A VARIATIONAL APPROACH TO THE BROWN-RAVENHALL OPERATOR FOR THE RELATIVISTIC ONE-ELECTRON ATOMS

VITTORIO COTI ZELATI AND MARGHERITA NOLASCO

Abstract. We use the Foldy–Wouthuysen (unitary) transformation to give an alternative characterization of the eigenvalues and eigenfunctions for the Brown-Ravenhall operator (the projected Dirac operator) in the case of a one-electron atom. In particular we transform the eigenvalues problem into an elliptic problem in the 4-dim half space $\mathbb{R}^4_+$ with Neumann boundary condition.

1. Introduction and main results

The Dirac operator is a first order operator acting on 4-spinors $\Psi: \mathbb{R}^3 \rightarrow \mathbb{C}^4$, given by

$$D_0 = -ic\hbar \alpha \cdot \nabla + mc^2 \beta$$

where $c$ denotes the speed of light, $m > 0$ the mass, $\hbar$ the Planck’s constant (from now on we choose a system of physical units such that $\hbar = 1$), $\alpha_k$, $k = 1, 2, 3$ and $\beta$ are the Pauli-Dirac $4 \times 4$-matrices,

$$\beta = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}, \quad \alpha_k = \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix} \quad k = 1, 2, 3$$

and $\sigma_k$ are the Pauli $2 \times 2$-matrices given by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$  

Denoting the Fourier transform (of a function in $u \in S(\mathbb{R}^3)$) by

$$F(u)(p) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} e^{-ip \cdot x} u(x) \, dx,$$

the free Dirac operator becomes in (momentum) Fourier space the multiplication operator $\hat{D}(p) = F D_0 F^{-1}(p)$ given, for each $p \in \mathbb{R}^3$, by an Hermitian $4 \times 4$-matrix which has the eigenvalues

$$\lambda_1(p) = \lambda_2(p) = -\lambda_3(p) = -\lambda_4(p) = \sqrt{c^2|p|^2 + m^2 c^4} \equiv \lambda(p).$$

The unitary transformation $U(p)$ which diagonalize $\hat{D}(p)$ is given explicitly by

$$U(p) = a_+(p) I_4 + a_-(p) \frac{\alpha \cdot p}{|p|}$$

$$U^{-1}(p) = a_+(p) I_4 - a_-(p) \frac{\alpha \cdot p}{|p|}$$

with $a_\pm(p) = \sqrt{\frac{1}{2}(1 \pm mc^2/\lambda(p))}$

$$U(p) \hat{D}(p) U^{-1}(p) = \beta \lambda(p) = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix} \sqrt{c^2|p|^2 + m^2 c^4}$$

We recall here the main properties of the free Dirac operator $D_0$.

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Remark 1.2. We recall that \( L^p_{\text{loc}}(\mathbb{R}^N) \), the weak \( L^p \) space, is the space of all measurable functions \( f \) such that
\[
\sup_{\alpha > 0} \left\{ x \mid |f(x)| > \alpha \right\}^{1/p} < +\infty.
\]
where \(|E|\) denotes the Lebesgue measure of a set \(E \subset \mathbb{R}^N\). Note that \(f(x) = |x|^{-1}\) does not belong to any \(L^q\)-space but it belongs to \(L^3_w(\mathbb{R}^3)\). (see e.g. [12] for more details).

**Remark 1.3.** The validity of (h2) when \(V\) is the Coulomb potential

\[
V(x) = -\frac{Ze^2}{|x|} \quad \text{(in cgs units)}
\]

follows from important inequalities. Let us recall them here.

**Hardy:** for all \(\psi \in H^1(\mathbb{R}^3)\)

\[
\|x|^{-1}\psi\|_{L^2} \leq 2\|\nabla \psi\|_{L^2} \leq \frac{2}{c}\|\sqrt{-c^2\Delta + m^2c^4}\psi\|_{L^2}
\]

**Kato, Herbst** [9]: for all \(\psi \in H^{1/2}(\mathbb{R}^3)\)

\[
(\psi, |x|^{-1}\psi)_{L^2} \leq \frac{\pi}{2}(\psi, \sqrt{-\Delta}\psi)_{L^2} \leq \frac{\pi}{2c}(\psi, \sqrt{-c^2\Delta + m^2c^4}\psi)_{L^2}
\]

**Tix** [18]: for all \(\psi \in H^{1/2}(\mathbb{R}^3, \mathbb{C}^4)\)

\[
(\Lambda_+\psi, |x|^{-1}\Lambda_+\psi)_{L^2} \leq \frac{1}{2c}\left(1 + \frac{2}{\pi}\right)^2(\Lambda_+\psi, \sqrt{-c^2\Delta + m^2c^4}\Lambda_+\psi)_{L^2}
\]

Note that (h2) is satisfied for the electrostatic potential provided \(0 < Z < 68\) by Hardy, \(0 < Z < 87\) by Kato and \(0 < Z < Z_c = 124\) by Tix’s inequality.

Many efforts have been devoted to the characterization and computation of the eigenvalues for the Dirac-Coulomb Hamiltonian (i.e. the operator \(D_0 + V\) when \(V\) is given by (1.4)), see [7] and references therein. Due to the unboundedness of the spectrum of the free Dirac operator, attention has been given also to approximate Hamiltonians constructed by using projectors. One of the first attempts in this direction was made by Brown and Ravenhall [2].

The Brown-Ravenhall Hamiltonian is defined as

\[
\mathcal{B} = \Lambda_+(D_0 - \frac{Ze^2}{|x|})\Lambda_+.
\]

This Hamiltonian \(\mathcal{B}\) has been considered also in the study of the “stability of matter” for relativistic multi-particle systems (see [13]).

In [3] it is proved that the operator \(\mathcal{B}\) is bounded from below if and only if \(Z \leq Z_c\). Then, in [18] (see also [3]) proved that the operator \(\mathcal{B}\) is strictly positive for \(Z \leq Z_c\).

Under our assumptions the quadratic form associate to \(\mathcal{B} = \Lambda_+(D_0 + V)\Lambda_+\) is positive definite. Hence, by the Friedrichs extension theorem, \(\mathcal{B}\) can be defined as a unique self-adjoint positive operator with domain contained in the form domain \(Q(D_0) = H^{1/2}(\mathbb{R}^3, \mathbb{C}^4)\). Moreover, by the KLMN theorem, \(\mathcal{B}\) may also be defined via quadratic forms as a form sum with form domain \(Q(\mathcal{B}) = H^{1/2}(\mathbb{R}^3, \mathbb{C}^4)\). The resulting self-adjoint extensions are equal (see [13]). Hence

\[
\Lambda_+(D_0 + V)\Lambda_+ = \Lambda_+D_0\Lambda_+ + \Lambda_+V\Lambda_+ = \Lambda_+\sqrt{-c^2\Delta + m^2c^4}\Lambda_+ + \Lambda_+V\Lambda_+,
\]

**Remark 1.5.** The assumptions (h1)-(h2) and (h3) are very similar to the ones given in [7]. Our assumption (h2) is slight more restrictive and it allows us to apply the KLMN theorem.

Follows from [2, 3] below that \(V\) is a compact operator from \(H^1\) to \(H^{-1}\) (but not from \(H^{1/2}\) to \(H^{-1/2}\)), and this is enough to guarantee that the perturbation \(\Lambda_+V\Lambda_+\) does not modify the essential spectrum. Namely, \(\sigma_{ess}(\mathcal{B}) = [mc^2, +\infty)\) (see [13] Corollary 4 to Weyl’s essential spectrum theorem XIII.14)).
Notation. To simplify the notation we will denote simply with $H^{1/2}$ the Hilbert space $H^{1/2}(\mathbb{R}^3, \mathbb{C}^2)$, with $L^2$ the space $L^2(\mathbb{R}^3, \mathbb{C}^2)$ or $L^2(\mathbb{R}^3, \mathbb{C}^4)$ as appropriate, and with $H^1$ the space $H^1(\mathbb{R}^3_+, \mathbb{C}^2)$ where $\mathbb{R}^3_+ = \{(x, y_1, \ldots, y_3) \in \mathbb{R}^4 \mid x > 0\}$.

In the FW-representation (since $U_{\mathbb{R}^3}^+ \Lambda_+ U_{\mathbb{R}^3}^- = \frac{1}{2}(\mathbb{1} + \beta)$) the associated quadratic form acting on $\mathcal{H}_+$ reduces to $2 \times 2$ (Hermitian) matrix form with domain $\mathcal{Q}(B_{\mathbb{R}^3}) = H^{1/2}(\mathbb{R}^3, \mathbb{C}^2)$ and for any $\psi, \phi \in \mathcal{Q}(B_{\mathbb{R}^3})$ is defined by

$$
(\phi, B_{\mathbb{R}^3} \psi)_{L^2} = (\phi, \sqrt{-c^2 \Delta + m^2 c^4 R} \psi)_{L^2} + (\phi, V_{\mathbb{R}^3}^{2x^2} \psi)_{L^2}
$$

where

$$
V_{\mathbb{R}^3}^{2x^2} \psi = Q^* U_{\mathbb{R}^3}^{-1} V U_{\mathbb{R}^3}^{-1} Q \psi, \quad \psi \in H^{1/2},
$$

$Q: \mathbb{C}^2 \to \mathbb{C}^4$, $Q(z_1, z_2) = (z_1, z_2, 0, 0)$

$Q^*: \mathbb{C}^4 \to \mathbb{C}^2$, $Q^*(z_1, z_2, z_3, z_4) = (z_1, z_2)$

so that

$$
(\phi, \sqrt{-c^2 \Delta + m^2 c^4 R} \psi)_{L^2} = (\Lambda_+ U_{\mathbb{R}^3}^{-1} Q \phi, D_0 \Lambda_+ U_{\mathbb{R}^3}^{-1} Q \psi)_{L^2(\mathbb{R}^3, \mathbb{C}^4)},
$$

and

$$
(\phi, V_{\mathbb{R}^3}^{2x^2} \psi)_{L^2} = (U_{\mathbb{R}^3}^{-1} Q \phi, V U_{\mathbb{R}^3}^{-1} Q \psi)_{L^2(\mathbb{R}^3, \mathbb{C}^4)},
$$

and

Note that $U_{\mathbb{R}^3}^{-1} Q \varphi = \Lambda_+ U_{\mathbb{R}^3}^{-1} \varphi \in \Lambda_+ L^2(\mathbb{R}^3, \mathbb{C}^4)$ for any $\varphi \in L^2$.

