The Haldane-Rezayi Quantum Hall State and Magnetic Flux

Kazusumi Ino
Institute for Solid State Physics, University of Tokyo, Roppongi 7-22-1, Minatoku, Tokyo, 106, Japan

We consider the general abelian background configurations for the Haldane-Rezayi quantum Hall state. We determine the stable configurations to be the ones with the spontaneous flux of $(2+1/2)\phi_0$ with $\phi_0 = \hbar c/e$. This gives the physical mechanism by which the edge theory of the state becomes identical to the one for the 331 state. It also provides a new experimental consequence which can be tested in the enigmatic $\nu = \frac{5}{2}$ plateau in a single layer system.
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In 87, Willet et al discovered a quantum Hall effect at the even-denominator filling fraction $\nu = 5/2$. This is so far the only even-denominator FQHE found in a single-layer system. While tilted field experiments suggest that the ground state of the plateau is a spin-unpolarized singlet state, numerical studies show that the ground state is a spin-polarized Pfaffian-like state. This discrepancy has not yet been resolved. The $\nu = 5/2$ plateau remains an enigma.

Soon after the discovery of the $\nu = 5/2$ plateau, Haldane and Rezayi proposed a variational ansatz for the ground state, the so-called Haldane-Rezayi (HR) state. It is a spin-singlet paired state. The HR state has peculiar physical properties such as the $5\phi$ degeneracy on the torus for the state with the filling fraction $\nu = 1/p$ ($p$ is an even integer). In the conformal field theory description, the spin and pairing degrees of freedom give a non-unitary $c = -2$ conformal field theory while the charge degrees of freedom are described by the $c = 1$ chiral boson $\varphi$. This theory is known to have physical inconsistency: the failure of the modular invariance for the state on the cylinder. There have been some arguments to secure the physical consistency, all of which propose the edge theory for the HR state to be the same as the one for the Halperin state ($331$ state). However no physical mechanism has been given for this to be hold.

Physically the HR state is a quantum Hall analog of the BCS superconductor. The pairing symmetry of the paired electrons is $d$-wave. Recently a phase transition driven by high magnetic field was discovered in a certain $d$-wave superconductor. Laughlin proposed that the phase transition is a consequence of the development of $T$-violating order parameter. The essential point of his arguments is the spontaneous induction of magnetic moment. There are some other discussions on this pairing order.

As the HR state is a quantum Hall and thus $T$-violating analog of $d$-wave superconductor, Laughlin’s argument suggests the possibility of induced moment or flux for the state. In this paper, we study the possible background configurations of the HR state. We show that there is indeed spontaneous induction of flux in the most stable configurations. The modular invariance is restored in these configurations. We also study the bulk excitations by a variant of Laughlin’s thought experiment and determine the composite laws of the excitations.

Let us first consider the edge conformal field theory of the HR state on the disk. The conformal field theory can be deduced by studying the wave function $\Psi$. The conformal field theory for the spin and pairing degrees of freedom is a $c = -2$ conformal field theory, while the charge degrees of freedom are described by the $c = 1$ chiral boson $\varphi$. In the free field representation, the $c = -2$ theory is realized by two anticommuting scalars ($\text{symplectic fermions}$) $\Psi, \tilde{\Psi}$. In the $\nu = 1/p$ state, the electron fields are given by $\partial \Psi e^{i\varphi}$, $\partial \tilde{\Psi} e^{i\varphi}$. They enlarge the chiral algebra of the state. The zero modes do not enter to the chiral algebra. The edge excitations are determined by the no monodromy condition with the electron fields. We treat the zero modes separately, then the primary fields for each sector are given by

$$I e^{i\varphi}, \partial \Psi e^{i\varphi}, \sigma e^{i\varphi}, \sigma^\alpha e^{i\varphi},$$

where $I$ is the identity and $\sigma, \sigma^\alpha$ are the spin and disorder field respectively, creating twisted sectors for $\partial \Psi e^{i\varphi}$. The fields also correspond to the quasiparticles in the bulk. The edge excitations in each sector are generated by acting the Virasoro algebra on the corresponding primary field. The complete description of edge excitations is conveniently summarized in the Virasoro characters. For the fermionic excitations, they are

