FI-⊕-J-supplemented modules

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Abstract
A Module M is called cofinite J-Supplemented Module if for every cofinite submodule L of M, there exists a submodule N of M such that M = L + N with \( N \cap L \ll J \). Initially, we gave the main properties of cof-J-supplemented modules. An R-module M is called fully invariant-J-supplemented if for every fully invariant submodule N of M, there exists a submodule K of M, such that M = N + K with N \( \ll K \). A condition under which the direct sum of FI-J-supplemented modules is FI-J-supplemented was given. Also, some types of modules that are related to the FI-J-supplemented module were discussed.

Keywords: cofinitely J-supplemented modules , fully invariant J-supplemented modules , fully invariant ⊕-J-supplemented modules.

1. Introduction
Throughout this paper, an arbitrary associative ring with identity is denoted by \( R \) and all modules are unitary left \( R \)-modules. Assume that C and D are submodules of M, a submodule C is called small submodule of M (\( C \ll M \)), if whenever \( M = C + D \), we have \( M = D \) [1]. A submodule \( C \) of a module M is called J-small submodule of M (\( C \ll J \)) if whenever \( M = C + D \), \( J(\frac{M}{D}) = \frac{M}{D} \) implies \( M = D \), were \( J(M) \) denotes the Jacobson radical of M [2]. A submodule \( C \) is a supplement of \( D \) in \( M \) if \( C \) is minimal with respect to \( M = C + D \). Equivalently, \( M = C + D \) with \( C \cap D \ll C \) [3]. A module M is called supplemented module if every submodule of M has a supplement in M [4]. A Submodule \( C \) is called J-supplement of \( D \) in M if \( M = C + D \) and \( \cap D \ll C \). M is called J-supplemented if every submodule of M has J-supplement in M [2]. A module M is called ⊕-supplemented module if every submodule of M has a direct summand supplement in M [5]. A Submodule C is called a ⊕-Jacobson-supplement of D in M (for short ⊕-J-supplement ) if \( M = C + D \), and C is a direct summand of \( M \)

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with $C \cap D \ll_J C$. It is called a $\oplus$-J-supplemented if every submodule of $M$ has a $\oplus$-J-supplement in $M$ [6]. A submodule $C$ of a module $M$ is called cofinite submodule of $M$ if $\frac{M}{C}$ is finitely generated. A module $M$ is called cofinitely supplemented if every cofinite submodule of $M$ has supplement submodule [7]. As a generalization of cofinitely supplemented, we define the cofinitely J-supplemented (for short cof-J-supplemented) as follows. A module $M$ is called cof-J-supplemented if for every cofinite submodule $C$ of $M$, there exists a submodule $D$ of $M$ such that $M = C + D$ and $C \cap D \ll_J D$. A submodule $C$ of a module $M$ is called a fully invariant submodule if $f(C) \subseteq C$ for every $f \in \text{End}_R(M)$ [8].

In section 2, we prove some properties of cof-J-supplemented and we show that any factor module of cof-J-supplemented module is cof-J-supplemented and any finite sum of cof-J-supplemented is cof-J-supplemented.

In section 3, we introduce the concept of fully invariant J-supplemented modules (for short FI-J-supplemented) as a generalization of J-supplemented, as follows. The module $M$ is said to be FI-J-supplemented if for every fully invariant submodule $C$ of $M$, there exists a submodule $D$ of $M$ such that $M = C + D$ and $C \cap D \ll_J D$. Clearly, the supplemented modules are J-supplemented and the J-supplemented modules are FI-J-supplemented. As a generalization of a $\oplus$-J-supplemented module, we introduce the concept of fully invariant $\oplus$-J-supplemented modules (FI-$\oplus$-J-supplemented). A module $M$ is called FI-$\oplus$-J-supplemented if for every fully invariant submodule $C$ of $M$, there exists a direct summand $D$ of $M$ such that $M = C + D$ and $C \cap D \ll_J D$. Clearly, FI-$\oplus$-J-supplemented modules are FI-J-supplemented.

2. Cofinitely J-supplemented modules

This section is devoted to introduce the cofinitely J-supplemented modules as a generalization of J-supplemented modules, and illustrate this concept by remarks and properties.

Definition(2.1): A module $M$ is called cofinitely J-supplemented module (for short cof-J-supplemented) if for every cofinite submodule $L$ of $M$, there exists a submodule $N$ of $M$ such that $M = L + N$ and $N \cap L \ll_J N$.

