I. INTRODUCTION

The physics of low-dimensional Fermi gases with attractive interactions has generated interest because of the possibility to realize exotic fermionic superfluids such as the Sarma [1] or Fulde-Ferrell-Larkin-Ovchinnikov (FFLO) state [2, 3] (see Refs. [4–6] for a review). In particular, the one-dimensional analogue of the FFLO state, a state with spatially oscillating pairing correlations, is, at a finite spin imbalance, the ground state of the Fermi-Hubbard model and its continuum analogue, the Yang-Gaudin model (see Refs. [7] and [8] for a review). An actual experimental proof for the presence of a spin-imbalanced Fermi gas in an array of one-dimensional systems and has demonstrated that the observed density profiles are in agreement with theoretical predictions [10–12]. An actual experimental proof for the presence of a spatially modulated quasi-condensate in this system, expected from theory [13–15], is still lacking, which has led to a large number of proposals for schemes to probe FFLO correlations [19–24]. Many of these proposals involve the presence of an optical lattice along the 1D direction suggesting that the Fermi-Hubbard model is the appropriate model. This system has been realized with ultra-cold atoms in several groups [25–27].

One-dimensional gases also provide a natural arena to study the non-equilibrium dynamics of closed, strongly correlated many-body systems and several experiments were specifically tailored to explore this physics [28–35]. For instance, theoretical and experimental research is aiming at understanding the conditions for the emergence of thermalization [36, 37] and the qualitative features in the relaxation dynamics such as time scales for the approach to the steady state (see Refs. [38–39] for recent reviews). Integrable one-dimensional models such as the Fermi-Hubbard model play a special role, since there, thermalization can often only occur with respect to so-called Generalized Gibbs ensembles [40], a subject addressed in a series of recent studies (see, e.g., [41–47]).

In this work we study the real-time dynamics of FFLO correlations and the double occupancy in interaction quantum quenches in the Fermi-Hubbard model with spin imbalance. The Hamiltonian of the system is given by

\[ H = -J \sum_{i=1}^{L-1} (c_{i\sigma}^\dagger c_{i+1\sigma} + h.c.) + U \sum_{i=1}^{L} n_{i\uparrow} n_{i\downarrow}, \]

where \( c_{i\sigma} \) annihilates a fermion with spin \( \sigma = \uparrow, \downarrow \) on site \( i \) of a chain of length \( L \), \( n_{i\sigma} = c_{i\sigma}^\dagger c_{i\sigma} \) is the density operator for the spin-\( \sigma \) component, \( U \) is the onsite interaction and \( J \) the hopping parameter. Moreover, we define the total density, \( n = (N_\uparrow + N_\downarrow)/L \), and the spin polarization, \( p = (N_\uparrow - N_\downarrow)/(N_\uparrow + N_\downarrow) \), \( N_\sigma \) being the number of particles with spin \( \sigma \). The non-equilibrium dynamics in the Fermi-Hubbard model has been studied in recent experiments [48–51]. Our main interest will be in the relaxation dynamics, the identification of relevant time scales for the formation of FFLO correlations and the dependence of observables on post-quench parameters.

Based on two numerical methods, time evolution in a Krylov subspace using exact diagonalization (see, e.g., [52–53]) and the time-dependent density matrix renormalization group technique [54–56], we investigate the behavior of the system following both instantaneous and slow quenches of the interaction strength. In the latter case, we are interested in the crossover to the adiabatic regime (see Refs. [57–58] for studies of slow quenches in the Bose-Hubbard model).

We first present results for quenches from the non-interacting case, \( U = 0 \), to the attractive regime, \( U < 0 \). In particular, we investigate how FFLO correlations emerge in the system and we propose that such a time
scale can be extracted by monitoring the dynamics of natural orbitals, namely the eigenstates of the pair density matrix, and of the pair quasi-momentum distribution function (MDF). This latter quantity is a key observable in the discussion of the FFLO state since the spatially inhomogeneous quasi-condensate translates into maxima in the pair MDF at finite momenta $q$, which is controlled via the polarization $\pi np$.

Moreover, we analyze the relaxation dynamics of the double occupancy, a much discussed quantity in studies of quantum quenches in the repulsive Fermi-Hubbard model, which has been measured in non-equilibrium experiments.

We then turn our attention to quenches from the attractive regime, $U < 0$, to the repulsive one, $U > 0$. In particular, we demonstrate how FFLO correlations of Cooper pairs can be imprinted onto repulsively bound pairs. We find that the visibility of the FFLO peak crucially depends on the final (post-quench) value of $U$ and on the ramp time.

The plan of this paper is as follows. In Sec. II, we introduce the quench schemes and provide definitions of quantities used throughout the paper and details on our numerical methods. Section III is devoted to the formation of FFLO correlations in quenches and ramps starting from the non-interacting case to negative values of $U$. In Sec. IV, we discuss the imprinting of FFLO correlations onto repulsively bound pairs in quenches and ramps from $U < 0$ to $U > 0$. In Sec. V we summarize our main results and discuss perspectives for future work on this exciting topic.

