Algorithmic Integrability Tests for Nonlinear Differential and Lattice Equations\textsuperscript{1}

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Abstract

Three symbolic algorithms for testing the integrability of polynomial systems of partial differential and differential-difference equations are presented. The first algorithm is the well-known Painlevé test, which is applicable to polynomial systems of ordinary and partial differential equations. The second and third algorithms allow one to explicitly compute polynomial conserved densities and higher-order symmetries of nonlinear evolution and lattice equations.

The first algorithm is implemented in the symbolic syntax of both Macsyma and Mathematica. The second and third algorithms are available in Mathematica. The codes can be used for computer-aided integrability testing of nonlinear differential and lattice equations as they occur in various branches of the sciences and engineering. Applied to systems with parameters, the codes can determine the conditions on the parameters so that the systems pass the Painlevé test, or admit a sequence of conserved densities or higher-order symmetries.

Key words: Integrability; Painlevé test; Conservation law; Invariant; Symmetry; Differential equation; Lattice

\textsuperscript{1} Research supported in part by NSF under Grant CCR-9625421.
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1 Introduction

During the last three decades, the study of integrability, invariants, symmetries, and exact solutions of nonlinear ordinary and partial differential equations (ODEs, PDEs) and differential-difference equations (DDEs) has been the topic of major research projects in the dynamical systems and the soliton communities.

Various techniques have been developed to determine whether or not PDEs belong to the privileged class of completely integrable equations [15]. One of the most successful and widely applied techniques is the Painlevé test, named after the French mathematician Paul Painlevé (1863-1933) [35], who classified second-order differential equations that are globally integrable in terms of elementary functions by quadratures or by linearization. In essence, the Painlevé test verifies whether or not solutions of differential equations in the complex plane are single-valued in the neighborhood of all their movable singularities.

To a large extent, the Painlevé test is algorithmic, yet very cumbersome when done by hand. In particular, the verification of the compatibility conditions is assessed by many practitioners of the Painlevé test as a painstaking computation. In addition to the tedious verification of self-consistency (or compatibility) conditions, computer programs are helpful at exploring all possibilities of balancing singular terms. Indeed, the omission of one or more choices of “dominant behavior” can lead to wrong conclusions [12,44].

We therefore developed the programs painsing.max and painsys.max [25] (both in Macsyma syntax [36]), and painsing.m and painsys.m in Mathematica language [53], that perform the Painlevé test for polynomial systems of ODEs and PDEs. In this paper we demonstrate the above codes by analyzing a few prototypical nonlinear equations and systems, such as the Boussinesq and nonlinear Schrödinger equations, a class of fifth-order KdV equations, and the Hirota-Satsuma and Lorenz systems.

Our computer code does not deal with the theoretical shortcomings of the Painlevé test as identified by Kruskal and others [30–32]. Thus far, we have implemented the traditional Painlevé test [26,27], and not yet incorporated the latest advances in Painlevé type methods, such as the poly-Painlevé test [31] or other generalizations [30,32]. Neither did we code the weak Painlevé test [43,44] or other variants [9–11,17]. Furthermore, we do not have code for the singularity confinement method [23], i.e. an adaptation of the Painlevé test that allows one to test the integrability of difference equations.

Among the various alternatives to establish the integrability [15] of nonlinear PDEs and DDEs, the search for conserved densities and higher-order symme-
tries is particularly appealing. Indeed, in this paper we will give algorithms that apply to both the continuous and semi-discrete cases. We implemented these algorithms [18–20] in Mathematica, but they are fairly simple to code in other computer algebra languages (see [18,20]).

Our algorithms are based on the concept of dilation (scaling) invariance. That inherently limits their scope to polynomial conserved densities and higher-order symmetries of polynomial systems. Although the existence of a sequence of conserved densities predicts integrability, the nonexistence of polynomial conserved quantities does not preclude integrability. Indeed, integrable PDEs or DDEs could be disguised with a coordinate transformation in DDEs that no longer admit conserved densities of polynomial type [46]. The same care should be taken in drawing conclusions about non-integrability based on the lack of higher-order symmetries, or equations failing the Painlevé test.

Apart from integrability testing, the knowledge of the explicit form of conserved densities and higher-order symmetries is useful. For instance, with higher-order symmetries of integrable systems, one can build new completely integrable systems, or discover connections between integrable equations and their group theoretic origin.

Explicit forms of conserved densities are useful in the numerical solution of PDEs or DDEs. In solving DDEs, which may arise from integrable discretizations of PDEs, one should check that conserved quantities indeed remain constant. In particular, the conservation of a positive definite quadratic quantity may prevent nonlinear instabilities in the numerical scheme.

Our integrability package InvariantsSymmetries.m [21] in Mathematica automates the tedious computation of closed-form expressions for conserved densities and higher-order symmetries for both PDEs and DDEs. Applied to systems with parameters, the package determines the conditions on these parameters so that a sequence of conserved densities or symmetries exists. The software can thus be used to test the integrability of classes of equations that model various wave phenomena. Our examples include a vector modified KdV equation, the extended Lotka-Volterra and relativistic Toda lattices, and the Heisenberg spin model.

The conserved densities and symmetries presented in this paper were obtained with InvariantsSymmetries.m.
2 Symbolic Program for the Painlevé Test

2.1 Purpose

We focus on PDEs. As originally formulated by Ablowitz et al. [2,3], the Painlevé conjecture asserts that all similarity reductions of a completely integrable PDE should be of Painlevé-type; i.e. its solutions should have no movable singularities other than “poles” in the complex plane.

A later version of the Painlevé test due to Weiss et al. [52] allows testing of the PDE directly, without recourse to the reduction(s) to an ODE. A PDE is said to have the Painlevé property [1] if its solutions in the complex plane are single-valued in the neighborhood of all its movable singularities. In other words, the equation must have a solution without any branching around the singular points whose positions depend on the initial conditions. For ODEs, it suffices to show that the general solution has no worse singularities than movable poles, or that no branching occurs around movable essential singularities.

A three step-algorithm, known as the Painlevé test, allows one to verify whether or not a given nonlinear system of ODEs or PDEs with (real) polynomial terms fulfills the necessary conditions for having the Painlevé property. Such equations are prime candidates for being completely integrable.

There is a vast amount of literature about the test and its applications to specific ODEs and PDEs. Several well-documented surveys [5,9,15,31,32,34,38,40] and books [8,10,48] discuss subtleties and pathological cases of the test that are far beyond the scope of this article. Other survey papers [15,16,39] deal with the many interesting by-products of the Painlevé test. For example, they show how a truncated Laurent series expansion of the type introduced below, allows one to construct Lax pairs, Bäcklund and Darboux transformations, and closed-form particular solutions of PDEs.

