Ground States for translationally invariant Pauli-Fierz Models at zero Momentum

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Abstract

We consider the translationally invariant Pauli-Fierz model describing a charged particle interacting with the electromagnetic field. We show under natural assumptions that the fiber Hamiltonian at zero momentum has a ground state.

1 Introduction

Non-relativistic qed has been a successful theory describing low energy aspects of quantum mechanical matter interacting with the quantized radiation field. Within this model many physical phenomena have been mathematically understood. In the present paper, we shall discuss an aspect which falls within the scattering problem of an electron interacting with the quantized radiation field. The main result which we prove is that the total system composed of electron and photon field at zero momentum is free of infrared divergences. In this situation the infrared singularity is critical. As soon as the momentum is nonzero a ground state ceases to exist, [14].

Starting with the work of Bloch and Nordsieck [4] the so called infrared catastrophe in scattering theory has been intensively investigated. Although the physical reasons for...
infrared divergences were well understood at that time and did not lead to any physical problems, the formal treatment was not satisfactory. In [20] Fadeev and Kulish introduced a new space of asymptotic states and gave a theoretical discussion of this phenomenon in the framework of relativistic quantum field theory. Fröhlich studied such asymptotic states in the so called Nelson Model [9, 10], which is mathematically well defined. Extending these results an iterative algorithm for constructing asymptotic states in Nelson’s model was developed in [23]. This construction was extended to the model of non-relativistic qed in [8]. Specifically, the case of zero momentum in nonrelativistic qed has previously been investigated in [2, 6].

In the present paper we consider the Hamiltonian $H$ of non-relativistic qed describing an electron, with or without spin, coupled to the quantized radiation field. The Hamiltonian commutes with the operator of total momentum. We are interested in the operator, $H(0)$, obtained by restricting the Hamiltonian to the subspace of total momentum zero. Based on a natural energy inequality, we prove that for all values of the coupling constant, $H(0)$ has a ground state. The energy inequality has been shown to hold in the spinless case for all values of the coupling constant [12], and in the case of spin the energy inequality follows in a related situation from the main theorem in [6]. The existence of a ground state can for example be used to obtain expansions on the binding energy of the hydrogen atom [3]. To the best of our knowledge the result, which we prove, has so far only been shown in the spinless case for small values of the coupling constant [2], see also [8] for related results. In contrast to the proofs given there, our proof is non-perturbative and independent of the magnitude of any ultraviolet cutoff parameter. The proof, which we give, uses a compactness argument and is not constructive. Nevertheless, once the existence of the ground state is established one can use other methods to obtain asymptotic expansions of the ground state as well as its energy [1, 5], in this context see also [13].

The idea of the proof given in the present paper follows closely the ideas introduced in [11], which were applied in a similar case in [21]. However, in the situation which we encounter the infrared singularity is more severe and subtler estimates are necessary. We believe that the estimates which we use in the present paper might establish alternative
proofs of existence of ground states in other critical cases, as for example [16].

In the next section we introduce the model and state the main result. The proofs are presented in Section 3.

2 Model and Statement of Results

Let $\mathfrak{h}$ be a complex Hilbert space. We introduce the symmetric Fock space

$$\mathcal{F}(\mathfrak{h}) = \bigoplus_{n=0}^{\infty} \mathfrak{h}^{(n)},$$

where $\mathfrak{h}^{(0)} = \mathbb{C}$ and where $\mathfrak{h}^{(n)} = \mathcal{P}_n(\bigotimes_{k=1}^{n} \mathfrak{h})$ for $n \geq 1$, with $\mathcal{P}_n$ denoting the orthogonal projection onto the subspace of totally symmetric tensors. Thus we can identify $\psi \in \mathcal{F}(\mathfrak{h})$ with the sequence $(\psi^{(n)})_{n \in \mathbb{N}_0}$ with $\psi^{(n)} \in \mathfrak{h}^{(n)}$. The vacuum is the vector $\Omega := (1, 0, 0, \ldots) \in \mathcal{F}(\mathfrak{h})$. We define for $f \in \mathfrak{h}$ the creation operator $a^*(f)$ acting on vectors $\psi \in \mathcal{F}$ by

$$(a^*(f)\psi)^{(n)} = \sqrt{n} \mathcal{P}_n (f \otimes \psi^{(n-1)})$$

with domain $D(a^*(f)) := \{\psi \in \mathcal{F} : a^*(f)\psi \in \mathcal{F}\}$. This yields a densely defined closed operator. For $f \in \mathfrak{h}$ we define the annihilation $a(f)$ as the adjoint of $a^*(f)$, i.e.,

$$a(f) = [a^*(f)]^*.$$ 

It follows from the definition that $a(f)$ is anti-linear, and $a^*(f)$ is linear in $f$. Creation and annihilation operators are well known to satisfy the so called canonical commutation relations

$$[a^*(f), a^*(g)] = 0 \ , \quad [a(f), a(g)] = 0 \ , \quad [a(f), a^*(g)] = \langle f, g \rangle_{\mathfrak{h}},$$

where $f, g \in \mathfrak{h}$, $[\cdot, \cdot]$ stands for the commutator, and $\langle f, g \rangle_{\mathfrak{h}}$ denotes the inner product of $\mathfrak{h}$. For a self-adjoint operator $A$ in $\mathfrak{h}$ we define the operator $d\Gamma(A)$ as follows. In $\mathfrak{h}^{(n)}$ we define

$$A^{(n)} := A \otimes 1 \otimes \cdots \otimes 1 + 1 \otimes A \otimes \cdots \otimes 1 + \cdots + 1 \otimes \cdots \otimes 1 \otimes A, \ n \in \mathbb{N},$$
in the sense of [26, VIII.10] and \( A^{(0)} := 0 \). By definition \( \psi \in \mathcal{F}(\mathfrak{h}) \) is in the domain of 
\( d\Gamma(A) \) if \( \psi(n) \in D(A^{(n)}) \) for all \( n \in \mathbb{N}_0 \) and

\[
(d\Gamma(A)\psi)(n) = A^{(n)}\psi(n), \quad n \in \mathbb{N}_0,
\]

is a vector in \( \mathcal{F}(\mathfrak{h}) \), in which case \( d\Gamma(A)\psi \) is defined by (1). The operator \( d\Gamma(A) \) is self-adjoint, see for example [26, VIII.10].

Henceforth, we shall consider specifically

\[
\mathfrak{h} := L^2(\mathbb{Z}_2 \times \mathbb{R}^3) \cong L^2(\mathbb{R}^3; \mathbb{C}^2)
\]

and write \( \mathcal{F} \) for \( \mathcal{F}(\mathfrak{h}) \). The Hilbert space \( \mathfrak{h} \) describes a so called transversally polarized photon. By physical interpretation the variable \( (\lambda, k) \in \mathbb{Z}_2 \times \mathbb{R}^3 \) consists of the wave vector \( k \) and the polarization label \( \lambda \). Because of [23], the elements \( \psi \in \mathcal{F} \) can be identified with sequences \( (\psi(n))_{n=0}^{\infty} \) of so called \( n \)-photon wave functions, \( \psi(n) \in L^{2\text{sym}}((\mathbb{Z}_2 \times \mathbb{R}^3)^n) \), where the subscript “sym” stands for the subspace of functions which are totally symmetric in their \( n \) arguments. Henceforth, we shall make use of this identification without mention. The Fock space inherits a scalar product from \( \mathfrak{h} \), explicitly

\[
\langle \psi, \varphi \rangle = \overline{\psi(0)}\varphi(0) + \sum_{n=1}^{\infty} \sum_{\lambda_1, \ldots, \lambda_n \in \{1, 2\}} \int \overline{\psi(n)(\lambda_1, k_1, \ldots, \lambda_n, k_n)} \varphi(n)(\lambda_1, k_1, \ldots, \lambda_n, k_n) dk_1 \ldots dk_n.
\]

We shall make use of the physics notation of the creation and annihilation operators. One defines for \( (\lambda, k) \in \mathbb{Z}_2 \times \mathbb{R}^3 \) and \( \psi \in \mathcal{F} \)

\[
[a_\lambda(k)\psi](n)(\lambda_1, k_1, \ldots, \lambda_n, k_n) = \sqrt{n+1}\psi(n+1)(\lambda, k, \lambda_1, k_1, \ldots, \lambda_n, k_n), \quad n \in \mathbb{N}_0.
\]

Now (3) defines a well defined operator \( a_\lambda(k) \) on

\[
D_S := \{ \psi \in \mathcal{F} : \psi(n) = 0 \text{ for all but finitely many }, \psi(n) \in \mathcal{S}((\mathbb{Z}_2 \times \mathbb{R}^3)^n) \},
\]

where \( \mathcal{S} \) stands for the Schwartz space. In the sense of quadratic forms on \( D_S \times D_S \) its adjoint \( a_\lambda^*(k) \) is well defined. Furthermore, in the sense of quadratic forms one has for all

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For details we refer the reader to [24, X.7]. The field energy operator denoted by $H_f$ is given by

$$H_f = d\Gamma(\omega),$$

where $\omega : \mathbb{Z}_2 \times \mathbb{R}^3 \to \mathbb{R}$ is defined by $\omega(\lambda, k) = |k|$. The operator of momentum $P_f$ is defined as a three dimensional vector of operators, where the $j$-th component is defined by

$$(P_f)_j := d\Gamma(\pi_j),$$

where $\pi_j : \mathbb{Z}_2 \times \mathbb{R}^3 \to \mathbb{R}$ is defined by $\pi_j(\lambda, k) = k_j$.

