Virial coefficients of trapped and un-trapped three-component fermions
with three-body forces in arbitrary spatial dimensions

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(Dated: March 12, 2020)

Using a coarse temporal lattice approximation, we calculate the first few terms of the virial expansion of a three-species fermion system with a three-body contact interaction in d spatial dimensions, both in homogeneous space as well as in a harmonic trapping potential of frequency ω. Using the three-body problem to renormalize, we report analytic results for the change in the fourth- and fifth-order virial coefficients Δb4 and Δb5 as functions of Δb3. Additionally, we argue that in the ω → 0 limit the relationship b3^T = n^{-d/2}b_n holds between the trapped (T) and homogeneous coefficients for arbitrary temperature and coupling strength (not merely in scale-invariant regimes). Finally, we point out an exact, universal (coupling- and frequency-independent) relationship between Δb3^T in 1D with three-body forces and Δb3^2 in 2D with two-body forces.

I. INTRODUCTION

Motivated by the recent interest in one-dimensional (1D) Fermi and Bose gases in the fine-tuned situation where only three-body interactions are present [10,11], we explore here the thermodynamics of fermions with a contact three-body interaction in the region of low fugacity (which corresponds to a dilute regime and therefore high temperatures in units of the energy scale set by the density). We focus on the fermionic case but explore the problem in arbitrary dimension d. To that end, we implement a semiclassical lattice approximation (SCLA) to calculate the virial coefficients b_n, and carry out their evaluation up to n = 5 at leading order (LO) in that approximation.

The LO-SCLA was introduced in Ref. [10] as a way to estimate virial coefficients in two-component Fermi gases. The approximation seems crude in its definition but performs surprisingly well when the lowest non-trivial term in the virial expansion is used as a renormalized coupling constant (b_3 for two-body forces, for example, and b_5 in this work). Not surprisingly, the approximation was seen to work better at weak coupling, which makes sense as the radius of convergence of the virial expansion was found to be quickly reduced as a result of the interaction. In Ref. [11], the NLO-SCLA was explored up to b_7, displaying the convergence properties up to the unitary point (in 3D) and in Ref. [12] the LO-SCLA was used for systems in a harmonic trap, showing that the approximation can capture the dependence on the trap frequency ω. In both cases, the analytic dependence of virial coefficients on the dimension was obtained, as will be the case here. This is to be contrasted with conventional methods to calculate virial coefficients, which can be very precise but are limited to specific situations (coupling strength, dimension, etc.) and are typically unable to provide analytic insight as they are entirely numerical.

Our analytic formulas for the virial coefficients, although approximate, support and shed light on the relationship b_3^T → n^{-d/2}b_n in the ω → 0 limit, where the superindex T indicates the harmonically trapped situation. This connection is well-known to be valid in the noninteracting limit and in the so-called unitary limit of spin-1/2 fermions in 3D, both of which feature temperature-independent coefficients b_n. As we will argue, that relationship is actually valid for all temperatures and coupling constants, and holds for three-body interactions just as well as for two-body interactions. Finally, we point out an exact, coupling- and frequency-independent relationship between the Δb3^T in 1D with three-body forces and Δb3^2 in 2D with two-body forces.

II. HAMILTONIAN AND VIRIAL EXPANSION

We focus on a non-relativistic Fermi system with a three-body contact interaction, such that the Hamiltonian for three flavors 1, 2, 3 is $H = T + V$, where

$$\hat{T} = \int d^d x \hat{\psi}^\dagger_s(x) \left(-\frac{\hbar^2 \nabla^2}{2m}\right) \hat{\psi}_s(x)$$

and

$$\hat{V} = -g_d \int d^d x \hat{n}_1(x) \hat{n}_2(x) \hat{n}_3(x),$$

where the field operators $\hat{\psi}_s, \hat{\psi}^\dagger_s$ are fermionic fields for particles of type 1, 2, 3 (summed over s above), and $\hat{n}_s(x)$ are the coordinate-space densities. In the remainder of this work, we will take $\hbar = k_B = m = 1$. Besides the above, we will also consider the case in which an external trapping potential term is added to the Hamiltonian, of the form

$$\hat{V}_{\text{ext}} = \frac{1}{2} m \omega^2 \int d^d x \; x^2 \left[ \hat{n}_1(x) + \hat{n}_2(x) + \hat{n}_3(x) \right].$$

