NEW Z-EIGENVALUE LOCALIZATION SETS FOR TENSORS WITH APPLICATIONS

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Abstract. In this paper, let $A = (a_{i_1i_2\cdots i_m}) \in \mathbb{R}^{[m,n]}$, when $m \geq 4$, based on the condition $||x||_2 = 1$, a new Z-eigenvalue localization set for tensors is given. And we extend the Geršgorin-type localization set for Z-eigenvalues of fourth order tensors to higher order tensors. As an application, a sharper upper bound for the Z-spectral radius of nonnegative tensors is obtained. Let $H$ be a $k$-uniform hypergraph with $k \geq 4$ and $A(H)$ be the adjacency tensor of $H$, a new upper bound for the Z-spectral radius $\rho(H)$ is also presented. Finally, a checkable sufficient condition for the positive definiteness of even-order tensors and asymptotically stability of time-invariant polynomial systems is also given.

1. Introduction. Let $A = (a_{i_1i_2\cdots i_m}) \in \mathbb{R}^{[m,n]}$ be an $m$-th order $n$ dimensional real square tensor, $x$ be a real $n$-vector. Then, let $N = \{1, 2, \ldots, n\}$, we define the following real $n$-vector:

$$A x^{m-1} = \left( \sum_{i_2, \cdots, i_m=1}^{n} a_{i_1i_2\cdots i_m} x_{i_2} \cdots x_{i_m} \right), \quad x^{[m-1]} = (x_{i}^{m-1})_{i \in N}.$$

If there exists a nonzero real vector $x$ and a real number $\lambda$ such that

$$A x^{m-1} = \lambda x^{[m-1]},$$

then $\lambda$ is called an H-eigenvalue of $A$ and $x$ is called an H-eigenvector of $A$ associated with $\lambda$. If there exists a real vector $x$ and a real number $\lambda$ such that

$$A x^{m-1} = \lambda x, \quad x^T x = 1,$$

then $\lambda$ is called a Z-eigenvalue of $A$ and $x$ is called a Z-eigenvector of $A$ associated with $\lambda$[15].
Recently, many people have focused on the Z-eigenvalue localization sets and the Z-spectral radius of nonnegative tensors in \([2, 21, 24, 20, 18, 13, 14, 5]\). Let \(\sigma(A)\) be the Z-spectrum of \(A\), 
\[
\Delta^k = \{(i_2, i_3, i_4) : \text{at least two of the indices } i_2, i_3, i_4 \text{ are equal to } k\},
\]
\[
\Delta = \Delta^1 \cup \Delta^2 \cup \ldots \cup \Delta^n, \quad \overline{\Delta} = \{(i_2, i_3, i_4) : i_2, i_3, i_4 \in N\} \setminus \Delta,
\]
For any \(k \in N\), let 
\[
\Delta^{kt} = \{(i_2, i_3, i_4) : (i_2, i_3, i_4) \in \Delta^k \text{ and } t \notin \{i_2, i_3, i_4\}\},
\]
\[
\Delta^{k\overline{t}} = \{(i_2, i_3, i_4) : (i_2, i_3, i_4) \in \Delta^k \text{ and } t \notin \{i_2, i_3, i_4\}\},
\]
\[
\Delta^t = \Delta^t \cup \Delta^{2t} \cup \ldots \cup \Delta^{nt}, \quad \overline{\Delta}^t = \overline{\Delta} \setminus \Delta^t,
\]
and 
\[
\overline{\Delta} = \{(i_2, i_3, i_4) : (i_2, i_3, i_4) \in \overline{\Delta} \text{ and } t \notin \{i_2, i_3, i_4\}\}, \quad \overline{\Delta}^t = \overline{\Delta} \setminus \overline{\Delta}^t.
\]

He, Liu and Xu presented the following Z-eigenvalue localization sets for fourth order tensors \([6]\).

**Theorem 1.1.** Let \(A = (a_{i_1 \ldots i_m}) \in \mathbb{R}^{[4, n]}\). Then 
\[
\sigma(A) \subseteq \bigcup_{i,j \in N, i \neq j} \Upsilon_{ij}(A) \cup \left( \bigcup_{i \in N} \mathcal{M}_i(A) \right),
\]
where 
\[
\Upsilon_{ij}(A) = \{z \in \mathbb{R} : \left| z - \beta^i_{ij}(A) - r^i_{ij}(A) \right| \leq \beta^i_{ij}(A) + r^i_{ij}(A) \},
\]
\[
\mathcal{M}_i(A) = \{z \in \mathbb{R} : \left| z - \beta^i_{ij}(A) - r^i_{ij}(A) \right| \leq 0 \},
\]
and 
\[
\beta^i_{ij}(A) = \max_{k \in N} \left\{ \sum_{(i_2, i_3, i_4) \in \Delta^{ki}} |a_{i_2i_3i_4i_i}| : r^i_{ij}(A) = \sum_{(i_2, i_3, i_4) \in \overline{\Delta}^i} |a_{i_2i_3i_4i_i}| \},
\]
\[
\beta^i_{ij}(A) = \max_{k \in N} \left\{ \sum_{(i_2, i_3, i_4) \in \Delta^{ki}} |a_{i_2i_3i_4i_i}| : r^i_{ij}(A) = \sum_{(i_2, i_3, i_4) \in \overline{\Delta}^i} |a_{i_2i_3i_4i_i}| \},
\]
\[
\beta^i_{ij}(A) = \max_{k \in N} \left\{ \sum_{(i_2, i_3, i_4) \in \Delta^{ki}} |a_{i_2i_3i_4i_i}| : r^i_{ij}(A) = \sum_{(i_2, i_3, i_4) \in \overline{\Delta}^i} |a_{i_2i_3i_4i_i}| \},
\]
\[
\beta^i_{ij}(A) = \max_{k \in N} \left\{ \sum_{(i_2, i_3, i_4) \in \Delta^{ki}} |a_{i_2i_3i_4i_i}| : r^i_{ij}(A) = \sum_{(i_2, i_3, i_4) \in \overline{\Delta}^i} |a_{i_2i_3i_4i_i}| \},
\]