II. SET-UP, DEFINITIONS, AND METHODS

A. Quench schemes

We consider two quench schemes. In the first case, we prepare the system in the ground state of Eq. (1) at $U = 0$ and at $t = 0$ we change the interaction strength to $U < 0$. This probes the formation of FFLO correlations in a previously non-interacting two-component Fermi gas. In the second scheme, the initial state is the ground state at some $U < 0$, and then we instantaneously change the sign of the interaction at $t = 0$. This scheme is supposed to imprint FFLO correlations onto the repulsively bound pairs present on the repulsive side $U > 0$.

Besides these quenches that change the value of $U$ abruptly from $U_i$ to $U_f$, we also consider slow, linear quenches, referred to as ramps, where in the time interval $t \in [0, t_{\text{ramp}}]$, $U$ takes the time dependence according to

$$U(t) = U_i + \frac{U_f - U_i}{t_{\text{ramp}}} t$$

where $t_{\text{ramp}}$ is the ramp time.

B. Observables

Our study mainly focuses on two quantities, namely the double occupancy and the pair momentum distribution function. The double occupancy $d$ is defined as

$$d = \sum_{i=1}^{L} (n_{i\uparrow}n_{i\downarrow})$$

The pair momentum distribution function $n_k^\text{pair}$ is the Fourier transform of pair correlations $\rho_{ij}$, where

$$\rho_{ij} = \langle \ell_i \uparrow \ell_j \downarrow \ell_j \uparrow \ell_i \downarrow \rangle$$

such that

$$n_k^\text{pair} = \frac{1}{L} \sum_{i,j} e^{i(t-j)k} \rho_{ij}$$

Notice that the two quantities defined above are related to each other by the normalization condition $d = \sum_k n_k^\text{pair}$.

While generally, there are established methods to measure the MDF of the fermionic components via time-of-flight and band-mapping techniques, the measurement of the pair MDF in the FFLO phase has not been accomplished yet, but its observation is a goal of future experiments.

We further define the visibility $V$ via

$$V = \frac{n_k^\text{pair} = 0 - n_k^\text{pair} = 0}{n_k^\text{pair} = 0 + n_k^\text{pair} = 0}$$

This particular definition is motivated by the fact that the maximum in $n_k^\text{pair}$ is initially at $k = 0$ for $U_i = 0$ and at $k = \pm q$ in the ground state for $U < 0$.

C. Numerical methods

We employ two numerical methods in our work, namely exact diagonalization (ED) and the time-dependent density matrix renormalization group (tDMRG) method. In the ED method, we use a Krylov-space method to propagate the wave function in time (for technical details, see, e.g., [68]). We use a Trotter-Suzuki method to propagate the wave function using tDMRG; the time step is typically chosen to be $\delta t \sim 0.02/J$. The discarded weight, which controls the accuracy of the truncation involved in tDMRG, is set to $10^{-6}$, which is sufficient to ensure negligible numerical errors for the times reached in the simulation. We perform all tDMRG and ED simulations for open boundary conditions.
III. FORMATION OF FFLO CORRELATIONS IN QUENCHES AND LINEAR RAMPS FROM $U = 0$ TO $U < 0$

In this section, we present our results for quenches and linear ramps from the non-interacting case $U_i = 0$ to attractive interactions $U_f < 0$. We first discuss the behavior of the double occupancy and the pair MDF for the quantum quench in Sec. IIIA, and then we turn to slow quenches in Sec. IIIB.

A. Quantum quench

1. Double occupancy

Example for the time evolution. Typical results for the time dependence of the double occupancy are shown in Fig. 1(a), where we display ED data for $L = 12$ and DMRG data for $L = 24, 36$, $U_f = -8J$, $p = 1/2$, and filling $n = 1/3$. Since the system is initially prepared in an uncorrelated state, the double occupancy at $t = 0$ is given by $d(0) = n_1 n_L$, that is,

$$\frac{d(0)}{N_1} = \frac{n(1+p)}{2}. \quad (8)$$

Similar to the behavior of the double occupancy in other interaction quantum quenches \cite{34, 12}, $d(t)$ undergoes a fast transient characterized by large oscillations and relaxes to a constant value $\bar{d}$ on a time scale $\tau_d$ of order $1/J$, as is evident from the data for $L = 36$. For small systems such as $L = 12$, there are revivals due to reflections from the boundary of the simulation box. These spurious oscillations quickly disappear as $L$ increases.

