REGULARIZATION OF 2d SUPERSYMMETRIC YANG-MILLS THEORY VIA NON COMMUTATIVE GEOMETRY.

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Abstract. The non commutative geometry is a possible framework to regularize Quantum Field Theory in a nonperturbative way. This idea is an extension of the lattice approximation by non commutativity that allows to preserve symmetries. The supersymmetric version is also studied and more precisely in the case of the Schwinger model on supersphere [14]. This paper is a generalization of this latter work to more general gauge groups.

1. Introduction

Formally the quantization (in Feynman’s point of view) of a field is represented by a path integral, but this integral is not well defined [1]. The lattice approximation was first proposed as a way to regularize this integral but it does not preserve the Lorentz invariance. Snyder has introduced non commutativity of the coordinates to conserve Lorentz symmetry [23]. In this approach the space time is not a manifold but is decomposed in cells of certain size (multiple of Planck constant). This approach introduces a natural (UV) cut-off and it can be non perturbative. At least in compact cases, this cut-off allows us to remove divergences. This fuzzy approach [19, 20, 7, 15, 4, 14] of the regularization is exposed in the case of sphere using Berezin quantization [2], the result is so-called fuzzy sphere. In this framework, there are lot of works [12, 7, 21, 16, 4] which are trying to include all the fields. But in the noncompact cases the (UV) divergences can persist [5].

The fuzzy sphere is introduced by quantization of the symplectic structure on the usual sphere. It replaces the commutative structure by non commutative one and the quantum version of the symplectic reduction introduces naturally the finiteness. The first step is the regularization of a scalar field on the sphere [19, 10, 3]. The scalar field on fuzzy sphere is just a matrix and the action (always invariant by $SO(3)$) is defined using the trace on finite matrices.

Other field theories (spinors fields, gauge fields and topologically nontrivial field configurations) are also defined on the fuzzy sphere [10, 7, 13, 4, 8, 14] and their regularization proved, thanks again to the finiteness of the matrices. These constructions needed the non commutative generalization of spinors, of the differential complex and of the topologically nontrivial configurations. To know more about noncommutative geometry and its applications, see [6]. In [11, 12], the definition of the spinors (element of a bimodule) on fuzzy sphere, which allows to construct Dirac operator and chiral operator. But the latter two didn’t anticommute, thus the previous assumption did not preserve the perfect analogy between fuzzy sphere

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and the ordinary sphere. Another approach consists in using supersymmetry [7]. The supersymmetric version of the fuzziness is very similar to the ordinary case: fuzzy superspheres are finite supermatrices, the scalars fields are just the even parts of the supermatrices (bosonic submanifold) and spinors are the odd ones. In fact, the scalar fields and spinors are both contained in superscalars fields in a canonical way. One can also construct gauge fields using a differential complex based on this concept [15].

If we want to consider supersymmetric gauge theories, all these constructions are constraint to be gauge invariant and supersymmetric invariant. Using this idea, C. Klimcik constructs the supersymmetric Schwinger model (analogue of the Euclidean Maxwell field in two dimension) on the ordinary sphere and on the fuzzy sphere [14]. He constructed an suitable invariant supersymmetric differential complex based on the super Lie algebra $sl(2, 1)$ and its sub super Lie algebra $osp(2, 1)$. He worked out in detail the abelian case and we aim to study the non abelian case in this paper.

For this purpose, we conserve the general form of the action defining the electromagnetic on the fuzzy supersphere but we need to modify the differential complex to incorporate the non abelian case. At the commutative limit, a long calculation allows us to describe (also in an original way) the supersymmetric Yang-Mills theory on the ordiary sphere. This paper is organized as follows:

In the section 2, we recall some preliminary notions that underlie our framework: supersphere, symplectic reduction, quantization of the supersphere, super Lie algebras $sl(2, 1)$ and $osp(2, 1)$ and integration over the fermionic variables.

In the section 3, we construct the analogue of the supersymmetric differential complex presented [14] in the bosonic case and we modify it to include the non abelian case. Then we apply this construction of the modified complex to the supersphere and fuzzy supersphere.

At the commutative limit, we obtain respectively the Schwinger model [14] and the ordinary Yang-Mills theory on the supersphere.

Last section is devoted to conclusions.

2. Preliminaries

2.1. Supersphere. To perform easily the quantization of the sphere as a phase space, we use the well known symplectic reduction of the complex plane $\mathbb{C}^2$ by the group $U(1)$. We consider the complex plane $\mathbb{C}^2$ generated by $\chi^\alpha$, $\alpha = 1, 2$, with the following Poisson structure:

$$\{ f, g \} = \partial_{\chi^\alpha} f \partial_{\chi^\alpha} g - \partial_{\chi^\alpha} f \partial_{\chi^\alpha} g.$$ (2.1)

We call $\omega$, the 2-form underlying this symplectic structure. We consider a moment map $J = \chi_1^2 + \chi_2^2 - 1$ then we can associate $U(1)$ vector field $X$ to $J$ by

$$dJ = \omega (X, \cdot).$$

In the submanifold $J^{-1}(0)$, the form $\omega$ is degenerated. We obtain the standard 2-sphere $S^2$ with its symplectic structure by considering the quotient on this submanifold $J^{-1}(0)$ by null-space of the 2-form $\omega$. In other words, the algebra of functions on the sphere consists of functions on $\mathbb{C}^2$ with the property

$$\{ f, \chi_1^2 + \chi_2^2 - 1 \} = 0.$$ (2.2)
Moreover such two functions are equivalent if their difference is a function of the following form $h(\chi_1^2 + \chi_2^2 - 1)$. This procedure is called the symplectic reduction with a moment map $\chi_1^2 + \chi_2^2 - 1$.

In analogy with the algebra of functions on the sphere defined by symplectic reduction with respect to a moment map $\chi_1^2 + \chi_2^2 - 1$ in the complex plan $\mathbb{C}^2$, the algebra of functions $A_\infty$ on the supersphere is defined by (super) symplectic reduction with respect to a moment map $\chi_1^2 + \chi_2^2 + \bar{a}a - 1$ in the complex superplane $\mathbb{C}^{2,1}$, with additional fermionic or grassmanian variables $\bar{a}, a$ [1]. The Poisson structure on $\mathbb{C}^{2,1}$ is the following

\begin{align}
(2.3) \quad \{f, g\} = \partial_\alpha f \partial_\beta g - \partial_\beta f \partial_\alpha g - (-1)^f [\partial_\alpha f \partial_\beta g + \partial_\beta f \partial_\alpha g]
\end{align}

applied to coordinates, seen as functions, it gives

\begin{align}
(2.4) \quad \{\chi^\alpha, \chi^\beta\} = \delta_{\alpha\beta}, \quad \{a, \bar{a}\} = 1, \quad \alpha, \beta = 1, 2.
\end{align}

The following parametrization simplifies our work

\begin{align}
z = \frac{\chi^1}{\chi^2}, \quad \tau = \frac{\chi^1}{\chi^2}, \quad b = \frac{\chi^3}{\chi^1}, \quad \bar{b} = \frac{\bar{\chi}^1}{\chi^1}.
\end{align}

The Berezin integral on this algebra is written as, $e$ is the unit of $A_\infty$

\begin{align}
(2.5) \quad I[f] = \frac{1}{2\pi i} \int \frac{d\tau d\bar{\tau} dz db d\bar{b}}{1 + \tau \bar{\tau} + b \bar{b}} f, \quad I[e] = 1.
\end{align}

This algebra is equipped with graded involution

\begin{align}
(\chi^\alpha)^* = \chi^{\bar{\alpha}}, \quad (\chi^{\bar{\alpha}})^* = \chi^\alpha, \quad a^* = \bar{a}, \quad \bar{a}^* = -a.
\end{align}

Like $sl(2)$ on the sphere, the Lie superalgebra $sl(2, 1)$ is naturally represented on $A_\infty$. First of all, we recall that $sl(2, 1)$ is generated by $R_\pm, R_3, \Gamma, V_\pm, D_\pm$ with the following super Lie algebra structure. We note $[\ldots, \ldots]$ the anti-commutator.

\begin{align}
[R_3, R_\pm] = \pm R_\pm, \quad [R_\pm, R_-] = 2R_3, \quad [R_i, \Gamma] = 0, \quad i = +, -, 3.

[D_\pm, V_\pm] = 0, \quad [D_\pm, V_\mp] = \pm \frac{1}{4} \Gamma, \quad [D_\pm, D_,] = \mp \frac{1}{2} R_3,

[D_\pm, D_\mp] = \frac{1}{2} R_3, \quad [V_\pm, V_\pm] = \pm \frac{1}{2} R_3, \quad [V_\pm, V_\mp] = - \frac{1}{2} R_3,

[R_3, V_\pm] = \pm \frac{1}{2} V_\pm, \quad [R_\pm, V_\pm] = 0, \quad [R_\pm, D_\mp] = D_\pm, \quad [\Gamma, V_\pm] = D_\pm, \quad [\Gamma, D_\pm] = V_\pm.
\end{align}

The representation of $sl(2, 1)$ on $A_\infty$ is defined in the following way

\begin{align}
(2.7) \quad V_\pm f = \{v_\pm, f\}, \quad \Gamma f = \{\gamma, f\}, \quad f \in A_\infty

D_\pm f = \{d_\pm, f\}, \quad R_3 f = \{r_3, f\},

R_+ f = \{r_+, f\}, \quad R_- f = \{r_-, f\},
\end{align}

with respect to the following charges

\begin{align}
(2.8) r_+ = \chi^\dagger \chi^2, \quad r_- = \bar{\chi}^\dagger \chi^1, \quad r_3 = \frac{1}{2} (\chi^\dagger \chi^1 - \bar{\chi}^\dagger \bar{\chi}^2), \quad \gamma = \bar{a}a + 1

2v_+ = \chi^\dagger a + \bar{a} \chi^2, \quad 2v_- = \bar{\chi}^\dagger a - \bar{\chi} \chi^1,

