DISPERSE TYPE ESTIMATES FOR FOURIER INTEGRALS
AND APPLICATIONS TO HYPERBOLIC SYSTEMS

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Abstract. In this note we provide dispersive estimates for Fourier integrals with parameter-dependent phase functions in terms of geometric quantities of associated families of Fresnel surfaces. The results are based on a multi-dimensional van der Corput lemma due to the first author.

Applications to dispersive estimates for hyperbolic systems and scalar higher order hyperbolic equations are also discussed.

1. Introduction. Dispersive estimates for solutions to linear evolution equations are vitally important for the study of a wide range of problems, including questions of stability of solutions to corresponding non-linear equations, descriptions of large-time asymptotic profiles of solutions or scattering results based on the time-dependent approach.

For the particular situation of hyperbolic evolution equations with a hyperbolic time-asymptotic profile, such estimates can be based on corresponding decay estimates for Fourier integral operators with real phases and certain well-controlled amplitude functions. The aim of this note is to provide some robust estimates for such Fourier integrals in terms of geometric properties of associated Fresnel surfaces.

2. Fourier integral operators, notation and main results. We are interested in $L^1-L^\infty$ estimates for a family of Fourier integral operators depending on a parameter $t \in \mathbb{R}_+$ (thought of as time later on)

$$ T_t u(x) = \int_{\mathbb{R}^n} e^{i(x \cdot \xi + t\phi(t,x,\xi))} a(t,x,\xi) \hat{u}(\xi) d\xi, $$

where $a(t,x,\xi)$ is uniformly in $t$ a (global) symbol of order zero,

$$ |\partial^\alpha \xi a(t,x,\xi)| \leq C_\alpha \langle \xi \rangle^{-|\alpha|}, $$

with $\langle \xi \rangle = \sqrt{1+|\xi|^2}$, and $\phi(t,x,\xi)$ is a homogeneous real-valued function satisfying

$$ C^{-1} |\xi| \leq \phi(t,x,\xi) \leq C|\xi|, \quad |\partial^\alpha \phi(t,x,\xi)| \leq C'_\alpha |\xi|^{1-|\alpha|} $$

uniform in $t$ and $x$. For simplicity, we always assume $u \in \mathcal{S}(\mathbb{R}^n)$, estimates extend by continuity arguments to larger spaces. If we are only interested in estimates of

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$T_t u$ in the $L^\infty$-norm, we can treat $t$ and $x$ both as parameters and derive estimates for the corresponding stationary model operator

\[ T : u \mapsto \int_{\mathbb{R}^n} e^{i(x \cdot \xi + \phi(\xi))} a(\xi) \hat{u}(\xi) d\xi. \tag{4} \]

A precise understanding of the appearing constants in the estimate translates to estimates of the $t$-behaviour of the full model (1).

2.1. Fresnel surfaces and contact indices. We turn to the analysis of (4) and define the geometric quantities which determine its $L^\infty$ behaviour. The consideration follows essentially Sugimoto [15], [16], for the non-degenerate situations see the earlier works of Strichartz [14] or Brenner [1].

We assume that the phase $\phi(\xi)$ is non-negative, 1-homogeneous and smooth, and define the associated Fresnel surface (also called slowness surface)

\[ \Sigma = \{ \xi \in \mathbb{R}^n : \phi(\xi) = 1 \}. \tag{5} \]

We introduce two indices for such a Fresnel surface $\Sigma$, assuming that it is of class $C^k$ with $k$ being sufficiently large. For $p \in \Sigma$ we denote by $T_p$ the tangent hyperplane to $\Sigma$ at $p$. Then for any plane $H$ of dimension 2 which contains $p$ and the normal of $\Sigma$ at $p$ we denote by $\gamma(\Sigma, p, H)$ the order of contact between the curve $\Sigma \cap H$ and its tangent $H \cap T_p$. Furthermore, we set

\[ \gamma(\Sigma) = \sup_p \sup_H \gamma(\Sigma, p, H), \quad \gamma_0(\Sigma) = \sup_H \gamma(\Sigma, p, H). \tag{6} \]

The definition implies directly that $2 \leq \gamma_0(\Sigma) \leq \gamma(\Sigma)$. Furthermore, for isotropic problems $\Sigma$ is a dilation of the sphere $S^{n-1}$ and $\gamma(S^{n-1}) = \gamma_0(S^{n-1}) = 2$. Moreover, if the Gaussian curvature of $\Sigma$ never vanishes $\Sigma$ is convex and we have $\gamma(\Sigma) = 2$.

