Causality constraints for charged particles

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Abstract

In quantum systems with short-range interactions, causality imposes nontrivial constraints on low-energy scattering parameters. We investigate these causality constraints for systems where a long-range Coulomb potential is present in addition to a short-range interaction. The main result is an upper bound for the Coulomb-modified effective range parameter. We discuss the implications of this bound to the effective field theory for nuclear halo systems. In particular, we consider several examples of proton–nucleus and nucleus–nucleus scattering. For the bound-state regime, we find relations for the asymptotic normalization coefficients (ANCs) of nuclear halo states. As an application of these relations, we extract the ANC of the excited 2+ and 1− states in 16O from α−12C scattering data.

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I. INTRODUCTION

The constraints of causality for two-body scattering with finite-range interactions were first derived by Wigner [1]. The causality bound can be understood as a lower bound on the time delay between the incoming and outgoing wave packets, $\Delta t$. When $\Delta t$ is negative, the outgoing wave packet departs earlier than for the non-interacting system. However, the incoming wave must first reach the interaction region before the outgoing wave can leave. In low-energy scattering this manifests itself as an upper bound on the effective range parameter. In Ref. [2], Phillips and Cohen derived this bound for S-wave scattering with finite-range interactions. Some constraints on nucleon–nucleon scattering and the chiral two-pion exchange potential were considered in Ref. [3], and relations between the scattering length and effective range have been explored for one-boson exchange potentials [4] and van der Waals potentials [5]. In Refs. [6, 7] the causality bounds for finite-range interactions were extended to an arbitrary number of spacetime dimensions and arbitrary angular momentum. The extension to systems with partial wave mixing was first studied in Ref. [8].

In this paper, we consider the causality constraints for the scattering of two charged particles with an arbitrary finite-range interaction. Our analysis is the first study of causality bounds that takes into account the long-range Coulomb force. The results presented here are relevant to studies of low-energy scattering of nuclei and nucleons using effective field theory (EFT). In particular, they are important for the application of effective field theory to nuclear halo systems (cf. Refs. [9, 10]).

This so-called halo EFT utilizes the separation of energy scales between the internal excitations of the core nucleus and soft halo physics. An effective Lagrangian is built order by order from local interactions and used to describe low-energy phenomena such as shallow bound states, charge radii, near-threshold resonances, radiative capture reactions, and soft photodissociation. Halo EFT has been used to describe neutron–alpha scattering [11, 12] and alpha–alpha scattering [13]. Bound single-neutron halo systems such as $^{11}\text{Be}$ [14] and $^{8}\text{Li}$ [15, 16] have been studied as well as various two-neutron halo systems [17–19].

There is an important connection between causality bounds and the convergence of effective field theory calculations with increasing order [8]. For local contact interactions, the range of the effective interaction is controlled by the momentum cutoff scale of the effective theory. In effective theories with non-perturbative renormalization, which typically occur in nuclear physics, exact cutoff-independence can generally not be achieved. There is a “natural” value of the cutoff at which all higher-order corrections scale as expected from dimensional analysis. If the cutoff is taken larger, “new physics” intervenes, the corrections scale unnaturally, and unitarity violations may occur. This is different from what one encounters in high-energy particle physics where the renormalization is typically perturbative and cutoff momenta can be taken arbitrarily large. For calculations using dimensional regularization, the renormalization scale plays a similar role in regulating ultraviolet physics.

The term “new physics”, in this context, refers to details left out (integrated out) in the effective theory. In the case of halo EFT, these details are the finite size of the core nucleus and its internal excitations as well as the exponential tail of the pion-exchange interaction. Problems with convergence of the effective theory can occur if the cutoff scale is set higher than the scale of the new physics. However, it is useful to have a more quantitative measure of when problems may appear, and this is where the causality bound provides a useful diagnostic tool. For each scattering channel we use the physical scattering parameters to compute a quantity called the causal range, $R_c$. It is the minimum range for finite-range
interactions consistent with the requirements of causality and unitarity. For any fixed cutoff scale, the causality bound marks a branch cut of the effective theory when viewed as a function of physical scattering parameters \cite{8}. The coupling constants of the effective theory become complex when scattering parameters violating the causality bound are enforced. These branch cuts do not appear in perturbation theory; however, a nearby branch point can spoil the absolute convergence of the perturbative expansion.

Our results can be viewed as a guide for improving the convergence of halo EFT calculations. In particular, if the cutoff momentum used in a calculation is too high, then problems with convergence may appear in some observables. Consequently, the causal range can be used to estimate the “natural” ultraviolet cutoff $\Lambda$ of the effective theory as $R_c^{-1}$. The natural cutoff is optimal in the sense that no known infrared physics is left out of the theory and that all corrections involving the ultraviolet cutoff scale naturally \cite{20, 21}. Increasing the cutoff beyond the natural value will not improve the accuracy of the calculation.

The causality bounds also have an impact in the regime of bound states. For two-body halo states, or more generally whenever there is a shallow two-body bound state close to threshold, the same integral identity that yields the causality bound for the effective range can be used to derive a relation between the asymptotic normalization constant (ANC) of the bound-state wave function, the binding momentum, and the effective range for the scattering of the two halo constituents. This relation can be shown to be equivalent to a result previously derived by Sparenberg et al. \cite{22}. Its significance lies in the fact that the ANC is an important input parameter for the calculation of near-threshold radiative capture and photodissociation reactions. The causality bounds also constrain the range of model potentials that are fitted to scattering data in order to extract ANCs.

The organization of this paper is as follows. We first review the general theory of scattering for two charged particles with additional short-range interactions. Our analysis includes both attractive and repulsive Coulomb forces. In the next section we derive the charged-particle causality bounds for arbitrary values of the orbital angular momentum. Using the causality bounds, we extract and discuss the causal range for several nuclear scattering processes including proton–proton, proton–deuteron, proton–$^3$He, proton–alpha, and alpha–alpha scattering. We then elucidate the relation for the ANC and extract the ANCs of the excited $2^+$ and $1^-$ states in $^{16}$O from $\alpha–^{12}$C scattering data as an application. We conclude with a summary of the main results and provide an outlook. In the appendices, we provide technical details of the derivation as well as numerical examples.

II. TWO CHARGED PARTICLES WITH SHORT-RANGE INTERACTIONS

In this section, we review some preliminaries that we will need later in order to derive the desired causality bounds. We consider a two-particle system with reduced mass $\mu$ interacting via a finite-range potential with range $R$. We write the interaction as a real symmetric operator with kernel $V(r, r')$ satisfying the finite-range condition,

$$V(r, r') = 0 \text{ if } r > R \text{ or } r' > R. \quad (1)$$

In particular, we assume that the interaction is energy-independent. After giving a detailed formal derivation of the causality bounds in what follows, we will come back to the question what the above assumptions mean for the application to (halo) EFT calculations in Sec. [V].
In the absence of Coulomb interactions the system is described by the radial Schrödinger equation,
\[ p^2 u_\ell(r) = -\frac{d^2}{dr^2} u_\ell(r) + \frac{\ell(\ell + 1)}{r^2} u_\ell(r) + 2\mu \int_0^R dr' V(r,r') u_\ell(r'), \quad (2) \]
where by \( u_\ell^{(p)} \) we explicitly indicate a solution for center-of-mass momentum \( p \) in the following. Following the conventions of Ref. [7], the normalization of \( u_\ell^{(p)} \) is chosen such that for \( r \geq R \) we have
\[ u_\ell^{(p)}(r) = p^\ell \left[ \cot \delta_\ell^{(p)} S_\ell^{(p)}(pr) + C_\ell^{(p)}(pr) \right], \quad (3) \]
where \( S_\ell \) and \( C_\ell \) are the Riccati-Bessel functions and \( \delta_\ell(p) \) is the scattering phase shift.

If the particles carry electromagnetic charges \( Z_1e \) and \( Z_2e \), respectively, there is a Coulomb potential in addition to the finite-range interaction. The radial Schrödinger equation now reads
\[ p^2 w_\ell(r) = -\frac{d^2}{dr^2} w_\ell(r) + \frac{\ell(\ell + 1)}{r^2} w_\ell(r) + 2\mu \int_0^R dr' V(r,r') w_\ell(r') + \frac{\gamma}{r} w_\ell(r) \quad (4) \]
with the Coulomb parameter
\[ \gamma = 2\mu \cdot \alpha Z_1 Z_2. \quad (5) \]
The normalization of \( w_\ell^{(p)} \) is now chosen such that for \( r \geq R \) we have
\[ w_\ell^{(p)}(r) = p^\ell C_{\eta,\ell} \left[ \cot \tilde{\delta}_\ell^{(p)} F_\ell^{(p)}(pr) + G_\ell^{(p)}(pr) \right], \quad (6) \]
where \( F_\ell^{(p)} \) and \( G_\ell^{(p)} \) are the regular and irregular Coulomb wave functions [23], respectively, and \( \tilde{\delta}_\ell \) is the phase shift of the full solution \( w_\ell^{(p)} \) compared to the regular Coulomb function \( F_\ell^{(p)} \) [24]. The factor \( C_{\eta,\ell} \) is given by
\[ \eta = \frac{\gamma}{2p}, \quad (7) \]
\[ C_{\eta,0}^2 = \frac{2\pi \eta}{e^{2\pi \eta} - 1}, \quad (8) \]
and
\[ C_{\eta,\ell}^2 = \frac{2^{2\ell}}{[(2\ell + 1)!]^2} \prod_{s=1}^{\ell} (s^2 + \eta^2) \cdot C_{\eta,0}^2. \quad (9) \]

\footnote{Note that for \( \ell = 0 \) our normalization is the same as chosen in Ref. [24], i.e., for \( r \geq R \) our solution \( w_0^{(p)} \) coincides with the function \( \varphi \) defined in Eq. (42) of that paper.}
A. Coulomb wave functions

For the Coulomb wave functions $F^{(p)}_{\ell}(r)$ and $G^{(p)}_{\ell}(r)$ we use the conventions introduced by Yost, Breit and Wheeler in Ref. [23]. Explicitly, we write them as [25, 26]

\begin{equation}
F^{(p)}_{\ell}(r) = \frac{1}{2} \left| \frac{e^{i\frac{\pi}{2}\kappa}}{\Gamma(2m + 1)} \right| e^{-i\frac{\pi}{2}(\ell + m)} M_{\kappa,m}(z)
\end{equation}

\begin{equation}
G^{(p)}_{\ell}(r) = \frac{\Gamma\left(\frac{1}{2} + m - \kappa\right)}{\Gamma\left(\frac{1}{2} + m - \kappa\right)} e^{-i\frac{\pi}{2}(\ell + m + \kappa)} W_{\kappa,m}(z) + i F^{(p)}_{\ell}(r),
\end{equation}

where

\begin{align}
\rho &= p \cdot r \text{ , } z = 2i\rho \text{ , } \kappa = i\eta \text{ , } m = \ell + \frac{1}{2} .
\end{align}

The functions $M_{\kappa,m}$ and $W_{\kappa,m}$ are Whittaker functions, which can be expressed in terms of hypergeometric functions as

\begin{align}
M_{\kappa,m}(z) &= e^{-\frac{1}{2}z\frac{1}{2} + m} {}_1F_1\left(\frac{1}{2} + m - \kappa, 1 + 2m; z\right), \\
W_{\kappa,m}(z) &= e^{-\frac{1}{2}z\frac{1}{2} + m} U\left(\frac{1}{2} + m - \kappa, 1 + 2m; z\right).
\end{align}

