Asymptotically flat black holes sourced by a massless scalar field

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We derive exact, asymptotically flat black hole solutions of Einstein-scalar gravity sourced by a non trivial scalar field with $1/r$ asymptotic behaviour. They are determined using an ansatz for the scalar field profile and working out, together with the metric functions, the corresponding form of the scalar self-interaction potential. Near to the singularity the black hole behaves as the Janis-Newmann-Winicour-Wyman solution. We also work out a consistent thermodynamical description of our black hole solutions. For large mass our hairy black holes have the same thermodynamical behaviour of the Schwarzschild black hole, whereas for small masses they differ substantially from the latter.

I. INTRODUCTION

In the past, the issue of the uniqueness of the Schwarzschild black hole has motivated the formulation of no-hair theorems [1, 2] forbidding the existence of black hole solutions endowed with a non trivial scalar field. Later, it was discovered that some low-energy string models allow for black hole solutions with scalar hair [3–7]. Nevertheless, the existence of these solutions remains limited to gravity theories with non-minimal couplings between the scalar field and the electromagnetic field.

Recently, the application of the AdS/CFT correspondence to condensed matter systems generated a flurry of activity on the search of new black hole and black brane solutions with AdS or domain wall [8–10] asymptotics endowed with scalar hair [11–19]. The main reason behind this interest is the holographic interpretation of the scalar field. In the dual QFT the scalar field plays the role of an order parameter whose non trivial profile generates symmetry breaking and/or phase transitions. Shifting to asymptotically AdS solutions allows to circumvent the standard no-hair theorems, relating the existence of hairy black holes to the violation of the positive energy theorem (PET) [20, 21]. In the AdS spacetime, differently from the flat case, a scalar field may have negative squared-mass, without destabilizing the vacuum [22].

Consequently, several numerical and analytical, black hole and black brane, solutions with scalar hair have been derived and used in the literature for holographic applications [11, 19, 23, 26].

In spite of this advances on the side of black hole solutions with AdS asymptotics, very few progress has been achieved in the search for asymptotically flat (AF) black holes with scalar hair (see, however, Refs. [27, 33]). On the other hand, it is becoming increasingly evident that scalar fields play a key role for understanding the physics of fundamental interactions. The recent discovery of the Higgs particle at LHC has confirmed experimentally the existence of a fundamental scalar particle responsible for the breaking of the electroweak symmetry [34, 35]. Observations of the Planck 2013 satellite give a striking confirmation of the existence of cosmological inflation generated by a scalar field coupled to gravity [36]. Finally, the present accelerated expansion of the universe could be also explained by Einstein gravity coupled to a scalar field.

An other important point to be mentioned here, is the universal scaling behaviour observed, numerically, years ago by Choptuik in black hole formation due to the collapse of a scalar field [37].

It is therefore of considerable interest to ask about the relevance of non trivial scalar field configurations for asymptotically flat black holes. In this paper we consider the simplest case of asymptotically flat black holes sourced by a massless scalar field $\phi$. Since at high energies the mass term can be neglected with respect to the kinetic term, we expect this situation to describe the short scale behaviour of any Einstein-scalar theory of gravity.

Static, spherically symmetric, AF, gravitational solutions sourced by a non trivial scalar field with an identically vanishing potential $V(\phi)$ are known since a long time [33, 10]. They are called the Janis-Newmann-Winicour or Wyman (JNWW) solutions. Consistently with the PET they do not describe black holes but naked singularities. To find AF hairy black hole solutions one has to relax the condition $V = 0$. We will therefore consider a potential

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II. EINSTEIN-SCALAR GRAVITY SOURCED BY A SCALAR FIELD

We consider Einstein gravity in four spacetime dimensions minimally coupled with a scalar field $\phi$:

$$A = \int d^4x\sqrt{-g} \left( R - 2(\partial\phi)^2 - V(\phi) \right),$$

where $R$ is the scalar curvature of the spacetime and we use natural units with $G = 1/16\pi$.

The field equations are

$$\nabla^2 \phi = \frac{1}{4} \frac{\partial V}{\partial \phi}, \quad (2.2a)$$

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 2 \left( \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} g_{\mu\nu} \partial^\rho \phi \partial_\rho \phi \right) - \frac{1}{2} V(\phi) g_{\mu\nu}. \quad (2.2b)$$

We are interested in static, spherically symmetric solutions of the field equations. We parametrize the spacetime metric using a Schwarzschild gauge:

$$ds^2 = -U(r)dt^2 + U^{-1}(r)dr^2 + R^2(r)d\Omega^2, \quad (2.3)$$

where $d\Omega^2$ is the metric element of the two-sphere $S^2$.

Using the parametrization $(2.3)$, the field equations take the form

$$\frac{R''}{R} = -(\phi')^2, \quad (2.4a)$$

$$(UR^2\phi')' = \frac{1}{4} R^2 \frac{\partial V}{\partial \phi}, \quad (2.4b)$$

$$(UR^2)'' = 2 - 2R^2V, \quad (2.4c)$$

$$(URR')' = 1 - \frac{1}{2} R^2 V. \quad (2.4d)$$

where $' \equiv d/dr$.

