Cramér-type moderate deviations for Euler-Maruyama scheme for SDE

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Abstract

In this paper, we establish normalized and self-normalized Cramér-type moderate deviations for Euler-Maruyama scheme for SDE. As a consequence of our results, Berry-Esseen’s bounds and moderate deviation principles are also obtained. Our normalized Cramér-type moderate deviations refines the recent work of [Lu, J., Tan, Y., Xu, L., 2022. Central limit theorem and self-normalized Cramér-type moderate deviation for Euler-Maruyama scheme. Bernoulli 28(2): 937–964].

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1. Introduction

Consider the following stochastic differential equation (SDE) on $\mathbb{R}^d$:

$$dX_t = g(X_t)dt + \sigma dB_t, \quad X_0 = x_0,$$

where $B_t$ is a $d$-dimensional standard Brownian motion, $\sigma$ is an invertible $d \times d$ matrix and $g : \mathbb{R}^d \to \mathbb{R}^d$ satisfies the following assumption. There exist constants $L, K_1 > 0$ and $K_2 \geq 0$ such that for every $x, y \in \mathbb{R}^d$,

$$\|g(x) - g(y)\|_2 \leq L\|x - y\|_2,$$

$$\langle g(x) - g(y), x - y \rangle \leq -K_1\|x - y\|_2 + K_2,$$

where $\langle x, y \rangle$ stands for inner product of $x$ and $y$, and $\| \cdot \|_2$ is the Euclidean norm. Moreover, assume that $g(x)$ is second order differentiable and the second order derivative of $g$ is bounded. Given step size $\eta \in (0, 1)$, the Euler-Maruyama scheme for SDE (1.1) is given by

$$\theta_{k+1} = \theta_k + \eta g(\theta_k) + \sqrt{\eta} \sigma \xi_{k+1}, \quad k \geq 0,$$

where $(\xi_k)_{k \geq 1}$ are independent and identically distributed (i.i.d.) standard $d$-dimensional normal random vectors. It is known that SDE (1.1) and $(\theta_k)_{k \geq 0}$ are both ergodic and admit invariant measures, denoted by $\pi$ and $\pi_\eta$ respectively; see Lemma 2.3 of Lu, Tan and Xu [8]. Moreover, when $\sigma$ is the identity matrix and $g(x)$ is third order differentiable with an appropriate growth condition, Fang,
Shao and Xu [5] have proved that the Wasserstein-1 distance between $\pi$ and $\pi_\eta$ is in order of $\eta^{1/2}$, up to a logarithmic correction.

Denote $C^2_b(\mathbb{R}^d, \mathbb{R})$ the collection of all bounded 2-th order continuously differentiable functions. Given an $h \in C^2_b(\mathbb{R}^d, \mathbb{R})$, denote $\varphi$ the solution to the following Stein’s equation:

$$h - \pi(h) = A \varphi,$$

where $A$ is the generator of SDE (1.1) defined as follows:

$$A \varphi(x) = \langle g(x), \nabla \varphi(x) \rangle + \frac{1}{2} \langle \sigma \sigma^T \nabla^2 \varphi(x) \rangle_{\text{HS}},$$

with $T$ the transport operator and $\langle A, B \rangle_{\text{HS}} := \sum_{i=1}^d \sum_{j=1}^d a_{ij} b_{ij}$ for $A = (a_{ij})_{d \times d}, B = (b_{ij})_{d \times d} \in \mathbb{R}^{d \times d}$. For a small $\eta \in (0, 1)$, define

$$\Pi_\eta(\cdot) = \frac{1}{[\eta^{-2}]^{[\eta^{-2}]}} \sum_{k=0}^{[\eta^{-2}] - 1} \delta_{\theta_k}(\cdot),$$

where $\delta_y(\cdot)$ is the Dirac measure of $y$. Then $\Pi_\eta$ is an asymptotically consistent statistic of $\pi$ as $\eta \to 0$. We also denote

$$W_\eta = \eta^{-1/2}(\Pi_\eta(h) - \pi(h)) \quad \text{with} \quad Y_\eta = \frac{1}{[\eta^{-2}]} \sum_{k=0}^{[\eta^{-2}] - 1} \|\sigma^T \nabla \varphi(\theta_k)\|_2^2.$$

Recently, Lu, Tan and Xu [8] proved the following normalized Cramér-type moderate deviation. If $\theta_0 \sim \pi_\eta$ and $h \in C^2_b(\mathbb{R}^d, \mathbb{R})$, then

$$P(W_\eta > x) = 1 - \Phi(x) = 1 + O((1 + x)\eta^{1/6})$$

holds uniformly for $c\eta^{1/6} \leq x = o(\eta^{-1/6})$ as $\eta \to 0$. In this paper, we give an improvement on (1.4). In particular, our result implies that

$$P(W_\eta > x) = 1 + o\left(x^{2} \eta^{1/2} + (1 + x)(\eta \ln \eta)^{1/2}\right)$$

(1.5)

holds uniformly for $0 \leq x = o(\eta^{-1/4})$ as $\eta \to 0$. Compared to (1.4), equality (1.5) holds for a much larger range. Moreover, from (1.5), we obtain the following Berry-Esseen bound

$$\sup_{x \in \mathbb{R}} \left| P(W_\eta \leq x) - \Phi(x) \right| \leq c(\eta \ln \eta)^{1/2}.$$

(1.6)

Notice that the limit $\lim_{\eta \to 0} \eta^{-1/2}(\Pi_\eta(h) - \pi(h))$ has a normal distribution. Thus the best possible convergence rate of Berry-Esseen’s bound is in order of $\eta^{1/2}$. Thus the convergence rate in the last Berry-Esseen bound (1.6) is close to the best possible one $\eta^{1/2}$, up to a logarithmic correction $|\ln \eta|^{1/2}$. 
In particular, we further establish the following self-normalized Cramér-type moderate deviation. Denote
\[
S_\eta = \frac{\eta^{-1/2}(\Pi_\eta(h) - \pi(h))}{\sqrt{\mathcal{V}_\eta}} \quad \text{with} \quad \mathcal{V}_\eta = \frac{1}{\eta [\eta^{-2}]} \sum_{k=0}^{[\eta^{-2}]-1} \| (\theta_{k+1} - \theta_k - \eta g(\theta_k))^T \nabla \varphi(\theta_k) \|^2.
\] (1.7)

We also show that (1.5) and (1.6) hold also when \( W_\eta \) is replaced by \( S_\eta \). As \((\theta_k)_{0 \leq k \leq [\eta^{-2}]-1}\) are observable, then \( S_\eta \) is a self-normalized process. The moderate deviation expansion with respect to \( S_\eta \) is called as self-normalized Cramér-type moderate deviation. Self-normalized Cramér-type moderate deviation plays an important role in statistical inference of \( \pi(h) \), because in practice one usually does not know the exact values of the matrix \( \sigma \) and the factor \( \mathcal{V}_\eta \) does not depend on the invertible matrix \( \sigma \).

Throughout the paper, denote \( C \) a positive constant, and denote \( c \) a positive constant depending only on \( L, K_1, K_2, g, \| g(0) \|_2 \) and \( \sigma \). The exact values of \( C \) and \( c \) may vary from line to line. All over the paper, \( \| \cdot \| \) stands for the Euclidean norm for higher rank tensors.

## 2. Main results

We have the following normalized Cramér-type moderate deviation.

**Theorem 2.1.** Let \( \theta_0 \sim \pi_\eta \) and \( h \in C^2_b(\mathbb{R}^d, \mathbb{R}) \). Then the following inequality
\[
\left| \ln \frac{\mathbb{P}(W_\eta > x)}{1 - \Phi(x)} \right| \leq c \left( x^3 \eta + x^2 \eta^{1/2} + (1 + x)(\eta |\ln \eta|)^{1/2} \right)
\]
holds uniformly for \( 0 \leq x \leq \eta^{-3/4} \). In particular, it implies that
\[
\frac{\mathbb{P}(W_\eta > x)}{1 - \Phi(x)} = 1 + o \left( x^2 \eta^{1/2} + (1 + x)(\eta |\ln \eta|)^{1/2} \right)
\]
holds uniformly for \( 0 \leq x = o(\eta^{-1/4}) \) as \( \eta \to 0 \). Moreover, the same results hold when \( W_\eta \) is replaced by \(-W_\eta\).

From Theorem 2.1, we have the following Berry-Esseen bound for \( W_\eta \).

**Corollary 2.1.** Assume the conditions of Theorem 2.1. Then the following inequality holds
\[
\sup_{x \in \mathbb{R}} \left| \mathbb{P}(W_\eta \leq x) - \Phi(x) \right| \leq c (\eta |\ln \eta|)^{1/2}.
\] (2.1)

Notice that from (1.4), one may obtain a Berry-Esseen’s bound of order \( \eta^{1/6} \), which is slower the one in Corollary 2.1. Moreover, the convergence rate of Berry-Esseen’s bound in Corollary 2.1 is close to the convergence rate in the Wasserstein-1 distance between \( \pi \) and \( \pi_\eta \), which is of order \( \eta^{1/2} \) up to a logarithmic correction.

