IRREDUCIBILITY OF NEWTON STRATA IN GU(1, n − 1) SHIMURA VARIETIES

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ABSTRACT. Let \( L \) be a quadratic imaginary field, inert at the rational prime \( p \). Fix an integer \( n \geq 3 \), and let \( \mathcal{M} \) be the moduli space (in characteristic \( p \)) of principally polarized abelian varieties of dimension \( n \) equipped with an action by \( \mathcal{O}_L \) of signature \((1, n - 1)\). We show that each Newton stratum of \( \mathcal{M} \), other than the supersingular stratum, is irreducible.

1. INTRODUCTION

For a complex abelian variety \( X \), the isomorphism class of its \( p \)-torsion group scheme \( X[p] \) and of its \( p \)-divisible group \( X[p^\infty] \) depend only on the dimension of \( X \). In contrast, in characteristic \( p \), there are different possibilities for the corresponding isomorphism (or even isogeny) class. Each such invariant provides a stratification of a family of abelian varieties in positive characteristic.

The isogeny class of \( X[p^\infty] \) is called the Newton polygon of \( X \). The goal of the present note is to prove that the space of abelian varieties with given Newton polygon and a certain, specified endomorphism structure is irreducible.

More precisely, let \( L \) be a quadratic imaginary field, inert at the rational prime \( p \). Fix an integer \( n \geq 3 \), and let \( \mathcal{M} \) be the moduli space (over \( \mathbb{F}_{p^2} \)) of principally polarized abelian varieties of dimension \( n \) equipped with an action by \( \mathcal{O}_L \) of signature \((1, n - 1)\). Our main result is:

Theorem 1.1. Let \( \xi \neq \sigma \) be an admissible Newton polygon for \( \mathcal{M} \) which is not supersingular. Then the corresponding stratum \( N^\xi \) is irreducible.

The proof of Theorem 1.1 is modelled on, but considerably easier than, that of [4, Thm. A]. This is possible because the Newton and Ekedahl-Oort stratifications on \( \mathcal{M} \) are much simpler than those of \( \mathcal{A}_g \).

In the special case where \( L = \mathbb{Q}(\zeta_3) \) and \( n \) is 3 or 4, \( \mathcal{M} \) essentially coincides with a component of the moduli space of cyclic triple covers of the projective line. Theorem 1.1 provides a crucial base case for forthcoming work of Ozman, Pries and Weir on such covers [11], and that work was the initial impetus for the present study.

For a topological space \( T \), let \( \Pi_0(T) \) denote the set of irreducible components of \( T \). If \( T \subset \mathcal{M} \), then \( \overline{T} \) is its closure in \( \mathcal{M} \). The symbol \( k \) will denote an arbitrary algebraically closed field of characteristic \( p \).

2. BACKGROUND ON \( \mathcal{M} \)

2.1. Moduli spaces. Let \( \mathcal{M} \) be the moduli stack (over \( \mathcal{O}_L/p \cong \mathbb{F}_{p^2} \)) of principally polarized abelian varieties of dimension \( n \) with an action by \( \mathcal{O}_L \) of signature \((1, n - 1)\). Somewhat more precisely, \( \mathcal{M}(S) \) consists of isomorphism classes of data \( (X, \iota, \lambda) \), where \( X \to S \) is an abelian variety of relative dimension \( n \), \( \iota : \mathcal{O}_L \to \text{End}_S(X) \) is an embedding taking \( 1_L \) to \( \text{id}_X \) such that \( \text{Lie}(X) \),

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as a module over $\mathcal{O}_L \otimes \mathcal{O}_S$, has signature $(1, n − 1)$; and $\lambda : X \rightarrow X^c$ is a principal polarization such that, if $(\dagger)$ is the induced Rosati involution on $\text{End}(X)$, then for each $a \in \mathcal{O}_L$ one has $\iota(\pi) = \iota(a)^{(\dagger)}$. It is standard that $\dim \mathcal{M} = 1 \cdot (n − 1) = n − 1$.

In fact, $\mathcal{M}$ is the moduli stack attached to the Shimura (pro-)variety constructed from a certain group $G$, as follows.

