Bispectral algebras of commuting ordinary differential operators

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Abstract

We develop a systematic way for constructing bispectral algebras of commuting ordinary differential operators of any rank $N$. It combines and unifies the ideas of Duistermaat–Gröbbaun and Wilson. Our construction is completely algorithmic and enables us to obtain all previously known classes or individual examples of bispectral operators. The method also provides new broad families of bispectral algebras which may help to penetrate deeper into the problem.

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\section{Introduction}

In this paper we reconsider the bispectral problem. As stated in \cite{DG}, it asks for which ordinary differential operators $L(x, \partial_x)$ there exists a family of eigenfunctions $\Psi(x, z)$ that are also eigenfunctions for another differential operator $\Lambda(z, \partial_z)$ but this time in the “spectral parameter” $z$, to wit

\begin{align}
L(x, \partial_x) \Psi(x, z) &= f(z) \Psi(x, z), \\
\Lambda(z, \partial_z) \Psi(x, z) &= \Theta(x) \Psi(x, z)
\end{align}

for some functions $f(z), \Theta(x)$. Both operators $L$ and $\Lambda$ are called bispectral.

This problem first appeared in \cite{G1} in connection with “limited angle tomography” (see also \cite{G2, G3, DG}). Later it turned out to be related with several, seemingly far from it, topics and in particular, with soliton mathematics. To be more specific, we have to mention the deep connection with some very actively developing areas of research in mathematics and theoretical physics like Calogero–Moser particle system \cite{W2, K} (see also \cite{H}), additional symmetries of KdV and KP hierarchies \cite{MZ, BHY4}, representation theory of $W_{1+\infty}$-algebra \cite{BHY4}, etc. These studies not only revealed the rich mathematical structure behind the bispectral problem, but also (if we use a remark by G. Wilson \cite{W2}) “deepened the mystery” around it. Thus, not only applications, but also purely mathematical questions motivated the great activity in the past few years in the bispectral problem.

In the present paper we construct new families of bispectral operators. In order to explain better our contribution, we need to review some of the achievements in the subject.

The first general result in the direction of classifying bispectral operators belongs to J. J. Duistermaat and F. A. Grünbaum \cite{DG}. They determined all second order operators $L$ admitting an operator $\Lambda$ such that the pair $(L, \Lambda)$ solves the bispectral problem \eqref{eq0}, \eqref{eq01}. Their answer is as follows. If we write the operator $L$ in the standard Schrödinger form

$$L = \frac{d^2}{dx^2} + u(x),$$

the bispectral potentials $u(x)$ are given (up to translations and rescalings of $x$ and $z$) by the following list:

\begin{enumerate}
    \item $u(x) = x$ \quad (Airy);
    \item $u(x) = cx^{-2}, \ c \in \mathbb{C}$ \quad (Bessel);
    \item $u(x), \ \text{which can be obtained by finitely many rational Darboux transformations from } u(x) = 0$;
    \item $u(x), \ \text{which can be obtained by finitely many rational Darboux transformations from } u(x) = -\frac{1}{4x^2}.$
\end{enumerate}

The family \eqref{eq05} has previously appeared in \cite{AMM, AM} and is known as “\textit{rational solutions of KdV}”. They can be obtained also by applying “\textit{higher KdV flows}” to potentials $v_k(x) = k(k+1)x^{-2}, \ k \in \mathbb{N}$. 

The second family \( \{0.3\} \) was interpreted by F. Magri and J. Zubelli \( [MZ] \) as potentials invariant under the flows of the "master symmetries" or Virasoro flows.

Besides the classification of the bispectral operators by their order, another scheme has been suggested in \( [DG] \) and used in \( [W1] \). Below we explain it in a general context as it will be used throughout this paper. One may consider an operator \( L(x, \partial_x) \) as an element of a maximal algebra \( \mathcal{A} \) of commuting ordinary differential operators \( [BC] \). Following G. Wilson \( [W1] \), we call such an algebra bispectral if there exists a joint eigenfunction \( \Psi(x, z) \) for the operators \( L \) in \( \mathcal{A} \) that satisfies also equation (0.2). The dimension of the space of eigenfunctions \( \Psi(x, z) \) is called rank of the commutative algebra \( \mathcal{A} \) (see e.g. \( [KnN] \)). This number coincides with the greatest common divisor of the orders of the operators in \( \mathcal{A} \). For example, the operators with potentials \( \{0.3\} \) belong to rank 1 algebras and those with potentials \( \{0.3, 0.4, 0.6\} \) to rank 2 algebras \( [DG] \).

All rank 1 maximal bispectral algebras were recently found by G. Wilson \( [W1] \). These algebras do not necessarily contain an operator of order two.

The methods of the above mentioned papers \( [DG] \) and \( [W1] \) may seem quite different. Indeed, while in \( [DG] \) the "rational" Darboux transformations play a decisive role, G. Wilson \( [W1] \) uses planes in Sato’s Grassmannian obtained from the standard \( H_+ = \text{span}\{z^k\}_{k \geq 0} \) by imposing a number of conditions on it. One of our main observations is that both methods, appropriately modified, can be looked upon as the two sides of one general theory.

From this new point of view in the present paper we construct nontrivial maximal bispectral algebras of any rank \( N \), thus extending the results from \( [DG, W1] \). For example, for any positive integer \( k \) we obtain bispectral algebras of rank \( N \) with the lowest order of the operators equal to \( kN \). Our method allows us to obtain all classes and single examples of bispectral operators known to us by a unique method. At the same time we suggest an effective procedure for constructing bispectral operators, despite the fact that the theory involves highly transcendental functions like Airy or Bessel ones. The point is that the latter are used in the proofs while the algorithm given at the end of Sect. 3 performs arithmetic operations and differentiations only on explicit rational functions.

In the rest of the introduction we describe in more detail the main results of the paper together with some of the ideas behind them.

The framework of our construction is Sato’s theory of KP–hierarchy \( [S, DJKM, SW, vM] \). In particular, our eigenfunctions are Baker or wave functions \( \Psi_V(x, z) \) corresponding to planes \( V \) in Sato’s Grassmannian \( Gr \) and our algebras of commuting differential operators are the spectral algebras \( \mathcal{A}_V \). We obtain our bispectral algebras by applying a version of Darboux transformations, introduced in our previous paper \( [BHY3] \), on specific wave functions which we call Bessel (and Airy) wave functions (see Sect. 1 and 4). As both notions are fundamental for the present paper we hold the attention of the reader on them. Bessel wave functions are the simplest functions which solve the bispectral problem (see \( [4] \) where they were introduced and \( [8] \)). They can be defined as follows. For \( \beta = (\beta_1, \ldots, \beta_N) \in \mathbb{C}^N \) \( \Psi_\beta(x, z) \) is the unique wave function satisfying
\[
x \partial_x \Psi_\beta(x, z) = z \partial_z \Psi_\beta(x, z)
\]
(i.e. \( \Psi_\beta(x, z) \) depends only on \( xz \)) and

\[
L_\beta(x, \partial_x)\Psi_\beta(x, z) = z^N\Psi_\beta(x, z),
\]

where \( L_\beta(x, \partial_x) = x^{-N}(x\partial_x - \beta_1) \cdots (x\partial_x - \beta_N) \) is the Bessel operator. Obviously, the above equations lead to

\[
L_\beta(z, \partial_z)\Psi_\beta(x, z) = x^N\Psi_\beta(x, z).
\]

Similarly, for \( \alpha = (\alpha_0, \alpha_1, \ldots, \alpha_{N-1}) \in \mathbb{C}^{N-1} \) consider the (generalized) Airy wave function (see [KS, Dij]) satisfying:

\[
\left( \partial_x^N + \sum_{i=2}^{N-1} \alpha_i \partial_x^{N-i} - \alpha_0 x \right) \Psi_\alpha(x, z) = z^N\Psi_\alpha(x, z).
\]

It depends only on \( \alpha_0 x + z^N \) and again gives a simple solution to the bispectral problem. The Airy case is in many respects similar to the Bessel one. As we find the latter case richer in properties, we pay more attention to it, contenting ourselves only with a sketch of the former.

Classically, a Darboux transformation [BC, Da] of a differential operator \( L \), presented as a product \( L = QP \), is defined by exchanging the places of the factors, i.e. \( \overline{L} = PQ \). Obviously, if \( \Psi(x, \lambda) \) is an eigenfunction of \( L \), i.e. \( L(x, \partial_x)\Psi(x, \lambda) = \lambda\Psi(x, \lambda) \), then \( P\Psi(x, \lambda) \) is an eigenfunction of \( \overline{L} \).

Here we introduce Darboux transformations not only on individual operators but also on the entire spectral algebra corresponding to a Bessel (or Airy) plane. In other words, we apply them on operators \( L \) which are polynomials \( h(L_\beta) \) of Bessel (or Airy) operators. These transformations may be considered as Bäcklund–Darboux transformations on the corresponding wave functions [AvM]. Such Darboux transformation is completely determined by a choice of a \( \mathbb{Z}_N \)-invariant operator \( P(x, \partial_x) \) with rational coefficients normalized appropriately by a factor \( g^{-1}(z) \) to ensure that

\[
\Psi_W(x, z) = \frac{1}{g(z)}P(x, \partial_x)\Psi_\beta(x, z)
\]

is a wave function. We call \( \Psi_W(x, z) \) (respectively \( W \)) a polynomial Darboux transformation of \( \Psi_\beta \) (respectively \( V_\beta \)). The definition of polynomial Darboux transformations of Airy planes is similar to that in the Bessel case with only minor modifications: \( P \) is not necessarily \( \mathbb{Z}_N \)-invariant and \( g(z) \) has to belong to \( \mathbb{C}[z^N] \).

Thus we come to our main result.

**Theorem 0.1** If the wave function \( \Psi_W(x, z) \) is a polynomial Darboux transformation of a Bessel or Airy wave function \( \Psi_\beta(x, z) \), then it is a solution to the bispectral problem, i.e. there exist differential operators \( L(x, \partial_x), \Lambda(z, \partial_z) \) and functions \( f(z), \Theta(x) \) such that [1.1] and [1.2] hold.

Note the difference between the classical definition and the definition introduced here. In contrast to [DG] where the authors make a finite number of “rational” Darboux transformations, we perform only one polynomial Darboux transformation to achieve the same result.
Our definition of polynomial Darboux transformation is constructive as $P(x, \partial_x)$ is determined by the finite dimensional space $\text{Ker} P$. For this reason one can explicitly present at least one operator $L \in A_W$; it can be given by $Ph(L_\beta)P^{-1}$. Usually it is of high order. But as it is only one element of the whole bispectral algebra $A_W$ there can be eventually operators of a lower order. For example, the bispectral operators of [DG] are of order two. There is a simple procedure (see [BHY3]) to produce the entire bispectral algebra $A_W$ of commuting differential operators. In addition, one can show that the spectral curve $\text{Spec} A_W$ (see e.g. [AMcD] for definition) is rational, unicursal and $\mathbb{Z}_N$–invariant.

In the course of our work we have widely used important ideas introduced by G. Wilson [W1]. Among them we mention first the idea of explicitly writing conditions on vectors of a plane $V \in Gr$ which define the new plane obtained by a Darboux transformation. Second is the notion of involutions on the Sato’s Grassmannian. In particular, we extend the bispectral involution $b$ introduced in [W1] to the manifolds of polynomial Darboux transformations. More precisely, we prove the following theorem, from which Theorem 0.1 is an obvious consequence.

**Theorem 0.2**

(i) The bispectral involution is defined for planes $W$ which are polynomial Darboux transformations of Bessel or Airy planes.

(ii) The image $bW$ of such a plane $W$ is again a polynomial Darboux transformation of the corresponding Bessel (respectively) Airy plane.

Our main concern in the present paper is to prove Theorem 0.2. Our second goal is to provide explicit formulae and examples (see Sect. 5), which are not only an illustration of our method but also show the existence of new families of bispectral operators with particular properties. Some of them generalize directly the well known ones like the Duistermaat–Gr"unbaum’s “even case” ([DG]). Other families exhibit quite different properties from the well known examples. In this respect Sect. 5 has also the role to supply diverse experimental material for new insights into the theory of bispectral algebras. We draw the attention of the reader also to the explicit formulae for the action of the bispectral involution on an important class of Darboux transformations (which we call monomial) of Bessel operators. As a particular case, we obtain such formulae for all second order bispectral operators found in [DG].

The class of monomial Darboux transformations has also other remarkable properties, e.g. they are connected to representation theory of $W_{1+\infty}$–algebra. We do not touch this matter here for lack of space. The interested reader can learn about it in [BHY4].

A natural question is if the operators found in this paper form the entire class of bispectral operators. The answer is negative as recently shown in [BHY3].

At the end for the reader’s convenience we give a brief description of the organization of the paper. Sect. 1 is intended only for reference. It reviews results connected with Sato’s theory, which we need for the treatment of the bispectral problem. Besides the general notions (see e.g. [S, DJKM, SW]) we recall the involutions, introduced by G. Wilson [W1] and in particular, the bispectral involution. In Sect. 2 we introduce our manifolds $Gr^{(N)}_A$ of polynomial Darboux transformations of Bessel
planes. We give two equivalent definitions (Definition 2.3 and the one provided by
the statement of Theorem 2.7). Sect. 3 contains our main results – Theorems 3.1 and
3.2 for the Bessel case. Sect. 4 deals with the analogs of Sect. 2 and 3 for the Airy
case (although in different order). The last Sect. 5 is devoted to explicit examples of
bispectral operators, which have been studied in other papers [DG, V2], as well as
new families (which we have not seen elsewhere). The emphasis in Sect. 5 is rather
on the simple algorithmic way of constructing bispectral operators (wave functions,
etc.) than on the novelty of the examples.

For readers who wish to see the main results as soon as possible we propose
another plan of reading the paper. They can start with Sect. 2 and read it up to
the statement of Theorem 2.7, returning to Sect. 1 for reference when needed. Then
skipping the (technical) proof of Theorem 2.7 they can go Sect. 3. After that, taking
for granted the proof of Theorem 3.2, they can look at the examples of bispectral
operators, originating from Bessel ones in Sect. 5. Thus they will have a complete
picture of the results in the Bessel case, and having this experience, they can easily
go through the Airy case.

More detailed information about the material included in each section can be
found in its beginning.

The present paper is a part of our project on the bispectral problem [BHY2]–
[BHY5]. The main results contained here were announced at the conference of
Geometry and Mathematical Physics, Zlatograd 95 (see [BHY1]).

After this paper was written, we got a paper [KR] where some of the results
about the Airy case were obtained independently.

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1 Preliminaries

In this section we have collected results about Sato’s theory, relevant to the bispectral
problem. For reader’s convenience we have divided the section into 4 subsections,
whose titles, hopefully, give an idea of their content. The reader, who is acquainted
with Sato’s theory may even skip this section and return to it for references when
needed. More detailed account of the material of the subsections can be found in
their beginnings.

1.1 Sato’s Grassmannian and KP–hierarchy

We shall recall some facts and notation from Sato’s theory of KP-hierarchy needed
in the paper. The survey below cannot be used as a systematic study. There are
several complete texts on Sato’s theory, starting with the original papers of M. Sato and his collaborators [S, DJKM] (see also [SW, vM]).

Consider the space of formal series
\[ V = \left\{ \sum_{k \in \mathbb{Z}} a_k v_k \middle| a_k = 0 \text{ for } k \gg 0 \right\}. \]

Sato’s Grassmannian \( Gr \) [S, DJKM, SW] consists of all subspaces (planes) \( W \subset V \) which have an admissible basis
\[ w_k = v_k + \sum_{i < k} w_{ik} v_i, \quad k = 0, 1, 2, \ldots \]

In Sato’s theory \( V \) is most often realized as the space of formal Laurent series in \( z^{-1} \) via \( v_k = z^k \). The Baker (or wave function) \( \Psi_W(x, z) \) of the plane \( W \) contains the whole information about \( W \) as the vectors \( w_k = \partial_x^k \Psi_W(x, z)|_{x=0} \) form an admissible basis of \( W \). We can expand \( \Psi_W(x, z) \) in a formal series
\[ \Psi_W(x, z) = e^{xz} \left( 1 + \sum_{k=0}^{\infty} a_k(x) z^{-k} \right). \] (1.1)

The wave function \( \Psi_W(x, z) \) can also be written in terms of the so-called wave operator \( K_W \). This is a pseudo-differential operator defined by
\[ K_W(x, \partial_x) = 1 + \sum_{k=1}^{\infty} a_k(x) \partial_x^{-k}. \] (1.2)

Then obviously
\[ \Psi_W(x, z) = K_W(x, \partial_x) e^{xz}. \]

Introduce also the pseudo-differential operator
\[ P = K_W \circ \partial_x \circ K_W^{-1}. \] (1.3)

For the treatment of the bispectral problem the following identity is crucial:
\[ P \Psi_W(x, z) = z \Psi_W(x, z). \] (1.4)

When it happens that some power of \( P \), say \( P^N \), is a differential operator \( L \) we get that \( \Psi_W(x, z) \) is an eigenfunction of an ordinary differential operator:
\[ L \Psi_W(x, z) = z^N \Psi_W(x, z). \] (1.5)

It is easy to show that \( P^N \) is differential iff the plane \( W \) is invariant under the multiplication by \( z^N \):
\[ z^N W \subset W. \] (1.6)

The submanifold of \( Gr \) consisting of planes \( W \) satisfying (1.6) is denoted by \( Gr^{(N)} \).