From Theorem 2.1, by an argument similar to the proof of Corollary 2.2 in [3], we easily obtain the following moderate deviation principle (MDP) result.
Corollary 2.2. Assume the conditions of Theorem 2.1. Let \((a_n)\) be any sequence of real numbers satisfying \(a_n \to \infty\) and \(\eta^{3/4} a_n \to 0\) as \(\eta \to 0\). Then for each Borel set \(B\),

\[
- \inf_{x \in B^o} \frac{x^2}{2} \leq \liminf_{\eta \to 0} \frac{1}{a_\eta^2} \ln \Pr \left( \frac{W_\eta}{a_\eta} \in B \right) \leq \limsup_{\eta \to 0} \frac{1}{a_\eta^2} \ln \Pr \left( \frac{W_\eta}{a_\eta} \in B \right) \leq - \inf_{x \in \overline{B}} \frac{x^2}{2},
\]

(2.2)

where \(B^o\) and \(\overline{B}\) denote the interior and the closure of \(B\), respectively.

Self-normalized limit theory for independent random variables has been studied in depth in the past twenty-five years. See, for instance, Shao [10] for self-normalized large deviations, and Jing, Shao and Wang [7] and Shao and Zhou [11] for self-normalized Cramér-type moderate deviations. A few results for dependent random variables, we refer to Chen, Shao, Wu and Xu [1] and [3]. In the next theorem, we present a self-normalized Cramér-type moderate deviation for the Euler-Maruyama scheme for SDE (1.1). Recall the definition of \(S_\eta\) in (1.7). We have the following moderate deviation expansions.

Theorem 2.2. Let \(\theta_0 \sim \pi_\eta\) and \(h \in C_b^2(\mathbb{R}^d, \mathbb{R})\). Then the following inequality

\[
\ln \frac{\Pr(S_\eta > x)}{1 - \Phi(x)} \leq c \left( x^3 \eta + x^2 \eta^{1/2} + (1 + x)(\eta | \ln \eta|)^{1/2} \right)
\]

holds uniformly for \(0 \leq x \leq \eta^{-3/4}\). In particular, it implies that

\[
\frac{\Pr(S_\eta > x)}{1 - \Phi(x)} = 1 + o \left( x^2 \eta^{1/2} + (1 + x)(\eta | \ln \eta|)^{1/2} \right)
\]

holds uniformly for \(0 \leq x = o(\eta^{-1/4})\) as \(\eta \to 0\). Moreover, the same results hold when \(S_\eta\) is replaced by \(-S_\eta\).

We call the results in Theorem 2.2 as self-normalized Cramér-type moderate deviations, because the normalized factor \(V_\eta\) does not depend on the invertible matrix \(\sigma\). Such type results play an important role in statistical inference of \(\pi(h)\), since in practice one usually does not know the exact values of the matrix \(\sigma\).

By arguments similar to the proofs of Corollaries 2.1 and 2.2, from Theorem 2.2, it is easy to see that the assertions in Corollaries 2.1 and 2.2 remain valid when \(W_\eta\) is replaced by \(S_\eta\).

Remark 2.1. The assumption \(\theta_0 \sim \pi_\eta\) in Theorems 2.1 and 2.2 is not essential. Thanks to the exponential ergodicity of the EM scheme, one can extend Theorems 2.1 and 2.2 to the case in which \(\theta_0\) is subgaussian distributed. Indeed, from the proof of (3.2) in Lu, Tan and Xu [8], one can see that Lemma 3.2 holds also when \(\theta_0\) is subgaussian distributed. The advantage of taking \(\theta_0 \sim \pi_\eta\) is that in their calculations, the terms describing the difference between the distribution of \(\theta_k\) and \(\pi_\eta\) will vanish, while in general case, one has to use exponential ergodicity of \(\theta_k\) to bound the difference. Since \(\theta_k\) converges to \(\pi_\eta\) exponentially fast, the difference will not put an essential difficulty. In our paper, we assumed the same condition as in Lu, Tan and Xu [8] and thus only considered the case of \(\theta_0 \sim \pi_\eta\).
3. Preliminary lemmas

In the proof of Theorem 2.1, we need the following three lemmas of Lu, Tan and Xu [8], see Lemmas 3.1, 3.3 and 5.1 therein.

**Lemma 3.1.** Let $h \in C^2_b(R^d, R)$. Then

$$|| \nabla^k \varphi || \leq c, \quad k = 0, 1, 2, 3, 4,$$

where $c$ depends on $g$ and $\sigma$.

**Lemma 3.2.** If $\theta_0 \sim \pi_\eta$, then there exists a constant $\gamma > 0$, depending on $L, K_1, K_2, \|g(0)\|_2$ and $\sigma$, such that

$$E\left(\exp\left\{\gamma \eta \sum_{k=0}^{m-1} \|g(\theta_k)\|_2^2\right\}\right) \leq c_1 e^{c_2 \eta^{-1}},$$

and for all $x > 0$,

$$P\left(\eta \sum_{k=0}^{m-1} \|g(\theta_k)\|_2^2 > x\right) \leq c_1 e^{c_2 \eta^{-1}} e^{-c_3 x}.$$

**Lemma 3.3.** For any $k \in \mathbb{N}$, it holds for all $y > 0$,

$$P\left(\left|\sum_{i=0}^{k-1} \|\sigma^T \nabla \varphi(\theta_i)\|_2^2 - k \pi_\eta(\|\sigma^T \nabla \varphi\|_2^2)\right| > y\right) \leq 2 e^{-c y^2 k^{-1}},$$

where $c$ depends on $g$ and $\sigma$.

In the proof of Theorem 2.1, we also make use of the following lemma.

**Lemma 3.4.** Let $(\zeta_i, F_i)_{i \geq 1}$ be a sequence of martingale differences. Assume there exist positive constants $c$ and $\alpha \in (0, 1)$ such that

$$1 \leq u_n := \sum_{i=1}^{n} \left\|E(c \zeta_i^2 \exp\{c|\zeta_i|^{\alpha}\}|F_{i-1})\right\|_\infty < \infty. \quad (3.1)$$

Then there exists a positive constant $c_\alpha$ such that for all $x > 0$,

$$P\left(\sum_{i=1}^{n} \zeta_i \geq x\right) \leq c_\alpha \exp\left\{-\frac{x^2}{c_\alpha (u_n + x^{2-\alpha})}\right\}. \quad (3.2)$$

**Proof.** We only give a proof for the case $\alpha \in (0, 1)$. For $\alpha = 1$, the proof is similar. For all $y > 0$, denote

$$\eta_i(y) = \zeta_i 1_{\{\zeta_i \leq y\}}.$$
Then $(\eta_i(y), F_i)_{i=1,\ldots,n}$ is a sequence of supermartingale differences satisfying $E(\exp\{\lambda\eta_i(y)\}) < \infty$ for all $\lambda \in [0, \infty)$ and all $i$. Define the exponential multiplicative martingale $Z(\lambda) = (Z_k(\lambda), F_k)_{k \geq 0}$, where

$$
\tilde{Z}_0(\lambda) = 1, \quad \tilde{Z}_k(\lambda) = \prod_{i=1}^{k} \frac{\exp\{\lambda \eta_i(y)\}}{E(\exp\{\lambda \eta_i(y)\} | F_{i-1})}.
$$

Then the random variable $\tilde{Z}_n(\lambda)$ satisfies $\int \tilde{Z}_n(\lambda) d\bar{P} = E\tilde{Z}_n(\lambda) = 1$. Define the conjugate probability measure

$$d\bar{P}_\lambda = \tilde{Z}_n(\lambda) d\bar{P}, \quad (3.3)$$

and denote by $\bar{E}_\lambda$ the expectation with respect to $\bar{P}_\lambda$. Notice that $\zeta_i = \eta_i(y) + \zeta_i I_{\{\zeta_i > y\}}$. It is easy to see that for all $x, y > 0$,

$$P\left(\sum_{i=1}^{n} \zeta_i \geq x\right) \leq P\left(\sum_{i=1}^{n} \eta_i(y) \geq x\right) + P\left(\sum_{i=1}^{n} \zeta_i I_{\{\zeta_i > y\}} > 0\right) =: P_1 + P\left(\max_{1 \leq i \leq n} \zeta_i > y\right), \quad (3.4)$$

By the change of measure defined by (3.3), we deduce that for all $x, y > 0$,

$$P_1 = \bar{E}_\lambda \left(\tilde{Z}_n(\lambda)^{-1} I_{\{\sum_{i=1}^{n} \eta_i(y) \geq x\}}\right)$$

$$= \bar{E}_\lambda \left(\exp\left\{-\lambda \left(\sum_{i=1}^{n} \eta_i(y)\right) + \hat{\Psi}_n(\lambda)\right\} I_{\{\sum_{i=1}^{n} \eta_i(y) \geq x\}}\right)$$

$$\leq \bar{E}_\lambda \left(\exp\left\{-\lambda x + \hat{\Psi}_n(\lambda)\right\} I_{\{\sum_{i=1}^{n} \eta_i(y) \geq x\}}\right),$$

where $\hat{\Psi}_n(\lambda) = \sum_{i=1}^{n} \ln E(e^{\lambda \eta_i(y)} | F_{i-1})$. Notice that $e^x \leq 1 + x + \frac{1}{2} x^2 e^{|x|}$, $x \in \mathbb{R}$. Then, we have for all $\lambda > 0$,

$$E(e^{\lambda \eta_i(y)} | F_{i-1}) \leq 1 + E(\lambda \eta_i(y) | F_{i-1}) + \frac{\lambda^2}{2} E(\eta_i(y)^2 | F_{i-1}) \exp\{|\lambda \eta_i(y)\}| | F_{i-1})$$

$$\leq 1 + \frac{\lambda^2}{2} E(\eta_i(y)^2 | F_{i-1}) \exp\{\lambda y^{1-\alpha} | \eta_i(y)^\alpha\} | F_{i-1})$$