Based upon above theorem, the authors presented Geršgorin-type results for Z-eigenvalues of fourth order tensors and Z-eigenvalues based sufficient conditions for the positive definiteness of structured fourth order tensors \([6]\). And they ask: can the Geršgorin-type results for Z-eigenvalues of fourth order tensors and Z-eigenvalues based sufficient conditions be extended to higher order tensors? The authors in \([11, 17]\) presented different answers for this question.
In this paper, under the assumption $m \geq 4$, we give a new Z-eigenvalue localization set for tensors in section 2. As an application, in section 3, a new upper bound for the Z-spectral radius of nonnegative tensors is obtained. Let $A(H)$ be the adjacency tensor of a $k$-uniform hypergraph $H$ with maximum degree $d_1$, a new upper bound for the Z-spectral radius $\rho(H)$ is also presented:

$$\rho(H) \leq \frac{1}{2}d_1,$$

which improve the existed upper bound: $\rho(H) \leq d_1$ [22]. A checkable sufficient condition for the positive definiteness of even-order tensors and asymptotically stability of time-invariant polynomial systems is also given based on the Geršgorin-type results for Z-eigenvalues of higher order tensors.

2. Main results. In this section, we give a new Z-eigenvalue localization set for tensors with $m \geq 4$. We need the following lemma [19].

Lemma 2.1. [19] For any $x \in \mathbb{R}^n$, if

$$x_1^2 + x_2^2 + \ldots + x_n^2 = 1,$$

then, \( \max_{i,j \in N, i \neq j} |x_i||x_j| \leq \frac{1}{2} \).

For any $k \in N$, let

$$\Phi^k = \{(i_2, \ldots, i_m) : \text{at least two of the indices } i_2, \ldots, i_m \text{ are equal to } k\},$$

$$\Phi = \Phi^1 \cup \Phi^2 \cup \ldots \cup \Phi^n, \text{ and } \Phi^i \cap \Phi^j = \emptyset, \text{ if } i \neq j,$$

$$\overline{\Phi} = \{(i_2, \ldots, i_m) : i_2, \ldots, i_m \in N\} \setminus \Phi,$$

we give our main result in this section.

Theorem 2.2. Let $A = (a_{i_1 \ldots i_m}) \in \mathbb{R}^{[m,n]}$ and $m \geq 4$. Then

$$\sigma(A) \subseteq \Theta(A) = \bigcup_{i \in N} \Theta_i(A),$$

where

$$\Theta_i(A) = \{z \in \mathbb{R} : |z| \leq \alpha_i + \frac{1}{2}r_{\overline{\Phi}}(A)\},$$

and

$$\alpha_i = \max_{k \in N} \left\{ \sum_{(i_2, \ldots, i_m) \in \Phi^k} |a_{i_1 i_2 \ldots i_m}|, \ r_{\overline{\Phi}}(A) = \sum_{(i_2, \ldots, i_m) \in \overline{\Phi}} |a_{i_1 i_2 \ldots i_m}| \right\}.$$

Proof. Let $\lambda$ be a Z-eigenvalue of $A$ with corresponding Z-eigenvector $x = (x_1, \ldots, x_n)^T \in \mathbb{R}^n \setminus \{0\}$, i.e.,

$$Ax^{m-1} = \lambda x, \text{ and } ||x||_2 = 1.$$

Let $|x_i| = \max_{i \in N} |x_i|$, we have

$$\lambda x_i = \sum_{(i_2, \ldots, i_m) \in \Phi} a_{i_1 i_2 \ldots i_m} x_{i_2} \ldots x_{i_m} + \sum_{(i_2, \ldots, i_m) \in \overline{\Phi}} a_{i_1 i_2 \ldots i_m} x_{i_2} \ldots x_{i_m}.$$
Taking modulus in the above equation, and using the triangle inequality and \( x^T x = 1 \), we get
\[
|\lambda| |x_1| \leq \sum_{(i_2, \ldots, i_m) \in \Phi} |a_{i_2 \ldots i_m}||x_{i_2}| \cdots |x_{i_m}| + \sum_{(i_2, \ldots, i_m) \in \Phi} |a_{i_2 \ldots i_m}||x_{i_2}| \cdots |x_{i_m}|
\]
\[
= \left( \sum_{(i_2, \ldots, i_m) \in \Phi^1} |a_{i_2 \ldots i_m}||x_1|^2 + \cdots + \sum_{(i_2, \ldots, i_m) \in \Phi^n} |a_{i_2 \ldots i_m}||x_n|^2 \right) |x_1| + \frac{1}{2} \sum_{(i_2, \ldots, i_m) \in \Phi} |a_{i_2 \ldots i_m}||x_1|
\]
\[
\leq \max_{k \in \mathbb{N}} \left\{ \sum_{(i_2, \ldots, i_m) \in \Phi^k} |a_{i_2 \ldots i_m}| \right\} \left( x_1^2 + \cdots + x_n^2 \right) |x_1| + \frac{1}{2} \sum_{(i_2, \ldots, i_m) \in \Phi} |a_{i_2 \ldots i_m}||x_1|
\]
\[
= \alpha_1 |x_1| + \frac{1}{2} r_\Phi^1(A)|x_1|.
\]
Then,
\[
|\lambda| \leq \alpha_1 + \frac{1}{2} r_\Phi^1(A).
\]
Thus, we complete the proof. \( \square \)

