q-ALGEBRAS and ARRANGEMENTS OF HYPERPLANES

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ABSTRACT. Varchenko’s approach to quantum groups, from the theory of arrangements of hyperplanes, can be usefully applied to q-algebras in general, of which quantum groups and quantum (super) Kac-Moody algebras are special cases. New results are obtained on the classification of q-algebras, and of the Serre ideals of generalized quantum (super) Kac-Moody algebras.

1. INTRODUCTION

1.1. Quantum Groups.

Drinfel’d, in his address to the International Congress of Mathematicians in Berkeley [D], defined what he proposed to call quantum universal enveloping algebras, a class of deformations of the enveloping algebras of Kac-Moody algebras. In fact, the much wider family of generalized Kac-Moody algebras can be similarly quantized, and there arises the new problem of classifying these objects. The generalized Kac-Moody algebras themselves have resisted classification till now, but because they have a singular position within the deformed family (as is always the case with essential deformations), there is some room for hoping that an approach from general position may be effective.

This section begins with a brief review of the structures defined by Drinfel’d, setting the stage for introducing the generalized quantum groups and for a statement of the problem addressed in this paper - in Subsection 1.1.4.
1.1.1. Drinfel’d’s Quantum Groups.

Let $g$ be a Kac-Moody algebra in the sense of Kac [K], defined in terms of a ‘generalized Cartan matrix’. A square, complex matrix $A$ is so called if

$$A_{ii} = 2, \quad i = 1, ..., N,$$

$$A_{ij} \text{ is a non-positive integer for } i \neq j,$$

$$A_{ij} = 0 \text{ implies } A_{ji} = 0.$$

Let $h$ be a complex parameter. For any generalized Cartan matrix $A$, the associated Drinfel’d Quantum Group (quantized Kac-Moody algebra) is the $\mathbb{C}[[h]]$-algebra generated by elements $\{H_a, e_i, f_i\}_{i=1, ..., N}^a=1, ..., M$ with relations

$$[H_a, H_b] = 0, \quad a, b = 1, ..., N,$$

$$[H_a, e_i] = H_a(i)e_i, \quad [H_a, f_i] = -H_a(i)f_i,$$

$$[e_i, f_j] = \delta_{ij} \frac{2}{h} \sinh\left(\frac{h}{2} H^\vee_i\right).$$

(1.1)

Here $H(1), ..., H(N)$ are the roots, $H^\vee_1, ..., H^\vee_N$ are the co-roots and $A_{ij} = H^\vee_i(H_j)$. Furthermore, for each pair $(i, j), i \neq j$, the quantum Serre relations

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k}_q q^{-k(n-k)/2} (e_i)^k (e_j)^{n-k} = 0,$$

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k}_q q^{-k(n-k)/2} (f_i)^k (f_j)^{n-k} = 0,$$

(1.2)

$$q = e^h, \quad n = 1 - A_{ij}.$$

That $A$ is a generalized Cartan matrix implies that $n$ is a positive integer. In [K], the author finds it convenient to begin without this condition, taking $A$ to be an arbitrary matrix, although “a deep theory can be developed only for the Lie algebra $g$ associated to a generalized Cartan matrix ...”. Our aim is to challenge that remark. The question is what replaces the Serre relations in the more general case.

1.1.2. Quantum Supergroups.

The algebras defined in Subsection 1.1.1 are deformations of Kac-Moody algebras. Super-Kac-Moody algebras can be deformed in a similar manner, but the Serre relations are more complicated and differ greatly from case to case. [Y] The difference between Kac-Moody algebras and super-Kac-Moody algebras has often been emphasized: they are different types of tensor categories. But both categories merge upon deformation; that is one of the attractive features of quantization.

1.1.3. Serre relations.

Let $A$ be the algebra generated by $\{H_a, e_i, f_i\}_{i=1, ..., N}^a=1, ..., M$ with relations (1.1). Let $A_+$ be the algebra generated by $\{e_i, H_a\}$ and the relations $[H_a, H_b] = 0, \quad [H_a, e_i] = H_a(i)e_i, \quad [H_a, f_i] = -H_a(i)f_i.$

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and let $A_-$ be the algebra generated by $\{f_i, H_a\}$ and the relations $[H_a, H_b] = 0$, $[H_a, f_i] = -H_a(i)f_i$. Finally, let $B_+$ be the $\mathbb{C}$-algebra freely generated by the $e_i$ and let $B_-$ be the $\mathbb{C}$-algebra freely generated by the $f_i$.

Define mappings

$$f_i^\#: B_+ \to A_+, \ e_j \mapsto \[e_j, f_i\],$$

$$e_i^\#: B_- \to A_-, \ f_j \mapsto [f_j, e_i].$$

An element of $B_+$ ($B_-$) is said to be invariant if it is annihilated by all the mappings $f_i^\#$ (all the mappings $e_i^\#$). Let $\mathcal{I}_+ \subset B_+$ and $\mathcal{I}_- \subset B_-$ be the two-sided ideals generated by the invariants.

**Definition.** The Serre ideal of $A$ is the direct sum $\mathcal{I}(A) = \mathcal{I}_+ \oplus \mathcal{I}_-$.

**Theorem.** The Serre ideal of $A$ is generated by the Serre relations.

This allows to define the Drinfel’d quantum group as the algebra

$$\mathcal{A}' = A/\mathcal{I}(A), \quad (1.3)$$

and this formulation allows to relax the condition that $A$ be a generalized Cartan matrix, and to define a generalized quantum group.

### 1.1.4. Generalized quantum groups.

**Definition.** Let $M, N$ be two countable sets, and $\phi, \psi$ two maps,

$$\phi : M \times M \to \mathbb{C}, \ a, b \mapsto \phi^{ab},$$

$$\psi : M \times N \to \mathbb{C}, \ a, i \mapsto H_a(i).$$

Let

$$\phi(i, \cdot) = \sum_{a, b \in M} \phi^{ab} H_a(i) H_b, \ \phi(\cdot, i) = \sum_{a, b \in M} \phi^{ab} H_a H_b(i),$$

and suppose that $e^{\phi(i, \cdot) + \phi(\cdot, i)} \neq 1, i \in N$.

Let $A = A(\phi, \psi)$ be the universal, associative, unital $\mathbb{C}$-algebra with generators $\{e_i, f_i\}_{i \in N}$ and $\{H_a\}_{a \in M}$ and relations

$$[H_a, H_b] = 0, \ a, b \in M,$$

$$[H_a, e_i] = H_a(i)e_i, \ [H_a, f_i] = -H_a(i)f_i,$$

$$[e_i, f_j] = \delta_{ij} (e^{\phi(i, \cdot)} - e^{-\phi(\cdot, i)}).$$

Then the generalized quantum group $\mathcal{A}' = \mathcal{A}'(\phi, \psi)$ is the quotient $\mathcal{A}' = A/\mathcal{I}(A)$, where $\mathcal{I}(A)$ is the Serre ideal of $A$, defined as in Subsection 1.1.4.
If the form $\phi$ is symmetric and $A_{ij} = \phi(i, j)$ is a generalized Cartan matrix, then $A'$ is a quantum group in the sense that this term is used in most of the literature. The partial generalization that consists of relaxing the symmetry requirement was studied by Reshetikhin [Re]. The program of this paper is the classification of the larger family of generalized quantum groups (no restrictions on the matrix $A$), in terms of their Serre ideals.

