Finite groups in which the $\mathcal{X}$-maximal subgroups are conjugate

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Abstract
Let $\mathcal{X}$ be a class of finite groups closed under taking the subgroups, homomorphic images and extensions. By $\mathcal{D}_\mathcal{X}$ denote the class of finite groups $G$ in which every two $\mathcal{X}$-maximal subgroups are conjugate. In the paper, the following statement is proven. Let $A$ be a normal subgroup of a finite group $G$. Then
$$G \in \mathcal{D}_\mathcal{X} \text{ if and only if } A \in \mathcal{D}_\mathcal{X} \text{ and } G/A \in \mathcal{D}_\mathcal{X}.$$ This statement implies that the $\mathcal{X}$-maximal subgroups are conjugate if and only if the so called $\mathcal{X}$-submaximal subgroups are conjugate. Thus we obtain an affirmative solution to a problem posed by H.Wielandt in 1964.

Key words: $\mathcal{X}$-maximal subgroup, $\mathcal{X}$-submaximal subgroup, $\pi$-Hall subgroup, Sylow $\pi$-theorem, $\mathcal{D}_\mathcal{X}$-group, $\mathcal{D}_\pi$-group.

1 Introduction
1.1 Mains concepts: $\mathcal{X}$-maximal and $\mathcal{X}$-submaximal subgroups. History and problems
In the paper we consider only finite groups, $G$ always denotes a finite group, and $\pi$ is a set of primes.

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According to H.Wielandt, a class of finite groups is said to be *complete* if it is non-empty and closed under taking subgroups, homomorphic images and extensions\footnote{Recall that a group $G$ is an *extension* of a group $A$ by a group $B$ if there is an epimorphism $G \to B$ with the kernel isomorphic to $A$. Thus a class $\mathcal{X}$ is closed under taking the extensions if $N \in \mathcal{X}$ and $G/N \in \mathcal{X}$ imply $G \in \mathcal{X}$ for any group $G$ and its normal subgroup $N$.}. Moreover, $\mathcal{X}$ always denotes some given complete class. Examples of complete classes are

- $\mathcal{G}$, the class of all finite groups;
- $\mathcal{S}$, the class of all finite solvable groups.

The following two classes are among the most important examples of complete classes:

- $\mathcal{G}_\pi$, the class of all $\pi$-groups for a set $\pi$ of primes (i.e. the class of all groups $G$ such that every prime divisor of $|G|$ belongs to $\pi$);
- $\mathcal{S}_\pi$, the class of all solvable $\pi$-groups for a set $\pi$ of primes.

In fact, these two cases are extremal for every $\mathcal{X}$. If we denote by $\pi(\mathcal{X}) = \pi(\mathcal{X})$ the union of the sets of prime divisors of $|G|$, where $G$ runs through $\mathcal{X}$, then\footnote{First inclusion follows from the Sylow theorem and from the solvability of groups of prime power order. The second one is obvious.}

$$\mathcal{S}_\pi \subseteq \mathcal{X} \subseteq \mathcal{G}_\pi.$$  

For given group $G$ we denote by $m_\mathcal{X}(G)$ the set of $\mathcal{X}$-*maximal* subgroups of $G$, i.e. the set of all maximal members of $\{H \leq G \mid H \in \mathcal{X}\}$ with respect to inclusion.

Thus, the $\mathcal{X}$-subgroups of $G$ (i.e. subgroups of $G$ belonging to $\mathcal{X}$) are exactly the subgroups of members of $m_\mathcal{X}(G)$. One of the fundamental problems in the finite group theory is: given a group $G$ and a complete class $\mathcal{X}$, to determine $m_\mathcal{X}(G)$.

In case $\pi(\mathcal{X}) = \{p\}$, this problem is solved by the Sylow theorem: the order of every $\mathcal{X}$-maximal subgroup of $G$ equals the greatest power of $p$ dividing $|G|$ and all $\mathcal{X}$-maximal subgroups of $G$ are conjugate. Recall that a $\pi$-subgroup $H$ of a group $G$ is called a $\pi$-*Hall* subgroup, if its index $|G:H|$ is not divisible by primes from $\pi$. The Hall theorem\footnote{First inclusion follows from the Sylow theorem and from the solvability of groups of prime power order. The second one is obvious.} says that a complete analogue of the Sylow theorem for $\pi$-Hall subgroups in solvable groups holds, i.e. for any set $\pi$ of primes, the $\pi$-maximal subgroups of a solvable group $G$ are exactly $\pi$-Hall subgroups and they are conjugate. Thus, for a solvable group $G$, the set $m_\mathcal{X}(G)$ coincides with the set $m_\pi(G)$ of $\pi$-maximal subgroups of $G$, where $\pi = \pi(\mathcal{X})$, and members of $m_\pi(G)$ are precisely $\pi$-Hall subgroups.

For a group $G$, the existence of $\pi$-Hall subgroups for all sets $\pi$ is equivalent to the solvability of $G$\footnote{Recall that a group $G$ is an *extension* of a group $A$ by a group $B$ if there is an epimorphism $G \to B$ with the kernel isomorphic to $A$. Thus a class $\mathcal{X}$ is closed under taking the extensions if $N \in \mathcal{X}$ and $G/N \in \mathcal{X}$ imply $G \in \mathcal{X}$ for any group $G$ and its normal subgroup $N$.}. This means that for every non-solvable group $G$ there exists $\pi$ such that $G$ has more than one conjugacy class of $\pi$-maximal subgroups and these subgroups are not $\pi$-Hall subgroups.

Although $\pi$-Hall subgroups in non-solvable groups may not exist, they have nice properties and are well studied by now (see survey \cite{survey}). In particular, it is known that

\begin{itemize}
  \item[(*)] if $N$ is a normal and $H$ is a $\pi$-Hall subgroup of $G$ then $H \cap N$ is a $\pi$-Hall subgroup of $N$ and $HN/N$ is a $\pi$-Hall subgroup of $G/N$ (see Lemma \cite{lemma} below).
\end{itemize}

Consequently, the existence of a $\pi$-Hall subgroup in a group implies that every composition factor of the group possesses a $\pi$-Hall subgroup as well. The converse statement is not true in general. A criterion for the existence of $\pi$-Hall subgroups in a group $G$ is formulated
in terms of so-called groups of $G$-induced automorphisms of composition factors of $G$ (see [11, 13]). A classification of $\pi$-Hall subgroups in almost simple groups \footnote{Recall that $G$ is almost simple if its socle is a nonabelian simple group} is required in order to apply the criterion of existence. There exist a lot of papers dedicated to the classification of $\pi$-Hall subgroups in almost simple groups. First steps were made by P.Hall \footnote{A subgroup $H$ of a group $G$ is said to be subnormal if there is a series of subgroups such that $G_i$ is normal in $G_{i-1}$ for $i = 1, \ldots, n$.} and J.Thompson \footnote{Hartley’s \cite{22} and Shemetkov’s \cite{21} results are proved for $\mathfrak{X} = \mathfrak{S}_\pi$.}, who classified solvable and nonsolvable Hall subgroups in symmetric groups respectively. The reader can find the bibliography and the results in the survey paper \cite{46}.

In contrast with $\pi$-Hall subgroups, $\mathfrak{X}$-maximal subgroups have no properties similar to (⋆) even for $\mathfrak{X} = \mathfrak{S}_\pi$. In fact, an analog of (⋆) is not true for homomorphic images, since H.Wielandt in \cite{18, 19} note: if $A$ contains more than one conjugacy class of $\mathfrak{X}$-maximal subgroups, $B$ is a group and $G = A \wr B$ is the regular wreath product, then every $\mathfrak{X}$-subgroup of $B$ is the image of an $\mathfrak{X}$-maximal subgroup under the natural epimorphism $G \to B$. Also the intersection of an $\mathfrak{X}$-maximal subgroup $H$ with a normal subgroup $N$ of $G$ is not an $\mathfrak{X}$-maximal in $N$ in general. For example, a Sylow 2-subgroup $H$ of $G = PGL_2(7)$ is \{2, 3\}-maximal in $G$ but $H \cap N \notin \mathfrak{m}_{\{2,3\}}(N)$ for $N = PSL_2(7)$.

In his lectures \cite{19} and in his plenary talk at the famous conference on finite groups in Santa-Cruz (USA) in 1979 \cite{18}, Wielandt put forward a program on how to study $\mathfrak{X}$-maximal subgroups of finite groups by using $\mathfrak{X}$-submaximal subgroups. Recall the Wielandt–Hartley theorem first.

**Proposition 1** (Wielandt and Hartley) Let $N$ be a subnormal \footnote{In \cite{48} $G^\phi$ is required to be normal in $G^*$.} subgroup of a group $G$ and $H \in \mathfrak{m}_\mathfrak{X}(G)$. Then $H \cap N = 1$ if and only if $N$ is a $\pi'$-group, where $\pi'$ is the complement to $\pi = \pi(\mathfrak{X})$ in the set of all primes.

In the case when $N$ normal, Wielandt’s proof of this statement can be found in \cite{13, 13.2], and Hartley’s proof in \cite{28, Lemmas 2 and 3}. For the general case see \cite{42, Theorem 7} and \cite{20, Proposition 8].

In light of Proposition 1, it is natural to consider the following concept.

**Definition 1** According to Wielandt (see \cite{18}), a subgroup $H$ of a group $G$ is called an $\mathfrak{X}$-submaximal subgroup, if there is a monomorphism $\phi: G \to G^*$ into a group $G^*$ such that $G^\phi$ is subnormal \footnote{In \cite{48} $G^\phi$ is required to be normal in $G^*$.} in $G^*$ and $H^\phi = K \cap G^\phi$ for an $\mathfrak{X}$-maximal subgroup $K$ of $G^*$. We denote the set of $\mathfrak{X}$-submaximal subgroups of $G$ by $\mathfrak{m}_\mathfrak{X}(G)$.

Evidently, $\mathfrak{m}_\mathfrak{X}(G) \subseteq \mathfrak{sm}_\mathfrak{X}(G)$ for any group $G$. The inverse inclusion does not hold in general: any Sylow 2-subgroup of $PSL_2(7)$ is \{2, 3\}-submaximal but is not \{2, 3\}-maximal.

The importance of the classification of $\mathfrak{X}$-submaximal subgroups in simple groups is explained in \cite{21, 24}: the classification would be a crucial ingredient in finding $\mathfrak{X}$-maximal subgroups in arbitrary nonsolvable group. In \cite{23} the classification of $\mathfrak{X}$-submaximal subgroups in minimal non-solvable groups is obtained.

As we mention above, if $M \in \mathfrak{m}_\mathfrak{X}(G)$ and $N \leq G$, then $MN/N$ may not lie in $\mathfrak{m}_\mathfrak{X}(G/N)$ in general. An important part of Wielandt’s program is to find necessary and sufficient conditions for such subgroups $N$. In \cite{48} Wielandt put forward a program on how to study $\mathfrak{X}$-maximal subgroups of finite groups by using $\mathfrak{X}$-submaximal subgroups. Recall the Wielandt–Hartley theorem first.
conditions on $N$ making the correspondence

$$M \mapsto MN/N$$

to be a (surjective) map from $m_X(G)$ to $m_X(G/N)$. Wielandt shows [19, 15.4] that conjugateness of all $X$-submaximal subgroups in $N$ is a sufficient condition. Moreover, if the $X$-submaximal subgroups in $N$ are conjugate, then $M \mapsto MN/N$ induces a bijection between the sets of conjugacy classes of $X$-maximal subgroups of $G$ and $G/N$. Immediately after this statement, Wielandt puts forward the following problem.

**Problem 1** (H. Wielandt, [19, offene Frage zu 15.4]) Whether the conjugateness of the $X$-maximal subgroups of a finite group $G$ implies the conjugateness of the $X$-submaximal subgroups $G$?

It was proved in [21, Theorem 2] that this problem can be equivalently reformulated as Problem 2 below. We need the concept of a $D_X$-group introduced in [21] which plays an important role in this article.

**Definition 2** A finite group $G$ is a $D_X$-group (we say also that $G$ belongs to $D_X$ and write $G \in D_X$) if every two $X$-maximal subgroups of $G$ are conjugate. If $X = \mathfrak{G}_\pi$ is the class of all $\pi$-groups, then we write $D_\pi$ instead of $D_X$.

**Problem 2** Is an extension of a $D_X$-group by a $D_X$-group always a $D_X$-group?

A particular case of Problem 2 for $X = \mathfrak{G}_\pi$ was first stated in the one-hour talk by Wielandt at 13th International Congress of Mathematicians in Edinburgh in 1958 [50]. The problem is mentioned in surveys [7, 43, 51] and in text-books [18, 19, 41, 44], and was also included by L. Shemetkov into the “Kourovka Notebook” [32, Problem 3.62]. Now, Problem 2 for $X = \mathfrak{G}_\pi$ is solved in the affirmative (see [46, Theorem 6.6]).

Wielandt [19, 15.6] note: if $G$ contains a nilpotent $\pi'$-Hall subgroup for $\pi = \pi(X)$, then $G$ has exactly one conjugacy class of $X$-submaximal subgroups. At the same section he asks [19, offene Frage, Seite 37 (643)]: does there exist a group containing a non-nilpotent maximal $X$-subgroup with the unique conjugacy class of $X$-submaximal subgroups? The following example allows to construct such groups with non-nilpotent maximal $X$-subgroups. Assume $G$ possesses a normal series

$$G = G_0 > G_1 > \cdots > G_n = 1$$

such that, for every $i = 1, \ldots, n$, either $G_{i-1}/G_i \in X$, or $G_{i-1}/G_i$ is a $\pi'$-group. Then all $X$-submaximal subgroups are conjugate in $G$ (this follows by Lemma 2.8 below), but $X$-maximal subgroups (in this case they appear to be the $\pi$-Hall subgroups) of $G$ are not nilpotent, in general. Therefore, the following interpretation of above mentioned Wielandt’s question seems to be more relevant:

**Problem 3** (H. Wielandt, [19, offene Frage, Seite 37 (643)]) In what groups all $X$-submaximal subgroups are conjugate?

The goal of this article is to solve Problems 1, 2, and 3.
1.2 Mains results

In [20, 21] the study of Problems 1 and 2 is reduced to the case of simple groups (see also [24]). We obtain the solutions to Problems 1, 2, and 3 as consequences of the following theorem.

**Theorem 1** Let $X$ be a complete class of finite groups, $\pi = \pi(X)$, and let $G$ be a finite simple group. Then $G \in D_X$ if and only if either $G \in X$ or $\pi(G) \not\subseteq \pi$ and $G \in D_\pi$. In particular, if $G \in D_X$, then $G \in D_\pi$.

The following statement solves Problem 2 in the affirmative.

**Corollary 1.1** Let $X$ be a complete class of finite groups. Assume, $N$ is a normal subgroup of $G$. Then $G \in D_X$ if and only if $N \in D_X$ and $G/N \in D_X$.

Since Problem 1 is equivalent to Problem 2, we obtain next Corollary which solves Problem 1 in the affirmative.

**Corollary 1.2** Let $X$ be a complete class of finite groups. Then the conjugateness of the $X$-maximal subgroups of a finite group is equivalent to the conjugateness of the $X$-submaximal subgroups.

In view of Corollary 1.2, $X$-maximal subgroups in a finite group are conjugate if and only if $X$-submaximal subgroups are conjugate. Thus $X$-maximal subgroups in a finite group are conjugate if and only if $X$-submaximal subgroups in the sense of [49] are conjugate, and so Corollary 1.2 provides an affirmative answer to Problem 1.

**Corollary 1.3** Let $X$ be a complete class of finite groups. Assume, $N \in D_X$ is a normal subgroup of $G$. Then

$$M \mapsto MN/M$$

surjectively maps $m_X(G)$ onto $m_X(G/N)$ and induces a bijection between the conjugacy classes of $X$-maximal subgroups of $G$ and $G/N$. Moreover,

$$M \mapsto M \cap N$$

surjectively maps $m_X(G)$ onto $m_X(N) = sm_X(N)$.

Note that, under assumption of Corollary 1.3, the set $m_X(N) = sm_X(N)$ coincides with Hall$_\pi(N)$, where $\pi = \pi(X)$ (see Corollary 1.6 below).

According to Wielandt [49], denote by $k_X(G)$ the number of conjugacy classes of $X$-maximal subgroups of $G$. We can join statements of Corollaries 1.2 and 1.3 in the following way.

**Corollary 1.4** Let $X$ be a complete class and $G$ be a finite group. Then the following statements are equivalent:

1. $G \in D_X$.
2. The $X$-submaximal subgroups of $G$ are conjugate.
3. $k_X(A) = k_X(A/B)$ for every finite group $A$ containing a normal subgroup $B$ isomorphic to $G$.
Corollary 1.5 Let $\mathcal{X}$ be a complete class of finite groups, $\pi = \pi(\mathcal{X})$. Then $\mathcal{D}_\mathcal{X} \subseteq \mathcal{D}_\pi$.

Corollary 1.6 Let $\mathcal{X}$ be a complete class of finite groups, $\pi = \pi(\mathcal{X})$. Then every $\pi$-subgroup of a $\mathcal{D}_\mathcal{X}$-group is an $\mathcal{X}$-group. In particular, $m_\mathcal{X}(G) = \text{Hall}_\pi(G)$.

