Magnetic branes in Gauss-Bonnet gravity with nonlinear electrodynamics: correction of magnetic branes in Einstein-Maxwell gravity

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In this paper, we are considering two first order corrections to both gravity and gauge sides of the Einstein-Maxwell gravity: Gauss-Bonnet gravity and quadratic Maxwell invariant as corrections. We obtain horizonless magnetic solutions by implying a metric which representing a topological defect. We analyze the geometric properties of the solutions and investigate the effects of both corrections, and find that these solutions may be interpreted as the magnetic branes. We study the singularity condition and find a nonsingular spacetime with a conical geometry. We also investigate the effects of different parameters on deficit angle of spacetime near the origin.

I. INTRODUCTION

Magnetic strings/branes may be interpreted as topological defects that were formed during phase transition of the early universe [1]. These defects contain information regarding early structure of the universe and also its evolution [2]. On the other hand, considering the AdS/CFT correspondence, these horizonless magnetic solutions in presence of negative cosmological constant may contain information regarding a quantum field theory on the boundary of the AdS spacetime [3]. These topological defects have wide variety of applications in quantum gravity and has been studied in different context such as hadron dynamics [4], anti-ferromagnetic crystals [5], Yang-Mills plasma [6] and also, in studying quantum criticality [7]. These magnetic branes/strings have been studied through some papers and their properties for different cases of gravity and nonlinear electromagnetic fields have been derived [8]. Motivated by these facts, we study magnetic branes in presence of two mentioned corrections and investigate the effects of these two corrections on properties of the magnetic branes.

From the other point of view, Einstein (EN) gravity has been a successful theory for describing many phenomena, whereas in some aspects it confronts some fundamental problems [9]. In order to overcome these problems, alternative theories of gravity or generalization of the EN gravity have been introduced [10–12]. One of these theories is generalization of the EN gravity to the well-known Gauss-Bonnet (GB) theory. This generalization solves some of shortcomings of the EN gravity and gives a renewed view and properties in gravitational context [13]. This theory of gravity has been studied for different astrophysical objects [14]. The properties of the GB gravity, may attract one to the idea that not to consider the GB gravity as a generalization, but as a correction to EN gravity. In other words, one can consider first order of the GB parameter, $\alpha$, as a correction and study its effects on the properties of solutions. This fact shows that one can take small values of the GB parameter into account and interpret the effects of the GB correction as a perturbation to EN gravity. This consideration enables us to study the effects of the GB parameter in more details. In this paper, we consider the GB gravity not as a generalization but as a correction or perturbation to EN theory [15].

Naturally, most of the systems that we are studying are nonlinear or they have nonlinear properties. In order to have more realistic results, one should take into account the nonlinear behavior of these systems. Maxwell theory of electrodynamics is a linear theory which works well in many aspects but fails regarding some important issues. In order to overcome its problems, different theories of nonlinear electrodynamics (NED) were introduced [10]. Among them, Born-Infeld (BI) type ones are quite interesting due to their properties and the fact that these theories may arise from low energy limits of the effective string theory [17]. For large values of nonlinearity parameter, these BI type theories lead to same behavior such as Maxwell theory. In order to avoid complexity that nonlinear electromagnetic fields pose, one can consider the effects of nonlinearity as a correction to the Maxwell field. In other words, one can add the quadratic Maxwell invariant to the Lagrangian of the Maxwell theory [18, 19] to obtain NED as a correction. On the other hand, it is arguable that in order to obtain physical results that are consistent with experiments, one should consider the weak effects of nonlinearity. Using the series expansion of BI type theories in weak field limit, one finds that the first term is Lagrangian of the Maxwell theory and the second term is proportional to the quadratic Maxwell invariant [20]. Therefore, in this paper, we are considering quadratic Maxwell invariant as a correction of

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the Maxwell electrodynamics and study its effects on the properties of solution.

The structure of the paper will be as follow. In next section, we will present fields equation and obtain metric function for the case of the magnetic branes. We will plot some graphs in order to study the effects of corrections on metric function. Also we will study the geometrical structure of obtained solutions and investigate the effects of different parameters on deficit angle and conical structure of the magnetic branes through graphs. Last section is devoted to closing remarks.

