An inverse problem for localization operators

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Abstract
A classical result of time–frequency analysis, obtained by Daubechies in 1988, states that the eigenfunctions of a time–frequency localization operator with circular localization domain and Gaussian analysis window are the Hermite functions. In this contribution, a converse of Daubechies’ theorem is proved. More precisely, it is shown that, for simply connected localization domains, if one of the eigenfunctions of a time–frequency localization operator with Gaussian window is a Hermite function, then its localization domain is a disc. The general problem of obtaining, from some knowledge of its eigenfunctions, information about the symbol of a time–frequency localization operator is denoted as the inverse problem, and the problem studied by Daubechies as the direct problem of time–frequency analysis. Here, we also solve the corresponding problem for wavelet localization, providing the inverse problem analogue of the direct problem studied by Daubechies and Paul.

1. Introduction

Most real-life signals of interest change their frequency properties over time. Therefore, a signal description by means of time–frequency analysis is often preferable to the signal’s Fourier transform, which reliably yields frequency information, but without any localization in time. The core purpose of time–frequency analysis is to represent a given signal as a function in the time–frequency or in the time-scale plane. However, in real world applications like optics and wireless communications, one can only ‘sense’ a signal within a certain region of those planes. This means that, in practice, the part of the signal outside the region of interest is neglected and only its ‘localized’ version is observed. Localization operators turn this observation process into rigorous mathematical terms. They transform a given signal into one that is localized in a given region by reducing the signal energy outside that region to a negligible amount.

The first approach to time–frequency localization, introduced in 1961, consists in separately selecting time- and frequency-content, and is described in a famous series of papers known as the ‘Bell labs papers’. We refer to Slepian’s review [20] for an account of this beautiful body of work. In 1988, Daubechies added a new perspective by introducing
operators, that localize directly in the time–frequency plane [4] and, together with Paul [5], extended the analysis to the time-scale plane. The time–frequency plane is associated with the short-time Fourier transform and the time-scale plane is associated with the wavelet transform. We begin our presentation by defining the short-time Fourier transform, which leads to the concept of time–frequency localization operators.

The short-time Fourier (or Gabor) transform of a function or distribution \( f \) with respect to a window function \( g \in L^2(\mathbb{R}) \) is defined to be, for \( z = (x, \xi) \in \mathbb{R}^2 \),

\[
V_{g,f}(z) = V_{g}f(x, \xi) = \int_{\mathbb{R}} f(t) \overline{g(t-x)} e^{-2\pi i t \xi} \, dt,
\]

where the overline denotes complex conjugation. We let \( \pi(z)g(t) = g(t-x) e^{2\pi i t \xi} \) and observe that \( f \) can be resynthesized from \( V_{g}f \) as

\[
f = \frac{1}{\|g\|_2^2} \int_{\mathbb{R}^2} V_{g}f(z) \pi(z) g \, dz.
\]

Given a symbol \( \sigma \in L^1(\mathbb{R}^2) \), time–frequency localization operators \( H_{\sigma,g} \) are defined by

\[
H_{\sigma,g}f = \int_{\mathbb{R}^2} \sigma(z) V_{g}f(z) \pi(z) g \, dz = V_{g}^* \sigma V_{g}f.
\]

In signal processing, it is very common to modify a signal \( f \) by acting on its time–frequency coefficients \( V_{g}f \), for example, in order to achieve noise reduction [14]; the corresponding localization operators have been the object of research in time–frequency analysis, [7, 3]. In [4], Daubechies considered the window \( g(t) = \phi(t) = 2^t e^{-\pi t^2} \), the symbol \( \sigma(z) = \chi_\Omega(z) \), i.e. the indicator function of a set \( \Omega \subset \mathbb{R}^2 \), and investigated the eigenvalue problem

\[
H_{\Omega}f := H_{\chi_\Omega,g}f = \lambda f
\]

for the case where \( \Omega \) is a disc centered at zero. She concluded that in this situation, the eigenfunctions of \( H_{\chi_\Omega} \) are the Hermite functions. Consequently, since, \( H_{\Omega_2 \setminus \Omega_1} = H_{\Omega_2} - H_{\Omega_1} \), for two sets \( \Omega_1 \subset \Omega_2 \), the Hermite functions are also eigenfunctions with respect to domains in the form of an annulus centered at zero and for any union of annuli.

Problem (2) is important in time–frequency analysis, because its solutions are the functions with best concentration in the subregion \( \Omega \) of the time–frequency plane, where we consider the time–frequency concentration of a function \( f \) in \( \Omega \subset \mathbb{R}^2 \) defined as

\[
C_\Omega(f) = \frac{\int_{\Omega} |V_{g}f(z)|^2 \, dz}{\|f\|_2^2}.
\]

In this paper, we will be concerned with the inverse situation of the one considered by Daubechies. This leads us to the following question.

- Given a localization operator with unknown localization domain \( \Omega \), can we recover the shape of \( \Omega \) from information about its eigenfunctions?

This is a new type of inverse problem, and we will call it the inverse problem of time–frequency localization. We solve the problem in the case where explicit computations can be made, which is the set-up of [4]. Our main contribution is the following.

**Theorem 1.** Let \( \Omega \subset \mathbb{R}^2 \) be simply connected. If one of the eigenfunctions of the localization operator \( H_{\Omega} \) is a Hermite function, then \( \Omega \) must be a disc centered at 0.

Let us briefly discuss some motivations for our studies and consequences of our result. Hermite functions have been proposed as modulating pulses in pulse-shape modulation for ultra-wideband (UWB) communication, mainly due to their maximal joint concentration in time and frequency, cf [19, 15, 9] and references therein. The receiver of the communication
system often applies a filter with the modulation pulses as eigenfunctions corresponding to large eigenvalues, in order to suppress random noise that accumulates during transmission. In this situation, theorem 1 shows that the filter must be designed with a circular localization domain. Furthermore, if the filter on the receiver side is known or designed to have one single Hermite function as an eigenfunction, it is possible to guarantee the location of the time–frequency plane that the filter is sensing. This is particularly important in UWB communication, where the permitted spectrum is officially prescribed, cp [6].

This last remark also hints at an additional possible application, which is system identification. Identification of linear time-variant systems is a notoriously difficult task in general, cp [13, 2, 17]. While a linear time-invariant system is straightforwardly identified by sending an impulse to the system and retrieving the impulse response, [10, section 4.2], no similar method exists for general linear time-variant systems. By our result, and for a system that is known to be a localization operator of the form \( H_\Omega \) for some time–frequency region \( \Omega \), we may send any Hermite function to the system and judge from the response whether \( \Omega \) can be a disc. In the positive case, one can then evaluate the size of the disc by subsequently sending additional Hermite functions and evaluating the resulting scaling factors. Obviously, this approach should be extended to other shapes and its feasibility will be the topic of further research.