The Hilbert space describing the system composed of a charged particle with spin $s \in \{0, \frac{1}{2}\}$ and the quantized field is

$$\mathcal{H}_0 := L^2(\mathbb{R}^3 \times \mathbb{Z}_{2s+1}) \otimes \mathcal{F}.$$

The Hamiltonian is

$$H = \frac{1}{2} (p + eA(x))^2 + eS \cdot B(x) + H_f,$$

with

$$A(x) = \sum_{\lambda=1,2} \int \frac{\varepsilon(\lambda)(k)}{\sqrt{2|k|}} \left( \rho(k)a_\lambda(k)e^{ik\cdot x} + \rho(k)a^*_\lambda(k)e^{-ik\cdot x} \right) dk,$$

$$B(x) = \sum_{\lambda=1,2} \int \frac{k \wedge \varepsilon(\lambda)(k)}{\sqrt{2|k|}} \left( \rho(k)a_\lambda(k)e^{ik\cdot x} - \rho(k)a^*_\lambda(k)e^{-ik\cdot x} \right) dk,$$

where the $\varepsilon(\lambda)(k) \in \mathbb{R}^3$ are vectors, depending measurably on $\hat{k} = k/|k|$, such that $(|k|, \varepsilon_1(k), \varepsilon_2(k))$ forms an orthonormal basis. For the proof we shall make an explicit choice in (24), below. We shall adopt the standard convention that for $v = (v_1, v_2, v_3)$ we
write \( v^2 := \sum_{j=1}^{3} v_j v_j \). By \( x \) we denote the position of the electron and its canonically conjugate momentum by \( p = -i \nabla_x \). If \( s = 1/2 \), let \( S = (\sigma_1, \sigma_2, \sigma_3) \) denote the vector of Pauli-matrices. If \( s = 0 \), let \( S = 0 \). The number \( e \in \mathbb{R} \) is called the coupling constant.

The so called form factor \( \rho : \mathbb{R}^3 \to \mathbb{C} \) is a measurable function for which we shall assume the following hypothesis for the main theorem.

**Hypothesis A.** For some \( 0 < \Lambda < \infty \) we have

\[
\rho(k) = \frac{1}{(2\pi)^{3/2} \chi_{\Lambda}(|k|)} , \quad k \in \mathbb{R}^3 ,
\]

where \( \chi_{\Lambda} = 1_{[0, \Lambda]} \).

We note that Hypothesis A is usually assumed in Pauli-Fierz type models. The Hamiltonian is translation invariant and commutes with the generator of translations, i.e., the operator of total momentum

\[
P_{\text{tot}} = p + P_f .
\]

Let

\[
W = \exp(ix \cdot P_f) .
\]

Note \( WP_{\text{tot}}W^* = p \) so that in the new representation \( p \) is the total momentum. One easily computes

\[
WHW^* = \frac{1}{2} (p - P_f + eA)^2 + eS \cdot B + H_f ,
\]

where set \( A := A(0) \) and \( B := B(0) \). Let \( F \) be the Fourier transform in the electron variable \( x \), i.e., on \( L^2(\mathbb{R}^3) \),

\[
(F\psi)(\xi) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} e^{-i\xi \cdot x} \psi(x) dx .
\]

Then the composition \( U = FW \) is a unitary operator

\[
U : \mathcal{H}_0 \to L^2(\mathbb{R}^3 \times \mathbb{Z}_{2s+1}) \otimes \mathcal{F} \cong \int_{\mathbb{R}^3} \mathbb{C}^{2s+1} \otimes \mathcal{F} d\xi ,
\]
yielding the so called fiber decomposition of the Hamiltonian,

\[ UHU^* = \int_{\mathbb{R}^3} H(\xi)d\xi, \]

where

\[ H(\xi) = \frac{1}{2}(\xi - P_f + eA)^2 + eS \cdot B + H_f \]

is an operator in the so called reduced Hilbert space

\[ \mathcal{H} := \mathbb{C}^{2s+1} \otimes \mathcal{F}. \]

The operator \( H(\xi) \) is self-adjoint on \( D(P_f^2) \cap D(H_f) \), see Theorem 3 in the next section. To prove that \( H(0) \) has a ground state, we use a compactness argument similar to [11]. The idea is to first introduce a positive photon mass in the field energy. For \( m \geq 0 \) we define

\[ H_{f,m} = d\Gamma(\omega_m), \]

where

\[ \omega_m(\lambda,k) = \sqrt{m^2 + k^2}, \]

and study the operator

\[ H_m(\xi) = \frac{1}{2}(\xi - P_f + eA)^2 + eS \cdot B + H_{f,m}. \tag{8} \]

We set

\[ E_m(\xi) := \inf \sigma(H_m(\xi)). \]

The proof is based on the following energy inequality. Specifically we can show our result for any \( e \in \mathbb{R} \) for which there exists an \( m_0 > 0 \) such that for all \( m \in (0, m_0) \)

\[ E_m(\xi) \geq E_m(0), \quad \forall \xi \in \mathbb{R}^3. \tag{9} \]

Inequality (9) has been investigated in the literature. In spinless case, \( s = 0 \), Inequality (9) has in fact been shown for all values \( m \geq 0 \) and \( e \in \mathbb{R} \) using functional integration, [12, 28, 17, 21]. In case of spin \( s = 1/2 \) Inequality (9) has to the best of our knowledge
not yet been shown by means of functional integration. For $s = 1/2$ an Inequality of the type (9) follows in a related situation for small $|e|$ from the main theorem stated in [6], which in turn is based on perturbative arguments. We now state the main result of this paper.

**Theorem 1.** Suppose Hypothesis A holds, and let $e \in \mathbb{R}$. If there exists an $m_0 > 0$ such that for all $m \in (0, m_0)$ the energy inequality (9) holds, then the operator $H(0)$ has a ground state.

By the results in the literature mentioned in the previous paragraph, we obtain immediately the following theorem as corollary.

**Theorem 2.** Suppose Hypothesis A holds. In the spinless case, i.e., $s = 0$, the operator $H(0)$ has a ground state for all values of the coupling constant $e$.

We note that for small values of the coupling constant the statement in Therorem 2 has been shown previously in [2], see also [8] for related work. In the present paper we extend that result to all values of the coupling constant and provide a rather short proof. We want to point out, that the question whether Inequality (9) holds for all values of the coupling constant in the case of spin $s = 1/2$ seems to be an open question. We would like to mention work [18] in that direction.

### 3 Proof of Results

We first state the following technical result about the domain of self-adjointness.

**Theorem 3.** Let $m \geq 0$ and $\rho \in L^2(\mathbb{R}^3; (|k| + \omega_m(k))^{-1}|k|^{-1})dk)$. Then for all $\xi \in \mathbb{R}^3$ and $e \in \mathbb{R}$ the operator $H_\xi (\xi)$ is self-adjoint on the natural domain of $\frac{1}{2}P_j^2 + H_{f,m}$.

Versions of this theorem have been shown in [17], [18] and [21]. Since we could not find the precise version, which we need, in the literature, we shall provide a short proof of Theorem 3 in Appendix A. The proof follows closely a proof given in [15]. Moreover, we shall use the following result to prove the main theorem.
Theorem 4. Let $\rho \in L^2(\mathbb{R}^3; (|k| + |k|^{-1})dk)$ with $\rho = \rho(-\cdot)$. Let $e \in \mathbb{R}$ and $m > 0$ and suppose (9) holds. If $|\xi| \leq 1$, then $E_m(\xi)$ is an eigenvalue of $H_m(\xi)$ isolated from the essential spectrum.

Versions of this theorem have been shown in the literature [10, 9, 28, 11, 21]. We could not find in the literature the precise version, which we need, so we provide a proof of Theorem 4 in Appendix B.

Let us now outline the strategy of the proof of Theorem 1 which follows closely ideas given in [11]. For $m > 0$ it follows from Theorem 4 that $E_m(0)$ is an eigenvalue of $H_m(0)$. Henceforth let $\psi_m$ denote a normalized eigenvector of $H_m(0)$ with eigenvalue $E_m(0)$. We will show in Proposition 7, below, that $(\psi_m)_{m>0}$ is a minimizing family for $H(0)$ as $m$ tends to zero, i.e.,

$$0 \leq \langle \psi_m, (H(0) - E(0))\psi_m \rangle \to 0 \quad (m \downarrow 0). \quad (10)$$

Finally we shall use a compactness argument, based on two infrared bounds, stated in Lemmas 11 and 12 to show that there exists a strongly convergent subsequence $(\psi_{m_j})_{j \in \mathbb{N}}$ which converges to a nonzero vector, say $\psi_0$. Using lower semicontinuity of nonnegative quadratic forms [27] (or alternatively the spectral theorem and Fatou’s Lemma), it will then follow from (10) that

$$0 \leq \langle \psi_0, (H(0) - E(0))\psi_0 \rangle \leq \liminf_{i \to \infty} \langle \psi_{m_i}, (H(0) - E(0))\psi_{m_i} \rangle = 0,$$

i.e., that $\psi_0$ is a ground state of $H(0)$.

Remark 1. We note that in contrast to [11, 21] the infrared bounds which we obtain have stronger infrared singularities. Therefore it is harder to prove compactness. This difficulty will be addressed in Lemma 13 below.