One way to characterize the thermodynamics is through the virial expansion [13], which is an expansion around the dilute limit $z \to 0$, where $z = e^{\delta_B}$ is the fugacity, i.e. it is a low-fugacity expansion. The corresponding coefficients accompanying the powers of $z$ in

Finally, we point out an exact, universal (coupling- and frequency-independent) relationship between the Δb3^T in 1D with three-body forces and Δb3^2 in 2D with two-body forces.
the expansion of the grand-canonical potential $\Omega$ are the virial coefficients; specifically,

$$-\beta\Omega = \ln Z = Q_1 \sum_{n=1}^{\infty} b_n z^n$$  \hspace{1cm} (4)

where

$$Z = \text{Tr} \left[ e^{-\beta(R - \mu N)} \right] = \sum_{N=0}^{\infty} z^N Q_N$$  \hspace{1cm} (5)

is the grand-canonical partition function, $Q_1$ is the one-body partition function, $b_1 = 1$, and the higher-order coefficients require solving the corresponding few-body problems:

$$Q_1 b_2 = Q_2 - \frac{Q_1^2}{2!},$$ \hspace{1cm} (6)

$$Q_1 b_3 = Q_3 - b_2 Q_1^2 - \frac{Q_1^3}{3!},$$ \hspace{1cm} (7)

$$Q_1 b_4 = Q_4 - \left( b_3 + \frac{b_2^2}{2} \right) Q_1^2 - b_2 Q_1^3 - \frac{Q_1^4}{4!},$$ \hspace{1cm} (8)

$$Q_1 b_5 = Q_5 - (b_4 + b_2 b_3) Q_1^2 - \left( b_2^2 + b_3 \right) \frac{Q_1^3}{2} - b_2 \frac{Q_1^4}{3!} - \frac{Q_1^5}{5!},$$ \hspace{1cm} (9)

and so forth.

Since $Q_1 \propto V$, the above expressions display precisely how the volume dependence cancels out in each $b_n$. In particular, the highest power of $Q_1$ will always involve single-particle (i.e. noninteracting) physics and will therefore cancel in the change due to interactions $\Delta b_n$, such that

$$Q_1 \Delta b_2 = \Delta Q_2$$ \hspace{1cm} (10)

$$Q_1 \Delta b_3 = \Delta Q_3 - \Delta b_2 Q_1^2,$$ \hspace{1cm} (11)

$$Q_1 \Delta b_4 = \Delta Q_4 - \Delta \left( b_3 + \frac{b_2^2}{2} \right) Q_1^2 - \Delta b_2 Q_1^3,$$ \hspace{1cm} (12)

$$Q_1 \Delta b_5 = \Delta Q_5 - \Delta (b_4 + b_2 b_3) Q_1^2 - \frac{1}{2} \Delta (b_2^2 + b_3) Q_1^3 - \frac{\Delta b_2}{3!} Q_1^4,$$ \hspace{1cm} (13)

and so on. Note that, when only three-body interactions are present, as is the case we consider here, there is no change in the two-body spectrum, i.e. $\Delta b_2 = 0$. Therefore, the above expressions simplify to

$$Q_1 \Delta b_3 = \Delta Q_3,$$ \hspace{1cm} (14)

$$Q_1 \Delta b_4 = \Delta Q_4 - \Delta b_3 Q_1^2,$$ \hspace{1cm} (15)

$$Q_1 \Delta b_5 = \Delta Q_5 - (\Delta b_4 + b_2 \Delta b_3) Q_1^2 - \frac{\Delta b_3}{2} Q_1^3.$$ \hspace{1cm} (16)

In terms of the partition functions $Q_{MNL}$ of $M$ particles of type 1, $N$ of type 2, and $L$ of type 3, we have

$$\Delta Q_3 = \Delta Q_{111},$$ \hspace{1cm} (17)

$$\Delta Q_4 = 3 \Delta Q_{211},$$ \hspace{1cm} (18)

$$\Delta Q_5 = 3 \Delta Q_{311} + 3 \Delta Q_{221}.$$ \hspace{1cm} (19)

From the above equations we see that there is only a small number of non-trivial contributions to each virial coefficient. The main task is calculating each of these terms and for that purpose we use a coarse lattice (or semiclassical) approximation, as explained next.