In general, the form of the solutions of these field equations depends on the class of potentials $V(\phi)$ one takes into consideration. Usually, one requires the existence of the Schwarzschild black hole solution sourced by a constant scalar $\phi = \phi_0$, which implies $V'(0) = 0$. Notice that without loss of generality we have chosen $\phi_0 = 0$. Additionally, we must impose boundary conditions on the $r = \infty$ asymptotic behaviour of the solution: if we require the spacetime to be asymptotically flat, it follows $V(0) = 0$, whereas for asymptotically anti de Sitter (AdS) spacetimes, $V(0) = \Lambda$, with $\Lambda$ strictly negative.

The existence of black hole solutions of the field equation $(2.4)$ sourced by a non trivial scalar field is strongly constrained by well-known no-hair theorems. Here, we investigate the simplest case of an AF black hole sourced by an asymptotically massless scalar field, and we will therefore consider the class of potentials satisfying

$$V(0) = V'(0) = V''(0) = 0. \quad (2.5)$$

Even by fixing the form of the potential $V(\phi)$ it is very difficult to find exact solutions of the field equations $(2.4)$ sourced by a non trivial scalar. We can improve the situation by imposing a boundary condition on the $r = \infty$
asymptotic behaviour of the scalar field. Because we are interested in massless scalar field the most natural boundary condition is that the scalar field behaves asymptotically as an harmonic function i.e. we require the $r \to \infty$ fall-off behaviour

$$\phi \sim 1/r.$$  \hfill (2.6)

As we will see in detail in Sect. IV this boundary condition strongly constrains the form of the potential $V(\phi)$. Moreover, starting from Eq. (2.6) one can use the general method proposed in Ref. [19] in order to solve the field equations. In fact, this method is particularly useful to generate solutions once the scalar field profile $\phi = \phi(r)$ is given. Using the variables introduced in Ref. [19]

$$R = e^Y, \quad u = UR^2,$$  \hfill (2.7)

the field equations (2.4) become

$$Y'' + Y^2 = -(\phi')^2,$$  \hfill (2.8a)

$$(u\phi')' = \frac{1}{4} \frac{\partial V}{\partial \phi} e^2 f Y,$$  \hfill (2.8b)

$$u'' - 4(uY)' = -2,$$  \hfill (2.8c)

$$u'' = 2 - 2V e^2 f Y.$$  \hfill (2.8d)

Eqs. (2.8c) and (2.8d) are second order linear differential equations in $u$ whereas (2.8a) is a first-order nonlinear equation for $Y$, known as the Riccati equation. One now starts from a given scalar field profile $\phi = \phi(r)$ and looks for solutions of the Riccati equation. Once the solution for $Y$ has been found one can integrate (2.8c) to obtain

$$u = R^4 \left[ - \int dr \left( \frac{2r + C_1}{R^4} \right) + C_2 \right],$$  \hfill (2.9)

where $C_{1,2}$ are integration constants.

The last step is to determine the potential using Eq. (2.8d)

$$V = \frac{1}{R^2} \left( 1 - \frac{u''}{2} \right).$$  \hfill (2.10)

### III. SPHERICALLY SYMMETRIC SOLUTIONS FOR $V = 0$

An important particular case of the Einstein-scalar gravity models we are considering is when $V = 0$, identically. As already mentioned, spherically symmetric solutions of the equations (2.4) in the case of a vanishing potential are known as the JNWW solutions and they represent naked singularities. Despite that, they have several interesting features: they are stable under scalar perturbations [41], they appear as the extremal limit of charged dilatonic black hole solutions [4, 41] and they have been used to construct traversable wormholes [42].

The JNWW solutions also appear as the extremal limit of the exact black hole solutions we will derive in Sect. V. For these reasons, in this section we will derive the JNWW solutions using our parametrization for the metric functions and discuss their most relevant physical properties.

We first solve the linear equation (2.8d) giving $u$ as a quadratic function of $r$, then we solve (2.8c) for $Y$. Finally, we use (2.8b) to determine $\phi$. The Riccati equation (2.8a) gives just an algebraic constraint among the integration constants $w$ and $\gamma$. We find:

$$U = \left( 1 - \frac{r_0}{r} \right)^{2w-1}, \quad R^2 = r^2 \left( 1 - \frac{r_0}{r} \right)^{2(1-w)}, \quad \phi = -\gamma \ln \left( 1 - \frac{r_0}{r} \right) + \phi_0, \quad w - w^2 = \gamma^2.$$  \hfill (3.1)

If we ignore the physically irrelevant constant shift of the scalar, Eqs. (3.1) give a two-parameter family of solutions. They are parametrized by the length scale $r_0$ and the dimensionless parameter $w$. As expected, the scalar field $\phi$ has the harmonic behaviour (2.6) for $r \to \infty$.

The constraint $w - w^2 = \gamma^2$ implies $0 \leq w \leq 1$. Note that it is invariant under the transformation $w \to 1 - w$. Under this transformation, the exponents in the metric functions change but the metric remains invariant if we simultaneously translate the coordinate $r$ according to $r \to r_0 - r$. Because of this discrete symmetry we are allowed to restrict the range of the parameter $w$ to $1/2 \leq w \leq 1$. 
The solution (3.1) with $r_0$ being a generic real number is therefore the most general solution. For $w = 1$ we get the usual Schwarzschild black hole solution (with constant scalar field) which reduces to the usual Minkowski vacuum solution in the $r_0 = 0$ limit. For $w \neq 1$ and $r_0$ positive, $r = r_0$ is a curvature singularity, whereas for $r_0$ negative, the curvature singularity is at $r = 0$.