3.1 Ground State Properties for massive Photons

Throughout this section we assume that $\rho \in L^2(\mathbb{R}^3; (|k| + |k|^{-2})dk)$. We will use the notation $N = d\Gamma(1)$.}

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Proposition 5. We have $E_m(0) \geq E(0)$ and

$$E(0) = \lim_{m \downarrow 0} E_m(0).$$  \hspace{1cm} (11)

Proof. For $0 \leq m' \leq m$ we have $\omega \leq \omega_{m'} \leq \omega_m$ and hence $H(0) \leq H_{m'}(0) \leq H_m(0)$. It follows that $E_m(0)$ is monotonically decreasing as $m$ tends to zero and $E(0) \leq E_m(0)$. This implies the existence of the limit and

$$\lim_{m \downarrow 0} E_m(0) \geq E(0).$$  \hspace{1cm} (12)

To show the opposite inequality we argue as follows. From Theorem 3 it follows that any core for $P_f^2 + H_f$ is a core for $H(0)$. Thus for any $\epsilon > 0$, there exists a normalized vector $\phi \in D(\mathbb{N}) \cap D(P_f^2 + H_f)$ such that

$$\langle \phi, H(0)\phi \rangle \leq E(0) + \epsilon.$$

On the other hand, since $H_m(0) \leq H(0) + mN$, it follows that for any $m \geq 0$,

$$E_m(0) \leq \langle \phi, H_m(0)\phi \rangle \leq \langle \phi, H(0)\phi \rangle + m\langle \phi, N\phi \rangle \leq E(0) + \epsilon + m\langle \phi, N\phi \rangle.$$

Hence

$$\lim_{m \downarrow 0} E_m(0) \leq E(0) + \epsilon.$$  \hspace{1cm} (13)

Since $\epsilon$ is arbitrary, (11) follows from (12) and (13).

Let us collect a basic inequality in the following lemma.

Lemma 6. Let $e \in \mathbb{R}$ and $m \geq 0$, and suppose (9) holds. Then for all $\xi \in \mathbb{R}^3$ we have

$$H_m(\xi) - E_m(0) \geq 0.$$

Proof. Follows from $H_m(\xi) \geq E_m(\xi) \geq E_m(0)$.

For later use we state the following Proposition.
Proposition 7. Assume $\rho(\cdot) = \rho$. Let $e \in \mathbb{R}$ and suppose there exists an $m_0 > 0$ such that (9) holds for all $m \in (0, m_0)$. Then

$$0 \leq \langle \psi_m, (H(0) - E(0))\psi_m \rangle \to 0,$$

in the limit $m \downarrow 0$.

Proof. Using that $H_m(0) \geq H(0)$ we find from Proposition 5 that

$$0 \leq \langle \psi_m, (H(0) - E(0))\psi_m \rangle \leq \langle \psi_m, (H_m(0) - E(0))\psi_m \rangle = E_m(0) - E(0) \to 0, \quad (m \downarrow 0).$$

3.2 Infrared Bounds

Throughout this section we assume that $\rho \in L^2(\mathbb{R}^3; (|k| + |k|^{-1})dk)$ with $\rho = \rho(\cdot)$. To simplify the notation we write

$$v := -P_f + eA,$$

$$h_m := H_m(0),$$

$$e_m := E_m(0).$$

Lemma 8. Let $e \in \mathbb{R}$ and $m > 0$, and suppose (9) holds. Then for $i = 1, 2, 3$, the vector $v_i\psi_m \in \mathcal{F}$ is orthogonal to $\psi_m$ and

$$0 \leq \langle v_i\psi_m, (H_m(0) - E_m(0))^{-1}v_i\psi_m \rangle \leq \frac{1}{2}.$$

Proof. For the proof we use analytic perturbation theory. For details we refer the reader to [19, 25]. On $\mathbb{C}^3$ the operator valued function $\zeta \mapsto H_m(\zeta) = H_m(0) + \zeta \cdot v + \frac{1}{2}\zeta^2$ is an analytic family of type (A) in each component. By Theorem 4 we know that $E_m(0)$ is an eigenvalue isolated from the essential spectrum. Let $P_m(0)$ be the orthogonal projection
onto the finite dimensional eigenspace of $E_m(0)$. By first order perturbation theory and the energy inequality (9) we conclude that $P_m(0)vP_m(0) = 0$. By second order perturbation theory and an application of the energy inequality (9) we conclude that for $i = 1, 2, 3$,

$$0 \leq \partial_\xi \partial_\xi E_m(\xi) \bigg|_{\xi = 0} \leq (1 - 2P_m(0)v_i(H_m(0) - E_m(0))^{-1}v_iP_m(0)),$$

where the second inequality is understood as an operator inequality on $\text{Ran} P_m(0)$. The second inequality in fact holds, since $E_m(\xi)$ is defined as an infimum. This shows the claim.

For notational convenience we set

$$R_m(k) := (H_m(-k) + \omega_m(k) - e_m)^{-1}, \quad k \in \mathbb{R}^3,$$

which by Lemma 6 is well defined and satisfies

$$\|R_m(k)\| \leq \omega_m(k)^{-1}. \quad (15)$$

The formula of the next Lemma is known as a so called Pull-through relation.

**Lemma 9.** Let $e \in \mathbb{R}$ and $m > 0$, and suppose (9) holds. Then for a.e. $k$, $a_\lambda(k)\psi_m \in \mathcal{F}$ and

$$a_\lambda(k)\psi_m = \frac{e\rho(k)}{\sqrt{2|k|}} R_m(k)(-\varepsilon_\lambda(k) \cdot v + S \cdot (ik \wedge \varepsilon_\lambda(k)))\psi_m. \quad (16)$$

**Proof.** The proof is similar to [7, Lemma 6.1], see also [14, Lemma 7]. By Theorem 3, we know that $\psi_m \in D(H_f)$. Hence using the standard expression of the free field energy in terms of annihilation operators

$$\sum_{n=0}^{\infty} \sum_{\lambda} \int |k| \left\| (a_\lambda(k)\psi_m)_{(n)} \right\|^2 dk = \langle \psi_m, H_f\psi_m \rangle < \infty,$$

which implies $a_\lambda(k)\psi_m \in \mathcal{F}$ for a.e. $k$. We write

$$f_A(k, \lambda) := \frac{\rho(k)}{\sqrt{2|k|}} \varepsilon_\lambda(k), \quad f_B(k, \lambda) := -ik \wedge f_A(k, \lambda). \quad (17)$$
By the canonical commutation relations of creation and annihilation operators we find

$$a_\lambda(k)H_m = (H_m(-k) + \omega_m(k)) a_\lambda(k) + ef_A(k) \cdot v + eS \cdot f_B(k),$$

which holds for a.e. $k$ as an identity of measurable functions. Applying this to $\psi_m$ and using that $H_m\psi_m = e_m\psi_m$ we find for a.e. $k$

$$((H_m(-k) - e_m + \omega_m(k)) a_\lambda(k)\psi_m = -(ef_A(k) \cdot v + eS \cdot f_B(k))\psi_m.$$  \hfill (18)

This implies that $a_\lambda(k)\psi_m$ is in the domain of $H_m(-k)$ for a.e. $k$. Indeed, the map

$$l: D_S \to \mathbb{C}, \quad \eta \mapsto \langle a_\lambda(k)\psi_m, H_m(-k)\eta \rangle$$

is bounded, since in view of (18) we can write

$$l(\eta) = \langle -(ef_A(k) \cdot v + eS \cdot f_B(k))\psi_m + (e_m - \omega_m(k))a_\lambda(k)\psi_m, \eta \rangle.$$  \hfill (19)

Now it follows that $a_\lambda(k)\psi_m \in D(H_m(-k))$, because $H_m(-k)$ is essentially self-adjoint on $D_S$, in view of Theorem 3. Hence the lemma follows by applying $R_m(k)$ to (18). \hfill \Box

**Lemma 10.** Suppose there exists an $m_0 > 0$ such that (9) holds for all $m \in (0, m_0)$. Then there exists a constant $C$ such that for all $e \in \mathbb{R}$, $m \in (0, m_0)$, $k \in \mathbb{R}^3$, and $j = 1, 2, 3$,

(a) $\|R_m(k)v_j\psi_m\| \leq C\omega_m^{-1/2}(k)(1 + |k|^{1/2}),$

(b) $\|R_m(k)v_j|_{D(|P_f|) \cap D(H_f^{1/2})} \| \leq C\omega_m(k)^{-1}(1 + |k|).$

**Proof.** (a) We start with the product inequality

$$\|R_m(k)v_j\psi_m\| \leq \|R_m(k)(h_m - e_m)^{1/2}\||(h_m - e_m)^{-1/2}v_j\psi_m\|.$$  \hfill (19)

By Lemma 8 the second factor on the right hand side can be estimated using

$$\|(h_m - e_m)^{-1/2}v_j\psi_m\| \leq 1/\sqrt{2}.$$  \hfill (19)

It remains to estimate the first factor in (19). First we use the trivial identity

$$h_m - e_m = \frac{1}{2}(v - k)^2 + H_{f,m} - e_m + \frac{1}{2}k^2 + (v - k)k.$$  \hfill (19)

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Estimating the last term using
\[(v - k)k \leq \frac{1}{2}|k| + \frac{1}{2}|k|(v - k)^2,\]
we find with \(\frac{1}{2}(v - k)^2 \leq H_m(-k)\) that
\[h_m - e_m \leq (1 + |k|)(H_m(-k) + \omega_m(k) - e_m) + \frac{1}{2}(|k| + k^2) + |k|e_m. \tag{20}\]
Now multiplying this inequality on both sides with the self-adjoint operator \(R_m(k)\) we obtain
\[R_m(k)(h_m - e_m)R_m(k) \leq (1 + |k|)R_m(k) + \left(\frac{1}{2}(|k| + k^2) + |k|e_m\right)R_m(k)^2.\]
Using this, we estimate
\[\|R_m(k)(h_m - e_m)^{1/2}\|^2 \leq \|R_m(k)\|^2\]
\[\leq \|(h_m - e_m)^{1/2}R_m(k)\|^2\]
\[= \sup_{\|\phi\|=1} \langle \phi, R_m(k)(h_m - e_m)R_m(k)\phi \rangle\]
\[\leq (1 + |k|)\|R_m(k)\| + \left(\frac{1}{2}(|k| + k^2) + |k|e_m\right)\|R_m(k)\|^2. \tag{22}\]
Using (15) and that \(e_m\) is bounded for \(0 \leq m \leq m_0\), we see that (22) inserted in (19) implies the bound stated in (a).

