Numerical modeling of the problem of indentation of elastic and elastic-plastic massive bodies

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Abstract. The paper considers the dynamic pressing-in indenter in a plane elastic-plastic statement. A general statement of the problem of a rigid body under a flexible stamp pressing-in is given; it allows considering the elastic and plastic properties of the material and the process of the stress-strain state formation. The problem of a flexible stamp action on the outer boundary is realized using the finite-difference method. By solving the problem for a quadrangular profile, the distribution of the stress-strain state of the body under consideration and the formation of stresses over time are shown. The effect of plastic properties of the material on the stress state of the body is assessed by comparative analysis.

1. Introduction

The contact tasks of the continuum mechanics are based on practical needs whose purpose is to increase reliability and reduce metal consumption in mechanical engineering, to conduct optimal design during metal forming, to determine the bearing capacity of foundations, etc. A rigorous statement of the contact problem is difficult to solve in analytical form. Various numerical schemes are used to obtain a solution to the problem. Due to mathematical difficulties, differences often arise between theoretical solutions and practical results.

For the first time, the decision to press-in a stamp was obtained by Prandtl. It was considered that the stamp is an absolutely rigid non-deformable body, and the medium was considered to be a perfect plastic one. There was no friction at the contact boundary between the stamp and the half-plane. This task was important for assessing the bearing capacity of the soil mass under a rigid foundation. Prandtl has also considered that elastic strains can be neglected.

As a rule, all numerical solutions are obtained in the presence of a static load under certain assumptions. For example, there is a numerical solution to the problem of pressing-in a rectangular stamp into an elastic rough half-space in the presence of Coulomb friction [1]. The problems of stamp pressing-in into anisotropic media under various boundary conditions are also widely implemented [2]. In all problems, the case of plane strains and, as a rule, the case of perfect plastic base material is realized.

2. Statement of the problem

Consider a sufficiently long deformable beam with a quadrangular cross section lying on a smooth undeformable surface. Let the normal components of the stress vector act on the side surfaces of the beam. Consider that the beam is experiencing a plane deformed state, so this problem can be formulated in a two-dimensional statement for the cross-sectional profile (figure 1). Study the stress-
strain state (SSS) of a $2a \times b$ quadrangular massive body and the SSS formation under external loads, figure 1. The solution of this boundary value problem in the Eulerian representation is expressed by the following equations:

equation of motion of the particles of body:

$$\rho \frac{dv_x}{dt} = \frac{\partial \sigma_{xx}}{\partial z} + \frac{\partial \tau_{xy}}{\partial y}, \quad \rho \frac{dv_y}{dt} = \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y};$$  \hspace{1cm} (1)

equations between particle velocities $v_x, v_y$ and strains $\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{xy}$:

$$\frac{d\varepsilon_{xx}}{dt} = \frac{\partial v_x}{\partial x}, \quad \frac{d\varepsilon_{yy}}{dt} = \frac{\partial v_y}{\partial y}, \quad \frac{d\varepsilon_{xy}}{dt} = \frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y};$$  \hspace{1cm} (2)

continuity equations:

$$\frac{1}{\rho} \frac{d\rho}{dt} + \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0;$$  \hspace{1cm} (3)

constitutive equations for elastic-plastic continua:

$$\frac{dP}{dt} = K \frac{1}{\rho} \frac{d\rho}{dt}, \quad \frac{dS_{xx}}{dt} + \lambda S_{xy} = 2G \left( \frac{d\varepsilon_{xx}}{dt} - \frac{1}{3\rho} \frac{d\rho}{dt} \right), \quad \frac{dS_{yy}}{dt} + \lambda S_{xy} = -2G \frac{d\varepsilon_{yy}}{dt}, \quad \frac{d\tau_{xy}}{dt} + \lambda \tau_{xy} = G \frac{d\varepsilon_{xy}}{dt}. \hspace{1cm} (4)$$

In equations (1)-(4), $\rho$ is the density of the body material, $\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \tau_{xy}$ are the stress components, expressed in terms of pressure $P$ and stress deviators $S_{xx}, S_{yy}, S_{zz}, S_{xy} = \tau_{xy}$:

$$\sigma_{xx} = S_{xx} - P, \quad \sigma_{yy} = S_{yy} - P, \quad \sigma_{zz} = S_{zz} - P. \hspace{1cm} (5)$$