Long-time average $\bar{d}$. We extract the stationary value $\bar{d}$ by averaging the double occupancy $d(t)$ at long times ($t \gtrsim 3/J$). These data are plotted in Fig. 2 as a function of the
post-quench interaction strength $U_f/J$ for two different values of the filling, namely $n = 1/3$ and $n = 2/3$, and polarization $p = 1/2$. As $|U_f|/J$ increases, we see that $\bar{d}$ initially increases from its $U = 0$ value until it reaches a maximum value as the interaction strength becomes of the order of the bandwidth, $|U_f|/J \sim 4$, and then it decreases (slowly) for larger values of $|U_f|$.

It is instructive to compare the steady-state double occupancy $\bar{d}$ to its ground-state value $d_{gs}$, calculated for $U = U_f$. The latter is shown in Fig. 2 as dashed lines. Different from $\bar{d}$, the ground-state double occupancy is a monotonically increasing function of $|U|/J$, approaching the largest possible value of $d_{gs} = N_\uparrow$ for $|U_f|/J \to \infty$ (note that we plot the ratio $d/N_\uparrow$ in the figure). In particular, taking into account that $d_{gs} = \partial E_{gs}/\partial U$, where $E_{gs} = -\sqrt{U^2 + 16J^2}$ is the ground-state energy of the two-body problem, for vanishing density, $n \to 0$, we find (dotted line in Fig. 2)

$$\frac{d_{gs}}{N_\downarrow} = \frac{-U}{\sqrt{U^2 + 16J^2}}. \quad (9)$$

We see in Fig. 2 that for sufficiently weak interactions, $|U_f| \ll J$, the steady-state double occupancy follows closely the post-quench ground-state value, while for stronger interactions it is well below that value. This effect can be understood by taking into account that during the time evolution the energy is conserved, i.e.,

$$E_{kin}(0) + U_f d(0) = E_{kin}(t) + U_f d(t), \quad (10)$$

where $E_{kin}$ is the expectation value of the kinetic energy. Equation (10) shows that an increase of the double occupancy has to be compensated by a corresponding change of the kinetic energy. Since the single-particle dispersion has a finite bandwidth $4J$, the conversion between interaction and kinetic energies is progressively inhibited for $|U_f| > 4J$ implying that $d$ decreases by further increasing the interaction strength. In particular, for $U_f/J \to -\infty$ the double occupancy remains frozen to its initial value, $\bar{d} = d(t) = d(0)$, with the latter given by Eq. (8).

In Fig. 3 we show the dependence of $\bar{d}$ on the spin polarization for fixed interaction strength $U_f = -8J$ and for two different fillings $n = 2/3$ (diamonds) and $n = 1/3$ (stars). $\bar{d}$ increases roughly linearly as a function of $p$, as is the case for the initial double occupancy [see Eq. (5)].

Relaxation towards the stationary regime. To further analyze the data, we observe that the following fit function

$$d(t) = -a \cos(\omega t + \phi) \exp(-t/\tau_d) + \bar{d} \quad (11)$$

provides a reasonably good description of the numerical data in a wide parameter regime. An example is shown in Fig. 1(b) for the largest system size, $L = 36$.

The frequency of the coherent oscillation is, for large values of $|U_f|/J$, given by $\omega \propto |U_f|$, similar to other studies of collapse and revival phenomena [50, 72, 74], since the dynamics is predominantly governed by the interaction term in that limit. By the same argument, $\omega$ increases with density, since the double occupancy increases with density, both in equilibrium and in the steady state. In the two-body limit (small $N_\uparrow = N_\downarrow$), the frequency is given by the binding energy $\epsilon_b$:

$$\omega = \epsilon_b = \sqrt{U^2 + 16J^2} - 4J. \quad (12)$$

In particular, the good agreement between Eq. (11) and the data shown in Fig. 1(b) suggests that there is an exponential decay towards the time-independent value $\bar{d}$, which allows to extract a relaxation time $\tau_d$.

We observe that $\tau_d$ varies only mildly with both polarization and post-quench interaction strength $U_f$, whereas the dependence on $n$ is much stronger. This is illustrated in Fig. 4 where we plot $\tau_d$ versus $n$. Starting from small densities, $\tau_d$ decreases as $n$ increases and then becomes as short as $\tau_d \sim 0.5/J$. For large polarizations, however, $\tau_d$ increases again for $n \to 1$. Several aspects play a role in the relaxation dynamics: (i) the presence of doublons defined by the initial state at $t = 0$, (ii) the formation of additional doublons and (iii) the scattering of doublons, which leads to the decay of $d(t)$ to the stationary value. The density dependence can be understood from the simple argument that the larger the density, the higher is the probability for scattering events. At a large density, these events can occur on times scales set by inverse hopping rate, resulting in $\tau_d \sim 1/J$.

In the opposite limit of small densities $n \lesssim 0.2$, the decay of the amplitude of the oscillations is actually better described by a power law than by an exponential [see the example shown in Fig. 1(c)]. One can therefore attempt to describe the crossover from a power-law, realized at small $n$, to an exponential relaxation at large $n$ with a function

$$d(t) = -a \cos(\omega t + \phi) \exp(-t/\tau_d)/t^\alpha + \bar{d}. \quad (13)$$

Therefore, the power-law decay and the coherent oscillations are inherited from the few-body dynamics, whereas
the many-body physics introduces an actual damping. Consistent with this picture, we observe that the exponent of the power-law \( \alpha \approx 0.5 \) very weakly depends on \( U/J \) and that \( \tau_d \) becomes very large for \( n \to 0 \).

