SOME NEW RAMANUJAN-SATO SERIES FOR $1/\pi$

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Abstract. We derive 10 new Ramanujan-Sato series of $1/\pi$ by using the method of Huber, Schultz and Ye. The levels of these series are 14, 15, 16, 20, 21, 22, 26, 35, 39.

Keywords: Hauptmodul; modular polynomial; Moonshine group.

1. Introduction

In Ramanujan’s paper [1], he announced 17 series for $1/\pi$, those series have the form of

$$\frac{1}{\pi} = \sum_{n=0}^{\infty} A_n(Bn + C)X^n. \quad (1.1)$$

For example, one of his series is

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801} \sum_{n=0}^{\infty} \frac{(4n)!}{(n!)^44^{4n}} \frac{1103 + 26390n}{99^{4n}}.$$ 

It was not until 1987 in [2] that all of his formulas were proved. Nowadays, series of the form (1.1) is called the Ramanujan-Sato series due to Sato’s research on this type of series. Moreover, Borweins and Chudnovskys derived some new $1/\pi$ series similar to Ramanujan’s independently in [3].

More recently, Chan, Chan and Liu provided a systematic classification of these series by the levels of the modular forms in [4], see [5] for a summary with levels less than 12. In an unpublished preprint [6], Huber, Schultz and Ye constructed a systematically method to find new families of Ramanujan-Sato series by using the Hauptmodul for some Moonshine groups. They also derived series of levels 17 and 20 in [7, 8].

We briefly introduce Huber et al.’s method. Let $\Gamma$ be a genus zero discrete subgroup of $\text{SL}_2(\mathbb{R})$ with Hauptmodul $x(\tau)$ (see Section 2 for definitions). There is a weight 2 modular form $z(\tau)$ that satisfies a third-order differential equation

$$2wz_{xxx} + 3w_xz_{xx} + (w_{xx} - 2R)z_x - Rxz = 0,$$

where $w, R$ are polynomials in $x$. From this equation, we can express $z$ as

$$z = \sum_{n=0}^{\infty} A_nx^n.$$

By picking some special CM-points $\tau_0 \in \mathbb{H}$ and using the modular equations satisfies by $x$, we can derive series of the form (1.1).

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In [6], $x, w, R$ that associated with some moonshine groups are listed in several huge tables, the Hauptmoduls that they list all have the Fourier expansion $x = q + O(q^2)$. If $x$ has Fourier expansion $x = 1/q + O(1)$, then we can derive the modular equation. It turns out that we can also find the modular equations for $x = q + O(q^2)$ in several cases. In this paper, we search the Hauptmoduls that has a modular equation and derive 10 new Ramanujan-Sato series.

2. Hauptmodul and Differential equations

Let $\Gamma$ be a genus zero discrete subgroup of $\text{SL}_2(\mathbb{R})$ commensurable with $\text{SL}_2(\mathbb{Z})$, i.e.

$$[\Gamma : \Gamma \cap \text{SL}_2(\mathbb{Z})], [\text{SL}_2(\mathbb{Z}) : \Gamma \cap \text{SL}_2(\mathbb{Z})] < \infty$$

and the modular curve $\Gamma \setminus \mathcal{H}^*$ is a compact Riemann surface of genus zero, where $\mathcal{H}^*$ is the upper half plane together with cusps. From the knowledge of Riemann surfaces, we know that the field of meromorphic functions on $\Gamma \setminus \mathcal{H}^*$ is generated by a single element $t_{\Gamma}$ that transcendental over $\mathbb{C}$, such $t_{\Gamma}$ is called a Hauptmodul for $\Gamma$.

For two natural numbers $N$ and $e||N$, i.e. $e|N$ and $\gcd(e, N/e) = 1$, let

$$W_e = \left\{ \left( \begin{array}{cc} a & b \\ Nc & ed \end{array} \right) \mid (a, b, c, d) \in \mathbb{Z}^4, ead - N = 1 \right\}$$

be the set of Atkin-Lehner involutions. For any set of indices $e$ closed under the following law of multiplication

$$W_e W_f \equiv W_{ef/\gcd(e, f)^2} \mod \Gamma_0(N),$$

the Moonshine group $\Gamma := \cup e W_e$ is a subgroup of the normalizer of $\Gamma_0(N)$. We denote such a $\Gamma$ by $N + e_1, e_2, \cdots$, or $N+$ if all of the indices are present. More details about Moonshine groups can be found in Conway’s Monstrous Moonshine paper [9].

From now on, let’s focus on the genus zero Moonshine group $\Gamma$. Since $\left( \begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array} \right) \in \Gamma$, $t_{\Gamma}$ has a Fourier expansion $t_{\Gamma} = \sum_{n=m}^{\infty} a_n q^n$ for some $m \in \mathbb{Z}$, where $q = e^{2\pi i \tau}$. Notice that for any Hauptmodul $t_{\Gamma}$ and $\left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \text{SL}_2(\mathbb{Z})$, the Möbius transformation $\frac{ad \tau + b}{cd \tau + d}$ is also a Hauptmodul for $\Gamma$. The Hauptmodul $t_{\Gamma}$ is said to be normalized if it has the Fourier expansion of the form

$$t_{\Gamma} = \frac{1}{q} + \sum_{n=1}^{\infty} a_n q^n.$$

Let $f(x)$ be a function, here and throughout the paper, denote $f_x = x \frac{df}{dx}$ be the same as [6].