At elastic case, the parameter $\lambda = 0$, and in elastic-plastic case, it is determined by using the von Mises criteria:

$$2J_2 = S_{xx}^2 + S_{yy}^2 + S_{zz}^2 + 2\tau_{xy}^2 \leq 2Y^2 / 3, \hspace{1cm} (6)$$

and relation $\lambda = 3W H(W) / \sqrt{2Y^2}$, where $H(W) = 0$ at $W < 0$, $H(W) = 1$ at $W \geq 0$,

$$W = 2G \left[ \sum_{k=1,2,3} S_{ik} \left( \frac{d\varepsilon_{ik}}{dt} - \frac{1}{3\rho} \frac{d\rho}{dt} \right) + \tau_{xy} \frac{d\varepsilon_{xy}}{dt} \right].$$

We formulate the boundary conditions (figure 1):

on the contact boundary on smooth rigid undeformable surface

$$v_x = 0, \quad \tau_{xy} = 0 \text{ at } \left| x \right| \leq a, \quad y = 0, \quad t \geq 0; \hspace{1cm} (7)$$

on lateral side boundaries of the body

$$\sigma_{xx} = -p(t), \quad \tau_{xy} = 0 \text{ at } x = \pm a, \quad 0 < y < b, \quad t \geq 0; \hspace{1cm} (8)$$

on the upper boundary

$$\sigma_{yy} = -q(t), \quad \tau_{xy} = 0 \text{ at } \left| x \right| \leq a, \quad y = b, \quad t \geq 0. \hspace{1cm} (9)$$
The initial conditions are considered to be zero. Thus, the problem under consideration is reduced to solving the systems of equations (1)-(6) with zero initial and boundary conditions (7)-(9).

3. Method of solution

The problem under consideration is solved numerically using the finite difference method according to the Wilkins scheme [3]. The body under consideration is divided into \( N \times M \) quadrangular cells. As in [3], the velocity and coordinates are determined for the nodal points of the cells, the remaining parameters - in the centre of the cells. Finite-difference relations for inner points are given in [3]. The velocities of the boundary nodal points are determined by introducing an imaginary cell, as shown in figures 2-3. Introduce the following notation: the value of the parameter \( \psi_{x,c}^{n} \) at a point \( \left( x_{c,j}, y_{c,k} \right) \) at a time \( t^{n} \) is denoted by \( \psi_{x,c}^{n} = F\left( x_{c,j}, y_{c,k}, t^{n} \right) \). Let all parameters of the problem be known at a time \( t^{n} \). We show the implementation of the boundary conditions (7)-(9).

![Figure 1](image1.png)  
**Figure 1.** Scheme of body loading.

![Figure 2](image2.png)  
**Figure 2.** Design scheme on the boundary \( y = 0 \).

At the boundary \( y = 0 \), imaginary cells are introduced by a mirror image through the boundary, as shown in figure 2. For all points \( \left( x_{c,j}, y_{c,k} \right), \ j \in \left[ 1, N - 1 \right], k = 0 \) located on the boundary \( y = 0 \) from equations (1) using the scheme [3] we have:

\[
\left( \frac{dv_{x}}{dt} \right)_{c,j,k} = -\frac{1}{2\psi_{n}^{x}} \left[ Q_{x,c}^{n} \left( \sigma_{yy}, y \right) - Q_{y,c}^{n} \left( \sigma_{xx}, x \right) \right],
\]

\[
\left( \frac{dv_{y}}{dt} \right)_{c,j,k} = 0,
\]

where \( Q_{x,c}^{n} \left( \sigma_{yy}, y \right) = \sigma_{y_{j+1/2,k+1/2}}^{n} \left( y_{j+1/2,k+1/2}^{n} - y_{j,k+1}^{n} \right) + \sigma_{y_{j+1/2,k+1/2}}^{n} \left( y_{j+1,k+1}^{n} - y_{j+1/2,k}^{n} \right), \)

