LARGEST EIGENVALUE OF THE LAPLACIAN MATRIX

BENJAMIN IRIARTE

Abstract. We study the eigenspace of the Laplacian matrix of a simple graph corresponding to the largest eigenvalue, subsequently arriving at the theory of modular decomposition of T. Gallai.

1. Introduction.

Let $G = G([n], E)$ be a simple (undirected) graph, where $[n] = \{1, 2, \ldots, n\}$, $n \in \mathbb{P}$. The adjacency matrix of $G$ is the $n \times n$ matrix $A = A(G)$ such that:

$$(A)_{ij} = a_{ij} := \begin{cases} 1 & \text{if } \{i, j\} \in E, \\ 0 & \text{otherwise}. \end{cases}$$

The Laplacian matrix of $G$ is the $n \times n$ matrix $L = L(G)$ such that:

$$(L)_{ij} = l_{ij} := \begin{cases} d_i & \text{if } i = j, \\ -a_{ij} & \text{otherwise}, \end{cases}$$

where $(d_G)_i = d_i$ is the degree of vertex $i$ in $G$.

The spectral theory of these matrices, i.e. the theory about their eigenvalues and eigenspaces, has been the object of much study for the last 40 years. The roots of this beautiful theory, however, can arguably be traced back to Kirchhoff’s matrix-tree theorem, whose first proof is often attributed to Borchardt (1860) even though at least one proof was already known by Sylvester (1857). A recollection of some interesting applications of the theory can be found in Spielman (2009), and more complete accounts of the mathematical backbone are Brouwer and Haemers (2011) and Chung (1997). Still, it would be largely inconvenient and prone to unfair omissions to attempt here a fair account of the many contributors and contributing papers that helped shape the state-of-the-art of our knowledge of graph spectra, and we refer the reader to our references for further inquiries of the literature.

This article aims to fill one (of the many) gap(s) in our current knowledge of the theory, namely, the lack of results about eigenvectors of the Laplacian with largest eigenvalue. We will answer the question: What information about the structure of a graph is carried in these eigenvectors? Our work follows the spirit of Fiedler (2011), who pioneered the use of eigenvectors of the Laplacian matrix to learn about a graph’s structure. One of the first observations that can be made about $L$ is that it is positive-semidefinite, a consequence of it being a product of incidence matrices.
We will thus let,

$$0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n = \lambda_{\text{max}} = \lambda_{\text{max}}(G),$$

be the (real) eigenvalues of $L$, and note that $\lambda_2 > 0$ if $G$ is a connected graph; we have effectively dropped $G$ from the notation for convenience but remark that eigenvalues and eigenvectors depend on the particular graph at question, which will be clear from the context. We will also let $E_{\lambda_i}$ be the eigenspace corresponding to $\lambda_i$. In its most primitive form, Fiedler's nodal domain's theorem \cite{Fiedler1975} states that when $G$ is connected and for all $x \in E_{\lambda_2}$, the induced subgraph $G\{i \in [n]: x_i \geq 0\}$ is connected. Related work, also relevant to the present writing, might be found in Merris \cite{Merris1998}.

We will go even further in the way in which we use eigenvectors of the Laplacian to learn properties of $G$. To explain this, let us firstly call a map,

$$O : E \to ([n] \times [n]) \cup E = [n]^2 \cup E,$$

such that $O(e) \in \{e, (i, j), (j, i)\}$ for all $e := \{i, j\} \in E$, an (partial) orientation of $E$ (or $G$), and say that, furthermore, $O$ is acyclic if $O(e) \neq e$ for all $e$ and the directed graph on vertex-set $[n]$ and edge-set $O(E)$ has no directed-cycles. On numerous occasions, we will somewhat abusively also identify $O$ with the set $O(E)$.

During this paper, eigenvectors of the Laplacian and more precisely, elements of $E_{\lambda_{\text{max}}}$, will be used to obtain orientations of certain (not necessarily induced) subgraphs of $G$. Henceforth, given $G$ and for all $x \in \mathbb{R}^{|n|}$, the reader should always automatically consider the orientation (map) $O_x = O_x(G)$ associated to $x$, $O_x : E \to [n]^2 \cup E$, such that for $e := \{i, j\} \in E$:

$$O_x(e) = \begin{cases} 
  e & \text{if } x_i = x_j, \\
  (i, j) & \text{if } x_i < x_j, \\
  (j, i) & \text{if } x_i > x_j.
\end{cases}$$

The orientation $O_x$ will be said to be induced by $x$ (e.g. Figure 1C).

Implicit above is another subtle perspective that we will adopt, explicitly, that vectors $x \in \mathbb{R}^{|n|}$ are real functions from the vertex-set of the graph in question (all our graphs will be on vertex-set $[n]$). In our case, this graph is $G$, and even though accustomed to do so otherwise, entries of $x$ should be really thought of as being indexed by vertices of $G$ and not simply by positive integers. Later on in Section 3 for example, we will regularly state (combinatorial) results about the fibers of $x$ when $x$ belongs to a certain subset of $\mathbb{R}^{|n|}$ (e.g. $E_{\lambda_{\text{max}}}$), thereby regarding these fibers as vertex-subsets of the particular graph being discussed at that moment.

Using this perspective, we will learn that the eigenspace $E_{\lambda_{\text{max}}}$ is closely related to the theory of modular decomposition of Gallai \cite{Gallai1967}; orientations induced by elements of $E_{\lambda_{\text{max}}}$ lead naturally to the discovery of modules. This connection will most concretely be exemplified when $G$ is a comparability graph, in which case these orientations iteratively correspond to and exhaust the transitive orientations of $G$. It will be instructive to see Figure 1 at this point.

In Section 2 we will introduce the background and definitions necessary to state the precise main contributions of this article. These punch line results will then be presented in Section 3. The central theme of Section 3 will be a stepwise proof of Theorem 3.1: our main result for comparability graphs, which summarily states that when $G$ is a comparability graph, elements of $E_{\lambda_{\text{max}}}$ induce transitive orientations of the copartition subgraph of $G$. It will be along the natural course of this
proof that we present our three main results that apply to arbitrary simple graphs: Propositions 3.10 and 3.11 and Corollary 3.12.

Finally, in Section 4, we will present a curious novel characterization of comparability graphs that results from the theory of Section 3.

2. Background and definitions.

2.1. The graphical arrangement.

**Definition 2.1.** Let \( G = (V,E) \) be a simple (undirected) graph. The graphical arrangement of \( G \) is the union of hyperplanes in \( \mathbb{R}^{|V|} \):

\[
\mathcal{A}_G := \{ x \in \mathbb{R}^{|V|} : x_i - x_j = 0, \forall \{i,j\} \in E \}.
\]

Basic properties of graphical arrangements and, more generally, of hyperplane arrangements, are presented in Chapter 2 of Stanley (2004).

