Eta Invariants for Even Dimensional Manifolds

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Abstract. In previous work, we introduced eta invariants for even dimensional manifolds. It plays the same role as the eta invariant of Atiyah-Patodi-Singer, which is for odd dimensional manifolds. It is associated to $K^1$ representatives on even dimensional manifolds, and is defined on a finite cylinder, rather than on the manifold itself. Thus it is an interesting question to find an intrinsic spectral interpretation of this new invariant. Using adiabatic limit technique, we give such an intrinsic interpretation.

1. Introduction

The $\eta$-invariant is introduced by Atiyah-Patodi-Singer in their seminal series of papers [APS1, APS2, APS3] as the correction term from the boundary for the index formula on a manifold with boundary. It is a spectral invariant associated to the natural geometric operator on the boundary and it vanishes for even dimensional manifolds (in this case the corresponding manifold with boundary will have odd dimension). In our previous work with Weiping Zhang [DZ1], we introduced an invariant of eta type for even dimensional manifolds. It plays the same role as the eta invariant of Atiyah-Patodi-Singer.

Any elliptic differential operator on an odd dimensional closed manifold will have index zero. In this case, the appropriate index to consider is that of Toeplitz operators. This also fits perfectly with the interpretation of the index of Dirac operator on even dimensional manifolds as a pairing between the even $K$-group and $K$-homology. Thus in the odd dimensional case one considers the odd $K$-group and odd $K$-homology. For a closed manifold $M$, an element of $K^{-1}(M)$ can be represented by a differentiable map from $M$ into the unitary group

$$g : M \rightarrow U(N, \mathbb{C})$$

where $N$ is a positive integer. As we mentioned the appropriate index pairing between the odd $K$-group and $K$-homology is given by that of the Toeplitz operator, defined as follows.

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Consider \( L^2(S(TM) \otimes E) \), the space of \( L^2 \) spinor fields (twisted by an auxiliary vector bundle \( E \)). It decomposes into an orthogonal direct sum

\[
L^2(S(TM) \otimes E) = \bigoplus_{\lambda \in \text{Spec}(D^E)} E_\lambda,
\]

according to the eigenvalues \( \lambda \) of the Dirac operator \( D^E \). The “Hardy space” will be

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L^2_+(S(TM) \otimes E) = \bigoplus_{\lambda \geq 0} E_\lambda.
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The corresponding orthogonal projection from \( L^2(S(TM) \otimes E) \) to \( L^2_+(S(TM) \otimes E) \) will be denoted by \( P^E_{\geq 0} \).

The Toeplitz operator \( T^E_g \) is then defined as

\[
T^E_g = P^E_{\geq 0} g P^E_{\geq 0} : L^2_+(S(TM) \otimes E \otimes C^N) \rightarrow L^2_+(S(TM) \otimes E \otimes C^N).
\]

This is a Fredholm operator whose index is given by

\[
\text{ind} T^E_g = - \left( \hat{A}(TM) \text{ch}(E) \text{ch}(g), [M] \right),
\]

where \( \text{ch}(g) \) is the odd Chern character associated to \( g \). It is represented by the differential form (cf. [Z1, Chap. 1])

\[
\text{ch}(g) = \sum_{n=0}^{\dim M-1} \frac{n!}{(2n+1)!} \text{Tr} \left[ (g^{-1} dg)^{2n+1} \right].
\]

In [DZ1] we establish an index theorem which generalizes (1.1) to the case where \( M \) is an odd dimensional spin manifold with boundary \( \partial M \). The definition of the Toeplitz operator now uses Atiyah-Patodi-Singer boundary conditions on \( \partial M \). The self adjoint Atiyah-Patodi-Singer boundary conditions depend on choices of Lagrangian subspaces \( L \subset \ker D^E_{\partial M} \). We will denote the corresponding boundary condition by \( P_{\partial M}(L) \). The resulting Toeplitz operator will then be denoted by \( T^E_g(L) \).

