EXOTIC GEOMETRIC STRUCTURES ON KODAIRA SURFACES

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ABSTRACT. On all compact complex surfaces (modulo finite unramified coverings), we classify all of the locally homogeneous geometric structures which are locally isomorphic to the “exotic” homogeneous surfaces of Lie.

CONTENTS

1. Introduction 1
2. Definitions 2

2.1. \((G, X)\)-structures 2
2.2. Pulling back 2
2.3. Developing maps and holonomy morphisms 3
2.4. Inducing structures from other structures 3
2.5. The definition of \(G_D\) 4
2.6. The definition of \(G_D'\) 4
3. Statements of the theorems 5
4. \(G_D\) and tori 5
5. \(G_D'\) on tori 7
6. Primary Kodaira surfaces 10
7. \(G_D\) and primary Kodaira surfaces 11
8. \(G_D'\) and primary Kodaira surfaces 16
9. Nonsingular holomorphic foliations on compact complex surfaces 19
10. Nothing else 23
11. Conclusion 24
References 24

1. INTRODUCTION

In Lie’s classification of Lie group actions on surfaces, there are two exotic cases, in which the definition of the Lie group depends not only on parameters, but on the set of solutions of a differential equation [12] p.767–773, [16]. We will classify, on all compact complex surfaces, all of the locally homogeneous structures which are locally isomorphic to these exotic surfaces of Lie. This paper is part of a larger programme to classify holomorphic locally homogeneous structures on low dimensional compact complex manifolds, in joint work with Sorin Dumitrescu and Alexey Pokrovskiy [14, 15].

Theorem 1. Let \(S\) be a compact complex surface. Suppose that \(S\) has a holomorphic locally homogeneous structure modelled on one of the exotic surfaces of Lie
Then, up to replacing $S$ by a finite unramified covering space, $S$ is a complex torus or primary Kodaira surface. Every complex torus and every primary Kodaira surface admits such structures. (We write out these structures explicitly in sections 4, 5, 7 and 8). On any complex torus, all holomorphic locally homogeneous structures modelled on Lie’s exotic surfaces are induced by the translation structure. (See section 2.4 for this terminology.)

Let $G_0$ be the group of complex affine transformations of $\mathbb{C}^2$ of the form 
\[
\begin{pmatrix} z \\ w \end{pmatrix} \mapsto \begin{pmatrix} z + b \\ w + az + c \end{pmatrix}.
\]

On any primary Kodaira surface, all holomorphic locally homogeneous structures modelled on Lie’s exotic surfaces are induced from a certain holomorphic locally homogeneous structure modelled on the $G_0$-action on $\mathbb{C}^2$ which we will write out explicitly in section 7 on page 11.

2. Definitions

2.1. $(G, X)$-structures. Suppose that $G$ is a Lie group and that $H \subset G$ is a closed subgroup and let $X = G/H$.

Definition 1. An $X$-chart on a manifold $M$ is a local diffeomorphism from an open subset of $M$ to an open subset of $X$.

Definition 2. Two $X$-charts $f_0$ and $f_1$ on a manifold are $G$-compatible if there is some element $g \in G$ so that $f_1 = gf_0$ wherever both $f_0$ and $f_1$ are defined.

Definition 3. An $X$-atlas on a manifold $M$ is a collection of mutually $G$-compatible $X$-charts whose domains cover $M$.

Definition 4. A $(G, X)$-structure on a manifold $M$ is a maximal $(G, X)$-atlas.

2.2. Pulling back.

Definition 5. If $F : M_0 \to M_1$ is a local diffeomorphism, and $H \subset G$ a closed subgroup of a Lie group, then every $(G, X)$-structure on $M_1$ has a pullback structure on $M_0$, whose charts are precisely the compositions $f \circ F$, for $f$ a chart of the $(G, X)$-structure. Conversely, if $F$ is a normal covering map, and $M_0$ has a $(G, X)$-structure which is invariant under the deck transformations, then it induces a $(G, X)$-structure on $M_1$.

2.3. Developing maps and holonomy morphisms.

Definition 6. Suppose that $(M, m_0)$ is a pointed manifold, with universal covering space $(\tilde{M}, \tilde{m}_0)$. Suppose that $H \subset G$ is a closed subgroup of a Lie group. Let $X = G/H$ and $x_0 = 1 \cdot H \in X$. A $(G, X)$-developing system is a pair $(h, \delta)$ of maps, where 
\[
\delta : (\tilde{M}, \tilde{m}_0) \to (X, x_0)
\]
is a local diffeomorphism and 
\[
h : \pi_1(M) \to G
\]
is a group homomorphism so that 
\[
\delta (\gamma \tilde{m}) = h (\gamma) \delta (\tilde{m}),
\]
for every $\gamma \in \pi_1(M)$ and $\tilde{m} \in \tilde{M}$. The map $\delta$ is called the developing map, and the morphism $h$ is called the holonomy morphism of the developing system.
Definition 7. Denote the universal covering map of a pointed manifold \((M, m_0)\) as
\[
\pi_M : (\bar{M}, \bar{m}_0) \to (M, m_0).
\]
Given a \((G, X)\)-developing system \((h, \delta)\) on a manifold \(M\), the *induced* \((G, X)\)-atlas on \(M\) is the one whose charts consist of all maps \(f\) so that \(\delta = f \circ \pi_M\). Every \((G, X)\)-atlas lies in a unique \((G, X)\)-structure, so the induced \((G, X)\)-structure is the \((G, X)\)-structure containing the induced \((G, X)\)-atlas.

**Remark 1.** Conversely, it is well known \([6]\) that if \(G\) acts faithfully on \(X = G/H\), then every \((G, X)\)-structure is induced by a developing system \((h, \delta)\), which is uniquely determined up to (1) conjugacy:
\[
(h, \delta) \mapsto (\text{Ad}(g)h, g\delta)
\]
and (2) choice of a point \(m_0 \in M\) to develop from.

As a general principle, geometric objects (i.e. tensors, foliations, maps, etc.) on \((G, X)\) which are invariant under the image of the holonomy morphism induce geometric objects of the same type on \(M\).

2.4. **Inducing structures from other structures.**

**Definition 8.** Suppose that \(h : G_0 \to G_1\) is a morphism of Lie groups, that \(H_0 \subset G_0\) and \(H_1 \subset G_1\) are closed subgroups. Let \(X_0 = G_0/H_0\) and \(X_1 = G_1/H_1\). Suppose that \(h(H_0) \subset H_1\). Define a smooth map
\[
\delta : X_0 \to X_1
\]
by
\[
\delta (g_0 H_0) = h (g_0) H_1.
\]
We call \((h, \delta)\) a *morphism* of homogeneous spaces. Suppose also that
\[
h^\prime(1) : g_0/H_0 \to g_1/H_1
\]
is a linear isomorphism. Then clearly \(\delta\) is a local diffeomorphism, and we call \((h, \delta)\) an *inducing* morphism of homogeneous spaces. If a manifold \(M\) is equipped with an \(X_0\)-chart \(f\), then \(\delta \circ f\) is clearly a \(X_1\)-chart. A \((G_0, X_0)\)-structure \(\{f_a\}\) has *induced* \((G_1, X_1)\)-structure \(\{\delta \circ f_a\}\). Every developing system \((h_0, \delta_0)\) on \(M\) has induced developing system \((h_1, \delta_1) = (h \circ h_0, \delta \circ \delta_0)\).

2.5. **The definition of \(G_D\).** Pick an effective divisor \(D\) on \(C\) of positive degree. Let \(p(z)\) be the monic polynomial with zero locus \(D\) (counting multiplicities). Let \(V_D\) be the set of all holomorphic functions \(f : C \to C\) so that
\[
p(\partial_z) f(z) = 0.
\]
If
\[
D = n_1 [\lambda_1] + n_2 [\lambda_2] + \cdots + n_\ell [\lambda_\ell],
\]
with distinct \(\lambda_j\), then the space \(V_D\) has as basis the functions
\[
\lambda^k e^{\lambda z}
\]
for \(\lambda = \lambda_j\) and \(0 \leq k \leq n_j - 1\), \(j = 1, 2, \ldots, \ell\). Let \(G_D = C \rtimes \tilde{V}_D\) with the group operation
\[
(t_0, f_0(z)) (t_1, f_1(z)) = (t_0 + t_1, f_0(z) + f_1 (z - t_0)),
\]
and inverse operation
\[
(t, f(z))^{-1} = (-t, -f (z + t)),
\]
and identity element \((0, 0)\).

Let \(G_D\) act on \(C^2\) by the faithful group action
\[
(t, f) (z, w) = (z + t, w + f (z + t)).
\]
The stabilizer of the origin of $\mathbb{C}^2$ is the group $H_D \subset G_D$ of pairs $(0, f)$ so that $f(0) = 0$. The surface $\mathbb{C}^2$ with this action of $G_D$ is the exotic Lie surface of the first kind.

Writing elements of $\mathbb{C}^2$ as $(z, w)$, the action of $G_D$ preserves the holomorphic vector field $\partial_w$, the holomorphic 1-form $dz$, the foliation $z = \text{constant}$, the holomorphic volume form $dz \wedge dw$. On each leaf of that foliation, the stabilizer of the leaf preserves the holomorphic 1-form $dw$.

If we add more points to our divisor, i.e. add an effective divisor $D'$, clearly
\[ V_D \subset V_{D+D'}, \]
\[ G_D \subset G_{D+D'}, \]
\[ H_D \subset H_{D+D'}, \]
\[ G_D/H_D = G_{D+D'}/H_{D+D'} = \mathbb{C}^2. \]
So any $(G_D, \mathbb{C}^2)$-structure is also a $(G_{D+D'}, \mathbb{C}^2)$-structure.