Let $V$ be an $n$-dimensional vector space over $L$, equipped with a Hermitian pairing of signature $(1, n − 1)$. Let $G / O$ be the group of unitary similitudes of $V$, and let $U$ be the unitary group of $V$. Fix a hyperspecial subgroup $\mathbb{K}_p \subset G(\mathbb{Q}_p)$. For each sufficiently small open compact subgroup $\mathbb{K}^p \subset G(\mathbb{A}_f^p)$, there is a moduli space $\mathcal{M}_{\mathbb{K}^p} = \mathcal{M}_{\mathbb{K}^p, \mathbb{K}^p}$ of abelian varieties of dimension $n$ as above with $\mathbb{K}^p$ structure; see [7] for more details. If $\mathbb{K}^p$ is sufficiently small, then $\mathcal{M}_{\mathbb{K}^p}$ is a smooth, quasiprojective variety; and $\mathcal{M}$ may be constructed as the quotient of any $\mathcal{M}_{\mathbb{K}^p}$ by an appropriate finite group.

2.2. Newton polygons in $\mathcal{M}$. Newton and Ekedahl-Oort stratifications on $GU(1, n − 1)$ Shimura varieties are well understood [2]. There are exactly $1 + \lfloor n/2 \rfloor$ (“admissible”) Newton polygons which occur, and the poset of admissible Newton polygons is actually totally ordered. Let $\sigma$ be the supersingular Newton polygon, so that $\sigma \preceq \xi$ for any admissible Newton polygon $\xi$ for $\mathcal{M}$. For a Newton polygon $\xi$, let $\mathcal{M}^\xi$ denote the locally closed locus corresponding to abelian varieties with Newton polygon $\xi$. Then $\mathcal{M}^\sigma$ is pure of dimension $\lfloor n/2 \rfloor$. By purity [5, 9], if $Z_\sigma \in \Pi_0(\mathcal{M}^\sigma)$ and $\sigma \preceq \xi$, then there exists some $Z_\xi \in \Pi_0(\mathcal{M}^\xi)$ such that $Z_\sigma \subseteq Z_\xi$, the closure of $Z_\xi$ in $\mathcal{M}$.

The Newton stratification of $\mathcal{M}$ is described in [2], as follows. Each admissible Newton polygon is determined by its smallest slope. For each integer $1 \leq j \leq \lfloor n/2 \rfloor$, there is a Newton polygon $\tilde{\xi}_{2j}$, with smallest slope

$$\lambda(2j) = \frac{1}{2} - \frac{1}{2(\lfloor n/2 \rfloor + 1 - j)};$$

then $\mathcal{M}^\tilde{\xi}_{2j}$ has codimension $\lfloor n/2 \rfloor − j$ in $\mathcal{M}$. (Admittedly, in many ways this normalization is more awkward than that of [2], in which $\mathcal{M}^\tilde{\xi}_{2j}$ is labeled $\mathcal{M}_{2(\lfloor n/2 \rfloor − j)}$; but it will be more convenient for the deformation theory below.)

Away from the supersingular locus $\mathcal{M}^\sigma$, the Newton, Ekedahl-Oort, and final stratifications coincide; a $p$-divisible group is determined by its mod $p$ truncation [2 Thm. 5.3]. This is recalled in greater detail in Section 2.3 below.

The Newton polygon and Ekedahl-Oort type of a polarized $\mathcal{O}_L$-abelian variety with prime-to-$p$ level structure do not depend on the level structure, and we set $\mathcal{M}^\xi_{\mathbb{K}^p} = \mathcal{M}_{\mathbb{K}^p} \times_\mathcal{M} \mathcal{M}^\xi$.

2.3. $p$-divisible groups. In contrast to the Siegel case, it is possible to write down a finite, explicit collection of those principally quasipolarized $p$-divisible groups with $\mathcal{O}_L$-action which occur as $(X, t, \lambda)[p^\infty]$ for $(X, t, \lambda) \in \mathcal{M}(k)$. Following Wedhorn, we describe such $p$-divisible groups in terms of their covariant Dieudonné modules, as follows.

For $m \in \mathbb{N}$, let $M(m)$ be the following Dieudonné module.

