On numbers not representable as $n + w(n)$

Petr Kucheriaviy
National Research University Higher School of Economics, Moscow, Russia

November 29, 2022

Abstract

Let $w(n)$ be an additive non-negative integer-valued arithmetic function which is equal to 1 on primes. We study the distribution of $n + w(n) \pmod{p}$ and give a lower bound for the density of the set of numbers which are not representable as $n + w(n)$.

1 Introduction

Let $w(n)$ be an additive non-negative integer-valued arithmetic function which is equal to 1 on primes. For example one can set $w(n)$ to be $\omega(n)$ – the number of distinct prime factors of $n$ or $\Omega(n)$ – the number of prime factors with multiplicity. By $E$ we denote the set of natural numbers not representable as $n + w(n)$. For the complement of $E$ we use the notation $E^c$. In this article we are concerned with the lower bound for

$$\Xi(N) = |[1, N] \cap E|.$$ 

Conjecture.

$$\Xi(N) \gg N.$$ 

We show that

Theorem 1.

$$\Xi(N) \gg \frac{N}{\log \log N}.$$ 

The main idea is to consider $n + w(n)$ modulo a prime number. For a prime number $p$ and $r \in \mathbb{Z}/p\mathbb{Z}$ let us denote

$$a(r) := \sum_{\substack{n \leq N \atop n+w(n) \equiv r \pmod{p}}} 1.$$ 

The author was supported by the Basic Research Program of the National Research University Higher School of Economics.
Lemma 1. Let $R$ be a subset of $\mathbb{Z}/p\mathbb{Z}$. Then

$$\Xi(N) \geq \sum_{r \in R} (N/p - a(r) - 1).$$

Proof. Let

$$N_r := \{n \mid n \leq N, n \equiv r \pmod{p}\}.$$ 

Note that

$$[1, N] \cap \mathbb{Z} = \bigsqcup_{r \in \mathbb{Z}/p\mathbb{Z}} N_r, \quad |N_r| \geq N/p - 1, \quad |N_r \cap E^c| \leq a(r).$$

Hence

$$|N_r \cap E| = |N_r| - |N_r \cap E^c| \geq (N/p - a(r) - 1).$$

Finally,

$$\Xi(N) = \sum_{r \in \mathbb{Z}/p\mathbb{Z}} |N_r \cap E| \geq \sum_{r \in R} |N_r \cap E| \geq \sum_{r \in R} (N/p - a(r) - 1).$$

In [3] M. E. Changa studied the distribution of $w(n)$ in a residue ring. Using the same method we study the asymptotic behavior of $a(r)$.

It turns out that for any fixed $p$

$$a(r) \sim N/p.$$ 

That is the reason we can’t prove the Conjecture directly using the observation of Lemma 1. Nevertheless, we are able to find the second term in the asymptotic expansion of $a(r)$. Then Theorem II follows by an appropriate choice of $p$ and $R$.

Let

$$G(N) := |\{(n, m) : n + w(n) = m + w(m), n \leq N, m \leq N, n \neq m\}|.$$

Proposition 1. If $w(n) \ll n^\varepsilon$ and $0 < \varepsilon < 1$, then

$$G(N) \gg \Xi(N).$$

Proof. We have $w(n) \leq Cn^\varepsilon$ for some constant $C$.

Let us denote

$$g(n) := |\{m : m + w(m) = n, m \leq N\}|.$$

Note that if $n \leq N - CN^\varepsilon$, then $n + w(n) \leq N$. Hence

$$\sum_{n \leq N} g(n) = \sum_{n \geq 1} 1 \geq N - CN^\varepsilon.$$

We have

$$G(N) \geq \sum_{n \leq N} \binom{g(n)}{2} = \sum_{n \in [1, N] \cap E^c} \binom{g(n)}{2} \geq \sum_{n \in [1, N] \cap E^c} (g(n) - 1) \geq (N - CN^\varepsilon) - |[1, N] \cap E^c| = \Xi(N) - CN^\varepsilon \gg \Xi(N).$$
In [1], P. Erdős, A. Sárközy, C. Pomerance showed that if \( w(n) \) is the number of distinct prime factors of \( n \), then
\[
G(N) \geq N \exp(-4000 \log \log N \log \log \log N)
\]
for \( N \) large enough.

