Research Article

Optimal Control of an SIR Model with Delay in State and Control Variables

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We will investigate the optimal control strategy of an SIR epidemic model with time delay in state and control variables. We use a vaccination program to minimize the number of susceptible and infected individuals and to maximize the number of recovered individuals. Existence for the optimal control is established; Pontryagin’s maximum principle is used to characterize this optimal control, and the optimality system is solved by a discretization method based on the forward and backward difference approximations. The numerical simulation is carried out using data regarding the course of influenza A (H1N1) in Morocco. The obtained results confirm the performance of the optimization strategy.

1. Introduction

For a long time, infectious diseases have caused several epidemics, leaving behind them not only millions of dead and infected individuals but also severe socioeconomic consequences. Nowadays, mathematical modeling of infectious diseases is one of the most important research areas. Indeed, mathematical epidemiology has contributed to a better understanding of the dynamical behavior of infectious diseases, its impacts, and possible future predictions about its spreading. Mathematical models are used in comparing, planning, implementing, evaluating, and optimizing various detection, prevention, therapy, and control programs. Many influential results related to the development and analysis of epidemiological models have been established and can be found in many articles and books (see, e.g., [1–3]).

Epidemiological models often take the form of a system of nonlinear, ordinary, and differential equations without time delay. However, for various biological reasons, the real dynamic behavior of an epidemic depends not only on its current state but also on its past history. Thus, to reflect the real behavior of some diseases, many researchers have proposed and analyzed more realistic models including delays to model different mechanisms in the dynamics of epidemics like latent period, temporary immunity and length of infection (see, e.g., [4–8] and the references therein).

To the best of our knowledge, including time delay in both state and control variables in the context of an epidemic controlled model has not been studied. There have been some works (like [9, 10]) which study an optimal control problem with time delay but only in the state variable. In this paper, we will investigate the effect of a vaccination program in the case of an SIR (susceptible-infected-recovered) epidemic model with time delay in the control and the state variables. To do this, we will consider an optimally controlled SIR epidemic model with time delay, where the control means the percentage of susceptible individuals being vaccinated per time unit, and the time delay represents the required time so that a vaccinated susceptible person moves from the susceptibles class to the recovered class. We use optimal control approach to minimize the number of susceptible and infected individuals and to maximize the number of recovered individuals during the course of an epidemic.

This paper is organized as follows. In Section 2, we will present a mathematical model with time delay and a control term. The analysis of optimization problem is presented in
In Section 4, we will give a numerical appropriate method and the corresponding simulation results. Finally, the conclusions are summarized in Section 5.

2. Mathematical Model

We consider the SIR epidemic model with constant total population size. The population is divided into three disease-state compartments: susceptible individuals (S), people who can catch the disease; infectious (infective) individuals (I), people who have the disease and can transmit the disease; recovered individuals (R), people who have recovered from the disease. We assume that an individual can be infected only through contacts with infectious individuals and that immunity is permanent. The transitions between different states are described by the following parameters:

(i) $\Lambda$ is the recruitment rate of susceptibles;

(ii) $\beta$ is the effective contact rate;

(iii) $d$ is the natural mortality rate;

(iv) $\gamma$ is the recovery rate;

(v) $\epsilon$ is the disease induced death rate.

The population dynamics is given by the following system of ordinary differential equations subject to nonnegative initial conditions:

$$
\begin{align*}
\frac{dS(t)}{dt} &= \Lambda - \beta S(t) \frac{I(t)}{N(t)} - dS(t), \\
\frac{dI(t)}{dt} &= \beta S(t) \frac{I(t)}{N(t)} - (\gamma + d + \epsilon) I(t), \\
\frac{dR(t)}{dt} &= \gamma I(t) - dR(t),
\end{align*}
$$

where $S(0) = S_0, I(0) = I_0, R(0) = R_0$, and $N(t) = S(t) + I(t) + R(t)$ is the total population number at time $t$.

The strategy of the control we adopt consists of a vaccination program; our goal is to minimize the level of susceptible and infected individuals and to maximize the recovered individuals. Into the model (1) we include a control $u$ that represents the percentage of susceptible individuals being vaccinated per time unit. In order to have a realistic model, we need to take into account that the movement of the vaccinated susceptible individuals from the class of susceptibles into the recovered class is subject to delay. Thus, the time delay is introduced in the system as follows: at time $t$ only a percentage of susceptible individuals that have been vaccinated $\tau$ time unit ago, that is, at time $t - \tau$, are removed from the susceptible class and added to the recovered class. So the mathematical system with time delay in state and control variables is given by the nonlinear retarded differential equations:

$$
\frac{dS(t)}{dt} = \Lambda - \beta S(t) \frac{I(t)}{N(t)} - dS(t) - u(t-\tau)S(t-\tau),
$$

$$
\frac{dI(t)}{dt} = \beta S(t) \frac{I(t)}{N(t)} - (\gamma + d + \epsilon) I(t),
$$

$$
\frac{dR(t)}{dt} = \gamma I(t) - dR(t) + u(t-\tau)S(t-\tau).
$$

In addition, for biological reasons, we assume, for $\theta \in [-\tau, 0]$, that $S(\theta), I(\theta), \text{and } R(\theta)$ are nonnegative continuous functions and $u(\theta) = 0$. Note that the control $u$ is assumed to be integrable in the sense of Lebesgue, bounded with $0 \leq u \leq b < 1$, and $b$ is a given constant.

