TRAVELING WAVE SOLUTIONS TO MODIFIED BURGERS AND DIFFUSIONLESS FISHER PDE’S

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ABSTRACT. We investigate traveling wave (TW) solutions to modified versions of the Burgers and Fisher PDE’s. Both equations are nonlinear parabolic PDE’s having square-root dynamics in their advection and reaction terms. Under certain assumptions, exact forms are constructed for the TW solutions.

1. Introduction. Our main goal is to construct traveling wave (TW) solutions to the following two nonlinear PDE’s

\[ u_t + a_1 \sqrt{u} u_x = D_1 u_{xx}; \quad D_1 > 0, \ a_1 > 0; \]  

\[ u_t + a_2 \sqrt{u} u_x = \lambda_1 \sqrt{u} - \lambda_2 u; \quad a_2 > 0, \ \lambda_1 > 0, \ \lambda_2 > 0; \]  

where \( u = u(x,t) \), and, in general, \(-\infty < x < \infty\), with \( t \geq 0 \). These PDE’s are modified versions, respectively, of the Burgers equation \([2, 4, 12]\)

\[ u_t + a_3 uu_x = D_2 u_{xx}, \]  

and the Fisher equation \([2, 5, 9]\)

\[ u_t = D_3 u_{xx} + \lambda_3 u - \lambda_4 u^2. \]

Note that Eq. (1.2) is a diffusionless extension of Eq. (1.4). We demonstrate that these two PDE’s have TW solutions by explicit construction of their mathematical forms.

The essential new feature, incorporated into Eqs. (1.1) and (1.2), is the using of square-root dynamics, i.e., the advection term is \( \sqrt{u} u_x \), rather than the usual expression \( uu_x \), and the reaction term in the Fisher equation is \( \lambda_1 \sqrt{u} - \lambda_2 u \), in place of the standard form \( \lambda_3 u - \lambda_4 u^2 \). The references provide several articles which also make use of terms containing \( \sqrt{u} \); see Buckmire et al [1], Jordan [3], and Mickens [6, 7].

2000 Mathematics Subject Classification. Primary: 34E05, 35K57; Secondary: 35B09, 35B40.  
Key words and phrases. Traveling wave solutions, parabolic PDE’s, Burgers’ equation, Fisher equations, asymptotics.

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For physical systems, \( u(x,t) \) is either a particle number or density and, as a consequence, must be non-negative, i.e.,

\[
\begin{align*}
  u(x,t) \geq 0.
\end{align*}
\]

(1.5)

In fact, for the general physical meaningful solutions to Eqs. (1.1) and (1.2), the condition

\[
\begin{align*}
  0 \leq u(x,0) \leq M \Rightarrow 0 \leq u(x,t) \leq M, \quad t > 0,
\end{align*}
\]

(1.6)

is required \([2, 5, 11]\) where \( M \) is a fixed positive constant. We assume, without proof, that this requirement holds.

In Section 2, we use the method of variable scaling \([8, 9]\) to rewrite Eqs. (1.1) and (1.2) in terms of dimensionless variables. Section 3 provides a brief summary of the important general properties of TW solutions. In the next two Sections 4 and 5, we respectively, derive the TW solutions to dimensionless forms of Eqs. (1.1) and (1.2), and discuss several of their mathematical properties. In the final Section 6, we give a summary of the obtained results and consider several possible extensions of this work.

The following abbreviations will be used in the remainder of the paper:

- MBESRD - modified Burgers equation with square-root dynamics
- MFESRD - modified, diffusionless Fisher equation with square-root dynamics
- PDE - partial differential equation
- TW - traveling wave
- WF - wave front

2. Dimensionless equations. First, consider the MBESRD, Eq. (1.1), and rescale the variables in the forms

\[
\begin{align*}
  t &= T\bar{t}, \quad x = L\bar{x}, \quad u = U\bar{u},
\end{align*}
\]

(2.1)

where \((T, L, U)\) are the respective time, space and dependent variable scales \([6, 7]\), and \((\bar{t}, \bar{x}, \bar{u})\) are the corresponding dimensionless new variables. Substitution into Eq. (1.1) and rewriting, gives

\[
\begin{align*}
  \frac{\partial \bar{u}}{\partial \bar{t}} + \left[ \frac{a_1 T \sqrt{U}}{L} \right] \sqrt{\bar{u}} \frac{\partial \bar{u}}{\partial \bar{x}} = \left( \frac{D_1 T}{L^2} \right) \frac{\partial^2 \bar{u}}{\partial \bar{x}^2}.
\end{align*}
\]

