Local well-posedness result for a class of non-local quasi-linear systems and its application to the justification of Whitham-Boussinesq systems

Louis Emerald

June 22, 2022

Abstract

In this paper we prove a local well-posedness result for a class of quasi-linear systems of hyperbolic type involving Fourier multipliers. Among the physically relevant systems in this class is a family of Whitham-Boussinesq systems arising in the modeling free-surface water waves. Our result allows to prove the rigorous justification of these systems as approximations to the general water waves system on a relevant time scale, independent of the shallowness parameter.

1 Introduction

1.1 Motivations

In this work we prove a well-posedness result and a stability result for systems of the form

\[
\begin{aligned}
\partial_t \zeta + (G_1)^2 \nabla \cdot v + \epsilon G_2 \nabla \cdot (\zeta G_2[v]) &= 0, \\
\partial_t v + \nabla \zeta + \epsilon (G_2[v] \cdot \nabla) G_2[v] &= 0,
\end{aligned}
\]

where \((G_1, G_2)\) are admissible Fourier multipliers (see definition below), and the unknowns \(\zeta\) and \(v\) are functions of time \(t\) and space \(x \in \mathbb{R}^d\) \((d \in \{1, 2\})\) valued in \(\mathbb{R}\) and \(\mathbb{R}^d\) respectively, and \(\epsilon \geq 0\).

Several standard equations fit into the above framework. For instance, setting \(G_1 = G_2 = \text{Id}\) one obtains the standard shallow-water system, which can be viewed as a special case of the \(d\)-dimensional isentropic, compressible Euler equation for ideal gases with quadratic pressure law, and whose well-posedness theory is quite standard (see e.g. [15]). This work extends this theory to the general class of equations above. Now, if we set \(G_2 = (\text{Id} - \mu b \Delta_x)^{-1}\) and \(G_1 = (\text{Id} + \mu a \Delta_x)^{1/2}(\text{Id} - \mu b \Delta_x)^{-1}\) with \(a, b, \mu \geq 0\), we obtain (setting \(u = G_2[v]\)) the so-called abcd-Boussinesq system with \(c = 0\) and \(d = b\). These systems model surface gravity waves in the long wave regime, and have been introduced in full generality by Bona, Chen and Saut in [2]. Concerning the well-posedness of the initial-value problem, some early results were proved by the energy method and using the regularization properties of the operator \((\text{Id} + \Delta_x)^{-1/2}\) in [3, 1], but the dependency of the result (and in particular the existence time) with respect to \(\epsilon\) and \(\mu\) were not specified. Following [9], Linares, Pilod and Saut [14] used dispersive techniques to prove the well-posedness of a large class of Boussinesq systems (when \(d = 2\)) on a time-interval of length \(T \gtrsim \epsilon^{-1/2}\) when \(\mu \approx \epsilon\). Subsequently, Saut
and Xu [18] (see also [16, 4, 19]) improved this result by proving well-posedness results on a time interval of length $T \gtrsim \epsilon^{-1}$ by using symmetrization and energy techniques. Importantly, the latter result holds for $d \in \{1, 2\}$ and on the full shallow-water regime, that is $\mu \in (0, 1]$, as the authors do not make use of the dispersive nature of the equations. This property is important as it allows to rigorously justify Boussinesq systems as an asymptotic model, improving the shallow-water system (which corresponds to setting $\mu = 0$) in the long wave regime, $\epsilon \lesssim \mu$; see the memoir of Lannes [13].

Our main interest in considering (1.1) stems from the so-called Whitham-Boussinesq systems. Indeed, setting $G_1^2 = G_2 = \frac{\tanh(\sqrt{\mu(D)})}{\sqrt{\mu(D)}}$, we obtain the system introduced by Dinvay, Dutykh and Kalisch in [7]. It has been recently proved by the author [11] that Whitham-Boussinesq systems are approximations of order $O(\epsilon \mu)$, as opposed to $O(\mu)$ for the shallow-water system and $O(\mu + \mu \epsilon)$ for Boussinesq systems, of the general water waves model, in the sense of consistency (see Proposition 1.9 below for a precise statement). The consistency property is the first step to prove the full justification of a model for surface gravity waves. The second step is to prove stability estimates and the local well-posedness of the model for sufficiently regular data on the relevant timescale. The well-posedness of the aforementioned Whitham-Boussinesq systems has been studied in [6, 8, 20, 5]. In [8], Dinvay diagonalizes linear terms and uses strongly the regularization properties of the operator $(G_2)^{-1}$, from which stems an existence time of length $T \gtrsim \epsilon \mu^{-1/2}$ (in dimension $d = 1$). In [8], Dinvay, Selberg and Tesfahun exploit the dispersive nature of the system. This allows them to prove the well-posedness at the energy level in dimension $d = 1$, from which global-in-time well-posedness (for sufficiently small initial data) follows. They also prove the local-in-time existence in dimension $d = 2$. Yet, due to the use of dispersive estimates, these two results do not provide the control of solutions and their derivatives uniformly with respect to $\mu \in (0, 1]$, as required for the rigorous justification of the system as an asymptotic model for water waves. The precise dependency of the time on which solutions exist and are controlled (in a ball of twice the size of the initial data in the relevant Banach space) is clarified in subsequent works: the result of Tesfahun [20] in dimension $d = 2$ exhibits a time interval of length $T \gtrsim \epsilon^{-2+\delta} \mu^{3/2-\delta}$ with $\delta > 0$ arbitrarily small, while the result of Deneke, Dufera and Tesfahun [5] in dimension $d = 1$ exhibits a time interval of length $T \gtrsim \epsilon^{-1}\mu^{1/2}$.

Let us conclude this state of the art by mentioning the recent work of Paulsen [17] where the well-posedness and control of some Whitham-Boussinesq systems, different from the one mentioned above and considered here, is proved on a time-interval of length $T \gtrsim \epsilon^{-1}$ for parameters on the shallow-water regime $(\epsilon, \mu) \in (0, 1]^2$. The strategy of the proof is, similarly to ours, based on the energy method and, as stated therein, the two works complete each other well.

1.2 Definitions and notations

**Definition 1.1.** For $u : \mathbb{R}^d \to \mathbb{R}$ a tempered distribution, denote $\widehat{u}$ its Fourier transform. Let $G : \mathbb{R}^d \to \mathbb{R}$ be a bounded function. The Fourier multiplier associated with $G(\xi)$ is denoted $G := G(D)$ and defined by

$$\forall u \in L^2(\mathbb{R}^d), \quad (G[u])(\xi) = G(\xi)\widehat{u}(\xi).$$

**Definition 1.2.** We say that a couple of Fourier multipliers $(G_1, G_2)$ is admissible if its symbols satisfy

- for $k \in \{1, 2\}$, $G_k \in L^\infty(\mathbb{R}^d)$ and $(\cdot)\nabla G_k \in L^\infty(\mathbb{R}^d)^d$;

- for all $\xi \in \mathbb{R}^d$, we have $G_1(\xi) > 0$;

- for all $\xi \in \mathbb{R}^d$, we have $|G_2(\xi)| \leq G_1(\xi)$.
Denoting $U := \begin{pmatrix} \zeta \\ v \end{pmatrix}$, we can write systems (1.1) under their matricial form

$$\partial_t U + \sum_{j=1}^d A_j(U)[\partial_j U] = 0,$$  

where (in dimension $d = 2$, the analogous definitions when $d = 1$ is straightforward)

$$A_1(U)[\zeta] = \begin{pmatrix} \epsilon G_2[G_2[v_1][\zeta] & (G_1)^2[\zeta] + \epsilon G_2[\zeta G_2[\zeta]] \\ 1 & \epsilon G_2[v_1][G_2[\zeta]] \\ 0 & 0 & \epsilon G_2[v_1][G_2[\zeta]] \end{pmatrix},$$

$$A_2(U)[v] = \begin{pmatrix} \epsilon G_2[G_2[v_2][\zeta] & 0 & (G_1)^2[\zeta] + \epsilon G_2[\zeta G_2[\zeta]] \\ 0 & \epsilon G_2[v_2][G_2[\zeta]] & 0 \\ 1 & 0 & \epsilon G_2[v_2][G_2[\zeta]] \end{pmatrix}.$$  

The natural functional setting is given by the energy norms.

**Definition 1.3.**  
- We denote by $\mathcal{S}'(\mathbb{R}^d)$ the set of tempered distributions.
- We denote by respectively $| \cdot |_2$ and $\langle \cdot, \cdot \rangle_2$, the norm and the scalar product in $L^2(\mathbb{R}^d)$.
- Let $s \geq 0$. We denote by $H^s(\mathbb{R}^d)$ the Sobolev spaces of order $s$ in $L^2(\mathbb{R}^d)$. Denoting $\Lambda^s := (1 - \Delta)^{s/2}$, where $\Delta$ is the Laplace operator in $\mathbb{R}^d$, the norm associated with $H^s(\mathbb{R}^d)$ is $| \cdot |_{H^s} := |\Lambda^s \cdot |_2$.
- Let $G_1$ be a Fourier multiplier of order 0, defined by positive function $G_1$. Let also $s \geq 0$. We define the Banach spaces $X^s(\mathbb{R}^d)$ and $Y^s(\mathbb{R}^d)$ by

$$X^s(\mathbb{R}^d) := \{ U = (\zeta, v) \in \mathcal{S}'(\mathbb{R}^d) \times \mathcal{S}'(\mathbb{R}^d), \ |U|_{X^s} < +\infty \},$$

$$Y^s(\mathbb{R}^d) := \{ U = (\zeta, v) \in \mathcal{S}'(\mathbb{R}^d) \times \mathcal{S}'(\mathbb{R}^d), \ |U|_{Y^s} < +\infty \},$$

where $|U|_{X^s} := |\zeta|_{H^s} + |G_1[v]|_{H^s}$ and $|U|_{Y^s} := |\zeta|_{H^s} + |G_1^{-1}[v]|_{H^s}$. These are the energy norms associated with system (1.2).