Corollary 1.7 Let $\mathcal{X}$ be a complete class of finite groups, $G \in \mathcal{D}_\mathcal{X}$, $H \in m_\mathcal{X}(G)$ and $H \leq M \leq G$. Then $M \in \mathcal{D}_\mathcal{X}$. In particular, $m_\mathcal{X}(M) \subseteq m_\mathcal{X}(G)$.

The next consequence of Theorem [4] gives an exhaustive solution to Problem 3.

Corollary 1.8 Let $\mathcal{X}$ be a complete class of finite groups. Then, for a finite group $G$, the following statements are equivalent:

(1) All $\mathcal{X}$-submaximal subgroups are conjugate;

(2) All $\mathcal{X}$-maximal subgroups are conjugate (i.e. $G \in \mathcal{D}_\mathcal{X}$);

(3) For every composition factor $S$ of $G$ either $S \in \mathcal{X}$ or pair $(S, \pi)$ satisfies one of Conditions I–VII below, where $\pi = \pi(\mathcal{X})$.

**Condition I.** We say that $(S, \pi)$ satisfies Condition I if $|\pi \cap \pi(S)| \leq 1$.

**Condition II.** We say that $(S, \pi)$ satisfies Condition II if one of the following cases holds.

(1) $S \cong M_{11}$ and $\pi \cap \pi(S) = \{5, 11\}$;

(2) $S \cong M_{12}$ and $\pi \cap \pi(S) = \{5, 11\}$;

(3) $S \cong M_{22}$ and $\pi \cap \pi(S) = \{5, 11\}$;

(4) $S \cong M_{23}$ and $\pi \cap \pi(S)$ coincide with one of the following sets $\{5, 11\}$ and $\{11, 23\}$;

(5) $S \cong M_{24}$ and $\pi \cap \pi(S)$ coincide with one of the following sets $\{5, 11\}$ and $\{11, 23\}$;

(6) $S \cong J_1$ and $\pi \cap \pi(S)$ coincide with one of the following sets $\{3, 5\}$, $\{3, 7\}$, $\{3, 19\}$, and $\{5, 11\}$;

(7) $S \cong J_4$ and $\pi \cap \pi(S)$ coincide with one of the following sets $\{5, 7\}$, $\{5, 11\}$, $\{5, 31\}$, $\{7, 29\}$, and $\{7, 43\}$;

(8) $S \cong O'N$ and $\pi \cap \pi(S)$ coincide with one of the following sets $\{5, 11\}$ and $\{5, 31\}$;

(9) $S \cong L_6$ and $\pi \cap \pi(S) = \{11, 67\}$;

(10) $S \cong R_6$ and $\pi \cap \pi(S) = \{7, 29\}$;

(11) $S \cong Co_1$ and $\pi \cap \pi(S) = \{11, 23\}$;

(12) $S \cong Co_2$ and $\pi \cap \pi(S) = \{11, 23\}$;

(13) $S \cong Co_3$ and $\pi \cap \pi(S) = \{11, 23\}$;

(14) $S \cong M(23)$ and $\pi \cap \pi(S) = \{11, 23\}$;

(15) $S \cong M(24)'$ and $\pi \cap \pi(S) = \{11, 23\}$;

(16) $S \cong B$ and $\pi \cap \pi(S)$ coincide with one of the following sets $\{11, 23\}$ and $\{23, 47\}$;
(17) \( S \cong M \) and \( \pi \cap \pi(S) \) coincide with one of the following sets \( \{23, 47\} \) and \( \{29, 59\} \).

**Condition III.** Let \( S \) be isomorphic to a group of Lie type over the field \( \mathbb{F}_q \) of characteristic \( p \in \pi \) and let \( \tau = (\pi \cap \pi(S)) \setminus \{p\} \). We say that \((S, \pi)\) satisfies Condition III if \( \tau \subseteq \pi(q-1) \) and every prime in \( \pi \) does not divide the order of the Weyl group of \( S \).

In order to formulate Conditions IV and V, we need the following notation. If \( r \) is an odd prime and \( q \) is an integer not divisible by \( r \), then \( e(q, r) \) is the smallest positive integer \( e \) with \( q^e \equiv 1 \mod r \).

**Condition IV.** Let \( S \) be isomorphic to a group of Lie type with the base field \( \mathbb{F}_q \) of characteristic \( p \). Let \( 2, p \not\in \pi \). Denote by \( r \) the minimum in \( \pi \cap \pi(S) \) and let \( \tau = (\pi \cap \pi(S)) \setminus \{r\} \) and \( a = e(q, r) \). We say that \((S, \pi)\) satisfies Condition IV if there exists \( t \in \tau \) with \( b = e(q, t) \neq a \) and one of the following statements holds.

1. \( S \cong A_{n-1}(q), a = r - 1, b = r, (q^{r-1} - 1)_r = r, [\frac{n}{r-1}] = a, \) and \( e(q, s) = b \) for every \( s \in \tau \);
2. \( S \cong A_{n-1}(q), a = r - 1, b = r, (q^{r-1} - 1)_r = r, [\frac{n}{r-1}] = a, n \equiv -1 \mod r, \) and \( e(q, s) = b \) for every \( s \in \tau \);
3. \( S \cong ^2A_{n-1}(q), r \equiv 1 \mod 4, a = r - 1, b = 2r, (q^{r-1} - 1)_r = r, [\frac{n}{r-1}] = a, \) and \( e(q, s) = b \) for every \( s \in \tau \);
4. \( S \cong ^2A_{n-1}(q), r \equiv 3 \mod 4, a = \frac{r-1}{2}, b = 2r, (q^{r-1} - 1)_r = r, [\frac{n}{r-1}] = a, n \equiv -1 \mod r, \) and \( e(q, s) = b \) for every \( s \in \tau \);
5. \( S \cong ^2A_{n-1}(q), r \equiv 1 \mod 4, a = r - 1, b = 2r, (q^{r-1} - 1)_r = r, [\frac{n}{r-1}] = a, n \equiv -1 \mod r, \) and \( e(q, s) = b \) for every \( s \in \tau \);
6. \( S \cong ^2D_n(q), a \equiv 1 \mod 2, n = b = 2a \) and for every \( s \in \tau \) either \( e(q, s) = a \) or \( e(q, s) = b \);
7. \( S \cong ^2D_n(q), b \equiv 1 \mod 2, n = a = 2b \) and for every \( s \in \tau \) either \( e(q, s) = a \) or \( e(q, s) = b \).

**Condition V.** Let \( S \) be isomorphic to a group of Lie type with the base field \( \mathbb{F}_q \) of characteristic \( p \). Suppose, \( 2, p \not\in \pi \). Let \( r \) be the minimum in \( \pi \cap \pi(S) \), let \( \tau = (\pi \cap \pi(S)) \setminus \{r\} \) and \( c = e(q, r) \). We say that \((S, \pi)\) satisfies Condition V if \( e(q, t) = c \) for every \( t \in \tau \) and one of the following statements holds.

1. \( S \cong A_{n-1}(q) \) and \( n < cs \) for every \( s \in \tau \);
2. \( S \cong ^2A_{n-1}(q), c \equiv 0 \mod 4 \) and \( n < cs \) for every \( s \in \tau \);
3. \( S \cong ^2A_{n-1}(q), c \equiv 2 \mod 4 \) and \( 2n < cs \) for every \( s \in \tau \);
4. \( S \cong ^2A_{n-1}(q), c \equiv 1 \mod 2 \) and \( n < 2cs \) for every \( s \in \tau \);
5. \( S \) is isomorphic to one of the groups \( B_n(q), C_n(q), \) or \( ^2D_n(q) \), \( c \) is odd and \( 2n < cs \) for every \( s \in \tau \);
(6) $S$ is isomorphic to one of the groups $B_n(q)$, $C_n(q)$, or $D_n(q)$, $c$ is even and $n < cs$ for every $s \in \tau$;
(7) $S \cong D_n(q)$, $c$ is even and $2n \leq cs$ for every $s \in \tau$;
(8) $S \cong \mathcal{S} D_n(q)$, $c$ is odd and $n \leq cs$ for every $s \in \tau$;
(9) $S \cong \mathcal{S} D_4(q)$;
(10) $S \cong E_6(q)$, and if $r = 3$ and $c = 1$ then $5, 13 \not\in \tau$;
(11) $S \cong \mathcal{S} E_6(q)$, and if $r = 3$ and $c = 2$ then $5, 13 \not\in \tau$;
(12) $S \cong E_7(q)$, if $r = 3$ and $c \in \{1, 2\}$ then $5, 7, 13 \not\in \tau$, and if $r = 5$ and $c \in \{1, 2\}$ then $7 \not\in \tau$;
(13) $S \cong E_8(q)$, if $r = 3$ and $c \in \{1, 2\}$ then $5, 7, 13 \not\in \tau$, and if $r = 5$ and $c \in \{1, 2\}$ then $7, 31 \not\in \tau$;
(14) $S \cong G_2(q)$;
(15) $S \cong F_4(q)$, and if $r = 3$ and $c = 1$ then $13 \not\in \tau$.

**Condition VI.** We say that $(S, \pi)$ satisfies Condition VI if one of the following statements holds.

1. $S$ is isomorphic to $\mathcal{S} B_2(2^{2m+1})$ and $\pi \cap \pi(S)$ is contained in one of the sets
   $$\pi(2^{2m+1} - 1), \quad \pi(2^{2m+1} \pm 2^{m+1} + 1);$$
2. $S$ is isomorphic to $\mathcal{S} G_2(3^{2m+1})$ and $\pi \cap \pi(S)$ is contained in one of the sets
   $$\pi(3^{2m+1} - 1) \setminus \{2\}, \quad \pi(3^{2m+1} \pm 3^{m+1} + 1) \setminus \{2\};$$
3. $S$ is isomorphic to $\mathcal{S} F_4(2^{2m+1})$ and $\pi \cap \pi(S)$ is contained in one of the sets
   $$\pi(2^{2(2m+1)} \pm 1), \quad \pi(2^{2m+1} \pm 2^{m+1} + 1),$$
   $$\pi(2^{2(2m+1)} \pm 2^{3m+2} \pm 2^{m+1} - 1), \quad \pi(2^{2(2m+1)} \pm 2^{3m+2} \pm 2^{m+1} \pm 2^{m+1} - 1).$$

**Condition VII.** Let $S$ be isomorphic to a group of Lie type with the base field $\mathbb{F}_q$ of characteristic $p$. Suppose that $2 \in \pi$ and $3, p \not\in \pi$, and let $\tau = (\pi \cap \pi(S)) \setminus \{2\}$ and $\varphi = \{t \in \tau \mid t$ is a Fermat number$\}$. We say that $(S, \pi)$ satisfies Condition VII if $\tau \subseteq \pi(q - \varepsilon)$, where the number $\varepsilon = \pm 1$ is such that $4$ divides $q - \varepsilon$, and one of the following statements holds.

1. $S$ is isomorphic to either $A_n(q)$ or $\mathcal{S} A_n(q)$, $s > n$ for every $s \in \tau$, and $t > n + 1$ for every $t \in \varphi$;
2. $S \cong B_n(q)$, and $s > 2n + 1$ for every $s \in \tau$;
3. $S \cong C_n(q)$, $s > n$ for every $s \in \tau$, and $t > 2n + 1$ for every $t \in \varphi$;
4. $S$ is isomorphic to either $D_n(q)$ or $\mathcal{S} D_n(q)$, and $s > 2n$ for every $s \in \tau$;
(5) $S$ is isomorphic to either $G_2(q)$ or $^2G_2(q)$, and $7 \notin \tau$;

(6) $S \cong F_4(q)$ and $5, 7 \notin \tau$;

(7) $S$ is isomorphic to either $E_6(q)$ or $^2E_6(q)$, and $5, 7 \notin \tau$;

(8) $S \cong E_7(q)$ and $5, 7, 11 \notin \tau$;

(9) $S \cong E_8(q)$ and $5, 7, 11, 13 \notin \tau$;

(10) $S \cong ^3D_4(q)$ and $7 \notin \tau$.

Conditions I–VII appear in [33–36], see also [46, Theorem 6.9 and Appendix 2], as necessary and sufficient ones for a simple group $S$ to satisfy $D_\pi$. Note that Condition I here differs from Condition I in [34,46]. In these articles Condition I include case $\pi(S) \subseteq \pi$. But a $\pi$-group is not an $X$-group, in general.

Now we return to the problem of determining of $X$-maximal subgroups in a finite group. It follows from results of the paper that every finite group $G$ has the $D_X$-radical, i.e. the greatest normal $D_X$-subgroup $N$. This subgroup coincides with the subgroup generated by all subnormal subgroups $U$ of $G$ such that every composition factor of $U$ either is an $X$-group or satisfies one of Conditions I–VII. In view of Corollary 1.3, there is a bijection between the sets conjugacy classes of $X$-maximal subgroups in $G$ and $G/N$, and we can study the members of $m_X(G/N)$ instead of $m_X(G)$.

2 Preliminaries

2.1 Notation

According to [1, 3, 10], we use the following notation.

$\varepsilon$ and $\eta$ always denote either $+1$ or $-1$ and the sign of this number. Sometimes (in the notation of orthogonal groups of odd dimension) $\eta$ can be used as an empty symbol.

$\varepsilon(q)$ denotes $\varepsilon \in \{+1, -1\}$ (or the sign of $\varepsilon$) such that $q \equiv \varepsilon \pmod{4}$ for given odd $q$.

$n$ denotes the cyclic group of order $n$, where $n$ is a positive integer.

$A^n$ denotes the direct product of $n$ copies of $A$. In particular,

$p^n$ denotes the elementary abelian group of order $p^n$, where $p$ is a prime.

$\text{Sym}(\Omega)$ denotes the symmetric group on $\Omega$.

$\text{Sym}_n$ is the symmetric group of degree $n$, i.e. $\text{Sym}_n = \text{Sym}(\Omega)$, where $\Omega = \{1, 2, \ldots, n\}$.

$\text{Alt}_n$ denotes the alternating group of degree $n$.

$\text{GL}_n(q)$ or $\text{GL}_n^+(q)$ denotes the general linear group of degree $n$ over a field of order $q$.

$\text{SL}_n(q)$ or $\text{SL}_n^+(q)$ denotes the special linear group of degree $n$ over a field of order $q$. 

$\text{PSL}_n(q)$ or $\text{PSL}_n^+(q)$ denotes the projective special linear group of degree $n$ over a field of order $q$.

$\text{PGL}_n(q)$ or $\text{PGL}_n^+(q)$ denotes the projective general linear group of degree $n$ over a field of order $q$.

$\text{GU}_n(q)$ or $\text{GL}_n^-(q)$ denotes the general unitary group of degree $n$ over a field of order $q$.

$\text{SU}_n(q)$ or $\text{SL}_n^-(q)$ denotes the special unitary group of degree $n$ over a field of order $q$.

$\text{PSU}_n(q)$ or $\text{PSL}_n^-(q)$ denotes the projective special unitary group of degree $n$ over a field of order $q$.

$\text{PGU}_n(q)$ or $\text{PGL}_n^-(q)$ denotes the projective general unitary group of degree $n$ over a field of order $q$.

$\text{O}_n^\varepsilon(q)$ is the orthogonal group of degree $n$ over a field of order $q$, where $\varepsilon \in \{ +1, -1 \}$ for $n$ even and $\varepsilon$ is an empty symbol for $n$ odd.

$\text{SO}_n^\varepsilon(q)$ is $\text{O}_n^\varepsilon(q) \cap \text{SL}_n(q)$, the special orthogonal group of degree $n$ over a field of order $q$.

$\Omega_n^\varepsilon(q)$ is the derived subgroup of $\text{SO}_n^\varepsilon(q)$.

$\text{PG}_n(q)$ is the reduction of $\Omega_n^\varepsilon(q)$ modulo scalars.

$\text{Sp}_n(q)$ denotes the symplectic group of even degree $n$ over a field of order $q$.

$\text{PSp}_n(q)$ denotes the projective symplectic group of even degree $n$ over a field of order $q$.

$r^{1+2n}$ denotes an extra special group of order $r^{1+2n}$, where $r$ is a prime.

$A : B$ means a split extension of a group $A$ by a group $B$ ($A$ is normal).

$A \cdot B$ means a non-split extension of a group $A$ by a group $B$ ($A$ is normal).

$A . B$ means an arbitrary (split or non-split extension) of a group $A$ by a group $B$ ($A$ is normal).

$A^{m+n}$ means $A^m : A^n$.

$\text{PG}$, for a linear group $G$, means the reduction of $G$ modulo scalars.

$\mathcal{X}$ is a complete class of groups.

$\mathcal{G}$ is a class of all solvable groups.

$\mathcal{D}_\mathcal{X}$ is a class of groups with all maximal $\mathcal{X}$-subgroups conjugate.

$\pi$ is a set of primes.

$\mathfrak{G}_\pi$ is a class of all solvable $\pi$-groups

$\mathfrak{G}_\pi$ is a class of all $\pi$-groups.
\( D_\pi \) is a class of groups with all maximal \( \pi \)-subgroups conjugate, i.e. \( D_\pi = D_{\phi_\pi} \).

\( E_\pi \) is a class of groups possessing \( \pi \)-Hall subgroups, i.e. \( G \in E_\pi \) if Hall_\( \pi \)(\( G \)) is nonempty.

