Closed operator functional calculus in Banach modules and applications

Anatoly G. Baskakov, Ilya A. Krishtal and Natalia B. Uskova

Abstract. We describe a closed operator functional calculus in Banach modules over the group algebra $L^1(\mathbb{R})$ and illustrate its usefulness with a few applications. In particular, we deduce a spectral mapping theorem for operators in the functional calculus, which generalizes some of the known results. We also obtain an estimate for the spectrum of a perturbed differential operator in a certain class.

Keywords. Functional Calculus, Banach modules, Asymptotic spectral analysis, Spectral mapping theorem.

1. Introduction

The goal of this paper is to describe a closed operator functional calculus in Banach modules over the group algebra $L^1(\mathbb{R})$ and to illustrate its usefulness with a few applications. The functional calculus was introduced in [9] in order to obtain several non-commutative extensions of Wiener’s $1/f$ lemma [27].

The first application discussed in this paper (see Theorems 3.5 and 3.12) gives several versions of the spectral mapping theorem [3, 5, 16, 24].

The second application of the functional calculus is an estimate of the spectrum $\sigma(\mathcal{L})$ of a differential operator $\mathcal{L} = A - V : D(A) \subseteq L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$, $A = -i \frac{d}{dt}$. The domain $D(A)$ is chosen to be the Sobolev space $W^{1,2}(\mathbb{R})$ of absolutely continuous functions with the (almost everywhere) derivative in $L^2(\mathbb{R})$, and the operator $V : D(A) \subseteq L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ is chosen to be of the form

$$\langle Vx\rangle(t) = v(t)x(-t), \ t \in \mathbb{R}, v \in L^2(\mathbb{R}).$$

In [10, 11], we performed the spectral analysis for an analogous operator on $L^2([0, \omega])$. In fact, in that case, $\sigma(\mathcal{L})$ is discrete and differs from $\sigma(A)$ by an $\ell^2$ sequence. Here, $\sigma(A) = \mathbb{R}$, and we end up estimating a region in $\mathbb{C}$ that contains $\sigma(\mathcal{L})$. We cite [10, 11, 14, 21, 26] and references therein for the
motivation of studying differential operators with an involution, such as the reflixion operator $V$.

The resulting estimate is contained in the following theorem.

**Theorem 1.1.** Consider the operator $\mathcal{L} = -i\frac{d}{dt} - V : W^{1,2}(\mathbb{R}) \subseteq L^2(\mathbb{R}) \to L^2(\mathbb{R})$ with $V$ of the form (1.1). Then there exists a continuous real-valued function $f \in L^2(\mathbb{R})$ such that for any $\lambda \in \sigma(\mathcal{L})$ one has $|\Im \lambda| \leq f(\Re \lambda)$.

Thus, in Theorem 1.1, the spectrum $\sigma(\mathcal{L})$ lies between the graphs of the functions $f$ and $-f$.

The remainder of the paper is organized as follows. In Section 2, we introduce the necessary notions and notation and describe the functional calculus. In Section 3, we formulate and prove a few novel versions of the spectral mapping theorem. In Section 4, we prove Theorem 1.1. Finally, Section 5 contains proofs of a few auxiliary results.

2. $\mathcal{F}L^1_{\text{loc}}$ functional calculus.

In our exposition of the closed operator functional calculus for generators of Banach $L^1(\mathbb{R})$-modules, we follow [9]. Let us introduce some notation.

We denote by $\mathcal{X}$ a complex Banach space and by $B(\mathcal{X})$ the Banach algebra of all bounded linear operators in $\mathcal{X}$. We also assume that $\mathcal{X}$ is endowed with a non-degenerate Banach module structure over the group algebra $L^1(\mathbb{R})$. The multiplication in $L^1(\mathbb{R})$ is the convolution

$$(f * g)(t) = \int_{\mathbb{R}} f(s)g(t-s)ds, \ f, g \in L^1(\mathbb{R}), \ t \in \mathbb{R}.$$  

**Definition 2.1.** A complex Banach space $\mathcal{X}$ is a Banach module over $L^1(\mathbb{R})$ if there is a bilinear map $(f, x) \mapsto fx : L^1(\mathbb{R}) \times \mathcal{X} \to \mathcal{X}$ which has the following properties:

1. $(f * g)x = f(gx)$, $f, g \in L^1(\mathbb{R})$, $x \in \mathcal{X}$;
2. $\|fx\| \leq \|f\|_1 \|x\|$, $f \in L^1(\mathbb{R})$, $x \in \mathcal{X}$.

As usual (see [7, 8] and references therein), by non-degeneracy of the module we mean that $fx = 0$ for all $f \in L^1(\mathbb{R})$ implies that $x = 0$. We only consider Banach module structures that are associated with an isometric representation $\mathcal{T} : \mathbb{R} \to B(\mathcal{X})$, that is we have

$$\mathcal{T}(t)(fx) = f_tx = f(\mathcal{T}(t)x), \ t \in \mathbb{R}, f \in L^1(\mathbb{R}), x \in \mathcal{X},$$  

where $f_t(s) = f(t + s)$, $t, s \in \mathbb{R}$. With a slight abuse of notation [8], given $f \in L^1(\mathbb{R})$, we shall denote by $\mathcal{T}(f)$ the operator in $B(\mathcal{X})$ defined by $\mathcal{T}(f)x = fx$, $x \in \mathcal{X}$. Observe that we have $\|\mathcal{T}(f)\| \leq \|f\|_1$, $f \in L^1(\mathbb{R})$, by Property 2 in Definition 2.1. For the Banach module $\mathcal{X}$, we will also use the notation $(\mathcal{X}, \mathcal{T})$ if we want to emphasize that the module structure is associated with the representation $\mathcal{T}$.

We use the Fourier transform of the form

$$(\mathcal{F}(f))(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}} f(t)e^{-it\xi}dt, \ f \in L^1(\mathbb{R}),$$
so that \( \| \hat{f} \|_2 = \sqrt{2\pi} \| f \|_2 \), \( f \in L^2(\mathbb{R}) \). We shall denote by \( \mathcal{F}L^1 = \mathcal{F}L^1(\mathbb{R}) \) the Fourier algebra \( \mathcal{F}(L^1(\mathbb{R})) \). The inverse Fourier transform of a function \( h \in \mathcal{F}L^1(\mathbb{R}) \) will be denoted by \( \hat{h} \) or \( \mathcal{F}^{-1}(h) \).

**Definition 2.2.** Let \((\mathcal{X}, \mathcal{T})\) be a non-degenerate Banach \( L^1(\mathbb{R}) \)-module, and \( N \) be a subset of \( \mathcal{X} \). The **Beurling spectrum** \( \Lambda(N) = \Lambda(N, \mathcal{T}) \) is defined by

\[
\Lambda(N, \mathcal{T}) = \{ \lambda \in \mathbb{R} : fx = 0 \text{ for all } x \in N \text{ implies } \hat{f}(\lambda) = 0, f \in L^1 \}.
\]

To simplify the notation we shall write \( \Lambda(x) \) instead of \( \Lambda(\{x\}) \), \( x \in \mathcal{X} \). We refer to \([8, \text{Lemma 3.3}]\) for the basic properties of the Beurling spectrum. We also define

\[
\mathcal{X}_{\text{comp}} = \{ x \in \mathcal{X} : \Lambda(x) \text{ is compact} \}, \quad \mathcal{X}_{\Phi} = \{ \mathcal{T}(f)x : f \in L^1(\mathbb{R}), x \in \mathcal{X} \}
\]

and

\[
\mathcal{X}_c = \{ x \in \mathcal{X} : \text{the function } t \mapsto \mathcal{T}(t)x : \mathbb{R} \to \mathcal{X} \text{ is continuous} \}.
\]

For any \( z \in \mathbb{C} \setminus \mathbb{R} \), consider the function \( f_z \in L^1(\mathbb{R}) \) whose Fourier transform is the function \( \phi_z : \mathbb{R} \to \mathbb{C} \) defined by \( \phi_z(\lambda) = (\lambda - z)^{-1} \), \( \lambda \in \mathbb{R} \). Hilbert’s resolvent identity holds for the operator-valued function \( R : \mathbb{C} \setminus \mathbb{R} \to \text{B}(\mathcal{X}) \) given by \( R(z) = \mathcal{T}(f_z) \), \( z \in \mathbb{C} \setminus \mathbb{R} \). Since the \( L^1(\mathbb{R}) \)-module \( \mathcal{X} \) is non-degenerate, we have \( \bigcap_{z \in \mathbb{C} \setminus \mathbb{R}} \ker R(z) = \{ 0 \} \). Therefore \([6] \), \( R \) is the resolvent of some linear operator \( \mathcal{A} : D(\mathcal{A}) \subseteq \mathcal{X} \to \mathcal{X} \). This operator \( \mathcal{A} \) is called the generator of the \( L^1(\mathbb{R}) \)-module \( \mathcal{X} \). We remark that if \( \mathcal{T} : \mathbb{R} \to \text{B}(\mathcal{X}) \) is a strongly continuous group representation, then \( i\mathcal{A} \) is its generator.