2d_+ = \bar{\chi}^\dagger a + \chi^1 a, \quad 2d_- = - \bar{\chi}^\dagger a - \bar{\chi} \chi^1.
\end{align}
This representation is called Hamiltonian because it can be defined by the super-Poisson structure (2.1). The derivatives $V_\pm, D_\pm, \Gamma, R_\pm, R_3$ can be also expressed in terms of the standard supersymmetric derivatives $D, \overline{D}, Q, \overline{Q}$ in two dimensions

$$D = \partial_b + b\partial_z, \quad \overline{D} = \partial_{\overline{b}} + \overline{b}\partial_{\overline{z}},$$

$$Q = \partial_b - b\partial_z, \quad \overline{Q} = \partial_{\overline{b}} - \overline{b}\partial_{\overline{z}}.$$ 

We write the generators of $sl(2,1)$ using these four derivatives:

$$D_+ = \frac{1}{2}(D - \overline{z}\overline{D}), \quad D_- = -\frac{1}{2}(\overline{D} + zD),$$

$$V_+ = \frac{1}{2}(Q + \overline{z}\overline{Q}), \quad V_- = \frac{1}{2}(\overline{Q} - zQ),$$

$$\Gamma = \overline{b}\partial_{\overline{b}} - b\partial_b, \quad R_3 = \overline{z}\partial_{\overline{z}} - z\partial_z + \frac{1}{2}\overline{b}\partial_{\overline{b}} - \frac{1}{2}b\partial_b,$$

$$R_+ = -\partial_z - \overline{z}^2\partial_{\overline{z}} - \overline{z}\partial_{\overline{b}}, \quad R_- = \partial_{\overline{z}} + z^2\partial_z - zb\partial_b.$$ 

In the supersymmetric framework the Taylor expansion of the functions is finite (because the nilpotency of the fermionic variables). An even element writes

$$f(\overline{z}, z, \overline{b}, b) = u(\overline{z}, z) + \overline{b}\psi(\overline{z}, z) + \overline{b}\phi(\overline{z}, z) + \overline{b}vw(\overline{z}, z)$$

with $u$ and $v$ belong to the even part of $P$, $P$ a graded commutative algebra. And $\psi$ and $\phi$ in the odd one. Thus it is globally even. It is same to the odd element.

We recall the integration on the fermionic variables

$$(2.9) \quad \int dB = 0, \quad \int d\overline{D} = 0, \quad \int dbb = 0,$$

$$(2.10) \quad \int dB\overline{b} = 0, \quad \int d\overline{D}\overline{b} = 0, \quad \int \overline{D} bb f(\overline{z}, z, \overline{b}, b) = \int d\overline{b} (\psi - \overline{v}) = -v.$$ 

2.2. Quantization of the supersphere. In the previous part, we introduced the symplectic reduction because its simplifies the quantization of the supersphere. Indeed, first we quantize the superplane and we perform the quantum symplectic reduction \[15\]. As in quantum mechanics, we transform the generators of the algebra in creation and annihilation operators with the standard replacement

$$(2.11) \quad \{.,.\} \rightarrow \frac{1}{\hbar} [.,.] \text{ with } \hbar \text{ is real parameter.}$$

Thus the generators $\chi^\alpha, \chi^\alpha, \overline{\alpha}, a$ become operators verifying

$$[\chi^\alpha, \chi^\beta] = \hbar\delta_{\alpha\beta}, \quad [a, \overline{\alpha}]_+ = \hbar, \quad \alpha, \beta = 1, 2$$

and acting on a Hilbert space which is constructed as follows

$$\chi^\alpha |0 > \text{ is an vector}$$

$$\overline{\alpha} |0 > \text{ is another vector}$$

$$\chi^\alpha |0 > = 0$$

$$a |0 > = 0$$

It means, one considers a vector (vacuum vector and the standard notation is $|0 >$) and one constructs an irreducible representation of this algebra. The space generated by this denumerable family of vectors is a Hilbert space, called Fock space.

\[1\] There is an another way to quantize it, which is equivalent to the previous one, using the representation theory of $sl(2,1)$ [7].
The analogue of symplectic reduction with moment map is just the restriction of the Hilbert space only to the vectors $\psi$ satisfying the constraint

$$(\chi_1 \chi_1 + \chi_2 \chi_2 + \bar{\alpha} a - 1) \psi = 0$$

as in the previous section we define the quantized version of $A_\infty$ by the operators $\hat{f}$ which verify

$$[\hat{f}, \chi_1^2 + \chi_2^2 + \bar{\alpha} a] = 0.$$ 

Let us determine the dimension of the kernel of the operator $\chi_1^2 + \chi_2^2 + \bar{\alpha} a - 1$. Let be $\psi$ an element of the Fock space, it is easy to show that

$$\psi = (\chi_1)^{n_1} (\chi_2)^{n_2} \mid 0 > \quad \text{or} \quad \psi = (\chi_1)^{n_1} (\chi_2)^{n_2} \bar{\alpha}^2 \mid 0 > \quad \text{with } n_1, n_2 \in \mathbb{N}$$

which implies that

$$(\chi_1 \chi_1 + \chi_2 \chi_2 + \bar{\alpha} a - 1) \psi = Nh - 1 \quad \text{with } N \in \mathbb{N}.$$ 

Thus the condition to fulfil (2.12) is that $h = \frac{1}{N}$ and in this case, the dimension of the kernel of $\chi_1^2 + \chi_2^2 + \bar{\alpha} a$ is just the number of possibilities to have $N = n_1 + n_2$ or $n_1 + n_2 + 1$, it is exactly $2N + 1$. The each admissible value of the parameter $h$ gives us a $(2N + 1)$-dimensional subspace $H_N$ of the Fock space and the deformed version of $A_\infty$ is then $A_N = M_{2N+1}(\mathbb{C})$. When $N \to \infty$ we have the constant $h$ approaching 0 and the algebra $A_N$ tends to the classical limit $A_\infty$ [8]. The Hilbert space $H_N$ is graded $H_N = H_{eN} \oplus H_{oN}$ where $H_{eN}$ generated by bosonic creation operators

$$(\chi_1)^{n_1} (\chi_2)^{n_2} \mid 0 > , \quad n_1 + n_2 = N$$

and $H_{oN}$ both bosonic and fermionic creation operators

$$(\chi_1)^{n_1} (\chi_2)^{n_2} \bar{\alpha}^2 \mid 0 > , \quad n_1 + n_2 + 1 = N.$$ 

The involution in $A_N$ is defined exactly as in (2,6), $A_N$ is also graded as follows [7]

$$\Phi = \left( \begin{array}{cc} \phi_R \in M_{n+1}(\mathbb{C}) & \psi_R \in M_{n+1,n}(\mathbb{C}) \\ \psi_L \in M_{n,n+1}(\mathbb{C}) & \phi_L \in M_{n}(\mathbb{C}) \end{array} \right) \in A_N$$

where even part is composed by diagonal blocks and odd by the off-diagonal blocks. The integration over $A_N$ is given by

$$I[\Phi] = STr[\Phi], \quad \Phi \in A_N.$$ 

The relations of the super Lie bracket with the super-Poisson structure for $N \to \infty$ is given by

$$(2.13) \quad \{X, Y\} = N[X, Y], \quad X, Y \in A_N.$$ 

The graduation of the commutator depends on the graduation of the elements: if $X$ and $Y$ are both odd, it is in fact the anti-commutator. The representation defined by (2.7) on $A_\infty$ is preserved by quantization and becomes a representation on $A_N$ in which we replace the Poisson bracket by the graded commutator. In the "quantum" case, the action is defined by

$$(2.14) \quad \begin{array}{ll} V_j f = [v_+, f], & \Gamma f = [\gamma, f], \\ D_\pm f = [d_\pm, f], & R_3 f = [r_3, f], \\ R_+ f = [r_+, f], & R_- f = [r_-, f]. \end{array}$$
The explicit form of the supermatrices $r_i, \gamma, v_\alpha, d_\beta$ are given in [14]. The representations of $sl(2,1)$ on $\mathcal{A}_N$ and $\mathcal{A}_\infty$ are completely reducible, their decompositions into irreducible ones are the following

$$\mathcal{A}_N = \bigoplus_{j=0}^{N} j, \quad \mathcal{A}_\infty = \bigoplus_{j=0}^{\infty} j$$

where $j$ means the $sl(2,1)$ superspin of the representation, for more details see [7, 14]. We recall that the quantization performed using the representation theory of $sl(2,1)$ is just the approximation at level $N$ of $\mathcal{A}_\infty = \bigoplus_{j=0}^{\infty} j$ by $\mathcal{A}_N = \bigoplus_{j=0}^{N} j$ endowed with a new multiplication rule. It is clear that at the limit $\mathcal{A}_N$ becomes $\mathcal{A}_\infty$, for more details see [7].

In [14] C. Klimcik constructed an action of the supersymmetric gauge theory for the finite $N$, at the limit it becomes the standard free supersymmetric electrodynamics in the ordinary sphere. In the following section we construct the modified differential complex that allows us to include the non abelian case.

3. DESCRIPTION OF THE MODIFIED DIFFERENTIAL COMPLEX

3.1. Bosonic case. Firstly, we construct a differential complex on the ordinary sphere in an invariant way and then we extend it to the supersphere [14]. The invariant complex on the ordinary sphere is obtained by another way in [16]. The differential complex constructed in [14] can be seen as a supersymmetric generalization of the following one.

**Definition 1.** A Poisson algebra $\mathcal{A}$ is an unital $\mathbb{C}$-algebra with a Poisson structure compatible with the product $m: \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$.

$$\{X, YZ\} = \{X, Y\}Z + Y\{X, Z\}, \quad X, Y, Z \in \mathcal{A}$$

$\mathcal{A}$ is equipped with a linear trace

$$\text{Trace}: \mathcal{A} \to \mathbb{C}$$

$$\text{Trace}(e) = 1 \text{ where } e \text{ is the unit of } \mathcal{A},$$

$$\text{Trace}(\{X, Y\}) = 0, \quad X, Y \in \mathcal{A}.$$

**Definition 2.** We say that $(\mathcal{A}, \mathcal{G})$ is a double over a Poisson $\mathbb{C}$-algebra $\mathcal{A}$ if $\mathcal{G}$ is a Lie subalgebra of $\mathcal{A}$ and a bilinear form $\text{Trace} \circ m$ restricted to $\mathcal{G}$ is non-degenerated. In this case the bilinear form $\text{Trace} \circ m$ determines an element $C^G \in \mathcal{G} \otimes \mathcal{G}$ called a quadratic Casimir element of the double $(\mathcal{A}, \mathcal{G})$.