A very important object connected to the plane \( W \) is the algebra \( A_W \) of polynomials \( f(z) \) that leave \( W \) invariant:
\[ A_W = \{ f(z) \mid f(z) W \subset W \}. \] (1.7)
For each \( f(z) \in A_W \) one can show that there exists a unique differential operator \( L_f(x, \partial_x) \), the order of \( L_f \) being equal to the degree of \( f \), such that

\[
L_f(x, \partial_x) \Psi_W(x, z) = f(z) \Psi_W(x, z).
\]  

(1.8)

Explicitly we have

\[
L_f = K_W \circ f(\partial_x) \circ K_W^{-1}.
\]  

(1.9)

We denote the commutative algebra of these operators by \( A_W \), i.e.

\[
A_W = \{ L_f | L_f \Psi_W = f \Psi_W, f \in A_W \}.
\]  

(1.10)

Obviously, \( A_W \) and \( A_W \) are isomorphic. We call \( A_W \) spectral algebra corresponding to the plane \( W \). Following I. Krichever (see e.g. [Kri]), we introduce a rank of \( A_W \) to be the dimension of the space of eigenfunctions \( \Psi_W \); this number is equal to the greatest common divisor of the orders of the operators \( L_f \). At the end we define the spectral curve corresponding to the plane \( W \) to be \( \text{Spec} A_W \) (for definition see e.g. [Am]). It is known that \( \text{Spec} A_W \) is an algebraic curve (see [BC, I, Kri]).

**Remark 1.1** If \( \Psi_W(x, z) \) is well defined for \( x = x_0 \) we set \( v_k = e^{x_0 z} z^k \) and consider the subspace \( W_{x_0} \) of \( V \) with an admissible basis \( w_k = \partial_x^k \Psi_W(x, z) \|_{x=x_0} \). The wave functions of \( W_{x_0} \) and \( W \) are connected by \( \Psi_W(x_0, z) = e^{-x_0 z} \Psi_W(x_0 + x_0, z) \) and obviously

\[
K_W(x_0, \partial_x) = K_W(x + x_0, \partial_x).
\]

These shifts are inessential for the bispectral problem and for our proofs. Throughout the paper we shall sometimes work with \( \Psi_{W_{x_0}} \) calling it by abuse of notation a wave function of \( W \) and denoting it \( \Psi_W \).

**Remark 1.2** In Sato’s theory of KP hierarchy one usually considers the wave function

\[
\Psi_W(t, z) = e^{\sum_{k=1}^{\infty} t_k z^k \left( 1 + \sum_{k=1}^{\infty} a_k(t) z^{-k} \right)}
\]

depending on all times \( t_1, t_2, \ldots \) (here \( t_1 = x \)). Then the operators \( K_W(t, \partial_x) \) and \( P(t, \partial_x) \) are given again by formulae (1.2, 1.3) with \( a_k(x) \) replaced with \( a_k(t) \) and \( P \) satisfies the following infinite system of non-linear differential equations

\[
\frac{\partial}{\partial t_k} P = [P^k, P],
\]  

(1.11)

where \( P^k \) stands for the differential part of the \( k \)-th power \( P^k \) of the operator \( P \), \([.,.,.\]\) is the standard commutator of pseudo-differential operators. The equations (1.11) are called the KP hierarchy. If the plane \( W \) lies in \( Gr^{(N)} \) then (1.11) can be written in the form

\[
\frac{\partial}{\partial t_k} L = [L^{k/N}, L]
\]  

(1.12)

\((L = P^N)\) and is called an \( N \)-th reduction of the KP-hierarchy or an \( N \)-th Gelfand–Dickey hierarchy [G].
1.2 Darboux transformations

In this subsection we recall the notion of Darboux transformations on objects connected to points of Sato’s Grassmannian, introduced in our recent paper [BHY3].

Recall that a Darboux transformation $[Da]$ of an ordinary differential operator $L$ is obtained by presenting it as a product and exchanging the places of the factors:

$$L = QP \mapsto L = PQ.$$ 

A (monic) operator $L$ is completely determined by its kernel: if $\{\Phi_0, \ldots, \Phi_{n-1}\}$ is a basis of $\text{Ker} L$ then (see e.g. [I])

$$L\Phi = \frac{\text{Wr}(\Phi_0, \ldots, \Phi_{n-1}, \Phi)}{\text{Wr}(\Phi_0, \ldots, \Phi_{n-1})}$$  \hspace{1cm} (1.13)

where Wr denotes the Wronski determinant. The next lemma answers the question when the factorization $L = QP$ is possible (see e.g. [I]).

**Lemma 1.3** (i) For a given basis $\{\Phi_0, \ldots, \Phi_{n-1}\}$ of $\text{Ker} L$ set

$$q_1(x) = \partial_x \log \Phi_0,$$  \hspace{1cm} (1.14)

$$q_k(x) = \partial_x \log \frac{\text{Wr}(\Phi_0, \ldots, \Phi_{k-1})}{\text{Wr}(\Phi_0, \ldots, \Phi_{k-2})}, \quad 2 \leq k \leq n.$$  \hspace{1cm} (1.15)

Then the operator $L$ can be factorized as follows

$$L = (\partial_x - q_n)(\partial_x - q_{n-1}) \cdots (\partial_x - q_1).$$  \hspace{1cm} (1.16)

(ii) $L$ can be factorized as

$$L = QP \text{ iff } \text{Ker} P \subset \text{Ker} L.$$  \hspace{1cm} (1.17)

In this case

$$\text{Ker} Q = P(\text{Ker} L).$$  \hspace{1cm} (1.18)

A slightly more general construction is the following one. For operators $L$ and $P$ such that the kernel of $P$ is invariant under $L$, i.e.

$$L(\text{Ker} P) \subset \text{Ker} P$$  \hspace{1cm} (1.19)

we consider the transformation

$$L \mapsto L' = PLP^{-1}.$$  \hspace{1cm} (1.20)

The fact that $L'$ is a differential operator follows from Lemma [L3] (ii). Indeed, $L(\text{Ker} P) \subset \text{Ker} P$ is equivalent to $\text{Ker} P \subset \text{Ker}(PL)$.

In [BHY3] we defined a version of Darboux transformation on points in Sato’s Grassmannian and on related objects – wave functions, tau-functions and spectral algebras.
**Definition 1.4** We say that a plane $W$ (or the corresponding wave function $\Psi_W(x, z)$) is a Darboux transformation of the plane $V$ (respectively wave function $\Psi_V(x, z)$) iff there exist monic polynomials $f(z)$, $g(z)$ and differential operators $P(x, \partial_x)$, $Q(x, \partial_x)$ such that

\begin{align*}
\Psi_W(x, z) &= \frac{1}{g(z)} P(x, \partial_x) \Psi_V(x, z), \\
\Psi_V(x, z) &= \frac{1}{f(z)} Q(x, \partial_x) \Psi_W(x, z).
\end{align*}

(1.21)

(1.22)

An equivalent definition is that $W$ is a Darboux transformation of $V$ iff

\[ fV \subset W \subset \frac{1}{g} V \]

(1.23)

for some polynomials $f(z)$, $g(z)$.

Simple consequences of Definition 1.4 are the identities

\begin{align*}
PQ \Psi_W(x, z) &= f(z) g(z) \Psi_W(x, z), \\
QP \Psi_V(x, z) &= f(z) g(z) \Psi_V(x, z).
\end{align*}

(1.24)

(1.25)

The operator $L = PQ \in A_W$ is a Darboux transformation of $L = QP \in A_V$.

Having in mind applications to the bispectral problem, the most important for our study is the case when

\[ A_V = \mathbb{C}[z^N], \quad A_V = \mathbb{C}[L_V] \]

(1.26)

for some natural number $N$ and a differential operator $L_V$ of order $N$. (This is the simplest case of rank $N$ spectral algebra with a spectral curve $\mathbb{C}$.) Then due to (1.25) we have

\begin{align*}
f(z) g(z) &= h(z^N), \\
QP &= h(L_V)
\end{align*}

(1.27)

(1.28)

for some polynomial $h(z)$. In [BHY3] we connected the spectral algebra $A_W$ (respectively $A_W$) with $A_V$ (respectively $A_V$).

**Proposition 1.5** (i) The Darboux transformations preserve the rank of the spectral algebras, i.e. if $W$ is a Darboux transformation of $V$ then $\text{rank} A_W = \text{rank} A_V$.

(ii) If $A_V = \mathbb{C}[L_V]$, $\text{ord} L_V = N$ then

\[ A_W = \{ u \in \mathbb{C}[z^N] \mid u(L_V) \text{Ker} P \subset \text{Ker} P \} \]

(1.29)

and

\[ A_W = \{ Pu(L_V) P^{-1} \mid u \in A_W \}. \]

(1.30)
1.3 Bessel operators, Bessel planes and related objects

Now we define the planes of the Sato’s Grassmannian on which we shall perform the Darboux transformations. For \( \beta \in \mathbb{C}^N \) such that
\[
\sum_{i=1}^{N} \beta_i = \frac{N(N-1)}{2}
\]  \hspace{1cm} (1.31)
we introduce the ordinary differential operator
\[
P_\beta(D_z) = (D_z - \beta_1)(D_z - \beta_2) \cdots (D_z - \beta_N),
\]  \hspace{1cm} (1.32)
where \( D_z = z \partial_z \), and consider the differential equation
\[
P_\beta(D_z) \Phi_\beta(z) = z^N \Phi_\beta(z).
\]  \hspace{1cm} (1.33)

For every sector \( S \) with a center at the irregular singular point \( z = \infty \) and an angle less than \( 2\pi \) the equation (1.33) has a solution \( \Phi_\beta \) with an asymptotics
\[
\Phi_\beta(z) \sim \Psi_\beta(z) = e^{z^2} \left( 1 + \sum_{k=1}^{\infty} a_k(\beta) z^{-k} \right)
\]  \hspace{1cm} (1.34)
for \( |z| \to \infty, z \in S \) (see e.g. [Wa]). Here \( a_k(\beta) \) are symmetric polynomials in \( \beta_i \).
The function \( \Phi_\beta(z) \) can be taken to be (up to a rescaling) the Meijer’s G-function
\[
\Phi_\beta(z) = G_{0N}^{N0} \left( \frac{-z}{N} \right)^N \left| \frac{1}{N} \beta \right).
\]  \hspace{1cm} (1.35)
– see [BE], §5.3.

The next definition is fundamental for the present paper.

**Definition 1.6** Bessel wave function is called the function \( \Psi_\beta(x, z) = \Psi_\beta(xz) \) (cf. [F, Z]). The Bessel operator \( L_\beta \) is defined as
\[
L_\beta(x, \partial_x) = x^{-N} P_\beta(D_x).
\]  \hspace{1cm} (1.36)
A Bessel wave function \( \Psi_\beta \) defines a plane \( V_\beta \in Gr \) (called Bessel plane) by the standard procedure:
\[
V_\beta = \text{span}\{ \partial_x^k \Psi_\beta(x, z) | x = 1 \}.
\]
(In fact \( \Psi_{V_\beta}(x, z) = e^{-z} \Psi_\beta(x+1, z) \), cf. Remark [13]; we took \( x_0 = 1 \) because \( \Psi_\beta(x, z) \) is singular at \( x = 0 \), arbitrary \( x_0 \neq 0 \) will do.)

Because \( \Psi_\beta(x, z) \) depends only on \( xz \), i.e.
\[
D_z \Psi_\beta(x, z) = D_x \Psi_\beta(x, z),
\]  \hspace{1cm} (1.37)
it gives the simplest solution to the bispectral problem:
\[
L_\beta(x, \partial_x) \Psi_\beta(x, z) = z^N \Psi_\beta(x, z),
\]  \hspace{1cm} (1.38)
\[
L_\beta(z, \partial_z) \Psi_\beta(x, z) = x^N \Psi_\beta(x, z).
\]  \hspace{1cm} (1.39)
In this subsection, following G. Wilson [W1], we define several involutions on points of Sato’s Grassmannian and on related objects – wave functions and wave operators. Besides the general properties of the involutions taken from [W1], we specify their action on Bessel planes.

Introduce after [DJKM] the non-degenerate form in $V$ (realized as the space of formal Laurent series in $z^{-1}$)

$$B(f, g) = -\text{Res}_\infty f(z)g(-z) \, dz, \quad f, g \in V.$$  \hfill (1.40)

If $V \in Gr$ is a plane, define $aV \in Gr$ to be the plane orthogonal to $V$ with respect to the form $B(., .)$, to wit

$$aV = \{g(z) \mid B(f, g) = 0, \text{ for all } f \in V\}. \hfill (1.40)$$

Obviously, $a(aV) = V$, i.e. the map $a$ is an involution. Following [W1], we call it the adjoint involution. On the wave operator $K_W$ (1.2) the involution $a$ acts as [W1]

$$K_{aV} = (K_V^*)^{-1}, \hfill (1.41)$$

where $* \text{ is the formal conjugation on pseudo-differential operators, i.e. the antiautomorphism defined by } \partial_x^* = -\partial_x, \quad x^* = x.$ For our purposes the most important property of the involution $a$ is that it inverses inclusions, i.e.:

$$\text{if } W \subset V, \text{ then } aW \supset aV. \hfill (1.42)$$

The following proposition will be used in the description of the action of $a$ on Darboux transformations. Its simple proof is similar to that of Corollary 7.7 from [W1].

**Proposition 1.7** (i) If $\Psi_W(x, z) = \frac{1}{g(z)}P(x, \partial_x)\Psi_V(x, z)$, then

$$\Psi_{aV}(x, z) = \frac{1}{\tilde{g}(z)}P^*(x, \partial_x)\Psi_{aW}(x, z) \hfill (1.43)$$

where $\tilde{g}(z) = g(-z)$.

(ii) [W1] Let $A_V$ be the algebra of operators (1.10). Then $A_{aV}$ consists of the conjugated operators of $A_V$.

**Proof.** We have $\Psi_W(x, z) = g^{-1}(z)P\Psi_V(x, z) = g^{-1}(z)PK_Ve^{xz} = PK_Vg^{-1}(\partial_x)e^{xz}$ which implies $K_W = PK_Vg^{-1}(\partial_x)$. Applying the involution $a$ we obtain

$$K_{aW} = (K_W^*)^{-1} = (P^*)^{-1}(K_V^*)^{-1}\tilde{g}(\partial_x),$$

yielding $P^*K_{aW} = K_{aV}\tilde{g}(\partial_x)$ and hence (1.43). \hfill \square

The sign involution $s$ [W1] is defined on the wave functions by the property

$$\Psi_{sV}(x, z) = \Psi_V(-x, -z). \hfill (1.44)$$
On the wave operators $K_V(x, \partial_x)$ (1.44) translates into

$$K_{sV}(x, \partial_x) = K_V(-x, -\partial_x).$$

(1.45)

The plane $sV$ is defined by

$$sV = \{f(-z) \mid f(z) \in V\}.$$  

(1.46)

In the subsequent chapters we shall need the action of $a$ and $s$ on the Bessel planes. We describe them in the next proposition.

**Proposition 1.8** The involutions $s$ and $a$ act on Bessel planes $V_\beta$ ($\beta \in \mathbb{C}^N$) as follows

$$sV_\beta = V_\beta,$$

(1.47)

$$aV_\beta = V_{a(\beta)},$$

(1.48)

where $a(\beta) = (N - 1)\delta - \beta$, $\delta = (1, 1, \ldots, 1)$.

**Proof.** We compute the action of the involutions on the corresponding wave functions. Obviously,

$$\Psi_{sV_\beta}(x, z) = \Psi_\beta(-x, -z) = \Psi_\beta(x, z),$$

showing (1.47).

Proposition 1.7, eq. (1.38) and the fact that $L_{a(\beta)}^* = (-1)^N L_{a(\beta)}$ imply $\Psi_{aV_\beta}(x, z) = \gamma(z)\Psi_{V_{a(\beta)}}(x, z)$ for some formal power series $\gamma(z)$ in $z$. To show that $\gamma(z) = 1$ we notice that $\Psi_{aV_\beta}(x, z)$ depends on $xz$. Indeed, (1.37) implies $K_{V_{a(\beta)}}(cx, c^{-1}\partial_x) = K_{V_\beta}(x, \partial_x)$ for all $c \neq 0$ and the same is true for $K_{aV_\beta} = (K_{V_\beta})^{-1}$.  

We end this section by recalling the bispectral involution $b$ which Wilson [W1] introduced for the purpose of the bispectral problem. Contrary to $a$ and $s$, the bispectral involution $b$ is not defined on the entire Grassmannian. Whenever one can define $b$, put

$$\Psi_{bV}(x, z) = \Psi_{V}(z, x),$$

(1.49)

i.e. the involution $b$ interchanges the places of the arguments $x$ and $z$.

A simple example of a point $V \in Gr$ where the involution $b$ is well defined is the Bessel plane $V_\beta$. It immediately follows from the definition that

$$\Psi_\beta(x, z) = \Psi_\beta(z, x), \quad \text{i.e. } bV_\beta = V_\beta.$$ 

In terms of the bispectral involution our approach to the bispectral problem can be formulated geometrically as follows:

Find points $V \in Gr$ such that

1) $g(z)V \subset V$ for some nontrivial polynomial $g(z)$;
2) $bV$ exists and $f(z)bV \subset bV$ for some nontrivial polynomial $f(z)$.