**Remark 1.** For special indices \((i_{02}, \ldots, i_{0_m})\), if at least two of them are equal to \(k_0\) and at least two of them are equal to \(k_1\), then, one natural question is: \((i_{02}, i_{02}, \ldots, i_{0_m}) \in \Phi^{k_0}\) or \((i_{02}, i_{02}, \ldots, i_{0_m}) \in \Phi^{k_1}\)? From the proof of Theorem 2.2, we can find that, smaller \(\alpha_i\) can obtain tighter inclusion set \(\Theta(A)\). Therefore, if \((i_{02}, i_{02}, \ldots, i_{0_m}) \in \Phi^{k_0}\), we have \(\alpha_{i_0}\), if \((i_{02}, i_{02}, \ldots, i_{0_m}) \in \Phi^{k_1}\), we have \(\alpha_{i_i}\), without loss of generality, we assume \(\alpha_{i_0} \leq \alpha_{i_1}\), then, \((i_{02}, i_{02}, \ldots, i_{0_m}) \in \Phi^{k_0}\).

**Remark 2.** From the proof of Theorem 2.2, we have
\[
\alpha_i + \frac{1}{2} r_\Phi^i(A) \leq \alpha_i + r_\Phi^i(A) \leq \sum_{(i_2, \ldots, i_m) \in N} |a_{i_2 \ldots i_m}|
\]
which means that, the result in Theorem 2.2 is always better than the results in Theorem 4 in [6] and Theorem 3.1 in [21].

**Example 2.1.** Consider the tensor \(A = (a_{i_1 \ldots i_m}) \in \mathbb{R}^{[6,3]}\) with entries defined as follows:
\[
a_{111111} = 1, \ a_{111122} = 1, \ a_{111133} = 1, \\
a_{222211} = 2, \ a_{222222} = 3, \ a_{222233} = 2, \\
a_{333311} = -3, \ a_{333211} = -3, \ a_{333333} = -3,
\]
and other \(a_{ijkl} = 0\). By computations, we get that, \(\sigma(A) = \{-3, 1, 3\}\).

When \(i_1 = 1\), let
\[
\Phi^1 = \{(i_{02}, \ldots, i_0) : (1, 1, 1, 1, 1)\},
\]
\[
\Phi^2 = \{(i_{02}, \ldots, i_0) : (1, 1, 1, 2, 2)\},
\]
\[
\Phi^3 = \{(i_{02}, \ldots, i_0) : (1, 1, 1, 3, 3)\},
\]
which implies \(\alpha_1 = 1\). When \(i_1 = 2\), let
\[
\Phi^1 = \{(i_{02}, \ldots, i_0) : (2, 1, 1, 1, 1)\},
\]
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\[ \Phi^2 = \{(i_2, \ldots, i_6) : (2, 2, 2, 2, 2)\}, \]
\[ \Phi^3 = \{(i_2, \ldots, i_6) : (2, 2, 2, 3, 3)\}, \]
which implies \( \alpha_2 = 3 \). When \( i_1 = 3 \), let
\[ \Phi^1 = \{(i_2, \ldots, i_6) : (3, 1, 1, 1, 1)\}, \]
\[ \Phi^2 = \{(i_2, \ldots, i_6) : (3, 2, 2, 1, 1)\}, \]
\[ \Phi^3 = \{(i_2, \ldots, i_6) : (3, 3, 3, 3, 3)\}, \]
which implies \( \alpha_3 = 3 \). And \( \Phi = \emptyset \), which means \( r_i^\Phi(A) = 0 \) for all \( i = 1, 2, 3 \).
Therefore, from Theorem 2.2, we have
\[ \Theta(A) = \{z \in \mathbb{R} : |z| \leq 3\}. \]

There are so many inclusion sets for Z-eigenvalues of tensors, we now compare our result with the first inclusion set \( K(A) \)(Theorem 3.1 in [21]) and the recent result \( L(A) \)(Theorem 5 in [16]), which are listed as follows:
\[ K(A) = \{z \in \mathbb{R} : |z| \leq 9\}, \]
and
\[ L(A) = \{z \in \mathbb{R} : |z| \leq 7\}. \]

The Z-eigenvalue inclusion sets \( K(A), L(A), \Theta(A) \) and the exact Z-eigenvalues are drawn in Figure 1, where \( K(A) \) is represented by black dashed boundary, \( L(A) \) is represented by blue solid boundary, and \( \Theta(A) \) is represented by green point line boundary. The exact eigenvalues are plotted by red “+”. From Figure 1, we can see that, \( \Theta(A) \) can capture all Z-eigenvalues of \( A \) more precisely than \( L(A) \) and \( K(A) \).