As expected from the PET, the solution with $w \neq 1$ does not represent a black hole, because the metric (3.1) has no event horizon. It interpolates between the Minkowski spacetime at $r = \infty$ and a power-law metric near the singularity. For $r_0 > 0$ after shifting $r \to r + r_0$, the metric behaves near the singularity as

$$U = \left(\frac{r}{r_0}\right)^{2w-1}, \quad R^2 = r_0^2 \left(\frac{r}{r_0}\right)^{2-2w}, \quad \phi = -\gamma \ln \frac{r}{r_0},$$

(3.2)

whereas for $r_0 < 0$ we have

$$U = \left(\frac{r}{|r_0|}\right)^{1-2w}, \quad R^2 = r_0^2 \left(\frac{r}{|r_0|}\right)^{2w}, \quad \phi = \gamma \ln \frac{r}{|r_0|}.$$

(3.3)

A. Energy of the solution

Let us now calculate the total energy $M$ of the solution. This is a very important point because it tells us whether the solution will be stable with respect to the Minkowski vacuum. Expanding $U$ in $1/r$ powers we get

$$U = 1 - \frac{(2w - 1) r_0}{r} + O(1/r^2).$$

(3.4)

The gravitational mass $M_0 = 8\pi (2w - 1) r_0$ is positive for $1/2 < w < 1$ when $r_0 > 0$. However, the total energy could also have a contribution coming from the scalar field [9, 26, 43, 44]. Let us verify that this is not the case using the Euclidean action formalism.

The variation of the boundary terms of the action has both a gravitational and a scalar contribution and we get

$$\delta M = 8\pi [-RR'\delta U + U'R\delta R - 2UR\delta R']^\infty - 16\pi [R^2 U'\delta \phi']^\infty.$$  

(3.5)

$M$ can be calculated by expanding the metric functions and the scalar field up to terms proportional to $1/r$,

$$U \approx 1 - \frac{(2w - 1) r_0}{r}, \quad R \approx r - (1 - w) r_0 - \frac{\gamma^2 r_0^2}{2r}, \quad \phi \approx \phi_0 + \frac{\gamma r_0}{r}. $$

(3.6)

The contribution of the scalar field to (3.5) as well as the second and third term behave like $1/r$ and therefore they vanish at $r = \infty$. Only the first term gives a non-vanishing contribution leading to

$$M = 8\pi (2w - 1) r_0 = M_0,$$

(3.7)

which means that the total energy of the solution is of purely gravitational origin.

We see from the previous equation that for $r_0 > 0$ the energy of the solution is positive (negative) for $w > 1/2$ ($w < 1/2$) and it vanishes for $w = 1/2$ (the exactly opposite holds for $r_0 < 0$). This means that for $r_0 > 0$ solutions with $w < 1/2$ are stable with respect to the Minkowski vacuum, whereas solutions with $w = 1/2$ are degenerate with respect to the same vacuum. However, it has to be stressed that the JNWW solutions represent naked singularities, therefore they can be ruled out by means of a cosmic censorship principle. An other interesting feature of the JNWW solutions is that their mass can be positive or zero even in the presence of a naked singularity. This is rather unusual and it is due to the back-reaction of the metric to the presence of a non trivial scalar field.

B. Zero mass and charge limit of dilatonic black hole solutions

An interesting point is that solutions (3.1) appear as limiting case of dilatonic, black hole solutions, i.e. solutions of non minimally coupled Einstein-Maxwell-dilaton gravity. This is already known for Garfinkle-Horowitz-Strominger (GHS) solutions [4]. In fact, by taking the zero charge (i.e. $r_+ = 0$) limit of the GHS black hole we obtain the JNWW solution (3.1).

Let us now show that solutions (3.1) with $w = 1/2$ appear as the $Q \to 0, M \to 0$ limit of the charged, dilatonic, black hole solutions of the S-duality model investigated in Ref. [6].
If we add to the Lagrangian in (2.1) a term $-(\cosh 2\phi) F^2$, where $F$ is the Maxwell tensor (and take $V = 0$) we get the S-duality model investigated in Ref. [6]. The model allows for charged, scalar-dressed, asymptotically flat black hole solutions of the form (2.3) with the metric functions and scalar field given by [6]

$$U = \frac{(r - r_-)(r - r_+)}{r(r - r_0)}, \quad R^2 = r(r - r_0), \quad \phi = \phi_0 + \frac{1}{2} \ln \left(1 - \frac{r_0}{r}\right).$$

(3.8)

The constants $r_{\pm}$ are related to the mass $M$, the magnetic charge $Q$ and the scalar charges $\sigma = -r_0/2, \phi_0$, through

$$r_{\pm} = M + \frac{r_0}{2} \pm \sqrt{M^2 + \frac{r_0^2}{4} - Q^2 \cosh 2\phi_0}, \quad r_0 = -\frac{Q^2}{M} \sinh 2\phi_0.$$

(3.9)

The solution (3.8) represents a three parameter family of black hole solutions generalizing the well-known Reissner-Nordström solution of general relativity for $M^2 + \frac{r_0^2}{4} - Q^2 \cosh 2\phi_0 \geq 0$. The extremal limit is reached when the previous inequality is saturated.

One can easily realize that the solutions (3.1) with $w = 1/2$ can be obtained from the dilatonic black hole solution (3.8) in the limit $M \to 0$, $Q \to 0$ keeping $Q^2/M$ finite. In this limit the inner horizon at $r = r_-$ is pushed to $r = 0$ whereas the outer horizon at $r = r_+$ coincides with the singularity at $r = r_0$.