**Theorem 2.1** (Theorem 2.2 and 2.3 of [6]). For any choice of $x(\tau)$ as a Möbius transformation of Hauptmodul $t_{\Gamma}$, there exists a polynomial $w(x)$ and a modular form $z = (\log(x))_q$ of weight 2, so that $R := \frac{2zzqq - 3zz^2}{z^4}$ is a polynomial in $x$, and we have a differential equation for $z$ with respect to $x$:

$$2wz_{xxx} + 3wz_{xx} + (w_{xx} - 2R)z_x - R_x z = 0.$$

(2.1)
By this theorem, we can write \( z \) as a power series in \( x \), i.e. \( z = \sum_{n=0}^{\infty} A_n x^n \) and the coefficients \( A_n \) can be obtained from the recurrence relation \( \mathcal{R} \) implied by the differential equation. In [6], there is a table that lists some Moonshine groups together with their Hauptmoduls. For every \( \Gamma \) and some choice of \( x(\tau) \) as a Möbius transformation of Hauptmodul, the authors also calculated the \( w(x) \) and \( R(x) \) associated with \( x(\tau) \). We give here an example of the calculation of the recurrence relation for \( \Gamma = 39 + 39 \). Before this example, recall that Dedekind eta-function \( \eta(\tau) = e^{2\pi i \tau / 24} \prod_{j=1}^{\infty} (1 - q^j) \), we define \( \eta(\tau) := \eta(l\tau) \).

**Example 2.2.** If \( \Gamma = 39 + 39 \), then \( t_\Gamma = \frac{\eta_3 \eta_{13}}{\eta_1 \eta_{39}} \) is a Hauptmodul for \( \Gamma \). Choose
\[
x(\tau) = \frac{1}{1 + t_\Gamma} = q - q^2 + q^3 - q^5 + 2q^7 + \cdots,
\]
According to [6], we have
\[
w(x) = (1 + x)^2(1 - 7x + 11x^2 - 7x^3 + x^4)(1 + x - x^2 + x^3 + x^4),
\]
\[
R(x) = x(2 + 17x - 48x^2 - 25x^3 + 194x^4 - 45x^5 - 168x^6 + 137x^7 + 82x^8 - 25x^9),
\]
and
\[
z(\tau) = \frac{(\log(x) + \sqrt{w(x)})}{q} = 1 + q + 3q^2 + q^3 + 5q^4 + 3q^5 + 7q^6 + 5q^7 + \cdots.
\]
Now assume \( z = \sum_{n=0}^{\infty} A_n x^n \), via Fourier coefficients of \( x \) and \( z \), we have
\[
A_0 = 1, A_1 = 1, A_2 = 4, A_3 = 10, A_4 = 38, A_5 = 140,
\]
\[
A_6 = 563, A_7 = 2315, A_8 = 9816, A_9 = 42432.
\]
Using (2.1), we can derive the recurrence relation
\[
( -250 + 150n - 30n^2 + 2n^3 ) A_{n-10} + ( 738 - 488n + 108n^2 - 8n^3 ) A_{n-9}
\]
\[
+ ( 1096 - 786n + 192n^2 - 16n^3 ) A_{n-8} + ( -1176 + 924n - 252n^2 + 24n^3 ) A_{n-7}
\]
\[
+ ( -270 + 234n - 72n^2 + 8n^3 ) A_{n-6} + ( 970 - 938n + 330n^2 - 44n^3 ) A_{n-5}
\]
\[
+ ( -100 + 114n - 48n^2 + 8n^3 ) A_{n-4} + ( -144 + 204n - 108n^2 + 24n^3 ) A_{n-3}
\]
\[
+ ( 34 - 66n + 48n^2 - 16n^3 ) A_{n-2} + ( 2 - 8n + 12n^2 - 8n^3 ) A_{n-1} + 2n^3 A_n = 0.
\]

**Remark 2.3.** 10 moonshine groups together with their Hauptmodul \( t_\Gamma \), \( x(\tau) \), \( w(x) \), \( R(x) \) that we used to derive \( 1/\pi \)-series are listed in Table 1 in Section 5. We also list the recurrence relations of \( A_n \) implied by (2.1) with initial values in Table 2.

3. Modular equations for \( x(\tau) \)

Recall that the \( j \)-invariant
\[
j(\tau) = \frac{1}{q} + 744 + 196884q + \cdots
\]
has the property that for any CM-point \( \tau_0 \in \mathcal{H} \) (i.e. There exist integers \( a, b, c \) with \( a \neq 0 \) such that \( a\tau_0^2 + b\tau_0 + c = 0 \)), \( j(\tau_0) \) is algebraic. This fact can be proved by using the modular equations for \( j(\tau) \), that are polynomials \( \Psi_n(X, Y) \in \mathbb{Z}[X, Y] \) such that

\[
\Psi_n(j(\tau), Y) = \prod_{i=1}^{r}(Y - j(\gamma_i\tau)),
\]

where \( \gamma_i \in \Gamma_n := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_{2 \times 2}(\mathbb{Z}) \middle| ad - bc = n \right\} \) and \( \Gamma_n = \bigcup_{i=1}^{r} \text{SL}_2(\mathbb{Z})\gamma_i \). Similarly, for each Moonshine group \( \Gamma \) and its Hauptmodul \( x(\tau) \) we have

