Plausibility and probability in deductive reasoning

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Abstract

We consider the problem of rational uncertainty about unproven mathematical statements, remarked on by Gödel and others. Using Bayesian-inspired arguments we build a normative model of fair bets under deductive uncertainty which draws from both probability and the theory of algorithms. We comment on connections to Zeilberger’s notion of “semi-rigorous proofs”, particularly that inherent subjectivity would be present. We also discuss a financial view with models of arbitrage where traders have limited computational resources.

Contents

1 A natural problem: Quantified deductive uncertainty 2
  1.1 The phenomenon . . . . . . . . . . . . . . . . . . . . . . . . 2
  1.2 The (modeling) problem . . . . . . . . . . . . . . . . . . . 3

2 Formal representation of plausibilities 4
  2.1 Plausibility functions . . . . . . . . . . . . . . . . . . . . . . 4
  2.2 Languages . . . . . . . . . . . . . . . . . . . . . . . . . . . . 5
  2.3 Epistemic quality vs. computation costs . . . . . . . . . . . . 5
  2.4 Conditional plausibility . . . . . . . . . . . . . . . . . . . . . 7

3 Rational plausibilities in mathematics 7
  3.1 Scoring rules . . . . . . . . . . . . . . . . . . . . . . . . . . . . 7
  3.2 Foundations of “semi-rigorous proofs” . . . . . . . . . . . . . 8

4 Arbitrage pricing in markets with computational constraints 10

5 Conclusion 11

References 11

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1 A natural problem: Quantified deductive uncertainty

1.1 The phenomenon

Epistemic uncertainty is usually defined as uncertainty among physical states due to lack of data or information. By information we mean facts which we observe from the external world. For example, whether it rains tomorrow is a piece of information we have not observed, so we are uncertain about its truth value. However, we also may have uncertainty about purely deductive statements, which are completely determined by the information we have, due to limited reasoning ability. That is, before we have proved or refuted a mathematical statement, we have some deductive uncertainty about whether there is a proof or refutation.

Under deductive uncertainty, there is a familiar process of appraising a degree of belief one way or the other, saying a statement has high or low plausibility. We may express rough confidence levels in notable open conjectures such as $P \neq NP$ or the Goldbach conjecture, and we also deal with plausibility in everyday mathematical reasoning. Sometimes general patterns show up across problems and we extrapolate them to new ones. If we have an algorithm and are told it runs in time $O(n \lg n)$, we usually assume that this implies good practical performance because this is a commonly observed co-occurrence. So the plausibility of the running time being $10^{10^n} n^{\lceil \lg n \rceil}$ is considered particularly low. Any mathematical result seen as “surprising” must have been a priori implausible. Etcetera. Many more examples of plausibility in mathematics may be found in [48, 41].

In some instances it may be natural to quantify deductive uncertainty, and perhaps speak of “probabilities”. For example, let $d$ be the $10^{100}$th decimal digit of $\pi$. If we have not computed $d$ and all we know is that, say, $d$ is odd, it feels like $d$ has a uniform probability distribution over \{1, 3, 5, 7, 9\}. Mazur [41, Sec. 2] would describe this as an application of the principle of insufficient reason. We use the same “symmetry” argument to state the probability that a given number $n$ is prime via the prime number theorem or Fermat primality test. Probabilities also show up in enumerative induction, where confidence in a universal quantification increases as individual instances are verified. The four color theorem is one of many cases where only positive instances could be found and eventually a proof was given. Furthermore, to this theorem and other similar claims there are relevant 0-1 laws which state that the “conditional probability” of a uniform random instance being a counterexample, given that counterexamples exist, goes to 1 asymptotically. With this fact one can use individual instances to “update” a Bayesian probability on the universal statement. Bayesianism in practical mathematics has been discussed previously in [12].