II. STATIC SOLUTIONS

In order to study horizonless magnetic branes, we consider the following metric for $d$-dimensions \[ ds^2 = \frac{\rho^2}{l^2} dt^2 + \frac{d\rho^2}{f(\rho)} + l^2 f^2 \rho^2 + \frac{\rho^2}{l^2} dX^2, \] (1)

where $l$ is a scale factor related to the cosmological constant, $f(\rho)$ is a function of coordinate $\rho$ and $dX^2 = \sum_{i=1}^{d-3} dx_i^2$ is the Euclidean metric on the $(d-3)$-dimensional submanifold. The angular coordinate $\phi$ is dimensionless and ranges in $[0, 2\pi]$, while $x_i$ ranges in $(-\infty, \infty)$. Due to fact that we are interested in solutions that contain the GB gravity and a correction to Maxwell field, we consider the following field equations $[15, 18]$

$$
\partial_\mu \left( \sqrt{-g} L_F F^{\mu\nu} \right) = 0, \tag{2}
$$

$$
A g_{\mu\nu} + G^{(1)}_{\mu\nu} + \alpha G^{(2)}_{\mu\nu} + O (\alpha^2) = \frac{1}{2} g_{\mu\nu} \mathcal{L}(\mathcal{F}) - 2 \mathcal{L}_F R_{\lambda\nu} F^{\lambda
\mu
\nu}, \tag{3}
$$

where $L_F = \frac{d\mathcal{L}(\mathcal{F})}{dF}$, in which $\mathcal{L}(\mathcal{F})$ is the Lagrangian of NED; $\Lambda = -(d-1)(d-2)$ and $G^{(3)}_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$ are, respectively the cosmological constant and the EN tensor; $\alpha$ is the GB coefficient and

$$
G^{(2)}_{\mu\nu} = 2(R_{\sigma\tau\nu\mu} R^{\sigma\tau} - 2R_{\mu\nu\sigma\rho} R^{\rho\sigma} - 2R_{\rho\sigma\nu\mu} + RR_{\mu\nu}) - \frac{\mathcal{L}^{(2)}}{2} g_{\mu\nu}, \tag{4}
$$

where $\mathcal{L}^{(2)}$ denotes the Lagrangian of the GB gravity, given as

$$
\mathcal{L}^{(2)} = R_{\mu\nu\gamma\delta} R^{\mu\nu\gamma\delta} - 4 R_{\mu\nu} R^{\mu\nu} + R^2. \tag{5}
$$

We consider the following Lagrangian for the electromagnetic field $[15, 18, 19]$

$$
\mathcal{L}(\mathcal{F}) = -\mathcal{F} + \beta \mathcal{F}^2 + O (\beta^2), \tag{6}
$$

where $\beta$ is the nonlinearity parameter and the Maxwell invariant $\mathcal{F} = F_{\mu\nu} F^{\mu\nu}$, in which $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field tensor and $A_\mu$ is the gauge potential. It is easy to show that the electric field comes from the time component of the vector potential ($A_t$), while the magnetic field is associated with the angular component ($A_\phi$). The black hole solutions of GB gravity in presence of this nonlinear electromagnetic field were obtained previously $[12]$. In this paper we are looking for horizonless solutions with conical singularity which are not interpreted as black holes but magnetic branes solutions. Since we are looking for the magnetic solutions, we consider the following form of gauge potential

$$
A_\mu = h(\rho) \delta_\mu^\phi. \tag{7}
$$

Using Eq. $[1]$ with the mentioned NED, one can show that the electromagnetic field equation, $[2]$, reduces to the following differential equation

$$
[(d - 2) E + E' \rho] l^2 + 4E^2 [(d - 2) E + 3E^2 \rho] \beta + O (\beta^2) = 0, \tag{8}
$$

where prime denotes the first derivative with respect to $\rho$ and $E = h'(\rho)$. Solving Eq. $[8]$, one obtains