In analogy with theorem 1, we will also consider an inverse problem for wavelet localization operators. Here, we show that the domain of localization of the localization operators investigated by Daubechies and Paul [5] is a pseudohyperbolic disc in the upper-half plane whenever one of the operator’s eigenfunctions is the Fourier transform of a Laguerre function. We will essentially use methods from complex analysis and our techniques are strongly influenced by the ideas contained in [1] and [18].

This paper is organized as follows. Section 2.1 collects some properties of the eigenfunctions of localization operators with respect to radially weighted measures and section 2.2 deduces the geometry of localization domains under the assumption of orthogonality of any single monomial to almost all monomials. The corresponding inverse problem for Gabor localization is studied in section 3, and section 4 is devoted to the investigation of the inverse problem for wavelet localization.

2. Double orthogonality and reproducing kernel Hilbert spaces

This section is devoted to the properties of complex monomials, namely their double orthogonality with respect to any radially weighted measures and the consequences of this property.

2.1. Eigenfunctions of localization operators

Let \( D_a \) denote a disc of radius \( a, 0 < a \leq \infty \), \( d\mu(z) = \mu(|z|) \, dz \) a radially weighted measure and \( dz \) a Lebesgue measure on \( \mathbb{C} \).

In the following, we will denote by \( \mathcal{H}_a = L^2(D_a, d\mu(z)) \) the Hilbert space of analytic functions \( F \) on \( \mathbb{C} \), such that

\[
\|F\|_{\mathcal{H}_a} = \int_{D_a} |F(z)|^2 \, d\mu(z)
\]

is finite.

In proposition 1 we collect the most important facts about the ‘direct problem’ studied in [4], [5] when transferred to the complex domain. This point of view is essentially contained in [18], but we have observed that both problems can be understood as special cases of a more
general formulation with general radial measures on complex domains. This viewpoint is later reflected in our derivation of the results about the inverse problems.

**Proposition 1.** Consider all radial measures on discs $D_R$ with radius $R$ in the complex plane, i.e. the measures constituted by the weighted measure $d\mu(z) = \mu(|z|) \, dz$, defined on $D_R$, whose weight $\mu(|z|)$ depends only on $r = |z|$. The following statements are true.

(a) The monomials are orthogonal on any disc $D_R$ centered at zero with radius $R$ in the complex plane and with respect to all concentric measures. Consequently, the monomials are also orthogonal on any annulus centered at zero.

(b) Assume $0 < c_{n,a} < \infty$ for all moments $c_{n,a}$ of $\mu(|z|)$ dz. Then, the normalized monomials $e_{n,a} = z^n/\sqrt{(c_{2n+1, a})}$ constitute an orthonormal basis for $H_a$.

(c) If, in addition, $\sum_{n \geq 0} (c_{2n+1, a})^{-1}|z|^{2n}$ is finite for all $z \in D_a$, then $H_a$ is a reproducing kernel Hilbert space with reproducing kernel

$$K(z, w) = \sum_{n \geq 0} (c_{2n+1, a})^{-1} |z|^n w^n.$$ 

(d) The functions $F(z) = e_{n,a}$ are eigenfunctions of the problem

$$\int_{D_R} F(z) K(z, w) \, d\mu(z) = \lambda F(w).$$ 

(4)

**Proof.**

(a) Orthogonality can directly be seen by

$$\int_{D_R} z^{n+m} \, d\mu(z) = \int_{0}^{R} r^{n+m+1} \int_{0}^{2\pi} e^{i(n-m)\theta} \, d\mu(r) \, dr = c_{2n+1, R} \delta_{n,m},$$

with $c_{n,R} = 2\pi \int_{0}^{R} r^n \, d\mu(r) \, dr$.

(b) Consider a domain $D_a, R < a \leq \infty$ such that $\lim_{r \to a} d\mu(r) = 0$. Since the power series $\sum_{n \geq 0} a_n z^n$ of an analytic function $F$ on $\mathbb{C}$ converges uniformly on every $D_R$, we may interchange integral and summation in the following equations: suppose that $\langle F, e_{n,a} \rangle = 0$ for all $n \in \mathbb{Z}$, then

$$0 = \frac{1}{\sqrt{(c_{2n+1, a})}} \lim_{R \to a} \int_{D_R} \sum_{n \geq 0} a_n z^n \, d\mu(|z|) \, dz$$

$$= \frac{1}{\sqrt{(c_{2n+1, a})}} \lim_{R \to a} \sum_{n \geq 0} a_n \int_{D_R} z^n \, d\mu(|z|) \, dz$$

$$= \frac{1}{\sqrt{(c_{2n+1, a})}} \lim_{R \to a} \sum_{n \geq 0} a_n c_{2n+1, R}$$

which implies $a_m = 0$ for all $m$ and hence $F \equiv 0$, which proves completeness of the functions $\{e_{n,a}\}$ in $H_a$.

(c) We need to show that point evaluations of $F \in H_a$ are bounded. Expanding $F$ in terms of $\{e_{n,a}\}$, we observe that

$$|F(z)| \leq \left| \sum_{n \geq 0} (F, e_{n,a}) \frac{z^n}{\sqrt{(c_{2n+1, a})}} \right| \leq \|F\|_{H_a} \cdot \left( \sum_{n \geq 0} \frac{1}{(c_{2n+1, a})} |z|^{2n} \right)^{1/2}.$$

Thus, by the assumption on the growth of the moments, $H_a$ is a reproducing kernel Hilbert space.
(d) Write $U$ for the operator which multiplies a function $F \in \mathcal{H}$ by the characteristic function of the circle $D_R$ and $P$ for the orthogonal projection onto $\mathcal{H}_a$, given by the reproducing kernel. Since $P(\sqrt{z_2^{(i+1)^2}}) = \sqrt{z_2^{(i+1)^2}}$, we note that

$$0 = \int_{D_R} e_{n,a} e_{n,a} \mu(z) \, dz = \int_{D_a} e_{n,a} P(U) e_{n,a} \mu(z) \, dz$$

and completeness of $e_{n,a}$ implies that $P(U) e_{n,a} = e_{n,a}$. Denoting by $K(z_2, w)$ the reproducing kernel of $\mathcal{H}_a$, the functions $F(z_2) = e_{n,a}$ are eigenfunctions of problem (4).

Using appropriate unitary operators (the so-called Bargmann and Bergman transform, to be defined later in this paper), the solution to the general problem just described can be shown to be equivalent to the solution of the ‘direct’ problems considered in [4] and [5]. Indeed, the $d \mu(z) = e^{-\pi |z|^2} \, dz$ case can be translated to the Gabor localization problem studied by Daubechies and the case $d \mu(z) = (1 - |z|^2)^\alpha \, dz$ to the wavelet localization studied by Daubechies and Paul. Details will be given in sections 3 and 4.

