PERMUTATION INVARIANT FUNCTIONALS OF LÉVY PROCESSES

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Abstract. We study natural invariance properties of functionals defined on Lévy processes and relate to them structural properties of the corresponding chaos expansion.

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INTRODUCTION

Itô's chaos expansion [7] for functionals on a Lévy process $X = (X_t)_{t \in [0,1]}$ is one of the fundamental representations of the Lévy-Wiener space, i.e. of $L_2(\mathcal{F}^X) := L_2(\Omega, \mathcal{F}^X, \mathbb{P})$ with $\mathcal{F}^X$ being the completion of $\sigma(X_t : t \in [0,1])$. The chaos expansion is an orthogonal decomposition

$$F = \sum_{n=0}^{\infty} I_n(f_n) \in L_2(\mathcal{F}^X),$$

where deterministic and symmetric kernel functions $f_n : ((0,1] \times \mathbb{R})^n \to \mathbb{R}$ are used and in $f_n((t_1, x_1), \ldots, (t_n, x_n))$ the variables $t_1, \ldots, t_n$ represent the time and $x_1, \ldots, x_n$ the state space. The expressions $I_n(f_n)$ are multiple integrals with respect to a random measure associated to the process $(X_t)_{t \in [0,1]}$. Various stochastic properties of $F$ transfer to or can be seen by the kernel functions $f_n$. In fact, the chaos expansion was used in [7] to investigate the spectral type of operators that are induced by a time shift of the underlying process (here with the two-sided unbounded time domain $(-\infty, \infty)$) and constitutes a basic example to investigate Lévy-Wiener type spaces by its chaos representation. In our paper we will continue this line of research.

Often the knowledge of the precise structure of the kernel functions $(f_n)_{n=0}^{\infty}$ of the chaos expansion is not needed as moment estimates are sufficient. This is the case for certain types of
Besov spaces obtained by real interpolation that describe the fractional smoothness of a random variable $F \in L_2(F^X)$, see for example [4].

One can represent the kernel functions in certain situations, for example by difference operators in the case of Poisson processes [9] or by Malliavin derivatives in the case of general Lévy processes [5].

However, in general the precise structure of the kernels $f_n$ is not explicitly known, which makes the investigation of the random variable $F$ by the deterministic sequence $(f_n)_{n=0}^\infty$ sometimes difficult. For example, in order to use the chaos expansion in the investigation of backward stochastic differential equations driven by Lévy processes in [6], a particular structure of a terminal condition $F$ is assumed beforehand, which transfers to the kernels $f_n$ (see Remark 6.7 below).

The starting point of this paper consists of the following two questions: Firstly, are there some simple and easy to check invariance properties of $F$ transferring into structural properties of the chaos kernels, and which simplify the chaos expansion and make it applicable in certain situations? Secondly, having a certain structure of the chaos expansion of some $F \in L_2(F^X)$ and applying a function $\varphi: \mathbb{R} \to \mathbb{R}$, does the chaos expansion of the new random variable $\varphi(F)$ share the same structural properties whenever $\varphi(F) \in L_2(F^X)$? For example, letting $(X_t)_{t \in [0,1]}$ be square-integrable and of mean zero, one knows that

\begin{equation}
S_t = 1 + \sum_{n=1}^{\infty} I_n \left( \frac{1}{n!} \mathbb{1}_{(0,t]}^\otimes n \right)
\end{equation}

with $\mathbb{1}_{(0,t]}((t_1,x_1),\ldots,(t_n,x_n)) := \mathbb{1}_{(0,t]}(t_1) \cdots \mathbb{1}_{(0,t]}(t_n)$ satisfies the equation for the Doléans-Dade exponential

\begin{equation}
S_t = 1 + \int_{(0,t]} S_{u-}dX_u.
\end{equation}

Taking a functional $\varphi(S_1) \in L_2(F^X)$, it turns out that we get an expansion of type

$\varphi(S_1) = \sum_{n=0}^{\infty} I_n(f_n)$ with $f_n((t_1,x_1),\ldots,(t_n,x_n)) = g_n(x_1,\ldots,x_n),$

where the $g_n: \mathbb{R}^n \to \mathbb{R}$ are symmetric. This might be checked by Malliavin calculus (cf. [5]). The results of the paper will provide a simple approach to check this structural property immediately by observing that $S_1$ is invariant with respect to elementary dyadic permutation operators.

The situation is more involved as it might seem in the beginning: In Section 6 we give a simple example of some $F \in L_2(F^X)$ that is invariant with respect to all periodic shifts of the underlying Lévy process, but these symmetries are not sufficient to guarantee a representation

$F = \sum_{n=0}^{\infty} I_n(g_n \mathbb{1}_{(0,1]}^\otimes n)$

with symmetric $g_n: \mathbb{R}^n \to \mathbb{R}$. This means, although $g \in L_2((0,1])$ is a.s. constant whenever $g$ is invariant with respect to all shifts, this phenomena does not transfer to the chaos kernels. The reason is that the groups acting on the $n$-dimensional chaos turn out to be some kind of diagonal groups instead of simple product groups. Taking this into the account, we introduce the concept of a \textit{locally ergodic set} in Definition 4.1 below. As part of Theorem 5.3 (implication (1) $\implies$ (3)) we will prove the following statement:
**Theorem.** Let $G$ be a group of dyadic permutations of $[0,1]$ and let $E_1, \ldots, E_L$ be a partition of $[0,1]$ consisting of locally ergodic sets with respect to $G$. If $F \in L_2(F^X)$ is invariant with respect to all permutations of the underlying Lévy process induced by $G$, then there is a representation $F = \sum_{n=0}^{\infty} I_n(f_n)$ with symmetric kernels $f_n : ((0,1] \times \mathbb{R})^n \to \mathbb{R}$ that are constant in the time-variables on all cuboids $E_{l_1} \times \cdots \times E_{l_n}$ for $l_1, \ldots, l_n \in \{1, \ldots, L\}$. In particular, if $L = 1$ and $E_1 = (0,1]$, then $F$ can be represented as $F = \sum_{n=0}^{\infty} I_n(g_n \mathbb{1}_{[0,1]})$ with symmetric $g_n : \mathbb{R}^n \to \mathbb{R}$.

**Outline of the paper.** In Section 1 we provide some preliminaries for Lévy processes. The permutation operators acting on functionals of Lévy processes are introduced in Section 2. The first abstract set of general invariance properties is obtained in Section 3, which is based on the general concepts recalled in Appendix A. The main results concerning Lévy processes are presented in Section 5. They are directly derived from the results in the more general setting given in Section 4, where we consider diagonal groups. We conclude with some examples in Section 6.

**Some notation.** The space of bounded continuous functions on a metric space $M$ is denoted by $C_b(M)$, the set of positive integers by $\mathbb{N}$. Given an $L > 0$ and $\xi \in \mathbb{R}$, we shall use the truncation function $\psi_L(\xi) := \max\{-L, \min\{\xi, L\}\}$.

### 1. Preliminaries for Lévy processes

We recall some facts about Lévy processes, for more information the reader is referred, for example, to [1] and [11]. Let $X = (X_t)_{t \in [0,1]}$, $X_t : \Omega \to \mathbb{R}$, be a Lévy process, where all paths are right-continuous and have left-limits, $X_0 \equiv 0$, and where we assume that $(\Omega, F, P)$ is a complete probability space and that $F = \sigma(X_t : t \in [0,1]) \vee \{A \in F : P(A) = 0\}$. To emphasize the minimality of $F$ we write $F = F^X$. There are some places where stochastic integration is formally used. Here we assume that as filtration the augmentation of the natural filtration of $X$ is taken. For $E \in \mathcal{B}((0,1] \times \mathbb{R})$ let

$$N(E) := \#\{t \in (0,1] : (t, \Delta X_t) \in E\}$$

be the Poisson random measure with values in $\{\emptyset, 0, 1, 2, \ldots\}$ associated to $X$. Assuming $B \in \mathcal{B}(\mathbb{R})$ with $B \cap (-\varepsilon, \varepsilon) = \emptyset$ for some $\varepsilon > 0$, we set

$$\nu(B) := \mathbb{E}N((0,1] \times B)$$

and by $\varepsilon \to 0$ we obtain the Lévy measure $\nu$ on $\mathcal{B}(\mathbb{R})$ with $\nu(\{0\}) = 0$ and $\int_{\mathbb{R}} [x^2 \wedge 1] d\nu(x) < \infty$. If $\sigma \geq 0$ is the parameter for the Brownian motion part of $X$, then we define the $\sigma$-finite measures

$$d\mu(x) := \sigma^2 d\delta_0(x) + x^2 d\nu(x),$$
$$d\mu(t,x) := d(\lambda \otimes \mu)(t,x)$$

on $\mathcal{B}(\mathbb{R})$ and $\mathcal{B}((0,1] \times \mathbb{R})$, respectively. The compensated Poisson random measure is defined by

$$\tilde{N} := N - \lambda \otimes \nu$$

on $E \in \mathcal{B}((0,1] \times \mathbb{R})$ with $\mathbb{E}(\tilde{N}) < \infty$. For such an $E$ one introduces

$$M(E) := \sigma \left(\int_{E \cap ((0,1] \times \{0\})} dW_t \right) + \lim_{N \to \infty} \int_{E \cap ((0,1] \times (\{0\} \times \mathbb{R}))} xd\tilde{N}(t,x),$$

where $W$ is the Brownian motion part of $X$ and the limit is taken in $L_2$. To recall Itô’s chaos expansion, that goes back to [7], we let

$$L^n_2 := L_2((((0,1] \times \mathbb{R})^n, \mathcal{B}(((0,1] \times \mathbb{R})^n), \mathbb{P}^\otimes n))$$

and define for pair-wise disjoint $E_1, \ldots, E_n \in \mathcal{B}((0,1] \times \mathbb{R})$ with $\mathbb{E}(E_i) < \infty$ the multiple integral

$$I_n(f_n) := M(E_1) \cdots M(E_n)$$

if $f_n((t_1, x_1), \ldots, (t_n, x_n)) := \mathbb{1}_{E_1}(t_1, x_1) \cdots \mathbb{1}_{E_n}(t_n, x_n)$. 


