Some aspect of perturbed extended general quasi variational inequality by using auxiliary principle technique

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ABSTRACT
Perturb Extended GQVI provide natural, unified, simple and novel context to study an extensive class of equilibrium problems arising in various fields. In this work, we familiarize and study a number of new and known computational approaches to solve new class of QVI with three nonlinear machinists $T_\varepsilon, g, h$ by using different techniques including auxiliary approach and projection approach. We also deal with the existence of the solution of this newly established class of VI, as well as discuss the convergent theorem of the new algorithm by using the projection principal approach, which is simple as compared with other known classes of VI. Our consequences present an important enhancement of earlier recognized techniques for explaining various classes of VI. We are also debating several similar techniques as special cases. Several real-life equilibrium undeveloped problems have been proposed for further open new exploration horizon.

Nomenclature
- FPP: Fixed Point Problems
- VI: Variational Inequality
- QVI: Quasi Variational Inequality
- PAEGQVI: Perturbed Auxiliary Extended General Quasi Variational Inequality
- PGQVI: Perturbed General Quasi Variational Inequality
- $S$: Hilbert space
- CP: Complementarity Problems
- s.t.: such that
- $K$: Convex set
- $\langle\cdot,\cdot\rangle$: Inner Product
- $\|\cdot\|$: Norm
- $T_\varepsilon$: Perturbed operator
- OR: Optimization Problems

1. Introduction
The VI concept presented by Stampacchia in 1964 may be well-thought-out as a postponement of variational principles and natural generalization [1]. Most of this VI is used as mathematical models to work for equilibrium problems for $n$th size problem, stability problems arise in finance, engineering science and optimization, see [2,3]. The performance of such equilibrium solution change when the statistics of the problematic are changed. Variational inequalities are an effective tool in studying the existence of solution of constraint problems arising in optimization (see [4–33]).

Another beneficial and generalization of VI is Quasi Variational Inequality, which play an important role in applied and pure science. It is famous that an extensive class of problem stand up in so many fields can be studied by Quasi-Variational inequality. Quasi-Variational inequality continuously benefits from exchange between computational analysis, nonconvex and convex analysis, numerical analysis [10,14,16,18,30,32,34].

The area of the variational inequalities is gaining the more and more attention of experts from various parts of natural science in general and applied mathematician in [6,8,9,11,12,18,31] particularly fluid mechanic when we deal the applications of computational techniques to solve the boundary value problem [34–36]. It turns out that it has a great connection with the different practical areas that arise in various sciences. The methods in variational techniques provide a simplistic, dynamic and natural way of dealing with the large variety of problem in both linear and nonlinear natural science of famous numerical methods [8,9,34] include the projection method [7,13], Weiner Hopf techniques [13,17], auxiliary principle [23,24] etc. provides an efficient way to solve numerical optimization problem [7,31–34,37].

Perturbed Auxiliary EGQVI are being used as mathematical programming models to study a huge number

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Equilibrium problems theory provides us with an integrated, natural, inventive and broad context to study an extensive class of problems rising in optimization, agricultural economics, the mathematical analysis of multifaceted working processes in terms of a network, transportation. This notion has observed a volatile expansion in hypothetical improvements and applications in sciences. Finally, we have a diversity of procedures to deal the existence of equilibrium problems comprise variational inequalities as special case [34–40]. Fixed point theory also has lot of applications in fluid mechanics [34–40], transport theory for growing cell populations [26].

One more imperative application of existence of a solution to the VI problem ascending in a prototypical for sand surface advancement has been an open issue concerns conclusive the dual variable, the flux of sand pouring down the developing sand surface, which is also of real-world concentration in a variability amplification of this model, see [5,18]. Earlier, these glitches were resolved for the exceptional case in which the inequality is simply variational [5]. The agriculture sustainability problem encompasses several mathematical glitches as a specific case, i.e. the VI problems, FPP, CP and OP, see [17,18,25]. Furthermore, computational methods are well-organized tools to estimate the result to an equilibrium problem of land use patterns in agriculture [30,41].

In view of the implication and significance idea of the GQVI in variational inequalities theory, we present a new discussion of VI is called Perturbed Auxiliary EGQVI. Fundamentally consuming the projection procedure, we create the resemblance between AEGQVI and fixed point problematic. Since Classical VI, GQVI and general quasi complementarity problems are special cases of AEGQVI. Nowadays VI theory is a very fast growing research field. In this paper, established new class of VI will open new horizon for researchers to explore fascinating applications in sustainability of agriculture, sustainability of water and other existing open equilibrium problems in various areas.