2. Pair correlations and quasi-momentum distribution function

**Time evolution of the pair MDF.** We next turn to the main quantity of interest in our study, namely the pair MDF \( n_k^{\text{pair}} \). A typical example for its time evolution in the quench from \( U_i = 0 \) to \( U_f = -8J \) is shown in Fig. 5(a). At \( t = 0 \), there is a maximum at \( k = 0 \), which after the quench and a fast transient time that is shorter than \( 1/J \) moves to finite momenta and then approaches \( k = q \), where \( q \) is given by Eq. (2).

The pair MDF is normalized to the double occupancy \( \sum_k n_k^{\text{pair}} = d \). Since the quench introduces both a redistribution of weight and generates additional pairs, as is evident from Fig. 2(b) and (c) the sum over \( n_k \) increases. Figures 5(b) and (c) show the individual time evolution of \( n_{k=0}^{\text{pair}} \) and \( n_{k=q}^{\text{pair}} \), respectively. Both numbers are always below the expectation values in the post-quench ground state (dotted lines). There are only small finite-size effects, comparing \( L = 12 \) (dashed lines) and \( L = 36 \) (solid lines). Compared to its initial value at \( t = 0 \), the long-time average of \( n_{k=0}^{\text{pair}} \) does not decrease much (this can be different for other parameter values) while \( n_{k=q}^{\text{pair}} \) increases significantly. In any case, the largest absolute increase of \( n_{k=q}^{\text{pair}} \) typically occurs at \( k = q \) (compare also Fig. 10).

**Natural orbitals and decay of correlations.** In order to further characterize the state after the fast quench dynamics, it is instructive to compute the natural orbitals \( \phi_\mu(i) \), which are the eigenvectors of the pair-pair correlation function \( \rho_{ij}^{\text{pair}} \) defined in Eq. (4). We are, in particular, interested in the spatial structure of the natural orbital \( \phi_0(i) \) that belongs to the largest eigenvalue of \( \rho_{ij}^{\text{pair}} \). The time average of this state (i.e., the time average over \( |\phi_0(i)|^2 \)) is plotted in Fig. 6(a), compared to the same quantity in the post-quench and pre-quench ground state (all for \( U_f = -8J \)). As expected we observe a spatial oscillation with \( k = q \) that was not present in...
the initial state (diamonds). The amplitude in the time-averaged state is smaller than in the post-quench ground state (squares).

The pair correlation function $\rho_{ij}^{\mathrm{pair}}$ itself, shown in Fig. 6(b), has a power-law decay in the post-quench ground state superposed with oscillations. The same characteristic FFLO oscillations emerge in the time-averaged post-quench state, yet the decay of the spatial correlations is much faster and roughly exponential.

From these results, we conclude that the time-averaged state indeed has the characteristic features of the FFLO state, albeit not with quasi-long-range order, which is expected since the system is at a finite energy above the ground state. Our next goal is to define and analyze a characteristic time scale for the formation of the FFLO correlations.

**Time scale for the formation of the FFLO correlations.**

In contrast to the double occupancy, $n_k^{\mathrm{pair}}(t)$ does not exhibit a single relaxation time, i.e., it is not well approximated by a simple fit function of the type of Eq. 11. In order to nevertheless define a time scale for the formation of FFLO correlations, we find it instructive to study the time evolution of the natural orbital $\phi_0(i)$. This quantity is shown in Fig. 7(a). Initially, apart from oscillations in $|\phi_0(i)|^2$ induced by the open boundaries, $\phi_0(i)$ is flat, yet at times $t \sim 0.5J$, it clearly develops a spatially modulated pattern, indicative of FFLO correlations.

The time $\tau_{\mathrm{FFLO}}$ at which this modulation appears in $|\phi_0(i)|^2$ is also reflected in the time dependence of the visibility $V(t)$ of the FFLO peak in $n_k^{\mathrm{pair}}$ defined in Eq. 7, namely, for $t \lesssim \tau_{\mathrm{FFLO}}$, $V(t) < 0$ while for $t \gtrsim \tau_{\mathrm{FFLO}}$, $V(t) > 0$, as is evident from Fig. 7. We therefore define the time scale characteristic for the formation of the FFLO correlations as the (first) zero of the visibility

$$V(\tau_{\mathrm{FFLO}}) = 0.$$  

Note that this could be measured in experiments, provided the pair MDF is accessible.