It is not hard to show that the operators \( \mathcal{T}(f), f \in L^1(\mathbb{R}) \), provide a functional calculus for the generator \( \mathcal{A} \). Via the isomorphism of \( L^1(\mathbb{R}) \) and \( \mathcal{F}L^1(\mathbb{R}) \), we also get the functional calculus \( \hat{\mathcal{T}}(\hat{f}) = \mathcal{T}(f), \hat{f} = \mathcal{F}(f) \in \mathcal{F}L^1 \).

It is useful to extend this functional calculus to the space \( \mathcal{F}L^1_{\text{loc}}(\mathbb{R}) = \{ h : \mathbb{R} \to \mathbb{C} \text{ such that } h\varphi \in \mathcal{F}L^1(\mathbb{R}) \text{ for any } \varphi \in L^1(\mathbb{R}) \text{ with } \text{supp } \varphi \text{ compact} \} \). Observe that \( \mathcal{F}L^1(\mathbb{R}) \subset \mathcal{F}L^1_{\text{loc}}(\mathbb{R}) \). Moreover, \( \mathcal{F}L^1_{\text{loc}} \) is also an algebra under pointwise multiplication.

For \( h \in \mathcal{F}L^1_{\text{loc}}(\mathbb{R}) \) we define a (closed) operator \( \hat{\mathcal{T}}(h) = h \circ : D(h\varphi) = D(\hat{\mathcal{T}}(h)) \subseteq \mathcal{X} \to \mathcal{X} \) in the following way. First, let \( x \in \mathcal{X}_{\text{comp}} \) and

\[
\hat{\mathcal{T}}(h)x = h \circ x := (h\varphi)^{\vee}x = \mathcal{T}((h\varphi)^{\vee})x, \tag{2.2}
\]

where \( \varphi \in L^1(\mathbb{R}) \) is such that \( \text{supp } \varphi \) is compact and \( \varphi \equiv 1 \) in a neighborhood of \( \Lambda(x) \). The vector \( \hat{\mathcal{T}}(h)x \) is well defined in this way because it is independent of the choice of \( \varphi \).

Next, we extend the definition of \( \hat{\mathcal{T}}(h) \) by taking the closure of the just defined operator on \( \mathcal{X}_{\text{comp}} \). In other words, if \( x_n \in \mathcal{X}_{\text{comp}}, n \in \mathbb{N}, x = \lim_{n \to \infty} x_n \), and \( y = \lim_{n \to \infty} h \circ x_n \) exists, we let \( \hat{\mathcal{T}}(h)x = h \circ x = y \). Lemma 2.7 in \([9] \) shows that \( \hat{\mathcal{T}}(h) \) is then a well-defined closed linear operator and we do, indeed, have \( \hat{\mathcal{T}}(\hat{f}) = \mathcal{T}(f), f \in L^1(\mathbb{R}) \). Moreover, applying \([9, \text{Proposition 2.8}] \), we get for a given \( x \in D(\hat{\mathcal{T}}(h)) \) that \( \mathcal{T}(t)(h \circ x) = h \circ (\mathcal{T}(t)x) \) and

\[
\mathcal{T}(f)(h \circ x) = h \circ (\mathcal{T}(f)x) = (\hat{f}h) \circ x, \tag{2.3}
\]
t ∈ ℝ, f ∈ L^1(ℝ), h ∈ F L^1_{loc}(ℝ). We note that D(\tilde{T}(h)) ⊆ X_c for all h ∈ F L^1_{loc}. We also note the following useful property that is implied by the definition of the operators \tilde{T}(h) and [8, Lemma 3.3]:

\[ \Lambda(\tilde{T}(h)x, T) \subseteq \sup h \cap \Lambda(x, T), \ h \in F L^1_{loc}, x \in X. \] (2.4)

It is not hard to see that the generator A of the module (X, T) satisfies

\[ A = \tilde{T}(\text{id}), \] (2.5)

where id ∈ F L^1_{loc}(ℝ) is the identity function id(ξ) = ξ, ξ ∈ ℝ. Thus, we have an F L^1_{loc} functional calculus for the generator A, which we will use to prove a few spectral mapping results and construct a similarity transform to obtain an estimate for the spectrum of the perturbed differential operators.

We will use the following sufficient condition for functions in L^2 to belong to F L^1. For completeness, we provide its proof in Section 5.

**Lemma 2.1.** Assume f ∈ L^2(ℝ) and \( \hat{f} \in W^{1,2}(ℝ) \). Then f ∈ L^1(ℝ) and

\[ \| f \|_1^2 \leq 2 \| \hat{f} \|_2 \| \hat{f}' \|_2. \] (2.6)

We illustrate the above lemma with the following two examples.

**Example 2.1.** For \( a > 0 \), consider the “trapezoid function” \( τ_a \) defined by

\[ τ_a(ξ) = \begin{cases} 1, & |ξ| \leq a, \\ \frac{1}{a}(2a - |ξ|), & a < |ξ| < 2a, \\ 0, & |ξ| \geq 2a. \end{cases} \]

Direct computations show that \( \| τ_a \|_2 = 2\sqrt{\frac{2}{3}a}, \| τ_a' \|_2 = \sqrt{\frac{2}{a}}, \) and \( τ_a = \varphi_a \), where

\[ \varphi_a(t) = \frac{2 \sin \frac{3at}{\pi} \sin \frac{at}{\pi}}{\pi at^2}, \ t ∈ ℝ. \]

From Lemma 2.1 we conclude that \( \| \varphi_a \|_1 \leq 2\frac{\pi}{3} \cdot 3^{-\frac{4}{3}}. \) We remark that [22, Lemma 1.10.1] yields a better estimate: \( \| \varphi_a \|_1 \leq \frac{4}{\pi} + \frac{2}{\pi} \ln 3. \) An even better estimate, \( \| \varphi_a \|_1 \leq \sqrt{3} \), follows from [25, Proposition 5.1.5].