### III. THE SEMICLASSICAL APPROXIMATION AT LEADING ORDER

To carry out our calculations of virial coefficients we introduce a Trotter-Suzuki (TS) factorization of the Boltzmann weight. In the lowest possible order, the TS factorization amounts to keeping only the leading term in the following formula:

$$e^{-\beta(T+V)} = e^{-\beta T} e^{-\beta V} \times e^{-\frac{\beta^2}{2} [T, V]} \times \ldots,$$ \hspace{1cm} (20)

where higher orders involve exponentials of nested commutators of $T$ with $V$. Taking the leading order in this expansion is equivalent to setting $[T, V] = 0$, which is why we refer to it as a semiclassical approximation. As Refs. [10][12] have shown, this seemingly crude approximation provides surprisingly good answers, especially at weak coupling, and is therefore useful toward examining the virial expansion in an analytic fashion. Below, we give two explicit examples of the application of our approximation to the calculation of virial coefficients.

#### A. A simple example: $\Delta b_3$

As the simplest example, we consider $Q_{111}$:

$$Q_{111} = \sum_{P_j} \langle P \rangle e^{-\beta T} e^{-\beta V} | P \rangle$$ \hspace{1cm} (21)

$$= \sum_{P_j} e^{-\beta (p_1^2 + p_2^2 + p_3^2)/2m} \langle P \rangle e^{-\beta V} | P \rangle,$$ \hspace{1cm} (22)

where we have used a collective momentum index $P = (p_1, p_2, p_3)$. Inserting a coordinate-space completeness relation to evaluate the potential energy factor, we obtain

$$e^{-\beta V} | X \rangle = \prod_{z} (1 + C \hat{\mu}_1(z) \hat{\mu}_2(z) \hat{\mu}_3(z)) | X \rangle$$ \hspace{1cm} (23)

$$= | X \rangle + C \sum_{z} \delta(x_1 - z) \delta(x_2 - z) \delta(x_3 - z) | X \rangle$$

$$= [1 + C \delta(x_1 - x_3) \delta(x_2 - x_3)] | X \rangle,$$

where $C = (e^{\beta g_{\ell}} - 1) \ell^2$, $\ell$ is an ultraviolet regulator in the form of a spatial lattice spacing, and we used the fermionic relation $\hat{\mu}_n^2 = \hat{n}_n$. We also introduced a collective index $X = (x_1, x_2, x_3)$. The $C$-independent term yields the noninteracting result, such that we may write

$$\Delta Q_{111} = C \sum_{P_j, X_k} e^{-\beta (p_1^2 + p_2^2 + p_3^2)/2m} \times \delta(x_1 - x_3) \delta(x_2 - x_3) | X \rangle | P \rangle^2,$$ \hspace{1cm} (24)
which simplifies substantially when using a plane wave basis since $|\langle \mathbf{X}|\mathbf{P}\rangle|^2 = 1/V^3$, where $V$ is the $d$-dimensional volume of the system. We then find

$$\Delta Q_{111} = C \frac{Q_{100}^3}{V^2}$$

(25)

where

$$Q_{100} = \sum_{\mathbf{p}_1} e^{-\beta \mathbf{p}_1^2/2m}.$$  

(26)

Thus,

$$\Delta b_3 = C \frac{Q_{100}^3}{V^2 Q_1} = C \frac{Q_1^3}{V^2} = \frac{C}{V^2} \frac{1}{\lambda_T^2 3^3},$$

(27)

where $Q_1 = 3Q_{100} = 3V/\lambda_T^2$, $\lambda_T = \sqrt{2\pi\beta}$ is the thermal wavelength, and $V$ is the system’s spatial volume. This relationship between the bare coupling constant $C$ and the physical quantity $\Delta b_3$ provides a way to renormalize the problem. In other words, $\Delta b_3$ will play the role of the renormalized dimensionless coupling constant.