For \( G \) as in Definition 2.1, let \( \mathcal{R}(\mathcal{A}_G) \) be the collection of all (open) connected components of the set \( \mathbb{R}^{|V|} \setminus \mathcal{A}_G \). An element of \( \mathcal{R}(\mathcal{A}_G) \) is called a region of \( \mathcal{A}_G \), and every region of \( \mathcal{A}_G \) is therefore an \( n \)-dimensional open convex cone in \( \mathbb{R}^{|V|} \).

Furthermore, the following is true about regions of the graphical arrangement:

**Proposition 2.2.** Let \( G \) be as in Definition 2.1. Then, for all \( x, y \in \mathcal{R}(\mathcal{A}_G) \), we have that:

\[
O_x \cap O_y = \emptyset.
\]

Moreover, the map \( R \mapsto O_R \) from the set of regions of \( \mathcal{A}_G \) to the set of orientations of \( E \) is a bijection between \( \mathcal{R}(\mathcal{A}_G) \) and the set of acyclic orientations of \( G \).

Motivated by Proposition 2.2 and the comments before, we will introduce special notation for certain subsets of \( \mathbb{R}^{|V|} \) obtained from \( \mathcal{A}_G \).

**Notation 2.3.** Let \( G \) be as in Definition 2.1. For an acyclic orientation \( O \) of \( E \), we will let \( C_O \) denote the \( n \)-dimensional closed convex cone in \( \mathbb{R}^{|V|} \) that is equal to the topological closure of the region of \( \mathcal{A}_G \) corresponding to \( O \) in Proposition 2.2.

2.2. Modular decomposition.

We need to concur on some standard terminology and notation from graph theory, so let \( G = (V,E) \) be a simple (undirected) graph and \( X \) a subset of \([n]\).

As customary, \( G \) denotes the complement graph of \( G \). The notation \( N(X) \) denotes the open neighborhood of \( X \) in \( G \):

\[
N(X) := \{ j \in [n] \setminus X : \text{there exists some } i \in X \text{ such that } \{i,j\} \in E \}.
\]

The induced subgraph of \( G \) on \( X \) is denoted by \( G[X] \), and the binary operation of graph disjoint union is represented by the plus sign \( + \). Lastly, for \( Y \subseteq [n] \), \( X \) and \( Y \) are said to be completely adjacent in \( G \) if:

\[
X \cap Y = \emptyset, \quad \text{and}
\]

for all \( i \in X \) and \( j \in Y \), we have that \( \{i,j\} \in E \).

The concepts of module and modular decomposition in graph theory were introduced by Gallai (1967) as a means to understand the structure of comparability graphs. The same work would eventually present a remarkable characterization of these graphs in terms of forbidden subgraphs. Section 3 of the present work will present an alternate and surprising route to modules.
Definition 2.4. Let $G = G([n], E)$ be a simple (undirected) graph. A module of $G$ is a set $A \subseteq [n]$ such that for all $i, j \in A$:

$$N(i) \setminus A = N(j) \setminus A = N(A).$$

Furthermore, $A$ is said to be proper if $A \subseteq [n]$, non-trivial if $|A| > 1$, and connected if $G[A]$ is connected.

Corollary 2.5. In Definition 2.4, two disjoint modules of $G$ are either completely adjacent or no edges exist between them.

Let us now present some basic results about modules that we will need.

Lemma 2.6 (Gallai (1967)). Let $G = ([n], E)$ be a connected graph such that $G$ is connected. If $A$ and $B$ are maximal (by inclusion) proper modules of $G$ with $A \neq B$, then $A \cap B = \emptyset$.

Corollary 2.7 (Gallai (1967)). Let $G = ([n], E)$ be a connected graph such that $G$ is connected. Then, there exists a unique partition of $[n]$ into maximal proper modules of $G$, and this partition contains more than two blocks.

From Corollary 2.7, it is therefore natural to consider the partition of the vertex-set of a graph into its maximal modules; the appropriate framework for doing this is presented in Definition 2.8. Hereafter, however, we will assume that our graphs are connected unless otherwise stated since (1) the results for disconnected graphs will follow immediately from the results for connected graphs, and (2) this will allow us to focus on the interesting parts of the theory.

Definition 2.8 (Ramírez-Alfonsín and Reed (2001)). Let $G = G([n], E)$ be a connected graph. We will let the canonical partition of $G$ be the set $\mathcal{P} = \mathcal{P}(G)$ such that:

a. If $G$ is connected, $\mathcal{P}$ is the unique partition of $[n]$ into the maximal proper modules of $G$.

b. If $G$ is disconnected, $\mathcal{P}$ is the partition of $[n]$ into the vertex-sets of the connected components of $G$.

Hence, in Definition 2.8 every element of the canonical partition is a module of the graph. Elements of the canonical partition of a graph on vertex-set $[8]$ are shown in Figure 1B.

Definition 2.9. In Definition 2.8, we will let the copartition subgraph of $G$ be the graph $G^\mathcal{P}$ on vertex-set $[n]$ and edge-set equal to:

$$E \setminus \{i, j\} \in E : i, j \in A \text{ for some } A \in \mathcal{P}.$$

2.3. Comparability graphs.

We had anticipated the importance of comparability graphs in this work, yet, we need to define what they are.

Definition 2.10. A comparability graph is a simple (undirected) graph $G = G(V, E)$ such that there exists a partial order on $V$ under which two different vertices $u, v \in V$ are comparable if and only if $\{u, v\} \in E$.

A comparability graph on vertex-set $[8]$ is shown in Figure 1B.

Comparability graphs are perfectly orderable graphs and more generally, perfect graphs. These three families of graphs are all large hereditary classes of graphs.
Note that, given a comparability graph \( G = G(V, E) \), we can find at least two partial orders on \( V \) whose comparability graphs (obtained as discussed in Definition 2.10) agree precisely with \( G \), and the number of such partial orders depends on the modular decomposition of \( G \). Let us record this idea in a definition.

**Definition 2.11.** Let \( G = G(V, E) \) be a comparability graph, and let \( O \) be an acyclic orientation of \( E \). Consider the partial order induced by \( O \) under which, for \( u, v \in V \), \( u \) is less than \( v \) iff there is a directed-path in \( O \) that begins in \( u \) and ends in \( v \). If the comparability graph of this partial order on \( V \) (obtained as in Definition 2.10) agrees precisely with \( G \), then we will say that \( O \) is a transitive orientation of \( G \).

![Figure 1](image)

**Figure 1.** (A) Hasse diagram of a poset \( P \) on [8]. (B) Comparability graph \( G = G([8], E) \) of the poset \( P \), where closed regions are maximal proper modules of \( G \). (C) Unit eigenvector \( x \in E_{\lambda_{\text{max}}} \) of \( G \) fully calculated, where \( \dim \langle E_{\lambda_{\text{max}}} \rangle = 1 \). Arrows represent the induced orientation \( O_x \) of \( G \). Notice the relation between \( O_x \), modules of \( G \), and poset \( P \).