**Example 2.2.** For \( a > 0 \), let

\[ ω_a(ξ) = \frac{1}{ξ}(1 - τ_a(ξ)) = \begin{cases} 0, & |ξ| \leq a, \\ -\frac{1}{a} - \frac{1}{ξ}, & -2a < ξ \leq -a, \\ \frac{1}{a} - \frac{1}{ξ}, & a < ξ \leq 2a, \\ \frac{1}{ξ}, & |ξ| > 2a. \end{cases} \]

Then \( \| ω_a \|_2 = \sqrt{\frac{4 - 4 \ln 2}{a}} \leq 1.11/\sqrt{a} \) and \( \| ω_a' \|_2 = \sqrt{\frac{2}{\ln a}} \leq 0.82/(a \sqrt{a}). \) It follows that the functions \( ψ_a \) defined by \( \hat{ψ}_a = ω_a \) satisfy \( \| ψ_a \|_1^2 = \frac{4}{a^2} \sqrt{\frac{2}{3}(1 - \ln 2)} \) so that \( \| ψ_a \|_1 \leq 1.35/a. \)
We will also need an estimate for $\|\psi_a\|_\infty$. Observe that for $t > 0$

$$\frac{1}{2\pi} \left| \int_{-\infty}^{\infty} \omega_a(\xi) e^{it\xi} d\xi \right| = \frac{1}{\pi} \left| \int_{0}^{\infty} \omega_a(\xi) \sin(t\xi) d\xi \right| \leq \frac{1}{\pi} \left( 1 + \left| \int_{a}^{\infty} \frac{\sin(t\xi)}{\xi} d\xi \right| \right) \leq \frac{1}{\pi} + 1. $$

Since $\omega_a$ is an odd function, it follows that

$$\|\psi_a\|_\infty \leq \frac{1}{\pi} + 1. \quad (2.7)$$

Now we use the functions from the above two examples in our $\mathcal{F}L^1_{loc}$ functional calculus. In view of (2.3) and (2.5), we get the following crucial relationship:

$$\mathcal{A}T(\psi_a)x = \mathcal{T}(id)\mathcal{T}(\psi_a)x = \mathcal{T}(id \cdot \omega_a)x = \mathcal{T}(\mathbb{1} - \tau_a)x = x - \mathcal{T}(\varphi_a)x, \quad (2.8)$$

which holds for every $x \in D(A)$; by $\mathbb{1} \in \mathcal{F}L^1_{loc}$ we denoted the function $\mathbb{1}(\xi) = 1, \xi \in \mathbb{R}$. In fact, since $D(\mathcal{T}(\mathbb{1})) = \mathcal{X}_c$ and $A$ is a closed operator, we get that (2.8) holds for all $x \in \mathcal{X}_c$. Moreover, non-degeneracy of the module $(\mathcal{X}, \mathcal{T})$ implies that the operator $\mathcal{A}T(\psi_a)$ extends uniquely to a bounded operator $I - \mathcal{T}(\varphi_a) \in B(\mathcal{X})$. To simplify the notation, given $\lambda \in \mathbb{C}$, we may write $\lambda - f$ instead of $\lambda \mathbb{1} - f$ for functions and $\lambda - A$ instead of $\lambda I - A$ for operators.

We note that the family $(\varphi_\alpha)$ from Example 2.1 has another useful property: it forms a bounded approximate identity.

**Definition 2.3.** A family of functions $(\varphi_a)_{a > 0}$ is called a bounded approximate identity or b.a.i. if $\|\varphi_a\|_1 \leq M$ for all $a > 0$ and $\lim_{a \to \infty} \|\varphi_a \ast f - f\|_1 = 0$ for all $f \in L^1(\mathbb{R})$.

Following [9], we call a b.a.i. $(\varphi_a)$ a cf-b.a.i. if supp $\hat{\varphi_a}$ is compact for each $a > 0$. Clearly, the family $(\varphi_\alpha)$ from Example 2.1 is a cf-b.a.i. Another useful cf-b.a.i. is given by the family

$$\gamma_a(t) = \frac{a}{2\pi} \left( \frac{\sin(at/2)}{at/2} \right)^2, \quad t \in \mathbb{R}, a > 0. \quad (2.9)$$

Observe that $\hat{\gamma}_a(\xi) = (1 - |\xi|/a)\mathbb{1}_{[-a,a]}(\xi) =: \Delta_a(\xi)$ is the so called triangle function and, since $\gamma_a$ is non-negative, $\|\gamma_a\|_1 = \Delta_a(0) = 1$.

We conclude this section by recalling the following result that contains the celebrated Cohen-Hewitt factorization theorem [15, 20].

**Proposition 2.2 ([8], Lemma 4.3).** For any b.a.i. $(\varphi_a)$, we have

$$\mathcal{X}_c = \mathcal{X}_\Phi = \overline{\mathcal{X}_{comp}} = \{ x \in \mathcal{X} : x = \lim_{a \to \infty} \varphi_a x = x \}.$$
3. Spectral mapping theorems.

We begin this section by recalling a spectral mapping theorem from [5] which we endeavor to extend (see also [16, 24]).

**Theorem 3.1 ([5], Corollary 1.5.3).** Let $X$ be a non-degenerate Banach $L^1(\mathbb{R})$-module with the structure associated with a representation $T$. For $f \in L^1(\mathbb{R})$, $\sigma(T(f)) = \hat{f}(\Lambda(X, T))$.

To prove our extensions we will need the following two lemmas that, in particular, give a special case of the above result.

**Lemma 3.2.** Assume that $K = \Lambda(X, T)$ is compact. Then, for $h \in FL^1_{\text{loc}}$, we have $\sigma(\hat{T}(h)) \subseteq h(K)$.

**Proof.** Observe that $h(K)$ is automatically compact as a continuous image of a compact set. Assume $\lambda \notin h(K)$. Then the function $u$ given by $u(t) = \frac{1}{\lambda - t}$ is analytic in a neighborhood of the compact set $h(K)$. The Wiener-Lévy theorem [25, Theorem 1.3.1] ensures existence of a function $f \in L^1(\mathbb{R})$ such that $\hat{f}(\xi) = \frac{1}{\lambda - h(\xi)}$ for $\xi$ in a neighborhood of $K$. For $x \in X = X_{\text{comp}}$, we use (2.3) and [8, Lemma 3.3] to obtain

$$(\lambda - T(f))\hat{T}(h)x = \hat{T}(h)(\lambda - T(f))x = \hat{T}(\lambda \hat{f} - \hat{f}h)x = \hat{T}(1)x = x,$$

so that $\lambda \in \rho(\hat{T}(h))$. Thus, $\sigma(\hat{T}(h)) \subseteq h(K)$.

**Lemma 3.3.** For $h \in FL^1_{\text{loc}}(\mathbb{R})$, we have $\sigma(\hat{T}(h)) \supseteq h(\Lambda(X, T))$.

**Proof.** Assume $\lambda = h(\xi) \in \rho(\hat{T}(h))$ for some $\xi \in \mathbb{R}$ and let $f \in L^1(\mathbb{R})$ be such that $\text{supp} \hat{f}$ is compact, $\xi \in \text{supp} \hat{f}$, and $\hat{f}(\text{supp} h) \subseteq \rho(\hat{T}(h))$. Let also $\mathcal{Y} = T(f)X$ and $B = \hat{T}(h)|_{\mathcal{Y}}$ be the restriction of the operator $\hat{T}(h)$ to the submodule $\mathcal{Y}$. Clearly, $B \in B(\mathcal{Y})$. We claim that $\rho(B) = \mathbb{C}$, which would imply $\mathcal{Y} = \{0\}$ yielding $\text{supp} \hat{f} \cap \Lambda(X, T) = \emptyset$. To prove the claim, we first observe that $\rho(\hat{T}(h)) \subseteq \rho(B)$. Indeed, since $\hat{T}(h)$ commutes with $T(f)$ by (2.3), we have that for $\lambda \in \rho(\hat{T}(h))$ the resolvent operator $(\lambda I - \hat{T}(h))^{-1}$ also commutes with $T(f)$ ensuring $(\lambda I - B)^{-1} = (\lambda I - \hat{T}(h))^{-1}|_{\mathcal{Y}}$. Using (2.4), we get

$$\hat{f}(\Lambda(\mathcal{Y})) \subseteq \hat{f}(\text{supp} h) \subseteq \rho(\hat{T}(h)) \subseteq \rho(B).$$

Secondly, Lemma 3.2 implies $\hat{f}(\Lambda(\mathcal{Y}))^c \subseteq \rho(B)$, and the claim is established.

Thus, to extend Theorem 3.5 to the $FL^1_{\text{loc}}$ setting we only need an analog of Lemma 3.2 for the case when $\Lambda(X)$ is not necessarily compact. This, however, may not always hold at this level of generality as we can no longer use the Wiener-Lévy theorem. We offer several ways to circumvent the problem.

