Logarithmic corrections to the RG flow for the two-dimensional bond disordered Ising model

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Using the mapping of the partition function of the two-dimensional Ising model onto a pfaffian we evaluate the domain wall free energy difference for the pure and disordered Ising model close to the pure fixed point. Using this method very large lattices can be studied exactly and we confirm that disorder even including frustrating interactions indeed are irrelevant close to the pure fixed point. The finite-size renormalization group flow shows a power-law behavior modified by a logarithmic term that dominates for small lattice sizes.

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I. INTRODUCTION

The influence of disorder on the pure two-dimensional Ising model has been a problem of long standing interest. The works of Ludwig [1], Shankar [2] and Dotsenko and Dotsenko [3] have shown that logarithmic corrections to the specific heat arise once disorder is present but the otherwise the critical exponents remain unchanged from the pure case. In a certain sense one might say that disorder is marginally irrelevant for the pure Ising model. From the Harris criterion [4] we know that disorder is relevant if the correlation length exponent, \( \nu > 2/d \). However, for the two-dimensional Ising model \( \nu = 1 \) and the Harris criterion gives no definite answer to relevance or irrelevance of disorder which in this case is a marginal perturbation. Most of the analytical treatments of this problem considers only non-frustrating disorder and one might ask the question if frustration could have a different influence on the system than non-frustrating disorder.

In this paper we revisit this problem and by mapping the partition function onto a pfaffian we are able to obtain large scale numerically exact results independent of the disorder distribution for various different quantities close the pure fixed point. To our knowledge pfaffian techniques have so far not been applied to fully disordered systems. Compared to more standard transfer matrix techniques this method presents significant advantages notably in the system sizes it is possible to treat. In some cases it is possible to obtain exact results for system sizes comparable to what has been treated with Monte Carlo methods.Using finite-size scaling ideas we show that disorder indeed is irrelevant at the pure fixed point even including frustration. However, the finite-size renormalization group flow has a novel logarithmic behavior recently predicted to be rather universal for many models by Cardy [5]. For small system sizes disorder appears to be relevant and only for very large system sizes does it become apparent that the flow is towards the pure fixed point.

II. METHOD

Our approach is inspired by the domain wall renormalization group (DWRG) developed by McMillan [6] and a recent modification of this, used to determine the fixed point structure of the two-dimensional disordered Potts model [7]. In this approach the stiffness of the system is calculated and a finite size scaling ansatz is used to determine the flow. We briefly recapitulate the central results.

In the absence of disorder it is known from hyperscaling that the singular part of the free energy density scales as the inverse of a correlation volume

\[
\frac{f_s}{k_b T_c} = \frac{C}{\xi^d}. \tag{1}
\]

Exploiting the point of view of the DWRG we assume that the critical part of the free energy, \( f_s \), comes from the difference in free energy between two configurations with different boundary conditions

\[
f_s = \Delta f = f_{\text{periodic}} - f_{\text{antiperiodic}}. \tag{2}
\]

Using standard finite-size scaling arguments [7] we find that the total free energy difference \( \Delta F \) scales as

\[
\frac{\Delta F}{k_b T} = A g(\delta L^{1/\nu}), \tag{2}
\]

with \( g \) a universal function and \( \delta = |T - T_c| \). At \( T_c \) the free energy difference is thus a universal amplitude.

\( A g(\delta L^{1/\nu}) \) can be regarded as the stiffness of the system, \( \rho(L,T) \). In an ordered phase we would expect \( \rho \) to diverge with the system size, whereas in a disordered phase \( \rho \) should scale to zero with the system size. Deep in the ordered phase it is known that the leading correction to \( \Delta F \) diverges linearly with the system size \( L \) [8].

\[
|\Delta F/T| \sim L \left| \ln \left( \frac{1}{|z|} \right) \right|, \quad z = \tanh \left( \frac{J}{T} \right). \tag{3}
\]

At \( T = T_c \) this term is rigourously zero and the finite universal free energy difference comes from higher order corrections. The critical point, \( T_c \), can now be located by
standard methods, i.e. by tracing $\Delta F$ as a function of $T$ for several different system sizes and locating the point where the lines cross.

