Dynamics of a qubit-oscillator system with periodically varying coupling

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Abstract

The dynamics of qubits coupled to a harmonic oscillator with time-periodic coupling is investigated in the framework of Floquet theory. This system can be used to model nonadiabatic phenomena that require a periodic modulation of the qubit/oscillator coupling. The case of a single qubit coupled to a resonator populated with $n = 0, 1$ photon is explicitly treated. The time-dependent Schrödinger equation describing the system’s dynamics is solved within the Floquet formalism and compared to a solution obtained with perturbative approaches in the time- and Laplace-domain. Excellent quantitative agreement is found between the analytical and numerical calculations within the Floquet approach, demonstrating the effectiveness of the technique for the study of the dynamical Lamb effect.

Keywords: qubit, dynamical Lamb effect, cavity

(Some figures may appear in colour only in the online journal)

1. Introduction

The dynamics of quantum systems, which are periodically driven by low- and high-frequency field, has been widely investigated in the framework of the Floquet approach [1–5]. The Floquet formalism was introduced in reference [6] to simplify the solution of ordinary differential equations with terms that show a certain periodicity in time. More generally, it allows to consider the case of solutions of a linear partial differential equation periodic with respect to several variables, for example, periodic with respect to a crystal lattice or time [7–9]. Clearly, this method can be applied to the Schrödinger equation for the study of the time-evolution of a quantum system with periodic Hamiltonian [10–14]. The approach turns the problem of solving a differential equation into the problem of finding the eigenvalues and the eigenvectors of a matrix. Depending on the problem at hand, this can make it easier to find its solution. The Floquet framework was first widely adopted for the study of atomic and optical phenomena [10–18]. Moreover, a generalized Floquet formalism beyond the conventional one for accurate non-perturbative treatment of a broad range of strong-field atomic and molecular processes in intense laser field was developed. (See review [5]) applications to condensed matter systems were also explored in references [19, 20]. Furthermore, references [21–23] present applications of the Floquet formalism to the study of the dynamics of superconducting circuits. Thus, Floquet’s approach provides a canonical form for the solution of periodically driven systems and has found wide applicability in various branches of quantum physics since 1965 [10], but this approach has not been considered for the analysis of the dynamics of a qubit/oscillator system which we propose below.

Here, we consider a system composed of $N$ two-level systems, also called qubits, coupled to a harmonic oscillator. This is a good model for many physical systems such as atoms coupled to the electromagnetic field inside a cavity or superconducting circuits with non-linear elements coupled to a superconducting coplanar waveguide. More specifically, we consider the case of a superconducting circuit system where...
the qubit/resonator coupling is modulated periodically. As found in reference [24–30], if the coupling is nonadiabatically modulated, the qubits and photons in the resonator can be excited from the ground state. Furthermore, periodic modulation of the coupling can greatly increase the probability of excitation of both qubits and resonator. This phenomenon, which involves the creation of excitation from the quantum vacuum and is caused by the nonadiabatic change of the boundary conditions of the system, is known as dynamical Lamb effect (DLE) [31].

Since the periodic modulation of the coupling is required to enhance the effects of the DLE, one can consider the Floquet approach to study the time-evolution of the system. As it turns out, the Floquet approach provides a framework where analytical calculations can be performed and, despite the approximations, give results which are in good agreement with the numerical calculations. The effects of the DLE is studied by considering the time-evolution of the state \( |e, 1 \rangle \). In fact, as discussed in [30], the latter can only be reached from the ground state if the counter-rotating terms \( \hat{\sigma}^+ \hat{a} \) and \( \hat{\sigma}^- \hat{a} \) in the interaction Hamiltonian of the system become relevant. This can happen, for example, when the qubit/resonator coupling is modulated at the sum frequency of the qubit and the resonator transition frequencies.

In our previous work [24, 25], we have investigated the dynamics of a system of \( N \) qubits coupled to a resonator using different approaches. In reference [24], the time-dependent Schrödinger equation is solved directly in the time-domain using a perturbative approach and considering an averaged time-independent coupling. In reference [25] the Laplace transform is used to show how ordinary differential equations with a complicated time-dependence can easily be solved in the Laplace-domain. We briefly introduce these different approaches and compare the results obtained using these methods to the Floquet approach. One of the advantages of the perturbative approaches is that one can obtain explicit analytical expression for the wavefunction of the system at any time \( t \). For instance, a perturbative approach was used in references [26, 27, 29]. Even though a perturbative approach is a standard way to solve the time-dependent Schrödinger equation, it is only valid under the assumption of a weak interaction Hamiltonian. On the other hand, the Floquet approach does not rely on such assumption and it can be used for any interaction strength.

In this work, we consider the case of a single qubit coupled to a resonator. The time evolution of the state \( |e, 1 \rangle \) can then easily be calculated both analytically and numerically to show the effects of the DLE. The different methods described above are used to study the dynamics of the system and their results are compared. Overall, all methods give comparable results and the solution obtained with each of them shows the same qualitative features although a non-negligible discrepancy with the direct numerical integration of the Schrödinger equation arises at long times in case of perturbative approaches. Excellent quantitative agreement is found between the analytical and numerical calculations within the Floquet approach.

The article is organized in the following way. Section 2 defines the Hamiltonian of the system considered. The general case of \( N \) qubits coupled to a resonator is described. The Floquet formalism is introduced and applied to the case of a single qubit coupled to a resonator in section 3. Analytical and numerical calculations within the Floquet approach are then compared. In section 4, other analytical and numerical methods are presented. These include the perturbative integration of the Schrödinger equation in time- and Laplace-domain. The time-evolution of the \( |e, 1 \rangle \) state is explicitly calculated in the framework of these methods. A comparison of all the results obtained within the different approaches is given in section 5. Conclusions follow in section 6.

