On injectors of Hartley set of a finite group

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Abstract

Let $G$ be a group and $\mathcal{H}$ be a Hartley set of $G$. In this paper, we prove the existence and conjugacy of $\mathcal{H}$-injectors of $G$ and describe the structure of the injectors. As application, some known results are directly followed.

1 Introduction

Throughout this paper, all groups are finite. In theory of classes of finite soluble groups, a basic result which generalizes fundamental theorems of Sylow and Hall is the theorem of Fischer, Gaschütz and Hartley [4] on existence and conjugacy of $\mathfrak{F}$-injectors in soluble groups for every Fitting class $\mathfrak{F}$.

Recall that a class $\mathfrak{F}$ is called a Fitting class if $\mathfrak{F}$ is closed under taking normal subgroups and products of normal $\mathfrak{F}$-subgroups. For any class $\mathfrak{F}$ of groups, a subgroup $V$ of a group $G$ is said to be $\mathfrak{F}$-maximal if $V \in \mathfrak{F}$ and $U = V$ whenever $V \leq U \leq G$ and $U \in \mathfrak{F}$. From the definition of Fitting class $\mathfrak{F}$, every group $G$ has the largest normal $\mathfrak{F}$-subgroup $G_\mathfrak{F}$, so called $\mathfrak{F}$-radical of $G$, which is the

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product of all normal $\mathfrak{F}$ subgroups. In particular, if $\mathfrak{F} = \mathfrak{N}$ is the Fitting class of all nilpotent groups, then $G_{\mathfrak{N}} = F(G)$ is the Fitting subgroup of $G$. A subgroup $V$ of a group $G$ is said to be an $\mathfrak{F}$-injector of $G$ if $V \cap N$ is an $\mathfrak{F}$-maximal subgroup of $N$ for every subnormal subgroup $N$ of $G$. Note that if $\mathfrak{F} = \mathfrak{N}_p$ is the Fitting class of all $p$-groups, then the $\mathfrak{F}$-injectors of a group $G$ are Sylow $p$-subgroups of $G$; if $\mathfrak{F}$ is the Fitting class of all groups with soluble Hall $\pi$-subgroups (i.e. $G$ is $E_\pi^s$-group [8, p.81]), where $\pi$ is a set of prime numbers, then the $\mathfrak{F}$-injectors of $G$ are Hall $\pi$-subgroups of $G$.

As a development of the theorem of Fischer, Gaschütz and Hartley [4], Shemetkov [15] (resp. Anderson [1]) proved that if $G$ is a $\pi$-soluble group (resp. soluble group) and $\mathcal{F}$ is a Fitting set of $G$, then $G$ possesses exactly one conjugacy class of $\mathcal{F}$-injectors, where $\pi$ is the set of all primes dividing orders of all subgroups of $G$ in $\mathcal{F}$.

Recall that a nonempty set $\mathcal{F}$ of subgroups of a group $G$ is called a Fitting set of $G$ [1, 15], if the following three conditions hold: (i) If $T \subseteq S \subseteq \mathcal{F}$, then $T \in \mathcal{F}$; (ii) If $S \in \mathcal{F}$ and $T \in \mathcal{F}$, $S \unlhd ST$ and $T \unlhd ST$, then $ST \in \mathcal{F}$; (iii) If $S \in \mathcal{F}$ and $x \in G$, then $S^x \in \mathcal{F}$. So from the definition of Fitting set $\mathcal{F}$, the $\mathcal{F}$-radical $G_{\mathcal{F}}$ of a group $G$ can also be defined as the product of all its normal $\mathcal{F}$-subgroups. For a Fitting set $\mathcal{F}$ of $G$, the $\mathcal{F}$-injector of $G$ is similarly defined as the $\mathcal{F}$-injector for Fitting class $\mathfrak{F}$ (see [1, Definition VIII. (2.5)]).

If $\mathfrak{F}$ is a Fitting class and $G$ is a group, then the set $\{H \leq G : H \in \mathfrak{F}\}$ is a Fitting set, which is denoted by $Tr_\mathfrak{F}(G)$ and called the trace of $\mathfrak{F}$ in $G$ (see [3, VIII, 2.2(a)]). Note that for a Fitting class $\mathfrak{F}$, the $\mathfrak{F}$-injectors and $Tr_\mathfrak{F}(G)$-injectors of $G$ coincide, but not every Fitting set of $G$ is the trace of a Fitting class ([3, VIII. Examples (2.2)(c)]). Hence, if $\mathcal{F} = Tr_\mathfrak{F}(G)$, then the theorem of Anderson [1] and the theorem of Fischer, Gaschütz and Hartley [4] are Corollaries of the theorem of Shemetkov [15]. Vorob’ev and Semenov [17] proved that for every set $\pi$ of primes and every Fitting set $\mathcal{F}$ of $\pi$-soluble group $G$, $G$ possesses an $\mathcal{F}$-injector and any two $\mathcal{F}$-injectors are conjugate if $\mathcal{F}$ is $\pi$-saturated, i.e. $\mathcal{F} = \{H \leq G : H/H_{\mathcal{F}} \in \mathfrak{C}_\pi\}$. In connection with these theorems, the following question naturally arise:

**Question 1.1** For an arbitrary Fitting set $\mathcal{F}$ of a group $G$ (in case $G$ is a non-$\pi$-soluble group), when $G$ possesses $\mathcal{F}$-injector and any two $\mathcal{F}$-injectors are conjugate?