2.2 Algorithm for a Single Equation

We briefly outline the three steps of the Painlevé test for a single PDE,

\[ \mathcal{F}(x, t, u(x, t)) = 0, \]  

(1)
in two independent variables $x$ and $t$. Our software can handle the four independent variables $(x, y, z, t)$. Throughout the paper we will use the notations

$$u_t = \frac{\partial u}{\partial t}, \quad u_{nx} = \frac{\partial^n u}{\partial x^n}, \quad u_{tx} = \frac{\partial^2 u}{\partial t \partial x}, \quad \text{etc.}$$

(2)

In the approach proposed by Weiss, the solution $u(x,t)$, expressed as a Laurent series

$$u(x,t) = g^\alpha(x,t) \sum_{k=0}^{\infty} u_k(x,t) g^k(x,t),$$

(3)

should be single-valued in the neighborhood of a non-characteristic, movable singular manifold $g(x,t)$, which can be viewed as the surface of the movable poles in the complex plane. In (3), $u_0(x,t) \neq 0$, $\alpha$ is a negative integer, and $u_k(x,t)$ are analytic functions in a neighborhood of $g(x,t)$.

Note that for ODEs the singular manifold is $g(x,t) = x - x_0$, where $x_0$ is the initial value for $x$. For PDEs, if $u(x,t)$ has simple zeros and $g_x(x,t) \neq 0$, one may apply the implicit function theorem near the singularity manifold and set $g(x,t) = x - h(t)$, for an arbitrary function $h(t)$ [33, 44]. This so-called the Kruskal simplification, considerably reduces the length of the calculations.

The Painlevé test proceeds in three steps:

**Step 1: Determine the dominant behavior**

Determine the negative integer $\alpha$ and $u_0$ from the leading order “ansatz”. This is done by balancing the minimal power terms after substitution of $u \propto u_0 g^\alpha$ into the given PDE. There may be several branches for $u_0$, and for each the next two steps must be performed.

**Step 2: Determine the resonances**

For a selected $\alpha$ and $u_0$, calculate the non-negative integer powers $r$, called the resonances, at which arbitrary functions $u_r$ enter the expansion. This is done by requiring that $u_r$ is arbitrary after substitution of $u \propto u_0 g^\alpha + u_r g^{\alpha+r}$ into the equation, only retaining its most singular terms. The coefficient $u_r$ will be arbitrary if its coefficient equals zero. The integer roots of the resulting polynomial must be computed. The number of roots, including $r = -1$, should match the order of the given PDE. The root $r = -1$ corresponds to the arbitrariness of the manifold $g(x,t)$.

**Step 3: Verify the correct number of free coefficients**
Verify that the correct number of arbitrary functions \( u_r \) indeed exists by substituting the truncated expansion

\[
  u(x, t) = g^\alpha \sum_{k=0}^{r_{\text{max}}} u_k(x, t)g^k(x, t)
\]  

into the PDE, where \( r_{\text{max}} \) is the largest resonance. At non-resonance levels, determine all \( u_k \). At resonance levels, \( u_r \) should be arbitrary, and since we are dealing with a nonlinear equation, a \textit{compatibility condition} must be verified.

An equation for which the above steps can be carried out consistently and unambiguously, is said to have the Painlevé property and is conjectured to be completely integrable. This entails that the solution has the necessary number of free coefficients \( u_r \), and that the compatibility condition at each of these resonances is unconditionally satisfied.

The reader should be warned that the above algorithm does not detect essential singularities and therefore cannot determine whether or not branching occurs about these. So, for an equation to be integrable it is \textit{not sufficient} that it passes the Painlevé test. Neither it is \textit{necessary}. Indeed, there are integrable equations, such as the Dym-Kruskal equation, \( u_t = u^3u_{3x} \), that do not pass the Painlevé test, yet, by a complicated change of variables can be transformed into an integrable equation.

### 2.3 Algorithm for Systems

The generalization of the algorithm to systems of ODEs and PDEs is obvious. Yet, it is non-trivial to implement. One of the reasons is that the major symbolic packages do not handle inequalities well.

With respect to systems, our code is based on the above three step-algorithm but generalized to systems, as it can be found in [33,44,48]. In these papers there is an abundance of worked examples that served as test cases.

For example, given a system of first-order ODEs,

\[
  \frac{du_i}{dx} = G_i(u_1, u_2, ..., u_n; x), \quad i = 1, 2, ..., n, 
\]  

one introduces a Laurent series for every dependent variable \( u_i(x) \):

\[
  u_i = (x - x_0)^{\alpha_i} \sum_{k=0}^{\infty} u_k^{(i)}(x - x_0)^k. 
\]
The computer program must carefully determine all branches of dominant behavior corresponding to various choices of $\alpha_i$ and/or $u_0^{(i)}$. For each branch, the single-valuedness of the corresponding Laurent expansion must be tested (i.e. the resonances must be computed and the compatibility conditions must be verified). All the details can be found in [15,33,44].

Singularity analysis for PDEs is nontrivial [32] and the Painlevé test should be applied with extreme care. Notwithstanding, our software automatically performs the formal steps of the Painlevé test for systems of ODEs and PDEs. The examples in section 3.2 illustrate how the code works. Careful analysis of the output and drawing conclusions about integrability should be done by humans. Some subtleties of the mathematics of the Painlevé test of systems of PDEs were also dealt with in [6,13,28,49].

3 Examples of the Painlevé Test

3.1 Single Equations

Numerous examples of the Painlevé test for ODEs can be found in the review papers. We turn our attention to PDEs. Using our software package painsing.max or painsing.m one can determine the conditions under which the equation

$$u_{tx} + a(t)u_x + 6uu_{2x} + 6u_x^2 + u_{4x} = 0,$$  \hspace{1cm} (7)

passes the Painlevé test.

For (7), $\alpha = -2$ and $u_0 = -2g_2^2$. Apart from $r = -1$, the roots are $r = 4, 5$, and 6. The latter three are resonances. Furthermore,

$$u_1 = 2g_{2x}, \quad u_2 = -\frac{1}{6g_x^2}(4g_xg_{3x} - 3g_{2x}^2 + g_xtg_x),$$ \hspace{1cm} (8)

$$u_3 = -\frac{1}{6g_x^4}(a(t)g_x^3g_x + g_x^2g_{4x} - 4g_xg_{2x}g_{3x} + 3g_{2x}^2 - g_xtg_{2x} + g_{tx}g_x^2),$$ \hspace{1cm} (9)

and $u_4$ and $u_5$ are indeed arbitrary since the compatibility conditions at resonances $r = 4$ and $r = 5$ are satisfied identically.