(b) Using that \(v^2 \leq h_m\) we see from (20) that
\[v^2 \leq (1 + |k|)(H_m(-k) + \omega_m(k) - e_m) + \frac{1}{2}(|k| + k^2) + (1 + |k|)e_m.\]
This implies
\[\left\|R_m(k)v_i\right\|_{D([P_f]) \cap D(H^1)}^2 \leq (1 + |k|)\|R_m(k)\| + \left(\frac{1}{2}(|k| + k^2) + (|k| + 1)e_m\right)\|R_m(k)\|^2\]
\[\leq C(1 + |k|^2)\omega_m(k)^{-2}, \tag{23}\]
hence (b) follows. \(\square\)
Estimating the expression in Lemma 9 using Lemma 10 (a) we obtain the next lemma.

**Lemma 11.** Suppose Hypothesis A holds. Suppose there exists an \( m_0 > 0 \) such that (9) holds for all \( m \in (0, m_0) \). Then there exists a finite constant \( C \) such that for all \( e \in \mathbb{R} \), \( m \in (0, m_0) \) we have

\[
\| (a_\lambda(k) \psi_m) \| \leq \frac{C|e\rho(k)|}{|k|} \quad \text{for a.e. } k.
\]

We still need an estimate involving derivatives. To this end, we shall henceforth make an explicit choice of the polarization vectors. After a possible unitary transformation on Fock space we can always achieve that the polarization vectors are given by

\[
\varepsilon_1(k) = \frac{(k_2, -k_1, 0)}{\sqrt{k_1^2 + k_2^2}} \quad \text{and} \quad \varepsilon_2(k) = \frac{k}{|k|} \wedge \varepsilon_1(k). \tag{24}
\]

**Lemma 12.** Suppose Hypothesis A holds. Suppose there exists an \( m_0 > 0 \) such that (9) holds for all \( m \in (0, m_0) \). Then there exists a finite constant \( C \) such that for all \( e \in \mathbb{R} \), \( m \in (0, m_0) \), and a.e. \( k \) with \( |k| < \Lambda \)

\[
\| \nabla_k (a_\lambda(k) \psi_m) \| \leq \frac{C|e\rho(k)|}{|k|\sqrt{k_1^2 + k_2^2}}.
\]

**Proof.** We want to calculate the derivative of the expression in Equation (16). Calculating the derivative with respect to the operator norm topology, we find by means of the resolvent identity, that

\[
\nabla_k R_m(k) = -R_m(k)(k - v + \nabla_k \omega_m(k)) R_m(k).
\]

Using this we can calculate the derivative for \( 0 < |k| < \Lambda \)

\[
\begin{align*}
\nabla_k (a_\lambda(k) \psi_m) &= -\frac{1}{2} \frac{e\rho(k)}{\sqrt{2|k|k^2}} R_m(k)(-\varepsilon_\lambda(k) \cdot v + S \cdot (k \wedge \varepsilon_\lambda(k))) \psi_m \quad &
\varepsilon_1(k) = \frac{(k_2, -k_1, 0)}{\sqrt{k_1^2 + k_2^2}} \quad \text{and} \quad \varepsilon_2(k) = \frac{k}{|k|} \wedge \varepsilon_1(k). \tag{24}
\end{align*}
\]
We now use that by Lemma 10, there exists a constant $C$ such that $\|\omega_m(k) R_m(k) v \psi_m\| \leq C \omega_m(k)^{1/2} (1 + |k|)$. Using this together with (15) the first and second term are estimated from above by a finite constant times $|k|^{-2}$. To estimate the third term we note that by the choice (24), we have for $\lambda = 1, 2$,

$$\left| \frac{\partial}{\partial k_l} \varepsilon_\lambda(k) \right| \leq \text{const.} \frac{1}{\sqrt{k_1^2 + k_2^2}}, \quad l = 1, 2, 3.$$ 

Lemma 13 (y-Bound). Suppose Hypothesis A holds. Suppose there exists an $m_0 > 0$ such that (19) holds for all $m \in (0, m_0)$, and let $e \in \mathbb{R}$. Then there exists a constant $C$, and a $\delta > 0$ such that for all $m \in (0, m_0)$ and all $n \in \mathbb{N}$,

$$\sum_{\lambda_1, \ldots, \lambda_n} \int \sum_{i=1}^n n^{-1} |y_i|^\delta \| (\hat{\psi}_m)_{(n)}(\lambda_1, y_1, \ldots, \lambda_n, y_n) \|^2 dy_1 \ldots dy_n \leq C,$$

where $(\hat{\psi}_m)_{(n)}$ denotes the Fourier transform of the $n$-photon component of $\psi_m$.

Proof. We drop the subscript $m$. Thus by $\hat{\psi}_{(n)}$ we denote the Fourier transform of $\psi_{(n)}$ in all its $n$-components. We define the functions

$$\psi_{(n)}(k) : (\lambda, k_1, \lambda_1, \ldots, k_{n-1}, \lambda_{n-1}) \mapsto \psi_{(n)}(k, \lambda, k_1, \lambda_1, \ldots, k_{n-1}, \lambda_{n-1})$$

$$\hat{\psi}_{(n)}(y) : (\lambda, y_1, \lambda_1, \ldots, y_{n-1}, \lambda_{n-1}) \mapsto \hat{\psi}_{(n)}(y, \lambda, y_1, \lambda_1, \ldots, y_{n-1}, \lambda_{n-1}).$$

Step 1: There exists a $\delta > 0$ and a constant $C$ such that for all $a \in \mathbb{R}^3$,

$$\int |1 - e^{-iay}|^2 \| \hat{\psi}_{(n)}(y) \|^2 dy \leq \begin{cases} C|a|^\delta & \text{if } |a| < \frac{1}{2} \Lambda, \\ C & \text{if } |a| \geq \frac{1}{2} \Lambda. \end{cases}$$

The claim follows easily for $|a| \geq \frac{1}{2} \Lambda$, since $\psi$ is a normalized state in Fock space and $|1 - e^{-iay}| \leq 2$. Now lets consider the case $|a| < \frac{1}{2} \Lambda$. By the Fourier transform, we have
the identity
\[
\int |1 - e^{-iy}|^2 \|\hat{\psi}_n(y)\|^2 dy = \int \|\psi_n(k + a) - \psi_n(k)\|^2 dk
\]
\[
= \int_{|k| < \Lambda - |a|} \|\psi_n(k + a) - \psi_n(k)\|^2 dk + \int_{\Lambda - |a| \leq |k|} \|\psi_n(k + a) - \psi_n(k)\|^2 dk.
\] (25)

To estimate the second integral we use Lemma 11 and observe that the integrand vanishes for \(|k| > \Lambda + |a|\),
\[
\int_{\Lambda - |a| \leq |k|} \|\psi_n(k + a) - \psi_n(k)\|^2 dk \leq \text{const.} \int_{\Lambda - |a| \leq |k| \leq \Lambda + |a|} \left( \frac{1}{|k + a|^2} + \frac{1}{|k|^2} \right) dk
\]
\[
\leq \text{const.} \int_{\Lambda - 2|a| \leq |k| \leq \Lambda + 2|a|} \frac{1}{|k|^2} dk
\]
\[
\leq \text{const.} |a|.
\] (26)

Next we estimate the first integral and assume \(|k| < \Lambda - |a|\). Using Lemma 12 we find
\[
\|\psi_n(k + a) - \psi_n(k)\| = \left\| \int_0^1 \left( \frac{d}{dt} \psi_n(k + ta) \right) dt \right\|
\]
\[
\leq |a| \int_0^1 \|\nabla_k \psi_n(k + ta)\| dt
\]
\[
\leq \text{const.} |a| \int_0^1 \frac{\rho(k + ta)}{|k + ta| |\pi_3(k + ta)|} dt ,
\] (27)

where \(\pi_3\) denotes the projection in \(\mathbb{R}^3\) along the 3-axis and const. denotes a finite constant independent of \(n\). Let \(\pi_a\) denote the projection in \(\mathbb{R}^3\) along the vector \(a\) and let \(\pi_{3,a}\) denote the projection in the (1,2)-plane along \(\pi_3 a\) (with convention that \(\pi_{3,a} = \pi_3\), if \(\pi_3 a = 0\)).