2.2. The localization domain of monomials

We now turn to the general problem, given by (4). The following, central proposition states that orthogonality of any monomial to almost all other monomials with respect to a bounded, simply connected domain $\Omega \subset \mathbb{C}$ forces $\Omega$ to be a disc centered at zero. We also consider more general domains as described in corollary 1. Note that we identify $\mathbb{R}^2$ with $\mathbb{C}$ for the geometric description. The proof is based on an idea of Zalcman [22] and is essentially similar to the proof given in [1], but in a more general setting, namely generalizing from area measure to general concentric measures. To adapt the original argument, we rely on proposition 1.

**Proposition 2.** Let $d \mu(z)$ be a positive, concentric measure on $D_a \subseteq \mathbb{C}$ and consider a simply connected set $\Omega \subset D_a$. Assume, for some $m$ and $k \geq 0$ that

$$\int_{\Omega} \frac{|z|^2m}{z^2 - w^2} \, d\mu(z) = \lambda \delta_{k,0}. \quad (6)$$

Then $\Omega$ must be a disc centered at zero.

**Proof.** Since

$$\frac{\pi w}{z - w} = - \frac{\pi}{1 - \frac{w}{z}} = - \sum_{n=1}^{\infty} \frac{\pi}{w^{n-1}},$$

we have for every $z \in \Omega$ and $w$ such that $|w| > \sup\{|z|: z \in \Omega\}$, the following expansion:

$$|z|^2m \frac{\pi w}{z - w} = -|z|2m \left( \frac{\pi}{w} + \frac{\pi^2}{w^2} + \frac{\pi^3}{w^3} + \cdots \right).$$

Integrating term wise and using (6) yields

$$\int_{\Omega} \frac{|z|^2m}{z - w} \, d\mu(z) = 0; \quad (7)$$

hence

$$\int_{\Omega} \frac{|z|^2m}{|z - w|^2} \, d\mu(|z|) = \int_{\Omega} |z|^2m \frac{\pi}{z - w} \, d\mu(z) \quad (8)$$

$$= \frac{1}{w} \int_{\Omega} \frac{|z|^2m \pi w}{z - w} \, d\mu(z) = 0. \quad (9)$$
The left expression in (8) is continuous as a function of \( w \) since the integrand is locally integrable in \( z \). Therefore, (9) holds on \( \overline{\Omega} \).

We next show that 0 is inside \( \Omega \). Begin by observing that, for \( |w| > \text{sup}|z| \cap \Omega \), we can expand and integrate term wise so that

\[
\int_{\Omega} \frac{1}{|z|^{2m}} \frac{1}{\overline{z} - w} d\mu(z) = \frac{1}{w} \int_{\Omega} \frac{1}{|z|^{2m}} \overline{w} \frac{1}{\overline{z} - w} d\mu(z) = \frac{1}{w} \lambda.
\]

(10)

Let \( C > \text{sup}|z| \cap \Omega \). We let \( d(w, \Omega) \) denote the Euclidean distance between \( w \) and \( \Omega \), i.e. \( d(w, \Omega) = \inf_{w' \in \Omega} |w - w'| \). Then the following pointwise estimate in \( \overline{\Omega} \) holds:

\[
\int_{\Omega} \left| \frac{1}{|z|^{2m}} \frac{1}{\overline{z} - w} \right| d\mu(z) \leq \frac{C^{2m}}{d(w, \Omega)}.
\]

This allows us to extend (10) by analytic continuation to \( \overline{\Omega} \).

Suppose now that 0 \( \not\in \overline{\Omega} \). Then we can find a sequence of points \( \{w_n\} \) contained in \( \overline{\Omega} \) such that \( w_n \rightarrow 0 \). This would give

\[
\lim_{n \to \infty} \int_{\Omega} \frac{1}{|z|^{2m}} \frac{1}{\overline{z} - w_n} d\mu(z) = \lim_{n \to \infty} \frac{1}{w_n} \lambda_m = \infty.
\]

On the other hand, because of the continuity of the left expression in \( \overline{\Omega} \),

\[
\lim_{n \to \infty} \int_{\Omega} \frac{1}{|z|^{2m}} \frac{1}{\overline{z} - w_n} d\mu(z) = \int_{\Omega} \frac{1}{|z|^{2m}} \frac{1}{\overline{z}} d\mu(z),
\]

and the integral on the right is bounded for every \( m \geq 0 \), since we are assuming that 0 \( \not\in \Omega \). This is a contradiction and we must have 0 \( \in \Omega \).

Finally, we can consider \( D_R \), the largest disc centered at zero and contained in \( \Omega \). Using proposition 1(a), we can repeat the steps leading to (10) with \( D_R \) instead of \( \Omega \). Pick a point \( w_0 \in \partial D_R \cap \partial \Omega \). Then

\[
\int_{\Omega \setminus D_R} |z|^{2m} \frac{|z|^2 - \text{Re} \Sigma w_0}{|z - w_0|^2} d\mu(z) = 0.
\]

Since for \( z \in \Omega \setminus D_R \), \( |z||w_0| \leq |z|^2 \), the integrand is positive on \( \Omega \setminus D_R \). This forces \( \Omega \setminus D_R \) to be of area measure zero, which implies \( \Omega = D_R \). \( \square \)

For the next statement, we consider a more general situation. Let \( \gamma_j \), \( j = 1, \ldots, n \) be a family of non-intersecting Jordan curves with interiors \( I^j \) such that \( I^{j-1} \subset I^j \) for all \( j > 1 \).

If \( n \) is even, set \( K = \frac{n}{2} \) and let \( \Omega_k = I^{2k-1} \setminus I^{2k-2} \) for \( k = 1, \ldots, K \).

If \( n \) is odd, set \( K = \frac{n-1}{2} \) and let \( \Omega_k = I^{2k} \setminus I^{2k-2} \) for \( k = 2, \ldots, K \). For the situation just described, we set \( \Omega = \bigcup_{k=1}^{K} \Omega_k \) and consider the corresponding localization operator. The next corollary shows that under the double orthogonality condition (6), all curves must contain 0 in their interior. Furthermore, for \( n = 2 \), if one of the two curves is a circle, then \( \Omega \) must be an annulus.

**Corollary 1.**

(a) Let (6) hold for \( \Omega = \bigcup_{k=1}^{K} \Omega_k \) defined by a family of nested Jordan curves as described above. Then all curves \( \gamma_j \) must contain zero.

(b) If \( n = 2 \) and \( \gamma_j \) is a circle centered at 0 for \( j = 1 \) or \( j = 2 \), then \( \Omega \) is an annulus, see figure 1.

