Rhythmic generation of infinite trees and languages

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Abstract

This work introduces the idea of breadth-first generation of infinite trees and languages. It is orthogonal to usual descriptions by classical objects such as automata and grammars which refer more naturally to a depth-first approach. This idea is brought into play with periodic inputs, a case that comes from the study of rational base number systems which correspond to specific inputs derived from Christoffel words. Conversely, we characterise languages generated in that way as representations of integers in such number systems with non-canonical alphabets of digits.

1 Introduction

This work introduces a breadth-first description or generation process for infinite trees and languages. Such process is in essence, or conceptually, orthogonal to the usual description of trees and languages that classically rely on objects such as automata or grammars which refer more naturally to a depth-first approach.

For instance, in the case of languages, the possible suffixes a word $u$ are entirely determined by some kind of summary about the word $u$ (e.g. the state of the respective DFA for rational language). If one consider the labelled tree of the prefixes of this language, the word $u$ labels a path from the root to some node. The successors of this node being determined by $u$, it becomes clear that the generation process is indeed, depth-first.

On the other hand-side, the breadth-first approach makes sense from the point of view of effectivity: it insures that every element of the infinite object under scrutiny will eventually be considered.

For languages, it corresponds to the enumeration in the radix order. For trees, nodes are processed in increasing order of distance to the root: the root is processed first, then all nodes at distance 1, then all nodes at distance 2, etc. Node labels of the tree shown at Figure 1 highlight this ordering. And the tree of that figure may be said to be the result of a breadth-first process as the number of successors

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of a node depends on information gathered from those nodes that are directly smaller in the breadth-first order. In this case, the number of successors is simply periodic: 2, 1, 2, 1, ... (nodes with even label have 2 successors, node with odd have 1), with the notable exception of the root, labelled by 0.

When we have a tree — in fact an embedded tree since the successors of every node are ordered — we assume that we have a labelling of the arcs that is consistent with that embedding, that is, the radix order on the branch language of the tree coincides with the ordering of the nodes.

The general case we consider is a generalisation of the tree and language of Fig. 1. We consider a tuple of integers: \( r = (r_0, r_1, \ldots, r_{q-1}) \) which we call rhythm, and generate a tree such that the \( n \)-th node (in the breadth-first order) has \( r_k \) successors, where \( k \) is congruent to \( n \) modulo \( q \) (plus a special rule for the root).

We call directing parameter of \( r \) the pair \( (q, p) \) where \( p \) is the sum of the \( r_i \)'s.

The main result of this paper reads then (below we give the definition of BLIP languages, which, roughly speaking, are languages that meet no kind of iteration lemma, and in particular are not context-free):

**Theorem** Let \( K_r \) be the branch language of the tree generated by a rhythm \( r \) of directing parameter \( (q, p) \).

- a. If \( \frac{p}{q} \) is an integer, then \( K_r \) is a rational language;
- b. If \( \frac{p}{q} \) is not an integer, then \( K_r \) is a BLIP language.

This process originally comes from the theory of rational base number systems,
that have been introduced and studied in [1], that we briefly recall now.

Let \( p \) and \( q \) be two coprime integers, with \( p > q \). In the \( \frac{p}{q} \)-system, every integer has a unique finite representation, but the set \( L_{\frac{p}{q}} \) of the \( \frac{p}{q} \)-representations of the integers is not a rational language. In fact, in a previous work [8], we even explained that \( L_{\frac{p}{q}} \) (or, indeed the representation of any finitely generated additive monoid) in a rational base is very hard to fit in the classical language theory, as it defeats any conceivable iteration lemma.

However it turns out that this language may be generated by the very simple breadth-first process previously described. The example we took to explain this process was indeed the representations of integers in base \( \frac{3}{2} \) (cf. Figure 1), and follows the rhythm \((2,1)\) as previously mentioned.

In the general case, the rhythm of \( L_{\frac{p}{q}} \) corresponds to the most equitable way of parting \( p \) objects into \( q \) cases (with a bias to the left when necessary). We call it the Christoffel rhythm associated with \( \frac{p}{q} \), as it can be derived from the more classical notion of Christoffel word of slope \( \frac{p}{q} \) (cf. [2]), that is, the canonical way to approximate the line of slope \( \frac{p}{q} \) on a \( \mathbb{Z} \times \mathbb{Z} \) lattice.

The proof of Theorem 11 is the purpose of Section 5 and consists of the reduction of any structure generated by a rhythm to the number system whose base is the growth ratio of this rhythm. In definitive, the language generated by a rhythm is simply a non-canonical representation of the integers in this base, in the sense that the integers are represented on a non-canonical alphabet. Using the existing work on alphabet conversion in rational base number systems (cf. [1] or [4]) it allows to conclude that both languages are basically as complicated (or as simple, in the degenerate case where the growth ratio happens to be an integer).

This article is organised as follows: in the preliminaries, we essentially present the number system in base \( \frac{p}{q} \). In Section 3 we give a precise definition of the breadth-first generation of infinite trees and language by a rhythm. Then in Section 4, we describe how this process can be used to generate the language of the representation of integers in a rational base number system. Finally, in Section 5, we prove that any language build by a rhythm is a non-canonical representation of the integers in some underlying rational base.

2 Preliminaries

2.1 Numbers, words

Given two positive integers \( n \) and \( m \), we denote by \( \frac{n}{m} \) their division in \( \mathbb{Q} \); by \( n \div m \) and \( n \% m \) respectively the quotient and the remainder of the Euclidean division of \( n \) by \( m \), that is \( n = (n \div m)m + (n \% m) \) and \( 0 \leq (n \% m) < m \). Additionally, we denote by \( [n, m] \) the integer interval \( \{n, (n + 1), \ldots , m\} \).

An alphabet \( A \) is a finite set of letters, the free monoid generated by \( A \), and denoted by \( A^* \), is the set of finite words over \( A \); the empty word is denoted by \( \varepsilon \). The concatenation of the words \( u \) and \( v \) is simply denoted by \( uv \), and if \( w = uv \) for some \( u, v \) and \( w \) then \( u \) is called a prefix of \( w \). A language over \( A \) is any subset
A language is said to be *prefix-closed* if it contains a word only if it contains each of its prefixes.

### 2.2 Relation, function

Given any set $S$, we denote by $\mathcal{P}(S)$ its power set, that is the set of all the subsets of $S$. Given two sets $S$ and $U$, we call (binary) relation $\theta : S \rightarrow U$ any subset of $\mathcal{P}(S \times U)$. For all elements $s$ of $S$, we denote by $\theta(s)$, the set of all the elements of $U$ in relation with $s$, that is $\theta(s) = \{ u \mid (s, u) \in \theta \}$. Since the relation we study are mostly ‘child’ relation of trees, we favour the denotation $\theta(s) \ni u$, or $s \rightarrow u$, or for short $s \rightarrow u$ when $\theta$ is clear, over the more classical $(s, u) \in \theta$.