The compatibility condition at resonance $r = 6$ is $a_t + 2a^2 = 0$. Ignoring the trivial solution, we get $a = \frac{1}{2(t-t_0)}$. Without loss of generality, we set $t_0 = 0$ and equation (7) becomes the cylindrical KdV equation which is indeed completely
integrable [1,4]. Painlevé based investigations for integrable PDEs with space and time dependent coefficients are given in [1,7,24,27].

Our Painlevé programs cannot automatically test a class of equations such as

\[ u_t + au^2u_x + bu_xu_{2x} + cuu_{3x} + u_{5x} = 0, \]  

(10)

with arbitrary (non-zero and real) parameters \(a, b\) and \(c\). The parameters affect the lowest coefficient in the Laurent expansion in such a way that the roots \((r)\) cannot be computed, and the integrability conditions can no longer be tested.

In (10) there are four cases that are of particular interest:

(i) \(a = \frac{3}{10}c^2\) and \(b = 2c\) (Lax equation),
(ii) \(a = \frac{1}{5}c^2\) and \(b = c\) (Sawada-Kotera equation),
(iii) \(a = \frac{1}{5}c^2\) and \(b = \frac{5}{2}c\) (Kaup-Kupershmidt equation), and
(iv) \(a = \frac{2}{9}c^2\) and \(b = 2c\) (Ito equation).

In Table 1 we list the results of the Painlevé test applied to these cases. For the first three equations the compatibility conditions are satisfied at all the resonances. These equations pass the test. For the Ito equation the compatibility conditions are only satisfied at some of the resonances. The Ito equation fails the test. The first three equations are known to be completely integrable. Ito’s equation is not completely integrable.

The two other algorithms presented in this paper can determine the conditions (i), (ii) and (iii) that assure the complete integrability of (10). The conserved densities and higher-order symmetries of (10) can be found in [18] and [20].
\begin{table}
\begin{center}
\begin{tabular}{|c|c|c|c|c|}
\hline
Lax & Sawada-Kotera & Kaup-Kupershmidt & Ito \\
\hline
\((a, b, c) = (30, 20, 10)\) & \((a, b, c) = (5, 5, 5)\) & \((a, b, c) = (20, 25, 10)\) & \((a, b, c) = (2, 6, 3)\) \\
\hline
\(\alpha = -2\) & \(\alpha = -2\) & \(\alpha = -2\) & \(\alpha = -2\) \\
Branch 1 & Branch 1 & Branch 1 & Branch 1 \\
\(u_0 = -2g_x^2\) & \(u_0 = -6g_x^2\) & \(u_0 = -\frac{3}{2}g_x^2\) & \(u_0 = -6g_x^2\) \\
\(r = -1, 2, 5, 6, 8\) & \(r = -1, 2, 3, 6, 10\) & \(r = -1, 3, 5, 6, 7\) & \(r = -1, 3, 4, 6, 8\) \\
OK for \(r \geq 2\) & OK for \(r \geq 2\) & OK for \(r \geq 3\) & OK at \(r = 3\) \\
& & & Not at \(r = 4, 6, 8\) \\
\hline
Branch 2 & Branch 2 & Branch 2 & Branch 2 \\
\(u_0 = -6g_x^2\) & \(u_0 = -12g_x^2\) & \(u_0 = -12g_x^2\) & \(u_0 = -30g_x^2\) \\
\(r = -3, -1, 6, 8, 10\) & \(r = -2, -1, 5, 6, 12\) & \(r = -7, -1, 6, 10, 12\) & \(r = -5, -1, 6, 8, 12\) \\
OK for \(r \geq 6\) & OK for \(r \geq 5\) & OK for \(r \geq 6\) & OK at \(r = 6, 8\) \\
& & & Not at \(r = 12\) \\
\hline
Passes Test & Passes Test & Passes Test & Fails Test \\
\hline
\end{tabular}
\end{center}
\caption{Painlevé analysis of fifth-order KdV equation}
\end{table}

\begin{equation}
u_t + au^2u_x + bu_xu_{2x} + cuu_{3x} + u_{5x} = 0
\end{equation}

\subsection{Simple Systems}

We start with a famous system of ODEs,

\begin{equation}
u_1' = a(u_2 - u_1), \quad u_2' = -u_1u_3 + bu_1 - u_2, \quad u_3' = u_1u_2 - cu_3,
\end{equation}

where \(a, b,\) and \(c\) are positive constants. System \((11)\) was proposed by the meteorologist E. N. Lorenz as a simplified model for atmospheric turbulence in a vertical air cell beneath a thunderhead \cite{47}.

Using our code \texttt{painsys.max} or \texttt{painsys.m}, with the series \((6)\) for \(i = 1, 2, 3\), we determine the leading orders \(\alpha_1 = -1, \alpha_2 = \alpha_3 = -2\). The first coefficients in the series are

\begin{equation}
u_0^{(1)} = \pm 2i, \quad \nu_0^{(2)} = \mp \frac{2i}{a}, \quad \nu_0^{(3)} = -\frac{2}{a}.
\end{equation}
The roots are \( r = -1, 2, 4 \). Furthermore,
\[
\begin{align*}
    u_1^{(1)} &= \pm \frac{1}{3} i (2c - 3a + 1), &
    u_1^{(2)} &= \pm 2i, &
    u_1^{(3)} &= \frac{2}{3a} (3a - c + 1). \\
\end{align*}
\] (13)

The compatibility conditions at resonances \( r = 2 \) and \( r = 4 \) are not satisfied. At resonance \( r = 2 \) we encounter the condition \( a(c - 2a)(c + 3a - 1) = 0 \). So, we consider two cases:

- For \( c = 2a \), the compatibility condition at \( r = 4 \) is not satisfied.
- For \( c = 1 - 3a \), the compatibility condition at \( r = 4 \) is satisfied provided that \( a = \frac{1}{3} \).

We conclude that the Lorenz system (11) passes the Painlevé test when \( a = \frac{1}{3} \) and \( c = 0 \). This special case may not be relevant in the context of meteorology since in (11) the parameters \( a, b, c \) in (11) are supposed to be positive.