This extends by linearity and continuity to $I_n: L^2_2 \to L_2(\mathcal{F}^X)$. For $n \neq m$ the integrals $I_n(f_n)$ and $I_m(f_m)$ are orthogonal for any kernels $f_n$ and $f_m$. A kernel $f_n$ is called symmetric provided that

$$f((t_1, x_1), \ldots, (t_n, x_n)) = f((t_{\pi(1)}, x_{\pi(1)}), \ldots, (t_{\pi(n)}, x_{\pi(n)}))$$

for all $(t_1, x_1), \ldots, (t_n, x_n)$ and $\pi \in S_n$, where $S_n$ is the set of all permutations acting on \{1, \ldots, n\}. The symmetrization of a general $f_n \in L^2_2$ is given by

$$\tilde{f}_n((t_1, x_1), \ldots, (t_n, x_n)) := \frac{1}{n!} \sum_{\pi \in S_n} f((t_{\pi(1)}, x_{\pi(1)}), \ldots, (t_{\pi(n)}, x_{\pi(n)}))$$

and shares the two important properties, $I_n(\tilde{f}_n) = I_n(f_n)$ a.s. and $\|I_n(\tilde{f}_n)\|_{L_2(\mathcal{F}^X)} = \sqrt{n} \|f_n\|_{L^2_2}$. By Itô’s orthogonal decomposition [7], for any $F \in L_2(\mathcal{F}^X)$ there exist unique symmetric kernels $f_n \in L^2_n$ such that

$$F = \sum_{n=0}^{\infty} I_n(f_n) \quad \text{in} \quad L_2(\mathcal{F}^X).$$

If $\mathcal{H}_n := I_n(L^2_n) \subseteq L_2(\mathcal{F}^X)$ and if $\tilde{L}^n_2$ are the (equivalence classes of) symmetric functions in $L^2_n$, then

$$\mathcal{J}: \bigoplus_{n=0}^{\infty} \tilde{L}^n_2 \longrightarrow L_2(\mathcal{F}^X) \cong \bigoplus_{n=0}^{\infty} \mathcal{H}_n$$

$$(f_n)_{n=0}^{\infty} \mapsto \sum_{n=0}^{\infty} I_n(f_n),$$

defines an isometric bijection, where $\bigoplus_{n=0}^{\infty} \mathcal{H}_n$ is the $\ell_2$-product and $\bigoplus_{n=0}^{\infty} \tilde{L}^n_2$ is equipped with the norm

$$\|(f_0, f_1, \ldots)\| := \left(\sum_{n=0}^{\infty} n! \|f_n\|^2\right)^{\frac{1}{2}}.$$

2. Dyadic permutations and Lévy processes

In this section we investigate measure preserving transformations on $L_2(\mathcal{F}^X)$ and on the chaos decomposition induced by dyadic measure preserving maps $g: (0, 1] \to (0, 1]$. The final commutative diagram will be

$$\begin{array}{ccc}
L_2(\mathcal{F}^X) & \xrightarrow{T_g} & L_2(\mathcal{F}^X) \\
\bigoplus_{n=0}^{\infty} \tilde{L}^n_2 & \xrightarrow{S_{-1}} & \bigoplus_{n=0}^{\infty} \tilde{L}^n_2 \\
\mathcal{J} & \downarrow & \mathcal{J} \\
\end{array}$$

and is verified in Theorem 2.9 below. This diagram transfers [7, Lemma 1], where shift operations are considered, to our setting. The diagram is based on the fact that by the definition of Lévy processes their increments are exchangeable. Later we investigate how this exchangeability transfers to certain functionals defined on the process $X$ or more generally, to $L^2(\mathcal{F}^X)$-random variables. In order to shorten the presentation, given $0 \leq a < b \leq 1$ and $I := (a, b)$, we let $X_I := X_b - X_a$. The dyadic intervals we denote by

$$I^d_k := \left[\frac{k-1}{2^d}, \frac{k}{2^d}\right] \quad \text{for} \quad d \geq 0 \text{ and } k \in \{1, \ldots, 2^d\}.$$
2.1. Construction of $T_g$. For an integer $d \geq 0$ we let
\[
\mathcal{H}^{X,d} := \left\{ F \in L_2(\mathcal{F}^X) : F = f(X_{t_1}, \ldots, X_{t_d}), f \in C_b(\mathbb{R}^{2d}) \right\} \quad \text{and} \quad \mathcal{H}^X := \bigcup_{d \geq 0} \mathcal{H}^{X,d}.
\]
All spaces $\mathcal{H}^{X,0} \subseteq \mathcal{H}^{X,1} \subseteq \cdots \subseteq \mathcal{H}^X$ are linear subspaces of $L_2(\mathcal{F}^X)$.

**Lemma 2.1.** $\mathcal{H}^X$ is dense in $L_2(\mathcal{F}^X)$.

**Proof.** It is known that $\mathbb{R}$ is dense in $\mathcal{H}^X$. All spaces $\mathcal{H}^{X,0} \subseteq \mathcal{H}^{X,1} \subseteq \cdots \subseteq \mathcal{H}^X$ are linear subspaces of $L_2(\mathcal{F}^X)$.

**Definition 2.2.**

1. For $d \geq 0$ and $\pi \in \mathcal{S}_{2^d}$ we define $g_{\pi} : (0, 1] \to (0, 1]$ by shifting $I^d_{\pi(k)}$, i.e.
\[
g_{\pi}(t) := \frac{\pi(k)}{2^d} - \frac{k}{2^d} \quad \text{if} \quad t \in \left[ \frac{k - 1}{2^d}, \frac{k}{2^d} \right].
\]

2. We let $\mathcal{M}^{\text{dyad}} := \{ g_{\pi} : (0, 1] \to (0, 1] : \pi \in \mathcal{S}_{2^d}, d \geq 0 \}$.

3. We say that $g \in \mathcal{M}^{\text{dyad}}$ is represented by $\pi \in \mathcal{S}_{2^d}$ for some $d \geq 0$ if $g = g_{\pi}$.

4. For $g \in \mathcal{M}^{\text{dyad}}$ we let $\deg(g) := \min d$, where the minimum is taken over all $d \geq 0$ such that $g$ can be represented by some $\pi \in \mathcal{S}_{2^d}$.

Note that for $d \geq \deg(g)$ the map $g$ can always be represented by some $\pi \in \mathcal{S}_{2^d}$ and that all $g \in \mathcal{M}^{\text{dyad}}$ preserve the Lebesgue measure.

**Definition 2.3.** For $g \in \mathcal{M}^{\text{dyad}}, d \geq \deg(g)$, and $\pi \in \mathcal{S}_{2^d}$ representing $g$, we define the operator $T_g : \mathcal{H}^{X,d} \to \mathcal{H}^{X,d}$ by
\[
T_g f(X_{t_1}, \ldots, X_{t_d}) := f(X_{t_{\pi(1)}}, \ldots, X_{t_{\pi(2^d)}}).
\]

**Lemma 2.4.**

1. For $d \geq \deg(g)$ the operator $T_g : \mathcal{H}^{X,d} \to \mathcal{H}^{X,d}$ is well defined.

2. For $d \geq \deg(g)$ the operators $T_g : \mathcal{H}^{X,d} \to \mathcal{H}^{X,d}$ and $T_g : \mathcal{H}^{X,e} \to \mathcal{H}^{X,e}$ are consistent.

3. For $F \in \mathcal{H}^{X,d}$ with $d \geq \deg(g)$ the random variables $F$ and $T_g F$ have the same distribution. In particular, $T_g$ is a linear isometry in $L_2(\mathcal{F}^X)$.

**Proof.** (1) Assume that $f_1(X_{t_1}, \ldots, X_{t_d}) = f_2(X_{t_1}, \ldots, X_{t_d})$ a.s. Because of the exchange-ability of the increments of the Lévy process, the permuted vector of increments has the same distribution as the original vector. Therefore we have that $f_1(X_{t_{\pi(1)}}, \ldots, X_{t_{\pi(2^d)}}) = f_2(X_{t_{\pi(1)}}, \ldots, X_{t_{\pi(2^d)}})$ a.s. and the equivalences classes coincide. Assertion (2) follows from the definition and assertion (3) follows by the same distributional argument as in (1).

Because of Lemma 2.4 we can extend $T_g$ to an $L_2$-isometry $T_g : \mathcal{H}^X \to \mathcal{H}^X$, and by Lemma 2.1 we obtain an isometry
\[
T_g : L_2(\mathcal{F}^X) \to L_2(\mathcal{F}^X).
\]

The operator $T_g$ acts on the jump-part of $X$ as follows:

**Lemma 2.5.** Let $g \in \mathcal{M}^{\text{dyad}}$, $N$ be the Poisson random measure associated to $X$, $I = (a, b]$ with $0 \leq a < b \leq 1$ being dyadic, and $E = (c, d)$ with $-\infty < c < d < \infty$ and $0 \notin \overline{E}$. Then,
\[
(2.1) \quad T_g \int_{I \times E} x dN(s, x) = \int_{g(I) \times E} x dN(s, x) \quad \text{a.s.}
\]
Proof. The proof follows an idea of [5]. We show that for $L \in \mathbb{N}$ and the truncation $\psi_L(\xi) = \max\{-L, \min\{\xi, L\}\}$ it holds that

$$T_g \psi_L \left( \int_{I \times E} x \, dN(s, x) \right) = \psi_L \left( \int_{g(I) \times E} x \, dN(s, x) \right) \text{ a.s.}$$

Then the assertion follows from the fact that $\psi_L(F)$ converges in $L_2(F^X)$ to $F$ whenever $F \in L_2(F^X)$. For $l \in \mathbb{N}$ with $2/l < d - c$ we define a continuous function $h_l$ such that $h_l(x) = x$ on $[c + (1/l), d - (1/l)]$, $h_l(x) = 0$ if $x \not\in [c, d]$ and on the remaining parts we take the linear interpolation. By construction, $\lim_{l \to \infty} h_l(x) = x \|E(x)|$ and $|h_l(x)| \leq |x| \|E(x)|$. By definition,

$$T_g \psi_L \left( \sum_{k=1}^{2^n} h_l \left( X^{t_k}_l \right) \right) = \psi_L \left( \sum_{k=1}^{2^n} h_l \left( X^{t_k}_{g(t_k)} \right) \right) \text{ a.s.,}$$

where we assume that $n \geq \deg(g) \lor n_0$, with $n_0 \geq 0$ chosen such that $a$ and $b$ belong to the dyadic grid with mesh-size $2^{-n_0}$. Using the fact that for a fixed càdlàg path $t \to \xi_t = X_t(\omega)$ one finds for any $\varepsilon > 0$ a partition $0 = t_0 < \cdots < t_N = 1$ such that for all $t_{i-1} \leq s < t_i$ one has that $|\xi_{t_i} - \xi_s| \leq \varepsilon$ (see [3, Lemma 1, Chapter 3]) one concludes by $n \to \infty$ with dominated convergence that

$$T_g \psi_L \left( \sum_{t \in (a, b]} h_l(\Delta X_t) \right) = \psi_L \left( \sum_{t \in (a, b]} h_l(\Delta X_{g(t)} \mathbb{1}_E(\Delta X_t)) \right) \text{ a.s.}$$

Letting $l \to \infty$ and using again dominated convergence finally give

$$T_g \psi_L \left( \sum_{t \in (a, b]} \Delta X_t \mathbb{1}_E(\Delta X_t) \right) = \psi_L \left( \sum_{t \in (a, b]} \Delta X_{g(t)} \mathbb{1}_E(\Delta X_{g(t)}) \right) \text{ a.s.}$$

In order to verify the Gaussian part of $X$ in Lemma 2.7 below, we need

**Lemma 2.6.** Let $g \in \mathbb{M}_d^{\text{dyad}}, F_1, \ldots, F_n \in L_2(F^X)$ and $f: \mathbb{R}^n \to \mathbb{R}$ be continuous such that $f(F_1, \ldots, F_n) \in L_2(F^X)$. Then $T_g f(F_1, \ldots, F_n) = f(T_g F_1, \ldots, T_g F_n)$ a.s.