\( G_X \) means the \( X \)-radical of \( G \), i.e. the subgroup generated by all normal \( X \)-subgroups of \( G \). In particular,

\( G_\text{sol} \) means the solvable radical of \( G \).

\( O_\pi(\ G \) means the \( \pi \)-radical of \( G \), the subgroup generated by all normal \( \pi \)-subgroups of \( G \), i.e. \( O_\pi(\ G = G_{\phi_\pi} \).

\( \mu(G) \) denotes the degree of the minimal faithful permutation representation of a finite group \( G \), i.e. the smallest \( n \) such that \( G \) is isomorphic to a subgroup of \( \text{Sym}_n \).

A \( X \)-Hall subgroup, is a subgroup \( H \) of \( G \) such that \( H \) is an \( X \)-subgroup and a \( \pi(X) \)-Hall subgroup.

Hall_\( X \)(\( G \)) is the set of all \( X \)-Hall subgroups of \( G \), i.e. \( \text{Hall}_X \)(\( G = X \cap \text{Hall}_\pi \)(\( G \)).

### 2.2 Known properties of \( \pi \)-Hall subgroups, \( D_\pi \)- and \( D_X \)-groups

**Lemma 2.1** [27, Lemma 1] Let \( N \) be a normal subgroup and \( H \) a \( \pi \)-Hall subgroup of \( G \). Then \( H \cap N \in \text{Hall}_\pi(\ N \) and \( HN/N \in \text{Hall}_\pi(\ G/N \).

**Lemma 2.2** [16, Theorem A] If \( 2 \notin \pi \) and \( G \in E_\pi \), then every two \( \pi \)-Hall subgroups of \( G \) are conjugate.

**Lemma 2.3** [38, Theorem 7.7], [10, Theorem 6.6] Let \( N \) be a normal subgroup of \( G \). Then \( G \in D_\pi \) if and only if \( N \in D_\pi \) and \( G/N \in D_\pi \).

**Lemma 2.4** [34, Theorem 3] Let \( \pi \) be a set of primes and \( G \) be a simple group. Then \( G \in D_\pi \) if and only if either \( G \) is a \( \pi \)-group or \( (G, \pi) \) satisfies one of Conditions I–VII above.

**Lemma 2.5** [21, Proposition 1] Let \( \mathcal{X} \) be a complete class, \( \pi = \pi(\mathcal{X}) \), and \( G \in D_{\mathcal{X}} \). Then

\[
\text{m}_\mathcal{X}(\ G = \text{Hall}_\mathcal{X}(\ G \subseteq \text{Hall}_\pi(\ G \).
\]

In particular, \( G \in E_\pi \).

**Lemma 2.6** [21, Theorem 1] If \( G \in D_{\mathcal{X}} \) and \( N \unlhd G \), then \( N \in D_{\mathcal{X}} \) and \( G/N \in D_{\mathcal{X}} \).

**Lemma 2.7** [21, Theorem 2] For a complete class \( \mathcal{X} \), the following statements are equivalent.

1. The elements of \( \text{sm}_\mathcal{X}(\ G \) are conjugate in any \( G \in D_{\mathcal{X}} \).
2. \( \text{sm}_\mathcal{X}(\ G) = \text{m}_\mathcal{X}(\ G \) for any \( G \in D_{\mathcal{X}} \).
3. \( D_{\mathcal{X}} \) is closed under taking extensions.
4. \( \text{Aut} \ S \in D_{\mathcal{X}} \) for every simple \( S \in D_{\mathcal{X}} \).
(5) The elements of $\text{sm}_X(S)$ are conjugate in any simple $S \in \mathcal{D}_X$.

By the analogy with Chunikhin’s concept of a $\pi$-separable group, we say that $G$ is $X$-separable\(^8\) if $G$ has a subnormal series

$$G = G_0 > G_1 > \cdots > G_n = 1$$

such that $G_{i-1}/G_i$ is either an $X$-group or a $\pi(X)'$-group. It is clear that every solvable group is $X$-separable for every complete class $X$.

**Lemma 2.8** [31, Theorem 1] Let $N$ be a normal $X$-separable subgroup of $G$. Then the map given by the rule $M \mapsto MN/N$ is a surjection between sets $m_X(G)$ and $m_X(G/N)$. Moreover, this map induces a bijection between the sets of conjugacy classes of $X$-maximal subgroups of $G$ and $G/N$. In particular, $G \in \mathcal{D}_X$ if and only if $G/N \in \mathcal{D}_X$.

**Lemma 2.9** Let $G \in \mathcal{D}_\pi, H \in \text{Hall}_\pi(G)$ and $H \leq M \leq G$. Then $M \in \mathcal{D}_\pi$.

**Lemma 2.10** [39, Lemma 2.1(e)] Let $N$ be a normal subgroup of $G$ and $\pi(G/N) \subseteq \pi$. Assume $N$ contains a $\pi$-Hall subgroup $H_0$. Then the following statements are equivalent.

1. There is $H \in \text{Hall}_\pi(G)$ such that $H_0 = H \cap N$.
2. For every $g \in G$ there exists $x \in N$ such that $H^g_0 = H^x_0$.

### 2.3 Arithmetic Lemmas

For an odd integer $q$, denote by $\varepsilon(q)$ the number $\varepsilon = \pm 1$ such that $q \equiv \varepsilon \pmod{4}$.

If $r$ is an odd prime and $k$ is an integer not divisible by $r$, then $e(k, r)$ is the smallest positive integer $e$ with $k^e \equiv 1 \pmod{r}$. So, $e(k, r)$ is the multiplicative order of $k$ modulo $r$.

For a natural number $e$ set

$$e^* = \begin{cases} 2e & \text{if } e \equiv 1 \pmod{2}, \\ e & \text{if } e \equiv 0 \pmod{4}, \\ e/2 & \text{if } e \equiv 2 \pmod{4}. \end{cases}$$

It follows from the definition that if $e$ divides an even number $n$ then $e^*$ divides $n$ again. Moreover, it follows from definition that $e^{**} = e$ for every $e$.

For a real $x$, the integer part of $x$ is denoted by $[x]$, i.e. $[x]$ is a unique integer such that

$$[x] \leq x < [x] + 1.$$

The following lemma is evident.

**Lemma 2.11** If $m$ is a positive integer and $x$ is a real, then

$$[[x]/m] = [x/m].$$

The next result may be found in [17].

---

\(^8\)Wielandt [13] named such groups by the German term ‘$X$-reihig’.
Lemma 2.12 (\cite{[13]}, Lemmas 2.4 and 2.5) Let $r$ be an odd prime, $k$ an integer not divisible by $r$, and $m$ a positive integer. Denote $e(k, r)$ by $e$.

Then the following identities hold.

\[(k^m - 1)_r = \begin{cases} (k^e - 1)_r(m/e)_r & \text{if } e \text{ divides } m, \\ 1 & \text{if } e \text{ does not divide } m; \end{cases} \]

\[(k^m - (-1)^m)_r = \begin{cases} (k^e - (-1)^e)^*(m/e^*_r)_r & \text{if } e^* \text{ divides } m, \\ 1 & \text{if } e^* \text{ does not divide } m. \end{cases} \]

Lemma 2.13 Let $q > 1$ and $n$ be positive integers, let $r$ be an odd prime such that $(q, r) = 1$, and let $e = e(r, q)$. Then the following statements hold:

(i) $(n!)_r = r^n$, where $\alpha = \sum_{i=1}^{\infty} [n/r^i]$;

(ii) $\prod_{i=1}^{n} (q^i - 1)_r = (q^e - 1)^{(n/e)!}(n/e)_r$;

(iii) $\prod_{i=1}^{m} (k^i - (-1)^i)_r = (k^e - (-1)^e)^*(m/e^*_r)!_r$;

(iv) $\prod_{i=1}^{n} (q^i - 1)_r$ if and only if $e = r - 1$, $(q^{r-1} - 1)_r = r$ and $[n/r] = [n/(r - 1)]$.

(v) $\prod_{i=1}^{m} (q^i - (-1)^i)_r = (n!)_r$ if and only if $e^* = r - 1$, $(q^{(r-1)^*} - (-1)^{(r-1)^*})_r = r$ and $[n/r] = [n/(r - 1)]$.

**Proof.** The statement (i) is well-known (see, for example, \cite{[13], Lemma 2}). Statements (ii) and (iii) follow from Lemma 2.12.

Now we prove (iv). Let $A = \prod_{i=1}^{n} (q^i - 1)_r$. Then by ii) and in view of the Little Fermat Theorem,

\[A = (q^e - 1)^{(n/e)!}(n/e)!_r \geq r^{[n/e]}([n/e]!)_r \geq r^{[n/(r-1)]}([n/(r-1)]!)_r \geq r^{[n/r]}([n/r]!)_r = r^\beta, \]

where by (i) and, in view of Lemma 2.11 for $x = n/r$ and $m = r^i$, we have

\[\beta = [n/r] + \sum_{i=1}^{\infty} \left[\frac{n/r}{r^i}\right] = [n/r] + \sum_{i=1}^{\infty} \left[\frac{[n/r]}{r^{i+1}}\right] = \sum_{i=1}^{\infty} \left[\frac{n/r}{r^i}\right] = \log_r(n/r)_r.\]

Therefore, $A \geq (n!)_r$. Moreover, this inequality becomes an equality if and only if all inequalities in (1) are equalities, i.e. if and only if

\[r - 1 = e, \quad (q^e - 1)_r = r, \quad \text{and } [n/(r - 1)] = [n/r].\]

This implies (iv).
Now we prove (v). Let $A' = \prod_{i=1}^{n}(q^i - (-1)^i)r_i$. Since $r$ is odd, in view of the Little Fermat Theorem $e$ divides the even number $r - 1$. Consequently, $e^*$ also divides $r - 1$ and, by (iii),

$$A' = (q^{e^*} - (-1)^{e^*})r_i \geq r^{[n/e^*]}(n/e^*!)r_i \geq r^{[n/(r - 1)]}(n/(r - 1)!)r_i \geq r^{[n/r]}(n/r) = r^\beta,$$

(2)

where $\beta$ is as above.

Therefore, $A' \geq (n!)r_i$. Again this inequality becomes an equality if and only if all the inequalities in (2) are equalities, i.e. if and only if one of the equalities

$$r - 1 = e^*, \quad (q^{e^*} - (-1)^{e^*})r_i = r_i, \quad \text{and} \quad [n/(r - 1)] = [n/r].$$

This implies (v).

\[\Box\]

\section{2.4 On Hall subgroups of finite simple groups}

\textbf{Lemma 2.14} \cite{27, Theorem A4 and the notices after it}, \cite{45, Main result}, \cite{46, Theorem 8.1} Suppose that $n \geq 5$ and $\pi$ is a set of primes with $|\pi \cap \pi(n!)| > 1$ and $\pi(n!) \not\subseteq \pi$. Then

(1) The complete list of possibilities for $\text{Sym}_n$ containing a $\pi$-Hall subgroup $H$ is given in Table 1.

(2) $M \in \text{Hall}_\pi(\text{Alt}_n)$ if and only if $M = H \cap \text{Alt}_n$ for some $H \in \text{Hall}_\pi(\text{Sym}_n)$.

In particular, every proper nonsolvable $\pi$-Hall subgroup of a symmetric group of degree $n$ is isomorphic to a symmetric group of degree $n$ or $n - 1$ and has a unique nonabelian composition factor isomorphic to an alternating group of the same degree.

\begin{center}
\begin{tabular}{|c|c|c|}
\hline
$n$ & $\pi$ & $H \in \text{Hall}_\pi(\text{Sym}_n)$ \\
\hline
prime & $\pi((n - 1)!)$ & $\text{Sym}_{n-1}$ \\
7 & $\{2, 3\}$ & $\text{Sym}_3 \times \text{Sym}_4$ \\
8 & $\{2, 3\}$ & $\text{Sym}_4 \wr \text{Sym}_2$ \\
\hline
\end{tabular}
\end{center}

\textbf{Lemma 2.15} \cite{21, Proposition 3} Let $\pi = \pi(\mathcal{X})$. Then for $G = \text{Alt}_n$ the following conditions are equivalent.

(1) $G \in \mathcal{D}_\mathcal{X}$.

(2) $G \in \mathcal{D}_\mathcal{X} \cap \mathcal{D}_\pi$.

(3) either $|\pi \cap \pi(G)| \leq 1$ or $G \in \mathcal{X}$.

(4) All submaximal $\mathcal{X}$-subgroups are conjugate in $G$.

\textbf{Lemma 2.16} \cite{44, Theorem 4.1}, \cite{46, Theorem 8.2} Let $G$ be either one of the 26 sporadic groups or a Tits group, $\pi$ be such that $2 \in \pi$, $\pi(G) \not\subseteq \pi$, and $|\pi \cap \pi(G)| > 1$, and $H$ be
a \pi\text{-Hall subgroup of } G. \text{ Then the corresponding intersections } \pi \cap \pi(G) \text{ and the structure of } H \text{ are indicated in Table } 2.

Table 2:

| G      | \pi \cap \pi(G) | Structure of H       |
|--------|------------------|----------------------|
| M_{11} | \{2, 3\}         | 3^2 : Q_8 \cdot 2    |
|        | \{2, 3, 5\}      | Alt_6 : 2           |
| M_{22} | \{2, 3, 5\}      | 2^4 : Alt_6         |
| M_{23} | \{2, 3\}         | 2^4 : (3 \times A_4) : 2 |
|        | \{2, 3, 5\}      | 2^4 : Alt_6         |
|        | \{2, 3, 5, 7\}   | 2^4 : (3 \times Alt_5) : 2 |
|        | \{2, 3, 5, 7\}   | 2^4 : Alt_6         |
|        | \{2, 3, 5, 7, 11\} | M_{22}             |
| M_{24} | \{2, 3, 5\}      | 2^6 : 3 \cdot \text{Sym}_6 |
| J_1    | \{2, 3\}         | 2 \times Alt_4      |
|        | \{2, 3, 5\}      | 2 \times Alt_5      |
|        | \{2, 3, 7\}      | 2^3 : 7 : 3         |
|        | \{2, 7\}         | 2^4 : 7            |
| J_4    | \{2, 3, 5\}      | 2^{11} : (2^6 : 3 \cdot \text{Sym}_6) |

Lemma 2.17 [53, Lemma 3.1], [55, Lemma 8.10] Let } \pi \text{ be a set of primes with } 2, 3 \in \pi. \text{ Assume that } G \cong \text{SL}_q(q) \cong \text{SL}_2^2(q) \cong \text{Sp}_2(q), \text{ where } q \text{ is a power of an odd prime } p \notin \pi, \text{ and } \varepsilon = \varepsilon(q). \text{ Recall that for a subgroup } A \text{ of } G \text{ we denote by } PA \text{ the reduction modulo scalars. Then the following statements hold.}

(A) \text{ If } G \in \mathcal{E}_\pi \text{ and } H \in \text{Hall}_\pi(G), \text{ then one of the following statements holds.}

(a) \pi \cap \pi(G) \subseteq \pi(q - \varepsilon), \text{ PH is a } \pi\text{-Hall subgroup in the dihedral subgroup } D_{q-\varepsilon} \text{ of order } q - \varepsilon \text{ of } PG. \text{ All } \pi\text{-Hall subgroups of this type are conjugate in } G.

(b) \pi \cap \pi(G) = \{2, 3\}, \ (q^2 - 1)_{(2,3)} = 24, \text{ PH } \cong \text{Alt}_4. \text{ All } \pi\text{-Hall subgroups of this type are conjugate in } G.

(c) \pi \cap \pi(G) = \{2, 3\}, \ (q^2 - 1)_{(2,3)} = 48, \text{ PH } \cong \text{Sym}_4. \text{ There exist exactly two classes of conjugate subgroups of this type, and } \text{PGL}_2^2(q) \text{ interchanges these classes.}

(d) \pi \cap \pi(G) = \{2, 3, 5\}, \ (q^2 - 1)_{(2,3,5)} = 120, \text{ PH } \cong \text{Alt}_5. \text{ There exist exactly two classes of conjugate subgroups of this type, and } \text{PGL}_2^2(q) \text{ interchanges these classes.}

(B) Conversely, if } \pi \text{ and } (q^2 - 1)_\pi \text{ satisfy one of statements (a)--(d), then } G \in \mathcal{E}_\pi.

(C) \text{ Every } \pi\text{-Hall subgroup of } PG \text{ can be obtained as } \text{PH} \text{ for some } H \in \text{Hall}_\pi(G). \text{ Conversely, if } PH \in \text{Hall}_\pi(PG) \text{ and } H \text{ is a full preimage of } PH \text{ in } G, \text{ then } H \in \text{Hall}_\pi(G).

Lemma 2.18 [53, Lemma 3.2], [55, Corollary 8.11] Let } G = \text{GL}_2^2(q), \text{ PG = G/Z(G) = PGL}_2^2(q), \text{ where } q \text{ is a power of a prime } p, \text{ and } \varepsilon = \varepsilon(q). \text{ Let } \pi \text{ be a set of primes such that } 2, 3 \in \pi \text{ and } p \notin \pi. \text{ A subgroup } H \text{ of } G \text{ is a } \pi\text{-Hall subgroup if and only if } H \cap \text{SL}_2(q)
is a $\pi$-Hall subgroup of $\text{SL}_2(q)$, $|H : H \cap \text{SL}_2(q)|_\pi = (q - \eta)_\pi$, and either statement (a), or statement (b) of Lemma 2.19 holds. More precisely, one of the following statements holds.

