ISOPERIMETRIC FUNCTIONS FOR GRAPH PRODUCTS

DANIEL E. COHEN

Queen Mary and Westfield College, London University

Abstract. Let \( \Gamma \) be a finite graph, and for each vertex \( i \) let \( G_i \) be a finitely presented group. Let \( G \) be the graph product of the \( G_i \). That is, \( G \) is the group obtained from the free product of the \( G_i \) by factoring out by the smallest normal subgroup containing all \([g, h]\) where \( g \in G_i \) and \( h \in G_j \) and there is an edge joining \( i \) and \( j \). We show that \( G \) has an isoperimetric function of degree \( k > 1 \) (or an exponential isoperimetric function) if each vertex group has such an isoperimetric function.

Graph Products

Let \( \Gamma \) be a finite graph; that is, \( \Gamma \) consists of a finite set of vertices and a finite set of edges, where each edge is an unordered pair of vertices. Let us be given a group \( G_i \) for each vertex \( i \). Then the graph product \( G \) of the \( G_i \) is the group obtained from the free product of the \( G_i \) by factoring out by the smallest normal subgroup containing all \([g, h]\) where \( g \in G_i \) and \( h \in G_j \) and there is an edge joining \( i \) and \( j \). Note that if \( G_i \) has presentation \( \langle A_i ; R_i \rangle \), where the \( A_i \) are disjoint, then \( G \) has presentation \( \langle \bigcup A_i ; \bigcup R_i \cup S \rangle \), where \( S \) is the set of commutators \([a, b]\) for \( a \in A_i, b \in A_j \), where \( i \) and \( j \) are joined by an edge. The free product and the direct product are examples of graph products (corresponding to graphs with no edges and complete graphs, respectively). All groups considered will be finitely presented.

Gersten [G] defines an isoperimetric function for a finite presentation \( \langle Y; S \rangle \) of a group \( H \) to be a function \( f \) such that if \( w \) is a word of length \( n \) in the free group on \( Y \) and \( w \) equals 1 in \( H \) then \( w \) is the product of at most \( f(n) \) conjugates of elements of \( S \) and their inverses. He shows that if we change to another finite presentation then there are positive constants \( a, \ldots, e \) such that the new presentation has an isoperimetric function \( g \) given by \( g(n) = af(bn + c) + dn + e \).

Consequently, we say that \( g \preceq f \) if there are positive constants \( a, \ldots, e \) such that \( g(n) \leq af(bn + c) + dn + e \) for all \( n \), and we call \( g \) equivalent to \( f \) is we have both \( g \preceq f \) and \( f \preceq g \). This is slightly different from Gersten’s definition of equivalence of functions. I prefer this definition because it makes all polynomials of a given degree equivalent, and also makes all exponentials equivalent.

When the free monoid \( Y^* \) maps onto \( H \) (and not just the free group on \( Y \)) we say that \( Y \) is a set of monoid generators of \( H \). It is particularly useful if \( Y \) has the property that to each \( y \in Y \) there is \( \bar{y} \in Y \) such that \( y\bar{y} \) equals 1 in \( H \). When this happens, it is easy to see that we can find a set of defining relators

Key words and phrases. graph products, isoperimetric functions, Thue systems.

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containing all the elements $y\bar{y}$ and lying in $Y^*$. It is also easy to check that any finite presentation can be changed to a finite presentation of this sort, and that, in looking for an isoperimetric function, we need only consider elements of $Y$ and not general elements of the free group on $Y$. In this paper we prove the following theorem.

**Theorem.** If each vertex group has an isoperimetric function which is polynomial of degree $k > 1$ (or an exponential isoperimetric function) then so does their graph product.

The theorem will also hold for other classes of isoperimetric functions (this follows immediately from the proof), but the precise condition is messy and these two cases are the most important. One requirement is that the function is at least quadratic. When this holds, it is sufficient that the equivalence class contains a function $f$ such that $f(m + n) \leq f(m) + f(n)$ for all $m$ and $n$. Ol’shanskii has proved [O] that groups whose isoperimetric function is subquadratic are hyperbolic and hence have linear isoperimetric function. Note that the graph product of groups with a linear isoperimetric function usually does not have a linear isoperimetric function.

The inspiration for this paper came from work on graph products by Hermiller and Meier [HM]. Her discussion of normal forms in graph products, and a similar discussion by Laurence [L], led me to the approach given here.

In proving the theorem we may take any finite presentations of the vertex groups. It will be convenient to take the $A_i$ to be disjoint finite sets which are monoid generators of $G_i$, so that there is a homomorphism $\pi_i : A_i^* \to G_i$ (where, for any set $Y$, $Y^*$ is the free monoid on $Y$).