These two definitions allow us to construct a invariant differential complex on $\mathcal{A}$ by the following way: The complex $\Omega(\mathcal{A}, \mathcal{G})$ over the double $(\mathcal{A}, \mathcal{G})$ is defined as follows

$$\Omega(\mathcal{A}, \mathcal{G}) = \bigoplus_{i=0}^{3} \Omega_i(\mathcal{A}, \mathcal{G})$$

where

$$\Omega_0(\mathcal{A}, \mathcal{G}) = \Omega_3(\mathcal{A}, \mathcal{G}) = (\mathcal{A})_0 \equiv e \otimes \mathcal{A}$$

$$\Omega_1(\mathcal{A}, \mathcal{G}) = \Omega_2(\mathcal{A}, \mathcal{G}) = \mathcal{G} \otimes \mathcal{A}$$
We note \( m \) the left regular action and \( ad \) the adjoint action of \( \mathcal{A} \) on itself. We explicit their actions

\[
\begin{align*}
ad(X)Y &= \{X,Y\} \\
m(X)Y &= XY
\end{align*}
\]

Using Sweedler notation, we note formally \( C^G \) as \( C^1_1 \otimes C^2_2 \in \mathcal{G} \otimes \mathcal{G} \). Let us introduce now the coboundary operator

\[
(3.3) \quad \delta^G : \Omega_i(\mathcal{A}, \mathcal{G}) \rightarrow \Omega_{i+1}(\mathcal{A}, \mathcal{G})
\]

which acts explicitly

\[
\begin{align*}
\delta^G(e \otimes X) &= m(C^1_1) e \otimes ad(C^2_2) X, \quad e \otimes X \in \Omega_0(\mathcal{A}, \mathcal{G}) \\
\delta^G(g \otimes X) &= \left( ad(C^1_1) \otimes ad(C^2_2) + \frac{1}{2} d_G \right) (g \otimes X), \quad g \otimes X \in \Omega_1(\mathcal{A}, \mathcal{G}) \\
\delta^G(k \otimes Y) &= e \otimes ad(k) Y, \quad k \otimes Y \in \Omega_2(\mathcal{A}, \mathcal{G}) \\
\delta^G(e \otimes W) &= 0, \quad e \otimes W \in \Omega_3(\mathcal{A}, \mathcal{G}).
\end{align*}
\]

with \( d_G \) the Dynkin index for the trace, which can be defined by

\[
\text{Trace}(XY) = \frac{1}{d_G} \text{Trace}(ad(X) ad(Y))
\]

We define also the associative graded product on this differential algebra which is compatible with \( \delta^G \)

\[
(3.4) \quad *_G : \Omega_i(\mathcal{A}, \mathcal{G}) \otimes \Omega_j(\mathcal{A}, \mathcal{G}) \rightarrow \Omega_{i+j}(\mathcal{A}, \mathcal{G})
\]

The multiplication is given by the following table

\[
\begin{pmatrix}
*G & e \otimes X' & g' \otimes X' & k' \otimes Z' & e \otimes W' \\
e \otimes X & m \otimes m & m \otimes m & m \otimes m & m \otimes m \\
g \otimes X & m \otimes m & ad \otimes m & (\text{Trace} \otimes \text{Id}) (m \otimes m) & 0 \\
k \otimes Z & m \otimes m & (\text{Trace} \otimes \text{Id}) \otimes (m \otimes m) & 0 & 0 \\
e \otimes W & m \otimes m & 0 & 0 & 0
\end{pmatrix}
\]

Finally we define a map, called Hodge triangle, which is the analogue of the Hodge star. It is just the identification between 2-forms and 1-forms, between 0-forms and 3-forms and we denote it \( \lhd \). This presentation is just the application of the one constructed in the supersymmetric case in [14] to the bosonic case. In [16] C. Klimcik showed that this complex applied to \( \mathcal{A} = C^\infty(S^2) \) and \( \mathcal{G} = su(2) \) is isomorphic to another one constructed with the de Rham complex of the 2-sphere [16]. Now we recall it :

\[
(3.5) \quad \omega = \sum_{i=0}^{3} \omega_i
\]

with

\[
(3.6) \quad \omega_0 = \Omega_0 \oplus \{0\}, \quad \omega_1 = \Omega_1 \oplus \Omega_2 \\
\omega_2 = \Omega_2 \oplus \Omega_1, \quad \omega_3 = \{0\} \oplus \Omega_0.
\]

We note \( \Omega_i \), the space of \( i \)-forms in the usual de Rham complex, \( d \) the de Rham differential operator and \( * \) the usual Hodge operator. The coboundary operator on
\( \omega \) is defined as follows
\[
\delta \equiv d \oplus *d \ast \text{ on } \omega_0, \omega_2, \omega_3
\]
\[
\delta (V \oplus v) = (dV + v) \oplus *d \ast v, \quad V \oplus v \in \Omega_1 \oplus \Omega_2
\]

We recall the definition of the Hodge triangle for this complex [16]

\[
\vartriangleleft \phi = \phi, \quad \phi \in \omega_0 \tag{3.8}
\]

\[
\vartriangleleft (V \oplus v) = v \oplus V, \quad V \oplus v \in \omega_1 \tag{3.9}
\]

\[
\vartriangleleft (v \oplus V) = V \oplus v, \quad v \oplus V \in \omega_2
\]

\[
\vartriangleleft \Phi = \Phi, \quad \Phi \in \omega_3 \tag{3.10}
\]

We define the integral of a 3-form by
\[
\text{Int}(\Phi) = \int_{S^2} *\Phi \tag{3.11}
\]

where \( \Phi \) is seen as a 0-form of the de Rham complex. Now we can write an action
\[
S(A) = \text{Int}(\delta A \cdot \vartriangleleft \delta A) + \text{Int}(A \cdot \delta A) \tag{3.12}
\]

where \( A = V \oplus v \in \omega_1 \) and \( A^\dagger = A \). This action is gauge invariant under the gauge transformation
\[
A \rightarrow A + \delta \phi, \quad \phi \in \omega_0. \tag{3.13}
\]

Explicitly, this action is written
\[
S(V, v) = \int_{S^2} dV \wedge *dV + \int_{S^2} d \ast v \wedge *d \ast v \tag{3.14}
\]

and the gauge transformation is written as follows
\[
V \rightarrow V + d\phi, \quad v \rightarrow v.
\]

The second term of the action (3.12) does not violate the gauge invariance and it is useful to separate the fields \( V \) and \( v \) in the action. The first term is the pure electrodynamic plus free massless scalar on \( S^2 \). Its interaction with a scalar matter field \( \Phi \in \omega_0^C \) is described by \([16]\]
\[
S_m(A, \Phi) = \text{Int} \left( (\delta \Phi - iA \Phi) \cdot \vartriangleleft (\delta \Phi - iA \Phi) \right) \tag{3.15}
\]

the bare means ordinary complex conjugation. In terms of fields \( V, v \), the matter action becomes
\[
S_m(\Phi, V, v) = \int (d\Phi - iV \Phi) \wedge *d(\Phi - iV \Phi) + \int (v \wedge *v) \Phi \overline{\Phi} \tag{3.16}
\]

This action is the standard interaction of the complex scalar matter with \( U(1) \) gauge field but the second term is a new one. With the convenient constraint which respects the gauge invariance, we can suppress \( v \). Thus we are able to construct the non commutative version without extra propagating fields.

This complex can be also viewed as a subcomplex of the de Rham complex on \( SU(2) \). Forms in this subcomplex are characterized by their invariance with respect to \( U(1) \) subgroup of \( SU(2) \). So they can be interpreted as objects living on \( S^2 = SU(2)/U(1) \) \([16]\). The \( SU(2) \) covariant formalism for the complex \( \omega \) is exactly our complex \( \Omega(A, G) \) in case of \( A = C^\infty(S^2) \) and \( G = su(2) \).
We note $R_i$, with $i = +, -, 3$, the generators of the Lie algebra $su(2)$ with the following relations

\begin{equation}
[R_+, R_-] = 2R_3, \quad [R_3, R_\pm] = \pm R_\pm
\end{equation}

It is easy to show that in this case

\begin{align}
C^g &= R_+ \otimes R_- + R_- \otimes R_+ + \frac{1}{2} R_3 \otimes R_3 \\
d_\varphi &= -2 \\
\Omega_0 (A, G) &= \Omega_3 (A, G) = e \otimes A = \Omega_0 \\
\Omega_1 (A, G) &= \Omega_2 (A, G) = G \otimes A = \Omega_0 \oplus \Omega_0 \oplus \Omega_0
\end{align}

with $r_i$ the following charges corresponding to vector field $R_i$

\begin{equation}
r_+ = \chi^1 \chi^2 = \frac{z}{1 + \overline{z} z}, \quad r_- = \overline{\chi}^2 \chi^1 = \frac{z}{1 + \overline{z} z}, \quad r_3 = (\chi^1 \chi^2 - \overline{\chi}^2 \chi^1) = \frac{1 - \overline{z} z}{2 (1 + \overline{z} z)}
\end{equation}

We recall that $C^\infty (S^2)$ is the 0-forms in the de Rham complex. Let us to note

\begin{align}
\varphi &= e \otimes \varphi \in \Omega_0 \cong C^\infty (S^2), \\
A_i &= R_i \otimes A_i \in \Omega_1 (A, G) \cong C^\infty (S^2) \oplus C^\infty (S^2) \oplus C^\infty (S^2), \\
a_i &= R_i \otimes a_i \in \Omega_3 (A, G) \cong C^\infty (S^2) \oplus C^\infty (S^2) \oplus C^\infty (S^2), \\
\Phi &= e \otimes \Phi \in \Omega_0, \cong C^\infty (S^2)
\end{align}

So the multiplication becomes explicitly

\begin{equation}
m \otimes m (e \otimes \varphi) (R_i \otimes A_i) = R_i \otimes \varphi A_i \equiv \varphi A_i,
\end{equation}

in this way we obtain the complete table

\begin{align*}
\varphi \ast \varphi &= \varphi \psi, \quad \varphi \ast a_i = \varphi A_i, \quad \varphi, \psi \in \Omega_0, A \in \Omega_1 \\
\varphi \ast b_i &= \varphi b_i, \quad \varphi \ast \Phi = \varphi \Phi, \quad \varphi \in \Omega_0, b \in \Omega_2 \\
A \ast B &= (A_3B_+ - A_+B_3, A_3B_- - A_-B_3, 2 (A_+ B_- - A_- B_+)), \quad A, B \in \Omega_1 \\
A \ast a &= A_+ a_+ + A_- a_- + A_3 a_3, \quad A \in \Omega_1, a \in \Omega_2.
\end{align*}