1.1.5. Generalized Drinfel’d-Jimbo algebras.

Technical difficulties that arise from the appearance of infinite series in $H_1, ..., H_M$ within the relations can be avoided. If $M = N$ and $H_i' = 2H_i$ replace the Cartan generators $H_1, ..., H_N$ by

$$K_i = e^{H_i}, \quad K^i = e^{-H_i}, \quad i = 1, ..., N.$$  

The relations are now

$$K_iK^i = K^iK_i = 1,$$

$$K_i e_j K^i = e^{H_i(j)} e_j, \quad K_i f_j K^i = e^{-H_i(j)} f_j,$$

$$[e_i, f_j] = \delta_{ij} \frac{2}{\hbar} (K_i - K^i),$$

and the Serre relations. A slight disadvantage is that the classical limit is no longer the underlying Kac-Moody algebra $g$; the difference arises from the fact that $K_i \to \pm 1$. However, since statements that are true for Drinfel’d’s quantized enveloping algebras usually imply analogous results for the Drinfel’d-Jimbo algebras we shall not be greatly concerned with the distinction.

In the general case set

$$K_i = e^{\phi(i.,.)}, \quad K^i = e^{-\phi(.,i)}.$$  

The relations $K_iK^i = K^iK_i = 1$ are omitted while the rest of the relations remain as written.

1.1.6. Generalized quantum supergroups.

An interesting aspect of super Lie algeras is the existence of two kinds of odd roots. In the case of a generalized quantum supergroups there are parameters $q_{ii}$ that are fixed and equal to -1; then $e_i$ is a ‘null root’ and one of the relations is $e_i^2 = 0$. The other odd roots are characterized by the fact that $q_{jj} \to -1$ in the classical limit.

1.2. Hopf structure of generalized quantum groups.

In Drinfel’d’s terminology a quantum group is a coboundary Hopf algebra. This Hopf structure plays a relatively minor role in this paper since the methods used are essentially algebraic. However, the differential operators $\partial_i$ (see Subsection 1.2.4.) first appeared in an investigation of Hopf structures; this justifies a short review. It is possible that a more direct use of the Hopf structure may lead to simpler proofs and, in the hands of an expert, to further results.
1.2.1. Hopf Structure.

Fix the sets $\mathcal{M}, \mathcal{N}$ and the maps $\phi, \psi$ and let $A, \mathcal{I}$ and $A'$ be defined as in 1.1.4.

**Proposition.** [F1] There exists a unique homomorphism $\Delta : A \to A \otimes A$, such that

\[
\Delta(H_a) = H_a \otimes 1 + 1 \otimes H_a, \quad A \in \mathcal{M},
\]
\[
\Delta(e_i) = 1 \otimes e_i + e_i \otimes \phi(i, \cdot),
\]
\[
\Delta(f_i) = e^{-\phi(\cdot, i)} f_i + f_i \otimes 1, \quad i \in \mathcal{N}.
\]

The homomorphism $\Delta$ induces a unique homomorphism $A' \to A' \otimes A'$, also denoted $\Delta$. The algebra $A$ becomes a Hopf algebra when endowed with the counit $E$ and the antipode $S$. The former is the unique homomorphism $A \to A$ that vanishes on all the generators. The antipode is the unique anti-homomorphism $A \to A$ such that

\[
S(H_a) = -H_a, \quad a \in \mathcal{M},
\]
\[
S(e_i) = -e_i e^{-\phi(i, \cdot)}, \quad S(f_i) = -e^{\phi(\cdot, i)} f_i, \quad i \in \mathcal{N}.
\]

The counit $E$ and the antipode $S$ induce analogous structures on $A'$.

1.2.2. Coboundary property.

Let $\Delta'$ denote the opposite coproduct: in Sweedler’s notation, if $\Delta(x) = \sum x_1 \otimes x_2$ then $\Delta'(x) = \sum x_2 \otimes x_1$.

We restrict our attention temporarily to the special case of Drinfel’d’s quantum groups. Then there exists an element $R \in A' \otimes A'$ that interpolates between $\Delta$ and $\Delta'$:

\[
\Delta(x)R - R\Delta'(x) = 0, \quad \forall x \in A'.
\]

This element is known as the Universal Yang-Baxter Matrix; it satisfies the Yang-Baxter relation

\[
R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12},
\]

and it has been calculated explicitly by Reshetikhin [Re] and others.

Let $\sigma$ be the operator in $A' \otimes A'$ that interchanges the two spaces, and let $P := \sigma \circ R$. Then

Then

\[
Q(x) := \Delta(x)P - P\Delta(x) = 0.
\]

Let $d : \text{Hom}(A'^{\otimes p}, A'^{\otimes q}) \to \text{Hom}(A'^{\otimes p+1}, A'^{\otimes q})$ be the Hochschild differential of $A'$. We have $P \in \text{Hom}(\mathbb{C}, A'^{\otimes 2})$, and $dP = Q \in \text{Hom}(A', A'^{\otimes 2})$. Let $U$ be the bialgebra topologically dual to $A'$. By duality, $Q(x)$ is interpreted as an element of $\text{Hom}(U^{\otimes 2}, \mathbb{C})$ and $Q(x) = 0$ determines the algebraic structure of $U$. We have

\[
dQ(x, y) = \Delta(x)Q(y) - Q(xy) + Q(x)\Delta(y) \in \text{Hom}(U^{\otimes 2}, \mathbb{C}).
\]

If $Q(x) = Q(y) = 0$, $x, y \in A'$, then the property $dQ = 0$ reduces to $Q(xy) = 0$. Thus $Q$ must be closed, while the existence of the Universal $R$-matrix tells us that $Q$ is exact. Accordingly, the Drinfel’d quantum groups are called coboundary Hopf algebras.
1.2.3. The R-matrix of a generalized quantum group.

The Hopf algebras introduced in Subsection 1.1.4 are also of the coboundary type, and the R-matrices have been calculated in [F1].

Proposition. [F1] The algebra $A'$ is a coboundary Hopf algebra with a Universal R-matrix in the form of a series

$$R = e^\phi (1 + f_i \otimes e_i + \sum_{n=2}^{\infty} t_{ij}^k f_i \otimes e_j),$$

with

$$\phi = \phi^{ab} H_a \otimes H_b, \ i = i_1, ..., i_n, \ f_i = f_{i_1}...f_{i_n},$$

and with complex coefficients $t_{ij}^k$.

Outline of proof. (a) Define elements $t_\underline{i} \in B_+$ by

$$t_\underline{i} = \sum_j t^j_\underline{i} e_j.$$

By direct calculation one finds that the above series satisfies the Yang-Baxter relation if and only if the following recursion relations hold,

$$[t_\underline{i}, f_k] = e^{\phi(k,\cdot)} \delta_{i_1}^k t_\underline{i} - t_\underline{i} \delta_{i_n}^k e^{-\phi(\cdot,k)} , \ k = 1, 2, ..., N.$$

(b) Define operators $\partial_k$ on $B_+$ by

$$\partial_k e_i x = \delta_{ik} x + e^{-\phi(k,i)} e_i \partial_k x, \ x \in A'.$$

then the above recursion relation is equivalent to

$$\partial_k t_\underline{i} = \delta_{i_1}^k t_\underline{i}.$$

(c) Define the matrix $S$ by

$$S^j_\underline{i} = \partial_{\underline{i}} e_j, \ \partial_{\underline{i}} = \partial_{j_n}...\partial_{j_1},$$

(1.4)

for multi-indices of equal length $n = 1, 2, ...,$, all other matrix elements zero. The projection of this matrix on $A'$ is invertible, and the inverse is the projection on $A'$ of the matrix $t$ with matrix elements $t^j_\underline{i}$.