**Proposition 3.1** (Proposition 3.2 of [6]). Assume \( x(\tau) \) has the Fourier expansion of the form \( x(\tau) = \frac{1}{q} + O(1) \). For any integer \( n \geq 2 \) with \( \gcd(n, N) = 1 \), there is a symmetric irreducible polynomial \( \Psi_n(X, Y) \in \mathbb{C}[X, Y] \) of degree \( \psi(n) = n \prod_{q \text{ prime}} (1 + \frac{1}{q}) \) in \( X \) and \( Y \) such that

\[
\Psi(x(\tau), Y) = \prod_{\substack{\alpha \beta = n \\ 0 \leq \beta < \delta \\ \gcd(\alpha, \beta, \delta) = 1}} \left( Y - x \left( \frac{\alpha \tau + \beta}{\delta} \right) \right).
\]

That is to say, if \( S_k \) is the elementary symmetric function of degree \( 1 \leq k \leq \psi(n) \) in \( \psi(n) \) variables, then \( S_k \left( x(n\tau), \cdots, x \left( \frac{\tau + n - 1}{n} \right) \right) \) is a polynomial in \( x(\tau) \).

To use Proposition 3.1, we require that \( x(\tau) \) has the Fourier expansion of the form \( \frac{1}{q} + O(1) \). However, the Möbius transformations \( x(\tau) = 1/\tau \) that we chosen in Section 2 all have the Fourier expansion \( x(\tau) = q + O(q^2) \). Fortunately, it turns out that in this cases, we can also find the Modular Equations \( \Psi(X, Y) \). For example, we have

**Theorem 3.2.** If \( \Gamma = 39 + 39, t_\Gamma = \frac{n_\Gamma h_3}{n_3 h_3} = \frac{1}{q} + O(1), x(\tau) = 1/t_\Gamma \). Let

\[
\Psi_2(X, Y) = Y^3 + (2X - X^2)Y^2 + (-X + 2X^2)Y + X^3,
\]

then we have

\[
\Psi_2(x(\tau), Y) = (Y - x(2\tau)) \left( Y - x \left( \frac{\tau}{2} \right) \right) \left( Y - x \left( \frac{\tau + 1}{2} \right) \right).
\]

**Proof.** Let \( t_1 = t_\Gamma(2\tau), t_2 = t_\Gamma \left( \frac{\tau}{2} \right), t_3 = t_\Gamma \left( \frac{\tau + 1}{2} \right) \). Proposition 3.1 says that, \( t_1 + t_2 + t_3, t_1t_2 + t_1t_3 + t_2t_3 \) and \( t_1t_2t_3 \) are all polynomial of degree small than \( \psi(2) = 3 \) in \( t_\Gamma \). By using the \( q \)-expansion, we find that

\[
t_1 + t_2 + t_3 = t_\Gamma^2 - 2t_\Gamma, \quad t_1t_2 + t_1t_3 + t_2t_3 = 2t_\Gamma^2 - t_\Gamma, \quad t_1t_2t_3 = -t_\Gamma^3.
\]
Since $x(\tau) = 1/t_\Gamma(\tau)$, we have
\[
x(2\tau) + x\left(\frac{\tau}{2}\right) + x\left(\frac{\tau + 1}{2}\right) = \frac{t_1t_2 + t_1t_3 + t_2t_3}{t_1t_2t_3}
\]
\[
= \frac{2t_1^2 - t_1}{-t_1^2}
\]
\[
= -2x + x^2.
\]
Similarly,
\[
x(2\tau) x\left(\frac{\tau}{2}\right) + x(2\tau) x\left(\frac{\tau + 1}{2}\right) + x\left(\frac{\tau}{2}\right) x\left(\frac{\tau + 1}{2}\right) = -x + 2x^2,
\]
\[
x(2\tau) x\left(\frac{\tau}{2}\right) x\left(\frac{\tau + 1}{2}\right) = -x^3.
\]
Thus, we have (3.2). \hfill \square

For each $\Gamma$ and $x(\tau) = 1/t_\Gamma(\tau)$, we choose $n = 2$ or $3$ and list their modular equations $\Psi_n(X,Y)$ in Table 3.

4. Series for $1/\pi$

Suppose that $\Phi_n(X,Y)$ is a modular equation for some $\Gamma$. If $\tau_0 \in \mathbb{H}$ such that
\[
\frac{a\tau_0 + b}{c\tau_0 + d} = \frac{\alpha\tau_0 + \beta}{\delta}
\]
for some $a\delta = n, 0 \leq \beta < \delta$ and $\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \in \Gamma$, then by (3.1), we know that
\[
\Psi_n\left(x(\tau_0), x(\tau_0)\right) = \Psi_n\left(x(\tau_0), x\left(\frac{a\tau_0 + b}{c\tau_0 + d}\right)\right)
\]
\[
= \Psi_n\left(x(\tau_0), x\left(\frac{\alpha\tau_0 + \beta}{\delta}\right)\right)
\]
\[
= 0.
\]
Thus $x(\tau_0)$ is a root of $\Psi_n(X, X) = 0$.