### IV. ASYMPTOTIC BEHAVIOUR OF THE SCALAR FIELD AND OF THE POTENTIAL

Solutions (3.1) are the most general, static, spherically symmetric solutions sourced by a scalar field with an identically vanishing potential. Consistently with the PET, they do not describe black holes but naked singularities. In order to have black hole solutions sourced by a non trivial scalar the PET must be violated. A simple way to achieve this is to consider a potential which is zero only asymptotically but becomes non zero and negative in the bulk spacetime. Focusing on black hole solutions sourced by an asymptotically massless scalar, we have to impose the conditions (2.5) on the potential and the boundary conditions (2.6) on the scalar field.

Conditions (2.5) imply that asymptotically, near $\phi = 0$ the potential must behave at leading order as $V(\phi) = \mu \phi^n$ where $\mu$ is a constant and $n \geq 3$. The corresponding asymptotic behavior of the scalar field for $r \to \infty$ can be determined by using the field equation for the scalar (2.8b) written in the form

$$(u\phi')' = n\mu r^2 \phi^{n-1},$$

(4.1)

and the conditions for asymptotic flatness of the spacetime: $u = r^2$, $R^2 = r^2$. Using these conditions, Eq. (4.1) gives the fall-off behaviour of $\phi$ at $r = \infty$. Note that for $n = 2$ (a massive scalar field) we get the well-known Yukawa behaviour $\phi = e^{-\sqrt{\mu r}}/r$. For $n = 3$ we have a scalar field decaying asymptotically as $\phi = 2/(3\mu r^2)$. For $n = 4$ the theory corresponds to a conformal field theory in flat spacetime, which allows for time dependent meron solutions $\phi \propto 1/\sqrt{r^2 - t^2}$.

The most interesting case is however obtained for $n = 5$. In this case the scalar field behaves asymptotically as in Eq. (3.1), i.e. as an harmonic function in 3D:

$$\phi = \frac{\beta}{r} + O(1/r^2),$$

(4.2)

where $\beta$ is a constant. It is important to stress that the presence of a term $O(1/r^2)$ is necessary to cancel the contribution of the $1/r$ term in the RHS of Eq. (4.1).

We have reached an important result. Compatibility of conditions (2.5) with condition (2.6) require a quintic asymptotic behaviour of potential $V(\phi)$. This condition translates immediately in a condition for the existence of asymptotically flat black hole solutions sourced by a scalar field behaving asymptotically as massless, i.e. decaying as an harmonic function.

### V. BLACK HOLE SOLUTIONS SOURCED BY AN ASYMPTOTICALLY MASSLESS SCALAR FIELD

In this section we will derive asymptotically flat black hole solutions sourced by a scalar field behaving asymptotically as $1/r$. If existing, we expect these solution to be closely related to solution (3.1). Moreover, in view of the results of Sect. IV we also expect the potential to behave asymptotically as $V \sim \phi^5$. 


We start with the scalar field profile one obtains in the case of a vanishing potential

$$
\phi = -\gamma \ln \left(1 - \frac{r_0}{r}\right),
$$

(5.1)

and we determine the metric functions and the potential using the method described in Sect. [4].

Notice that solutions for the scalar field expressed in terms of harmonic functions like Eq. (5.1) have been also used to derive black branes and black hole solutions with asymptotic anti de Sitter behaviour [19]. The solution for \( R \) is obtained by integrating the Riccati equation (2.8a) and it can be read directly from Eq. (3.1),

$$
R^2 = r^2 \left(1 - \frac{r_0}{r}\right)^{2(1-w)}, \quad w - w^2 = \gamma^2.
$$

(5.2)

The other metric function \( U \) is obtained performing the integration in (2.9). For generic values of the integration constants \( C_1 \) and \( C_2 \) the corresponding solutions are not asymptotically flat.

For \( w \neq 1/4, 1/2, 3/4 \) we get AF solutions by choosing

$$
C_2 = \frac{C_1 - r_0 + 4r_0w}{r_0(2w-1)(4w-3)(4w-1)},
$$

(5.3)

and the metric function \( U \) reads

$$
U(r) = X^{2w-1} \left[1 - \Lambda \left(r^2 + (4w-3)rr_0 + (2w-1)(4w-3)r_0^2\right)\right] + \Lambda r^2 X^{2(1-w)}, \quad X = 1 - \frac{r_0}{r},
$$

(5.4)

where \( \Lambda = C_2 \). Using Eq. (2.10) and inverting \( \phi = \phi(r) \) given by Eq. (5.1) we are now able to write down the corresponding potential \( V(\phi) \). We get

$$
V(\phi) = 4\Lambda \left[-w(1-4w) \sinh \left(\frac{2w-2}{\gamma} \phi \right) + 8\gamma^2 \sinh \left(\frac{2w-1}{\gamma} \phi \right) + (1-w)(3-4w) \sinh \frac{2w\phi}{\gamma}\right].
$$

(5.5)

Similarly to the \( V = 0 \) case, the solutions (5.1), (5.2), (5.4) and the potential (5.5) are invariant under the transformation \( w \to 1-w \). In fact, the solutions remain invariant if we simultaneously translate the radial coordinate \( r \to r_0 - r \). Again, we can restrict the range of \( w \) to \( 1/2 \leq w \leq 1 \).