The general form of the change $\Delta Q_{MNL}$ in the partition function for $M$ type-1 particles, $N$ type-2 particles and $L$ type-3 particles, with a contact interaction, is given by

$$\Delta Q_{MNL} = \sum_{\mathbf{p}, \mathbf{X}} e^{-\beta \mathbf{P}^2/2m} |\langle \mathbf{X}|\mathbf{p}\rangle|^2 (C f_a(\mathbf{X}) + C^2 f_b(\mathbf{X}) + \ldots),$$

(28)

where $\mathbf{P}, \mathbf{X}$ represent all momenta and positions of the $M + N + L$ particles, and the functions $f_a, f_b, \ldots$, which encode the matrix element of $e^{-\beta \mathbf{P}^2}$, depend on the specific $MNL$ case being considered. The wavefunction $\langle \mathbf{X}|\mathbf{P}\rangle$ is a product of three Slater determinants which, if using a plane-wave single-particle basis, leads to Gaussian integrals over the momenta $\mathbf{P}$.

### B. Another example: $\Delta b_4$ in a harmonic trap.

In this section we consider the case in which the system is held in a harmonic trapping potential of frequency $\omega$. As the expressions for the virial coefficients in terms of the canonical partition functions carry over to this case, we will simply add the superindex ‘T’ to denote quantities in the trapped system. To calculate $\Delta b_4^T$, we need $\Delta b_3^T$ and $Q_1^T$. The latter is of course trivial as there is no interaction in that case (see Ref. [12]):

$$Q_1^T = 3 \sum_n e^{-\beta E_n} = 3 e^{-\beta \omega d/2} \left(1 + \frac{1}{1 - e^{-\beta \omega}} \right)^d,$$

(29)

$$= 3 \left(\frac{1}{2 \sinh(\beta \omega/2)} \right)^d,$$

(30)

where $E_n$ is the single-particle energy level of the harmonic oscillator (separated in $d$-dimensional cartesian coordinates such that $n$ represents a $d$-dimensional vector of harmonic oscillator quantum numbers).

To obtain $\Delta b_3^T$, we proceed as in the previous example to obtain the analogue of Eq. (21) for the trapped case:

$$\Delta Q_{111}^T = C \sum_{n_1, n_2} e^{-\beta (E_{n_1} + E_{n_2} + E_{n_3})} \times \delta(x_1 - x_3) \delta(x_2 - x_3) |\langle x_1 x_2 x_3 | n_1 n_2 n_3 \rangle|^2,$$

(31)

The sums over $x_3, x_2$ can be carried out right away, and moreover

$$|\langle x_1 x_2 x_3 | n_1 n_2 n_3 \rangle|^2 = |\phi_{n_1}(x_1)|^2 |\phi_{n_2}(x_2)|^2 |\phi_{n_3}(x_3)|^2,$$

(32)

where $\phi_n(x)$ is the single-particle harmonic oscillator wavefunction in $d$-dimensional cartesian coordinates. Using the above, we obtain

$$\Delta Q_{111}^T = C \sum_x \rho^2(x; \omega),$$

(33)

where

$$\rho(x; \omega) = \sum_n e^{-\beta E_n} |\phi_n(x)|^2.$$

(34)

Note that $\sum_x \rho(x; \omega) = Q_1^T / 3$.

Using the Mehler kernel (see Ref. [12]) evaluated at equal spatial arguments, we find that

$$\rho(x; \omega) = \omega^d e^{-x^2 / (2\pi \sinh(\beta \omega))} \int \frac{d^d x^2}{(2\pi \sinh(\beta \omega))^2},$$

(35)

where we note that $\text{tanh}(\beta \omega) > 0$ for all $\beta \omega > 0$. Carrying out the resulting Gaussian integrals and simplifying,

$$\Delta b_3^T = \frac{\Delta Q_{111}^T}{Q_1^T} = C \lambda_T^2 3^3 \frac{(\beta \omega)^d}{\sinh(\beta \omega)}^d,$$

(36)

where $\lambda_T = \sqrt{2\pi\beta}$.

Note that, as $\beta \omega \to 0$, we obtain

$$\Delta b_3^T = C \frac{1}{\lambda_T^2 3^3} = \frac{1}{\lambda_T^2 3^3} \Delta b_3,$$

(37)

where in the last equality we have used Eq. (27).

