Survival exponents for some Gaussian processes

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Abstract. The problem is a power-law asymptotics of the probability that a self-similar process does not exceed a fixed level during long time. The exponent in such asymptotics is estimated for some Gaussian processes, including the fractional Brownian motion (FBM) in \((-T_-, T)\), \(T \geq T_- >> 1\) and the integrated FBM in \((0,T)\), \(T >> 1\).

1. The problem

Let \(x(t), x(0) = 0\) be a real-valued stochastic process with the following asymptotics:

\[
P(x(t) < 1, t \in \Delta_T) = T^{-\theta_x + o(1)}, \ T \to \infty,
\]

where \(\theta_x\) is the so-called survival exponent of \(x(t)\). Below we focus on estimating \(\theta_x\) for some self-similar Gaussian processes in extended intervals \(\Delta_T = (0,T)\) and \((-T_-, T)\), \(T \geq T_- >> 1\). Usually the estimation of the survival exponents is based on Slepian’s lemma. To do this we need reference processes with explicit or almost explicit values of \(\theta\). Unfortunately, the list of such processes is very short. This includes the fractional Brownian motion (FBM), \(w_H(t)\), of order \(0 < H < 1\) both with one- and multidimensional time. According to Molchan (1999)

\[
\theta_{w_H} = 1 - H \text{ for } \Delta_T = (0,T) \quad \text{and} \quad \theta_{w_H} = d \text{ for } \Delta_T = (-T,T)^d.
\]

Another important example is the integrated Brownian motion \(I(t) = \int_0^t w(s)ds\) with the exponent

\[
\theta_I = 1/4, \quad \Delta_T = (0,T) \quad (\text{Sinai, 1992})
\]

The nature of this result is best understood in terms of a series of generalizations where the integrand is random walk with discrete or continuous time (see, e.g., Isozaki and Watanabe, 1994; Isozaki and Kotani, 2000; Simon, 2007; Vysotsky, 2010; Aurzada and Dereich, 2011; Dembo and Gao, 2011). The extension of (3) to include the case of the integrated fractional Brownian motion, \(I_H(t) = \int_0^t w_H(s)ds\), remains an important, but as yet unsolved problem.

Below we consider the survival exponents for the following Gaussian processes:

\(I_H(t), t \in (0,T); \chi_H(t) = \text{sign}(t)w_H(t), t \in (-T,T); \text{FBM in } \Delta_T = (-T^\alpha, T)\), \(0 \leq \alpha \leq 1\); the Laplace transform of white noise with \(\Delta_T = (0,T)\), and the fractional Slepian’s stationary process whose correlation function is \(B_{\tilde{H}}(t) = (1 - |t|^{2H})\), \(0 < H \leq 1/2\).

Our approach to estimation of \(\theta_I\) is more or less traditional. Namely, any self-similar process \(x(t)\) in \(\Delta_T = (0,T)\) generates a dual stationary process \(\tilde{x}(s) = e^{-hs}x(e^s)\), \(s < \ln T := \tilde{T}\), where \(h\) is the self-similarity index of \(x(t)\). For a large class of Gaussian processes, relation (1) induces the dual asymptotics...
\[ P(\tilde{x}(s) \leq 0, 0 < s < \tilde{T}) = \exp(-\tilde{\theta}_x \tilde{T}(1 + o(1))) \quad \tilde{T} \to \infty \]  

(4)

with the same exponent \( \tilde{\theta}_x = \theta_x \) (Molchan, 1999, 2008). More generally, the dual exponent is defined by the asymptotics

\[ P(x(t) \leq 0, t \in \Delta_T \setminus (-1,1)) = \exp(-\tilde{\theta}_x \tilde{T}(1 + o(1))) . \]

To formulate the simplest condition of the exponent equality, we define one more exponent \( \tilde{\theta}_x \) by means of the asymptotics

\[ P(|x_t^*| \leq 1) = T^{-\tilde{\theta}_x+o(1)} , \]

where \( x_t^* \) is the position of the maximum of \( x(t) \) in \( \Delta_T \), i.e., \( x(t^*_T) = \sup(x(t), t \in \Delta_T) \).

**Lemma 1.** Let \( x(t), x(0) = 0 \) be a self-similar continuous Gaussian process in \( \Delta_T = (-T_-, T), \ T_- \leq T \), and \( (H_x(\Delta_T), \| . \|_1 \) be the reproducing kernel Hilbert space associated with \( x(t) \). Suppose there exists such an element \( \varphi_x \) of \( H_x(\Delta_T) \) that \( \varphi_x(t) \geq 1, \| t \| > 1 \) and \( \| \varphi_x \|^2_1 = o(\ln T) \). Then \( \theta_x, \tilde{\theta}_x \), and \( \tilde{\theta}_x \) can exist simultaneously only; moreover, the exponents are equal to each other.

The equality \( \theta = \tilde{\theta} \) reduces the original problem to the estimation of \( \tilde{\theta} \). Non-negativity of the correlation function of \( \tilde{x}(s) \) guarantees the existence of the exponent \( \tilde{\theta} \) (Li & Shao, 2004). In turn, the inequality of two correlation functions, \( B_t(s) \leq B_{2t}(s) \) , \( B_t(0) = 1 \), implies, by Slepian’s lemma, the inverse inequality for the corresponding exponents: \( \tilde{\theta}_1 \geq \tilde{\theta}_2 \).

An essentially different approach is required to find the explicit value of \( \tilde{\theta} \) for FBM in \( \Delta_T = (-T^a, T) \) and to estimate \( \tilde{\theta} \) in (4) for the fractional Slepian process with a small H parameter.

### 2. Examples

#### 2.1. Integrated fractional Brownian motion

Consider the process

\[ I_H(t) = \int_0^t w_H(s)ds , \]

where \( w_H(t) \) is the fractional Brownian motion, i.e., a Gaussian random process with the stationary increments: \( E[w_H(t) - w_H(s)]^2 = |t - s|^{2H} \), \( w_H(0) = 0 \). Molchan and Khokhlov (2003, 2004) analyzed theoretically and numerically the exponent \( \theta_{I_H} \) in the general case of \( H \) and formulated the following

**Hypothesis:** \( \theta_{I_H} = H(1-H) \) for \( \Delta_T = (0,T) \) and \( \theta_{I_H} = 1 - H \) for \( \Delta_T = (-T,T) \).

The unexpected symmetry \( \theta_{I_H} = \theta_{I_{-H}} \) of the exponents for \( \Delta_T = (0,T) \) caused some doubt as to the numerical results. To support the hypothesis, Molchan (2008) derived the following estimates:

\[ \rho H(1 - H) \leq \theta_{I_H}^{+} \leq \theta_{I_H}^{-} \leq (1 - H) , \]  

(5)
where \( \rho \) is a small constant and (+) and (−/+) are indicators of the intervals \( \Delta_T = (0, T) \) and \( \Delta_T = (-T, T) \), respectively. Note that, in the case of \( H < 1/2 \) and \( \Delta_T = (-T, T) \), it is unknown whether the exponent exists. In such cases we have to operate with upper \( \bar{\theta} \) and lower \( \bar{\theta} \) exponents. Therefore, \( \theta_T^{−/+/} \) in (5) for \( H < 1/2 \) is any number from the interval \((\bar{\theta}, \bar{\theta})\). The relation (5) can be improved as follows:

**Proposition 1.** For the intervals \( \Delta_T = (0, T) \)

(a) \( \theta_T \geq \theta_{T-h} \), \( 0 < H \leq 0.5 \),

(b) \( 0.5(\hat{H} \wedge \hat{H}) \leq \theta_T \leq \hat{H} \), \( \hat{H} = 1 - H \),

(c) \( \theta_T \leq \sqrt{(1 - (H \wedge \hat{H})^2)}/12 \).