The particular cases \( 1/2 \) and \( 3/4 \) must be treated separately. For \( w = 1/2 \) the solutions are AF when

$$
C_2 = -\frac{2}{r_0^2},
$$

(5.6)

The solutions and the potential read

$$
U(r) = \frac{r^2}{r_0^2} X \left[(1 + r_0^2\Lambda) X - 2r_0^2\Lambda \ln X + (1 - r_0^2\Lambda) X^{-1} - 2\right],
$$

(5.7)

$$
V(\phi) = 4\Lambda \left[3 \sinh(2\phi) - 2\phi \cosh(2\phi) + 2\right], \quad \Lambda = -\frac{C_1 + r_0}{r_0^2}.
$$

(5.8)

For \( w = 3/4 \) we have AF solutions for

$$
C_2 = -\frac{3C_1}{2r_0^2} - \frac{2}{r_0^2},
$$

(5.9)

The solutions and the potential take the form

$$
U(r) = \frac{r^2}{r_0^2} X^{1/2} \left[(1 + \frac{r_0^2\Lambda}{2}) X^2 - 2(1 + r_0^2\Lambda) X + r_0^2\Lambda \ln X + 1 + \frac{3r_0^2\Lambda}{2}\right],
$$

(5.10)

$$
V(\phi) = \Lambda \left(8\sqrt{3}\phi \cosh \frac{2\phi}{\sqrt{3}} - 9\sinh \frac{2\phi}{\sqrt{3}} - \sinh 2\sqrt{3}\phi\right), \quad \Lambda = -\frac{C_1 + 2r_0}{r_0^3}.
$$

(5.11)

Solutions (5.4), (5.7) and (5.10) represent an one-parameter family of AF, spherically symmetric solutions of the Einstein-scalar gravity theory (2.1) with potential specified, respectively, in Eqs. (5.5), (5.8) and (5.11) and sourced by a non trivial scalar field given by Eq. (5.1). The solutions (5.4), (5.7) and (5.10) have a curvature singularity at \( r = r_0 \) (for \( r_0 > 0 \)) or at \( r = 0 \) (for \( r_0 < 0 \)). One can easily show that near the singularity the solutions have the same
scaling behaviour of the JNWW solution given in Eqs. (3.2) and (3.3). Hence our solution share the same singularity structure with the JNWW solution. Moreover, similarly to the latter, they interpolate between an AF spacetime and a power-law metric near to the singularity and are characterized by a scalar field behaving asymptotically as a harmonic function.

The potential for $w$ generic (5.5) arises also from the study of a general class of Petrov type D solutions [30, 31] and the solution (5.4), albeit in a different form, has been already derived in Ref. [31].

In view of the results of Sect. [IV] asymptotically $r \to \infty$ ($\phi \to 0$) the potentials (5.5), (5.8) and (5.11) and their $n$-order derivatives vanish at $\phi = 0$ till $n = 5$, i.e. the potential behaves near $\phi = 0$ respectively for $w$ generic, $w = 1/2$ and $w = 3/4$ as

$$V(\phi \approx 0) = -32\Lambda \frac{(2w - 1)(4w - 1)(4w - 3)}{(w - w^2)^{3/2}} \phi^3 + O(\phi^5), \quad (5.12)$$

$$V(\phi \approx 0) = -256\Lambda \phi^5 + O(\phi^7), \quad V(\phi \approx 0) = - \frac{1856\Lambda}{3\sqrt{w}} \phi^5 + O(\phi^7). \quad (5.13)$$

For all values of the parameter $1/2 \leq w < 1$ the potential is always antisymmetric under $\phi \to -\phi$ and diverges for $\phi \to \infty$, which means that it is always not limited from below. Since $(\partial V/\partial \phi)|_{\phi=0} = 0$ all the three models allow for the Schwarzschild black hole solution endowed with a identically trivial scalar, $\phi = 0$. However, $\phi = 0$ is not a minimum of the potential so that we naturally expect this solution to be unstable. Moreover, because the potential is unlimited from below, the PET is violated and black hole solutions with non trivial scalar profile are in principle allowed. Solutions (5.4), (5.7) and (5.10) represent black hole if they have an event horizon. In the next subsections we will show that this is indeed the case. We will first consider the particular cases $w = 1/2$ and $w = 3/4$ and then the general case.

### A. Black hole solutions for $w = 1/2$

The solution of the transcendental equation

$$X \ln X = \frac{1}{2}(1 + \lambda)X^2 - \lambda X - \frac{1}{2}(1 - \lambda), \quad \lambda = 1/(r_h^2\Lambda) \quad (5.14)$$

give the position of the event horizon $r_h$ of the metric function (5.7). For $0 < \lambda \leq 1$, corresponding to $r_h^2 = 1/\Lambda$, \quad (5.15)

Eq. (5.14) has always an acceptable solution, i.e. a solution $0 \leq X(r_h) < 1$, corresponding to $r_0 \leq r_h < \infty$.

The spacetime represents a black hole with event horizon at $r = r_h$ and a curvature singularity at $r = r_0$. For $X(r_h) \to 1$ (corresponding to $r_h \to \infty$), $\lambda \to 0$ and then we have large black holes. Conversely, for $X(r_h) = 0$ (corresponding to $r_h = r_0$) the horizon disappears and we are left with a naked singularity.

The black hole mass can be easily evaluated from the coefficient of the $1/r$ term in the $1/r$ expansion of the metric function $U$. We have

$$M = \frac{8\pi r_0}{3\Lambda}. \quad (5.16)$$

Because of the bound (5.15) there is a minimum value for the black hole mass,

$$M_{\text{min}} = \frac{8\pi}{3\sqrt{\Lambda}}. \quad (5.17)$$

under which black hole solutions cannot exist. Because of the bound (5.15), the continuous part of the black hole mass spectrum is separated from the Minkowski vacuum, attained for $r_0 = 0$, by a mass gap.