Henceforth, we denote $C(\lambda_1, \lambda_2, \ldots)$ a positive constant depending non-decreasingly on its parameters. Unless it is essential, the dependency with respect to the regularity index $s$ is omitted.

We write $a \lesssim b$ for $a \leq Cb$ with $C > 0$ a positive number whose dependency is unessential or clear from the context. We denote $a \asymp b$ when $a \lesssim b$ and $b \lesssim a$. We denote $\langle \cdot \rangle := (1 + |\cdot|^2)^{1/2}$.

### 1.3 Main results

The systems (1.2) are symmetrizable quasi-linear hyperbolic. The main key to prove the local well-posedness of such systems is the following energy estimates on the linearized system.

**Proposition 1.4.** Let $s \geq 0$, $t_0 > d/2$ and $(G_1, G_2)$ be admissible Fourier multipliers. For any $\epsilon \in (0, 1]$, $T > 0$, $\underline{U} = (\zeta, v) \in W^{1,\infty}(\{0, T/\epsilon\}, X^{1}_{t}((0, T/\epsilon), X^\infty((0, T/\epsilon), X^{\max(t_0+1,s)}((0, T/\epsilon), \mathbb{R}^d)))$ for which there exists $\lambda_{\min} > 0$ such that for all $(t, x) \in [0, T/\epsilon] \times \mathbb{R}^d$

$$1 + \epsilon \zeta \geq \lambda_{\min},$$

(1.4)
and for any \( U = (\zeta, v) \in W^{1,\infty}([0, T/\epsilon], X^s(\mathbb{R}^d)) \cap L^\infty([0, T/\epsilon], X^{s+1}(\mathbb{R}^d)) \) satisfying the system

\[
\partial_t U + \sum_{j=1}^d A_j(U) \partial_j U = \epsilon R,
\]

where \( R \in L^\infty([0, T/\epsilon], X^s(\mathbb{R}^d)) \), and for \( j = 1, 2, A_j(U) \) is defined by (1.3), we have for any \( t \in [0, T/\epsilon] \),

\[
|U|_{X^s} \leq \kappa_0 e^{\lambda_0 t} |U|_{X^s}|_{t=0} + \epsilon \nu_s \int_0^t |R(t')|_{X^s} dt',
\]

(1.5)

where \( \lambda_s, \nu_s := C(\frac{1}{\eta_{\min}}, T, |U|_{W^{2,\infty}_s X^0}, |U|_{L^\infty_t X^{max(t_0+1, s)}}) \) and \( \kappa_0 := C(\frac{1}{\eta_{\min}}, |U|_{X^0}|_{t=0}). \)

Using a Picard iteration scheme and the regularization method from Chapter 7 in [15] we infer the following well-posedness result on the systems (1.2).

**Theorem 1.5.** Let \( s > d/2 + 1, h_{\min} > 0 \) and \( M > 0 \). Let also \((G_1, G_2)\) be a couple of admissible Fourier multipliers. There exist \( T > 0 \) and \( C > 0 \) such that for all \( \epsilon \in (0, 1], U_0 \in X^s(\mathbb{R}^d) \) with \( |U_0|_{X^s} \leq M \) and satisfying (1.4), there exists a unique solution \( U \in C^0([0, T/\epsilon], X^s(\mathbb{R}^d)) \) of the Cauchy problem

\[
\begin{cases}
\partial_t U + \sum_{j=1}^d A_j(U) \partial_j U = 0, \\
U|_{t=0} = U_0.
\end{cases}
\]

Moreover \( |U|_{L^{\infty}([0, T/\epsilon], X^s)} \leq C|U_0|_{X^s}. \)

We also have the following stability result.

**Proposition 1.6.** Let the assumptions of Theorem 1.5 be satisfied and use the notations therein. Assume also that there exists \( \widetilde{U} \in C([0, \widetilde{T}/\epsilon], X^s(\mathbb{R}^d)) \) solution of

\[
\partial_t \widetilde{U} + \sum_{j=1}^d A_j(\widetilde{U}) \partial_j \widetilde{U} = \widetilde{R},
\]

where \( \widetilde{R} \in L^\infty([0, T/\epsilon], X^{s-1}(\mathbb{R}^d)) \). Then, the error with respect to the solution \( U \in C^0([0, T/\epsilon], X^s(\mathbb{R}^d)) \) given by Theorem 1.5 satisfies for all times \( t \in [0, \min(\widetilde{T}, T)/\epsilon] \),

\[
|\epsilon|_{L^{\infty}([0, t], X^{s-1})} \leq C(\frac{1}{\eta_{\min}}, |U|_{L^{\infty}([0, t], X^s)}, |\widetilde{U}|_{L^{\infty}([0, t], X^s)}, |\epsilon|_{X^{s-1}}|_{t=0} + t |\widetilde{R}|_{L^{\infty}([0, t], X^{s-1})}),
\]

where \( \epsilon := U - \widetilde{U} \).

**Remark 1.7.** In all previous statement, the dependency of the constants and the existence time with respect to the admissible pair of Fourier multipliers \((G_1, G_2)\) occurs only through \( |\langle G_k, \cdot \rangle \nabla G_k \rangle|_{L^\infty} \) for \( k \in \{1, 2\} \).

Let us now turn to the rigorous justification of the systems (1.2) in the context of irrotational free surface flows. We first recall the notations and the physical meaning of the different variables (see [11]).

- The free surface elevation is the graph of \( \zeta \), which is a function of time \( t \) and horizontal space \( x \in \mathbb{R}^d \).
- \( \nu(t, x) \) is the gradient of the trace at the surface of the velocity potential.
Moreover every variable and function in (1.2) is compared with physical characteristic parameters of the same dimension. Among those are the characteristic water depth $H_0$, the characteristic wave amplitude $a_{surf}$ and the characteristic wavelength $L$. These physical characteristic parameters define two dimensionless parameters of main importance:

$$\mu := \frac{H_0^2}{L^2}, \quad \epsilon := \frac{a_{surf}}{H_0}.$$  

The first parameter, $\mu$, is called the shallow water parameter. The second parameter, $\epsilon$, is called the nonlinearity parameter. In the following we restrict ourselves to the shallow-water regime: $(\mu, \epsilon) \in (0,1]^2$.

**Notation 1.8.** From now on the Fourier multipliers are denoted $G_1^\mu$ and $G_2^\mu$ as they depend on $\mu \in (0,1]$ and are of the form $G_k^\mu = G_k(\sqrt{\mu}D)$, $k = 1, 2$ where the symbols $G_1$ and $G_2$ are independent of $\mu$.

Proposition 1.15 in [11] can be easily extended to the following result.

**Proposition 1.9.** Let $s \geq 0$. In (1.2), let $G_1^\mu := \sqrt{\frac{\tanh(\sqrt{\mu}D)}{\sqrt{\mu}D}}$. Let also $G_2^\mu$ be a Fourier multiplier such that $|G_2^\mu(\xi)| - 1 \lesssim |\mu \xi|^2$. Then any classical solution $(\zeta, \psi)$ of the water waves equations satisfying the non-cavitation hypothesis (1.4), with $U = (\zeta, \nabla \psi) \in C^0([0, T/\epsilon], H^{s+4}(\mathbb{R}^d))$, satisfy the system (1.2) up to a remainder term of order $O(\mu \epsilon)$, i.e. for any $t \in [0, T/\epsilon)$,

$$\partial_t U + \sum_{j=1}^d A_j(U)\partial_j U = \mu \epsilon R,$$

where $|R(t, \cdot)|_{H^s} \leq C(\frac{1}{t_{min}}, |\zeta|_{H^{s+4}}, |\nabla \psi|_{H^{s+4}})$, uniformly with respect to $(\mu, \epsilon) \in (0,1]^2$.

We say that the water waves equations are consistent at order $O(\mu \epsilon)$ with the systems (1.2) in the shallow water regime.

**Remark 1.10.** Within the assumptions of Proposition 1.9, the systems (1.2) are Whitham-Boussinesq systems (see [11]). As aforementioned, setting $G_2^\mu = (G_1^\mu)^2$, we obtain the system by Dinvay, Dutykh and Kalisch in [7]. Previously existing well-posedness results use the regularizing effect of $(G_1^\mu)^2$ which gives the system a semi-linear structure (namely the system can be solved through the Duhamel formula). Theorem 1.5 allows us to set, for instance, $G_2^\mu = G_1^\mu$, thus proving the local well-posedness for a Whitham-Boussinesq system with a genuinely quasi-linear structure. When $G_2^\mu = \text{Id}$, the system (1.2) is a Whitham-Boussinesq system, yet the question of local well-posedness for this system is open.

From Theorem 1.5, Proposition 1.6 and Proposition 1.9 we infer the full justification of a class of Whitham-Boussinesq systems.

**Theorem 1.11.** Under the assumption and using the notation of Proposition 1.9, and provided $(G_1, G_2)$ are admissible Fourier multipliers, then for any $U = (\zeta, \nabla \psi) \in C^0([0, T/\epsilon], H^{s+4}(\mathbb{R}^d))$ classical solution of the water waves equations and satisfying the non-cavitation assumption (1.4), there exists a unique $U_{WB} = (\zeta_{WB}, \psi_{WB}) \in C^0([0, T/\epsilon], X^{s+4}(\mathbb{R}^d))$ classical solution of the Whitham-Boussinesq systems (1.2) with initial data $U_{WB}|_{t=0} = (\zeta|_{t=0}, \nabla \psi|_{t=0})$, and one has for all times $t \in [0, \min (T, T/\epsilon)]$

$$|U - U_{WB}|_{L^\infty([0, T/\epsilon], X^\cdot)} \leq C \mu \epsilon t,$$

with $T$ (provided by Theorem 1.5) and $C = C(\frac{1}{t_{min}}, |U|_{L^\infty([0, T/\epsilon], H^{s+4})})$ uniform with respect to $(\mu, \epsilon) \in (0,1]^2$.