Write the divisor $D$ as
\[ D = n_1 [\lambda_1] + n_2 [\lambda_2] + \cdots + n_k [\lambda_k], \]
where $n_j \in \mathbb{Z}_{>0}$ is a multiplicity, and $\lambda_j \in \mathbb{C}$ is a point. Pick any nonzero complex number $\mu$. Let
\[ D' = n_1 [\mu \lambda_1] + n_2 [\mu \lambda_2] + \cdots + n_k [\mu \lambda_k]. \]
There is an obvious isomorphism
\[ (t, f) \in G_D \mapsto \left( \frac{t}{\mu}, f(\mu z) \right) \in G_{D'} \]
and equivariant biholomorphism
\[ (z, w) \in \mathbb{C}^2 \mapsto \left( \frac{z}{\mu}, w \right). \]

The Lie algebra $g_D$ of $G_D$ is spanned by $\partial_z$ together with all of the vector fields $f(z) \partial_w$ where $f \in V_D$. The adjoint action is
\[ (t, f)_* \partial_z = \partial_z + f'(z) \partial_w, \]
\[ (t, f)_* g(z) \partial_w = g(z-t) \partial_w. \]

2.6. The definition of $G_D'$. Let $G_D' = \mathbb{C} \times \mathbb{C}^* \times V_D$ with group operation
\[ (t_0, \mu_0, f_0(z)) (t_1, \mu_1, f_1(z)) = (t_0 + t_1, \mu_0 \mu_1, f_0(z) + \mu_0 f_1(z-t_0)), \]
inverse operation
\[ (t, \mu, f(z))^{-1} = \left( -t, \frac{1}{\mu}, -\frac{f(z+t)}{\mu} \right), \]
identity element
\[ (0, 1, 0), \]
and action on $\mathbb{C}^2$
\[ (t, \mu, f)(z, w) = (z + t, \mu w + f(z + t)). \]
The stabilizer $H_D'$ of $(0, 0) \in \mathbb{C}^2$ is the group of triples $(0, \mu, f(z))$ for which $f(0) = 0$. The surface $\mathbb{C}^2$ with this action of $G_D'$ is the exotic Lie surface of the second kind.

Writing elements of $\mathbb{C}^2$ as $(z, w)$, the action of $G_D'$ preserves the holomorphic 1-form $dz$, the foliation $z = \text{constant}$. On each leaf of that foliation, the stabilizer of that leaf preserves the holomorphic affine connection $\nabla = \partial_w$. The action of $G_D'$ on $\mathbb{C}^2$ preserves the standard flat connection on the canonical bundle of $\mathbb{C}^2$.
The Lie algebra of $G'_D$ is generated by $\partial_z, w\partial_w$ and the vector fields $f(z)\partial_w$ for $f \in V_D$. The adjoint action is

$$(t, \mu, f)_\ast \partial_z = \partial_z + f'(z)\partial_w,$$

$$(t, \mu, f)_\ast w\partial_w = (w - f(z))\partial_w,$$

$$(t, \mu, f)_\ast \frac{g(z)}{w}\partial_w = \mu g(z - t)\partial_w.$$  

Clearly $G_D$ is a subgroup of $G'_D$.

3. Statements of the theorems

In the remainder of this paper we will prove the following theorems.

**Theorem 2.** For any effective divisor $D$ on $\mathbb{C}$, either (1) 0 does not lie in the support of $D$ and no complex torus admits any holomorphic $(G_D, \mathbb{C}^2)$-structure or (2) 0 lies in the support of $D$ and every complex torus admits a 1-parameter family of pairwise nonisomorphic holomorphic $(G_D, \mathbb{C}^2)$-structures. Any holomorphic $(G_D, \mathbb{C}^2)$-structure on any complex torus is isomorphic to precisely one of these.

**Theorem 3.** For any effective divisor $D$ on $\mathbb{C}$, and for each distinct point $\lambda$ in the support of $D$, every complex torus admits a 1-parameter family of pairwise nonisomorphic holomorphic $(G'_D, \mathbb{C}^2)$-structures and an exceptional $(G'_D, \mathbb{C}^2)$-structure. Any holomorphic $(G'_D, \mathbb{C}^2)$-structure on any complex torus is isomorphic to precisely one of these.

**Theorem 4.** For any effective divisor $D$ on $\mathbb{C}$, either (1) 0 has multiplicity less than 2 in $D$ and no Kodaira surface admits any holomorphic $(G_D, \mathbb{C}^2)$-structure or (2) 0 has multiplicity at least 2 in $D$ and every primary Kodaira surface admits a 1-parameter family of pairwise nonisomorphic holomorphic $(G_D, \mathbb{C}^2)$-structures. Any holomorphic $(G_D, \mathbb{C}^2)$-structure on any primary Kodaira surface is isomorphic to precisely one of these.

**Theorem 5.** For any effective divisor $D$ on $\mathbb{C}$, and for each $\lambda$ with multiplicity 2 or more in $D$, any primary Kodaira surface admits a 1-parameter family of pairwise nonisomorphic holomorphic $(G'_D, \mathbb{C}^2)$-structures. Any holomorphic $(G'_D, \mathbb{C}^2)$-structure on any primary Kodaira surface is isomorphic to precisely one of these.

4. $G_D$ and tori

**Example 1.** Pick an effective divisor $D$ on $\mathbb{C}$. Let $n_0$ be the order of 0 in $D$, and assume that $n_0 \geq 1$. Let $G_0 = (\mathbb{C}^2, +)$, and $H_0 = \{0\} \subset G_0$. Pick any $k \in \mathbb{C}$. Define a complex Lie group morphism

$$h: (\lambda, \mu) \in \mathbb{C}^2 \rightarrow (\lambda, \mu + k(z^{n_0} - (z - \lambda)^{n_0})) \in G_D.$$  

Check that this induces the biholomorphism

$$\delta: (s, t) \in \mathbb{C}^2 \rightarrow G_0/H_0 \rightarrow (s, t + ks^{n_0}) \in \mathbb{C}^2,$$

so that $(h, \delta)$ is an inducing morphism of homogeneous spaces. Consequently every holomorphic $(G_0, \mathbb{C}^2)$-structure, i.e. translation structure on a complex surface, induces a holomorphic $(G_D, \mathbb{C}^2)$-structure. In particular, every complex torus bears a holomorphic $(G_D, \mathbb{C}^2)$-structure. The inducing morphisms are conjugate just when the induced $(G_D, \mathbb{C}^2)$-structures are conjugate. We will prove below that all $(G_D, \mathbb{C}^2)$-structures on the torus are induced by an inducing morphism of this form, up to choice of affine coordinates on the universal covering space of the torus. So the moduli space of $(G_D, \mathbb{C}^2)$-structures modulo conjugation is identified with $\mathbb{C}$, and does not depend on the particular choice of complex torus.
Proposition 1. Suppose that $S$ is a complex torus bearing a $(G_D, \mathbb{C}^2)$-structure for some effective divisor $D$ on $\mathbb{C}$. Then, for some choice of affine coordinates on the universal covering space of $S$, the $(G_D, \mathbb{C}^2)$-structure is one of those in example [example on the previous page]

Proof. Pick any $(G_D, \mathbb{C}^2)$-structure on any complex torus $S = \mathbb{C}^2/\Lambda$. Write the developing map as

$$\delta(s, t) = (z(s, t), w(s, t))$$

and the holonomy morphism as

$$h(\lambda, \mu) = (t_{\lambda, \mu}, f_{\lambda, \mu}),$$

for each $(\lambda, \mu) \in \Lambda$. Hence

$$z(s + \lambda, t + \mu) = z(s, t) + t_{\lambda, \mu}.$$

Consequently, $dz$ is a holomorphic 1-form on $S$, so

$$dz = a \, ds + b \, dt$$

for some complex numbers $a, b \in \mathbb{C}^2$. We can arrange, by linear change of variables on $\mathbb{C}^2 = \tilde{S}$ that $z = s$. Therefore $t_{\lambda, \mu} = \lambda$. So $w(s, t)$ must satisfy

$$w(s + \lambda, t + \mu) = w(s, t) + f_{\lambda, \mu}(s + \lambda),$$

for every $(\lambda, \mu) \in \Lambda$ and $(s, t) \in \mathbb{C}^2$. So

$$\frac{\partial w}{\partial t}(s + \lambda, t + \mu) = \frac{\partial w}{\partial t}(s, t),$$

i.e. $\frac{\partial w}{\partial t}$ is a holomorphic function on the compact complex surface $S$, so a constant, which we rescale to be 1, so

$$w(s, t) = t + W(s),$$

for some holomorphic function $W(s)$. The holonomy equivariance of $\delta$ is then precisely

$$W(s + \lambda) - W(s) = f_{\lambda, \mu}(s + \lambda) - \mu.$$

Let $P_D$ be the differential operator

$$P_D = \prod_j (\partial_s - \lambda_j)^{n_j}. $$

So a holomorphic function $F(s)$ lies in $V_D$ just when $P_D F(s) = 0$. Applying $P_D$ to both sides,

$$P_D W(s + \lambda) = P_D W(s) - P_D \mu.$$

Therefore $P_D W'$ is a holomorphic function on the surface $S$, and so is constant, say

$$P_D W'(s) = k_0,$$

and

$$P_D W(s) = k_0 s + k_1.$$

Therefore

$$k_0 (s + \lambda) + k_1 = k_0 s + k_1 - P_D \mu.$$

Clearly

$$P_D \mu = \mu \prod_j (-\lambda_j)^{n_j}. $$

So

$$k_0 \lambda = -\mu \prod_j (-\lambda_j)^{n_j}. $$
for every $(\lambda, \mu) \in \Lambda$. Since $\Lambda \subset \mathbb{C}^2$ is a lattice, so spans $\mathbb{C}^2$, we must have $k_0 = 0$ and
\[
\prod_j (-\lambda_j)^{n_j} = 0,
\]
i.e. $\lambda_j = 0$ for some $j$, i.e. 0 lies in the support of $D$. Now $P_D W(s) = k_1$ is a constant, so
\[
W(s) \in V_{D+\{0\}},
\]
i.e.
\[
W(s) = f(s) + ks^{n_0},
\]
for some $f \in V_D$. We can replace our developing system $(h, \delta)$ by the equivalent system $(ghg^{-1}, g\delta)$ for any $g \in G_D$, and thereby arrange $f = 0$, so
\[
\delta(s, t) = (s, t + ks^{n_0}).
\]