$$
E(\rho) = \frac{2q l^2}{\rho^{d-2}} - \frac{32q^3 l^4 \beta}{\rho^{3(d-2)}} + O (\beta^2), \tag{9}
$$

where $q$ is an integration constant related to the electric charge. We should note that for small values of $\beta$ all relations reduce to the corresponding relations of the Maxwell theory.
In order to obtain the metric function, \( f(\rho) \), one should solve all components of the gravitational field equation \( \Box \), simultaneously. After cumbersome calculations, we find that there are two different differential equations with the following explicit forms

\[
e_{\rho \rho} = K_1 + \alpha K_2 = 0, \tag{10}
\]

\[
e_{tt} = K_{11} + \alpha K_{22} = 0, \tag{11}
\]

where

\[
K_1 = \Lambda + \frac{(d-2)(d-3)}{2 \rho^2} f + \frac{4 q^2 l^2}{\rho^{d-4}} f' + \frac{(d-2) f'}{2 \rho} - \frac{32 l^4 q^4 \beta}{\rho^{d-8}} + O(\beta^2),
\]

\[
K_2 = -\frac{(d-2)(d-3)(d-4)}{2 \rho^4} f f' - \frac{(d-2)(d-3)(d-4)(d-5) f^2}{2 \rho^4},
\]

\[
K_{11} = -\Lambda - \frac{(d-3)(d-4)}{2 \rho^2} \frac{f'}{\rho} - \frac{(d-3) f'}{2} + \frac{4 l^2 q^2}{\rho^{d-8}} - \frac{96 l^4 q^4 \beta}{2 \rho^{d-12}} + O(\beta^2),
\]

\[
K_{22} = \frac{(d-3)(d-4)(d-5)}{2 \rho^4} \left( (d-6) f^2 + 4 \rho f f'' + \frac{2 \rho^2}{d-5} (f'^2 + f f''') \right),
\]

Eqs. (10) and (11) are corresponding to \( \rho \rho \) and \( tt \) components of the gravitational field equation \( \Box \). It is easy to show that \( \phi \phi \) and \( x_i x_i \) components of Eq. \( \Box \) are, respectively, similar to \( e_{\rho \rho} \) and \( e_{tt} \), and therefore, it is sufficient to solve Eqs. (10) and (11), simultaneously. Since we desire to obtain higher dimensional magnetic brane solutions with GB as a correction for EN gravity and quadratic Maxwell invariant as a correction to the Maxwell theory, we ignored \( \alpha \beta, \alpha^2 \) and \( \beta^2 \) terms and higher orders. Interestingly, the results for consideration of these two corrections will be as follow

\[
f(\rho) = f_{EN} - \frac{64 q^4 l^4}{(d-2)(3d-7)} \frac{\rho^{4d-10}}{\rho^2} \beta + \frac{(d-3)(d-4) f_{EN}^2}{\rho^2} \alpha + O \left( \alpha \beta, \alpha^2, \beta^2 \right), \tag{12}
\]

with

\[
f_{EN} = \frac{2 m l^3}{\rho^{d-3}} - \frac{2 \Lambda}{(d-1)(d-2) \rho^2} + \frac{8 q^2 l^2}{(d-2)(d-3) \rho^{2d-6}}, \tag{13}
\]

where \( m \) is an integration constant related to the mass. As one can see for the case of \( \alpha = \beta = 0 \), the effects of corrections are cancelled and obtained results will be magnetic solutions of EN gravity. In order to study the effects of these two corrections on the obtained metric function, we plot some graphs in the presence (absence) of these two corrections (see Fig. 1).

