Transcendence of polynomial canonical heights

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Abstract
There are two fundamental problems motivated by Silverman’s conversations over the years concerning the nature of the exact values of canonical heights of $f(z) \in \bar{\mathbb{Q}}(z)$ with $d := \deg(f) \geq 2$. The first problem is the conjecture that $\hat{h}_f(a)$ is either 0 or transcendental for every $a \in \mathbb{P}^1(\bar{\mathbb{Q}})$; this holds when $f$ is linearly conjugate to $z^d$ or $\pm C_d(z)$ where $C_d(z)$ is the Chebyshev polynomial of degree $d$ since $\hat{H}_f(a)$ is algebraic for every $a$. Other than this, very little is known: for example, it is not known if there exists even one rational number $a$ such that $\hat{h}_f(a)$ is irrational where $f(z) = z^2 + \frac{1}{2}$. The second problem asks for the characterization of all pairs $(f, a)$ such that $\hat{H}_f(a)$ is algebraic. In this paper, we solve the second problem and obtain significant progress toward the first problem in the case of polynomial dynamics. These are consequences of our main result concerning algebraic numbers that can be expressed as a multiplicative combination of values of Böttcher coordinates. The proof of our main result uses a certain auxiliary polynomial and the powerful Medvedev–Scanlon classification of preperiodic subvarieties of split polynomial maps.

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1 Introduction

For many decades, the theory of Weil heights and canonical heights has been not only an indispensable tool in diophantine geometry and arithmetic dynamics but also a highly interesting subject of its own. Yet the nature of the values of canonical height functions remains mysterious. For dynamics of univariate rational functions, we have the following:
Conjecture 1.1 (Silverman) Let $f(z) \in \bar{\mathbb{Q}}(z)$ such that $d := \deg(f) \geq 2$. For every $a \in \mathbb{P}^1(\bar{\mathbb{Q}})$, we have that $\hat{h}_f(a)$ is either 0 or transcendental.

As explained in Silverman’s comments [30], this conjecture originates from his conversations over the years about canonical heights of non-torsion points on elliptic curves. Conjecture 1.1 together with some related problems and comments also appear in [23]. Conjecture 1.1 is in stark contrast to known results over function fields by Chatzidakis-Hrushovski [7, Lemma 4.21] in which values of canonical heights are usually algebraic (it appears that in [7], the authors use the notation $H_D$ to denote a certain logarithmic canonical height) as well as more recent rationality results by DeMarco-Ghioca [10].

There is an analogue of Conjecture 1.1 for the multiplicative canonical height $\hat{H}_f = \exp(\hat{h}_f)$. A variant of the following has been suggested in Silverman’s comments [30] as well:

Problem 1.2 (Silverman) Characterize pairs $(f, a)$ such that $f(z) \in \bar{\mathbb{Q}}(z)$ has degree $d \geq 2$, $a \in \mathbb{P}^1(\bar{\mathbb{Q}})$, and $\hat{H}_f(a)$ is algebraic.

Throughout this paper, $\mathbb{N}$ denotes the set of positive integers and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Let $C_d(z)$ be the Chebyshev polynomial of degree $d$: it satisfies the functional equation $C_d\left(z + \frac{1}{z}\right) = z^d + \frac{1}{z^d}$. When $f(z) \in \bar{\mathbb{Q}}(z)$ is linearly conjugate to $z^d$ or $\pm C_d(z)$, we have that $\hat{H}_f(a)$ is algebraic for every $a \in \mathbb{P}^1(\bar{\mathbb{Q}})$ and hence $\hat{h}_f(a) = \log(\hat{H}_f(a))$ is either 0 or transcendental thanks to Lindemann’s theorem [16]. Other than this, very little is known about Conjecture 1.1. It appears that we do not know even one example of a rational number $a$ such that $\hat{h}_f(a)$ is irrational where $f(z) = z^2 + \frac{1}{2}$.

As immediate consequences of our main result, we resolve Problem 1.2 and make first significant progress to Conjecture 1.1 in the case of polynomial dynamics. From now on, we consider the case $f(z) \in \bar{\mathbb{Q}}[z]$. We use $J_f$ and $K_f$ respectively to denote the Julia set and filled Julia set of $f$. Let $a \in \bar{\mathbb{Q}}$. A very artificial way to force the algebraicity of $\hat{H}_f(a)$ is to require that $\sigma(a) \in K_{\sigma(f)}$ for every $\sigma \in G := \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$. Let $K$ denote a number field containing $a$ and the coefficients of $f$. The above property is equivalent to having $\sup_v |f^n(a)|_v < \infty$ for every archimedean place $v$ of $K$.

Algebraicity of $\hat{H}_f(a)$ follows from the fact that the non-archimedean contribution to $\hat{H}_f(a)$ is an algebraic number while the (logarithm of the) archimedean contribution to $\hat{H}_f(a)$ vanishes. This is reminiscent of the situation over function fields in which there is no archimedean contribution at all. Moreover if $a$ is not $f$-preperiodic then we have an example in which $\hat{h}_f(a)$ is transcendental thanks to Lindemann’s theorem again. As mentioned in [23], this strategy works under the assumption that the interior of $K_{\sigma(f)}$ is non-empty for every $\sigma \in G$. On the other hand, when there exists $\sigma \in G$ such that the interior of $K_{\sigma(f)}$ is empty (equivalently $K_{\sigma(f)} = J_{\sigma(f)}$), it is a difficult problem in general to determine whether a fractal like $J_{\sigma(f)}$ contains an algebraic number that is not $\sigma(f)$-preperiodic. This problem is analogous to the conjecture that all irrational numbers in the Cantor set are transcendental.