Proof. As in the proof of Lemma 2.5 it is enough to prove that

$$T_g \psi_L(f(F_1, \ldots, F_n)) = \psi_L(f(T_g F_1, \ldots, T_g F_n)) \text{ a.s.}$$

so that we can assume that $f \in C_0(\mathbb{R}^n)$. By Lemma 2.1 we find $\mathcal{H}^X \ni F_{i,k} \to F_i$ in $L_2(F^X)$ as $k \to \infty$. By a diagonal argument we find a sub-sequence $(k_i)_{i=1}^{\infty}$ such that, for $l \to \infty$, $F_{i,k_i} \to F_i$ a.s. and $T_g F_{i,k_i} \to T_g F_i$ a.s. for $i = 1, \ldots, n$. Therefore, as $l \to \infty$,

$$f(F_1, k_1, \ldots, F_n, k_n) \to f(F_1, \ldots, F_n) \quad \text{and} \quad f(T_g F_1, k_1, \ldots, T_g F_n, k_n) \to f(T_g F_1, \ldots, T_g F_n)$$

a.s. and therefore by the boundedness of $f$ we have convergence in $L_2(F^X)$. We conclude by

$$T_g f(F_1, \ldots, F_n) = \lim_{l \to \infty} T_g f(F_{1,k_1}, \ldots, F_{n,k_1}) = \lim_{l \to \infty} f(T_g F_{1,k_1}, \ldots, T_g F_{n,k_1}) = f(T_g F_1, \ldots, T_g F_n),$$

where the limits are taken in $L_2(F^X)$.
Lemma 2.7. Let $g \in \mathcal{M}^{\text{dyad}}$ and let $(\sigma B_t)_{t \in [0,1]}$ be the Brownian motion part of $X$, where we assume that $\sigma > 0$ and that $t \in (0,1)$ is dyadic. Then,

$$T_g B_t = \int_{g((0,t])} dB_s \text{ a.s.}$$

Proof. From the Lévy-Itô decomposition [11, Theorem 19.2] we know that there is a set $\Omega_0$ of measure one and a sequence $(\alpha_N)_{N=2}^{\infty} \subseteq \mathbb{R}$, such that for all $\omega \in \Omega_0$, $r \in [0,1]$, and $E_N := (-N, -\frac{1}{N}) \cup (\frac{1}{N}, N)$, one has

$$\sigma B_r(\omega) = X_r(\omega) - \lim_{N \to \infty} N \geq 2 \left[ \left( \int_{(0,r] \times E_N} xdN(s,x) \right)(\omega) - \alpha_N r \right].$$

Using the truncations $\psi_L, L \in \mathbb{N}$, we get therefore

$$\sigma B_t = \lim_{L \to \infty} \psi_L \left( X_t - \lim_{N \to \infty} N \geq 2 \left[ \left( \int_{(0,t] \times E_N} xdN(s,x) \right) - \alpha_N t \right] \right) \text{ a.s.},$$

$$\sigma B_{g((0,t])} = \lim_{L \to \infty} \psi_L \left( X_{g((0,t])} - \lim_{N \to \infty} N \geq 2 \left[ \left( \int_{g((0,t]) \times E_N} xdN(s,x) \right) - \alpha_N t \right] \right) \text{ a.s.},$$

where we assume that $g$ is represented by some fixed permutation of dyadic intervals and $B_{g((0,t])}$ and $X_{g((0,t])}$ are obtained by finite differences over these intervals in the canonical way. Moreover, the term $\alpha_N t$ in the second equation appears due to the fact that $g$ is measure preserving. Therefore it is sufficient to prove that

$$T_g \psi_L \left( X_t - \lim_{N \to \infty} N \geq 2 \left[ \left( \int_{(0,t] \times E_N} xdN(s,x) \right) - \alpha_N t \right] \right)$$

$$= \psi_L \left( X_{g((0,t])} - \lim_{N \to \infty} N \geq 2 \left[ \left( \int_{g((0,t]) \times E_N} xdN(s,x) \right) - \alpha_N t \right] \right) \text{ a.s.}$$

Because of the almost sure convergence in $N \to \infty$ it is sufficient to verify that

$$T_g \psi_L \left( X_t - \int_{(0,t] \times E_N} xdN(s,x) + \alpha_N t \right)$$

$$= \psi_L \left( X_{g((0,t])} - \int_{g((0,t]) \times E_N} xdN(s,x) + \alpha_N t \right) \text{ a.s.}$$

for $N \geq 2$, or

$$T_g \psi_L \left( \psi_K(X_t) - \int_{(0,t] \times E_N} xdN(s,x) + \alpha_N t \right)$$

$$= \psi_L \left( \psi_K(X_{g((0,t])}) - \int_{g((0,t]) \times E_N} xdN(s,x) + \alpha_N t \right) \text{ a.s.}$$

for $K, L \in \mathbb{N}$. As the integral terms belong to $L_2(F_N)$, this follows from Lemmas 2.5 and 2.6. $\square$
2.2. Construction of $S_g$. For $g \in \mathbb{M}^{\text{dyad}}$ we define the operator

\begin{equation}
S_g: \prod_{n=0}^{\infty} L^2_n \rightarrow \prod_{n=0}^{\infty} L^2_n \quad \text{by} \quad (f_n)_{n=0}^{\infty} \mapsto (S_g, n(f_n))_{n=0}^{\infty},
\end{equation}

where $S_{g,n}: L^2_n \rightarrow L^2_n$ is given by

$$f_n((t_1, x_1), \ldots, (t_n, x_n)) \mapsto f_n((g(t_1), x_1), \ldots, (g(t_n), x_n)).$$

The distributions of $f_n$ and $S_{g,n} f_n$ coincide, so that the operators $S_{g,n}$ and $S_g$ are isometries. The next lemma shows that we can restrict ourselves to symmetric functionals in $L^2_{\infty}$ when investigating $S_g$.

**Lemma 2.8.**

(1) For $f_n, h_n \in L^2_\infty$ with $I_n(f_n) = I_n(h_n)$ one has $I_n(S_{g,n} f_n) = I_n(S_{g,n} h_n)$.

(2) For $f_n \in L^2_\infty$ one has $I_n(S_{g,n} f_n) = I_n(S_{g,n} f_n)$.

**Proof.** (2) follows from (1) by the property $I_n(f_n) = I_n(f_n)$. (1) We know that for the symmetrizations $f_n$ and $h_n$ we have $I_n(f_n) = I_n(h_n)$ a.e. if and only if $\widetilde{f}_n = \widetilde{h}_n$ a.e. Hence it suffices to show that $\widetilde{f}_n = \widetilde{h}_n$ a.e. implies $S_{g,n} f_n = S_{g,n} h_n$ a.e. Using the transformation $r = g(s)$, this follows from

$$S_{g,n} f_n((s_1, x_1), \ldots, (s_n, x_n)) = \frac{1}{n!} \sum_{\varrho \in S_n} f_n((g(s_{\varrho(1)}), x_{\varrho(1)}), \ldots, (g(s_{\varrho(n)}), x_{\varrho(n)}))$$

$$\quad = \frac{1}{n!} \sum_{\varrho \in S_n} f_n((r_{\varrho(1)}, x_{\varrho(1)}), \ldots, (r_{\varrho(n)}, x_{\varrho(n)}))$$

$$\quad = \widetilde{f}_n((r_1, x_1), \ldots, (r_n, x_n))$$

$$\quad = h_n((r_1, x_1), \ldots, (r_n, x_n))$$

$$\quad = \left(S_{g,n} h_n\right)((s_1, x_1), \ldots, (s_n, x_n))$$

for every $((r_1, x_1), \ldots, (r_n, x_n))$ for which $\widetilde{f}_n$ and $\widetilde{h}_n$ coincide. This concludes the proof. \qed

2.3. The commutative diagram.

**Theorem 2.9.** For $g \in \mathbb{M}^{\text{dyad}}$ the following diagram is commutative:

$$
\begin{array}{ccc}
L_2(F^X) & \xrightarrow{T_g} & L_2(F^X) \\
\bigotimes_{n=0}^{\infty} L^2_n & \xrightarrow{\mathcal{J}} & \bigotimes_{n=0}^{\infty} L^2_n \\
\end{array}
$$

**Proof.** As all linear combinations of

$$f_n((s_1, x_1), \ldots, (s_n, x_n)) = \mathbb{I}_{(a_1, b_1) \times E_1}(s_1, x_1) \cdot \ldots \cdot \mathbb{I}_{(a_n, b_n) \times E_n}(s_n, x_n),$$

where the $(a_1, b_1), \ldots, (a_n, b_n)$ are dyadic and pair-wise disjoint and the $E_i$ are of form $E_i = (c_i, d_i)$ with $c_i, d_i > 0$ or $E_i = \{0\}$, are dense in $L^2_n$, and therefore the symmetrizations $\widetilde{f}_n$ are dense in $\widetilde{L}^2_\infty$, it suffices to show that $\mathcal{J} S_{g,n} f_n = T_{g,n} f_n$ for all $n \in \mathbb{N}$. For this it is sufficient to check that $I_n S_{g,n} f_n = T_{g,n} I_n f_n$, which follows from Lemmas 2.5, 2.7, and 2.6, where we use that the sets $g((a_i, b_i))$ are pair-wise disjoint as well. \qed
3. Invariances for Lévy processes

Throughout this section we let \( \mathbb{G} \subseteq \mathbb{M}_{\text{dyad}} \) be a subgroup of the group of dyadic measure preserving maps. For \( n \in \mathbb{N} \) we derive the group \( \mathbb{G}[n] \) of the measure-preserving \((0,1] \times \mathbb{R})^n\)-automorphisms

\[
g[n] : ((t_1, x_1), \ldots, (t_n, x_n)) \mapsto (g(t_1), x_1), \ldots, (g(t_n), x_n)) \quad \text{with} \quad g \in \mathbb{G}.
\]

Now we introduce the main concepts of invariance we are interested in.

