Testing Clauser-Horne-Shimony-Holt inequalities using observables with arbitrary spectrum

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The Clauser-Horne-Shimony and Holt inequality applies when measurements with binary outcomes are performed to physical systems under the assumption of local realism. We establish the general conditions observables with an arbitrary spectrum must satisfy to test, in the quantum mechanical realm, this inequality. We show that Pauli matrices and dichotomic observables are particular cases of the general observables we obtain, which consist of positive operator valued measurements to the system. Our results lead to a general method of revealing binary correlations in bipartite systems applying to any measurable quantum system. It does not require prior knowledge of its Hilbert space dimension, the states to be measured or any pre defined binning procedure.

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The Bell inequality was derived in 1964 in response to the Einstein, Podolski and Rosen paradox, that is itself a consequence of the authors’ attempts to answer the question: can quantum-mechanical description of physical reality be considered complete? By using the hypotheses of locality and realism, as in John S. Bell, and later on, Clauser-Horne-Shimony and Holt (CHSH) could establish an inequality that must be observed for measurements with binary outcomes performed to classical systems, even in the case where one admits the hypotheses of hidden variables evoked in [2]. The CHSH inequalities involve correlations between measurements of two physical quantities, \( A_s_a \) and \( B_{s_b} \), by observers \( a \) and \( b \) in different experimental settings \( s_a = \phi^{(i)}_a \) \((i = 1, 2)\). Each value of \( i \) corresponds to a different setting and \( \phi \) corresponds to the observers \( a, b \). To create a different experimental setting, one modifies some physical parameter of the experiment without modifying the fact that only two values, \( \pm 1 \), can be assigned to the measured quantity. The correlations appearing in the CHSH inequality are in the form:

\[
\langle A_{s_a} B_{s_b} \rangle = E(\phi^{(i)}_a, \phi^{(j)}_b) = P^{i,j}_{++} + P^{i,j}_{--} - P^{i,j}_{+-} - P^{i,j}_{-+}, \tag{1}
\]

where \( P_{k,l} \) is the joint conditional probability that \( a \) and \( b \) observe the value \( k, l = \pm 1 \) when \( A_{\phi^{(i)}_a} \) and \( B_{\phi^{(j)}_b} \) are measured in the \( i, j \)-th setting. We have here that \( P^{i,j}_{++} + P^{i,j}_{--} + P^{i,j}_{+-} + P^{i,j}_{-+} = 1 \). Using (1), the CHSH inequality can be written as:

\[
|E(\phi^{(1)}_a, \phi^{(1)}_b) + E(\phi^{(1)}_a, \phi^{(2)}_b) + E(\phi^{(2)}_a, \phi^{(1)}_b) - E(\phi^{(2)}_a, \phi^{(2)}_b)| = |\langle \hat{C} \rangle| \leq 2. \tag{2}
\]

Following these rules, it was shown in [3] and [4] that the particular form of the CHSH inequality demands that the statistics of the measurement outcomes when measuring \( A_{s_a} \) and \( B_{s_b} \) are associated to some binning. By such, we mean that it is mandatory that correlation functions can be written as in (1), which is a direct consequence of the fact that, for each party, the expectation value of, say, \( A_{s_a} \) can be written as \( \langle A_{s_a} \rangle = P_+ - P_- \). We insist on the subtlety of the results of [3, 4]: they show that (2) requires correlation functions of the type (1), since continuous variables correlation function easily lead to a local hidden variable theory violating the threshold in (2). However, such binary like correlations can be obtained using measurements on different types of physical quantities \( A_{s_a} \) and \( B_{s_b} \), provided that one can associate the measurement results to the binary correlation function in (1) and build the analogous to a dichotomic system. Inequalities analogous to (2) can be derived for higher dimensional discrete systems [5, 6]. Even though the results presented here can be generalized to such systems, they will not be discussed in detail in the present work.

