BUS OPERATORS IN COMPETITION: A DIRECTED LOCATION APPROACH

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ABSTRACT. We present a directed variant of Salop (1979) model to analyze bus transport dynamics. The players are operators competing in cooperative and non-cooperative games. Utility, like in most bus concession schemes in emerging countries, is proportional to the total fare collection. Competition for picking up passengers leads to well documented and dangerous driving practices that cause road accidents, traffic congestion and pollution. We obtain theoretical results that support the existence and implementation of such practices, and give a qualitative description of how they come to occur. In addition, our results allow to compare the current or base transport system with a more cooperative one.

1. INTRODUCTION

In this work, we model the competition of bus operators for passengers in a public transport concession scheme. The models -which are directed variants of the Salop model [18], in turn a circuit adaptation of the classic Hotelling model [13]- are a characterization of Mexico City’s transport system, where 74.1% of the trips made by public transport are carried out on buses with concession contracts [15].

Much like in other Latin American cities, the contracts laying the responsibilities, penalties and service areas are rarely enforced by the corresponding authorities, and in these instances, the main driver determining the planning and operations tend to be the operator’s profit margins [12] p. 9. Leaving such task to the companies or in some cases individuals, has lead to what [10] refer to as curious old practices: driving habits adopted by bus operators, whose salary is proportional to the fare collection, to maximize the number of users boarding the unit. While these practices were observed and recorded in the United Kingdom in the 1920s, they are very much present in current times, particularly in cities with emerging economies and suboptimal concession plans. We mention the practices enlisted by [10] pertaining to driving:

1. Hanging back or Crowling. Buses go slowly to pick up as much people possible that would otherwise take the bus behind. A variant is to stop altogether until the bus is fully loaded, or the bus behind appears in the horizon.
2. Racing. A driver decides there are too few passengers in a station to make a stop, rendering it more profitable to continue and collect passengers ahead.
3. Overtaking, Tailing or Chasing. Attempting to pass the bus ahead, trying to cut in and pick up the users ahead.
4. Turning. When a bus is empty or nearly empty it turns around before the end of the route, and drives back in the opposite direction.
Many of these practices have negative consequences on the perception of public transport, and in particular, that with concession contracts. The 2019 Survey on victimization in public transport [5] carried out in Mexico City and its metropolitan area, showed that 50% of the interviewees perceived the quality of concession transport to be Bad, and 15% to be Very Bad, while 27% labeled traveling in concessioned transport Somewhat Dangerous, and 60% labeled it Very Dangerous. In both dimensions, public transport concession schemes did worse than any other public or private form of transport. The matter is pressing enough that the current administration of Mexico City stressed in its Strategic Mobility Plan of 2019 [19] p. 9: The business model that governs this (transport) sector, in which profits are individual and exclusively per transported passenger, produces competition in the streets for users, which results in the pick up and drop off of passengers in unauthorized places, increased congestion and a large number of traffic incidents each year.

To reduce public expenditure on transport, Margaret Thatcher introduced the Transport Act 1985 [1], a privatization scheme that demanded companies to keep its vehicles in good condition, avoid dangerous driving, establish service routes, and publish timetables, in addition to forcing drivers to adhere to routes and timetables through regulatory mechanisms established by the companies. This type of regulation and its implementation appear distant for Mexico, so with this work we aim to shed new light on the implications of a transport system where operators compete for passengers without much regulation. We obtain a simple interpretation of the results, consistent with the driving practices mentioned above.

To the best of our knowledge, our approach where bus operators are independent players competing to maximize their utility, which is proportional to the potential advantage of driving at different speeds is novel, and it allows us to model a variety of scenarios and obtain explicit descriptions of their equilibria. Furthermore, we are able to explore the time-evolution of the adopted strategies. All the results are expressed in terms of the behavior of the operators, and not in terms of global economic variables. Given the tractability of our models, some natural theoretical questions emerge.

Relevant literature on transport problems includes [17] modeling of the optimal headway bus service from the point of view of a central dispatcher. In the historical context of Transport Act 1985 [1], several scientific articles analyzed the effect of the privatization. Under the assumption of the existence of an economic equilibrium in the competition system, [10] classify the driving practices into two categories: those consistent with the equilibrium, and those who are not. They analyzed the expected timetables in the deregulated scenario. In [9] is presented a comparative analysis of fare and timetable allocation in competition, monopoly and net benefit maximization (both restricted and unrestricted to a zero profit) models. Building on from this, [16] introduces the consumer’s perspective and obtains the equilibria prices and number of services offered by transport companies. The possibility of predatory behavior between two enterprises competing through fares and service level, is analyzed by [7], using the data from the city of Inverness. In [8] the authors study the optimal policies of competing enterprises in terms of fares, and the bus service headway, in a unique bus stop and destination scenario. They also introduce the concept of demand coordination which can be implemented through timetables. Assuming a spatial directed model with a single enterprise, [6] finds the timetable that minimizes the costs associated to service delays. The work of [4] analyzes flight time data and finds empirical evidence to support Hotelling models. From a non-economic perspective, [3] models competing buses in a circuit behaving like random particles with repulsion between them (meaning they could not pass each other). A contemporary review on transport market models using game theory is given by [2], and a general review of control problems which arise in buses transport systems is presented in [14].
2. The model

The assumptions of the game are the following. There are \( n \leq 2 \) buses being driven by \( n \) drivers along a route. There is only one type of bus and one type of driver, meaning that the buses have identical features, and that the drivers are homogeneous in terms of skill and other relevant characteristics.

The speed of a bus \( v \), is bounded throughout every time and place of the road by:

\[
0 < v_{\text{min}} \leq v \leq v_{\text{max}},
\]

where the constants \( v_{\text{min}} \) and \( v_{\text{max}} \) are fixed, and determined by exogenous factors like the condition of the bus, Federal and State laws and regulations, the infrastructure of the road, etc.

Drivers can pick up passengers along any point on the route at any given time. In other words, there are no designated bus stations, nor interval-based time schedules in place. This scenario is an approximation to a route with a large number of homogeneously distributed bus stops.

We allow for infinite bus capacity, so drivers can pick up any number of passengers they come across. Alternatively, one can assume that passengers alight from the bus almost right after boarding it, so the bus is virtually empty and ready to pick up users at any given time. The important point to note is that passengers that have boarded a bus will not hop on the next, either because they never descended it in the first place, or because they already reached their final destination if they did.

Bus users reach their pick up point at random times, so demand for transport is proportional to the time elapsed between bus arrivals. Let \( \lambda > 0 \) denote the mean number of passengers boarding a bus per unit of time, and let \( p \geq 0 \) denote the fixed fare paid by each user. We assume that there is a fixed driving cost, \( c \geq 0 \) per unit of time. This cost summarizes fuel consumption, maintenance, bus and passenger protection insurance, etc.