$_1F_1(a,b;z)$ is Kummer’s function of the first kind,

\begin{equation}
_1F_1(a,b;z) = \sum_{n=0}^{\infty} \frac{a^{(n)} z^n}{b^{(n)} n!} , \quad a^{(n)} = a(a + 1) \cdots (a + n - 1),
\end{equation}

and

\begin{equation}
U(a,b;z) = \frac{\Gamma(1 - b)}{\Gamma(1 + a - b)} {}_1F_1(a,b;z) + \frac{\Gamma(b - 1)}{\Gamma(a)} z^{1-b} {}_1F_1(a-b+1,2-b;z).
\end{equation}

A comprehensive discussion of the functions $F^{(p)}_{\ell}(r)$ and $G^{(p)}_{\ell}(r)$ can be found in Ref. [27]. For asymptotically large $\rho = pr$, the Coulomb wave functions behave as

\begin{align}
F^{(p)}_{\ell}(r) &\sim \sin(\rho - \ell \pi/2 - \eta \log(2\rho) + \sigma_{\ell}) , \\
G^{(p)}_{\ell}(r) &\sim \cos(\rho - \ell \pi/2 - \eta \log(2\rho) + \sigma_{\ell})
\end{align}

with the Coulomb phase shift

\begin{equation}
\sigma_{\ell} = \arg \Gamma(\ell + 1 + i\eta).
\end{equation}

In the limit $\rho \to 0$, their behavior is

\begin{align}
F^{(p)}_{\ell}(r) &\sim C_{\eta,\ell} \rho^{\ell+1} , \\
G^{(p)}_{\ell}(r) &\sim \frac{\rho^{-\ell}}{C_{\eta,\ell}(2\ell + 1)}.
\end{align}
B. Effective range expansions

For a system without Coulomb force and only a short-range interaction, one has the well-known effective range expansion

$$p^{2\ell+1} \cot \delta_\ell(p) = -\frac{1}{a_\ell} + \frac{1}{2} r_\ell p^2 + \cdots,$$

(20)

where $a_\ell$ and $r_\ell$ are the scattering and effective range parameters, respectively. For a system of charged particles we have instead the Coulomb-modified effective range expansion

$$C_{n,\ell}^2 p^{2\ell+1} \cot \tilde{\delta}_\ell(p) + \gamma h_\ell(p) = -\frac{1}{a_{\ell}^C} + \frac{1}{2} r_{\ell}^C p^2 + \cdots,$$

(21)

where

$$h_\ell(p) = p^{2\ell} \frac{C_{n,\ell}^2}{C_{n,0}^2} h(\eta),$$

(22)

$$h(\eta) = \text{Re} \psi(i\eta) - \log |\eta|,$$

(23)

and $\psi(z) = \Gamma'(z)/\Gamma(z)$ is the logarithmic derivative of the Gamma function, also called digamma function. For $\ell = 0$, the expansion simplifies to

$$C_{n,0}^2 p \cot \tilde{\delta}_0(p) + \gamma h(\eta) = -\frac{1}{a_0^C} + \frac{1}{2} r_0^C p^2 + \cdots.$$

(24)

A derivation of Eq. (24) for the case of proton–proton scattering can be found, for example, in Ref. [24]. See also Ref. [28] for a detailed discussion. The analytic properties of the $\ell = 0$ modified effective range function are investigated in Ref. [29].

In Ref. [30], Bollé and Gesztesy derived a very general form of the Coulomb-modified effective range expansion for an arbitrary number of spatial dimensions. Specializing their result to the three-dimensional case, a version of Eq. (21) can be written as

$$C_{n,\ell}^2 p^{2\ell+1} \left( \cot \tilde{\delta}_\ell(p) - i \right) + \gamma \tilde{h}_\ell(p) = -\frac{1}{a_{\ell}^C} + \frac{1}{2} r_{\ell}^C p^2 + \cdots$$

(25)

with

$$\tilde{h}_\ell(p) = \frac{(2p)^{2\ell}}{\Gamma(2\ell + 2)^2} \frac{\Gamma(\ell + 1 + i\eta)^2}{|\Gamma(1 + i\eta)|^2} \left( \psi(i\eta) + \frac{1}{2i\eta} - \log(i\eta) \right).$$

(26)

As a remark we note that on first sight the expansion given in Eq. (51) of Bethe’s paper [24] seems to be different from the one given here in Eq. (24), which is the same as given in later publications referring to Bethe’s result. The $\eta$-dependent function on the left hand side of Bethe’s expansion appears to differ from our $h(\eta)$ by two times the Euler-Mascheroni constant $\gamma_E$. This apparent conflict can be resolved by noting that the $g(\eta)$ in Eq. (51) of Ref. [24] is not the function defined in Eq. (47a) of the same paper, but rather given by $\lim_{\eta \to \infty} [g(\eta) - g(\eta_1)]$, where in this latter expression the $g$ from Eq. (47a) is meant. The limiting process then yields exactly the term $-2\gamma_E$.

This definition essentially comes from combining Eqs. (4.1) and (4.2) in Ref. [30], with the correction that the exponent in Eq. (4.2) should be $-2$ rather than 2.
The latter function can be rewritten using
\[
C_{\eta,\ell}^2 = \frac{2^{2\ell}}{\Gamma(2\ell + 2)^2} \frac{|\Gamma(\ell + 1 + i\eta)|^2}{|\Gamma(1 + i\eta)|^2} \cdot C_{\eta,0}^2,
\] (27)

with \( C_{\eta,0}^2 \) as defined in Eq. [8]. The expressions given here reproduce Eqs. (21) and (9) when one explicitly assumes that the momentum \( p \) is real. In fact, one has to rewrite Eq. (26) in this manner in order to get an effective range function that is analytic in \( p^2 \) around threshold.

The conventions for the Coulomb-modified effective range expansion for general \( \ell \) that we use throughout this work, Eqs. [21] and [25], are the same as in Ref. [30]. We note, however, that a different convention is also used in the literature which differs from ours by an overall momentum-independent factor. The effective range expansion given in Refs. [31–33] uses the normalization
\[
\left( \frac{\Gamma(2\ell + 2)}{2\ell \Gamma(\ell + 1)} \right)^2 \left[ C_{\eta,\ell}^2 p^{2\ell+1} \left( \cot \delta_{\ell}(p) - i \right) + \gamma \tilde{h}_{\ell}(p) \right] = -\frac{1}{\hat{a}_{\ell}^2} + \frac{1}{2} \tilde{r}_{\ell}^C p^2 + \cdots.
\] (28)

This choice of normalization has the advantage of having a more direct correspondence to the ordinary effective range expansion without Coulomb effects. For \( \ell = 0 \), both conventions give the same expression.

### III. CAUSALITY BOUNDS FOR CHARGED PARTICLES

With the Coulomb wave functions and Coulomb-modified effective range expansions at our hands, we can now closely follow the derivation presented in Ref. [7] for scattering in the absence of Coulomb interactions.

#### A. Wronskian identities

We consider solutions of the radial Schrödinger equation [4] for two different momenta \( p_A \) and \( p_B \). Introducing the short-hand notation
\[
w_{A,B}(r) = w_{(pA,B)}^{(\ell)}(r),
\] (29)
i.e., suppressing the angular-momentum subscript \( \ell \) for convenience, we get
\[
(p_B^2 - p_A^2) \int_{\epsilon}^r \, dr' w_A(r') w_B(r') = (w_B w_A' - w_A w_B') \bigg|_{\epsilon}^r
- 2\mu \int_{\epsilon}^r \, dr' \int_0^R \, dr \, (w_B(r)V(r,r') w_A(r') - w_A(r)V(r,r') w_B(r'))
\] (30)

by subtracting \( w_A \) times the equation for \( w_B \) from that for \( w_B \) multiplied by \( w_A \), as it is done in Ref. [7], and integrating from a small radius \( \epsilon \) to \( r \).

We assume that our interaction \( V(r,r') \) is such that it alone permits a solution that is sufficiently regular at the origin, i.e., \( u_\ell(0) = 0 \) and \( \partial_r u_\ell \) stays finite as \( r \rightarrow 0 \), where \( u_\ell \) is a solution of Eq. [2]. As boundary condition for the solutions \( w_{A,B} \) of the full radial Schrödinger equation we can then demand as well that they vanish with finite derivative.
at the origin. If we only had the Coulomb potential and no additional interaction, this is fulfilled by the regular Coulomb function $F_{\ell}^{(p)}(r)$, cf. Eq. (19a). We can thus take the limit $\epsilon \to 0$ in Eq. (30) and get the relation

$$W[w_B, w_A](r) = (p_B^2 - p_A^2) \int_0^r dr' w_A(r')w_B(r'),$$

(31)

where the Wronskian $W[w_B, w_A]$ is defined as

$$W[w_B, w_A](r) = w_B(r)w_A'(r) - w_B'(r)w_A(r).$$

(32)

**B. Rewriting the wave functions**

Following further the derivation presented in Ref. [7], we re-express the solutions $w_{\ell}^{(p)}(r)$ in terms of functions $f(p, r)$ and $g(p, r)$ such that

$$w_{\ell}^{(p)}(r) = p^{2\ell+1}C_{\eta,\ell}^2 \cot \bar{\delta}(p) f(p, r) + g(p, r)$$

(33)

for $r \geq R$, with $f(p, r)$ analytic in $p^2$,

$$f(p, r) = f_0(r) + f_2(r)p^2 + O(p^4),$$

(34)

and

$$g(p, r) = \bar{g}(p, r) + \phi(p) \cdot f(p, r).$$

(35a)

The $g(p, r)$ contains a term which is non-analytic in $p^2$ and is proportional to $f(p, r)$. The remainder \(\bar{g}(p, r)\), however, is analytic in $p^2$,

$$\bar{g}(p, r) = g_0(r) + g_2(r)p^2 + O(p^4).$$

(35b)

Explicit expressions for these functions can be obtained from the results of Bollé and Gesztesy [30]; they are discussed further in Appendix B. Combining Eqs. (6) and (33), we find

$$f(p, r) = \frac{1}{p^{\ell+1}C_{\eta,\ell}} F_{\ell}^{(p)}(r)$$

(36a)

and

$$g(p, r) = p^{\ell}C_{\eta,\ell} G_{\ell}^{(p)}(r).$$

(36b)

We insert now the modified effective range expansion (21) into the asymptotic Coulomb wave function (33). Note also that

$$\phi(p) = \gamma \tilde{h}_\ell(p) - ip^{2\ell+1}C_{\eta,\ell}^2 = \gamma h_\ell(p),$$

(37)

where the last step follows from the relations in Appendix B. The term involving $h_\ell(p)$ exactly drops out and we are left with

$$w_{\ell}^{(p)}(r) = \left( -\frac{1}{a_\ell^2} + \frac{1}{2} r_\ell^C p^2 + \cdots \right) f(p, r) + \bar{g}(p, r), \quad r \geq R.$$  

(38)

Thus it is possible to choose a normalization such that $w_{\ell}^{(p)}(r)$ is analytic in $p^2$. Combining this with the expansions (34) and (35), we arrive at

$$w_{\ell}^{(p)}(r) = -\frac{1}{a_\ell^2} f_0(r) + g_0(r) + p^2 \left[ \frac{1}{2} r_\ell^C f_0(r) - \frac{1}{a_\ell^2} f_2(r) + g_2(r) \right] + O(p^4).$$