**Theorem 1.1** (Dai-Zhang). The Toeplitz operator \( T^E_g(L) \) is Fredholm with index given by

\[
\text{ind} T^E_g(L) = - \left( \frac{1}{2\pi \sqrt{-1}} \right)^{\dim M/2} \int_M \hat{A}(R^{TM}) \text{Tr} \left[ \exp \left( -R^E \right) \right] \text{ch}(g)
\]

\[
- \eta(\partial M, E, g) + \tau_{\mu} \left( g P_{\partial M}(L) g^{-1}, P_{\partial M}(L), P_M \right).
\]

Here \( \eta(\partial M, E, g) \) denotes the invariant of \( \eta \)-type for even dimensional manifold \( \partial M \) and the \( K^1 \) representative \( g \). The third term is an interesting new integer term here, a triple Maslov index introduced in [KL], see [DZ1] for details.

This paper is organized as follows. In Section 2, we review the definition of the eta invariant for an even dimensional closed manifold introduced in [DZ1]. In Section 3, we discuss some general properties of the invariant. In Section 4, we give an intrinsic spectral interpretation of the eta invariant. And we end with a conjecture and a few remarks in the last section.

\[\text{In this paper, for simplicity, we will generally assume that our manifolds are spin, although our discussion extends trivially to the case of Clifford modules}\]
Acknowledgement: This is a survey about our previous joint work, as well as the recent new work with Weiping Zhang. I would like to thank my collaborator Weiping Zhang for constant inspiration. Thanks are also due to Matthias Lesch for interesting conversations.

2. An invariant of $\eta$ type for even dimensional manifolds

For an even dimensional closed manifold $X$ and a $K^1$ representative $g : X \to U(N)$, the eta invariant will be defined in terms of an eta invariant on the cylinder $[0, 1] \times X$ with appropriate APS boundary conditions.

In general, for a compact manifold $M$ with boundary $\partial M$ with the product structure near the boundary, the Dirac operator $D^E$ twisted by an hermitian vector bundle $E$ decomposes near the boundary as

$$D^E = c(\frac{\partial}{\partial x})(\frac{\partial}{\partial x} + D^E_{\partial M}).$$

The APS projection $P_{\partial M}$ is an elliptic global boundary condition for $D^E$. However, for self adjoint boundary conditions, we need to modify it by a Lagrangian subspace of $\ker D^E_{\partial M}$, namely, a subspace $L$ of $\ker D^E_{\partial M}$ such that $c(\frac{\partial}{\partial x})L = L^\perp \cap (\ker D^E_{\partial M})$. Since $\partial M$ bounds $M$, by the cobordism invariance of the index, such Lagrangian subspaces always exist.

The modified APS projection is then obtained by adding the projection onto the Lagrangian subspace. Let $P_L$ denote the orthogonal projection from $L^2((S(TM) \otimes E)|_{\partial M})$ to $L^2((S(TM) \otimes E)|_{\partial M}) \oplus L$:

$$P_{\partial M}(L) = P_{\partial M} + P_L,$$

where $P_L$ denotes the orthogonal projection from $L^2((S(TM) \otimes E)|_{\partial M})$ to $L$.

The pair $(D^E, P^E_{\partial M}(L))$ forms a self-adjoint elliptic boundary problem, and $P_{\partial M}(L)$ is called an Atiyah-Patodi-Singer boundary condition associated to $L$. We will denote the corresponding elliptic self-adjoint operator by $D^E_{P_{\partial M}(L)}$.

In [DZ1], we originally intend to consider the conjugated elliptic boundary value problem $D^E_{gP_{\partial M}(L)g^{-1}}$. However, the analysis turns out to be surprisingly subtle and difficult. To circumvent this difficulty, a perturbation of the original problem was constructed.

Let $\psi = \psi(x)$ be a cut off function which is identically 1 in the $\epsilon$-tubular neighborhood of $\partial M$ ($\epsilon > 0$ sufficiently small) and vanishes outside the $2\epsilon$-tubular neighborhood of $\partial M$. Consider the Dirac type operator

$$D^\psi = (1 - \psi)D^E + \psi D^E g^{-1}.$$

The motivation for considering this perturbation is that, near the boundary, the operator $D^\psi$ is actually given by the conjugation of $D^E$, and therefore, the elliptic boundary problem $(D^\psi, gP_{\partial M}(L)g^{-1})$ is now the conjugation of the APS boundary problem $(D^E, gP_{\partial M}(L))$, i.e., this is now effectively standard APS situation and we have a self adjoint boundary value problem $(D^\psi, gP_{\partial M}(L)g^{-1})$ together with its associated self adjoint elliptic operator $D^\psi_{gP_{\partial M}(L)g^{-1}}$.