Following the terminology by Favre-Gauthier book [13], we have:
**Definition 1.3** A polynomial of degree \( d \geq 2 \) is called **integrable** if it is linearly conjugate to \( z^d \) or \( \pm C_d(z) \), otherwise it is said to be **non-integrable**. We let \( D_d \) denote the set of non-integrable polynomials of degree \( d \).

Non-integrable polynomials are called disintegrated by Medvedev-Scanlon paper [21]. For a polynomial \( f(z) \in \mathbb{C}[z] \) of degree \( d \geq 2 \), a Böttcher coordinate of \( f \) is a Laurent series:

\[
\phi_f(z) = a_1 z + a_0 + \frac{a_{-1}}{z} + \frac{a_{-2}}{z^2} + \cdots \in z\mathbb{C}[[1/z]]
\]

with \( a_1 \neq 0 \) such that \( \phi_f(f(z)) = \phi_f(z)^d \). A Böttcher coordinate exists and is unique up to multiplication by a \((d-1)\)-th root of unity. More details about Böttcher coordinates and polynomial canonical heights will be given in the next section. The basin of infinity of \( f \) is the set of \( a \in \mathbb{C} \cup \{ \infty \} \) such that \( \lim_{n \to \infty} f^n(a) = \infty \), i.e. the set \((\mathbb{C} \cup \{ \infty \}) \setminus K_f \). By a **functional domain of convergence** of \( \phi_f \), we mean a connected neighborhood \( D \) of \( \infty \) inside the basin of infinity such that \( f(D) \subseteq D \) and \( \phi_f \) is convergent on \( D \). Then we have \( \phi_f(f(a)) = \phi_f(a)^d \) for every \( a \in D \). At first sight, our main result does not seem to have anything to do with canonical heights:

**Theorem 1.4** Suppose \( d \geq 2 \), \( r \in \mathbb{N}_0 \), \( f_1, \ldots, f_r \in D_d \), \( a_i \) for \( 1 \leq i \leq r \) are algebraic numbers in a functional domain of convergence of \( \phi_{f_i} \), and \( n_1, \ldots, n_r \) are integers. If \( \alpha := \phi_{f_1}(a_1)^{n_1} \cdots \phi_{f_r}(a_r)^{n_r} \) is algebraic, then \( a \) is a root of unity.

**Remark 1.5** In this paper, the superscript \( n \) means both the \( n \)-th iterate and the \( n \)-th power maps. We believe that this will not cause any confusion.

The first consequence of Theorem 1.4 resolves Problem 1.2 in the case of polynomial dynamics:

**Corollary 1.6** Let \( f \in \overline{\mathbb{Q}}[z] \) be a non-integrable polynomial of degree \( d \geq 2 \) and let \( a \in \overline{\mathbb{Q}}. \) The following are equivalent:

(i) \( \hat{H}_f(a) \) is algebraic.

(ii) \( \sigma(a) \in K_{\sigma(f)} \) for every \( \sigma \in \mathcal{G} \).

(iii) For every archimedean place \( v \) of \( \overline{\mathbb{Q}} \), we have \( \sup_n |f^n(a)|_v < \infty \).

**Remark 1.7** Our formulation of Corollary 1.6 does not involve a number field \( K \) over which \( f \) and \( a \) are defined. If we pick such a \( K \) then in part (ii) we can replace \( \sigma \in \mathcal{G} \) by “embedding \( \sigma : K \to \mathbb{C} \).” Similarly in part (iii), we can replace “archimedean place \( v \) of \( \overline{\mathbb{Q}} \)” by “archimedean place \( v \) of \( K \).” Corollary 1.6 amounts to the rather surprising fact that the artificial way to force the algebraicity of \( \hat{H}_f(a) \) mentioned earlier is indeed the only way! In other words, algebraicity of \( \hat{H}_f(a) \) happens exactly when there is no archimedean contribution. This is in stark contrast to the case when \( f \) is linearly conjugate to \( z^d \) or \( \pm C_d(z) \) in which \( \hat{H}_f(a) \) is always algebraic.

The second consequence of Theorem 1.4 is that there exist many numbers having irrational logarithmic canonical height:
Corollary 1.8 Let \( r, d \geq 2 \), let \( K \) be a number field, and let \( f_1(z), \ldots, f_r(z) \in K[z] \) be non-integrable polynomials of degree \( d \). Let \( S = \{ p_1, \ldots, p_r \} \) be a set of \( r \) distinct prime numbers. Let \( a_1, \ldots, a_r \in K \) such that the following property holds. For every \( 1 \leq i \leq r \), there exists a place \( v_i \) of \( K \) lying above \( p_i \) such that 
\[
\lim_{n \to \infty} |f_i^n(a_i)|_{v_i} = \infty
\]
while \( \sup_n |f_i^n(a_i)|_w < \infty \) for every place \( w \) of \( K \) lying above any of the \( p_j \) with \( j \neq i \). Then the numbers \( \hat{h}_{f_1}(a_1), \ldots, \hat{h}_{f_r}(a_r) \) are linearly independent over \( \mathbb{Q} \). Consequently, all but at most one of them are irrational.

Example 1.9 Consider the example \( f_1(z) = \cdots = f_r(z) = f(z) = z^2 + \frac{1}{2} \) earlier and consider \( \hat{h}_f(1/p) \) for odd prime numbers \( p \). Corollary 1.8 implies that at most one such value is rational, at most two such values belong to any given quadratic field such as \( \mathbb{Q}(\sqrt{2}) \), at most three of them belong to any given cubic field, etc.

