ON THE GLOBAL CONVERGENCE OF FREQUENCY SYNCHRONIZATION FOR KURAMOTO AND WINFREE OSCILLATORS

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Abstract. We investigate the collective behavior of synchrony for the Kuramoto and Winfree models. We first prove the global convergence of frequency synchronization for the non-identical Kuramoto system of three oscillators. It is shown that the uniform boundedness of the diameter of the phase functions implies complete frequency synchronization. In light of this, we show, under a suitable condition on the coupling strength and deviation of the intrinsic frequencies, that the diameter function of the phases is uniformly bounded. In a similar spirit, we also prove the global convergence of phase-locked synchronization for the Winfree model of $N$ oscillators for $N \geq 2$.

1. Introduction. In this paper, we investigate the collective behavior of synchrony for the Kuramoto and Winfree models. More specifically, we prove that both models exhibit the global frequency synchronization (independent of initial phase configurations), under certain condition on the coupling strength and deviation of the intrinsic frequencies. First we consider the Kuramoto model:

$$\dot{\theta}_i = \omega_i + \frac{K}{N} \sum_{j=1}^{N} \sin(\theta_j - \theta_i), \quad t > 0, \quad i = 1, 2, \ldots, N,$$

(1.1)

where $\theta_i = \theta_i(t) \in \mathbb{R}$ is the phase of $i$-th oscillator with the intrinsic frequency $\omega_i$, $K$ is the coupling strength constant, $N$ is the number of the oscillators, and $\theta$ denotes the first derivative of $\theta$.

(1.1) is called the Kuramoto model of the first order, and it is first introduced by Yoshiki Kuramoto in [11, 10] to describe synchronization phenomena observed in the systems of chemical and biological oscillators. This model makes assumptions that (i) the oscillators are all-to-all, weakly coupled, (ii) the interactions between two oscillators depends sinusoidally on the phase difference. There have been extensive studies for the Kuramoto model, and we refer the interested readers to [1, 3, 5, 6, 14, 13] for more details of the model. Especially, in [13], the motivation and...
background of the works by Kuramoto [11, 10] and Winfree [15] have been clearly illustrated. The infinite-dimensional oscillator networks are surveyed in detail in [1, 13].

Among many, we address some interesting results for (1.1). In [2], they extend the existing result of frequency synchronization with the initial configuration confined to the half circle, to a class of initial configurations lying on an arc of length slightly greater than $\pi$. In [7], the frequency synchronization is proved for a generic large initial configuration set. More specifically they prove that for a fixed initial configuration, if the coupling strength is large enough (depending on the choice of initial data), the system achieves frequency synchronization.

In order to discuss our results, we shall introduce some notations and functions. Let
$$\Theta(t) := (\theta_1(t), \theta_2(t), \theta_3(t), \cdots, \theta_N(t))$$
be a vector-valued function $\Theta : [0, \infty) \to \mathbb{R}^N$, and
$$\Omega := (\omega_1, \omega_2, \omega_3, \cdots, \omega_N)$$
be a vector in $\mathbb{R}^N$. For any finite dimensional vector $X = (x_1, x_2, x_3, \cdots, x_N)$, we define
$$D(X) := \max_{i \neq j} (x_i - x_j).$$

Let $\omega$ be the mean of $\Omega$, i.e.,
$$\omega := \frac{1}{N} \sum_{i=1}^{N} \omega_i. \quad (1.2)$$

**Definition 1.** We say the Kuramoto oscillator ensemble, $\Theta(t)$, achieves the complete-frequency synchronization asymptotically (depending on the initial data, $\Theta(0)$) if
$$\lim_{t \to \infty} \dot{\theta}_i(t) = \omega \quad \text{for all } 1 \leq i \leq N. \quad (1.3)$$

In general, in order to show synchronization, some restriction on the initial data, $\Theta(0)$, is required. For instance, $\max_{i \neq j} (\theta_i(0) - \theta_j(0)) < \alpha$ for some number $\alpha \in [0, \pi)$. I.e., the initial phases are confined in an arc of certain angle. (1.3) means that the differences of the velocities of any two different oscillators tend to zero as time tends to infinity. In fact each velocity of the $i$-th oscillator tends to the same constant velocity, i.e., the oscillators will have the same frequencies in the limit of $t \to \infty$.

**Definition 2.** The Kuramoto system (1.1) is said to exhibit the global convergence of frequency synchronization if for any $\Theta(0) \in \mathbb{R}^N$ (no restriction on the initial data), the solution, $\Theta(t)$, achieves the complete frequency synchronization, i.e., it satisfies
$$\lim_{t \to \infty} \dot{\theta}_i(t) = \omega \quad \text{for all } 1 \leq i \leq N.$$
Definition 4. The diameter function is defined by

$$D(\Theta(t)) := \max_{1 \leq i, j \leq N} (\theta_i(t) - \theta_j(t)). \quad (1.4)$$

Here the right hand side is maximized by a pair of oscillators, say $\theta_m$ and $\theta_n$, i.e., $D(\Theta(t)) = \theta_m(t) - \theta_n(t)$ for some $m, n \in \{1, 2, ..., N\}$, and it is called a representation of $D(\Theta(t))$. When the collision occurs at a certain moment $t = t_0$, there are more than one representation of the diameter functions.