The Hamiltonian that we consider is the bond disordered two dimensional Ising model:

$$ H = - \sum_{<i,j>} J_{ij} S_i S_j, \quad S_i = \pm 1. \quad (4) $$

To facilitate some of our calculations we restrict our treatment to a bimodal distribution of the bonds:

$$ P(J) = x \delta(J - 1) + (1 - x) \delta(J - 1), \quad (5) $$

where $x$ is the concentration of antiferromagnetic bonds. However, one should note that the present techniques can be used for any distribution of disorder, continuous as well as discrete.

$$ Twisted \ Boundary \ Condition $$

Since the works of Kac and Ward [9] it has been known that the partition function for some two-dimensional models can be expressed as a pfaffian. A general theorem due to Kasteleyn [10] shows that this is possible for all planar graphs. For the problem at interest here this means that a pfaffian formulation is not possible in higher dimensions. The Pfaffian formulation for the two-dimensional Ising model was later developed in detail by several authors [9,11,12,8]. We refer the reader to the litterature for the derivation of this mapping. What is very noteworthy is that the the mapping onto a pfaffian can be exploited without problems even for the fully disordered case. For the disordered Ising model on a torus one finds a sum of four pfaffians:

$$ Z = 2^{L^2} (\cosh(J/T))^{2L^2-N} (\cosh(-J/T))^N 
\quad (-Pf_1 + Pf_2 + Pf_3 + Pf_4)/2, \quad (6) $$

with $N$ the number of AF bonds. In order for this expression to be useful it is necessary to be able to evaluate the pfaffians for rather large matrices. Usually this is done by using the well known relation for a skew-symmetric matrix $D$, $|\text{Pf}(D)| = \text{Det}(D)^{1/2}$. However, even though efficient methods exists for evaluating determinants, this expression only determines the absolute value of the pfaffian. Fortunately, recent developments in combinatorial algorithms have shown that it is possible to evaluate the pfaffian directly in polynomial time [13], without making use of the above formula. This polynomial algorithm is relatively slow compared to the more efficient methods for evaluating the determinant and for the larger systems sizes we have considered we have calculated the determinant and obtained the sign using continuity arguments. This can be done since the sign of the four pfaffians is positive at sufficiently high temperatures.

III. RESULTS - PURE SYSTEM

As an illustration of how well this works we first consider the pure Ising model without disorder and try to locate the known critical point, $T_c = 2/ \log(1 + \sqrt{2})$, using the above mentioned techniques. Introducing a line of antiferromagnetic bonds along one line through the system, as illustrated in Fig. 1, we can calculate the total free energy difference, $\Delta F$, between periodic and antiperiodic boundary conditions. Our results are shown in Fig. 2 for system sizes of $L = 16, 24, 32, 40, 48$ as a function of $T$.

![figure 2: $|\Delta F/T|$ for $L = 16, 24, 32, 40, 48$ as a function of $T$.](image)
critical point can be located to four decimal places. The slope of $\Delta F$ at $T_c$ should scale as $L^{1/\nu}$ and can be used for a very precise determination of the exponent $\nu$. Since

$$|\Delta F(T) - F_c| \approx 0.986485 - 0.303151L^{-2.00853}. \quad (7)$$

From conformal invariance it is known that for the geometry of an infinite long cylinder $\Delta F/T = \pi \eta$ [14], where $\eta$ is the magnetic critical exponent. Hence, for this geometry, $\Delta F$ directly measures a bulk critical exponent. However, for the toroidal geometry used here we are not aware of any results for the universal number, 0.986485, even though many exact properties are known for the Ising model. We would welcome any information as to where and if such a calculation has been performed. The corrections to scaling follow a standard power-law form with an exponent extremely close to 2 as expected.

