Small-time global approximate controllability for incompressible MHD with coupled Navier slip boundary conditions

Manuel Rissel * Ya-Guang Wang †

Abstract

We study the small-time global approximate controllability for incompressible magnetohydrodynamic (MHD) flows in smoothly bounded two- or three-dimensional domains. The controls act on arbitrary open portions of each connected boundary component, while linearly coupled Navier slip-with-friction conditions are imposed along the uncontrolled parts of the boundary. Certain choices for the friction operators at the boundary give rise to interacting velocity and magnetic field boundary layers. We obtain dissipation properties of these layers by a detailed analysis of the corresponding asymptotic expansions. If the boundary controls are not compatible with the equation for the magnetic field, a corrector term appears in the induction equation, and we show that such a term does not exist for problems defined in planar simply-connected domains with all uncoupled and many coupled Navier slip-with-friction boundary conditions.

Keywords and phrases: magnetohydrodynamics; global approximate controllability; Navier slip-with-friction boundary conditions; boundary layers

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1 Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be any bounded domain of dimension $N \in \{2, 3\}$ with $\Gamma := \partial \Omega$ being smooth and $n$ denoting the outward unit normal vector to $\Omega$ along $\Gamma$. The goal of this article is to steer incompressible magnetohydrodynamic (MHD) flows from a
prescribed initial state approximately towards a desired terminal state, without placing restrictions on the control time. This is accomplished by acting on the system via boundary controls within a small open subset $\Gamma_c \subseteq \Gamma$ which non-trivially intersects all connected components of $\Gamma$.

When it comes to nonlinear evolution equations with controls localized in arbitrary open subsets of the boundary or an interior sub-domain, establishing global approximate controllability usually constitutes a challenging task and few methods for tackling such questions are available. In the context of fluid dynamics, but not limited to, one successful approach is known as the return method, which has first been introduced by Coron in [13] for the stabilization of certain mechanical systems and shall also be employed here. An introduction to the return method and its applications to nonlinear partial differential equations may be found in [16, Part 2, Chapter 6]. In contrast to the Navier-Stokes equations, for which global approximate controllability has been actively investigated in the past, nothing seems to be known regarding similar questions for viscous MHD in the presence of boundaries.

In this article, we focus on incompressible flows of viscosity $\nu_1 > 0$ and resistivity $\nu_2 > 0$, for which the velocity $u \in \mathbb{R}^N$, the magnetic field $B \in \mathbb{R}^N$ and the total pressure $p \in \mathbb{R}$ are described until a given terminal time $T_{\text{ctrl}} > 0$ as a solution to the initial boundary value problem

$$\begin{cases}
\partial_t u - \nu_1 \Delta u + (u \cdot \nabla)u - \mu (B \cdot \nabla)B + \nabla p = 0 & \text{in } \Omega \times (0, T_{\text{ctrl}}), \\
\partial_t B - \nu_2 \Delta B + (u \cdot \nabla)B - (B \cdot \nabla)u = 0 & \text{in } \Omega \times (0, T_{\text{ctrl}}), \\
\nabla \cdot u = \nabla \cdot B = 0 & \text{in } \Omega \times (0, T_{\text{ctrl}}), \\
u \cdot n = B \cdot n = 0 & \text{on } (\Gamma \setminus \Gamma_c) \times (0, T_{\text{ctrl}}), \\
\mathcal{N}_1(u, B) = \mathcal{N}_2(u, B) = 0 & \text{on } (\Gamma \setminus \Gamma_c) \times (0, T_{\text{ctrl}}), \\
u(\cdot, 0) = u_0, \ B(\cdot, 0) = B_0 & \text{in } \Omega,
\end{cases} \tag{1.1}$$

while the underlying state space for both velocity and magnetic field is taken as

$$L_c^2(\Omega) := \left\{ f \in L^2(\Omega; \mathbb{R}^2) \left| \nabla \cdot f = 0 \text{ in } \Omega, f \cdot n = 0 \text{ on } \Gamma \setminus \Gamma_c \right. \right\}.$$  

In (1.1), the vector fields $u_0 \in L_c^2(\Omega)$ and $B_0 \in L_c^2(\Omega)$ represent the given initial data. The parameter $\mu > 0$ stands for the magnetic permeability. No boundary conditions are prescribed along the controlled boundary $\Gamma_c$, which implies that (1.1) is underdetermined as explained in Remark 1.1 below. Before introducing the general boundary operators $\mathcal{N}_1$ and $\mathcal{N}_2$ acting along $\Gamma \setminus \Gamma_c$, it is emphasized that this article includes all boundary conditions of the form

$$\left( \nabla \times u \right) \times n = [Mu]_{\text{tan}}, \quad \left( \nabla \times B \right) \times n = [LB]_{\text{tan}}, \quad u \cdot n = B \cdot n = 0, \tag{1.2}$$

with symmetric $M, L \in C^\infty(\Gamma_c; \mathbb{R})$ and $[\cdot]_{\text{tan}}$ denoting the tangential part.

**Remark 1.1.** The notations for the case $N = 3$ are employed whenever possible. When $N = 2$, the curl $\nabla \times h$ has to be replaced by $\nabla \wedge h := \partial_1 h_2 - \partial_2 h_1$, the curl of a
scalar function \( h \) refers to \( \nabla^\perp h := [\partial_2 h, -\partial_1 h]^\top \) and the cross product \( h \times g \) becomes \( h \wedge g := h_1 g_2 - g_2 h_1 \). As a result, some objects denoted here as vectors might be scalars and \((\nabla \times h) \times n\) means \((\nabla \wedge h)[n_2, -n_1]^\top\).

**The Navier slip-with-friction boundary conditions.** Let \( \mathcal{D} \subseteq \mathbb{R}^N \) be a smoothly bounded domain with outward unit normal vector \( n_\mathcal{D} : \partial \mathcal{D} \rightarrow \mathbb{R}^N \), and denote by \( T_x \) the tangent space of \( \partial \mathcal{D} \) at \( x \). At each \( x \in \partial \mathcal{D} \), the Weingarten map \( W_\mathcal{D}(x) : T_x \rightarrow T_x \) is defined by \( \tau \mapsto W_\mathcal{D}(x) \tau := \nabla_\tau n_\mathcal{D} \). Then, for \( h_1, h_2 : \overline{\mathcal{D}} \rightarrow \mathbb{R}^N \) and friction matrices

\[
L_1, L_2, M_1, M_2 \in C^\infty(\Gamma \setminus \Gamma_c; \mathbb{R}^{N \times N}),
\]

the linearly coupled Navier slip-with-friction operators in (1.1) are defined as

\[
\mathcal{N}_i(h_1, h_2) := [D(h_i)n(x) + W_\mathcal{D} h_i + M_i(x) h_1 + L_i h_2]_{\text{tan}}, \quad i = 1, 2,
\]

wherein the tangential part and the symmetrized gradient are respectively denoted by

\[
[h]_{\text{tan}} := h - (h \cdot n_\mathcal{D}) n_\mathcal{D}, \quad D(h) := \frac{1}{2} [\nabla h + (\nabla h)^\top].
\]

The Weingarten map \( W_\mathcal{D} \) is smooth, cf. [17, Lemma 1] and [11, 22], and when \( h \) is tangential to \( \partial \mathcal{D} \) one has the relation

\[
[D(h(x, t)) n_\mathcal{D}(x) + W_\mathcal{D}(x) h(x, t)]_{\text{tan}} = -\frac{1}{2} (\nabla \times h(x, t)) \times n_\mathcal{D}(x).
\]

As a consequence of (1.5), the boundary conditions in (1.1) along \((\Gamma \setminus \Gamma_c) \times (0, T_{\text{ctrl}})\) are equivalent to

\[
(\nabla \times u) \times n = \rho_1(u, B) := 2 [M_1(x) u + L_1(x) B]_{\text{tan}}, \quad u \cdot n = 0,
\]

\[
(\nabla \times B) \times n = \rho_2(u, B) := 2 [M_2(x) u + L_2(x) B]_{\text{tan}}, \quad B \cdot n = 0.
\]

Our main motivation is to treat simply-connected domains and the boundary conditions given in (1.2). This already constitutes a general setup which has not been studied in terms of controllability but recently attracted attention in view of inviscid limit problems. However, similarly to the Navier-Stokes equations studied in [17], the constructions of the controls extend also to the more general boundary conditions (1.6). In particular, the situation \( M_2 \neq 0 \) introduces interesting challenges.

Navier slip-with-friction boundary conditions, as already proposed by Navier [36] two centuries ago, are relevant to a range of applications, thus have been studied in the context of the Navier-Stokes equations from various points of view. For instance, inviscid limit problems are treated in [11, 29, 30, 43], regularity questions are investigated in [1–3, 39, 40] and control problems are tackled in [17, 25, 33]. Concerning MHD, several singular limit problems have been studied under Navier slip-with-friction boundary conditions in [26, 35, 44]. Compared with the above mentioned references, the here employed boundary conditions are more general in that the shear stresses of the velocity and the magnetic field at the boundary are linearly coupled with tangential
velocity and magnetic field contributions. While (1.6) includes the classical Navier slip conditions studied before, also more complex interactions can be captured.

From the global approximate controllability point of view, several difficulties appear however when the magnetic shear stress is coupled with the tangential velocity: a magnetic field boundary layer potentially enters the analysis of Section 3. This in turn challenges the construction of magnetic field boundary controls without generating a pressure gradient term or additional control forces in the induction equation.

1.1 Main results

The statements of the main theorems anticipate the notion of a weak controlled trajectory from Section 2.4.

**Theorem 1.2.** Assume one of the following configurations (some examples of which are sketched in Figure 1):

a) The domain \( \Omega \subseteq \mathbb{R}^2 \) is simply-connected and the open subset \( \Gamma_c \subseteq \Gamma \) is connected. The friction operators in (1.4) satisfy \( L_1, L_2, M_1 \in C^\infty(\Gamma \setminus \Gamma_c; \mathbb{R}^{2 \times 2}) \) and \( M_2 = 0 \).

b) For \( r_2 > r_1 > 0 \) and \( D_r := \{ x \in \mathbb{R}^2 | |x| < r \} \), the domain \( \Omega \subseteq A_r^\gamma := D_{r_2} \setminus \overline{D_{r_1}} \) is simply-connected and bounded by a closed Lipschitz curve \( \Gamma \), while the controlled part is \( \Gamma_c := \Gamma \setminus \partial A_r^\gamma \). The operators \( M_1, L_1, L_2 \in C^\infty(\Gamma \setminus \Gamma_c; \mathbb{R}^{2 \times 2}) \) are arbitrary and \( M_2 := \rho I \), for \( \rho \in \mathbb{R} \) and \( I \in \mathbb{R}^{2 \times 2} \) being the identity matrix.

Then, for any given initial states \( u_0, B_0 \in L^2_{\text{a}} \), target states \( u_1, B_1 \in L^2_{\text{a}}(\Omega) \), \( T_{\text{ctrl}} > 0 \) and \( \delta > 0 \), there exists at least one weak controlled trajectory

\[
(u, B) \in \left[ C^0_w([0, T_{\text{ctrl}}]; L^2_{\text{a}}(\Omega)) \cap L^2((0, T_{\text{ctrl}}); H^1(\Omega)) \right]^2
\]

to the MHD equations (1.1) which obeys the terminal condition

\[
\|u(\cdot, T_{\text{ctrl}}) - u_1\|_{L^2(\Omega)} + \|B(\cdot, T_{\text{ctrl}}) - B_1\|_{L^2(\Omega)} < \delta.
\] (1.7)

(a) A general planar simply-connected domain with \( \Gamma_c \) being connected. (b) An annulus section with controls acting along the cuts.

Figure 1: Two examples for domains \( \Omega \) that are covered by Theorem 1.2. The controls act along the dashed boundaries which represent \( \Gamma_c \).
Remark 1.3. The assumption $M_2 = 0$ in Theorem 1.2 a) rules out certain coupled boundary conditions, all of which would lead to magnetic field boundary layers in the analysis of Section 3. As seen in Theorem 1.2 b), these assumptions are not sharp and possible improvements are left as open question.

Remark 1.4. Under additional assumptions on the normal traces at $\Gamma_c$ of the initial data, in Theorem 1.2 a) one can choose $\Gamma_c$ as an arbitrary open subset of $\Gamma$, see Remark 3.7.

The next theorem is valid for all $L_1, L_2, M_1, M_2 \in C^{00}(\Gamma \setminus \Gamma_c; \mathbb{R}^{N \times N})$. It involves a pressure-like unknown $q$ and a control $\zeta$, which however satisfies $\zeta \equiv 0$ when $M_2 = 0$. For the case $N = 3$, we will need additional assumptions, since to our knowledge there does not exist a $L^\infty(\mathbb{R}^3; \mathbb{R}^3)$ and $L^\infty(\mathbb{R}^3; \mathbb{R}^3)$ strong solution theory for MHD with general Navier slip-with-friction conditions in the literature, which prevents us in general from justifying the regularization argument from Lemma 3.3 when $N = 3$. Thus, when $N = 3$, we introduce the following class of the initial data:

The class $S$. The states $u_0, B_0 \in \mathbb{L}^2(\Omega)$ belong to the class $S$ if one of the following holds:

- $M_1, L_2$ are symmetric, $L_1 = M_2 = 0$, and $\Omega \subseteq \mathbb{R}^3$ is simply-connected.
- $u_0, B_0$ are restrictions of divergence-free functions $\tilde{u}_0, \tilde{B}_0 \in H^3(\mathbb{E})$ which are tangential to $\partial \mathbb{E}$, where $\mathbb{E} \subseteq \mathbb{R}^3$ is a smoothly bounded domain extension for $\Omega \subseteq \mathbb{R}^3$ of the type introduced in Section 2.1.

Example 1.5. All $u_0, B_0 \in \mathbb{L}^2(\Omega) \cap H^3(\Omega)$ which vanish at $\Gamma_c$ and have vanishing normal derivatives of all defined orders at $\Gamma_c$ belong to $S$.

Theorem 1.6. For any given time $T_{\text{ctrl}} > 0$, fixed initial states $u_0, B_0 \in \mathbb{L}^2(\Omega)$, which belong to the class $S$ when $N = 3$, target states $u_1, B_1 \in \mathbb{L}^2(\Omega)$, and $\delta > 0$, there exists a smooth function $\zeta : \overline{\Omega} \times [0, T_{\text{ctrl}}] \rightarrow \mathbb{R}^N$, with $\zeta \equiv 0$ when $M_2 = 0$, such that the MHD system

\[
\begin{aligned}
\partial_t u - v_1 \Delta u + (u \cdot \nabla) u - \mu (B \cdot \nabla) B + \nabla p &= 0 & \text{in } \Omega \times (0, T_{\text{ctrl}}), \\
\partial_t B - v_2 \Delta B + (u \cdot \nabla) B - (B \cdot \nabla) u &= \nabla q + \zeta & \text{in } \Omega \times (0, T_{\text{ctrl}}), \\
\nabla \cdot u &= \nabla \cdot B = 0 & \text{in } \Omega \times (0, T_{\text{ctrl}}), \\
\nabla \cdot n &= B \cdot n = 0 & \text{in } (\Gamma \setminus \Gamma_c) \times (0, T_{\text{ctrl}}), \\
N_1(u, B) &= N_2(u, B) = 0 & \text{in } (\Gamma \setminus \Gamma_c) \times (0, T_{\text{ctrl}}), \\
\end{aligned}
\]

(1.8)

admits at least one weak controlled trajectory

\[ (u, B) \in [C^0_{\text{w}}([0, T_{\text{ctrl}}]; \mathbb{L}^2(\Omega)) \cap \mathbb{L}^2(\mathbb{R}^N)]^2 \]

which obeys the terminal condition

\[ \|u(\cdot, T_{\text{ctrl}}) - u_1\|_{\mathbb{L}^2(\Omega)} + \|B(\cdot, T_{\text{ctrl}}) - B_1\|_{\mathbb{L}^2(\Omega)} < \delta. \]

(1.9)
Remark 1.7. For $M_2 \neq 0$, the control $\zeta$ enters (1.8) when the magnetic field boundary layer described in Section 3.4.1 is not divergence-free. In order to illustrate that this statement is not sharp, we consider as in Figure 2b a cylinder $\Omega := (a, b) \times D$, for a smoothly bounded connected open set $D \subseteq \mathbb{R}^2$ and $-\infty < a < b < +\infty$, with controlled part $\Gamma_c := \{a, b\} \times D$. In this case, Theorem 1.6 is valid for all $L_1, L_2, M_1, M_2 \in C^\infty(\Gamma \setminus \Gamma_c; \mathbb{R}^{N \times N})$ with $\zeta = 0$. This will be illustrated by means of Example 3.4 and the discussion in Section 3.4.1.

![Figure 2: Two examples for domains $\Omega$ that are covered by Theorem 1.6.](image)

(a) A general multiply-connected domain as considered in Theorem 1.6. The sketch is two-dimensional only for simplicity. (b) A multiply-connected cylindrical domain as in Remark 1.7, with controls at the base faces. In this case, one can take $\zeta = 0$.

Remark 1.8. The systems (1.1) and (1.8) are under-determined since no boundary condition is prescribed along $\Gamma_c$. Once a weak controlled trajectory is found via Theorem 1.2 or 1.6, one obtains explicit boundary controls by taking traces along $\Gamma_c$, see also [15, 17, 20, 24].

Remark 1.9. Since the proofs for Theorems 1.2 and 1.6 will be carried out in a certain extended domain, one can allow the interior of $\Gamma_c$ to be merely Lipschitz regular.

1.2 Related literature and organization of the article

Global controllability for viscous- and resistive MHD in non-periodic domains has to our knowledge not been studied, neither for incompressible- nor for compressible models. Therefore, the present work constitutes a first step in this direction. As a possible continuation, it would be interesting to generalize Theorem 1.2 such that $N \in \{2, 3\}$ and arbitrary $M_2 \in C^\infty(\Gamma \setminus \Gamma_c; \mathbb{R}^{N \times N})$ are allowed without introducing an additional force in the induction equation. Also, the topic of global exact controllability to zero or towards trajectories is widely open.

Concerning local exact controllability for MHD, where the initial state lies in the vicinity of the target trajectory, there have been some interesting works when the velocity satisfies the no-slip boundary condition. For incompressible viscous MHD, Badra obtained in [7] the local exact controllability to trajectories, while maintaining truly localized and solenoidal interior controls. However, since the boundary conditions studied in [7] are different from those employed here, one cannot deduce the small-time global exact controllability towards trajectories by combining the approaches given in
with our global approximate results. A variety of previous local exact controllability results may also be found in \cite{Barbu} by Barbu et al. and in \cite{Havarneanu, Havarneanu2} by Havarneanu et al., while approximate interior controllability for certain toroidal configurations without boundary has been investigated by Galan in \cite{Galan}. Moreover, Anh and Toi studied in \cite{Anh} the local exact controllability to trajectories for magneto-micropolar fluids, while Tao considered the local exact controllability for planar compressible MHD in the recent work \cite{Tao}.

Recently, we have studied the global exact controllability for the ideal incompressible MHD in \cite{Kukavica}, in which the small-time global exact controllability in rectangular channels is obtained in the presence of a harmonic unknown like \( q \) in (1.8). Subsequently, Kukavica et al. demonstrated in \cite{Kukavica2}, likewise restricted to a rectangular domain, how to find boundary controls such that \( \nabla q \) either vanishes or is explicitly characterized.