Now we present our main theorem on the global frequency synchronization for (1.1) with $N = 3$. Define $\lambda_3$ and $\beta_3 \in [0, \pi]$ by

$$\lambda_3 := \max_{\beta \in [0, \pi]} \left( \frac{2}{3} \sin \left( \frac{\beta}{2} \right) \left( 2 \cos \left( \frac{\beta}{2} \right) - 1 \right) \right)$$

$$= \frac{2}{3} \sin \left( \frac{\beta_3}{2} \right) \left( 2 \cos \left( \frac{\beta_3}{2} \right) - 1 \right) \approx 0.24601.$$  \quad (1.5)

Here $\beta_3 \approx 1.1357 \approx 65.07^\circ$.

Theorem 1.1 (Global convergence of frequency synchronization). Let $N = 3$. Suppose that

$$D(\Omega)/K < \lambda_3. \quad (1.6)$$

Then the oscillators achieve the global frequency synchronization asymptotically. I.e., for any $\Theta(0) \in \mathbb{R}^3$, the initial value problem for (1.1) admits a unique solution, $\Theta \in C^\infty([0, \infty))$, satisfying

$$\lim_{t \to \infty} \dot{\theta}_i(t) = \omega \quad \text{for } i = 1, 2, 3,$$

where

$$\omega := \frac{1}{3}(\omega_1 + \omega_2 + \omega_3).$$

The proof of Theorem 1.1 will be given in section 2. We do not know how sharp the number $\lambda_3$ given in (1.5) is to ensure the global convergence of synchronization. Finding the optimal condition seems a quite challenging problem.

We remark that Theorem 1.1 does not require any condition on the initial phase, $\Theta(0)$. I.e., for any initial phase, the complete frequency synchronization is achieved time-asymptotically. This is referred to as the global convergence of synchronization.

It is easy to see that, for $N = 2$ and $\lambda_2 = 1$, the system exhibits the global frequency synchronization, and that the number $\lambda_2$ is sharp. For instance, if $D(\Omega) = |\omega_1 - \omega_2| > K$, one can easily see that the oscillators are never synchronized. Moreover, it is numerically demonstrated that the coupling strength constant $K$ must be large enough, compared to the variation of frequencies, for a group of non-identical oscillators to synchronize. For this reason, the coupling strength condition in Theorem 1.1 seems an inevitable condition for the global convergence of synchrony. Interesting numerical examples of unsynchronized oscillators when the coupling strength condition fails, are presented in Section 4 (See Figure 4.2 and (I)', (II)' and (III)' in Table 2 and Table 3).

It is naturally conjectured, by numerical tests and also by a simple example of two-oscillator, that $N$ Kuramoto oscillators will synchronize globally in a certain large coupling regime, i.e., $D(\Omega)/K$ is suitably small. For the related numerical tests, see the example of (III) in Table 2 and Table 3 whose initial phases are distributed over the circle, i.e., no restriction on the phase, but it exhibits the frequency synchronization. However, up to date, we are not aware of any rigorous
result on global convergence of synchronization beyond the case of $N = 2$. For some related work, we refer the readers to [7].

For any initial data, $\Theta(0) \in \mathbb{R}^N$, the solution, $\Theta(t)$, to (1.1) is analytic in $t \geq 0$. Thanks to this, it is easy to see that the diameter function $D(\Theta(t))$ is a continuous function. However, $D(\Theta(t))$ may not be $C^1$ for all $t > 0$ due to the collision of oscillators. In fact, for (1.1) of the non-identical oscillators, multiple collisions occur, in general. This is a significantly different feature from the identical oscillator case, i.e., $\omega_i = \omega$ for all $i$, for which no collision occurs thanks to the nature of the first order autonomous ODEs (one can prove it by the uniqueness argument). This causes singularities of $D(\Theta(t))$, i.e., $C^1$ regularity of the diameter function may break down upon the collision. Hence, to investigate the rate of change of the diameter function, we will need to consider the first derivative of all the representations for $D(\Theta(t))$. We emphasize that our proofs do not require $C^1$ regularity of $D(\Theta(t))$.

Recall that $\omega$ is the mean of $\Omega$, i.e., (1.1). When it comes to the analysis of the frequency synchronization, by replacing $\theta_i$ by $\theta_i - \omega t$, it is always handy to assume the mean zero condition on the natural frequencies

$$\omega = \frac{1}{N} \sum_{i=1}^{N} \omega_i = 0.$$  \hfill (1.7)

One of the key features of the proofs is, by using an appropriate Lyapunov functional, to establish a simple sufficient condition for the frequency synchronization, that is the uniform boundedness of “phases”, i.e., $\sup_{t \geq 0} D(\Theta(t)) < \infty$. More specifically, we define

$$E_N(t) := \sum_{i=1}^{N} |\dot{\theta}_i(t)|^2.$$  \hfill (1.10)

We observe, using symmetry of the equations and (1.7), that the center of phases is conserved:

$$\bar{\theta}(t) := \frac{1}{N} \sum_{i=1}^{N} \theta_i(t) = \frac{1}{N} \sum_{i=1}^{N} \theta_i(0),$$  \hfill (1.8)

which can be easily checked by

$$\frac{d}{dt} \sum_{i=1}^{N} \theta_i(t) = \sum_{i=1}^{N} \omega_i + \frac{K}{N} \sum_{i,j=1}^{N} \sin(\theta_i(t) - \theta_j(t)) = 0.$$  \hfill (1.9)