(2.2)

If we replace \( U \) by the positive constant \( M \) appearing in Eq. (1.6) and then set the coefficients to one, then we obtain

\[
\begin{align*}
  \frac{a_1 \sqrt{MT}}{L} = 1, \quad \frac{D_1 T}{L^2} = 1,
\end{align*}
\]

(2.3)

for which solving for \( T \) and \( L \) gives the following time and space scales

\[
\begin{align*}
  T = \frac{D_1}{a_1^2 M^{3/2}}, \quad L = \frac{D_1}{a_1 M}.
\end{align*}
\]

(2.4)

Note that \( M \) is the scale for the \( u \) variable. Using the results of Eq. (2.3) and dropping the bars, gives the dimensionless form for the MBESRD

\[
\begin{align*}
  u_t + \sqrt{u}u_x = u_{xx}
\end{align*}
\]

(2.5)

In a similar manner, the MFESRD, Eq. (1.2), can be rewritten to the form

\[
\begin{align*}
  u_t + \sqrt{u}u_x = \sqrt{u} - u,
\end{align*}
\]

(2.6)
with the rescaling variables

\[ T = \frac{1}{\lambda_2}, \quad L = \frac{a_2 \lambda_1}{\lambda_2^2}, \quad M = \left( \frac{\lambda_1}{\lambda_2} \right)^2, \]

and the dropping of bars on the new variables.

In the calculations to follow the dimensionless PDE's, Eqs. (2.5) and (2.6), are used. Further note that for both PDE’s there are no free parameters, i.e., the original PDE’s having, respectively, two and three dimensional parameters, have been rescaled such that the dimensionless equations do not contain any parameters.

Finally, it should be noted that an earlier publication by Soluyan and Khokhov [10] derived an equation similar to our Eqs. (2.5) and (2.6), using the approximate equations of relaxation gas dynamics; see their Eq. (20). However, these calculations are not directly relevant to our current paper and, as a consequence, no discussion will be given of their findings.

3. Properties of TW solutions. Consider a general parabolic PDE [2, 4, 8, 9]

\[ u_t = H(u, u_x, u_{xx}), \quad -\infty < x < \infty. \]  

(3.1)

We assume \( H(\ldots) \) has properties such that solutions exact, whether unique or not. A TW solution to Eq. (3.1) has the following properties [2, 5, 11]

(i) \[ u(x, t) = f(z), \quad z = x - ct \]  

(3.2)

(ii) \[ \lim_{z \to -\infty} f(z) = f_1, \quad \lim_{z \to +\infty} f(z) = f_2 \]  

(3.3a)

where

\[ 0 \leq f_2 \leq f_1 \]  

(3.3b)

(iii) \[ f_2 \leq f(z) \leq f_1 \]  

(3.4)

(iv) \[ f'(z) \equiv df(z)/dz \leq 0, \quad -\infty < z < +\infty. \]  

(3.5)

For the purposes of this paper, we select the following values for \( f_1 \) and \( f_2 \)

\[ f_1 = f(-\infty) = 1, \quad f_2 = f(+\infty) = 0. \]  

(3.6)

TW solutions generally exist in two forms. First, there does not exist a finite \( z_0 \), such that

\[ f(z_0) = 0, \quad -\infty < z_0 < +\infty. \]  

(3.7)

Second, there exists a \( z_0 \), such that \( f(z_0) = 0 \) and

\[ \begin{cases} 
  f(z) > 0, & z \leq z_0; \\
  f(z) = 0, & z > z_0.
\end{cases} \]  

(3.8)

Note that at \( z = z_0 \), the derivative may or may not exist [2, 9].