**Remark 1.12.** Regular solutions of the water waves equations as in Proposition 1.9 and Theorem 1.11 are provided by Theorem 4.16 in [12].
1.4 Outline

Section 2 is dedicated to the proof of Proposition 1.4. In Subsection 2.1 we focus on the symmetrization of the systems (1.1). In Subsection 2.2, we prove the energy estimates of Proposition 1.4 in the case of \( s = 0 \). Finally in Subsection 2.3, we prove the general case \( s \geq 0 \).

Section 3 is dedicated to the proof of Theorem 1.5. In Subsection 3.1 we prove a local well-posedness result for the systems (1.1) linearized around a sufficiently regular state. In Subsection 3.2 we focus on the proof of Theorem 1.5. In Subsection 3.3 we establish a blow-up criterion for the local well-posedness of the systems (1.1).

In Section 4 we prove Proposition 1.6.

In Section 5 we prove Theorem 1.11.

Appendix A collects useful technical results.

2 Energy estimates

2.1 Symmetrization

In this subsection we focus on the symmetrization of the systems (1.2). We perform the computations in the setting \( d = 2 \), the case \( d = 1 \) is obtained in the same way.

Property 2.1. Let \( s \geq 0 \) and \( t_0 > d/2 \). For any \( U \in X^{t_0+1}(\mathbb{R}^d) \), let

\[
S_0(U)[\cdot] := \begin{pmatrix}
1 & 0 \\
0 & (G_1)^2[\cdot] + \epsilon G_2[\partial G_2[\cdot]]
\end{pmatrix},
\]

be an operator defined in \( X^0(\mathbb{R}^d) \). Applying the latter operator to system (1.2) we get

\[
S_0(U)[\partial_t U] + \sum_{j=1}^d \tilde{A}_j(U)[\partial_j U] = \epsilon \sum_{j=1}^d F_j(U)[\partial_j U],
\]

where for \( j = 1, 2 \), \( \tilde{A}_j(U) \) is a symmetric operator defined by (denoting by * the adjoint in \( L^2(\mathbb{R}^d) \))

\[
\tilde{A}_j(U) = \frac{B_j(U) + B_j(U)^*}{2},
\]

with

\[
B_1(U)[\cdot] = \begin{pmatrix}
B_1^{1,1}(U)[\cdot] & B_1^{1,2}(U)[\cdot] & 0 \\
B_1^{2,1}(U)[\cdot] & B_1^{2,2}(U)[\cdot] & 0 \\
0 & 0 & B_1^{2,2}(U)[\cdot]
\end{pmatrix},
\]

where

\[
\begin{cases}
B_1^{1,1}(U)[\cdot] = \epsilon G_2[G_2[v_1][\cdot]], \\
B_1^{1,2}(U)[\cdot] = (G_1)^2[\cdot] + \epsilon G_2[\partial G_2[\cdot]], \\
B_1^{2,2}(U)[\cdot] = \epsilon (G_1)^2[G_2[v_1][G_2[\cdot]]] + \epsilon^2 G_2[\partial G_2[\partial G_2[\cdot]]] + \epsilon G_2[\partial G_2[\partial G_2[\cdot]]], \\
B_1^{2,2}(U)[\cdot] = \epsilon (G_1)^2[G_2[v_1][G_2[\cdot]]] + \epsilon^2 G_2[\partial G_2[\partial G_2[\cdot]]] + \epsilon G_2[\partial G_2[\partial G_2[\cdot]]].
\end{cases}
\]
and

\[ B_2(U)[o] = \begin{pmatrix}
B_2^{1,1}(U)[o] & 0 & B_2^{1,3}(U)[o] \\
0 & B_2^{2,2}(U)[o] & 0 \\
B_2^{1,3}(U)[o] & 0 & B_2^{2,2}(U)[o]
\end{pmatrix}, \]

where

\[
\begin{align*}
B_2^{1,1}(U)[o] &= \epsilon G_2[G_2[v_2][o], \\
B_2^{1,3}(U)[o] &= (G_1)^2[o] + \epsilon G_2[\zeta G_2[o]], \\
B_2^{2,2}(U)[o] &= \epsilon(G_1)^2[G_2[v_2][o] + \epsilon^2 G_2[\zeta G_2[G_2[v_2][o]]].
\end{align*}
\]

Finally, for \( j = 1, 2 \),

\[ F_j(U)[o] = -\frac{B_j(U)[o] - B_j(U)^*[o]}{2\epsilon}. \]

Proof. Applying the matricial operator \( S_0(U)[o] \) to the system (1.2), we immediately get

\[ S_0(U)[\partial_t U] + \sum_{j=1}^d B_j(U)[\partial_j U] = 0. \]

Then we write

\[ B_j(U) = \frac{B_j(U) + B_j(U)^*}{2} + \frac{B_j(U) - B_j(U)^*}{2} = \tilde{A}_j(U) - \epsilon F_j(U). \]

Proposition 2.2. Let \( s \geq 0 \) and \( t_0 > d/2 \). For any \( U \in X^{\max(t_0+1,s)}(\mathbb{R}^d) \) and any \( U \in X^s(\mathbb{R}^d) \), for \( j = 1, 2 \), we have

\[ |F_j(U)[\partial_t U]|_{Y^s} \leq C(|U|_{X^{\max(t_0+1,s)}}) |U|_{X^s}. \]

(2.3)

It makes it a term of order 0 with respect to the energy norm.

Proof. First remark that \((B_1^{1,2})^* = B_1^{1,2}\) and \((B_2^{1,3})^* = B_2^{1,3}\). So that we need to estimate the terms of \( F_j(U) \), \( j = 1, 2 \), defined by \( B_1^{1,1}, B_1^{2,2}, B_2^{1,1} \) and \( B_2^{2,2} \). We shall only estimate the contributions from \( B_2^{2,2} \), the other terms being similar.

For the first contribution, we have

\[
|(G_1)^{-1}[(G_1)^2[G_2[v_2][G_2[\partial_t v_2]]] - G_2[G_2[v_2][G_1][\partial_t v_2]]]|_{H^s} \\
\leq |G_1[G_2[v_2][G_2[\partial_t v_2]] - G_2[G_2[v_2][G_1][\partial_t v_2]]]|_{H^s} + |(G_1)^{-1}G_2[(G_1, G_2[v_2][G_1][\partial_t v_2]]]|_{H^s} := I_1 + I_2.
\]

We decompose

\[ I_1 = |[G_1, G_2[v_2]][G_2[\partial_t v_2]] - [G_2, G_2[v_2]][G_1[\partial_t v_2]]|_{H^s}, \]

so that using the commutator estimates of Proposition A.4 and the assumption \(|G_2| \leq G_1 \in L^\infty(\mathbb{R}^d)\) we get

\[ I_1 \lesssim |G_2[v_2]|_{H^{\max(t_0+1,s)}} |G_1[\partial_t v_2]|_{H^{s+1}} \lesssim |U|_{X^{\max(t_0+1,s)}} |U|_{X^s}. \]

Using the same tools, we get

\[ I_2 \lesssim |U|_{X^{\max(t_0+1,s)}} |U|_{X^s}. \]
For the second contribution, using the same tools in addition to the product estimates of Proposition A.1, we have

\[ (G_1)^{-1}[G_2[\xi G_2[G_2[v_1]G_2[\partial_1 v_2]]] - G_2[G_2[v_1]G_2[\xi G_2[\partial_1 v_2]]]|_{H^s} \]

\[ \leq |\xi|_H^{max} (t_0 + 1, \varepsilon) |G_2[v_1]G_2[\partial_1 v_2]|_{H^{s-1}} + |G_2[v_1]|_{H^{max}(t_0 + 1, \varepsilon)} |\xi G_2[\partial_1 v_2]|_{H^{s-1}} \]

\[ \leq |U|^2_{\max(t_0 + 1, \varepsilon)} |U|_{X^s}. \]

As aforementioned, all other terms are estimated in the same way. \(\square\)

### 2.2 Energy estimates of order 0

In this subsection we prove the energy estimates of order 0 on the systems (1.2) linearized around a sufficiently regular state, specifically Proposition 1.4 with \(s = 0\).

**Proposition 2.3.** Let \(t_0 > d/2\) and \((G_1, G_2)\) be admissible Fourier multipliers. For any \(\epsilon \in (0, 1]\), \(T > 0\), \(\mathcal{U} = (\xi, \nu) \in W^{1, \infty}([0, T/\epsilon], X^{0}(\mathbb{R}^d)) \cap L^{\infty}([0, T/\epsilon], X^{0}(\mathbb{R}^d))\) for which there exists \(h_{\min} > 0\) such that for all \((t, x) \in [0, T/\epsilon] \times \mathbb{R}^d\), (1.4) holds; and for any \(U = (\xi, \nu) \in W^{1, \infty}([0, T/\epsilon], X^{0}(\mathbb{R}^d)) \cap L^{\infty}([0, T/\epsilon], X^{1}(\mathbb{R}^d))\) satisfying the system

\[ \partial_t U + \sum_{j=1}^d A_j(U)[\partial_j U] = \epsilon R, \]

where \(R \in L^{\infty}([0, T/\epsilon], X^{0}(\mathbb{R}^d))\), and for \(j = 1, 2\), \(A_j(U)\) is defined by (1.3), we have for any \(t \in [0, T/\epsilon]\),

\[ |U|_{X^0} \leq \kappa_0 e^{\lambda_0 t}|U|_{X^0}|_{t=0} + \epsilon \nu_0 \int_0^t |R(t')|_{X^0}dt', \]

where \(\lambda_0, \nu_0 := C(\frac{1}{h_{\min}}, T, |U|_{W^{1, \infty}X^{0}, U} L^{\infty}X^{0+1})\) and \(\kappa_0 := C(\frac{1}{h_{\min}}, |U|_{X^{0}}|_{t=0}).\)

To prove this result we need some properties on the symmetrizer \(S_0(U)\).