By multiplying (1.1) by $\dot{\theta}_i$, summing over $i = 1, \cdots, N$, and integrating it over $[0, t]$, we obtain

$$\int_{0}^{t} E_N(s)ds = \int_{0}^{t} \Lambda_N(s)ds + \int_{0}^{t} H_N(s)ds,$$  \hfill (1.10)

where

$$\Lambda_N(t) := \sum_{i=1}^{N} \omega_i \dot{\theta}_i(t) \quad \text{and} \quad H_N(t) := \frac{K}{N} \sum_{i,j=1}^{N} \sin(\theta_j - \theta_i) \dot{\theta}_i(t).$$  \hfill (1.11)

Here, by symmetry, we find that

$$H_N(t) = \frac{K}{N} \frac{d}{dt} \sum_{1 \leq i < j \leq N} \cos(\theta_j - \theta_i)(t).$$  \hfill (1.12)
In order to prove the frequency synchronization, it suffices to establish the uniform bounds for the right-hand side of (1.10), i.e.,
\[
\sup_{t \geq 0} \left| \int_0^t \Lambda_N(s) \, ds \right| < \infty \quad \text{and} \quad \sup_{t \geq 0} \left| \int_0^t H_N(s) \, ds \right| < \infty. \tag{1.13}
\]
In fact, if (1.13) holds, the following limit exists and is finite:
\[
\sup_{t \geq 0} \int_0^t E_N(s) \, ds = \int_0^\infty E_N(s) \, ds < \infty.
\]
This together with the fact that \(E_N(t)\) is uniformly continuous on \([0, \infty)\) implies that \(E_N(t) \to 0\) as \(t \to \infty\). This immediately implies the complete frequency synchronization, i.e., \(\dot{\Theta}(t) \to 0\) as \(t \to \infty\). A more detailed discussion of this argument is presented inLemma 1 and its proof.

On the other hand, it is easy to see that the second term of the right-hand side of (1.10) is uniformly bounded, i.e.,
\[
\int_0^t H_N(s) \, ds = \frac{K}{N} \sum_{1 \leq i < j \leq N} \cos(\theta_j - \theta_i)(s) \bigg|_{s=0}^{s=t} \leq K(N - 1), \tag{1.14}
\]
and that the first term is given by
\[
\int_0^t \Lambda_N(s) \, ds = \sum_{i=1}^N \omega_i (\theta_i(t) - \theta_i(0)). \tag{1.15}
\]

**Remark 1.** It is obvious that \(\sup_{t \geq 0} \int_0^t \Lambda_N(s) \, ds < \infty\) if
\[
\sup_{t \geq 0} |\theta_i(t)| < \infty, \tag{1.16}
\]
for all \(i = 1, 2, 3, \ldots, N\). By (1.8) together with the definition of \(D(\Theta(t))\), one has that (1.16) is equivalent to
\[
\sup_{t \geq 0} D(\Theta(t)) < \infty. \tag{1.17}
\]
This is a sufficient condition for the complete frequency synchronization. When we prove our theorems in what follows, we will verify this sufficient condition, i.e., establishing the uniform bound of \(D(\Theta(t))\).

Next, we discuss the global asymptotic complete-frequency synchronization for the following Winfree model:
\[
\dot{\theta}_i = \omega_i - \sum_{j=1}^N k_{ij} \sin \theta_i(1 + \cos \theta_j), \quad i = 1, 2, 3, \ldots, N, \tag{1.18}
\]
where, similarly as before, \(\theta_i = \theta_i(t) \in \mathbb{R}\) is the phase of \(i\)th oscillator with the intrinsic natural frequency \(\omega_i\), \(N \geq 2\) is the number of the oscillators and \(k_{ij}\) is the coupling constant between \(i\)-th and \(j\)-th oscillators satisfying
\[
k_{ij} = k_{ji} \geq 0 \quad \text{for all} \ i, j = 1, 2, \ldots, N. \tag{1.19}
\]

The Winfree model (1.18)-(1.19) is first proposed in [15] to describe the synchronization phenomena of a population of organisms or oscillators. It is assumed that the oscillators are all-to-all and weakly coupled. Unlike the Kuramoto oscillators, the zero mean condition of natural frequencies, i.e., \(\sum_{i=1}^N \omega_i = 0\), is irrelevant due to the structure of the model, and the total phase is not conserved, in general.
Due to this, we will need to define the notion of synchronization for the Winfree model slightly differently.

**Definition 5.** The Winfree system (1.18) is said to exhibit the global convergence of frequency synchronization if for any \( \Theta(0) \in \mathbb{R}^N \), the solution, \( \Theta(t) \), satisfies

\[
\lim_{t \to \infty} \dot{\theta}_i(t) = 0 \quad \text{for all } 1 \leq i \leq N.
\]

To state our result, we set \( \alpha^* = \pi/3 \in (0, \pi/2) \) such that

\[
\frac{3\sqrt{3}}{4} = \sin \alpha^*(1 + \cos \alpha^*) = \max_{0 \leq \alpha \leq \pi/2} (\sin \alpha(1 + \cos \alpha)).
\]  

(1.20)

Our main result for the Winfree model is the following theorem.