Our results for the dependence of $\tau_{\mathrm{FFLO}}$ on filling, polarization and post-quench interaction strength are summarized in Figs. 8(a)-(c), respectively. The overall density dependence is similar to the one of $\tau_d$ discussed in Sec. III A 1 (compare Fig. 4): starting from small densities, $\tau_{\mathrm{FFLO}}$ decreases. It also monotonously decreases with $p$, yet exhibits a weaker overall variation.

The processes behind the formation of the FFLO correlations and the formation of the oscillatory structure in the natural orbital $|\phi_0(i)|^2$ are the pair formation, redistribution of momentum among pairs, and the spatial redistribution of excess fermions. The time scale for the formation of the natural orbital is defined by

$$|\phi_0(i)|^2 \equiv \frac{1}{L} \sum_{i} |\phi_0(i)|^2.$$  

FIG. 7. (Color online) (a) Time evolution of the lowest natural orbital $|\phi_0(i)|^2$ (for better color contrast, we plot $|\phi_0(i)|$) and (b) time evolution of the FFLO visibility $V$ [see Eq. (7)]. The time at which the natural orbital $|\phi_0(i)|^2$ develops an oscillatory pattern roughly coincides with the time at which the visibility goes through zero. For the pre-quench system with $U_i = 0$ the lowest natural orbital is two-fold degenerate where one of them has a deviation from the constant background near the left and the other near the right boundary. Numerically, we select one of the degenerate orbitals, which explains the spatial anisotropy of $|\phi_0(i)|^2$ for $t \lesssim \tau_{\mathrm{FFLO}}$. 

FIG. 8. (Color online) Time scale $\tau_{\mathrm{FFLO}}$ for the formation of FFLO correlations in the quench from $U_i = 0$ to $U_f < 0$, versus (a) filling $n$, (b) polarization $p$ and (c) post-quench interaction strength $U_f$. The time scale is extracted as the first zero of the visibility $V(t)$. 

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pair formation is similar to the formation of doublons [in excess of those present in the initial state, see Eq. (8)], which are essentially a measure of the pairs, while the excess fermions have to rearrange themselves over distances of \( t \sim 1/(2q) \).

The probability of a minority fermion finding an excess fermion is proportional to the density of majority fermions \( n_\uparrow = N_\uparrow /L \). Therefore, we expect a leading dependence for the formation of additional pairs given by \( \tau_{FFLO} \propto 1/(1+\frac{1}{n_\uparrow}) \). Our numerical results for \( \tau_{FFLO} \) extracted from the zero of the visibility shown in Fig. 8 are in reasonable qualitative agreement with this simple argument.

There is also an interesting dependence on \( U_f \) [see Fig. 8(c)]: (i) the smaller \(|U_f|\), the larger \( \tau_{FFLO} \) and (ii) \( \tau_{FFLO} \) depends more strongly on \( U_f \) the lower the density is. The first aspect is intuitive, since for \(|U_f| \to 0\), no FFLO correlations are ever formed, and hence \( \tau_{FFLO} \to \infty \). Note also that for large \(|U|/J\), \( \tau_{FFLO} \) seems to become independent of density.

**Time average of the visibility.** We finally discuss the long-time average of the visibility as a function of post-quench interaction strength. The results are shown in Fig. 9 both for the dependence of \( V \) on \( U/J \) in the ground state [panel (a)] and on the post-quench interaction strength \( U_f/J \) [panel (b)]. In both cases, \( V \) increases monotonically with \( U \) and \( U_f \), respectively, however, the comparison of different system sizes shows only very small variations for the time averages, while in the ground state, \( V \) increases with \( L \). This is consistent with the fact that in the ground state we have an actual quasi-condensate.

**B. Ramps from \( U = 0 \) to \( U < 0 \): Crossover to adiabatic dynamics**

To conclude our discussion of the real-time dynamics starting from a non-interacting gas, we turn to slow quenches with a linear ramp from \( U_i = 0 \) to \( U_f \) according to Eq. (3). We are, in particular, interested in the crossover to adiabatic behavior, for which we expect to obtain the ground-state expectation values of observables in the time average.

Our results for the time average of \( n^\text{pair}_k \) versus quasi-momentum \( k \) for ramps and quenches to \( U_f = -8J \) are presented in Figs. 10(a)-(d), for polarizations \( p = 0, 1/4, 1/2 \) and \( 3/4 \), respectively. These plots also include \( n^\text{pair}_k \) calculated in the ground state at \( U_i = 0 \) and \( U_f = -8J \) and the results for instantaneous quenches. For the quenches, the time averages at \( p > 0 \) result in maxima at \( k = q \) but the distribution is mostly reduced in height compared to the post-quench ground state (squares).

Moreover, we see that, both for quenches and ramps, for large values of the quasi-momentum \( k \), the time-averaged pair momentum distribution is already very close to its ground-state value for \( U = U_f \) and therefore does not depend much on the sweep rate. For small quasi-momenta, the averaged pair momentum distribution increases by decreasing the sweep rate. Neverthe-
less, a comparably short ramp time of $t_{\text{ramp}} \sim 4/J$ is enough to obtain time averages that are very close to the ground-state correlations also for small quasi-momenta.

