ROTATION TOPOLOGICAL FACTORS OF MINIMAL 
$Z^d$-ACTIONS ON THE CANTOR SET

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Abstract. In this paper we study conditions under which a free minimal $Z^d$-action on the Cantor set is a topological extension of the action of $d$ rotations, either on the product $T^d$ of $d$ 1-tori or on a single 1-torus $T^1$. We extend the notion of linearly recurrent systems defined for $Z$-actions on the Cantor set to $Z^d$-actions and we derive in this more general setting, a necessary and sufficient condition, which involves a natural combinatorial data associated with the action, allowing the existence of a rotation topological factor of one these two types.

1. Introduction

Let $(X, A)$ be a $Z^d$-action (by homeomorphisms) on a compact metric space $X$. The action is free if $A(\bar{n}, x) = x$ for some $\bar{n} \in Z^d$ and $x \in X$ implies $\bar{n} = 0$ and is minimal if the orbit of any point $x \in X$, $O_A(x) = \{A(\bar{n}, x) : \bar{n} \in Z^d\}$, is dense in $X$.

The simplest non trivial examples of free minimal $Z^d$-actions on a compact metric space are given by “rotation-type” actions on compact topological groups. This type of factors play a central role in topological dynamics of $Z^d$-actions since in particular they determine weak mixing property through the existence of continuous eigenvalues. In this paper, we focus on two kinds of “rotation-type” factors that we describe now.

- First consider the $Z^d$-action generated by $d$ rotations on the product $d$-torus $T^d = \mathbb{R}^d/Z^d = T^1 \times \cdots \times T^1$, each rotation acting on $T^1$. More precisely, take $\bar{\theta} = (\theta_1, \ldots, \theta_d) \in \mathbb{R}^d$ and let $A^d_{\bar{\theta}} : Z^d \times T^d \to T^d$ be the map defined by:

$$A^d_{\bar{\theta}}(\bar{n}, x) = x + [\bar{n}, \bar{\theta}] \mod Z^d,$$

for $\bar{n} = (n_1, \ldots, n_d) \in Z^d$, $x \in T^d$ and where $[\bar{n}, \bar{\theta}] = (n_1 \cdot \theta_1, \ldots, n_d \cdot \theta_d)$. This construction yields a minimal $Z^d$-action $(O^d, A^d_{\bar{\theta}})$ on the closure $O^d$ of the orbit of 0 in the $d$-torus $T^d$. When the coordinates of $\bar{\theta}$ are rationally independent, the set $O^d$ is the $d$-torus $T^d$ and the action is free.

- The same $\bar{\theta}$ can be used to define a $Z^d$-action on $T^1$. Consider the map $A^1_{\bar{\theta}} : Z^d \times T^1 \to T^1$ given by

$$A^1_{\bar{\theta}}(\bar{n}, t) = t + \langle \bar{n}, \bar{\theta} \rangle \mod Z,$$

where $\langle \cdot, \cdot \rangle$ is the usual inner product in $\mathbb{R}^d$. The $Z^d$-action $(O^1, A^1_{\bar{\theta}})$ on the closure $O^1$ of the orbit of 0 in the 1-torus $T^1$ is again minimal. When
the coordinates of \( \bar{\theta} \) are independent on \( Q \), the set \( \mathbb{O}^1 \) is the 1-torus \( T^1 \) and the action is free.

Assume \( X \) is a Cantor set, i.e., it has a countable basis of closed and open (clopen) sets and has no isolated points (or equivalently, it is a totally disconnected compact metric space with no isolated points).

The main question we address in this paper is to determine whether a free minimal \( \mathbb{Z}^d \)-action \( \mathcal{A} \) on the Cantor set \( X \) is an extension of an action of type \((\mathbb{O}^d, \mathcal{A}_d^\theta)\) or \((\mathbb{O}^1, \mathcal{A}_1^\bar{\theta})\) for some \( \bar{\theta} \in \mathbb{R}^d \).

Notice that a complete combinatorial answer to this question is given in [BDM] in the particular case when the dimension \( d = 1 \) and when the free minimal \( \mathbb{Z} \)-action is linearly recurrent. The linear recurrence of a given \( \mathbb{Z} \)-action is a property that involves the combinatorics of return times associated with a nested sequence of clopen sets (for further references on linearly recurrent \( \mathbb{Z} \)-actions see [CDHM], [Du1] and [Du2]).