**Theorem 1.2.** Let \( N \geq 2 \) be any integer. Suppose that

\[
\max_{1 \leq i \leq N} |\omega_i| < \frac{3\sqrt{3}}{4} k_{ii}.
\]  

(1.21)

Then, (1.18) exhibits the global convergence of frequency synchronization, i.e., for any \( \Theta(0) \in \mathbb{R}^N \), the initial value problem for (1.18) admits a unique global solution, \( \Theta \in C^\infty([0, \infty)) \), satisfying

\[
\lim_{t \to \infty} \dot{\Theta}(t) = 0.
\]

We notice that our theorem does not require any condition on the initial phase configuration. Similarly as the three-oscillator of Kuramoto, this is the global convergence of synchronization. To our best knowledge, this is the first result on the global synchronization for the Winfree model (1.18). We refer the readers to [8] for some relevant work.

For the related numerical tests, we refer to the examples of (III) in Table 1 and Table 4, whose initial phases are uniformly distributed over the circle, i.e., no restriction, but exhibiting the synchronization. See also (I)', (II)' and (III)' in Table 1 and Table 4 for desynchronization when the conditions in our theorem fail.

In order to give a brief illustration on how we prove this, we proceed similarly as in (1.10) to have

\[
\int_0^t \sum_{i=1}^N \dot{\theta}_i^2(s)ds = \int_0^t \Lambda_N(s)ds + \int_0^t W_N(s)ds,
\]  

(1.22)

where

\[
\Lambda_N(s) := \sum_{i=1}^N \omega_i \dot{\theta}_i(s),
\]

(1.23)

\[
W_N(s) := \frac{d}{ds} \left( \frac{1}{2} \sum_{i,j=1}^N k_{ij}(1 + \cos \theta_i(s))(1 + \cos \theta_j(s)) \right).
\]

(1.24)

It is obvious that, using (1.24), the second integral of the right-hand side of (1.22) is uniformly bounded for all \( t \geq 0 \). Thanks to a similar argument as in Lemma 1 (see Section 2), it is remained to show that the first integral in the right-hand side of (1.22) is uniformly bounded for all \( t \geq 0 \). This implies the desired synchronization for the Winfree model. The proof is given in Section 3.
1.1. Discussion and open questions. By a number of numerical experiments, a large group of the nonidentical Kuramoto oscillators, $N \geq 4$ is also believed to globally synchronize, provided that a similar condition as (1.6) holds. It is not clear, at least in our framework, if we can extend the global convergence of synchronization result to the cases where the number of oscillators is greater than 3. At this point, we do not see any reason (except the technical ones) why it has to be restricted only to the cases up to $N = 3$. This is an interesting open question. For some related work, we refer the readers to [7].

Another challenging problem is to investigate the global convergence of frequency synchronization for the second order Kuramoto model of nonidentical oscillators:

$$m \ddot{\theta}_i + \dot{\theta}_i = \omega_i + \frac{K}{N} \sum_{j=1}^{N} \sin(\theta_j - \theta_i), \quad t > 0, \quad i = 1, 2, \ldots, N, \quad (1.25)$$

where $m > 0$ is a mass constant. This model has been first proposed in [6] to describe the slow synchronization of certain biological systems, e.g., fireflies of Pteroptyx malacca. For this model, it still remains true that the uniform boundedness of $\Theta(t)$ implies the frequency synchronization as in the first order case. Due to a more complicated dynamics of the second order ODEs, a more delicate analysis is required. For some related work, see [9].

In this work, we prove the global convergence of synchronization in Theorem 1.1 and Theorem 1.2 for Kuramoto model and Winfree model, respectively. On the other hand, interesting numerical examples of (un)synchronized oscillators when the suggested coupling strength conditions fail (by far), are presented in Section 4 (See Figure 4.2). Thanks to the numerical experiments, we notice that the coupling strength conditions in Theorem 1.1 and Theorem 1.2 are not sharp. Finding the optimal coupling conditions for the both cases are intriguing questions.

2. Global convergence of synchronization for the Kuramoto model with $N = 3$. In this section, we shall prove Theorem 1.1. As discussed in the introduction, when it comes to the frequency synchronization, without loss of generality, we may assume the mean zero condition on the natural frequencies

$$\omega = \frac{1}{N} \sum_{i=1}^{N} \omega_i = 0. \quad (2.1)$$

We first illustrate how energy methods lead to the complete frequency synchronization. Multiplying (1.1) by $\dot{\theta}_i$, summing over $i = 1, 2, \ldots, N$, and integrating the resultant over $(0, t)$, we obtain

$$\int_{0}^{t} \sum_{i=1}^{N} \dot{\theta}_i^2(s)ds = \int_{0}^{t} \Lambda_N(s)ds + \int_{0}^{t} H_N(s)ds, \quad (2.2)$$

where $\Lambda_N(t)$ and $H_N(t)$ are given as in (1.11) and (1.12). As we have discussed in the introduction, it is easy to see that $\sup_{t \geq 0} \int_{0}^{t} H_N(s)ds < \infty$. We shall see in the following lemma that if $\int_{0}^{t} \Lambda_N(s)ds$ is uniformly bounded in $t \geq 0$, then the frequency synchronization will be achieved.

