Chiral geometries of (2+1)-d AdS gravity

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Abstract

Pure gravity in (2+1)-dimensions with negative cosmological constant is classically equivalent to Chern-Simons gauge theory with gauge group $SO(2,2)$, which may be realized on chiral and antichiral gauge connections. This paper looks at half-AdS geometries i.e. those with a trivial right-moving gauge connection while the left-moving connection is a standard (Bañados-Teitelboim-Zanelli) BTZ connection. These are shown to be related by diffeomorphism to a BTZ geometry with different mass and angular momentum. Generically this is over-spinning, leading to a naked closed timelike curves. Other closely related solutions are also studied. These results suggest that the measure of the Chern-Simons path integral cannot factorize in a chiral way, if it is to represent a sum over physically sensible states.
I. INTRODUCTION

General Relativity in (2+1)-dimensions has long been considered an arena for investigating issues in gravity without the additional complications of gravitational dynamics in higher dimensions. The reason for this is that in (2+1)-dimensions there are no local propagating degrees of freedom in the bulk i.e. there are no gravitational waves in the bulk and the dynamics arises purely from global topology - defects and boundaries [1, 2] (for a comprehensive review consult [3]). Although the theory does not have gravitational waves i.e. no gravitational attraction, interestingly, it does have black hole solutions [4, 5] for the case of negative cosmological constant. These (BTZ) black hole metrics are locally isomorphic to the maximally symmetric solution, namely anti de Sitter space and share the same asymptotic symmetry group - the conformal group in 2-dimensions [6]. The infinite set of Virasoro charges of this asymptotic conformal group parametrize all asymptotically AdS metrics [7]. In particular for BTZ metrics, the Virasoro charges \((L_0, \bar{L}_0)\) are simply linear combinations of the mass \((M)\) and spin \((J)\) of the black hole \((L_0 = \frac{M+J}{2}, \bar{L}_0 = \frac{M-J}{2})\).

There is a long history of efforts formulating gravity in general \(d\)-dimensions as a gauge theory by combining the vierbein and spin connection into a single \(ISO(d-1,1)\) gauge connection, since small diffeomorphisms can be expressed as local Lorentz rotations and translations [8] on shell. However this program was abortive since the Einstein-Hilbert action could not be expressed in terms of the gauge connection. This obstacle was circumvented for the \(d = 2 + 1\) case [9, 10] for general nonvanishing cosmological constant and the Einstein-Hilbert action was expressed as a Chern-Simons action for the \(ISO(2,1)\) connection for zero cosmological constant, and a \(SO(2,2)\) connection for negative cosmological constant. Since \(SO(2,2) \equiv SL(2,\mathbb{R}) \times SL(2,\mathbb{R})/\mathbb{Z}_2\) (the \(\mathbb{Z}_2\) acts as \(-1\) on the \(SL(2,\mathbb{R})\) connections), the action can be written as a sum of two Chern-Simons terms with independent \(SL(2,\mathbb{R})\) gauge connections. By adding an additional topological term to the Einstein action [10], the coefficients of the left/right Chern-Simons terms become independent. Based on this motivation, it has been conjectured the path integral for pure \(2 + 1\)-d gravity with negative cosmological constant factors holomorphically [11, 12, 13, 14, 15, 16, 17, 18]. Also in other extensions of \(2 + 1\)-d AdS gravity with a gravitational Chern-Simons terms there are sectors where the left (or right) gauge connection can be pure gauge i.e. geometries that are chiral [19, 20]. In such cases again the path integral is expected to be holomorphic.
factorizable. Of course there are several issues - the classical equivalence of Einstein’s theory and the Chern-Simons theory might not extend to the quantum realm in such a simple manner. This is certainly true for large diffeomorphisms when the vierbein is noninvertible and the metric interpretation is unclear.

The aim of this paper is to study geometries where one of the CS gauge fields is set to be globally trivial (anti-de Sitter) with the other gauge field being nontrivial (BTZ-like). If holomorphic factorization holds at the level of the Chern-Simons path integral, such geometries should have a meaningful metric interpretation. We find that such metrics generically have naked closed timelike curves (CTCs) and hence do not respect causality.

The plan of the paper is as follows. In section 2 we review the Chern-Simons formulation of the BTZ solution and note the relationship between the holonomies and the casimirs (mass and spin). In section 3, we construct ”hybrid” metrics made of a left(+) connection of a \((M, J)\) BTZ solution and a right(−) connection of a \((m, j)\) BTZ solution. The hybrid metric after a suitable change of coordinates is shown to be another BTZ solution with charges \((\frac{M+m}{2} + \frac{J-j}{2l}, \frac{(M-m)l}{2} + \frac{J+j}{2})\). In section 4 we set the right connection to pure AdS\(_3\) and note that these geometries are super-rotating BTZ solutions which necessarily have naked CTCs thus violating causality. The right connection when set to zero instead of pure AdS\(_3\) gives rise to singular metrics.