There has been substantial research on characterizations of $\mathfrak{F}$-injector for various types of soluble Fitting classes $\mathfrak{F}$ (see [5, 6, 7, 9, 10, 12, 14, 15]). It is well known that the product of any two Fitting classes is also Fitting class and multiplication of Fitting classes satisfies associative law (see [3, Theorem IX. (1.12)(a),(c)]). Hartley [9] proved that for the Fitting class of type $\mathfrak{N}$ (where $\mathfrak{N}$ is a nonempty Fitting class and $\mathfrak{N}$ is the Fitting class of all nilpotent groups), a subgroup $V$ of a soluble group $G$ is an $\mathfrak{N}$-injector of $G$ if and only if $V/G_{\mathfrak{N}}$ is a nilpotent subgroup of $G$. As a further improvement, Guo and Vorob’ev [6] proved that for the Hartley class $\mathfrak{H}$, the set of all $\mathfrak{H}$-injectors of soluble group $G$ coincide with the set of all $\mathfrak{H}$-maximal subgroups of $G$ containing $\mathfrak{H}$-radical of $G$.

Let $\mathbb{P}$ be the set of all prime numbers. Following [9], a function $h : \mathbb{P} \to \{\text{nonempty Fitting classes}\}$ is a Hartley function (or in brevity $H$-function). Let $LH(h) = \cap_{p \in \mathbb{P}} h(p) \mathfrak{N}_p$, where $\mathfrak{N}_p$ is the class
of all \( p \)-groups and \( \mathcal{E}_{p'} \) is the class of all soluble \( p' \)-groups. A Fitting class \( \mathcal{H} \) is called a Hartley class if \( \mathcal{H} = LH(h) \) for some \( H \)-function \( h \).

We need to develop and extend the local method of Hartley [9] (for soluble Fitting classes) for Fitting sets of groups (not necessary in the universe of soluble groups). For a Fitting set \( \mathcal{H} \) of a group \( G \) and a nonempty Fitting class \( \mathcal{F} \), we call the set \( \{ H \leq G : H/H \mathcal{H} \in \mathcal{F} \} \) of subgroups of \( G \) the product of \( \mathcal{H} \) and \( \mathcal{F} \), and denote it by \( \mathcal{H} \circ \mathcal{F} \) is a Fitting set of \( G \) (see Lemma 2.1).

Following [16], a function \( h : \mathcal{P} \to \{ \text{Fitting sets of } G \} \) is called a Hartley function of \( G \) (or in brevity an \( H \)-function of \( G \)).

**Definition 1.2** Let \( h \) be an \( H \)-function of a group \( G \) and \( HS(h) = \bigcap_{p \in \mathcal{P}} h(p) \circ (\mathcal{E}_{p'}\mathcal{H}_p) \), where \( \mathcal{E}_{p'} \) is the class of all \( p' \)-groups. A Fitting set \( \mathcal{H} \) of \( G \) called the Hartley set of \( G \) if \( \mathcal{H} = HS(h) \) for some \( H \)-function \( h \).

**Definition 1.3** Let \( \mathcal{H} = HS(h) \) be a Hartley set of a group \( G \). Then \( h \) is said to be:

1. integrated if \( h(p) \subseteq \mathcal{H} \) for all \( p \);
2. full if \( h(p) \subseteq h(q) \circ \mathcal{E}_{q'} \) for all different primes \( p \) and \( q \);
3. full integrated if \( h \) is full and integrated as well.

It is easy to see that every Hartley set of a group \( G \) can be defined by an integrated \( H \)-function. Moreover we prove that every Hartley set of \( G \) can be defined by a full integrated \( H \)-function in Lemma 3.4.

In connection with above, the following question naturally arise:

**Question 1.4** Let \( G \) be a group (in particular, \( G \) is a soluble group), and \( \mathcal{H} \) be a Hartley set of \( G \), what’s the structure of \( \mathcal{H} \)-injectors of \( G \)?

For a Hartley set \( \mathcal{H} = HS(h) \) of \( G \), \( h \) is a full integrated \( H \)-function of \( \mathcal{H} \), we call the subgroup \( G_h = \prod_{p \in \mathcal{P}} G_{h(p)} \) the \( h \)-radical of \( G \). A group \( G \) is said to be \( \mathcal{H} \)-constrained if \( C_{G}(F(G)) \leq F(G) \).

The following theorem resolved the Questions 1.1 and 1.4.

**Theorem 1.5** Let \( \mathcal{H} \) be a Hartley set of a group \( G \) defined by a full integrated \( H \)-function \( h \) and \( G_h \) the \( h \)-radical of \( G \). If \( G/G_h \) is \( \mathcal{H} \)-constrained, then the following statements hold:

1. A subgroup \( V \) of \( G \) is an \( \mathcal{H} \)-injector of \( G \) if and only if \( V/G_h \) is a nilpotent injector of \( G/G_h \);
2. \( G \) possesses an \( \mathcal{H} \)-injector and any two \( \mathcal{H} \)-injectors are conjugate in \( G \);
3. A subgroup \( V \) of \( G \) is an \( \mathcal{H} \)-injector of \( G \) if and only if \( V \) is an \( \mathcal{H} \)-maximal subgroup of \( G \) and \( G\mathcal{H} \leq V \).

Theorem 1.5 give the new theory of \( \mathcal{H} \)-injectors for Fitting sets of non-soluble groups. From Theorem 1.5, a series of famous results can be directly generalized. For example, Fischer [5, Corollary IX.4.13], Hartley [9, section 4.1], Mann [12, Theorem IX.4.12], Guo and Vorob’ev [8, Theorem 5.6.8].
All unexplained notion and terminology are standard. The reader is referred to [3, 8, 2].

2 Preliminaries

Note that if all groups in a class $\mathcal{X}$ are soluble groups (that is $\mathcal{X} \subseteq \mathcal{S}$), then $\mathcal{X}$ is said to be a soluble class.