A very important general property of $b$, which we intend to use, is its connection to the other involutions [W1]:

$$ab = bas.$$

(1.50)

For completeness we also point out that the involution $s$ commutes with $a$ and $b$:

$$as = sa, \quad bs = sb.$$  

(1.51)
2 Polynomial Darboux transformations of Bessel wave functions

The main purpose of this section is to introduce the submanifolds (denoted below by $Gr^{(N)}_B$) on which, as we prove in the next section, the bispectral involution $b$ is well-defined, and whose points correspond to bispectral operators. The points of $Gr^{(N)}_B$ are obtained by a version of Bäcklund–Darboux transformation performed on Bessel wave functions (or equivalently, on polynomials $h(L_\beta)$ of Bessel operators). Below we call these transformations polynomial Darboux transformations. Definition 2.5, where this is done and the statement of Theorem 2.7, where we provide an equivalent definition, form the heart of the present section. Definition 2.5 has the advantage to be more natural and to supply an algorithmic procedure for constructing bispectral operators. The second definition (from Theorem 2.7) is more suitable for the proof of our bispectrality theorem in Sect. 3.

The reader who wishes to see as soon as possible the main results of the paper can use the second definition, the one from the statement of Theorem 2.7, skipping its technical proof, which occupies half of the section.

In the first half of this section we describe the kernel of the operator $P$ from Definition 1.4 and (which is equivalent) – the conditions of the type as in [W1], imposed on a Bessel plane, which define the corresponding Darboux transformation. To do so, we first need a description of the kernels of the operators $h(L_\beta)$ which are polynomials of $L_\beta$.

Fix $\beta \in \mathbb{C}^N$ satisfying (1.31) and let $V_\beta$ be the corresponding Bessel plane (see Subsect. 1.3). Throughout this section $W$ will be a Darboux transformation of $V_\beta$ (we shall use the notation of Definition 1.4 with $V = V_\beta$). We shall need a lemma describing the kernel of the operator $h(L_\beta)$ for an arbitrary polynomial $h$.

Lemma 2.1. Let $h(z)$ be a polynomial

$$h(z) = z^{d_0} (z - \lambda_1^N)^{d_1} \cdots (z - \lambda_r^N)^{d_r}, \quad \lambda_i^N \neq \lambda_j^N, \quad \lambda_0 = 0, \quad d_i \geq 0. \quad (2.1)$$

Then we have

(i) $\text{Ker} h(L_\beta) = \bigoplus_{i=0}^r \text{Ker} (L_\beta - \lambda_i^N)^{d_i}$.

(ii) $(L_\beta)^d = L_{\beta^d}$, where

$$\beta^d = (\beta_1, \beta_1 + N, \ldots, \beta_1 + (d - 1)N, \ldots, \beta_N, \ldots, \beta_N + (d - 1)N). \quad (2.2)$$

(iii) If $\{\beta_1, \ldots, \beta_N\} = \{\alpha_1, \ldots, \alpha_1, \ldots, \alpha_s, \ldots, \alpha_s\}$ with distinct $\alpha_1, \ldots, \alpha_s$, then

$$\text{Ker} L_\beta = \text{span} \left\{ x^{\alpha_i}(\ln x)^k \right\}_{1 \leq i \leq s, \ 0 \leq k \leq k_i - 1}.$$

(iv) For $\lambda \neq 0$

$$\text{Ker} (L_\beta - \lambda^N)^d = \text{span} \left\{ \partial_z^k \Psi_\beta(x, z) |_{z = \lambda \varepsilon} \right\}_{0 \leq k \leq d - 1, \ 0 \leq j \leq N - 1},$$

where $\varepsilon = e^{2\pi i/N}$ is an $N$-th root of unity.
The proof being obvious is omitted (cf. Lemma 1.3). □

Let us consider the simplest example of a Darboux transformation. Set
\[
h(z) = z^d, \quad g(z) = z^n, \quad f(z) = z^{dN-n}
\]
and \( \gamma = \beta^d \), i.e.
\[
\gamma_{(k-1)d+j} := \beta_k + (j-1)N, \quad 1 \leq k \leq N, \; 1 \leq j \leq d.
\]
For an \( n \)-element subset \( I \) of \( \{1, \ldots, dN\} \) such that \( \gamma_i \neq \gamma_j \) for \( i \neq j \in I \), we put
\[
\text{Ker} P = \text{span}\{x^n\}_{i \in I}.
\]
Such \( P \) corresponds to a Darboux transformation \( \Psi_I(x, z) \) of \( \Psi_\beta(x, z) \). The following simple fact will be useful in the sequel.

**Lemma 2.2** \( \Psi_I(x, z) \) is again a Bessel wave function:
\[
\Psi_I(x, z) = \Psi_{\gamma+dN\delta_I-n\delta}(x, z).
\]

Here and further we use the vectors \( \delta_I, \delta \) defined by
\[
(\delta_I)_i = \begin{cases} 1, & \text{if } i \in I \\ 0, & \text{if } i \notin I \end{cases}
\]
and
\[
\delta_i = 1 \quad \text{for all } i \in \{1, \ldots, dN\}.
\]

**Proof.** By definition
\[
\Psi_I(x, z) = z^{-n}L_{\gamma_I}\Psi_{\gamma}(x, z),
\]
where \( \gamma_I = \{\gamma_i\}_{i \in I} \). Then \( \Psi_I(x, z) \) is an eigenfunction of the differential operator \( L_{\gamma_I}L_\gamma L_{\gamma_I}^{-1} \), which is straightforwardly computed to be equal to \( L_{\gamma+dN\delta_I-n\delta} \). □

Our next step is to study the spectral algebra \( A_\beta \equiv A_{V_\beta} \) of a Bessel plane \( V_\beta \) (see (1.7, 1.10)).

**Lemma 2.3** If \( L(x, \partial_x)\Psi_\beta(x, z) = u(z)\Psi_\beta(x, z) \) for some operator \( L \in A_\beta \) and some polynomial \( u(z) \in A_\beta \), then \( L \) is a linear combination of Bessel operators \( L_\alpha \), \( \alpha \in \mathbb{C}^k \) such that \( L_\alpha\Psi_\beta(x, z) = z^k\Psi_\beta(x, z) \).

**Proof.** Let \( u(z) = \sum u_k z^k \), \( u_k \neq 0 \), \( 0 \leq k \leq M \) for some \( M \). Then for arbitrary \( c \neq 0 \) we have
\[
u(cz)\Psi(x, z) = u(cz)\Psi(c^{-1}x, cz) = L(c^{-1}x, c\partial_x)\Psi_\beta(x, z).
\]
This implies that \( u(cz) \in A_\beta \) and thus \( z^k \in A_\beta \). On the other hand \( D_x V_\beta \subset V_\beta \) and the compatibility condition is of the form \( (D_x - \alpha_1) \cdots (D_x - \alpha_k)\Psi_\beta(1, z) = z^k\Psi_\beta(1, z) \) which implies \( L_\alpha\Psi_\beta(x, z) = z^k\Psi_\beta(x, z) \). □

Let us introduce the following terminology. We say that \( \beta \in \mathbb{C}^N \) is generic if \( V_\beta \) is not a Darboux transformation of another Bessel plane \( V_{\beta'} \) with \( \beta' \in \mathbb{C}^{N'}, \; N' < N \). The next proposition seems obvious but we do not know a simpler proof.
Proposition 2.4 For a generic $\beta \in \mathbb{C}^N$ we have

$$A_\beta = \mathbb{C}[z^N], \quad A_\beta = \mathbb{C}[L_\beta].$$

(2.10)

Proof. We shall prove that if $\text{rank} A_\beta = r < N$ then there exists $\beta' \in \mathbb{C}^r$ (with $A_{\beta'} = \mathbb{C}[z^r]$) such that $V_\beta$ is a Darboux transformation of $V_{\beta'}$. The main idea is to apply to $V_\beta$ Darboux transformations which lead again to Bessel planes but reduce the order of the operator $L_\beta$. Note that according to Proposition 1.5 they preserve the rank $r$ of the spectral algebra.

Split the set $\beta$ into congruent modNZ classes

$$\beta = \left( \beta^{(1)}_1, \ldots, \beta^{(1)}_{N_1}, \ldots, \beta^{(p)}_1, \ldots, \beta^{(p)}_{N_p} \right)$$

such that

$$\beta^{(i)}_s - \beta^{(j)}_t \not\in N\mathbb{Z} \text{ for } i \neq j \text{ and all } s, t$$

($N = N_1 + \cdots + N_p$).

By a Darboux transformation this $\beta$ can be changed to

$$\beta' = \left( \beta^{(1)}_1, \ldots, \beta^{(1)}_{N_1}, \ldots, \beta^{(p)}_1, \ldots, \beta^{(p)}_{N_p} \right)$$

such that

$$|\text{Re}(\beta^{(i)} - \beta^{(j)})| < N \quad \text{and} \quad \beta^{(i)} \neq \beta^{(j)} \text{ for } i \neq j$$

(2.11) (see Lemma 2.2).

Suppose that $A_{\beta'} \neq \mathbb{C}[z^N]$. Then by Lemma 2.3 there exists a Bessel operator $L_\alpha$ such that

$$L_\alpha \Psi_{\beta'}(x, z) = z^M \Psi_{\beta'}(x, z), \quad \alpha \in \mathbb{C}^M$$

(2.12)

and $L_\alpha \neq L_{\beta'}^k$ for any $k$. It is clear that

$$L_\alpha L_{\beta'} = L_{\beta'} L_\alpha,$$

(2.13)

which is equivalent to

$$\left\{ \alpha_1 + N, \alpha_2 + N, \ldots, \alpha_M + N, \beta^{(1)}_1, \ldots, \beta^{(p)}_1, \ldots, \beta^{(1)}_N, \ldots, \beta^{(p)}_N \right\}$$

$$= \left\{ \beta^{(1)}_1 + M, \ldots, \beta^{(1)}_N + M, \ldots, \beta^{(p)}_1 + M, \ldots, \beta^{(p)}_N + M, \alpha_1, \ldots, \alpha_M \right\}.$$

Now if $M > N$ this imply that $\beta' \subset \alpha$ and therefore there exists a Bessel operator $L_{\alpha'}$ such that

$$L_\alpha = L_{\alpha'} L_{\beta'} \quad \text{and} \quad L_{\alpha'} L_{\beta'} = L_{\beta'} L_{\alpha'}.$$

Repeating the same argument with $\alpha'$, we obtain that there exists $L_\alpha$ satisfying (2.13) with $M < N$. But then (2.13) is equivalent to $V_{\beta'} = V_\alpha$. By Proposition 1.5 $r = \text{rank} A_\beta = \text{rank} A_{\beta'} = \text{rank} A_\alpha$ divides $M$ and $N$. If $V_\alpha = V_{\beta'} = \mathbb{C}[z^r]$ this finishes the proof. Otherwise we can repeat the above argument with $V_\alpha$ instead of $V_{\beta'}$. $\Box$
Now we come to the main purpose of this section: the definition of manifolds of Darboux transformations, which will give solutions to the bispectral problem. To get some insight we shall consider, following Wilson [W1], the geometrical meaning of Darboux transformations, provided by the so-called conditions $C$.

Proposition 2.4 implies that for generic $\beta \in \mathbb{C}^N$ (1.27, 1.28) hold with $V = V_\beta$ and $\text{Ker} P$ is a subspace of $\text{Ker} h(L_\beta)$. Each element $f$ of $\text{Ker} P$ corresponds to a condition $c$ (a linear functional on $V_\beta$), such that

$$f(x) = \langle c, \Psi_\beta(x,z) \rangle,$$  \hspace{1cm} (2.14)

$c$ acts on the variable $z$. These linear functionals form an $n$-dimensional linear space $C$ (space of conditions) where $n = \text{ord} P$. In this terminology the definition of Darboux transformation can be reformulated as

$$W = \frac{1}{g(z)} \left\{ v \in V_\beta \mid \langle c, v \rangle = 0 \text{ for all } c \in C \right\}$$

(see [W1, BHY3]). Following Wilson [W1], we call the condition $c$ supported at $\lambda$ if it is of the form (cf. Lemma 2.1 (iv))

$$c = \sum_k a_k \partial^k_z |_{z=\lambda}$$ \hspace{1cm} (2.15)

(the sum is over $k \in \mathbb{Z}_{\geq 0}$ and only a finite number of $a_k \neq 0$). For Bessel wave functions this definition does not make sense when $\lambda = 0$ (since $\Psi_\beta(x,z)$ has a singularity at $z = 0$ for $N > 1$). In this case we say that $c$ is supported at $z = 0$ if it is of the form (cf. Lemma 2.1 (ii, iii))

$$\langle c, \Psi_\beta(x,z) \rangle = \sum_\alpha \sum_j b_{\alpha j} x^\alpha (\ln x)^j.$$  \hspace{1cm} (2.16)

The sums are over $\alpha \in \bigcup_{i=1}^N \{ \beta_i + N \mathbb{Z}_{\geq 0} \}$ and $0 \leq j \leq \text{mult}(\alpha) - 1$ where $\text{mult}(\alpha)$ is the multiplicity of $\alpha$ in the above union (only a finite number of $b_{\alpha j} \neq 0$). The space of conditions $C$ is called homogeneous iff it has a basis of homogeneous conditions $c$ (i.e. the support of $c$ is a point).

It is easy to see that if $C$ is homogeneous then the spectral curve $\text{Spec} A_W$ is rational and unicursal [W1] (i.e. its singularities can be only cusps) – the condition $c$ supported at $\lambda$ “makes” a cusp at $\lambda$. For rank one algebras rationality and unicursality of $\text{Spec} A_W$ are necessary and sufficient for bispectrality [W1]. For rank $N > 1$ another necessary condition is that $\text{Spec} A_W$ be $\mathbb{Z}_N$-invariant, i.e.

$$A_W \subset \mathbb{C}[z^N].$$  \hspace{1cm} (2.17)

When $W$ is a Darboux transformation of a Bessel plane $V_\beta$, with generic $\beta \in \mathbb{C}^N$, this condition is satisfied because of Propositions 2.4, 1.3. It is natural to demand that the space of conditions $C$ (or equivalently $\text{Ker} P$) also be $\mathbb{Z}_N$-invariant.

The $\mathbb{Z}_N$-invariance of $\text{Ker} P$ simply means that

$$f(x) \in \text{Ker} P \Rightarrow f(\varepsilon x) \in \text{Ker} P, \quad \varepsilon = e^{2\pi i/N}.$$  \hspace{1cm} (2.18)
It is easy to see that $C$ is \textit{homogeneous and $\mathbb{Z}_N$-invariant} iff $\text{Ker}P$ has a basis which is a union of:

(i) Several groups of elements supported at 0 of the form:

$$\partial^l_y \left( \sum_{k=0}^{k_0} \sum_{j=0}^{\text{mult}(\beta_i+kN)-1} b_{kj} x^{\beta_i+kN} y^l \right) \bigg|_{y=\ln x}, \quad 0 \leq l \leq j_0,$$  \hspace{1cm} (2.18)

where $j_0 = \max\{j \mid b_{kj} \neq 0 \text{ for some } k\}$;

(ii) Several groups of elements supported at the points $\varepsilon^i \lambda$ ($0 \leq i \leq N-1, \lambda \neq 0$) of the form:

$$\sum_{k=0}^{k_0} a_k z^{\varepsilon^i \lambda} \partial^k_z \Psi_{\beta}(x, z) \big|_{z=\varepsilon^i \lambda}, \quad 0 \leq i \leq N-1.$$  \hspace{1cm} (2.19)

Instead of (2.19) we can also take

$$\sum_{k=0}^{k_0} a_k D^k_z \Psi_{\beta}(x, z) \big|_{z=\varepsilon^i \lambda}, \quad 0 \leq i \leq N-1.$$  \hspace{1cm} (2.20)

Denote by $n_0$ the number of conditions $c$ supported at 0 (i.e. the number of elements of the form (2.18) in the above basis of $\text{Ker}P$). For $1 \leq j \leq r$ denote by $n_j$ the number of conditions $c$ supported at each of the points $\varepsilon^i \lambda_j$, $0 \leq i \leq N-1$ (i.e. the number of groups of elements of the form (2.19) with $\lambda = \lambda_j$).

We have at last arrived at our fundamental definition.

\textbf{Definition 2.5} We say that the wave function $\Psi_W(x, z)$ is a \textit{polynomial Darboux transformation} of the Bessel wave function $\Psi_{\beta}(x, z)$, $\beta \in \mathbb{C}^N$, iff (1.21) holds (for $V = V_{\beta}$) with $P(x, \partial_x)$ and $g(z)$ satisfying:

(i) The corresponding space of conditions $C$ is homogeneous and $\mathbb{Z}_N$-invariant, or equivalently $\text{Ker}P$ has a basis of the form (2.18, 2.19).

(ii) The polynomial $g(z)$ is given by

$$g(z) = z^{n_0} (z^N - \lambda_1^N) \cdots (z^N - \lambda_r^N)^{n_r}$$  \hspace{1cm} (2.21)

where $n_j$ are the numbers defined above.

We denote the set of planes $W$ satisfying (i), (ii) by $\text{Gr}_B(\beta)$ and put $\text{Gr}_B^{(N)} = \bigcup_{\beta} \text{Gr}_B(\beta), \beta \in \mathbb{C}^N$-generic.