To show the existence of solutions for the control system (2), we first prove that the system (2) is dissipative; that is, all solutions are uniformly bounded in a proper subset $\Omega \subset \mathbb{R}_+^3$.

Let $(S, I, R) \in \mathbb{R}_+^3$ be any solution with nonnegative initial conditions. Adding equations of (2) we get

$$
\frac{dN}{dt} = \Lambda - dN - \epsilon I < \Lambda - dN.
$$

After integration, using the constant variation formula, we have

$$
N(t) \leq \Lambda \frac{d}{d} + N(0) e^{-d t}.
$$

It then follows that

$$
0 \leq N(t) \leq \frac{\Lambda}{d} \quad \text{as } t \to \infty.
$$

Therefore all feasible solutions of the system (2) enter into the region

$$
\Omega = \left\{(S, I, R) \in \mathbb{R}_+^3: N \leq \frac{\Lambda}{d}\right\}.
$$

Then we can rewrite (2) in the following form:

$$
\frac{dX}{dt} = AX + F(X, X_\tau) = G(X, X_\tau),
$$

where

$$
X(t) = \begin{bmatrix} S(t) \\ I(t) \\ R(t) \end{bmatrix},
A = \begin{bmatrix} -d & 0 & 0 \\ 0 & -(d + \gamma + \epsilon) & 0 \\ 0 & \gamma & -d \end{bmatrix},
F(X(t), X_\tau(t)) = \begin{bmatrix} \Lambda - \beta S(t) I(t) \\ \beta S(t) I(t) \\ u_\tau(t) S_\tau(t) \end{bmatrix},
G(X(t), X_\tau(t)) = \begin{bmatrix} u_\tau(t) S_\tau(t) \\ u_\tau(t) S_\tau(t) \end{bmatrix},
u_\tau(t) = u(t-\tau), \quad X_\tau(t) = X(t-\tau).
$$
The second term on the right-hand side of (7) satisfies
\[
|F(X_1(t), X_{1\tau}(t)) - F(X_2(t), X_{2\tau}(t))| \\
\leq M_1 |X_1(t) - X_2(t)| \\
+ M_2 |X_{1\tau}(t) - X_{2\tau}(t)|,
\]
where $M_1$ and $M_2$ are some positive constants, independent of the state variables $S(t), I(t)$, and $R(t)$, and
\[
|X_1(t) - X_2(t)| = |S_1(t) - S_2(t)| \\
+ |I_1(t) - I_2(t)| + |R_1(t) - R_2(t)|,
\]
\[
|X_{1\tau}(t) - X_{2\tau}(t)| = |S_{1\tau}(t) - S_{2\tau}(t)| \\
+ |I_{1\tau}(t) - I_{2\tau}(t)| + |R_{1\tau}(t) - R_{2\tau}(t)|.
\]
where $\lambda_i$ is the right side of the differential equation of the $i$th state variable.

3.1. Existence of an Optimal Control. The existence of the optimal control can be obtained using a result by Fleming and Rishel in [12].

**Theorem 1.** Consider the control problem with system (2). There exists an optimal control $u^* \in \mathcal{U}$ such that
\[
J(u^*) = \min_{u \in \mathcal{U}} J(u).
\]

**Proof.** To use an existence result in [12], we must check the following properties.

1. The set of controls and corresponding state variables is nonempty.
2. The control set $\mathcal{U}$ is convex and closed.
3. The right-hand side of the state system is bounded by a linear function in the state and control variables.
4. The integrand of the objective functional is convex on $\mathcal{U}$.
5. There exist constants $c_1, c_2 > 0$ and $\rho > 1$ such that the integrand $L(S, I, R, u)$ of the objective functional satisfies
\[
L(S, I, R, u) \geq c_2 + c_1 \left( u^2 \right)^{\frac{\rho}{2}}.
\]