2. Basic results

Let $S$ be a real Hilbert space whose inner product and norm are denoted by $(\cdot, \cdot)$ and $\| \cdot \|$, respectively. Let $K \subset S$, and $T, g : S \to S$ are nonlinear machinists with mapping $K : x \to K(x)$, associated with closed convex set $K(x)$ of $S$ for all $x \in S$, s. t.

$$\langle T_{\varepsilon} x, g(y) - h(x) \rangle \geq 0 \forall y \in S : g(y), h(x) \in K(x), \quad (1)$$

where $T_{\varepsilon} = (T + \varepsilon I)$ is Perturbed operator and inequality (1) is called Perturbed EGGQVI with three nonlinear machinists $T_{\varepsilon}, h$ and $g$ is wide-ranging and integrated one [15,21].

We remark that:

**Case 1:** If $T_{\varepsilon} \equiv T$ and $g \equiv h \equiv I$, then Equation (1), we have

$$\langle Tx, x - y \rangle \leq 0 \forall x \in S : y \in K(x), \quad (2)$$

inequality (2.2) is the problem introduced by Bensoussan and Lions [5].

**Case 2:** If $K \equiv K(x)$ the set $K(x)$ is free from $x$, then Equation (1) is called PEGQVI involving three machinists introduced by Noor [21].

$$\langle h(x) - g(y), T_{\varepsilon} x \rangle \leq 0 \forall y \in S, g(y), h(x) \in K, \quad (3)$$

**Case 3:** For $g \equiv h \equiv I$, then Equation (3) breakdowns to problem is called Perturbed VI introduce by Noor [15].

$$\langle T_{\varepsilon} x, y - x \rangle \geq 0 \forall x, y \in K \subset S \quad (4)$$

**Case 4:** For $T_{\varepsilon} = T$, problem (4) become

$$\langle T_{\varepsilon} x, y - x \rangle \geq 0 \forall x, y \in K \subset S \quad (5)$$

called Classical VI introduce by Stampacchia [1].

**Case 5:** If $K^* = \{ x \in S : (x,y) \geq 0, \forall x \in K \}$, then we have

$$x \in S, T_{\varepsilon} x K^* \langle Tx, g(x) \rangle = 0, \quad (6)$$

called general complementarity problem introduce by Noor [21] and Stampacchia [1], where $K^*$ and $K \subset S$ are polar cone and convex cone respectively.

We use above results to proof the convergence analysis by using contraction mapping.

**Lemma 2.1 (1):** Let $K(x) \subset S$ for a given $x \in K(x)$, we $S$ satisfies the inequality

$$0 \leq \langle -w + x, -x + y \rangle, \forall y \in K(x),$$

if

$$p_{K(x)}w = x.$$ The projection operator $p_{K(x)}$ is not non-expansive.

**Lemma 2.2 (Noor [15]):** For all $x, y, z \in S$, the implicit operator $p_{K(x)}$ fulfills the condition:

$$2\gamma \| x - y \| \geq \| p_{K(x)}z - p_{K(x)}y \|, \forall x, y, z \in S, \quad (7)$$

where $\gamma > 0$. 

of equilibrium problems get up in finance, economics, transportation, optimization, engineering and sciences.
**Proof:** Let convex-value set

\[ K(x) = K + m(x) \]

where \( m(x) \) is a point-to-point mapping and is Lipschitz continuous with constant \( \gamma > 0 \), and \( P_{K(x)} \) is not non-expensive operator written as:

\[ P_{K}[-m(x) + z] + m(x) = P_{K(z)} \]

\[
\|P_{K(z)} - P_{K(y)}z\| = \|P_{K}[-m(x) + z] + m(x) - P_{K}[-m(y) + z] - m(y)\|,
\]

\[
\leq \|m(y) - m(x)\| + \|P_{K}[-m(x) + z] - P_{K}[-m(y) + z]\|,
\]

using non-expensive condition, we have

\[
= 2\|m(y) - m(x)\|, \forall y, z \in S.
\]

**Lemma 2.3:** If \( T : S \rightarrow S \) be strongly monotone operator with constant \( \alpha > 0 \) and Lipschitz continuous with constant \( \beta \) then show that Perturbed operator \( T_{\varepsilon} = (T + \varepsilon I) \) is also strongly monotone \( \alpha + \varepsilon \) and Lipschitz continuous with constant \( \beta + \varepsilon \), where \( I \) is identity operator.