For $\Delta b_4^T$, we need $\Delta Q_{211}^T$, which is easily seen to be given by

$$\Delta Q_{211}^T = C \sum_{x, x'} \rho^2(x; \omega) \left[ \rho(x; \omega) \rho(x'; \omega) - \rho^2(x, x'; \omega) \right],$$

(38)

$$= \Delta Q_{111}^T Q_1^T / 3 - C \sum_{x, x'} \rho^2(x; \omega) \rho^2(x, x'; \omega),$$

where

$$\rho(x, x'; \omega) = \sum_n e^{-\beta E_n} \phi_n(x) \phi_n(x'),$$

(39)

which, using the Mehler kernel, becomes

$$\rho(x, x'; \omega) = \frac{\omega^d e^{-\omega \coth(\beta \omega)(x^2 + x'^2)/2 + \omega \cosh(\beta \omega)x \cdot x'}}{(2\pi \sinh(\beta \omega))^{d/2}}.$$

(40)
Thus, in the continuum limit,
\[
\Delta b_4^T = 3 \frac{\Delta Q_{211}^T}{Q_{11}^T} - \frac{\Delta b_3^T Q_1^T}{Q_{11}^T}
\]
\[
= -3C' \sum_{x \neq x'} \rho^2(x; \beta \omega) \rho^2(x, x'; \beta \omega)
\]
\[
= - \frac{C}{\lambda_T^d} \frac{1}{2^d} \left[ 1 \beta \omega \frac{1}{\sinh(\beta \omega) (1 + 3 \cosh(\beta \omega))} \right]^d
\]
\[
= -3^{\frac{d-1}{2}} \frac{1}{2^d} \frac{1}{(1 + 3 \cosh(\beta \omega))^d} \Delta b_3^T.
\]  
(42)

Note that, in the \( \beta \omega \to 0 \) limit, our approximation yields

\[
\Delta b_4^T = -3^{\frac{d}{2}} \frac{3}{2^d} \frac{\Delta b_3^T}{2^d} = -\frac{1}{2^d} \frac{3}{2^d} \Delta b_3,
\]  
(43)

which we will use below.

IV. RESULTS IN HOMOGENEOUS SPACE

A. Virial coefficients

Using the steps outlined above, we have calculated \( \Delta b_4 \) and \( \Delta b_5 \), and obtained

\[
\Delta b_4 = -C \frac{Q_1 Q_2 (2 \beta)}{9 V^2} = -3 \frac{Q_1 (2 \beta)}{Q_1} \Delta b_3,
\]  
(44)
\[
\Delta b_5 = C \left( \frac{(Q_1 (2 \beta))^2}{9 V^2} + \frac{Q_1 Q_2 (3 \beta)}{9 V^2} \right)
\]
\[
= \left( \frac{3 (Q_1 (2 \beta))^2}{Q_1^2} + \frac{3 Q_1 Q_2 (3 \beta)}{Q_1} \right) \Delta b_3.
\]  
(45)

for the fermionic three-species system with a three-body contact interaction in \( d \) spatial dimensions. In the last equation, the first term on the right-hand side represents the contribution of \( Q_{221} \), and the second term that of \( Q_{311} \).

In the continuum limit, it is easy to perform the resulting Gaussian integrals that determine \( Q_1 \) and obtain

\[
\Delta b_4 = -\frac{3}{2^d} \Delta b_3,
\]  
(46)
\[
\Delta b_5 = 3 \left( \frac{1}{2^d} + \frac{1}{3^d} \right) \Delta b_3.
\]  
(47)

Using these results, one may calculate the pressure, density, compressibility and even Tan’s contact (with knowledge of \( \Delta b_3 \) as a function of the interaction strength, e.g. \( \beta \epsilon_B \) in 1D or 2D, where \( \epsilon_B \) is the trimer binding energy). To provide a description of the thermodynamics that is as universal as possible across spatial dimensions, we will use \( \Delta b_3 \) as the measure of the interaction strength and display our results in terms of that parameter. Furthermore, one may also define a (dimensionless) contact density as

\[
\mathcal{C} = \frac{\lambda_T^d}{V} \frac{\partial \ln Z}{\partial \Delta b_3},
\]  
(48)

which differs from the conventional definition by a chain-rule factor \( \partial \Delta b_3 / \partial \lambda \) (which in turn can be determined by solving the three-body scattering problem), where \( \lambda \) is the \( d \)-dimensional coupling constant. To make the expression dimensionless, we have used the thermal wavelength \( \lambda_T = \sqrt{2 \pi \beta} \).