An existence result by Lukes [13] was used to give the existence of solution of system (2) with bounded coefficients, which gives condition 1. The control set is convex and closed by definition. Since the state system is linear in $u$, the right side of (2) satisfies condition 3, using the boundedness of the solution. The integrand in the objective functional (14) is convex on $\mathcal{U}$. In addition, we can easily see that there exist a constant $\rho > 1$ and positive numbers $c_1$ and $c_2$ satisfying
\[
L(S, I, R, u) \geq c_2 + c_1 \left( u^2 \right)^{\frac{\rho}{2}}.
\]

3.2. Characterization of the Optimal Control. In order to derive the necessary condition for the optimal control, Pontryagin’s maximum principal with delay given in [14] was used. This principal converts (2), (14), and (15) into a problem of minimizing a Hamiltonian, $H$, defined by
\[
H = A_1 S(t) + A_2 I(t) - A_3 R(t) + \frac{A_4}{2} u^2(t) + \sum_{j=1}^{\infty} \lambda_j f_i,
\]
where $f_i$ is the right side of the differential equation of the $i$th state variable.
Theorem 2. Given an optimal control $u^* \in U$ and solutions $S^*$, $I^*$, and $R^*$ of the corresponding state system (2), there exist adjoint functions $\lambda_1$, $\lambda_2$, and $\lambda_3$ satisfying

\begin{align*}
\dot{\lambda}_1 &= -A_1 + \lambda_1 d + (\lambda_1 - \lambda_2) \beta N^* (S^*)^- \\
&\quad + \chi_{[0,t_f-\tau]} (t) \left( \lambda_1^- - \lambda_1^+ \right) u^*, \\
\dot{\lambda}_2 &= -A_2 + (\lambda_1 - \lambda_2) \beta S^* N^* + (d + \gamma + \epsilon) \lambda_2 - \gamma \lambda_3, \\
\dot{\lambda}_3 &= A_3 + \lambda_3 d
\end{align*}

(21)

with the transversality conditions

\begin{align*}
\lambda_1 (t_f) = \lambda_2 (t_f) = \lambda_3 (t_f) = 0.
\end{align*}

(22)

Furthermore, the optimal control $u^*$ is given by

\begin{align*}
u^* (t) = \min \left( b, \max \left( 0, \frac{(\lambda_1^+ - \lambda_1^-)}{A_k} \chi_{[0,t_f-\tau]} (t) S^* \right) \right),
\end{align*}

(23)

where $\lambda_i^0 = \lambda_i (t + \tau)$ for $i = 1, \ldots, 3$.

Proof. The adjoint equations and transversality conditions can be obtained by using Pontryagin’s maximum principle with delay in the state and control variables [14] such that

\begin{align*}
\dot{\lambda}_1 &= -\frac{\partial H}{\partial S (t)} - \chi_{[0,t_f-\tau]} (t) \frac{\partial H (t + \tau)}{\partial (t - \tau)}, \quad \lambda_1 (t_f) = 0, \\
\dot{\lambda}_2 &= -\frac{\partial H}{\partial I}, \quad \lambda_2 (t_f) = 0, \\
\dot{\lambda}_3 &= -\frac{\partial H}{\partial R}, \quad \lambda_3 (t_f) = 0.
\end{align*}

(24)

The optimal control $u^*$ can be solved from the optimality condition:

\begin{align*}
\frac{\partial H}{\partial u} + \chi_{[0,t_f-\tau]} (t) \frac{\partial H (t + \tau)}{\partial u_t} = 0.
\end{align*}

(25)

That is,

\begin{align*}
A_k u + \chi_{[0,t_f-\tau]} (t) \left( \lambda_1^+ - \lambda_1^- \right) S = 0.
\end{align*}

(26)

By the bounds in $U$ of the control, it is easy to rewrite $u^*$ in the form (23).

4. Numerical Simulation

In this section, we give a numerical method to solve the optimality system which is a two-point boundary value problem, with separated boundary conditions at times $t_0 = 0$ and $t_f$.

Let there exist a step size $h > 0$ and $(n,m) \in \mathbb{N}^2$ with $\tau = mh$ and $t_f - t_0 = nh$. For reasons of programming, we consider $m$ knots to left of $t_0$ and right of $t_f$, and we obtain the following partition:

\begin{align*}
\Delta = \left( t_{-m} = -\tau < \cdots < t_{-1} < t_0 < t_1 < \cdots < t_n = t_f < t_{n+1} < \cdots < t_{m+m} \right).
\end{align*}

(27)

Then, we have $t_i = t_0 + ih (-m \leq i \leq n + m)$. Next, we define the state and adjoint variables $S(t), I(t), R(t), \lambda_1, \lambda_2, \lambda_3$, and the control $u$ in terms of nodal points $S_i, I_i, R_i, \lambda_1^i, \lambda_2^i, \lambda_3^i$, and $u_i$. Now, using combination of forward and backward difference approximations, we obtain the Algorithm 1.