The same thing can be said about the conjugation of $D^\psi$:

$$D^{\psi,g} = g^{-1}D^\psi g = D^E + (1 - \psi)g^{-1}[D^E, g].$$

We will in fact use $D^{\psi,g}$. 
We are now ready to construct the eta invariant for even dimensional manifolds. Given an even dimensional closed spin manifold $X$, we consider the cylinder $[0, 1] \times X$ with the product metric. Let $g : X \to U(N)$ be a map from $X$ into the unitary group which extends trivially to the cylinder. Similarly, $E \to X$ is an Hermitian vector bundle which is also extended trivially to the cylinder. We assume that $\text{ind} \ D_{[0,1]}^\psi = 0$ on $X$ which guarantees the existence of the Lagrangian subspaces $L$.

Consider the analog of $D_{[0,1]}^{\psi,g}$ as defined in [21], but now on the cylinder $[0, 1] \times X$ and denote it by $D_{[0,1]}^{\psi,g}$. Here $\psi = \psi(x)$ is a cut off function on $[0, 1]$ which is identically 1 for $0 \leq x \leq \epsilon$ ($\epsilon > 0$ sufficiently small) and vanishes when $1 - 2\epsilon \leq x \leq 1$. We equip it with the boundary condition $P_X(L)$ on one of the boundary component $\{0\} \times X$ and the boundary condition $\text{Id} - g^{-1}P_X(L)g$ on the other boundary component $\{1\} \times X$ (Note that the Lagrangian subspace $L$ exists by our assumption of vanishing index). Then $(D_{[0,1]}^{\psi,g}, P_X(L), \text{Id} - g^{-1}P_X(L)g)$ forms a self-adjoint elliptic boundary problem. For simplicity, we will still denote the corresponding elliptic self-adjoint operator by $D_{[0,1]}^{\psi,g}$.

Let $\eta(D_{[0,1]}^{\psi,g}, s)$ be the $\eta$-function of $D_{[0,1]}^{\psi,g}$ which, when $\text{Re}(s) >> 0$, is defined by

$$
\eta(D_{[0,1]}^{\psi,g}, s) = \sum_{\lambda \neq 0} \frac{\text{sgn}(\lambda)}{|\lambda|^s},
$$

where $\lambda$ runs through the nonzero eigenvalues of $D_{[0,1]}^{\psi,g}$.

By [DW], one knows that the $\eta$-function $\eta(D_{[0,1]}^{\psi,g}, s)$ admits a meromorphic extension to $C$ with $s = 0$ a regular point (and only simple poles). One then defines, as in [APS1], the $\eta$-invariant of $D_{[0,1]}^{\psi,g}$, $\eta(D_{[0,1]}^{\psi,g}) = \eta(D_{[0,1]}^{\psi,g}, 0)$, and the reduced $\eta$-invariant by

$$
\overline{\eta}(D_{[0,1]}^{\psi,g}) = \frac{\dim \ker D_{[0,1]}^{\psi,g} + \eta(D_{[0,1]}^{\psi,g})}{2}.
$$

We can also consider the invariant $\overline{\eta}(D_{[0,a]}^{\psi,g})$, similarly constructed on a cylinder $[0, a] \times X$, and it turns out to not depend on the radial size of the cylinder $a > 0$. This can be seen by a rescaling argument (cf. [Mii, Proposition 2.16]).