Drivers get a share of the total revenue, and consequently seek to maximize it. Since they cannot control the number of passengers on the route, the fare, or the driving costs, their only available resource is to set the driving speed, which we assume remains constant throughout the trip. The strategy space of a bus driver (player) is then

\[
\Gamma = \{ v \geq 0 : v_{\text{min}} \leq v \leq v_{\text{max}} \},
\]

where \( v_{\text{min}} \) and \( v_{\text{max}} \) are given in \((2.1)\). We define a mixed strategy to be a random variable taking values in the space \( \Gamma \).

In what follows we define the expected utility of drivers given a set of assumptions on the number of players and their starting positions, the fixed variables of the models, and route characteristics. Relevant notation and concepts are introduced when deemed necessary.

2.1. Single player games. We first consider a game with only one driver picking up passengers along the road. Importantly, the fact that only one bus is covering the route implies that commuters have no option but to wait for its arrival, the player is aware of this.

- **Fixed-distance game**

A single bus departs the origin of a route of length \( D \). We adopt the convention that the initial time is whenever the bus departs the origin. We define the expected utility of driving at a given speed \( v \) to be

\[
u(v) := (p\lambda)T - cT,
\]

where \( T = \frac{D}{v} \) is the time needed to travel the distance \( D \) at speed \( v \).
Note that since there is no other bus picking up passengers, the expected number of people waiting for the bus in a fixed interval of the road increases proportionally with time. From this, one infers that the expected total number of passengers taking the bus is proportional to the time it takes the bus to reach its final destination. The conclusion and its implication can be expressed rigorously using a space-time Poisson process, see for example [11] pp. 283-296.

- **Fixed-time game**

Suppose now that the driver chooses a constant speed $v$ satisfying (2.1) in order to drive for $T$ units of time. The bus then travels the distance $D = Tv$, which clearly depends on $v$. We define the expected utility of driving at a given speed $v$ to be

$$u(v) := (p\lambda)D - cT.$$  

The underlying assumption is that for sufficiently small $T$, there are virtually no new arrivals of commuters to the route, so effectively, the number of people queuing for the bus remains the same as that of the previous instant. The requirement is that $T$ is small compared to the expected interarrival times of commuters. It follows that the total amount of money collected by the driver is proportional to the total distance traveled by the bus.

2.2. **Two player games.** There are two buses picking up passengers along a route, which we assume is a one-way traffic circuit. An advantageous feature of circuits is that buses that return from any point on the route to the initial stop may remain in service; this is generally not the case in other types of routes. In particular, we assume that the circuit is a one-dimensional torus of length $D$. For illustration purposes and without loss of generality, from now on we require the direction of traffic to be clockwise.

We define the $D$-module of any real number $r$ as

$$(r)_{mod D} := \frac{r}{D} - \left\lfloor \frac{r}{D} \right\rfloor,$$

where $\lfloor z \rfloor$ is the greatest integer less than or equal to $z$.

The interpretation of $(r)_{mod D}$ is the following: if starting from the origin, a bus travels the total distance $r$, then $(r)_{mod D}$ denotes its relative position on the torus. Indeed, $r$ may be such that the bus loops around the circuit many times, nonetheless $(r)_{mod D}$ is in $[0, D)$ for all $r$. We refer to $r$ as the absolute position of the bus, and to $(r)_{mod D}$ as the relative position (with respect to the torus). Note that the origin and the end of the route share the same relative position, since $(0)_{mod D} = 0 = (D)_{mod D}$.

Let $x$ and $y$ denote the two players of the game, and let $x, y$ be their respective relative positions. The directed distance function $d_x$ is given by

$$d_x(x, y) := \begin{cases} 
    y - x & \text{if } x \leq y, \\
    D + y - x & \text{if } x > y.
\end{cases}$$

Equation (2.5) has a key geometrical interpretation: it gives the distance from $x$ to $y$ considering that traffic is one-way. The interest of this is that the potential amount of commuters $x$ picks up is proportional to the distance between $x$ and $y$, namely $d_x(x, y)$. See Figure 1.

A straightforward observation is that for any real number $r$, we have

$$d_x((x + r)_{mod D}, (y + r)_{mod D}) = d_x(x, y).$$

This justifies the first summand in (2.3).
This asserts that if we shift the relative position of the two players by \( r \) units (either clockwise or counterclockwise, depending on the sign of \( r \)), then the directed distance \( d_x \) is unchanged.

One can define the directed distance \( d_y \) analogously,

\[
d_y(x, y) := \begin{cases} 
  x - y & \text{if } y \leq x, \\
  D + x - y & \text{if } y > x.
\end{cases}
\]

By definition, there is an intrinsic symmetry between \( d_x \) and \( d_y \): we have \( d_x(x, y) = d_y(y, x) \) and \( d_y(x, y) = d_y(y, x) \). Roughly speaking, this means that if one were to swap all the labels, namely \( x \) to \( y \), \( x \) to \( y \), and vice versa, then it suffices to plug the new labels into the previous definitions to obtain the directed distances.

Another immediate observation is that for any pair of different positions \((x, y)\), the sum of the two directed distances gives the total length of the circuit,

\[
d_x(x, y) + d_y(x, y) = D.
\]

This is portrayed in Figure 1.

**FIGURE 1. Directed distances**

Let us assume that players \( x \) and \( y \) have starting positions \( x_0 \) and \( y_0 \) in \([0, D)\). The initial minimal distance is defined to be

\[
d_0 := \min\{d_x(x_0, y_0), d_y(x_0, y_0)\}.
\]

Now suppose that starting from \( x_0 \) and \( y_0 \), operators drive at the respective speeds \( v_x \) and \( v_y \), with \( v_x, v_y \) in \( \Gamma \), for \( T \) units of time. Their final relative positions are then

\[
x_T = (x_0 + T v_x) \mod D \quad \text{and} \quad y_T = (y_0 + T v_y) \mod D.
\]

Let us orient the maximum displacement of buses by requiring \( T v_{\max} \), with \( v_{\max} \) given in (2.1), to be small compared to \( D \). The reason for this is to be consistent with our assumption of constant speed strategies, since they are short-term. More precisely, we require

\[
T v_{\max} < \frac{D}{2}.
\]

Lastly, we define the escape distance by

\[
d := T(v_{\max} - v_{\min}).
\]

\[\text{Importantly, this switches the relative positions of the players.}\]
This gives a threshold such that if the distance between the players is shorter than $d$, then the buses can catch up to the other, given an appropriate pair of strategies. If the distance is greater than $d$, this cannot occur.