5. $G'_D$ on tori

Example 2. Suppose that $D$ is an effective divisor on $\mathbb{C}$. Let $G_0 = (\mathbb{C}^2, +)$, and $H_0 = \{0\} \subset G_0$. Define
\[
\delta: (s, t) \in \mathbb{C}^2 \rightarrow (s, e^t) \in \mathbb{C}^2,
\]
and
\[
\lambda: (\lambda, \mu) \in G_0 \rightarrow (\lambda, e^\mu, 0) \in G'_D.
\]
Then $(h, \delta)$ is an inducing morphism. Consequently every holomorphic $(G_0, \mathbb{C}^2)$-structure, i.e. translation structure on a complex surface, induces a holomorphic $(G'_D, \mathbb{C}^2)$-structure. In particular, every complex torus bears a holomorphic $(G'_D, \mathbb{C}^2)$-structure.

Example 3. Suppose that $D$ is an effective divisor on $\mathbb{C}$. Pick a complex number $a$ in the support of $D$. Let $n_a$ be the order of $a$ in $D$. Let $G_0 = (\mathbb{C}^2, +)$, and $H_0 = \{0\} \subset G_0$. Pick any $k \in \mathbb{C}$. For any $(\lambda, \mu) \in \mathbb{C}^2$, let
\[
f_{\lambda, \mu}(z) = e^{az}(\mu + k(z^{n_a} - (z - \lambda)^{n_a})).
\]
Define a morphism of complex Lie groups
\[
h: (\lambda, \mu) \in \mathbb{C}^2 \rightarrow (\lambda, e^{\lambda}, f_{\lambda, \mu}) \in G'_D.
\]
This morphism induces the biholomorphism
\[
\delta: (s, t) \in \mathbb{C}^2 = G_0/H_0 \rightarrow (s, e^{az}(t + ks^{n_a})) \in \mathbb{C}^2 = G'_D/H'_D
\]
so that $(h, \delta)$ is an inducing morphism. Consequently every holomorphic $(G_0, \mathbb{C}^2)$-structure, i.e. translation structure on a complex surface, induces a holomorphic $(G'_D, \mathbb{C}^2)$-structure. In particular, every complex torus bears a $(G'_D, \mathbb{C}^2)$-structure. Two induced $(G'_D, \mathbb{C}^2)$-structures are conjugate if and only if they are induced from conjugate inducing morphisms. We will prove below that all $(G'_D, \mathbb{C}^2)$-structures on the torus are induced by such an inducing morphism (except those induced as in example 2). So the $(G'_D, \mathbb{C}^2)$-structures on any complex 2-torus $\mathbb{C}^2/\Lambda$ are identified, modulo conjugation and modulo the choice of affine coordinates on the universal covering space of the torus, with
\[
* \sqcup \bigcup_{a \in \text{supp } D} V_D/\left(\frac{d}{dz} - a\right) V_D,
\]
a disjoint union of 1-dimensional complex vector spaces and one point for example 2 (The same moduli space parameterizes the inducing morphisms $(G_0, \mathbb{C}^2) \rightarrow$
Note that this moduli space is independent of the choice of complex 2-torus, but depends on the support of the divisor $D$.

**Proposition 2.** Suppose that $S$ is a complex torus bearing a $(G_D', \mathbb{C}^2)$-structure for some effective divisor $D$ on $\mathbb{C}$. Then, for some choice of affine coordinates on the universal covering space of $S$, the $(G_D', \mathbb{C}^2)$-structure is that of example 2 on the preceding page or one of those in example 3 on the previous page.

**Proof.** Pick any $(G_D', \mathbb{C}^2)$-structure on any complex torus $S = \mathbb{C}^2 / \Lambda$. Write the developing map as

$$\delta(s, t) = (z(s, t), w(s, t))$$

and the holonomy morphism as

$$h(\lambda, \mu) = (t_{\lambda, \mu}, g_{\mu}, f_{\lambda, \mu}),$$

for each $(\lambda, \mu) \in \Lambda$.

So

$$z(s + \lambda, t + \mu) = z(s, t) + t_{\lambda, \mu}.$$ 

Consequently, $dz$ is a holomorphic 1-form on $S$, so

$$dz = a \, ds + b \, dt$$

for some complex numbers $a, b \in \mathbb{C}^2$. We can arrange, by linear change of variables on $\mathbb{C}^2 = \tilde{S}$ that $z = s$. Therefore $t_{\lambda, \mu} = \lambda$. So $w(s, t)$ must satisfy

$$w(s + \lambda, t + \mu) = g_{\mu}w(s, t) + f_{\lambda, \mu}(s + \lambda),$$

for every $(\lambda, \mu) \in \Lambda$ and $(s, t) \in \mathbb{C}^2$. So

$$\frac{\partial w}{\partial t}(s + \lambda, t + \mu) = g_{\mu} \frac{\partial w}{\partial t}(s, t).$$

If $h = \frac{\partial w}{\partial t}$, then $dh/h$ is a holomorphic 1-form on $S$, so

$$\frac{dh}{h} = a \, ds + b \, dt,$$

say. Therefore

$$h(s, t) = Ae^{as + bt},$$

for some constant $A \in \mathbb{C}^\times$. This forces

$$g_{\lambda, \mu} = e^{a\lambda + b\mu}$$

for every $(\lambda, \mu) \in \Lambda$. Integrate:

$$w(s, t) = e^{as} \begin{cases} W(s) + Be^{bt}, & \text{if } b \neq 0, \\ W(s) + Bt, & \text{if } b = 0, \end{cases}$$

for some constant $B \in \mathbb{C}^\times$ and some holomorphic function $W(s)$. We can rescale and translate $t$ to arrange i.e.

$$w(s, t) = e^{as} \begin{cases} W(s) + e^t, & \text{or} \\ W(s) + t. \end{cases}$$

Suppose for the moment that $w(s, t) = e^{as}(W(s) + Be^t)$. The holonomy equivariance of $\delta$ is then precisely

$$e^{a(s+\lambda)}(W(s + \lambda) - e^sw(s)) = f_{\lambda, \mu}(s + \lambda).$$

Let $P_D$ be the differential operator

$$P_D = \prod_j (\partial_s - \lambda_j)^{n_j}.$$
So a holomorphic function \( F(s) \) lies in \( V_D \) just when \( P_D F(s) = 0 \). Applying \( P_D \) to both sides,
\[
P_D e^{as} W(s + \lambda) = e^{\mu} P_D e^{as} W(s).
\]
Suppose that
\[
D = \sum_j n_j [\lambda_j].
\]
Let
\[
D_a = \sum_j n_j [\lambda_j - a].
\]
Then for any holomorphic function \( f(s) \),
\[
P_D (e^{as} f(s)) = e^{as} P_D_a f(s).
\]
So
\[
P_D_a W(s + \lambda) = e^{\mu} P_D_a W(s).
\]
Let \( F = P_{D_a} W \): \( F(s + \lambda) = e^{\mu} F(s) \), for every \((\lambda, \mu) \in \Lambda\). So \( F \) is a holomorphic section of a flat line bundle on a complex torus. Any holomorphic section of a degree 0 line bundle must be everywhere 0, unless the bundle is holomorphically trivial, in which case the holomorphic sections are everywhere 0 or everywhere nonzero. So \( F \) is either everywhere 0 or everywhere nonzero. If \( F \) is everywhere nonzero, then \( \frac{dF}{F} \) is a holomorphic 1-form, so
\[
\frac{dF}{F} = p \, ds + q \, dt,
\]
for some constants \( p, q \in \mathbb{C} \). But \( F = F(s) \) is independent of \( t \), so
\[
\frac{dF}{F} = p \, ds.
\]
Therefore
\[
F(s) = Ce^{ps},
\]
for some constant \( C \neq 0 \). Therefore the holomorphic function
\[
e^{ps-t}
\]
is defined on \( S \), and so must be constant, a contradiction to our hypothesis that \( F \neq 0 \). Therefore \( F = 0 \).