Set $\lambda = cy^{\alpha-1}$. By the last inequality and the inequality $\ln(1 + t) \leq t$ for all $t \geq 0$, it is easy to see that for all $y > 0$,

$$\hat{\Psi}_n(\lambda) \leq \sum_{i=1}^{n} \ln \left(1 + \frac{\lambda^2}{2} E(\eta_i(y)^2 | F_{i-1}) \exp\{\lambda y^{1-\alpha} | \eta_i(y)^\alpha\} | F_{i-1})\right)$$

$$\leq \sum_{i=1}^{n} \frac{\lambda^2}{2} E(\eta_i(y)^2 | F_{i-1}) \exp\{\lambda y^{1-\alpha} | \eta_i(y)^\alpha\} | F_{i-1})$$

$$\leq \frac{1}{2} e^{2y^{2\alpha-2} u_n}.$$
Hence, we get for all \( x, y > 0 \),

\[
P_1 \leq \exp \left\{ -c y^{\alpha-1} x + \frac{1}{2} c^2 y^{2\alpha-2} u_n \right\}.
\]

From (3.4), it follows that for all \( x, y > 0 \),

\[
P \left( \sum_{i=1}^{n} \zeta_i \geq x \right) \leq \exp \left\{ -c y^{\alpha-1} x + \frac{1}{2} c^2 y^{2\alpha-2} u_n \right\} + P \left( \max_{1 \leq i \leq n} \zeta_i > y \right). \tag{3.5}
\]

By exponential Markov’s inequality, we deduce that for all \( y > 0 \),

\[
P \left( \max_{1 \leq i \leq n} \zeta_i > y \right) \leq \sum_{i=1}^{n} P \left( c^{1/\alpha} \zeta_i > c^{1/\alpha} y \right)
\]

\[
\leq \frac{1}{c^{2/\alpha} y^2} \exp \{ -c y^\alpha \} \sum_{i=1}^{n} E(\zeta_i^2 \exp \{ c (\zeta_i^*)^\alpha \})
\]

\[
\leq \frac{u_n}{c^{2/\alpha} y^2} \exp \{ -c y^\alpha \}. \tag{3.6}
\]

Taking

\[
y = \begin{cases} 
\left( \frac{c u_n}{x} \right)^{1/(1-\alpha)} & \text{if } 0 < x < (c u_n)^{1/(2-\alpha)} \\
x & \text{if } x \geq (c u_n)^{1/(2-\alpha)}
\end{cases}
\]

from (3.5) and (3.6), we obtain

\[
P \left( \sum_{i=1}^{n} \zeta_i \geq x \right) \leq \begin{cases} 
\exp \left\{ -\frac{x^2}{2u_n} \right\} + \frac{x^{2-\alpha}}{u_n^{1-\alpha} c^{\alpha(1+\alpha)}} \exp \left\{ -c \left( \frac{c u_n}{x} \right)^{\frac{\alpha}{1-\alpha}} \right\} & \text{if } 0 < x < (c u_n)^{\frac{1}{2-\alpha}} \\
\exp \left\{ -c x^\alpha (1 - \frac{c u_n}{2x^{2-\alpha}}) \right\} + \frac{u_n}{c^{2/\alpha} x^2} \exp \left\{ -c x^\alpha \right\} & \text{if } x \geq (c u_n)^{\frac{1}{2-\alpha}}.
\end{cases} \tag{3.7}
\]

From (3.7), we get the following rough bounds for all \( x > 0 \),

\[
P \left( \sum_{i=1}^{n} \zeta_i \geq x \right) \leq \begin{cases} 
c_\alpha \exp \left\{ -\frac{x^2}{2c_{1,\alpha} u_n} \right\} & \text{if } 0 \leq x < (c u_n)^{1/(2-\alpha)} \\
c_\alpha \exp \left\{ -\frac{c}{2} x^\alpha \right\} & \text{if } x \geq (c u_n)^{1/(2-\alpha)}
\end{cases}
\]

\[
\leq c_\alpha \exp \left\{ -\frac{x^2}{2c_{\alpha} (u_n + x^{2-\alpha})} \right\}.
\]

This completes the proof of Lemma 3.4. \( \square \)
In the proof of Theorem 2.1, we also need the following normalized Cramér-type moderate deviations for martingales; see [2, 4]. Let \((\xi_i, \mathcal{F}_i)_{i=0,...,n}\) be a finite sequence of martingale differences. Set \(X_k = \sum_{i=1}^{k} \xi_i, k = 1,...,n\). Denote by \(\langle X \rangle\) the quadratic characteristic of the martingale \(X = (X_k, \mathcal{F}_k)_{k=0,...,n}\), that is
\[
\langle X \rangle_0 = 0, \quad \langle X \rangle_k = \sum_{i=1}^{k} \mathbb{E}(\xi_i^2 | \mathcal{F}_{i-1}), \quad k = 1,...,n.
\]

In the sequel we shall use the following conditions:

(A1) There exists a number \(\epsilon_n \in (0, \frac{1}{2}]\) such that
\[
|\mathbb{E}(\xi_i^k | \mathcal{F}_{i-1})| \leq \frac{1}{2} k! k^{-2} \mathbb{E}(\xi_i^2 | \mathcal{F}_{i-1}), \quad \text{for all } k \geq 2 \text{ and } 1 \leq i \leq n;
\]

(A2) There exist a number \(\delta_n \in (0, \frac{1}{2}]\) and a positive constant \(C_1\) such that for all \(x > 0\),
\[
\mathbb{P}(\langle X \rangle_n - 1 \geq x) \leq C_1 \exp\{-x^2 \delta_n^{-2}\}.
\]

Lemma 3.5. Assume that conditions (A1) and (A2) are satisfied. Then the following inequality holds for all \(0 \leq x = o(\min\{\epsilon_n^{-1}, \delta_n^{-1}\})\),
\[
\left| \ln \frac{\mathbb{P}(X_n/ \sqrt{\langle X \rangle_n} > x)}{1 - \Phi(x)} \right| \leq C \left( x^3(\epsilon_n + \delta_n) + (1 + x)(\delta_n |\ln \delta_n| + \epsilon_n |\ln \epsilon_n|) \right).
\]

In the proof of Theorem 2.2, we make use of the following lemma.

Lemma 3.6. It holds for all \(y > 0\),
\[
\mathbb{P}\left( \left| \sum_{i=0}^{[\eta^{-2}] - 1} \eta^{-1} \left( (\theta_{i+1} - \theta_i - \eta g(\theta_i))^T \nabla \varphi(\theta_i) \right)^2 - \|\sigma^T \nabla \varphi(\theta_i)\|_2^2 \right) > y \right) \leq c \exp\left\{- \frac{y^2}{c_1(\eta^{-2} + cy)} \right\},
\]
where \(c_1\) and \(c\) depend on \(g\) and \(\sigma\).

Proof. Denote by \(\psi_{k+1} = \eta^{-1} \left( (\theta_{k+1} - \theta_k - \eta g(\theta_k))^T \nabla \varphi(\theta_k) \right)^2 - \|\sigma^T \nabla \varphi(\theta_k)\|_2^2\). By (1.2), the random variable \(\psi_{k+1}\) can be rewritten as
\[
\psi_{k+1} = \| (\sigma \xi_{k+1})^T \nabla \varphi(\theta_k) \|_2^2 - \|\sigma^T \nabla \varphi(\theta_k)\|_2^2.
\]
Set \(\mathcal{F}_n = \sigma(\theta_0, \xi_k, 1 \leq k \leq n)\). It is easy to see that \(\mathbb{E}(\psi_{k+1} | \mathcal{F}_k) = 0\), and thus \((\psi_i, \mathcal{F}_i)_{i \geq 1}\) is a sequence of martingale differences. By Lemma 3.1, we deduce that
\[
|\psi_{k+1}| \leq |\nabla^2 \varphi(\theta_k)| \cdot \| (\sigma \xi_{k+1})(\sigma \xi_{k+1})^T - \sigma \sigma^T \| \\
\leq c (1 + \| \xi_{k+1} \|_2^2).
\]
The last line implies that there exists a small positive constant \(c\) such that
\[
\mathbb{E}(\psi_{k+1}^2 \exp\{c|\psi_{k+1}|\} | \mathcal{F}_k) < \infty.
\]
Therefore, by Lemma 3.4, we have for all $y \geq c \eta^{1/2}$,

$$
P \left( \left| \sum_{i=0}^{[\eta^{-2}] - 1} \left( \eta^{-1} \left\| \left( \theta_{i+1} - \theta_i - \eta g(\theta_i) \right)^T \nabla \varphi(\theta_i) \right\|_2^2 - \| \sigma^T \nabla \varphi(\theta_i) \|_2^2 \right) \right| < y \right) 
\leq P \left( \left( \sum_{i=0}^{[\eta^{-2}] - 1} \psi_i \right) \right) > y 
\leq c \exp \left( - \frac{y^2}{c_1 (\eta^{-2} + cy)} \right).
$$