The operator $\sqrt{-c^2 \Delta + m^2 c^4}$, exactly as for the fractional Laplacian, can be related to a Dirichlet to Neumann operator (see for example [4] for problems involving the fractional laplacian, and [5] for more closely related models).

For any given function $u \in \mathcal{S}(\mathbb{R}^3)$ we consider the following Dirichlet boundary problem

\[
\begin{align*}
-\partial^2_{xx} v - c^2 \Delta_y v + m^2 c^4 v &= 0 \quad \text{in } \mathbb{R}^4_+ = \{(x, y) \in \mathbb{R} \times \mathbb{R}^3 \mid x > 0\} \\
v(0, y) &= u(y) \quad \text{for } y \in \mathbb{R}^3 = \partial \mathbb{R}^4_+.
\end{align*}
\]

Solving the equation via Fourier transform (w.r.t. $y \in \mathbb{R}^3$) we get

$$
v(x, y) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} e^{ip \cdot y} \hat{u}(p) e^{-\sqrt{c^2 |p|^2 + m^2 c^4} x} \, dp.
$$

Let us define

$$
\mathcal{T} u(y) = \frac{\partial v}{\partial y}(0, y) = -\frac{\partial v}{\partial x}(0, y);
$$

hence

$$
\mathcal{T} u(y) = -\frac{\partial v}{\partial x}(0, y) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} e^{ip \cdot y} \sqrt{c^2 |p|^2 + m^2 c^4} \hat{u}(p) \, dp
$$

namely $\mathcal{T} = -\sqrt{-c^2 \Delta_y + m^2 c^4}$ on the dense domain $\mathcal{S}(\mathbb{R}^3)$.

Our aim is to prove a variational characterization of the eigenvalues and eigenvectors of $B_{\mathbb{R}^3}$ different from the classical Rayleigh quotient and which gives rise, as we will see later, to an alternative eigenvalues problem (see $(E_1)$ below) for $B_{\mathbb{R}^3}$ involving the Dirichlet to Neumann operator. We believe that such a characterization can be useful for a fine analysis of the properties —such as regularity and exponential decay— of eigenfunctions, which have been object of investigation with different techniques in [1].
Lemma 2.1. For \( \phi \in H^1(\mathbb{R}^4, \mathbb{C}^2) \)
\[
I(\phi) = \int_{\mathbb{R}^4} \left( |\partial_x \phi|^2 + c^2 |\nabla_y \phi|^2 + m^2 c^4 |\phi|^2 \right) dx dy + \int_{\mathbb{R}^3} \left( \varphi_v, V_{\varphi}^2 \varphi_v \right) dy
given \varphi_v \in H^{1/2} \) denotes the trace of \( \phi \in H^1 \) on \( \partial \mathbb{R}^4 = \mathbb{R}^3 \).

We have the following result.

Theorem 1.8. Let (h1)-(h2)-(h3) hold. Then there exist \( \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_k \leq \ldots \) and \( \varphi_1, \varphi_2, \ldots, \varphi_k, \ldots \in H^1(\mathbb{R}^4, \mathbb{C}^2) \) such that, for all \( k \in \mathbb{N} \)
\[
\lambda_k = \inf_{X_k} I(\varphi_k)
\]
where
\[
X_k = \{ \phi \in H^1 \mid |\varphi_{tr}|_{L^2} = 1 \}
\]
and, for \( 1 < k \in \mathbb{N} \)
\[
X_k = \{ \phi \in H^1 \mid |\varphi_{tr}|_{L^2} = 1, (\varphi_{tr}, (\varphi_i)_{tr})_{L^2} = 0, i = 1, \ldots, k-1 \}.
\]
Moreover \( \{ \lambda_k \}_{k \geq 1} \in \sigma_{\text{disc}}(B_{\varphi v}) = \sigma_{\text{disc}}(B) \) and
\[
0 < \lambda_1 \leq \ldots \leq \lambda_k \leq \lambda_{k+1} \to \inf\{ \sigma_{\text{ess}}(B_{\varphi v}) \} = mc^2 \text{ for } k \to +\infty.
\]
The corresponding eigenfunctions are \( \varphi_k = (\varphi_k)_{tr} \in H^{1/2}(\mathbb{R}^3, \mathbb{C}^2) \), and \( \varphi_k \in H^1(\mathbb{R}^4, \mathbb{C}^2) \) are weak solutions of the Neumann problem
\[
(\mathcal{L}_k)
\]
\[
\begin{cases}
-\partial^2_{\varphi} \phi_k - c^2 \Delta \varphi_k + m^2 c^4 \varphi_k = 0 & \text{in } \mathbb{R}^4 \\
\partial_{\varphi} \varphi_k + V_{\varphi}^2 \varphi_k = \lambda_k \varphi_k & \text{on } \partial \mathbb{R}^4 = \mathbb{R}^3.
\end{cases}
\]

2. Proof of Theorem 1.8.

It is convenient to introduce the following (equivalent) norm in \( H^1(\mathbb{R}^4, \mathbb{C}^2) \)
\[
\|\varphi\|_{n_1}^2 = \int_{\mathbb{R}^4} \left( |\partial_x \varphi|^2 + c^2 |\nabla_y \varphi|^2 + m^2 c^4 |\varphi|^2 \right) dx dy.
\]
The following property can be easily verified.

Lemma 2.1. For \( w \in H^1(\mathbb{R}^4) \), let \( u = w_{tr} \in H^{1/2}(\mathbb{R}^3) \) be the trace of \( w, \hat{u} = F(u) \) and
\[
v(x, y) = F^{-1}_y \left[ \hat{u}(p) e^{-\sqrt{c^2 |p|^2 + m^2 c^4 x}} \right].
\]
Then \( v \in H^1(\mathbb{R}^4), \|v\|_{H^1(\mathbb{R}^4)} = \|u\|_{H^{1/2}(\mathbb{R}^3)}, \) and
\[
\int_{\mathbb{R}^3} \sqrt{c^2 |p|^2 + m^2 c^4} |\hat{u}|^2 dp = \int_{\mathbb{R}^4} \left( |\partial_x v|^2 + c^2 |\nabla_y v|^2 + m^2 c^4 |v|^2 \right) dx dy
\]
\[
\leq \int_{\mathbb{R}^4} \left( |\partial_x w|^2 + c^2 |\nabla_y w|^2 + m^2 c^4 |w|^2 \right) dx dy.
\]

Let introduce also the following norm in the weak \( L^q \)-space:
\[
|f|_{L^q} = \sup \{ |A|^{-1/r} \int_A |f(x)| dx \mid A \subset \mathbb{R}^3, \text{measurable}, 0 < |A| < +\infty \}
\]
where \( 1/q + 1/r = 1 \).

We have the following fact:

Lemma 2.2. Let \( V \in L^2_0(\mathbb{R}^3) \) and \( f \in H^{1/2}(\mathbb{R}^3) \).
We have that
\[
|V|^{1/2} f|_{L^2} \leq C |V|^{1/2} |f|_{H^{1/2}}.
\]
Proof. Follows from [11, (42)] that the Green function $G_0^w$ of $(-\Delta + \mu^2)^{\alpha/2}$ belongs to $L_w^{(3-\alpha)}(\mathbb{R}^3)$ if $\mu \geq 0$ and $0 < \alpha < 3$.

Then, given $f \in H^{1/2}(\mathbb{R}^3)$, let $h = (-\Delta + \mu^2)^{1/2}f \in L^2(\mathbb{R}^3)$, $f = G_0^w * h$. From the weak Young’s inequality (see Proposition A.1 in appendix A), we deduce
\[
|V^{1/2}f|_{L^2} = |V^{1/2}(G_0^w * h)|_{L^2} \leq C|V^{1/2}|_{L^3}^1 \|h\|_{L^6}^\alpha
\]
\[
\leq C|V|_{L^3}^1 \|(-\Delta + \mu^2)^{1/2}f\|_{L^2} \leq C|V|_{L^3}^1 \|f\|_{H^{1/2}}.
\]