First, we present a result that is immediate from the proof of Lemma 3.2.
Proposition 3.4. Let $h \in \mathcal{F}L_{\text{loc}}^1$ and $\lambda \in \mathbb{C}$. Assume that there exists a function $g_\lambda \in \mathcal{F}L_{\text{loc}}^1$ such that $g_\lambda = (\lambda - h)^{-1}$ in a neighborhood of $\Lambda(\mathcal{X}, T)$ and $\mathcal{T}(g_\lambda)$ belongs to $B(\mathcal{X})$. Then $\lambda \in \rho(\mathcal{T}(h))$.

This motivates the following definition.

Definition 3.1. Let $\mathcal{X} = (\mathcal{X}, T)$ be a non-degenerate Banach $L^1(\mathbb{R})$-module. A function $h \in \mathcal{F}L_{\text{loc}}^1(\mathbb{R})$ is called $\mathcal{X}$-regular if for any $\lambda \notin h(\Lambda(\mathcal{X}, T))$ there exists a function $g_\lambda \in \mathcal{F}L_{\text{loc}}^1(\mathbb{R})$ such that

$$g_\lambda(\xi)(\lambda - h(\xi)) = 1,$$

for every $\xi$ in a neighborhood of $\Lambda(\mathcal{X}, T)$, and $\mathcal{T}(g_\lambda)$ belongs to $B(\mathcal{X})$.

Clearly, it would be sufficient for the functions $g_\lambda$ in the above definition to belong to $\mathcal{F}L^1$. Hence, by the Wiener-Lévy theorem, if $\Lambda(\mathcal{X})$ is compact, every $h \in \mathcal{F}L_{\text{loc}}^1(\mathbb{R})$ is $\mathcal{X}$-regular.

The next result is now immediate.

Theorem 3.5. Assume that $h \in \mathcal{F}L_{\text{loc}}^1(\mathbb{R})$ is $\mathcal{X}$-regular. Then $\sigma(\mathcal{T}(h)) = h(\Lambda(\mathcal{X}, T))$. Moreover, given $\lambda \notin h(\Lambda(\mathcal{X}, T))$, we have $(\lambda - \mathcal{T}(h))^{-1} = \mathcal{T}(g_\lambda)$, where $g_\lambda \in \mathcal{F}L_{\text{loc}}^1(\mathbb{R})$ is defined by (3.1).

The following result shows that Theorem 3.5 does indeed generalize Theorem 3.1.

Proposition 3.6. Any function $h \in \mathcal{F}L^1(\mathbb{R})$ is $\mathcal{X}$-regular for any $\mathcal{X}$.

Proof. Observe that by the Riemann-Lebesgue lemma if $0 \notin h(\Lambda(\mathcal{X}, T))$ then $\Lambda(\mathcal{X}, T)$ is compact, and the result follows.

Assume now that $0 \neq \lambda \notin h(\Lambda(\mathcal{X}, T))$. Then, without loss of generality we may assume that $\lambda \notin h(\mathbb{R})$. Indeed, if that was not the case, we would have $0 \neq \lambda \in h(\mathbb{R})$ and $h^{-1}(\{\lambda\})$ would be a compact set disjoint from $\Lambda(\mathcal{X}, T)$. We could then find $\phi \in \mathcal{F}L^1$ with compact support that is disjoint from $\Lambda(\mathcal{X}, T)$ and such that $\lambda \notin (h + \phi)(\mathbb{R})$. We would then apply the following argument to $h + \phi$ instead of $h$.

A modification of the Wiener-Lévy theorem (see [25, Theorem 1.3.4] or [17]) or a special case of the Bochner-Phillips theorem (see [9, Theorem 10.3] or [13]) show that

$$g_\lambda = (\lambda - h)^{-1} = \lambda^{-1} + \tilde{h}$$

for some $\tilde{h} \in \mathcal{F}L^1$. Then $\mathcal{T}(g_\lambda) = \lambda^{-1}I + \mathcal{T}(\tilde{h}) \in B(\mathcal{X})$, and the result follows.

Another sufficient condition for $\mathcal{X}$-regularity follows from Lemma 2.1.

Proposition 3.7. Assume that $h \in \mathcal{F}L_{\text{loc}}^1(\mathbb{R})$ is such that for every $\lambda \notin h(\Lambda(\mathcal{X}, T))$ there exists a function $g_\lambda \in W^{1,2}(\mathbb{R})$ that satisfies (3.1) in a neighborhood of $h(\Lambda(\mathcal{X}, T))$. Then $h$ is $\mathcal{X}$-regular. In particular, every polynomial is $\mathcal{X}$-regular for any $\mathcal{X}$. 

Proof. The first assertion follows immediately from Lemma 2.1. To prove the second one, we note that for every polynomial \( p \) and \( \lambda \notin p(\mathbb{R}) = p(\mathbb{R}) \) the function \( (\lambda - p(\cdot))^{-1} \) belongs to \( W^{1,2}(\mathbb{R}) \). This shows that \( p \) is \( \mathcal{X} \)-regular for any \( \mathcal{X} \) such that \( \Lambda(\mathcal{X}, T) = \mathbb{R} \). In case \( \Lambda(\mathcal{X}, T) \neq \mathbb{R} \), given \( \lambda \in p(\mathbb{R}) \setminus p(\Lambda(\mathcal{X}, T)) \), we observe that \( p^{-1}\{\lambda\} \) is a finite set that does not intersect \( \Lambda(\mathcal{X}, T) \). Hence, there is an infinitely many times differentiable function \( \phi_\lambda \in FL^1 \) with compact support that is disjoint from \( \Lambda(\mathcal{X}, T) \) and such that \( \lambda \notin (p + \phi_\lambda)(\mathbb{R}) \). Then \( g_\lambda = (\lambda - p - \phi_\lambda)^{-1} \in W^{1,2}(\mathbb{R}) \), and the result follows.

The following well-known result now follows immediately from (2.5) and the fact that the Beurling spectrum is a closed set.

**Corollary 3.8.** The generator \( A \) of a non-degenerate Banach \( L^1(\mathbb{R}) \)-module \( \mathcal{X} \) satisfies \( \sigma(A) = \Lambda(\mathcal{X}) \).

In the context of Proposition 3.7, Lemma 2.1 also allows us to estimate the resolvent of the operators \( \tilde{T}(h) \).

**Corollary 3.9.** Assume that \( h \in FL^1_{loc}(\mathbb{R}), \lambda \notin h(\mathbb{R}) \), and the function \( g_\lambda \) defined by (3.1) belongs to \( W^{1,2}(\mathbb{R}) \). Then

\[
\left\| (\lambda - \tilde{T}(h))^{-1} \right\| \leq \sqrt{2\parallel g_\lambda \parallel_2 \parallel g_\lambda' \parallel_2}.
\]

The following definition allows us to provide yet another example of \( \mathcal{X} \)-regularity.

**Definition 3.2.** We say that \( h \in FL^1_{loc}(\mathbb{R}) \) is an almost periodic function with a summable Fourier series if

\[
h(\xi) = \sum_{n \in \mathbb{Z}} c_n e^{i \xi n}, \quad \sum_{n \in \mathbb{Z}} |c_n| < \infty, \quad \xi, t_n \in \mathbb{R}, n \in \mathbb{Z}.
\]

The set of all such functions is denoted by \( \mathcal{AP}_1 \) or \( \mathcal{AP}_1(\mathbb{R}) \).

