The Ashtekar-Hansen universal structure at spatial infinity is weakly pseudo-Carrollian.

G W Gibbons
D.A.M.T.P.,
Cambridge University,
Wilberforce Road,
Cambridge CB3 0WA,
U.K.

February 26, 2019

Abstract

It is shown that Ashtekar and Hansen’s Universal Structure at Spatial Infinity (SPI), which has recently been used to establish the conservation of supercharges from past null infinity to future null infinity, is an example of a (pseudo-) Carollian structure. The relation to Kinematic Algebras is clarified and the associated $\mathbb{Z}_2$ symmetry is related to the Gouy effect in optics and the Grgin discontinuity of Twistor theory.

1 Introduction

In a recent paper [1] it has been argued that the conservation of supercharges from past to future null infinity holds provided the spacetime is asymptotically flat in the sense of the Ashtekar and Hansen [2]. The purpose of the present paper is to show that the universal structure at infinity that such spacetimes admit is an example of what has been defined as a Carollian structure [3, 4] under the novel assumption that the degenerate Carollian metric is pseudo-Riemannian rather than Riemannian. The relation of this structure to older [5, 6, 7] and more recent [8, 9, 10, 11, 12, 13] work on Kinematic Algebras is described.

A further purpose is to relate (in the appendix) the associated $\mathbb{Z}_2$ symmetry to some old observations going back to Gouy [33] and to Grgin [31, 32].
2 The Ashtekar-Hansen universal structure on Spi.

In [2] the authors constructed a 4-manifold Spi with a universal structure associated to the spatial infinity, \( i^0 \) for spacetimes which are asymptotically empty and flat at null and spatial infinity (AEFANSI). Spi should be construed as a blow up of the point \( i^0 \) in the conformal embedding of the physical spacetime which captures the idea that \( i^0 \) is the ultimate destination of spatial curves. The equivalences class of curves is specified by an ultimate direction and a notion of ultimate acceleration.

The construction was later revisited in terms of a timelike boundary of an asymptotically flat spacetime whose normal is only defined up to a direction [17].

The four-manifold \( \text{Spi} = \{ E, \pi, B \} \) is a principal line bundle \( E \) over the set \( B \) of unit spacelike four vectors in the tangent space of \( i^0 \) with structural group \( \mathbb{R} \). The base space \( B \) of the line bundle Spi may be identified with three dimensional de-Sitter spacetime \( dS_3 \) equipped with its Lorentzian metric \( g_3 \) of constant curvature. Using the projection map \( \pi : \text{Spi} \to B = dS_3 \), the Lorentzian metric \( g_3 \) may be pulled back to the 4-manifold Spi to give a degenerate bilinear form \( g_4 = \pi^* g_3 \) on the tangent space of Spi with kernel tangent vectors to the fibres.

The automorphisms of this structure is an infinite dimensional group \( G \) analogous to the BMS groups of future and past null infinity \( \mathcal{I}^\pm \).

The group of such diffeomorphism obviously includes the Lorentz group \( SO(3,1) \) acting on the base \( B \). In addition, since there is no natural coordinate on the fibres of the bundle, that is no natural section, the space of all sections, i.e. functions on the base space, form an infinite dimensional abelian subgroup group under addition called Spi supertranslations which we call \( T \). In fact

\[
G = SO(3,1) \ltimes T, \quad G/T = SO(3,1). \tag{1}
\]

Moreover there is a 4 dimensional normal subgroup of translations \( T_4 \).

3 The Weakly Carrollian structures

The universal structure on Spi described above, especially its degenerate metric closely resembles what has been called a Carrollian structure [3, 4] and it has been shown how the BMS group acts the automorphism group of the Carrollian structure on \( \mathcal{I} \) which is a principal line bundle over \( S^2 \) with fibre \( \mathbb{R} \) and a degenerate bilinear form whose kernel consists of tangent vectors to the fibres. Apart from its dimension the main difference is that the metric on the base space of the Carrollian structures considered previously was positive definite. However the general ideas go through of one merely requires that the metric on the base is non-degenerate, i.e. pseudo-Riemannian. It seems reasonable to refer to such structures as pseudo-Carrollian . In [3] by analogy with a Newton-Cartan
structure, a strong Carrollian structure which included an affine connection was defined. But in \[4\] weaker definition was adopted and it is this weaker definition which seems to be more appropriate in the present case.

Thus a pseudo-Carrollian manifold may be defined as a triple \((C,g,\xi)\) where \(C\) is a smooth \((d+1)\) dimensional twice covariant degenerate symmetric tensor field \(g\) whose kernel is generated by the nowhere vanishing, complete vector field \(\xi\). The associated Carroll group \(\text{Carr}(C,g,\xi)\) are its automorphisms and consist of all diffeomorphisms of \(C\) preserving the bilinear form \(g\) and the vector field \(\xi\). The pseudo-Carroll Lie algebra, \(\text{carr}(C,g,\xi)\) is then identified with the Lie algebra of those vector fields \(X\) on \(C\) such that

\[
L_Xg = 0, \quad L_X\xi = 0. \tag{2}
\]

In fact (2) are the same equations which were used in §4 of \[2\] to determine the asymptotic symmetries at infinity of AEFANSI spacetimes admitting a Spi.