Now, we introduce the differential $dI(\phi) : H^1 \to \mathbb{R}$ of the quadratic functional $I$
\[
dI(\phi)[h] = 2 \text{Re} \int_{\mathbb{R}^3} ((\partial_\phi h, \phi)_{c^2} + c^2 \langle \nabla_y \phi, \nabla_y h \rangle_{c^2} + m^2 c^4 \langle \phi, h \rangle_{c^2})
\]
\[
+ 2 \text{Re}(\phi, V^{2\chi^2}h)_{L^2}
\]
and also we compute $dI(\phi)[\chi^2 \phi]$ for any $\chi = \chi(y) \in C_0^\infty(\mathbb{R}^3)$ and $\phi \in H^1$. We have
\[
\text{Re} \int_{\mathbb{R}^3} (\partial_\phi \phi, \partial_\phi (\chi^2 \phi))_{c^2} = \int_{\mathbb{R}^3} |\partial_\phi (\chi \phi)|^2
\]
\[
\text{Re} \int_{\mathbb{R}^3} (\partial_\phi \phi, \partial_\phi (\chi^2 \phi))_{c^2} = \int_{\mathbb{R}^3} |\partial_\phi (\chi \phi)|^2
\]
and we compute, adding and substracting $|\nabla_y (\chi \phi)|^2$,
\[
\text{Re} \int_{\mathbb{R}^3} (\nabla_y \phi, \nabla_y (\chi^2 \phi))_{c^2} = \int_{\mathbb{R}^3} |\nabla_y (\chi \phi)|^2
\]
\[
- \int_{\mathbb{R}^3} (\chi^2 |\nabla_y \phi|^2 + |\phi|^2 |\nabla_y \chi|^2 + 2 \text{Re}(\phi \nabla_y \chi, \chi \nabla_y \phi)_{c^2})
\]
\[
+ 2 \text{Re} \int_{\mathbb{R}^3} ((\nabla_y \phi, \chi \phi \nabla_y \chi)_{c^2} + \chi^2 |\nabla_y \phi|^2)
\]
\[
= \int_{\mathbb{R}^3} (|\nabla_y (\chi \phi)|^2 - |\phi|^2 |\nabla_y \chi|^2).
\]
and, letting $\varphi = \phi_{tr} \in H^{1/2}$,
\[
\text{Re}(\varphi, V^{2\chi^2}\chi^2 \varphi)_{L^2} = (\varphi \varphi, V^{2\chi^2}\chi \varphi)_{L^2} + \text{Re}(\varphi, [V^{2\chi^2}, \chi] \varphi)_{L^2}
\]
where $[\cdot, \cdot]$ denotes the commutator of operators. Then, we have that
\[
dI(\phi)[\chi^2 \phi] = dI(\chi \phi)[\chi \phi] - 2c^2 \int_{\mathbb{R}^3} |\nabla_y \chi|^2 |\phi|^2 + 2 \text{Re}(\varphi, [V^{2\chi^2}, \chi] \varphi)_{L^2}.
\]
Now, we use the commutator identity
\[
[ABC, D] = AB[C, D] + A[B, D]C + [A, D]BC
\]
and the fact that $[V, \chi] = 0$ to deduce that (we let $Q \varphi = U_{rw} \varphi$)
\[
(\varphi, [V^{2\chi^2}, \chi] \varphi)_{L^2} = (\varphi, Q^*U_{rw} V U_{rw}^{-1} \chi |Q \varphi|)_{L^2} = (Q^*U_{rw} V U_{rw}^{-1} \chi |Q \varphi|)_{L^2}
\]
\[
= (Q^*U_{rw} V U_{rw}^{-1} \chi |Q \varphi|)_{L^2} + (Q^*U_{rw} \chi |Q^{-1} U_{rw} \varphi|)_{L^2}
\]
\[
= (U_{rw} \varphi, U_{rw} V U_{rw}^{-1} \chi |U_{rw} \varphi|)_{L^2} + (U_{rw} \varphi, U_{rw} \chi |U_{rw}^{-1} U_{rw} \varphi|)_{L^2}
\]
\[
= (V \varphi, [U_{rw}^{-1}, \chi] U_{rw} \varphi)_{L^2} + ([\chi, U_{rw}^{-1}] U_{rw} \varphi, V U_{rw} \varphi)_{L^2}.
\]
In addition
\[
dI(\chi \phi)[\chi \phi] = 2I(\chi \phi).\]
Finally we get

\[
(2.4) \quad d\mathcal{I}(\phi)[\chi^2\phi] = 2\mathcal{I}(\chi\phi) - 2c^2 \int_{\mathbb{R}^3} |\nabla_y\chi|^2 |\phi|^2 + 2 \text{Re}(V\psi, [U_{PW}^{-1}, \chi]\psi)_{L^2} \\
+ 2 \text{Re}([\chi, U_{PW}^{-1}]U_{PW}\psi, VU_{PW}^{-1}\psi)_{L^2}
\]

We divide the proof of Theorem 1.8 in several steps. Let us begin with the existence of the ground state.

We consider the following minimization problem:

\[
(\mathcal{P}_i) \quad \lambda_1 = \inf_{\phi \in S} \mathcal{I}(\phi),
\]

where \( S = \{ \phi \in H^1 \mid |\phi_{\text{ts}}|^2_{L^2} = 1 \} \).

**Lemma 2.5.** The following holds:

(i) \( \mathcal{I}(\phi) \) is bounded by below and coercive on \( H^1 \).

(ii) \( 0 < \lambda_1 < mc^2 \).

**Proof.** (i) Let \( \phi \in H^1 \), \( \varphi = \phi_{\text{ts}} \) and define \( \psi = U_{PW}^{-1}Q\varphi \), then \( \Lambda_\varphi = \psi \), hence by (h2), (L0), (L7) and lemma 2.1, there exists \( a \in (0, 1) \) such that

\[
(\Lambda_\varphi, V_{PW}^2\varphi)_{L^2} = (\Lambda_\varphi, \Lambda_\varphi \psi)_{L^2} \geq -a(\Lambda_\varphi, D_0\Lambda_\varphi \psi)_{L^2} = -a(\varphi, \sqrt{-}\Delta + mc^2\chi \varphi)_{L^2} \geq -a \int_{\mathbb{R}^3} (|\partial_x\varphi|^2 + c^2|\nabla_y\varphi|^2 + mc^2|\varphi|^2) \, dx \, dy
\]

Therefore, we may conclude that there exists \( \delta > 0 \) such that \( \mathcal{I}(\phi) \geq \delta \|\phi\|^2_{H^1} \).

(ii) From (i) immediately follows that \( \lambda_1 > 0 \). Now take \( \phi(x, y) = e^{-mc^2x}\varphi(y) \), with \( \varphi \in C_0^\infty(\mathbb{R}^2, \mathbb{C}^2) \), and \( |\varphi|_{L^2} = 1 \), we have

\[
\mathcal{I}(\phi) - mc^2 = \frac{1}{2m} \int_{\mathbb{R}^2} |\nabla\varphi|^2 + \int_{\mathbb{R}^2} (\varphi, V_{PW}^2\varphi)_{L^2} = \mathcal{E}(\varphi)
\]

Take, now \( \varphi_n(y) = \eta^{1/2}\varphi(\eta y) \), we have \( |\varphi_n|_{L^2} = 1 \), for any \( \eta > 0 \) and setting \( \phi_n(x, y) = e^{-mc^2x}\varphi_n(y) \)

\[
\lambda_1 - mc^2 \leq \inf_{\eta > 0} \mathcal{I}(\phi_n) - mc^2 = \inf_{\eta > 0} \mathcal{E}(\varphi_n) = \frac{\eta^2}{2m} \int_{\mathbb{R}^2} |\nabla\varphi|^2 + \int_{\mathbb{R}^2} (\varphi, V_{PW}^2\varphi_n)_{L^2}
\]

We claim that by (h3),

\[
\limsup_{\eta \to 0^+} \frac{1}{\eta^2} (\varphi_n, V_{PW}^2\varphi_n)_{L^2} = -\infty,
\]

which implies that \( \lambda_1 - mc^2 < 0 \).

Indeed, denoting \( \tilde{f}(p) = F(f)(p) \) and \( U_{PW, 0} = F^{-1}U(\eta p)F \) we have

\[
(\varphi_n, V_{PW}^2\varphi_n)_{L^2} = (Q\varphi_n, U_{PW}V(\eta y)U_{PW}^{-1}Q\varphi_n)_{L^2} = (FQ\varphi_n, FU_{PW}V(\eta y)U_{PW}^{-1}Q\varphi_n)_{L^2} = (Q\varphi_n, U(\eta p)FV(\eta y)F^{-1}U(p)^{-1}Q\varphi_n)_{L^2} = (Q\varphi_n, U_{PW, 0}V(\eta y)U_{PW, 0}^{-1}Q\varphi_n)_{L^2} = (\varphi, V(\eta y)\varphi)_{L^2} + (Q\varphi_n, U_{PW, 0}V(\eta y)U_{PW, 0}^{-1}Q\varphi_n)_{L^2}.
\]
Hence, since by \((h1)\) and lemma \([2.2]\) for any \(f \in H^{1/2}\) we have
\[
|V^{1/2}(\eta^{-1}y)f|_{L^2} \leq C\eta^{1/2}|V|_{L^1}^{1/2} |f|_{H^{1/2}}.
\]
We get
\[
|Q\varphi, [U_{VW,\eta}^{-1}, V(\eta^{-1}y)]U_{VW,\eta}^{-1} Q\varphi|_{L^2}
= |Q\varphi, [U_{VW,\eta}^{-1} - I_4, V(\eta^{-1}y)]U_{VW,\eta}^{-1} Q\varphi|_{L^2}
\leq |V^{1/2}(\eta^{-1}y)(U_{VW,\eta}^{-1} - I_4)Q\varphi|_{L^2} |V|_{L^1}^{1/2}(\eta^{-1}y)U_{VW,\eta}^{-1} Q\varphi|_{L^2}
+ |V^{1/2}(\eta^{-1}y)Q\varphi|_{L^2} |V^{1/2}(\eta^{-1}y)(U_{VW,\eta}^{-1} - I_4)U_{VW,\eta}^{-1} Q\varphi|_{L^2}
\leq C\eta |V|_{L^2} |U_{VW,\eta}^{-1} - I_4| Q\varphi |_{H^{1/2}} |\varphi|_{H^{1/2}}.
\]
Then recalling that
\[
U^{-1}(\eta p) = a_+ (\eta p) I_4 - a_- (\eta p) \beta \frac{\alpha \cdot p}{|p|}
\]
with \(a_\pm (\eta p) = \sqrt{\frac{2}{3}} (1 \pm mc^2/\lambda(\eta p))\), and \(\lambda(\eta p) = \sqrt{\eta^2c^2|p|^2 + m^2c^4}\), we have
\[
U^{-1}(\eta p) - I_4 = (a_+ (\eta p) - 1) I_4 - a_- (\eta p) \beta \frac{\alpha \cdot p}{|p|}
\]
and we estimate
\[
|a_+ (\eta p) - 1| \leq \frac{|mc^2 - \lambda(\eta p)|}{2\lambda(\eta p)} \leq \frac{\eta^2 |p|^2}{2mc^2}
\]
and
\[
|a_- (\eta p)| \leq \left( \frac{\lambda(\eta p) - mc^2}{2\lambda(\eta p)} \right)^{1/2} \leq \frac{\eta |p|}{\sqrt{2mc}}
\]
Therefore we may conclude
\[
\sup_{p \in \mathbb{R}^3} \frac{|a_+ (\eta p) - 1|}{(1 + |p|)^2} \leq C\eta^2 \quad \text{and} \quad \sup_{p \in \mathbb{R}^3} \frac{|a_- (\eta p)|}{1 + |p|} \leq C\eta
\]
for some constant \(C > 0\).