B. Thermodynamics and contact across dimensions

The interaction change in the pressure \( \Delta P \) can be written in dimensionless form in arbitrary dimension as

\[
\beta V \Delta P = Q_1 \sum_{k=1}^{\infty} \Delta b_k z^k.
\]  
(49)

Similarly, the interaction change in the density can be written as

\[
\lambda_T^d \Delta n = 3 \sum_{k=1}^{\infty} k \Delta b_k z^k,
\]  
(50)

and, using our definition of the contact in Eq. (48),

\[
\Delta \mathcal{C} = 3 \sum_{k=1}^{\infty} \frac{\partial \Delta b_k}{\partial \Delta b_3} z^k.
\]  
(51)

Implementing our LO-SCLA results, we obtain

\[
\beta \lambda_T^d \Delta P \simeq 3 \Delta b_3 z^3 \left[ 1 - \frac{3}{2^d} z + 3 \left( \frac{1}{2^d} + \frac{1}{3^d} \right) z^2 \right],
\]  
(52)
\[
\lambda_T^d \Delta n \simeq 9 \Delta b_3 z^3 \left[ 1 - \frac{4}{2^d} z + 5 \left( \frac{1}{2^d} + \frac{1}{3^d} \right) z^2 \right],
\]  
(53)
\[
\Delta \mathcal{C} \simeq 3 z^3 \left[ 1 - \frac{3}{2^d} z + 3 \left( \frac{1}{2^d} + \frac{1}{3^d} \right) z^2 \right].
\]  
(54)

FIG. 1. Density, in units of \( \lambda_T^d = (2 \pi \beta)^{d/2} \) as a function of \( \ln z = \beta \mu \), at \( \Delta b_3 = 0.25 \).
As an example, in Fig. 1, we display the density as a function of the logarithm of the fugacity \( \ln z = \beta \mu \) for \( \Delta b_3 = 0.25 \) and for \( d = 1, 2, 3 \).

The behavior of \( \Delta n \) as a function of \( \beta \mu \) in Fig. 1 is as expected for a system with attractive interactions, namely the interaction-induced change in the density is positive and enhanced by increasing \( \beta \mu \) (or, equivalently, washed out at low densities, i.e. for large and negative \( \beta \mu \)). Also as expected (and as observed in Refs. [10] and [11] for two-body interactions), interaction effects are more pronounced in lower dimensions at fixed \( \Delta b_3 \).

V. RESULTS IN A HARMONIC TRAP

A. Fourth- and fifth-order virial coefficients

We have generalized our example of \( \Delta b_4^T \), discussed in a previous section, to \( \Delta b_5^T \). For future reference, we show both results:

\[
\Delta b_4^T = -\frac{3^{\frac{d}{2}}+1}{2^d} \frac{1}{(1 + 3 \cosh(\beta \omega))^{\frac{d}{2}}} \Delta b_3^T, \quad (55)
\]

\[
\Delta b_5^T = 3^{\frac{d}{2}+1} \left[ \frac{1}{12 \cosh^2(\beta \omega) + 4 \cosh(\beta \omega) - 1} \right]^{\frac{d}{2}}
+ \left[ \frac{1}{12 \cosh^2(\beta \omega) + 8 \cosh(\beta \omega)} \right]^{\frac{d}{2}} \Delta b_3^T. \quad (56)
\]

In Fig. 2, we show these results in \( d = 1, 2, 3 \) as a function of \( \beta \omega \). In contrast to the behavior of \( \Delta b_4^T \) for the case of two-body interactions, explored in Refs. [12][13], here both \( \Delta b_4^T \) and \( \Delta b_5^T \) display monotonic behavior. Furthermore, at this order in the SCLA, both \( \Delta b_4^T \) and \( \Delta b_5^T \) are proportional to \( \Delta b_3^T \), such that the results of Fig. 2 are universal predictions in the sense of being coupling-independent.