The approach of this paper will be based on a combination of the return method and the well-prepared dissipation method as laid out in \cite{Coron} by Coron et al., where the small-time global exact controllability to trajectories has been studied for incompressible Navier-Stokes equations in general smooth two- and three-dimensional domains under Navier slip-with-friction conditions along the uncontrolled boundaries. Meanwhile, we shall also extend the asymptotic expansions obtained by Ifimie and Sueur in \cite{Ifimie} for the incompressible Navier-Stokes equations to the system of MHD. When it comes to MHD, the return method needs to be carefully applied in order to avoid generating pressure-like and additional forcing terms in the induction equation. To this end, under the assumptions of Theorem 1.2, there seem to be at least two options:

1) Modifying the return method trajectory from \cite{Coron} to be everywhere divergence-free, but not everywhere curl-free.

2) Modifying the definition for weak controlled trajectories, compared to our choice in Section 2.4, in order to maintain a divergence free magnetic field in the extended domain from Section 2.1 while employing the same return method trajectory as in \cite{Coron}. This might be possible by replacing the induction equation in (2.7) below by

\[
\partial_t B - \nu_2 \Delta B + \nabla \times (B \times u) = \eta.
\]

In this article, we implement the first strategy for planar simply-connected domains. This seems to be a new approach which has not been discussed in previous literature. In particular, using an everywhere solenoidal return method profile might be useful for further studies on the controllability of the ideal MHD equations. Moreover, in the case of Theorem 1.2 b), seeking a special return method trajectory allows the choice \( M_2 = \rho I \). However, our arguments in this direction appear not to extend easily to more general domains. The second possible option mentioned above is not investigated in this article, but might be interesting for further research. Another possibility for treating general domains without having \( \nabla q \) could involve a weaker form of Lemma 3.5 for \( N = 3 \) combined with arguments from \cite{Iftimie} for the case where \( \Gamma_c \) does not intersect each connected component of \( \Gamma \).
Let us also mention other recent works treating global controllability problems in fluid dynamics by means of the return- and well-prepared dissipation methods. For instance, an incompressible Boussinesq system with Navier slip-with-friction boundary conditions for the velocity is considered by Chaves-Silva et al. in [10]. Moreover, the question of smooth controllability for the Navier-Stokes equations with Navier slip-with-friction boundary conditions is investigated in [32]. Further, Coron et al. obtain in [18] global exact controllability results for the Navier-Stokes equations under the no-slip condition in a rectangular domain.

The remainder of this article is organized as follows. In Section 2, several preliminaries are collected. In Section 3, the global approximate controllability from regular initial data towards an arbitrarily smooth state is shown. Subsequently, the main theorems are concluded in Section 4. In Appendices A and B, higher order boundary layer estimates and a proof of Lemma 4.1 are provided respectively.

2 Preliminaries

In this section, we shall present certain preliminaries, which will be used later. A domain extension is introduced in Section 2.1, several function spaces and norms are defined in Section 2.2, initial data extensions are discussed in Section 2.3, notions of weak controlled trajectories are defined in Section 2.4, and Section 2.5 contains supplementary explanations. Throughout the whole article, if not indicated otherwise, constants of the form \( C > 0 \) are generic and can change from line to line during the estimates.

2.1 Domain extensions

In what follows, the sets \( \Gamma^1, \ldots, \Gamma^{K(\Omega)} \) denote the connected components of \( \Gamma \) and \( \Gamma^1_c, \ldots, \Gamma^{K(\Omega)}_c \) stand for the respective intersections \( \Gamma^1 \cap \Gamma_c, \ldots, \Gamma^{K(\Omega)} \cap \Gamma_c \), hence

\[
\Gamma = \bigcup_{i \in \{1, \ldots, K(\Omega)\}} \Gamma^i, \quad \Gamma_c = \bigcup_{i \in \{1, \ldots, K(\Omega)\}} \Gamma^i_c.
\]

Let \( \mathcal{E} \subseteq \mathbb{R}^N \) be a smoothly bounded domain, which is an extension of \( \Omega \) as shown in Figure 3, with \( \Omega \subseteq \mathcal{E}, \Gamma^i \subseteq \overline{\mathcal{E}}, \Gamma^i \cap \mathcal{E} \neq \emptyset \) for \( i \in \{1, \ldots, K(\Omega)\} \), and \( \Gamma \setminus \Gamma_c \subseteq \partial \mathcal{E} \). Such an extension exists by the requirements on \( \Omega \). Further, the outward unit normal field to \( \mathcal{E} \) along \( \partial \mathcal{E} \) is denoted by \( \mathbf{n}_{\partial \mathcal{E}} \), or simply by \( \mathbf{n} \) if no confusion can arise.

**Remark 2.1.** Since the initial data are allowed to have nonzero normal traces at \( \Gamma_c \), it is implicit that the extension \( \mathcal{E} \) is selected such that \( \mathbf{u}_0 \) and \( B_0 \) are tangential at \( \partial \Omega \cap \partial \mathcal{E} \). Also, for simplifying the notations, it is assumed that to each connected component of \( \Gamma_c \) at most one connected component of \( \mathcal{E} \setminus \Omega \) is attached to. However, one connected component of \( \mathcal{E} \setminus \Omega \) may be attached to multiple connected components of \( \Gamma_c \) at the same time, as for instance in the case of Theorem 1.2 b).}

When \( \mathcal{E} \) is multiply-connected, there is a number \( L(\mathcal{E}) \in \mathbb{N} \) of smooth \((N-1)\)-dimensional and mutually disjoint cuts \( \Sigma_1, \ldots, \Sigma_{L(\mathcal{E})} \subseteq \mathcal{E} \), which meet \( \partial \mathcal{E} \) trans-
versely, such that one obtains a simply-connected set by means of
\[ \hat{E} := E \setminus (\Sigma_1 \cup \cdots \cup \Sigma_{L(E)}). \]

Additional details may for instance be found in [42, Appendix I]. In what follows, for each \( i \in \{1, \ldots, L(E)\} \), a unit normal field to \( \Sigma_i \) is denoted by \( \vec{n}_i \). When \( E \) is simply-connected, we set \( L(E) := 0 \).

![Figure 3: A multiply-connected domain \( \Omega \subseteq \mathbb{R}^2 \) with two controlled boundary components and extension \( E \). The dashed lines mark the controlled boundary parts.](image)

**Lemma 2.2.** There exists a constant \( C > 0 \) such that for any \( h \in H^1(E) \), one has the estimate
\[
\| h \|_{H^1(E)} \leq C \left( \| \nabla \cdot h \|_{L^2(E)} + \| \nabla \times h \|_{L^2(E)} + \| h \cdot n \|_{H^{1/2}(\partial E)} \right) + C \sum_{i=1}^{L(E)} \left| \int_{\Sigma_i} h \cdot \vec{n}_i \, d\Sigma_i \right| \leq C \left( \| \nabla \cdot h \|_{L^2(E)} + \| \nabla \times h \|_{L^2(E)} + \| h \cdot n \|_{H^{1/2}(\partial E)} + \| h \|_{L^2(E)} \right).
\]

**Proof.** The estimate (2.1) can for instance be derived from the exposition in [42, Appendix I]. Here, for the sake of brevity we directly resort to [5] where it is shown that all \( f \in H^1(E) \) with \( f \cdot n = 0 \) along \( \partial E \) obey
\[
\| f \|_{H^1(E)} \leq C \left( \| \nabla \cdot f \|_{L^2(E)} + \| \nabla \times f \|_{L^2(E)} + \sum_{i=1}^{L(E)} \left| \int_{\Sigma_i} f \cdot \vec{n}_i \, d\Sigma_i \right| \right).
\]

As demonstrated in [9, Theorem III.4.3], there exists a function \( \psi \in H^2(E) \) which solves the Neumann problem
\[
\begin{cases}
\Delta \psi = \nabla \cdot h & \text{in } E, \\
\partial_n \psi = h \cdot n & \text{on } \partial E,
\end{cases}
\]
and satisfies
\[ \| \psi \|_{H^1(\mathcal{E})} \leq C \left( \| \nabla \cdot h \|_{L^2(\mathcal{E})} + \| h \cdot n \|_{H^{1/2}(\partial \mathcal{E})} \right). \] (2.3)

Therefore, by employing trace estimates and the properties of \( \psi \), the vector field \( g = \nabla \psi \) is seen to satisfy
\[ \nabla \cdot g = \nabla \cdot h, \quad \nabla \times g = 0, \quad g \cdot n = h \cdot n, \quad \sum_{i=1}^{L(\mathcal{E})} \left| \int_{\Sigma_i} g \cdot \vec{n}^i \, d\Sigma_i \right| \leq C \| g \|_{H^1(\mathcal{E})} . \]

Consequently, by means of (2.2) and (2.3), the first inequality in (2.1) follows with \( f := h - g \) from
\[ \| h \|_{H^1(\mathcal{E})} \leq \| f \|_{H^1(\mathcal{E})} + \| g \|_{H^1(\mathcal{E})} \]
\[ \leq C \| \nabla \times h \|_{L^2(\mathcal{E})} + C \sum_{i=1}^{L(\mathcal{E})} \left| \int_{\Sigma_i} f \cdot \vec{n}^i \, d\Sigma_i \right| + \| g \|_{H^1(\mathcal{E})} \]
\[ \leq C \| \nabla \times h \|_{L^2(\mathcal{E})} + C \sum_{i=1}^{L(\mathcal{E})} \left| \int_{\Sigma_i} h \cdot \vec{n}^i \, d\Sigma_i \right| + C \| g \|_{H^1(\mathcal{E})} \]
\[ \leq C \left( \| \nabla \cdot h \|_{L^2(\mathcal{E})} + \| \nabla \times h \|_{L^2(\mathcal{E})} + \| h \cdot n \|_{H^{1/2}(\partial \mathcal{E})} \right) \]
\[ + C \sum_{i=1}^{L(\mathcal{E})} \left| \int_{\Sigma_i} h \cdot \vec{n}^i \, d\Sigma_i \right| . \]

Concerning the second inequality in (2.1), let the multi-valued functions \( q_1, \ldots, q_{L(\mathcal{E})} \) be chosen such that \( \{ \nabla q_1, \ldots, \nabla q_{L(\mathcal{E})} \} \) is a basis for the space of curl-free and divergence-free vector fields that are tangential at \( \partial \mathcal{E} \). As shown in [42, Appendix I], one can select this basis such that \([q_i]_j = \delta_{i,j}\), where \([f]_j\) denotes the jump of a multi-valued function \( f \) across \( \Sigma_j \) and \( \delta_{i,j} \) is the usual Kronecker symbol. Therefore,
\[ \int_{\Sigma_i} h \cdot \vec{n}^i \, d\Sigma_i = \int_{\mathcal{E}} h \cdot \nabla q_i \, dx + \int_{\mathcal{E}} (\nabla \cdot h) q_i \, dx - \int_{\partial \mathcal{E}} (h \cdot n) q_i \, dS. \]

\( \square \)

Let \( d > 0 \) be sufficiently small so that \( \mathcal{V} := \{ x \in \mathbb{R}^N : \text{dist}(x, \partial \mathcal{E}) < d \} \) denotes a thin tubular neighborhood in \( \mathbb{R}^N \) of the boundary \( \partial \mathcal{E} \). Further, let \( \varphi_{\mathcal{E}} \in C^\infty(\mathbb{R}^N, \mathbb{R}) \) satisfy \( |\nabla \varphi_{\mathcal{E}}(x)| = 1 \) for all \( x \in \mathcal{V} \) and
\[ \mathcal{E} \cap \mathcal{V} = \{ \varphi_{\mathcal{E}} > 0 \} \cap \mathcal{V}, \quad (\mathbb{R}^N \setminus \overline{\mathcal{E}}) \cap \mathcal{V} = \{ \varphi_{\mathcal{E}} < 0 \} \cap \mathcal{V}, \quad \partial \mathcal{E} = \{ \varphi_{\mathcal{E}} = 0 \}. \]

This implies \( \varphi_{\mathcal{E}}(x) = \text{dist}(x, \partial \mathcal{E}) \) for all \( x \in \mathcal{V} \) and a smooth extension of \( n_{\partial \mathcal{E}} \) to \( \overline{\mathcal{E}} \) is provided by
\[ n(x) = n_{\mathcal{E}}(x) := \begin{cases} n_{\partial \mathcal{E}}(x) & \text{if } x \in \partial \mathcal{E}, \\ -\nabla \varphi_{\mathcal{E}}(x) & \text{if } x \in \mathcal{E}. \end{cases} \]
In this sense, the tangential part \([h]_{\text{tan}} = h - (h \cdot n) n\) of \(h : \mathbb{E} \to \mathbb{R}^N\) is now defined everywhere in \(\mathbb{E}\). Moreover, the Weingarten map \(W_E\) and the general friction matrices \(M_1, M_2, L_1, L_2\) are smoothly continued to \(\mathbb{E}\) such that

\[
W_E, M_1, M_2, L_1, L_2, \in C^\infty(\mathbb{E}, \mathbb{R}^{N \times N}),
\]

while also extending any assumptions (such as \(M_2 = 0\) or \(M_2 = \rho I\)) that might have been made depending of the considered version of Theorems 1.2 and 1.6.

For describing boundary layers in the vicinity of \(\partial \mathbb{E}\), when a parameter \(\epsilon > 0\) is assumed small, some functions will depend on a slow variable \(x \in \mathbb{E}\), the time \(t \geq 0\) and a fast variable \(z = \varphi_E(x)/\sqrt{\epsilon} \in \mathbb{R}_+\). In this case, for a map \((x, t, z) \mapsto h(x, t, z)\) we denote

\[
\|h\|_\epsilon(x, t) := h(x, t, \varphi_E(x)/\sqrt{\epsilon}).
\]

By convention, differential operators are always taken with respect to \(x \in \mathbb{E}\) only, if not indicated otherwise by the notation. Therefore, as also remarked in [17, 29], one has the commutation formulas

\[
\begin{align*}
\nabla \cdot ([h]_\epsilon) = & \|\nabla \cdot h\|_\epsilon - n \cdot \|\partial_n h\|_\epsilon / \sqrt{\epsilon}, \\
\nabla (\|[h]_\epsilon\|_\epsilon) = & \|\nabla h\|_\epsilon - \|\partial_n h\|_\epsilon n^T / \sqrt{\epsilon}, \\
[D([h]_\epsilon)n]_{\text{tan}} = & \|[D(h)n]_{\text{tan}}\|_\epsilon - \|[\partial_n h]_{\text{tan}}\|_\epsilon / \sqrt{4\epsilon}, \\
\epsilon \Delta [h]_\epsilon = & \epsilon \|[\Delta h]_\epsilon\|_\epsilon + \sqrt{\epsilon} \|\Delta \varphi_E [\partial_n h]_\epsilon\|_\epsilon - 2 \sqrt{\epsilon} \|[n \cdot \nabla] \partial_n h\|_\epsilon \\
& + |n|^2 \|[\partial_n h]_\epsilon\|_\epsilon,
\end{align*}
\]

and consequently

\[
\mathcal{N}_i ([h_1]_\epsilon, [h_2]_\epsilon) = \|[\mathcal{N}_i(h_1, h_2)]_\epsilon\|_\epsilon - \|[\partial_n h_i]_{\text{tan}}\|_\epsilon / \sqrt{4\epsilon}, \quad i \in \{1, 2\}.
\]

### 2.2 Function spaces and norms

The Hilbert spaces \(H(\mathbb{E})\) and \(W(\mathbb{E})\) of divergence-free and tangential vector fields are defined by means of

\[
H(\mathbb{E}) := \text{clos}_{L^2(\mathbb{E})} \left( \left\{ f \in C^1(\mathbb{E}) \mid \nabla \cdot f = 0 \text{ in } \mathbb{E}, f \cdot n = 0 \text{ on } \partial \mathbb{E} \right\} \right),
\]

\[
W(\mathbb{E}) := H(\mathbb{E}) \cap H^1(\mathbb{E}),
\]

where \(\text{clos}_{L^2(\mathbb{E})}\) denotes the closure in \(L^2(\mathbb{E})\). In addition, for any fixed \(T > 0\), denote by \(C^0_w([0, T]; H(\mathbb{E}))\) the space of weakly continuous functions from \([0, T]\) to \(H(\mathbb{E})\). The solution space \(\mathcal{X}_T\) and the control space \(\mathcal{U}_T\) are given by

\[
\mathcal{X}_T := C^0_w([0, T]; H(\mathbb{E})) \cap L^2((0, T); W(\mathbb{E})),
\]

\[
\mathcal{U}_T := C^1([0, T]; H^1(\mathbb{E})) \cap C^0([0, T]; H^2(\mathbb{E})).
\]
Moreover, for $m, p, k, s \in \mathbb{N}_0$ we employ the weighted Sobolev spaces

$$H^{k,m,p}_E := \left\{ f \in L^2(\mathcal{E} \times \mathbb{R}^+) \left| \|f\|_{H^{k,m,p}_E} := \left( \sum_{r=0}^{p} |f|_{k,m,r,E}^2 \right)^{\frac{1}{2}} < +\infty \right. \right\},$$

$$H^{k,s}(\mathbb{R}) := \left\{ f \in L^2(\mathbb{R}) \left| \|f\|_{H^{k,s}(\mathbb{R})} := \left( \sum_{l=0}^{s} \int_{\mathbb{R}} (1 + z^{2k})|\partial_x^l f(z)|^2 dz \right)^{\frac{1}{2}} < +\infty \right. \right\},$$

where $|f|_{k,m,r,E}$ denotes for functions $(x, z) \mapsto f(x, z)$ the seminorm

$$|f|_{k,m,r,E} := \left( \sum_{|\beta| \leq m} \int_{E} \int_{\mathbb{R}^n} (1 + z^{2k})|\partial_x^\beta \partial_z^\gamma f|^2 dz dx \right)^{\frac{1}{2}}.$$

### 2.3 Initial data extensions

Depending on the normal traces of the initial states $(u_0, B_0) \in L^2_c(\Omega) \times L^2_c(\Omega)$, there either exist extensions of the type

$$\tilde{u}_0, \tilde{B}_0 \in H(\mathcal{E}), \quad \tilde{u}_0|_\Omega = u_0, \quad \tilde{B}_0|_\Omega = B_0$$

or there are extensions

$$\tilde{u}_0, \tilde{B}_0 \in L^2(\mathcal{E}), \quad \tilde{u}_0 \cdot n = \tilde{B}_0 \cdot n = 0 \text{ at } \partial \mathcal{E}, \quad \tilde{u}_0|_\Omega = u_0, \quad \tilde{B}_0|_\Omega = B_0.$$

This statement is summarized by the following version of [10, Proposition 2.1]. Hereto, for each $i \in \{1, \ldots, K(\Omega)\}$, the sets $\Gamma_{c,1}^i, \ldots, \Gamma_{c,m_i}^i$ denote the connected components of the controlled boundary part $\Gamma_c^i$. Moreover, the set $\Omega_j^i$ is the extension attached to $\Omega$ at $\Gamma_j^i$, namely the maximal union of connected components of $\mathcal{E} \setminus \Omega$ with $(\partial \Omega_j^i \cap \Gamma) \subseteq \Gamma_{c,1}^i$. Then, let $\Omega_j^i \subseteq \Omega_j^j$ be the connected set attached to $\Gamma_{c,j}^i$. Hereby, the case $\Omega_j^i = \Omega_j^j$ occurs for $j \neq l$ when $\Omega_j^l$ is attached to more than one connected part of $\Gamma_{c,j}^i$.