There is an associated Conformal Carroll group of level \(N\), \(\text{CCarr}_N(C,g,\xi)\) whose transformations preserve the tensor field \(g \otimes \xi \otimes N\) canonically associated with a Carroll manifold, that is all diffeomorphisms \(f\) satisfying

\[
f^*g = \Omega^2 g, \quad f^*\xi = \Omega^{-\frac{2}{N}} \xi \tag{3}
\]

for some positive function \(\Omega\) and integer \(N\). The Lie algebra of infinitesimal conformal Carroll transformations \(\text{ccarr}_n(C,g,\xi)\) is spanned by vector fields \(X\) such that

\[
L_Xg = \omega g, \quad L_X\xi = -\frac{\omega}{N} \xi, \tag{4}
\]

for some function \(\omega\) on \(C\).

4 Kinematical algebras and Kinematical Space-times

In \[5\] (see also \[6, 7, 8, 9\].) Levy-Leblond and Bacry introduced the idea of a kinematical algebra, a 10-dimensional algebra \(\mathfrak{k} = J \oplus H \oplus P \oplus B\) where \(J = \mathfrak{so}(3)\), \(H = \mathbb{R}\), \(P = \mathbb{R}^3\), \(B = \mathbb{R}^3\) are rotations, time translations, space translations and boosts respectively, and is a deformation of the so-called Static kinematic algebra for which the only non-vanishing brackets are

\[
[J, J] \in J, \quad [H, J] = 0, \quad [J, P] \in P, \quad [J, B] \in B. \tag{5}
\]

Bacry and Levy-Leblond \[5\] found 12 such algebras (or 11 if one insists that the boosts \(B\) be non-compact generators). All may be obtained by contractions starting either from \(\mathfrak{so}(3,2)\) or \(\mathfrak{so}(4,1)\), both of which contract to the Poincaré algebra \(\mathfrak{p}\) in the limit that the curvature goes to zero. Performing a further contraction one finds that as the velocity light goes to infinity one obtains the Galilei group \(\mathfrak{g}\). On the other hand as it goes to zero one obtains the finite,
10 dimensional, Carroll subalgebra algebra \( \mathfrak{c} \) of the infinite dimensional Carroll algebra \( \mathfrak{Carr}(3, 1) \). Starting from \( \mathfrak{so}(3, 2) \) one obtains an algebra isomorphic to the the Poincaré algebra \( \mathfrak{p} \), called by Bacry and Levy-Leblond the Para-Poincaré algebra \( \mathfrak{p}' \) in which the boosts \( \mathbf{B} \) and translations \( \mathbf{P} \) are interchanged.

Each kinematical algebra \( \mathfrak{k} \) gives rise to a kinematical group \( K \). Figueroa-O’Farrill and Prohazka have shown \([12]\) that associated with each such kinematical group are one or more Kinematical Spacetimes, that is 4 dimensional homogeneous spacetimes \( M = K/H \) where the group \( H \) has lie algebra \( \mathfrak{h} \) and the pair \( \mathfrak{k}, \mathfrak{h} \) are subject to certain admissibility conditions.

In the case of the para-Poincaré Group \( \mathfrak{P}' \), which \([12]\) call \( \text{AdSC} \), the structure induced on the associated Kinematic spacetime is Carrollian. If we write the Anti-De-Sitter metric as

\[
ds^2 = c^2(1 + r^2/R^2)dt^2 + \frac{dr^2}{1 + r^2/R^2} + r^2(d\theta^2 + \sin^2 d\phi^2) \tag{6}
\]

and take the limit \( c^2 \downarrow 0 \) we obtain a Carrollian metric with \( g_{ij} \) the metric on hyperbolic three space \( \mathbb{H}^3 \).

In fact it was shown in \([3, 4]\) that one may obtain the Carroll structures as the data induced on a null hypersurface in a \( d + 2 \) pseudo Riemann manifold endowed with a Bargmann structure, in particular on the null hyperplane \( x^+ = \) constant from five-dimensional Minkowski spacetime with metric.

\[
ds^2 = 2dx^+dx^- + dx^2. \tag{7}
\]

In \([12]\) this procedure was adapted to obtain this Carrollian structure by considering 5-dimensional anti-de-Sitter spacetime \( \text{AdS}_5 \) with metric.

\[
ds^2 = \frac{1}{z^2}\{2dx^+dx^- + dz^2 + (dx^1)^2 + (dx^2)^2\}. \tag{8}
\]

A null hypersurface is obtained by setting \( xk = \) constant and the degenerate bilinear form is

\[
ds^2 = \frac{1}{z^2}\{dz^2 + (dx^1)^2 + (dx^2)^2\}, \tag{9}
\]

which is the upper half space model of hyperbolic three space \( \mathbb{H}^3 \). By thinking of the \( \text{AdS}_5 \) as a quadric in \( \mathbb{E}^{4,2} \) one sees that action of \( \text{AdSC} \) is clearly the subgroup of \( \text{SO}(4, 2) \) leaving invariant the intersection with the hypersurface \( x^+ = \) constant.