We note that \( \mathcal{AP}_1 \) is a Banach space with the norm

\[
\|h\|_{\mathcal{AP}_1} = \sum_{n \in \mathbb{Z}} |c_n|,
\]

where \( h \in \mathcal{AP}_1 \) is given by (3.2). We also mention [9, Proposition 2.11], which states that for such \( h \) we have

\[
\tilde{T}(h) = \sum_{n \in \mathbb{Z}} c_n T(t_n) \in B(\mathcal{X}).
\]

**Proposition 3.10.** Any function \( h \in \mathcal{AP}_1 \) is \( \mathcal{X} \)-regular for any \( \mathcal{X} \). Moreover, if \( \lambda \notin h(\mathbb{R}) \), then

\[
\left\| (\lambda - \tilde{T}(h))^{-1} \right\| \leq \left\| \frac{1}{\lambda - h} \right\|_{\mathcal{AP}_1}.
\]

**Proof.** Let \( \mathbb{R}_d \) be the group of real numbers with the discrete topology and \( \mathbb{R}_c \) be its Pontryagin dual – the Bohr compactification of \( \mathbb{R} \). It is well-known that a function in \( \mathcal{AP}_1(\mathbb{R}) \) has a unique continuous extension to \( \mathbb{R}_c \) and can be identified with an element of \( FL^1(\mathbb{R}_d) \) – the Fourier algebra of the
group $\mathbb{R}_d$. The closure of $\Lambda(\mathcal{X}, T)$ in $\mathbb{R}_c$ is then a compact subset of $\mathbb{R}_c$ and, given $\lambda \notin h(\Lambda(\mathcal{X}, T))$, the Wiener-Lévy theorem for locally compact Abelian groups ([25, Theorem 6.1.1]) establishes existence of $g_\lambda \in \mathcal{AP}_1(\mathbb{R})$ that satisfies (3.1) in a neighborhood of $\Lambda(\mathcal{X}, T)$ in $\mathbb{R}$. Hence, $\mathcal{X}$-regularity follows from (3.3), i.e. [9, Proposition 2.11].

For $\lambda \notin h(\mathbb{R})$, it suffices to apply the almost periodic version of Wiener’s $1/f$ lemma [2, 23] that shows that $g_\lambda = \frac{1}{\lambda - h} \in \mathcal{AP}_1$. The desired estimate then follows from (3.3).

**Example 3.1.** Let $h(\xi) = e^{i\xi t_0}$ for some $t_0 \in \mathbb{R}$ and $\lambda \in \mathbb{C} \setminus \mathbb{T}$, where $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. From (3.3), we get $\tilde{T}(h) = T(t_0)$. Using the estimate from Proposition 3.10, we get by direct computation that

$$
\left\| (\lambda - T(t_0))^{-1} \right\| \leq \left\| \frac{1}{\lambda - h} \right\|_{\mathcal{AP}_1} = \frac{1}{|1 - |\lambda||} = (\text{dist}(\lambda, \mathbb{T}))^{-1}.
$$

In general, it may be hard to check if a given function is $\mathcal{X}$-regular. We cite [7] and references therein for related results. We note that the notion of regularity at infinity discussed in [7] is more restrictive than $\mathcal{X}$-regularity.

The following theorem provides a special case when the assumption of $\mathcal{X}$-regularity is not needed.

**Theorem 3.11.** Assume that $\mathcal{X} = H$ is a Hilbert space and the representation $T$ is unitary. Then for any $h \in \mathcal{FL}_{\text{loc}}^1(\mathbb{R})$ we have $\sigma(\tilde{T}(h)) = h(\Lambda(\mathcal{X}, T))$. Moreover, given $\lambda \notin h(\Lambda(\mathcal{X}, T))$, we have

$$
\left\| (\lambda - \tilde{T}(h))^{-1} \right\| = \left( \text{dist} \left( \lambda, \overline{h(\Lambda(\mathcal{X}, T))} \right) \right)^{-1}. \tag{3.4}
$$

**Proof.** In view of Lemma 3.3, we only need to prove $\sigma(\tilde{T}(h)) \subset h(\Lambda(\mathcal{X}, T))$, $h \in \mathcal{FL}_{\text{loc}}^1(\mathbb{R})$. Pick $\lambda \notin h(\Lambda(\mathcal{X}, T))$. We will show that $\lambda \in \rho(\tilde{T}(h))$.

Let $\{\phi_a\}$ be a cf-b.a.i. and $\mathcal{X}_a = \mathcal{T}(\phi_a)\mathcal{X}$, $a > 0$, be the corresponding submodules of $\mathcal{X}$. From (2.4), we have $\Lambda(\mathcal{X}_a, T) \subseteq \text{supp} \, \hat{\phi}_a \cap \Lambda(\mathcal{X}, T)$. Hence, $\mathcal{X}_a \subseteq \mathcal{X}_{\text{comp}} \subseteq \mathcal{D}(\tilde{T}(h))$ and (2.3) implies that $\mathcal{X}_a$ is invariant for $\tilde{T}(h)$. Therefore, the restrictions of $\tilde{T}(h)$ to $\mathcal{X}_a$, $a > 0$, are well defined. We will denote these restrictions by $B_a$.

Since $h$ is $\mathcal{X}_a$-regular, Theorem 3.5 applies for $B_a$ yielding $\sigma(B_a) = h(\Lambda(\mathcal{X}_a, T)) \subseteq h(\overline{\Lambda(\mathcal{X}, T)})$. It follows that $\lambda \in \rho(B_a)$. Moreover, since the representation $T$ is unitary, the operators $B_a$ are normal. Therefore, the norms of their resolvents satisfy

$$
\| R(\lambda; B_a) \| = (\text{dist}(\lambda, \sigma(B_a))^{-1} \leq \left( \text{dist} \left( \lambda, h(\overline{\Lambda(\mathcal{X}, T)}) \right) \right)^{-1}. \tag{3.5}
$$

Now, since the representation $T$ is strongly continuous, Proposition 2.2 implies that

$$
x = \lim_{a \to \infty} \mathcal{T}(\phi_a)x
$$
for an arbitrary $x \in \mathcal{X}$. From (3.5) and the Banach-Steinhaus theorem, we get that $C = C_{\lambda}$ given by
\[
x = \lim_{a \to \infty} R(\lambda; B_a) \mathcal{T}(\phi_a)x
\] is a well-defined bounded linear operator. By direct computation, it follows that
\[
\mathcal{T}(f)C(\lambda - \hat{T}(h))x = \mathcal{T}(f)(\lambda - \hat{T}(h))Cx,
\]
for any $x \in D(\hat{T}(h))$ and $f \in L^1(\mathbb{R})$ with supp $\hat{f}$ compact. Since the module $\mathcal{X}$ is non-degenerate, we get $C = (\lambda - \hat{T}(h))^{-1}$. Finally, the equality in (3.4) follows since $C = (\lambda - \hat{T}(h))^{-1}$ is a normal operator.

**Remark 3.1.** Often [1, 4, 19] a representation $\mathcal{T}$ and operators of the form $\hat{T}(h)$ act not just in a single Banach module but in a whole chain $(\mathcal{X}_p)$ of such modules. For example, matrices with sufficient off-diagonal decay define bounded operators on all $\ell^p$, $p \in [1, \infty)$. In this case, it is not unusual for $\sigma(\hat{T}(h))$ to be independent of $p$. If also one of the modules $\mathcal{X}_p$ happened to be a Hilbert space, Theorem 3.11 would then yield a spectral mapping theorem for all Banach modules $\mathcal{X}_p$ in the chain.

The proof of Theorem 3.11 leads us to define the following notion of regularity for functions in $\mathcal{F}L^1_{loc}(\mathbb{R})$.

**Definition 3.3.** Let $(\varphi_a)$ be the cf-b.a.i. from Example 2.1. A function $h \in \mathcal{F}L^1_{loc}(\mathbb{R})$ is called spectrally admissible if for each $\lambda \notin h(\Lambda(\mathcal{X}, \mathcal{T}))$ there exist functions $g_{\lambda}^a \in \mathcal{F}L^1(\mathbb{R})$ such that
\[
g_{\lambda}^a = \frac{\hat{\varphi}_a}{(\lambda - h)}\text{ in a neighborhood of } h(\Lambda(\mathcal{X}, \mathcal{T)})
\]
and
\[
M_h(\lambda) := \sup_{a > 0} \| \mathcal{F}^{-1}(g_{\lambda}^a) \|_1 < \infty.
\] (3.7)

**Remark 3.2.** In the above definition, instead of the functions from Example 2.1 we may use a cf-b.a.i. $(\varphi_{a,n})$, $n > 1$, given by
\[
\hat{\varphi}_{a,n}(\xi) = \tau_{a,n}(\xi) = \begin{cases} 1, & |\xi| \leq a, \\
\frac{1}{(n - 1)a}(na - |\xi|), & a < |\xi| < na, \\
0, & |\xi| \geq na. \end{cases}
\]
From [25, Proposition 5.1.5] we get $\| \varphi_{a,n} \|_1 \leq \sqrt{n + 1},$ which may give a smaller $M_h(\lambda)$ in (3.7). We also note that Lemma 2.1 may often be used to prove spectral admissibility.