**Definition 3.1.**

1. **\( \mathbb{H}_G \)-invariance.** An \( F \in L_2(F^X) \) is \( \mathbb{G} \)-invariant if \( T_g F = F \) a.s. for all \( g \in \mathbb{G} \). The set of all \( \mathbb{G} \)-invariant (equivalence classes of) random variables is denoted by \( \mathbb{H}_G \).

2. **\( \mathbb{H}_G \)-measurability.** A symmetric chaos kernel \( f_n : ((0,1] \times \mathbb{R})^n \to \mathbb{R} \) is \( \mathbb{G}[n] \)-invariant if \( f_n = f_n \circ g[n] \) a.e. for all \( g \in \mathbb{G} \). We let

\[
\mathbb{H}_G := \sigma(I_n(f_n) : f_n \in \mathbb{G}[n]-\text{invariant}, \ n \in \mathbb{N}) \vee \mathcal{N} \quad \text{with} \quad \mathcal{N} := \{ A \in \mathcal{F}^X : P(A) = 0 \}.
\]

3. **\( \mathbb{G} \)-invariant chaos expansion.** An \( F \in L_2(F^X) \) has a \( \mathbb{G} \)-invariant chaos expansion if all chaos kernels \( f_n \) are symmetric and \( \mathbb{G}[n] \)-invariant.

The definition of \( \mathbb{H}_G \) can understood in the way that we take particular representatives of \( I_n(f_n) \) to define the \( \sigma \)-algebra. By adding the null-sets all representatives become measurable with respect to \( \mathbb{H}_G \). The next theorem is the main result of this section:

**Theorem 3.2.** For a group of dyadic measure preserving maps \( \mathbb{G} \subseteq \mathbb{M}_{\text{dyad}} \) and \( F \in L_2(F^X) \) the following assertions are equivalent:

1. \( F \in \mathbb{H}_G \).
2. \( F \) is measurable with respect to \( \mathbb{H}_G \).
3. \( F \) has a \( \mathbb{G} \)-invariant chaos expansion.
4. \( F \) has symmetric chaos kernels \( f_n, n \in \mathbb{N} \), which are constant on the orbits of \( \mathbb{G}[n] \) on \((0,1] \times \mathbb{R})^n\).

**Definition 3.3.** If \( F \in L_2(F^X) \) satisfies one of the conditions of Theorem 3.2, then we will say that \( F \) is \( \mathbb{G} \)-invariant.

In order to prove Theorem 3.2 we start with the following lemma, which continues Lemma 2.6:

**Lemma 3.4.** Let \( F_1, \ldots, F_n \in \mathbb{H}_G \) and \( \varphi: \mathbb{R}^n \to \mathbb{R} \) be Borel measurable with \( \varphi(F_1, \ldots, F_n) \in L_2(F^X) \). Then, \( \varphi(F_1, \ldots, F_n) \in \mathbb{H}_G \).

**Proof.** A Borel measurable function \( \varphi \) can be approximated by truncation by bounded Borel measurable functions \( \varphi_L := \psi_L(\varphi), \ L \in \mathbb{N} \), and \( \varphi_L(F_1, \ldots, F_n) \in \mathbb{H}_G \) implies \( \varphi(F_1, \ldots, F_n) \in \mathbb{H}_G \) by monotone convergence and the completeness of \( \mathbb{H}_G \) (which is easy to check as the operators \( T_g : L_2(F^X) \to L_2(F^X) \) are isometries). Assuming that \( \varphi \) is bounded, we approximate \( \varphi \) pointwise by simple functions \( \varphi_k \) with \( \| \varphi_k \|_\infty \leq \| \varphi \|_\infty \). It follows that \( \varphi_k(F_1, \ldots, F_n) \to \varphi(F_1, \ldots, F_n) \) in \( L_2(F^X) \) by dominated convergence. Therefore, it is sufficient to check the statement for \( \varphi = 1_B \) where \( B \) is a Borel set from \( \mathbb{R}^n \). Using the outer regularity of the law of \((F_1, \ldots, F_n)\) we can verify this in turn by using \( \varphi \in C_b(\mathbb{R}^n) \). But this case follows from Lemma 2.6. \( \square \)

**Proof of Theorem 3.2.** (1) \( \iff \) (3) follows from Theorem 2.9 and the uniqueness of symmetric kernels in the chaos expansion.

(3) \( \implies \) (2) follows by definition and the completeness of \((\Omega, \mathcal{H}_G, P)\).

(4) \( \implies \) (3) is a consequence of Lemma A.2.
(3) $\implies$ (4) First we use Lemma A.4 to obtain a chaos kernel that is constant on the orbits. This new kernel will be symmetrized which keeps the property that the kernel is constant on the orbits.

(2) $\implies$ (1) As $H_G$ is a closed subspace of $L_2(A^X)$, it is sufficient to check that $\mathbb{I}_A \in H_G$ for all $A \in H_G$. Here it is sufficient to take $A$ such that there exists a sequence $(I_{i_k}(f_{i_k}))_{k \in \mathbb{N}}$ with $(i_k)_{k \in \mathbb{N}} \subseteq \mathbb{N}$ and $G[i_k]$-invariant kernels $f_{i_k}$ such that

$$A \in G := \sigma(I_{i_k}(f_{i_k}) : k \in \mathbb{N}).$$

By martingale convergence $\mathbb{I}_A$ can be approximated in $L_2$ by $G_n$-measurable functions, where

$$G_n := \sigma(I_{i_k}(f_{i_k}) : k \in \{1, \ldots, n\}).$$

By Doob’s factorization lemma (see [2, Lemma II.11.7]) there are Borel functions $\varphi_n : \mathbb{R}^n \to \mathbb{R}$ such that

$$\mathbb{E}(\mathbb{I}_A|G_n) = \varphi_n(I_{i_1}(f_{i_1}), \ldots, I_{i_n}(f_{i_n})) \text{ a.s.,}$$

so that

$$\lim_n \mathbb{E}|\mathbb{I}_A - \varphi_n(I_{i_1}(f_{i_1}), \ldots, I_{i_n}(f_{i_n}))|^2 = 0.$$  

Because of the equivalence (1) $\iff$ (3) we have that $I_{i_k}(f_{i_k}) \in H_G$ for all $k \in \mathbb{N}$ and Lemma 3.4 implies that

$$\varphi_n(I_{i_1}(f_{i_1}), \ldots, I_{i_n}(f_{i_n})) \in H_G.$$  

Because $H_G$ is closed in $L_2(A^X)$, we derive that $\mathbb{I}_A \in H_G$.  

\section{4. Diagonal Groups and Locally Ergodic Sets}

Let $(T, \tau, (T_N)_{N=0}^\infty)$ be a filtered probability space such that there are refining partitions

$$T = T_{N,1} \cup \cdots \cup T_{N,L_N}, \quad N = 0, 1, 2, \ldots,$$

satisfying the following assumptions:

- (1) $T_N = \sigma(T_{N,1}, \ldots, T_{N,L_N})$,
- (2) $\tau(T_{N,l}) > 0$ for all $(N,l)$,
- (3) $\lim_{N \to \infty} \sup_{l=1, \ldots, L_N} \tau(T_{N,l}) = 0$,
- (4) $\mathcal{T} = \bigvee_{N=0}^\infty T_N$.

We let $\mathcal{O}(T)$ be the system of countable unions of elements from $\bigcup_{N=0}^\infty T_N$ (including the empty set). The system forms a topology, in particular a set $G \subseteq T$ is open provided that it is empty or for each $x \in G$ there is a $T_{N,l}$ with $x \in T_{N,l} \subseteq G$.

Finally, we suppose that there is a countable group $G$ of bijective bi-measurable $g : T \to T$.

\textbf{Definition 4.1.}  

(1) A set $E \subseteq T$ of positive measure is called \textit{finite locally ergodic} with respect to $G$ provided that there is an $N_E \geq 0$ such that $E \in T_{N,E}$ and for all $A := T_{N,1} \cup T_{N,m} \subseteq E$ with $l \neq m$ and $N \geq N_E$ there is a subgroup $\mathbb{H} \subseteq G$ such that

\begin{enumerate}
  \item[(a)] $g|_{\mathbb{H}} = \text{id}_{\mathbb{H}}$ for all $g \in \mathbb{H}$,
  \item[(b)] the probability space $(A, \mathcal{I}(\mathbb{H}|A), \tau_A)$ is trivial, i.e., contains only sets of measure one or zero, where $\mathbb{H}|A$ is the restriction of $\mathbb{H}$ to $A$ and $\tau_A$ the normalized restriction of $\tau$ to $A$,
\end{enumerate}

(2) A set $E \subseteq T$ is called \textit{locally ergodic} with respect to $G$ provided that there is a sequence $E^j$ of finite locally ergodic sets with respect to $G$ such that

$$E^1 \subseteq E^2 \subseteq \cdots \subseteq E \quad \text{and} \quad E = \bigcup_{j=1}^\infty E^j.$$
Remark 4.2.  
(1) By definition locally ergodic sets belong to $\mathcal{O}(T)$.
(2) The local ergodicity is stable with respect to passing to open subsets: If $\emptyset \neq F \subseteq E$, where $F \in \mathcal{O}(T)$ and where $E$ is locally ergodic, then $F$ is locally ergodic.

