CONSTRUCTION OF DIFFERENCE SCHEMES
FOR NONLINEAR SINGULAR PERTURBED
EQUATIONS BY APPROXIMATION OF
COEFFICIENTS

L. V. Rozanova

Scientific Advisor
A. I. Zadorin

Omsk – 2000
## Contents

Relevance 1

1 Nonlinear differential equation with exponential boundary layer 4
   1.1 Analysis of the original problem .............................................. 5
   1.2 Estimate of the derivative ....................................................... 8
   1.3 Construction of a difference scheme ........................................ 9
   1.4 Evaluation of convergence .................................................... 12

2 Nonlinear differential equations with power-law boundary layer 14
   2.1 Boundedness of solutions ........................................................ 14
   2.2 Estimation of the derivative .................................................... 15
   2.3 Construction of the scheme ..................................................... 17
   2.4 Justification of convergence .................................................... 18

Results 19

References: 20
Relevance

Mathematical modeling of many physical processes such as diffusion, viscosity of fluids and combustion involves differential equations with small coefficients of higher derivatives. These may be small diffusion coefficients for modeling the spreading of impurities, small coefficients of viscosity in fluid flow simulation etc.

The difficulty with solving such problem is that if you set the small parameter at higher derivatives to zero, the solution of the degenerate problem doesn’t correctly approximate the original problem, even if the small parameter approaches zero; the solution of the original problem exhibits the emergency of a boundary layer. As a result, the application of classical difference schemes for solving such equations produces great inaccuracies. Therefore, numerical solution of differential equations with small coefficients at higher derivatives demands special difference schemes exhibiting uniform convergence with respect to the small parameters involved.

Current status of this problem

The development of numerical methods for problems with exponential and power-law boundary layers introduced among others the following methods:

i) densification of grids in boundary layers;

ii) scheme adjustment on the boundary layer component of solutions.

The first approach includes the works of N. S. Bakhvalov, V. D. Liseikin, G. I. Shishkin, V. B. Andreyev, R. Vulanovich and other authors. In P. S. Bakhvalov’s approach, the grid nodes distribution is constructed in such a way that approximation inaccuracy on the grid nodes of the boundary layers stays equal while outside of the boundary layer the mesh remains uniform.

It is shown that application of such a grid yields second-order accuracy with respect to the number of grid points. V. D. Liseikin purposes to perform change of variables in such a way that derivatives up to some order are uniformly bounded. In terms of original variables, this is equivalent to mesh
densification. G. I. Shishkin defined an approach for building grids that are uniform both within the boundary layer and outside of the boundary layer. In the works of G. I. Shishkin, V. B. Andreev, E. A. Savin, N. V. Kopteva it is shown that on such a grid, a number of difference schemes (including non-monotone scheme of central differences) are uniformly convergent. The significance of this approach for partial differential equations is strengthened by the fact that in the case of a parabolic boundary layer, as shown by G. I. Shishkin, there is no uniformly convergent adjustment schemes on a uniform grid.

The second approach is presented in the works of A. M. Ilyin, G. I. Shishkin, K. V. Emelyanov, D. Miller, R. Kellogg and others. The main idea of this approach is isolating the boundary layer component of solutions and constructing a difference scheme which is exact on the boundary layer function. The advantage of this approach is that it does not impose restrictions on the mesh gauge, while the drawback is that the boundary layer function must be expressed explicitly and the scheme must be specifically adjusted to the function, which is not always possible.

In this article author investigates two nonlinear boundary value problems on a finite interval, resulting in exponential and power-law boundary layers.

In the first chapter author constructs a first-order accurate finite difference scheme for an exponential boundary layer problem. Author then introduces a certain grid where on each interval the coefficients are replaced by constants. The second chapter provides a construction of a first-order accurate difference scheme for a problem with power-law boundary layer using a similar technique with additional mesh densification on the boundary layer for scheme convergence.
Chapter 1

Nonlinear differential equation with exponential boundary layer

The purpose of this chapter is to construct a difference scheme for a nonlinear equation with exponential boundary layer, uniformly convergent in the small parameter. Since the presence of a small parameter $\varepsilon$ at the second-order derivative causes an exponential growth of solutions near the boundaries of the interval, application of classical difference schemes for solution of equations with small coefficient at the highest derivative produces great inaccuracies. This follows from the fact that the accuracy of known schemes is estimated as a product of some power of the mesh size $h$ and maximum absolute value of some derivative of the solution, and derivatives of the solutions grow unbounded in boundary layer with decreasing $\varepsilon$. Thus the classical accuracy estimation of difference schemes for solving equations with small parameter is not acceptable.

Let us consider the boundary value problem:

\[
\begin{align*}
T_{\varepsilon} u &= -\varepsilon u'' + a(x) u' + g(u) = 0 \\
u(0) &= A, \quad R_{\varepsilon} u &= \varepsilon u(L) + f(u(L)) = 0
\end{align*}
\] (1.0.1)

assuming that:

\[
\begin{align*}
a(x) &\geq \alpha, \quad \frac{\partial g}{\partial u} \geq -\beta, \quad \frac{\partial f}{\partial u} \geq 0, \\
\alpha &> 0, \quad \beta > 0, \quad \varepsilon > 0, \quad \alpha^2 - 4\varepsilon\beta \geq \gamma > 0.
\end{align*}
\] (1.0.2)

Let’s assume the function $g(s)$ to be twice continuously differentiable for all $s \in \mathbb{R}$, $\varepsilon \in (0, 1]$, and $a(x)$ continuously differentiable.
1.1 Analysis of the original problem

Define an auxiliary linear operator $L$:

$$Lu = -\varepsilon u'' + a(x) u' + c(x) u$$

with boundary conditions:

$$u(0), \quad Du = \varepsilon u'(L) + d(x) u.$$ 

Let us find out whether for the operator $L$ the Principle of Maximum holds. 