We now explain in detail the method in the proof of Theorem 1.4. Generally speaking, there are two common methods to prove that a given number is transcendental. The first one is to use the Schmidt’s Subspace Theorem. Recent examples include the paper [25] resolving the transcendence counterpart of an open problem by Erdős–Graham and the paper [5] by Bell–Diller–Jonsson giving the first examples of dominant rational self-maps having transcendental dynamical degrees as well as their recent work with Krieger [2] on birational self-maps. Roughly speaking, in the above applications of the Subspace Theorem, the given number has a special form so that it has a strong diophantine approximation property. Unfortunately, this does not seem to be the case for numbers of the form \( \phi_{f_1}(a_1)^{n_1} \cdots \phi_{f_r}(a_r)^{n_r} \).

The other method is to construct certain auxiliary polynomials vanishing at certain points. This is an old and extremely useful method in transcendental number theory and diophantine approximation. In recent years, it has been used to prove several surprising results in combinatorics [1, 9, 12] and coined the Polynomial Method [14]. Perhaps the most relevant idea to our current work dates back to a series of papers by Mahler in the late 1920s [17–19] concerning the transcendence and algebraic independence of values of functions satisfying certain functional equations. The ideas in Mahler’s papers have been developed further since the 1970s and the readers are referred to Nishioka’s notes [26] for a survey of results up to the mid 1990s. Unlike these modern results in which Siegel’s lemma is needed in order to control the size of the coefficients of the auxiliary polynomials, our construction relies on much simpler dimension counting arguments as in [1, 9, 12] and the powerful Medvedev-Scanlon classification [21].

Let \( m \in \mathbb{N} \) and let \( f_1, \ldots, f_m \in \bar{\mathbb{Q}}[z] \). We use \( (f_1, \ldots, f_m) \) to denote the split polynomial map \((\mathbb{P}^1)^m \to (\mathbb{P}^1)^m\) given by 
\[
(x_1, \ldots, x_m) \mapsto (f_1(x_1), \ldots, f_m(x_m)).
\]

We will use the “coarse structure” of preperiodic subvarieties under the above split polynomial map [21, Sect. 2]. Some of its aspects have appeared in earlier work of Chatzidakis–Hrushoski–Peterzil [8] and Medvedev [20]. The “finer structure” [21, Sects. 3–6] (also see [24, 27]) boils down to the description of invariant curves of \((\mathbb{P}^1)^2\) under self-maps of the form \((f, f)\) and involves mostly elementary yet highly
technical arguments in the theory of polynomial decomposition but this is not needed
in the proof of Theorem 1.4.

2 Notations and preliminary results for polynomial dynamics

2.1 Canonical height

Define $\log^+(x) = \log \max(|x|, 1)$ for every $x$. Throughout this subsection, let $K$
be a number field, let $f(z) \in K[z]$ with $d := \deg(f) \geq 2$, and let $a \in K$. Let
$M_K = M_K^\infty \cup M_K^0$ where $M_K^\infty$ is the set of archimedean places and $M_K^0$
is the set of finite places of $K$. For each $v \in M_K$, we normalize the absolute value
$|\cdot|_v$ on $K$ to be the unique extension of the usual $|\cdot|_p$ on $Q$ where $p$ is the restriction of $v$ to $Q$
and let $n_v = [K_v : Q_p]$, this normalization follows [29, Chapter 3] and differs from
[3, p. 11]. We define the Green function and canonical height functions:

$$g_{f,v}(a) = \lim_{n \to \infty} \frac{1}{d^n} \log^+ |f^n(a)|_v,$$

$$\hat{h}_f(a) = \frac{1}{[K : Q]} \sum_{v \in M_K} n_v g_{f,v}(a),$$

$$\hat{H}_f(a) = \exp(\hat{h}_f(a)).$$

It is well-known that the limit in the definition of $g_{f,v}$ exists, the sum in the definition
of $\hat{h}_f(a)$ is a finite sum (i.e. $g_{f,v}(a) = 0$ for all but finitely many $v \in M_K$), and $\hat{h}_f(a)$
is independent of the choice of $K$, see [29, Chapter 5]. We have:

**Lemma 2.1** Suppose $v \in M_K^0$ restricts to $p \in M_Q^0$. Then $g_{f,v}(a) = c \log p$ with $c \in Q$.

**Proof** This is well-known, we include the proof here for the convenience of the readers.
We can write $|f^n(a)|_v = p^{c_n}$ with $c_n \in Q$. If the $c_n$’s are bounded from above then
$g_{f,v}(a) = 0$. Otherwise, let $n_0$ be such that $|f^{n_0}(a)|_v = p^{c_{n_0}}$ is sufficiently large.
Let $\ell$ be the leading coefficient of $f$ and let $c' \in Q$ be such that $|\ell|_v = p^{c'}$, then
$c_{n+1} = d c_n + c'$ for every $n \geq n_0$. Put $c'' = c'/(d - 1)$, then the above recurrence
relation gives:

$$c_n = d^{n-n_0}(c_{n_0} + c'') - c''$$

for $n \geq n_0$.

Therefore $c = \lim_{n \to \infty} \frac{c_n}{d^n} = \frac{c_{n_0} + c''}{d^{n_0}} \in Q$. 

2.2 Böttcher coordinates

The theory of Böttcher coordinates is an important tool for polynomial dynamics. Early
contributors are Böttcher [6] and Ritt [28]. We refer the readers to [22, Chapter 9] for
classical results over $\mathbb{C}$. There have been several modern treatments recently [11, 13, 15] and we follow [13, Section 2.4] here.