Given a relation $\theta : S \rightarrow U$, it is said to be
- **injective** if, for all elements $s, s'$ of $S$ and $u$ of $U$, if $\theta(s') \ni u$ and $\theta(s) \ni u$, then $s = s'$;
- **surjective** if, for all elements $s$ of $S$, there exists an element $u$ of $U$ such that $\theta(u) \ni s$;
- **functional** if for all elements $s$ of $S$, there exists a unique element $u$ of $U$ such that $\theta(s) \ni u$; or, equivalently that $\theta^{-1}$ is injective;
- **fully defined** if for all elements $s$ of $S$, there exists an element $u$ of $U$ such that $\theta(s) \ni u$; or, equivalently that $\theta^{-1}$ is surjective.

A (partial) *function* is a functional relation while an *application* is a fully-defined and functional relation. In both case, by abuse of language, we write $\theta(u) = s$ instead of $\theta(u) = \{ s \}$.

Additionally, in the case where $S$ and $U$ are ordered sets, a relation $\theta : S \rightarrow U$ is said to be monotonous if $\theta$ conserves the order, that is if for all elements $s' > s$ of $S$ and $u \in \theta(s')$, $u' \in \theta(u')$ then $u' \geq u$.

### 2.3 Trees, labels, languages

The trees we are considering in this article are infinite, rooted; and ordered, that is in which the children are ordered. By convention, when drawing a tree, we assume that the children of a node are ordered from bottom to top and that each node is labelled by the ordering of a breadth-first traversal. For instance, the nodes of the tree shown at Figure 1 in introduction, are labelled by this ordering.

Since we use a non-standard definition of trees, they will be defined at the beginning of Section 3.

Additionally, we only consider in this article languages that are prefix-closed. Therefore, given a finite alphabet $A$, a language over $A$ can be viewed (and will be considered) as a tree whose edges are labelled over the alphabet $A$. 

4
2.4 Integer base number systems

Given an integer \( p \geq 2 \) as a base, the evaluation function \( \pi_p \) associate a number to every word over the alphabet \( A_p = [0, p-1] \):

\[
\forall a_n a_{n-1} \cdots a_0 \in A_p^*, \quad \pi_p(a_n a_{n-1} \cdots a_0) = \sum_{i=0}^{k} a_i p^i.
\]

A word \( u \) is called a \( p \)-representation of an integer \( n \) if \( \pi_p(u) = n \). The representation is unique up to leading 0’s, hence we denote by \( \langle n \rangle_p \) the \( p \)-representation of \( n \) which does not start with a 0. It can be computed, from right to left, by the Euclidean algorithm given below. Let us define \( N_0 = n \) and, for all \( i > 0 \)

\[
N_i = pN_{i+1} + a_i,
\]

where \( a_i \) is the remainder of the Euclidean division of \( N_i \) by \( p \); and \( \langle n \rangle_p = a_n a_{n-1} \cdots a_0 \). The language \( L_p = \langle \mathbb{N} \rangle_p \) of the \( p \)-representations of integers is equal to the rational language \( (A_p \setminus \{0\}) A_p^* \) of the words that do not start with a 0.

2.5 Rational base number systems

The Modified Euclidean Division Algorithm. Let \( p \) and \( q \) be two co-prime integers such that \( p > q > 1 \). Given a positive integer \( N \), let us define \( N_0 = N \) and, for all \( i > 0 \),

\[
qN_i = pN_{i+1} + a_i,
\]

where \( a_i \) is the remainder of the Euclidean division of \( qN_i \) by \( p \), hence in \( A_p = [0, p-1] \). Since \( p > q \), the sequence \( (N_i)_{i \in \mathbb{N}} \) is strictly decreasing and eventually stops at \( N_{k+1} = 0 \). Moreover, it holds that

\[
N = \sum_{i=0}^{k} \frac{a_i}{q} \left( \frac{p}{q} \right)^i.
\]

The evaluation function \( \pi \) is derived from this formula. Given a word \( a_n a_{n-1} \cdots a_0 \) over \( A_p \), and indeed over any alphabet of digits, its value is defined by

\[
\pi(a_n a_{n-1} \cdots a_0) = \sum_{i=0}^{n} \frac{a_i}{q} \left( \frac{p}{q} \right)^i.
\]

The \( L_{\frac{p}{q}} \) language. Conversely, a word \( u \) in \( A_p^* \) is called a \( \frac{p}{q} \)-representation of an integer \( x \) if \( \pi(u) = x \). Since the representation is unique up to leading 0’s (see [1] Theorem 1) the \( \frac{p}{q} \)-representation of \( x \) which does not starts with a 0 is denoted by \( \langle x \rangle_{\frac{p}{q}} \) (or \( \langle x \rangle \) for short) and can be computed with the modified Euclidean division algorithm above. By convention, the representation of 0 is the empty word \( \varepsilon \). The set of \( \frac{p}{q} \)-representations of integers is denoted by \( L_{\frac{p}{q}} \):

\[
L_{\frac{p}{q}} = \{ \langle n \rangle_{\frac{p}{q}} \mid n \in \mathbb{N} \}
\].
It should be noted that a rational base number systems is not a \(\beta\)-numeration — where the representation of a number is computed by the (greedy) Rényi algorithm (cf. [11, Chapter 7]) — in the special case where \(\beta\) is a rational number. In such a system, the digit set is \(\{0, 1, \ldots, \lfloor \frac{p}{q} \rfloor\}\) and the weight of the \(i\)-th leftmost digit is \(\frac{1}{q^i}\); whereas in the rational base number system, they are \(\{0, 1 \ldots (p-1)\}\) and \(\frac{1}{q} (\frac{1}{q})^i\) respectively.

It is immediate that \(L_{\frac{p}{q}}\) is prefix-closed (since, in the modified Euclidean division algorithm \(\langle N \rangle = \langle N_1 \rangle a_0\)) and right-extendable (for every representation \(\langle n \rangle\), there exists (at least) an \(a\) in \(A_p\) such that \(q\) divides \((np + a)\) and then \(\langle np + a \rangle = \langle n \rangle a\)). As a consequence, \(L_{\frac{p}{q}}\) can be represented as an infinite tree (cf. Figure 1 in introduction).

By abuse of language, in the following we will write that \(n \xrightarrow{u} m\) (or \(n \xrightarrow{u} m\), for short) to signify that \(\langle m \rangle = \langle n \rangle u\); it should be noted that, with this notation, the following equation hold.

\[
\forall n \in \mathbb{N}, \forall m \in \mathbb{N}, \forall a \in A_p \quad n \xrightarrow{a} m \iff a = qm - pn .
\]  

**BLIP languages.** It is known that \(L_{\frac{p}{q}}\) is not rational and not even context-free (cf. [1]). In fact \(L_{\frac{p}{q}}\) defeats any kind of iteration lemma as it is a BLIP language, defined below (cf. [8]).

**Definition 1.** We say that a language \(L\), over an alphabet \(A\) has the Bounded Left Iteration Property, or is a BLIP language for short, if

\[
\forall u, v \in A^*, \quad uv^i \text{ is prefix of a word of } L \quad \text{for only finitely many } i \in \mathbb{N} .
\]

BLIP is a quite robust property, as it is stable by finite union, intersection, sublanguage (but neither by complementation nor concatenation).