### 2.4. Linear algebra.

Some standard terminology of linear algebra and other related conventions that we adopt are presented here. Firstly, we will always be working in Euclidean space \( \mathbb{R}^n \), and all (Euclidean-normed real) vector spaces considered are assumed to live therein. Euclidean norm is denoted by \( \| \cdot \| \). The standard basis of \( \mathbb{R}^n \) will be \( \{e_i\}_{i \in [n]} \), as customary. Generalizing this notation, for all \( I \subseteq [n] \), we will also let:

\[
e_I := \sum_{i \in I} e_i.
\]

The orthogonal complement in \( \mathbb{R}^n \) to \( \text{span}_{\mathbb{R}} \langle e_{[n]} \rangle \) will be of importance to us, so we will use special notation to denote it:

\[
\mathbb{R}^*_{[n]} := \left( \text{span}_{\mathbb{R}} \langle e_{[n]} \rangle \right)^\perp.
\]

For an arbitrary vector space \( V \) and a linear transformation \( T : V \to V \), we will say that a set \( U \subseteq V \) is invariant under \( T \), or that \( T \) is \( U \)-invariant, if \( T(U) \subseteq U \).
Lastly, a key concept of this paper:

For a vector \( x \in \mathbb{R}^n \) and a set \( \xi \subseteq [n] \),

we will say that \( \xi \) is a fiber of \( x \)

if there exists \( \alpha \in \mathbb{R} \) such that \( x_i = \alpha \) if and only if \( i \in \xi \).

The notion of being a generic vector in a certain vector space, to be understood
from the point of view of Lebesgue measure theory, is a central ingredient in many
of our results. We now make this notion precise.

**Definition 2.12.** Let \( V \) be a linear subspace of \( \mathbb{R}^n \) with \( \dim x V \neq 0 \).
We will say that a vector \( x \in V \) is a uniformly chosen at random unit vector or u.c.u.v. if
\( x \) is uniformly chosen at random from the set \( \{ y \in V : ||y|| = 1 \} \).

For \( x \in V \) a u.c.u.v., a certain event or statement about \( x \) is said to occur or hold true almost surely if it is true with probability one.

### 2.5. Spectral theory of the Laplacian.

We will need only a few background results on the spectral theory of the Laplacian matrix of a graph. We present these below in a single statement, but refer the reader to Brouwer and Haemers (2011) for additional background and history.

**Lemma 2.13.** Let \( G = G([n], E) \) be a simple (undirected) graph. Let \( L = L(G) \) be the Laplacian matrix of \( G \) and \( 0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n = \lambda_{\max} = \lambda_{\max}(G) \) be the eigenvalues of \( L \). Then:

1. The number of connected components of \( G \) is equal to the multiplicity of the eigenvalue 0 in \( L \).
2. If \( \overline{G} \) is the complement of \( G \) and \( \overline{L} \) is the Laplacian matrix of \( \overline{G} \), then \( \overline{L} = nI - J - L \), where \( I \) is the \( n \times n \) identity matrix and \( J \) is the \( n \times n \) matrix of all-1’s. Consequently, \( \lambda_{\max} \leq n \).
3. If \( H \) is a (not necessarily induced) subgraph of \( G \) on the same vertex-set \([n]\), and if \( \mu_1 \leq \mu_2 \leq \cdots \leq \mu_n \) are the eigenvalues of the Laplacian of \( H \), then \( \lambda_i \geq \mu_i \) for all \( i \in [n] \).

**Lemma 2.13** Part 1’s proof was discussed during the Introduction (Section 1), and Part 2 is a straightforward verification, but Part 3 is a more advanced result.

### 3. Largest Eigenvalue of a Comparability Graph.

The main goal of this section is to prove the following theorem:

**Theorem 3.1.** Let \( G = G([n], E) \) be a connected comparability graph with Laplacian matrix \( L = L(G) \) and canonical partition \( P = P(G) \). Let \( \lambda_{\max} = \lambda_{\max}(G) \) be the largest eigenvalue of \( L \) and \( E_{\lambda_{\max}} \) its associated eigenspace. Then, the following are true:

i. If \( O \) is a transitive orientation of \( G \), then:

\[
\dim \langle CO \cap E_{\lambda_{\max}} \rangle = \dim \langle E_{\lambda_{\max}} \rangle.
\]

ii. \( E_{\lambda_{\max}} \subseteq \bigcup_O CO \), where the union is over all transitive orientations of \( G \).

iii. Let \( x \in E_{\lambda_{\max}} \) be a u.c.u.v.. Almost surely:

1. If \( A \in P \), then \( A \) belongs to a fiber of \( x \).
2. If \( A, A' \in P \) are completely adjacent in \( G \), then \( A \) and \( A' \) belong to different fibers of \( x \).
3. $x$ induces a transitive orientation of $G^P$. In particular, $G^P$ is a comparability graph.
4. All transitive orientations of $G^P$ can be induced by $x$ with positive probability.
5. If $x$ is a fiber of $x$, then:
   \[ G[x] = G[B_1] + \cdots + G[B_k], \]
   where for all $i \in [k]$, $B_i$ is a connected module of $G$ and $G[B_i]$ is a comparability graph.
6. $G$ has exactly two transitive orientations if and only if $\dim \langle E_{\lambda_{\text{max}}} \rangle = 1$ and every fiber of $x$ is an independent set of $G$.

iv. If $G$ is connected, then $\dim \langle E_{\lambda_{\text{max}}} \rangle = 1$. If $G$ is disconnected, then $\dim \langle E_{\lambda_{\text{max}}} \rangle$ is equal to the number of connected components of $G$ minus one.

**Remark 3.2** (to Theorem 3.1). In fact, as it will be explained, all transitive orientations of $G$ can be obtained with the following procedure: Select an arbitrary transitive orientation for $G^P$, and select arbitrary transitive orientations for (the connected components of) each $G[A], A \in \mathcal{P}$. Therefore, i-iii imply an iterative algorithm that obtains every transitive orientation of $G$ with positive probability.

The proof of Theorem 3.1 will be stepwise and its notation and conventions will carry over to the next results, unless otherwise stated. Let us begin with this work.

**Proposition 3.3.** Let $G = G([n], E)$ be a connected comparability graph and let $C_0$ be the (closed convex) cone corresponding to a transitive orientation $O$ of $G$. Then, $C_0$ contains a non-zero eigenvector of $L$ with eigenvalue $\lambda_{\text{max}}$. Furthermore:

\[ \dim \langle C_0 \cap E_{\lambda_{\text{max}}} \rangle = \dim \langle E_{\lambda_{\text{max}}} \rangle. \]

**Proof.** The cases $n = 1$ and $n = 2$ are easy to verify, so we assume that $n > 2$.

The proof consists of two main steps. Firstly, we will prove that $C_0$ is invariant under left-multiplication by $L$. Then, we will prove that $\dim \langle C_0 \cap E_{\lambda_{\text{max}}} \rangle = \dim \langle E_{\lambda_{\text{max}}} \rangle$.