**Lemma 2.3.** There exists a constant $C > 0$, such that for each $h \in L^2_c(\Omega)$ there is a function $\sigma \in C_0^\infty(\mathcal{E})$ with $\text{supp}(\sigma) \subseteq \mathcal{E} \setminus \overline{\Omega}$ and an extension $\tilde{h} \in L^2(\mathcal{E})$ satisfying

$$\tilde{h} = h \text{ in } \Omega, \quad \nabla \cdot \tilde{h} = \sigma \text{ in } \mathcal{E}, \quad \tilde{h} \cdot n = 0 \text{ on } \partial \mathcal{E}, \quad \|\tilde{h}\|_{L^2(\mathcal{E})} \leq C\|h\|_{L^2(\Omega)}.$$

When the vector field $h$ additionally obeys along $\Gamma_c$ the conditions

$$\forall i \in \{1, \ldots, K(\Omega)\}, \forall j \in \{1, \ldots, m_i\}:

\langle h \cdot n, 1 \rangle_{H^{-1/2}(\partial \Omega_j^i \cap \Gamma)} , H^{1/2}(\partial \Omega_j^i \cap \Gamma) = 0, \quad (2.5)$$

then one can build such an extension $\tilde{h} \in H(\mathcal{E})$. 

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Proof. Let \( n \) denote the outward unit normal to \( \Gamma \), while the outward unit normal to \( \partial E \) is written as \( n_{\partial E} \). It is known (cf. [9, Chapter IV, Section 3.2]) that there exists a continuous normal trace operator

\[
\gamma_n : L^2_c(\Omega) \rightarrow H^{-1/2}(\Gamma), \quad \forall w \in C^\infty(\overline{\Omega}) : \gamma_n(w) = w \cdot n.
\]

Then, for each \( i \in \{1, \ldots, K(\Omega)\} \) a smooth function \( \sigma^i \in C^\infty_0(\Omega^i; \mathbb{R}) \) is fixed with

\[
\forall i \in \{1, \ldots, m_1\} : \left| \int_{\Omega^i} \sigma^i(x) \, dx - \int_{\partial \Omega^i \cap \Gamma^i_c} \gamma_n(h) \, dS \right| = 0.
\]

This guarantees that one can solve for each \( i \in \{1, \ldots, K(\Omega)\} \) a weak formulation of the respective elliptic problem

\[
\begin{cases}
\Delta \varphi^i = \sigma^i & \text{in } \Omega^i, \\
\nabla \varphi^i \cdot n = \gamma_n(h) & \text{on } \Gamma_c^i \cap \partial \Omega^i, \\
\nabla \varphi^i \cdot n_{\partial E} = 0 & \text{on } \partial \Omega^i \setminus \Gamma_c^i.
\end{cases}
\tag{2.6}
\]

In particular, if (2.5) holds for \( i \in \{1, \ldots, K(\Omega)\} \), then \( \sigma^i = 0 \) can be chosen. Accordingly, the proof is concluded by means of the definitions \( \tilde{h} := h \) in \( \Omega \) and \( \tilde{\nabla} := \nabla \varphi^i \) in \( \Omega^i \). The continuity of the extension operator follows from (2.6) and the divergence-free condition encoded in \( L^2_c(\Omega) \). \( \square \)

**Remark 2.4.** In the context of Theorem 1.2, the condition (2.5) is automatically satisfied by \( u_0, B_0 \in L^2_c(\Omega) \). Indeed, since \( \Gamma_c \) is connected, the definition of \( L^2_c(\Omega) \) yields (2.5). This might not be the case for Theorem 1.2 b), but here the property (2.5) is ensured by the particular construction of \( E \) as an annulus.

### 2.4 Weak controlled trajectories

Given a time \( T > 0 \), we denote the space-time cylinder \( \mathcal{E}_T := E \times (0, T) \) and its mantle \( \Sigma_T := \partial E \times (0, T) \). Moreover, it is assumed that \( \xi \in \mathcal{U}_T(E) \) and \( \eta = \tilde{\eta} + \zeta \in \mathcal{U}_T(E) \) are two forces with

\[
\bigcup_{t \in [0, T]} (\text{supp}(\xi(\cdot, t)) \cup \text{supp}(\tilde{\eta}(\cdot, t))) \subseteq \overline{E} \setminus \overline{\Omega},
\]

where \( \zeta \) denotes a smooth function \( \overline{E} \times [0, T] \rightarrow \mathbb{R}^N \).

#### 2.4.1 The case of Theorem 1.2

Even though Theorem 1.2 is concerned with \( N = 2 \), the following formulations are valid for \( N \in \{2, 3\} \). For arbitrary fixed initial data \( u_0, B_0 \in L^2_c(\Omega) \) which admit extensions to \( H(\mathcal{E}) \), a weak controlled trajectory for (1.1) is any pair

\[
(u, B) \in \left[ C^0_\mu([0, T]; L^2_c(\Omega)) \cap L^2((0, T); H^1(\Omega)) \right]^2,
\]
which is the restriction \((u, B) = (\tilde{u}|_{\Omega}, \tilde{B}|_{\Omega})\) of a weak Leray-Hopf solution \((\tilde{u}, \tilde{B})\) to the following viscous and resistive incompressible MHD system

\[
\begin{align*}
\partial_t u - v_1 \Delta u + (u \cdot \nabla) u - \mu (B \cdot \nabla) B + \nabla p &= \xi \quad \text{in } \mathcal{E}_T, \\
\partial_t B - v_2 \Delta B + (u \cdot \nabla) B - (B \cdot \nabla) u &= \eta \quad \text{in } \mathcal{E}_T, \\
\nabla \cdot u &= \nabla \cdot B = 0 \quad \text{in } \mathcal{E}_T, \\
u \cdot n &= B \cdot n = 0 \quad \text{on } \Sigma_T, \\
\mathcal{N}_1(u, B) &= \mathcal{N}_2(u, B) = 0 \quad \text{on } \Sigma_T, \\
u(\cdot, 0) &= u_0, B(\cdot, 0) = B_0 \quad \text{in } \mathcal{E}
\end{align*}
\] (2.7)

Hereby, a pair \((u, B) \in \mathcal{X}^2_T\) is called a weak Leray-Hopf solution to (2.7) if it satisfies for all \(\varphi, \psi \in C_0^\infty(\overline{\mathcal{E}} \times [0, T]; \mathbb{R}^N) \cap C^\infty([0, T]; H(\mathcal{E}))\) and almost all \(t \in [0, T]\) the variational formulation

\[
\begin{align*}
\int_\mathcal{E} (u(x, t) \cdot \varphi(x, t) + B(x, t) \cdot \psi(x, t) - u_0(x) \cdot \varphi(x, 0) - B_0(x) \cdot \psi(x, 0)) \, dx \\
&\quad - \int_0^t \int_\mathcal{E} (\nabla \times u) \cdot (\nabla \times \varphi) \, dx \, dt \\
&\quad + \int_0^t \int_\mathcal{E} (\nabla \times B) \cdot (\nabla \times \psi) \, dx \, dt \\
&\quad + \int_0^t \int_\mathcal{E} ((u \cdot \nabla) B - (B \cdot \nabla) u) \cdot \psi \, dx \, dt \\
&\quad - \int_0^t \int_{\partial \mathcal{E}} \rho_2(u, B) \cdot \psi \, dS \, dt = \int_0^t \int_\mathcal{E} (\xi \cdot \varphi + \eta \cdot \psi) \, dx \, dt,
\end{align*}
\] (2.8)

together with the following energy inequality for almost all \(0 \leq s < t \leq T:\)

\[
\begin{align*}
\|u(\cdot, t)\|^2_{L^2(\mathcal{E})} + \mu \|B(\cdot, t)\|^2_{L^2(\mathcal{E})} + 2 \int_s^t \int_\mathcal{E} \left( v_1 |\nabla \times u|^2 + v_2 \mu |\nabla \times B|^2 \right) \, dx \, dt \\
\leq \|u(\cdot, s)\|^2_{L^2(\mathcal{E})} + \mu \|B(\cdot, s)\|^2_{L^2(\mathcal{E})} + 2 \int_s^t \int_\mathcal{E} \xi \cdot u \, dx \, dt + 2 \mu \int_s^t \int_\mathcal{E} \eta \cdot B \, dx \, dt \\
&\quad + 2 v_1 \int_s^t \int_{\partial \mathcal{E}} \rho_1(u, B) \cdot u \, dS \, dt + 2 v_2 \mu \int_s^t \int_{\partial \mathcal{E}} \rho_2(u, B) \cdot B \, dS \, dt.
\end{align*}
\] (2.9)

For deriving the weak form (2.8), one has used the identities \(\Delta u = -\nabla \times (\nabla \times u)\) and \(\Delta B = -\nabla \times (\nabla \times B)\), by employing \(\nabla \cdot u = \nabla \cdot B = 0\), together with the integration by parts formula

\[
\int_\mathcal{E} g \cdot (\nabla \times h) \, dx = \int_\mathcal{E} (\nabla \times g) \cdot h \, dx - \int_{\partial \mathcal{E}} (g \times h) \cdot n \, dS,
\]

where \(dS\) is the induced surface measure on \(\partial \mathcal{E}\), and the vector calculus identities

\[
(g \times h) \cdot n = (h \times n) \cdot g = -(g \times n) \cdot h.
\]
It is important to verify that when a weak solution is regular enough, it is a classical solution to the original problem. On the one hand, if \((u, B)\) possesses the necessary regularity and satisfies (2.8), then \((u, B)\) also classically obeys (2.7) with the induction equation replaced by

\[
\partial_t B - \nu_2 \Delta B + (u \cdot \nabla) B - (B \cdot \nabla) u + \nabla q = \eta. \tag{2.10}
\]

On the other hand, because the test function \(\psi\) in (2.8) is divergence-free and tangential, one cannot immediately conclude that such \((u, B)\) satisfies (2.7). However, assuming enough regularity, one may take the \(L^2\) inner product of (2.10) with \(\nabla q\) and also with arbitrary \(\psi = \nabla \phi\) for any \(\phi \in C_0^\infty([0, T] \times \overline{E})\), in order to obtain a priori that

\[
\|\nabla q\|_{L^2(E)}^2 = 0 \tag{2.11}
\]

as long as the force \(\eta\) given in (2.7) satisfies

\[
\nabla \cdot \eta = 0 \text{ in } E, \quad \text{and} \quad \eta \cdot n = 0 \text{ on } \partial E. \tag{2.12}
\]

Thus, one needs to have (2.12) in order that (2.8) is a reasonable weak formulation for (2.7).

The existence of weak Leray-Hopf solutions \((u, B) \in \mathcal{X}_T^2\) satisfying (2.8) and (2.9) can be obtained by analysis similar to the Navier-Stokes equations, for instance via the Galerkin method explained in [42, Chapter 3]. Regarding the strong energy inequality (2.9), we refer to [29, Section 3] for a strategy that carries over to the present MHD model. Let us briefly mention why the boundary integrals in (2.8) and (2.9) do not cause additional difficulties comparing to the references mentioned above. Indeed, assume that \(M \in C^\infty(\overline{E}; \mathbb{R}^{N \times N}), h \in L^2((0, T); W(E))\) and \(g \in L^2((0, T); H(E))\). Then, for any \(\delta \in (0, 1)\) and a constant \(C = C(E, \|M\|_{C^1(\overline{E})}) > 0\) one has

\[
\left| \int_s^t \int_{\partial E} [M(x)h]_{\tan} \cdot g \, dS \, dr \right| \\
\leq \frac{C}{\delta} \int_s^t \|g(\cdot, r)\|_{L^2(E)}^2 \, dr + \delta \int_s^t \|g(\cdot, r)\|_{H^1(E)}^2 \, dr \\
+ \frac{C}{\delta} \int_s^t \|h(\cdot, r)\|_{L^2(E)}^2 \, dr + \delta \int_s^t \|h(\cdot, r)\|_{H^1(E)}^2 \, dr. \tag{2.13}
\]

Combined with the estimate (2.1), this argument allows to employ the Galerkin approach described in [42, Chapter 3] for the construction of approximate solutions \((u^k, B^k)_{k \in \mathbb{N}}\) to (2.8) bounded in \(L^2((0, T); H^1(E))\), satisfying a discrete version of (2.9) and converging to a weak Leray-Hopf solution \((u, B) \in \mathcal{X}_T^2\) as \(k \to +\infty\). For passing the limit \(k \to +\infty\) in the discrete version of the energy inequality (2.9), one needs to show

\[
\nu_1 \int_s^t \int_{\partial E} \rho_1(u^k, B^k) \cdot u^k \, dS \, dt + \nu_2 \mu \int_s^t \int_{\partial E} \rho_2(u^k, B^k) \cdot B^k \, dS \, dt \\
\rightarrow \nu_1 \int_s^t \int_{\partial E} \rho_1(u, B) \cdot u \, dS \, dt + \nu_2 \mu \int_s^t \int_{\partial E} \rho_2(u, B) \cdot B \, dS \, dt, \quad \text{as } k \to +\infty.
\]
Here, if \( h^k \) denotes either \( u^k \) or \( B^k \), and \( h \) represents either \( u \) or \( B \), then by means of trace theorems one has
\[
\|h^k - h\|_{L^2((0,T);L^2(\partial\Omega))}^2 \leq \|h^k - h\|_{L^2((0,T);L^2(\partial\Omega))} \|h^k - h\|_{L^2((0,T);H^1(\Omega))} \\
\leq C \|h^k - h\|_{L^2((0,T);L^2(\partial\Omega))} \rightarrow 0, \quad k \rightarrow +\infty,
\]
which implies for \((g^1, g^2) = (u, B)\) and \((g^{1,k}, g^{2,k}) = (u^k, B^k)\) that
\[
\int_s^t \int_{\partial\Omega} \left| [M_i u^k + L_i B^k]_\tan \cdot g^{i,k} - [M_i u + L_i B]_\tan \cdot g_i \right| \, dS \, dt \\
\leq \int_s^t \int_{\partial\Omega} \left| \left( [M_i (u^k - u)] + L_i (B^k - B) \right)_\tan \cdot g^{i,k} \right| \, dS \, dt \\
+ \int_s^t \int_{\partial\Omega} \left| [M_i u + L_i B]_\tan \cdot (g_i - g^{i,k}) \right| \, dS \, dt 
\]
\( \rightarrow 0 \) as \( k \rightarrow +\infty, \ i = 1, 2. \)

### 2.4.2 Change of unknowns

A few exceptions aside, the subsequent analysis can be streamlined significantly by introducing the symmetrized notations
\[
z^\pm := u \pm \sqrt{\mu}B, \quad \xi^\pm := \xi \pm \sqrt{\mu}\eta, \quad \lambda^\pm := \frac{\nu_1 \pm \nu_2}{2}, \quad z^+_0 := u_0 \pm \sqrt{\mu}B_0,
\]
as well as
\[
N^\pm(h^+, h^-) := [D(h^+) n(x) + W h^+ + M^+(x) h^+ + L^+ h^-]_\tan
\]
and
\[
\rho^\pm(h^+, h^-) := 2 \left[ M^\pm(x) h^+ + L^\pm(x) h^- \right]_\tan,
\]
wherein
\[
M^\pm := \frac{\sqrt{\mu} M_1 \pm \sqrt{\mu} M_2 + L_1 \pm L_2}{2\sqrt{\mu}}, \quad L^\pm := \frac{\sqrt{\mu} M_1 \pm \sqrt{\mu} M_2 - L_1 \pm L_2}{2\sqrt{\mu}}.
\]

By utilizing the inner product structure of \( L^2(\Omega) \) one obtains for the energy
\[
E(t) := \|u(\cdot, t)\|_{L^2(\Omega)}^2 + \mu\|B(\cdot, t)\|_{L^2(\Omega)}^2 + 2 \int_s^t \int_{\partial\Omega} \left( \nu_1 |\nabla \times u|^2 + \nu_2 |\nabla \times B|^2 \right) \, dx \, dt
\]
that
\[
E(t) = \frac{1}{2} \sum_{\Delta \in \{+, -\}} \|z^\Delta(\cdot, t)\|_{L^2(\Omega)}^2 + \lambda^+ \sum_{\Delta \in \{+, -\}} \int_s^t \int_{\partial\Omega} \nabla \times z^\Delta \cdot \nabla \times z^\Delta \, dx \, ds \\
+ \lambda^- \sum_{(\Delta, \rho) \in \{(+, -), (-, +)\}} \int_s^t \int_{\partial\Omega} \nabla \times z^\Delta \cdot \nabla \times z^\Delta \, dx \, ds.
\]
Therefore, if \((u, B) \in \mathcal{X}_T^2\) is a weak Leray-Hopf solution to (2.7), by using (2.16) in (2.9) and the above transformation (2.15), it follows for almost all \(0 \leq s \leq t \leq T\) the inequality
\[
\frac{1}{2} \sum_{\partial \in \{+, -\}} \|z^\partial(\cdot, t)\|_{L^2(\Omega)}^2 + \lambda^+ \sum_{\partial \in \{+, -\}} \int_s^t \int_{\Omega} \nabla \times z^\partial \cdot \nabla \times z^\partial \, dx \, dr \\
+ \lambda^- \sum_{\partial \in \{+, -\}} \int_s^t \int_{\Omega} \nabla \times z^\partial \cdot \nabla \times z^\partial \, dx \, dr \\
\leq \sum_{\partial \in \{+, -\}} \left( \frac{1}{2} \|z^\partial(\cdot, s)\|_{L^2(\Omega)}^2 + \int_s^t \int_{\Omega} \xi \cdot z^\partial \, dx \, dr + \lambda^+ \int_s^t \int_{\partial \Omega} \rho^\partial(z^+ z^-) \cdot z^\partial \, dS \, dr \right) \\
+ \lambda^- \sum_{\partial \in \{+, -\}} \int_s^t \int_{\partial \Omega} \rho^\partial(z^+ z^-) \cdot z^\partial \, dS \, dr.
\]
(2.17)

### 2.4.3 The case of Theorem 1.6

Under the assumptions of Theorem 1.6, Lemma 2.3 provides extensions \(u_0, B_0 \in L^2(\Omega)\) with \(u_0 \cdot n = B_0 \cdot n = 0\) along \(\partial \Omega\) and \(\nabla \cdot u_0, \nabla \cdot B_0 \in C^\infty(\overline{\mathcal{E}} \times [0, T]; \mathbb{R})\) supported in \(\mathcal{E} \setminus \overline{\Omega}\). A weak controlled trajectory for (1.8) is then defined as any pair \((u, B) \in \left[ C^0_0([0, T]; L^2_\Omega) \cap L^2((0, T); H^1(\Omega)) \right]^2\), with \(u = \frac{z^+ + z^-}{2}|_\Omega\), \(B = \frac{z^+ - z^-}{\sqrt{\mu^2}}|_\Omega\),

where \(z^+ \in C^0_0([0, T]; L^2(\mathcal{E})) \cap L^2((0, T); H^1(\mathcal{E}))\) solve in the below specified weak Leray-Hopf sense the Elsasser system\(^1\)

\[
\begin{align*}
\partial_t z^\pm - \Delta (\lambda^\pm z^\pm + \lambda^\pm z^-) + (z^\pm \cdot \nabla) z^\pm + \nabla p^\pm &= \xi^\pm \quad \text{in } \mathcal{E}_T, \\
\nabla \cdot z^\pm &= \sigma^\pm \quad \text{in } \mathcal{E}_T, \\
z^\pm \cdot n &= 0 \quad \text{on } \Sigma_T, \\
\mathcal{N}^\pm(z^\pm, z^-) &= 0 \quad \text{on } \Sigma_T, \\
z^\pm(\cdot, 0) &= z^\pm_0 = u_0 \pm \sqrt{\mu} B_0 \quad \text{in } \mathcal{E}_T,
\end{align*}
\]

with \(\xi^\pm := \xi \pm \sqrt{\mu} \eta, \quad p^\pm := p \mp \sqrt{\mu} q\).