The interest of Equation (2) with respect to the original Bell inequality is that it is easier to be tested experimentally in quantum physical systems that can be produced and measured in a controlled way. In the quantum formalism, the physical quantities \( A_{s_a} \) and \( B_{s_b} \) become hermitian operators (observables): \( A_{s_a} \rightarrow \hat{A}_{s_a} \) and \( B_{s_b} \rightarrow \hat{B}_{s_b} \). What are then the conditions such observ-
ables must satisfy so as to be suitable to test the CHSH inequality and, in some cases, violate it, showing that quantum correlations cannot be explained by a local hidden variable theory? The first observables that appeared as natural candidates to perform experiments testing the inequality (2) were quantum two-dimensional systems, that can be completely described by the identity operator and the three Pauli matrices, $\hat{\sigma}_x, \alpha = x, y, z$. Some examples are the projections of a spin 1/2 system or the orthogonal polarizations of a photon. Indeed, measurement of the Pauli operators in any direction of space can only return the values $\pm 1$. Thus, the correspondence between the measurement results of such observables and of the aforementioned classical physical quantities $A_{s_A}$ and $B_{s_B}$ is direct: one can associate to $P_{\pm}$ the probability of obtaining an outcome $\pm 1$ when measuring $A_{s_A}$ or $B_{s_B}$. Using this fact, it was shown experimentally that quantum states entangled in degrees of freedom represented by two-dimensional Hilbert spaces can violate (2). A natural question is whether quantum states in dimension higher than two can be used to test and violate an inequality as (2), as for instance, states that are described by continuous variables. In [9], it was shown that this is indeed possible, if one measures a dichotomic property of these states, as for instance, their parity. It is known that an arbitrary state can be written as a superposition of odd and even states. Furthermore, parity is analogous to a spin 1/2 system: one can define two conjugate observables that, with parity, fulfill the same commutation relations as the Pauli matrices [10, 11]. Thus, by using parity measurements in arbitrary quantum systems, we see that some parity entangled states can violate (2). However, it is clear that requiring that observables $A_{i}$ and $B_{j}$ have a discrete dichotomic spectrum is a too strong demand. Indeed, equation (2) merely involves binary correlations between measurement results, and only the expectation values of observables are relevant, not their spectrum. Consequently, when aiming at violating an inequality as (2), one should design experiments that are capable of reproducing the same type of statistics as in (1). Such an experiment can, in principle, involve systems in arbitrary Hilbert space dimension. This fact was used, for instance, in [12], where the values of $P_{++}, P_{-\pm}, P_{\pm-}$ and $P_{-\pm}$ in (1) were obtained, using a binning procedure, from continuous measurements. However, in this case, the observers need to have an a priori knowledge of the non trivial binning procedure, that, moreover, is state dependent. In [13], a method based on the measurement of auxiliary dichotomic states entangled to the continuous variables ones was proposed, enabling the indirect measurement of arbitrary continuous variables systems leading to the violation of CHSH inequalities.

In the present contribution we address the following question: what are the conditions an observable must satisfy to test Eq. (2), violating it for some particular quantum states and showing that quantum correlations are stronger than all local realistic hidden variables theories? We demonstrate that a proper choice of measurements can lead to the violation of a CHSH inequality using direct measurements on quantum systems with arbitrary spectrum (not necessarily known), without an a priori definition of a binning or dichotomization procedure. We show that generalized quantum measurements, represented by a set of 2 Positive Operator Valued Measures (POVM) for each observer, can naturally create a binary statistics leading to the possibility of testing, and, for some states, violating a CHSH inequality.

To show our main result, we use the fact that, in order to test the CHSH inequality (2), observables $A_{s_A}$ and $B_{s_B}$ must be such that

$$\langle \hat{A}_{s_A} \hat{B}_{s_B} \rangle = E(\phi^{(i)}_{a}, \phi^{(j)}_{b}) = P^{i,j}_{++} + P^{i,j}_{-\pm} - P^{i,j}_{\pm-} - P^{i,j}_{-\pm}$$

(3)

$\forall \ i, j$. This means that it is possible to associate, to the expectation value of the product of observables $A_{s_A} B_{s_B}$, a correlation function that is typical of binary systems. A combination of the four possible values of the settings $i, j (i, j = 1, 2)$, as in [2], must lead to $|\langle \hat{C} \rangle | > 2$ for some quantum states in order to observe violation of (2). We will see that the condition such observables must satisfy does not necessarily require a known a priori binning process, and that the previous cited examples of dichotomization or spin 1/2 particles can be seen as particular cases of the general operators we obtain.