(39)
C. Causality bounds

From here, we can proceed exactly as in Ref. [7]. For the Wronskian of two solutions $w_A$ and $w_B$ for $r \geq R$ we find

$$W[w_B, w_A](r) = (p_B^2 - p_A^2) \left\{ \frac{1}{2} C^C W[f_0, g_0](r) + \left( \frac{1}{a_C^2} \right)^2 W[f_2, f_0](r) \right.$$ 

$$\left. - \frac{1}{a_C^2} [W[f_2, g_0](r) - W[g_2, f_0](r)] + W[g_2, g_0](r) \right\} + O(p_{A,B}^4). \quad (40)$$

Note that in the $O(p_{A,B}^4)$ we have also included terms of the form $p_A^2 p_B^2$. We set $p_A = 0$ in Eq. (31) and furthermore take the limit $p_B \to 0$. Using the expansion (40), we get

$$- r_C^C W[f_0, g_0](r) = b_C^C(r) - 2 \int_0^r dr' \left[ w^{(0)}_C(r') \right]^2 \quad (41)$$

for $r \geq R$, with $w^{(0)}_C(r) = \lim_{p \to 0} w^{(p)}_C(r)$ and

$$b_C^C(r) = 2W[g_2, g_0](r) - \frac{2}{a_C^2} \left( W[f_2, g_0](r) + W[g_2, f_0](r) \right) + \frac{2}{(a_C^2)^2} W[f_2, f_0](r). \quad (42)$$

Written as a function of $\rho = p \cdot r$, the Wronskian of the Coulomb wave functions is\footnote{See, for example, Eq. (14.2.4) in Ref. [34].}

$$W[F^{(p)}_C, G^{(p)}_C](\rho) = -W[G^{(p)}_C, F^{(p)}_C](\rho) = -1. \quad (43)$$

Since $d/dr = p \cdot d/d\rho$ and $W[f, f] \equiv 0$, we also have

$$W[f, g](r) = W[f, \bar{g}](r) = -1. \quad (44)$$

Plugging in the expansions (34) and (35), we see that $W[f_0, g_0](r) = -1$ for the leading-order functions, and $W[f_2, g_0](r) = W[g_2, f_0](r)$ for the terms at $O(p^2)$. Inserting these relations into Eq. (41), we get

$$r_C^C = b_C^C(r) - 2 \int_0^r dr' \left[ w^{(0)}_C(r') \right]^2, \quad (45)$$

where $b_C^C(r)$ has been simplified to

$$b_C^C(r) = 2W[g_2, g_0](r) - \frac{4}{a_C^2} W[f_2, g_0](r) + \frac{2}{(a_C^2)^2} W[f_2, f_0](r). \quad (46)$$

Since the integral in Eq. (45) is positive definite, the resulting causality bound is

$$r_C^C \leq b_C^C(r), \ \forall \ r \geq R. \quad (47)$$
D. Calculating the Wronskians

We now want to derive the explicit form of the function $b_C^\ell(r)$. To do this, we need expressions for the Wronskians that appear in Eq. (46). We can obtain them by first noting that $f(p, r)$ and $\tilde{g}(p, r)$, being linear combinations of Coulomb wave functions (with $p$-dependent coefficients), are solutions of the Coulomb Schrödinger equation,

$$\left[ -\frac{d^2}{dr^2} + \ell(\ell + 1) \frac{r^2}{r^2} + \gamma \frac{r}{r} - p^2 \right] x(p, r) = 0,$$

which, of course, corresponds to setting $V(r, r') = 0$ in Eq. (4). Here and in the following, $x$ stands for either $f$ or $\tilde{g}$. Inserting the expansion

$$x(p, r) = x_0(r) + p^2 x_2(r) + \mathcal{O}(p^4)$$

into Eq. (48) and comparing orders in $p^2$, we find that

$$\left[ -\frac{d^2}{dr^2} + \ell(\ell + 1) \frac{r^2}{r^2} + \gamma \right] x_0(r) = 0,$$

i.e., $x_0$ is a solution of the zero-energy Coulomb Schrödinger equation, and

$$\left[ -\frac{d^2}{dr^2} + \ell(\ell + 1) \frac{r^2}{r^2} + \gamma \right] x_2(r) = x_0(r).$$

From this we readily obtain the differential equations

$$\frac{d}{dr} W[f_2, f_0](r) = |f_0(r)|^2,$$  
(52a)

$$\frac{d}{dr} W[g_2, g_0](r) = |g_0(r)|^2,$$  
(52b)

$$\frac{d}{dr} W[f_2, g_0](r) = f_0(r)g_0(r)$$  
(52c)

for the desired Wronskians. Put together, this yields a simple first-order differential equation for $b_C^\ell(r)$,

$$\frac{d}{dr} b_C^\ell(r) = 2 \left( g_0(r) - \frac{1}{a_C f_0(r)} \right)^2.$$

(53)

From Eqs. (A.7) and (A.8) in Ref. [30] we have the explicit expressions

$$f_0(r) = \frac{(2l + 1)!}{\sqrt{\gamma^{2l+1}}} \sqrt{r} I_{2\ell+1}(2\sqrt{|\gamma r|}),$$  
(54a)

$$g_0(r) = \frac{2\sqrt{\gamma^{2l+1}}}{(2l + 1)!} \sqrt{r} K_{2\ell+1}(2\sqrt{|\gamma r|})$$  
(54b)

for $\gamma > 0$, where $I_\alpha$ and $K_\alpha$ are modified Bessel functions, and

$$f_0(r) = \frac{(2l + 1)!}{\sqrt{(-\gamma)^{2l+1}}} \sqrt{r} J_{2\ell+1}(2\sqrt{-\gamma r}),$$  
(55a)

$$g_0(r) = \frac{-\pi \sqrt{(-\gamma)^{2l+1}}}{(2l + 1)!} \sqrt{r} Y_{2\ell+1}(2\sqrt{-\gamma r})$$  
(55b)
for $\gamma < 0$, where $J_\alpha$ and $Y_\alpha$ are the ordinary Bessel functions.$^5$,$^6$

Using these expressions for $f_0$ and $g_0$ and Eq. (53) we can determine $b^C_\ell(r)$ up to an integration constant. In order to fix this constant, we must work directly with the Wronskians in Eq. (42). Before we do that, however, we first discuss the general form of $b^C_\ell(r)$. Let us break apart the function as a sum of two functions, $X_\ell(r)$ and $Y_\ell(r)$, and a constant term $Z_\ell$,

$$b^C_\ell(r) = X_\ell(r) + Y_\ell(r) + Z_\ell.$$  

We take $X_\ell(r)$ to be a function consisting entirely of a sum of terms that have a pole at $r = 0$, ranging from order 1 to $\ell$,

$$X_\ell(r) = \sum_{m=1}^\ell X_{\ell,m} r^{-m}.  \quad (57)$$

By furthermore requiring the function $Y_\ell(r)$ to vanish at $r = 0$, the decomposition in Eq. (56) is unique.

Where exactly the contributions to the three terms in the decomposition come from can be inferred from the behavior of $f(p,r)$ and $g(p,r)$ at the origin. From Eq. (19a) combined with Eq. (36) we find that $f(p,r) \sim r^{\ell+1}$ as $r \to 0$. Therefore,

$$\lim_{r \to 0} W[f_2, f_0](r) = 0 \quad (59)$$

for all $\ell$, which means that this Wronskian only yields contributions to the $Y_\ell(r)$.

Furthermore, from Eq. (17) in Ref. [23] we know that the irregular Coulomb wave function has the asymptotic behavior

$$G_\ell^{(p)} \sim D_{\eta,\ell} \rho^{-\ell} \quad \text{as} \quad \rho \to 0, \quad (60)$$

with $D_{\eta,\ell}$ fulfilling $C_{\eta,\ell} D_{\eta,\ell} = 2\ell + 1$. Using Eq. (36b) then yields

$$g(p,r) \sim \frac{r^{-\ell}}{2\ell + 1} \quad \text{as} \quad r \to 0. \quad (61)$$

We note that $g_0(r)$ has exactly the same behavior near $r = 0$ and can thus show that $g_2(r)$ is subleading as $r \to 0$, $g_2(r) \sim r^{-\ell+c}$ for $c > 0$. From this we infer that

$$\lim_{r \to 0} W[f_0, g_2](r) = 0 \quad (62)$$

for all $\ell$, so also from this Wronskian we only get contributions to $Y_\ell(r)$. Both the singular $X_\ell(r)$ and the constant $Z_\ell$, therefore, only come from the Wronskian $W[g_2, g_0](r)$.

$^5$From Eq. (9.1.50) and the remark above Eq. (9.6.41) in Ref. [34] it is clear that these $f_0$ and $g_0$ are indeed solutions of (50).

$^6$Bollé and Gesztesy actually give an expression for $g_0$ in the attractive case ($\gamma < 0$) that involves the Hankel function $H^{(2)}$ times $i$ instead of $Y$. With that, however, $g_0$ would not be real, which it should be. Our $g_0$ as in Eq. (55b) is taken from the results of Lambert [35].
For \( \ell = 0 \) the situation is still simple because the above analysis also tells us that

\[
\lim_{r \to 0} W[g_2, g_0](r) = 0 \quad \text{for} \quad \ell = 0,
\]

i.e., \( b^C_0(r) \) is given entirely by \( Y_0(r) \). With the knowledge that it vanishes at the origin, it is actually straightforward to give an explicit expression for \( b^C_0(r) \) in terms of antiderivatives of the right hand side of Eq. \((53)\), where one has to insert the \( f_0(r) \) and \( g_0(r) \) from Eqs. \((54)\) and \((55)\). The result, obtained by integrating from 0 to \( r \), is

\[
b^C_0(r) = \frac{2r^3}{3} \left( a^C_0 \right)^{-2} {}_1F_2 \left( \frac{3}{2}, 2, 4; 4\gamma r \right) - \frac{4r^2}{\sqrt{\pi}} \left( a^C_0 \right)^{-1} G^{2,1}_{1,3} \left( 4\gamma r \right| 0, 1, -2 \right) \\
+ 4\sqrt{\pi} \gamma r^2 G^{3,1}_{2,4} \left( 4\gamma r \right| -1, \frac{1}{2}, 0, 1, -2 \right)
\]

\[(64)\]

for the repulsive case, and

\[
b^C_0(r) = \frac{2r^3}{3} \left( a^C_0 \right)^{-2} {}_1F_2 \left( \frac{3}{2}, 2, 4; 4\gamma r \right) + 4\sqrt{\pi} r^2 \left( a^C_0 \right)^{-1} G^{2,1}_{2,4} \left( -4\gamma r \right| 0, 1, -2, -\frac{1}{2} \right) \\
+ 2\pi^2 \left[ \gamma^2 r^3 \right] \left[ {}_1F_2 \left( \frac{3}{2}, 2, 4; 4\gamma r \right) - \frac{2\gamma r^2}{\sqrt{\pi}} G^{3,1}_{3,5} \left( -4\gamma r \right| -1, \frac{1}{2}, -\frac{1}{2}, 0, 1, -2, -\frac{1}{2} \right) \right]
\]

\[(65)\]

for an attractive Coulomb interaction. In the above equations, \( pF_q \) and \( G_{p,q}^{m,n} \) denote the (generalized) hypergeometric and Meijer \( G \)-functions, respectively.