Remark(2.2): It is clear that every J-supplemented module is cof-J-supplemented. The converse in general is not true. For instance, $Q$ as $Z$-module is cof-J-supplemented module, but $Q$ is not J-supplemented.

Proposition(2.3): Let $M$ be a finitely generated $R$-module. Then $M$ is J-supplemented module if and only if $M$ is cof-J-supplemented.

Proof: Let $L$ be a submodule of $M$. Since $M$ is a finitely generated $R$-module, then $\frac{M}{L}$ is finitely generated, hence $L$ is a cofinite submodule of $M$. But $M$ is cof-J-supplemented, therefore $L$ is J-supplemented in $M$. Thus $M$ is J-supplemented module. The converse is clear.

Proposition(2.4): Let $M$ be a cof-J-supplemented module, and let $B$ be a submodule of $M$, then $\frac{M}{B}$ is a cof-J-supplemented.

Proof: Let $B$ be a submodule of $M$ and let $\frac{K}{B}$ be any cofinite submodule of $\frac{M}{B}$, then $\frac{M}{K} \cong \frac{M}{B} \cdot \frac{K}{B}$. Therefore $\frac{M}{K}$ is finitely generated, then $K$ is cofinite submodule of $M$. Since $M$ is a cof-J-supplemented, then there exists a submodule $C$ of $M$ such that $M = K + C$, $K \cap C \ll_J C$. Now, $\frac{M}{B} = \frac{K+C}{B} \cdot \frac{K \cap C}{B} \cdot \frac{K+C}{B} \cong \frac{B}{B} \cdot \frac{K \cap C}{B} \cong \frac{B}{B}$ (by modular law), but $K \cap C \ll_J C$, then $\frac{K}{B} \cap \frac{C}{B} \ll_J \frac{C}{B}$ [2]. Therefore $\frac{M}{B}$ is a cof-J-supplemented.

The converse in general is not true. For example, $Z$ as $Z$-module $\frac{Z}{2Z} \cong Z_3$ is cof-J-supplemented, but $Z$ is not cof-J-supplemented.

Corollary(2.5): The homomorphic image of a cof-J-supplemented module is a cof-J-supplemented module.

Proof: Since every homomorphic image is isomorphic to a quotient module.

Corollary(2.6): The direct summand of a cof-J-supplemented module is cof-J-supplemented.

Proof: Clear.
Proposition (2.7): Let $M = M_1 \oplus M_2$, then $M_1$ and $M_2$ are cof-J-supplemented modules if and only if $M$ is cof-J-supplemented.

Proof: ($\Rightarrow$) Let $L$ be a cofinite submodule of $M$, then $M = L + M_1 + M_2$. Now, $\frac{M_2}{M_2 \cap (L + M_1)} \cong \frac{M_2 + L + M_1}{L + M_1}$, which is finitely generated, hence $M_2 \cap (L + M_1)$ is a cofinite submodule of $M_2$. Since $M_2$ is a cof-J-supplemented, then there exists a submodule $H$ of $M_2$ such that $M_2 = H + [M_2 \cap (L + M_1)]$ with $H \cap (L + M_1) \ll_j H$. We have $M = L + M_1 + M_2 = L + M_1 + M_2 \cap (L + M_1) + H = M_1 + L + H$ and since $M_1 \cap (L + H)$ is a cofinite submodule of $M_1$, then $M_1$ is a cof-J-supplemented, then there exists a submodule $G$ of $M_1$ such that $M_1 = G + [M_1 \cap (L + H)]$ and $G \cap (L + H) \ll_j H$. Then $M = G + M_1 \cap (L + H) + L + H = L + H + G$ and $(H + G) \cap L \leq [H \cap (L + M_2)] + [G \cap (L + H)] \ll_j H + G$. Therefore $M$ is a cof-J-supplemented module.

($\Leftarrow$) by Corollary (2.4).

3. FI-J-supplemented and FI-$\oplus$-J-supplemented modules

In this section, the concept of FI-J-supplemented modules as a generalization of J-supplemented and some properties of this type of modules are given. Also, as a generalization of FI-J-supplemented modules, FI-$\oplus$-J-supplemented modules are introduced.

Definition (3.1): An $R$-module $M$ is called fully invariant-J-supplemented (for short FI-J-supplemented) if for every fully invariant submodule $N$ of $M$, there exists a submodule $K$ of $M$, such that $M = N + K$ and $N \ll_j K$.