As one can see, in the absence of Maxwell and GB corrections, the plotted graph for metric function versus \( \rho \) is quite different comparing to when one of the considered corrections (GB or Maxwell) is present. Considering at least one of these corrections will modify the behavior of metric function. This modification is more evident and stronger for small values of \( \rho \). The metric function will have root(s) for specific value of \( \rho \) (namely \( \rho_0 \)). The function \( f(\rho) \) has two extrema at \( \rho_{ext1} \) and \( \rho_{ext2} \) (\( \rho_{ext1} < \rho_{ext2} \)) in which for \( \rho_0 \leq \rho \leq \rho_{ext1} \), metric function is an increasing function of \( \rho \). For \( \rho_{ext1} \leq \rho \leq \rho_{ext2} \), \( f(\rho) \) is a decreasing function of \( \rho \) and finally, in case of \( \rho \geq \rho_{ext2} \), it is an increasing function of radial coordinate (Fig. 1 left and middle). The root of \( f(\rho) \) is an increasing (or decreasing) function of \( \beta \) (or \( \alpha \)) parameter (Fig. 1 left and middle). \( \rho_{ext1} \) is decreasing functions of GB parameter and \( f(\rho_{ext1}) \) is increasing function of functions of GB parameter. As for nonlinearity parameter, \( \rho_{ext1} \) is an increasing function of \( \beta \) but interestingly \( f(\rho_{ext1}) \) is a decreasing function of it. It is worthwhile to mention that variation of \( \alpha \) and \( \beta \) do not have reasonable effect on \( f(\rho_{ext2}) \) and \( f(\rho_{ext3}) \). It is notable that for small values of the nonlinearity parameter the metric function has no root. Contrary to this effect, for large values of GB parameter, metric function will be without any root. It simply shows the fact that GB and nonlinearity parameters have opposite effects on the behavior of metric function.

Next, we are going to discuss the geometric properties of the solutions. To do this, we look for possible black hole solutions with obtaining the curvature singularities and their horizons. We usually calculate the Kretschmann scalar, \( R_{\alpha \beta \gamma \delta} R^{\alpha \beta \gamma \delta} \), to achieve essential singularity. Considering the mentioned spacetime, it is easy to show that

\[
R_{\alpha \beta \gamma \delta} R^{\alpha \beta \gamma \delta} = f'' + 2 (d-2) \left( f' \rho \right)^2 + 2 (d-2) (d-3) \left( \frac{f}{\rho^2} \right)^2. \tag{14}
\]
Inserting the metric function, $f(\rho)$, in Eq. (14) and using numerical analysis, one finds that the Kretschmann scalar diverges at $\rho = 0$ and it is finite for $\rho > 0$ and naturally one may think that there is a curvature singularity located at $\rho = 0$. In what follows, we state an important point, in which confirms that the spacetime never reaches $\rho = 0$. As one can confirm, easily, the metric function has positive value for $\rho > r_+$. So two cases may occur. For the first case, $f(\rho)$ is a positive definite function with no root and therefore the singularity is called a naked singularity which we are not interested in. We consider the second case, in which the metric function has one or more real positive root(s). We denote $r_+$ as the largest real positive root of $f(\rho)$. The metric function is negative for $\rho < r_+$ and positive for $\rho > r_+$ and hence the metric signature may change from $(-++++)$ to $(-+++...+)$ in the range $\rho < r_+$. We should note that this situation is different from that of black hole solutions. Considering a typical $d$-dimensional black hole metric $ds^2 = -f(\rho)dt^2 + \frac{dr^2}{f(\rho)} + r^2d\Omega^2$ with $(-+++...+)$ signature. Denoting $r_+$ as largest real positive root of $f(\rho)$, we know that for $\rho > r_+$ the metric function is positive definite and the signature does not change. For $\rho < r_+$, although the mentioned signature changes to $(+-+++...+)$, the number of positive and negative signs remain unchange. In other words, for entire spacetime, black hole metric has one negative (temporal coordinates) sign and one positive (spatial coordinates) signs. In this case, the change in sign merely signifies that we are not interested in. We consider the second case, in which the metric function has one or more real positive root(s). We denote $r_+$ as the largest real positive root of $f(\rho)$. The metric function is negative for $\rho < r_+$ and positive for $\rho > r_+$ and hence the metric signature may change from $(-++++)$ to $(-+++...+)$ in the range $\rho < r_+$.