**Lemma 1.1.1.** Assume $\exists \phi(x) \geq 0 : L\phi(x) > 0$, $D\phi > 0$. Then for the operator $L$ the Principle of Maximum holds, i.e. for an arbitrary twice differentiable function $\psi(x)$ the condition:

$$L\psi(x) \geq 0, \quad D\psi \geq 0, \quad \psi(0) \geq 0$$ 

implies that: $\psi(x) \geq 0$.

**Proof.** Assume $\psi(x)$ to be some differentiable function. Express $\psi(x)$ as the product: $\psi(x) = v(x)\phi(x)$.

Assume there is $x_0: \psi(x_0) < 0$.

Then $v(x_0) < 0$.

$$v(0) \geq 0, \quad D\psi = (v\phi)'(L) + dv\phi(L) \geq 0 \quad v'\phi + v\phi' + dv\phi \geq 0 \quad v'\phi + [\phi' + d\phi]v \geq 0.$$ 

Consequently,

$$v'\phi + D\phi v \geq 0.$$ 

$$L\psi = L(v\phi) = -\varepsilon(v\phi)'' + a(v\phi)' + cv\phi = -\varepsilon(v'\phi + v\phi')' + a(v'\phi + v\phi') + cv\phi = -\varepsilon(v''\phi + 2v'\phi' + v\phi'') + a(v'\phi + v\phi') + cv\phi = -\varepsilon(v''\phi + 2v'\phi') + v(-\varepsilon\phi'' + a\phi + c\phi) + av'\phi = -\varepsilon v''\phi + 2\varepsilon v'\phi' + av'\phi + vL\phi = -\varepsilon \phi v'' - v'(2\varepsilon\phi' - a\phi) + vL\phi \geq 0 \quad (1.1.1)$$ 

Consider two cases:
i) Let \( v(L) \geq 0 \). Then there is \( \eta \), such that \( v(\eta) < 0 \) holds on the interval \([0,L]\). Then \( \eta \) is a minimum point for the function \( v(x) \). At the point of minimum first derivative equals zero, therefore, given the conditions of Lemma, in \((1.1.1)\) \( \varepsilon \phi v'' \geq 0 \), \( vL\phi > 0 \), and the term \( v'(2\varepsilon\phi' - a\phi) \) vanishes. So \( L\psi < 0 \), but from condition of the Lemma \( L\psi \geq 0 \), a contradiction follows.

ii) Let \( v(L) < 0 \). Then due to the inequality
\[
v'\phi + D\phi v \geq 0
\]
we obtain
\[
v'(L) > 0
\]
and, consequently, there is a minimum point \( \tilde{\eta} \). Arguing similarly to the first case, we arrive at a contradiction.

Suppose that
\[
d(x) \geq 0, \ c(x) \geq -\beta, \ \alpha^2 - 4\beta\varepsilon \geq \gamma > 0, \ \beta > 0.
\]
We show that for \( L \) the Principle of Maximum holds. Define the function \( \phi(x) \) as:
\[
\phi(x) = e^{\frac{\delta}{\varepsilon}(x-L)}, \text{ where } \delta = \frac{2\beta\varepsilon}{\alpha}.
\]
\[
D\phi(L) = \delta + d > 0.
\]
\[
L\phi(x) = \left( -\frac{\delta^2}{\varepsilon} + a\frac{\delta}{\varepsilon} + c \right) \phi(x)
\]
\[
\geq \left( -\frac{\delta^2}{\varepsilon} + \frac{\delta}{\varepsilon} - \beta \right)
\]
\[
= \frac{\beta}{\alpha^2} (\alpha^2 - 4\beta\varepsilon) \phi(x) > 0.
\]
Thus,
\[
\phi > 0, \ D\phi > 0, \ L\phi > 0,
\]
i.e. the Principle of Maximum holds.

Now we obtain an estimate for stability for the problem \((1.0.1)\).

**Lemma 1.1.2.** Let \( p(x) \) and \( q(x) \) be two arbitrary functions. Then
\[
\|p(x) - q(x)\| \leq c \|T_\varepsilon p - T_\varepsilon q\| + c |p(0) - q(0)| + c |R_\varepsilon p - R_\varepsilon q|.
\]
Proof. Define \( z = p - q \). Then with respect to \( z \) we obtain the boundary problem:

\[
Lz = -\varepsilon z'' + a(x) z' + c(x) z = T_\varepsilon p - T_\varepsilon q,
\]
where

\[
c(x) = \frac{g(p) - g(q)}{p - q} \geq -\beta.
\]

\[
z(0) = p(0) - q(0).
\]

Introduce the function

\[
\psi(x) = c_1 e^{\frac{2\beta x}{\alpha}} \| T_\varepsilon p - T_\varepsilon q \| + c_2 e^{\frac{\alpha(x-L)}{2\varepsilon}} \| R_\varepsilon p - R_\varepsilon q \| + e^{\frac{2\beta}{\alpha}} \| p(0) - q(0) \| \pm z(x).
\]