The numerical simulations were carried out using data regarding the course of the influenza A (H1N1) in Morocco. The initial conditions and parameters of the system (2) are taken from [15, 16], while the time delay value is taken from [17]:

\begin{align*}
\beta &= 0.3095, \quad \Lambda = 1174.17, \\
d &= 3.9139 \times 10^{-5}, \quad \gamma = 0.2, \\
\epsilon &= 0.0063, \quad \tau = 10,
\end{align*}

(28)

the initial conditions for the ordinary differential system were

\begin{align*}
S(0) = 30 \times 10^6, \quad I(0) = 30, \quad R(0) = 28,
\end{align*}

(29)

and the transversality conditions for the ordinary differential system were

\begin{align*}
\lambda_i (t_f) = 0 \quad (i = 1, \ldots, 3).
\end{align*}

(30)

Figure 1 indicates that the number of susceptible individuals ($S$) decreases more rapidly in the case with control. It reaches $7.399 \times 10^5$ at the end of the vaccination period against $6.732 \times 10^6$ in case without control, that is, a reduction of $5.992 \times 10^6$. 

![Figure 1: The function $S$ with and without control.](image-url)
Step 1: For $i = -m, \ldots, 0$ do
  $S_i = S_0$, $I_i = I_0$, $R_i = R_0$, $u_i = 0$
End for
For $i = n, \ldots, n + m$ do
  $\lambda_i^1 = 0$, $\lambda_i^2 = 0$, $\lambda_i^3 = 0$
End for
Step 2: For $i = 0, \ldots, n - 1$, do
  $S_{i+1} = S_i + h(A - u_{i-m}S_{i-m})$
  $I_{i+1} = I_i + h(y_{i+1} + u_{i-m}S_{i-m})$
  $R_{i+1} = R_i + h(\gamma I_{i+1} + u_{i-m}S_{i-m})$
\[\lambda_i^{n+i-1} = \frac{\lambda_i^{n+i} + h(\Lambda + \beta(I_{i+1}/N)\lambda_i^{n+i} + (\lambda_i^{n+i-m} - \lambda_i^{n+i-1-m})X(t_{i+1})S_{i+1})}{1 + h(d + \beta(S_{i+1}/N))}\]
\[\lambda_i^{n+i-1} = \frac{\lambda_i^{n+i} + h(A_2 + \lambda_i^{n+i}\gamma - \lambda_i^{n+i-1}\beta(S_{i+1}/N))}{1 + h(\gamma + d + \epsilon - \beta(S_{i+1}/N))}\]
\[\lambda_i^{n+i-1} = \frac{\lambda_i^{n+i} - A_{3}h}{1 + dh}\]
\[T_{i+1}^{n+i} = \frac{(\lambda_i^{n+i-1} - \lambda_i^{n+i-1-m})X(t_{i+1})S_{i+1}}{A_{4}}\]
  $u_{i+1} = \min(b, \max(0, T_{i+1}^{n+i}))$
Step 3: For $i = 0, \ldots, n$, write
  $S^*(t_i) = S_i$, $I^*(t_i) = I_i$, $R^*(t_i) = R_i$, $u^*(t_i) = u_i$
End for

Algorithm 1

**Figure 2:** The function $I$ with and without control.

**Figure 3:** The function $I$ with control.

**Figure 4:** The function $I$ without control.

Figures 2, 3, and 4 represent the number of infected individuals ($I$) with control (solid curve) and without control (dashed curve). It shows that in the presence of a control, the number of infected individuals ($I$) decreases greatly. The maximum number of infected individuals in the case with control is $1.787 \times 10^4$ and is $4.404 \times 10^6$ in the case without control; then the efficiency of our strategy in reducing the spread of infection is nearly 99.52%.

Figure 5 shows that the number of people removed begins to grow notably from 24th day instead of 48th day in the absence of control. Moreover the number of recovered individuals at the end of the vaccination period is $2.926 \times 10^7$ instead of $2.342 \times 10^7$, which represents an increase of $5.840 \times 10^6$ cases.

Figures 6 and 7 represent, respectively, the optimal control and optimal value of the cost. The curves start to increase during the first month because of the high infection level;
5. Conclusion

The work in this paper contributes to a growing literature on applying optimal control techniques to epidemiology. We proposed a more realistic controlled model by including time delay which represents the needed time for the migration from the susceptible class to the recovered class after vaccination. The optimal control theory has been applied in the context of an SIR model with time delay in state and control variables, and that includes a control $\mu$ that represents the percentage of susceptible individuals being vaccinated per time unit. By using Pontryagin’s maximum principle, the explicit expression of the optimal controls was obtained. Simulation results indicate that the proposed control strategy is effective in reducing the number of susceptible and infected individuals and maximizing the recovered individuals.

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