A non-trivial element of $A_i^*$ will be called an $i$-word. To each $i$-word we take a symbol $[u]$. Let $X$ be the set of all such symbols. Then there is a homomorphism $\rho$ from $X^*$ onto $G$ which sends $[u]$ to $\pi_i u$ when $u$ is an $i$-word. An element of $X^*$ will just be called a word. We say that $[u]$ is in the star of $i$ if $u$ is a $j$-word where $i$ and $j$ are joined by an edge. We say that the word $W$ is in the star of $i$ if $W$ is $[u_1] . . . [u_m]$ with each of $[u_1], . . . , [u_m]$ in the star of $i$.

A sequence of words $W_1, . . . , W_n = \varepsilon$, where $\varepsilon$ is the empty word, will be called a reduction sequence if, for all $m < n$, $W_{m+1}$ is obtained from $W_m$ by one of the following moves:

1. replace $P[u][v]Q$ by $P[uv]Q$, for any words $P, Q$ and, for any $i$, any $i$-words $u$ and $v$;
2. replace $P[u]Q[v]T$ by $P[uv]QT$ for any words $P, T$, any $i$-words $u, v$, any $i$, and any word $Q$ in the star of $i$;
3. replace $P[u]Q$ by $PQ$ for any words $P, Q$ any $i$, and any $i$-word $u$ such that $\pi_i u = 1$.

We refer to $i$-moves if there is a need to mention $i$ explicitly.

The following lemma will be proved in the next section.

**Lemma.** If $\rho W = 1$ then there is a reduction sequence starting with $W$.

Let $W = W_1, . . . , W_n = \varepsilon$ be a reduction sequence. We show how to replace it by another reduction sequence with nice properties.

Since the sequence ends with $\varepsilon$, a move of type 3 must be used at some point. Let the first such move be an $i$-move, going from $W_m$ to $W_{m+1}$. Since all earlier moves are of types 1 and 2, it is easy to check that, in the sequence $W_1, . . . , W_{m+1}$,
a $j$-move followed by an $i$-move can be replaced by an $i$-move followed by a $j$-move. Thus we may assume that each of the first $m$ moves is an $i$-move.

We can now see easily (by induction, looking at the reduction sequence beginning with $W_2$) that $W$ must be of the form $P[u_1]Q_1[u_2] \cdots Q_{r-1}[u_r]P'$, where $u_1, \ldots, u_r$ are $i$-words, $Q_1, \ldots, Q_{r-1}$ are (possibly empty) words in the star of $i$, and $\pi(u_1 \ldots u_r) = 1$.

For an arbitrary word $V = [v_1] \ldots [v_s]$, define $\|V\|$ to be $|v_1| + \cdots + |v_s|$, where $|v|$ is the length of $v$. We define the weight of a move of type 1 to be 0, the weight of a move of type 2 to be $|Q| \cdot |v|$, and the weight of a move of type 3 to be $f(|u|)$, where $f$ is an isoperimetric function for all of the groups $G_i$. We define the weight of a reduction sequence to be the sum of the weights of its moves, and we define the weight of a word $W$ for which $\rho W = 1$ to be the minimum weight of the reduction sequences beginning with $W$.

Let $g(n)$ be the maximum of $f(n_1) + \cdots + f(n_s)$ over all $s$ and all positive integers $n_1, \ldots, n_s$ whose sum is $n$. Note that if $f$ is polynomial of degree $k$ (or exponential) then so is $g$.

We next show that the weight of a word $W$ with $\rho W = 1$ is at most $\|W\|^2 + g(\|W\|)$. As already remarked, we can write $W$ as $P[u_1]Q_1[u_2] \cdots Q_{r-1}[u_r]P'$, where $u_1, \ldots, u_r$ are $i$-words, $Q_1, \ldots, Q_{r-1}$ are (possibly empty) words in the star of $i$, and $\pi(u_1 \ldots u_r) = 1$. Then there is a reduction sequence beginning with $W, P[u_1u_2]Q_1Q_2[u_3] \cdots [u_r]P' \ldots P[u_1 \ldots u_r]Q_1 \cdots Q_{r-1}, W'$, where $W'$ is $PQ_1 \cdots Q_{r-1}P'$.

Since the sum of the weights of the moves from $W$ to $W'$ is at most $\|W\| \cdot (|u_1| + \cdots + |u_r| + f(|u_1\ldots u_r|))$, the required result holds by induction.