From the asymptotic expansion of Eq. (5.1) one can read off the scalar charge $\sigma = r_0/2$. Mass and scalar charge are not independent but satisfy $M = (64\pi\Lambda/3)^{1/2}$. This is consistent with the no-hair theorem, which forbids solutions with independent scalar hair.
B. Black hole solutions for $w = 3/4$

In this case, the position of the event horizon $r_h$ is given by the solutions of the equation
\[ \ln X = -\left(\lambda + \frac{1}{2}\right)X^2 + 2(\lambda + 1)X - \lambda - \frac{3}{2}. \]  
(5.18)

Solutions of this equation with $0 \leq X(r_h) < 1$ always exist for $\lambda \geq 0$. Also in this case we have large black holes ($X(r_h) \to 1$) when $\lambda \to 0$ and naked singularities ($X(r_h) = 0$) for $\lambda \to \infty$.

The black hole mass is
\[ M = 4\pi r_0 + \frac{8\pi r_0}{3\lambda}. \]  
(5.19)

Notice that also in this case the mass and scalar charge are not independent. Since $\lambda$ has no upper bound, differently from the previous case, black holes exist for arbitrarily small values of the mass whereas the naked singularity has zero mass.

C. Black hole solutions for $w$ generic

The position of the event horizon $r_h$ is given by the zeros with $0 \leq X < 1$ of the following equation
\[ f(X) := \left\{ \left[\lambda - (2w - 1)(4w - 3)\right](1 - X)^2 - (4w - 3)(1 - X) - 1 \right\}X^{4w-3} + 1 = 0. \]  
(5.20)

The solutions of this equation can be found graphically by determining for which values of the parameter $\lambda$ the function $f(X)$ intersects the $X$ axis at $0 \leq X < 1$.

We have to distinguish between the two cases $1/2 < w < 3/4$ and $3/4 < w < 1$.

$1/2 < w < 3/4$ Taking into account that $f'(1) = 0$, necessary conditions for the existence of the solution are $f(X) \to +\infty$ for $X \to 0^+$ and $f(X) \to -\infty$ for $X \to +\infty$, requiring $(2w - 1)(4w - 3) < \lambda \leq (2w - 1)(4w - 1)$.

On the other hand $f(X)$ has a local minimum for
\[ X = X_2 = \frac{(4w - 3)[\lambda - (2w - 1)(4w - 1)]}{(4w - 1)[\lambda - (2w - 1)(4w - 3)]}, \]  
(5.21)

$f(X)$ intersects the $X$-axis at $0 < X < 1$ only if $0 \leq X_2 < 1$, which in turn implies $\lambda > 0$. Thus Eq. (5.20) admits a solution only for $0 < \lambda \leq (2w - 1)(4w - 1)$. This case is similar to the $w = 1/2$ case. Black hole solutions exist only for
\[ r_0^2 \geq \frac{1}{(2w - 1)(4w - 1)\Lambda}. \]  
(5.22)

The black hole mass is given by
\[ M = (2w - 1)8\pi r_0 \left[ 1 - \frac{(4w - 3)(4w - 1)}{3\lambda} \right], \]  
(5.23)

whereas the scalar charge is determined by the mass. We have large black holes for $\lambda \to 0$ and a naked singularity for $\lambda = (2w - 1)(4w - 1)$. Owing to Eq. (5.22) the black hole mass has a lower bound given by
\[ M_{\text{min}} = \frac{16\pi}{3\sqrt[3]{\Lambda}} \sqrt{\frac{w}{(4w - 1)(2w - 1)}}. \]  
(5.24)

$3/4 < w < 1$ In this case we always have $f(0) = 1$, so that a necessary condition for a solution to Eq. (5.20) to exist is $f(X) \to -\infty$ for $X \to +\infty$, requiring $\lambda < (2w - 1)(4w - 1)$. On the other hand, $0 \leq X < 1$ implies $\lambda < 0$. It follows that the solutions always exist for $\lambda < 0$.

We see that here, analogously to the $w = 3/4$ case, we do not have any lower bound for $\lambda$ nor for the black hole mass (5.22). Choosing $\lambda$ negative, in the potential (5.5) we have a continuous black hole mass spectrum without a lower bound. Black hole exist for arbitrarily small values of the mass (corresponding to $\lambda \to -\infty$) and the mass of the solution describing naked singularity is zero.
VI. THERMODYNAMICS

In this section we investigate the thermodynamics of the black hole solutions we have found in the previous section. The mass of the black hole has already been calculated. The temperature \( T \) and the entropy \( S \) will be calculated using the well-known formulæ involving the surface gravity and the area law:

\[
T = \frac{\partial U}{4\pi |_{r=r_h}}, \quad S = 16\pi^2 R^2 |_{r=r_h}. \tag{6.1}
\]

We will also show that consistently with the non-existence of an independent scalar hair the thermodynamical parameters \( M, T \) and \( S \) satisfy the first principle \( dM = TdS \). The same results can be derived using the Euclidean action formalism, but we will omit the calculations here. As usual we discuss separately the three cases \( w = 1/2, w = 3/4 \) and \( w \) generic with \( 1/2 < w < 1 \)