**Proof.**

(a) We will show by induction that 0 must be inside all curves \( \gamma_j, j = 1, \ldots, n \).
Case $n = 1$. Then $\Omega$ is the interior of $\gamma_1$, therefore simply connected, and it follows from the proof of proposition 2, that $0 \in \Omega$.

Case $n = 2$. Then $\Omega = I^{\gamma_2} \setminus I^{\gamma_1}$ and $I^{\gamma_1}$ is simply connected. We apply, by assuming that $0 \in (I^{\gamma_1})^c$, the argument used in the first paragraph of case $n = 1$ to show that $0 \in (\Omega \cup I^{\gamma_1})$. Then, either $0 \in \Omega$ or $0 \in I^{\gamma_1}$. In the first case, we consider again $D_R$, the largest disc centered at zero contained in $\Omega$ and argue as in case $n = 1$ to show that $\Omega = D_R$, which contradicts the assumption that $n = 2$. Therefore, $0 \in I^{\gamma_1}$.

Arbitrary $n \in \mathbb{N}$. Assume that, for $n - 1$ curves, 0 is inside all curves. For $n$ curves, we first show that $0 \in I^{\gamma_n}$, assume that $0 \in \Omega_k$ and use, as before, the argument from case $n = 1$ to show that this leads to $n = 1$. Consequently, 0 must be inside the remaining $n - 1$ curves and, by induction hypothesis, inside all curves $\gamma_j$, $j = 1, \ldots, n$.

(b) First assume that $\Omega$ is a disc, centered at zero, with a hole—in other words, that $\gamma_2$ is a circle. Then, $I^{\gamma_n}$ is a disc centered at zero, such that (6) holds for $I^{\gamma_n}$ and therefore also for $I^{\gamma_1}$. Since the latter is simply connected, it must be a disc centered at 0.

Now let $I^{\gamma_n}$ enclose a disc centered at 0. We then consider the largest annulus $\Pi$ contained in $\Omega$; it is given by $\Pi = D_R \setminus I^{\gamma_1}$ where $D_R$ is the largest disc centered at zero and contained in $I^{\gamma_1}$. Due to proposition 1(a), condition (6) holds on $\Pi$ and we obtain (10) with $\Pi$ instead of $\Omega$. Pick a point $w_0 \in \partial D_R \cap \gamma_2$. Then

$$
\iint_{\Omega \setminus \Pi} |z|^{2m} \frac{|z|^2 - \Re \bar{w}_0}{|z - w_0|^2} d\mu(z) = 0.
$$

and $|\Re \bar{w}_0| \leq |z||w_0| \leq |z|^2$ and the integrand is positive on $z \in \Omega \setminus \Pi$, which implies $\Omega = \Pi$. \hfill \Box
3. An inverse problem for Gabor localization

In this section, we prove theorem 1 and derive the complete solution of the classical eigenvalue problem (2) from the assumption that any single solution is a Hermite function.

In the following, we identify \((x, \xi)\) with \(z = x + i\xi\) and we recall that \(\pi(z)\varphi(t) = \pi(x, \xi)\varphi(t) = \varphi(t - x)e^{2\pi i\xi t}\).

3.1. Bargmann transform

In the Gabor case, the choice of the Gaussian function \(\varphi(t) = 2^{\frac{1}{4}}e^{-\pi t^2}\) allows the translation of the time–frequency localization operator \(H_{\mu}\), to the complex analysis set-up via the Bargmann transform \(\mathcal{B}\). \(\mathcal{B}\) is defined for functions of a real variable as

\[
\mathcal{B}f(z) = \int_{\mathbb{R}} f(t) e^{2\pi i z t - \frac{1}{2}z^2} \, dt = e^{-i\pi z^2 + \pi \frac{z^2}{2}} \mathcal{V}\varphi f(x, -\xi).
\]

\(\mathcal{B}\) maps \(L^2(\mathbb{R})\) unitarily onto \(\mathcal{F}^2(\mathbb{C})\), the Bargmann–Fock space of analytic functions with the inner product obtained by choosing the measure \(d\mu(z) = e^{-|z|^2} \, dz\).

3.2. The Hermite functions

The normalized monomials \(e_n = (\pi^n/n!) \cdot z^n = B h_n(z) = e^{-i\pi z^2 + \pi \frac{z^2}{2}} \mathcal{V}\varphi h_n(z)\) form an orthonormal basis for \(\mathcal{F}^2(\mathbb{C})\). Here, \(h_n(t) = c_n e^{\pi t^2} \left(\frac{dt}{\sqrt{\pi}}\right)^n (e^{-2\pi t^2})\) are the Hermite functions, which, by appropriate choice of \(c_n\), provide an orthonormal basis of \(L^2(\mathbb{R})\). As a direct consequence of the unitarity of \(\mathcal{B}\) and \(\mathcal{V}\varphi\), the set \(\{\mathcal{V}\varphi h_n, n \in \mathbb{N}\}\) is orthogonal over all discs \(D_R\).

3.3. Proof of theorem 1

We first deduce the equivalent formulation of the eigenvalue problem (2) in the Bargmann domain. Since the Bargmann transform is unitary, (2) is equivalent to

\[
\int_{\Omega} \mathcal{V}\varphi f(\xi) \mathcal{B}(\pi(z)\varphi)(w) \, dz = \lambda \mathcal{B}f(w).
\]

Now, since \(\mathcal{B}(\pi(z)\varphi)(w) = e^{-i\pi z^2} e^{-|z|^2/2} e^{2\pi i w^2}\), we write the previous equation as

\[
\int_{\Omega} \mathcal{V}\varphi f(\xi) e^{-i\pi z^2} e^{-|z|^2/2} e^{2\pi i w^2} \, dz = \lambda \mathcal{B}f(w).
\]

Thus, by (11), we have

\[
\int_{\Omega} \mathcal{B}f(z) e^{\pi z\xi - \pi |z|^2} \, dz = \lambda \mathcal{B}f(w).
\]

By the unitarity of the Bargmann transform, we conclude that the eigenvalue problem (2) on \(L^2(\mathbb{R})\) is equivalent to

\[
\int_{\Omega} F(z) e^{\pi z\xi - \pi |z|^2} \, dz = \lambda F(w),
\]

on \(\mathcal{F}^2(\mathbb{C})\). We may now expand the kernel \(e^{\pi z\xi}\) in its power series which transforms the eigenvalue equation to

\[
\lambda F(w) = \sum_{n=0}^{\infty} \frac{\pi^n}{n!} w^n \int_{\Omega} F(z) e^{-\pi |z|^2} \, dz.
\]
Now we use the assumption that $z^m$ solves (13) for $\lambda = \lambda_m$—in other words, that any of the solutions of (2) is a Hermite function. Setting $F(z) = z^m$ then gives

$$\lambda_m w^m = \sum_{n=0}^{\infty} \frac{n!}{n!} \int_{\Omega} z^m e^{-\pi|z|^2} \, dz.$$ 

By the identity theorem for analytic functions, this implies

$$\int_{\Omega} z^m e^{-\pi|z|^2} \, dz = \lambda_m \frac{m!}{\pi m} \delta_{n,m}.$$ 

In particular, setting $n = m + k$,

$$\int_{\Omega} |z|^{2m-k} e^{-\pi|z|^2} \, dz = \lambda_m \delta_{k,0}, \text{ for all } k \geq 1. \quad (14)$$

Now proposition 2 can be applied and we conclude that $\Omega$ must be the union of $\frac{n}{2}$ annuli centered at 0 for even $n$ and the union of a disc and $\frac{n-1}{2}$ annuli centered at 0 for odd $n$. In particular, for simply connected $\Omega$, we obtain a disc centered at zero.