An equivalent definition of a BLIP language \(L\) is that the language of the prefixes of \(L\) meets the IRS condition (Infinite Regular Subset), that is, does not contain any infinite rational sublanguage ([3]). We only consider prefix-closed languages in this paper and such language is BLIP if and only if it does not contain any infinite rational sublanguage.

**Remark 2.** Even though we separated the notions of rational and integer base number systems in order to give specific statements, it should be noted that the former extends naturally the latter. Indeed, in the case where \(q = 1\), the definitions of \(\pi_{\frac{p}{q}}\), \(\langle \cdot \rangle_{\frac{p}{q}}\) and \(L_{\frac{p}{q}}\) respectively coincide with those of \(\pi_p\), \(\langle \cdot \rangle_p\) and \(L_p\). In the sequel, we will consider the base \(\frac{p}{q}\) such that \(p > q \geq 1\), that is indifferently one number system or the other.

### 3 Breadth-first generation of infinite trees and languages

We study here a way of building structures in a breadth-first manner. When building an infinite tree in such a way, a node is created when it is reached for the
first time but will only be processed (that is, its outgoing edges will be created) after every other unprocessed states. It is similar to putting unprocessed states on a FIFO (first in first out) list.

3.1 Trees and breadth-first signature

As said in the preliminaries, we consider infinite rooted and ordered trees. A tree of this particular class is canonically associated with its child relation, that is with an injective relation \( \mathbb{N} \to \mathbb{N} \) verifying that for all integers \( n \), there exists an integer \( m > n \), \( \theta ([0, n]) = [1, m] \) . (5)

Intuitively, if the biggest child of the node \( n \) is \( m \), and the node \( (n + 1) \) has \( k \) children, then they are labelled by the successive integers \( m + 1, m + 2, \ldots, m + k \). This 1) prevents to have any node \( n > 0 \) without predecessor; 2) the nodes are labelled according the breadth-first search (it is unique since our trees are ordered). Finally, the condition ‘\( m > n \)’ is there to ensures connexity.

**Definition 3** (Breadth-first signature of a tree). Given a tree of child relation \( \theta \), we call breadth-first signature of \( \theta \) the infinite sequence

\[ n_0, n_1, \ldots, n_k, \ldots \quad \text{where } n_i = |\theta(i)| \quad \text{for all integers } i > 0 \]
\[ n_0 = |\theta(0)| + 1 \] (6a)

It follows directly from this definition that the breadth-first signature is characteristic of its tree, as stated below.

**Proposition 4.** Two trees with the same breadth-first signature are equal.

The intuition behind the special case of 0 (cf. Equation (6b)) is an artefact of the non-surjectivity of \( \theta \) is not surjective, the node 0 having no predecessor.

We define an i-tree as a tree (with the definition above) augmented with a loop on the root (it corresponds to add the element 0 to the set \( \theta(0) \)). It is common, in the concrete implementation of trees, to have this loop on the root, for instance the file system tree, or more technically when using an array to implement a tree. More formally, an i-tree is associated with its child relation, that is an injective relation \( \theta : \mathbb{N} \to \mathbb{N} \) such that for all integers \( m > n \), \( \theta([0, n]) = [0, m] \).

Since a (prefix-closed) language over an alphabet \( A \) is considered as a tree labelled over \( A \), it is formally a relation \( \mathbb{N} \times A \times \mathbb{N} \) whose projection to \( \mathbb{N} \times \mathbb{N} \) is a tree. With this notation, the \( (n + 1) \)-th word in radix ordering of this language corresponds to the node labelled \( n \) in the underlying tree. We write \( n \xrightarrow{a} m \) if \((n, a, m)\) is part of this relation.

3.2 Rhythm

**Definition 5.** Let \( p \) and \( q \) be two integers with \( p > q \geq 1 \).
1. We call rhythm of directing parameter \((q, p)\), a \(q\)-tuple
\[
\mathbf{r} = (r_0, r_1, \ldots, r_{q-1})
\]
of non-negative integers whose sum is \(p\).

2. We say that the rhythm \(\mathbf{r}\) is valid if it satisfies the following equation:
\[
\forall j \in [0, q - 1] \quad \sum_{i=0}^{j} r_i > j + 1.
\]

3. We call growth ratio of \(\mathbf{r}\) the rational number \(z = \frac{p}{q}\), also written \(z = \frac{p'}{q'}\) where \(p'\) and \(q'\) are the quotients of \(p\) and \(q\) by their \(\gcd\), hence coprime.

From the definition follows that the growth ratio is always greater than 1. Examples of rhythms of growth ratio \(\frac{5}{3}\) are \((2, 2, 1)\), \((3, 0, 2)\), \((1, 2, 2)\), \((2, 2, 1, 2, 2, 1)\), \((2, 1, 3, 0, 0, 4)\); all but the third one are valid; the directing parameter is \((3, 5)\) for the first three, and \((6, 10)\) for the last two. Unless otherwise specified, every rhythm is assumed in the sequel to be valid.

**Word and path representations**
Rhythms may be given a convenient geometric representation as paths in the \(\mathbb{Z} \times \mathbb{Z}\)-lattice and such paths are coded by words of \(\{x, y\}^*\) where \(x\) denotes a unit horizontal segment and \(y\) a unit vertical segment. Hence the name path given to a word associated with a rhythm.

**Definition 6.** With a rhythm \(\mathbf{r} = (r_0, r_1, \ldots, r_{q-1})\) of directing parameter \((q, p)\), we associate the word of \(\{x, y\}^*\) path\((\mathbf{r}) = y^{r_0}x y^{r_1}x y^{r_2}x \cdots y^{r_{q-1}}x\) which corresponds to a path from \((0, 0)\) to \((q, p)\) in the \(\mathbb{Z} \times \mathbb{Z}\)-lattice.

For instance, the path associated with the rhythm \((2, 2, 1)\) is \(y^2x y^2x y^2x\). Figure 2 shows the path associated with three of the above rhythms. Condition (7) can then be restated as ‘a rhythm is valid if the associated path is strictly above the line of slope 1’.

### 3.3 Generating infinite trees by rhythm

A sequence of integers \(n_0, n_1, \ldots, n_i, \ldots\) is said to be (purely) periodic if there exists an integer \(k\) (called a period) such that for all \(i\), \(n_i = n_{j}\), where \(j = i \mod k\). The rhythm \((n_0, n_1, \ldots, n_{k-1})\) is then called the residual of this sequence.

**Definition 7.** Given a rhythm \(\mathbf{r}\), we call tree generated by \(\mathbf{r}\), denoted by \(J_{\mathbf{r}}\), the tree whose breadth-first signature is periodic, of residual \(\mathbf{r}\).

A more intuitive way of defining the same object is through its construction, similar to a breadth-first unravelling of the tree. It is shown, for instance at Figure 3 for the tree \(J_{(3,1,1)}\). Procedure 1 implements this construction.

The next lemma justify the fact that \(J_{\mathbf{r}}\) is indeed a tree. It is a direct consequence of Equation (7) (and is false for invalid rhythm) which insures that in Procedure 2, \(n\) is always smaller than \(m\).
The next statement justify the term growth associated to the ratio $\frac{p}{q}$. Indeed, it says that if one takes $q$ successive (in radix ordering) nodes, the total number of their successors is $p$, hence the number of successors of $n$ successive nodes is roughly $n \frac{p}{q}$.