These indices are directly related to mapping properties of (4). The following theorem is a combination of results from [15], [16] in the analytic setting, improved by the author [9], [10] to the smooth setting as well as to the limited regularity cases. Moreover, the corresponding results in [9], [10] do not require the positive homogeneity of the phase in $\xi$ and allow the dependence on parameters.

As usual, $B^m_{p,q}(\mathbb{R}^n)$ denotes the Besov space of regularity $r$ modelled over $L^p$.

**Theorem 2.1.** Assume that $\phi$ is smooth on $\mathbb{R}^{1+n} \times (\mathbb{R}^n \setminus \{0\})$, positive and 1-homogeneous in $\xi$, $\Sigma$ smooth and $a \in S^0_{1,0}(\mathbb{R}^n)$. Then the operator $T$ defined by (4) maps

\[ B^m_{p,q}(\mathbb{R}^n) \to L^\infty(\mathbb{R}^n) \tag{7} \]

where $r = n - \frac{n-1}{\gamma(\Sigma)}$ if $\Sigma$ is convex and $r = n - \frac{1}{\gamma_0(\Sigma)}$ if $\Sigma$ is arbitrary.

The operator norm in this estimate depends on finitely many symbol seminorms of $a$, the estimates of $\phi$ and its derivatives together with some quantitative measures of the contact between $\Sigma$ and tangent lines.

To control the order of contact of $\Sigma$ by tangent lines quantitatively, we will introduce another quantity $\kappa(\Sigma)$. First, assume that $\Sigma$ is convex and that it is of class $C^{\gamma(\Sigma)+1}$. For $p \in \Sigma$, rotating $\Sigma$ if necessary, we may assume that it is parameterised by points $\{(y, h(y)), y \in \Omega\}$ near $p$ for an open set $\Omega \subset \mathbb{R}^{n-1}$. For $p = (y, h(y))$, let us define

\[ \kappa(\Sigma, p) = \inf_{|\omega|=1} \left| \sum_{j=2}^{\gamma(\Sigma)} \sum_{\nu=0}^{\gamma_j(\Sigma)} \frac{\partial^j}{\partial \rho^j} h(y + \rho \omega)|_{\rho=0} \right|. \tag{8} \]
From the definition of $\gamma(\Sigma)$ it follows that $\kappa(\Sigma, p) > 0$ for all $p \in \Omega$. Indeed, from the definition of $\gamma(\Sigma, p, H)$ it follows that if $\omega$ is such that $y + \rho \omega \in H$, then

$$\kappa(\Sigma, p, H) = \left| \frac{\partial \gamma(\Sigma, p, H)}{\partial \rho} h(y + \rho \omega) \right|_{\rho = 0} > 0. \quad (9)$$

Now, we clearly have $\sum_{j=2}^{\gamma(\Sigma)} \left| \frac{\partial^j}{\partial \rho^j} h(y + \rho \omega) \right|_{\rho = 0} \geq \kappa(\Sigma, p, H)$, and hence we have $\kappa(\Sigma, p) > 0$ since the set $|\omega| = 1$ is compact. Noticing that $\kappa(\Sigma, p)$ is a continuous function of $p$, by compactness of $\Sigma$ it follows that if we define

$$\kappa(\Sigma) = \min_{p \in \Sigma} \kappa(\Sigma, p), \quad (10)$$

then $\kappa(\Sigma) > 0$.

