Localization theorems for eigenvalues of quaternionic matrices

Sk. Safique Ahmad∗  Istkhar Ali†

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Abstract

Ostrowski type and Brauer type theorems are derived for the left eigenvalues of quaternionic matrix. We see that the above theorems for the left eigenvalues are also true for the case of right eigenvalues, when the diagonals of quaternionic matrix are real. Some distribution theorems are given in terms of ovals of Cassini that are sharper than the Ostrowski type theorems, respectively, for the left and right eigenvalues of quaternionic matrix. In addition, generalizations of the Gerschgorin type theorems are discussed for both the left and right eigenvalues of quaternionic matrix, and finally, we see that our framework is so developed that generalizes the existing results in the literatures.

Keywords. Quaternionic matrices, left and right eigenvalue, Gerschgorin type theorems, Brauer type theorem.

AMS subject classification. 15A18, 15A66.

1 Introduction

Localization theorems of quaternionic matrices have been becoming a wide range of research by several authors due to their various applications in the fields of Science and Engineering, for instance, see e.g., [1,2,4,6,7,9,12,13,15,17,19,21] and references therein. Gerschgorin type theorems for quaternionic matrices were proposed by F. Zhang [21]. Ostrowski and the Brauer type theorems have been found, see e.g., [10,22]. The above localization theorems are well known in the literature in the case of complex field [3,5,8,14,18]. Unlike complex field, two different Gerschgorin type theorems have been found due to non-commutativity of quaternions [21]. Similarly, other localization theorems are of two kinds as two different left and right eigenvalues exist in the case of quaternionic matrix. Ostrowski and the Brauer type theorems have been found in [22] for the right eigenvalues of quaternionic matrix with real diagonal entries and the Brauer type theorem for the left eigenvalues of quaternionic matrix has been derived in [10]. On the contrary, there is no literature on Ostrowski type theorem for the left eigenvalues and further, the Brauer type theorem that has been derived in [10] for the left eigenvalues is found to be incorrect for the case of deleted absolute column sums of quaternionic matrix. Our goal in this paper is to provide a general setup on localization theorems and the derivation of the Ostrowski type theorem for the left eigenvalues of quaternionic matrix.

∗Corresponding author: School of Basic Sciences, Discipline of Mathematics, IET DAVV Campus Khandwa Road Indore, MP 452017; email: safique@iiti.ac.in. Phone: +91-731-2438731, Fax: +91-731-2364182
†School of Basic Sciences, Discipline of Mathematics, Indian Institute of Technology, Indore, IET, DAVV Campus, Khandwa Road, Indore-452017, India, email: istkhara@iiti.ac.in. Research work funded by the CSIR, Govt. of India.
quaternionic matrix are given. Next, the derivation of localization theorems for the left and right eigenvalues are derived in terms of the ovals of Cassini that provide better estimation than the Ostrowski type theorems for the left and right eigenvalues of a quaternionic matrix, respectively, which generalize some existing results on this direction.

Consider a matrix $A := (a_{ij}) \in M_n(\mathbb{H})$, where $M_n(\mathbb{H})$ is the set of $n$-by-$n$ quaternionic matrices. The Ostrowski type theorem for the left eigenvalues are discussed, i.e., all the left eigenvalues of $A$ are located in the union of $n$ balls $T_i(A) := \{z \in \mathbb{H} : |z - a_{ii}| \leq R_i^\gamma C_i^{1-\gamma}\}$, where $\gamma \in [0, 1], R_i := \sum_{i \neq j=1}^n |a_{ij}|$, and $C_i := \sum_{i \neq j=1}^n |a_{ji}|$. If the diagonals of $A$ are real, then we show that Ostrowski type theorems for the left and right eigenvalues are same. As a consequence, we provide the sufficient conditions for a quaternionic matrix $A$ to be nonsingular. Since 2008, Junliang et al. [10] have been introduced the Brauer type theorem for the left eigenvalues, see [10] Theorem 4.7 and a sharper result to the existing result given in [22, Theorem 4.3]. Moreover, we obtain the generalizations of some existing results given in [19, 21] for the case of generalized Hölder’s inequality, i.e., all the left eigenvalues of $A \in M_n(\mathbb{H})$ are contained in the union of $n$ generalized balls: $B_i(A) := \{z \in \mathbb{H} : |z - a_{ii}| \leq (n - 1)\frac{1}{\gamma} R_i^\gamma N_i^{1-\gamma}\}$, where $\gamma \in [0, 1]$ and $N_i := \left(\sum_{i \neq j=1}^n |a_{ij}|^p\right)^{\frac{1}{p}} \frac{1}{p} + \frac{1}{q} = 1, p, q \in (1, \infty)$. In case of right eigenvalues, the above result is not true, while we show that for every right eigenvalue $\lambda$ of $A \in M_n(\mathbb{H})$ there exists a non-zero quaternion $\beta$ such that $\beta^{-1} \lambda \beta$ is contained in the union of $n$ generalized balls $B_i(A)$, i.e., $\{\rho^{-1} \lambda \rho : \emptyset \neq \rho \in \mathbb{H}\} \cap \bigcup_{i=1}^n B_i(A) \neq \emptyset$. As a consequence, we find that all the right eigenvalues of $A \in M_n(\mathbb{H})$ with real diagonal entries are contained in the union of $n$ generalized balls $B_i(A)$.