2. \( N \) superconducting qubits coupled to a resonator

Let us consider a system of \( N \) superconducting qubits coupled to a resonator with a time-dependent coupling. This system is well described by the Hamiltonian of \( N \) two-level systems coupled to a single-mode of the electromagnetic field with a time-dependent coupling, which is used to describe an atom interacting with the electromagnetic field of a cavity with variable atom/cavity coupling strength

\[
\hat{H}(t) = \hbar \omega_0 \hat{a}^\dagger \hat{a} + \sum_{i=1}^{N} \left[ \hbar \omega_i \hat{\sigma}_i^+ \hat{\sigma}_i^- + \hbar g(t) \hat{\sigma}_i^z \left( \hat{a}^\dagger + \hat{a} \right) \right]. \tag{1}
\]

For the case of a superconducting circuit setup, \( \omega_0 \) is the transition frequency of the \( i \)-th qubit, \( \omega_i \) is the frequency of the photons in the resonator, \( \hat{\sigma}_i^+ = \frac{\hat{\sigma}_i^x + i \hat{\sigma}_i^y}{2} \), \( \hat{\sigma}_i^- = \frac{\hat{\sigma}_i^x - i \hat{\sigma}_i^y}{2} \) and \( \hat{a}^\dagger, \hat{a} \) are the creation and annihilation operators for excitations of qubits and photons, respectively, \( \hat{\sigma}_i^x, \hat{\sigma}_i^y \) and \( \hat{\sigma}_i^z \) are the Pauli \( x, y \) and \( z \) operators for each qubit, while \( g(t) \) is the time-dependent coupling strength between the qubit and the resonator. Differently from the case of atoms in a cavity, the parameters of superconducting circuits can be engineered over a wide range of values. This allows, for instance, to increase or decrease the coupling strength between qubits and the resonator to very low or high values, effectively turning ‘off’ or ‘on’ their coupling. Let us mention that a possible experimental setup where this can be achieved was proposed in reference [30].

The Hamiltonian (1) can be split into two parts:

\[
\hat{H}(t) = \hat{H}_0 + \hat{H}_1(t), \tag{2}
\]

where

\[
\hat{H}_0 = \hbar \omega_0 \hat{a}^\dagger \hat{a} + \sum_{i=1}^{N} \hbar \omega_i \hat{\sigma}_i^+ \hat{\sigma}_i^- \tag{3}
\]

\[
\hat{H}_1(t) = \sum_{i=1}^{N} \left[ \hbar g(t) \hat{\sigma}_i^z \left( \hat{a}^\dagger + \hat{a} \right) \right] \tag{4}
\]

are a time-independent and a time-dependent term, respectively. As discussed and demonstrated in references [24, 26], the DLE is maximum when the qubit/resonator coupling is modulated periodically. Furthermore, in reference [30] we found that a sinusoidal modulation of the right frequency can
be used. Therefore, we take the qubit/resonator coupling as

$$g(t) = g_0 \cos(\omega_s t),$$  \hspace{1cm} (5)

where $g_0$ is the qubit/resonator coupling strength and $\omega_s$ is the frequency of switching of the coupling. Thus, the time-dependent Hamiltonian term is also periodic $\hat{H}_f(t) = \hat{H}_f(t + T)$, with period $T = \frac{2\pi}{\omega_s}$.

3. Floquet theory

Given a quantum system described by a Hamiltonian periodic in time with a period $T$, that is $\hat{H}(t) = \hat{H}(t + T)$, one can investigate the dynamics of the system within the Floquet approach \[10, 14\]. Let us now consider the Schrödinger equation with the Hamiltonian (2), which describes the time-evolution of the state $|\psi(t)\rangle$ of a system

$$i\hbar \frac{\partial}{\partial t}|\psi(t)\rangle = \hat{H}(t)|\psi(t)\rangle.$$  \hspace{1cm} (6)

Since $\hat{H}(t)$ is periodic, equation (6) is a differential equation with periodic coefficients. Thus, one can find its solutions using the Floquet theorem \[6\] in the following form

$$|\psi_n(t)\rangle = |\phi_n(t)\rangle e^{-i\epsilon_n t},$$  \hspace{1cm} (7)

where $|\phi_n(t)\rangle$ is called Floquet mode and is a periodic function with period $T$ and $\epsilon_n$ is called quasienergy or Floquet characteristic exponent. Clearly, different values of the quasienergy $\epsilon_n = \epsilon_n + n\frac{2\pi \hbar}{T}$, where $n = 0, \pm 1, \pm 2, \ldots$, correspond to the same solution $|\psi_n(t)\rangle$.

It is important to note that the Floquet modes and the quasienergies are the eigenfunctions and eigenvalues, respectively, of the operator $\hat{H}(t) = \hat{H}(t) - i\hbar \frac{\partial}{\partial t}$

$$\hat{H}(t)|\phi_n(t)\rangle = \epsilon_n|\phi_n(t)\rangle.$$  \hspace{1cm} (8)