To illustrate the generalization of the algorithm to systems of PDEs, consider the system
\[
\begin{align*}
    u u_{xt} - u_x u_t + v_x v_t &= 0, &
    u v_{xt} - u_t v_x - u_x v_t &= 0, \\
\end{align*}
\] (14)

which plays a role in elementary particle physics [49]. For any of the dependent variables we introduce a Laurent expansion
\[
\begin{align*}
    u &= g^\alpha \sum_{k=0}^{\infty} u_k g^k, &
    v &= g^\beta \sum_{k=0}^{\infty} v_k g^k. \\
\end{align*}
\] (15)

Analysis of the dominant behavior leads to \( \alpha = \beta = -1 \) and \( u_0^2 + v_0^2 = 0 \). Let \( v_0 \) be arbitrary, then \( u_0 = \pm iv_0 \). Next, we look for powers of \( g \) at which arbitrary functions \( u_r, v_r \) can enter. Equating the coefficients of \( g^r\), \( r > 0 \), we obtain the system
\[
\begin{pmatrix}
    2r & i r (r - 1) \\
    i (r^2 - r + 2) & -2(r - 1)
\end{pmatrix}
\begin{pmatrix}
    u_r \\
    v_r
\end{pmatrix}
= 0. \\
\] (16)

The determinant of the coefficient matrix vanishes provided \( r = -1, 0, 1, 2 \). Note that \( r = 0 \) confirms that \( v_0 \) can be chosen freely. Upon substitution of the Laurent series (truncated at level \( k = 2 \)) and setting the terms in \( g^{-2} \) and \( g^{-3} \) equal to zero, one finds that \( u_1 \) is completely determined, whereas \( v_1 \) and \( u_2 \) (or \( v_2 \)) are arbitrary. The compatibility conditions are satisfied. The Laurent expansions have the required number of arbitrary functions \( u_k \). Thus, the system (14) passes the Painlevé test.
Hirota and Satsuma [1] proposed a coupled system of KdV equations,

\[
\begin{align*}
    u_t - 6auu_x + 6vv_x - au_{3x} &= 0, \\
    v_t + 3uv_x + v_{3x} &= 0,
\end{align*}
\]

(17)

where \(a\) is a nonzero parameter. System (17) describes interactions of two long waves with different dispersion relations. It is known to be completely integrable for \(a = \frac{1}{2}\). This is confirmed by the Painlevé test. Indeed, with (15) we obtain \(\alpha = \beta = -2\) and \(r = -2, -1, 3, 4, 6\) and \(8\). Furthermore, \(u_0 = -4\) and \(v_0 = \pm 2\sqrt{2a}\), determine the coefficients \(u_1, v_1, u_2, v_2\) unambiguously. At resonances 3 and 4 there is one free function and no condition for \(a\). The coefficients \(u_5\) and \(v_5\) are unique determined, but at resonance 6, the compatibility condition is only satisfied if \(a = \frac{1}{2}\). For this value, the compatibility condition at resonance 8 is also satisfied.

For the integrable version of the Boussinesq system [45]

\[
\begin{align*}
    u_t + v_x + uu_x &= 0, \\
    v_t + u_{3x} + (uv)_x &= 0,
\end{align*}
\]

(18)

with the Laurent expansions in (15), the leading order is \(\alpha = -1, \beta = -2\). Careful investigation of the recursion relations linking the \(u_k\) and \(v_k\) allows one to conclude that there are resonances at levels 2, 3 and 4. Finally, the three compatibility conditions are seen to hold after lengthy computations. System (18) thus passes the test.

Our codes \(\text{painsys.max}\) are also applicable to complex equations such as the nonlinear Schrödinger (NLS) equation [1],

\[
iq_t - g_{2x} + 2|q|^2 q = 0,
\]

(19)

where \(q(x, t)\) is a complex function. First, (19) must be rewritten as a system:

\[
\begin{align*}
    u_t - v_{2x} + 2v(u^2 + v^2) &= 0, \\
    v_t + u_{2x} - 2u(u^2 + v^2) &= 0,
\end{align*}
\]

(20)

where \(q = u + iv\). For simplicity, we use the Kruskal simplification, setting \(g(x, t) = x - h(t)\) in (15). Then, the leading order is \(\alpha = \beta = -1\), and \(r = -1, 0, 3\) and \(4\). Furthermore, \(v_0 = \pm \sqrt{1 - u_0^2}\), with \(u_0\) arbitrary, and

\[
\begin{align*}
    u_1 &= \frac{1}{2}v_0 h_t, \\
    v_1 &= -\frac{1}{2}u_0 h_t, \\
    u_2 &= -\frac{1}{12v_0}u_0(v_0 h_t^2 + 2u_0 t), \\
    v_2 &= -\frac{1}{12}(v_0 h_t^2 + 2u_0 t), \\
    u_3 &= -\frac{1}{8u_0}(h_{tt} + 8v_0 v_3), \\
    u_4 &= -\frac{1}{24v_0^3}(2v_0^3 h_t h_{tt} - v_0^2 u_{0,tt} - u_0 u_{0,t}^2 - 24u_0 v_0^3 v_4),
\end{align*}
\]

(21)
where $v_3$ and $v_4$ are arbitrary. Note that at every resonance there is one free function in the Laurent expansion. The NLS equation passes the test.

In searching the literature we found numerous examples of systems for which the Painlevé test was carried out by hand. Some of the most elaborate examples of Painlevé analysis involve three-wave interactions [37], the mixmaster universe model [12], and chaotic star pulsations [51].

4 Symbolic Programs for Conserved Densities and Symmetries

4.1 The Key Concepts

The key observation behind our algorithms is that conserved densities and higher-order symmetries of a PDE or DDE system abide by the dilation symmetry of that system. We illustrate this for prototypical examples of single PDEs and DDEs.

Dilation Invariance of PDEs. The ubiquitous Korteweg-de Vries (KdV) equation from soliton theory [1],

\[ u_t = 6uu_x + u_{3x}, \]  

(22)

is invariant under the dilation (scaling) symmetry

\[ (t, x, u) \rightarrow (\lambda^{-3}t, \lambda^{-1}x, \lambda^2 u), \]  

(23)

where $\lambda$ is an arbitrary parameter. Obviously, $u$ corresponds to two derivatives in $x$, i.e. $u \sim \partial^2 / \partial x^2$. Similarly, $\partial / \partial t \sim \partial^3 / \partial x^3$. Introducing weights, denoted by $w$, we have $w(u) = 2$ and $w(\partial / \partial t) = 3$, provided we set $w(\partial / \partial x) = 1$. The rank $R$ of a monomial equals the sum of all of its weights. Observe that (22) is uniform in rank since all the terms have rank $R = 5$.