We now derive Ramanujan-Sato series for $1/\pi$, the following argument follows [6]. If
\[
M = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} := \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}^{-1}\begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix}
\]
has $c' \neq 0$ (this is equivalent to $c \neq 0$), then $M\tau_0 = \tau_0$ and $y(\tau) := x(M\tau)$ satisfies
\[
\Psi_n(x(\tau), y(\tau)) = 0
\] (4.1)
by (3.1). We can apply (4.1) to expand $y$ about $x(\tau_0)$:
\[
y(\tau) = \sum_{k=0}^{\infty} y^{(k)}(x(\tau_0)) \frac{(x(\tau) - x(\tau_0))^k}{k!},
\] (4.2)
where \( y^{(k)} = \frac{d^k y}{dx^k} \). Now applying \( \frac{1}{2\pi i} \frac{d}{d\tau} \) to both side of (4.2), we have
\[
\frac{1}{2\pi i (c'\tau + d')^2} \frac{d}{d\tau} (M\tau) = \frac{1}{2\pi i} \left( y^{(1)}(x(\tau_0)) \frac{dx}{d\tau}(\tau) + y^{(2)}(x(\tau_0))(x(\tau) - x(\tau_0)) \frac{dx}{d\tau}(\tau) + \cdots \right).
\]

Let \( z = \frac{(\log(x))_q}{\sqrt{w(x)}} \) as in Theorem 2.1. Then
\[
\frac{1}{2\pi i} \frac{dx}{d\tau}(\tau) = x(\tau)z(\tau)\sqrt{w(\tau)}.
\] (4.3)

Denote \( \sqrt{w(\tau)} \) by \( W(\tau) \), thus
\[
\frac{a'd' - b'c'}{(c'\tau + d')^2} x(M\tau)z(M\tau)W(M\tau) = \left( y^{(1)}(x(\tau_0)) + y^{(2)}(x(\tau_0))(x - x(\tau_0)) + \cdots \right) xzW. \] (4.4)

Applying \( \frac{1}{2\pi i} \frac{d}{d\tau} \) again to both side of (4.4) and use (4.3), we have
\[
\frac{ic'}{\pi} \frac{a'd' - b'c'}{(c'\tau + d')^2} x(M\tau)z(M\tau)W(M\tau) + \left( \frac{(a'd' - b'c')^2}{(c'\tau + d')^4} \right) z(M\tau)W(M\tau) + \left( \frac{d}{dx}(M\tau)z(M\tau)W(M\tau) \right) x(\tau)z(M\tau)W(M\tau) + \cdots = \left( y^{(2)}(x(\tau_0)) + \cdots \right) x^2z^2W^2 + \left( y^{(1)}(x(\tau_0)) + y^{(2)}(x(\tau_0))(x - x(\tau_0)) \right)
\]
\[
+ \cdots \right) \left( z(\tau)w(\tau) + x(\tau) \frac{dz}{dx}(\tau)W(\tau) + x(\tau)z(\tau) \frac{dW}{dx}(\tau) \right) xzW.
\] (4.5)

Let \( \tau = \tau_0 \) in (4.5) and make some simplification, we have
\[
\frac{1}{\pi} = W(1 - y^{(1)}(c'\tau_0 + d')) \left( \frac{dz}{dx} + \left( 1 + \frac{xW}{W} + \frac{xy^{(2)}}{y^{(1)}(1 - y^{(1)})} \right) \right) \bigg|_{\tau=\tau_0}.
\]

Since \( z = \sum_{n=0}^{\infty} A_n x^n \), we get
\[
\frac{1}{\pi} = \sum_{n=0}^{\infty} A_n (Bn + C)x(\tau_0)^n,
\]
where
\[
B = \frac{W(x(\tau_0))(1 - y^{(1)}(x(\tau_0))(c'\tau_0 + d'))}{ic'},
\]
\[
C = B \left( 1 + \frac{x(\tau_0)\frac{dW}{dx}(x(\tau_0))}{W(x(\tau_0))} + \frac{x(\tau_0)y^{(2)}(x(\tau_0))}{y^{(1)}(x(\tau_0))(1 - y^{(1)}(x(\tau_0)))} \right). \] (4.6)

Let’s continue with the example \( \Gamma = 39 + 39 \) to illustrate how the above calculation works.

**Example 4.1.** Let \( n = 2 \), we know from Theorem 3.2 that the modular equation for \( x = \frac{\eta_1 \eta_9}{\eta_3 \eta_5} \) is
\[
\Psi_2(X, Y) = Y^3 + (2X - X^2)Y^2 + (-X + 2X^2)Y + X^3.
\] (4.7)
Thus, the roots of \( \Psi(X, X) = 0 \) are \( 0, 3 \pm 2\sqrt{2} \). Let \( \tau_0 = i\sqrt{\frac{2}{39}} \), notice that \( \tau_0 \) satisfies

\[
\begin{pmatrix}
0 & -1 \\
39 & 0
\end{pmatrix}
\tau_0 = \begin{pmatrix}
1 & 0 \\
0 & 2
\end{pmatrix}
\tau_0,
\]

where \( \begin{pmatrix}
0 & -1 \\
39 & 0
\end{pmatrix} \in \Gamma \). Choose \( M = \begin{pmatrix}
0 & -1 \\
39 & 0
\end{pmatrix}^{-1} \begin{pmatrix}
1 & 0 \\
0 & 2
\end{pmatrix} = \begin{pmatrix}
0 & 2/39 \\
-1 & 0
\end{pmatrix} \), we have \( c' = -1, d' = 0 \), set \( y(\tau) = x(M\tau) \). By numerical approximation, we have \( x(\tau_0) = 3 - 2\sqrt{2} \). Since \( \Psi_2(x(\tau), y(\tau)) = 0 \), we can calculate from (4.7) that

\[
y^{(1)}(3 - 2\sqrt{2}) = -1, \quad y^{(2)}(3 - 2\sqrt{2}) = -16 - \frac{25}{\sqrt{2}}.
\]