Examples (3.2):

(1) Every semi simple is FI-J-supplemented, for example $Z_6$ as $Z$-module is FI-J-supplemented.

(2) $Q$ as $Z$-module is not FI-J-supplemented, by [2, proposition (2.5)].

(3) It is clear that every J-supplemented is FI-J-supplemented.

The following proposition gives a condition under which the J-supplemented and FI-J-supplemented are equivalent.

Proposition (3.3): Let $M$ be a duo module. Then $M$ is J-supplemented if and only if $M$ is FI-J-supplemented.

Proof: Clear.

Proposition (3.4): Let $M$ be FI-J-supplemented module and let $N$ be fully invariant submodule of $M$. Then the factor $M/N$ is FI-J-supplemented.

Proof: Let $M$ be FI-J-supplemented, and let $\frac{B}{N}$ be any fully invariant submodule of $\frac{M}{N}$, then $B$ is a fully invariant submodule in $M$, by [9, Lemma (2.2)]. Since $M$ is FI-J-supplemented module, then there exists a submodule $C$ of $M$ such that $M = C + B$, $C \cap B \ll_j C$ and $\frac{M}{N} = \frac{B + C}{N} = \frac{B}{N} + \frac{C + N}{N}$. Let $\left(\frac{B}{N} \cap \frac{C + N}{N}\right) + \frac{V}{N} = \frac{C + N}{N}$ with $J(\frac{C + N}{V}) = \frac{C + N}{V}$, $\frac{B \cap (C + N)}{N} = \frac{N + (B \cap C)}{N}$ (by modular law), then $\frac{N + (B \cap C)}{V} = \frac{C + N}{V}$ and $N + (B \cap C) + V = C + N$, and $N \subseteq V$, then $(B \cap C) + V = C + N$, and $J(\frac{N}{V}) = \frac{C + N}{V}$. But $B \cap C \ll_j C \subseteq C + N$ and by [2, Proposition (2.6(4))], $B \cap C \ll_j C + N$, thus $V = C + N$.
and \( \frac{V}{N} = \frac{C+N}{N} \). Then \( \frac{C+N}{N} \) is J-supplement of \( \frac{B}{N} \) in \( \frac{M}{N} \), \( \frac{C+N}{N} \) is FI-J-supplement of \( \frac{B}{N} \) in \( \frac{M}{N} \).

Therefore \( \frac{M}{N} \) is FI-J-supplemented.

**Proposition (3.5):** Let \( M_1 \) and \( U \) be fully invariant submodules of \( M \), and let \( M_1 \) be FI-J-supplemented module. If \( M_1 + U \) has FI-J-supplement in \( M \), then so does \( U \).

**Proof:** Since \( M_1 + U \) has FI-J-supplement in \( M \), then there exists a fully invariant \( X \subseteq M \), such that \( X + (M_1 + U) = M \), and \( X \cap (M_1 + U) \prec \prec X \). Since \( M_1 \) is FI-J-supplement module, then there exists \( Y \subseteq M_1 \) such that \( (X + U) \cap M_1 = Y = M_1 \) and \( (X + U) \cap Y \prec Y \). Thus we have \( M = M_1 + U + X = (X + U) \cap M_1 + Y + U + X = X + U + Y \), and \( (X + U) \cap Y \prec Y \), that is \( X \) is FI-J-supplement of \( X + U \) in \( M \). It is clear that \((X + Y) + U = M\), so it suffices to show that \((X + Y) \cap U \prec \prec X + Y \) since \( X + U \subseteq M_1 + U \), then \( X \cap (Y + U) \subseteq X \cap (M_1 + U) \prec \prec X \) and \( X \cap (Y + U) \prec \prec X \).

**Proof:** Let \( M = M_1 \oplus M_2 \), then \( M_1 \) and \( M_2 \) are FI-J-supplemented modules if and only if \( M \) is FI-J-supplemented module.

**Proposition (3.6):** Let \( M = M_1 \oplus M_2 \), then \( M_1 \) and \( M_2 \) are FI-J-supplemented modules if and only if \( M \) is FI-J-supplemented module.

**Proof:** (\( \Rightarrow \)) Let \( K \) be a fully invariant submodule of \( M \), then since \( M_1 + M_2 + K = M \), it trivially has FI-J-supplement in \( M \), by Proposition (3.5), then \( M_2 + K \) and \( K \) have FI-J-supplement in \( M \). Also, by Proposition (3.5) again, \( K \) has FI-J-supplement in \( M \), so \( M \) is FI-J-supplemented module.