**Lemma 9.** Given a rhythm $r$ of directing parameter $(q,p)$ generating the tree $J_r$, for all integers $n$ and $m$ such that $n \to m$, then $(n + q) \to (m + p)$.

**Proof.** Indeed, every $q$-modulo class is visited exactly once among the nodes $n, n+1, \ldots, (n + q - 1)$. Since a node congruent to $i$ modulo $q$ has $r_i$ successors, the total number of successors of the states belonging
\[ 3 - 1 = 2, \]

Figure 3: Building \( J_{(3,1,1)} \) from the rhythm \((3,1,1)\)

\[ \ldots, 1, 3, 1, 1, 3, 1 \]

Figure 4a shows the language \( K_{(3,1,1)} \) while Figure 5a presents \( K_{(3,1)} \). The next statement is the main contribution of this article: a language built by a rhythm is either rational or BLIP, depending of the growth ratio of this rhythm.

3.4 Generating languages by rhythm

**Definition 10.** Given a rhythm \( r \), we denote by \( K_r \), the language over \( A_p \) whose underlying tree is \( J_r \), augmented with the following labels

\[ \forall n, m \in \mathbb{N} \quad n \xrightarrow{a} m \quad \text{with } a = m \% p. \]

The growth of \( n \) to \( \{n, n+1, \ldots, (n+q-1)\} \) is \( (\sum_{i=0}^{q-1} r_i) = p \), and more precisely consists of the set \( \{m, m+1, \ldots, (m+p-1)\} \). Hence \( n + q \to m + p \). 

\[ \text{□} \]
**Theorem 11.** Let $r$ be a rhythm of directing parameter $(q,p)$.

a. If $\frac{p}{q}$ is an integer, then $K_r$ is a rational language.

b. If $\frac{p}{q}$ is not an integer, then $K_r$ is a BLIP language.

The proof of the whole statement consists of a reduction to the case of the representation language of rational base number systems which takes indeed the remainder of the paper (Theorem 22 and Corollary 29). However, proof of Statement 11a is established below in a much simpler and direct way.

**Proof of Theorem 11a.** Let $p$ and $q$ be two positive integers where $q$ is greater than 1 and divides $p$: $p = kq$, and let $r = (r_0, r_1, \ldots, r_{q-1})$ be a rhythm of directing parameter $(q,p)$. The proof consists of considering the underlying tree of $K_r$ as an infinite automaton and then proving that it has a finite number of suffix languages. More precisely, we will prove that two states $n$ and $m$ congruent modulo $q$ are Nerode-equivalent.

Claim. For all $n > 0$ and $m > 0$, if $n \equiv m \ [q]$, then $n \sim m$.

Let us consider two integer $n$ and $m$ such that $n \equiv m \ [q]$. By induction. The relation $\sim_0$ is trivial and has only one equivalency class, hence $n \sim_0 m$.

Let now be $i$ an integer strictly greater than 0. Since $n \equiv m \ [q]$, it follows from Lemma 3 that for every $n'$ such that $n \xrightarrow{a} n'$, then there exists an integer $m'$ such that $m \xrightarrow{b} m'$ and $n' \equiv m' \ [p]$. Hence, from Definition 10 $a = (n' \% p) = (m' \% p) = b$.

Moreover, by hypothesis $q \mid p$, hence $n' \equiv m' \ [q]$. By induction hypothesis, $n' \sim_{(i-1)} m'$, hence $n \sim_i m$. 

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Figure 4: Two languages generated by the rhythm $(3, 1, 1)$
The automaton accepting $K_r$ has then $q + 1$ states: one for each congruency class modulo $q$ for positive integers, plus one special state for 0 which is initial. See Figure 5 for the case of rhythm $(3, 1)$. \[\square\]

![Figure 5: The case of the rhythm $(3, 1)$ of integer growth ratio](image)

### 3.5 Generating language by rhythm and labelling

In the context of a rhythm of directing parameter $(q, p)$, we call labelling, denoted by $\lambda$, any $p$-tuple of letters $(\lambda_0, \lambda_1, \ldots, \lambda_{p-1})$. We will only consider the case where letters are digits.

**Definition 12.** Given rhythm $r$ and a labelling $\lambda$, the language generated by $r$ and $\lambda$ is defined as the language of underlying tree $J_r$, augmented with the labels

$$\forall n, m \in \mathbb{N} \quad n \xrightarrow{a} m \quad \text{with} \quad a = \lambda(m \mod p).$$

\[1\] If one considers the i-tree generated by rhythm $r$ (instead of the tree) then the special case is unnecessary, there is only $q$ states and the congruency class $(0 \mod q)$ is initial. For instance, in Figure 5c, if there were a self loop on the state 0, it would be Nerode-equivalent to the state even.
Figure 4b shows the language built by the rhythm (3, 1, 1) and the labelling (0, 3, 6, 4, 2). Procedure 2 gives a more technical definition.

**Procedure 2** Generate by rhythm & label(rhythm[0..q−1], label[0..p−1])

1. int $n := 0$ // $n$ is the node currently processed
2. int $m := 1$ // $m$ is the smallest node with no predecessor
3. for $i$ from 1 to (rhythm[0]−1) do // Special case for $n = 0$
   4. add the transition $0 → _{label[m]} m$
5. $m := m + 1$
6. end for
7. $n := 1$
8. while true do
9. for $i$ from 1 to rhythm[$n \% q$] do // General case $n > 0$
10. add the transition $n → _{label[m \% p]} m$
11. $m := m + 1$
12. end for
13. $n := n + 1$
14. end while

The careful reader would have noted that the language $K_{r_1}$, defined in the previous section is built by rhythm $r$ and the labelling (0, 1, 2,...,p − 1); since it uses the most naive $p$-tuple, we call it the naive labelling. The next proposition, stating that the naive labelling can be projected to any other labelling, follows directly from the definitions. It should be noted that the reverse is not true: a given labelling cannot be projected to the naive labelling if two of its components are equal

**Proposition 13.** Given any language $L$ generated by a rhythm $r$ and a labelling $\lambda$, then there exists a strictly alphabetical morphism $\varphi$ such that $\varphi(K_r) = L$.

**Remark 14.** Procedure 2 was built in order to be as naive as possible. It results that the states are accessed through their labels, which is unnecessary.

- The only operation affecting the variables $n$ and $m$ are incrementations;
- once the variable $n$ surpasses some integer $k$, the state $k$ will never be accessed again;
- any newly reachable state (labelled by $m$) will be processed after any other state $k$ between $n$ and $m$.

It follows that a simple FIFO (first in first out) list $n → n+1 → \cdots → m$ would suffice. This is implemented in Procedure 3.