**Proof.** The identity of dual exponents for \( I_H(t) \) follows from (Molchan and Khokhlov, 2004); the dual survival exponent exists, because the dual correlation function

\[
\widetilde{B}_{\tilde{t}_H}(s) = (2 + 4H)^{-1}[(2 + 2H)(e^{th} + e^{-th}) - e^{(H+H)s} - e^{-(H+H)s} + (e^{i/2} - e^{-i/2})^2H^2],
\]

is positive. The inequality (a) is a consequence of the relation

\[
\widetilde{B}_{\tilde{t}_H}(t) \leq \widetilde{B}_{\tilde{t}_{1-H}}(t), \quad 0 < H \leq 1/2.
\]

To prove (b, c), we use the correlation function of the process \( \tilde{I}_{1/2}(ps) \), i.e.,

\[
\widetilde{B}_{\tilde{t}_{1/2}}(ps) = 1/2(3\exp(-ps|s|/2) - \exp(-3ps|s|/2)),
\]

and the respective exponent \( \widetilde{\theta} = p/4 \) (see (3)). The relation

\[
\widetilde{B}_{\tilde{t}_H}(t) \leq \widetilde{B}_{\tilde{t}_{1/2}}(pt), \quad H \geq 1/2, \quad p = 2(1 - H)
\]

implies \( \theta_T \geq (1 - H)/2 \) for \( H \geq 1/2 \). Using (a) in addition, we come to the lower bound in (b) because \( \theta_{T-h} \geq \theta_{T-h} \geq H/2 \) for \( H \leq 1/2 \).

Similarly, the relation

\[
\widetilde{B}_{\tilde{t}_H}(t) \geq \widetilde{B}_{\tilde{t}_{1/2}}(pt), \quad H \leq 1/2, \quad p = 2\sqrt{(1 - H^2)/3}
\]

implies (c) for all \( H \). A test of the pure analytical facts (7, 9, and 10) is given in Appendix.

**Remark 1.** The proposition 1a follows from the more informative relation

\[
P(I_H(s) \leq 0, s \in (0, \tilde{T})) \leq P(I_{1-H}(s) \leq 0, s \in (0, \tilde{T})).
\]

This inequality is important for understanding the numerical result by Molchan and Khokhlov(2003) represented in the form of the empirical estimates of \( \tilde{\theta}_{T-h} \) in Figure 1. We can see that the empirical estimates show small but one-sided deviations from the hypothetical curve \( \theta = H(1 - H) \) before and after
The signs of these deviations are consistent with (11), while the amplitudes are compatible with the model

\[
P(\bar{T}_a(s) \leq 0, s \in (0, \bar{T})) \approx C_{\bar{T}^a(H)} \exp(-H (1-H) \bar{T}), \quad \bar{T} \gg 1, \quad \text{sgn} \, \alpha(H) = \text{sign}(H - 0.5),
\]

and \( \alpha(H) = H - 0.5 \) (see more in Molchan and Khokhlov, 2003).

### 2.2 Laplace transform of white noise.

Consider the process \( L(t) = \int_0^t e^{w(t)} \, dt \), where \( w(u) \) is Brownian motion. The dual stationary process \( \tilde{L}(s) \) has the correlation function \( \tilde{B}_L(s) = 1/\cosh(s/2) \). Using (8) as a majorant of \( \tilde{B}_L(s) \), we improve the lower bound of \( \tilde{\theta}_L \) as follows:

**Proposition 2.** \( 3^{-1/2} \leq 4 \tilde{\theta}_L \leq 1 \).

**Proof.** The exponent equality for the dual processes \( L \) and \( \tilde{L} \) follows from Lemma 1 with \( \phi_L(t) = (1 + \epsilon_L)/(t + \epsilon_L) \), where \( \epsilon_L = 1/\sqrt{\ln T} \). For indeed, \( \phi_L(t) = E_L(t) \eta \), where \( \eta = (1 + \epsilon_L^{-1}) L(\epsilon_L) \). By definition of the Hilbert space \( H_e(\Delta_L) \), we have the desired estimate:

\[
\|\phi_L\|^2 = E \eta^2 = (\epsilon_L^{-1} (\epsilon_L + 1)^2 / 2 = O(\sqrt{\ln T}).
\]

By (3) and Slepian’s lemma, the relation

\[
\tilde{B}_{h_2/2} (t) \leq \tilde{B}_L(pt), \quad p \leq 1
\]

has as a consequence the estimate \( 4p \tilde{\theta}_L \leq 1 \). The opposite inequality

\[
\tilde{B}_{h_2/2} (t) \geq \tilde{B}_L(pt), \quad p^2 \geq 3
\]

implies \( 4p \tilde{\theta}_L \geq 1 \). The test of (13, \( p = 1 \)) and (14, \( p = 2 \)) is very simple and yields the Li and Shao (2004) estimates: \( 0.5 < 4 \tilde{\theta}_L < 1 \). The Appendix contains a proof of (13, 14) for all interesting values of \( p : 1, 2, \) and \( \sqrt{3} \).

**Remark 2.** The dual survival exponent of \( L(t) \) is of interest as a parameter of the following asymptotic relation

\[
P(\sum_{i=1}^{2n} \xi_i x_i \neq 0, x \in R^1) = (2n)^{-4 \tilde{\theta}_L + o(1)}, \quad n \to \infty
\]

for random polynomials with the standard Gaussian independent coefficients (Dembo et al., 2002). A continuous analogue of the polynomial on any of four intervals \( 0 < x \leq 1 \) is the Laplace transform of white noise that partially explains the appearance of \( \tilde{\theta}_L \) in the asymptotic relation (15). Simulations suggest \( 4 \tilde{\theta}_L = 0.76 \pm 0.03 \) (Dembo et al, 2002) and \( 4 \tilde{\theta}_L \approx 0.75 \) (Newman and Loinaz, 2001).