(\( \Leftarrow \)) \( M_2 \cong \frac{M}{M_1} \), since \( M \) is FI-J-supplemented module, and by Proposition (3.4), \( \frac{M_2}{M_1} \) is FI-J-supplemented module. Thus \( M_2 \) is FI-J-supplemented module. Similarly, \( M_1 \) is FI-J-supplemented module.

**Corollary(3.7):** Let \( M = \bigoplus_{i=1}^{n} M_i \) be a direct sum of FI-J-supplemented R-modules. Then \( M \) is FI-J-supplemented.

**Proof:** Let \( n \) be any positive integer and let \( M_1 \) be FI-J-supplemented R-module for each \( 1 \leq i \leq n \). Let \( M = M_1 \oplus ... \oplus M_n \). To prove that \( M \) is FI-J-supplemented R-module, it is sufficient by the induction on \( n \) to prove that this is the case when \( n = 2 \). Thus suppose that \( n = 2 \).

Let \( A \) be any fully invariant submodule of \( M \), then \( M_1 + M_2 + A \). Then since \( M_1 \) and \( M_2 \) are FI-J-supplemented, then by proposition(3.5), we have \( M_1 + A \) and \( M_2 + A \) have FI-J-supplement in \( M \), and by proposition(3.5) again, \( A \) has FI-J-supplement in \( M \). Therefore \( M = M_1 \oplus M_2 \) is FI-J-supplemented.

**Corollary (3.8):** Let \( M = M_1 \oplus M_2 \) be a duo module, \( N \) and \( L \) are fully invariant submodules of \( M_1 \), if \( N \) is FI-J-supplement of \( L \) in \( M_1 \), then \( N \oplus M_2 \) is FI-J-supplement of \( L \) in \( M \).

**Proof:** Let \( N \) be FI-J-supplement of \( L \) in \( M_1 \), then \( M_1 = N + L \) and \( N \cap L \prec \prec N \). Since \( M_1 = M_1 \oplus M_2 \), then \( M = (N + L) \oplus M_2 \), hence \( M = L^{\prime} \oplus M_2 \). But \( N \oplus M_2 \cap L = (N \oplus M_2) \cap M_1 \cap L = N \cap L \prec \prec N \), then \( N \oplus M_2 \) is FI-J-supplement of \( L \) in \( M \).

**Proposition (3.9):** Let \( U \) and \( V \) be fully invariant submodules of an R-module \( M \) and let \( V \) be FI-J-supplement of \( U \) in \( M \). If \( K \prec \prec M \), then \( V \) is FI-J-supplement of \( U + K \) in \( M \).

**Proof:** Let \( V \) be FI-J-supplement of \( U \) in \( M \), then \( M = V + U \) and \( V \cap U \prec \prec V \). Let \( V + (U + K) = M \), and let \( V \cap (U + K) = X = V \), with \( J \left( \frac{V}{X} \right) = \frac{V}{X} \) and \( M = V + (U + K) = V \cap (U + K) + X = (U + K) = X + (U + K) = (U + X) + K \). Since \( M = \frac{V}{X} \) and \( X = \frac{V}{U+X} \), then \( \frac{V}{X} \) is FI-J-supplement of \( U + K \) in \( M \).

**Proposition (3.10):** Let \( M \) be any R-module, \( V \) be FI-J-supplement of \( W \) in \( M \), and \( K \) be fully invariant of \( M \) such that \( K \subseteq V \). Then \( K \prec \prec M \) if and only if \( K \prec \prec V \).

**Proof:** (\\( \Rightarrow \)) Let \( K \subseteq V \), with \( J \left( \frac{V}{X} \right) = \frac{V}{X} \). Since \( V + W = M \) and \( V \cap W \prec \prec V \), then \( M = (K + X) + W \). Hence \( M = K + (X + W) \) to show that \( J \left( \frac{M}{X+W} \right) = \frac{M}{X+W} \), since \( \frac{M}{X+W} = \frac{V}{X+W} = \frac{V}{X+W} = \frac{V}{X+W} = \frac{V}{X+W} \) by the second isomorphism and modular law. But \( J \left( \frac{V}{X} \right) = \frac{V}{X} \), then we get
J(\frac{V}{X+(V \cap W)}) = \frac{V}{X+(V \cap W)} \text{ [2, Proposition(2.2)]}. Hence \( J(\frac{M}{X+W}) = \frac{M}{X+W} \). Since \( K \ll T M \) then \( M = X + W \). Now \( M = V + W, X \subseteq V \), and \( J(\frac{V}{X}) = \frac{V}{X} \), then \( V = X \). Hence \( K \ll T V \).