Let $K$ be a field of characteristic 0 and let $f(z) \in K[z]$ with $d := \deg(f) \geq 2$. A Böttcher coordinate of $f$ is a Laurent series:

$$
\phi_f(z) = a_1 z + a_0 + \frac{a_{-1}}{z} + \frac{a_{-2}}{z^2} + \ldots \in \bar{K}[[1/z]]
$$

with $a_1 \neq 0$ such that $\phi_f(f(z)) = \phi_f(z)^d$. We have the following:

**Proposition 2.2** Suppose $K$ contains a $(d-1)$-th root of the leading coefficient $\ell$ of $f$. Then there exists $\phi(z) = a_1 z + a_0 + \cdots \in zK[[1/z]]$ that is a Böttcher coordinate of $f$. Moreover, we have:

(a) $a_1^{d-1} = \ell$.

(b) $\psi(z) \in \bar{K}((1/z))$ is a Böttcher coordinate of $f$ if and only if $\psi(z) = \zeta \phi(z)$ where $\zeta$ is a $(d-1)$-th root of unity.

**Proof** See [13, Sect. 2.4].

For the rest of this subsection, we assume that $K$ is a number field containing a $(d-1)$-th root of the leading coefficient of $f$. Let $\phi(z) \in zK[[1/z]]$ be a Böttcher coordinate of $f$.

**Proposition 2.3** The following hold:

(a) For each $v \in M_K$, there exists a positive number $B_v > 0$ such that for every $z \in K_v$ with $|z|_v > B_v$, we have $|f^n(z)|_v = \infty$ and $\phi(z)$ is convergent. If $|z|_v > B_v$ and $|f(z)|_v > B_v$ then $\phi(f(z)) = \phi(z)^d$.

(b) If $v \in M_K$ and $|a|_v > B_v$, then $g_{f,v}(a) = \log |\phi(a)|_v$.

(c) Let $\sigma$ be an embedding of $K$ into $\mathbb{C}$ and let $v \in M_K^\infty$ be given by $|z|_v = |\sigma(z)|$ for every $z \in K$. Then $\sigma(\phi)$ is a Böttcher coordinate of $\sigma(f)$. Moreover, if $|a|_v > B_v$ then $\sigma(\phi)$ is convergent at $\sigma(a)$ and $g_{f,v}(a) = \log |\sigma(\phi)(\sigma(a))|$.

**Proof** Part (a) follows from [13, Sect. 2.4]. Part (b) is well-known but it is only stated in [13, Proposition 2.13] when $f$ has a special form, so we include the short proof here for the sake of completeness. Replacing $a$ by some $f^k(a)$ if necessary, we may assume that $|f^n(a)|_v > B_v$ for $n \geq 0$. Write $\phi(z) = a_1 z + \cdots$, then as $n \to \infty$ we have

$$
|\phi(a)|_v^{d^n} = |\phi(f^n(a))|_v = |a_1|_v \cdot |f^n(a)|_v + O(1)
$$

and get the desired result. For part (c), we have that $\sigma(\phi)$ satisfies the functional equation in the definition of a Böttcher coordinate for $\sigma(f)$:

$$
\sigma(\phi)(\sigma(f)) = \sigma(\phi(f)) = \sigma(\phi)^d = \sigma(\phi)^d,
$$
hence $\sigma(\phi)$ is a Böttcher coordinate of $\sigma(f)$. Finally, $\sigma$ is an isomorphism from $(K, | \cdot |_v)$ to $((\sigma(K), | \cdot |)$. Hence it can be extended to an isomorphism between $(K_v, | \cdot |_v)$ and $((\sigma(K), | \cdot |)$ where $\sigma(K) = \mathbb{R}$ or $\mathbb{C}$ is the completion of $\sigma(K)$ with respect to $| \cdot |$. Then we have

$$g_{f,v}(a) = \log |\phi(a)|_v = \log |\sigma(\phi)(\sigma(a))|.$$

\[\square\]

2.3 The Medvedev–Scanlon classification

Throughout this subsection, all polynomials and algebraic varieties are over $\mathbb{C}$.

**Definition 2.4** A curve in $\mathbb{P}^1 \times \mathbb{P}^1$ is called non-fibered if its projection to each of the coordinates $\mathbb{P}^1$ is non-constant.

We have the following properties of the coarse structure of preperiodic subvarieties of a split polynomial map given in [21, Section 2]:

**Proposition 2.5** Let $f_1, \ldots, f_m$ be non-integrable polynomials of degree $d \geq 2$ and let $g_1, \ldots, g_n$ be integrable polynomials of degree $d$. We have:

(a) Every subvariety of $(\mathbb{P}^1)^{m+n}$ that is preperiodic under $(f_1, \ldots, f_m, g_1, \ldots, g_n)$ has the form $V \times W$ where $V$ is a subvariety of $(\mathbb{P}^1)^m$ that is preperiodic under $(f_1, \ldots, f_m)$ and $W$ is a subvariety of $(\mathbb{P}^1)^n$ that is preperiodic under $(g_1, \ldots, g_n)$.

(b) Let $V$ be a subvariety of $(\mathbb{P}^1)^m$ that is preperiodic under $(f_1, \ldots, f_m)$. Suppose $m \geq 2$ and $\dim(V) < m$ then there exist an $(f_i, f_j)$-preperiodic curve $C$ in $(\mathbb{P}^1)^2$ such that $V \subseteq \pi_i^{-1}(C)$ where $\pi_{ij} : (\mathbb{P}^1)^m \to (\mathbb{P}^1)^2$ is the projection onto the $i$-th and $j$th coordinate factors.