5 Lorentzian Kinematic Algebras and Kinematical spacetimes

It is clear that at the formal level that much of the previous section will go through with \( \mathfrak{so}(3) \) replaced by \( \mathfrak{so}(2, 1) \). The analogue of the spacetime associated to \( \text{AdSC} \) is Spi.
6 Acknowledgement

The author thanks Gary Horowitz and Jorge Santos for an enquiry and subsequent discussions about geometric structures at spatial infinity which stimulated my initial interest on some of the material described in this paper.

References

[1] K. Prabhu, Conservation of asymptotic charges from past to future null infinity: Supermomentum in general relativity, arXiv:1902.08200 [gr-qc].

[2] A. Ashtekar and R. O. Hansen, A unified treatment of null and spatial infinity in general relativity. I - Universal structure, asymptotic symmetries, and conserved quantities at spatial infinity,” J. Math. Phys. 19 (1978) 1542. doi:10.1063/1.523863

[3] C. Duval, G. W. Gibbons, P. A. Horvathy and P. M. Zhang, Carroll versus Newton and Galilei: two dual non-Einsteinian concepts of time, Class. Quant. Grav. 31 (2014) 085016 doi:10.1088/0264-9381/31/8/085016 arXiv:1402.0657 [gr-qc].

[4] C. Duval, G. W. Gibbons and P. A. Horvathy, Conformal Carroll groups and BMS symmetry,’ Class. Quant. Grav. 31 (2014) 092001 doi:10.1088/0264-9381/31/9/092001

[5] H. Bacry and J. Levy-Leblond, Possible kinematics, J. Math. Phys. 9 (1968) 1605. doi:10.1063/1.1664490

[6] H. Bacry and J. Nuyts, ‘Classification of Ten-dimensional Kinematical Groups With Space Isotropy,’ J. Math. Phys. 27 (1986) 2455. doi:10.1063/1.527306

[7] J. Nzotungicimpaye, ‘Kinematical versus Dynamical Contractions of the de Sitter Lie algebras, arXiv:1406.0972 [math-ph].

[8] J. Figueroa-O’Farrill, Classification of kinematical Lie algebras, arXiv:1711.05676 [hep-th].

[9] J. M. Figueroa-O’Farrill, Kinematical Lie algebras via deformation theory, J. Math. Phys. 59 (2018) no.6, 061701 doi:10.1063/1.5016288 arXiv:1711.06111 [hep-th].

[10] J. M. Figueroa-O’Farrill, Higher-dimensional kinematical Lie algebras via deformation theory, J. Math. Phys. 59 (2018) no.6, 061702 doi:10.1063/1.5016616 arXiv:1711.07363 [hep-th].

[11] T. Andrziejewski and J. M. Figueroa-O’Farrill, ‘Kinematical lie algebras in 2 + 1 dimensions, J. Math. Phys. 59 (2018) no.6, 061703 doi:10.1063/1.5025785 arXiv:1802.04048 [hep-th].
[12] J. Figueroa-O’Farrill and S. Prohazka, Spatially isotropic homogeneous spacetimes, [arXiv:1809.01224 [hep-th]].

[13] J. M. Figueroa-O’Farrill, Conformal Lie algebras via deformation theory, [arXiv:1809.03603 [hep-th]].

[14] G. Güoy, C. R. Acad. Sci. Paris 110 (1890) 125.

[15] E. Grgin, A global technique for the study of spinor fields Ph.D. Thesis Syracuse University, Syracuse, New York (1966)

[16] E. Grgin, J. Math. Phys. 9 (1968) 1595.

[17] A. Ashtekar and J. D. Romano, ‘Spatial infinity as a boundary of spacetime, Class. Quant. Grav. 9 (1992) 1069

[18] A. Strominger, Asymptotic Symmetries of Yang-Mills Theory, JHEP 1407 (2014) 151 doi:10.1007/JHEP07(2014)151 [arXiv:1308.0589 [hep-th]].

[19] T. He, P. Mitra, A. P. Porfyriadis and A. Strominger, ‘New Symmetries of Massless QED, JHEP 1410 (2014) 112 doi:10.1007/JHEP10(2014)112 [arXiv:1407.3789 [hep-th]].