**Figure 1.** Comparisons of Z-eigenvalue inclusion sets.
3. Bounds for the Z-spectral radius of tensors and hypergraphs. A tensor \( \mathcal{A} \) is called *weakly symmetric* [2] if the associated homogeneous polynomial \( \mathcal{A}x^m \) satisfies
\[
\nabla \mathcal{A}x^m = m\mathcal{A}x^{m-1}.
\]
The Z-spectral radius \( \rho(\mathcal{A}) \) of \( \mathcal{A} \) is defined as
\[
\rho(\mathcal{A}) := \sup\{ |\lambda| : \lambda \in \sigma(\mathcal{A}) \}.
\]

There are many applications for the Z-eigenpair of a nonnegative tensor, such as high order Markov chain, best rank-one approximations in statistical data analysis [3, 7]. Obviously, by the inclusion set \( \mathcal{K}(\mathcal{A}) \) (Theorem 3.1 in [21]) and the recent result \( \mathcal{L}(\mathcal{A}) \) (Theorem 5 in [16]), the authors obtained the following upper bounds for \( \rho(\mathcal{A}) \):
\[
\rho(\mathcal{A}) \leq \max_{i \in \mathbb{N}} R_i(\mathcal{A}), \quad R_i(\mathcal{A}) = \sum_{(i_2, \ldots, i_m) \in \mathbb{N}} |a_{i_2 \ldots i_m}|, \quad (2)
\]
\[
\rho(\mathcal{A}) \leq \max_{i \in \mathbb{N}} \min_{j \in \mathbb{N}, j \neq i} \Omega_{i,j}(\mathcal{A}), \quad (3)
\]
where
\[
\Omega_{i,j}(\mathcal{A}) = \left\{ \min\{R_i(\mathcal{A}) - a_{i_2 \ldots j}, r_j^{\Pi}(\mathcal{A})\}, \min\{R_i(\mathcal{A}), \tilde{\Omega}_{i,j}(\mathcal{A})\} \right\},
\]
\[
\tilde{\Omega}_{i,j}(\mathcal{A}) = \frac{1}{2} \left\{ R_i(\mathcal{A}) - a_{i_2 \ldots j} + r_j^{\Pi}(\mathcal{A}) + \bar{\Omega}_{i,j}(\mathcal{A}) \right\},
\]
\[
\bar{\Omega}_{i,j}(\mathcal{A}) = \sqrt{(R_i(\mathcal{A}) - a_{i_2 \ldots j} - r_j^{\Pi}(\mathcal{A}))^2 + 4a_{i_2 \ldots j}r_j^{\Pi}(\mathcal{A})},
\]
\[
\Pi^k = \{(i_2, \ldots, i_m) : i_k = j \text{ for some } k \in \{2, 3, \ldots, m\} \text{ where } j, i_2, \ldots, i_m \in \mathbb{N}\},
\]
\[
\tilde{\Pi}^k = \{(i_2, \ldots, i_m) : i_k \neq j \text{ for any } k \in \{2, 3, \ldots, m\} \text{ where } j, i_2, \ldots, i_m \in \mathbb{N}\},
\]
and
\[
r_j^{\Pi}(\mathcal{A}) = \sum_{(i_2, \ldots, i_m) \in \Pi_j} |a_{i_2 \ldots i_m}|, \quad r_j^{\tilde{\Pi}}(\mathcal{A}) = \sum_{(i_2, \ldots, i_m) \in \tilde{\Pi}_j} |a_{i_2 \ldots i_m}|.
\]

Using a new technique, He and Huang [5] presented the following upper bound for a weakly symmetric positive tensor:
\[
\rho(\mathcal{A}) \leq R - l(1 - \theta), \quad (4)
\]
where \( r_i = \sum_{(i_2, \ldots, i_m) \in \mathbb{N}} |a_{i_2 \ldots i_m}|, \quad R = \max r_i, \quad r = \min r_i, \quad l = \min_{i_1, \ldots, i_m} a_{i_1 \ldots i_m}, \quad \theta = \frac{r}{R}. \)

Recently, the authors provided the following upper bounds for the Z-spectral radius of a weakly symmetric nonnegative tensor [13]:
\[
\rho(\mathcal{A}) \leq \max\{r_i + a_{i_2 \ldots j}(\omega^{\frac{m-1}{m}} - 1)\}, \quad (5)
\]
where \( \omega = \frac{\min_{i, j} a_{i_2 \ldots j}}{r - \min_{i, j} a_{i_2 \ldots j}} + \gamma, \quad \gamma = \frac{R - \min_{i, j} a_{i_2 \ldots j}}{r - \min_{i, j} a_{i_2 \ldots j}}. \)

Based on the above Lemma, we give the main result of this section.

**Theorem 3.1.** Suppose that an \( m \)-order \( n \)-dimensional tensor \( \mathcal{A} \) is a nonnegative tensor. Then
\[
\rho(\mathcal{A}) \leq \max_{i \in \mathbb{N}}\left\{ \alpha_i + \frac{1}{2} \bar{\gamma}(\mathcal{A}) \right\}.
\]
Proof. Since $A$ is a nonnegative tensor, then we have $\sigma(A) \neq \emptyset$ by Theorem 2.5 in [2]. Let $(\lambda, x)$ be a $Z$-eigenpair of $A$. And we have $\lambda \in \Theta(A)$ from Theorem 2.2, there is $i_0 \in N$, such that
$$|\lambda| \leq \alpha_{i_0} + \frac{1}{2}r_i^a(A).$$

We now show the efficiency of the new upper bound in Theorem 3.1 by the following examples.