**Definition 2.2.** We define an invariant of $\eta$ type for the complex vector bundle $E$ on the even dimensional manifold $X$ (with vanishing index) and the $K^1$ representative $g$ by

$$
\overline{\eta}(X, E, g) = \overline{\eta}(D_{[0,1]}^{\psi,g}) - \text{sf} \left\{ D_{[0,1]}^{\psi,g}(s); 0 \leq s \leq 1 \right\},
$$

where $D_{[0,1]}^{\psi,g}(s)$ is a path connecting $g^{-1}D_E g$ with $D_{[0,1]}^{\psi,g}$ defined by

$$
D_{[0,1]}^{\psi,g}(s) = D_E + (1 - s\psi)g^{-1}[D_E, g]
$$

on $[0, 1] \times X$, with the boundary condition $P_X(L)$ on $\{0\} \times X$ and the boundary condition $\text{Id} - g^{-1}P_X(L)g$ at $\{1\} \times X$.

It was shown in [DZT] that $\overline{\eta}(X, E, g)$ does not depend on the cut off function $\psi$. 

3. Some properties of the eta invariant

In this section we look at the properties of our invariant \( \eta(X, E, g) \), which depends first of all on the geometry of the even dimensional manifold \( X \), as well as the hermitian vector bundle \( E \) on \( X \), and also the \( K^1 \) representative \( g : X \to U(N) \). An immediate consequence of our Toeplitz index theorem, Theorem 1.1, is that, when \( X = \partial M \) is the boundary of an odd dimensional compact manifold \( M \), the mod \( Z \) reduction of \( \eta(X, E, g) \) is related to some Chern-Simons invariants:

\[
\eta(\partial M, E, g) \equiv -\left( \frac{1}{2\pi \sqrt{-1}} \right)^{(\dim M + 1)/2} \int_M \hat{A}(R^T M) \text{Tr} \left[ \exp(-R^E) \right] \text{ch}(g) \mod Z.
\]

In this section we will continue to denote the mod \( Z \) reduction of \( \eta(X, E, g) \) by the same notation. One can first study the behavior of \( \eta(X, E, g) \) under the metric changes of \( X \). Thus let \( X_1 \) and \( X_2 \) denote the same manifold \( X \) but with two different Riemannian metrics. Then \( X_1 - X_2 = \partial M \) where \( M = [0, 1] \times X \) with suitable Riemannian metrics. Applying the above formula to the current situation yields

\[
\eta(X_1, E, g) - \eta(X_2, E, g) \equiv -\left( \frac{1}{2\pi \sqrt{-1}} \right)^{(\dim M + 1)/2} \int_M \hat{A}(R^T M) \text{Tr} \left[ \exp(-R^E) \right] \text{ch}(g).
\]

In particular, when \( E \) is a flat bundle,

\[
\eta(X_1, E, g) - \eta(X_2, E, g) \equiv -\text{rank} E \left( \frac{1}{2\pi \sqrt{-1}} \right)^{(\dim M + 1)/2} \int_M \hat{A}(R^T M) \text{ch}(g) \mod Z
\]

depends only on the rank of the vector bundle \( E \). Thus, if we define

\[
\rho(X, E, g) = \eta(X, E, g) - \eta(X, \mathbb{C}^{\text{rank} E}, g), \quad \text{in } \mathbb{R}/\mathbb{Z}
\]

then we have deduced that

**Theorem 3.1.** The invariant \( \rho(X, E, g) \) is independent of the Riemannian metric on \( X \), and hence is an invariant associated to the manifold \( X \), the flat hermitian vector bundle \( E \), and the \( K^1 \) representative \( g \). It is a cobordism invariant in the sense that when \( X = \partial M \) is a boundary and \( E, g \) extends to the interior, then

\[
\rho(X, E, g) = 0.
\]

If \( E \) is not assumed to be flat but \( X = \partial M \) is the boundary of an odd dimensional compact manifold, the above discussion can be further refined, as was pointed out in \cite{DZ1}. We recall it in the following.