It is important to emphasize that the time-averaged pair momentum distribution, for most parameter values, exhibits the typical FFLO peak at $k = q$, which is expected in the ground state for attractive interactions, $U = U_f < 0$.

C. Summary: Quenches and linear ramps from $U_i = 0$ to $U_f < 0$

As a main result of our analysis of quantum quenches from $U_i = 0$ to attractive interactions, we showed that the time average of the double occupancy has a maximum at $|U_f| \sim 4J$, while the value at small $|U_f|/J$ is close to the post-quench ground-state values. At large $|U_f|/J$, the initial double occupancy simply gets frozen in on the accessible time scales. Interestingly, we observe that, primarily for low densities, the approach to the stationary regime is exponential in time. This relaxation process is superimposed with coherent oscillations with $\omega \sim |U_f|$ for large $|U_f|/J$.

Even though the quantum quench puts the system at high energies, the pair momentum distribution function still develops a maximum at the characteristic FFLO momentum $q$. We analyzed the eigenstates of the pair density matrix $\rho_{ij}^{\text{pair}}$ and showed that the emergence of a maximum in $n_{k}^{\text{pair}}$ is accompanied by the eigenstate of $\rho_{ij}^{\text{pair}}$ belonging to the largest eigenvalue developing a spatially oscillatory structure. This coincides with the point in time at which the visibility of the FFLO peak in $n_{k}^{\text{pair}}$ becomes positive, meaning that the maximum is at $k = q$. We therefore extracted a time scale for the formation of oscillatory pair correlations from the first zero of the visibility. In ramps from $U_i = 0$ to $U_f < 0$, we observe that ramp times of $t_{\text{ramp}} \sim 3/J$ are enough to be adiabatic in the sense that the time average of $n_{k}^{\text{pair}}$ after the ramp cannot be discerned from the post-quench ground-state form of $n_{k}^{\text{pair}}$. Overall, both quenches and ramps produce the largest visibilities of the FFLO maximum in $n_{k}^{\text{pair}}$ at large polarizations (compare Fig. 10).

It is tempting to relate the time average to a temperature by comparison with thermodynamic ensembles, as is often done in studies of thermalization in interaction quantum quenches (see 72 and references therein). The Fermi-Hubbard model being integrable, we expect that an infinitely large number of Lagrange parameters (order of $L$ many) is necessary to describe the time averages with a generalized Gibbs ensemble along the lines of 71. While it is not the purpose of this study to clarify the validity of the generalized Gibbs ensemble for this particular quench, we note that we verified that the time averages of $d(t)$ cannot be described by the canonical ensemble, as expected for an integrable model.

Various extensions of our analysis such as the formation of a polaron (i.e., $N_{i} = 1$), the dynamics in the low-density limit, or the dynamics starting from product states constitute interesting problems for future research. Some of these questions may permit an analytical solution or are commonly studied in quantum gas experiments (namely the dynamics starting from product states in real space, see, e.g., 73 74 75 76).

IV. IMPRINTING FFLO CORRELATIONS ONTO REPULSIVELY BOUND PAIRS

By carrying out a quench from negative to positive values of $U$, we aim at imprinting FFLO correlations onto repulsively bound pairs present on the $U > 0$ side. In one dimension, repulsively and attractively bound pairs exist for all values of $U > 0$ and $U < 0$, respectively. Figure 11 shows a sketch of the spectrum of repulsively and attractively bound pairs. The former were observed in a seminal experiment with bosons in optical lattices 75. We expect that, if $U_f \gg 4J$, the repulsively bound pairs will be very long-lived implying that FFLO correlations will be preserved for a long time. This is related to the observation that in Hubbard models, the double occupancy is approximately conserved for values of $U/J$ larger than the band-width 76, which is at the heart of many intriguing transient and meta-stable phenomena in quantum quenches such as quantum distillation 23 70 71 72 or the exponentially long thermalization

![Figure 11. (Color online) Spectra of the scattering continuum and the bound states for different on-site interaction strength $U$ from a two-particle calculation for pairs with total momentum $K$ in the Fermi-Hubbard model (spin-up and -down pairs). The shading of the scattering continuum is proportional to the density of states (DOS) where darker gray indicates higher DOS (note that the DOS diverges at the edges of the continuum region). The bandwidth of the scattering continuum is 8$J$. The bound states are discrete and symmetric for $U \to -U$. There is no energy overlap of repulsively bound states with the continuum for $U > 4J$ (dotted black line).](image)
times for quenches into the large \( U/J \) regime that involve finite densities of doublons \cite{63,79,80}.

We first discuss quenches from \( U_i < 0 \) to \( U_f > 0 \) (Sec. IV A) and then turn our attention to linear ramps (Sec. IV B).