We now proceed to define the expected utility of players given the type of game being played, namely, whether it is cooperative or non-cooperative.

- **Non-cooperative game**

  We define the utility of $x$ given the initial positions of players $x_0$ and $y_0$, and the strategies $v_x$ and $v_y$, to be

  \[
  u_x(x_0, v_x, y_0, v_y) := \begin{cases} 
  p\lambda d_x(x_T, y_T) - cT & \text{if } x_T \neq y_T, \\
  p\lambda \frac{D^2}{2} - cT & \text{if } x_T = y_T.
  \end{cases}
  \]

  The definition above includes two summands: the first one gives the (gross) expected income of $x$, since the factor $p\lambda$ is the expected income per unit of distance. The second factor gives the total driving cost.

  It is worth pointing out that for simplicity, we have assumed that the expected income depends only on the relative final positions $x_T$ and $y_T$. A more precise account would consider the entire trajectory of the buses. Nevertheless, even if this could be described with mathematical precision, the model would grow greatly in complexity without adding to the economic interpretation.

  Similarly, we define

  \[
  u_y(x_0, v_x, y_0, v_y) := \begin{cases} 
  p\lambda d_y(x_T, y_T) - cT & \text{if } x_T \neq y_T, \\
  p\lambda \frac{D^2}{2} - cT & \text{if } x_T = y_T.
  \end{cases}
  \]

  By equation (2.7) and the definition of the utility functions (2.11), (2.12), we have that the sum $u_x + u_y$ is a constant that does not depend on the driving speeds nor on the initial positions. For this reason, we analyze the game as a zero-sum game.

- **Cooperative game**

  Players aim to maximize the collective payoff, and this amounts to solving the global optimization of the sum $U_x + U_y$, which includes the utility functions in the non-cooperative game (2.11) and (2.12). Since the non-cooperative game is a zero-sum game, we introduce an extra term in the utility, which gives the discomfort players derive from payoff inequality. This assumption can be imagined in a situation where equity in payments is desirable, specially since players have complete information.

  We define the utility function to be

  \[
  u(x_0, v_x, y_0, v_y) := u_x(x_0, v_x, y_0, v_y) + u_y(x_0, v_x, y_0, v_y) - k|u_x(x_0, v_x, y_0, v_y) - u_y(x_0, v_x, y_0, v_y)|,
  \]

  where $k$ is a non-negative constant, and all the other elements being as in the non-cooperative game.

  **2.2.1. Mixed strategies and $\varepsilon$-equilibria.**

  For the solution of two player games it is convenient to define the expected utility of randomizing over the set of strategies. We also introduce the definition of $\varepsilon$-equilibrium.

  Suppose that players $x$ and $y$ use the mixed strategies $X$ and $Y$.\footnote{Recall that a mixed strategy is a random variable taking values in the set $\Gamma = \{v \geq 0 : v_{\min} \leq v \leq v_{\max}\}$.} We define the utility of player $x$ to be

  \[
  U(x_0, X, y_0, Y) := E[u(x_0, X, y_0, Y)].
  \]
An analogous definition can be derived for player \( y \).

Let \( \varepsilon > 0 \). We say that a pair of pure strategies \( (v_x^*, v_y^*) \) is an \( \varepsilon \)-equilibrium if for every \( v_x \) and \( v_y \) we have

\[
 u_x(x_0, v_x, y_0, v_y^*) \leq u_x(x_0, v_x^*, y_0, v_y^*) + \varepsilon,
\]

and

\[
 u_y(x_0, v_x^*, y_0, v_y) \leq u_y(x_0, v_x^*, y_0, v_y^*) + \varepsilon.
\]

This means that any unilateral deviation from the equilibrium strategy leads to a gain of no more than \( \varepsilon \); this is why an \( \varepsilon \)-equilibrium is also called near-Nash equilibrium. Note that in particular, an \( \varepsilon \)-equilibrium with \( \varepsilon = 0 \) gives the standard definition of Nash equilibrium.

A mixed strategies \( \varepsilon \)-equilibrium \( (X, Y) \) is similarly defined by replacing the utility functions with the expected utility functions in the last definition.

3. Results

In what follows, we analyze the speeds that drivers choose, both in the short-run and in longer time periods. Results on the short-term are crucial for the latter analysis, which involves the implementation of the obtained equilibria.

3.1. Single player games. The single player games have pure strategy Nash equilibria. We state the driver’s optimal strategy in each game.

**Theorem 1.** Let \( v^* \) in \( \Gamma \) be the driving speed that maximizes the utility of the driver. We provide an explicit description of \( v^* \).

a) Fixed-distance game. Given the utility function defined in (2.3), we have

\[
 v^* = \begin{cases} 
 v_{\text{min}} & \text{if } p\lambda > c, \\
 v_{\text{min}} \leq v \leq v_{\text{max}} & \text{if } p\lambda = c, \\
 v_{\text{max}} & \text{if } p\lambda < c.
\end{cases}
\]

b) Fixed-time game. Given the utility function defined in (2.4), we have \( v^* = v_{\text{max}} \).

**Proof.** The proofs are straightforward, nevertheless we outline the main ideas. Note that in the fixed-distance game, \( p\lambda - c \) gives the driver’s expected net income per unit of time. If this amount is positive, then the player maximizes her utility by driving for the longest time, or equivalently, by driving at the lowest possible speed. Conversely, a negative expected net income leads to driving at the highest speed. Lastly, a null expected income makes the driver indifferent between any given speed in the range.

In the fixed-time game, the total revenue is proportional to the traveled distance, so the driver maximizes her utility by driving at the highest speed.

3.2. Two-player games. The strategies adopted by the players strongly depend on the initial minimal distance defined in (2.8). We cover all the cases.