[20] V. Lysov, S. Pasterski and A. Strominger, Lows Subleading Soft Theorem as a Symmetry of QED, Phys. Rev. Lett. 113 (2014) no.11, 111601 doi:10.1103/PhysRevLett.113.111601 [arXiv:1407.3814 [hep-th]].

[21] D. Kapec, V. Lysov and A. Strominger, ‘Asymptotic Symmetries of Massless QED in Even Dimensions, [arXiv:1412.2763 [hep-th]].

[22] D. Kapec, M. Pate and A. Strominger, ‘New Symmetries of QED, [arXiv:1506.02906 [hep-th]].

[23] A. Strominger, Magnetic Corrections to the Soft Photon Theorem, Phys. Rev. Lett. 116 (2016) no.3, 031602 doi:10.1103/PhysRevLett.116.031602 [arXiv:1509.00543 [hep-th]].

[24] S. Pasterski, S. H. Shao and A. Strominger, Flat Space Amplitudes and Conformal Symmetry of the Celestial Sphere, [arXiv:1701.00049 [hep-th]].

[25] A. Strominger, Lectures on the Infrared Structure of Gravity and Gauge Theory, [arXiv:1703.05448 [hep-th]].

[26] D. Kapec, M. Perry, A. M. Raclariu and A. Strominger, ‘Infrared Divergences in QED, Revisited, [arXiv:1705.04311 [hep-th]].

[27] D. Lerner, Twistor Newsletter 3 (1976) 7;

[28] N. Woodhouse, Twistor Newsletter 6 (1977) 1

[29] C. Clarke and D. Lerner, Comm. Math. Phys. 55 (1977) 179.
[30] R. Penrose and W. Rindler *Spinors and Space-Time* Cambridge Monographs on Mathematical Physics Cambridge, UK: Cambridge Univ. Press (1985)

[31] E. Grgin, *A global technique for the study of spinor fields* Ph.D. Thesis Syracuse University, Syracuse, New York (1966)

[32] E. Grigin, *J. Math. Phys.* 9 (1968) 1595.

[33] G. Gouy, *C. R. Acad. Sci. Paris* 110 (1890) 125.

[34] A. Uhlmann, The Closure of Minkowski Space *Acta Physica Polonica* 23 (1963) 295-296

[35] E. Farhi, V. Khoze, and R. Singleton, *Phys. Rev. D* 47 (1993) 5551;

[36] E. Farhi, V. Khoze, and R. Singleton, *Phys. Rev. D* 50 (1994) 4162..

[37] O. Bertolami, J. M. Mouro, R. F. Picken, I. P. Volobujev, *Int. J. Mod. Phys.* A6 (1991) 4149.

[38] A. Hosoya and W. Ogura, *Phys. Lett.* B225 (1989) 117.

[39] R. Penrose, Conformal Treatment of infinity, in *Relativity Groups and Topology* ed. C. M. de Witt and B. de Witt, Les Houches Summer School 1963. Gordon and Breach, New York (1964) reprinted as R. Penrose, Conformal treatment of infinity, Gen. Rel. Grav. 43 (2011) 901. doi:10.1007/s10714-010-1110-5

7 Compactified Minkowski Spacetime and the Gouy-Grigin effect

The recent re-kindling of interest in the BMS group and the general interest in “soft-graviton theorems” is very much motivated by the case of electrodynamics in Minkowski spacetime. Of particular interest has been existence of whose conservation is a consequence of $Z_2$ symmetry relating the behaviour of the ingoing and outgoing radiation fields at future and past null infinity [18, 19, 20, 21, 22, 23, 24, 25, 26]. In this appendix I relate this symmetry to some facts, long well known in the Twistor Theory literature [27, 28, 29, 30] as the Grgin discontinuity [31, 32] and even longer in the optics literature as the Gouy effect [33].

I shall begin by reviewing some global aspects of the conformal compactification of Minkowski spacetime both in the traditional fashion familiar in general relativity due to Penrose [39], and then in a less familiar but equivalent formulation due to Uhlmann [34] which makes use of the special properties of four

---

Footnotes:

1. See [25] for a review.
2. See especially volume 2 pp. 324-332 of [30] whose notation and conventions we shall follow.
spacetime dimensions. I shall then illustrate how this works out for the scattering of fields of various spins.

7.1 Compactified Minkowski spacetime

In this appendix we review the conformal compactification of Minkowski space to $S^1 \times S^3/\pm$. Conformal compactification is the Lorentzian analogue of stereographic projection. Stereographic projection and conformal compactification in any dimension can be described in a unified way and globally well defined fashion by considering the light cone in a space of two higher dimensions. The two spaces mapped to one another are then hyperplanar cross-sections of the light cone, with the mapping achieved by projection along the null generators of the light cone. The mapping is necessarily conformal.