\( \psi_{n}^{x} = \frac{4}{M_{j+1/2,k+1/2}} \left( M_{j+1/2,k+1/2} + M_{j+1/2,k+1/2} + M_{j+1/2,k+1/2} + M_{j+1/2,k+1/2} \right) / 4 \), \( M_{j+1/2,k+1/2} \) - mass of a quadrangular cell whose centre is a point with coordinates \( \left( x_{j+1/2,k+1/2}, y_{j+1/2,k+1/2} \right). \)

To calculate the velocity at the boundary \( y = b \), we use the scheme shown in figure 3(a), taking into account the acting load:

\[
\left( \frac{dv_{x}}{dt} \right)_{c,j,k} = -\frac{1}{2\psi_{n}^{x}} \left[ R_{x,c}^{n} \left( \sigma_{yy}, y \right) + Z_{y,c}^{n} \left( q_{T}, y \right) - R_{y,c}^{n} \left( \sigma_{xx}, x \right) - Z_{x,c}^{n} \left( q_{N}, x \right) \right],
\]

\[
\left( \frac{dv_{y}}{dt} \right)_{c,j,k} = \frac{1}{2\psi_{n}^{x}} \left[ R_{x,c}^{n} \left( \sigma_{yy}, y \right) + Z_{y,c}^{n} \left( q_{T}, y \right) - R_{y,c}^{n} \left( \sigma_{xx}, x \right) - Z_{x,c}^{n} \left( q_{N}, x \right) \right],
\]

where \( \psi_{n}^{x} = \frac{4}{M_{j+1/2,k+1/2}} \left( M_{j+1/2,k+1/2} + M_{j+1/2,k+1/2} \right) / 4 \).
\[
R_{j,k}^n (\sigma, y) = \sigma_{j-1/2,k-1/2}^n \left( y_{j,k-1}^n - y_{j-1,k}^n \right) + \sigma_{j+1/2,k-1/2}^n \left( y_{j,k+1}^n - y_{j+1,k}^n \right),
\]
\[
Z_{j,k}^n (\rho, x) = q_{j+1/2,k-1/2}^n \left( x_{j+1,k}^n - x_{j,k}^n \right) + q_{j-1/2,k-1/2}^n \left( x_{j-1,k}^n - x_{j,k}^n \right).
\]

Here, the fact is taken into account that in the process of strain, the boundary under consideration can deviate from the straight line, which is observed both for the outer and inner boundaries between the core and the rock-fill part when studying the stress-strain state of the dam [4]. At inner points, these deviations are taken into account with the correction of the element rotation at each time step according to [3]. In equations (10) and (11), the normal and tangential outer impacts for the boundary are defined as follows:

\[
(q_N)_{j-1/2,k}^n = q_n \cos \alpha_{(j-1/2,k)}^n, \quad (q_T)_{j-1/2,k}^n = q_n \sin \alpha_{(j-1/2,k)}^n,
\]

where \( \alpha_{(j-1/2,k)}^n \) is the angle of inclination of a straight line passing through the nodal points \((x_{j-1,k}, y_{j-1,k})\) and \((x_{j,k}, y_{j,k})\) at time \( t^n \). Particle velocities for inner points at the boundaries \( x = \pm a \) are calculated similarly. For example, for the boundary \( x = a \), according to figure 3(b), the expressions for the nodal points acceleration correspond to equations (10) and (11) when \( q_N, q_T \) substituted by \( p_T, p_N \), taking into account

\[
\varphi_{j,k}^n = (M_{j-1/2,k-1/2} + M_{j+1/2,k+1/2}) / 4,
\]
\[
R_{j,k}^n (\sigma, y) = \sigma_{j+1/2,k+1/2}^n \left( y_{j,k+1}^n - y_{j,k-1}^n \right) + \sigma_{j+1/2,k+1/2}^n \left( y_{j,k+1}^n - y_{j,k+1}^n \right),
\]
\[
Z_{j,k}^n (\rho, x) = p_{j+1/2,k+1/2}^n \left( x_{j+1,k}^n - x_{j,k}^n \right) + p_{j+1/2,k+1/2}^n \left( x_{j+1,k}^n - x_{j+1,k}^n \right).
\]