4 Breadth-first generation of a rational base number system

Let $p$ and $q$ be two coprime integers, $p > q \geq 1$, which then define the number system with base $\frac{p}{q}$. We first introduce a special, canonical, rhythm $r_{\frac{p}{q}}$ of parameter $(q, p)$,
Procedure 3 Generate_by_rhythm & label(rhythm[0..q-1],label[0..p-1])

1. FifoList list = [] // list contains the unprocessed states
2. int n := 0 // n is the current index of rhythm
3. int m := 1 // m is the current index of permutation
4. for i from 1 to (rhythm[0] - 1) do // Special case for n = 0
5. Node newState := new Node()
6. list.push_to_tail(newState)
7. add the transition list.pop_head() → newState
8. m := (m + 1) % q
9. end for
10. n := 1
11. while true do
12. for i from 1 to rhythm[n] do
13. Node newState := new Node()
14. list.push_to_tail(newState)
15. add the transition list.pop_head() → newState
16. m := (m + 1) % q
17. end for
18. n := (n + 1) % q
19. end while

hence of growth ratio $z = \frac{p}{q}$ which turns out to be a classical notion. We then define a special labelling $\gamma_{p/q}$, which is the permutation resulting from the generation of $\mathbb{Z}/p\mathbb{Z}$ by $q$. Then the remarkable fact is that the representation language in the $\frac{p}{q}$-number system is generated by the rhythm $r_{p/q}$ and the permutation $\gamma_{p/q}$ (Theorem 22).

4.1 The rhythm of a rational base

The rhythm that we associate with $\frac{p}{q}$ corresponds to the most equitable way of parting $p$ items into $q$ boxes (with a bias to the left when necessary). Moreover, the path associated with this rhythm $r_{p/q}$ is the best approximation in the $\mathbb{Z} \times \mathbb{Z}$-lattice of the segment from $(0,0)$ to $(q,p)$.

**Definition 15.** Let $p$ and $q$ be two coprime integers, $p > q \geq 1$. The Christoffel rhythm associated with the rational number $\frac{p}{q}$ is the sequence $r_{\frac{p}{q}} = (r_0, r_1, \ldots, r_{q-1})$ where every $r_j$ is the smallest integer such that the partial sum $r_0 + r_1 + \cdots + r_j$ surpasses $(j+1)\frac{p}{q}$, that is:

$$\forall j \in [0, q - 1] \quad q \sum_{i=0}^j r_i \leq (j + 1) p < q \left(1 + \sum_{i=0}^j r_i \right) . \quad (8)$$

Note that for $j = q - 1$, Equation (8) implies that $r_{\frac{p}{q}}$ is indeed a rhythm of directing parameter $(q,p)$, that is, the sum $r_0 + r_1 + \cdots + r_{q-1}$ is equal to $p$.

For instance, the Christoffel rhythm of base $\frac{2}{3}$ is $(2,2,1)$; it is respectively the 1-tuple $(2)$ for base 2.
Together with the Christoffel rhythm \( r_{\frac{p}{q}} = (r_0, r_1, \ldots, r_{q-1}) \), we define the sequence of integers \( e_0, e_1, \ldots, e_{q-1} \) such that \( e_{j+1} \) is the difference between the two right-hand side, that is:

\[
\forall j \in [0, q-1] \quad e_j = q \left( \sum_{i=0}^{j-1} r_i \right) - jq. \tag{9}
\]

See Figure 6 for a more visual interpretation of the definition of Christoffel rhythm.

![Figure 6: Visual interpretation of the rhythm (2, 2, 1) of base \( \frac{5}{3} \)](image)

The following lemma gathers immediate consequences of the definition of the Christoffel rhythm and of the \( e_j \)'s.

**Lemma 16.** Let \( r_{\frac{p}{q}} = (r_0, r_1, \ldots, r_{q-1}) \) be the Christoffel rhythm of the base \( \frac{p}{q} \) and an integer \( j \) of \( [0, q-1] \),

a. \( \sum_{i=0}^{j-1} r_i = \lfloor jq \rfloor \);

b. \( \sum_{i=q-j}^{q-1} r_i = \lfloor j \frac{p}{q} \rfloor \);

c. \( e_j \) belongs to \( [0, q-1] \);

d. \( r_j \) is the smallest integer such that \( q r_j + e_j \geq p \).

### 4.2 Christoffel words

Christoffel rhythms are more commonly defined (in computer science) through the notion of Christoffel words which corresponds, when translated into paths in the \( \mathbb{Z} \times \mathbb{Z} \)-lattice, to the best approximation of segments (with the coding introduced in Section 3.2).

**Definition 17 (2).** Let \( p \) and \( q \) be two coprime positive integers. The (upper) Christoffel word associated with \( \frac{p}{q} \), and denoted by \( w_{\frac{p}{q}} \), is the label of the path from \( (0,0) \) to \( (q, p) \) on the \( \mathbb{Z} \times \mathbb{Z} \) lattice, such that

- the path is above the line of slope \( \frac{p}{q} \) passing through the origin;
- the region of the plane enclosed by the path and the line contains no point of \( \mathbb{Z} \times \mathbb{Z} \).
Figure 7: Christoffel word associated with three rational numbers

Figure 7 shows the Christoffel words associated with three rational numbers: $\frac{3}{2}$, $\frac{5}{2}$, and $\frac{5}{3}$.

That Definitions 15 and 17 are consistent is the matter of a statement, the proof of which is sketched in the appendix.

**Proposition 18.** Let $p$ and $q$ be two coprime positive integers. The word associated with the Christoffel rhythm of $\frac{p}{q}$ is equal to the Christoffel word of $\frac{p}{q}$:

\[ \text{path}(r_{\frac{p}{q}}) = w_{\frac{p}{q}}. \]

**Sketch of Proof.** Fig. 8 gives an idea for the proof of Proposition 18: all parameters computed in the proof appear in this drawing. The unit segments in the $\mathbb{Z} \times \mathbb{Z}$-lattice are divided into $q$ subunits; the $e_j$ defined at Equation (9) are seen at the crossing of the line of slope $\frac{p}{q}$ with the unit lines parallel to the $y$ axis.

The next lemma gives properties of Christoffel rhythms highlighting that they are well-balanced. They are directly inherited from similar properties of Christoffel words, given at Proposition 20 (cf. Proposition 4.2 of [2]) and Theorem 21 (cf. [9]).

**Lemma 19.** Given a rhythm $r = (r_0, r_1, \ldots, r_{q-1})$,

a. if $r$ is a Christoffel rhythm, then $(r_0 - 1, r_1, \ldots, r_{q-1})$ is a palindrome;

b. $r$ is a Christoffel rhythm if, and only if, $(r_0 - 1, r_1, \ldots, r_{q-1} + 1)$ is conjugated with $r$.

**Proposition 20.** Any Christoffel word is of the form $yux$ with $u$ a palindrome.

**Theorem 21 ([9]).** A word $yux$ over $\{x,y\}$ is a Christoffel word if, and only if, $xuy$ and $yux$ are conjugate.
4.3 Generation of $L_{\frac{p}{q}}$ by rhythm and labelling

Let $p$ and $q$ be two coprime integers. Then, $q$ is a generator of the (additive) group $\mathbb{Z}/p\mathbb{Z}$; the sequence induced by this process: $(0, q \% p, (2q) \% p, \ldots, ((p-1)q) \% p)$ is a $p$-tuple, denoted by $\gamma_{\frac{p}{q}}$, that we call the canonical labelling associated with base $\frac{p}{q}$.