Moreover, since \(\hat{\varphi}\) is the Fourier transform of a compact support \(C^\infty\)-function, it decays at infinity faster than any power, namely for any \(\alpha > 0\) there exists a positive constant \(C_\alpha > 0\) such that
\[
|\hat{\varphi}(p)| \leq \frac{C_\alpha}{(1 + |p|)^\alpha}
\]
Then, we have
\[
|U_{VW,\eta}^{-1} - I_4)Q\varphi|^2 |_{L^2} \leq \int_{\mathbb{R}^3} (1 + |p|) |\mathcal{F}((U_{VW,\eta}^{-1} - I_4)Q\varphi|^2 dp
\leq 2 \int_{\mathbb{R}^3} (1 + |p|) |a_+ (\eta p) - 1| |\hat{\varphi}|^2 + 2 \int_{\mathbb{R}^3} (1 + |p|) |a_- (\eta p) \beta \frac{\alpha \cdot p}{|p|} Q\varphi|^2 dp
\leq 2 \int_{\mathbb{R}^3} (1 + |p|) |a_+ (\eta p)| - 1 |\hat{\varphi}|^2 dp + 2 \int_{\mathbb{R}^3} (1 + |p|) |a_- (\eta p)|^2 |\hat{\varphi}|^2 dp
\leq 2 \sup_{p \in \mathbb{R}^3} \frac{|a_+ (\eta p)| - 1 |^2}{(1 + |p|)^4} \int_{\mathbb{R}^3} (1 + |p|)^5 |\hat{\varphi}|^2 dp
+ 2 \sup_{p \in \mathbb{R}^3} \frac{|a_- (\eta p)|^2}{(1 + |p|)^2} \int_{\mathbb{R}^3} (1 + |p|)^3 |\hat{\varphi}|^2 dp \leq C\eta^2.
\]
for some constant \(C > 0\) depending only on \(\varphi\) and \(\eta\) sufficiently small. We get
\[
(\varphi, V_{VW}^{x,y} \varphi)_{L^2} = (\varphi, V(\eta^{-1}y)\varphi)_{L^2} + O(\eta^2).
\]
By (h3) for any $K > 0$ there exists $R > 0$ such that for any $|y| > R$ we have $V(y) \leq -K/|y|^2$ a.e.. We have
\[
(\varphi, V(\eta^{-1}y)\varphi)_{L^2} = \int_{\{\eta^{-1}|y| \leq R\}} V(\eta^{-1}y)|\varphi|^2 + \int_{\{\eta^{-1}|y| > R\}} V(\eta^{-1}y)|\varphi|^2 \\ \leq \eta \sup_{|y| \leq R} |V(y)|^2 \int_{\{|y| \leq R\}} |V(y)| - K\eta^2 \int_{\{|y| > R\}} \frac{1}{|y|^2} |\varphi|^2 \\ \leq C(\eta^2 - K\eta^2)
\]
where the constant $C > 0$ depends on $\varphi$ and $R$, and $K > 0$ is arbitrarily large.

As claimed above we may then conclude that given $\varphi \in C_0^\infty(\mathbb{R}^3; \mathbb{C}^2)$
\[
\limsup_{\eta \to 0^+} \frac{1}{\eta^2} (\varphi, V^{2+2}_\phi \varphi)_{L^2} = \limsup_{\eta \to 0^+} \frac{1}{\eta^2} (\varphi, V(\eta^{-1}y)\varphi)_{L^2} + \frac{O(\eta^2)}{\eta^2} = -\infty
\]

\[\Box\]

We will minimize $\mathcal{I}$ on the set
\[S = \{ \phi \in H^1 \mid G(\phi) = |\phi_{tr}|^2_{L^2} = 1 \} \]
We recall that the tangent space at $S$ at the point $\phi \in S$ is the set
\[T_0 S = \{ h \in H^1 \mid dG(\phi)[h] = 2 \text{Re}(\phi_{tr}, h_{tr})_{L^2} = 0 \} \]
and that $\nabla S \mathcal{I}(\phi)$, the projection of the gradient on the tangent space $T_0 S$ to $S$ at the point $\phi$ is given by
\[\nabla S \mathcal{I}(\phi) = \nabla \mathcal{I}(\phi) - \mu(\phi) \nabla G(\phi)\]
where $\nabla \mathcal{I}(\phi) \in H^1$ is such that
\[(\nabla \mathcal{I}(\phi), h)_{H^1} = 2 \text{Re}(\phi, h)_{H^1} + 2 \text{Re}(\phi_{tr}, V_{\phi_{tr}}^2 h_{tr})_{L^2} \quad \text{for all } h \in H^1,
\]
$\nabla G(\phi) \in H^1$ is such that
\[(\nabla G(\phi), h)_{H^1} = 2 \text{Re}(\phi_{tr}, h_{tr})_{L^2} \quad \text{for all } h \in H^1,
\]
and $\mu(\phi) \in \mathbb{R}$ is such that $\nabla S \mathcal{I}(\phi) \in T_0 S$. Then
\[0 = (\nabla G(\phi), \nabla S \mathcal{I}(\phi))_{H^1} = (\nabla G(\phi), \nabla \mathcal{I}(\phi))_{H^1} - \mu(\phi) \|\nabla G(\phi)\|^2_{H^1}
\]
and
\[\mu(\phi) = \frac{(\nabla G(\phi), \nabla S \mathcal{I}(\phi))_{H^1}}{\|\nabla G(\phi)\|^2_{H^1}}\]

From
\[(\nabla S \mathcal{I}(\phi), \phi)_{H^1} = (\nabla \mathcal{I}(\phi), \phi)_{H^1} - \mu(\phi)(\nabla G(\phi), \phi)_{H^1} \\ = 2\mathcal{I}(\phi) - 2\mu(\phi)G(\phi) = 2\mathcal{I}(\phi) - 2\mu(\phi)
\]
we also deduce that
\[(2.8) \quad \mu(\phi) = \mathcal{I}(\phi) - \frac{1}{2}(\nabla S \mathcal{I}(\phi), \phi)_{H^1}
\]

We now recall the following well known result

**Lemma 2.9.** There exists a Palais-Smale minimizing sequence $\phi_n$ for $\mathcal{I}$ on the set
\[S = \{ \phi \mid |\phi_{tr}|^2_{L^2} = 1 \}, \quad \text{that is a sequence such that, denoting } \phi_n = (\phi_n)_{tr},
\]
\[
\mathcal{I}(\phi_n) \to \lambda_1, \quad \nabla S \mathcal{I}(\phi_n) \to 0, \quad |\varphi_n|^2_{L^2} = 1
\]
Proof. Assuming that the result does not hold, one deduces that there exist $\epsilon > 0$, $\delta > 0$ such that $\|\nabla S I(\phi)\| \geq \delta > 0$ for all $\phi \in S$ such that $\lambda_1 - \epsilon < I(\phi) < \lambda_1 + \epsilon$. Building a gradient flow $\gamma' = \nabla_S I(\gamma)$, which leaves $S$ invariant and pushes $\{I < \lambda_1 + \epsilon\} \cap S$ into $\{I < \lambda_1 - \epsilon\} \cap S$, one easily reaches a contradiction.

One can prove the lemma also using Ekeland’s variational principle. \hfill \square

Lemma 2.10. Let $\phi_n$ be a Palais Smale sequence at some level $\lambda \geq 0$ for $I$ on $S$.

Let $\varphi_n = (\phi_n)_{tr}$.

If $\varphi_n \to 0$ in $H^{1/2}$ then

$$(\varphi_n, \varphi_{n,2}^2)_{L^2} \to 0.$$  

Proof. Since $I$ is coercive, $\varphi_n$ is bounded $H^1$, $\varphi_n$ is bounded in $H^{1/2}$ and, by Sobolev embedding, relatively compact in $L^p_{tr}$ for $p \in [2, 3)$. From (2.8) follows that also $\mu_n$ is bounded.

By (h8) $V \in L^\infty(\mathbb{R}^3 \setminus B_R)$ and for any $\sigma > 0$, the set $A_\sigma = \{ y \in \mathbb{R}^3 \setminus B_R | \|V(y)\| \geq \sigma \}$ is bounded.

Take a radial function $\chi \in C_0^\infty(\mathbb{R}^3)$, with values in $[0, 1]$ such that $\chi(y) = 1$ for $y \in B_1$ and $\chi(y) = 0$ for $y \in \mathbb{R}^3 \setminus B_2$ and let $\chi_R(y) = \chi(R^{-1}y)$.

Step 1: $|(\varphi_n, \varphi_{n,2}^2)_{L^2} - (\chi_R \varphi_n, \varphi_{n,2}^2)_{L^2}| \to 0$ as $R \to +\infty$.

Let $\psi_n = U_{-1}^{-1} Q \varphi_n$, we have that $\psi_n \to 0$ in $H^{1/2}$ and also that

$$(\varphi_n, \varphi_{n,2}^2)_{L^2} = (\psi_n, \varphi_{n,2}^2)_{L^2} = (\psi_n, (1 - \chi_R^2)\varphi_n)_{L^2} + (\psi_n, \chi_R^2 \varphi_n)_{L^2}.$$  

Taking $R > R_0$ in such a way that $A_\sigma \subset B_R$ we have

$$|(\psi_n, (1 - \chi_R^2)\varphi_n)_{L^2}| \leq \epsilon \|\psi_n\|^2_{L^2} \leq C \epsilon.$$