**Theorem 3.12.** Let $(\mathcal{X}, \mathcal{T})$ be a non-degenerate Banach $L^1(\mathbb{R})$-module such that the representation $\mathcal{T}$ is strongly continuous. Assume that a function $h \in \mathcal{F}L^1_{loc}(\mathbb{R})$ is spectrally admissible. Then $\sigma(\hat{T}(h)) = h(\Lambda(\mathcal{X}, \mathcal{T}))$. Moreover, given $\lambda \notin h(\Lambda(\mathcal{X}, \mathcal{T}))$, we have
\[
\| (\lambda - \hat{T}(h))^{-1} \| \leq M_h(\lambda).
\]
Proof. As in the proof of Theorem 3.11, given a cf-b.a.i. \(\phi_a\), we let \(X_a = \mathcal{T}(\phi_a)\mathcal{X}\) and \(B_a = \mathcal{T}(h)|_{X_a}\). Observe that for a sufficiently large \(b > 0\), the function \(g^b_\lambda\) from Definition 3.3 satisfies \(g^b_\lambda(\xi) = \frac{1}{b - h(\xi)}\) for every \(\xi\) in a neighborhood of \(h(\Lambda(X_a, \mathcal{T}))\). From Theorem 3.5, we deduce that
\[
\|R(\lambda, B_a)\| \leq \|\mathcal{F}^{-1}(g^b_\lambda)\|_1 \leq M_h(\lambda) < \infty, \lambda \notin h(\Lambda(\mathcal{X}, \mathcal{T})).
\]

The remainder of the proof of Theorem 3.11 now goes through in this setting. An application of the Banach-Steinhaus theorem shows that (3.6) defines an operator \(C_\lambda \in \mathcal{B}(H)\) satisfying \(\|C_\lambda\| \leq M_h(\lambda)\sup_a \|\phi_a\|\), and, choosing \(\phi_a = \gamma_a\) defined by (2.9) gives \(\|C_\lambda\| \leq M_h(\lambda)\). It is then verified by direct computation that \(C_\lambda = (\lambda - \mathcal{T}(h))^{-1}\).

4. Spectral estimates for the operator \(\mathcal{L}\).

In this section, we prove Theorem 1.1. The approach we pursue is based on the following result which holds for Hilbert-Schmidt perturbations of general self-adjoint operators on an abstract complex Hilbert space \(H\). The ideal of all Hilbert-Schmidt operators in \(H\) will be denoted by \(\mathcal{S}_2(H)\).

**Theorem 4.1.** Let \(A : D(A) \subseteq H \rightarrow H\) be a self-adjoint operator and \(B \in \mathcal{S}_2(H)\). Then there exists a continuous real-valued function \(f \in L^2(\mathbb{R})\) such that for any \(\lambda \in \sigma(A + B)\) one has \(|\Im \lambda| \leq f(\Re \lambda)|\).

We believe that the above result has been known for a long time. Since we didn’t find the reference, however, we provide its proof in Section 5. We cite [18] for related results.

Clearly, the perturbation \(V\) of the form (1.1) may not be Hilbert-Schmidt; it is not even a bounded operator, in general. We will, however, construct a similarity transform which will allow us to use the above result.

**Definition 4.1.** Two linear operators \(A_i : D(A_i) \subset H \rightarrow H\), \(i = 1, 2\), are called similar if there exists an invertible operator \(U \in \mathcal{B}(H)\) such that \(UD(A_2) = D(A_1)\) and \(A_1 Ux = U A_2 x, x \in D(A_2)\). We call the operator \(U\) the similarity transform of \(A_1\) into \(A_2\).

It is immediate that for similar operators \(A_1\) and \(A_2\) one has \(\sigma(A_1) = \sigma(A_2)\). Thus, to prove Theorem 1.1, it suffices to construct a similarity transform of \(\mathcal{L}\) into \(-i \frac{d}{dt} - B\) with \(B \in \mathcal{S}_2(H)\). In order to do it, we apply the \(\mathcal{FL}^1_{loc}\) functional calculus in the space \(\mathcal{X} = \mathcal{L}_A(H)\) of closed linear \(A\)-bounded operators that is defined as follows.

**Definition 4.2.** Let \(A : D(A) \subset H \rightarrow H\) be a closed linear operator. A linear operator \(X : D(X) \subset H \rightarrow H\) is \(A\)-bounded if \(D(X) \supseteq D(A)\) and \(\|X\|_A = \inf\{c > 0 : \|X x\| \leq c(\|x\| + \|Ax\|), x \in D(A)\} < \infty\).

The space \(\mathcal{L}_A(H)\) of all \(A\)-bounded linear operators with the domain equal to \(D(A)\) is a Banach space with respect to the norm \(\|\cdot\|_A\). For densely defined operators \(A\), restricting the domain of bounded operators to \(D(A)\), allows us to view \(B(H)\) as a subspace of \(\mathcal{L}_A(H)\).
Now we need to define the Banach module structure in $X = L_A(H)$ with $A = -i \frac{d}{dt}$. We begin with a Banach module structure in $H = L^2(\mathbb{R})$.

The operator $A = -i \frac{d}{dt} : W^{1,2}(\mathbb{R}) \subseteq L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is self-adjoint, and the operator $iA$ generates an isometric strongly continuous group of translations $T : \mathbb{R} \to B(H)$, $T(t)x(s) = x(t + s)$, $x \in L^2(\mathbb{R})$. The non-degenerate $L^1$-module structure in $H$ is then given by convolution:

$$T(f)x = \int_\mathbb{R} f(t)T(-t)x dt, \quad x \in H.$$  

Next, we let $\mathcal{T} : \mathbb{R} \to B(X) = B(L_A(H))$ be defined by $\mathcal{T}(t)x = T(t)XT(-t)$, $t \in \mathbb{R}$. Since $T$ is an isometric representation, we get that $\mathcal{T}$ also has this property. We then have that

$$(\mathcal{T}(f)X)x = \int_\mathbb{R} f(t)(\mathcal{T}(-t)X)x dt = \int_\mathbb{R} f(t)T(-t)XT(t)x dt, \quad (4.1)$$

$X \in X$, $x \in D(A)$, $f \in L^1(\mathbb{R})$, defines a non-degenerate $L^1$-module structure in $X$ that is associated with the representation $\mathcal{T}$. Moreover, the generator $\mathcal{A}$ of the module $(X, \mathcal{T})$ satisfies

$$\mathcal{A}X = AX -XA, \quad X \in D(A),$$

see e.g. [16]. We now apply (2.8) in this Banach module $(X, \mathcal{T})$ to get

$$A(\mathcal{T}(\psi_a)X) - (\mathcal{T}(\psi_a)AX) = X - \mathcal{T}(\varphi_a)X, \quad X \in D(A), \quad (4.2)$$

where the functions $\varphi_a$ and $\psi_a$, $a > 0$, are defined in Examples 2.1 and 2.2. Moreover, the discussion following (2.8) shows that for any $X \in L_A(H)$ and $x \in D(A)$ we have

$$A(\mathcal{T}(\psi_a)X)x - (\mathcal{T}(\psi_a)X)Ax = Xx - (\mathcal{T}(\varphi_a)X)x. \quad (4.3)$$

**Lemma 4.2.** Consider the functions $\varphi_a$ and $\psi_a$, $a > 0$, defined in Examples 2.1 and 2.2. An operator $V$ of the form (1.1) has the following properties.