(d) Finally it is easy to verify that this R-matrix satisfies the relation $\Delta(x)R = R\Delta'(x)$.

The complete proof makes extensive use of the properties of the algebra $B_+$ endowed with the differential structure introduced by the action of the operators $\partial_k$. Some of these properties will be summarized below.
1.2.4. Proposition.

The space $I_+ \subset B_+$, generated by the invariants in $B_+$, coincides with the space generated by the "constants"; namely, the elements $x \in B_+$ that satisfy the relations $\partial_i x = 0$, $i = 1, \ldots, N$.

The problem of determining the Serre ideal of $A'$ is thus reduced to the calculation of the space of constants in $B_+$.

1.3. Classification of $q$-algebras.

It is proposed to determine the Serre ideals of the algebras $A' = A'(\phi, \psi)$ defined in Subsection 1.1.5. These algebras are parameterized by the values of the maps $\phi$ and $\psi$, the ideals by the parameters

$$q_{ij} = e^{-\phi(i,j)} \neq 0, \quad i, j = 1, \ldots, N.$$ 

By Proposition 1.2.4 the problem reduces to a study of $q$-algebras, that we now define.

1.3.1. Definition; $q$-algebras.

On the freely generated algebra $B = \mathcal{C}[e_1, \ldots, e_N]$, introduce differential operators $\partial_1, \ldots, \partial_N$ with the action defined by $\partial_i e_j = \delta_{ij}$ and

$$\partial_i (e_j x) = \delta_{ij} x + q_{ij} e_j \partial_i x, \quad x \in B.$$ 

Let $B_q$ be the same algebra $B$ with this differential structure.

1.3.2. Definition; constants.

A "constant" in $B_q$ is a polynomial $C \in B$, having no term of order 0, such that $\partial_i C = 0$, $i = 1, \ldots, N$. Let $I_q$ denote the ideal in $B_q$ that is generated by the constants.

Theorem. ([F1]) The ideal $I_q$ of $B$, via the identification of $B$ with the subalgebra $B_+ \subset A$, is precisely the component $I_+$ of the Serre ideal of $A$.

The interest focuses on the quotient,

$$B'_q := B_q/I_q.$$ (1.5)

These are the ‘$q$-algebras’ of the title.

1.3.3. Remarks.

(a) One can introduce a second set of differential operators $\partial'_i$, acting on $B$ from the right. Kharchenko has shown [Kh1] that $(B, \partial_i, \partial'_i)$ is a bicovariant differential structure in the sense of Woronowicz [W]. (b) The operators $\partial_i$ and $\partial'_i$ have been introduced by Kashiwara, in his work [Ka] on crystal bases. Kashiwara showed that these operators can be constructed inside the quantum group (with one parameter). Whether this remains true for generalized quantum groups is not known, nor is it directly relevant for the problematics of this paper.
1.3.4. Essential parameters.

It is shown in [FG] that the essential parameters, that determine the existence and the coefficients of constants, are

\[ \sigma_{ij} := q_{ij} q_{ji}, \quad i \neq j, \quad i, j = 1, \ldots, N, \]

and \( q_{ii}, \quad i = 1, \ldots, N. \) For generic values of these parameters there are no constants and the Serre ideal of \( \mathcal{A} \) is empty.

1.3.5. Gradings.

The algebra \( \mathcal{B} \) has a natural grading by the total polynomial degree, and this grading is passed on to \( \mathcal{B}_q \) and to \( \mathcal{B}'_q. \) A finer grading is the map that takes \( e_{i_1} \cdots e_{i_n} \) to the unordered set \( \{i_1, \ldots, i_n\}. \) Under this grading, the monomials of \( \mathcal{B} \) (or \( \mathcal{B}_q \) or \( \mathcal{B}'_q \)) are partially ordered by the relation of inclusion of sets, which gives a sense to the term ‘lower degree’. The space of constants has a basis of polynomials that are homogeneous in this finer grading; that is, linear combinations of the permutations of a single monomial.

Definition. A homogeneous constant is called ‘primitive’ if it is not in the ideal generated by constants of lower degree. The ‘space of primitive constants’ is defined via filtration.

The ideal \( \mathcal{I}_q \) is generated by a set of primitive constants.

The first general result was this.

1.3.6. Theorem. [FG]) Fix the degree \( G = \{1, \ldots, n\} \), and suppose that there are no constants of lower degree. Then the space of constants of degree \( G \) has dimension

\[ (n - 2)! \text{, if } \sigma_{1 \ldots n} := \prod q_{ij} = 1, \]

\[ 0, \quad \text{otherwise}. \]

The product runs over all pairs \( i \neq j, \quad i, j = 1, \ldots, n. \)

An essentially equivalent result was obtained by Kharchenko [Kh2].

1.3.7. Comparison with other work.

In addition to references already quoted we mention the work of Rosso [R]. He gives a nice direct presentation of the \( q \)-algebras \( \mathcal{B}'_q \) in which the ideal vanishes identically. This is equivalent to a result in [FG], where it was shown that the homomorphism from \( \mathcal{B}_q \) to the algebra \( \mathcal{B}_q^* \) of quantum differential operators on \( \mathcal{B}_q \), defined by \( e_i \mapsto \partial_i \), induces an isomorphism between \( \mathcal{B}'_q \) and \( \mathcal{B}_q^* \). The Hopf structure is prevalent in the work of Flores de Chelia and Greene [FCG], who have recently arrived independently at a result that is equivalent to Theorem 1.3.6. In our work the Hopf structure is represented by the matrix \( S \) (Section 2). This matrix is intimately related to the universal R-matrix (Subsection 1.2.3); it appears in almost all work in this area, notably in Varchenko [V] (who regards it as a form and calls it \( B \)) and in the paper [FCG] (where it is denoted \( \Omega \)).
1.4. Summary.

General results for the case of arbitrary degree $G$, but with the essential proviso that there be no constants of lower degree, have been reported [F2]. In this paper we return to the multilinear case $G = \{1, \cdots, n\}$ (no repetitions). In Section 2 we reduce the problem to a study of a determinant, and set up a scheme for the classification of $q$-algebras in terms of determinantal varieties. In Section 3 we explain the results of Varchenko that will be used. In Sections 4 and 5 we limit our study to the case when there may be any number of primitive constants of lower order, but all of total degree 2. These constants are generated by polynomials of the form

$$e_ie_j - q_{ji}e_je_i, \quad i \neq j,$$

and these polynomials are constants if and only if $\sigma_{ij} := q_{ij}q_{ji} = 1$.

Results for this special case are obtained in Section 4 and presented as Theorem 4.2. As I do not know how or if the method of arrangements of hyperplanes can be adapted to a more general situation, I present in Section 5 an alternative and completely algebraic proof of Theorem 4.2. Though it owes much to the paper [V], it makes no use of geometric concepts.