We start by considering a single party operator, $\hat{A}_{s_A}$ and the same reasoning will be applied to $B_{s_B}$. Since we require that $\langle \hat{A}_{s_A} \rangle = P_+ - P_- = 1$, there exists a $\phi$ such that $\langle \hat{A}_{s_A} \rangle = \cos \phi, P_- = \sin^2 \frac{\phi}{2}$ and $P_+ = \cos^2 \frac{\phi}{2}$. We can create a general measurement strategy for obtaining the required results by associating $\langle \hat{A}_{s_A} \rangle = \cos \phi = \langle \cos \phi \rangle$, where $\phi$ is a measurable observable for $a$, with eigenstates defined as $|\phi \rangle$ and eigenvalues $\phi$ such that $\phi \mod 2\pi = \phi$. Thus, we can write

$$\hat{A}_{s_A} = \int_0^{2\pi} d\phi \cos \phi \hat{P}(\phi),$$

(4)

where $\hat{P}(\phi)$ are the projectors associated to a given value of $\phi$.

In (4) we considered the most general case, where $\hat{A}_{s_A}$ has a continuous spectrum. Operators with a discrete spectrum can be written in a form analogous to (4) where instead of an integral, one would have a sum over the discrete eigenstates of $\hat{A}_{s_A}$. As an example, we discuss in detail the case of Pauli matrices, that are commonly used to test CHSH-type inequalities, showing that they are a particular case of (4). In this case, there are only two eigenstates $|\phi_j \rangle, j = 0, 1$, with eigenvalues $\cos (\phi + (-1)^j \pi/2)$. For any value of $\phi$, a proper normalization recovers the Pauli matrices with spectrum $\pm 1$.

Our results can be interpreted in terms of POVMs in
the following way:

\[ P_{\pm} = \langle \hat{E}_{\pm} \rangle = \frac{1}{2} (1 \pm \langle \hat{A}_{s_2} \rangle), \]

(5)

where \( \hat{E}_{\pm} \) are two POVM. Since \( \hat{A}_{s_2} \) is hermitian, we can write \( 2\hat{A}_{s_2} = \hat{D}_{s_2} + \hat{D}_{s_2}^\dagger \), where \( \hat{D}_{s_2} \) is an unitary operator, leading to

\[ \hat{E}_{\pm} = \frac{1}{4} (\mathbb{1} \pm \hat{D}_{s_2})(\mathbb{1} \pm \hat{D}_{s_2}). \]

(6)

with \( \hat{E}_+ + \hat{E}_- = \mathbb{1} \).

Starting simply from the fact that one can realize measurements in a given basis \( |\phi\rangle \) of arbitrary dimension (not necessarily known), Eq. (4) provides the general form an observable must have to be a good candidate to test a CHSH type inequality. Correlations between the outcomes of measurements of \( \hat{A}_{s_2} \) and \( \hat{B}_{s_3} \) in the form (4) can be constructed as:

\[ P_{i,j} = \frac{1}{4} \left( 1 + (-1)^i \langle \hat{A}_{s_2} \rangle + (-1)^j \langle \hat{B}_{s_3} \rangle + (-1)^{i+j} \langle \hat{A}_{s_2} \hat{B}_{s_3} \rangle \right) \]

(7)

directly leading to (3), without the need of a prior choice of a binning procedure. Any binning procedure can be recast as (7).

Considering such general form is interesting because it opens the prospect of testing CHSH inequalities in any measurable quantum system: once one knows how to measure the probability of finding a quantum system in a given state \( |\phi\rangle \), operators as (4) can be constructed, leading to the test of a CHSH inequality using (7). Alternatively, knowing the general form (4) has important consequences if one is aiming at testing a CHSH inequality without having a priori information about the Hilbert space dimension of the system to be measured and on possible binning or dichotomization procedures. In other words, by measuring an observable in the form (4) in some experimentally accessible basis (that will determine \( |\phi\rangle \)), one is sure to be able, at least in principle, to test a CHSH inequality without requiring any other information about the system, as its state or Hilbert space dimension.