**Theorem 22.** Given a base $\frac{p}{q}$, the language $L_{\frac{p}{q}}$ is generated by the associated Christoffel rhythm $r_{\frac{p}{q}}$ and canonical labelling $\gamma_{\frac{p}{q}}$.

For instance, one can verify that $L_{\frac{3}{2}}$, showed at Figure 1 in introduction, is built with rhythm $(2, 1)$ and labelling $(0, 2, 1)$. The proof of Theorem 22 relies mostly on the following statement itself being a consequence of the technical Lemma 16d.

**Proposition 23.** For every integer $n > 0$ (resp. $n = 0$), there is exactly $r(n \% q)$ (resp. $(r_0 - 1))$ letters $a$ of $A_p$ such that $(n).a$ is in $L_{\frac{p}{q}}$.

**Proof.** We denote by $j$ the congruency class of $n$ modulo $q$. From Lemma 16d, $r_j$ is the smallest integer such that $qr_j + e_j > p$. It follows that for all $k$ in $[0, r_j - 1]$ $(e_j + qk) < p$ and $e_j + qr_j > p$.

From Lemma 16d, $e_j$ is the smallest label of the state $n$, hence the state $n$ has exactly $r_j$ outgoing transitions, respectively labelled by $e_j, e_j + q, \ldots, (e_j + q(r_j - 1))$.

**Lemma 24.** Given a base $\frac{p}{q}$ and an integer $n$, the smallest letter $a$ of $A_p$ such that $(n).a$ is in $L_{\frac{p}{q}}$ is $e(n \% q)$.

Figure 8: Link between Christoffel rhythm and Christoffel word for $\frac{5}{3}$
Proof. Let us denote by \( n \) an integer and by \( j \) its congruency modulo \( q \). Since \( e_j \) is in \( A_q \) (from Lemma 16c), it is enough to prove that \( e_j \) is an outgoing label of \( n \), or (from Equation (4)) that \( np + e_j \) is a multiple of \( q \); or, equivalently that \( jp + e_j \) is a multiple of \( q \). From Equation (9), \( jp + e_j = (q \sum_{i=0}^{j-1} r_i) \), that is, a multiple of \( q \).

To achieve the proof of Theorem 22, it remains to prove that for every integer \( n \), the last letter of \( \langle n \rangle \) is \((nq) \% p\), which directly results from the definition of the modified Euclidean division algorithm (Equation (2)).

5 Reduction to rational base number systems

The last step to characterise the languages generated by rhythm (and labelling), and then to establish Theorem 11 consists in showing that they can be mapped onto the representation language of integers in a rational base number system.

5.1 Smart labelling and the language \( L_r \)

We consider a rhythm \( r \) of directing parameter \((q, p)\) fixed in the following and the associated tree \( J_r \). We recall that we denote by \( p' \) and \( q' \) the integers such that \( \frac{p'}{q'} \) is the irreducible fraction of \( \frac{p}{q} \).

We use here a smart labelling \( \sigma \), that is best defined indirectly. Its \( p \) components will label the \( p \) ingoing transitions of the nodes of \( J_1 \), \( p' \). Given an integer \( m \) of this interval, let us denote by \( n \) the integer such that \( n \rightarrow m \), the label associated with this transition is \( \sigma_m = (q'm - p'n) \). Intuitively, the smart labelling replicates the canonical labelling (as in \( \mathbb{L}_{\frac{p}{q}} \), cf. Equation (4)) but only for small nodes; it will be proven later on that it is also the case for greater nodes.

**Definition 25.** We denote by \( \mathbb{L}_r \) the language generated by a given rhythm \( r \) and the associated smart labelling \( \sigma \).

Let us for a moment assume that \( r \) is a Christoffel rhythm\(^2\). Both \( \mathbb{L}_r \) and \( \mathbb{L}_{\frac{p}{q}} \) are generated by the rhythm \( r \) and the respective labellings \( \gamma_{\frac{p}{q}} \) and \( \sigma \). It follows from the definition of \( \sigma \) that they must coincide, hence \( \mathbb{L}_r = \mathbb{L}_{\frac{p}{q}} \).

In the general case, every integer \( n \) labels a unique node of \( J_r \) by definition. We call \( r \)-representation of \( n \), denoted by \( \langle n \rangle_r \), the word labelling the path \( 0 \rightarrow n \) of \( \mathbb{L}_r \). The next proposition, as already hinted through the case where \( r \) is a Christoffel rhythm, states that the smart labelling is a generalisation of the canonical labelling of rational base number system (cf. Equation (4)).

**Proposition 26.** Given a rhythm \( r \) of directing parameter \((q, p)\) and two integers \( n \) and \( m \) such that \( n \rightarrow m \) in \( \mathbb{L}_r \), then \( a = q'm - p'n \).

\(^2\) If \( r \) is a Christoffel rhythm, \( p \) and \( q \) are coprime, hence \( p = p' \) and \( q = q' \).

\(^3\) This defines an abstract numeration system (cf. [6, Definition 3.1.10]) associated with the language \( \mathbb{L}_r \) (if we relax the condition that \( \mathbb{L}_r \) must be a rational language). Indeed \( \langle n \rangle_r \) is the \((n+1)\)-th word of \( \mathbb{L}_r \) in the radix (or genealogical) ordering.
Proof. By Induction. It is true for \( m < p \), from the definition of \( \sigma \).

Let us consider any edge of the form \((n + q) \xrightarrow{a} (m + p)\). It follows from Lemma \( \text{[9]} \) that the edge \( n \xrightarrow{b} m \) exists for some letter \( b \); and from Definition \( \text{[12]} \) \( a = \sigma_{(m+p)\pi_p} = \sigma_{m\pi_p} = b \). Moreover, by induction hypothesis, \( b = q'm - p'n \); and since \( p'q = p'\gcd(p, q)q' = pq' \), it follows that \( q'(m + p) - p'(n + q) = q'm - p'n = b = a \). \( \square \)

5.2 From \( \langle n \rangle_r \) to \( \langle n \rangle_{\mathcal{U}_q} \)

The next proposition links any rhythm of directing parameter \((q, p)\) with the \( \mathcal{U}_q \)-number systems, by stating that the \( r \)-representation of an integer \( n \) is simply a non-canonical representation of \( n \) in the system \( \mathcal{U}_q \).

**Proposition 27.** For all integer \( n \) of \( r \)-representation \( \langle n \rangle_r = a_ka_{k-1}\ldots a_0 \),

\[
\pi_{\mathcal{U}_q}(\langle n \rangle_r) = \sum_{i=0}^{k} \frac{a_i}{q'} \left( \frac{p'}{q} \right)^i = n.
\]

Proof. By induction over \( n \). It is obviously true for \( \langle 0 \rangle_r = \varepsilon \).

Let \( m \) be an integer and \( \langle m \rangle_r = a_{k+1}a_{k-1}\ldots a_0 a_0 \) its representation, that is a word of \( L_r \). It follows that the word \( a_k a_{k-1}\ldots a_1 a_0 \) is also in \( L_r \), and the representation of some integer \( n \) strictly smaller than \( m \), such that \( n \xrightarrow{a_0} m \).