In (2.18), the boundary operators \(\mathcal{N}^\pm\) are as defined in Section 2.4.2. Moreover, the functions \(\sigma^\pm: \overline{\mathcal{E}} \times [0, T] \rightarrow \mathbb{R}\) are assumed to be smooth, supported in \(\overline{\mathcal{E}} \setminus \overline{\Omega}\), and of

\(^1\)Systems of the form (2.18), but with different boundary conditions, have been derived by Elsasser in [19].
In view of deriving a weak formulation for (2.18), it is convenient to introduce the ansatz
\[u_{\sigma} = u_{\sigma}^1 + u_{\sigma}^2,\]
where the right-hand side is given by
\[\sum_{j=1}^{K(\Omega)} \int_{\Omega_j} \sigma^j \, dx = 0.\]

Since \(M^\pm\) and \(L^\pm\) are likewise smooth, a notion for weak solutions to (2.18) can now be introduced similarly to the Navier-Stokes case in [17]. In order to lift the possibly nonzero divergence constraints, let \(z_{\sigma}^\pm\) solve the linear Elsasser system
\[
\begin{align*}
\partial_t z_{\sigma}^\pm - \Delta (\lambda^\pm z_{\sigma}^\pm + \lambda^\mp z_{\sigma}^\mp) + \nabla p_{\sigma}^\pm &= 0 & \text{in } \mathcal{E}_T, \\
\nabla \cdot z_{\sigma}^\pm &= \sigma^\pm & \text{in } \mathcal{E}_T, \\
\left(\nabla \times z_{\sigma}^\pm\right) \times n - \rho^\pm (z_{\sigma}^+, z_{\sigma}^-) &= 0 & \text{on } \Sigma_T, \\
z_{\sigma}^\pm(\cdot, 0) &= z_{\sigma}^\pm_0 & \text{in } \mathcal{E}.
\end{align*}
\]  
(2.19)

In view of deriving a weak formulation for (2.19), it would be of advantage to first eliminate the inhomogeneous divergence data \(\sigma^\pm\). Hereto, one may decompose \(z_{\sigma}^\pm := Z_{\sigma}^\pm + \nabla \theta_{\sigma}^\pm\), with \(\theta_{\sigma}^\pm(\cdot, t)\) being for each \(t \in [0, T]\) smooth solutions to the elliptic Neumann problems
\[
\begin{align*}
\Delta \theta_{\sigma}^\pm(\cdot, t) &= \sigma^\pm(\cdot, t) & \text{in } \mathcal{E}, \\
\nabla \theta_{\sigma}^\pm(\cdot, t) \cdot n(x) &= 0 & \text{on } \partial \mathcal{E},
\end{align*}
\]  
(2.20)
while \(Z_{\sigma}^\pm\) obey the inhomogeneous system
\[
\begin{align*}
\partial_t Z_{\sigma}^\pm - \Delta (\lambda^\pm Z_{\sigma}^\pm + \lambda^\mp Z_{\sigma}^\mp) + \nabla p_{\sigma}^\pm &= \Theta^\pm & \text{in } \mathcal{E}_T, \\
\nabla \cdot Z_{\sigma}^\pm &= 0 & \text{in } \mathcal{E}_T, \\
\left(\nabla \times Z_{\sigma}^\pm\right) \times n - \rho^\pm (Z_{\sigma}^+, Z_{\sigma}^-) &= \rho^\pm (\nabla \theta_{\sigma}^+, \nabla \theta_{\sigma}^-) & \text{on } \Sigma_T, \\
Z_{\sigma}^\pm(\cdot, 0) &= z_{\sigma}^\pm_0 - \nabla \theta^\pm(\cdot, 0) & \text{in } \mathcal{E}.
\end{align*}
\]  
(2.21)

where the right-hand side is given by
\[\Theta^\pm := -\partial_t \nabla \theta_{\sigma}^\pm + \Delta (\lambda^\pm \nabla \theta_{\sigma}^+ + \lambda^\mp \nabla \theta_{\sigma}^-).\]

A weak formulation for (2.21) can now be derived similarly to that for the nonlinear system (2.22) below, and a Galerkin method can be employed for constructing a weak solution to the linear problem (2.21). By the same idea as used in the proof of Lemma B.1, the regularity of weak solutions to (2.21) can be investigated by means of a reduction to the Navier-Stokes case, and one finds that \(Z_{\sigma}^\pm\) are smooth for \(t > 0\). Finally, the ansatz \(z^\pm = \tilde{z}^\pm + z_{\sigma}^\pm\) for the solutions of (2.18) provides a description of \(\tilde{z}^\pm\) by means of the perturbed Elsasser equations
\[
\begin{align*}
\partial_t \tilde{z}^\pm - \Delta (\lambda^\pm \tilde{z}^\pm + \lambda^\mp \tilde{z}^-) + (\tilde{z}^+ + z_{\sigma}^+) \cdot \nabla (\tilde{z}^\pm + z_{\sigma}^\pm) + \nabla \tilde{p}^\pm &= \xi^\pm & \text{in } \mathcal{E}_T, \\
\nabla \cdot \tilde{z}^\pm &= 0 & \text{in } \mathcal{E}_T, \\
\Lambda^\pm (\tilde{z}^+, \tilde{z}^-) &= 0 & \text{on } \Sigma_T, \\
\tilde{z}^\pm(\cdot, 0) &= 0 & \text{in } \mathcal{E}.
\end{align*}
\]
A weak Leray-Hopf solution to the above system is then understood as any pair 
\((\overline{z}^+, \overline{z}^-)\) which satisfies for all \(\varphi^\pm \in C^\infty \left(\overline{E} \times [0, T); \mathbb{R}^N\right) \cap C^\infty ([0, T]; H(E))\) and almost all \(t \in [0, T]\) the variational formulation

\[
\int_E \overline{z}^\pm (x, t) \cdot \varphi^\pm (x, t) \, dx - \int_0^t \int_E \overline{z}^\pm \cdot \partial_t \varphi^\pm \, dx \, dt
+ \lambda^\pm \int_0^t \int_{\partial E} \left(\nabla \times \overline{z}^+ \right) \cdot \left(\nabla \times \varphi^+\right) \, dx \, dt + \lambda^- \int_0^t \int_{\partial E} \left(\nabla \times \overline{z}^- \right) \cdot \left(\nabla \times \varphi^-\right) \, dx \, dt
- \lambda^- \int_0^t \int_{\partial E} \rho^-(\overline{z}^+, \overline{z}^-) \cdot \varphi^- \, dS \, dt - \lambda^+ \int_0^t \int_{\partial E} \rho^+(\overline{z}^+, \overline{z}^-) \cdot \varphi^+ \, dS \, dt
+ \int_0^t \int_{\partial E} \left((\overline{z}^\pm \cdot \nabla)\overline{z}^\pm + (\overline{z}^\pm \cdot \nabla)\overline{z}^\pm + (\overline{z}^\pm \cdot \nabla)\varphi^\pm \right) \cdot \varphi^\pm \, dx \, dt
= \int_0^t \int_E \overline{z}^\pm \cdot \varphi^\pm \, dx \, dt,
\]
and for almost all \(0 \leq s \leq t \leq T\) the energy inequality

\[
\sum_{\Delta \in \{+,-\}} \|\overline{z}^\Delta(t, s)\|_{L^2(E)}^2 + \sum_{\Delta \in \{+,-\}} (\lambda^+ \square \lambda^-) \int_s^t \int_E \left|\nabla \times (\overline{z}^+ \square \overline{z}^-)\right|^2 \, dx \, dt
\leq \sum_{\Delta \in \{+,-\}} \left(\|\overline{z}^\Delta(s, t)\|^2_{L^2(E)} + 2 \lambda^+ \int_s^t \int_{\partial E} \rho^\Delta(\overline{z}^+, \overline{z}^-) \cdot \overline{z}^\Delta \, dS \, dx \right)
+ \sum_{\Delta, \sigma \in \{(+,-), (-,+), (0,0), (2,2)\}} \left(2 \lambda^- \int_s^t \int_{\partial E} \rho^\Delta(\overline{z}^+, \overline{z}^-) \cdot \overline{z}^\Delta \, dS \, dx + \int_s^t \int_E |\overline{z}^\Delta|^2 \, dx \, dt \right)
- 2 \sum_{\Delta, \sigma \in \{(+,-), (-,+), (0,0), (2,2)\}} \left(\int_s^t \int_E (\overline{z}^\Delta \cdot \nabla)\overline{z}^\Delta + (\overline{z}^\Delta \cdot \nabla)\varphi^\Delta - \varphi^\Delta \right) \cdot \varphi^\Delta \, dx \, dt.
\]

Since the profiles \(\overline{z}^\pm\) are smooth and \(L^\pm, M^\pm \in C^\infty (\overline{E}; \mathbb{R}^{N \times N})\), the existence of \((\overline{z}^+, \overline{z}^-)\) in \(X^2_T\) satisfying (2.22) and (2.23) can be obtained through a Galerkin method as explained in Section 2.4.1.

**Remark 2.5.** In view of Section 2.4.2, the weak formulation from Section 2.4.1 for (2.7) can be regarded as a special case of (2.22) and (2.23). The two formulations are presented separately in order to make a stronger distinction between the case without a gradient term \(\nabla q\) and the non-physical case where even \(\nabla \cdot B \neq 0\) is possible in \(\overline{E} \setminus \overline{Q}\).

### 2.5 Preliminary description of the strategy

Let us describe the strategy for proving Theorems 1.2 and 1.6 first, see also Section 4 for the details and [17] for the case of the Navier-Stokes equations. First, the interval \([0, T]\) will be divided into two sub-intervals \([0, T_{\text{reg}}]\) and \((T_{\text{reg}}, T_{\text{ctrl}}]\), which correspond to the two stages of the control strategy:
Stage 1) A weak Leray-Hopf solution to (2.7) or (2.18) with \( \xi = \eta = 0 \) is selected such that its state at \( t = T_{\text{reg}} \) belongs to \( H^3(\mathcal{E}) \cap W(\mathcal{E}) \) at a time \( T_{\text{reg}} \in [0, T_{\text{ctrl}}] \).

Stage 2) During \((T_{\text{reg}}, T_{\text{ctrl}}]\), appropriate forces \( \xi \) and \( \eta \) are applied in \( \overline{\mathcal{E}} \setminus \overline{\Omega} \), while in the case of Theorem 1.6 the force \( \xi \) might act in the whole domain. It will then be shown that each weak Leray-Hopf solution to (2.7) or (2.18), which starts from \( H^3(\mathcal{E}) \cap H(\mathcal{E}) \) at \( t = T_{\text{reg}} \), approaches the final state in \( L^2(\mathcal{E}) \) at \( t = T \). In view of the remarks given in Section 2.4, for proving Theorem 1.2 we need that

\[
\nabla \cdot \eta = 0 \text{ in } \mathcal{E}, \quad \eta \cdot n = 0 \text{ on } \partial \mathcal{E}.
\]  

(2.24)

The second stage makes up the main part of this article and is carried out in Section 3 below. The construction of the controls and related estimates are of \textit{a priori} type, depending only on the regularized initial data, the terminal time, the final state, the approximation accuracy, and the fixed geometry.

3 Approximate controllability between regular states

Let a time \( T > 0 \), a small constant \( \delta > 0 \) and the initial data \( u_0, B_0 \in H^3(\mathcal{E}) \cap W(\mathcal{E}) \) be arbitrarily fixed. Moreover, assume that \((\xi, \eta) \in \mathcal{U}_T \times \mathcal{U}_T \) and \( \sigma^\pm \in C_0^\infty(\overline{E} \times [0, T]) \) are given functions. The different configurations in Theorems 1.2 and 1.6 are treated simultaneously as follows:

- In the context of Theorem 1.2, the weak Leray-Hopf solution \((u, B) \in \mathcal{X}_E^T \times \mathcal{X}_E^T \) to (2.7) with the data \((u_0, B_0, \xi, \eta) \) is fixed and by means of Section 2.4.2 rewritten in the variables \( z^\pm \).

- In the case of Theorem 1.6, any weak Leray-Hopf solution \((z^+, z^-) \in \mathcal{X}_E^T \times \mathcal{X}_E^T \) to (2.18) with the data \((z^0_0 = u_0 \pm \sqrt{\mu}B_0, \xi^\pm = \xi \pm \sqrt{\mu}\eta, \sigma^\pm) \) is fixed.

It will be shown that, if the already selected functions \((\xi, \eta, \sigma^\pm) \) are of a certain form, then \( z^\pm \) satisfy

\[
\|z^+(\cdot, T)\|_{L^2(\mathcal{E})} + \|z^-(\cdot, T)\|_{L^2(\mathcal{E})} < \delta. \tag{3.1}
\]

More generally, for arbitrary \( \overline{z}_i^\pm \in C_0^\infty(\overline{E}) \cap H(\mathcal{E}) \) it will be demonstrated that for suitable choices \((\xi, \eta, \sigma^\pm) \) one has

\[
\|z^+(\cdot, T) - \overline{z}_i^+\|_{L^2(\mathcal{E})} + \|z^-(\cdot, T) - \overline{z}_i^-\|_{L^2(\mathcal{E})} < \delta. \tag{3.2}
\]

The estimate (3.2), together with the representations

\[
2u = (z^+ + z^-), \quad 2\sqrt{\mu}B = (z^+ - z^-), \quad 2\overline{\mu}B_1 = (\overline{z}_i^+ + \overline{z}_i^-), \quad 2\sqrt{\mu}B_1 := (\overline{z}_i^+ - \overline{z}_i^-),
\]

implies

\[
\|u(\cdot, T) - \overline{B}_i\|_{L^2(\mathcal{E})} + \|B(\cdot, T) - \overline{B}_1\|_{L^2(\mathcal{E})} < \delta.
\]

3.1 Asymptotic expansions

The systems (2.7) or (2.18) are now simultaneously reformulated as a small-dissipation perturbation of an ideal MHD system\(^2\) in the variables \((z^+, z^-)\). Hereby, for any small

\(^2\)The here considered limit equations correspond in fact to an Euler system.
\( \epsilon \in (0, 1) \) the following scaling is performed
\[
    z^{\pm, \epsilon}(x, t) := \epsilon z^{\pm}(x, \epsilon t), \quad p^{\pm, \epsilon}(x, t) := \epsilon^2 p^{\pm}(x, \epsilon t), \quad \sigma^{\pm, \epsilon}(x, t) := \epsilon \sigma^{\pm}(x, \epsilon t),
\]
(3.3)
and for the controls
\[
    \xi^{\pm, \epsilon}(x, t) := \epsilon^2 \xi^{\pm}(x, \epsilon t).
\]
(3.4)

As a result, the functions \( z^{\pm, \epsilon} \) are seen to satisfy a weak formulation and strong energy inequality for the following problem
\[
    \begin{cases}
        \partial_t z^{\pm, \epsilon} - \epsilon \Delta (\lambda^{\pm} z^{\pm, \epsilon} + \lambda^z z^{-, \epsilon}) + (z^{-, \epsilon} \cdot \nabla) z^{\pm, \epsilon} + \nabla p^{\pm, \epsilon} = \xi^{\pm, \epsilon} & \text{in } \mathcal{E}_{T/\epsilon}, \\
        \nabla \cdot z^{\pm, \epsilon} = \sigma^{\pm, \epsilon} & \text{in } \mathcal{E}_{T/\epsilon}, \\
        z^{\pm, \epsilon} \cdot n = 0 & \text{on } \Sigma_{T/\epsilon}, \\
        \nabla^{\pm}(z^{+, \epsilon}, z^{-, \epsilon}) = 0 & \text{on } \Sigma_{T/\epsilon}, \\
        z^{\pm, \epsilon}(\cdot, 0) = \epsilon z^0_{\pm} & \text{in } \mathcal{E}.
    \end{cases}
\]
(3.5)

In order to achieve the desired estimate (3.2), it shall be verified that, for \( \xi^{\pm, \epsilon} \) and \( \sigma^{\pm, \epsilon} \) being of specific forms, all solutions \( (z^+, z^-) \in \mathcal{X}^T \times \mathcal{X}^T \) to (3.5) obey
\[
    \|z^{\pm, \epsilon}(x, T) - z^+_{\pm}(\cdot, T)\|_{L^2(\mathcal{E})} = O(\epsilon^{9/8}).
\]
(3.6)

Hence, after choosing \( \epsilon = \epsilon(\delta) > 0 \) sufficiently small, the asymptotic behavior (3.6) implies (3.2). To prove (3.6) with \( z^+_{\pm} = 0 \), see Section 3.6 for the general case, the above fixed solution to the problem (3.5) is expanded according to the ansatz
\[
    z^{\pm, \epsilon} = z^0 + \sqrt{\epsilon} \| v^\pm \|_\epsilon + \epsilon z^{\pm, 1} + \epsilon \nabla \theta^{\pm, \epsilon} + \epsilon \| w^\pm \|_\epsilon + \epsilon r^{\pm, \epsilon},
\]
\[
p^{\pm, \epsilon} = p^0 + \epsilon \| q^\pm \|_\epsilon + \epsilon p^{\pm, 1} + \epsilon \theta^{\pm, \epsilon} + \epsilon \pi^{\pm, \epsilon},
\]
\[
    \sigma^{\pm, \epsilon} = \sigma^0,
\]
(3.7)
and for the controls
\[
    \xi^{\pm, \epsilon} = \xi^0 + \sqrt{\epsilon} \| \mu^\pm \|_\epsilon + \epsilon \xi^{\pm, 1} + \epsilon \tilde{\zeta}^{\pm, \epsilon}.
\]
(3.8)

For large times \( t \geq T \), we already fix at this point
\[
    z^0(\cdot, t) = z^{\pm, 1}(\cdot, t) = \xi^{\pm, 1}(\cdot, t) = \xi^{\pm, \epsilon}(\cdot, t) = \mu^{\pm}(\cdot, t, \cdot) = 0
\]
and
\[
p^0(\cdot, t) = p^{\pm, 1}(\cdot, t) = \sigma^0(\cdot, t) = 0.
\]

On the time interval \([0, T] \), the profiles
\[
    z^0: \mathcal{E}_T \rightarrow \mathbb{R}^N, \quad p^0: \mathcal{E}_T \rightarrow \mathbb{R}, \quad \xi^0: \mathcal{E}_T \rightarrow \mathbb{R}^N, \quad \sigma^0: \mathcal{E}_T \rightarrow \mathbb{R}
\]
are chosen in the following way:

- If \( \Omega \) is a general smoothly bounded domain as in Theorem 1.6, then \( z^0, p^0, \xi^0, \) and \( \sigma^0 \) are determined by Lemma 3.2 below.
• If $\Omega \subseteq \mathbb{R}^2$ is the simply-connected domain as assumed by Theorem 1.2 a), then $z^0, p^0, \xi^0$, and $\sigma^0$ are determined by Lemma 3.5 below.