Finally, we use this result to obtain an isoperimetric function for $G$. We use the set of monoid generators $\bigcup A_i$. There are homomorphisms $\pi : (\bigcup A_i)^* \to G$, $\alpha : (\bigcup A_i)^* \to X^*$, and $\beta : X^* \to (\bigcup A_i)^*$, defined by $\pi a = \pi a$ for $a \in A_i$, $aa = [a]$, and $\beta [u] = u$. Plainly, for any $w \in (\bigcup A_i)^*$, we have $\beta \alpha w = w$ and $\rho \omega w = \pi w$. Also $|w| = \|\alpha w\|$. It is easy to see that if $W'$ is obtained from $W$ by a move of weight $k$ then $\beta W$ is the product of $\beta W'$ and $k$ conjugates of the defining relators (and their inverses) of the finite presentation of $G$. By induction on the length of the reduction sequence, if $\rho W = 1$ then $\beta W$ is the product of at most weight $(W)$ conjugates of the defining relators and their inverses.

Applying this to $\alpha w$, where $\pi w = 1$, and using the formula for the weight, we see that $g(n) + n^2$ is an isoperimetric function for our presentation of $G$, proving the theorem.

**Thue Systems**

Let $X$ be an arbitrary set. A *Thue system*, or *rewriting system* on $X$ is a subset $S$ of $X^* \times X^*$. Such a system induces an equivalence relation on $X^*$; namely, the smallest equivalence relation such that $ulv \equiv urv$ for all words $u, v$ and all pairs $(l, r)$ in $S$. The quotient of $X^*$ by this equivalence is called the *monoid presented by $(X; S)$*.

When we look for normal forms for the equivalence classes, there are two ways to proceed. One treats all members of $S$ alike, and compares the equivalence relation with the non-symmetric relation in which we can replace $ulv$ by $urv$ but not vice versa. We then endeavour to see if this terminates, and whether it provides a unique normal form.
The other approach, which is more convenient in our situation, begins by assuming that \( S \) consists only of pairs for which \(|l| \geq |r|\) and that, for any \((l, r)\) in \( S \) with \(|l| = |r|\) we also have \((r, l)\) in \( S \). This can be done without loss of generality, since we get the same equivalence relation if we replace a pair \((l, r)\) by \((r, l)\) and also if we add pairs \((l, r)\) for which \((l, r)\) are already in \( S \).

If we do this, then, when we consider replacing \( ulv \) by \( urv \) but not vice versa, if \(|l| = |r|\) we can use the further pair \((r, l)\) to return from \( urv \) to \( ulv \). Consequently, we treat such pairs differently from those pairs for which \(|l| > |r|\). It is quite common in computer science to distinguish between the two approaches by using the phrase ‘rewriting system’ for the first one and the phrase ‘Thue system’ for the second.

We write \( ulv \to urv \) for a pair \((l, r)\) with \(|l| > |r|\), and \( ulv \sim urv \) for a pair with \(|l| = |r|\). We let \( \to \) and \( \sim \) be the reflexive transitive closures of these.

We say that the pair \( u, v \) is — it almost confluent if there are \( u_1, v_1 \) such that \( u \to u_1, v \to v_1 \), and \( u_1 \sim v_1 \). Plainly, almost confluent words are equivalent, and we call \( S \) almost confluent if every pair of equivalent words is almost confluent. In searching for nice representatives of the equivalence classes, it is particularly helpful if \( S \) is almost confluent. Clearly, when this holds, if \( u \) is equivalent to \( \epsilon \) then \( u \to \epsilon \).

This situation is very familiar to computer scientists. A sufficient condition for the property to hold can be given in terms of the behaviour of certain critical pairs of words, which arise from certain words in which two of the elements of \( S \) may be used. The situation is less well-known to group theorists, but results sometimes referred to as ‘Peak Reduction Lemmas’ are essentially of this form.

Huet [H] showed that \( S \) is almost confluent whenever almost confluence holds for all pairs \( u, v \) such that there is some \( w \) with \( w \to u \) and either \( w \to v \) or \( w \sim v \). It is not difficult to prove this directly using peak reduction arguments. (If readers want to look at [H], they should note that Huet’s \( \sim \) is our \( \tilde{\sim} \).)

Huet also showed that we do not even have to consider all such pairs. It is enough to consider those \( w \) of form \( abc \) for some words \( a, b, c \) such that \( S \) either has elements \( (ab, r_1) \) and \( (bc, r_2) \) or has elements \( (abc, r_1) \) and \( (b, R_2) \), with \( u \) and \( v \) (or \( v \) and \( u \)) obtained from \( w \) by applying these two elements. These pairs \( u, v \) of words are called the critical pairs. Huet’s proof applies in much more generality, and it is probably simpler to prove this directly in our situation.