**Step 1:** $Lx \in C_0$ whenever $x \in C_0$.

Take an arbitrary vector $x \in C_0$ and let $(i, j) \in E$ with $(i, j) \in O$. Hence, $x_i \leq x_j$. If we consider the vector $Lx$, then:

\[
(Lx)_j - (Lx)_i = (x_j \deg j - \sum_{k \in N(j)} x_k) - (x_i \deg i - \sum_{\ell \in N(i)} x_\ell)
\]

\[
= \sum_{k \in N(j)} (x_j - x_k) - \sum_{\ell \in N(i)} (x_i - x_\ell)
\]

\[
= |N(i) \cap N(j)| (x_j - x_i) + \sum_{\ell \in N(j) \setminus N(i)} (x_j - x_\ell)
\]

\[
- \sum_{m \in N(i) \setminus N(j)} (x_i - x_m).
\]

Now, since $O$ is transitive and $G$ is comparability, if $\ell \in N(j) \setminus N(i)$, then we must have that $(\ell, j)$ is an edge in $O$, so that $x_\ell \leq x_j$ since $x \in C_0$. Otherwise, we would require that $(i, \ell) \in E$, which is false. Similarly, if $m \in N(i) \setminus N(j)$, we must have that $(i, m)$ is an edge in $O$, so $x_m \geq x_i$. Since also $x_j \geq x_i$ then, we see that $(Lx)_j - (Lx)_i \geq 0$. Verification of the analogous condition for every edge of $E$ shows that indeed $Lx \in C_0$. 


Step 2: \( \dim \langle C_0 \cap E_{\lambda_{\text{max}}} \rangle = \dim \langle E_{\lambda_{\text{max}}} \rangle \).

Suppose on the contrary that \( \dim \langle C_0 \cap E_{\lambda_{\text{max}}} \rangle < \dim \langle E_{\lambda_{\text{max}}} \rangle \). Then, there exists \( x^* \in E_{\lambda_{\text{max}}} \setminus \text{span}_\mathbb{R} \langle C_0 \cap E_{\lambda_{\text{max}}} \rangle \). Since \( C_0 \) is full-dimensional in \( \mathbb{R}^{[n]} \), we can write \( x^* = x - y \) for some \( x, y \in C_0 \), where necessarily either \( x \notin E_{\lambda_{\text{max}}} \) or \( y \notin E_{\lambda_{\text{max}}} \). In fact, we must have that \( x, y \notin E_{\lambda_{\text{max}}} \). Otherwise, if \( y \in E_{\lambda_{\text{max}}} \), then
\[
  x^* = \lim_{N \to \infty} L^N (x - y)/||L^N (x - y)|| = \lim_{N \to \infty} L^N x/||L^N x|| = E_{\lambda_{\text{max}}} \text{ from Step 1, and similarly, if } x \in E_{\lambda_{\text{max}}} \text{, then } x^* \in C_0, \text{ so in both cases } x^* \in \text{span}_\mathbb{R} \langle C_0 \cap E_{\lambda_{\text{max}}} \rangle.
\]

Hence, \( 0 < ||L^N x||, ||L^N y|| \leq \lambda_{\text{max}} \max(||x||, ||y||) \) for all \( N \geq 1 \) and, moreover, since both \( L^N x/||L^N x|| \) and \( L^N y/||L^N y|| \) can be made arbitrarily close to \( \text{span}_\mathbb{R} \langle C_0 \cap E_{\lambda_{\text{max}}} \rangle \) (in particular, using Step 1, each gets close to \( C_0 \cap E_{\lambda_{\text{max}}} \)) for large \( N \), then the same will be true for
\[
  \frac{L^N x - L^N y}{\lambda_{\text{max}} \max(||x||, ||y||)} = c \frac{L^N x^*}{\lambda_{\text{max}} ||x^*||} = cx^*,
\]
where \( c = \frac{||x^*||}{\max(||x||, ||y||)} \neq 0 \). Therefore, letting \( N \to \infty \), we obtain that \( x^* \in \text{span}_\mathbb{R} \langle C_0 \cap E_{\lambda_{\text{max}}} \rangle \). This contradicts our choice of \( x^* \), so:
\[
  E_{\lambda_{\text{max}}} \setminus \text{span}_\mathbb{R} \langle C_0 \cap E_{\lambda_{\text{max}}} \rangle = \emptyset.
\]

\[\square\]

Lemma 3.4. Let \( G = G([n], E) \) be a connected comparability graph and let \( O \) be a transitive orientation of \( G \). If \( x \in C_0 \cap E_{\lambda_{\text{max}}} \), \( x \neq 0 \), satisfies that \( x_u = x_v = \alpha \) for some \( \{u, v\} \in E \) and \( \alpha \in \mathbb{R} \), then there must exist \( A \subset [n] \) such that:

i. \( A \) is a (proper non-trivial) connected module of \( G \) and \( u, v \in A \).

ii. \( x_i = \alpha \) for all \( i \in A \).

Proof. That such an \( x \) may exist is the content of Proposition 3.3, but we are assuming here that indeed, such an \( x \) exists with the stated properties.

Consider the maximal (by inclusion) set \( A \subset [n] \) such that \( G[A] \) is connected, \( u, v \in A \), and \( x_k = \alpha \) for all \( k \in A \). Primarily, \( G[A] \) cannot be equal to \( G \), since that would imply that \( x \) is equal to \( \alpha e_{[n]} \), which is impossible. Hence, \( G[A] \) is a proper non-trivial connected induced subgraph of \( G \).

We will show that \( A \) is a (proper non-trivial connected) module of \( G \). Suppose on the contrary, that \( A \) is not a module of \( G \). Then, there must exist two vertices \( i, j \in A \) such that \( N(i) \setminus A \neq N(j) \setminus A \). Consequently, \( N(i) \Delta N(j) \setminus A \neq \emptyset \). Furthermore, considering a path in \( G[A] \) connecting \( i \) and \( j \), we observe that we may assume that \( i \) and \( j \) are adjacent in \( G[A] \), so that \( \{i, j\} \in E \). Under this assumption, suppose now that \( \{i, j\} \) is an edge in \( O \). As \( O \) is transitive, we must have that \( \{i, k\} \) is an edge in \( O \) whenever \( (j, k) \) is. Similarly, \( (k, j) \) must be an edge in \( O \) whenever \( (k, i) \) is. As such, since \( N(i) \setminus A \neq N(j) \setminus A \), then it must be the case that for \( k \in N(i) \Delta N(j) \setminus A \):