**Example 3.1.** [2, 13, 12, 5] Consider the tensor $A = (a_{i_1i_2i_3i_4}) \in \mathbb{R}^{[4,2]}$ with entries defined as follows:
$$a_{1111} = \frac{1}{2}, \quad a_{2222} = 3,$$
and other $a_{i_1i_2i_3i_4} = \frac{1}{3}$. Let
$$\Phi_1 = \{(i_2, i_3, i_4) : (1, 1, 1), (1, 1, 2), (1, 2, 1), (2, 1, 1)\},$$
$$\Phi_2 = \{(i_2, i_3, i_4) : (2, 2, 1), (2, 1, 2), (1, 2, 2), (2, 2, 2)\},$$
which implies $\alpha_1 = \frac{3}{2}, \alpha_2 = 4$ and $r_i^a(A) = 0$ for all $i = 1, 2$. Therefore, from Theorem 3.1, we have, $\rho(A) \leq 4$.

**Example 3.2.** Consider the tensor $A = (a_{i_1i_2i_3i_4i_5}) \in \mathbb{R}^{[5,2]}$ with entries defined as follows:
$$a_{11112} = 10, \quad a_{11122} = 5, \quad a_{11212} = 5, \quad a_{21111} = 15, \quad a_{21112} = 5, \quad a_{21122} = 20,$$
and other $a_{i_1i_2i_3i_4i_5} = 0$. When $i_1 = 1$, let
$$\Phi_1 = \{(i_2, i_3, i_4, i_5) : (1, 1, 1, 2)\},$$
$$\Phi_2 = \{(i_2, i_3, i_4, i_5) : (1, 1, 2, 2), (1, 2, 2, 2)\},$$
which implies $\alpha_1 = 10$.

When $i_1 = 2$, let
$$\Phi_1 = \{(i_2, i_3, i_4, i_5) : (1, 1, 1, 1), (1, 1, 1, 2)\},$$
$$\Phi_2 = \{(i_2, i_3, i_4, i_5) : (1, 1, 2, 2)\},$$
which implies $\alpha_2 = 20$, and $r_i^a(A) = 0$ for all $i = 1, 2$. Therefore, from Theorem 3.1, we have, $\rho(A) \leq 20$.

**Table 1.** Comparisons with the existed upper bounds.

|        | Example 2 | Example 3 |
|--------|-----------|-----------|
| $\rho(A)$ | 3.1092    | 7.3525    |
| Bound (2) | 5.3333  | 40        |
| Bound (3) | 5.0437  | 25        |
| Bound (4) | 5.2846  | -         |
| Bound (5) | 5.1935  | -         |
| Theorem 4.2 of [12] | 4.4632 | -         |
| Theorem 3.1 | 4       | 20        |
Let $\mathcal{H}$ be a $k$-uniform hypergraph with vertex set $V(\mathcal{H})$ and edge set $E(\mathcal{H})$ [1, 22, 23]. The adjacency tensor $A(\mathcal{H}) = (a_{i_1, \ldots, i_k})$ of $\mathcal{H}$ is defined as follows:

$$a_{i_1, \ldots, i_k} = \begin{cases} \frac{1}{(k-1)!}, & \text{if } \{i_1, \ldots, i_k\} \in E(\mathcal{H}), \\ 0, & \text{otherwise}. \end{cases}$$

The $Z$-spectral radius of the adjacency tensor $A(\mathcal{H})$ is denoted by $\rho(\mathcal{H})$. For a $k$-uniform hypergraph $\mathcal{H}$, let $d_1$ be the maximum degree of $\mathcal{H}$. In 2013, Xie and Chang [22] obtained the following upper bound for the $Z$-spectral radius $\rho(\mathcal{H})$:

$$\rho(\mathcal{H}) \leq d_1. \quad (6)$$

By Theorem 3.1, we present a bound on the $Z$-spectral radius of the adjacency tensor $A(\mathcal{H})$.

**Theorem 3.2.** Let $\mathcal{H}$ be a $k$-uniform hypergraph with maximum degree $d_1$ and $k \geq 4$. Then

$$\rho(\mathcal{H}) \leq \frac{1}{2} d_1. \quad (7)$$

**Proof.** The result can be obtained immediately by the definition of the adjacency tensor $A(\mathcal{H})$. \qed

4. **$Z$-eigenvalues-based sufficient condition for the positive definiteness of even-order tensors.** In this section, based on the inclusion sets for $Z$-eigenvalues of a tensor, a Geršgorin-type theorems for $Z$-eigenvalues of even-order tensors is given. When $m$ is even, the $Z$-identity tensor $I_Z = (e_{i_1, \ldots, i_m}) \in \mathbb{R}^{[m, n]}$ is defined as follows [8]:

$$e_{i_1, \ldots, i_m} = \frac{1}{m!} \sum_{p \in \pi_m} \delta_{p(1)}^{i_1} \delta_{p(2)}^{i_2} \cdots \delta_{p(m)}^{i_m},$$

where $\pi_m$ is the set of all permutations of $(1, \ldots, m)$, $\delta$ is the standard Kronecker delta. First, let us recall the $H$-eigenvalues based sufficient conditions for the positive definiteness of even-order tensors [10, 9].