(a) $\pi \cap \pi(G) \subseteq \pi(q - \varepsilon)$, where $\varepsilon = \varepsilon(q)$, $PH$ is a $\pi$-Hall subgroup in the dihedral group $D_{2(q-\varepsilon)}$ of order $2(q-\varepsilon)$ of $PG$. All $\pi$-Hall subgroups of this type are conjugate in $G$.

(b) $\pi \cap \pi(G) = \{2, 3\}$, $(q^2 - 1)_{(2,3)} = 24$, $PH \cong \text{Sym}_4$. All $\pi$-Hall subgroups of this type are conjugate in $G$.

**Lemma 2.19** [14, Theorem 3.2], [17, Theorem 3.1], [37, Theorem 1.2], [46, Theorems 8.3–8.7] Let $\pi$ be a set of primes and $G$ a group of Lie type over the field $\mathbb{F}_q$ of characteristic $p \in \pi$. Assume that $G \in \mathcal{E}_\pi$ and $H \in \text{Hall}_\pi(G)$. Then one of the following statements holds.

1. $H = G$.

2. $\pi \cap \pi(G) \subseteq \pi(q - 1) \cup \{p\}$, $H$ is contained in a Borel subgroup of $G$ (in particular, $H$ is solvable) and any prime in $\pi \setminus \{p\}$ does not divide the order of the Weyl group of $G$.

3. $p = 2$, $G = D_l(q)$, the Dynkin diagram of the fundamental root system $\Pi^1$ of $G$ is on Pic. 1, $l$ is a Fermat prime, $(l, q - 1) = 1$ and $H$ is conjugate to the canonic parabolic maximal subgroup corresponding to the set $\Pi \setminus \{r_1\}$ of fundamental roots.

4. $p = 2$, $G = 2 D_l(q)$, the Dynkin diagram of the fundamental root system $\Pi$ of $G$ is on Pic. 2, $l - 1$ is a Mersenne prime, $(l - 1, q - 1) = 1$ and $H$ is conjugate to the canonic parabolic maximal subgroup corresponding to the set $\Pi^1 \setminus \{r_1^1\}$ of fundamental roots;

5. $G$ is isomorphic to the quotient by the center of $\text{SL}(V)$, where $V$ is a vector space of a dimension $n$ over $\mathbb{F}_q$ and $H$ is the image in $G$ under the natural epimorphism of the stabilizer in $\text{SL}(V)$ of a series

$$0 = V_0 < V_1 < \cdots < V_s = V$$

such that $\dim V_i/V_{i-1} = n_i$, $i = 1, 2, \ldots, s$, and one of the following conditions holds:

(a) $n$ is a prime, $(n, q - 1) = 1$, $s = 2$, $n_1, n_2 \in \{1, n - 1\}$;

(b) $n = 4$, $(2 \cdot 3, q - 1) = 1$, $s = 2$, $n_1 = n_2 = 2$;

(c) $n = 5$, $(2 \cdot 5, q - 1) = 1$, $s = 2$, $n_1, n_2 \in \{2, 3\}$;

Pic. 1. Dynkin diagram of the root system of $D_l(q)$. 

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Lemma 2.20 [39, Lemma 4.3], [39, Theorem 8.12] Assume $G = \text{SL}_n^q(q)$ is a special linear or unitary group with the base field $\mathbb{F}_q$ of characteristic $p$ and $n \geq 2$. Let $\pi$ be a set of primes such that $2, 3 \in \pi$ and $p \not\in \pi$. Then the following statements hold.

(A) Suppose $G \in \mathcal{E}_\pi$, and $H$ is a $\pi$-Hall subgroup of $G$. Then for $G$, $H$, and $\pi$ one of the following statements holds.

(a) $n = 2$ and one of the statements (a)–(d) of Lemma 2.17 holds.
(b) either $q \equiv \eta \pmod{12}$, or $n = 3$ and $q \equiv \eta \pmod{4}$, $\text{Sym}_n$ satisfies $\mathcal{E}_\pi$, $\pi \cap \pi(G) \subseteq \pi(q-\eta) \cup \pi(n!)$, and if $r \in (\pi \cap \pi(n!)) \setminus \pi(q-\eta)$, then $|G|_r = |\text{Sym}_n|_r$; $H$ is included in

$$M = L \cap G \cong \mathbb{Z}^n \cdot \text{Sym}_n,$$

where $L = \mathbb{Z} \cdot \text{Sym}_n \leq \text{GL}_n^q(q)$ and $Z = \text{GL}_1^q(q)$ is a cyclic group of order $q-\eta$. All $\pi$-Hall subgroups of this type are conjugate in $G$.
(c) $n = 2m + k$, where $k \in \{0, 1\}$, $m \geq 1$, $q \equiv -\eta \pmod{3}$, $G \cap \pi(G) \subseteq \pi(q^2 - 1)$, the groups $\text{Sym}_m$ and $\text{GL}_2^q(q)$ satisfy $\mathcal{E}_\pi$, and

$$M = L \cap G \cong (\underbrace{\text{GL}_2^q(q) \circ \cdots \circ \text{GL}_2^q(q)}_{m \text{ times}}) \cdot \text{Sym}_m \circ Z,$$

where $L = \text{GL}_2^q(q) \cap \text{Sym}_m \times Z \leq \text{GL}_n(q)$ and $Z$ is a cyclic group of order $q-\eta$ for $k = 1$ and $Z$ is trivial for $k = 0$. The subgroup $H$ acting by conjugation on the set of factors in the central product

$$\underbrace{\text{GL}_2^q(q) \circ \cdots \circ \text{GL}_2^q(q)}_{m \text{ times}}$$

has at most two orbits. The intersection of $H$ with each factor $\text{GL}_2^q(q)$ in (3) is a $\pi$-Hall subgroup in $\text{GL}_2^q(q)$. The intersections with the factors from the same
orbit all satisfy the same statement (a) or (b) of Lemma 2.18. Two \( \pi \)-Hall subgroups of \( M \) are conjugate in \( G \) if and only if they are conjugate in \( M \). Moreover \( M \) possesses one, two, or four classes of conjugate \( \pi \)-Hall subgroups, while all subgroups \( M \) are conjugate in \( G \).

(d) \( n = 4 \), \( \pi \cap \pi(G) = \{2, 3, 5\} \), \( q \equiv 5 \eta \pmod{8} \), \( (q + \eta)_3 = 3 \), \( (q^2 + 1)_5 = 5 \), and \( H \cong 4 \cdot 2^4 \cdot \text{Alt}_6 \). In this case, \( G \) possesses exactly two classes of conjugate \( \pi \)-Hall subgroups of this type and \( \text{GL}_2^\pm(q) \) interchanges these classes.

(e) \( n = 11 \), \( \pi \cap \pi(G) = \{2, 3\} \), \( (q^2 - 1)_{2,3} = 24 \), \( q \equiv -\eta \pmod{3} \), \( q \equiv \eta \pmod{4} \).

H is included in a subgroup \( M = L \cap G \), where \( L \) is a subgroup of \( G \) of type \(((\text{GL}_2^\pm(q) \wr \text{Sym}_4) \downarrow (\text{GL}_2^\pm(q) \lhd \text{Sym}_3)) \), and

\[
H = (((Z \circ 2 \cdot \text{Sym}_4) \lhd \text{Sym}_4) \times (Z \lhd \text{Sym}_3)) \cap G,
\]

where \( Z \) is a Sylow 2-subgroup of a cyclic group of order \( q - \eta \). All \( \pi \)-Hall subgroups of this type are conjugate in \( G \).

(B) Conversely, if the conditions on \( \pi \), \( n \), \( \eta \), and \( q \) in one of statements (a)–(e) are satisfied, then \( G \in \mathcal{E}_\pi \).

Lemma 2.21 [43, Lemma 4.4], [46, Theorem 8.13] Let \( G = \text{Sp}_{2n}(q) \) be a symplectic group over a field \( \mathbb{F}_q \) of characteristic \( p \). Assume that \( \pi \) is a set of primes such that \( 2, 3 \in \pi \) and \( p \not\in \pi \). Then the following statements hold.

(A) Suppose \( G \in \mathcal{E}_\pi \) and \( H \in \text{Hall}_\pi(G) \). Then both \( \text{Sym}_n \) and \( \text{SL}_2(q) \) satisfy \( \mathcal{E}_\pi \) and \( \pi \cap \pi(G) \subseteq \pi(q^2 - 1) \). Moreover, \( H \) is a \( \pi \)-Hall subgroup of

\[
M = \text{Sp}_2(q) \wr \text{Sym}_n \cong \left( \frac{\text{SL}_2(q) \times \cdots \times \text{SL}_2(q)}{n \text{ times}} \right) : \text{Sym}_n \leq G.
\]

(B) Conversely, if both \( \text{Sym}_n \) and \( \text{SL}_2(q) \) satisfy \( \mathcal{E}_\pi \) and \( \pi \cap \pi(G) \subseteq \pi(q^2 - 1) \), then \( M \in \mathcal{E}_\pi \) and every \( \pi \)-Hall subgroup \( H \) of \( M \) is a \( \pi \)-Hall subgroup of \( G \).

(C) Two \( \pi \)-Hall subgroups of \( M \) are conjugate in \( G \) if and only if they are conjugate in \( M \).

Lemma 2.22 [33, Lemma 6.7], [46, Theorem 8.14] Assume that \( G = \Omega_n^\varepsilon(q), \eta \in \{+, -, o\}, q \) is a power of a prime \( p, n \geq 7, \varepsilon = \varepsilon(q) \). Let \( \pi \) be a set of primes such that \( 2, 3 \in \pi, p \not\in \pi \). Then the following statements hold.

(A) If \( G \) possesses a \( \pi \)-Hall subgroup \( H \), then one of the following statements holds.

(a) \( n = 2m + 1 \), \( \pi \cap \pi(G) \subseteq \pi(q - \varepsilon) \), \( q \equiv \varepsilon \pmod{12} \), \( \text{Sym}_m \in \mathcal{E}_\pi \), and \( H \) is a \( \pi \)-Hall subgroup in \( M = (\text{O}_2^\varepsilon(q) \wr \text{Sym}_m \times \text{O}_1(q)) \cap G \). All \( \pi \)-Hall subgroup of this type are conjugate.

(b) \( n = 2m \), \( \eta = \varepsilon^m, \pi \cap \pi(G) \subseteq \pi(q - \varepsilon), q \equiv \varepsilon \pmod{12} \), \( \text{Sym}_m \in \mathcal{E}_\pi \), and \( H \) is a \( \pi \)-Hall subgroup in \( M = (\text{O}_2^\varepsilon(q) \lhd \text{Sym}_m) \cap G \). All \( \pi \)-Hall subgroup of this type are conjugate.

(c) \( n = 2m \), \( \eta = -\varepsilon^m, \pi \cap \pi(G) \subseteq \pi(q - \varepsilon), q \equiv \varepsilon \pmod{12} \), \( \text{Sym}_{m-1} \in \mathcal{E}_\pi \), and \( H \) is a \( \pi \)-Hall subgroup of \( M = (\text{O}_2^\varepsilon(q) \times \text{O}_2^{-\varepsilon}(q)) \cap G \). All \( \pi \)-Hall subgroup of this type are conjugate.
(d) \( n = 11, \pi \cap \pi(G) = \{2, 3\}, q \equiv \varepsilon \pmod{12}, (q^2 - 1)_{11} = 24, \) and \( H \) is a \( \pi \)-Hall subgroup of \( M = \{O_5^+(q) \wr \text{Sym}_4 \wr \text{O}_1(q) \wr \text{Sym}_3\} \cap G. \) All \( \pi \)-Hall subgroup of this type are conjugate.

(e) \( n = 12, \eta = -, \pi \cap \pi(G) = \{2, 3\}, q \equiv \varepsilon \pmod{12}, (q^2 - 1)_{12} = 24, \) and \( H \) is a \( \pi \)-Hall subgroup of \( M = \{O_5^+(q) \wr \text{Sym}_4 \wr \text{O}_1(q) \wr \text{Sym}_3\} \cap G. \) There exist precisely two classes of conjugate subgroups of this type in \( G, \) and the automorphism of order 2 induced by the group of similarities of the natural module interchanges these classes.

(f) \( n = 7, \pi \cap \pi(G) = \{2, 3, 5, 7\}, |G|_\pi = 2^9 \cdot 3^4 \cdot 5 \cdot 7, \) and \( H \cong \Omega_7(2). \) There exist precisely two classes of conjugate subgroups of this type in \( G, \) and \( \text{SO}_9(q) \) interchanges these classes.

(g) \( n = 8, \eta = +, \pi \cap \pi(G) = \{2, 3, 5, 7\}, |G|_\pi = 2^{13} \cdot 3^5 \cdot 5^2 \cdot 7, \) and \( H \cong 2 \cdot \Omega_8^+(2). \) There exist precisely four classes of conjugate subgroups of this type in \( G. \) The subgroup of \( \text{Out}(G) \) generated by diagonal and graph automorphisms is isomorphic to \( \text{Sym}_4 \) and acts on the set of these classes as \( \text{Sym}_4 \) in its natural permutation representation, and every diagonal automorphism acts without fixed points.

(h) \( n = 9, \pi \cap \pi(G) = \{2, 3, 5, 7\}, |G|_\pi = 2^{14} \cdot 3^5 \cdot 5^2 \cdot 7, \) and \( H \cong 2 \cdot \Omega_9^+(2). \) There exist precisely two classes of conjugate subgroups of this type in \( G, \) and \( \text{SO}_9(q) \) interchanges these classes.

(B) Conversely, if one of the statements (a)–(h) holds, then \( G \) possesses a \( \pi \)-Hall subgroup with the given structure.

**Lemma 2.23** [3] Lemmas 7.1–7.6, [10] Theorem 8.13 | Assume that

\[
G \in \{E_6^0(q), E_7(q), E_8(q), F_4(q), G_2(q), 3D_4(q)\},
\]

where \( q \) is a power of a prime \( p. \) Let \( \varepsilon = \varepsilon(q). \) Let \( \pi \) be a set of primes such that \( 2, 3 \in \pi, \) \( p \notin \pi. \) Then \( G \) contains a \( \pi \)-Hall subgroup \( H \) if and only if one of the following statements hold:

(a) \( G \) is a group in Table 3 and the values for the untwisted Lie rank \( l \) of \( G, \delta \) and the structure of the Weyl group \( W \) are given in the Table 3; if \( G = E_6^0(q) \) then \( \eta = \varepsilon; \)

\( \pi(W) \subseteq \pi \cap \pi(G) \subseteq \pi(q - \varepsilon), \) \( H \) is a \( \pi \)-Hall subgroup of a group \( T \cdot W, \) where \( T \) is a maximal torus of order \( (q - \varepsilon)^l/\delta. \) All \( \pi \)-Hall subgroups of this type are conjugate in \( G; \)

(b) \( G = 3D_4(q), \pi \cap \pi(G) \subseteq \pi(q - \varepsilon) \) and \( H \) is a \( \pi \)-Hall subgroup in \( T \cdot W(G_2), \) where \( T \) is a maximal torus of order \( (q - \varepsilon)(q^3 - \varepsilon). \) All \( \pi \)-Hall subgroups of this type are conjugate in \( G; \)

(c) \( G = E_8^\varepsilon(q), \pi \cap \pi(G) \subseteq \pi(q - \varepsilon) \) and \( H \) is a \( \pi \)-Hall subgroup in \( T \cdot W(F_4), \) where \( T \) is a maximal torus of order \( (q^2 - 1)^2(q - \varepsilon)^3/(3, q + \varepsilon). \) All \( \pi \)-Hall subgroups of this type are conjugate in \( G; \)

(d) \( G = G_2(q), \pi \cap \pi(G) = \{2, 3, 7\}, (q^2 - 1)_{2,3,7} = 24, (q^4 + q^2 + 1), = 7, H \cong G_2(2), \) and all \( \pi \)-Hall subgroups of this type are conjugate in \( G. \)
Lemma 2.24 \cite[Theorem 3.1]{17} Let $G$ be a group of Lie type with base field $\mathbb{F}_q$ of some characteristic $p$. Assume that $\pi$ is such that $2, p \in \pi$, and $3 \notin \pi$. Suppose $G \in \mathcal{E}_\pi$ and $H \in \text{Hall}_\pi(G)$. Then $p = 2$ and one of the following statements holds.

1. $\pi \cap \pi(G) \subseteq \pi(q - 1) \cup \{2\}$, a Sylow 2-subgroup $P$ of $H$ is normal in $H$ and $H/P$ is Abelian.

2. $p = 2, G \cong 2B_2(2^{2m+1})$ and $\pi(G) \subseteq \pi$.

Lemma 2.25 \cite[Lemma 4]{17} Let $G$ be a nonabelian simple group. Then $G \in \mathcal{D}_{\{2,3\}}$ if and only if $G$ is a Suzuki group $2B_2(q)$. In this case every $\pi$-subgroup of $G$ is 2-group.

Lemma 2.26 \cite[Lemma 5.1 and Theorem 5.2]{17} Let $G$ be a group of Lie type over a field of characteristic $p$. Assume that $\pi$ is such that $3, p \notin \pi$ and $2 \in \pi$. Suppose $G \in \mathcal{E}_\pi$ and $H \in \text{Hall}_\pi(G)$. Then either $H$ possesses a normal abelian 2'-Hall subgroup or $G \cong 2G_2(3^{2m+1})$ and $\pi \cap \pi(G) = \{2, 7\}$.