Conserved Densities of PDEs. For PDEs like (22), the conservation law

\[ D_t \rho + D_x J = 0 \]  

(24)

connects the conserved density $\rho$ and the associated flux $J$. As usual, $D_t$ and $D_x$ are total derivatives. With few exceptions, polynomial density-flux pairs only depend on $u, u_x$, etc., and not explicitly on $t$ and $x$.

The first three (of infinitely many) independent conservation laws for (22) are
\[
D_t(u) - D_x(3u^2 + u_{2x}) = 0, \quad D_t(u^2) - D_x(4u^3 - u_x^2 + 2uu_{2x}) = 0, \quad (25)
\]
\[
D_t(u^3 - \frac{1}{2}u_{x}^2) - D_x(\frac{9}{2}u^4 - 6uu_{x}^2 + 3u^2u_{2x} + \frac{1}{2}u_{2x}^2 - u_xu_{3x}) = 0. \quad (26)
\]

The densities \(\rho = u, u^2, u^3 - \frac{1}{2}u_{x}^2\) have ranks 2, 4 and 6, respectively. The associated fluxes have ranks 4, 6 and 8. The terms in the conservation laws have ranks 5, 7, and 9.

Integration of both terms in the conservation law with respect to \(x\) yields

\[
P = \int_{-\infty}^{+\infty} \rho \, dx = \text{constant in time}, \quad (27)
\]

provided \(J\) vanishes at infinity. \(P\) is the true conserved quantity. The first two conservation laws correspond to conservation of momentum and energy. For ODEs, the quantities \(P\) are called constants of motion.

**Symmetries of PDEs.** As summarized in Table 2, \(G(x, t, u, u_x, u_{2x}, \ldots)\) is a symmetry of a PDE system if and only if it leaves it invariant for the change \(u \rightarrow u + \epsilon G\) within order \(\epsilon\). Hence, \(D_t(u + \epsilon G) = F(u + \epsilon G)\) must hold up to order \(\epsilon\). Thus, \(G\) must satisfy the linearized equation \(D_tG = F'(u)[G]\), where \(F'\) is the Fréchet derivative: \(F'(u)[G] = \frac{\partial}{\partial \epsilon} F(u + \epsilon G)|_{\epsilon=0}\).

| System | Continuous Case (PDEs) | Semi-discrete Case (DDEs) |
|--------|------------------------|---------------------------|
| System | \(u_t = F(u, u_x, u_{2x}, \ldots)\) | \(\dot{u}_n = F(\ldots, u_{n-1}, u_n, u_{n+1}, \ldots)\) |
| Conservation Law | \(D_t\rho + D_xJ = 0\) | \(\dot{\rho}_n + J_{n+1} - J_n = 0\) |
| Symmetry | \(D_tG = F'(u)[G]\) | \(D_tG = F'(u_n)[G]\) |
| | \[= \frac{\partial}{\partial \epsilon} F(u + \epsilon G)|_{\epsilon=0}\] | \[= \frac{\partial}{\partial \epsilon} F(u_n + \epsilon G)|_{\epsilon=0}\] |

Table 2: Conservation Laws and Symmetries
Two nontrivial higher-order symmetries [42] of (22) are

\[
G^{(1)} = 30u^2 u_x + 20u_x u_{2x} + 10u u_{3x} + u_{5x}, \quad (28)
\]

\[
G^{(2)} = 140u^3 u_x + 70u^3 + 280u u_x u_{2x} + 70u^2 u_{3x} + 70u_{2x} u_{3x} + 42u_x u_{4x} + 14u u_{5x} + u_{7x}. \quad (29)
\]

The recursion operator [42, p. 312],

\[
\Phi = D^2 + 4u + 2u_x D^{-1}, \quad (30)
\]

connects the above symmetries, \( \Phi G^{(1)} = G^{(2)} \), and also the lower order symmetries as shown in [20].

Note that the recursion operator (30) is also uniform in rank with \( R = 2 \) since \( w(D^{-1}) = -1 \). Currently, we are working on an algorithm and symbolic code to compute recursion operators based on the knowledge of a few higher-order symmetries and conserved densities.

Higher-order symmetries lead to new integrable PDEs. Indeed, the evolution equations \( u_t = G^{(1)} \) and \( u_t = G^{(2)} \) are the well-known fifth and seventh-order equations in the completely integrable KdV hierarchy [1].

**Dilation Invariance of DDEs.** We now turn to the semi-discrete case. As prototype, consider the Volterra lattice [1],

\[
\dot{u}_n = u_n (u_{n+1} - u_{n-1}), \quad (31)
\]

which is one of the discretizations of (22). Note that (31) is invariant under

\[
(t, u_n) \rightarrow (\lambda^{-1} t, \lambda u_n). \quad (32)
\]

So, \( u_n \sim d/dt \), or \( w(u_n) = 1 \) if we set \( w(d/dt) = 1 \). Every term in (31) has rank \( R = 2 \), thus (31) is uniform in rank.

**Conserved Densities of DDEs.** For DDEs like (31), the conservation law

\[
\dot{\rho}_n + J_{n+1} - J_n = 0 \quad (33)
\]

connects the conserved density \( \rho_n \) and the associated flux \( J_n \). For (31) the first two conservation laws (of ranks 2 and 3) are

\[
\frac{d}{dt} (u_n) + u_n u_{n-1} - u_{n+1} u_n = 0, \quad (34)
\]
\[
\frac{d}{dt}\left(\frac{1}{2}u_n^2 + u_n u_{n+1}\right) + u_{n-1}u_n^2 + u_{n-1}u_n u_{n+1} - u_n u_{n+1}^2 - u_n u_{n+1} u_{n+2} = 0. \tag{35}
\]

The densities \(\rho_n = u_n\) and \(\rho_n = u_n \left(\frac{1}{2}u_n + u_{n+1}\right)\) have ranks 1 and 2. Their fluxes \(J_n = -u_n u_{n-1}\) and \(J_n = -u_n u_{n-1}^2 - u_{n-1} u_n u_{n+1}\) have ranks 2 and 3.

**Symmetries of DDEs.** For DDEs of type (31), \(G(\ldots, u_{n-1}, u_n, u_{n+1}, \ldots)\) is a symmetry if and only if the infinitesimal transformation \(u_n \rightarrow u_n + \epsilon G\) leaves the DDE invariant within order \(\epsilon\). Consequently, \(G\) must satisfy \(\frac{dG}{dt} = F'(u_n)[G]\), where \(F'\) is the Fréchet derivative, \(F'(u_n)[G] = \frac{\partial}{\partial \epsilon} F(u_n + \epsilon G)|_{\epsilon=0}\). See Table 2.