On the other hand, (note that $\Lambda - (\chi_R \psi_n) \neq 0$)

$$(\psi_n, \chi_R^2 \varphi_n)_{L^2} = (\chi_R U_{-1}^{-1} Q \varphi_n, V \chi_R U_{-1}^{-1} Q \varphi_n)_{L^2}$$

$$= (\chi_R \varphi_n, Q^{*} U_{fw} V \chi_R U_{-1}^{fw} Q \varphi_n)_{L^2}$$

$$= (\chi_R \varphi_n, U_{fw} V \chi_R U_{-1}^{fw} Q \varphi_n)_{L^2}$$

$$= (\chi_R \varphi_n, U_{fw}^{2} \chi_R \varphi_n)_{L^2} + (\chi_R \varphi_n, Q^{*} U_{fw} Q \varphi_n)_{L^2}$$

$$= (\chi_R \varphi_n, U_{fw}^{2} \chi_R \varphi_n)_{L^2} + (U_{fw}^{2} U \chi_R U_{fw}^{2} U \varphi_n)_{L^2}$$

$$= (\chi_R \varphi_n, U_{fw}^{2} \chi_R \varphi_n)_{L^2} = |V_{-1/2}^{fw} |U_{fw}^{2} \chi_R U_{fw} \varphi_n|_{L^2}$$

$$= (\chi_R \varphi_n, U_{fw}^{2} \chi_R \varphi_n)_{L^2} = |V_{-1/2}^{fw} |U_{fw}^{2} \chi_R U_{fw} \varphi_n|_{L^2}$$

$$= (\chi_R \varphi_n, U_{fw}^{2} \chi_R \varphi_n)_{L^2} = |V_{-1/2}^{fw} |U_{fw}^{2} \chi_R U_{fw} \varphi_n|_{L^2}$$

$$= (\chi_R \varphi_n, U_{fw}^{2} \chi_R \varphi_n)_{L^2} = |V_{-1/2}^{fw} |U_{fw}^{2} \chi_R U_{fw} \varphi_n|_{L^2}$$

$$+ 2 \text{Re}(\chi_R \varphi_n, V \chi_R U_{-1}^{fw} U_{fw} \psi_n)_{L^2}$$

Hence, we have

$$|(\psi_n, \chi_R^2 \varphi_n)_{L^2} - (\chi_R \varphi_n, \varphi_{n,2}^2)_{L^2}| \leq |V_{-1/2}^{fw} |U_{fw}^{2} \chi_R U_{fw} \varphi_n|_{L^2}$$

$$+ 2 \text{Re}(\chi_R \varphi_n, V \chi_R U_{-1}^{fw} U_{fw} \varphi_n)_{L^2}$$

$$\leq |V_{-1/2}^{fw} |U_{fw}^{2} \chi_R U_{fw} \varphi_n|_{L^2}$$

$$+ 2 V_{-1/2}^{fw} |U_{fw}^{2} \chi_R U_{fw} \varphi_n|_{L^2} |V_{-1/2}^{fw} \chi_R \varphi_n|_{L^2}$$
Using lemma 2.2 we have
\[ |V^{1/2} \chi_n \psi_n|_{L^2} \leq C |V|^{1/2} \psi_n|_{H^{1/2}} \]
\[ |V^{1/2} [\chi_n, U_{FW}^{-1}] U_{FW} \psi_n|_{L^2} \leq C |V|^{1/2} [\chi_n, U_{FW}] U_{FW} \psi_n|_{H^{1/2}} \]
and it follows from lemma 2.1 (Appendix B) that
\[ ||[\chi_n, U_{FW}^{-1}] U_{FW} \psi_n|_{H^{1/2}} \leq \frac{C}{R} \psi_n|_{H^{1/2}} \]
and hence
\[ (2.11) \]
\[ |(\psi_n, \chi_n^2 V \psi_n)|_{L^2} \leq (\chi_n \varphi_n, V_{FW} \chi_n \varphi_n)|_{L^2} | \leq \frac{C}{R} |V|_{L^2} \psi_n^2|_{H^{1/2}} \rightarrow 0 \quad \text{as } R \rightarrow +\infty. \]

Step 1 follows.

**Step 2:** \[ (\chi_n \varphi_n, V_{FW}^{2 \times 2} \chi_n \varphi_n)|_{L^2} \rightarrow 0 \quad \text{as } n \rightarrow +\infty. \]

We have, by assumption, \( \mathcal{I}(\phi_n) \rightarrow \lambda \), \( G(\phi_n) = |\varphi_n|_{L^2}^2 = 1 \) and
\[ ||\nabla G \mathcal{I}(\phi_n)|| = ||d \mathcal{I}(\phi_n) - \mu_n dG(\phi_n)|| \rightarrow 0. \]
where \( \mu_n = \mu(\phi_n) \) and also, by (2.8) in particular,
\[ (2.12) \]
\[ \mu_n = \mathcal{I}(\phi_n) - \frac{1}{2} (\nabla G \mathcal{I}(\phi_n), \phi_n)_{H^1} \rightarrow \lambda. \]

Using (2.11) we have
\[ o_n(1) = ||\nabla \mathcal{I}(\phi_n) - \mu_n \nabla G(\phi_n)||_{H^1} \geq ||\nabla \mathcal{I}(\phi_n) - \mu_n \nabla G(\phi_n, \chi_n^2 \phi_n)_{H^1}|| \]
\[ \geq |d \mathcal{I}(\phi_n)|_{H^1}^{\chi_n^2 \phi_n} - \mu_n 2 \Re (\varphi_n, \chi_n^2 \varphi_n)_{L^2} | \]
\[ \geq 2 \mathcal{I}((\chi_n \varphi_n) - 2 |\mu_n||\chi_n \varphi_n|_{L^2}^2 - 2c^2 |\phi_n \nabla y \chi_n|_{L^2}^2 \]
\[ - 2 |(V \psi_n, U_{FW}^{-1}) \chi_n U_{FW} \psi_n|_{L^2} | - 2 |(\chi_n, U_{FW}^{-1}) U_{FW} \psi_n, V U_{FW} \chi_n U_{FW} \psi_n)|_{L^2} | \]
\[ \geq 2 \mathcal{I}(\chi_n \varphi_n) - 2 |\mu_n||\chi_n \varphi_n|_{L^2}^2 - (I) - (II) - (III) \]
Now, by Sobolev compact embedding, for any given \( R > 0 \),
\[ |\chi_n \varphi_n|_{L^2} \rightarrow 0 \quad \text{as } n \rightarrow +\infty. \]
Moreover,
\[ (I) = 2c^2 |\phi_n \nabla \chi_n|_{L^2}^2 \leq 2c^2 \sup_{y \in \mathbb{R}^2} |\nabla \chi_n|^2 \leq \frac{C}{R^2} \]
and from lemma 2.1 (Appendix B) we have
\[ (II) = 2 |(V \psi_n, U_{FW}^{-1}) \chi_n U_{FW} \psi_n|_{L^2} | \]
\[ \leq |V^{1/2} [\chi_n, U_{FW}^{-1}] \chi_n U_{FW} \psi_n|_{L^2} | V^{1/2} \psi_n|_{L^2} | \]
\[ \leq \frac{C}{R} |V|_{L^2} \psi_n^2|_{H^{1/2}} \]
\[ (III) = 2 |(\chi_n, U_{FW}^{-1}) U_{FW} \psi_n, V U_{FW} \chi_n U_{FW} \psi_n)|_{L^2} | \]
\[ \leq |V^{1/2} [\chi_n, U_{FW}^{-1}] U_{FW} \psi_n|_{L^2} | V^{1/2} U_{FW} \chi_n U_{FW} \psi_n|_{L^2} | \]
\[ \leq \frac{C}{R} |V|_{L^2} \psi_n^2|_{H^{1/2}}. \]
Since by Lemma 2.3 (i) we have
\[ \mathcal{I}(\chi_n \phi_n) \geq \delta ||\chi_n \phi_n||_{H^1}^2 \]
we may conclude (recalling that \( \mu_n \) is bounded) that
\[
\| \chi_n \phi_n \|_{\mu_1}^2 \leq \epsilon_n + \frac{C}{R}
\]
and hence by \((h2)\) and lemma 2.1 we get
\[
| (\chi_n \varphi_n, V_{pw}^{2s} \chi_n \varphi_n)_{L^2} | \leq a \| \chi_n \phi_n \|_{\mu_1}^2 \leq \epsilon_n + \frac{C}{R}
\]
for some \( \epsilon_n \to 0 \) and \( R \) arbitrarily large, and step 2 follows. Then the lemma follows from step 1 and 2. \( \square \)

**Remark 2.13.** We recall that for all \( v \in C_0^\infty (\mathbb{R}^4) \)
\[
\int_{\mathbb{R}^3} |v(0,y)|^2 dy = \int_{\mathbb{R}^3} dy \int_{+\infty}^0 \partial_x |v|^2 dx \leq 2 \|v\|_{L^2(\mathbb{R}_x^+)} \|\partial_x v\|_{L^2(\mathbb{R}_x^+)}
\]
and by density we get for all \( \phi \in H^1 \)
\[
mc^2 \int_{\mathbb{R}^3} |\phi_n|^2 dy \leq \int_{\mathbb{R}_x^+} (|\partial_x \phi|^2 + m^2 c^4 |\phi|^2) dxdy
\]
Hence the quadratic form (kinetic energy)
\[
\mathcal{T}(\phi) = \int_{\mathbb{R}_x^+} (|\partial_x \phi|^2 + m^2 c^4 |\phi|^2) dxdy - mc^2 \|\phi_n\|_{L^2}^2
\]
is positive definite.

Now we may conclude the existence of a minimizer for \( \mathcal{P}_1 \). We have the following proposition:

**Proposition 2.14.** Let \( \phi_n \) be a minimizing Palais Smale sequence at level \( \lambda_1 > 0 \) for \( \mathcal{I} \) with \( |(\phi_n)_T|_{L^2} = 1 \) (as in Lemma 2.10).

Then \( \phi_n \to \phi \not\equiv 0 \) in \( H^1 \) and \( \hat{\phi} = |\phi|^{-1} \phi \) is a minimizer for \( \mathcal{I} \) on \( S \), that is
\[
\mathcal{I}(\hat{\phi}) = \lambda_1, \quad |\hat{\phi}|_{L^2} = 1.
\]

Moreover \( \hat{\phi} \) (and hence also \( \phi \)) is a weak solution of the Neumann problem \((\mathcal{E}_1)\).

**Proof.** Since \( \mathcal{I} \) is coercive, \( \phi_n \) is bounded (and weakly convergent) in \( H^1 \), \( \varphi_n = (\phi_n)_T \) is bounded (and weakly convergent) in \( H^{1/2} \).

If by contradiction \( \varphi_n \to \varphi \equiv 0 \), then by lemma 2.10 we have
\[
(\varphi_n, V_{pw}^{2s} \varphi_n)_{L^2} \to 0.
\]
Now, by Remark 2.13 we get
\[
\mathcal{I}(\phi_n) - mc^2 |\varphi_n|_{L^2}^2 \geq (\varphi_n, V_{pw}^{2s} \varphi_n)_{L^2} \to 0.
\]
On the other hand, by Lemma 2.3 (ii)
\[
\mathcal{I}(\phi_n) - mc^2 |\varphi_n|_{L^2}^2 = \mathcal{I}(\phi_n) - mc^2 \to \lambda_1 - mc^2 < 0
\]
a contradiction, that is \( \varphi_n \to \varphi \not\equiv 0 \).