Lemma 2.27 Let $G$ be a simple nonabelian group. Assume that $\pi$ is such that $\pi(G) \not\subseteq \pi$, $|\pi \cap \pi(G)| > 1, 2 \in \pi$ and $3 \notin \pi$. Suppose $G \in \mathcal{E}_\pi$ and $H \in \text{Hall}_\pi(G)$. Then the following statements hold.

1. $H$ is solvable.

2. If every solvable $\pi$-subgroup of $G$ is conjugate to a subgroup of $H$, then $G$ is a group of Lie type over a field of characteristic $p \notin \pi$ and $G \in \mathcal{D}_\pi$.

Proof. Statement (1) is proved in \cite[Lemma 10]{17}.

Prove (2). Consider all possibilities for $G$, according to the classification of finite simple groups (see \cite[Theorem 0.1.1]{3}).

Case 1: $G \cong \text{Alt}_n, n \geq 5$. This case is impossible by Lemma \ref{lem:alt}.

Case 2: $G$ is either a sporadic group or a Tits group. By Lemma \ref{lem:tits} it follows that $G \cong J_1$ and $\pi \cap \pi(G) = \{2, 7\}$. Now, by the Burnside theorem \cite[Ch. I, 2]{10}, every $\pi$-subgroup of $G$ is solvable and is conjugate to a subgroup of $H$, that is $G \in \mathcal{D}_\pi$. It contradicts Lemma \ref{lem:j1}.

Case 3: $G$ is a group of Lie type over a field $\mathbb{F}_q$ of characteristic $p \in \pi$. By Lemma \ref{lem:lie} $p = 2$, $H$ is solvable and a Sylow 2-subgroup $P$ of $H$ is normal in $H$. Moreover, $\pi \cap \pi(G) \subseteq \pi(q - 1) \cup \{2\}$. Condition $|\pi \cap \pi(G)| > 1$ implies that $q > 2$. It is known that $G$ has a subgroup which is a homomorphic image of $\text{SL}_2(q) = \text{PSL}_2(q)$. Since $\text{PSL}_2(q)$ is simple, we assume that $\text{PSL}_2(q) \leq G$. Take $r \in \pi \cap \pi(q - 1)$. Then $\text{PSL}_2(q)$ contains a dihedral subgroup $U$ of order $2r$ and $U$ has no normal Sylow 2-subgroups. Hence, $U$ is not conjugate to any subgroup of $H$.

Case 4: $G$ is a group of Lie type over a field of characteristic $p \notin \pi$. Lemma \ref{lem:lie} implies that either $H$ possesses a normal abelian 2'-Hall subgroup or $G \cong 2G_2(3^{2m+1})$, \cite{17}.

Table 3: Weyl groups of exceptional root systems

| $G$     | $\ell$ | $\delta$ | $W$                  | $|W|$          |
|---------|--------|----------|----------------------|----------------|
| $E_6^7(q)$ | 6      | $(3, q - \eta)$ | $W(E_6) \cong \text{Spin}_4(3)$ | $2^7.3^1.5$  |
| $E_7(q)$  | 7      | 2        | $W(E_7) \cong 2 \times \text{Spin}_7(2)$ | $2^{10}.3^4.5.7$ |
| $E_8(q)$  | 8      | 1        | $W(E_8) \cong 2 \cdot \text{Spin}_7^+(2).2$ | $2^{14}.3^5.5^2.7$ |
| $F_4(q)$  | 4      | 1        | $W(F_4)$                  | $2^7.3^2$     |
| $G_2(q)$  | 2      | 1        | $W(G_2)$                  | $2^2.3$       |
\[ \pi \cap \pi(G) = \{2,7\}. \] In the last case every \(\pi\)-subgroup of \(G\) is solvable by the Burnside theorem [14, Ch. I, 2] and \(G \in \mathcal{D}_\pi\). Suppose, \(H\) possesses a normal abelian \(2'\)-Hall subgroup. It is sufficient to prove that every \(\pi\)-subgroup of \(G\) is solvable by the Burnside theorem [10, Ch. I, 2]. Take in \(U\) the full preimage \(V\) of a Borel subgroup of \(2^B_2(2^{2m+1})\). Then \(V\) is solvable and is conjugate to a subgroup of \(H\). In particular, \(V\) and a Borel subgroup \(V/U_\mathcal{E}\) of \(2^B_2(2^{2m+1})\) possesses a normal \(2'\)-Hall subgroup, but this is not true. \(\blacksquare\)

**Lemma 2.28** Let \(G \in \mathcal{D}_\pi\) be a nonabelian simple group. Then either \(G\) is a \(\pi\)-group or every \(\pi\)-Hall subgroup of \(G\) is solvable. In particular, if \(G\) is not a \(\pi\)-group, then \(G \in \mathcal{D}_\tau\) for every \(\tau \subseteq \pi\).

**Proof.** Lemma follows from Lemmas 2.25 and 2.27 and the solvability of groups of odd order [12]. \(\blacksquare\)

### 2.5 Degrees of minimal faithful permutation representation

In the following Lemma we collect some statements about minimal degrees of faithful permutation representations of some groups.

**Lemma 2.29** The following statements hold.

1. If \(H \leq G\), then \(\mu(H) \leq \mu(G)\).
2. [13, Theorem 2] Let \(\mathcal{L}\) be a complete class of finite groups. Let \(N\) be the \(\mathcal{L}\)-radical of \(G\), that is the maximal normal \(\mathcal{L}\)-subgroup of \(G\). Then \(\mu(G) \geq \mu(G/N)\).
3. [14, Theorem 3.1] If \(G = L_1 \times L_2 \times \cdots \times L_r\) and \(L_1, L_2, \ldots, L_r\) are simple, then \(\mu(G) = \mu(L_1) + \mu(L_2) + \cdots + \mu(L_r)\).
4. If \(G\) is simple, then \(\mu(G)\) is equal to the minimum of indices of maximal subgroups in \(G\).
5. \(\mu(\text{Sym}_n) = \mu(\text{Alt}_n) = n\).

### 2.6 Some subgroups of quasisimple and almost simple groups

**Lemma 2.30** [4, Tables 8.1 and 8.2] Assume that \(q^2 \equiv 1 \pmod{5}\) and \(q\) is a power of an odd prime. Then \(\text{SL}_2(q)\) contains a subgroup isomorphic to \(\text{SL}_2(5)\) and \(\text{PSL}_2(q)\) contains a subgroup isomorphic to \(\text{PSL}_2(5) \cong \text{Alt}_5\).

**Lemma 2.31** [4, Tables 8.8 and 8.10] Assume that \(q \equiv \eta \pmod{4}\), where \(q\) is a power of an odd prime and \(\eta = \pm 1\). Then \(\text{SL}_2^4(q)\) contains a subgroup isomorphic to \(4 \circ 2^{1+4} \cdot \text{Alt}_6\).
Lemma 2.32 [14, Lemma 1.24] Let $l$ be an odd prime, $q > 2$ be a power of a prime. Assume $G = \langle D, x \rangle$, where $D$ is the group of all diagonal matrices of determinant 1, and

$$x = \begin{pmatrix}
0 & 1 & & \\
0 & 1 & & \\
& & \ddots & \\
1 & 0 & & 0
\end{pmatrix} \in \text{SL}_r(q).$$

Then $|G| = (q - 1)^{r-1}r$ and $G$ is absolutely irreducible.

Lemma 2.33 Let $q$ be a power of an odd prime. Let pair $(G^*, m)$, where $G^*$ is a quasisimple group and $m$ is an positive integer, appear in the following list:

1. $(G^*, m) = (\text{SL}^n(q), [n/2]), n > 2, \eta = \pm$;
2. $(G^*, m) = (\text{Sp}_n(q), n/2), n > 2$ is even;
3. $(G^*, m) = (\Omega_n(q), 2[n/4]), n > 5$ is odd;
4. $(G^*, m) = (\Omega_n^+(q), 2[n/4]), n > 6$ is even;
5. $(G^*, m) = (\Omega_n^-(q), 2[(n - 1)/4]), n > 6$ is even;
6. $(G^*, m) = (E_6^q(q), 4), G^*$ is a quotient of the universal group by a central subgroup;
7. $(G^*, m) = (E_7(q), 7), G^*$ is a quotient of the universal group by a central subgroup;
8. $(G^*, m) = (E_8(q), 8)$.

Then $G^*$ contains a collection $\Delta$ of subgroups such that

(a) every member of $\Delta$ is isomorphic to $\text{SL}_2(q)$,
(b) if $K^*, L^* \in \Delta$ are distinct, then $[K^*, L^*] = 1$, and
(c) $|\Delta| = m$.

Proof. Lemma follows from Aschbacher’s theory of fundamental subgroups. Recall that, if $G^*$ is a group from the lemma, then $K^*$ is a fundamental subgroup, if $K^*$ is conjugate to a subgroup generated by a long root subgroup $U$ and its opposite $U^-$. Every fundamental subgroup of $G^*$ is isomorphic to $\text{SL}_2(q)$. Fix a Sylow 2-subgroup $S^*$ of $G^*$ and consider $\Delta = \text{Fun}(S^*)$ consisting of all fundamental subgroups $L^*$ such that $S^* \cap L^*$ is a Sylow 2-subgroup of $L^*$. It follows from [14, 6.2] that distinct elements of $\Delta$ elementwise commute and follows from [4, Theorem 2] that $|\Delta| = m$. \hfill $\Box$

Lemma 2.34 Let $G = \text{Sym}_n$, where $n \geq 5$, and $G$ contains a subgroup $H$ having $\text{Alt}_m$ as a homomorphic image for some $m \in \{n - 1, n\}$. Then $H \cong \text{Alt}_m$.

Proof. Suppose that $m = n$. In this case $|G : H| \leq 2$ and $H \in \{\text{Sym}_n, \text{Alt}_n\}$. Since $\text{Sym}_n$ has no $\text{Alt}_n$ as a homomorphic image, we have $H = \text{Alt}_n$.

Suppose that $m = n - 1$. First of all, note the following well-known fact: every subgroup $H_0$ of $G$ of index $n$ is isomorphic to $\text{Sym}_{n-1}$. Indeed, let $K$ be the kernel of the action of $G$ by right multiplication on the set $\Omega$ of right cosets of $H$ in $G$. Then $K \leq H_0$
and \( |G : K| \geq |G : H_0| = n > 2 \). Since \( G \) has a unique minimal normal subgroup \( \text{Alt}_n \) and its index equals 2, we have \( K = 1 \). Therefore, \( G \) is embedded in \( \text{Sym}(\Omega) \cong \text{Sym}_n \) and \( G \cong \text{Sym}(\Omega) \). Since \( H_0 \) is a point stabilizer in \( G \), we have \( H_0 \cong \text{Sym}_{n-1} \).

Now let \( L \) be the kernel of an epimorphism \( H \to \text{Alt}_{n-1} \). We need to show that \( L = 1 \). If not, then

\[
|H| = |L| |\text{Alt}_{n-1}| \geq 2 |\text{Alt}_{n-1}| = (n-1)! \text{ and } |G : H| \leq n.
\]

Since \( G \) has no proper subgroups of index less than \( n \), except \( \text{Alt}_n \), we have either \( H = G = \text{Sym}_n \), or \( H \cong \text{Sym}_{n-1} \). But \( \text{Alt}_{n-1} \) is not a homomorphic image of both \( S_n \) and \( \text{Sym}_{n-1} \).

**Lemma 2.35** Let \( G = \text{Sym}_n \). Then \( H = N_G(H) \) for every subgroup \( H \) of \( G \) such that \( |G : H| \) is odd.

**Proof.** Let \( S \) be a Sylow \( 2 \)-subgroup of \( H \). Then \( S \) is a Sylow \( 2 \)-subgroup of \( G \) and \( S = N_G(S) \) by [5, Lemma 4]. So \( S = N_G(S) \leq H \), and \( H = N_G(H) \) by the Frattini Argument. \( \square \)

**Lemma 2.36** Let \( H \) be a \( \pi \)-Hall subgroup of \( G = L \wr \text{Sym}_n \). Denote by \( L_i \times \ldots \times L_n \) the base of the wreath product. Assume that \( L_i \) possesses a \( \pi \)-Hall subgroup that is isomorphic to \( H \cap L_i \) and is not conjugate with \( H \cap L_i \) in \( L_i \). Then \( G \) possesses a \( \pi \)-Hall subgroup \( K \) such that \( H \) and \( K \) have the same composition factors and are not conjugate in \( G \).

**Proof.** We can assume for the simplicity that \( i = 1 \). We set \( A = L_1 \times \ldots \times L_n \) and \( H_1 = H \cap L_1 \) and denote by \( K_1 \) a subgroup of \( L_1 \) that is isomorphic to \( H_1 \) but is not conjugate to \( H_1 \). Note that \( G \) acts on the set \( \Omega = \{L_1, \ldots, L_n\} \) via conjugation and \( A \) is the kernel of this action. Moreover, it follows from the definition of a wreath product that \( N_G(L_1) = C_G(L_1)L_1 \).

Renumbering \( \{L_1, \ldots, L_n\} \), if necessary, we may choose a right transversal \( h_1 = 1, \ldots, h_m \) of \( N_H(L_1) \) in \( H \) so that \( L_1^{h_1} = L_1 \). Then \( L_i^{h_i} \neq L_i^{h_j} \) if \( i \neq j \). In particular, \( m \leq n \). So \( \{L_1, \ldots, L_m\} \) is an orbit of \( H \) on \( \Omega \). Thus both \( \Delta = \{L_1, \ldots, L_m\} \) and \( \Gamma = \{L_{m+1}, \ldots, L_n\} \) are \( H \)-invariant. Set

\[
K_i = \left\{ \begin{array}{ll}
K_1^{h_i} & \text{for } i = 1, \ldots, m \\
H \cap L_i & \text{for } i = m + 1, \ldots, n
\end{array} \right.
\]

and \( K_0 = \langle K_i \mid i = 1, \ldots, n \rangle = K_1 \times \ldots \times K_n \). By construction, \( K_0 \leq A \) and \( K_0 \cong H \cap A \in \text{Hall}_r(A) \).

We claim that for every \( h \in H \) there exists \( a \in A \) such that \( K_0^h = K_0^a \).

Indeed, take \( h \in H \). Then there exists \( \sigma \in \text{Sym}_n \) such that \( L_i^h = L_i^{i\sigma} \) for \( i = 1, \ldots, n \). Since \( \Delta \) and \( \Gamma \) are both \( H \)-invariant, we obtain that \( i\sigma \in \{1, \ldots, m\} \) for \( i = 1, \ldots, m \) and \( i\sigma \in \{m + 1, \ldots, n\} \) for \( i = m + 1, \ldots, n \).

Take \( i \leq m \). Then \( h_i h = xh_{i\sigma} \) for some \( x \in N_H(L_1) \). In this case

\[
K_i^h = K_1^{h_i} = K_1^{xh_{i\sigma}}.
\]

Since \( x \in N_H(L_1) \leq N_G(L_1) = C_G(L_1)L_1 \) and \( K_1 \leq L_1 \), \( K_1^b = K_1^c \) for some \( b \in L_1 \). Set \( b_i = b_1^{h_i} \in L_i \). Then we have

\[
K_i^h = K_1^{xh_{i\sigma}} = K_1^{bh_{i\sigma}} = \left(K_1^{xh_{i\sigma}}ight)^{bh_{i\sigma}} = K_1^{h_{i\sigma}}.
\]
Thus, we see that there are $b_1 \in L_1, \ldots, b_m \in L_m$ such that 

$$K^h_i = K^b_i$$

for every $i \leq m$.

Let $a = b_1 \ldots b_m$. We show that $K^h_0 = K^a_0$. Indeed, we have seen that $K^h_i = K^b_i = K^a_i$ if $i \leq m$. If $i > m$ then

$$K^h_i = H \cap L^h_i = H \cap L_i = K_i = K_i^a,$$

since $a$ centralizes $K_j$ for all $j > m$. Hence,

$$K^h_0 = \langle K^h_i \mid i = 1, \ldots, n \rangle = \langle K^a_i \mid i = 1, \ldots, n \rangle = K^a_0.$$

Now Lemma 2.11 implies that there is $K \in \text{Hall}_\pi(HA) \subseteq \text{Hall}_\pi(G)$ such that $K_0 = K \cap A$.

The groups $H$ and $K$ have the same composition factors, since

$$K/K \cap A \cong K/A = HA/A \cong H/H \cap A \text{ and } K \cap A = K_0 \cong H \cap A.$$

Suppose, $K = H^g$ for some $g \in G$. Then the image of $g$ in $G/A$ normalizes $HA/A = KA/A$. Note that $2 \in \pi$ in view of Lemma 2.2. Therefore, the index of $HA/A$ in $G/A \cong \text{Sym}_n$ is odd. Lemma 2.35 implies that $g \in HA$, and so we may assume that $g \in A$. Thus $K \cap A = H^g \cap A$, i.e. $K_0 = K \cap A = K_1 \times \ldots \times K_n$ and $H \cap A = H_1 \times \ldots \times H_n$ are conjugate in $A = L_1 \times \ldots \times L_n$. In particular, $K_1$ and $H_1$ are conjugate in $L_1$, a contradiction. □

3 Proof of Theorem 1

Theorem 1 says that, for a finite simple group $G$, the following two statements are equivalent.