B. A universal relation in the \( \beta \omega \to 0 \) limit

Note that, in the \( \beta \omega \to 0 \) limit, where the homogeneous system is recovered,

\[
\Delta b_5^T \to 3^{\frac{d}{2}}+1 \left( \frac{1}{2^d} + \frac{1}{3^{\frac{d}{2}}} \right) \Delta b_3^T = \frac{3}{5^\frac{d}{2}} \left( \frac{1}{2^d} + \frac{1}{3^{\frac{d}{2}}} \right) \Delta b_3, \quad (57)
\]

Using Eqs. (43), (46), and (57), we find that trapped and un-trapped virial coefficients are related, in the \( \beta \omega \to 0 \) limit, as follows:

\[
\begin{align*}
\Delta b_3^T &= 3^{-\frac{d}{2}} \Delta b_3, \quad (58) \\
\Delta b_4^T &= 4^{-\frac{d}{2}} \Delta b_4, \quad (59) \\
\Delta b_5^T &= 5^{-\frac{d}{2}} \Delta b_5. \quad (60)
\end{align*}
\]

Although we have only explored \( \Delta b_n^T \) for \( n = 3, 4, 5 \) here (the cases \( n = 1, 2 \) are trivially satisfied as well), the fact that the above relationship holds points us to conjecture that the relation

\[
b_n^T|_{\beta \omega \to 0} = n^{-\frac{d}{2}} b_n, \quad (61)
\]

is universally valid for all \( n \), couplings, and temperatures (it is well known to be satisfied by noninteracting gases). Other authors, see e.g. [13][15][16] have noted (and proven using the local density approximation) that this relationship is satisfied in the unitary limit (where the \( b_n \) are temperature-independent), and the same connection was found for \( n = 3, 4 \) in systems with two-body forces in Ref. [12] for arbitrary couplings (within the LO-SCLA).

In principle, there is no special reason why \( b_n^T \) should not approach \( b_n \) when the trapping potential is removed. That there is a \( d \)- and \( n \)-dependent factor connecting those two quantities in the noninteracting case is merely a geometrical artifact of the choice of basis in which the calculations are performed (namely the harmonic oscillator basis in the trapped case and plane waves in the homogeneous case), which has no impact on physical quantities. Based entirely on dimensional analysis, however, the natural guess is that \( b_n^T \) may approach \( b_n \) times a dimensionless function of temperature and other dynamical scales. [That would actually change the partition function in a non-trivial way; in particular concerning Tan’s contact, but let us put that aside for the moment.]

Such a dimensionless function could only result from the interplay between the trapping potential \( V_{\text{ext}} \) and the interaction \( V \), possibly leading to subtleties in the \( \omega \to 0 \) limit (similar to those arising from degenerate perturbation theory). However, the fact that \( \{V_{\text{ext}}, V\} = 0 \) suggests that there should be no such subtlety and therefore no residual dependence on interaction-related scales in the relationship between \( b_n^T \) and \( b_n \) as \( \beta \omega \to 0 \). In that limit, the dimensionless quantities \( b_n^T \) and \( b_n \) should be related by a coupling- and temperature-independent function; their connection should be entirely geometrical and fully determined by the noninteracting case, for which \( b_n^T = n^{-\frac{d}{2}} b_n \) when \( \beta \omega \to 0 \). We therefore conclude that the conjecture is true for all \( n \), coupling strengths, and temperatures.
C. An exact relation across systems and dimensions

Finally, we point out a coupling-independent relationship between the 1D case with a three-body interaction (i.e., the 1D case of the system studied in this work) and the 2D case with only two-body interactions (denoted below by the superindex “2b2D”). As pointed out in Ref. [17], there exists an exact relationship between the three-body problem of the former situation and the two-body problem of the latter. That relationship yields a simple proportionality rule between the corresponding two-body problem of the latter. That relationship yields the three-body problem of the former situation and the relationship between our three-body 1D problem and its two-body counterpart in 2D with two-body interactions.

\[
\Delta b_3 = \frac{Q_{\text{cm}}^{1D}}{Q_{\text{cm}, 2b2D}^{1D}} \frac{Q_{\text{2b2D}}^{1D}}{Q_{1}^{2b2D}} \Delta b_{2b2D},
\]

where the superscript “cm” indicates the partition function associated with the center-of-mass motion, which is not affected by the interactions and completely factorizes (both in the spatially homogeneous as well as in the harmonically trapped case). In the spatially homogeneous case, the proportionality factor between \(\Delta b_3\) and \(\Delta b_{2b2D}\) is \(1/\sqrt{3}\), as shown in Ref. [17]. On the other hand, in the harmonically trapped case, the relationship becomes

\[
\Delta b_{1T}^T = \frac{2}{3} \Delta b_{2b2D}^{T, 2b2D}.
\]

We stress that while this relationship is restricted to the 1D \(\Delta b_{1T}^T\) and \(\Delta b_{2b2D}^{T, 2b2D}\), it is valid for all couplings and all values of \(\beta \omega\) and is in that sense universal.