**Lemma 2.4.** Let \(t_0 > d/2\). Let \(U \in X^{0}(\mathbb{R}^d)\) be such that (1.4) is satisfied. The symmetrizer \(S_0(U)\) satisfies for any \(U \in X^{0}(\mathbb{R}^d)\)

\[ (S_0(U)U, U)_2 \geq |\varsigma|^2 + h_{\min}|G_1|v|^2 \geq h_{\min}|U|^2_{X^0}. \quad (2.4) \]

and

\[ \begin{cases} |S_0(U)U|_{X^0} \leq C(|U|_{X^{0}})|U|_{X^0}, \\ (S_0(U)U, U)_2 \leq C(|U|_{X^{0}})|U|^2_{X^0}. \end{cases} \quad (2.5) \]

**Proof.** Given the definition of \(S_0(U)\) (see (2.1)), the assumption \(G_2 \leq G_1\) and the Sobolev embedding \(H^{t_0} \subset L^\infty\), the estimates of (2.5) are obvious. We prove here the inequality (2.4).

\[ (S_0(U)U, U)_2 = |\varsigma|^2 + ((G_1)^2[v] + G_2[\xi G_2[v]], v)_2 \]

\[ = |\varsigma|^2 + (((G_1)^2 - (1 - h_{\min})(G_2)^2)[v], v)_2 + ((1 - h_{\min})(G_2)^2[v] + G_2[\xi G_2[v]], v)_2 \]
Lemma 2.5. With the same assumptions as in Proposition 2.3, we have the following estimates

\[
\frac{d}{dt}(S_0(U)|U, U) \leq cC(|U|_{W^{1, \infty} X''}, |U|_{L^{\infty} X''}) (S_0(U)|U, U) + cC(|U|_{L^2 X''} |R|_{X''} \sqrt{(S_0(U)|U, U)}.
\]

Proof. Using the self-adjointness of \(S_0(U)\), we have

\[
\frac{1}{2} \frac{d}{dt}(S_0(U)|U, U) = \frac{1}{2} ((\partial_t S_0(U)|U, U) + (S_0(U)|\partial_t U, U),
\]

where

\[
\partial_t S_0(U)[o] = \begin{pmatrix} 0 & 0 \\ 0 & \epsilon G_2[(\partial_t G_2)[o]] \end{pmatrix}.
\]

Then using the symmetrization (2.2), we get

\[
\frac{1}{2} \frac{d}{dt}(S_0(U)|U, U) = \frac{1}{2} ((\partial_t S_0(U)|U, U) - \sum_{j=1}^{d} (\tilde{A}_j(U)|\partial_j U, U) + \epsilon (S_0(U)|R, U) + \epsilon \sum_{j=1}^{d} (F_j(U)|\partial_j U, U).
\]

But using the symmetry of \(\tilde{A}_j\), \(j = 1, 2\), we have

\[
(\tilde{A}_j(U)|\partial_j U, U) = (\partial_j U, \tilde{A}_j(U)|U) = -(U, (\partial_j \tilde{A}_j(U)|U) - (U, \tilde{A}_j(U)|\partial_j U),
\]

where

\[
(\partial_1 \tilde{A}_1(U))[o] = \frac{(\partial_1 B_1(U))[o] + (\partial_1 (B_1(U)^*))}[o],
\]

with

\[
(\partial_1 B_1(U))[o] = \begin{pmatrix} (\partial_1 B_1^{1,1}(U))[o] & (\partial_1 B_1^{1,2}(U))[o] & 0 \\ (\partial_1 B_1^{2,1}(U))[o] & (\partial_1 B_1^{2,2}(U))[o] & 0 \\ 0 & 0 & (\partial_1 B_1^{2,2}(U))[o] \end{pmatrix},
\]

where

\[
\begin{align*}
(\partial_1 B_1^{1,1}(U))[o] &= \epsilon G_2[G_2[\partial_1 v_1]|o], \\
(\partial_1 B_1^{1,2}(U))[o] &= \epsilon G_2[(\partial_1 G_2)[o]], \\
(\partial_1 B_1^{2,1}(U))[o] &= \epsilon (G_1)^2[G_2[\partial_1 v_1]G_2[o]] + \epsilon^2 G_2[(\partial_1 G_2)[o] G_2] + \epsilon^2 G_2[G_2[\partial_1 v_1]G_2[o]], \\
(\partial_1 B_1^{2,2}(U))[o] &= \epsilon (G_2)^2[G_2[\partial_1 v_2]G_2[o]] + \epsilon^2 G_2[G_2[\partial_1 v_2]G_2[o]], 
\end{align*}
\]

and

\[
(\partial_2 \tilde{A}_2(U))[o] = \frac{(\partial_2 B_2(U))[o] + (\partial_2 (B_2(U)^*))}{2},
\]
with
\[
(\partial_2 B_j(U))[\cdot] = \begin{pmatrix}
(\partial_2 B_j^{1,1}(U))[\cdot] & 0 & (\partial_2 B_j^{1,3}(U))[\cdot] \\
0 & (\partial_2 B_j^{2,2}(U))[\cdot] & 0 \\
(\partial_2 B_j^{1,3}(U))[\cdot] & 0 & (\partial_2 B_j^{2,2}(U))[\cdot]
\end{pmatrix},
\]
where
\[
\begin{aligned}
(\partial_2 B_j^{1,1}(U))[\cdot] &= \epsilon G_2[G_2[\partial_2 G_2][\cdot]], \\
(\partial_2 B_j^{1,3}(U))[\cdot] &= \epsilon G_2[(\partial_2 \zeta)G_2[\cdot]], \\
(\partial_2 B_j^{2,2}(U))[\cdot] &= \epsilon(G_1^2[G_2[\partial_2 G_2][\cdot]] + \epsilon^2 G_2[(\partial_2 \zeta)G_2[G_2[\partial_2 G_2][\cdot]]] + \epsilon^2 G_2[\zeta G_2[G_2[\partial_2 G_2][\cdot]]].
\end{aligned}
\]
For any \(j = 1, 2\), \( (\partial_j B_j(U^*))[\cdot] \) is also easily computed and have the same mathematical structure as \( (\partial_j B_j(U))[\cdot] \).

So
\[
\frac{1}{2} \frac{d}{dt} \left( S_0(U) + \sum_{j=1}^d \partial_j A_j(U) \right) U \leq \epsilon \sum_{j=1}^d (F_j(U)[\partial_j U], U) + \epsilon S_0(U)[R], U) + \epsilon \sum_{j=1}^d (F_j(U)[\partial_j U], U).
\]
(2.6)

The term \( \epsilon S_0(U)[R], U \) is easily estimated by Cauchy-Schwarz inequality and (2.5):
\[
\epsilon S_0(U)[R], U \leq \epsilon |S_0(U)[R]|_{Y^0}|U|_{X^0} \leq \epsilon C(|U|_{L^\infty_t X^0})|R|_{Y^0}|U|_{X^0}.
\]
(2.7)

The term \( \epsilon \sum_{j=1}^d (F_j(U)[\partial_j U], U) \) is estimated using the Proposition 2.2. Indeed, we have
\[
\epsilon \sum_{j=1}^d (F_j(U)[\partial_j U], U) \leq \epsilon \sum_{j=1}^d |F_j(U)[\partial_j U]|_{Y^0}|U|_{X^0} \leq \epsilon C(|U|_{L^\infty_t X^{0+1}})|U|_{X^0}^2.
\]
(2.8)

Then, the result comes from the following estimates
\[
|((\partial_1 S_0(U) + \sum_{j=1}^d \partial_j A_j(U))[U], U)| \leq \epsilon C(|U|_{W^{1,\infty}_t X^{0+1}})|U|_{X^0}^2.
\]
(2.9)

We provide below some examples of controls needed to get the latter estimates:
\[
\left\{
\begin{aligned}
|G_2[|\partial_1 \zeta|G_2[v], v)| & \leq \epsilon |\partial_1 \zeta|_{L^\infty_t H^{0+1}}|G_1[v]|_{X^0}^2, \\
|G_2[|\partial_1 \zeta|G_2[|v_1]], v_1)| & \leq \epsilon |\zeta|_{L^\infty_t H^{0+1}}|G_1[v_1]|_{X^0}^2, \\
|G_2[|\partial_1 \zeta|G_2[v_1]G_2[v_1]], v_1)| & \leq \epsilon^2 |\zeta|_{L^\infty_t H^{0+1}}|G_2[G_2[v_1]G_2[v_1]]|_{X^0}^2, \\
|\partial_1 \zeta|G_2[|v_1]|_{L^\infty_t H^{0+1}}|G_1[v_1]|_{X^0}^2, \\
\end{aligned}\right.
\]
where we used the assumption \( |G_2| \leq G_1 \) and the boundedness of the latter Fourier multiplier in \( H^s(\mathbb{R}^d) \) (see Proposition A.3). All other contributions are estimated using the same tools.