\((\Leftarrow)\) Clearly by [2, Proposition(2.6(4))].

**Proposition (3.11):** Let \( M \) be any \( R \)-module and let \( V \) be FI-J-supplement of \( U \) in \( M \), \( K \) and \( T \) are fully invariant submodules of \( M \) such that \( K \subseteq T \subseteq V \). Then \( T \) is FI-J-supplement of \( K \) in \( V \) if and only if \( T \) is FI-J-supplement of \( U + K \) in \( M \).

**Proof:** \((\Rightarrow)\) Let \( T \) be FI-J-supplement of \( K \) in \( V \), then \( V = T + K \) and \( T \cap K \ll T \). Let \((U + K) + L = M \) for \( L \subseteq T \) with \( J(\frac{T}{L}) = \frac{T}{L} \). Now \( K + L \subseteq V \). Since \( J(\frac{V}{K+L}) = \frac{V}{K+L} \), by the second isomorphism and modular law, \( J(\frac{V}{K+L}) = \frac{V}{K+L} \) [2].

Hence \( J(\frac{V}{K+L}) = \frac{V}{K+L} \) and because \( V \) is FI-J-supplement of \( U \) in \( M \), then \( M = U + V \), and by [2], \( K + L = V \). Since \( L \subseteq T \) and \( T \) is FI-J-supplement of \( K \) in \( V \), then \( T = L \). Hence \( T \) is FI-J-supplement of \( U + K \) in \( M \).

\((\Leftarrow)\) Let \( T \) be FI-J-supplement of \( U + K \) in \( M \). Then \( T = (U + K) \) and \( T \cap (U + K) \ll T \). Let \( T + K = V \). Since \( T \cap K \subseteq T \cap (U + K) \ll T \), then by [2, Proposition(2.6(1))], \( T \cap K \ll T \). Hence \( T \) is FI-J-supplement of \( K \) in \( V \).

Let \( U \), \( V \) be submodules of a module \( M \). We will say that \( U \) and \( V \) are mutual FI-J-supplements, if \( U \) is FI-J-supplement of \( V \) in \( M \) and \( V \) is FI-J-supplement of \( U \) in \( M \).

**Corollary (3.12):** Let \( M \) be any \( R \)-module and let \( U \) and \( V \) be mutual FI-J-supplements in \( M \). Let \( L \) be FI-J-supplement of \( S \) in \( U \) and \( T \) be FI-J-supplement of \( K \) in \( V \), then \( L + T \) is FI-J-supplement of \( K + S \) in \( M \).

**Proof:** Since \( U = S + L \) and \( V = T + K \) is FI-J-supplement of \( U \) in \( M \), then by Proposition(3.11), \( T \) is FI-J-supplement of \( S + L + K \) in \( M \) and then \( (S + L + K) \cap T \ll T \). Since \( V = K + T \) and \( U \) is FI-J-supplement of \( V \) in \( M \), then by Proposition (3.11), \( L \) is J-supplement of \( S + K + T \) in \( M \) and then \( (S + K + T) \cap L \ll L \). Because \( U = S + L \), \( V = K + T \), and \( M = U + V \), then we have \( M = S + L + K + T = S + K + L + T \). Then by [2, Proposition(2.6(2))], \( (S + K) \cap (L + T) \subseteq L \cap (S + K + T) + T \cap (S + K + L) \ll L + T \). And since \( L \) and \( T \) are fully invariant in \( M \), then \( L + T \) is fully invariant in \( M \) [10]. Therefore \( L + T \) is FI-J-supplement of \( K + S \) in \( M \).

**Definition (3.13):** An \( R \)-module \( M \) is called fully invariant \( \oplus \)-J-supplemented (for short FI-\( \oplus \)-J-supplemented) if for every fully invariant submodule \( N \) of \( M \), there exists a direct summand \( K \) of \( M \), such that \( M = N + K \) and \( N \cap K \ll K \).