2 Notation and preliminaries

Throughout the paper, we adopt the following notations. Denote by $\mathbb{R}$ and $\mathbb{C}$ the fields of real and complex numbers, respectively. The set of real quaternions is defined by $\mathbb{H} := \{q = a_0 + a_1i + a_2j + a_3k : a_0, a_1, a_2, a_3 \in \mathbb{R}\}$ with $i^2 = j^2 = k^2 = ijk = -1$. For $q \in \mathbb{H}$, we denote the conjugate of $q$ by $\overline{q} := a_0 - a_1i - a_2j - a_3k$ and $|q| := \sqrt{a_0^2 + a_1^2 + a_2^2 + a_3^2}$ as the modulus of $q$. Let $\mathbb{H}^n$ be the right vector space over $\mathbb{H}$. For $x, y \in \mathbb{H}^n$, define $\langle x, y \rangle := y^H x$ as inner product and $\|x\| := \sqrt{\langle x, x \rangle}$, the norm on $\mathbb{H}^n$. The set of $n$-by-$n$ real, complex, and quaternionic matrices are given by $M_n(\mathbb{R}), M_n(\mathbb{C})$, and $M_n(\mathbb{H})$, respectively. For $A \in M_n(K), K := \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$, the transpose and conjugate transpose are given by $A^T$ and $A^H$, respectively. We denote by $[q]$ the equivalence class containing $q \in \mathbb{H}$. For $A \in M_n(\mathbb{H})$, we write $A = A_1 + A_2j$, where $A_1, A_2 \in M_n(\mathbb{C})$, and define $\Psi : M_n(\mathbb{H}) \to M_{2n}(\mathbb{C})$ by $\Psi_A := \Psi(A) := \begin{bmatrix} A_1 & A_2 \\ -A_2 & A_1 \end{bmatrix}$ as a adjoint complex matrix.

**Definition 2.1** Let $A \in M_n(\mathbb{H})$. Then the left, right, and the standard eigenvalues, respectively, are given by

\[
\Lambda_l(A) := \{\lambda \in \mathbb{H} : Ax = \lambda x \text{ for some non-zero } x \in \mathbb{H}^n\},
\]

\[
\Lambda_r(A) := \{\lambda \in \mathbb{H} : Ax = x\lambda \text{ for some non-zero } x \in \mathbb{H}^n\}, \text{ and}
\]

\[
\Lambda_s(A) := \{\lambda \in \mathbb{C} : Ax = x\lambda \text{ for some non-zero } x \in \mathbb{H}^n, \Im(\lambda) \geq 0\}.
\]
Definition 2.2 Let $A \in M_n(\mathbb{H})$. Then the matrix $A$ is said to be stable if and only if $\Lambda_r(A) \subset \mathbb{H}^- := \{ q \in \mathbb{H} : \Re(q) < 0 \}$.

Definition 2.3 Let $A \in M_n(\mathbb{H})$. Then $A$ is said to be $\eta$-Hermitian if $A = A^H \eta$, $\eta \in \{i, j, k\}$, where $A^H = \eta^H A \eta$.

Theorem 2.4 [20, Theorem 4.3]. Let $A \in M_n(\mathbb{H})$. Then the following statements are equivalent:
(a) $A$ is invertible, (b) $Ax = 0$ has a unique solution, (c) $\det(\Psi_A) \neq 0$, (d) $\Psi_A$ is invertible, (e) $A$ has non-zero eigenvalues either left or right.

We now extend Hölder’s inequality from the complex field to quaternion field which is as follows, however generalized Cauchy Schwartz’s inequality on quaternion vectors has been proved in [19].

Lemma 2.5 (Generalized Hölder’s inequality) For arbitrarily quaternion vectors $z := (z_1, \ldots, z_n) \in \mathbb{H}^n$, and $w := (w_1, \ldots, w_n) \in \mathbb{H}^n$, the following inequality holds:
$$\sum_{k=1}^{n} |z_k w_k| \leq \left( \sum_{k=1}^{n} |z_k|^p \right)^{\frac{1}{p}} \left( \sum_{k=1}^{n} |w_k|^q \right)^{\frac{1}{q}}, \quad \frac{1}{p} + \frac{1}{q} = 1, \quad p, q \in (1, \infty).$$

Proof. Proof follows directly from the Hölder’s inequality for the complex case.

3 Distribution for the left and right eigenvalues of quaternionic matrix

Let $A := (a_{ij}) \in M_n(\mathbb{H})$ and define the deleted absolute row and column sums:
$$R_i := \sum_{i \neq j=1}^{n} |a_{ij}| \text{ and } C_i := \sum_{i \neq j=1}^{n} |a_{ji}|.$$

Throughout this section we would be using the above notations in our theory.

It has been proved in [22] that $A$ and $A^H$ have the same right eigenvalues. However, this is not true for the case of left eigenvalues which follows from the following example.

Example 3.1 Consider $A = \begin{bmatrix} i & 0 \\ 0 & j \end{bmatrix}$. Then $A^H = \begin{bmatrix} -i & 0 \\ 0 & -j \end{bmatrix}$.

This example shows, $A$ and $A^H$ have different left eigenvalues. We now give the following theorem for the left eigenvalues of $A$ and $A^H$.

Theorem 3.2 Let $A \in M_n(\mathbb{H})$. Then $\lambda$ is a left eigenvalue of $A$ if and only if $\lambda^\ast$ is a left eigenvalue of $A^H$.

Proof. Let $\lambda$ be a left eigenvalue of $A$, then by the definition, there exists $0 \neq x \in \mathbb{H}^n$ such that $(A - \lambda I)x = 0$ if and only if $\Psi_{(A - \lambda I)} \Psi x = 0$. It follows that $\lambda$ is a left eigenvalue of $A$ if and only if $\det [\Psi_{(A - \lambda I)}] = 0$. Then the following results hold:
\( \lambda \) is a left eigenvalue of \( A \) if and only if \( \det \left[ \Psi^H_{(A-\lambda I)} \right] = 0 \)

\( \lambda \) is a left eigenvalue of \( A \) if and only if \( \det \left[ \Psi_{(A-\lambda I)}^H \right] = 0 \)

\( \lambda \) is a left eigenvalue of \( A \) if and only if \( \det \left[ \Psi_{(A^H-\lambda I)} \right] = 0 \).

Thus \( \overline{\lambda} \) is a left eigenvalue of \( A^H \). ■

It has been found from the literature that the Gershgorin type theorem for the left eigenvalues of matrix \( A \in M_n(\mathbb{H}) \) in terms of deleted absolute row sums [21]. However, there is no literature on the Gershgorin type theorem for the left eigenvalues in terms of the deleted absolute column sums of \( A \). For deriving the generalized Ostrowski type theorem, we need to derive the Gershgorin type theorem for the deleted absolute column sums of \( A \) which is as follows.