This completes the proof of Lemma 3.6. □

4. Proof of Theorem 2.1

Now we are in position to prove Theorem 2.1. Without loss of generality, we assume from now on that $\eta^{-2}$ is an integer. From equality (3.1) of Lu, Tan and Xu [8], we have

$$
\eta^{-1/2} (\Pi_\eta (h) - \pi (h)) = \mathcal{H}_\eta + \mathcal{R}_\eta,
$$

where

$$
\mathcal{H}_\eta = - \eta \sum_{k=0}^{m-1} \langle \nabla \varphi(\theta_k), \sigma \xi_{k+1} \rangle 
\text{ and } \mathcal{R}_\eta = - \sum_{i=1}^{6} \mathcal{R}_{\eta,i},
$$

with $m = \eta^{-2}$,

$$
\mathcal{R}_{\eta,1} = \sqrt{\eta} (\varphi(\theta_0) - \varphi(\theta_m)),
$$

$$
\mathcal{R}_{\eta,2} = \frac{\eta}{2} \sum_{k=0}^{m-1} \langle \nabla^2 \varphi(\theta_k), (\sigma \xi_{k+1})^T - \sigma^T \rangle_{\mathcal{H}^2},
$$

$$
\mathcal{R}_{\eta,3} = \frac{\eta}{2} \sum_{k=0}^{m-1} \langle \nabla^2 \varphi(\theta_k), g(\theta_k)(\sigma \xi_{k+1})^T \rangle_{\mathcal{H}^2} + \langle \nabla^2 \varphi(\theta_k), \sigma \xi_{k+1} g(\theta_k)^T \rangle_{\mathcal{H}^2},
$$

$$
\mathcal{R}_{\eta,4} = \frac{\eta}{6} \sum_{k=0}^{m-1} \int_0^1 \sum_{i_1, i_2, i_3 = 1}^d \nabla^3_{i_1, i_2, i_3} \varphi(\theta_k + t \triangle \theta_k)(\sigma \xi_{k+1})_{i_1}(\sigma \xi_{k+1})_{i_2}(\sigma \xi_{k+1})_{i_3} dt,
$$

$$
\mathcal{R}_{\eta,5} = \frac{\eta^{3/2}}{2} \sum_{k=0}^{m-1} \langle \nabla^2 \varphi(\theta_k), g(\theta_k)g(\theta_k)^T \rangle_{\mathcal{H}^2}
$$

$$
+ \frac{\eta^{7/2}}{6} \sum_{k=0}^{m-1} \int_0^1 \sum_{i_1, i_2, i_3 = 1}^d \nabla^3_{i_1, i_2, i_3} \varphi(\theta_k + t \triangle \theta_k)(g(\theta_k))_{i_1}(g(\theta_k))_{i_2}(g(\theta_k))_{i_3} dt,
$$

$$
\mathcal{R}_{\eta,6} = \frac{\eta^{5/2}}{2} \sum_{k=0}^{m-1} \int_0^1 \sum_{i_1, i_2, i_3 = 1}^d \left[ \nabla^3_{i_1, i_2, i_3} \varphi(\theta_k + t \triangle \theta_k)(\sigma \xi_{k+1})_{i_1}(\sigma \xi_{k+1})_{i_2}(\sigma \xi_{k+1})_{i_3}
$$

$$
+ \sqrt{\eta} \nabla^3_{i_1, i_2, i_3} \varphi(\theta_k + t \triangle \theta_k)(g(\theta_k))_{i_1}(g(\theta_k))_{i_2}(\sigma \xi_{k+1})_{i_3} \right] dt.
$$
Notice that for all $0 \leq x = o(\eta^{-1})$ and $y = C_0(\eta^{1/2}x + (\eta|\ln \eta|)^{1/2})$ with $C_0$ large enough, we have
\[
P(W_\eta \geq x) = P\left(\frac{\mathcal{H}_\eta + \mathcal{R}_\eta}{\sqrt{\mathcal{V}_\eta}} \geq x\right) \leq P\left(\frac{\mathcal{H}_\eta}{\sqrt{\mathcal{V}_\eta}} \geq x - y\right) + P\left(\frac{\mathcal{R}_\eta}{\sqrt{\mathcal{V}_\eta}} \geq y\right).
\]

Next, we give an estimation for the first term in the r.h.s. of the last inequality. Set $\mathcal{F}_n = \sigma(\theta_0, \xi_k, 1 \leq k \leq n)$. Then $(-\eta(\nabla \varphi(\theta_k), \sigma \xi_{k+1}), \mathcal{F}_{k+1})_{k \geq 0}$ is a sequence of martingale differences. Since the normal random variable satisfies the Bernstein condition, by the boundedness of $||\nabla \varphi||$ (cf. Lemma 3.1), it holds for all $k \geq 2$,
\[
E((-\eta(\nabla \varphi(\theta_k), \sigma \xi_{k+1}))^k | \mathcal{F}_k) \leq \frac{1}{2} k! (c||\sigma||\eta)^k - 2 E((-\eta(\nabla \varphi(\theta_k), \sigma \xi_{k+1}))^2 | \mathcal{F}_k)
\]
and
\[
\langle \mathcal{H}_\eta \rangle_m \equiv \sum_{k=0}^{m-1} E((-\eta(\nabla \varphi(\theta_k), \sigma \xi_{k+1}))^2 | \mathcal{F}_k) = \mathcal{Y}_\eta.
\]

By Lemma 3.3 with $k = \eta^{-2}$, we have for all $y > 0$,
\[
P\left(|\mathcal{Y}_\eta/E\mathcal{Y}_\eta - 1| \geq y\right) \leq 2 \exp\left\{-c y^2 \eta^{-2}\right\}.
\]

By Theorem 3.5, we get for all $0 \leq x = o(\eta^{-1})$,
\[
P\left(\frac{\mathcal{H}_\eta}{\sqrt{\mathcal{V}_\eta}} \geq x - y\right) = P\left(\frac{\mathcal{H}_\eta}{\sqrt{\mathcal{V}_\eta}} \geq x - y\right) \leq 1 - \Phi(x - y)
\]
\[
\leq \exp\left\{c_1 \left(\frac{1}{2} x^3 + (1 + x)\eta|\ln \eta|\right)\right\} \exp\left\{c_2 C_0x \left(\eta^{1/2} + (\eta|\ln \eta|)^{1/2}\right)\right\}
\]
\[
\leq \exp\left\{c_3 \left(\frac{1}{2} x^3 + x^2\frac{1}{2} + x(\eta|\ln \eta|)^{1/2} + \eta|\ln \eta|\right)\right\}
\]
\[
\leq \exp\left\{c_4 \left(\frac{1}{2} x^3 + x^2\frac{1}{2} + (1 + x)(\eta|\ln \eta|)^{1/2}\right)\right\}.
\]

Next, we give some estimations for the tail probability $P(|\mathcal{R}_\eta| \geq y)$ for $0 < y = o(\eta^{-1})$. Clearly, it holds
\[
P(|\mathcal{R}_\eta| \geq y) \leq \sum_{i=1}^{6} P(|\mathcal{R}_{\eta,i}| \geq y/6) =: \sum_{i=1}^{6} I_i.
\]

We now give estimates for $I_i, i = 1, 2, ..., 6$.

**a) Control of $I_1$.** First, by the boundedness of $|\varphi|$ (cf. Lemma 3.1), we have $|\mathcal{R}_{\eta,1}| \leq c_1 \eta^{1/2}$, and thus for all $y \geq \eta^{1/2},$
\[
I_1 \leq c_2 e^{-y^2 \eta^{-1}}.
\]

**b) Control of $I_2$.** Denote $\zeta_{k+1} = (\nabla \varphi(\theta_k), (\sigma \xi_{k+1})(\sigma \xi_{k+1})^T - \sigma \sigma^T)_{HS}$. Then it is easy to see that $E(\zeta_{k+1} | \mathcal{F}_k) = 0$ and, by Lemma 3.1, that
\[
|\zeta_{k+1}| \leq ||\nabla \varphi(\theta_k)|| \cdot ||(\sigma \xi_{k+1})(\sigma \xi_{k+1})^T - \sigma \sigma^T||
\]
\[
\leq c (1 + ||\xi_{k+1}||^2_2).
\]
The last line implies that there exists a small positive constant $c$ such that
\[
E(|\xi_{k+1}|^2 \exp\{c|\xi_{k+1}|\} | \mathcal{F}_k) < \infty.
\]

Therefore, we have for all $y \geq c\eta^{1/2}$,
\[
I_2 \leq P\left( \sum_{k=0}^{m-1} |\xi_{k+1}| \geq c_1 y \eta^{-3/2} \right) \leq c \exp\left\{ - \frac{(c y \eta^{-3/2})^2}{c_1 (\eta^{-2} + c y \eta^{-3/2})} \right\}
\]
\[
\leq c_1 \exp\left\{ - \frac{y^2 \eta^{-1}}{c_2 (1 + \eta^{1/2} y)} \right\}.
\]