So Theorem \( \mathbf{1} \) in view of Proposition \( \mathbf{1} \) improves this bound and gives
\[
G(N) \gg N \frac{\log \log N}{\log \log \log \log N}.
\]

2 Lemmas

Lemma 2 (Perron’s formula). Let
\[
F(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s},
\]
where the series converges absolutely for \( \text{Re} \ s > a \geq 0 \), and let
\[
\sum_{n=1}^{\infty} \frac{|a_n|}{n^\sigma} \ll (\sigma - a)^{-\alpha}
\]
as \( \sigma \to a + 0 \) for some \( \alpha > 0 \). Then for every \( b > a \), \( x \geq 2 \) and \( T \geq 2 \) we have
\[
\sum_{n \leq x} a(n) = \frac{1}{2\pi i} \int_{b-iT}^{b+iT} F(s) \frac{x^s}{s} ds + R(x),
\]
where
\[
R(x) \ll \frac{x^b}{T(b-a)^\alpha} + 2^b \left( \frac{x \log x}{T} + \log \frac{T}{b} + 1 \right) \max_{x/2 \leq n \leq 3x/2} |a(n)|.
\]

Proof. See, for example, [2, Theorem 7, p. 20] or [1, Theorem 1, p. 64].

Lemma 3 (M. E. Changa). Let \( f(n) \) be a complex valued arithmetic function such that \( |f(n)| \leq 1 \), let the following equality hold for \( \text{Re} \ s > 1 \):
\[
\sum_{n=1}^{\infty} \frac{f(n)}{n^s} = (\zeta(s))^\alpha H(s),
\]
where \( \alpha \in \mathbb{C} \), \( |\alpha| \leq 1 \), and the function \( H(s) \), \( s = \sigma + it \), is analytic in the domain \( \sigma > 1 - \frac{c_1}{\log T} \), \( |t| \leq T \) for every sufficiently large \( T \), and let the bound \( H(s) \ll \log^{c_2} T \) hold in this domain. Then
\[
\sum_{n \leq x} f(n) = \frac{H(1)}{\Gamma(\alpha)} x \log^{\alpha-1} x + O(x \log^{\Re \alpha - 2} x).
\]

Here constant in \( O \) depends only on \( c_1, c_2, C \).

Proof. See [3, Lemma 3.1].
Lemma 4. Let $T \geq 3$. If $\chi$ is a complex character modulo $k$ and $s = \sigma + it$, then $L(s, \chi)$ has no zeroes in the domain

$$\text{Re } s = \sigma \geq 1 - \frac{c_3}{\log kT}, \ |t| \leq T.$$ 

If $\chi$ is a real character modulo $k$ and $s = \sigma + it$, then $L(s, \chi)$ has no zeroes in the domain

$$\text{Re } s = \sigma \geq 1 - \frac{c_3}{\log kT}, \ 0 < |t| \leq T.$$ 

Proof. See [1] Theorem 2, p. 124]

Lemma 5. Let $3 \leq X$. There exists at most one $3 \leq k \leq X$ and at most one real primitive character $\chi_1$ modulo $k$ for which $L(s, \chi_1)$ has a simple real zero $\beta_1$ such that

$$\beta_1 \geq 1 - \frac{c_4}{\log X}.$$ 

Proof. See [1] Corollary 2, p. 131].

We call a prime number $p < X$ unexceptional if $L(s, \chi)$ has no real zeroes on the interval

$$\left[ 1 - \frac{\min(c_3, c_4)}{\log X}, +\infty \right)$$

for every character $\chi$ modulo $p$. Otherwise we call $p$ exceptional. Note that the notion of being exceptional depends on $X$. Lemma 5 implies that there exist at least one exceptional $p < X$.

Lemma 6. For every sufficiently large $X$, the interval

$$[X - X^{4/5}, X]$$

contains at least two prime numbers.

Proof. See [5] or [1] Theorem 2, p. 98].