• If $\Omega \subseteq \mathbb{R}^2$ is a domain as considered in the case b) of Theorem 1.2, then $z^0, p^0, \xi^0$, and $\sigma^0$ are fixed in (3.12).

• If $\Omega$ is the cylinder from Remark 1.7, then $z^0, p^0, \xi^0$, and $\sigma^0$ are given by Example 3.4.

The profiles $z^{\pm,1}: E_T \to \mathbb{R}^N$ are later on defined on $[0,T]$ via Lemma 3.8 as solutions to (3.14), together with associated pressure terms $p^{\pm,1}: E_T \to \mathbb{R}$ and interior controls $\xi^{\pm,1}: E_T \to \mathbb{R}^N$. The vector field $z^0$ fails in general to obey $N^\pm(z^0, z^0) = 0$ along $\partial E$, which gives rise to weak amplitude boundary layers in the zero dissipation limit $\epsilon \to 0$. These boundary layers are of the same nature as those studied in [17, 29]. In the particular case $N^+(z^0, z^0) \neq N^-(z^0, z^0)$, there appears not only a velocity boundary layer but also one for the magnetic field. The profiles

$$v^\pm, w^\pm: E \times \mathbb{R}^+ \times \mathbb{R}_+ \to \mathbb{R}, \quad \theta^\pm, q^\pm, \theta^\pm: E \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R},$$

which are related to such boundary layers, will be described in Section 3.4. In the presence of magnetic field boundary layers, or due to certain coupled boundary conditions, the controls $\tilde{\xi}^{\pm, \epsilon}: E_T \to \mathbb{R}^N$ become important and will for $t \in [0,T]$ be defined at the end of Section 3.4.1. On $[0,T]$, the boundary layer dissipation control profiles $\mu^\pm: E_T \times \mathbb{R}_+ \to \mathbb{R}^N$ shall be defined in Section 3.4.2. In Section 3.5, the remainder terms $r^\pm, \epsilon$ are estimated. Then, concerning approximate null controllability, the asymptotic behavior (3.6) with $\Omega_1 = 0$ is shown in Corollary 3.28. The approximate controllability towards arbitrary smooth states is concluded in Section 3.6.

### 3.2 A return method trajectory

The zero order profiles $z^0, p^0, \xi^0$, and $\sigma^0$ are chosen for $t \in [0,T]$ as a special solution to the controlled Euler system

\[
\begin{align*}
\partial_t z^0 + (z^0 \cdot \nabla) z^0 + \nabla p^0 &= \xi^0 \quad \text{in } E_T, \\
\nabla \cdot z^0 &= \sigma^0 \quad \text{in } E_T, \\
z^0 \cdot n &= 0 \quad \text{on } \Sigma_T, \\
z^0(\cdot, 0) &= 0 \quad \text{in } E, \\
z^0(\cdot, T) &= 0 \quad \text{in } E.
\end{align*}
\]  

(3.9)

Given a smooth vector field $z^0$, let $Z^0(x, s, t)$ denote for $(x, s, t) \in \overline{E} \times [0,T] \times [0,T]$ the unique flow which solves the ordinary differential equation

\[
\frac{d}{dt} Z^0(x, s, t) = z^0(Z^0(x, s, t), t), \quad Z^0(x, s, s) = x.
\]  

(3.10)

**Remark 3.1.** As in [17], the ansatz (3.7)–(3.8) is based on the idea that the states $(z^{\pm, \epsilon}, p^{\pm, \epsilon}, \sigma^{\pm, \epsilon})$ should be near the return method profile $(z^0, p^0, \sigma^0)$, which starts from $(0, 0, 0)$ at $t = 0$ and returns back to $(0, 0, 0)$ at time $t = T$. 

\[22\]
Under the assumptions of Theorem 1.6, the profiles \( z^0, p^0, \xi^0, \) and \( \sigma^0 \) are chosen by means of Lemma 3.2 below, which is taken from [17, Lemma 2] and has been proved in [12,14,23,24].

**Lemma 3.2** ([17, Lemma 2]). There exists a solution \((z^0, p^0, \xi^0, \sigma^0) \in C^\infty(\overline{E} \times [0,T]; \mathbb{R}^{2N+2})\) to the controlled system (3.9) such that the flow \( Z^0 \) obtained via (3.10), satisfies

\[
\forall x \in E, \exists t_x \in (0,T): \ Z^0(x,0,t_x) \not\in \overline{\Omega}.
\] (3.11)

Moreover, all functions \((z^0, p^0, \xi^0, \sigma^0)\) are compactly supported in the time interval \((0,T), the control \( \xi^0 \) obeys

\[
\text{supp}(\xi^0) \subseteq (E \setminus \overline{\Omega}) \times (0,T),
\]

and \( z^0 \) can be chosen with \( \nabla \cdot z^0 = 0 \) in \( \overline{\Omega} \times [0,T] \) as well as \( \nabla \times z^0 = 0 \) in \( E \times [0,T] \).

**Remark 3.3.** As shown in [14], for two-dimensional simply-connected domains one can replace (3.11) by a uniform flushing property. That is to say, there exists a smoothly bounded open set \( \Omega_1 \subseteq E \) such that \((\overline{\Omega} \setminus \partial E) \subseteq \Omega_1\) and for all \( x \in \overline{\Omega}_1 \) one has \( Z^0(x,0,T) \not\in \overline{\Omega}_1 \).

**Example 3.4.** In order to illustrate Remark 3.3 by means of a very specific example, and to provide more details regarding Remark 1.7, let us consider as in Figure 4 for a smoothly bounded connected open set \( D \subseteq \mathbb{R}^2 \) the cylindrical setup

\[
\Omega := (0,1) \times D, \quad \Gamma_c := \{0,1\} \times D.
\]

Let \( \widetilde{D} \supseteq D \) be the planar simply-connected extension of \( D \) with \( \partial \widetilde{D} \subseteq \partial D \). One can extend \( \Omega \) through \( \Gamma_c \), in the sense of Section 2.1, to a smoothly bounded domain \( E \subseteq \mathbb{R}^3 \) with

\[
(-1,2) \times D \subseteq E \subseteq \mathbb{R} \times \widetilde{D}.
\]

Now, let \( \chi_1 \in C^\infty((-1,2); [0,1]) \) with \( \chi_1(s) = 1 \) for \( s \in [-1/2,3/2] \) and extend \( \chi_1 \) by zero to \( \mathbb{R} \). Moreover, for some large number \( M_1 > 0 \), take \( \gamma_1 \in C^\infty((0,T); [0,1]) \) such that \( \gamma_1(t) = M_1 \) when \( t \in [T/8,7T/8] \). Then, for \( x = [x_1,x_2,x_3]^T \in \overline{E} \) and \( t \in [0,T] \) choose

\[
\begin{align*}
Z^0(x,t) &:= \nabla \left( \gamma_1(t) \chi_1(x_1)x_1 \right), \\
p^0(x,t) &:= -\partial_t \gamma_1(t) \chi_1(x_1)x_1, \\
\xi^0(x,t) &:= (z^0(x,t) \cdot \nabla)z^0(x,t), \\
\sigma^0(x,t) &:= \nabla \cdot z^0(x,t).
\end{align*}
\]

Since \( z^0 \) does not depend on the spatial variables in \((-1/2,3/2) \times \overline{D} \), the above profiles solve the controllability problem (3.9) with the support of \( \sigma^0 \) and \( \xi^0 \) being located away from \( \overline{\Omega} \). Also, for \( M_1 > 0 \) large enough, one has a uniform flushing property as mentioned in Remark 3.3, see for instance the proof of [38, Lemma 3.1] which carries over to a three-dimensional channel.
In the context of Theorem 1.2, the zeroth order profiles \((z^0, p^0, \xi^0, \sigma^0 = 0)\) in (3.7)–(3.8) will be chosen by Lemma 3.5 below.

**Lemma 3.5.** When \(\Omega \subseteq \mathbb{R}^2\) is simply-connected and \(\Gamma_c\) is connected, there exists \(\eta > 0\) and profiles \((z^0, p^0, \xi^0, \sigma^0)\) ∈ \(C^\infty(\bar{E}; \mathbb{R}^6)\) which solve (3.9), are compactly supported in \((0, T)\), and satisfy

\[
\sigma^0 = 0 \text{ in } \bar{E} \times [0, T], \quad \text{dist}(\text{supp}(\nabla \wedge z^0), \bar{\Omega}) > \eta.
\]

Moreover, \(\xi^0(\cdot, t)\) is supported in \(\bar{E} \setminus \bar{\Omega}\) for all \(t \in [0, T]\) and the flow \(Z^0\) determined via (3.10) obeys the flushing property (3.11).

**Proof.** Let \((z^0, p^0, \xi^0, \sigma^0)\) be obtained from Lemma 3.2. Then, as sketched in Figure 5, there exist \(\eta > 0\) and a regular set \(B \subseteq \mathbb{R}^2\) satisfying

\[
B \cap \partial E \neq \emptyset, \quad \text{dist}(\overline{B}, \overline{\Omega}) > \eta
\]

such that

\[
\text{supp}(\sigma^0) \subseteq B \times (0, T).
\]

It is not restrictive to assume that the set \(B\) can be chosen such that \(V := E \setminus \overline{B}\) is simply-connected. Thus, there exists a scalar potential \(\phi \in C^\infty(\overline{\nabla \times [0, T]}; \mathbb{R})\) satisfying

\[
z^0(x, t) = \text{curl}(\phi)(x, t) := \nabla^\perp \phi(x, t), \quad (x, t) \in V \times [0, T].
\]

Indeed, if \(R : \mathbb{R}^2 \rightarrow \mathbb{R}^2\) denotes a rotation by \(\pi/2\), then one has the equivalences

\[
\nabla \cdot z^0 = 0 \iff \nabla \wedge (Rz^0) = 0,
\]

and

\[
Rz^0 = \nabla \phi \iff z^0 = \text{curl}(\phi).
\]

Let \(\tilde{\phi} : \mathbb{R}^2 \times [0, T] \rightarrow \mathbb{R}\) be an arbitrary extension of \(\phi\) to \(\mathbb{R}^2 \times [0, T]\) and take any smooth cutoff function \(\psi \in C^\infty(\mathbb{R}^2; [0, 1])\) with

\[
\psi(x) := \begin{cases} 
0 & \text{if } x \in \overline{B}, \\
1 & \text{if } \text{dist}(x, \Omega) < \eta/2.
\end{cases}
\]
Since $\tau := n^\perp := [n_2, -n_1]^T$ is a tangential field at $\partial E$, one can observe along $\Sigma_T \cap (\overline{V} \times [0, T])$ the vanishing tangential derivatives

$$\nabla \phi \cdot \tau = \text{curl}(\phi) \cdot n = z^0 \cdot n = 0.$$  

Therefore, there is a time-dependent constant $c(t)$ such that along the boundary section $\partial E \setminus \overline{B}$ one has $\tilde{\phi} \cdot c(t) = c(t)$ for each $t \in [0, T]$. Finally, in $E_T$ we define

$$\tilde{z}^0(x, t) := \text{curl} \left( \psi(x) \left( \phi(x, t) - c(t) \right) \right).$$

In this way, $\nabla \cdot \tilde{z}^0 = 0$ in $\overline{E} \times [0, T]$ and $\tilde{z}^0 \cdot n = 0$ along $\partial E \times [0, T]$. Since $\tilde{z}^0$ possibly differs from $z^0$ only if $\text{dist}(x, \Omega) \geq \eta/2$, one has $\text{supp}(\nabla \wedge \tilde{z}^0) \in \overline{E} \setminus \overline{\Omega}$ and the flushing property (3.11) remains valid. The proof is thus concluded by renaming $\tilde{z}^0$ as $z^0$, $\nabla \cdot z^0$ as $\sigma^0$, and adjusting $\tilde{z}^0$ inside $\overline{E} \setminus \overline{\Omega}$ such that (3.9) holds. \qed

**Remark 3.6.** In the proof of Lemma 3.5, assuming for simplicity that the domain extension is simply-connected, and by first introducing a suitable intermediate extension $\tilde{E} \subseteq E$ with $\Omega \subseteq \tilde{E}$, followed by applying Lemma 3.2 to this new situation, one can explicitly choose the size and location of $B$, as long as $B$ is open and $B \cap \partial E \neq \emptyset$.

**Remark 3.7.** When $\Gamma_c$ is not connected, the proof of Lemma 3.5 remains valid at least if the extension $E$ is simply-connected. Here, one selects $z^0$, hence the flow $Z^0$, such that $\overline{E}$ is flushed under $Z^0$ into one single connected component of $\overline{E} \setminus \Omega$. In order to avoid a gradient term $\nabla q$ during the regularization stage, the initial data should obey relations of the type (2.5).

---

**Figure 5:** A sketch of the situation considered in the proof of Lemma 3.5. The (red) circle indicates the set $B$. To the right of the (green) line, which crosses the whole domain, one has $\psi = 1$. The thin dashed line stands for the controlled boundary $\Gamma_c$. Along the boundary part of $\partial (\Omega^1 \cup \Omega)$ which is highlighted by thick dashes, it holds $\tilde{\phi}(\cdot, t) = \phi(\cdot, t) = c(t)$.  

---
The case of domains being a part of an annulus. Concerning Theorem 1.2 b), we introduce now an explicit return method trajectory. While Lemma 3.5 applies in view of Remark 3.7 to this situation as well, it would not be helpful for treating the boundary conditions allowed by Theorem 1.2 b), which are more general than those considered in Theorem 1.2 a). Indeed, now we have to ensure that the conditions for $z^0$ stated below in (3.52) are satisfied with the choice $M_1 = \rho I$ in (1.4) for all $\rho \in \mathbb{R}$. In particular, $z^0$ should be curl-free, divergence-free and tangential, requiring a multiply-connected extension $E$. By the assumptions of Theorem 1.2 b), there is a Lipschitz

\begin{align*}
\Omega = O_1, \quad \Gamma_c = \partial O_1 \setminus \partial A_{r_1}^2, \quad A_{r_1}^2 \setminus \Gamma_A = O_1 \cup O_2, \quad D_{r_1} \subseteq (D_{r_2} \setminus \overline{O_1}).
\end{align*}

As sketched in Figure 6a, a smooth extension for $\Omega$ in the sense of Section 2.1 is given by $E := A_{r_1}^2$. Notably, $E$ is multiply-connected while $\Omega$ is simply-connected. For the purpose of building a return method trajectory, we define

\begin{align*}
\tilde{\varphi}: E \to \mathbb{R}_+, \quad x \mapsto \tilde{\varphi}(x) := \ln |x|
\end{align*}

and choose for a constant $M_e > 0$ a smooth function $\gamma_{M_e} \in C_0^\infty((0, 1); \mathbb{R}_+)$ satisfying

\begin{align*}
\gamma_{M_e}(t) \geq M_e
\end{align*}

for all $t \in (T/8, 7T/8)$. Then, in $E \times [0, T]$ we denote the vector field

\begin{align*}
y^*(x, t) := [y_1^*, y_2^*]^\top(x, t) := \gamma_{M_e}(t) \begin{bmatrix} -\partial_2 \tilde{\varphi}(x) \\ \partial_1 \tilde{\varphi}(x) \end{bmatrix},
\end{align*}

Figure 6: Two examples for $\Omega$ and $\Gamma_c$ in the case b) of Theorem 1.2. The left picture also shows the domain extension $E$. The controlled boundaries are indicated by a line with additional dashes.
which possesses in all of \( \overline{E} \times [0, T] \) the properties
\[
\nabla \cdot y^* = 0, \quad \partial_t y_1^* - \partial_1 y_2^* = 0, \quad y^* \cdot n = 0.
\]
Due to the radial symmetry of \( E \), and because a neighborhood of the origin does not belong to \( E \), one can take the extended unit normal field \( n \) defined on \( \overline{E} \) everywhere orthogonal to \( y^* \). Now, by selecting \( M_E > 0 \) sufficiently large, one obtains profiles \( (z^0, p^0, \xi^0) \) in the sense of Lemma 3.5 by means of the definitions
\[
\begin{align*}
    z^0 &:= y^* & \text{in} & \overline{E} \times [0, T], \\
p^0 &:= -\partial_t y^* - \frac{1}{2} |y^*|^2 & \text{in} & \overline{E} \times [0, T], \\
\xi^0 &:= \partial_t y^* + (y^* \cdot \nabla) y^* + \nabla p^0 & \text{in} & \overline{E} \times [0, T],
\end{align*}
\]
(3.12)
where \( \psi^* \in C^\infty(\overline{E} \times [0, T]; \mathbb{R}) \) is a scalar potential in the sense that
\[
y^*(x, t) = \nabla \psi^*(x, t)
\]
for \( \text{dist}(x, \overline{\Omega}) < \eta \) with some fixed \( \eta > 0 \). The latter choice is possible, since, for \( \eta > 0 \) being sufficiently small, the set \( \{ x \in \overline{E} \mid \text{dist}(x, \overline{\Omega}) < \eta \} \) is simply-connected. This definition implies in particular that \( \xi^0(x, t) = 0 \) if \( x \in \overline{\Omega} \). For sufficiently large \( M_E > 0 \), a flushing property of the type (3.11) can be shown. Indeed, \( z^0 \) never vanishes in \( \overline{E} \times (T/8, 7T/8) \) and the associated flow \( Z^0 \) must move all particles along circular trajectories around the annulus. Finally, fix \( \rho \in \mathbb{R} \) and set \( M_2 = \rho I \in C^\infty(\overline{E}; \mathbb{R}^{2 \times 2}) \) together with general friction operators \( M_1, L_1, L_2 \in C^\infty(\overline{E}; \mathbb{R}^{2 \times 2}) \) in (1.4). Then, \( z^0 \) satisfies the relations
\[
\begin{align*}
    \nabla \cdot (N^+(z^0, z^0) - N^-(z^0, z^0)) &= 0 \quad \text{in} \overline{E}_T, \\
    z^0 \cdot n &= 0 \quad \text{in} \overline{E}_T, \\
    \nabla \cdot z^0 &= 0 \quad \text{in} \overline{E}_T, \\
    \nabla \wedge z^0 &= 0 \quad \text{in} \overline{E}_T,
\end{align*}
\]
(3.13)
because
\[
\nabla \cdot (N^+(z^0, z^0) - N^-(z^0, z^0)) = 2 \nabla \cdot [M_2 z^0]_{\text{tan}} = 2 \rho \nabla \cdot [z^0]_{\text{tan}} = 2 \rho \nabla \cdot z^0 = 0.
\]

### 3.3 Flushing the initial data

Due to the scaling introduced in (3.3) and (3.4), the contributions of \( z^0_{\pm} \) are at \( O(\epsilon) \). In order to avoid that \( z^0_{\pm} \) enter the remainder estimates in Section 3.5, the goal is to cancel the influence of this data until \( t = T \) by using the controls \( \xi^{\pm, 1} \) which are supported only in \( \overline{E} \setminus \overline{\Omega} \). After inserting the ansatz (3.7)–(3.8) into (3.5), and motivated by the approach from [17], one observes that a good strategy consists of defining \( z^{\pm, 1} \) as the
solutions to the linear problem

\[
\begin{aligned}
\partial_t z^{\pm,1} + (z^{\pm,1} \cdot \nabla) z^0 + (z^0 \cdot \nabla) z^{\pm,1} + \nabla p^{\pm,1} &= \xi^{\pm,1} + (\lambda^{\pm} + \lambda^\mp) \Delta z^0 & \text{in } \mathcal{E}_T, \\
\nabla \cdot z^{\pm,1} &= 0 & \text{in } \mathcal{E}_T, \\
z^{\pm,1} \cdot n &= 0 & \text{on } \Sigma_T, \\
z^{\pm,1}(\cdot, 0) &= z^0 & \text{in } \mathcal{E}.
\end{aligned}
\]

In the following, we determine the controls \(\xi^{\pm,1}\) such that the corresponding solution \((z^{+1}, z^{-1})\) to (3.14) satisfies \(z^{\pm,1}(\cdot, T) = 0\). This is achieved via [17, Lemma 3], combined with new ideas concerning the case of Theorem 1.2, where we have to maintain the properties

\[
\nabla \cdot (\xi^{+1} - \xi^{-1}) = 0 \quad \text{in } \mathcal{E}_T, \\
(\xi^{+1} - \xi^{-1}) \cdot n = 0 \quad \text{on } \Sigma_T.
\]

**Lemma 3.8.** There exist \(\xi^{\pm,1} \in \mathcal{U}_T\) such that the solution \((z^{+1}, z^{-1})\) to (3.14) obeys \(z^{\pm,1}(x, T) = 0\) for all \(x \in \mathcal{E}\) and is bounded in \(L^\infty((0, T); H^3(\mathcal{E})^2)\). Moreover, for all \(t \in (0, T)\) one has

\[
\text{supp}(\xi^{\pm,1}(\cdot, t)) \subseteq \overline{\mathcal{E} \setminus \Omega}.
\]

Given the assumptions of Theorem 1.2, one can choose the controls \(\xi^{\pm,1}\) satisfying (3.15).