We point out that the form (2.21) of $g(z)$ was introduced for $N = 1$ by Wilson \cite{W}. (Note that $g(z) = z^{n_0} \prod_{j=1}^{r} \prod_{\lambda_j=0}^{N-1} (z - \varepsilon^i \lambda_j)^{n_j}$.) We make this normalization in order that $\Psi_W(x, z) = \Psi_W(z, x)$ be a wave function; for the bispectral problem it is inessential.

\textbf{Definition 2.6} We say that the polynomial Darboux transformation $\Psi_W(x, z)$ of $\Psi_{\beta}(x, z)$ is \textit{monomial} iff

$$g(z) = z^{n_0}$$

(i.e. iff all conditions $c$ are supported at 0). Denote the set of the corresponding planes $W$ by $\text{Gr}_{MB}(\beta)$ and put $\text{Gr}_{MB}^{(N)} = \bigcup_{\beta} \text{Gr}_{MB}(\beta), \beta \in \mathbb{C}^N$-generic.
The next theorem provides another equivalent definition of $Gr_B(\beta)$ and is used essentially in the proof of the bispectrality in the next section.

**Theorem 2.7** The wave function $\Psi_W(x, z)$ is a polynomial Darboux transformation of the Bessel wave function $\Psi_\beta(x, z)$, $\beta \in \mathbb{C}^N$, iff (1.21, 1.22, 1.27, 1.28) hold (for $V = V_\beta$) and

(i) The operator $P$ has the form

$$P(x, \partial_x) = x^{-n} \sum_{k=0}^{n} p_k(x^N)(x \partial_x)^k,$$

(2.22)

where $p_k$ are rational functions, $p_n \equiv 1$.

(ii) There exists the formal limit

$$\lim_{x \to \infty} e^{-xz} \Psi_W(x, z) = 1.$$  

(2.23)

The proof will be split into three lemmas. Before giving it we shall make a few comments.

The rationality of $P$ is always necessary for bispectrality [DG, W1]. Also imposes the $\mathbb{Z}_N$-invariance. The condition (2.23) is necessary in order that $\Psi_{bW}(x, z) = \Psi_W(z, x)$ be a wave function. The limit in (2.23) is formal in the sense that it is taken in the coefficient at any power of $z$ in the formal expansion (1.1) separately, i.e.

$$\lim_{x \to \infty} a_j(x) = 0 \quad \text{for all} \quad j \geq 1.$$  

(2.24)

Our first lemma is similar to Proposition 5.1 ((i) $\Rightarrow$ (ii)) from [W1].

**Lemma 2.8** If $P$ has rational coefficients and is $\mathbb{Z}_N$-invariant (see (2.22)) then the conditions $C$ are homogeneous and $\mathbb{Z}_N$-invariant (see (2.18, 2.19))

**Proof.** If $\text{Ker} P = \text{span}\{f_0, \ldots, f_{n-1}\}$, the second coefficient of $P$ is

$$-\partial_x \log \text{Wr}(f_0, \ldots, f_{n-1})$$

and is rational. Lemma 2.1 implies that $\text{Wr}(f_0, \ldots, f_{n-1})$ is of the form

$$x^\alpha e^{\lambda x} \times \text{(Laurent series in } x^{-1}).$$

In particular each element of $\text{Ker} P$ is a sum of terms of the form

$$e^{\lambda x} \times \text{(Laurent series in } x^{-1}) \text{ or } x^\alpha (\ln x)^k.$$

We order the (finite) set of all such $e^{\lambda x}$ and $x^\alpha (\ln x)^k$ occurring in $\text{Ker} P$. The highest term in $\text{Wr}(f_i)$ is just the Wronskian of the highest terms of the $f_i$. If it vanishes then the highest terms of the $f_i$ are linearly dependent, so by a linear combination we can obtain a new basis with lower highest terms. So we can suppose that the highest term of $\text{Wr}(f_i)$ is non-zero. Repeating the same argument with the lowest term, we shall finally obtain a basis whose elements consist of only one term, i.e. are homogeneous (cf. [W1]).
Because the coefficients of $P$ are rational, (1.13) implies that it does not matter which branch of the functions $x^\alpha (\ln x)^k$ in $\text{Ker} P$ we take for $x \in \mathbb{C}$. Let
\[ \sum_{j=0}^{j_0} f_j(x)(\ln x)^j \in \text{Ker} P \]
with $f_j(x) = \sum_\alpha b_{\alpha j} x^\alpha$. Then \( \sum f_j(x)(\ln x + 2l\pi i)^j \in \text{Ker} P \) for arbitrary $l \in \mathbb{Z}$ and also for $l \in \mathbb{C}$ since it is polynomial in $l$. Taking the derivative with respect to $l$ we obtain that
\[ \sum_{j=0}^{j_0} f_j(x)(\ln x)^j \]
also belongs to $\text{Ker} P$.

On the other hand the $\mathbb{Z}_N$-invariance of $P$ (see (2.17)) implies \( \sum f_j(\varepsilon x)(\ln x + 2\pi i/N)^j \in \text{Ker} P \) and
\[ \sum_{j=0}^{j_0} f_j(\varepsilon x)(\ln x)^j \in \text{Ker} P \]
for $\varepsilon = e^{2\pi i/N}$.

Now it is obvious that $\text{Ker} P$ has a basis of the form (2.18, 2.19).

\[ \textbf{Lemma 2.9} \]
If $\text{Ker} P$ has a basis of the form (2.18, 2.19) then $P$ has rational coefficients and is $\mathbb{Z}_N$-invariant (see (2.22)).

\[ \textbf{Proof.} \]
Consider first the case when the basis of $\text{Ker} P$ is
\[ f_i(x) = \sum_k a_{k j} \partial_z^k \Psi_\beta(\varepsilon^i x, z)|_{z=\lambda}, \quad 0 \leq i \leq N - 1, \quad \lambda \neq 0. \]

We shall show that \( \det(\partial_x^{n_j} f_i(x)) \) is a rational function of $x$ for arbitrary $n_j \in \mathbb{Z}_{\geq 0}$. Using (1.37, 1.38) we can express all derivatives of $\Psi_\beta(x, z)$ (both with respect to $z$ and $x$) only by $\partial_z^k \Psi_\beta(x, z)$, $0 \leq k \leq N - 1$, to obtain
\[ \partial_x^{n_j} f_i(x) = \sum_{k=0}^{N-1} \alpha_{kj}(x, \lambda) \partial_x^k \Psi_\beta(e^i x, \lambda) \]  (2.25)
with rational coefficients $\alpha_{kj}$. Therefore
\[ \det(\partial_x^{n_j} f_i(x)) = \det(\alpha_{kj}(x, \lambda)) \det(\partial_x^k \Psi_\beta(e^i x, \lambda)). \]

But \( \det(\partial_x^k \Psi_\beta(e^i x, \lambda)) = \text{const} \) because the second coefficient of $L_\beta - \lambda$ is 0. If the basis $f_0, \ldots, f_{mN-1}$ of $\text{Ker} P$ contains $m$ groups of the type considered above (i.e. (2.19)) we can represent the matrix
\[ (\partial_x^{n_j} f_i(x))_{0 \leq i,j \leq mN-1}, \quad n_j \in \mathbb{Z}_{\geq 0}, \]
20
in the block-diagonal form

\[
\begin{pmatrix}
W_1 & 0 & \cdots & 0 \\
0 & W_2 & \cdots & 0 \\
\vdots & & \ddots & \vdots \\
0 & 0 & \cdots & W_m
\end{pmatrix}
\]

where each block \(W_s\) has the form already considered above. This can be achieved by columns and rows operations, using the representation \(2.23\).

If in addition there are some groups of elements of the form \((2.18)\), we kill the logarithms by columns operations and then cancel the powers \(x^{s_i}\) from the numerator and the denominator of \((1.13)\). □

**Lemma 2.10** If \(C\) is homogeneous and \(h, g\) are as in \((2.21), (2.24)\), then \((2.23)\) is satisfied. Conversely, \((2.23)\) implies \((2.21)\).

**Proof.** The second part of the lemma is an obvious consequence of the first one.

For a basis \(\{\Phi_i(x)\}_{0 \leq i \leq dN - 1}\) of \(\text{Ker}L_\beta\) \((d = \deg h)\) we consider the basis of \(\text{Ker}P\)

\[
f_k(x) = \sum_{i=0}^{dN-1} a_{ki} \Phi_i(x), \quad 0 \leq k \leq n - 1.
\]

Formulae \((1.21), (1.13)\) imply

\[
\Psi_W(x, z) = \frac{\text{Wr}(f_0(x), \ldots, f_{n-1}(x), \Psi_\beta(x, z))}{g(z)\text{Wr}(f_0(x), \ldots, f_{n-1}(x))}
\]

\[
= \frac{\sum \det A^I \text{Wr}(\Phi_I(x)) \Psi_I(x, z)}{\sum \det A^I \text{Wr}(\Phi_I(x))}.
\]

The sum is taken over all \(n\)-element subsets

\[
I = \{i_0 < i_1 < \ldots < i_{n-1}\} \subseteq \{0, 1, \ldots, dN - 1\}
\]

and here and further we use the following notation: \(A\) is the matrix from \((2.26)\) and \(A^I = (a_{ki})_{0 \leq k, i \leq n - 1}\) is the corresponding minor of \(A\), \(\Phi_I(x) = \{\Phi_{i_0}(x), \ldots, \Phi_{i_{n-1}}(x)\}\) is the corresponding subset of the basis \(\{\Phi_i(x)\}\) of \(\text{Ker}L_\beta\) and

\[
\Psi_I(x, z) = \frac{\text{Wr}(\Phi_I(x), \Psi_\beta(x, z))}{g(z)\text{Wr}(\Phi_I(x))}
\]

is a Darboux transformation of \(\Psi_\beta(x, z)\) with a basis of \(\text{Ker}P\) \(f_k = \Phi_{i_k}\).

Using \((2.28)\) it is sufficient to prove \((2.23)\) for \(\Psi_I(x, z)\), hence we can take \(\text{Ker}P\) consisting of functions

\[
f_i(x) = \partial_{x^i}^k \Psi_\beta(x, z)|_{z=\lambda_i}, \quad 0 \leq i \leq p - 1,
\]

\[
f_i(x) = x^{\alpha_i} (\ln x)^i, \quad p \leq i \leq n - 1.
\]

We shall consider the case when \(\lambda_i \neq \lambda_j\) for \(i \neq j\). The general case can be reduced to this by taking a limit. In the formula \((2.27)\) we expand the determinants in the last \(n - p\) columns (using the Laplace rule):

\[
\text{Wr}(f, \Psi_\beta) = \sum_{0 \leq s \leq p} \prod_{0 \leq i \leq p-1} \det (\partial_{x^i}^j f_i(x))_{p+1 \leq s \leq n} \cdot \det (\partial_{x^i}^j f_i(x))_{0 \leq i \leq p-1}.
\]
\[
\text{Wr}(f) = \sum_{s,i} \pm \det \left( \partial^j_{x} f_i(x) \right)_{0 \leq s, i \leq p-1} \cdot \det \left( \partial^j_{x} f_i(x) \right)_{p \leq s, i \leq n-1},
\]

where the sums are over the permutations \((j_0, \ldots, j_n)\) (resp. \((j_0, \ldots, j_{n-1})\)) of \((0, \ldots, n)\) (resp. \((0, \ldots, n-1)\)) such that \(j_0 < \ldots < j_p\) and \(j_{p+1} < \ldots < j_n\) (resp. \(j_0 < \ldots < j_{p-1}\) and \(j_p < \ldots < j_{n-1}\)). We extract the terms with the highest power of \(x\) in the numerator and in the denominator of (2.27). Obviously,

\[
\det \left( \partial^j_{x} f_i(x) \right)_{p \leq i \leq n-1} = \text{const} \cdot x^{\sum_{i=p}^{n-1} \alpha_i - \sum_{i=1}^{p} \alpha_i} \cdot R_J(\ln x)
\]

for some polynomials \(R_J(\ln x)\) (\(J\) is the permutation \((j_s)\)). On the other hand for \(0 \leq i \leq p-1\)

\[
\partial^j_{x} f_i(x) = \partial^j_{x} \partial^k_{z} \Psi_\beta(x,z)|_{z=\lambda_i} = x^k_i e^{\lambda_i x} \left( \lambda_i^j + O(x^{-1}) \right)
\]

and

\[
\partial^j_{x} \Psi_\beta(x,z) = e^{xz} \left( z^j + O(x^{-1}) \right).
\]

Now it is easy to see that the leading terms are obtained for the permutations

\[
(n-p, n-p+1, \ldots, n, 0, 1, \ldots, n-p-1),
\]

respectively

\[
(n-p, n-p+1, \ldots, n-1, 0, 1, \ldots, n-p-1).
\]

Substituting (2.34, 2.35, 2.36) in (2.32, 2.33) and canceling the determinant (2.34) for \(J = (0,1,\ldots, n-p)\), we derive that

\[
\lim_{x \to \infty} e^{-xz} P(x, \partial_x) \Psi_\beta(x,z)
\]

is a fraction of two van der Monde determinants and therefore is equal to \(g(z)\). \(\square\)

## 3 Bispectrality of polynomial Darboux transformations

In this section we prove the main result of the paper, Theorem 3.3, claiming that polynomial Darboux transformations (see Definition 2.5), performed on Bessel operators, produce bispectral operators. On its hand Theorem 3.3 is an almost obvious consequence of Theorem 3.2 in which we prove that the bispectral involution is well-defined on the submanifolds \(G_{TB}(\beta)\) and maps them into themselves. The importance of Theorem 3.3 is not only to provide a proof of our main result (Theorem 3.3) but also to enlighten the bispectral involution. Its proof uses only the definition of polynomial Darboux transformation from Theorem 2.7 (i.e. it does not use Definition 2.5). On the other hand, the proof is completely constructive and together with Definition 2.5 it provides an algorithmic procedure to compute bispectral wave functions and the corresponding bispectral operators. This procedure is described at the end of the section. Many examples computed by making use of it are presented in Sect. 5.
Let $V_\beta$ be a Bessel plane for a generic $\beta \in \mathbb{C}^N$ (i.e. $V_\beta$ is not a Darboux transformation of $V_{\beta'}$ with $\beta' \in \mathbb{C}^N$, $N' < N$). In this section $W$ will be a polynomial Darboux transformation of $V_\beta$, i.e.

$$W \in Gr_B(\beta).$$

We use the notation from (1.21, 1.22) with $V = V_\beta$. In the next proposition we show that the manifold of polynomial Darboux transformations is preserved by the involutions $a$ and $s$ (introduced in Subsect. 1.4).

**Proposition 3.1** If $W \in Gr_B(\beta)$, then

(i) $sW \in Gr_B(\beta)$;

(ii) $aW \in Gr_B(a(\beta))$, where $a(\beta) = (N - 1)\delta - \beta$, $\delta = (1, 1, \ldots, 1)$.

**Proof.** First recall that (Proposition 1.8) $sV_\beta = V_\beta$ and $aV_\beta = V_{a(\beta)}$. We shall study the action of the involutions on $\Psi_W(x,z)$ and check that the conditions of Theorem 2.7 are satisfied.

(i) is trivial because

$$\Psi_{sW}(x,z) = \Psi_W(-x, -z) = \frac{1}{g(-z)}P(-x, -\partial_x)\Psi_\beta(x,z).$$

To prove (ii) we note that the $\mathbb{Z}_N$-homogeneity of $P$ (see (2.22)) is equivalent to

$$P(\varepsilon x, \varepsilon^{-1}\partial_x) = \varepsilon^{-n}P(x, \partial_x),$$

for $n = \text{ord}P$, $\varepsilon = e^{2\pi i/N}$. It follows from (1.28) that the operator $Q$ (from (1.22)) has the same property and also that $Q = h(L_\beta)P^{-1}$ has rational coefficients. Proposition 1.7 implies that $\Psi_{aW}$ is a Darboux transformation of $\Psi_{a(\beta)}$ with

$$\Psi_{aW}(x,z) = \frac{1}{g(z)}Q^*(x, \partial_x)\Psi_{a(\beta)}(x,z).$$

Obviously, $Q^*$ also satisfies (3.1). To check (2.28), we set

$$K_W = 1 + \sum_{j=1}^{\infty} a_j(x)\partial_x^{-j}$$

(see (1.1, 1.2)). Recalling that

$$K_W' = 1 + \sum_{j=1}^{\infty} (-\partial_x)^{-j}a_j(x)$$

and

$$K_{aW} = 1 + \sum_{j=1}^{\infty} b_j(x)\partial_x^{-j} = (K_W^*)^{-1},$$

we compute the coefficients $b_j(x)$ inductively and find that all of them are polynomials in $a_j(x)$ and their derivatives. But by Theorem 2.7 all $a_j(x)$ are rational
functions of \( x \) and \( \lim_{x \to \infty} a_j(x) = 0 \), which leads to \( \lim_{x \to \infty} b_j(x) = 0 \) for all \( j \geq 1 \). This proves (2.23) for \( aW \) (cf. (2.24)). \( \square \)

Proposition 3.1 shows that the involutions \( a \) and \( s \) preserve \( Gr_B^{(N)} \). The central result of the present paper is that the bispectral involution \( b \) has the same property. It immediately implies that wave functions \( \Psi_W \) with \( W \in Gr_B^{(N)} \) give solutions to the bispectral problem. Our next theorem addresses this issue.