- As a $W(k)$-module, $M(m)$ admits basis $\{u_1, \cdots, u_m, v_1, \cdots, v_m\}$.
2.4. Hecke operators. An inclusion $\mathbb{K}_1^p \to \mathbb{K}_2^p$ of open compact subgroups of $G(\mathbb{A}_f^p)$ induces a cover of Shimura varieties $\mathcal{M}_{\mathbb{K}_1^p} \to \mathcal{M}_{\mathbb{K}_2^p}$. More generally, an element $g \in G(\mathbb{A}_f^p)$ induces, for each open compact $\mathbb{K}^p$, a natural morphism $\mathcal{M}_{\mathbb{K}^p} \to \mathcal{M}_{\mathbb{K}^p g}$.

Let $z \in \mathcal{M}_{\mathbb{K}_0^p}(k)$. Its prime-to-$p$ (unitary) Hecke orbit, $\mathcal{H}^p(z)$, is defined as follows. Consider the pro-variety $\tilde{\mathcal{M}}_{\mathbb{K}_0^p} = \lim_{\mathbb{K}^p \subset \mathbb{K}_0^p} \mathcal{M}_{\mathbb{K}^p}$. Choose a lift $\tilde{z}$ of $z$ to $\tilde{\mathcal{M}}_{\mathbb{K}_0^p}$. Then $\mathcal{H}^p(z)$ is the projection to $\mathcal{M}_{\mathbb{K}_0^p}$ of $U(\mathbb{A}_f^p)\tilde{z}$. (One can also construct the “similitude” Hecke orbit of $\tilde{z}$, by replacing the orbit $U(\mathbb{A}_f^p)\tilde{z}$ with $G(\mathbb{A}_f^p)\tilde{z}$. However, the unitary Hecke orbit is both the output of [12] Thm. 4.6 and the input to [6] Thm. 1.4], and thus better suited to the task at hand.)
3. Closures of Newton strata

Let $\xi$ be an admissible Newton polygon for $\mathcal{M}$ such that $\xi \neq \sigma$.

**Lemma 3.1.** The locus $\mathcal{M}^\xi$ is smooth.

*Proof.* The isomorphism class of $(X[p^\infty], \iota[p^\infty], \lambda[p^\infty])$ for $(X, \iota, \lambda) \in \mathcal{M}^\xi(k)$ is independent of the choice of point (Theorem 2.1). By the Serre-Tate theorem, the formal neighborhoods of all points of $\mathcal{M}^\xi$ are thus isomorphic. Since $\mathcal{M}^\xi$ is by definition equipped with the reduced subscheme structure, it must be smooth. \hfill \Box

**Lemma 3.2.** If $Z_\xi \in \Pi_0(\mathcal{M}^\xi)$, then there exists $Z_\sigma \in \Pi_0(\mathcal{M}^\sigma)$ such that $Z_\sigma \subset Z_\xi$.

*Proof.* We prove the following apparently stronger result. Suppose $\nu$ and $\xi$ are admissible Newton polygons with $\nu \prec \xi$, and $Z_\xi \in \Pi_0(\mathcal{M}^\xi)$. We show that there exists $Z_\nu \in \Pi_0(\mathcal{N}^\sigma)$ such that $Z_\nu \subset Z_\xi$. It suffices to prove this statement under the assumption that $\nu$ is the immediate predecessor of $\xi$, so that $\dim \mathcal{M}^\nu = \dim \mathcal{M}^\xi - 1$. The statement is trivially true if $\xi = \xi_{2|n/2}$ is the locus with positive $p$-rank; henceforth, we assume that $\xi$ is strictly smaller than $\xi_{2|n/2}$.

It is slightly more convenient to work with fine moduli schemes. Let $\mathcal{K}^p \subset G(\mathcal{A}_f^p)$ be an open compact subgroup which is small enough that $\mathcal{M}_{\mathcal{K}^p}$ is a smooth, quasiprojective variety. Let $W_\xi \in \Pi_0(Z_\xi \times \mathcal{M}_{\mathcal{K}^p})$ be an irreducible component of $\mathcal{M}_{\mathcal{K}^p}^\xi$ lying over $Z_\xi$. It suffices to show that the closure of $W_\xi$ in $\mathcal{M}_{\mathcal{K}^p}$ contains an irreducible component of $\mathcal{M}_{\mathcal{K}^p}^\nu$.