If \( k \in N(i) \), then \( \{i, k\} \) is an edge in \( O \);

and if \( k \in N(j) \), then \( \{k, j\} \) is an edge in \( O \).
Left-Multiplying $x$ by the Laplacian of $G$, we obtain:

$$0 = \lambda_{\max} x - \lambda_{\max} x = \lambda_{\max} x_j - \lambda_{\max} x_i$$

$$= (Lx)_j - (Lx)_i = \sum_{k \in N(j)} (x_j - x_k) - \sum_{\ell \in N(i)} (x_i - x_\ell)$$

$$= \sum_{k \in N(j) \setminus A \cup N(i)} (x_j - x_k) - \sum_{\ell \in N(i) \setminus A \cup N(j)} (x_i - x_\ell)$$

$$= \sum_{k \in N(j) \setminus A \cup N(i)} |x_j - x_k| + \sum_{\ell \in N(i) \setminus A \cup N(j)} |x_i - x_\ell|.$$

Since $N(i) \triangle N(j) \setminus A \neq \emptyset$ and $A$ was chosen maximal, then at least one of the terms in the last summations must be non-zero and we obtain a contradiction. This proves that $A$ is a module of $G$ with the required properties.

\[ \Box \]

**Theorem 3.5.** Let $G = G([n], E)$ be a connected comparability graph without proper non-trivial connected modules. Then:

i. Any $x \in \mathbf{E}_{\lambda_{\max}} \setminus \{0\}$ induces a transitive orientation of $G$.

ii. $\dim \langle \mathbf{E}_{\lambda_{\max}} \rangle = 1$.

iii. $G$ has exactly two transitive orientations.

**Proof.** The cases $n = 1$ and $n = 2$ are easy to check, so we assume that $n > 2$.

Fix a transitive orientation $O$ of $G$ and consider the cone $C_O$. Per Proposition 3.3 we can find at least one $x \in C_O \cap \mathbf{E}_{\lambda_{\max}}, x \neq 0$. By Lemma 3.4 and since $G$ does not have proper non-trivial connected modules, $x$ must belong to the interior of $C_O$. This establishes i.

To prove ii, assume on the contrary, that $\dim \langle \mathbf{E}_{\lambda_{\max}} \rangle > 1$. Consider two dual transitive orientations $O$ and $O_{\text{dual}}$ of $G$, i.e. $O_{\text{dual}}$ is obtained from $O$ by reversion of the orientation of all the edges. Using i, let $y, z \in \mathbf{E}_{\lambda_{\max}} \setminus \{0\}$ be such that $y \in \text{int}(C_O), z \in \text{int}(C_{O_{\text{dual}}})$, and $z \notin \text{span}_R \langle y \rangle$. Then, there exists $\alpha \in (0, 1)$ such that $0 \neq \alpha y + (1 - \alpha)z \in \partial \langle C_O \cap \mathbf{E}_{\lambda_{\max}} \rangle$, contradicting i.

Finally, iii follows easily from i-ii and Proposition 3.3.

\[ \Box \]

The remaining part of the theorem will rely heavily on some standard results of the spectral theory of the Laplacian (Section 2.5). These will be of central importance to establish Proposition 3.10, Proposition 3.11 and Corollary 3.12, which deal with arbitrary simple graphs.

**Lemma 3.6.** Let $G = G([n], E)$ be a complete $p$-partite graph with maximal independent sets $A_1, \ldots, A_p$. Then, $\lambda_{\max} = n$ and:

$$\mathbf{E}_{\lambda_{\max}} = \{ x \in \mathbb{R}^*[n] : \text{if } i, j \in A_q \text{ for some } q \in [p], \text{ then } x_i = x_j \}$$

$$= \text{span}_R \langle \{ e_{A_q} \}_{q \in [p]} \rangle \cap \mathbb{R}^*[n].$$

In particular, $\dim \langle \mathbf{E}_{\lambda_{\max}} \rangle = p - 1$.

**Proof.** The complement of $G$ has $p$ connected components, so by Parts 1 and 2 in Lemma 2.13, $\lambda_{\max} = n$ and $\dim \langle \mathbf{E}_{\lambda_{\max}} \rangle = p - 1$. Let $b_1, \ldots, b_p \in \mathbb{R}$ and let $x \in \mathbb{R}^*[n]$ be such that $x_i = b_q$ for all $i \in A_q, q \in [p]$. For any $i \in [n]$, if $i \in A_q$ then $(Lx)_i = (n - |A_q|)b_q - (0 - |A_q|b_q) = nb_q = nx_i$. The set of all such $x$ has dimension $p - 1$. 

\[ \Box \]
Lemma 3.7. Let $G = G([n], E)$ be a connected bipartite graph with bipartition $\{X, Y\}$. Then, $\dim \langle \mathbf{E}_{\lambda_{\text{max}}} \rangle = 1$. Furthermore, if $x \in \mathbf{E}_{\lambda_{\text{max}}} \setminus \{0\}$, then either $x_i < 0$ for all $i \in X$ and $x_j > 0$ for all $j \in Y$, or vice-versa.

Proof. If $G$ is complete 2-partite, this is a consequence of Lemma 3.6. Otherwise, as a connected bipartite graph, $G$ is also a comparability graph and $G$ does not have connected proper non-trivial modules, so Theorem 3.5 shows that $\dim \langle \mathbf{E}_{\lambda_{\text{max}}} \rangle = 1$ and that $x \in \mathbf{E}_{\lambda_{\text{max}}} \setminus \{0\}$ induces a transitive orientation of $G$. So take $x \in \mathbf{E}_{\lambda_{\text{max}}} \setminus \{0\}$ and suppose that $x_i = 0$, $i \in X$. Then, $(Lx)_i \neq 0$ as $x$ induces a transitive orientation of $G$ and since $G$ is connected.

□

We have not found an agreed-upon notation in the literature for the following objects, so we will need to introduce it here.

Definition 3.8. Let $G = G([n], E)$ be a simple connected graph, and let $Q = \{X_1, \ldots, X_m\}$ be a partition of $[n]$ with non-empty blocks. Then, for all $k \in [m]$:

a. $G_{X_k}$ will denote the graph on vertex-set $[n]$ and edge-set:
$$\{i, j \in E : i, j \in X_k\}.$$  

b. $R_{X_k} := \{x \in \mathbb{R}^*[n] : x_i = 0 \text{ if } i \notin X_k, i \in [n]\}.$

Also,
$$R^Q := \{x \in \mathbb{R}^*[n] : x \text{ is constant on each } X_k, k \in [m]\}$$
$$= \text{span}_\mathbb{R} \langle \{e_{X_k}\} \rangle_{X_k \in [m]} \cap \mathbb{R}^*[n].$$

Observation 3.9. In Definition 3.8, the linear subspaces $R^Q$ and $R_{X_k}$ for all $k \in [m]$, are mutually orthogonal.

Furthermore, any vector $x \in \mathbb{R}^*[n]$ can be uniquely written as:
$$x = y + x_1 + x_2 + \cdots + x_m,$$
with $y \in R^Q$ and $x_k \in R_{X_k}$, $k \in [m]$.