**Theorem 3.3** Let \( A := (a_{ij}) \in M_n(\mathbb{H}) \). Then all the left eigenvalues of \( A \) are located in the union of \( n \) Gershgorin balls \( \Omega_i(A) := \{ z \in \mathbb{H} : |z - a_{ii}| \leq C_i \} \), i.e.,

\[ \Lambda_l(A) \subseteq \Omega(A) := \bigcup_{i=1}^n \Omega_i(A). \]

**Proof.** Let \( \lambda \) be a left eigenvalue of \( A \). Then by Theorem 3.2. \( \overline{\lambda} \) is a left eigenvalue of \( A^H \).

Thus \( A^H x = \overline{\lambda} x \) for some non-zero \( x := [x_1, \ldots, x_n]^T \in \mathbb{H}^n \). Let \( x_i \) be an element of \( x \) such that \( |x_i| \geq |x_i| \forall i (1 \leq i \leq n) \). Then \( |x_i| > 0 \). From \( t \)-th equation of \( A^H x = \overline{\lambda} x \), we have

\[ \sum_{j=1}^n \overline{a}_{ij} x_j = \overline{\lambda} x_i. \]

Since \( |\overline{p} - \overline{q}| = |p - q|, \forall p, q \in \mathbb{H} \) and \( |x_i| \geq |x_i| \forall i (1 \leq i \leq n) \), then

\[ |\lambda - a_{ii}| \leq \sum_{t \neq j=1}^n |a_{jt}| := C_t. \] ■

Next we derive the localization theorem in terms of the deleted absolute row and column sums of \( A \) and this is known as generalized Ostrowski type theorem which is as follows.

**Theorem 3.4** (Ostrowski type theorem for the left eigenvalues) Let \( A := (a_{ij}) \in M_n(\mathbb{H}) \) and let \( \gamma \in [0, 1] \) be given. Then all the left eigenvalues of \( A \) are located in the union of \( n \) balls \( T_i(A) := \{ z \in \mathbb{H} : |z - a_{ii}| \leq R_i \gamma C_i^{1-\gamma} \} \), i.e.,

\[ \Lambda_l(A) \subseteq T(A) := \bigcup_{i=1}^n T_i(A). \]

**Proof.** Let \( \lambda \) be a left eigenvalue of \( A \). Then by [21] Theorem 6, and for any \( \gamma \in [0, 1] \),

\[ |\lambda - a_{ii}|^\gamma \leq R_i \gamma, \] (1)

similarly, from Theorem 3.3 we obtain

\[ |\lambda - a_{ii}|^{1-\gamma} \leq C_i^{1-\gamma}. \] (2)

Combining (1) and (2)

\[ |\lambda - a_{ii}| \leq R_i \gamma C_i^{1-\gamma}. \]

Thus, all the left eigenvalues of \( A \) are located in the union of \( n \) balls \( T_i(A) \). ■

From the above Theorem it is clear that when the diagonals of \( A \in M_n(\mathbb{H}) \) are real then all the right eigenvalues will lie in the union of \( n \) balls \( T_i(A) \) and is known as the Ostrowski type theorem for the right eigenvalues of \( A \in M_n(\mathbb{H}) \).
Theorem 3.5 Let \( A := (a_{ij}) \in M_n(\mathbb{H}) \) with \( a_{ii} \in \mathbb{R} \) and let \( \gamma \in [0, 1] \) be given. Then all the right eigenvalues of \( A \) are located in the union of \( n \) Gerschgorin balls \( G_i(A) := \left\{ z \in \mathbb{H} : |z - a_{ii}| \leq R_i \gamma C_i^{1-\gamma} \right\} \), i.e.,
\[
\Lambda_r(A) \subseteq G(A) := \bigcup_{i=1}^n G_i(A).
\]

We can find the above Theorem in [22, Theorem 4.3].

Corollary 3.6 For any \( A := (a_{ij}) \in M_n(\mathbb{H}) \), \( n \geq 2 \) and for any \( \gamma \in [0, 1] \), assume that
\[
|a_{ii}| > R_i \gamma C_i^{1-\gamma}, \forall i \ (1 \leq i \leq n). \tag{3}
\]
Then \( A \) is nonsingular.

Proof. On the contrary, suppose that \( A \) is singular. Then by Theorem 2.4, there is a left eigenvalue \( \lambda = 0 \) of \( A \). Then from Theorem 3.4, we obtain
\[
|a_{ii}| \leq R_i \gamma C_i^{1-\gamma},
\]
which contradicts our assumption (3). Hence \( A \) is nonsingular.

[10, Theorem 7] has been stated for the central closed quaternionic matrix, now we generalize for all quaternionic matrices as follows.

Theorem 3.7 Let \( A := (a_{ij}) \in M_n(\mathbb{H}) \) and let \( \gamma \in [0, 1] \). Then all the left eigenvalues of \( A \) are located in the union of \( \frac{n(n-1)}{2} \) ovals of Cassini;
\[
K_{ij}(A) := \left\{ z \in \mathbb{H} : |z - a_{ii}| \ |z - a_{jj}| \leq R_i \gamma R_j \gamma C_i^{1-\gamma} C_j^{1-\gamma} \right\},
\]
i.e.,
\[
\Lambda_l(A) \subseteq K(A) := \bigcup_{i,j=1\atop i \neq j}^n K_{ij}(A).
\]

Proof. Let \( \lambda \) be a left eigenvalue of \( A \). Then by Theorem 3.4, for any \( i \) and \( j \), \( i \neq j \), we have
\[
|\lambda - a_{ii}| \leq R_i \gamma C_i^{1-\gamma} \tag{4}
\]
and
\[
|\lambda - a_{jj}| \leq R_j \gamma C_j^{1-\gamma}. \tag{5}
\]
Combining (4) and (5)
\[
|\lambda - a_{ii}| |\lambda - a_{jj}| \leq R_i \gamma R_j \gamma C_i^{1-\gamma} C_j^{1-\gamma}. \tag{6}
\]

Thus Theorem 3.7 generalizes [10, Theorem 7].