This can be seen by substituting the general solution given in equation (7) into the Schrödinger equation (6). Then, equation (8) provides an alternative way of determining the state $|\psi(t)\rangle$ of the system. To demonstrate this, let us consider the decomposition of the generic state $|\psi(t)\rangle$ in terms of Floquet modes

$$|\psi(t)\rangle = \sum_{\alpha} c_{\alpha} |\phi_{\alpha}(t)\rangle e^{-i\epsilon_{\alpha} t},$$  \hspace{1cm} (9)

where the coefficients $c_{\alpha} = \langle \psi(0)|\phi_{\alpha}(0)\rangle$ quantify the overlap of the wavefunction with the Floquet modes at time $t = 0$. Following references \[32, 33\], one can find the Floquet modes by noting that they are eigenfunctions of the time-evolution operator $\hat{U}(t, t_0)$. In fact, the Schrödinger equation written in terms of the time-evolution operator,

$$\hat{U}(t_0 + T, t_0)|\psi(t_0)\rangle = |\psi(t_0 + T)\rangle,$$  \hspace{1cm} (10)

can be rewritten in terms of a generic Floquet mode as

$$\hat{U}(t_0 + T, t_0)|\phi_{\alpha}(t_0)\rangle e^{-i\epsilon_{\alpha} T} = |\phi_{\alpha}(t_0 + T)\rangle e^{-i\epsilon_{\alpha} T},$$  \hspace{1cm} (11)

Using the periodicity of the Floquet modes $\phi_{\alpha}(t_0 + T) = \phi_{\alpha}(t_0)$, equation (11) can be reduced to the following eigenvalue problem

$$\hat{U}(t_0 + T, t_0)|\phi_{\alpha}(t_0)\rangle = e^{-i\epsilon_{\alpha} T} |\phi_{\alpha}(t_0)\rangle.$$  \hspace{1cm} (12)

If we take the initial time $t_0 = 0$, then the Floquet mode $|\phi_{\alpha}(0)\rangle$ and the quasienergies $\epsilon_{\alpha}$ can be found by finding the eigenfunctions and eigenvalues of $U(T, 0)$.

To summarize, in order to describe the time-evolution of a system we use the following approach: (i). Find the one-period time-evolution operator $U(T, 0)$; (ii). Determine its eigenfunctions $|\phi_{\alpha}(0)\rangle$ and eigenvalues $e^{-i\epsilon_{\alpha} T}$ by solving the eigenvalue problem (12); (iii). Find the decomposition of $|\psi(0)\rangle$ in terms of Floquet modes: $c_{\alpha} = \langle \phi_{\alpha}(0)|\psi(0)\rangle$; (iv). Calculate the state of the system at time $t$ using the time-evolution operator $|\psi(t)\rangle = \sum_{\alpha} c_{\alpha} e^{-i\epsilon_{\alpha} T} U(t, 0)|\phi_{\alpha}(0)\rangle$. Therefore, the use of the Floquet theorem allows us to reduce the problem of solving a differential equation into the problem of finding the eigenvalues and eigenfunctions of a matrix.

3.1. Analytical results

As a first step, we solve the problem analytically using the approach described in the previous section. In general, the time-evolution operator $U(T, 0)$ can be obtained from the infinite series \[34\]

$$U(T, 0) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!}\left(-\frac{i}{\hbar}\right)^n \int_0^T d\tau_1 \int_0^{\tau_1} d\tau_2 \cdots \int_0^{T-n\tau} d\tau_n \hat{H}(\tau_1)\hat{H}(\tau_2)\cdots\hat{H}(\tau_n).$$  \hspace{1cm} (13)

Since the Hamiltonian (1) is time-dependent and its terms do not all commute with each other, the analytical expression of the time-evolution operator (13) for Hamiltonian (1) can only be found approximately. Using the Trotter–Suzuki formula \[35–37\], in the case of a time-dependent Hamiltonian \[38\], we write

$$U(T, 0) \approx U((N_t - 1)\tau + \tau, (N_t - 1)\tau) \cdots U(\tau + \tau, \tau)U(\tau, 0),$$  \hspace{1cm} (14)

where the dynamics of the system is decomposed in a discrete number of steps $N_t$, with $\tau = T/N_t$ being the infinitesimal time-step of the Trotter decomposition. Equation (14) becomes exact as $N_t \to \infty$. At each step, the time-evolution operator can be factorized in two parts

$$U(\tau, 0) = U_0(\tau, 0)U_1(\tau, 0),$$  \hspace{1cm} (15)

where $U_0(\tau, 0)$ and $U_1(\tau, 0)$ are the time-evolution operators corresponding to the Hamiltonian $\hat{H}_0$ and $\hat{H}_1$, respectively. Assuming the coupling to be on for half of the period $T$ with strength $g_0$, and ‘off’ otherwise, as in reference \[39\], equation (14) reduces to
operator. By noting that $U$agate the initial state by using the one-period time-evolution $\omega$ is the time-dependence of the probability of exciting the system to the $\{e, 1\}$ state, where different values of the frequency of switching of the qubit/resonator coupling $\omega_s$ are plotted on the $y$-axis. (b) The slice for the time-dependence of the probability at switching frequency $\omega_s = \omega_0 + \omega_r$. Figure 1(a) shows the time dependence of the probability of finding the system in the $\{e, 1\}$ state for different values of the switching frequency $\omega_s$, in the case of a system initially in the ground state $|\psi(0)\rangle = |g, 0\rangle$. The particular case where the switching frequency $\omega_s$ equals the sum of the qubit and resonator frequencies $\omega_0 + \omega_r$ is depicted in figure 1(b). When $\omega_s$ takes this value, figure 1(b) shows that the probability reaches its maximum. In this calculation, and throughout the rest of the paper, the following values where chosen for the parameters of the system: $\omega_0 = 5$ GHz, $\omega_r = 6$ GHz, $g_0 = 0.1$ GHz and $\omega_s$ varies in the range $[\omega_0 + \omega_r, \omega_0 + 2\omega_r]$. The Floquet approach can now be used to study the Trotterized dynamics of the system. As proven in the previous section, the eigenvalues and the eigenvectors of the one-period time-evolution operator $U(T, 0)$ give an exponential function of the Floquet quasienergies and the Floquet modes, respectively. Their full expressions are given by equations (A1) in appendix A. With these results, we can now find the wavefunction of the system at any time $t$ by following the procedure outlined in section 3. First, we decompose the initial state of the system $|\psi(0)\rangle$ in terms of Floquet modes at time $t = 0$, then we propagate the initial state by using the one-period time-evolution operator. By noting that $U(T, 0) \approx U(\tau, 0)^N$, as proved in reference [14], we can write the wavefunction as