The notion of return time to a clopen set can be generalized to \( \mathbb{Z}^d \)-actions when \( d \geq 2 \). In this case, the combinatorics of the return times associated with a nested sequence of clopen sets inherits a richer structure than in the case \( d = 1 \). However, as for \( d = 1 \), there exists a natural definition of linearly recurrent \( \mathbb{Z}^d \)-action. These generalizations are developed in Section 2 which is devoted to the combinatorics of return times (for further references on the structure of return times associated with a \( \mathbb{Z}^d \)-action see [BG] where the hierarchical ideas used in this paper are introduced, see also [S] and [SW] for related topics).

This combinatorial approach allows us to derive a necessary condition on the action to be an extension of an action of one of the two rotations described above. In the case of a linearly recurrent action this condition is sufficient. This result is given in Section 3 (Theorem 3.1) together with its proof.

2. Combinatorics of return times

Let us start this section with some general considerations.

Let \( \mathbb{R}^d \) be the Euclidean \( d \)-space and \( \| - \| \) its Euclidean norm. Consider two positive numbers \( r \) and \( R \). An \((r, R)\)-Delone set is a subset \( \mathcal{D} \) of the \( d \)-space \( \mathbb{R}^d \) equipped with the Euclidean norm \( \| - \| \), which satisfies the following two properties:

(i) *Uniformly Discrete:* each open ball with radius \( r \) in \( \mathbb{R}^d \) contains at most one point in \( \mathcal{D} \);

(ii) *Relatively Dense:* each open ball with radius \( R \) contains at least one point in \( \mathcal{D} \).

When the constants \( r \) and \( R \) are not explicitly used, we will say in short *Delone set* for an \((r, R)\)-Delone set. We refer to [LP] for a more detailed approach of the theory of Delone sets.

A patch of a Delone set \( \mathcal{D} \) is a finite subset of \( \mathcal{D} \). A Delone set is of *finite type* if for each \( M > 0 \), there exist only finitely many patches in \( \mathcal{D} \) of diameter smaller than \( M \) up to translation. Finally, a Delone set of finite type is repetitive if for each patch \( P \) in \( \mathcal{D} \), there exists \( M > 0 \) such that each ball with radius \( M \) in \( \mathbb{R}^d \) contains a translated copy of \( P \) in \( \mathcal{D} \).

Let \( x \) be a point of a Delone set \( \mathcal{D} \). The *Voronoi cell* \( V_x \) associated with \( x \) is the convex closed set in \( \mathbb{R}^d \) defined by:
The set of return vectors associated with $D$ is a cover of $\mathbb{R}^d$. We say that two points $x$ and $x'$ in $D$ are neighbors if $V_x \cap V_{x'} \neq \emptyset$.

The set of return vectors associated with $D$ is defined by:

$$\mathcal{D} = \{ x - y : (x, y) \in D \times D \}.$$ 

**Lemma 2.1.** Let $D$ be a Delone set of finite type. Then, there exists a finite collection $\mathcal{F}$ of vectors in $\mathcal{D}$ such that:

- any vector in $\mathcal{D}$ is a linear combination with non negative integer coefficients of vectors in $\mathcal{F}$.

**Proof.** When $D$ is a Delone set of finite type, the set of vectors

$$\mathcal{F} = \bigcup_{(x,x') \in D \times D, (x,x') \text{ neighbors}} (x - x')$$

is finite, satisfies $\mathcal{F} = -\mathcal{F}$ and clearly any vector in $\mathcal{D}$ is a linear combination with non negative integer coefficients of vectors in $\mathcal{F}$. \qed

Given such a set $\mathcal{F}$, we can define the $\mathcal{F}$-distance $d_{\mathcal{F}}(x,x')$ as the minimal number of vectors in $\mathcal{F}$ (counted with multiplicity) needed to write $x - x'$ for $x,x' \in D$. The $\mathcal{F}$-diameter of a patch $P$, denoted by $diam_{\mathcal{F}}(P)$, is the maximal $\mathcal{F}$-distance of pair of points in $D$.

Consider now a free minimal $\mathbb{Z}^d$-action $A$ on the Cantor set $X$. Let $C$ be a clopen set in $X$ and $y$ a point in $C$. The set of return times of the orbit of $y$ in $C$ is defined by

$$\mathcal{R}_C(y) = \{ \bar{n} \in \mathbb{Z}^d : A(\bar{n}, y) \in C \}.$$ 

**Proposition 2.2.** The set of return times $\mathcal{R}_C(y)$ is a repetitive Delone set of finite type in $\mathbb{Z}^d$. Furthermore, if $y$ and $y'$ are two points in $C$, the sets $\mathcal{R}_C(y)$ and $\mathcal{R}_C(y')$ have the same patches up to translation.