For general \( \ell \geq 1 \), \( W[g_2, g_0](r) \) is singular at \( r = 0 \) and the analysis becomes more complicated. For practical purposes, one can simply use power-series expansions for the Bessel functions that appear in the expressions for the zero-energy functions and integrate these term by term until a desired precision is reached. The only additional ingredients needed are the values for the constant terms \( Z_\ell \) because these are obviously not generated by the integration. In Table I we list these constants for \( \ell = 0, 1, 2 \).

| \( \ell \) | \( 0 \) | \( 1 \) | \( 2 \) |
|---|---|---|---|
| \( Z_\ell \) | \( 0 \) | \( \gamma \left( \frac{1}{6} - \frac{2\gamma E}{9} \right) \) | \( \gamma^3 \left( \frac{79}{21600} - \frac{\gamma E}{360} \right) \) |

TABLE I: Constant term \( Z_\ell \) in Eq. \((56)\) for \( \ell = 0, 1, 2 \). \( \gamma \approx 0.577216 \ldots \) is the Euler-Mascheroni constant. The values are the same for repulsive (\( \gamma > 0 \)) and attractive (\( \gamma < 0 \)) Coulomb potentials.

In Appendix A we describe how the values in Table I can be obtained with the help of computer algebra software. In that appendix we also give explicit expressions for the causality bound functions \( b^C_1(r) \) and \( b^C_2(r) \).

IV. EXAMPLES AND RESULTS

A. The causal range

The causality bound \((47)\) can be rewritten as

\[
b^C_\ell(r) - r^C_\ell \geq 0 \quad \forall \quad r \geq R.
\]

\[(66)\]
In cases where the details of the interaction (in particular its range, assuming that a description with finite-range potentials is applicable) is not known, one can use Eq. (66) to define the causal range $R_c$ of a scattering system as that value of $r$ for which the bound is just satisfied, i.e.,

$$b^C_\ell(R_c) - r^C_\ell = 0.$$  

We note from Eq. (53) that the derivative of $b^C_\ell(r)$ is non-negative,

$$\frac{d}{dr} b^C_\ell(r) = 2 \left( g_0(r) - \frac{1}{a_\ell^C} f_0(r) \right)^2 \geq 0.$$  

Hence, $b^C_\ell$ is an increasing function of $r$ and the causal range is defined uniquely. For the case that Eq. (67) does not have a solution (i.e., if $b^C_\ell(r)$ is positive already for $r = 0$), we define the causal range to be zero. Note that the causal range is a function only of the scattering length and effective range. It can thus be calculated from observables in a well-defined way.

The importance of the causal range is given by the fact that it can be interpreted as the minimum range a potential is allowed to have to be consistent with causality. If for a given system the values in individual partial waves differ significantly, the maximum value should be taken as the causal range of the underlying potential. Alternatively, one can model the interaction with an $\ell$-dependent potential. For effective field theories with short-range interactions such as halo EFT, the causal range constrains the allowed values of the momentum-space cutoff or the lattice spacing used in numerical calculations.

At this point we recall that our derivation of the causality bounds was based on the assumption that the concrete system under consideration is described by a finite-range (though possibly non-local) two-body interaction which is energy independent. In EFT calculations one frequently obtains effective interactions that explicitly depend on the energy. We note that this energy dependence can be traded for momentum dependence at any given order in the power counting (EFT expansion) by using the equations of motion obtained from the effective Lagrangian. However, the energy dependence introduces another length scale into the system, and so the conversion to momentum-dependent interactions could produce an interaction range so large that the causality bounds may not be useful in practice.

There are also other theoretical frameworks, e.g., Feshbach reaction theory, that explicitly use energy-dependent interactions. Here again the energy dependence introduces a length scale which acts as an interaction range. This can be seen from the time delay of the scattered wavepacket, which is proportional to the derivative of the phase shift with respect to energy. By setting a very strong energy dependence for the interactions, it is possible to produce a time delay which is arbitrarily large and negative. This has the same effect as interactions at arbitrarily large separation distances.

Furthermore, the assumption of a strict finite range certainly is an idealization that is only applicable to a varying degree of validity to concrete physical systems. For example, there can be exchange forces arising from the Pauli principle. Consider for example nuclear halo systems with a tightly bound core and a halo nucleon which is only weakly bound to the core. The exchange of a nucleon from the core and the halo nucleon that is necessary to anti-symmetrize the system can only give a sizeable contribution if there is spatial overlap between the wavefunction of the core and the wavefunction of the halo nucleon. This yields a short-range exponential tail that, within the domain of validity of the effective theory, can be subsumed in the effective range parameters of the halo-core interaction. The same
analysis would apply to low-energy nucleon-nucleus scattering upon the core nucleus. Another more prominent effect is given by exponential tails generated by simple pion-exchange contributions; see Ref. [8] and the discussion below.

We now calculate explicit values for causal ranges in few-nucleon systems. In Fig. 1 we plot the left-hand side of Eq. (66) as a function of $r$ for the case of proton–proton S-wave scattering. The causal range can then be read off as the point where the function becomes zero. Fig. 2 shows analogous plots for $\alpha-\alpha$ S- and D-wave scattering. In this system, there are visible error bands due to the larger uncertainties in the effective range parameters.

![Graph](image1)

**FIG. 1:** Causal range plot for S-wave proton–proton scattering.

![Graph](image2)

**FIG. 2:** Causal range plot for S-wave $\alpha-\alpha$ scattering.
In Table II we give a summary of the causal ranges that one finds for various two-body systems of light nuclei where low-energy scattering parameters and/or phase shifts are available from experiments. The results are briefly discussed in the following subsections.

| System | Reference | Channel | $a_C / \text{fm}^{2\epsilon+1}$ | $r_C / \text{fm}^{-2\epsilon+1}$ | Causal range / fm |
|--------|-----------|---------|-------------------------------|-------------------------------|------------------|
| $p-p$  | [36]      | $^1S_0$ | $-7.828 \pm 0.008$            | $2.80 \pm 0.02$              | $1.38 \pm 0.01$  |
| $p-p$  | [37]      | $^3P_0$ | $-3.03 \pm 0.11$             | $4.22 \pm 0.11$              | $2.33 \pm 0.05$  |
| $p-p$  | [37]      | $^3P_1$ | $2.013 \pm 0.053$            | $-7.92 \pm 0.17$             | $\approx 0.03$   |
| $p-d$  | [38]      | $^2S_1/2$ | $2.73 \pm 0.10$            | $2.27 \pm 0.12$              | $3.90 \pm 0.15$  |
| $p-d$  | [39]      | $^2S_1/2$ | $4$                        | $-2.8$                       | $0$              |
| $p-d$  | [38]      | $^4S_3/2$ | $11.88 \pm 0.40$           | $2.63 \pm 0.02$              | $2.20^{+0.07}_{-0.06}$ |
| $p-d$  | [39]      | $^4S_3/2$ | $11.11$                    | $2.64$                       | $2.29$           |
| $p-{^3He}$ | [40] | $^1S_0$ | $11.1 \pm 0.4$            | $1.58 \pm 0.12$              | $1.32^{+0.21}_{-0.17}$ |
| $p-{^3He}$ | [40] | $^3S_1$ | $9.04 \pm 0.14$           | $1.50 \pm 0.06$              | $1.27^{+0.10}_{-0.09}$ |
| $p-\alpha$ | [41] | $S_{1/2}$ | $4.97 \pm 0.12$          | $1.295 \pm 0.082$             | $1.32^{+0.40}_{-0.21}$ |
| $p-\alpha$ | [41] | $P_{1/2}$ | $-19.36 \pm 0.50$       | $0.349 \pm 0.021$             | $2.65 \pm 0.07$  |
| $p-\alpha$ | [41] | $P_{3/2}$ | $-44.83 \pm 0.51$       | $-0.365 \pm 0.113$            | $0.49^{+0.17}_{-0.10}$ |
| $\alpha-\alpha$ | [42] | $S$ | $(-1.65 \pm 0.17) \cdot 10^3$ | $1.084 \pm 0.011$            | $2.58^{+0.14}_{-0.13}$ |
| $\alpha-\alpha$ | [42] | $D$ | $(-7.23 \pm 0.61) \cdot 10^3$ | $(-1.31 \pm 0.22) \cdot 10^{-3}$ | $2.03^{+0.09}_{-0.07}$ |

TABLE II: Summary of causal-range results obtained from experimental input for various few-nucleon systems.

1. Proton–proton scattering

For $p-p$ S-wave scattering one finds a causal range of about 1.38 fm. This value is very close to the range estimate obtained by assuming that the typical length scale of the $N-N$ interaction is set by the inverse pion mass, $\hbar c/M_\pi \approx 1.4$ fm. The value one finds in the $^3P_0$-channel is somewhat larger ($R_c \approx 2.3$ fm), whereas the $^3P_1$ effective range parameters impose almost no constraint on the range of the nuclear potential in this channel. As we will discuss in more detail below, this suggests some significant differences in the radial dependence of the interactions for the $^3P_1$ channel.

For effective field theory calculations with purely local interactions (i.e., pionless effective field theory), our results suggest to keep the cutoff momentum smaller than $M_\pi$ for the $^1S_0$ and $^3P_0$ channel. However, there is more freedom to take a higher cutoff for the $^3P_1$ channel.

In Ref. [8], causality bounds were investigated for neutron–proton scattering. The results ($R_c = 1.27$ fm for $^1S_0$, $R_c = 3.07$ fm for $^3P_0$, and $R_c = 0.23$ fm for $^3P_1$) are qualitatively very similar to what we find for the $p-p$ system. This would indicate only a moderate amount of isospin breaking.

In the same publication [8], the influence of the shape of the potential upon the neutron–proton causal range was also studied numerically. When the potential is repulsive at shorter distances (less than $\sim 1$ fm) and attractive at larger distances (greater than $\sim 1$ fm), the causal range comes out on the larger side, about 2 fm or more. When the potential is attractive at intermediate distances and repulsive at larger distances, then the causal range
is smaller, about 1 fm or less. The pion tail determines the sign of the potential at larger
distances. For both the \( n-p \) and the \( p-p \) interaction, the one-pion exchange tail is repulsive
in the \( 3P_1 \)-channel while it is attractive in the \( 3P_0 \)-channel.

Note that causality bounds in the presence of pion-exchange contributions were also
discussed by Phillips and Cohen in Ref. \[2\]. Ideally, one would account for the one-pion
exchange tail explicitly in the calculation of the causal range, as it was done in this work for
the long-range Coulomb potential. Without knowing analytical solutions of the Schrödinger
equation involving a Yukawa-like potential (plus a Coulomb part, in the \( p-p \) case), however,
such a procedure can at best be implemented numerically. For an example, see Ref. \[43\].

2. Proton–deuteron scattering

There are several experimental determinations of \( p-d \) effective range parameters. In
Table \[11\] we have included results from Arvieux \[38\] and Huttel et al. \[39\]. While for the
quartet-channel there is a good agreement between the scattering lengths and effective ranges
(and, of course, of the resulting causal ranges, which come out as \( 2.2 - 2.3 \) fm), there is a
large discrepancy for the doublet-channel results.

The difficulty of determining the proton–deuteron doublet-channel scattering length has
previously been discussed by Orlov and Orevkov \[44\]. Comparing different models, the
authors conclude that \( a_C = 0.024 \) fm is currently the best theoretical estimate for the
doublet-channel \( p-d \) scattering length. From Table 3 in Ref. \[44\] one reads off that the
corresponding value for the Coulomb-modified effective range is as huge as \( r_C = 8.23 \times 10^5 \) fm.
Inserting these numbers into the causal-range calculation one gets a very large value of
\( R_c \approx 8.15 \) fm. As a consequence, no definite conclusion can be reached with the currently
available data.