For the angular points of the body, we use the schemes shown in figures 3(c) and 3(d). The acceleration of the angular point \((x_{j,k}^n, y_{j,k}^n) = (a, b)\) (figure 3(c)) is calculated by the formula:
\[
\begin{align*}
\frac{dv_{x}}{dt}_{j,k} &= -\frac{1}{2\varphi_{n}^{1-j,k}} \left[ p_{n}^{j-k}(\sigma_{x},x) + X_{j,k}^{n}(q_{T},x) + Y_{j,k}^{n}(p_{T},x) - p_{n}^{j-k}(\tau_{y},x) - X_{j,k}^{n}(q_{T},x) - Y_{j,k}^{n}(p_{T},x) \right], \\
\frac{dv_{y}}{dt}_{j,k} &= \frac{1}{2\varphi_{n}^{1-j,k}} \left[ p_{n}^{j-k}(\sigma_{y},x) + X_{j,k}^{n}(q_{T},x) + Y_{j,k}^{n}(p_{T},x) - p_{n}^{j-k}(\tau_{y},x) - X_{j,k}^{n}(q_{T},x) - Y_{j,k}^{n}(p_{T},x) \right].
\end{align*}
\]

(13)  \quad (14)  

Here \( \varphi_{n}^{j,k} = (M_{1-j/2,k+1/2})/4 \), \( p_{n}^{j,k}(\sigma,y) = \sigma_{n}^{j-k,1/2,j} \left( y_{j,k}^{n} - y_{j,k-1}^{n} \right) \), \( X_{j,k}^{n}(q,x) = q_{n}^{j-k,1/2,j} \left( x_{j,k}^{n} - x_{j,k-1}^{n} \right) \)

and \( Y_{j,k}^{n}(p,x) = p_{n}^{j-k,1/2,j} \left( x_{j,k}^{n} - x_{j,k+1}^{n} \right) \).

Accordingly, the acceleration of the angular point \( (x_{j,k},y_{j,k}) = (a,0) \) shown in figure 3(d) is determined from the relations

\[
\begin{align*}
\frac{dv_{x}}{dt}_{j,k} &= -\frac{1}{2\varphi_{n}^{1-j,k}} \left[ p_{n}^{j-k}(\sigma_{x},x) + Y_{j,k}^{n}(p_{T},x) - p_{n}^{j-k}(\tau_{y},x) - Y_{j,k}^{n}(p_{T},x) \right], \\
\frac{dv_{y}}{dt}_{j,k} &= \frac{1}{2\varphi_{n}^{1-j,k}} \left[ p_{n}^{j-k}(\sigma_{y},x) + Y_{j,k}^{n}(p_{T},x) - p_{n}^{j-k}(\tau_{y},x) - Y_{j,k}^{n}(p_{T},x) \right].
\end{align*}
\]

(15)  \quad (16)

here \( \varphi_{n}^{j,k} = (M_{j-k+1/2,k+1/2})/4 \), \( p_{n}^{j,k}(\sigma,y) = \sigma_{n}^{j+k+1/2,j-k} \left( y_{j,k+1}^{n} - y_{j,k}^{n} \right) \) and \( Y_{j,k}^{n}(p,x) = p_{n}^{j-k+1/2,j-k} \left( x_{j,k}^{n} - x_{j,k+1}^{n} \right) \).

The particle velocities of the body are found similarly, at other boundaries and angular points. New coordinate positions, strains and stresses are determined from the relations given in [3]. We have developed an algorithm and a program for solving the problem using the relations (10)-(16) and [3]. The problem at \( p = 0 \) and \( q = \text{const} \) was considered as a test example.

4. Calculation results and their analysis

For the model problem, the initial density, elastic modulus and Poisson's ratio of the body material under consideration are taken as follows: 2000 kg \( \cdot \) m\(^{-3} \), 200 MPa and 0.3 respectively. Geometrical dimensions are: \( a = b = 1 \) m, and acting loads (8) and (9): \( p(t) = p_{0} \cdot t^{-1} \) at \( t \leq t_{0} \), \( p(t) = p_{0} \) at \( t > t_{0} \); \( q(t) = q_{0} \cdot t^{-1} \) at \( t \leq t_{0} \), \( q(t) = q_{0} \) at \( t > t_{0} \), where \( p_{0} = 1 \) MPa, \( q_{0} = 0.5 \) MPa and \( t_{0} = 0.002 \) sec. In the case of elastic-plastic model of strain in relation (6), \( Y = G \cdot 200^{-1} \) is taken.