Lemma 2.1 [13, Proposition 3.1] Let $\mathcal{F}$ be a Fitting set of a group $G$ and $\mathcal{X}$ is a nonempty Fitting class. Then the product $\mathcal{F} \circ \mathcal{X}$ is a Fitting set of $G$.

Lemma 2.2 Let $\mathcal{F}$ and $\mathcal{K}$ be Fitting sets of $G$, and $\mathcal{X}$, $\mathcal{Y}$ be nonempty Fitting formations. Then

(a) [13, Proposition 3.4 (3)] $\mathcal{F} \circ (\mathcal{X} \cap \mathcal{Y}) = \mathcal{F} \circ \mathcal{X} \cap \mathcal{F} \circ \mathcal{Y}$.

(b) [13, Proposition 3.2 (1)] If $\mathcal{M}$ is nonempty Fitting class, then $\mathcal{F} \subseteq \mathcal{F} \circ \mathcal{M}$.

(c) [13, Proposition 3.4 (2)] $(\mathcal{F} \cap \mathcal{K}) \circ \mathcal{X} = \mathcal{F} \circ \mathcal{X} \cap \mathcal{K} \circ \mathcal{X}$.

(d) [13, Proposition 3.4 (1)] If $\mathcal{F} \subseteq \mathcal{M}$, then $\mathcal{F} \circ \mathcal{X} \subseteq \mathcal{M} \circ \mathcal{X}$.

Lemma 2.3 [13, Proposition 3.3] Let $\mathcal{F}$ be a Fitting set of a group $G$ and $\mathcal{X}$, $\mathcal{Y}$ be Fitting formations. Then $$(\mathcal{F} \circ \mathcal{X}) \circ \mathcal{Y} = \mathcal{F} \circ (\mathcal{X} \circ \mathcal{Y})$$

Lemma 2.4 [3, Theorem IV. (1.8)] Let $\mathcal{F}$ and $\mathcal{K}$ be nonempty formations. If $\mathcal{F} \subseteq \mathcal{K}$, then $G^\mathcal{F} \leq G^\mathcal{K}$ for every group $G$.

Lemma 2.5 [3, Proposition VIII. (2.4) (d)] Let $\mathcal{F}$ be a Fitting set of a group $G$. If $N \trianglelefteq G$, then $N^\mathcal{F} = N \cap G^\mathcal{F}$.

Let $\mathcal{F}$ be a nonempty Fitting class. A group $G$ is said to be $\mathcal{F}$-constrained if $C_G(G_\mathcal{F}) \leq G_\mathcal{F}$.

Lemma 2.6 [3, Remark p.624] or [11] The class of all $\mathcal{R}$-constrained groups is a Fitting class strictly large than $\mathcal{S}$.

Lemma 2.7 [3, Theorem IX.(4.12) (c)-(d)] Let $G$ be a group. If $G$ is $\mathcal{R}$-constrained, then $G$ possesses exactly one conjugacy class of nilpotent injectors.

The following properties follow at once from definition of an $\mathcal{F}$-injector of a group $G$ and [3] Remarks IX, (1.3), VIII, (2.6)(2.7)].

Lemma 2.8 Let $\mathcal{F}$ be a Fitting set of a group $G$ and $\mathcal{K}$ a class of finite groups. Then

(a) If $K \trianglelefteq G$ and $V$ is an $\mathcal{F}$-injector of $G$, then $V \cap K$ is an $\mathcal{F}$-injector (or $\mathcal{F}_K$-injector) of $K$;

(b) If $V$ is an $\mathcal{F}$-injector of $G$, then $G_\mathcal{F} \leq V$ and $V$ is an $\mathcal{F}$-maximal subgroup of $G$;

(c) If $V$ is an $\mathcal{F}$-maximal of $G$ and $V \cap M$ is an $\mathcal{F}$-injector $M$ for any maximal normal subgroup $M$ of $G$, then $V$ is an $\mathcal{F}$-injector of $G$.

(d) If $V \in \text{Inj}_\mathcal{F}(G)$ and $\alpha : G \to G_\alpha$ an isomorphism, then $V_\alpha \in \text{Inj}_\mathcal{F}(G_\alpha)$; in particular, $\text{Inj}_\mathcal{F}(G)$ is a union of $G$-conjugacy classes.
Lemma 2.9 [12] or [3, Theorem IX. (4.12)] If $G$ is a $\mathcal{N}$-constrained group, then a subgroup $V$ of $G$ is a nilpotent injector of $G$ if and only if $F(G) \subseteq V$ and $V$ is an $\mathcal{N}$-maximal subgroup of $G$.

3 Hartley set and $h$-radical

In this section we give some results about Hartley sets and $h$-radical of a group $G$, which are also main steps in the proof of Theorem 1.5. Recall that by Lemma 2.1 for a Fitting set $\mathcal{H}$ of $G$ and a nonempty Fitting class $\mathcal{F}$, the set $\mathcal{H} \circ \mathcal{F} = \{ H : H/H\mathcal{H} \in \mathcal{F} \}$ is a Fitting set of $G$. Firstly, we give some following examples of Hartley set.

Example 3.1 (a) Let $\mathcal{N}$ be the trace of the Fitting class $\mathcal{N}$ in group $G$ and let $h$ be an $H$-function defined as follows: $h(p) = \{ 1 \}$ for all $p \in \mathbb{P}$, where 1 is an identity subgroup of $G$. Then by Lemma 2.2 (a), we have $HS(h) = \cap_{p \in \mathbb{P}} \{ 1 \} \circ (\mathcal{E}_p \mathcal{N}_p) = \{ 1 \} \circ \left( \cap_{p \in \mathbb{P}} \mathcal{E}_p \mathcal{N}_p \right) = \{ 1 \} \circ \mathcal{N} = \mathcal{N}$. Hence the set of all nilpotent subgroups of $G$ is a Hartley set of $G$.

