Symmetry protected topological order in magnetization plateau states

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A symmetry protected topologically ordered phase is a short-range entangled state, for which some imposed symmetry prohibits the adiabatic deformation into a trivial state which lacks entanglement. In this paper we argue that magnetization plateau states of one-dimensional antiferromagnets which satisfy the conditions $S - m \in \mathbb{Z}$, where $S$ is the spin quantum number and $m$ the magnetization per site, can be identified as symmetry protected topological states if an inversion symmetry about the link-center is present. This assertion is reached by mapping the antiferromagnet into a nonlinear sigma model type effective field theory containing a novel Berry phase term (a total derivative term) with a coefficient proportional to the quantity $S - m$, and then analyzing the topological structure of the ground state wave function which is inherited from the latter term. A boson-vortex duality transformation is employed to examine the topological stability of the ground state in the absence/presence of a perturbation violating link-inversion symmetry. Our prediction based on field theory is verified by means of a numerical study of the entanglement spectra of actual spin chains, which we find to exhibit two-fold degeneracies when the aforementioned condition is met. We complete this study with a rigorous analysis using matrix product states.

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I. INTRODUCTION

Classification of phases is an important and fundamental problem in statistical physics. In Ginzburg-Landau theories, different phases are distinguished by local order parameters. It has come to be realized in the past few decades, however, that there are phases which cannot be characterized by local order parameters but are nevertheless nontrivial. Dubbed topologically ordered phases, they have in recent years become an intensively studied subject in condensed matter physics [1]. There are largely two known types of topological orders. One is characterized by states with long-range entanglement that sustain anyonic excitations. The other arises in states with short-range entanglement and become robust as a phase when some specific symmetries are imposed; the imposed symmetry condition forbids perturbative terms, whose incorporation would otherwise smoothly deform the state into a direct product (i.e., topologically trivial) state. States characterized by the latter type of order are said to belong to a symmetry protected topological (SPT) phase.

A typical example of a SPT phase is the Haldane state [2, 3] of $S=1$ quantum magnets, which is conveniently characterized as a valence bond solid (VBS) [Fig. 1]. Haldane conjectured that Heisenberg antiferromagnets composed of integer spin have a nonzero excitation gap while those with half-integer spin are gapless. Decades after this prediction was verified through many numerical calculations and experiments [4–6], recent studies motivated by the quest for new topological phases of matter have made it apparent that even within the gapped ground states for integer spin systems, there is a qualitative difference between the $S = \text{odd}$ and even cases. For example, in the $S = 1$ case, it was shown that a phase transition must always intervene between the Haldane phase and a topologically trivial phase (a typical example of the latter being the large-$D$ phase [Fig. 2], which can be expressed in the $S^2$ basis as $| \ldots 0000 \ldots \rangle$) provided one of the following three symmetries is imposed onto the system [7]: (i) the dihedral group of $\pi$ rotations about the $x, y, z$ axes, (ii) time-reversal, and (iii) link-center inversion. For $S = 2$ chains, in contrast, it is possible to connect the Haldane and large-$D$ phases adiabatically [8]. Phases in one-dimensional gapped spin chains can thus be enriched by symmetry protection [9] and their classification using group cohomology has been put forth [3]. SPT phases in higher dimensions have also been proposed [10, 11].

Gapped phases are also observed in antiferromagnets under an applied external magnetic field. These are the magnetization plateaus, i.e., the regions within the magnetization curve where the magnetization remains unchanged with increasing field strength. The Oshikawa-Yamanaka-Affleck (OYA) theory generalizes the celebrated Lieb-Schultz-Mattis theorem [3] and summarizes the necessary condition for the appearance of a plateau in the form $r(S - m) \in \mathbb{Z}$, where $r$ is the number of sites in one unit cell and $S$ is the spin quantum number and $m$ the magnetization per site [17, 12]. The stability of this gapped plateau phase has also been explained in terms of an effective field theory where Berry phase terms play a crucial role [13]. Strong analogies between the Haldane gap and magnetization plateau states have been noted early on [14], and a VBS picture similar to Fig. 1 can be employed to depict the latter (see e.g., Fig. 3). In view of the

![Figure 1](https://example.com/fig1.png)

**FIG. 1:** (Color online) Schematic pictures of $S = 1$ Haldane (VBS) and large-$D$ phases. The links connecting spin-1/2’s (balls) represent singlet bonds.
current interest in physical realizations of SPT states, it would thus seem important to search for magnetization plateau states which can be characterized as a SPT phase. This is the principal purpose of the present work. We note that there are several previous studies on magnetization plateaux which have also been conducted in the light of topological phases (though the explicit connection with SPT phases was not addressed): for instance the Chern number and nontrivial edge excitations associated with plateau state in periodically modulated chains was discussed in Ref. [17], while the entanglement spectra of plateau states which occur in ferro-ferro-antiferromagnetic chains was investigated in Ref. [18].

An additional motivation comes from the correspondence [16,19] between antiferromagnets subjected to a magnetic field and bosons with a tunable chemical potential (which can be casted into a Bose-Hubbard model [20]). SPT states which appear in the phase diagram of the Bose-Hubbard model have been proposed [21,22], and new developments in cold atom experiments enable one to directly measure string orders [23] which in principle detect the subtle topological orders which characterize such states. It is our hope that the present study will contribute some new insights to this closely related problem.

This paper is organized as follows. We begin by developing in Sec. II an effective field theory for magnetization plateau states. In particular, we elic a spin Berry phase term which will be central to the discussion that follows in Sec. III. This term is a surface contribution which was not considered in the earlier field theoretical work described in Ref. [16]. We then go on to investigate, following the path integral formalism of Ref. [24], the topological structure of the ground state wave functional of our spin system. Here we will see that the surface Berry phase term will contribute a phase factor to the wave functional through the presence of a temporal edge state, i.e., a temporal analog of the boundary Berry phase [25] which represents the fractional spins which appear at the end of open spin chains. We show that this factor will govern whether or not the wave function belongs to a SPT phase: our finding is that a plateau state can be in a SPT phase if $S - m$ is an odd integer. In Sec. IV we verify affirmatively this prediction by presenting numerical calculations for model spin systems giving rise to plateaus satisfying $S - m \in \mathbb{Z}$ and/or $S - m \in \mathbb{N}$. This is carried out by examining the degeneracy of the entanglement spectrum, which provides a direct fingerprint of topological order. Finally, in Sec. V we present a matrix product state (MPS) construction for the $m = 1/2$ plateau in $S = 3/2$ chains, which enables us to confirm in a rigorous manner that this state is indeed a SPT phase. Section VI is devoted to discussions and a summary.