Let $\overline{\mathcal{M}}_{\mathcal{K}^p}$ be a toroidal compactification of $\mathcal{M}_{\mathcal{K}^p}$ (e.g., [8] 6.4.1.1). It is a smooth, projective variety. Let $\overline{W}_\xi$ be the closure of $W_\xi$ in $\overline{\mathcal{M}}_{\mathcal{K}^p}$, and let $\partial W_\xi = \overline{W}_\xi \setminus W_\xi$. Newton strata (other than the supersingular stratum) coincide with Ekedahl-Oort strata, and the latter are known to be affine (e.g., [9]). Because $W_\xi$ is positive dimensional, $\partial W_\xi$ is nonempty. The first slope of $\xi$ is positive, while the boundary of $\overline{\mathcal{M}}_{\mathcal{K}^p}$ parametrizes semiabelian varieties with nontrivial toric part. Consequently, $\partial W_\xi \cap (\overline{\mathcal{M}}_{\mathcal{K}^p} \setminus \mathcal{M}_{\mathcal{K}^p})$ is empty, and $\partial W_\xi \subset \mathcal{M}_{\mathcal{K}^p}$. Again by purity ([9]), $\dim \partial W_\xi = \dim W_\xi - 1$. By semicontinuity of Newton polygons, there is an a priori containment $\partial W_\xi \subset \cup_{\tau \prec \nu} \mathcal{M}_{\mathcal{K}^p}^\tau$. The result now follows from dimension counts: $\mathcal{M}_{\mathcal{K}^p}^\nu$ is pure of dimension $\dim W_\xi - 1$, while if $\tau \prec \nu$ then $\dim \mathcal{M}_{\mathcal{K}^p}^\tau < \dim W_\nu = \dim \partial W_\xi$. \hfill \Box

Conversely,

**Lemma 3.3.** If $Z_\sigma \in \Pi_0(\mathcal{M}^\sigma)$, then there is a unique $Z_\xi \in \Pi_0(\mathcal{M}^\xi)$ such that $Z_\sigma \subset Z_\xi$.

*Proof.* The existence of such a $Z_\xi$ follows from purity and dimension-counting. If there were two such components, then they would intersect along $Z_\sigma \cap \mathcal{M}^{\sigma^0}$, which would contradict the smoothness shown in Lemma 4.1. \hfill \Box

4. Local calculations

**Lemma 4.1.** Let $\xi$ be an admissible Newton polygon which is not supersingular, and suppose $z \in \mathcal{M}^{\sigma^0}(k)$. Then $\overline{\mathcal{M}}^\xi$ is smooth at $z$.

*Proof.* This follows directly from the explicit calculation (Lemmas 4.3 and 4.8) of the Newton stratification on the formal neighborhood $\mathcal{M}^{iz}$ of $z$. \hfill \Box

The necessary calculations are somewhat sensitive to the parity of $n$. We first work out the details when $n$ is odd, and then indicate the changes necessary to accommodate even $n$. 

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4.1. The case of $n$ odd. Throughout this section, assume that $n$ is odd.

4.1.1. Explicit deformations. Suppose $z = (X, t, \lambda) \in \mathcal{M}^{\omega}(k)$. Our goal is to understand the Newton stratification on the formal neighborhood $\mathcal{M}/z = \text{Spf} \mathcal{R}$ of $z$ in $\mathcal{M}$. This will be accomplished using (covariant) Dieudonné theory. Suppose $z = (X, t, \lambda) \in \mathcal{M}^{\omega}(k)$. Then the Dieudonné module $\mathcal{D}_s(X[p^\omega])$ is isomorphic to $\mathcal{M} := M(n)$ (Theorem 2.1).