**Proof.** $\bullet$ $\mathcal{R}_C(y)$ is a Delone set of finite type.

The minimality of the action implies that the orbit of any point in $X$ visits $C$. For each $x \in X$ consider $\bar{n}_x \in \mathbb{Z}^d$ be such that $A(\bar{n}_x, x)$ is in $C$. Since $C$ is open, there exists a small neighborhood $U_x$ of $x$ such that for any $x'$ in $U_x$ we also have $A(\bar{n}_x, x') \in C$. Therefore $\{ U_x : x \in X \}$ is a cover of $X$. Since $X$ is compact, we can extract a finite cover $\{ U_{x_i} : i \in I \}$. Let us choose $R > \max_{i \in I} \| \bar{n}_{x_i} \|$. It is clear that any ball with radius $R$ in $\mathbb{R}^d$ intersects $\mathcal{R}_C(y)$. Thus, $\mathcal{R}_C(y)$ is relatively dense. Since it is a subset of $\mathbb{Z}^d$, it is a Delone set of finite type.

$\bullet$ $\mathcal{R}_C(y)$ is repetitive$^1$.

Consider a patch $P$ in $\mathcal{R}_C(y)$, choose $\bar{n}_0$ in $P$ and let $z = A(\bar{n}_0, y) \in C$. Choose now a clopen set $C_z$ containing $z$, small enough so that for any $z'$ in $C_z$, $A(\bar{n} - \bar{n}_0, z')$ is in $C$ for each $\bar{n}$ in $P$. The set $\mathcal{R}_{C_z}(z)$ is relatively dense, let $R_1$ be its $R$-constant. Let $M$ stand for the diameter of $P$ and let us prove that any ball with radius $R_1 + M$

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$^1$The proof that minimality implies repetitivity is classical and works in a more general situation. However, for sake of completeness, we fix it here for our specific context.
in $\mathbb{R}^d$ contains a translation of the patch $P$. Indeed, given such a ball $B$, choose an element $\bar{m} \in \mathcal{R}_c(z)$ in the corresponding centered sub-ball of radius $R_1$, then by construction $\bar{m} + P$ belongs to $\mathcal{R}_c(y)$ and to the ball $B$.

- $\mathcal{R}_c(y)$ and $\mathcal{R}_c(y')$ have the same patches up to translation.

Let $P$ be a patch of $\mathcal{R}_c(y)$ and $\bar{n}_0$ be a point in $P$. The minimality of the action implies that the orbit of $y'$ accumulates on $z = A(\bar{n}_0, y)$. This means that there exists $\bar{n}_1 \in \mathbb{Z}^d$ such that $A(\bar{n}_1 + \bar{n} - \bar{n}_0, y')$ is in $\mathcal{C}$ when $\bar{n}$ is in $P$. Thus a translation of the patch $P$ is in $\mathcal{R}_c(y')$.

\[ \mathcal{R}_c = \mathcal{R}_c(y) - \mathcal{R}_c(y) = \{ \bar{n} - \bar{m} : (\bar{n}, \bar{m}) \in \mathcal{R}_c(y) \times \mathcal{R}_c(y) \}. \]

The fact that for any pair of points $y$ and $y'$ in $\mathcal{C}$, the patches of $\mathcal{R}_c(y)$ and $\mathcal{R}_c(y')$ fit up to translation, implies that $\mathcal{R}_c$ does not depend on $y$ in $\mathcal{C}$, as suggested by the notation. Lemma 2.4 and Proposition 2.2 yield the following corollary.

**Corollary 2.3.** There exists in $\mathcal{R}_c$ a finite collection of vectors $\mathcal{F}_c$ such that:

- $\mathcal{F}_c = -\mathcal{F}_c$;
- any vector in $\mathcal{R}_c$ is a linear combination with non negative integer coefficients of vectors in $\mathcal{F}_c$.

Such a set $\mathcal{F}_c$ is called a set of first return vectors associated with $\mathcal{C}$.