### 2.3 Fractional Slepian’s process.

We reserve this term for a Gaussian stationary process \( S_H(t) \) with correlation function

\[
H = 1/2.
\]
\[ B_{S_H}(t) = (1 - \|\|^2 H)_{+}, \quad 0 < H \leq 1/2, \]  \hspace{1cm} (16)

because \( S_{1/2}(t) \) is known as the Slepian process and \( S_H(t) - S_H(0), 0 < t \leq 1, \) is equal in distribution to the fractional Brownian motion on the interval \((0,1)\). By the Polya criterion, the fractional Slepian process exists because \( B_{S_H}(t) \) is a non-increasing and convex function on the semi-axis \( t \geq 0 \). The fact of the correlation function being non-negative guarantees the existence of the exponent \( \tilde{\vartheta}_{S_H} \) in (4). \( S_H(t) \) can be useful as a reference process in estimation of the survival exponents. Therefore it is important to have accurate estimates of the exponent for \( S_H(t) \). The case of small \( H \) is the most interesting because it describes a transition of \( S_H(t) \) to white noise. Our estimates of \( \tilde{\vartheta}_{S_H} \) are based on two lemmas, where we use the following notation

\[ \tilde{\vartheta}(f, \Delta) = -|\Delta|^{-1} \log P(x(t) \leq f(t), t \in \Delta). \]  \hspace{1cm} (17)

**Lemma 2.** (Li and Shao, 2004). Let \( x(t) \) be a centered Gaussian stationary process with a finite non-negative correlation function, i.e., \( B_x(t) \geq 0 \) and \( B_x(t) = 0 \) for \( |t| \geq T_0 \). Then the limit

\[ \tilde{\vartheta}(a) = \lim_{T \to \infty} \tilde{\vartheta}(a, (0, T)) \]

exists for every \( a \in R^1 \). Moreover,

\[ (1 + 1/k)^{-1} \tilde{\vartheta}(a, k\Delta_0) \leq \tilde{\vartheta}(a) \leq \tilde{\vartheta}(a, k\Delta_0), \quad \Delta_0 = (0, T_0). \]  \hspace{1cm} (18)

**Remark 3.** Lemma 1 was derived by Li and Shao (2004) for the Slepian process, \( S_{1/2}(t) \), but the proof remains valid for the general case. There is an explicit but very complicated formula for \( \tilde{\vartheta}_{S_H}(0, \Delta) \) with \( H = 1/2 \) (Shepp, 1971). In case of \( \Delta = (0,2) \), the Shepp result reduces to

\[ P(S_{1/2}(t) \leq 0, t \in (0,2)) = 1/6 - (2 + \sqrt{3})/(8\pi), \]

and gives \( 1.336 < \tilde{\vartheta}_{S_{1/2}} < 2.004 \).

**Lemma 3.** (Aurzada&Dereich, 2011). Let \( x(t) \) be a centered Gaussian process in an interval \( \Delta \) with a correlation function \( B(t,s) \) and \( (H_x(\Delta), \|\|_1) \) be the Hilbert space with the reproducing kernel \( B(t,s) \) on \( \Delta \times \Delta \). If \( 0 < \tilde{\vartheta}(a, \Delta) < \infty \), then

\[ \sqrt[2]{\tilde{\vartheta}(a+f,\Delta)} - \sqrt[2]{\tilde{\vartheta}(a,\Delta)} \leq \|f\|_1 / \sqrt[2]{2|\Delta|}. \]  \hspace{1cm} (19)

**Remark 4.** Lemma 3 is a version of Proposition 1.6 from the paper by Aurzada and Dereich (2011); relation (19) successfully supplements the original Lemma 1.

**Proposition 3.** The persistence exponent of process \( S_H(t) \) has the following estimates

\[ (1 - H)H^{-1} \log 1/(2H) \leq \tilde{\vartheta}_{S_H} \leq 49H^{-2}, \]  \hspace{1cm} (20)

where the left inequality holds for \( 0 < H \leq e^{-2}/2 \).
**Corollary:** odd component of the fractional Brownian motion.

Consider \( w_H(t) = (w_H(t) - w_H(-t)) / 2 \). Its dual stationary process \( \widetilde{w}_H \) has the following correlation function:

\[
\widetilde{B}_{\widetilde{w}_H}(t) = (\cosh \frac{t}{2})^{2H} - (\sinh \frac{t}{2})^{2H}.
\]

The exponent \( \widetilde{\theta}_{\widetilde{w}_H} \) exists because \( \widetilde{B}_{\widetilde{w}_H}(t) \geq 0 \). By comparing \( \widetilde{B}_{\widetilde{w}_H}(t) \) with \( \widetilde{B}_{\widetilde{w}_H}(pt) \), Krug et al (1997) estimated the exponent as follows:

\[
\begin{align*}
\widetilde{\theta}_{\widetilde{w}_H} &\geq \min((1 - H)^2 / H, (1 - H)^{2(1/H) - 1}), \quad 0 < H < 0.5, \\
\widetilde{\theta}_{\widetilde{w}_H} &
\leq (1 - H)^2 / H, \quad 0.1549 < H < 0.5.
\end{align*}
\]

For small \( H \) these estimates are one-sided only. The following inequality

\[
\widetilde{B}_{\widetilde{w}_H}(2t) = (\cosh t)^{2H} (1 - (\tanh t)^{2H}) \geq (\cosh t)^{2H} (1 - |t|^{2H})_+ \geq \widetilde{B}_{\widetilde{w}_H}(t)
\]

and Proposition 3 immediately yield

\[
\widetilde{\theta}_{\widetilde{w}_H} \leq (7 / H)^2 / 2, \quad 0 < H < 0.5.
\]

**Remark 5.** A considerable difference in the behavior of \( \widetilde{\theta}_{\widetilde{w}_H} \) and \( \widetilde{\theta}_{\widetilde{w}_H} = 1 - H \) for small \( H \) is expected. Heuristically this can be explained as follows. As \( H \to 0 \), the discrete processes \( \widetilde{w}_H(k\Delta) \) and \( \widetilde{w}_H(k\Delta) \) have different weak limits: \( \{\xi_k\} \) and \( \{(\xi_k - \eta) / \sqrt{2}\} \), respectively, where \( \{\xi_k\} \) and \( \eta \) are independent standard Gaussian variables. The probability (4) for the limiting processes are quite different:

\[
P\{\xi_k < 0, k = 1 \div N\} = 2^{-N} \quad \text{and} \quad P\{\xi_k - \eta < 0, k = 1 \div N\} = (N + 1)^{-1}.
\]

Unfortunately, this argument is insufficient to predict the behavior of \( \widetilde{\theta}_{\widetilde{w}_H} \) for small \( H \), because the step \( \Delta \) cannot be arbitrary and is a function of \( H \).

**2.4 Khanin’s problem.**

The survival exponent for fractional Brownian motion in the intervals \( \Delta_T = (-T, T) \) is independent of the parameter \( H : \theta_{\widetilde{w}_H} = 1 \). This interesting fact follows from both self-similarity of \( w_H \) and the stationarity of its increments (Molchan, 1999).

In the case \( H < 0.5 \), variables \( w_H(t) \) and \( w_H(-t) \) are positive correlated. Therefore, a possible power-law asymptotics

\[
P(w_H(t) < 1, w_H(-t) < 1, t \in (0, T)) = T^{-\theta_{w_H}(1)},
\]

where we change sign before \( w_H(t) \) for negative \( t \) only, may have a radically different exponent compared with \( \theta_{\widetilde{w}_H} = 1 \). The question of finding bounds on the exponent \( \theta_{\widetilde{w}_H} \) for the process
\[
\chi_H(t) = \text{sign}(t)w_H(t) , \quad \Delta_T = (-T, T)
\]
was asked by K. Khanin. The next proposition contains a partial answer to this question.