Deformations of $X[p^\omega]$ are parametrized by $\text{Hom}(VM/pM, M/VM)$; those which preserve the $\mathcal{O}_L$-structure are classified by $\text{Hom}_{\mathcal{O}_L \otimes k}(VM/pM, M/VM)$ (e.g., [1]). The display we have chosen gives coordinates on $VM/pM$ and $M/VM$:

$$VM/pM = k\{v_1, u_2, u_3, \cdots, u_n\}$$

$$M/VM = k\{u_1, v_2, v_3, \cdots, v_n\}$$

Consequently,

$$\text{Hom}_{\mathcal{O}_L \otimes k}(VM/pM, M/VM) = k\{v_1^i v_2, v_1^i v_3, \cdots, v_1^i v_n, u_2^i u_1, u_3^i u_1, \cdots, u_n^i u_1\}$$

$$\subset \text{Hom}_k(VM/pM, M/VM) = (VM/pM)^* \otimes (M/VM),$$

and the universal equicharacteristic deformation ring of $(X[p^\omega], t[p^\omega])$ is

$$\mathcal{R}' = k[[t(v_1 v_2), t(v_1 v_3), \cdots, t(v_1 v_n), t(u_2 u_1), \cdots, t(u_n u_1)]].$$

For $t(xy) \in \mathcal{R}'$, let $t(xy)$ be its Teichmuller lift to $W(\mathcal{R}')$. Then $X[p^\omega]$ is displayed over $\mathcal{R}'$ by

$$\tilde{F}u_1 = -v_n$$

$$\tilde{F}v_2 = u_1$$

$$\tilde{F}v_3 = u_2 + t(u_2 u_1)u_1$$

$$\vdots$$

$$\tilde{F}v_n = u_{n-1} + t(u_{n-1} u_1)u_1$$

The pairing $\langle \cdot, \cdot \rangle$ extends to $\tilde{M} = M \otimes_{W(k)} W(\mathcal{R}')$ by linearity. We would like to identify the largest quotient $\tilde{R}$ of $\mathcal{R}'$ to which $\langle \cdot, \cdot \rangle$ extends as a pairing of Dieudonné modules; for then $\mathcal{M}/z \cong \text{Spf} \mathcal{R}$.

The quasipolarization extends to a ring $R$ if and only if, for each $x, y \in \tilde{M}_R := \tilde{M} \otimes_{W(\mathcal{R}')} W(R)$, one has

$$\langle \tilde{F}x, y \rangle = \langle x, \tilde{V}y \rangle^\sigma.$$ 

Suppose $3 \leq j \leq n$, and let $(x, y) = (v_j, v_1 + \sum_{2 \leq k \leq n} t(v_1 v_k) v_k)$. Then

$$\langle \tilde{F}x, y \rangle = \langle u_{j-1} + t(u_{j-1} u_1) u_1, v_1 + \sum_{2 \leq k \leq n} t(v_1 v_k) v_k \rangle$$

$$= t(v_1 v_{j-1}) \langle u_{j-1}, v_{j-1} \rangle + t(u_{j-1} u_1) \langle u_1, v_1 \rangle$$

$$= (-1)^{j-1} t(v_1 v_{j-1}) - t(u_{j-1} u_1),$$

while

$$\langle x, \tilde{V}y \rangle = \langle v_j, u_2 \rangle$$

$$= 0.$$

Consequently, if $\tilde{M}_R$ is quasipolarized by $\langle \cdot, \cdot \rangle$, then for each $2 \leq k \leq n-1$, the image of $(-1)^k t(v_1 v_k) - t(u_k u_1)$ in $R$ is zero.
Similarly, by considering \((x, y) = (v_1, v_2 + \sum f(v_2 v_j)v_j)\), we see that the image of \(f(v_1 v_n)\) in such an \(R\) must be zero.

The quotient \(\overline{R}'\) by these relations is a smooth, local ring of dimension \(n - 1\), and thus we identify \(\overline{R}\) with

\[
\overline{R} = k[[s_2, \ldots, s_n]],
\]

where \(s_j\) is the image of \(t(u_j u_1)\) in \(\overline{R}\). We record these calculations as follows.