(4.3)

\[\text{c) Control of } I_3. \text{ The following inequality holds for all } y \geq c\eta^{1/2},\]
\[
I_3 \leq c_1 \exp\left\{ - c_2 y \eta^{-3/2} \right\};
\]

(4.4)

see inequality (1.10) in [9].

\[\text{d) Control of } I_4. \text{ It is easy to see that for all } y > 0,\]
\[
I_4 := P\left(|R_{\eta,4}| \geq y/6\right)
\]
\[
\leq P\left(\sum_{k=0}^{m-1} \int_0^1 \sum_{i_1,i_2,i_3=1}^{d} \nabla^3_{i_1,i_2,i_3} \varphi(\theta_k + t \triangle \theta_k)(\sigma \xi_{k+1})_{i_1}(\sigma \xi_{k+1})_{i_2}(\sigma \xi_{k+1})_{i_3} dt \geq y \eta^{-2}\right)
\]
\[
\leq P\left(\sum_{k=0}^{m-1} \int_0^1 \sum_{i_1,i_2,i_3=1}^{d} (\nabla^3_{i_1,i_2,i_3} \varphi(\theta_k + t \triangle \theta_k) - \nabla^3_{i_1,i_2,i_3} \varphi(\theta_k))(\sigma \xi_{k+1})_{i_1}(\sigma \xi_{k+1})_{i_2}(\sigma \xi_{k+1})_{i_3} dt \geq \frac{1}{2} y \eta^{-2}\right)
\]
\[
+ P\left(\sum_{k=0}^{m-1} \int_0^1 \sum_{i_1,i_2,i_3=1}^{d} \nabla^3_{i_1,i_2,i_3} \varphi(\theta_k)(\sigma \xi_{k+1})_{i_1}(\sigma \xi_{k+1})_{i_2}(\sigma \xi_{k+1})_{i_3} dt \geq \frac{1}{2} y \eta^{-2}\right)
\]
\[
\leq P\left(\sum_{k=0}^{m-1} \int_0^1 \sum_{i_1,i_2,i_3=1}^{d} \nabla^4_{i_1,i_2,i_3,i_4} \varphi(\theta_k + tt' \triangle \theta_k)(t \triangle \theta_k)_{i_4}(\sigma \xi_{k+1})_{i_1}(\sigma \xi_{k+1})_{i_2}(\sigma \xi_{k+1})_{i_3} dt' dt \geq \frac{1}{2} y \eta^{-2}\right)
\]
\[
+ P\left(\sum_{k=0}^{m-1} \int_0^1 \sum_{i_1,i_2,i_3=1}^{d} \nabla^3_{i_1,i_2,i_3} \varphi(\theta_k)(\sigma \xi_{k+1})_{i_1}(\sigma \xi_{k+1})_{i_2}(\sigma \xi_{k+1})_{i_3} dt \geq \frac{1}{2} y \eta^{-2}\right)
\]
\[=: I_{\eta,4.1} + I_{\eta,4.2}.\]  

(4.5)

We first estimate $I_{\eta,4.1}$. Denote \( \triangle \theta_k = \theta_{k+1} - \theta_k \). By the boundedness of \( ||\nabla^4 \varphi|| \) and the fact \( \triangle \theta_k = \)
\( \eta g(\theta_k) + \sqrt{\eta} \sigma \xi_{k+1} \), we deduce that for all \( y > 0 \),
\[
I_{\eta,4,1} \leq P \left( \sum_{k=0}^{m-1} \sum_{i_1,i_2,i_3=1}^d |(\Delta \theta_k)_{i_4}(\sigma \xi_{k+1})_{i_1}(\sigma \xi_{k+1})_{i_2}(\sigma \xi_{k+1})_{i_3} | \geq c_1 y \eta^{-2} \right) 
\leq P \left( \sum_{k=0}^{m-1} \| \eta g(\theta_k) + \sqrt{\eta} \sigma \xi_{k+1} \|_2 \sigma \xi_{k+1} \|_2^2 \geq 2 c_2 y \eta^{-2} \right) 
\leq P \left( \sum_{k=0}^{m-1} \sqrt{\eta} \sigma \xi_{k+1} \|_2^2 \geq c_2 y \eta^{-2} \right) + P \left( \sum_{k=0}^{m-1} \| \eta g(\theta_k) \|_2 \sigma \xi_{k+1} \|_2^2 \geq c_2 y \eta^{-2} \right) 
=: I'_{\eta,4,1} + I''_{\eta,4,1}.
\]
Next, we estimate \( I'_{\eta,4,1} \). It is easy to see that
\[
I'_{\eta,4,1} \leq P \left( \sum_{k=0}^{m-1} \| \xi_{k+1} \|_2^2 \geq c_2 y \eta^{-5/2} \right) 
\leq P \left( \sum_{k=0}^{m-1} \left( \| \xi_{k+1} \|_2^2 - E \| \xi_{k+1} \|_2^4 \right) \geq \left( c_2 y \eta^{-5/2} - \sum_{k=0}^{m-1} E \| \xi_{k+1} \|_2^4 \right) \right).
\]
Clearly, there exists a small positive constant \( c \) such that
\[
E \left( \| \xi_{k+1} \|_2^4 - E \| \xi_{k+1} \|_2^4 \right)^2 \exp \left\{ c \| \xi_{k+1} \|_2^4 - E \| \xi_{k+1} \|_2^4 \right\} | \mathcal{F}_k | < \infty.
\]
Using Lemma 3.4 with \( \alpha = 1/2 \), we have for all \( y \geq c' \eta^{1/2} \) with \( c' \) large enough,
\[
I'_{\eta,4,1} \leq c \exp \left\{ - \frac{(y \eta^{-5/2})^2}{c_1 (\eta^{-2} + (y \eta^{-5/2})^{3/2})} \right\} 
\leq c \exp \left\{ - c_2 y^{1/2} \eta^{-5/4} \right\}.
\]
In the sequel, we estimate \( I''_{\eta,4,1} \). Using Hölder’s inequality, we get for all \( y > 0 \),
\[
I''_{\eta,4,1} \leq P \left( \eta \left( \sum_{k=0}^{m-1} \| g(\theta_k) \|_2^6 \right)^{1/2} \left( \sum_{k=0}^{m-1} \| \sigma \xi_{k+1} \|_2^6 \right)^{1/2} \geq c_2 y \eta^{-2} \right) 
\leq P \left( \eta \left( \sum_{k=0}^{m-1} \| g(\theta_k) \|_2^6 \right)^{1/2} \left( \sum_{k=0}^{m-1} \| \sigma \xi_{k+1} \|_2^6 \right)^{1/2} \geq c_2 y \eta^{-2} \right) + \eta \sum_{k=0}^{m-1} \| g(\theta_k) \|_2^2 \geq C y^{1/2} \eta^{-5/4} \right) 
\leq P \left( \eta \sum_{k=0}^{m-1} \| g(\theta_k) \|_2^2 \geq C y^{1/2} \eta^{-5/4} \right) + P \left( \left( C y^{1/2} \eta^{-9/4} \right)^{1/2} \sum_{k=0}^{m-1} \| \sigma \xi_{k+1} \|_2^6 \right)^{1/2} \geq c_2 y \eta^{-2} \right) 
\leq \eta \sum_{k=0}^{m-1} \| g(\theta_k) \|_2^2 \geq C y^{1/2} \eta^{-5/4} \right) + \sum_{k=0}^{m-1} \| \sigma \xi_{k+1} \|_2^6 \geq c_2 y^{3/2} \eta^{-15/4} \right). \quad (4.6)
\]
Next, we give an estimation for the second term in the last inequality. Denote $T_{k+1} = \|\sigma\xi_k + 1\|_2^6 - E\|\sigma\xi_k + 1\|_2^6$. It is easy to see that there exists a small positive constant $c$ such that

$$E(T_{k+1}^2 \exp\{c |T_{k+1}|^{1/3}\} | F_k) < \infty.$$ 

Using Lemma 3.4 with $\alpha = 1/3$, we have for all $y \geq c' \eta^{1/2}$ with $c'$ large enough,

$$P\left( \sum_{k=0}^{m-1} \|\sigma\xi_k + 1\|_2^6 \geq c_2 y^{3/2} \eta^{-15/4} \right) = P\left( \sum_{k=0}^{m-1} T_{k+1} \geq \left( c_2 y^{3/2} \eta^{-15/4} - \sum_{k=0}^{m-1} E\|\sigma\xi_k + 1\|_2^6 \right) \right)
\leq c \exp\left\{ - \frac{(y^{3/2} \eta^{-15/4})^2}{c_1 (\eta^2 + (y^{3/2} \eta^{-15/4})^{5/3})} \right\}
\leq c \exp\left\{ - c_2 y^{1/2} \eta^{-5/4} \right\}.$$

