \end{cases}
$$

For $\ell = 1$, (5.3) holds by (5.1). Assume that (5.3) holds for $\ell$.

**Case 1.** $h(\sigma_1 \ldots \sigma_\ell) = X$.

In this case, (5.2) holds, which means that the right neighbor of $\varphi_{\sigma_1 \ldots \sigma_\ell}([0,1]^2)$ belongs to $C_\ell$. If $h(\sigma_1 \ldots \sigma_\ell \sigma_{\ell+1}) = X$, then the right neighbor of $\varphi_{\sigma_1 \ldots \sigma_\ell \sigma_{\ell+1}}([0,1]^2)$ belongs to $C_{\ell+1}$ and we have $\sigma_{\ell+1} \in J_{XX}$. Hence $\sigma_1 \ldots \sigma_\ell \sigma_{\ell+1} \in J_{Y_X}^{(\ell+1)}$. Similarly, if $h(\sigma_1 \ldots \sigma_\ell \sigma_{\ell+1}) = Y$, then $\sigma_{\ell+1} \in J_{XY}$ and $\sigma_1 \ldots \sigma_\ell \sigma_{\ell+1} \in J_{YY}^{(\ell+1)}$.

**Case 2.** $h(\sigma_1 \ldots \sigma_\ell) = Y$.

In this case, the right neighbor of $\varphi_{\sigma_1 \ldots \sigma_\ell}([0,1]^2)$ is not contained in $C_\ell$. By a similar argument as Case 1, we have $\sigma_1 \ldots \sigma_\ell \sigma_{\ell+1} \in J_{Y_X}^{(\ell+1)}$ if $h(\sigma_1 \ldots \sigma_\ell \sigma_{\ell+1}) = X$, and $\sigma_1 \ldots \sigma_\ell \sigma_{\ell+1} \in J_{YY}^{(\ell+1)}$ if $h(\sigma_1 \ldots \sigma_\ell \sigma_{\ell+1}) = Y$.

Therefore, (5.3) holds for $\ell+1$. Clearly, (5.3) implies that $C_\ell = Y_\ell$. Statement (i) is proved.

(ii) Notice that $\varphi_i(K) \cap \varphi_j(K) \subset C$ for each $i, j \in \Sigma$ with $i \neq j$. Let $C'$ be a non-trivial connected component of $K$. Let $\omega$ be the longest word in $\Sigma^*$ such that $C' \subset \varphi_\omega(K)$. Then $\varphi_\omega^{-1}(C') \subset K$ and there exists $i, j \in \Sigma$ such that $\varphi_\omega^{-1}(C') \cap \varphi_i(K) \cap \varphi_j(K) \neq \emptyset$. It follows that $\varphi_\omega^{-1}(C') \subset C$, hence $C' \subset \varphi_\omega(C)$. Since $C'$ is a connected component, we have $C' = \varphi_\omega(C)$. Statement (ii) is proved. \(\square\)

By Lemma 5.1 we have $\mathcal{I}(K) = \dim H C = \dim H Y = \frac{\log \lambda}{\log 5}$, where $\lambda = \frac{16+\sqrt{132}}{2}$ is the maximal eigenvalue of the matrix $\begin{bmatrix} 11 & 8 \\ 3 & 5 \end{bmatrix}$. Let $K'$ be the fractal square in Example 1.2. It is obvious that $\mathcal{I}(K') = \frac{\log 13}{\log 5}$. 


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