On Concavity of the Monopolist’s Problem
Facing Consumers with Nonlinear Price Preferences

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Abstract
A monopolist wishes to maximize her profits by finding an optimal price policy. After she announces a menu of products and prices, each agent $x$ will choose product $y(x)$ which maximizes his own utility, if positive. The principal’s profits are the sum of the net earnings produced by each product sold. These are determined by the costs of production and the distribution of products sold, which in turn are based on the distribution of anonymous agents and the choices they make in response to the principal’s price menu. In this paper, we provide a necessary and sufficient condition for the convexity or concavity of the principal’s (bilevel) optimization problem, assuming each agent’s disutility is a strictly increasing but not necessarily affine (i.e., quasilinear) function of the price paid. Concavity, when present, makes the problem more amenable to computational and theoretical analysis; it is key to obtaining uniqueness and stability results for the principal’s strategy in particular. Even in the quasilinear case, our analysis goes beyond previous work by addressing convexity as well as concavity, by establishing conditions which are not only sufficient but necessary, and by requiring fewer hypotheses on the agents’ preferences. © 2019 Wiley Periodicals, Inc.

1 Introduction
As one of the central problems in microeconomic theory, the principal-agent framework characterizes the type of noncompetitive decision-making problems that involve aligning incentives so that one set of parties (the agents) finds it beneficial to act in the interests of another (the principal) despite holding private information. It arises in a variety of different contexts. Besides nonlinear pricing [1, 26, 39, 42], economists also use this framework to model many different types of transactions, including tax policy [12, 25, 31], contract theory [29], regulation of monopolies [3], product line design [33], labour market signaling [38], public utilities [30], and mechanism design [15, 21, 22, 24, 27, 41]. Many of these share the same mathematical model. In this paper, we use nonlinear pricing to
motivate the discussion, in spite of the fact that our conclusions may be equally pertinent to many other areas of application.

Consider the problem of a monopolist who wants to maximize her profits by selecting the dependence of the price \( v(y) \) on each type \( y \in \text{cl}(Y) \) of product sold. An agent of type \( x \in X \) will choose to buy that product which maximizes his benefit

\[
(1.1) \quad u(x) := \max_{y \in \text{cl}(Y)} G(x, y, v(y)),
\]

where \((x, y, z) \in X \times \text{cl}(Y) \times \mathbb{R} \mapsto G(x, y, z) \in \mathbb{R}\) is the given direct utility function for agent type \( x \) to choose product type \( y \) at price \( z \), and \( X, Y \) are open and bounded subsets in \( \mathbb{R}^m \) and \( \mathbb{R}^n \) (\( m \geq n \)), respectively, with closures \( \text{cl}(X) \) and \( \text{cl}(Y) \). After agents, whose distribution \( d \mu(x) \) is known to the monopolist, have chosen their favorite items to buy, the monopolist calculates her profit to be given by the functional

\[
(1.2) \quad \Pi(v, y) := \int_X \pi(x, y(x), v(y(x)))d\mu(x),
\]

where \( y(x) \) denotes the product type \( y \) which agent type \( x \) chooses to buy (and which maximizes (1.1)), \( v(y(x)) \) denotes the selling price of type \( y(x) \), and \( \pi \in C^0(\text{cl}(X) \times \text{cl}(Y) \times \mathbb{R}) \) denotes the principal’s net profit of selling product type \( y \in \text{cl}(Y) \) to agent type \( x \in X \) at price \( z \in \mathbb{R} \). The monopolist wants to maximize her net profit among all lower semicontinuous pricing policies. Individual agents accept to contract only if the benefits they earn are no less than their outside option; we model this by assuming the existence of a distinguished point \( y_\emptyset \in \text{cl}(Y) \) that represents the outside option, and whose price cannot exceed some fixed value \( z_\emptyset \in \mathbb{R} \) beyond the monopolist’s control. This removes any incentive for the monopolist to raise the prices of other options too high. (We can choose normalizations such as \( \pi(x, y_\emptyset, z_\emptyset) = 0 = G(x, y_\emptyset, z_\emptyset) \) and \((y_\emptyset, z_\emptyset) = (0, 0)\), or not, as we wish.)

For the quasilinear case, where the utility \( G(x, y, z) \) depends linearly on its third variable, and net profit \( \pi(x, y, z) = z - a(y) \) represents the difference of selling price \( z \) and manufacturing cost \( a \) of product type \( y \), theories of existence \([4,5,24,34]\), uniqueness \([6,9,26,33]\), and robustness \([4,9]\) have been well studied. The equivalence of concavity to the corresponding nonnegative cross-curvature condition revealed by Figalli-Kim-McCann \([9]\) directly inspires our work. In addition to the quasilinearity of \( G(x, y, z) = b(x, y) - z \) essential to their model, they require additional restrictions such as \( m = n \) and \( b \in C^4(\text{cl}(X \times Y)) \), which are not economically motivated and which we shall relax or remove. Moreover, our results allow for the monopolist’s profit \( \pi \) to depend in a general way both on monetary transfers and on the agents’ types \( x \), revealed after contracting. Such dependence plays an important role in applications such as insurance marketing.
The following is a table of notation:

| Mathematical Expression | Economic Meaning |
|-------------------------|------------------|
| $x$                      | agent type       |
| $y$                      | product type     |
| $X \subset \mathbb{R}^m$ | (open, bounded) domain of agent types |
| $\text{cl}(Y) \subset \mathbb{R}^n$ | domain of product types, closure of $Y$ |
| $v(y)$                   | selling price of product type $y$ |
| $v(\varnothing) \leq z, \varnothing$ | price normalization of the outside option $\varnothing \in \text{cl}(Y)$ |
| $u(x)$                   | indirect utility of agent type $x$ |
| $\text{dom } Du$        | points in $X$ where $u$ is differentiable |
| $G(x, y, z)$             | direct utility of buying product $y$ at price $z$ for agent $x$ |
| $H(x, y, u)$             | price at which $y$ brings $x$ utility $u$, so that $H(x, y, G(x, y, z)) = z$ |
| $\pi(x, y, z)$           | the principal’s profit for selling product $y$ to agent $x$ at price $z$ |
| $d\mu(x)$               | Borel probability measure giving the distribution of agent types on $X$ |
| $\mu \ll \mathcal{L}^m$ | $\mu$ vanishes on each subset of $\mathbb{R}^m$ having zero Lebesgue volume $\mathcal{L}^m$ |
| $\Pi(v, y)$              | monopolist’s profit facing agents’ responses $y(\cdot)$ to her chosen price policy $v(\cdot)$ |
| $\Pi(u)$                 | monopolist’s profit, viewed instead as a function of agents’ indirect utilities $u(\cdot)$ |

For a particular case, where $\text{cl}(X) = \text{cl}(Y) = [0, \infty)^n$, $G(x, y, z) = (x, y) - z$, $a(y) = (y, y)/2$, and $(\varnothing, \varnothing, \varnothing) = (0, 0)$, Rochet and Choné [33] not only obtained existence results but also partially characterized optimal solutions and interpreted them economically. Here $(\cdot, \cdot)$ denotes the euclidean inner product.

More generally, Carlier [5] proved existence for the general quasilinear utility, where $X$ is a bounded open convex subset in $\mathbb{R}^m$ with $C^1$ boundary, $\text{cl}(Y) = [0, \infty)^n$, $G(x, y, z) = b(x, y) - z$, and the manufacturing cost $a$ is linear; see also Monteiro and Page [24].

The generalization of quasilinear to nonlinear preferences has many potential applications. For example, the benefit function $G(x, y, v(y)) = b(x, y) - v^2(y)$ models agents who are more sensitive to high prices, while another function, $G(x, y, v(y)) = b(x, y) - v^{1/2}(y)$, models agents who are less sensitive to high prices. Very few results are known for such nonlinearities due to the complications that they entail. However, we
shall eventually show that under certain conditions the concavity or convexity of $G$ and $\pi$ (or their derivatives) with respect to $v$ tends to be reflected by concavity or convexity of $\Pi$, not with respect to $v$ or $y$, but rather with respect to the agents indirect utility $u$, in terms of which the principal’s maximization is reformulated below.

The generalized existence problem was mentioned as a conjecture by Basov [4, chap. 8]. Recently Nöldeke and Samuelson [28] provided a general existence result for $\text{cl}(X)$, $\text{cl}(Y)$ being compact and the utility $G$ being decreasing with respect to its third variable, by implementing a duality argument based on Galois connections. Initiated independently of [28], Zhang [43] found another general existence result for the principal-agent framework where the utility function is decreasing with respect to its third variable but on potentially unbounded domains. In Zhang’s work, $G$-convexity plays a crucial role.

In 2013, Trudinger’s lecture at the optimal transport program at MSRI inspired us to try generalizing Figalli-Kim-McCann [9] to the nonquasilinear case. With the tools developed by Trudinger [40] and others [2, 37], we are able to provide concavity and convexity theorems for general utility and net profit functions. For an application of $G$-convexity to geometric optics, see [14].

Section 2 lists most of the hypotheses we need in this paper. Section 3 introduces $G$-convexity, $G$-subdifferentiability, and various equivalent forms of the principal’s program. Section 4 presents a variety of necessary and sufficient conditions for concavity (and convexity) of the principal’s problem, and the resulting uniqueness of her optimal strategy. Several examples are developed, based on an analytic criterion for concavity, whose proof is deferred to Section 5. Section 6 gives a differential criterion for the crucial hypotheses—(G3) of the next section—clarifying its relation to that of Figalli, Kim, and McCann [9], and the Ma-Trudinger-Wang criteria for regularity of optimal maps [19], which inspired it. Its proof is relegated to Appendix A. In Appendix B, we assume the monopolist’s utility does not depend on the agent’s private information, which in certain circumstances allows us to provide a necessary and sufficient condition for concavity of her profit functional.

## 2 Hypotheses

For notational convenience, we adopt the following technical hypotheses, inspired by those of Trudinger [40] and Figalli-Kim-McCann [9]. Here we use $G_x = (\frac{\partial G}{\partial x_1}, \frac{\partial G}{\partial x_2}, \ldots, \frac{\partial G}{\partial x_m})$, $G_y = (\frac{\partial G}{\partial y_1}, \frac{\partial G}{\partial y_2}, \ldots, \frac{\partial G}{\partial y_n})$, and $G_z = \frac{\partial G}{\partial z}$ to denote derivatives with respect to $x \in X \subset \mathbb{R}^m$, $y \in Y \subset \mathbb{R}^n$, and $z \in \mathbb{R}$, respectively.
Also, for second partial derivatives, we adopt the following notations:

\[
G_{x,y} = \begin{bmatrix}
\frac{\partial^2 G}{\partial x^1 \partial y^1} & \frac{\partial^2 G}{\partial x^1 \partial y^2} & \cdots & \frac{\partial^2 G}{\partial x^1 \partial y^n} \\
\frac{\partial^2 G}{\partial x^2 \partial y^1} & \frac{\partial^2 G}{\partial x^2 \partial y^2} & \cdots & \frac{\partial^2 G}{\partial x^2 \partial y^n} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial^2 G}{\partial x^m \partial y^1} & \frac{\partial^2 G}{\partial x^m \partial y^2} & \cdots & \frac{\partial^2 G}{\partial x^m \partial y^n}
\end{bmatrix}
\]

\[
G_{x,z} = \left( \frac{\partial^2 G}{\partial x^1 \partial z^1}, \frac{\partial^2 G}{\partial x^2 \partial z^1}, \ldots, \frac{\partial^2 G}{\partial x^m \partial z^1} \right).
\]

We say \(G \in C^1(\text{cl}(X \times Y \times Z))\) if all the partial derivatives exist and are continuous. Also, we say \(G \in C^2(\text{cl}(X \times Y \times Z))\) if all the partial derivatives up to second order (i.e., \(\frac{\partial^2 G}{\partial \alpha \partial \beta}\), where \(\alpha, \beta = x^1, \ldots, x^m, y^1, \ldots, y^n, z\)) exist and are continuous. Any bijective continuous function whose inverse is also continuous is called a homeomorphism (aka “bicontinuous”).

The following hypotheses will be relevant: (G1)–(G3) represent partial analogues of the twist, domain convexity, and nonnegative cross-curvature hypotheses from the quasilinear setting \([9, 18]\); (G4) encodes a form of the desirability of money to each agent, while (G5) quantifies the assertion that the maximum price \(\bar{z}\) is high enough that no agent prefers paying it for any product \(y\) to the outside option.

\begin{enumerate}
\item[(G0)] \(G \in C^1(\text{cl}(X \times Y \times Z))\), where \(X \subset \mathbb{R}^m, Y \subset \mathbb{R}^n\) are open and bounded and \(Z = (z, \bar{z})\) with \(-\infty < z < \bar{z} \leq +\infty\).
\item[(G1)] For each \(x \in X\), the map \((y, z) \in \text{cl}(Y \times Z) \mapsto (G_x, G)(x, y, z)\) is a homeomorphism onto its range.
\item[(G2)] Its range \((\text{cl}(Y \times Z))_x := (G_x, G)(x, \text{cl}(Y \times Z)) \subset \mathbb{R}^{m+1}\) is convex.
\item[(G3)] For each \(x_0 \in X\) and \((y_0, z_0), (y_1, z_1) \in \text{cl}(Y \times Z)\), define \((y_t, z_t) \in \text{cl}(Y \times Z)\) such that the following equation holds:
\[
(G_x, G)(x_0, y_t, z_t) = (1-t)(G_x, G)(x_0, y_0, z_0) + t(G_x, G)(x_0, y_1, z_1),
\] for each \(t \in [0, 1]\).
\end{enumerate}

By (G1) and (G2), \((y_t, z_t)\) is uniquely determined by (2.1). We call \(t \in [0, 1] \mapsto (x_0, y_t, z_t)\) the \(G\)-segment connecting \((x_0, y_0, z_0)\) and \((x_0, y_1, z_1)\).

\begin{enumerate}
\item[(G3)] For each \(x, x_0 \in X\), assume \(t \in [0, 1] \mapsto G(x, y_t, z_t)\) is convex along all \(G\)-segments (2.1).
\item[(G4)] For each \((x, y, z) \in X \times \text{cl}(Y) \times \text{cl}(Z)\), assume \(G_z(x, y, z) < 0\).
\item[(G5)] If \(\pi \in C^0(\text{cl}(X \times Y \times Z))\) and \(u_\varnothing(x) := G(x, y_\varnothing, z_\varnothing)\) for some fixed \((y_\varnothing, z_\varnothing) \in \text{cl}(Y \times Z)\) satisfying
\[
G(x, y, z) := \lim_{z \to \bar{z}} G(x, y, z) \leq G(x, y_\varnothing, z_\varnothing)
\] for all \((x, y) \in X \times \text{cl}(Y)\).
\end{enumerate}
When \( \tilde{z} = +\infty \) assume this inequality is strict, and moreover that \( z \) sufficiently large implies
\[
G(x, y, z) < G(x, y_{\emptyset}, z_{\emptyset}) \quad \text{for all } (x, y) \in X \times \text{cl}(Y).
\]
For each \( u \in \mathbb{R} \), (G4) allows us to define \( H(x, y, u) := z \) if \( G(x, y, z) = u \), i.e., \( H(x, y, \cdot) = G^{-1}(x, y, \cdot) \).

3 Principal’s Program, \( G \)-Convexity, and \( G \)-Subdifferentiability

In this section, we will introduce the principal’s program and reformulate it in the language of \( G \)-convexity and \( G \)-subdifferentiability. The purpose of this section is to fix terminology and prepare the preliminaries for the main results of the next section.

In economic models, incentive compatibility is needed to ensure that all the agents report their preferences truthfully. According to the revelation principle, this costs no generality. Decisions made by the monopolist according to the information collected from agents then lead to the expected market reaction (as in [5], [33]). Individual rationality is required to ensure full participation, so that each agent will choose to play, possibly by accepting the outside option.

**Definition 3.1 (Incentive compatible and individually rational).** A measurable map \( x \in X \mapsto (y(x), z(x)) \in \text{cl}(Y \times Z) \) of agents to (product, price) pairs is called **incentive compatible** if and only if \( G(x, y(x), z(x)) \geq G(x, y(x'), z(x')) \) for all \( (x, x') \in X^2 \). Such a map offers agent \( x \) no incentive to pretend to be \( x' \). It is called **individually rational** if and only if \( G(x, y(x), z(x)) \geq G(x, y_{\emptyset}, z_{\emptyset}) \) for all \( x \in X \), meaning no individual \( x \) strictly prefers the outside option to his assignment \((y(x), z(x))\).