Proof. Let us check (2). By definition we find finite locally ergodic sets such that

$$E^1 \subseteq E^2 \subseteq \cdots \subseteq E \quad \text{and} \quad E = \bigcup_{j=1}^{\infty} E^j.$$ 

At the same time we find an increasing sequence $F^j \in \mathcal{T}_{N_j}$, $j \in \mathbb{N}$, such that $F = \bigcup_{j=1}^{\infty} F^j$. One obtains

$$F = F \cap E = \bigcup_{j=1}^{\infty} (F^j \cap E^j)$$

and that $F^j \cap E^j$ is finite locally ergodic because $E^j$ is of this type and $F^j \cap E^j \subseteq E^j$. \hfill $\square$

Now we define our diagonal group. We fix $n \in \mathbb{N}$ and consider an auxiliary $\sigma$-finite measure space $(R, \mathcal{R}, \rho)$ with $\rho(R) > 0$ and the group $\mathbb{G}[n]$ that consists of all maps $g[n]: (T \times R)^n \to (T \times R)^n$ given by

$$((t_1, x_1), \ldots, (t_n, x_n)) \mapsto ((g(t_1), x_1), \ldots, (g(t_n), x_n)) \quad \text{with} \quad g \in \mathbb{G}.$$ 

To formulate our main result, we recall that $\mathcal{I}(\mathbb{G}[n])$ denotes the invariant $\sigma$-algebra with respect to the group $\mathbb{G}[n]$, see Definition A.1 below.

Theorem 4.3. Let $n \in \mathbb{N}$, $E_1, \ldots, E_L \in \mathcal{T}$ be pairwise disjoint and locally ergodic with respect to $\mathbb{G}$,

$$\mathcal{T}_E := \mathcal{T}_{|T \setminus (\bigcup_{i=1}^{L} E_i)} \vee \sigma(E_1, \ldots, E_L), \quad \text{and} \quad \mathcal{N}_n := \{A \in (\mathcal{T} \otimes \mathcal{R})^{\otimes n} : (\tau \otimes \rho)^{\otimes n}(A) = 0\}.$$ 

Then $\mathcal{I}(\mathbb{G}[n]) \subseteq (\mathcal{T}_E \otimes \mathcal{R})^{\otimes n} \vee \mathcal{N}_n$.

Lemma 4.4. Assume a probability space $(\mathcal{M}, \mathcal{M}, m)$, a decreasing sequence of measurable sets $D_0 \supseteq D_1 \supseteq \ldots$, a sub-$\sigma$-algebra $\mathcal{I} \subseteq \mathcal{M}$ and

$$\mathbb{G}_N := \mathcal{I} \vee \sigma(A_N \in \mathcal{M} \text{ with } A_N \subseteq D_N).$$

Assume that $m(D_N) \to 0$ as $N \to \infty$. Then

$$\bigcap_{N=0}^{\infty} (\mathbb{G}_N \vee \mathcal{N}) \subseteq \mathcal{I} \vee \mathcal{N} \quad \text{with} \quad \mathcal{N} := \{A \in \mathcal{M} : m(A) = 0\}.$$ 

Proof. The $\sigma$-algebra $\mathbb{G}_N$ consists of all

$$B_N = (I_N \cap D_N) \cup A_N$$

with $A_N \in \mathcal{M}$, $A_N \subseteq D_N$ and $I_N \in \mathcal{I}$. Therefore $B \in \bigcap_{N=0}^{\infty} (\mathbb{G}_N \vee \mathcal{N})$ gives $I_N \in \mathcal{I}$ and $A_N \in \mathcal{M}$ with $A_N \subseteq D_N$ such that

$$B_N := (I_N \cap D_N) \cup A_N \quad \text{satisfies} \quad B_N \Delta B \in \mathcal{N} \quad \text{for all} \quad N \geq 0.$$ 

Defining $C := \bigcup_{N=0}^{\infty} (B_N \Delta B) \in \mathcal{N}$, this implies on $C^c$ that

$$B = B_N = (I_N \cap D_N) \cup A_N.$$ 

Let

$$I := \bigcup_{N=0}^{\infty} \bigcap_{k=N}^{\infty} I_k \in \mathcal{I}.$$
By construction, \( I_N = B_N \) on \( D_N^c \) and \( D_N^c \subseteq D_j^c \subseteq \cdots \). Therefore, \( I \Delta B \subseteq D_N \cup C \) which implies \( \mathbb{P}(I \Delta B) \leq \lim_N \mathbb{P}(D_N) = 0 \) and proves the lemma.

**Proof of Theorem 4.3.** We assume a partition \( R = \bigcup_{j \in J} R_j \) with \( \rho(R_j) \in (0, \infty) \). Choosing \( \lambda_j \in (0, \infty) \) we can arrange that \( \rho^\prime(A) := \sum_{j \in J} \lambda_j \rho(A \cap R_j) \) becomes a probability measure which has a strictly positive density with respect to \( \rho \). As our statement only concerns null-sets we can replace \( \rho \) by \( \rho^\prime \), or we can assume w.l.o.g. that \( \rho \) itself is a probability measure.

I. First we assume that \( E_1, \ldots, E_L \) are finite locally ergodic. Let us fix a set \( B \in \mathcal{I}(\mathcal{G}[n]) \) of positive measure.

(a) We observe that \( \bigvee_{N \geq 0} (T_N \otimes \mathcal{R})^{\otimes n} = (T \otimes \mathcal{R})^{\otimes n} \), so that martingale convergence yields
\[
\lim_{N \to \infty} f_N = \mathbb{1}_B \quad (\tau \otimes \rho)^{\otimes n} \text{-a.s.,}
\]
where, for \((t_1, \ldots, t_n) \in Q_{l_1, \ldots, l_n}^N := T_{N, t_1} \times \cdots \times T_{N, t_n}, \)
\[
f_N((t_1, x_1), \ldots, (t_n, x_n)) := \int_{Q_{l_1, \ldots, l_n}^N} \mathbb{1}_B((s_1, x_1), \ldots, (s_n, x_n)) d\tau(s_1) \cdots d\tau(s_n) \]
\[
\otimes^{\otimes n} \rho(Q_{l_1, \ldots, l_n}^N).
\]
(b) For \( N \geq 0 \) we let
\[
\Delta_N := \bigcup_{l_1, \ldots, l_n \in \{1, \ldots, L_N\}} Q_{l_1, \ldots, l_n}^N
\]
which is empty for \( n = 1 \). For \( n \geq 2 \) the size of \( \Delta_N \) can be upper bounded by
\[
\tau^{\otimes n}(\Delta_N) \leq \left( \begin{array}{c} n \\ 2 \end{array} \right) \max_{l=1, \ldots, N} \tau(T_{N, l}) \quad \text{so that } \lim_N \tau^{\otimes n}(\Delta_N) = 0.
\]
Define
\[
\mathcal{G}_N := (T_E \otimes \mathcal{R})^{\otimes n} \vee \sigma \left( D \times G : D \in T^{\otimes n}, D \subseteq \Delta_N, G \in \mathcal{R}^{\otimes n} \right)
\]
with a slight abuse of notation concerning the order of components, which gives the \( \sigma \)-algebra \( T_E \otimes \mathcal{R} \) in the case \( n = 1 \). As \( \Delta_0 \supseteq \Delta_1 \supseteq \cdots \) we have \( \mathcal{G}_0 \supseteq \mathcal{G}_1 \supseteq \cdots \).

(c) Let \( N_0 := \max\{N_{E_1}, \ldots, N_{E_L}\} \geq 0 \), where the \( N_{E_i} \) are taken from Definition 4.1(1). The main observation of the proof is that \( f_N \) is \( \mathcal{G}_N \)-measurable for \( N \geq N_0 \). By definition, \( f_N \) is constant on all cuboids \( Q_{l_1, \ldots, l_n}^N \). Assume two cuboids
\[
Q_{l_1, \ldots, l_n}^N \text{ and } Q_{m_1, l_2, \ldots, l_n}^N
\]
such that \((l_1, \ldots, l_n)\) are distinct, \((m_1, l_2, \ldots, l_n)\) are distinct, \( l_1 \neq m_1 \), and that \( T_{N, l_1}, T_{N, m_1} \subseteq E_l \), where \( l \in \{1, \ldots, L\} \) is now fixed. By assumption, there is a sub-group \( \mathbb{H} \) of \( \mathcal{G} \) such that \( A_l := T_{N, l_1} \cup T_{N, m_1} \) is a quasi-atom of \((A_l, \mathcal{I}(\mathbb{H}|A_l), \tau_{A_l})\) and that \( \mathbb{H} \) acts as an identity outside \( A_l \). Because \( B \in \mathcal{I}(\mathcal{G}[n]) \) we have that
\[
\mathbb{1}_B g[n] = \mathbb{1}_B \quad \text{for all } g \in \mathcal{G},
\]
so that, for all \( g \in \mathbb{H}, \)
\[
\mathbb{1}_B((g t_1, x_1), (t_2, x_2), \ldots, (t_n, x_n)) = \mathbb{1}_B((t_1, x_1), (t_2, x_2), \ldots, (t_n, x_n))
\]
on \( (A_l \times R) \times (T_{N, L_2} \times R) \times \cdots \times (T_{N, L_n} \times R) \). This implies that the subset \( A_l \) of the section of \( B \), taken at
\[
(x_1, (t_2, x_2), \ldots, (t_n, x_n)) \in R \times (T_{N, t_2} \times R) \times \cdots \times (T_{N, t_n} \times R),
\]
is invariant with respect to \( \mathbb{H}|A_l \) and therefore the function
\[
t_1 \to \mathbb{1}_B((t_1, x_1), (t_2, x_2), \ldots, (t_n, x_n))
\]
is almost surely constant on $A_l$ under the condition (4.1). Consequently,

$$
\int_{Q_{j_1 \ldots j_n}} I_A((t_1, x_1), (t_2, x_2), \ldots, (t_n, x_n)) \frac{d\tau(t_1) \cdots d\tau(t_n)}{\tau^{\otimes n}(Q_{j_1 \ldots j_n})} = \int_{Q_{j_1 \ldots j_n}} I_A((t_1, x_1), (t_2, x_2), \ldots, (t_n, x_n)) \frac{d\tau(t_1) \cdots d\tau(t_n)}{\tau^{\otimes n}(Q_{j_1 \ldots j_n})}
$$

for $(x_1, \ldots, x_n) \in R^n$. We can repeat the argument, where we replace the exchange of the first component of the cuboid by any other component. This implies that $f_N$ is $G_N$-measurable.