$$|\psi(T)\rangle \approx \sum_{\alpha} c_\alpha(0)U(\tau, 0)^Nj_{\alpha}(\tau, 0)^{N_1}|\phi_\alpha(0)\rangle \left( e^{-\frac{g_0 T}{2}} \right)^{N_1} \left( e^{-i\omega_0 T} \right)^{N_1} \left( e^{-i\omega_1 T} \right)^{N_1} \left( e^{-i(\omega_0 + \omega_r) T} \right)^{N_1}. \quad (18)$$

The results of our calculations are presented in figure 1. Figure 1(a) shows the time dependence of the probability of finding the system in the $\{e, 1\}$ state for different values of the switching frequency $\omega_s$, in the case of a system initially in the ground state $|\psi(0)\rangle = |g, 0\rangle$. The particular case where the switching frequency $\omega_s$ equals the sum of the qubit and resonator frequencies $\omega_0 + \omega_r$ is depicted in figure 1(b). When $\omega_s$ takes this value, figure 1(b) shows that the probability reaches its maximum. In this calculation, and throughout the rest of the paper, the following values where chosen for the parameters of the system: $\omega_0 = 5$ GHz, $\omega_r = 6$ GHz, $g_0 = 0.1$ GHz and $\omega_s$ varies in the range $[\omega_0 + \omega_r, \omega_0 + 2\omega_r]$. The same procedure can be used for the numerical calculations. First, one finds the one-period time-evolution operator by numerically integrating the Schrödinger equation. In this way, up to numerical error, the true propagator $U(T, 0)$ can be obtained. Then, the eigenvalues of time-evolution of the system in the Floquet approach are also numerically calculated. The results obtained using the QuTip package [33, 40] are depicted in figure 2. A direct comparison of the two results can be seen in figure 3. Clearly, the analytical results converge
Figure 2. Numerical calculations of the time-dependence of the probability of finding the system in the $|e, 1\rangle$ state using the Floquet approach. (a) The contour-plot where different values of the frequency of switching of the qubit/resonator coupling $\pi s$ are plotted on the $y$-axis. (b) The slice for the switching frequency $\pi s = \omega_0 + \omega_r$.

Figure 3. Comparison of the time-dependence of the probability of exciting the $|e, 1\rangle$ state for the switching frequency $\pi s = \omega_0 + \omega_r$ calculated analytically and numerically for different number of Trotter steps $N_t$. The analytical solution starts converging to the numerical one as the number of Trotter steps increases from (a) $N_t = 200$ to (b) $N_t = 400$. (c) Excellent agreement is found for $N_t = 600$.

Figure 4. Dependence of the difference between the value of the quasienergies calculated analytically and numerically as a function of the number of Trotter steps $N_t$.

to the numerical results as the number of steps in the Trotter decomposition increases.

To show the dependence of the accuracy of the analytical solution compared to the numerical one, in figure 4 we plot the difference between the value of each of the quasienergies calculated analytically and numerically as a function of the number of Trotter steps $N_t$. As the number of Trotter steps increases, the difference between the analytical and the numerical value of each of the quasienergies $|\varepsilon^{(an)} - \varepsilon^{(num)}|$ goes to zero. Thus, as the number of Trotter steps increases the analytically calculated quasienergies converge to the value of the ones calculated numerically.

4. Comparison with other methods

The dynamics of the system can be studied by solving the time-dependent Schrödinger equation within the framework of other approaches as well. Let us show two of such approaches which make use of perturbation theory to arrive to an analytical expression for the solution of the Schrödinger equation. First, we will solve the set of linear differential equations for the system using a perturbative approach in the time-domain. This is the most widely adopted approach which was used to study the DLE in references [26, 27, 29]. Second, we will use another equivalent perturbative approach in the Laplace-domain. This can sometimes simplify the expression of time-dependent quantities, making it easier to describe the system’s dynamics. One should note that while both perturbative approaches allow to obtain the analytical expression for the solution of the Schrödinger equation, they can be
effectively used only when the interaction part of the Hamiltonian is small compared to the unperturbed Hamiltonian.

4.1. Perturbative approach in the time-domain

The dynamical behavior of the system can be found by solving the Schrödinger equation (6). For a system of \( N = 1 \) qubit and \( n = 0, 1 \) photons described by Hamiltonian (1) we can write the wavefunction

\[
|\psi(t)\rangle = \alpha_{g,0}(t) |g,0\rangle + \alpha_{e,0}(t) |e,0\rangle + \alpha_{e,1}(t) |e,1\rangle,
\]

(19)

where indices \( g \) and \( e \) correspond to ground and excited state of the qubit, 0, 1 counts the number of photons and \( \alpha(t) \) are the time-dependent amplitudes. The Schrödinger equation then gives a set of coupled differential equations

\[
\begin{align*}
\frac{d\alpha_{g,0}(t)}{dt} &= g(t)\alpha_{e,1}(t), \\
\frac{d\alpha_{e,1}(t)}{dt} &= \omega_1\alpha_{g,1}(t) + g(t)\alpha_{e,0}(t), \\
\frac{d\alpha_{g,0}(t)}{dt} &= \omega_0\alpha_{e,0}(t) + g(t)\alpha_{g,1}(t), \\
\frac{d\alpha_{e,1}(t)}{dt} &= \Omega_\| \alpha_{e,1}(t) + g(t)\alpha_{e,0}(t).
\end{align*}
\]