Note that three-body forces have been found to be very important for theoretical calcu-
lations of the \( p-d \) and \( n-d \) threshold scattering parameters. Our analysis here, however,
is independent of the microscopic origin of the effective interaction between proton and
deuteron. In a detailed picture, the force might arise from two-nucleon forces or three-
nucleon forces, but the result is always some effective two-body interaction between the
proton and the deuteron. The causal range we calculate is the minimum range that this
effective interaction has to have in order to be able to reproduce the experimentally deter-
mined scattering parameters.

3. Proton–helion scattering

For the scattering of protons off a helium nucleus we were able to find data for both \( p-^3\)He
and \( p-\alpha \) scattering. In the first case, there was only enough data available to calculate the
causal range for the S-wave channels. Since both the scattering lengths and effective ranges
are very similar for the singlet and the triplet channel, so are the resulting causal ranges,
which come out as approximately 1.3 fm.

Incidentally, one finds almost the same value for the S-wave in \( p-\alpha \) scattering. For this
system, it is interesting to compare to the neutron–alpha system, where there is no Coulomb
repulsion in the scattering process. Results for the \( n-\alpha \) causal ranges can be read off from
Fig. 5 in Ref. \[7\] (obtained using effective range parameters from Ref. \[11\]). Even though
from the plot one only gets quite rough estimates, one clearly sees that the results for the \( S_{1/2} \)
and $P_{1/2}$ channels agree very well between $p-\alpha$ and $n-\alpha$ scattering, which as in the nucleon–nucleon case discussed above could be interpreted as only a moderate amount of isospin breaking. However, the causal ranges for the $P_{3/2}$ channels are very different ($\sim 0.5$ fm for $p-\alpha$, $\sim 2$ fm for $n-\alpha$). It is an interesting question if this discrepancy hints at an error in the extraction of the effective range parameters (either for one of the systems or possibly both), or if there actually is a physical reason behind the difference in the causal ranges.

4. Alpha–alpha scattering

For $\alpha-\alpha$ S-wave scattering we use the values given by Higa et al. (see Ref. [13] and experimental references therein), $a^C_0 = (-1.65 \pm 0.17) \cdot 10^3$ fm and $r^C_0 = (-1.084 \pm 0.011)$ fm to find a causal range of about 2.58 fm.

For the $\ell = 2$ channel, no effective range parameters could be found in the literature. We have thus used the phase-shift data collected in the review article by Afzal et al. [42] to perform the fit to the effective range expansion (Eq. (21) with $\ell = 2$) ourselves. By including the phase-shift data up to $E_{\text{lab}} \approx 6.5$ MeV we find $a^C_2 = (-7.23 \pm 0.61) \cdot 10^3$ fm$^5$ and $r^C_2 = (-1.31 \pm 0.22) \cdot 10^{-3}$ fm$^{-3}$. However, the fit is strongly dominated by the $O(p^4)$ shape parameter, so the actual uncertainties of $a^C_2$ and $r^C_2$ should probably be somewhat larger. For the causal range in this channel we find a value of about 2 fm, which is just slightly smaller than the S-wave result.

V. RELATION FOR ASYMPTOTIC NORMALIZATION CONSTANTS

We now discuss the connection between the integral relations we have derived above and asymptotic normalization constants (ANC), denoted in the following as $A$. They are defined by writing the bound-state wave function in the asymptotic region as

$$w_\ell(r) = A \cdot W_{-\eta,\ell+\frac{1}{2}}(2\kappa r) \quad \text{for} \quad r > R, \quad (69)$$

where $w_\ell(r)$ is a bound-state solution (with momentum $p = i\kappa$, $\kappa > 0$) of the Schrödinger equation [4], normalized according to

$$\int_0^\infty dr \, |w_\ell(r)|^2 = 1. \quad (70)$$

$W_{-\eta,\ell+\frac{1}{2}}$ is the Whittaker function defined in Eq. [13]. At very low-energies, the radiative capture cross section to a two-body halo nucleus is proportional to $|A|^2$.

A. Derivation of the relation

Our starting point is Eq. [45] derived in Sec. [III] from which we already obtained the causality bounds for the effective range parameter. For bound-state momenta, we write it as

$$r^C_\ell = b^C_\ell(r) - 2 \lim_{\kappa \to 0} \int_0^r dr' \left[w^{(i\kappa)}_\ell(r')\right]^2 \quad \text{for all} \quad r > R, \quad (71)$$
with \( b_\ell^C(r) \) as defined in Eq. (46). In order to derive from this a relation for the ANC, we need the precise connection between the radial wave functions \( w^{(i\kappa)}_\ell \) and the solution \( w_\ell(r) \) used in Eq. (69). Formally, we can write

\[
  w^{(i\kappa)}_\ell(r) = (i\kappa)^\ell C_{\eta,\ell} \left[ \cot \delta_\ell(i\kappa) F^{(i\kappa)}_\ell(r) + G^{(i\kappa)}_\ell(r) \right] \quad \text{for } r > R ,
\]

but we need to be careful with the way the analytic continuation to the bound-state regime is done. We start with the assumption that \( \cot \delta_\ell(i\kappa) = i \) for a bound state and thus consider the linear combination

\[
  G^{(p)}_\ell(r) + i F^{(p)}_\ell(r) ,
\]

for \( r > R \), (72)

A short calculation shows that

\[
  G^{(p)}_\ell(r) + i F^{(p)}_\ell(r) = e^{i\sigma_\ell} e^{-i\pi/2} (\ell + i\eta) W - i\eta,\ell + \frac{1}{2} (2\kappa r)
\]

with the Coulomb phase shift \( \sigma_\ell \) defined via (74).

Note that \( i\eta \) will be real for bound-state momenta. In the following, we only consider the case of a repulsive Coulomb potential (i.e., \( \gamma > 0 \)), so \( i\eta \) is in fact a positive number. For the \( C_{\eta,\ell} \), we use the expression (46),

\[
  C_{\eta,\ell} = 2^\ell e^{-\eta} \Gamma(\ell + 1 + i\eta) / \Gamma(2\ell + 2) ,
\]

combining Eqs. (74) and (75) we find that the problematic terms \( \Gamma(\ell + 1 - i\eta) \) and \( \exp(\pi\eta/2) \) drop out, and we can finally write

\[
  w^{(i\kappa)}_\ell(r) = \kappa^\ell \tilde{C}_{\eta,\ell} W - i\eta,\ell + \frac{1}{2} (2\kappa r) \quad \text{for } r > R
\]

where we have defined

\[
  \tilde{C}_{\eta,\ell} = 2^\ell \Gamma(\ell + 1 + i\eta) / \Gamma(2\ell + 2) .
\]

Comparing with Eq. (69) we readily infer that

\[
  w^{(i\kappa)}_\ell(r) = \frac{\kappa^\ell}{A} \tilde{C}_{\eta,\ell} \cdot w_\ell(r) .
\]

Using the overall normalization of the wave function, we can rewrite Eq. (71) as

\[
  r^C_\ell = b_\ell^C(r) - 2 \lim_{\kappa \to 0} \left\{ \frac{\kappa^2}{A^2} \tilde{C}_{\eta,\ell}^2 - \int_r^\infty dr' \left[ w^{(i\kappa)}_\ell(r') \right]^2 \right\} , \quad r > R .
\]

Canceling the \( r \)-dependence

At this point it is important to note that Eq. (79) can be rigorously valid only for zero-energy bound states, i.e., if one strictly considers the limit \( \kappa \to 0 \). Since the left hand side is a constant, the \( r \)-dependence of \( b_\ell^C(r) \) has to cancel that of the integral for any \( r > R \).
In the scattering regime this is ensured by taking the zero-energy limit, i.e., considering scattering directly at threshold. In the bound-state regime, on the other hand, one is of course interested in finding a relation for the case where there is a bound state close to threshold, i.e., where $\kappa$ is small but finite. It is obvious that in principle this poses a problem because the $b^{(\ell)}_{\kappa}(r)$ in Eq. (79) is a strictly increasing function of $r$, whereas the integral is always bounded and in fact becomes smaller when $r$ is made larger.

However, up to corrections of higher order in $\kappa$, the cancellation still works, i.e., the leading $r$-dependence drops out. In Ref. [47], where we derived an analogous ANC relation for the case without any long-range force, we established the cancellation by carrying out the integral over the asymptotic wave function (a Riccati-Hankel function, in that case) analytically and then expanding the result in powers of $\kappa$. To show it here in a more general way, we recall from Eq. (39) that

$$w^{(\ell)}_{\kappa}(r) = -\frac{1}{a^{\ell}_{C}}f_{0}(r) + g_{0}(r) + \mathcal{O}(p^{2}).$$

(80)

It is straightforward to assume that this is valid also in the bound-state regime since we are working with a wave function explicitly analytic in $p^{2}$. Hence,

$$\left[w^{(\ell)}_{\kappa}(r)\right]^{2} = \left(-\frac{1}{a^{\ell}_{C}}f_{0}(r) + g_{0}(r)\right)^{2} + \mathcal{O}(\kappa^{2}) = \frac{1}{2}\frac{d}{dr}b^{(\ell)}_{\kappa}(r) + \mathcal{O}(\kappa^{2}),$$

(81)

where in the last step we have inserted Eq. (53). This directly tells us that $b^{(\ell)}_{\kappa}(r)$ in Eq. (79) cancels with the integral up to higher-order terms and, possibly, an integration constant. This situation is already familiar from the calculation of the Wronskian $W[g_{2}, g_{0}] (r)$. Again the constant term can only come from the integral over $g_{0}(r)^{2}$. To determine it, we note that

$$\int_{r}^{\infty} dr' [g_{0}(r')]^{2} < \infty \text{ if } r > 0,$$

(82)

i.e., the integral is actually convergent and gives a Meijer $G$-function, as already encountered in Sec. [IITD]. The constant term coming from this function has to be combined with the already known $Z_{\ell}$ in $b^{(\ell)}_{\kappa}(r)$. We define

$$\Delta Z_{\ell} = W[g_{2}, g_{0}] (r) + \int_{r}^{\infty} dr' [g_{0}(r')]^{2},$$

(83)

which is explicitly $r$-independent, to finally arrive at

$$r_{C}^{\ell} + \frac{2\kappa^{2\ell}}{A^{2}}\tilde{C}_{\eta,\ell} + 2\Delta Z_{\ell} = \mathcal{O}(\kappa^{2})$$

(84)

or, equivalently,

$$|A| = \frac{\kappa^{\ell}}{\tilde{C}_{\eta,\ell}} \left(-\frac{r_{C}^{\ell}}{2} - \Delta Z_{\ell} + \mathcal{O}(\kappa^{2})\right)^{-1/2},$$

(85)

both valid in the limit where $\kappa \to 0$. In Table [III] we give the resulting values for the $\Delta Z_{\ell}$ for $\ell = 0, 1, 2$.

Finally, we note that Sparenberg et al. [22] have previously derived an ANC relation equivalent to ours, but written in terms of the scattering length instead of the effective range parameter. The equivalence of the two relations up to the contributions of higher order shape parameters is shown in Appendix [I].
\[
\begin{array}{c|c|c|c}
\ell & 0 & 1 & 2 \\
\hline
\Delta Z_\ell & -1/(3\gamma) & -\gamma/108 & -17\gamma^3/10800 \\
\end{array}
\]

TABLE III: Integration constant $\Delta Z_\ell$ in Eqs. \[84\] \[85\] for $\ell = 0, 1, 2$.

B. Application to the oxygen-16 system

The $^{16}\text{O}$ nucleus has two excited states lying just below the $\alpha-^{12}\text{C}$ threshold, a $2^+$ at about $-245$ keV and an even more shallow $1^-$ at only $-45$ keV. The properties of these states play an important role in astrophysical helium burning processes \[48\] \[50\]. In the following, we calculate asymptotic normalization constants for both states under the assumption that they can be described in a $\alpha +^{12}\text{C}$ halo picture.