A. The case \( w = 1/2 \)

In order to simplify the discussion we write Eq. \((5.14)\) in terms of the dimensionless parameter \( \omega = r_0/r_h \), with \( 0 < \omega \leq 1 \),

\[
2(1 - \omega) \ln(1 - \omega) - \omega^2(1 + \lambda) + 2\omega = 0. \tag{6.2}
\]

The temperature and the entropy \((6.1)\) can be easily written as functions of \( \omega \) and \( \lambda \),

\[
T(\omega) = \frac{\sqrt{3\pi}}{4\pi \Lambda} \left[ 2 \left( 1 - \frac{2}{\omega} \right) \ln(1 - \omega) - 4 \right], \quad S(\omega) = \frac{16\pi^2}{\Lambda \omega^2} \left( \frac{1}{\omega^2} - \frac{1}{\omega} \right). \tag{6.3}
\]

where \( \lambda \) is a function of \( \omega \), obtained by solving Eq. \((6.2)\) for \( \lambda \)

\[
\lambda(\omega) = \frac{2(1 - \omega) \ln(1 - \omega)}{\omega^2} + \frac{2}{\omega} - 1. \tag{6.4}
\]

Being \( 0 < \lambda, \omega \leq 1 \), the temperature, the mass and the entropy are always positive. For \( \omega = \lambda = 1 \) we have an extremal state with zero entropy and infinite temperature saturating the inequality \((5.15)\). Near to the singularity, the temperature diverges logarithmically \( T \sim - \ln(1 - \omega) \). For this state the horizon coincides with the singularity, nevertheless the mass is not zero but it is given by the minimum value \((5.17)\). The behaviour of this singular extremal state has to be compared with that of the Schwarzschild black hole, for which the extremal, infinite temperature state has zero mass.

Large black holes are obtained for \( \lambda, \omega \to 0 \). In this limit the mass and entropy diverge whereas the temperature tends to zero.

The general thermodynamical relations \( M(T), S(T) \) characterizing the black hole cannot be found analytically, however one can easily check by differentiating \( \lambda(\omega), M(\omega) \) the validity of the first law of thermodynamics \( dM = TdS \). This a rather non trivial consistency check, because the scalar charge of the solution is not independent, therefore we cannot have a thermodynamical potential associated to it.

We can derive the explicit form of the thermodynamical potentials in the limit of large black holes, \( \lambda, \omega \to 0 \) corresponding to \( r_0^2 \gg 1/\Lambda \). Expanding Eq. \((6.4)\) about \( \omega = 0 \) we get \( \lambda = \omega/3 + \mathcal{O}(\omega^2) \), which inserted in Eqs. \((6.3)\) and \((5.16)\) gives at leading order

\[
M = \frac{8\sqrt{3\pi}}{\sqrt{\Lambda} \omega^{3/2}}, \quad T = \frac{\sqrt{\Lambda}}{4\sqrt{3\pi}} \omega^{3/2}, \quad S = \frac{48\pi^2}{\Lambda \omega^3}. \tag{6.5}
\]

From these equations one easily finds the thermodynamical potentials

\[
M = \frac{2}{T}, \quad S = \frac{1}{T^2}, \quad F = M - TS = \frac{1}{T}, \tag{6.6}
\]

where \( F \) is the free energy. The previous thermodynamical relations are exactly those satisfied by the Schwarzschild black hole. Thus for large mass our scalar dressed black hole is thermodynamically indistinguishable from a Schwarzschild one at the same temperature. This is an interesting result, particularly if one considers that, for the model under consideration, the Schwarzschild solution sourced by a constant scalar is unstable.
B. The case \( w = 3/4 \)

Also in this case we begin writing the temperature and the entropy as functions of the dimensionless parameters \( \omega \) and \( \lambda \),

\[
T(\omega) = \frac{\sqrt{\Lambda}}{4\pi\sqrt{\lambda}} \left( \frac{1}{\omega} + 2\lambda \right) \omega^2 - 2\lambda \omega, \quad S(\omega) = \frac{16\pi^2}{\lambda} \frac{\sqrt{1-\omega}}{\omega^2}. \tag{6.7}
\]

where \( \lambda(\omega) \) is obtained solving Eq. \( \text{(5.18)} \) with respect to \( \lambda \),

\[
\lambda(\omega) = -\frac{\omega(\omega + 2) + 2\ln(1-\omega)}{2\omega^2}, \tag{6.8}
\]

where now \( \lambda > 0 \) and \( 0 < \omega \leq 1 \).

Although the explicit general form of \( M(T), S(T) \) characterizing the black hole cannot be found analytically, one can easily check, using the same procedure as before, the validity of the first principle \( dM = TdS \), consistently with the absence an independent thermodynamical potential associated to the scalar charge.

The extremal, singular, black hole state is obtained for \( \omega = 1 \), \( \lambda = \infty \). We now have an extremal state with \( M = S = 0 \) and \( T = \infty \). In this state the horizon coincides with the singularity and the mass is zero analogously to the Schwarzschild black hole. Conversely, large black holes are obtained for \( \lambda, \omega \to 0 \) when the mass and entropy diverge whereas the temperature tends to zero. In this limit we get again approximate solution for \( \lambda = \omega/3 \), at leading order in \( \omega \), the temperature and the entropy satisfy the same relations as in Eq. \( \text{(6.5)} \) and we have the same thermodynamical potentials \( \text{(6.6)} \).

