Totally Nonnegative Tropical Flags and the Totally Nonnegative Flag Dressian

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Abstract

We study the totally nonnegative part of the complete flag variety and of its tropicalization. We show that Lusztig’s notion of nonnegative complete flag variety coincides with the flags in the complete flag variety which have nonnegative Plücker coordinates. This mirrors the characterization of the totally nonnegative Grassmannian as those points in the Grassmannian whose Plücker coordinates are all non-negative. We then study the tropical complete flag variety and complete flag Dressian, which are two tropical versions of the complete flag variety, capturing realizable and abstract flags of tropical linear spaces, respectively. In general, the complete flag Dressian properly contains the tropical complete flag variety. However, we show that the totally nonnegative parts of these spaces coincide.

Contents

1 Introduction 1

2 Background 3

2.1 The Totally nonnegative Complete Flag Variety 3

2.2 Flag Positroids 5

2.3 The Complete Flag Dressian and the Tropical Complete Flag Variety 6

2.4 Lindström-Gessel-Viennot Construction 7

3 Parametrization of the Totally nonnegative Complete Flag Variety 9

3.1 The Marsh-Rietsch Parametrization 9

3.2 A Graphical Description of the Marsh-Rietsch Parametrization 10

3.3 Extremal Non-Zero Plücker Coordinates 16

4 The Totally Positive Complete Flag Variety and its Tropicalization 18

5 The Totally nonnegative Complete Flag Variety and its Tropicalization 22

5.1 Graphical Description of Extremal Plücker Indices 22

5.2 Determining Non-Extremal Plücker Coordinates 38

5.3 Characterizing the TNN Complete Flag Variety and Dressian 41

1 Introduction

In this paper, we look to build upon and unite two perspectives on flag varieties. On the one hand, there has been progress in understanding the totally nonnegative parts of flag varieties and how we can characterize them, including [19][27][35][34][17][18] and [4]. On the other hand, there has been interest in the tropicalizations of flag varieties and various ways to understand the resulting tropical variety, including [30][33][7] and [8]. The Grassmannian is a particularly nice flag variety where these two mathematical notions have been brought together, by studying the totally nonnegative tropical Grassmannian and showing that
it equals the totally nonnegative Dressian \cite{31}\cite{1}\cite{32}. We extend these ideas to the complete flag variety, showing that the totally nonnegative tropical complete flag variety equals the totally nonnegative complete flag Dressian.

The real Grassmannian of \( k \)-planes in \( n \)-space, \( \text{Gr}_{k,n} \), is the variety where each point corresponds to a \( k \)-dimensional linear subspace of \( \mathbb{R}^n \). A natural generalization of the Grassmannian is the flag variety \( \text{Fl}_{r,n} \) of rank \( r = (r_1, r_2, \ldots, r_k) \) in \( n \)-space. The points of this space correspond to collections of linear subspaces \( L_1 \subseteq L_2 \subseteq \cdots \subseteq L_k \subseteq \mathbb{R}^n \) such that \( \dim(L_i) = r_i \). Two notable examples of flag varieties are \( \text{Gr}_{k,n} \), of rank \( k \), and the complete flag variety \( \text{Fl}_n \), of rank \( (1, 2, \ldots, n) \).

In \cite{19}, the totally nonnegative part of a flag variety is defined. A number of authors, among them \cite{28}, \cite{34}, \cite{17} and \cite{18}, have proven that the totally nonnegative Grassmannian consists precisely of those points in the Grassmannian where each of its Plücker coordinates is nonnegative. We extend this result to the setting of the complete flag variety. A construction based on the parameterization of the totally nonnegative complete flag variety, \( \text{Fl}_{r,n}^{\geq 0} \), by Marsh and Rietsch \cite{21} will allow us to understand explicitly the Plücker coordinates of an arbitrary flag \( F \) in \( \text{Fl}_{r,n}^{\geq 0} \). These coordinates will be indexed by subsets \( I \subseteq [n] \) and denoted \( P_I(F) \). We will give a combinatorial description of a maximal algebraically independent subset of the Plücker coordinates of \( F \) and use it to prove our first main result, Theorem 5.25:

**Theorem.** The totally nonnegative complete flag variety \( \text{Fl}_{r,n}^{\geq 0} \) equals the set \( \{ F \in \text{Fl}_n | P_I(F) \geq 0 \ \forall \ I \subseteq [n] \} \).

We learned recently that a more general result has been independently proven in \cite{4}, where they show that the totally nonnegative part of the flag variety \( \text{Fl}_{r,n} \) can be characterized as the set of flags with nonnegative Plücker coordinates if and only if \( r = (a, a+1, \ldots, b) \) consists of consecutive integers. This includes the complete flag variety and the Grassmannian as special cases. In the case of the complete flag variety, our proof leads to a more direct understanding of the dependencies between Plücker coordinates, which will prove to be an important feature for proving our second main result.

Tropical geometry is the geometry of the tropical semiring \( \mathbb{T} = \mathbb{R} \cup \{ \infty \} \) where multiplication is replaced by addition, and addition is replaced by minimization. Thus, roughly, if we tropicalize a polynomial, we get a minimization over a collection of sums of variables. We say a point is a solution of a tropical polynomial if that minimum is achieved at least twice. In this setting, varieties become polyhedral cell complexes, making them amenable to combinatorial study. While we will reserve precise definitions for Section 2.3, tropical geometry has indeed proven a useful tool in algebraic combinatorics, as in \cite{22}\cite{14}, and most notably for our purposes, \cite{31} and \cite{32}.

For \( k \leq n \), \( \text{Gr}_{k,n} \) is an algebraic variety cut out by the Plücker relations, which generate an ideal called the Plücker ideal. The set of points satisfying the tropicalizations of all the Plücker relations is called the Dressian and is the parameter space of abstract tropical linear spaces \cite{33}. The set of points satisfying the tropicalizations of all polynomials in the Plücker ideal is called the tropical Grassmannian and is the parameter space of realizable tropical linear spaces \cite{30}\cite{11}. In general, the Dressian properly contains the tropical Grassmannian (see, for instance, \cite{12}). However, independently in \cite{32} and \cite{1}, it is shown that if we restrict to positive solutions, for an appropriate notion of positivity, the situation is simpler: the positive (resp. totally nonnegative) Dressian equals the positive (resp. totally nonnegative) tropical Grassmannian. More explicitly, this means that a common positive solution to the tropicalizations of all the Plücker relations is also a positive solution to the tropicalization of any polynomial in the ideal generated by the Plücker relations. Our goal is to generalize this fact to the setting of the complete flag variety.

We begin with a set up analogous to the previous paragraph. The variety \( \text{Fl}_n \) is cut out by the incidence Plücker relations, a set of polynomials which extends the Plücker relations. These polynomials generate an ideal called the incidence Plücker ideal. We consider the set of points satisfying the tropicalizations of the incidence Plücker relations, called the complete flag Dressian, \( \text{FlDr}_n \), and the set of points satisfying the tropicalizations of all polynomials in the incidence Plücker ideal, called the tropical complete flag variety, \( \text{TrFl}_n \). These can be thought of as parameterizing abstract flags of tropical linear spaces and realizable flags of tropical linear spaces, respectively \cite{8}.

In general, \( \text{FlDr}_n \) strictly contains \( \text{TrFl}_n \) \cite{8}. Motivated by the example of the tropical Grassmannian, we investigate the nonnegative parts of these spaces. We define the totally nonnegative complete flag Dressian to be the set of simultaneous positive solutions to the tropicalizations of the incidence Plücker relations and the totally nonnegative tropical complete flag variety to be the set of simultaneous positive solutions to the tropicalizations of all the polynomials in the incidence Plücker ideal. Our second main result, Theorem 5.25\text{trop},
saying the following:

**Theorem.** The totally nonnegative tropical complete flag variety, $\text{TrFl}_{n}^{≥0}$, equals the totally nonnegative complete flag Dressian, $\text{FlDr}_{n}^{≥0}$.

At the outset, the two results highlighted in this introduction may feel fairly different from one another. However, their proofs are virtually identical. In fact, as we go through this paper, we will present two parallel stories. The two settings will be the real and tropical worlds. Many of our results and definitions about the complete flag variety will immediately be followed by an analogous statement about the complete flag Dressian, which will be indicated by a superscript “trop” in the numbering of the statement, as demonstrated in the previous paragraph.

Let $S_n$ be the symmetric group on $[n]$. Define the permutahedron $\text{Perm}_n$ to be the convex hull of the points $\{(w(1), w(2), \ldots, w(n)) \mid w \in S_n\}$ in $\mathbb{R}^n$. For $u, v \in S_n$ with $u \leq v$ in Bruhat order, define a Bruhat interval polytope $P_{u,v}$ to be the convex hull of the points $\{(w(1), w(2), \ldots, w(n)) \mid w \in S_n$ and $u \leq w \leq v$ in Bruhat order$\}$ in $\mathbb{R}^n$ [15]. Note that $\text{Perm}_n$ is the Bruhat interval polytope $P_{\text{id}, w_0}$, where $\text{id}$ is the identity and $w_0$ is the maximal element of $S_n$. In [13], the authors showed that a point of the positive flag Dressian can be thought of as a height function on the permutahedron, which induces a coherent subdivision into Bruhat interval polytopes.

In this paper, we study the totally nonnegative flag Dressian, which extends the positive flag Dressian by allowing points to have coordinates whose values are $\infty$. As we will see in this paper, each such point can be seen as a height function on some Bruhat interval polytope $P_{u,v}$. In our follow up paper [5], we will study the details of a parameterization of it. In Section 2, we present two parallel results highlighted in this introduction may feel fairly different from one another. However, their proofs are virtually identical. In fact, as we go through this paper, we will present two parallel stories. The two settings will be the real and tropical worlds. Many of our results and definitions about the complete flag variety will immediately be followed by an analogous statement about the complete flag Dressian, which will be indicated by a superscript “trop” in the numbering of the statement, as demonstrated in the previous paragraph.

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The structure of the paper is as follows: In Section 2, we introduce the totally nonnegative complete flag variety and tropical versions of this space. We also fill in some important combinatorial background. In Section 3, we delve deeper into the combinatorics of the totally nonnegative complete flag variety and explore the details of a parameterization of it. In Section 4, we present our main results in the context of the totally positive flag variety, where all Plücker coordinates are strictly positive. Our results are significantly easier to prove in this setting. As such, the totally positive flag variety offers an illustrative application of our proof method which avoids many of the technical details. In Section 5, we extend the methods from the previous section to show our main results in full generality.

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## 2 Background

### 2.1 The Totally nonnegative Complete Flag Variety

We begin by defining the complete flag variety, which will be the space we primarily focus on in this paper.

**Definition 2.1.** The complete flag variety $\text{Fl}_n$ is the collection of all complete flags in $\mathbb{R}^n$, which are collections $(V_i)_{i=0}^n$ of linear subspaces satisfying $\{0\} = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_n = \mathbb{R}^n$.

We first observe that $\text{Fl}_n$ is a multi-projective variety. We can represent a flag $(V_i)_{i=1}^n$ by a full rank $n$ by $n$ matrix $M$ such that $V_i$ equals the span of the topmost $i$ rows of $M$. Let $\text{GL}_n$ be the group of invertible $n$ by $n$ matrices and $\text{SL}(n, \mathbb{R})$ be the special linear group of real matrices with determinant 1. Let $B_+$ be the Borel subgroup of $\text{GL}_n$ consisting of lower triangular matrices. One can check that two matrices $M$ and $M'$ represent the same flag if and only if they are related by left multiplication by some $B \in B_+$. Thus, we
can think of the complete flag variety as \( \text{Fl}_n = \{ B \_u \mid u \in SL(n, \mathbb{R}) \} \), where a flag in \( \text{Fl}_n \) represented by a matrix \( u \) is identified with the set \( B \_u \).

For \( I \subset [n] = \{1, \ldots, n\} \) and \( M \) an \( n \) by \( n \) matrix, the Plücker coordinate (or, alternatively, flag minor) \( P_I(M) \) is the determinant of the submatrix of \( M \) in rows \( \{1, 2, \ldots, |I|\} \) and columns \( I \). To any flag \( F \), associate the collection of Plücker coordinates \( (P_I(F))_{I \subset [n]} \), defined to be the Plücker coordinates of any matrix representative of that flag. By [23, Proposition 14.2], this is an embedding of \( \text{Fl}_n \) in \( \mathbb{RP}^{(n-1) \times \cdots \times \mathbb{RP}^{(n-1)}} \).

The Plücker coordinates of flags in \( \text{Fl}_n \) are cut out by multi-homogeneous polynomials, as shown in the following statements. Note that we will often use shorthand notation such as \( (S \setminus \{a, b\}) \cup \{c, d\} \).

**Definition 2.2** ([9]). Consider \( \mathbb{RP}^{(n-1) \times \cdots \times \mathbb{RP}^{(n-1)}} \), with coordinates indexed by proper subsets of \([n]\). For \( 1 \leq r \leq s \leq n \), the **incidence Plücker relations** for indices of size \( r \) and \( s \) are

\[
\mathcal{P}_{r,s; n} = \left\{ \sum_{j \in J \cup I} \text{sign}(j, I, J) P_{I \cup J} P_{J \cup I} \mid I \in \binom{[n]}{r-1}, J \in \binom{[n]}{s+1} \right\},
\]

where \( \text{sign}(j, I, J) = (-1)^{|\{k \in J \mid k < j\}| + |\{i \in I \mid i < j\}|} \).

The full set of incidence Plücker relations is \( \mathcal{P}_{IP;n} = \bigcup_{1 \leq r \leq s \leq n} \mathcal{P}_{r,s; n} \). The ideal generated by \( \mathcal{P}_{IP;n} \), denoted \( IP_{n} \), is called the **incidence Plücker ideal**.

**Remark:** Note that the above definition allows for the option of \( r = s \). The incidence Plücker relations for which \( r = s \) are called the (Grassmann) Plücker relations. When we want to emphasize that we are interested in the incidence Plücker relations for which \( r \neq s \), we will call them incidence relations.

**Proposition 2.3** ([9, Section 9, Proposition 1] and discussion following its proof). Let \( P \in \mathbb{RP}^{(n-1) \times \cdots \times \mathbb{RP}^{(n-1)}} \). Then \( P = P(F) \) for some \( F \in \text{Fl}_n \) if and only if \( P \) satisfies the incidence Plücker relations \( \mathcal{P}_{IP;n} \).

In particular, this means the incidence Plücker relations are precisely the relations between the topmost minors of a generic full rank matrix.

Lusztig introduced the notion of non-negativity for flag varieties. We outline here the definition of the **totally nonnegative complete flag variety**, following [19]. We work in type A and so the appropriate simplifications will be made in presenting the definition. Let \( s_i \) be the transposition \((i, i+1)\) in the symmetric group \( S_n \) and let \( w_0 \) be the longest permutation in \( S_n \). For \( 1 \leq k < n \), let \( x_k(a) \) be the \( n \) by \( n \) matrix which is the identity matrix with an \( a \) added in row \( k \) of column \( k + 1 \). Explicitly,

\[
x_k(a) = \begin{pmatrix}
1 & & & & \cdots & k & k + 1 \\
& 1 & & & & \cdots & \\
& & \ddots & & & & \\
& & & 1 & a & & \\
& & & & 0 & 1 & & \\
& & & & & \ddots & & \\
& & & & & & 1
\end{pmatrix},
\]

where unmarked off-diagonal matrix entries are 0.

**Definition 2.4** ([20]). Let \( N = \binom{n}{2} \). Pick \((i_1, i_2, \ldots, i_N)\) such that \( s_{i_1} \cdots s_{i_N} = w_0 \). Then let

\[
U_{i_0}^{+} = \{ x_{i_1}(a_1) \cdots x_{i_N}(a_N) \mid \forall i, a_i \in \mathbb{R}_{>0} \}
\]

One can show that this definition is independent of the choice of sequence \((i_1, \ldots, i_N)\).
Definition 2.5 ([20]). Let $B_{>0} = \{B_u \mid u \in U^+_0 \} \subset Fl_n$. The totally nonnegative (TNN) complete
flag variety (of type A), $Fl^0_n$, is the closure of $B_{>0}$.

In Section 3, we give a cell decomposition and a parameterization of the TNN complete flag variety and
investigate the Plücker coordinates of TNN complete flags.

2.2 Flag Positroids

We now introduce flag matroids as well as two notions of flag positroids. We will show that these
notions coincide in section 5.2. We assume basic familiarity with the definitions of matroids and
positroids and direct the reader to [24] for background on matroids and to [27] for background on positroids.

Definition 2.6 ([16, Lemma 8.1.7]). A (complete) flag matroid on a ground set $E$ is a sequence of matroids
$M = (M_1, M_2, \ldots, M_n)$ on the ground set $E$, with the rank of $M_i$ equal to $i$, such that for any $j < k$,

- each basis of $M_j$ is contained in some basis of $M_k$.
- each basis of $M_k$ contains some basis of $M_j$.

The matroids $M_i$ are called the constituent matroids of the flag matroid.

As with many matroid theoretic concepts, there are numerous cryptomorphic descriptions of flag matroids,
some of which can be found in [6]. Note that the indices of the non-zero Plücker coordinates of an invertible
square matrix are easily seen to be the bases of a flag matroid.

Definition 2.7. A flag matroid on $[n]$ is realizable if its bases are the non-zero Plücker coordinates of
$P(F)$ for some $F \in Fl_n$. Similarly, a realizable flag positroid on $[n]$ is a flag matroid whose bases are the
non-zero Plücker coordinates of a flag $F \in Fl^0_n$.

While this is a natural definition of flag positroids, there is another notion which one could argue is an
equally natural extension of the definition of a positroid.

Definition 2.8. A synthetic flag positroid on $[n]$ is a flag matroid on $[n]$ whose bases are the non-zero
Plücker coordinates of a flag $F \in Fl_n$ such that $P_I(F) \geq 0$ for all $I \subseteq [n]$.

Remark: The above definition is slightly imprecise. Since the Plücker coordinates are multi-projective,
we should really insist that for each $k \in [n]$, $P_k$ has a fixed sign for all $I$ with $|I| = k$. However, we will use
the projective degree of freedom to fix $P_{[k]} = 1$ and then we will indeed want all Plücker coordinates to be
nonnegative.

We will show in Lemma 5.21 that these two notions of flag positroid coincide. Note that both of the above
definitions define stronger conditions than just being a flag of matroids in which each constituent matroid is a
positroid. For instance, the definition of a synthetic flag positroid requires each of the constituent positroids
to be simultaneously realizable, with nonnegative Plücker coordinates, by a single matrix. By contrast, there
exist flags of positroids which are not realizable in that way.

Example 2.9. Consider the flag matroid $\mathcal{M} = (M_1, M_2, M_3)$ on the ground set $[3]$ with bases $\{1, 3\}$,
$\{12, 13, 23\}$ and $\{123\}$, respectively. Each of these is clearly a positroid. Any three by three matrix with
nonnegative Plücker coordinates which represents $\mathcal{M}$ must have the first row

\[
\begin{pmatrix}
a & 0 & b
\end{pmatrix}
\]  

with $P_1 = a$ and $P_3 = b$ positive.

Using row reduction operations which do not change the rowspan of either the top row or top two rows,
we can write the full matrix as

\[
\begin{pmatrix}
a & 0 & b \\
0 & c & * \\
* & * & *
\end{pmatrix}
\]  

(3)
where the \( * \) entries are arbitrary and not of interest to us. Note that there is no value of \( c \) which allows both \( P_{12} > 0 \) and \( P_{23} > 0 \). Thus, \( \mathcal{M} \) is a flag of positroids which cannot be realized by a single matrix with all nonnegative Plücker coordinates. As a result, \( \mathcal{M} \) is not a synthetic flag positroid.

### 2.3 The Complete Flag Dressian and the Tropical Complete Flag Variety

We now introduce some notation and terminology needed for discussing tropical varieties, as well as the precise definitions of the totally nonnegative (positive) tropical complete flag variety and the totally nonnegative (positive) complete flag Dressian.

Let \( \mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n \). For \( \mathbf{b} = (b_1, \ldots, b_n) \in \mathbb{N}^n \), we write \( \mathbf{x}^\mathbf{b} = x_1^{b_1} \cdots x_n^{b_n} \). We now define the tropicalization of a polynomial \( p \) which is, roughly speaking, the expression obtained by replacing additions by minimizations and multiplications by additions. Definition 2.10 and Definition 2.15 are modified from [8]. Proposition 2.16 will offer further motivation for some of the terminology employed.

**Definition 2.10.** Let \( \mathbb{T} = \mathbb{R} \cup \infty \) and \( p = \sum_i a_i x^{b_i} \) be a (Laurent) polynomial, where each \( a_i \in \mathbb{R} \) and each \( b_i \in \mathbb{Z}^n \). We define the **tropicalization** of \( p \) by

\[
\text{Trop} (p) = \min_i \{ a_i + \mathbf{x} \cdot \mathbf{b}_i \} = \min_i \{ a_i + x_1(b_i)_1 + \cdots + x_n(b_i)_n \},
\]

if \( p \) is non-zero. We define \( \text{Trop} (0) = \infty \). For an \( m \)-tuple of polynomials \( p = (p_1, \ldots, p_m) : \mathbb{R}^n \to \mathbb{R}^m \), we define \( \text{Trop} (p) \) componentwise.

**Definition 2.11.** For a polynomial \( p = \sum_i a_i x^{b_i} : \mathbb{R}^n \to \mathbb{R} \), we say that a point \( \mathbf{x} \in \mathbb{T}^n \) is a **solution of the tropicalization of** \( p \) if \( \min_i \{ a_i + \mathbf{x} \cdot \mathbf{b}_i \} \) is achieved at least twice. We say that a point in \( \mathbb{T}^n \) is a **positive solution of the tropicalization of** \( p \) if, additionally, at least one of the minima comes from a term of \( p \) with a positive real coefficient, and at least one of the minima comes from a term of \( p \) with a negative real coefficient. Equivalently, if we rewrite \( p = 0 \) in the form \( \sum c_j x^{b_j} = \sum d_i x^{b_i} \) with all \( c_j \) and \( d_i \) positive, then we want at least one minimum to occur in a term coming from each side of the equality.

The tropical objects we are interested will live in **projective tropical spaces**, which are spaces that interact nicely with homogeneous polynomials.

**Definition 2.12.** Projective tropical space, denoted \( \mathbb{T}P^n \), is given by \( (\mathbb{T}^{n+1} \setminus (\infty, \ldots, \infty)) / \sim \) where \( \mathbf{x} \sim \mathbf{y} \) if there exists \( c \in \mathbb{R} \) such that \( x_i = y_i + c \) for all \( i \in [n] \).

The following is immediate from the definition:

**Proposition 2.13.** If \( p \) is a homogeneous polynomial, then \( \mathbf{x} \) is a (positive) solution of \( \text{trop}(p) \) if and only if \( \mathbf{y} \sim \mathbf{x} \) is a (positive) solution of \( \text{trop}(p) \) for all \( \mathbf{y} \sim \mathbf{x} \).

**Example 2.14.** Consider the homogeneous polynomial \( p(x, y) = x^2 - xz + y^2 \). Then \( \text{trop}(p) \) is given by \( \min \{ 2x, x + z, 2y \} \). The point \( (1, 1, 2) \) is a solution to \( \text{Trop} (p) \) since \( 2x = 2y = 2 \leq x + z = 3 \). However, this is not a positive solution to \( \text{Trop} (p) \) since the minima, \( 2x \) and \( 2y \), originate from the terms \( x^2 \) and \( y^2 \) of \( p \), which have coefficients of the same sign. Since \( p \) is homogeneous and \( (3, 3, 4) \sim (1, 1, 2) \), we know that \( (3, 3, 4) \) is also a solution of \( \text{Trop}(p) \).

**Definition 2.15.** Given a set of multi-homogeneous polynomials \( \mathcal{P} \), each of which is homogeneous with respect to sets of variables of sizes \( \{ n_i \}_{i=1}^t \), and the ideal \( I \) which they generate, we define the following sets in \( \mathbb{T}P_1^{n_1-1} \times \cdots \times \mathbb{T}P_t^{n_t-1} \):

- **The tropical previarity** \( \overline{\text{trop}}(\mathcal{P}) \) or \( \overline{\text{trop}}(I) \) is the set of simultaneous solutions to the tropicalizations of all the polynomials in \( \mathcal{P} \) or in \( I \), respectively.