(c) Let $f$ and $g$ be non-integrable polynomials of degree $d \geq 2$ and let $C$ be a non-fibered curve that is invariant under $(f, g)$. Then there exist non-constant polynomials $A$, $B$, $Q$ such that $C = \{(A(t), B(t)) : t \in \mathbb{P}^1(\overline{\mathbb{Q}})\}$, $f \circ A = A \circ Q$, and $g \circ B = B \circ Q$.

**Proof** Parts (a) and (b) follow from [21, Theorem 2.30] while part (c) follows from [21, Proposition 2.34].

Our next result connects the functional equation in part (c) of Proposition 1.2 to a relationship between Böttcher coordinates:

**Proposition 2.6** Let $f(z)$ be a polynomial of degree $d \geq 2$. Let $A(z)$ and $Q(z)$ be non-constant polynomials such that $f \circ A = A \circ Q$. Let $\phi(z)$ and $\psi(z)$ be respectively a Böttcher coordinate of $f$ and $Q$ and let $\delta = \deg(A)$. Then $\frac{\phi(A(z))}{\psi(z)^{\delta}}$ is a root of unity.
**Proof** First, we observe that if \( S(z) \in \frac{1}{z} \mathbb{C}[[1/z]] \) and \( m \) is a positive integer then

\[
(1 + S(z))^{1/m} = 1 + \frac{1}{m} S(z) + \frac{1}{2} \cdot \frac{1}{m} \cdot \left( \frac{1}{m} - 1 \right) S(z)^2 + \cdots
\]

gives a well-defined element of \( \mathbb{C}[[1/z]] \) and the \( m \)th power of this element is \( 1 + S(z) \).

Write

\[
\phi(A(z)) = \alpha z^\delta + T(z)
\]

with \( \alpha \neq 0 \) and \( T(z) \in z^{\delta-1} \mathbb{C}[[1/z]] \). Choose a \( \delta \)-th root \( \beta \) of \( \alpha \). Then we have

\[
\eta(z) := \beta z \left( 1 + \frac{T(z)}{z^\delta} \right)^{1/\delta}
\]
satisfies \( \eta(z)^\delta = \phi(A(z)) \). Thanks to the given functional equation, we have:

\[
\eta(Q(z))^\delta = \phi(A(Q(z))) = \phi(f(A(z))) = \phi(A(z))^d = \eta(z)^{d\delta}.
\]

Therefore \( \eta(Q(z)) = \zeta \eta(z)^d \) for some \( \delta \)-th root of unity \( \zeta \). Put \( \mu(z) = \zeta_1 \eta(z) \) where \( \zeta_1^{d-1} = \zeta \). Then we have:

\[
\mu(Q(z)) = \zeta_1 \eta(Q(z)) = \zeta_1 \zeta \eta(z)^d = \zeta_1 \zeta (\mu(z)/\zeta_1)^d = \mu(z)^d.
\]

Since \( \deg(Q) = \deg(f) = d \), the above functional equation implies that \( \mu(z) \) is a Böttcher coordinate of \( Q(z) \). Proposition 2.2 gives that \( \frac{\mu(z)}{\psi(z)} \) is a root of unity. Raising this to the \( \delta \)-th power, we get the desired result.

\[\square\]

### 3 Proof of Theorem 1.4

We prove this theorem by contradiction. Suppose there exist \( \alpha, d, r \), the \( f_i \)'s, \( a_i \)'s, and \( n_i \)'s as in the statement of Theorem 1.4 with

\[
\alpha = \phi_{f_1}(a_1)^{n_1} \cdots \phi_{f_r}(a_r)^{n_r}
\]

where \( \alpha \) is an algebraic number that is *not* a root of unity and \( r \) is as small as possible. To simplify the notation, we write \( \phi_i \) instead of \( \phi_{f_i} \). Let \( K \) be a number field such that \( \alpha \), the \( f_i \)'s, \( a_i \)'s, and \( \phi_i \)'s are defined over \( K \) and let \( \mathcal{O}_K \) be its ring of integers. Obviously, \( r > 0 \) otherwise the RHS of (1) is 1. Moreover each \( n_i \neq 0 \) thanks to the minimality of \( r \). We also have that \( \alpha \neq 0 \) since each \( \phi_i(a_i) \neq 0 \) thanks to the identities \( \phi_i(f_i^k(a_i)) = \phi_i(a_i)^{d^k}, f_i^k(a_i) \to \infty \) as \( k \to \infty \), and \( \phi_i(\infty) = \infty \). In the various constructions and estimates below, \( C \) denotes a large positive integer depending on the initial data and \( L \) denotes a large positive integer depending on \( C \) and the initial
data. These $C$ and $L$ are fixed and we will describe how to choose them later. After fixing $C$ and $L$, we use $k$ to denote a sufficiently large integer.

Put $\Phi(X_1, \ldots, X_r) = \phi_1(X_1)^{n_1} \cdots \phi_r(X_r)^{n_r}$. Consider an auxiliary function of the form:

$$
\mathcal{A}(X_1, \ldots, X_r) := P_1 \Phi + P_2 \Phi^2 + \cdots + P_L \Phi^L
$$

where each $P_\ell \in \mathcal{O}_K[X_1, \ldots, X_r]$ has degree at most $L$ in each of the variables $X_1, \ldots, X_r$ for every $1 \leq \ell \leq L$.