Given \( \pi \) and \((y_{\emptyset}, z_{\emptyset})\) from (G5), the principal’s program can be described as follows:

\[
\begin{cases}
\sup \Pi(v, y) = \int_X \pi(x, y(x), v(y(x)))d\mu(x) \quad \text{among} \\
\quad x \in X \mapsto (y(x), v(y(x))) \text{ incentive compatible, individually rational,} \\
\quad \text{and } v:\text{cl}(Y) \to \text{cl}(Z) \text{ lower semicontinuous with } v(y_{\emptyset}) \leq z_{\emptyset}.
\end{cases}
\]

In this section we’ll also see that incentive compatibility is conveniently encoded via the \( G \)-convexity defined below of the agents’ indirect utility \( u \). We’ll then reformulate the principal’s program using \( u \) as a proxy for the prices \( v \) controlled by the principal, thus generalizing Carlier’s approach [5] to the nonquasilinear setting. Moreover, the agent’s indirect utility \( u \) and product selling price \( v \) are \( G \)-dual to each other in the sense of [40].

**Definition 3.2 (G-convexity).** A function \( u \in C^0(X) \) is called **\( G \)-convex** if for each \( x_0 \in X \), there exists \( y_0 \in \text{cl}(Y) \) and \( z_0 \in \text{cl}(Z) \) such that \( u(x_0) = G(x_0, y_0, z_0) \) and \( u(x) \geq G(x, y_0, z_0) \) for all \( x \in X \).
From the definition, we know if \( u \) is a \( G \)-convex function, for any \( x \in \text{dom } Du \), there exists \( y \in \text{cl}(Y) \) and \( z \in \text{cl}(Z) \) such that

\[(3.1) \quad u(x) = G(x, y, z), \quad Du(x) = D_x G(x, y, z).\]

Conversely, when (3.1) holds, one can identify \((y, z) \in \text{cl}(Y \times Z)\) in terms of \( u(x) \) and \( Du(x) \), according to condition (G1). We denote this \( \tilde{y} := (y, z) \) by

\[\tilde{y}_G(x, u(x), Du(x)) := (y_G, z_G)(x, u(x), Du(x)).\]

and drop the subscript \( G \) when it is clear from context. Under our hypotheses, \( \tilde{y}_G \) is a continuous function on the relevant domain of definition. It will often prove convenient to augment the types \( x \) and \( y \) with an extra real variable; here and later we use the notation \( \tilde{x} \in \mathbb{R}^{m+1} \) and \( \tilde{y} \in \mathbb{R}^{n+1} \) to signify this augmentation. In addition, the set \( X \setminus \text{dom } Du \) has Lebesgue measure zero, which will be shown in the proof of Theorem 3.8.

While \( G \)-convexity acts as a generalized notion of convexity, the \( G \)-subdifferentiability defined below generalizes the concept of differentiability/subdifferentiability.

**Definition 3.3 (G-subdifferentiability).** The \( G \)-subdifferential of a function \( u : X \to \mathbb{R} \) is defined by

\[\partial_G^G u(x) := \{ y \in \text{cl}(Y) \mid u(x') \geq G(x', y, H(x, y, u(x))) \text{ for all } x' \in X \}.
\]

A function \( u \) is said to be \( G \)-subdifferentiable at \( x \) if and only if \( \partial_G^G u(x) \neq \emptyset \).

In [40], this point-to-set map \( \partial_G^G u \) is also called \( G \)-normal mapping. For more properties related to \( G \)-convexity, see [40].

Lemma 3.4 shows an equivalent relationship between \( G \)-convexity and \( G \)-subdifferentiability, a special case of which is that a function is convex if and only if it is subdifferentiable on (the interior of) its domain.

**Lemma 3.4 (G-subdifferentiability characterizes G-convexity).** Allowing hypothesis (G4), a function \( u : X \to \mathbb{R} \) is \( G \)-convex if and only if it is \( G \)-subdifferentiable everywhere.

**Proof.** Assume \( u \) is \( G \)-convex; we want to show \( u \) is \( G \)-subdifferentiable everywhere; i.e., we need to prove \( \partial_G^G u(x_0) \neq \emptyset \) for all \( x_0 \in X \).

Since \( u \) is \( G \)-convex, by definition for each \( x_0 \in X \), there exists \( y_0 \in \text{cl}(Y) \) and \( z_0 \in \text{cl}(Z) \) such that \( u(x_0) = G(x_0, y_0, z_0) \), and

\[u(x) \geq G(x, y_0, z_0) = G(x, y_0, H(x_0, y_0, u(x_0)))\]

for all \( x \in X \). By the definition of \( G \)-subdifferentiability, \( y_0 \in \partial_G^G u(x_0) \), i.e., \( \partial_G^G u(x_0) \neq \emptyset \).

\[1\]Namely, \((\text{id}_X, G, G_x)((x, y, z) \in \text{cl}(X \times Y \times Z) \mid G(x, y, z) \geq G(x, y \varnothing, z \varnothing))).\]
On the other hand, assume \( u \) is \( G \)-subdifferentiable everywhere; then for each \( x_0 \in X \), there exists \( y_0 \in \partial^G u(x_0) \). Set \( z_0 := H(x_0, y_0, u(x_0)) \) so that \( u(x_0) = G(x_0, y_0, z_0) \).

Since \( y_0 \in \partial^G u(x_0) \), we have

\[
  u(x) \geq G(x, y_0, H(x_0, y_0, u(x_0))) = G(x, y_0, z_0)
\]

for all \( x \in X \). By definition, \( u \) is \( G \)-convex. \( \square \)

We now show each \( G \)-convex function defined in Definition 3.2 can be achieved by some price menu \( v \), and conversely each price menu yields a \( G \)-convex indirect utility [40]. We require either (G5) or (3.2), which asserts all agents are repelled by the maximum price and insensitive to which contract they receive at that price.

**Proposition 3.5 (Duality between prices and indirect utilities).** Assume (G0) and (G4).

(a) If

\[
  (3.2) \quad G(x, y, z) := \lim_{z \to z} G(x, y, z) = \inf_{(\tilde{y}, \tilde{z}) \in \text{cl}(Y \times Z)} G(x, \tilde{y}, \tilde{z}).
\]

for all \((x, y) \in X \times \text{cl}(Y)\), then a function \( u \in C^0(X) \) is \( G \)-convex if and only if there exists a lower semicontinuous \( v : \text{cl}(Y) \to \text{cl}(Z) \) such that

\[
  u(x) = \max_{y \in \text{cl}(Y)} G(x, y, v(y)).
\]

(b) If instead of (3.2) we assume (G5), then \( u_\emptyset \leq u \in C^0(X) \) is \( G \)-convex if and only if there exists a lower semicontinuous function \( v : \text{cl}(Y) \to \text{cl}(Z) \) with \( v(y_\emptyset) \leq z_\emptyset \) such that \( u(x) = \max_{y \in \text{cl}(Y)} G(x, y, v(y)) \).

**proof.**

1. Suppose \( u \) is \( G \)-convex. Then for any agent type \( x_0 \in X \), there exists a product and price \((y_0, z_0) \in \text{cl}(Y \times Z)\) such that \( u(x_0) = G(x_0, y_0, z_0) \) and \( u(x) \geq G(x, y_0, z_0) \) for all \( x \in X \).

   Let \( A := \bigcup_{x \in X} \partial^G u(x) \) denote the corresponding set of products. For \( y_0 \in A \), define \( v(y_0) = z_0 \), where \( z_0 \in \text{cl}(Z) \) and \( x_0 \in X \) satisfy \( u(x_0) = G(x_0, y_0, z_0) \) and \( u(x) \geq G(x, y_0, z_0) \) for all \( x \in X \). We shall shortly show this makes \( v : A \to \text{cl}(Z) \) (i) well-defined and (ii) lower semicontinuous. Taking (i) for granted, our construction yields

\[
  (3.3) \quad u(x) = \max_{y \in A} G(x, y, v(y)) \quad \forall x \in X.
\]

(i) Now for \( y_0 \in A \), suppose there exist \((x_0, z_0), (x_1, z_1) \in X \times \text{cl}(Z)\) with \( z_0 \neq z_1 \) such that \( u(x_i) = G(x_i, y_0, z_i) \) and \( u(x) \geq G(x, y_0, z_i) \) for all \( x \in X \) and \( i = 0, 1 \). Without loss of generality, assume \( z_0 < z_1 \). By (G4), we know \( u(x_1) = G(x_1, y_0, z_1) < G(x_1, y_0, z_0) \), contradicting \( u(x) \geq G(x, y_0, z_0) \) for all \( x \in X \). Having shown \( v : A \to \text{cl}(Z) \) is well-defined, we now show it is lower semicontinuous.
(ii) Suppose \( \{y_k\} \subset A \) converges to \( y_0 \in A \) and
\[
z_\infty := \lim_{k \to \infty} v(y_k) = \lim_{y \to y_0} \inf_{y \in y_0} v(y).
\]
We need to show \( v(y_0) \leq z_\infty \). Letting \( z_k := v(y_k) \) for each \( k \), there exists \( x_k \in X \) such that
\[
(3.4) \quad u(x) \geq G(x, y_k, z_k) \quad \forall x \in X \text{ and } k = 0, 1, \ldots,
\]
with equality holding at \( x = x_k \). In case (b) we deduce \( z_\infty < \infty \) from
\[
G(x_k, y_k, z_k) = u(x_k) \geq G(x_k, y_0, z_0)
\]
and (G5). Taking \( k \to \infty \), (G0) (or (3.2) in case (a) when \( z_\infty = +\infty \)) implies
\[
(3.5) \quad u(x) \geq G(x, y_0, z_\infty) \quad \forall x \in X.
\]
Applying (G4) to \( G(x_0, y_0, z_0) = u(x_0) \geq G(x_0, y_0, z_\infty) \) yields the desired semicontinuity: \( z_0 \leq z_\infty \).

(iii) We extend \( v \) from \( A \) to \( \text{cl}(Y) \) by taking its lower semicontinuous hull; this does not change the values of \( v \) on \( A \), but satisfies \( v(y_0) = \inf_{y \in A} v(y) \). We now show this choice of price menu \( v \) yields (1.1).

Recall for each \( x \in X \) that there exists \( y_0 \in \text{cl}(Y) \) such that
\[
(3.3) \quad u(x) = G(x, y_0, z_\infty)
\]
in view of (3.2) (or (G5)) and the fact that \( v(y) = z_0 \) for each \( y \) outside \( \text{cl}(A) \). Thus to establish (1.1), we need only show that (3.3) remains true when the domain of the maximum is enlarged from \( A \) to \( \text{cl}(A) \). Since we have chosen the largest lower semicontinuous extension of \( v \) outside of \( A \), each \( y_0 \in \text{cl}(A) \setminus A \) is approximated by a sequence \( \{y_k\} \subset A \) for which \( z_k := v(y_k) \) converges to \( z_\infty := v(y_0) \). As before, (3.3) holds and implies (3.5), showing (3.3) indeed remains true when the domain of the maximum is enlarged from \( A \) to \( \text{cl}(A) \), and establishing (1.1). Finally, if \( v(y_0) > z_\infty \) in case (b), then (G4) yields
\[
(3.6) \quad u(x) \geq u_\infty(x) > G(x, y_\infty, v(y_0)),
\]
and we may redefine \( v(y_\infty) := z_\infty \) without violating either (1.1) or the lower semicontinuity of \( v \).

2. Conversely, suppose there exists a lower semicontinuous function
\[
v: \text{cl}(Y) \to \text{cl}(Z) \quad \text{such that } u(x) = \max_{y \in \text{cl}(Y)} G(x, y, v(y)).
\]
Then for any \( x_0 \in X \), there exists \( y_0 \in \text{cl}(Y) \) such that \( u(x_0) = G(x_0, y_0, v(y_0)) \). Let \( z_0 := v(y_0) \); then \( u(x_0) = G(x_0, y_0, z_0) \), and for all \( x \in X \),
\[
u(x) \geq G(x, y_0, z_0).
\]
By definition, \( u \) is \( G \)-convex. If \( v(y_\infty) \leq z_\infty \) then
\[
\begin{align*}
u(\cdot) & \geq G(\cdot, y_\infty, v(y_\infty)) \\
& \geq u(\cdot)
\end{align*}
\]
by (1.1) and (G4). □
Remark 3.6 (Optimal agent strategies). Assume (G0) and (G4). When $\bar{z} < \infty$, lower semicontinuity of $v : \text{cl}(Y) \to \text{cl}(Z)$ is enough to ensure the maximum (1.1) is attained. However, when $\bar{z} = +\infty$, we can reach the same conclusion either by assuming that the limit (3.2) converges uniformly with respect to $y \in \text{cl}(Y)$ or that $v(y_\emptyset) \leq z_\emptyset$ and (G5).

Proof. For any fixed $x \in X$ let $u(x) = \sup_{y \in \text{cl}(Y)} G(x, y, v(y))$. We will show that the maximum is attained. Since $\text{cl}(Y)$ is compact, suppose $\{y_k\} \subset \text{cl}(Y)$ converges to $y_0 \in \text{cl}(Y)$,

$$ z_\infty := \limsup_{k \to \infty} v(y_k) \quad \text{and} \quad u(x) = \lim_{k \to \infty} G(x, y_k, v(y_k)). $$

By extracting a subsequence of $\{y_k\}$ and relabeling, without loss of generality, assume $\lim_{k \to \infty} v(y_k) = z_\infty$.

1. If $z_\infty < \bar{z}$, then lower semicontinuity of $v$ yields $v(y_0) \leq z_\infty < +\infty$. By (G4), one has

$$ G(x, y_0, v(y_0)) \geq G(x, y_0, z_\infty) = \lim_{k \to \infty} G(x, y_k, v(y_k)) $$

(3.6)

$$ = u(x) = \sup_{y \in \text{cl}(Y)} G(x, y, v(y)). $$

Therefore, the maximum is attained by $y_0$.

2. If $z_\infty = \bar{z}$, then $\lim_{k \to \infty} v(y_k) = \bar{z} = +\infty$.

2.1. By assuming the limit (3.2) converges uniformly with respect to $y \in \text{cl}(Y)$, we have

$$ \inf_{(\bar{y}, \bar{z}) \in \text{cl}(Y \times Z)} G(x, \bar{y}, \bar{z}) = G(x, y_0, \bar{z}) = \lim_{k \to \infty} G(x, y_k, v(y_k)) $$

$$ = u(x) = \sup_{y \in \text{cl}(Y)} G(x, y, v(y)). $$

In this case, the maximum is attained by $y_0$.

2.2. By assuming (G5), for sufficiently large $k$, we have $G(x, y_k, v(y_k)) < G(x, y_\emptyset, z_\emptyset)$. Taking $k \to \infty$, by $v(y_\emptyset) \leq z_\emptyset$ and (G4), one has

$$ \sup_{y \in \text{cl}(Y)} G(x, y, v(y)) = u(x) = \lim_{k \to \infty} G(x, y_k, v(y_k)) $$

$$ \leq G(x, y_\emptyset, z_\emptyset) \leq G(x, y_\emptyset, v(y_\emptyset)). $$

Thus, the maximum is attained by $y_\emptyset$.

Lemma 3.7 plays the role of a bridge connecting incentive compatibility in the economic context with $G$-convexity and $G$-subdifferentiability in mathematical analysis, generalizing the results of Rochet [32] and Carlier [5].

Lemma 3.7 ($G$-convex utilities characterize incentive compatibility). Assume (G4). A measurable map $x \in X \mapsto (y(x), z(x)) \in \text{cl}(Y \times Z)$ represents an incentive compatible contract if and only if $u(\cdot) := G(\cdot, y(\cdot), z(\cdot))$ is $G$-convex on $X$ and $y(x) \in \partial^G u(x)$ for each $x \in X$. 
PROOF. Suppose $(y, z)$ is incentive compatible. For each $x_0 \in X$, let $y_0 = y(x_0)$, $z_0 = z(x_0)$; then $u(x_0) = G(x_0, y(x_0), z(x_0)) = G(x_0, y_0, z_0)$. Moreover, for all $x \in X$, $u(x) = G(x, y(x), z(x)) \geq G(x, y(x_0), z(x_0)) = G(x, y_0, z_0)$. By definition, $u$ is $G$-convex.

Since $u(x_0) = G(x_0, y_0, z_0)$, one has $z_0 = H(x_0, y_0, u(x_0))$. For all $x \in X$, $u(x) \geq G(x, y_0, z_0)$; thus $u(x) \geq G(x, y_0, H(x_0, y_0, u(x_0)))$. By definition, $y(x_0) = y_0 \in \partial^G u(x_0)$.