- **The nonnegative tropical previarity** \( \overline{\text{trop}}^{\geq 0}(\mathcal{P}) \) or \( \overline{\text{trop}}^{\geq 0}(I) \), is the set of simultaneous positive solutions of the tropicalizations of all the polynomials in \( \mathcal{P} \) or in \( I \), respectively.

- **The positive tropical previarity** \( \overline{\text{trop}}^{> 0}(\mathcal{P}) \) or \( \overline{\text{trop}}^{> 0}(I) \), is the set of simultaneous real positive solutions of the tropicalizations of all the polynomials in \( \mathcal{P} \) or in \( I \), respectively. Equivalently, \( \overline{\text{trop}}^{> 0}(\mathcal{P}) = \overline{\text{trop}}^{\geq 0}(\mathcal{P}) \cap \mathbb{R}^n \) and \( \overline{\text{trop}}^{> 0}(I) = \overline{\text{trop}}^{\geq 0}(I) \cap \mathbb{R}^n \).
Solutions of tropicalizations of polynomials can be described in a different way, which clarifies the origin of the term “positive solution”. Let $\mathcal{C} = \bigcup_{n=1}^{\infty} \mathbb{C}((t^{1/n}))$ be the field of Puiseux series over $\mathbb{C}$. A Puiseux series $p(t) \in \mathcal{C}$ has a term with a lowest exponent, say $at^u$ with $a \in \mathbb{C}^*$ and $u \in \mathbb{Q}$. We define $\text{val}(p(t)) = u$. Further, we will say $p(t) \in \mathbb{R}^+$ if $a \in \mathbb{R}^+$.

Given an ideal $I \subseteq \mathbb{C}[x_1, \ldots, x_n]$, let $V(I) \subseteq \mathbb{C}^n$ be the variety where all polynomials in $I$ vanish. Thinking of the semifield $\mathbb{R}^+$ in the field $\mathcal{C}$ as being analogous to the semifield $\mathbb{R}^+$ in the field $\mathbb{C}$, we define the positive part of this variety to be $V^+(I) = V(I) \cap (\mathbb{R}^+)^n$. Our use of the term positive tropical prevariety is motivated by the following:

**Proposition 2.16** ([26, Proposition 3.7] and [31, Proposition 2.2]). Let $I$ be an ideal of $\mathbb{C}[x_1, \ldots, x_n]$. Then $\text{trop}(I) \cap \mathbb{R}^n = \text{val}(V(I))$ and $\text{trop}^0(I) = \text{val}(V^+(I))$, where $\text{val}(V(I))$ and $\text{val}(V^+(I))$ are the closures of $\text{val}(V(I))$ and $\text{val}(V^+(I))$, respectively.

We now focus in on the tropical spaces we will study in the rest of this paper. We define two tropical analogues of $\text{Fl}_n$ along with their totally nonnegative and totally positive parts.

**Definition 2.17.** We define spaces relating to the complete flag Dressian as follows:

- The complete flag Dressian is $\text{FlDr}_n = \overline{\text{trop}(\mathbb{C}_{IP:n})}$, the tropical variety defined by the incidence Plücker relations.
- The totally nonnegative complete flag Dressian is $\text{FlDr}^{\geq 0}_n = \overline{\text{trop}^{\geq 0}(\mathbb{C}_{IP:n})}$.
- The totally positive complete flag Dressian $\text{FlDr}_{n}^{> 0} = \overline{\text{trop}^{> 0}(\mathbb{C}_{IP:n})}$.

**Definition 2.18.** We define spaces relating to the tropical complete flag variety as follows:

- The tropical complete flag variety is $\text{TrFl}_n = \overline{\text{trop}(I_{IP:n})}$, the tropical prevariety defined by the entire incidence Plücker ideal.
- The totally nonnegative tropical complete flag variety is $\text{TrFl}^{\geq 0}_n = \overline{\text{trop}^{\geq 0}(I_{IP:n})}$.
- The totally positive tropical complete flag variety is $\text{TrFl}^{> 0}_n = \overline{\text{trop}^{> 0}(I_{IP:n})}$.

One can think of $\text{FlDr}_n$ and $\text{TrFl}_n$ as parameterizing abstract flags of tropical linear spaces and realizable flags of tropical linear spaces, respectively [8]. Expanding on this, one can think of $\text{FlDr}^{\geq 0}_n$ and $\text{TrFl}^{\geq 0}_n$ as parameterizing abstract flags of positive tropical linear spaces and positively realizable flags of tropical linear spaces, respectively. We note that for a realizable tropical flag to be “positively realizable”, it is not enough for it to be a flag consisting of positive tropical linear spaces. Rather, there must be some positivity conditions satisfied between the constituent linear spaces, as captured by the positive tropical incidence relations. This is analogous to the fact that not all flags of positroids are synthetic or realizable flag positroids.

Our main result, Theorem 5.25, will show that $\text{TrFl}^{\geq 0}_n$ and $\text{FlDr}^{\geq 0}_n$ coincide. Note that this is not obvious since a priori a point in $\text{TrFl}^{\geq 0}_n$ satisfies more relations than a point in $\text{FlDr}^{\geq 0}_n$. In fact, in general, the tropical prevariety of a collection of polynomials will properly contain the tropical prevariety of the ideal those polynomials generate. In the specific case of the complete flag variety, it is shown in [8, Example 5.2.4] that for $n \geq 6$, $\text{FlDr}_n$ properly contains $\text{TrFl}_n$.

### 2.4 Lindström-Gessel-Viennot Construction

The Lindström-Gessel-Viennot (LGV) construction is a construction yields a matrix containing information about the paths in a given weighted digraph. We will use this to study the coordinates of flags in $\text{Fl}^{\geq 0}_n$.

The construction works as follows: Let $G$ be a finite cycle-free digraph with an edge weight $w_e$ associated to each directed edge $e$. Within $G$, fix any collection of vertices $A = \{a_1, \ldots, a_n\}$ to be called sources and any collection of vertices $B = \{b_1, \ldots, b_n\}$, disjoint from $A$, to be called sinks. The weight of a directed path $P$ from $a_i$ to $b_j$ is defined to be the product of the weights of the edges in that path. It is denoted $w(P)$. We define $\omega(i,j) = \sum_{P:a_i \rightarrow b_j} w(P)$ to be the sum of the weights of all paths from $a_i$ to $b_j$. 

Definition 2.19. Let $|A| = |B| = n$. Then a collection of non-intersecting paths from $A$ to $B$ is a collection of paths from $a_i$ to $b_{\sigma(i)}$, where $\sigma$ is any permutation of $[n]$, such that no two paths have a common vertex. Such a collection is denoted $P = (P_1, \ldots, P_n)$ where $P_i$ is the path originating from $a_i$. We will denote by $\sigma(P)$ the permutation of $[n]$ such that $P_i$ terminates at $b_{\sigma(i)}$.

We are now ready to state the LGV construction.

Theorem 2.20 ([10]). Let $G$ be a finite acyclic directed graph with weighted edges. Choose a source set $A$ and a sink set $B$ disjoint from $A$, each of size $n$. Consider the matrix

$$N = (\omega(i,j))_{i,j}.$$ 

Then

$$\det(N) = \sum_{P: A \to B} \text{sgn}(\sigma(P)) \prod_{i=1}^n w(P_i),$$

where the sum is over all collections of non-intersecting paths $P = (P_1, \ldots, P_n)$ from $A$ to $B$.

Corollary 2.21. In the setting of Theorem 2.20, let $I \subseteq A$ and $J \subseteq B$ with $|I| = |J|$. Let $I'$ and $J'$ be the sets of indices of the corresponding sources in $A$ and sinks in $B$, respectively. Let $N_{I',J}$ denote the submatrix of $N$ consisting of rows $I'$ and columns $J'$. Then

$$\det(N_{I',J}) = \sum_{P: I' \to J} \text{sgn}(\sigma'(P)) \prod_{i=1}^{|I|} w(P_i),$$

(5)

where the sum is now over all collections of non-intersecting paths from $I'$ to $J$ and where the bijection $\sigma'(P) : I' \to J'$ is such that $P_i$ is a path from $a_i$ to $b_{\sigma'(P)(i)}$. In this context, $\text{sgn}(\sigma') = (-1)^t$ where $t$ is the number of inversions in $\sigma'$, which is to say, the number of pairs $i < j \in I'$ such that $\sigma'(j) < \sigma'(i)$.

Example 2.22. Consider the weighted digraph in Fig. 1 with source set $\{1', 2', 3'\}$ and sink set $\{1, 2, 3\}$.

![Figure 1: A weighted directed graph. The unmarked diagonal edges have weight 1.](image)

We construct the matrix $N$. There is a unique path from $1'$ to 1, consisting of the diagonal path, which has weight 1. Thus, $N_{1',1} = 1$. There are two paths from $1'$ to 2, one using the weighted vertical edge $a$ and the other using the weighted vertical edge $c$. Thus, $N_{1',2} = a + c$. There is a unique path from $1'$ to 3 using the weighted vertical edges $a$ and $b$. Thus, $N_{1',3} = ab$. Note that there are no paths from $2'$ to 1, so $N_{2',1} = 0$. Continuing in this manner, we obtain

$$N = \begin{pmatrix}
1 & a + c & ab \\
0 & 1 & b \\
0 & 0 & 1
\end{pmatrix}.$$

There are two path collections from $\{1', 3'\}$ to $\{2, 3\}$. The first uses the weighted vertical edge $a$ and the second uses the weighted vertical edge $c$. Both these path collections connect $1'$ to 2 and $3'$ to 3, which preserves the relative orders of the sources and the sinks. Correspondingly, $\det(N_{\{1',3'\},\{2,3\}}) = a + c$. There is a unique path collection from $\{1', 2', 3'\}$ to $\{1, 2, 3\}$, consisting entirely of diagonal edges. This has weight 1, which is reflected by the fact that $\det(N) = 1$. 

8
3 Parametrization of the Totally nonnegative Complete Flag Variety

3.1 The Marsh-Rietsch Parametrization

As shown by Rietsch [28], $F_{\mathfrak{t}}^\geq 0$ is a cell complex, whose cells $\mathcal{R}_{v, w}^\geq 0$ are indexed by pairs of permutations $v \leq w$ in a partial order called Bruhat order on $S_n$. Each such $\mathcal{R}_{v, w}^\geq 0$ is given an explicit parameterization in [21]. We will describe this parameterization here, making some choices that in principle are arbitrary but will be convenient for our purposes, and invite the reader to look at the cited references for full generalities.

Any permutation $w$ in $S_n$ has at least one expression, which is to say, a way to write it as a product of simple transpositions $s_i = (i, i + 1)$. The length of $w$, $\ell(w)$, is the fewest number of simple transpositions in any expression for $w$. An expression for $w$ consisting of $\ell(w)$ transpositions is called reduced. We now define Bruhat order.

**Definition 3.1.** Let $v, w \in S_n$. If there exists a reduced expression $w = s_{i_1}s_{i_2} \cdots s_{i_k}$ for $w$ which admits a reduced subexpression $v = s_{i_{j_1}}s_{i_{j_2}} \cdots s_{i_{j_m}}$ for $v$ where $1 \leq j_1 < j_2 < \cdots < j_m \leq k$, we say $v \leq w$ in Bruhat order.

**Proposition 3.2** ([3, Corollary 2.2.3]). If $v \leq w$ in Bruhat order, then any expression for $w$ admits a subexpression for $v$.

We will be interested in a special choice of subexpression which is called the positive distinguished subexpression. Intuitively, this can be thought of as the leftmost reduced subexpression.

**Definition 3.3.** Let $v \leq w$. Choose a reduced expression $w = s_{i_1}s_{i_2} \cdots s_{i_k}$ for $w$. Then a reduced subexpression $v = s_{i_{j_1}}s_{i_{j_2}} \cdots s_{i_{j_m}}$ for $v$ in $w$ is a *positive distinguished subexpression* if, whenever $\ell(s_{i_{p}}s_{i_{r}} \cdots s_{i_{m}}) < \ell(s_{i_{p}}s_{i_{r}} \cdots s_{i_{m}})$ for $j_r - 1 < p < j_r$, we have $p = j_r - 1$.

**Lemma 3.4.** For every $v \leq w$, and every reduced expression $w$ of $w$, there is a unique positive distinguished subexpression for $v$ in $w$.

*Proof.* Given an expression $w$ for $w$, define $w^{-1}$ to be the expression for $w^{-1}$ obtained by reversing the order of the transpositions in $w$. Observe that [21] defines positive distinguished subexpressions as rightmost reduced subexpressions. However, $v$ is a positive distinguished subexpression in $w$ by our convention if and only if $v^{-1}$ is a positive distinguished subexpressions of $w^{-1}$ by the reverse convention. Thus, the result follows from [21, Lemma 3.5].

**Example 3.5.** Let $n = 4$. Fix the expression $w = s_1s_2s_3s_1s_2s_1$ for $w$ and let $v = s_1s_2s_1$. There are a number of subexpressions for $v$ in $w$. For instance, $v$ can be written as $s_{i_{j_1}}s_{i_{j_2}}s_{i_{j_3}}$ with $j_1 = 1$, $j_2 = 2$ and $j_3 = 6$. Note that $\ell(s_{i_2}s_{i_2}s_{i_3}) = \ell(s_2s_1s_3) = 1$ is less than $\ell(s_{i_{j_2}}s_{i_{j_3}}) = \ell(s_2s_1) = 2$, but $j_1 \neq 2$. Thus, this is not the positive distinguished subexpression for $v$.

The leftmost subexpression is $j_1 = 1$, $j_2 = 2$ and $j_3 = 4$. Indeed, one can verify that this choice satisfies the definition of a positive distinguished subexpression.

For $1 \leq k < n$, let $\hat{s}_k$ be the $n$ by $n$ identity matrix with the $2 \times 2$ submatrix in rows $\{k, k + 1\}$ and columns $\{k, k + 1\}$ replaced by the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Explicitly,$$
\hat{s}_k = \begin{pmatrix} 1 & \cdots & 0 \\ k & \cdots & 1 \\ k + 1 & \cdots & 0 \end{pmatrix},$

where $\hat{s}_k$ is the identity matrix except for the $2 \times 2$ submatrix in rows and columns $\{k, k + 1\}$.

\[9\]
where unmarked off-diagonal matrix entries are 0.

**Definition 3.6.** Fix $v \leq w$ in the Bruhat order. Fix a vector $a \in \mathbb{R}^{\ell(w) - \ell(v)}$. Consider the reduced expression $w_0 = (s_1 s_2 \cdots s_{n-1})(s_1 s_2 \cdots s_{n-2})(\cdots)(s_1 s_2)(s_1)$ for $w_0$, the longest permutation in the Bruhat order in $S_n$. Choose the positive distinguished subexpression $v$ for $w$ in $w_0$, and the positive distinguished subexpression $v$ for $v$ in $w$, and write them as $w = s_{i_1} \cdots s_{i_k}$ and $v = s_{i_1} \cdots s_{i_m}$, respectively. Let $J = \{ j \mid j = j_t \text{ for some } t \}$. In other words, $J$ are the positions of transpositions which are used in $v$. Let

$$M_{v,w}(a) := M_1 \cdots M_k,$$

where $M_j = \begin{cases} s_{i_j}, & j \in J \\ x_{i_j}(a_j), & j \notin J \end{cases}$.

**Theorem 3.7** (Marsh-Rietsch Parametrization, [21]). The TNN complete flag variety $Fl_n^{>0}$ admits a cell decomposition into cells $R_{v,w}^{>0}$ for $v \leq w$, each of which can be parameterized as

$$R_{v,w}^{>0} = \left\{ M_{v,w}(a) \mid a \in \mathbb{R}^{\ell(w) - \ell(v)} \right\}.$$

In particular, each $R_{v,w}^{>0}$ is homeomorphic to an open ball and each flag $F \in Fl_n^{>0}$ is uniquely represented in some unique $R_{v,w}^{>0}$.

Note that we adopt a slightly different convention than [21]. The matrices in their $R_{v,w}^{>0}$ would be the transposes of the matrices in our $R_{v,w}^{>1}$, $w^{-1}$. This difference is accounted for by the fact that we define positive distinguished subexpressions as leftmost reduced subexpressions rather than rightmost, as noted in the proof of Lemma 3.4. This convention makes notation a bit cleaner in the rest of this paper. Also note that we slightly abuse notation, as we should be describing $R_{v,w}^{>0}$ as consisting of flags, not matrices. We will continue to implicitly identify a flag $F \in R_{v,w}^{>0}$ with the unique matrix of the form $M_{v,w}(a)$ representing it.

**Example 3.8.** Let $n = 4$, $w = s_1 s_3 s_2 s_1$ and $v = s_2$. The positive distinguished subexpression for $v$ in $w$ is the subexpression where $j_1 = 3$, so $J = \{ 3 \}$. Thus, $M_1 = x_1(a_1)$, $M_2 = x_3(a_2)$, $M_3 = s_2$, $M_4 = x_1(a_4)$ and $a = (a_1, a_2, a_4)$. The cell $R_{v,w}^{>0}$ is given by

$$M_{v,w}(a) = M_1 M_2 M_3 M_4 = \begin{pmatrix} 1 & a_4 & a_1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & a_2 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where the $a_i$ range over all positive real numbers. Note that the Plücker coordinates of $M_{v,w}(a)$ are all nonnegative, as our next result will tell us is true for any cell of $Fl_n^{>0}$.

**Lemma 3.9** ([15, Lemma 3.10]). For any flag $F \in Fl_n^{>0}$ and any $I \subset [n]$, $P_I(F) \geq 0$.

We establish some notation that we will use going forward.

**Definition 3.10.** Let $\Phi_{v,w} : \mathbb{R}^{\ell(w) - \ell(v)} \rightarrow \mathbb{R}^{n+1} + \cdots + \left( \begin{smallmatrix} n \end{smallmatrix} \right)$ be the map which sends $a$ to the Plücker coordinates of $M_{v,w}(a)$. This map is a bijection, and thus it admits an inverse, which we denote $\Psi_{v,w}$.

### 3.2 A Graphical Description of the Marsh-Rietsch Parametrization

Fix $v \leq w$. Our main goal in this section is to construct a digraph $G_{v,w}(a)$ which offers a combinatorial way to determine precisely which Plücker coordinates are positive in $R_{v,w}^{>0}$.

Recall that in defining $R_{v,w}$, we had to choose specific subexpressions $v$ and $w$ of

$$w_0 = (s_1 \cdots s_{n-1})(s_1 \cdots s_{n-2})(\cdots)(s_1 s_2)(s_1).$$

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1This choice of expression is arbitrary in the context of the Marsh-Rietsch parameterization, but plays an important role in the rest of this paper.
We will think of \( w_0 \) as consisting of a series of runs of transpositions, with the first run being \( s_1 \cdots s_{n-1} \), the second run being \( s_1 \cdots s_{n-2} \), all the way up to the \((n-1)\)'st run, being just \( s_1 \). Each simple transposition in \( w \) or \( v \) can thus be associated with some run of \( w_0 \).

We start by constructing a skeleton digraph \( S \) on top of which we will build our graphs \( G_{v,w} \) by adding edges and permuting certain vertex labels. The skeleton \( S \) has \( 2n \) vertices, with vertices labelled from 1’ to \( n' \) lying vertically, with \( n' \) on top and 1’ on the bottom, and vertices labeled from 1 to \( n \) lying horizontally, with 1 to the right and \( n \) to the left. It also has a directed edge from \( i' \) to \( i \) for each \( i \in [n] \), as illustrated in Fig. 2. For any graph obtained from \( S \) by adding edges and permuting vertex labels, we will say “strand \( r \)” to refer to the diagonal edge that is \( r \) from the bottom, so that the bottom-most of these diagonal edges is strand 1 and the top-most is strand \( n \). We will also say “column \( c \)” to refer to the vertical slice of the graph between the vertices \( c \) and \( c + 1 \).

We adopt the notation \( (r,c)_G \) to refer to the intersection of strand \( r \) and column \( c \) in graph \( G \), dropping the \( G \) from the notation when it is unambiguous.

**Definition 3.11.** Let \( v \leq w \in S_n \), with \( \ell(w) = k \). We present an inductive construction of \( G_{v,w} \) starting from \( G_{v,w}^{(0)} = S \). Recall the product expansion for a matrix \( M_{v,w}(a) \) as \( M = M_1 \cdots M_k \) described in Definition 3.6. Each \( M_j \) corresponds to some transposition \( s_{i_j} \) in \( w \), and can thus be associated to a specific run of \( w_0 \). For \( 1 \leq j \leq k \), we obtain \( G_{v,w}^{(j)} \) from \( G_{v,w}^{(j-1)} \) as follows:

1. If \( M_j = x_{i_j}(a_{i_j}) \) comes from run \( r \) of \( w_0 \), then we obtain \( G_{v,w}^{(j)} \) from \( G_{v,w}^{(j-1)} \) by adding a vertical edge from \( (i_j, n+1-r) \) to \( (i_j+1, n+1-r) \).

2. If \( M_j = s_j \), then we obtain \( G_{v,w}^{(j)} \) from \( G_{v,w}^{(j-1)} \) by swapping the primed vertices attached to strands \( i_j \) and \( i_j' \) (but not the unprimed vertices at the other end). As part of this swap, we maintain the incidence of any vertical arrows. In other words, if a vertical arrow started (terminated) on strand \( i_j \) in \( G_{v,w}^{(j-1)} \), then it now starts (terminates) on strand \( i_j + 1 \) in \( G_{v,w}^{(j)} \).

We define \( G_{v,w} = G_{v,w}^{(k)} \) to be the end result of this procedure.

**Example 3.12.** Let’s continue with Example 3.8, where \( n = 4 \), \( w = s_1 s_3 s_2 s_1 \) is the positive distinguished subexpression of \( w = 4213 \) in \( w_0 \), and \( v = s_2 \) is the positive distinguished subexpression of \( v = 1324 \) in \( w \). Note that as a subexpression of \( w_0 \), the first two transpositions in \( w \) come from the first run, the third transposition comes from the second run and the final transposition comes from the third run.

Let \( a = (a_1, a_2, a_3) \). We had \( M_{v,w}(a) = x_{i_1}(a_1) x_{i_3}(a_2) s_2 x_{i_1}(a_4) \). The first term, \( x_{i_1}(a_1) \), comes from the first run of \( w_0 \) and tells us to add a vertical edge from strand 1 to strand 2 in column 4, obtaining Fig. 3.
The next term, $x_3(a_2)$, also comes from the first run of $w_0$ and tells us to add an edge from strand 3 to strand 4, still in column 4. This is shown in Fig. 4.

The next term, $s_2$, tells us to swap the strands 2 and 3. Note that the vertical edge terminating on strand 2 in $G_{v,w}^{(2)}$ should terminate on strand 3 in $G_{v,w}^{(3)}$, as in Fig. 5. Similarly, the edge originating on strand 3 in $G_{v,w}^{(2)}$ should originate on strand 2 in $G_{v,w}^{(3)}$. Observe that while the primed vertices are permuted, the unprimed vertices remain unchanged.

The final term, $x_1(a_4)$, comes from the third run of $w_0$ and tells us to add an edge from strand 1 to strand 2 in column 2, as illustrated in Fig. 6.
We now state the main theorem which highlights the combinatorial value of the graphs $G_{v,w}$.

**Theorem 3.13.** For $v \leq w \in S_n$, and a flag $F \in \mathbb{R}^{\geq 0}$, the Plücker coordinate $P_I(F)$ is non-zero if and only if there is a non-intersecting path collection from $\{1', 2', \ldots, |I'|\}$ to $I$ in $G_{v,w}$.

To prove this, we add weights on $G_{v,w}$ to obtain a weighted digraph $G_{v,w}(a)$ which, through the LGV construction, will provide the key link to the Marsh-Rietsch parameterization of $\mathcal{F}_{v,w}^{\geq 0}$.