It follows from (2.8) that
\[
\mu_n = \mathcal{I}(\phi_n) - \frac{1}{2} (\nabla_S \mathcal{I}(\phi_n), \phi_n)_{\mu_1} \to \lambda_1
\]
and hence, by weak convergence, we have
\[
d\mathcal{I}(\phi_n)[h] - \mu_n d\mathcal{G}(\phi_n)[h] \to d\mathcal{I}(\phi)[h] - \lambda_1 d\mathcal{G}(\phi)[h] = 0 \quad \forall h \in H^1
\]
hence in particular
\[
0 = d\mathcal{I}(\phi)[\phi] - \lambda_1 d\mathcal{G}(\phi)[\phi] = 2\mathcal{I}(\phi) - 2\lambda_1 \mathcal{G}(\phi)\]
and we may conclude that $\hat{\phi} = G(\phi)^{-1/2}\phi$ is a minimizer for $\mathcal{I}$ on $S$, namely

$$\lambda_1 = \frac{\mathcal{I}(\hat{\phi})}{G(\phi)} = \mathcal{I}(G(\phi)^{-1/2}\phi) = \mathcal{I}(\phi)$$

$$G(\phi) = G(G(\phi)^{-1/2}\phi) = 1$$

□

Now, we look for the existence of higher eigenvalues and corresponding eigenfunctions. We proceed by induction.

Let $\lambda_1$ be defined by $(\mathcal{P}_1)$ and $\phi_1$ be the corresponding minimizer given by Proposition 2.14.

Assume we have defined, for $j = 1, \ldots, k - 1$, $\lambda_1 \leq \ldots \leq \lambda_j < mc^2$ and $\phi_j \in H^1$, $\varphi_j = (\phi_j)_{tr} \in H^{1/2}$ such that

$$(\varphi_i, \varphi_j)_{L^2} = \delta_{ij}, \quad i, j = 1, \ldots, k - 1,$$

and

$$(\mathcal{P}_j) \quad \lambda_j = \mathcal{I}(\phi_j) = \inf_{\phi \in X_j} \mathcal{I}(\phi) \quad j = 1, \ldots, k - 1$$

where,

$$X_j = \{ \phi \in H^1 \mid G(\phi) = |\phi_{tr}|_{L^2}^2 = 1, \ (\phi_{tr}, \varphi_i)_{L^2} = 0 \ \text{for} \ i = 1, \ldots, j - 1 \}.$$

We define

$$(\mathcal{P}_j) \quad \lambda_k = \inf_{\phi \in X_k} \mathcal{I}(\phi)$$

**Remark 2.15.** Setting $G_j(\phi) = (\varphi_j, \phi_{tr})_{L^2}$, for $j \geq 1$, we have that the linear functionals $G_j$ are bounded on $H^1$ and for any $\phi, h \in H^1$

$$dG_j(\phi)[h] = (\nabla G_j(\phi), h)_{H^1} = (\varphi_j, h_{tr})_{L^2} = G_j(h) \quad j = 1, \ldots, k - 1.$$

Then $X_k = \{ \phi \in X_1 \mid G(\phi) = 1, \ G_j(\phi) = 0, \ j = 1, \ldots, k - 1 \}$,

$$T_{\phi}X_k = \{ h \in H^1 \mid (\nabla G(\phi), h)_{H^1} = 0, \ G_j(h) = 0, \ j = 1, \ldots, k - 1 \}$$

and the constrained gradient (i.e. the projection of the gradient of $\mathcal{I}$ on the tangent space $T_{\phi}X_k$) is given by

$$\nabla_{X_k} \mathcal{I}(\phi) = \nabla \mathcal{I}(\phi) - \mu_0(\phi)\nabla G(\phi) - \sum_{j=1}^{k-1} \mu_j(\phi)\nabla G_j(\phi).$$

From

$$(\nabla_{X_k} \mathcal{I}(\phi), \phi)_{H^1} = (\nabla \mathcal{I}(\phi), \phi)_{H^1} - \mu_0(\phi)(\nabla G(\phi), \phi)_{H^1} - \sum_{j=1}^{k-1} \mu_j(\phi)(\nabla G_j(\phi), \phi)_{H^1}$$

$$= 2\mathcal{I}(\phi) - 2\mu_0(\phi)G(\phi) - \sum_{j=1}^{k-1} \mu_j(\phi)G_j(\phi) = 2\mathcal{I}(\phi) - 2\mu_0(\phi)$$

for $\phi \in X_k$, we deduce that

$$(2.16) \quad \mu_0(\phi) = \mathcal{I}(\phi) - \frac{1}{2} (\nabla_{X_k} \mathcal{I}(\phi), \phi)_{H^1}$$

while for $\phi \in X_k$ and $\varphi_i = (\phi_i)_{tr}$, for $i = 1, \ldots, k - 1$, from
we have that

\[
(\nabla X_k \mathcal{I}(\phi), \phi_i)_{H^1} = (\nabla \mathcal{I}(\phi), \phi_i)_{H^1} - \mu_0(\phi)(\nabla G(\phi), \phi_i)_{H^1} - \sum_{j=1}^{k-1} \mu_j(\phi)(\nabla G_j(\phi), \phi_i)_{H^1}
\]

\[
= d\mathcal{I}(\phi)[\phi_i] - \mu_0(\phi)2 \text{Re} \langle \phi_{tr}, \varphi_i \rangle_{L^2} - \sum_{j=1}^{k-1} \mu_j(\phi)(\varphi_j, \varphi_i)_{L^2}
\]

\[
= d\mathcal{I}(\phi)[\phi_i] - \mu_i(\phi)
\]

We say that \(\phi_n \in X_k\) is a (constrained) Palais Smale sequence for \(\mathcal{I}\) on \(X_k\) at level \(\lambda_k\) if \(\phi_n \in X_k\),

\[
\mathcal{I}(\phi_n) \to \lambda_k \quad \text{and} \quad \|\nabla X_k \mathcal{I}(\phi_n)\| \to 0.
\]

The proof of existence of a minimizer for \((\mathcal{I})\) proceeds as the proof of the existence of the ground state \(\phi_1\). The key points are the following two lemmas.

**Lemma 2.18.** \(\lambda_1 \leq \lambda_k < mc^2\).

**Proof.** Let us consider any \(k\)-dimensional linear subspace \(G_k \subset C_0^\infty(\mathbb{R}^3; \mathbb{C}^2)\).

For \(\varphi \in G_k \cap S\) and \(\eta > 0\) we let \(\varphi_\eta(y) = \eta^{-1/2} \varphi(\eta y) \in S\) and

\[
F^n_k = \{ \phi_\eta \in H^1 \mid \phi_\eta(x, y) = e^{-mc^2 x} \varphi_\eta(y), \varphi \in G_k \cap S \}.
\]

Then, for any \(\phi_\eta \in F^n_k\)

\[
\mathcal{I}(\phi_\eta) - mc^2 = \frac{1}{2m} \int_{\mathbb{R}^3} |\nabla \varphi_\eta|^2 + \int_{\mathbb{R}^3} (\varphi_\eta, V_{pw}^{2\times 2} \varphi_\eta)_{L^2} = \frac{\eta^2}{2m} \int_{\mathbb{R}^3} |\nabla \varphi|^2 + \int_{\mathbb{R}^3} (\varphi_\eta, V_{pw}^{2\times 2} \varphi_\eta)_{L^2}
\]

Arguing as in Lemma 2.16 (ii) and by compactness of the set \(G_k \cap S\), there exists \(\bar{\eta} > 0\) such that for any \(\phi_\eta \in F^n_k\), we have

\[
\bar{\eta}^2 \frac{1}{2m} \int_{\mathbb{R}^3} |\nabla \varphi|^2 + \int_{\mathbb{R}^3} (\varphi_\eta, V_{pw}^{2\times 2} \varphi_\eta)_{L^2} < 0.
\]

Since \(X_k \cap F^n_k \neq \emptyset\), we have \(\lambda_k \leq \sup_{F^n_k} \mathcal{I}(\phi_\eta) < mc^2\).

**Lemma 2.19.** Let \(\zeta_n \in X_k\) be a (constrained) Palais Smale sequence at level \(\lambda_k\) for \(\mathcal{I}\) on \(X_k\).

Then, as \(n \to +\infty\)

\[
\mu_0(\zeta_n) \to \lambda_k \quad \mu_j(\zeta_n) \to 0 \quad (j = 1, \ldots, k - 1).
\]

Moreover, if \(\xi_n = (\zeta_n)_{tr} \to 0\) in \(H^{1/2}\) then

\[
(\xi_n, V_{pw}^{2\times 2} \xi_n)_{L^2} \to 0.
\]

**Proof.** We have that \(\zeta_n \in X_k\) is such that

\[
\mathcal{I}(\zeta_n) \to \lambda_k \quad \text{and} \quad \|\nabla X_k \mathcal{I}(\zeta_n)\| \to 0.
\]

Then \(\zeta_n\) is bounded and from \((2.16)\) and \((2.17)\) we have, as \(n \to +\infty\)

\[
\mu_0(\zeta_n) = \mathcal{I}(\zeta_n) - \frac{1}{2}(\nabla X_k \mathcal{I}(\zeta_n), \zeta_n)_{H^1} \to \lambda_k \quad \mu_j(\zeta_n) = d\mathcal{I}(\zeta_n)[\phi_j] - (\nabla X_k \mathcal{I}(\zeta_n), \phi_j)_{H^1} = d\mathcal{I}(\phi_j)[\zeta_n] - (\nabla X_k \mathcal{I}(\zeta_n), \phi_j)_{H^1} = 2\lambda_j \text{Re} \langle \xi_n, \varphi_j \rangle_{L^2} - (\nabla X_k \mathcal{I}(\zeta_n), \phi_j)_{H^1} \to 0
\]
for \( j = \ldots, k - 1 \).