Lemma 7 (Borel–Carathéodory theorem). Let $R > 0$, and let the function $f(s)$ be analytic in the circle $|s - s_0| \leq R$. Moreover assume that $\text{Re } f(s) \leq M$ on the circle $|s - s_0| = R$. Then in the disk $|s - s_0| \leq r < R$ we have

$$|f(s) - f(s_0)| \leq 2(M - \text{Re } f(s_0)) \frac{r}{R - r}.$$ 

Proof. See, for example, [1] Lemma 4, p. 35].

Lemma 8. Let $\chi$ be a non-principal character modulo $k$. Then the following bound holds in the domain $|\sigma - 1| \leq c/\log kT, |t| \leq T$:

$$L(s, \chi) = O(\log kT).$$
Proof. Let us denote \( S(t) = \sum_{n \leq t} \chi(n) \). We have
\[
\sum_{n=N+1}^{M} \frac{\chi(n)}{n^s} = \int_{N}^{M} \frac{dS(t)}{t^s} = \frac{S(M)}{M^s} - \frac{S(N)}{N^s} + s \int_{N}^{M} \frac{S(t) \, dt}{t^{s+1}}.
\]
As \( M \) approaches infinity, we find that
\[
L(s, \chi) = \sum_{n=1}^{N} \frac{\chi(n)}{n^s} - \frac{S(N)}{N^s} + s \int_{N}^{\infty} \frac{S(t) \, dt}{t^{s+1}}.
\]
Hence
\[
L(s, \chi) - \sum_{n=1}^{N} \frac{\chi(n)}{n^s} = O(kN^{-\sigma}) + O(kT \sigma^{-1} N^{-\sigma}).
\]
If \( n \leq kT \), then
\[
|n^{-s}| = n^{-\sigma} \leq n^{-1} (kT)^{c/\log kT} \ll n^{-1}.
\]
If we take \( N = kT \), we obtain
\[
L(s, \chi) = O(\log kT).
\]

Lemma 9. Let \( T \geq 3 \), let \( \chi \) be a non-principal character modulo \( k \), and let \( L(s, \chi) \) be nonzero in the domain
\[
\sigma \geq 1 - \frac{3c}{\log kT}, \ |t| \leq 2T;
\]
then on the boundary of the domain
\[
\sigma \geq 1 - \frac{c}{\log kT}, \ |t| \leq T
\]
we have \( \log L(s, \chi) \ll \log \log kT \).

Proof. If \( \sigma \geq 1 + c/\log kT \), then
\[
|\log L(s, \chi)| = \left| \sum_{p} \sum_{m=1}^{\infty} \frac{\chi_{m}(p)}{mp^{ms}} \right| \leq \log \zeta(\sigma) \ll \log \frac{1}{\sigma - 1} \ll \log \log kT.
\]
Using Lemma 8, we obtain
\[
\text{Re} \log L(s, \chi) = \log |L(s, \chi)| \leq C'' \log \log kT.
\]
for \(|\sigma - 1| \leq 3c/\log kT, \ |t| \leq 2T\).

Thus, setting \( s_0 = 1 + c/\log kT + it, \ R = 3c/\log kT, \ r = 2c/\log kT \) in Lemma 7, we find that
\[
|\log L(s, \chi) - \log L(s_0, \chi)| \leq 2(C'' \log \log kT - O(\log \log kT)) \ll \log kT.
\]
for \(|s - s_0| = r\). \qed
Lemma 10. Let \( \chi \) be a non-principal character modulo \( k \) and \( \log k = o(\sqrt{\log x}) \). Assume that \( L(s, \chi) \) has no zeroes in the domain \( \sigma \geq 1 - \frac{3c}{\log kT}, |t| \leq 2T \) for \( T > C \). Let \( \alpha \in \mathbb{C}, |\alpha| \leq 1 \). Let \( f(n) \) be a complex valued arithmetic function, such that \( |f(n)| \leq 1 \) and let

\[
\sum_{n=1}^{\infty} \frac{f(n)}{n^s} = (L(s, \chi))^\alpha H(s),
\]

for \( \text{Re} s > 1 \). Let \( H(s), s = \sigma + it \) be analytic in the domain \( \sigma > 1 - \frac{3c}{\log kT}, |t| \leq T \) for each \( T > C \). Moreover, let \( H(s) \ll \log^{55} T \) in this domain.