(20)

Here we have defined \( \Omega_\| \equiv \omega_0 + \omega_1 \) to simplify readability. One way to find a solution to the system of linear differential equations is to take a perturbative approach [24, 26, 27, 29]. Consider the following Hamiltonian

\[
\hat{H}(t) = \hat{H}_0 + \delta \hat{H}_1(t),
\]

(21)

where \( \hat{H}_0 \) is taken as the non-interacting Hamiltonian, \( \hat{H}_1 \) is taken as the perturbation and \( \delta \) is a dimensionless parameter between 0 and 1. If the coupling strength \( \delta \) between the qubit and the resonator is small compared to the spacing of the energy levels of the unperturbed Hamiltonian \( \hat{H}_0 \), then the wavefunction can be expanded in powers of \( \delta \)

\[
|\psi(t)\rangle = |\psi(0)\rangle + \delta |\psi(1)\rangle + \delta^2 |\psi(2)\rangle + \cdots.
\]

(22)

As a result one obtains the following differential equations for any other order \( (j) \) of the perturbation

\[
\frac{d|\psi(t)\rangle^{(j)}}{dt} = \hat{H}_0|\psi(t)\rangle^{(j)} + \hat{H}_1(t)|\psi(t)\rangle^{(j-1)},
\]

(23)

One can then solve the Schrödinger equation order by order. For the case considered here, an analytical perturbative solution to the Schrödinger equation up to second order in the perturbation \( \delta \) is found. The calculation of the values of the coefficients of the wavefunction (19) is presented in appendix

B. The resulting wavefunction reads

\[
|\psi(t)\rangle = |g,0\rangle^{(0)} - \delta |g,0\rangle e^{-i\Omega_\| t} \times \left\{ \Omega_\| + e^{i\Omega_\| t} \left[ i\varepsilon_\| \sin(\varepsilon_\| t) - \Omega_\| \cos(\varepsilon_\| t) \right] \right\} \times \left[ \varepsilon_\| - \Omega_\| \right] \\
\times |e,1\rangle^{(1)} + (+\delta g_0)^2 \\
\times \left\{ i\varepsilon_\| \left[ -1 + 2i\Omega_\| t + \cos(2\varepsilon_\| t) - \Omega_\| \sin(2\varepsilon_\| t) \right] \right\} \times \frac{4\varepsilon_\|}{4\varepsilon_\| + \Omega_\|} \left[ \varepsilon_\| - \Omega_\| \right] \\
\times |g,0\rangle^{(2)}.
\]

(24)

Using the expression above, one can study the dynamics of the system. In figures 5(a) and (b), the time-evolution of the probability of finding the system in the state \( |e,1\rangle \) using the perturbative solution (24) is presented along with the results obtained from numerical integration of the Schrödinger equation shown in figures 5(c) and (d).

A direct comparison between the time-evolution of the probability that the system is in the \( |e,1\rangle \) state obtained using the perturbative solution (24) is presented along with the results obtained by direct numerical integration of the Schrödinger equation for a system initially in the ground state is shown in figure 6.

4.2. Perturbative approach in the Laplace-domain

The perturbative approach in the Laplace-domain presents an alternative approach that can be used when time-dependent parameters make the direct solution of the Schrödinger equation too difficult. Similarly to the Fourier transform, the Laplace transform can be used to reduce the problem at hand into a different problem, which is sometimes easier to solve. In fact, a set of linear differential equations in the time-domain can be turned into a set of algebraic equations in the Laplace-domain. The price to pay is the cost of the transformation from the time-domain to the Laplace-domain and back. The Laplace transform of a function \( f(t) \) is defined as

\[
F(s) = \mathcal{L}\{f(t)\}(s) = \int_0^\infty dt f(t)e^{-st},
\]

(25)

where \( s \) is a complex number. Thus the Laplace transform requires us to compute an integral. On the other hand, the possibility of using Cauchy’s residue theorem reduces the calculation of the inverse Laplace transform to the calculation of residues

\[
\mathcal{L}^{-1}\{F(s)\}(t) = \frac{1}{2\pi i} \int_{b-i\infty}^{b+i\infty} ds F(s)e^{st},
\]

(26)

where \( b \) is a point on the real axis on the right of the rightmost pole of \( F(s) \). To find the dynamics of the system considered above within this approach, one starts from the Hamiltonian (21) and the perturbative expansion of the wavefunction (22). Then applying the Laplace transform to the set of linear differential equations (B1), (B2) and (B4) given in
Figure 5. The time-dependence of the probability of finding the system in the $|e, 1\rangle$ state is determined using a perturbative approach and by direct numerical integration of Schrödinger’s equation. The results in (a) and (b) are obtained by solving the Schrödinger equation within a perturbative approach. (c) and (d) Show the results obtained by direct numerical integration of Schrödinger’s equation. (a) and (c) Show a two-dimensional plot for different values of the frequency of switching of the qubit/resonator coupling $\omega_s$. (b) and (d) Show a one-dimensional slice for $\omega_s = \Omega +$.