We can also calculate from (2.2) that

\[
W(3 - 2\sqrt{2}) = -6(-2 + \sqrt{2})\sqrt{7501 - 5304\sqrt{2}},
\]

\[
\frac{dW}{dx}(3 - 2\sqrt{2}) = -3(-5753 + 4068\sqrt{2})\sqrt{7501 + 5304\sqrt{2}}.
\]

Finally, via (4.6), we get

\[
\frac{1}{\pi} = \sum_{n=0}^{\infty} A_n(Bn + C)(3 - 2\sqrt{2})^n
\]

with

\[
B = -4(-2 + \sqrt{2})\sqrt{6(577 - 408\sqrt{2})},
\]

\[
C = 44\sqrt{3(577 - 408\sqrt{2}) - 22\sqrt{6(577 - 408\sqrt{2}) + 2(-47420 + 33531\sqrt{2})\sqrt{3(577 + 408\sqrt{2})}}}
\]

and \( A_n \) as in Example 2.2.

In Table 4 we list \( \tau_0, M, x(\tau_0) \) that used to derive the series. Also in Table 5, we list \( B, C \) for each series.

5. Tables

In this section, we list the tables of modular equations of Huaptmodul \( t_{\Gamma} \), the \( a, b, c \) for \( 1/\pi \) series \( \frac{1}{\pi} = \sum_{n=0}^{\infty} A_n(an + b)c^n \), and recursion formula of \( A_n \) for some given level \( N \).

The details of \( \tilde{R}(x) \) and \( w(x) \) defining the differential equation come from [6]. The notation \( \eta_a \) means

\[
\eta_a = q^{a/24} \prod_{n=1}^{\infty} (1 - q^{an}).
\]

Moreover, we write \( N + e \) for \( \Gamma_0(N) + e \).
Table 1. Hauptmoduls, \(x(\tau), w(x)\) and \(R(x)\) for 10 moonshine groups.

| \(\Gamma\) | \(t_\Gamma\) | \(x\) | \(w(x)\) | \(R(x)\) |
|---|---|---|---|---|
| 14 + 7 | \((\eta_1 \eta_7, \eta_2 \eta_{14})^3\) | \((1 + x)(1 + 8x)(1 + 5x + 8x^2)\) | \(-8x(1 + 4x)(1 + 7x + 8x^2)\) |
| 14 + 14 | \((\eta_1 \eta_7, \eta_2 \eta_{14})^4\) | \((1 - 14x + 19x^2 - 14x^3 + x^4)\) | \(x(6 - 25x + 34x^2 - 4x^3)\) |
| 15 + 15 | \((\eta_1 \eta_7, \eta_3 \eta_{15})^3\) | \((-1 - x + x^2)(-1 + 11x + x^2)\) | \(4x(1 + 4x - 6x^2 - x^3)\) |
| 16+ | \((\eta_2 \eta_{18})^6, (\eta_1 \eta_4 \eta_{10})^2\) | \((1 - 2x)^2(1 - 12x + 4x^2)\) | \(8x(1 - 2x)(1 - 8x + 4x^2)\) |
| 20 + 20 | \((\eta_1 \eta_2, \eta_2 \eta_{20})^2\) | \((1 + x)^2(1 - 8x - 2x^2 - 8x^3 + x^4)\) | \(x(1 + x)(2 + 25x + 31x^2 + 47x^3 - 9x^4)\) |
| 21 + 21 | \((\eta_1 \eta_7, \eta_1 \eta_{21})^2\) | \((1 - x)^2(1 - 6x - 17x^2 - 6x^3 + x^4)\) | \(4x + 4x^2 - 70x^3 + 16x^4 + 52x^5 - 9x^6\) |
| 22 + 11 | \((\eta_1 \eta_2, \eta_2 \eta_{22})^2\) | \((1 + 4x + 8x^2 + 4x^3)(1 + 8x + 16x^2 + 16x^3)\) | \(-8x(1 + 12x + 57x^2 + 132x^3 + 160x^4 + 72x^5)\) |
| 26 + 26 | \((\eta_1 \eta_3, \eta_1 \eta_{26})^2\) | \((1 - x)(1 - 8x + 8x^2 - 18x^3 + 8x^4 - 8x^5 + x^6)\) | \((x/4)(20 - 109x + 339x^2 - 521x^3 + 445x^4 - 335x^5 + 49x^6)\) |
| 35 + 35 | \((\eta_1 \eta_{35})\) | \((1 + x - x^2)(1 - 5x - 9x^3 - 5x^5 - x^6)\) | \(-x(-2 - 9x - 14x^2 - 47x^3 + 30x^4 - 57x^5 + 50x^6 + 16x^7)\) |
| 39 + 39 | \((\eta_1 \eta_{39})\) | \((1 + x)^2(1 - 7x + 11x^2 - 7x^3 + x^4)\) | \(x(2 + 17x - 48x^2 - 25x^3 + 194x^4 - 45x^5 - 168x^6 + 137x^7 + 82x^8 - 25x^9)\) |
Table 2. The recurrence relations and initial values of $A_n$.