### 3.4. Consequences of theorem 1

**Corollary 2.** Let $\Omega$ be simply connected. If the Gabor transform of one of the eigenfunctions of the localization operator $H_\Omega$ has Gaussian growth, $O(e^{-\pi|z|^2})$, then $\Omega$ must be a disc. The same conclusion holds if some eigenfunction has Gaussian growth in both the time and the frequency domains.

**Proof.** This is a consequence of the version of Hardy’s uncertainty principle for the Gabor transform proved by Gröchenig and Zimmermann [11]. They showed that, if $V_{g,f}(z) = O(e^{-\pi|z|^2})$, then both $f$ and $g$ must be time–frequency shifts of a Gaussian function. Therefore, under the hypotheses of the corollary, the Gaussian (which is the first Hermite function) is an eigenfunction of the localization operator $H_\Omega$ and by theorem 1, $\Omega$ must be a disc. The second statement follows in a similar fashion from the classical Hardy uncertainty principle [12].

The result of theorem 1 immediately implies that the complete solution of (2) is given by the orthonormal basis of Hermite functions.

**Corollary 3.** Assume that an orthonormal basis of $L^2(\mathbb{R})$ has doubly orthogonal Gabor transform with respect to the Gaussian window $\varphi$ and some domain $\Omega$:

$$\int_{\Omega} \mathcal{V}_{\varphi} \psi_j(z) \overline{\mathcal{V}_{\varphi} \psi_{j'}}(z) \, dz = c_{j,j'} \delta_{j,j'}.$$ 

Let $\Omega$ be simply connected or of the form stated in corollary 1(b). If, for any $j_0$, $\varphi_{j_0} = h_{j_0}$ is a Hermite function, then for every $j \geq 0$,

$$\psi_j = h_j.$$ 

**Proof.** Note that an orthonormal basis of $L^2(\mathbb{R})$ satisfies (15) if and only if it consists of eigenfunctions of the localization operator $H_\Omega$. Hence, we are in the situation of theorem 1, and $\Omega$ must be disc centered at zero, the union of a disc and a finite number of annuli centered at zero or an annulus centered at zero, respectively. This, in turn, implies that all eigenfunctions are Hermite functions.
Remark 1. Note the following consequence of theorem 1: if the localization domain $\Omega$ is not a disc, then the function of optimal concentration inside $\Omega$, in the sense of (3), cannot be a Gaussian window. On the other hand, it is well-known that Gaussian windows uniquely minimize the Heisenberg uncertainty relation. In this sense, disks seem to be the optimal domain for measuring time–frequency concentration.

Gaussian windows $\varphi$ (and also higher order Hermite functions) are a popular choice for the basic atom in the generation of Gabor frames, whose members are given as $\pi(\lambda)\varphi$, $\lambda \in \Lambda$ for some discrete subgroup $\Lambda \subset \mathbb{R}^2$. A popular choice of $\Lambda$ is $\Lambda = a\mathbb{Z} \times b\mathbb{Z}$, i.e. a rectangular lattice. In this case, the fundamental domain of $\Lambda$ in $\mathbb{R}^2$ is rectangular and thus, according to theorem 1, no Hermite function can be maximally concentrated inside the fundamental domain. This observation suggests that Gaussian or Hermitian windows are not an ideal choice for generating Gabor frames along a rectangular lattice. Although no proof for a precise statement exists, this observation has been made before and it is consistently confirmed in numerical experiments. In particular, in [21] it is mentioned that Gabor frames generated by time–frequency shifting Gaussian pulses over a hexagonal lattice have better condition number than frames obtained via a corresponding rectangular lattice. It is well-known and was shown by Gauss in 1840 that hexagonal lattices provide the densest packing of circles in the plane\footnote{Weisstein E W ‘Circle Packing’, from MathWorld—A Wolfram Web Resource \url{http://mathworld.wolfram.com/CirclePacking.html}.}. On the other hand, it is known that a Gabor frame with a Gaussian basic window is never tight [8].

Motivated by our observations we formulate the following conjecture:

Given a fixed redundancy $\text{red} > 1$, the condition number of a Gabor frame with Gaussian window $\varphi$ is optimal for a hexagonal lattice.

3.5. Remark (due to Karlheinz Gröchenig)

Since Daubechies’ results also extend to localization operators with symbols $\sigma$ other than indicator functions, stating that any radial symbol equally leads to localization operators diagonalized by the Hermite functions, one may ask the obvious question, whether a similar inverse statement to theorem 1 can be expected for more general symbol classes than indicator functions. The following example shows that this is not true.

Let $H_{\varphi}$ be a time–frequency localization operator. Then, for every $N \in \mathbb{N}_0$ there exist non-negative, non-radial symbols $\sigma$, such that $H_{\varphi}h_N = \lambda_N h_N$.

To construct $\sigma$, we proceed as in section 3.3 and consider the equivalent operator on $\mathcal{F}^2(\mathbb{C})$, i.e.

$$T_{\sigma}F(w) = \int_{\mathbb{C}} \sigma(z)F(z)e^{\pi z w - \pi |z|^2} \, dz.$$  