Corollary 3.8 Let \( A := (a_{ij}) \in M_n(\mathbb{H}) \) and \( |a_{ss}| |a_{tt}| > R_s \gamma R_t \gamma C_s^{1-\gamma} C_t^{1-\gamma}, \ (1 \leq s, t \leq n, \ s \neq t) \). Then \( A \) is invertible.

Proof. It is easy to prove by using the Theorem 3.7, so we omit the proof. ■

Next we extend the Brauer theorem from complex matrix to quaternionic matrix for the deleted absolute column sums. The quaternionic version of Brauer type theorem for the deleted absolute column sums of \( A \) has been given in [10]. In this report [10] it has been found that if \( \lambda \in \Lambda_l(A) \), then its conjugate \( \overline{\lambda} \) lies in the union of \( \frac{n(n-1)}{2} \) ovals of Cassini and is found to be incorrect, follows from the following example.

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Example 3.9 Consider $A := \begin{bmatrix} i & j \\ -j & i \end{bmatrix}$. Then by [10, Theorem 5], oval of Cassini is given by \( \{ z \in \mathbb{H} : |z - i| \leq 1 \} \). In this example, \( i - k \) is a left eigenvalue of \( A \) and its conjugate \( -i + k \) is not contained in the above oval of Cassini.

Now we derive the corrected version of the Brauer type theorem in terms of deleted absolute column sums of \( A \) as follows.

Corollary 3.10 Let \( A := (a_{ij}) \in M_n(\mathbb{H}) \). Then all the left eigenvalues of \( A \) are located in the union of \( \frac{n(n-1)}{2} \) ovals of Cassini;

(a) \( \Lambda_l(A) \subseteq E(A) := \bigcup_{i,j=1 \atop i \neq j}^n \{ z \in \mathbb{H} : |z - a_{ii}| |z - a_{jj}| \leq R_i R_j \} \).

(b) \( \Lambda_l(A) \subseteq F(A) := \bigcup_{i,j=1 \atop i \neq j}^n \{ z \in \mathbb{H} : |z - a_{ii}| |z - a_{jj}| \leq C_i C_j \} \).

Proof. Substituting \( \gamma = 0, 1 \), in the Theorem 3.7 we obtain the required results. The result \( (a) \) can be found in [10, Theorem 4].

Now, we present the following result which is sharper than the Theorem 3.3.

Theorem 3.11 Let \( A := (a_{ij}) \in M_n(\mathbb{H}) \) with \( a_{ii} \in \mathbb{R} \) and \( \gamma \in [0, 1] \) be given. Then all the right eigenvalues of \( A \) are located in the union of \( \frac{n(n-1)}{2} \) ovals of Cassini \( G_{ij}(A) := \{ z \in \mathbb{H} : |z - a_{ii}| |z - a_{jj}| \leq R_i^\gamma R_j^\gamma C_i^{1-\gamma} C_j^{1-\gamma} \} \), i.e.,

\[ \Lambda_r(A) \subseteq G(A) = \bigcup_{i,j=1 \atop i \neq j}^n G_{ij}(A). \]

Proof. Proof is follows from the proof method of Theorem 3.1 and using Theorem 3.3 so we omit the proof.

Corollary 3.12 Let \( A := (a_{ij}) \in M_n(\mathbb{H}) \) with \( a_{ii} \in \mathbb{R} \). Then all the right eigenvalues of \( A \) are located in the union of \( \frac{n(n-1)}{2} \) ovals of Cassini;

(a) \( \Lambda_r(A) \subseteq L(A) := \bigcup_{i,j=1 \atop i \neq j}^n \{ z \in \mathbb{H} : |z - a_{ii}| |z - a_{jj}| \leq R_i R_j \} \).

(b) \( \Lambda_r(A) \subseteq Q(A) := \bigcup_{i,j=1 \atop i \neq j}^n \{ z \in \mathbb{H} : |z - a_{ii}| |z - a_{jj}| \leq C_i C_j \} \).

Proof. Substituting \( \gamma = 0, 1 \), in the Theorem 3.11 we obtain the required results. These can be found in [22, Theorem 4.1, Corollary 4.1].

Now, the following result shows that the Theorem 3.7 is sharper than the Theorem 3.4.

Theorem 3.13 Let \( A := (a_{ij}) \in M_n(\mathbb{H}) \) with \( n \geq 2 \) and let \( \gamma \in [0, 1] \) be given. Then \( K(A) \subseteq T(A) \).

Proof. Let \( z \in K_{ij}(A) \) and fix any \( i \) and \( j \) \( (1 \leq i, j \leq n, i \neq j) \), then from Theorem 3.7, we have

\[ |z - a_{ii}| |z - a_{jj}| \leq R_i^\gamma R_j^\gamma C_i^{1-\gamma} C_j^{1-\gamma}. \] (6)

Then following two cases are possible,
3.5 and 3.11, so we skip the proof. □

Case 1: If $R_i^\gamma R_j^\gamma C_i^{1-\gamma} C_j^{1-\gamma} = 0$, then $z = a_{ii}$ or $z = a_{jj}$. However from Theorem 3.4, we have $a_{ii} \in T_i(A)$ and $a_{jj} \in T_j(A)$. Thus $z \in T_i(A) \cup T_j(A)$.

Case 2: If $R_i^\gamma R_j^\gamma C_i^{1-\gamma} C_j^{1-\gamma} > 0$, then by (8)

$$\left(\frac{|z - a_{ii}|}{R_i^\gamma C_i^{1-\gamma}}\right) \left(\frac{|z - a_{jj}|}{R_j^\gamma C_j^{1-\gamma}}\right) \leq 1. \quad (7)$$

As the left side of (7) cannot exceed unity, then one of the factors of the left side is at most unity, i.e., $z \in T_i(A)$ or $z \in T_j(A)$. Hence $z \in T_i(A) \cup T_j(A)$, so

$$K_{ij} \subseteq T_i(A) \cup T_j(A). \quad (8)$$

From Theorem 3.4 and Theorem 3.7, (8) implies

$$K(A) := \cup_{i,j=1}^n K_{ij}(A) \subseteq \cup_{i,j=1}^n \{T_i(A) \cup T_j(A)\} = \cup_{k=1}^n T_k(A) =: T(A).$$

We derive the following theorem which states that Theorem 3.11 is sharper than the Theorem 3.5.