Consider now the particular case of the conformal compactification of Minkowski spacetime, $M = M^{1,3}$. The space of two higher dimensions is the six-dimensional flat space, $M_2^1,4$,

$$ds^2 = -(dX^0)^2 - (dX^5)^2 + (dX^1)^2 + (dX^2)^2 + (dX^3)^2 + (dX^4)^2 \quad (10)$$

and the light cone, $K$, centered at the origin is given by

$$(X^0)^2 + (X^5)^2 - (X^1)^2 - (X^2)^2 - (X^3)^2 - (X^4)^2 = 0. \quad (11)$$

Four-dimensional Minkowski spacetime, $M$, is the intersection of $K$ with the null hyperplane, $X^5 = -X^4 + 1$, where $(X^0, X^i) \equiv (t, \vec{x}) \in M$ and $X^4 = (t^2 - r^2 + 1)/2$, $X^5 = (-t^2 + r^2 + 1)/2$, $(r \equiv |\vec{x}|)$. Similarly, the intersection of $K$ with the 5-sphere,

$$(X^0)^2 + (X^5)^2 + (X^1)^2 + (X^2)^2 + (X^3)^2 + (X^4)^2 = 1 \quad (12)$$

is the periodically identified Einstein static universe (ESU)

$$S^1 \times S^3 = \left\{ (X^0)^2 + (X^5)^2 = 1, (X^1)^2 + (X^2)^2 + (X^3)^2 + (X^4)^2 = 1 \right\}. \quad (13)$$

On $S^1 \times S^3$ define coordinates $(\eta, \chi, \theta, \phi)$

$$X^0 + iX^5 = \exp i\eta \quad (14)$$
$$X^1 + iX^2 = \sin \chi \sin \theta \exp i\phi \quad (15)$$
$$X^3 = \sin \chi \cos \theta \quad (16)$$
$$X^4 = \cos \chi. \quad (17)$$

The mapping from ESU to $M$ is obtained by projection along the null generators of $K$ and takes the form

$$v \equiv t + r = \tan \frac{\eta + \chi}{2} \quad (18)$$
$$u \equiv t - r = \tan \frac{\eta - \chi}{2}. \quad (19)$$
The mapping is 2 → 1 since $X^A$ and $-X^A$ (or $(\eta, \chi, \theta, \phi)$ and $(\eta + \pi, \pi - \chi, \pi - \theta, \phi + \pi)$) map to the same point of $M$. Identifying these points and defining $M^# \equiv S^1 \times S^3 / \pm 1$, we obtain a 1−1 conformal compactification of $M$ onto $M^#$. We note that the antipodal maps on odd spheres are connected to the identity diffeomorphism so that topologically $M^# \equiv S^1 \times S^3$. The mapping takes the flat metric on $M$ to the metric

$$ds^2 = \Omega^{-2} \left( -d\eta^2 + d\chi^2 + \sin^2 \chi (d\theta^2 + \sin^2 \theta d\phi^2) \right)$$

$$\Omega = \cos \eta + \cos \chi = \frac{2}{\sqrt{(1 + u^2)(1 + v^2)}}$$

which is indeed conformal to the standard product metric on ESU. Future and past null infinity $I^\pm$ correspond to $\eta \pm \chi = \pi$. One should note that under the $Z_2$ action, $I^\pm$ are identified with one another. The conformal group $Conf(3,1)$ of Minkowski spacetime acts smoothly and transitively on compactified identified Minkowski spacetime $M^#$. It now follows easily that $Conf(3,1) \equiv SO(4,2)/\pm 1$.

### 7.2 $M^#$ and the Cayley map

The conformal compactification can be described in a more group theoretic way and will be useful when discussing the Yang-Mills solutions. This is standard, but the special feature that we shall exploit is perhaps not as well known as it should be, i.e. the correspondences

$$M^# \equiv U(2) \equiv U(1) \times SU(2)/Z_2 \equiv S^1 \times S^3 / \pm 1 \equiv S^1 \times S^3. \quad (21)$$

where the last two equivalences are topological not metrical.

To see how this comes about we identify points in Minkowski spacetime $M$ with $2 \times 2$ Hermitian matrices $X$:

$$(t, x^i) \in M \rightarrow X = t + \sigma^i x^i \quad (22)$$

with metric

$$ds^2 = -\det dX. \quad (23)$$

The Cayley map $h : M \rightarrow U(2)$ is given by

$$X \rightarrow U = \frac{1 + iX}{1 - iX} \quad (24)$$

and provides a conformal embedding of $M$ into $U(2)$.