In Section 6 it is shown that this new approach is applicable to a much more general case, allowing for any number and any type of constraints (and constants) of lower degree. The result, Theorem 6.5, is a solution for the multilinear case, $G = \{1, \cdots, n\}$, under the stipulation that there be at least one pair $\{i, j\}$, such that there is no constraint on $\sigma_{ij}$. This last stipulation is important; unfortunately it is violated by ordinary quantum (super) groups.

2. The matrix $S$ and the form $B$.

2.1. The matrix $S$.

We continue to use the multi index notation, $\underline{i} := i_1 \cdots i_n$, $\underline{i}' = i_n \cdots i_1$ and

$$\partial_{\underline{i}'} = \partial_{i_n} \cdots \partial_{i_1}, \quad e_j = e_{j_1} \cdots e_{j_n}.$$  

A matrix $S = (S_{\underline{i} \underline{j}})$ is defined by

$$S_{\underline{i} \underline{j}} = \partial_{\underline{i}'} e_{\underline{j}|0},$$

where $x|_0$ is the term of total order 0 in the polynomial $x \in B$. This matrix commutes with the grading,

$$S = \oplus_G S_G, \quad (S_G)_{\underline{i} \underline{j}} = \partial_{\underline{j}'} e_{\underline{i}},$$

where $\underline{i}, \underline{j}$ run over the orderings of the unordered set $G$.

The matrix $S$ is singular if and only if there is a constant in $B_q$, and $S_G$ is singular if and only if there is a constant (primitive or not) of degree $G$. The existence of constants can thus be decided by inspection of the determinants. For example, if $\sigma_{12} := q_{12}q_{21} = 1$, then there is a constant of degree $G = \{1, 2\}$, namely $e_1 e_2 - q_{21} e_2 e_1$, and

$$S_G = \begin{pmatrix} 1 & q_{12} \\ q_{21} & 1 \end{pmatrix}, \quad \det S_G = 1 - \sigma_{12} = 0.$$
2.2. The determinant.

2.2.1. Parameters in general position.

The family \{B_q\} of algebras is parameterized by \( q = \{q_{ij}\}_{i,j=1,...,N} \in V := \mathbb{C}^{N^2} \). There is an open subset \( V_{\text{gen}} \) of \( V \) such that for \( q \in V_{\text{gen}} \) there are no constants in \( B_q \), namely, the subspace defined by \( \det S \neq 0 \). We shall say that parameters in this open set are in general position. Until further notice suppose that the parameters are in general position.

Let \( B_G \) be the subspace of \( B_q \) that consists of all polynomials of degree \( G \). From now on in this paper \( G = \{1, \cdots, n\} \), \( n \) fixed. Set

\[
w_{n,k} = u_{n,k} v_k,
\]

(2.1)

where

\[
u_{n,k} = (n + 1 - k)!
\]

(2.2)

and

\[
v_k = (k - 2)!
\]

(2.3)

Then it is a result of Varchenko that

\[
det S_G = \prod_k \prod_{i_1, \cdots, i_k} (1 - \sigma_{i_1 \cdots i_k})^{w_{n,k}}.
\]

(2.4)

The inner product is over all subsets of cardinality \( k \geq 2 \) of the set \( \{1, \cdots, n\} \). The total degree in \( q \)'s of \( det S_G \) is \( \binom{n}{2} n! \), and the formula implies the sum rule

\[
\sum_{k=2}^{n} k(k-1)w_{n,k} \binom{n}{k} = \binom{n}{2} n!\]

(2.5)

Since all \( \sigma_{i_j} \) appear symmetrically, the total degree in \( \sigma_{12} \), say, is

\[
\sum_{k=2}^{n} w_{n,k} \binom{n-2}{k-2} = n!/2.
\]

(2.6)

The numbers (2.2) and (2.3) have the following interpretation. Fix the integer \( k \leq n \) and let \( G_k = \{1, \cdots, k\} \). Let the parameters approach a portion of the boundary of \( V_{\text{gen}} \) where \( \sigma_{1 \cdots k} = 1 \) but \( \sigma_{i_k} \neq 1 \) for all \( i_k \neq 1 \cdots k \) (as un-ordered sets). Then primitive constants appear in \( B_{G_k} \); \( v_k \) is the dimension of the space of (primitive) constants in \( B_{G_k} \) and \( u_{n,k} \) is the dimension of the ideal in \( B_G \) generated by each constant in \( B_{G_k} \). A geometrical interpretation will follow.

2.2.2. Example.

Let \( G = \{1, 2, 3\} \) and suppose that there are no constants of lower degree, then

\[
det S_G = (1 - \sigma_{12})^2 (1 - \sigma_{23})^2 (1 - \sigma_{13})^2 (1 - \sigma_{123}).
\]
The surface on which $S_G$ is singular has four components, and in particular $S_G$ is singular on the surface $\sigma_{123} = 1$. On this surface the algebra $\mathcal{B}_q$ is characterized by the existence of a primitive constant of degree $G = \{123\}$.

2.2.3. Example.

Let $G = \{1, 2, 3, 4\}$ and suppose that there are no constants of lower degree, then

$$S_G = \prod_{i<j}(1 - \sigma_{ij})^6 \prod_{i<j<k}(1 - \sigma_{ijk})^2 (1 - \sigma_{1234})^2.$$

On the surface $\sigma_{1234} = 1$ there is a 2-dimensional subspace of constants in $\mathcal{B}_G$.

2.3. Cell decomposition of parameter space.

The space of parameters is the space $V = \mathbb{C}^{N^2}$ in which the $N^2$ parameters $q_{ij}$ take their values, with the natural analytic structure defined by these parameters. This space is the disjoint union of its $G$-cells ($G$ fixed), defined as follows.

2.3.1. Definition.

A $G$-cell in $V$ is a connected subset of $V$ on which the rank of each matrix $S_{G'}$, $G' \leq G$, is constant. A regular function on a $G$-cell is the restriction to the cell of a polynomial on $V$.

There is a space of constants associated to each point $q \in V$, and a regular field of constants on each $G$-cell.

2.3.2. Definition.

Two algebras $\mathcal{B}'_q$ and $\mathcal{B}'_{q'}$ are of the same $G$-type if $q$ and $q'$ belong to the same $G$-cell. They are of the same multilinear type if they are of the same $G$-type for every degree $G = \{i, j, \ldots\}$ without repetition, and of the same type if they are of the same $G$-type for every degree $G$.