Now we shall construct a combinatorial data associated to a $\mathbb{Z}^d$-action. Let $x$ be a point in $X$ and consider a sequence of nested clopen sets $X = C_0 \supseteq C_1 \supseteq \cdots \supseteq C_n$ such that

\[ \bigcap_{n \geq 0} C_n = \{ x \}. \]

Consider also the associated sets of return times $\mathcal{R}_{C_n}(x)$, of return vectors $\mathcal{R}_{C_n}$ and of first return vectors $\mathcal{F}_{C_n}$ that we denote respectively (in short) by $\mathcal{R}_n(x)$, $\mathcal{R}_n$ and $\mathcal{F}_n$.

**Proposition 2.4.** For each $n \geq 0$, there exist a constant $k(n) > 0$ and a partition of $\mathcal{R}_n(x)$ in disjoint patches $\{ \mathcal{P}_n(m) \}_{m \in \mathcal{R}_{n+1}(x)}$ such that, for each $\bar{m} \in \mathcal{R}_{n+1}(x)$:

1. $\mathcal{P}_n(\bar{m}) \cap \mathcal{R}_{n+1}(x) = \{ \bar{m} \}$;
2. $\text{diam}_{\mathcal{F}_n}(\mathcal{P}_n(\bar{m})) \leq k(n)$.

**Proof.** For any point $\bar{m}$ in $\mathcal{R}_{n+1}(x)$ consider its Voronoi cell $\mathcal{V}_{\bar{m},n+1}$. The intersection of this Voronoi cell with $\mathcal{R}_n(x)$ defines a patch $\mathcal{P}_n(\bar{m})$ which intersects $\mathcal{R}_{n+1}(x)$ at $\bar{m}$. It may occasionally happen that a point $\bar{l}$ in $\mathcal{R}_n(x)$ belongs to more than one Voronoi cell $\mathcal{V}_{\bar{m},n+1}$. In this case, we make an arbitrary choice to exclude the point $\bar{l}$ from all the patches it belongs to but one. This surgery done, the collection of patches $\{ \mathcal{P}_n(\bar{m}) \}_{m \in \mathcal{R}_{n+1}(x)}$ realizes a partition of $\mathcal{R}_n(x)$. Furthermore, since $\mathcal{R}_{n+1}(x)$ and $\mathcal{R}_n(x)$ are repetitive Delone sets, the Euclidean diameters of the cells $\mathcal{V}_{\bar{m},n+1}$ are bounded independently of $\bar{m}$, and thus their $\mathcal{F}_n$-diameters are bounded independently of $\bar{m}$. \[\square\]
The data \((\bar{x}, \{c_n\}_{n \geq 0}), \{(\bar{m}_n)\}_{\bar{m} \in \mathcal{R}_{n+1}(x)}\) is called a combinatorial data associated with the action \((X, \mathcal{A})\).

We remark that Proposition 2.4 does not require any condition on the nested sequence of clopen sets. By forgetting some \(c_n\)'s in the sequence, it is always possible to insure the following two extra properties for the combinatorial data:

(iii) for each \(n \geq 0\) and for each \(\bar{m} \in \mathcal{R}_{n+1}(x)\)
\[
\bar{F}_n \subseteq \mathcal{P}_n(\bar{m}) - \mathcal{P}_n(\bar{m});
\]

(iv) for each \(n \geq 0\) and for each \(\bar{m} \in \mathcal{R}_{n+2}(x)\), all the patches \(\mathcal{P}_n(\bar{m})\) are identical up to translation.

In this case, we say that the combinatorial data
\[
(\bar{x}, \{c_n\}_{n \geq 0}) \text{, } \{(\bar{m}_n)\}_{\bar{m} \in \mathcal{R}_{n+1}(x)} \text{, } \{\bar{F}_n\}_{n \geq 0}
\]
is well distributed.

Let \(m\) and \(n\) be two integers such that \(0 \leq n \leq m\), and let \(\bar{p}\) be a point in \(\mathcal{R}_m(x)\).

We denote by \(\mathcal{P}_n^m(\bar{p})\) the patch in \(\mathcal{R}_n(x)\) defined recursively by:
\[
\mathcal{P}_n^m(\bar{p}) = \mathcal{P}_{n-1}(\bar{p});
\]
and
\[
\mathcal{P}_n^m(\bar{p}) = \bigcup_{\bar{q} \in \mathcal{P}_{n+1}^m(\bar{p})} \mathcal{P}_n(\bar{q});
\]
We adopt the convention \(\mathcal{P}_0^m(\bar{p}) = \{\bar{p}\}\). The proof of the following result is plain.