Notwithstanding the examples above, mathematicians generally leave their uncertainty unquantified. This may be due to haziness about, for example, what
a “60% chance” means, and how probability should be used, in the context of deductive reasoning. One point to emphasize is that we are referring to subjective uncertainty rather than any due to inherent “randomness” of mathematics. Of course there is nothing random about whether a proof exists of a given mathematical statement. Frege, speaking on mathematical reasoning, appears to note this lack of randomness as a problem: “the ground [is] unfavorable for induction; for here there is none of that uniformity which in other fields can give the method a high degree of reliability” [21]. I.e. a set of propositions may be homogeneous in some ways but their truth values cannot be seen as merely IID samples because the propositions are actually distinct and distinguishable. However, if we model bounded reasoning we may indeed have an analogy to other forms of uncertainty.

1.2 The (modeling) problem

Gödel mentions deductive probabilities in a discussion of empirical methods in mathematics [28]:

It is easy to give examples of general propositions about integers where the probability can be estimated even now. For example, the probability of the proposition which states that for each \( n \) there is at least one digit \( \neq 0 \) between the \( n \)-th and \( n^2 \)-th digits of the decimal expansion of \( \pi \) converges toward 1 as one goes on verifying it for greater and greater \( n \).

In commentary, Boolos naturally asks how such probabilities would be computed [9]:

One may, however, be uncertain whether it makes sense to ask what the probability is of that general statement, given that it has not been falsified below \( n = 1000000 \), or to ask for which \( n \) the probability would exceed .999.

With Boolos, we want to know, how would subjective deductive probabilities work in general? Are there right and wrong ways to assign these probabilities? Do they even make sense? These questions have both positive and normative versions; we focus on the normative.

Bayesianism is the theory of probabilities for physical uncertainty [58]. It gives an interpretation of probability, where we take a probability space, and interpret the probability measure as assigning subjective degrees of belief to events which represent expressible physical states. Looking from the reverse direction, Bayesianism argues from the nature of physical uncertainty to reach a conclusion that degrees of belief should form a probability space. In this second sense we can think of Bayesianism as a proof, where the premise is physical uncertainty, the inferences are rationality arguments, and the conclusion is probability theory. There are two ways to make use of a proof to learn something new. First, we can apply the theorem if we are able to satisfy
the premise. Here this would mean reducing deductive uncertainty to physical uncertainty by defining virtual information states. This general problem of deductive probabilities has received some attention in the literature (the sample space is taken to consist of complete formal theories) as we mention in later sections. But what if there is no valid way to define virtual information for deductive uncertainty, i.e. what if probability theory is not appropriate for deductive uncertainty in this manner? What if something else is? The second way to learn from a proof is to imitate the proof technique. Here we would start with the premise of deductive uncertainty, proceed using analogous but adapted rationality arguments, and reach a conclusion which is a set of mathematical rules possibly different from probability theory. We focus on this approach.

The first step is to fix a concrete and unambiguous way to quantify uncertainty. If we assign a number to our belief in a statement, what does that mean? And is it always possible for us to do so? In Bayesianism, uncertainty is quantified by a single real number from [0, 1] and a prominent operational definition of this quantification is based on fair bets [52, Ch. 13], [58, Sec. 2.2.1]. This operationalization appears to work equally well for deductive uncertainty. That is, anyone can express their uncertainty about a mathematical statement \( \phi \) using a number in [0, 1] which encodes the betting payoffs they consider acceptable if they were to bet on \( \phi \). We call these values plausibilities. (This is not to be confused with other usages such as “plausibility measures” [29, Sec. 2.8]). We assume this operationalization is meaningful and understood.

In the context of physical uncertainty, Bayesianism adds constraints on what plausibilities/probabilities should be for a rational agent, namely coherence (so plausibilities are probabilities), conditionalization, regularity and other guidance for selecting priors [58, 40]. Also, probabilistic forecasts may be rated on accuracy using loss functions [38].

However, the assumptions of Bayesianism on which these constraints are based do not necessarily still apply to deductive plausibilities and indeed we may have additional or different requirements. Thus the precise question to answer is, what constraints should be put on deductive plausibilities and what mathematical structure results?