We use the recent data obtained by Tischhauser et al. \[51\] (for the actual phase shifts see their Ref. \[32\]) in order to extract the Coulomb-modified effective range for the $\alpha-^{12}\text{C}$ P- and D-wave channels. Focusing first on the D-wave, we note that the combination of the strong Coulomb repulsion and the $\ell = 2$ centrifugal barrier makes the low-energy phase shifts very small over a wide energy range. Moreover, there is a narrow resonance at a center-of-mass energy of about 2.7 MeV, which strongly constrains the energy region for a straightforward fit to the effective range function. From a simple fit to the data up to the narrow resonance one only obtains an effective range parameter with an uncertainty too large (50%) for an extraction of the ANC since the latter depends on $r_C$ very sensitively.

To mitigate this problem, we use the position of the $2^+$ oxygen state as an additional input parameter for a self-consistent extraction of the effective range. In the following, we describe this procedure in more detail.

At the position of the bound state, where $\cot \delta_\ell(i\kappa) = i$, the Coulomb-modified effective range expansion \[25\] reduces to

\[
\gamma \tilde{h}_\ell(i\kappa) = -\frac{1}{a^2_C} - \frac{1}{2} r_C \kappa^2 + \cdots ,
\]

where now

\[
\tilde{h}_\ell(i\kappa) = \frac{(2i\kappa)^{2\ell}}{(2\ell + 1)^2} \prod_{s=1}^{\ell} (s^2 + \eta^2) \times \left( \psi(i\eta) + \frac{1}{2i\eta} - \log(i\eta) \right) , \quad \eta = \frac{\gamma}{2i\kappa} .
\]

A straightforward calculation shows that for the prefactor we have

\[
\frac{(2i\kappa)^{2\ell}}{(2\ell + 1)^2} \prod_{s=1}^{\ell} (s^2 + \eta^2) = \frac{\gamma^{2\ell}}{(2\ell + 1)^2} + \mathcal{O}(\kappa^2) .
\]

Furthermore, for the digamma function we have the asymptotic expansion (Eq. (6.3.18) in Ref. \[34\])

\[
\psi(z) = \log z - \frac{1}{2z} - \sum_{n=1}^{\infty} \frac{B_{2n}}{z^{2n}}
\]

for $|z| \to \infty$, where the $B_{2n}$ are the Bernoulli numbers,

\[
B_0 = 1, \quad B_1 = -\frac{1}{2}, \quad B_2 = \frac{1}{6}, \quad \ldots .
\]
Noting that the sum in Eq. 89 only starts at \( n = 1 \), we find that \( \gamma \tilde{h}_\ell(i\kappa) \) only starts at order \( \kappa^2 \), and inserting the precise relation into Eq. 86, we arrive at

\[
\frac{1}{a^C_\ell} = \left( -\frac{1}{2} r^C_\ell + \frac{\gamma^{2\ell-1}}{3(2\ell+1)^2} \right) \kappa^2 + \mathcal{O}(\kappa^4). \quad (91)
\]

Neglecting the effective range contribution at leading order, we insert the binding momentum \( \kappa \approx 0.187 \text{ fm}^{-1} \) of the \( 2^+ \) state to get a first approximation for the scattering length parameter. This is then used to constrain a subsequent fit to the phase shift data up to about 2.6 MeV, i.e., just below the narrow resonance. We include a single \( \mathcal{O}(p^4) \) shape parameter in the fit since some curvature is clearly necessary. Iterating this procedure a couple of times yields a well-converged self-consistent result for \( r^C_\ell \). After eight iterations, we find

\[
r^C_2 = (5.94 \pm 0.35) \cdot 10^{-4} \text{ fm}^{-3} \quad (92)
\]

for the \( ^{12}\text{C} \) D-wave. Including a second shape parameter in the fitting procedure only changes this result within the given uncertainty, so we conclude that for the energy range we have been fitting, a single shape parameter really is sufficient to account for the curvature. Inserting the fit result into Eq. 85 yields

\[
|A(2^+)| = (2.41 \pm 0.38) \cdot 10^4 \text{ fm}^{-1/2} \quad (93)
\]

for the \( 2^+ \) state in \(^{16}\text{O}\). Including in Eq. 84 an \( \mathcal{O}(\kappa^4) \) term of the order of the shape parameter gives a consistent result for \( |A(2^+)| \) within the error given above.

When we apply the same same procedure to the \( 1^- \) state just 45 keV below the \( ^{12}\text{C} \) threshold, we also see a nice convergence and obtain the results

\[
r^C_1 = (4.546 \pm 0.002) \cdot 10^{-2} \text{ fm}^{-1} \quad (94)
\]

and

\[
|A(1^-)| = (1.188 \pm 0.024) \cdot 10^{14} \text{ fm}^{-1/2}. \quad (95)
\]

The overall picture for the \( 1^- \) state is somewhat more complicated, however. On the one hand, the extraction of the effective range parameters is easier in this case because there is no narrow resonance limiting the fit range (we have used the data up to \( E_{\text{cm}} = 3.75 \text{ MeV} \)). By doing a simple fit without the self-consistent iteration we get values for the effective range and the ANC that are slightly smaller, but overlap with the results given above when the respective uncertainties are taken into account. On the other hand, allowing for a second shape parameter changes the value of the effective range quite dramatically to 0.046 fm\(^{-1}\), which leads to an imaginary ANC. This could indicate that the effective range parameters from the simple fit violate the causality bound. The fact that \( r^C \) has to be such that Eq. 84 yields a real value for the ANC can be interpreted as a weaker remnant of the original causality bound relation. Alternatively, the cluster picture might not be applicable for the shallow \( 1^- \) state in \(^{16}\text{O}\).

Our result for the ANC of the \( 2^+ \) state is about a factor five smaller than the value \( |A(2^+)| = (1.11 \pm 0.11) \cdot 10^5 \text{ fm}^{-1/2} \) obtained by Brune et al. 48, while the value for \( |A(1^-)| \) is about a factor two smaller. Other, more recent determinations 52, 53 have found even larger
values. For the interpretation of this discrepancy, note that our calculations are predictions of the ANCs based only on alpha–carbon elastic scattering data and the assumption that the system can be approximately described in an effective two-body picture with a finite-range interaction. In the references mentioned above, the ANCs are extracted from alpha–carbon transfer measurements. Note furthermore that a comparison of the experimental extractions in Ref. [52] exhibits quite some discrepancy (factors of two up to roughly an order of magnitude) also between the cited individual experimental determinations of the ANCs.

Sparenberg et al. [54] have carried out a similar analysis for the $2^+$ state based on their ANC relation (cf. Ref. [22]). They subtracted the narrow resonance from the $\alpha^{12}\text{C}$ D-wave phase shift data in order to extract a set of higher-order shape parameters and concluded that present day data are not sufficient to constrain the ANC strongly. As discussed above, our approach of performing a self-consistent fit to the data below the resonance constrained by the separation energy and using the effective range instead of the scattering length as input in the ANC relation improves the stability of the extraction. However, compared to other determinations our ANC values are generally smaller by a factor two to five. We thus conclude that this issue requires further study.

VI. CONCLUSION

In this paper, we have investigated the constraints imposed by causality on the low-energy scattering parameters of charged particles interacting via a short-range interaction and a long-range Coulomb potential. Similar to the case of neutral particles without Coulomb interaction [2–8], our considerations lead to a constraint on the maximum value of the Coulomb-modified effective range.

While conceptionally straightforward, the calculation of the Wronskians of the wave functions required for the derivation of the bound is intricate. We have calculated them through term-by-term integration of the power series expansion of the zero-energy wave functions and additionally determining the integration constants that are not generated by this process.

We define the causal range as the minimum value of the interaction range consistent with the causality bound. In effective field theories with contact interactions such as the halo EFT, the natural momentum cutoff is of the order of the inverse of the causal range. If the natural cutoff is not known from other considerations, its size can be estimated from the causal range. If the momentum cutoff used in a calculation is too high, then problems with convergence of higher order corrections can appear. For example, the convergence pattern might be such that an improvement in higher orders of the EFT can only be sustained through large cancellations between individual terms. Such an unnatural pattern would be especially undesirable for the stability of numerical (lattice) calculations. Our results can thus be viewed as a guide for improving the convergence pattern of EFT calculations with contact interactions. In lattice simulations of halo EFT, the lattice spacing should not be taken smaller than the causal range.

We have analyzed the causal ranges for a variety of systems ranging from proton–proton scattering to alpha–alpha scattering. Our results for causal ranges in different partial waves in these systems typically vary by factors of 2-3. The precise values are quite sensitive to small uncertainties in the effective range parameters. In channels with a large negative effective range the causal range is very close to zero, which implies that causality provides almost no constraints on the range of the interaction in this case. Thus the causal range
provides a good order of magnitude estimate of the range of interaction, but drawing more quantitative conclusions about the structure of the underlying potentials is difficult.

After an analytic continuation to the bound state regime, the integral relations for the causality bound can also be used to derive a model-independent expression for the ANC for shallow bound states. If the state is a two-body halo state, the relation can be used to extract the ANC from low-energy scattering parameters. Up to higher order shape parameters, our relation is equivalent to the one previously derived by Sparenberg et al. in Ref. [22] (see Appendix D for details). One difference is that we express the ANC in terms of the binding momentum and the effective range rather than the binding momentum and the scattering length. We find this form more suitable for the extraction of ANCs from scattering data since the effective range is typically more precisely determined than the scattering length for shallow states. Moreover, extracting the effective range in a self-consistent fit from the scattering data that reproduces the correct separation energy improves the stability of the extraction.

Finally, we illustrate our relation by extracting the ANCs of the excited $2^+$ and $1^-$ states in $^{16}$O from $\alpha-^{12}$C scattering data. Compared to previous extractions [48, 52, 53], our values are generally smaller. Whether this difference is physically significant requires further study. The application of our relation to other shallow cluster states and a benchmark against model calculations would also be very interesting.

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Appendix A: Calculating the constant terms in the causality bound function

To determine the constants $Z_\ell$ introduced in Sec. III D we consider the explicit form of $\tilde{g}(p,r)$. From the results of Bollé and Gesztesy \[30\], we have\(^7\)

$$\tilde{g}(p,r) = N_\ell(p) \cdot \gamma \log (|\gamma| r) \cdot f(p,r) + \gamma \Re \left\{ \mathcal{N}_\ell(p) \cdot r^{\ell+1} e^{-ipr} \cdot \sum_{n=0}^{\infty} [a_{\ell,n}(p) + b_{\ell,n}(p)] r^n \right\} + \Re \left\{ \frac{r^{-\ell}}{2\ell + 1} e^{-ipr} \cdot \sum_{n=0}^{2\ell} d_{\ell,n}(p) r^n \right\}, \quad (A1)$$

where

$$\mathcal{N}_\ell(p) = \frac{(2p)^{2\ell}}{\Gamma(2\ell + 2)^2} \prod_{s=1}^{\ell} (s^2 + \eta^2), \quad (A2)$$

$$a_{\ell,n}(p) = \frac{-\Gamma(2\ell + 2)}{\Gamma(n+1)\Gamma(n+2\ell + 2)} (2ip)^n \prod_{s=1}^{n} (s + \ell - in) \cdot [\psi(n+1) + \psi(n+2\ell + 2)]^2, \quad (A3a)$$

$$b_{\ell,n}(p) = \frac{\Gamma(2\ell + 2)}{\Gamma(n+1)\Gamma(n+2\ell + 2)} (2ip)^n \prod_{s=1}^{n} (s + \ell - in) \cdot \sum_{j=1}^{n+\ell} \frac{1}{j - in}; \quad (A3b)$$

and

$$d_{\ell,n}(p) = \frac{1}{\Gamma(n+1)} (2ip)^n \prod_{s=1}^{n} \left( \frac{s - \ell - 1 - in}{s - 2\ell - 1} \right). \quad (A4)$$

With this result and the appropriate expression for $g_0(r)$ from Eq. (54) or (55), one can use the following procedure to calculate the $Z_\ell$, i.e., the terms of order $r^0$ in the Wronskian $W[g_2,g_0](r)$.