Let \( g^T M \) resp. \( g^T M \), \( g^E \) resp. \( g^E \), and \( \nabla^E \) resp. \( \nabla^E \) be two Riemannian metrics on \( M \), two Hermitian metrics on \( E \), and two connections on \( E \). Let \( D^E \) resp. \( \bar{D}^E \) be the corresponding (twisted) Dirac operators. In order to emphasize the dependence on the particular geometry of the manifold, we will denote our eta invariant \( \eta(\partial M, E, g) \) by the more explicit notation \( \eta(D^E_{\partial M}, g) \)
Let $\omega$ be the Chern-Simons form which transgresses the $\hat{A} \wedge \text{ch}$ forms:

$$d\omega = \left( \frac{1}{2\pi \sqrt{-1}} \right)^{\dim M + 1} \left( \hat{A} \left( R^{TM} \right) \exp \left( -\hat{E} R^{TM} \right) - \hat{A} \left( R^{TM} \right) \exp \left( -R^{E} \right) \right).$$

Then the following formula describing the variation of $\eta(D_{DE} \partial M, g)$, when $g|\partial M$, $g^E|\partial M$, and $\nabla^E|\partial M$ change, is proved in [DZ1].

**Theorem 3.2** (Dai-Zhang). The following identity holds,

$$\eta\left( \tilde{D}^E_{\partial M}, g \right) - \eta\left( D^E_{\partial M}, g \right) \equiv - \int_{\partial M} \omega \text{ch}(g) \mod \mathbb{Z}.$$

One can also study the behavior of the invariant $\eta(X, E, g)$ under the deformations of the $K^1$ representative, as in [DZ1]. Let $g_t$, $0 \leq t \leq 1$ be a smooth family of $K^1$ representatives $g : X \to U(N)$. Then

$$\tilde{\text{ch}} \left( g_t, 0 \leq t \leq 1 \right) = \sum_{n=0}^{(\dim M - 1)/2} \frac{n!}{(2n)!} \int_0^1 \text{Tr} \left[ g_t^{-1} \frac{\partial g_t}{\partial t} \left( g_t^{-1} \frac{d g_t}{d t} \right)^{2n} \right] dt.$$

transgress the odd Chern character:

$$\text{ch}(g_1) - \text{ch}(g_0) = d \tilde{\text{ch}} \left( g_t, 0 \leq t \leq 1 \right).$$

**Theorem 3.3** (Dai-Zhang). If $\{g_t\}_{0 \leq t \leq 1}$ is a smooth family of maps from $X$ to $U(N)$ and $X$ is a closed even dimensional manifold with vanishing index, then

$$\eta(X, E, g_1) - \eta(X, E, g_0) \equiv - \left( \frac{1}{2\pi \sqrt{-1}} \right)^{\dim X + 1} \int_X \hat{A} \left( R^X \right) \text{Tr} \left[ \exp \left( -R^E \right) \right] \tilde{\text{ch}} \left( g_t, 0 \leq t \leq 1 \right) \mod \mathbb{Z}.$$ 

In particular, if $g_0 = \text{Id}$, that is, $g = g_1$ is homotopic to the identity map, then

$$\eta(X, E, g) \equiv - \left( \frac{1}{2\pi \sqrt{-1}} \right)^{\dim X + 1} \int_X \hat{A} \left( R^X \right) \text{Tr} \left[ \exp \left( -R^E \right) \right] \tilde{\text{ch}} \left( g_t, 0 \leq t \leq 1 \right) \mod \mathbb{Z}. \tag{3.4}$$

**Remark** The eta invariant $\eta(\partial M, g)$ gives an intrinsic interpretation of the Wess-Zumino term in the WZW theory. When $\partial M = S^2$, the Bott periodicity tells us that every $K^1$ element $g$ on $S^2$ can be deformed to the identity (adding a trivial bundle if necessary). Hence, (3.3) gives another intrinsic form of the Wess-Zumino term, which is purely local on $S^2$.

Finally, there is also an interesting additivity formula for $\eta(X, E, g)$, as we recall from [DZ1].

**Theorem 3.4** (Dai-Zhang). Given $f, g : X \to U(N)$, the following identity holds in $\mathbb{R}/\mathbb{Z}$,

$$\eta(X, E, fg) = \eta(X, E, f) + \eta(X, E, g).$$
4. An intrinsic spectral interpretation

The usefulness of the eta invariant of Atiyah-Patodi-Singer comes, at least partially, from the spectral nature of the invariant, i.e. that it is defined via the spectral data of the Dirac operator on the (odd dimensional) manifold. Our eta invariant for even dimensional manifold is defined via the eta invariant on the corresponding odd dimensional cylinder by imposing APS boundary conditions. Thus, it will be desirable to have a direct spectral interpretation in terms of the spectral data of the original manifold (and the $K^1$ representative). In this section we give such an interpretation using the adiabatic limit.