## 2 Formal representation of plausibilities

### 2.1 Plausibility functions

Fix an encoding of deductive statements into finite strings so that there is a decision problem \( \Pi \subseteq \{0, 1\}^* \) corresponding to the true statements. We take an association of plausibilities to encoded statements as a function \( p : \{0, 1\}^* \rightarrow [0, 1] \). We call \( p \) a plausibility function. A plausibility function represents an agent’s uncertainty about \( \Pi \). One could also have defined a plausibility function to take a finite sequence of input strings and return a
finite sequence of plausibilities, that is, working at the level of bet systems instead of bets.

2.2 Languages

Finding proofs is a matter of computation, so our reasoning abilities are equivalent to our computational resources; and generally we will experience deductive uncertainty when faced with any intractable computational problem. Importantly, we cannot meaningfully talk about problems with only one input, since obviously the best output is the actual truth value. So we must talk of uncertainty about an entire set of inputs simultaneously.

Probability spaces consist of a measurable space and a probability measure. In Bayesianism, the measurable space may be substituted by a “sentence space” which is closed under logical operations. In the deductive case, any nontrivial problem Π has an input set that is trivially closed under logical operations, since any input is logically equivalent to “true” or “false”. We conclude that the problem Π need not a priori have any particular syntactic structure and we may consider standard problems from theoretical computer science.

There is a line of research on deductive uncertainty where the inputs come from a first-order language. Typically this work aims at finding composites of logic and probability theory, and there is less focus on practicality. The most recent work is by Garrabrant et al. [23] and [23, Sec. 1.2] reviews previous literature. In the present work we instead restrict our attention to decidable problems. We do this because inconsistent logics do not make sense in terms of betting. So first-order logics are problematic due to Gödel’s first and second incompleteness theorems.

2.3 Epistemic quality vs. computation costs

For a given problem Π ⊆ {0, 1}∗ we seek an epistemic improvement relation ≺Π on plausibility functions, where q ≺Π p iff p is a strictly better uncertainty assignment for Π than q, ignoring any computational costs of the functions. For example, if we decide to require regularity, we would say that for all functions p and q, if p is regular and q is not then q ≺Π p. Guidance for selecting plausibility functions is then based on weighing ≺Π against computational costs. If we are given a probability distribution on inputs, we take the distributional decision problem (Π, D) and consider a distribution-specific relation ≺_{(Π,D)}. We refer to the relation as an order but it is not necessarily total.

Improvement relations can be found in other modeling contexts. For algorithm running time sequences we use asymptotic relations such as big O or polynomial growth vs. non-polynomial growth. Fine-grained relations may be used if the model of computation is fixed. The Nash equilibrium represents a kind of ordering which expresses that a strategy profile can be improved from one player’s perspective. Pareto optimality is similar but for group rationality.
Another example is the Marshall/Kaldor-Hicks social welfare improvement relation in economics [19, 22]. This last relation is useful even though it cannot be defined to be both transitive and antisymmetric.

In general we have a tradeoff between epistemic quality (whatever we determine that to be) and computational complexity. A theory of deductive uncertainty must not only define gradients of epistemic quality but dictate how to make this tradeoff. If we allow arbitrary computations, the problem immediately disappears. E.g. there is a temptation to look into Solomonoff induction [40] as a model of inductive reasoning applied to mathematical knowledge. This would be an attempt to formalize, e.g. Pólya’s patterns of plausible reasoning [48, 41], such as “A analogous to B, B more credible ⇒ A somewhat more credible”. However we must be cautious, because an incomputable prior cannot be the correct tradeoff between quality and efficiency.

Computation costs may or may not be measured asymptotically. Asymptotic means no finite set of inputs can make a difference. If we use asymptotic complexity this forces us to define \( \prec_{\Pi} \) so that it is compatible, i.e. also asymptotic. As an example, utility in game theory is generally not compatible with asymptotic computation costs. There are, however, game models which trade off running time and other considerations in a non-trivial way, for example using discounted utility. In economics and game theory, the concept of “bounded rationality” refers to decision making with limitations on reasoning/optimization power, such as imperfect recall, time constraints, etc. [51]. We note some economic models which incorporate bounded computation: game-playing Turing machines with bounded state set [42], automata models [46], machines as strategies and utility affected by computation cost [30, 20], information asymmetry in finance [1]. If a game model uses a practically awkward criterion for algorithm performance, simple models of computation may be used, or equilibria may be reasoned about without analyzing a particular algorithm.