Figure 6. Comparison of the solution of the Schrödinger equation within a perturbative approach (dashed curve) with the numerical integration of this equation (solid curve).

appendix B, one can solve the algebraic system of equation order by order. A solution in the time-domain is then obtained by taking the inverse Laplace transform. Using the results obtained in appendix C, one can write the approximate analytical solution of the Schrödinger equation within the Laplace approach as

$$|\psi(t)\rangle = |g, 0\rangle^{(0)} - \frac{\delta g_0}{2} \times \left[ \frac{e^{i\omega_s t}}{\omega_s - \Omega} - \frac{e^{i\omega_s t}}{\omega_s + \Omega} - \frac{2 e^{-i\Omega t} \Omega}{\omega_s^2 - \Omega^2} \right] |e, 1\rangle^{(1)} +$$

$$+ \frac{1}{16} (\delta g_0)^2 \left( \frac{1}{\omega_s (\omega_s - \Omega)} + \frac{e^{i\omega_s t}}{\omega_s (\omega_s + \Omega)} \right) |g, 0\rangle^{(2)}.$$  \hspace{1cm} (27)

Results of the calculations for the time-dependence of the probability of finding the system in the $|e, 1\rangle$ state found in the framework of the Laplace approach are presented in figure 7.

5. Results and discussion

We presented three approaches for the description of the dynamics of a periodically driven quantum system of $N$ qubits coupled to a harmonic oscillator. A final comparison of all methods used to compute the dynamics of the $|e, 1\rangle$ state of the system is shown in figure 8. The comparison of the results shows that the analytical perturbative solutions of the
Figure 7. Results for the time-dependence of the probability of finding the system in the $|e, 1\rangle$ state found using the Laplace approach. The analytical perturbative solution in equation (27) is used to make the plots. (a) Is a two-dimensional plot where different frequencies of switching of the coupling $\omega_s$ are considered and (b) is a one-dimensional slice for $\omega_s = \Omega_\omega$.

Figure 8. Comparison of the results obtained using all methods presented in the paper.

Schrödinger equation obtained in the time-domain and the Laplace-domain coincide with only a negligibly small difference that does not change with the time. The agreement between these two perturbative approaches should not be surprising. In sections 4.1 and 4.2, perturbation theory was used in the time-domain and the Laplace-domain, respectively, and the infinite series was truncated at the second order in terms of the perturbation parameter. The Laplace method takes as a starting point the set of differential equations that can be obtained with the Schrödinger method using perturbation theory and transforms it to the Laplace-domain. In the Laplace-domain this set of differential equations becomes a set of algebraic equations. The Laplace transform is calculated analytically, no approximations are made in this step. After solving the system of algebraic equations in the Laplace domain, the solution is transformed back to the time-domain by analytically calculating its inverse Laplace transform without making any approximations. As no further approximations were made within the Laplace method, other than the perturbative expansion, the two solutions should coincide. Both analytical perturbative solutions tend to be accurate only at small time and then start to diverge from the direct numerical solution of the Schrödinger equation. While there is disagreement with the direct numerical solution of the Schrödinger equation with the analytical perturbative solutions, nonetheless, all qualitative features of the numerically solved solutions are displayed in both analytical perturbative solutions as well.

The analytical and numerical solutions obtained within the framework of the Floquet approach are in excellent agreement with direct numerical solution of the Schrödinger equation. Even though we have approximated the time-evolution in a discrete number of steps $N_t$, as $N_t \to \infty$ the analytical solution converges to the numerical one. Thus, we demonstrate the advantage of the Floquet approach adopted here, compared to the more standard perturbative approaches seen in the literature, see, for example, references [24, 26, 27, 29], for the study of the DLE.

6. Conclusions

In conclusion, we have used the Floquet approach to give an outline of an analytical solution for the dynamics of a system of $N$ qubits coupled to a harmonic oscillator with a periodically varying coupling. The case of a single qubit coupled to a resonator populated with $n = 0, 1$ photons was explicitly treated. The analytical solution of the Floquet problem found in this way closely resembles the solution that can be found numerically at all times.

We have also studied the same problem adopting different analytical and numerical approaches. Whenever the Hamiltonian contains terms that can be considered as a perturbation, that is their effect is small compared to other terms in the Hamiltonian, analytical expressions for the solution of the Schrödinger equation can be found using a perturbative approach. We presented the analytical perturbative solutions of the Schrödinger equation up to second order in the perturbation within the time-domain and the Laplace-domain. Since the Laplace transform was calculated analytically, the results coincide with the ones obtained by perturbatively
solving Schrödinger’s equation in the time-domain. However, the Laplace transform can be used to solve the Schrödinger equation when particular time-dependence in the parameters of the system make a perturbative approach in the time-domain difficult. The analytical perturbative solutions show similar qualitative features as the direct numerical integration of the Schrödinger equation and tend to be accurate only at small time. The discrepancy between the analytical perturbative solutions and the direct numerical integration of the Schrödinger equation increases with the increasing of time.

Because the problem considered is explicitly a time-periodic problem, the Floquet approach will work best as it provides a natural framework for periodic problems. Perturbative approaches provide an explicit analytical solution of the time-dependent Schrödinger equation, in principle, at any order of the perturbative parameter and can be used for periodic and aperiodic systems. Overall, we showed that different analytical and numerical methods can be used to study the time-evolution and dynamics of a qubit/oscillator system with periodically varying coupling. The perturbative methods investigated give a good description of the dynamics of the system at short time, but a non-negligible discrepancy with the numerical methods arises at long times. Nonetheless, the time- or Laplace-domain perturbative approaches can be used successfully, under the assumption of a weak interaction Hamiltonian, as well as in the presence of aperiodic time-dependent terms in the Hamiltonian. On the other hand, the Floquet approach provides a framework where analytical and numerical methods give comparable results despite the approximations needed to obtain an analytical solution of the problem and can be applied for any strength of the interaction.