We find from (27)
\[
\|\psi_n(k + a) - \psi_n(k)\| \leq \text{const.} \frac{|a|}{|\pi_a(k)| |\pi_{3,a}(k)|} .
\] (28)

On the other hand using Lemma 11 we obtain
\[
\|\psi_n(k + a) - \psi_n(k)\| \leq \text{const.} \left( \frac{\rho(k + a)}{|k + a|} + \frac{\rho(k)}{|k|} \right) .
\] (29)
Introducing Inequalities (28) and (29) into the first integral of (25), we find for any $\theta$ with $0 \leq \theta \leq 1$,
\[
\int_{|k|<\Lambda-|a|} \|\psi_n(k+a) - \psi_n(k)\|^2 dk = \text{const.} |a|^{2\theta} \int_{|k|<\Lambda-|a|} \frac{1}{|\pi_a(k)|^{2\theta}|\pi_{3,a}(k)|^{2\theta}} \left( \frac{\rho(k+a)}{|k+a|} + \frac{\rho(k)}{|k|} \right)^{2(1-\theta)} dk
\]

Now we use Young’s inequality: $bc \leq b^p/p + c^q/q$, whenever $p, q > 1$ and $p^{-1} + q^{-1} = 1$; and the convexity of $x \mapsto x^{2(1-\theta)q}$ on $\mathbb{R}_+$, for $0 < \theta < 1/2$. Thus for $0 < \theta < 1/2$,
\[
\int_{|k|<\Lambda-|a|} \|\psi_n(k+a) - \psi_n(k)\|^2 dk \leq |a|^{2\theta} \text{const.} \int_{|k|<\Lambda} \left( \frac{1}{|\pi_a(k)|^{4\theta p}} + \frac{1}{|\pi_{3,a}(k)|^{4\theta p}} \right)^{2(1-\theta)q} \left( 1 + \left( \frac{1}{|k+a|} \right)^{2(1-\theta)q} + \left( \frac{1}{|k|} \right)^{2(1-\theta)q} \right) dk . \tag{30}
\]

For any $q$ with $1 < q \leq 3/2$, we can choose $\theta > 0$ sufficiently small such that the right hand side is finite. Inserting (26) und (30) into (25) we obtain the desired estimate.

Step 2: Step 1 implies the statement of the Lemma.

From Step 1 we know that there exists a finite constant $C$ such that
\[
\int \frac{|1 - e^{-iay}|^2}{|a|^{3/2}} \frac{|\widehat{\psi_n}(y)|^2}{|a|^3} dy \, da \, \frac{da}{|a|^3} \leq C .
\]

After interchanging the order of integration and a change of integration variables $b = |y|a$, we find
\[
C \geq \int \|\widehat{\psi_n}(y)\|^2 \int \frac{|1 - e^{-iay}|^2}{|a|^{3/2}} \frac{da}{|a|^3} dy = \int \|\widehat{\psi_n}(y)\|^2 |y|^{3/2} \int \frac{|1 - e^{-iby}|^2}{|b|^{3/2}} \frac{db}{|b|^3} dy ,
\]

where $c$ is nonzero and does not depend on $y$. 

\[\square\]
3.3 Existence of the Ground State

Proof of Theorem 1. Fix a positive $m_0$ such that for all $m \in (0, m_0)$ the energy inequality (9) holds.

Step 1: All $\psi_m$, with $m_0 \geq m > 0$, lie in a compact subspace of the reduced Hilbert space $\mathcal{H}$.

Let $T$ be the self-adjoint operator associated to the nonnegative and closed quadratic form $q$ in $\mathcal{H}$ defined by

$$q(\phi) := \langle \phi, N\phi \rangle + \sum_{n=1}^{\infty} n^{-3} \langle \hat{\phi}(n), \sum_{i=1}^{n} |y_i|^{4}\hat{\phi}(n) \rangle + \langle \phi, Hf\phi \rangle,$$

for all $\phi \in D(q)$, the natural form domain of $q$. We choose $\delta > 0$ such that Lemma 13 holds. By this and Lemma 11 and Proposition 7, there exists a finite $C$ such that for all $m$ with $0 < m < m_0$,

$$\psi_m \in K := \{ \phi \in D(q) : \|\phi\| \leq 1, q(\phi) \leq C \}.$$

The set $K$ is a compact subset of $\mathcal{H}$, provided $T$ has compact resolvent [25, Theorem XIII.64]. Hence it remains to show that $T$ has compact resolvent. The operator $T$ preserves the $n$-photon sectors. Let $T_n$ denote the restriction of $T$ to the $n$-photon sector. From Rellich's criterion [25, Theorem XIII.65] it follows that $T_n$ has compact resolvent. Therefore $\mu_l(T_n) \to \infty$ as $l$ tends to infinity, where $\mu_l$ denotes the $l$-th eigenvalue obtained by the min-max principle. Moreover since $\mu_l(T_n) \geq n$ for all $l, n$, it follows that $\mu_l(T) \to \infty$ as $l \to \infty$. Hence $T$ has a compact resolvent.

Step 2: There exists a nonzero vector $\psi_0$ such that $\langle \psi_0, H(0)\psi_0 \rangle = \inf \sigma(H(0))$.

Here we use the argument outlined at the beginning of this section. By Step 1, we know that all $\psi_m$, with $m_0 \geq m > 0$, lie in a compact subspace of $\mathcal{H}$. It follows that
there exists a subsequence \((\psi_{m_i})_{i \in \mathbb{N}}\), with \(m_i \to 0\) as \(i \to \infty\), which converges strongly to a normalized vector \(\psi_0\). By lower semicontinuity of non-negative quadratic forms we see from Proposition 7 that

\[
\langle \psi_0, (H(0) - E(0))\psi_0 \rangle 
\leq \liminf_{i \to \infty} \langle \psi_{m_i}, (H(0) - E(0))\psi_{m_i} \rangle = 0.
\]

This shows Step 2.

\[\square\]

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### A Self-adjointness

In this section we prove Theorem 3. The proof is based on an inequality similar to \[15\]. In contrast to the proof given in that paper, where the domain of self-adjointness is determined by means of quadratic forms, we use the following abstract proposition, which can be derived from a theorem of Wüst, similar to \[22\].

**Proposition 14.** Let \(T\) be a self-adjoint operator on a Hilbert space and let \(T^{(n)}\), \(n = 1, 2\) be symmetric and \(T\)-bounded operators. For \(\kappa \in \mathbb{C}\) let \(T(\kappa) = T + \kappa T^{(1)} + \kappa^2 T^{(2)}\) be the operator with domain \(D(T)\). If \(T(\kappa)\) is closed for all \(\kappa \in [0, t]\), then \(T(t)\) is self-adjoint.

**Proof.** Let \(Z = \{\kappa \in \mathbb{C} : T(\kappa)\) is closed \}. We claim that \(Z\) is open. If \(\kappa_0 \in Z\), then \(T(\kappa_0)\) is closed and \(D(T(\kappa_0)) = D(T)\). The operators \(T^{(j)}\) are closable operators such that \(D(T(\kappa_0)) = D(T) \subset D(T^{(j)})\). Then by the closed graph theorem \(T^{(j)}\) are also \(T(\kappa_0)\) bounded \[29\] Theorem 5.9]. It follows that \(T(\kappa)\) is closed for \(\kappa\) close to \(\kappa_0\) \[29\] Theorem 5.5]. Thus we have shown that \(Z\) is open. It follows that \(\kappa \mapsto T(\kappa)\) is on \(Z\) a holomorphic family of type (A) \[19\]. Since \(T^{(j)}\) are symmetric and \(T\)-bounded, we have \(T(\overline{\kappa}) \subset T(\kappa)^*\).
for all $\kappa \in \mathbb{C}$. Let $Z_0$ denote the connected component of $Z \cap \overline{Z}$ containing 0. Since $T(0) = T$ is self-adjoint, and hence $0 \in Z_0$, it follows from a Theorem of Wüst, [30, Theorem 1], that $T(\kappa) = T(\kappa)^*$ for all $\kappa \in Z_0$. Since $[0, t] \subset Z_0$, it follows that $T(\kappa)$ is self-adjoint for all $\kappa \in [0, t]$.

We apply the above Proposition to $T(e) = T + eT^{(1)} + e^2T^{(2)}$, where $T = \frac{1}{2}(\xi - P_f)^2 + H_{f,m}$, with natural domain, $T^{(1)} = \frac{1}{2}(P_f - \xi) \cdot A + \frac{1}{2}A \cdot (P_f - \xi) + S \cdot B$, and $T^{(2)} = \frac{1}{2}A^2$.

First we note the following. Since $A$ is divergence free we have $A \cdot P_f = P_f \cdot A$, and so $T^{(1)} = A \cdot P_f - \xi \cdot A + S \cdot B$. Let $T_0 = \frac{1}{2}P_f^2 + H_{f,m}$, with natural domain. Since $T$ and $T_0$ are non-negative multiplication operators, it is easy to see that the domains of $T_0$ and $T$ agree, and that the operators are mutually bounded. Thus, as $T(e) = H_m(\xi)$, Theorem will follow as a consequence of Proposition [14] and the following two lemmas.

Lemma 15. Let $m \geq 0$, $\xi \in \mathbb{R}^3$, and $\rho \in L^2(\mathbb{R}^3; (|k| + \omega_m(k)^{-1}|k|^{-1})dk)$. Then $T^{(1)}$ and $T^{(2)}$ are $T$-bounded.

Proof. Standard estimates, see for example [24], show that $A^2$ is $H_{f,m}$-bounded and that the components of $A$ and $B$ are $H_{1/2}^{1/2}$-bounded. It follows that for some $C$ we have $\|A_i P_f \psi\| \leq C(\|H_{f,m}^{1/2} P_f \psi\| + \|P_f \psi\|)$. Using $\|H_{f,m}^{1/2} P_f \psi\|^2 \leq \|(P_f^2 + H_{f,m}) \psi\|^2$ and collecting estimates shows the claim.

Lemma 16. Let $m \geq 0$ and $\rho \in L^2(\mathbb{R}^3; (|k| + \omega_m(k)^{-1}|k|^{-1})dk)$. Then for any $e \in \mathbb{R}$ and $\xi \in \mathbb{R}^3$ there exist constants $C_1, C_2$ such that for all $\varphi \in D(T)$,

\[
\|T \varphi\|^2 \leq C_1\|T(e) \varphi\|^2 + C_2\|\varphi\|^2.
\] (31)

Proof. Let $e \in \mathbb{R}$ and $m \geq 0$ be fixed. For notational compactness we set $Q := P_f - \xi$ and $F := eA$.