1. Note that

$$W[\tilde{g},g_0](r) = p^2 W[g_2,g_0](r) + \mathcal{O}(p^4) \quad (A5)$$

and calculate this Wronskian using a truncated version (including terms up to the order $2\ell + 1$ in $r$ is sufficient) of $\tilde{g}(p,r)$ as given in Eq. (A1).

2. From the result, extract the terms that are of the order $r^0$.

3. From that expression then extract the terms that are of the order $p^2$. They constitute the $\mathcal{O}(r^0)$-contributions in a series expansion of $W[g_2,g_0](r)$ which cannot be obtained from a term-by-term integration of $g_0(r)^2$.

\(^7\)One gets this form from Eq. (B6) in Appendix B by inserting $n = 2\ell + 3$, $\eta = \gamma/(2p)$ and assuming that the momentum $p$ is real and positive.
With a computer algebra software, this prescription is straightforward to implement. The results for \( \ell = 0, \ldots, 2 \) are shown in Table I in Sec. III D.

At this point we remark that there is a recent publication by Seaton [55] discussing Coulomb wave functions that are explicitly analytic in the energy (or the momentum squared). In principle, it is possible to use these results to get explicit expressions for \( f_2(r) \) and \( g_2(r) \) (in addition to the already known zero-energy functions), and then simply calculate all the Wronskians directly. However, the analytic irregular Coulomb function defined by Seaton is slightly different from our \( \tilde{g}(p, r) \). More importantly, not all coefficients needed for the expansions in energy are given explicitly. Finally, writing everything in terms of the wave functions of Bollé and Gesztesy paves the way for a generalization of the results presented in the current paper to an arbitrary number of spatial dimensions.

Knowing now what the constant terms \( Z_\ell \) that come from \( W[g_2, g_0](r) \) should be, it is possible to write down explicit expressions for the causality bound functions \( b_\ell^C(r) \). To do that we use that the antiderivative of the right hand side of Eq. (53) can be expressed in terms of (generalized) hypergeometric functions \( _pF_q \) and Meijer \( G \)-functions \( G_{m,n}^{p,q} \). The only additional point to be taken into account is that the antiderivative of \( g_0(r)^2 \) in general includes a constant term that is different from the desired \( Z_\ell \). Hence, one has to determine this term and add another constant such that the sum is exactly equal to the \( Z_\ell \) given in Table I. For the antiderivative of \( g_0(r)^2 \) that we are using in the following, up to a minus sign the correction term is just the \( \Delta Z_\ell \) introduced in Sec. V.

a. Repulsive case, \( \gamma > 0 \)

The \( \ell = 0 \) result for a repulsive Coulomb potential is given by Eq. (64). For \( \ell = 1 \), we get

\[
b_1^C(r) = \frac{6r^2}{5\gamma^3} \left(a_1^C\right)^{-2} \left[ 3\gamma^2 r^2 \, _1F_2 \left(\frac{5}{2}, 3, 6; 4\gamma r\right) - 20\gamma r \, _1F_2 \left(\frac{3}{2}, 2, 5; 4\gamma r\right) \right. \\
+ 120 \, _1F_2 \left(\frac{1}{2}, 1; 4; 4\gamma r\right) - 120 \, _2F_3 \left(\frac{1}{2}, 2; 1, 3, 4; 4\gamma r\right) \left. \right] \\
- \frac{4r^2}{\sqrt{\pi}} \left(a_1^C\right)^{-1} \, G_{2,4}^{2,2} \left(4\gamma r, -1, \frac{1}{2}, \frac{1}{2}, 0, 3, -3, -2\right) \\
- \frac{\sqrt{\pi} \gamma^3 r^2}{9} \, G_{2,4}^{4,0} \left(4\gamma r, -1, \frac{1}{2}, -3, -2, 0, 3\right) + \frac{\gamma}{54},
\]

\( A6 \)

In other words, our choice for the antiderivative corresponds to using \( \int_0^\infty dr' g_0(r')^2 \), at least for the case of a repulsive Coulomb interaction, where this integral is convergent.
and for $\ell = 2$ the result is

$$b_2^C(r) = \frac{50r^2}{63\gamma^5} (a_2^C)^{-2} \left[ 7\gamma^4 r^4 \text{ } _1F_2 \left( \frac{9}{2}; 5, 10; 4\gamma r \right) - 108\gamma^3 r^3 \text{ } _1F_2 \left( \frac{7}{2}; 4, 9; 4\gamma r \right) \\
+1296\gamma^2 r^2 \text{ } _1F_2 \left( \frac{5}{2}; 3, 8; 4\gamma r \right) - 12096\gamma r \text{ } _1F_2 \left( \frac{3}{2}; 2, 7; 4\gamma r \right) \\
+108864\text{ } _1F_2 \left( \frac{1}{2}; 1, 6; 4\gamma r \right) - 108864\text{ } _2F_3 \left( \frac{1}{2}; 1, 3, 6; 4\gamma r \right) \right] \\
- \frac{4r^2}{\sqrt{\pi}} (a_2^C)^{-1} G_{2,4}^{2,2} \left( 4\gamma r \left| \begin{array}{c}
-1, \frac{1}{2} \\
0, 5, -5, -2
\end{array} \right. \right) \\
- \frac{\sqrt{\pi} \gamma^5 r^2}{3600} G_{2,4}^{4,0} \left( 4\gamma r \left| \begin{array}{c}
-1, \frac{1}{2} \\
-5, -2, 0, 5
\end{array} \right. \right) + \frac{\gamma^3}{21600}.$$

(A7)

b. Attractive case, $\gamma < 0$

For an attractive Coulomb potential, the $\ell = 0$ result is given by Eq. (65). The $\ell = 1$ result reads

$$b_1^C(r) = \frac{6r^2}{5\gamma^3} (a_1^C)^{-2} \left[ 3\gamma^2 r^2 \text{ } _1F_2 \left( \frac{5}{2}; 3, 6; 4\gamma r \right) - 20\gamma r \text{ } _1F_2 \left( \frac{3}{2}; 2, 5; 4\gamma r \right) \\
+120\text{ } _1F_2 \left( \frac{1}{2}; 1, 4; 4\gamma r \right) - 120\text{ } _2F_3 \left( \frac{1}{2}; 2, 1, 3, 4; 4\gamma r \right) \right] \\
+ 4\sqrt{\pi} r^2 (a_1^C)^{-1} G_{3,5}^{2,2} \left( -4\gamma r \left| \begin{array}{c}
-1, \frac{1}{2}, -\frac{1}{2} \\
0, 3, -3, -2, -\frac{1}{2}
\end{array} \right. \right) + \frac{\pi^{3/2} \gamma^3 r^2}{18} \left[ 2 G_{3,5}^{4,0} \left( -4\gamma r \left| \begin{array}{c}
-\frac{5}{2}, -1, \frac{1}{2} \\
-3, -2, 0, 3, -\frac{5}{2}
\end{array} \right. \right) \\
- G_{2,4}^{4,2} \left( -4\gamma r \left| \begin{array}{c}
-1, \frac{1}{2} \\
3, -3, -2, 0
\end{array} \right. \right) \right] + \frac{\gamma}{54}.$$

(A8)
and for $\ell = 2$ one finds

$$b_2^C(r) = \frac{50r^2}{63\gamma^5} (a_2^C)^{-2} \left[ 7\gamma^4 r^4 F_2 \left( \frac{9}{2}; 5, 10; 4\gamma r \right) - 108\gamma^3 r^3 F_2 \left( \frac{7}{2}; 4, 9; 4\gamma r \right)
+ 1296\gamma^2 r^2 F_2 \left( \frac{5}{2}; 3, 8; 4\gamma r \right) - 12096\gamma r F_1 \left( \frac{3}{2}; 2, 7; 4\gamma r \right)
+ 108864 F_2 \left( \frac{1}{2}; 1, 6; 4\gamma r \right) - 108864 F_2 \left( \frac{1}{2}; 1, 3, 6; 4\gamma r \right) \right]$$

$$+ 4\gamma r^2 (a_2^C)^{-1} G_{3,5}^{2,2} \left( -4\gamma r \begin{array}{c}
-1, \frac{1}{2}, -\frac{1}{2} \\
0, 5, -5, -2, -\frac{1}{2}
\end{array} \right)
+ \frac{\pi^{3/2}\gamma r^2}{7200} \left[ 2 G_{3,5}^{4,0} \left( -4\gamma r \begin{array}{c}
-\frac{9}{2}, -1, \frac{1}{2} \\
-5, -2, 0, 5, -\frac{9}{2}
\end{array} \right)
- G_{2,4}^{1,2} \left( -4\gamma r \begin{array}{c}
-1, \frac{1}{2} \\
5, -5, -2, 0
\end{array} \right) \right] + \frac{\gamma^3}{21600}.$$  

(A9)

Appendix B: The Coulomb wave functions of Bollé and Gesztesy

Bollé and Gesztesy define the Coulomb wave functions

$$F_n^{(0)}(p, r) = r^{\frac{1}{2} + m} e^{-i\nu r} F_1 \left( \frac{1}{2} + m - \kappa, 1 + 2m; z \right)$$  

(B1a)

and

$$G_n^{(0)}(p, r) = \frac{\Gamma \left( \frac{1}{2} + m - \kappa \right)}{\Gamma(2m + 1)} (2i)^{2m} r^{\frac{3}{2} + m} e^{-i\nu r} U \left( \frac{1}{2} + m - \kappa, 1 + 2m; z \right),$$  

(B1b)

where, in our case, $n = 2\ell + 3$ and $m, \kappa, z$ are as defined in Eq. (11). Using the formula

$$C_{\eta, \ell} = \frac{2\ell e^{-\pi\eta/2} |\Gamma(\ell + 1 + i\eta)|}{(2\ell + 1)!},$$  

(B2)

Eq. (5.1) in Ref. [26], one finds that

$$F_n^{(0)}(p, r) = \frac{1}{p^{\ell+1} C_{\eta, \ell}} F_{\ell}^{(p)}(r)$$  

(B3a)

and

$$G_n^{(0)}(p, r) = p^{\ell} C_{\eta, \ell} \left[ G_{\ell}^{(p)}(r) - iF_{\ell}^{(p)}(r) \right].$$  

(B3b)

It is shown in Ref. [30] that $F_n^{(0)}(p, r)$ is analytic in $p^2$. Furthermore, from Eqs. (3.16), (3.17) and (4.1) in that paper it follows that

$$G_n^{(0)}(p, r) = \tilde{G}_n^{(0)}(p, r) + \left( \gamma \bar{h}_\ell(p) - i p^{2\ell+1} C_{\eta, \ell} \right) \cdot F_n^{(0)}(p, r),$$  

(B4)

---

From Eq. (3.1) in Ref. [26] one directly sees that this is consistent with Eq. (27) in Sec. IIIB.
where $\tilde{h}_k(p)$ is the function defined in Eq. (26), and where $\tilde{G}_n^{(0)}(p,r)$ is analytic in $p^2$. The functions $f(p,r)$ and $\tilde{g}(p,r)$ that we introduce in Sec. III B are exactly the analytic wave functions defined above, i.e.,

$$f(p,r) = F_n^{(0)}(p,r) \quad \text{and} \quad \tilde{g}(p,r) = \tilde{G}_n^{(0)}(p,r).$$

(B5)

It should be noted that Lambert [35] already defines Coulomb wave functions that are analytic in $p^2$, but only for the most common case of three spatial dimensions. Bollé and Gesztesy extend Lambert’s results to an arbitrary number of dimensions $\geq 2$.