**Lemma 4.1.** [10, 9] Let $A = (a_{i_1, \ldots, i_m}) \in \mathbb{R}^{[m, n]}$ be an even symmetric tensor with positive diagonal entries, if for $i \in N$,

$$a_{i, i} > \check{R}_i(A),$$

where $\check{R}_i(A) = \sum_{(i_2, \ldots, i_m) \in N} |a_{i_2, \ldots, i_m}| - a_{i, i}$. Then $A$ is positive definite, and $A$ is called strictly diagonally(SDD) dominant.

**Lemma 4.2.** [10, 9] Let $A = (a_{i_1, \ldots, i_m}) \in \mathbb{R}^{[m, n]}$ be an even symmetric tensor with positive diagonal entries, if for $i, j \in N, i \neq j$,

$$(a_{i, i} - \check{R}_i(A))a_{j, j} > |a_{i, j} - \check{R}_j(A)|,$$

where $\check{R}_i(A) = \sum_{(i_2, \ldots, i_m) \in N} |a_{i_2, \ldots, i_m}| - a_{i, i} - |a_{i, j}|$. Then $A$ is positive definite, and $A$ is called quasi-doubly(QDSDD) strictly dominant.

Let $m$ be even, $m_1, \ldots, m_n$ be nonnegative integers, and $m_1 + \ldots + m_n = \frac{m-2}{2}$, $D_i^{[m_1, \ldots, m_n]} = \{(i_2, \ldots, i_m) : (i_2, \ldots, i_m) \in \pi(i, 1, \ldots, 1, 2, \ldots, n, \ldots, n)\}$,

$$D_i = \bigcup_{m_1 + \ldots + m_n = \frac{m-2}{2}} D_i^{[m_1, \ldots, m_n]}.$$
and
\[ r_i^{D_i}(A) = \sum_{(i_2,\ldots,i_m) \in D_i} |a_{ii_2 \ldots i_m}|, \]
\[ r_i^{\bar{D}_i}(A) = \sum_{(i_2,\ldots,i_m) \notin D_i} |a_{ii_2 \ldots i_m}|, \]
a Z-eigenvalues based sufficient condition for the positive definiteness of even-order tensors is given as follows [17].

Lemma 4.3. [17] Let \( A = (a_{i_1 \ldots i_m}) \in \mathbb{R}^{[m,n]} \) be an even symmetric tensor with positive diagonal entries, if for \( i \in \mathbb{N} \),
\[ a_{i \ldots i} > r_i^{D_i}(A) + r_i^{\bar{D}_i}(A). \]
Then \( A \) is positive definite.

We give our main results in this section as follows.

Theorem 4.4. Let \( A = (a_{i_1 \ldots i_m}) \in \mathbb{R}^{[m,n]} \) with \( m \geq 4 \) being even. For any real vector \( \mu = (\mu_1,\ldots,\mu_n)^T \in \mathbb{R}^n \), then
\[ \sigma(A) \subseteq \bar{\Theta}(A) = \bigcup_{i \in \mathbb{N}} \bar{\Theta}_i(A), \]
where
\[ \bar{\Theta}_i(A) = \{ z \in \mathbb{R} : |z - \mu_i| \leq \bar{\alpha}_i + \frac{1}{2} r_i^\Phi(A) \}, \]
and
\[ \bar{\alpha}_i = \max_{k \in \mathbb{N}} \left\{ \sum_{(i_2,\ldots,i_m) \in \Phi_k} |a_{i_2 \ldots i_m} - \mu_i e_{i_2 \ldots i_m}| \right\}. \]

Proof. Let \( \lambda \) be a Z-eigenvalue of \( A \) with corresponding Z-eigenvector \( x = (x_1,\ldots,x_n)^T \in \mathbb{R}^n \setminus \{0\} \), i.e.,
\[ Ax^{m-1} = \lambda x, \quad \text{and} \quad ||x||_2 = 1. \quad \text{(8)} \]
Let \( |x_i| = \max_{i \in \mathbb{N}} |x_i| \), then for any \( k \in \mathbb{N} \), we have
\[ (\lambda - \mu_i)x_i = \sum_{(i_2,\ldots,i_m) \in \Phi} (a_{i_2 \ldots i_m} - \mu_i e_{i_2 \ldots i_m})x_i_2 \ldots x_i_m \]
\[ + \sum_{(i_2,\ldots,i_m) \in \Phi} a_{i_2 \ldots i_m}x_i_2 \ldots x_i_m. \]

Similar to the proof of Theorem 2.2, we get
\[ |\lambda - \mu_i||x_i| \leq \bar{\alpha}_i |x_i| + \frac{1}{2} r_i^\Phi(A) |x_i|. \]
Then,
\[ |\lambda - \mu_i| \leq \bar{\alpha}_i + \frac{1}{2} r_i^\Phi(A), \]
which implies \( \lambda \in \bar{\Theta}(A). \)

Remark 3. Obviously, from the proof of Theorem 4.4, we have
\[ \bar{\alpha}_i = \max_{k \in \mathbb{N}} \left\{ \sum_{(i_2,\ldots,i_m) \in \Phi_k} |a_{i_2 \ldots i_m} - \mu_i e_{i_2 \ldots i_m}| \right\} \]
\[ \leq \sum_{(i_2,\ldots,i_m) \in \Phi_k} |a_{i_2 \ldots i_m} - \mu_i e_{i_2 \ldots i_m}|, \]
which implies that, if choosing same $\mu$, our inclusion set in Theorem 4.4 is always better than the results in [11, 17].