**Theorem 2.** Non-cooperative game. Assume, without lose of generality, that \( d_0 = d(x_0, y_0) \). We then have:

a) If \( d_0 = 0 \), that is, if the initial positions of the players are the same, then the pair of strategies \( (v_{\text{max}}, v_{\text{max}}) \) is the only Nash equilibrium.
3.3. If \( 0 < d_0 < d < d_k(x_0, y_0) \), with \( d \) the escape distance in (2.14), then for sufficiently small \( \varepsilon \), the mixed strategies \( \varepsilon \)-equilibria \((X, Y)\) are of the form

\[
P(X = V) = \begin{cases} 
1 - \frac{d-d_0}{D} & \text{if } V = v_{\min} \\
\frac{d-d_0}{D} & \text{if } V = U_X
\end{cases}
\] and

\[
P(Y = W) = \begin{cases} 
q_1 & \text{if } W = v_{\min} \\
q_2 & \text{if } W = U_Y \\
1 - \frac{d}{D} & \text{if } W = v_{\max} - \frac{d_0}{T} + \frac{\varepsilon}{T}
\end{cases}
\]

where \( U_X \) is a uniform random variable on \((v_{\min} + \frac{d_0}{T}, v_{\max})\), \( q_1 \) and \( q_2 \) are non-negative numbers such that \( q_1 + q_2 = \frac{d}{D} \) and \( q_2 \leq \frac{d-d_0}{D} \), and \( U_Y \) is a uniform random variable on \((v_{\max} - \frac{d_0}{T} - q_2 \frac{D}{T}, v_{\max} - \frac{d_0}{T})\).

c) If \( 0 < d = d_0 < d_k(x_0, y_0) \), then for sufficiently small \( \varepsilon \), the mixed strategies \( \varepsilon \)-equilibrium \((X, Y)\) are of the form

\[
P(X = V) = \begin{cases} 
1 - \frac{2d}{D} & \text{if } V = v_{\min} \\
\frac{2d}{D} & \text{if } V = v_{\max}
\end{cases}
\] and

\[
P(Y = W) = \begin{cases} 
\frac{2d}{D} & \text{if } W = v_{\min} \\
1 - \frac{2d}{D} & \text{if } W = v_{\min} + \frac{\varepsilon}{T}
\end{cases}
\]

d) If \( d < d_0 \), then the pair of strategies \((v_{\min}, v_{\min})\) is the unique Nash equilibrium.

Proof. The proof is in Appendix A.\[\square\]

By assumption (2.9), this result covers all the possible initial positions \((x_0, y_0)\), so we have a complete and explicit characterization of the equilibria. Simply put, the theorem asserts that if the players have the same starting point, they drive at the maximum speed. If their positions differ by at most the escape distance, then they play mixed strategies. Lastly, if the distance between them is greater than the escape one, they drive at the minimum speed. See Figure 2 for an illustration of the result and its cases.

**Theorem 3. Cooperative game.** Without loss of generality we assume that \( d_0 = d_k(x_0, y_0) \).

a) If \( d_0 = 0 \), then the optimal driving speeds are \((v_{\min}, v_{\max})\), and \((v_{\max}, v_{\min})\).

b) If \( 0 < d_0 \) and \( d_0 + d < \frac{D}{2} \), then the only optimal strategies are \((v_{\min}, v_{\max})\).

c) If \( d_0 + d > \frac{D}{2} \), then any pair \((v_X, v_Y)\) such that \( T(v_Y - v_X) = \frac{D}{2} \) is an optimal strategy.

Proof. The proof is direct. Since the sum \( u_X(x_0, v_X, y_0, v_Y) + u_Y(x_0, v_X, y_0, v_Y) \) is equal to a constant for any pair \((v_X, v_Y)\), the only quantity left to optimize is \(-k|v_X(x_0, v_X, y_0, v_Y) - v_Y(x_0, v_X, y_0, v_Y)|\). Minimization occurs when the distance between the final positions \( x_F \) and \( y_F \) is the greatest possible. It is easy to check that the driving speeds listed above do just this.\[\square\]

An important observation is that in the case where \( d_0 = \frac{D}{2} \), accounted for in c), all the optimal strategies are of the form \((v, v)\) for feasible speed \( v \). Intuitively, this means that if the players have diametrically opposite initial positions, then any speed is optimal, as long as both adopt it.

### 3.3. Time evolution.
Let us recall that the previous results are obtained for small enough \( T \), the formal requirement being stated in (2.9). It is of interest to know what happens in longer periods, and in particular, in the long-run. To this end, we repeat the base game, but update the location of the players accordingly. It is convenient to define a recursive process, and to introduce a few variables.

Consider the initial positions of \( x \) and \( y \), \((x_0, y_0)\) with \( d_0 \) defined in (2.8). Let \( \{(x_n, y_n)\}_{n \geq 1} \) be a stochastic process with the following property: the pair \((x_{k+1}, y_{k+1})\) gives the final locations of the
Figure 2. Time-space depictions of Theorem 2. On the right (y-axis at time $T$), are the final positions of players: blue for $x$ and red for $y$, when they implement the equilibrium strategies. Points represent probability mass atoms, while continuous intervals give the range of the uniform random variables.

Given that equilibria in Theorem 2 involve mixed strategies, randomness is present in the process.

We may define the distance between the buses at any (non-negative integer) time as:

$$d_n := \min\{d_x(x_n, y_n), d_y(x_n, y_n)\} \quad \forall n \geq 0.$$  

We also consider the first time in which $d_n$ exceeds the escape distance $d$ (defined in (2.10)), $N$ defined as follows

$$N = \min\{n \geq 0 : d_n > d\}.$$

**Theorem 4. Non-cooperative game.** If $d_0 \neq 0, d$, we have

$$\mathbb{P}(N > k) \leq \left(\frac{d}{D}\right)^k \quad \text{for all } k \geq 1.$$  

If $d_0 = d$, then there exists a geometrically distributed random time $M$ with parameter $1 - (1 - \frac{d}{2D})(\frac{d}{D})$, taking values in the natural numbers, with $\varepsilon$ satisfying the $\varepsilon$-equilibrium conditions in Theorem 3, with the property that $d_k = d$ for all $k < M$, and
\[ d_M = \begin{cases} 
0 & \text{with probability } \frac{4\epsilon d}{D^2 \left(1 - \left(1 - \frac{2d}{D} \right) \left( \frac{2d}{D} \right) \right)}, \\
> d & \text{with complementary probability.} 
\end{cases} \]

Proof. For the proof we refer the reader to Appendix A. \qed

Explicitly, this means that for most starting points, playing the game repeatedly leads to a bus gap greater than the escape distance. From Theorem 2, we conclude that in this case, drivers end up driving at the minimum speed. There are two exceptions to this: if the drivers have the same starting position, or if the initial distance between them is exactly that of escape. In the former case, the drivers choose to go at the maximum speed forever, and in the latter, they maintain their distance for some random time, and from then on reach the escape distance, and drive at the minimum speed. It is with very little probability (proportional to \( \epsilon \)) that this scenario does not occur. Figure 3 shows the evolution of the distance process \( \{d_n : n \geq 0\} \) given a few initial distances \( d_0 \).

![Figure 3. Evolution of the process \( \{d_n : n \geq 0\} \) for different initial positions.](image)

\textbf{Theorem 5. Cooperative game.} We have \( N \leq \lceil \frac{D}{2d} \rceil \), where \( N = \min \left\{ n \geq 0 : d_n = \frac{D}{2} \right\} \), and \( \lceil z \rceil \) is the least integer greater than or equal to the real number \( z \).