(d) From (c) we immediately get that $f_M$ is $G_N$-measurable for $M \geq N \geq N_0$. Therefore $I_B$ is $G_N \vee N_n$-measurable for all $N \geq N_0$. Applying Lemma 4.4 we get that $I_B$ is $(T_E \otimes \mathcal{R})^{\otimes n} \vee N_n$-measurable.

II. Now we assume general locally ergodic sets $E_1, \ldots, E_L$. By definition we find monotone sequences of finite locally ergodic sets $(E_j^i)_{j=1}^{\infty}$ with

$$
\bigcup_{j=1}^{\infty} E_j^i = E_i.
$$

We proved in step I that $\mathcal{I}(G[n]) \subseteq (T_{E^j} \otimes \mathcal{R})^{\otimes n} \vee N_n$ with

$$
T_{E^j} := T|_{\mathcal{T}(\bigcup_{k=1}^{L} E_k^i) \vee \sigma(A^j_1, \ldots, A^j_L)},
$$

so that

$$
\mathcal{I}(G[n]) \subseteq \bigcap_{j=1}^{\infty} \left( (T_{E^j} \otimes \mathcal{R})^{\otimes n} \vee N_n \right).
$$

Observing $$
(T_{E^j} \otimes \mathcal{R})^{\otimes n} \subseteq (T_E \otimes \mathcal{R})^{\otimes n} \vee \sigma(A^j \in (T \otimes \mathcal{R})^{\otimes n} : A^j \subseteq D^j)
$$

with $$
D^j := \left\{ ((t_1, x_1), \ldots, (t_n, x_n)) \in (T \times R)^n : t_k \in E_{k}^i \setminus E_{k}^j \text{ for some } k \in \{1, \ldots, n\} \right\},
$$
gives that

$$
\mathcal{I}(G[n]) \subseteq \bigcap_{j=1}^{\infty} \left( (T_E \otimes \mathcal{R})^{\otimes n} \vee \sigma(A^j \in (T \otimes \mathcal{R})^{\otimes n} : A^j \subseteq D^j) \vee N_n \right).
$$

Finally, because of $D^1 \supseteq D^2 \supseteq \cdots$ and

$$
(\tau \otimes \rho)^{\otimes n}(D^j) \leq n \left[ \sum_{l=1}^{L} \tau(E_l^i \setminus E_l^j) \right] \to 0 \text{ as } j \to \infty,
$$

we can again apply Lemma 4.4. \hfill \Box

5. Reduced chaos expansions for Lévy processes

In this section we apply the results from Section 4 to Lévy processes. For this purpose we let

(1) $(T, T, \tau) := ((0, 1], B((0, 1]), \lambda)$ with $T_N = \mathcal{F}_N^{\text{dyad}} := \sigma \left( \left\{ (\frac{i-1}{2^n}, \frac{i}{2^n}) : i = 1, \ldots, 2^N \right\} \right)$,

(2) $M_E^{\text{dyad}} := \{ g \in M^\text{dyad} : g|_{E^c} = \text{id}_{E^c} \}$ for $E \subseteq (0, 1]$,

(3) $(R, R, \rho) := (\mathbb{R}, B(\mathbb{R}), \mu)$,

(4) and $\mathcal{N}_n$ be the null-sets in $((0, 1] \times \mathbb{R})^n$ with respect to $(\lambda \otimes \mu)^{\otimes n}$. 
Let us begin with a prototype of a locally ergodic set.

**Lemma 5.1.** Let $E \in \mathcal{O}((0,1])$ be non-empty. Then $E$ is locally ergodic with respect to $\mathcal{M}^\text{dyad}_E$.

**Proof.** It is enough to show the following: If $A \in \mathcal{F}^\text{dyad}_{N_0}$ is a non-empty subset of $E$, then $(A, \mathcal{I}(\mathcal{M}^\text{dyad}_A | A), \lambda_A)$ is trivial. Take any $B \in \mathcal{I}(\mathcal{M}^\text{dyad}_A | A)$. Using the dyadic filtration restricted to $A$, we start with the level $N_0$, we interpret $\mathbb{1}_B$ as closure of a martingale in $(A, \mathcal{B}((0,1]) | A, \lambda_A)$ along this filtration. By the invariance of $B$, the random variables, that form this martingale, are individually constant. Therefore we get a sequence of constants that converge to $\mathbb{1}_B$ in $L_2(A, \lambda_A)$ and $\lambda_A$-a.s. Hence $\mathbb{1}_B$ is a constant almost surely which implies the statement. \hfill $\square$

**Remark 5.2.** One can find groups $G$ such that for example $E = (0,1]$ is locally ergodic but $G \subset \mathcal{M}^\text{dyad}$. Take for example all permutations that leave the first interval $(0, 2^{-N}]$ invariant on each dyadic level $N$. It would be of interest to characterize those sub-groups $G \subset \mathcal{M}^\text{dyad}$ such that a given $E \in \mathcal{O}((0,1])$ gets locally ergodic.

Now we let $G$ be a group like in Section 3. The main result is the following simplification of the chaos decomposition:

**Theorem 5.3.** For pair-wise disjoint $E_1, \ldots, E_L \in \mathcal{B}((0,1])$, that are locally ergodic with respect to $G$, and $F \in L_2(\mathcal{F})$ consider the following conditions:

1. $F$ is $G$-invariant.
2. One has $F = \sum_{n=0}^{\infty} I_n(f_n)$ with symmetric, $G[n]$-invariant, and $(\mathcal{B}((0,1])_E \otimes \mathcal{B}(\mathbb{R}))^\otimes n$ $\mathcal{N}_n$-measurable $f_n$.
3. One has $F = \sum_{n=0}^{\infty} I_n(f_n)$ with symmetric $(\mathcal{B}((0,1])_E \otimes \mathcal{B}(\mathbb{R}))^\otimes n$-measurable $f_n$.
4. $F$ is invariant with respect to the group $\mathbb{H}$ generated by $\mathcal{M}^\text{dyad}_{E_1}, \ldots, \mathcal{M}^\text{dyad}_{E_L}$.

Then it holds that $(1) \iff (2) \iff (3) \iff (4)$.

**Proof.** (2) $\implies$ (1) follows from Theorem 3.2 and (1) $\implies$ (2) from Theorems 3.2 and 4.3.

(2) $\implies$ (3) We find a $f'_n = f_n$ a.e. that is $(\mathcal{B}((0,1])_E \otimes \mathcal{B}(\mathbb{R}))^\otimes n$-measurable. By symmetrizing this $f'_n$ we get a symmetric kernel that is $(\mathcal{B}((0,1])_E \otimes \mathcal{B}(\mathbb{R}))^\otimes n$-measurable.

(3) $\implies$ (4) follows again from Theorem 3.2.

(4) $\implies$ (3) By Theorem 3.2 we get symmetric kernels that are $\mathbb{H}[n]$-invariant. On the other side, Lemma 5.1 yields that $E_1, \ldots, E_L$ are locally ergodic with respect to $\mathbb{H}$ so that $\mathcal{I}(\mathbb{H}[n]) \subseteq (\mathcal{B}((0,1])_E \otimes \mathcal{B}(\mathbb{R}))^\otimes n$ $\mathcal{N}_n$-measurable by Theorem 4.3. One can finish as in $(2) \implies (3)$. \hfill $\square$

**Corollary 5.4.** If $E_1, \ldots, E_L$ from Theorem 5.3 form a partition of $(0,1]$, then the symmetric kernels $f_n$ in Theorem 5.3(3) are constant on all cuboids $E_{l_1} \times \cdots \times E_{l_n}$ where $l_1, \ldots, l_n \in \{1, \ldots, L\}$.

**Proof.** This follows from the fact that $\mathcal{B}((0,1])_E = \sigma(E_1, \ldots, E_L)$ due to $E_1, \ldots, E_L$ forming a partition of $(0,1]$. \hfill $\square$

6. **Examples**

6.1. **A negative example:** Shift operators. First we motivate the need of locally ergodic sets. We do this by considering the group generated by shifts, which is inspired by the work of Itô [7]. Assume that

$$F = I_2(f_2) \quad \text{where} \quad f_2((s,x),(t,y)) := g_2(|s-t|)h_2(x,y)$$
with a measurable function \( g_2 : [0, 1] \to \mathbb{R} \) such that \( g_2(1/2 - s) = g_2(1/2 + s) \) for \( s \in [0, 1/2] \) and a symmetric Borel function \( h_2 : \mathbb{R}^2 \to \mathbb{R} \) such that \( f_2 \in L^2_{\mathbb{F}} \). It is straightforward to check that \( F \) is invariant with respect to all shifts \( s_h : (0, 1] \to (0, 1], 0 < h < 1 \), defined by \( s_h(t) := t + h \) if \( t + h \leq 1 \) and \( s_h(t) := t + h - 1 \) if \( t + h > 1 \). Obviously, the measure \( \mu \) and the functions \( g_2 \) and \( h_2 \) can be chosen such that there is no symmetric \( \tilde{f}_2((s, x), (t, y)) \) not depending on \((s, t)\), but with \( f_2 = \tilde{f}_2 \) a.e. (take for example \( \mu \) as the Dirac measure in 1).

6.2. Positive examples. Our positive examples are based on Proposition 6.2 below for which we need the notion of \textit{weak} \( G \)-invariance:

**Definition 6.1.** Given a subgroup \( G \subseteq \mathbb{M}^{\text{dyad}} \), we say that an \( \mathcal{F}^X \)-measurable random variable \( Z : \Omega \to \mathbb{R} \) is \textit{weakly} \( G \)-invariant provided that \( f(Z) \) is \( G \)-invariant for all \( f \in \mathcal{C}_b(\mathbb{R}) \).

Lemma 2.6 gives that \( G \)-invariant implies \textit{weakly} \( G \)-invariant, but the converse does not need to be true because of a possibly missing integrability. To consider our examples, let us fix a sequence of time-points

\[
0 \leq s_1 < t_1 \leq \ldots \leq s_L < t_L \leq 1
\]

for the rest of this section.