Put $n := \max n_i$. Then $\mathcal{A}$ is a sum of terms of the form $X_1^{\delta_1} \cdots X_r^{\delta_r}$ where $\delta_i \in \mathbb{Z}$ and $\delta_i \leq (n + 1)L$ for $1 \leq i \leq r$. The number of tuples $(\delta_1, \ldots, \delta_r)$ satisfying the property:

$$
\delta_i \in [-CL, (n + 1)L] \quad \text{for every } 1 \leq i \leq r
$$

is at most $(2CL)^r$ as long as $C > n + 1$. Having the coefficient of $X_1^{\delta_1} \cdots X_r^{\delta_r}$ vanish for every $(\delta_1, \ldots, \delta_r)$ satisfying (2) is the same as having the coefficients of the $P_\ell$’s satisfy a homogeneous system of at most $(2CL)^r$ many linear equations defined over $K$. Since there are more than $L^{r+1}$ such coefficients, when $L > (2C)^r$ we can always find $P_1, \ldots, P_L$ not all of which are zero such that the coefficient of every $X_1^{\delta_1} \cdots X_r^{\delta_r}$ in $\mathcal{A}$ where the $\delta_i$’s satisfy (2) is zero. Therefore for every $(x_1, \ldots, x_r) \in \mathbb{C}^r$ such that $|x_i|$ is sufficiently large for every $i$, we have:

$$
|\mathcal{A}(x_1, \ldots, x_r)| \ll \max \left\{ 1 \leq i \leq r : |x_i|^{-CL} \prod_{j \neq i} |x_j|^{(n+1)L} \right\}.
$$

Let $P(X_1, \ldots, X_r, Y) = P_1 Y + P_2 Y^2 + \cdots + P_L Y^L$ which is a non-zero element of $K[X_1, \ldots, X_r, Y]$ since some $P_1$ is non-zero. Identity (1) together with the identity $\phi_i(f_r^k(a_i)) = \phi_i(a_i)^{d_k}$ for $1 \leq i \leq r$ yield:

$$
\alpha^{d_k} = \Phi(f_1^k(a_1), \ldots, f_r^k(a_r)) \quad \text{for every } k \in \mathbb{N}_0.
$$

Therefore

$$
|P(f_1^k(a_1), \ldots, f_r^k(a_r), \alpha^{d_k})| = |\mathcal{A}(f_1^k(a_1), \ldots, f_r^k(a_r))|
$$

$$
\ll \max \left\{ 1 \leq i \leq r : |f_i^k(a_i)|^{-CL} \prod_{j \neq i} |f_j^k(a_j)|^{(n+1)L} \right\}
$$

when $k$ is sufficiently large so that each $|f_i^k(a_i)|$ is sufficiently large. Let $C_1, C_2 > 1$ be real numbers depending only on the $f_i$’s, the $a_i$’s, and $\alpha$ such that

$$
C_1^{d_k} < |f_i^k(a_i)|, \quad |\sigma(f_i^k(a_i))| < C_2^{d_k} \quad \text{and } |\sigma(\alpha)| < C_2
$$

(4)
for $1 \leq i \leq r$, $\sigma \in \mathcal{G}$, and for every large integer $k$. Combining this with the previous inequality, we now have:

$$|P(f_1^k(a_1), \ldots, f_r^k(a_r), \alpha^{d^k})| \ll C_1^{-C L d^k} C_2^{r(n+1)L d^k}$$

for every large integer $k$ where the implied constants are independent of $k$. Put:

$$\|P\| = \max\{|\sigma(c)| : \sigma \in \mathcal{G} \text{ and } c \text{ is a coefficient of a monomial term in } P\}$$

and let $C_3$ be a positive integer such that $C_3^{d^k} \alpha^{d^k}$ and the $C_3^{d^k} f_i^k(a_i)$'s are algebraic integers for $1 \leq i \leq r$.

For every $\sigma \in \mathcal{G}$ and for $1 \leq i \leq L$, we have

$$|\sigma(P_i(f_1^k(a_1), \ldots, f_r^k(a_r)))| \leq \|P\|(L + 1)^r C_2^{L r d^k}$$

thanks to (4), the definition of $\|P\|$, and the given properties of $P_i$. Then we have:

$$|\sigma(P(f_1^k(a_1), \ldots, f_r^k(a_r), \alpha^{d^k}))| = \left| \sum_{i=1}^L \sigma(P_i(f_1^k(a_1), \ldots, f_r^k(a_r))) \sigma(\alpha)^{i d^k} \right| \leq L \|P\|(L + 1)^r C_2^{(r+1)L d^k}.$$  

Let $N_{K/\mathbb{Q}}$ denote the norm function of $K/\mathbb{Q}$. Put $D = [K : \mathbb{Q}]$ then (5) and (6) yield:

$$|N_{K/\mathbb{Q}}(P(f_1^k(a_1), \ldots, f_r^k(a_r), \alpha^{d^k}))| \ll C_1^{-C L d^k} C_2^{r(n+1)+(r+1)D L d^k}$$

for all sufficiently large $k$; we emphasize again that the implied constants are independent of $k$. From the choice of $C_3$ and the given properties of the $P_i$'s, we have:

$$C_3^{D(r+1)L d^k} \cdot N_{K/\mathbb{Q}}(P(f_1^k(a_1), \ldots, f_r^k(a_r), \alpha^{d^k})) \in \mathbb{Z}$$

for every large $k$. We now choose $C$ and $L$ such that:

the earlier inequalities $C > n + 1$ and $L > (2C)^r$ hold and

$$C_4 := C_1^{-C} C_2^{r(n+1)+(r+1)D} < \frac{1}{C_3^{D(r+1)}}.$$  

With this choice, (7) and (8) implies that there exists a positive integer $N$ such that

$$P(f_1^k(a_1), \ldots, f_r^k(a_r), \alpha^{d^k}) = 0 \text{ for every integer } k \geq N.$$  