First we recall the setup and result from [D], which is an extension of [BC] to manifolds with boundary. More precisely, let

\[ Y \to X \xrightarrow{\pi} B \]

be a fibration where the fiber $Y$ is closed but the base $B$ may have nonempty boundary. Let $g_B$ be a metric on $B$ which is of the product type near the boundary $\partial B$. Now equip $X$ with a submersion metric $g$,

\[ g = \pi^*g_B + g_Y \]

so that $g$ is also product near $\partial X$. This is equivalent to requiring $g_Y$ to be independent of the normal variable near $\partial B$.

The adiabatic metric $g_x$ on $X$ is given by

\[ g_x = x^{-2}\pi^*g_B + g_Y, \]

where $x$ is a positive parameter.

Associated to these data we have in particular the total Dirac operator $D^X_x$ on $X$, the boundary Dirac operator $D^Y_x$ on $\partial X$, and the family of Dirac operators $D_Y$ along the fibers. If the family $D_Y$ is invertible, then, according to [BC], the boundary Dirac operator $D^Y_x$ is also invertible for all small $x$, therefore the eta invariant of $D_x$ with the APS boundary condition, $\eta(D_x)$, is well-defined.

**Theorem 4.1.** Consider the fibration $Y \to X \to B$ as above. Assume that the Dirac family along the fiber, $D_Y$, is invertible. Consider the total Dirac operator $D^X_x$ on $X$ with respect to the adiabatic metric $g_x$ and let $\eta(D^X_x)$ denote the eta invariant of $D^X_x$ with the APS boundary condition. Then the limit

\[ \lim_{x \to 0} \bar{\eta}(D^X_x) = \lim_{x \to 0} \frac{1}{2} \bar{\eta}(D^X_x) \]

exists in $\mathbb{R}$ and

\[ \lim_{x \to 0} \bar{\eta}(D_x) = \int_B \bar{A}(\frac{R^B}{2\pi}) \wedge \bar{\eta}, \]

where $R^B$ is the curvature of $g_B$, $\bar{A}$ denote the the $\bar{A}$-polynomial and $\bar{\eta}$ is the $\eta$-form of Bismut-Cheeger [BC].

We apply this result to our current situation where $M = [0,1] \times X$ fibers over $[0,1]$ with the fibre $X$. The operator

\[ D^{\psi,s} = D^E + (1 - \psi)g^{-1}[D^E, g] = D^E + (1 - \psi)c(g^{-1}dg) \]

will be of Dirac type, and of product type near the boundaries. In order to apply the adiabatic limit result, we will assume the invertibility condition that

\[ \ker[D_X + s c(g^{-1}dg)] = 0, \quad \forall \ 0 \leq s \leq 1. \]
Under this assumption there is no spectral flow contribution and hence
\[
\tilde{\eta}(X, g) = \lim_{a \to \infty} \eta(D_{[0,a]}^{\psi,g}) = \text{the adiabatic limit of } \eta(D_{[0,1]}^{\psi,g})
\]
is given by the adiabatic limit formula.

By using Theorem 4.1, we obtain

**Theorem 4.2** (Dai-Zhang). Under the assumption that \(\ker[D_X + s c(g^{-1}dg)] = 0, \forall 0 \leq s \leq 1\),
\[
\tilde{\eta}(X, E, g) = \frac{i}{4\pi} \int_0^1 \int_0^\infty \operatorname{tr}_s [c(g^{-1}dg) (D_X + s c(g^{-1}dg)) e^{-t(D_X+s c(g^{-1}dg))^2}] dt \, ds.
\]

For details and further generalization without invertibility assumption, we refer to [DZ2].