(b) Let $\mathcal{F}$ be a Fitting set of $G$ and $\mathcal{H} = \mathcal{F} \circ \mathcal{N}$. Let $h$ be an $H$-function such that $h(p) = \mathcal{F}$ for all $p \in \mathbb{P}$. Then by Lemma 2.2 (a) we obtain $HS(h) = \mathcal{F} \circ \left( \cap_{p \in \mathbb{P}} \mathcal{E}_p \mathcal{N}_p \right) = \mathcal{F} \circ \mathcal{N}$ and so $\mathcal{H}$ is a Hartley set of $G$ (the least equality follows from [3, Lemma 2.7 (a)]).

(c) If $k \in \mathbb{N}$, let $N^k(k \geq 1)$ the set of all subgroups of soluble group $G$ of the nilpotent length at most $k$. If $k \geq 1$, take the $H$-function $h$ such that $h(p) = Tr_{\mathbb{N}^k-1}(G)$ for all $p \in \mathbb{P}$. Then by Example (b) above we have $HS(h) = \mathcal{N}^k$ is a Hartley set of $G$.

(d) Let $\mathcal{H}$ be the trace of Fitting class $\mathcal{E}_p \mathcal{N}_p$ in group $G$, i.e. $\mathcal{H}$ is the set of all $p$-nilpotent subgroups of $G$. Let $h$ be an $H$-function defined as follows: $h(p) = \{ 1 \}$ and $h(q) = \mathcal{H}$ for all primes $q \neq p$. Then by Lemma 2.2 (a), we have $HS(h) = (\{ 1 \} \circ \mathcal{E}_p \mathcal{N}_p) \cap (\mathcal{H} \circ (\cap_{p \neq q} \mathcal{E}_p \mathcal{N}_p)) = \mathcal{H} \cap \mathcal{H} \circ (\mathcal{E}_p \mathcal{N}_p)$. Now by Lemma 2.2 (b) $HS(h) = \mathcal{H}$ and $\mathcal{H}$ is a Hartley set.

Lemma 3.2 Every Hartley set can be defined by an integrated $H$-function.

Proof. Let $\mathcal{H}$ be a Hartley set of a group $G$. Then $\mathcal{H} = HS(h_1)$ for some $H$-function $h_1$. Let $h$ be an $H$-function defined as follows: $h(p) = h_1(p) \cap \mathcal{H}$ for all $p \in \mathbb{P}$. By Lemma 2.2 (c), we have $HS(h) = \cap_{p \in \mathbb{P}} (h_1(p) \cap \mathcal{H}) \circ \mathcal{E}_p \mathcal{N}_p = (\cap_{p \in \mathbb{P}} h_1(p) \circ \mathcal{E}_p \mathcal{N}_p) \cap (\cap_{p \in \mathbb{P}} \mathcal{H} \circ (\mathcal{E}_p \mathcal{N}_p))$.

Hence by Lemma 2.2 (a), (b), $HS(h) = \mathcal{H} \cap \mathcal{H} \circ (\cap_{p \in \mathbb{P}} \mathcal{E}_p \mathcal{N}_p) = \mathcal{H} \cap \mathcal{H} \circ \mathcal{N} = \mathcal{H}$. The Lemma is proved.

Let $\mathcal{H}$ be a set of subgroups of a group $G$. For $\mathcal{H}$ and nonempty Fitting class $\mathcal{F}$, we call the set $\{ H \leq G : H$ has a normal subgroup $L \in \mathcal{H}$ with $H/L \in \mathcal{F} \} \subseteq \mathbb{N}$ of subgroups of $G$ the product of $\mathcal{H}$ and $\mathcal{F}$ and denote it by $\mathcal{H}\mathcal{F}$.

Remark 3.3 Let $\mathcal{H}$ be a Fitting set of a group $G$. Then it is clear that $\mathcal{H} \circ \mathcal{F} \subseteq \mathcal{H}\mathcal{F}$. Assume that $\mathcal{F}$ is the Fitting class and $\mathcal{F}$ is a homomorph, i.e. $\mathcal{F}$ is quotient closed. Put $H \leq G$ and $H \in \mathcal{H}\mathcal{F}$. Then $H/L \in \mathcal{F}$ for some normal subgroup $L \in \mathcal{H}$ of $H$. Since $L \leq H_{\mathcal{H}}$ and $H/L/H_{\mathcal{H}}L \cong H/H_{\mathcal{H}}$,
$H \in \mathcal{H} \circ \mathfrak{F}$. Thus $\mathcal{H}\mathfrak{F} = \mathcal{H} \circ \mathfrak{F}$.

Let $G$ be a group and $\mathfrak{Y}$ is a set of subgroups of a group $G$. Then $\text{Fitset}\mathfrak{Y}$ will denote the intersection of all Fitting sets of $G$ that contain $\mathfrak{Y}$ (see [3, Definition VIII. 3.1 (b)]).

**Lemma 3.4** Every Hartley set can be defined by a full integrated $H$-function.

**Proof.** Let $\mathcal{H}$ be a Hartley set of a group $G$. By Lemma 3.2, $\mathcal{H} = HS(h_1)$ for some integrated $H$-function $h_1$. We define a set of subgroups of $G$ by $\overline{h_1}(p) = \{H \leq G : H$ is conjugate with $R^{p^{p'}}$ in $G$ for some $R \in h_1(p)\}$ for all $p \in \mathbb{P}$. Note that if $H \in \overline{h_1}(p)$, then $H \in h_1(p)$ and so $\overline{h_1}(p) \subseteq h_1(p)$ for all $p \in \mathbb{P}$.