**Lemma 1.** If $\Theta(t)$ is a solution of (1.1) such that

$$\sup_{t \geq 0} \left| \int_{0}^{t} \Lambda_N(s)ds \right| < M, \quad (2.3)$$
for some $M > 0$, then\[
\lim_{t \to \infty} D(\dot{\Theta}(t)) = 0.
\]

Proof. Using (2.3) and (1.14), we have\[
\int_0^t \sum_{i=1}^N \dot{\theta}_i^2(s)ds = \int_0^t \Lambda_N(s)ds + \int_0^t H_N(s)ds < M + K(N - 1). \tag{2.4}
\]
Here, using (1.1), one can easily check that\[
|\dot{\theta}_i(t)| = |\omega_i + \frac{K}{N} \sum_{j=1}^N \sin(\theta_j - \theta_i)| \leq |\omega_i| + K. \tag{2.5}
\]
From this, we see that $\sup_{t \geq 0} |\dot{\theta}_i(t)| < \infty$. By differentiating (1.1) in $t$, we also have\[
|\ddot{\theta}_i| = \frac{K}{N} \sum_{j=1}^N \cos(\theta_j - \theta_i)(\dot{\theta}_j - \dot{\theta}_i) \leq |\omega_i| + K(|\dot{\theta}_i| + |\dot{\theta}_j|).
\]
Using (2.5), we find that $\sup_{t \geq 0} |\ddot{\theta}_i(t)| < \infty$. Using these bounds, one has that\[
\sup_{t \geq 0} \frac{d}{dt} \sum_{i=1}^N \dot{\theta}_i^2(t) = \sup_{t \geq 0} \left[2 \sum_{i=1}^N \dot{\theta}_i(t) \ddot{\theta}_i(t)\right] < \infty.
\]
This specifically implies that the integrand of (2.4), $\sum_{i=1}^N \dot{\theta}_i^2(t)$ is uniformly continuous on $[0, \infty)$. Thus, we have the desired result (See Lemma 9 in [9]). ∎

In light of Remark 1, with the mean zero condition on the natural frequencies (2.1), the boundedness of $\sup_{t \geq 0} |\int_0^t \Lambda_N(s)ds|$ is equivalent to the uniform boundedness of $D(\Theta(t))$.

Recall that $\lambda_3$ and $\beta_3 \in [0, \pi]$ are defined in (1.5) by\[
\lambda_3 := \max_{\beta \in [0, \pi]} \left(\frac{2}{3} \sin\left(\frac{\beta}{2}\right) \left(2 \cos\left(\frac{\beta}{2}\right) - 1\right)\right) = \frac{2}{3} \sin\left(\frac{\beta_3}{2}\right) \left(2 \cos\left(\frac{\beta_3}{2}\right) - 1\right) > 0. \tag{2.6}
\]

Lemma 2. Let $N = 3$. Suppose $D(\Omega)/K < \lambda_3$. If $\Theta(t)$ is any solution of (1.1) with (2.1), then\[
\sup_{t \geq 0} D(\Theta(t)) < \infty. \tag{2.7}
\]

Proof. Without loss of generality, we may assume\[
0 \leq \theta_i(0) < 2\pi \quad \text{for } i = 1, 2, 3,
\]
and hence $D(\Theta(0)) < 2\pi$. Suppose that $t = t_0 > 0$ is the first moment at which $D(\Theta(t))$ hits $2\pi + \beta_3$ and $\theta_1(t_0) - \theta_2(t_0)$ is one of the representations of $D(\Theta(t_0))$, i.e.,\[
D(\Theta(t_0)) = \theta_1(t_0) - \theta_2(t_0) = 2\pi + \beta_3. \tag{2.8}
\]
Then we see that\[
\dot{\theta}_1(t_0) - \dot{\theta}_2(t_0) \geq 0.
\]
On the other hand, we have at $t = t_0$ that
\[
\dot{\theta}_1(t_0) - \dot{\theta}_2(t_0)
= \omega_1 - \omega_2 - \frac{2K}{3} \sin\left(\frac{\theta_1 - \theta_2}{2}\right) \left(2 \cos\left(\frac{\theta_1 - \theta_2}{2}\right) + \cos\left(\frac{\theta_1 + \theta_2}{2}\right)\right)
\leq D(\Omega) - \frac{2K}{3} \sin\left(\frac{\beta_3}{2}\right) \left[2 \cos\left(\frac{\beta_3}{2}\right) - 1\right]
\leq D(\Omega) - \lambda_3 K < 0, \tag{2.9}
\]
which is a contradiction. This completes the proof.

**Proof of Theorem 1.1.** As we mentioned in Remark 1, we find that (2.7) implies (2.3). To see this, we first note that
\[
\int_0^t \Lambda(s) ds = \sum_{i=1}^N \omega_i (\theta_i(t) - \theta_i(0))
= \sum_{i=1}^N \omega_i (\theta_i(t) - \bar{\theta}(t)) - \sum_{i=1}^N \omega_i \theta_i(0),
\]
where we have used (1.8), i.e., $\bar{\theta}(t) = \bar{\theta}(0)$ for all $t \geq 0$, and the fact that $\sum_{i=1}^N \omega_i \bar{\theta}(t) = \sum_{i=1}^N \omega_i \bar{\theta}(0) = 0$. Then it is easy to see that
\[
\left| \int_0^t \Lambda(s) ds \right| \leq \sum_{i=1}^N |\omega_i| |\theta_i(t) - \bar{\theta}(t)| + \sum_{i=1}^N |\omega_i||\theta_i(0)|
\leq \sum_{i=1}^N |\omega_i| D(\Theta(t)) + \sum_{i=1}^N |\omega_i||\theta_i(0)| < \infty \tag{2.10}
\]
if $\sup_{t \geq 0} D(\Theta(t)) < \infty$. By Lemma 2, we have $\sup_{t \geq 0} D(\Theta(t)) < \infty$. Then Theorem 1.1 immediately follows from (2.10) and Lemma 1.