It is worthwhile to mention that with this new coordinate, the metric function will be in the following form

$$f(\rho) = f_{EN} - \frac{64q^4l^4}{(d-2)(3d-7)(r^2 + r_+^2)^{2d-5}\beta} + \frac{(d-3)(d-4)f_{EN}^2}{r^2 + r_+^2} + O(\alpha\beta, \alpha^2, \beta^2).$$

where

$$f_{EN} = \frac{2ml^3}{(r^2 + r_+^2)^{-\frac{d-3}{2}}} - \frac{2\Lambda}{(d-1)(d-2)(r^2 + r_+^2)} + \frac{8q^2l^2}{(d-2)(d-3)(r^2 + r_+^2)^{d-3}}.$$
We should note that function \( f(r) \) given in Eqs. (17) is a non negative function in the whole spacetime. Although the Kretschmann scalar does not diverge in the range \( 0 \leq r < \infty \), one can show that there is a conical singularity at \( r = 0 \). One can investigate the conic geometry by using the circumference/radius ratio. Using the Taylor expansion, in the vicinity of \( r = 0 \), we find

\[
f(r) = f(r) \big|_{r=0} + \left( \frac{df}{dr} \big|_{r=0} \right) r + \frac{1}{2} \left( \frac{d^2 f}{dr^2} \big|_{r=0} \right) r^2 + O(r^3) + \ldots, \tag{19}\]

where

\[
f(r) \big|_{r=0} = \frac{df}{dr} \big|_{r=0} = 0, \tag{20}\]

\[
\frac{d^2 f}{dr^2} \big|_{r=0} = f'' = \frac{-2\Lambda}{d-2} - \frac{8l^2q^2}{(d-2)r^2} + \frac{64l^2q^4}{(d-2)r^4} \beta + O(\beta^2), \tag{21}\]

and hence

\[
\lim_{r \to 0^+} \frac{1}{r} \sqrt{\frac{g_{\phi\phi}}{g_{rr}}} = \lim_{r \to 0^+} \frac{\sqrt{r^2 + r^2_f f(r)}}{r^2} = \frac{lr_+}{2} f'' \neq 1, \tag{22}\]

which confirms that as the radius \( r \) tends to zero, the limit of the circumference/radius ratio is not \( 2\pi \) and, therefore, the spacetime has a conical singularity at \( r = 0 \). This conical singularity may be removed if one identifies the coordinate \( \phi \) with the period

\[
\text{Period}_\phi = 2\pi \left( \lim_{r \to 0^+} \frac{1}{r} \sqrt{\frac{g_{\phi\phi}}{g_{rr}}} \right)^{-1} = 2\pi (1 - 4\mu), \tag{23}\]

where \( \mu \) is given by

\[
\mu = \frac{1}{4} \left( 1 - \frac{2}{lr_+ f''} \right). \tag{24}\]

In other words, the near origin limit of the metric (16) describes a locally flat spacetime which has a conical singularity at \( r = 0 \) with a deficit angle \( \delta \phi = 8\pi \mu \). Using the Vilenkin procedure, one can interpret \( \mu \) as the mass per unit volume of the magnetic brane [22]. It is obvious that the nonlinearity of electrodynamics can change the value of deficit angle \( \delta \phi \). Taking into account Eqs. (21) and (24), we can write

\[
\frac{d (\delta \phi)}{d\beta} = -4\pi \frac{d \left( f''^{-1} \right)}{d\beta} = \frac{256\pi l^3 q^4}{(d-2)r^4 f''^2} > 0. \tag{25}\]

Eq. (25) indicates that \( \delta \phi \) is an increasing function of \( \beta \). In addition, considering Eqs. (21) and (24), one finds deficit angle does not depend the GB coefficient. In order to investigate the effects of nonlinearity, \( r_+ \), \( q \), \( l \) and dimensionality, we plot \( \delta \phi \) versus \( \beta \) and \( r_+ \).

As one can see, deficit angle can be affected by changing the values of \( q \), \( \beta \), \( d \) and \( r_+ \). In order to provide additional clarification, we plot Figs. 2-4. Considering these figures, left panels indicate the variation of deficit angle versus \( \beta \), whereas the right panels correspond to the behavior of deficit angle with respect to \( r_+ \). Left panel of Figs. 2-4 confirm that deficit angle is an increasing function of \( \beta \). Moreover Fig. (2) left) shows that for \( \beta \to 0 \) (Maxwell case), deficit angle is a decreasing function of \( q \). In addition, there is a \( q_c \), in which for \( q > q_c \) deficit angle is negative for \( \beta = 0 \) (one can obtain \( q_c \) in such a way that \( \delta \phi \big|_{\beta=0,q=q_c} = 0 \)).