Proof. First note that \( N \) gives the time in which the buses reach diametrically opposite positions in the circuit. It is also worth noting that playing the optimal strategies in Theorem 3 increases the distance between the buses by \( d \). Hence, repeating the game eventually leads to reaching the diametric distance. This means that \( N \) is at most the number of steps of size \( d \) necessary to go over \( \frac{D}{2} \). Once that some diametrical positions are reached, that distance is preserved. \qed
3.4. Extension. It is possible to account for perturbations like traffic lights, congestion, or accidents in the model, by introducing a random noise to the displacement of buses. One could do this defining

\[
x_T = (x_0 + T v_x + \sigma Z_x) \mod D \quad \text{and} \quad y_T = (y_0 + T v_y + \sigma Z_y) \mod D,
\]

where \( Z_x \) and \( Z_y \) are independent standard normal random variables and \( \sigma \geq 0 \) is a fixed parameter.

- **Non-cooperative game.** Given that the expected value of the final positions is unchanged, Theorem 2 remains valid. However, the repetition of this new game leads to an interesting result. Since the probability of maintaining a null or escape distance \( d \) at any positive time is zero, the long-run analysis is reduced to two distinct cases: \( 0 < d_0 < d \) and \( d_0 < d \).

  Arguments similar to that in the proof of Theorem 4 show that if \( 0 < d_0 < d \), we have \( d_N \geq d \) in an exponentially fast time. If \( d < d_0 \), then the process \( \{d_n\}_{n \geq 1} \) is above \( d \) for a random time \( M \), but falls below it eventually. The expected time above is inversely proportional to \( \sigma \).

- **Cooperative game.** The analysis collapses to the cases b) and c) of Theorem 3. So, the players try to reach the diametrically opposite positions, though with probability one this does not occur.

4. Concluding remarks

Our theoretical results are consistent with the driving practices mentioned in the Introduction. In particular, Theorem 2a corresponds to (2) Racing, Theorem 2b, c to (2) Racing and (3) Overtaking, Tailing or Chasing, and Theorem 2d to (1) Hanging back or Crowling. It is worth noting that all of the aforementioned are short-term strategies. As far as time-evolution goes, Theorem 4 asserts that in the long run and with high probability, both operators end up hanging back. Theorems 3 and 5 are intended to contrast the drivers’ optimal strategies and ultimately the equilibria when cooperation is desired.

In subsection 3.4, we extended the model to allow for randomness in displacement. In this scenario no equilibrium is lasting, so the operators alternate between racing, hanging back and chasing from time to time. We believe this is precisely what happens in Mexico City, although proving this would require a data driven approach analysis.

There are a few open problems worth exploring. First, one could increase the number of players, and investigate whether equilibria still exists, and if so, try to characterize it. Second, one may vary the distribution of passengers along the route, dispensing with the homogeneous assumption. Along these lines, one may introduce traffic congestion by making the utility function depend on space in a non-homogeneous manner. This would potentially require strategies to depend on the player’s position. Lastly, one could introduce decision variables like tariffs and timetables; doing so would allow to compare the results with some that have already been addressed in the literature.

**Declaration of interest.** None.

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To prove Theorem 2, it is convenient to introduce the following Lemma.

**Lemma 1.** Let \( X \) be a mixed strategy of \( x \) and \( Y \) be a mixed strategy of \( y \). We define \( Z \) to be a mixed random variable in the Probability theory sense: it has both discrete and continuous components. In particular, \( Z \) is of the form

\[
Z = \begin{cases} 
  z_i & \text{with probability } p_i, \text{ for } i \in I, \\
  W & \text{with probability } 1 - \sum_{i \in I} p_i,
\end{cases}
\]

where \( I \) is a finite or numerable set, and \( W \) is a continuous random variable with density \( f_W(t) \) on its support, denoted by \( \text{supp}(f_W) \). Then,

\[
U_x(x_0, X, y_0, Y) = \sum_{i \in I} \mathbb{E}(u_x(x_0, X, y_0, Y)|Z = z_i) p_i + \left(1 - \sum_{i \in I} p_i\right) \int_{\text{supp}(f_W)} \mathbb{E}(u_x(x_0, X, y_0, Y)|Z = w)f_W(w)dw.
\]

(A.1)

If \( Z = X \) and \((X, Y)\) is a mixed strategy Nash equilibrium, then

\[
\mathbb{E}(u_x(x_0, X, y_0, Y)|X = z_i) = \int_{\text{supp}(f_W)} \mathbb{E}(u_x(x_0, X, y_0, Y)|X = w)f_W(w)dw \quad \forall i \in I,
\]

(A.2)

and

\[
\mathbb{E}(u_x(x_0, X, y_0, Y)|X = w_1) = \mathbb{E}(u_x(x_0, X, y_0, Y)|X = w_2) \quad \forall w_1, w_2 \in \text{supp}(f_W).
\]

(A.3)
Proof. Equation (A.1) is straightforwardly obtained by computing the conditional expectancy of the random variable $u_x(x_0, X, y_0, Y)$ given the values of $Z$.

Note that if (A.2) does not occur, then there exist two different values $z_i$ and $z_j$, such that $E(u_x(x_0, X, y_0, Y)|X = z_i) \neq E(u_x(x_0, X, y_0, Y)|X = z_j)$. This means that $U_x$ can be increased by placing all the probability on the value that gives the highest expectation. This leads to a contradiction with the form of the mixed strategy $X$. Similar arguments apply to the case where (A.2) is violated through the continuous component.

Likewise, if condition (A.3) is not fulfilled, then there are two values $w_1$ and $w_2$ such that $E(u_x(x_0, X, y_0, Y)|X = w_i)$ are different. Then, $U_x$ can be increased by restricting the support of $f_W$ to the points where the maximum of the function $g(w) = E(u_x(x_0, X, y_0, Y)|X = w)$ is reached. Here, the form of the mixed strategy $X$ is violated.

Proof of Theorem 2

First, note that for optimizing the utility function (2.11), (2.12) the terms $p\lambda$ and $c$ are irrelevant, since the arg min of any function is invariant under linear transformations. Thus, there is no loss of generality in assuming that $p\lambda = 1$ and $c = 0$.

By equation (2.6), we may actually assume that $0 = x_0 \leq y_0 < D$. We then have

$$d_0 = d_x(x_0, y_0) = y_0 \quad \text{and} \quad d_y(x_0, y_0) = D - y_0.$$  

Under the above assumption and using (2.1), (2.9) in cases a), b), c) and d), it happens that $0 < x_T, y_T < D$, so we can get rid of all the $D$-modules in the computations.