We then claim that $T_{\sigma}(e^N) = \lambda_N e^N$ for some non-radial $\sigma$. We fix $N \in \mathbb{N}_0$ and let

$$\sigma(z) = \sigma(r \, e^{2\pi i \gamma}) = \sigma_0(r) + \sigma_1(r) \cdot (e^{2\pi i(N+1)\gamma} + e^{-2\pi i(N+1)\gamma}),$$  

where $\sigma_0(r) \geq 2|\sigma_1(r)| \quad \forall r \geq 0$ and $\int_0^{\infty} \sigma_1(r) r^{2N+1} e^{-\pi r^2} \, r \, dr = 0$. Observe that $\sigma_1$ can be chosen to be bounded, compactly supported and real-valued. Then we have $\sigma(z) \geq \sigma_0(r) - 2|\sigma_1(r)| \geq 0$. Since $\sigma_0$ is radial, we have $T_{\sigma_0}(e^N) = \lambda_N e^N$ with $\lambda_N > 0$. Therefore, it is enough to show that $T_{\sigma_1}(e^N) = 0$. However, this is easy to see by considering $\sigma_+ = \sigma_1(r) e^{2\pi i(N+1)\gamma}$ and $\sigma_- = \sigma_1(r) e^{-2\pi i(N+1)\gamma}$ separately and noting that, since $T_{\sigma_0}$ is...
entire, the task is reduced to showing that \( \frac{df}{ds} T_{\sigma_+}(z^N) |_{w=0} = 0 \) for all \( l \in \mathbb{N}_0 \). A straightforward calculation shows that, setting \( F(z) = z^N \) and writing \( (T_{\sigma_+}(f))^{(l)} = \frac{df}{ds} T_{\sigma_+}(f) \), we have

\[
(T_{\sigma_+}(F))^{(l)}(0) = \int_{\mathbb{C}} \alpha_{\pm}(z) e^{N\pi\alpha} e^{-\pi|z|^2} \, dz.
\]

We finally substitute polar coordinates \( z = r e^{i\theta} \) to obtain, for \( \sigma_+ \):

\[
(T_{\sigma_+}(F))^{(l)}(0) = \pi^l \int_0^\infty \sigma_1(z) r^{N+l} e^{-\pi r^2} r \, dr \int_0^{\pi} e^{2\pi i (2N+1-l) \theta} \, d\theta.
\]

The integral over \( t \) is zero for \( l \neq 2N+1 \) by orthogonality of the Fourier basis and the integral over \( r \) is zero for \( l = 2N+1 \) by assumption. The argument for \( \sigma_- \) is similar.

4. An inverse problem for wavelet localization

By replacing ‘Gabor transform’ by ‘wavelet transform’ in the formulation of the inverse problem for time–frequency localization, we may define a completely analogous inverse problem for wavelet localization. The corresponding direct problem has been treated by Daubechies and Paul in [5] and by Seip in [18]. This section is related to the previous one in the same way as the direct problem studied in [5] is related to the problem studied in [4]. It is quite remarkable that, after an appropriate reformulation of the eigenvalue problem, we can apply proposition 2 to wavelet localization operators. Since our arguments depend on the connection to complex variables, it is essential to consider the Hardy space of the upper-half plane as the domain of the wavelet transform. Then, we choose a certain analyzing wavelet which plays the role of the Gaussian, and the localization problem can be reformulated in certain weighted Bergman spaces. This basic strategy follows the lines which lead to the Bargmann–Fock space formulation in the Gabor case.

One relevant difference between the wavelet and the Gabor case stems from the hyperbolic geometry of the upper-half plane. Since the set-up of proposition 1 is not visible in the spaces defined on the half-plane, we will translate the problem to a conformally equivalent hyperbolic region: the unit disc. There, the problem finds a natural formulation and proposition 2 applies. This point of view is suggested by Seip’s approach in [18]. In short, while in the Gabor case the Bargmann transform maps \( L^2(\mathbb{R}) \) to the Bargmann–Fock space, where the monomials are orthogonal,

\[
\mathcal{B} : L^2(\mathbb{R}) \rightarrow \mathcal{F}^2(\mathbb{C}),
\]

we now need to further transform the images of the so-called Bergman transform (\( \text{Ber}_{\mathcal{D}} \)) to a space defined in the unit disc. This transformation is given by a Cayley transform \( \mathcal{D} \) as defined in section 4.2:

\[
H^2(\mathbb{C}^+) \xrightarrow{\text{Ber}_{\mathcal{D}}} A_0(\mathbb{C}^+) \xrightarrow{\mathcal{D}} A_0(\mathcal{D}).
\]

The role of the Hermite functions is taken over by special functions, whose Fourier transforms are the Laguerre functions. This is possible, since the Laguerre functions constitute an orthogonal basis for \( L^2(0, \infty) \) and the Fourier transform provides a unitary isomorphism \( H^2(\mathbb{C}^+) \rightarrow L^2(0, \infty) \).

4.1. The wavelet transform

Since analyticity will play a fundamental role, in this section we restrict ourselves to functions in a subspace of \( L^2(\mathbb{R}) \), namely to \( f \in H^2(\mathbb{C}^+) \), the Hardy space in the upper-half plane. \( H^2(\mathbb{C}^+) \) is constituted by analytic functions \( f \) such that

\[
\sup_{0 \leq s < \infty} \int_{-\infty}^{\infty} |f(x + is)|^2 \, dx < \infty.
\]
The functions in the space $H^2(\mathbb{C}^+)$ may be considered as being of ‘positive frequency’ since a well-known Paley–Wiener theorem says that $\mathcal{F}(H^2(\mathbb{C}^+)) = L^2(0, \infty)$. For this reason it is common to study $H^2(\mathbb{C}^+)$ on the ‘frequency side’, where many calculations become easier.

For convenience, we will use a different normalization of the Fourier transform in this section, namely $(\mathcal{F}f)(\xi) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} e^{-i\xi t} f(t) \, dt$. Now consider $\mathbb{C}^+ = \{z \in \mathbb{C} : \text{Re}(z) > 0\}$. For every $x \in \mathbb{R}$ and $s \in \mathbb{R}^+$, let $z = x + is \in \mathbb{C}^+$ and define

$$\pi_\gamma(t) = s^{-\frac{1}{2}} g(s^{-1} (t-x)).$$

Fix a function $g \neq 0$ such that

$$0 < \|\mathcal{F}g\|_{L^2(\mathbb{R}^+,x^{-1})}^2 = C_g < \infty.$$  

Such functions are called admissible and the constant $C_g$ is the admissibility constant. Then the continuous wavelet transform of a function $f$ with respect to a wavelet $g$ is defined, for every $z = x + is \in \mathbb{C}$ as

$$W_z f(z) = \langle f, \pi_z g \rangle_{H^2(\mathbb{C}^+)}.$$  

Let $d\mu^+(z)$ denote the standard normalized area measure in $\mathbb{C}^+$. The orthogonal relations for the wavelet transform

$$\int_{\mathbb{C}^+} W_{z_1} f_1(x,s) W_{z_2} f_2(x,s) s^{-2} d\mu^+(z) = \langle \mathcal{F}g_1, \mathcal{F}g_2 \rangle_{L^2(\mathbb{R}^+,x^{-1})}(f_1, f_2)_{H^2(\mathbb{C}^+)},$$  

are valid for all $f_1, f_2 \in H^2(\mathbb{C}^+)$ and $g_1, g_2 \in H^2(\mathbb{C}^+)$ admissible. As a result, the continuous wavelet transform provides an isometric inclusion

$$W_g : H^2(\mathbb{C}^+) \rightarrow L^2(\mathbb{C}^+, s^{-2} \, dx \, ds),$$

which is an isometry for $C_g = 1$.