From (4.6), by the last inequality and Lemma 3.2, we deduce that for all $y \geq c' \eta^{1/2}$,

$$I''_{\eta,4,1} \leq \exp\left\{ - c_4 y^{1/2} \eta^{-5/4} \right\} + c \exp\left\{ - c_2 y^{1/2} \eta^{-5/4} \right\}
\leq c_1 \exp\left\{ - c_2 y^{1/2} \eta^{-5/4} \right\} + c \exp\left\{ - c_2 y^{1/2} \eta^{-5/4} \right\}
\leq c_3 \exp\left\{ - c_2 y^{1/2} \eta^{-5/4} \right\},$$

with $c'$ and $C$ large enough. Combining the estimations of $I'_{\eta,4,1}$ and $I''_{\eta,4,2}$, we have the following estimation for $I_{\eta,4,1}$: for all $y \geq c' \eta^{1/2}$ with $c'$ large enough,

$$I_{\eta,4,1} \leq I'_{\eta,4,1} + I''_{\eta,4,2} \leq c_1 \exp\left\{ - c_2 y^{1/2} \eta^{-5/4} \right\}.$$

Next, we estimate $I_{\eta,4,2}$. Denote

$$h_{k+1} = \int_0^1 \sum_{i_1,i_2,i_3=1}^d \nabla^3_{i_1,i_2,i_3} \varphi(\theta_k)(\sigma\xi_k + 1)_{i_1}(\sigma\xi_k + 1)_{i_2}(\sigma\xi_k + 1)_{i_3} dt.$$

Then $(h_{k+1}, F_{k+1})_{k \geq 0}$ is a sequence of martingale differences. Moreover, by the boundedness of $\|\nabla^3 \varphi\|$, we have

$$|h_{k+1}| \leq c \sum_{k=0}^{m-1} \|\xi_k + 1\|_2^3.$$

Therefore, there exists a small positive constant $c$ such that $E(|h_{k+1}|^2 \exp\{c |h_{k+1}|^{2/3}\} | F_k) < \infty$. Using
Lemma 3.4 with $\alpha = 2/3$, we have for all $y \geq c' \eta^{1/2}$ with $c'$ large enough,

$$
I_{\eta,4,2} \leq P \left( \sum_{k=0}^{m-1} h_{k+1} \geq c\eta^{-2} \right) = P \left( \sum_{k=0}^{m-1} \left( h_{k+1} - Eh_{k+1} \right) \geq \left( c\eta^{-2} - \sum_{k=0}^{m-1} Eh_{k+1} \right) \right) 
\leq c \exp \left\{ - \frac{(y\eta^{-2})^2}{c_1(\eta^{-2} + (y\eta^{-2})^{4/3})} \right\} 
\leq c \exp \left\{ - c_2 y^{2/3} \eta^{-4/3} \right\} 
\leq c \exp \left\{ - c_3 y^{1/2} \eta^{-5/4} \right\}.
$$

Hence, from (4.5), we get for all $y \geq c' \eta^{1/2}$ with $c'$ large enough,

$$
I_4 \leq I_{\eta,4,1} + I_{\eta,4,2} \leq c_1 \exp \left\{ - c_2 y^{1/2} \eta^{-5/4} \right\}.
$$

e) Control of $I_5$. By the boundedness of $\|\nabla^2 \varphi\|$ and $\|\nabla^3 \varphi\|$ and Hölder’s inequality, we deduce that for all $y > 0$,

$$
I_5 \leq P \left( \eta^{5/2} \sum_{k=0}^{m-1} \|g(\theta_k)\|^2_2 \geq cy \right) + P \left( \eta^{7/2} \sum_{k=0}^{m-1} \|g(\theta_k)\|^3_2 \geq cy \right) 
\leq P \left( \eta \sum_{k=0}^{m-1} \|g(\theta_k)\|^2_2 \geq cy \eta^{-3/2} \right) + P \left( \eta^{7/2} m^{1/4} \left( \sum_{k=0}^{m-1} \|g(\theta_k)\|^2_2 \right)^{3/2} \geq cy \right) 
\leq P \left( \eta \sum_{k=0}^{m-1} \|g(\theta_k)\|^2_2 \geq cy \eta^{-3/2} \right) + P \left( \eta \sum_{k=0}^{m-1} \|g(\theta_k)\|^3_2 \geq cy \right) 
\leq P \left( \eta \sum_{k=0}^{m-1} \|g(\theta_k)\|^2_2 \geq cy \eta^{-3/2} \right) + P \left( \eta \sum_{k=0}^{m-1} \|g(\theta_k)\|^3_2 \geq cy^{2/3} \eta^{-2} \right).
$$

By Lemma 3.2, we get for all $y \geq c' \eta^{1/2}$ with $c'$ large enough,

$$
I_5 \leq c \exp \left\{ c_1 \eta^{-1} - c_2 y \eta^{-3/2} \right\} + c \exp \left\{ c_1 \eta^{-1} - c_3 y^{2/3} \eta^{-2} \right\} 
\leq c \exp \left\{ - c_2 y^{1/2} \eta^{-5/4} \right\}.
$$

f) Control of $I_6$. By the boundedness of $\|\nabla^3 \varphi\|$, one has for all $y > 0$,

$$
I_6 \leq P \left( \eta^{5/2} \sum_{k=0}^{m-1} \left( \|g(\theta_k)\|_2 \|\sigma \xi_{k+1}\|_2 + \sqrt{\eta} \|g(\theta_k)\|_2^2 \|\xi_{k+1}\|_2 \right) \geq cy \right) 
\leq P \left( \sum_{k=0}^{m-1} \|g(\theta_k)\|_2 \|\sigma \xi_{k+1}\|_2 \geq c y \eta^{-5/2} \right) + P \left( \sum_{k=0}^{m-1} \|g(\theta_k)\|_2^2 \|\xi_{k+1}\|_2 \geq c y \eta^{-3} \right) 
=: I_{6,1} + I_{6,2}.
$$

(4.7)
By Hölder’s inequality, we have for $C$ large enough and all $y > 0$,

$$I_{6,1} \leq P\left(\left(\sum_{k=0}^{m-1} \|g(\theta_k)\|_2^2\right)^{1/2} \left(\sum_{k=0}^{m-1} \|\sigma \xi_{k+1}\|_2^2\right)^{1/2} \geq cy^{-5/2}\right)$$

$$\leq P\left(\left(\sum_{k=0}^{m-1} \|g(\theta_k)\|_2^2\right)^{1/2} \left(\sum_{k=0}^{m-1} \|\sigma \xi_{k+1}\|_2^2\right)^{1/2} \geq c_2y^{-5/2}, \quad \eta \sum_{k=0}^{m-1} \|g(\theta_k)\|_2 \geq C y^{1/2} \eta^{-5/4}\right)$$

$$+ P\left(\left(\sum_{k=0}^{m-1} \|g(\theta_k)\|_2^2\right)^{1/2} \left(\sum_{k=0}^{m-1} \|\sigma \xi_{k+1}\|_2^2\right)^{1/2} \geq c_2y^{-5/2}, \quad \eta \sum_{k=0}^{m-1} \|g(\theta_k)\|_2 < C y^{1/2} \eta^{-5/4}\right)$$

$$\leq P\left(\eta \sum_{k=0}^{m-1} \|g(\theta_k)\|_2 \geq C y^{1/2} \eta^{-5/4}\right) + P\left(C^{1/2} \left(\sum_{k=0}^{m-1} \|\sigma \xi_{k+1}\|_2^2\right)^{1/2} \geq c_2y^{3/4} \eta^{-19/8}\right)$$

$$\leq c_1 \exp\left\{-c_2 y^{1/2} \eta^{-5/4}\right\} + P\left(\sum_{k=0}^{m-1} \|\sigma \xi_{k+1}\|_2^2 \geq c_3 y^{3/2} \eta^{-19/4}\right).$$

There exists a small positive constant $c$ such that

$$E(\|\sigma \xi_{k+1}\|_2^2 - E[\|\sigma \xi_{k+1}\|_2^2])^2 \exp\{c \|\sigma \xi_{k+1}\|_2^2 - E[\|\sigma \xi_{k+1}\|_2^2]\} | \mathcal{F}_k) < \infty.$$