II. EFFECTIVE FIELD THEORY OF MAGNETIZATION PLATEAU PHASES: TOPOLOGICAL TERMS

A field-theoretical description of the magnetization plateau state which emphasizes the role played by Berry phases was formulated in Ref. [16]. In the following we refine this treatment in such a way that exposes the relation of this state to

![FIG. 2: (Color online) Schematic pictures of spin Berry phase for magnetization plateau states in spherical coordinates.](image)

SPT phases. For simplicity, we hereon assume that a unit cell contains only one site, in which case the OYA condition reads $S - m \in \mathbb{Z}$. Following Ref. [16], we shall start with the following Hamiltonian which describes an antiferromagnetic spin chain in an external magnetic field,

$$H = J \sum_j S_j \cdot S_{j+1} + D \sum_j (S_j^z)^2 - H \sum_j S_j^z. \quad (1)$$

We assume a canted spin configuration:

$$S_j = S_n_j = \left( \frac{\sqrt{S^2 - m^2} \cos(kx)}{\sqrt{S^2 - m^2} \sin(kx)} \right)_{mj} \quad (x = ja), \quad (2)$$

where $a$ is a lattice constant and $k = \pi/a$. Spins are aligned in an anti-parallel fashion within the $xy$ plane while the $z$ component is uniform. As depicted in Fig. 2, we parametrize the corresponding unit vector $n_j$ using spherical coordinates

$$n_j(\tau) = \frac{(-1)^j \cos \phi_j(\tau) \sin \theta_0}{\cos \theta_0} \quad \frac{(-1)^j \sin \phi_j(\tau) \sin \theta_0}{\cos \theta_0} \quad \frac{(-1)^j \cos \phi_j(\tau) \sin \theta_0}{\cos \theta_0} \quad \frac{(-1)^j \sin \phi_j(\tau) \sin \theta_0}{\cos \theta_0} \quad (3)$$

The magnetization is $m = S \cos \theta_0$. For the classical solution, $\cos \theta_0 = H / (2S(D + 2J))$.

The effective action for (1) can be divided into the kinetic term $S_{\text{kin}}$ and the Berry phase contribution $S_{\text{BP}}^\text{tot}$,

$$S = S_{\text{kin}} + S_{\text{BP}}^\text{tot}, \quad S_{\text{kin}} = \int d\tau H. \quad (3)$$

The continuum limit of the kinetic term is [16]

$$S_{\text{kin}} \rightarrow \int dx d\tau \zeta \left( \frac{1}{2} a_j \phi_j \right)^2 + \left( \frac{1}{4} a_j \phi_j \right)^2 \frac{1}{2},$$

where

$$\zeta = a Js^2 \left( 1 - \frac{H^2}{4S^2(D + 2J)^2} \right),$$

and

$$u = J \sqrt{\frac{4S^2(D + 2J)^2 - H^2}{2J(D + 2J)}}.$$
The Berry phase part of the action (3) is the summation of the contribution from each site. For the sake of the following discussion, it proves convenient to introduce an auxiliary vector

\[ N_j(\tau) \equiv \left( \begin{array}{c} \cos \phi_j(\tau) \sin \theta_0 \\ \sin \phi_j(\tau) \sin \theta_0 \end{array} \right). \] (4)

We note that the spin Berry phase term for \( n(\tau) \) coincides with that for \( N_j(\tau) \). This follows since both spins precess in the same direction around the z-axis along the same constant latitude. \( S_{BP}^{\text{tot}} = \sum_j S_{BP}[N_j(\tau)] \), where

\[ S_{BP}[N_j(\tau)] = iS(1 - \cos \theta_0) \int d\tau \partial_\tau \phi_j \]

\[ = i(S - m) \int d\tau \partial_\tau \phi_j. \] (5)

Applying the identity

\[ S_{BP}[N_j(\tau)] = 2i(S - m) \int d\tau \partial_\tau \phi_j - S_{BP}[N_j(\tau)] \]

only to terms associated with \( j = \text{even sites} \), we recast \( S_{BP}^{\text{tot}} \) as

\[ S_{BP}^{\text{tot}} = \sum_{j: \text{odd}} S_{BP}[N_j(\tau)] \\
+ \sum_{j: \text{even}} \left[ 2i(S - m) \int d\tau \partial_\tau \phi - S_{BP}[N_j(\tau)] \right] \\
= \sum_j (-1)^j S_{BP}[N_j(\tau)] + \sum_j i(S - m) \int d\tau \partial_\tau \phi_j. \] (6)

In the continuum limit, the second term of the last line of (6) reads

\[ \sum_j i(S - m) \int d\tau \partial_\tau \phi_j \rightarrow i \int dx d\tau \frac{S - m}{a} \partial_\tau \phi. \] (7)

This is the Berry phase term that was derived in Ref. [24]. There it was argued, by incorporating a a boson-vortex duality transformation, that for the case \( S - m \notin \mathbb{Z} \), this term has a nontrivial effect and will generally lead to a gapless theory by prohibiting vortex condensation. Meanwhile, for \( S - m \in \mathbb{Z} \), it was found that this term does not affect the low energy physics, allowing for vortex proliferation and hence the formation of a gapped (i.e., the magnetization plateau) state. As we are focused on the latter situation, this term can be discarded.

We now turn to the first term of the last line of (6), which was previously not considered. By placing the system on a space-time grid, the summation over spin Berry phases with an alternating sign can conveniently be converted into a net space-time vorticity [25] as schematically shown in Fig. 3. We therefore have

\[ S_{BP}^{\text{tot}} = \sum_j (-1)^j S_{BP}[N_j(\tau)] \\
= \sum_j (-1)^j (S - m) \int d\tau \partial_\tau \phi_j \\
= i2\pi(S - m) \sum_{\text{odd \ column}} \text{(spacetime vorticity of } \phi). \]

Taking the continuum limit, we obtain

\[ S_{BP}^{\text{tot}} \rightarrow i \frac{S - m}{2} \int d\tau dx \partial_\tau (\partial_x - \partial_x \partial_\tau) \phi(\tau, x) \]

\[ \equiv i\pi(S - m)Q_v, \] (8)

where the factor 1/2 in the first line can be understood by observing that we are to add up the total vorticity in every other row, and \( Q_v \) is the net vorticity throughout the entire space-time.

In order to gain an understanding on the physics which is represented by the action (3), it is insightful to compare this topological term with that which appears in the effective theory of a planar antiferromagnetic chain (without an applied magnetic field). It is well known that the O(3) nonlinear \( \sigma \) (NL\( \sigma \)) model with a topological \( \theta \)-term captures the low energy property of an antiferromagnetic spin chain. One way to incorporate the planar nature of the order parameter into this action while preserving the global topological properties of the theory, is to employ the \( CP^1 \) representation [27], which re-expresses the planar unit vector

\[ N_{pl}(\tau, x) \equiv \left( \begin{array}{c} \cos \phi(\tau, x) \\ \sin \phi(\tau, x) \\ 0 \end{array} \right) \]

in terms of a spinor field \( z \) through the relation \( N_{pl}^a = z^\dagger \sigma^a z \).
where the vacuum angle \( \theta \) (not to be confused with the spherical coordinate) is \( \theta = 2\pi S \), we arrive at

\[
S_{\theta}^{\text{pl}} = i\pi SQ_v, \tag{9}
\]

which is consistent with the lattice-based results of Ref. \([24]\). Also, as explained later in this section, it is possible to start from Eq. \((8)\) to derive a dual vortex field theory which, when treated with due care, correctly discriminates the behavior of integer- and half-integer- \(S\) systems in agreement with the Haldane conjecture. This fact lends credibility to the use of this particular expression in the continuum theory.