Step 1: There exist constants $c_1, c_2, c_3$ such that

\[
\|Q^2 \varphi\|^2 \leq c_1\|(Q + F)^2 \varphi\|^2 + c_2\|H_{f,m} \varphi\|^2 + c_3\|\varphi\|^2, \quad \forall \varphi \in D(T).
\]
In the following we denote by $C$ a constant which may change from line to line. We have
\[
\|Q^2\varphi\|^2 = \|(Q + F)^2 - 2F \cdot (Q + F) + F^2\varphi\|^2 
\leq 3\|(Q + F)^2\varphi\|^2 + 12\|F \cdot (Q + F)\varphi\|^2 + 3\|F^2\varphi\|^2. \tag{32}
\]
The second term in (32) is estimated as follows
\[
\|F \cdot (Q + F)\varphi\|^2 \leq C \sum_j \|F_j(Q_j + F_j)\varphi\|^2 \leq C \sum_j \|(H_{f,m} + 1)^{1/2}(Q_j + F_j)\varphi\|^2.
\]
Further, using a commutator
\[
\sum_j \|(H_{f,m} + 1)^{1/2}(Q_j + F_j)\varphi\|^2 
= \sum\langle(Q_j + F_j)\varphi, (H_{f,m} + 1)(Q_j + F_j)\varphi\rangle 
= \sum\langle(Q_j + F_j)^2\varphi, (H_{f,m} + 1)\varphi\rangle + \langle(Q + F)_{j}\varphi, [H_{f,m}, F_j]\varphi\rangle 
\leq C(\|(Q + F)^2\varphi\|^2 + \sum_j \|(Q_j + F_j)\varphi\|^2 + \|(H_{f,m} + 1)\varphi\|^2) 
\leq C(\|(Q + F)^2\varphi\|^2 + \|(H_{f,m} + 1)\varphi\|^2).
\]
Collecting the above estimates yields Step 1.

\underline{Step 2}: There exists a constant $C$ such that
\[
\|\frac{1}{2}(Q + F)^2\varphi\|^2 + \|H_{f,m}\varphi\|^2 \leq \|\frac{1}{2}(Q + F)^2 + H_{f,m}\varphi\|^2 + C\|\varphi\|^2, \quad \forall \varphi \in D(T).
\]
Calculating a double commutator, we see that
\[
\frac{1}{2}\langle H_{f,m}\varphi, (Q + F)^2\varphi\rangle + \langle(Q + F)^2\varphi, H_{f,m}\varphi\rangle 
= \sum\langle(Q_j + F_j)\varphi, H_{f,m}(Q_j + F_j)\varphi\rangle + \frac{1}{2}\sum\langle\varphi, [F_j, [F_j, H_{f,m}]]\varphi\rangle 
\geq -b\|\varphi\|^2, \tag{33}
\]
for some $b$. Step 2 follows by adding Inequality (33) to the left hand side and completing
the square.

Step 3: Inequality (31) holds.

First observe that
\[
\|\left(\frac{1}{2}Q^2 + H_{f,m}\right)\varphi\|^2 \leq \frac{1}{2}\|Q^2\varphi\|^2 + 2\|H_{f,m}\varphi\|^2, \quad \forall \varphi \in D(T).
\] (34)

Inserting on the right hand side of (34) first the inequality of Step 1 and then the inequality
of Step 2, we find for some constant $C$
\[
\|\left(\frac{1}{2}Q^2 + H_{f,m}\right)\varphi\| \leq C\left(\frac{1}{2}(Q + F)^2 + H_{f,m}\right)\varphi + \|\varphi\|, \quad \forall \varphi \in D(T).
\] (35)

By standard estimates $S \cdot B$ is infinitesimally bounded with respect to $H_{f,m}$. It follows
by (35) that it is also infinitesimally bounded with respect to $\frac{1}{2}(Q + F)^2 + H_{f,m}$. Thus
$\frac{1}{2}(Q + F)^2 + H_{f,m}$ is $T(e)$-bounded. Now (31) follows from (35).

B Ground State for positive Photon Mass

In this section we provide a proof of Theorem 4. It follows closely the proofs given in
[11, 21]. Theorem 4 will follow directly from Propositions 17 and 24, belo w. For $\xi \in \mathbb{R}^3$ and $m \geq 0$ we define
\[
\Delta_m(\xi) = \inf_{k \in \mathbb{R}^3} \{E_m(\xi - k) - E_m(\xi) + \omega_m(k)\}.
\]

Proposition 17. Let $\rho \in L^2(\mathbb{R}^3; (|k| + |k|^{-1})dk)$ and $m > 0$. Then for all $\xi \in \mathbb{R}^3$ we have
\[
\inf_{\sigma_{\text{ess}}(H_m(\xi)) \geq E_m(\xi) + \Delta_m(\xi)}.
\]

To prove the proposition we need the following notation. Let $\mathfrak{h}_1$ and $\mathfrak{h}_2$ be two Hilbert
spaces. If $A : \mathfrak{h}_1 \to \mathfrak{h}_2$ is a partial isometry, we define $\Gamma(A)$ to be the linear operator
$\mathcal{F}(\mathfrak{h}_1) \to \mathcal{F}(h_2)$ which equals $\bigotimes_{k=1}^n A$ when restricted to $\mathfrak{h}_1^{(n)}$, $n \geq 1$, and which equals to
the identity on $\mathfrak{h}_1^{(0)}$. The following two lemmas are straightforward to verify.
Lemma 18. Let $h_1$ and $h_2$ be two Hilbert spaces, and $A : h_1 \rightarrow h_2$ a partial isometry. Then $\Gamma(A)^* = \Gamma(A^*)$, and
\[ \Gamma(A)a^*(f) = a^*(Af)\Gamma(A). \]
If $A$ is an isometry, then so is $\Gamma(A)$ and $\Gamma(A)a(f) = a(Af)\Gamma(A)$.

Lemma 19. Let $h_1$ and $h_2$ be two Hilbert spaces. Then there exists a unique bounded linear map $U : \mathcal{F}(h_1 \oplus h_2) \rightarrow \mathcal{F}(h_1) \otimes \mathcal{F}(h_2)$ such that
\[ U\Omega = \Omega \otimes \Omega, \]
\[ U(a^*(h_1, h_2)) = (a^*(h_1) \otimes 1 + 1 \otimes a^*(h_2))U, \quad \forall (h_1, h_2) \in h_1 \oplus h_2. \]
It follows that $U$ is unitary.

Let $\chi_1, \chi_2 \in C^\infty(\mathbb{R}^3; [0, 1])$ with $\chi_1^2 + \chi_2^2 = 1$ and $\chi_1(x) = 1$, if $|x| < 1$, and $\chi(x) = 0$, if $|x| > 2$. We define $j_l = \chi_l(-i\nabla_k/L)$ for $l = 1, 2$ and $L > 0$. Define
\[ j : h \rightarrow h \oplus h, \quad f \mapsto (j_1 f, j_2 f), \]
with $h$ given by (2). Henceforth we denote the identity map of $h$ by 1. One readily verifies that $j$ is an isometry. Hence $j^*$ is a partial isometry and $j^*j = 1$. Explicitly, one finds $j^*(f_1, f_2) = j_1 f_1 + j_2 f_2$ for $f_j \in h$, $j = 1, 2$. Henceforth, let $U$ denote the isometry as in Lemma 19 with $h_1 = h_2 = h$, and let $J = U\Gamma(j)$. Recall that $N = d\Gamma(1)$. In the following let $a^\#$ stand for $a^*$ or $a$.

Lemma 20. The following holds.

(a) $J^*J = 1$.

(b) $Ja(f)^\# = \{a(j_1 f)^\# \otimes 1 + 1 \otimes a(j_2 f)^\#\}J$.

(c) We have
\[ J^*(a^*(f_1) \otimes 1 + 1 \otimes a^*(f_2)) = a^*(j_1 f_1 + j_2 f_2)J^*, \]
\[ (a(f_1) \otimes 1 + 1 \otimes a(f_2))J = Ja(j_1 f_1 + j_2 f_2). \]
Proof. The Lemma follows directly from the definition of $J$ and the properties of Lemmas 18 and 19.

Lemma 21. Let $f \in \mathfrak{h}$.

(a) For $\psi \in D(N^{1/2})$ we have
\[
\|(Ja(f) - (a(f) \otimes 1)J)\psi\| \leq \|(1 - j_1)f\|N^{1/2}\psi, \quad (36)
\]
\[
\|(Ja^*(f) - (a^*(f) \otimes 1)J)\psi\| \leq \|(1 - j_1)f\| + \|j_2f\|(N + 1)^{1/2}\psi. \quad (37)
\]

(b) We have
\[
Ja^#(f) - (a^#(f) \otimes 1)J = (a^#((j_1 - 1)f) \otimes 1 + 1 \otimes a^#(j_2f))J.
\]

Proof. Part (b) follows directly from Lemma 20 (b). To show (36), we insert in (b) the identity from Lemma 20 (c) and find
\[
Ja(f) - (a(f) \otimes 1)J = Ja(j_1(j_1 - 1)f + j_2^2f) = Ja(1 - j_1)f).
\]
Now the inequality follows from standard estimates. To show (37) we again use (b),
\[
Ja^*(f) - (a^*(f) \otimes 1)J = (a^*((j_1 - 1)f) \otimes 1 + 1 \otimes a^*(j_2f))J.
\]
To estimate the second term on the right hand side we first use the canonical commutation relations and then Lemma 20 (c) and find
\[
J^*(1 \otimes a(j_2f)a^*(j_2f))J = \|j_2f\|^2 J^*J + J^*(1 \otimes a^*(j_2f)a(j_2f))J
= \|j_2f\|^2 + a^*(j_2^2f)J^*Ja(j_2^2f)
\]
\[
\leq \|j_2f\|^2 + \|j_2^2f\|^2 N
\leq \|j_2f\|^2(1 + N).
\]
To estimate the first term on the right hand side we find similarly
\[
J^*(a((j_1 - 1)f))a^*((j_1 - 1)f) \otimes 1)J
= \|(j_1 - 1)f\|^2 J^*J + J^*a^*((j_1 - 1)f)a((j_1 - 1)f) \otimes 1)J
= \|j_2f\|^2 + a^*(j_1(j_1 - 1)f)J^*Ja(j_1(j_1 - 1)f)
\]
\[
\leq \|(j_1 - 1)f\|^2 + \|j_1(j_1 - 1)f\|^2 N \leq \|(j_1 - 1)f\|^2(1 + N).
\]
Collecting estimates shows \[37\].