**Proof:** We know that an operator \( T : S \rightarrow S \) is:

**Strongly monotone**, if \( 0 < \alpha \) exists, s.t.

\[
\alpha \|x - y\|^2 \leq \langle Tx - Ty, x - y \rangle, \forall y, x \in S.
\]

**Lipschitz continuous**, if \( \beta \) exists, s.t.

\[
\|Tx - Ty\| \leq \beta \|x - y\|, \forall y, x \in S.
\]

Now we prove that operator \( T_{\varepsilon} \) is also strongly monotone with constant \( \alpha + \varepsilon > 0 \)

\[
\langle x - y, T_{\varepsilon}x - T_{\varepsilon}y \rangle = \langle (T + \varepsilon I)x - (T + \varepsilon I)y, x - y \rangle
\]

\[
= \langle Tx + \varepsilon Ix - Ty - \varepsilon Iy, x - y \rangle
\]

\[
= \langle x - y, Tx - Ty \rangle + \varepsilon \langle y - x - y, x - y \rangle
\]

\[
= (\alpha + \varepsilon) \| y - x\|^2, \text{ for all } x, y \in S.
\]

Similarly see [15], operator \( T : S \rightarrow S \) is:

**Lipschitz continuous**, if \( \beta > 0 \) exists, s.t.

\[
\|Tx - Ty\| \leq \beta \|x - y\|, \forall y, x \in S.
\]

Now we prove that operator \( T_{\varepsilon} \) is also Lipschitz continuous with constant \( 0 < \beta + \varepsilon \), we have

\[
\|T_{\varepsilon}x - T_{\varepsilon}y\| = \|T(x + \varepsilon I)x - (T + \varepsilon I)y\|
\]

\[
= \| - Ty - \varepsilon y + Tx + \varepsilon x\|
\]

\[
\leq (\beta + \varepsilon) \|y - x\|, \text{ for all } y, x \in S.
\]

Hence, Perturbed operator \( T_{\varepsilon} = (T + \varepsilon I) \) is also strongly monotone with \( 0 < \alpha + \varepsilon \) and Lipschitz continuous with constant \( 0 < \beta + \varepsilon \), where \( I \) is the identity operator.

3. **Main results**

3.1. **Existence theory by using auxiliary technique**

We apply the auxiliary technique to expression that Equation (1) equivalent of implicit fixed point problem introduce in Noor [15,21] and Baloch and Noor [24].

**Theorem 3.1:** Let \( T_{\varepsilon} = (T + \varepsilon I) \) be the operator descried in Lemma 2.3 and \( g \) be strongly monotone with constant \( \sigma > 0 \) and Lipschitz continuous with constant \( \delta > 0 \). Let \( h \) is firmly expanding machinist s.t.

\[
\|w_{1} - w_{1}\| \leq \|h(w_{1}) - h(w_{2})\|.
\]

If Lemma 2.1 to Lemma 2.3 holds, then unique solution exists of PAEGQVI define in Equation (1). For

\[
\beta \frac{\alpha + \varepsilon}{(\beta + \varepsilon)^{2}} < \frac{(\alpha + \varepsilon)^{2} + (\beta + \varepsilon)^{2}k(2 - k)}{(\beta + \varepsilon)^{2}}, \quad k < 1
\]

\[
(\alpha + \varepsilon) > (\beta + \varepsilon)\sqrt{(2 - k)k}
\]

\[
k = 2\sqrt{1 - 2\delta + \sigma^{2}}
\]

**Proof:** Let \( x \in S : g(x) \in K(x) \) be the solution satisfying the new class of Variational Inequalities (1), we assume \( h(w) \in K(x), w \in S \) s.t.

\[
\langle g(y) - h(w), h(w) - g(x) + \rho T_{\varepsilon}x \rangle
\]

\[
\geq 0, \forall g(y) \in K(x), y \in S,
\]

where \( \rho > 0 \) is a constant. This variational inequality is called the Perturbed AEGQVI defines a mapping \( x \mapsto w \) with \( w \in S \) satisfying relation (1).