We are now ready to present the results about the space $\mathbf{E}_{\lambda_{\text{max}}}$ for simple graphs. Their proofs will use the same language and main ideas, so we will present them contiguously to make this resemblance clear.

Proposition 3.10. Let $G = G([n], E)$ be a connected simple graph such that $G$ is connected. For any fixed proper module $A$ of $G$, the following is true: If $x \in \mathbf{E}_{\lambda_{\text{max}}}$, then $A$ belongs to a fiber of $x$.

Proposition 3.11. Let $G = G([n], E)$ be a connected simple graph such that $G$ is disconnected. Then, $\lambda_{\text{max}} = n$ and:
$$\mathbf{E}_{\lambda_{\text{max}}} = \{x \in \mathbb{R}^*[n] : x_i = x_j, \text{ whenever } i \text{ and } j \text{ belong to the same connected component of } G\}.$$  

In particular, $\dim \mathbf{E}_{\lambda_{\text{max}}}$ is equal to the number of connected components of $G$ minus one, and $G^P$ is a complete $p$-partite graph, where $p$ is the number of connected components of $G$. 

□
LARGEST EIGENVALUE OF THE LAPLACIAN

Proof of Proposition 3.10. Let $I$ be the $n \times n$ identity matrix. As usual, $P = \{A_1, \ldots, A_p\}$ will be the canonical partition of $G$. Let $L$ be the Laplacian matrix of $G$, $L^P$ be the Laplacian matrix of the copartition subgraph $G^P$ of $G$, and $L_{A_q}$ be the Laplacian matrix of $G_{A_q}$ for $q \in [p]$. Firstly, we observe that $L = L^P + \sum_{q=1}^{p} L_{A_q}$.

Proof of Proposition 3.10. The plan of the proof is to show that the eigenspace of $L^P$ corresponding to its largest eigenvalue lives inside $R^P$, and then to show that this eigenspace is precisely equal to $E_{\lambda_{\text{max}}}$. This will be sufficient since $A \subseteq A_q$ for some $q \in [p]$. To prove the first claim, first note that left-multiplication by $L^P$ is $R^P$-invariant, where the condition that the $A_q$'s are modules is fundamental to prove this. Now, for any $x \in \mathbb{R}^{s[n]}$, and writing $x = y + x_1 + \cdots + x_p$ with $y \in R^P$ and $x_q \in R_{A_q}$, $q \in [p]$, we have that:

$$L^P x = L^P y + \sum_{q=1}^{p} |N(A_q)| x_q.$$ 

Hence, by Observation 3.9 if we can show that the largest eigenvalue of $L^P$ is strictly greater than $\max\{|N(A_q)|, q \in [p]\}$, the claim will follow. This is what we will do now.

In fact, we will prove that the largest eigenvalue of $L^P$ is strictly greater than $\max\{|N(A_q)| + |A_q|, q \in [p]\}$. To check this, first note that both $G^P$ and its complement are connected graphs, and that for $q \in [p]$, $A_q$ is both a maximal proper module and an independent set of $G^P$. For an arbitrary $q \in [p]$, consider the (not necessarily induced) subgraph $H_{-q}$ of $G^P$ on vertex-set $A_q \cup N(A_q)$ and whose edge-set is $\{(i, j) \in E : i \in A_q$ and $j \in N(A_q)\}$. Firstly, $H_{-q}$ is a complete 2-partite graph, so its largest eigenvalue is precisely $|N(A_q)| + |A_q|$ from Lemma 3.6. Secondly, since both $G^P$ and its complement are connected, there exists a (not necessarily induced) connected bipartite subgraph $H$ of $G^P$ such that $H_{-q} = H[A_q \cup N(A_q)]$ and $H \neq H_{-q}$. By Lemma 2.13 Part 3 and Lemma 3.7, the largest eigenvalue of the Laplacian matrix of $H$ must be strictly greater than that of $H_{-q}$, since any non-zero eigenvector for this eigenvalue must be non-zero on the vertices of $H$ that are not vertices of $H_{-q}$. Also, by the same Lemma 2.13 Part 3, the largest eigenvalue of $L^P$ must be at least equal to the largest eigenvalue of the Laplacian matrix of $H$. This proves the first claim.

To prove the second claim, note that for $q \in [p]$, left-multiplication by $L_{A_q}$ is $R_{A_q}$-invariant. Also, for an arbitrary $x \in \mathbb{R}^{s[n]}$ decomposed as above, we have that:

$$Lx = L^P y + \sum_{q=1}^{p} (|N(A_q)| I + L_{A_q}) x_q,$$

and this gives the unique decomposition of $Lx$ of Observation 3.9. But then, from the proof of the first claim, we note that it suffices to prove that the largest eigenvalue of $L^P$ is strictly greater than that of $|N(A_q)| I + L_{A_q}$ for any $q \in [p]$. However, from Lemma 2.13 Part 1, we know that the largest eigenvalue of $L_{A_q}$ is at most $|A_q|$, so the largest eigenvalue of $|N(A_q)| I + L_{A_q}$ is at most $|N(A_q)| + |A_q|$. We have already proved that the largest eigenvalue of $L^P$ is strictly greater than $\max\{|N(A_q)| + |A_q|, q \in [p]\}$, so the second claim follows. □
Proof of Proposition 3.11. That $G^P$ is a complete $p$-partite graph is clear, so from Lemma 3.6 it will suffice to prove that $E_{\lambda_{\text{max}}}$ is exactly equal to the eigenspace of $L^P$ corresponding to its largest eigenvalue ($= n$). This is what we do.

As in the proof of Proposition 3.10 we observe that left-multiplication by $L^P$ is $R^P$-invariant, and that for $q \in [p]$, left-multiplication by $L_{A_q}$ is $R_{A_q}$-invariant. For an arbitrary $x \in \mathbb{R}^{|P|}$ with $x = y + x_1 + \cdots + x_p$, where $y \in \mathbb{R}^P$ and $x_q \in R_{A_q}$, $q \in [p]$, and noting that $|N(A_q)| = n - |A_q|$ in this case, we have that:

$$Lx = L^P y + \sum_{q=1}^p ((n - |A_q|)I + L_{A_q})x_q,$$

and this gives the unique decomposition of $Lx$ of Observation 3.9. Hence, we will be done if we can show that the largest eigenvalue of any of the matrices $L_{A_q}$, $q \in [p]$, is strictly less than $|A_q|$. However, since by construction (from the definition of canonical partition), $G_{[A_q]}$ satisfies that its complement is connected, then Lemma 2.13 Parts 1 and 2 imply that the largest eigenvalue $L_{A_q}$ is strictly less than $|A_q|$, and this holds for all $q \in [p]$. This completes the proof.

Corollary 3.12. Let $G = G([n], E)$ be a connected simple graph with canonical partition $P$ (with $L$ and $E_{\lambda_{\text{max}}}$ as usual). If $L^P$ denotes the Laplacian matrix of $G^P$, then the eigenspace of $L^P$ corresponding to the largest eigenvalue coincides with $E_{\lambda_{\text{max}}}$.