For computing the $\varepsilon$-equilibrium, we will consider the $\varepsilon$-best reply, defined as follows. Let $\varepsilon$ be a positive number. We say that a strategy $v^*_x$ is $x$’s $\varepsilon$-best reply to $y$’s strategy $v_y$, if

$$u_x(x_0, v_x, y_0, v_y) \leq u_x(x_0, v^*_x, y_0, v_y) + \varepsilon,$$

for all strategies $v_x$.

To simplify notation, we write $u_x(v_x, v_y)$ and $u_x(X, Y)$ in the case of mixed strategies, instead of $u_x(x_0, v_x, y_0, v_y)$ and $u_x(x_0, X, y_0, Y)$ if the computations do not depend on the fixed initial positions.

• Case a)  

We assume that $x_0 = y_0 = 0$. Let player $y$ pick the strategy $v_y = v_{\text{max}}$. Then,

$$u_x(x_0, v_x, y_0, v_{\text{min}}) = d_x(Tv_x, Tv_{\text{max}}) = Tv_{\text{max}} - Tv_x \leq Tv_{\text{max}}.$$  

Using (2.9), we obtain the bound

$$u_x(x_0, v_x, y_0, v_{\text{min}}) \leq \frac{D}{2} = u_x(x_0, v_{\text{max}}, y_0, v_{\text{max}}).$$

Explicitly, this means that the strategy $v_x = v_{\text{max}}$ is the best reply to $v_y = v_{\text{max}}$. By symmetry, we conclude that $(v_{\text{max}}, v_{\text{max}})$ is a Nash equilibrium.

To check the uniqueness of the equilibrium, we note that $y$’s $\varepsilon$-best reply to a given speed $v_x < v_{\text{max}}$ chosen by $x$, is $v_y = v_x + \varepsilon$ for sufficiently small $\varepsilon$. On the other hand, $x$’s $\varepsilon$-best reply to $v_y = v_x + \varepsilon$ is $v_x = v_y + \varepsilon$. Therefore the only equilibrium is $(v_{\text{max}}, v_{\text{max}})$.  

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• Case b)

Let us denote by \(B_X(v)\) X’s best reply when \(y\) plays \(v\). It is straightforward to show that

\[
B_X(v) = \begin{cases} 
  v + \frac{d_0}{T} + \frac{x}{T} & \text{if } v_{\min} \leq v < v_{\max} - \frac{d_0}{T}, \\
  v_{\max} & \text{if } v = v_{\max} - \frac{d_0}{T}, \\
  v_{\min} & \text{if } v_{\max} - \frac{d_0}{T} < v,
\end{cases}
\]

and

\[
B_Y(v) = \begin{cases} 
  v_{\min} & \text{if } v < v_{\min} + \frac{d_0}{T}, \\
  v - \frac{d_0}{T} + \frac{x}{T} & \text{if } v_{\min} + \frac{d_0}{T} \leq v \leq v_{\max},
\end{cases}
\]

under hypothesis b).

If \((X, Y)\) is a mixed strategy Nash equilibrium, then the support of the random variable \(X\) should be contained in the set of \(X\)’s best replies, the corresponding is true for variable \(Y\). In this particular case, \(X\) has support on \(\{v_{\min}\} \cup (v_{\min} + \frac{d_0}{T}, v_{\max}) \cup \{v_{\max}\}\), while \(Y\) has support on \(\{v_{\min}\} \cup (v_{\min}, v_{\max} - \frac{d_0}{T}) \cup \{v_{\max} - \frac{d_0}{T} + \frac{x}{T}\}\).

Hence, a mixed strategy \(X\) with the support obtained is of the form

\[
X = \begin{cases} 
  v_{\min} & \text{with probability } p_1, \\
  U & \text{with probability } p_2, \\
  v_{\max} & \text{with probability } 1 - p_1 - p_2,
\end{cases}
\]

where \(p_1, p_2 \in [0, 1]\), and \(U\) is a continuous random variable with density \(f_U(u)\) and support contained in \((v_{\min} + \frac{d_0}{T}, v_{\max})\). Similarly, a mixed strategy \(Y\) with the desired support is

\[
Y = \begin{cases} 
  v_{\min} & \text{with probability } q_1, \\
  V & \text{with probability } q_2, \\
  v_{\max} - \frac{d_0}{T} + \frac{x}{T} & \text{with probability } 1 - q_1 - q_2,
\end{cases}
\]

where \(q_1, q_2 \in [0, 1]\), and \(V\) is a continuous random variable with density \(f_V(v)\) with support contained in \((v_{\min}, v_{\max} - \frac{d_0}{T})\).

To compute the density of \(U\), we apply \((A.3)\) to \(Y\). Let us compute \(E(u_Y(X, Y)|Y = v)\) when \(v \in (v_{\min} - \frac{d_0}{T}, v_{\max} - \frac{d_0}{T})\):

\[
E(u_Y(X, Y)|Y = v) = \mathbb{E}(u_Y(X, v)) = p_1 u_Y(v_{\min}, v) + p_2 \mathbb{E}(u_Y(U, v)) + (1 - p_1 - p_2) u_Y(v_{\max}, v) = p_1 (D + T v_{\min} - T v - d_0) + p_2 \int_{v_{\min}}^{v+\frac{d_0}{T}} (D + Tu - T v - d_0) f_U(u) du + \int_{v+\frac{d_0}{T}}^{v_{\max}} (T u - T v - d_0) f_U(u) du + (1 - p_1 - p_2) (Tv_{\max} - T v - d_0) = p_1 D + p_2 DF_U(v + \frac{d_0}{T}) + p_1 T v_{\min} + (1 - p_1 - p_2) T v_{\max} + p_2 T \mathbb{E}(U) - T v - d_0,
\]

where \(F_U(u)\) is the cumulative probability distribution function of the random variable \(U\).

By \((A.3)\), we have

\[
(A.4) \quad p_1 D + p_2 DF_U(v + \frac{d_0}{T}) + p_1 T v_{\min} + (1 - p_1 - p_2) T v_{\max} + p_2 T \mathbb{E}(U) - T v - d_0 = k,
\]

for some constant \(k\).

Since \(F_U(v_{\max}) = 1\), when we plug \(v = v_{\max} - \frac{d_0}{T}\), we obtain its value
On substituting $k$ into (A.4) we obtain

$$F_U\left(v + \frac{d_0}{T}\right) = 1 - \frac{T(v_{max} - v) - d_0}{p_2D}.$$ 

Let $u = v + \frac{d_0}{T}$. Then, $u \in (v_{min} + \frac{d_0}{T}, v_{max})$ and $F_U(u) = 1 - \frac{T(v_{max} - u)}{p_2D}$. From this we have $u^* = v_{max} - \frac{p_2D}{2}$ is the value such that $F_U(u^*) = 0$.