C. \( w \) generic

For a generic \( 1/2 < w < 1 \), \( w \neq 3/4 \) the functions \( T(\omega), S(\omega) \) are given by

\[
T(\omega) = \frac{\sqrt{\Lambda}}{4\pi\sqrt{\lambda}} \left[ \left( \frac{2}{\omega} + \frac{3-4w}{1-\omega} \right)(1-\omega)^{2-2w} - \left( \frac{2}{\omega} + 4w - 3 \right)(1-\omega)^{2w-1} \right], \quad S(\omega) = \frac{16\pi^2}{\lambda} \frac{(1-\omega)^{2-2w}}{\omega^2}. \tag{6.9}
\]

whereas for \( \lambda(\omega) \) we have

\[
\lambda(\omega) = \frac{1-(1-\omega)^{3-4w}}{\omega^2} + \frac{4w-3}{\omega} + (2w-1)(4w-3). \tag{6.10}
\]

Again, \( 0 < \omega \leq 1 \), and as discussed in Sect. \( \text{V.C} \), \( 0 < \lambda \leq (2w-1)(4w-1) \) for \( 1/2 < w < 3/4 \) whereas \( \lambda < 0 \) for \( 3/4 < w < 1 \). Using Eqs. \( \text{(6.9)}, \text{(6.10)} \) and \( \text{(5.23)} \), one can check the validity of the thermodynamical relation \( dM = TdS \), hence the absence of a thermodynamical potential associated with the scalar charge.

The large mass limit can be discussed expanding \( T, S, M \) and \( \lambda \) about \( \omega = 0 \). At leading order one gets from \( \text{(6.10)} \), \( \lambda = \frac{1}{2}(1-2w)(3-4w)(1-4w) \) which inserted into the expansion of \( \text{(6.9)} \) gives

\[
T \approx \frac{\sqrt{\Lambda}\omega^{1/2}}{4\sqrt{3\pi}} \frac{1}{(1-2w)(3-4w)(1-4w)}, \quad S \approx \frac{48\pi^2}{\Lambda\omega^3} (1-2w)^2(3-4w)^2(1-4w)^2. \tag{6.11}
\]

The leading term for in the mass \( \text{(5.23)} \) is

\[
M \approx \frac{8\sqrt{3\pi}}{\sqrt{\Lambda}\omega^{3/2}} (1-2w)(3-4w)(1-4w). \tag{6.12}
\]

Using these equations one can easily get the thermodynamical potentials \( \text{(6.6)} \).

The behaviour near the singular state is different for the two cases \( 1/2 < w < 3/4 \) and \( 3/4 < w < 1 \). The first one is very similar to the \( w = 1/2 \) case: we have a singular extremal state with \( S = 0 \), \( T = \infty \) and non vanishing mass given by the minimal mass \( \text{(5.24)} \) for \( \lambda = (2w-1)(4w-1) \), corresponding to \( \omega = 1 \). The second case is akin to the \( w = 3/4 \) case: we have a singular extremal state with \( S = 0 \), \( T = \infty \) and vanishing mass for \( \lambda \to -\infty \).
VII. CONCLUSION

In this paper we have derived, using the solution-generating method of Ref. [19], exact, AF, black hole solutions sourced by a non trivial scalar field behaving asymptotically as 1/r. We have shown that these solutions have several interesting features. Near to the singularity they behave as the JNWW solutions, whereas in the large mass limit they have the same thermodynamical behaviour of the Schwarzschild solution. Although characterized by a non trivial scalar field profile, the corresponding scalar charge is not independent, implying the absence of a corresponding thermodynamical potential. The infrared behaviour of the mass spectrum for the black hole solutions with 1/2 ≤ w < 3/4 is characterized by the presence of a mass gap. Differently from the Schwarzschild solution, the extremal, singular solution (a naked singularity) is reached for zero entropy, infinite temperature but for a minimum, non vanishing, value of the black hole mass.

On the other hand, the model has some troublesome features, related to the behaviour of the potential V(φ) both at φ = 0 and φ → ∞. Because φ = 0 is not a minimum of the potential but only an inflection point, the φ = 0 Schwarzschild black hole, although solution of the field equation, is most likely unstable. Moreover, the potential is unlimited from below, V(φ) → −∞ for φ → −∞, and behaves near φ = 0 as V(φ) ∼ φ4, hence it is not renormalizable from the quantum field theory point of view.

For these reasons our model cannot be fundamental but can give only an effective description valid in the region φ ≳ 0. It is well-known that the renormalization group flow may drive the scalar field potential in regions of instability. An important example of this kind of behaviour is given by the coefficient of the quartic term in the Higgs potential, which at short distances could become negative making the usual Higgs vacuum unstable [15, 16].

A rather intriguing possibility comes in to the play if we consider the parameter Λ in the potentials (5.5), (5.8) and (7.11) as dynamical. This can be the case if we regard the model as an effective description (e.g. resulting from some renormalization group flow) of some fundamental microscopic theory. If this is the case, focusing on the case w = 1/2, the vacuum can be obtained at Λ = 0, corresponding to r0 = ∞. For λ = 0 we get the solution (3.1) for a massless field with the value w = 1/2, i.e. a solution with zero mass, endowed with a non trivial scalar field.

An important point we have not addressed in this paper is the stability of the black hole solutions we have found. For all our models we have the Minkowski vacuum solution for φ = 0. On the other hand we have already argued about the instability of the Schwarzschild solution. The stability of solution (5.4) has been investigated in Ref. [32], where it has been shown that it presents mode instability against linear radial perturbations.

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