Comparing equations \((8)\) and \((9)\), and noting in addition that the kinetic term of our action bears the form of an \(O(3)\) NL\(\sigma\) model in the \(XY\) limit, we find that the two effective theories are identical in form. In particular, Eq. \((8)\) corresponds exactly to the theta term \(S_{\theta}^{\text{pl}}\) with an effective vacuum angle of

\[
\theta_{\text{eff}} = 2\pi(S - m).
\]

This coincidence is quite natural when we recall the VBS-construction of a magnetization plateau state, such as depicted in Fig. \([18]\). At each site, the dynamics of the polarized portion of the spin moment (of magnitude \(m\)) is pinned down by the magnetic field, while the subsystem consisting of the remaining “active” moment of magnitude \(S - m\) form a VBS-like state. The low energy physics of this state is therefore essentially that of (the planar limit of) a Haldane gap state of a spin \(S - m\) antiferromagnet (recall that we are confining our attention to the case where \(S - m \in \mathbb{Z}\)). It is also worthwhile to note that the Berry phase action, Eq. \((8)\), is a total derivative term, which will have important consequences in the following section.

In the remainder of this section, we discuss how the application of a duality transformation on the effective action obtained above:

\[
S = S_{\text{kin}} + S_{\text{BdG}}^{\text{tot}},
\]

\[
S_{\text{kin}} = \int dx \frac{1}{2} \left\{ \frac{1}{\mu^2} (\partial_\tau \phi)^2 + (\partial_x \phi)^2 \right\}, \tag{10}
\]

\[
S_{\text{BdG}}^{\text{tot}} = i(S - m) \int d\tau dx (\partial_\tau a_x - \partial_x a_\tau),
\]

will enable us to seek additional insight from the viewpoint of vortex condensates. Since (i) the action \((10)\) is basically a quantum \(XY\) model, and (ii) vortex proliferation is allowed in the magnetization plateau phase, it is not difficult to guess that the dual vortex-field theory will come out as some variant of the quantum sine-Gordon action, a fact which will be verified shortly. Though the technicalities involved in carrying out the “dual-izing” procedure were described in some details in Ref. \([24]\), we briefly sketch the main steps as there are differences stemming from use of a different topological term. For the sake of simplicity, the mapping will be performed in the continuum limit. The corresponding procedures can however be carried out on the lattice as well, which can easily be seen to lead to identical results.

We start then, with a slight rewriting of the effective Lagrangian density:

\[
\mathcal{L} = \frac{1}{2g} (\partial_\mu \phi)^2 + i\pi(S - m) \rho_v,
\]

where \(\rho_v = (\partial_\tau a_x - \partial_x a_\tau \rho_v/(2\pi))\) is the density of space-time vortices. Here we have set \(v = 1\) and \(g = 1/\zeta\) for notational simplicity. An Hubbard-Stratonovich transformation recasts the kinetic term as \((\partial_\mu \phi)^2/(2g) \rightarrow (g/2)J_\mu^2 + iJ_\mu \partial_\mu \phi\). After decomposing \(\phi\) into a vortex-free portion \(\phi_0\) and a portion with vorticity \(\phi_v\), i.e., \((\partial_\tau \partial_x - \partial_x \partial_\tau) \phi_v = 0\) and \((\partial_\tau \partial_x - \partial_x \partial_\tau) \phi_v \neq 0\), the integration over \(\phi_v\) yields a delta function contribution \(\delta(\partial_\mu J_\mu)\). The constraint \(\partial_\mu J_\mu = 0\) is formally solved by introducing a new vortex-free scalar field \(\varphi\) and putting \(J_\mu \equiv \epsilon_{\mu\nu} \partial_\nu \varphi/(2\pi)\), which leads to

\[
\mathcal{L} = \frac{g}{8\pi^2} (\partial_\mu \varphi)^2 + i\pi (S - m - \varphi) \rho_v.
\]

Integrating out the dual field \(\varphi\), we obtain

\[
\mathcal{L}_{\text{dual}} = \frac{\pi^2}{g} \rho_v \frac{1}{\rho^2} \rho_v + i\pi(S - m) \rho_v.
\]

This is the dual action for vortices written at the first quantization level, describing the inter-vortex logarithmic interaction (first term) and the vortex Berry phase (second term).

Alternatively one can proceed to derive a vortex field theory, i.e., a dual theory at the second quantization level. For this purpose, we submit the system to a standard fugacity expansion and restrict the vorticity to \(\pm 1\). Denoting the fugacity as \(z = e^{-\mu}\), where \(\mu\) is the creation energy of a vortex, the grand canonical partition function of the vortex gas becomes
Having started with a new Berry phase term which only applies to the plateau state, our vortex field theory differs from the one found in Ref. [1] in that a phase-soliton of height \( \varphi \), and \((\partial_\mu \varphi)\) then is equivalent to the following massive Dirac fermion

\[
\mathcal{L}_{\text{fer}} = \bar{\psi} i\gamma^\mu \partial_\mu \psi + M \bar{\psi} \psi,
\]

where \( \psi^T = (R, L), \gamma_0 = \sigma_1, \gamma_1 = -\sigma_2, \gamma_5 = \sigma_3 = i\gamma_0 \gamma_1, \psi = \psi^\dagger \gamma_0, \) and \( M = 16\pi z \cos (\pi(S - m)) (\sigma_1, 2, 3) \) are Pauli matrices. Through the relations:

\[
i(\partial_\tau - i \partial_x) \varphi / (4\pi) \leftrightarrow R^\dagger R;
\]

\[
- i(\partial_\tau + i \partial_x) \varphi / (4\pi) \leftrightarrow L^\dagger L ;
\]

the charge density of fermions is represented as \( \rho \equiv R^\dagger R + L^\dagger L := \partial_\varphi / (2\pi). \) For \( S - m = \) even (odd) systems, the fermion mass \( M \) is positive (negative) and \( \varphi \) is pinned at \( \pi \) (0). If \( S - m = \) even (odd) in \( x > 0 \) (\( x \leq 0 \)), the fermion charge is

\[
\int_{-\infty}^{\infty} dx \rho = \int_{-\infty}^{\infty} dx \frac{\partial_\varphi}{2\pi} = \frac{1}{2}.
\]

This implies that a soliton with fractionalized charge exists at the boundary between \( S - m \) = odd and even systems. This soliton corresponds to the boundary spin-1/2 stated above. The action \( \mathcal{L}_{\text{dual}} \) also tells us that, even for the same sign of \( z \), we may have different types of plateaus depending on the parity of \( S - m \). We will see examples of this in Sec. [11].