If $\Sigma$ is not convex, we define similarly

$$\kappa_0(\Sigma) = \min_{p \in \Sigma} \sup_{|\omega| = 1} \sum_{j=2}^{\gamma(\Sigma)} \left| \frac{\partial^j}{\partial \rho^j} h(y + \rho \omega) \right|_{\rho = 0} \quad (11)$$

Again, we have $\kappa_0(\Sigma) > 0$.

considering again the example of spheres $\Sigma = rS^{n-1}$ of radius $r$, we see that $\gamma(\Sigma) = \gamma_0(\Sigma) = 2$ and that $\kappa(\Sigma) = \kappa_0(\Sigma) \sim \frac{1}{r}$ is a measure of scalar curvature.

2.2. Main results. Estimates of $(1)$ depend on indices of a family of Fresnel surfaces defined in terms $\phi(t, x, \xi)$,

$$\Sigma_{t, x} = \{ \xi \in \mathbb{R}^n : \phi(t, x, \xi) = 1 \}, \quad t \geq t_0, x \in \mathbb{R}^n. \quad (12)$$

For a first result we assume that $\Sigma_{t, x}$ is convex for all $t \geq t_0$ and all $x \in \mathbb{R}^n$.

**Theorem 2.2.** Assume that $\phi(t, x, \xi)$ is real-valued, continuous in $t$ and $x$ and smooth in $\xi \in \mathbb{R}^n \setminus \{0\}$, homogeneous of order one in $\xi$ and such that for some $t_0 > 0$ and $C > 0$ we have

$$C^{-1}|\xi| \leq \phi(t, x, \xi) \leq C|\xi| \quad \text{and} \quad |\partial^\alpha \phi(t, x, \xi)| \leq C|\xi|^{1-|\alpha|} \quad (13)$$

for all $t \geq t_0$, all $x$, $\xi \neq 0$ and all multi-indices $\alpha$ with $|\alpha| \leq \max\{2, \lfloor (n - 1)/2 \rfloor \}$. Assume further that all the sets $\Sigma_{t, x}$ defined by $(12)$ are convex for all $t \geq t_0$ and assume that

$$\limsup_{t \to \infty} \sup_x \gamma(\Sigma_{t, x}) \leq \gamma \quad \text{together with} \quad \liminf_{t \to \infty} \kappa(\Sigma_{t, x}) > 0. \quad (14)$$

Suppose further that the amplitude $a(t, x, \xi)$ satisfies

$$|\partial^\alpha a(t, x, \xi)| \leq C_{a(\xi)}|\xi|^{-|\alpha|} \quad \text{for all} \quad |\alpha| \leq \lfloor (n - 1)/2 \rfloor + 1. \quad (15)$$

Then the operator family $T_t$ defined by $(1)$ satisfies for all $t \geq t_0$ the estimate

$$\|T_t u\|_{L^\infty(\mathbb{R}^n)} \leq C t^{-\frac{n-1}{\gamma}} \|u\|_{B^{\gamma}_{1, 0}(\mathbb{R}^n)} \quad (16)$$

with $r = n - \frac{n-1}{\gamma}$.

In the non-convex situation decay rates can be much lower. Results can be improved by using a more detailed analysis of the geometric situation near points of high contact order.

**Theorem 2.3.** Assume that $\phi(t, x, \xi)$ is real-valued, continuous in $t$ and $x$ and smooth in $\xi \in \mathbb{R}^n \setminus \{0\}$, homogeneous of order one in $\xi$ and such that for some $t_0 > 0$ and $C > 0$ we have

$$C^{-1}|\xi| \leq \phi(t, x, \xi) \leq C|\xi| \quad \text{and} \quad |\partial^\alpha \phi(t, x, \xi)| \leq C|\xi|^{1-|\alpha|} \quad (17)$$
for all \( t \geq t_0 \), all \( x, \xi \neq 0 \) and all multi-indices \( \alpha \) with \( |\alpha| \leq \gamma_0 + 1 \). Assume further that all the sets \( \Sigma_{t,x} \) defined by (12) satisfy
\[
\limsup_{t \to \infty} \sup_x \gamma_0(\Sigma_{t,x}) \leq \gamma_0 \quad \text{together with} \quad \liminf_{t \to \infty} \inf_x \gamma_0(\Sigma_{t,x}) > 0.
\]
(18)