**Definition 3.14.** Let $G_{v,w}(a)$ be the weighted digraph on the underlying digraph $G_{v,w}$ obtained by modifying the inductive steps in the construction of $G_{v,w}$ in Definition 3.11 as follows:

1'. If $M_j = x_{ij}(a_j)$, assign the vertical edge which is added in Item 1 of Definition 3.11 a weight of $a_j$.

2'. If $M_j = \dot{s}_{ij}$ comes from run $r$ of $w_0$, then, after appropriately modifying the strands and incident edges as in Item 2 of Definition 3.11, assign the diagonal edge between $(ij, n - r + 1)$ and $(ij, n - r + 2)$ a weight of $-1$.

Any edges not already assigned a weight by this procedure will be of weight 1.

**Example 3.15.** We continue with Example 3.12, with $a = (a_1, a_2, a_4)$ and $M_{v,w}(a) = x_1(a_1)x_3(a_2)\dot{s}_2x_1(a_4)$. Modifying the steps applied in Example 3.12 appropriately, we find the graph $G_{v,w}(a)$ as illustrated in Fig. 7.

**Proposition 3.16.** For any $v \leq w$ in $S_n$, and $a = (a_1, a_2, \ldots, a_{\ell(v)})$, the matrix obtained from $G_{v,w}(a)$ via the LGV construction will coincide with $M_{v,w}(a)$.
Proof. To verify this, we use induction on $\ell(w)$. If $\ell(w) = 0$, both constructions yield the identity matrix. Let $N = \ell(w) > 0$. Let $M'$ be the matrix whose product expansion is the same as that of $M_{v,w}(a)$, but with the last term removed. If that last term was an $x_i(a_N)$, then $M'$ represents a flag in $R_{v,w}^0$ where $w'$ is just $w$ with the final transposition removed from its positive distinguished subexpression $w$ in $w_0$. Let $a'$ be $a$ with its last coordinate removed. By induction, we obtain $M'$ from the LGV construction on $G_{v,w}(a')$. To obtain $G_{v,w}(a)$ from $G_{v,w}(a')$, we add a single vertical edge $e$ of weight $a_N$ between strands $i$ and $i+1$ in some column $c$. Let $M$ be the matrix obtained from $G_{v,w}(a)$ by the LGV construction so that the matrix entry $M_{b,d}$ is the sum of weights of paths in $G_{v,w}(a)$ from $b'$ to $d$. Note that by construction, $e$ is the last edge which gets added to $G_{v,w}(a)$ and, as result, there are no vertical edges weakly above and to the right of it. Thus, any path which uses $e$ must continue along diagonal strand $i+1$ until vertex $i+1$. It follows that if $d \neq i+1$, then $M_{b,d} = M_{b',d}$. Once again using the fact that there are no vertical edges weakly above and to the right of $e$, we note that any path terminating at $i$ in $G_{v,w}$ must be on strand $i$ when it passes through column $c$. Thus, for each path $p$ from $b'$ to $i$ in $G_{v,w}(a')$, we obtain a path from $b'$ to $i+1$ in $G_{v,w}(a)$ which is identical to $p$ until reaching the bottom of $c$, uses edge $e$, and continues to the end of strand $i+1$. This new path has weight $a_N w(p)$. Also, any path from $b'$ to $i+1$ in $G_{v,w}(a')$ is still a path from $b'$ to $i+1$ in $G_{v,w}(a)$. Thus, $M_{b',i+1} = a_N M_{b',i} + M_{b',i+1}$ and we may conclude that $M = M' s_i = M_{v,w}(a)$.

Similarly, if the last term was $s_i$, then $M'$ is in $R_{v,w}^{0,c}$, where $w'$ is as before and $v'$ is obtained form $v$ by removing the final transposition from its positive distinguished subexpression in $w$. Let $a'$ be as before. We obtain $G_{v,w}(a)$ from $G_{v,w}(a')$ by swapping strands $i$ and $i+1$ and by adding a weight of $-1$ to strand $i$ between columns $c$ and $c+1$ for some $c$. Let $M$ be the matrix obtained from $G_{v,w}(a)$ by the LGV construction. Since we maintain the incidences of edges which start or end on strands $i$ and $i+1$, it is clear that $M_{b,d} = M_{b',d}$ for $d \neq i, i+1$. Further, for the same reason, paths terminating at vertex $i$ in $G_{v',w}(a')$ are in bijection with paths terminating at $i+1$ in $G_{v,w}(a)$. Thus, $M_{b,i+1} = M_{b',i}$. Similarly, paths terminating at vertex $i+1$ in $G_{v',w}(a')$ are in bijection with paths terminating at $i$ in $G_{v,w}(a)$. However, since there are no edges weakly above and to the right of the intersection $(i, c)$ in $G_{v,w}(a)$, these paths must pass through the weight $-1$ section of diagonal strand $i$. Thus, $M_{b,i} = M_{b,i+1}$ and we may conclude that $M = M' s_i = M_{v,w}(a)$.

We now state and prove a previously-known result [29, Proposition 5.1], but we give a new proof using the machinery we developed in this section. We adopt the notation $[k]:= \{1', 2', \ldots, k'\}$.

**Lemma 3.17.** Let $v \leq w$, $k \in [n]$ and $a \in R_{v,w}^{0,-\ell(v)}$. Then, for any $I \in \binom{[n]}{k}$, the Plücker coordinate $P_l(M_{v,w}(a))$ is a subtraction free polynomial combination of the weights $a$.

**Proof.** What needs to be verified is that all path collections contribute a positive expression in Eq. (5). It is clear that the sets of vertical edges used in two path collections with the same source and sink sets cannot properly contain one another. If a path collection $P$ contributed a negative expression in Eq. (5), then we could set the weights of all vertical edges not involved in $P$ close to 0 and make the weights of the vertical edges which are involved in $P$ large. Then Eq. (5) would evaluate to a negative number, meaning that there is a Plücker coordinate which is negative. However, given Proposition 3.16, this contradicts Lemma 3.9.

**Proof of Theorem 3.13.** This is immediate from Lemma 3.17.

The following corollary is stated in a more detailed form in [15, Lemma 3.11]. We will revisit the full result in section 5.1, but for now just need this simpler form which follows easily from our graphical framework.

**Corollary 3.18.** Each cell $R_{v,w}^{0} \text{Fl}_n^0$ consists entirely of flags for which some fixed collection of Plücker coordinates is strictly positive and the rest are 0.

**Lemma 3.19.** The image of $\text{Trop} \Phi_{v,w}$, the tropicalization of $\Phi_{v,w}$, lies in $\text{TrFl}_n^0$.

**Proof.** In the case of $\Phi_{a,w}$, this result is a direct application of [25, Theorem 2] and [31]. For other cells, we make use of Corollary 3.18 to conclude that for some $S \subset 2^n$, $P_l = 0$ if and only if $I \in S$. We can work with the lower dimensional space where we remove the coordinates $P_l$. This space is cut out by the ideal obtained from the incidence Plücker ideal by setting $P_l = 0$ for all $I \in S$ in each polynomial in the incidence Plücker
ideal. The (non-zero) coordinates in the resulting variety are still the images of subtraction free polynomials, as in Lemma 3.17, allowing us to directly apply the same proof as in the case of \( \Phi_{id,w_0} \).

**Definition 3.20.** We will denote the image of \( \text{Trop } \Phi_{v,w} \) by \( \text{TrFL}_{v,w}^{\geq 0} \).

It will be helpful for us, going forward, to have convenient ways to differentiate positive distinguished subexpressions of \( w_0 \) from other subexpressions. We now offer a straightforward necessary condition for a subexpression of

\[
w_0 = (s_1 \cdots s_{n-1})(s_1 \cdots s_{n-2})(\cdots)(s_1 s_2)(s_1)
\]
to be positive distinguished. Recall that we think of \( w_0 \) as consisting of a series of runs of transpositions, with the first run being \( s_1 \cdots s_{n-1} \), the second run being \( s_1 \cdots s_{n-2} \), all the way up to the \( (n-1) \)'st run, being just \( s_1 \).

**Lemma 3.21.** Let \( w \) be any positive distinguished subexpression in

\[
w_0 = (s_1 \cdots s_{n-1})(s_1 \cdots s_{n-2})(\cdots)(s_1 s_2)(s_1).
\]

If, for \( k \geq 2 \), \( w \) uses an \( s_i \) from the \( k \)'th run of \( w_0 \) then it also uses an \( s_{i+1} \) from the \( (k-1) \)'st run of \( w_0 \).

**Proof.** Suppose \( w \) is an expression for \( w \). We will prove this lemma by induction on \( i \). For \( i = 1 \), suppose that \( w \) uses the \( s_1 \) in run \( k \) but not the \( s_2 \) in run \( k-1 \). Then either \( w \) uses the \( s_1 \) in run \( k-1 \) and so \( w \) is not reduced, or we can add the \( s_1 \) in run \( k-1 \) which would cancel with the \( s_1 \) in run \( k \), contradicting Definition 3.3. For the induction step, assume to the contrary that \( w \) uses the \( s_1 \) in run \( k \) but not the \( s_{i+1} \) in run \( k-1 \). Suppose that \( w \) uses the transpositions \( s_{i-r}, s_{i-r+1}, \ldots, s_i \) in run \( k \) but does not use the \( s_{i-r-1} \) in run \( k \), with \( r \) possibly 0. By induction, \( w \) has the transpositions \( s_{i-r+1} \cdots s_i \) in \( w \) in run \( k-1 \). Since \( w \) does not use the \( s_{i+1} \) in run \( k-1 \), we can obtain a new expression for \( w \), no longer necessarily a subexpression of \( w_0 \), containing a sequence of consecutive transpositions of the form \((s_{i-r+1} \cdots s_i) (s_{i-r} \cdots s_{i-1}) \). In this expression, the transpositions in the first pair of parentheses started in run \( k-1 \) and the transpositions in the second pair of parentheses started in run \( k \). Note that the terms in parentheses represent the permutation which maps \( i+1 \) to \( i-r \), \( i \) to \( i-r+1 \), \( j \) to \( j+2 \) for \( i-r \leq j \leq i-1 \), and fixes all other numbers. This permutation can be rewritten as \((s_{i-r} s_{i-r+1} \cdots s_i) (s_{i-r} \cdots s_{i-1}) \). Again due to the fact that \( w \) did not use the \( s_{i+1} \) in run \( k-1 \), we can rearrange this into a subexpression of \( w_0 \) where the transpositions in the first pair of parentheses wind up in run \( k-1 \) and the transpositions in the second pair of parentheses wind up in run \( k \). If \( w \) had an \( s_i \) from run \( k-1 \), then this new subexpression shows that \( w \) is not reduced and thus is not positive distinguished. If not, then this new subexpression shows that \( w \) violates Definition 3.3.

**Example 3.22.** We give an example of the argument in the induction step of the previous proof. Consider the expression \( w = s_1 s_2 s_3 s_5 s_1 s_2 s_3 s_4 s_1 s_2 \) for a permutation \( w \). This uses \( s_1, s_2, s_3, s_5 \) from run 1, \( s_1, s_2, s_3, s_4 \) from run 2, and \( s_1, s_2 \) from run 3. In particular, \( w \) uses \( s_3 \) from run 2 but not \( s_4 \) from run 1. By Lemma 3.21, this is not a positive distinguished subexpression. To see this, note that the permutation \( w \) can be expressed as

\[
w = s_1 s_2 s_3 s_5 s_1 s_2 s_3 s_4 s_1 s_2 = s_1 (s_2 s_3) s_5 (s_1 s_2 s_3) s_4 s_1 s_2 = s_1 s_5 (s_2 s_3) (s_1 s_2 s_3) s_4 s_1 s_2 = s_1 s_5 (s_1 s_2 s_3) (s_1 s_2) s_4 s_1 s_2 = s_1 (s_1 s_2 s_3) s_5 (s_1 s_2) s_4 s_1 s_2.
\]

Note that the last expression has two copies of \( s_1 \) next to each other, showing that \( w \) was not reduced.
3.3 Extremal Non-Zero Plücker Coordinates

We define a special subset of the Plücker coordinates of a flag which we call extremal non-zero Plücker coordinates. For a flag $F \in \text{Fl}^n_{\leq 0}$, the indices of these coordinates will depend only on which cell $R_{v,w}^0$ contains $F$. In future sections, we will prove a number of useful facts about the extremal non-zero Plücker coordinates of a flag, including that the extremal non-zero Plücker coordinates determine all of the other Plücker coordinates and also form a positivity test in $R_{v,w}^0$. We will also see the beginnings of the parallel story about the complete flag Dressian take shape in this section.

For any $1 \leq k < n$ and any $P \in \mathbb{R}P^{(1)}_k \times \cdots \times \mathbb{R}P^{(n-1)}_k$, we define a map $\Xi_P : \{[n]_k \} \rightarrow \{[n]_k \}$. Intuitively, when applied to the index of a non-zero Plücker coordinate $F_j$, this map finds the largest member of $I$ that can be increased without making the corresponding Plücker coordinate 0 and increases it maximally.

**Definition 3.23.** Let $P \in \mathbb{R}P^{(1)}_k \times \cdots \times \mathbb{R}P^{(n-1)}_k$ and $I \in \{[n]_k \}$. Let $B = \{i \mid \exists j \notin I, i < j, P_{(I \setminus i) \cup j} \neq 0 \}$. If $B$ is non-empty, define $b = \max_{i \in I} B$ and $a = \max_{j \notin I} \{j \mid P_{(I \setminus b) \cup j} \neq 0 \}$. Then,

$$\Xi_P(I) = \begin{cases} (I \setminus b) \cup a & \text{if } I \text{ is the index of a non-zero Plücker coordinate and } B \text{ is non-empty,} \\ I & \text{otherwise.} \end{cases}$$

Recall that the indices of non-zero Plücker coordinates of a point in the TNN flag variety can be thought of as bases of a flag matroid. It follows that $\Xi_P$ acts by basis exchange on each of the constituent matroids. Also note that, by Corollary 3.18, for a TNN flag $F$, the map $\Xi_P(F)$ depends only on the cell $R_{v,w}^0$ in which $F$ lies. We denote this map by $\Xi_{v,w}$.

We similarly define $\Xi^*_P$ to be the map going the other way. Applied to the index of a non-zero Plücker coordinate $I$, it tries to find the smallest member of $I$ which can be lowered without making the corresponding Plücker coordinate 0 and lowers it maximally.

**Definition 3.24.** Let $P \in \mathbb{R}P^{(1)}_k \times \cdots \times \mathbb{R}P^{(n-1)}_k$ and $I \in \{[n]_k \}$. If $B^* = \{i \mid \exists j \notin I, i > j, P_{(I \cup i) \setminus j} \neq 0 \}$ is non-empty, define $b^* = \min_{i \in I} B$ and $a^* = \min_{j \notin I} \{j \mid P_{(I \cup b^*) \setminus j} \neq 0 \}$. Then,

$$\Xi^*_P(I) = \begin{cases} (I \setminus b^*) \cup a^* & \text{if } I \text{ is the index of a non-zero Plücker coordinate and } B \text{ is non-empty,} \\ I & \text{otherwise.} \end{cases}$$

Again, this can be thought of as a basis exchange. Also, as above, if $P = P(F)$ for a flag $F$ in some cell $R_{v,w}^0$, we can denote the corresponding map by $\Xi_{v,w}$.

We remark that the definitions of $\Xi$ and $\Xi^*$ are closely related to the notions of internal and external activity as defined in [2]. Specifically, note that when we compute $\Xi(I)$, $a$ is the largest externally passive element for the basis $I$ and $b^*$ is an internally passive element of the basis $I$. The values $b$ and $a^*$ can similarly be related to dual notions of internal and external activity where minimums are replaced by maximums.

As we continue through the next few sections, we will translate much of what we say about the complete flag variety into the language of the tropical complete flag variety and the complete flag Dressian. We will emphasize such “translations” by labeling the relevant statements with the superscript “trop”.

**Definition 3.23$^{\text{trop}}$.** Let $P \in \text{TFl}^{(1)}_k \times \cdots \times \text{TFl}^{(n-1)}_k$ and $I \in \{[n]_k \}$ for some $k \in [n]$. If $B = \{i \mid \exists j \notin I, i < j, P_{(I \setminus i) \cup j} \neq \infty \}$ is non-empty, define $b = \max_{i \in I} B$ and $a = \max_{j \notin I} \{j \mid P_{(I \setminus b) \cup j} \neq \infty \}$. Then,

$$\Xi_P(I) = \begin{cases} (I \setminus b) \cup a & \text{if } I \text{ is the index of a non-zero Plücker coordinate and } B \text{ is non-empty,} \\ I & \text{otherwise.} \end{cases}$$

We observe that if $P \in \text{TFl}^0_{v,w} = \text{Im}(\text{Trop} \Phi_{v,w})$, then $\Xi_P = \Xi_{v,w}$, since the non-infinite coordinates of $P$ are exactly the same as the non-zero coordinates of a flag $F \in R_{v,w}^0$.

The extremal non-zero Plücker coordinates will be given as the coordinates indexed by certain $\Xi$ orbits. To properly define them, we will first need a few preliminary ideas related to positroids:

**Definition 3.25.** Let $I = \{i_1 < \cdots < i_k \}$ and $J = \{j_1 < \cdots < j_k \}$ be subsets of $[n]$. Then $I \leq J$ in the Gale order if $i_r \leq j_r$ for every $r \in [k]$. 

16
Lemma 3.26 ([6, Theorem 1.3.1]). Any matroid (and in particular, positroid) has a unique Gale minimal basis and a unique Gale maximal basis.

Note that the Gale minimal and maximal bases referenced in the previous lemma must simply be the lexicographically minimal and maximal bases, respectively. For \( F \in \text{Fl}_n \), we have observed that the indices of fixed size of non-zero Plücker coordinates of \( F \) form the bases of a matroid. We exploit this fact in defining the extremal Plücker coordinates.

**Definition 3.27.** Given a flag \( F \in \text{Fl}_n \) such that \( P_I(F) \geq 0 \) for all \( I \subset [n] \), let \( I_k \) be the Gale minimal index of size \( k \) such that \( P_{I_k}(F) \neq 0 \). The set of indices of the extremal non-zero Plücker coordinates (referred to as extremal indices) of \( P(F) \) consists of all sets lying in the \( \Xi_{P(F)} \) orbit of some \( I_k \) with \( k \in [n-1] \).

Before stating our next translated definition, we note that it has been shown that the support, which is to say the indices of the non-infinite coordinates, of any point in the flag Dressian is a flag matroid [8]. Thus, the following is well-defined:

**Definition 3.27**\textsuperscript{top}. Given a point \( P \in \text{Fl}_{n}^{>0} \), let \( I_k \) be the Gale minimal index of size \( k \) such that \( P_{I_k}(F) \neq \infty \). The set of indices of the extremal non-infinite Plücker coordinates (referred to as extremal indices) of \( P \) consists of all sets lying in the \( \Xi_P \) orbit of some \( I_k \) with \( k \in [n-1] \).

As we remarked after Definition 3.23, the extremal indices of a flag in \( \text{Fl}_{n}^{>0} \) depend only on the cell \( \mathcal{R}_{v,w}^{>0} \) containing that flag. Similarly, all points in \( \text{TrFl}_{v,w}^{>0} \) have the same extremal indices as a flag in \( \mathcal{R}_{v,w}^{>0} \).

Intuitively, \( \Xi_{v,w} \) iteratively increases indices in the Gale order. We will use it as a way to interpolate between the non-zero (or non-infinite, in the tropical case) Plücker coordinates of a flag with Gale-minimal and Gale-maximal indices. In particular, we note at this point that basis exchange guarantees that repeated applications of \( \Xi_{v,w} \) must eventually yield the unique Gale-maximal index of a non-zero (non-infinite) Plücker coordinate of a flag in \( \mathcal{R}_{v,w}^{>0} \) (\( \text{TrFl}_{v,w}^{>0} \)).

We note that one can define a dual notion of extremal non-zero (non-infinite) Plücker coordinates using the \( \Xi^* \) orbit of the Gale maximal indices of non-zero (non-infinite) Plücker coordinates. While these dual extremal indices are in fact different in some cases, every result which we prove would work equally with the dual version of extremal non-zero (non-infinite) Plücker coordinates.

**Example 3.28.** We continue with Example 3.8. Recall that we had

\[
M = \begin{pmatrix}
1 & a_4 & a_1 & 0 \\
0 & 0 & 1 & 0 \\
0 & -1 & 0 & a_2 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]

Let \( P = P(M) \) and \( \Xi = \Xi_P \). The non-zero Plücker coordinate with Gale minimal index of size 1 is \( P_1 = 1 \). Applying \( \Xi \) to \( \{1\} \), we replace 1 with the maximal single element index of a non-zero Plücker coordinate, obtaining \( \Xi(1) = 3 \). Thus, \( P_3 = a_1 \) is also an extremal non-zero Plücker coordinate.

The non-zero Plücker coordinate with Gale minimal index of size 2 is \( P_{13} = 1 \). Applying \( \Xi \) to \( \{1,3\} \) first tries to replace 3 with something bigger. However, that is not possible since \( P_{14} = 0 \). Thus, it tries to replace 1 with something bigger, yielding \( \Xi(13) = 23 \). Thus \( P_{23} = a_4 \) is also an extremal non-zero Plücker coordinate. The index 23 is the Gale maximal index of size 2 of a non-zero Plücker coordinate.

The non-zero Plücker coordinate with Gale minimal index of size 3 is \( P_{123} = 1 \). Applying \( \Xi \), we obtain \( \Xi(123) = 134 \) and \( \Xi(134) = 234 \). Thus \( P_{134} = a_2 \) and \( P_{234} = a_2a_4 \) are both extremal non-zero Plücker coordinates.

**Example 3.29.** For any \( F \in \text{Fl}_n \), any non-zero Plücker coordinate with index of size \( n-1 \) is an extremal non-zero Plücker coordinate. To see this, note that the Gale ordering is a total ordering on subsets of size \( n-1 \) and in this case, \( \Xi_{P(F)} \) simply acts by replacing the index of a non-zero Plücker coordinate by the next largest such index in the Gale order.
4 The Totally Positive Complete Flag Variety and its Tropicalization

In the cell decomposition of $\text{Fl}_{n}^{\geq 0}$, the top cell is $\mathcal{R}_{\text{id},w_{0}}^{\geq 0}$. It consists of those points where all of the Plücker coordinates are positive, as is easy seen from the graph $G_{\text{id},w_{0}}$. In this section, we study this space carefully, along with its tropicalization, the totally positive tropical complete flag variety. In the next section, we will generalize the arguments presented here to address the TNN complete flag variety and the TNN tropical complete flag variety more generally.

We first describe the extremal coordinates of flags $F \in \mathcal{R}_{\text{id},w_{0}}^{> 0}$. We start with an example.

Example 4.1. Let $n = 5$ and $F \in \mathcal{R}_{\text{id},w_{0}}^{> 0}$, so $P_{1}(F) \neq 0$ for all $I \subset [5]$. Then $P_{1} \neq 0$ is the non-zero Plücker coordinate with Gale-minimal index of size 2, and so it is an extremal non-zero Plücker coordinate. Applying $\Xi_{\text{id},w_{0}}$, we increase the 2 in the index maximally. Since all Plücker coordinates are non-zero, we can replace the index 12 by the index 15. Thus, $P_{15}$ is also an extremal non-zero Plücker coordinate. Applying $\Xi_{\text{id},w_{0}}$ again, we increase the 1 in the index maximally to obtain that $P_{15}$ is an extremal non-zero Plücker coordinate as well. Similarly, the extremal indices of size 3 are $\{123, 125, 145, 345\}$.

We can extend this example to a general fact:

Proposition 4.2. The extremal indices of a flag $F \in \mathcal{R}_{\text{id},w_{0}}^{> 0} \subset \text{Fl}_{n}$ are precisely the subsets $\{1, 2, \ldots, m - 1, m, k, k + 1, \ldots, n\}$ for $m < k - 1$.