In case of deficit angle versus \( r_+ \) for different values of charge parameter (Fig. 2 right), the plotted graph is divided into three distinguishable regions. We find that variation of \( q \), \( \beta \) and \( q \) do not affect, significantly, the behavior of deficit angle for sufficiently small (large) \( r_+ \) (see right panel of Figs. 2-4). In addition Fig. 2(right) shows that there is an extremum point \( r_{+ext} \), in which for \( r_+ \leq r_{+ext} \) (\( r_{+ext} \leq r_+ \)), deficit angle is a decreasing (increasing) function of \( r_+ \). The mentioned \( r_{+ext} \) and its corresponding deficit angle are increasing functions of charge parameter. In other words, for adequately small charge the minimum value of deficit angle is negative (Fig. 2 right). By increasing charge the region of negativity and hence the distance between roots of deficit angle decreases.

Now, we plot deficit angle versus \( r_+ \) for various \( \beta \) (Fig. 3 right) to show an interesting behavior. This figure indicates two divergency for deficit angle when \( \beta < \beta_c \). In other words, these singularities, \( r_{+1} \) and \( r_{+2} \) (\( r_{+1} < r_{+2} \)),
FIG. 2: $\frac{\delta \phi}{\pi}$ versus $\beta$ (left) and $\frac{\delta \phi}{\pi}$ versus $r_+$ (right) for $d = 5$ and $l = 1$.
left diagram: $r_+ = 2$, $q = 8$ (dotted line), $q = 8.5$ (continuous line) and $q = 9$ (dashed line).
right diagram: $\beta = 0.05$, $q = 0.4$ (dotted line), $q = 0.73$ (continuous line) and $q = 1$ (dashed line).

can be removed by increasing $\beta$. For $\beta > \beta_c$, deficit angle has a minimum and for $\beta < \beta_c$, we cannot obtain physical ($\delta \phi \leq 2\pi$) deficit angle for $r_{+1} < r_+ < r_{+2}$. Deficit angle before first singular point is a decreasing function of $r_+$ whereas, it is an increasing function after second singular point. Variation of the nonlinearity parameter changes the distance between these two singular points. In other words, increasing $\beta$ leads to decrease the mentioned interval. Fig. 3 (left), indicates that deficit angle is an increasing function of $r_+$ in this parameter region.

Finally, we are considering the effects of dimensions on deficit angle (Fig. 4). As one can see, for certain dimensions, there is a region for $\beta$ where, the deficit angle is negative (Fig. 4 left) and increasing dimensions leads to vanishing this region. Fig. 4 shows that deficit angle is an increasing function of dimensions in this parameter region. As for
deficit angle versus $r_+$, (Fig. 4 right), there is a similar behavior with (Fig. 2 right) and deficit angle has a minimum.

In order to explain the negative deficit angle, we first describe positive one. Cutting segment of a certain angular size of a two dimensional plane and then sewing together the edges to obtain a conical surface. This conical space is flat but has a singular point corresponding to the apex of the cone. The deleted segment from the plan is known as deficit angle with positive values. According to the previous statement, here, we imagine a new situation when a segment is added to the new plane to obtain a flat surface with a saddle-like cone (for more details see Fig. 2 in Ref. [23]). This added segment is corresponding to a negative deficit angle (or surplus angle).

It is worthwhile to mention that although the deleted segment is bounded by the value of $2\pi$, the added segment is unbounded. Therefore, one can conclude that the range of deficit angles is from $-\infty$ to $2\pi$. Positive/negative deficit angles may be related to the positive/negative torsion of space or the attractive-type/repulsive-type of gravitational potentials and more details of negative deficit angle with its physical interpretations can be found in [23, 24].