Combining (2.6), (2.7), (2.8), (2.9) and (2.4) in Lemma 2.4 we get the desired differential inequality.

We now have all the elements needed to prove the Proposition 2.3.
Proof of Proposition 2.3. By Lemma 2.5, we have
\[
\sqrt{(S_0(U)[U], U)} \frac{d}{dt} \sqrt{(S_0(U)[U], U)} \leq \frac{1}{2} \frac{d}{dt} (S_0(U)[U], U) + \epsilon C \left( \frac{1}{h_{\min}} \| U \|_{L^2} \right) (S_0(U)[U], U) + \epsilon C \left( \frac{1}{h_{\min}} \| U \|_{L^2} \right) (S_0(U)[U], U).
\]
Dividing by \( \sqrt{(S_0(U)[U], U)} \), we get
\[
\frac{d}{dt} \sqrt{(S_0(U)[U], U)} \leq \epsilon \lambda_0 \sqrt{(S_0(U)[U], U)} + \epsilon C \left( \frac{1}{h_{\min}} \| U \|_{L^2} \right) (S_0(U)[U], U).
\]
with \( \lambda_0 = C \left( \frac{1}{h_{\min}} \| U \|_{W^{1, \infty}} \right) \). We can integrate this inequality in time between 0 and \( t \) to get
\[
\sqrt{(S_0(U)[U], U)} \leq e^{\epsilon \lambda_0 t} \sqrt{(S_0(U)[U], U)} \bigg|_{t=0} + \epsilon C \left( \frac{1}{h_{\min}} \| U \|_{L^2} \right) \int_0^t e^{(t-t') \lambda_0} \| R(t') \|^2_{X^0} dt',
\]
And using Lemma 2.4 yields the desired estimate. □

2.3 Energy estimates of higher order

We now prove Proposition 1.4 which we recall here for the sake of clarity.

Proposition 2.6. Let \( s \geq 0, t_0 > d/2 \) and \((G_1, G_2)\) be admissible Fourier multipliers. For any \( \epsilon \in (0, 1] \), \( T > 0, U = (\zeta, v) \in W^{1, \infty}([0, T/\epsilon], X^{t_0}(\mathbb{R}^d)) \cap L^\infty([0, T/\epsilon], X^{\max(t_0+1, s)}(\mathbb{R}^d)) \) for which there exists \( h_{\min} > 0 \) such that for all \( (t, x) \in [0, T/\epsilon] \times \mathbb{R}^d \), (1.4) holds; and for any \( U = (\zeta, v) \in W^{1, \infty}([0, T/\epsilon], X^s(\mathbb{R}^d)) \cap L^\infty([0, T/\epsilon], X^{s+1}(\mathbb{R}^d)) \) satisfying the system
\[
\partial_t U + \sum_{j=1}^d A_j(U) \partial_j U = \epsilon R,
\]
where \( R \in L^\infty([0, T/\epsilon], X^s(\mathbb{R}^d)) \), and for \( j = 1, 2 \), \( A_j(U) \) is defined by (1.3), we have for any \( t \in [0, T/\epsilon] \),
\[
\| U \|_{X^s} \leq \kappa_0 e^{\lambda_0 t} \| U \|_{X^s} \bigg|_{t=0} + \epsilon \nu_0 \int_0^t |R(t')|_{X^s} dt',
\]
where \( \lambda_0, \nu_0 := C \left( \frac{1}{h_{\min}}, T, \| U \|_{W^{1, \infty}} \right) \) \( \| U \|_{L^\infty X^{\max(t_0+1, s)}} \) and \( \kappa_0 := C \left( \frac{1}{h_{\min}}, \| U \|_{X^{t_0}} \right) \).

We begin by proving the following lemma.

Lemma 2.7. With the same assumptions as Proposition 2.6, there exists \( R(s) \in L^\infty([0, T/\epsilon], X^0(\mathbb{R}^d)) \) such that
\[
\partial_t \Lambda^s U + \sum_{j=1}^d A_j(U) \partial_j \Lambda^s U = \epsilon R(s),
\]
with \( \Lambda^s := (1 - \Delta)^{s/2} \) and
\[
|R(s)|_{X^0} \leq C \left( \frac{1}{h_{\min}}, \| U \|_{L^\infty X^{\max(t_0+1, s)}} \right) (\| U \|_{X^s} + |R|_{X^s}).
\]
Proof. We know that
\[ \partial_t U + \sum_{j=1}^{d} A_j(U) \partial_j U = \epsilon R. \]
Applying the operator \( \Lambda^s = (1 - \Delta)^{s/2} \) to this equation, we get
\[ \partial_t \Lambda^s U + \sum_{j=1}^{d} A_j(U) [\partial_j \Lambda^s U] = \epsilon \Lambda^s R - \sum_{j=1}^{d} [\Lambda^s, A_j(U)] [\partial_j U]. \]
But because \( \Lambda^s \) commutes with \( G_1 \) and \( G_2 \), we have
\[
\begin{align*}
[\Lambda^s, A_1(U)] &= \begin{pmatrix}
\epsilon G_2[[\Lambda^s, G_2[v_1]]] & \epsilon G_2[[\Lambda^s, G_2[v_2]]] & 0
\end{pmatrix}, \\
&= \begin{pmatrix}
0 & \epsilon [\Lambda^s, G_2[v_1]] & 0
\end{pmatrix}, \\
&= \begin{pmatrix}
0 & 0 & \epsilon [\Lambda^s, G_2[v_2]]
\end{pmatrix}
\end{align*}
\]
So that using the commutator estimates of Proposition A.2 and \( |G_2| \leq G_1 \in L^\infty(\mathbb{R}^d) \), we get
\[
| \sum_{j=1}^{d} [\Lambda^s, A_j(U)] [\partial_j U] |_{X^0} \leq \epsilon C(\|U\|_{L^\infty_t X^{max} (t_0 + 1, \epsilon)}) |U|_{X^s}.
\]
At the end, denoting \( R_{(s)} = \Lambda^s R - \frac{1}{2} \sum_{j=1}^{d} [\Lambda^s, A_j(U)] [\partial_j U] \), we get the result.
\( \square \)

We now prove Proposition 2.6.

Proof of Proposition 2.6. Using the energy estimates and notations of Proposition 2.3 and Lemma 2.7 we get
\[
|U|_{X^s} \leq \kappa_0 e^{\lambda_0 t} |U|_{X^s}|_{t=0} + \nu_0 \int_0^t C(\frac{1}{\tau_{\min}}, |U|_{L^\infty_t X^{max} (t_0 + 1, \epsilon)}) |U|_{X^s} dt'
+ \epsilon \nu_0 \int_0^t C(\frac{1}{\tau_{\min}}, |U|_{L^\infty_t X^{max} (t_0 + 1, \epsilon)}) |R(t')|_{X^s} dt',
\]
and using Gronwall's lemma, we get
\[
|U|_{X^s} \leq (\kappa_0 e^{\lambda_0 t} |U|_{X^s}|_{t=0} + \epsilon \nu_0 C_s \int_0^t |R(t')|_{X^s} dt') e^{\epsilon \nu_0 C_s t}
\]
with \( C_s = C(\frac{1}{\tau_{\min}}, |U|_{L^\infty_t X^{max} (t_0 + 1, \epsilon)}) \), and hence
\[
|U|_{X^s} \leq \kappa_0 e^{\lambda_0 t} |U|_{X^s}|_{t=0} + \epsilon \nu_0 \int_0^t |R(t')|_{X^s} dt',
\]
where \( \nu_s, \lambda_s := C(\frac{1}{\tau_{\min}}, T, |U|_{W^{1,\infty}_t X^{10}}, |U|_{L^\infty_t X^{max} (t_0 + 1, \epsilon)}) \). This concludes the proof. \( \square \)
3 Local well-posedness and stability

In this section we prove Theorem 1.5, following the regularization technique employed for instance in the Chapter 7 of [15] to symmetrizable quasi-linear hyperbolic systems of conservation laws.

3.1 Well-posedness of the linearized systems

In this subsection we study the local well-posedness of the systems (1.2) linearized around a sufficiently regular state.

Theorem 3.1. Let \( s > d/2 + 1, h_{\text{min}} > 0 \) and \( 0 \leq \epsilon \leq 1 \). Let also \((G_1, G_2)\) be a couple of admissible Fourier multipliers. Let \( V \in W^{1,\infty}([0,T/\epsilon], X^{s-1}(\mathbb{R}^d)) \cap L^\infty([0,T/\epsilon], X^s(\mathbb{R}^d)) \) be such that (1.4) is satisfied. Let \( R \in L^\infty([0,T/\epsilon], X^s(\mathbb{R}^d)) \) and \( U_0 \in X^s(\mathbb{R}^d) \). The Cauchy problem

\[
\begin{cases}
\partial_t U + \sum_{j=1}^d A_j(U) [\partial_j U] = \epsilon R, \\
U|_{t=0} = U_0,
\end{cases}
\]

where \( A_j(U) \) is defined by (1.3), has a unique solution in \( C^0([0,T/\epsilon], X^s(\mathbb{R}^d)) \). Moreover, the solution satisfies the energy estimates (2.10).

As said previously, to prove this result, we use a regularization method.

Let \( J_\alpha = (1 - \alpha \Delta)^{-1/2} \), \( \alpha \in [0,1] \), be a regularizing Fourier multiplier. We have the following properties:

- For \( \alpha > 0 \), \( J_\alpha \) is a regularizing operator of order \(-1\).
- For any \( s \geq 0 \), the family \( \{J_\alpha, \alpha \in [0,1]\} \) is uniformly bounded in \( X^s(\mathbb{R}^d) \).
- For any \( s \geq 0 \), and for all \( v \in X^s(\mathbb{R}^d) \), \( J_\alpha v \to v \) in \( X^s(\mathbb{R}^d) \) as \( \alpha \to 0 \).