Assume that $X \in \overline{h_1}(p)e_{p'}$. Then $X$ has a normal subgroup $K \in \overline{h_1}(p)$ with $X/K \in e_{p'}$. Since $\overline{h_1}(p) \subseteq h_1(p)$, $K \leq X_{h_1(p)}$. Hence by the isomorphism $X/K/X_{h_1(p)}/K \cong X/X_{h_1(p)}$, we have $X/X_{h_1(p)} \in e_{p'}$ and so $X \in h_1(p) \circ e_{p'}$. Thus $\overline{h_1}(p)e_{p'} \subseteq h_1(p)e_{p'}$.

On other hand, let $Y \in h_1(p) \circ e_{p'}$. Then $Y/Y_{h_1(p)} \in e_{p'}$ and $Y^{e_{p'}} \leq h_1(p)$. Hence $Y^{e_{p'}} \in h_1(p)$. Since $(Y^{e_{p'}})^{e_{p'}} = Y^{e_{p'}}$ and obviously, $Y^{e_{p'}}$ is a subgroup conjugate with $Y^{e_{p'}}$ in $G$, we have $Y^{e_{p'}} \in \overline{h_1}(p)$ and so $Y \in \overline{h_1}(p)e_{p'}$. Thus we obtain the following equation:

$$\overline{h_1}(p)e_{p'} = h_1(p) \circ e_{p'}.$$ 

Now, let $h$ be a function such that $h(p) = \text{Fitset}(\overline{h_1}(p))$ for all $p \in \mathbb{P}$. We prove that $HS(h) = \mathcal{H}$. Since $\overline{h_1}(p) \subseteq h_1(p)$, $\text{Fitset}(\overline{h_1}(p)) \subseteq \text{Fitset}(h_1(p)) = h_1(p)$ and so $h(p) \subseteq h_1(p)$. Hence by Lemma 2.2 (d), we have $h(p) \circ e_{p'} \subseteq h_1(p) \circ e_{p'}$. Note that by Lemma 2.3 $(h(p) \circ e_{p'}) \circ \mathfrak{N}_p = h(p) \circ (e_{p'} \mathfrak{N}_p)$ and $(h(p) \circ e_{p'}) \circ \mathfrak{N}_p = h_1(p) \circ (e_{p'} \mathfrak{N}_p)$. Therefore by Lemma 2.2 (d) $h(p) \circ (e_{p'} \mathfrak{N}_p) \subseteq h_1(p) \circ (e_{p'} \mathfrak{N}_p)$ for all $p \in \mathbb{P}$. Consequently, $HS(h) \subseteq \mathcal{H}$.

Further more, by (*), we have $h_1(p) \circ e_{p'} = \text{Fitset}(\overline{h_1}(p)e_{p'})$. Since $\overline{h_1}(p) \subseteq \text{Fitset}(\overline{h_1}(p))$, by Lemma 2.2 (d) $\overline{h_1}(p)e_{p'} \subseteq (\text{Fitset}(\overline{h_1}(p))) \circ e_{p'}$. Hence $\text{Fitset}(\overline{h_1}(p)e_{p'}) \subseteq (\text{Fitset}(\overline{h_1}(p))) \circ e_{p'}$. Thus, for every $p \in \mathbb{P}$, we have the inclusion:

$$\overline{h_1}(p)e_{p'} \subseteq (\text{Fitset}(\overline{h_1}(p))) \circ e_{p'} = h(p) \circ e_{p'}$$

By Lemma 2.3 and Lemma 2.2, we have $h_1(p) \circ (e_{p'} \mathfrak{N}_p) \subseteq h(p) \circ (e_{p'} \circ \mathfrak{N}_p)$. Hence $\mathcal{H} \subseteq HS(h)$ and $\mathcal{H} = HS(h)$.

Since $\overline{h_1}(p) \subseteq h_1(p)$ for all $p \in \mathbb{P}$ and $h_1$ is an integrated $H$-function of $\mathcal{H}$, $h(p) \subseteq \mathcal{H}$ for all $p \in \mathbb{P}$ and so $h$ is an integrated $H$-function of $\mathcal{H}$.

Now, we show that $h(p) \subseteq h(q) \circ e_{q'}$ for all $p \neq q$.

Let $H$ be an arbitrary subgroup in $h_1(p)$ and $p \neq q$. Since $h_1$ is an integrated $H$-function, $H \in \mathcal{H}$ and so $H \in h_1(q) \circ (e_{q'} \mathfrak{N}_q)$. By Lemma 2.3 $h_1(q) \circ (e_{q'} \mathfrak{N}_q) = (h_1(q) \circ e_{q'}) \circ \mathfrak{N}_q$. Hence $H^{\mathfrak{N}_q} \in h_1(q) \circ e_{q'}$. Since $p \neq q$, $H^{e_{q'}} \leq H^{\mathfrak{N}_q}$ by Lemma 2.4. Consequently, $H^{e_{q'}} \in h_1(q)e_{q'}$. Then by (**), $H^{e_{q'}} \in h(q) \circ e_{q'}$ for every $H \in h_1(p)$ and all primes $p \neq q$. 


Let $R \in \overline{\mathbf{r}_1}(p)$. Then by definition of the set $\overline{\mathbf{r}_1}(p)$, we have that $R$ is a conjugate subgroup of $G$ with $S^{\mathbf{r}_1}$ for some subgroup $S \in h_1(p)$. Therefore $R \in h(q)\mathbf{r}_q'$ and so $h_1(p) \subseteq h(q) \circ \mathbf{r}_q'$. Thus $h(p) = \text{Fitset}(\overline{h_1}(p)) \subseteq \text{Fitset}(h(q) \circ \mathbf{r}_q') = h(q) \circ \mathbf{r}_q'$ for all primes $q \neq p$. This complete the proof.