2.3.3. Classification of the algebras by type.

It is our final aim to classify $q$-algebras by type; in this paper we have the more modest goal of a preliminary classification by multilinear type. The proposed strategy is inductive. For $G = \{1, 2\}$ there are are two cells:

$$C_1 : \sigma_{12} \neq 1, \quad C_2 = dC_1 : \sigma_{12} = 1,$$

where $dC$ denotes the boundary of the cell $C$. Suppose the cells have been determined for all multilinear degrees lower than $G = \{1, 2, \ldots, n\}$. Fix the $G'$-type for each $G' < G$; this amounts to fixing a certain set $Q$ of constraints of lower order. Let $V^Q$ be the closed subspace of $V$ defined by these constraints, let $I(Q)$ be the ideal generated by the associated constants, $\mathcal{B}_G(Q) = \mathcal{B}_G / I(Q) \cap \mathcal{B}_G$ and $S_G(Q)$ the projection of $S_G$ on $\mathcal{B}_G(Q)$. There is an open subset of $V^Q$ on which $\det S_G(Q) \neq 0$ and $\mathcal{B}_G(Q)$ has no constant, and this determines the $G$-type of $\mathcal{B}'_q$ for these parameter values. There remains the boundary $dV^Q$ of $V^Q$, the hypersurface $\det S_G(Q) = 0$. The points of this boundary are of two kinds.
First, those characterized by the appearance of one or more additional constant of lower degree, each determined by a constraint that involves a proper subset of \( \{ \sigma_{ij} \} \), this places the parameter in a \( V^Q \) of lower dimension. By treating the spaces \( V^Q \) in order of non-increasing dimension we avoid having to take these points into account. The complement in \( dV^Q \) of this first part of the boundary, if any, will be called the ‘primitive boundary of \( V^Q \);, it consists of points where \( \sigma_{1...n} = 1 \). The classification of types reduces to the question of the existence of primitive boundaries.

In Example 2.2.2 above the set \( Q \) is empty and \( S_G(Q) = S_G \). The expression for the determinant shows that there is a primitive boundary characterized by the constraint \( \sigma_{123} = 1 \) (and \( \sigma_{12}, \sigma_{13}, \sigma_{23} \neq 1 \)). The \( G \)-cells \( C_1, C_2 \) are the subsets of \( V^Q \) defined by \( \sigma_{12}, \sigma_{13}, \sigma_{23} \neq 1 \) and

\[
C_1 : \sigma_{123} \neq 1, \quad C_2 : \sigma_{123} = 1.
\]

This situation is further illustrated by Example 2.2.3. Here too there are two \( G \)-cells of interest, on which all \( \sigma_{ij} \neq 1 \), all \( \sigma_{ijk} \neq 0 \) and \( \sigma_{1234} \) is either equal to 0 or different from zero.

### 2.3.3. Example.

Let \( Q \) be the constraint \( \sigma_{12} = 1 \) associated with the constant \( e_1e_2 - q_{21}e_2e_1 \) and \( G = \{1, 2, 3\} \). Then \( V^Q \) is the surface in \( V \) on which \( \sigma_{12} = 1 \), and

\[
\text{det} S_G(Q) = (1 - \sigma_{13})^2(1 - \sigma_{23})^2 \neq 0.
\]

There is no primitive boundary and only one classifying \( G \)-type in this case. In the generic case a new constant appears on the surface \( \sigma_{123} = 1 \), but in the present special case, when there is a constant of lower order, this surface is not singular for \( S_G(Q) \).

### 2.3.4. Example.

Let \( G = \{1234\} \) and let \( Q \) be the set \( q_{12}q_{21} = q_{34}q_{43} = 1 \). The associated space of constants is generated by \( e_1e_2 - q_{21}e_2e_1 \) and \( e_3e_4 - q_{43}e_4e_3 \). One finds that

\[
\text{det} S_G(Q) = (1 - \sigma_{13})^6(1 - \sigma_{16})^6(1 - \sigma_{23})^6(1 - \sigma_{24})^6(1 - \sigma_{1234})^6.
\]

There is a 1-dimensional subspace of \( B_G(Q) \) of primitive constants, on the primitive boundary of \( V^Q \) on which \( \sigma_{1234} = 1 \).

### 2.3.5. Classification by multilinear type in Case \( N = 3 \).

The constraints of order 2 are, up to permutations of the indices,

\[
Q^1 : \sigma_{23} = 1, \quad Q^{12} : \sigma_{13} = \sigma_{23} = 1, \quad Q^{123} : \sigma_{23} = \sigma_{12} = \sigma_{13} = 1,
\]

and the empty set (no constraint). The classification by multilinear types of total degree 2 yields 4 types. The discussion of Example 2.3.3 shows that there is a distinguished boundary only if \( Q \) is the empty set. The complete classification by multilinear type thus yields 5 distinct types (up to a permutation of the generators).
2.3.5. Classification by multilinear type in Case $N = 4$.

At total order 2 there are 11 possibilities (always up to a permutation of the generators). We list the set of parameters that are fixed at unity in each case.

1. None,
2. $\sigma_{12}$,
3. $\sigma_{12}, \sigma_{34}$,
4. $\sigma_{12}, \sigma_{13}$,
5. $\sigma_{12}, \sigma_{13}, \sigma_{14}$,
6. $\sigma_{12}, \sigma_{13}, \sigma_{23}$,
7. $\sigma_{12}, \sigma_{23}, \sigma_{34}$,
8. $\sigma_{12}, \sigma_{23}, \sigma_{34}, \sigma_{14}$,
9. $\sigma_{12}, \sigma_{23}, \sigma_{34}, \sigma_{13}$,
10. $\sigma_{12}, \sigma_{23}, \sigma_{34}, \sigma_{13}, \sigma_{14}$,
11. all $\{\sigma_{ij}\}_{i<j}$.

This give rise to 11 varieties $V_{Q_1}^1, ..., V_{Q_{11}}^1$. A distinguished boundary at degree 3 appears in the first five cases only. Up to total order 3 there are 16 possibilities: the 11 cases listed and in addition the following,

12. $\sigma_{123}$,
13. $\sigma_{123}, \sigma_{124}$,
14. $\sigma_{123}, \sigma_{124}, \sigma_{134}$,
15. all $\{\sigma_{ijk}\}_{i<j<k}$,
16. $\sigma_{12}, \sigma_{134}$,
17. $\sigma_{12}, \sigma_{14}, \sigma_{234}$,
18. $\sigma_{12}, \sigma_{13}, \sigma_{14}, \sigma_{234}$.

Finally, we examine each of $V_{Q_1}^1, ..., V_{Q_{16}}^1$ and find that there is a primitive boundary in cases 1,2,3 and 12 only. the complete list of multilinear types for $N = 4$ is given by the 16 possibilities already listed, plus the following,

19. $\sigma_{1234}$,
20. $\sigma_{12}, \sigma_{1234}$,
21. $\sigma_{12}, \sigma_{34}, \sigma_{1234}$,
22. $\sigma_{123}, \sigma_{1234}$.

2.3.6. The general case.

Our problem can be solved by calculating the determinant of $S_G(Q)$ for all degrees $G$ and for any set $Q$ of constraints $\sigma_{i_1 \cdots i_k} = 1$, $k < n$. Eq. (2.2) gives the answer in the simplest case, when there are no constants of degree lower than $G$. Our first result, Theorem 4.2, gives the condition for $V^Q$ to have a primitive boundary and for the existence of a primitive constant of degree $\{1, 2, ..., n\}$ in the case that all primitive constants of lower degree have total degree 2. The most far reaching result obtained is Theorem 6.5, which applies whenever there is at least one $\sigma_{ij}$ that is not subject to any constraint.