Let $\varphi : (\mathbb{P}^1)^{r+1} \to (\mathbb{P}^1)^{r+1}$ be given by

$$\varphi(x_1, \ldots, x_r, y) = (f_1(x_1), \ldots, f_r(x_r), y^{d^k})$$
and let \( x = (a_1, \ldots, a_r, \alpha) \). Then (10) means \( \varphi^k(x) \) belongs to the proper Zariski closed set defined by \( P = 0 \) for every \( k \geq N \). At this point, it is an easy exercise to show that \( x \) belongs to a proper \( \varphi \)-preperiodic subvariety of \((\mathbb{P}^1)^{r+1}\) and we include the short proof here for the convenience of the readers. Let \( Z \) be the Zariski closure of the \( \varphi^k(x) \)'s with \( k \geq N \). Among all the irreducible components of \( Z \), let \( Z' \) be one with the largest dimension. Then the set \( \{ \varphi^k(x) : k \geq N \} \cap Z' \) is Zariski dense in \( Z' \); otherwise we could replace \( Z' \) by the Zariski closure of this set and have a smaller Zariski closed set than \( Z \) containing all the \( \varphi^k(x) \) for \( k \geq N \). Then for every \( m \geq 0 \), the set \( \{ \varphi^{k+m}(x) : k \geq N \} \cap \varphi^m(Z') \) is Zariski dense in \( \varphi^m(Z') \) and this implies \( \varphi^m(Z') \subseteq Z \). By the maximality of \( \dim(Z') \), we must have that \( \varphi^m(Z') \) is an irreducible component of \( Z \) for every \( m \geq 0 \). This proves that \( Z' \) is \( \varphi \)-preperiodic.

Since \( \alpha \neq 0 \) and \( \alpha \) is not a root of unity, Proposition 2.5(a) implies that \( Z' = V \times \mathbb{P}^1 \) where \( V \) is a proper subvariety of \((\mathbb{P}^1)^r \) that is preperiodic under \((f_1, \ldots, f_r)\). We must have \( r \geq 2 \) since otherwise \( V \) is a preperiodic point and this is a contradiction since \( a_1 \) is not \( f_1 \)-preperiodic. The projection from \( V \) to each coordinate factor \( \mathbb{P}^1 \) is non-constant since each \( a_i \) is not \( f_i \)-preperiodic. Proposition 2.5(b) now implies that there exists \( 1 \leq i \neq j \leq r \) such that \((a_i, a_j)\) belongs to a non-fibered curve in \((\mathbb{P}^1)^2 \) that is preperiodic under \((f_i, f_j)\).

For the rest of this section, fix \( k \in \mathbb{N}_0 \) and \( m \in \mathbb{N} \) such that \((f_i^k(a_i), f_j^k(a_j))\) belongs to a non-fibered curve \( \Gamma \) in \((\mathbb{P}^1)^2 \) that is invariant under \((f_i^m, f_j^m)\). By Proposition 2.5(c) there exist non-constant polynomials \( A, B, \) and \( Q \) such that \( \Gamma \) is parametrized by \((A, B)\) and \( f_i^m \circ A = A \circ Q \) and \( f_j^m \circ B = B \circ Q \). Let \( t \in \overline{\mathbb{Q}} \) such that \( A(t) = f_i^k(a_i) \) and \( B(t) = f_j^k(a_j) \). Let \( \psi \) be a Böttcher coordinate of \( Q \).

Proposition 2.6 implies that \( \frac{\phi_i(f_i^k(a_i))}{\psi(t)^{\deg(A)}} \) and \( \frac{\phi_j(f_j^k(a_j))}{\psi(t)^{\deg(B)}} \) are roots of unity. Hence there is a relation of the form:

\[
\phi_i(a_i)^{dk \deg(B)} = \xi \phi_j(a_j)^{dk \deg(A)}
\]

where \( \xi \) is a root of unity. We now use this relation to eliminate \( \phi_i(a_i) \) from (1) to lower the value of \( r \). This contradicts the minimality of \( r \) and we finish the proof.

4 Proof of the corollaries and comments about further work

4.1 Proof of Corollary 1.6

Let \( K \) be a number field such that \( K/\mathbb{Q} \) is Galois, \( f \in K[z], a \in K, \) and \( K \) contains a \((d - 1)\)-th root of the leading coefficient of \( f \). Although this is not strictly necessary, we enlarge \( K \) so that it has no real embedding. Let \( \phi(z) \in zK[[1/z]] \) be a Böttcher coordinate of \( f \).

As explained before, (ii) and (iii) are equivalent to each other since archimedean places of \( K \) correspond to pairs of complex conjugate embeddings. We have (iii) implies (i) since there is no archimedean contribution to \( \hat{H}_f(a) \) while the non-archimedean contribution is algebraic thanks to Lemma 2.1.
It remains to prove that (i) implies (iii). We prove this by contradiction: suppose that the set

\[ S = \left\{ v \in M_K^\infty : \sup_n |f^n(a)|_v = \infty \right\} \]

is non-empty. For each \( v \in S \), let \( B_v \) be as in Proposition 2.3. Replacing \( a \) by some \( f^k(a) \) if necessary, we may assume that \( |f^n(a)|_v > B_v \) for every \( n \in \mathbb{N}_0 \) and every \( v \in S \). List elements of \( S \) as \( v_1, \ldots, v_r \); note that \( n_{v_i} = 2 \) for every \( i \) since each \( v_i \) is complex. For \( 1 \leq i \leq r \), let \( \sigma_i \) and \( \overline{\sigma}_i \) be the pair of complex conjugate embeddings in \( \text{Gal}(K/\mathbb{Q}) \) that correspond to \( v_i \) meaning \( |x|_{v_i} = |\sigma_i(x)| \) for every \( x \in K \). For \( 1 \leq i \leq r \), we choose the Böttcher coordinates \( \phi_{\sigma_i(f)} \) and \( \phi_{\overline{\sigma}_i(f)} \) of \( \sigma_i(f) \) and \( \overline{\sigma}_i(f) \) so that they are complex conjugate Laurent series. Let

\[ H_0 = \exp \left( \frac{1}{[K : \mathbb{Q}]} \sum_{v \in M_K^0} n_v g_{f,v}(a) \right) \]

be the non-archimedean contribution to \( \hat{H}_f(a) \) which is algebraic thanks to Lemma 2.1.