**Theorem 3.2** If \( W \in Gr_B(\beta) \) then \( bW \) exists and \( bW \in Gr_B(\beta) \).

**Proof.** Before proving the existence of \( bW \), we shall find an analog of (1.21) for \( \Psi_{bW}(x, z) = \Psi_W(z, x) \), i.e. we shall show the existence of an operator \( P_b(x, \partial_x) \) and a polynomial \( g_b(z) \) such that

\[
\Psi_{bW}(x, z) = \frac{1}{g_b(z)} P_b(x, \partial_x) \Psi_{\beta}(x, z).
\]  

(3.2)

From (2.22) it follows that the operator \( P \) can be written as

\[
P(x, \partial_x) = \frac{1}{x^n P_n(x^N)} \sum_{k=0}^{n} p_k(x^N)(x \partial_x)^k,
\]  

(3.3)

where now \( p_k(x^N) \) are polynomials. Use (1.37–1.39) to obtain

\[
\Psi_W(x, z) = \frac{1}{x^n P_n(x^N) g(z)} \sum k p_k(x^N)(x \partial_x)^k \Psi_{\beta}(x, z)
\]  

\[
= \frac{1}{x^n P_n(x^N) g(z)} \sum (z \partial_z)^k p_k(L_\beta(z, \partial_z)) \Psi_{\beta}(x, z).
\]

This implies (3.2) with

\[
P_b(x, \partial_x) = \frac{1}{g(x)} \sum_{k=0}^{n} (x \partial_x)^k p_k(L_\beta(x, \partial_x)),
\]  

(3.4)

\[
g_b(z) = z^n p_n(z^N).
\]  

(3.5)

Now we can prove the existence of \( bW \), i.e. that \( \Psi_{bW}(x, z) \) is a wave function (see (1.1)). Indeed, using (3.2) we can differentiate the formal expansion (1.34) of \( \Psi_{\beta}(x, z) = \Psi_{\beta}(xz) \); expanding \( g_b^{-1}(z) \) at \( z = \infty \) we obtain

\[
\Psi_{bW}(x, z) = e^{xz} \sum_{k \geq k_0} b_k(x) z^{-k}
\]

for some finite \( k_0 \). Note that the coefficients \( b_k(x) \) are rational. On the other hand

\[
\Psi_{bW}(x, z) = \Psi_W(z, x) = e^{xz} \sum_{j \geq 0} a_j(z) x^{-j}
\]

with rational \( a_j(z) \) such that (see (2.24))

\[
\lim_{z \to \infty} a_j(z) = 0, \ j \geq 1; \ a_0(z) \equiv 1.
\]  

(3.6)
These two (formal) expansions of $\Psi_{bW}(x, z)$ are connected by

$$a_j(z) = \sum_{k \geq k_0} b_{kj} z^{-k},$$

where

$$b_k(x) = \sum_j b_{kj} x^{-j}, \quad b_{kj} = 0 \text{ for } j < 0.$$  

Now (3.6) implies

$$b_{kj} = 0 \text{ for } k < 0, j \geq 1.$$  

This shows that

$$\Psi_{bW}(x, z) = e^{xz} \left( 1 + \sum_{k \geq 1} b_k(x) z^{-k} \right)$$

is a wave function. It is clear that it satisfies (2.23) as well.

To show an analog of (1.22), i.e. that

$$\Psi_{\beta}(x, z) = f_{b}(z) Q_{b}(x, \partial_x) \Psi_{bW}(x, z) \quad (3.7)$$

with an operator $Q_{b}$ and a polynomial $f_{b}$, we shall use the above proven identity (3.2) with $a_{W}$ instead of $W$. It follows from Proposition 1.7 that

$$\Psi_{a_{W}}(x, z) = f(z) \sum_{s=0}^{m} \overline{q}_s(x^N)(x \partial_x)^s \Psi_{a_{(\beta)}}(x, z). \quad (3.8)$$

Proposition 3.1 and Theorem 2.7 (i) allow us to present $Q^*(-x, -\partial_x)$ in the form

$$Q^*(-x, -\partial_x) = \frac{1}{x^m \overline{q}_m(x^N)} \sum_{s=0}^{m} \overline{q}_s(x^N)(x \partial_x)^s$$

with polynomials $\overline{q}_s(x^N)$. Then

$$\Psi_{ba_{W}}(x, z) = \frac{1}{f(z) x^m \overline{q}_m(z^N)} \sum_{s=0}^{m} (x \partial_x)^s \overline{q}_s(L_{a_{(\beta)}}(x, \partial_x)) \Psi_{a_{(\beta)}}(x, z). \quad (3.10)$$

The identity $ab = bas \ [W1]$ and Proposition 1.7 now lead to (3.7) with

$$Q_{b}(x, \partial_x) = \left( \frac{1}{f(x)} \sum_{s=0}^{m} (x \partial_x)^s \overline{q}_s(L_{a_{(\beta)}}(x, \partial_x)) \right)^*$$

$$= \sum_{s=0}^{m} \overline{q}_s \left( (-1)^N L_{\beta}(x, \partial_x) \right) (-x \partial_x - 1)^s \frac{1}{f(x)} \quad (3.11)$$

and

$$f_{b}(z) = (-z)^m \overline{q}_m \left( (-z)^N \right). \quad (3.12)$$

From (2.21) and (3.4) it is obvious that $P_{b}$ is $\mathbb{Z}_N$-homogeneous. This completes the proof of Theorem 3.2. \hfill $\square$

An immediate corollary is the following result, which we state as a theorem because of its fundamental character.
Theorem 3.3 If \( W \in \text{Gr}^{(N)}_B \) then the wave function \( \Psi_W(x,z) \) solves the bispectral problem, i.e. there exist operators \( L(x, \partial_x) \) and \( \Lambda(z, \partial_z) \) such that

\[
L(x, \partial_x)\Psi_W(x,z) = h(z^N)\Psi_W(x,z), \quad (3.13)
\]
\[
\Lambda(z, \partial_z)\Psi_W(x,z) = \Theta(x^N)\Psi_W(x,z), \quad (3.14)
\]

Moreover,

\[
\text{rank}A_W = \text{rank}A_{bW} = N. \quad (3.15)
\]

Proof. \((3.13, 3.14)\) follow from \((1.21, 1.22, 3.2, 3.7)\) if we set

\[
L(x, \partial_x) = P(x, \partial_x)Q(x, \partial_x), \quad h(z^N) = f(z)g(z); \quad (3.16)
\]
\[
\Lambda(z, \partial_z) = P_b(z, \partial_z)Q_b(z, \partial_z), \quad \Theta(x^N) = f_b(x)g_b(x). \quad (3.17)
\]

The eq. \((3.15)\) follows from Propositions \((1.5 \text{ (i)} \text{ and } 2.4)\). \(\blacksquare\)

Example 3.4 All bispectral algebras of rank 1 are polynomial Darboux transformations of the plane \( H_+ = \{z^k\}_{k \geq 0} \) (see \([W1]\)). This corresponds to the \( N = 1 \) Bessel with

\[
\beta = (0), \quad L_0 = \partial_x, \quad V_0 = H_+ = \{z^k\}_{k \geq 0}, \quad \psi_0(x,z) = e^{xz}.
\]

Every linear functional on \( H_+ \) is a linear combination of

\[
e(k, \lambda) = \partial_x^k |_{x=\lambda}
\]

and \( h(L_0) = h(\partial_x) \) is an operator with constant coefficients. The “adelic Grassmannian” \( \text{Gr}^{ad} \), introduced by Wilson \([W1]\), coincides with \( \text{Gr}_B((0)) = \text{Gr}_B^{(1)} \). In our terminology the result of \([W1]\) can be reformulated as follows.

All bispectral operators belonging to rank one bispectral algebras are polynomial Darboux transformations of operators with constant coefficients. \(\blacksquare\)

Remark 3.5 The eigenfunction \( \Psi_W(x,z) \) from eq. \((1.21)\) is a formal series. Let \( \Phi_\beta(x,z) = \Phi_\beta(xz) \), where \( \Phi_\beta(z) \) is the Meijer’s \( G \)-function \((1.33)\) (or any convergent solution of \((1.33)\) in arbitrary domain) and set

\[
\Phi_W(x,z) = \frac{1}{g(z)}P(x, \partial_x)\Phi_\beta(x,z). \quad (3.18)
\]

Then

\[
\Phi_W(x,z) = \frac{1}{g_b(x)}P_b(z, \partial_z)\Phi_\beta(x,z) \quad (3.19)
\]

because of \((1.33)\) and \( x\partial_x\Phi_\beta(x,z) = z\partial_z\Phi_\beta(x,z) \). The equations \(QP = h(L_\beta) \) and \( Q_bP_b = \Theta(L_\beta) \) imply

\[
\Phi_\beta(x,z) = \frac{1}{f(z)}Q(x, \partial_x)\Phi_\beta(x,z), \quad (3.20)
\]
\[
\Phi_\beta(x,z) = \frac{1}{f_b(x)}Q_b(z, \partial_z)\Phi_W(x,z). \quad (3.21)
\]
So, we proved that $\Phi_W(x, z)$ is a convergent bispectral eigenfunction of the same operators $L(x, \partial_x)$ and $\Lambda(z, \partial_z)$ as $\Psi_W(x, z)$. The involutions $a$, $s$ and $b$ can be defined on the manifold of “convergent” polynomial Darboux transformations (1.18) by the equations (1.43) [1.44] [1.49] in which $\Psi$ is replaced by $\Phi$ and they preserve it (Proposition 1.7 (i) now becomes a definition). The validity of the equation $ab = bas$ in the “convergent” case is a consequence of that in the “formal” one (see the proof of Theorem 3.2). The rationality of the coefficients of the operator $P(x, \partial_x)$ implies that its kernel has one and the same form (see eqs. (2.18, 2.19)) in $\Psi$- and in $\Phi$-bases. 

It is not difficult to provide an explicit algorithm for producing bispectral pairs $L(x, \partial_x)$, $\Lambda(z, \partial_z)$. Although obvious we have collected the steps of this algorithm as they are scattered in the present and the previous sections.

**Step 1.** Choose an arbitrary set of conditions based in some points $\lambda_0 = 0, \lambda_1, \ldots, \lambda_r$ of the form (2.18, 2.19), i.e. a basis of $\text{Ker} P$. The proof of Lemma 2.9 provides an explicit computation of the coefficients of $P$ in terms of the coefficients $a_k$, $b_k$ in $\text{Ker} P$. The polynomial $g(z)$ is given by Definition 2.3 (ii).

**Step 2.** Take $h(z^N) = z^{d_0}N \prod_{j=1}^{r} \left(z^N - \lambda_j^N\right)^{d_j}$ with high enough powers $d_0, \ldots, d_r$ such that $\text{Ker} P \subset \text{Ker} h(L_\beta)$ (cf. Lemma 2.1). The minimal such $d_j$’s can be computed as follows.

(i) For a condition, supported at 0, of the form (2.18) set $j(k) = \max\{j | b_{kj} \neq 0\}$, $0 \leq k \leq k_0$. Let $\beta_k + kN = \beta_j + p_jN$ for $0 \leq s \leq \text{mult}(\beta_k + kN) - 1$ with $0 \leq p_0 \leq \ldots \leq p_{\text{mult}(\beta_k + kN)-1}$ and $i_s \neq i_t$ for $s \neq t$. Then set

$$d_0 = 1 + \max p_{j(k)},$$

the maximum is over all $k$ and all conditions of the form (2.18).

(ii) For a condition, supported at $\lambda_j \neq 0$, of the form (2.19) let $k_0 = \max\{k | a_k \neq 0\}$. Then set

$$d_j = 1 + \max k_0,$$

the maximum is over all conditions of the form (2.19) supported at $\lambda_j$.

Then put $f(z) = h(z^N)/g(z)$.

**Step 3.** Find the coefficients of the operator $Q(x, \partial_x)$ recursively out of the equation $Q(x, \partial_x)P(x, \partial_x) = h(L_\beta(x, \partial_x))$. Then $L(x, \partial_x) = P(x, \partial_x)Q(x, \partial_x)$. A lower order operator $L$ can be constructed using Proposition [1.3, i.e. find $u(L_\beta)$ such that $\text{Ker} P$ is invariant under $u(L_\beta)$ and then $L$ out of the equation $LP = Pu(L_\beta)$.

**Step 4.** Compute by (3.4) $P_b(x, \partial_x)$ and by (3.5) $g_b(z)$. Also (3.11) and (3.12) give $Q_b(x, \partial_x)$ and $f_b(z)$. All expressions are explicit in terms of the coefficients of the operators $P$ and $Q$. Then $\Lambda(z, \partial_z) = P_b(z, \partial_z)Q_b(z, \partial_z)$ and $\Theta(x) = f_b(x)g_b(x)$.

4 \hspace{1cm} \textbf{Polynomial Darboux transformations of Airy planes}

This section contains analogs of the results from Sections 2 and 3 but here the building blocks are (generalized) Airy operators (see [KS, Dij]) instead of Bessel
ones. There is a minor difference in the organization of the present section compared to that of Sections 2 and 3. Here we give the definition of polynomial Darboux transformations on Airy wave functions (see Definitions 4.2, 4.3) in the spirit of the one provided by Theorem 2.7. Then we prove our main result Theorem 4.5 (which is an analog of Theorem 3.2). As in Sect. 2, it automatically implies bispectrality of the polynomial Darboux transformations. At the end, in Proposition 4.9 we show that Definition 4.3 is equivalent to a second one (analog of Definition 2.5) in terms of conditions on Airy planes. This is again important for algorithmic computations, some of which are presented in the next section.

First we recall the definition of (generalized higher) Airy functions. For \( \alpha = (\alpha_0, \alpha_2, \alpha_3, \ldots, \alpha_{N-1}) \in \mathbb{C}^{N-1}, \alpha_0 \neq 0 \), consider the Airy operator

\[
L_\alpha(x, \partial_x) = \partial_x^N - \alpha_0 x + \sum_{i=2}^{N-1} \alpha_i \partial_x^{N-i} \equiv P_{\alpha'}(\partial_x) - \alpha_0 x
\]

(4.1)

where \( \alpha' = (\alpha_2, \alpha_3, \ldots, \alpha_{N-1}) \). The Airy equation is

\[
L_\alpha(x, \partial_x)\Phi(x) = 0, \quad \text{i.e.} \quad P_{\alpha'}(\partial_x)\Phi(x) = \alpha_0 x \Phi(x).
\]

(4.2)

**Example 4.1** When \( \alpha_0 = 1 \), \( \alpha' = 0 \) eq. (4.2) becomes the classical higher Airy equation (cf. [KS])

\[
\partial_x^N \Phi(x) = x \Phi(x).
\]

(4.3)

In every sector \( S \) with a center at \( x = \infty \) and an angle less than \( N\pi/(N+1) \), it has a solution with an asymptotics of the form (see e.g. [Wa])

\[
\Phi(x) \sim x^{-\frac{N-1}{2N}} e^{\frac{N+1}{N} x} \left( 1 + \sum_{i=1}^{\infty} a_i x^{-i/N} \right), \quad |x| \to \infty, \ x \in S.
\]

(4.4)

\[ \square \]

Similarly, in each sector \( S \) as in Example 4.1 eq. (4.2) has a solution with an asymptotics of the form

\[
\Phi(x) \sim \Psi_\alpha(x) := x^{d/N} e^{Q(x^{1/N})} \left( 1 + \sum_{i=1}^{\infty} a_i x^{-i/N} \right), \quad |x| \to \infty, \ x \in S
\]

(4.5)

for some \( d \in \mathbb{C} \) and a polynomial \( Q(x) \) of degree \( N+1 \) with leading coefficient \( \mu_0 \mu_{N+1} x^{N+1} \), where \( \alpha_0 = \mu_0^N \). The solution \( \Phi \) is by no means unique, but \( d, Q \) and all \( a_i \) are uniquely determined and do not depend on \( S \). In the sequel we shall deal only with \( \Psi_\alpha \), which is a formal solution of eq. (4.2).

**Definition 4.2** For each \( \alpha \in \mathbb{C}^{N-1} \) we call an Airy wave function the following function

\[
\psi_\alpha(x, z) := \mu_0^d z^{-d} e^{-Q(z^{-1})} \Psi_\alpha(x, z),
\]

(4.6)

where

\[
\Psi_\alpha(x, z) := \Psi_\alpha(\alpha_0^{-1} z^N + x).
\]
It is easy to see that $\psi_\alpha$ is indeed a wave function if we expand $\Psi_\alpha(\alpha_0^{-1}z^N + x)$ at $x = 0$:

$$(\alpha_0^{-1}z^N + x)^{-i/N} = \sum_{k \geq 0} \left(\frac{-i/N}{k}\right)(\mu_0^{-1}z)^{-i-kN}x^k$$  \hspace{1cm} (4.7)

(we shall always use $\mu_0$ as an $N$-th root of $\alpha_0$).

The plane in Sato’s Grassmannian corresponding to $\psi_\alpha(x, z)$ will be called an Airy plane and will be denoted by $V_\alpha$.

Obviously, $\Psi_\alpha(x, z)$ solves the bispectral problem

$$L_\alpha(x, \partial_x)\Psi_\alpha(x, z) = z^N\Psi_\alpha(x, z)$$  \hspace{1cm} (4.8)

$$L_\alpha(\alpha_0^{-1}z^N, \partial_{\alpha_0^{-1}z^N})\Psi_\alpha(x, z) = \alpha_0 x \Psi_\alpha(x, z)$$  \hspace{1cm} (4.9)

because

$$\partial_x \Psi_\alpha(x, z) = \partial_{\alpha_0^{-1}z^N} \Psi_\alpha(x, z).$$  \hspace{1cm} (4.10)

It is clear that $\psi_\alpha$ satisfies (4.8) and analogs of (4.9, 4.10) obtained by conjugating by $z^{-d}e^{-Q(\mu_0^{-1}z)}$. (Up to this conjugation $\Psi_\alpha$ and $\psi_\alpha$ give one and the same solution to the bispectral problem.)