In order to illustrate this fact and the power of our results, we consider the following situation: suppose that each one of the observers \( a \) and \( b \) has a measuring apparatus that correlates some physical property of a quantum system to the position of the same system and then measures this position. In order to measure this property, one needs to collect statistics about the quantum system’s position. This scenario is probably the most current in physics, corresponding to Stern-Gerlach (SG) type experiments [14], measurements of the spectrum of a multi-mode field using a diffraction grating, Mach-Zender type interferometric measurements, among many others. If the referred physical property has unknown Hilbert space dimension, as was the case of the first SG experiment performed in 1922, no assumption can be made, a priori, on the possible outcomes of the position measurements that will reveal such informations. In order to be even more general, one can suppose that the way correlations are created between position and the physical quantity to be measured is also unknown. Thus, observers \( a \) and \( b \), when inspecting the position probability distribution that is measured by their apparatuses, have simply no idea of what type of results they can find. Using the present formulation, even in this extremal case, there exists a strategy enabling, in principle, the violation of (3).

The considered scenario can be expressed mathematically as follows: we consider first the case of a single party system and then extend our results to the bipartite one. The initial state of the system is given by

\[ |\psi\rangle |x_o\rangle = \int_{-\infty}^{\infty} ds f(s) |s\rangle |x_o\rangle, \]

(8)

where we labeled \( |s\rangle \) a complete basis describing the property that is being measured. \( x_o \) is the position measurement result if no correlation is created by the measuring apparatus that correlates some physical property of the system. This scenario describes the SG experiment or diffraction by a grating. Another situation would be the one of parity measurements, i.e., the case where position becomes correlated to the parity of the state. Then, by writing \( f(s) = f_{o,e}(s) + f_{\psi}(s) \), where \( f_{o,e}(s) \) are the odd/even parts of \( f(s) \), we have that the system’s state is split in two according to its parity, and there are only two states \( |\psi_s\rangle: f_{\psi}(s) = f_{o,e}(s) \). Correlations between parity and position can be created in optical systems, for instance, by using an orbital angular momentum sorter [15]. The previous cases are extreme examples, illustrating perfect and binary only correlation. Both extremes, and all the intermediate cases are covered by the present formalism, even of one does not know what type of correlations are created between position and some property of the system. The important aspect in the discussed situation is that \( a \) and \( b \) are realizing position measurements. Hence,
in order to be able to test (2), they can use an operator as (4) in the position basis, irrespectively of the type of information position measurements bring from the quantum system.

An example of an unitary operator in the position basis is \( \hat{U} = \int_{-\infty}^{\infty} e^{i\hat{x}x} |x\rangle \langle x| \), where \( \Delta x \) is some number with the dimension of a length that will be set equal to one. It is clear that \( \hat{U} \) is infinitely degenerated, since its eigenvalues are \( 2\pi \) periodic. The integral over the whole space can be replaced by an integral over one period and an infinite sum if one defines the following notation: let \( \tilde{x} = x \mod 2\pi \) and \( |\tilde{x}\rangle = \sum_{n=0}^{\infty} |\tilde{x} + 2\pi n\rangle |\tilde{x} + 2\pi n| \) [10]. Then, \( \hat{U} \) can be written as:

\[
\hat{U} = \int_{0}^{2\pi} d\tilde{x} e^{i\tilde{x}} |\tilde{x}\rangle \langle \tilde{x}|. \tag{10}
\]

By identifying \( \hat{U} \) to \( \hat{D}_{s_a} \), we have that

\[
\int_{0}^{2\pi} d\tilde{x} \cos(\tilde{x}) |\tilde{x}\rangle \langle \tilde{x}| = \int_{0}^{\pi} d\tilde{x} \cos(\tilde{x}) \hat{D}_{s_a} = \hat{A}_{s_a}, \tag{11}
\]

where \( \hat{D}_{s_a} = |\tilde{x}\rangle \langle \tilde{x}| - |\tilde{x} + \pi\rangle \langle \tilde{x} + \pi| \) [17]. This is an example of observable in the form (4) in the position basis that can reveal the binary statistics required for testing CHSH inequalities. We now compute its expectation value using the general state (9). For this, we re-express (9) using that \( x(\psi_{s_a}) = \bar{x}(\psi_{s_a}) + 2\pi n(\psi_{s_a}) \), leading to

\[
\langle \hat{A}_{s_a} \rangle = \int_{0}^{2\pi} d\bar{x} \cos(\bar{x}) p(\bar{x}(\psi_{s_a})) = \sqrt{\sum_{n(\psi_{s_a})} n(\psi_{s_a}) \int_{-\infty}^{\infty} ds_a |f_{\psi_{s_a}}(s_a)|^2}.
\tag{12}
\]