It follows that

$$\det dX = \Omega^{-2} \left( -(dX^0)^2 - (dX^5)^2 + (dX^1)^2 + (dX^2)^2 + (dX^3)^2 + (dX^4)^2 \right)$$

$$= \Omega^{-2} \left( -d\eta^2 + d\chi^2 + \sin^2 \chi (d\theta^2 + \sin^2 \theta d\phi^2) \right) \quad (25)$$
with
\[ \Omega = \cos \eta + \cos \chi = \frac{1}{2} |\det(1 + U)| \] (26)
thus reproducing (20). The left hand side of (25) is the standard Minkowski metric, and the right hand side of (25) is the standard product metric on the Einstein cylinder \( R \times S^3 \) which is the universal covering space \( \tilde{M}^# \) of compactified identified Minkowski spacetime \( M^# \). Thus the Cayley embedding \( h \) given by (24) is a conformal one. The Cayley map \( h \) ceases to be invertible when \( \det(1 + U) \) vanishes, i.e. when the conformal factor \( \Omega \) vanishes. These points correspond to null infinity \( I^\pm \) (also called “scri”). From \( \tilde{M} \) we may express \( U \) as
\[ U = \exp(i(\eta + \chi n \cdot \sigma)), \quad n^i = x^i/r. \] (27)

Note that at \( t = 0 \), (i.e. \( \eta = 0 \)) ?? is just the standard stereographic projection of \( S^3 \) onto \( R^3 \).

### 7.3 The Topology of \( U(2) \)

Here we review some facts about the group \( U(2) \). To begin with we note that \( U(2) \equiv U(1) \times SU(2)/\pm 1 \). To see why we note that any \( 2 \times 2 \) unitary matrix \( U \) may be expressed as
\[ U = g_1 g_2 \left( \begin{array}{cc} X_5 + iX^0 & 0 \\ 0 & X_5 + iX^0 \end{array} \right) \left( \begin{array}{cc} X^4 + iX^3 & iX^1 + X^2 \\ iX^1 - X^2 & X^4 - iX^3 \end{array} \right) \] (28)
with
\[ |X^5 + iX^0|^2 = 1, \] (29)
and
\[ |X^4 + iX^3|^2 + |X^1 + iX^2|^2 = 1. \] (30)
The first factor \( g_1 \) is an element of \( U(1) \) and the second \( g_2 \) as is an element of \( SU(2) \). However the pair \( \{g_a, g_2\} \) and \( \{-g_1, -g_2\} \) produce the same element of \( U(2) \) and so we must factor by the \( Z_2 \) action which sends \( \{g_a, g_2\} \) to \( \{g_a, g_2\} \).

Topologically we have from (29) that \( U(1) \equiv S^1 \) and \( SU(2) \equiv S^3 \). Thus topologically \( U(2) \equiv (S^1 \times S^3)/\pm \) where \( Z_2 \) acts as the antipodal map on each factor. Now topologically \( S^1/\pm \equiv S^1 \) and \( S^3/\pm \equiv RP^3 \) and so topologically \( U(2) \equiv S^1 \times RP^3 \).

### 7.4 Conformally Invariant Equations

Given the Yang-Mills solutions on \( S^1 \times S^3 \) and the conformal compactification from Minkowski space to \( S^1 \times S^3 \) discussed above, we can obtain Yang-Mills solutions in Minkowski space. This makes use of the conformal invariance of the \( 3 + 1 \) dimensional Yang-Mills field equations. When one speaks of conformally invariant equations, one may mean invariance under the infinite dimensional abelian group of Weyl rescalings of the metric or the 15 parameter Bateman-Cunningham group \( \text{Conf}(3,1) \) of conformal isometries of Minkowski spacetime.
or possibly just invariance under its 11 dimensional causality preserving Zee-man subgroup consisting of the semi-direct product of the Poincaré group with spacetime homotheties also called dilatations. This latter group is sometimes, slightly misleadingly, also called the Weyl group.

Equations which depend only upon the conformal equivalence class of the spacetime metric $g_{\alpha\beta}$ so that given a solution with respect to a metric $g_{\alpha\beta}$ it is also (possibly after multiplying by some function of the conformal factor $\Omega$) a solution with respect to a conformally rescaled metric $\Omega^2 g_{\alpha\beta}$ are said to be Weyl invariant. Important examples are: the massless scalar field

$$-\nabla^2 \varphi + \frac{1}{6} R \varphi = 0, \quad \varphi \to \varphi/\Omega,$$

(31)

the massless Dirac equation

$$-i \gamma^\alpha \nabla_\alpha \psi = 0, \quad \psi \to \psi/\Omega^2,$$

(32)

the Yang-Mills equation

$$g^{\alpha\beta} D_\alpha F_{\beta\gamma} = 0, \quad F_{\alpha\beta} \to F_{\alpha\beta},$$

(33)

and the motion of a perfect radiation fluid

$$T^{\alpha\beta ; \beta} = 0, \quad u^\alpha \to \Omega^{-1} u^\alpha, \quad \rho \to \Omega^{-4} \rho,$$

(34)

with

$$T^{\alpha\beta} = \frac{4}{3} \rho u^\alpha u^\beta - \frac{1}{3} \rho g^{\alpha\beta}$$

(35)

In all cases the energy momentum tensor $T^{\alpha\beta}$ is traceless and Weyl rescales as $T^{\alpha\beta} \to \Omega^{-6} T^{\alpha\beta}$.