After the preliminaries given in Sections 2 and 3, we are now at a point to examine whether the MBESRD and MFESRD have TW solutions.
4. TW’s for MBESRD. The substitution \( u(x,t) = f(x-ct) \) into Eq. (2.5), with a rearrangement of the terms, gives

\[
\frac{d^2}{dz^2} f(x) = \left(\frac{2}{3}\right) \left(\frac{f^{3/2}}{2}\right) - cf, \tag{4.1}
\]

where \( f' = \frac{df}{dz} \) and \( f'' = \frac{d^2f}{dz^2} \). We also assume the following conditions to hold

\[
\begin{align*}
\lim_{z \to -\infty} f(z) &= 1, & \lim_{z \to +\infty} f(z) &= 0 \tag{4.2a} \\
\lim_{|z| \to +\infty} \left( \frac{f'(z)}{f''(z)} \right) &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}. & \tag{4.2b}
\end{align*}
\]

Integrating Eq. (4.1) once gives

\[
\frac{df}{dz} = \left(\frac{2}{3}\right) \left(\frac{f^{3/2}}{2}\right) - cf + A, \tag{4.3}
\]

where \( A \) is a constant. If the \( \lim z \to +\infty \) is taken, then

\[
0 = \left(\frac{2}{3}\right) (0) - c(0) + A \Rightarrow A = 0. \tag{4.4}
\]

Likewise, taking the \( \lim z \to -\infty \), gives

\[
0 = \left(\frac{2}{3}\right) (1) - c(1) \Rightarrow c = \frac{2}{3}, \tag{4.5}
\]

and the TW speed has been determined.

The imposition of the results from Eqs. (4.4) and (4.5) allows the rewriting of Eq. (4.3) to take the form

\[
\frac{df}{dz} = \left(\frac{2}{3}\right) f \left( f^{1/2} - 1 \right). \tag{4.6}
\]

This equation is a separable first-order ordinary differential equation and it can be solved using elementary methods, as follows:

\[
\int \frac{df}{f(f^{1/2} - 1)} = \int \left(\frac{2}{3}\right) dz = \left(\frac{2}{3}\right) z + B, \tag{4.7}
\]

where \( B \) is an integration constant. Making the transformation of dependent variable

\[
v(z) = \sqrt{f(z)} \tag{4.8}
\]

gives

\[
\int \frac{2v dv}{v^2(v - 1)} = 2 \int \frac{dv}{v(v - 1)} = \left(\frac{2}{3}\right) z + B. \tag{4.9}
\]

The integral on the left-side can be easily calculated using the method of partial fractions. Carrying out this calculation and selecting \( v(0) = \frac{1}{2} \), an arbitrary choice, we finally obtain

\[
f(z) = \left[ \frac{1}{1 + e^{z/3}} \right]^2. \tag{4.10}
\]

Putting in \( z = x - ct = x - (2/3)t \) gives the TW solution

\[
u(x,t) = f \left[ x - \left(\frac{2}{3}\right) t \right] = \left\{ 1 + \exp \left[ \left(\frac{1}{3}\right) \left( x - \left(\frac{2}{3}\right) t \right) \right] \right\}^{-2}. \tag{4.11}
\]
Inspection of the result in Eq. (4.11) immediately shows that
\[ f(-\infty) = 1, \quad f(0) = \frac{1}{4}, \quad f(+\infty) = 0. \] (4.12a)
\[ 0 \leq f(z) \leq 1, \quad f'(z) > 0, \quad -\infty < z < +\infty. \] (4.12b)

5. **TW’s for MFESRD.** The substitution \( u(x,t) = f(x-ct) \) into Eq. (2.6) gives
\[ -cf' + \left( \frac{2}{3} \right) \left( f^{3/2} \right)' = \sqrt{f} - f. \] (5.1)
Making the transformation of variable \( v = \sqrt{f} \),
\[ v' = -\left( \frac{1}{2} \right) \left( \frac{1-v}{c-v} \right). \] (5.2)
Note that the original differential equation, Eq. (5.1), has two fixed-points or constant solutions,
\[ \bar{f}_1 = 1, \quad \bar{f}_2 = 0. \] (5.3)
Consequently, we have
\[ f(-\infty) = \bar{f}_1 = 1, \quad f(+\infty) = \bar{f}_2 = 0, \] (5.4)
along with the requirement
\[ f'(z) \leq 0, \quad -\infty < z < +\infty. \] (5.5)
There are three cases to consider for the solutions of Eq. (5.3): (i) \( c < 1 \), (ii) \( c = 1 \), and (iii) \( c > 1 \). The case, \( c < 1 \), is ruled out since the condition given in Eq. (5.6) may not hold for all values of \( z \). For case (ii), Eq. (5.3) becomes
\[ v' = -\left( \frac{1}{2} \right), \] (5.6)
and has the solution
\[ v(z) = A - \left( \frac{1}{2} \right) z. \] (5.7)
Since \( v(z) \) has negative slope, the non-negativity of \( v(z) \), Eq. (5.2) and the satisfaction of the boundary conditions, Eq. (5.5) gives
\[ 0 \leq v(z) \leq 1, \] (5.8)
from which it follows that \( v(z) \) is a piece-wise continuous function whose mathematical structure takes the form
\[ v(z) = \begin{cases} 
1, & z \leq z^*, \\
\frac{1}{3}(z_0 - z), & z^* < z \leq z_0, \\
0, & z > z_0, 
\end{cases} \] (5.9)
where \( z_0 \) and \( z^* \) are values of \( z \) such that
\[ v(z_0) = 0, \quad v(z^*) = 1, \] (5.10)
as derived from Eq. (5.8). Observe that
\[ z^* = z_0 - 2. \] (5.11)
Thus, the TW solution, for \( c = 1 \), is