Now I explicit the coboundary operator

\begin{align*}
\delta^g (e \otimes \varphi) &= (R_-, \varphi, R_+ \varphi, R_3 \varphi) \\
\delta^g (r_i \otimes A_i) &= (-R_- A_3 - R_3 A_+ - A_+, \\
& \quad R_+ A_3 + R_3 A_+ - A_-, \\
& \quad 2 R_- A_+ - 2 R_+ A_+ - A_3) \\
\delta^g (r_i \otimes a_i) &= R_+ A_+ + R_- A_- + R_3 A_3
\end{align*}

The identification between the two descriptions in the case of the 1-forms is

\begin{align}
A_+ &= - i V - i \overline{z}^2 \overline{v} + \frac{\overline{z}}{1 + \overline{z} z} \ast v \\
A_- &= i V + i \overline{z}^2 \overline{v} + \frac{z}{1 + \overline{z} z} \ast v \\
A_3 &= i \overline{z} \overline{v} - i \overline{z} V + \frac{1 - \overline{z} z}{2 (1 + \overline{z} z)} \ast v
\end{align}
where $Vdz + \overline{V} d\overline{z}$ and $v$ are de Rham 1-form and 2-form respectively with $\overline{V}, V$ functions on $S^2$ verify
\[
\int_{S^2} dz d\overline{z} \overline{V} < \infty.
\]

The first integral of the matter action (3.16) is the standard interaction of the complex scalar matter with $U(1)$ gauge field. We impose certain constraint to eliminate the second term [16], using the isomorphism (3.22) the constraint is
\[
(r_+ A_- + r_- A_+ + r_3 A_3 = 0).
\]

Using (3.19) and (3.22) it is easy to show that this constraint eliminate $v$. In the invariant description this constraint is written
\[
C^\mathcal{G} * \mathcal{G} A = 0, \quad A \in \Omega_1 (\mathcal{A}, \mathcal{G})
\]

It is important to note that constraint (3.24) is gauge and $su(2)$ invariant. In this constraint $C^\mathcal{G}$ is viewed as a 2-form. All these constructions are extensible to the fuzzy sphere [20]. Briefly we recall it
\[
(3.25)
\]

The product between forms is defined as in (3.4). The coboundary operator is
\[
\delta (\varphi) = \{ [r_-, \varphi], [r_+, \varphi], [r_3, \varphi] \}
\]
\[
\delta (A) = (- [r_-, A_3] + [r_3, A_+] - A_+, \quad [r_+, A_3] + [r_3, A_-] - A_-, \quad 2r_- A_- - 2r_+ A_+ - A_3)
\]
\[
\delta (a) = [r_+, A_+] + [r_-, A_-] + [r_3, A_3]
\]

with quantized charges of the Hamiltonian vectors $r_i$ which are operators defined as in (3.19). The Int (3.11) is becomes $1 \over N+1 \text{Trace}$ in the noncommutative case. Then the action is written
\[
S_N(A) = {1 \over N+1} \text{Trace} (F \lhd F + A \delta A)
\]

The natural way to consider the non abelian case is to introduce a new gauge transformation law of the 1-fields as follows
\[
(3.30)
\]

To preserve to gauge invariance of the action (3.29), we modify it as follows
\[
(3.31)
\]

The analogue of the constraint (3.23) for this action is
\[
(3.32)
\]

In a invariant description
\[
(3.33)
\]
In the commutative limit, $N \to \infty$, terms $A.A$ and $A.A.A$ vanished and we obtain (3.29). Thus one obtains the noncommutative version of the scalar Maxwell theory on the sphere [16]. Now we can naturally incorporate the Yang-Mills system into this framework. It implies that $A_N$ should be replaced by $A_N = A_N \otimes M_n(\mathbb{C})$. The gauge group $U$ can be viewed as the unitary elements of $A_N \otimes M_n(\mathbb{C})$. Since $A_N = M_N(\mathbb{C})$, we have $U = U_n(\mathbb{C})$. In this formalism the only thing to modify is the coboundary operator, we recall that coboundary operator is defined using the charges $r_i$, therefore the modification concern them. We set
\[
\delta' (\phi \otimes m) = ([r_-, \phi] \otimes m, [r_+ \phi, \phi] \otimes m, [r_3, \phi] \otimes m), \quad \phi \otimes m \in A_N
\]
\[
\delta' (A_i \otimes n_i) = (-[r_-, A_3] \otimes n_3 - [r_3, A_+] \otimes n_+ - A_+ \otimes n_+, [r_+, A_3] \otimes n_3 + [r_3, A_-] \otimes n_- - A_- \otimes n_-,
\]
\[
2[r_-, A_-] \otimes n_- - 2[r_+, A_+] \otimes n_3 - A_3 \otimes n_3). \]
\[
\delta' (a_i \otimes p_i) = [r_+, a_+] \otimes p_+ + [r_-, a_-] \otimes p_- + [r_3, A_3] \otimes p_3.
\]
\[
\delta' (\Phi \otimes q) = 0. \quad \Phi \otimes q \in A_N'
\]
with $(A_i \otimes n_i), (a_i \otimes p_i) \in A_N' \otimes \mathbb{C}^3$. In the same manner, the Casimir element $C^g$ becomes
\[
C^g = C_1^g \otimes C_2^g \otimes \mathbb{I}_n
\]
The gauge invariant analogue of the constraint (3.24) is
\[
C^g * \zeta A + \zeta A * C^g + \frac{1}{N} \zeta A * \zeta A = 0
\]
This previous bosonic work allowed us to understand the way to incorporate abelian and non abelian theories in a same framework. Now we introduce the modified differential complex which will be used in the supersymmetric framework.

3.2. Description of the modified differential complex. Now we will construct a differential complex on the supersphere and the supergauge abelian and non abelian theories on it. This complex is slightly different from the complex constructed in [14] in order to incorporate the non abelian theory on the supersphere. For a general propose, we consider $A$, a $\mathbb{Z}_2$-graded unital $\mathbb{C}$-algebra with a super-Poisson structure and $A' = A \otimes M_n(\mathbb{C})$. The product on $A'$ is
\[
(X \otimes m) \cdot (Y \otimes n) = XY \otimes mn, \quad X \otimes m, Y \otimes n \in A'
\]
Now we define a bilinear map on $A'$ as follows
\[
A' \times A' \rightarrow A'
\]
\[
\{X \otimes m, Y \otimes n\} = \{X, Y\} \otimes mn, \quad X \otimes m, Y \otimes n \in A'
\]
with $\{\ldots\}$ the super Poisson structure on $A$ compatible with the product on $A$. The restriction of this map on the subalgebra $A \equiv A' \otimes \mathbb{I}_n$ is a super Poisson structure compatible with the product. But the map is not a super Poisson on $A \otimes M_n(\mathbb{C})$. Before giving our definitions, let us list those appropriated for the abelian case.

Definition 3 (14). $A$ is a $\mathbb{Z}_2$-graded unital $\mathbb{C}$-algebra with a super-Poisson structure $\{\ldots\}$ compatible with the product and equipped with a linear supertrace
\[
STrace : A \rightarrow \mathbb{C}
\]
\[
STrace(e) = 1 \text{ where } e \text{ is the unit of } A.
\]
\[
STrace(\{X, Y\}) = 0, \quad X, Y \in A
\]
Definition 4 (14). We say that \((A,G)\) is a supersymmetric double over a super-Poisson \(\mathbb{F}\)-algebra \(A\) with \(\mathbb{P}\) a graded commutative algebra, if \(G = G_0 \oplus G_1\) is a super-Lie subalgebra of \(A\) and a bilinear form \(STrace \circ m\) restricted to \(G\) is non-degenerated. In this case the bilinear form \(STrace \circ m\) determines an element \(C^G \in G \otimes G\) called a quadratic Casimir element of the double \((A,G)\).

Example 1. The algebras \(A_{\infty}\) and \(A_N\) with these super-Poisson structures (2.3) (2.13) and Berezin integral or supertrace on the matrices. For \(G\) we take naturally \(G\) imbedded as super-Lie subalgebra on \(A\) via (2.8).

Definition 5 (14). We say that \((A,G,H)\) is a supersymmetric triple, if it exists a subspace \(H\) of \(A\) such that
1) \(H\) is a super-Lie subalgebra of \(G\),
2) \((A,H)\) is the supersymmetric double with the Casimir element \(C^H \in H \otimes H\),
coboundary \(\delta^H\) and product \(\ast_H\),
3) An element \(C \equiv C^G - C^H\) fulfils \(m(C) \in \mathbb{C}e\),
4) \(ad (H^\perp \otimes H^\perp) \subset H\) where \(H^\perp\) is an orthogonal complement of \(H\) in \(G\) with respect to \(STrace \circ m\).