On the other hand one says that an equation has Bateman-Cunningham invariance or is Conf(3, 1) invariant if given a solution $\Phi(x^\alpha)$ then the pull-back

$$J^w \Phi(c^{-1} x^\alpha)$$

(36)

is a solution. $c \in $ Conf(3, 1) is a conformal isometry of Minkowski space-time, $w$ a suitable conformal weight and $J$ the Jacobian of $c$. Every Weyl invariant equation is Conf(3, 1) invariant (provided it is diffeomorphism invariant) but the converse is not true. Counter-examples are provided by various higher spin wave equations for zero-mass particles. Similarly any Conf(3, 1) invariant equation is necessarily invariant under its causality preserving subgroup but again the converse is not true. Essentially any Poincaré invariant equation without dimension-full coupling constants will be invariant under dilatations. Examples include a polytropic perfect fluid with a constant ratio of pressure to density, equations involving the eponymous dilaton, which may be thought of as a gauge field for dilatations, and the non-linear $\sigma$-model.

A well known strategy for obtaining solutions of Weyl invariant equations on Minkowski spacetime, and one which we have adopted here, is to solve them on
the Einstein cylinder, i.e. on the universal covering space $\tilde{\mathcal{M}}$ of compactified identified Minkowski spacetime $\mathcal{M}^\#$, and then to project down onto Minkowski spacetime $\mathcal{M}$. Provided the solutions are well behaved with finite energy (with respect to the Killing vector $\frac{\partial}{\partial \eta}$ on $\tilde{\mathcal{M}}^\#$) they will remain well behaved and of finite energy on $\mathcal{M}$. Note that it is neither necessary nor indeed always possible to insist that the solutions are single valued on compactified identified Minkowski spacetime $\mathcal{M}^\#$. This is related in part to the so-called Grgin phenomenon [32]. Consider for example solutions of the free conformally invariant wave equation (31). One might consider superpositions of solutions of the form:

$$\varphi = \exp(i(l + 1)\eta) \, Y_l(\chi, \theta, \phi)$$

(37)

where $Y_l(\chi, \theta, \phi)$ is a hyper-spherical harmonic of degree $l$ on the 3-sphere $S^3$. To obtain $\mathcal{M}^\#$, points $(\eta, \chi, \theta, \phi)$ and $(\eta + \pi, \pi - \chi, \pi - \theta, \phi + \pi)$ must be identified, but from (37) and the properties of harmonics

$$\varphi(\eta, \chi, \theta, \phi) = -\varphi(\eta + \pi, \pi - \chi, \pi - \theta, \phi + \pi).$$

(38)

Thus solutions $\varphi$ of the form (37) are not single-valued, in fact they may be thought of as sections of a flat bundle over $\mathcal{M}^\#$ twisted by a representation of the conformal group $\text{Conf}(3,1)$ in $\pm 1$ [27, 28, 29]. This is consistent with the fact that $\varphi$ scales as one power of $\Omega$ and that if one extends $\Omega$ smoothly over the Einstein cylinder $R \times S^3$ it is anti-periodic under the identification.

Physically, the solutions (37) may be thought of as an incoming wave which starts off from past null infinity $I^-$, i.e. from $\eta - \chi = -\pi$ comes to a focus at the origin of spherical polars, (i.e. $r = 0$) re-expands and reaches future null infinity $I^+$, i.e. $\eta + \chi = \pi$. Because it has passed once through a spherical focus or caustic, its amplitude is, by the well-known Gouy effect [33], multiplied by minus one. Note that $I^-$ and $I^+$ are identified in $\mathcal{M}^\#$.

### 7.5 Yang-Mills Solutions in Minkowski Space

We now obtain the Yang-Mills solutions on Minkowski space by projecting the $SO(2) \times SO(4)$ invariant $f = 1/2$ solution from $\mathcal{M}^\#$. Note that by contrast with the general $SO(4)$ invariant solution, the $SO(2) \times SO(4)$ invariant solution is in fact well defined on identified compactified Minkowski spacetime $\mathcal{M}^\#$. Substituting $g = \exp i\chi n \cdot \sigma$ yields

$$A^a_u = \frac{2}{er} \frac{n^a}{1 + u^2}, \quad u = t - r$$

$$A^a_v = -\frac{2}{er} \frac{n^a}{1 + v^2}, \quad v = t + r$$

$$A^i = -\frac{2}{e(1 + u^2)(1 + v^2)} \left(2\epsilon_{iab}x^b + (1 + uv)P^{ia}\right), \quad P^{ij} = \delta^{ij} - n^i n^j.$$  

(39)