3. **Global convergence of phase-locked synchronization for the Winfree model.** In this section, we shall give a proof of Theorem 1.2. We consider the Winfree model:
\[
\dot{\theta}_i = \omega_i - \sum_{j=1}^N k_{ij} \sin \theta_i (1 + \cos \theta_j), \quad i = 1, 2, 3, \cdots N, \tag{3.1}
\]
where the coupling strength constants are positive and symmetric, i.e.,
\[
k_{ij} = k_{ji} \geq 0. \tag{3.2}
\]
By multiplying (3.1) by $\dot{\theta}_i$, summing over $i = 1, 2, \cdots, N$, and integrating it over $(0, t)$, we have
\[
\int_0^t \sum_{i=1}^N \dot{\theta}_i^2(s) ds = \int_0^t \Lambda_N(s) ds + \int_0^t H_N(s) ds, \tag{3.3}
\]
where
\[
\Lambda_N(s) := \sum_{i=1}^N \omega_i \dot{\theta}_i(s), \tag{3.4}
\]
\[ H_N(s) := \frac{d}{ds} \left( \frac{1}{2} \sum_{i,j=1}^{N} k_{ij}(1 + \cos \theta_i(s))(1 + \cos \theta_j(s)) \right). \quad (3.5) \]

It is obvious that the second integral of the right-hand side of (3.3) is uniformly bounded for all \( t \geq 0 \). Here we present a Lemma similar to Lemma 1.

**Lemma 3.** If \( \Theta(t) \) is a solution of (3.1) such that
\[
\sup_{t \geq 0} \left| \int_0^t \Lambda_N(s) ds \right| < M, \quad (3.6)
\]
for some \( M > 0 \), then
\[
\lim_{t \to \infty} D(\dot{\Theta}(t)) = 0.
\]

Since the proof is similar to that of Lemma 1, we omit it here.

In light of this, in order to show the phase-locked synchronization for the Winfree model, it suffices to show that the first integral is uniformly bounded for all \( t \geq 0 \). To this end, we first provide a preliminary Lemma.

**Lemma 4.** Let \( 0 < \alpha < \frac{\pi}{2} \). Assume the coupling strength constant \( k_{ij} \)'s satisfy
\[
|\omega_i| < k_{ii} \sin \alpha (1 + \cos \alpha), \quad (3.7)
\]
for \( i = 1, 2, 3, \ldots, N \). Suppose that \( \Theta(t) \) is a solution of (3.1)-(3.2) with initial configuration \( \Theta(0) \) such that
\[
2(m_i - 1)\pi - \alpha \leq \theta_i(0) \leq 2m_i\pi + \alpha \quad (3.8)
\]
for some integer \( m_i \). Then there holds
\[
2(m_i - 1)\pi - \alpha \leq \theta_i(t) \leq 2m_i\pi + \alpha \quad (3.9)
\]
for all \( t \geq 0 \).

**Proof.** Suppose that there is a moment \( t_0 > 0 \) such that \( \theta_i(t_0) = 2m_i\pi + \alpha \). Evaluating the \( i \)-th equation of (3.1) at \( t = t_0 \) gives
\[
\dot{\theta}_i(t_0) = \omega_i - k_{ii} \sin \alpha (1 + \cos \alpha) - \sum_{j \neq i} k_{ij} \sin \alpha (1 + \cos(\theta_j(t_0)))
\leq \omega_i - k_{ii} \sin \alpha (1 + \cos \alpha)
< 0. \quad (3.10)
\]
This implies that \( \theta_i(t) \leq 2m_i\pi + \alpha \) for all \( t \geq 0 \). On the other hand, if there is a moment \( t_1 > 0 \) such that \( \theta_i(t_1) = 2m_i - 1)\pi - \alpha \). Direct calculation gives
\[
\dot{\theta}_i(t_1) = \omega_i + k_{ii} \sin \alpha (1 + \cos \alpha) + \sum_{j \neq i} k_{ij} \sin \alpha (1 + \cos(\theta_j(t_1)))
\geq \omega_i + k_{ii} \sin \alpha (1 + \cos \alpha)
> 0. \quad (3.11)
\]
This implies that \( \theta_i(t) \geq 2(m_i - 1)\pi - \alpha \) for all \( t \geq 0 \). This completes the proof. \( \square \)

Now we are ready to prove Theorem 1.2.

**Proof of Theorem 1.2.** In view of Lemma 3 and the energy estimate (3.3), we only need to prove that under the assumption (1.21) for any initial condition \( \Theta(0) \), the first integral on the right-hand side of (3.3) is uniformly bounded for all \( t \geq 0 \). It is
easy to see that by Lemma 4 the above statement holds true since for each initial configuration \( \Theta(0) \), we can find integer \( M \) such that
\[
2(M - 1)\pi - \alpha^* \leq \theta_i(0) \leq 2M\pi + \alpha^*
\]
for \( i = 1, 2, 3, \ldots, N \). This completes the proof of Theorem 1.2.