By Theorem 4.4, the following $Z$-eigenvalues-based sufficient condition for the positive definiteness of even-order tensors can be obtained.

**Theorem 4.5.** Let $A = (a_{i_1...i_m}) \in \mathbb{R}^{[m,n]}$ be a symmetric tensor with $m \geq 4$ being even. If there exists a positive real vector $\mu = (\mu_1, \ldots, \mu_n)^T \in \mathbb{R}^n$ such that

$$\mu_i > \bar{\alpha}_i + \frac{1}{2} r_i(A), \ i \in N$$

then $A$ is positive definite.

**Proof.** Assume that $\lambda \leq 0$ is a Z-eigenvalue of $A$. From Theorem 4.4, we have $\lambda \in \underline{\Theta}(A)$, hence, there is an $i_0 \in N$ such that

$$|\lambda - \mu_{i_0}| \leq \bar{\alpha}_{i_0} + \frac{1}{2} r_{i_0}(A).$$

From $\mu_{i_0} > 0$ for all $i_0 \in N$, we get

$$|\lambda - \mu_{i_0}| \geq \mu_{i_0} > \bar{\alpha}_{i_0} + \frac{1}{2} r_{i_0}(A),$$

This is a contradiction. Hence, $\lambda > 0$. Then, the symmetric tensor $A$ is positive definite. \hfill \qed

**Example 4.1.** Consider the symmetric tensor $A = (a_{i_1...i_6}) \in \mathbb{R}^{[6,2]}$ of Example 4.1 in [17], where

$$
a_{111111} = a_{222222} = 35, \ a_{111112} = 2, \ a_{111122} = 7, \\
a_{111222} = 1, \ a_{112222} = 7, \ a_{122222} = 4.
$$

Taking $\mu_1 = \mu_2 = 35$, by Theorem 3 in [17], we obtain

$$\sigma(A) \subseteq \bar{G}(A) = \{z \in \mathbb{R} : |z - 35| \leq 32\}.$$

Taking $\mu_1 = \mu_2 = 35$, when $i_1 = 2$, let

$$\Phi^1 = \{(i_2, i_3, i_4, i_5, i_6) : (1, 1, 1, 1, 1), \pi_5(2, 2, 1, 1, 1)\}, \\
\Phi^2 = \{(i_2, i_3, i_4, i_5, i_6) : \pi_5(1, 2, 2, 2, 2)\},$$

then by Theorem 4.4, we have

$$\sigma(A) \subseteq \bar{\Theta}(A) = \{z \in \mathbb{R} : |z - 35| \leq 20\}.$$ 

From Figure 2, we can see that: $\Theta(A) \subseteq \bar{G}(A)$.

$Z$-eigenvalues-based sufficient conditions for the positive definiteness of even-order tensors are also studied in [17], an example is given to show the effectiveness of Theorem 4.5.

**Example 4.2.** Consider the symmetric tensor $A = (a_{i_1...i_6}) \in \mathbb{R}^{[6,2]}$, where

$$
a_{111111} = a_{222222} = 35, \ a_{111112} = 6, \ a_{111122} = 7, \\
a_{111222} = 1, \ a_{112222} = 7, \ a_{122222} = 4.
$$

Let $\mu_1 = \mu_2 = 35$, by Proposition 1 in [17], we obtain

$$\mu_1 = a_{111111} = 35 < r_{i_1}^{D_1}(A) = 44, \ \mu_2 = a_{222222} = 35 < r_{i_2}^{D_2}(A) = 36,$$

which means that, Proposition 1 in [17] (Lemma 4.3) is not suitable to identify the positive definiteness of $A$. 

Taking $\mu_1 = \mu_2 = 35$, when $i_1 = 1$, let
\[
\Phi_1 = \{(i_2, i_3, i_4, i_5, i_6) : \pi_5(1, 1, 1, 1, 2)\},
\]
\[
\Phi_2 = \{(i_2, i_3, i_4, i_5, i_6) : (2, 2, 2, 2, 2), \pi_5(1, 1, 2, 2, 2)\},
\]
then we have
\[
\mu_1 = 35 > \bar{\alpha}_1 + r_{\Phi_1}(A) = 30.
\]
When $i_1 = 2$, let
\[
\Phi_1 = \{(i_2, i_3, i_4, i_5, i_6) : (1, 1, 1, 1, 1), \pi_5(2, 2, 1, 1, 1)\},
\]
\[
\Phi_2 = \{(i_2, i_3, i_4, i_5, i_6) : \pi_5(1, 2, 2, 2, 2)\},
\]
then we have
\[
\mu_2 = 35 > \bar{\alpha}_2 + r_{\Phi_2}(A) = 20.
\]
Therefore, from Theorem 4.5, $A$ is positive definite. In fact, $\sigma(A) = \{23.9534, 46.0466\}$.

And by computations, we have,
\[
a_{111111} = 35 < \bar{R}_1(A) = 149, \ a_{222222} = 35 < \bar{R}_2(A) = 141.
\]
Therefore, $A$ is not an SDD tensor. Then, Lemma 4.1 can not be used to identify the positive definiteness of $A$.