On the other hand, suppose $u(x) = G(x, y(x), z(x))$ is $G$-convex, and for each $x \in X$, $y(x) \in \partial^G u(x)$, so that $u(x') \geq G(x', y(x), H(x, y(x), u(x))) = G(x', y(x), z(x))$ for all $x' \in X$. Then $G(x', y(x'), z(x')) \geq G(x', y(x), z(x))$ for all $x, x' \in X$. By definition, $(y, z)$ is incentive compatible.

The following proposition not only reformulates the principal’s problem, but manifests the existence of maximizer(s). For other existence results guaranteeing this supremum is attained in the nonquasilinear setting, see Nöldeke-Samuelson [28], who require mere continuity of the direct utility $G$, and Zhang [43], who relaxes relative compactness of the domain.

In this paper, we use $L^m$ to denote Lebesgue measure, which characterizes $m$-dimensional volume. A nonnegative measure $\mu$ is said to be absolutely continuous with respect to $L^m$ if for every measurable set $A$, $L^m(A) = 0$ implies $\mu(A) = 0$. This is written as $\mu \ll L^m$.

THEOREM 3.8 (Reformulating the principal’s program using the agents’ indirect utilities). Assume hypotheses (G0)–(G1) and (G4)–(G5), $\bar{z} < +\infty$, and $\mu \ll L^m$. Setting $\bar{\Pi}(u, y) = \int_X \pi(x, y(x), H(x, y(x), u(x)))d\mu(x)$, the principal’s problem $(P_0)$ is equivalent to

$$
(P) \begin{cases} 
\max \bar{\Pi}(u, y) \text{ among } G\text{-convex } u(x) \geq u_\emptyset(x) \\
\text{with } y(x) \in \partial^G u(x) \text{ for all } x \in X.
\end{cases}
$$

This maximum is attained. Moreover, $u$ determines $y(x)$ uniquely for a.e. $x \in X$.

PROOF.

1. Proposition 3.5 encodes a bijective correspondence between lower semicontinuous price menus $v: \text{cl}(Y) \to \text{cl}(Z)$ with $v(y_\emptyset) \leq z_\emptyset$ and $G$-convex indirect utilities $u \geq u_\emptyset$; it also shows (1.1) is attained. Fix a $G$-convex $u \geq u_\emptyset$ and the corresponding price menu $v$. For each $x \in X$ let $y(x)$ denote the point achieving the maximum (1.1) so that $u(x) = G(x, y(x), z(x))$ with $z(x) := v(y(x)) = H(x, y(x), u(x))$ and $\Pi(v, y) = \bar{\Pi}(u, y)$. From (1.1) we see

$$
G(\cdot, y(\cdot), v \circ y(\cdot)) = u(\cdot) \geq G(\cdot, y(x), H(x, y(x), u(x))),
$$

so that $y(x) \in \partial^G u(x)$. Apart from the measurability established below, Lemma 3.7 asserts incentive compatibility of $(y, v \circ y)$, while $u \geq u_\emptyset$ shows individual rationality, so $(P) \leq (P_0)$. 


2. The reverse inequality begins with a lower semicontinuous price menu \( v \): cl\((Y) \to \text{cl}(Z) \) with \( v(y_\emptyset) \leq z_\emptyset \) and an incentive compatible, individually rational menu \( (y, v \circ \gamma) \) on \( X \). Lemma 3.7 then asserts \( G \)-convexity of \( u(\cdot) := G(\cdot, y(\cdot), v(y(\cdot))) \) and that \( y(x) \in \partial^G u(x) \) for each \( x \in X \). Choosing \( \cdot = x \) in the corresponding inequality (3.7) produces equality, whence (G4) implies \( v(y(x)) = H(x, y(x), u(x)) \) and \( \Pi(v, y) = \tilde{\Pi}(u, y) \). Since \( u \geq u_\emptyset \) follows from individual rationality, we have established equivalence of (P) to (P0). Let us now argue that the supremum \( \Pi \) is attained.

3. Let us first show \( \pi(x, y(x), H(x, y(x), u(x))) \) is measurable on \( X \) for all \( G \)-convex \( u \) and \( y(x) \in \partial^G u(x) \).

   By (G0), we know \( G \) is Lipschitz; i.e., there exists \( L > 0 \), such that
   \[
   |G(x_1, y_1, z_1) - G(x_2, y_2, z_2)| \leq L\|x_1 - x_2, y_1 - y_2, z_1 - z_2\| \quad \text{for all} \quad (x_1, y_1, z_1), (x_2, y_2, z_2) \in \text{cl}(X \times Y \times Z). \]
   Since \( u \) is \( G \)-convex, for any \( x_1, x_2 \in X \), there exist \( (y_1, z_1), (y_2, z_2) \in \text{cl}(Y \times Z) \) such that \( u(x_i) = G(x_i, y_i, z_i) \) for \( i = 1, 2 \). Therefore,
   \[
   u(x_1) - u(x_2) \geq G(x_1, y_1, z_1) - G(x_2, y_2, z_2) \geq -L\|x_1 - x_2\|, \\
   u(x_1) - u(x_2) \leq G(x_1, y_1, z_1) - G(x_2, y_2, z_2) \leq L\|x_1 - x_2\|. 
   \]
   That is to say, \( u \) is also Lipschitz with Lipschitz constant \( L \). By Rademacher’s theorem and \( \mu \ll \mathcal{L}^m \), we have \( \mu(X \setminus \text{dom } Du) = \mathcal{L}^m(X \setminus \text{dom } Du) = 0 \). Moreover, since \( u \) is continuous,

   \[
   \frac{\partial u(x)}{\partial x_j} = \lim_{h \to 0} \frac{u(x + he_j) - u(x)}{h}
   \]

   is measurable on \( \text{dom } Du \) for \( j = 1, 2, \ldots, m \), where \( e_j = (0, \ldots, 0, 1, 0, \ldots, 0) \) is the unit vector in \( \mathbb{R}^m \) with \( j \)-th coordinate nonzero. Thus, \( Du \) is also Borel on \( \text{dom } Du \).

   For any \( x \in \text{dom } Du \), since \( y(x) \in \partial^G u(x) \), we have
   \[
   u(x) = G(x, y(x), H(x, y(x), u(x))), \\
   Du(x) = D_x G(x, y(x), H(x, y(x), u(x))).
   \]

   By (G1), there exists a continuous function \( y_G \) such that
   \[
   y(x) = y_G(x, u(x), Du(x)).
   \]

   Thus \( y(x) \) is Borel on \( \text{dom } Du \), which implies \( \pi(x, y(x), H(x, y(x), u(x))) \) is measurable on \( X \), given \( \pi \in C^0(\text{cl}(X \times Y \times Z)) \) and \( \mu \ll \mathcal{L}^m \). Here we use the fact that \( H \) is also continuous since \( G \) is continuous and strictly decreasing with respect to its third variable.

4. To show that the supremum is attained, let \( \{u_k\}_{k \in \mathbb{N}} \) be a sequence of \( G \)-convex functions, \( u_k(x) \geq u_\emptyset(x) \) and \( y_k(x) \in \partial^G u_k(x) \) for any \( x \in X \) and \( k \in \mathbb{N} \), such that

   \[
   \lim_{k \to \infty} \tilde{\Pi}(u_k, y_k) = \sup \tilde{\Pi}(u, y)
   \]
among all feasible \((u, y)\). Below we construct a feasible pair \((u_\infty, y_\infty)\) attaining the maximum.

**Claim 1.** There exists \( M > 0 \) such that \(|u(x)| < M\) for any \(G\)-convex \(u\) and any \(x \in X\). Thus \(\{u_k\}_{k \in \mathbb{N}}\) is uniformly bounded.

**Proof.** Since \(u\) is \(G\)-convex, for any \(x \in X\) there exists \((y, z) \in \text{cl}(Y \times Z)\) such that \(u(x) = G(x, y, z)\). Notice that \(G\) is bounded, since \(G\) is continuous on a compact set. Thus, there exists \( M > 0 \) such that \(|u(x)| = |G(x, y, z)| < M\) is also bounded. \(\square\)

4.1. From part 1, we know \(\{u_k\}_{k \in \mathbb{N}}\) are uniformly Lipschitz with Lipschitz constant \(L\); thus \(\{u_k\}_{k \in \mathbb{N}}\) are uniformly equicontinuous.

4.2. By the Arzelà-Ascoli theorem, there exists a subsequence of \(\{u_k\}_{k \in \mathbb{N}}\), again denoted as \(\{u_k\}_{k \in \mathbb{N}}\), and \(u_\infty : X \to \mathbb{R}\) such that \(\{u_k\}_{k \in \mathbb{N}}\) converges uniformly to \(u_\infty\) on \(X\).

**Claim 2.** \(u_\infty\) is also Lipschitz.

**Proof.** For any \(\varepsilon > 0\), any \(x_1, x_2 \in X\), since \(\{u_k\}_{k \in \mathbb{N}}\) converges to \(u_\infty\) uniformly, there exists \(K > 0\) such that for any \(k > K\), we have \(|u_k(x_i) - u_\infty(x_i)| < \varepsilon\) for \(i = 1, 2\). Therefore,

\[
|u_\infty(x_1) - u_\infty(x_2)| 
\leq |u_k(x_1) - u_\infty(x_1)| + |u_k(x_2) - u_\infty(x_2)| + |u_k(x_1) - u_k(x_2)| 
< 2\varepsilon + L\|x_1 - x_2\|.
\]

Since the above inequality is true for all \(\varepsilon > 0\), thus \(u_\infty\) is also Lipschitz. \(\square\)

4.3. For any \(x \in X\), since \(u_k(x) \geq u(\emptyset)(x)\) and \(\lim_{k \to \infty} u_k(x) = u_\infty(x)\), we have \(u_\infty(x) \geq u(\emptyset)(x)\). Therefore, \(u_\infty\) satisfies the participation constraint.

4.4. For any fixed \(x \in X\), since \(\{y_k(x)\}_{k \in \mathbb{N}} \subset \text{cl}(Y)\), which is compact, there exists a subsequence \(\{y_{k_l}(x)\}_{l \in \mathbb{N}}\) that converges. Define

\[
y_\infty(x) := \lim_{l \to \infty} y_{k_l}(x) \in \text{cl}(Y).
\]

For each \(l \in \mathbb{N}\), because \(y_{k_l}(x) \in \partial^G u_{k_l}(x)\), by definition we have

\[
u_{k_l}(x_0) \geq G(x_0, y_{k_l}(x), H(x, y_{k_l}(x), u_{k_l}(x))) \text{ for any } x_0 \in X.
\]

This implies, for all \(x_0 \in X\),

\[
u_\infty(x_0) = \lim_{l \to \infty} u_{k_l}(x_0) \geq \lim_{l \to \infty} G(x_0, y_{k_l}(x), H(x, y_{k_l}(x), u_{k_l}(x))) 
\geq G(x_0, y_\infty(x), H(x, y_\infty(x), u_\infty(x))).
\]

Thus, \(y_\infty(x) \in \partial^G u_\infty(x)\).

Therefore, \(\partial^G u_\infty(x) \neq \emptyset\) for any \(x \in X\). By Lemma 3.4 this implies \(u_\infty\) is \(G\)-convex.

At this point, we have found a feasible pair \((u_\infty, y_\infty)\) satisfying all the constraints in \((P)\).
Claim 3. For any \( x \in \text{dom } Du_{\infty} \), the sequence \( \{y_k(x)\}_{k \in \mathbb{N}} \subset \text{cl}(Y) \) converges to \( y_{\infty}(x) \).

**Proof.** Since \( u_{\infty} \) is Lipschitz, by Rademacher’s theorem, \( u_{\infty} \) is differentiable almost everywhere in \( X \), i.e., \( \mu(X \setminus \text{dom } Du_{\infty}) = \mathcal{L}^m(X \setminus \text{dom } Du_{\infty}) = 0 \).

For any \( x \in \text{dom } Du_{\infty} \) and any \( \tilde{y} \in \partial^G u_{\infty}(x) \), we have
\[
\tilde{y}(x) = y_G(x, u_{\infty}(x), Du_{\infty}(x)),
\]
according to equation (3.8) and hypothesis (G1). This implies \( \partial^G u_{\infty}(x) \) is a singleton for each \( x \in \text{dom } Du_{\infty} \), i.e., \( \partial^G u_{\infty}(x) = \{y_{\infty}(x)\} \).

For any \( x \in \text{dom } Du_{\infty} \), by a similar argument to that in part 4.4, we can show that any (other) accumulation points of \( \{y_k(x)\}_{k \in \mathbb{N}} \) are elements in the set \( \partial^G u_{\infty}(x) = \{y_{\infty}(x)\} \); i.e., the sequence \( \{y_k(x)\}_{k \in \mathbb{N}} \) converges to \( y_{\infty}(x) \). \( \square \)

4.5. Finally, since \( \mu \ll \mathcal{L}^m \), by Fatou’s lemma, we have
\[
\lim_{k \to \infty} \mathcal{P}(u_k, y_k) = \sup_{(u, y)} \lim_{k \to \infty} \mathcal{P}(u_k, y_k)
\]
among all feasible \((u, y)\). Thus, the supremum is attained. \( \square \)

Remark 3.9 (More singular measures). If \( G \in C^2 \) (uniformly in \( z \in Z \)) the same conclusions extend to \( \mu \), which need not be absolutely continuous with respect to Lebesgue, provided \( \mu \) vanishes on all hypersurfaces parametrized locally as a difference of convex functions \([9, 11]\). This is because \( G \)-convexity then implies semiconvexity of \( u \). On the other hand, apart from its final sentence, the proposition extends to all probability measures \( \mu \) if \( G \) is merely continuous, according to Nöldke-Samuelson \([28]\). Our argument is simpler than theirs on one point however: Borel measurability of \( y(x) \) on \( \text{dom } Du \) follows automatically from \((G0)–(G1)\); in the absence of these extra hypotheses, they are required to make a measurable selection from among each agent’s preferred products to define \( y(x) \).

Remark 3.10 (Tie-breaking rules for singular measures). When an agent \( x \) finds more than one product that maximize his utility, in order to reduce the ambiguity, it is convenient to assume the principal has satisfactory persuasion to convince the agent to choose one of those products that maximize the principal’s profit. According to equation (3.1) and condition (G1), this scenario would occur only for \( x \in X \setminus \text{dom } Du \), which has Lebesgue measure zero. Thus this convention has
4 Concavity and Convexity Results

The advantage of the reformulation from Section 3 is to make the principal’s objective $\Pi$ depend on a scalar function $u$ instead of a vector field $y$. By (G1), the optimal choice $y(x)$ of Lebesgue almost every agent $x \in X$ is uniquely determined by $u$. Recall that $\bar{y}_G(x, u(x), Du(x))$ is the unique solution $(y, z)$ of the system (3.1) for any $x \in \text{dom } Du$. Then the principal’s problem (P) can be rewritten as maximizing a functional depending only on the agents’ indirect utility $u$:

$$\max_{u \in U} \Pi(u) := \max_{u \in U} \int_X \pi(x, \bar{y}_G(x, u(x), Du(x))) d\mu(x).$$

Define $U := \{u : X \to \mathbb{R} \mid u \text{ is } G\text{-convex}\}$ and $U_\varnothing := \{u \in U \mid u \geq u_\varnothing\}$. Then the problem becomes to maximize $\Pi$ on $U_\varnothing$. In this section, we give conditions under which the function space $U_\varnothing$ is convex and the functional $\Pi$ is concave, often strictly. Uniqueness and stability of the principal’s maximizing strategy follow from strict concavity as in [9]. We also provide conditions under which $\Pi$ is convex. In this situation, the maximizers of $\Pi$ may not be unique, but are attained at extreme points of $U_\varnothing$. (Recall that $u \in U$ is called extreme if $u$ does not lie at the midpoint of any segment in $U$.)

**Theorem 4.1** (G-convex functions form a convex set). If $G : \text{cl}(X \times Y \times Z) \to \mathbb{R}$ satisfies (G0)–(G2), then (G3) becomes necessary and sufficient for the convexity of the set $U$.

**Proof.** Assume (G0)–(G2). For any $u_0, u_1 \in U$, define

$$u_t(x) := (1 - t)u_0(x) + tu_1(x) \quad \text{for all } t \in (0, 1).$$

We want to show $u_t$ is $G$-convex as well for each $t \in (0, 1)$.

For any fixed $x_0 \in X$, since $u_0, u_1$ are $G$-convex, there exist $(y_0, z_0), (y_1, z_1) \in \text{cl}(Y \times Z)$ such that $u_0(x_0) = G(x_0, y_0, z_0), u_1(x_0) = G(x_0, y_1, z_1), u_0(x) \geq G(x, y_0, z_0), \text{ and } u_1(x) \geq G(x, y_1, z_1)$ for all $x \in X$.