**Proposition 6.2.** Assume \( \mathcal{F}^X \)-measurable \( Z_1, \ldots, Z_N : \Omega \to \mathbb{R} \) that are weakly \( \mathbb{M}^{\text{dyad}}_{(s_l,t_l)} \)-invariant for \( l \in \{1, \ldots, L\} \). If \( f : \mathbb{R}^N \to \mathbb{R} \) is a Borel function with \( F = f(Z_1, \ldots, Z_N) \in L_2(\mathcal{F}^X) \), then one can choose symmetric chaos kernels \( f_n \) for \( F \) that are constant on the cuboids

\[
\prod_{j=1}^n(s_{l_j}, t_{l_j}) \quad \text{for} \quad l_1, \ldots, l_n \in \{1, \ldots, L\}.
\]

**Proof.** The variables \( \varphi(Z_k), k = 1, \ldots, N, \) are \( \mathbb{M}^{\text{dyad}}_{(s_l,t_l)} \)-invariant, where \( \varphi(x) := \arctan(x) \). Letting \( \psi(y) := \tan(y) \) for \( y \in (-\pi/2, \pi/2) \) and \( \psi(y) := 0 \) otherwise, and using the change of variables \( g(y_1, \ldots, y_N) := f(\varphi(y_1), \ldots, \varphi(y_N)) \), Lemma 3.4 implies that \( F = g(\varphi(Z_1), \ldots, \varphi(Z_N)) \) is \( \mathbb{M}^{\text{dyad}}_{(s_l,t_l)} \)-invariant. The sets \( E_l := (s_l, t_l] \) if \( t_l \) is dyadic, and \( E_l := (s_l, t_l) \) otherwise, belong to \( \mathcal{O}(0, 1) \). According to Lemma 5.1 the set \( E_l \) is locally ergodic with respect to \( \mathbb{M}^{\text{dyad}}_{E_l} \) and therefore with respect to the group generated by \( \mathbb{M}^{\text{dyad}}_{E_1}, \ldots, \mathbb{M}^{\text{dyad}}_{E_L} \). Furthermore, observing that \( \mathbb{M}^{\text{dyad}}_{E_l} = \mathbb{M}^{\text{dyad}}_{(s_l,t_l)} = \mathbb{M}^{\text{dyad}}_{(s_l,t_l)} \) if \( t_l \) is not dyadic, and applying Theorem 5.3 gives the existence of symmetric kernels \( f_n \) that are constant on all cuboids \( E_{l_1} \times \cdots \times E_{l_n} \) where \( l_1, \ldots, l_n \in \{1, \ldots, L\} \). Modifying the kernels on a null set yields the assertion.

6.2.1. Doléans-Dade stochastic exponential. We follow [4] and assume \( X \) to be \( L_2 \)-integrable and of mean zero. For \( 0 \leq a \leq t \leq 1 \) we let

\[
S^a_t := 1 + \sum_{n=1}^\infty \frac{I_n(1^{\otimes n})}{n!},
\]

where we can assume that all paths of \((S^a_t)_{t \in [a, 1]}\) are càdlàg for any fixed \( a \in [0, 1] \). Then we get that

\[
S^a_t = 1 + \int_{(a,t]} S^a_u \, dX_u \quad \text{a.s. and} \quad S_t = S^0_t S_a \quad \text{a.s. with} \quad S_t := S^0_t.
\]

Therefore we get from the chaos representation of \( S^a_t \):

**Lemma 6.3.** Each random variable \( S^a_{t_k} \) is \( \mathbb{M}^{\text{dyad}}_{(a,t_k)} \)-invariant for \( k = 1, \ldots, L \).

One could continue the investigation by using more general Doléans-Dade exponential formulas (see for example [10, Chapter II, Theorem 37]), which is not done here.
6.2.2. Limit functionals. Behind the next examples there is common idea formulated in

**Definition 6.4.** For $0 \leq s < t \leq 1$ a random variable $Z : \Omega \to \mathbb{R}$ belongs to the class $C(s,t]$ provided that there exists a sequence $0 \leq N_1 < N_2 < \ldots$ of integers and Borel functions $\Phi_k : \mathbb{R}^{N_k} \to \mathbb{R}$ such that

$$Z = \lim_{k \to \infty} Z^k := \lim_{k \to \infty} \Phi_k \left( \frac{X_{\frac{a_k}{2^k}} - X_{\frac{a_{k+1}}{2^{k+1}}}, \ldots, X_{\frac{b_k}{2^k}} - X_{\frac{b_{k+1}}{2^{k+1}}}}{2^k} \right) \text{ a.s.,}$$

where $\frac{a_k}{2^k}$ is the smallest grid point greater than or equal to $s$ and $\frac{b_k}{2^k}$ is the largest grid point smaller than or equal to $t$, $M_k := b_k - a_k + 2$, and the function $\Phi_k$ is symmetric in its arguments where the first and last coordinate are excluded.

**Lemma 6.5.** Let $Z_1, \ldots, Z_L : \Omega \to \mathbb{R}$ be random variables such that $Z_l$ belongs to the class $C(s_l,t_l]$ for $l = 1, \ldots, L$, and let $f : \mathbb{R}^L \to \mathbb{R}$ be a Borel function with $F := f(Z_1, \ldots, Z_L) \in L_2(\mathcal{F}^X)$. Then one can choose symmetric chaos kernels $f_n$ for $F$ that are constant on the cuboids

$$\prod_{j=1}^n \{s_{lj}, t_{lj}\} \text{ for } l_1, \ldots, l_n \in \{1, \ldots, L\}.$$

**Proof.** By Proposition 6.2 it is sufficient to show that $Z_1, \ldots, Z_L$ are weakly $M_{\{s_l,t_l\}}^{\text{dyad}}$-invariant for $l \in \{1, \ldots, L\}$, i.e. that $\varphi(Z_m)$ is $M_{\{s_l,t_l\}}^{\text{dyad}}$-invariant for $\varphi \in C_b(\mathbb{R})$ and $m, l \in \{1, \ldots, L\}$. Let $g \in M_{\{s_l,t_l\}}^{\text{dyad}}$ (be not the identity). Then there exists an integer $M \geq 0$ such that $g$ acts as a permutation of the dyadic intervals of length $2^{-M}$ and as an identity on $(s_l,t_l]$. Therefore, there exist integers $0 \leq a < b \leq 2^M$ such that

$$(s_l,t_l] := \left( \frac{a}{2^M}, \frac{b}{2^M} \right] \subseteq (s_l,t_l]$$

and $g$ can be described by permuting dyadic intervals on $(s_l,t_l]$ of length $2^{-M}$. By Definition 6.4 there is an approximation $Z_m = \lim_{k \to \infty} Z_m^k$ a.s. By construction, there is a $k_0 \geq 1$ such that for all $k \geq k_0$ one has that $\varphi(Z_m^k)$ is $T_g$-invariant (here one has to distinguish between the cases $m = l$ and $m \neq l$). By dominated convergence, $\lim_{k \to \infty} \varphi(Z_m^k) = \varphi(Z_m)$ in $L_2(\mathcal{F}^X)$ so that $\varphi(Z_m)$ is invariant with respect to $T_g$ as well and the proof is complete.

**Example 6.6.** For $0 \leq s < t \leq 1$ the following random variables belong to the class $C(s,t]$:

1. $X_t - X_s$.
2. $[X,X]_t - [X,X]_s$, where $[X,X]$ is the quadratic variation process of $X$.
3. $\sup_{r \in [s,t]} |X_t - X_r|$.

**Proof.** (1) is obvious. (2): Here we first take $\Phi_k(x_1, \ldots, x_{M_k}) := |x_1|^2 + \cdots + |x_{M_k}|^2$ with $N_k := k \geq k_0$, use [10, Chapter II, Theorem 22] to get a sequence that converges in probability, and extract a sub-sequence that converges almost surely.

(3) Taking $\Phi_k(x_1, \ldots, x_{M_k}) := \max\{|x_1|, \ldots, |x_{M_k}|\}$ and $N_k := k$ with $k \geq k_0$ and the uniformity result for cádlág paths [3, Chapter 3, Lemma 1] yields the assertion.

**Remark 6.7.** Combining Lemma 6.5 with Example 6.6(1) yields that the symmetric chaos kernels $f_n$ of $F = f(X_{t_1} - X_{s_1}, \ldots, X_{t_L} - X_{s_L})$ can be chosen to be constant on the cuboids

$$\prod_{j=1}^n \{s_{lj}, t_{lj}\} \text{ for } l_1, \ldots, l_n \in \{1, \ldots, L\}.$$

\footnote{Here and in the following it is implicitly assumed that the partitions are taken always in a way that $\frac{a_k}{2^k} < \frac{b_k}{2^k}$ by choosing $N_k$ large enough.}
This was used in [6] in the investigation of variational properties of backward stochastic differential equations driven by Lévy processes.

**Appendix A. Invariant sets**

We recall concepts related to classical ergodic theory (see [8, Chapter 10] or [12, Chapter V]) and adapt them to our setting. For the convenience of the reader we include the proofs in this version of the article.

We assume a measurable space \((S, \Sigma)\) and a group \(\mathbb{A}\) of automorphisms of \(S\), i.e. bijective \(\mathbb{A}\)-measurable functions \(T: S \to S\).

**Definition A.1.** The invariant \(\sigma\)-algebra w.r.t. \(\mathbb{A}\) is given by
\[
\mathcal{I}(\mathbb{A}) := \{ B \in \Sigma : B = T^{-1}(B) \text{ for all } T \in \mathbb{A} \}.
\]

**Lemma A.2.** For a function \(\xi: S \to \mathbb{R}\) the following assertions are equivalent:

1. \(\xi\) is \(\mathcal{I}(\mathbb{A})\)-measurable.
2. \(\xi\) is \(\Sigma\)-measurable and constant on the orbits \(\{Ts : T \in \mathbb{A}\}, s \in S\).
3. \(\xi\) is \(\Sigma\)-measurable and \(\xi \circ T = \xi\) for all \(T \in \mathbb{A}\).

**Proof.** (2) \(\iff\) (3) is obvious. (1) \(\implies\) (3) Let \(\xi\) be a simple \(\mathcal{I}(\mathbb{A})\)-measurable function
\[
\xi(s) := \sum_{k=1}^{N} \alpha_k \mathbb{1}_{B_k}(s) \quad \text{with} \quad B_k \in \mathcal{I}(\mathbb{A}) \text{ and } \alpha_k \in \mathbb{R}.
\]

In this case, the assertion follows from \(\mathbb{1}_{B_k}(T(s)) = \mathbb{1}_{T^{-1}(B_k)}(s) = \mathbb{1}_{B_k}(s)\). For a general \(\mathcal{I}(\mathbb{A})\)-measurable \(\xi\) one finds a point-wise approximating sequence \(\xi_n\) of simple \(\mathcal{I}(\mathbb{A})\)-measurable functions with \(\xi_n \circ T(s) = \xi_n(s)\) for all \(n \in \mathbb{N}, T \in \mathbb{A}\) and \(s \in S\). This implies \(\xi \circ T(s) = \xi(s)\) by \(n \to \infty\).