Suppose further that the amplitude \( a(t,x,\xi) \) satisfies
\[
|\partial_\xi^\alpha a(t,x,\xi)| \leq C_\alpha |\xi|^{-|\alpha|} \quad \text{for all} \quad |\alpha| \leq 1.
\]
(19)

Then the operator family \( T_t \) defined by (1) satisfies for all \( t \geq t_0 \) the estimate
\[
\|T_t u\|_{L^\infty(\mathbb{R}^n)} \leq C t^{-\frac{1}{n}} \|u\|_{B_t^r(\mathbb{R}^n)}
\]
(20)

with \( r = n - \frac{1}{\gamma_0} \).

The proofs of both theorems are similar to the ones in [13] and full details are omitted here. The dependence on \( x \) is parametric, and the assumption assure that the resulting estimates are uniform in \( x \).

If we make further regularity assumptions with respect to the \( x \)-variable it follows that the operators \( T_t \) are uniformly \( L^2 \)-bounded such that interpolation implies the usual \( L^p-L^q \) decay estimates. For completeness, we include a corresponding statement. The assumptions are chosen such that after substituting \( x = ty \) and \( t\xi = \eta \) Theorem 2.5 of [11] applies:

**Theorem 2.4** ([11]). Let \( \Gamma_y, \Gamma_\xi \subset \mathbb{R}^n \) be open cones. Let operator \( T \) be defined by
\[
T u(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(x \cdot \xi + \varphi(y,\xi))} a(y,\xi) u(y) \, dy \, d\xi,
\]
(21)

where \( a(y,\xi) \in C^\infty(\mathbb{R}^n_y \times \mathbb{R}^n_\xi) \), \( \text{supp} \, a \subset \Gamma_y \times \Gamma_\xi \), and \( \varphi(y,\xi) \in C^\infty(\Gamma_y \times \Gamma_\xi) \) is a real-valued function. Assume that
\[
|\partial_\xi^\alpha \partial_\xi^\beta a(y,\xi)| \leq C_{\alpha,\beta},
\]
for \( |\alpha|, |\beta| \leq 2n + 1 \). Also assume that
\[
|\text{det} \, \partial_\gamma \partial_\xi \varphi(y,\xi)| \geq C > 0 \quad \text{on} \quad \Gamma_y \times \Gamma_\xi,
\]
that
\[
|\partial_\xi \varphi(x,\xi) - \partial_\xi \varphi(y,\xi)| \geq C |x - y| \quad \text{for} \quad x, y \in \Gamma_y, \xi \in \Gamma_\xi,
\]
and that
\[
|\partial_\xi^\gamma \partial_\xi \varphi(y,\xi)| \leq C_\gamma \quad \text{on supp} \, a
\]
for \( 1 \leq |\alpha|, |\beta| \leq 2n + 2 \). Then the operator \( T \) is \( L^2(\mathbb{R}^n) \)-bounded, and satisfies
\[
\|T\|_{L^2 \to L^2} \leq C \sup_{|\alpha|,|\beta| \leq 2n+1} \|\partial_\gamma^\alpha \partial_\xi^\beta a(y,\xi)\|_{L^\infty(\mathbb{R}^n_y \times \mathbb{R}^n_\xi)}.
\]

We can now formulate the corresponding \( L^2 \)-result:

**Theorem 2.5.** Let
\[
T_t u(x) = \int_{\mathbb{R}^n} e^{i(x \cdot \xi + t \phi(t,x,\xi))} a(t,x,\xi) \tilde{u}(\xi) d\xi,
\]
(22)