By induction hypothesis, \( \sum_{i=1}^{k} \frac{a_i}{q} \left( \frac{p'}{q} \right)^i = n \). It follows from Proposition \( \text{[26]} \) that \( a_0 = q'm - p'n \), or equivalently that \( \frac{np' + a_0}{q'} = m \), hence

\[
m = \frac{np' + a_0}{q'} = \frac{p'}{q'} \left( \sum_{i=1}^{k} \frac{a_i}{q} \left( \frac{p'}{q} \right)^i \right) + \frac{a_0}{q} = \left( \sum_{i=1}^{k} \frac{a_i}{q} \left( \frac{p'}{q} \right)^i \right) + \frac{a_0}{q}.
\]

\( \square \)

The work of \( \text{[1]} \) teaches us that there is a representation converter (indeed a finite right letter-to-letter sequential transducer) from any finite alphabet to the canonical alphabet. Given a finite alphabet \( B \), we define the partial normalisation function \( \chi_B : B^* \rightarrow A_p^* \) such that

1. \( \chi_B \) conserves the value, that is, for all word \( w \), \( \pi(\chi_B(w)) = \pi(w) \);
2. \( \chi_B \) is defined on the larger sensical domain, that is on any word \( u \) of \( B^* \) such that there exists a word of \( A_p^* \) with the same value.

**Theorem 28 (\( \text{[1]} \) \[4\]).** For all alphabets \( B \), the map \( \chi_B \) is a rational function.

The alphabet labelling \( L_r \) is finite (it is the set of the components of \( \sigma \)), hence Theorem \( \text{[28]} \) can be applied in our special case, 

and allows to finish the proof of Theorem \( \text{[1]} \)
**Corollary 29.** For every rhythm \( r \) of growth ratio \( \frac{p'}{q'} \), the function mapping \( \langle n \rangle_r \) to \( \langle n \rangle_{\frac{p'}{q'}} \) is rational.

**Proof of Theorem 11b.** Since \( K_r \) is prefix closed, it is a BLIP language if, and only if it contains no infinite rational sublanguage. Ab absurdo, let us assume that it does contain a rational sublanguage \( K \). From Proposition 13 there exists a strict alphabetical morphism \( \varphi \) such that \( \varphi(K_r) = L_r \), hence \( L = \varphi(K) \) is a rational sublanguage of \( L_r \). From Corollary 29, there exists a rational function \( \gamma \) such that \( \gamma(L) \) is a rational sublanguage of the BLIP language \( L_{\frac{p'}{q'}} \), a contradiction. \( \square \)

### 6 On the representation of negative integers

The purpose of this section is to shed a new light on the representations of negative integer in a rational able number system. They were introduced in [3] by Frougny and Klouda, as a special case of the representations of any \( p \)-adic number. We here apply rhythm to the representations of negative integer to 1) prove that the language of the representations of negative integer is generated by rhythm (Theorem 33) and 2) give a new proof of one of the result from [3] (Proposition 30).

Let us denote by \( \mathcal{T}_{\frac{p}{q}} \) a labelled oriented graph which is the pendant of \( \mathcal{T}_{\frac{p}{q}} \) for negative integers.

\[
\mathcal{T}_{\frac{p}{q}} = \left( \mathbb{Z} \setminus \mathbb{N}, A_p, \tau_{\frac{p}{q}} \right)
\]

As can be seen at Figure 9, showing the structure \( \mathcal{T}_{\frac{2}{3}}, \mathcal{T}_{\frac{3}{4}} \) may feature multiple connected components. They however always are in finite number and consists of some self-looping states (for instance the state \(-1\), in Figure 9), plus a unique infinite connected components (a tree with a loop on the root).

In the following, using the same the denotation as in [3], we denote by \( B \) the integer \( B = \left\lceil \frac{p-1}{p-q} \right\rceil \). The purpose of this section is to give a new proof of the following statement.

**Proposition 30 ([3]).** The mapping which maps state \( k \) of \( \mathcal{T}_{\frac{p}{q}} \) to state \( \left( -\left\lfloor \frac{p-1}{p-q} \right\rfloor - k \right) \) of \( \mathcal{T}_{\frac{p}{q}} \) is an automaton isomorphism if, and only if, \( \frac{p-1}{p-q} \) is an integer.

#### 6.1 \( \mathcal{T}_{\frac{p}{q}} \) follows a rhythm

The first step is to prove that \( \mathcal{T}_{\frac{p}{q}} \) is generated by rhythm and labelling computable from those generating \( L_{\frac{p}{q}} \) (Theorem 33). We denote by \( \mu \) the mirror function that is: \( \mu(r_0, r_1, \ldots, r_{q-1}) = (r_{q-1}, \ldots, r_1, r_0) \); and by \( \sigma \) the circular rotation that is: \( \mu(r_0, r_1, \ldots, r_{q-1}) = (r_1, r_2, \ldots, r_{q-1}, r_0) \). The graph \( \mathcal{T}_{\frac{p}{q}} \) follows the invalid rhythm \( \mu(r_{\frac{p}{q}}) \) as stated below.

**Lemma 31.** Given a base \( \frac{p}{q} \), the state \( (-n) \) of \( \mathcal{T}_{\frac{p}{q}} \) has \( r_k \) outgoing edges where \( k = (-n) \% q \).
Lemma 32. Given a base $\frac{p}{q}$, for all $i < (B - 1)$, $\sigma^i(\mu(r_{\frac{p}{q}}))$ is an invalid rhythm and $\sigma^{(B-1)}(\mu(r_{\frac{p}{q}}))$ is a valid rhythm.

Proof. Let us denote by $\frac{p}{q}$ a base. It should be noted that in case one considers a rhythm with positive components (which is the case for $r_{\frac{p}{q}}$, and therefore for $\sigma^i(\mu(r_{\frac{p}{q}}))$ for all $i$), it is valid if and only if its first component is greater than 1. The statement is then equivalent to prove that the last $(B - 1)$ components of $r_{\frac{p}{q}}$ are all equal to 1, and the one before that is not. It is done below.

Let us denote by $j$ the number of 1’s at the end of $r$; that is the integer such that $r_{q-j} = r_{q-j+1} = \ldots = r_{q-1} = 1$, and $r_{q-j-1} > 1$.

From Lemma 16, it follows that for all integers $i \leq j$, $\Sigma_{k=(q-i)}^{q-1} r_k = \lfloor \frac{i p}{q} \rfloor$ that is $i = \lfloor \frac{i p}{q} \rfloor$. And this equation is false for $i > j$, from the same lemma, hence $j$ is the biggest integer $i$ satisfying $i = \lfloor \frac{i p}{q} \rfloor$. 

$$
\frac{ip}{q} - 1 < i \leq \frac{ip}{q}
$$

$$
-1 < i \left(1 - \frac{p}{q}\right) \leq 0
$$

$$
\frac{1}{q-p} > i \geq 0
$$

$$
\frac{q}{p-q} > i \geq 0
$$

$$
\frac{q-1}{p-q} > i \geq 0
$$
It follows that $j$ is the biggest (non-negative) integer which is smaller than $\frac{q-1}{p-q}$, hence $j = \left\lfloor \frac{q-1}{p-q} \right\rfloor = B - 1$.