Continuing to suppose that \( w(s, t) = e^{as} (W(s) + e^t) \), we can say that \( F = 0 \) i.e. \( P_{D_a} W = 0 \). By replacing the developing map and holonomy by action of an element of \( G_D \), we can arrange that \( W(s) = 0 \), i.e.
\[
\delta(s, t) = (s, e^t).
\]
The holonomy morphism must be
\[
h(\lambda, \mu) = (\lambda, e^\mu, 0).
\]
Next we can suppose that
\[
(z(s, t), w(s, t)) = (s, e^{as} (W(s) + t)).
\]
The holonomy equivariance says that
\[
e^{a(s+\lambda)} (W(s + \lambda) + \mu - W(s)) = f_{\lambda, \mu}(s + \lambda).
\]
Taking \( P_D \) of both sides,
\[
P_{D_a} W(s + \lambda) - P_{D_a} W(s) + P_{D_a} \mu = 0.
\]
As above, \( P_{D_a} W'(s) \) is a holomorphic function on \( S \) so constant, say
\[
P_{D_a} W'(s) = k_0.
So

\[ P_D W(s) = k_0 s + k_1 \]

for some constants \( k_0, k_1 \in \mathbb{C} \). Plugging this in to equation 1,

\[ 0 = k_0 \lambda + (-1)^{\deg D} \prod_j (\lambda_j - a)^{n_j} \mu. \]

This linear equation between \( \lambda \) and \( \mu \) holds for every \((\lambda, \mu) \in \Lambda\). Since \( \Lambda \subset \mathbb{C}^2 \) is a lattice, there is no linear relation satisfied by its elements, so \( k_0 = 0 \) and \( \lambda_j = a \) for some \( j \). Moreover \( P_D W(s) = k_1 \) for some constant \( k_1 \), i.e. \( P_{D_n+0} W(s) = 0 \). Therefore

\[ W(s) = e^{-as} f(s) + ks^{n_a}, \]

for a unique \( f \in V_D \), where \( n_a \) is the order of \( a \) in \( D \). We can conjugate the holonomy to arrange that \( f = 0 \). \( \square \)

6. Primary Kodaira Surfaces

A primary Kodaira surface is a compact complex surface \( S \) of odd first Betti number which occurs as the total space of a principal bundle

\[
\begin{array}{ccc}
E_1 & \longrightarrow & S \\
\downarrow & & \downarrow \\
E_0 & & \\
\end{array}
\]

where \( E_0 \) and \( E_1 \) are elliptic curves. The canonical bundle of \( S \) is trivial \cite{2} p. 147. Up to finite covering, all holomorphic elliptic curve fibrations over an elliptic curve base are principal \cite{2} p. 147. Kodaira \cite{9} p. 788 theorem 19 shows that primary Kodaira surfaces have the form \( S = \Gamma \backslash \mathbb{C}^2 \) where \( \Gamma \) is a discrete group of affine transformations acting properly discontinuously without fixed points on \( \mathbb{C}^2 \), preserving \( dz_1 \land dz_2 \). The classification of Suwa \cite{17} pp. 247–249 says that a compact complex surface of the form \( \Gamma \backslash \mathbb{C}^2 \) is an elliptic fiber bundle with nonzero first Chern class if and only if it has first Betti number 3. The description \( S = \Gamma \backslash \mathbb{C}^2 \) makes explicit that \( S \) has an affine structure. It is convenient to write each affine transformation \( z \mapsto az + b \) as a matrix

\[
\begin{pmatrix}
a_{11} & a_{12} & b_1 \\
a_{21} & a_{22} & b_2 \\
0 & 0 & 1
\end{pmatrix}.
\]

In fact, Suwa proves that we can arrange all of the elements of \( \Gamma \) to be affine transformations of the form

\[
g = \begin{pmatrix}
1 & g_1 & g_2 \\
0 & 1 & g_3 \\
0 & 0 & 1
\end{pmatrix},
\]

after perhaps replacing \( S \) by a finite covering space. Every element of \( \Gamma \) clearly preserves the holomorphic foliation \( dz_2 = 0 \). The leaves of this foliation quotient to become the fibers of the elliptic surface \( S \).

Fillmore and Schueneman \cite{5} main theorem, Vitter \cite{18} p. 238 and Dürr \cite{4} lemma 4.8 provide the following even more explicit description. Suppose that \( S \) is
a primary Kodaira surface

\[
\begin{array}{c}
E_0 \\
\downarrow \\
E_1 \\
\end{array}
\rightarrow S
\]

where \(E_0\) and \(E_1\) are elliptic curves. The fundamental group of \(S\) admits a presentation

\[\pi_1(S) = \Gamma = \langle a, b, c, d | c, d \text{ central}, [a, b] = c^r \rangle\]

where \(r\) is a positive integer. The kernel of the morphism \(\pi_1(S) \to \pi_1(E_0)\) is precisely the center of \(\pi_1(S)\). The surface \(S\) is explicitly the quotient \(\Gamma \setminus \mathbb{C}^2\), where \(a, b, c, d \in \Gamma\) act as the affine transformations

\[
a = \begin{pmatrix}
1 & a_1 & 0 \\
0 & 1 & 1 \\
0 & 0 & 1
\end{pmatrix},
b = \begin{pmatrix}
1 & b_1 & 0 \\
0 & 1 & b_3 \\
0 & 0 & 1
\end{pmatrix},
c = \begin{pmatrix}
1 & 0 & c_2 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix},
d = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix},
\]

for some complex numbers \(a_1, b_1, c_1, b_3\). The equation \([a, b] = c^r\) forces precisely \(b_1 = a_1b_3 - rc_2\). Moreover we can arrange that \(c_2\) and \(b_3\) both lie in the upper half plane. (We can even arrange that \(c_2\) and \(b_3\) both lie in a given fundamental domain for the modular group in the upper half plane.) Conversely, for any choice of complex constants \(a_1, c_2, b_3\), if we set \(b_1 = a_1b_3 - rc_2\), and if \(c_2\) and \(b_3\) both lie in the upper half plane, then we can define \(a, b, c, d\) and \(\Gamma\) as above, and the surface \(S = \Gamma \setminus \mathbb{C}^2\) is a primary Kodaira surface.

**Example 4.** For our purposes, we will need to add in two extra parameters to this family: every primary Kodaira surface can be written in many ways as the the quotient \(\Gamma \setminus \mathbb{C}^2\), where \(a, b, c, d \in \Gamma\) act as the affine transformations

\[
a = \begin{pmatrix}
1 & a_1 & 0 \\
0 & 1 & a_3 \\
0 & 0 & 1
\end{pmatrix},
b = \begin{pmatrix}
1 & b_1 & 0 \\
0 & 1 & b_3 \\
0 & 0 & 1
\end{pmatrix},
c = \begin{pmatrix}
1 & 0 & c_2 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix},
d = \begin{pmatrix}
1 & 0 & d_2 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix},
\]

for some complex numbers \(a_1, a_3, b_1, b_3, c_2, d_2\), with \(a_1b_3 - b_1a_3 = rc_2\), where \(a_3, b_3\) are \(\mathbb{R}\)-linearly independent, and \(c_2, d_2\) are \(\mathbb{R}\)-linearly independent. This extra freedom will allow us to normalize various constants appearing later on.

**7. \(G_D\) and Primary Kodaira Surfaces**

**Example 5.** Let \(G_0\) be the group of all complex affine transformations of the form

\[
g = \begin{pmatrix}
1 & g_1 & g_2 \\
0 & 1 & g_3 \\
0 & 0 & 1
\end{pmatrix}
\]

for any \(g_1, g_2, g_3 \in \mathbb{C}\), and let \(G_0\) act on \(\mathbb{C}^2\) as complex affine transformations

\[(z_1, z_2) \mapsto (z_1 + g_1 z_2 + g_2, z_2 + g_3)\]

Let \(H_0 \subset G_0\) be the subgroup fixing the origin in \(\mathbb{C}^2\). By construction, every primary Kodaira surface has a holomorphic \((G_0, \mathbb{C}^2)\)-structure with developing map the identity \(\mathbb{C}^2 \to \mathbb{C}^2\) and and holonomy morphism the inclusion \(\Gamma \to G_0\). We leave the reader to check that the group \(\Gamma = \langle a, b, c, d \rangle \subset G_0\) is dense in the complex algebraic Zariski topology of \(G_0\), so that there is no complex analytic reduction of structure group of this structure to any complex algebraic proper subgroup of \(G_0\).
Remark 2. The real Lie subgroup of $G_0$ consisting of the matrices
\[
\begin{pmatrix}
1 & g_1 & g_2 \\
0 & 1 & -\sqrt{-1}g_1 \\
0 & 0 & 1
\end{pmatrix}
\]
acts transitively on $\mathbb{C}^2$, and contains a conjugate of each subgroup $\Gamma$ defined in example 4 on the preceding page, so we can reduce the structure group $G_0$ this real Lie group.

Example 6. Define a complex Lie group isomorphism
\[
h: g = \begin{pmatrix}
1 & g_1 & g_2 \\
0 & 1 & g_3 \\
0 & 0 & 1
\end{pmatrix} \in G_0 \mapsto (g_3, g_1 (z - g_3) + g_2) \in G_{2[0]}
\]
and biholomorphism
\[
\delta: (z_1, z_2) \in \mathbb{C}^2 = G_0/H_0 \mapsto (z_1, z_2) \in \mathbb{C}^2 = G_{2[0]}/H_{2[0]}
\]
so that $(h, \delta)$ is an inducing morphism. Any holomorphic $(G_0, \mathbb{C}^2)$-structure on any complex surface, induces a holomorphic $(G_{2[0]}, \mathbb{C}^2)$-structure.

Example 7. We generalize the last example. Pick any constant $k \in \mathbb{C}$. Define a complex Lie group isomorphism
\[
h: g = \begin{pmatrix}
1 & g_1 & g_2 \\
0 & 1 & g_3 \\
0 & 0 & 1
\end{pmatrix} \in G_0 \mapsto \left( g_3, (g_1 + kg_3) (z - g_3) + g_2 + \frac{k}{2} g_3^2 \right) \in G_{2[0]}.
\]
This morphism defines a biholomorphism
\[
\delta: (z_1, z_2) \in \mathbb{C}^2 = G_0/H_0 \mapsto \left( z_1 + \frac{k}{2} z_2^2, z_2 \right) \in \mathbb{C}^2 = G_{2[0]}/H_{2[0]}
\]
so that $(h, \delta)$ is an inducing morphism. Any holomorphic $(G_0, \mathbb{C}^2)$-structure on any complex surface has a 1-parameter family of deformations given by varying $k$ above. Since $G_0$ lies in the complex affine group, any holomorphic $(G_0, \mathbb{C}^2)$-structure determines a holomorphic affine structure. Vitter \cite{18} proves that every holomorphic affine structure on a primary Kodaira surface is among one of these, depending only on the arbitrary choice of constant $k \in \mathbb{C}$.