(1) $G \in \mathcal{D}_\pi$ and

(2) either $G \in \mathfrak{X}$ or $\pi(G) \nsubseteq \pi$ and $G \in \mathcal{D}_\pi$.

(2) $\Rightarrow$ (1). Obviously $\mathfrak{X} \subseteq \mathcal{D}_\pi$. So we need to prove that if $\pi(G) \nsubseteq \pi$ and $G \in \mathcal{D}_\pi$ then $G \in \mathcal{D}_\pi$. Lemma 2.28 implies that every $\pi$-Hall subgroup (hence every $\pi$-subgroup of $G$, since $G \in \mathcal{D}_\pi$) is solvable, thus it belongs to $\mathfrak{X}$. On the other hand, every $\mathfrak{X}$-subgroup of $G$ is a $\pi$-subgroup and so is contained in a $\pi$-Hall subgroup (we again use $G \in \mathcal{D}_\pi$ here). Therefore, $m_{\mathfrak{X}}(G) = \text{Hall}_\pi(G)$. Hence every two $\mathfrak{X}$-maximal subgroups of $G$ are conjugate, i.e. $G \in \mathcal{D}_\pi$.

(1) $\Rightarrow$ (2). This implication is much harder to prove. The proof of the implication requires case by case consideration and we organize it in a series of steps, and divide it in the following subsections.

3.1 Proof of the implication (1) $\Rightarrow$ (2): general remarks

Assume that $G \in \mathcal{D}_\pi$ and $G \nsubseteq \mathfrak{X}$. We need to show that $G \in \mathcal{D}_\pi$.

Lemma 2.3 implies that
(i) \( m_X(G) = X \cap \text{Hall}_\pi(G) = \text{Hall}_\pi(G) \). In particular, \( G \in \mathcal{E}_\pi \) and all elements of \( \text{Hall}_X(G) \) are conjugate.

Suppose by contradiction that \( G \notin \mathcal{D}_\pi \). Then

(ii) There exists a \( \pi \)-subgroup of \( G \) which does not belong to \( \mathcal{X} \).

Otherwise the \( \pi \)-subgroups of \( G \) are exactly the \( \mathcal{X} \)-subgroups, thus the \( \pi \)-maximal subgroups of \( G \) are conjugate, i.e. \( G \in \mathcal{D}_\pi \).

The inclusion \( \mathcal{G}_\pi \subseteq \mathcal{X} \) and (ii) immediately imply

(iii) There exists a non-solvable \( \pi \)-subgroup in \( G \).

The solvability of primary and biprimary groups [10, Ch. I, 2] and (iii) implies

(iv) \( |\pi \cap \pi(G)| > 2 \).

The Feit–Thompson theorem [12] implies

(v) \( 2 \in \pi \cap \pi(G) \).

Moreover, it follows from (v) and Lemma 2.27 that

(vi) \( 3 \in \pi \cap \pi(G) \).

Now we prove that

(vii) \( G \) has no solvable \( \pi \)-Hall subgroups.

Indeed, if \( G \) has a solvable \( \pi \)-Hall subgroup \( H \), then \( H \in \mathcal{X} \cap \text{Hall}_\pi(G) = m_X(G) \). In view of (v), (vi) and the Hall theorem, \( H \) contains a \( \{2, 3\} \)-Hall subgroup \( H_0 \) and \( H_0 \in \text{Hall}_{\{2,3\}}(G) \). Take an arbitrary \( \{2, 3\} \)-subgroup \( U \) in \( G \). Since \( U \) is solvable and in view of (v) and (vi) we have \( U \in \mathcal{X} \). Now \( G \in \mathcal{D}_\mathcal{X} \) implies that \( U \) is conjugate to a subgroup of \( H \). Moreover, the solvability of \( H \) means that \( U \) is conjugate to a subgroup of \( H_0 \) by the Hall theorem. Hence \( G \in \mathcal{D}_{\{2,3\}} \), a contradiction with Lemma 2.23.

Now we exclude all possibilities for \( G \), considering finite simple groups case by case, according to the Classification of the finite simple groups.

3.2 Alternating groups

The following statement follows from Lemma 2.19.

(viii) \( G \) is not isomorphic to an alternating group.
3.3 Sporadic groups and Tits group

Now, exclude any possibilities for $G$ to be a sporadic group.

(ix) $G$ is not isomorphic to the Mathieu group $M_{11}$.

Suppose that, $G = M_{11}$. According to Lemma 2.10 and Table 2 and in view of (v)–(vii) it is sufficient to consider the situation $\pi \cap \pi(G) = \{2, 3, 5\}$ and a Hall $\mathfrak{X}$-subgroup $H$ of $G$ is $M_{10} = \text{Alt}_6 \cdot 2$. Take a $\{2, 3\}$-Hall subgroup $U$ of $G$ (this group appears in Table 4). Since $U$ is a solvable $\pi$-group, we have $U \in \mathfrak{X}$. Now $G \in \mathfrak{D}_X$ implies that $U$ is conjugate to a subgroup of $H$. But this means that $H$ and its unique nonabelian composition factor $\text{Alt}_6$ satisfy $\delta_{\{2,3\}}$. A contradiction with Lemma 2.14.

(x) $G$ is not isomorphic to the Mathieu group $M_{22}$.

According to Lemma 2.16 and Table 2 if $G = M_{22}$, then an $\mathfrak{X}$-Hall subgroup $H$ of $G$ is isomorphic to $2^4 : \text{Alt}_6$. But $G$ contains a maximal subgroup $U \simeq 2^4 : \text{Sym}_5$ which is an $\mathfrak{X}$-group and is not isomorphic to a subgroup of $H$.

(xi) $G$ is not isomorphic to the Mathieu group $M_{23}$.

Suppose that $G = M_{23}$ and $H \in \text{Hall}_{\mathfrak{X}}(G)$. Lemma 2.10, Table 2 and (vii) imply that one of the following cases holds.

(a) $\pi \cap \pi(G) = \{2, 3, 5\}$ and $H \cong 2^4 : \text{Alt}_6$;
(b) $\pi \cap \pi(G) = \{2, 3, 5\}$ and $H \cong 2^4 : (3 \times \text{Alt}_5)$;
(c) $\pi \cap \pi(G) = \{2, 3, 5, 7\}$ and $H \cong \text{PSL}_3(4) : 2_2$;
(d) $\pi \cap \pi(G) = \{2, 3, 5, 7\}$ and $H \cong 2^4 : \text{Alt}_7$;
(e) $\pi \cap \pi(G) = \{2, 3, 5, 7, 11\}$ and $H \cong M_{22}$.

In Case (a), we consider an $\mathfrak{X}$-subgroup $U \cong 2^4 : (3 \times \text{Alt}_5)$ which is a $\pi$-Hall subgroup (and appears in Case (b)) and is not isomorphic to $H$.

Suppose that Case (b) holds. In $G$, consider a $\{2,3\}$-subgroup $U \cong 3^2 : Q_8$, a Frobenius group which is contained in $\text{PSL}_3(4)$, see [3]. Suppose, $U$ is a subgroup of $H$. Let

$$\pi : H \to H/O_2(H)$$

be the natural epimorphism. Since $U$ has no non-trivial normal 2-subgroups, we have

$$U \cong \overline{U} \leq H \cong 3 \times \text{Alt}_5.$$

Now $|U|_3 = |H|_3 = 3^2$, i.e. $\overline{U}$ contains a Sylow 3-subgroup of $\overline{H}$ and the cyclic subgroup $O_3(\overline{H})$ of order 3 must be a normal subgroup in $\overline{U}$. But $U \ncong \overline{U}$ has no normal subgroups of order 3. A contradiction.

We exclude Cases (c), (d) and (e), since the subgroup $H$ does not contain elements of order 15 in these cases while $M_{23}$ has a cyclic subgroup $U$ of order 15 and $U \in \mathfrak{S}_n \subseteq \mathfrak{X}$.

(xii) $G$ is not isomorphic to the Mathieu group $M_{24}$.

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If $G = M_{24}$, then an $\mathfrak{X}$-Hall subgroup $H$ is isomorphic to $2^6 : 3 \cdot \text{Sym}_6$. Consider an $\mathfrak{X}$ subgroup $U \cong 2^4 : \text{Alt}_6$ which is included in a maximal subgroup $M = M_{23}$ of $G$ and is a $\{2, 3, 5\}$-Hall subgroup of $M$. Since $G \in \mathcal{D}_{\mathfrak{X}}$, without loss of generality, we can assume that $U \leq H$. Now $U$ contains a subgroup $U_0 \cong \text{Alt}_6$ and, clearly, $U_0 \cap O_2(H) = 1$. Let $\pi : H \to H/O_2(H)$ be the natural epimorphism. We have

$$\text{Alt}_6 \cong U_0 \cong \overline{U}_0 \leq \overline{H} \cong 3 \cdot \text{Sym}_6.$$ 

But this means that $\text{Alt}_6 \cong U_0 = U'_0 \leq H'_0 \cong 3 \cdot \text{Alt}_6$ and we have a contradiction.

(xiii) $G$ is not isomorphic to the Janko group $J_1$.

Suppose that $G = J_1$ and $H \in \text{Hall}_\mathfrak{X}(G)$. It follows from Lemma 2.16, Table 2 and (vii) that $H \cong 2 \times \text{Alt}_5$. Clearly, $H$ contains no elements of order 15. But $G$ has a cyclic subgroup $U$ of order 15 (see [9]) and $U \in \mathfrak{S}_x \subseteq \mathfrak{X}$, a contradiction.

(xiv) $G$ is not isomorphic to the Janko group $J_4$.

Suppose, $G = J_4$ and $H \in \text{Hall}_\mathfrak{X}(G)$. Lemma 2.16 and Table 2 imply that

$$H \cong 2^{11} : 2^6 : 3 \cdot \text{Sym}_6.$$ 

We exclude this possibility arguing exactly as in (xii), because $G$ contains a subgroup isomorphic to $M_{24}$.

(xv) $G$ is not isomorphic to any sporadic group or a Tits group.

This statement follows from (v) Lemma 2.10 and (ix)–(xiv).

3.4 Groups of Lie type of characteristic $p \in \pi(\mathfrak{X})$

Now, according to Lemma 2.19, we exclude the possibilities for $G$ to be isomorphic to a group of Lie type whose characteristic belongs to $\pi$.

(xvi) If $G$ is a group of Lie type, then $G$ has no $\pi$-Hall subgroups contained in a Borel subgroup.

Since every Borel subgroup of $G$ is solvable, (xvi) follows from (vii).

(xvii) $G$ is not isomorphic to $D_l(q)$, where $q$ is a power of some $p \in \pi$.

Suppose that $G \cong D_l(q)$ and the numeration of the roots in a fundamental root system $\Pi$ of $G$ is chosen as in the Dynkin diagram on Pic. 1. It follows from Lemma 2.19 that $q$ is a power of 2, $l$ is a Fermat prime (in particular, $l \geq 5$), and $(l, q-1) = 1$. Moreover, if $H \in \text{Hall}_\mathfrak{X}(G)$, then $H$ is conjugate to the canonic parabolic maximal subgroup corresponding
to the set $\Pi \setminus \{r_1\}$ of fundamental roots. This parabolic subgroup has a composition factor isomorphic to $D_{l-1}(q)$. Since $\mathfrak{X}$ is a complete class, we obtain that

$$D_i(q) \in \mathfrak{X} \quad \text{for} \quad i \leq l - 1, \quad \text{and} \quad A_1(q) \in \mathfrak{X}.$$ 

Moreover, $\pi(q - 1) \subseteq \pi$. Consider the canonic parabolic maximal subgroup $P_J$ of $G$, corresponding to the set $J = \Pi \setminus \{r_2\}$. Above remarks and the completeness of $\mathfrak{X}$ under extensions implies that $P_J \in \mathfrak{X}$: the nonabelian composition factors of $P$ are isomorphic to $D_{l-2}(q)$ and, possibly, $A_1(q)$, while the orders of abelian composition factors belong to $\pi(q - 1) \cup \{2\} \subseteq \pi$. But the maximality of $P$ means that $P_J$ is not conjugate to any subgroup of $H$, a contradiction with $G \in \mathcal{D}_\mathfrak{X}$.

(xviii) $G$ is not isomorphic to $^2D_l(q)$, where $q$ is a power of some $p \in \pi$.

Suppose that $G \cong ^2D_l(q)$ and the numeration of the roots in a fundamental root system $\Pi^1$ of $G$ is chosen as in the Dynkin diagram on Pic. 2. It follows from Lemma 2.19 that $q$ is a power of $2$, $l - 1$ is a Mersinne prime, and $(l-1, q-1) = 1$. Take $H \in \text{Hall}_\mathfrak{X}(G)$. Then $H$ is conjugate to the canonic parabolic maximal subgroup corresponding to the set $\Pi^1 \setminus \{r_1\}$ of fundamental roots. This parabolic subgroup has a composition factor isomorphic to $^2D_{l-1}(q)$ if $l > 4$ or isomorphic to $^2A_3(q)$ if $l = 4$. Consider the canonic parabolic maximal subgroup $P_J$ of $G$ which corresponds to the set $J = \Pi^1 \setminus \{r_1\}$ of fundamental roots. Arguing as in (xviii), we see that $P_J \in \mathfrak{X}$ and $P$ is not conjugate to any subgroup of $H$, a contradiction with $G \in \mathcal{D}_\mathfrak{X}$.

(xix) $G$ is not isomorphic to $A_{l-1}(q) \cong \text{PSL}_l(q)$, where $q$ is a power of some $p \in \pi$.

Suppose that $G = \text{PSL}_n(q)$, where $q$ is a power of some $p \in \pi$, and let $G^* = \text{SL}_n(q)$. Lemma 2.8 implies that $G \in \mathcal{D}_\mathfrak{X}$ if and only if $G^* \in \mathcal{D}_\mathfrak{X}$. Thus, $G^* \in \mathcal{D}_\mathfrak{X}$ and, moreover, it follows from (vii) and Lemma 2.8 that there are no solvable $\pi$-Hall subgroups in $G^*$.

Identify $G^*$ with $\text{SL}(V)$, where $V = \mathbb{F}_q^n$ is the natural $n$-dimensional module for $G^*$. Let $H^* \in \text{Hall}_\mathfrak{X}(G^*)$. By Lemma 2.19, $H^*$ is the stabilizer in $G^*$ of a series

$$0 = V_0 < V_1 < \cdots < V_s = V$$

of subspaces such that $\dim V_i/V_{i-1} = n_i, i = 1, 2, \ldots, s$, and one of the following conditions holds:

(a) $n$ is a prime, $s = 2, n_1, n_2 \in \{1, n - 1\}$;
(b) $n = 4, s = 2, n_1 = n_2 = 2$; moreover, $q = 2^{2t+1}$;
(c) $n = 5, s = 2, n_1, n_2 \in \{2, 3\}$;
(d) $n = 5, s = 3, n_1, n_2, n_3 \in \{1, 2\}$;
(e) $n = 7, s = 2, n_1, n_2 \in \{3, 4\}$;
(f) $n = 8, s = 2, n_1 = n_2 = 4$; moreover, $q = 2^{2t}$;
(g) $n = 11, s = 2, n_1, n_2 \in \{5, 6\}$. 

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In cases (a), (c), (e), and (g), \( H^* \) is the stabilizer of a subspace of some dimension \( m \neq n - m \) and the stabilizer \( K^* \) of a subspace of dimension \( n - m \) is isomorphic to \( H^* \) (in particular, \( K^* \in \mathfrak{X} \)) but is not conjugate to \( H^* \). It contradicts \( G^* \in \mathcal{D}_X \).

If case (d) holds, then there are exactly three conjugacy classes of \( \pi \)-Hall subgroups with the same composition factors and \( H^* \) belongs to one of them. Thus, case (d) is impossible for \( G^* \in \mathcal{D}_X \).

Now consider cases (b) and (f). In these cases \( n = 4 \) and \( n = 8 \), respectively. Moreover, if \( q = 2 \) then case (b) holds and \( G = \text{PSL}_4(2) \cong \text{Alt}_8 \notin \mathcal{D}_X \) in view of (viii). Therefore, we assume that \( q > 2 \) if \( n = 4 \). Define \( r = n - 1 = 3 \) in (b) and \( r = n - 1 = 7 \) in (f). It is easy to check that \( r \in \pi \) in both cases. Consider the subgroup \( U^* \) of \( G^* \), consisting of all matrices of type

\[
\begin{pmatrix}
a & 1 \\
1 & 0
\end{pmatrix},
\]

where \( a \in \langle D, x \rangle \leq \text{SL}_r(q) \), \( D \) is the group of all diagonal matrices in \( \text{SL}_r(q) \) and

\[
x = \begin{pmatrix}
0 & 1 & & \\
0 & 1 & & \\
& & & \\
1 & 0 & & 
\end{pmatrix} \in \text{SL}_r(q).
\]

By Lemma 2.32 it follows that there is a subspace \( W \) of \( V \) of dimension \( r \), such that \( U^* \) acts irreducibly on \( W \). Clearly, \( U^* \) cannot stabilize any subspace of dimension \( n/2 = 2 \) in case (b) or \( n/2 = 4 \) in case (f). Therefore, \( U^* \) is not conjugate to any subgroup of \( H^* \). By Lemma 2.32, \( U^* \) is a solvable \( \pi \)-group, so \( U^* \in \mathfrak{X} \), a contradiction with \( G^* \in \mathcal{D}_X \).