The first nontrivial higher-order symmetry of (31) is

\[
G = u_n u_{n+1} (u_n + u_{n+1} + u_{n+2}) - u_{n-1} u_n (u_{n-2} + u_{n-1} + u_n), \tag{36}
\]

and, similar to the continuous case, an integrable lattice \(\dot{u}_n = G\) follows.

Both (22) and (31) have infinitely many polynomial conserved densities \([18,22]\) and symmetries \([20]\).

**Remarks:**

(i) For scaling invariant systems like (22) and (31), it suffices to consider the dilation symmetry on the space of independent and dependent variables.

(ii) For systems that are inhomogeneous under a suitable scaling symmetry we use the following trick: We introduce one (or more) auxiliary parameter(s) with appropriate scaling. In other words, we extend the action of the dilation symmetry to the space of independent and dependent variables, including the parameters. These extra parameters should be viewed as additional dependent variables, with the caveat that their derivatives are zero. With this trick we can apply our algorithms to a larger class of polynomial PDE and DDE systems. Examples can be found in \([18–20,22]\).

Scaling (dilation) invariance, which is a special Lie-point symmetry, is common to many integrable nonlinear PDEs and DDEs. In the next two sections we show how the scaling invariance can be explicitly used to compute polynomial conserved densities and higher-order symmetries of PDEs and DDEs.

### 4.2 Algorithms for the Computation of Conserved Densities

**PDE Case.** Conserved densities of PDEs can be computed as follows:
• Require that each equation in the system of PDEs is uniform in rank. Solve the resulting linear system to determine the weights of the dependent variables. For instance for (22), solve \( w(u) + w(\partial/\partial t) = 2w(u) + 1 = w(u) + 3 \), to get \( w(u) = 2 \) and \( w(\partial/\partial t) = 3 \).

• Select the rank \( R \) of \( \rho \), say, \( R = 6 \). Make a linear combination of all the monomials in the components of \( u \) and their \( x \)-derivatives that have rank \( R \). Remove all monomials that are total \( x \)-derivatives (like \( u^4 \)). Remove equivalent monomials, that is, those that only differ by a total \( x \)-derivative. For example, \( uu_2x^2 \) and \( u^2x \) are equivalent since\( uu_2x^2 = \frac{1}{2}(u^2)_{2x} \). For (22), one gets \( \rho = c_1u^3 + c_2u^2 \) of rank \( R = 6 \).

• Substitute \( \rho \) into the conservation law (24). Use the PDE system to eliminate all \( t \)-derivatives, and require that the resulting expression \( E \) is a total \( x \)-derivative. Apply the Euler operator (see [18] for the general form),

\[
\mathcal{L}_u = \frac{\partial}{\partial u} - D_x(\frac{\partial}{\partial u_x}) + D_x^2(\frac{\partial}{\partial u_{2x}}) + \cdots + (-1)^n D_x^n(\frac{\partial}{\partial u_{nx}})
\]

(37)

to \( E \) to avoid integration by parts. If any terms remain, they must vanish identically. This yields a linear system for the constants \( c_i \). Solve the system. For (22), one gets \( c_1 = 1, c_2 = -1/2 \).

See [18] for the complete algorithm and its implementation.

**DDE Case.** The computation of conserved densities proceeds as follows:

• Compute the weights in the same way as for PDEs. For (31), one gets \( w(u_n) = 1 \) by solving \( w(u_n)+1 = 2w(u_n) \). Note that \( w(d/dt) = 1 \) and weights are independent of \( n \).

• Determine all monomials of rank \( R \) in the components of \( u_n \) and their \( t \)-derivatives. Use the DDE to replace all the \( t \)-derivatives. Monomials are *equivalent* if they belong to the same equivalence class of shifted monomials. For example, \( u_{n-1}v_{n+1}, u_{n+2}v_{n+4} \) and \( u_{n-3}v_{n-1} \) are equivalent. Keep only the main representatives (centered at \( n \)) of the various classes.

• Combine these representatives linearly with coefficients \( c_i \), and substitute the form of \( \rho_n \) into the conservation law \( \dot{\rho}_n = J_n - J_{n+1} \).

• Remove all \( t \)-derivatives and pattern-match the resulting expression with \( J_n - J_{n+1} \). To do so use the following equivalence criterion: if two monomials \( m_1 \) and \( m_2 \) are equivalent, \( m_1 \equiv m_2 \), then \( m_1 = m_2 + [M_n-M_{n+1}] \) for some polynomial \( M_n \) that depends on \( u_n \) and its shifts. For example, \( u_{n-2}u_n \equiv u_{n-1}u_{n+1} \) since \( u_{n-2}u_n = u_{n-1}u_{n+1} + [u_{n-2}u_n - u_{n-1}u_{n+1}] = u_{n-1}u_{n+1} + [M_n - \)
Set the non-matching part equal to zero, and solve the linear system for the $c_i$. Determine $J_n$ from the pattern $J_n - J_{n+1}$. For (31), the first three (of infinitely many) densities $\rho_n$ are listed in Table 3.

| Equation | KdV Equation | Volterra Lattice |
|----------|--------------|------------------|
| $u_t = 6uu_x + u_{3x}$ | $u_n = u_n (u_{n+1} - u_{n-1})$ |
| Densities | $\rho = u$, $\rho = u^2$ | $\rho_n = u_n$, $\rho_n = u_n (\frac{1}{2} u_n + u_{n+1})$ |
| | $\rho = u^3 - \frac{1}{2} u_x^2$ | $\rho_n = \frac{1}{2} u_{n}^2 + u_n u_{n+1} (u_n + u_{n+1} + u_{n+2})$ |
| Symmetries | $G = u_x$, $G = 6uu_x + u_{3x}$ | $G = u_n u_{n+1} (u_n + u_{n+1} + u_{n+2})$ |
| | $G = 30u^2 u_x + 20u_x u_{2x}$ | $-u_{n-1} u_n (u_{n-2} + u_{n-1} + u_n)$ |
| | $+ 10 uu_{3x} + u_{5x}$ | |

Table 3: Prototypical Examples

Details about this algorithm and its implementation are in [19,22]. See [21] for an integrated Mathematica Package that computes conserved densities (and also symmetries) of PDEs and DDEs.