4. **Numerical experiments.** We will use the so-called order parameter \( r \) to measure the synchrony of the phases,
\[
re^{i\psi} = \frac{1}{N} \sum_{j=1}^{N} e^{i\theta_j}.
\]
Thus, if the modulus of the order parameter \( r, |r| \), is close to 1, all the phases tend to be synchronised. The phase \( \psi \) gives the average phase of all the oscillators.

4.1. **Experiments on Kuramoto model (1.1).** The numerical experiment in Figure 4.1 satisfying the coupling strength condition (1.6), i.e. \( D(\Omega)/K < \lambda_3 \approx 0.24601 \), demonstrates the frequency synchronization which supports Theorem 1.1. The three oscillators all are synchronized to zero in frequency and locked in phase. In particular, we can observe that the diameter function \( D(\Theta(t)) \) is continuous (with two cusps), but not \( C^1([0, \infty)) \) (see (c),(d) in Figure 4.1).

On the other hand, for the experiments in Figure 4.2, we have set \( D(\Omega)/K = 1.23691 \), violating the condition (1.6), for which the oscillators are not synchronized. Interestingly, they display some periodic patterns in frequency.

The numerical results in Table 3 well support Theorem 1.1. The parameters for Table 3 are given in Table 2. The cases (I),(II),(III) display the complete-frequency synchronization and the oscillators are phase-locked. On the other hand, the oscillators in the cases (I)',(II)',(III)' are not synchronized where we have set \( D(\Omega) \) and \( K \) so that the condition (1.6) by far fails.

4.2. **Experiments on Winfree model (3.1).** The examples in Figures 4.3 and 4.4 are tested for Theorem 1.2 where we have set \( N = 5 \), i.e., five oscillators and the entries of coupling strength matrix \( K \) are given in (4.2). For Figure 4.3, we set |\( \omega_i \)| and \( k_{ii} \) satisfy the condition (1.21). The results support Theorem 1.2 stating that all oscillators are synchronized in frequency and they are phase-locked, which are shown in Figure 4.3. When we set the parameters violate the condition (1.21), Figure 4.4 shows non-synchrony. The oscillators in this experiment asymptotically behave in a periodic pattern. This is an interesting pattern to be studied further for the future work.

The parameters for Table 4 are given in Table 1. For (I), (II) and (III), we have set the parameters which do not satisfy (1.21), but they are still moderate. Then the results in Table 4 display that the oscillators are still synchronized in frequency and are locked in phase.

On the other hand, the oscillators in the cases (I)',(II)',(III)', where the condition (1.21) fail to satisfy by far, are not synchronized.

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Figure 4.1. The Kuramoto model (1.1) with $N = 3$, $K = 1$, $D(\Omega)/K = 0.0201916$. The plots are in log scale in $t$.

Figure 4.2. The Kuramoto model (1.1) with $N = 3$, $K = 1$, $D(\Omega)/K = 1.23691$. 

(a) frequencies $\dot{\Theta}(t)$; (b) $D(\dot{\Theta}(t))$

(c) phases $\Theta(t)$; (d) $D(\Theta(t))$
Figure 4.3. The Winfree model (3.1) with $N = 5$, $\max_{1 \leq i \leq N} |\omega_i| = 1.15405$ where the matrix $K = K_1$ is given in (4.2). The plots are in log scale in $t$.

$$K_1 = \begin{pmatrix}
1.94207 & 1.15238 & 1.07042 & 1.99377 & 1.54267 \\
1.15238 & 1.56275 & 1.36392 & 1.43537 & 1.92062 \\
1.07042 & 1.36392 & 1.97026 & 1.98971 & 1.43525 \\
1.99377 & 1.43537 & 1.98971 & 1.21312 & 1.7247 \\
1.54267 & 1.92062 & 1.43525 & 1.7247 & 1.11864 
\end{pmatrix}$$  \hspace{1cm} (4.2a)

$$K_2 = \begin{pmatrix}
1.60084 & 1.73353 & 1.50676 & 1.14228 & 1.04362 \\
1.73353 & 1.06379 & 1.31342 & 1.8695 & 1.71068 \\
1.50676 & 1.31342 & 1.93056 & 1.61179 & 1.91045 \\
1.14228 & 1.8695 & 1.61179 & 1.4734 & 1.14045 \\
1.04362 & 1.71068 & 1.91045 & 1.14045 & 1.11695 
\end{pmatrix}$$  \hspace{1cm} (4.2b)

Figure 4.4. The Winfree model (3.1) with $N = 5$, $K =$, $\max_{1 \leq i \leq N} |\omega_i| = 13.8456$ where the matrix $K = K_2$ is given in (4.2).
The entries of $K_3, K_4, K_6, K_7$ are given as

$$K_3 = \begin{pmatrix}
0.783798 & 0.79908 & 0.744986 \\
0.79908 & 0.675993 & 0.934825 \\
0.744986 & 0.934825 & 0.662168 \\
\end{pmatrix}$$

$$K_4 = \begin{pmatrix}
0.973939 & 0.608799 & 0.57724 & 0.995599 & 0.589132 \\
0.608799 & 0.674581 & 0.728719 & 0.841889 & 0.953886 \\
0.57724 & 0.728719 & 0.587554 & 0.723902 & 0.680716 \\
0.995599 & 0.841889 & 0.723902 & 0.831634 & 0.622991 \\
0.589132 & 0.953886 & 0.680716 & 0.622991 & 0.718457 \\
\end{pmatrix}$$

$$K_6 = \begin{pmatrix}
0.573173 & 0.75495 & 0.816262 & 0.458612 \\
0.75495 & 0.816262 & 0.951424 & 0.455176 \\
0.816262 & 0.951424 & 0.458612 & 0.455176 \\
\end{pmatrix}$$

$$K_7 = \begin{pmatrix}
0.674774 & 0.82574 & 0.86530 & 0.57354 & 0.534592 \\
0.82574 & 0.653069 & 0.890512 & 0.835276 & 0.943237 \\
0.86530 & 0.890512 & 0.738833 & 0.515758 & 0.523214 \\
0.57354 & 0.835276 & 0.515758 & 0.821713 & 0.533673 \\
0.534592 & 0.943237 & 0.523214 & 0.533673 & 0.705871 \\
\end{pmatrix}$$