Denote by $(x_0, y_t, z_t)$ the $G$-segment connecting $(x_0, y_0, z_0)$ and $(x_0, y_1, z_1)$. Then

$$u_t(x_0) = (1 - t)u_0(x_0) + tu_1(x_0)$$

$$= (1 - t)G(x_0, y_0, z_0) + tG(x_0, y_1, z_1) = G(x_0, y_t, z_t),$$

where the last equality comes from (2.1).

In order to prove $u_t$ is $G$-convex, it remains to show $u_t(x) \geq G(x, y_t, z_t)$ for all $x \in X$.

By (G3), $G(x, y_t, z_t)$ is convex in $t$, i.e., $G(x, y_t, z_t) \leq (1 - t)G(x, y_0, z_0) + tG(x, y_1, z_1)$. So, $u_t(x) = (1 - t)u_0(x) + tu_1(x) \geq (1 - t)G(x, y_0, z_0) + tG(x, y_1, z_1)$.
\[ tG(x, y_t, z_t) \geq G(x, y_t, z_t) \] for each \( x \in X \). By definition, \( u_t \) is \( G \)-convex, i.e., \\
\[ u_t \in \mathcal{U} \text{ for all } t \in (0, 1) \]. Thus, \( \mathcal{U} \) is convex.

Conversely, assume \( \mathcal{U} \) is convex. For any fixed \( x_0 \in X \), \( (y_t, z_t) \in \text{cl}(Y \times Z) \) with \( (x_0, y_t, z_t) \) being a \( G \)-segment, we would like to show that \( G(x, y_t, z_t) \leq (1-t)G(x, y_0, z_0) + tG(x, y_1, z_1) \) for any \( x \in X \).

Define \( u_t(x) := G(x, y_t, z_t) \) for \( i = 0, 1 \). Then by definition of \( G \)-convexity, \( u_0, u_1 \in \mathcal{U} \). Denote \( u_t := (1-t)u_0 + tu_1 \) for all \( t \in (0, 1) \). Since \( \mathcal{U} \) is a convex set, \( u_t \) is also \( G \)-convex. For this \( x_0 \) and each \( t \in (0, 1) \), there exists \( (\tilde{y}_t, \tilde{z}_t) \in \text{cl}(Y \times Z) \) such that \( u_t(x) \geq G(x, \tilde{y}_t, \tilde{z}_t) \) for all \( x \in X \), and equality holds at \( x_0 \). Thus, \( Du_t(x_0) = D_xG(x_0, \tilde{y}_t, \tilde{z}_t) \).

Since \((x_0, y_t, z_t)\) is a \( G \)-segment, from (2.1) we know

\[
D_xG(x_0, y_t, z_t) = (1-t)D_xG(x_0, y_0, z_0) + tD_xG(x_0, y_1, z_1) \\
= (1-t)Du_0(x_0) + tDu_1(x_0) = Du_t(x_0).
\]

Thus, by (G1), \( (\tilde{y}_t, \tilde{z}_t) = (y_t, z_t) \) for each \( t \in (0, 1) \). Therefore,

\[
(1-t)G(x, y_0, z_0) + tG(x, y_1, z_1) = u_t \geq G(x, \tilde{y}_t, \tilde{z}_t) = G(x, y_t, z_t)
\]

for all \( x \in X \), i.e., \( G(x, y_t, z_t) \) is convex in \( t \) along any \( G \)-segment \((x_0, y_t, z_t)\).

The following theorem provides a sufficient and necessary condition for the functional \( \Pi(u) \) to be concave. It reveals the relationship between linear interpolations on the function space \( \mathcal{U} \) and \( G \)-segments on the underlying type space \( \text{cl}(Y \times Z) \).

**Theorem 4.2** (Concavity of the principal’s objective). If \( G \) and \( \pi : \text{cl}(X \times Y \times Z) \rightarrow \mathbb{R} \) satisfy (G0)–(G5), the following statements are equivalent:

(i) \( t \in [0, 1] \mapsto \pi(x, y_t, z_t) \) is concave along all \( G \)-segments \((x, y_t, z_t)\).

(ii) \( \Pi(u) \) is concave in \( \mathcal{U} \) for all \( \mu \ll L^m \).

**Proof.**

(i) \( \Rightarrow \) (ii). For any \( u_0, u_1 \in \mathcal{U} \), \( t \in (0, 1) \), define \( u_t = (1-t)u_0 + tu_1 \). We want to prove \( \Pi(u_t) \geq (1-t)\Pi(u_0) + t\Pi(u_1) \) for any \( \mu \ll L^m \).

Equations (3.1) implies that there exist

\[
y_0, y_1 : \text{dom } Du \to \text{cl}(Y) \quad \text{and} \quad z_0, z_1 : \text{dom } Du \to \text{cl}(Z)
\]
such that

\begin{align}
(G_x, G)(x, y_0(x), z_0(x)) &= (Du_0, u_0)(x), \\
(G_x, G)(x, y_1(x), z_1(x)) &= (Du_1, u_1)(x).
\end{align}

(4.1)

For each \( x \in \text{dom } Du \), \( (y_0(x), z_0(x)), (y_1(x), z_1(x)) \in \text{cl}(Y \times Z) \), let \( t \in [0, 1] \mapsto (x, y_t(x), z_t(x)) \) be the \( G \)-segment connecting \((x, y_0(x), z_0(x))\) and \((x, y_1(x), z_1(x))\). Combining (4.1) and (2.1), we have

\[
(G_x, G)(x, y_t(x), z_t(x)) = (Du_t, u_t)(x).
\]

(4.2)
Thus, by concavity of $\pi$ on $G$-segments, for every $t \in [0, 1],$
\[
\Pi(u_t) = \int_X \pi(x, y_t(x), z_t(x))d\mu(x)
\geq \int_X (1-t)\pi(x, y_0(x), z_0(x)) + t\pi(x, y_1(x), z_1(x))d\mu(x)
= (1-t)\Pi(u_0) + t\Pi(u_1).
\]
Thus, $\Pi$ is concave in $\mathcal{U}$.

(ii) $\Rightarrow$ (i). To derive a contradiction, assume (i) fails. Then there exists a $G$-segment $(x_0, y_t(x_0), z_t(x_0))$ and $t_0 \in (0, 1)$ such that
\[
\pi(x_0, y_{t_0}(x_0), z_{t_0}(x_0)) < (1-t_0)\pi(x_0, y_0(x_0), z_0(x_0)) + t_0\pi(x_0, y_1(x_0), z_1(x_0)).
\]

Let $u_0(x) := G(x, y_0(x_0), z_0(x_0)), u_1(x) := G(x, y_1(x_0), z_1(x_0)),$ and $u_{t_0} = (1-t_0)u_0 + t_0u_1.$ Then $u_0, u_1, u_{t_0} \in \mathcal{U}.$ From (4.1) we know $y_t(x) \equiv y_i(x_0), z_t(x) \equiv z_i(x_0)$ for $i = 0, 1.$ Let $t \in [0, 1] \mapsto (x, y_t(x), z_t(x))$ be the $G$-segment connecting $(x, y_0(x_0), z_0(x_0))$ and $(x, y_1(x), z_1(x)).$ Combining (3.1) and (2.1), we have
\[
(G_x, G)(x, y_0(x_0), z_0(x_0)) = (Du_0, u_0)(x),
(G_x, G)(x, y_1(x_0), z_1(x_0)) = (Du_1, u_1)(x),
(G_x, G)(x, y_{t_0}(x_0), z_{t_0}(x)) = (Du_{t_0}, u_{t_0})(x).
\]

Since $\pi, y_{t_0},$ and $z_{t_0}$ are continuous, there exists $\varepsilon > 0$ such that for all $x \in B_\varepsilon(x_0),$ one has
\[
\pi(x, y_{t_0}(x), z_{t_0}(x)) < (1-t_0)\pi(x, y_0(x_0), z_0(x_0)) + t_0\pi(x, y_1(x_0), z_1(x_0)).
\]

Here we use $B_\varepsilon(x_0)$ to denote the open ball in $\mathbb{R}^m$ centered at $x_0$ with radius $\varepsilon.$

Take $d\mu = dL^m|_{B_\varepsilon(x_0)} / L^m(B_\varepsilon(x_0))$ to be a uniform measure on $B_\varepsilon(x_0).$ Thus,
\[
\Pi(u_{t_0}) = \int_X \pi(x, y_{t_0}(x), z_{t_0}(x))d\mu(x)
\leq \int_X (1-t_0)\pi(x, y_0(x_0), z_0(x_0)) + t_0\pi(x, y_1(x_0), z_1(x_0))d\mu(x)
= (1-t_0)\Pi(u_0) + t_0\Pi(u_1).
\]
This contradicts the concavity of $\Pi.$

A similar proof shows the following result. Corollary 4.3 implies that concavity of the principal’s profit is equivalent to concavity of the principal’s utility along qualified $G$-segments. Moreover, Theorem 4.1 and Corollary 4.3 together imply that the principal’s profit $\Pi$ is a concave functional on a convex space under assumptions (G0)–(G5), $\mu \ll L^m,$ and (i)' below.
Corollary 4.3. If $G$ and $\pi$ satisfy (G0)–(G5), the following are equivalent:

(i) $t \in [0, 1] \mapsto \pi(x, y_t(x), z_t(x))$ is concave along all $G$-segments $(x, y_t(x), z_t(x))$ whose endpoints satisfy

$$\min\{G(x, y_0(x), z_0(x)), G(x, y_1(x), z_1(x))\} \geq u_{\emptyset}(x);$$

(ii) $\Pi(u)$ is concave in $U_{\emptyset}$ for all $\mu \ll \mathcal{L}^m$.

To obtain uniqueness and stability of optimizers requires a stronger form of convexity. Recall that a function $f$ defined on a convex subset of a normed space is said to be strictly convex if

$$f((1-t)x + ty) > (1-t)f(x) + tf(y)$$

whenever $0 < t < 1$ and $x \neq y$. It is said to be (2-)uniformly concave if there exists $\lambda > 0$ such that for any $x, y$ in the domain of $f$ and $t \in [0, 1]$, the following inequality holds:

$$f((1-t)x + ty) - (1-t)f(x) - tf(y) \geq t(1-t)\lambda\|x - y\|^2.$$

For such strengthenings, it is necessary to view indirect utilities $u \in U$ as equivalence classes of functions that differ only on sets of $\mu$ measure zero. More precisely, it is natural to adopt the Sobolev norm

$$\|u\|^2_{W^{1,2}(X, d\mu)} := \int_X (|u|^2 + |Du|^2)d\mu(x)$$

on $U$ and $U_{\emptyset}$. We then have the following results:

Corollary 4.4. Let $\pi$ and $G$ satisfy (G0)–(G5). If

(iii) $t \in [0, 1] \mapsto \pi(x, y_t(x), z_t(x))$ is strictly concave along all $G$-segments $(x, y_t(x), z_t(x))$ whose endpoints satisfy

$$\min\{G(x, y_0(x), z_0(x)), G(x, y_1(x), z_1(x))\} \geq u_{\emptyset}(x),$$

then

(iv) $\Pi(u)$ is strictly concave in $U \subset W^{1,2}(X, d\mu)$ for all $\mu \ll \mathcal{L}^m$. If

(iii) $t \in [0, 1] \mapsto \pi(x, y_t(x), z_t(x))$ is strictly concave along all $G$-segments $(x, y_t(x), z_t(x))$ whose endpoints satisfy

$$\min\{G(x, y_0(x), z_0(x)), G(x, y_1(x), z_1(x))\} \geq u_{\emptyset}(x),$$

then

(iv) $\Pi(u)$ is strictly concave in $U_{\emptyset} \subset W^{1,2}(X, d\mu)$ for all $\mu \ll \mathcal{L}^m$.

In addition, Theorem 4.1 and Corollary 4.4 together imply strict concavity of the principal’s profit on a convex space, which guarantees a unique solution to the monopolist’s problem.

Define $\bar{G}(\bar{x}, \bar{y}) = \bar{G}(x, x_0, y, z) := x_0G(x, y, z)$, where $\bar{x} = (x, x_0)$, $\bar{y} = (y, z)$, and $x_0 \in X_0$, where $X_0 \subset (-\infty, 0)$ is an open bounded interval containing $-1$. Hereafter, except in Appendix A we use $x_0$ to denote a number in $X_0$. For further applications, we need the following nondegeneracy assumption:

(G6) $G \in C^2(\text{cl}(X \times Y \times Z))$, and $D_{\bar{x}, \bar{y}}(\bar{G})(x, -1, y, z)$ has full rank for each $(x, y, z) \in \text{cl}(X \times Y \times Z)$.

Since (G1) implies $m \geq n$, full rank means $D_{\bar{x}, \bar{y}}(\bar{G})(x, -1, y, z)$ has rank $n + 1$. 

THEOREM 4.5 (Uniform concavity of the principal’s objective). Assume $G \in C^2(\text{cl}(X \times Y \times Z))$ and $\pi$ satisfy (G0)–(G6). In case $\overline{z} = +\infty$, assume the homeomorphisms of (G1) are uniformly bi-Lipschitz. Then the following statements are equivalent:

(v) Uniform concavity of $\pi$ along $G$-segments; i.e., there exists $\lambda > 0$ such that for any $G$-segment $(x, y_1(x), z_1(x))$ and any $t \in [0, 1]$,

$$\pi(x, y_1(x), z_1(x)) - (1 - t)\pi(x, y_0(x), z_0(x)) - t\pi(x, y_1(x), z_1(x))$$

$$\geq t(1 - t)\lambda \| (y_1(x) - y_0(x), z_1(x) - z_0(x)) \|_{\mathbb{R}^{n+1}}^2.$$ (4.3)

(vi) $\Pi(u)$ is uniformly concave in $U \subset W^{1,2}(X, d\mu)$, uniformly for all $\mu \ll \mathcal{L}^m$.

PROOF. (v) $\Rightarrow$ (vi). With the same notation as in the last proof, we want to prove there exists $\tilde{\lambda} > 0$ such that

$$\Pi(u_t) - (1 - t)\Pi(u_0) - t\Pi(u_1) \geq t(1 - t)\tilde{\lambda} \| u_1 - u_0 \|_{W^{1,2}(X, d\mu)}^2$$

for any $\mu \ll \mathcal{L}^m, u_0, u_1 \in U$, and $t \in (0, 1)$.

Similarly to the last proof, we have (4.1) and (4.2). Denote $\text{Lip}(G_x, G)$ to be the uniform Lipschitz constant of the map $(x, y, z) \in X \times Y \times Z \mapsto (G_x, G)(x, y, z)$.

Thus by uniform concavity of $\pi$ on $G$-segments, there exists $\lambda > 0$ such that for every $t \in [0, 1]$,

$$\Pi(u_t) - (1 - t)\Pi(u_0) - t\Pi(u_1)$$

$$= \int_X \pi(x, y_1(x), z_1(x)) - (1 - t)\pi(x, y_0(x), z_0(x))$$

$$- t\pi(x, y_1(x), z_1(x))d\mu(x)$$

$$\geq \int_X t(1 - t)\lambda \| (y_1(x) - y_0(x), z_1(x) - z_0(x)) \|_{\mathbb{R}^{n+1}}^2 d\mu(x)$$

$$\geq \int_X t(1 - t)\lambda \| (Du_1(x) - Du_0(x), u_1(x) - u_0(x)) \|_{\mathbb{R}^{n+1}}^2 d\mu(x)$$

$$\geq t(1 - t)\frac{\lambda}{\text{Lip}^2(G_x, G)} \| u_1 - u_0 \|_{W^{1,2}(X, d\mu)}^2.$$ (4.3)

Thus, $\Pi$ is uniformly concave in $U$, with $\tilde{\lambda} = \lambda / \text{Lip}^2(G_x, G) > 0$.

(vi) $\Rightarrow$ (v). To derive a contradiction, assume (v) fails. Then for any $\lambda > 0$, there exists a $G$-segment $(x^0, y_\tau(x^0), z_\tau(x^0))$ and some $\tau \in (0, 1)$ such that

$$\pi(x^0, y_\tau(x^0), z_\tau(x^0)) - (1 - \tau)\pi(x^0, y_0(x^0), z_0(x^0))$$

$$- \tau\pi(x^0, y_1(x^0), z_1(x^0))$$

$$< \tau(1 - t)\lambda \| (y_1(x^0) - y_0(x^0), z_1(x^0) - z_0(x^0)) \|_{\mathbb{R}^{n+1}}^2.$$
Take $u_0(x) := G(x, y_0(x^0), z_0(x^0))$, $u_1(x) := G(x, y_1(x^0), z_1(x^0))$, and for $t \in (0, 1)$, assign $u_t := (1 - t)u_0 + tu_1$. Then $u_t \in \mathcal{U}$ for $t \in [0, 1]$. From (4.6), we know $y_i(x) \equiv y_i(x^0)$, $z_i(x) \equiv z_i(x^0)$ for $i = 0, 1$. Let $t \in [0, 1]$ imply that the principal’s profit for such $(x, y_0(x), z_0(x))$ and $(x, y_1(x), z_1(x))$.