We write \((A)_0 ((A)_1)\) is even (odd) part with respect to the sum of grading of \(A\) and of \(\mathbb{P}\). \(\mathbb{P}\) can be Grassmanian algebras or graded matrix algebras. In the non abelian case any element of \(A \otimes M_n(\mathbb{C})\) is a matrix in which each component is a element \(A\) with respect to previous graduation. We note \(m\) the left regular action and \(ad\) the adjoint action of \(A\) on itself. We have
\[
(3.36) \quad ad(X)Y = (-1)^X \{X,Y\} \\
(3.37) \quad m(X)Y = (-1)^Y XY
\]
\[
(3.38) \quad \widetilde{ad} (X \otimes n) (Y \otimes p) \equiv (-1)^X \{X,Y\} \otimes np \\
(3.39) \quad \widetilde{m} (X \otimes n) (Y \otimes p) \equiv (-1)^Y XY \otimes np
\]
We call modified Casimir the following element which is written formally as
\[
(3.40) \quad \tilde{C}^G \equiv C_1^G \otimes C_2^G \otimes I_n \in G \otimes G \otimes I_n
\]
We note \(d_{\varphi}\), analogue of the Dynkin index for the supertrace, which can be defined by
\[
(3.41) \quad STrace(XY) = \frac{1}{d_{\varphi}} STrace(ad(X) ad(Y))
\]

Proposition 1. The modified complex \(\tilde{\Omega}\) over \(A' = A \otimes M_n(\mathbb{C})\) is defined as follows
\[
\tilde{\Omega}(A',G) = \bigoplus_{i=0}^3 \tilde{\Omega}_i (A',G)
\]
where
\[
\tilde{\Omega}_0 (A',G) = \tilde{\Omega}_3 (A',G) = e \otimes (A)_0 \otimes M_n(\mathbb{C}) \\
\tilde{\Omega}_1 (A',G) = \tilde{\Omega}_2 (A',G) = (G_0 \otimes (A)_0 \otimes M_n(\mathbb{C})) \oplus (G_1 \otimes (A)_1 \otimes M_n(\mathbb{C}))
\]
Let us introduce now the coboundary operator
\[
\delta^\mathcal{G} : \tilde{\Omega}_i (\mathcal{A}', \mathcal{G}) \longrightarrow \tilde{\Omega}_{i+1} (\mathcal{A}', \mathcal{G})
\]
\[
\delta^\mathcal{G} (e \otimes X \otimes n) = (-1)^g m (C^\mathcal{G}_1) e \otimes ad (C^\mathcal{G}_2) X \otimes n,
\]
\[
\delta^\mathcal{G} (g \otimes Y \otimes n) = \text{ad} (C^\mathcal{G}_1) (g) \otimes \text{ad} (C^\mathcal{G}_2) (Y) \otimes n + \frac{1}{2} d^\mathcal{G} (g \otimes Y \otimes n),
\]
\[
\delta^\mathcal{G} (k \otimes Z \otimes n) = (-1)^k e \otimes \text{ad} (k) Z \otimes n,
\]
\[
\delta^\mathcal{G} (e \otimes W \otimes n) = 0,
\]
and the associative product between the forms compatible with \(\delta^\mathcal{G}\)
\[
\ast^\mathcal{G} : \tilde{\Omega}_i (\mathcal{A}', \mathcal{G}) \otimes \tilde{\Omega}_j (\mathcal{A}', \mathcal{G}) \longrightarrow \tilde{\Omega}_{i+j} (\mathcal{A}', \mathcal{G})
\]
is given by the following table
\[
\begin{pmatrix}
\ast^\mathcal{G} & e \otimes X' & g' \otimes X' & k' \otimes Z' & e \otimes W' \\
\tilde{\Omega}_i (\mathcal{A}', \mathcal{G}) & \tilde{\Omega}_j (\mathcal{A}', \mathcal{G}) & \tilde{\Omega}_k (\mathcal{A}', \mathcal{G}) & \tilde{\Omega}_l (\mathcal{A}', \mathcal{G}) & \tilde{\Omega}_m (\mathcal{A}', \mathcal{G})
\end{pmatrix}
\]
\[
\begin{pmatrix}
\ast^\mathcal{G} & e \otimes X' & g' \otimes X' & k' \otimes Z' & e \otimes W' \\
e \otimes X & m \otimes m & m \otimes m & m \otimes m & m \otimes m \\
g \otimes X & m \otimes m & \text{ad} \otimes \tilde{m} & (\text{STrace} \otimes \text{Id}) \otimes (m \otimes \tilde{m}) & 0 \\
k \otimes Z & m \otimes m & (\text{STrace} \otimes \text{Id}) \otimes (m \otimes \tilde{m}) & 0 & 0 \\
e \otimes W & m \otimes m & 0 & 0 & 0
\end{pmatrix}
\]

Remark 1. The complex of Klimcik complex [14] is exactly the previous one in the case \(\mathcal{A}' = \mathcal{A}\). We obtain it from the modified differential complex in a natural way
\[
\tilde{C}^\mathcal{G} = C^\mathcal{G}_1 \otimes C^\mathcal{G}_2 \in \mathcal{G} \otimes \mathcal{G}
\]
and
\[
\Omega (\mathcal{A}, \mathcal{G}) = \bigoplus_{i=0}^3 \Omega_i (\mathcal{A}, \mathcal{G})
\]
where
\[
\begin{align*}
\Omega_0 (\mathcal{A}, \mathcal{G}) &= \Omega_3 (\mathcal{A}, \mathcal{G}) = e \otimes (\mathcal{A})_0 \\
\Omega_1 (\mathcal{A}, \mathcal{G}) &= \Omega_2 (\mathcal{A}, \mathcal{G}) = \mathcal{G}_0 \otimes (\mathcal{A})_0 \oplus \mathcal{G}_1 \otimes (\mathcal{A})_1.
\end{align*}
\]
The coboundary operator is
\[
\delta^\mathcal{G} : \tilde{\Omega}_i (\mathcal{A}, \mathcal{G}) \longrightarrow \tilde{\Omega}_{i+1} (\mathcal{A}, \mathcal{G})
\]
\[
\delta^\mathcal{G} (e \otimes X) = (-1)^g m (C^\mathcal{G}_1) e \otimes ad (C^\mathcal{G}_2) X, \quad e \otimes X \in \Omega_0
\]
\[
\delta^\mathcal{G} (g \otimes X) = \left( \text{ad} (C^\mathcal{G}_1) \otimes \text{ad} (C^\mathcal{G}_2) + \frac{1}{2} d^\mathcal{G} \right) (g \otimes X), \quad g \otimes X \in \Omega_1
\]
\[
\delta^\mathcal{G} (k \otimes Y) = (-1)^k e \otimes \text{ad} (k) Y, \quad k \otimes Y \in \Omega_2
\]
\[
\delta^\mathcal{G} (e \otimes W) = 0, \quad e \otimes W \in \Omega_3
\]
and the associative product between the forms
\[
\ast^\mathcal{G} : \tilde{\Omega}_i (\mathcal{A}, \mathcal{G}) \otimes \tilde{\Omega}_j (\mathcal{A}, \mathcal{G}) \longrightarrow \tilde{\Omega}_{i+j} (\mathcal{A}, \mathcal{G})
\]
compatible with \(\delta^\mathcal{G}\) with The multiplication is given by the same table.
We construct a canonical complex $\Omega (\mathcal{A}, \mathcal{G}, \mathcal{H})$ over $(\mathcal{A}, \mathcal{G}, \mathcal{H})$ as follows [14]

\begin{equation}
\forall i = 0, 1, 2, 3 : \Omega_i (\mathcal{A}, \mathcal{G}, \mathcal{H}) = \Omega_i (\mathcal{A}, \mathcal{G})
\end{equation}

and we define the exterior derivative $\delta$ on $(\mathcal{A}, \mathcal{G}, \mathcal{H})$ as follows

$$\delta = \delta^0 \text{ on } \Omega_i (\mathcal{A}, \mathcal{G}) \text{ for } i = 0, 2, 3$$

$$\delta (g \otimes X + h \otimes Y) = \delta^0 (g \otimes X + h \otimes Y) - \delta^H (g \otimes X + h \otimes Y) \text{ on } \Omega_1 (\mathcal{A}, \mathcal{G}, \mathcal{H})$$

Before acting $\delta^H$ on $g \otimes X + h \otimes Y$, we do the projection this element on $\mathcal{H} \otimes \mathcal{A}$. The product $\ast$ in $\Omega (\mathcal{A}, \mathcal{G}, \mathcal{H})$ is defined by

\begin{align}
(3.45) \ast &= \ast^\mathcal{G} \text{ on } \Omega_i (\mathcal{A}, \mathcal{G}) \text{ for } i = 0, 2, 3, \\
(3.46) \ast &= \ast^\mathcal{G} - \ast^H \text{ on } \Omega_1 (\mathcal{A}, \mathcal{G}, \mathcal{H})
\end{align}

The product $\ast$ and the coboundary $\delta$ verify the Leibniz rule.

**Proposition 2.** We can also construct a modified complex $\tilde{\Omega} (\mathcal{A}', \mathcal{G}, \mathcal{H})$ on $\mathcal{A}'$ in the following way

$$\tilde{\delta} = \tilde{\delta}^\mathcal{G} \text{ on } \tilde{\Omega}_i (\mathcal{A}', \mathcal{G}) \text{ for } i = 0, 2, 3$$

$$\tilde{\delta} = \tilde{\delta}^\mathcal{G} - \tilde{\delta}^H \text{ on } \tilde{\Omega}_1 (\mathcal{A}', \mathcal{G}, \mathcal{H})$$

Before acting $\tilde{\delta}^H$ on an 1-form, we do the projection of this element on $\mathcal{H} \otimes \mathcal{A}'$. The product $\ast$ in $\tilde{\Omega} (\mathcal{A}', \mathcal{G}, \mathcal{H})$ is defined by

\begin{align}
(3.47) \tilde{\ast} &= \tilde{\ast}^\mathcal{G} \text{ on } \tilde{\Omega}_i (\mathcal{A}', \mathcal{G}) \text{ for } i = 0, 2, 3, \\
\tilde{\ast} &= \tilde{\ast}^\mathcal{G} - \tilde{\ast}^H \text{ on } \tilde{\Omega}_1 (\mathcal{A}', \mathcal{G}, \mathcal{H})
\end{align}

**Proof.** To prove the previous two propositions, it is sufficient to prove it in the case $\mathcal{A}' = \mathcal{A}$ with $\mathcal{A}$ is as in the definitions (3)(4). To illustrate this idea, we compute $\tilde{\delta}^\mathcal{G} \circ \tilde{\delta}^\mathcal{G} (g \otimes X \otimes m)$ with $g \otimes X \otimes m \in \tilde{\Omega}_1 (\mathcal{A}', \mathcal{G})$,

$$\tilde{\delta}^\mathcal{G} \circ \tilde{\delta}^\mathcal{G} (g \otimes X \otimes m) = \delta^\mathcal{G} \left( \{ C_1^g, g \} \otimes \{ C_2^g, X \} \otimes m + \frac{1}{2} d_{\mathcal{G}} (g \otimes X \otimes m) \right)$$

$$= (-1)^g \left( \{ C_1^g, g \} \otimes \{ C_2^g, X \} \right) \otimes m$$

$$+ \frac{1}{2} (-1)^g d_{\mathcal{G}} \otimes \{ g, X \} \otimes m$$

$$= \left[ (-1)^g \{ C_1^g, g \} \otimes \{ C_2^g, X \} \right] + \frac{1}{2} (-1)^g d_{\mathcal{G}} \otimes \{ g, X \} \otimes m$$

It is clear that is equivalent to prove the nilpotency of $\tilde{\delta}^\mathcal{G}$ in the abelian case. We use the same trick to prove the other assertions.