It is easy to verify that:

\[
L e^{\frac{\alpha(x-L)}{2\varepsilon}} \geq \left( -\frac{\alpha^2}{4\varepsilon} + \frac{\alpha^2}{2\varepsilon} + c(x) \right) e^{\alpha(x-L)2\varepsilon} \geq \left( -\frac{\alpha^2}{4\varepsilon} - \beta \right) e^{\alpha(x-L)2\varepsilon}
\]

\[
= \left( -\frac{\alpha^2 - 4\beta \varepsilon}{4\varepsilon} \right) e^{\alpha(x-L)2\varepsilon} \geq \gamma e^{\alpha(x-L)2\varepsilon} > 0.
\]

\[
L e^{\frac{2\beta x}{\alpha}} \geq \left( -\varepsilon \frac{4\beta^2}{\alpha^2} + 2\beta + c(x) \right) e^{2\beta x} \alpha \geq \left( -\varepsilon \frac{4\beta^2}{\alpha^2} + \beta \right) e^{2\beta x} \alpha
\]

\[
= \beta e^{2\beta x} \alpha \left( -\varepsilon \frac{4\beta^2}{\alpha^2} + \frac{\alpha^2}{\alpha^2} \right) \geq \frac{\beta \gamma}{\alpha^2} e^{2\beta x} \alpha > 0.
\]

Consequently,

\[
L \psi \geq c_1 \frac{\beta \gamma}{\alpha^2} e^{2\beta x} \alpha \| T_\varepsilon p - T_\varepsilon q \| - \| T_\varepsilon p - T_\varepsilon q \| \geq 0,
\]

where \( c_1 = \frac{\alpha^2}{\beta \gamma} \).

\[
D e^{\frac{2\beta x}{\alpha}} = \left( \frac{2\beta}{\alpha} + d \right) \frac{2\beta x}{\alpha} \geq 0,
\]

\[
D e^{-\frac{\alpha(x-L)}{2\varepsilon}} = \alpha + d > 0.
\]
1.2. ESTIMATE OF THE DERIVATIVE

Consequently,

\[ L\psi \geq (\alpha + d)c_2 |R_\varepsilon p - R_\varepsilon q| - |R_\varepsilon p - R_\varepsilon q| \geq 0, \]

where \( c_2 = \frac{1}{\alpha + d} \).

Thus, defining

\[ \psi(x) = \beta \gamma \alpha^2 e^{\frac{2\beta x}{\alpha}} \| T_\varepsilon p - T_\varepsilon q \| + \frac{1}{a + d} e^{\frac{a - \varepsilon}{\alpha}} \pm z(x), \]

we obtain \( \psi(0) \geq 0, \; D\psi \geq 0, \; L\psi \geq 0 \), hence by Lemma 1.1.1 \( \psi(x) \geq 0. \) □

**Corollary 1.1.3.** Lemma 1.1.2 implies uniqueness and boundedness of problem (1.0.1).

Proof of the boundedness:
Let \( p(x) = u(x), \; q(x) = 0 \), then by Lemma 1.1.2,

\[ |u(x)| \leq \frac{\alpha^2}{\beta \gamma} e^{\frac{2\beta x}{\alpha}} |g(0)| + \frac{1}{a + d} e^{\frac{a - \varepsilon}{\alpha}} |f(0)| + e^{\frac{2\beta x}{\alpha}} |u(0)|. \]

The uniqueness of this solution is obvious.

### 1.2 Estimate of the derivative

We obtain an estimate of the derivative problem (1.0.1).

**Lemma 1.2.1.**

\[ |u'(x)| \leq c \left[ 1 + \frac{1}{\varepsilon} e^{\frac{a(t - L)}{\varepsilon}} \right]. \]

**Proof.** We express the equation (1.0.1) in the form:

\[ \left( -\varepsilon u' \exp \left[ \int_x^L \frac{a(t)}{\varepsilon} \, dt \right] \right)' + g(u(x)) \exp \left[ \int_x^L \frac{a(t)}{\varepsilon} \, dt \right] = 0 \]

Integrating from \( x \) to \( L \):

\[ -\varepsilon u'(L) + \varepsilon u'(x) \exp \left[ \int_x^L \frac{a(t)}{\varepsilon} \, dt \right] + \int_x^L g(u(s)) \exp \left[ \int_x^s \frac{a(t)}{\varepsilon} \, dt \right] ds = 0 \]

\[ u'(x) = u'(L) \exp \left[ -\int_x^L \frac{a(t)}{\varepsilon} \, dt \right] - \frac{1}{\varepsilon} \int_x^L g(u(s)) \exp \left[ -\int_x^s \frac{a(t)}{\varepsilon} \, dt \right] ds \]

8
1.3. **Construction of a difference scheme**

It is clear that:

\[
\frac{1}{\varepsilon} \int_{x}^{L} g(u(s)) \exp \left[ - \int_{x}^{L} \frac{a(t)}{\varepsilon} dt \right] ds \leq \frac{c}{\varepsilon} \int_{x}^{L} \exp \left[ - \int_{x}^{L} \frac{a(t)}{\varepsilon} dt \right] ds
\]

\[
= \frac{c}{\varepsilon} \int_{x}^{L} e^{\frac{\alpha}{\varepsilon} (x-L)} ds
\]

\[
= \frac{c}{\varepsilon} \left[ 1 - e^{\frac{\alpha}{\varepsilon} (x-L)} \right]
\]

\[
\leq \frac{c}{\alpha} = c_1
\]

From the boundary condition we conclude that

\[
|u'(x)| \leq \left| \frac{f(u(L))}{\varepsilon} \right| \leq \frac{c}{\varepsilon},
\]

consequently,

\[
|u'(x)| \leq c + \frac{c}{\varepsilon} e^{\frac{\alpha}{\varepsilon} (x-L)}.
\]

The obtained estimate of the derivative characterizes the boundary layer.