Recall that $G_h = \prod_{p \in \mathbb{P}} G_{h(p)}$, where $h$ is a full integrated $H$-function of a Hartley set of a group $G$, i.e. $h$ is an $h$-radical of $G$.

**Lemma 3.5** Let $\mathcal{H}$ be a Hartley set of a group $G$ and $h$ is a full integrated $H$-function of $\mathcal{H}$. If $H$ is a subgroup of $G$ such that $G_h \leq H$ and $H/G_h$ is a nilpotent subgroup of $G/G_h$, then $H \in \mathcal{H}$.

**Proof.** Let $q$ be an arbitrary prime number. Since $G_{h(q)} \leq H$, $G_{h(q)} \leq H_{h(q)}$ for all $q \in \mathbb{P}$. Let $p \in \mathbb{P}$ and $p \neq q$. Note that $G_{h(q)} G_{h(p)}/G_{h(q)} \cong G_{h(p)}/G_{h(q)} \cap G_{h(p)} = G_{h(p)}/(G_{h(p)}/G_{h(q)}).$ Since $h$ is a full integrated $H$-function of $\mathcal{H}$, $h(p) \subseteq h(q)\mathbf{r}_q'$. Hence $G_{h(p)} \subseteq h(q)\mathbf{r}_q'$. Therefore $G_{h(q)} G_{h(p)}/G_{h(q)} \subseteq \mathbf{r}_q'$ for all primes $p$ and $q$. Consequently, $G_h/G_{h(q)} \subseteq \mathbf{r}_q'$ and by using the isomorphisms $H_{h(q)} G_h/H_{h(q)} \cong G_{h(q)}/G_{h(q)} \cap G_{h(p)} \cong (G_{h(q)}/G_{h(q)})/(H_{h(q)} \cap G_{h(p)}),$ we obtain that $H_{h(q)} G_h/H_{h(q)}$ is a $q'$-group. Since $H/G_h$ is a nilpotent subgroup of $G/G_h$, $H/H_{h(q)} G_h$ is also a nilpotent subgroup of $G/G_h$ and so $H/H_{h(q)} G_h \in \cap_{q \in \mathbb{P}} \mathbf{r}_q'$. Therefore, by the isomorphism $H/H_{h(q)} G_h \cong (H/H_{h(q)})/(H_{h(q)} G_h/H_{h(q)}),$ we have that $H/H_{h(q)} G_h \in \cap_{q \in \mathbb{P}} \mathbf{r}_q'$ for all $q \in \mathbb{P}$. Hence $H \in \cap_{q \in \mathbb{P}} h(q)\mathbf{r}_q' = \mathcal{H}$. The Lemma is proved.

**Lemma 3.6** Let $\mathcal{H}$ be a Hartley set of a group $G$. If $h$ is a full integrated $H$-function of $\mathcal{H}$, then $G_{\mathcal{H}}/G_h = F(G/G_h)$.

**Proof.** Let $F(G/G_h) = R/G_h$. Since $h$ is a integrated $H$-function of $\mathcal{H}$, $(G_{\mathcal{H}})_{h(p)} = G_{h(p)}$. Hence $G_{\mathcal{H}} G_{h(p)} \subseteq \mathbf{r}_p' \mathbf{r}_p$ for all $p \in \mathbb{P}$ and so $G_{\mathcal{H}}/G_h$ is a nilpotent subgroup of $G/G_h$. Hence $G_{\mathcal{H}}/G_h \leq F(G/G_h)$ and we have that $G_{\mathcal{H}} \leq R$.

On the other hand, since $R/G_h$ is a nilpotent subgroup of $G/G_h$, by Lemma 3.5, $R \in \mathcal{H}$. Hence $R \leq G_{\mathcal{H}}$. Thus $F(G/G_h) = G_{\mathcal{H}}/G_h$. The Lemma is proved.

**Lemma 3.7** Let $h$ be a full integrated $H$-function of a Hartley set $\mathcal{H}$ of a group $G$. If $G/G_h$ is $\mathcal{H}$-constrained, $G_{\mathcal{H}} \leq H \leq G$ and $H \in \mathcal{H}$, then $H/G_h$ is nilpotent.

**Proof.** Since $G_{\mathcal{H}} \leq H$ and $h$ is an integrated $H$-function of $\mathcal{H}$, by Lemma 2.5, $G_{h(p)} = (G_{\mathcal{H}})_{h(p)} = H_{h(p)} \cap G_{\mathcal{H}}$. Hence $[H_{h(p)}, G_{\mathcal{H}}] \leq H_{h(p)} \cap G_{\mathcal{H}} = G_{h(p)}$ and so $H_{h(p)} \leq C_G(G_{\mathcal{H}}/G_{h(p)}) \leq C_G(G_{\mathcal{H}}/G_h)$. Since $G/G_h$ is constrained and by Lemma 3.6 $G_{\mathcal{H}} G_h = F(G/G_h)$, $C_G(G_{\mathcal{H}}/G_h) \leq G_{\mathcal{H}}$. Thus $H_{h(p)} \leq G_{\mathcal{H}}$. Therefore, $G_{h(p)} = (G_{\mathcal{H}})_{h(p)} = H_{h(p)} \cap G_{\mathcal{H}} = H_{h(p)}$ for all $p \in \mathbb{P}$. Hence, $H/G_h = H/H_{h(p)} \in \mathbf{r}_p' \mathbf{r}_p$ for all $p \in \mathbb{P}$ and so $H/G_h$ is a nilpotent group. This complete the proof.