**Proposition 4.** 1. In the case \(0.5 \leq H < 1\), the exponent \(\theta_{\chi_H}\) for \(\Delta_T = (-T, T)\) exists and admits of the following estimates

\[
1 < \theta_{\chi_H} (1-H)^{-1} \leq 2 , \quad 0.5 \leq H < 1 ,
\]
in addition \(\theta_{\chi_{1/2}} = 1\).

2. Let \(\theta_{w_H}\) be the lower exponent in (22); then

\[
\theta_{\chi_H} (1-H)^{-1} \geq (H^{-1} - 1) \wedge 2^{1/(2H)-1} , \quad 0 < H < 0.25 ,
\]
\[
\theta_{w_H} (1-H)^{-1} \geq 2 , \quad 0.25 < H \leq 0.5 .
\]

**Remark 6.** To clarify why \(\theta_{\chi_H} / \theta_{w_H}\) is unbounded for small \(H\) in the case \(\Delta_T = (-T, T)\), we consider again the limiting sequence for \(w_H(k\Delta)\) as \(H \to 0\). This is \(\{(\xi_k - \xi_0) / \sqrt{2}\}\), where the \(\{\xi_k\}\) are independent standard Gaussian variables. The probability (1) for the limit sequence is

\[
P\{\xi_k < \xi_0 + \sqrt{2}, |k| \leq N\} = (2N + 1)^{-1} l(N),
\]
where \(l(N)\) is a slowly varying function, whereas for the limit sequence of \(\chi_H(k\Delta)\) we have

\[
P\{\xi_k - \sqrt{2} < \xi_0 < \xi_k + \sqrt{2}, 0 < k \leq N\} \approx \sqrt{\pi}eN^{-1/2}\Phi(\sqrt{2})^{2N} ,
\]
where \(\Phi(x)\) is the Gaussian distribution function. As in Remark 5, we have non-trivial exponential asymptotics where the threshold for \(\{\xi_k\}\) is constant or bounded. Indeed, the event in (23) yields the inequality

\[
|\xi_0| < \sqrt{2} + \max\left(\sum_{k=1}^{N} |\xi_k|, \sum_{k=1}^{N} |\xi_k| \right) / N = \sqrt{2} + O(1) / \sqrt{N}.
\]

**2.5 Explicit value of \(\theta_{w_H}\).**

We have two explicit but isolated results for the fractional Brownian motion: \(\theta_{w_H} = (1-H)\) for \(\Delta_T = (0, T)\) and \(\theta_{w_H} = 1\) for \(\Delta_T = (-T, T)\). These results can be combined as follows:

**Proposition 5.** If \(\Delta_T = (-T^\alpha , T)\), \(0 \leq \alpha \leq 1\), then \(\theta_{w_H} = \alpha H + (1-H)\).

**Remark 7.** The result is based on the following properties of the position \(t_\Delta^*\) of maximum of \(w_H(t)\) in \(\Delta = [0,1]: t_\Delta^*\) has continuous probability density \(f_\Delta^* (t)\) in \((0, 1)\) and \(f_\Delta^* (t) \approx O(t^{-H})\) as \(t \to 0\). In the case of multidimensional time, the behavior of \(f_\Delta^* (t, \Delta = (0,1)^d\), near \(t = 0\) is a key to the survival exponent for \(w_H(t)\) in \(\Delta_T = (-T^\alpha , T)^d\), if \(0 < \alpha < 1\) and \(H < 1\). By (2), \(\theta_{w_H} = d\) in the case \(\alpha = 1\), and \(\theta_{w_H} = \alpha d\) in the degenerate case: \(H = 1\).
3. Proofs.

Proof of Proposition 3.

Lower bound. Let \( \tilde{w}_H(t) \) be the dual fractional Brownian motion with the parameter \( H \), i.e., a Gaussian stationary process with correlation function \( \tilde{B}_{wH}(t) = \cosh(Ht) - 0.5(2 \sinh(t/2)^2)^H \). We prove in the Appendix that for \( 0 < H \leq e^{-2}/2 \),

\[
\tilde{B}_{wH}(pt) \geq B_S(t), \quad p = -H^{-1} \ln(2H) .
\]  

(24)

Applying Slepian's lemma, one has \( \tilde{\theta}_{S_H} \geq p(1 - H) \) because \( \tilde{\theta}_{wH} = (1 - H) \).

Upper bound. The random variable \( \eta = \int_0^1 S_H(t)dt \) corresponds to an element \( f_\eta(t) \) of the Hilbert space, \( H_\Delta, \Delta = (0,1) \), with the reproducing kernel \( B(t,s) = 1 - |t - s|^{2H} \). By definition of \( H_\Delta \), we have

\[
f_\eta(t) = ES_H(t)\eta = 1 - \left(t^{1+2H} + (1 - t)^{1+2H}\right)/(1 + 2H) ,
\]

\[
\|f_\eta\|_S^2 = E\eta^2 = H(3 + 2H)(1 + H)^{-1}(1 + 2H)^{-1} .
\]

It is easy to see that \( f_\eta(0) \leq f_\eta(t) \leq f_\eta(1/2) \). Therefore,

\[
H < f_\eta(t) < 2H \ln(2e) \quad \text{and} \quad 4H/3 < \|f_\eta\|_S^2 < 3H .
\]

(25)

Let \( m_H \) be the median of the random variable \( M = \max \{S_H(t), t \in \Delta\} \), where \( \Delta = (0,1) \). Then

\[
0.5 = P(S_H(t) < m_H, t \in \Delta) < P(S_H(t) < m_H H^{-1} f_\eta(t), t \in \Delta) ,
\]

because \( H^{-1} f_\eta(t) > 1 \). Setting \( x(t) = S_H(t) \) in Lemma 2 and using notation (17), one has

\[
\tilde{\theta}(m_H H^{-1} f_\eta(t), \Delta) < \ln 2
\]

and

\[
\sqrt{\tilde{\theta}(0, \Delta)} < \sqrt{\ln 2 + m_H H^{-1}\|f_\eta\|_S} / \sqrt{2} .
\]

Using Lemma 1 and the inequality \( \|f_\eta\|_S < \sqrt{3H} \), we have

\[
\tilde{\theta}_{S_H} < \tilde{\theta}(0, \Delta) < (\sqrt{\ln 2 + m_H \sqrt{1.5/H}})^2 .
\]

It is well known (see, e.g., Lifshits, 1995) that \( m_H < 4\sqrt{2} D(\Delta, \sigma/2) \), where \( \sigma^2 = \max_x ES_H(t) \) and \( D \) is the Dudley entropy integral related to the semi-metrics on \( \Delta \): \( \rho^2(t,s) = E(S_H(t) - S_H(s))^2 \).