The conclusion is that $U$ is uniformly distributed on the interval $(v_{max} - \frac{p_2D}{T}, v_{max})$, thus

(A.6) $$E(U) = v_{max} - \frac{p_2D}{2T}.$$ 

In the same manner we can see that $V$ has uniform distribution on the interval $(v_{max} - \frac{d_0}{T} - \frac{q_2D}{T}, v_{max} - \frac{d_0}{T})$, with expectancy given by

(A.7) $$E(V) = v_{max} - \frac{d_0}{T} - \frac{q_2D}{2T}.$$ 

To compute the values of $p_1$ and $p_2$ necessary for the $\varepsilon$-equilibrium, we use (A.2). We first compute the conditional expectancy of $u_y(X,Y)$ given $Y$,

$$E(u_y(X,Y)|Y = v_{min}) = p_1u_y(v_{min}, v_{min}) + p_2E(u_y(U, v_{min})) + (1 - p_1 - p_2)u_y(v_{max}, v_{min})$$

$$= p_1(D - d_0) + p_2\left[\int_{v_{min}}^{v_{max}}(Tu - Tv_{min} - d_0)f_U(u) \, du\right]$$

$$+ (1 - p_1 - p_2)(Tv_{max} - Tv_{min} - d_0)$$

$$= p_1(D - d_0) + p_2(TE(U) - Tv_{min} - d_0) + (1 - p_1 - p_2)(Tv_{max} - Tv_{min} - d).$$

By (A.6), we have

(A.8) $$E(u_y(X,Y)|Y = v_{min}) = p_1D - d_0 + (1 - p_1)(Tv_{max} - Tv_{min}) - \frac{p_2D}{2}.$$ 

Computing $E(u_y(X,Y)|Y = V)$ yields

$$E(u_y(X,Y)|Y = V) = \int_{v_{max} - \frac{d_0}{T} - \frac{q_2D}{T}}^{v_{max} - \frac{d_0}{T}} E(u_y(X,Y)|Y = v)f_Y(v) \, dv.$$ 

Since we know that the integrand is constant and its value is given by equations (A.5) and (A.6), we directly obtain

(A.9) $$E(u_y(X,Y)|Y = V) = (p_1 + p_2)D - p_1(Tv_{max} - Tv_{min}) - \frac{p_2D}{2}.$$
We are left with the task of determining the expected value of \( u_Y(X, Y) \) conditioned on the value \( Y = v_{\max} - \frac{d_0}{T} + \frac{\varepsilon}{T} \),

\[
\mathbb{E}(u_Y(X, Y)|Y = v_{\max} - \frac{d_0}{T} + \frac{\varepsilon}{T}) = p_1 u_Y(v_{\min}, v_{\max} - \frac{d_0}{T} + \frac{\varepsilon}{T}) + p_2 \mathbb{E}(u_Y(U, v_{\max} - \frac{d_0}{T} + \frac{\varepsilon}{T}))
\]

\[
+ (1 - p_1 - p_2) u_Y(v_{\max}, v_{\max} - \frac{d_0}{T} + \frac{\varepsilon}{T})
\]

\[
= p_1(D - T(v_{\max} - v_{\min}) - \varepsilon)
\]

\[
+ p_2 \int_{v_{\max} - \frac{d_0}{T}}^{v_{\max}} (D - T(v_{\max} - u) - \varepsilon)f_U(u) \, du + (1 - p_1 - p_2)(D - \varepsilon)
\]

\[
= p_1(D - T(v_{\max} - v_{\min}) - \varepsilon)
\]

\[
+ p_2(D - T v_{\max} + T \mathbb{E}(U) - \varepsilon) + (1 - p_1 - p_2)(D - \varepsilon)
\]

\[
(A.10)
\]

where we used (A.6) in the last equality.

Lemma (A.2) implies that in order to have an \( \varepsilon \)-equilibrium, the expressions (A.8), (A.9) and (A.10) must be equal. This system of equations has the unique solution

\[
p_1 = 1 - \frac{T(v_{\max} - v_{\min}) - d_0}{D}, \quad p_2 = \frac{T(v_{\max} - v_{\min}) - d_0}{D}, \quad 1 - p_1 - p_2 = 0.
\]

We now apply this argument again, to obtain the expectancy of the random variable \( u_Y(X, Y) \) conditioned on the values of \( X \), as well as the values \( q_1, q_2 \) necessary to have an \( \varepsilon \)-equilibrium. In this case, there are many solutions. Indeed, any combination \( q_1, q_2 \) satisfying

\[
0 \leq q_1, q_2, \quad q_1 + q_2 = \frac{T(v_{\max} - v_{\min})}{D}, \quad 1 - q_1 - q_2 = 1 - \frac{T(v_{\max} - v_{\min})}{D}
\]

fulfills equation (A.2).

Given that the support of \( V \) is \( (v_{\min} - \frac{d_0}{T} - q_2 \frac{d_0}{T}, v_{\max} - \frac{d_0}{T}) \subseteq (v_{\min}, v_{\max} - \frac{d_0}{T}) \), it is necessary to impose the condition \( q_2 \leq \frac{d_0 - \frac{d_0}{T}}{D} \).

- **Case c)**

From the conditions stated in c), it follows that

\[
B_X(v) = \begin{cases} 
  v_{\max} & \text{if } v = v_{\min}, \\
  v_{\min} & \text{if } v > v_{\min}.
\end{cases}
\]

Intuitively, under hypothesis c), it always happens that \( x_T \leq y_T \) for every pair of strategies \( v_X, v_Y \). Equality holds only when \( v_X = v_{\max} \) and \( v_Y = v_{\min} \).

Similarly, one can check that

\[
B_Y(v) = \begin{cases} 
  v_{\min} & \text{if } v < v_{\max}, \\
  v_{\max} + \frac{\varepsilon}{T} & \text{if } v = v_{\max},
\end{cases}
\]

where last case is an \( \varepsilon \)-best reply.

To find the \( \varepsilon \)-equilibria, we define \( X \) to be a random variable such that

\[
\mathbb{P}(X = v_{\min}) = p, \quad \mathbb{P}(X = v_{\max}) = 1 - p, \quad \text{for some probability } p \in [0, 1].
\]
Similarly, we define a random variable $Y$ such that
\[
\mathbb{P}(Y = v_{\min}) = q, \quad \mathbb{P}(Y = v_{\min} + \frac{\varepsilon}{T}) = 1 - q, \quad \text{for } q \in [0, 1].
\]

An $\varepsilon$-equilibrium requires $\mathbb{E}(u_x(v_{\min}, Y)) = \mathbb{E}(u_x(v_{\max}, Y))$, which is exactly the condition \[A.2\] when there is no continuous part for $X$.