And we can get
\[
(|a_{111111} - \bar{R}_1^2(A)|a_{222222}) = -3850 < |a_{122222}|\bar{R}_2(A) = 564,
\]
\[
(|a_{222222} - \bar{R}_2^2(A)|a_{111111}) = -3500 < |a_{211111}|\bar{R}_1(A) = 894.
\]
Therefore, $A$ is not a QSDD tensor. Then, Lemma 4.2 can not be used to identify the positive definiteness of $A$.

From analyses above, it is easy to see that, the Z-eigenvalues-based sufficient condition, which is introduced in Theorem 4.5, has advantages on identifying the positive definiteness of even-order tensors in some cases.
5. Asymptotically stability of time-invariant polynomial systems. Consider the following time-invariant polynomial system \(4\):

\[
\Sigma : \dot{x} = A_2 x + A_4 x^2 + \ldots + A_{2k} x^{2k-1},
\]

where \(A_i = (a_{ij\ldots i}) \in \mathbb{R}^{[i,n]}\), \(t = 2, 4, \ldots, 2k\) and \(x = (x_1, \ldots, x_n)^T\). Based on the Lyapunov stability theorem and positive definiteness of tensors, the stability of the following time-invariant polynomial system is presented.

**Lemma 5.1.** [4] For the nonlinear system \(\Sigma : \dot{x} = A_2 x + A_4 x^2 + \ldots + A_{2k} x^{2k-1}\), if \(-A_i\) is positive definite, \(t = 2, 4, \ldots, 2k\), then the equilibrium point of \(\Sigma\) is asymptotically stable.

By Theorem 4.5 and Lemma 5.1, a checkable sufficient condition for the asymptotically stability is presented as follows.

**Corollary 1.** For the nonlinear system \(\Sigma : \dot{x} = A_2 x + A_4 x^2 + \ldots + A_{2k} x^{2k-1}\), if \(-A_i\) satisfies all conditions of Theorem 4.5, then the equilibrium point of \(\Sigma\) is asymptotically stable.

**Example 5.1.** Consider the following time-invariant polynomial system:

\[
\Sigma : \begin{align*}
\dot{x}_1 &= -4x_1 + x_2 + 3x_3 - 3x_1^3 - 2.7x_1^2x_2 - 3x_1x_3^2 - 4.8x_1^2x_3^2 - 2.7x_2x_3^2, \\
\dot{x}_2 &= x_1 - 4x_2 + x_3 - 3.9x_1^3 - 4.8x_1^2x_2 - 2.7x_1x_3^2 - 1.2x_2^2x_3 - 3x_2x_3^2, \\
\dot{x}_3 &= x_1 + x_2 - 4x_3 - 3x_1^3 - 3x_1^2x_3 - 204x_1x_2x_3 - 0.4x_1^3 - 3x_2^2 - 3x_3^2,
\end{align*}
\]

then \(\Sigma\) can be written as \(\dot{x} = A_2 x + A_4 x^3\), where \(x = (x_1, x_2, x_3)^T\),

\[
A_2 = \begin{bmatrix}
-4 & 1 & 1 \\
1 & -4 & 1 \\
1 & 1 & -4
\end{bmatrix},
\]

and \(A_4 = a_{ijkl} \in \mathbb{R}^{[4,3]}\), where

\[
\begin{align*}
a_{1111} &= -3; a_{2222} = -3; a_{3333} = -3; \\
a_{1112} &= a_{1121} = a_{1211} = a_{2111} = -0.9, \\
a_{1122} &= a_{1212} = a_{1221} = a_{2112} = a_{2121} = a_{2211} = -1.6; \\
a_{1133} &= a_{1313} = a_{1331} = a_{3113} = a_{3131} = a_{3311} = -1; \\
a_{1233} &= a_{1323} = a_{1332} = a_{2313} = a_{2331} = a_{2331} = -0.9; \\
a_{3123} &= a_{3132} = a_{3213} = a_{3231} = a_{3321} = a_{3321} = -0.4; \\
a_{2223} &= a_{2232} = a_{2322} = a_{3222} = -0.4; \\
a_{2233} &= a_{2323} = a_{2332} = a_{3232} = a_{3322} = -1; \\
a_{ijkl} &= 0, \text{ otherwise.}
\end{align*}
\]

Clearly, \(-A_2\) is positive definite. Consider the symmetric tensor \(-A_4\), let \(\mu = (3, 3, 3)^T\), by direct computations, we have

\[
\mu_1 = 3 > \bar{a}_1 + \bar{r}_1^T(-A_4) = 2.7,
\]
\[
\mu_2 = 3 > \alpha_2 + \frac{3}{2}(-A_4) = 2.7, \\
\mu_3 = 3 > \alpha_3 + \frac{3}{2}(-A_4) = 2.8,
\]
therefore, from Corollary 1, \(-A_4\) is positive definite, which means that, the equilibrium point of \(\Sigma\) is asymptotically stable.

6. Conclusion. In this paper, when \(m \geq 4\), based on the condition \(||x||_2 = 1\), we get a new Z-eigenvalue localization set for tensors. As an application, a upper bound associated with the Z-eigenvalue localization set can also be obtained. Choosing parameter \(\mu\), the Geršgorin-type localization set for Z-eigenvalues is also obtained. A checkable sufficient condition for the asymptotically stability of time-invariant polynomial system is presented.

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