### 4.2. Bergman spaces

Let $\alpha > -1$. The Bergman space in the upper-half plane, $A_\alpha(\mathbb{C}^+)$, is constituted by the analytic functions in $\mathbb{C}^+$ such that

$$\int_{\mathbb{C}^+} |f(z)|^2 s^\alpha \, d\mu^+(z) < \infty,$$

where $d\mu^+(z)$ stands for the standard normalized area measure in $\mathbb{C}^+$. Now consider $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. The Bergman space in the unit disc, denoted by $A_\alpha(\mathbb{D})$, is constituted by the analytic functions in $\mathbb{D}$ such that

$$\int_{\mathbb{D}} |f(w)|^2 (1-|w|)^\alpha \, dA(w) < \infty,$$

with $dA(w)$ being the normalized area measure in $\mathbb{D}$. The map $T_\alpha : A_\alpha(\mathbb{C}^+) \rightarrow A_\alpha(\mathbb{D})$, defined as

$$(T_\alpha f)(w) = \frac{2^{\frac{\alpha+1}{2}}}{(1-w)^{\alpha+2}} f \left( \frac{w+1}{i(w-1)} \right),$$

provides a unitary isomorphism between the two spaces. The reproducing kernel of $A_\alpha(\mathbb{C}^+)$ is

$$K^\alpha_{\mathbb{C}^+}(z,w) = \left( \frac{1}{w-z} \right)^{\alpha+2}.$$  

Now observe that, letting $T_\alpha$ act on the reproducing kernel of $A_\alpha(\mathbb{C}^+)$, first as a function of $w$ and then as a function of $z$, we are led to the reproducing kernel of $A_\alpha(\mathbb{D})$,

$$K^\alpha_{\mathbb{D}}(z,w) = \frac{1}{(1-wz)^{\alpha+2}}.$$  

(21)
4.3. The Bergman transform

If we choose the window \( \psi_\alpha \) as

\[
\mathcal{F} \psi_\alpha(t) = \frac{1}{c_\alpha} \mathbf{1}_{(0, \infty)} t^\alpha e^{-t},
\]

then we can relate the wavelet transform to Bergman spaces of analytic functions. Here,

\[
c_\alpha^2 = \int_0^\infty x^{2\alpha-1} e^{-2x} \, dx = 2^{2\alpha-1} \Gamma(2\alpha),
\]

where \( \Gamma \) is the Gamma function. The choice of \( c_\alpha \) implies \( C_{\psi_\alpha} = 1 \) and the corresponding wavelet transform is isometric. The \textit{Bergman transform} of order \( \alpha \) is the unitary map \( \text{Ber}_\alpha : H(\mathbb{C}^+) \rightarrow A_\alpha(\mathbb{C}^+) \) given by

\[
\text{Ber}_\alpha f(z) = s^{-\alpha/2} W_{-\frac{\alpha}{2}} f(-x, s) = c_\alpha \int_0^\infty t^{\alpha/2} (\mathcal{F} f)(t) e^{i\beta t} \, dt.
\] (23)

4.4. The Laguerre and other related systems of functions

We define the Laguerre functions

\[
L_n^\alpha(x) = \mathbf{1}_{(0, \infty)}(x) e^{-x/2} x^{\alpha/2} L_n^\alpha(x)
\]
in terms of the Laguerre polynomials

\[
L_n^\alpha(x) = \frac{e^{x-\alpha} x^\alpha}{n!} \frac{d^n}{dx^n} [e^{-x} x^{\alpha+n}] = \sum_{k=0}^n (-1)^k \binom{n+\alpha}{n-k} \frac{x^k}{k!}.
\] (24)

By repeated integration by parts, one sees that the polynomials \( L_n^\alpha(x) \) are orthogonal on \((0, \infty)\) with respect to the weight function \( e^{-x} x^\alpha \). Thus, for \( \alpha \geq 0 \), the Laguerre functions \( L_n^\alpha \) constitute an orthogonal basis for the space \( L^2(0, \infty) \). We will use a related system of functions \( \psi_n^\alpha \) defined as

\[
(\mathcal{F} \psi_n^\alpha)(t) = \left( \frac{(-1)^n n!}{2^{n+2\alpha+1} \Gamma(n+2+\alpha) \Gamma(2+\alpha)} \right)^{1/2} t^{\alpha+1}(2t).
\]

Now consider the monomials

\[
e_n^\alpha(w) = \left( \frac{\Gamma(n+2+\alpha)}{n! \Gamma(2+\alpha)} \right)^{1/2} w^n.
\]

We can apply proposition 1 with \( \mu(|z|) = (1-|w|^2)^\alpha \). We conclude that \( \{e_n^\alpha\}_{n=0}^\infty \) forms an orthonormal basis for \( A_\alpha(\mathbb{D}) \) and that they are orthogonal on every disc \( D_r \subset \mathbb{D} \) for every \( r > 0 \),

\[
\int_{D_r} e_n^\alpha(w) \overline{e_m^\alpha(w)} (1-|w|^2)^\alpha \, dA(w) = C(r, m) \delta_{nm}.
\] (25)

The normalization constant \( C(r, m) \) depends on \( r \) and \( m \) and satisfies \( \lim_{r \to 1^-} C(r, m) = 1 \). Now, the functions \( \Psi_n^\alpha \), for every \( n \geq 0 \) and \( \alpha > -1 \),

\[
\Psi_n^\alpha(z) = \frac{1}{4^{\alpha+\frac{1}{2}}} \left( \frac{\Gamma(n+2+\alpha)}{n! \Gamma(2+\alpha)} \right)^{1/2} \left( \frac{z-i}{z+i} \right)^n \left( \frac{1}{z+i} \right)^{\alpha+2}, \quad z \in \mathbb{C}^+,
\]

are conveniently chosen such that

\[
(T_n \Psi_n^\alpha)(w) = e_n^\alpha(w).
\] (26)
Thus, a change of variables $w = \frac{z-i}{z+i}$ in (25) leads to
\[
\int_{\phi(z_1, z_2)} \psi^\alpha_n(z) \overline{\psi^\alpha_n(z)} \, d\mu^+(z) = C(r, m) \delta_{nm},
\]
where $\phi(z_1, z_2) = \left| \frac{z_1 - z_2}{z_1 + z_2} \right|$ is the pseudohyperbolic metric on $C^+$. Moreover, the unitarity of the operator $T_\alpha$ translates the basis property of $\{e_n^\alpha(z)\}_{n=0}^\infty$ in $A_n(\mathbb{D})$ to $A_n(C^+)$. In other words, (26) shows that $\{\psi^\alpha_n(z)\}_{n=0}^\infty$ is an orthogonal basis of $A_n(C^+)$. Finally, we observe that (23) together with the special function formula
\[
\int_0^\infty x^\alpha L_n^\alpha(x) \, e^{-s x} \, dx = \frac{\Gamma(\alpha + n + 1)}{n!} s^{-\alpha - n - 1} (s - 1)^n
\]
gives
\[
\text{Ber}_n \psi^\alpha_n = \psi^\alpha_n.
\]
For an intuitive grasp of this section, keep in mind that with the composition of transforms (16), one associates the transformations of the basis functions:
\[
\psi^\alpha_n \in H(C^+) \xrightarrow{\text{Ber}} \psi^\alpha_n \in A_n(C^+) \xrightarrow{T} \psi^\alpha_n \in A_n(\mathbb{D}).
\]