III. CLOSING REMARKS

In this paper, we have considered two different kinds of corrections to both matter and gravitational fields. For gravitational aspect, we have considered GB gravity as a correction to EN gravity whereas for electromagnetic aspect, we have regarded a quadratic power of Maxwell invariant in addition to Maxwell Lagrangian as a correction to electromagnetic field. Remarkably it was seen that, in absence of these two corrections, the behavior of the metric function is completely different comparing to consideration of at least one of them.

Interestingly, the root(s) of metric function was also modified by considering these corrections. The place of this root(s) was a decreasing (an increasing) function of GB (nonlinearity) parameter. For small values of the nonlinearity parameter and large values of the GB parameter, the behavior of the metric function was similar to the case of Einstein-Maxwell. In addition, we found that the contribution of considered matter field was opposite of the gravitational field. We found that for large values of radial coordinate these corrections do not have significant effect on the metric function. In other words, the dominant region in which these two corrections modify metric function meaningfully was for small values of $\rho$.

Next, in order to avoid change of signature, we used a suitable radial transformation and found that there is no curvature singularity through the whole of spacetime, but there is a conical singularity located at the origin. We have studied the behavior of deficit angle and the effects of different parameters on it. We have plotted two kinds of diagrams. One is for deficit angle versus $\beta$ and the other one is deficit angle versus $r_+$. For the case of deficit angle versus $\beta$, due to fact that we considered nonlinearity as a correction, we only have taken small values of it into
account. In general the behavior of the left graphs were monotonic and the calculated deficit angles were increasing function of nonlinearity parameter. As for deficit angle versus \( r_+ \), interestingly, the behavior was completely different to the other case. In this case, for variation of \( q \) and \( d \), we found that for sufficiently small or large \( r_+ \) deficit angle is independent of the value of other parameters. In addition, we showed that there is a minimum value for deficit angle in a specific \( r_+ \). Moreover, we found that the only parameter that modified the general behavior of these graphs, significantly, were nonlinearity parameter. For small values of \( \beta \) two singular points were seen. Interestingly, as one increases the nonlinearity parameter the distance between these two singular points decreases. In other word, by increasing nonlinearity parameter, a compactification happens which decreases the region between two singular points to a level that the mentioned singularities vanish. This behavior shows the fact that small values of \( \beta \) have stronger contribution comparing to other parameters drastically.

The existence of root, region of negativity and divergency for deficit angle are other important issues that must be taken into account. In studying deficit angle, we are considering second order derivation of metric function with respect to radial coordinate. Considering the fact that metric function could be interpreted as a potential (see for example chapter 9 of Ref. [25]), it is arguable to state that the singular point may be indicated as a phase transition. On the other hand, geometrical structure of the solutions in case of positive and negative values of deficit angle is different. In case of positive deficit angle, the geometrical structure of the object is cone like with a deficit angle whereas in case of the negative angle the structure will be saddle-like with surplus angle. We found that for positive deficit angle, there is an upper limit whereas for negative deficit angle there is no limit. Therefore, one may state that due to these differences in the structure of the solutions, the root of deficit angle may represent a phase transition. These two arguments could be discussed in more details if the physical concept of negative deficit angle have become more clear. Moreover, we should note that, spacetime has no deficit angle for vanishing \( \delta \phi \). In other words, the geometrical structure of the solutions in this case represents no defect. Therefore, one may argue that these cases are representing the magnetic brane without conical structure.

Finally, we will be interested in analyzing the theory of gravity that was proposed in this paper in more details and calculating conserved quantities of this theory. Also one can consider higher orders of the Lovelock gravity as corrections to the EN gravity and study their effects on magnetic branes and their deficit angle. Phase transition, structural properties and physical behavior of different objects have been studied through these defects. Since obtained solutions are asymptotically AdS, it is worthwhile to consider these solutions in context of AdS/CFT correspondence and study different phenomena through these solutions. In addition, one can make some modifications regarding the geometry of the solutions and remove the mentioned conical singularity, and then, use the copy-and-paste method to obtain geodesically complete spacetime with a minimum value for \( \rho_{\text{min}} = r_+ \) as a throat [27]. We left these issues for the future works.

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