\((xx)\) \( G \) is not isomorphic to any group of Lie type of characteristic \( p \in \pi \).

This statement follows from (xvi)–(xix) and Lemma 2.19.

### 3.5 Classical groups of characteristic \( p \notin \pi(\mathfrak{X}) \)

In view of (xx), \( G \) is a group of Lie type over a field of an order \( q \) and characteristic \( p \notin \pi \). In particular, \( p \neq 2, 3 \).

We start with the smallest case \( G = \text{PSL}_2(q) \).

\((xxi)\) \( G \) is not isomorphic to \( \text{PSL}_2(q) \).

Suppose \( G = \text{PSL}_2(q) \), and denote \( G^* = \text{SL}_2(q) \). Then \( G^* \in \mathcal{D}_X \) and \( G^* \) has no solvable \( \pi \)-Hall subgroups by (vii) and Lemmas 2.1 and 2.8, so statement (d) of Lemma 2.17 holds. Therefore if \( H^* \) is an \( \mathfrak{X} \)-Hall subgroup of \( G^* \) then the image of \( H^* \) in \( G^*/Z(G^*) \cong G \) is isomorphic to \( \text{Alt}_5 \). But in this case there are exactly two conjugacy classes of \( \mathfrak{X} \)-Hall subgroup in \( G \). It contradicts \( G \in \mathcal{D}_X \).

Now we show that \( G \) is not isomorphic to a classical group. First we consider the most transparent case of symplectic groups. Similar, but more complicated, arguments appear in the consideration of the other types of classical groups: linear, unitary and orthogonal.

\((xxii)\) \( G \) is not isomorphic to \( \text{PSp}_{2n}(q) \).
Suppose \( G = \text{PSp}_{2n}(q) \) and denote \( G^* = \text{Sp}_{2n}(q) \). By (viii) and Lemma 2.21, we have \( G^* \in \mathcal{D}_X \) and \( G^* \) has no solvable \( \pi \)-Hall subgroups. Consider \( H^* \in \text{Hall}_X(G^*) \). We claim that

- \( \pi \cap \pi(G^*) \subseteq \pi(q^2 - 1) \);
- \( H^* \) is included in a subgroup
  \[ M^* \cong \text{SL}_2(q) \wr \text{Sym}_n, \]
  we denote by \( B^* \) the base of this wreath product;
- \( H^*/(H^* \cap B^*) \) is isomorphic to a \( \pi \)-Hall subgroup of \( \text{Sym}_n \);
- \( H^* \cap B^* \) is solvable.

First two items can be found in Lemma 2.21, the third item follows by Lemma 2.1. The last item follows by Lemma 2.33, since if \( H^* \cap B^* \) is nonsolvable, then, for some component \( L^* = \text{SL}_2(q) \), \( H^* \cap L^* \) is a nonsolvable \( \pi \)-Hall subgroup of \( L^* \). Now Lemma 2.11 implies that \( L^* \) possesses \( \pi \)-Hall subgroup that is isomorphic and nonconjugate to \( H^* \cap L^* \). Finally Lemma 2.33 implies that \( M^* \) possesses a \( \pi \)-Hall subgroup that is nonconjugate to \( H^* \) but have the same composition factors. Lemma 2.21(C) implies that \( G^* \) possesses nonconjugate \( X \)-Hall subgroups, a contradiction with \( G^* \in \mathcal{D}_X \).

The nonsolvability of \( H^* \) and Lemma 2.14 imply that \( H^*/(H^* \cap B^*) \) is isomorphic to a symmetric group of degree \( n \) or \( n - 1 \) and this degree is at least 5. In particular, \( 5 \in \pi \cap \pi(G^*) \), \( \text{Alt}_5 \in X \), and 5 divides \( q^2 - 1 \). Moreover, \( H^* \cap B^* \) coincides with the solvable radical \( H_5^* \) of \( H^* \).

Lemma 2.33 implies that \( G^* \) possesses a collection \( \Delta \) of subgroups isomorphic to \( \text{SL}_2(q) \) such that \( |\Delta| = n \) and \( [K^*, L^*] = 1 \) for every \( K^*, L^* \in \Delta \) and \( K^* \neq L^* \). Since 5 divides \( q^2 - 1 \), Lemma 2.30 implies that \( \text{SL}_2(q) \) possesses a subgroup isomorphic to \( \text{SL}_2(5) \). For every \( K^* \in \Delta \) fix some \( U(K^*) \leq K^* \) such that \( U(K^*) \cong \text{SL}_2(5) \). Set
\[
U^* = \langle U(K^*) \mid K^* \in \Delta \rangle.
\]
It follows from the definition that
\[
U^*/U^*_\mathcal{D} \cong \text{Alt}_5 \times \cdots \times \text{Alt}_5\]
and \( U^*_\mathcal{D} \) is a 2-group. Thus, \( U^* \in X \).

We show that \( U^* \) is not conjugate to a subgroup of \( H^* \), and this contradicts \( G^* \in \mathcal{D}_X \). Indeed, if \( U^* \) is conjugate to a subgroup of \( H^* \), then we can assume that \( U^* \leq H^* \). Denote by \( R^* = H^* \cap B^* \) the solvable radical of \( H^* \) and let
\[
\phi : H^* \to H^*/R^* = \overline{H}^*
\]
be the natural epimorphism. We have seen above that \( \overline{H}^* \) is isomorphic to a subgroup of \( \text{Sym}_n \). Therefore,
\[
\mu(\overline{U}^*) \leq \mu(\overline{H}^*) \leq n.
\]
On the other hand, \( \overline{U}^*/U^*_\mathcal{D} \cong U^*/U^*_\mathcal{D} \) and Lemma 2.29 implies that
\[
\mu(\overline{U}^*) \geq \mu(U^*/U^*_\mathcal{D}) = 5n > n.
\]
It contradicts the previous inequality.

Thus, (xxii) is proved.
(xxiii) $G$ is not isomorphic to $\text{PSL}_n^\gamma(q)$, $\eta = \pm$.

Suppose $G = \text{PSL}_n^\gamma(q)$ and denote $G^* = \text{SL}_n^\gamma(q)$. By (viii) and Lemma 2.33 we have $G^* \in \mathcal{D}_X$ and $G^*$ has no solvable $\pi$-Hall subgroups. Let $H^* \in \text{Hall}_X(G^*)$. Consider all possibilities for $H^*$ given in statements (a)--(e) of Lemma 2.20.

In case (a) $n$ is equal to 2, and this case is excluded in view of (xxi).

In case (d) $H^*$ is isomorphic to $4.2^4.\text{Alt}_6$. In this case $G^*$ has two conjugacy classes of $\pi$-Hall subgroups isomorphic $4.2^4.\text{Alt}_6$. So if $H^*$ satisfies (d), then this contradicts $G^* \in \mathcal{D}_X$.

In case (e) we have $\pi \cap \pi(G^*) = \{2, 3\}$. So $H^*$ is solvable and this case is excluded in view of (vii).

Thus, one of the following statements holds.

(b) $q \equiv \eta \pmod{4}$, $\text{Sym}_n$ satisfies $\mathcal{E}_\pi$, $\pi \cap \pi(G) \subseteq \pi(q - \eta) \cup \pi(n!)$, and if $r \in (\pi \cap \pi(n!)) \setminus \pi(q - \eta)$, then $|G^*|_r = |\text{Sym}_n|_r$. In this case $H^*$ is included in

$$M^* = L^* \cap G^* \cong (q - \eta)^{n-1} \cdot \text{Sym}_n,$$

where $L^* = \text{GL}_1^\gamma(q) \cdot \text{Sym}_n \leq \text{GL}_n^\gamma(q)$.

(c) $n = 2m + k$, where $k \in \{0, 1\}$, $m \geq 1$, $q \equiv -\eta \pmod{3}$, $\pi \cap \pi(G) \subseteq \pi(q^2 - 1)$, the groups $\text{Sym}_m$ and $\text{GL}_2^\gamma(q)$ satisfy $\mathcal{E}_\pi$. In this case $H^*$ is contained in

$$M^* = L^* \cap G^* \cong \left(\text{GL}_1^\gamma(q) \circ \cdots \circ \text{GL}_2^\gamma(q)\right) \cdot \text{Sym}_m \circ Z,$$

$m$ times

where $L^* = \text{GL}_2^\gamma(q) \cdot \text{Sym}_m \times Z \leq \text{GL}_n(q)$ and $Z$ is a cyclic group of order $q - \eta$ for $k = 1$, and $Z$ is trivial for $k = 0$. The intersection of $H^*$ with each factor $\text{GL}_2^\gamma(q)$ is a $\pi$-Hall subgroup in $\text{GL}_2^\gamma(q)$.

By Lemma 2.13 $\pi$-Hall subgroups of $\text{GL}_2^\gamma(q)$ are solvable. Since $H^*$ is nonsolvable, every nonabelian composition factor of $H^*$ is a composition factor of a $\pi$-Hall subgroup of a symmetric group of degree at most $n$. In both cases (b) and (c), it follows from Lemma 2.14 that

- every nonabelian composition factor of $H^*$ is isomorphic to an alternating group; in particular
- $5 \in \pi \cap \pi(G^*)$, $\text{Alt}_5 \in \mathcal{X}$ and $n \geq 5$; and
- $H^*/H^*_5$ is isomorphic to a subgroup of $\text{Sym}_n$.

Now we consider two cases: 5 divides $q^2 - 1$ and 5 does not divide $q^2 - 1$.

Suppose, 5 divides $q^2 - 1$. In this case we argue similarly the case of symplectic groups above. Lemma 2.33 implies that $G^*$ possesses a collection $\Delta$ of subgroups isomorphic to $\text{SL}_2(q)$ such that $|\Delta| = [n/2]$ and $[K^*, L^*] = 1$ for every $K^*, L^* \in \Delta$ and $K^* \neq L^*$. Since 5 divides $q^2 - 1$, Lemma 2.30 implies that $\text{SL}_2(q)$ possesses a subgroup isomorphic to $\text{SL}_2(5)$. For every $K^* \in \Delta$ fix some $U(K^*) \leq K^*$ such that $U(K^*) \cong \text{SL}_2(5)$. Set

$$U^* = \langle U(K^*) \mid K^* \in \Delta \rangle.$$
Indeed, if \( U^* \) is not conjugate to a subgroup of \( H^* \) and this contradicts \( G^* \in \mathcal{D}_X \).

We claim that \( U^* \) is not conjugate to a subgroup of \( H^* \) and this contradicts \( G^* \in \mathcal{D}_X \).

On the other hand, Lemma 2.29 implies that it contradicts the previous inequality. Hence 5 does not divide \( q^2 - 1 \).

Suppose that 5 does not divide \( q^2 - 1 \). It means that case (b) holds (in particular, the solvable radical \( H^*_\mathbb{S} \) of \( H^* \) is abelian) and \( |G^*|_5 = |\text{Sym}_n|_5 \). We have

\[ |G^*|_5 = \prod_{i=1}^n (q^i - \eta^i)_5 \quad \text{and} \quad |\text{Sym}_n|_5 = (n!)_5. \]

Lemma 2.13 implies that \( [n/4] = [n/5] \). Since \( n \geq 5 \), this means \( n \in \{5, 6, 7, 10, 11, 15\} \).

Assume that \( n \in \{5, 6, 7\} \) first. Since \( \text{Sym}_n \in \mathcal{D}_\pi \) and a \( \pi \)-Hall subgroup of \( \text{Sym}_n \) belongs to \( \mathcal{X} \), it follows from Lemma 2.14 that \( \text{Alt}_5 \in \mathcal{X} \). Moreover, if \( n = 6 \) or \( n = 7 \), then \( \text{Alt}_6 \in \mathcal{X} \).

The group \( G^* \) has a subgroup isomorphic to \( SL_2^\eta(q) \). Moreover, (b) implies that \( q \equiv \eta \pmod{4} \) and by Lemma 2.31, \( G^* \) has a subgroup

\[ W^* \cong 4 \circ 2^{1+4} \cdot \text{Alt}_6. \]

Define \( U^* \leq G^* \) in the following way. If \( n = 6, 7 \), then \( U^* = W^* \). If \( n = 5 \), then \( W^*/W^*_\mathbb{S} \cong \text{Alt}_6 \) contains a subgroup isomorphic to \( \text{Alt}_5 \), and we set \( U^* \) to be equal to its full preimage in \( W^* \). By construction \( U^* \in \mathcal{X} \).

We claim that \( U^* \) is not conjugate to any subgroup of \( H^* \) and this contradicts \( G^* \in \mathcal{D}_X \).

Indeed, if \( U^* \leq H^* \) and \( R^* = H^*_\mathbb{S} \) is the solvable radical of \( H^* \) then \( U^*/(U^* \cap R^*) \) is isomorphic to a subgroup of \( H^*/R^* \leq \text{Sym}_n \). We have that \( U^*/U^*_\mathbb{S} \cong \text{Alt}_m \) for some \( m \in \{n, n-1\} \). Since \( U^*/U^*_\mathbb{S} \) is a homomorphic image of \( U^*/(U^* \cap R^*) \), it follows by Lemma 2.34 that \( U^*/(U^* \cap R^*) \cong \text{Alt}_m \). Therefore, \( U^*_\mathbb{S} \cong U^* \cap R^* \). This is impossible, since \( R^* \) is abelian, while \( U^*_\mathbb{S} \cong 4 \circ 2^{1+4} \) contains an extra special 2-subgroup of order \( 2^5 \).

Assume finally that \( n \in \{10, 11, 15\} \). Lemma 2.14 implies that \( \text{Alt}_{10} \in \mathcal{X} \). Therefore \( \text{Alt}_6 \in \mathcal{X} \). It is clear that \( G^* \) has a subgroup

\[ SL_4^\eta(q) \circ SL_4^\eta(q) \text{ if } n \in \{10, 11\}, \text{ and } SL_4^\eta(q) \circ SL_4^\eta(q) \circ SL_4^\eta(q) \text{ if } n = 15. \]
By Lemma 2.31 and in view of \( q \equiv \eta \pmod{4} \), we can find a subgroup \( U^* \) in \( G^* \) such that \( U^* = O_2(U^*) \) and

\[
U^*/U^*_S \cong \begin{cases} 
\text{Alt}_6 \times \text{Alt}_6, & \text{if } n \in \{10, 11\}, \\
\text{Alt}_6 \times \text{Alt}_6 \times \text{Alt}_6, & \text{if } n = 15.
\end{cases}
\]

Clearly, \( U^* \in \mathcal{X} \). But \( U^* \) is not conjugate to a subgroup of \( H^* \). Indeed, if \( U^* \leq H^* \), then \( U^*/(U^* \cap R^*) \) is isomorphic to a subgroup of \( \text{Sym}_n \), where \( R^* = H^*_S \). Therefore, by Lemma 2.29 we have

\[
n \geq \mu(U^*/(U^* \cap R^*)) \geq \mu(U^*/U^*_S) = \begin{cases} 
\mu(\text{Alt}_6 \times \text{Alt}_6) = 12, & \text{if } n \in \{10, 11\}, \\
\mu(\text{Alt}_6 \times \text{Alt}_6 \times \text{Alt}_6) = 18, & \text{if } n = 15,
\end{cases}
\]

a contradiction.

Thus, (xxiii) is proven.

(xxiv) \( G \) is not isomorphic to \( P\Omega_3^n(q), \eta \in \{+, -, \circ\} \).

Suppose \( G = P\Omega_3^n(q), n \geq 7 \) and denote \( G^* = \Omega_3^n(q) \). By (vii) and Lemma 2.8, we have \( G^* \in \mathcal{D}_X \), and \( G^* \) has no solvable \( \pi \)-Hall subgroups and has exactly one class of \( \mathcal{X} \)-Hall subgroups. Let \( H^* \in \text{Hall}_X(G^*) \). Consider all possibilities for \( H^* \) given in statements (a)–(h) of Lemma 2.22.

In cases (d) and (e) we have \( \pi \cap \pi(G^*) = \{2, 3\} \), and we exclude these cases in view of (vii) and the solvability of \( \{2, 3\} \)-groups.

We exclude cases (f), (g) and (h), since in all these cases there are at least two conjugacy classes of \( \mathcal{X} \)-Hall subgroups of \( G^* \) isomorphic to \( H^* \).

Thus, one of the following statements holds.

(a) \( n = 2m + 1, \pi \cap \pi(G^*) \subseteq \pi(q - \varepsilon), q \equiv \varepsilon \pmod{12}, \text{Sym}_m \in \mathcal{E}_n \), and \( H^* \) is a \( \pi \)-Hall subgroup in

\[
M^* = (O_2^+(q) \wr \text{Sym}_m \times O_1(q)) \cap G^*.
\]

(b) \( n = 2m, \eta = \varepsilon^m, \pi \cap \pi(G^*) \subseteq \pi(q - \varepsilon), q \equiv \varepsilon \pmod{12}, \text{Sym}_m \in \mathcal{E}_n \), and \( H \) is a \( \pi \)-Hall subgroup in

\[
M^* = (O_2^{-}(q) \wr \text{Sym}_m \cap G^*.
\]

(c) \( n = 2m, \eta = -\varepsilon^m, \pi \cap \pi(G^*) \subseteq \pi(q - \varepsilon), q \equiv \varepsilon \pmod{12}, \text{Sym}_{m-1} \in \mathcal{E}_n \), and \( H^* \) is a \( \pi \)-Hall subgroup of

\[
M^* = (O_2^{-}(q) \wr \text{Sym}_{m-1} \times O_2^{-\varepsilon}(q)) \cap G^*.
\]

Here \( \varepsilon = \pm 1 \) and \( q - \varepsilon \) is divisible by 4.