We shall define polynomial Darboux transformations of Airy planes as in the Bessel case (see Definition 2.5 and Theorem 2.7). Before that we shall define a bispectral involution $b_1$ on them. Note that the involution $b$ from [W1] (see Subsect. 1.4) is not well defined on $V_\alpha$ (i.e. $\psi_\alpha(z, x)$ is not a wave function). The properties of $b$ we would like $b_1$ to have, are:

1) it has to interchange the roles of $x$ and $z$;
2) it has to preserve Airy planes.

Therefore we define

$$b_1 \Psi_\alpha(x, z) := \Psi_\alpha(x, z) = \Psi_\alpha(\alpha_0^{-1}z^N, \mu_0 x^{1/N}),$$  \hspace{1cm} (4.11)

or equivalently,

$$b_1 \psi_\alpha(x, z) := \psi_\alpha(x, z) = \mu_0^d x^{-d/N}z^{-d}e^{Q(\mu_0 x^{1/N})-Q(\mu_0^{-1}z)}\psi_\alpha(\alpha_0^{-1}z^N, \mu_0 x^{1/N}).$$  \hspace{1cm} (4.12)

For a Darboux transformation $W$ of $V_\alpha$ we define $\psi_{b_1 W}$ and $\Psi_{b_1 W}$ in a similar way. (We still do not know whether $b_1 W \in Gr$, the notation $\psi_{b_1 W}$ is still formal.)

**Definition 4.3** A Darboux transformation $W$ of an Airy plane $V_\alpha$ is called polynomial iff (in the notation of Definition 1.4)

(i) the operator $P$ has rational coefficients;
(ii) $g(z) = g_1(z^N)$, $g_1 \in \mathbb{C}[z]$;
(iii) $\lim_{z \to \infty} e^{-xz} \psi_{b_1 W}(x, z) = 1$.

(The limit is formal and has the same meaning as in [2.23].)

Denote the set of all such $W \in Gr$ by $Gr_A(\alpha)$ and put $Gr_A^{(N)} = \bigcup_{\alpha \in \mathbb{C}^{N-1}} Gr_A(\alpha)$.  \hspace{29} (29)
Remark 4.4 The parts (i) and (ii) of the above definition remain the same if we substitute $\psi_\alpha$ and $\psi_W$ by $\Psi_\alpha$ and $\Psi_W$, where
\[
\psi_W(x, z) := \mu_0^d z^{-d} e^{-Q(\mu_0^{-1}z)}\Psi_W(x, z).
\] (4.13)

The main result of this section is that $Gr_A(\alpha)$ is preserved by the involution $b_1$.

**Theorem 4.5** (i) If $W \in Gr_A(\alpha)$, then $\psi_{b_1W}(x, z)$ is a wave function corresponding to a plane $b_1W \in Gr_A(\alpha)$.

(ii) For $\alpha \in \mathbb{C}^{N-1}$ the spectral algebra $A_{V_\alpha}$ is $\mathbb{C}[L_\alpha]$.

An immediate **corollary** is that the planes $W \in Gr_A(\alpha)$ give solutions to the bispectral problem of rank $N$:

\[
\text{rank} A_W = \text{rank} A_{b_1W} = N.
\]

The **proof of Theorem 4.5** is completely parallel to that of Theorem 3.2. We shall be very brief, indicating only the major differences and the most important steps.

We start with a lemma illuminating the purpose of the constraints (i) and (ii) in Definition 4.3 (cf. (3.2)).

**Lemma 4.6** If $W \in Gr_A(\alpha)$, then
\[
\Psi_{b_1W}(x, z) = \frac{1}{g_b(z)} P_b(x, \partial_x) \Psi_\alpha(x, z),
\] (4.14)

$P_b$ is with rational coefficients and $g_b$ is polynomial in $z^N$.

**Proof.** We compute
\[
\Psi_{b_1W}(x, z) = \Psi_W(\alpha_0^{-1}z^N, \mu_0 x^{1/N})
\]
\[
= \frac{P(\alpha_0^{-1}z^N, \partial_{\alpha_0^{-1}z^N}) \Psi_\alpha(\alpha_0^{-1}z^N, \mu_0 x^{1/N})}{g(\mu_0 x^{1/N})} = \frac{1}{g_b(z)} P_b(x, \partial_x) \Psi_\alpha(x, z),
\]

where if
\[
P(x, \partial_x) = \frac{1}{p_b(x)} \sum_{k=0}^n p_k(x) \partial_x^k, \quad g(z) = g_1(z^N)
\] (4.15)

with polynomials $p_k$ and $g_1$, then (using (4.8, 4.10))
\[
P_b(x, \partial_x) = \frac{1}{g_1(\alpha_0 x)} \sum_{k=0}^n \partial_x^k p_k(\alpha_0^{-1}L_\alpha(x, \partial_x)),
\] (4.16)

\[
g_b(z) = p_n(\alpha_0^{-1}z^N).
\] (4.17)

The proof that $\psi_{b_1W}(x, z)$ is a wave function is the same as in the Bessel case, using the above lemma and the condition (iii) of Definition 4.3. Now the identity $ab = ba$ [\text{W}3] is modified in the following way.
Introduce the maps $p$ and $p^{-1}$ as follows

$$
\Psi_{pW}(x, z) := \Psi_W(\alpha_0^{-1}x^N, \mu_0 z^{1/N})
$$

$$
\Psi_{p^{-1}W}(x, z) := \Psi_W(\mu_0 x^{1/N}, \alpha_0^{-1}z^N).
$$

The notation $pW$, $p^{-1}W$ is formal – these are not planes in $Gr$. But

$$
\Psi_{b_1W}(x, z) = \Psi_{bpW}(x, z) = \Psi_{p^{-1}bW}(x, z)
$$
corresponds to the wave function $\psi_{b_1W}(x, z)$ and to $b_1W \in Gr$.

Multiplying the identity $ab = bas$ on the right by $p$, we obtain

$$
ab_1 = b_1a_1, \quad \text{where } a_1 = p^{-1}asp. \quad (4.18)
$$

Note that for $W \in Gr_A(\alpha)$, $aW, b_1W$ and hence $a_1W$ are planes in $Gr$.

The next lemma gives the action of the involutions on the Airy planes (the proof
is the same as that of Proposition 3.3).

Lemma 4.7 (i) $sV\alpha = V_{s(\alpha)}$, where $s(\alpha) = ((-1)^{N+1}a_0, \alpha_2, -\alpha_3, \ldots, (-1)^{N-1}a_{N-1})$;
(ii) $aV\alpha = a_1V\alpha = V_{a(\alpha)}$, where $a(\alpha) = ((-1)^{N}a_0, \alpha_2, -\alpha_3, \ldots, (-1)^{N-1}a_{N-1})$.

We also need an analog of Proposition 3.4.

Lemma 4.8 If $W \in Gr_A(\alpha)$, then $aW$ and $a_1W$ belong to $Gr_A(a(\alpha))$.

For the proof we need an analog of Proposition 2.7 for $a_1$. A simple computation shows that if

$$
\Psi_V(x, z) = \frac{1}{f(z)}Q(x, \partial_x)\Psi_W(x, z)
$$

for $V, W \in Gr_A(\alpha)$ and

$$
Q(x, \partial_x) = \sum q_k(x)\partial_x^k
$$

then

$$
\Psi_{a_1W}(x, z) = \frac{1}{f(z)}Q^{*1}(x, \partial_x)\Psi_{a_1V}(x, z)
$$

with

$$
Q^{*1} = \sum \left( \partial_x + \frac{1 - N}{N}x^{-1} \right)^k (-1)^{(N-1)k}q_k((-1)^N x).
$$

The rest of the proof is left to the reader. □

The proof of part (i) of Theorem 4.3 is completed exactly as in the Bessel case. For the part (ii), we note that while the Bessel wave functions are “multiplication invariant”, the Airy ones are “translation invariant”. More precisely, for arbitrary $c \in \mathbb{C}$

$$
\Psi_\alpha(x + c, (z^N - \alpha_0 c)^{1/N}) = \Psi_\alpha(\alpha_0^{-1}(z^N - \alpha_0 c) + x + c) = \Psi_\alpha(\alpha_0^{-1}z^N + x) = \Psi_\alpha(x, z)
$$

(expand $(z^N - \alpha_0 c)^{1/N} = \sum_{k \geq 0} \binom{1/N}{k} z^{1-kN}(-\alpha_0 c)^k$). Let $u(z) \in A_\alpha$, $L(x, \partial_x) \in A_\alpha$ and

$$
L(x, \partial_x)\Psi_\alpha(x, z) = u(z)\Psi_\alpha(x, z)
$$
Inserting (4.21) in (4.20), we obtain another formal series of the kernel of the operator \( L \) considered as a formal power series in \( \psi \) wave function but some more explanation is needed. For fixed \( \lambda \) fact, most of the proofs in Sect. 2 are valid in a more general situation. (In fact, most of the proofs in Sect. 2 are valid in a more general situation.)

**Proposition 4.9** The Darboux transformation \( W \) of \( V_\alpha \) is polynomial iff Ker\( \) has a basis of the form

\[
 f_{ij}(x) = \sum_k a_{ki} \partial_z^k \psi_\alpha(x, \varepsilon^j z) \big|_{z = \lambda_i} \\
 = \sum_k a_{ki} \varepsilon^{-jk} \partial_z^k \psi_\alpha(x, z) \big|_{z = \varepsilon^j \lambda_i}, \tag{4.19}
\]

\( 0 \leq j \leq N - 1, 1 \leq i \leq r \) (for some \( r \), \( \lambda_i \neq 0 \)).

(ii) The polynomial \( g(z) \) has the form \( a_{k0} = \) \( \frac{a_{k0} \lambda^N}{(\varepsilon^k \lambda^N)} \) \( \varepsilon^k \lambda^N \), \( 0 \leq k \leq N - 1 \).

The proof of the “if” part is the same as in the Bessel case and will be omitted. (In fact, most of the proofs in Sect. 2 are valid in a more general situation.)

The “only if” part is also similar to the corresponding result in the Bessel case but some more explanation is needed. For fixed \( \lambda \neq 0 \) we shall use representations of the kernel of the operator \( L_\alpha - \lambda^N \) in three different linear spaces of formal power series. First set

\[
 \varphi_\alpha(x, \lambda) = \mu_0^d \lambda^{-d} e^{-Q(\lambda)} \Psi_\alpha(y), \quad y = \alpha_0^{-1} \lambda^N + x, \tag{4.20}
\]

considered as a formal power series in \( y^{-1/N} \) (where \( \Psi_\alpha \) is from \( 4.3 \)). The Airy wave function \( \psi_\alpha(x, \lambda) \) (see \( 4.6 \)) is given by the same formula after expanding \( y^{-1/N} \) at \( x = 0 \) as in \( 4.7 \). The other possibility is to expand \( y^{-1/N} \) at \( x = \infty \):

\[
 (x + \alpha_0^{-1} \lambda^N)^{-i/N} = \sum_{k \geq 0} \left( -\frac{i}{N} k \right) x^{-i/\lambda^N - k} (\alpha_0^{-1} \lambda^N)^k. \tag{4.21}
\]

Inserting \( 4.21 \) in \( 4.20 \), we obtain another formal series \( \chi_\alpha(x, \lambda) \). Denote by \( \varphi_\alpha^{(i)}, \psi_\alpha^{(i)}, \chi_\alpha^{(i)} \) the images of \( \varphi_\alpha, \psi_\alpha, \chi_\alpha \) under the transformations

\[
 y^{1/N} \mapsto \varepsilon^j y^{1/N}, \quad \lambda \mapsto \varepsilon^j \lambda, \quad x^{1/N} \mapsto \varepsilon^j x^{1/N},
\]

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respectively \((\varepsilon = e^{2\pi i/N})\). Then \(\psi^{(j)}_\alpha\) and \(\chi^{(j)}_\alpha\) are obtained by expanding \(\varphi^{(j)}_\alpha\) and in the corresponding spaces of formal series \(\text{Ker}(L_\alpha - \lambda^N)^d\) has bases
\[
\{\partial^k_{x^j} \psi^{(j)}_\alpha\}, \{\partial^k_{x^j} \chi^{(j)}_\alpha\}, \quad 0 \leq k \leq d - 1, \; 0 \leq j \leq N - 1.
\]

Our observation is that if \(\text{Ker}P\) has a basis \(f(x) = \sum_{k,j} a_{ij}^{k} x^i y^j\), then the same formula gives a basis of \(\text{Ker}P\) when \(\psi\)'s are substituted by \(\varphi\)'s or \(\chi\)'s and vice versa. Indeed, this follows from \(1.13\) and the fact that \(P\) has rational coefficients. We complete the proof of Proposition \(4.9\) noting that while \(P\) depends rationally on \(x\), \(\chi^{(j)}_\alpha\) are formal series in \(x^{1/N}\) and the same argument as in the Bessel case gives that \(\text{Ker}P\) has a \(\chi\)-basis of the form \(4.19\). \(\square\)

5 Explicit formulae and examples

In this section we have collected several classes of examples. We wanted at least to include all previously known examples (unless by ignorance we miss some of them) – see \([DG, W1, Z, G3, LP]\). We hope that we have elucidated and unified them.

For monomial transformations we derive formulae expressing the operators \(L\) and \(\Lambda\), solving the bispectral problem, only in terms of the matrix \(A\) and the vector \(\gamma\) (see Proposition \(5.1\) below). This explicit expression for \(\Lambda\) (though possibly of high order) to the best of our knowledge is new even for \(N = 2\) (see \([DG]\)). In other examples we illustrate the properties of the operator of minimal order from a bispectral algebra: when does its order coincide with the rank of the algebra and when this operator is a Darboux transformation of a power of a Bessel operator. We also point out that the classical Bessel potentials \(u(x) = cx^{-2}\) \([DG]\) can produce new solutions of the bispectral problem for any \(c\).

We describe in detail the polynomial Darboux transformations from \((L_\alpha - \lambda^N)^2\) where \(L_\alpha\) is an arbitrary Airy or Bessel operator of order \(N\). We do not want simply to show that our procedure of constructing bispectral operators works but to point out that the involutions \(a\) and \(b\) (\(b_1\) in Airy case) possess some very interesting properties which deserve further study.

5.1 Monomial Darboux transformations of Bessel planes

Let \(\beta \in \mathbb{C}^N\) and \(W \in Gr_{MB}(\beta)\). We use the notation from \(1.21, 1.22\) (with \(V = V_\beta\)) and from \(3.2, 3.7\). When the Darboux transformation is monomial
\[
g(z) = z^n, \quad h(z) = z^d
\]
for some \(n, d\). We shall consider only the case when there are no logarithms in the basis \(2.18\) of \(\text{Ker}P\). The general case can be reduced to this one by taking a limit in all formulae (see Example \(5.2\) below). Now \(\text{Ker}P\) has a basis of the form
\[
f_k(x) = \sum_{i=1}^{dN} a_{ki} x^{\gamma_i}, \quad 0 \leq k \leq n - 1,
\]
such that
\[ \gamma_i - \gamma_j \in N\mathbb{Z} \setminus 0 \quad \text{if} \quad a_{ki}a_{kj} \neq 0, \quad i \neq j, \quad (5.3) \]
where \( \gamma = \beta^d \) is from (2.4).

Let \( A \) be the matrix \((a_{ki})\). We shall use multi-index notation for subsets \( I = \{i_0 < \ldots < i_{n-1}\} \) of \( \{1, \ldots, dN\} \) and \( \delta_I \) from (2.7). We also put \( \gamma_I = \{\gamma_i\}_{i \in I} \),
\[
A^I = (a_{k,i})_{0 \leq k, l \leq n-1}
\]
and
\[
\Delta_I = \prod_{r<s}(\gamma_{i_r} - \gamma_{i_s}).
\]

Let \( I_{\text{min}} \) be the subset of \( \{1, \ldots, dN\} \) with \( n \) elements such that \( \det A^{I_{\text{min}}} \neq 0 \) and \( \sum_{i \in I_{\text{min}}} \gamma_i \) be the minimum of all such sums, and set
\[
p_I = \sum_{i \in I} \gamma_i - \sum_{i \in I_{\text{min}}} \gamma_i.
\]
Eq. (5.3) implies that these numbers are divisible by \( N \). Finally, for a subset \( I \) of \( \{1, \ldots, dN\} \) denote by \( I^0 \) its complement.

In the following proposition we express everything entering (1.21, 1.22, 3.2, 3.7) only in terms of the matrix \( A \) and the vector \( \gamma \). Therefore for each \( A \) and \( \beta \in \mathbb{C}^N \) satisfying (5.3) (with \( \gamma = \beta^d \)) we give an explicit solution to the bispectral problem (cf. (3.16, 3.17)).