**Theorem 3.14** Let $A := (a_{ij}) \in M_n(\mathbb{H})$, $n \geq 2$ with $a_{ii} \in \mathbb{R}$ and let $\gamma \in [0, 1]$ be given, then

$$G(A) \subseteq G(A),$$

where $G(A)$ and $G(A)$ are from Theorem 3.3 and Theorem 3.11 respectively.

**Proof.** Proof is immediate from the proof method of Theorem 3.13 and using Theorems 3.5 and 3.11 so we skip the proof. □

Now we define for $A := (a_{ij}) \in M_n(\mathbb{H})$

$$N_i := \left(\sum_{i \neq j=1}^n |a_{ij}|^p\right)^{\frac{1}{p}} , \quad N_i' := \sqrt{\sum_{i \neq j=1}^n |a_{ij}|^2}.$$

Some different distribution theorems for the left and right eigenvalues of $A \in M_n(\mathbb{H})$ are given as follows by using H"older’s inequality.

**Theorem 3.15** Let $A := (a_{ij}) \in M_n(\mathbb{H})$ and let $\gamma \in [0, 1]$ be given. Then all the left eigenvalue of $A$ are contained in the union of $n$ generalized balls

$$B_i(A) := \left\{z \in \mathbb{H} : |z - a_{ii}| \leq (n - 1)^{-\frac{1}{p}} R_i^\gamma N_i^{1-\gamma}\right\},$$

i.e.,

$$\Lambda_l(A) \subseteq B(A) := \cup_{i=1}^n B_i(A).$$

**Proof.** Let $\mu$ be a left eigenvalue of $A$. Then $Ax = \mu x$ for some non-zero vector $x := [x_1, \ldots, x_n]^T \in \mathbb{H}^n$. Let $x_i$ be an element of $x$ such that $|x_i| \geq |x| \forall i (1 \leq i \leq n)$. Then $|x_i| > 0$. Thus from $Ax = \mu x$, we have

$$a_{ii}x_i + \sum_{t \neq j=1}^n a_{ij}x_j = \mu x_i,$$
\[ |\mu - a_{tt}| |x_t| = \left| \sum_{t \neq j=1}^{n} a_{tj} x_j \right| \leq \sum_{t \neq j=1}^{n} |a_{tj}| |x_j|. \]  

(9)

Applying Hölder's inequality [Lemma 2.5] to (9)

\[ |\mu - a_{tt}| |x_t| \leq \left( \sum_{t \neq j=1}^{n} |a_{tj}|^p \right)^{\frac{1}{p}} \left( \sum_{t \neq j=1}^{n} |x_j|^q \right)^{\frac{1}{q}}, \]

since \(|x_t| \geq |x_i| \forall i \ (1 \leq i \leq n)\), we conclude

\[ |\mu - a_{tt}| |x_t| \leq N_t ((n - 1)|x_i|^q)^{\frac{1}{q}}, \]

i.e.,

\[ |\mu - a_{tt}| \leq N_t (n - 1)^{\frac{1}{2}}. \]  

(10)

From (9) and using \(|x_t| \geq |x_i| \forall i \ (1 \leq i \leq n)\), we have

\[ |\mu - a_{tt}| \leq \sum_{t \neq j=1}^{n} |a_{tj}| = R_t. \]  

(11)

Therefore, for any \( \gamma \in [0, 1] \), and from (10), (11)

\[ |\mu - a_{tt}|^{1-\gamma} \leq N_t^{1-\gamma} (n - 1)^{\frac{1-\gamma}{q}} \]

and \(|\mu - a_{tt}|^\gamma \leq R_t^\gamma\), i.e.,

\[ |\mu - a_{tt}| \leq (n - 1)^{\frac{1-\gamma}{q}} N_t^{1-\gamma} R_t^\gamma. \]  

(12)

Thus theorem is proved. ■

Following example is now given to illustrate the above result.

**Example 3.16** Consider a matrix \( A := \begin{bmatrix} -i - j & 1 - 2k \\ 1 & -i + j \end{bmatrix} \). Consider \( p = q = 2, \gamma = 1/2 \) in Theorem 3.15, we obtain the following ball

\[ B(A) := \{ z \in \mathbb{H} : |z + i + j| \leq 2.2361 \} \cup \{ z \in \mathbb{H} : |z + i - j| \leq 1 \}. \]

Hence the above Example illustrates the Theorem 3.15 and hence it is verified.

Now for the special cases we find the results available in the literatures.

**Corollary 3.17** Let \( A := (a_{ij}) \in M_n(\mathbb{H}) \). Then all the left eigenvalue of \( A \) are contained in the union of \( n \) balls \( \Gamma_i(A) := \{ z \in \mathbb{H} : |z - a_{ii}| \leq R_i \} \), i.e.,

\[ \Lambda_l(A) \subseteq \Gamma(A) := \bigcup_{i=1}^{n} \Gamma_i(A). \]

**Proof.** Assume \( p = q = 2, \gamma = 1 \), in the previous Theorem, we obtain the required result. This can be found in [21, Theorem 6]. ■

**Corollary 3.18** Let \( A := (a_{ij}) \in M_n(\mathbb{H}) \). Then all the left eigenvalues of \( A \) are contained in the union of \( n \) balls \( S_i(A) := \{ z \in \mathbb{H} : |z - a_{ii}| \leq (n - 1)^{1/2} N_i^l \} \), i.e.,

\[ \Lambda_l(A) \subseteq S(A) := \bigcup_{i=1}^{n} S_i(A). \]
Proof. Assume \( p = q = 2, \gamma = 0 \), in the previous Theorem, we obtain the required result. This can be found in [19, Theorem 1]. 

We now present the generalization of [21, Theorem 7] and [22, Theorem 3.1] for the case of Hölder’s inequality and \( \gamma \in [0, 1] \). In case of general quaternionic matrix, all the right eigenvalues may not lie in the generalized balls \( B_1(A) \), however, we show that every connected region of the generalized balls \( B_1(A) \) contains some right eigenvalues of \( A \).