Then

\[
\sum_{n \leq x} f(n) \ll x \exp\left(-c_6 \sqrt{\log x}\right).
\]

Here the constant in Vinogradov symbol depends only on \( c, c_5, C \) and does not depend on \( \chi \).

Proof. Substituting \( b = 1 + 1/\log x, \log T = \sqrt{\log x} \) into Perron’s formula (Lemma 2), we obtain

\[
\sum_{n \leq x} f(n) = \frac{1}{2\pi i} \int_{b-iT}^{b+iT} (L(s, \chi))^\alpha H(s) \frac{x^s}{s} ds + R(x),
\]

where

\[
R(x) \ll x \exp(-c_7 \sqrt{\log x}).
\]

Let \( \beta := 1 - c/\log kT \) and consider the contour \( \Gamma \) consisting of segments joining in succession the points \( \beta + iT, \beta - iT, b - iT, b + iT, \beta + iT \).

Note that \( (L(s, \chi))^\alpha H(s) \frac{x^s}{s} \) is well-defined and holomorphic in the interior and on the sides of \( \Gamma \).

On the sides of \( \Gamma \) we have

\[
|(L(s, \chi))^\alpha| = |e^{\alpha \log L(s, \chi)}| \leq e^{\log L(s, \chi)} \ll \log^{55} kT,
\]

here we used Lemma 9.

By residue theorem, we have

\[
\int_{\Gamma} (L(s, \chi))^\alpha H(s) \frac{x^s}{s} ds = 0.
\]

Estimating the integral trivially on the sides of \( \Gamma \), we obtain

\[
\int_{b-iT}^{b+iT} (L(s, \chi))^\alpha H(s) \frac{x^s}{s} ds \ll x \exp\left(-c_6 \sqrt{\log x}\right).
\]

This concludes the proof.

Let us set \( e(z) := \exp(2\pi i z) \). By \( \mathbb{P} \) we denote the set of prime numbers.

Lemma 11. Let \( u(n) \) be a completely multiplicative function such that \( |u(n)| \leq 1 \) for all \( n \). Set

\[
L(s, u) := \sum_{n \geq 1} \frac{u(n)}{n^s}.
\]
Then for \( \text{Re} \, s > 1 \) we have

\[
\sum_{n \geq 1} e \left( \frac{w(n)t}{p} \right) \frac{u(n)}{n^s} = (L(s, u))^{e(t/p)} F_{t,p,u}(s).
\]

Here \( F_{t,p,u}(s) \) is holomorphic on \( \text{Re}(s) > 1/2 \). Moreover, for every \( \epsilon > 0 \), there are \( 0 < b_{1,\epsilon} < b_{2,\epsilon} \), such that in the half-plane \( \text{Re}(s) > 1/2 + \epsilon \) we have

\[
b_{1,\epsilon} < |F_{t,p,u}(s)| < b_{2,\epsilon}.
\]

And \( b_{1,\epsilon} \) and \( b_{2,\epsilon} \) do not depend on \( p, t, \) and \( u \).

In particular there exist constants \( 0 < b_1 < b_2 \) such that

\[
b_1 \leq |F_{t,p,u}(1)| \leq b_2
\]

for every \( p, t \) and \( u \).

**Proof.** Set

\[
F_{t,p,u}(s) = (L(s, u))^{-e(t/p)} \sum_{n \geq 1} e \left( \frac{w(n)t}{p} \right) \frac{u(n)}{n^s}.
\]

By assumption, \( w(n) \) is an additive function and \( w(q) = 1 \) for \( q \in \mathbb{P} \). Thus for \( \text{Re} \, s > 1 \), we have

\[
\sum_{n \geq 1} e \left( \frac{w(n)t}{p} \right) \frac{u(n)}{n^s} = \prod_{q \in \mathbb{P}} \left( 1 + e(t/p)u(q) + \frac{e \left( \frac{w(q^2)t}{p} \right) u(q)^2}{q^{2s}} + \ldots \right)
\]

and

\[
L(s, u) = \prod_{q \in \mathbb{P}} (1 - u(q)q^{-s})^{-1}.
\]