We then proceed as in the proof of lemma 2.10. In particular the first step in the proof of that lemma holds also here, that is

\[
|\langle \xi_n, V_{pw}^2 \xi_n \rangle_{L^2} - \langle \chi_n \xi_n, V_{pw}^2 \chi_n \xi_n \rangle_{L^2} | \to 0 \text{ as } R \to +\infty.
\]

We now use again 2.14 and similar to step 2 of lemma 2.10 and the fact that \( \zeta_n \) is a constrained Palais Smale sequence, we have

\[
o_n(1) = \| \nabla \chi_2 E(\zeta_n) \|_{L^2} \| \zeta_n \|_{H^1} \geq \| \langle \nabla \chi_2 E(\zeta_n), \chi_n^2 \zeta_n \rangle \|_{H^1} \]

\[
\geq \langle dE(\zeta_n) [\chi_n^2 \zeta_n] \rangle - \mu_0 (\zeta_n) \langle \nabla G(\zeta_n), \chi_n^2 \zeta_n \rangle_{H^1} - \sum_{j=1}^{k-1} \mu_j (\zeta_n) \langle \nabla G_j(\zeta_n), \chi_n^2 \zeta_n \rangle_{H^1} \| \zeta_n \|_{L^2} - (I) - (II) - (III)
\]

where as in the proof of lemma 2.10 (see also lemma 13.1, Appendix B) we have

\[
(I) = 2e^2 |\zeta_n \nabla \chi_n|_{L^2}^2 \leq 2e^2 \sup_{\xi \in R^3} \| \nabla \chi_n \|_{L^2}^2 \leq \frac{C}{R^2}
\]

\[
(II) = 2 \| V \psi_n, [U^{-1}_{pw}, \chi_n \chi_n U_{pw} \psi_n] \|_{L^2} \]
\[
\leq \| V^{1/2} [\chi_n, U^{-1}_{pw}] \chi_n U_{pw} \psi_n \|_{L^2} \| V^{1/2} \psi_n \|_{L^2} \]
\[
\leq \frac{C}{R} \| \psi_n \|_{H^{1/2}}^2
\]

\[
(III) = 2 \| [\chi_n, U^{-1}_{pw}] U_{pw} \psi_n, V U^{-1}_{pw} \chi_n U_{pw} \psi_n \|_{L^2} \]
\[
\leq \| V^{1/2} [\chi_n, U^{-1}_{pw}] U_{pw} \psi_n \|_{L^2} \| V^{1/2} U^{-1}_{pw} \chi_n U_{pw} \psi_n \|_{L^2} \]
\[
\leq \frac{C}{R} \| \psi_n \|_{H^{1/2}}^2
\]

(here \( \psi_n = U_{pw}^{-1} Q \xi_n \))

Then by Sobolev compact embedding, for any given \( R > 0 \),

\[
|\chi_n \xi_n|_{L^2} \to 0 \text{ as } n \to +\infty.
\]

Moreover, \( |\mu_j (\zeta_n)| \leq C \) for \( j = 0, \ldots, k - 1 \). More precisely,

Now, since \( E \) is coercive, exactly as in lemma 2.10 we may conclude

\[
\| \chi_n \xi_n \|_{H^1}^2 \leq \epsilon_n + \frac{C}{R}
\]

and by (h2) and lemma 2.1

\[
|\langle V_{pw}^2 \chi_n \xi_n, \chi_n \xi_n \rangle_{L^2} | \leq a \| \chi_n \xi_n \|_{H^1}^2 \leq \epsilon_n + \frac{C}{R}
\]

for \( \epsilon_n \to 0 \) as \( n \to +\infty \), \( R \) arbitrary large, and the lemma follows.

We are now ready to prove the following proposition for the existence of a minimizer for \( P_k \).

**Proposition 2.20.** Let \( \zeta_n \in X_k \) be a minimizing Palais Smale sequence for \( P_k \).

Then \( \zeta_n \to \phi_k \) in \( H^1 \) and \( \langle \phi_n, \zeta_n \rangle |_{L^2} = \langle \phi_n, \zeta_n \rangle \) is a minimizer for problem \( P_k \), and a weak solution of the Neumann problem \( E_k \).

**Proof.** We proceed as in the proof of lemma 2.14 to conclude that \( \zeta_n \to \phi_k \neq 0 \).

We clearly have that \( G_j (\phi_k) = 0 \) for \( j = 1, \ldots, k - 1 \). We do not know if \( |\phi_k|_{L^2} = 1 \) (where \( \phi_k = (\phi_k)_l \)).

By lemma 2.10 we have that

\[
\mu_0 (\zeta_n) \to \lambda_k \quad \mu_j (\zeta_n) \to 0 \quad (j = 1, \ldots, k - 1)
\]
then by weak convergence we then have that for all $h \in H^1$, as $n \to +\infty$

$$(\nabla X_h I(\zeta_0), h)_{H^1} = dI(\zeta_0)[h] - 2\mu_0(\zeta_0) \operatorname{Re}(\zeta_0, h_{tr})_{L^2} = \sum_{j=1}^{k-1} \mu_j(\zeta_0)(\varphi_j, h_{tr})_{L^2}$$

$$\to dI(\phi_k)[h] - 2\lambda_k \operatorname{Re}(\varphi_k, h_{tr})_{L^2} = 0.$$ We deduce, taking $h = \phi_k$

$$0 = dI(\phi_k)[\phi_k] - \lambda_k 2|\varphi_k|^2_{L^2} = 2I(\phi_k) - 2\lambda_k |\varphi_k|^2_{L^2}$$

and we conclude that $|\varphi_k|^{-1}\phi_k \in X_k$ is a minimizer for $(P_k)$. \hfill \Box

To conclude the proof of Theorem 1.8 we prove that $\{\lambda_k\}_{k \geq 1} \in \sigma_{disc}(B_{pv})$ namely that $\lambda_k$ has finite multiplicity.

Indeed suppose that there exists an eigenvalue $\lambda_k$ with infinite multiplicity. Then there exist a corresponding sequence $\{\varphi_n^{(k)}\}_{n \in \mathbb{N}} \subset H^{1/2}$ of eigenfunctions corresponding to the same eigenvalue $\lambda_k$. We will assume that $|\varphi_n^{(k)}|_{L^2} = 1$ for all $n \in \mathbb{N}$. Letting

$$\varphi_n^{(k)} = F^{-1}_v \left[ e^{-x}\sqrt{m^2+e^2}\nu^{2} F_{\varphi_n^{(k)}} \right] \in X_k,$$

by lemma 2.1 we have $\nabla X_h I(\varphi_n^{(k)}) = 0$ and $I(\varphi_n^{(k)}) = \lambda_k$. We deduce from this that $\varphi_n^{(k)}$ is a bounded sequence in $H^{1/2}$, since by orthogonality $\varphi_n^{(k)} \to 0$ in $L^2$, we have $\varphi_n^{(k)} \to 0$ in $H^{1/2}$, therefore by lemma 2.10 we get

$$\langle \varphi_n^{(k)}, V^{2+\varepsilon} \varphi_n^{(k)} \rangle_{L^2} \to 0 \quad \text{as } n \to +\infty$$

and from this we get a contradiction, namely $\lambda_k = I(\varphi_n^{(k)}) \geq mc^2$.

Finally since eigenvalues can accumulate only on the essential spectrum, we may conclude that

$$0 < \lambda_1 \leq ... \leq \lambda_{k-1} \leq \lambda_k \to \inf \{\sigma_{ess}(B_{pv})\} = mc^2 \quad \text{for } k \to +\infty.$$

\section*{Appendix A. Multiplication and convolutions on Lorentz spaces}

We refer to [19] for the definition of the Lorentz space $L(p,q)$, for $1 \leq p, q \leq \infty$ and the corresponding norm $\|f\|_{(p,q)}$. Let us recall here the following facts and inequalities (see e.g. [19, 16] and references therein for more details)

- for $p = q$ the Lorentz spaces $L(p,q)$ coincide with the usual $L^p$-space.
- for $p > 1$ and $q = \infty$ the Lorentz spaces $L(p,\infty)$ correspond to the weak $L^p$-space $L^w_p(\mathbb{R}^N)$ (also known as Marcinkiewicz spaces $M_p(\mathbb{R}^N)$).
- \textbf{multiplication:} Let $f \in L(p_1,q_1)$ and $g \in L(p_2,q_2)$ then $fg \in L(r,s)$ with $\frac{1}{r} + \frac{1}{p_1} = \frac{1}{r} + \frac{1}{q_1} = \frac{1}{s}$ for $1 < p_1, p_2 < \infty, 1 \leq q_1, q_2 \leq \infty$.
- \textbf{convolution:} Let $f \in L(p_1,q_1)$ and $g \in L(p_2,q_2)$ then $f \ast g \in L(r,s)$ with $\frac{1}{r} + \frac{1}{p_1} = \frac{1}{r} + \frac{1}{q_1} = \frac{1}{s}$ for $1 < p_1, p_2 < \infty, 1 \leq q_1, q_2 \leq \infty$ and $\frac{1}{p_1} + \frac{1}{p_2} > 1$. Moreover

$$\|f \ast g\|_{(r,s)} \leq 3r\|f\|_{(p_1,q_1)}\|g\|_{(p_2,q_2)}.$$

Finally let us point out the following generalization of the weak Young inequality.

\textbf{Proposition A.1} (see [10] thm. 2.10). Let $f \in L^w_\infty(\mathbb{R}^N)$, $g \in L^w_\infty(\mathbb{R}^N)$ and $h \in L^p(\mathbb{R}^N)$ with $\frac{1}{q} + \frac{1}{q'} = 1$ and $1 < p < q$. Then

$$(A.2) \quad \|f(g \ast h)\|_p \leq C\|f\|_{q,w}\|g\|_{q',w}\|h\|_p.$$
Appendix B. Estimates on commutators

Lemma B.1. Let \( \chi \in C_0^\infty(\mathbb{R}^3) \) and for \( R > 0 \) let define \( \chi_R(y) = \chi(R^{-1}y) \). Then the operator \( [\chi_R, U_{fw}^{-1}]U_{fw} : H^{1/2}(\mathbb{R}^3; \mathbb{C}^4) \to H^{1/2}(\mathbb{R}^3; \mathbb{C}^4) \) satisfies

\[
\|[\chi_R, U_{fw}^{-1}]U_{fw}\| = O(R^{-1}) \quad \text{as } R \to +\infty.
\]

Proof. We have

\[
\|[\chi_R, U_{fw}^{-1}]U_{fw}\psi\|^2_{H^{1/2}} = \int_{\mathbb{R}^3} dp \left(1 + |p|\right) \left|\mathcal{F}(\|[\chi_R, U_{fw}^{-1}]U_{fw}\psi\|)\right|^2
\]

where

\[
\mathcal{F}(\|[\chi_R, U_{fw}^{-1}]U_{fw}\psi\|)(p) = \mathcal{F}(\chi_R\psi)(p) - \mathcal{F}(U_{fw}^{-1}\chi_R U_{fw}\psi)(p)
\]

\[
= \hat{\chi}_n * \hat{\psi}(p) - U^{-1}(p)\mathcal{F}(\chi_R U_{fw}\psi)(p)
\]

\[
= \int_{\mathbb{R}^3} R^3 \hat{\chi}(R q) \hat{\psi}(p - q) dq - U^{-1}(p)(\hat{\chi}_n * (U \hat{\psi}))(p)
\]

\[
= U^{-1}(p) \int_{\mathbb{R}^3} dq R^3 \hat{\chi}(R q) (U(p) - U(p - q)) \hat{\psi}(p - q) dq
\]

where \( \hat{\chi} \) is the Fourier transform of \( \chi \in C_0^\infty(\mathbb{R}^3) \) (and hence \( \hat{\chi}_n(p) = R^3 \hat{\chi}(R p) \)).