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Appendix A.

The quasienergies and the Floquet modes of the Hamiltonian (1) for the case of a single qubit coupled to a resonator with \(n = 0\), 1 photons can be found by calculating the eigenvalues and eigenvectors of \(U(\tau, 0)\), the one-step time-evolution operator (17). We get the following eigenvalues

\[
e^{-\frac{i\omega_1}{2}\tau} = \frac{1}{2} \cos \left(\frac{\omega_1}{2}\tau\right) e^{-i\Omega_1\tau},
\]

\[
e^{-\frac{i\omega_2}{2}\tau} = \frac{1}{2} \cos \left(\frac{\omega_2}{2}\tau\right) e^{-i\Omega_2\tau}.
\]

\[
e^{-\frac{i\omega_3}{2}\tau} = \frac{1}{2} \cos \left(\frac{\omega_3}{2}\tau\right) e^{-i\Omega_3\tau},
\]

\[
e^{-\frac{i\omega_4}{2}\tau} = \frac{1}{2} \cos \left(\frac{\omega_4}{2}\tau\right) e^{-i\Omega_4\tau}.
\]

\[
\left[ e^{i\Omega_1\tau} + 1 - i\sqrt{\left(e^{i\Omega_1\tau} + 1\right)^2 + 4 \sec^2\left(\frac{\omega_1}{2}\right) e^{i\Omega_1\tau}} \right],
\]

\[
\left[ e^{i\Omega_2\tau} + 1 - i\sqrt{\left(e^{i\Omega_2\tau} + 1\right)^2 + 4 \sec^2\left(\frac{\omega_2}{2}\right) e^{i\Omega_2\tau}} \right],
\]

\[
\left[ e^{i\Omega_3\tau} + 1 - i\sqrt{\left(e^{i\Omega_3\tau} + 1\right)^2 + 4 \sec^2\left(\frac{\omega_3}{2}\right) e^{i\Omega_3\tau}} \right],
\]

\[
\left[ e^{i\Omega_4\tau} + 1 - i\sqrt{\left(e^{i\Omega_4\tau} + 1\right)^2 + 4 \sec^2\left(\frac{\omega_4}{2}\right) e^{i\Omega_4\tau}} \right].
\]

While for the eigenvectors we get

\[
|\phi_1(0)\rangle = \left\{ \frac{1}{2} i \cot \left(\frac{\omega_1}{2}\right) \left[ 1 - e^{-i\Omega_1\tau} + e^{-i\Omega_1\tau} \right] \right\} , 0, 0, 1, 0, \right\},
\]

\[
|\phi_2(0)\rangle = \left\{ \frac{1}{2} i \cot \left(\frac{\omega_2}{2}\right) \left[ 1 - e^{-i\Omega_2\tau} + e^{-i\Omega_2\tau} \right] \right\} , 0, 0, 1, 0, \right\},
\]

\[
|\phi_3(0)\rangle = \left\{ \frac{1}{2} i \cot \left(\frac{\omega_3}{2}\right) \left[ 1 - e^{-i\Omega_3\tau} - e^{-i\Omega_3\tau} \right] \right\} , 0, 0, 1, 0, \right\},
\]

\[
|\phi_4(0)\rangle = \left\{ \frac{1}{2} i \cot \left(\frac{\omega_4}{2}\right) \left[ 1 - e^{-i\Omega_4\tau} + e^{-i\Omega_4\tau} \right] \right\} , 0, 0, 1, 0, \right\}.
\]

Appendix B.

The perturbative approach allows one to rewrite the system of coupled differential equation (20) at each order of \(\delta\). For the case of a system with the wavefunction (19), at the zeroth order in terms of \(\delta\), equation (23) gives

\[
\frac{d\alpha_{0,0}(t)}{dt} = 0,
\]

\[
\frac{d\alpha_{1,0}(t)}{dt} = i\hbar \kappa \alpha_{0,1}(t),
\]

\[
\frac{d\alpha_{0,1}(t)}{dt} = i\hbar \kappa \alpha_{0,0}(t),
\]

\[
\frac{d\alpha_{0,0}(t)}{dt} = \Omega_{+} \alpha_{0,1}(t).
\]

Given that the system is initially in the ground state \(|\psi(0)\rangle = |\text{g}, 0\rangle\), thus the only non-zero coefficient is \(\alpha_{0,0}(0) = 1\), one finds that \(\alpha_{0,0}(t) = 1\) and \(\alpha_{0,1}(t) = \alpha_{0,0}(t) = \alpha_{0,0}(0) = 0\). At first order in terms of \(\delta\) the one finds

\[
\frac{d\alpha_{1,0}(t)}{dt} = g(t) \alpha_{0,1}(t),
\]

\[
\frac{d\alpha_{1,1}(t)}{dt} = i\hbar \kappa \alpha_{1,0}(t) + g(t) \alpha_{0,0}(t),
\]

\[
\frac{d\alpha_{1,1}(t)}{dt} = \omega_{0} \alpha_{1,0}(t) + g(t) \alpha_{0,0}(t).
The Laplace approach can simplify the solution of time-dependent equations. Before taking the Laplace transform, one can solve for the first order coefficients. The only non-zero coefficients at first order is
\[
\alpha_{e,1}(t) = -g_0 e^{-i\Omega_+ t} \left\{ \Omega_+ + e^{i\Omega_+ t} i\omega_s \sin \left( \omega_s t - \Omega_+ \cos \left( \omega_s t \right) \right) \right\},
\]
(B2)