**Corollary 2.5.** For any \(n_0 \geq 0\) and any \(\bar{p}\) in \(\mathcal{R}_{n_0}(x)\), there exists a unique \(m_0 \geq n_0\) and a unique sequence \(\{\bar{p}_l\}_{0 \leq l \leq m_0 - n_0}\) of points in \(\mathbb{Z}^d\) such that:

- \(m_0\) is the smallest \(m \geq n_0\) for which \(\bar{p} \in \mathcal{P}_m^0(0)\);
- \(\bar{p}_0 = 0\);
- \(\bar{p}_l \in \mathcal{P}_{m_0-l}(\bar{p}_{l-1})\) and \(\bar{p} \in \mathcal{P}_{m_0-l}(\bar{p}_l)\) for all \(1 \leq l \leq m_0 - n_0\);
- \(\bar{p}_{m_0-n_0} = \bar{p}\).

When the constant \(k(n)\) in Proposition 2.4 is bounded independently on \(n\), we say that the free minimal \(\mathbb{Z}^d\)-action \(\mathcal{A}\) on the Cantor set \(X\) is linearly recurrent. In this case, the combinatorial data \((\bar{x}, \{c_n\}_{n \geq 0}), \{(\bar{m}_n)\}_{\bar{m} \in \mathcal{R}_{n+1}(x)}\) is said adapted to the action.

### 3. Main results

To each vector \(\bar{\theta}\) in \(\mathbb{R}^d\) we associate the linear maps \(c^l_{\bar{\theta}} \in \mathcal{L}(\mathbb{Z}^d, \mathbb{R}^l)\) and \(c^d_{\bar{\theta}} \in \mathcal{L}(\mathbb{Z}^d, \mathbb{R}^d)\) defined by
\[
c^l_{\bar{\theta}}(\bar{p}) = <\bar{\theta}, \bar{p}> \mod \mathbb{Z} \quad \text{and} \quad c^d_{\bar{\theta}}(\bar{p}) = [\bar{\theta}, \bar{p}] \mod \mathbb{Z}^d
\]
for each \(\bar{p}\) in \(\mathbb{Z}^d\).

Consider a minimal free \(\mathbb{Z}^d\)-action \((X, \mathcal{A})\) on the Cantor set \(X\) and a combinatorial data \((\bar{x}, \{c_n\}_{n \geq 0}), \{(\bar{m}_n)\}_{\bar{m} \in \mathcal{R}_{n+1}(x)}\) associated with this action.

For any \(n \geq 0\) and any \(\bar{\theta} \in \mathbb{R}^d\) we define the \(\bar{\theta}\)-length of \(\bar{F}_n\) of dimension 1 and \(d\) respectively by:
\[
l^1_{n,\bar{\theta}} = \max_{r_n \in \bar{F}_n} |||c^{l}_{\bar{\theta}}(r_n)||| \quad \text{and} \quad l^d_{n,\bar{\theta}} = \max_{r_n \in \bar{F}_n} |||c^{d}_{\bar{\theta}}(r_n)|||
\]
where $||| \cdot |||$ stands for the Euclidean distance to 0 on the $k$-torus, $\mathbb{T}^k = \mathbb{R}^k / \mathbb{Z}^k$, $k = 1, d$. The following theorem is the main result of this paper.

**Theorem 3.1.** Let $(X, A)$ be a free minimal $\mathbb{Z}^d$-action on the Cantor set $X$, $(x, \{C_n\}_{n \geq 0}, \{\{P_n(\tilde{m})\}_{m \in R_{n+1}(x)}\}_{n \geq 0}, \{\bar{F}_n\}_{n \geq 0})$ be an associated combinatorial data and $k = 1$ or $k = d$.

(i) Assume that for some $\bar{\theta} \in \mathbb{R}^d$, $(X, A)$ is an extension of the action $(\mathcal{O}^k, A^k_{\bar{\theta}})$. Assume furthermore that the combinatorial data is well distributed. Then the series $\sum_{n \geq 0} l^k_{n, \bar{\theta}}$ converges.

(ii) Conversely assume that the action is linearly recurrent, that the combinatorial data is adapted to the action and that, for some $\bar{\theta} \in \mathbb{R}^d$, the series $\sum_{n \geq 0} l^k_{n, \bar{\theta}}$ converges. Then $(X, A)$ is an extension of the action $(\mathcal{O}^k, A^k_{\bar{\theta}})$.