**Theorem 3.19** Let \( A := (a_{ij}) \in M_n(\mathbb{H}) \) and let \( \gamma \in [0, 1] \) be given. For every right eigenvalue \( \mu \) of \( A \) there exists a quaternion \( \beta \) such that \( \beta^{-1}\mu \beta \) (which is also a right eigenvalue) is contained in the union of \( n \) generalized balls;

\[
B_i(A) := \left\{ z \in \mathbb{H} : |z - a_{ii}| \leq (n - 1)^{\frac{1-\gamma}{\gamma}} R_i^\gamma N_i^{1-\gamma} \right\},
\]

i.e., \( \{ \beta^{-1}\mu \beta : 0 \neq \beta \in \mathbb{H} \} \cap \bigcup_{i=1}^n B_i(A) \neq \emptyset \).

Proof. Let \( \mu \) be a right eigenvalue of \( A \). Then there exists a non-zero vector \( x := [x_1, \ldots, x_n]^T \in \mathbb{H}^n \) such that \( Ax = \mu x \). Choose \( x_i \) from \( x \) as in Theorem 3.15 and consider \( \rho \) such that \( x_i \mu = \rho x_i \). Then we have the following

\[
|\rho - a_{ii}| x_i = \sum_{t \neq j=1}^n a_{tj} x_j \leq \sum_{j=1, j \neq t}^n |a_{tj}| |x_j|.
\]

Now according to the proof method of the Theorem 3.15, we have

\[
|\rho - a_{ii}| \leq (n - 1)^{\frac{1-\gamma}{\gamma}} R_i^\gamma N_i^{1-\gamma}.
\]

Thus the theorem is proved. 

Next we see the following results are available in the literatures for special cases.

**Corollary 3.20** Let \( A := (a_{ij}) \in M_n(\mathbb{H}) \). For every right eigenvalue \( \lambda \) of \( A \) there exists a quaternion \( \alpha \) such that \( \alpha^{-1}\lambda \alpha \) (which is also a right eigenvalue) is contained in the union of \( n \) Gerschgorin balls \( \{ z \in \mathbb{H} : |z - a_{ii}| \leq R_i \} \), i.e.,

\[
\{ \alpha^{-1}\lambda \alpha : 0 \neq \alpha \in \mathbb{H} \} \cap \bigcup_{i=1}^n \{ z \in \mathbb{H} : |z - a_{ii}| \leq R_i \} \neq \emptyset.
\]

Proof. Substituting \( p = q = 2, \gamma = 1 \), in the previous Theorem, we obtain the required result. This can be found in [21, Theorem 7].

**Corollary 3.21** Let \( A := (a_{ij}) \in M_n(\mathbb{H}) \). For every right eigenvalue \( \lambda \) of \( A \) there exists a quaternion \( \alpha \) such that \( \alpha^{-1}\lambda \alpha \) is contained in the union of \( n \) balls \( \{ z \in \mathbb{H} : |z - a_{ii}| \leq \sqrt{n - 1}^\gamma N_i^{1-\gamma} \} \), i.e.,

\[
\{ \alpha^{-1}\lambda \alpha : 0 \neq \alpha \in \mathbb{H} \} \cap \bigcup_{i=1}^n \{ z \in \mathbb{H} : |z - a_{ii}| \leq \sqrt{n - 1}^\gamma N_i^{1-\gamma} \} \neq \emptyset.
\]

Proof. Substituting \( p = q = 2, \gamma = 0 \), in the previous Theorem, we obtain the required result. This can be found in [22, Theorem 3.1].

Now we derive the following theorem when the diagonals of square matrix \( A \in M_n(\mathbb{H}) \) are real.
Theorem 3.22 Let $A := (a_{ij}) \in M_n(\mathbb{H})$ with $a_{ii} \in \mathbb{R}$ and let $\gamma \in [0, 1]$ be given. Then all the right eigenvalues of $A$ are contained in the union of $n$ generalized balls

$$B_i(A) := \left\{ z \in \mathbb{H} : |z - a_{ii}| \leq (n - 1)^{1-\gamma} R_i \gamma N_i^{1-\gamma} \right\},$$

i.e., $\Lambda_r(A) \subseteq B(A) := \cup_{i=1}^n B_i(A)$.

Proof. Proof is immediate from Theorem 3.15 so we skip the proof. ■

The above theorem is important because the Hermitian and $\eta$-Hermitian matrices have real diagonal entries. Now we present a sufficient condition for the matrix $A$ to be stable.

Proposition 3.23 Let $A := (a_{ij}) \in M_n(\mathbb{H})$ with $a_{ii} \in \mathbb{R}$ and $\gamma \in [0, 1]$ be given. Assume that $a_{ii} + (n - 1)^{1-\gamma} R_i \gamma N_i^{1-\gamma} < 0$, for all $i (1 \leq i \leq n)$, then the matrix $A$ is stable.

Proof. By Theorem 3.22 we know that all the right eigenvalues of $A$ lie in the union of $n$-generalized balls $B_i(A) := \left\{ z \in \mathbb{H} : |z - a_{ii}| \leq (n - 1)^{1-\gamma} R_i \gamma N_i^{1-\gamma} \right\}$, i.e.,

$$\Lambda_r(A) \subseteq B(A) := \cup_{i=1}^n B_i(A).$$

Since $a_{ii} + (n - 1)^{1-\gamma} R_i \gamma N_i^{1-\gamma} < 0$, for all $i (1 \leq i \leq n)$, then all the balls $B_i(A)$ lie in the left half plane $\mathbb{H}^-$. This implies all its right eigenvalues lie in the left half plane. This completes the proof. ■

Example 3.24 Let $A := \begin{bmatrix} 1 & j & k \\ 1 + i & 2 & i+j \\ 1 - j & j+k & 4 \end{bmatrix}$.