### 1.3 Construction of a difference scheme

Let us introduce a non-uniform grid and replace the coefficients by constants in each interval of that grid. That allows us to write the solution explicitly. Matching derivatives on adjacent intervals will lead to a difference scheme.

So, let \( \Delta_n = [x_{n-1}, x_n] \).

Now we turn to a problem with piecewise constant coefficients:

\[
\begin{align*}
\varepsilon V'' - \tilde{a} V' + \tilde{g}(V) &= 0, \\
V(0) &= A, \\
R \varepsilon V &= 0,
\end{align*}
\]

where \( \tilde{a} = a_n = a(x_{n-1}), \tilde{g}(V(x)) = g_n = g(V(x_{n-1})) \), at \( x \in \Delta_n \).

On an arbitrary interval \( \Delta_n \) we have:

\[
\begin{align*}
\varepsilon V'' - a_n V' + g_n &= 0, \\
V(x_{n-1}) &= V_{n-1}^h, \\
V(x_n) &= V_n^h, \\
x &= \Delta_n
\end{align*}
\]

9
1.3. CONSTRUCTION OF A DIFFERENCE SCHEME

where \( \{V^h_n\} \) is not yet defined.

The solution on the interval \( \Delta_n \) has the form:

\[
V(x) = c_1 + c_2 e^{\frac{a_n}{\varepsilon} (x - x_n)} + \frac{g_n}{a_n} x. \tag{1.3.1}
\]

Let \( h_n \) be the grid size: \( h_n = x_n - x_{n-1} \).

Define \( c_2 \) from the boundary conditions:

\[
\begin{cases}
    c_1 + c_2 e^{\frac{a_n h_n}{\varepsilon}} + \frac{g_n}{a_n} x_{n-1} = V^h_{n-1}, \\
    c_1 + c_2 + \frac{g_n}{a_n} x_n = V^h_n.
\end{cases}
\]

Let us find \( c_2 \):

\[
V^h_{n-1} - c_2 e^{\frac{a_n h_n}{\varepsilon}} - \frac{g_n}{a_n} x_{n-1} = V^h_n - c_2 - \frac{g_n}{a_n} x_n,
\]

\[
V^h_{n-1} - V^h_n + \frac{g_n}{a_n} h_n = c_2 \left( e^{\frac{a_n h_n}{\varepsilon}} - 1 \right),
\]

\[
c_2 = \frac{V^h_{n-1} - V^h_n + \frac{g_n}{a_n} h_n}{\left( e^{\frac{a_n h_n}{\varepsilon}} - 1 \right)}.
\]

Then let us find \( V'(x) \):

\[
V'(x) = \frac{a_n}{\varepsilon} c_2 e^{\frac{a_n}{\varepsilon} (x - x_n)} + \frac{g_n}{a_n}.
\]

To ensure that the solution is continuously differentiable, the derivatives of solutions must be matched on the adjacent the intervals. This requires

\[
\lim_{x \to x_{n-0}} V'(x) = \lim_{x \to x_{n+0}} V'(x)
\]

\[
= \frac{g_n}{a_n} + \frac{a_n}{\varepsilon} c_2 = \frac{g_{n+1}}{a_{n+1}} + \frac{a_{n+1}}{\varepsilon} c_2 e^{-\frac{a_n h_n + 1}{\varepsilon}},
\]

\[
= \frac{g_n}{a_n} + \frac{a_n}{\varepsilon} \left( V^h_{n-1} - V^h_n + \frac{g_n}{a_n} h_n \right) / \left( e^{\frac{a_n h_n}{\varepsilon}} - 1 \right) = \frac{g_{n+1}}{a_{n+1}} + \frac{a_{n+1}}{\varepsilon} \left( V^h_{n-1} - V^h_n + \frac{g_{n+1}}{a_{n+1}} h_{n+1} \right) / \left( e^{\frac{a_{n+1} h_{n+1}}{\varepsilon}} - 1 \right)
\]

\[
= \frac{g_n}{a_n} + \left( \frac{V^h_n - V^h_{n-1}}{h_n} - \frac{g_n}{a_n} \right) / \frac{a_n}{h_n a_n} \left( 1 - e^{\frac{-a_n h_n}{\varepsilon}} \right)
\]

\[
= \frac{g_{n+1}}{a_{n+1}} + \left( \frac{V^h_{n+1} - V^h_n}{h_{n+1}} - \frac{g_{n+1}}{a_{n+1}} \right) e^{-\frac{a_{n+1} h_{n+1}}{\varepsilon}} / \frac{a_{n+1}}{h_{n+1} a_{n+1}} \left( 1 - e^{\frac{-a_{n+1} h_{n+1}}{\varepsilon}} \right).
\]
1.3. CONSTRUCTION OF A DIFFERENCE SCHEME

Let \( s_n = \frac{\varepsilon}{h_n a_n} \left( 1 - e^{-\frac{a_n h_n}{\varepsilon}} \right) \).