where \( p(\bar{x}(\psi_{s_a})) = \sum_{n(\psi_{s_a})} n(\psi_{s_a}) \int_{-\infty}^{\infty} ds_a |f_{\psi_{s_a}}(s_a)|^2 \). Correlations between measurements in different settings can be combined, as in (2), so as to test the CHSH inequality for a system whose Hilbert space dimension is unknown. In this scenario, if one supposes that observers blindly compute the quantity (13) for each setting without necessarily observing the statistics of position measurements in detail, it would be possible to violate the CSHS inequality without discovering that the spin projections are two-valued quantities, in the case of a SG experiment.

In practical situations, position measurements are realized in a given finite interval defined as \( 2l \), even in the case where the spectra \( s_a \) and \( s_b \) are arguably infinite. We can thus set \( \Delta x >> 2l \), which is equivalent to consider \( n(\psi_{s_a}) = 0 \) and \( x(\psi_{s_a}) = \bar{x} \) and \( y(\psi_{s_a}) = \bar{y} \), that will simplify the notation and interpretation of the results.

Equation (15) corresponds to an arbitrary setting \( s_a, s_b \). In (2) one must be able to change settings and compare correlations between the measurement results in different settings. In the present situation, the changing of settings is realized by applying an unitary operation to the initial state \( |\Psi\rangle \). This is equivalent to changing the measuring apparatus. By keeping the role of the measuring apparatuses the same, i.e., by keeping the same rule to create correlations between position and some property of the state to be measured, one then achieves the equivalent to a modification of the measurement setting.

We now discuss the natural binning procedure that is realized by operator (4). We see that \( P_\pm = \pm 1 \) when we compute \( \langle A_{s_a} \rangle \) for an eigenstate with maximum(minimum) eigenvalue. This state will be called from now on \( |1(0)\rangle \). The values \pm 1 were chosen due to the imposed normalization condition of the correlation function. Of course, an infinity of states that will be denoted as \{ |\alpha\rangle \} can be associated to a same value of \( P_+ - P_- \). These states cannot be distinguished as far as the expectation value \( \langle \hat{A}_{s_a} \rangle \) is concerned. They are all
equivalent to a state $|\tilde{\alpha}\rangle = \cos \alpha |\bar{0}\rangle + e^{i\beta} \sin \alpha |\bar{1}\rangle$, where $\beta$ is an arbitrary phase, such that

$$
\langle \hat{A}_s \rangle_{\{\alpha\}} = P_+ - P_- = \langle \hat{A}_s \rangle_{|\tilde{\alpha}\rangle} = \cos^2 \alpha - \sin^2 \alpha = \cos (2\alpha).
$$

(16)

These results can be straightforwardly extended to the correlations $\langle \hat{A}_s \hat{B}_s \rangle$, giving the properties of the states that are associated in this natural binning procedure realized by the POVMs (6).

From the previous discussion, we can find some examples of states violating (2). For a maximal violation, one should look for eigenstates of operators $\hat{A}_s$ and $\hat{B}_s$ in the form of Bell states, as $rac{1}{\sqrt{2}} |\bar{1}, \bar{1}\rangle \pm |\bar{0}, \bar{0}\rangle$ and $rac{1}{\sqrt{2}} |\bar{1}, \bar{0}\rangle \pm |\bar{0}, \bar{1}\rangle$. However, in the case where observables have a continuous spectrum, such states are non physical, i.e., they correspond to a $\delta$ type function. Nevertheless, we can find physical states arbitrarily close to these ones by using other types of distributions. It is clear that, in the known cases of Pauli matrices, such states will converge to the usual Bell states.

In conclusion, we provided the general conditions observables with an arbitrary spectrum must satisfy to be suitable to test a CHSH type inequality. We have shown that the specific cases of dichotomic observables is a particular case of the formal solution developed here. We also detailed the situation where observers have access to position measurements only and position is correlated in an unknown way to some physical property of the system to be measured. Our formalism provides a general strategy enabling the test of CHSH inequalities without demanding prior knowledge of the spectrum of the observable to be measured. Our results help creating a general environment for CHSH inequalities tests without the need of complicated state dependent binning procedures or prior knowledge of the physical properties of the system, rendering CHSH inequalities tests accessible to a broader class of experimental systems.

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