So every primary Kodaira surface admits a 1-parameter family of $(G_{2[0]}, \mathbb{C}^2)$-structures. Vitter proves that the induced affine structures are distinct. Therefore they are distinct as $(G_{2[0]}, \mathbb{C}^2)$-structures, and nonisomorphic on every primary Kodaira surface.

Lemma 1. Every holomorphic $(G_{2[0]}, \mathbb{C}^2)$-structure on any primary Kodaira surface is isomorphic to one of those constructed in example 5 on the previous page, for some values of the constants $a_1, a_3, b_1, b_3, d_2, r$

used to construct the primary Kodaira surface and for some constant $k$ used to construct the structure.

Proof. Take the standard structure, say $\{f_0\}$, as defined in example 5, i.e. the one with $k = 0$. Then consider some other structure, say $\{g_0\}$. By lemma \cite{8} on page 20, we will see that the induced foliations of the two structures must be identical, because the only holomorphic foliation of any primary Kodaira surface is the fibration by elliptic curves. Every holomorphic vector field on any primary Kodaira surface is a constant multiple of the vector field generating the principal bundle action. Therefore the holomorphic vector fields corresponding to $\partial_w$ in
$G_{2[0]}/H_{2[0]}$ must be constant multiples of one another. Similarly the holomorphic 1-forms corresponding to $dz$ in the model must agree up to constant multiple, since they have to vanish on the leaves of the same fibration. We can also assume, following Vitter’s classification of affine structures on primary Kodaira surfaces (and perhaps after deforming the other structure by precisely one of the inducing morphisms in Vitter’s deformation) that the two structures induce the same affine structure.

Pick $p, q \neq 0$ any complex constants. We can change variables according to $$(Z, W) = (pz, qw)$$
and transform $G_{2[0]}$ by the morphism $$(t', f') = (pt, qf(z/p)).$$
This will then alter the values of the constants used to construct the primary Kodaira surface, by

$$a_1 \mapsto \frac{q}{p} a_1,$$
$$b_1 \mapsto \frac{q}{p} b_1,$$
$$c_2 \mapsto q c_2,$$
$$d_2 \mapsto q d_2,$$
$$a_3 \mapsto p a_3,$$
$$b_3 \mapsto p b_3.$$  

For suitable choice of $p$ and $q$ this rescaling will ensure that these holomorphic vector fields and 1-forms match precisely, and require us only to change the choices of the constants used to define the primary Kodaira surface. So we can assume that these vector fields and 1-forms agree, and that the affine structure agrees. Take $(z, w)$ a local chart for the standard structure, and $(Z, W)$ a local chart for the other structure. Since they have the same affine structure, the matrix

$$
\begin{pmatrix}
\frac{\partial Z}{\partial z} & \frac{\partial Z}{\partial w} \\
\frac{\partial W}{\partial z} & \frac{\partial W}{\partial w}
\end{pmatrix}
$$

is a constant invertible matrix. Since the 1-forms match, $dZ = dz$, so $Z = z + c$. Since the vector fields match, $\partial W = \partial w$, so $\partial_w W = 1$. Therefore

$$Z = z + c, W = w + az,$$

which is a chart for the standard structure. □

**Example 8.** Pick any constant $k \in \mathbb{C}$, and any integer $n \geq 2$. Let

$$(t, f) = \left(0, \frac{k}{n} z^n\right) \in G_{(n+1)[0]}.$$  

Define a complex Lie group homomorphism $h = Ad(t, f): (t_1, f_1) \in G_{2[0]} \to \left(t_1, f_1(z) + \frac{k}{n} \left(z^n - (z - t_1)^n\right)\right) \in G_{n[0]}.$

(N.B. the cancellation of the $z^n$ leading terms here ensures that this morphism of complex Lie groups is valued in $G_{n[0]}$, not merely in $G_{(n+1)[0]}$.) This morphism defines a biholomorphism $\delta = (t, f): (z, w) \in \mathbb{C}^2 = G_{2[0]}/H_{2[0]} \to \left(z, w + \frac{k}{n} z^n\right) \in \mathbb{C}^2 = G_{n[0]}/H_{n[0]}$.
so that \((h, \delta)\) is an inducing morphism of homogeneous spaces. Any holomorphic \((G_{2[0]}, \mathbb{C}^2)\)-structure on any complex surface induces a 1-parameter family of holomorphic \((G_n[0], \mathbb{C}^2)\)-structures by varying the parameter \(k\). Clearly this generalizes Vitter’s construction of affine structures from example 5 on page 11 which is precisely the case \(n = 2\).

**Lemma 2.** Pick an effective divisor \(D\) on \(\mathbb{C}\). Suppose that \(S\) is a primary Kodaira surface bearing a \((G_D, \mathbb{C}^2)\)-structure. Then the \((G_D, \mathbb{C}^2)\)-structure is a unique one of those in example 8 on the preceding page, up to holomorphic isomorphism.

**Proof.** Suppose that \(S\) is a primary Kodaira surface, with universal covering space \(\tilde{S} = \mathbb{C}^2\), with coordinates \((s, t)\), and each element

\[
\gamma = \begin{pmatrix} 1 & \gamma_1 & \gamma_2 \\ 0 & 1 & \gamma_3 \\ 0 & 0 & 1 \end{pmatrix} \in \pi_1(S)
\]

acts on \(\mathbb{C}^2\) as

\[
\gamma(s, t) = (s + \gamma_3, t + \gamma_1 s + \gamma_2).
\]

Suppose that \(S\) has a \((G_D, \mathbb{C}^2)\)-structure with developing map \(\delta\) and holonomy morphism \(h\). Write \(\delta = (z, w)\) and \(h(\gamma) = (\tau_\gamma, f_\gamma)\). Then, for each \(\gamma \in \pi_1(S)\),

\[
z(s + \gamma_3, t + \gamma_1 s + \gamma_3) = z(s, t) + \tau_\gamma.
\]

Therefore \(\frac{\partial z}{\partial t}\) is a holomorphic function on \(S\), so constant. By rescaling the \(t\) coordinate on \(\tilde{S}\), we can arrange that \(\frac{\partial z}{\partial t} = 1\) or \(\frac{\partial z}{\partial t} = 0\).

First, consider the case where \(\frac{\partial z}{\partial t} = 1\), say

\[
z(s, t) = t + Z(s),
\]

for some holomorphic function \(Z(s)\). Equivariance of the developing map under the holonomy action implies

\[
Z(s + \gamma_3) - Z(s) = t_\gamma - \gamma_1 s - \gamma_2.
\]

Therefore \(Z''(s)\) is a holomorphic function on \(S\), so constant, so

\[
Z(s) = k_0 + k_1 s + k_2 s^2
\]

for some constants \(k_0, k_1, k_2 \in \mathbb{C}\). Equivariance now says

\[
2k_2 \gamma_3 s + k_2 \gamma_3^2 + k_1 \gamma_3 = t_\gamma - \gamma_1 s - \gamma_2.
\]

This holds for every \(s \in \mathbb{C}\), so

\[
2k_2 \gamma_3 = \gamma_1 \quad \text{and} \quad k_2 \gamma_3^2 + k_1 \gamma_3 = t_\gamma - \gamma_2
\]

for every

\[
\gamma = \begin{pmatrix} 1 & \gamma_1 & \gamma_2 \\ 0 & 1 & \gamma_3 \\ 0 & 0 & 1 \end{pmatrix} \in \pi_1(S).
\]

In particular, plugging in the generators for \(\pi_1(S)\) which we described in example 4 on page 11 we find

\[
2k_2 a_3 = a_1
\]

and

\[
2k_2 b_3 = b_1
\]

where \(a_1, a_3, b_1, b_3\) are the various constants appearing in our construction of the primary Kodaira surface in example 3. This implies, again in terms of those constants,

\[
rc_2 = a_1 b_3 - b_1 a_3 = 0.
\]
But the constant $c_2$ has to be in the upper half plane, a contradiction. Therefore we can assume that $\frac{\partial z}{\partial t} = 0$. 

Write $z(s,t) = z(s)$. Equivariance of the developing map under the action of the fundamental group implies 

$$z(s + \gamma_3) = z(s) + t\gamma,$$

so that $z'(s)$ is a holomorphic function on the surface $S$, so a constant, say $z(s) = k_0s + k_1$. Since the developing map is a local biholomorphism, $k_0 \neq 0$. We can arrange by translating the coordinates on $\tilde{S}$ that $z(s) = k_0s$. We can rescale the affine coordinates $(s,t)$ as we like, perhaps changing the expression of the fundamental group as affine transformations, but keeping the same form of a primary Kodaira surface. Therefore we can arrange $k_0 = 1$, so $z(s) = s$.