1. $\mathcal{T}(\varphi_a)V \in \mathcal{S}_2(H)$ and $\|\mathcal{T}(\varphi_a)V\|_2 = 2a \sqrt{2\pi} \|v\|_2$.
2. $\mathcal{T}(\psi_a)V \in \mathcal{S}_2(H)$ and $\|\mathcal{T}(\psi_a)V\|_2 = \frac{1-\ln 2}{a\pi} \|v\|_2$.
3. $\mathcal{T}(\psi_a)V(W^{1,2}(\mathbb{R})) \subseteq W^{1,2}(\mathbb{R})$.
4. $VT(\psi_a)V \in \mathcal{S}_2(H)$ and $\|VT(\psi_a)V\|_2 \leq \frac{\pi+1}{\pi \sqrt{2}} \|v\|_2^2$.
5. Given $\epsilon > 0$, there is $\lambda_\epsilon \in \mathbb{C} \setminus \mathbb{R}$ such that $\|V(\lambda_\epsilon - A)^{-1}\| < \epsilon$.

**Proof.** Observe that for any $h \in L^1 \cap L^2$ we have

$$(\mathcal{T}(h)V)x(s) = \int_\mathbb{R} h(t)v(s-t)x(-s+2t) dt. \quad (4.4)$$

Hence, $\mathcal{T}(h)V \in \mathcal{S}_2(H)$ and

$$\|\mathcal{T}(h)V\|_2^2 = \frac{1}{2} \int_\mathbb{R} \int_\mathbb{R} |h(t)v(s-t)|^2 dsdt = \frac{1}{2} \|h\|_2^2 \|v\|_2^2.$$ 

Plugging in the norms $\|\varphi_a\|_2$ and $\|\psi_a\|_2$ from Examples 2.1 and 2.2 establishes Properties 1 and 2.
To prove Property 3, pick $z > 0$ and let $R = R(z; A) = (z - A)^{-1}$. Using the definition of the generator of a Banach module, we have

$$Rx = \int_{\mathbb{R}} f_{z}(t)T(-t)xdt, x \in H,$$

(4.5)

where $\hat{f}_{z}(\lambda) = (\lambda - z)^{-1}$. Then for any $h \in L^{1} \cap L^{2}$, letting $h_{t} = T(t)h$, one easily gets

$$(T(h)V)Rx = R(T(h_{t})V)x,$$

after plugging in (4.1) and (4.5). Hence, Property 3 follows.

Next, observe that for any $h \in L^{1} \cap L^{2}$ we have

$$V(T(h)V)x(s) = \int_{\mathbb{R}} v(s)h(t)v(-s - t)x(s + 2t)dt.$$

(4.6)

Hence, $VT(h)V \in \mathcal{G}_{2}(H)$ and

$$\|VT(h)V\|_{2}^{2} = \frac{1}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} |v(s)h(t)v(-s - t)|^{2}dsdt \leq \frac{1}{2}\|h\|_{\infty}\|v\|_{2}^{4},$$

and the estimate for $\|\psi_{a}\|_{\infty}$ from Example 2.2 yields Property 4.

Finally, observe that (4.5) yields

$$VR(\lambda_{c}; A)x(s) = \int_{\mathbb{R}} v(s)f_{i\lambda_{c}}(t)x(-s - t)dt, x \in H, \lambda_{c} \in \mathbb{R} \setminus \{0\},$$

and, hence,

$$\|VR(\lambda_{c}; A)\|_{2}^{2} = \frac{1}{2\pi} \|v\|_{2}^{2} \int_{\mathbb{R}} \frac{dt}{|t - i\lambda_{c}|^{2}} = \frac{1}{2\lambda_{c}} \|v\|_{2}^{2},$$

which implies Property 5.

From the above lemma, it is clear that we can choose $a > 0$ such that $\|T(\psi_{a})V\|_{2} < 1$. Then operator $U = I + T(\psi_{a})V \in B(H)$ is invertible and the estimates in the lemma together with (4.3) allow us to use [12, Theorem 3.3] to obtain the following result.

**Theorem 4.3.** Consider an operator $\mathcal{L}$ with $V$ of the form (1.1) and the functions $\varphi_{a}$ and $\psi_{a}$, $a > 0$, defined in Examples 2.1 and 2.2. Pick $a = 4^{1 - \ln 2}\|v\|_{2}^{2}$. Then $\|T(\psi_{a})V\|_{2} = \frac{1}{2}$, $U = I + T(\psi_{a})V \in B(H)$, $U^{-1} \in B(H)$, and $\|U^{-1} - I\|_{2} \leq 1$. Moreover, $U$ is the similarity transform of $\mathcal{L}$ into $-i\frac{d}{dt} - B$, where

$$B = T(\varphi_{a})V + U^{-1}(VT(\psi_{a})V - (T(\psi_{a})V)(T(\varphi_{a})V),$$

$$= U^{-1}(VT(\psi_{a})V + T(\varphi_{a})V) \in \mathcal{G}_{2}(H),$$

and we have $\|B\|_{2} \leq \frac{\sqrt{\pi}}{\pi} \left(4\sqrt{\frac{1 - \ln 2}{3}} + \pi + 1\right)\|v\|_{2} \leq 2.45\|v\|_{2}^{2}$.

**Proof.** Even though the assumptions of [12, Theorem 3.3] are slightly different, its proof applies nearly verbatim to establish the similarity of $\mathcal{L}$ and $-i\frac{d}{dt} - B$. The postulated estimates are then easily obtained by direct computation.
Theorems 4.1 and 4.3 immediately yield Theorem 1.1.

**Remark 4.1.** We observe that analogs of Theorem 1.1 hold for any self-adjoint operator $A$ and a perturbation $V \in \mathcal{L}_A(H)$ for which the properties of Lemma 4.2 hold without the specific estimates of the Hilbert-Schmidt norms. Moreover, Properties 1, 2, and 4 may be replaced by the following weaker assumptions:

- $T(\psi_a)V \in B(H)$ and there is $a > 0$ such that $\|T(\psi_a)V\| < 1$.
- $VT(\psi_a)V + T(\varphi_a)V \in \mathfrak{S}_2(H)$.

5. Appendix.

In this section, we collect the proofs that we include for completeness of the exposition.

**Proof of Theorem 4.1.** Let $E_n = E([-n, n])$ be the spectral projection corresponding to $A$ and the interval $[-n, n]$, $n \in \mathbb{N}$, and $\tilde{E}_n = I - E_n$. Similarly, let $A_n = E_nA = AE_n$, $\tilde{A} = A - A_n$, $B_n = E_nBE_n$ and $\tilde{B}_n = B - B_n$. Observe that $\|B\|_2^2 = \|B_n\|_2^2 + \|\tilde{B}_n\|_2^2$, and the sequence $(b_n)$ with $b_n = \|\tilde{B}_n\|_2$, $n \in \mathbb{N}$, is in $\ell^2(\mathbb{N})$. Observe also that for $\lambda \in \mathbb{C} \setminus \mathbb{R}$ such that $|\Re \lambda| \geq n + 2\|B\|_2$ or $|\Im \lambda| \geq 2\|B\|_2$ we have

\[
(\lambda - A - B_n)^{-1} = (\lambda - A)^{-1} \tilde{E}_n + E_n(\lambda - A - B_n)^{-1}E_n = (\lambda - A)^{-1} \left( \tilde{E}_n + \sum_{k=0}^{\infty} (B_n(\lambda - A_n)^{-1})^k \right),
\]

where the series converges absolutely since $\|(B_n(\lambda - A_n)^{-1})\| \leq \frac{1}{2}$ due to $\text{dist}(\lambda, \sigma(A_n)) > 2\|B\|_2 \geq 2\|B_n\|$. Hence for $\lambda \in \mathbb{C} \setminus \mathbb{R}$ with $\text{dist}(\lambda, [-n, n]) > 2\|B\|_2$ we have

\[
\|(\lambda - A - B_n)^{-1}\| \leq \frac{1}{|3m\lambda|} \left( 1 + \sum_{k=0}^{\infty} 2^{-k} \right) = \frac{3}{|3m\lambda|}.
\]

For any $n \in \mathbb{N}$ let

\[
Q_n = \{ \lambda \in \mathbb{C} : |3m\lambda| > 3\|\tilde{B}_n\|_2 \text{ and } |3m\lambda| > 2\|B\|_2, \text{ if } |\Re \lambda| \leq n + 2\|B\|_2 \}.
\]

Then for any $\lambda \in Q_n$ we have

\[
(\lambda - A - B)^{-1} = (\lambda - A - B_n)^{-1} \left( I - \tilde{B}_n(\lambda - A - B_n)^{-1} \right)^{-1} = (\lambda - A - B_n)^{-1} \left( \sum_{k=0}^{\infty} (\tilde{B}_n(\lambda - A - B_n)^{-1})^k \right) \in B(H),
\]

and the result follows by considering the union of $Q_n$, $n \in \mathbb{N}$.
Remark 5.1. We note that for an explicitly known operator $B$ the above proof essentially yields an algorithm for constructing a function $f \in L^2$ that envelops the spectrum $\sigma(A+B)$.