Combining (3.1) and (2.1), we have

$$(G_x, G)(x, y_0(x^0), z_0(x^0)) = (Du_0, u_0)(x),$$

$$(G_x, G)(x, y_1(x^0), z_1(x^0)) = (Du_1, u_1)(x),$$

$$(G_x, G)(x, y_t(x), z_t(x)) = (Du_t, u_t)(x).$$

Since $\pi$, $y_t$, and $z_t$ are continuous, there exists $\varepsilon > 0$ such that for all $x \in B_\varepsilon(x^0)$,

$$\pi(x, y_t(x), z_t(x)) - (1 - \tau)\pi(x, y_0(x^0), z_0(x^0)) - \tau\pi(x, y_1(x^0), z_1(x^0))$$

$$< \tau(1 - \tau)\lambda\|\{y_1(x^0) - y_0(x^0), z_1(x^0) - z_0(x^0)\}\|_{\mathbb{R}^{n+1}}^2.$$

Here we use $B_\varepsilon(x^0)$ to denote the open ball in $\mathbb{R}^m$ centered at $x^0$ with radius $\varepsilon$. Take $d\mu = d\mathcal{L}^m |_{B_\varepsilon(x^0)} / \mathcal{L}^m(B_\varepsilon(x^0))$ to be the unique measure on $B_\varepsilon(x^0)$. By (G6), the map $\bar{y}_G : (x, p, q) \mapsto (y, z)$, which solves equation (3.1), is uniformly Lipschitz on $X \times \mathbb{R} \times \mathbb{R}^m$. Denote by $\text{Lip}(\bar{y}_G)$ its Lipschitz constant.

Thus for such $\tau$, $u_0$, $u_1$, and $\mu$, we have

$$\Pi(u_t) - (1 - \tau)\Pi(u_0) - \tau\Pi(u_1)$$

$$= \int_X \pi(x, y_t(x), z_t(x)) - (1 - \tau)\pi(x, y_0(x^0), z_0(x^0))$$

$$- \tau\pi(x, y_1(x^0), z_1(x^0))d\mu(x)$$

$$< \int_X \tau(1 - \tau)\lambda\|\{y_1(x^0) - y_0(x^0), z_1(x^0) - z_0(x^0)\}\|_{\mathbb{R}^{n+1}}^2 d\mu(x)$$

$$\leq \tau(1 - \tau)\lambda\text{Lip}^2(\bar{y}_G)\|u_1 - u_0\|_{W^{1,2}(X, d\mu)}^2.$$

This contradicts the uniform concavity of $\Pi$. □

A similar argument implies the following equivalence. Theorems 4.1 and Corollary 4.6 together imply that the principal’s profit $\Pi$ is a uniformly concave functional on a convex space under assumptions (G0)–(G6), $\mu \ll \mathcal{L}^m$, and (v)’.

**Corollary 4.6.** Under the same assumptions as in Theorem 4.5, the following are equivalent:

(v)’ Uniform concavity of $\pi$ (in the sense of equation (4.3)) along $G$-segments $(x, y_t(x), z_t(x))$ whose endpoints satisfy

$$\min\{G(x, y_0(x), z_0(x)), G(x, y_1(x), z_1(x))\} \geq u_\emptyset(x).$$

(vi)’ $\Pi(u)$ is uniformly concave in $\mathcal{U}_\emptyset \subset W^{1,2}(X, d\mu)$ uniformly for all $\mu \ll \mathcal{L}^m$. 
The preceding concavity results also have convexity analogues. Unlike strict concavity, strict convexity does not imply uniqueness of the principal’s profit-maximizing strategy, though it suggests it should only be attained at extreme points of the strategy space $U$, where extreme point needs to be interpreted appropriately.

**Remark 4.7 (Convexity of principal’s objective).** If $\pi$ and $G$ satisfy (G0)–(G5), the equivalences (i) $\Leftrightarrow$ (ii) and (i) $\Rightarrow$ (ii)$'$ and implications (iii) $\Rightarrow$ (iv) and (iii)$'$ $\Rightarrow$ (iv)$'$ remain true when all occurrences of concavity are replaced by convexity. Similarly, the equivalences (v) $\Leftrightarrow$ (vi) and (v)$'$ $\Leftrightarrow$ (vi)$'$ remain true when both occurrences of uniform concavity are replaced by uniform convexity in Theorem 4.5.

We close with several examples, which are established by computing two derivatives of $\pi(x, y_t, z_t)$ along an arbitrary $G$-segment $t \in [0, 1] \mapsto (x, y_t, z_t)$. These computations are tedious but straightforward.

Assuming (G6), we denote by $(\tilde{G}_{\vec{x}, \vec{y}})^{-1}$ the left inverse of $D_{\vec{x}, \vec{y}}(\tilde{G})(x, x_0, y, z)$. We will use Einstein notation for simplifying expressions including summations of vectors, matrices, and general tensors for higher-order derivatives. There are essentially three rules of Einstein summation notation, namely: (1) repeated indices are implicitly summed over; (2) each index can appear at most twice in any term; and (3) both sides of an equation must contain the same nonrepeated indices. For example, $a_{ij}v_i = \sum_j a_{ij}v_i, a_{ij}b^{kj}v_k = \sum_j \sum_k a_{ij}b^{kj}v_k$. We also use commas to separate subscripts: the subscripts before a comma represent derivatives with respect to the first variable and those after a comma represent derivatives with respect to the second variable. For instance, for $b = b(x, y)$, $b_{kl}$ represents second derivatives with respect to $y$ only. And for $G = G(x, y, z)$, where $z \in \mathbb{R}$, $G_{i,jz}$ denotes third-order derivatives with respect to $x, y$, and $z$, instead of using another comma to separate subscripts corresponding to $y$ and $z$. Starting from now, for subscripts, we use $i, k, j, l, \alpha, \beta$ denoting integers from either $\{1, \ldots, m\}$ or $\{1, \ldots, n\}$, and $\vec{i}, \vec{k}, \vec{j}, \vec{l}$ denoting augmented indices from $\{1, \ldots, m+1\}$ or $\{1, \ldots, n+1\}$. For instance, $\pi_{ij}$ denotes a first-order derivative with respect to $x$ only, $\pi_{\vec{i}j}$ represents a Hessian matrix with respect to $\vec{y}$ only, and $\tilde{G}_{\vec{i}j\vec{k}}$ denotes a third-order derivative tensor that can be viewed as taking an $\vec{x}$-derivative of $\tilde{G}_{\vec{i}j\vec{k}}$.

The following remark reformulates concavity of $\pi$ on $G$-segments using nonpositive definiteness of a matrix. This equivalent form provides a simple method to verify a concavity condition stated in Theorem 4.2. We will apply this matrix form to establish Corollary 4.10 and Examples 4.11–4.13.

**Lemma 4.8 (Characterizing concavity of principal’s profit in the smooth case).** When $G \in C^3(\text{cl}(X \times Y \times Z))$ satisfies (G0)–(G6) and $\pi \in C^2(\text{cl}(X \times Y \times Z))$, then differentiating $\pi$ along an arbitrary $G$-segment $t \in [0, 1] \mapsto (x, y_t, z_t)$ yields

$$
\frac{d^2}{dt^2} \pi(x, y_t, z_t) = \left( \pi_{\vec{i}j\vec{k}} - \pi_{\vec{i}\vec{j}\vec{k}} \tilde{G}_{\vec{i}\vec{j}\vec{k}} \right) \vec{y}^k \vec{y}^j
$$

(4.4)
where \( \tilde{G}^{i,j} \) denotes the left inverse of the matrix \( \tilde{G}^{i,j}_{k,l} \) and \( \tilde{y}^{k} = (\frac{d}{dt})\tilde{y}^{k} \). Thus (i) in Theorem 4.2 is equivalent to nonpositive definiteness of the quadratic form \( \pi_{i,j} - \pi_{i,j}^{G^{i,j}} \tilde{G}^{i,j}_{i,j} \) on \( T_{\tilde{y}^{k}}(Y \times Z) = \mathbb{R}^{n+1} \) for each \((x, \tilde{y}) \in X \times Y \times Z\). Similarly, Theorem 4.5 (v) is equivalent to uniform negative definiteness of the same form.

Before we state another result, we need the following definition, which is a generalized Legendre transform (see Moreau [25], Kutateladze-Rubinov [17], Elster-Nehse [8], Balder [2], Dolecki-Kurcyusz [7], Gangbo-McCann [10], Singer [37], Rubinov [35, 36], and Martínez-Legaz [20] for more references).

**Definition 4.9** (\( G \)-concavity, \( G^{*} \)-concavity). A function \( \phi: \text{cl}(X \times X_{0}) \to \mathbb{R} \) is called \( G \)-concave if \( \phi = (\phi^{G^{*}})^{\tilde{G}} \) and a function \( \psi: \text{cl}(Y \times Z) \to \mathbb{R} \) is called \( G^{*} \)-concave if \( \psi = (\psi^{\tilde{G}})^{G^{*}} \), where

\[
\psi^{\tilde{G}}(\tilde{x}) = \min_{\tilde{y} \in \text{cl}(Y \times Z)} \tilde{G}(\tilde{x}, \tilde{y}) - \psi(\tilde{y}),
\]

\[
\phi^{\tilde{G}^{*}}(\tilde{y}) = \min_{\tilde{x} \in \text{cl}(X \times X_{0})} \tilde{G}(\tilde{x}, \tilde{y}) - \phi(\tilde{x}).
\]

We say \( \psi \) is strictly \( G^{*} \)-concave if in addition \( \psi^{\tilde{G}} \in C^{1}(X \times X_{0}) \).

Note that, apart from an overall sign and the extra variables, Definition 4.9 coincides with a quasilinear version \( G(\tilde{x}, \tilde{y}, z) = \tilde{G}(\tilde{x}, \tilde{y}) - z \) of Definition 3.2.

The following corollary characterizes concavity of the principal’s profit when her utility on one hand is not influenced by the agents’ identity, and on the other hand has adequate generality to encompass a tangled nonlinear relationship between products and selling prices. It generalizes the convexity result in Figalli-Kim-McCann [9], where \( G(x, y, z) = \tilde{b}(x, y) - z \) and \( \pi(x, y, z) = z - a(y) \). See Appendix B for a (partial) converse.

**Corollary 4.10** (Concavity of the principal’s objective with her utility not depending on agents’ types). If \( G \in C^{3}(\text{cl}(X \times Y \times Z)) \) satisfies (G0)-(G6), \( \pi \in C^{2}(\text{cl}(Y \times Z)) \) is \( G^{*} \)-concave, and \( \mu \ll \mathcal{L}^{m} \), then \( \Pi \) is concave.

For specific nonquasilinear agent preferences, we use the explicit expression above for the desired second derivative to establish the following examples, which assume the principal is indifferent to whom she transacts business with and that her preferences depend linearly on payments. These examples give conditions under which the principal’s program inherits concavity or convexity from the agents’ price sensitivity. Although the resulting conditions appear complicated, they illustrate the subtle interplay between the preferences of agents and the principal for products in the first example and between the preferences of agents for products as opposed to prices in the second.

**Example 4.11** (Nonlinear yet homogeneous sensitivity of agents to prices). Take \( \pi(x, y, z) = z - a(y) \), \( G(x, y, z) = \tilde{b}(x, y) - f(z) \), satisfying (G0)-(G6), \( G \in C^{3}(\text{cl}(X \times Y \times Z)) \), \( \pi \in C^{2}(\text{cl}(X \times Y \times Z)) \), and assume \( \tilde{z} < +\infty \).
1. If \( f(z) \) is convex (respectively, concave) in \( \text{cl}(Z) \), then \( \Pi(u) \) is concave (respectively, convex) for all \( \mu \ll \mathcal{L}^m \) if and only if there exists \( \varepsilon \geq 0 \) such that each \( (x, y, z) \in X \times Y \times Z \) and \( \xi \in \mathbb{R}^n \) satisfy

\[
\pm \left\{ d_{kj}(y) - \frac{b_{kj}(x, y)}{f'(z)} \right\} + \left( \frac{b_{j}(x, y)}{f'(z)} - a_j(y) \right) b_{i,j}(x, y) b_{i,kj}(x, y) \right\} \xi^k \xi^j \geq \varepsilon |\xi|^2. \tag{4.6}
\]

2. Also, \( \Pi(u) \) is uniformly concave (respectively, uniformly convex) on \( W^{1,2}(X, d\mu) \) uniformly for all \( \mu \ll \mathcal{L}^m \) if and only if \( \varepsilon > 0 \).

Although the next two examples are not completely general, they have the following economic interpretation. The same selling price impacts utility differently for different types of agents. In other words, it models the situation where agents have different sensitivities to the same price. In Example 4.12, the principal’s utility is linear and depends exclusively on her revenue, which is a simple special case of Example 4.13.

**Example 4.12 (Inhomogeneous sensitivity of agents to prices, zero cost).** Take \( \pi(x, y, z) = z, G(x, y, z) = b(x, y) - f(x, z) \) satisfying \( G \in C^3(\text{cl}(X \times Y \times Z)) \), \( \pi \in C^2(\text{cl}(X \times Y \times Z)) \), and \( (G0) - (G6) \), and assume \( \bar{z} < +\infty \). Suppose \( D_{x,y} b(x, y) \) has full rank for each \( (x, y) \in X \times Y \), and denote its left inverse \( b^{-1}(x, y) \).

1. If \( (x, y, z) \mapsto h(x, y, z) := f(x, z) - b_{j}(x, y) b_{i,j}(x, y) f_{i}(x, z) \) is strictly increasing and convex (respectively, concave) with respect to \( z \), then \( \Pi(u) \) is concave (respectively, convex) for all \( \mu \ll \mathcal{L}^m \) if and only if there exists \( \varepsilon \geq 0 \) such that each \( (x, y) \in X \times Y \) and \( \xi \in \mathbb{R}^n \) satisfy

\[
\pm \left\{ -b_{kj}(x, y) + b_{j}(x, y) b_{i,j}(x, y) b_{i,kj}(x, y) \right\} \xi^k \xi^j \geq \varepsilon |\xi|^2. \tag{4.7}
\]

2. In addition, \( \Pi(u) \) is uniformly concave (respectively, uniformly convex) on \( W^{1,2}(X, d\mu) \) uniformly for all \( \mu \ll \mathcal{L}^m \) if and only if \( \varepsilon > 0 \) and \( (4.7) \) holds with \( \varepsilon > 0 \).

**Example 4.13 (Inhomogeneous sensitivity of agents to prices).** Take \( \pi(x, y, z) = z - a(y), G(x, y, z) = b(x, y) - f(x, z) \), satisfying \( (G0) - (G6) \), \( G \in C^3(\text{cl}(X \times Y \times Z)) \), \( \pi \in C^2(\text{cl}(X \times Y \times Z)) \), and assume \( \bar{z} < +\infty \). Suppose \( D_{x,y} b(x, y) \) has full rank for each \( (x, y) \in X \times Y \), and \( 1 - (f_x)^{-1} b \beta b^\alpha \beta f_{\alpha,z} \neq 0 \) for all \( (x, y, z) \in X \times Y \times Z \).

1. If

\[
(x, y, z) \mapsto h(x, y, z) := a_i b_{i,j} f_{i,z} + \frac{(a_\beta b^\alpha \beta f_{\alpha,z} - 1)(b_j b^{i,j} f_{i,zz} - f_{zz})}{f_z - b \beta b^\alpha \beta f_{\alpha,z}}
\]

\[
\geq 0 \ (\leq 0),
\]
then \( \Pi(u) \) is concave (respectively, convex) for all \( \mu \ll L^m \) if and only if there exists \( \varepsilon \geq 0 \) such that each \( (x, y, z) \in X \times Y \times Z \) and \( \xi \in \mathbb{R}^n \) satisfy

\[
\pm \left\{ a_{kj} - a_l b_{i,l} b_{i,kj} \right\} + \frac{1 - a_{\beta} b_{\alpha, \beta} f_{a,z} - b_{k,j} f_{a,z} + b_{i,l} b_{i,k,j}}{1 - (f_{z}^{-1})b_{k,j} b_{a,z} f_{a,z}} \xi^k \xi^j \geq \varepsilon |\xi|^2.
\]

(4.8)

2. In addition, \( \Pi(u) \) is uniformly concave (respectively, uniformly convex) for all \( \mu \ll L^m \) if and only if \( \varepsilon > 0 \) and (4.8) holds with \( \varepsilon > 0 \).