![Figure 4. Stress isolines \( \sigma_{yy} \) (MPa) and \( \tau_{xy} \) (kPa).](image)

Figures 4-5 show the stress distribution \( \sigma_{yy} \) (figure 4(a)), \( \tau_{xy} \) (figure 4(b)) and strain distribution \( \varepsilon_{xx} \) (figure 5(a)), \( \varepsilon_{yy} \) (figure 5(b)) at time \( t = 0.07 \) sec.

Figure 6 shows the isolines of body particles motion at a fixed point in time \( (t = 0.07 \) sec). Here the solid lines refer to \( u_{y} \) (cm), and the dashed lines refer to \( u_{x} \) (cm). In all the presented figures, the symmetry of parameters distribution is observed.
Figure 5. Strain isolines $\varepsilon_{xx}$ and $\varepsilon_{yy}$.

Figure 6. Isoline of particle displacements at $t = 0.07$ sec.

Figure 7. Stress changes in time.

Figure 7 shows the change in the stress state of some points of the considered elastic body in time. Figure 7(a) corresponds to $\sigma_{xx}(t)$, and figure 7(b) - to $\sigma_{yy}(t)$. Curves 1-6 correspond to changes in stresses at the points $y = 0$, 0.2 m, 0.4 m, 0.6 m, 0.8 m, and 0.99 m in the cross section $x = 0$. It can be observed that with the arrival of stress wave, increasing, they come to a stable value. As is known, under dynamic loads, the values of stresses in the continua almost double its static values. In this case, a linearly increasing effective load within 0.002 sec reaches a constant value. In solution, almost 2 times greater values were obtained and the process of solution smoothing was obtained using artificial
viscosity [3,4]. As seen from figure 7, to form a stable state, it was enough to carry out a calculation up to time of 0.7 seconds, further calculations differed quantitatively by no more than 3%.

The results for the elastic-plastic strain of the body under consideration are shown in figures 8 and 9. An account for plastic properties of the body material led to an increase in parameters, especially in displacements $u_y$ (cm) presented in figure 8(a) due to double value of vertical load compared to the horizontal one; in $u_x$ the changes were observed near the boundary (figure 8(b)).

![Figure 8. Isolines of vertical and horizontal displacements at $t = 0.07$ sec.](image)

![Figure 9. Stress changes in time.](image)

Plastic properties of the material led not only to an increase in displacements and strains, but to a redistribution of the stress state (figure 9).

5. Discussion

The solution to the problem is considered in a dynamic statement. This approach allows us to evaluate the formation of a stress-strain state and the finite form of a rigid body under static load. The method is similar to the approach considered in [5] when determining the loess soils settlement under their own weight. The implemented relations (10)-(16) for the boundary conditions correspond to the flexible stamp load or a remote pulse-shock load. Thus, using a numerical experiment, we consider the process of pressing-in of rigid bodies under static (setting a constant value of stresses) and dynamic loads. In this statement of the problem and the implementation of the method, the spherical and deviator parts of stress are used separately in equations. This makes it possible to evaluate the body forming as a result of calculation independently on its volume change. Also in the future, this approach allows the use of more complex equations of state that take into account plastic properties under various conditions, complex loading processes, anisotropy of the material, etc. Cauchy relations (2) allow us to take into account finite strains, although at each step of calculations, this method takes into account virtual significant (large) strains. We also note that with the proposed approach, the prospect of a theoretical study of flat stamping formation of materials opens up.
6. Conclusions
A general statement of the problem of a rigid body pressing-in under a flexible stamp is given, which allows one to take into account the elastic and plastic properties of the material and the process of formation of a stress-strain state. The problem of the action of a flexible stamp on the outer boundary is realized using the finite-difference method. By solving the problem for a quadrangular profile, the distribution of the stress-strain state of the body under consideration and the formation of stresses in time are shown. By comparative analysis, the effect of plastic properties of the material on the stress state of the body is assessed.

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