Then the scheme takes the form:

\[
\frac{g_n}{a_n} + \frac{\left( V^h_n - V^h_{n-1} \right)}{s_n h_n} - \frac{g_n}{a_n} = \frac{g_{n+1}}{a_{n+1}} + \left( \frac{V^h_{n+1} - V^h_n}{s_{n+1} h_{n+1}} - \frac{g_{n+1}}{a_{n+1}} \right) e^{-\frac{a_{n+1} h_{n+1}}{\varepsilon}} / s_{n+1} .
\]

Rewrite the scheme in the form:

\[
\frac{g_n}{a_n} + \frac{V^h_n - V^h_{n-1}}{s_n h_n} = \frac{g_{n+1}}{a_{n+1}} + \left( \frac{V^h_{n+1} - V^h_n}{s_{n+1} h_{n+1}} - \frac{g_{n+1}}{a_{n+1}} \right) e^{-\frac{a_{n+1} h_{n+1}}{\varepsilon}} .
\]

So,

\[
\frac{V^h_n}{s_n h_n} - \frac{V^h_{n+1}}{s_{n+1} h_{n+1}} e^{\frac{a_{n+1} h_{n+1}}{\varepsilon}} - \frac{V^h_n - V^h_{n-1}}{s_n h_n} = \frac{g_{n+1}}{a_{n+1}} e^{\frac{a_{n+1} h_{n+1}}{\varepsilon}} - \frac{g_{n}}{s_n a_n} + \frac{g_{n}}{a_n} - \frac{g_{n+1}}{a_{n+1}} .
\]

Supplement this scheme by an approximation of the boundary conditions:

\[
V^h_0 = A, \; \varepsilon V^f + f(V(L)) = 0.
\]

In the last interval \( V'(x) = c(N) a_N \varepsilon e^{\frac{a_N}{\varepsilon} (x-L)} + \frac{g_N}{a_N} \), therefore, boundary conditions take the following form:

\[
V^h_0 = A, \quad c^N a_N + \frac{g_N a_N}{\varepsilon} + f(V^h_0) = 0, \; i.e.
\]

\[
\frac{(V^h_0 - V^h_{N-1}) a_N}{1 - e^{-\frac{a_N h_N}{\varepsilon}}} - \frac{g_N h_N}{1 - e^{-\frac{a_N h_N}{\varepsilon}}} + \frac{g_N a_N}{\varepsilon} + f(V^h_N) = 0
\]

So, as a result of the difference scheme has the form:

\[
\frac{V^h_{n+1} - V^h_n}{s_{n+1} h_{n+1}} e^{-\frac{a_{n+1} h_{n+1}}{\varepsilon}} - \frac{V^h_n - V^h_{n-1}}{s_n h_n} = \frac{g_{n+1}}{a_{n+1}} e^{-\frac{a_{n+1} h_{n+1}}{\varepsilon}} - \frac{g_n}{s_n a_n} + \frac{g_n}{a_n} - \frac{g_{n+1}}{a_{n+1}} .
\]

\[
V^h_0 = A, \; n = 1, 2, \ldots, (N - 1)
\]

\[
\frac{(V^h_N - V^h_{N-1}) a_N}{1 - e^{-\frac{a_N h_N}{\varepsilon}}} - \frac{g_N h_N}{1 - e^{-\frac{a_N h_N}{\varepsilon}}} + \frac{g_N a_N}{\varepsilon} + f(V^h_N) = 0 \quad (1.3.2)
\]

Due to \( g_n = g(V^h_{n-1}) \), the scheme (1.3) is nonlinear and can be linearized by the method of iterations [6], and solved by sweeping at each iteration [7].
1.4 Evaluation of convergence

**Theorem 1.4.1.** Let $u(x)$ be solution of the problem \((1.0.1)\), $u^h$ the solution of the scheme \((1.3.2)\). Then there exists a constant $C$, independent of $\varepsilon$, such that

$$\max_n |u^h_n - u(x_n)| \leq C \cdot \max_n h_n.$$  

**Proof.** Rewrite the problem \((1.0.1)\) and auxiliary problem:

$$T_{\varepsilon} u = -\varepsilon u'' + a(x) u' + g(u), \quad u(0) = A, \quad \varepsilon u'(L) + f(u(L)) = 0,$$

$$\tilde{T}_{\varepsilon} v = -\varepsilon v'' + a(x) v' + \tilde{g}(v), \quad v(0) = A, \quad \varepsilon v'(L) + f(v(L)) = 0. \tag{1.4.1}$$

We use the fact that the scheme \((1.3.2)\) is exact for the problem \((1.4.1)\).

Therefore it is sufficient to estimate the proximity of these problems.

Let $z = u - v$.

Write the problem on $z$:

$$\begin{cases} 
-\varepsilon z'' + az' + g(u) - \tilde{g}(v) = 0, \\
z(0) = 0, \quad \varepsilon z'(L) + f(u(L)) - f(v(L)) = 0.
\end{cases} \tag{1.4.2}$$

Rewrite the problem in the form:

$$-\varepsilon z'' + az' + g(u) - \tilde{g}(v) + \tilde{g}(u) - \tilde{g}(u) = 0$$

$$-\varepsilon z'' + az' + \frac{\tilde{g}(u) - \tilde{g}(v)}{u - v} \cdot z = \tilde{g}(u) - g(u)$$

For $z(x)$ we obtain the boundary problem:

$$\begin{cases} 
L z = -\varepsilon z'' + a(x) z' + g'_u(\theta) z = \tilde{g}(u) - g(u), \\
z(0) = 0, \quad Dz = \varepsilon z'(L) + f'_u(\theta) z(L) = 0.
\end{cases}$$

Estimate $g(u) - \tilde{g}(u)$.