Finally we mention that a quantum XY model with the Berry phase term of Eq. \( \mathcal{L}_{\text{dual}} \), which describes planar AF chains in the absence of external fields, can be submitted to the same duality procedures of the proceeding paragraphs. The final vortex field theory is identical in form to the sine-Gordon action \( \mathcal{L}_{\text{dual}} \), but with \( S - m \) replaced by \( S \). (Here a slight subtlety must be accounted for to reach this form. Namely, we need to assume for this purpose that the system is an easy-plane antiferromagnet, and the spin would prefer to escape into the out-of-plane direction at the vortex core, which would be less costly than creating a singular core. It is important to notice at this point that for a given winding sense of a vortex configuration, there are two possible (up and down) directions in which the spin at the core can point. Taking all four topological (meron) configurations into account, we arrive at the dual theory mentioned above [23].) If \( S \) is an half-odd integer, the coefficient \( \cos(\pi S) \) vanishes identically, yielding a massless theory, while the cosine term can generate a mass gap for integer \( S \). The consistency of this result with the Haldane conjecture provides a useful check on the validity of the type of field theory that we have discussed in this section.

III. TEMPORAL SURFACE TERMS AND THE SPT ORDER IN MAGNETIZATION PLATEAU STATES

Having arrived at our effective action \( \mathcal{L}_{\text{dual}} \), we are in a position to discuss, along the lines of Ref. [24], the possible emergence of SPT order in a magnetization plateau state. The basic strategy is to express the ground state wave functional using a Feynman path integral representation, and thereby isolate the phase factor which comes from a temporal surface contribution generated by the Berry phase term. (Imposing a periodic boundary condition along the spatial direction, as we will do below, implies that the action does not contain the more familiar spatial surface contributions.) One finds that this temporal edge state encodes into the wave functional the information necessary to discriminate between states with and without SPT order.

Here we will take the strong coupling limit \( \zeta \rightarrow 0 \), and only consider the topological part of the action, although this is not an absolute necessity and will not effect the conclusions. Focusing primarily on bulk properties, we will employ spatial periodic boundary conditions. The state vector for the ground state can be expanded with respect to the ba-
sis $|N_{pl}(x)|$, which we use as a shorthand notation for the spin coherent state corresponding to the snapshot configuration $\{N_{pl}(x)\}$,

$$|\Psi_{GS}\rangle = \sum_{N_{pl}(x)} \Psi[N_{pl}(x)]|N_{pl}(x)\rangle.$$  

Each coefficient (wave functional) $\Psi[N_{pl}(x)]$ stands for the probability amplitude that the configuration $\{N_{pl}(x)\}$ occurs in the ground state, and can formally be expressed in path integral language as an evolution in imaginary time starting from some initial configuration:

$$\Psi[N_{pl}(x)] \propto \int_{N_{pl,i}}^{N_{pl,f}} D N_{pl}'(\tau, x) e^{-S[N_{pl}'(\tau, x)]}.$$  

(12)

Here, $N_{pl,i(f)} \equiv N_{pl}'(\tau_i(f), x)$ represents spin configurations at initial (final) imaginary time $\tau_i(f)$. Substituting the expression for the Berry phase term in (11) into (12), and taking into account the spatial periodic boundary condition yields

$$\Psi[N_{pl}(x)] \propto \int_{N_{pl,i}}^{N_{pl,f}} D N_{pl}'(\tau, x) e^{-i(S-m) \int dx (a_x(\tau_i) - a_x(\tau_f))}.$$  

Localized at the two ends of the interval on which the imaginary-time integration is performed, the exponent in the right hand side of the above equation may be viewed, as already mentioned, as the action contributed by temporal edge states. Since $a_x(\tau)$ is fixed by the initial condition and can be placed outside the path integral (a Feynman-sum over initial configurations is to be performed afterwards), we need only concern ourselves with the term involving $a_x(\tau)$, and we are left with

$$\Psi[N_{pl}(x)] \propto e^{-i(S-m)\pi W},$$  

(13)

where $W \equiv (1/2\pi) \int dx \partial_x \phi \in \mathbb{Z}$ is the winding number which records the number of times the planar vector $N_{pl}(x)$ wraps around its circular target space as we follow its orientation along the spatial extent of the system.

For $S-m = \text{even}$, $e^{i(S-m)\pi W} \equiv 1$ and the wave functional (13) reduces to that in the absence of the topological term. The case where $S-m = \text{odd}$, meanwhile, yields the nontrivial factor $(-1)^W$. This suggests that the ground states break up into two sectors, i.e., that odd-$(S-m)$ plateaus are SPT states that are topologically distinct from the even-$(S-m)$ states, which we expect to be topologically trivial. To verify this assertion it remains to identify, as we will address in a moment, the symmetry (or symmetries) which can protect this distinction (i.e., prevent the addition of perturbations that will cause the wave functional to adiabatically interpolate between the above two sectors).

We mention in passing that we have restricted our attention to the case where the unit cell consists of one site. The extension of our treatment to a system with $r$ sites per cell is straightforward. There the parity of

$$r(S-m) \in \mathbb{Z},$$  

(14)

FIG. 4: (Color online) Application of a staggered magnetic field along the $z$ axis alters the local magnetization from $S-m + (\pm)\delta m$, which clearly results in the breakdown of the structure (12). This perturbation can be prohibited by imposing a link-center inversion symmetry on the system.

(where $m$ still stands for the magnetization per site) will replace the role of $S-m$ of the present argument.

Let us now consider the effect of applying a staggered magnetic field along the $z$ axis. Repeating the derivation of the total Berry phase for this case, it is clear that this is the generic perturbation that directly affects the first (sign-alternating) term of (6), leading to the modification $S-m \to S-m - \delta m$, while leaving unchanged the second (uniform) term. The latter contribution can therefore be discarded as before, and we are left with

$$S_{BP}^\text{tot} = i(S-m-\delta m) \int d\tau d\phi (\partial_x a_x - \partial_z a_z).$$

Accordingly, the wave functional formerly expressed by (13) now takes on the form

$$\Psi[N_{pl}(x)] \propto e^{-i(S-m-\delta m)\pi W}.$$  

By varying $\delta m$, it is now possible to continuously interpolate between the two aforementioned dependencies of the wave functional on the winding number $W$. Additional information comes from revisiting the derivation of the vortex field theory (14); upon applying the staggered magnetic field, the action modifies to

$$L_{\text{dual}} = \frac{g}{2} (\partial_\mu \varphi)^2 + 2z \cos (\pi(S-m-\delta m) - \varphi).$$

It is clear from this action that the optimal value of the field $\varphi$ will change continuously as $\delta m$ is varied, without ever closing the energy gap. We therefore find that the application of a staggered magnetic field ruins the topological distinction between the even- and odd-$(S-m)$ cases. Since this perturbation can be prohibited by requiring that the system respect link-inversion symmetry, our observations strongly suggest that the odd-$(S-m)$ plateaus are SPT states protected by link-inversion symmetry and are distinct from the even-$(S-m)$ plateaus, which are topologically trivial. We will arrive at the same conclusion both through a numerical study in Sec. 13 and by analyzing a MPS representation for magnetization plateaus in Sec. 14.