| $\Gamma$ | $A_n$ |
|---------|-------|
| 14+7   | $(-1024 + 1536n - 768n^2 + 128n^3)A_{n-4} + (-864 + 1584n - 1008n^2 + 224n^3)A_{n-3} + (-176 + 420n - 366n^2 + 122n^3)A_{n-2} + (-8 + 30n - 42n^2 + 28n^3)A_{n-1} + 2n^2A_n = 0, A_0 = 1, A_1 = -4, A_2 = 16, A_3 = -72.$ |
| 14+14  | $(-16 + 24n - 12n^2 + 2n^3)A_{n-4} + (102 - 194n + 126n^2 - 28n^2)A_{n-3} + (-50 + 126n - 114n^2 + 38n^3)A_{n-2} + (6 - 26n + 42n^2 - 28n^3)A_{n-1} + 2n^3A_n = 0, A_0 = 1, A_1 = 2, A_2 = 16, A_3 = 117.$ |
| 15+15  | $(-16 + 24n - 12n^2 + 2n^3)A_{n-4} + (-72 + 138n - 90n^2 + 20n^3)A_{n-3} + (32 - 84n + 78n^2 - 26n^3)A_{n-2} + (4 - 18n + 30n^2 - 20n^3)A_{n-1} + 2n^3A_n = 0, A_0 = 1, A_1 = 2, A_2 = 11, A_3 = 72.$ |
| 16+    | $(-256 + 384n - 192n^2 + 32n^3)A_{n-4} + (480 - 896n + 576n^2 - 128n^3)A_{n-3} + (-160 + 384n - 336n^2 + 112n^3)A_{n-2} + (8 - 32n + 48n^2 - 32n^3)A_{n-1} + 2n^3A_n = 0, A_0 = 1, A_1 = 4, A_2 = 20, A_3 = 128.$ |
| 20+20  | $(-54 + 54n - 18n^2 + 2n^3)A_{n-6} + (190 - 226n + 90n^2 - 12n^3)A_{n-5} + (312 - 428n + 204n^2 - 34n^3)A_{n-4} + (168 - 292n + 180n^2 - 40n^3)A_{n-3} + (54 - 122n + 102n^2 - 34n^3)A_{n-2} + (2 - 10n + 18n^2 - 12n^3)A_{n-1} + 2n^3A_n = 0, A_0 = 1, A_1 = 1, A_2 = 6, A_3 = 30, A_4 = 115, A_5 = 1087.$ |
| 21+21  | $(-54 + 54n - 18n^2 + 2n^3)A_{n-6} + (260 - 304n + 120n^2 - 16n^3)A_{n-5} + (64 - 96n + 48n^2 - 8n^3)A_{n-4} + (-210 + 338n - 198n^2 + 44n^3)A_{n-3} + (8 - 24n + 24n^2 - 8n^3)A_{n-2} + (4 - 16n + 24n^2 - 16n^3)A_{n-1} + 2n^3A_n = 0, A_0 = 1, A_1 = 2, A_2 = 8, A_3 = 37, A_4 = 204, A_5 = 1218.$ |
| 22+11  | $(-3456 + 3456n - 1152n^2 + 128n^3)A_{n-6} + (-6400 + 7360n - 2880n^2 + 384n^3)A_{n-5} + (-4224 + 5696n - 2688n^2 + 448n^3)A_{n-4} + (-1368 + 2244n - 1332n^2 + 296n^3)A_{n-3} + (-192 + 416n - 336n^2 + 112n^3)A_{n-2} + (-8 + 28n - 36n^2 + 24n^3)A_{n-1} + 2n^3A_n = 0, A_0 = 1, A_1 = -4, A_2 = 12, A_3 = -36, A_4 = 124, A_5 = -496.$ |
| 26+26  | $$\left(\frac{343}{4} - \frac{147n}{2} + 21n^2 - 2n^3\right)A_{n-7} + \left(-\frac{1005}{2} + \frac{983n}{2} - 162n^2 + 18n^3\right)A_{n-6} + \left(\frac{225}{2} - \frac{125n}{2} + 240n^2 - 32n^3\right)A_{n-5} + \left(-\frac{521}{2} + \frac{135n}{2} - 312n^2 + 52n^3\right)A_{n-4} + \left(\frac{1017}{4} - \frac{807n}{2} + 234n^2 - 52n^3\right)A_{n-3} + \left(-\frac{109}{2} + \frac{237n}{2} - 96n^2 + 32n^3\right)A_{n-2} + \left(5 - 19n + 27n^2 - 18n^3\right)A_{n-1} + 2n^3A_n = 0, A_0 = 1, A_1 = \frac{2}{3}, A_2 = \frac{59}{8}, A_3 = \frac{497}{16}, A_4 = \frac{19539}{256}, A_5 = \frac{207051}{1024}, A_6 = \frac{4023151}{1024}.$$ |
| 35+35  | $$(-128 + 96n - 24n^2 + 2n^3)A_{n-8} + (-350 + 296n - 8n^2 + 8n^3)A_{n-7} + (342 - 330n + 108n^2 - 12n^3)A_{n-6} + (-150 + 160n - 60n^2 + 8n^3)A_{n-5} + (188 - 238n + 108n^2 - 18n^3)A_{n-4} + (42 - 64n + 36n^2 - 8n^3)A_{n-3} + (18 - 42n + 36n^2 - 12n^3)A_{n-2} + (2 - 8n + 12n^2 - 8n^3)A_{n-1} + 2n^3A_n = 0, A_0 = 1, A_1 = 1, A_2 = 3, A_3 = 10, A_4 = 38, A_5 = 150, A_6 = 627, A_7 = 2703.$$ |
Table 3. The modular equation of Hauptmodul.