For $x \in \Delta_n$ we have:

$$\tilde{a}(x) = a(x_{n-1}), \quad \tilde{g}(u) = g(u(x_{n-1})).$$
1.4. EVALUATION OF CONVERGENCE

We obtain:

\[ |g(u(x)) - \tilde{g}(u(x))| = |g(u(x_n)) - g(u(x))| \]
\[ \leq \left| \frac{\partial g}{\partial u} \right| |u(x_n) - u(x)| \]
\[ \leq C |u(x_n) - u(x)| \leq C \left| \int_{x_{n-1}}^{x} u'(s) \, ds \right| \]
\[ \leq C \int_{x_{n-1}}^{x} |u'(s)| \, ds \]
\[ \leq \tilde{C} \int_{x_{n-1}}^{x} \left[ 1 + \frac{1}{\varepsilon} e^{\frac{a(s-L)}{\varepsilon}} \right] ds \leq C h_n + \frac{Ch_n e^{a(x-L)}}{\varepsilon} \]

Let \( h = \max_n h_n \).
Define function \( \psi(x) \):

\[ \psi(x) = \tilde{C} h \left[ e^{\frac{a(x-L)}{2\varepsilon}} + e^{\frac{2a_2}{\varepsilon}} \right] \pm z(x) \]

Then:

\[ \psi(0) \geq 0, \ D\psi \geq 0, \ L\psi \geq \tilde{C} h \frac{C_1}{\varepsilon} e^{\frac{a(x-L)}{2\varepsilon}} + \tilde{C} \cdot C_2 h - Ch_n - \frac{Ch_n e^{a(x-L)}}{\varepsilon} \geq 0 \]

if \( \tilde{C} \leq \frac{C}{C_1}, \tilde{C} \leq \frac{C}{C_1} \).

By the Principle of Maximum \( \psi(x) \geq 0 \).
Consequently, \( |z(x)| \leq Ch \). \( \square \)

Now we have constructed a difference scheme for a nonlinear second-order equation with exponential boundary layer and proved its uniform convergence in the small parameter.
Chapter 2

Nonlinear differential equations with power-law boundary layer

Consider the following boundary problem:

\begin{equation}
\begin{cases}
(\varepsilon + x)^2 u'' - f(x, u) = 0, \\
u(0) = A, \quad u(1) = B,
\end{cases}
\end{equation}

(2.0.1)

under assumption that \( \partial f / \partial u \geq \alpha > 0 \), \( \varepsilon \in (0, 1] \) and the function \( f \) is continuously differentiable in its arguments.

Let’s investigate the problem (2.0.1). We will prove that the solution contains exponential boundary layer, construct a difference scheme and prove its convergence.

\( C \) and \( C_1 \) will everywhere designate the positive constants independent of \( \varepsilon \) and the mesh gauge.

2.1 Boundedness of solutions

We prove the boundedness for the solution of the problem (2.0.1).

Lemma 2.1.1.

\[ \|u(x)\| \leq \frac{1}{\alpha} \|f(x, 0)\|, \]

where \( \|u(x)\| = \max_a |u(x)| \).
2.2. ESTIMATION OF THE DERIVATIVE

Proof. Write the equation (2.0.1) in the following form:

\[ Lu = (\varepsilon + x)^2 u'' - \frac{f(x, u) - f(u, 0)}{u} u = f(x, 0) \]
\[ Lu = (\varepsilon + x)^2 u'' - c(x) u = f(x, 0), \]

where \( c(x) \geq \alpha > 0 \).

Introduce the function \( \psi(x) = \frac{\|f(x, 0)\|}{\alpha} \pm u(x) \).

Hence, according to the Principle of Maximum \( \psi(x) \geq 0 \).

\[ \|f(x, 0)\| \geq \frac{1}{\alpha} \|f(x, 0)\|. \]

2.2 Estimation of the derivative

Lemma 2.2.1.

\[ |u'(x)| \leq \frac{C}{\varepsilon + x}. \]

Proof. From the equation (2.0.1) we conclude that

\[ |u''(x)| \leq \frac{C}{(\varepsilon + x)^2} \]
\[ u''(x) = \frac{f(x, u)}{(\varepsilon + x)^2}. \quad (2.2.1) \]

Integrate (2.2.1) from \( \xi \) to \( x \):

\[ u'(x) - u'(\xi) = \int_{\xi}^{x} \frac{f(x, u(x))}{(\varepsilon + x)^2} \, dx, \]

consequently,

\[ |u'(x) - u'(\xi)| = \left| \int_{\xi}^{x} \frac{f(x, u(x))}{(\varepsilon + x)^2} \, dx \right|. \]

i) Let \( \xi \leq x \). Then

\[ |u'(x) - u'(\xi)| \leq \left[ \frac{-C}{\varepsilon + x} \right]_{\xi}^{x} = C \left[ \frac{1}{\varepsilon + x} - \frac{1}{\varepsilon + \xi} \right]. \]

15
2.2. ESTIMATION OF THE DERIVATIVE

ii) Let \( \xi > x \). Then

\[
|u'(x) - u'(_\xi)| = \int_{_\xi}^x \frac{C}{(\varepsilon + x)^2} dx,
\]

consequently,

\[
|u'(x) - u'(_\xi)| \leq -C \left[ \frac{1}{\varepsilon + \xi} - \frac{1}{\varepsilon + x} \right].
\]

Thus for any \( \xi \)

\[
|u'(x) - u'(_\xi)| \leq C \left[ \frac{1}{\varepsilon + \xi} - \frac{1}{\varepsilon + x} \right].
\]