**Lemma 4.2.** The formal neighborhood \(\mathcal{M}^{/z}\) of \(z\) is isomorphic to \(\overline{R} = \text{Spf} k[[s_2, \ldots, s_n]]\). Over \(\overline{R}\), the Dieudonné module \(\overline{M} = \overline{M}_k\) of the universal deformation of \((X[p^\infty], \iota[p^\infty], \lambda[p^\infty])\) is displayed by

\[
\begin{align*}
\tilde{F}u_1 &= -v_n & v_1 &= \tilde{V}(u_n - s_n u_1) \\
\tilde{F}v_2 &= u_1 & u_2 &= \tilde{V}(v_1 + \sum_{2 \leq j \leq n} (-1)^j s_j v_j) \\
\tilde{F}v_3 &= u_2 + s_2 u_1 & u_3 &= \tilde{V}v_2 \\
\vdots & & \vdots \\
\tilde{F}v_n &= u_{n-1} + s_{n-1} u_1 & u_n &= \tilde{V}v_{n-1}
\end{align*}
\]

4.1.2. **Newton strata in local coordinates.** In this choice of coordinates, it is easy to calculate the Newton stratification on \(\mathcal{M}^{/z}\). For \(1 \leq j \leq \lfloor n/2 \rfloor\), let

\[
\mathcal{M}^{/z}_{< j} := \mathcal{M}^{/z} \cap (\mathcal{M}^o \cup \bigcup_{1 \leq i < j} \mathcal{M}^{\tilde{s}_{2i}})
\]

be the locus in \(\mathcal{M}^{/z}\) parametrizing those deformations whose first slope is strictly larger than \(\lambda(2j)\).

**Lemma 4.3.** Suppose \(1 \leq j \leq \lfloor n/2 \rfloor\). Then

\[
\mathcal{M}^{/z}_{\leq j} = \text{Spf} \frac{k[[s_2, \ldots, s_n]]}{(s_{2j}, s_{2(j+1)}, \ldots, s_{2\lfloor n/2 \rfloor})} \subseteq \mathcal{M}^{/z} = \text{Spf} k[[s_2, \ldots, s_n]].
\]

Before proceeding with the proof, we construct a graph to encode part of the structure of (a deformation of) \(M\). Initially, construct a graph \(\Gamma\) as follows (see Figure 4.1.2). With a slight abuse of notation, let the vertex set be \(\{u_1, \ldots, u_n, v_1, \ldots, v_n\}\). For \(2 \leq i \leq n\), draw a (light) gray arrow from \(v_i\) to \(u_{i-1}\), to encode the fact that \(Fv_i = u_{i-1}\). Similarly, draw a gray arrow from \(u_1\) to \(v_n\).

Also, for each \(2 \leq i \leq n\), draw a black arrow from \(u_i\) to \(v_{i-1}\), to encode the fact that \(Fu_i = pv_{i-1}\). Similarly, draw a black arrow from \(v_1\) to \(u_n\).

Note that \(\Gamma\) is a (colored) cycle. In fact, starting from vertex \(u_1\), one successively visits

\[
\{u_1, v_n, u_{n-1}, v_{n-2}, u_{n-3}, \ldots, v_1, u_n, v_{n-1}, u_{n-2}, \ldots, v_2, u_1\}.
\]
Now let $S$ be an integral domain equipped with a surjection $\phi: k[[s_2, \ldots, s_n]] \to S$, and let $K$ be the field of fractions of $S$. Construct a graph $\Gamma_S$ by (possibly) augmenting the edge set of $\Gamma$, as follows.

For each $2 \leq i \leq n - 1$, if $\phi(s_i) \neq 0$, then add a gray edge from $v_i$ to $u_1$. (For the sake of completeness, if $\phi(s_n) \neq 0$, then add a black edge from $v_n$ to $u_1$. For each $2 \leq i \leq n - 1$, if $\phi(s_i) \neq 0$, then add a black edge from $u_2$ to $v_i$. These additional black edges will not affect the final calculation.)

Let $C$ be a cycle or path in $\Gamma_S$. The length of $C$ is the number of edges in $C$, while the weight of $C$ is the number of black edges in $C$. Define the slope of $C$ to be

$$\lambda(C) = \frac{\text{weight}(C)}{\text{length}(C)}.$$

Note that for the trivial deformation, corresponding to $\Gamma$ itself, we have $\lambda(\Gamma) = \frac{n-1}{2n} = \frac{1}{2}$.

**Lemma 4.4.** If $C \subset \Gamma_S$ is a cycle through $u_1$, then the smallest slope of the Newton polygon is at most $\lambda(C)$.