### 5. Final remarks

Finally we end with a conjecture and some remarks. As we mentioned, the eta type invariant \(\tilde{\eta}(X, E, g)\), which we introduced using a cut off function, is in fact independent of the cut off function. This leads naturally to the question of whether \(\tilde{\eta}(X, E, g)\) can actually be defined directly. The following conjecture is stated in [DZ1].

Let \(D^{[0,1]}\) be the Dirac operator on \([0, 1] \times X\). We equip the boundary condition \(gP_X(L)g^{-1}\) at \(\{0\} \times X\) and the boundary condition \(\text{Id} - P_X(L)\) at \(\{1\} \times X\).

Then \((D^{[0,1]}, gP_X(L)g^{-1}, \text{Id} - P_X(L))\) forms a self-adjoint elliptic boundary problem. We denote the corresponding elliptic self-adjoint operator by \(D_{gP_X(L)g^{-1}, P_X(L)}^{[0,1]}\).

Let \(\eta(D_{gP_X(L)g^{-1}, P_X(L)}^{[0,1]}, s)\) be the \(\eta\)-function of \(D_{gP_X(L)g^{-1}, P_X(L)}^{[0,1]}\). By [KL, Theorem 3.1], which goes back to [Gr], one knows that the \(\eta\)-function \(\eta(D_{gP_X(L)g^{-1}, P_X(L)}^{[0,1]}, s)\) admits a meromorphic extension to \(C\) with poles of order at most 2. One then defines, as in [KL, Definition 3.2], the \(\eta\)-invariant of \(D_{gP_X(L)g^{-1}, P_X(L)}^{[0,1]}\), denoted by \(\tilde{\eta}(D_{gP_X(L)g^{-1}, P_X(L)}^{[0,1]}, s)\), to be the constant term in the Laurent expansion of \(\eta(D_{gP_X(L)g^{-1}, P_X(L)}^{[0,1]}, s)\) at \(s = 0\).

Let \(\tilde{\eta}(D_{gP_X(L)g^{-1}, P_X(L)}^{[0,1]}\) be the associated reduced \(\eta\)-invariant.

**Conjecture:**
\[
\tilde{\eta}(X, E, g) = \tilde{\eta}(D_{gP_X(L)g^{-1}, P_X(L)}^{[0,1]}).
\]

We would also like to say a few words about the technical assumption that \(\text{ind } D_X^{E} = 0\) imposed in order to define the eta invariant \(\tilde{\eta}(X, E, g)\). The assumption guarantees the existence of the Lagrangian subspaces \(L\) which are used in the boundary conditions. In the Toeplitz index theorem, this assumption is automatically satisfied since \(X = \partial M\) is a boundary. In general, of course, it may not. However, if one is willing to overlook the integer contribution (as one often does in applications), this technical issue can be overcome by using another eta invariant,
this time on $S^1 \times X$, as follows. Note that we now have no boundary, hence no need for boundary conditions!

Consider $S^1 \times X = [0,1] \times X / \sim$ where $\sim$ is the equivalence relation that identifies $0 \times X$ with $1 \times X$. Let $E_g \rightarrow S^1 \times X$ be the vector bundle which is $E \otimes C^N$ over $(0,1) \times X$ and the transition from $0 \times X$ to $1 \times X$ is given by $g : X \rightarrow U(N)$. Denote by $D_{E_g}$ the Dirac operator on $S^1 \times X$ twisted by $E_g$.

**Proposition 5.1.** One has 

$$\eta(X, E, g) \equiv \eta(D_{E_g}) \mod \mathbb{Z}.$$ 

This is an easy consequence of the so called gluing law for the eta invariant, see [Bu, BL, DF].

**Remark** It might be interesting to note the duality that $\eta(D_{\partial M}, g)$ is a spectral invariant associated to a $K^1$-representative on an even dimensional manifold, while the usual Atiyah-Patodi-Singer $\eta$-invariant ([APS1]) is a spectral invariant associated to a $K^0$-representative on an odd dimensional manifold.

Finally, we would like to mention a recent paper of Zizhang Xie [X] in which he uses our eta invariant to prove, among other things, an odd index theorem for even dimensional closed manifolds as well as an odd analog of the relative index pairing formula of Lesch, Moscovici and Pflaum [LMP].

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