\[
\begin{align*}
    f(z) &= \begin{cases} 
    1, & z \leq z^* = z_0 - 2, \\
    \left(\frac{1}{4}\right)(z_0 - z)^2, & (z_0 - 2) < z \leq z_0, \\
    0, & z > z_0,
    \end{cases} 
\end{align*}
\]

(5.13)

where \( z = x - t \). Figure 1 provides a sketch of \(v(z)\) and \(f(z)\).

![Figure 1](image-url)

**Figure 1.** a) \(v(z)\) vs \(z\), b) \(f(z) = v(z)^2\) vs \(z\). See Eqs. (5.10) and (5.13).

The third case is for \( c > 1 \). Since Eq. (5.3) is a separable equation, it can be rewritten to the expression

\[
\left(\frac{c - v}{1 - v}\right) dv = -\left(\frac{1}{2}\right) dz,
\]

(5.14)

where, using the method of partial fractions [8], its solution is

\[
v + (1 - c) \ln |(v - 1)| = -\left(\frac{1}{2}\right)(z - z_0),
\]

(5.15)

and where we imposed the requirement

\[
v(z_0) = 0.
\]

(5.16)

Using the mathematical properties of \(v(z)\) and \(f(z) = v(z)^2\), allows the general features of these functions to be determined. Figure 2 is a representation of these results. Note, in particular, that \(v(z)\) has a discontinuous derivative at \( z = z_0 \), while \(f(z)\) has a continuous derivative with the value \(f(z_0) = 0\).

Remarkably Eq. (5.15) can be explicitly solved for \(v(z)\) and expressed in terms of the Lambert-W function [4, 8, 11]. This exact solution is \((c > 1)\)

\[
\begin{align*}
    v(z) &= 1 - (c - 1)W_0 \left\{ \left(\frac{1}{c-1}\right)e^{\frac{z_0}{(c-1)}} \right\}, & z \leq z_0 \\
    v(z) &= 0, & z > z_0.
\end{align*}
\]

(5.17)
Figure 2. a) $v(z)$ vs $z$, b) $f(z)$ vs $z$. See Eq. (5.15).

Note that $W_0$ is the principal branch of the $W$-function.

Let us now calculate the magnitude of the jump discontinuities in $dv/dz$ and $df/dz$ for Eqs. (5.10) and (5.13). Note, however, that both $v(z)$ and $f(z)$ are continuous for all finite values of $z$.

If a function $g(z)$ is discontinuous at $z = \bar{z}$, then the magnitude of the jump discontinuity is defined as follows

$$\Delta(g, \bar{z}) \equiv \lim_{\epsilon \to 0} \left[ g(\bar{z} - \epsilon) - g(\bar{z} + \epsilon) \right].$$

(5.18)

An elementary application of this definition to Eqs. (5.10) and (5.13) gives, respectively,

$$\Delta(v, z^*) = \frac{1}{2}, \quad \Delta(v, z_0) = -\left(\frac{1}{2}\right),$$

(5.19a)

$$\Delta(f, z^*) = 1, \quad \Delta(f, z_0) = 0.$$  
(5.19b)

Finally, we complete this section by examining certain of the properties of $v(z)$ as represented in Eq. (5.17). Let

$$\lambda = \frac{1}{c - 1}, \quad c > 1,$$  
(5.20)

and take the limit $z \to -\infty$. It follows that

$$v(\infty) = \lim_{z \to -\infty} v(z) = 1 - \left(\frac{1}{\lambda}\right) W_0\{\lambda e^\lambda(0)\}$$

$$= 1 - \left(\frac{1}{\lambda}\right) W_0(0).$$

(5.21)

Since $W_0(0) = 0$ [4, 8, 11], we obtain

$$v(\infty) = 1,$$  
(5.22)
an expected result based on our prior discussion.