**Proof.** It is harmless, and convenient, to replace $K$ by its perfection. Suppose there is a cycle $C$ of length $b$ and weight $a$; let $\tilde{N}_K = W(K)\{u_2, \ldots, u_n, v_1, \ldots, v_n\}$. Then $F^b u_1 \in p^a W(K)\{u_1\} + \tilde{N}_K$ but $F^b u_1 \notin \tilde{N}_K$, and $\tilde{M}_K/\tilde{N}_K$ is an $F$-$\sigma^a$-crystal of slope at most $a/b$. Therefore, the smallest slope of $\tilde{M}_K$ is at most $a/b$.

**Remark 4.5.** Let $B(K) = \text{Frac} W(K)$; then the $B(K)[F]$-span of $u_1$ in $\tilde{M}_K \otimes B(K)$ is all of $\tilde{M}_K \otimes B(K)$. Therefore, one can in fact show that the smallest slope of $\tilde{M}_K$ is

$$\min\limits_{C \subset \Gamma \text{ a cycle through } u_1} \lambda(C).$$

**Lemma 4.6.** If $\phi(s_{2j}) \neq 0$, then there is a cycle in $\Gamma_S$ of length $n + 1 - 2j$ and weight $\frac{n-1}{2} - j$.

**Proof.** In $\Gamma$, the unique path $P$ from $u_1$ to $v_{2j+1}$ has length $n + 1 - (2j + 1) = n - 2j$ and weight $\frac{n-(2j+1)}{2} = \frac{n-1}{2} - j$. If $\phi(s_{2j}) \neq 0$, then in $\Gamma_S$ there is a cycle, obtained by concatenating $u_1$ to $P$, of length $\text{length}(P) + 1$ and weight $\text{weight}(P)$.

**Lemma 4.7.** If the smallest slope of $\tilde{M}_K$ is greater than $\lambda(2j)$, then

$$\phi(s_{2j}) = \phi(s_{2(j+1)}) = \cdots = \phi(2\lceil n/2 \rceil) = 0.$$

**Proof.** The contrapositive follows immediately from Lemmas 4.6 and 4.4 if there is some $i \geq j$ with $\phi(s_{2i}) \neq 0$, then the smallest slope of $\tilde{M}_K$ is at most $\lambda(2i)$.

**Proof of Lemma 4.3** By Lemma 4.7, the sought-for neighborhood $\mathcal{N}_{<j}/\mathcal{M}_{<j}$ is the formal spectrum of a quotient of $R_{<j} := k[[s_1, \ldots, s_n]]/(s_{2j}, s_{2(j+1)}, \ldots, s_{2\lceil n/2 \rceil})$. We thus have $\mathcal{N}_{<j} \hookrightarrow \text{Spf } R_{<j} \hookrightarrow \mathcal{M}_{<j}/\mathcal{N}_{<j}$. Since both $\mathcal{M}_{<j}/\mathcal{N}_{<j}$ and $\text{Spf } R_{<j}$ have codimension $\lceil n/2 \rceil - j + 1$, the result follows.
4.2. The case of \( n \) even. We now indicate the changes which must be made in order to perform the calculations of Section 4.1 in the case where \( n \) is even.

Suppose \( z = (X, \iota, \lambda) \in \mathcal{M}^{\infty}(k) \). The quasipolarized Dieudonné module \( M \) of \( X[p^{\infty}] \), as a \( p \)-divisible group with \( O_{L} \)-action, is \( M(n - 1) \oplus N \) (Theorem 2.1). A calculation exactly like that in Section 4.1.1 shows \( \mathcal{M}^{/z} \cong \text{Spf} R = \text{Spf} k[[s_{0}, s_{2}, \cdots, s_{n-1}]] \); the corresponding deformation \( \bar{M} \) of \( M \) is displayed by

\[
\begin{align*}
\bar{F}v_{0} &= -u_{0} - \bar{s}_{0}u_{1} & u_{0} &= \bar{V}v_{0} \\
\bar{F}u_{1} &= -v_{n-1} & v_{1} &= \bar{V}(u_{n-1} - \bar{s}_{n-1}u_{1}) \\
\bar{F}v_{2} &= u_{1} & u_{2} &= \bar{V}(v_{1} + \sum_{2 \leq j \leq n-1} (-1)^{j}\bar{s}_{j}v_{j}) \\
\bar{F}v_{3} &= u_{2} + \bar{s}_{2}u_{1} & u_{3} &= \bar{V}v_{2} \\
\vdots \\
\bar{F}v_{n-1} &= u_{n-2} + \bar{s}_{n-2}u_{1} & u_{n-1} &= \bar{V}v_{n-2}
\end{align*}
\]