**Examples (3.14):**

1. It is clear that every FI- \( \oplus \)-J-supplemented is FI- J-supplemented. But the converse in general is not true, for example \( Z \) as \( Z \)-module.
2. \( Z \) as \( Z \)-module is FI- \( \oplus \)-J-supplemented.
3. It is clear that every \( \oplus \)-J-supplemented is FI- \( \oplus \)-J-supplemented.
4. \( Q \) as \( Z \)-module is not FI- \( \oplus \)-J-supplemented.

The following proposition gives a condition under which the \( \oplus \)-J-supplemented and FI- \( \oplus \)-J-supplemented are equivalent.

**Proposition (3.15):** Let \( M \) be a duo module. Then \( M \) is a \( \oplus \)-J-supplemented if and only if \( M \) is FI-\( \oplus \)-J-supplemented module.

**Proof:** We have to show that \( M \) is a \( \oplus \)-J-supplemented module. Let \( A \) be a submodule of \( M \). Since \( M \) is a duo module, then \( A \) is a fully invariant submodule of \( M \). But \( M \) is FI-\( \oplus \)-J-supplemented module. Hence \( A \) has a \( \oplus \)-J-supplement in \( M \). Therefore \( M \) is a \( \oplus \)-J-supplemented module. The converse is clear.

**Proposition (3.16):** Let \( M \) be an \( R \)-module. Then \( M \) is FI-\( \oplus \)-J-supplemented module if and only if for every fully invariant submodule \( N \) of \( M \), there exists a direct summand \( K \) of \( M \) such that \( M = N + K \) and \( N \cap K \ll K \).

**Proof:** See [2, Proposition(2.7)].
Proposition (3.17): Let $M$ be FI-$\oplus$-J-supplemented module and let $A$ be fully invariant submodule of $M$. Then the factor \( \frac{M}{A} \) is FI-$\oplus$-J-supplemented module.

Proof: Let $B$ be any fully invariant submodule of $\frac{M}{A}$. Then $B$ is a fully invariant submodule of $M$ by [9, Lemma(2.1)]. Since $M$ is FI-$\oplus$-J-supplemented module, then there exists a direct summand $C$ of $M$ such that $M = C + B$, $C \cap B \ll_{J} C$, $M = C \oplus \hat{C}$, $\hat{C} \leq M$ and $\frac{M}{A} = \frac{B + C}{A} = \frac{B}{A} + \frac{C}{A}$. Let \( \frac{B}{A} \cap \frac{C}{A} = \frac{V}{A} \). Then \( \frac{B}{A} \cap \frac{C}{A} \supseteq \frac{V}{A} \). Let \( \frac{B}{A} \cap \frac{C}{A} \cap \frac{V}{A} = \frac{W}{A} \). If \( \frac{B}{A} \cap \frac{C}{A} \cap \frac{V}{A} \supseteq \frac{W}{A} \), then \( \frac{B}{A} \cap \frac{C}{A} \cap \frac{V}{A} = \frac{W}{A} \). But \( \frac{B}{A} \cap \frac{C}{A} = \frac{V}{A} \) and \( \frac{B}{A} \cap \frac{C}{A} \cap \frac{V}{A} \supseteq \frac{W}{A} \). Therefore, \( \frac{W}{A} \) is a direct summand of \( \frac{M}{A} \).

The converse is not true in general. For example $Z$ as $Z$-module, \( \frac{Z}{A} \oplus \frac{Z}{B} \) is FI-$\oplus$-J-supplemented but $Z$ is not FI-$\oplus$-J-supplemented.

Proposition (3.18): Let $M_1$ and $K$ are fully invariant submodules of $M$, and let $M_2$ be FI-$\oplus$-J-supplemented module. If $M_1 + K$ has FI-$\oplus$-J-supplement in $M$, then so does $K$.

Proof: Since $M_1 + K$ has FI-$\oplus$-J-supplement in $M$, then there exists a direct summand fully invariant $X$ of $M$, such that $(M_1 + K) + X = M$, and $(M_1 + K) \cap X \ll_{J} X$. Since $M_1$ is FI-$\oplus$-J-supplemented module, then there exists a direct summand $Y$ of $M_1$ such that $(X + K) \cap M_1 + Y = M$, and $(X + K) \cap Y \ll_{J} Y$. We have $M = M_1 + K + X = (X + K) \cap M_1 + Y + K + X = Y + K + X$, and $M = Y + K + X$, and $(X + K) \cap Y \ll_{J} Y$, that is $Y$ is FI-$\oplus$-J-supplement of $X + K$ in $M$. Next, we show that $X + Y$ is FI-$\oplus$-J-supplement of $K$ in $M$. It is clear that $M = K + (X + Y)$, so it suffices to show that $(X + Y) \cap K \ll_{J} X + Y$. Suppose that $(X + Y) \cap K \subseteq M_1 + K$, then $X \cap (Y + K) \subseteq X \cap (M_1 + K) \ll_{J} X$, and by [2, Proposition(2.6(1))] then $X \cap (Y + K) \ll_{J} X$. Thus by [2, Proposition(2.6(5))], $(X + Y) \cap K \subseteq X \cap (Y + K) \subseteq X + Y$.