**Corollary 1.** There is a simple relation between the complex in the abelian case and the non abelian case

$$\tilde{\delta}^\mathcal{G} (e \otimes X \otimes m) = \delta^\mathcal{G} (e \otimes X) \otimes m, \quad e \otimes X \otimes m \in \Omega_0,$$

$$\tilde{\delta}^\mathcal{G} (g \otimes X \otimes m) = \delta^\mathcal{G} (g \otimes X) \otimes m, \quad g \otimes X \otimes m \in \Omega_1,$$

$$\tilde{\delta}^\mathcal{G} (k \otimes Y \otimes n) = \delta^\mathcal{G} (k \otimes Y) \otimes n, \quad k \otimes Y \otimes n \in \Omega_2.$$
and the product on 1-forms can be written
\[ (e \otimes X \otimes m) \tilde{\ast}_G (e \otimes Y \otimes n) = e \otimes XY \otimes mn \]
\[ = (e \otimes X) \ast_G (e \otimes Y) \otimes mn \]
in the same way we can formally we note \( \tilde{\ast}_G \equiv \ast_G \otimes \times \), where \( \times \) is the matrix product.

3.3. Modified complex on supersphere and on fuzzy supersphere. We consider the previous complex on the fuzzy supersphere \( A_N \) (for \( N = \{1,2, \ldots, \infty\} \) for a particular choice of \( G = sl(2,1) \), \( H = ops(2,1) \). Recall the \( sl(2,1) \) is generated by \( R_\pm, R_3, \Gamma, V_\pm, D_\pm \) and \( osp(2,1) \) by \( R_\pm, R_3, V_\pm \). Thus we have

1) Abelian case
\[ \Omega_0 = \Omega_3 = A_N \]
\[ \Omega_1 = \Omega_2 = \bigoplus_{i=0}^{8} (A_N)_i \]

2) Non abelian
\[ \tilde{\Omega}_0 = \tilde{\Omega}_3 = A_N \otimes M_n(\mathbb{C}) \]
\[ \tilde{\Omega}_1 = \tilde{\Omega}_2 = \bigoplus_{i=0}^{8} (A_N \otimes M_n(\mathbb{C}))_i \]

In details, all the forms are written as follows using a basis of the \( G \) :
\[ \phi \]
\[ r_+ \otimes C_+ + r_- \otimes C_- + r_3 \otimes C_3 + \gamma \otimes W, \]
\[ + v_+ \otimes A_+ + v_- \otimes A_- + d_+ \otimes B_+ + d_- \otimes B_- \]
\[ = (A_+, A_-, B_+, B_-, C_+, C_-, C_3, W) \]
in the same way
\[ 2\text{-form} \equiv (a_+, a_-, b_+, b_-, c_+, c_-, c_3, w) \]

All these elements are in \( A_N \) or in \( A_N \otimes M_n(\mathbb{C}) \). In the second case \( A_+ = A^i_+ E_i \) with \( E_i \) a basis of \( M_n(\mathbb{C}) \). And the Casimir element \( C^G - C^H \) in this basis is
\[ C = 2d_- \otimes d_+ - 2d_+ \otimes d_- - \frac{1}{2} \gamma \otimes \gamma \]
or
\[ C = 2d_- \otimes d_+ \otimes I_n - 2d_+ \otimes d_- \otimes I_n - \frac{1}{2} \gamma \otimes \gamma \otimes I_n \]

In first we have to explicit the product between forms
\[ \phi_1 \ast \phi_2 = \phi \]
\[ \phi_\ast (A_\pm, W, C_i, B_\pm) = (\phi A_\pm, \phi W, \phi C_i, \phi B_\pm) \]
\[ \phi_\ast (a_\pm, w, c_i, b_\pm) = (\phi a_\pm, \phi w, \phi c_i, \phi b_\pm) \]
\[ \phi_\ast \Phi = \phi \Phi \]
\[ (A_\pm, W, C_i, B_\pm) \tilde{\ast} \psi = (\psi A_\pm, \psi W, \psi C_i, \psi B_\pm) \]
\[ (A^1_\pm, W^1, C^1_i, B^1_\pm) \tilde{\ast} (A^2_\pm, W^2, C^2_i, B^2_\pm) = (a'_\pm, w', c'_i, b'_\pm) \]
The product between 1-forms and 2-forms is written

\[(a'_+, w', c'_i, b'_\pm) = (W^1 B^2_+ - B^1_+ W^2 - 2C^1_+ A^2_+ + 2A^1_+ C^2_+ - 2C^1_+ A^2_+ + 2A^1_+ C^3_+, W^1 B^2_+ - B^1_+ W^2 - 2C^1_+ A^2_+ + 2A^1_+ C^2_+ - 2C^1_+ A^2_+ + 2A^1_+ C^3_+),\]

\[-4B^1_+ A^2_+ + 4B^1_+ A^2_+ - 4A^1_+ B^2_+ + 4A^1_+ B^2_+ - 4A^1_+ A^2_+, 2A^1_+ A^2_+ + 2A^1_+ A^2_+, 4A^1_+ A^2_+ - A^1_+ W^2, W^1 A^2_+ - A^1_+ W^2);\]

The product between 1-forms and 2-forms is written

\[(A_{\pm}, W, C_i, B_{\pm}) \equiv (a_{\pm}, w, c_i, b_{\pm}) = A_{\pm} a_{\pm} - A_{\pm} a_{\pm} + \frac{1}{4} W w - \frac{1}{2} C_+ c_- - \frac{1}{2} C_- c_+ - C_3 c_3 - B_+ b_- + B_- b_+\]

All other products vanish. Now we are ready to explicit the coboundary

\[\delta \phi = (D_\pm \phi, \Gamma \phi, R_+, R_- \phi, R_3 \phi, V_\pm \phi)\]

\[\delta (A_{\pm}, W, C_i, B_{\pm}) = (\Gamma B_{\pm} - V_{\pm} W - 2R_{\pm} A_{\pm} + 2D_{\pm} C_+ \pm 2R_{\pm} A_{\pm} \mp 2D_{\pm} C_+ + 2A_{\pm} A_{\pm} - 4V_{\pm} A_{\pm} - 4V_{\pm} A_{\pm} + 4D_{\pm} B_{\pm} - 4D_{\pm} B_{\pm} + 2W, -4D_{\pm} A_{\pm} - C_+ , -C_3 + 2D_{\pm} A_{\pm} + 2D_{\pm} A_{\pm}, 4D_{\pm} A_{\pm} - C_-, -B_{\pm} - D_{\pm} W + \Gamma A_{\pm});\]

\[\delta (a_{\pm}, w, c_i, b_{\pm}) = D_+ a_- - D_- a_+ + \frac{1}{4} \Gamma w - \frac{1}{2} R_+ c_- - \frac{1}{2} R_- c_+ - R_3 c_3 - V_+ b_- + V_- b_+ .\]

For example $\Gamma B_{\pm}$ means $(\Gamma B_{\pm}) E_i$ The action of the operators $R_i, V_\pm, D_{\pm}, \Gamma$ is given in (2.7)(2.14) and $\delta$ is $osp(2, 1)$ invariant [14].

We say that the 1-form $V = (A_{\pm}, W, C_i, B_{\pm})$ satisfies the reality condition $V^* = V$ when we have

\[(3.58) \quad A^*_{\pm} = A_{\pm}, \quad A^*_{\pm} = -A_{\pm}, \quad B^*_{\pm} = -B_{\pm}, \quad B^*_{\pm} = B_{\pm} \]

\[C^*_{\pm} = C_{\pm}, \quad C^*_{\pm} = C_{\pm}, \quad C^*_{\pm} = C_{\pm}, \quad W^* = W.\]

in the non abelian case, the reality condition means we consider only 1-forms $\bar{V} = V \otimes h$ with $V = V^*$ and $h$ a hermitian $n \times n$ matrix.

4. Fields theories

4.1. The noncommutative pure gauge and supersymmetric fields over $A_N$. The noncommutative pure supersymmetric electrodynamics$^2$ (respectively Yang-Mills) over $A_N$ (resp. $A'_N$) is a theory of 1-forms in the complex $\Omega (A_N, \mathcal{G}, \mathcal{H})$

---

$^2$This case is studied in [14].
\[
\left(\text{resp. } \tilde{\Omega}(A_N, G, H)\right) \text{ defined by an action}
\]
\[
(4.1) \quad S(V) = \frac{1}{g^2} \text{Trace} \left[ \text{STrace} \left[ \alpha \preceq F^* F + \beta \left( V^\dagger \delta V + \frac{2}{3} V^\dagger V^2 V \right) \right] \right].
\]
where \( \text{trace} \) is the usual trace on the matrices which is used in the non abelian case, \( F = \delta V + V^* V \) is the field strength of \( V \), \( \alpha, \beta \) are parameters, \( g \) a coupling constant and the Hodge triangle \( \preceq \) is the identification map between \( \tilde{\Omega}_1(A_N, G, H) \) and \( \tilde{\Omega}_2(A_N, G, H) \). The connection \( V \) is real 1-form \( V^* = V \), verifying
\[
(4.2) \quad \left( \delta V + V^* V \right)_H = 0,
\]
for the field theoretical application we need moreover constraint
\[
(4.3) \quad C^* V_H + V_H^* C + \frac{1}{N} \left( \langle V_H^* \rangle_N \right) = 0.
\]
Constraint (4.2) implies that the theory contains only 1-forms only \( V_H^* \) constrained moreover by (4.3). We can write the interaction with matter as follows \[11\].
\[
(4.4) \quad S_{\text{matter}}(V) = \text{STrace} \left[ \left( \tilde{\delta} G - \tilde{\delta} H + V_H^* \right) \Phi^* \tilde{\delta} G - \tilde{\delta} H + V_H^* \Phi \right]
\]
\[S_{\text{matter}}(V) + S(V)\] gives the \( H \)-supersymmetric \[3\] Schwinger model (resp. Yang-Mills theory) over \( A_N \) (resp. \( A'_N \)). This action (4.1) and constraints (4.2)(4.3) are invariant by gauge transformation
\[
(4.5) \quad V \mapsto UVU^{-1} - \tilde{\delta} UU^{-1}, \quad U \in U(A_N \otimes M_n(\mathbb{C})).
\]
and by action of \( H \). For the details of the \( H \)-action see \[14\].