In our case \( \rho(t,s) = \sqrt{2}|t - s|^H \), \( \sigma = 1 \) and therefore

\[
m_H < c_H / \sqrt{H} ,
\]

where
\[ c_H = 4\sqrt{(1 - H)\ln 2 + 2^{3-H}\sqrt{\pi\Phi(-\sqrt{1 - H} \ln 4)} < 5.36 \quad , \quad H < 1/2 \],
and \( \Phi(x) \) is the standard Gaussian distribution. Hence,
\[ \tilde{\theta}_{x_H} < (\sqrt{\ln 2 + 5.36}\sqrt{1.5 / H}) < (7 / H)^2. \]

**Proof of Proposition 4.**

**Proposition 4.1.** In the case of \( H \geq 0.5 \), the process \( \chi_H(t) = \text{sign}(t)w_H(t) \) has non-negative correlations on \( R^1 \). In the standard manner, this implies the existence of \( \theta_{\chi_H} \) for \( \Delta_T = (-T, T) \). More precisely, starting from a self-similar 2-D process \( x(t) = (w_H(t), -w_H(-t)) \) on \( R^1 \), we consider the dual 2-D stationary process \( \tilde{x}(t) = x(t)\exp(-Ht) \) whose correlation matrix has positive elements. Therefore, by Li and Shao (2004), we conclude that the exponent \( \tilde{\theta}_{\chi_H} \) for \( \tilde{x}(t) \) exists.

**Equality \( \tilde{\theta}_{\chi_H} = \theta_{\chi_H} \) for \( \Delta_T = (-T, T) \).** We will use Lemma 1. By the relation \( \chi_H(t) = \text{sign}(t)w_H(t) \), the map \( \phi(t) \mapsto \text{sign}(t)\phi(t) \) is an isometry between the reproducing kernel Hilbert spaces \( H_{\chi_H}(\Delta_T) \) and \( H_{w_H}(\Delta_T) \) associated with \( \chi_H(t) \) and \( w_H(t) \) on \( \Delta_T = (-T, T) \), respectively. To prove that the dual exponents are equal, it is enough to find \( \phi(t) \in (H_{w_H}(R^1), \| \cdot \|_{w_H}) \) such that \( \text{sgn}(t)\phi(t) \geq 1 \) for \( |t| \geq 1 \). We can use
\[ \phi(t) = \text{sgn}(t) \min(1, |t|) = \int (e^{\|t\|\lambda} - 1) \frac{\sin \lambda}{\pi \lambda^2} d\lambda, \]
because
\[ \|\phi\|^2_{w_H} = k_H \int \left( \frac{\sin \lambda}{\lambda^2} \right)^2 |\lambda|^{2H} d\lambda < \infty, \]
(see Molchan and Khokhlov, 2004).

**Estimation of \( \theta_{\chi_H}, H > 1/2 \).** Since \( E\chi_H(t)\chi_H(s) \geq 0 \) for any \( t, s \), we have, by Slepian's lemma,
\[ p_T := P(w_H(t) < 1, -w_H(-t) < 1, t \in (0, T)) \geq [P(w_H(t) < 1, t \in (0, T))]^2. \]
Using (2), one has \( \theta_{\chi_H} \leq 2(1 - H) \).

Obviously, \( p_T \leq P(w_H(t) < 1, t \in (0, T)) \). Therefore, \( \theta_{\chi_H} \geq (1 - H) \) for any \( H \).

**Proposition 4.2.** Let \( 0 < H \leq 1/2 \), then \( Ew_H(t)(-w_H(-s)) \leq 0 \) for \( t, s > 0 \). Hence,
\[ p_T \leq [P(w_H(t) < 1, t \in (0, T))]^2 \quad \text{and} \quad \theta_{\chi_H} \geq 2(1 - H) \).

Finally,
\[ p_T \leq P(w_H(t) - w_H(-t) < 2, t \in (0, T)) = P(w^-_H(t) < 1, t \in (0, T)). \]
But then, \( \tilde{\theta}_{\chi_H} \geq \theta_{w_H} \) for all \( H \). If \( \tilde{\theta}_{w_H} = \tilde{\theta}_{w^-_H} \), then we get the lower bound of \( \tilde{\theta}_{\chi_H} \) for \( 0 < H \leq 1/4 \).
The equality \( \tilde{\theta}_{w_H} = \tilde{\theta}_{-w_H} \). Let \( H_{w_H}(\Delta) \) and \( H_{-w_H}(\Delta) \) be the reproducing kernel Hilbert spaces associated with \( w_H(t) \) and \( w_H(t) \), respectively. By the definition of \( w_H(t) \), the map \( (\varphi(t), t > 0) \mapsto (\text{sign}(t)\varphi(t), |t| < \infty) \) is an isometric embedding of \( H_{w_H}(R^1_+) \) in \( H_{w_H}(R^1) \). To prove that the exponents are equal, it is enough to find \( \varphi(t), t \geq 0 \) such that \( \text{sign}(t)\varphi(t) \in (H_{w_H}(R^1), \| \cdot \|_{w_H}) \), \( \varphi(t) \geq 1 \) for \( t \geq 1 \), and \( \| \varphi \|_{w_H} < \infty \). As we showed above, this can be \( \varphi(t) = \min(t,1), t > 0 \).

**Proof of Proposition 5.** Consider the fractional Brownian motion in \( \Delta_{T} = (-T^\alpha, T), \ 0 \leq \alpha \leq 1 \). By lemma 1, we can focus on the exponent related to the position of the maximum of \( w_H(t) \) in \( \Delta_{T}, T_{\star T} \).

**Distribution of \( \Delta_{T} \).** We remind the main properties of the distribution function, \( F^*(x) \), of \( \Delta_{T} \) related to the normalized interval \( \Delta = (0,1) \) (see Molchan, 1999; Molchan and Khokhlov, 2004):

- \( F^*(x) \) has continuous density \( f^*_\Delta(x) > 0, 0 < x < 1 \) such that \( (1 - x)f^*_\Delta(x) \) decreases and \( xf^*_\Delta(x) \) increases on \( \Delta \);

- \( F^*(x) \) have the following estimates:

\[
x^{1-H}l^{-1}(x) \leq F^*(x) \leq x^{1-H}l(x),
\]

where \( l(x) = \exp(c\sqrt{-\ln x}), c > 0 \).