II. CHERN-SIMONS FORMULATION OF THE BTZ BLACK HOLE

To start with, let us very briefly review the Chern Simons formulation of Lorentzian 2+1d gravity with negative cosmological constant \(\Lambda = -\frac{1}{l^2}\). The details can be found in [7, 10, 20]. The vierbein, \(e^a\) and the spin connection one form \(\omega^a\) are combined into a \(SO(2, 2)\) gauge connection one form,

\[
A = e^a P_a + \omega^a J_a
\]

with the algebra,

\[
[J_a, J_b] = \epsilon_{ab}^c J_c, \ [P_a, P_b] = \frac{1}{l^2} \epsilon_{ab}^c J_c, \ [J_a, P_b] = \epsilon_{ab}^c P_c.
\]

The \(SO(2, 2)\) generators can be split into two \(SL(2, \mathbb{R})\) copies,

\[
T_a^\pm = \frac{1}{2} (J_a \pm iP_a)
\]
satisfying

\[ [T_a^+, T_b^+] = \epsilon_{ab} c T_c^+, \quad [T_a^-, T_b^-] = \epsilon_{ab} c T_c^-, \quad [T_a^+, T_b^-] = 0. \]

Here \( \epsilon^{012} = 1 \) and \( \eta = \text{diag} \ (-1, 1, 1) \). The gauge connection accordingly factorizes to,

\[
A = A^a T_a^+ + A^a T_a^-
\]

\[
A^\pm = \omega \pm \frac{c}{l}.
\]

The Einstein-Hilbert action with a negative constant term can then be cast as the difference of two Chern-Simons terms,

\[ I = I_{CS}[A^+] - I_{CS}[A^-], \]

where the Chern-Simons action is,

\[ I_{CS}[A] = \frac{k}{4\pi} \int \text{Tr}(dA \wedge A + \frac{2}{3} A \wedge A \wedge A), \]

where the trace is over the 2-dimensional representation of \( SL(2, \mathbb{R}) \),

\[ T_0 = -i \frac{\sigma^2}{2}, \quad T_1 = -i \frac{\sigma^3}{2}, \quad T_2 = \frac{\sigma^1}{2}, \]

and the Chern-Simons coupling constant or level number,

\[ k = -2l. \]

Following [4, 5, 7] we choose the convention \( 8G = 1 \). The solutions to Einstein’s equations are given by flat connections

\[ dA^\pm + A^\pm \wedge A^\pm = 0. \]

In fact the above pair of flat connection equations of motion can also be obtained by any action which is the difference of two Chern-Simons actions with different Chern-Simons couplings i.e. \( k^+ \neq k^- \). This corresponds to adding a topological term to the Einstein action with a coupling proportional to the difference \( (k^+ - k^-) \), which does not change the local equations of motion, but will be important if the full path integral is considered.

Flat connections are classified according to their holonomy, i.e. every flat connection can be gauge transformed to the form

\[ A^\pm = U^{\pm 1} dU^\pm \quad (1) \]
where $U \in SL(2, \mathbb{R})$ is a gauge transformation element. The connection is globally pure gauge only if $U$ is single valued. The holonomy, $w$, for a connection is given by the Wilson loop operator along noncontractible cycles

$$W^\pm = \exp \left( \oint A^\pm \right) = \exp(w^\pm).$$

The local pure gauge condition is a reflection of the fact that in 2+1d any solution of Einstein equation with a negative cosmological constant is locally anti-de Sitter space, which is the maximally symmetric solution. Globally distinct solutions can be obtain by orbifolding the maximally symmetric AdS$_3$ spacetime, i.e. identifying points along an orbit of some killing vector, thus introducing nontrivial cycles. This leads to black hole solutions [4, 5] with mass $M$ and spin $J$ given by the metric,

$$ds^2 = -N(r)^2 dt^2 + N(r)^{-2} dr^2 + (r^2 N^\phi dt + d\phi)^2$$

where,

$$N^2(r) = -M + \frac{r^2}{l^2} + \frac{J^2}{4r^2}$$

and

$$N^\phi = -\frac{J}{2r^2}$$

and $t \in (-\infty, \infty)$, $r \in (0, \infty)$ and $\phi \in [0, 2\pi)$. Now quotienting the maximally symmetric AdS$_3$ space reduces the number of symmetries (or Killing vectors) by demanding bonafide tensor fields respect single-valuedness after the periodic identifications. The necessary and sufficient condition for which is that tensor fields must commute with the Killing vectors belonging to the identification subgroup (along the orbits of which the identifications were made). This is true only for $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial \phi}$ for the BTZ. Thus the symmetry group of BTZ is not $SO(2, 2)$ but $R \times SO(1, 1)$.