**Lemma 22.** Let \( h \) be a selfadjoint operator in \( \mathfrak{h} \). Suppose there exists a dense subspace \( \mathcal{D} \subset D(h) \) such that \( j_l(\mathcal{D}) \subset D(h) \) for \( l = 1, 2 \).

(a) Suppose that \([j_l, h]\) is bounded for \( l = 1, 2 \). Then for \( \psi \in D(d\Gamma(h)) \cap D(N) \) we have

\[
\| (Jd\Gamma(h) - (d\Gamma(h) \otimes 1 + 1 \otimes d\Gamma(h)))J \psi \| \leq (\| [j_1, h] \|^2 + \| [j_2, h] \|^2)^{1/2}\| N\psi \|.
\]

(b) Let \( (e_l)_{l \in \mathbb{N}} \) be an orthonormal basis of \( \mathfrak{h} \) which lies in \( \mathcal{D} \). Then

\[
Jd\Gamma(h) - (d\Gamma(h) \otimes 1 + 1 \otimes d\Gamma(h))J = \sum_l (a^*([j_1, h]e_l) \otimes 1 + 1 \otimes a^*([j_2, h]e_l))Ja(e_l),
\]

where the expression on the right hand side is understood in the weak sense.

**Proof.** In the proof let \( \langle \cdot, \cdot \rangle \) denote the scalar product in \( \mathfrak{h} \). First we show (b). We find using Lemma 20 (b)

\[
Jd\Gamma(h) = J\sum_{l,k} \langle e_l, he_k \rangle a^*(e_l)a(e_k)
\]

\[
= \sum_{l,k} \langle e_l, he_k \rangle (a^*([j_1, h]e_l) \otimes 1 + 1 \otimes a^*([j_2, h]e_l))Ja(e_k)
\]

\[
= \sum_{l,k} (\langle e_l, j_1 he_k \rangle a^*(e_l) \otimes 1 + \langle e_l, j_2 he_k \rangle 1 \otimes a^*(e_l))Ja(e_k).
\]

Using Lemma 20 (c) we find

\[
(d\Gamma(h) \otimes 1 + 1 \otimes d\Gamma(h))J
\]

\[
= \sum_{l,k} \langle e_l, he_k \rangle (a^*(e_l)a(e_k) \otimes 1 + 1 \otimes a^*(e_l)a(e_k))J
\]

\[
= \sum_{l,k} \langle e_l, he_k \rangle ((a^*(e_l) \otimes 1)Ja(j_1 e_k) + (1 \otimes a^*(e_l))Ja(j_2 e_k))
\]

\[
= \sum_{l,k} (\langle e_l, hj_1 e_k \rangle (a^*(e_l) \otimes 1)Ja(e_k) + \langle e_l, hj_2 e_k \rangle (1 \otimes a^*(e_l))Ja(e_k)).
\]
Taking the difference (b) follows. Now (a) follows from (b) using standard estimates. For example using Lemmas 18 and 19 we find

Right hand side of (b)
\[
= U \sum_l a^*([j_1, h]e_l, [j_2, h]e_l) a(j_1 e_l, j_2 e_l) \Gamma(j)
\]
\[
= Ud \Gamma(A) \Gamma(j),
\]
where we defined the following operator on \( h \otimes h \)

\[
A = \left( \begin{array}{cc}
[j_1, h]j_1 & [j_1, h]j_2 \\
[j_2, h]j_1 & [j_2, h]j_2
\end{array} \right).
\]

Now the bound follows since the operator preserves the \( n \)-particle sector and satisfies the following estimate

\[
\|d \Gamma(A) \Gamma(j)|_{h^{(n)}}\| \leq \|(A_j) \otimes j \otimes \cdots \otimes j\| + \|j \otimes (A_j) \otimes \cdots \otimes j\| + \cdots + \|j \otimes \cdots \otimes j \otimes (A_j)\| \leq n\|A_j\|n^{-1} = n\|A_j\| = n(\|[j_1, h]\|^2 + \|[j_2, h]\]|^2)^{1/2}.
\]

Recall that \( \mathcal{H} = \mathbb{C}^{2s+1} \otimes \mathcal{F} \). We consider the map

\[
\mathbb{1} \otimes J : \mathbb{C}^{2s+1} \otimes \mathcal{F} \to \mathbb{C}^{2s+1} \otimes (\mathcal{F} \otimes \mathcal{F})
\]

from \( \mathcal{H} \) to \( \mathcal{H} \otimes \mathcal{F} \). By abuse of notation we shall henceforth denote this map again by \( J \). We introduce the operator

\[
\tilde{H}_m(\xi) := \frac{1}{2}(\xi - P_f \otimes \mathbb{1} - \mathbb{1} \otimes P_f - eA \otimes \mathbb{1})^2 + eS \cdot B \otimes \mathbb{1} + H_{f,m} \otimes \mathbb{1} + \mathbb{1} \otimes H_{f,m}
\]
on \( \mathcal{H} \otimes \mathcal{F} \), with domain given by the natural domain of \( \tilde{H}_m(\xi)|_{e=0} \).

**Lemma 23.** Let \( m > 0 \) and \( \rho \in L^2(\mathbb{R}^3; (|k| + |k|^{-1})dk) \). Then the following holds.
(a) For $\varphi \in D(H_m(\xi))$ we have
\[|\langle \varphi, H_m(\xi)\varphi \rangle - \langle J\varphi, \tilde{H}_m(\xi)J\varphi \rangle| \leq o(L^0)(\|H_m(\xi)\varphi\|^2 + \|\varphi\|^2) \quad (L \to \infty),\]
where $o(L^0)$ does not depend on $\varphi$.

(b) For $\varphi \in D(\tilde{H}_m(\xi))$ we have
\[\langle \varphi, \tilde{H}_m(\xi)\varphi \rangle \geq \langle \varphi, \{E_m(\xi) + \Delta_m(\xi)(\mathbb{1} - P_{\Omega,2})\} \varphi \rangle,\]
where $P_{\Omega,2}$ denotes the orthogonal projection in $\mathcal{H} \otimes \mathcal{F}$ onto $\mathcal{H} \otimes \Omega$.

**Proof.** (a) Defining the operators
\[Q = J(\xi + v) - (\xi - P_f \otimes \mathbb{1} - \mathbb{1} \otimes P_f + eA \otimes \mathbb{1})J,\]
we can write
\[H_m(\xi) - J^*\tilde{H}_m(\xi)J = \frac{1}{2}\{((\xi + v)J^*Q + Q^*J(\xi + v) - Q^*Q)\]
\[+ eS \cdot (B - J^*(B \otimes \mathbb{1})J)\]
\[+ H_{f,m} - J^*(H_{f,m} \otimes \mathbb{1} + \mathbb{1} \otimes H_{f,m})J.\]

Now using that $J$ is an isometry, it follows from Lemma 22 (a) (choosing for example $\mathcal{D} = C_c(\mathbb{Z}_2 \times \mathbb{R}^3)$) that for $\varphi \in D(N)$ we have,
\[\|H_{f,m} - J^*(H_{f,m} \otimes \mathbb{1} + \mathbb{1} \otimes H_{f,m})J\| \varphi\| \leq (\|[j_1, \omega_m]\| + \|[j_2, \omega_m]\|)\|N\varphi\|.

Using again that $J$ is an isometry it follows from Lemma 21 (a) that for $\varphi \in D(N^{1/2})$ we have, recalling the notation (17),
\[\|(S \cdot (B - J^*(B \otimes \mathbb{1})J))\| \varphi\| \leq (\|[j_1 - 1]f_B\| + \|[j_2]_{f_B}\|)\|(N + 1)^{1/2}\varphi\|.