Proof. Start with the Gale minimal extremal index of size $m + (n - k) + 1$, which is $\{1, 2, \ldots, m + n - k + 1\}$. Now, bearing in mind that all Plücker coordinates are non-zero in this case, apply $\Xi_{\text{id},w_{0}}$ to this index $n - k + 1$ times to obtain the desired result. \hfill $\square$

Proposition 4.2$^{\text{top}}$. The extremal indices of a point $P \in \text{TrFl}_{\text{id},w_{0}}^{> 0}$ are precisely the subsets $\{1, 2, \ldots, m - 1, m, k, k + 1, \ldots, n\}$ for $m < k$.

Proof. The extremal indices of a point in $\text{TrFl}_{\text{id},w_{0}}^{> 0}$ are identical to those of a flag in $\mathcal{R}_{\text{id},w_{0}}^{> 0}$. \hfill $\square$

Definition 4.3. We call a collection of paths in $G_{v,w}$ diagonal if none of the paths it contains use any vertical edges.

Definition 4.4. Consider a source set $A$ in $G_{v,w}$. Label the sources in $A$ from top to bottom as $a_{1}', \ldots, a_{m}'$. Construct a path collection as follows: Start with the path originating from $a_{1}'$ which takes every left turn it can. After having added a path originating at $a_{i}'$ to the path collection, we add in the path originating at $a_{i+1}'$ which takes every left turn it can without intersecting any of the paths already in the collection. The path collection obtained after adding in the path originating at $a_{m}'$ will be called the greedy left path collection originating at $A$. Less formally, this is a path collection with source set $A$ where each path takes every left turn available to it without intersecting a path that originates above it.

From our explicit description of the extremal indices, we can immediately make a few observations.

Proposition 4.5. Let $I$ be an extremal index of a flag in $\mathcal{R}_{\text{id},w_{0}}^{> 0}$ (or, equivalently, of a point in $\text{TrFl}_{\text{id},w_{0}}^{> 0}$).

1. Then there is a unique path collection originating from $|I|$ and terminating at $I$ in the graph $G_{\text{id},w_{0}}$.

2. The unique path collection from $|I|$ to $I$ in $G_{\text{id},w_{0}}$ consists of the union of a diagonal path collection and a left greedy path collection.

Proof. Note that the graph $G_{\text{id},w_{0}}$ has all possible vertical edges. By Proposition 4.2, we are interested in path collections with origin set $\{1', 2', \ldots, (m + n - k + 1)\}$ and sink set $\{1, 2, \ldots, m - 1, m, k, k + 1, \ldots, n\}$ for $m < k$. The only path which terminates at 1 is the diagonal path originating at $1'$. Once we have added this path to our path collection, the only path which avoids it and terminates at 2 is the diagonal path originating at $2'$. Continuing in this way, we obtain a unique path collection consisting of diagonal paths which originates at $\{1', \ldots, m'\}$ and terminates at $\{1, \ldots, m\}$. Now, given that there must be a path terminating at $n$ and that the topmost path in our collection must originate at $(m + n - k + 1)'$, we must
include the unique path between these two vertices. This happens to be a left greedy path. Similarly, once we have added this path to our collection, the only way to have a path originating at \((m + n - k)'\) and a path terminating at \(n - 1\) is to have the unique, left greedy path connecting them. Continuing in this way we obtain the left greedy path collection which originates at \(\{(m + 1)', \ldots, (m + n - k + 1)\}\) and terminates at \(\{k, \ldots, n\}\). Putting these two collections together gives the desired unique path collection. See Fig. 8 for an example of this construction.

Figure 8: A path collection, indicated by dotted edges, in \(G_{id,w_0}\) for \(n = 5\) whose sink set is an extremal index. Note that this is the union of two diagonal paths and one left greedy path.

Let \(e_{id,w_0}(S)\) denote the set \(\Xi_{id,w_0}^k(S)\), where \(k\) is chosen to be minimal such that \(e_{id,w_0}(S)\) is extremal. One can think of \(e_{id,w_0}(S) \setminus S\) as providing a notion of how far an index set \(S\) is from being an extremal index for \(F\). In that spirit, the proof that all Plücker coordinates are determined by the extremal non-zero Plücker coordinates will involve an induction on \(|e_{id,w_0}(S) \setminus S|\).

**Corollary 4.6.** For each extremal index \(I\) for \(R_{id,w_0}^\geq 0\), the Plücker coordinate \(P_I\) is a monomial in the weights appearing on the vertical edges of \(G_{id,w_0}\).

**Proof.** This is immediate from the existence of a unique path collection with sink set \(I\). □

**Theorem 4.7.** For any flag \(F\) in \(R_{id,w_0}^\geq 0\), the extremal non-zero Plücker coordinates of \(F\) uniquely determine the other non-zero Plücker coordinates of \(F\) by three-term incidence Plücker relations.

**Proof.** Let \(P = P(F)\). Our goal is to determine \(P_S\) for any \(S \subset [n]\) using just the extremal non-zero Plücker coordinates and the three term incidence Plücker relations. This proof will proceed by triple induction. First, we will work by reverse induction on the size of the index \(S\). For subsets of a fixed size, we will use induction on \(t_{id,w_0}(S) := |e_{id,w_0}(S) \setminus S|\). For subsets with equal values of \(t_{id,w_0}(S)\), we will work with induction on the Gale order. All non-zero\(^2\) Plücker coordinates of size \(n - 1\) are extremal, as are non-zero Plücker coordinates with \(t_{id,w_0}(S) = 0\). These serve as base cases.

Let \(S\) be a subset of \([n]\) of size \(k\) which is the index of a non-zero, non-extremal Plücker coordinate. Assume that \(P_I\) is either 0 or uniquely determined by the extremal non-zero Plücker coordinates via three-term incidence Plücker relations for all \(I\) of size larger than \(k\), or of size \(k\) but with \(t_{id,w_0}(I) < t_{id,w_0}(S)\), or of size \(k\) with \(t_{id,w_0}(I) = t_{id,w_0}(S)\) but \(I < S\) in the Gale order. We need to show that \(P_S\) is determined by a three-term incidence Plücker relation from Plücker coordinates which are already known. Since \(S\) is not extremal, \(t_{id,w_0}(S) > 0\). Let \(b\) be the minimal element of \(S \setminus e_{id,w_0}(S)\). Since \(b \notin e_{id,w_0}(S)\), after some number \(r\) many applications of \(\Xi_{id,w_0}\), \(b\) will no longer be in \(\Xi_{id,w_0}(S)\). Pick \(r\) minimal. Using matroid basis exchange between \(S\) and \(\Xi_{id,w_0}(S)\), we have \(S' = (S \setminus b) \cup c\) is the index of a non-zero Plücker coordinate for some \(c \in \Xi_{id,w_0}(S) \setminus S\). Clearly, we must have \(c > b\).

\(^2\)The adjective non-zero is redundant here, but we will wish to make use of this proof in more general circumstances later.
Since \( b \notin e_{id,w_0}(S) \), there exists \( a < b \) with \( a \notin S \). Then, we have a three-term incidence Plücker relation that says \( P_S P_{S \setminus \{b\} \cup ac} = P_{S \setminus \{b\} \cup c} + P_{S \setminus \{b\} \cup a} P_{S \cup c} \). Since \( F \in \mathcal{R}_{id,w_0}^> \), all the Plücker coordinates are positive. Note that \( (S \setminus b) \cup ac, S \cup a \) and \( S \cup c \) are all larger than \( S \) and so the corresponding coordinates are determined by induction. Since \( b \notin e_{id,w_0}(S) \), we have that \( t_{id,w_0}((S \setminus b) \cup a) \leq t_{id,w_0}(S) \) (the inequality being strict if and only if \( a \in e_{id,w_0}(S \setminus b) \cup a) \). Also, \( (S \setminus b) \cup a < S \) in the Gale order, so the coordinate with index \( (S \setminus b) \cup a \) is also already determined by induction. Finally, since we chose \( b \notin e_{id,w_0}(S) \) and \( c \in e_{id,w_0}(S) \setminus S \), it is easy to see that \( t_{id,w_0}((S \setminus b) \cup c) < t_{id,w_0}(S) \). Thus, we are done by induction.

**Corollary 4.8.** For any flag \( F \) in \( \mathcal{R}_{id,w_0}^> \), the extremal Plücker coordinates serve as a positivity test. Explicitly, if the extremal non-zero Plücker coordinates of \( F \) are positive, then so are all the other Plücker coordinates of \( F \).

**Proof.** One must simply observe that in the previous proof, \( P_S \) always appears on the monomial side of a three-term incidence Plücker relation, when all terms are written with positive coefficients.

**Theorem 4.7** \( \text{trop} \). For any point \( P \) in \( \text{TrFl}_{id,w_0}^> \), the extremal non-infinite Plücker coordinates of \( P \) uniquely determine the other non-infinite Plücker coordinates of \( P \) by three-term tropical incidence Plücker relations.

**Proof.** The tropical three-term incidence Plücker relation corresponding to the relation \( P_S P_{S \setminus \{b\} \cup ac} = P_{S \setminus \{b\} \cup c} + P_{S \setminus \{b\} \cup a} P_{S \cup c} \) is \( P_S + P_{S \setminus \{b\} \cup c} = \min\{P_{S \setminus \{b\} \cup c}, P_{S \setminus \{b\} \cup a} P_{S \cup c}\} \). Observe that a coordinate appearing on the left hand side of this equation is uniquely determined by the other five coordinates appearing in the equation. Since \( P_S \) always appears on the left hand side of equations in the proof of Theorem 4.7, the proof carries through to the tropical case.

**Proposition 4.9.** The map \( \Psi_{id,w_0} \), which is inverse to \( \Phi_{id,w_0} \), can be expressed as Laurent monomials in the extremal non-zero Plücker coordinates of a flag in \( \mathcal{R}_{id,w_0}^> \).

**Proof.** Fix an order \( \prec \) such that \( I \prec J \) means \( |I| > |J| \) or \( |I| = |J| \) and \( I \) is Gale less than \( J \). Note that since all extremal indices are Gale comparable, this is a total order on extremal indices. We claim that if one goes through the extremal non-zero Plücker coordinates in \( \prec \) order, then each corresponding path collection uses at most one weight which had not been used previously\(^3\). For the moment, assume the claim. Then for each extremal index \( I \), we obtain an equation of the form \( P_I = \prod \text{weights} \). Going through the extremal indices in \( \prec \) order, each such equation introduces at most one new weight. Thus, we can solve each equation for the new weight to obtain an expression for it as a Laurent monomial in \( P_I \) and a bunch of previously determined weights. We are then done by induction.

We now prove the claim. Let \( G = G_{id,w_0} \). Start with the diagonal path collection \( C \) with source set \( [n-1] \) in \( G \). This corresponds to the extremal non-zero Plücker coordinate \( P_{[n-1]} \), which has value 1. Consider \( \Xi_{id,w_0}([n-1]) \). The unique corresponding path collection will involve a single weighted vertical edge. Continue applying \( \Xi_{id,w_0} \) to move through the path collections whose sink sets are extremal indices of size \( n-1 \). Each new path collection involves at most one new vertical edge.

Suppose \( I \) is an extremal index of size \( t < n-1 \). Further, suppose that the unique path collection \( C_I \) in \( G \) with sink set \( I \) contains a path originating on vertex \( v' \) and using an edge \( e \) which had not previously been used by \( C_I \) for any \( J \prec I \). Note that for any \( J \prec I \), \( C_J \) must have a path originating at \( v' \). Moreover, by Proposition 4.5, the non-diagonal part of any such \( C_J \) is left greedy. Fix \( J \) to be the extremal index closest to \( I \) in \( \prec \) order such that the path originating on \( v' \) in \( C_J \) is left greedy. In \( C_J \), the path originating on \( v' \) must have been blocked from turning onto edge \( e \) by some other path \( p' \). After using the edge \( e \), the path originating on \( v' \) in \( C_I \) should continue in the left greedy way, which will coincide with the path \( p' \). It follows that this path does not use any additional new vertical edges.

Let \( \pi_{id,w_0} \) be the map which projects a collection of Plücker coordinates of a flag in \( \mathcal{R}_{id,w_0}^> \) to just the extremal non-zero Plücker coordinates.

**Proposition 4.9** \( \text{trop} \). The map \( \text{Trop} \Psi_{id,w_0} \) is inverse to \( \text{Trop} \left( \pi_{id,w_0} \circ \Phi_{id,w_0} \right) \).

\(^3\) In this case, precisely one. We will want to use this proof in more general circumstances later.
4.6. Let $F \in \mathcal{R}_{id,w_0}^{>0}$. By Corollary 4.6, the map $\Phi_{id,w_0}$ expresses the extremal non-zero Plücker coordinates of $F$ as monomials in the weights $a_i$ which one puts on the graph $G_{id,w_0}$. By Proposition 4.9, the map $\Psi_{id,w_0}$ expresses the weights $a_i$ as Laurent monomials in the extremal non-zero Plücker coordinates of $F$. We know that $\pi_{id,w_0} \circ \Phi_{id,w_0}$ and $\Psi_{id,w_0}$ are inverses. Since tropicalizing converts products and quotients to sums and differences, respectively, we will also have that $\text{Trop} \ (\pi_{id,w_0} \circ \Phi_{id,w_0})$ are inverses.

Note that, for a flag $F \in \mathcal{R}_{id,w_0}^{>0}$, the extremal non-zero Plücker coordinates must be algebraically independent. To see this, observe that the dimension of $\mathcal{R}_{id,w_0}^{>0}$ is $\ell(w_0) - \ell(id) = \binom{n}{2}$. Also, we have $\binom{n}{2}$ extremal non-zero Plücker coordinates. We may ignore the $n$ extremal non-zero Plücker coordinates $P_{[k]}$, since we fix these to be 1 in order to specify the projective scaling of our coordinates. Arbitrarily specifying the remaining $\binom{n}{2}$ many extremal non-zero Plücker coordinates uniquely determines a point in the $\binom{n}{2}$ dimensional space $\mathcal{R}_{v,w}^{>0}$ and so they must be algebraically independent.

**Theorem 4.10.** The totally positive complete flag variety $\mathcal{R}_{id,w_0}^{>0}$ equals the set $\{F \in \mathcal{F}_n| P_I(F) > 0 \forall I \subseteq [n]\}$.

**Proof.** We already established in Lemma 3.9 that for any $F \in \mathcal{R}_{id,w_0}^{>0}$, we have $P_I(F) > 0$ for any $I \subseteq [n]$. We are left to prove the reverse direction.

Let $F$ be any flag in $\mathcal{F}_n$ such that $P_I(F) > 0$ for all $I \subseteq [n]$. We prove that $F \in \mathcal{R}_{id,w_0}^{>0}$. To do so, we show that there exist some positive real weights $a$ such that $\Phi_{id,w_0}(a)$ yields the Plücker coordinates of $F$. Apply the map $\Psi_{v,w}$ to the extremal non-zero Plücker coordinates of $F$ to get a collection of weights $a$. Let $\Phi_{id,w_0}(a) = Q \in \mathbb{R}P(\binom{n}{2})^{-1} \times \cdots \times \mathbb{R}P(\binom{n-i}{i})^{-1}$. By construction, $P(F)$ and $Q$ agree on their extremal non-zero Plücker coordinates. By Theorem 4.7, the three-term incidence Plücker relations determine all the other Plücker coordinates in terms of the extremal non-zero Plücker coordinates and so $P(F)$ and $Q$ agree on all Plücker coordinates. Thus, $F$ lies in $\mathcal{R}_{id,w_0}^{>0}$. □

Before tropicalizing this result, we present an example.

**Example 4.11.** For $n = 3$, the graph $G_{id,w_0}$ is given in Fig. 9. The extremal indices in $\mathcal{R}_{id,w_0}^{>0}$ are $\{1\}, \{3\}, \{1, 2\}, \{1, 3\}$ and $\{2, 3\}$. Let $P = P(F)$ for $F$ a flag in $\mathcal{R}_{id,w_0}^{>0}$.

![Figure 9: The graph $G_{id,w_0}$ for $n = 3$.](image)

First, we will determine the weights $a_i$ in terms of the extremal Plücker coordinates. We start with the Gale minimal extremal index of size 2, which is $\{1, 2\}$. The unique path collection with source set $\{2\}'$ and sink set $\{1, 2\}$ has weight 1. The Gale-next extremal index of size 2 is $\{1, 3\}$. The weight of the path collection with source set $\{2\}'$ and sink set $\{1, 3\}$ is just $a_2$. Thus, we have $a_2 = P_{13}$. The Gale-next extremal index of size 2 is $\{2, 3\}$. The unique path collection with source set $\{2\}'$ and sink set $\{2, 3\}$ has weight $a_2a_3$. Thus, $a_3 = \frac{P_{23}}{P_{13}}$. Next, we move on to the extremal indices of size 1. The Gale minimal extremal index of size 1 is $\{1\}$. The unique path from $\{1\}'$ to $\{1\}$ has weight 1. The Gale-next extremal index of size 1 is $\{3\}$. The unique path from $\{1\}'$ to $\{3\}$ has weight $a_1a_2$, so $a_1 = \frac{P_{13}}{P_{12}}$. This determines all the weights in $G_{id,w_0}$ as Laurent monomials in the extremal non-zero Plücker coordinates. We now check that if we apply the LGV construction, the Plücker coordinates $Q$ which we obtain are the same as the Plücker coordinates $P$ we started with. The only interesting coordinate to check is the non-extremal Plücker coordinate $Q_2$. Indeed, $Q_2 = a_1 + a_3 = \frac{P_{13}}{P_{12}} + \frac{P_{23}}{P_{22}} = P_2$, where the last equality follows from the three-term incidence Plücker relation.
Theorem 4.10\textsuperscript{trop}. The totally positive tropical complete flag variety $\text{TrFl}_{n}^{>0}$ equals the totally positive complete flag Dressian $\text{FlDr}_{n}^{>0}$.

Proof. It is clear by definition that $\text{TrFl}_{n}^{>0} \subseteq \text{FlDr}_{n}^{>0}$. It is also clear that $\text{TrFl}_{id,w_{0}}^{>0} \subseteq \text{TrFl}_{n}^{>0}$. We are left to show that $\text{FlDr}_{n}^{>0} \subseteq \text{TrFl}_{id,w_{0}}^{>0}$.

Let $P \in \text{FlDr}_{n}^{>0}$. We must show that there exists some real weights $a$ such that $\text{Trop} \Phi_{id,w_{0}}(a) = P$. Apply the map $\text{Trop} \Psi_{v,w}$ to the extremal non-infinite Plücker coordinates of $P$ to get a collection of weights $a$. Consider $Q = \text{Trop} \Phi_{id,w_{0}}(a) \in \mathbb{T}^{\mathbb{P}(1)}_{1} \times \cdots \times \mathbb{T}^{\mathbb{P}(n-1)}_{n-1}$. By Proposition 4.9\textsuperscript{trop}, $P$ and $Q$ agree on their extremal non-infinite Plücker coordinates. By Theorem 4.7\textsuperscript{trop}, the three-term incidence Plücker relations determine all the other Plücker coordinates in terms of the extremal non-infinite Plücker coordinates and so $P$ and $Q$ agree on all Plücker coordinates. Thus, $F$ lies in $\text{TrFl}_{id,w_{0}}^{>0}$. \hfill \Box

We now take a moment to comment on similarities and differences between the extremal non-zero Plücker coordinates and the standard chamber minors defined and studied in [21]. Marsh and Rietsch express the parameters in their parameterization of the TNN complete flag variety as Laurent monomials in the standard chamber minors, much as we do for the extremal non-zero Plücker coordinates. They also express the standard chamber minors as monomials in the parameters. However, in general, the standard chamber minors differ from our extremal non-zero Plücker coordinates. These differences are best illustrated by an example.

Example 4.12. We consider $\text{Fl}_{4}^{>0}$. We provide the weighted digraph $G_{id,w_{0}}(a)$ in Fig. 10.

![Graph](image-url)

Figure 10: The graph $G_{id,w_{0}}(a)$ for $n = 4$.

Reading off the extremal non-zero Plücker coordinates and ignoring those which are simply 1, we obtain $\{a_{1}a_{2}a_{3}, a_{2}a_{3}, a_{3}a_{4}a_{5}, a_{3}a_{5}, a_{3}a_{5}a_{6}\}$. Meanwhile, in the notation of [21], one can calculate the standard chamber minor $\Delta_{v(k-1)w_{k}}(z)$ for $k = 5$. In this case, $v(4) = id$, $w(4) = s_{1}s_{2}s_{3}s_{1}$ and $i_{5} = 2$. Inputting this data and working through the computation, we obtain the result $a_{2}a_{3}a_{4}$. Note that this is not an extremal non-zero Plücker coordinate and the simplest way to express it as a Laurent monomial in the extremal non-zero Plücker coordinates requires three terms. Moreover, there is no Plücker coordinate (not even a non-extremal one) which equals $a_{2}a_{3}a_{4}$. This highlights, even in the relatively simple setting of the top cell, that there are subtle but potentially important differences between the extremal non-zero Plücker coordinates and the standard chamber minors of Marsh and Rietsch.

5 The Totally nonnegative Complete Flag Variety and its Tropicalization

5.1 Graphical Description of Extremal Plücker Indices

In this section, we prove results analogous to those in the previous section but in the more general setting of $\mathcal{R}_{v,w}^{>0}$ for any $v \leq w \in S_{n}$. We begin by understanding the extremal Plücker coordinates of a point in an
arbitrary cell $R_{v,w}^{>0}$ of the TNN complete flag variety. This is where we will make real use of the graphs $G_{v,w}$ introduced earlier. At a basic level, what changes is that we can now have coordinates which are zero. Thus, it will be helpful to introduce the following terminology:

**Definition 5.1.** For a point $P \in \mathbb{RP}((1)^{-1} \times \cdots \times \mathbb{RP}((n-1)^{-1}$, the support of $P$, denoted $\text{Supp}(P)$, will refer to the set of indices of the non-zero coordinates of $P$.

**Definition 5.1** trop. For a point $P \in \mathbb{TP}((1)^{-1} \times \cdots \times \mathbb{TP}((n-1)^{-1}$ the support of $P$, denoted $\text{Supp}(P)$, will refer to the set of indices of the non-zero coordinates of $P$.

Recall that by tropicizing $\Phi_{v,w}$, we obtain the parameterization of a cell $\text{TrFl}_{v,w}^{>0}$ of $\text{TrFl}_{v,w}^{>0}$. Note that the points in this tropical cell have the same support as those flags in $R_{v,w}^{>0}$. Since $\text{TrFl}_{v,w}^{>0} \subset \text{FlDr}_{v,w}^{>0}$, there exist points in $\text{FlDr}_{v,w}^{>0}$ with that same support as well. We denote by $\text{FlDr}_{v,w}^{>0}$ the set $\{ P \in \text{FlDr}_{v,w}^{>0} \mid \text{Supp}(P) = \text{Supp}(F) \}$ for any $F \in R_{v,w}^{>0}$.

**Lemma 5.2** ([15, Lemma 3.11]). Let $F \in R_{v,w}^{>0}$ and $K = \{ k_1, \ldots, k_m \}$. Then, $P_K(F) \neq 0$ if and only if $K$ consists of $\{ u(1), \ldots, u(m) \}$ for some $v^{-1} \leq u \leq w^{-1}$.

Note that the above lemma differs from the statement cited by the inclusion of inverses on the permutations $v$ and $w$. This accounts for the different convention we used in defining $R_{v,w}^{>0}$.

Let $w \in S_n$. Note that adding an inversion in $w$ can not decrease the set $\{ w^{-1}(1), w^{-1}(2), \ldots, w^{-1}(t) \}$ in Gale order for any $t \in [n]$. Thus, it is clear that in $R_{v,w}^{>0}$, the Gale minimal indices of non-zero Plücker coordinates are $\{ v^{-1}(1), \ldots, v^{-1}(t) \}$ for $t \in [n]$ while the Gale maximal indices of non-zero Plücker coordinates, which one obtains after repeated application of $\Xi_{v,w}$, are $\{ w^{-1}(1), \ldots, w^{-1}(t) \}$ for $t \in [n]$.

Our next goal is to understand a little bit better how the extremal indexes relate to one another.