More importantly, at least for our application, in Eq. (4.3) of Ref. [30], Bollé and Gesztesy give an explicit expression for $\tilde{G}_n^{(0)}(p,r)$. Since there are two typos in their original equation,\footnote{The $i\gamma/k$ in the first line should be $i\gamma/(2k)$ and the $(q+1)$ in the last line should be $\Gamma(q+1)$.} we quote the whole expression for completeness. Slightly altering the notation to match our conventions, we have

$$\tilde{G}_n^{(0)}(p,r) = \left[\Gamma(n-1)^{-2}\right] (2p)^{n-3} \left| \Gamma\left(\frac{n-1}{2} + \frac{i\gamma}{2p}\right) \right|^2 \left| \Gamma\left(1 + \frac{i\gamma}{2p}\right) \right|^2 \gamma \log (|\gamma| r) \cdot F_n^{(0)}(p,r)$$

$$- \gamma \Re \left\{ \left[\Gamma(n-1)^{-2}\right] (2p)^{n-3} \left| \Gamma\left(\frac{n-1}{2} + \frac{i\gamma}{2p}\right) \right|^2 \left| \Gamma\left(1 + \frac{i\gamma}{2p}\right) \right|^2 r^{-n(n-1)/2} e^{-ipr} \right\}$$

$$\times \sum_{k=0}^{\infty} \frac{\Gamma\left(\frac{n-1}{2} + k - \frac{i\gamma}{2p}\right) \Gamma(n-1)(2ipr)^k}{\Gamma\left(\frac{n-1}{2} - \frac{i\gamma}{2p}\right) \Gamma(n-1 + k)\Gamma(k+1)} \left[ \psi(k+1) + \psi(k+n+1) \right]$$

$$+ \gamma \Re \left\{ \left[\Gamma(n-1)^{-2}\right] (2p)^{n-3} \left| \Gamma\left(\frac{n-1}{2} + \frac{i\gamma}{2p}\right) \right|^2 \left| \Gamma\left(1 + \frac{i\gamma}{2p}\right) \right|^2 r^{-n(n-1)/2} e^{-ipr} \right\}$$

$$\times \sum_{k=0}^{\infty} \frac{\Gamma\left(\frac{n-1}{2} + k - \frac{i\gamma}{2p}\right) \Gamma(n-1)(2ipr)^k}{\Gamma\left(\frac{n-1}{2} + \frac{i\gamma}{2p}\right) \Gamma(n-1 + k)\Gamma(k+1)} \sum_{s=1}^{n-1} \left( s - \frac{i\gamma}{2p} \right)^{-1}$$

$$+ \Re \left\{ \left( n - 2 \right)^{-1} (2ip)^{n-2} r^{-n(n-1)/2} e^{-ipr} \right\}$$

\[(n-2)^{-1} \sum_{q=0}^{n-3} \frac{\Gamma\left(\frac{3-n}{2} + q - \frac{i\gamma}{2p}\right) \Gamma(3-n)(2ipr)^{q+2-n}}{\Gamma\left(\frac{3-n}{2} - \frac{i\gamma}{2p}\right) \Gamma(q + 3 - n)\Gamma(q+1)} \right\}, \quad (B6)

valid for any odd $n \geq 3$ and where the sum over $s$ in the fourth line is defined to give zero for $n = 3$ and $k = 0$.

**Appendix C: Numerical calculations**

In order to check our relations and to get a better understanding of the values for the causal range, we consider some explicit examples numerically.

By cutting off the singular parts of the potential (i.e., the Coulomb potential and the angular momentum term for $\ell \geq 0$) at very small distances, it is a simple task to numerically solve the radial Schrödinger equation \[4\] in configuration space. From the radial wave
functions one can extract the Coulomb-modified phase shifts by looking for a zero at some large (i.e., much larger than the range $R$ of the short-range potential) distance,

$$w_\ell^{(p)}(r_0) = 0 \quad r_0 \gg R,$$

(C1)

and then calculating

$$\cot \tilde{\delta}_\ell(p) = -\frac{G_\ell(p)(r_0)}{F_\ell(p)(r_0)}.$$  

(C2)

For the simplest case of a local step potential,

$$V(r, r') = V_{\text{step}}(r) \cdot \delta(r - r') \equiv V_0 \theta(R - r) \cdot \delta(r - r'),$$

(C3)

one can of course also obtain the phase shift directly by matching the wave functions at $r = R$. The effective range parameters are then obtained by repeating the calculation for several (small) momenta and fitting Eq. (21) to the results.

In order to test Eq. (45) directly one needs the wave function to calculate the integral

$$\int_0^r dr' \left[w_\ell^{(0)}(r') \right]^2 = \lim_{p \to 0} \int_0^r dr' \left[w_\ell^{(p)}(r') \right]^2.$$  

(C4)

Even if we do not actually take the limit $p \to 0$ but rather just insert some small $p_0 = 0.1$ (in units of an arbitrary inverse length scale), we find that the relation

$$r^C_\ell \approx b^C_\ell(R) - 2 \int_0^R dr' \left[w_\ell^{(p_0)}(r') \right]^2,$$

(C5)

is typically fulfilled to better than one percent accuracy for the simple step potential defined above.

For illustration, in the following we choose units where the radial distance is measured in fm. The potential range is set to 1 fm and its strength is measured in MeV.\(^{11}\) Furthermore, the reduced mass and Coulomb parameter are set to the values for the proton–proton system, i.e., $2\mu = m_N \approx 940$ MeV and $\gamma = \gamma_{p-p} \approx 0.035$ fm\(^{-1}\).

In Figs. 3 and 4 we show the results (for $\ell = 0, 1, 2$) for both repulsive and attractive step potentials. Quite interestingly, the $\ell = 0$ causal range for the repulsive potential stays at zero (meaning that one could reproduce the underlying values of the scattering length and the effective range even with a contact interaction) until a potential strength of about 100 MeV. For higher partial waves the causal range takes a nonzero value for much weaker potentials, but the rise is less steep. In general, it is remarkable that the causal range is typically considerably smaller than the actual potential range ($R = 1$ fm).

For attractive potentials the causal range grows much faster as the potential strength (now negative) increases. In contrast to what one might expect, no special features are seen in the causal ranges as the potential becomes strong enough to support a new bound state close to threshold, i.e., when there is a pole in the scattering length parameter.

To conclude this section, we show the general dependence of the causal range on both the scattering length and the effective range, which has the advantage of not depending on

\(^{11}\)Note that with these conventions, the quantity that is used in the numerical calculation is $v_0 = 2\mu V_0/(\hbar c)^2$, where $\hbar c \approx 197.33$ MeV · fm is used for the unit conversion.
FIG. 3: Causal range for a repulsive step potential and $\gamma = \gamma_{p-p}$.

FIG. 4: Causal range (thick lines) for an attractive step potential and $\gamma = \gamma_{p-p}$. The thin lines show the corresponding scattering lengths.

For illustration, we again measure distances in fm and set the Coulomb parameter to the value of the proton–proton system. In Figs. 5 and 6 we show the results for $\ell = 0$ and $\ell = 1$. For negative effective range $r^C$, the causal range stays essentially zero. For positive effective range, it increases as the absolute value of $a^C$ becomes larger. If one gradually turns off the Coulomb interaction by letting $\gamma \to 0$, the $\ell = 1$ plot stays almost unchanged, whereas the $\ell = 0$ result remains qualitatively the same, but with a much steeper rise in the quadrant where $a^C > 0$ and $r^C > 0$. 
FIG. 5: Causal range for $\gamma = \gamma_{p-p}$ and $\ell = 0$ in dependence of $a^C_0$ and $r^C_0$, both measured in fm.

FIG. 6: Causal range for $\gamma = \gamma_{p-p}$ and $\ell = 1$ in dependence of $a^C_1$ and $r^C_1$, measured in fm$^3$ and fm$^{-1}$, respectively.

Appendix D: Equivalence to relation by Sparenberg et al.

In this appendix, we demonstrate the equivalence of our ANC relation to the one derived by Sparenberg et al. [22].
Their ANC relation, translated to our notational convention, reads

\[ |A| \frac{\ell!}{\Gamma(\ell + 1 + \hat{\eta})} \approx \kappa^{\ell+1} \sqrt{\tilde{a}_\ell^C}. \]  

(D1)

The tilde on the \( \tilde{a}_\ell^C \) is there to indicate that Sparenberg et al. use the convention for the Coulomb-modified effective range expansion as in Refs. [31–33], which, as noted at the end of Sec. II B, differs from the one used in this paper by an overall factor \[ \frac{2\ell!}{(2\ell + 1)!}. \]

Combining equations in Ref. [22] and again matching to our notation, one finds that

\[ \frac{1}{\ell!^2} \prod_{s=1}^{\ell} (s^2 + \eta^2) \left[ C_{\eta,0}^2 \rho^{2\ell+1} \cot \tilde{\delta}_\ell(p) + \gamma p^{2\ell} h(p) \right] = -\frac{1}{\tilde{a}_\ell^C} + \frac{1}{2} \tilde{r}_\ell^C p^2 + \cdots. \]  

(D2)

Note that this is just Eq. (28) with the prefactors combined. More explicitly, we have

\[ \tilde{a}_\ell^C = \left( \frac{\ell! 2^\ell}{(2\ell + 1)!} \right)^2 a_\ell^C, \quad \tilde{r}_\ell^C = \left( \frac{(2\ell + 1)!}{\ell! 2^\ell} \right)^2 r_\ell^C, \quad \text{etc.}. \]  

(D3)

The final step of the derivation by Sparenberg et al. eliminates the effective range in favor of the scattering length. Without invoking this final step their relation reads

\[ |A| \frac{2^{\ell}(2\ell + 1)!}{\Gamma(\ell + 1 + \hat{\eta})} \approx \kappa^{\ell} \left( -\frac{r_\ell^C}{2} + \frac{\gamma^{2\ell-1}}{3(2\ell + 1)!} \right)^{-1/2}, \]  

(D4)

where we are now using our convention for the effective range expansion. With the definition of \( \tilde{C}_{\eta,0}^\ell \) from Eq. (77) and the values for \( \Delta Z_\ell \) from Table III one sees that at least for \( \ell = 0, \ldots, 2 \) this is exactly equivalent to our Eq. (85) with the \( O(\kappa^2) \) set to zero. In order to prove the equivalence for arbitrary \( \ell \) one would need a general expression for \( \Delta Z_\ell \). This, in turn, requires knowledge of the constant terms in the Wronskians. It would thus probably be more interesting to turn the argument around, i.e., take the equivalence for granted and derive from it a general expression for the constant terms in the Wronskians. The only additional ingredient one would need for this procedure is a general series expansion for the Meijer \( G \)-functions that arise from the integral of \( g_0(r)^2 \).

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