Equivariance of the developing map under the fundamental group action is now expressed as the equation 

$$w(s + \gamma_3, t + \gamma_1 s + \gamma_2) = w(s, t) + f_\gamma (s + \gamma_3).$$

Therefore $\frac{\partial w}{\partial t}$ is a holomorphic function on $S$, so a constant, and we can rescale the $t$ variable to arrange that this constant is 1, say 

$$w(s, t) = t + W(s).$$

Equivariance of the developing map under the fundamental group action is now expressed as the equation 

(2) 

$$W(s + \gamma_3) - W(s) = f_\gamma (s + \gamma_3) - \gamma_1 s - \gamma_2.$$

Let $n_0$ be the order of $D$ at 0. Let $P_D$ be the constant coefficient differential operator 

$$P_D = \prod_j (\partial s - \lambda_j)^{n_j}$$

with kernel $V_D$. Then clearly 

$$P_{D+2[0]} W(s + c) - P_{D+2[0]} W(s) = 0.$$

Therefore $P_{D+2[0]} W$ is a holomorphic function on $S$, so constant, i.e. 

$$P_{D+3[0]} W(s) = 0.$$

Therefore 

$$W(s) = f(s) + k_0s^{n_0} + k_1s^{n_0+1} + k_2s^{n_0+2},$$

for some constants $k_0, k_1, k_2 \in \mathbb{C}$ and some $f \in V_D$. By conjugacy of the holonomy morphism inside $G_D$, we can arrange that $f(s) = 0$, so 

$$W(s) = k_0s^{n_0} + k_1s^{n_0+1} + k_2s^{n_0+2}.$$

We need 

$$W(s + c) - W(s) + \gamma_1 s + \gamma_2 \in V_D$$

for every 

$$\gamma = \begin{pmatrix} 1 & \gamma_1 & \gamma_2 \\ 0 & 1 & \gamma_3 \\ 0 & 0 & 1 \end{pmatrix} \in \pi_1(S),$$

i.e., 

$$(n_0 + 2) k_2 \gamma_3 s^{n_0+1} + (n_0 + 1) \gamma_3 \left( k_1 + \frac{(n_0 + 2) k_2 \gamma_3}{2} \right) s^{n_0} + \gamma_1 s + \gamma_2 \in V_D.$$ 

If $n_0 = 0$, then this expands to 

$$0 = (2k_2 \gamma_3 + \gamma_1) s + \left( k_2 \gamma_3^2 + k_1 \gamma_3 + \gamma_2 \right)$$
so that $2k_2\gamma_3 + \gamma_1 = 0$ for every

$$\gamma = \begin{pmatrix} 1 & \gamma_1 & \gamma_2 \\ 0 & 1 & \gamma_3 \\ 0 & 0 & 1 \end{pmatrix} \in \pi_1(S).$$

Examining generators, we find

$$a_1 = -2k_0a_3$$
$$b_1 = -2k_0b_3$$

which contradicts

$$0 \neq re_2 = a_1b_3 - b_1a_3.$$

Therefore $n_0 \geq 1$.

If $n_0 = 1$, then the same equation expands to require that

$$3k_2\gamma_3s^2 + (3k_2\gamma_3^2 + 2k_1\gamma_3 + \gamma_1)s$$

must be constant. Examining generators of $\pi_1(S)$, we find $k_2 = 0$ and $a_1 = -2k_1a_3$ and $b_1 = -2k_1b_3$ which again contradicts the requirement

$$0 \neq re_2 = a_1b_3 - b_1a_3$$

for the constants appearing in the generators of the fundamental group of a primary Kodaira surface. So $n_0 \geq 2$.

Plugging in that $n_0 \geq 2$, then equivariance under the holonomy morphism expands to

$$k_2(n_0 + 2)\gamma_3s^{n_0+1} + (n_0 + 1)\gamma_3\left(k_1 + \frac{n_0 + 2}{2}k_2\gamma_3\right)s^{n_0} \in V_D$$

modulo the terms in $V_D$, so we need $k_2 = 0$, and then clearly $k_1 = 0$ as well. So finally $W(s) = k_0 s^{n_0}$.

\hfill \Box

8. $G_D'$ and primary Kodaira surfaces

**Example 9.** As in example 5 on page 11, write elements of $G_{2[0]}$ as

$$(t, f) = (g_3, g_1(z - g_3) + g_2).$$

Pick an integer $n \geq 2$ and any two complex numbers $k, \lambda \in \mathbb{C}$. Consider the homomorphism of complex Lie groups

$$h: (t, f) \in G_{2[0]} \mapsto (t', \mu', f') \in G_{\mu[\lambda]}',$$

where $t' = g_3, \mu' = e^{\lambda g_3}$ and

$$f'(z) = e^{\lambda z}(g_1(z - g_3) + g_2 + k(z^n - (z - g_3)^n)).$$

This morphism takes $H_{2[0]}$ to $H_{\mu[\lambda]}'$ and thereby induces the biholomorphism

$$\delta: (z, w) \in \mathbb{C}^2 \mapsto (z, e^{\lambda z}(w + k z^n)) \in \mathbb{C}^2,$$

so that $(h, \delta)$ is an inducing morphism, giving a 1-parameter family of $(G_D', \mathbb{C}^2)$-structures (depending on $k$) for any effective divisor $D$ with multiplicity at least $n$ at $\lambda$, and for any $(G_{2[0]}, \mathbb{C}^2)$-structure. In particular, all at once we obtain families of these structures on all primary Kodaira surfaces.

**Lemma 3.** Pick an effective divisor $D$ on $\mathbb{C}$. Suppose that $S$ is a primary Kodaira surface. Then every $(G_D', \mathbb{C}^2)$-structure on $S$ is a unique one of those in example 9 up to holomorphic isomorphism.
Proof. Suppose that the developing map is $\delta = (z, w)$. As in the proof of lemma on page 14, we can arrange $z(s, t) = s$. Write the holonomy morphism as

$$h: \gamma \in \pi_1(S) \to (\tau_\gamma, g_\gamma, f_\gamma) \in G_D'.$$

Then for each

$$\gamma = \begin{pmatrix} 1 & \gamma_1 & \gamma_2 \\ 0 & 1 & \gamma_3 \\ 0 & 0 & 1 \end{pmatrix} \in \pi_1(S),$$

we have

$$w(s + \gamma_3, t + \gamma_1 s + \gamma_2) = g_\gamma w(s, t) + f_\gamma(s + \gamma_3).$$

Let $h = \frac{\partial w}{\partial t}$. Since $(z, w) = (s, w)$ is a local biholomorphism, $h \neq 0$. Moreover, $\frac{\partial h}{\partial t}h$ is invariant under $\pi_1(S)$, so is a holomorphic function on $S$, so constant, say $\frac{\partial h}{\partial t}h = k_0 h$,

and so

$$h(s, t) = e^{k_0 t} H(s),$$

for some nowhere vanishing holomorphic function $H$. Invariance of $h$ implies

$$H(s + \gamma_3) = g_\gamma e^{-k_0(\gamma_1 s + \gamma_2)} H(s).$$

So

$$H'(s + \gamma_3) = g_\gamma e^{-k_0(\gamma_1 s + \gamma_2)} (-k_0 \gamma_1 H(s) + H'(s)).$$

Dividing by $H(s + \gamma_3)$,

$$\frac{H'(s + \gamma_3)}{H(s + \gamma_3)} = -k_0 \gamma_1 + \frac{H'(s)}{H(s)},$$

It then follows that

$$\left( \frac{H'}{H} \right)' = k_1,$$

some $k_1 \in \mathbb{C}$. But then

$$\frac{H'(s)}{H(s)} = k_1 s + k_2,$$

some $k_2 \in \mathbb{C}$. The equation

$$\frac{H'(s + \gamma_3)}{H(s + \gamma_3)} = -k_0 \gamma_1 + \frac{H'(s)}{H(s)}$$

tells us that

$$k_1 \gamma_3 = -k_0 \gamma_1,$$

for all

$$\gamma = \begin{pmatrix} 1 & \gamma_1 & \gamma_2 \\ 0 & 1 & \gamma_3 \\ 0 & 0 & 1 \end{pmatrix} \in \pi_1(S).$$

There is no linear relation between the $\gamma_1$ and $\gamma_2$ components of $\gamma$ among the generators of $\pi_1(S)$, as we see above, so $k_0 = k_1 = 0$. Therefore

$$\frac{H'(s)}{H(s)} = k_2,$$
so that
\[ H(s) = e^{k_2 s + k_3} \]
for some constant \( k_3 \in \mathbb{C} \). Going backward,
\[ \frac{\partial w}{\partial s} = e^{k_2 s + k_3} , \]
so
\[ w(s, t) = e^{k_2 s + k_3} + W(s) , \]
for some holomorphic function \( W(s) \) and \( g_\gamma = e^{k_2 \gamma_3} \).

The equation for \( w \) forces
\[ e^{k_2 (s + \gamma_3) + k_3} (\gamma_1 s + \gamma_2) + W(s + \gamma_3) = e^{k_2 \gamma_3} W(s) + f_\gamma (s + \gamma_3) . \]
Therefore if we let
\[ F = P_{D+2[k_2]} W , \]
then
\[ F (s + \gamma_3) = e^{k_2 \gamma_3} F(s) . \]
So \( F \) is a section of a degree 0 line bundle on the elliptic curve \( E_0 \) (the base of the elliptic fibration \( E_1 \to S \to E_0 \) of the primary Kodaira surface \( S \)), and therefore either \( F \) is everywhere 0 or everywhere nonzero.