In the abelian case, by the non commutativity of the algebra \( A_N \) we have the term \( V^* V \) but in the commutative case this term disappears. In the non commutative case the operator \( \delta \) commutes only with elements of the form \( U = \exp(it\rho)e \) where \( e \) is the unit of \( A_N \). Thus the action (4.1) is the noncommutative deformation of an \( U(1) \) gauge theory.

In the non abelian case, using corollary (1) it is easy to show that \( \delta \) commute with elements of \( U(A_N \otimes M_n(\mathbb{C})) \). Thus the action (4.1) is also the noncommutative deformation of an \( U(n) \) gauge theory. Now we are going to study commutative limit \( N \to \infty \) of (4.1) in the two cases

4.1.1. Commutative abelian case. In the case, we have a pure gauge field action with \( V = (A_\pm, W, C_\pm, C_i, B_\pm) \) satisfying (3.58).
\[
(4.6) \quad S_\infty(V) = I \left[ \alpha' \delta V \star \delta V + \beta' V \star \delta V \right],
\]
the constraint (4.2) becomes
\[
(4.7) \quad V_H^* = (A_+, A_-, W, 0, 0, 0, 0, 0)
\]
and (4.3) becomes
\[
(4.8) \quad d_+ A_- - d_- A_+ + \frac{1}{4} \gamma W = 0
\]
It follows
\[
(4.9) \quad F = \delta V = (F_+, F_-, f, 0, 0, 0, 0, 0) \in \bigoplus_{i=0}^8 (A_\infty),
\]
\(^3\text{invariant with respect to } H\)-action.
where $\alpha', \beta'$ are real parameters. The constraints (4.8) gives the following constraints on the "additional" superfields $C_\pm, C_i, B_\pm$

(4.10) \[ C_\pm = \mp D_\pm A_\pm, \quad C_3 = 2D_- A_+ + 2D_+ A_- \quad \text{with a new parametrization} \]

(4.11) \[ A_+ = \frac{1}{2} (A - \bar{\sigma} A), \quad A_- = -\frac{1}{2} (zA + \bar{A}), \quad W = \bar{b}A - bA \]

A long calculation gives us the following result obtained in [14]

**Lemma 1.** We have

(4.12) \[ F_+ = -\frac{3}{2} [\bar{\sigma}D(n\omega) + D(n\omega)] - 4d_+ n\omega \]

(4.13) \[ F_+ = \frac{3}{2} [zD(n\omega) - \bar{D}(\omega)] - 4d_+ n\omega \]

(4.14) \[ f = 3 [bD(n\omega) + bD(n\omega)] - 4\gamma n\omega \]

with $n = (1 + \bar{z}z + \bar{b}b)$, $\omega = \bar{D}A + D\bar{A}$.

The action (4.5) becomes

(4.15) \[ S_\infty (V) = \frac{1}{2\pi i} \int d\bar{z}dz db \left\{ \alpha \bar{D}(n\omega)D(n\omega) + \beta n\omega^2 \right\} . \]

the parameters $\alpha, \beta$ are linear combinations of $\alpha', \beta'$. This action is $osp(2,1)$ supersymmetric.

The gauge symmetry $A \rightarrow A + iDA, \bar{A} \rightarrow \bar{A} + i\bar{D}A$ gives the following expressions for $A, \bar{A}$ by gauge fixation which eliminates some components

(4.16) \[ iA = bv + \frac{1}{2}\bar{b} + \frac{iu}{1 + \bar{z}z} + b\eta \left( \frac{\eta}{1 + \bar{z}z} \right) \]

(4.17) \[ i\bar{A} = \frac{1}{2}\bar{b} + \frac{iu}{1 + \bar{z}z} + \bar{b}\eta + \bar{b}\eta \left( \frac{\eta^*}{1 + \bar{z}z} \right) \]

with $u$ real, $v, \bar{v}$ mutually complex conjugate and $\eta^* = \bar{\eta}$. We obtain

(4.18) \[ n\omega = iv + b\eta - \bar{b}\eta + \bar{b}(1 + \bar{z}z)(\partial\bar{v} - \partial_z \bar{v}) + \frac{iu}{1 + \bar{z}z} \]

To finish we obtain by taking $\alpha = -\beta$

(4.19) \[ S_\infty (V) = -\frac{\alpha}{2\pi i} \int d\bar{z}dz \left\{ - \left( 1 + \bar{z}z \right)^2 (\partial \bar{v} - \partial_z \bar{v})^2 + \partial \sigma u \partial_z u + \frac{u^2}{(1 + \bar{z}z)^2} + \gamma \partial \eta + \eta \partial_z \bar{\eta} + \frac{2\eta}{1 + \bar{z}z} \right\} . \]

Hence we conclude that the commutative limit of the (4.1) is indeed standard supersymmetric Schwinger model on the ordinary sphere.

4.1.2. **Commutative non abelian case.** We consider the commutative limit of the action (4.1) and we obtain the pure non abelian gauge field with $V$ satisfies the reality condition.

(4.19) \[ S_\infty (V) = \frac{1}{g^2} \int \text{Trace} \left[ \alpha' \phi F \ast F + \beta' \left( V \ast \delta V + \frac{2}{3} V \ast V \ast V \right) \right] \]
with

\[ F = \delta V + V^2 = (F_+, F_-, f, 0, 0, 0, 0, 0). \]

Using (4.26) We shall go to give explicitly all the components of \( F \). Let us set

\[ V = (A_\pm, W, C_i, B_{\pm}) \]

constrained by (4.7) and (4.8). We have

\[ \delta V = (\Gamma B_{\pm} - V_{\pm} W - 2R_{\pm} A_+ + 2D_{\pm} C_{\pm} \mp 2R_3 A_\pm \pm 2D_{\pm} C_3 \\
+ 2A_{\pm}, 4V_+ A_+ - 4V_- A_+ + 4D_- B_+ - 4D_+ B_- + 2W, \\
- 4D_+ A_+ - C_+, \\
- 4D_+ A_+ - 2D_+ A_+, \\
4D_- A_+ - C_-, \\
- B_\pm - D_{\pm} W + \Gamma A_{\pm}); \]

and

\[ V^2 = ([W, B_\pm] + 2 [A_-, C_+] + 2 [A_+, C_3], \\
[W, B_-] + 2 [A_+, C_-] + 2 [A_-, C_3], \\
- 4 \{B_+, A_-\} + 4 \{A_+, B_-\}, \\
- 4A_+ A_+ + 2A_- A_+ + 2A_+ A_-, 4A_- A_- \\
[W, A_+], [W, A_-]); \]

In components

\[ F_{\pm} = (\Gamma B_{\pm} - V_{\pm} W - 2R_{\pm} A_+ + 2D_{\pm} C_{\pm} \mp 2R_3 A_\pm \pm 2D_{\pm} C_3 + 2A_{\pm} \\
+ [W, B_{\pm}] + 2 [A_+, C_{\pm}] + 2 [A_{\pm}, C_3], \\
4V_+ A_+ - 4V_- A_+ + 4D_- B_+ - 4D_+ B_- + 2W + 4 [B_-, A_+] + 4 [A_+, B_-], \\
- 4D_+ A_+ - C_+ - 4A_+ A_+, \\
- 4D_- A_+ - 4A_A_+ + 2A_- A_+ + 2A_+ A_-, \\
4D_- A_+ - C_+ + 4A_- A_+, \\
- B_\pm - D_{\pm} W + \Gamma A_{\pm} + [W, A_{\pm}]); \]

The constraint (4.9) implies the following constraints on the ”extra” super fields \( C_i, B_{\pm} \)

\[ C_{\pm} = - 4 (D_{\pm} A_{\pm} + A_{\pm} A_{\pm}) \]
\[ C_3 = + 2D_- A_+ + 2D_+ A_- + 2A_- A_+ + 2A_+ A_- \]
\[ B_{\pm} = - D_{\pm} W + \Gamma A_{\pm} \]

using (4.10), \( \delta V \) and \( V^2 \) become
\[ \delta V = (\Gamma (-D_\pm W + \Gamma A_{\pm}) - V_\pm W - 2R_\pm A_\mp \mp \mp D_\mp (D_\mp A_\mp + A_\pm A_\pm)) \mp 2R_\mp A_\pm \] 
\[ \pm 2D_\pm (2D_- A_+ + 2D_+ A_- + 2 (A_- A_+ + A_+ A_-)) + 2 A_\pm, \] 
\[ 4V_+ A_- - 4V_- A_+ + 4D_- (-D_+ W + \Gamma A_+) - 4D_+ (-D_- W + \Gamma A_-) + 2W, \] 
\[ + 4A_+ A_+, \] 
\[ - 2 (A_- A_+ + A_+ A_-), \] 
\[ - 4A_- A_-, 0, 0) \] 

and

\[ V^2 = ([W, (-D_\pm W + \Gamma A_{\pm})] + 2 [A_+, (-4 (D_\pm A_{\pm} + A_\pm A_\pm))]) \] 
\[ + 2 [A_\pm, 2D_- A_+ + 2D_+ A_- + 2 (A_- A_+ + A_+ A_-)], \] 
\[ - 4 \{(-D_+ W + \Gamma A_+), A_-\} + 4 \{A_+, (-D_- W + \Gamma A_-)\}, \] 
\[ - 4A_+ A_+, 2A_- A_+ + 2A_+ A_-, 4A_- A_-, \] 
\[ 0, 0): \] 