Example 4.14 asserts the concavity of the monopolist’s maximization in the zero-sum setting, where the agent’s utilities are relatively general but the principal’s profit is extremely special. In addition, more nonquasilinear examples could be discovered by applying Lemma 4.8.

Example 4.14 (Zero sum transactions). Take \( \pi(x, y, z) = -G(x, y, z) \) satisfying (G0)–(G5) and \( \mu \ll L^m \), which means the monopolist’s profit in each transaction coincides exactly with the agent’s loss. From (2.1), since \( G \) is linear on \( G \)-segments, we know \( u \) is linear.

5 Proofs

In this section, we prove Lemma 4.8, Corollary 4.10, and Examples 4.11 and 4.13.

5.1 Proof of Lemma 4.8

For any \( G \)-segments \( (x, y, t, z, t) \) satisfying equation (4.2) and \( \pi \in C^2(\text{cl}(X \times Y \times Z)), t \in [0, 1] \mapsto \pi(x, y, t, z, t) \) is concave (respectively, uniformly concave) if and only if \( \frac{d^2}{dt^2} \pi(x, y, t, z, t) \leq 0 \) (respectively, \( \leq -\lambda \| (\dot{y}, \dot{t}, \dot{z}) \|_{\mathbb{R}_{n+1}}^2 < 0 \)) for all \( t \in [0, 1] \).

On the one hand, since \( \frac{d}{dt} \pi(x, y, t, z, t) = \pi \cdot \dot{z} \cdot \dot{y} \), taking another derivative with respect to \( t \) gives

\[
\frac{d^2}{dt^2} \pi(x, y, t, z, t) = \pi \cdot \ddot{z} \cdot \dot{y} \cdot \dot{y} + \pi \cdot \ddot{y} \cdot \dot{z} \cdot \dot{y}.
\]

(5.1)

On the other hand, taking the second derivative with respect to \( t \) on both sides of equation (4.2), which is equivalent to \( \tilde{G}_{t,x} \cdot (x, y_0, y_t(x), z_t(x)) = (x_0 D u_t, u_t(x)) \) for some fixed \( x_0 \in X_0 \), implies

\[
\tilde{G}_{t,k,j} \cdot \dot{y} \cdot \dot{y} \cdot \dot{y} + \tilde{G}_{t,k} \cdot \ddot{y} \cdot \dot{y} = 0.
\]

(5.2)

Combining equations (5.1) with (5.2) yields (4.4). For \( x \in X \), there is a \( G \)-segment with any given tangent direction through \( \tilde{y} = (y, z) \in Y \times Z \). Thus, the nonpositivity of \( \frac{d^2}{dt^2} \pi(x, y, t, z, t) \) along all \( G \)-segments \( (x, y, t, z, t) \) is equivalent to
nonpositive definiteness of the matrix \((\pi_{k,j} - \pi_{j} \overline{G}_{i,k,j})\) on \(T_{\gamma}(Y \times Z) = \mathbb{R}^{n+1}\).

In addition, the uniform concavity of \(\pi_{i,j}(x,y,z)\) along all \(G\)-segments \((x,y,z)\) is equivalent to uniform negative definiteness of \((\pi_{k,j} - \pi_{j} \overline{G}_{i,k,j})\) on \(\mathbb{R}^{n+1}\). □

5.2 Proof of Corollary 4.10

According to Lemma 4.8, for concavity we only need to show nonpositive definiteness of \((\pi_{k,j} - \pi_{j} \overline{G}_{i,k,j})\) on \(\mathbb{R}^{n+1}\), i.e., for any \(\bar{x} = (x,x_0) \in X \times X_0\), \(\bar{y} \in Y \times Z\), and \(\xi \in \mathbb{R}^{n+1}\),

\[
(\pi_{k,j}(\bar{y}) - \pi_{j}(\bar{y}) \overline{G}_{i,k,j}(\bar{x},\bar{y}) \overline{G}_{i,k,j}(\bar{x},\bar{y})) \xi^T \xi \leq 0.
\]

For any fixed \(\bar{x} = (x,x_0) \in X \times X_0\), \(\bar{y} \in Y \times Z\), \(\xi \in \mathbb{R}^{n+1}\), there exist \(\delta > 0\) and \(t \in (-\delta, \delta) \mapsto \bar{y}_t \in Y \times Z\) such that \(\bar{y}_t|_{t=0} = \bar{y}\), \(\bar{y}_t|_{t=0} = \xi\), and \(\frac{d^2}{dt^2} \overline{G}_{i,k,j}(\bar{x},\bar{y}_t)|_{t=0} = 0\). Thus,

\[
(\pi_{k,j}(\bar{y}) - \pi_{j}(\bar{y}) \overline{G}_{i,k,j}(\bar{x},\bar{y}) \overline{G}_{i,k,j}(\bar{x},\bar{y})) \xi^T \xi \leq 0.
\]

Since \(\pi\) is \(\overline{G}\)-concave, we have \(\pi(\bar{y}) = \min_{\bar{x} \in \text{cl}(X \times X_0)} \overline{G}(\bar{x},\bar{y}) - \phi(\bar{x})\) for some \(\overline{G}\)-concave function \(\phi\). Since \(\text{cl}(X \times X_0)\) is compact, for this \(\bar{y}\) there exists \(\bar{x}^* = (x^*,x_0^*) \in \text{cl}(X \times X_0)\) such that \(\pi_{i,j}(\bar{y}) = \overline{G}_{i,j}(\bar{x}^*,\bar{y})\) for each \(i = 1,2,\ldots,n+1\) and \(\pi_{k,j}(\bar{y}) \xi^T \xi \leq \overline{G}_{k,j}(\bar{x}^*,\bar{y}) \xi^T \xi \) for each \(\xi \in \mathbb{R}^{n+1}\). Combined with (5.3), this yields

\[
(\pi_{k,j}(\bar{y}) - \pi_{j}(\bar{y}) \overline{G}_{i,k,j}(\bar{x}^*,\bar{y}) \overline{G}_{i,k,j}(\bar{x}^*,\bar{y})) \xi^T \xi \leq \overline{G}_{k,j}(\bar{x}^*,\bar{y}) \xi^T \xi
\]

\[
\leq \overline{G}_{k,j}(\bar{x}^*,\bar{y}) \xi^T \xi + \overline{G}_{j}(\bar{x}^*,\bar{y}) \cdot (\bar{y}_t)|_{t=0}
\]

\[
= \frac{d^2}{dt^2} \overline{G}(\bar{x}^*,\bar{y}_t)|_{t=0}
\]

\[
= x_0^* \cdot \frac{d^2}{dt^2} G(x^*,\bar{y}_t)|_{t=0}
\]

\[
\leq 0.
\]

The last inequality comes from \(x_0^* \leq 0\) and (G3). □
5.3 Proof of Example 4.11

From Lemma 4.8, \( \Pi(u) \) is concave for all \( \mu \ll L^m \) if and only if

\[
(\pi_{\bar{i},\bar{j}} - \pi_{f}(\bar{G}_{i,k}\bar{j}))|_{x_0=-1}
\]

is nonpositive definite, and uniformly concave uniformly for all \( \mu \ll L^m \) if and only if this matrix is uniform negative definite.

In this example, we have \( \pi(x, y, z) = z - a(y), \bar{G}(x, x_0, y, z) = x_0G(x, y, z) = x_0(b(x, y) - f(z)) \). Thus,

\[
\pi_{\bar{i}j} = \begin{pmatrix} -a_{k}\bar{j} & 0 \\ 0 & 0 \end{pmatrix}, \quad \pi_{\bar{j}} = (-a_I, 1), \quad \bar{G}_{i,k}\bar{j}|_{x_0=-1} = \begin{pmatrix} -b_{i,l} & 0 \\ b_{i} & -f'(z) \end{pmatrix}.
\]

By (G4), \( f'(z) > 0 \) for all \( z \in \text{cl}(Z) \). By (G6), since \( \bar{G}_{i,k}\bar{j}|_{x_0=-1} \) has the full rank, the matrix \( (b_{i,l}) \) also has its full rank. Taking \( b_{i,l} \) as its left inverse, we have

\[
\bar{G}_{i,k}\bar{j}|_{x_0=-1} = \begin{pmatrix} -b_{i,l} & 0 \\ b_{i} & f'(z) \end{pmatrix}, \quad \bar{G}_{i,k}\bar{j}|_{x_0=-1} = \begin{pmatrix} -b_{i,l} & 0 \\ b_{i} & f'(z) \end{pmatrix}.
\]

Therefore,

\[
(\pi_{\bar{i}j} - \pi_{f}(\bar{G}_{i,k}\bar{j}))|_{x_0=-1}
\]

\[
= \begin{pmatrix} -a_{k}\bar{j} & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} -a_I b_{i,l} + b_{i} f'(z) b_{i,k}\bar{j} - b_{k}\bar{j} & 0 \\ 0 & 0 \end{pmatrix}
\]

\[
= -\begin{pmatrix} a_{k}\bar{j} + (-a_I + b_{i}\frac{f'(z)}{f'(z)}) b_{i,k}\bar{j} - b_{k}\bar{j} & 0 \\ 0 & 0 \end{pmatrix}
\]

Since (G4) and \( f \) is convex, we have \( f'(z) > 0 \) and \( f''(z) \geq 0 \) for all \( z \in \text{cl}(Z) \). Thus, \( \pi_{\bar{i}j} - \pi_{f}(\bar{G}_{i,k}\bar{j}) \) is nonpositive definite if and only if

\[
ak_{kj} + (-a_I + \frac{b_{i}}{f'(z)}) b_{i,k}\bar{j} - \frac{b_{k}\bar{j}}{f'(z)}
\]

is nonnegative definite; i.e., there exists \( \varepsilon \geq 0 \) such that each \( (x, y, z) \in X \times Y \times Z \) and \( \xi \in \mathbb{R}^n \) satisfy

\[
\left\{ a_{kj}(y) - \frac{b_{k}\bar{j}(x, y)}{f'(z)} + \left( \frac{b_{i}(x, y)}{f'(z)} - a_I(y) \right) b_{i,k}\bar{j}(x, y) \right\} \xi_k \xi_j \geq \varepsilon |\xi|^2.
\]

In addition, \( \pi_{\bar{i}j} - \pi_{f}(\bar{G}_{i,k}\bar{j}) \) is uniform negative definite if and only if \( f'' > 0 \) and \( \varepsilon > 0 \), which is equivalent to \( \Pi(u) \) being uniformly concave independently of \( \mu \ll L^m \). Similarly, one can show equivalent conditions for \( \Pi(u) \) being convex or uniformly convex. \( \square \)
5.4 Proof of Example 4.13

Similar to the proof of Example 4.11, \( \Pi(u) \) is concave for all \( \mu \ll L^m \) if and only if \( (\pi_{ikj} - \pi_{ij}G_{i,k}G_{j,k}) \) is nonpositive definite, and uniformly concave uniformly for all \( \mu \ll L^m \) if and only if this tensor is uniform negative definite.

Since \( D_{x,y}b(x, y) \) has full rank for each \( (x, y) \in X \times Y \) and

\[
1 - (f_z)^{-1}b_{\beta}b_\alpha \beta f_{\alpha,z} \neq 0 \quad \text{for all} \quad (x, y, z) \in X \times Y \times Z,
\]

for \( \pi(x, y, z) = z - a(y) \) and \( \tilde{G}(x, x_0, y, z) = x_0(b(x, y) - f(x, z)) \), we have

\[
- (\pi_{ikj} - \pi_{ij}G_{i,k}G_{j,k}) = \begin{pmatrix}
  a_{kj} - a_1b^{i,l}b_{i,kj} + \frac{(a_\beta b_\alpha \beta f_{\alpha,z} - 1)(b_{k,j} - b_{i,l}b^{i,l}b_{i,kj})}{f_z - b_\beta b_\alpha \beta f_{\alpha,z}} & 0 \\
  0 & h(x, y, z)
\end{pmatrix}
\]

where

\[
h(x, y, z) = a_1b^{i,l}f_{i,z} + \frac{(a_\beta b_\alpha \beta f_{\alpha,z} - 1)(b_{k,j} - b_{i,l}b^{i,l}b_{i,kj})}{f_z - b_\beta b_\alpha \beta f_{\alpha,z}}.
\]

Since \( h(x, y, z) \geq 0 \), then \( (\pi_{ikj} - \pi_{ij}G_{i,k}G_{j,k}) \) is nonpositive definite if and only if there exists \( \varepsilon \geq 0 \) such that each \( (x, y, z) \in X \times Y \times Z \) and \( \xi \in \mathbb{R}^n \) satisfy

\[
\left\{ a_{kj} - a_1b^{i,l}b_{i,kj} + \frac{(a_\beta b_\alpha \beta f_{\alpha,z} - 1)(b_{k,j} - b_{i,l}b^{i,l}b_{i,kj})}{f_z - b_\beta b_\alpha \beta f_{\alpha,z}} \right\} \xi^{k} \xi^{j} \geq \varepsilon |\xi|^2.
\]

In addition, \( \Pi(u) \) is uniformly concave on \( W^{1,2}(X, d\mu) \) uniformly for all \( \mu \ll L^m \) if and only if \( h > 0 \) and \( \varepsilon > 0 \).

6 A Fourth-Order Differential Re-expression of (G3)

In our convexity argument, hypothesis (G3) plays a crucial role. In this final section, we localize this hypothesis using differential calculus. Inspired by and strongly connected with Trudinger’s theory of generalized prescribed Jacobian equations, this form is analogous to the nonnegative cross-curvature condition (B3) of Figalli-Kim-McCann [9], a fourth-order condition in the spirit of the Ma-Trudinger-Wang [19]. For another formulation; see, e.g., [13].

Apart from the assumptions of Section 2, we shall need the nondegeneracy condition assumed in Section 4:

(G6) \( G \in C^2(\text{cl}(X \times Y \times Z)) \), and \( D_{x,y}c(\tilde{G})(x, -1, y, z) \) has full rank for each \( (x, y, z) \in \text{cl}(X \times Y \times Z) \).

For this section only, we assume the dimensions of spaces \( X \) and \( Y \) are equal, i.e., \( m = n \), so that the matrix mentioned in (G6) becomes square. We shall also need to extend the twist and convex range hypotheses (G1) and (G2) to the function \( H \) in place of \( G \). This is equivalent to assuming:

(G7) For each \( (y, z) \in \text{cl}(Y \times Z) \) the map \( x \in X \mapsto \frac{G_x}{G_z}(\cdot, y, z) \) is one-to-one.
(G8) The range \( X_{(y,z)} := \frac{G_x}{G_z}(X, y, z) \subset \mathbb{R}^n \) of this map is convex.

We can now state:

**Theorem 6.1.** Assume (G0)–(G2) and (G4)–(G8). If, in addition, \( G \in C^4(\text{cl}(X \times Y \times Z)) \), then the following statements are equivalent:

(i) (G3).

(ii) For any given \( x_0, x_1 \in X \) and any curve \((y_t, z_t) \in \text{cl}(Y \times Z)\) connecting \((y_0, z_0)\) and \((y_1, z_1)\), we have

\[
\frac{\partial^2}{\partial s^2}\left( \frac{1}{G_z(x_s, y_t, z_t)} \frac{\partial^2}{\partial t^2} G(x_s, y_t, z_t) \right) \bigg|_{t=t_0} \leq 0,
\]

whenever \( s \in [0, 1] \mapsto \frac{G_z}{G_x}(x_s, y_{t_0}, z_{t_0}) \) forms an affinely parametrized line segment for some \( t_0 \in [0, 1] \).

(iii) For any given curve \( x_s \in X \) connecting \( x_0 \) and \( x_1 \), any \((y_0, z_0), (y_1, z_1) \in \text{cl}(Y \times Z)\), we have

\[
\frac{\partial^2}{\partial s^2}\left( \frac{1}{G_z(x_s, y_t, z_t)} \frac{\partial^2}{\partial t^2} G(x_s, y_t, z_t) \right) \bigg|_{s=s_0} \leq 0,
\]

whenever \( t \in [0, 1] \mapsto (G_x, G)(x_{s_0}, y_t, z_t) \) forms an affinely parametrized line segment for some \( s_0 \in [0, 1] \).

Its proof and embellishments are relegated to Appendix A.

**Appendix A**  Proof and Variations on Theorem 6.1

For any vectors \( p, w \in \mathbb{R}^n \), we denote \( p \parallel w \) if \( p \) and \( w \) are parallel.