Using Lemma 3.4 with $\alpha = 1$, we have for all $y \geq \eta^{1/2}$,

$$P\left(\sum_{k=0}^{m-1} \|\sigma \xi_{k+1}\|_2^2 \geq c_3 y^{3/2} \eta^{-19/4}\right)$$

$$= P\left(\sum_{k=0}^{m-1} (\|\sigma \xi_{k+1}\|_2 - E[\|\sigma \xi_{k+1}\|_2]) \geq (c_1 y^{3/2} \eta^{-19/4} - \sum_{k=0}^{m-1} E[\|\sigma \xi_{k+1}\|_2]))\right)$$

$$\leq c \exp\left\{-\frac{(y^{3/2} \eta^{-19/4})^2}{c_2(\eta^{-2} + (y^{3/2} \eta^{-19/4}))}\right\} \leq c \exp\left\{-c_3 y^{3/2} \eta^{-19/4}\right\}$$

$$\leq c \exp\left\{-c_3 y^{1/2} \eta^{-5/4}\right\}.$$

Hence, we get for all $y \geq \eta^{1/2}$,

$$I_{6,1} \leq c_1 \exp\left\{-c_2 y^{1/2} \eta^{-5/4}\right\}.$$
For $I_{6,2}$, by lemma 3.2, we have the following estimation for all $y \geq C\eta^{1/2}$ with $C$ large enough,

$$I_{6,2} = \mathbb{P} \left( \sum_{k=0}^{m-1} \|g(\theta_k)\|_2^2 \sigma_{\xi_{k+1}} \geq cy\eta^{-3} \right)$$

$$\leq \mathbb{P} \left( \sum_{k=0}^{m-1} \|g(\theta_k)\|_2^2 Cy^{1/2}\eta^{-3/4} \geq cy\eta^{-3} \right) + \mathbb{P} \left( \|\sigma_{\xi_{k+1}}\|_2 \geq Cy^{1/2}\eta^{-3/4} \right)$$

$$\leq \mathbb{P} \left( \eta \sum_{k=0}^{m-1} \|g(\theta_k)\|_2^2 \geq Cy^{1/2}\eta^{-5/4} \right) + \eta^{-2} \exp \left\{ -C_1 y\eta^{-3/2} \right\}$$

$$\leq c_1 \exp \left\{ -c_2 y^{1/2}\eta^{-5/4} \right\} + \eta^{-2} \exp \left\{ -C_2 y\eta^{-3/2} \right\}$$

$$\leq c_3 \exp \left\{ -c_4 y^{1/2}\eta^{-5/4} \right\}.$$ 

Hence, we have for all $y \geq c'\eta^{1/2}$ with $c'$ large enough,

$$I_6 \leq I_{6,1} + I_{6,2} \leq c_1 \exp \left\{ -c_2 y^{1/2}\eta^{-5/4} \right\}. \quad (4.8)$$

Thus, by the estimations of $I_i, 1 \leq i \leq 6$, we have for all $y \geq c'\eta^{1/2}$ with $c'$ large enough,

$$\mathbb{P}(|R_\eta| \geq y) \leq c_1 \left[ \exp \left\{ -\frac{y^{2}\eta^{-1}}{c_2 (1 + \eta^{1/2}y)} \right\} + \exp \left\{ -c_3 y^{1/2}\eta^{-5/4} \right\} \right].$$

Next, we give an estimation for $\mathbb{P} \left( \frac{R_\eta}{\sqrt{E\eta}} \geq y \right)$. Clearly, it holds

$$\mathbb{P} \left( \frac{R_\eta}{\sqrt{E\eta}} \geq y \right) \leq \mathbb{P} \left( \frac{R_\eta}{\sqrt{E\eta}} \geq y, \mathcal{Y}_\eta \leq E\mathcal{Y}_\eta - \frac{1}{2}E\mathcal{Y}_\eta \right) + \mathbb{P} \left( \mathcal{Y}_\eta \leq E\mathcal{Y}_\eta - \frac{1}{2}E\mathcal{Y}_\eta \right)$$

$$\leq \mathbb{P} \left( \frac{R_\eta}{\sqrt{E\eta}} \geq y \right) + \mathbb{P} \left( \mathcal{Y}_\eta \leq E\mathcal{Y}_\eta - \frac{1}{2}E\mathcal{Y}_\eta \right).$$

By stationarity of $\theta_k$ and $y = C_0(x\eta^{1/2} + (\eta|\ln\eta|)^{1/2})$, we have for all $y \geq c\eta^{1/2}$ with $c$ large enough,

$$\mathbb{P} \left( \frac{R_\eta}{\sqrt{E\eta}} \geq y \right) \leq c_1 \left[ \exp \left\{ -\frac{y^{2}\eta^{-1}}{c_2 (1 + \eta^{1/2}y)} \right\} + \exp \left\{ -c_3 y^{1/2}\eta^{-5/4} \right\} \right]$$

$$\leq 2c_1 \left[ \exp \left\{ -\frac{y^{2}\eta^{-1}}{c_2 (1 + \eta^{1/2}y)} \right\} I_{\{c_0^{1/2} \leq \eta^{-1/6}\}} + \exp \left\{ -c' y^{1/2}\eta^{-5/4} \right\} I_{\{y > \eta^{-1/6}\}} \right]$$

$$\leq c_4 \left[ \exp \left\{ -c_5 C_0^2 (x + \sqrt{|\ln\eta|}) \right\} I_{\{0 \leq x \leq \eta^{-2/3}\}} \right]$$

$$\quad + \exp \left\{ -c_6 x^{1/2}\eta^{-1} \right\} I_{\{x > \eta^{-2/3}\}}.$$
and, by Lemma 3.3,
\[ P\left( \mathcal{Y}_\eta \leq \mathcal{E}_\eta - \frac{1}{2} \mathcal{E}_\eta \right) = P\left( \frac{1}{2} \mathcal{E}_\eta \leq \mathcal{E}_\eta - \mathcal{Y}_\eta \right) \leq \exp \left\{ -c \eta^{-2} \right\}. \]

Hence, for all \( 0 < x \leq \eta^{-2/3} \),
\[ P\left( \frac{\mathcal{R}_\eta}{\sqrt{\mathcal{Y}_\eta}} \geq y \right) \leq c \exp \left\{ -c_5 C_0^2 \left( x + \sqrt{\ln \eta} \right)^2 \right\}. \] (4.9)

Take \( C_0 \) such that \( c_5 C_0^2 \geq 4 \). Combining the inequalities (4.1) and (4.9) together, we get for all \( 0 \leq x \leq \eta^{-2/3} \),
\[ P(W_\eta \geq x) \leq \frac{P(\mathcal{H}_\eta/\sqrt{\mathcal{Y}_\eta} \geq x - y)}{1 - \Phi(x)} + \frac{P(\mathcal{R}_\eta/\sqrt{\mathcal{Y}_\eta} \geq y)}{1 - \Phi(x)} \]
\[ \leq \exp \left\{ c_2 \left( x^3 \eta + x^2 \eta^{1/2} + x(\eta \ln \eta)^{1/2} + \eta \ln \eta \right) \right\} \]
\[ + \frac{c}{1 - \Phi(x)} \exp \left\{ -c_5 C_0^2 \left( x + \sqrt{\ln \eta} \right)^2 \right\}, \] (4.10)

Similarly, we have for all \( 0 \leq x \leq \eta^{-2/3} \),
\[ P(W_\eta \geq x) \geq \frac{P(\mathcal{H}_\eta/\sqrt{\mathcal{Y}_\eta} \geq x + y)}{1 - \Phi(x)} - \frac{P(\mathcal{R}_\eta/\sqrt{\mathcal{Y}_\eta} \leq -y)}{1 - \Phi(x)} \]
\[ \geq \exp \left\{ - c_2 \left( x^3 \eta + x^2 \eta^{1/2} + (1 + x)(\eta \ln \eta)^{1/2} \right) \right\} \]
\[ - \frac{c}{1 - \Phi(x)} \exp \left\{ -c_5 C_0^2 \left( x + \sqrt{\ln \eta} \right)^2 \right\}, \] (4.11)

Combining the inequalities (4.10) and (4.11) together, we obtain the first desired inequality for all \( 0 \leq x \leq \eta^{-2/3} \). For the case \( \eta^{-2/3} < x \leq \eta^{-3/4} \), the assertion of Theorem 2.1 follows by a similar argument by taking \( y = C_0 x^2 \eta \), instead of \( y = C_0 (x \eta^{1/2} + (\eta \ln \eta)^{1/2}) \), and accordingly in the subsequent statements.

The result for \( -W_\eta \) follows by the first inequality applying to \( -W_\eta \). This completes the proof of Theorem 2.1.

5. Proof of Corollary 2.1

It is easy to see that
\[ \sup_{x \in \mathbb{R}} \left| \mathbb{P}(W_\eta \leq x) - \Phi(x) \right| \leq \sup_{x > \eta^{-1/8}} \left| \mathbb{P}(W_\eta \leq x) - \Phi(x) \right| + \sup_{0 \leq x \leq \eta^{-1/8}} \left| \mathbb{P}(W_\eta \leq x) - \Phi(x) \right| \]
\[ + \sup_{-\eta^{-1/8} \leq x \leq 0} \left| \mathbb{P}(W_\eta \leq x) - \Phi(x) \right| + \sup_{x < -\eta^{-1/8}} \left| \mathbb{P}(W_\eta \leq x) - \Phi(x) \right| \]
\[ =: H_1 + H_2 + H_3 + H_4. \] (5.1)
It is known that
\[
\frac{1}{\sqrt{2\pi(1 + \lambda)}} e^{-\lambda^2/2} \leq 1 - \Phi (\lambda) \leq \frac{1}{\sqrt{\pi(1 + \lambda)}} e^{-\lambda^2/2}, \quad \lambda \geq 0,
\]
see [6]. By Theorem 2.1, we deduce that
\[
H_1 \leq \sup_{x > \eta^{-1/8}} P(W_\eta > x) + \sup_{x > \eta^{-1/8}} (1 - \Phi (x)) \leq P(W_\eta > \eta^{-1/8}) + (1 - \Phi(\eta^{-1/8}))
\]
\[
\leq (1 - \Phi(\eta^{-1/8})) e^c + \exp \left\{ - \frac{1}{2} \eta^{-1/4} \right\}
\]
\[
\leq c_1 (\eta |\ln \eta|)^{1/2}
\]
and
\[
H_4 \leq \sup_{x < -\eta^{-1/8}} P(W_\eta \leq x) + \sup_{x < -\eta^{-1/8}} \Phi (x) \leq P(W_\eta \leq -\eta^{-1/8}) + \Phi(-\eta^{-1/8})
\]
\[
\leq \Phi(-\eta^{-1/8}) e^c + \exp \left\{ - \frac{1}{2} \eta^{-1/4} \right\}
\]
\[
\leq c_2 (\eta |\ln \eta|)^{1/2}.
\]
Using Theorem 2.1 and the inequality $|e^x - 1| \leq |x|e^{|x|}$, we get
\[
H_2 = \sup_{0 \leq x \leq \eta^{-1/8}} \left| P(W_\eta > x) - (1 - \Phi (x)) \right|
\]
\[
\leq \sup_{0 \leq x \leq \eta^{-1/8}} c_1 (1 - \Phi(x)) \left( x^3 \eta + x^2 \eta^{1/2} + (1 + x)(\eta |\ln \eta|)^{1/2} \right)
\]
\[
\leq c_3 (\eta |\ln \eta|)^{1/2}
\]
and
\[
H_3 = \sup_{-\eta^{-1/8} \leq x \leq 0} \left| P(W_\eta \leq x) - \Phi (x) \right|
\]
\[
\leq \sup_{-\eta^{-1/8} \leq x \leq 0} c_1 \Phi(x) \left( |x|^3 \eta + x^2 \eta^{1/2} + (1 + |x|)(\eta |\ln \eta|)^{1/2} \right)
\]
\[
\leq c_4 (\eta |\ln \eta|)^{1/2}.
\]
Applying the bounds of $H_1, H_2, H_3$ and $H_4$ to (5.1), we obtain the desired inequality of Corollary 2.1.
6. Proof of Theorem 2.2