Therefore,

\[
\log F_{t,p,u}(s) = \sum_{q \in \mathbb{P}} e(t/p) \log(1 - u(q)q^{-s}) + \log \left( 1 + \frac{e(t/p)u(q)}{q^s} + \frac{e \left( \frac{w(q^2)t}{p} \right) u(q)^2}{q^{2s}} + \ldots \right).
\]

Let us denote

\[
h_{q,t,p,u}(s) := \sum_{j \geq 2} e \left( \frac{w(q^j)t}{p} \right) \frac{u(q)^j}{q^{js}}.
\]

This series converges uniformly on \( \text{Re}(s) > 1/2 + \epsilon \). So \( h_{q,t,p,u}(s) \) is analytic in the half-plane \( \text{Re}(s) > 1/2 \). In addition, \( h_{q,t,p,u}(s) = O(q^{-2s}) \).

For \( \text{Re} \, s > 1 \) we have

\[
\log F_{t,p,u}(s) = \sum_{q \in \mathbb{P}} e(t/p)(-u(q)q^{-s} - u(q)^2q^{-2s}/2 - \ldots) +
\]

\[
(e(t/p)u(q)q^{-s} + h_{q,t,p,u}(s)) + \sum_{j \geq 2} \frac{(-1)^{j+1}}{j} (e(t/p)u(q)q^{-s} + h_{q,t,p,u}(s))^j.
\]
If \( \Re s \geq 1/2 + \epsilon \), we obtain

\[
\log F_{t,p,u}(s) \ll \sum_{q \in \mathcal{P}} q^{-2\sigma} \ll \epsilon.
\]

So \( |\log F_{t,p,u}(s)| \leq C \epsilon \). Hence

\[
e^{-C \epsilon} \leq F_{t,p,u}(s) \leq e^{C \epsilon}.
\]

### 3 Main results

Fix a large \( N \). Let \( X = X(N) \) be a number that we will choose later. Assume that \( \log X(N) = o(\sqrt{\log N}) \). Let \( p \) be an unexceptional prime lying in the interval \([X - X^{4/5}, X]\). By Lemmas 5 and 6 such \( p \) exists.

**Theorem 2.** Under assumptions above,

\[
a(r) = \frac{N}{p} + 2 \frac{N}{p} \Re \left( e \left( \frac{-r}{p} A_t \right) \right) + B_1(r) + B_2(r) + B_3(r).
\]

Here

\[
A_t = \frac{F_{t,p}(1) (1 - \frac{p}{p-1} d_{t,p}(1))}{\Gamma(e(t/p))} \log e(t/p) - 1 \quad N,
\]

\[
d_{t,p}(s) := \left( 1 + \frac{e \left( \frac{t}{p} \right)}{p^s} + \frac{e \left( \frac{w(p^2)}{p} \right)}{p^{2s}} + \ldots \right)^{-1},
\]

\[
F_{t,p}(s) = \prod_{q \in \mathcal{P}} \left( 1 + \frac{e \left( \frac{t}{q} \right)}{q^s} + \frac{e \left( \frac{w(q^2)}{q} \right)}{q^{2s}} + \ldots \right) \left( 1 - \frac{1}{q^s} \right)^{e(t/p)} + 1,
\]

\[
B_1(r) \ll B_1 := pN \exp \left( -c_6 \sqrt{\log N} \right),
\]

\[
B_2(r) \ll B_2 := N \log^{\cos(2\pi/p) - 2} N
\]

and

\[
B_3(r) = 2 \frac{N}{p} \sum_{t=2}^{(p-1)/2} \Re \left( e \left( \frac{-r}{p} \right) A_t \right) \ll B_3 := \frac{N}{p^3} \sum_{t=2}^{(p-1)/2} t \log^{\cos(2\pi t/p) - 1} N.
\]

Moreover

\[
|A_1| \asymp p^{-2} \log^{\cos(2\pi/p) - 1} N.
\]