Proposition (3.19): Let $M = M_1 \oplus M_2$, and $M_1$ and $M_2$ are FI-$\oplus$-J-supplemented modules if and only if $M$ is FI-$\oplus$-J-supplemented.

Proof: ($\Rightarrow$) Suppose that $M = M_1 \oplus M_2$, and $M_1$ and $M_2$ are FI-$\oplus$-J-supplemented modules. Let $K$ be a fully invariant submodule of $M$. Since $M_1 + M_2 + K = M$, it trivially has FI-$\oplus$-J-supplement in $M$. By proposition (3.18), then $M_2 + K$ has FI-$\oplus$-J-supplement in $M$, and by proposition (3.18) again, $K$ has FI-$\oplus$-J-supplement in $M$, so $M$ is FI-$\oplus$-J-supplemented module.

($\Leftarrow$) Suppose that $M = M_1 \oplus M_2$, and $M$ is FI-$\oplus$-J-supplemented module. To show that $M_1$ and $M_2$ are FI-$\oplus$-J-supplemented modules. Since $M_2 \cong \frac{M}{M_1}$ and $M$ is FI-$\oplus$-J-supplemented module, then by Proposition (3.17), $\frac{M}{M_1}$ is FI-$\oplus$-J-supplemented module. Thus $M_2$ is FI-$\oplus$-J-supplemented module.

Similarity $M_1$ is FI-$\oplus$-J-supplemented module.

Corollary (3.20): Let $M = M_1 \oplus M_2$ be a duo module, and $K$ and $L$ are fully invariant submodules of $M_1$. If $K$ is FI-$\oplus$-J-supplement of $L$ in $M_1$, then $K \oplus M_2$ is FI-$\oplus$-J-supplement of $L$ in $M$.

Proof: Let $K$ be FI-$\oplus$-J-supplement of $L$ in $M_1$, then $M_1 = K + L$, $K$ is a direct summand of $M_1$ and $K \cap L \ll_{J} K$. Since $M = M_1 \oplus M_2$, then $M = (K + L) \oplus M_2$, hence $M = L + (K \oplus M_2)$ but $(K \oplus M_2) \cap L = (K \oplus M_2) \cap M_1 \cap L = K \cap L \ll_{J} K$. And by [2, Proposition(2.6(4))], then $K \cap L \ll_{J} K \oplus M_2$, hence $K \oplus M_2$ is FI-$\oplus$-J-supplement of $L$ in $M$.

Theorem (3.21): Let $M$ be a module such that $M = M_1 \oplus M_2$ is a direct sum of submodules $M_1$ and $M_2$. Then $M_2$ is FI-$\oplus$-J-supplemented module if and only if there exists a direct summand $Y$ of $M$ such that $Y \subseteq M_2$, $M = X + Y$ and $X \cap Y \ll_{J} Y$, for every fully invariant submodule \( \frac{X}{M_1} \) of \( \frac{M}{M_1} \).

Proof: ($\Rightarrow$) Let $\frac{X}{M_1}$ be any fully invariant submodule of $\frac{M}{M_1}$. Then $X \cap M_2$ is fully invariant submodule of $M_2$ by [12, Lemma(2.3)]. Since $M_2$ is FI-$\oplus$-J-supplemented module, then there exists a
direct summand $Y$ of $M_2$ such that $M_2 = (X \cap M_2) + Y$ and $X \cap M_2 \cap Y = X \cap Y \ll_j Y$. Clearly, $Y$ is a direct summand of $M$ and $M = M_1 + M_2 = M_1 + (X \cap M_2) + Y \subseteq M_1 + X + Y$, but $M_1 \subseteq X$, therefore $M = X + Y$. So we get the result.