The components of the gauge field are peaked on the future and past light cone $u = v = 0$. The stress-energy tensor for the gauge field can be obtained
from the stress tensor of the gauge field on ESU. Under a conformal transformation \( \tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu} \), the energy density transforms as \( \tilde{\rho} = \Omega^{-4} \rho \). Weyl rescaling the energy density of the solution on ESU to Minkowski spacetime \( M \), we obtain the energy density

\[
\tilde{\rho} = \frac{192E}{e^2} \cdot \frac{1}{(1 + u^2)^2(1 + v^2)^2}
\]

which is indeed peaked on the past and future light cone. The Chern-Simons number \( N_{CS} \) of the solution evaluated on any Cauchy surface in Minkowski spacetime \( M \) equals \( \frac{1}{2} \). Hence, by itself this solution does not lead to a change of Chern-Simons number. Indeed, we are envisioning this solution as an approximation to the latter part of a process in which one begins with the static sphaleron configuration. Solutions to the pure Yang-Mills equations which do involve a change of Chern-Simons number were discussed in [35, 36]. Thus we have a time-symmetric shell (moving at the speed of light) which carries in an amount \( N_{CS} = \frac{1}{2} \) from past null infinity \( I^- \) and carries out the same amount back to future null infinity \( I^+ \). The solution has a moment of time symmetry, i.e. it is momentarily at rest on the spacelike hyperplane \( t = 0 \). The total energy of the shell carrying this Chern-Simons number is finite because the field falls off rapidly near infinity. The flow vector \( u^\alpha \) in Minkowski spacetime will be orthogonal to the surfaces \( \eta = \text{constant} \), i.e. to the surfaces

\[
\tan \eta = \frac{u + v}{1 - uv} = \frac{2t}{1 + r^2 - t^2} = \text{constant}
\]

and is given by

\[
u^\mu = \Omega \left( \frac{\partial}{\partial \eta} \right)^\mu = \left( \frac{1 + u^2}{1 + v^2} \right)^{1/2} \left( \frac{\partial}{\partial u} \right)^\mu + \left( \frac{1 + v^2}{1 + u^2} \right)^{1/2} \left( \frac{\partial}{\partial v} \right)^\mu.
\]

### 7.6 Other Gauge Groups

It is possible to consider \( SO(4) \) invariant solutions for other gauge groups \( G \). Bertolami, et. al. [37] study the gauge group \( G = SO(n) \). Setting \( A_0 = 0 \), then their gauge field can be expressed in terms of the time-dependent functions \( (\chi_0, \chi_i) \), \( i = 1, \ldots, n - 3 \). \( \chi_0 \) corresponds to the \( SU(2) \) subgroup and is related to our \( f \) by \( \chi_0 = 2f - 1 \). Like the \( SU(2) \) solutions, these \( SO(n) \) solutions can be expressed in terms of a particle moving in the potential:

\[
V(\chi_0, \chi_i) = \frac{1}{8} \left( (1 - \chi_0^2 - \chi^2)^2 + 4\chi_0^2\chi^2 \right), \quad \chi^2 = \chi^i \chi^i.
\]

The associated Chern-Simons number (in their normalization) is given by

\[
N_{CS} = 2 + 3\chi_0 - \chi_0^3 - 3\chi_0\chi^2.
\]

For \( \chi^i = 0 \), the previous results are recovered (taking into account a difference of normalization). These solutions can be mapped to solutions in Minkowski space.
space which have qualitatively similar properties to the $SU(2)$ cases. The energy density of the resulting gauge field will have the same form, but with a different numerical factor which can be determined from (43). There are four vacua on $\tilde{C}$ having zero energy and which are absolute minima. These are at $(\chi_0, \chi) = (1, 0), (-1, 0), (0, 1), (0, -1)$ with $N_{CS} = 4, 0, 2, 2$ respectively. There is a local maximum at $(0, 0)$ with energy $\frac{1}{8}$ and $N_{CS} = 2$ and four saddle points $(\frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, -\frac{1}{2}), (-\frac{1}{2}, \frac{1}{2}), (-\frac{1}{2}, -\frac{1}{2})$ having energy $\frac{1}{16}$ and $N_{CS} = 3, 3, 1, 1$.

It is interesting to note the existence of sphaleron configurations with lower energy than the pure $SU(2)$ one that we have considered above.

### 7.7 Spinors

It is straightforward to solve the massless Dirac equation on the Einstein cylinder in the background of the $SO(2) \times SO(4)$ invariant $f = \frac{1}{T}$ solution [38]. In particular one can find time-independent (i.e. $\eta$ independent) zero energy solutions of the coupled Dirac equation on the Einstein cylinder. Scalar quantities such as $\psi \psi$ are homogeneous for the zero-energy solutions. These solutions may be Weyl rescaled to give time dependent solutions of the Dirac equation in the time-dependent sphaleron. From (32), scalar quantities such as $\tilde{\psi} \tilde{\psi}$ will now be proportional to $\Omega^3$ and will therefore be small except where $\Omega$ is large. In other words these special solutions of the Dirac equation will have their support peaked on the shell and may be thought of as carried by the shell.