Finally, we have

\[ F = (\Gamma (-D_\pm W + \Gamma A_{\pm}) - V_\pm W - 2R_\pm A_\mp \mp \mp D_\mp (D_\mp A_\mp + A_\pm A_\pm)) \mp 2R_\mp A_\pm \] 
\[ \pm 2D_\pm (2D_- A_+ + 2D_+ A_- + 2 (A_- A_+ + A_+ A_-)) + 2 A_\pm \] 
\[ + ([W, (-D_\pm W + \Gamma A_{\pm})] + 2 [A_+, (-4 (D_\pm A_{\pm} + A_\pm A_\pm))]) \] 
\[ + 2 [A_\pm, 2D_- A_+ + 2D_+ A_- + 2 (A_- A_+ + A_+ A_-)], \] 
\[ 4V_+ A_- - 4V_- A_+ + 4D_- (-D_+ W + \Gamma A_+) - 4D_+ (-D_- W + \Gamma A_-) + 2W \] 
\[ - 4 \{(-D_+ W + \Gamma A_+), A_-\} + 4 \{A_+, (-D_- W + \Gamma A_-)\} + 0, 0, 0, 0) \] 

with

\[ F_\pm = (\Gamma^2 + 2) A_\pm - (\Gamma D_\pm + V_\pm) W \mp 12D_\mp D_\pm A_\mp \mp \mp 12D_\pm D_\pm A_\mp \] 
\[ \mp 4D_\pm (A_- A_+ + A_+ A_-) \mp 8D_\pm (A_\pm A_\pm) \] 
\[ \mp 8 [A_+, (D_\pm A_{\pm})] \mp 4 [A_\pm, D_- A_+ + D_+ A_-] \] 

\[ f = 2W + 4 (D_+ D_- - D_- D_+) W + 4 (V_+ - D_+ \Gamma) A_- - 4 (V_- - D_- \Gamma) A_+ \] 
\[ + 4 [D_+ W - \Gamma A_+, A_-] + 4 [A_+, D_- W - \Gamma A_-] \] 

using the parametrization (4.11), we obtain

\[ F_+ = - \frac{3}{2} [\mathbb{D} (n \omega) + D (n \omega)] - 4d_+ n \omega \] 
\[ - \frac{1}{2} (1 + z \mathbb{D}) D (A \mathbb{A} + \mathbb{A} A) - \frac{1}{2} (1 + z \mathbb{D}) \mathbb{D} (A \mathbb{A} + \mathbb{A} A) \] 
\[ + \frac{3}{2} (1 + z \mathbb{D}) \mathbb{D} A^2 + \frac{3}{2} (1 + z \mathbb{D}) D \mathbb{A} \mathbb{A} \] 
\[ \mathbb{D} (1 + z \mathbb{D}) [A, D \mathbb{A} - \frac{1}{2} (1 + z \mathbb{D}) [A, D \mathbb{A}] \] 
\[ + (1 + z \mathbb{D}) [\mathbb{A}, D \mathbb{A}] - \frac{1}{2} (1 + z \mathbb{D}) [\mathbb{A}, D \mathbb{A}] , \]
and
\[
F_- = \frac{3}{2} \left[ z D (n \omega) - \overline{D} (\omega) \right] - 4d_- n \omega \\
- \frac{1}{2} (1 + z \overline{\tau}) \overline{D} (A \overline{A} + \overline{A} A) + \frac{1}{2} z (1 + z \overline{\tau}) D (A \overline{A} + \overline{A} A) \\
- \frac{3}{2} (1 + z \overline{\tau}) z \overline{D} A^2 + \frac{3}{2} (1 + z \overline{\tau}) D \overline{A}^2 \\
+ (1 + z \overline{\tau}) [A, \overline{D} A] + \frac{1}{2} z (1 + z \overline{\tau}) [A, D \overline{A}] \\
- \frac{1}{2} (1 + z \overline{\tau}) [\overline{A}, D A] - (1 + z \overline{\tau}) z [\overline{A}, DA],
\]
and
\[
f = -2 (1 + z \overline{\tau}) [\overline{A}, A]_+.
\]
Thus the action (4.19) is written as

**Proposition 3.**

\[
S = \int \alpha d \bar{D} (n \bar{\omega}) D (n \bar{\omega}) + [A, n \bar{\omega}] + \beta n \bar{\omega} \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) 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Lemma 2. We found the variation of the following terms under $\text{osp}(2,1)$ infinitesimal action

\begin{align}
\Delta(n\omega) &= eV(n\omega) \\
\Delta(\{A,\overline{A}\}) &= eV(\{A,\overline{A}\}) \\
\Delta([A,n\tilde{\omega}]) &= eV([A,n\tilde{\omega}]) + \frac{1}{2}e^{-b}[A,n\tilde{\omega}] \\
\Delta([A,n\tilde{\omega}]) &= eV([A,n\tilde{\omega}]) - \frac{1}{2}e^{-b}[A,n\tilde{\omega}] \\
\Delta([\overline{A},n\tilde{\omega}]) &= eV([\overline{A},n\tilde{\omega}]) - \frac{1}{2}e^{-b}[\overline{A},n\tilde{\omega}] \\
\end{align}

and we obtain

\begin{align}
\Delta S &= \text{Trace} \int \frac{dxdyb}{n} eV \left( \{\alpha (D(n\tilde{\omega}) + [\overline{A},n\tilde{\omega}]) (D(n\tilde{\omega}) + [A,n\tilde{\omega}]) + \beta n\tilde{\omega}^2\} \right)
\end{align}

Lemma 3. The following property holds in this framework

\begin{align}
\int \frac{dxdyb}{n} eV(f) = 0
\end{align}

with $f \in \mathcal{A}_\infty$.

Proof. Easy computation

The supergauge symmetry $A \rightarrow A + iDA + i[A,\Lambda]$, $\overline{A} \rightarrow \overline{A} + iD\overline{A} + i[\overline{A},\Lambda]$ with $\Lambda \in \mathcal{A}_\infty$, is also evident. We recall the fields $A,\overline{A}$ are odd elements of the algebras $\mathcal{A}_\infty$ with values in an arbitrary Lie algebra ($u(n)$ or $su(n)$). It means that $A = A_i T^i$ with $T^i$ generators of the Lie algebras and $A_i$ elements of the functions algebra on the supersphere. After the gauge fixation which allows us to eliminate same components, we have

\begin{align}
iA &= bv + \frac{i}{2}b\eta \frac{\eta}{1 + \bar{z}z} + \frac{1}{2}b\frac{\eta}{1 + \bar{z}z} \\
i\overline{A} &= \frac{1}{2}b\eta \frac{\eta}{1 + \bar{z}z} + \frac{1}{2}b\eta \frac{\eta}{1 + \bar{z}z}
\end{align}

which allow us to compute explicitly the action

\begin{align}
S &= \frac{1}{2\pi i} \int \text{Trace}[-\alpha (\eta \partial_z \overline{\eta} + \partial_z u \partial_z u + \eta \partial_z \eta) + \alpha \left( (1 + \bar{z}z)^2 (i [v,\overline{\eta}] + (\partial_\overline{\eta} - \partial_z \overline{\eta}))^2 \right) \\
&\quad + i\alpha \left( \overline{\eta} \eta \right) + \frac{1}{2}i\bar{z}z u [\eta,\eta] + 2\beta \frac{u^2}{(1 + \bar{z}z)^2} + 2\beta \frac{\eta \overline{\eta}}{1 + \bar{z}z} \\
&\quad + 2\alpha i u [\eta,\eta] + (\partial_\overline{\eta} - \partial_z \overline{\eta}) + 2\beta i u [\eta,\eta] + (\partial_\overline{\eta} - \partial_z \overline{\eta})]
\end{align}

Certain terms of this action merit some explanations, we note the component of the fields in an explicit way

\begin{align}
v &= v_i T^i, \quad u = u_j T^j, \quad \overline{v} = \overline{v}_h T^h \quad \eta = \eta_k T^k, \quad \overline{\eta} = \overline{\eta}_g T^g
\end{align}

and the product is defined as follows
uv = u_j v_i T^i T^j

where $u_j v_i$ is product in the function algebra on the supersphere and $T^i T^j$ is product in the enveloping algebra of the Lie algebra. For example

$$\{\eta, \eta\} = \{\eta_k T^k, \eta_g T^g\}
= \eta_k T^k \eta_g T^g + \eta_g T^g \eta_k T^k
= \eta_k \eta_g [T^k, T^g]
= \eta_k \eta_g C^i_{kg} T^i$$

with $C^i_{kg}$ constants structure of the Lie algebra. Now we explicit the product of the type $\{\eta, \eta\} v$

$$v \{\eta, \eta\} = v_j \eta_k \eta_g C^i_{kg} T^j T^i$$

(4.36)

It is an element on the enveloping algebra. The parameters $\alpha, \beta$ are arbitrary and choosing $\alpha = -\beta$, we obtain the Yang-Mills action on the ordinary sphere with some extras mass terms as in the abelian case [14]. The action is very close to the standard Yang-Mills in the flat euclidean space.

$$S = -\frac{\alpha}{2\pi i} \int d\bar{z} dz \text{Trace} \{-(1 + \bar{z} z)^2 (i [v, \bar{v}] + (\partial z v - \partial \bar{z} \bar{v}))^2 + \partial z u \partial \bar{z} u
+ \frac{u^2}{(1 + \bar{z} z)^2} + \eta_i \partial \bar{z} \eta_i + 2 \eta_i \eta_i \frac{1}{1 + \bar{z} z}
- iv \{\eta, \eta\} + iv \{\eta, \eta\} + \frac{1}{1 + \bar{z} z} u \{\eta, \eta\} \}.$$  

In component the action (4.46), with the choice $\text{Trace} (T^i T^j) = \delta_{ij}$, becomes

$$S = -\frac{\alpha}{2\pi i} \int d\bar{z} dz \{-(1 + \bar{z} z)^2 (i C^k_{ij} v_i \bar{v}_j + (\partial z v_k - \partial \bar{z} \bar{v}_k))^2
+ \frac{u^2}{(1 + \bar{z} z)^2} + \eta_i \partial \bar{z} \eta_i + \eta_i \partial \eta_i + 2 \eta_i \eta_i \frac{1}{1 + \bar{z} z}
- iC^k_{ij} v_j \eta_k \bar{v}_j - iC^l_{mn} v_m \eta_n + \frac{1}{1 + \bar{z} z} C^s_{lx} u_s \eta_s \}.$$  

5. CONCLUSIONS

We have constructed the supersymmetric electrodynamics and Yang-Mills theories on the noncommutative sphere using a modified differential complex: These theories possess only finite number of degrees of freedom. They are respectively supersymmetric and supergauge invariant such that these commutative limits become the supersymmetric Schwinger model and supersymmetric Yang-Mills on the ordinary sphere.

This is a new step towards the understanding of the role of the noncommutative geometry in the nonperturbative regularization of QFT. The supersymmetry approach allows us to consider scalar fields, gauge fields and spinors fields in a canonical set-up and the supersymmetric and supergauge invariance single the good constraints which give us the correct theory.
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