Due to monotonicity of \( (1 - x)f^*_\Delta(x) \) and \( xf^*_\Delta(x) \), one has

\[
(1 - x)f^*_\Delta(x) \leq x^{-1}\int_0^x (1 - u)f^*_\Delta(u)du \leq x^{-1}F^*(x),
\]

\[
xf^*_\Delta(x) \geq x^{-1}\int_x^1 uf^*_\Delta(u)du \geq q(F^*(x) - F^*(xq)), 0 < q < 1.
\]

By (26, 27),

\[
f^*_\Delta(x) \leq x^{-H}l(x)(1 - x)^{-1}.
\]

Using (26, 28), one has

\[
f^*_\Delta(x) \geq qx^{-H}l^{-1}(x)(1 - l(x))(xq)q^{1-H}.
\]

If we set \( q^{1-H} = l^{-1}(x) / 2 \), then

\[
f^*_\Delta(x) \geq qx^{-H}l^{-1}(x) / 2 = c_Hx^{-H}l^{-\nu_H}(x),
\]

where \( \nu_H = (3 - H)/(1 - H), \ c_H = 2^{-(2-H)/(1-H)}. \)

**Distribution of \( \Delta_{T'} \).** Let \( T' = T_+ + T \), where \( T_+ = T^\alpha \), then the following processes

\( w_H(T' \tau - T_+) - w_H(-T_+) \) and \( w_H(T' \tau_{T'}) \) on \( \Delta = (0,1) \) are equal in distribution. Hence, \( \Delta_{T'} \) and \( T'f^*_\Delta - T_- \) have the same distribution as well. Therefore,
\[ p_T := P\left( \frac{1}{\sqrt{\lambda}} \leq 0.5 \right) = P\left( \frac{f_\lambda^+ (T_+ \pm \varepsilon)}{T_1} \leq 0.5 / T_1 \right) = T_1^{-1} f_\lambda^+ (T_+ + \varepsilon) / T_1, \]  

where \( \varepsilon \leq 0.5 \). We have used here the existence and continuity of \( f_\lambda^+ (x) \).

Exponent \( \bar{\theta}_w \). Suppose \( \alpha = 1 \). Then (31) implies \( \lim_{T \to \infty} T p_T = 0.5 f_\lambda^+ (0.5) \).

Let \( \alpha < 1 \), then \( (T_+ + \varepsilon) / T_1 = o(1) \) as \( T \to \infty \), and (30, 31) give a lower bound of \( p_T \):

\[ T_1 p_T \geq c_H (a_T^+)^{-H} l^{-\alpha} (a_T^+). \]

Here and below \( a_T^\pm = (T_\pm 0.5) / T_1 \).

Using (29, 31), we get an upper bound on \( p_T \):

\[ T_1 p_T = f_\lambda^+ \left( (T^\alpha + \varepsilon) / T_1 \right) \leq (a_T^+)^{-H} l(a_T^+) T_1 / (T_1 + 1) \leq 2 (a_T^+)^{-H} l(a_T^+). \]

By substituting \( T_\pm = T^\alpha \), we have

\[ \ln a_T^\pm = -(1 - \alpha) \ln T + O(T^{-\beta}), \beta = \alpha \wedge (1 - \alpha) \quad \text{and} \quad \ln l(a_T^+) = O(\sqrt{\ln T}). \]

Hence,

\[ - \ln p_T = (1 - (1 - \alpha) H) \ln T + O(\sqrt{\ln T}), \]

i.e., \( \bar{\theta}_w = \alpha H + (1 - H) \).

The equality \( \bar{\theta}_w = \theta_w \). Consider the Hilbert space \( (H_w (R^1), \| \|) \) related to FBM and a function

\[ \varphi(t) = \min(\| f \|) = \int \left( e^{ix} - 1 \right) \left( \frac{\sin \lambda / 2}{\sqrt{2 \pi \lambda / 2}} \right)^2 d\lambda. \]  

The standard spectral representation of the kernel \( Ew_H (t)w_H (s) \) and the representation (32) yield

\[ \| \varphi \|^2 = k_H \int \left( \frac{\sin \lambda / 2}{\sqrt{2 \pi \lambda / 2}} \right)^2 |x|^{-2H} d\lambda < \infty, \]

where \( k_H = \int \left| e^{ix} - 1 \right|^2 \left| x \right|^{-2H} \). Setting \( \varphi_t := \{ \varphi(t), t \in \Delta_T \} \), the desired statement follows from Lemma1 because \( \varphi_t \in (H_w (\Delta_T), \| \|) \) and \( \| \varphi_t \|, \| \| \leq \| \| \).

11
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Appendix.

Relation (7): \( \widetilde{B}_{I_H}(t) \leq \widetilde{B}_{I_{H,\alpha}}(t) \).

By (6), one has for small and large \( t \)

\[
\widetilde{B}_{I_H}(t) = 1 - (1 - H^2) t^2 / 2 + (2 + 4H)^{-1} t^{2+2H} (1 + o(1)) , \quad t \to 0 ,
\]

(A1)

\[
\widetilde{B}_{I_H}(t) = (1 + H)(1 + 2H)^{-1} e^{-tH} (1 - e^{-t}) + 0.5(1 + H)e^{-tH} (1 + O(e^{-t})) , \quad t \to \infty ,
\]

(A2)

where \( H = 1 - H \). Therefore, we have the following asymptotics for \( \Delta(t) = \widetilde{B}_{I_H}(t) - \widetilde{B}_{I_{H,\alpha}}(t) \):

\[
\Delta(t) = -(1 - 2H)t^2 / 2 + O(t^{2+2H}) , \quad t \to 0 ,
\]

\[
\Delta(t) = -(1 - 2H)H(2 + 4H)^{-1} e^{-tH} - (1 - 2H)H(2 + 4H)^{-1} e^{-tH} + O(e^{-t}) , \quad t \to \infty .
\]

These relations support (7) both for small and large enough \( t \). To verify (7) in the general case, we consider the following test function: \( (2 + 4H)(2 + 4H)\Delta(t) \exp(-1.5t) \). Using new variables: \( x = \exp(-t) , \quad \alpha = 1 - 2H \), the test function is transformed in a function \( \psi \) on square \( S = (0,1) \times (0,1) \).

Namely, \( \psi = U(x,\alpha) - U(x,-\alpha) \), where

\[
U(x,\alpha) = (4 - \alpha^2)x^{3/2}(3 - \alpha) \int_0^x [(x-u)((1-u)^{1-\alpha} - u^{1-\alpha}) + u^{1-\alpha}] du .
\]

We have to show that \( \psi \leq 0 \). It is easy to see that \( \psi = 0 \) at the boundary of \( S \). By (A1, A2), \( \psi \leq 0 \) in a neighborhood of two sides of \( S \); \( x=0 \) and \( x=1 \). The same is true for the other sides: \( \alpha = 0 \) and \( \alpha = 1 \) because

\[
\frac{\partial \psi}{\partial \alpha}(x,0) = -4(1-x^2) \int_{1-x}^1 \ln(1/u) du < 0 ,
\]

and

\[
\frac{\partial \psi}{\partial \alpha}(x,1) = (1-x)x^{-1/2} f(x) > 0 .
\]

Here

\[
f(x) = -x(1-x) + x^3 \ln 1/x + (1-x^3) \ln 1/(1-x) .
\]

To verify \( f(x) > 0 , 0 < x < 1 \), note that \( f'(x) = 3x^2(1 + v + \ln v) \), where \( v = (1-x)/x \). Obviously, \( f' \) has a single zero in \( (0,1) \), i.e. \( f \) has a unique extreme point. But \( f(0) = 0 = f(1) \) and \( f(x) > 0 \) for small \( x \). Therefore \( f(x) \geq 0, 0 < x < 1 \).

Numerical testing supports the desired inequality \( \psi < 0 \) for interior points of \( S \).

Comment. Our preliminary numerical test was concerned with points of grid with step 0.005. The first derivatives of \( \psi \) are uniformly bounded from above on \( S \). This fact helps to find a final grid step to prove \( \psi < 0 \) for all interior points of \( S \). The corresponding analysis is unwieldy and so is omitted.