Assume \( \phi(t,x,\xi) \) is real-valued, \( 1 \)-homogeneous in \( \xi \) and satisfies
\[
|\text{det} \, (1 + t \partial_x \partial_\xi \phi(t,x,\xi))| \geq C_0 > 0, \quad |t^{|\alpha|} \partial_\xi^\alpha \partial_\xi^\beta \phi(t,x,\xi)| \leq C_{\alpha,\beta}
\]
(23)

uniform in \( t \geq t_0, x, \xi \neq 0 \) and for \( 1 \leq |\alpha|,|\beta| \leq 2n + 2 \). Assume further that \( a(t,x,\xi) \) is supported in \( t|\xi| \geq C \) for some constant \( C \) and that
\[
|t^{|\alpha|} \partial_\gamma^\alpha \partial_\xi^\beta a(t,x,\xi)| \leq C_{\alpha,\beta}
\]
(24)
for all $|\alpha|, |\beta| \leq 2n + 2$ and $t \geq t_0$. Then the family $T_t$ in (22), $t \geq t_0$, is uniformly $L^2$-bounded,

$$\|T_t u\|_{L^2(\mathbb{R}^n)} \leq C\|u\|_{L^2(\mathbb{R}^n)}. \quad (25)$$

Let us sketch how Theorem 2.4 implies Theorem 2.5. Using homogeneity, we can rewrite (22) as

$$T_t u(x) = t^{-n} \int_{\mathbb{R}^n} e^{i(x \cdot \eta + \phi_t(t, z, \eta))} a_t(t, z, \eta) \hat{u}(\eta/t) d\eta, \quad (26)$$

with $x = tz, t\xi = \eta$, and

$$\phi_t(t, z, \eta) = \phi(t, tz, \eta), \quad a_t(t, z, \eta) = a(t, tz, \eta/t).$$

By rescaling, the powers of $t$ in the $L^2$-estimate cancel, and it can be shown that the adjoint to the operator in (26) satisfies assumptions on Theorem 2.4 uniformly for $t \geq t_0$.

We also note that the assumptions on the mixed derivatives of $\phi$ as in (23) can be relaxed by looking only at $|\alpha| = 1$ or $\beta = 1$, as in Theorem 2.4.

2.3. Appendix: Fourier transform of surface carried measures. Let $\Sigma \subseteq \mathbb{R}^n$ be a closed hypersurface. Then the indices $\gamma(\Sigma)$ and $\gamma_0(\Sigma)$ are related to decay estimates for Fourier transforms of surface carried measures. Consider for $f \in C^\infty(\Sigma)$ the integral

$$u(x) = \int_{\Sigma} e^{ix \cdot \xi} f(\xi) d\xi, \quad (27)$$

where $d\xi$ denotes the $(n-1)$-dimensional surface measure on $\Sigma$. Then the following theorem is valid.

**Theorem 2.6.**

1. Assume $\Sigma$ is a smooth hypersurface and convex. Then

$$|u(x)| \leq C(x)^{-n + 1/\gamma(\Sigma)} \|f\|_{C^r(\Sigma)} \quad (28)$$

with $r \geq (n-1)/\gamma(\Sigma) + 1$.

2. For general (non-convex) smooth hypersurface $\Sigma$ it follows that

$$|u(x)| \leq C(x)^{-1/\gamma_0(\Sigma)} \|f\|_{C^1(\Sigma)}. \quad (29)$$

The proof of the first statement in [15] uses real analyticity of the surface, the stronger results are due to methods of [9] and [10]. Smoothness of the surface can be further relaxed to the class $C^\gamma$ with $\gamma = \gamma(\Sigma)$ (or $\gamma_0(\Sigma)$, respectively).

3. Applications to hyperbolic evolution equations. The estimates discussed so far have applications for dispersive estimates of hyperbolic evolution equations. Without aiming for completeness we will discuss some possible applications.