### A. Interaction quench from \( U < 0 \) to \( U > 0 \)

#### 1. Double occupancy

On the attractive side, because of pair formation, a large double occupancy is favored, whereas \( U/J > 0 \) suppresses \( d \) in the ground state. The fraction of quench-induced doublons that survive in the quench from \( U_i < 0 \) to \( U_f = |U_i| > 0 \) is therefore a good measure for the stability of repulsively bound pairs. Figure 12(a) shows tDMRG results for the time evolution of the relative fraction of doublons, which we define as (\( o \) representing any observable)

\[
f_o(t) = \frac{o(t) - o_{gs}}{o_{0} - o_{gs}},
\]

where \( o_{gs} \) is the expectation value in the post-quench ground state and \( o_{0} = o(t = 0) \) is the value in the pre-quench ground state. Considering \( o = d \), a value of \( f_d = 1 \) means that all doublons survive the quench. The figure unveils that for all values of \( U_i \), after a short transient, the double occupancy settles into a constant, superposed with oscillations. Moreover, the larger \( |U_i|/J \), the more doublons survive and \( f_d \to 1 \) for \( |U_i|/J \gg 4 \). Note that typical quantum gas experiments with interacting atoms in optical lattices reach time scales of the order of \( tJ \lesssim 20 \) \cite{31}, hence our results apply to the experimentally accessible time window.

Figure 12(b) shows the time average \( \bar{d} \) of the double occupancy after the quench, compared to the initial value \( d(t = 0) \) and the post-quench ground-state value \( d_{gs} \). \( \bar{d} \) exceeds \( d_{gs}(U_f) \) and approaches \( d_{gs}(U_i) \) for \( |U_i| \gg 4J \), at least on the accessible time scales. The minimum of \( \bar{d} \) at small \( |U_f|/J \) has a similar origin as the maximum of \( \bar{d} \) in the quenches from \( U_i = 0 \) to \( U_f < 0 \) (compare Fig. 2); for small excess energies, \( \bar{d} \) is only slightly above the post-quench ground-state values, but ultimately approaches the value set by the initial conditions for large \( |U_f|/J \).

#### 2. Pair momentum distribution function

Our results for the pair MDF in the quench from \( U_i < 0 \) to \( U_f = -U_i \) are presented in Fig. 13, which shows the time averages of \( n^\text{pair}_k \) versus \( k \) for four different values

\[\text{FIG. 13. (Color online) Time average of the pair momentum distribution function } n^\text{pair}_k \text{, calculated for } t > 3/J, \text{ for the quench from attractive interactions } U_i < 0 \text{ to repulsive interactions } U_f = -U_i > 0 \text{, for different } U_f: (a) } U_f/J = 32, \text{ (b) } U_f/J = 8, \text{ (c) } U_f/J = 4, \text{ (d) } U_f/J = 2 \text{ (circles); for the ground state of the attractive system (diamonds); and for the ground state of the repulsive system at } U_f \text{ (squares), all for } n = 2/3, p = 1/2, L = 36.\]
of \( U_i / J = -32, -8, -4, -2 \) at polarization \( p = 1/2 \) (circles). The figure also includes the respective \( n_{k \pi}^{\text{pair}} \) curves in the pre- and post-quench ground states (diamonds and squares, respectively). As expected, for large \( |U_i|/J \gg 4 \) the initial \( n_{k \pi}^{\text{pair}} \) is perfectly preserved after the quench (at least for \( fJ \leq 5 \)). Upon decreasing \( |U_i|/J \), the overall height of \( n_{k \pi}^{\text{pair}} \) decreases and the maximum in \( n_{k \pi}^{\text{pair}} \) shifts towards larger values of \( k > q \). This is related to the fact that for large quasi-momenta the time-averaged pair momentum distribution is always larger than its pre-quench counterpart.

As the interaction strength becomes of the order of the bandwidth, \( |U_i|/J = 4 \), we see that the time-averaged \( n_{k \pi}^{\text{pair}} \) approaches the post-quench ground-state distribution also for small momenta. By further decreasing the interaction at \( |U_i|/J = 2 \), the pair momentum distribution becomes already practically identical to the post-quench ground-state curve for all quasi-momenta.

**B. Slow quenches: Linear ramps from \( U_i < 0 \) to \( |U_i| > 0 \)**

In any experiment, a quench is a fast parameter change which nonetheless takes place over a finite time window, such that one has to worry about quench-type behavior versus adiabatic changes. To account for that, we investigate slow quenches with linear ramps [see Eq. (3)] from \( U_i < 0 \) to \( U_f = -U_i \).