Suppose that \( F \) is everywhere nonzero. Clearly \( dF/F \) is a holomorphic 1-form on the elliptic curve \( E_0 \), so \( dF/F = k_4 ds \), for some constant \( k_4 \in \mathbb{C} \), i.e.
\[ F(s) = k_5 e^{k_4 s} . \]
By the \( \mathbb{R} \)-linear independence of the periods of the elliptic curve, \( k_4 = k_2 \), so
\[ F(s) = k_5 e^{k_2 s} . \]
Therefore
\[ P_{D+3[k_2]} W = 0 . \]
Suppose that \( n_0 \) is the order of \( k_2 \) in \( D \). Then
\[ W(s) = f(s) + e^{k_2 s} (K_0 s^{n_0} + K_1 s^{n_0+1} + K_2 s^{n_0+2}) \]
for some \( f \in V_D \) and constants \( K_0, K_1, K_2 \in \mathbb{C} \). We can conjugate the holonomy in \( G_D \) to arrange that \( f = 0 \), so
\[ W(s) = e^{k_2 s} (K_0 s^{n_0} + K_1 s^{n_0+1} + K_2 s^{n_0+2}) . \]
Similarly if \( F \) is everywhere 0 then \( W(s) \) has this form, but with \( K_2 = 0 \).

Equivariance of the developing map under the holonomy morphism then implies
\[ e^{k_3} (\gamma_1 s + \gamma_2) + (n_0 + 1) \gamma_3 \left( K_1 + \frac{(n_0 + 2) K_2 \gamma_3}{2} \right) s^{n_0} + K_2 (n_0 + 2) \gamma_3 s^{n_0+1} \in V_{D_{\mathbb{C}}} \]
for all
\[ \gamma = \begin{pmatrix} 1 & \gamma_1 & \gamma_2 \\ 0 & 1 & \gamma_3 \\ 0 & 0 & 1 \end{pmatrix} \in \pi_1(S) . \]
If \( n_0 = 0 \), then this says
\[ 0 = (e^{k_3 \gamma_1} + 2K_2 \gamma_3) s + e^{k_3} + K_1 \gamma_3 + K_2 \gamma_3^2 . \]
So
\[ e^{k_3 \gamma_1} + 2K_2 \gamma_3 = 0 \]
for all \( \gamma \in \pi_1(S) \). There can be no complex linear relation between the \( \gamma_1 \) and \( \gamma_3 \) components of \( \gamma \in \pi_1(S) \), a contradiction, so \( n_0 \geq 1 \).

If \( n_0 = 1 \), then this says that
\[ 3K_2 \gamma_3 s^2 + (e^{k_3 \gamma_1} + 2K_1 \gamma_3 + 3K_2 \gamma_3^2) s \]
is constant, for all $\gamma \in \pi_1(S)$. Therefore $K_2 = 0$ and again a linear relation

$$e^{k_3}\gamma_1 + 2K_1\gamma_3 = 0$$

which is impossible. So $n_0 \geq 2$. We can rescale the $t$ variable, possibly changing the representation of the fundamental group of our surface, but still keeping it a primary Kodaira surface, to arrange $k_3 = 0$.

Now assuming $n_0 \geq 2$, we easily see that $K_1 = K_2 = 0$. □

9. Nonsingular holomorphic foliations on compact complex surfaces

Lemma 4. Suppose that a compact complex surface $S$ is the total space of an elliptic fibration, and that $S$ contains no rational curves. Then $S$ has no singular fibers and $S$ is an elliptic fiber bundle. Up to replacing $S$ by a finite covering space, $S$ is a principal bundle

\[
\begin{array}{ccc}
E & \longrightarrow & S \\
\downarrow & & \downarrow \\
C & & C
\end{array}
\]

where $E$ is some elliptic curve and $C$ is a compact complex curve. Moreover the following are then equivalent:

1. the first Chern class vanishes: $c_1(C, S) = 0$,
2. the first Betti number $b_1(S)$ is even,
3. $S$ is Kähler,
4. $S \to C$ admits a holomorphic flat connection,
5. $S \to C$ is topologically trivial (though perhaps not holomorphically trivial).

Proof. The singular fibers must have lower genus than the generic fiber, so must be rational. The $j$-invariant of the elliptic curve fibers is a holomorphic function on the base, so must be constant. The fibration must be a holomorphic fiber bundle by the Grauert–Fischer local triviality theorem; see Barth et. al. [2] p. 29. The fiber bundle has transition functions valued in the biholomorphism group of the elliptic curve fibers. This group is a finite extension of the elliptic curve, so taking a finite covering space we can reduce the structure group to the elliptic curve. The equivalence of the various conditions is well known; see [2] p. 145–149. □

Lemma 5 (Klingler [3]). Suppose that

\[
\begin{array}{ccc}
E & \longrightarrow & S \\
\downarrow & & \downarrow \\
C & & C
\end{array}
\]

is a principal bundle, $E$ is an elliptic curve and $C$ is a compact complex curve of genus $g \geq 1$. Suppose that $b_1(S)$ is odd. Then the universal covering space $\tilde{S}$ of $S$ is biholomorphic to $\mathbb{C}^2$ (if $g \geq 1$) or $\mathbb{H} \times \mathbb{C}$ (if $g \geq 2$). The fundamental group of $S$ admits a presentation

$$\pi_1(S) = \left\langle a_1, b_1, a_2, b_2, \ldots, a_g, b_g, c, d \mid c, d \text{ central, } \prod_{i=1}^{g} [a_i, b_i] = c^r \right\rangle$$

where $r$ is a positive integer, equal to the Chern number of the bundle $S \to C$. 

Lemma 6. Suppose that \( S \) is an elliptic fiber bundle, with a nonsingular holomorphic foliation transverse to its fibers. Then, after perhaps replacing \( S \) by a finite covering space, \( S \) becomes a principal elliptic fiber bundle and there is a unique holomorphic flat connection for \( S \to C \), say with connection form \( \eta \), so that the foliation is \( \eta = 0 \). Conversely, if \( \eta \) is one such holomorphic connection form, then every transverse holomorphic foliation on \( S \) has a unique expression as \( \eta = p^*\xi \) where \( \xi \) is an arbitrary holomorphic 1-form on \( C \).

Proof. Replace \( S \) with a finite covering space to arrange that \( S \) is principal, \( E \to S \to C \). We can average over the action of the elliptic curve \( E \), to find a transverse foliation which is \( E \)-invariant, i.e. a holomorphic connection. Therefore the fibration has a holomorphic connection as a principal bundle over the curve \( C \). So the first Chern class of the bundle must vanish (see Atiyah [1]). Let \( \eta \) be a holomorphic connection 1-form. Define a holomorphic section \( \xi \in H^0(S, p^*\Omega^1_C) \) by \( \eta(v) = \xi(p_*v) \) for any tangent vector \( v \) tangent to a leaf of the foliation. Since \( \eta \) is a connection, \( E \)-invariance of \( \eta \) forces \( \xi \) to be the pullback of a 1-form on \( C \), which we also call \( \xi \). Replace \( \eta \) by \( \eta - \xi \), which is also a holomorphic connection.

Definition 9. A nonsingular holomorphic foliation \( F \) of a complex surface \( S \) is turbulent if there is an elliptic fibration \( S \to B \) so that finitely many fibers are tangent to \( F \), while all others fibers are everywhere transverse to \( F \).

Lemma 7. [Brunella [3]] Every holomorphic nowhere singular foliation on any compact complex surface is one of the following:

1. a holomorphic fibration by compact complex curves,
2. everywhere transverse to an elliptic or rational fibration,
3. a turbulent foliation on an elliptically fibered surface,
4. a linear foliation on a complex torus,
5. a holomorphic foliation of a Hopf surface, linear or nonlinear (see Mall [13]),
6. a holomorphic foliation of an Inoue surface (see Inoue [7], Kohler [10, 11]),
7. a transversely hyperbolic foliation with dense leaves on a surface whose universal covering space is a holomorphic disk fibration over a disk.

These categories are not quite exclusive. For example, on a product of compact Riemann surfaces, the “horizontal” fibration is transverse to the obvious “vertical” fibration.

Proposition 3. Suppose that \( S \) is the total space of an elliptic fiber bundle over a curve of genus \( g \geq 2 \). Suppose that \( b_1(S) \) is odd. Any nonsingular holomorphic foliation on \( S \) either is turbulent or coincides with the fibration.

Proof. Brunella’s classification shows us that every nonsingular holomorphic foliation on \( S \) either (1) is turbulent or (2) coincides with the fibration or (3) is everywhere transverse to the fibration. Suppose that the foliation is everywhere transverse. By lemma 6, we can suppose, after replacing \( S \) with a finite covering space of \( S \), that \( S \) has a holomorphic flat connection. But the first Chern class does not vanish by lemma 5 on the preceding page.

Definition 10. A holomorphic foliation with transverse translation structure on a complex surface is a closed nowhere zero holomorphic 1-form. The associated foliation is the one on whose tangent lines the 1-form vanishes.

Lemma 8. Suppose that \( S \) is a compact complex surface, containing no rational curves. Suppose that \( S \) admits a holomorphic nonsingular foliation with a transverse translation structure. Then up to replacing \( S \) with a finite covering space, \( S \) is
(1) a complex torus with a linear foliation, or
(2) a principal elliptic bundle with a holomorphic flat connection $\eta$ with foliation $\eta = 0$, or
(3) a primary Kodaira surface, i.e. a holomorphic principal bundle

$$
\begin{array}{ccc}
E_1 & \rightarrow & S \\
\downarrow & & \\
E_0 & & 
\end{array}
$$

with elliptic curve structure group over an elliptic curve base, with foliation equal to the fibration.

Proof. Suppose that $S$ is a compact complex surface with nowhere vanishing closed holomorphic 1-form $\eta$, and consider the foliation $\eta = 0$. Our foliation must be on Brunella’s list in lemma 7 on the preceding page.

Every holomorphic 1-form on a complex torus is translation invariant, and therefore any foliation of the torus with transverse translation structure must be a linear foliation. So we can assume that $S$ is not a torus.