**Proof of Theorem 6.1.**

(i) \( \Rightarrow \) (ii). Suppose for some \( t_0 \in [0, 1], s \in [0, 1] \mapsto \frac{G_z}{G_x}(x_s, y_{t_0}, z_{t_0}) \) forms an affinely parametrized line segment.

For any fixed \( s_0 \in [0, 1] \), consider \( x_{s_0} \in X \). There is a \( G \)-segment \((x_{s_0}, y_{t_0}^{s_0}, z_{t_0}^{s_0})\) passing through \((x_{s_0}, y_{t_0}, z_{t_0})\) at \( t = t_0 \) with the same tangent vector as \((x_{s_0}, y_{t_0}^{s_0}, z_{t_0}^{s_0})\) at \( t = t_0 \); i.e., there exists another curve \((y_{t_0}^{s_0}, z_{t_0}^{s_0}) \in \text{cl}(Y \times Z)\) such that \((y_{t_0}^{s_0}, z_{t_0}^{s_0}) \big|_{t=t_0} = (y_{t_0}, z_{t_0}) \big|_{t=t_0} \parallel (\dot{y}_{t_0}, \dot{z}_{t_0}) \big|_{t=t_0}, \) and

\[
(G_x, G)(x_{s_0}, y_{t_0}^{s_0}, z_{t_0}^{s_0}) = (1-t)(G_x, G)(x_{s_0}, y_{t_0}^{s_0}, z_{t_0}^{s_0}) + t(G_x, G)(x_{s_0}, y_{t_0}^{s_0}, z_{t_0}^{s_0}).
\]

Computing the fourth mixed derivative yields

\[
\frac{\partial^2}{\partial s^2}\left( \frac{1}{G_z(x_s, y_t, z_t)} \frac{\partial^2}{\partial t^2} G(x_s, y_t, z_t) \right)
= \frac{\partial^2}{\partial s^2}\left( \frac{1}{G_z} \right) \frac{\partial^2}{\partial t^2} G + 2 \frac{\partial}{\partial s} \frac{\partial}{\partial s} \frac{\partial^2}{\partial t^2} G + \frac{1}{G_z} \frac{\partial^4}{\partial s^2 \partial t^2} G
= \left[ -(G_z)^{-2}G_i \dddot{x}_i^j - (G_z)^{-2}G_{ij, z} \dot{x}_i^j \dot{x}_i^j + 2(G_z)^{-3}G_{ij, z, x_i} \dot{x}_i^j \dot{x}_i^j \right]
\cdot [G_{kl} \dot{y}_t^k + G_z \dddot{z}_t + G_{kl} \dot{y}_t^k \dddot{z}_t + 2G_{kl} \dddot{z}_t \dot{y}_t^k + G_{kl} \dddot{z}_t \dot{z}_t^k + G_{kl} \dddot{z}_t \dddot{z}_t^k] +
\]
\[
\begin{aligned}
+ 2 \left[ -(G_x)^{-2} G_{i,z} x^i_s \right] & \cdot \left[ G_{j,k} x^j_s \dot{y}^k_t + G_{j,z} x^j_s \dot{z}_t + G_{j,kl} x^j_s \dot{y}^k_t \dot{y}^l_t \\
+ 2 G_{j,kz} x^j_s \dot{y}^k_t \dot{z}_t + G_{j,zz} x^j_s (\dot{z}_t)^2 \right] \\
+ (G_z)^{-1} \left[ G_{i,k} x^i_s \dot{y}^k_t + G_{ij,k} x^i_s x^j_s \dot{y}^k_t + G_{i,z} x^i_s \dot{z}_t + G_{ij,z} x^i_s x^j_s \dot{z}_t \\
+ G_{i,kl} x^i_s \dot{y}^k_t \dot{y}^l_t + G_{ij,kl} x^i_s x^j_s \dot{y}^k_t \dot{y}^l_t + 2 G_{i,kz} x^i_s \dot{y}^k_t \dot{z}_t \\
+ 2 G_{i,kz} x^i_s \dot{z}_t + G_{i,zz} x^i_s (\dot{z}_t)^2 \right] \\
= \left[ ((G_x)^{-1} G_{i,k} - (G_x)^{-2} G_{i,z} G_{i,k}) \dot{x}^i_s \right] + \left[ ((G_z)^{-1} G_{ij,k} - (G_z)^{-2} G_{ij,z} G_{i,k}) \right] \dot{y}^k_t \\
+ \left[ (G_x)^{-1} G_{i,kl} - (G_x)^{-2} G_{i,kl} \right] \dot{x}^i_s \dot{y}^k_t \dot{y}^l_t \\
+ \left[ (G_z)^{-1} G_{ij,k} - (G_z)^{-2} G_{ij,z} G_{i,k} \right] \dot{x}^i_s \dot{x}^j_s \dot{y}^k_t \dot{y}^l_t \\
+ \left[ 2 (G_x)^{-1} G_{i,k} - (G_x)^{-2} G_{i,k} \right] \dot{x}^i_s \dot{y}^k_t \dot{z}_t \\
+ \left[ 2 (G_x)^{-1} G_{i,k} - (G_x)^{-2} G_{i,k} \right] \dot{x}^i_s \dot{y}^k_t \dot{z}_t \\
+ \left[ (G_z)^{-1} G_{ij,k} - (G_z)^{-2} G_{ij,z} G_{i,k} \right] \dot{x}^i_s \dot{x}^j_s (\dot{z}_t)^2 \\
+ \left[ (G_z)^{-1} G_{ij,k} - (G_z)^{-2} G_{ij,z} G_{i,k} \right] \dot{x}^i_s \dot{x}^j_s \dot{z}_t \\
+ 2 (G_x)^{-1} G_{i,kl} \dot{x}^i_s \dot{x}^j_s \dot{y}^k_t \dot{y}^l_t .
\end{aligned}
\]

The coefficient of \( \dot{y}^k_t \) vanishes when this expression is evaluated at \( t = t_0 \), due to the assumption that \( s \in [0, 1] \mapsto \frac{G_x}{G_z} (x_s, y_{s_0}, z_{s_0}) \) forms an affinely parametrized line segment, which implies

\[
0 = \frac{\partial^2}{\partial s^2} G_z (x_s, y_{s_0}, z_{s_0})
= \left[ ((G_x)^{-1} G_{i,y} - (G_x)^{-2} G_{i,z} G_{i,y}) \right] \dot{x}^i_s \\
+ \left[ (G_x)^{-1} G_{ij,y} - (G_x)^{-2} G_{ij,z} g_{ij,z} - 2 (G_x)^{-2} G_{ij,z} G_{j,y} \\
+ 2 (G_x)^{-3} G_{ij,y} G_{i,j,z} \right] \dot{x}^i_s \dot{x}^j_s
\]
for all \( s \in [0, 1] \).

Since \( (\dot{y}^0_t, \dot{z}^0_t)|_{t=t_0} \parallel (\hat{y}^0_t, \hat{z}^0_t)|_{t=t_0} \), there exists some constant \( C_1 > 0 \) such that \( (\dot{y}^0_t, \dot{z}^0_t)|_{t=t_0} = C_1 (y^0_t, z^0_t)|_{t=t_0} \). Moreover, since \( (y_t, z_t)|_{t=t_0} = (y^0_t, z^0_t)|_{t=t_0} \), we have

\[
\frac{\partial^2}{\partial s^2} \left( \frac{1}{G_z (x_s, y_{s_0}, z_{s_0})} \frac{\partial^2}{\partial t^2} G(x_s, y_t, z_t) \right)|_{t=t_0}
= C_1^2 \frac{\partial^2}{\partial s^2} \left( \frac{1}{G_z (x_s, y^0_t, z^0_t)} \frac{\partial^2}{\partial t^2} G(x_s, y^0_t, z^0_t) \right)|_{t=t_0} .
\]
Define \( g(s) : = \frac{\partial^2}{\partial t^2} \bigg|_{t = t_0} G(x_s, y_t, z_t) \) for \( s \in [0, 1] \). Since \((x_{s_0}, y_{t_0}, z_{t_0})\) is a \( G \)-segment, by (G3) we have \( g(s) \geq 0 \) for all \( s \in [0, 1] \). By definition of \((y_{t_0}^{s_0}, z_{t_0}^{s_0})\), it is clear that \( g(s_0) = 0 \). The first- and second-order conditions for an interior minimum then give \( g'(s_0) = 0 \leq g''(s_0) \) (in fact, \( g'(s_0) = 0 \) also follows directly from the definition of a \( G \)-segment).

By the assumption (G4), we have \( G_z < 0 \), thus,

\[
\frac{\partial^2}{\partial s^2} \left( \frac{1}{G_z(x_s, y_t, z_t)} \frac{\partial^2}{\partial t^2} G(x_s, y_t, z_t) \right) \bigg|_{(s,t) = (s_0, t_0)} = C_1^2 \frac{\partial^2}{\partial s^2} \left( \frac{1}{G_z(x_s, y_t^{s_0}, z_t^{s_0})} \frac{\partial^2}{\partial t^2} G(x_s, y_t^{s_0}, z_t^{s_0}) \right) \bigg|_{(s,t) = (s_0, t_0)} \\
+ 2C_1^2 \frac{\partial}{\partial s} \bigg|_{(s,t) = (s_0, t_0)} \left( \frac{1}{G_z(x_s, y_t^{s_0}, z_t^{s_0})} g(s_0) \right) + \frac{C_1^2}{G_z(x_s, y_t^{s_0}, z_t^{s_0})} g''(s_0) \leq 0.
\]

(ii) \( \Rightarrow \) (i). For any fixed \( x_0 \in X \) and \( G \)-segment \((x_0, y_t, z_t)\), we need to show \( \frac{\partial^2}{\partial t^2} G(x_1, y_t, z_t) \geq 0 \) for all \( t \in [0, 1] \) and \( x_1 \in X \).

For any fixed \( t_0 \in [0, 1] \) and \( x_1 \in X \), define \( x_s \) as the solution \( \widehat{x} \) to the equation

(A.1) \( \frac{G_y}{G_z} (\widehat{x}, y_{t_0}, z_{t_0}) = (1 - s) \frac{G_y}{G_z} (x_0, y_{t_0}, z_{t_0}) + s \frac{G_y}{G_z} (x_1, y_{t_0}, z_{t_0}) \).

By (G7) and (G8), \( x_s \) is uniquely determined for each \( s \in (0, 1) \). In addition, \( x_0 \) and \( x_1 \) satisfy the above equation for \( s = 0 \) and \( s = 1 \), respectively. Define

\[
g(s) : = \frac{1}{G_z(x_s, y_t, z_t)} \frac{\partial^2}{\partial t^2} G(x_s, y_t, z_t) \bigg|_{t = t_0} \text{ for } s \in [0, 1].
\]

Then \( g(0) = 0 = g'(0) \) from the two conditions defining a \( G \)-segment.

In our setting, \( s \in [0, 1] \mapsto \frac{G_y}{G_z} (x_s, y_{t_0}, z_{t_0}) \) forms an affinely parametrized line segment; thus

\[
0 \geq \frac{\partial^2}{\partial s^2} \left( \frac{1}{G_z(x_s, y_t, z_t)} \frac{\partial^2}{\partial t^2} G(x_s, y_t, z_t) \right) \bigg|_{t = t_0} = g''(s) \text{ for all } s \in [0, 1]
\]

by hypothesis (ii).

Hence \( g \) is concave in \([0, 1]\) and \( g(0) = 0 \) is a critical point, thus \( g(1) \leq 0 \).

Since \( G_z < 0 \), this implies

\[
\frac{\partial^2}{\partial t^2} G(x_1, y_t, z_t) \geq 0
\]

for any \( t_0 \in [0, 1] \) and \( x_1 \in X \), as desired.
(i) \(\Rightarrow\) (iii). For any fixed \(s_0 \in [0, 1]\), suppose \(t \in [0, 1] \mapsto (G_x, G)(x_{s_0}, y_t, z_t)\) forms an affinely parametrized line segment. For any fixed \(t_0 \in [0, 1]\), define

\[
g(s) := \left( \frac{1}{G_z(x_s, y_t, z_t)} \frac{\partial^2}{\partial t^2} G(x_s, y_t, z_t) \right)_{t=t_0} \quad \text{for all } s \in [0, 1].
\]

By (G3)–(G4), we know \(g(s) \leq 0\) for all \(s \in [0, 1]\). By the definition of \((y_t, z_t)\), we have \(g(s_0) = g'(s_0) = 0\). Thus \(g''(s_0) \leq 0\).

(iii) \(\Rightarrow\) (i). For any fixed \(x_0 \in X\), suppose \((x_0, y_0^t, z_0^t)\) is a \(G\)-segment; then we need to show \(\frac{\partial^2}{\partial t^2} G(x_1, y_0^t, z_0^t) \geq 0\) for all \(t \in [0, 1]\), \(x_1 \in X\).

For any fixed \(t_0 \in [0, 1]\), \(x_1 \in X\), define \(x_s\) as the solution \(\hat{x}\) of equation

\[
\frac{G_y}{G_z}(\hat{x}, y_0^0, z_0^0) = (1 - s)\frac{G_y}{G_z}(x_0, y_0^0, z_0^0) + s\frac{G_y}{G_z}(x_1, y_0^0, z_0^0).
\]

By (G7) and (G8), \(x_0\) is uniquely determined for each \(s \in (0, 1)\). In addition, \(x_0\) and \(x_1\) satisfy the above equation for \(s = 0\) and \(s = 1\), respectively. Define

\[
g(s) := \left( \frac{1}{G_z(x_s, y_0^t, z_0^t)} \frac{\partial^2}{\partial t^2} G(x_s, y_0^t, z_0^t) \right)_{t=t_0} \quad \text{for } s \in [0, 1].
\]

Then \(g(0) = g'(0) = 0\) by the two conditions defining a \(G\)-segment.

For any fixed \(s_0 \in [0, 1]\), there is a \(G\)-segment \((x_{s_0}, y_{s_0}^t, z_{s_0}^t)\) passing through \((x_0, y_0^0, z_0^0)\) at \(t = t_0\) with the same tangent vector as \((x_{s_0}, y_{s_0}^0, z_{s_0}^0)\) at \(t = t_0\); i.e., there exists another curve \((y_{s_0}^t, z_{s_0}^t) \in \text{cl}(Y \times Z)\) and some constant \(C_2 > 0\) such that

\[
(y_{s_0}^t, z_{s_0}^t)_{t=t_0} = (y_0^0, z_0^0)_{t=t_0}, \quad (y_{s_0}^t, z_{s_0}^t)_{t=t_0} = \frac{1}{C_2}(y_0^0, z_0^0)_{t=t_0},
\]

and

\[
(G_x, G)(x_{s_0}, y_{s_0}^t, z_{s_0}^t) = (1 - t)(G_x, G)(x_{s_0}, y_0^0, z_0^0) + t(G_x, G)(x_{s_0}, y_{s_0}^0, z_{s_0}^0).
\]

Computing the mixed fourth derivative yields

\[
\frac{\partial^2}{\partial s^2} \left( \frac{1}{G_z(x_s, y_0^t, z_0^t)} \frac{\partial^2}{\partial t^2} G(x_s, y_0^t, z_0^t) \right)_{(s,t)=(s_0,t_0)} = C_2 \frac{\partial^2}{\partial s^2} \left( \frac{1}{G_z(x_s, y_{s_0}^t, z_{s_0}^t)} \frac{\partial^2}{\partial t^2} G(x_s, y_{s_0}^t, z_{s_0}^t) \right)_{(s,t)=(t_0,t_0)},
\]

where the equality is derived from the condition that \(s \in [0, 1] \mapsto \frac{G_x}{G_z}(x_s, y_{s_0}^t, z_{s_0}^t)\) forms an affinely parametrized line segment, \((y_{s_0}^0, z_{s_0}^0)_{t=t_0} = (y_0^0, z_0^0)_{t=t_0}\) and \((y_{s_0}^0, z_{s_0}^0)_{t=t_0} = C_2(y_{s_0}^0, z_{s_0}^0)_{t=t_0}\). Moreover, the latter expression is nonpositive by assumption (iii).
Thus \( g''(s_0) \leq 0 \) for all \( s_0 \in [0, 1] \). Since \( g \) is concave in \([0, 1]\) and \( g(0) = 0 \) is a critical point, we have \( g(1) \leq 0 \). Thus \( G_z < 0 \) implies

\[
\frac{\partial^2}{\partial t^2} \bigg|_{t=t_0} G(x_1, y_1^0, z_1^0) \geq 0 \quad \text{for all } t_0 \in [0, 1] \text{ and } x_1 \in X,
\]

as desired. \( \square \)

For strict concavity of the profit functional, one might need a strict version of hypothesis (G3):

\((G3)_s\). For each \( x, x_0 \in X \) and \( x \neq x_0 \), assume \( t \in [0, 1] \mapsto G(x, y_t, z_t) \) is strictly convex along all \( G \)-segments \((x_0, y_t, z_t)\) defined in (2.1).