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### Table 3. The Kuramoto phases $\Theta(t)$ and the modulus of the order parameter, $|r|$, given in (4.1).

| $t$  | (I)     | (II)    | (III)   | (I)'    | (II)'   | (III)'  |
|------|---------|---------|---------|---------|---------|---------|
| 0    | 3.80000 | 4.70000 | 4.60170 | 2.30000 | 4.70000 | 5.67130 |
| $D(\Theta(t))$ | 0.23108 | 1.93260 | 0.67878 | 0.61164 | 1.88670 | 2.41560 |
| $|r|$ | 0.34515 | 0.30511 | 0.27291 | 0.60397 | 0.30511 | 0.03716 |
| 5    | 6.05170 | 6.96100 | 0.48507 | 7.56320 | 8.10750 | 13.97020 |
| $D(\dot{\Theta}(t))$ | 0.15616 | 0.00375 | 0.41549 | 0.57795 | 1.22680 | 3.08550 |
| $|r|$ | 0.98882 | 0.91483 | 0.99814 | 0.86806 | 0.61695 | 0.60180 |
| 20   | 6.18270 | 6.96220 | 0.21578 | 20.38000 | 28.04320 | 22.71800 |
| $D(\dot{\Theta}(t))$ | 0.00000 | 0.00000 | 0.00000 | 0.38512 | 1.01360 | 0.94404 |
| $|r|$ | 0.99664 | 0.91483 | 0.99814 | 0.86806 | 0.61695 | 0.60180 |
| 150  | 13.18640 | 0.40287 | 14.95700 | 30.70820 | 81.84340 | 27.60650 |
| $D(\dot{\Theta}(t))$ | 0.00000 | 0.00000 | 0.00000 | 11.39260 | 19.27490 | 0.98300 |
| $|r|$ | 0.99664 | 0.91483 | 0.99814 | 0.86806 | 0.61695 | 0.60180 |
| 500  | 13.18640 | 0.40287 | 14.95700 | 109.18220 | 316.97400 | 78.33540 |
| $D(\dot{\Theta}(t))$ | 0.00000 | 0.00000 | 0.00000 | 3.66710 | 9.38180 | 0.88459 |
| $|r|$ | 0.99664 | 0.91483 | 0.99814 | 0.86806 | 0.61695 | 0.60180 |

### Table 4. The Winfree phases $\Theta(t)$ and the modulus of the order parameter, $|r|$, given in (4.1).

| $t$  | (I)     | (II)    | (III)   | (I)'    | (II)'   | (III)'  |
|------|---------|---------|---------|---------|---------|---------|
| 0    | 5.70000 | 4.00000 | 5.70000 | 5.70000 | 4.00000 | 5.74710 |
| $D(\Theta(t))$ | 4.27740 | 9.39060 | 54.67330 | 9.42280 | 15.11920 | 76.17700 |
| $|r|$ | 0.45537 | 0.17338 | 0.03257 | 0.45537 | 0.17338 | 0.24487 |
| 5    | 13.18640 | 0.40287 | 14.95700 | 30.70820 | 81.84340 | 78.33540 |
| $D(\dot{\Theta}(t))$ | 0.00000 | 0.00000 | 0.00000 | 11.39260 | 19.27490 | 0.98300 |
| $|r|$ | 0.99664 | 0.91483 | 0.99814 | 0.86806 | 0.61695 | 0.60180 |
| 20   | 13.18640 | 0.40287 | 14.95700 | 109.18220 | 316.97400 | 109.18220 |
| $D(\dot{\Theta}(t))$ | 0.00000 | 0.00000 | 0.00000 | 3.66710 | 9.38180 | 0.88459 |
| $|r|$ | 0.99664 | 0.91483 | 0.99814 | 0.86806 | 0.61695 | 0.60180 |
| 150  | 13.18640 | 0.40287 | 14.95700 | 818.10600 | 2347.20000 | 549.54340 |
| $D(\dot{\Theta}(t))$ | 0.00000 | 0.00000 | 0.00000 | 10.20440 | 16.58980 | 0.72854 |
| $|r|$ | 0.99664 | 0.91483 | 0.99814 | 0.86806 | 0.61695 | 0.60180 |
| 500  | 13.18640 | 0.40287 | 14.95700 | 2725.00000 | 7813.20000 | 1818.60000 |
| $D(\dot{\Theta}(t))$ | 0.00000 | 0.00000 | 0.00000 | 5.57280 | 11.35220 | 0.35697 |
| $|r|$ | 0.99664 | 0.91483 | 0.99814 | 0.86806 | 0.61695 | 0.60180 |

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