A simple approach to the tradeoff is to fix a resource bound and consider as “feasible” only functions that can be computed within the bound. Then, given \( \prec_{\Pi} \), we optimize over the subset of feasible plausibility functions. This is the method we focus on in the remainder. E.g. we may assume the Cobham-Edmonds thesis and consider \( \prec_{\Pi} \) restricted to polynomial-time-computable functions.

We make the assumption that we, as modelers, are always capable of analyzing given plausibility functions however is necessary to evaluate \( \prec_{\Pi} \) and analyze computational complexity. This is of course not true, as discussed in [2, 43] which consider bounded rationality in economic systems. However this is a worthy assumption since building meta-uncertainty into the model creates a regress which would add significant complexity. Thus we can say that optimizing \( \prec_{\Pi} \) is the rational way to select a plausibility function even if we are not currently able to do so constructively. Particularly, when we analyze functions according to an input distribution, the business of defining the distribution is that of the unbounded analyst. In practice, e.g. approximation algorithms are analyzed, even if the problem they attempt to approximate is
2.4 Conditional plausibility

If we select plausibility functions by optimizing $\prec_{\Pi}$ over feasible functions, the definition of feasibility could change over time or simply vary across contexts, so in general we speak of conditional plausibility functions $p(\cdot|S)$, where $S$ is an oracle or complexity class or other representation of the resources available to compute the function. Another interpretation is that, over time, computation costs may effectively come down, enlarging the budget set of feasible functions. This notation and terminology indicates an analogy where knowledge, in the computational sense (roughly that of [25 Sec. 9.2.3], [26 Sec. 7.2]), takes the place of information.

In Bayesianism, conditionalization is the process by which updates on new information must occur. I.e. after observing $A$, our new probability of $B$ is $P(B|A) = P(A \cap B)/P(A)$. We note that conditionalization is a uniform process in that there is a finite rule that performs updates for all events $A$ at all times. If there is an infinite set of possible $S$, we could restrict to uniform-updating plausibility functions, i.e. those which take a representation of $S$ as a parameter. In, for example, Garrabrant’s model [23], the plausibility function takes an additional parameter $n$, which is the number of stages to run. However this level of analysis is beyond our scope.

3 Rational plausibilities in mathematics

3.1 Scoring rules

As a method of eliciting quantified uncertainty, fair bet odds are equivalent to asking “what is an objective chance $\bar{p}$ such that your uncertainty of this event is equivalent to that of an event with objective chance $\bar{p}$?” These are also equivalent to strictly proper scoring rules, which go back to de Finetti and beyond [15]. This refers to a situation where the agent is asked for a number $x \in [0, 1]$ representing uncertainty about a proposition $\phi$, and then the agent receives a score of $B([\phi], x)$. Here we use Iverson brackets where $[\phi]$ is 1 if $\phi$ is true, and 0 otherwise. (The word “score” is a bit of a misnomer because lower scores are better.) The scoring function $B$ is strictly proper iff

$$\arg \min_x yB(1, x) + (1 - y)B(0, x) = y,$$

i.e. if we assumed $[\phi]$ is a $\text{Bernoulli}(y)$-distributed random variable, then we obtain the optimal expected score by choosing $x = E[\phi] = y$. We note that with both betting and scoring, the agent is presented with a simple decision where the only computational difficulty comes from the one instance of some decision problem. This allows us to conclude that their equivalence holds in the computational setting as well.
Since plausibilities encode how we make simple decisions, more desirable plausibility values are those that lead to better outcomes from decisions. Dutch book arguments justifying Bayesianism are essentially saying if you can avoid losing money in a bet, you should. On the other hand, scoring rules are generally considered to index epistemic quality. In fact, the concepts of betting and scoring rules are essentially the same for classical uncertainty. It is shown in [47] that Dutch books exist iff an agent’s forecasts are strictly dominated, where domination means there is another plausibility assignment that obtains a better score in all outcomes, according to a continuous proper scoring rule. We take this scoring rule characterization of Bayesianism (which led to probability theory in that case) and apply it to deductive plausibilities via analysis of plausibility functions.