4.3 Algorithm for the Computation of Symmetries

**PDE Case.** Higher-order (or generalized symmetries) of PDEs can be computed as follows:

- Determine the weights of the dependent variables in the system.
- Select the rank $R$ of the symmetry. Make a linear combination of all the monomials involving $u$ and its $x$-derivatives of rank $R$. For example, for (22), $G = c_1 u^2 u_x + c_2 u_x u_{2x} + c_3 uu_{3x} + c_4 u_{5x}$ is the form of the generalized symmetry of rank $R = 7$. In contrast to the computation of conserved densities, no terms are removed here.
- Compute $D_t G$. Use the PDE system to remove all $t$-derivatives. Equate the result to the Fréchet derivative $F'(u) [G]$. Treat the different monomial terms in $u$ and its $x$-derivatives as independent, to get the linear system for $c_i$. Solve that system. For (22), one obtains

$$G = 30u^2 u_x + 20u_x u_{2x} + 10 uu_{3x} + u_{5x}.$$ (38)

The symmetries of the Lax family of rank 3, 5, and 7 are listed in Table 3. They are the first three of infinitely many. See [20] for the details about the
algorithm.
**DDE Case.** Higher-order symmetries of DDEs can be computed as follows:

- First determine the weights of the variables in the DDE the same way as for conserved densities.

- Determine all monomials of rank $R$ in the components of $u_n$ and their $t$-derivatives. Use the DDE to replace all the $t$-derivatives. Make a linear combination of the resulting monomials with coefficients $c_i$.

- Compute $D_t G$ and remove all $\dot{u}_{n-1}, \dot{u}_n, \dot{u}_{n+1}$, etc. Equate the resulting expression to the Fréchet derivative $F'(u_n)[G]$ and solve the system for the $c_i$, treating the monomials in $u_n$ and its shifts as independent.

For (31), the symmetry $G$ of rank $R = 3$ is listed in Table 3. There are infinitely many symmetries, all with different ranks.

See [20] for the complete algorithm and its implementation in Mathematica, and [21] for an integrated Mathematica Package that computes symmetries of PDEs and DDEs.

**Notes:**

(i) A slight modification of these methods allows one to find conserved densities and symmetries of PDEs that explicitly depend on $t$ and $x$. See the first example in the next section.

(ii) Applied to systems with free parameters, the linear system for the $c_i$ will depend on these parameters. A careful analysis of the eliminant leads to conditions on these parameters so that a sequence of conserved densities or symmetries exists. Details about this type of analysis and its computer implementation can be found in [18].

5 **Examples of Densities and Symmetries**

5.1 **Vector Modified KdV Equation**

In [50], Verheest investigated the integrability of a vector form of the modified KdV equation (vmKdV),

$$ B_t + (B^2 B)_x + B_{xxx} = 0, \quad (39) $$

or component-wise for $B = (u, v)$,
\[ u_t + 3u^2 u_x + v^2 u_x + 2uvv_x + u_{3x} = 0, \]
\[ v_t + 3v^2 v_x + u^2 v_x + 2uvu_x + v_{3x} = 0. \]

(40)

With our software InvariantsSymmetries.m [21] we computed

\[ \rho_1 = u, \quad \rho_2 = v, \quad \rho_3 = u^2 + v^2, \quad \rho_4 = \frac{1}{2}(u^2 + v^2)^2 - (u_x^2 + v_x^2), \]
\[ \rho_5 = \frac{1}{3}x(u^2 + v^2) - \frac{1}{2}t(u^2 + v^2)^2 + t(u_x^2 + v_x^2). \]

(41)

(42)

Note that the latter density depends explicitly on \( x \) and \( t \). Verheest [50] has shown that (40) is non-integrable for it lacks a bi-Hamiltonian structure and recursion operator. We were unable to find additional polynomial conserved densities. Polynomial higher-order symmetries for (40) do not appear to exist.

5.2 Heisenberg Spin Model

The continuous Heisenberg spin system [14] or Landau-Lifshitz equation,

\[ S_t = S \times \Delta S + S \times DS, \]

(43)

models a continuous anisotropic Heisenberg ferromagnet. It is considered a universal integrable system since various known integrable PDEs, such as the NLS and sine-Gordon equations, can be derived from it. In (43), \( S = [u, v, w]^T \) with real components, \( \Delta = \nabla^2 \) is the Laplacian, \( D \) is a diagonal matrix, and \( \times \) is the standard cross product of vectors.

Split into components, (43) reads

\[ u_t = vw_{2x} - wv_{2x} + (\beta - \alpha)vw, \]
\[ v_t = wu_{2x} - uw_{2x} + (1 - \beta)uw, \]
\[ w_t = w^2 - vw_{2x} + (\alpha - 1)uv. \]

(44)

In Table 4 we list the conserved densities for three typical cases; other cases are similar.

Note that for all the cases we considered

\[ \rho = u^2 + v^2 + w^2 = ||S||^2 \]

(45)

is constant in time (since \( J = 0 \)). Hence, all even powers of \( ||S|| \) are also conserved densities, but they are dependent of (45).
\begin{tabular}{|c|c|c|}
\hline
$D = \text{diag}(1, \alpha, \beta)$ & $D = \text{diag}(1, \alpha, 0)$ & $D = \text{diag}(0, 0, 0)$ \\
$\alpha \neq 0, \beta \neq 0$ & $\alpha \neq 0$ & \\
$\rho = u$ if $\alpha = \beta$ & $\rho = w$ if $\alpha = 1$ & $\rho = u$ \\
$\rho = v$ if $\beta = 1$ & $\rho = u^2 + v^2 + w^2$ & $\rho = v$ \\
$\rho = w$ if $\alpha = 1$ & $\rho = (1 - \alpha)v^2 + w^2 + u_x^2 + v_x^2 + w_x^2$ & $\rho = w$ \\
$\rho = u^2 + v^2 + w^2$ & $\rho = u^2 + v^2 + w^2$ & $\rho = u_x^2 + v_x^2 + w_x^2$ \\
$\rho = (1 - \alpha)v^2 + (1 - \beta)w^2 + u_x^2 + v_x^2 + w_x^2$ & & \\
\hline
\end{tabular}

Table 4: Conserved Densities for the Heisenberg Spin Model

Furthermore, the sum of two conserved densities is a conserved density. Hence, after adding (45),

$$\rho = (\alpha - 1)v^2 + (\beta - 1)w^2 - (u_x^2 + v_x^2 + w_x^2)$$

(46)

can be replaced by

$$\rho = u^2 + \alpha v^2 + \beta w^2 - (u_x^2 + v_x^2 + w_x^2).$$

(47)

Note that $u_x^2 = D_x(uu_x) - uu_{2x}$ and recall that densities are equivalent if they only differ by a total $x$-derivative. So, (47) is equivalent with

$$\rho = u^2 + \alpha v^2 + \beta w^2 - uu_{2x} + vv_{2x} + wv_{2x},$$

(48)

which can be compactly written as $\rho = S \cdot \Delta S + S \cdot DS$, where $D = \text{diag}(1, \alpha, \beta)$. Consequently, the Hamiltonian of (43)

$$\mathcal{H} = -\frac{1}{2} \int S \cdot \Delta S + S \cdot DS \, dx$$

(49)

is constant in time. The dot ($\cdot$) refers to the standard inner product of vectors.