Given the Christoffel rhythm $r_\frac{p}{q}$ we define the rhythm $s_\frac{p}{q}$ as the $q$-tuple shifted $(B - 1)$ times from the mirror of $r_\frac{p}{q}$ that is $s_\frac{p}{q} = \sigma((B-1)(\mu(r_\frac{p}{q}))$. The labelling $\mu_\frac{p}{q}$ is defined analogously, that is $s_\frac{p}{q} = \sigma((B-1)(\lambda \gamma_\frac{p}{q}))$. The next statement follows directly from Lemma 31.

**Theorem 33.** The graph of the vertices reachable from $-B$ in $\overline{T_\frac{p}{q}}$ is an i-tree generated by the rhythm $s_\frac{p}{q}$ and labelling $\mu_\frac{p}{q}$.

Since $s_\frac{p}{q}$ is a permutation of $r_\frac{p}{q}$, they have the same growth ratio, hence Theorem 33 allows to apply Proposition 35, resulting in the following statement.

**Corollary 34.** The function mapping $(-n-B)\frac{p}{q}$ to $(n)\frac{p}{q}$ is rational.

### 6.2 the remarkable cases where $(p-q)|(p-1)$

The purpose of this section is to study the bases $\frac{p}{q}$ where $(p-q)|(p-1)$. It corresponds to the cases where $s_\frac{p}{q}$ and $r_\frac{p}{q}$ are equal (cf. Proposition 35), and as a consequence $\overline{L_\frac{p}{q}}$ and $L_\frac{p}{q}$ are equal up to a permutation of letters (cf. Corollary 36).

**Proposition 35.** Given a base $\frac{p}{q}$, the rhythm $s_\frac{p}{q}$ and $r_\frac{p}{q}$ are equal if and only if $\frac{p-1}{p-q}$ is an integer.

**Corollary 36.** Given a base $\frac{p}{q}$, such that $\frac{p-1}{p-q}$ is an integer, then there exists a permutation $\sigma$ of $A_p$ such that $L_\frac{p}{q} = \sigma(L_\frac{q}{q})$.

The proof of Proposition 35 breaks down into the two technical Lemmas 37 and 38 both heavily relying on the properties of Christoffel rhythm given at Lemma 19. Lemma 37 characterise the Christoffel rhythms satisfying $(p-1)|(p-q)$ while 38 characterise the Christoffel rhythm $r_\frac{p}{q}$ being equal to $s_\frac{p}{q}$. Both characterisation coincide.

**Lemma 37.** A rhythm is the Christoffel rhythm of a base $\frac{p}{q}$ verifying $(p-q)|(p-1)$ if and only if it is of the form $(2^{1^{(j-1)}})k1$ for some $j$ and $k$.

**Proof.** Forward direction. We denote by $j$ the integer such that $(p-q)(j+1) = (p-1)$, and by $k = (p-q)$. Let us prove that the rhythm $(2^{1^{(j-3)}})1$ is the Christoffel rhythm of $\frac{p}{q}$. It as the correct growth: there is $(p-q)j + 1 = q$ letters, whose sum is equal to $(p-q)(j+1) + 1 = p$. Besides, it seems to be an equitable way to part $p$ objects into $q$ cases: it is of the form $(2^{1^{(j-3)}})1(2^{1^{(j-3)}})1(2^{1^{(j-3)}})$.

More formally, the characterisation of Christoffel rhythm from Lemma 19 applies: $1^{(j-3)}(2^{1^{(j-3)}})(p-q-1)2$ is obviously conjugated with $(2^{1^{(j-3)}})(2^{1^{(j-3)}})$, with the dot representing the cutting point.

Backward direction. A rhythm of this form, is necessarily a Christoffel rhythm, as it satisfies the conditions of Lemma 19. Moreover, its corresponding $q = k \times j + 1$ and $p = k(j + 1) + 1$ hence $(p-1) = (k(j+1))$ is a multiple of $(p-q) = k$. 

22
Lemma 38. Given a base \( \frac{p}{q} \), \( r_\frac{p}{q} = s_\frac{p}{q} \) if and only if \( r_\frac{p}{q} \) is for the form \( (21^{(j-1)})^k 1 \) for some \( j \) and \( k \).

Proof. Backward direction is immediate, as \( (21^{(j-1)})^k 2 \) is obviously a palindrome.

Forward direction. Let \( \frac{p}{q} \) be a base such that \( r_\frac{p}{q} = s_\frac{p}{q} \). It should be noted that a rhythm satisfying this equality is necessarily of the form \( u 1^j \) for some integer \( j \) and some palindrome \( u \). The idea of the proof is to use alternatively the characterisation of Christoffel rhythm from Lemma 19b and the previous remark to bring out the form of \( r_\frac{p}{q} \). We denote by \( j \) the number of trailing 1’s in \( r_\frac{p}{q} \) hence \( r_\frac{p}{q} \) is of the form \( 2u21^j \).

Step 1 - From Lemma 19a, \( 1u21^j \) must be a palindrome, hence \( r \) is either

a. of the form \( 21^{(j-2)}u 21^j \) for some \( u' \) shorter than \( u \):

b. equal to \( 21^{(j-2)}21^j \).

Step 2 - From the hypothesis \( r = \tilde{r} \), \( 21^{(j-2)}2u'2 \) is a palindrome, hence \( r \) is either

a. of the form \( 21^{(j-2)}u'' 21^{(j-2)}21^j \), for some \( u'' \) shorter than \( u' \):

b. equal to \( 21^{(j-2)}21^{(j-2)}21^j \).

By iterating this process, it will necessarily finish in step 1b or 2b, concluding the forward direction.

\( \square \)

7 Conclusion and future work

In this work, we studied a new way of generating infinite trees and languages. Usual ways to build such structures all use depth-first processes, while in this article, we use a rhythm (that is a tuple of integer) to define a breadth-first process.

It is both remarkable and astonishing that the resulting structure is either very simple (e.g. for language, rational) or very complicated (e.g. for language, BLIP) with no middle ground. The difference between both behaviour directly depends on the growth ratio (that is, the average of the components) of the used rhythm; if it is an integer, the structure is simple, otherwise, the structure is complicated. This separation is due to a direct link between any structure generated from a rhythm \( r \) and the number system whose base is the growth ratio of \( r \). Indeed, rational base number systems feature structures that are ostensibly more complicated than the respective structure in an integer base number system.

This link allows to revisit the work of [3] about the representation of negative integers in rational base number system; it is the subject of ongoing work and will probably be integrated in an extended version of this work. Moreover, since every rhythm of integer growth ratio generates a rational language, it immediately raise the question of characterising the class of rational languages that may be be generated by rhythm.

In the higher scheme, this work presents a framework for breadth-first process that is completely unexplored and opens a lot of perspectives. For starters, the use of a rhythm is not mandatory, it simply corresponds to a periodic approach that we originally observed in rational base number system; we plan to study aperiodic
cases in the future. Moreover, with simple automata-like structure, one could generate languages of trees (by opposition to the languages of words generated by rhythms).

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