**Proof.** Applying discrete Fourier transform, we obtain

\[
a(r) = \sum_{n \leq N} \sum_{n + w(n) \equiv r \pmod{p}} \frac{1}{p} e \left( \frac{(n + w(n))t}{p} \right) e \left( \frac{-rt}{p} \right).
\]
Let \( k, t \in \mathbb{Z}/p\mathbb{Z} \). Set
\[
S_t := \sum_{n \leq N} e\left( \frac{w(n)t}{p} \right),
\]
\[
S_{k,t} := \sum_{n \leq N, n \equiv k \pmod{p}} e\left( \frac{w(n)t}{p} \right),
\]
\[
S_{t,\chi} := \sum_{n \leq N} e\left( \frac{w(n)t}{p} \right) \chi(n).
\]
Then
\[
a(r) = \frac{1}{p} \sum_{t,k \in \mathbb{Z}/p\mathbb{Z}} e\left( \frac{(k-r)t}{p} \right) S_{k,t}.
\]
Note that
\[
S_{k,0} = \frac{N}{p} + O(1),
\]
\[
S_{0,t} = S_t - S_{t,\chi_0},
\]
\[
S_{k,t} = \frac{1}{p-1} \sum_{\chi} \chi(k) S_{t,\chi}, \text{ if } k,t \neq 0.
\]
Putting \( u \equiv 1 \) in Lemma 11, we obtain
\[
\sum_{n \geq 1} e\left( \frac{w(n)t}{p} \right) \frac{n^s}{n} = (\zeta(s))^{e(t/p)} F_{t,p}(s).
\]
Here \( F_{t,p}(s) := F_{t,p,u}(s) \).
Now applying Lemma 3 we see that
\[
S_t = \frac{F_{t,p}(1)}{\Gamma(e(t/p)-1)} N \log e^{(t/p)-1} N + O(N \log^{\cos(2\pi t/p)-2} N).
\]
Lemma 10 gives us
\[
\sum_{n \geq 1} e\left( \frac{w(n)t}{p} \right) \frac{\chi(n)}{n^s} = (L(s, \chi))^{e(t/p)} F_{t,p,\chi}(s).
\]
If \( \chi \) is non-principal, then using Lemma 10 we see that
\[
S_{t,\chi} = O\left( N \exp\left( -c_6 \sqrt{\log N} \right) \right).
\]
Note that \( c_6 \) and constant in \( O \) do not depend on \( \chi \).
Therefore, if \( k,t \neq 0 \), then we have
\[
S_{k,t} = \frac{S_{t,\chi_0}}{p-1} + O\left( N \exp\left( -c_6 \sqrt{\log N} \right) \right),
\]
there \( \chi_0 \) is the principal character modulo \( p \). Let us denote
\[
d_{t,p}(s) := \left( 1 + \frac{e\left( \frac{i}{p} \right)}{p^s} + \frac{e\left( \frac{w(p^2)t}{p} \right)}{p^{2s}} + \ldots \right)^{-1}.
\]
We have
\[
\sum_{n \geq 1} e\left(\frac{w(n)t}{p} \right) \chi_0(n) = d_{t,p}(s) \sum_{n \geq 1} e\left(\frac{w(n)t}{p} \right) = (\zeta(s))^e(t/p) F_{t,p}(s) d_{t,p}(s).
\]

Using Lemma 3 we obtain
\[
S_{t,\chi_0} = \frac{F_{t,p}(1)d_{t,p}(1)}{\Gamma(e(t/p))} N \log^{e(t/p)-1} N + O(N \log^{\cos(2\pi t/p)-2} N),
\]

Putting all together we obtain
\[
a(r) = \frac{N}{p} + \frac{1}{p} \sum_{t \neq 0 \pmod{p}} e(-rt/p) S_{0,t} + \frac{1}{p(p-1)} \sum_{t \neq 0 \pmod{p}} S_{t,\chi_0} \sum_{k \neq 0 \pmod{p}} e\left(\frac{(k-r)t}{p}\right) + B_1(r).
\]

Here
\[
B_1(r) \ll pN \exp\left(-c_6 \sqrt{\log N}\right).
\]