From Proposition 2.3, we have:

\[ \hat{H}_f(a) = H_0 \cdot \left( \prod_{i=1}^r \phi_{\sigma_i(f)}(\sigma_i(a))\phi_{\overline{\sigma}_i(f)}(\overline{\sigma}_i(a)) \right)^{1/[K:\mathbb{Q}]} \]

Theorem 1.4 implies that \( \hat{H}_f(a)/H_0 \) is a root of unity but this is impossible since each

\[ |\phi_{\sigma_i(f)}(\sigma_i(a))| = |\phi_{\overline{\sigma}_i(f)}(\overline{\sigma}_i(a))| = |\phi_f(a)|_{v_i} > 1 \]

and we finish the proof.

4.2 Proof of Corollary 1.8

We assume that there is a non-trivial linear relation \( \sum_{i=1}^r c_i \hat{H}_{f_i}(a_i) = 0 \) with \( c_i \in \mathbb{Z} \) for every \( i \) and arrive at a contradiction. This yields the multiplicative relation

\[ \prod_{i=1}^r \hat{H}_{f_i}(a_i)^{c_i} = 1. \]

For each \( 1 \leq i \leq r \), let

\[ H_{i,0} = \exp \left( \frac{1}{[K : \mathbb{Q}]} \sum_{v \in M_K^0} n_v g_{f_i,v}(a_i) \right) \quad \text{and} \quad H_{i,\infty} = \exp \left( \frac{1}{[K : \mathbb{Q}]} \sum_{v \in M_K^\infty} n_v g_{f_i,v}(a_i) \right) \]
be respectively the non-archimedean and archimedean contribution to $\hat{H}_{f_i}(a_i)$. The earlier multiplicative relation becomes:

$$\prod_{i=1}^{r} H_{i, \infty}^{c_i} = \prod_{i=1}^{r} H_{i, 0}^{-c_i}. \tag{11}$$

The given local condition implies that the $H_{i,0}$’s are multiplicatively independent, hence the RHS of (11) is a positive real algebraic number that is not 1. As in the previous subsection, we can express the LHS of (11) into the form

$$\prod_{j=1}^{m} \phi_{g_j}(b_j)^{\gamma_j}$$

where each $(g_j, b_j)$ is Galois conjugate to some $(f_i, a_i)$ and each $\gamma_j$ is a rational number; we allow the possibility that $\sup_v |f_i^n(a_i)|_v < \infty$ for $1 \leq i \leq r$ and $v \in M_K$. In which $m = 0$ and the above expression is the empty product. Theorem 1.4 now implies that the RHS of (11) is a root of unity, contradiction.

**4.3 Further comments**

Results in this paper are just the beginning and we end with comments on some further directions. The most natural continuation of Theorem 1.4 is to characterize pairs $(f_1, a_1), \ldots, (f_r, a_r)$ satisfying

$$\phi_{f_1}(a_1)^{n_1} \cdots \phi_{f_r}(a_r)^{n_r} = 1$$

for some non-zero integers $n_1, \ldots, n_r$. Our work indicates that the answer should be when there exist $1 \leq i \neq j \leq r$ such that $(a_i, a_j)$ belongs to an $(f_i, f_j)$-preperiodic curve. In the proof of Theorem 1.4, one has the auxiliary polynomial $P = P_1 Y + \cdots + P_L Y^L$ and it is obvious that $P$ is non-zero when one of the $P_i$’s is non-zero. However, if we imitate the same construction in this further problem, we will have $P = P_1 + \cdots + P_r$ and it is possible that $P = 0$ even when some $P_i \neq 0$. Therefore some further machinery or even an entirely different construction of auxiliary function is needed. Once the above problem is solved, we can characterize the $(f_i, a_i)$’s so that the $\hat{h}_{f_i}(a_i)$’s are linearly dependent over $\mathbb{Q}$ without the further local conditions of Corollary 1.8. Professor Jason Bell also suggests the possibility of strengthening existing results for the Dynamical Mordell–Lang problem [4] for the split polynomial map $(f, g)$ on $\mathbb{P}^1 \times \mathbb{P}^1$.

Beyond these, there is yet another interesting direction. It is usually the case that results in diophantine geometry motivate those in arithmetic dynamics. On the other hand once we establish the above results we can speculate what might happen in diophantine geometry. Consider points $A_1$ and $A_2$ on elliptic curves $E_1$ and $E_2$ respectively and let $\hat{h}_i$ be the Néron–Tate canonical height on $E_i$ and the question is when $\hat{h}_1(A_1)$ and $\hat{h}_2(A_2)$ are linearly dependent over $\mathbb{Q}$. Based on what happens in arithmetic dynamics, one might speculate that except for the special case when either $A_1$ or
$A_2$ is torsion and up to applying an automorphism $\sigma \in G$ the answer is when $(A_1, A_2)$ belongs to a non-fibered torsion translate of an elliptic curve in $E_1 \times E_2$.

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