Proper scoring rules conveniently associate a real number to each of our plausibilities. The Brier score has some desirable properties [55]. The logarithmic score also has desirable properties and is closely related to information-theoretic entropy. Given any proper scoring rule, one can always construct a decision environment such that performance in the environment is precisely performance according to the scoring rule. However, scoring rules are equivalent in the context of conditional expectation [4] and by analogy we may expect rational plausibilities to approximately share this property and others [39, 49].

Worst-case scoring of plausibility functions leads to trivialities since \( p \equiv 1/2 \) is optimal unless the problem can be exactly solved. An alternative in some cases would be to consider inputs of length \( \leq n \) rather than \( n \), but we focus on the standard practice of considering inputs of the same length. In the context of professional mathematics, we may take an average-case approach with a fixed input distribution. This is a non-adversarial model where the distribution reflects inputs that come up in practice and the agent is betting “against nature”.

If inputs are distributed according to an ensemble \( D \), we may say that \( q \prec_{(\Pi, D)} p \) if the expected score of \( p \) is less than that of \( q \). The comparison may be asymptotic. This is the model used in [37] which develops some relevant mathematical theory. More general background includes average-case complexity theory [33, 8] and probabilistic numerics [50, 11]. (A very similar view is considering \( p \) as an unnormalized distribution on strings of length \( n \), and finding the statistical distance between the normalized distribution and the normalized distribution corresponding to \( 1_\Pi \). There we would require at least that \( E(p) = E(1_\Pi) \).

### 3.2 Foundations of “semi-rigorous proofs”

Zeilberger presented the concept of a “semi-rigorous proof” and predicted that it might become acceptable by the mathematical community: “I can envision an abstract of a paper, c. 2100, that reads: ‘We show, in a certain precise sense, that the Goldbach conjecture is true with probability larger than 0.99999’ ”
In order for such a result to be useful, perhaps there must be cases where plausibilities are objective.

According to $≺_{(\Pi,D)}$, are plausibilities objective or subjective? Should we expect people to agree on plausibilities? There are various sources of subjectivity: how to embed individual questions in problems, first-order logic issues, and problems $\Pi$ where $≺_{(\Pi,D)}$ has no unique optimum. For example, take Gödel’s $\pi$ problem from Sec. 1.2. Let $\phi$ represent the sentence, for each $n$ there is at least one digit $\neq 0$ between the $n$-th and $n^2$-th digits of the decimal expansion of $\pi$. To condition on observed digits of $\pi$, we can allow access to an oracle that checks $\phi$ for large ranges of the digits of $\pi$. First, there may not be a proof or refutation of $\phi$, in which case betting outcomes are undefined. Second, if $\phi$ is decideable, its truth value can always be hard coded in $p$ even if we embed $\phi$ in a class of problem instances which is standard and universal. Analyzing general methods of computing plausibilities, rather than individual values, does makes sense since people typically use heuristics in a consistent manner across problems.

In traditional Bayesianism there is a seemingly ineradicable source of subjectivity from the choice of prefix Turing machine used to define Solomonoff’s prior. Any one string can be assigned (almost) an arbitrary probability. Perhaps we are left with an analogous but different kind of subjectivity for mathematical plausibilities.

We stated at the end of Sec. 1.1 that mathematicians may be uncomfortable with putting too much focus on plausibilities. Gödel says, “I admit that every mathematician has an inborn abhorrence to giving more than heuristic significance to such inductive arguments” [28]. Also, Corfield notes, “Pólya himself had the intuition that two mathematicians with apparently similar expertise in a field might have different degrees of belief in the truth of a result and treat evidence for that result differently” [12]. However, the physics community has had notable success using non-rigorous mathematical methods.

One practical issue with publishing probabilities for mathematical problems is error amplification. If we take the conjunction of two “independent” uncertain statements we end up with uncertainty greater than that of either of the original statements, which means confidence erodes over time. In mathematics this is undesirable since we are used to taking arbitrarily long sequences of conjunctions and implications with no loss of validity.