We decompose the proof of Theorem 3.1 into several lemmas.

Lemma 3.2. With the same assumptions as in Theorem 3.1, the Cauchy problem (3.1) has a weak solution \( U \in L^\infty([0,T/\epsilon], X^s(\mathbb{R}^d)) \).

Proof. We consider the Cauchy problem

\[
\begin{cases}
\partial_t U_\alpha + \sum_{j=1}^d A_j(U) \partial_j U_\alpha = \epsilon R, \\
U_\alpha|_{t=0} = U_0.
\end{cases}
\]

For \( j = 1,2 \) the operator \( A_j(U) \partial_j J_\alpha \) is bounded in \( X^s(\mathbb{R}^d) \), so that the Cauchy-Lipschitz theorem gives the existence of a solution \( U_\alpha \in C^0([0,T/\epsilon], X^s(\mathbb{R}^d)) \). Because \( J_\alpha \) is a Fourier multiplier of order 0 uniformly in \( \alpha \), and is bounded in \( X^s(\mathbb{R}^d) \) uniformly in \( \alpha \), we can get the same energy estimates for (3.2) as the ones for (3.1). It implies that the sequence \( U_\alpha \) is bounded in \( L^\infty([0,T/\epsilon], X^s(\mathbb{R}^d)) \).

Now recall that \( L^\infty([0,T/\epsilon], X^s(\mathbb{R}^d)) \) is the dual of \( L^1([0,T/\epsilon], Y^{-s}(\mathbb{R}^d)) \). So by the weak* compactness of the closed balls of the dual of a normed space, there exists a subsequence, still denoted \( U_\alpha \), which converges weak* as \( \alpha \to 0 \) to an element \( U \). Passing to the limit in the sense of distributions in (3.2) we get \( U \in L^\infty([0,T/\epsilon], X^s(\mathbb{R}^d)) \) is a weak solution of (3.1).
It still remains to prove that \( U|_{t=0} = U_0 \) makes sense (and holds). Remark that from the equation \( \partial_t U \in L^\infty([0, T/\epsilon], X^{s-1}(\mathbb{R}^d)) \). Hence \( U \in C^0([0, T/\epsilon], X^{s-1}(\mathbb{R}^d)) \), and it makes sense to take the trace at \( t = 0 \) of \( U \) in \( X^{s-1}(\mathbb{R}^d) \), and we do have \( U|_{t=0} = U_0 \) from the limiting process. \( \square \)

**Lemma 3.3.** Let \( R \in L^\infty([0, T/\epsilon], X^s(\mathbb{R}^d)) \) and suppose that \( U \in L^\infty([0, T/\epsilon], X^s(\mathbb{R}^d)) \) satisfies the linearized system (3.1) with \( U_0 = U|_{t=0} \in X^s(\mathbb{R}^d) \). Then \( U \in C^0([0, T/\epsilon], X^s(\mathbb{R}^d)) \) and satisfies the energy estimates (2.10).

**Proof.** Applying \( J_\alpha \) to the system (3.1), we get

\[
\partial_t J_\alpha U + \sum_{j=1}^d A_j(U)[\partial_j J_\alpha U] = \epsilon J_\alpha R - \sum_{j=1}^d [J_\alpha, A_j(U)][\partial_j U]. \tag{3.3}
\]

We denote \( R(\alpha) = J_\alpha - \frac{1}{\epsilon} \sum_{j=1}^d [J_\alpha, A_j(U)][\partial_j U] \). We easily see that \( R(\alpha) \in L^\infty([0, T/\epsilon], X^s(\mathbb{R}^d)) \) using the same argument as for the proof of Lemma 2.7 and the fact that \( J_\alpha \) is of order 0.

Moreover from the density of \( X^{s+1}(\mathbb{R}^d) \) in \( X^s(\mathbb{R}^d) \), we get

\[
[J_\alpha, A_j(U)][\partial_j U] \to 0 \quad \text{in} \quad L^\infty([0, T/\epsilon], X^s(\mathbb{R}^d)),
\]
as \( \alpha \to 0 \). It implies that \( R(\alpha) \to R \) in \( L^\infty([0, T/\epsilon], X^s(\mathbb{R}^d)) \).

We know that \( J_\alpha U \in L^\infty([0, T/\epsilon], X^{s+1}(\mathbb{R}^d)) \), so using the equation (3.3), we get \( \partial_t J_\alpha U \in L^\infty([0, T/\epsilon], X^s(\mathbb{R}^d)) \). The Sobolev embedding in dimension 1 gives \( J_\alpha U \in C^0([0, T/\epsilon], X^s(\mathbb{R}^d)) \). Using the energy estimates of order \( s \) (2.10) on \( J_\alpha U = J_\alpha U \) we get that \( (J_\alpha U)_{\alpha \geq 0} \) is a Cauchy sequence in \( C^0([0, T/\epsilon], X^s(\mathbb{R}^d)) \) as \( \alpha \to 0 \). So \( J_\alpha U \) converges in \( C^0([0, T/\epsilon], X^s(\mathbb{R}^d)) \). But \( J_\alpha U \) converges to \( U \) in \( L^\infty([0, T/\epsilon], X^s(\mathbb{R}^d)) \). Thus \( J_\alpha U \to U \) in \( C^0([0, T/\epsilon], X^s(\mathbb{R}^d)) \) as \( \alpha \to 0 \).

Using again the energy estimates of order \( s \) (2.10) but this time on \( J_\alpha U \) and passing to the limit \( \alpha \to 0 \), we get that \( U \) satisfies the energy estimates of order \( s \). \( \square \)

We now complete the proof of Theorem 3.1.

**Proof.** The two previous lemmas provide the existence of a solution \( U \in C^0([0, T/\epsilon], X^s(\mathbb{R}^d)) \) which satisfies the energy estimates (2.10).

It only remains to prove the uniqueness. For two solutions \( U_1 \) and \( U_2 \) with the same initial condition \( U_0 \in X^s(\mathbb{R}^d) \), the difference \( V = U_1 - U_2 \) satisfies the system

\[
\begin{cases}
\partial_t V + \sum_{j=1}^d A_j(U)[\partial_j V] = 0, \\
V|_{t=0} = 0.
\end{cases}
\]

There remains to use the energy estimates of Proposition 2.3 to infer \( V = 0 \). \( \square \)

### 3.2 Well-posedness of the non-linear systems

This subsection is dedicated to the proof of Theorem 1.5 which we recall here for the sake of clarity.
Theorem 3.4. Let \( s > d/2 + 1 \), \( h_{\text{min}} > 0 \) and \( M > 0 \). Let also \( (G_1, G_2) \) be a couple of admissible Fourier multipliers. There exist \( T > 0 \) and \( C > 0 \) such that for all \( \epsilon \in (0, 1] \), \( U_0 \in X^s(\mathbb{R}^d) \) with \( |U_0|_{X^s} \leq M \) and satisfying (1.4), there exists a unique solution \( U \in C^0([0, T/\epsilon], X^s(\mathbb{R}^d)) \) of the Cauchy problem

\[
\begin{aligned}
\partial_t U + \sum_{j=1}^{d} A_j(U) \partial_j U &= 0, \\
U|_{t=0} &= U_0.
\end{aligned}
\tag{3.4}
\]

Moreover \( |U|_{L^\infty([0,T/\epsilon],X^s)} \leq C|U_0|_{X^s} \).

Proof. Consider the iterative scheme \( U_0(t, x) = U_0(x) \) and for \( n \in \mathbb{N} \)

\[
\begin{aligned}
\partial_t U_{n+1} + \sum_{j=1}^{d} A_j(U_n) \partial_j U_{n+1} &= 0, \\
U_{n+1}|_{t=0} &= U_0.
\end{aligned}
\tag{3.5}
\]

Lemma 3.5. There exists \( T > 0 \) as in Theorem 3.4 such that the sequences \( U_n \) and \( \partial_t U_n \) are well defined and are bounded in respectively \( C^0([0, T/\epsilon], X^s(\mathbb{R}^d)) \) and \( C^0([0, T/\epsilon], X^{s-1}(\mathbb{R}^d)) \).

Proof. We prove by induction that there exists \( C_1, C_2 > 0 \) and \( T > 0 \) as in Theorem 3.4 for all \( n \in \mathbb{N} \)

\[
\sup_{t \in [0,T/\epsilon]} |U_n|_{X^s} \leq C_1 |U_0|_{X^s}, \quad \sup_{t \in [0,T/\epsilon]} |\partial_t U_n|_{X^{s-1}} \leq C_2 |U_0|_{X^s}, \quad \inf_{t \in [0,T/\epsilon], x \in \mathbb{R}^d} (1 + \epsilon \zeta_n(t, x)) \geq h_{\text{min}}/2.
\]

By Theorem 3.1 \( U_{n+1} \) is well-defined and satisfies (2.10). Specifically, on the time interval \([0, T/\epsilon]\) we have

\[
|U_{n+1}|_{X^s} \leq \kappa_0 e^{\lambda_n t} |U_0|_{X^s}.
\]

where \( \lambda_n = C(T, |U_n|_{W^{1,\infty}X^0}, |U_n|_{L^\infty X^\text{max}(t_0+1,s)}) \) and \( \kappa_0 = C(|U_0|_{X^0}|_{t=0}) \).