**Lemma 5.3.** For $F \in \text{Fl}_{v,w}$, let $P = P(F)$ and suppose $P_I \geq 0$ for all $I \subset [n]$. Let $S$ be any subset of $[n]$ of size at most $n - 4$. Let $a < b < c < d$ with $a, b, c, d \notin S$. Then, both $P_{S \cup ab} \neq 0$ and $P_{S \cup ac} \neq 0$ if and only if either both $P_{S \cup ab} \neq 0$ and $P_{S \cup bd} \neq 0$, or both $P_{S \cup ad} \neq 0$ and $P_{S \cup bc} \neq 0$.

**Proof.** Since $P_I \geq 0$ for all $I \subset [n]$, the lemma follows from the three term Plücker relation Eq. (1) with $I = S \cup ab$ and $J = S \cup bd$. \qed

**Lemma 5.4.** Let $F \in \text{Fl}_{v,w}$ such that $P_I(F) \geq 0$ for all $I \subset [n]$ and let $\Xi = \Xi_{P(F)}$. Let $I_k$ denote the Gale minimal index of a non-zero Plücker coordinate such that $|I_k| = k$. Let $s$ be such that $\Xi^{-1}(I_k) = \Xi^r(I_k)$. For $1 \leq r \leq s$, define $\alpha_r$ and $\beta_r$ by $\Xi^{-1}(I_k) = (\Xi^{-1}(I_k) \setminus \alpha_r) \cup \beta_r$. If $1 \leq r < s$, then $\alpha_r < \alpha_s$.

**Proof.** It suffices to show that $\alpha_{r+1} < \alpha_r$ for $1 \leq r < s$. Note that $r > s$ simply ensures that $\Xi^{r+1}(I_k) \neq \Xi^r(I_k)$. By definition, we have $\beta_r > \alpha_r$ and $\beta_{r+1} > \alpha_{r+1}$. Also, $\alpha_{r+1} \neq \beta_r$ by the definition of $\beta_r$. Let $S = \Xi^{-1}(I_k)$, so that $S$ and $\Xi^2(S) = (S \setminus (\alpha_r \cup \alpha_{r+1}) \cup \beta_r \cup \beta_{r+1})$ are both indices of non-zero Plücker coordinates. Suppose $\alpha_r < \alpha_{r+1}$. Then we are in one of the following three cases:

Case 1: If $\alpha_r < \alpha_{r+1} < \beta_r < \beta_{r+1}$, then Lemma 5.3 applied to $S \setminus (\alpha_r \cup \alpha_{r+1})$ tells us that $(S \setminus (\alpha_r \cup \alpha_{r+1}) \cup \beta_r \cup \beta_{r+1}$ is the index of a non-zero Plücker coordinate as well. However, this contradicts the definition of $\beta_r$.

Case 2: If $\alpha_r < \alpha_{r+1} < \beta_{r+1} < \beta_r$, then Lemma 5.3 applied to $S \setminus (\alpha_r \cup \alpha_{r+1})$ tells us that $(S \setminus (\alpha_r \cup \alpha_{r+1}) \cup \beta_r$ is the index of a non-zero Plücker coordinate as well. This contradicts the definition of $\alpha_r$.

Case 3: If $\alpha_r < \beta_r < \alpha_{r+1} < \beta_{r+1}$ then Lemma 5.3 applied to $S \setminus (\alpha_r \cup \alpha_{r+1})$ tells us that either $(S \setminus (\alpha_r \cup \alpha_{r+1}) \cup \beta_{r+1}$ or $(S \setminus (\alpha_r \cup \alpha_{r+1}) \cup \beta_r \cup \beta_{r+1}$ must be the index of a non-zero Plücker coordinate. These both lead to contradictions, as in cases 1 and 2, respectively. \qed

**Lemma 5.5.** Let $F \in R_{v,w}^{>0}$. Let $I$ be an extremal index with $|I| = k$ and $\Xi(I) = (I \setminus a) \cup b$. Then $b \in \{ w^{-1}(1), \ldots, w^{-1}(k) \}$.

**Proof.** As we noted after Definition 3.27 trop, repeated applications of $\Xi_{v,w}$ eventually yield the Gale maximal index of a non-zero Plücker coordinate. The result then follows immediately from Lemma 5.2 and Lemma 5.4. \qed


We now present a graphical description of extremal coordinates, which will be useful through the rest of this paper. This offers a more combinatorial way to think about the extremal indices of a TNN flag. The key fact will be that for any \( v \leq w \) and any extremal index \( I \) of \( R^{>0}_{v,w} \), there is a unique path collection in \( G_{v,w} \) with source set \( \{1',2',\ldots,[I]\} \) and sink set \( I \). Before tackling the general case, we look at the Gale minimal Plücker coordinates. Fix some \( v \leq w \) and let \( I_k \) be the unique Gale minimal index of size \( k \) of a non-zero Plücker coordinate for a flag in \( R^{>0}_{v,w} \). The next two results show that in the appropriate \( G_{v,w} \) there is a unique path collection, consisting of diagonal paths, with sink set \( I_k \).

**Lemma 5.6.** For any \( v \leq w \), there are no vertical arrows in \( G_{v,w} \) directed downwards.

**Proof.** Let \( w \) be the positive distinguished subexpression for \( w \) in \( w_0 \) and let \( v \) be the positive distinguished subexpression for \( v \) in \( w \). Recall that strand \( k \) is the strand which is \( k \)th from the bottom. To flip an edge \( e \), we must have it directed from strand \( k \) to strand \( k+1 \) before being acted on by an \( s_k \) in \( v \).

One possibility is to have such an edge \( e \) be added by a transposition \( s^{(1)} = s_k \) in \( w \) but not \( v \). So long as we keep its origin and endpoint on strands \( k \) and \( k+1 \), respectively, we can flip \( e \) using an \( s^{(0)} = s_k \) in \( v \). To accomplish this, we would need to have no \( s_{k+1} \) or \( s_{k-1} \) in \( v \) intervening between \( s^{(1)} \) and \( s^{(0)} \). However, this would contradict the positive distinguishedness of \( v \).

Thus, to flip \( e \), we would need to move it up or down first. To do so, we must use some transpositions \( s_i \) which are in \( v \). To flip an edge \( e \) using \( s^{(0)} = s_k \) in \( v \), we must wind up with \( e \) directed from strand \( k \) to strand \( k+1 \). Observe that the transposition \( s^{(1)} \) in \( v \) which causes \( e \) to be directed from strand \( k \) to strand \( k+1 \) must be either an \( s_k \) or an \( s_{k+1} \). In either case, we may assume there is no intervening \( s_{k+1} \) or \( s_{k-1} \) in \( v \) between \( s^{(1)} \) and \( s^{(0)} \) (this must be the case for some choice of \( s^{(1)} \), since otherwise \( s_k \) would not actually flip the edge). If \( s^{(1)} = s_k \), this would mean \( v \) is not reduced. So, \( s^{(1)} = s_{k+1} \). Note that this means \( s^{(1)} \) swapped the endpoint of \( e \) from strand \( k+2 \) to strand \( k+1 \).

Thus, at some point we had \( e \) terminating on strand \( k+2 \). To achieve this, there must have been either an \( s_{k+1} \) in \( w \) but not \( v \), or an \( s_{k+1} \) in \( v \), or an \( s_{k+2} \) in \( v \). Call this transposition \( s^{(2)} \). In any event, there is an appropriate choice of \( s^{(2)} \) such that there is no \( s_k \) or \( s_{k+2} \) in \( v \) between \( s^{(1)} \) and \( s^{(2)} \). The first two possibilities violate positive distinguishedness and reducedness of \( v \) in \( w \), respectively, so we must have that \( s^{(2)} = s_{k+2} \) is a transposition in \( v \). Continuing this way, we eventually reach \( s^{(i)} = s_{k+i} \) for \( i = n-k \), which yields a contradiction.

**Corollary 5.7.** For \( v \leq w \), there exists a unique path collection in \( G_{v,w} \) with sources \( [k] \)' whose sink set is the lexicographically minimal attainable sink set, \( I_k \). Moreover, this path collection is diagonal with source set \( [k] \)'.

**Proof.** Since there are no vertical arrows pointing down, it is straightforward to see that the unique path collection with lexicographically minimal sink set is the path collection consisting of diagonal paths.

Recall the greedy left path collections of Definition 4.4. We now introduce a related notion. These concepts are in general different, although we will see that they coincide in cases of interest.

**Definition 5.8.** Consider a source set \( A = \{a_i\}_{i=1}^m \) ordered from top to bottom in \( G_{v,w} \). Choose the leftmost possible sink \( b_1 \) such that there exists a path from \( a_1 \) to \( b_1 \). Having determined \( b_i \) for \( i \leq j \), choose the leftmost possible sink \( b_{j+1} \) subject to the constraint that there is a non-intersecting path collection from \( \{a_i\}_{i=1}^{j+1} \) to \( \{b_i\}_{i=1}^{j+1} \). Then, any path collection from \( A = \{a_i\}_{i=1}^m \) to \( \{b_i\}_{i=1}^m \) is called left extreme. See Fig. 11 for an illustration.
Figure 11: The dashed paths form a left extreme path collection with source set \( \{a_1' = 5', a_2' = 3', a_3' = 1'\} \), and sink set \( \{b_1 = 5, b_2 = 4, b_3 = 3\} \). Note that this is not a left greedy path collection.

**Definition 5.9.** Let \( v \leq w \in S_n \). Fix \( 0 \leq i \leq k < n \) and let \( A = [k]' \). Consider the union of a left extreme path collection with source set the topmost \( i \) vertices of \( A \) and the diagonal path collection with source set the bottom-most \( k - i \) vertices of \( A \) in the graph \( G_{v,w} \). Such a path collection will be called a **graph extremal path collection** and its sink set will be called a **graph extremal index**.

It is clear that the above definition is formulated precisely such that the following holds:

**Proposition 5.10.** Let \( v \leq w \in S_n \). The extremal indices of a flag in \( R_{v,w}' \) coincide precisely with graph extremal indices of \( G_{v,w} \).

Let \( I \) be an extremal index of a flag in some \( R_{v,w}' \). As in Proposition 4.5, we now want to show that there exists a unique path collection with sources \( |I|' = \{1', \ldots, I'\} \) and sinks \( I \). Let \( S \) be a graph extremal index of \( G_{v,w} \) such that

1. In the unique path collection \( C \) from \( \{1', \ldots, |S|'\} \) to \( S \), the path originating on strand \( r \) terminates at vertex \( r \) and the path originating on strand \( r + 1 \) terminates at vertex \( r + 1 \) (or stated otherwise, both are diagonal paths).

2. For some \( r + 1 \leq b < d \), we have \( \Xi_{v,w}(S) = (S \setminus r + 1) \cup d \) and \( \Xi_{v,w}^2(S) = (\Xi_{v,w}(S) \setminus r) \cup b \).

3. In the unique path collection \( C' \) from \( \{1', \ldots, |S|'\} \) to \( \Xi_{v,w}^2(S) \), the path originating on strand \( r \) terminates at vertex \( b \) and the path originating on strand \( r + 1 \) terminates at vertex \( d \).

Then, there is no path originating on strand \( r \) and terminating at \( d \) which avoids \( C(r + 1) \). Moreover, any path originating on strand \( r \) and terminating at \( d \) which avoids \( C(r + 2) \) must pass through strand \( r + 1 \). All such paths leave strand \( r + 1 \) weakly to the right of column \( c \) and then terminate identically to \( p_{r+1}' \).
Figure 12: An illustration of the set up of Lemma 5.11. Here, $C$ is indicated with dashed lines, $S = \{1, 2, 5\}$ and $r = 1$. Applying $\Xi_{v,w}$ once, we replace the end of the path from $2'$ to 2 by the densely dotted path terminating at 4. Applying $\Xi_{v,w}$ a second time, we replace the end of the path from $1'$ to 1 by the loosely dotted path terminating at 3. Accordingly, $b = 3$ and $d = 4$.

Proof. The set up of this lemma is illustrated in Fig. 12.

We first consider the case where $r + 1 < b$. Consider the Plücker relation $P_{(S \setminus r + 1) \cup d} P_{(S \setminus r) \cup b}$. We can express this as a polynomial equation in the weights $a$ of the vertical edges of $G_{v,w}$. Each monomial is a product of the weights obtained form the vertical edges appearing in some particular pair of path collection. Thus, this equation tells us there is a weight-preserving bijection $\phi$ between pairs of path collections appearing on the left and those appearing on the right. Let $\langle x, y \rangle$ denote the set of all non-intersecting pairs of paths such that one originates on strand $r$ and terminates at $x$, the second originates on strand $r + 1$ and terminates at $y$, and they both avoid $C(r + 2)$. This can be identified with a path collection whose sink set is $(S \setminus r(r + 1)) \cup xy$, where paths originating above strand $r + 1$ or below strand $r$ are the same as the paths originating on those strands in $C$. We will implicitly make this identification going forward. Suppose we apply $\phi$ to a pair of path collections of the form $(P_1, P_2) \in \langle x_1, y_1 \rangle \times \langle x_2, y_2 \rangle$. Then, since $C(r + 2)$ is a subcollection of both $P_1$ and $P_2$, all the vertical edges contained in $C(r + 2)$ must be used in each of the two path collections in the image. Is is straightforward to check that if a path collection originating from $\langle l \rangle'$ uses the vertical edges from $C(r + 2)$, it must in fact contain $C(r + 2)$. Thus, $\phi$ restricts to a bijection between those pairs of path collections which lie in any of the $\langle (x_1, y_1), (x_2, y_2) \rangle$.

In Table 1, we break down the terms appearing in our Plücker relation. Note that, for example, there is no $(b, r)$ in the first row since the path originating on strand $r + 1$ can never terminate at $r$. There is similarly no $(d, b)$ in the fourth row since by assumption, the unique path collection with that sink set is of the form $(b, d)$.

| Plücker Coordinate | Path Collections |
|--------------------|------------------|
| $P_{(S \setminus r + 1) \cup b}$ | $\langle r, b \rangle$ |
| $P_{(S \setminus r) \cup d}$ | $\langle r + 1, d \rangle \cup \langle d, r + 1 \rangle$ |
| $P_S$ | $\langle r, r + 1 \rangle$ |
| $P_{(S \setminus r(r + 1)) \cup b}$ | $\langle b, d \rangle$ |
| $P_{(S \setminus r + 1) \cup d}$ | $\langle r, d \rangle$ |
| $P_{(S \setminus r) \cup b}$ | $\langle b, r + 1 \rangle \cup \langle r + 1, b \rangle$ |

Table 1: The path collections corresponding to the monomials in each Plücker coordinate appearing in our incidence Plücker relation.
The term on the left side of our Plücker equation corresponds to pairs of path collections in \((\langle r, b \rangle \times \langle r + 1, d \rangle) \sqcup (\langle r, d \rangle \times \langle b, r + 1 \rangle)\) and the terms on the right correspond to those in \((\langle r, r + 1 \rangle \times \langle b, d \rangle) \sqcup (\langle r, d \rangle \times \langle r + 1, b \rangle)\). We first claim that \(\phi(\langle r, b \rangle \times \langle r + 1, d \rangle) = (\langle r, r + 1 \rangle \times \langle b, d \rangle) \sqcup (\langle r, d \rangle \times \langle r + 1, b \rangle)\). Indeed, we observe that, \((Q_1, Q_2) = \phi(P_1, P_2) \in (\langle r, d \rangle \times \langle r + 1, b \rangle)\) if and only if \((P_1, P_2) \in (\langle r, b \rangle \times \langle r + 1, d \rangle)\) and the path \(P_{r+1}^{Q_1}\), terminating at \(b\), does not intersect with the path \(P_{r+1}^{Q_2}\). Specifically, \(P_r^{Q_1} = P_r^{P_1}\) and \(P_r^{Q_1} = P_{r+1}^{P_2}\), while \(P_r^{Q_2} = P_r^{P_2}\) and \(P_{r+1}^{Q_2} = P_{r+1}^{P_1}\). This is illustrated in Fig. 13. Similarly, \((Q_1, Q_2) = \phi(P_1, P_2) \in (\langle r, r + 1 \rangle \times \langle b, d \rangle)\) if and only if \((P_1, P_2) \in (\langle r, b \rangle \times \langle r + 1, d \rangle)\) and the above-described paths do intersect. In this case, \(Q_1\) contains the unique diagonal paths originating on strands \(r\) and \(r + 1\). By assumption, \(P_{r+1}^{Q_1}\) and \(P_{r+1}^{Q_2}\) intersect. Then \(P_{r}^{Q_2}\) is identical to \(P_{r+1}^{Q_1}\) until the point of intersection and then to \(P_{r+1}^{Q_2}\) after the point of intersection. Also, \(P_{r+1}^{Q_1} = P_{r+1}^{P_1}\). The fact that the two paths we just described lying in \(Q_2\) don’t intersect is a consequence of the fact that the sink set of a path collection in \(\langle r, d \rangle\) is extremal and so \(|\langle r, d \rangle| = 1\) by assumption. This is illustrated in Fig. 14. We have shown that \(\phi(\langle r, b \rangle \times \langle r + 1, d \rangle) = (\langle r, r + 1 \rangle \times \langle b, d \rangle) \sqcup (\langle r, d \rangle \times \langle r + 1, b \rangle)\). Consequently, \(\phi(\langle r, b \rangle \times \langle d, r + 1 \rangle) = (\langle r, d \rangle \times \langle b, r + 1 \rangle)\).

Figure 13: Continuing with the setup of Fig. 12, the left figure illustrates \(P_1\) with dashed edges and \(P_2\) with red edges such that the dashed path originating on strand \(r + 1 = 2\) and terminating at \(b = 3\) does not intersect the red path originating on strand \(r = 1\) and terminating at \(r + 1 = 2\). The right figure shows a path collection \((Q_1, Q_2) \in (\langle r, d \rangle \times \langle r + 1, b \rangle) = (\langle 1, 4 \rangle \times \langle 2, 3 \rangle)\), where \(Q_1\) has dashed edges and \(Q_2\) has red edges. Observe that \((Q_1, Q_2)\) uses the exact same multiset of vertical edges as \((P_1, P_2)\).
Figure 14: Continuing with the setup of Fig. 12, the left figure illustrates $P_1$ with dashed edges and $P_2$ with red edges such that the dashed path originating on strand $r+1 = 2$ and terminating at $b = 3$ intersects the red path originating on strand $r = 1$ and terminating at $r+1 = 2$. The right figure shows a path collection $(Q_1, Q_2) \in (r, r+1) \times (b, d) = (1, 2) \times (3, 4)$, where $Q_1$ has dashed edges and $Q_2$ has red edges. Observe that $(Q_1, Q_2)$ uses the exact same multiset of vertical edges as $(P_1, P_2)$.

Note that the first claim of the lemma is that $(d, r + 1)$ is empty. Suppose otherwise. Observe that $(r, b)$ is necessarily non-empty since otherwise the Plücker equation would be saying $0 = 0$, in contradiction to the choice of $S, b$ and $d$. Thus, the bijection between $(r, b) \times (d, r + 1)$ and $(r, d) \times (b, r + 1)$ is non-trivial. Consider a path collection $C$ in $(r, b)$. Let $e$ be the edge used by $P_{r+1}^C$ to leave strand $r + 1$ en route to $d$. It must be used by some path in some pair of path collections in $(r, d) \times (b, r + 1)$. The only path coming from such a pair of path collections which can use $e$ is a path originating on strand $r + 1$ and terminating at $d$ which is part of a path collection in $(r, d)$. Since path collections in $(r, d)$ terminate at an extremal index, $|\langle r, d \rangle| = 1$ and it follows that for any path collection in $(r, b)$, the path originating on strand $r + 1$ and terminating at $b$ must leave strand $r + 1$ using the same edge $e$.

Since $\mathcal{P}_P(S, (r, r+1))_{\setminus b, d} \neq 0$ by choice of $S, b$ and $d$, we must have that there is a pair of path collection $Q = (Q_1, Q_2) \in (r, r+1) \times (b, d)$. We have shown that our bijection $\phi$ maps a subset of $\langle r, b \rangle \times (r + 1, d)$ onto $(r, r+1) \times (b, d)$. Let $P = (P_1, P_2) \in (r, b) \times \langle r + 1, d \rangle$ be such that $\phi(P) = Q$. A path collection in $\langle r, r + 1 \rangle$ does not use any vertical edges. Thus, all the vertical edges used by $P$ must be used by $Q_2 \in (b, d)$. Since $P_1 \in (r, b)$, it uses edge $e$. Further, note that $P_2 \in \langle r + 1, d \rangle$ must use a superset of the vertical edges used by the unique path collection in $(r, d)$, and in particular, it uses edge $e$. Thus, $P$ uses $e$ twice. Since $e$ cannot be used twice in $Q_2$, we have reached a contradiction. This shows the first claim of the lemma.

As for the second claim of the lemma, any path originating on strand $r$ and terminating at $d$ which avoids $C(r + 2)$ must actually visit strand $r + 1$ by the first statement of the lemma. By assumption, $|\langle r, d \rangle| = 1$ and so there is a unique path $\tilde{p}$ originating on strand $r + 1$ and terminating at $d$. We know one obvious instance of such a path, which is $p_{r+1}^C$. Thus, any path originating on strand $r$ and terminating at $d$ which avoids $C(r + 2)$ must terminate identically to $p_{r+1}^C$ after leaving strand $r + 1$. Moreover, paths originating on strand $r$ and terminating at $d$ use a vertical strand to get from strand $r$ to strand $r + 1$. This is only possible weakly to the right of column $c$. It follows that certainly any such path leaves strand $r + 1$ weakly to the right of column $c$.

We now modify this argument for the case where $r + 1 = b$. We claim that there cannot be a path $p$ which originates on strand $r$, terminates at $d$ and avoids $C(r + 1)$. To understand why this is, note that the paths originating on strands $r$ and $r + 1$ in $C$ are diagonal paths. If $p$ existed, we could define the path collection $\check{C}$ which is the same as $C$ except the diagonal path originating on strand $r$ is replaced by $p$. This has the same sink set as $\Xi_{v, w}^2(S)$, contradicting the induction hypothesis that in $G_{v, w}$ there is a unique path collection whose sink set is any given graph extremal index.

Thus, any path in originating from strand $r$ and terminating at $d$ while avoiding $C(r + 2)$ must actually
pass through strand \( r + 1 \). We can then conclude as in the previous case.

We now aim to prove that for any extremal index \( I \) of \( \mathcal{R}^{\geq 0}_{v,w} \), there is a unique path collection in \( G_{v,w} \) with sink set \( I \). Our proof will proceed by induction on \( \ell(v) \) and the following lemma will establish our base case.

**Lemma 5.12.** For any \( w \in S_n \), \( F \in \mathcal{R}^{\geq 0}_{v,w} \) and any extremal Plücker index \( I \) of \( F \), there is exactly one path collection in \( G_{id,w} \) with source set \( ||I||' = \{ 1', 2', \ldots, |I|' \} \) and sink set \( I \). Moreover, this unique path collection is the union of a diagonal path collection and a greedy left path collection.

**Proof.** Let \( G = G_{id,w} \). There are no vertical edges skipping strands in \( G \), because strands are never permuted. By Lemma 3.21, if we have a vertical edge from \((r,c)\) to \((r+1,c)\), then we also have a vertical edge from \((r+1,c+1)\) to \((r+2,c+1)\).

A graph extremal index in \( G \) is the sink set of a path collection consisting of a left extreme path collection originating from \( \{ l', (l+1)', \ldots, q' \} \) union the unique diagonal path collection originating from \(|l-1|'\). Let \( C \) be the greedy left path collection originating from \( \{ l', (l+1)', \ldots, q' \} \). Since there are no vertical edges in \( G \) which skip strands, it is clear that this is an example of a left extreme path collection. We must show it is the unique left extreme path collection originating from \( \{ l', (l+1)', \ldots, q' \} \). First, we show that \( p^C_q \) is unique in that any path \( p_q \) originating from \( q' \) that has the same sink as the path \( p^C_q \) satisfies \( p_q = p^C_q \). Whenever \( p_q \) reaches some column \( c \), it must go up as many vertical edges as it can. If it does not go all the way up to the top strand, it must take a diagonal strand, say strand \( r \), to column \( c - 1 \). Suppose \( p_q \) also reaches \((r,c-1)\), but differs from \( p^C_q \) immediately before reaching that intersection. Then \( p_q \) would necessarily need to take a vertical edge from \((r-1,c-1)\) to \((r,c-1)\). This means there was a vertical edge from \((r,c)\) to \((r+1,c)\), which means that \( p^C_q \) should not have taken the diagonal strand that it did, leading to a contradiction. Thus, if \( p_q \) ever differs from \( p^C_q \), we reach a contradiction.