| $\Gamma$  | $\psi_n(X, Y), n$                                                                                       |
|--------|---------------------------------------------------------------------------------------------------------|
| 14+7  | $Y^4 + (-18X - 72X^2 - 64X^3)Y^3 + (-9X - 54X^2 - 72X^3)Y^2 + (-X - 9X^2 - 18X^3)Y + X^4, n = 3$      |
| 14+14 | $Y^4 + (-18X + 12X^2 - X^3)Y^3 + (12X + 9X^2 + 12X^3)Y^2 + (-X + 12X^2 - 18X^3)Y + X^4, n = 3$      |
| 15+15 | $Y^3 + (6X + X^2)Y^2 + (-X + 6X^2)Y + X^3, n = 2$                                                      |
| 16+   | $Y^4 + (-24X + 48X^2 - 16X^3)Y^3 + (12X - 42X^2 + 48X^3)Y^2 + (-X + 12X^2 - 24X^3)Y + X^4, n = 3$   |
| 20+20 | $Y^4 + (3X + 6X^2 - X^3)Y^3 + (6X + 18X^2 + 6X^3)Y^2 + (-X + 6X^2 + 3X^3)Y + X^4, n = 3$           |
| 21+21 | $Y^3 + (4X - X^2)Y^2 + (-X + 4X^2)Y + X^3, n = 2$                                                      |
| 22+11 | $Y^4 + (-9X - 24X^2 - 16X^3)Y^3 + (-6X - 24X^2 - 24X^3)Y^2 + (-X - 6X^2 - 9X^3)Y + X^4, n = 3$     |
| 26+26 | $Y^4 + (-3X + 6X^2 - X^3)Y^3 + (6X - 9X^2 + 6X^3)Y^2 + (-X + 6X^2 - 3X^3)Y + X^4, n = 3$           |
| 35+35 | $Y^3 + (2X + X^2)Y^2 + (-X + 2X^2)Y + X^3, n = 2$                                                      |
| 39+39 | $Y^3 + (2X - X^2)Y^2 + (-X + 2X^2)Y + X^3, n = 2$                                                      |
Table 4. $\tau_0, M$ and $x(\tau_0)$ that used to derive the series for $1/\pi$.

| $\Gamma$ | $\tau_0$ | $(a \ b) \ (c \ d) \tau_0 = (\alpha \ \beta) \tau_0$ | $M$ | $x(\tau_0)$ |
| --- | --- | --- | --- | --- |
| 14+7 | $\frac{-7 + i\sqrt{21}}{14}$ | $(\begin{array}{cc} -7 & -3 \\ -14 & -17 \end{array}) \tau_0 = (\begin{array}{cc} 1 & 2 \\ 0 & 3 \end{array}) \tau_0$ | $(\begin{array}{cc} 1 & -5/7 \\ 2 & 1 \end{array})$ | $\frac{1}{4}(-3 + \sqrt{7})$ |
| 14+14 | $i\sqrt{\frac{3}{14}}$ | $(\begin{array}{cc} 0 & 1 \\ -14 & 0 \end{array}) \tau_0 = (\begin{array}{cc} 1 & 0 \\ 0 & 3 \end{array}) \tau_0$ | $(\begin{array}{cc} 0 & -3/14 \\ 1 & 0 \end{array})$ | $\frac{2}{\frac{23 + 5\sqrt{21}}{1}}$ |
| 15+15 | $i\sqrt{\frac{2}{15}}$ | $(\begin{array}{cc} 0 & -1 \\ 15 & 0 \end{array}) \tau_0 = (\begin{array}{cc} 1 & 0 \\ 0 & 2 \end{array}) \tau_0$ | $(\begin{array}{cc} 0 & 2/15 \\ -1 & 0 \end{array})$ | $-7 + 5\sqrt{2}$ |
| 16+ | $\frac{8 + i\sqrt{3}}{4}$ | $(\begin{array}{cc} -16 & 33 \\ -16 & -32 \end{array}) \tau_0 = (\begin{array}{cc} 1 & 1 \\ 0 & 3 \end{array}) \tau_0$ | $(\begin{array}{cc} 2 & -67/16 \\ 1 & -2 \end{array})$ | $\frac{1}{2}(5 - 2\sqrt{6})$ |
| 20+20 | $\frac{1}{2}i\sqrt{\frac{3}{5}}$ | $(\begin{array}{cc} 0 & -1 \\ 20 & 0 \end{array}) \tau_0 = (\begin{array}{cc} 1 & 0 \\ 0 & 3 \end{array}) \tau_0$ | $(\begin{array}{cc} 0 & 3/20 \\ -1 & 0 \end{array})$ | $7 - 4\sqrt{3}$ |
| 21+21 | $i\sqrt{\frac{2}{21}}$ | $(\begin{array}{cc} 0 & 1 \\ -21 & 0 \end{array}) \tau_0 = (\begin{array}{cc} 1 & 0 \\ 0 & 2 \end{array}) \tau_0$ | $(\begin{array}{cc} 0 & -2/21 \\ 1 & 0 \end{array})$ | $5 - 2\sqrt{6}$ |
| 22+11 | $\frac{-33 + i\sqrt{33}}{22}$ | $(\begin{array}{cc} -11 & -17 \\ 22 & 33 \end{array}) \tau_0 = (\begin{array}{cc} 1 & 0 \\ 0 & 3 \end{array}) \tau_0$ | $(\begin{array}{cc} 3 & 51/11 \\ -2 & -3 \end{array})$ | $\frac{1}{2}(-2 + \sqrt{3})$ |
| 26+26 | $i\sqrt{\frac{3}{26}}$ | $(\begin{array}{cc} 0 & -1 \\ 26 & 0 \end{array}) \tau_0 = (\begin{array}{cc} 1 & 0 \\ 0 & 3 \end{array}) \tau_0$ | $(\begin{array}{cc} 0 & 3/26 \\ -1 & 0 \end{array})$ | $\frac{1}{2}(11 - 3\sqrt{13})$ |
| 35+35 | $i\sqrt{\frac{2}{35}}$ | $(\begin{array}{cc} 0 & -1 \\ 35 & 0 \end{array}) \tau_0 = (\begin{array}{cc} 1 & 0 \\ 0 & 2 \end{array}) \tau_0$ | $(\begin{array}{cc} 0 & 2/35 \\ -1 & 0 \end{array})$ | $\sqrt{10} - 3$ |
| 39+39 | $i\sqrt{\frac{2}{39}}$ | $(\begin{array}{cc} 0 & -1 \\ 39 & 0 \end{array}) \tau_0 = (\begin{array}{cc} 1 & 0 \\ 0 & 2 \end{array}) \tau_0$ | $(\begin{array}{cc} 0 & 2/39 \\ -1 & 0 \end{array})$ | $3 - 2\sqrt{2}$ |
Table 5. The coefficients $B, C$ of $1/\pi$ series.