Now, let define \( U_{fw}^{-1} = \mathcal{F}^{-1} U(R^{-1}p) \mathcal{F} \) and \( \psi_R(q) = R^{-3/2} \psi(R^{-1}q) \) where \( \psi \) is the Fourier transform of \( \psi \in H^{1/2}(\mathbb{R}^3) \). By rescaling variables we get

\[
\|[\chi_R, U_{fw}^{-1}]U_{fw}\psi\|^2_{H^{1/2}} = \int_{\mathbb{R}^3} dp \left(1 + R^{-1}|p|\right) \left|U^{-1}(R^{-1}p) \int_{\mathbb{R}^3} dq \hat{\chi}(q) K_R(p, q) \hat{\psi}_R(p - q)\right|^2
\]

where

\[
K_R(p, q) = U(R^{-1}p) - U(R^{-1}(p - q))
\]

\[
= (a_+ (R^{-1}p) - a_+ (R^{-1}(p - q))) \cdot 4 + 3 \left( a_{-1} (R^{-1}p) \frac{p_i}{|p|} - a_{-1} (R^{-1}(p - q)) \frac{p_i - q_i}{|p - q|}\right) \beta \alpha_i
\]

\[
= K_{+, R}(p, q) \cdot 4 + \sum_{i=1}^{3} K_{-i, R}(p, q) \beta \alpha_i
\]

and \( a_+(R^{-1}p) = \sqrt{\frac{1}{2}}(1 + mc^2/\lambda(R^{-1}p)) \).

Letting \( \lambda_n(p) = \lambda(R^{-1}p) = \sqrt{c^2 R^{-2}|p|^2 + mc^2} \) we have

\[
|K_{+, R}(p, q)| = |a_+ (R^{-1}p) - a_+ (R^{-1}(p - q))| = \frac{|a_+^2 (R^{-1}p) - a_+^2 (R^{-1}(p - q))|}{a_+ (R^{-1}p) + a_+ (R^{-1}(p - q))}
\]

\[
\leq \frac{mc^2}{2\sqrt{2}} \left|\frac{1}{\lambda_n(p)} - \frac{1}{\lambda_n(p - q)}\right| \leq \frac{mc^2}{2\sqrt{2}} \frac{1}{\lambda_n(p)} \lambda_n(p - q) \left(\lambda_n(p) + \lambda_n(p - q)\right)
\]

\[
\leq \frac{mc^2}{2\sqrt{2}} \frac{c^2 R^{-2}|p - q|^2}{|p|^2} \leq \frac{\sqrt{2}|q|}{4mcR}
\]

and analogously,
exists a positive constant $C$ and, in particular we get

\[ |K_{\perp, R}(p, q)| \leq \left| a_-(R^{-1}(p - q)) \frac{(p_i - q_i)}{|p - q|} - a_-(R^{-1}p) \frac{p_i}{|p|} \right| \]

\[ \leq |p| \left| a_-(R^{-1}(p - q)) \frac{(p_i - q_i)}{|p - q|} - a_-(R^{-1}p) \frac{p_i}{|p|} \right| + |q| a_-(R^{-1}(p - q)) \frac{p_i}{|p - q|} \]

\[ \leq \frac{1}{\sqrt{2}} \left| p \right| \left( \frac{\lambda_n(p - q) - mc^2}{\lambda_n(p - q)} \right)^{\frac{1}{2}} - \left( \frac{\lambda_n(p) - mc^2}{\lambda_n(p)} \right) \right| \]

\[ \leq \frac{1}{\sqrt{2}} \left| q \right| \left( \frac{\lambda_n(p) - mc^2}{\lambda_n(p)} \right) \right| \]

\[ \leq \frac{1}{\sqrt{2}} \frac{|p|}{R} \lambda_n(p) \lambda_n(p - q) \]

\[ + \frac{1}{\sqrt{2}} \frac{c|q|}{R mc^2} \]

\[ \leq \frac{1}{\sqrt{2}} \frac{|p|^2 - |p - q|^2}{|p| + |p - q|} + \frac{1}{\sqrt{2}} \frac{|q|}{mcR} \leq \frac{3\sqrt{2} |q|}{2mcR}. \]

Therefore we may conclude that,

\[ |K_R(p, q)| \leq |K_{\perp, R}(p, q)| + \sum_{i=1}^{3} |K_{\perp, R}(p, q)\beta_{ai}| \leq \frac{5\sqrt{2} |q|}{mcR} \]

and, in particular we get

\[ \sup_{(p, q) \in \mathbb{R}^3 \times \mathbb{R}^3} \frac{|K_R(p, q)|}{1 + |q|} \leq C. \]

Moreover, since $\hat{\chi}$ is the Fourier transform of the compact support function $\chi \in C_0^\infty(\mathbb{R}^3)$, it decays at infinity faster than any power, namely for any $\alpha > 0$ there exists a positive constant $C_\alpha > 0$ such that

\[ (B.2) \quad |\hat{\chi}(q)| \leq \frac{C_\alpha}{(1 + |q|)^{\alpha}}. \]
Then, by using Hölder inequality we get
\[
\|\chi_R, U_{\nu \omega}^{-1} U_{\nu \omega} \psi\|_{L^{1/2}}^2 = \int_{R^3} dp \left( 1 + R^{-1}|p| \right) \left( \int_{R^3} dq \, \hat{\chi}(q) K_R(p, q) \hat{\psi}_R(p - q) \right)^2 \leq \int_{R^3} dp \left( 1 + R^{-1}|p| \right) \left( \int_{R^3} dq \, \hat{\chi}(q) K_R(p, q) \hat{\psi}_R(p - q) \right)^2 \leq \sup_{(p, q) \in R^3 \times R^3} \frac{|K_R(p, q)|^2}{(1 + |q|)^2} \int_{R^3} dq \, dq_2 (1 + |q_1|) |\hat{\chi}(q_1)|(1 + |q_2|) |\hat{\chi}(q_2)| \times \int_{R^3} dp \left( 1 + \frac{|p - q_1|}{R} + \frac{|q_1|}{R} \right)^{1/2} |\hat{\psi}_R(p - q_1)| \left( 1 + \frac{|p - q_2|}{R} + \frac{|q_2|}{R} \right)^{1/2} |\hat{\psi}_R(p - q_2)| \leq \frac{C}{R^2} \left( \int_{R^3} dq \, |\hat{\chi}(q)| \left( \int_{R^3} dp \left( 1 + \frac{|p - q|}{R} + \frac{|q|}{R} \right)^{1/2} |\hat{\psi}_R(p - q)| \right)^2 \right)^{1/2}.
\]

Note that,
\[
\int_{R^3} dp \left( 1 + R^{-1}|p - q| + R^{-1}|q| \right) |\hat{\psi}_R(p - q)|^2 = \int_{R^3} dp \left( 1 + R^{-1}|p - q| \right) |\hat{\psi}_R(p - q)|^2 + R^{-1}|q| \int_{R^3} dp |\hat{\psi}_R(p)|^2 \leq \int_{R^3} dp (1 + R^{-1}|p - q|) |\hat{\psi}_R(p - q)|^2 \leq \int_{R^3} dp (1 + R^{-1}|p - q|) |\hat{\psi}_R(p - q)|^2 \leq \int_{R^3} dp (1 + R^{-1}|p - q|) |\hat{\psi}_R(p - q)|^2 \leq (1 + R^{-1}|q|) \|\psi\|_{L^2}^2 \leq (1 + R^{-1}|q|) \|\psi\|_{L^2}^2.
\]

Hence, for \( R > 1 \), we may conclude
\[
\|\chi_R, U_{\nu \omega}^{-1} U_{\nu \omega} \psi\|_{L^{1/2}} \leq \frac{C}{R} \left( \int_{R^3} dq \left( 1 + |q| \right) |\hat{\chi}(q)| \left( \int_{R^3} dp \left( 1 + \frac{|p - q|}{R} + \frac{|q|}{R} \right)^{1/2} |\hat{\psi}_R(p - q)|^2 \right)^{1/2} \right).
\]

\[
\|\chi_R, U_{\nu \omega}^{-1} U_{\nu \omega} \psi\|_{L^{1/2}} \leq \frac{C}{R} \|\psi\|_{H^{1/2}} \int_{R^3} dq \left( 1 + |q| \right) |\hat{\chi}(q)| \left( 1 + R^{-1}|q| \right)^{1/2} \|\psi\|_{L^2} \int_{R^3} dq \left( 1 + |q| \right)^{3/2} |\hat{\chi}(q)| \leq \frac{C}{R} \|\psi\|_{H^{1/2}} \|\psi\|_{L^2}.
\]

\[\square\]

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E-mail address, Coti Zelati: zelati@unina.it

(Coti Zelati) **Dipartimento di Matematica Pura e Applicata “R. Caccioppoli”, Università di Napoli “Federico II”, via Cintia, M.S. Angelo, 80126 Napoli (NA), Italy**

E-mail address, Nolasco: nolasco@univaq.it

(Nolasco) **Dipartimento di Ingegneria e Scienze dell’ informazione e Matematica, Università dell’Aquila via Vetoio, Loc. Coppito 67010 L’Aquila AQ Italia**