**Remark 1:** In the particular case when the $\mathbb{Z}^d$-action $A$ is the product of $d$ linearly recurrent $\mathbb{Z}$-actions on $X$, Theorem 3.1 for $k = d$ is a direct corollary of its $d = 1$ version proved in [BDM].

**Remark 2:** The lie group structure of $\mathbb{T}^k$ allows us to construct a continuous surjective map $\phi : \mathbb{T}^d \to \mathbb{T}^k$ defined by $\phi(\alpha_1, \ldots, \alpha_d) = \alpha_1 + \cdots + \alpha_d$. Assume that $h : (X, A) \to (\mathcal{O}^d, A^d_{\bar{\theta}})$ is an extension, then the map $\phi \circ h : (X, A) \to (\mathcal{O}^k, A^k_{\bar{\theta}})$ is also an extension. This is coherent with the fact that the convergence of the series $\sum_{n \geq 0} l^d_{n, \bar{\theta}}$ implies the convergence of the series $\sum_{n \geq 0} l^k_{n, \bar{\theta}}$.

**Proof of Theorem 3.1.** The proofs of both assertions of Theorem 3.1 for $k = 1$ or $k = d$ follow the same scheme and will be gathered in a single demonstration. Let $<< \cdot, \cdot >>$ stand for $[\cdot, \cdot] \mod \mathbb{Z}^d$ when $k = d$ and for $\langle \cdot, \cdot \rangle \mod \mathbb{Z}$ when $k = 1$.

(i) Assume that the free minimal $\mathbb{Z}^d$-action $(X, A)$ is an extension of the action $A^k_{\bar{\theta}}$ on the closure $\bar{O}^k$ of the orbit of the point 0 in the $k$-torus $\mathbb{T}^k$ for some $\bar{\theta} \in \mathbb{R}^d$.

Let us denote by $h : X \to \bar{O}^k$ the extension. Choose a well distributed associated combinatorial data

$$(x, \{C_n\}_{n \geq 0}, \{\{P_n(\tilde{m})\}_{m \in R_{n+1}(x)}\}_{n \geq 0}, \{\bar{F}_n\}_{n \geq 0})$$

and fix $h(x) = 0 \in \mathbb{T}^k$.

For each $n \geq 0$ let $v_n$ be the first return vector in $\bar{F}_n$ such that:

$$l^k_{n, \bar{\theta}} = \max_{u_n \in \bar{F}_n} |||c^k_{\bar{\theta}}(u_n)||| = |||c^k_{\bar{\theta}}(v_n)|||.$$

The following observation is a direct consequence of the continuity of $h$.

**Lemma 3.1.** The quantity $l^k_{n, \bar{\theta}}$ goes to 0 as $n$ goes to $\infty$. Furthermore, for each $\epsilon > 0$ there exists $N > 0$ such that for any pair of points $\langle \tilde{n}, \tilde{m} \rangle$ in $R_N(x) \times R_N(x)$, we have:

$$|||h(A(\tilde{n}, x)) - h(A(\tilde{m}, x))||| \leq \epsilon.$$
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Let $S_{\epsilon_1, \ldots, \epsilon_k} = \{(x_1, \ldots, x_k) \in B : x_i \cdot \epsilon_i \geq 0, \forall i \in \{1, \ldots, k\}\}.$

Let $I_{\epsilon_1, \ldots, \epsilon_k}$ be the set of integers $n$ such that $c^k_n(v_n)$ is in $S_{\epsilon_1, \ldots, \epsilon_k}$ and let us prove that the series $\sum_{n \in I_{\epsilon_1, \ldots, \epsilon_k}} t^k_{n, \bar{\theta}}$ converges. Actually, we only need to prove that the series $\sum_{n \in I_{\epsilon_1, \ldots, \epsilon_k}} t^k_{n, \bar{\theta}}$ converges, a similar proof works for the other cases. This sum can be split into two parts:

$$\sum_{n \in I_{\epsilon_1, \ldots, \epsilon_k}} t^k_{n, \bar{\theta}} = \sum_{n \in I_{\epsilon_1, \ldots, \epsilon_k}, \text{even}} t^k_{n, \bar{\theta}} + \sum_{n \in I_{\epsilon_1, \ldots, \epsilon_k}, \text{odd}} t^k_{n, \bar{\theta}}.$$

Here again we only need to prove that the series $\sum_{n \in I_{\epsilon_1, \ldots, \epsilon_k}, \text{even}} t^k_{n, \bar{\theta}}$ converges, a similar proof works also for the case where $n$ is odd. Observe that we are assuming $I_{\epsilon_1, \ldots, \epsilon_k}$ is infinite.