$$\chi_I(\tau) = \frac{1}{2} \left( \frac{\vartheta_2(\tau)}{2\eta(\tau)} + \eta(\tau)^2 \right),$$

$$\chi_\Psi(\tau) = \frac{1}{2} \left( \frac{\vartheta_2(\tau)}{2\eta(\tau)} - \eta(\tau)^2 \right),$$

$$\chi_\sigma(\tau) = \frac{1}{2} \left( \frac{\vartheta_3(\tau)}{\eta(\tau)} + \frac{\vartheta_4(\tau)}{\eta(\tau)} \right),$$

$$\chi_{\sigma^\alpha}(\tau) = \frac{1}{2} \left( \frac{\vartheta_3(\tau)}{\eta(\tau)} - \frac{\vartheta_4(\tau)}{\eta(\tau)} \right),$$

while the characters for the $U(1)$ sector are given by $\chi_R(\tau) = \frac{\vartheta_3(\tau)}{4\pi \tau}, \chi_{R^{(1/2)}}(\tau) = \frac{\vartheta_2(\tau)}{4\pi \tau}$. Here $\vartheta$'s are the Jacobi theta functions and $\eta$ is the Dedekind function $\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^\infty (1 - q^n)$ with $q = e^{2\pi i \tau}$ . By taking $\tau = iT_0/k_B T$ where $k_B$ is the Boltzmann constant (we set $k_B = 1$ below) and $T_0$ is the level spacing of the system, combinations of these Virasoro characters represent contributions from each sector to the grand partition.
function. By summing up these terms, the grand partition function becomes

$$Z_{HR}(T) = \frac{1}{\eta^2} \left[ \varphi_2(\tau) \varphi_3(\nu \tau) + \varphi_3(\tau) \varphi_2(\nu \tau) \right]$$  \hspace{1cm} (6)

where we have included the fermionic zero modes.

We now consider the general abelian background configurations of the state. The system has a U(1) symmetry generated by $J^r = \sqrt{\eta} \partial \varphi$. We will denote this U(1) symmetry as U(1)$_p$. The background configuration coupling to the U(1) current is the magnetic flux $\Phi_c$. There is also another U(1) symmetry in this system which we will denote as U(1)$_s$. It is twice the projected (z-component) spin current $J^s = \Psi^\dagger \partial \Psi - \Psi \partial \Psi^\dagger$. One can also couple magnetic flux $\Phi_s$ to this current. With the introduction of $\Phi_c$ and $\Phi_s$, the HR state is parametrized by the two dimensional space $(\Phi_c, \Phi_s)$. Under the presence of the fluxes $\Phi_c$ and $\Phi_s$, there are spectral flows of the edge excitations. For example, $\chi_p$ changes as $\chi_p(\tau) \rightarrow \chi_p(\tau, \Phi_c) = \sqrt{\eta} \varphi_{2q(\Phi_c)}(\tau)$ where $t = -1/\tau = i k_B T/\eta T_0$ and we measure $\Phi_c$ with the unit $\phi_0 = \hbar c/e$. After some modular transformations, we end up with the grand partition function for the $(\Phi_c, \Phi_s)$ HR state as

$$Z_{HR} = \sqrt{\eta} \left[ \varphi_4(\Phi_c, t) \varphi_3(\Phi_c, pt) + \varphi_3(\Phi_c, t) \varphi_4(\Phi_c, pt) \right].$$  \hspace{1cm} (7)

$Z_{HR}(T, \Phi_c, \Phi_s)$ has periods 1 for both $\Phi_c$ and $\Phi_s$. The free energy for the edge excitations of the HR state becomes $F_{HR}(T, \Phi_c, \Phi_s) = -k_B T \ln Z_{HR}$.