The $SO(2, 2)$ gauge connection was worked out in [21] but these have the unpleasant feature that the connections for the extreme BTZ are singular. So we use a different gauge in which the left and right connections appear as

$$A^{+0} = -N(r) \left( \frac{dt}{l} - d\phi \right), \quad A^{+1} = \frac{1 - \frac{N}{N(r)}dr}{l}, \quad A^{+2} = -\left( \frac{J}{2r} + \frac{r}{l} \right) \left( \frac{dt}{l} - d\phi \right),$$

$$A^{-0} = -N(r) \left( \frac{dt}{l} + d\phi \right), \quad A^{-1} = -\frac{1 + \frac{N}{N(r)}dr}{l}, \quad A^{-2} = \left( \frac{J}{2r} - \frac{r}{l} \right) \left( \frac{dt}{l} + d\phi \right).$$
and the corresponding gauge transformation elements are obtained by solving (1)

\[ U^\pm = e^{\theta_0 \pm T_0} e^{\theta_1 \pm T_1} e^{\theta_2 \pm T_2} \]

\[ \theta_0^\pm = 0 \]

\[ \sinh \theta_1^\pm = \frac{N(r)}{\sqrt{M \pm \frac{J}{l}}} \]

\[ \cosh \theta_1^\pm = \frac{\frac{J}{2r} \pm \frac{r}{l}}{\sqrt{M \pm \frac{J}{l}}} \]

\[ \theta_2^\pm = \sqrt{M \pm \frac{J}{l}} \left( \phi \mp \frac{t}{l} \right) . \]

Clearly these transformation are not suitable for the extreme case \( J = \pm Ml \). For \( J = Ml \), the nonsingular gauge group element is,

\[ U^- = e^{\theta(T_0 + T_2)} e^{\theta_1 T_1}, \]

\[ \theta = -ln \frac{r}{l} - \frac{Mx}{2r}, \]

\[ \theta_1 = - \left( \phi + \frac{t}{l} \right) . \]

Finally the Wilson loop operator along a constant \( t, \phi \) loop,

\[ W^\pm = \begin{pmatrix} \cosh \pi \sqrt{M \pm \frac{J}{l}} e^{-\theta_1} \sinh \pi \sqrt{M \pm \frac{J}{l}} \\ e^{\theta_1} \sinh \pi \sqrt{M \pm \frac{J}{l}} \cosh \pi \sqrt{M \pm \frac{J}{l}} \end{pmatrix} . \]

So the eigenvalues are \( e^\lambda, \lambda = \pm \sqrt{M \pm \frac{J}{l}} \) and the holonomy matrices turn out to be,

\[ w^\pm = \begin{pmatrix} \pm \pi \sqrt{M \pm \frac{J}{l}} & 0 \\ 0 & \mp \pi \sqrt{M \pm \frac{J}{l}} \end{pmatrix} \]

This gives the quadratic Casimirs,

\[ Tr(w^2_+ + w^2_-) = 4\pi^2 M \]

\[ Tr(w^2_+ - w^2_-) = 4\pi^2 \frac{J}{l} . \]

III. HYBRID GEOMETRIES

In this section we are interested in metrics resulting from combining the left and right \( SL(2, \mathbb{R}) \) connections for different BTZ solutions, say we combine a left connection for a
mass $M$, spin $J$ and a right connection for mass $m$ and spin $j$ BTZ metric. The metric for such a hybrid geometry is then,

$$ds^2 = g_{tt}(r)dt^2 + 2g_{t\phi}(r)dtd\phi + g_{rr}(r)dr^2 + g_{\phi\phi}(r)d\phi^2,$$

with,

$$\begin{align*}
\frac{g_{tt}}{l^2} &= \frac{M + m}{4} + \frac{Jj}{8r^2} - \frac{r^2}{2l^2} - \frac{1}{2}N_1(r)N_2(r) \\
\frac{g_{\phi\phi}}{l^2} &= \frac{m + M}{4} - \frac{jJ}{8r^2} + \frac{J - j}{2l} + \frac{r^2}{2l^2} + \frac{1}{2}N_1(r)N_2(r) \\
\frac{g_{rr}}{l^2} &= \left(\frac{1 - \frac{Jl}{2r^2}}{2N_1(r)} + \frac{1 + \frac{jl}{2r^2}}{2N_2(r)}\right)^2 \\
\end{align*}$$

where $N_1(r) = \sqrt{-M + \frac{r^2}{l^2} + \frac{j^2}{4l^2}}$ and $N_2(r) = \sqrt{-m + \frac{r^2}{l^2} + \frac{j^2}{4l^2}}$. In these coordinates the metric or the geometry looks highly unintuitive since the metric components are quartic in $r$ and the determinant seems to have multiple non-coincident zeroes making it hard to analyze the location of horizons, surfaces bounding regions with CTCs, etc. for the general $M$, $m$, $j$, $J$.