3.1. Wave models. As a first example we consider wave models with time-dependent coefficients,

$$\partial_t^2 u - \sum_{i,j=1}^{n} a_{ij}(t) \partial_{x_i} \partial_{x_j} u = 0 \quad (30)$$

in $\mathbb{R} \times \mathbb{R}^n$ and with Cauchy data $u(0, \cdot) = u_0$, $u_t(0, \cdot) = u_1$, both assumed to be Schwartz for simplicity. Denoting

$$T\{\ell\} = \{a \in C^\infty(\mathbb{R}) : |\partial_t^k a(t)| \leq c_k(t)^{\ell-k}\}, \quad (31)$$
we assume \( a_{ij} \in \mathcal{T}\{0\} \), \( a_{ij}(t) = \overline{a_{ji}(t)} \), together with the strict hyperbolicity assumption
\[
a(t, \xi) = \sum_{i,j=1}^n a_{ij}(t)\xi_i\xi_j \geq C|\xi|^2,
\]
for some \( C > 0 \) uniform in \( t \). Within the zone \( \{(1 + t)|\xi| \gtrsim 1\} \) solutions to (30) are
given by Fourier integrals of the form (1) with phase functions
\[
\phi_{\pm}(t, \xi) = \pm \frac{1}{t} \int_0^t \sqrt{a(\theta, \xi)}d\theta.
\]
In this situation the Fresnel surfaces are given by
\[
\Sigma_t = \left\{ \xi \in \mathbb{R}^n : t = \int_0^t \sqrt{a(\theta, \xi)}d\theta \right\}
\]
and satisfy \( \gamma(\Sigma_t) = 2 \) together with uniform upper and lower bounds for \( \varkappa(\Sigma_t) \).
This follows directly from \([6, \text{Lemma 3.1}]\). Thus Theorem 2.2 is applicable and in
combination with a uniform bound on the \( L^2\)-energy one concludes that solutions to (30) satisfy
\[
||\partial_t u(t, \cdot)||_q + ||\nabla u(t, \cdot)||_q \leq C_{pq}(1 + t)^{-\frac{\alpha(q - 1)}{2}} \left(||u_1||_{W^{p,r}+1} + ||u_2||_{W^{p,r}}\right)
\]
with \( 1 < p \leq 2 \leq q < \infty, \ pq = p + q \) and \( r_p = n(1/p - 1/q) \).
For a detailed consideration of such models we refer to Reissig \([6, \text{7}]\). It is possible to include further lower order terms \( b(t)\partial_t u, \ \sum_j c_j(t)\partial_{\xi_j} u \) and \( d(t)u \) with
\( b, c_j \in \mathcal{T}\{-1\} \) and \( d \in \mathcal{T}\{-2\} \) (together with further assumptions to control the
low-frequency behaviour). We refer to \([17]\) for an example of such a treatment. If
lower order terms belong to different classes, the asymptotic behaviour of solutions
might change its type, see, e.g., \([18]\) a discussion about dissipative wave equations
and effective dissipation.

The assumptions on coefficients can be weakened to classes
\[
\mathcal{T}_\nu\{t\} = \{ a \in C^\infty(\mathbb{R}) : |\partial_t^k a(t)| \leq c_k(t \log(e + t)^\nu)^{t-k}\}
\]
with parameter \( \nu \in (0, 1] \) where \( \nu \in (0, 1) \) implies a small loss of decay and \( \nu = 1 \)
yields a polynomial loss. It can be shown that without further assumptions \( \nu > 1 \)
leads generically to supra-polynomial growth of the energy. Additional assumptions
to compensate this behaviour for wave equations with variable speed of propagation
were considered by Hirosawa \([3]\), see also \([19]\), but the estimates of Theorems 2.2
and 2.3 need to be refined in order to obtain dispersive estimates in this situation.