Let \( w_{1} \neq w_{2} \) associated to \( x_{1}, x_{2} \in S \) respectively.

\[
\|w_{1} - w_{2}\| \leq \theta\|x_{1} - x_{2}\|, \text{with } 0 < \theta < 1.
\]

Take \( g(y) = g(w_{2}) \) (respectively \( g(y) = g(w_{2}) \)) in (14) related to \( g(x_{1}) \) (respectively \( g(x_{2}) \)), and adding the
resultant, we get
\[
\langle g(w_1) - h(w_2), \rho T_e x_2 - g(x_2) \rangle
+ h(w_2) + \langle \rho T_e x_1 + h(w_1) - g(x_1), -h(w_1) + g(w_2) \rangle \geq 0,
\forall y \in S : g(y) \in K(x).
\] (16)

Simplify Inequality (16), we get
\[
\langle h(w_1) - h(w_2), \rho (T_e x_1 - T_e x_2) + g(x_1) - g(x_2) \rangle
\geq \langle h(w_2) - h(w_1), h(w_2) - h(w_1) \rangle
\forall y \in S, g(y) \in K(x).
\] (17)

Expression (17), can be written as:
\[
\| h(w_1) - h(w_2) \|
\leq \| \rho T_e x_1 + T_e x_2 + g(x_1) - g(x_2) \|, \\
\leq \| g(x_1) - g(x_2) \| + x_1 - x_2 \|
+ \| x_1 - x_2 - \rho (T_e x_1 - T_e x_2) \|.
\] (18)

Using Lemma 2.3, we have
\[
\| x_1 - x_2 - \rho (T_e x_1 - T_e x_2) \|
\leq \sqrt{1 - 2\rho(\alpha + \epsilon) + \rho^2(\beta + \epsilon)^2}\| x_1 - x_2 \|.
\] (19)

and
\[
\| g(x_1) - g(x_2) \| + x_1 - x_2 \|
\leq \sqrt{1 - 2\delta + \sigma^2}\| x_1 - x_2 \|.
\] (20)

Substitute Equations (19), (20) in Equation (18), we get
\[
\| h(w_1) - h(w_2) \|
\leq \sqrt{1 - 2\rho(\alpha + \epsilon) + \rho^2(\beta + \epsilon)^2}\| x_1 - x_2 \|
+ \| x_1 - x_2 \| \sqrt{1 - 2\delta + \sigma^2}.
\]
\[
\| h(w_1) - h(w_2) \|
\leq \sqrt{1 - 2\rho(\alpha + \epsilon) + \rho^2(\beta + \epsilon)^2} \| x_1 - x_2 \|
+ \| x_1 - x_2 \| \sqrt{1 - 2\delta + \sigma^2}.
\]

and
\[
\sqrt{1 - 2\rho(\alpha + \epsilon) + \rho^2(\beta + \epsilon)^2} < (1 - k),
\]
\[
(1 - 2\rho(\alpha + \epsilon) + \rho^2(\beta + \epsilon)^2) < (1 - k)^2,
\]
\[
(1 - 2\rho(\alpha + \epsilon) + \rho^2(\beta + \epsilon)^2) < (1 - 2k + k^2),
\]
\[
-2\rho(\alpha + \epsilon) + \rho^2(\beta + \epsilon)^2 < -2k + k^2,
\]
\[
\frac{(\alpha + \epsilon)^2}{(\beta + \epsilon)^2} < 2k^2 + \frac{(\alpha + \epsilon)^2}{(\beta + \epsilon)^2},
\]
\[
\frac{(\alpha + \epsilon)^2}{(\beta + \epsilon)^2} < 2k^2 + \frac{(\alpha + \epsilon)^2}{(\beta + \epsilon)^2},
\]
\[
\left( \frac{\rho(\beta + \epsilon) - \alpha + \epsilon}{\beta + \epsilon} \right)^2 < 2k^2 + \frac{(\alpha + \epsilon)^2}{(\beta + \epsilon)^2},
\]
\[
\left( \frac{\rho - \alpha + \epsilon}{(\beta + \epsilon)^2} \right)^2 < -2k^2 + \frac{(\alpha + \epsilon)^2}{(\beta + \epsilon)^2},
\]
\[
\left( \frac{\rho}{\beta + \epsilon} \right)^2 < 2k^2 + \frac{(\alpha + \epsilon)^2}{(\beta + \epsilon)^2},
\]
\[
\left| \rho \frac{\alpha + \epsilon}{(\beta + \epsilon)^2} \right| < \frac{\sqrt{(\alpha + \epsilon)^2 + (\beta + \epsilon)^2 k(2 - k)}}{(\beta + \epsilon)^2}.
\]