The proof splits in five steps:



**Step 1:** Fix an even integer $N_0$ big enough in $I_{\epsilon_1, \ldots, \epsilon_k}$, and let $N < n_1 < n_2 < \cdots < n_1 < N_0$ be the ordered sequence of even integers bigger than $N$ that belong to $I_{\epsilon_1, \ldots, \epsilon_k}$.

**Step 2:** Consider two points $\bar{m}_1$ and $\bar{p}_1$ in $\mathcal{R}_{n_1}(x)$ such that

$$v_{n_1} = \bar{p}_1 - \bar{m}_1.$$

Since the combinatorial data is well distributed, the two patches $\mathcal{P}_{n_2} (\bar{m}_1)$ and $\mathcal{P}_{n_2} (\bar{p}_1)$ are identical up to translation and there exists a pair of points $(\bar{m}_2', \bar{m}_2)$ in $\mathcal{P}_{n_2} (\bar{m}_1) \times \mathcal{P}_{n_2} (\bar{m}_1)$ such that

$$v_{n_2} = \bar{m}_2' - \bar{m}_2.$$

We define $\bar{p}_2$ in $\mathcal{P}_{n_2} (\bar{p}_1)$ by $\bar{p}_2 - \bar{p}_1 = \bar{m}_2' - \bar{m}_1 + v_{n_2}$. We have:

$$\bar{p}_2 - \bar{m}_2 = v_{n_1} + v_{n_2}.$$

**Step 3:** Since the combinatorial data is well distributed, the two patches $\mathcal{P}_{n_3} (\bar{m}_2)$ and $\mathcal{P}_{n_3} (\bar{p}_2)$ are identical up to translation and there exists a pair of points $(\bar{m}_3', \bar{m}_3)$ in $\mathcal{P}_{n_3} (\bar{m}_2) \times \mathcal{P}_{n_2} (\bar{m}_2)$ such that

$$v_{n_3} = \bar{m}_3' - \bar{m}_3.$$

We define $\bar{p}_3$ in $\mathcal{P}_{n_3} (\bar{p}_2)$ by $\bar{p}_3 - \bar{p}_2 = \bar{m}_3' - \bar{m}_2 + v_{n_3}$. We have:

$$\bar{p}_3 - \bar{m}_3 = v_{n_1} + v_{n_2} + v_{n_3}.$$

**Step 4:** We iterate this construction until we get the points $\bar{m}_l$ and $\bar{p}_l$ which satisfy:

$$\bar{p}_l - \bar{m}_l = \sum_{j=1}^{l} v_{n_j}.$$
Step 5: We have:
\[
||h(A(\bar{p}_l, x)) - h(A(\bar{m}_l, x))|| = ||<\sum_{j=1}^l v_{n_j}, \bar{\theta}>>||
\]
\[
= ||\sum_{j=1}^l c_k^j(v_{n_j})||
\]
Since $\bar{p}_l$ and $\bar{m}_l$ are in $R_N(x)$, Lemma 3.1 implies that:
\[
||\sum_{j=1}^l c_k^j(v_{n_j})|| \leq \epsilon.
\]
Let $\pi: B \to B'$ be the canonical isometric identification of the ball $B$ with the open ball $B'$ in the Euclidean space $\mathbb{R}^d$ centered at 0 with radius $\sqrt{k}/2$. Through this identification, it is clear that for all $x$ in $B$: $||x|| = ||\pi(x)||$. Moreover, for any pair of points $x, x'$ in $S_{1,...,1}$ such that $x + x'$ is also in $S_{1,...,1}$, we have: $\pi(x + x') = \pi(x) + \pi(x')$. It follows that
\[
||\sum_{j=1}^l c_k^j(v_{n_j})|| = ||\sum_{j=1}^l \pi(c_k^j(v_{n_j}))||.
\]
Finally, since for $1 \leq j \leq l$, $c_k^j(v_{n_j})$ is in $S_{1,...,1}$, we have:
\[
\sum_{j=1}^l ||\pi(c_k^j(v_{n_j}))|| \leq 1/\sqrt{k} \cdot \sum_{j=1}^l ||\pi(c_k^j(v_{n_j}))||,
\]
which implies
\[
\sum_{N \leq n, n \in I_{1,...,1}, \text{even}} \sum_{k=0}^l t_{n,\theta}^k \leq 1/\sqrt{k} \cdot \epsilon.
\]
This insures that the series $\sum_{n \in I_{1,...,1}, \text{even}} t_{n,\theta}^k$ converges, and consequently the series $\sum_{n \geq 0} t_{n,\theta}^k$ converges too.