**Remark A.1.** Strict inequality in (ii) (or (iii)) implies \((G3)_s\), but the reverse is not necessarily true; i.e., \((G3)_s\) is intermediate in strength between (G3) and the strict inequality version of (ii) (or (iii)). Besides, strict inequality versions of (ii) and (iii) are equivalent, and denoted by \((G3)_u\).

Note inequality \([A.2]\) below and its strict and uniform versions \((G3)_s\) and \((G3)_u\) precisely generalize the analogous hypotheses (B3), (B3)_s, and (B3)_u from the quasilinear case \([9]\).

**Proof.** We only show that the strict inequality of (ii) implies that of (iii) here, since the other direction is similar.

For any fixed \( s_0 \in [0, 1] \), suppose \( t \in [0, 1] \mapsto (G_x, G)(x_{s_0}, y_t, z_t) \) forms an affinely parametrized line segment. For any fixed \( t_0 \in [0, 1] \), define \( x_{s_0}^0 \) as a solution to the equation

\[
\frac{G_y}{G_z}(x_{s_0}^0, y_{t_0}, z_{t_0}) = (1 - s) \frac{G_y}{G_z}(x_0, y_{t_0}, z_{t_0}) + s \frac{G_y}{G_z}(x_1, y_{t_0}, z_{t_0})
\]

with initial conditions \( x_{s_0}^0 |_{s=s_0} = x_{s_0} \) and \( x_{s_0}^0 |_{s=s_0} = C_1 \hat{x}_t |_{s=s_0} \) for some constant \( C_1 > 0 \). Thus, by strict inequality of (ii), we have

\[
0 > \frac{\partial^2}{\partial s^2} \left( \frac{1}{G_z(x_{s_0}^0, y_t, z_t)} \frac{\partial^2}{\partial t^2} G(x_{s_0}^0, y_t, z_t) \right) \bigg|_{(s,t)=(s_0,t_0)}
= \frac{\partial^2}{\partial s^2} \left( \frac{1}{G_z(x_{s_0}^0, y_t, z_t)} \frac{\partial^2}{\partial t^2} G(x_{s_0}^0, y_t, z_t) \right) \bigg|_{(s,t)=(s_0,t_0)}
+ \frac{\partial}{\partial s} \left( \frac{1}{G_z(x_{s_0}^0, y_t, z_t)} \frac{\partial^3}{\partial s \partial t^2} G(x_{s_0}^0, y_t, z_t) \right) \bigg|_{(s,t)=(s_0,t_0)}
+ \frac{1}{G_z(x_{s_0}^0, y_t, z_t)} \frac{\partial^4}{\partial s^2 \partial t^2} G(x_{s_0}^0, y_t, z_t) \bigg|_{(s,t)=(s_0,t_0)} = \]
Besides, since connecting each fixed principal’s maximization problem, not only for the specific example as above, 

\[ - \frac{G_{x,z}(x_{t_0}^f, y_t, z_t)}{G_z^2(x_{t_0}^f, y_t, z_t)} \frac{\partial^2}{\partial t^2} G_x(x_{t_0}^f, y_t, z_t)(\dot{x}_t^0)^2 \bigg|_{(s,t) = (s_0, t_0)} \\
+ \frac{1}{G_z(x_{t_0}^f, y_t, z_t)} \frac{\partial^2}{\partial t^2} G_{xx}(x_{t_0}^f, y_t, z_t)(\dot{x}_t^0)^2 \bigg|_{(s,t) = (s_0, t_0)} \\
= C_1^2 \left[ - \frac{G_{x,z}(x_s, y_t, z_t)}{G_z^2(x_s, y_t, z_t)} \frac{\partial^2}{\partial t^2} G_x(x_s, y_t, z_t)(\dot{x}_s)^2 \bigg|_{(s,t) = (s_0, t_0)} \\
+ \frac{1}{G_z(x_s, y_t, z_t)} \frac{\partial^2}{\partial t^2} G_{xx}(x_s, y_t, z_t)(\dot{x}_s)^2 \bigg|_{(s,t) = (s_0, t_0)} \right] \\
= C_1^2 \frac{\partial^2}{\partial s^2} \left( \frac{1}{G_z(x_s, y_t, z_t)} \frac{\partial^2}{\partial t^2} G(x_s, y_t, z_t) \right) \bigg|_{(s,t) = (s_0, t_0)}. \]

Here we use the initial condition \( x_{t_0}^f|_{s=s_0} = x_{t_0} \) and \( \dot{x}_{t_0}^f|_{s=s_0} = C_1 \dot{x}_s|_{s=s_0} \). Besides, since \((x_{s_0}, y_t, z_t)\) forms a \(G\)-segment, we have

\[ \frac{\partial^2}{\partial t^2} G(x_{t_0}^f, y_t, z_t) \bigg|_{(s,t) = (s_0, t_0)} = \frac{\partial^2}{\partial t^2} G(x_s, y_t, z_t) \bigg|_{(s,t) = (s_0, t_0)} = 0, \]

and

\[ \frac{\partial^2}{\partial t^2} G_x(x_{t_0}^f, y_t, z_t) \bigg|_{(s,t) = (s_0, t_0)} = \frac{\partial^2}{\partial t^2} G_x(x_s, y_t, z_t) \bigg|_{(s,t) = (s_0, t_0)} = 0. \]

From the above inequality and \( C_1 > 0 \), one has

\[ \frac{\partial^2}{\partial s^2} \left( \frac{1}{G_z(x_s, y_t, z_t)} \frac{\partial^2}{\partial t^2} G(x_s, y_t, z_t) \right) \bigg|_{(s,t) = (s_0, t_0)} < 0, \]

whenever \( \dot{x}_s|_{s=s_0} \) and \( (y_t, \dot{z}_t)|_{t=t_0} \) are nonzero. Since this inequality holds for each fixed \( t_0 \in [0,1] \), the strict version of (iii) is proved.

**Corollary A.2.** Assume (G0)–(G2), (G4)–(G8), and \( G \in C^4(\text{cl}(X \times Y \times Z)) \). Then (G3) is equivalent to the following statement: For any given curve \( x_s \in X \) connecting \( x_0 \) and \( x_1 \), and any curve \( (y_t, z_t) \in \text{cl}(Y \times Z) \) connecting \( (y_0, z_0) \) and \( (y_1, z_1) \), we have

\[ (A.2) \quad \frac{\partial^2}{\partial s^2} \left( \frac{G_z(x_s, y_t, z_t)}{G_z^2(x_s, y_t, z_t)} \frac{\partial^2}{\partial t^2} G(x_s, y_t, z_t) \right) \bigg|_{(s,t) = (s_0, t_0)} \leq 0, \]

whenever either of the two curves \( t \in [0,1] \mapsto (G_{x_s}, G)(x_{s_0}, y_t, z_t) \) and \( s \in [0,1] \mapsto \frac{G_z}{G_z^2}(x_s, y_{t_0}, z_{t_0}) \) forms an affinely parametrized line segment.

**Appendix B** Improved Concavity of Principal’s Objective in the Absence of Direct Dependence on Agents’ Types

In this section, we reveal a necessary and sufficient condition for the concavity of the principal’s maximization problem, not only for the specific example as above,
but for many other private-value circumstances, where the principal’s utility only directly depends on the products sold and their selling prices, but not the buyer’s type.

**Proposition B.1** (Concavity of the principal’s objective when her payoff is independent of agents’ types). Suppose $G \in C^3(\text{cl}(X \times Y \times Z))$ satisfies (G0)–(G6), $\pi \in C^2(\text{cl}(Y \times Z))$, and assume there exists a set $J \subset \text{cl}(X)$ such that for each $\bar{y} \in Y \times Z$, $0 \in (\pi_{\bar{y}} + G_{\bar{y}})(\text{cl}(J), \bar{y})$. Then the following statements are equivalent:

(i) local $G^*$-concavity of $\pi$: i.e., $\pi_{\bar{y}, \bar{y}}(x, \bar{y}) + G_{\bar{y}}(x, \bar{y})$ is nonpositive definite whenever $(x, \bar{y}) \in \text{cl}(J) \times Y \times Z$ satisfies $\pi_{\bar{y}}(\bar{y}) + G_{\bar{y}}(x, \bar{y}) = 0$;

(ii) $\Pi$ is concave on $\mathcal{U}$ for all $\mu \ll L^m$.

**Remark B.2.** If we choose $J = \text{cl}(X)$, then the hypothesis of Proposition B.1 becomes: for each $\bar{y} \in Y \times Z$, there exists $x \in \text{cl}(X)$ such that $(\pi_{\bar{y}} + G_{\bar{y}})(x, \bar{y}) = 0$. One of its economic interpretations is that for each product-price type, there exists a customer type such that his marginal disutility, the gradient with respect to product type (e.g., quality, quantity, etc.) and price type, coincides with the marginal utility of the monopolist.

**Proof of Proposition B.1.**

(i) $\Rightarrow$ (ii). Similar to the proof of Corollary 4.10 we only need to show non-positive definiteness of $(\pi_{k,j} - \pi_T G_{l,k,j}^{I,I} \bar{G}_{I,j,l}^{I,I})$, i.e., for any $\bar{x} = (x, x_0) \in X \times X_0$, $\bar{y} \in Y \times Z$, and $\xi \in \mathbb{R}^{n+1}$,

$$
(\pi_{k,j}(\bar{y}) - \pi_T(\bar{y})) G_{l,k,j}^{I,I} (\bar{x}, \bar{y}) \bar{G}_{I,j,l}(\bar{x}, \bar{y})) \xi^\top \xi \leq 0.
$$

For any fixed $\bar{x} = (x, x_0) \in X \times X_0$, $\bar{y} \in Y \times Z$, $\xi \in \mathbb{R}^{n+1}$, there exist $\delta > 0$ and a curve $t \in (-\delta, \delta) \mapsto \bar{y}_t \in Y \times Z$ such that $\bar{y}_t|_{t=0} = \bar{y}$, $\bar{y}_t|_{t=0} = \xi$, and
d

$$
\frac{d^2}{dt^2} \bar{G}_{I,j,l}^{I,I}(\bar{x}, \bar{y}_t) = \bar{G}_{I,k,j}^{I,I}(\bar{x}, \bar{y}_t) \xi^\top \xi \leq 0.
$$

For this $\bar{y}$, since $0 \in (\pi_{\bar{y}} + G_{\bar{y}})(\text{cl}(J), \bar{y})$, there exists $x^* \in \text{cl}(J)$ such that $(\pi_{\bar{y}} + G_{\bar{y}})(x^*, \bar{y}) = 0$. By property (i), one has $(\pi_{\bar{y}, \bar{y}}(\bar{y}) + G_{\bar{y}}(x^*, \bar{y})) \xi^\top \xi \leq 0$. Let $\bar{x}^* = (x^*, -1)$, then

$$
\pi_{k,j}(\bar{y}) = \bar{G}_{j}(\bar{x}^*, \bar{y}) \quad \text{and} \quad \pi_{k,j}(\bar{y}) \xi^\top \xi \leq \bar{G}_{k,j}(\bar{x}^*, \bar{y}) \xi^\top \xi
$$

for each $\bar{l} = 1, 2, \ldots, n$. Thus, combining (B.1) and (G3), we have

$$
(\pi_{k,j}(\bar{y}) - \pi_T(\bar{y})) G_{l,k,j}^{I,I} (\bar{x}, \bar{y}) \bar{G}_{I,j,l}(\bar{x}, \bar{y})) \xi^\top \xi \leq (\bar{G}_{k,j}(\bar{x}^*, \bar{y}) - \bar{G}_{j}(\bar{x}^*, \bar{y})) G_{l,k,j}^{I,I} (\bar{x}, \bar{y}) \bar{G}_{I,j,l}(\bar{x}, \bar{y})) \xi^\top \xi
$$

$$
= \bar{G}_{k,j}(\bar{x}^*, \bar{y}) \xi^\top \xi + \bar{G}_{j}(\bar{x}^*, \bar{y}) \xi^\top \xi.$$
we would like to show \( \pi_j(y) + G_j(x, y) = 0 \), we would like to show \( \pi_j(y) + G_j(x, y) = 0 \), \( \pi_j(y) = 0 \) for any \( \xi \in \mathbb{R}^{n+1} \). Let \( \bar{x} = (x, -1) \); there exist \( \delta > 0 \) and a curve \( t \in (-\delta, \delta) \mapsto y_t \in Y \times Z \) such that \( y_{t=0} = \bar{y}, y_{t=0} = \bar{y} \), and \( \frac{d^2}{dt^2} G_j(x, y_t) = 0 \). Thus, equation (B.1) holds.

Since \( \Pi \) is concave, by Theorem 4.2 and Lemma 4.8 as well as equation (B.1), we have

\[
0 \geq \left( \pi_j(y) - G_j(x, y) \right) \frac{d}{dt} G^j(x, y) \frac{d^2}{dt^2} G_j(x, y) + \frac{d^2}{dt^2} G_j(x, y)
\]

which completes the proof. \( \square \)

The following remark provides an equivalent condition for uniform concavity of the principal’s maximization problem. Its proof is very similar to that of the above proposition.

**Remark B.3.** In addition to the hypotheses of Proposition B.1, when \( \bar{z} = +\infty \) assume the homeomorphisms of (G1) are uniformly bi-Lipschitz. Then the following statements are equivalent:

(iii) \( \pi_j(y) + G_j(x, y) \) is uniformly negative definite for all \( (x, y) \in \text{cl}(J) \times Y \times Z \) such that \( \pi_j(y) + G_j(x, y) = 0 \);

(iv) \( \Pi \) is uniformly concave on \( \mathcal{U} \subset W^{1,2}(X, d\mu) \), uniformly for all \( \mu \ll L^m \).

When \( m = n \), \( G(x, y, z) = b(x, y) - z \in C^3(\text{cl}(X \times Y \times Z)) \) satisfies (G0)–(G8), and \( \pi(y, z) = z - a(y) \in C^2(\text{cl}(Y \times Z)) \), then Corollary 4.10 shows \( b^* \)-convexity of \( a \) is a sufficient condition for concavity of \( \Pi \) for all \( \mu \ll L^m \). One may wonder under what hypotheses it would become a necessary condition as well. From theorem A.1 in [16], under the same assumptions as above, the manufacturing cost \( a \) is \( b^* \)-convex if and only if it satisfies the following local \( b^* \)-convexity hypothesis: \( D^2a(y) \geq D^2_{yy}b(x, y) \) whenever \( Da(y) = D_yb(x, y) \).
Combined with Proposition B.1, we have the following corollary:

**Corollary B.4.** If we adopt the terminology of Figalli-Kim-McCann [9], i.e., (B0)–(B3), \( \Gamma(x, y, z) = b(x, y) - z \in C^3(\text{cl}(X \times Y \times Z) \times \pi(x, y, z) = z - a(y) \in C^2(\text{cl}(Y \times Z)) \), then \( a(y) \) is \( b^* \)-convex if and only if \( \Pi \) is concave on \( U \) and for every \( y \in Y \), there exists \( x \in \text{cl}(X) \) such that \( Da(y) = D_y b(x, y) \).

**Proof.** Assume \( a \) is \( b^* \)-convex; by definition there exists a function \( a^*: \text{cl}(X) \rightarrow \mathbb{R} \) such that for any \( y \in Y \), \( a(y) = \max_{x \in \text{cl}(X)} b(x, y) - a^*(x) \). Therefore, for any \( y_0 \in Y \), there exists \( x_0 \in \text{cl}(X) \) such that \( a(y_0) \geq b(x_0, y) - a^*(x_0) \) for all \( y \in Y \), with equality holding at \( y = y_0 \). This implies \( Da(y_0) = D_y b(x_0, y_0) \). Taking \( J = \text{cl}(X) \) and applying Proposition B.1 we have concavity of \( \Pi \), since local \( b^* \)-convexity of \( a \) is automatically satisfied by a \( b^* \)-convex function \( a \).

On the other hand, assuming \( \Pi \) is concave on \( U \) and for every \( y \in Y \), there exists \( x \in \text{cl}(X) \) such that \( Da(y) = D_y b(x, y) \), Proposition B.1 implies local \( b^* \)-convexity of \( a \). Together with theorem A.1 in [16], we know \( a \) is \( b^* \)-convex. □

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