Let \( \varepsilon \) be so that:

\[
\frac{1}{2} u'(_\xi) = u(1) - u \left( \frac{1}{2} \right).
\]

Then

\[
|u'(_\xi)| \leq C_0 \text{ and } \frac{1}{2} \leq \xi \leq 1.
\]

Since

\[
|u'(_\xi)| \leq C_0 + C \left[ \frac{1}{\varepsilon + \xi} - \frac{1}{\varepsilon + x} \right],
\]

then

\[
|u'(_\xi)| \leq \frac{C_1}{\varepsilon + x},
\]

We took into account that

\[
\frac{1}{\varepsilon + \xi} \leq \frac{1}{\varepsilon + \frac{1}{2}} \leq \frac{1}{\frac{1}{2}} = 2
\]

\[
0 \leq \frac{1}{\varepsilon + \xi} \leq 2,
\]

\[
|u'(_\xi)| \leq C_0 + 2 + \frac{C}{\varepsilon + x} \leq \frac{C_1}{\varepsilon + x},
\]

Hence the lemma follows. \( \square \)

According to Lemma 2.2.1, at the boundary \( x = 0 \) we have the power boundary layer.
2.3 CONSTRUCTION OF THE SCHEME

2.3 Construction of the scheme

Let’s proceed to the problem with piecewise constant coefficients.

\[
\begin{align*}
(\varepsilon + x)^2 v'' - \tilde{f}(x, v) &= 0, \\
v(0) &= A, \quad v(1) = B,
\end{align*}
\] (2.3.1)

where \( \tilde{f}(x, u) = f_n = f(x_{n-1}, V(x_n)), \) at \( x \in \Delta_n = [x_{n-1}, x_n]. \)

To construct the scheme of equation (2.3.1) let’s express \( v(x) \) as:

\[
\begin{align*}
v''(x) &= \frac{f_n}{(\varepsilon + x)^2}, \\
v'(x) &= -\frac{f_n}{\varepsilon + x} + C_1, \\
v(x) &= -f_n \cdot \ln(\varepsilon + x) + C_1 x + C_2
\end{align*}
\]

Let’s label \( v(x_{n-1}) = V_{n-1}^h, v(x_n) = V_n^h. \)

Let’s find \( C_1 \) from the boundary conditions:

\[
\begin{align*}
- f_n \cdot \ln(\varepsilon + x_{n-1}) + C_1 x_{n-1} + C_2 &= V_{n-1}^h, \\
- f_n \cdot \ln(\varepsilon + x_n) + C_1 x_n + C_2 &= V_n^h,
\end{align*}
\]

\[
C_1 h_n - f_n \ln \left( \frac{\varepsilon + x_n}{\varepsilon + x_{n-1}} \right) = V_n^h - V_{n-1}^h
\]

Then,

\[
C_1 = \frac{f_n}{h_n} \ln \left( \frac{\varepsilon + x_n}{\varepsilon + x_{n-1}} \right) + \frac{V_n^h - V_{n-1}^h}{h_n}
\]

We match the derivatives of solutions to the ends of adjacent intervals:

\[
\begin{align*}
&\lim_{x \to x_{n-0}} V'(x) = \lim_{x \to x_{n+0}} V'(x) \\
&- \frac{f_n}{\varepsilon + x_n} + C_1 = -\frac{f_{n+1}}{\varepsilon + x_{n+1}} + \tilde{C}_1 \\
&- \frac{f_n}{\varepsilon + x_n} + \frac{f_n}{h_n} \times \\
&\times \ln \left( \frac{\varepsilon + x_n}{\varepsilon + x_{n-1}} \right) + \\
&\quad + \frac{V_n^h - V_{n-1}^h}{h_n} = -\frac{f_{n+1}}{\varepsilon + x_{n+1}} + \frac{f_{n+1}}{h_{n+1}} \times \\
&\times \ln \left( \frac{\varepsilon + x_{n+1}}{\varepsilon + x_n} \right) + \frac{V_{n+1}^h - V_n^h}{h_{n+1}}
\end{align*}
\]
Thus we obtain:

\[
\frac{V_h^n - V_h^{n-1}}{h_n} - \frac{V_h^{n+1} - V_h^n}{h_{n+1}} =
\]

\[
= \frac{f_n}{\varepsilon + x_n} - \frac{f_n}{h_n} \ln \frac{\varepsilon + x_n}{\varepsilon + x_{n-1}} - \frac{f_{n+1}}{h_{n+1}} + \frac{f_{n+1}}{h_{n+1}} \ln \frac{\varepsilon + x_{n+1}}{\varepsilon + x_n}
\]

\[
(2.3.2)
\]

\[
V_h^0 = A, \quad V_h^N = B, \quad n = 1, 2, 3, \ldots, (N-1); \quad f_n = f(x_{n-1}, V_h^{n-1})
\]

### 2.4 Justification of convergence

Define the mesh Ω which densifies in the boundary layer in such a way that

\[
\ln \left( \frac{\varepsilon + x_n}{\varepsilon + x_{n-1}} \right) = C_N.
\]

Let \( x_n = \lambda \left( \frac{n}{N} \right) \), then

\[
\ln \left( \varepsilon + \lambda \left( \frac{n}{N} \right) \right) - \ln \left( \varepsilon + \lambda \left( \frac{n-1}{N} \right) \right) = \frac{C}{N}.
\]

Consequently,

\[
\frac{\lambda'(\frac{n}{N})}{\varepsilon + \lambda \left( \frac{n}{N} \right)} \cdot \frac{1}{N} = \frac{C}{N},
\]