**Proof.** In the general case where \(z^0\) is determined via Lemma 3.2, the proof consists of applying [17, Lemma 3] to the uncoupled systems solved individually by \(z^{+1} \pm z^{-1}\). Consequently, during the following steps, only the simply-connected case with \(N = 2\) and \(z^0\) from Lemma 3.5, or \(z^0\) from (3.12) are treated. Another difference to [17, Lemma 3] is the possibility that \(\nabla \wedge z^0 \neq 0\) is true in \(\overline{\mathcal{E} \setminus \Omega}\), which happens in the context of Theorem 1.2 a) and shall be addressed below.

**Step 1. Preliminaries.** By means of Lemma 3.5 or (3.12) one has \(\nabla \cdot z^0 = 0\) in \(\mathcal{E} \times [0, T]\). Thus, the smooth vector field

\[
(\lambda^+ + \lambda^-) \Delta z^0 = -(\lambda^+ + \lambda^-)(\nabla \perp (\nabla \wedge z^0))
\]

is supported in \(\overline{\mathcal{E} \setminus \Omega}\) and can be moved into the control terms

\[
f^\pm := \xi^{\pm,1} + (\lambda^\pm + \lambda^\mp) \Delta z^0.
\]

For showing the remaining statements of Lemma 3.8, it is of advantage to employ notations proportional to the original MHD unknowns, as for instance

\[
E^\pm := z^{+1} \pm z^{-1}, \quad F^\pm := f^+ \pm f^-.
\]

(3.16)
Step 2. A partition of unity. We employ several constructions from the proof of [17, Lemma 3]. Due to the regularity of $\mathcal{Z}^0$ and the flushing property (3.11), as provided by Lemma 3.2, Lemma 3.5, or the definition for $z^0$ in (3.12), there exists a small number $a > 0$ such that

$$\forall x \in \tilde{E}, \exists t_x \in (0, T): \text{dist}(\mathcal{Z}^0(x, 0, t_x), \tilde{\Omega}) \geq a.$$  

Hence, one can select a smoothly bounded closed set $S \subseteq \tilde{E}$ with $S \cap \tilde{\Omega} = \emptyset$ and

$$\forall x \in \tilde{E}, \exists t_x \in (0, T): \mathcal{Z}^0(x, 0, t_x) \in S.$$  

Moreover, for some $L \in \mathbb{N}$, we fix a finite covering $c_1, \ldots, c_L$ of $S$ which consists of interior and boundary squares in the following way. The boundary squares are centered in points of $\partial E \cap S$, fully included inside $\mathbb{R}^2 \setminus \tilde{\Omega}$, and one side lies in the interior of $E$. The interior squares are centered in points of $S \setminus \partial E$ and belong to $E \setminus \tilde{\Omega}$. Consequently, there exists $b > 0$ and a number $M \in \mathbb{N}$ of balls $B_1, \ldots, B_M \subseteq \mathbb{R}^2$ which cover $\tilde{E}$ such that for each $l \in \{1, \ldots, M\}$ one has

$$\exists t_l \in (b, T - b), \exists r_l \in \{1, \ldots, L\}, \forall t \in (t_l - b, t_l + b): \mathcal{Z}^0(B_l, 0, t) \in c_{r_l}. \quad (3.17)$$  

With respect to the balls $B_1, \ldots, B_M$, let $(\mu_l)_{l=1,\ldots,M} \subseteq C^\infty_0(\mathbb{R}^2; \mathbb{R}^2)$ be any fixed smooth partition of unity in the sense that

$$\forall l \in \{1, \ldots, M\}: \text{supp}(\mu_l) \subseteq B_l, \quad \forall x \in \tilde{E}: \sum_{l=1}^M \mu_l(x) = 1. \quad (3.18)$$  

Remark 3.9. Due to the special geometric setting considered at the moment, one could use a simplified partition of unity and only few squares, see also Remark 3.3. Here, a more general formulation is used in order to indicate how the arguments extend to other domains, see also Figure 7 for some examples.

Step 3. Flushing the initial magnetic field. The initial magnetic field can be flushed without pressure term by utilizing that the zero-divergence of the initial data is conserved. Essential at this point is that $\nabla \cdot z^0 = 0$ in $\tilde{E} \times [0, T]$. By means of (3.16), the vector field $E^-$ satisfies the following linear problem

$$\begin{cases}
\partial_t E^- + (z^0 \cdot \nabla) E^- - (E^- \cdot \nabla) z^0 + \nabla q^1 = F^- = \xi^+ - \xi^{-1} & \text{in } E_T, \\
\nabla \cdot E^- = 0 & \text{in } E_T, \\
E^- \cdot n = 0 & \text{on } \Sigma_T, \\
E^-(\cdot, 0) = z^+_0 - z^-_0 & \text{in } \tilde{E},
\end{cases} \quad (3.19)$$  

with $q^1 := p^+ - p^{-1}$. By employing the notations from the previous step, particularly the partition of unity $(\mu_l)_{l \in \{1, \ldots, M\}}$ given in (3.18), we first solve for $l \in \{1, \ldots, M\}$
(a) One boundary square is enough if each particle outside of the square is uniformly flushed by $Z^0$ into the square.

(b) In general multiply-connected domains, more than one boundary square is required in the proof of Lemma 3.8 based on [17, Lemma 3].

(c) Even a two-dimensional version of Example 3.4 might, due to the special choice of $z^0$, require more than one boundary square.

(d) When using $z^0$ from (3.12), two boundary squares are sufficient. Every particle from $E$ will move at least through one square.

Figure 7: A sketch of several boundary and interior squares for different situations. Some examples indicate situations covered by Theorem 1.6, for which here the proof for Lemma 3.8 refers to [17, Lemma 3].

the homogeneous problems

$$
\begin{aligned}
&\partial_t E^-_i + (z^0 \cdot \nabla) E^-_i - (E^-_i \cdot \nabla) z^0 + \nabla q_i^1 = 0 \quad \text{in } E_T, \\
&\nabla \cdot E^-_i = 0 \quad \text{in } E_T, \\
&E^-_i \cdot n = 0 \quad \text{on } \Sigma_T, \\
&E^-_i (\cdot, 0) = \mu_i (z^*_0 - z^-_0) \quad \text{in } E.
\end{aligned}
$$

(3.20)

Concerning the pressure gradient, after multiplying in (3.20) with $\nabla q_i^1$ and integrating by parts one finds

$$
\|\nabla q_i^1\|^2_{L^2(E)} = \int_{\partial E} (z^0 \wedge E^-_i) \nabla q_i^1 \wedge n \, dS = 0.
$$

Finally, we construct a force $F^- = \xi^{t,1} - \xi^{-,1} \in \mathcal{U}_T$ supported in $\overline{E} \setminus \overline{\Omega}$, which satisfies (3.15) and guarantees that the corresponding unique solution $E^-$ to (3.19) obeys $E^- (\cdot, T) = 0$. Indeed, such a couple $(E^-, F^-)$ may explicitly be defined via

$$
E^- (x, t) := \sum_{l=1}^M \beta (t - t_l) E^-_i (x, t), \quad F^- (x, t) := \sum_{l=1}^M \frac{d\beta}{dt} (t - t_l) E^-_i (x, t),
$$

(3.21)
with \( \beta : \mathbb{R} \rightarrow [0, 1] \) being a smooth cut-off function satisfying
\[
\beta(t) = \begin{cases} 
1 & \text{if } t \in (-\infty, -b), \\
0 & \text{if } t \in (b, +\infty) 
\end{cases}
\] (3.22)
for \( b > 0 \) from (3.17).

**Step 4. Flushing the initial velocity: the idea.** The vector field \( E^+ \) obeys together with \( p^1 := p^{+1} + p^{-1} \) the linear problem
\[
\begin{align*}
\partial_t E^+ + (z^0 \cdot \nabla) E^+ + (E^+ \cdot \nabla) z^0 + \nabla p^1 &= F^+ \quad \text{in } \mathcal{E}_T, \\
\nabla \cdot E^+ &= 0 \quad \text{in } \mathcal{E}_T, \\
E^+ \cdot n &= 0 \quad \text{on } \Sigma_T, \\
E^+(:, 0) &= z_0^+ + z_0^- \quad \text{in } \mathcal{E}.
\end{align*}
\] (3.23)

The pressure gradient is eliminated by taking the curl in (3.23), which leads to a transport equation with non-local terms for \( \nabla \wedge E^+ \). The goal is to determine \( F^+ \in \mathcal{U}_T \) supported in \( \overline{\mathcal{E}} \setminus \overline{\Omega} \), such that the corresponding solution \( E^+ \) to (3.23) satisfies
\[
\begin{align*}
\nabla \wedge E^+(:, T) &= 0 \quad \text{in } \mathcal{E}, \\
\nabla \cdot E^+(:, T) &= 0 \quad \text{in } \mathcal{E}, \\
E^+(:, T) \cdot n &= 0 \quad \text{on } \partial \mathcal{E}.
\end{align*}
\] (3.24)

For the case a) of Theorem 1.2, where \( \mathcal{E} \) can be assumed simply-connected, (3.24) implies by means of (2.1) that \( E^+(:, T) = 0 \) in \( \mathcal{E} \). For the case b) of Theorem 1.2, where \( \Omega \) is simply-connected, but the extension \( \mathcal{E} \) is multiply-connected, from (3.24) one can only conclude that
\[
E^+(:, T) = \lambda_1 Q,
\]
with \( \lambda_1 \in \mathbb{R} \) and \( Q \) spanning the one-dimensional space of divergence-free, curl-free, and tangential vector fields on the annulus \( A^0_\delta \). In fact, one can take \( Q = \nabla \wedge \ln |x| \).

Therefore, in this case we also need to show how one can steer any state of the form \( \lambda_1 Q \) to zero. To this end, note that \( z^0 \) chosen via (3.12) is divergence-free and curl-free in \( \mathcal{E} \), and of the form \( z^0 = \gamma M Q \). Moreover, since \( \Omega \) is simply-connected, for \( \tilde{\gamma} \in C^\infty([0, T]; [0, 1]) \) with \( \tilde{\gamma}(0) = 1 \) and \( \tilde{\gamma}(t) = 0 \) when \( t \geq T/4 \), there exists \( \tilde{\psi} \in C^\infty(\overline{\Omega} \times [0, T]; \mathbb{R}) \) such that
\[
\forall (x, t) \in \overline{\Omega} \times [0, T]: \lambda_1 \frac{d\tilde{\gamma}}{dt}(t)Q(x) = \nabla \tilde{\psi}(x, t).
\]

Thus, the function \( A := \tilde{\gamma}(t) \lambda_1 Q \) solves together with a smooth pressure \( \tilde{p} \) and a smooth control \( \tilde{F} \) supported in \( \overline{\mathcal{E}} \setminus \overline{\Omega} \) the controllability problem
\[
\begin{align*}
\partial_t A + (z^0 \cdot \nabla) A + (A \cdot \nabla) z^0 + \nabla \tilde{p} &= \tilde{F} \quad \text{in } \mathcal{E}_T, \\
\nabla \cdot A &= 0 \quad \text{in } \mathcal{E}_T, \\
A \cdot n &= 0 \quad \text{on } \Sigma_T, \\
A(:, 0) &= \lambda_1 Q \quad \text{in } \mathcal{E}, \\
A(:, T) &= 0 \quad \text{in } \mathcal{E}.
\end{align*}
\] (3.25)
Indeed, one can take \( \tilde{\rho} = -\tilde{\psi} - \gamma M_e \tilde{\gamma} \lambda |\mathbf{Q}|^2 \) and \( \mathbf{F} = 0 \) in \( \overline{\Omega} \). In \( \overline{E} \setminus \overline{\Omega} \), one may choose any smooth continuation for \( \tilde{\rho} \) and subsequently fix
\[
\tilde{\mathbf{F}} = \partial_t \mathbf{A} + (z^0 \cdot \nabla) \mathbf{A} + (\mathbf{A} \cdot \nabla) z^0 + \nabla \tilde{\rho}.
\]

Therefore, if (3.24) is shown, one can first use the controls \( (\mathbf{F}^+, \mathbf{F}^-) \) in order to steer \( (\mathbf{E}^+, \mathbf{E}^-) \) to \( (\lambda_1 \mathbf{Q}, 0) \) and then the controls \( (\tilde{\mathbf{F}}, 0) \) to steer \( (\lambda_1 \mathbf{Q}, 0) \) to \( (0, 0) \).

**Step 5. Flushing the initial velocity: showing (3.24).** It remains to find a force \( \mathbf{F}^+ \in \mathcal{U}_T \) such that (3.24) holds. Hereto, after writing (3.23) in vorticity form with the unknown \( \omega^+ := \nabla \wedge \mathbf{E}^+ \), and by employing the partition of unity \( (\mu_l)_{l \in \{1, \ldots, M\}} \) from (3.18), we make the ansatz
\[
\omega^+ = \sum_{l=1}^M \omega^+_l, \quad \mathbf{E}^+ = \sum_{l=1}^M \mathbf{E}^+_l, \quad \mathbf{F}^+ = \sum_{l=1}^M \mathbf{F}^+_l,
\]
wherein the \( \omega^+_l \) solves for each \( l \in \{1, \ldots, M\} \) the problem
\[
\begin{align*}
\partial_t \omega^+_l + (z^0 \cdot \nabla) \omega^+_l &= \nabla \wedge \mathbf{F}^+_l - (\tilde{\mathbf{E}}^+_l \cdot \nabla)(\nabla \wedge z^0) \quad \text{in} \ \mathcal{E}_T, \\

\nabla \wedge \tilde{\mathbf{E}}^+_l &= \omega^+_l \quad \text{in} \ \mathcal{E}_T, \\

\nabla \cdot \mathbf{E}^+_l &= 0 \quad \text{in} \ \mathcal{E}_T, \\

\tilde{\mathbf{E}}^+_l \cdot n &= 0 \quad \text{on} \ \Sigma_T, \\

\omega^+_l (\cdot, 0) &= \nabla \wedge \left( \mu_l (z_0^+ + z_0^-) \right) \quad \text{in} \ \mathcal{E},
\end{align*}
\]

Due to \( \nabla \wedge z^0 \) being supported inside \( \overline{E} \setminus \overline{\Omega} \), one can decouple the transport problem from the div-curl system without generating unwanted interior control terms in \( \overline{\Omega} \). Hereto, let \( \overline{\omega}^+_l \) solve (3.27) with zero right-hand side, that is
\[
\begin{align*}
\partial_t \overline{\omega}^+_l + (z^0 \cdot \nabla) \overline{\omega}^+_l &= 0 \quad \text{in} \ \mathcal{E}_T, \\

\overline{\omega}^+_l (\cdot, 0) &= \nabla \wedge \left( \mu_l (z_0^+ + z_0^-) \right) \quad \text{in} \ \mathcal{E}.
\end{align*}
\]

Subsequently, for each \( l \in \{1, \ldots, M\} \) the vector field \( \tilde{\mathbf{E}}^+_l \) is defined via
\[
\begin{align*}

\nabla \wedge \tilde{\mathbf{E}}^+_l &= \beta(t - t_l) \overline{\omega}^+_l \quad \text{in} \ \mathcal{E}_T, \\

\nabla \cdot \tilde{\mathbf{E}}^+_l &= 0 \quad \text{in} \ \mathcal{E}_T, \\

\tilde{\mathbf{E}}^+_l \cdot n &= 0 \quad \text{on} \ \partial \mathcal{E} \times (0, T),
\end{align*}
\]
where \( \beta \) is defined in (3.22). Now, take \( \omega^+_l \) as the function
\[
\omega^+_l := \beta(t - t_l) \overline{\omega}^+_l.
\]

As a consequence of this definition, the statement (3.17), and the properties of \( \beta \) from (3.22), for all \( l \in \{1, \ldots, M\} \) one has
\[
\forall \mathbf{x} \in \overline{E}: \omega^+_l (\mathbf{x}, T) = 0, \quad \forall t \in [0, T]: \sup \left( \frac{d}{dt} (t - t_l) \overline{\omega}^+_l (\cdot, t) \right) \subseteq c_{t_l}.
\]
Furthermore, \( \vec{E}_l^+ \) from (3.29) and \( \omega_l^+ \) solve (3.27) for each \( l \in \{1, \ldots, M\} \), provided that each \( F_l^+ \) satisfies
\[
\nabla \land F_l^+ = \frac{d\beta}{dt}(t - t_l)\omega_l^+ + (\vec{E}_l^+ \land \nabla)(\nabla \land z^0). \tag{3.30}
\]

The right-hand side of (3.30) is supported in \( \vec{E} \setminus \bar{\Omega} \). When \( z^0 \) is obtained from Lemma 3.5, in view of Remark 3.6, one can for simplicity assume that \( \text{supp}(\nabla \land z^0) \) is contained in a single boundary cube. In order to construct for each \( l \in \{1, \ldots, M\} \) a function \( F_l^+ \in \mathcal{U}_T \) supported in \( \vec{E} \setminus \bar{\Omega} \) and obeying (3.30), the right-hand side in (3.30) has to be of zero average on each interior cube. This is true since, for each \( l \in \{1, \ldots, L\} \) with \( B_l \subseteq E \), the average of \( \omega_l^+ \) on \( B_l \) vanishes at \( t = 0 \) and is transported along \( z^0 \), noting that \( \omega_l^+ \) satisfies the homogeneous transport equation (3.28).