IV. RESULTS - DISORDERED SYSTEM

We now turn to a discussion of our results in the presence of disorder. In this case we need to consider not only $\Delta F$ but also its standard deviation $\sigma(\Delta F)$. The relevant variable to consider is then $r = \sigma(\Delta F)/|\Delta F|$ where $\langle \rangle$ denotes the disorder average. We are interested in studying the behavior of this quantity close to the pure fixed point where $\sigma(\Delta F)$ is trivially zero. In order to determine the relevance or irrelevance of the disorder we need to calculate:

$$\text{slope} = \frac{d}{dx} \left[ \frac{\sigma(\Delta F)}{|\Delta F|} \right]_{x=0}. \quad (8)$$

Here $x$ denotes the concentration of antiferromagnetic bonds. At a standard fixed point we would expect the slope to follow a power-law behavior $L^{\lambda}$ and in the case of $\lambda < 0$ we qualify the perturbation as irrelevant and as relevant if $\lambda > 0$. The calculation of the derivative with respect to $x$ is performed by introducing a single antiferromagnetic bond into the lattice. For the case of the bimodal distribution that we consider it turns out that there are only two non-equivalent positions on the torus. Hence, we can evaluate both $|\Delta F|$ and $\sigma(\Delta F)$ exactly. A single antiferromagnetic bond corresponds to a given value of $x$ and since $r$ at $x = 0$ is trivially zero we can numerically determine the derivative. One should note that the error introduced by numerically determining the derivative is extremely small due to the large lattice sizes and the corresponding tiny value of $x$.

Our results for the slope at $T = T_c$ are shown in Fig. 4 for lattice sizes of $L = 16, 24, 32, 40, 48$. Surprisingly the slope increases with the lattice size $L$ and one could be led to the conclusion that in the presence of this frustrating disorder the perturbation is marginally relevant instead of marginally irrelevant as is known to be the case for unfrustrating disorder. However, since our results are

![Figure 3: $|\Delta F/T|$ at $T = T_c$ for $L = 16, 24, 32, 40, 48$. The open circles (○) indicate the exact numerical results and the solid circles (●) indicate a power-law fit to the data of the form $0.986485 - 0.303151L^{-2.00853}$.](image)

![Figure 4: The slope at $T = T_c$ for $L = 16, 24, 32, 40, 48$. The open circles (○) indicate the exact numerical results and the solid circles (●) indicate a power-law fit to the data of the form $2.94 \log(L)^{0.36} L^{-0.070}$. In the inset we show the same results as well as the extrapolated form out to $L = 10^3$.](image)
numerically exact we can verify that the functional form is not a simple power-law. Instead, we find that the only acceptable form is:

\[ A \log(L)^\alpha L^{-\beta}, \tag{9} \]

with \( \alpha \simeq 0.36, \beta \simeq -0.07 \) and \( A = 2.94 \). This is a functional form quite well-known for the correlation functions [1,2] in such disordered systems and as recently shown by Cardy [5] it is common for many low-dimensional models. Considering the precision of our numerical results we believe this form to be correct up to small errors in the exponents. If this is indeed the case the ‘flow’ will not reverse until lattice sizes of approximately \( L=200 \) are reached, a system size not accessible by our methods. We do not attach any importance to this length scale since it depends on the form of the disorder and various other microscopic parameters. In the inset in Fig. 4 we show our numerical results along with the extrapolation out to lattice sizes of \( L=10000 \).

### V. CONCLUSION

Applying the well-know mapping of the partition function of the Ising model onto a pfaffian we have succeeded in evaluating the finite-size renormalization group flow around the pure fixed point in an almost exact numerical manner. We have used a bimodal distribution of the disorder which includes frustration for which the domain wall free energy as well as its standard deviation can be evaluated exactly. Our results show that very large lattice sizes have to be used before it becomes clear that disorder, even frustrating, is irrelevant. It is possible that the functional form for the finite-size renormalization group flow Eq. (9) can be verified using analytical methods.

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