Proof of Lemma 2.1. Observe that for any $a > 0$ we have

$$
\|f\|_1 = \int \left| \frac{1}{a+it} (a+it) f(t) \right| dt \leq \left( \int \frac{dt}{a^2 + t^2} \right)^{\frac{1}{2}} \left( \int |(a+it) f(t)|^2 dt \right)^{\frac{1}{2}}
$$

$$
\leq \sqrt{\frac{\pi}{a}} \left( a \|f\|_2 + \frac{1}{\sqrt{2\pi}} \|\hat{f}\|_2 \right) = \frac{1}{\sqrt{2}} \left( \sqrt{a} \|\hat{f}\|_2 + \frac{1}{\sqrt{a}} \|\hat{f}'\|_2 \right),
$$

by the Cauchy-Schwarz inequality. Plugging in $a = \frac{\|\hat{f}\|_2}{\|f\|_2}$, yields the desired result.

Acknowledgement

The first and third authors were supported in part by the RFBR grant 19-01-00732.

References

[1] A. Aldroubi, A. Baskakov, and I. Krishtal, Slanted matrices, Banach frames, and sampling, J. Funct. Anal., 255 (2008), pp. 1667–1691.

[2] R. Balan and I. Krishtal, An almost periodic noncommutative Wiener’s lemma, J. Math. Anal. Appl., 370 (2010), pp. 339–349.

[3] A. G. Baskakov, Bernstein-type inequalities in abstract harmonic analysis, Sibirsk. Mat. Zh., 20 (1979), pp. 942–952, 1164. English translation: Siberian Math. J. 20 (1979), no. 5, pp. 665–672 (1980).

[4] ———, Asymptotic estimates for elements of matrices of inverse operators, and harmonic analysis, Sibirsk. Mat. Zh., 38 (1997), pp. 14–28, i. English translation: Siberian Math. J. 38 (1997), no. 1, pp. 10–22.

[5] ———, Theory of representations of Banach algebras, and abelian groups and semigroups in the spectral analysis of linear operators, Sovrem. Mat. Fundam. Napravl., 9 (2004), pp. 3–151 (electronic). English translation: J. Math. Sci. (N. Y.) 137 (2006), no. 4, pp. 4885–5036.

[6] ———, Analysis of linear differential equations by methods of the spectral theory of difference operators and linear relations, Uspekhi Mat. Nauk, 68 (2013), pp. 77–128. English translation: Russian Math. Surveys 68 (2013), no. 1, pp. 69–116.

[7] A. G. Baskakov and E. E. Dikarev, Spectral theory of functions in studying partial differential operators, Ufa Math. J., 11 (2019), pp. 3–18.

[8] A. G. Baskakov and I. A. Krishtal, Harmonic analysis of causal operators and their spectral properties, Izv. Ross. Akad. Nauk Ser. Mat., 69 (2005), pp. 3–54. English translation: Izv. Math. 69 (2005), no. 3, pp. 439–486.

[9] ———, Memory estimation of inverse operators, J. Funct. Anal., 267 (2014), pp. 2551–2605.
[10] A. G. Baskakov, I. A. Krishtal, and E. Y. Romanova, Spectral analysis of a differential operator with an involution, Journal of Evolution Equations, 17 (2017), pp. 669–684.

[11] A. G. Baskakov, I. A. Krishtal, and N. B. Uskova, Linear differential operator with an involution as a generator of an operator group, Oper. Matrices, 12 (2018), pp. 723–756.

[12] , Similarity techniques in the spectral analysis of perturbed operator matrices, J. Math. Anal. Appl., 477 (2019), pp. 930–960.

[13] S. Bochner and R. S. Phillips, Absolutely convergent Fourier expansions for non-commutative normed rings, Ann. of Math. (2), 43 (1942), pp. 409–418.

[14] M. S. Burlutskaya and A. P. Khromov, The Fourier method in a mixed problem for a first-order partial differential equation with involution, Zh. Vychisl. Mat. Mat. Fiz., 51 (2011), pp. 2233–2246. English translation: Comput. Math. Math. Phys. 51 (2011), no. 12, pp. 2102–2114.

[15] P. J. Cohen, Factorization in group algebras, Duke Math. J, 26 (1959), pp. 199–205.

[16] K.-J. Engel and R. Nagel, One-parameter semigroups for linear evolution equations, vol. 194 of Graduate Texts in Mathematics, Springer-Verlag, New York, 2000. With contributions by S. Brendle, M. Campiti, T. Hahn, G. Metafune, G. Nickel, D. Pallara, C. Perazzoli, A. Rhandi, S. Romanelli and R. Schnaubelt.

[17] I. Gelfand, D. Raikov, and G. Shilov, Commutative normed rings, Translated from the Russian, with a supplementary chapter, Chelsea Publishing Co., New York, 1964.

[18] I. C. Gohberg and M. G. Kreĭn, Introduction to the theory of linear non-selfadjoint operators, Translated from the Russian by A. Feinstein. Translations of Mathematical Monographs, Vol. 18, American Mathematical Society, Providence, R.I., 1969.

[19] K. Gröchenig, Localization of frames, Banach frames, and the invertibility of the frame operator, J. Fourier Anal. Appl., 10 (2004), pp. 105–132.

[20] E. Hewitt, The ranges of certain convolution operators, Math. Scand., 15 (1964), pp. 147–155.

[21] L. V. Kritskov and A. M. Sarsenbi, Riesz basis property of system of root functions of second-order differential operator with involution, Differ. Equ., 53 (2017), pp. 33–46. Translation of Differ. Uravn. 53 (2017), no. 1, pp. 35–48.

[22] B. M. Levitan, Počti-periodičeskie funkcii, Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1953.

[23] L. H. Loomis, An introduction to abstract harmonic analysis, D. Van Nostrand Company, Inc., Toronto-New York-London, 1953.

[24] Y. I. Lyubich, V. I. Mäcaev, and G. M. Fel’’dman, Representations with a separable spectrum, Funkcional. Anal. i Priložen., 7 (1973), pp. 52–61. English translation: Functional Anal. Appl. 7 (1973), pp. 129–136.

[25] H. Reiter and J. D. Stegeman, Classical harmonic analysis and locally compact groups, vol. 22 of London Mathematical Society Monographs. New Series, The Clarendon Press Oxford University Press, New York, second ed., 2000.
[26] M. A. Sadybekov, G. Dildabek, and M. B. Ivanova, *On an inverse problem of reconstructing a heat conduction process from nonlocal data*, Adv. Math. Phys., (2018), pp. Art. ID 8301656, 8.

[27] N. Wiener, *Tauberian theorems*, Ann. of Math. (2), 33 (1932), pp. 1–100.

Anatoly G. Baskakov  
Department of Applied Mathematics and Mechanics  
Voronezh State University  
Voronezh 394693  
Russia  
e-mail: anatbaskakov@yandex.ru

Ilya A. Krishtal  
Department of Mathematical Sciences  
Northern Illinois University  
DeKalb, IL 60115  
USA  
e-mail: ikrishtal@niu.edu

Natalia B. Uskova  
Department of Higher Mathematics and Mathematical Physical Modeling  
Voronezh State Technical University  
Voronezh 394026  
Russia  
e-mail: nat-uskova@mail.ru