Now, \( F_l^+ \in \mathcal{U}_T \) can be constructed explicitly by the formulas provided in [17, Section A.2] and the proof of Lemma 3.8 is concluded after returning via (3.26) to (3.23). Hereby, due to \( z_0^+ \in H^1(E) \cap H(E) \), the constructions imply that \( z^{\pm,1} \) are bounded in \( L^\infty((0, T); H^3(E)) \) and \( \xi^{\pm,1} \in \mathcal{U}_T \).

\[\square\]

### 3.4 Boundary layers and technical profiles

In this subsection, the boundary layers and related technical profiles given in the ansatz (3.7)–(3.8) will be described. In addition to \( \mathcal{V} \), as defined in Section 2.1, another tubular region is denoted by
\[
\mathcal{V}_{d^*} := \{ x \in \vec{E} | \text{dist}(x, \partial E) < d^* \} \subseteq \mathcal{V} \cap \vec{E},
\]
wherein \( d^* \in (0, d) \) is a small number to be fixed in Lemma 3.19 below. Moreover, given a function \( \psi_{d^*} \in C^\infty(\overline{\mathbb{R}}; [0, 1]) \) with
\[
\psi_{d^*}(s) = \begin{cases} 1 & \text{for } s \leq d^*/2, \\ 0 & \text{for } s \geq 2d^*/3, \end{cases}
\]
a smooth cutoff \( \chi_{\partial E} \in C^\infty(\overline{\mathbb{R}}; [0, 1]) \) is defined by
\[
\chi_{\partial E}(x) := \psi_{d^*}(\varphi_E(x)), \tag{3.31}
\]
where \( \varphi_E(x) \) is for \( x \in \mathcal{V} \) the distance \( \varphi_E(x) = \text{dist}(x, \partial E) \) introduced in Section 2.1. By construction, \( \chi_{\partial E} = 1 \) holds in the vicinity of \( \partial E \). Furthermore, one has \( \text{supp}(\chi_{\partial E}) \subseteq \mathcal{V}_{d^*} \) and the gradient of \( \chi_{\partial E} \) can be calculated as
\[
(\nabla \chi_{\partial E})(x) = \left( \frac{d}{ds} \psi_{d^*} \right) (\varphi_E(x)) \nabla \varphi_E(x) = -\left( \frac{d}{ds} \psi_{d^*} \right) (\varphi_E(x)) n(x). \tag{3.32}
\]

**Remark 3.10.** The property (3.32) of \( \chi_{\partial E} \) is employed later on for stating a condition under which the profiles \( \mu^\pm \) in (3.8) can be chosen with \( \nabla \cdot \mu^+ = \nabla \cdot \mu^- \).
3.4.1 Boundary layer equations

Our choice for the boundary layer correctors is motivated by [17, 29]. After plugging (3.7) and (3.8) into (3.5), one may observe a term at order $O(1)\epsilon$ which is not absorbed by (3.9). However, after using the idea from [29] to write

$$\left\|(z^0 \cdot n)\partial_\tau v^\pm\right\|_\epsilon = \sqrt{\epsilon}\left\|\frac{z^0 \cdot n}{\varphi_\mathcal{E}} z \partial_\tau v^\pm\right\|_\epsilon,$$

this contribution is seen to behave as $O(\sqrt{\epsilon})$. In order to also offset the mismatching boundary values $\mathcal{N}^\pm(z^0, z^0) \neq 0$, the boundary layer profiles $(v^+, v^-)$ in (3.7) are introduced in $\overline{\mathcal{E}} \times \mathbb{R}_+ \times \mathbb{R}_+$ as the solution to the coupled linear problem

$$\partial_t v^\pm - \partial_z(z^\pm v^+ + \lambda^\pm v^-) + \left[ (z^0 \cdot \nabla) v^\pm + (v^\mp \cdot \nabla) z^0 \right]_{\text{tang}} + \mathcal{F} \partial_\tau v^\pm = \mu^\pm \quad (3.33)$$

with boundary and initial conditions

$$\begin{align*}
\begin{cases}
\partial_z v^\pm(x, t, 0) &= g^\pm(x, t), \quad x \in \overline{\mathcal{E}}, t \in \mathbb{R}_+, \\
v^\pm(x, t, z) &\to 0, \text{ as } z \to +\infty, \quad x \in \overline{\mathcal{E}}, t \in \mathbb{R}_+, \\
v^\pm(x, 0, z) &= 0, \quad x \in \overline{\mathcal{E}}, z \in \mathbb{R}_+.
\end{cases}
\end{align*} \quad (3.34)$$

Above, the functions $\mathcal{F}$ and $g^\pm$ are for all $(x, t) \in \overline{\mathcal{E}} \times \mathbb{R}_+$ given by

$$\mathcal{F}(x, t) := -\frac{z^0(x, t) \cdot n(x)}{\varphi_\mathcal{E}(x)}, \quad g^\pm(x, t) := 2\chi_\partial \mathcal{E}(x) \mathcal{N}^\pm(z^0, z^0)(x, t). \quad (3.35)$$

Since $z^0 \cdot n = 0$ on $\partial \mathcal{E}$ and $\nabla \varphi_\mathcal{E} = -n$ in $\mathcal{E}$, a Taylor expansion shows that $\mathcal{F}$ is smooth, see also the arguments from [29, Lemma 4]. Due to the regularity of $z^0$ and $\varphi_\mathcal{E}$, the functions $g^\pm$ are smooth as well.

**Remark 3.11.** It would be sufficient for our purpose to define the boundary layer profiles $v^\pm$ only for $t \in [0, T/\epsilon]$. By stating (3.33) and (3.34) for all $t \in \mathbb{R}_+$, it is emphasized that $v^\pm$ are independent of $\epsilon$.

Now, several properties of the solutions to (3.33) and (3.34) are summarized.

**Lemma 3.12.** Assume that $\mu^\pm : \mathcal{E} \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}^N$ are smooth and satisfy $\mu^\pm \cdot n = 0$. Then, there exists a unique solution $(v^+, v^-)$ to (3.33) and (3.34) which possesses for all $k, l, r, s \in \mathbb{N}_0$, and any $T^* > 0$, the regularity

$$\partial_t^s v^\pm \in L^\infty((0, T^*); H^{k+l+r}_{\mathcal{E}}) \cap L^2((0, T^*); H^{k+l+r+1}_{\mathcal{E}}). \quad (3.36)$$

In addition, for each $(x, t, z) \in \overline{\mathcal{E}} \times \mathbb{R}_+ \times \mathbb{R}_+$ the profiles $v^\pm$ verify the orthogonality relation

$$v^\pm(x, t, z) \cdot n(x) = 0. \quad (3.37)$$
Proof. The well-posedness of the linear problem (3.33) and (3.34) is inferred from a priori estimates for the solutions to (3.33)–(3.34), and for a corresponding adjoint problem. Since \( \mathbf{\mu}^\pm \) are assumed smooth, the regularity stated in (3.36) is obtained from Lemma A.1. The relation (3.37) follows after multiplying the equations satisfied by \( \mathbf{v}^\pm \cdot \mathbf{n} \) with \( \mathbf{n} \), which leads to a priori estimates for \( (\mathbf{v}^\pm + \mathbf{v}^-) \cdot \mathbf{n} \) similar to that given in [29, Section 5].

For the sake of having \( \mathbf{v}^\pm \cdot \mathbf{n} = 0 \) in \( \partial \mathcal{E} \times \mathbb{R}_+ \times \mathbb{R}_+ \), the normal contributions of \( (\mathbf{z}^0 \cdot \nabla) \mathbf{v}^\pm + (\mathbf{v}^\mp \cdot \nabla) \mathbf{z}^0 \), which appear at \( \mathcal{O}(\sqrt{\varepsilon}) \) when inserting (3.7) into (3.5), have been omitted in (3.33). This and the commutation formula

\[
\nabla\left[ \langle q^\pm \rangle \right]_e = \left[ \nabla q^\pm \right]_e - \frac{1}{\varepsilon} \left[ \partial_\mathbf{\nu} q^\pm \right]_e \mathbf{n}
\]

motivate introducing the profiles \( q^\pm \) in (3.7) as the solutions to

\[
\begin{cases}
\partial_\mathbf{\nu} q^\pm = \left[ (\mathbf{z}^0 \cdot \nabla) \mathbf{v}^\pm + (\mathbf{v}^\mp \cdot \nabla) \mathbf{z}^0 \right] \cdot \mathbf{n} & \text{in } \partial \mathcal{E} \times \mathbb{R}_+ \times \mathbb{R}_+,
\lim_{z \to \pm \infty} q^\pm(x, t, z) = 0, & x \in \mathcal{E}, t \in (0, T).
\end{cases}
\]

Next, let us define the second boundary layer correctors \( \mathbf{w}^\pm \). The normal parts of \( \mathbf{w}^\pm \) will compensate for the non-vanishing divergence of \( \mathbf{v}^\pm \), while their tangential parts constitute a lifting for \( \mathcal{N}^\pm(\mathbf{v}^+, \mathbf{v}^-)(x, t, 0) \) and later on enable sufficient remainder estimates. Namely,

\[
\begin{align*}
\mathbf{w}^\pm(x, t, z) := \mathbf{\bar{w}}^\pm(x, t, z) &\mathbf{n}(x) - 2 e^{-z} \mathcal{N}^\pm(\mathbf{v}^+, \mathbf{v}^-)(x, t, 0), & x \in \mathcal{E}, t \in \mathbb{R}_+, z \in \mathbb{R}_+, \\
\mathbf{\bar{w}}^\pm(x, t, z) &:= - \int_z^{+\infty} \nabla \cdot \mathbf{v}^\pm(x, t, s) \, ds, & x \in \mathcal{E}, t \in \mathbb{R}_+, z \in \mathbb{R}_+,
\end{align*}
\]

noting that \( \mathbf{w}^\pm \) satisfy under the assumption

\[
\text{supp}(\mathbf{v}^\pm(\cdot, t, z)) \subseteq \mathcal{V}
\]

the relations

\[
\nabla \cdot \mathbf{v}^\pm = \mathbf{n} \cdot \partial_\mathbf{\nu} \mathbf{w}^\pm \quad \text{in } \partial \mathcal{E} \times \mathbb{R}_+ \times \mathbb{R}_+,
\]

\[
\mathcal{N}^\pm(\mathbf{v}^+, \mathbf{v}^-)(x, t, 0) = \frac{1}{2} \left[ \partial_\mathbf{\nu} \mathbf{w}^\pm \right]_{\| \nu \|} (x, t, 0), & x \in \mathcal{E}, t \in (0, T).
\]

Remark 3.13. The constructions in (3.38) and (3.39) only serve their purpose if \( \mathbf{n} \) is nonzero in the \( x \)-support of \( \mathbf{v}^\pm \). Therefore, by means of a sufficiently small choice for \( d^* > 0 \) in the definition of \( \chi_{\partial \mathcal{E}} \), it will be ensured by Lemma 3.19 that \( \text{supp}(\mathbf{v}^\pm(\cdot, t, z)) \subseteq \mathcal{V} \).

For balancing nonzero divergence contributions and normal fluxes generated by \( \mathbf{w}^\pm \), the correctors \( \theta^\pm, e \) are introduced as solutions to

\[
\begin{cases}
\Delta \theta^\pm, e = -\left[ \nabla \cdot \mathbf{w}^\pm \right]_e & \text{in } \mathcal{E} \times \mathbb{R}_+,
\partial_n \theta^\pm, e = -\mathbf{w}^\pm(x, t, 0) \cdot \mathbf{n}(x), & x \in \partial \mathcal{E} \times \mathbb{R}_+.
\end{cases}
\]
For each $t \in \mathbb{R}_+$, the equations in (3.41) are well-posed Neumann problems, as stated in Lemma 3.24 below.

It remains to introduce the profiles $\tilde{\theta}^{\pm,\varepsilon}$ and the forces $\tilde{\zeta}^{\pm,\varepsilon}$. Inserting the ansatz (3.7) into (3.5) gives rise to the terms

$$
\partial_t \nabla \theta^{\pm,\varepsilon} + (\zeta^0 \cdot \nabla) \nabla \theta^{\pm,\varepsilon} + (\nabla \theta^{\mp,\varepsilon} \cdot \nabla) \zeta^0,
$$

(3.42)

which are not behaving well regarding the remainder estimates that we seek later. In particular, the $L^2(\partial E)$ norms of the boundary data in (3.41) are $O(1)$. For this reason, the pressure correctors $\theta^{\pm,\varepsilon}$ are chosen as

$$
\theta^{\pm,\varepsilon} := -\partial_t \theta^{\pm,\varepsilon} - \zeta^0 \cdot \nabla \theta^{\pm,\varepsilon}.
$$

While in this way one cannot cancel (3.42) in general, the time derivatives $\partial_t \nabla \theta^{\pm,\varepsilon}$ are absorbed by this pressure corrector and, at least in the regions of $E$ where $\nabla \times z^0 = 0$ is valid, one has the representation

$$
\partial_t \nabla \theta^{\pm,\varepsilon} + (\zeta^0 \cdot \nabla) \nabla \theta^{\pm,\varepsilon} + (\nabla \theta^{\mp,\varepsilon} \cdot \nabla) \zeta^0 + \nabla \theta^{\pm,\varepsilon} = (\nabla (\theta^{\mp,\varepsilon} - \theta^{\pm,\varepsilon}) \cdot \nabla) \zeta^0.
$$

In view of Lemmas 3.2 or 3.5, or due to (3.12), or by the constructions in Example 3.4, the property $\nabla \times z^0 = 0$ always holds in the physical domain $\overline{\Omega}$. Therefore, we will split the unwanted terms into a good part that can be estimated, a part that can be absorbed by smooth controls acting in $E \setminus \overline{\Omega}$, and (if necessary) a bad part that is canceled by a magnetic field control acting in $\overline{\Omega}$. That is, depending on the chosen domain and boundary conditions, we shall now define the profiles $\tilde{\zeta}^{\pm,\varepsilon}$ as follows.\(^3\)

**The first case of the definition for $\tilde{\zeta}^{\pm,\varepsilon}$.** When $(w^+ - w^-)_{|z=0} \cdot n = 0$ is satisfied on $\partial E \times (0, T)$, then we choose $\tilde{\zeta}^{\pm,\varepsilon}$ as the smooth functions

$$
\tilde{\zeta}^{\pm,\varepsilon} := (\zeta^0 \cdot \nabla) \nabla \theta^{\pm,\varepsilon} + (\nabla \theta^{\mp,\varepsilon} \cdot \nabla) \zeta^0 - \nabla (\zeta^0 \cdot \nabla \theta^{\pm,\varepsilon}) - (\nabla (\theta^{\mp,\varepsilon} - \theta^{\pm,\varepsilon}) \cdot \nabla) \zeta^0.
$$

There are at least the following two situations with $(w^+ - w^-)_{|z=0} \cdot n = 0$.

- If $u^+ - u^- = 0$ in $E \times \mathbb{R}_+ \times \mathbb{R}_+$, then also $(w^+ - w^-) \cdot n = 0$. This happens for instance when $M_2 = 0$ is chosen in (1.4) and $\mu^\pm$ are determined such that $\mu^+ - \mu^- = 0$, cf. Lemma 3.15 and the proof of Lemma 3.18 below.

- In the case of Theorem 1.2 b), the definition of $z^0$ in (3.12) likewise guarantees that $(w^+ - w^-) \cdot n = 0$. Hereto, one employs $\nabla \cdot (v^+ - v^-) = 0$, which is established below in the proof of Lemma 3.18. Indeed, (3.57) conserves the divergence-free condition satisfied at the initial time and the controls $\mu^\pm$ are constructed such that $\nabla \cdot (\mu^+ - \mu^-) = 0$.

By the definition for $\tilde{\zeta}^{\pm,\varepsilon}$ above, one always has $\tilde{\zeta}^{\pm,\varepsilon}(x, t) = 0$ for $(x, t) \in \overline{\Omega} \times [0, T]$. Therefore, in this case, no external control forces will enter (1.1). However, for concluding Theorem 1.2 later on, we also need in the extended domain that

$$
\nabla \cdot (\tilde{\zeta}^{+,\varepsilon} - \tilde{\zeta}^{-,\varepsilon}) = 0, \quad (\tilde{\zeta}^{+,\varepsilon} - \tilde{\zeta}^{-,\varepsilon}) \cdot n = 0.
$$

\(^3\)As a motivation for the definitions, one may compare with (3.77) and the proof of (3.26) below.
This is true under the assumptions of Theorem 1.2. Either it follows from $M_2 = 0$ that $\nabla (\theta^+ - \theta^-) = 0$, hence $\tilde{\zeta}^+ - \tilde{\zeta}^- = 0$ in $\overline{E} \times [0, T]$. Or one has $\nabla \cdot z^0 = 0$ by means of (3.12), which even provides $\tilde{\zeta}^\pm = 0$.

**The second case of the definition for $\tilde{\zeta}^\pm$.** When $\nabla \times z^0 = 0$ in $\overline{E} \times (0, T)$ and $(w^+ - w^-)|_{z=0} \cdot n \neq 0$ at some points, then we define

$$
\tilde{\zeta}^\pm := (\nabla (\theta^+ - \theta^-) \cdot \nabla) z^0.
$$

This applies to the situation of Theorem 1.6, where $z^0$ is obtained from Lemma 3.2 in general. However, by this definition, $\tilde{\zeta}^+ - \tilde{\zeta}^- \neq 0$ is generally possible in the physical region $\Omega \times (0, T)$. In the special case of Remark 1.7, see also Example 3.4, one has $\tilde{\zeta}^\pm = 0$ in $\overline{\Omega} \times [0, T]$, as $z^0$ only depends on time in $\overline{\Omega} \times [0, T]$.

**Remark 3.14.** Under the assumptions made in Theorems 1.2 and 1.6, one can always employ one of the above definitions for $\tilde{\zeta}^\pm$.

The fact that $M_2 = 0$ rules out magnetic field boundary layers, given the universal definition of $v^\pm$ via (3.33) and (3.34), is now emphasized by the next lemma.