3. Varchenko’s method.

3.1 Arrangements of hyperplanes.

Following Varchenko [V], we consider an arrangement of hyperplanes $H_1, \cdots, H_k$ in $\mathbb{R}^n$. An edge is a non-empty intersection of hyperplanes and a domain is a connected part of the complement of the set of hyperplanes. Complex weights $a_1, a_2, \cdots$ are attached
to the hyperplanes, the weight \( a(L) \) of an edge \( L \) is the product of the weights of the hyperplanes that contain \( L \). A bilinear form is defined by

\[
B(D, D') = \prod a_i,
\]

where \( D, D' \) are any two domains and the product runs over the hyperplanes that separate them. Varchenko gives the following formula \([V]\),

\[
\det B = \prod_L (1 - a(L)^2)^{n(L)p(L)}.
\]

(3.1)

The product runs over all edges and \( n(L), p(L) \) are certain natural numbers or zero.

3.2. Interpretation.

Let the parameters \( \{q_{ij}\} \) be in general position. For a special choice of hyperplanes and weights, \( B \) is identified with the matrix \( S_G \), \( G = \{1, \cdots, n\} \). Namely, let the hyperplanes be

\[
H_{ij} = \{x_i = x_j\}, \quad i < j, \quad i, j = 1, \cdots, n.
\]

The domains are then

\[
D_{ij} = \{x_{i_1} < x_{i_2} < \cdots < x_{i_n}\},
\]

in natural correspondence with the monomials \( e_{i_1} \cdots e_{i_n} \) of degree \( G = \{i_1, \cdots, i_n\} \). Choose \( q_{ij} \) symmetric (this is harmless since the zeros of \( \det S_G \) depend only on the products \( q_{ij}q_{ji} = \sigma_{ij} \)) and set \( q_{ij} = a_{ij} \), the weight of the hyperplane \( H_{ij} \). Then

\[
B(D_{ij}, D_{jk}) = S_{ij}.
\]

For this case, Varchenko’s formula (3.1) for \( \det B \) coincides with the Eq.(2.4). The contributing edges are all those of the form \( L = \{x_{i_1} = x_{i_2} = \cdots = x_{i_k}\} \). In this interpretation the integer \( n(L) \) is the number of domains in the arrangement in \( L \) for which the hyperplanes are the intersections of \( L \) with those of the original planes that do not contain \( L \). This number is the same as the number \( u(n, k) = (n + 1 - k)! \) in Eq.(2.2); that is, the dimension of the intersection between \( B_G \) and the ideal generated by any primitive constant of degree \( \{1, \cdots, k\} \).

3.3. The number \( p(L) \).

The calculation of the number \( p(L) \), in the exponent in Eq.(3.1), is more subtle and at the center of interest. Any edge of the configuration under consideration is a hyperplane of the form (up to a renaming of the coordinates)

\[
\{x_1 = \cdots = x_k, \ x_{k+1} = \cdots = x_{k+l}, \ \cdots\, , \ \cdots\}.
\]

The number \( p(L) \) is defined as follows. Let \( N \) be the normal to \( L \); in our case it is

\[
N = \{\xi_1, \cdots, \xi_k; \ \eta_1, \cdots, \eta_l; \ \cdots\} , \quad \sum \xi_i = \sum \eta_j = \cdots = 0.
\]
There is the arrangement \( \{ H \cup N; L \subset H \} \) of hyperplanes in \( N \). The planes are \( \xi_i = \xi_j, \eta_i = \eta_j, \cdots \). Consider the projectivization of this arrangement. Fix any one of the hyperplanes, \( H \), say. Then the number \( p(L) \) is the number of projective domains the closures of which do not intersect \( H \).

**Proposition.** The number \( p(L) \) is zero unless \( L = \{ x_1 = \cdots = x_k \} \) for some \( k = 2, \cdots, n \), up to a permutation of the index set.

**Proof.** Let \( H = \{ \xi_1 = \xi_k \} \). If the set of coordinates of \( N \) includes one or more sets beyond the initial set \( \xi_1, \cdots, \xi_k \), then all the projective domains include some lines on which \( \xi_1 = \cdots = \xi_k = 0 \), and then \( p(L) = 0 \). The proposition is proved.

**Corollary.** The determinant of \( S_G, \; G = \{ 1, \cdots n \} \), is a product of factors of the form 

\[
1 - \sigma_{1\cdots i_k}, \; k = 2, \cdots, n.
\]

We are reduced to the case when, up to a renaming of the coordinates,

\[
L = \{ x_1 = \cdots = x_k \}, \quad N = \{ \xi_1, \cdots, \xi_k, 0, \cdots, 0 \}, \quad \sum_i \xi_i = 0.
\] (3.2)

Remember that the parameters are in general position, no constraints.

Consider the closure \( \xi_1 \leq \cdots \leq \xi_k \) of the domain \( \xi_1 < \cdots < \xi_k, \sum \xi_i = 0 \). It touches any hyperplane \( \{ \xi_i = \xi_j \} \) at points where \( \xi_i = \cdots = \xi_j = 0 \). What saves us from the conclusion that \( p(L) \) is always zero is the fact that this domain fails to touch the hyperplane \( \xi_1 = \xi_k \). This is because the point at the origin of \( N \) does not have a projective image. We conclude that for such edges, \( p(L) = (k - 2)! \), the number of domains of the type \( \xi_1 \leq \cdots \leq \xi_k \). This number is the same as the number \( v_k = (k - 2)! \) in (2.3); that is, the dimension of the space of primitive constants of degree \( G_k \), so Varchenko’s formula (3.1) reduces to Eq.(2.4) in this case.

**4. Constants of lower degree, each of total order 2.**

**4.1. A special case.**

We shall determine under what conditions there are primitive constants of degree \( G = \{ 1, \cdots, n \} \) in the case that there is any number of primitive constants of lower degree, but all of them of total order 2. The parameters are thus in general position, except that they satisfy a set of constraints,

\[
\sigma_{ij} = 1, \quad \{ i, j \} \in P,
\]

where \( P \) is a fixed subset of the set of pairs \( \{ i, j \} \), \( i \neq j, \; i, j = 1, \cdots, n \). The primitive constants of lower degree are \( e_i e_j - q_{ij} e_j e_i, \quad \{ i, j \} \in P \).

In the idiom of arrangements of hyperplanes, the constraint \( \sigma_{ij} = 1 \) means that the weight \( a_{ij} \) of the plane \( x_i = x_j \) is equal to unity.
Let $B(Q)$ be the matrix $B$ for the arrangement obtained by removing hyperplanes with weight 1. The identification $S_G(Q) = B(Q)$ holds in this case as well. The arrangement is in $\mathbb{R}^n$, with hyperplanes $x_i = x_j$, $\{i,j\} \notin P$.

Varchenko’s formula (3.1) applies and the only edges for which the number $p(L)$ is different from zero are the ones of the form (3.2). The determinant still has the form (2.4), but some of the exponents are diminished. Of importance for the classification problem is the question whether the factor $1 - \sigma_1 \cdots \sigma_n$ appears with non-zero exponent: if $L_0 := \{x_1 = \cdots = x_n\}$, when is $p(L_0)$ different from zero?

The space $N$ normal to $L_0$ is $\{\xi_1, \cdots, \xi_n\}$, $\sum \xi_i = 0$. The domains are defined by inequalities,

$$\xi_i < \xi_j, \quad \{i,j\} \notin P.$$

A domain the closure of which does not intersect a given hyperplane, $\xi_1 = \xi_n$, say, must bracket all the other variables, $\xi_2, \cdots, \xi_{n-1}$, between $\xi_1$ and $\xi_n$.