4.5. The inverse problem

We now consider the wavelet localization operator $P_{\Delta, \alpha}$ defined as
\[
P_{\Delta, \alpha} f = \int_\Delta W_{\Phi^\alpha} f(z) \psi_{\alpha+2} \, d\mu^+(z)
\]
and set up the corresponding eigenvalue problem
\[
P_{\Delta, \alpha} f = \lambda f. \tag{29}
\]

**Theorem 2.** If one of the eigenfunctions of the localization operator $P_{\Delta, \alpha}$ belongs to the family $\{\psi^\alpha_n\}$, then $\Delta$ must be a pseudohyperbolic disc centered at $i$.

**Proof.** We first rewrite the eigenvalue problem (29). A simple change of variables on the ‘Fourier’ side of the wavelet representation gives
\[
\text{Ber}_n (\pi, \psi^\alpha_n)(w) = m_n s^{\alpha+2} \left( \frac{1}{z - w} \right)^{\alpha+2} = s^{\alpha+2} K^\alpha_{\mathbb{C}^+}(z, w),
\]
where $m_n = \frac{\alpha+1}{2\pi}$. Now apply the Bergman transform and use (23) to rewrite (29) as
\[
\int_\Delta \text{Ber}_n f(z) K^\alpha_{\mathbb{C}^+}(z, w) \, d\mu^+(z) = \lambda \text{Ber}_n f(w).
\]
By the unitarity $\text{Ber}_n : H(C^+) \to A_n(C^+)$, we conclude that our eigenvalue problem is equivalent to
\[
\int_\Delta F(z) K^\alpha_{\mathbb{C}^+}(z, w) \, d\mu^+(z) = \lambda F(w),
\]
with $F \in A_n(C^+)$. Making the change of variables
\[
\begin{align*}
\zeta_D &= \frac{z - i}{z + i}, \quad \omega_D &= \frac{w - i}{w + i},
\end{align*}
\]
we move the eigenvalue problem to the unit disc
\[
\int_{\Omega = T_\alpha} (T_\alpha F)(\zeta_D) \left( \frac{1 - |\zeta_D|^2}{(1 - \omega_D \overline{\zeta_D})^{2\alpha}} \right) dA_\Omega (\zeta_D) = \lambda (T_\alpha F)(\omega_D),
\]
where $\Omega = T_\alpha$.
where \( T_c F(w) \in A_n(\mathbb{D}) \). We simplify the notation writing \( z_D = z, w_D = z \). Now, using the uniformly convergent expansion of the reproducing kernel,

\[
\frac{1}{(1 - wz)^{2+\alpha}} = \sum_{n=0}^{\infty} e_n^\alpha(w) \varphi_n^\alpha(z),
\]

we can transform the eigenvalue equation into

\[
\lambda(T_a F)(w) = \sum_{n=0}^{\infty} \frac{\Gamma(n + 2 + \alpha)}{n! \Gamma(2 + \alpha)} w^n \int_{\Omega} (T_a F)(z) \mathcal{F}(1 - |z|^\alpha) dA_D(z). \tag{30}
\]

If one of the eigenfunctions of the localization operator \( P_{\Delta,a} \) belongs to the family \( \{ \psi_n^\alpha \} \), then

\[
T_a (\text{Ber}_a \psi_n^\alpha)(z) = T_a (\Psi_n^\alpha)(z) = e_n^\alpha(z)
\]

solves (30) for \( \lambda = \lambda_n \). Setting \((T_a F)(z) = e_n^\alpha(z)\) gives

\[
\lambda_n w_m = \sum_{n=0}^{\infty} \frac{\Gamma(n + 2 + \alpha)}{n! \Gamma(2 + \alpha)} w^n \int_{\Omega} \mathcal{F}(1 - |z|^\alpha) dA_D(z).
\]

Comparison of coefficients yields

\[
\int_{\Omega} \mathcal{F} e^{-\pi |z|^2} (1 - |z|^\alpha) dA_D(z) = \lambda_n \frac{n! \Gamma(2 + \alpha)}{\Gamma(2 + \alpha) \delta_{m,n}}
\]

and further, with \( n = m + k \), the condition of proposition 2

\[
\int_{\Omega} |z|^{2m} \mathcal{F}(1 - |z|^\alpha) dA_D(z) = \lambda_k \delta_{k,0}, \text{ for all } k \geq 1. \tag{31}
\]

Hence, proposition 2 can be applied and \( \Omega \) must be a disc centered at zero. Now we can go back to the upper-half plane by the change of variables

\[
u = \frac{z + 1}{1 - z} \in \mathbb{C}^+,
\]

which maps \( 0 \in \mathbb{D} \) to \( i \in \mathbb{C}^+ \) and leaves the pseudohyperbolic metric invariant. Finally, note that the condition \( |z| < r \) can be written in terms of the pseudohyperbolic metric of the disc as \( \varrho_D(z, 0) < r \). Hence, the disc \( \Omega \) centered at zero is mapped to the pseudohyperbolic disc \( \Delta = \{ \varrho_{\mathbb{C}^+}(u, i) < r \} \) centered at \( u = i \).

\[\square\]

**Remark 2.** We can draw a conclusion similar to the one in remark 1 after theorem 1. Indeed, it follows from theorem 2, that if the localization domain \( \Delta \) is not a pseudohyperbolic disc, then the function \( f \) providing optimal concentration in the sense of maximizing

\[
\mathcal{C}_\Delta(f) = \int_\Delta |W_{\psi_\alpha}(f)(z)|^2 dz \| f \|^2_{L^2_{\mathcal{F}}(\mathbb{C}^+)} \tag{32}
\]

cannot be the function \( \psi_\alpha \) in (22)—the so-called Cauchy wavelet. On the other hand, it is known that the functions \( \psi_\alpha \) minimize the affine uncertainty principle as first mentioned in [16]. In this sense, pseudohyperbolic discs seem to be optimal domains to measure wavelet localization.

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