Thus
\[
a(r) = \frac{N}{p} + \frac{1}{p} \sum_{t \neq 0 \pmod{p}} e(-rt/p) (S_t - S_{t,\chi_0} - S_{t,\chi_0}/(p-1)) + B_1(r) =
\]
\[
\frac{N}{p} + \frac{1}{p} \sum_{t \neq 0 \pmod{p}} \left(S_t - \frac{p}{p-1} S_{t,\chi_0}\right) e(-rt/p) + B_1(r).
\]

Now we have
\[
a(r) = \frac{N}{p} + \frac{N}{p} \sum_{t \neq 0 \pmod{p}} \frac{e(-rt/p) F_{t,p}(1)(1 - \frac{p}{p-1} d_{t,p}(1))}{\Gamma(e(t/p))} \log^{e(t/p)-1} N + B_1(r) + B_2(r).
\]

Here
\[
B_2(r) = O\left(N \log^{\cos(2\pi/p)-2} N\right).
\]

Let us denote
\[
A_t := \frac{F_{t,p}(1)(1 - \frac{p}{p-1} d_{t,p}(1))}{\Gamma(e(t/p))} \log^{e(t/p)-1} N,
\]
\[
B_3(r) = \frac{N}{p} \sum_{t=2}^{p-2} e(-rt/p) A_t.
\]

Then
\[
a(r) = \frac{N}{p} + 2\frac{N}{p} \Re\left(e\left(\frac{-r}{p}\right) A_1\right) + B_1(r) + B_2(r) + B_3(r).
\]

We have
\[
B_3(r) = 2 \frac{N}{p} \sum_{t=2}^{(p-1)/2} \Re\left(e\left(\frac{-rt}{p}\right) A_t\right).
\]

Note that
\[
\left|1 - \frac{p}{p-1} d_{t,p}(1)\right| \asymp \left|1 - \frac{e(t/p)}{p} + \frac{e(t/p)^2 - e(w^2 t/p)}{p^2} + O(p^{-3})\right| =
\]

10
\[ |(1 - p^{-1}) - \left(1 - \left(1 + \frac{2\pi it}{p} + O(t^2/p^2)\right)p^{-1} + \theta p^{-2}\right)| = 2\pi i \frac{t}{p^2} + 2\theta p^{-2} + O\left(\frac{t^2}{p^3}\right) \ll \frac{|t|}{p^2}, \]

here \(|\theta| \leq 1\). If \(t = 1\), we obtain

\[ \left|1 - \frac{p}{p-1} d_{t,p}(1)\right| \gg p^{-2}. \]

Since \(\Gamma(s)^{-1}\) is an entire function and \(|F_{t,p}(1)| \leq b_2\), it follows that

\[ |A_1| \ll \frac{t}{p^2} \log^2(2\pi t/p) - 1 N. \]

Hence

\[ B_3 \ll \frac{N}{p^3} \left(\frac{p-1}{2}\right) \sum_{t=2} t \log^2(2\pi t/p) - 1 N. \]

Since \(|e(1/p) - 1| \ll p^{-1}\) and \(\Gamma(1) = 1\), it follows that

\[ |\Gamma(e(1/p))^{-1}| > C \Gamma \]

for some constant \(C \Gamma > 0\) and \(p\) large enough. Further we have \(|F_{t,p}(1)| \geq b_1\). Hence

\[ |A_1| \gg p^{-2} \exp((\cos(2\pi t/p) - 1) \log \log N). \]

This concludes the proof.

\[ \Box \]

**Proof of Theorem 1.** Let \(r\) belong to \(R \subset \mathbb{Z}/p\mathbb{Z}\) iff

\[ 2\frac{N}{p} \operatorname{Re} \left( e \left( -\frac{r}{p} \right) A_1 \right) < 0. \]

Note that

\[ \left|\left\{ r : \operatorname{Re} \left( e \left( -\frac{r}{p} \right) A_1 \right) \leq -\frac{1}{2} |A_1| \right\}\right| \asymp p. \]

Thus, substituting such \(R\) into Lemma 1, we obtain

\[ \Xi(N) \gg \sum_{r \in R} \frac{N}{p} - a(r) - 1 \gg - \sum_{r \in R} \left(2\frac{N}{p} \operatorname{Re} \left( e \left( -\frac{r}{p} \right) A_1 \right) + O(B_1 + B_2 + B_3 + 1)\right) \gg \]

\[ N |A_1| + pO(B_1 + B_2 + B_3 + 1). \]