3.2. Higher order scalar equations. In a similar way we can treat homogeneous
higher order equations of the form
\[
D_t^m u + \sum_{k=0}^{m-1} \sum_{|\alpha|=m-k} a_{k,\alpha}(t)D_t^kD_\xi^\alpha u = 0,
\]
\[
D_t^j u(0, \cdot) = u_j, \quad j = 0, 1, \ldots, m - 1,
\]
where \( D = -i\partial_t, \ a_{k,\alpha} \in \mathcal{T}\{0\} \) and the characteristic equation
\[
\tau^m + \sum_{k=0}^{m-1} \sum_{|\alpha|=m-k} a_{k,\alpha}(t)\tau^k\xi^{\alpha} = 0
\]
has \( m \) uniformly distinct real solutions \( \tau_1(t, \xi), \ldots, \tau_m(t, \xi) \), \( |\tau_i(t, \xi) - \tau_j(t, \xi)| \geq C|\xi| \) for \( i \neq j \) and some \( C > 0 \). Under the stronger assumption that \( a_{k,\alpha}(t) \in L^1(\mathbb{R}_+) \), this model was treated by Matsuyama and the first author, \[5\], the full form follows from \[13\]. If we assume
\[
\left| \sum_{k \neq j} \int_0^t \frac{\partial \tau_j(\theta, \xi)}{\tau_j(\theta, \xi) - \tau_k(\theta, \xi)} d\theta \right| < C \tag{40}
\]
uniform in \( j, t \) and \( \xi \neq 0 \), solutions are given as sums of Fourier integrals of the form \( (1) \) with phases
\[
\phi_j(t, \xi) = \frac{1}{t} \int_0^t \tau_j(\theta, \xi) d\theta \tag{41}
\]
and uniformly bounded amplitudes. Again, the results from Section 2.2 provide dispersive estimates for solutions to such problems. We remark that for higher order equations the associated Fresnel surfaces \( \Sigma_i \) need not be convex and therefore the full generality of Theorems 2.2 and 2.3 is of importance here.

We note that the analysis in \[5\] is based on the asymptotic integration method for constructing solutions for equations with homogeneous symbols, and it requires only \( C^1 \)-regularity of coefficients, i.e., only that \( a_{k,\alpha} \in C^1 \) in terms of the regularity in time, which is crucial in applications to the Kirchhoff equations, see e.g. \[4\].

3.3. Hyperbolic systems. In \[12\], the authors announced results on dispersive estimates for a general class of pseudo-differentiable hyperbolic systems with time-dependent coefficients and arbitrary \( \gamma \). D’Abbico, Lucente and Taglialatela, \[2\], considered differential hyperbolic systems with \( \gamma = 2 \), while in \[13\] the authors gave a rigorous treatment for general pseudo-differentiable hyperbolic systems with time-dependent coefficients and arbitrary \( \gamma \). For the sake of simplicity we formulate the results for the homogeneous situation. We consider
\[
D_t U = A(t, D_x) U, \quad U(0, \cdot) = U_0 \tag{42}
\]
for a \( m \times m \) matrix \( A(t, D_x) \) of \( t \)-dependent Fourier multipliers. We denote
\[
\mathcal{S}\{m_1, m_2\} = \{a(t, \xi) \in C^\infty(\mathbb{R} \times (\mathbb{R}^n \setminus \{0\})) : |\xi|^m |a|^{-m_1} D_\xi^\alpha a(t, \xi) \in \mathcal{T}\{m_1\}\}, \tag{43}
\]
the condition being uniform in \( \xi \). We assume that the symbol \( A(t, \xi) \in \mathcal{S}\{1,0\} \) is 1-homogeneous in \( \xi \), a self-adjoint matrix and its eigenvalues \( \tau_1(t, \xi), \ldots, \tau_m(t, \xi) \) are uniformly distinct, \( |\tau_i(t, \xi) - \tau_j(t, \xi)| \geq C|\xi| \) for \( i \neq j \) and some \( C > 0 \). Without further assumptions it follows that the solutions to \( (42) \) can be represented in the form \( (1) \) with phases given by \( (41) \).

Assumptions can be relaxed, without assuming self-adjointness we have to require real eigenvalues together with a uniform symmetrisability of the matrix \( A(t, \xi) \) together with further assumptions to control the large-time influence of lower order terms. We refer to \[13\] for details.

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