We can obtain more detailed information, on the $z \to -\infty$ limit, by using $[4, 8, 11]$ the relation

$$W_0(x) = x + O(x^2), \quad x \to 0^+, \quad (5.23)$$

with

$$x = (\lambda e^{\lambda}) e^{(\lambda/2)(z-z_0)}. \quad (5.24)$$

Note that

$$z \to -\infty \quad \text{gives} \quad x \to 0^+ \quad (5.25)$$

and for $z_0$ finite, we have

$$v(z) = 1 - (e^\lambda)e^{-(\lambda/2)|z|} + O(e^{-\lambda|z|}). \quad (5.26)$$

At $z = z_0$, we have

$$v(z_0) = \lim_{\epsilon \to 0^+} v(z_0 - \epsilon) = 1 - \left(\frac{1}{\lambda}\right) W_0\{\lambda e^\lambda(1)\}. \quad (5.27)$$

Since $[8, 11]$

$$W_0(\lambda e^\lambda) = \lambda, \quad (5.28)$$

it follows that

$$v(z_0) = 0. \quad (5.29)$$

Returning to Eq. (5.3), it should be indicated that this relationship only holds for $z < z_0$, since $v(z) = 0$ for $z \geq z_0$. Thus there is a jump discontinuity in $dv/dz = v'$ at $z = z_0$. Consequently, in more detail, Eq. (5.3) should be written as

$$\begin{cases} v'(z) = -\left(\frac{1}{2}\right) \left[\frac{1-v(z)}{1-v'(z)}\right], & z < z_0 \\ v'(z) = 0, & z \geq z_0 + \epsilon, \quad \epsilon > 0. \end{cases} \quad (5.30)$$

From this expression it follows that the magnitude of the jump discontinuities in $v'(z)$ and $f(z) = 2v(z)v'(z)$, are

$$\Delta(v', z_0) = -\left(\frac{1}{2\epsilon}\right), \quad \Delta(f', z_0) = 0. \quad (5.31)$$

In summary, $f(z) = v(z)^2$ and $f'(z) = 2v(z)v'(z)$ are continuous for all values of $z$, including $z = z_0$. However, $v(z)$ is continuous for all $z$, but its derivative, $v'(z)$, is discontinuous at $z = z_0$. Mathematically, we have

$$\begin{cases} v(z) = \left(\frac{1}{2\epsilon}\right) (z_0 - z) + O[(z_0 - z)^2], & z < z_0; \\ v(z) = 0, & z \geq z_0 \quad (5.32) \end{cases}$$

and

$$\begin{cases} f(z) = \left(\frac{1}{4\epsilon^2}\right) (z_0 - z)^2 + O[(z_0 - z)^3], & z < z_0; \\ f(z) = 0, & z \geq z_0. \quad (5.33) \end{cases}$$
6. **Discussion.** We have investigated two nonlinear PDE’s with regard to their TW solutions. One was a modified version of the standard Burgers equation, while the other corresponded to a modified Fisher equation, but for which the diffusion term is absent. An important feature of these two equations is the fact that the usual nonlinear advection term, $u u_x$, is replaced by $\sqrt{u} u_x$. The major consequence of this change is that the TW stops at some value of $z = z_0$, and for $z > z_0$, the TW is zero. This is another confirmation of the, in general, “finite dynamics” of nonlinear differential equations containing fractional powers of the dependent variable and/or their derivatives [1, 2, 6, 7, 8]. A significant finding is that both of our PDE’s can be solved to yield exact solutions which can be expressed in terms of either the elementary or Lambert-W functions.

A future project is to investigate the following Fisher type, nonlinear PDE

$$u_t + a \sqrt{u} u_x = Du_{xx} + \lambda_1 \sqrt{u} - \lambda_2 u.$$  \hspace{1cm} (6.1)

This differs from one of the PDE’s examined in this work by the inclusion of a diffusion term.

**Acknowledgments.** R.E.M. thanks Ms. Imani Beverly and Mr. Bryan Briones, Atlanta University Center, Robert W. Woodruff Library, for their timely and productive assistance in securing documents and books related to the research reported in this paper.

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Received October 2017; revised January 2018.

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