Relation (9): \( \widetilde{B}_{I_H}(t) \leq \widetilde{B}_{I_{H,p}}(pt) , \quad H \geq 1/2 , \quad p = 2(1 - H) \).
To verify the inequality \( \Delta(t) = \tilde{B}_t^{(i)}(t) - \tilde{B}_{t^{(i)}}(2(1 - H)t) \leq 0 \), we consider the following test function: 
\[
(2 + 4H)\Delta(t) \exp(-(1 + H)t). 
\]
Using (6, 8) and new variables \((x = \exp(-t), \alpha = 2H - 1) \in S = (0,1) \times (0,1)\), we will have the following representation for the test function:
\[
\psi(x, \alpha) = (3 + \alpha)(x + x^{\alpha+2}) - 1 - x^{\alpha+3} + (1-x)^{\alpha+3} - 3(\alpha + 2)x^2 + (\alpha + 2)x^{3-\alpha} \quad \text{(A3)}
\]
One has \( \psi(x, \alpha) \leq 0 \) in a neighborhood of two sides of \( S: x=0 \) and \( x=1 \), because
\[
\psi(x, \alpha) = -(\alpha + 2)(3 - \alpha)x^2 / 2 + O(x^{\alpha+2})(3-\alpha) < 0, x \to 0 ,
\]
\[
\psi(x, \alpha) = -2\alpha(1 - \alpha)(3 - \alpha)(1-x)^2 / 2 + O((1-x)^3) \leq 0, x \to 1 .
\]
The same is true for other sides: \( \alpha = 0 \) and \( \alpha = 1 \).

**Side \( \alpha = 0 \).** One has \( \psi(x,0) = 0 \) and 
\[
\frac{\partial \psi}{\partial \alpha}(x,0) = (1-x)[x(1-x) + 3x^2 \ln x + (1-x)^2 \ln(1-x)] = (1-x)\varphi_3(x) \leq 0
\]
because 
\[
\varphi_3(x) = x(1-x) + ax^2 \ln x + (1-x)^2 \ln(1-x) \leq 0 , \quad a > 1 \quad \text{(A4)}
\]
To prove (A4), note that \( \varphi_3(0) = \varphi_3(1) = 0 \) and \( \varphi_3(x) = ax^2 \ln x + O(x^2) \leq 0 \) as \( x \to 0 \). Hence, (A4) holds if \( \varphi_3(x) \) has unique extremum in \((0,1)\). By
\[
\varphi_3^{(4)}(x) = -2ax^2 - 2(1-x)^2 \leq 0 ,
\]
we conclude that 
\[
\varphi_3^{(4)}(x) = (3a + 1) + 2a \ln x + 2 \ln(1-x)
\]
is a concave function with two zeroes in \((0,1)\), because \( \varphi_3^{(4)}(1/2) > 0 \) and \( \varphi_3^{(4)}(x) \to -\infty \) as \( x \to 0 \) or 1.

It means that 
\[
\varphi_3'(x) = (a - 1)x + 2ax \ln x - 2(1-x) \ln(1-x)
\]
has two extremums in \((0,1)\) only. But \( \varphi_3'(0) = 0, \varphi_3'(1) = a - 1 > 0 \), and \( \varphi_3'(x) \leq 0 \) for small \( x \) because \( \varphi_3^{(4)}(x) \to -\infty \) as \( x \to 0 \). Hence \( \varphi_3'(x) \) has unique zero in \((0,1)\) and \( \varphi_3(x) \) has unique extremum.

So, we prove that \( \psi(x, \alpha) \leq 0 \) for small \( \alpha \).

**Side \( \alpha = 1 \).** Here \( \psi(x,1) = 0 \) and 
\[
\frac{\partial \psi}{\partial \alpha}(x,1) = (1-x)(3-x)x^2 \ln(1/x) + (1-x)^2[x + (1-x)^2 \ln(1-x)] \geq 0 ,
\]
because 
\[
[x + (1-x)^2 \ln(1-x)] \geq x + (1-x) \ln(1-x) = -\int_0^1 \ln(1-u)du \geq 0 .
\]
Hence,
\[
\psi(x, \alpha) = \psi_3'(x,1)(\alpha - 1)(1 + o(1-\alpha)) \leq 0 , \quad \alpha \to 1 .
\]
As a result $\psi(x, \alpha) \leq 0$ near the boundary of $S = (0,1) \times (0,1)$. Numerical testing supports the desired inequality $\psi < 0$ for interior of $S$ (see more in the Comment from the Appendix section ‘Relation 7’).

Relation (10): $\tilde{B}_{i_1}(t) \geq \tilde{B}_{i_2}(pt)$, $H \leq 1/2$, $p = 2\sqrt{(1-H^2)/3}$.

Let $\psi = (2 + 4H)(\tilde{B}_{i_1}(t) - \tilde{B}_{i_2}(pt)) e^{-(1-H)t}$. By change of variables: $x = \exp(-t)$ and $\alpha = 2H$, we get a test function

$$
\psi(x, \alpha) = (2 + \alpha)(x + x^\alpha) - 1 - x^\alpha + (1 - x)^\alpha - 3(\alpha + 1)x^{1+(\alpha+p)/2} + (\alpha + 1)x^{1+(\alpha+3p)/2}
$$

on $S = (0,1) \times (0,1)$ and the relation between $p$ and $\alpha$:

$$
3(p/2)^2 + (\alpha/2)^2 = 1.
$$

One has

$$
\psi(x, \alpha) = (2 + \alpha)x^{1+\alpha} - 3(1+\alpha)x^{1+(\alpha+p)/2} + O(x^2) \geq 0, x \to 0,
$$

$$
\psi(x, \alpha) = (1-x)^{2+\alpha} + O((1-x)^3) \geq 0, x \to 1.
$$

In addition,

$$
\psi(x, 0) = x(2 - 3x^{-3/2} + x^{3/2}) \geq 0
$$

Finally, $\psi(x, 1) = 0$ and

$$
\frac{\partial \psi}{\partial \alpha}(x, 1) = \bar{x}(\alpha \bar{x} + 2x^2 \ln x + x^2 \ln \bar{x}) = \bar{x} \varphi_2(x),
$$

where $\bar{x} = 1-x$. By (A4), $\varphi_2(x) \leq 0$.

Therefore $\psi(x, \alpha) \leq 0$ near the boundary of $S = (0,1) \times (0,1)$. The numerical testing supports this conclusion for interior of $S$ (see more in the Comment from the Appendix section ‘Relation 7’).

Relations (13, 14).

Consider $\Delta(t) = \tilde{B}_{i_1}(t) - \tilde{B}_{i_2}(pt)$, where $\tilde{B}_{i_1}(t) = 1/\cosh(t/2)$ and $\tilde{B}_{i_2}(t)$ is given in (8). By the change of variables $x = e^{-t/2}$, we transform the test function $2(1 + e^{-p})\Delta(t)$ in a function $\psi$ on $(0,1)$ such that

$$
\psi(x) = (3x - x^3)(1 + x^2p) - 4x^p.
$$

Taking into account the asymptotics of $\psi$ near 0, we come to a necessary condition for $\psi$ to be negative, namely: $p \leq 1$. Let $p = 1$, then $\psi = -(1-x^2)^2 x \leq 0$, i.e. $4\theta_2 \leq 1$.