More on probabilistic proofs is found in [18]. The potential for models of uncertainty in mathematics to explain the use of large computations to increase confidence in conjectures is noted in [12].
4 Arbitrage pricing in markets with computational constraints

Here we use the language of finance; see the paper [44] for some discussion on the equivalence of Dutch books and arbitrage. Suppose instead of a binary function $\Pi$ we estimate a continuous-valued function $F$. Consider a 2-period market where at time $t = 0$ the seller(s) offer a price $f(x)$ for each asset $x \in \{0, 1\}^*$. Also at time $t = 0$, a buyer buys a quantity $g(x)$ of the asset $x$ at the prices given by $f$, where $g(x) < 0$ indicates short selling. At time $t = 1$, each asset $x$ has terminal payoff determined by $F(x)$. Ignoring time discounting, the buyer’s gain from $x$ is

$$F(x)g(x) - f(x)g(x).$$

The function $F$ may be of two possible kinds. If the payoffs are deterministic, then computation of $F$ is presumably non-trivial but ultimately must be performed. On the other hand, the payoffs represented by $F$ may be expected values of some random variables, in which case there is no need for $F$ to be computationally tractable, but sampling from the relevant probability distributions should be. Roughly speaking, these situations we describe could be found where information is fixed throughout the market history $t = 0, 1$ and $F$ is a reference model used to price the assets $x$, and where $F$ has nontrivial computational complexity. In the more general and likely more realistic case where information is not fixed, there would be a tradeoff between waiting for more information and performing lengthier computations which is not included in this model.

Computational aspects of economic price models such as Arrow-Debreu equilibria [56] and combinatorial auctions [13] have received interest in the literature [17, 35, 3]. Pricing is often based on probabilistic models. Computational difficulty leads to the use of Monte Carlo methods e.g. for numerical solutions to stochastic differential equations, Bayesian posteriors, and other simulations [54, 24, 53]. Probabilistic inference in Bayesian networks is known to be computationally hard in general [14]. The hardness of pricing arbitrary exotic derivatives is explored in [10].

What does arbitrage mean in this context? Suppose $C^g$ is a class of functions $g$. Then we would say the prices $f$ admit arbitrage from $C^g$ based on the buyer’s gain

$$b_n(g) = \sum_{x \in \{0, 1\}^n} F(x)g(x) - f(x)g(x)$$

for different $g \in C^g$. In addition to computational constraints, we may restrict $C^g$ to polynomial growth and polynomial-sized support to reflect practical trading limits. We could also allow the arbitrageur to randomly generate an asset $x$, subject to computational constraints, so $b_n$ would be an expected value. The classical definition of arbitrage [16] suggests requiring there exist $g \in C^g$ such that $b_n(g) \geq 0$ for all $n$ and $b_n(g) > 0$ infinitely often. We say a sequence $a_n$ is negligible iff $a_n = n^{-o(1)}$. A relaxation of strict arbitrage would
require having $b_n(g)$ positive and non-negligible infinitely often and not having $-b_n(g)$ positive and non-negligible infinitely often.

In current complexity theory, it is not clear whether this form of arbitrage can or cannot be avoided. The paper [27] shows that for the SAT problem, if $f$ is computable in polynomial time and $C^g$ is the set of functions computed in polynomial time, then we can always get $b_n > K$ infinitely often for some fixed $K > 0$ unless $P = NP$.

### 5 Conclusion

Even within traditional Bayesianism, the assumption of computational unboundedness can be undesirable; this is known as the problem of logical omniscience [34, 57, 31]. Some work has been done on formal models for logical non-omniscience, including a resource-bounded AIXI [36] and resource-bounded Solomonoff prior [40], although these are somewhat ad-hoc. As mentioned in the previous section, in general an agent must make decisions based on the available information and the computational costs of reasoning. A powerful model is given in [45], although this model may be hard to integrate with computational complexity theory. Ideally a general decision theory would build on probability and computational complexity in a way that allows us to exploit our mathematical understanding of those fields. Still, even if we have a definition of the optimal decision rule, there may be research remaining in algorithm design and analysis to actually construct it. And that research in turn may utilize a Bayesian perspective as in the pseudo-distributions of the sum-of-squares algorithm [5, 6].

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