IV. NUMERICAL CALCULATIONS

In this section, we numerically provide an independent confirmation of the prediction by field theories that the parity of $S-m$ determines whether the system is in SPT phase or not.
A simple way to distinguish SPT and trivial phases is to investigate the degeneracy of the entanglement spectrum [3]. A numerical means suited for this purpose is the infinite time-evolving block decimation (iTEBD) [3]. This method utilizes the ability of MPS to approximate gapped states and enables one to simulate infinite-size systems by assuming a certain sort of spatial periodicity in the ground state. Through the imaginary-time evolution, the state approaches the ground state optimally approximated within the MPS representation with a fixed matrix dimension $\chi$. In general, the true ground state is better approximated with larger $\chi$.

An ideal quantum spin model for studying SPT phases in plateaus would be a Heisenberg model with single-ion anisotropy $D$ [Eq. (1)]. In fact, the existence of two different kinds of $m = 1/2$ plateaus has been pointed out in this model with $S = 3/2$. It has been also argued there that the one appearing for small $D$ (i.e., $0.387 \leq D/J \leq 0.943$) is characterized by the VBS-like state (where magnetization appears in plateaus) shown in Fig. 5(a). In addition, when $D$ further increases, a second-order Gaussian transition occurs and the system enters another plateau phase reminiscent of the large-$D$ phase, in which the unmagnetized background spin moments are quenched [Fig. 5(b)], and the short-range entanglement is absent.

It is easy to understand this fact in terms of our effective action (11). Observing that the sign of the coefficient $z \cos (\pi(S - m))$ of the cosine term in (11) determines whether the system in question is topological or not, one immediately sees that the transition between the VBS-like and large-$D$ plateaus studied in Ref. [30] should be characterized by $z = 0$. This explains why the phase transition is of the Gaussian type. We note that this mechanism is essentially the same as that for the transition between the Haldane and large-$D$ phases in $S = 1$ chains [31, 32]. Unfortunately, this plateau appears only in a tiny window of the applied magnetic field $H$ and it is difficult to approach this state by iTEBD simulations with fixed $H$.

In order to circumvent the technical difficulties mentioned above, we study the following ferro-ferro-antiferromagnetic (FFAF) Heisenberg chain [33] [see Fig. 5]:

$$
\mathcal{H}_{\text{FFAF}} = \sum_{j=1}^{L} (-J_F S_{3j-2} \cdot S_{3j-1} - J_F S_{3j-1} \cdot S_{3j} + J_{AF} S_{3j} \cdot S_{3j+1}) - HS_{z,\text{tot}}^2,
$$

where $L$ is the number of unit cells and $S_{z,\text{tot}}^2 = \sum_{j=1}^{3L} S_j^z$. Coupling constants $J_F$ and $J_{AF}$ are both positive. Each unit cell consists of three spins which are coupled through ferromagnetic bonds [Fig. 5]. Due to the Hund rule coupling $-J_F$, we regard this unit cell as one spin-$3S$. Magnetization $m$ corresponds to $(S_{z,\text{tot}}^z)/L$. Thus, the situation is the same as considering plateaus with magnetization $(S_{z,\text{tot}}^z)/L$ in spin-$3S$ chains. We can give equivalent description by substituting $r = 3$ and $m = (S_{z,\text{tot}}^z)/(3L)$ into (14).

In particular, we performed calculations for $S = 1/2$ and $S = 1$ FFAF chains. We set the couplings at $J_F = J_{AF} = J$ for $S = 1/2$ and $J_F = J$, $J_{AF} = 2J$ for $S = 1$. The magnetization curves of the $S = 1/2$ and $S = 1$ are shown in Fig. 5(a), (b). We employed the iTEBD method with MPS dimensions of $\chi = 150$ and 100 for $S = 1/2$ and $S = 1$, respectively. Magnetization plateaus appear at $(S_{z,\text{tot}}^z)/L = 1/2$ for $S = 1/2$ and at $(S_{z,\text{tot}}^z)/L = 1.2$ for $S = 1$ satisfying the OYA condition $3S - (S_{z,\text{tot}}^z)/L \in \mathbb{Z}$.

In order to verify that the above plateaus are topologically nontrivial, we investigate the entanglement spectrum [34] which is known to give the fingerprints of topological phases. The entanglement spectrum of a quantum state $|\Psi\rangle$ is defined through the bipartition of the system into regions A and B. The state $|\Psi\rangle$ can be Schmidt-decomposed as a superposi-
tion of direct products $|\Psi\rangle = \sum_{\alpha} \lambda_{\alpha} |\Psi_{A}\rangle_{\alpha} \otimes |\Psi_{B}\rangle_{\alpha}$ using the orthonormal basis sets $|\Psi_{A}\rangle_{\alpha}$ and $|\Psi_{B}\rangle_{\alpha}$ of the subsystems. The entanglement spectrum is defined by the logarithm $\{-\ln(\lambda_{\alpha})\}$ ($\alpha = 1, \ldots, \chi$) of the Schmidt eigenvalues $\lambda_{\alpha}$ normalized as $\sum_{\alpha} \lambda_{\alpha}^2 = 1$. As will be discussed in Sec. [5], if our system is in the SPT phase, the entanglement spectrum is two-fold degenerate, in other words, all the values $\lambda_{\alpha}$ should appear in pairs. We emphasize that the bipartition of the system should be made at an antiferromagnetic bond since ferromagnetically coupled three spins are considered as one site. In Fig. [3c], we present the entanglement spectra obtained for the plateau states at magnetic fields $H/J = 0.4$ for $S = 1/2$ and $H/J = 1.1$ and $2.5$ for $S = 1$ [shown by arrows in Fig. [3a, b)]. We can clearly see that entanglement spectra at $H/J = 0.4$ ($\langle S_{tot}^z \rangle / L = 1/2$) for $S = 1/2$ and at $H/J = 2.5$ ($\langle S_{tot}^z \rangle / L = 2$) for $S = 1$ exhibit two-fold degeneracy while the one at $H/J = 1.1$ ($\langle S_{tot}^z \rangle / L = 1$) for $S = 1$ does not. These results imply that the system is in the SPT (trivial) phase for $3S - \langle S_{tot}^z \rangle / L = \text{odd (even)}$, thus confirming the prediction from quantum field theories discussed in Sec. [11].

In closing this section, we briefly remark on the connection between the findings of this section with the field theoretical study of the earlier sections. In Sec. [11], we saw that the global structure of the ground state wave functional is determined by a temporal surface contribution coming from the topological term of the effective action. We can formally express the reduced density matrix $\rho_{A} = \text{Tr}_{B}|\Psi\rangle\langle\Psi|$ (where $|\Psi\rangle$ is the ground state and the trace operation is to be restricted to the region B), an object from which the entanglement spectrum can directly be extracted, along the same lines by incorporating a path integral representation [12]. Once again the topological term will give rise to a surface contribution, this time along a segment of the imaginary time axis bounded by the spatial edge of region A. While we expect that this will play an essential role in determining the entanglement spectrum, we leave the details for future work.