Case $p > 1$. In this case $\psi \geq 0$ as $x \to 0$. An additional condition on $p > 1$ we can get from the relation $\psi \geq 0$ as $x \to 1$. One has $\psi = xQ(x)$, where

$$
Q(x) = (3 - x^2)(1 + x^2p) - 4x^{p-1}.
$$
By \( Q(0) = 3, Q(1) = Q'(1) = 0 \), we have \( Q(x) = (1-x)^2 P(x) \) and \( P(1) = 0.5Q'(1) = 2(p^2 - 3) \). Thus \( Q(1) \geq 0 \) if \( p^2 \geq 3 \).

**Case** \( p = 2 \). Here, \( P(x) \) is a polynomial, \( P(x) = 3 + 2x - 2x^3 - x^4 \), and \( P''(x) = -12x(1+x) \leq 0 \), i.e. \( P(x) \) is a concave function with \( P(0) = 3, P(1) = 2 \). Therefore \( P(x) \geq 0 \) and as a result, \( 4\vartheta \geq 1/p = 0.5 \).

Consider \( p = \sqrt{3} \). One has \( Q(x) = 8(1-x)^3(1+o(1)), x \to 1 \) and \( Q(0) = 3 > 0 \). Therefore \( Q(x) \geq 0 \), if \( Q(x) \) is convex, i.e. \( Q''(x) \geq 0 \). To verify this property, note that
\[
0.5x^2Q''(x) = 2(3p - 5)x^{p-1} + 3(6 - p)x^{2p} - x^2 - (7 + 3p)x^{2+p}
= (7 + 3p)x^{2p}(1 - x^2) + \rho x^{p-1} + (1 - \rho)x^2 - x = \varphi(x),
\]
where \( \rho = 6p - 10 \).

Obviously, \( \varphi(x) \geq 0 \) if \( \rho x^{p-1} - x^2 \geq 0 \). This holds for \( 0 < x < x_0 = 0.478 \).

For \( x > x_0 \),
\[
\rho x^{p-1} + (1 - \rho)x^2 - x^2 \geq (\rho + (1 - \rho)x_0^{p-1})x^{p-1} - x^2.
\]
The right part here is positive for \( x < 0.55 \), i.e. \( \varphi(x) \geq 0 \) for \( x \leq 0.5 \).

Let \( x > 0.5 \). Then
\[
\varphi(x) \geq (7 + 3p)2^{-2p}(1-x^2) + \rho \varphi(x) + (1 - \rho)x^2 - x^2
= C - (C + 1)x^2 + \rho \varphi(x) + (1 - \rho)x^2 \equiv u(x),
\]
where \( C = (7 + 3p)2^{-2p} \). We have \( u(0) = C, u(1) = 0 \) and
\[
u'(x) = -2(C + 1)x + \rho(p - 1)x^{p-2} + 2(1 - \rho)x^{2p-1}
= -(C + 1 - 2(1 - \rho)x^{2p-2})x - ((C + 1)x^{2p} - \rho(p - 1))x^{p-2}.
\]
It is easy to see, that both terms in parentheses are positive on \( (0.5, 1) \).

Thus, \( u(x) \) decreases to \( u(1) = 0 \). This means that \( Q''(x) \geq 0 \). Q.E.D.

**Relation (24):** \( \widetilde{B}_{\alpha}(pt) \geq B_{\alpha}(t) \), \( pH = -\ln(2H), 0 < H < e^{-2}/2 \).

The difference of the correlation functions is the following
\[
\Delta(t) = \left(cosh(Hp) - 0.5(2 \sinh(pt/2))^{2H} \right) - (1 - |t|^{2H}),
\]
Let \( t > 1 \), then \( \Delta(t) = \widetilde{B}_{\alpha}(pt) \geq 0 \).

Let \( 2H < t < 1 \). It is enough to show that the first term, \( \varphi \), in the following representation
\[
\Delta(t) = [0.5e^{-Hp} - 1 + t^{2H}] + 0.5e^{Hp}(1 - (1 - e^{-pt})^{2H}) \equiv \varphi + R
\]
is non-negative. Setting \( Hp = -\ln(2H) \), \( \alpha = 2H \) one has
\[
\varphi(t) = 0.5\alpha^t + t^\alpha - 1.
\]

Let us show that \( \varphi \) is decreasing. Then \( \varphi \) is positive because \( \varphi(1) = \alpha/2 \).
We have
\[
\varphi'(t) = \alpha^t(-0.5 \ln(1/\alpha) + \psi(t)) ,
\]
where \( \psi(t) = \alpha^{-1 - t}/t^{\alpha} \). The function \( \psi(t) \) has a single extreme point in the interval:
\( t^* = (1 - \alpha)/\ln(1/\alpha) \). But \( \psi'(t) = \min \), because \( \psi(t) \) decreases near \( t = \alpha \):
\[
\psi(\alpha) = 1 \quad \text{and} \quad \psi'(\alpha) = (\alpha \ln(e/\alpha) - 1)/\alpha \leq 0 \quad \text{for} \ 0 < \alpha < 1.
\]

\[
\]
Hence, \( \psi(t) \leq \max(\psi(\alpha), \psi(1)) = 1 \). As a result,
\[ \varphi'(t) \leq \alpha'(-0.5 \ln(1/\alpha) + 1) \leq 0.\]
The last inequality holds for \( 0 < \alpha < e^{-2} \).
So, we have,
\[ \Delta(t) \geq 0, \ 2H < t < 1 \text{ for } 0 < \alpha < e^{-2}.\]

Let \( 0 < t < 2H \). Use
\[ \Delta(t) = \cosh(\text{Hpt}) - 1 + t^{2H} [1 - 0.5(2t^{-1} \sinh(pt/2))^{2H}] \]
then \( \Delta(t) \geq 0 \) if
\[ 2^{1/2H} \geq \max_{(0, 2H)}(2t^{-1} \sinh(pt/2)) = H^{-1} \sinh(pH) = (2H)^{-1} - 1.\]
This inequality holds for \( 0 < 2H < 1/4 \).
Putting the above inequalities together yields (24) for \( 2H \leq e^{-2} \wedge 1/4 \).
Figure 1
The survival exponents $\tilde{\theta}_{1_H}$ for the integrated fractional Brownian motion in $\Delta_T = (-T, T)$: hypothetical values (parabolic line), empirical estimations (small circles, squares), and theoretical bounds (shaded zone given by Proposition 1(b,e)).

The empirical exponents are based on the model (12, $\alpha(H) = 0$) in three time intervals of $\tilde{T} = \ln T$: $\ln(1/\varepsilon) \leq \tilde{T}(1 - H)H \leq \ln(10/\varepsilon)$ where $\varepsilon = 0.01$, 0.003, and 0.001 (see more in Molchan and Khohlov, 2003).