Also, using the equation and the product estimates of Proposition A.1,

\[
|\partial_t U_{n+1}|_{X^{s-1}} \leq \tilde{C}(|U_n|_{X^s})|U_{n+1}|_{X^s}.
\]

Moreover

\[
\epsilon \zeta_{n+1}(t, x) = \epsilon \zeta_0(x) + \epsilon \int_0^t \partial_t \zeta_{n+1}(T, x)dt'.
\]

But from the Sobolev embedding there exists \( C_s > 0 \) such that

\[
|\partial_t \zeta_{n+1}(t, x)| \leq C_s |\partial_t \zeta|_{L^\infty X^{s-1}}.
\]

So

\[
1 + \epsilon \zeta_{n+1} \geq h_{\text{min}} - TC_s |\partial_t \zeta|_{L^\infty X^{s-1}}.
\]

Let \( C_1 > \kappa_0 \). Let \( C_2 \) be such that \( \tilde{C}(C_1|U_0|_{X^s})C_1 \leq C_2 \). And let \( T \) be sufficiently small so that, \( \kappa_0 e^{\lambda T} \leq C_1 \)

where \( \lambda_n \leq \lambda = C(T, C_1|U_0|_{X^s}, C_2|U_0|_{X^s}) \), and \( TC_1 C_2 |U_0|_{X^s} \leq h_{\text{min}}/2 \).

\[
\square
\]

Lemma 3.6. The sequence \( U_n \) is a Cauchy sequence in \( C^0([0, T/\epsilon], X^0(\mathbb{R}^d)) \).
Proof. Let $V_n := U_{n+1} - U_n$. For $n \geq 1$, it satisfies

$$
\begin{align*}
\partial_t V_n + \sum_{j=1}^d A_j(U_n) \partial_j V_n &= \epsilon R_n, \\
V_{n+1}|_{t=0} &= 0,
\end{align*}
$$

where

$$
R_n = -\frac{1}{\epsilon} \sum_{j=1}^d (A_j(U_n) - A_j(U_{n-1})) \partial_j U_n.
$$

But from the expression of $A_j$ (1.3) and the uniform bounds of the sequence $U_n$ (see Lemma 3.5) and the product estimates of Proposition A.1 it is easy to see that there exists a constant $M > 0$ independent of $n$ such that

$$
|R_n|_{X^0} \leq M|V_{n-1}|_{X^0}.
$$

And from the energy estimates (2.10) and the uniform bounds of $U_n$ and $\partial_t U_n$ (see again Lemma 3.5), there exists a constant $M > 0$ independent of $n$ such that

$$
|V_n|_{X^0} \leq \epsilon M \int_0^t |V_{n-1}(t')|_{X^0} dt'.
$$

So

$$
|V_n|_{X^0} \leq M^n t^n \sup_{t \in [0,T/\epsilon]} |V_0|_{X^0}.
$$

Thus, the series $\sum V_n$ converges in $C^0([0,T/\epsilon],X^0(\mathbb{R}^d))$. \(\square\)

We now complete the proof of Theorem 3.4.

From Lemma 3.6, the sequence $U_n$ converges in $C^0([0,T/\epsilon],X^0(\mathbb{R}^d))$. From Lemma 3.5, the sequence $U_n$ is uniformly bounded in $C^0([0,T/\epsilon],X^s(\mathbb{R}^d))$. So for any $s' < s$, $U_n$ converges in $C^0([0,T/\epsilon],X^{s'}(\mathbb{R}^d))$. Take $s' > t_0 + 1$ and denote by $U$ the limit. The sequences $U_n$, $\partial_t U_n$ and for $j = 1, 2, \partial_j U_n$ converge uniformly in $C^0$ to respectively $U$, $\partial_t U$ and $\partial_j U$. Hence $U$ is solution to (1.2). Moreover, from Lemma 3.5, $U \in L^\infty([0,T/\epsilon],X^s(\mathbb{R}^d))$, $\partial_t U \in L^\infty([0,T/\epsilon],X^{s-1}(\mathbb{R}^d))$, and $U$ satisfies the estimates of Theorem 3.4. So we can consider $U$ as a solution of the linearized system (3.1) taking $\underline{U}$ as $U$. The Theorem 3.1 gives $U \in C^0([0,T/\epsilon],X^s(\mathbb{R}^d))$ and its uniqueness as a solution of the Cauchy problem (3.4). This concludes the proof of Theorem 3.4. \(\square\)

### 3.3 Blow up criterion

From Theorem 3.4, one can define the maximal time existence $T^* > 0$ of the solution $U \in C^0([0,T^*/\epsilon],X^s(\mathbb{R}^d))$ of the Cauchy problem (3.4) associated to an initial condition $U_0 \in X^s(\mathbb{R}^d)$ such that (1.4) holds.

**Proposition 3.7.** We have

$$
T^* < +\infty \implies \lim_{t \to T^*/\epsilon} |U|_{L^\infty([0,t],X^s)} = +\infty.
$$
Proof. Suppose for the sake of contradiction that $T_* = +\infty$ and there exists $M > 0$ such that

$$|U|_{L^\infty([0,T^*/\epsilon], X^{s})} = M.$$ 

Then from Theorem 3.4, there exists $T > 0$ such that for any $\beta > 0$, and $T_\beta = T^* - \beta$, the Cauchy problem (3.4) with initial condition $U(T_\beta)$ has a unique solution in $C^0([T_\beta/\epsilon, T_1/\epsilon], X^s(\mathbb{R}^d))$ with $T_1 = T_\beta + T > T^*$. Taking $\beta = T/2$, by uniqueness, $U$ has an extension $\tilde{U} \in C^0([0, T_1/\epsilon], X^s(\mathbb{R}^d))$ solution of the Cauchy problem (3.4). Thus, necessarily, $T^* = +\infty$.

\[\square\]

4 Stability

In this section we prove the stability result of Proposition 1.6.

**Proposition 4.1.** Let the assumptions of Theorem 3.4 be satisfied and use the notations therein. Assume also that there exists $\tilde{U} \in C([0, 0, T/\epsilon], X^s(\mathbb{R}^d))$ solution of

$$\partial_t \tilde{U} + \sum_{j=1}^d A_j(\tilde{U}) \partial_j \tilde{U} = \tilde{R},$$

where $\tilde{R} \in L^\infty([0, T/\epsilon], X^{s-1}(\mathbb{R}^d))$. Then, the error with respect to the solution $U \in C^0([0, T/\epsilon], X^s(\mathbb{R}^d))$ given by Theorem 3.4 satisfies for all times $t \in [0, \min(T,T)/\epsilon]$,

$$|e|_{L^\infty([0,0], X^{s-1})} \leq C \left(\frac{1}{\nu_{min}} |U|_{L^\infty([0,t], X^s)} \right) \left(\frac{1}{\nu_{max}} |\tilde{U}|_{L^\infty([0,t], X^s)} \right) (|e|_{X^{s-1}}|t=0 + t|\tilde{R}|_{L^\infty([0,t], X^{s-1}}),$$

where $\epsilon := U - \tilde{U}$.

**Proof.** We know that

$$\begin{align*}
\partial_t U + \sum_{j=1}^d A_j(U) \partial_j U &= 0, \\
\partial_t \tilde{U} + \sum_{j=1}^d A_j(\tilde{U}) \partial_j \tilde{U} &= \tilde{R}.
\end{align*}$$

(4.1)

Subtracting both equations we get

$$\partial_t e + \sum_{j=1}^d A_j(U) \partial_j e = \epsilon F,$$

(4.2)

where

$$F = -\frac{1}{\epsilon} \tilde{R} - \frac{1}{\epsilon} \sum_{j=1}^d (A_j(U) - A_j(\tilde{U})) \partial_j \tilde{U}.$$

We can easily estimate $F$ using the product estimates of Proposition A.1 ($s - 1 > d/2$):

$$|F|_{X^{s-1}} \leq \frac{1}{\epsilon} |\tilde{R}|_{X^{s-1}} + |\tilde{U}|_{X^s} |e|_{X^{s-1}}.$$

We use the energy estimates of Proposition 2.6 on (4.2) to get

$$|e|_{X^{s-1}} \leq \kappa_0 \epsilon \lambda_{X^{s-1}} |e|_{X^{s-1}} |t=0 + \nu_{s-1} \int_0^t |\tilde{R}(t')|_{X^{s-1}} dt' + \epsilon \nu_{s-1} \int_0^t |\tilde{U}|_{X^s} |e|_{X^{s-1}} dt'.$$
Using Gronwall’s lemma, we then have
\[ |e|_{X^s} \leq (\kappa_0 e^{\lambda_0 -1}|e|_{X^{s-1}}|_{t=0} + \nu s-1|\tilde{R}(t')|_{L^\infty([0,t],X^{s-1})}|dt') e^{\nu s-1|\tilde{R}|_{L^\infty([0,t],X^s)}}. \]

It only remains to see that using the equation on \( U \) of (4.1) and the product estimates of Proposition A.1, we have for all times \( t \in [0, \min(T,T')/\epsilon] \),
\[ |\partial_t U|_{X^{s-1}} \leq C(|U|_{X^s}), \]
to get the result. \( \square \)

5 Full justification of a class of Whitham-Boussinesq systems

In this section we prove the full justification of a class of Whitham-Boussinesq systems, Theorem 1.11, recalled below.