Let us now turn back our attention to comparability graphs and to the proofs of Theorem 4.1 and Theorem 3.1. Comparability graphs are, as anticipated, specially amenable to apply the previous two propositions and their corollary. In fact, the following result already establishes most of Theorem 3.1.

Proposition 3.13. Let $G = G([n], E)$ be a connected comparability graph with canonical partition $P$.

i. For $x \in E_{\lambda_{\text{max}}}$ a u.c.u.v., the following hold true almost surely:
   1. If $A \in P$, then $A$ belongs to a fiber of $x$.
   2. If $A, A' \in P$ are completely adjacent in $G$, then $A$ and $A'$ belong to different fibers of $x$.
   3. $x$ induces a transitive orientation of $G^P$. In particular, $G^P$ is a comparability graph.
   4. If $\xi$ is a fiber of $x$, then:
      $$G[\xi] = G[B_1] + \cdots + G[B_k],$$
      where for all $i \in k$, $B_i$ is a connected module of $G$ and $G[B_i]$ is a comparability graph.

ii. If $\overline{G}$ is connected, then $\dim (E_{\lambda_{\text{max}}}) = 1$. Also, $G^P$ has exactly two transitive orientations and each can be obtained with probability $\frac{1}{2}$ in i.

iii. If $\overline{G}$ is disconnected, then $\dim (E_{\lambda_{\text{max}}}) = p - 1$, where $p$ is the number of connected components of $\overline{G}$. Also, $G^P$ has exactly $p!$ transitive orientations and each can be obtained with positive probability in i.

Proof. We will work on each case, whether $\overline{G}$ is connected or disconnected, separately.

Case 1: $\overline{G}$ is connected.
From Proposition 3.3. take any \( x \in C_O \cap E_{\lambda_{\text{max}}}, \) \( x \neq 0, \) for some transitive orientation \( O \) of \( G. \) From Proposition 3.10, we know that \( x \) is constant on each \( A \in \mathcal{P}, \) so i.1 holds. Moreover, since the elements of \( \mathcal{P} \) are the maximal proper modules of \( G, \) then Lemma 3.4 shows that for completely adjacent \( A, A' \in \mathcal{P}, \) \( x_i \neq x_j \) whenever \( i \in A \) and \( j \in A', \) so i.2 holds. Now, since the orientation of \( G^P \) induced by \( x \) is then equal to the restriction of \( O \) to the edges of \( G^P, \) we observe that for \( A, A' \) as above, the edges \( \{i, j\} \in E : i \in A \) and \( j \in A' \) are oriented in \( O \) in the same direction (either from \( A \) to \( A' \), or vice-versa). Since \( O \) is transitive, this immediately implies that its restriction to \( G^P \) is transitive, so \( G^P \) is a comparability graph and i.3 holds. Notably, this holds for any choice of \( O. \) If \( \xi \) is a fiber of \( x, \) then we can write \( G[\xi] \) as a disjoint union of its connected components, say \( G[\xi] = G[B_1] + \cdots + G[B_k]. \) On the one hand, the restriction of \( O \) to any induced subgraph of \( G \) is transitive, so \( G[\xi] \) is a comparability graph, and also each of its connected components. On the other hand, from i.2, each \( B_i \) with \( i \in [k], \) satisfies that \( B_i \subseteq A \) for some \( A \in \mathcal{P}, \) and moreover, \( G[B_i] \) is a connected component of \( G[A], \) so \( B_i \) is a module \( G \) since \( B_i \) is a module of \( A \) and \( A \) is a module of \( G. \) This proves i.4.

As \( G^P \) does not have proper non-trivial connected modules, from Theorem 3.5 and Corollary 3.12, we obtain that \( \dim \langle E_{\lambda_{\text{max}}} \rangle = 1. \) Also, \( G^P \) has exactly two transitive orientations and each can be obtained with probability \( 1/2 \) from \( x \in E_{\lambda_{\text{max}}} \) a.u.c.u.v., proving ii.

Note: In fact, then, it follows that for any \( x \in E_{\lambda_{\text{max}}} \setminus \{0\}, \) necessarily \( x \in C_O \) or \( x \in C_{O_{\text{dual}}}, \) where \( O \) is the orientation used in the proof, and \( O_{\text{dual}} \) is the dual orientation to \( O. \)

Case 2: \( G \) is disconnected.

This is precisely the setting of Proposition 3.11 so i.1-3 and iii follow after noting that, firstly, \( p \)-partite graphs are comparability graphs, and secondly, their transitive orientations are exactly the acyclic orientations of their edges such that:

For every pair of maximal independent sets, all the edges between them (or having endpoints on both sets), are oriented in the same direction.

The proof of i.4 goes exactly as in Case 1.

\[ \square \]

**Corollary 3.14.** Let \( G = G([n], E) \) be a connected comparability graph with canonical partition \( \mathcal{P}, \) and let \( O \) be a transitive orientation of \( G. \) Then, (1) the restriction of \( O \) to each of \( G^P \) and \( G[A], A \in \mathcal{P}, \) is transitive.

Conversely, (2) if we select arbitrary transitive orientations for each of \( G^P \) and \( G[A], A \in \mathcal{P}, \) and then take the union of these, we obtain a transitive orientation for \( G. \)

**Proof.** Statement (1) follows from Proposition 3.13 and Proposition 3.3 since \( \dim \langle C_O \cap E_{\lambda_{\text{max}}} \rangle = \dim \langle E_{\lambda_{\text{max}}} \rangle. \)

For (2), select transitive orientations for each of \( G^P \) and \( G[A], A \in \mathcal{P}, \) and let \( O \) be the orientation of \( E \) so obtained. Since each element of \( \mathcal{P} \) is independent in \( G^P \) and since the restriction of \( O \) to \( G^P \) is transitive, then:

(\(*\)) For \( A, A' \in \mathcal{P} \) completely adjacent, the edges between \( A \) and \( A' \) must be oriented in \( O \) in the same direction.

This rules out the existence of directed cycles in \( O, \) so \( O \) is acyclic. Now, if \( O \) is not transitive, then there must exist \( i, j, k \in [n] \) such that \( (i, j) \) and \( (j, k) \) are in \( O \) but
not \((i,k)\). By the choice of \(O\), it must be the case that exactly two among \(i, j, k\) belong to the same \(A \in \mathcal{P}\), and the other one to a different \(A' \in \mathcal{P}\). The former cannot be \(i\) and \(k\), per the argument above \((\ast)\). Hence, without loss of generality, we can assume that \(i, j \in A\) and \(k \in A'\). But then, \(A\) and \(A'\) must be completely adjacent and \((i,k)\) must exist in \(O\), so we obtain a contradiction.