**Proposition 5.1** In the above notation the operators and the polynomials from (3.16, 3.17) are given by the following formulae:

(a) \( g(z) = z^n, \)
\[
P = \left( \sum \det A^I \Delta_I x^{p_I} \right)^{-1} \left( \sum \det A^I \Delta_I x^{p_I} L_{\gamma_I} \right).
\]

(b) \( f(z) = z^{dN-n}, \)
\[
Q = \left( \sum \det A^I \Delta_I L_{\gamma_{\beta} - n\delta_{\beta}} x^{p_I} \right) \left( \sum \det A^I \Delta_I x^{p_I} \right)^{-1}.
\]

(c) \( g_b(z) = z^n \sum \det A^I \Delta_I z^{p_I}, \)
\[
P_b = \sum \det A^I \Delta_I L_{\gamma_I} (L_{\beta})^{p_I/N}.
\]

(d) \( f_b(z) = z^{dN-n} \sum \det A^I \Delta_I z^{p_I}, \)
\[
Q_b = \sum \det A^I \Delta_I (L_{\beta})^{p_I/N} L_{\gamma_{\beta} - n\delta_{\beta}}.
\]

**Proof.** Note that (c) and (d) follow from (a) and (b) (see the proof of Theorem 3.2). To prove (a) we note that
\[
\Psi_W(x, z) = \frac{\Psi_W(f_0(x), \ldots, f_{n-1}(x), \Psi_{\beta}(x, z))}{z^n \Psi_W(f_0(x), \ldots, f_{n-1}(x))} = \frac{\sum \det A^I \Psi_W(x^{\gamma_I})}{\sum \det A^I \Psi_W(x^{\gamma_I})}.
\]

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The sum is taken over all \( n \)-element subsets \( I = \{ i_0 < i_1 < \ldots < i_{n-1} \} \subset \{0,1,\ldots,dN-1\} \), \( x^\gamma = \{ x^\gamma_i \}_{i \in I} \) and \( \Psi_I(x,z) \) are the Bessel wave functions (2.6). Using (2.9) and the simple fact

\[
\text{Wr}(x^\gamma) = \Delta_I x^\gamma \sum_{i \in I} \frac{\gamma_i - n(n-1)}{2},
\]  

(5.5)

we obtain (a).

To prove (b) we shall apply the involution \( a \) directly on the tau-function \( \tau_W \) of the plane \( W \). Recall that

\[
\Psi_W(t,z) = e^{\sum_{k=1}^{\infty} t_k z_k \frac{\tau(t-[z^{-1}])}{\tau(t)}},
\]  

(5.6)

where \([z^{-1}]\) is the vector \((z^{-1},z^{-2}/2,\ldots)\). The action of \( a \) is given by

\[
\tau_{aW}(t_1,t_2,\ldots,t_k,\ldots) = \tau_W(t_1,-t_2,\ldots,(-1)^{k-1}t_k,\ldots).
\]

We shall need the formulae [BHY3]

\[
\tau_W(t) = \frac{\sum \det A^I \Delta_I \tau_I(t)}{\sum \det A^I \Delta_I}
\]  

(5.7)

and

\[
\tau_I(x) = \frac{1}{\Delta_I} \text{Wr}(x^\gamma) \tau_\gamma(x)
\]  

(5.8)

where \( \tau_I(t) \) is the tau-function corresponding to the wave function \( \Psi_I(x,z) \) and \( \tau(x) = \tau(x,0,0,\ldots) \). Applying \( a \) to both sides of (5.7) and using (5.6) and (2.6) we obtain

\[
\Psi_{aW}(x,z) = \sum \det A^I \Delta_I \tau_{a(\gamma+dN\delta_I-n\delta)}(x) \Psi_{a(\gamma+dN\delta_I-n\delta)}(x,z) \frac{\sum \det A^I \Delta_I \tau_{a(\gamma+dN\delta_I-n\delta)}(x)}{\sum \det A^I \Delta_I \tau_{a(\gamma+dN\delta_I-n\delta)}(x)}.
\]  

(5.9)

We compute

\[
a(\gamma + dN\delta_I - n\delta) = a(\gamma) + dN\delta_{I_0} - (dN - n)\delta,
\]  

(5.10)

which is a Darboux transformation of \( a(\gamma) \).

The eqs. (5.8) and (5.3) imply

\[
\frac{\tau_I(x)}{\tau_J(x)} = \frac{x^{p_I}}{x^{p_J}}
\]  

(5.11)

To apply (5.11) in (5.9) we have to compute \( p_{I_0} \) but for \( a(\gamma) \) instead of \( \gamma \). It is a simple exercise to see that

\[
p_{I_0}(a(\gamma)) = p_I(\gamma) \equiv p_I.
\]

Using this we obtain

\[
\Psi_{aW}(x,z) = z^{-dN+n} \frac{\sum \det A^I \Delta_I x^{p_I} L(a(\gamma)) \Psi(a(\gamma))(x,z)}{\sum \det A^I \Delta_I x^{p_I} \Psi(a(\gamma))(x,z)}.
\]

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Now Proposition $[\text{L.4}]$ gives (b) because
\[(L_{\beta})^* = (-1)^N L_{a(\beta)} \quad \text{for } \beta \in \mathbb{C}^N\]
and
\[a \left((a(\gamma))_p^o\right) = \gamma_p^o - n \delta_p^o.\]
\[\square\]

In the following example we consider the case when there are logarithms in the basis (2.18) of Ker $P$.

**Example 5.2** Let $d = 2$, $\beta = (1, 1, 1)$, $\gamma = \beta^2 = (1, 1, 1, 4, 4, 4)$ and Ker $P$ has a basis

\[
\begin{align*}
  f_0(x) &= x^4, \\
  f_1(x) &= a_1 x + 2a_2 x^4 \ln x, \\
  f_2(x) &= a_0 x + a_1 x \ln x + a_2 x^4 \ln^2 x.
\end{align*}
\]

Using that $\ln^k x = \partial^k_{\epsilon} x^\epsilon |_{\epsilon = 0}$ we approximate the above functions with

\[
\begin{align*}
  f_0(x, \epsilon) &= x^4, \\
  f_1(x, \epsilon) &= a_1 x^{1+\epsilon} + 2a_2 \epsilon^{-1} (x^{4+\epsilon} - x^4), \\
  f_2(x, \epsilon) &= a_0 x^{1+2\epsilon} + a_1 \epsilon^{-1} (x^{1+2\epsilon} - x^{1+\epsilon}) + a_2 \epsilon^{-2} (x^{4+2\epsilon} - 2x^{4+\epsilon} + x^4).
\end{align*}
\]

Consider the Darboux transformation $W(\epsilon)$ of $V_{\beta(\epsilon)}$, where $\beta(\epsilon) = (1, 1 + \epsilon, 1 + 2\epsilon)$, with a basis of the operator $P(\epsilon)$ consisting of the functions $f_k(x, \epsilon)$. After changing the basis this corresponds to a matrix (cf. (5.2))

\[
A(\epsilon) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \epsilon a_1 & 2a_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \epsilon(\epsilon a_0 + a_1) & a_2 \end{pmatrix}.
\]

We apply (5.7) for $\tau_W(\epsilon)$. To make the limit $\epsilon \to 0$ we note that the numerator and the denominator depend polynomially on $\epsilon$ and that (in the notation of (5.7)) both $\tau_{(2,3,6)}$ and $\tau_{(2,4,5)}$ tend to one and the same Bessel tau-function. So after canceling $\epsilon^3$ and setting $\epsilon = 0$ we obtain that $\tau_W$ is a linear combination of 3 Bessel tau-functions:

\[
\tau_W = \frac{9a_1^2 \tau_{(-2,1,1,4,4,7)} + 18a_2(a_0 - a_1)\tau_{(-2,2,1,4,7,7)} + 4a_2^2 \tau_{(-2,-2,-2,7,7,7)}}{9a_1^2 + 18a_2(a_0 - a_1) + 4a_2^2}. \quad (5.12)
\]

As in the proof of Proposition 5.1 from this formula one can compute the operators $P$, $Q$, $P_\beta$ and $Q_\beta$. It is clear that they also can be obtained by taking the limit $\epsilon \to 0$ directly in the corresponding expressions for $W(\epsilon)$. \[\square\]

From here to the end of the subsection we shall restrict ourselves to the case when $\beta^d = \gamma$ has different coordinates. We choose the following basis of Ker$L^d_{\beta}$(cf. [MZ]):

\[
\Phi_{(k-1)d+j}(x) := \mu_{kj} x^{\beta_k + (j-1)N}, \quad 1 \leq k \leq N, \quad 1 \leq j \leq d, \quad (5.13)
\]
Proof.

Proposition 1.5 implies that form (2.20). The action of $L_W \in$ I.5 implies that form (2.20). The action of $L_W$ on the operator of order $\beta$ is quite simple:

$$L_\beta \Phi_{(k-1)d+j} = \begin{cases} \Phi_{(k-1)d+j-1}, & \text{for } 2 \leq j \leq d \\ 0, & \text{for } j = 1. \end{cases}$$ (5.14)

Let a basis of $\text{Ker}P$ be

$$f_k(x) = \sum_{i=1}^{dN} a_{ki} \Phi_i(x), \quad k = 0, \ldots, n - 1. \quad (5.15)$$

**Example 5.3** Let $n = d$, $\beta_i - \beta_j \in N\mathbb{Z}$ for all $i$, $j$ and the matrix $A = (a_{ki})$ has the form:

$$A = \begin{pmatrix} t_0^{(1)} & \ldots & t_0^{(N)} \\ t_1^{(1)} & \ldots & t_1^{(N)} \\ \vdots & \ddots & \vdots \\ t_{n-1}^{(1)} & \ldots & t_{n-1}^{(N)} \end{pmatrix}$$ (5.16)

The type of the matrix is tantamount to the identities $L_\beta f_0 = 0$, $L_\beta f_{k+1} = f_k$, $k = 1, \ldots, n - 1$. Then $\text{Ker}P$ is invariant under the action of $L_\beta$ and by Proposition 5.3 the operator $L = PL_\beta P^{-1}$ is differential of order $N$ and solves the bispectral problem. For a generic $\beta \in \mathbb{C}^N$ the spectral algebra has rank $N$ (i.e. it is $\mathbb{C}[L]$). This family can be considered as the most direct generalization of the “even case” of J. J. Duistermaat and F. A. Grünbaum [DG] (see also [MZ]). When $N = 2$ our example coincides with it but for $N > 2$ here we present a completely new class of bispectral operators. □

In connection with the above example we prove the following proposition.

**Proposition 5.4** Let $W \in \text{Gr}_B(\beta)$ ($\beta \in \mathbb{C}^N$–generic) be such that $A_W$ contains an operator of order $N$. Then $W$ is a monomial Darboux transformation of $V_\beta$, i.e. $W \in \text{Gr}_{MB}(\beta) \cap \text{Gr}^{(N)}$.

**Proof.** Proposition 1.3 implies that $W \in \text{Gr}_B(\beta)$ belongs to $\text{Gr}^{(N)}$ iff

$$L_\beta(\text{Ker}P) \subseteq \text{Ker}P.$$ (5.17)

If we suppose that $W \not\in \text{Gr}_{MB}(\beta)$ then $\text{Ker}P$ would contain some elements of the form (2.20). The action of $L_\beta$ on them is easily computed:

$$L_\beta D_\lambda^k \Psi_\beta(x, \varepsilon^i \lambda) = D_\lambda^k L_\beta \Psi_\beta(x, \varepsilon^i \lambda) = D_\lambda^k (\lambda^N \Psi_\beta(x, \varepsilon^i \lambda)) = \lambda^N(D_\lambda + N)^k \Psi_\beta(x, \varepsilon^i \lambda).$$
Thus the linear space span \( \{ D_k^\lambda \psi_\beta(x, \varepsilon^i \lambda) \}_{0 \leq k \leq m} \) can be identified with the space of polynomials in \( D \) of degree \( \leq m \), with the action of \( L_\beta \) corresponding to \( P(D) \mapsto \lambda^N P(D + N) \). It is clear that all the \( L_\beta \)-invariant subspaces are of the form
\[
\text{span} \left\{ D_k^\lambda \psi_\beta(x, \varepsilon^i \lambda) \right\}_{0 \leq k \leq k_0}
\]
for some \( k_0 \). The corresponding polynomial Darboux transformation is trivial in the sense that it leads again to the same plane \( V_\beta \) (the operator \( P = (L_\beta - \lambda)^{k_0} \) commutes with \( L_\beta \)). Therefore \( W \in \text{Gr}_{MB}(\beta) \). \( \square \)

In the same manner as in Example 5.3, one can build for arbitrary \( k \) rank \( N \) bispectral algebras with the lowest order of the operators equal to \( kN \). It is clear that when the matrix \( A \) is not of the form (5.16) (or a direct sum of such matrices) then \( \ker P \) (given by (5.15)) is not invariant under the action of \( L_\beta \). Proposition 1.5 implies that in this case the spectral algebra does not contain operators of order \( N \). The following example is one of the simplest of this type.

**Example 5.5** Let \( N = 2 \), \( \beta = (\beta_1, \beta_2) \), \( \beta_1 + \beta_2 = 1 \), \( d = n = 2 \). We take \( \ker P \) with a basis (5.15) where
\[
A = \begin{pmatrix} 1 & a & 0 & 0 \\ 0 & 0 & 1 & b \end{pmatrix}
\]
for some \( a, b \in \mathbb{C} \), i.e.
\[
f_0(x) = \Phi_1(x) + a \Phi_2(x) = x^{\beta_1} + \frac{a}{2(\beta_1 - 2) + 2} x^{\beta_1 + 2},
\]
\[
f_1(x) = \Phi_3(x) + b \Phi_4(x) = x^{\beta_2} + \frac{b}{2(\beta_2 - 1) + 2} x^{\beta_2 + 2}.
\]
Then
\[
L_\beta f_0(x) = ax^{\beta_1}, \quad L_\beta f_2(x) = bx^{\beta_2}
\]
and \( \ker P \) is not invariant under \( L_\beta \) when \( ab \neq 0 \). The spectral algebra \( A_W = \text{PL}_\beta^2 \mathbb{C}\{L_\beta\} P^{-1} \) consists of operators of orders 4, 6, 8, 10, \ldots

This example is also interesting for the fact that it does not require \( \beta_1 - \beta_2 \in 2\mathbb{Z} \). The generalization for arbitrary \( N \) is obvious. \( \square \)

Another example illustrating Proposition 1.5 is the following one.

**Example 5.6** Let \( N = 2 \), \( \beta = (\beta_1, \beta_2) \in \mathbb{C}^2 \), \( \beta_1 + \beta_2 = 1 \), \( \beta_1 - \beta_2 \in 2\mathbb{Z} \), \( d = 4 \), \( n = 2 \). We take \( \ker P \) with a basis (5.15) where
\[
A = \begin{pmatrix} \lambda & 0 & 0 & 0 & \lambda a + b & \lambda b & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & a & b \end{pmatrix}
\]
for some \( a, b, \lambda \in \mathbb{C} \).

Then it is easy to see that \( \ker P \) is invariant under the operator \( L_\beta^3 \beta \lambda L_\beta^2 \) but it is not invariant under any polynomial of \( L_\beta \) of degree \( \leq 2 \). On the other hand \( \ker P \subset \ker L_\beta^3 \beta \) obviously implies \( L_\beta^4 \beta \ker P \subset \ker P \) for \( k \geq 0 \). Therefore the spectral algebra \( A_W \) is the linear span of the operators
\[
P \left( L_\beta^3 \beta \lambda L_\beta^2 \right) P^{-1}, \quad PL_\beta^{1+k} P^{-1}, \quad k \geq 0.
\]
This example is interesting for the fact that (for \( \lambda \neq 0 \)) the operator of minimal order in the spectral algebra is not a Darboux transformation of a power of \( L_{\beta} \), although the Darboux transformation is monomial. \( \square \)

In the last example of this subsection we show that for \( d = n = 1 \) our results agree with those of [Z].

**Example 5.7** Let \( d = n = 1 \), \( \text{Ker} P = \mathbb{C} f_0 \),

\[
f_0(x) = \sum_{i=1}^{N} a_i x^{\beta_i}, \quad P = \partial_x - \frac{f'_0(x)}{f_0(x)}
\]

and \( \beta_i - \beta_j \in \mathbb{N} \mathbb{Z} \) if \( a_i a_j \neq 0 \). Then

\[
L = P L_{\beta} P^{-1} = PQ = (\sum a_i x^{p_i})^{-1} \left( \sum a_i x^{p_i} \left( \partial_x - \frac{\beta_i}{x} \right) \right) \times \\
\times \left( \sum a_i \frac{P_{\beta}(D_x + N)}{D_x + N - \beta_i} x^{p_i - N + 1} \right) \left( \sum a_i x^{p_i} \right)^{-1},
\]

\[
\Lambda = P_b Q_b = \left( \sum a_i \left( \partial_z - \frac{\beta_i}{z} \right) (L_{\beta})^{p_i/N} \right) \times \\
\times \left( \sum a_i (L_{\beta})^{p_i/N} x^{-N+1} \frac{P_{\beta}(D_z + 1)}{D_z + 1 - \beta_i} \right)
\]

where \( p_i = \beta_i - \beta_{\min}, \beta_{\min} = \min_{a_i \neq 0} \beta_i \),

\[
P_{\beta}(D) = \prod_{i=1}^{N} (D - \beta_i), \quad D_x = x \partial_x.
\]

We have

\[
\Theta(x) = x^N \left( \sum_{i=1}^{N} a_i x^{p_i} \right)^2, \quad \deg \Theta = N + 2(\beta_{\max} - \beta_{\min})
\]

where \( \beta_{\max} = \max_{a_i \neq 0} \beta_i \). When \( f_0(x) = tx^{\beta_1} + x^{\beta_2}, \beta_2 - \beta_1 = N \alpha, \alpha \in \mathbb{Z} \geq 0 \)

\[
\Theta(x) = x^N (t + x^{N\alpha})^2
\]

and we obtain the operator \( \Lambda \) from [Z]. \( \square \)

### 5.2 Polynomial Darboux transformations

In this subsection we shall consider the simplest case of polynomial Darboux transformation of an operator of order \( N \), namely when the polynomial \( h(z) \) from (1.27) is equal to \( (z - \lambda^N)^2 \) for some \( \lambda \in \mathbb{C} \setminus \{0\} \). Using the kernels of the operators \( P, Q^*, P_b \) and \( Q_b^* \) from (1.21, 1.22, 3.2, 3.7), we describe the action of the involutions \( a \) and \( b \) (\( b_1 \) in the Airy case). The Propositions 5.10, 5.12 below raise some interesting questions and conjectures.