Now we’d like to determine the stable configurations. We investigate the behavior of $F_{HR}(T, \Phi_c, \Phi_s)$ on the $(\Phi_c, \Phi_s)$ plane. Let us first consider the behavior at $T \gg T_0$. The minima are determined by the minimum conditions $j = \frac{\partial^2 F_{HR}}{\partial \Phi^a \partial \Phi^\beta} > 0$. Here $j$ is the induced persistent edge current for $\Phi_c$. By solving these conditions, the space of the minima is determined to be

$$\Gamma = \{(x, y) | (x, y) \in (Z/2, Z/2), x + y \in Z + 1/2 \}. \hspace{1cm} (8)$$

The lattice structure of $\Gamma$ is due to an extra periodicity of $Z_{HR}$ with period $\left(\frac{1}{2}, \frac{1}{2} \right)$. Thus the stable background configurations for the HR state are not the ones with no flux or integer flux, but the ones with the half integer net flux. Actually the half integer flux lattice is the set of maxima for $T \gg T_0$. We also note that the unitarity of the edge theory is restored in these configurations. Thus we conclude that the HR state is unstable by itself, and it has to generate the half integer flux for either $\Phi_c$ or $\Phi_s$. Spontaneously. The half integer flux is physically observable, while the integer flux is not physically observable.

Let us next consider the HR state on the cylinder. We restrict the excitations to the edge excitations. Then there are constraints between the excitations on two edges. The non-modular invariant cylinder partition function was obtained in Ref. [14]. It is the sum of the combinations of the characters for the left and right movers under the constraints between two edges. The partition function with the flux $\Phi_c$ and its modular behavior were studied in [13]. Its extension to include $\Phi_s$ is readily done. It can be written in terms of theta functions as

$$Z_{HR}^{cyl}(T, \Phi_c, \Phi_s) = \frac{1}{2p \eta(t)^4} \sum_{r=1}^{p} \left\{ \varphi_4(\Phi_c, t) \varphi_3(\Phi_c + r/p, t/p)^2 + \varphi_3(\Phi_c, t) \varphi_4(\Phi_c + r + 1/2/p, t/p)^2 \right\} .$$  \hspace{1cm} (9)

$Z_{HR}^{cyl}$ has the same periodicity in $(\Phi_c, \Phi_s)$ as $Z_{HR}$. We plot the free energy as a function of $(\Phi_c, \Phi_s)$ in Fig. 4. By using the explicit form above, it is shown that the free energy $F_{HR}^{cyl} = -k_B T \ln Z_{HR}^{cyl}$ shows a similar behavior with $F_{HR}$ and shares the same space of minima $\Gamma$ and the space of maxima. Thus there is spontaneously generated induced flux along the center of the cylinder.

Let us next consider the modular behavior of the partition function $Z_{HR}^{cyl}$. By using the analogous methods used in Ref. [13], it is shown that there is a relation between the Haldane-Rezayi state and the Halperin 331 state

$$Z_{HR}^{cyl}(T, x, y) = Z_{331}^{cyl}(T), \hspace{1cm} (x, y) \in \Gamma,$$$$

where $Z_{331}^{cyl}(T)$ is the cylinder partition function for the 331 state. The identity generalizes the previous results of Refs. [3-14]. As $Z_{331}(T)$ is modular invariant, modular invariance is restored on the minimum free-energy configurations. The failure of the modular invariance of $\Gamma$ occurs in its non-minimum free-energy configurations. Thus the generation of flux gives the physical mechanism that makes the edge theory of the Haldane-Rezayi state to be identical to the one for the 331 state.

We next consider the temperature region enough below the level spacing $T_0$. In this case it is shown that $Z_{HR}$ has an enlarged space of minima $\Gamma_0 = \{(x, y) | (x, y) \in (Z/2, Z/2) \}$. This property results in the 1/2 period in the induced edge current at zero temperature [21]. One can also prove the same behavior for $Z_{HR}^{cyl}$. $\Gamma$ remains the set of the most stable configurations also in this case.