4.2. Theorem.

Let $G = \{1, \cdots, n\}$, $n > 2$, $Q$ the set of constraints $\sigma_{ij} = 1$, $\{i,j\} \in P$. We may suppose that $\sigma_{1n} \neq 1$; that is, that $\{1,n\} \notin P$. The following condition is necessary and sufficient for $V^Q$ to have a primitive boundary. For any $i$, $1 < i < n$, there is a sequence $1, \cdots, i, \cdots, n$, a subsequence of a permutation of $1, \cdots, n$, such that no pair of neighbours in it belongs to $P$.

Examples. See the list of constraints in 2.3.5. In the case that $P = \{(12), (34)\}$ the projective domain $1 < 3 < 2 < 4$ does not touch the plane $x_1 = x_4$ since $P$ does not contain the pairs $(1,3)$, $(3,2)$ or $(2,4)$. There is only one such domain, so the determinant contains the factor $1 - \sigma_{1234}$ with exponent 1. There is at least one (actually exactly one) primitive constant of degree $G = \{1,2,3,4\}$ when $Q$ is the set $\sigma_{12} = \sigma_{34} = 1$. But if the constraint is $\sigma_{12} = \sigma_{13} = 1$, then there is no primitive constant of this degree. See [Kh3].

5. Algebraic proof of Theorem 4.2.

It was seen that the case of constraints of a very special type lies within the range of the theory of arrangements of hyperplanes. But the direct application of this theory to more general situations does not appear to be straightforward. For that reason it will be useful to reformulate the proof of Theorem 4.2 in purely algebraic terms.

By stipulation, the relations of $B(q)(Q)$ are generated by $e_i e_j = q_{ji} e_j e_i$, $\{i,j\} \in P$, and the constraints are $Q : \sigma_{ij} = 1$, $\{i,j\} \in P$. We may suppose that $\{1,n\} \notin P$.

5.1. Basis.

All bases used for $B_G(Q)$ and its subspaces will be monomial. A basic monomial will be called a word. If $e_\lambda$ is a word then so is $e_\lambda^*$ (the same word read backwards), unless the two are proportional to another. Since $\sigma_{1n} \neq 1$, this cannot happen in the context.
Example. If $\sigma_{12} = 1$, then a basis for $B_{\{1,2,3\}}(Q)$ is

$$e_1e_3e_2, \ e_2e_1e_3, \ e_2e_3e_1, \ e_3e_1e_2.$$  

We shall say that a word in $B_G(Q)$ is ‘positive’ if $e_1$ precedes $e_n$, and proceed to choose the positive words of a basis.

5.2. Factors and classes.

A positive word in $B_G(Q)$ has the form $xe_1ye_nz$. The degree of $y$ defines a filtration of $B_G(Q)$. Choose a monomial basis that respects this filtration. An element of the basis, of the form $xe_1ye_nz$, will be said to have the ‘factor’ $y$ and to be of ‘class’ $g =$ the degree of $y$. Remember that the degree of $e_i$ is the unordered set $\{i_1, \ldots, i_k\}$ of indices.

Lemma. The number of words with factor $y$ depends only on the degree of $y$.

Proof. The degree of $y$ selects a subset of the generators $e_2, \ldots, e_{n-1}$ and reduces the construction of the basis of $B_G(Q)$ to that of a basis for the subspace of polynomials in $e_0 := e_1ye_n$ and the supplementary set of generators. The relations are all of the type $e_ie_j = ke_je_i$, $k \in \mathbb{C}$ and are independent of the order of factors in the monomial $y$.

5.3. First sum rule.

Let $u_n(g)$ be the number of positive words for $B_G(Q)$ that contain some fixed factor $y$ of degree $g$, and let $v(g)$ be the number of (linearly independent) factors of degree $g$. Then

$$\sum_g u_n(g)v(g) = \frac{1}{2} \dim B_G(Q).$$

Example. $P = \{\{1,2\}\}$.

| Type | $u$ | $v$ | Basis, positive part, $i \neq j = 2,3$ |
|------|-----|-----|-------------------------------------|
| (·)  | 6   | 1   | $e_i e_j (e_1 e_4)$, $e_i (e_1 e_4) e_j$, $(e_1 e_4) e_i e_j$ |
| (3)  | 0   |     |                                      |
| (4)  | 2   | 1   | $e_2 (e_1 e_4)$, $(e_1 e_3 e_4) e_2$ |
| (34) | 0   |     |                                      |
| (32) | 1   | 1   | $(e_1 e_3 e_2 e_4)$ |

5.4. Second sum rule.

Lemma. Let $\tilde{v}(g)$ be the exponent of $(1 - \sigma_{11}^n)$ in $\det S_{\hat{g}}(Q)$, $\hat{g} = \{1,g,n\}$, $g$ the degree of $e_i$; then the exponent of the same factor in $\det S_G(Q)$ is $u_n(g)\tilde{v}(g)$.

Proof. Fix $\hat{g} = \{1,g,n\} = \{1,i_2,\ldots,i_k,n\}$. In the matrix $S_G(Q)$ replace by zero all $\sigma_{ij}, \{i,j\} \notin P$, that do not appear in $\sigma_{11}^n$. Then $S_G(Q)_{\hat{g}k}$ vanishes unless $e_j$ and $e_k$ are equal up to a reordering of the generators $e_1, e_{i_2}, \ldots, e_{i_k}, e_n$ only; that is, unless $e_j =
$xe_1ye_nz$, $e_k = x e_1 y' e_n z$ with $y$ and $y'$ of the same degree $g$. The matrix takes the block form and $\det S_G(Q)$ reduces to a power of $\det S_g(Q)$. The exponent is the number of blocks and is equal to the number $u_n(g)$ of words that contain some fixed $y$ of grade $g$.

Every $\sigma_{1n}$ is linear in $\sigma_{1n}$, and the terms of highest power of $\sigma_{1n}$ in $\det S_G(Q)$

$$\prod_i \partial_\nu e_i \propto (\sigma_{1n})^\kappa, \quad \kappa = \frac{1}{4} \dim B_G(Q),$$

where the product runs over all words. Hence

$$\sum_g u_n(g) \tilde{v}(g) = \frac{1}{4} \dim B_G(Q).$$

Clearly, $\tilde{v}(g) = v(g)$ when $n = 1$; therefore by induction in $n$, it follows from the two sum rules that $\tilde{v}(g) = v(g)$ and Theorem 4.2 is proved (again).

6. General constraints.

6.1. A proviso.

So far we have allowed primitive relations of order 2 only, associated with constraints $\sigma_{ij} = 1$, $\{i, j\} \in P$, where $P$ is a collection of pairs. The key to both proofs of Theorem 3.2 was the counting of powers of $\sigma_{1n}$ in $\det S_G(Q)$, and for this reason it was essential that $\{1, n\} \notin P$. This is not a real limitation, for another pair will do just as well, as long as there is at least one that is not in $P$; that is, except in the case that $\det S_G(Q) = 1$.