($\Leftarrow$) To show that $M_2$ is $\oplus$-$J$-supplemented, let $X$ be a fully invariant submodule of $M_2$. Then $X$ is a direct summand of $M$. Since $M_2$ is a direct summand of $M$, $X$ is a direct summand of $M$. Since $H$ is a direct summand of $M$, $H$ is a direct summand of $M$.

Let $X$ be any fully invariant submodule of $M$. Then $M = (X + Y) + H$ such that $(X + Y)$ is a direct summand of $X$ in $M_2$. Thus $M_2$ is $\oplus$-$J$-supplemented submodule.

**Theorem (3.22):** Let $M_2$ be a direct summand of $\oplus$-$J$-supplemented module $M$, such that for every direct summand $K$ of $M$ with $M = K + M_2$, $K \cap M_2$ is a direct summand of $M$. Then $M_2$ is $\oplus$-$J$-supplemented module.

**Proof:** Suppose that $M = M_1 \oplus M_2$ and let $N$ be a fully invariant submodule of $M$. Since $M$ is an $\oplus$-$J$-supplemented module, there exists a direct summand $K$ of $M$ such that $M = (N \cap M_2) + K$ and $N \cap M_2 \cap K \ll_j K$. By [3, Lemma(1.2)], $M = (K \cap M_2) + N$. Since $M = K + M_2$, then $K \cap M_2$ is a direct summand of $M$ by hypothesis, and by theorem(3.21), $M_2$ is $\oplus$-$J$-supplemented module.

**Lemma (3.23):** Let $X$ and $Y$ be fully invariant submodules of a module $M$ such that $X + Y$ has a $\oplus$-$J$-supplement $H$ in $M$ and $X \cap (H + Y)$ has a $\oplus$-$J$-supplement $G$ in $X$. Then $H + G$ is a $\oplus$-$J$-supplement of $Y$ in $M$.

**Proof:** Let $H$ be a $\oplus$-$J$-supplement of $X + Y$ in $M$ and let $G$ be a $\oplus$-$J$-supplement of $X \cap (H + Y)$ in $X$. Then $M = (X + Y) + H$ such that $(X + Y) \cap H \ll_j H$, $X = [X \cap (H + Y)] + G$ such that $(H + Y) \cap G \ll_j G$. Since $M = X + Y + H = X \cap (H + Y) + G + Y + H = Y + H + G$, then $M = Y + (H + G)$. But $G + Y \subseteq X + Y$, then $(G + Y) \cap H \subseteq (X + Y) \cap H \ll_j H$, and by [2, Proposition(2.6(1))], $(G + Y) \cap H \ll_j H$. Thus $H \cap (H + G) \subseteq (G + Y) \cap H + (H + Y) \cap G \ll_j H + G$.

**Theorem (3.24):** For any ring $R$, any finite direct sum of $\oplus$-$J$-supplemented R-modules is $\oplus$-$J$-supplemented.

**Proof:** Let $n$ be any positive integer and let $M_i$ be an $\oplus$-$J$-supplemented R-module for each $1 \leq i \leq n$. Let $M = M_1 \oplus \ldots \oplus M_n$. To prove that $M$ is an $\oplus$-$J$-supplemented R-module, it is sufficient by the induction on $n$ to prove this is the case when $n = 2$. Thus suppose that $n = 2$. Let $X$ be any fully invariant submodule of $M$. Then $M = M_1 + M_2 + X$ so that $M_1 + M_2 + X$ has an $\oplus$-$J$-supplement $H$ in $M_2$ such that $H$ is a direct summand of $M_2$. Let $M_2$ be an $\oplus$-$J$-supplemented module. Then $M_2 \cap (M_1 + X)$ has an $\oplus$-$J$-supplement $H$ in $M_2$. By lemma(3.23), $H$ is an $\oplus$-$J$-supplement of $M_2 + X$ in $M$. Since $M_2$ is an $\oplus$-$J$-supplemented module, $M_2 \cap (X + H)$ has an $\oplus$-$J$-supplement $K$ in $M_2$ such that $K$ is a direct summand of $M_2$. Again, by lemma(3.23), $H + K$ is an $\oplus$-$J$-supplement of $X$ in $M$. Since $H$ is a direct summand of $M_2$ and $K$ is a direct summand of $M_2$, it follows that $H + K = H \oplus K$ is a direct summand of $M$. Thus $M = M_1 \oplus M_2$ is $\oplus$-$J$-supplemented.

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