(ii) Let $(X, A)$ be a linearly recurrent $\mathbb{Z}^d$-action on the Cantor set $X$. Assume that the combinatorial data is adapted to the action and that the series of $\bar{\theta}$-lengths $\sum_{n \geq 0} t_{n,\theta}^k$ converges for some $\bar{\theta}$ in $\mathbb{R}^d$. Fix $\epsilon > 0$ and choose $n_0 \in \mathbb{N}$ big enough so that
\[
\sum_{n \geq n_0} t_{n,\theta}^k < \epsilon.
\]
Let us define the map $h$ on the $\mathbb{Z}^d$-orbit of $x$ by,
\[
h(A(\bar{n}, x)) = <\bar{n}, \bar{\theta}>= A_\bar{\theta}(\bar{n}, 0)
\]
for each $\bar{n}$ in $\mathbb{Z}^d$. In order to prove that the map $h$ extends to a continuous map on the closure $\mathbb{Q}^k$ of the orbit of 0 in $\mathbb{T}^k$, it is enough to prove that $h$ is uniformly continuous, which follows from the continuity of $h$ at $x$. Consider a point $\bar{p}$ in $R_{n_0}(x)$ and apply Corollary 2.5. There exists a unique $m_0 \geq n_0$ and a unique sequence $\{\bar{p}_l\}_{0 \leq l \leq m_0 - n_0}$ of points in $\mathbb{Z}^d$ such that:
Rotation factors of minimal $Z^d$-actions on the Cantor set

- $m_0$ is the smallest $m \geq n_0$ for which $\bar{p} \in P_{m_0}^n(0)$;
- $\bar{p}_0 = 0$;
- $\bar{p}_l \in P_{m_0-l}(\bar{p}_{l-1})$ and $\bar{p} \in P_{m_0-l}(\bar{p}_l)$, $\forall 1 \leq l \leq m_0 - n_0$;
- $\bar{p}_{m_0-n_0} = \bar{p}$.

Let us write:

$$h(A(\bar{p}, x)) = \sum_{l=1}^{m_0-n_0} (h(A(\bar{p}_l, x)) - h(A(\bar{p}_{l-1}, x))).$$

For any $1 \leq l \leq m_0 - n_0$ both points $\bar{p}_l$ and $\bar{p}_{l-1}$ are in $P_{m_0-l}(\bar{p}_{l-1})$. Consequently there exists a collection $\{v_{m_0-l,i}\}_{1 \leq i \leq q(m_0-l)}$ of vectors in $\mathcal{F}_{m_0-l}(\bar{p}_{l-1})$ such that:

- $q(m_0 - l) \leq k(m_0 - l)$;
- the sequence of points $\{\bar{p}_{l-1,i}\}_{0 \leq i \leq q(m_0-l)}$ defined by:
  - $\bar{p}_{l-1,0} = \bar{p}_{l-1}$;
  - $\bar{p}_{l-1,i} = \bar{p}_{l-1,i-1} + v_{m_0-l,i}$ for $1 \leq i \leq q(m_0-l)$;
  - $\bar{p}_{l-1,q(m_0-l)} = \bar{p}_l$;
  - belongs to $R_{m_0-l}(x)$.

This yields

$$h(A(\bar{p}, x)) = \sum_{l=1}^{m_0-n_0} \sum_{i=1}^{q(m_0-l)} (h(A(\bar{p}_{l-1,i}, x)) - h(A(\bar{p}_{l-1,i-1}, x))).$$

Now we use the fact that the action is linearly recurrent and that the combinatorial data is adapted to this action. We denote by $L$ a uniform upper bound for the sequence $\{k(n)\}_{n \geq 0}$. We get,

$$||h(A(\bar{p}, x))|| \leq L \cdot \sum_{i=1}^{m_0-n_0} y_{m_0-l,\delta}^k \leq L \cdot \sum_{n=n_0}^{\infty} y_{n,\delta}^k \leq \epsilon.$$

This proves the continuity of $h$ at $x$. 

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