Groups \( O_2^+(q) \) and \( O_2^-(q) \) are solvable. As in the proofs of (xxiii) and (xxviii), we see that the symmetric group of degree \( m \), in cases (a) and (b), and of degree \( m - 1 \) in case (c) has nonsolvable \( \mathcal{X} \)-Hall subgroup which is isomorphic to a symmetric group. Moreover, this \( \mathcal{X} \)-Hall subgroup is isomorphic to \( H^*/H^*_S \). Thus,

- \( 5 \in \pi \cap \pi(G^*) \subseteq \pi(q - \varepsilon) \subseteq \pi(q^2 - 1) \),
- \( \text{Alt}_5 \in \mathcal{X} \) and
• $H^*/H^*_\Delta$ is isomorphic to a subgroup of $\text{Sym}_m$ in cases (a) and (b) and of $\text{Sym}_{m-1}$ in case (c). Therefore,

$$\mu(H^*/H^*_\Delta) \leq m = \lceil n/2 \rceil.$$ 

Lemma 2.33 implies that $G^*$ possesses a collection $\Delta$ of subgroups isomorphic to $\text{SL}_2(q)$ such that $|\Delta| = k \geq 2[(n - 1)/4]$ and $[K^*, L^*] = 1$ for every $K^*, L^* \in \Delta$ and $K^* \neq L^*$. Since $5$ divides $q^2 - 1$, Lemma 2.30 implies that $\text{SL}_2(q)$ possesses a subgroup isomorphic to $\text{SL}_2(5)$. For every $K^* \in \Delta$ fix some $U(K^*) \leq K^*$ such that $U(K^*) \cong \text{SL}_2(5)$. Set

$$U^* = \langle U(K^*) \mid K^* \in \Delta \rangle.$$ 

It follows from the definition that

$$U^*/U^*_\Delta \cong \underbrace{\text{Alt}_5 \times \cdots \times \text{Alt}_5}_{k \text{ times}}$$

and $U^*_\Delta$ is a 2-group. Thus, $U^* \in \mathfrak{X}$.

We show that $U^*$ is not conjugate to a subgroup of $H^*$. Otherwise we can assume that $U^* \leq H^*$. Let $R^* = H^*_\Delta$ and let

$$\varphi: H^* \to H^*/R^* = \overline{H}^*$$

be the natural epimorphism. Therefore,

$$\mu(\overline{U}^*) \leq \mu(\overline{H}^*) \leq \lceil n/2 \rceil.$$ 

On the other hand, since $n \geq 7$ and in view of Lemma 2.29, we have

$$\mu(\overline{U}^*) \geq \mu(U^*/U^*_\Delta) = 5k \geq 10 \left\lceil \frac{n - 1}{4} \right\rceil \geq \frac{10(n - 4)}{4} = \frac{5(n - 4)}{2} > \frac{n + 1}{2} \geq \left\lceil \frac{n}{2} \right\rceil,$$

a contradiction.

\textit{(xxv)} $G$ is not isomorphic to a classical group.

This statement follows from (xx) if characteristic of a group belongs to $\pi$ and from \textit{(xxii)}–\textit{(xxiv)} in over cases.

### 3.6 Exceptional groups of Lie type of characteristic $p \notin \pi(\mathfrak{X})$

\textit{(xxv)} $G$ is not isomorphic to one of groups $2B_2(2^{2m+1})$, $2G_2(3^{2m+1})$, and $2F_4(2^{2m+1})$.

This statement follows from (v), (vi) and (xx).

\textit{(xxvi)} $G$ is not isomorphic to $G_2(q)$.

Suppose that $G = G_2(q)$ and $H \in \text{Hall}_x(G)$. By Lemma 2.23, either $H$ is solvable, which contradicts (vii), or statement (d) of Lemma 2.23 holds:

\textit{(d) $G = G_2(q)$, $\pi \cap \pi(G) = \{2, 3, 7\}$, $(q^2 - 1)_{(2,3,7)} = 24$, $(q^4 + q^2 + 1)_7 = 7$, and $H \cong G_2(2)$.}
By [9], \( H' \cong \text{PSU}_3(3) \) has a maximal subgroup isomorphic to \( \text{SL}_3(2) \cong \text{PSL}_2(7) \) and every maximal subgroup of \( H' \) not isomorphic to \( \text{SL}_3(2) \) is solvable. This implies that \( \text{SL}_3(2) \in \mathcal{X} \) and \( H \) has no subgroups isomorphic to \( 2^3 \cdot \text{SL}_3(2) \), which belongs to \( \mathcal{X} \). On the other hand, it follows from \([8}\ Table 1\) that \( G \) has a subgroups isomorphic to \( 2^3 \cdot \text{SL}_3(2) \).

\textbf{(xxvii)} \( G \) is not isomorphic to one of groups \( 3D_4(q) \) and \( F_4(q) \).

By Lemma \( \text{2.23} \), every Hall \( \mathcal{X} \)-subgroup of \( 3D_4(q) \) and \( F_4(q) \) is solvable, which contradicts (vii), if \( G \in \{ 3D_4(q), F_4(q) \} \).

\textbf{(xxviii)} \( G \) is not isomorphic to one of groups \( E_6(q) \) and \( 2E_6(q) \).

Suppose \( G = E_6^\eta(q), \eta = \pm 1 \) and \( H \in \text{Hall}_\chi(G) \). Since \( H \) is not solvable, statement (c) of Lemma \( \text{2.23} \) does not hold, and we have case (a) of this Lemma for \( E_6^\eta(q) \):

- 4 divides \( q - \eta \), \( \{2, 3, 5\} \subseteq \pi \cap \pi(G) \subseteq \pi(q - \eta) \), \( H \) is a \( \pi \)-Hall subgroup of a group \( T, \text{Sp}_4(3) \), where \( T \) is a maximal torus of order \( (q - \eta)^6/3 \).

Note that \( \text{Sp}_4(3) \) is a \( \pi \)-group. This implies that \( \text{Sp}_4(3) \) is a homomorphic image of \( H \) and \( \text{Sp}_4(3) \in \mathcal{X} \). Furthermore, \( H/H_\chi \cong \text{Sp}_4(3) \). By information in \([9]\), \( \text{Sp}_4(3) \) has a subgroup isomorphic to \( \text{Alt}_5 \). Therefore, \( \text{Alt}_5 \in \mathcal{X} \).

Lemma \( \text{2.33} \) implies that \( G \) possesses a collection \( \Delta \) of subgroups isomorphic to \( \text{SL}_2(q) \) such that \( |\Delta| = 4 \) and \( [K, L] = 1 \) for every \( K, L \in \Delta \) and \( K \neq L \). Since 5 divides \( q^2 - 1 \), Lemma \( \text{2.31} \) implies that \( \text{SL}_2(q) \) possesses a subgroup isomorphic to \( \text{SL}_2(5) \). For every \( K \in \Delta \) fix some \( U(K) \leq K \) such that \( U(K) \cong \text{SL}_2(5) \). Set

\[ U = \langle U(K) \mid K \in \Delta \rangle. \]

It follows from the definition that

\[ U/U_\chi \cong \text{Alt}_5 \times \text{Alt}_5 \times \text{Alt}_5 \times \text{Alt}_5 \]

and \( U_\chi \) is a 2-group. Thus, \( U \in \mathcal{X} \). Suppose that \( U \) is conjugate to a subgroup of \( H \). Then \( H/H_\chi \cong \text{Sp}_4(3) \) contains a subgroup for which \( U/U_\chi \) is a homomorphic image. But

\[ |H/H_\chi|_5 = |\text{Sp}_4(3)|_5 = 5 \times 5^4 = |\text{Alt}_5|_5^4 = |U/U_\chi|_5, \]

and this is impossible.

\textbf{(xxix)} \( G \) is not isomorphic to \( E_7(q) \).

Suppose \( G = E_7(q) \) and \( H \in \text{Hall}_\chi(G) \). By Lemma \( \text{2.23} \) we have:

- \( \{2, 3, 5, 7\} \subseteq \pi \cap \pi(G) \subseteq \pi(q - \varepsilon) \), where \( \varepsilon = \pm 1 \) is such that 4 divides \( q - \varepsilon \), \( H \) is a \( \pi \)-Hall subgroup of a group \( T, (2 \times \mathcal{P}_7(2)) \), where \( T \) is a maximal torus of order \( (q - \varepsilon)^7/2 \).

This implies that \( \mathcal{P}_7(2) \in \mathcal{X} \) and \( H/H_\chi \cong \mathcal{P}_7(2) \). In \( \mathcal{P}_7(2) \) there is a maximal subgroup \( \mathcal{P}_7^4(2) \cong \text{Sym}_8 \). Therefore, \( \text{Alt}_5 \in \mathcal{X} \).

Now we argue as in (xxviii). Lemma \( \text{2.33} \) implies that \( G \) possesses a collection \( \Delta \) of subgroups isomorphic to \( \text{SL}_2(q) \) such that \( |\Delta| = 7 \) and \( [K, L] = 1 \) for every \( K, L \in \Delta \) and \( K \neq L \). Since 5 divides \( q^2 - 1 \), Lemma \( \text{2.31} \) implies that \( \text{SL}_2(q) \) possesses a subgroup
isomorphic to \( \text{SL}_2(5) \). For every \( K \in \Delta \) fix some \( U(K) \trianglelefteq K \) such that \( U(K) \cong \text{SL}_2(5) \). Set
\[
U = \langle U(K) \mid K \in \Delta \rangle.
\]
It follows from the definition that
\[
U/U_\varnothing \cong \underbrace{\text{Alt}_5 \times \cdots \times \text{Alt}_5}_{7 \text{ times}}
\]
and \( U_\varnothing \) is a 2-group. Thus, \( U \in \mathcal{X} \). We claim that \( U \) is not conjugate to a subgroup of \( H \). It is sufficient to show that \( |U/U_\varnothing|_5 > |H/H_\varnothing|_5 \). Indeed,
\[
|H/H_\varnothing|_5 = |P\Omega_7(2)|_5 = 5 < 5^7 = |\text{Alt}_5|^7 = |U/U_\varnothing|_5,
\]
a contradiction with \( G \in \mathcal{D}_X \).

\( (xxx) \) \( G \) is not isomorphic to \( E_8(q) \).

Suppose \( G = E_7(q) \) and \( H \in \text{Hall}_\mathcal{X}(G) \). By Lemma 2.23 we have:

\begin{itemize}
  \item \( \{2, 3, 5, 7\} \subseteq \pi \cap \pi(G) \subseteq \pi(q - \varepsilon) \), where \( \varepsilon = \pm 1 \) is such that 4 divides \( q - \varepsilon \), \( H \) is a \( \pi \)-Hall subgroup of a group \( T \). \( P\Omega_8^+(2).2 \), where \( T \) is a maximal torus of order \( (q - \eta)^8 \).
\end{itemize}

This implies that \( P\Omega_8^+(2) \in \mathcal{X} \) and \( H/H_\varnothing \cong P\Omega_8^+(2).2 \). In \( P\Omega_8^+(2) \) there is a maximal subgroup \( \Omega_7(2) \). Therefore, \( \text{Alt}_5 \in \mathcal{X} \).

Now we argue as in (xxviii). Lemma 2.33 implies that \( G \) possesses a collection \( \Delta \) of subgroups isomorphic to \( \text{SL}_2(q) \) such that \( |\Delta| = 8 \) and \( [K, L] = 1 \) for every \( K, L \in \Delta \) and \( K \neq L \). Since 5 divides \( q^2 - 1 \), Lemma 2.30 implies that \( \text{SL}_2(q) \) possesses a subgroup isomorphic to \( \text{SL}_2(5) \). For every \( K \in \Delta \) fix some \( U(K) \leq K \) such that \( U(K) \cong \text{SL}_2(5) \).

Set
\[
U = \langle U(K) \mid K \in \Delta \rangle.
\]
It follows from the definition that
\[
U/U_\varnothing \cong \underbrace{\text{Alt}_5 \times \cdots \times \text{Alt}_5}_{8 \text{ times}}
\]
and \( U_\varnothing \) is a 2-group. Thus, \( U \in \mathcal{X} \). We claim that \( U \) is not conjugate to a subgroup of \( H \). It is sufficient to show that \( |U/U_\varnothing|_5 > |H/H_\varnothing|_5 \). Indeed,
\[
|H/H_\varnothing|_5 = |P\Omega_7(2)|_5 = 5 < 5^7 = |\text{Alt}_5|^7 = |U/U_\varnothing|_5,
\]
a contradiction with \( G \in \mathcal{D}_X \).

\( (xxxi) \) \( G \) is not isomorphic to any exceptional group of Lie type.

This statement follows from (xxvi)–(xxx).

3.7 Final proof of the implication \((1) \Rightarrow (2)\)

\( (xxxi) \) \( G \) does not exist.

Indeed, according to the classification of finite simple groups [3, Theorem 0.1.1], in (viii), (xv), (xx), (xxv), and (xxxi) we have excluded for \( G \) all possibilities to be a finite simple group.

Theorem [3] is proven.
4 Proofs of Corollaries

4.1 Proofs of Corollaries 1.1 and 1.2

In view of Lemma 2.6, in order to prove Corollary 1.1 it is sufficient to prove that any extension of a $D_X$-group by a $D_X$-group is a $D_X$-group. Now by Lemma 2.7, to prove Corollaries 1.1 and 1.2, it is sufficient to show that if $G \in D_X$ is a simple group then $\hat{G} = \text{Aut}(G) \in D_X$.

By Theorem 1 we need to consider two cases: $G \in X$ and $G \in D_X \cap D_\pi$, where $\pi = \pi(X)$, and in the last case $G$ is not a $\pi$-group.

In the first case, since $\text{Aut}(G)/\text{Inn}(G)$ is solvable and $\text{Inn}(G) \cong G \in X$, we conclude that $\hat{G}$ is $X$-separable and $\hat{G} \in D_X$ follows from Lemma 2.8.

In the last case, every $\pi$-Hall subgroup of $G$ is solvable by Lemma 2.28. Consequently, the $X$-subgroups of $G$ are exactly the solvable $\pi$-subgroups. Since $\text{Aut}(G)/\text{Inn}(G)$ is solvable, the same statement holds for the $X$-subgroups of $\hat{G} = \text{Aut}(G)$. In particular $m_X(G) = \text{Hall}_\pi(G)$. Now Lemma 2.3 implies that $G \in D_\pi$. Hence the elements of $m_X(\hat{G}) = \text{Hall}_\pi(\hat{G})$ are conjugate and $\hat{G} \in D_X$.

\[\square\]

4.2 Proof of Corollary 1.3

In fact, Corollary 1.3 is induced from Corollary 1.2 in [49, 15.4].

4.3 Proof of Corollary 1.4

The equivalency of (1) and (2) follows from Corollary 1.2 and the inclusion

$$m_X(G) \subseteq \text{sm}_X(G).$$

The implication (2) $\Rightarrow$ (3) is proved in Corollary 1.3.

Prove (3) $\Rightarrow$ (1). Take $A = B = G$. It follows from (3) that

$$k_X(G) = k_X(A) = k_X(A/B) = k_X(1) = 1$$

or, equivalently, $G \in D_X$.

\[\square\]

4.4 Proofs of Corollaries 1.5 and 1.6

Let $G \in D_X$. Then Corollary 1.1 implies that $S \in D_X$ for every composition factor $S$ of $G$. In view of Theorem 1, this means that $S \in D_\pi$. By Lemma 2.3, we have $G \in D_\pi$, and so Corollary 1.3 is proved. Now, it follows from Lemma 2.3 that some $\pi$-Hall subgroup $H$ of $G$ belongs to $X$. If $U$ is a $\pi$-subgroup of $G$, then $U$ is conjugate to a subgroup of $H$, since $G \in D_\pi$. The completeness of $X$ under taking subgroup means that $U \in X$. Hence Corollary 1.6 is proved.

\[\square\]

4.5 Proof of Corollary 1.7

Let $G \in D_X$, $H \in m_X(G)$ and $H \leq M \leq G$. Then $H \in \text{Hall}_\pi(G)$, where $\pi = \pi(X)$ and Lemma 2.9 implies that $M \in D_\pi$, i.e. every $\pi$ of $M$ is conjugate to $H$ in $M$. This implies that every $X$-maximal subgroup of $M$ is conjugate to $H$ in $M$ and $M \in D_X$. Now clearly, $m_X(M) \subseteq m_X(G)$.

\[\square\]
4.6 Proof of Corollary 1.8

Corollary 1.1 means that $G \in \mathcal{D}_X$ if and only if every composition factor $S$ of $G$ is a $\mathcal{D}_X$-group. Now Corollary 1.8 immediately follows from Theorem 1 and Lemma 2.4. □

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