Now let us consider the more general situation, when there is a family of constraints,

$$\sigma_i = 1, \quad i \in P, \tag{6.1}$$

where $P$ is any collection of proper subsets of $\{1, \cdots, n\}$. To apply the method of counting powers of $\sigma_{1n}$, we need for this parameter to be unconstrained, and this amounts to the limitation that, for any $g$,

$$\{1, g, n\} \notin P. \tag{6.2}$$

Proceeding as in Section 5, we encounter no difficulties in choosing a monomial basis based on the concepts of ‘factors’ and ‘classes’. But the Lemmas in 5.2 and 5.4 need to be re-examined. The first one is:

6.2. Lemma. The number of words with factor $y$ depends only on the degree of $y$.

Let us consider the process of choosing the basis in somewhat more detail. Begin with elements of the form $e_1 y e_n$, $y$ a permutation of $e_2 \cdots e_{n-1}$. If there are no relations involving either $e_1$ or $e_n$, then any set of independent ’$y$’s will do. It is enough to consider primitive constants involving $e_1$, say. (By stipulation, there are no primitive constants involving both $e_1$ and $e_n$.) Any constant of this type is a polynomial

$$e_1 A + \sum_{j=2}^{n-1} e_j e_1 A^j + \cdots + B e_1.$$
If there is only one such constant, with $A \neq 0$, then the filtration replaces $y$ by $y$ modulo the right ideal $\{Az\}$. This affects the number $v(g)$ of factors of this type, but the relation has no further effect on the construction of the basis, and the lemma still stands. If there is another constant, of the same type, then the argument still applies. But it may happen that there are two constants with the same $A$, and then there is a constant of the type

$$
\sum_j e_j e_1 A^j + \sum_{j,k} e_j e_k e_1 A^{jk} + \cdots + Be_1.
$$

In this case, suppose $A^2 \neq 0$, then differentiation with $\partial_2$ gives

$$
e_1 A^2 + \partial_2 \left( \sum_{j>3} e_j e_1 A^j + \sum_{j,k} e_j e_k e_1 A^{jk} + \cdots + Be_1 \right) = 0.
$$

Now this relation implies that the factor $y$ is defined modulo the ideal $\{A^2z\}$. Continuing in this manner we conclude that any relation that involves $e_1$ leads to a reduction in the number of factors, but it does not affect the number of basis elements containing a given factor. In fact, once the factors have been determined, then the enumeration of basis vectors is independent of the factor and depends only on the set of generators in it; that is, on its degree. The Lemma is proved.

This implies that the first sum rule remains valid.

**Example.** $P = \{\{1, 2, 3\}\}$, thus $\sigma_{123} = 1, n = 4$.

| Factors | $u_4$ | $v$ | words |
|---------|-------|-----|-------|
| ( )     | 6     | 1   | $e_i e_j (e_1 e_4)$, $e_i (e_1 e_4) e_j$, $(e_1 e_4) e_i e_j$ |
| ($e_2$) | 2     | 1   | $e_3 (e_1 e_3 e_4)$, $(e_1 e_2 e_4) e_3$ |
| ($e_3$) | 2     | 1   | $e_2 (e_1 e_3 e_4)$, $(e_1 e_3 e_4) e_2$ |
| ($e_2 e_3$) | 1 | 1 | $(e_1 e_2 e_3 e_4)$ |

There is only one constraint and its only effect is to exclude the monomial $(e_1 e_2 e_3 e_4)$ with factor $e_3 e_2$ from the basis.

The second sum rule also remains in force in the more general case, but our proof of the Lemma in 5.4 does not, since we cannot replace by zero parameters that are constrained. It must be replaced by the following two lemmas.

**Fix a natural number $k$, $1 < k < n$ and a subset $\{i_2, \ldots, i_k\} \subset G = \{1, \ldots, n\}$. Fix a set $Q$ of constraints so as to leave the parameter $\sigma_{1n}$ free, and let $B(Q)$ be the associated quotient algebra. Write $\{i_2, \cdots, i_k\} = g$.**

**6.3. Lemma** Fix all parameters except $\sigma_{1n}$. The exponent of $(1 - \sigma_{1i_2 \ldots i_k n})$ in $\det S_G(Q)$ is equal to the dimension of $\ker S_G(Q)$ at the value of $\sigma_{1n}$ that makes $\sigma_{1i_2 \ldots i_k n} = 1$.

**Proof.** Introduce a monomial basis as above, consisting of ‘positive’ words in which $e_1$ precedes $e_n$, and the same set taken in reverse order. Set $q_{1n} = q_{n1} = q$. Then the matrix
element $S_{ij}(Q)$ of $S_G(Q)$ is independent of $q$ if $i, j$ are both positive or both negative, linear in $q$ otherwise, and $S_G(Q)$ takes the form

$$S_G(Q) = \begin{pmatrix} A & qB^t \\ qB & C \end{pmatrix},$$

with $A, C$ symmetric (take $q_{ij} = q_{ji}$) and invertible (there are no constants in the subalgebra of $B_G$ generated by $e_2, ..., e_{n-1}$). Interpreting $S_G$ as a form, we transform $A \oplus C$ to a unit matrix without affecting the kernel of $S_G(Q)$, converting this form to $I_qD$ with $I_{ij} = \delta_{ij}$ and $D$ symmetric. Finally, interpreting $I_qD$ as a symmetric (and hence diagonalizable) matrix one obtains the result.

6.4. Lemma. If $(1 - \sigma_{1i_2...i_k})$ appears with exponent $\tilde{v}(g)$ in $\det S_{\hat{g}}(Q)$, $\hat{g} = \{1, g, n\}$, then it appears with exponent $u_n(g)\tilde{v}(g)$ in $\det S_G(G)$.

Proof. If $xe_1ye_nz$ is a word of class $g = \{i_2, ..., i_k\}$, and $C \in B_{\{1, g, n\}}(Q)$ is a constant, then $xCz$ is in the ideal generated by $C$, and this correspondence extends to a bijection (for fixed $y$ and $C$). The dimension of the ideal generated in $B_G(Q)$ by the constants in $B_{\hat{g}}(Q)$ (which is the same as the dimension of ker $S_G(Q)$ at $\sigma_{1i_2...i_k} = 1$) and by Lemma 6.1 equal to the exponent of $1 - \sigma_{1i_2...i_k}$ in $\det S_G(Q)$ is thus $u_n(g)$ times the dimension of the space of constants in $B_{\hat{g}}$. The lemma is proved.

This gives the second sum rule, and by induction, $v(k) = \tilde{v}(k)$. We have thus proved:

6.5. Theorem. Assume parameters as before, with Eqs (6.1) and (6.2). Then the factor $(1 - \sigma_{1i_1...i_k})$ appears in $\det S_G(Q)$ with an exponent that is equal to the number of words of class $g = \{2, \cdots, k\}$ in $B_G(Q)$ and the number of linearly independent constants that appear in $B_G(Q)$ when $\sigma_{1...n}$ tends to 1 is equal to the number of words of class $\{2, \cdots, n\}$.

Consequently, a necessary and sufficient condition for the appearance of at least one primitive constant in $B_G(Q)$ when $\sigma_{1...n}$ tends to 1, and for the existence of primitive boundary of $V^Q$, constant is that there is $e_1ye_n \in B_G(Q)$ that cannot be expressed in terms of monomials $xe_1y'e_nz$ with $xz$ of non-zero degree.

It remains to understand the case when all the parameters are constrained. This can happen with as few as 3 independent constraint, for example $\sigma_{1...n-1} = \sigma_{2...n} = 1$ and $\sigma_{1n} = 1$. Present methods fail because it makes no sense to count the powers of any one of the parameters.

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