| $\Gamma$ | $B$ | $C$ |
|----------|-----|-----|
| 14+7     | $\frac{3}{2} \sqrt{4 - \frac{3\sqrt{7}}{2}}$ | $\frac{1}{14} \left( 588 - 223\sqrt{7} + 13\sqrt{889} - 336\sqrt{7} \right) \sqrt{4 + \frac{3\sqrt{7}}{2}}$ |
| 14+14    | $8 \sqrt{\frac{6}{527 + 115\sqrt{21}}}$ | $\frac{4(747 + 163\sqrt{21})}{(23 + 5\sqrt{21})^2} \sqrt{\frac{2}{3}(527 - 115\sqrt{21})}$ |
| 15+15    | $2\sqrt{6(99 - 70\sqrt{2})}$ | $6\sqrt{3(99 - 70\sqrt{2}) + 2(-536 + 379\sqrt{2})\sqrt{3(99 + 70\sqrt{2})}}$ |
| 16+      | $2(-2 + \sqrt{6})\sqrt{15 - 6\sqrt{6}}$ | $2(-12 + 5\sqrt{6})\sqrt{\frac{1}{3}(5 - 2\sqrt{6})}$ |
| 20+20    | $-16(-2 + \sqrt{3}) \cdot \sqrt{3(97 - 56\sqrt{3})}$ | $4 \left( 14\sqrt{97 - 56\sqrt{3}} - 7\sqrt{3(97 - 56\sqrt{3})} + 3(-3064 + 1769\sqrt{3}) \sqrt{97 + 56\sqrt{3}} \right)$ |
| 21+21    | $\frac{4(-2 + \sqrt{6})}{3 \cdot \sqrt{98 - 40\sqrt{6}}}$ | $\frac{2}{3} \left( -26\sqrt{147 - 60\sqrt{6}} + 39\sqrt{98 - 40\sqrt{6}} + (-7035\sqrt{2} + 5744\sqrt{3}) \sqrt{49 + 20\sqrt{6}} \right)$ |
| 22+11    | $\sqrt{39 - \frac{45\sqrt{3}}{2}}$ | $\frac{1}{4} \left( 7\sqrt{52 - 30\sqrt{3}} + 3(-149 + 86\sqrt{3}) \sqrt{52 + 30\sqrt{3}} \right)$ |
| 26+26    | $12\sqrt{-8574 + 2378\sqrt{13}}$ | $2(-41828 + 11601\sqrt{13}) \sqrt{8574 + 2378\sqrt{13}}$ |
| 35+35    | $2\sqrt{14(721 - 228\sqrt{10})}$ | $2(2\sqrt{2} - \sqrt{5}) \sqrt{7(721 - 228\sqrt{10})}$ |
| 39+39    | $-4(-2 + \sqrt{2}) \cdot \sqrt{6(577 - 408\sqrt{2})}$ | $44\sqrt{3(577 - 408\sqrt{2}) - 22\sqrt{6(577 - 408\sqrt{2})}} + 2(-47420 + 33531\sqrt{2}) \sqrt{3(577 + 408\sqrt{2})}$ |
SOME NEW RAMANUJAN-SATO SERIES FOR $1/\pi$

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