**Corollary 3.8** Let $h$ be a full integrated $H$-function of a Hartley set of a group $G$. Let $G/G_h$ be an $\mathcal{H}$-constrained group and $G_{\mathcal{H}} \leq H \leq G$. Then $H \in \mathcal{H}$ if and only if $H/G_h$ is nilpotent.
4 Proof and Some Applications of Theorem 1.5

Proof of Theorem 1.5 (1) We first prove that if $V$ is a subgroup of a group $G$ such that $V/G_h$ is a nilpotent injector of $G/G_h$, then $V$ is an $\mathcal{H}$-injector of $G$. We prove this statement by induction on the order of $G$.

Let $M$ be an arbitrary maximal normal subgroup of $G$ and $M_h$ is an $h$-radical of $M$.

Since $h$ is a full integrated $H$-function of Hartley set $\mathcal{H}$, $h(p) \subseteq h(q) \circ \mathcal{E}_q$ and $h(p) \subseteq \mathcal{H}$, for all different $p, q \in \mathbb{P}$. Then, by the isomorphism $G_{h(q)}G_{h(p)}/G_{h(q)} \cong G_{h(p)}/G_{h(q)} \cap G_{h(q)} = G_{h(p)}/(G_{h(p)})_h(q)$, we see that $G_{h(q)}G_{h(p)}/G_{h(q)}$ is a $q'$-group of $G/G_{h(q)}$ for any prime $q$. Hence $G_{h(q)}$ is also a $q'$-group. Since $(G_h \cap M)G_{h(q)}/G_{h(q)} \leq G_h/G_{h(q)}$, by the isomorphism $(G_h \cap M)(G_{h(q)}/G_{h(q)} \cong (G_h \cap M)/(G_h \cap M) \cap G_{h(q)} = (G_h \cap M)/G_{h(q)} \cap M$, by Lemma 2.5, we obtain that $G_h \cap M/M_{h(q)}$ is a $q'$-group, for all $q \in \mathbb{P}$. Now, note that $(G_h \cap M)/M_{h(q)}/M_h/M_{h(q)} \cong G_h \cap M/M_h$. Hence $G_h \cap M/M_h \in \cap q \in \mathbb{P} \mathcal{E}_q' = (1)$ and so $G_h \cap M = M_h$.

If $G_h \leq M$. Then $G_h = M_h$.

Since $V/G_h$ is a nilpotent injector of $G/G_h$, $V \cap M/G_h$ is a nilpotent injector of $M/G_h$ and consequently $V \cap M/M_h$ is a nilpotent injector of $M/M_h$. Hence, by induction, $V \cap M$ is an $\mathcal{H}$-injector of $M$.

Now, in order to complete the proof of the statement, we only need to prove that $V$ is an $\mathcal{H}$-maximal subgroup of $G$. Since $V/G_h$ is nilpotent and $G_{\mathcal{H}} \leq V$, by Lemma 3.5 $V \in \mathcal{H}$. Assume that $V < V_1$, where $V_1$ is an $\mathcal{H}$-maximal subgroup of $G$. Since $V \cap M$ is an $\mathcal{H}$-maximal subgroup of $M$, $V \cap M = V_1 \cap M$. Hence $V_1$ is an $\mathcal{H}$-maximal subgroup of $G$ and $V \cap M$ is an $\mathcal{H}$-injector of $M$ for any maximal normal subgroup $M$ of $G$. Consequently, by Lemma 2.8 (b), (c), $V_1$ is an $\mathcal{H}$-injector of $G$ and $G_{\mathcal{H}} \leq V_1$. Then, by Corollary 3.8, we have $V_1/G_h$ is a nilpotent subgroup of $G/G_h$, contrary to the fact that $V/G_h$ is $\mathcal{H}$-maximal in $G/G_h$. Hence $V = V_1$ and so by Lemma 2.8 (c), $V$ is an $\mathcal{H}$-injector of $G$.

If $G_h \not\leq M$. In this case, by the maximality of $M$, we have that $G = G_hM$. Since, by Lemma 2.8(d), $G_h \cong M/G_h \cap M = M/M_h$ and so $V \cap M/M_h$ is a nilpotent injector of $M/M_h$. Then, by induction, $V \cap M$ is an $\mathcal{H}$-injector of $M$. By Lemma 3.5, we know that $V \in \mathcal{H}$. If $V < F_1$, where $F_1$ is an $\mathcal{H}$-maximal subgroup of $G$, then $V \cap M = F_1 \cap M$. Since $G_h \leq V$, $VM = G$. Consequently, $F_1 = F_1 \cap VM = V(F_1 \cap M) = V(V \cap M) = V$ and so $V$ is an $\mathcal{H}$-maximal subgroup of $G$. Therefore $V$ is an $\mathcal{H}$-injector of $G$.

Conversely, let $V$ be an $\mathcal{H}$-injector of $G$. We prove that $V/G_h$ is a nilpotent injector of $G/G_h$. By Lemma 2.8 (b),(c) $G_{\mathcal{H}} \leq V$ and $V$ is $\mathcal{H}$-maximal in $G$. Hence by Lemma 3.5, $V/G_h$ is nilpotent. Since $V$ is $\mathcal{H}$-maximal in $G$, we obtain that $V/G_h$ is $\mathcal{H}$-maximal subgroup of $G/G_h$ containing the nilpotent radical of $G/G_h$. Consequently, by Lemma 2.9, $V/G_h$ is a nilpotent injector of $G/G_h$. Thus, statement (1) hold.

(2) The existence of $\mathcal{H}$-injectors has proved in (1). Let $F/G_h$ and $V/G_h$ are nilpotent injectors of
Then by Lemma 2.7 \( F/G_h \) and \( V/G_h \) are conjugate in \( G/G_h \). Hence \( F \) and \( V \) are conjugate in \( G \).