For completeness and future reference, we provide here details on the origin of this correspondence for the trapped case. The Schrödinger equation for this system takes the form

\[
\left[ -\frac{\nabla^2}{2m} + \frac{g}{m} \delta(x-y)\delta(y-z) + \frac{1}{2} m \omega^2 r^2 \right] \psi(r) = E \psi(r)
\]

where \(x, y,\) and \(z\) again indicate the different-flavor particles, \(r = (x, y, z)\), and

\[
\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.
\]

Factoring out the center-of-mass (c.m.) motion by defining \(Q = \frac{1}{\sqrt{3}}(x+y+z)\), \(q_1 = \frac{1}{\sqrt{2}}(y-x)\), \(q_2 = \frac{1}{\sqrt{6}}(x+y-2z)\), and \(\psi(x, y, z) = \Phi(Q) \phi(q)\), with \(q = (q_1, q_2)\), we obtain

\[
\left[ -\frac{1}{2m} \frac{\partial^2}{\partial Q^2} + \frac{1}{2} m \omega^2 Q^2 \right] \Phi(Q) = E_{\text{c.m.}}\Phi(Q),
\]

for the c.m. motion, and

\[
\left[ -\frac{\nabla^2}{2m} + \frac{\hat{g}}{m} \delta(q) + \frac{1}{2} m \omega^2 q^2 \right] \phi(q) = E_r \phi(q),
\]

where \(\hat{g} = g/\sqrt{3}\) is the effective coupling and \(E_r\) is the energy of relative motion, which is identical to that of a single particle in a 2D harmonic oscillator potential with a \(\delta\)-potential at the origin. This establishes the exact relationship between our three-body 1D problem and its two-body counterpart in 2D with two-body interactions.

As in the spatially homogeneous case, the eigenvalues \(\epsilon_{\omega} = E_r/\omega\) of the harmonically trapped system are determined implicitly, in this case as solutions to

\[
\frac{1}{\hat{g}} = \frac{1}{\pi} \sum_{n=0}^{\Lambda_{\omega}} \frac{1}{\epsilon_{\omega} - (2n + 1)} \rightarrow \frac{1}{2\pi} \left[ \psi_0 \left( \frac{1 - \epsilon_{\omega}}{2} \right) - \ln \Lambda_{\omega} \right],
\]

where \(\psi_0(z)\) is the digamma function, where \(\Lambda_{\omega}\) is a UV cutoff. Unlike in the untrapped problem, with its unique bound state, the trapped problem admits an infinite set of discrete excited states (all with positive energy). The problem is renormalized by relating the bare coupling to the \(\epsilon_{\omega}\) occurring in the lowest energy branch.

VI. SUMMARY AND CONCLUSIONS

In this work we have calculated the high-temperature thermodynamics of three-flavored Fermi gases with a contact three-body interaction in \(d\) spatial dimensions, as determined by the virial expansion. We carried out calculations in homogeneous space as well as in a harmonic trapping potential of frequency \(\omega\). To that end, we implemented a coarse temporal lattice approximation at leading order (the LO-SCLA) and calculated the change in the virial coefficients \(\Delta b_n\) due to interaction effects. In that context, we established a relation between the first two non-trivial virial coefficients, namely \(\Delta b_4\) and \(\Delta b_5\), as functions of \(\Delta b_3\). In addition, we argued that in the \(\beta \omega \to 0\) limit, the relationship \(\Delta b_{1T}^T = n^{-d/2} \Delta b_n\) holds between the trapped and homogeneous coefficients for arbitrary \(n\), coupling strengths, and temperatures; furthermore, it is valid for systems with two- and three-body interactions. We showed that our calculations reproduce that relationship for \(n = 3, 4, 5\). Finally, we showed a relation between the harmonically trapped case in 1D with three-body interactions and its analogue in 2D with two-body interactions, namely \(\Delta b_{1T}^T = \frac{2}{3} \Delta b_{2b2D}^{T, 2b2D}\).

ACKNOWLEDGMENTS

This material is based upon work supported by the National Science Foundation under Grant No. PHY1452635 (Computational Physics Program).
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