To gain insight we switch over to a different set of coordinates as follows. It has been shown \[7\] that the most general solution to Einstein equations with negative cosmological constant in (2+1)-dimensions with asymptotically anti-de Sitter boundary conditions is of the form

$$ds^2 = \frac{lL(w)}{2}dw^2 + \frac{l\bar{L}(\bar{w})}{2}d\bar{w}^2 + \left(l^2e^{2\rho} + \frac{L(w)\bar{L}(\bar{w})}{4}e^{-2\rho}\right)dwd\bar{w} + l^2d\rho^2,$$

for arbitrary functions $L(w)$ and $\bar{L}(\bar{w})$. Here $w = \phi + \frac{t}{l}$, $\bar{w} = \phi - \frac{t}{l}$ are the boundary coordintes and $\rho$ is a radial coordinate. $\phi \in [0, 2\pi)$ is an angular coordinate, while $t$ is the time coordinate. In particular the BTZ solution is a special case with constant metric coefficients,

$$\begin{align*}
L(w) &= L_0 = \frac{Ml + J}{2} \\
\bar{L}(\bar{w}) &= \bar{L}_0 = \frac{Ml - J}{2}
\end{align*}$$

So after effecting the following change of coordinates,

$$\begin{align*}
w &= \phi + \frac{t}{l} \\
\bar{w} &= \phi - \frac{t}{l}
\end{align*}$$
\[ \rho = \frac{1}{2} \ln \left| \left( \frac{N_1 + \frac{r}{l} + \frac{j}{2l}}{4} \right) \left( \frac{N_2 + \frac{r}{l} - \frac{j}{2l}}{4} \right) \right| \]

we arrive at the metric in the form of (3) with,

\[
L(w) = \frac{Ml + J}{2} \\
\bar{L}(\bar{w}) = \frac{ml - j}{2}.
\]  

(4)

So it is just another BTZ black hole metric with ADM mass, \( M_{ADM} \) and spin, \( J_{ADM} \) given by,

\[
M_{ADM} = \frac{M + m}{2} + \frac{J - j}{2l} \\
J_{ADM} = \frac{(M - m)l}{2} + \frac{J + j}{2}.
\]

A crucial observation is that if the extremality bounds \( Ml \geq |J| \) and \( ml \geq |j| \) are satisfied then the resulting black hole also satisfies an extremality bound,

\[
M_{ADM}^2 l^2 - J_{ADM}^2 = (Ml + J)(ml - j) \geq 0.
\]  

(5)

IV. CHIRAL GEOMETRIES

A. Left BTZ - Right AdS\(_3\) geometries

For this case we set \( m = -1, j = 0 \) i.e. pure AdS\(_3\) space instead of a black hole right connection. So the resulting black hole has mass and spin,

\[
M_{ADM} = \frac{M - 1}{2} + \frac{J}{2l} \\
J_{ADM} = \frac{M + 1}{2} - l + \frac{J}{2}
\]

which clearly violates the extremality condition \( M_{ADM}^2 l^2 - J_{ADM}^2 = -(Ml + J)l \) i.e. the regions having CTCs are naked since there are no horizons to mask them. Clearly for such a spacetime causality breaks down.

B. Left BTZ - Right zero geometries

In this case we set,

\[ A^- = 0 \]
and keep the $A^+$ same as in the last case. So, the metric in this case becomes

$$ds^2 = \frac{ML + J}{4}l(d\phi - \frac{dt}{l})^2 + \left(1 - \frac{J}{2\pi^2}\right)^2 dr^2$$

Clearly the metric determinant vanishes, it is non-invertible. So these holomorphic geometries do not make sense either.

V. CONCLUSION AND OUTLOOK

If the path integral for pure gravity involves something like a holomorphically factorized path integral over Chern-Simons gauge fields, we have seen that large families of seemingly sensible gauge connections give rise to geometries with naked closed timelike curves. Eliminating these connections using some kind of physical state conditions that respect holomorphicity would conversely eliminate the basic vacuum solution, pure AdS. It seems then that physical state conditions must violate holomorphicity, by for example, requiring that the inequality (5) be satisfied. Alternatively it may simply be the case that the Chern-Simons formulation of pure gravity is not a good starting point for the quantization of the theory. The study of limits of string theory compactifications that give rise to pure gravity should provide useful clues to aid further progress.

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