Note: The argument for \((2)\) is essentially found in Ramírez-Alfonsín and Reed (2001).

\[\]

Corollary 3.15. Let \(G = G([n], E)\) be a connected comparability graph with at least one proper non-trivial connected module \(B\), and canonical partition \(\mathcal{P}\). Then, \(G\) has more than two transitive orientations.

**Proof.** Suppose, on the contrary, that \(G\) has only two transitive orientations. We will prove that, then, \(G\) cannot have proper non-trivial connected modules and so \(B\) does not exist.

From Corollary 3.14 and Proposition 3.13 ii-iii, a necessary condition for \(G\) to have no more than two transitive orientations is:

\((\ast)\) \(G = G^P\), and either \(\overline{G}\) is connected or it has exactly two connected components.

Now, if \(\overline{G}\) is connected, then \(B \subseteq A\) for some \(A \in \mathcal{P}\) by Corollary 2.7 so \(B\) is an independent set of \(G\) since \(A\) is independent. This contradicts the choice of \(B\). Also, if \(\overline{G}\) has two connected components, then \(G\) is a complete bipartite graph. However, it is clear that no such \(B\) can exist in a complete bipartite graph.

**Proof of Theorem 3.1.** The different numerals of this result have, for the most part, already been proved.

- i was proved in Proposition 3.3.
- ii was proved in Proposition 3.13 for the case when \(\overline{G}\) is connected (See Note). In the general case, ii follows from Proposition 3.13 i.1-3 and Corollary 3.14 Statement \((2)\) for \(x \in E_{\lambda_{\text{max}}} \) a u.c.u.v., and then for all \(x \in E_{\lambda_{\text{max}}}\) since the cones \(C_O\) (with \(O\) an acyclic orientation of \(E\)) are closed.
- iii.1-5 and iv are precisely Proposition 3.13.
- For iv, from Corollary 3.15 and Theorem 3.3 iii, \(G\) has exactly two transitive orientations if and only if \(G\) has no proper non-trivial connected modules. Now, if \(G\) has no proper non-trivial connected modules, then Proposition 3.13 i.4 shows that the fibers of \(x\) are independent sets of \(G\) and Theorem 3.3 ii gives \(\dim \langle E_{\lambda_{\text{max}}} \rangle = 1\). Conversely, if the fibers of \(x\) are independent sets of \(G\), then \(G = G^P\). Furthermore, per Proposition 3.13 ii-iii, if \(\dim \langle E_{\lambda_{\text{max}}} \rangle = 1\), then \(\overline{G}\) has at most two connected components. Hence, \(G = G^P\) and \(\overline{G}\) has at most two connected components, so we obtain precisely the setting of \((\ast)\) in Corollary 3.15. Consequently, \(G\) cannot have proper non-trivial connected modules.

\[\]

4. A characterization of comparability graphs.

This section offers a curious novel characterization of comparability graphs that results from our theory in Section 3.
Theorem 4.1. Let $G = G([n], E)$ be a simple undirected graph with Laplacian matrix $L$, and let $I$ be the $n \times n$ identity matrix.

Then, $G$ is a comparability graph if and only if there exists $\alpha \in \mathbb{R}_{\geq 0}$ and an acyclic orientation $O$ of $E$, such that $C_O$ is invariant under left-multiplication by $\alpha I + L$.

If $G$ is a comparability graph, the orientations that satisfy the condition are precisely the transitive orientations of $G$, and we can take $\alpha = 0$ for them.

Proof. If $G$ is a comparability graph and $O$ is a transitive orientation of $G$, then Step 1 of Proposition 3.3 shows that indeed, $Lx \in C_O$ whenever $x \in C_O$. Clearly then, for all $\alpha \in \mathbb{R}_{\geq 0}$, $\alpha I \cdot (\alpha I + L)x \in C_O$ whenever $x \in C_O$.

Suppose now that $G$ is an arbitrary simple graph, and let $O$ be an acyclic orientation (of $E$) that is not a transitive orientation of $G$. Then, there exist $i, j, k \in [n]$ such that $(i, j)$ and $(j, k)$ are in $O$ but not $(i, k)$, and the following set is non-empty:

$$X := \{ k \in [n] : \text{there exist } i, j \in [n] \text{ and directed edges } (i, j), (j, k) \text{ in } O, \text{ but } (i, k) \text{ is not in } O \}.$$

In the partial order on $[n]$ induced by $O$, take some $\ell \in X$ maximal, and consider the principal order filter $\ell^\vee$ whose unique minimal element is $\ell$. The indicator vector of $\ell^\vee$ is $e_\ell$. Then, $e_\ell \in C_O$. Now, choose $i, j \in [n]$ so that $(i, j)$ and $(j, \ell)$ are in $O$ but not $(i, \ell)$. As $\ell$ was chosen maximal in $X$, for every $k \in \ell^\vee, k \neq \ell$, then both $(i, k)$ and $(j, k)$ are in $O$. Therefore, we have:

$$L(e_\ell) = -|\ell^\vee| + 1, \text{ and}$$

$$L(e_\ell) = -|\ell^\vee|.$$ 

Hence, $(L(e_\ell))_i > (L(e_\ell))_j$ and $Le_\ell \not\in C_O$ since $(i, j)$ is in $O$. Since actually $e_\ell \in \ell C_O$, then $(\alpha I + L)e_\ell \not\in C_O$ for $\alpha \in \mathbb{R}_{\geq 0}$.

□

References

C. W. Borchardt. Ueber eine der interpolation entsprechende darstellung der eliminations-resultante. Journal für die reine und angewandte Mathematik, 57: 111–121, 1860.

A. E. Brouwer and W. H. Haemers. Spectra of graphs. Springer, 2011.

F. R. Chung. Spectral graph theory, volume 92. American Mathematical Soc., 1997.

M. Fiedler. A property of eigenvectors of nonnegative symmetric matrices and its application to graph theory. Czechoslovak Mathematical Journal, 25(4):619–633, 1975.

M. Fiedler. Matrices and graphs in geometry, volume 139. Cambridge University Press, 2011.

T. Gallai. Transitiv orientierbare graphen. Acta Mathematica Hungarica, 18(1): 25–66, 1967.

R. Merris. Laplacian graph eigenvectors. Linear algebra and its applications, 278 (1):221–236, 1998.

R. H. Möhring. Almost all comparability graphs are upo. Discrete mathematics, 50:63–70, 1984.

J. L. Ramírez-Alfonsín and B. A. Reed. Perfect graphs. Series in Discrete Mathematics and Optimization. Wiley-Interscience, 2001.
D. Spielman. Spectral graph theory. *Lecture Notes, Yale University*, pages 740–0776, 2009.

R. P. Stanley. *Enumerative Combinatorics, Vol. 2*: Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2001.

R. P. Stanley. Introduction to hyperplane arrangements. 2004.

R. P. Stanley. *Enumerative Combinatorics, Vol. 1*: Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2011.

J. Sylvester. On the change of systems of independent variables. *Quarterly Journal of Mathematics*, 1:42–56, 1857.