At second order in terms of $\delta$, we have
\[
i \frac{d\alpha_{e,2}(t)}{dt} = g(t)\alpha_{e,1}(t),
\]
\[
\frac{d\alpha_{g,0}(t)}{dt} = \omega_0\alpha_{e,1}(t),
\]
\[
\frac{d\alpha_{g,1}(t)}{dt} = \omega_0\alpha_{e,2}(t) + g(t)\alpha_{g,0}(t),
\]
\[
\frac{d\alpha_{g,2}(t)}{dt} = \omega_0\alpha_{g,1}(t) + g(t)\alpha_{g,2}(t),
\]
\[
\frac{d\alpha_{g,3}(t)}{dt} = \Omega_+ \alpha_{g,2}(t) + g(t)\alpha_{g,3}(t).
\]
(B4)

Substituting the value for the first order coefficients $\alpha^{(1)}(t)$, one can find the second order coefficients. The only non-zero coefficient is the following
\[
\alpha_{g,0}^{(2)}(t) = g_0 \left\{ i\omega_s \left[ 2\Omega_+ t + \cos \left( 2\omega_s t \right) - 1 \right] + \Omega_+ \sin \left( 2\omega_s t \right) \right\}.
\]
(B5)

**Appendix C.**

The Laplace approach can simplify the solution of time-dependent differential equations. In this case, we consider the perturbative dynamics described by equations (B1), (B2) and (B4) and use the Laplace approach to find a solution. At zeroth order in $\delta$ we have
\[
is A^{(0)}_{g,0}(s) = 1,
\]
\[
is A^{(0)}_{g,1}(s) = \omega_0 A^{(0)}_{g,0}(s),
\]
\[
is A^{(0)}_{g,2}(s) = \omega_0 A^{(0)}_{g,1}(s),
\]
\[
is A^{(0)}_{g,3}(s) = \Omega_+ A^{(0)}_{g,2}(s),
\]
(C1)

where $A(s) = \mathcal{L}[a(t)](s)$ is the Laplace transform of the time-dependent coefficients of the wavefunction. Considering a system initially in the ground state $|\psi(0)\rangle = |g, 0\rangle$, the only non-zero coefficient is $A^{(0)}_{g,0}(s) = 1/s$. Which gives $\alpha_{g,0}^{(0)}(0) = 1$. For simplicity we substitute the values for the zeroth order coefficients before taking the Laplace transform. At first order in terms of $\delta$, one finds
\[
is A^{(1)}_{g,0}(s) = 0,
\]
\[
is A^{(1)}_{g,1}(s) = \omega_0 A^{(1)}_{g,0}(s),
\]
\[
is A^{(1)}_{g,2}(s) = \omega_0 A^{(1)}_{g,1}(s),
\]
\[
is A^{(1)}_{g,3}(s) = \Omega_+ A^{(1)}_{g,2}(s),
\]
\[
is A^{(1)}_{e,0}(s) = \omega_0 A^{(1)}_{e,0}(s),
\]
\[
is A^{(1)}_{e,1}(s) = \omega_0 A^{(1)}_{e,1}(s),
\]
\[
is A^{(1)}_{e,2}(s) = \omega_0 A^{(1)}_{e,2}(s),
\]
\[
is A^{(1)}_{e,3}(s) = \Omega_+ A^{(1)}_{e,2}(s),
\]
\[
is A^{(1)}_{e,4}(s) = \Omega_+ A^{(1)}_{e,3}(s),
\]
(C2)

where $G(s) = \frac{g_0}{s^2 + \omega_0^2}$ is the Laplace transform of the time-dependent coupling (5). The only non-zero coefficients at first order is
\[
A_{e,1}(s) = -g_0 \left[ \frac{s}{s^2 + \omega_s^2} - \frac{1}{s - \Omega_+} \right],
\]
(C3)

or in the time-domain
\[
\alpha_{e,1}(t) = -\frac{g_0}{2} \left[ \frac{e^{i\omega_s t} - e^{-i\omega_s t}}{s - \Omega_+} - \frac{2 e^{-i\Omega_+ t} - e^{i\Omega_+ t}}{s^2 + \omega_s^2} \right].
\]
(C4)

At second order in terms of $\delta$, we have
\[
is A^{(2)}_{g,0}(s) = \mathcal{L}\left[ g(t)\alpha_{e,1}(t) \right](s),
\]
\[
is A^{(2)}_{g,1}(s) = \omega_0 A^{(2)}_{g,0}(s),
\]
\[
is A^{(2)}_{g,2}(s) = \omega_0 A^{(2)}_{g,1}(s),
\]
\[
is A^{(2)}_{g,3}(s) = \Omega_+ A^{(2)}_{g,2}(s).
\]
(C5)

The only non-zero coefficient is the following
\[
A_{g,2}(s) = \frac{i g_0 \left( \omega_0^2 + \Omega_+^2 \right) \left( s^2 + 4\omega_s^2 \right) \Omega_+ (s - i(\omega_s - \Omega_+))(s + i(\omega_s + \Omega_+))}{2 \left( s^2 + 4\omega_s^2 \right) \Omega_+ (s - i(\omega_s - \Omega_+))(s + i(\omega_s + \Omega_+))},
\]
which in the time-domain gives
\[
\alpha_{g,2}(t) = \frac{1}{16} g_0 \left\{ \frac{i\omega_s \left( 4 - 4i\Omega_+ t + 2i\Omega_+^2 \Omega_+ (2\Omega_+ t + 7i) \right)}{\left( \omega_s - \Omega_+ \right)^2 \left( \omega_s + \Omega_+ \right)^2} + \frac{e^{-2i\omega_s t}}{\omega_s (\omega_s - \Omega_+) + \omega_s (\omega_s + \Omega_+)} \right\}.
\]
(C7)

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