The path \( p^C_{q-1} \) can be described similarly to how we described \( p^C_q \). The difference is that when it takes a diagonal strand \( r \) from \((r,c)\) to \((r,c-1)\), it may be because the path \( p^C_q \) takes the diagonal strand \( r + 1 \) from \((r+1,c+1)\) to \((r+1,c)\), in the process blocking \( p^C_{q-1} \), as illustrated in Fig. 15. If this is the case, we suppose there is some path \( p_{q-1} \) which (1) has the same sink as \( p^C_{q-1} \), (2) avoids \( p^C_q \), (3) reaches \((r,c-1)\), and (4) differs from \( p^C_{q-1} \) immediately before that. This is only possible if \( p_{q-1} \) can use the vertical edge from \((r-1,c-1)\) to \((r,c-1)\). If this were the case, then there must be an edge from \((r+1,c+1)\) to \((r+2,c+1)\). This contradicts the fact that \( p^C_q \) took the diagonal strand \( r + 1 \) from \((r+1,c+1)\) to \((r+1,c)\). For an illustration of this argument, see Fig. 16. We can iterate this argument to show that \( C \) is the unique left extreme path collection originating from \( \{ l', (l+1)', \cdots, q' \} \). Therefore, in the graph \( G_{id,w} \), for each graph extremal index \( I \), there is a unique path collection with sink set \( I \) and it consists of a union of a greedy left path collection and a diagonal path collection.

Figure 15: The dotted path \( p^C_{q-1} \) does not turn left in column \( c \). Rather, it takes the diagonal strand \( r = q - 1 \) to column \( c + 1 \) since the dashed path \( p^C_q \) blocks it from turning left.
Figure 16: Suppose the semi-opaque edge in column 5 were not part of this graph. Then the dashed path combined with the dotted path using either the red or the blue dotted edges, would give two different left extreme path collections originating at \( \{1', 2'\} \). However, the existence of the vertical red edge in column 3 means, by Lemma 3.21, we must have the semi-opaque edge in the graph. This is in accordance with the claim of Lemma 5.14 that there is a unique left extreme path collection with source set \( \{1', 2'\} \).

The next statement is a clear consequence of the argument in the previous proof.

**Corollary 5.13.** For any \( w \in S_n \), there is a unique left extreme path originating from any source in \( G_{vd,w} \).

**Lemma 5.14.** For any \( v \leq w \), \( F \in R^{>0}_{n,w} \) and any extremal Plücker index \( I \) of \( F \), there is a unique path collection in \( G_{v,w} \) with source set \( \{1', 2', \ldots, |I'|\} \) and sink set \( I \). Moreover, this unique path collection is the union of a diagonal path collection and a greedy left path collection.

Before proving this result, we establish useful notation and further preliminary results which use that notation.

Let us fix \( v \leq w \). Let \( w \) be the positive distinguished subexpression for \( w \) in \( w_0 \). Suppose that \( v \) has positive distinguished subexpression \( v = s_{i_1} \cdots s_{i_m} \) in \( w \). Let \( v_d = s_{i_1} \cdots s_{i_d} \) for \( 1 \leq d \leq m \) be an expression for a permutation \( v_d \). Note that each \( v_d \) is a positive distinguished subexpression in \( w \). Our strategy is to show by induction that the lemma is true for each \( v_d \leq w \). Let \( G = G_{vd,w} \). Suppose \( v_{d+1} = v_d s_r \) and let \( G = G_{vd+1,w} \). Note that \( G \) differs from \( G' \) according to the following three operations, to which we give names for easy reference in the future:

- **(Rem)** A single vertical edge is removed from \( G' \) between strands \( r \) and \( r + 1 \) in some column, say in column \( c \).
- **(Int)** The labels on the left of strands \( r \) and \( r + 1 \) are interchanged.
- **(Swap)** Any vertical edge of \( G' \) strictly left of column \( c \) or in column \( c \) and strictly below strand \( r + 1 \) with an endpoint on either strand \( r \) or strand \( r + 1 \) has that endpoint swapped to strand \( r + 1 \) or \( r \), respectively\(^4\).

We now introduce notation further notation we will use in this proof. Fix some \( k \in [n] \). We study extremal path collections. These originate on the vertices \( \{1', \ldots, k'\} \) and, for some \( q \), are a union of a left extreme path collection from the top most \( q \) vertices and diagonal paths from the remaining vertices. Let \( I'_k \) (resp. \( I_k \)) be the Gale-minimal graph extremal index of \( G' \) (resp. \( G \)) of size \( k \). Let \( O'_t \) (resp. \( O_t \)) denote the subset of \( [k]' \) lying on or above strand \( t \). Let \( C'_t \) (resp. \( C_t \)) be any graph extremal path collection in \( G' \) (\( G \)) which consists of a left extreme path collection originating from \( O'_t \) (resp. \( O_t \)) union the diagonal path.

\(^4\)It is impossible for such an edge to have endpoints on both strands \( r \) and \( r + 1 \) by Lemma 5.6.
collection originating from \([k'] \setminus O'_i\) (resp. \(O \setminus O_i\)). Let \(S'_t\) (resp. \(S_t\)) be the sink set of the path collection \(C'_i\) (resp. \(C_i\)). By construction, these are precisely the graph extremal indices of \(G'\) (resp. \(G\)). We will use a number \(x\) in parentheses to restrict our attention to the top \(x\) paths of a path collection. Explicitly, \(C'_i(x)\) will be the sub collection of the path collection \(C'_i\) with source set \(O'_i\), and \(S'_i(x)\) will be the sink set of \(C'_i(x)\) (and similarly for unprimed notations). In parts of this proof, we will need to simultaneously consider extremal path collections consisting of different numbers of paths. At this point, we will denote by \(O'^{\kappa}\) the source set of \([k']\) lying on or above strand \(t\), and define \(O'^{\kappa}\) and \(S'^{\kappa}\) (as well as all their unprimed counterparts) in terms of \(O'^{\kappa}\). When we omit the superscript, as we will throughout most of this proof, we will be assuming that \(\kappa = k\), is fixed.

As a result of \((\text{Swap})\), there is a natural map from paths \(p'\) in \(G'\) originating on strand \(r\) to paths \(p\) in \(G\) originating on strand \(r + 1\) which use the same vertical edges, except for possibly an edge from strand \(r\) to strand \(r + 1\), if such an edge is used by \(p'\). Note that in this statement we are identifying edges which have their endpoints modified by \((\text{Swap})\). If \(p\) leaves strand \(r + 1\) left of column \(c\), there is also an obvious inverse map. Otherwise, the inverse may not be uniquely defined, but there will be at least one inverse. Similarly, for \(s \neq r, r + 1\), there is a natural map from paths \(p'\) in \(G'\) originating on strand \(s\) to paths \(p\) in \(G\) originating on strand \(s\) which use the same vertical edges, except for possibly an edge from strand \(r\) to strand \(r + 1\), if such an edge is used by \(p'\). Again, as long as \(p\) does not intersect strand \(r + 1\) to the right of column \(c\), there is also an obvious inverse. Otherwise, the inverse may not be uniquely defined, but there will be at least one inverse. In a similar spirit, there is a map from paths \(p\) in \(G\) originating on strand \(r\) to paths \(p'\) in \(G'\) originating on strand \(r + 1\) which use the same vertical edges except for possibly a vertical edge between strands \(r\) and \(r + 1\). We will refer to paths related in this way as being the same up to \((\text{Swap})\).

**Lemma 5.15.** In any \(G_{v,w}\), there exists a unique path from \(1'\) to \(b_1\), where \(b_1\) is the maximal sink attainable by a path originating at \(1'\), and this path is left greedy.

***Proof.*** This is certainly true if \(v = id\) by Lemma 5.12. We prove this by induction on \(d\). Thus, we can assume that the result holds for \(G' = G_{v, d}\) and prove it for \(G = G_{v, d, r, w}\). We consider a few cases.

1. If \(1'\) lies above strand \(r + 1\), then the claim clearly holds since \(G\) and \(G'\) have precisely the same set of edges with both endpoints lying above strand \(r + 1\).

2. Suppose \(1'\) lies below strand \(r\).

   (a) If \(b_1 \neq r\), then it is straightforward to check that two different paths between \(1'\) and \(b_1\) in \(G\) correspond to (at least) two different paths between \(1'\) and \(b_1\) in \(G'\), up to \((\text{Swap})\), contradicting our induction hypothesis. Thus, the claim holds for \(G\), with the unique path from \(1'\) to \(b_1\) being the same in \(G\) as it is in \(G'\), up to \((\text{Swap})\).

   (b) If \(b_1 = r\), it is conceivable that there are at least two paths from \(1'\) to \(r\) in \(G\) with one path, \(p_1\), reaching strand \(r\) left of column \(c\) and another, \(p_2\), achieving strand \(r\) right of column \(c\). In such a case, if there are no edges from strand \(r\) to strand \(r + 1\) in \(G\), then the path in \(G'\) corresponding to \(p_1\) up to \((\text{Swap})\) will terminate at \(r + 1\) while the path in \(G'\) corresponding to \(p_2\) up to \((\text{Swap})\) will terminate at \(r\). In all other cases, we can argue as in the previous subcases, so we focus on this situation. In particular, we show that it never occurs. In \(G'\), there is a path terminating at \(r + 1\), so by Lemma 5.2, there exists \(u'^{-1} \leq u' \leq w^{-1}\) with \(u'(1) = r + 1\). We will show that \(s_r v'^{-1} \leq u' \leq w^{-1}\), which will show that in fact the sink \(r + 1\) is achievable in \(G\) as well, contradicting the maximality of \(b_1 = r\). Suppose that \(s_r v'^{-1} \not\leq u'\). There must be no expression for \(u'\) which begins with an \(s_r\). If there were, choose any expression \(u'\) for \(u'\) and any subexpression \(v'^{-1}\) for \(v'^{-1}\). Then the initial \(s_r\) would not be in \(v'^{-1}\) since \(s_r v'^{-1} > v\) and so \(s_r v'^{-1}\) would be a subexpression for \(s_r v'^{-1}\) in \(u'\), which we are assuming to be false. Thus, \(s_r u' > u'\). However, \(s_r u'\) replaces \(u'(1) = r + 1\) by \(s_r u'(1) = r\), so that \(s_r u'\) has fewer inversions than \(u'\). Thus, \(s_r u' < u'\) and we have reached a contradiction.

3. Note that \(1'\) actually cannot lie on strand \(r\) in \(G\), since \((\text{Int})\) should add an inversion in the vertex labels of \(G\) by the reduceness of \(v\).

31
4. If \(1'\) lie on strand \(r + 1\), then let \(p\) be the unique left extreme path originating from \(1'\) in \(G'\). Then \(p\) originates on strand \(r\). Note that \(p\) cannot be diagonal because of the existence of the edge \(e\) removed by (Rem) in \(G'\). We can then consider the left greedy path originating from \(1'\) in \(G\). This is clearly the same as \(p\), up to (Swap), and also left extreme. We now show it is unique. Any other path \(\tilde{p}\) from \(1'\) to \(b_1\) must either leave strand \(r + 1\) to the left of column \(c\) or weakly to the right of column \(c\). If it leaves to the left of column \(c\), then there is a path in \(G'\) which originates on strand \(r\) and takes the same edge, up to (Swap), and terminates identically to \(\tilde{p}\), contradicting the uniqueness of \(p\). If it leaves weakly to the right of column \(c\), there is a path in \(G'\) that originates on strand \(r\), uses the edge \(e\) removed by (Rem) and terminates identically to \(\tilde{p}\), again contradicting the uniqueness of \(p\).

This proves the result for \(G\) and we are done by induction.

Lemma 5.16. Let \(q \in [n]\). Assume \(S'_q(q)\) is Gale maximal amongst the sink sets of all path collections originating at \(O'_q\) in \(G'\). Then there is no path collection in \(G\) from \(O'_q\) to a sink set which is Gale larger than \(S'_q(q)\).

Proof. Let \(C\) be a path collection in \(G\) originating on strands \(O'_q\) and with sink set \(S\). Then there exists at least one path collection \(C'\) in \(G'\) originating on strands \(O'_q\) and with sink set \(S'\) such that \(S' \geq S\) in the Gale order. To construct this, we let \(p'_C = p_C\) for \(q \neq r, r + 1\), up to (Swap). If \(O'_q\) does not contain the vertex on strand \(r\) or \(r + 1\) of \(G\), we are done. If \(O'_q\) contains the vertex on strand \(r\) of \(G\) but not the vertex on strand \(r + 1\) of \(G\), we set \(p_{r+1}' = p_{r+1}\), up to (Swap). In this case, it is clear that the sink set of \(C'\) is either \(S\) or \(S' = (S \setminus r) \cup r + 1\) (if \(p_C\) was diagonal). Note that the vertex on strand \(r + 1\) is labeled by a larger number than the vertex on strand \(r\) in \(G\) by (Int) and the reducedness of \(\nu\), so it is impossible for \(O'_q\) to contain the vertex on strand \(r + 1\) of \(G\) but not the vertex on strand \(r\) of \(G\). If \(O'_q\) contains both the vertex on strand \(r\) and the vertex on strand \(r + 1\) of \(G\), we distinguish based on where these paths leave the strands on which they originate.

1. If \(p_{C_{r+1}}'\) leaves strand \(r + 1\) left of column \(c\), we set \(p_{C_r}'\) to be the same as \(p_{C_{r+1}}'\) and \(p_{C_{r+1}}'\) to be the same as \(p_{r+1}'\), up to (Swap). This path collection has sink set \(S\) unless \(p_C\) was diagonal, in which case it has sink set \(S' = (S \setminus r) \cup r + 1\).

2. If \(p_{r+1}'\) leaves strand \(r + 1\) weakly to the right of column \(c\) and \(p_{r}'\) leaves strand \(r\) to the left of column \(c\), we set \(p_{C_{r+1}}'\) to be the same as \(p_{r+1}'\), up to (Swap). We would like to set \(p_{C_r}'\) to be the same as \(p_{C_{r+1}}'\) up to (Swap), but we are in one of the cases where saying “up to (Swap)” is not perfectly unambiguous. However, we can specify that this means that \(p_{C_r}'\) uses the edge removed by (Rem) and terminates exactly like \(p_{C_{r+1}}'\). The sink set of this path collection is again \(S\).

3. If \(p_{r}'\) and \(p_{C_{r+1}}'\) leave strands \(r\) and \(r + 1\), respectively, weakly to the right of column \(c\), we may set \(p_{C_r}'\) to be the same as \(p_r\) and \(p_{C_{r+1}}'\) to be the same as \(p_{C_{r+1}}\), up to (Swap). The sink set of this path collection is \(S\).

It follows that any path collection originating from \(O'_q\) must terminate at a sink set no larger than \(S'_q(q)\) in Gale order, since \(S'_q(q)\) is the Gale largest sink set of a path collection with that source set.

Lemma 5.17. When both paths exist, it is impossible for \(p_{C_r}'\) to terminate to the left of the endpoint of \(p_{r+1}'\).

Proof. Suppose \(p_{r+1}'\) and \(p_{r}'\) terminate at \(a\) and \(b\), respectively, with \(a < b\). Consider the set \(S = S' \setminus a \cup r\). The path collection from \(O'_r + 1\) to \(S'_r + 1(r + 1)\) has Gale maximal sink set and so there is no path collection in \(G'\) from \(O'_r + 1\) to \((S'_r + 1(r + 1) \setminus a) \cup b\). Adding in diagonal paths from \([k]' \setminus O'_r + 1\), we conclude that there is no path collection in \(G'\) from \([k]'\) to \(S\). Thus, by Lemma 5.2, there is no \(u \in S_n\) with \(v_{a-1} \leq u \leq v_{b-1}\) such that \(S = \{u(1), \cdots, u(k)\}\). However, in \(G\), consider the path collection consisting of:

1. All the paths in \(C_r\) which do not originate on strand \(r\) up to (Swap).
2. The diagonal path on strand $r$.

The sink set of this path collection is $S$. Thus, again by Lemma 5.2, there does exist $u \in S_n$ with $s_r v_d^{-1} \leq u \leq w^{-1}$ such that $S = \{u(1), \cdots, u(k)\}$. Since $s_r v_d^{-1} > v_d^{-1}$ by the reduceness of $v$, we also have $v_d^{-1} < u$, contradicting what we deduced previously.

With these preliminaries taken care of, we now move on to the proof of Lemma 5.14. The proof involves a careful induction, so we start by explaining the induction and their base cases. The bulk of the proof consists of proving the induction step, which involves a number of cases. Many of the steps of this proof mirror the logic used in Lemma 5.15, although the arguments here are more difficult.

**Proof of Lemma 5.14.** This proof proceeds by induction on a number of dimensions. To make our work easier, we will additionally prove the following result by induction: For each $q \in [u]$, $S_q (q)$ is the Gale largest sink set of any path collection with source set $O_q$. Recall that $v = v_1 v_2 \cdots v_{w+1}$ and $v_d = v_{i_1} v_{i_2} \cdots v_{i_{d-1}}$. We will induct on $d$, with the base case $v_0 = id$ considered in Lemma 5.12. Equivalently, we assume the result holds for $G'$ and prove it holds for $G$ as well. We also induct on $k = |I|$, with the base case of $k = 1$ considered in Lemma 5.15 for any $v \leq w$. Although it is not proven explicitly in the cited theorem, for both these base cases, it is evident that $S_q (q)$ is indeed the Gale largest sink set of any path collection with source set $O_q$.

Now fix $k$ and $d$. We will want to prove the result for $C_q$. Note that $C_n$ must consist entirely of diagonal paths and by Corollary 5.7, this is the unique path collection with sink set $I_k$. The left extreme part is $S_n (u) = \emptyset$, so the conditions of the theorem hold automatically, as does the fact that $S_n (u)$ is Gale maximal. Thus, we may induct backwards on $q$ and assume the result for $\tau > q$.

We now begin to prove the induction claim. We may assume that for any graph extremal index $I'$ of $G' = G_{v_d, w}$, there is a unique path collection with sink set $I'$ which is the union of diagonal paths and a greedy left path collection, and that the left greedy part of this path collection attains a Gale maximal sink. We may assume the same about graph extremal indices $I$ of $G$ with $|I| < k$. Our goal is to show that the same holds for graph extremal indices of $G = G_{v_d, s_r, w}$ of size $k$. We prove the result for each $S_q (q)$. We will distinguish three different cases, namely $q > r + 1$, $q = r + 1$ and $q < r$ and within each case, we will often be able to make use of our induction hypothesis that the result holds for $S_\tau (\tau)$ when $\tau > q$.

**Remark:** From the construction in the proof of Lemma 5.16, it is clear that if there exist two distinct path collections in $G$ with sink sink set $S \neq r$, there exist two distinct path collections in $G'$ with sink set $S$.

For the remainder of this proof, we assume that $C_q$ is different from $C_{q+1}$, as otherwise, by induction, there is nothing to show. We use an enumerated list to distinguish between different cases.

1. If $q > r + 1$, we claim that the unique choice of $C_q$ is the path collection in which the non-diagonal paths are exactly the same as the non-diagonal paths in the unique path collection $C'_q$ in $G'$, up to *(Swap)*. This will have sink set $S_q = S'_q$ which shows that $S_q (q)$ is as Gale large as it can possibly be, by Lemma 5.16. By construction, it is also clear that these paths remain part of the unique greedy left and extreme left path collection.

2. We now simultaneously consider $q = r + 1$ and $q = r$. There are three large possibilities to consider, the first of which will itself have a number of subcases.

   (a) First, suppose $C'_{r+1}$ differs from $C'_{r+2}$. In this case, the vertex on strand $r + 1$ in $G'$ has a label in $[k]'$. By *(Int)*, so does the vertex on strand $r$ in $G$. By the reduceness of $v_d$, the vertex labels interchanged by *(Int)* must introduce an inversion and so it must be the case that the label of the vertex on the left end of strand $r$ in $G$ is larger than the label of the vertex on the left end of strand $r + 1$ in $G$. Thus, we also have that the vertex label on strand $r + 1$ in $G$ is in $[k]'$.

   By Lemma 5.17, we know that $p_{r-1}^{C'}$ terminates to the right of $p_{r+1}^{C'}$. We are precisely in the situation specified by the hypotheses of Lemma 5.11, with $S = S_{r+2}'$, $p_{r+1}^{C'}$ terminating at $d$ and $p_{r}^{C'}$ terminating at $b$. The only thing that is not entirely immediate is Item 3 in the statement of Lemma 5.11, but this holds by the left greediness of the relevant path collections in $G'$.
i. Suppose there exists a path in $G'$ originating on strand $r$ and terminating at $d$ while avoiding $C_{r+2}(r+2)$. We conclude the following from Lemma 5.11: This path passes through strand $r+1$ and leaves strand $r+1$ using the same edge $e'$ as is used by $p_{r+1}^C$ to leave strand $r+1$, weakly to the right of column $c$. Note that this path cannot subsequently jump over any other paths in $G'$ since there are no edges skipping strands weakly above and to the right of column $c$. Moreover, $p_{r+1}^C$ attains sink $b < d$, and so it must actually leave strand $r+1$ to the right of $e'$ and consequently must leave strand $r$ to the right of $e'$ as well. In particular, it does not use the edge removed by (Rem) to get from strand $r$ to strand $r+1$. Now, define the path collection $\tilde{C}$ as follows: Let $p_i^\tilde{C} = p_i^C$ for all $i \in O$ other than $r$ and $r+1$, up to (Swap). We define $p_i^\tilde{C}$ and $p_i^\tilde{C} + 1$, where in this case we mean the paths are actually identical (not just up to (Swap)). This is well defined since neither path uses the edge removed by (Rem). It is clear that $\tilde{C}(r+1)$ is a path collection from $O_{r+1}$ to $S'(r+1)(r+1)$. Thus, this is the Gale maximal possible sink set of a path collection in $G$ originating from $O_{r+1}$ and thus must be the left extreme part of the extremal path collection $C_{r+1}$. Uniqueness of this path collection follows from the remark following Lemma 5.16. The left greediness follows from the left greediness of the identical path collection $C_{r+1}$. We similarly conclude that the left extreme part of $C_r$ is uniquely defined as $\tilde{C}(r)$.

ii. If such a path does not exist, then $p_{r+1}^C$ must leave strand $r+1$ to the left of column $c$; otherwise we could consider the path originating on strand $r$, using the edge removed by (Rem) and concluding in the same way as $p_{r+1}^C$, which would terminate at $d$. We then define $\tilde{C}$ to be the path collection with $p_i^\tilde{C} = p_i^C$ for all $i \in O$ different from $r$ and $r+1$, with $p_i^\tilde{C} = p_{r+1}^C$, and with $p_i^\tilde{C} + 1 = p_{r+1}^C$, all up to (Swap). This notion is not in general well defined for paths originating on strand $r+1$ but since our path in $C_r$ leaves strand $r+1$ left of column $c$, it is well defined in this case. It is clear that the sink set of $\tilde{C}(r)$ is $S'(r)$. As in the previous subcase, we use Lemma 5.16 to argue that the left extreme part of $C_r$ is uniquely defined as $\tilde{C}(r)$, and this path is left greedy and Gale maximal.

By construction, we know that we must have $S_{r+2}(r+2) \subseteq S_{r+1}(r+1) \subseteq S_r(r)$. From this it follows that either $S_{r+1}(r+1) = S_{r+2}(r+2) \cup d$ or $S_{r+1}(r+1) = S_{r+2}(r+2) \cup b$. If we can show that the latter holds, we can again conclude similarly to the previous case that the unique left extreme path collection with source set $O_{r+1}$ is left greedy and has Gale maximal sink set.