If $S$ is a fibration, and the foliation coincides with the fibration, then the 1-form is semibasic and so basic, and therefore arises from a nowhere vanishing 1-form on the base, so the base is an elliptic curve. The fibration must be a smooth fiber bundle, by lemma 1 on page 19. If the fibers are rational, then $S$ contains a rational curve, contradicting our hypotheses. If the fibers are elliptic curves, then the classification of elliptic curve bundles over an elliptic curve base (see [2] p. 146) tells us that (after perhaps replacing $S$ by a finite covering space) $S$ is a torus (and therefore the foliation is linear) or a primary Kodaira surface. If the fibers are curves of genus $g \geq 2$, then replacing $S$ by a finite covering space, we can arrange that $S = E \times C_{g \geq 2}$, and the fibration is the trivial product fibration (see [2] p. 149).

So from now on we can suppose that the foliation does not coincide with a fibration. Suppose that $S$ is an elliptic fiber bundle, $E \rightarrow S \rightarrow C$, and the holomorphic nowhere vanishing 1-form is $\eta$. On each fiber $S_c \cong E$, for any $c \in C$, we have a distinguished Maurer–Cartan 1-form, say $dw$, for $w$ any affine coordinate on the universal covering space of $E$. On $S_c$,

$$
\eta|_{S_c} = f \, dw,
$$

for some holomorphic function $f : S \rightarrow \mathbb{C}$. But then $f$ must be constant. If $f = 0$, then the foliation coincides with the fiber bundle, a case already covered. So without loss of generality,

$$
\eta|_{S_c} = dw,
$$

over every point $c \in C$ and the foliation is transverse. Apply lemma 6 on the preceding page to find that $S$ is, after replacement by a finite covering space, a holomorphic elliptic fiber bundle with holomorphic flat connection $\eta$ and, without loss of generality, the foliation is $\eta = 0$.

No Hopf surface carries any nonzero holomorphic 1-form, nor does any Inoue surface (see [2] p. 172 for the proof for linear Hopf surfaces, and p. 176 for the proof for Inoue surfaces. The argument on p. 172, without alteration, proves the result also for nonlinear Hopf surfaces.)
Suppose that $S$ has a transversely hyperbolic foliation with dense leaves, and $\tilde{S} \to S$, the universal covering space, is a holomorphic disk fibration over a disk:

$$\mathbb{D} \longrightarrow \tilde{S} \longrightarrow \mathbb{D}.$$

The holomorphic 1-form on $S$, say $\eta$, must lift to $\tilde{S}$ to be closed and vanish precisely on the leaves of the foliation. The fibers are simply connected, so there is no monodromy and $\eta$ must be pulled back from a 1-form on the base disk, which we also call $\eta$. The group $\pi_1(S)$ must preserve the form $\eta$ and the fibration, and therefore must act on the base disk. Let $\pi = \pi_1(S)$. The function $|\eta|^2$ (using the hyperbolic metric on the base disk) is a $\pi$-invariant smooth function on the base disk. It lifts to a $\pi$-invariant smooth function on $\tilde{S}$, constant on each leaf. Therefore it descends to a smooth function on $S$, constant on each leaf. But this function must therefore be constant. So $\eta$ has constant norm in the hyperbolic metric on the base disk. The hyperbolic metric is

$$ds^2 = \frac{|dz|^2}{(1 - |z|^2)^2},$$

so $\eta$ must have the form

$$\eta = \frac{e^{i\alpha}}{1 - |z|^2} dz,$$

(after perhaps replacing $\eta$ by a real constant rescaling) for some real valued smooth function $\alpha$. Since $\eta$ is closed, we can take exterior derivative to find

$$0 = \frac{\partial}{\partial \bar{z}} \frac{e^{i\alpha}}{1 - |z|^2},$$

$$= \frac{e^{i\alpha}}{(1 - |z|^2)^2} \left( i \frac{1 - |z|^2}{1 - |z|^2} \frac{\partial \alpha}{\partial z} + z \right).$$

Therefore

$$\frac{\partial \alpha}{\partial \bar{z}} = \frac{i z}{1 - |z|^2}.$$

Since $\alpha$ is real,

$$\frac{\partial \alpha}{\partial \bar{z}} = -\frac{i \bar{z}}{1 - |z|^2}.$$

But then

$$d\alpha = \frac{i}{1 - |z|^2} (\bar{z} dz - \bar{z} d\bar{z}).$$

Taking exterior derivative,

$$d^2 \alpha = \frac{2i}{(1 - |z|^2)^2} dz \wedge d\bar{z}.$$

But $d^2 = 0$, so a contradiction. Therefore there is no holomorphic 1-form of constant nonzero norm on the disk. Therefore there is no foliation on $S$ with holomorphic transverse translation structure. $\square$
10. Nothing else

Lemma 9. Let $S$ be a compact complex surface bearing a $(G_D, \mathbb{C}^2)$-structure or a $(G_D', \mathbb{C}^2)$-structure. Then $S$ contains no rational curves.

Proof. Suppose that $S$ contains a rational curve. The complete classification of all holomorphic Cartan geometries on any compact complex surface containing a rational curve appears in [14]. The connected complex homogeneous spaces $(G, X)$ which appear as model spaces for those geometries (with $G$ a connected complex Lie group acting holomorphically and effectively) are of three distinct types:

1. $(\mathbb{P} \text{SL}(3, \mathbb{C}), \mathbb{P}^2)$,
2. $((\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{P} \text{SL}(2, \mathbb{C}) \times \mathbb{P} \text{SL}(2, \mathbb{C})), \mathbb{P}^1 \times \mathbb{P}^1)$,
3. products $(G_0 \times \mathbb{P} \text{SL}(2, \mathbb{C}), X_0 \times \mathbb{P}^1)$.

We want to claim that $(G_D, \mathbb{C}^2)$ and $(G_D', \mathbb{C}^2)$ do not appear in that classification, for any divisor $D$. Clearly $G_D$ and $G_D'$ are connected, and have normal subgroup $V_D$, so not simple. Therefore we only have to prove that $G_D$ and $G_D'$ are not of the form $G_0 \times \mathbb{P} \text{SL}(2, \mathbb{C})$. The Lie algebras have

$[g_D, g_D] = V_D$, $[V_D, V_D] = 0$,

and similarly

$[g_D', g_D'] = V_D$, $[V_D, V_D] = 0$,

so that neither $g_D$ nor $g_D'$ contain any copy of $\mathfrak{sl}(2, \mathbb{C})$. □

We prove theorem 1 on page 1.

Proof. By lemma 9 we can assume that $S$ contains no rational curves. The surface $S$ inherits a nowhere vanishing holomorphic closed 1-form $\omega$ (coming from $dz$), and this 1-form is a transverse translation structure. The classification given in lemma 8 on page 20 of those foliations shows that, up to replacement by a finite covering space, $S$ is either

1. a complex torus with linear foliation, or
2. a holomorphic principal bundle of elliptic curves $E \to S \to C$ with a holomorphic flat connection, or
3. $S$ is a primary Kodaira surface, i.e. the total space of an elliptic fiber bundle over an elliptic curve base, and the bundle has nonzero first Chern class, and $S$ has odd first Betti number; the foliation is the fibration.

Suppose that $S \to C$ is a principal elliptic curve bundle with holomorphic flat connection $\eta$. We only need to prove that $C$ is an elliptic curve, so that $S$ is a complex torus or primary Kodaira surface. Suppose that $S$ has a holomorphic $(G_D, \mathbb{C}^2)$-structure. But $G_D$ preserves a holomorphic volume form on $\mathbb{C}^2$, so $S$ is equipped with a holomorphic volume form, which easily implies that $C$ is an elliptic curve and $S$ is a complex torus.

Therefore we can assume that $S$ has a $(G_D', \mathbb{C}^2)$-structure. If $C$ has genus 0, then $S$ contains a rational curve, contradicting our assumptions. If $C$ has genus 1, then $S$ is a complex torus and the relevant structures are classified in sections [4] and [5] on page 7. Therefore we can assume that $C$ has genus at least 2.

The group $G_D'$ acts on $\mathbb{C}^2$ preserving the holomorphic 2-form $dz \wedge dw$ up to constant rescaling, so preserving a holomorphic connection on the canonical bundle of $S$. Therefore the canonical bundle has vanishing Atiyah class [1], and if $S$ is Kähler then $c_1(S) = 0$. By lemma 1 on page 19 the bundle $S \to C$ is topologically trivial: $c_1(S) = c_1(E) + c_1(C) = c_1(C) \neq 0$. Therefore $S$ has no $(G_D', \mathbb{C}^2)$-structure. □
11. Conclusion

There are no interesting examples of geometric structures on compact complex surfaces modelled on Lie’s exotic surfaces. The discovery that this is so is an essential step in the classification of holomorphic geometric structures on low dimensional compact complex manifolds. Complex tori have obvious translation structures, while Kodaira surfaces have similar canonical geometric structures, and all of the exotic geometric structures on compact complex surfaces are induced from these more elementary structures. Therefore we can ignore the exotic surfaces of Lie in the search for locally homogeneous holomorphic geometric structures on compact complex surfaces.

The main difficulty in classifying locally homogeneous structures is that the model $X$ might be very flexible, i.e. the group $G$ might be very large, and so it becomes difficult to see the constraints on putting together a $(G,X)$-structure on a given surface. In particular, since the groups acting on Lie’s exotic surfaces are of arbitrarily large dimension, the exotic surfaces present an exceptionally difficult case study, which we had to face before we can develop the general theory of holomorphic geometric structures on compact complex surfaces.

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