We now turn to study the bulk excitations of the HR state by refining Laughlin’s thought experiment on quantum Hall state on the cylinder in terms of the edge partition function [22].

Let us consider the Laughlin state with $\nu = 1/p$ on the cylinder without flux. The edge state are in a minimum of the free energy $F_L = -k_B T \ln Z_L$ where $Z_L = \frac{1}{p \eta(t)^4} \sum_{r=1}^{p} \left\{ \varphi_4(\Phi_c + r, t/p)^2 \right\}$. Imagine that we turn on the unit flux through the cylinder. This causes the
transportation of the elementary quasihole $e^{-i\phi_{\sigma}^\nu}$ from one edge to the other. Indeed the flux $\Phi_c = 1$ is another minimum of $\mathcal{F}_L$. Thus the elementary quasihole is the excitation which corresponds to the transition between the nearest minima of the free energy of the edge excitation. The successive transitions between minima caused by turning on another flux give the $\mathbb{Z}$ structure of the bulk excitations, which is nothing but the fusion rules of the bulk CFT.

We generalize this argument to the HR state to determine the bulk excitation structure. When the temperature is far below the level spacing, the transition between minima becomes difficult. So we consider the region $T \gtrsim T_0$. First consider the HR state with flux $A (-\frac{1}{2}, 0)$ (Fig. 3). Imagine that we turn on the flux $\delta \Phi_c = 1$. This causes a transition from $A$ to another minimum $E (\frac{1}{2}, 0)$. This is the same transition as in the Laughlin case. The Laughlin quasihole $e^{-i\phi_{\sigma}^\nu}$ is transported from one edge to the other.

Next imagine that we turn on the flux $\delta \Phi_s = 1$. This causes a transition from $A$ to $C (0, \frac{1}{2})$. From the changes in the $\text{U}(1)$ quantum numbers on two edges, the bulk excitation transported is $\sigma \frac{1}{2} e^{-i\phi_{\sigma}^\nu}$ where $\sigma \frac{1}{2}$ is the $\text{U}(1)$ charge $\frac{1}{2}$ component of spin field $\sigma$. As we see in Fig. 4, this transition corresponds to the valley between the minima and energetically most favorable. Therefore the corresponding bulk excitation is the lightest quasiparticle in this system.

Next imagine that we turn on the flux $\delta \Phi_s = 1$. This causes a transition from $C$ to $B (\frac{1}{2}, 1)$. The corresponding bulk excitation should have the $\text{U}(1)_y$ charge 1. For example $\partial \Psi^{\nu_s}$ has $\text{U}(1)$ charge 1, but this field is not suitable to be transported because this transition should map the sectors of the Hilbert space $\mathcal{H}_{m,n} \rightarrow \mathcal{H}_{m,n+1}$ by one-to-one $(m, n): \text{U}(1)_x \times \text{U}(1)_y$ charges. As the fermionic edge excitations have the zero modes, the map must involve the fermionic zero modes. Thus the appropriate field for the excitation which accompanies the transition $A$ to $B$ is $\Psi^{\nu_s}$.

Let us next consider the successive transitions $A \rightarrow C \rightarrow D, \quad A \rightarrow C \rightarrow E$ (Fig. 4). In these transitions, two cases are possible. One is the successive occurrences of the transitions in order. In this case, two bulk excitations are created in order. The other is the simultaneous occurrence of the two transitions. The two bulk excitations will form an excitation corresponding to $A \rightarrow D$ or $A \rightarrow E$. These cases give the same consequence, thus physically equivalent.

One can relate the first case to the product or composition $\times$ of the two quasiparticles, which is fusing in the language of conformal field theory. Also the latter case gives the resulting representations of chiral algebra by the composition of the two quasiparticles.