That is

\[
\frac{\lambda'(\frac{n}{N})}{\varepsilon + \lambda \left( \frac{n}{N} \right)} = C, \quad \lambda(0) = 0, \lambda(1) = 1,
\]

Consequently,

\[
\lambda(t) = \varepsilon \left[ \left( 1 + \frac{1}{\varepsilon} \right)^t - 1 \right].
\]

Thus,

\[
\Omega = \left\{ x_n = \varepsilon \left[ \left( 1 + \frac{1}{\varepsilon} \right)^{\frac{n}{N}} - 1 \right], \quad n = 0, 1, \ldots, N \right\}.
\]

**Theorem 2.4.1.**

\[
\left\| V^h - [u]_H \right\| \leq \frac{C}{N} \ln \left( 1 + \frac{1}{\varepsilon} \right)
\]

**Proof.** Let \( z = u - v \), where \( u \) is the solution of problem (2.0.1). Since the scheme (2.3.2) is exact for the problem (2.3.1) by its construction, it is sufficient to estimate \( \|v - v\| \).
2.4. JUSTIFICATION OF CONVERGENCE

Problem for $z$ has the form:
\[
\begin{align*}
(\varepsilon + x)^2 z'' - f'(\theta(x))z &= f(x,v) - \bar{f}(x,v), \\
\quad z(0) = 0, \quad z(1) = 0,
\end{align*}
\] (2.4.1)

We estimate the right side:
\[
\left| f(x,v) - \bar{f}(x,v) \right| = \left| f(x,v) - f(x_{n-1},v_{n-1}) \right| \leq C |x - x_{n-1}| + C_1 |v(x) - v(x_{n-1})| \leq Ch_n + C_1 \int_{x_{n-1}}^{x_n} |v'(s)| \, ds. \tag{2.4.2}
\]

Since $|v'(x)| \leq \frac{C_2}{x + \varepsilon}$,
\[
\left| f(x,v) - \bar{f}(x,v) \right| \leq Ch_n + C_1 \int_{x_{n-1}}^{x_n} \frac{C_2}{x + \varepsilon} \, ds \leq Ch_n + C_1 C_2 \ln \left( \frac{\varepsilon + x_n}{\varepsilon + x_{n-1}} \right)
\]

In accordance with the construction of the mesh:
\[
\left| f(x,v) - \bar{f}(x,v) \right| \leq Ch_n + \frac{C_3}{N} \ln \left( 1 + \frac{1}{\varepsilon} \right)
\]

We estimate $h_n$.

i) From
\[
x_n = \varepsilon \left[ \left( 1 + \frac{1}{\varepsilon} \right)^{\frac{N}{\alpha}} - 1 \right]
\]
we conclude that
\[
h_N = x_N - x_{N-1} = 1 - \varepsilon \left[ \left( 1 + \frac{1}{\varepsilon} \right)^{\frac{N}{\alpha}} - 1 \right] \leq C_5 \frac{\ln \left( 1 + \frac{1}{\varepsilon} \right)}{N}
\]

ii) $h_n$ increases. Therefore it is sufficient to estimate $h_N$.

Then \[
\left| f(x,v) - \bar{f}(x,v) \right| \leq \frac{C_3}{N} \ln \left( 1 + \frac{1}{\varepsilon} \right).
\]

Let $\psi(x) = \frac{C_3}{N} \ln \left( 1 + \frac{1}{\varepsilon} \right) \pm z(x)$.

\[
\psi(0) \geq 0, \quad \psi(1) \geq 0, \quad L\psi(x) \leq 0.
\]

Consequently,
\[
|z(x)| \leq \frac{1}{\alpha} \cdot \frac{C}{N} \ln \left( 1 + \frac{1}{\varepsilon} \right).
\]

Thus,
\[
|u(x) - v(x)| \leq \frac{C}{N} \ln \left( 1 + \frac{1}{\varepsilon} \right).
\]

Since $[V]_{\Omega}$ coincides with $V^h$, we obtain the theorem. \qed
Results

At the result of the work, a first-order accurate difference scheme for a nonlinear equation with exponential boundary layer and nonlinear boundary conditions is constructed and its uniform convergence with respect to the small parameter is proven; a first-order accurate difference scheme for a nonlinear equation with power-law boundary layer and boundary conditions of first kind is constructed on a special mesh, and its convergence is proven as well.
Bibliography

[1] A.M. Ilyin: Difference scheme for differential equation with small parameter at the highest derivative // Mathematical Notes. - 1969, T.6, #2, pp.237-243.

[2] N.S. Bakhvalov: To optimization methods for solving boundary problems in the presence of the boundary layer // Journal of Computational Mathematics and Mathematical Physics. - 1969, T.9, #4, pp. 841-890.

[3] E. Doolan, D. Miller, U. Shilders: Uniform numerical methods for solving the problems with boundary layer. - Mir, Moscow, 1976.

[4] A. I. Zadorin: The numerical solution of ordinary differential equations with small parameter. - Omsk, 1997.

[5] A. I. Zadorin: Numerical solution of quasilinear singularly perturbed equation // Numerical methods of continuum mechanics. ITAM SB AS USSR, Novosibirsk, T.17, #6, pp. 35-44.

[6] D. Ortega, B. Reinboldt: Iterative methods for solving nonlinear systems of equations with many unknowns. - Mir, Moscow, 1975.

[7] A.A. Samarskiy: Theory of difference schemes. - Nauka, Moscow, 1983.