(2) \(\implies\) (1) We approximate \(\xi\) by
\[
\xi_n(s) := \sum_{k=-2^n}^{2^n - 1} \frac{k}{2^n} \mathbb{1}_{B^n_k}(s) \quad \text{with} \quad B^n_k := \xi^{-1} \left( \left[ \frac{k}{2^n}, \frac{k+1}{2^n} \right] \right).
\]

As \(\xi\) is constant on the orbits, they are entirely contained in one of the sets \(B^n_k\), so that \(B^n_k \in \mathcal{I}(\mathbb{A})\). Therefore \(\xi_n\) is \(\mathcal{I}(\mathbb{A})\)-measurable and, by the point-wise convergence, \(\xi\) as well. 

Let \((S, \Sigma, \gamma)\) be a \(\sigma\)-finite measure space with \(\gamma(S) > 0\), \(\mathbb{A}\) be a group of automorphisms acting on \(S\), and
\[
\overline{\mathcal{I}(\mathbb{A})} := \mathcal{I}(\mathbb{A}) \vee \mathcal{N} \quad \text{where} \quad \mathcal{N} := \{ B \in \Sigma : \gamma(B) = 0 \}.
\]
The equivalence class of \(\xi\) w.r.t. to the \(\gamma\)-a.e.-equivalence is denoted by \(\left[ \xi \right]\).

**Definition A.3.** The measure \(\gamma\) is called quasi-invariant w.r.t. \(\mathbb{A}\), if \(\gamma(T^{-1}B) = 0\) for all \(B \in \mathcal{N}\) and \(T \in \mathbb{A}\).

**Lemma A.4.** Let \((S, \Sigma, \gamma)\) be a \(\sigma\)-finite measure space with \(\gamma(S) > 0\) and \(\mathbb{A}\) be a group of automorphisms acting on \(S\). Then one has the following assertions:

1. The operation \(\left[ \xi \right] \circ T := [\xi \circ T] \) is well-defined for all \(T \in \mathbb{A}\) and \(\Sigma\)-measurable \(\xi: S \to \mathbb{R}\) if and only if \(\gamma\) is quasi-invariant w.r.t. \(\mathbb{A}\).
2. Let \(\gamma\) be quasi-invariant w.r.t. \(\mathbb{A}\) and \(\mathbb{A}\) be countable. Then \([\xi] \circ T = [\xi]\) for all \(T \in \mathbb{A}\) if and only if \(\xi\) is \(\overline{\mathcal{I}(\mathbb{A})}\)-measurable.
Thus, the definition, has measure zero as well so that the operator $\vartheta B \varphi$ which is a set of co-measure zero because $I \varphi$ have a finite collection we are looking for. Assume now that we obtain a infinite sequence of we take one with maximal measure. We continue in this way. If the procedure stops, then we

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any quasi-atom. Assume

Proof. (1) Assume that $\gamma$ is quasi-invariant and that $\xi \colon S \to \mathbb{R}$ is $\Sigma$-measurable. Then for $\xi_1, \xi_2 \in [\xi]$ it holds that $\gamma(\xi_1 \neq \xi_2) = 0$ and the set

$$\{ s \colon \xi_1(Ts) \neq \xi_2(Ts) \} = \{ T^{-1}t \colon \xi_1(t) \neq \xi_2(t) \}$$

has measure zero as well so that the operator $[\xi] \mapsto [\xi] \circ T$ is well-defined. For the other implication let $B$ be of measure zero and $\xi \colon := 1_B$ so that $[\xi] = 0$. By assumption, $[\xi] \circ T = 0$ and $0 = \gamma(\{ s \colon 1_B(Ts) \neq 0 \}) = \gamma(T^{-1}(B))$.

(2) If there exists an $I(\mathcal{A})$-measurable $\xi_0 \in [\xi]$ it is obvious that the equivalence class is invariant by (1) and Lemma A.2. Conversely, let $[\xi] \circ T = [\xi]$ for all $T \in \mathcal{A}$. Define

$$S_0 := \{ s \in S \colon \xi \circ T(s) = \xi(s) \text{ for all } T \in \mathcal{A} \} = \bigcap_{T \in \mathcal{A}} \{ s \in S \colon \xi \circ T(s) = \xi(s) \},$$

which is a set of co-measure zero because $\mathcal{A}$ is countable. Let us first prove that $S_0 \in I(\mathcal{A})$. By definition, $S_0 \in \Sigma$ and

$$T(S_0) = \{ Ts \in S \colon \xi \circ T'(s) = \xi(s) \text{ for all } T' \in \mathcal{A} \}$$

$$= \{ t \in S \colon \xi \circ T'(T^{-1}t) = \xi(T^{-1}t) \text{ for all } T' \in \mathcal{A} \}$$

$$= \{ t \in S \colon \xi \circ T'(T^{-1}t) = \xi(t) \text{ for all } T' \in \mathcal{A} \text{ and } \xi(T^{-1}t) = \xi(t) \}$$

$$= \{ t \in S \colon \xi \circ T'(t) = \xi(t) \text{ for all } T' \in \mathcal{A} \text{ and } \xi(T^{-1}t) = \xi(t) \}$$

$$= \{ t \in S \colon \xi \circ T'(t) = \xi(t) \text{ for all } T' \in \mathcal{A} \} = S_0,$$

thus $S_0 \in I(\mathcal{A})$. Setting $\xi_0(s) := \xi(s) \mathbb{1}_{S_0}(s)$, we obtain from Lemma A.2 that $\xi_0$ is $\mathcal{A}(\mathcal{A})$-measurable and $\gamma$-a.e. equal to $\xi$.

Definition A.5. Let $(S, \mathcal{I}, \gamma)$ be a $\sigma$-finite measure space with $\gamma(S) > 0$. A set $A \in \mathcal{I}$ with $\gamma(A) > 0$ is called quasi-atom provided that $B \subseteq A$ with $B \in \mathcal{I}$ implies that

$$\gamma(B) = 0 \quad \text{or} \quad \gamma(A \setminus B) = 0.$$

Lemma A.6. Let $(S, \mathcal{I}, \gamma)$ be a $\sigma$-finite measure space with $\gamma(S) > 0$ and $A, A_1, A_2$ be quasi-atoms.

(1) If $B \in \mathcal{I}$ and $\gamma(A \Delta B) = 0$, then $B$ is a quasi-atom.

(2) If $A_1 \subseteq A_2$, then $\gamma(A_2 \setminus A_1) = 0$.

(3) Either $\gamma(A_1 \cap A_2) = 0$ or $\gamma(A_1 \Delta A_2) = 0$.

(4) There exist countably many pairwise disjoint quasi-atoms $(A_i)_{i \in \mathbb{I}}$ such that $S \setminus \bigcup_{i \in \mathbb{I}} A_i$ does not contain any quasi-atom. For any quasi-atom $A$ there is an $i \in \mathbb{I}$ such that $\gamma(A \Delta A_i) = 0$.

Proof. The assumption $\sigma$-finite implies that all quasi-atoms have finite measure. Assertions (1), (2) and (3) are easy to prove and we skip the details.

(4a) First we assume that $\gamma$ is a finite measure. In this case we prove the statement in a constructive way. If there is no quasi-atom, then we are done. If there are quasi-atoms, then we take one with maximal measure (which exists) and call it $A_1$. Now we look for a quasi-atom $A_2 \subseteq A_1$. If there is no such atom, then the proof is again complete. If there is an atom, then we take one with maximal measure. We continue in this way. If the procedure stops, then we have a finite collection we are looking for. Assume now that we obtain an infinite sequence of disjoint quasi-atoms $(A_i)_{i=1}^\infty$ and let $A := \bigcup_{i=1}^\infty A_i$. We have to check that $A'$ cannot contain any quasi-atom. Assume $B \subseteq A'$ is a quasi-atom. Because $\lim_i \gamma(A_i) = 0$ there is some $i$ such that $\gamma(A_i) < \gamma(B)$. But this would be a contradiction to our construction.
In general, assume $S = \bigcup_{j \in J} S_j$ to be a disjoint union with $\gamma(S_j) \in (0, \infty)$. We apply our construction to each $S_j$ and obtain a countable collection $(A_{i,j})_{i \in I,j \in J}$ of quasi-atoms where $I_j$ might be empty. Denote $R_j := S_j \setminus \bigcup_{i \in I_j} A_{i,j}$ and $R := \bigcup_{j \in J} R_j$. Assume that $A \subseteq R$ is a quasi-atom. Then there is exactly one index $j_A$ such that $\gamma(R_{j_A} \cap A) = \gamma(A)$. Letting $B := R_{j_A} \cap A$ gives that $\gamma(A \Delta B) = 0$ and that $B \subseteq R_{j_A}$ is a quasi-atom. But this is again a contradiction to our construction.

The remaining part of (4) is obvious. □

Lemma A.7. Let $(S, \Sigma, \gamma)$ be a $\sigma$-finite measure space with $\gamma(S) > 0$ and $\mathbb{A}$ be a group of automorphisms of $S$ such that $(S, \mathcal{I}(\mathbb{A}), \gamma)$ is $\sigma$-finite. Assume that $(A_i)_{i \in I} \subseteq \mathcal{I}(\mathbb{A})$ is a countable collection of quasi-atoms like in Lemma A.6(4). Then for a function $\xi : S \to \mathbb{R}$ the following assertions are equivalent:

1. $\xi$ is $\mathcal{I}(\mathbb{A})$-measurable.
2. There exists a $\Sigma$-measurable $\eta$ which is constant on the orbits and the quasi-atoms $(A_i)_{i \in I}$ and such that $\eta = \gamma_{\mathbb{A}}(A)$-a.e.

Proof. (2) $\implies$ (1) Using Lemma A.2 we get that $\eta$ is $\mathcal{I}(\mathbb{A})$-measurable, so that $\xi$ is $\mathcal{I}(\mathbb{A})$-measurable.

(1) $\implies$ (2) First we find an $\xi_0 \in [\xi]$ that is $\mathcal{I}(\mathbb{A})$-measurable. It can be easily seen that $\xi_0$ can be modified to an $\mathcal{I}(\mathbb{A})$-measurable random variable $\eta$ satisfying the claimed properties. □

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