The transition $A \rightarrow C$ and $C \rightarrow D$ are caused by the transportation of the quasihole $\sigma \frac{1}{2} e^{-i\phi_{\sigma}^\nu}$ from one edge to the other, while $C \rightarrow E$ is from the transportation of the quasihole $\sigma \frac{1}{2} e^{-i\phi_{\sigma}^\nu}$. On the other hand, $A \rightarrow D$ and $A \rightarrow E$ make the transportation of $\sigma \frac{1}{2} e^{-i\phi_{\sigma}^\nu}$ and $\Psi^{\nu_s} e^{-i\phi_{\sigma}^\nu}$ respectively. As these processes are physically equivalent, we have thus the following identities:

$$\sigma \frac{1}{2} e^{-i\phi_{\sigma}^\nu} \times \sigma \frac{1}{2} e^{-i\phi_{\sigma}^\nu} \sim \Psi^{\nu_s} e^{-i\phi_{\sigma}^\nu}.$$  \hspace{1cm} (11)

$$\sigma \frac{1}{2} e^{-i\phi_{\sigma}^\nu} \times \sigma \frac{1}{2} e^{-i\phi_{\sigma}^\nu} \sim I e^{-i\phi_{\sigma}^\nu}.$$  \hspace{1cm} (12)

By combining these identities and omitting the charge part, we have the following identity for the neutral spin field $\sigma$, $\sigma \times \sigma \sim I + I$ where $I = \epsilon_{\alpha\beta} \Psi^{\nu_s} \Psi^{\nu_s}$ with $\epsilon_{\alpha\beta} = -\epsilon_{\beta\alpha} = 1/2$ and zero otherwise. This is nothing but the fusion rules of the spin field and the logarithmic operator in the $c = -2$ logarithmic CFT. Other fusion rules are deduced similarly. As the chiral algebra of the bulk does not include the fermionic zero modes and only has $\partial \Psi e^{-i\phi_{\sigma}^\nu}$ as extending field, the field $I e^{-i\phi_{\sigma}^\nu}$ cannot be generated from the fields in $\{\}$. Thus $I e^{-i\phi_{\sigma}^\nu}$ results in a new sector. It leads to the $5p$ sectors in the bulk, thus the $5p$ degeneracy of the HR state on the torus. On the other hand, in the edge theory where $\Psi$ effectively has a half integer monodromy by the presence of half integer flux, $I$ does not lead to a new sector, which results in the $4p$ sectors.

Thus we have determined the structure of the bulk excitations of the HR state from the space of minima of the edge free energy. There have been some arguments on other possibilities on the bulk theory of the HR state [24][14]. Our discussion shows that the bulk of the HR state is consistently described by the $c = -2$ logarithmic CFT [23].

Let us next discuss implications of our result for the $\nu = 5/2$ plateau. Recent numerical studies [9] for the small electron system on the sphere show that the $\nu = 5/2$ state may be spin-polarized, while a spin-unpolarized state is not favored by its relatively higher energy per one electron, and the energy of the system is scales as $O(N)$ where $N$ is the number of electrons. Our result shows that the generation of flux in the HR state in a finite system with edges lowers the energy of the its edge state. However its reduction only scales as $O(1)$. Thus, in the thermodynamic limit, our result does not by itself implies the bulk state of the $\nu = 5/2$ plateau to be the HR state with the stable background.

On the other hand, experimental studies [12][13] indeed favor a spin-unpolarized singlet state. The results of the tilted field experiment are naturally explained by the HR state. Our result provides a new experimental test on this. If the state is the HR state, then for the sample with shape like the annulus, the flux through the center should be quantized as $(\mathbb{Z} + \frac{1}{2}) \phi_0$. This is a unique property for the HR state and its detection in the $\nu = 5/2$ plateau gives an clear evidence that the plateau realizes the Haldane-Rezayi state.

Finally the physical mechanism we describe in this paper suggests the possibility of larger class of vacua in the FOH systems, which we will discuss in detail elsewhere [25].

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FIG. 1. Free energy for the cylinder as function of $\Phi_c$ and $\Phi_s$ at $T/T_0 = 1.36$ ($p = 2$). The terms not dependent on flux are omitted.
FIG. 2. The lattice of minima of $F_{HR}$ at $T \gtrsim T_0$ and transitions among the stable configurations.