**Theorem 5.1.** Under the assumption and using the notation of Proposition 1.9, and provided \((G_1, G_2)\) are admissible Fourier multipliers, then for any \( U = (\zeta, \nabla \psi) \in C^0([0, T]/\epsilon, H^{s+4}(\mathbb{R}^d)) \) classical solution of the water waves equations and satisfying the non-cavitation assumption (1.4), there exists a unique \( U_{WB} = (\zeta_{WB}, \nabla \psi_{WB}) \in C^0([0, T]/\epsilon, X^{s+4}(\mathbb{R}^d)) \) classical solution of the Whitham-Boussinesq systems (1.2) with initial data \( U_{WB}|_{t=0} = (\zeta|_{t=0}, \nabla \psi|_{t=0}) \), and one has for all times \( t \in [0, \min(T,T')/\epsilon] \),
\[ |U - U_{WB}|_{L^\infty([0,t],X^{s+1})} \leq C \mu \epsilon t, \]
with \( T \) (provided by Theorem 3.4) and \( C = \frac{1}{\min(|G_1|, |G_2|)} \|U|_{L^\infty([0,T/\epsilon],H^{s+4})} \) uniform with respect to \((\mu, \epsilon) \in (0, 1]^2\).

**Proof.** It is important to remark that, as pointed out in Remark 1.7, the dependency with our previous results with respect to admissible pairs of Fourier multipliers \((G_1, G_2)\) occurs only through Proposition A.3 and Proposition A.4 (in addition to \( |G_2| \leq G_1 \)), and hence through the quantity \(|(G_k, \cdot) \nabla G_k|_{L^\infty}\) for \( k \in \{1, 2\} \). Considering Fourier multipliers of the form \( G^1_0 = G_1(\sqrt{\mu}D) \) and \( G^2_0 = G_2(\sqrt{\mu}D) \) (see Notation 1.8), we can remark that the above quantity is non-increasing as \( \mu \) decreases, and hence all estimates proved in this paper hold uniformly with respect to \( \mu \in (0, 1] \). In particular, the existence time in Theorem 3.4 and the energy estimates of Proposition 2.6 are independent of \( \mu \in (0, 1] \).

Now, from the continuous embedding \( H^s(\mathbb{R}^d) \subset X^{s'}(\mathbb{R}^d) \) (for any \( s' \in \mathbb{R} \)) and Theorem 3.4 we have the existence and uniqueness of \( U_{WB} \in C^0([0,T/\epsilon], X^{s+4}(\mathbb{R}^d)) \) with the control
\[ |U_{WB}|_{L^\infty([0,T/\epsilon],X^{s+1})} \leq |(\zeta|_{t=0}, \nabla \psi|_{t=0})|_{X^{s+1}} \leq |(\zeta|_{t=0}, \nabla \psi|_{t=0})|_{H^{s+4}}, \]
with \( T \) independent of \( \mu \). From Proposition 1.9, we know that \( U \) satisfies
\[ \partial_t U + \sum_{j=1}^d A_j(U) \partial_j U = \mu \epsilon R, \]
where, for any \( t \in [0, T]/\epsilon \), \( |R(t, \cdot)|_{H^{s+4}} \leq C \left( \frac{1}{\min(|G_1|, |G_2|)} \|U|_{L^\infty([0,t],X^{s+1})}, |\nabla \psi|_{H^{s+4}} \right) \). From the stability result of Proposition 4.1, we infer that for all times \( t \in [0, \min(T,T')/\epsilon] \), one has
\[ |U - U_{WB}|_{L^\infty([0,t],X^{s+1})} \leq \mu \epsilon t C \left( \frac{1}{\min(|G_1|, |G_2|)} \|U|_{L^\infty([0,t],X^{s+1})}, |U_{WB}|_{L^\infty([0,t],X^{s+1})}, |R|_{L^\infty([0,t],X^s)} \right). \]
The result follows from combining the previous estimates and using once again the continuous embedding \( H^s(\mathbb{R}^d) \subset X^{s'}(\mathbb{R}^d) \) for \( s' = s \) and \( s' = s + 1 \). \( \square \)
A Technical tools

**Proposition A.1** (Product estimates). Let $t_0 > d/2$, $s \geq -t_0$ and $f \in H^s \cap H^{t_0}(\mathbb{R}^d)$, $g \in H^s(\mathbb{R}^d)$. Then $fg \in H^s(\mathbb{R}^d)$ and

$$|fg|_{H^s} \leq C |f|_{H^{\max(t_0, s)}} |g|_{H^s}$$

with $C$ depending uniquely on $s$ and $t_0$.

*Proof.* See Proposition B.2 in [12].

**Proposition A.2** (Commutator estimates with symbols of order $s$). Let $t_0 > d/2$, $s \geq 0$, and denote $\Lambda^s = (\text{Id} - \Delta)^{s/2}$. Then for any $f \in H^s \cap H^{t_0+1}(\mathbb{R}^d)$ and for all $g \in H^{s-1}(\mathbb{R}^d)$,

$$|[\Lambda^s, f]g|_{L^2} \leq C |f|_{H^{\max(t_0+1, s)}} |g|_{H^{s-1}}$$

with $C$ depending uniquely on $s$ and $t_0$.

*Proof.* See Corollary B.9 in [12].

**Proposition A.3.** Let $G \in L^\infty(\mathbb{R}^d)$. Then for any $s \in \mathbb{R}$ and $f \in H^s(\mathbb{R}^d)$, then $G(D)f \in H^s(\mathbb{R}^d)$ and

$$|G(D)f|_{H^s} \leq |G|_{L^{\infty}} |f|_{H^s}. $$

*Proof.* The result is immediate by Parseval’s theorem.

**Proposition A.4** (Commutator estimates with symbols of order 0). Let $t_0 > d/2$, $s \geq 0$ and $G \in W^{1,\infty}(\mathbb{R}^d)$ be such that $(\cdot) \nabla G \in L^\infty(\mathbb{R}^d)$ and for any $f \in H^s \cap H^{t_0+1}(\mathbb{R}^d)$ then, for all $g \in H^{s-1}(\mathbb{R}^d)$,

$$|[G(D), f]g|_{H^s} \leq C |f|_{H^{\max(t_0+1, s)}} |g|_{H^{s-1}}$$

with $C$ depending uniquely on $s$ and $t_0$, and $|(G, (\cdot) \nabla G)|_{L^{\infty}}$.

*Proof.* See Lemma 2.5 in [10].

References

[1] Cung The Anh. On the Boussinesq/full dispersion systems and Boussinesq/Boussinesq systems for internal waves. *Nonlinear Anal.*, 72(1):409–429, 2010.

[2] J. L. Bona, M. Chen, and J.-C. Saut. Boussinesq equations and other systems for small-amplitude long waves in nonlinear dispersive media. I. Derivation and linear theory. *J. Nonlinear Sci.*, 12(4):283–318, 2002.

[3] J. L. Bona, M. Chen, and J.-C. Saut. Boussinesq equations and other systems for small-amplitude long waves in nonlinear dispersive media. II. The nonlinear theory. *Nonlinearity*, 17(3):925–952, 2004.

[4] Cosmin Burtea. New long time existence results for a class of Boussinesq-type systems. *J. Math. Pures Appl. (9)*, 106(2):203–236, 2016.
[5] Tilahun Deneke, Tamirat T. Dufera, and Achenef Tesfahun. Comparison between Boussinesq-and Whitham–Boussinesq-type systems. *Mathematical Methods in the Applied Sciences*, 2022.

[6] Evgueni Dinvay. On well-posedness of a dispersive system of the Whitham–Boussinesq type. *Applied Mathematics Letters*, 88:13–20, 2019.

[7] Evgueni Dinvay, Denys Dutykh, and Henrik Kalisch. A comparative study of bi-directional systems. *Appl. Numer. Math.*, 141:248–262, 2019.

[8] Evgueni Dinvay, Sigmund Selberg, and Achenef Tesfahun. Well-posedness for a dispersive system of the Whitham-Boussinesq type. arXiv:1902.09438v3, 2019.

[9] Vassilios A. Dougalis, Dimitrios E. Mitsotakis, and Jean-Claude Saut. On some Boussinesq systems in two space dimensions: theory and numerical analysis. *M2AN Math. Model. Numer. Anal.*, 41(5):825–854, 2007.

[10] Vincent Duchêne and Benjamin Melinand. Rectification of a deep water model for surface gravity waves. arXiv preprint: 2203.03277, March 2022.

[11] Louis Emerald. Rigorous derivation from the water waves equations of some full dispersion shallow water models. arXiv:2004.09240, 2020.

[12] Lannes. *The water waves problem: mathematical analysis and asymptotics*. Mathematical surveys and monographs; volume 188. American Mathematical Society, Rhode Island, United-States, 2013.

[13] David Lannes. *The water waves problem*, volume 188 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2013. Mathematical analysis and asymptotics.

[14] Felipe Linares, Didier Pilod, and Jean-Claude Saut. Well-posedness of strongly dispersive two-dimensional surface wave Boussinesq systems. *SIAM J. Math. Anal.*, 44(6):4195–4221, 2012.

[15] Guy Métivier. *Para-differential calculus and applications to the Cauchy problem for nonlinear systems*, volume 5 of *Centro di Ricerca Matematica Ennio De Giorgi (CRM) Series*. Edizioni della Normale, Pisa, 2008.

[16] Mei Ming, Jean Claude Saut, and Ping Zhang. Long-time existence of solutions to Boussinesq systems. *SIAM J. Math. Anal.*, 44(6):4078–4100, 2012.

[17] Martin Oen Paulsen. Long time well-posedness of Whitham-Boussinesq systems. arXiv preprint: 2203.13336, March 2022.

[18] Jean-Claude Saut and Li Xu. The Cauchy problem on large time for surface waves Boussinesq systems. *J. Math. Pures Appl. (9)*, 97(6):635–662, 2012.

[19] Jean-Claude Saut and Li Xu. Long time existence for a two-dimensional strongly dispersive Boussinesq system. *Comm. Partial Differential Equations*, 46(11):2057–2087, 2021.

[20] Achenef Tesfahun. Long-time existence for a Whitham–Boussinesq system in two dimensions. arXiv preprint: 2201.03628, January 2022.