(3) Let \( V \) be an \( \mathcal{H} \)-injector of \( G \). Then by Lemma 2.8 (b), (c), \( V \geq G_2^\mathcal{H} \) and \( V \) is \( \mathcal{H} \)-maximal in \( G \).

Conversely, let \( V \) be an \( \mathcal{H} \)-maximal subgroup of \( G \) and \( V \geq G_h \). We prove that \( V \) is an \( \mathcal{H} \)-injector of \( G \). Clearly, \( G_h \leq V \). Then by Lemma 3.5, \( V/G_h \) is nilpotent. Since \( V \) is \( \mathcal{H} \)-maximal in \( G \), \( V/G_h \) is \( \mathcal{H} \)-maximal in \( G/G_h \). Now by Lemma 3.6, \( V/G_h \geq F(G/G_h) \). Hence by Lemma 2.9, \( V/G_h \) is a nilpotent \( \mathcal{H} \)-injector of \( G/G_h \). This complete the proof of the theorem.

Let \( X \) be a Fitting set of a group \( G \) and \( X \odot \mathcal{S} = \{ H \leq G : H/H_X \in \mathcal{S} \} \), where \( \mathcal{S} \) is a class of all soluble groups. Note that the set \( X \odot \mathcal{S} \) is a Fitting set by Lemma 2.1.

We give some applications of our main results

**Corollary 4.1** Let \( G \in X \odot \mathcal{S} \) and \( \mathcal{H} = X \odot N \) is a Fitting set of \( G \). Then:

(1) A subgroup \( V \) of \( G \) is an \( \mathcal{H} \)-injector of \( G \) if and only if \( V/G_X \) is nilpotent injector of \( G/G_X \).

(2) The set of all \( \mathcal{H} \)-injectors of \( G \) is exactly the subgroups \( V \) of \( G \) such that \( V \geq G_h \) and \( V \) is \( \mathcal{H} \)-maximal in \( G \).

**Proof.** By Example 3.1 (b), \( \mathcal{H} \) is a Hartley set of \( G \), which can be defined by full integrated \( H \)-function \( h \) such that \( h(p) = X \) for all \( p \in \mathbb{P} \). Since \( G/G_X \) is soluble, \( G/G_X \) is \( \mathcal{H} \)-constrained.

Let \( X \) be a nonempty Fitting class and \( \mathcal{H} = X \odot M \) is a Fitting product of \( X \) and \( M \), then we have the following result immediately from our Theorem.

**Corollary 4.2** (1) A subgroup \( V \) of a group \( G \in \mathcal{F} \odot \mathcal{S} \) is an \( \mathcal{H} \)-injector of \( G \) if and only if \( V/G_X \) is a nilpotent injector of \( G/G_X \).

(2) A subgroup \( V \) of a group \( G \in \mathcal{F} \odot \mathcal{S} \) is an \( \mathcal{H} \)-injector if and only if \( V \geq G_h \) and \( V \) is \( \mathcal{H} \)-maximal in \( G \).

**Proof.** Since the set of all \( X \odot M \)-injectors of \( G \) and the set of all \( Tr_{X \odot M}(G) \)-injectors of \( G \) are coincide, so Corollary 4.2 holds from Corollary 4.1.

**Corollary 4.3** (Hartley [9]) Let \( X \) be a nonempty soluble Fitting class and \( \mathcal{H} = X \odot M \). A subgroup \( V \) of soluble group \( G \) is an \( \mathcal{H} \)-injector if and only if \( V/G_X \) is a nilpotent injector of \( G/G_X \).

**Corollary 4.4** (Fischer [5]) A subgroup \( V \) of soluble group \( G \) is a nilpotent injector of \( G \) if and only if \( F(G) \leq V \) and \( V \) is \( \mathcal{H} \)-maximal in \( G \).

**Corollary 4.5** (Guo and Vorob’ev [6]) Let \( \mathcal{H} \) be a soluble Hartley class and \( G \) is a soluble group. Then a subgroup \( V \) of \( G \) is an \( \mathcal{H} \)-injector of \( G \) if and only if \( V/G_h \) is a nilpotent injector of \( G/G_h \).

**Corollary 4.6** Let \( N^k(k \geq 1) \) be a Fitting set of all subgroup of a soluble group \( G \) with a nilpotent length at most \( k \). Then the set of all \( N^k \)-injectors of \( G \) is exactly the set of all subgroups \( V \) of \( G \) such that \( V/G_{N^k-1} \) is a nilpotent of \( G/G_{N^k-1} \). In particular, a subgroup \( V \) is an \( N^2 \)-injector of \( G \) if
and only if $V/F(G)$ is a nilpotent injector of $G/F(G)$.

**Proof.** By Example 3.1 (c), $N^k$ is a Hartley set of $G$ and the function $h$ such that $h(p) = Tr_{N^k-1}(G)$ for all $p \in \mathbb{P}$ is a full integrated $H$-function of $N^k$. Since a group $G$ is soluble, $G/G_h$ is $H$-constrained and so the Corollary hold from our Theorem.

**Corollary 4.7** (Guo and Vorob’ev [6]) Let $\mathfrak{N}^k$ ($k \geq 1$) be the class of all groups with nilpotent length at most $k$ and $G$ a soluble group. Then the set of all $\mathfrak{N}^k$-injectors of $G$ is exactly the set of all subgroups $V$ of $G$ such that $V/G_{\mathfrak{N}^k-1}$ is a nilpotent injector of $G/G_{\mathfrak{N}^k-1}$. In particular, a subgroup $V$ of a soluble group $G$ is metanilpotent injector if and only if $V/F(G)$ is a nilpotent injector of $G/F(G)$.

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