Recall that $|I| = k$ and also that by induction, we may assume that the lemma holds for all extremal indices of size strictly less than $k$. If vertex $k'$ lies on a strand below $r+1$, then in $C_{r+1}$, we know that the path originating at $k'$ is diagonal. By simply removing this strand from the path collection, we have an extremal path collection consisting of $k-1$ paths. This is by induction uniquely defined, with its left extreme part being left greedy and having Gale maximal sink set. Adding back in the diagonal path originating at $k'$ implies the same for $C_{r+1}$.

Note that it is impossible for $k'$ to lie on strand $r+1$ since, by the reducedness of $\mathfrak{v}_d$, the label on strand $r$ should be larger than the label on strand $r+1$, and we are assuming both lie in $O$.

We now suppose $k'$ lies on a strand above $r+1$. In this section, we will use superscripts on path collections to indicate the number of paths they contain. As a preliminary remark, we note that $b$ must be the rightmost sink amongst all the sinks of the non-diagonal part of $\tilde{C}$. To see this, we observe that by Lemma 5.11, the path terminating at $d$ in $C_{r+1}^{kb}$ must leave strand $r+1$ weakly to the right of column $c$, whence it must not skip over any strands. Thus, for a path originating on strand $r$ to terminate at $b < d$, it must leave strand $r$ even further to the right of column $c$ and also must not skip any strands. Since it originates from the bottom-most source, it must attain the rightmost sink.

Suppose, in $\tilde{C}(r)$, the path $\tilde{p}$ originating at vertex $k'$ terminates at $k_1$. Now, we consider the left greedy path collection $C_{r+1}^{k_1-1}(r)$. Any path originating above the vertex labeled $k'$, will be the same as the corresponding paths in the left greedy path collection $\tilde{C}(r)$. Now, we look
at paths of $C^{k-1}_{r}(r)$ originating below vertex $k'$. Each subsequent path $	ilde{p}_i$ in $\tilde{C}(r)$ is either blocked from making a turn by $\tilde{p}$ or is not. If not, then the corresponding path in $C^{k-1}_{r}(r)$ is identical to $\tilde{p}_i$. If it is blocked, then in $C^{k-1}_{r}(r)$ the corresponding path is no longer blocked and should, by left greediness, actually make that turn. It then must follow the conclusion of $\tilde{p}$ to $\kappa_1$. We can then say the sink of $\tilde{p}_i$ is $\kappa_2$ and continue repeating this logic. Since $S^b_r$ is the sink set of $\tilde{C}$, we conclude from this argument that the sink set of $C^{k-1}_{r}(r)$ is of the form $S^b_r \setminus \kappa$ for some $\kappa$. We also note that since $b$ is the rightmost sink amongst the non-diagonal paths, we must have $\kappa \geq b$. This setup, with $|\kappa| = b$, is illustrated in Fig. 17. For the moment, suppose $\kappa > b$. Then applying the three-term incidence Plücker relation involving $S = (S^{b}_{r+2} \cup r + 1) \cup d$ and $\kappa > b > r$, we obtain

$$P_S P_{(S \setminus \kappa) \cup b} P_{S \setminus r} + P_{(S \setminus r) \cup b} P_{S \setminus \kappa}.$$ 

If we could obtain an extremal path collection whose left extreme part had sink set $S^{b}_{r+2} \cup r + 1 \cup d$, then by adding diagonal paths we would obtain an extremal path collection with sink set $S$, so it suffices to prove $P_S = 0$. Note that $P_{(S \setminus \kappa) \cup b} \neq 0$ since $(S \setminus \kappa) \cup b$ is precisely the sink set of the path collection $\tilde{C}$ when we remove the path originating from vertex $k'$. On the other hand, we claim $P_{S \setminus r} = P_{S \setminus \kappa} = 0$, which would show that $P_S = 0$, as desired. To see this, we note that by construction, the path collection in $G$ originating from $O^{k-1}_{r+1}$ with Gale maximal sink terminates at $S^b_r \setminus \kappa$. Thus, by induction, there is no path collection from $O^{k-1}_{r+1}$ to $S^b_r \setminus \kappa$. Such a path collection could be obtained from any path collection with sink set $S \setminus r$ by deleting diagonal paths, proving $P_{S \setminus r} = 0$. Similarly, to achieve $S \setminus \kappa$, there must be a path attaining the sink $r$ and so the path originating on strand $r$ must be diagonal. However, the left greedy path collection originating from $O^{k-1}_{r+1}$ terminates at $S^b_r \setminus r + 1 \cup \kappa$. Thus, by induction, there is no path collection from $O^{k-1}_{r+1}$ with sink set $S^b_r \setminus r + 1 \cup \kappa$. Such a path collection could be obtained from any path collection with sink set $S \setminus r$ by deleting diagonal paths, proving $P_{S \setminus r} = 0$. Note that we use the fact that we have already shown that the extremal path collection $C^b_k$ is uniquely defined and that in its left greedy part, the path originating on strand $r + 1$ terminates at $b$. Finally, we consider $\kappa = b$. By definition of $\kappa$, in such a case, the path originating on strand $r$ must still terminate at $d$ in the left extreme (and by induction, left greedy) path collection $C^{k-1}_{r}(r)$ originating from $O^{k-1}_{r+1}$ in $G$. Consider the endpoint $\kappa_0$ of the path originating from strand $r + 1$ in $C^{k-1}_{r}(r)$. If $\kappa_0$ is to the left of $d$, then consider the left greedy path collection $C'$ in $G'$ with the same sources. It is straightforward to check that since the path originating on strand $r$ in $G$ leaves strand $r$ left of column $c$, $C'$ will have each path terminating at the same sink as the path originating from the vertex with the same label in $C^{k-1}_{r}(r)$. Thus, in this path collection, the path originating on strand $r$ terminates left of the path originating on strand $r + 1$, contradicting Lemma 5.17. If $\kappa_0$ is to the right of $d$, we define $S$ to be $S^{b}_{r+2} \cup \kappa$, the sink set of $C^{k-1}_{r+2}$. Note that $S$ does not include $\kappa_0$. In this language, we want to show that $P_{S \setminus \kappa_0 d} = 0$, since by adding in diagonal paths to any path collection in $G$ from $O^{k}_{r+1}$ to $S^{k}_{r+2} \cup (r + 2) \cup d$, we would obtain a path collection from $[k']$ to $S \cup \kappa_0 d$. We will do this by considering the three-term incidence Plücker equation for $S$ and $r + 1 < \kappa_0 < d$:

$$P_{S \setminus \kappa_0} P_{S \setminus \kappa_0 (r+1)} = P_{S \setminus d} P_{S \setminus (r+1)} + P_{S \setminus (r+1)} P_{S \setminus \kappa_0 d}.$$ 

Observe that $P_{S \setminus d} = 0$ since the left greedy path originating from $O^{k-1}_{r+1}$ has sink set $S^{b}_{r+2} \cup (r + 2) \cup \kappa_0$ and by induction, we know this is Gale maximal. Similarly, to attain a sink set $S \setminus (r + 1) \cup d$, we must have the path originating on strand $r + 1$ be diagonal. Thus, the path collection must contain paths from $O^{k}_{r+2}$ to $S^{b}_{r+2} \cup (r + 2) \cup d$. However, the left greedy path collection with source set $O^{k}_{r+2}$ has sink set $S^{b}_{r+2} \cup \kappa_0 < S^{b}_{r+2} \cup d$. By induction, this is Gale maximal amongst all path collections originating from $O^{k}_{r+2}$, and so $P_{S \setminus (r+1) d} = 0$. It is clear that $P_S \setminus (r + 1) \neq 0$ by considering the left greedy path from $O^{k}_{r+2}$ union the diagonal path on strand $r + 1$. From all this it follows that $P_{S \setminus \kappa_0 d} = 0$, as desired.
(b) Second, suppose $C_{r+1}' = C_{r+2}'$ and both are different from $C_r'$. This happens if either the vertex label on strand $r + 1$ in $G'$ is in $[k]'$ but the path originating from strand $r + 1$ in $C_{r+1}'$ is diagonal, or the vertex label on strand $r + 1$ in $G'$, and equivalently the vertex label on strand $r$ in $G$, is not in $[k]'$. In the former case, we argue exactly as in the previous subcase. In the latter case, set $p_{r+1}'$ to be the path which is the same as $p_{r+1}''$, up to (Swap). With this choice, $S_r(r) = S_r''(r)$ and by Lemma 5.16, we conclude as before that this is the unique left extreme path collection, that it is left greedy, and that its sink set is Gale maximal. Similarly, we see from this that $C_{r+1} = C_r$ is uniquely defined and $C_{r+1}(r + 1)$ is a greedy left path collection with Gale maximal sink set.

(c) Finally, suppose $C_r' = C_{r+1}' = C_{r+2}'$. We are assuming either $C_{r+1}$ differs from $C_{r+2}$ or from $C_r$ since otherwise there is nothing to check. If $C_{r+1}$ differs from $C_{r+2}$, then the vertex label on strand $r + 1$ in $G$ is in $[k]'$. On the other hand, if $C_{r+1}$ differs from $C_r$, then the vertex label on strand $r$ in $G$ is in $[k]'$. However, by the reducedness of $v_d$, the swap of labels in (Int) causes the label of the vertex on strand $r$ in $G$ to be larger than the label of the vertex on strand $r + 1$ in $G$ and so the vertex label on strand $r + 1$ in $G$ is in $[k]'$ as well. Thus, in any event, vertex label on strand $r + 1$ in $G$ is in $[k]'$.

It follows that the vertex label on strand $r$ in $G'$ is in $[k]'$, and so we must have that $p_{r+1}''$ is diagonal to ensure $C_r' = C_{r+1}'$. This can only happen if there is a path in $C_r'$ originating on strand $r + 1$, since otherwise the greedy left path $p_{r+1}''$ would be able to use the edge $e$ removed by (Rem). Thus, the vertex label on strand $r + 1$ in $G'$, and equivalently the vertex label on strand $r$ in $G$, is in $[k]'$. We also must have that $p_{r+1}'$ is diagonal to ensure $C_{r+1}' = C_{r+2}'$. Consider $p_{r+1}''$. We claim it must be diagonal as well. If it could leave strand $r + 1$ left of column $c$, $p_{r+1}''$ would be
able to as well, by (Swap). If it could leave strand $r+1$ weakly to the right of column $c$, $p_{r+1}^{C_q}$ would be able to as well. Both of these would be contradictions. We also claim that $p_{r}^{C_q}$ must be diagonal. If it were not, it must leave strand $r$ left of column $c$ by skipping strand $r+1$, but then $p_{r}^{C_q}$ would be able to leave strand $r+1$ as well, by (Swap). This would be a contradiction.

3. If $q < r$, we study the structure of $S'_q$. By Lemma 5.2, $S'_q$ is of the form $\{u(1), \ldots, u(k)\}$ for some $v^{-1}_d \leq u \leq v^{-1}$. If $s_rv^{-1}_d \leq u$, then $S'_q$ is also the sink set of a path collection in $G$. The only possible way there can exist a path collection in $G$ with sink set $S'_q$ is if we can define $p_{i}^{C_q}$ to be the path which is the same as $p_{i}^{C_q}$ up to (Swap) for each strand $i$ whose vertex label lies in $[q]'$. Arguing as we have before, this choice of $p_{i}^{C_q}$ is easily seen to be the unique left extreme path originating from strand $q$ and also left greedy.

If it is not true that $s_rv^{-1}_d \leq u$, it must be the case that there is no expression for $u$ which begins with an $s_r$. If there were, choose such an expression $u$ for $u$ and any subexpression $v^{-1}_d$ for $v^{-1}$. Then the initial $s_r$ is not in $v^{-1}_d$ since $s_rv^{-1}_d > v^{-1}_d$ and so $s_rv^{-1}_d$ is a subexpression for $s_rv^{-1}_d$ in $u$. Thus, we conclude that $s_rv^{-1}_d \leq s_ru$. Here again, if $\{u(1), \ldots, u(k)\} = \{s_ru(1), \ldots, s_ru(k)\}$, then $S'_q$ is the sink set of a path collection in $G$. Then, as above, we can set $p_{i}^{C_q}$ to be the same as $p_{i}^{C_q}$ up to (Swap), and we are done.

Otherwise, since $s_ru > u$, it must be the case that $\{s_ru(1), \ldots, s_ru(k)\} = (\{u(1), \ldots, u(k)\} \setminus r) \cup r + 1$. Even in this case, we can again set $p_{i}^{C_q}$ to be the path which is the same as $p_{i}^{C_q}$ up to (Swap). This clearly makes $C_q(u)$ left extreme and left greedy. It is also clearly the unique such choice unless some $p_{i}^{C_q}$ terminates at $r$. In this case, we must do a bit more work to show that $p_{i}^{C_q}$ is the unique path originating on strand $i$, avoiding $C_q(u - 1)$ and terminating at $r + 1$. First, we claim $p_{i}^{C_q}$ does not reach strand $r$ weakly to the right of column $c$. If it did, say in column $c' \leq c$, then by left greediness, there are no vertical edges from strand $r$ to strand $r + 1$ weakly to the right of column $c'$ which are not blocked by other paths. But then it is impossible for $p_{i}^{C_q}$ to terminate at $r + 1$, which is a contradiction.

Thus, $p_{i}^{C_q}$ reaches strand $r$ left of column $c$. Then, by the uniqueness of $p_{i}^{C_q}$ and by (Swap), there is a unique path in $G$ originating on strand $i$ and terminating at strand $r + 1$ which skips strand $r$. Also, there can not be any paths in $G$ originating on strand $i$ and terminating at $r + 1$ which use a vertical edge to get from strand $r$ to strand $r + 1$, since any such vertical edge would necessarily be to the right of column $c$ and also would be available for use in $G'$, contradicting the left extremity of $p_{i}^{C_q}$.

Example 5.18. Continuing with our example of $w = 4213$ and $v = 1324$ from earlier, let's see how the extremal indices of size 2 arise from the graph $G = G_{v,w}$. The Gale minimal extremal index is the set of sinks of the diagonal path collection with source set $[2]'$. As indicated with dashed lines in Fig. 18, this is $I = \{1, 3\}$.

Figure 18: The minimal extremal index of size 2 is the set of sinks of the dashed path collection.
To find the next extremal index, we find the topmost source vertex for which the path originating at that vertex does not attain the leftmost possible sink. In this case, that is \( S' \). Swapping the diagonal path originating at \( S' \) for a path which does attain the leftmost possible sink, we get the dashed paths in Fig. 19. Thus, the next extremal index is \( \Xi_{v,w} (I) = \{2,3\} \), the set of sinks of these paths.

5.2 Determining Non-Extremal Plücker Coordinates

Our next goal is to prove that the extremal non-zero and non-infinite Plücker coordinates determine the rest of the Plücker coordinates, in the TNN flag variety and TNN Dressian, respectively.

Note that applying \( \Xi_{v,w} \) sufficiently many times to any index \( I \) of a non-zero Plücker coordinate of a flag \( F \in \mathcal{R}_{v,w} \) (or of a non-infinite Plücker coordinate of a point \( P \) in the TNN Dressian) will eventually yield the index of an extremal non-zero (non-infinite) Plücker coordinate of \( F \) (\( P \)). Analogous to our earlier notation, we define \( e_{v,w} (S) = \Xi_{v,w}^k (S) \) to be the first extremal index arising from repeated applications of \( \Xi_{v,w} \) to \( S \).

**Theorem 5.19.** For any flag \( F \in \mathcal{F}_{v,w} \), the extremal non-zero Plücker coordinates of \( F \) uniquely determine the other non-zero Plücker coordinates of \( F \) by three-term incidence Plücker relations.

**Proof.** Let \( F \in \mathcal{R}_0 \). This proof begins identically to the proof of Theorem 4.7, with \( v \) in the place \( id \) and \( w \) in the place \( w_0 \). Where things differ is in the third paragraph of that proof. There, we assert that there is \( a < b \) with \( a \notin S \) and, since we are in the totally positive case, we know that \( P_{S \cup a} \), \( P_{(S \backslash \{b\}) \cup a} \) and \( P_{S \cup c} \) are non-zero. In general, this may not be true. If it so happens that \( P_{S \cup a} \neq 0 \) for some \( a \notin S \) with \( a < b \), then the proof goes through as presented there. Otherwise, we need a bit more caution. We continue with the same notation.

Suppose \( P_{S \cup a} = 0 \) for all \( a \notin S \) with \( a < b \), but there exists such an \( a \) satisfying \( P_{(S \backslash \{b\}) \cup a} \neq 0 \). Since the non-zero Plücker coordinates of \( F \) form a complete flag matroid, there must be some \( d \) such that \( P_{S \cup d} \neq 0 \) and by hypothesis \( d > b \). We again have a three-term incidence Plücker relation that says \( P_{S \cup d} = P_{(S \backslash \{b\}) \cup d} P_{S \cup a} + P_{(S \backslash \{b\}) \cup i} P_{S \cup a} \). Now the first term on the right is 0 but the second term is not. Moreover, as earlier, we can observe that \( (S \backslash b) \cup ad \) and \( S \cup d \) are larger than \( S \) whereas \( t_{v,w} ((S \backslash b) \cup a) \leq t_{v,w} (S) \) and \( (S \backslash b) \cup a \) is smaller than \( S \) in the Gale order. Thus, we are done by induction.

Finally, we must address the case where \( P_{S \cup a} = P_{(S \backslash \{b\}) \cup a} = 0 \) for all \( a \notin S \) with \( a < b \). In such a case, we look at any path collection \( C \) in the graph \( G_{v,w} \) which attains the sink set \( S \) and note that the path terminating at \( b \) in \( C \) must be diagonal. If it were not, one could try to replace the path \( p_1 \) terminating at \( b \) with a diagonal path terminating at \( a_1 \). If this were possible, we would have \( P_{(S \backslash \{b\}) \cup a_1} \neq 0 \) with \( a_1 < b \). So it must be the case that a path \( p_2 \), which originates below strand \( a_1 \), uses some part of the diagonal strand \( a_1 \). Denote by \( T \) the point where \( p_2 \) first hits strand \( a_1 \). We could try to replace \( p_2 \) by a diagonal path terminating at \( a_2 < b \) and to replace \( p_1 \) by a path that is diagonal until \( T \) and then follows the end of the path \( p_2 \). Again, if this were possible, we would have that \( P_{S \backslash v.a_2} \neq 0 \) and this would contradict our assumption. It follows that there must be a path \( p_1 \) originating below strand \( a_2 \) which uses strand \( a_2 \). This logic continues indefinitely but there are only finitely many vertices in \( G_{v,w} \). Thus, the path terminating at \( b \) in \( C \) must have been diagonal. Now, consider \( \Xi_{v,w}^{k-1} (S) \). By assumption, it is not extremal. Let \( C' \) be any
path collection with sink set $\Xi_{v,w}^{k-1}(s)$. By the minimality of $b$, we have $b \in \Xi_{v,w}^{k-1}(S)$. Every path originating above strand $b$ has been made left extreme in $C'$, and the path originating on strand $b$ in $C'$ is diagonal. By our characterization of paths whose sink sets are extremal indices in Lemma 5.14, there must be some path originating on a strand below strand $b$ which is not diagonal in $C'$. Let strand $x$ be the strand farthest below strand $b$ with this property. It is not hard to see that the path $p$ originating on strand $x$ has the same property in $C$, which is to say it is the path with the lowest origin which is not diagonal. Say $p$ terminates at $y$. Let $c$ be as before. This set up is illustrated in Fig. 20. Note that in this notation, $b, y \in S$ and $c, x \notin S$. Also, by construction, it must be true that $P(S \setminus y) \cup x \neq 0$, since any path originating below strand $x$ is diagonal and thus cannot block $p$ from being replaced by the diagonal path on strand $x$. We must have $c > b > x$ and also $y > x$. We consider a number of cases depending on how $y$ relates to $b$ and $c$.

1. If $c > b > y > x$, then we have the following Plücker relation:

$$P(S \setminus b) \cup S = P(S \setminus x) \cup c + P(S \setminus y) \cup x.$$ 

The left side is non-zero and since $x < b$, $P(S \setminus b) \cup x = 0$. We have $t_{v,w}((S \setminus b) \cup c) < t_{v,w}(S)$. Also, we have $t_{v,w}((S \setminus y) \cup x) \leq t_{v,w}(S)$ since, by the minimality of $x$, it must be true that $x \in e_{v,w}((S \setminus y) \cup x)$. In this case we can have equality if $y \in e_{v,w}(S)$. Since $(S \setminus y) \cup x$ is less than $S$ in the Gale order, this term is already determined. Similarly, $t_{v,w}((S \setminus b) \cup cx) < t_{v,w}(S)$. Thus, by induction, $P_S$ is determined by a monomial relation coming from a three-term incidence Plücker relation.

2. If $c > y > b > x$, then we have the following Plücker relation:

$$P(S \setminus y) \cup S = P(S \setminus b) \cup c + P(S \setminus x) \cup y.$$ 

However, we know that $P(S \setminus b) \cup x = 0$ whereas $P(S \setminus y) \cup x P(S \setminus b) \cup c \neq 0$. Thus, this case can not actually occur.

Figure 20: A sample illustration of the set up in the proof of the final case of Theorem 5.19 when $n = 8$, with the path collection $C'$ marked using dashed lines. Note that from this set up we can see that if the diagonal path on the strand terminating at $x$ were in $C$, then that would mean $x$ is in the sink set of $C$ but not of $C'$, which is impossible since $x < b$. 

1. If $c > b > y > x$, then we have the following Plücker relation:

$$P(S \setminus b) \cup x = P(S \setminus y) \cup c + P(S \setminus y) \cup x.$$ 

The left side is non-zero and since $x < b$, $P(S \setminus b) \cup x = 0$. We have $t_{v,w}((S \setminus b) \cup c) < t_{v,w}(S)$. Also, we have $t_{v,w}((S \setminus y) \cup x) \leq t_{v,w}(S)$ since, by the minimality of $x$, it must be true that $x \in e_{v,w}((S \setminus y) \cup x)$. In this case we can have equality if $y \in e_{v,w}(S)$. Since $(S \setminus y) \cup x$ is less than $S$ in the Gale order, this term is already determined. Similarly, $t_{v,w}((S \setminus b) \cup cx) < t_{v,w}(S)$. Thus, by induction, $P_S$ is determined by a monomial relation coming from a three-term incidence Plücker relation.

2. If $c > y > b > x$, then we have the following Plücker relation:

$$P(S \setminus y) \cup x = P(S \setminus b) \cup c + P(S \setminus x) \cup y.$$ 

However, we know that $P(S \setminus b) \cup x = 0$ whereas $P(S \setminus y) \cup x P(S \setminus b) \cup c \neq 0$. Thus, this case can not actually occur.
3. If \( y > c > b > x \), then we have the following Plücker relation:

\[
P_S P_{(S \setminus \{y\}) \cup x} = P_{(S \setminus \{y\}) \cup x} P_{(S \setminus \{y\}) \cup x} + P_{(S \setminus \{y\}) \cup x} P_{(S \setminus \{y\}) \cup x}.
\]

We conclude exactly as in the first case (note that the non-zero expressions just swapped sides of the equation).

\[\square\]

**Corollary 5.20.** For any flag \( F \) in \( \text{Fl}_{n}^{\geq 0} \), the extremal Plücker coordinates serve as a positivity test for the non-zero Plücker coordinates. Explicitly, if the extremal non-zero Plücker coordinates of \( F \) are all the other non-zero Plücker coordinates of \( F \), then so are all the other non-zero Plücker coordinates of \( F \).

**Proof.** One must simply observe that in the previous proof, \( P_S \) always appears in one of two situations: (1) On the monomial side of a three-term incidence Plücker relation, when all terms are written with a positive sign or (2) in an expression equating two monomials.

\[\square\]

**Theorem 5.19**. For any point \( P \) in \( \text{Fl}_{v,w}^{\geq 0} \) for some \( v \leq w \), the extremal non-infinite Plücker coordinates of \( F \) uniquely determine the other non-infinite Plücker coordinates of \( F \).