An example for the time average of the pair momentum distribution function (calculated after the end of the ramp) is shown in Fig. 14 for various ramp times compared to the (instantaneous) quench and the expectation values in the pre- and post-quench ground state (for \( U_i = -8J \)). In this example, the quench already leads to a significant reduction of \( n_{k \pi}^{\text{pair}} \) (circles) compared to the initial state (diamonds), and a finite ramp time further suppresses the maximum in \( n_{k \pi}^{\text{pair}} \) at finite momentum, plus shifts the position of the maximum to larger values. Already at \( t_{\text{ramp}} = 2/J \), the time average coincides with the post-quench ground-state expectation values (compare the different curves with squares). It is interesting to notice that as the ramp time increases, the resulting pair-momentum distribution approaches the post-quench ground-state distribution for small momenta first, in contrast to Fig. 10 where the quench is from the non-interacting to the attractive regime.

The tendency of these quenches and ramps to dominantly populate repulsively bound pairs with large momenta can be understood from the fact that the minimum of the dispersion of the (anti-)bound state is at \( k = \pi \) (compare Fig. 11). Note, though, that \( k = \pi \) never gets populated, in fact, \( n_{k=\pi}^{\text{pair}} \) also vanishes in the pre-quench state. In other words, the so-called \( \eta \)-condensate \([76, 81, 82]\), an exact eigenstate of the Fermi-Hubbard model with a macroscopic occupation of doublons (or pairs) at center-of-mass momentum \( k = \pi \) has no overlap with our initial state.

Our results for \( f_d \) and \( f_V \) [see Eq. (15)] are displayed in Fig. 15 as a function of ramp time for \( U_i = -8J, n = 2/3, p = 1/2 \). Both \( f_d \) and \( f_V \) decrease monotonously with \( t_{\text{ramp}} \) and a value of \( t_{\text{ramp}} \sim 2/J \) is enough to suppress any excess pairs in the time average after the ramp, while the relative visibility \( f_V \) remains non-zero for \( t_{\text{ramp}} \lesssim 3/J \) for these parameters. Note, though, that the maximum of \( n_{k \pi}^{\text{pair}} \) is not necessarily at \( k = q \) in the post-quench state [see Fig. 14].
V. SUMMARY AND CONCLUSION

In this work we presented an extensive numerical analysis of quantum quenches and linear ramps of the interaction strength in the one-dimensional Fermi-Hubbard model. We were particularly interested in quenches that involve the FFLO state: we considered quenches from the non-interacting case to negative values of $U$ at finite spin imbalances as well as quenches from negative to positive $U$.

For the first example, we presented results for the double occupancy and the pair momentum distribution function. We highlight four key observations: (i) The time average of the post-quench double occupancy takes a maximum at $|U_f|/J \sim 4$, (ii) there are parameter regimes where the relaxation towards the steady state is exponential, (iii) the emergence of FFLO correlations in the post-quench state leaves clear fingerprints in the pair-momentum distribution function, and (iv) we further demonstrated that linear ramps with ramp times of $3/J$ are sufficient to be essentially adiabatic. These aspects should in principle be accessible in experiments. The current obstacle is that an experimental measurement of the pair momentum distribution has not been accomplished yet for one-dimensional quantum gases. The measurement of the double occupancy is a well-established technique.

In the second part of the paper, where we considered quenches from negative to positive values of the interaction strength, we demonstrated that FFLO correlations can be imprinted onto repulsively bound pairs if the post-quench interaction strength is large. This generates a metastable state due to the exponentially long life-time of doublons generated in such quenches \[76\]. In linear ramps, one quickly approaches the adiabatic case with ramp times as short as $t_{\text{ramp}} \sim 2/J$ and FFLO signatures are rapidly lost.

There are several interesting extensions of our work. For instance, for quenches of a spin-imbalanced gas from $U = 0$ to $U < 0$ in an inhomogeneous systems such as realized in a harmonic trap, there need to be macroscopic spin currents in the dynamical emergence of the shell structures characteristic of a 1D spin-imbalanced Fermi gas with attractive contact interactions \[10\] [11] [13] [17] [83]. In that system, the core of the system is partially polarized in one dimension and either fully paired or fully polarized, depending on the global polarization. The small polarization regime should be the most interesting one. Such an experiment could potentially give access to the transport coefficients of such a system (see \[84\] for work in that direction).

To further elucidate the relaxation dynamics of the Fermi-Hubbard model, the dynamics starting from product states in real space (see, e.g., \[31\] [51] and \[85\] for first numerical results) or the time-dependent formation of a polaron constitute potentially very instructive limiting cases for future studies.

For the question of thermalization in the Fermi-Hubbard model, where in general one would expect thermalization with respect to the generalized Gibbs ensemble because of the integrability of the model \[40\], at present the complicated structure of the integrals of motion presents an obstacle for testing this conjecture. The effect of integrability breaking can, in experiments, be studied by either using two-leg ladders \[86\], realized with super-lattices \[87\], in fermionic superfluids with resonant interactions \[88\] [89] or with mass-imbalanced systems, realizable with two-component gases of different fermionic species (see the discussion in \[90\] [91]) or using spin-dependent optical lattices \[95\]. A recent study \[96\] discussed quasi-many-body localization in such a system.

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