Consider $p = q = 2, \gamma = 1/2$ in Proposition 3.22 we obtain

$$B(A) := \{ z \in \mathbb{H} : |z - 1| \leq 2 \} \cup \{ z \in \mathbb{H} : |z - 2| \leq 2.8284 \} \cup \{ z \in \mathbb{H} : |z - 4| \leq 2.8284 \}.$$ 

Thus the right eigenvalues of $A$ are contained in $B(A)$.

4 Bounds for the absolute sum of left and right eigenvalues

In this section, consider $A := (a_{ij}) \in M_n(\mathbb{H})$ and define

$$\Upsilon_1 := \sum_{i=1}^n |a_{ii}| + (n - 1)^{1-\gamma} \sum_{i=1}^n R_i \gamma N_i^{1-\gamma}$$

$$\Upsilon_2 := \sum_{i=1}^n |a_{ii}| + (n - 1)^{1-\gamma} \sum_{i=1}^n R_i \gamma N_i^{1-\gamma} + \sum_{i=1}^n \left( \left| a_{ii} - \frac{\text{trace}(A)}{n} \right| \right) + |\text{trace}(A)|,$$

where $\frac{1}{p} + \frac{1}{q} = 1$ and $p, q \in (1, \infty), \gamma \in [0, 1]$. Now we derive bounds for the absolute sum of the left and right eigenvalue of quaternionic matrix.

Theorem 4.1 Let $A := (a_{ij}) \in M_n(\mathbb{H})$ and let $\gamma \in [0, 1]$ be given. Suppose $\lambda_i (1 \leq i \leq n)$, are distinct left eigenvalues of $A$ which lie in $n$ distinct generalized balls $B_i(A)$, respectively, then

$$(a). \sum_{i=1}^n |\lambda_i| \leq \Upsilon_1, \quad (b). \sum_{i=1}^n |\lambda_i| \leq \Upsilon_2.$$
Proof. (a) Consider \( \lambda_i \) are \( n \) distinct left eigenvalues of \( A \) which lie in \( n \) distinct generalized balls \( B_i(A) \). Then, without loss of generality, we consider \( \lambda_i \in B_i(A) := \{ z \in \mathbb{H} : |z - a_{ii}| \leq (n - 1)^{\frac{1-\gamma}{\gamma}} R_i^\gamma N_i^{1-\gamma} \} \), where \( B_i(A) \neq B_j(A) \) (1 \( \leq i, j \leq n, i \neq j \)).

Then by Theorem 3.15, we have \( |\lambda_i - a_{ii}| \leq (n - 1)^{\frac{1-\gamma}{\gamma}} R_i^\gamma N_i^{1-\gamma} \). Hence, we obtain \( |\lambda_i| \leq |a_{ii}| + (n - 1)^{\frac{1-\gamma}{\gamma}} R_i^\gamma N_i^{1-\gamma} \). Then,

\[
\sum_{i=1}^{n} |\lambda_i| \leq \sum_{i=1}^{n} |a_{ii}| + (n - 1)^{\frac{1-\gamma}{\gamma}} \sum_{i=1}^{n} R_i^\gamma N_i^{1-\gamma}.
\]

(b) Consider \( \lambda_i \) are \( n \) distinct left eigenvalues of \( A \) which lie within \( n \) distinct generalized balls \( B_i(A) \). Then, without loss of generality, we consider \( \lambda_i \in B_i(A) = \{ z \in \mathbb{H} : |z - a_{ii}| \leq (n - 1)^{\frac{1-\gamma}{\gamma}} R_i^\gamma N_i^{1-\gamma} \} \), where \( B_i(A) \neq B_j(A) \) (1 \( \leq i, j \leq n, i \neq j \)). Based on the particle and center gravity theorem, each \( B_i(A) \) can be treated as a particle or a rigid body. Then the center of all particles or rigid bodies is \( \frac{1}{n} \sum_{i=1}^{n} a_{ii} = \frac{\text{trace}(A)}{n} \). Now, we have

\[
\left| \lambda_i - \frac{\text{trace}(A)}{n} \right| = \left| \lambda_i - a_{ii} + a_{ii} - \frac{\text{trace}(A)}{n} \right| \leq |\lambda_i - a_{ii}| + \left| a_{ii} - \frac{\text{trace}(A)}{n} \right|
\]

\[
\left| \lambda_i - \frac{\text{trace}(A)}{n} \right| \leq (n - 1)^{\frac{1-\gamma}{\gamma}} R_i^\gamma N_i^{1-\gamma} + \left| a_{ii} - \frac{\text{trace}(A)}{n} \right|.
\]

This implies \( |\lambda_i| \leq (n - 1)^{\frac{1-\gamma}{\gamma}} R_i^\gamma N_i^{1-\gamma} + \left| a_{ii} - \frac{\text{trace}(A)}{n} \right| + \left| \frac{\text{trace}(A)}{n} \right| \), and

\[
\sum_{i=1}^{n} |\lambda_i| \leq (n - 1)^{\frac{1-\gamma}{\gamma}} \sum_{i=1}^{n} R_i^\gamma N_i^{1-\gamma} + \sum_{i=1}^{n} \left( \left| a_{ii} - \frac{\text{trace}(A)}{n} \right| \right) + |\text{trace}(A)|.
\]

Now, suppose \( \gamma = 1 \) in the previous Theorem 4.2(a), we obtain

\[
\sum_{i=1}^{n} |\lambda_i| \leq \sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}|.
\]

This has been derived in [11] Theorem 3.1.

Assume \( p = q = 2, \gamma = 0 \) in the previous Theorem 4.2(a), we obtain

\[
\sum_{i=1}^{n} |\lambda_i| \leq \sqrt{(n - 1)} \sum_{i=1}^{n} \left( \sqrt{\sum_{i \neq j=1}^{n} |a_{ij}|^2} \right) + \sum_{i=1}^{n} |a_{ii}|.
\]

This can be found in [11] Theorem 3.2.