V. MPS REPRESENTATION OF PLATEAU PHASES

In this section, we present a simple model ground state (MPS) that exhibits the SPT properties at finite magnetic fields and show that the degenerate structure found above in the entanglement spectrum is indeed closely tied to the underlying topological properties. We consider the $m = 1/2$ plateau in a $S = 3/2$ chain for example. On top of the trivial product state

$$|\otimes_{j} |S_{j}^z = 1/2\rangle\rangle,$$

we can think of an entangled state described by the VBS picture, schematically shown in Fig. [3]. Using the auxiliary (Schwinger) bosons, this state is represented as

$$|\Psi\rangle = \prod_{j} a_j^\dagger (a_{j+1}^\dagger b_{j+1}^\dagger - b_{j+1}^\dagger a_{j+1}^\dagger) \otimes |0\rangle_j,$$  \hspace{1cm} (15)

where $a_{j}^\dagger$ and $b_{j}^\dagger$ are the bosonic creation operators of spin-1/2 up and down on the $j$-th site, respectively. Note that there are exactly three bosons at each site which guarantee a local spin-3/2 at each site. The MPS representation of the above state is given by

$$|\Psi\rangle = C \sum_{S_{j} = -3/2}^{3/2} \cdots A[S_{j-1}]A[S_{j}]A[S_{j+1}] \otimes |S_{j}\rangle,$$  \hspace{1cm} (16)

where

$$A[3/2] = \begin{pmatrix} 0 & 0 \\ -\sqrt{6} & 0 \end{pmatrix}, \quad A[1/2] = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & -\sqrt{2} \end{pmatrix}$$

$$A[-1/2] = \begin{pmatrix} 0 & \sqrt{2} \\ 0 & 0 \end{pmatrix}, \quad A[-3/2] = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

and $C$ is a normalization constant. The transfer matrix of the MPS [16] is readily obtained as

$$T_{(\alpha_1, \alpha_2), (\beta_1, \beta_2)} = \sum_{S_{j} = -3/2}^{3/2} A_{\alpha_1, \beta_1}^\dagger [S_{j}]A_{\alpha_2, \beta_2} [S_{j}^\dagger]$$

$$= \begin{pmatrix} 2 & 0 & 0 & 2 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 6 & 0 & 0 & 2 \end{pmatrix}.$$  

A MPS representation is said to assume the canonical form [54, 55] when its transfer matrix satisfies the condition

$$\sum_{\alpha_1, \alpha_2} \delta_{\alpha_1, \alpha_2} T_{(\alpha_1, \alpha_2), (\beta_1, \beta_2)} = \delta_{\beta_1, \beta_2}$$

$$\sum_{\beta_1, \beta_2} T_{(\alpha_1, \alpha_2), (\beta_1, \beta_2)} \delta_{\beta_1, \beta_2} = \delta_{\alpha_1, \alpha_2}.$$  \hspace{1cm} (17)

In order to discuss the SPT phase, it is convenient to work in the canonical form of the MPS. Since the MPS representation [16] does not satisfy (17), we first render it canonical using a gauge transformation $A \rightarrow M^{-1}AM$ ($M$ is some matrix). It is obvious that this transformation does not change the MPS. Taking

$$M = \begin{pmatrix} 3^{-1/4} & 0 \\ 0 & 1 \end{pmatrix},$$

FIG. 7: (Color online) (a) The VBS picture of a model plateau state [45] at $m = 1/2$ in a $S = 3/2$ spin chain. (b) Large-$D$ plateau.
we obtain the following canonical form of the MPS

$$|\Psi\rangle = \sum_{S_j = -3/2}^{3/2} \ldots \Lambda|S_{j-1}\rangle \Lambda|S_j\rangle \Lambda|S_{j+1}\rangle \lambda \ldots \otimes |S_f\rangle,$$

where the matrices are given by

$$\begin{align*}
\Gamma[3/2] &= (1 + \sqrt{3})^{-1/2} \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{3} & 0 & 0 \\ 0 & \sqrt{3} & 0 \end{pmatrix} \\
\Gamma[1/2] &= (1 + \sqrt{3})^{-1/2} \begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & -\sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix} \\
\Gamma[-1/2] &= (1 + \sqrt{3})^{-1/2} \begin{pmatrix} 0 & \sqrt{3} & 0 \\ 0 & 0 & \sqrt{3} \\ \sqrt{3} & 0 & 0 \end{pmatrix} \\
\Gamma[-3/2] &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
\Lambda &= \begin{pmatrix} 1 & \sqrt{2} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.
\end{align*}$$

(18)

The new MPS is related to the original one through the transformation

$$M^{-1}A|S_f\rangle M = (2 + 2\sqrt{3})^{1/2} \Lambda|S_f\rangle.$$

We can check the condition (17) for the transfer matrix of the MPS (18)

$$T_{\alpha_1,\alpha_2,\beta_1,\beta_2}^{\text{can}} = \sum_{S_j = -3/2}^{3/2} \langle \Lambda|_{\alpha_1,\beta_1}|S_j\rangle (\Lambda|_{\alpha_2,\beta_2}|S_j\rangle = \sum_{S_j = -3/2}^{3/2} \langle \Gamma \Lambda|_{\alpha_1,\beta_1}|S_j\rangle (\Gamma \Lambda|_{\alpha_2,\beta_2}|S_j\rangle = \frac{1}{1 + \sqrt{3}} \begin{pmatrix} 1 & 0 & 0 & \sqrt{3} \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ \sqrt{3} & 0 & 0 & 1 \end{pmatrix},$$

and confirm that the MPS (18) is indeed in a canonical form.

Following the discussion in Refs. (19), a projective representation $U_{\mathcal{I}} (\chi \times \chi$ unitary matrix) of link-inversion $\mathcal{I}$ satisfies

$$\Gamma = e^{i\theta_{\mathcal{I}}} U_{\mathcal{I}}^\dagger \Gamma^T U_{\mathcal{I}}.$$

(19)

Using (19) twice, we see that the fundamental property $\mathcal{I}^2 = 1$ of the link-inversion implies

$$\Gamma = e^{2i\theta_{\mathcal{I}}} (U_{\mathcal{I}}^2 U_{\mathcal{I}})^{1/2} \Gamma (U_{\mathcal{I}}^2 U_{\mathcal{I}})$$

$$\Leftrightarrow (U_{\mathcal{I}}^2 U_{\mathcal{I}}) \Gamma = e^{2i\theta_{\mathcal{I}}} \Gamma (U_{\mathcal{I}}^2 U_{\mathcal{I}}).$$