Construction and analysis of graphs \( \Gamma \) and \( \Gamma_{S} \), for quotients \( S \) of \( \bar{R} \), shows that Lemma 4.3 holds for even \( n \), too:

**Lemma 4.8.** Suppose \( 1 \leq j \leq n/2 \). Then

\[
\mathcal{M}^{/z} \cap (\cup_{1 \leq i \leq j} \mathcal{M}^{s_{2i}}) = \text{Spf} \frac{k[[s_{0}, s_{2}, s_{3}, \cdots, s_{n}]]}{(s_{2j}, s_{2(j+1)}, \cdots, s_{n})}.
\]

5. HECKE ORBITS FOR THE SUPERSINGULAR LOCUS

**Lemma 5.1.** Let \( \mathbb{K}^{p} \subset G(A_{f}^{p}) \) be a compact open subgroup. The \( \text{U}(A_{f}^{p}) \)-Hecke operators act transitively on \( \Pi_{0}(\mathcal{M}^{p}_{K^{p}}) \).

**Proof.** Let \( (X, \iota, \lambda) = \bar{\eta} \) be a geometric generic point of \( \mathcal{M}^{p}_{K^{p}} \). The central leaf \( C([\bar{\eta}]) \), which in a general PEL Shimura variety context parametrizes those \((Y, j, \mu)\) with \((Y[p^{\infty}], j[p^{\infty}], \mu[p^{\infty}]) \cong (X[p^{\infty}], j[p^{\infty}], \lambda[p^{\infty}])\), in this case coincides with (the union of a choice of geometric point over each generic point of) \( \mathcal{M}^{\infty} \). There is an a priori inclusion \( \mathcal{H}^{p}(\bar{\eta}) \subseteq C([\bar{\eta}]) \). Since \( \bar{\eta} \) is basic and \( G \), the reductive group defining \( \mathcal{M} \), has simply connected derived group, the prime-to-\( p \) Hecke orbit of \( \bar{\eta} \) coincides with the central leaf \( C([X[p^{\infty}], j[p^{\infty}], \lambda[p^{\infty}]] \) [12] Thm. 4.6(1) and Rem. 4.7(3)].

6. IRREDUCIBILITY OF NEWTON STRATA

**Proof of Theorem 1.1.** Chai and Oort identify nine steps in their proof of [4] Thm. 3.1], which is the analogue for \( A_{g}^{p} \) of Theorem 1.1. We proceed here in a similar fashion. Fix an open compact subgroup \( \mathbb{K}^{p} \subset G(A_{f}^{p}) \); it suffices to prove that \( \mathcal{M}^{p}_{K^{p}} \) is irreducible.

**Steps 1-6:** By Lemma 3.3 there is a well-defined map of sets

\[
\Pi_{0}(\mathcal{M}^{p}) \longrightarrow \Pi_{0}(\mathcal{M}^{\infty}).
\]

It is surjective, by Lemma 3.2. From it, we deduce the existence of a surjective

\[
\Pi_{0}(\mathcal{M}^{p}_{K^{p}}) \longrightarrow \Pi_{0}(\mathcal{M}^{\infty}_{K^{p}}),
\]

visibly \( \text{U}(A_{f}^{p}) \)-equivariant.

**Steps 7-8:** By Lemma 5.1 the action of \( \text{U}(A_{f}^{p}) \) on \( \Pi_{0}(\mathcal{M}^{p}_{K^{p}}) \) is transitive.
Step 9: Taken together, this shows that $U(A_f^p)$ acts transitively on $\Pi_0(\mathcal{M}_{K_0}^\epsilon)$. By [6, Thm. 1.4], which is the PEL analogue of [3], $\mathcal{M}_{K_0}^\epsilon$ is connected. Since $\mathcal{M}_{K_0}^\epsilon$ is also smooth (Lemma [3.1]), it is irreducible.

□

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