Analogously, we find from Lemmas 22 and 21 for $\varphi \in D(N)$
\[\|Q\| \varphi\| \leq (\|[j_1, \nu]\| + \|[j_2, \nu]\|)\|N\varphi\| + |e|(\|[j_1 - 1]f_A\| + \|[j_2]_{f_A}\|)\|(N + 1)^{1/2}\varphi\|,
\]
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where $\nu : \mathbb{R}^3 \to \mathbb{R}^3$ with $\nu(k) = k$. Now we will use that for $m > 0$

$$\|N\varphi\| \leq C(\|H_m(\xi)\varphi\| + \|\varphi\|),$$

and that

$$\|(j_1 - 1) f_B\|, \|j_2 f_B\|, \|(j_1 - 1) f_A\|, \|j_2 f_A\| = o(L^0),$$

which follows by dominated convergence in Fourier space. Furthermore, we note

$$\|[j_1, \nu]\|, \|[j_2, \nu]\| = O(L^{-1}),$$

$$\|[j_1, \omega_m]\|, \|[j_2, \omega_m]\| = O(L^{-1}),$$

where the first line is easy to see and the second line can be seen as follows. Let $\chi_{1,L} = \chi_1(\cdot/L)$. For normalized $\varphi_1, \varphi_2$ we find using the usual notation and convention for the Fourier transform

$$|\langle \varphi_1, [j_1, \omega_m] \varphi_2 \rangle| = (2\pi)^{-3/2} \int_{\mathbb{R}^3 \times \mathbb{R}^3} \varphi_1(p) \overline{\chi_{1,L}(p-q)(\sqrt{q^2 + m^2} - \sqrt{p^2 + m^2})} \varphi_2(q) dp dq$$

$$\leq (2\pi)^{-3/2} \int_{\mathbb{R}^3 \times \mathbb{R}^3} |\varphi_1(p)| \sum_{s=1}^3 |\overline{\chi_{1,L}(p-q)}| |p_s - q_s| |\varphi_2(q)| dp dq$$

$$= (2\pi)^{-3/2} \int_{\mathbb{R}^3 \times \mathbb{R}^3} |\varphi_1(p)| \sum_{s=1}^3 |\overline{\partial_s \chi_{1,L}(p-q)}| |\varphi_2(q)| dp dq$$

$$\leq \sum_{s=1}^3 \|(|\overline{\partial_s \chi_{1,L}})| \|_{\infty} = L^{-1} \sum_{s=1}^3 \|(|\overline{\partial_s \chi_{1}})| \|_{\infty}.$$

To treat the term involving $j_2$ we apply a similar estimate to the function $1 - \chi_2$. Inserting the above estimates Part (a) now follows.

To show (b) we use the canonical isomorphism

$$\mathcal{H} \otimes \mathcal{F} \cong \bigoplus_{n \in \mathbb{N}_0} L^2_{\text{sym}}((\mathbb{Z}_2 \times \mathbb{R}^3)^n; \mathcal{H}),$$

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where the summand with \( n = 0 \) is by convention \( \mathcal{H} \). With respect to this fiber decomposition the Hamiltonian \( \tilde{H}_m(\xi) \) fibrates and we have for \( n \in \mathbb{N}_0 \),

\[
(\tilde{H}_m(\xi)\psi)_{(n)}(\lambda_1, k_1, \ldots, \lambda_n, k_n) = \left( H_m(\xi - \sum_{j=1}^{n} k_j) + \sum_{j=1}^{n} \omega_m(k_j) \right) \psi_{(n)}(\lambda_1, k_1, \ldots, \lambda_n, k_n).
\]

This yields the expectation

\[
\langle \psi, \tilde{H}_m(\xi)\psi \rangle = \sum_{n=0}^{\infty} \langle \psi_{(n)}, (\tilde{H}_m(\xi)\psi)_{(n)} \rangle.
\]

To calculate the summands on the right hand side we again use (38) and find for \( n \geq 1 \)

\[
\langle \psi_{(n)}, (\tilde{H}_m(\xi)\psi)_{(n)} \rangle \geq (E_m(\xi) + \Delta_m(\xi)) \| \psi_{(n)} \|^2,
\]

where we employed the following operator inequality. Using \( \omega_m(p) + \omega_m(q) \geq \omega_m(p+q) \) we find

\[
H_m \left( \xi - \sum_{j=1}^{n} k_j \right) + \sum_{j=1}^{n} \omega_m(k_j) \geq E_m \left( \xi - \sum_{j=1}^{n} k_j \right) + \sum_{j=1}^{n} \omega_m(k_j) \\
\geq E_m \left( \xi - \sum_{j=1}^{n} k_j \right) + \omega_m \left( \sum_{j=1}^{n} k_j \right) \\
\geq E_m(\xi) + \Delta_m(\xi).
\]

The above now implies

\[
\langle \psi, \tilde{H}_m(\xi)\psi \rangle \geq E_m(\xi)\| \psi(0) \|^2 + (E_m(\xi) + \Delta_m(\xi)) \sum_{n=1}^{\infty} \| \psi_{(n)} \|^2.
\]

Thus we have shown (b). \( \square \)

**Proof of Proposition 17.** From Lemma 23 we find with \( \| \varphi \|_{H_m(\xi)} := (\| H_m(\xi)\varphi \|^2 + \| \varphi \|^2)^{1/2} \)

\[
\langle \varphi, H_m(\xi)\varphi \rangle \geq (E_m(\xi) + \Delta_m(\xi))\| \varphi \|^2 - \Delta_m(\xi)\| \Gamma(j_1)\varphi \|^2 - o(L^0)\| \varphi \|^2_{H_m(\xi)},
\]

(39)
where we used that $\langle J\varphi, (I \otimes P_{\Omega})J\varphi \rangle = \|\Gamma(j_1)\varphi\|^2$. Let $\lambda \in \sigma_{\text{ess}}(H_m(\xi))$. Then there exists a normalized sequence $\psi_n, n \in \mathbb{N}$, converging weakly to zero such that

$$\lim_{n \to \infty} \|(H_m(\xi) - \lambda)\psi_n\| = 0.$$ 

Thus,

$$\langle \psi_n, H_m(\xi)\psi_n \rangle \geq E_m(\xi) + \Delta_m(\xi) - \Delta_m(\xi)\|\Gamma(j_1)\psi_n\|^2 - o(L^0)\|\psi_n\|^2_{H_m(\xi)}.$$ 

Taking the limit $n \to \infty$ we find

$$\|\Gamma(j_1)\psi_n\|^2 = \langle (1 + H_{f,m})\psi_n, (1 + H_{f,m})^{-1}\Gamma(j_1^2)\psi_n \rangle \to 0,$$

since $(1 + H_{f,m})^{-1}\Gamma(j_1^2)$ is compact (it is compact on every finite particle space and, since $m > 0$, it is given by $\lim_{n \to \infty}(1 + H_{f,m})^{-1}\Gamma(j_1^2)1_{N \leq n}$ in operator norm). Thus we find

$$\lambda \geq E_m(\xi) + \Delta_m(\xi) + o(L^0)(\lambda^2 + 1).$$

Taking $L \to \infty$ yields the claim. $\square$

**Proposition 24.** Let $\rho \in L^2(\mathbb{R}^3; (|k| + |k|^{-1})dk)$ with $\rho = \rho(-\cdot)$. Let $e \in \mathbb{R}$ and $m > 0$, and suppose (9) holds. Then $\Delta_m(\xi) > 0$, whenever $|\xi| \leq 1$.

**Proof.** First we show that the function $\xi \mapsto E_m(\xi)$ has the properties

(i) $E_m(0) \leq E_m(\xi)$,

(ii) $E_m(\xi) \leq \frac{1}{2}\xi^2 + E_m(0)$,

(iii) $G_m : \xi \mapsto \frac{1}{2}\xi^2 - E_m(\xi)$ is convex.

Property (i) follows from the assumption, (iii) follows since the pointwise supremum of a set of convex functions is convex. The symmetry $E_m(-\xi) = E_m(\xi)$, which follows form Lemma 23 below, implies $G_m(-\xi) = G_m(\xi)$. Thus by convexity $G_m(0) \leq G_m(\xi)$,
which implies (ii). It follows from a lemma about convex functions (see Lemma A2 in [LossMiyaoSpohn]) that properties (i)-(iii) imply

\[
E_m(\xi - k) - E_m(\xi) \geq \begin{cases} 
-|k||\xi| + \frac{1}{2}k^2, & \text{if } |k| \leq |\xi|, \\
-\frac{1}{2}\xi^2, & \text{if } |k| \geq |\xi|.
\end{cases}
\]

Thus we find using \( \omega_m(k) > |k| \)

\[
E_m(\xi - k) - E_m(\xi) + \omega_m(k) > \begin{cases} 
-|k||\xi| + |k|, & \text{if } |k| \leq |\xi|, \\
-\frac{1}{2}\xi^2 + |\xi|, & \text{if } |k| \geq |\xi|,
\end{cases}
\]

\[\geq 0,\]

provided \(|\xi| \leq 1.\]

\[\Box\]

**Lemma 25.** Let \( D_{\lambda,\lambda'}(k) = \varepsilon_\lambda(k) \cdot (-\varepsilon_{\lambda'}(-k)). \) Define \( I : L^2(\mathbb{Z}_2 \times \mathbb{R}^3) \rightarrow L^2(\mathbb{Z}_2 \times \mathbb{R}^3) \) by \((I\psi)(\lambda, k) = \sum_{\lambda'} D_{\lambda,\lambda'}(k)\psi(\lambda', -k)\) for \( \psi \in L^2(\mathbb{Z}_2 \times \mathbb{R}^3) \). Then \( I \) and \( \Gamma(I) \) are unitary operators. For \( m > 0 \) and \( \rho \in L^2(\mathbb{R}^3; (|k| + |k|^{-1})dk) \), we have \( \Gamma(I)H_m(\xi)\Gamma(I)^* = H_m(-\xi) \).

**Proof.** It is straightforward to verify that \( I \) is unitary. Hence \( \Gamma(I) \) is also unitary. It is easy to see that \( \Gamma(I)H_f\Gamma(I)^* = H_f \) and \( \Gamma(I)P_f\Gamma(I)^* = -P_f \). Using the properties of the polarization vectors and the parity symmetry of \( \rho \), an elementary calculation shows that \( \Gamma(I)A_j\Gamma(I)^* = -A_j \). The claim of the lemma now follows. \( \Box \)

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