We now return to graph products, with the set \( X \) as in the first section. We shall prove the lemma by applying this theory of Thue systems. The set \( S \) will consist of the following pairs:

1. \((u[v], u[w])\), where \( u \) and \( v \) are \( i \)-words for some \( i \);
2. \((u[P[v]], u[e]P)\), where \( u \) and \( v \) are \( i \)-words for some \( i \) and \( P \) is in the star of \( i \);
3. \((u, e)\), where \( u \) is an \( i \)-word such that \( \pi_i u = 1 \);
4. \((u[v], u[i])\), where \( u \) and \( v \) are \( i \)-words for some \( i \) and \( \pi_i v = \pi_i u \);
5. \((u[v], v[y][u])\), where \( u \) is an \( i \)-word, \( v \) is a \( j \)-word, and there is an edge of the graph joining \( i \) and \( j \).

It is clear that the set of \( i \)-words for a given \( i \), together with the corresponding pairs of types 1, 3, and 4, form a monoid presentation for \( G_i \); this is just a variant of the multiplication table presentation. If we take all pairs of types 1, 3, 4, and 5 we then clearly obtain a monoid presentation for \( G \). We can then add the pairs of type 2 and still get a monoid presentation for \( G \), since the two elements of a pair of type 2 clearly give the same element of \( G \).
To prove the lemma, we need only show that the criterion mentioned above is satisfied.

First look at \([u]P[v]Q[w]\), where \(u, v, \) and \(w\) are \(i\)-words for some \(i\), and \(P\) and \(Q\) are (possibly empty) words in the star of \(i\). We have \([u]P[v]Q[w] \rightarrow [uv]PQ[w]\) and also \([u]P[v]Q[w] \rightarrow [u]PvwQ\). Here we find that \([uv]PQ[w] \rightarrow [uvw]PQ\) and also \([u]P[v]P[w] \rightarrow [uvw]PQ\).

Next, look at \([u]P[v]\), where \(u\) and \(v\) are \(i\)-words, \(P\) is in the star of \(i\), and \(v_i v = 1\). Then \([u]P[v] \rightarrow [uv]P\) and also \([u]P[v] \rightarrow [u]P\). Since \(v_i (uv) = v_i u\), we have \([uv]P \sim [u]P\), using a pair of type 4. If we have \(v_i u = 1\) instead of \(v_i v = 1\), then \([u]P[v] \rightarrow [uv]P\) and \([u]P[v] \rightarrow P[v]\). Here we have \([uv]P \sim [v]P\), using a pair of type 4, and \([v]P \sim P[v]\), using pairs of type 5 (since \(P\) is in the star of \(i\)).

Suppose we have a word \([u]P[v]\), where \(u\) and \(v\) are \(i\)-words for some \(i\) and \(P\) is a (possibly empty) word in the star of \(i\). Let \(w\) be an \(i\)-word such that \(v_i w = v_i u\). Then \([u]P[v] \rightarrow [uv]P\) and \([u]P[v] \sim [w]P[v]\). We then have \([w]P[v] \rightarrow [uv]P\) and \([uv]P \sim [vw]P\). A similar argument works when, instead of \(v_i w = v_i u\), we have \(v_i w = v_i v\).

Let \(u\) be an \(i\)-word such that \(v_i u = 1\), and let \(w\) be an \(i\)-word such that \(v_i w = v_i u\). Then \([u] \rightarrow \epsilon\) and \([u] \sim [w]\), and we also have \([w] \rightarrow \epsilon\). Let \(v\) be a \(j\)-word, where there is an edge joining \(i\) and \(j\). Then we have \([u][v] \sim [v][u]\) and also \([u][v] \rightarrow [v]\). Here we have \([v][u] \rightarrow [v]\).

Suppose we have a word \([u]P[v]\), where \(u\) and \(v\) are \(i\)-words for some \(i\), and \(P\) is a (possibly empty) word in the star of \(i\). Let \(w\) be a \(j\)-word, where there is an edge joining \(i\) and \(j\). Then \([u]P[v][w] \rightarrow [uv]P[w]\) and \([u]P[v][w] \sim [u]P[w][v]\). Since \(P[w]\) is in the star of \(i\), we have \([u]P[w][v] \rightarrow [uv]P[w]\). Finally, we have \([w][u]P[v] \rightarrow [w][uv]P\) and \([w][u]P[v] \sim [u][w]P[v]\). We then have \([u][w]P[v] \rightarrow [uw][w]P\) and \([w][u]P[w] \sim [uw]P\).

We have now shown that all critical pairs satisfy the required criterion, and the lemma is proved.

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School of Mathematical Sciences, Queen Mary and Westfield College, Mile End Road, London E1 4NS, England

E-mail address: D.E.Cohen@uk.ac.qmw.maths