5.3 Extended Lotka-Volterra and Relativistic Toda Lattices

Itoh [29] studied this extended version of the Lotka-Volterra equation (31):

$$\dot{u}_n = \sum_{r=1}^{k-1} (u_{n-r} - u_{n+r}) u_n.$$
For $k = 2$, (50) is (31), for which three conserved densities and one symmetry are listed in Table 3. In [19], we gave two additional densities and in [20] we listed two more symmetries.

For (50), we computed 5 densities and 2 higher-order symmetries for $k = 3$ through $k = 5$. Here is a partial list of the results:

For $k = 3$:

\[
\begin{align*}
\rho_1 &= u_n, \\
\rho_2 &= \frac{1}{2} u_n^2 + u_n(u_{n+1} + u_{n+2}), \\
\rho_3 &= \frac{1}{3} u_n^3 + u_n^2(u_{n+1} + u_{n+2}) + u_n(u_{n+1} + u_{n+2})^2 \\
&\quad + u_n(u_{n+1}u_{n+3} + u_{n+2}u_{n+3} + u_{n+2}u_{n+4}), \\
G &= u_n^2(u_{n+1} + u_{n+2} - u_{n-2} - u_{n-1}) + u_n[(u_{n+1} + u_{n+2})^2 \\
&\quad - (u_{n-2} + u_{n-1})^2] + u_n[u_{n+1}u_{n+3} + u_{n+2}u_{n+3} + u_{n+2}u_{n+4} \\
&\quad - (u_{n-4}u_{n-2} + u_{n-3}u_{n-2} + u_{n-3}u_{n-1})].
\end{align*}
\]

For $k = 4$:

\[
\begin{align*}
\rho_1 &= u_n, \\
\rho_2 &= \frac{1}{2} u_n^2 + u_n(u_{n+1} + u_{n+2} + u_{n+3}), \\
\rho_3 &= \frac{1}{3} u_n^3 + u_n^2(u_{n+1} + u_{n+2} + u_{n+3}) + u_n(u_{n+1} + u_{n+2} + u_{n+3})^2 \\
&\quad + u_n(u_{n+1}u_{n+4} + u_{n+2}u_{n+4} + u_{n+3}u_{n+4} + u_{n+2}u_{n+5} \\
&\quad + u_{n+3}u_{n+5} + u_{n+3}u_{n+6}), \\
G &= u_n[u_{n+1}u_{n+4} + u_{n+2}u_{n+4} + u_{n+3}u_{n+4} + u_{n+2}u_{n+5} \\
&\quad + u_{n+3}u_{n+5} + u_{n+3}u_{n+6} - (u_{n-6}u_{n-3} + u_{n-5}u_{n-3} + u_{n-4}u_{n-3} \\
&\quad + u_{n-5}u_{n-2} - u_{n-4}u_{n-2} + u_{n-4}u_{n-1})] + u_n[(u_{n+1} + u_{n+2} + u_{n+3})^2 \\
&\quad - u_n(u_{n-3} + u_{n-2} + u_{n-1})^2] + u_n^2[u_{n+1} + u_{n+2} + u_{n+3} \\
&\quad - (u_{n-3} + u_{n-2} + u_{n-1})].
\end{align*}
\]

Our last example involves the integrable relativistic Toda lattice [41]:

\[
\begin{align*}
\dot{u}_n &= u_n(v_{n+1} - v_n + u_{n+1} - u_{n-1}), \\
\dot{v}_n &= v_n(u_n - u_{n-1}).
\end{align*}
\]

We computed the densities of rank 1 through 5. The first three are

\[
\begin{align*}
\rho_1 &= u_n + v_n, \\
\rho_2 &= \frac{1}{2}(u_n^2 + v_n^2) + u_n(u_{n+1} + v_n + v_{n+1}), \\
\rho_3 &= \frac{1}{3}(u_n^3 + v_n^3) + u_n^2(u_{n+1} + v_n + v_{n+1}) + u_n[(u_{n+1} + v_{n+1})^2.
\end{align*}
\]
We computed the symmetries for ranks \((2, 2)\) through \((4, 4)\). The first two are:

\[
G^{(1)}_1 = u_n (u_{n+1} - u_{n-1} + v_{n+1} - v_n), \quad G^{(1)}_2 = v_n (u_n - u_{n-1}),
\]

\[
G^{(2)}_1 = u_n^2 (u_{n+1} - u_{n-1} + v_{n+1} - v_n) + u_n [(u_{n+1} + v_{n+1})^2 - (u_{n-1} + v_n)^2]
+ u_{n+1} (u_{n+2} + v_{n+2}) - u_{n-1} (u_{n-2} + v_{n-1}),
\]

\[
G^{(2)}_2 = v_n^2 (u_n - u_{n-1}) + v_n (u_n^2 - u_{n-1} u_{n-2} - u_{n-1}^2 + u_n u_{n+1}
- u_{n-1} v_{n-1} + u_n v_{n+1}).
\]

Conserved densities and symmetries of other relativistic lattices are in [20,22].

6 Conclusions

We presented three methods to test the integrability of differential equations and difference-differential equations. One of these methods is the Painlevé test, which is applicable to polynomial systems of ODEs and PDEs.

The two other methods are based on the principle of dilation invariance. Thus far, they can only be applied to polynomial systems of evolution equations. As shown, it is easy to adapt these methods to the DDE case.

Although restricted to polynomial equations, the techniques presented in this paper are algorithmic and have the advantage that they are fairly easy to implement in symbolic code.

Applied to systems with parameters, the codes allow one to determine the conditions on the parameters so that the systems pass the Painlevé test, or have a sequence of conserved densities or higher-order symmetries. Given a class of equations, the software can thus be used to pick out the candidates for complete integrability.

Currently, we are extending our algorithms to the symbolic computation of recursion operators of evolution equations. In the future we will investigate generalizations of our methods to PDEs and DDEs in multiple space dimensions. The potential use of Lie-point symmetries other than dilation (scaling) symmetries will also be studied.
Acknowledgements

We acknowledge helpful discussions with Profs. B. Herbst, M. Kruskal, S. Mikhailov, C. Nucci, J. Sanders, E. Van Vleck, F. Verheest, P. Winternitz, and T. Wolf. We also thank C. Elmer and G. Erdmann for help with parts of this project.

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