Multiplying $(\Gamma \Lambda)^\dagger \Lambda$ from the left to this relation and taking the trace over the suffix $S^2$ in $\Gamma$, we obtain

$$\langle U_{\mathcal{I}}^2 U_{\mathcal{I}} |_{(\alpha_1,\alpha_2), (\beta_1,\beta_2)} T^{\text{can}}_{(\alpha_1,\alpha_2), (\beta_1,\beta_2)} = e^{2i\theta_{\mathcal{I}}} (U_{\mathcal{I}}^2 U_{\mathcal{I}})_{(\beta_1,\beta_2)}.$$

Here, note that $[U_{\mathcal{I}}, \Lambda] = 0$. Thus, $(U_{\mathcal{I}}^2 U_{\mathcal{I}})$ is the left eigenvector of the transfer matrix $T^{\text{can}}$. If we assume that unity is the unique largest eigenvalue of $T^{\text{can}}$, then $e^{2i\theta_{\mathcal{I}}} = 1$ and $U_{\mathcal{I}} U_{\mathcal{I}} = e^{i\phi_{\mathcal{I}}} E$ (E is unit matrix). Using $U_{\mathcal{I}} = e^{i\phi_{\mathcal{I}}} U_{\mathcal{I}}^T$, twice, $e^{2i\phi_{\mathcal{I}}} = 1$. Thus, the phase $\phi_{\mathcal{I}}$ is quantized to 0 and $\pi$, which characterizes the SPT order protected by $\mathcal{I}$. In fact, $\phi_{\mathcal{I}} = 0$ corresponds to a direct product (trivial) state. On the other hand, the state with $\phi_{\mathcal{I}} = \pi$, being characterized by a discrete integer, cannot be continuously deformed to the trivial one ($\phi_{\mathcal{I}} = 0$) without a phase transition. From $U_{\mathcal{I}} = \pm U_{\mathcal{I}}^T$, we can see that the unitary matrix $U_{\mathcal{I}}$ is either symmetric (trivial) or antisymmetric (topological).

The structure of the entanglement spectrum also reflects the property of $U_{\mathcal{I}}$. Here we consider the topological case $\phi_{\mathcal{I}} = \pi$ (i.e., $U_{\mathcal{I}} = -U_{\mathcal{I}}^T$). Since $U_{\mathcal{I}}$ and $\Lambda$ are commuting, the matrix $U_{\mathcal{I}}$ should be block diagonal according to subspaces labeled by the singular values $\lambda_\alpha$ (diagonal elements of $\Lambda$) which is equivalent to the entanglement eigenvalues. If we represent the dimension of the block $\alpha$ as $d_\alpha$, then

$$\det(U_{\mathcal{I}}) = \det(U_{\mathcal{I}}^T) = \det(-U_{\mathcal{I}}) = (-1)^{d_\alpha} \det(U_{\mathcal{I}}).$$

Therefore, $d_\alpha$ should be even for any $\alpha$. This indicates that the entanglement spectrum is two-fold degenerate for SPT phases.

For the model VBS state (19) for the $m = 1/2$ plateau phase in the $S = 3/2$ chain, we find that the following $\chi = 2$

$$U_{\mathcal{I}} = i\sigma_y = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

(21)

satisfies Eq. (20). As this is antisymmetric, we see that $\phi_{\mathcal{I}}$ is equal to $\pi$ confirming that this plateau state is in the topological (Haldane) phase protected by link-inversion symmetry $\mathcal{I}$. In contrast to the $S = 1$ Haldane phase, the two other protecting symmetries, i.e., time-reversal and $Z_2 \times Z_2$ are explicitly broken due to the presence of an external magnetic field.

A detailed comparison with Ref. (19) shows that the magnetization plateaus in an SPT phase falls into a category of SPT states classified by the group $Z_2 \times Z_2$. In other words, within this category there should be one trivial and three SPT phases, and the SPT plateau state that we have identified corresponds to one of the latter. The fine details on this “larger picture” is rather involved and is provided in Appendix A.

VI. DISCUSSIONS AND SUMMARY

Following a summary of what has been achieved thus far, we conclude by making several clarifying remarks on loose ends, and putting the work in context with related developments.

We began by deriving a semiclassical effective field theory which describes a magnetization plateau state. The action obtained was that of an XY model equipped with a topological term associated with space-time vortex events. Employing a path-integral formalism, we found that the latter term governs the structure of the ground state wave function, and induces a topological distinction between plateau states with even and odd values of $S - m$. The source of this distinction may be tracked down to the fact that at the level of the original lattice model, only one in every two spatial links contributed to
the total Berry phase, which in the continuum limit resulted in a crucial factor of \(1/2\). We further observed that an addition of a staggered magnetic field, which breaks the link-inversion symmetry enjoyed by the original theory, destructs this topological distinction. A dual vortex field theory was derived which showed explicitly that with this perturbation the two previously distinct wave functions could be smoothly deformed into each other without closing the energy gap. We were thus lead to conclude that the \(S - m = \text{odd} \) case is a SPT protected by link-center inversion symmetry, while the \(S - m = \text{even} \) case is topologically trivial. An independent support for this expectation was provided through numerical calculations for \(S = 1/2 \) and \(S = 1 \) FFAF chains, as well as a rigorous treatment which incorporates a MPS representation of the magnetization plateau state.

Readers may find it puzzling that we had started out with a lattice model which is identical to the one studied in Ref. [23], and yet we arrived at a different final expression for the topological term, which was crucial for what followed. The difference appeared from our adaption of the methods of Ref. [20] originally devised for treating antiferromagnetic spin chains in the XY limit (and in the absence of a magnetic field) and retaining the surface contribution that inevitably arises once we employ this setup. We also took advantage of the fact that the bulk contribution to the Berry phase, being trivial in a magnetization plateau state, could safely be discarded. In short, the difference in outcome between the two treatments stems from the fact that in the present work we made use of a mathematical procedure which picks up the correct surface contributions to the spin Berry phases. Apart from the presence/absence of these surface terms the two theories are equivalent.

Our treatment of the ground state wave functional basically extends the prescriptions of Ref. [23] to (1+1)-dimensions. The authors of Ref. [23], in their investigation of SPT states arising out of a (3+1)-dimensional O(5) NL\(\sigma\) model, needed to employ a limit in which the magnitude of one of the five components of their unit-length field was sent to zero. This intermediate step was necessary to derive a wave functional whose structure is governed by a topological obstruction at the (temporal) surface: in terms of the integer-component to two. While this step of reducing the number of components appears somewhat artificial from a physical point of view, it extends the prescriptions of Ref. [23] to (1+1)-dimensions.