**Proof.** Observe that when an unknown \( P_S \) appears in an equation as described in the proof of Corollary 5.20, it is uniquely determined by the other coordinates appearing in the positive tropicalization of that equation. Moreover, that proof only uses three term incidence Plücker relations, the positive tropicalizations of which hold in \( \text{Fl}_{v,w}^{\geq 0} \).

\[\square\]

Using the fact that the extremal non-zero (non-infinite) Plücker coordinates determine all other Plücker coordinates in the TNN complete flag variety, we revisit the two notions of flag positroid we introduced earlier, namely, realizable and synthetic.

We first establish a few pieces of terminology and notation. Note that the support of a flag in \( \text{Fl}_{[n]} \) is a realizable flag positroid, which we denote by \( \mathcal{M}_{v,w}^{v,w} = (\mathcal{M}_{1}^{v,w}, \ldots, \mathcal{M}_{n}^{v,w}) \). A collection of subsets of the base set \( E = [n] \) of the form \( \{R_1 \subseteq R_2 \subseteq \cdots \subseteq R_n = [n]\} \) will be called a flag of subsets of \([n]\).

**Lemma 5.21.** The set of synthetic flag positroids on \([n]\) equals the set of realizable flag positroids on \([n]\).

**Proof.** We have established in Lemma 3.9 that if \( F \in \text{Fl}_{n}^{\geq 0} \) then \( P_I(F) \geq 0 \) for all \( I \subseteq [n] \), so it is clear that all realizable flag positroids are synthetic flag positroids.

Let \( F \) be a flag such that \( P_I(F) \geq 0 \) for all \( I \subseteq [n] \). Let \( \mathcal{M} = (\mathcal{M}_1, \ldots, \mathcal{M}_n) \) be the synthetic flag positroid given by the indices of the non-zero coordinates of \( P \). For any flag matroid, the Gale minimal bases of each size form a flag of subsets in \([n]\) and similarly for the Gale maximal bases [6]. Consider the two flags of subsets \( F_I = \{ i_1 \} \subset \{ i_1, i_2 \} \subset \cdots \subset \{ i_1, \ldots, i_n \} = [n] \) and \( F_J = \{ j_1 \} \subset \{ j_1, j_2 \} \subset \cdots \subset \{ j_1, \ldots, j_n \} = [n] \) corresponding to the Gale minimal and Gale maximal bases of \( \mathcal{M} \), respectively. Let \( v^{-1} = i_1 i_2 \cdots i_n \) and \( w^{-1} = j_1 \cdots j_n \). We will show that \( \mathcal{M} = \mathcal{M}_{v^{-1}, w^{-1}}^{v^{-1}, w^{-1}} \).

As a first step, we will show that the extremal indices of \( P \) coincide with the extremal indices of a point \( Q = P(F') \) for a flag \( F' \in \mathcal{R}_{v,w}^{\geq 0} \). Since the extremal non-zero coordinates determine all the rest by three term incidence Plücker relations, we will then be able to conclude that \( \mathcal{M} = \mathcal{M}_{v,w}^{v,w} \) and so \( \mathcal{M} \) is a realizable flag positroid.

In a complete flag matroid, by definition, each basis lies in a flag of subsets of \([n]\) consisting of bases of the constituent matroids. Thus, if we are to have \( P_K \neq 0 \) for some index \( K \), we must (at least) have that \( K = \{k_1, \ldots, k_m\} \) lies in some flag of subsets \( F = \{ \{k_1 \} \subset \{k_1, k_2 \} \subset \cdots \subset [n]\} \) satisfying \( i_1, \ldots, i_t \leq k_1, \ldots, k_t \leq j_1, \ldots, j_t \) for each \( t \in [n] \). By Lemma 5.2, each such Plücker coordinate is non-zero for \( Q \).

Thus, any basis of \( \mathcal{M} \) is guaranteed to be a basis of \( \mathcal{M}_{v,w}^{v,w} \). We are left to show that for any extremal index \( L \) of \( Q \), we have \( P_L \neq 0 \). For indices of size 1, the extremal indices are simply the Gale minimal and Gale maximal (equivalently, the minimal and maximal) indices of non-zero Plücker coordinates. By hypothesis, these are \( v_1 \) and \( w_1 \), respectively, for both \( P \) and \( Q \). Pick \( l > 1 \) minimal such that the extremal indices of size \( l \) are different for \( P \) and \( Q \). Let \( L_1 \) be the Gale minimal extremal index of \( Q \) of size \( l \) such that \( P_{L_1} = 0 \).

Note that is is clear from our graphical description of extremal indices that each extremal index of \( Q \) lies in a flag of extremal indices. Since \( L_1 \) is an extremal index of \( Q \), \( L_1 \setminus d \) is extremal for some \( d \in L_1 \). By
the minimality of \( l \), we must have that \( L_1 \setminus d \) is extremal for \( P \) as well and in particular, \( P_{L_1 \setminus d} \neq 0 \). Note that since, for each \( t \in [n] \) the Gale minimal indices of size \( t \) of non-zero Plücker coordinates of \( P \) and of \( Q \) coincide by construction, \( L_1 \) cannot be the Gale minimal extremal index of size \( l \). Thus, we know that \( L_1 \) is of the form \( \bigcup_{v \leq w} (L_2 \setminus b) \cup c \) for some \( c > b \) and some other extremal index \( L_2 \) which is Gale less than \( L_1 \). In particular, \( P_{L_2} \neq 0 \). For any \( a < b \) in \( L_1 \), we have the three term incidence Plücker relation \( P_{L_1} P_{(L_1 \setminus c) \cup b} = P_{(L_1 \setminus c) \cup b} P_{L_1 \setminus c} + P_{L_2} P_{L_1 \setminus a} \). Since \( P_{L_1} = 0 \) and \( P_{L_2} \neq 0 \), we must have \( P_{L_1 \setminus a} = 0 \) for any \( a < b \). Thus, \( d \) must satisfy \( d > b \).

Note that by Lemma 5.5 applied to \( Q \), \( c \) must be in the Gale maximal extremal index \( M = \{ j_1, \cdots j_k \} \) of size \( k \) of \( Q \). Again, by construction, for each \( t \in [n] \) the Gale maximal indices of size \( t \) of non-zero Plücker coordinates of \( P \) and of \( Q \) coincide. Thus, \( P_M \neq 0 \). By dual basis exchange between \( M \) and \( L_2 \) in the matroid \( M_L \) of size \( l \) bases of \( P \), there must be some \( b' \in L_2 \setminus M \) such that \( P_{(L_2 \setminus b') \cup c} \neq 0 \). Clearly, \( b' \neq b \). Moreover, we cannot have \( b' > b \) since this would contradict the fact that \( \Xi_{v,w} (L_2) = (L_2 \setminus b) \cup c \). Thus, \( b' < b \). We define \( L_3 = (L_1 \setminus b') \cup b \) and consider the incidence Plücker relation \( P_{L_1} P_{L_3 \setminus d} = P_{L_3} P_{L_1 \setminus d} + P_{L_1 \setminus d} P_{L_1 \setminus b'} \). We know that \( P_{L_1} = 0 \) but \( P_{L_3} \neq 0 \) and \( P_{L_1 \setminus d} \neq 0 \), which leads to a contradiction since all \( P_I \) are nonnegative.

Note that the previous theorem says that the support of any TNN complete flag is a realizable flag positroid. The following result will analogously say that the support of any point in the TNN complete flag Dressian is a realizable flag positroid.

**Lemma 5.21** \( \text{trop} \). The TNN complete flag Dressian decomposes as \( \text{FID}^{>0}_n = \bigcup_{v \leq w} \text{FID}^{>0}_{v,w} \), where the disjoint union is over all pairs \( v \leq w \in S_n \).

**Proof.** By [8, Section 4.2], the support of any point in \( \text{FID}^0_n \) is a flag matroid. With this in mind, the previous proof can be modified as follows: \( P \) should lie in \( \text{FID}^{>0}_n \), \( Q \) should lie in \( \text{FID}^{>0}_{v,w} \), and all equations should be positively tropicalized. Any time we use incidence Plücker relations to determine a previously unknown coordinate, we do so in such a way that the analogous tropical coordinate is also uniquely determined by the corresponding positive tropical incidence Plücker relation.

### 5.3 Characterizing the TNN Complete Flag Variety and Dressian

Our next goal is to show that any flag in \( F_n \) with Plücker coordinates which are all nonnegative lies in \( \text{FID}^{>0}_n \). As was the case for the top cell in Proposition 4.9, the main idea will be to work with the map \( \Psi_{v,w} \) which is inverse to \( \Phi_{v,w} \). In this more general setting, we will have to exercise a bit more caution because our extremal non-zero Plücker coordinate are not in general algebraically independent.

Recall the order \( \prec \) on \( 2^n \) defined by \( I \prec J \) if \( |I| > |J| \) or if \( |I| = |J| \) and \( I \) is lexicographically smaller than \( J \). We inductively define a subset \( S_{v,w} \) of the extremal non-zero indices of a flag in \( \text{FID}^{>0}_v \). Start with \( S_{v,w} = \emptyset \). Then, going through the extremal non-zero Plücker coordinates according to the total order \( \prec \), add \( I \) to \( S_{v,w} \) if \( P_I \) is algebraically independent of the Plücker coordinates \( P_J \) with \( J \prec I \).

We first claim that the Plücker coordinates with index in \( S_{v,w} \) determine the other extremal non-zero Plücker coordinates by three-term incidence Plücker relations. These are algebraically independent by definition and so, we will be able to prove that any flag with nonnegative coordinates is in fact a TNN flag in a way that is very similar to Theorem 4.10. Note that it is clear by construction that the coordinates whose indices lie in \( S_{v,w} \) determine the other extremal non-zero Plücker coordinates. What will be important here (for the tropical version of our results) is that we can deduce the other extremal non-zero Plücker coordinates using just the three-term incidence Plücker relations.

**Lemma 5.22.** For any flag \( F \in \text{FID}^{>0}_v \), any extremal non-zero Plücker coordinate of \( F \) whose index is not in \( S_{v,w} \) is determined from the coordinates whose indices are in \( S_{v,w} \) by three-term incidence Plücker relations.

**Proof.** Let \( P = P(F) \). Suppose \( I \) with \( |I| = t \) is the index of an extremal non-zero Plücker coordinate of \( F \) which is not in \( S_{v,w} \). It suffices by induction to show that \( P_I \) can be expressed, via three-term incidence Plücker relations, in terms of extremal non-zero Plücker coordinates \( P_J \) with \( J \prec I \).

By Lemma 5.14, we can let \( C \) be the unique path collection in \( G_{v,w} \) with sink set \( I \). Let \( K \) be the set of vertices such that the paths in \( C \) originating from \( K \) are not diagonal (which is to say, the topmost \( k \) sources of \( C \) for some \( k \)). Again by Lemma 5.14, the paths originating from \( K \) form a left extreme path.
collection. We may assume \( I \) is not Gale minimal since if it were, \( P_I = 1 \) and so \( P_I \) is determined without any equations. Suppose the bottom-most vertex of \( K \) lies on strand \( a \), and the path originating on strand \( a \) in \( C \) terminates at \( b > a \). Then, \( I' = (I \setminus b) \cup a \) is also an extremal index and \( I = \Xi_{v,w}(I') \).

If the vertices in \( K \) all lie above vertex \((t+1)\)' then consider the path collection \( C' \) with source set \([t+1]'\) such that the path collection originating from \( K \) is a left extreme path collection and the path collection originating from the remaining vertices is diagonal. The sink set will be the extremal index \( I \cup f \), where vertex \((t+1)\)' lies on strand \( f \). In this case, \( f < a < b \). Note that \( I' \cup f \) is an extremal index as well. Thus, we have the three-term incidence Plücker relation \( P_{I'}P_{I \cup f} = P_I P_{I' \cup f} + P_{(I' \setminus a) \cup f} P_{I \cup b} \). Note that, \( P_{(I' \setminus a) \cup f} = 0 \) since in the unique path collection with sink set \( I' \), the path with sink \( a \) must be diagonal and so it cannot be replaced by a path with sink \( f < a \). Since \( I' \) is lexicographically less than \( I \), we have determined \( P_I \) according to the desired condition and in fact, we have determined it by a monomial relation. This case is illustrated in Fig. 21.

**Figure 21:** Illustration of the case where \((t+1)\)' lies below \( K \) in the case \( v = 34125, w = w_0 \) and \( t = 2 \). The dashed path collection is the unique path collection whose sink set is the extremal index \( I = \{4, 5\} \). In this case, the bottom-most vertex in \( K \) is \( 1 \)' lying on strand \( a = 3 \) and the path originating at \( 1 \)' terminates at \( b = 4 \). The dotted path shows the path originating at \( 3 \)' and terminating at \( f = 1 \). Note that \( I \cup f \), and \( I' \cup f \) are both easily seen to be extremal indices by our characterization of extremal path collections as unions of left extreme and diagonal path collections.

Otherwise, \( K \) contains vertices below vertex \((t+1)\)' . Consider the path collection \( C' \) of size \( t+1 \) where the paths originating at \( K \cup (t+1) \) are left greedy and the rest are diagonal. The sink set of this path collection is an extremal index \( L \) of the form \( I \cup d \) for some \( d \). Observe that since \((t+1)\)' lies above strand \( a \), we must have \( d > a \).

We first consider the case where \( d > b > a \). In this case, we have a three-term incidence Plücker relation which says that \( P_I P_{I \cup d} = P_I P_{I \cup d} + P_{(I \setminus b) \cup d} P_{I \cup a} \). Observe that since \( d > b \) and the paths originating from \( K \) are already left extreme in \( C \), \( P_{(I \setminus b) \cup d} = 0 \). Also, \( I \cup d \) and \( I' \cup d \) can easily be seen to be extremal indices. Since \( I' \) is lexicographically less than \( I \), we have determined \( P_I \) according to the desired condition and in fact, we have determined it by a monomial relation. This situation is illustrated in Fig. 22.
We claim that which are in \( S \), the path originating at \( b = 3 \). The dotted path shows the path originating at \( b' \) and terminating at \( d = 4 > b \). Note that \( I \cup d, (I \setminus b) \cup a \) and \( (I \setminus b) \cup ad \) are all easily seen to be extremal indices by our characterization of extremal path collections as unions of left extreme and diagonal path collections.

Next, we consider the case where \( b > d > a \). Let \( C' \) denote the unique path collection originating from \( [t + 1]' \) and with sink set \( I \cup d \). Denote by \( v' \) the bottom-most vertex of \( K \). We consider two subcases. If the path originating at \( v' \) in \( C \) is identical to the path originating at \( v' \) in \( C' \), then we study the three-term relation \( P_{(v \setminus b) \cup d} P_{t \cup a} = P_{t \cup d} P_{t} + P_{t} P_{t \cup ad} \). However, we note that \( P_{t \cup a} = 0 \). To see this, observe that since all the paths originating below strand \( a \) are already diagonal in \( C \), then in a path collection \( C' \) with sink set \( I \cup a \), the path originating on strand \( a \) must be diagonal and so the same must be true in \( C \). This contradicts the assumption that \( v' \in K \). By assumption, \( P_{t} P_{t \cup ad} \neq 0 \), so this case can never occur.

Finally, we consider the case where the path \( p \) originating at \( v' \) in \( C \) differs from the path \( p' \) originating at \( v' \) in \( C' \). It must be the case that \( p \) takes a left turn on a vertical edge \( e \) which \( p' \) is blocked from using. We claim that \( e \) cannot be used by any graph extremal path collection consisting of at least \( t + 1 \) paths. To show this, observe that if \( L_{t} \) is the left greedy path collection with source set \( [v]' \), then any vertical edge used by an extremal path collection consisting of \( r \) paths is used by \( L_{t} \). Thus, \( e \) is used in \( L_{t} \) but not in \( L_{t+1} \). Now, let \( t_{3} > t_{2} > t_{1} \) and observe that \( L_{t_{1}} \) and \( L_{t_{2}} \) are constructed from \( L_{t_{3}} \) by removing certain paths and adjusting as necessary to make the remaining paths left greedy. From this description, it is clear that any edge used in both \( L_{t_{1}} \) and \( L_{t_{2}} \) must also be used in \( L_{t_{3}} \). We conclude that \( e \) is not used in \( L_{t} \) for \( t' > t + 1 \) and consequently, \( e \) is not used in any extremal path collection consisting of at least \( t + 1 \) paths.

Further, consider the unique path collection \( C' \) with sink set \( I \) for some \( I \) which is of size \( t \) and is Gale less than \( I \). The path collection \( C' \) is obtained as the union of a diagonal path collection and the left extreme path collection originating from the top \( v \) vertices of \( K \) for some \( v < |K| \). Since it is the path originating from the bottom-most vertex \( v' \) of \( K \) that \( e \) does not use the vertical edge \( e \). This means that \( C \) uses an edge that was not used by any path collections with sink set \( J \prec I \) and thus the extremal non-zero Plücker coordinate corresponding to this path collection must be algebraically independent of those earlier than it in \( \prec \) order. This contradicts the fact that \( I \notin S_{v,w} \).

Thus, we have determined any extremal non-zero Plücker coordinate which is not in \( S_{v,w} \) from those which are in \( S_{v,w} \) by three-term incidence Plücker relations.

**Lemma 5.22**

**Lemma 5.22**

For any point \( P \in \text{FID}_{v,w}^{\geq 0} \), any extremal non-infinite Plücker coordinate of \( P \) whose index is not in \( S_{v,w} \) is determined from the coordinates whose indices are in \( S_{v,w} \) by three-term incidence Plücker relations.

**Proof.** Argue exactly as in the proof of Lemma 5.22, observing that any time we use incidence Plücker relations to determine a previously unknown coordinate, we do so in such a way that the analogous tropical coordinate is also uniquely determined by the corresponding positive tropical incidence Plücker relation. \( \Box \)
Proposition 5.23. For any \( v \leq w \), the map \( \Psi_{v,w} \) consists of Laurent monomials in the Plücker coordinates in \( S_{v,w} \).

Proof. Argue exactly as in the proof of Proposition 4.9, with the reference to Proposition 4.5 replaced by a reference to Lemma 5.14.

We provide an example of the algorithm which Proposition 5.23 describes.

Example 5.24. We recall the graph \( G_{v,w} = G_{1324,4213} \), which we have used as a running example, in Fig. 23. We also recall the extremal indices in the cell \( R_{1324,4213}^0 \) are \{1\}, \{3\}, \{1,3\}, \{2,3\}, \{1,2,3\}, \{1,3,4\} and \{2,3,4\}.

![Figure 23: The graph \( G_{1324,3241} \).](image)

We start with the Gale minimal extremal index of size 3, which is \{1,2,3\}. The unique path collection with source set \( \{3\}' \) and sink set \( \{1,2,3\} \) has weight 1. The Gale-next extremal index of size 3 is \{1,3,4\}. The unique path collection with source set \( \{3\}' \) and sink set \( \{1,3,4\} \) has weight \( a_2 \), so \( a_3 = P_{23} \). Next, we look at \{2,3,4\}. The unique path collection with source set \( \{3\}' \) and sink set \( \{2,3,4\} \) has weight \( a_2 a_4 \), so \( a_4 = P_{34}/P_{23} \). Now we move on to the Gale minimal extremal indices of size 2, which is \{1,3\}. The corresponding path collection has weight 1. The Gale-next extremal index of size 2 is \{2,3\}. The unique path collection with source set \( \{2\}' \) and sink set \( \{2,3\} \) has weight \( a_3 \). We have already determined \( a_3 \) as a Laurent monomial in the extremal non-zero Plücker coordinates, so we have nothing further to do here. However, we know that the relation \( P_{23} = a_4 = P_{34}/P_{23} \) follows using only three-term incidence Plücker relations. Indeed, \( P_{23} P_{43} = P_{13} P_{234} + P_{34} P_{123} \), where \( P_{13} = 1 \) and \( P_{34} = 0 \). Finally, we move to the minimal extremal index of size 1, which is just \{1\}. The corresponding path collection has weight 1. The Gale-next extremal index of size 1 is \{3\}. The unique path collection with source set \( \{1\}' \) and sink set \( \{3\} \) has weight \( a_1 \), so \( a_1 = P_{3} \). Thus, we have determined all the weights as Laurent monomials in the extremal non-zero Plücker coordinate.

Theorem 5.25. The totally nonnegative complete flag variety \( Fl_{n}^{\geq 0} \) equals the set \( \{ F \in Fl_{n} | P_I(F) \geq 0 \forall I \subseteq [n] \} \).

Proof. We already established in Lemma 3.9 that for any \( F \) in \( Fl_{n}^{\geq 0} \), we have \( P_I(F) \geq 0 \) for any \( I \subseteq [n] \). We are left to prove the reverse direction.

Let \( F \) be any flag in \( Fl_{n} \) such that \( P_I(F) \geq 0 \) for any \( I \subseteq [n] \). Our goal is to show that \( F \) has a representative in one of the Marsh-Rietsch cells.

By Lemma 5.21, the flag matroid corresponding to the non-zero Plücker coordinates of \( P \) is \( M^{v,w} \) for some \( v \leq w \). We prove that \( F \) has a representative in \( R_{v,w}^{\geq 0} \). To do so, we show that there exist some weights \( \alpha \) on \( G_{v,w} \) such that the matrix obtained from \( G_{v,w}(\alpha) \) through the LGV construction represents \( F \). Using Proposition 5.23, apply the map \( \Psi_{v,w} \) to the subset \( S_{v,w} \) of the extremal non-zero Plücker coordinates of \( F \) to get a collection of weights \( \alpha \). Let \( Q = \Phi_{v,w}(\alpha) \). This means \( Q = P(F') \) for some \( F' \in R_{v,w}^{\geq 0} \). By construction, \( P(F) \) and \( Q \) have the same support and also agree on \( S_{v,w} \). By Lemma 5.22 and Theorem 5.19, the three-term incidence Plücker relations determine all the other Plücker coordinates in terms of those coordinates in \( S_{v,w} \) and so \( P(F) \) and \( Q \) agree on all Plücker coordinates. Thus, \( F \) lies in \( R_{v,w}^{\geq 0} \). \( \square \)
Theorem 5.25\textsuperscript{trop}. The totally nonnegative tropical complete flag variety $\text{TrFl}^{\geq 0}_n$ equals the totally nonnegative complete flag Dressian $\text{FlDr}^{\geq 0}_n$.

Proof. It is clear by definition that $\text{TrFl}^{\geq 0}_n \subseteq \text{FlDr}^{\geq 0}_n$. We are left to prove the reverse inclusion. Let $P \in \text{FlDr}^{\geq 0}_n$. By Lemma 5.21\textsuperscript{trop}, $P \in \text{FlDr}^{\geq 0}_{v,w}$ for some $v \leq w \in S_n$. Using Proposition 5.23, apply the map $\text{Trop} \Psi_{v,w}$ to the subset $S_{v,w}$ of the extremal non-infinite Plücker coordinates of $P$ to get a collection of real numbers $a$. Consider $Q = \text{Trop} \Phi_{v,w}(a) \in \text{TrFl}^{\geq 0}_n$. By construction, $P$ and $Q$ have the same support. Let $\pi_{v,w}$ be the projection to those Plücker coordinates whose indices lie in $S_{v,w}$. Observe that $\pi_{v,w} \circ \Phi_{v,w}$ and $\Psi_{v,w}$ are inverse maps and both consist of Laurent monomials. Thus, it is clear that $\text{Trop} (\pi_{v,w} \circ \Phi_{v,w})$ and $\text{Trop} \Psi_{v,w}$ are inverse maps each consisting of sums and differences (and not minimizations). Thus, $P$ and $Q$ agree on those Plücker coordinates whose indices lie in $S_{v,w}$. By Lemma 5.22\textsuperscript{trop} and Theorem 5.19\textsuperscript{trop}, the positive tropicalizations of the three-term incidence Plücker relations determine all the other Plücker coordinates in terms of those coordinates in $S_{v,w}$. Since both $\text{FlDr}^{\geq 0}_n$ and $\text{TrFl}^{\geq 0}_n$ satisfy the positive tropicalizations of the three-term incidence Plücker relations, $P$ and $Q$ agree on all Plücker coordinates. Thus, $P$ lies in $\text{TrFl}^{\geq 0}_n$. \hfill \Box

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