Now \( \pi(\rho) \) assumes its minimum value at \( \rho = \frac{\alpha + \epsilon}{(\beta + \epsilon)^2} \) with \( \pi(\rho) = \sqrt{1 - \frac{(\alpha + \epsilon)^2}{(\beta + \epsilon)^2}} \). For \( \rho = \frac{\alpha + \epsilon}{(\beta + \epsilon)^2} < 1 \), implies that \( k < 1 \). Thus, \( \theta = k + \pi(\rho) < 1 \) for all \( \rho \) with
\[
\left| \rho \frac{\alpha + \epsilon}{(\beta + \epsilon)^2} \right| < \frac{\sqrt{(\alpha + \epsilon)^2 + (\beta + \epsilon)^2 k(2 - k)}}{(\beta + \epsilon)^2},
\]
and
\[
\alpha + \epsilon > (\beta + \epsilon)\sqrt{2 - k}k.
\]
The mapping in Equation (14) has well define contraction mapping with satisfies the condition (11–13).

3.2. Existence theory by using projection technique

Let \( g \equiv h x \in S : g(x) \in K(x) \) is a root of Equation (21) if \( x \in S, g(x) \in K(x) \) fulfills
\[
x = x - g(x) + P_K[g(x) - \rho T_e x], \rho > 0.
\] (21)

Iterative scheme can be written as:

Algorithm 3.1:
\[
x_{n+1} = x_n - g(x_n) + P_K[g(x_n) - \rho T_e x_n].
\]
3.3. Convergence analysis

Theorem 3.2: Let $T_e = (T + eI)$ be the operator described in Lemma 2.2–2.3 is strongly monotone with constant $\alpha + \varepsilon > 0$ and Lipschitz continuous with constant $\gamma > 0$. Let $g$ be strongly monotone with constant $\sigma > 0$ and Lipschitz continuous with constant $\delta > 0$. Let $K(x)$ be a closed convex set in if Lemma 2.2 to Lemma 2.3 holds and $\rho > 0$, then there exists approximate solutions $x_{n+1}$ convergences to a solution $x \in K(x)$.

Proof: Using Equation (21) and Algorithm 3.1, we have

$$
\|x_{n+1} - x\| \leq \|x_n - x\| + \|g(x_n) - g(x)\| + \|P_{K(x)}[g(x_n) - \rho T_ex]\|,
$$

where $P_{K(x)}$ is not non-expansion operator and using Lemma 2.2, we have

$$
\|x_n - x\| - \|g(x_n) - g(x)\| - \|P_{K(x)}[g(x_n) - \rho T_ex]\|,
$$

Using Equations (19), (20) and Lemma 2.2, we have

$$
\begin{align*}
\|x_{n+1} - x\| & \leq \sqrt{1 - 2\delta + \sigma^2} \|x_n - x\| + 2\gamma \|x_n - x\| \\
& + \sqrt{1 - 2\rho(\alpha + \varepsilon) + \rho^2(\beta + \varepsilon)^2} \|x_n - x\|,
\end{align*}
$$

\[\leq \left(\sqrt{1 - 2\delta + \sigma^2} + 2\gamma \right) \|x_n - x\|,

\|x_{n+1} - x\| \leq \theta \|x_n - x\|.

\[\theta < 1,

\] where $\theta = k + \pi(\rho)$ where $k = 2\sqrt{1 - 2\delta + \sigma^2} + 2\gamma$, and $\pi(\rho) = \sqrt{1 - 2\rho(\alpha + \varepsilon) + \rho^2(\beta + \varepsilon)^2}$ and

$$
\rho \left(\alpha + \varepsilon\right) \left(\beta + \varepsilon\right) \leq \sqrt{(\alpha + \varepsilon)^2 + (\beta + \varepsilon)^2k(2 - k)},
$$

Thus, $x_n$ convergence to the unique solution $x$ by choosing very large value of $n$. 

4. Conclusion

In this study, a new class of VI with three non-linear operators with Perturbed technique is called PEAGQVI. We established equivalence between perturbed extended auxiliary general quasi variational inequality and implicit fixed point problems by using auxiliary technique and projection technique. We proposed an algorithm and used it to prove existence and convergence theorem under suitable conditions. The broadside existence theory by using auxiliary technique and existence theory of projection technique will encourage and instigate to sightsee its solicitations in several areas of research in various fields.

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