Since
\[
\mathbb{E}(u_x(v_{\min}, Y)) = qu_x(v_{\min}, v_{\min}) + (1 - q)u_x\left(v_{\min}, v_{\min} + \frac{\varepsilon}{T}\right) = qd + (1 - q)(d + \varepsilon),
\]
and
\[
\mathbb{E}(u_x(v_{\max}, Y)) = qu_x(v_{\max}, v_{\min}) + (1 - q)u_x\left(v_{\max}, v_{\min} + \frac{\varepsilon}{T}\right) = q\left(\frac{D}{2}\right) + (1 - q)(\varepsilon),
\]
we can equalize the two equations and solve to obtain $q = \frac{2d}{D}$. Note that \[2.1\] implies that $0 < q < 1$.

Similarly, we should have $\mathbb{E}(u_y(X, v_{\min})) = \mathbb{E}\left(u_y\left(X, v_{\min} + \frac{\varepsilon}{T}\right)\right)$. The explicit formulas being
\[
\mathbb{E}(u_y(X, v_{\min})) = pu_y(v_{\min}, v_{\min}) + (1 - p)u_y(v_{\max}, v_{\min}) = p(D - d) + (1 - p)\frac{D}{2},
\]
and
\[
\mathbb{E}\left(u_y\left(X, v_{\min} + \frac{\varepsilon}{T}\right)\right) = p\left(u_y\left(v_{\min}, v_{\min} + \frac{\varepsilon}{T}\right) + (1 - p)u_y\left(v_{\max}, v_{\min} + \frac{\varepsilon}{T}\right)\right) = p(D - d - \varepsilon) + (1 - p)(D - \varepsilon).
\]

Matching and solving the two yields $0 < 1 - p = \frac{2\varepsilon}{T} < 1$.

• Case d)

Assume that player $y$ chooses strategy $v_y$ satisfying \[2.1\]. Then
\[
(A.11) \quad u_x(x_0, v_x, y_0, v_y) = d_x(Tv_x, y_0 + Tv_y).
\]
By assumption $d)$, we have
\[
T(v_x - v_y) \leq T(v_{\max} - v_{\min}) < d_x(x_0, y_0) = y_0,
\]
so $y_0 + Tv_y - Tv_x > 0$ for every $v_x, v_y$. Then, \[A.11\] is equal to
\[
u_x(x_0, v_x, y_0, v_y) = y_0 + T(v_y - v_x),
\]
which is bounded by
\[
u_x(x_0, v_x, y_0, v_y) \leq y_0 + T(v_y - v_{\min}) = u_x(x_0, v_{\min}, y_0, v_y).
\]
We conclude that $v_x = v_{\min}$ is $x$’s best reply to any strategy $v_y$ played by $y$.

Similarly, if $x$ chooses strategy $v_x$, then
\[
(A.12) \quad u_y(x_0, v_x, y_0, v_y) = d_y(Tv_x, y_0 + Tv_y).
\]
We have already proven that $y_0 + Tv_y - Tv_x > 0$ for every $v_x, v_y$, so \[A.12\] is equal to
\[
u_y(x_0, v_x, y_0, v_y) = D + Tv_x - y_0 - Tv_y.
\]
We can bound the last expression by
\[
u_y(x_0, v_x, y_0, v_y) = D - y_0 + Tv_x - v_y \leq D - y_0 + T(v_x - v_{\min}) = u_y(x_0, v_x, y_0, v_{\min}).
\]
This implies $v_y = v_{\min}$ is $y$’s best reply to any strategy $v_x$ played by $x$. The conclusion is that $(v_{\min}, v_{\min})$ is the unique Nash equilibrium.
\[\square\]
Proof of Theorem 4

First, note that $d_0 > d$ implies $N \equiv 0$, and the result holds trivially.

Assume that $0 < d_0 < d$, and suppose that $0 < d_k < d$ for some $k \geq 0$. Then, the strategies $(U, v_{\min}), (U, V), (U, v_{\max} - \frac{d_k}{T} + \frac{\epsilon}{T}), (v_{\min}, v_{\min}), (v_{\min}, V)$ lead to $0 < d_{k+1} < 0$ with probability one.

If the strategies of $x$ and $y$ are instead $(v_{\min}, v_{\max} - \frac{d_k}{T} - \frac{\epsilon}{T})$, then $d_{k+1} = d + \epsilon$. We can uniformly bound from below the probability that the players adopt these strategies by

$$P((X, Y) = (v_{\min}, v_{\max} - \frac{d_k}{T} - \frac{\epsilon}{T})) = \left(1 - \frac{d - d_k}{D}\right)\left(1 - \frac{d_k}{D}\right) \geq \left(1 - \frac{d}{D}\right), \quad \forall \ 0 < d_k < d,$$

where the inequality can be obtained by calculus (or by noting that this probability is an inverted parabola, as a function of $d_k$). Therefore,

$$P(N > k) \leq P(G > k),$$

where $G$ is a geometric random variable with parameter $1 - \frac{d}{D}$, and the result follows.

Finally, assume that $d_0 = d$. If players $x$ and $y$ choose $(v_{\min}, v_{\min})$, then $d_1 = d$. Any other strategy choice yields $d_1 \neq d$.

Define $M = \min\{n \geq 1 : d_n \neq d\}$. By the above remark, $M$ has geometric distribution on the natural numbers with parameter $1 - \left(1 - \frac{2\epsilon}{D}\right)\left(\frac{2d}{D}\right)$. After $M$ trials, we are on the conditional space where $x$ and $y$ do not play $(v_{\min}, v_{\min})$, instead they choose

$$\begin{align*}
(v_{\max}, v_{\min}) & \quad \text{with probability} \quad \frac{\left(\frac{2\epsilon}{D}\right)\left(\frac{2d}{D}\right)}{1 - \left(1 - \frac{2\epsilon}{D}\right)\left(\frac{2d}{D}\right)}, \\
(v_{\min}, v_{\min} + \frac{\epsilon}{T}) & \quad \text{with probability} \quad \frac{\left(1 - \frac{2\epsilon}{D}\right)\left(1 - \frac{2d}{D}\right)}{1 - \left(1 - \frac{2\epsilon}{D}\right)\left(\frac{2d}{D}\right)}, \\
v_{\max}, v_{\min} + \frac{\epsilon}{T} & \quad \text{with probability} \quad \frac{\left(\frac{2\epsilon}{D}\right)\left(1 - \frac{2d}{D}\right)}{1 - \left(1 - \frac{2\epsilon}{D}\right)\left(\frac{2d}{D}\right)}.
\end{align*}$$

The first election leads to $d_{M+1} = 0$, while the other two give $d_{M+1} > d$. This concludes the proof.

\[ \square \]

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