Assume that \( \varepsilon_x \in (0, 1/2] \). It is easy to see that for all \( x \geq 0 \),

\[
P(S_\eta > x) = P(\eta^{-1/2} (\Pi_\eta(h) - \pi(h)) > x \sqrt{\nu_\eta}, \ \nu_\eta \geq (1 - \varepsilon_x) \nu_\eta)
\]

\[
+ P(\eta^{-1/2} (\Pi_\eta(h) - \pi(h)) > x \sqrt{\nu_\eta}, \ \nu_\eta < (1 - \varepsilon_x) \nu_\eta)
\]

\[
\leq P(W_\eta \geq x \sqrt{1 - \varepsilon_x}) + P(\nu_\eta - \nu_\eta < -\varepsilon_x \nu_\eta, \ \nu_\eta \geq \frac{1}{2} \nu_\eta)
\]

\[
+ P(\nu_\eta - \nu_\eta < -\varepsilon_x \nu_\eta, \ \nu_\eta < \frac{1}{2} \nu_\eta)
\]

\[
\leq P(W_\eta \geq x \sqrt{1 - \varepsilon_x}) + P(\nu_\eta - \nu_\eta < -\frac{1}{2} \varepsilon_x \nu_\eta) + P(\nu_\eta - \nu_\eta < -\frac{1}{2} \nu_\eta)
\]

\[
=: P_1 + P_2 + P_3. \quad (6.1)
\]

By Theorem 2.1, we have for all \( 0 \leq x \leq \eta^{-3/4} \),

\[
P_1 \leq \left(1 - \Phi(x \sqrt{1 - \varepsilon_x})\right) \exp \left\{ c \left(x^3 \eta + x^2 \eta^{1/2} + (1 + x)(\eta \left| \ln \eta \right|)^{1/2}\right)\right\}
\]

\[
\leq \left(1 - \Phi(x)\right) \exp \left\{ c \left(x^3 \eta + x^2 \eta^{1/2} + (1 + x)(\eta \left| \ln \eta \right|)^{1/2}\right)\right\}. \quad (6.2)
\]

Using Lemma 3.3, we get for all \( x \geq 0 \),

\[
P_2 \leq 2 \exp \left\{ -c \varepsilon_x^2 \eta^{-2}\right\}. \quad (6.3)
\]

By Lemma 3.6, we deduce that for all \( x \geq 0 \),

\[
P_3 \leq c_1 \exp \left\{ -c \eta^{-2}\right\}. \quad (6.4)
\]

Taking \( \varepsilon_x = c_0 x \eta + \eta^{1/2} \) with \( c_0 \) large enough, by (6.1)-(6.4), we deduce that for all \( 0 \leq x \leq \eta^{-3/4} \),

\[
P(S_\eta > x) \leq \left(1 - \Phi(x)\right) \exp \left\{ c \left(x^3 \eta + x^2 \eta^{1/2} + (1 + x)(\eta \left| \ln \eta \right|)^{1/2}\right)\right\}
\]

\[
+ 2 \exp \left\{ -c \left(c_0^2 x^2 + \eta^{-1}\right)\right\} + c_1 \exp \left\{ -c \eta^{-2}\right\}
\]

Applying (5.2) to the last inequality, we obtain for all \( 0 \leq x \leq \eta^{-3/4} \),

\[
P(S_\eta > x) \leq \left(1 - \Phi(x)\right) \exp \left\{ c \left(x^3 \eta + x^2 \eta^{1/2} + (1 + x)(\eta \left| \ln \eta \right|)^{1/2}\right)\right\}, \quad (6.5)
\]
which gives the upper bound for the tail probability $P(S_\eta > x), x \geq 0$. Notice that for all $x \geq 0$,

$$
P(S_\eta > x) \geq P(\eta^{-1/2}(\Pi_\eta(h) - \pi(h))) > x \sqrt{\mathcal{V}_\eta}, \ \mathcal{V}_\eta < (1 + \varepsilon_x)\mathcal{V}_\eta
$$

$$
\geq P(W_\eta \geq x\sqrt{1 - \varepsilon_x}) - P(\mathcal{V}_\eta - \mathcal{V}_\eta \geq \varepsilon_x\mathcal{V}_\eta, \ \mathcal{V}_\eta \leq \frac{1}{2}\mathcal{E}\mathcal{V}_\eta)
$$

$$
- P(\mathcal{V}_\eta - \mathcal{V}_\eta \geq \varepsilon_x\mathcal{V}_\eta, \ \mathcal{V}_\eta > \frac{1}{2}\mathcal{E}\mathcal{V}_\eta)
$$

$$
\geq P(W_\eta \geq x\sqrt{1 - \varepsilon_x}) - P(\mathcal{V}_\eta - \mathcal{E}\mathcal{V}_\eta \leq \frac{1}{2}\mathcal{E}\mathcal{V}_\eta) - P(\mathcal{V}_\eta - \mathcal{V}_\eta \geq \frac{1}{2}\varepsilon_x\mathcal{E}\mathcal{V}_\eta)
$$

$$
=: P_4 - P_5 - P_6. \quad (6.6)
$$

By Theorem 2.1, we have for all $0 \leq x \leq \eta^{-3/4}$,

$$
P_4 \geq (1 - \Phi(x\sqrt{1 + \varepsilon_x})) \exp \left\{ - c \left( x^3 \eta + x^3 \eta^{1/2} + (1 + x)(\eta|\ln \eta|)^{1/2} \right) \right\}
$$

$$
\geq (1 - \Phi(x)) \exp \left\{ - c \left( x\varepsilon_x + x^3 \eta + x^2 \eta^{1/2} + (1 + x)(\eta|\ln \eta|)^{1/2} \right) \right\}. \quad (6.7)
$$

Using Lemma 3.3 with $k = \eta^{-2}$, we get for all $x \geq 0$,

$$
P_5 \leq 2 \exp \left\{ - c \eta^{-2} \right\}. \quad (6.8)
$$

By Lemma 3.6, we deduce that for all $x \geq 0$,

$$
P_6 \leq c_1 \exp \left\{ - \frac{(\varepsilon_x \eta^{-2})^2}{c_1 (\eta^{-2} + c \varepsilon_x \eta^{-2})} \right\} \leq c_1 \exp \left\{ - c_2 \varepsilon_x \eta^{-2} \right\}. \quad (6.9)
$$

Taking $\varepsilon_x = c_0 x \eta + \eta^{1/2}$ with $c_0$ large enough, by (6.6)-(6.9), we deduce that for all $0 \leq x \leq \eta^{-3/4}$,

$$
P(S_\eta > x) \geq (1 - \Phi(x)) \exp \left\{ - c \left( x^3 \eta + x^2 \eta^{1/2} + (1 + x)(\eta|\ln \eta|)^{1/2} \right) \right\}
$$

$$
- 2 \exp \left\{ - c \eta^{-2} \right\} - c_1 \exp \left\{ - c \left( \frac{c_0^2 x^2 + \eta^{-1}}{c_0} \right) \right\}.
$$

Applying (5.2) to the last inequality, we obtain for all $0 \leq x \leq \eta^{-3/4}$,

$$
P(S_\eta > x) \geq (1 - \Phi(x)) \exp \left\{ - c \left( x^3 \eta + x^2 \eta^{1/2} + (1 + x)(\eta|\ln \eta|)^{1/2} \right) \right\}, \quad (6.10)
$$

which gives the lower bound for the tail probability $P(S_\eta > x), x \geq 0$. The proof for $-S_\eta$ follows by a similar argument. This completes the proof of Theorem 2.2.

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