Consider \( \gamma = 1 \) in the previous Theorem 4.2(b), we obtain

\[
\sum_{i=1}^{n} |\lambda_i| \leq \sum_{i=1}^{n} \sum_{i \neq j=1}^{n} |a_{ij}| + \sum_{i=1}^{n} \left( \left| a_{ii} - \frac{\text{trace}(A)}{n} \right| \right) + |\text{trace}(A)|.
\]

This has been discussed in [11] Theorem 3.3. ■

Now we show that the above bounds are same for the right eigenvalues of quaternionic matrix as well, which are as follows
Theorem 4.2 Let $A := (a_{ij}) \in M_n(\mathbb{H})$ and let $\gamma \in [0, 1]$ be given. Suppose $\rho_i^{-1}\lambda_i\rho_i$, where $\lambda_i \in \Lambda_r(A)$ and $0 \neq \rho_i \in \mathbb{H}, (1 \leq i \leq n)$, are right eigenvalues of $A$ that lie in $n$ distinct generalized balls $B_i(A)$, then

$$(a). \sum_{i=1}^{n} |\lambda_i| \leq \Upsilon_1, \quad (b). \sum_{i=1}^{n} |\lambda_i| \leq \Upsilon_2.$$

Proof. (a) Consider $\rho_i^{-1}\lambda_i\rho_i$ are $n$ right eigenvalues of $A$ which lie in $n$ distinct generalized balls $B_i(A)$. Then, without loss of generality, we consider $\rho_i^{-1}\lambda_i\rho_i \in B_i(A) := \{z \in \mathbb{H} : |z - a_{ii}| \leq (n-1)^{1-\gamma} R_i^\gamma N_i^{-1-\gamma}\}$, where $B_i(A) \neq B_j(A), 1 \leq i, j \leq n, i \neq j$.

Then by Theorem 3.19 we have $|\rho_i^{-1}\lambda_i\rho_i - a_{ii}| \leq (n-1)^{1-\gamma} R_i^\gamma N_i^{-1-\gamma}$. Hence, we obtain $|\rho_i^{-1}\lambda_i\rho_i| = |\lambda_i| \leq |a_{ii}| + (n-1)^{1-\gamma} R_i^\gamma N_i^{-1-\gamma}$. Then,

$$\sum_{i=1}^{n} |\lambda_i| \leq \sum_{i=1}^{n} |a_{ii}| + (n-1)^{1-\gamma} \sum_{i=1}^{n} R_i^\gamma N_i^{-1-\gamma}. \blacksquare$$

(b) Consider $\rho_i^{-1}\lambda_i\rho_i$ are $n$ right eigenvalues of $A$ which lie within $n$ distinct generalized balls $B_i(A)$. Then, without loss of generality, we consider $\rho_i^{-1}\lambda_i\rho_i \in B_i(A) = \{z \in \mathbb{H} : |z - a_{ii}| \leq (n-1)^{1-\gamma} R_i^\gamma N_i^{-1-\gamma}\}$, where $B_i(A) \neq B_j(A), 1 \leq i, j \leq n, i \neq j$. Based on the particle and center gravity theorem, each $B_i(A)$ can be treated as a particle or a rigid body. Then the center of all particles or rigid bodies is $\frac{1}{n} \sum_{i=1}^{n} a_{ii} = \frac{\text{trace}(A)}{n}$. Now, we have

$$|\rho_i^{-1}\lambda_i\rho_i - \frac{\text{trace}(A)}{n}| = |\rho_i^{-1}\lambda_i\rho_i - a_{ii} + a_{ii} - \frac{\text{trace}(A)}{n}| \leq |\rho_i^{-1}\lambda_i\rho_i - a_{ii}| + |a_{ii} - \frac{\text{trace}(A)}{n}|$$

$$|\rho_i^{-1}\lambda_i\rho_i - \frac{\text{trace}(A)}{n}| \leq (n-1)^{1-\gamma} R_i^\gamma N_i^{-1-\gamma} + \left|a_{ii} - \frac{\text{trace}(A)}{n}\right|.$$

Hence $|\rho_i^{-1}\lambda_i\rho_i| = |\lambda_i| \leq (n-1)^{1-\gamma} R_i^\gamma N_i^{-1-\gamma} + \left|a_{ii} - \frac{\text{trace}(A)}{n}\right| + \left|\frac{\text{trace}(A)}{n}\right|$. Thus, we have

$$\sum_{i=1}^{n} |\lambda_i| \leq (n-1)^{1-\gamma} \sum_{i=1}^{n} R_i^\gamma N_i^{-1-\gamma}(A) + \sum_{i=1}^{n} \left|\frac{a_{ii} - \text{trace}(A)}{n}\right| + \left|\frac{\text{trace}(A)}{n}\right|. \blacksquare$$

Now we have the following table to verify the above bounds for $\gamma = 1/2, p = q = 2$.

**Example 4.3** Let

$$A := \begin{bmatrix} i + j & j & k/2 \\ 0 & k & j/2 \\ 0 & 0 & -i + j + k \end{bmatrix}.$$

Then we have the following table which gives bounds $\Upsilon_1$ and $\Upsilon_2$.

| Matrix: $A$ | $\sum_{i=1}^{3} |\lambda_i|$ | $\Upsilon_1$ | $\Upsilon_2$ |
|-------------|-----------------|--------------|--------------|
| $\lambda_i \in \Lambda_l(A)$ | 3.8284 | 5.9630 | 11.1504 |
| $\lambda_i \in \Lambda_r(A)$ | 3.8284 | 5.9630 | 11.1504 |
5 Conclusions

In this paper, we have derived Ostrowski type theorem for the left eigenvalues of quaternionic matrix. The corrected version of the Brauer type theorem for left eigenvalues in terms of deleted absolute column sums has been given. We have shown that the developed Ostrowski type theorem for the left eigenvalues generalizes the Ostrowski type theorem for the right eigenvalues when the diagonals of $A \in M_n(\mathbb{H})$ are real. In addition, the generalizations of the Gerschgorin type theorems and some other localization theorems have been discussed for the left as well as for right eigenvalues of $A \in M_n(\mathbb{H})$. Sharper results than Ostrowski type theorems have been presented. Finally, we have derived the generalizations of the bounds for the absolute sum of the $n$ distinct left and right eigenvalues of a quaternionic matrix.

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