The Bessel and Airy cases are very similar. We shall consider first the Airy one since it is simpler.
Let \( \text{Gr}_A(\alpha), \alpha \in \mathbb{C}^{N-1} \). Set
\[
h(z) = (z - \lambda^N)^2, \quad g(z) = f(z) = z^N - \lambda^N. \tag{5.18}
\]
Then \( \text{Ker}h(L_\alpha) \) has a basis of the form
\[
\left\{ \partial_x^k \Psi_\alpha(x, \varepsilon^j \lambda) \right\}_{0 \leq j \leq N-1, k=0,1} \tag{5.19}
\]
and \( \text{Ker}P \) has a basis
\[
f_j(x) = \Psi_\alpha(x, \varepsilon^j \lambda) + a \partial_x \Psi_\alpha(x, \varepsilon^j \lambda), \quad 0 \leq j \leq N - 1 \tag{5.20}
\]
for some \( a \in \mathbb{C} \).

We shall start with the case \( N = 2 \). The following example is due to \([\text{G3}, \text{LP}]\).

**Example 5.8** Let \( N = 2 \) and \( \alpha = (\alpha_0) \in \mathbb{C}^1 \). For fixed \( \alpha_0, a \in \mathbb{C} \setminus 0 \) we take the basis \( f_k \) of \( \text{Ker}P \):
\[
f_k(x) = \psi_k(x) + a \partial_x \psi_k(x), \quad k = 0, 1 \tag{5.21}
\]
where \( \psi_k(x) = \Psi_\alpha(x, (-1)^k \lambda) \). Using that
\[
\begin{align*}
\partial_x f_k &= a(\alpha_0 x + \lambda^2) \psi_k + \partial_x \psi_k, \\
\partial_x^2 f_k &= (a \alpha_0 + \alpha_0 x + \lambda^2) \psi_k + a(\alpha_0 x + \lambda^2) \partial_x \psi_k
\end{align*}
\]
we compute \( P \) from
\[
P \varphi = \frac{\text{Wr}(f_0, f_1, \varphi)}{\text{Wr}(f_0, f_1)} = \begin{vmatrix} 1 & a & \varphi \\ a(\alpha_0 x + \lambda^2) & 1 & \partial_x \varphi \\ a \alpha_0 + \alpha_0 x + \lambda^2 & a(\alpha_0 x + \lambda^2) & \partial_x^2 \varphi \end{vmatrix}.
\]
The result is
\[
P = \partial_x^2 + \frac{a^2 \alpha_0}{1 - a^2(\alpha_0 x + \lambda^2)} \partial_x + \frac{a^2(\alpha_0 x + \lambda^2)^2 - (\alpha_0 x + \lambda^2) - a \alpha_0}{1 - a^2(\alpha_0 x + \lambda^2)}. \tag{5.22}
\]
This expression coincides with that given in \([\text{G3}]\) if we set
\[
\alpha_0 = \frac{2}{2 + 3s} = \frac{2}{s}, \quad a = \frac{s}{2y}, \quad \lambda = 0.
\]
We compute the operators \( P, Q \) and \( Q^* \) as follows. If we write
\[
P = \partial_x^2 + p_1(x) \partial_x + p_0(x) \\
Q = \partial_x^2 + q_1(x) \partial_x + q_0(x) \\
Q^* = \partial_x^2 + \tilde{q}_1(x) \partial_x + \tilde{q}_0(x)
\]
then
then the identity $QP = h(L_\alpha)$ imply
\[ q_1 + p_1 = 0, \quad 2p'_1 + q_1p_1 + p_0 + q_0 = -2(\alpha_0x + \lambda^2) \]
and
\[ \tilde{q}_1 = -q_1, \quad \tilde{q}_0 = -q'_1 + q_0. \]

Our observation is that because $P^*Q^* = h(L_\alpha(a))$ and $\Psi_{aW} = f^{-1}Q^*\Psi_{a(\alpha)}$, the operator $Q^*$ has a basis of the form (5.21) with some $b \in \mathbb{C}$ instead of $a$ and $a(\alpha)$ instead of $\alpha$. Comparing the above expressions for $Q^*$ with (5.22) we obtain that $b = -a$. By Theorem 4.5 the operator $P_b$ also has a basis (5.19) with some $c$ instead of $a$ and $\mu$ instead of $\lambda$. On the other hand we can compute $P_b$ directly using eqs. (4.15, 4.16). Then
\[ g_b(z) = 1 - a^2(z^2 + \lambda^2) \]
which on the other hand is up to a constant $z^2 - \mu^2$. This gives
\[ \mu^2 = \frac{1 + a^2\lambda^2}{a^2}. \] (5.23)

The other coefficients give a surprising result: $c = a$. In conclusion, if we denote the operator $P$ from (5.22) with $P(a, \lambda)$ then
\[ P = P(a, \lambda), \quad Q = P^*(-a, \lambda), \quad P_b = P(a, \mu), \quad Q_b = P^*(-a, \mu) \] (5.24)
where $\mu$ and $\lambda$ are connected by (5.23).

Example 5.9 For $N = 3$ the Airy operator is
\[ L_\alpha = \partial_x^3 + \alpha_2 \partial_x - \alpha_0 x, \]
$\alpha = (\alpha_0, \alpha_2) \in \mathbb{C}^2$, $\alpha_0 \neq 0$. We take $P$ with a basis (5.20) ($N = 3$). Then using the eq. (5.20) we compute
\[ P = \partial_x^3 - \frac{a^3\alpha_0}{a^3(\alpha_0 x + \lambda^3) + (1 + a^2\alpha_2)} \partial_x^2 \]
\[ + \frac{a^3\alpha_2(\alpha_0 x + \lambda^3) + (1 + a^2\alpha_2)\alpha_2 + a^2\alpha_0}{a^3(\alpha_0 x + \lambda^3) + (1 + a^2\alpha_2)} \partial_x \]
\[ - \frac{a^3(\alpha_0 x + \lambda^3)^2 + a\alpha_0(1 + a^2\alpha_2)(1 + a^2\alpha_2)(\alpha_0 x + \lambda^3)}{a^3(\alpha_0 x + \lambda^3) + (1 + a^2\alpha_2)}. \]

A direct computation using Proposition 1.7, Theorem 4.5 and $QP = h(L_\alpha)$ leads to
\[ P = P(a, \lambda), \quad Q = -P^*(-a, -\lambda), \quad P_b = P(a, \mu), \quad Q_b = -P^*(-a, -\mu) \] (5.25)
with $\mu$ given by
\[ \mu^3 + \lambda^3 = -\frac{1 + a^2\alpha_2}{a^3} \] (5.26)

The above examples can be generalized for arbitrary $N$ as follows.
Proposition 5.10 Denote by $P = P(a, \lambda)$ the operator $P$ with a basis \eqref{5.20}. Then in the above notation we have

$$Q = (-1)^N P^*(-a, -\lambda), \quad P_b = P(a, \mu), \quad Q_b = (-1)^N P^*(-a, -\mu) \tag{5.27}$$

with $\lambda$ and $\mu$ connected by

$$\lambda^N + \mu^N = P_{\alpha'}(-1/a) \tag{5.28}$$

where $P_{\alpha'}$ is the polynomial from \eqref{1.1}. The spectral algebras

$$A_W = P(L_\alpha - \lambda^N)^2 \mathbb{C}[L_\alpha]^{-1} \tag{5.29}$$

$$A_{bW} = P_{b}(L_\alpha - \mu^N)^2 \mathbb{C}[L_\alpha]^{-1} \tag{5.30}$$

consist of operators of orders $2N, 3N, 4N, \ldots$

Proof. Because

$$(-1)^N P^* (-1)^N Q^* = (L_{\alpha}(\alpha) - (-\lambda)^N)^2, \quad \Psi_{aW}(x, z) = \frac{(-1)^N Q^* \Psi_{a(\alpha)}(x, z)}{z^N - (-\lambda)^N}$$

$$Q_b P_b = (L_\alpha - \mu^N)^2, \quad \Psi_{bW}(x, z) = \frac{P_{b} \Psi_{a(\alpha)}(x, z)}{z^N - \mu^N}$$

we see that $Q = (-1)^N P^*(b, -\lambda)$ and $P_b = P(c, \mu)$ for some $b, c, \mu$. Using the eq. $L_\alpha(x, \partial_x) \Psi_\alpha(x, \xi^j \lambda) = \lambda^N \Psi_\alpha(x, \xi^j \lambda)$ we compute $p_N(x) = \text{Wr}(f_0, f_1, \ldots, f_{N-1})$ as in the proof of Lemma 2.9. We obtain

$$p_N(x) = -(a)^N (\alpha_0 x + \lambda^N - P_{\alpha'}(-1/a)).$$

Eq. \eqref{5.28} leads to \eqref{5.29} because $g_b(z) = \text{const} \cdot (z^N - \mu^N)$. Applying \eqref{5.28} for $P_b$ instead of $P$, we obtain

$$P_{\alpha'}(-1/a) = P_{\alpha'}(-1/c).$$

Note that the map $a \mapsto c$ is an automorphism of $\mathbb{C}^p$ since it is an involution. The only solution of the above equation with this property is $c = a$.

To compute $(-1)^N Q^*$, we note that its second coefficient is equal to that of $P$ which is equal to $-p_N'(x)/p_N(x)$. This, Proposition \ref{1.7} and Lemma \ref{1.7} imply

$$\frac{\alpha_0}{\alpha_0 x + \lambda^N - P_{\alpha'}(-1/a)} = \frac{a(\alpha)_0}{a(\alpha)_0 x + (-\lambda)^N - P_{\alpha(a)'}(-1/b)}$$

which leads to a polynomial equation for $b$ in terms of $a$ and $\alpha$. Because $a \mapsto b$ is an automorphism of $\mathbb{C}^p$ we obtain that $b = -a$.

The eqs. \eqref{5.29}, \eqref{5.30} follow from Proposition 5.4. \hfill \Box

We shall find the analog of Proposition \ref{5.10} in the Bessel case. We use the notation from the beginning of the subsection with $\beta \in \mathbb{C}^N$ instead of $\alpha$ and eq. \eqref{5.20} modified as follows (cf. \eqref{2.20})

$$f_{j}(x) = \Psi_\beta(x, \xi^j \lambda) + a D_x \Psi_\beta(x, \xi^j \lambda) \tag{5.31}$$

$(j = 0, \ldots, N - 1, \ D_x = x \partial_x)$.

In the next example we shall study the simplest case $N = 2$. 

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Example 5.11 For $N = 2$, $\beta = (1 - \nu, \nu)$ the corresponding Bessel operator is

$$L_\beta = x^{-2}(D_x - (1 - \nu))(D_x - \nu) = \partial_x^2 + \frac{\nu(1 - \nu)}{x^2}, \quad D_x = x\partial_x.$$ 

Using (5.38) we compute the operator $P$ from

$$P\varphi = \frac{1}{x^2} \begin{vmatrix} f_0 & f_1 & \varphi \\ D_x f_0 & D_x f_1 & D_x \varphi \\ D_x^2 f_0 & D_x^2 f_1 & D_x^2 \varphi \end{vmatrix}.$$ 

The answer is the following. If we set

$$\mu^2 = \frac{a + 1 + a^2\nu(1 - \nu)}{a^2\lambda^2}$$

then

$$P = \frac{1}{x^2 p_2(x^2)} \left\{ p_2(x^2)D_x^2 + p_1(x^2)D_x + p_0(x^2) \right\}$$

with $p_2(x^2) = x^2 - \mu^2$, $p_1(x^2) = \mu^2 - 3x^2$ and $p_0(x^2) = -\lambda^2x^4 + (2\lambda^2\mu^2 + (a + 1)(2a - 1)a^{-2})x^2 + ((a + 1)a^{-2} - \lambda^2\mu^2)\mu^2$. The operator $P_b$ is (cf. (5.34))

$$P_b = \frac{1}{g(x)} \left\{ D_x^2 p_2(L_\beta) + D_x p_1(L_\beta) + p_0(L_\beta) \right\}$$

and $g_b(z) = z^2(z^2 - \mu^2)$. A straightforward computation shows that if we set

$$Q = P^*\left(-a/(a + 1), \lambda, \mu\right), \quad P_b = P(a, \mu, \lambda)L_\beta, \quad Q_b = L_\beta P^*(-a/(a + 1), \mu, \lambda).$$

Therefore we can take

$$P_b = P(a, \mu, \lambda), \quad Q_b = P^*(-a/(a + 1), \mu, \lambda)$$

i.e. the involution $b$ acts simply by exchanging $\lambda$ with $\mu$ and vice versa, while the involution $a$ acts as $a \mapsto -a/(a + 1)$. □

The action of the involutions for arbitrary $N$ is given in the next proposition.

**Proposition 5.12** Denote by $P = P(a, \lambda)$ the operator $P$ with a basis (5.31). Then we can take $P_b$ and $Q_b$ such that

$$Q = (-1)^N P^*(b, -\lambda), \quad P_b = P(a, \mu), \quad Q_b = (-1)^N P^*(b, -\mu)$$

with $\lambda$, $\mu$ and $a$, $b$ connected by

$$\lambda N \mu^N = P_\beta(-1/a), \quad \frac{1}{a} + \frac{1}{b} + N - 1 = 0$$

where $P_\beta$ is the polynomial from (5.32). The spectral algebras

$$\mathcal{A}_W = P \left(L_\beta - \lambda^N\right)^2 C[L_\beta]P^{-1}$$

$$\mathcal{A}_b W = P_b \left(L_\beta - \mu^N\right)^2 C[L_\beta]P_b^{-1}$$

consist of operators of orders $2N, 3N, 4N, \ldots$
Proof. We have \( g_b(z) = \text{const} \cdot z^N(z^N - \mu^N) \) for some \( \mu \in \mathbb{C} \). Using (3.5) we compute
\[
g_b(z) = z^N \det(D^T f_j(z))_{i,j = 0, \ldots, N-1} = (-a)^N z^N (z^N \lambda^N - P_\beta(-1/a))
\]
which gives the value of \( \mu \). We have to prove that \( P_b \) given by (3.4) (which is of order \( 2N \)) is divisible by \( L_\beta \) from the right. Indeed, it is easy to see that
\[
P_b(x, \partial_x) x^{\delta_1} = P(x, \partial_x) x^{\delta_1} \big|_{\lambda=0}
\]
and the proof of Lemma 2.9 implies \( P|_{\lambda=0} = L_\beta \). Thus we can take \( P_b = P(c, \mu) \) for some \( c \in \mathbb{C} \). Now (5.33) implies \( P_\beta(-1/a) = P_\beta(-1/c) \) leading to \( c = a \).

Finally, as in the Airy case, if \( Q = (-1)^N P^*(b, -\lambda) \) for some \( b \in \mathbb{C} \) then
\[
P_\beta(-1/a) = (-1)^N P_\alpha(-1/b) = P_\beta(1/b + N - 1)
\]
showing that \( a^{-1} + b^{-1} + N - 1 = 0. \)

The eqs. (5.36, 5.37) follow from Proposition 5.4. \( \Box \)

In conclusion we want to make some comments. In the case \( N = 1 \) the adjoint involution \( a \) has a simple and beautiful geometric interpretation (see [W1]): in terms of Krichever’s construction it preserves the spectral curve and maps the “sheaf of eigenfunctions” into some kind of a dual sheaf. In [W1] G. Wilson also posed the problem of describing the action of the bispectral involution on \( \text{Gr}_{ad} \).

We think that in the general case the study of the action of the involutions \( a \) and \( b \) on the bispectral manifolds of polynomial Darboux transformations of Bessel and Airy planes is equally interesting and difficult task.

The above examples lead us to the conjecture that the involutions \( a \) and \( b \) (\( b_1 \) in the Airy case) possess some universality property. Any polynomial Darboux transformation \( W \) of a Bessel plane \( V_\beta \) (respectively an Airy plane \( V_\alpha \)) is determined by the points \( \lambda_1, \ldots, \lambda_N \neq 0 \) at which the conditions \( C \) are supported (see (2.1)), by the matrix \( A \) defined by (2.20) (resp. (2.19)), and of course by the vector \( \beta \) (resp. \( \alpha \)). Then the corresponding matrices for \( aW \) and \( bW \) (resp. \( b_1W \) ) depend only on the matrix \( A \). The point is that they do not depend on the points \( \lambda_1, \ldots, \lambda_N \) at which the conditions \( C \) are supported nor on the vector \( \beta \) (resp. \( \alpha \)).

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