Note that

\[ \frac{X^2}{p^2} \leq 1 + 5X^{-1/5}. \]

Let us choose \(X = \alpha^{-1} \sqrt{\log \log N}\), we obtain

\[ N |A_1| \gg Np^{-2} \exp((\cos(2\pi/p) - 1) \log \log N) \gg Np^{-2} \exp\left(\frac{-2\pi^2}{p^2} \log \log N\right) \gg \]

\[ N X^{-2} \exp\left(-\frac{2\pi^2}{X^2} (1 + 5X^{-1/5}) \log \log N\right) \gg \]
\[ NX^{-2} \exp \left(-\frac{2\pi^2}{X^2} \log \log N\right) \gg \alpha^2 \exp \left(-2\pi^2 \alpha^2 \right) \frac{N}{\log \log N}. \]

Let us choose \( \alpha \) such that \( pB_3 \leq 0.1 N |A_1| \). That is

\[ \tilde{C} \left( \sum_{t=2}^{(p-1)/2} t \log \cos(2\pi t/p)^{-1} N \right) \leq \exp \left(-2\pi^2 \alpha^2 \right) \]

for some constant \( \tilde{C} \).

We have

\[ \sum_{t=2}^{(p-1)/2} t \log \cos(2\pi t/p)^{-1} N = \sum_{2 \leq t < p/4} t \log \cos(2\pi t/p)^{-1} N + O(p^2 \log^{-1} N). \]

Note that

\[ \frac{5\pi^2}{X^2} \leq \cos \frac{2\pi}{p} - \cos \frac{4\pi}{p} \leq \cos \frac{4\pi}{p} - \cos \frac{6\pi}{p} \leq \ldots \]

Therefore,

\[ \sum_{2 \leq t < p/4} t \log \cos(2\pi t/p)^{-1} N \leq \sum_{2 \leq t < p/4} t \log \cos(2\pi t/p)^{-1} N \leq \]

\[ 2 \sum_{t=1}^{\infty} t \exp \left(-5\pi^2 \alpha^2 t\right) = \frac{2 \exp(-5\pi^2 \alpha^2)}{(1 - \exp(-5\pi^2 \alpha^2))^2}. \]

Clearly, for \( \alpha \) large enough, we have

\[ \frac{2 \exp(-5\pi^2 \alpha^2)}{(1 - \exp(-5\pi^2 \alpha^2))^2} \leq 0.5 \tilde{C}^{-1} \exp \left(-2\pi^2 \alpha^2 \right). \]

Obviously,

\[ p^2 \log^{-1} N = o(1). \]

Thus we obtain

\[ pB_3 \leq 0.1 N |A_1| \]

for \( \alpha \) large enough. Note that \( \alpha \) does not depend on \( N \). Hence

\[ N |A_1| \gg \frac{N}{\log \log N}. \]

Obviously, \( pB_1 = o \left( N / \log \log N \right) \) and \( pB_2 = o \left( N / \log \log N \right) \).

Therefore,

\[ \Xi(N) \gg N |A_1| \gg \frac{N}{\log \log N}. \]

I am grateful to Vitalii V. Iudelevich for setting the problem and Alexander B. Kalmynnin for valuable discussions.
References

[1] A. A. Karatsuba, Melvyn B. Nathanson. Basic Analytic Number Theory, Springer-Verlag Berlin Heidelberg, 1993

[2] M. E. Changa. Methods of analytic number theory, RCD, Moscow–Izhevsk 2013. (Russian)

[3] M. E. Changa. On integers whose number of prime divisors belongs to a given residue class 2019 Izv. Math. 83-173

[4] P. Erdős, A. Sárközy, C. Pomerance. On locally repeated values of certain arithmetic function, I, 1983

[5] R. C. Baker, G. Harman, J. Pintz. The Difference Between Consecutive Primes, II, Proceedings of the London Mathematical Society, Volume 83, Issue 3, November 2001, Pages 532–562