In this regard the magnetization plateau phase in spin chains of local spin moment that is quenched. Our treatment of the ground state wave functional basically applying the “Haldane insulators” in bosonic systems [24]. We start from a one-dimensional Bose-Hubbard model:

\[
\mathcal{H}_\text{BH} = -t \sum_j (b_j^\dagger b_{j+1} + \text{H.c.}) + U/2 \sum_j n_j(n_j-1) - \mu \sum_j n_j, \]

where \(b_j, b_j^\dagger\) are boson annihilation and creation operators, and \(n_j = b_j^\dagger b_j\) is a number operator. Switching to a coherent-state path integral language via the substitution \(b_j(\tau) = \rho_j^{1/2} e^{i\phi_j(\tau)}\) and writing \(\rho_j = \rho_0 + \delta\rho_j\) (\(\rho_0\) is the number of bosons per site), we obtain, upon integrating out \(\delta\rho_j\) the effective Lagrangian [22]

\[
\mathcal{L}_\text{BH} = \frac{1}{2U} \sum_j \left( \partial_\tau \phi_j(\tau) \right)^2 - t \rho_0 \sum_j \cos \left( \phi_j(\tau) - \phi_{j+1}(\tau) \right) + i \rho_0 \sum_j \partial_\tau \phi_j(\tau). \tag{22} \]

In the continuum limit the kinetic terms (the first two terms on

FIG. 8: (Color online) Schematic pictures of \(S = 2\) Haldane, intermediate-\(D\) and large-\(D\) phases. The links connecting spin-1/2’s (balls) represent singlet bonds.
the r.h.s.) in (2) assume the form

\[ \mathcal{L}_{\text{BH,kin}} \sim \int dx \left[ c_r \left( \partial_x \phi_j(x, \tau) \right)^2 + c_x \left( \partial_x \phi_j(x, \tau) \right)^2 \right], \]

where \( c_r = 1/(2Ua) \), \( c_x = t \rho_0/2 \) (a is the lattice constant). Moreover, the term \( i \rho_0 \sum_j \partial_x \phi_j(\tau) \) becomes identical with (3) upon the replacement \( \rho_0 \to S - m \). Therefore, the effective action for Bose-Hubbard model takes the same form as (10). Essentially repeating the arguments of Secs. I and II we are lead to deduce that Haldane and trivial insulators in bosonic systems each correspond to \( \rho_0 = 0 \) and even, and that the addition of a staggered chemical potential (which is the counterpart of the staggered magnetization of our previous discussion) will destroy this topological distinction.

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### Appendix A: SPT phases protected by U(1)×Z^P_2 symmetry

SPT phases protected by the symmetry U(1)×Z^P_2 is classified by \( Z_2 \times Z_2 \); as can be read off from the classification table [1] for the mathematically equivalent entry U(1)×Z^T_2. (Here \( Z^T_2 \) and \( Z^P_2 \) stand for time and parity reversal symmetries, respectively.) One of these two \( Z_2 \) groups corresponds to whether \( U^Z_2 = +1 \) or \(-1 \) as is discussed in the main text, where \( U_P \) is a unitary matrix corresponding to the projective representation of the parity operation \( P \):

\[ P \Gamma = e^{i\theta_P} U^\dagger_P \Gamma \Gamma^T U_p. \quad (A1) \]

For our SPT plateau case, \( P \) is represented as \( \mathcal{I} \), since here the parity reversal is the link-center inversion.

Let us define a projective representation of the U(1) group, which consists of rotations around the z axis with angle \( \alpha \) \( (R^z_\alpha) \) such as

\[ R^z_\alpha \Gamma = e^{i\theta_{\alpha}} U^\dagger_\alpha \Gamma U^{\dagger \alpha}. \quad (A2) \]

We consider the effect of exchanging the order in which the two operators \( R^z_\alpha \) and \( P \) act. From (A1) and (A2),

\[ R^z_\alpha P \Gamma = e^{i\theta_P + \theta_{\alpha}} U^{\dagger_\alpha} U^{\dagger \alpha} \Gamma U^\dagger_{\alpha} U^{\dagger \alpha}, \]

\[ P R^z_\alpha \Gamma = e^{i\theta_P + \theta_{\alpha}} U^{\dagger_\alpha} (U^{\dagger \alpha} \Gamma U^{\dagger \alpha})^T U_{\alpha}. \]

The relation \( R^z_\alpha P \Gamma = P R^z_\alpha \Gamma \) leads to

\[ \Gamma = U P U^\dagger_\alpha U^\dagger_{\alpha} \Gamma U^{\dagger \alpha} U^\dagger_{\alpha} U^\dagger \alpha. \]

Hence, \( U P U^{\dagger_\alpha} U^\dagger_{\alpha} \Gamma \) should be the left eigenvector of the transfer matrix and is equal to \( e^{i\phi_\alpha} \Gamma E \) (\( E \) is unit matrix). This implies that

\[ U P U^{\dagger_\alpha} = e^{i\phi_\alpha} P U^{\dagger_\alpha} U^\dagger \alpha. \quad (A3) \]

Since the action of U(1) is diagonal \( (U^\dagger_{\alpha} T = U^{\dagger_\alpha})* \), \( U^{\dagger_\alpha} = U^{\dagger \alpha} = U^\dagger_{\alpha} = U^{\dagger \alpha} \alpha = U^{\dagger \alpha} \). Using this relation and (A3), we can also derive

\[ U^{\dagger_\alpha} U^\dagger_{\alpha} = e^{-i\phi_\alpha} U^\dagger_{\alpha} U^{\dagger_\alpha} \alpha. \quad (A4) \]

by the replacement \( \alpha \to -\alpha \). From (A3) and (A4), \( e^{i\phi_\alpha} \alpha = \pm 1 \) is proved, i.e., \( U_P U^{\dagger_\alpha} \alpha = \mp U_{\alpha} \). Therefore, the \( Z_2 \times Z_2 \) classification of SPT phases protected by U(1)×Z^P_2 proceeds according to (i) \( U^2 = +1 \) or \(-1 \), and (ii) \( U_P U^{\dagger_\alpha} \alpha = U^{\dagger_\alpha} \alpha \) or \( U_P U^\dagger_{\alpha} \alpha = -U^{\dagger_\alpha} \alpha \).

For the SPT plateau state discussed in Sec. V (see Eq. (13)), the U(1) rotation acts as

\[ R^z_\alpha \Gamma [3/2] = e^{i\theta_{\alpha}} \Gamma [3/2], \]

\[ R^z_\alpha \Gamma [1/2] = e^{i\theta_{\alpha}} \Gamma [1/2], \]

\[ R^z_\alpha \Gamma [-1/2] = e^{-i\theta_{\alpha}} \Gamma [-1/2], \]

\[ R^z_\alpha \Gamma [-3/2] = e^{-i\theta_{\alpha}} \Gamma [-3/2]. \]

We can find that

\[ U^{\dagger_\alpha} \alpha = e^{i\alpha \pi/2}, \quad \theta_{\alpha} = \alpha/2 \]

satisfies Eq. (A2). From (A1) and (A3), we can confirm that \( U_T U^{\dagger_\alpha} \alpha = U^{\dagger_\alpha} \alpha U_T \alpha \) holds. Therefore, the SPT plateau belongs to the category with (i) \( U^2 = -1 \) and (ii) \( U_P U^{\dagger_\alpha} \alpha = U^{\dagger_\alpha} \alpha \) \( \alpha \). The search for SPT phases with \( U_P U^{\dagger_\alpha} \alpha = -U^{\dagger_\alpha} \alpha \) \( \alpha \) is an interesting future problem.

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