**Lemma 3.15.** $M_2 = 0$ and $\mu^+ - \mu^- = 0$ imply $v^+ - v^- = 0$.

**Proof.** The condition $M_2 = 0$ implies $\mathcal{N}^+(z^0, z^0) - \mathcal{N}^-(z^0, z^0) = [2M_2 z^0]_\text{tan} = 0$. Therefore, if $v^\pm$ are obtained from (3.33) and (3.34), then their difference $W$ obeys

$$
\partial_t W - (\lambda^+ - \lambda^-) \partial_z W + [(z^0 \cdot \nabla) W - (W \cdot \nabla) z^0]_\text{tan} + \int z \partial_z W = 0,
$$

with vanishing boundary and initial conditions

$$
\begin{align*}
\partial_z W(x, t, 0) &= 0, & x \in \overline{E}, t \in \mathbb{R}_+, \\
W(x, t, z) &\to 0, \text{ as } z \to +\infty, & x \in \overline{E}, t \in \mathbb{R}_+, \\
W(x, 0, z) &= 0, & x \in \overline{E}, z \in \mathbb{R}_+.
\end{align*}
$$

Due to $W \cdot n = 0$, one can show by means of direct energy estimates that $W$ must be the unique solution to (3.43) and (3.44), thus $W = 0$. □

### 3.4.2 Boundary layer dissipation via vanishing moment conditions

The boundary layer controls $\mu^\pm$ appearing in (3.33) are now determined. For large times $t \geq T$, in Section 3.1 we already fixed

$$
\mu^\pm(x, t, z) = 0, \quad (x, t, z) \in \overline{E} \times [T, +\infty) \times \mathbb{R}_+.
$$

For $t \in [0, T)$, the controls $\mu^\pm$ will be chosen such that $v^\pm$ obey improved decay rates as $t \to +\infty$. This is part of the well-prepared dissipation method, described in [17] for the Navier-Stokes equations and previously in [34] for a viscous Burgers’ equation. We will employ two different constructions for $\mu^\pm$. The first one, namely Lemma 3.17, shall be reduced directly to known results and is suitable for showing Theorem 1.6 with nonzero $\zeta$. The second one, namely Lemma 3.18, will allow concluding Theorem 1.2 and the case $M_2 = 0$ of Theorem 1.6. To begin with, the following modification of [17, Lemma 6] is stated.

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Lemma 3.16. Let \( s, r \in \mathbb{N} \) and suppose that \( f_0^\pm \in H^{s+1}(\mathbb{R}) \) satisfy for all integers \( 0 \leq k < r \) the vanishing moment conditions

\[
\int_{\mathbb{R}} z^k f_0^\pm \, dz = 0.
\]  

(3.45)

Furthermore, assume that \((z, t) \mapsto f^\pm(z, t)\) solve the coupled parabolic system

\[
\begin{aligned}
\partial_t f^\pm - \partial_{zz} (\lambda^\pm f^\mp + \lambda^\mp f^-) &= 0 & \text{in } \mathbb{R} \times \mathbb{R}_+,
\ f^\pm(\cdot, 0) &= f_0^\pm & \text{in } \mathbb{R}.
\end{aligned}
\]

Then, for all \( k \in \{0, \ldots, r\} \) one has the decay estimate

\[
\|f^\pm(\cdot, t)\|_{H^{s+1}(\mathbb{R})} \leq C \max \limits_{\Delta, \Omega \in \{+,-\}} \|f_0^\pm\|_{H^{s+1}(\mathbb{R})} \left[ \frac{\ln(2 + (\lambda^\pm \square) t)}{2 + (\lambda^\pm \square) t} \right]^{\frac{s+1}{2}},
\]

(3.46)

where \( C = C(s, r) \) is a constant independent of \( t \geq 0 \) and the initial data \( f_0^\pm \).

Proof. We define the functions \( F^\pm := f^+ \pm f^- \) and introduce the scaled versions

\[
F_0^\pm(z, t) := F^\pm(\lambda^\pm \square)^{-1} t),
\]

which obey for the initial data \( F_0^\pm := f_0^+ \pm f_0^- \) the uncoupled heat equations

\[
\begin{aligned}
\partial_t F^\pm - \partial_{zz} F^\pm &= 0 & \text{in } \mathbb{R} \times \mathbb{R}_+,
\ F^\pm(\cdot, 0) &= F_0^\pm & \text{in } \mathbb{R}.
\end{aligned}
\]

(3.47)

By applying [17, Lemma 6] to (3.47), the bound (3.46) follows first for \( F^\pm \) and then by the triangle inequality also for \( f^\pm \). \( \square \)

The following result, which is a consequence of Lemma 3.16 and [17, Lemma 7], provides the controls \( \mu^\pm \) on the time interval \([0, T]\). It will be applied in the general situation of Theorem 1.6.

Lemma 3.17. For any \( r \in \mathbb{N} \), there exist \( \mu^\pm \in C^\infty(\overline{E} \times [0, T] \times \mathbb{R}_+; \mathbb{R}^N) \) satisfying

\[
\forall (x, t, z) \in E \times \mathbb{R}_+ \times \mathbb{R}_+: \mu^\pm(x, t, z) \cdot n(x) = 0,
\]

\[
\forall z \in \mathbb{R}_+: \text{supp} \mu^\pm(\cdot, \cdot, z) \subseteq \left( \overline{E \setminus \overline{E}} \right) \times (0, T)
\]

such that \( v^\pm \) obey the decay rate

\[
\|v^\pm(\cdot, t, \cdot)\|_{H^{s+1}(\mathbb{R})} \leq C \max \limits_{\Delta, \Omega \in \{+,-\}} \left[ \frac{\ln(2 + (\lambda^\pm \square) t)}{2 + (\lambda^\pm \square) t} \right]^{\frac{s+1}{2}},
\]

(3.48)

for all \( 0 \leq k \leq r \) and \( s, p, k \in \mathbb{N}_0 \), with a constant \( C = C(s, r, p, k) > 0 \) not depending on the time \( t \).
Proof. Consider the even extensions of \( v^\pm \) to \( \overline{E} \times \mathbb{R}_+ \times \mathbb{R} \) plus lifted boundary data, defined via

\[ V^\pm(x, t, z) := v^\pm(x, t, |z|) + g^\pm(x, t) e^{-|z|}, \quad (x, t, z) \in \overline{E} \times \mathbb{R}_+ \times \mathbb{R}. \tag{3.49} \]

For \( t \geq T \), one has \( z^0(\cdot, t) = g^\pm(\cdot, t) = \mu^\pm(\cdot, t, \cdot) = 0 \) by construction. Thus, \( V^\pm \) are governed by the parabolic system

\[ \partial_t V^\pm - \partial_{zz}(\lambda^+ V^+ + \lambda^- V^-) = 0 \quad \text{in} \quad \overline{E} \times [T, +\infty) \times \mathbb{R}, \tag{3.50} \]

wherein \( x \in \overline{E} \) is a parameter. Therefore, in view of Lemma 3.16 and (3.49), the decay estimate (3.48) follows if enough vanishing moment conditions of the type (3.45) are satisfied. Hereto, the proof of [17, Lemma 7]\(^4\) can be applied individually to the equations satisfied by \( V^+ + V^- \) and \( V^+ - V^- \). This provides controls \( \mu^\pm \) such that for each \( k \in \{1, \ldots, r-1\} \) the Fourier transformed functions

\[ \hat{V}^\pm(\cdot, \cdot, \xi) := \int_{\mathbb{R}} e^{-i\xi z} V^\pm(\cdot, \cdot, z) \, dz \]

obey the relations

\[ \partial_\xi^k \hat{V}^+(\cdot, T, \xi)_{|\xi = 0} \pm \partial_\xi^k \hat{V}^-(\cdot, T, \xi)_{|\xi = 0} = 0. \tag{3.51} \]

Since (3.51) implies sufficient vanishing moment conditions and Lemma 3.12 provides uniform bounds for \( v^\pm \) on the time interval \([0, T]\), the proof is complete. \( \square \)

In view of (2.24), we are now concerned with the possibility of obtaining magnetic field boundary layer controls that are not only tangential but also divergence-free in the extended domain \( \mathcal{E} \times (0, T) \times \mathbb{R}_+ \). The following lemma will be employed in the case of Theorem 1.2 for obtaining \( \mu^\pm \). But also in the more general situation of Theorem 1.6 with \( M_2 = 0 \), the for Lemma 3.18 below will explain how the controls \( \mu^\pm \) are selected with \( \mu^+ - \mu^- = 0 \).

Lemma 3.18. Given any \( r \in \mathbb{N} \), the controls \( \mu^\pm \) with the properties from Lemma 3.17 can under additional assumptions be chosen as follows:

1) When \( \mathcal{N}^+(z^0, z^0)(x, t) = \mathcal{N}^-(z^0, z^0)(x, t) \) is valid for all \( (x, t) \in \text{supp}(\chi_{\partial \mathcal{E}}) \times (0, T) \), then \( \mu^\pm \) can be fixed with

\[ \mu^+ - \mu^- = 0. \]

2) When the profile \( z^0 \) selected in Section 3.1 satisfies

\[ \nabla \cdot (\mathcal{N}^+(z^0, z^0) - \mathcal{N}^-(z^0, z^0)) = 0 \quad \text{in} \quad (\mathcal{V} \cap \overline{\mathcal{E}}) \times (0, T), \]

\[ z^0 \cdot n = 0 \quad \text{in} \quad (\mathcal{V} \cap \overline{\mathcal{E}}) \times (0, T), \tag{3.52} \]

\[ \nabla \cdot z^0 = 0 \quad \text{in} \quad \mathcal{E}_T, \]

\(^4\)The arguments presented below for showing Lemma 3.18 also contain a version of the ideas from [17, Lemma 7], applied to a different equation.
then after fixing the number \(d^* \in (0, d)\) from the definition of \(\chi_{\partial E}\) sufficiently small, one can construct \(\mu^\pm\) with

\[
\nabla \cdot (\mu^+ - \mu^-) = 0, \quad (x, t, z) \in \mathcal{E} \times \mathbb{R}_+ \times \mathbb{R}_+.
\]

(3.53)

3) If in addition to the conditions (3.52) one has \(z^0 \cdot n = 0\) in \(\mathcal{E}_T\), then \(\mu^\pm\) can be chosen with (3.53) for any choice \(d^* \in (0, d)\).

Proof. Regarding the first statement, one may take \(\mu^+ - \mu^- = 0\) because the magnetic field boundary layer \(v^+ - v^-\) solves in this case a well-posed linear problem with zero data, which yields \(v^+ - v^- = 0\) in \(\mathcal{E} \times \mathbb{R}_+ \times \mathbb{R}_+\) as shown by Lemma 3.15. Concerning the second and third statement, the control \(\mu^+ - \mu^-\) shall be determined through the steps presented below. In any case, we additionally apply the arguments from Lemma 3.17 for determining of which form \(\mu^+ + \mu^-\) must be such that

\[
\partial_\zeta^k \tilde{V}^+(\cdot, T, \zeta)|_{\zeta=0} + \partial_\zeta^k \tilde{V}^-(\cdot, T, \zeta)|_{\zeta=0} = 0, \quad k \in \{1, \ldots, r - 1\}.
\]

Thus, given the assumptions (3.52), it remains to identify a suitable choice for the control \(\mu^+ - \mu^-\) acting in the equation satisfied by \(v^+ - v^-\) in a way that

\[
\partial_\zeta^k \tilde{V}^+(\cdot, T, \zeta)|_{\zeta=0} - \partial_\zeta^k \tilde{V}^-(\cdot, T, \zeta)|_{\zeta=0} = 0, \quad k \in \{1, \ldots, r - 1\}.
\]

Step 1. Preliminaries. In view of (3.49), the vector field \(W := V^+ - V^-\) can with \(g := g^+ - g^-\) be written as

\[
W(x, t, z) = v^+(x, t, |z|) - v^-(x, t, |z|) + g(x, t) e^{-|z|}.
\]

(3.54)

Under the \textit{a priori} assumption \(\mu^\pm \cdot n = 0\), noting that \(g^\pm \cdot n = 0\) is true by construction, it follows from (3.37) that \(W \cdot n = 0\) holds as well. Also, (3.32) and (3.52) imply \(\nabla \cdot g = 0\) by means of

\[
\nabla \cdot (g^+ - g^-) = 2\chi_{\partial E} \nabla \cdot (\mathcal{N}^+(z^0, z^0) - \mathcal{N}^-(z^0, z^0)) \\
- 2 \left[ \left( \frac{d}{ds} \psi_{\mathcal{E}} \right) \circ \varphi_{\mathcal{E}} \right] n \cdot (\mathcal{N}^+(z^0, z^0) - \mathcal{N}^-(z^0, z^0)) \\
= 2\chi_{\partial E} \nabla \cdot (\mathcal{N}^+(z^0, z^0) - \mathcal{N}^-(z^0, z^0)) \\
= 0.
\]

(3.55)

Even more, \(n = -\nabla \varphi_{\mathcal{E}}\) being a gradient, hence \(\nabla \times n = 0\), ensures for two vector fields \(h_1\) and \(h_2\) on \(\mathcal{E}\) that

\[
(h_1 \cdot n = 0 \text{ and } h_2 \cdot n = 0) \implies ((h_1 \cdot \nabla) h_2 - (h_2 \cdot \nabla) h_1) \cdot n = 0.
\]

Therefore, one can infer at all points where \(z^0 \cdot n = 0\) holds the relation

\[
[z^0 \cdot \nabla] (v^+ - v^-) - ([v^+ - v^-] \cdot \nabla) z^0 \big|_{\text{tan}} = (z^0 \cdot \nabla) (v^+ - v^-) - ([v^+ - v^-] \cdot \nabla) z^0.
\]
Next, while from (3.52) it only follows that \( \mathfrak{f} = 0 \) in \((\mathcal{V} \cap \overline{E}) \times \mathbb{R}_+\), we temporarily guarantee that \( \mathfrak{f} = 0 \) in all of \( \overline{E} \times \mathbb{R}_+ \) by means of the artificially strong assumption\(^5\)

\[
z^0(x, t) \cdot n(x) = 0 \text{ in } \overline{E} \times \mathbb{R}_+. \tag{3.56}
\]

As a result of the foregoing considerations, and by understanding the yet unspecified functions \( \mu^\pm \) as extended evenly to all \( z \in \mathbb{R} \), the function \( W(x, t, z) \) satisfies the following problem:

\[
\begin{aligned}
& \partial_t W - \partial_z (\lambda^+ - \lambda^-) W + (z^0 \cdot \nabla) W - (W \cdot \nabla) z^0 = \mathcal{G} e^{-|z|} + \eta \quad \text{in } \overline{E} \times \mathbb{R}_+ \times \mathbb{R}, \\
& W|_{t=0} = 0 \quad \text{in } \overline{E} \times \mathbb{R},
\end{aligned}
\]

with

\[
\mathcal{G} := \partial_z g - (\lambda^+ - \lambda^-) g + (z^0 \cdot \nabla) g - (g \cdot \nabla) z^0, \quad \eta := \mu^+ - \mu^-.
\]

Consequently, the partial Fourier transform

\[
\tilde{W}(x, t, \zeta) := \int_{\mathbb{R}} W(x, t, z) e^{-i\zeta z} \, dz
\]

satisfies the problem

\[
\begin{aligned}
& \partial_t \tilde{W} + \zeta^2 (\lambda^+ - \lambda^-) \tilde{W} + (z^0 \cdot \nabla) \tilde{W} - (\tilde{W} \cdot \nabla) z^0 = \frac{2}{1 + \zeta^2} \mathcal{G} + \tilde{\eta} \quad \text{in } \mathcal{E} \times \mathbb{R}_+ \times \mathbb{R}, \\
& \tilde{W}|_{t=0} = 0 \quad \text{in } \mathcal{E} \times \mathbb{R}.
\end{aligned}
\]

Therefore, during the time interval \([0, T]\), for each \( k \in \mathbb{N}_0 \) the evolution of the evaluated derivatives

\[
Q^k(x, t) := \partial_{\zeta}^k \tilde{W}(x, t, \zeta)|_{\zeta=0}
\]

is governed by the transport equation

\[
\begin{aligned}
& \partial_t Q^k + (z^0 \cdot \nabla) Q^k - (Q^k \cdot \nabla) z^0 = P^k \quad \text{in } \mathcal{E}_T, \\
& Q^k(\cdot, 0) = 0 \quad \text{in } \overline{E},
\end{aligned}
\]

which contains the source term

\[
P^k := \partial_{\zeta}^k \left( \tilde{\eta} + \frac{2 \mathcal{G}}{1 + \zeta^2} \right)|_{\zeta=0} - k(k-1)(\lambda^+ - \lambda^-) Q^{\text{max}(0,k-2)}. \tag{3.60}
\]

**Step 2. Determining \( \mu^+ - \mu^- \).** Let \( \widetilde{r} \in \mathbb{N} \) denote the integer part of \((r - 1)/2\). Since the function \( W \) is symmetric about the \( z = 0 \) axis, it remains to steer for \( l \in \{0, \ldots, \widetilde{r}\} \) the even moments \( Q^{2l} \) to zero. Hereof, we make for \( \eta \) the ansatz

\[
\eta(x, t, z) = \sum_{i=0}^{\widetilde{r}} \eta_i(x, t) \phi_i(z), \tag{3.61}
\]

\(^5\)This assumption will be removed in the last step by choosing \( d^0 \in (0, \overline{d}) \) small. However, all cases for which Lemma 3.18 is employed with (3.52) in this article would also satisfy (3.56), see (3.13).
wherein the even functions \((\phi_j)_{j \in \{0, \ldots, r\}} \subseteq C^\infty(\mathbb{R}) \cap L^2(\mathbb{R})\) are chosen such that their Fourier transforms \(\hat{\phi}_j(\xi) := \int_\mathbb{R} \phi_j(z) e^{-i\xi z} \, dz\) obey for all \(l \in \{0, \ldots, r\}\) the relations
\[
\partial_\xi^{2l} \hat{\phi}_j(0) := \begin{cases} 
1 & \text{if } j = l, \\
0 & \text{otherwise}. 
\end{cases}
\]

For instance, for any even smooth cutoff \(\bar{\varphi} \in C^\infty_0(\mathbb{R})\) with \(\bar{\varphi} = 1\) in a neighborhood of the origin one may take as \(\phi_0, \ldots, \phi_r\) the inverse Fourier transforms of the functions
\[
\bar{\varphi}, \frac{1}{2} \xi^2 \bar{\varphi}, \ldots, \frac{1}{(2r)!} \xi^{2r} \bar{\varphi}.
\]

Subsequently, inserting the ansatz (3.61) for each even choice \(k \in \{1, \ldots, r - 1\}\) into (3.59) provides the cascade system of transport equations
\[
\begin{aligned}
\partial_t Q^0 + (z^0 \cdot \nabla) Q^0 - (Q^0 \cdot \nabla) z^0 &= \eta^0 + 2\Theta, \\
\partial_t Q^2 + (z^0 \cdot \nabla) Q^2 - (Q^2 \cdot \nabla) z^0 &= \eta^1 - 4\Theta - 2(\lambda^+ - \lambda^-) Q^0, \\
\quad & \quad \vdots \\
\partial_t Q^{2r} + (z^0 \cdot \nabla) Q^{2r} - (Q^{2r} \cdot \nabla) z^0 &= \eta^{2r} + \frac{2(2r)!}{(-1)^r} 6 - (4r^2 - 2r)(\lambda^+ - \lambda^-) Q^{2r-2}
\end{aligned}
\]
with zero initial conditions
\[
Q^0(\cdot, 0) = Q^2(\cdot, 0) = \cdots = Q^{2r}(\cdot, 0) = 0.
\]

Let us begin with determining the control \(\eta^0\) in (3.62). Hereto, let \(\overrightarrow{Q}^0\) be the unique solution to
\[
\begin{aligned}
\partial_t \overrightarrow{Q}^0 + (z^0 \cdot \nabla) \overrightarrow{Q}^0 - (\overrightarrow{Q}^0 \cdot \nabla) z^0 &= 2\Theta \quad \text{in } E_T, \\
\overrightarrow{Q}^0(\cdot, 0) &= 0 \quad \text{in } E.
\end{aligned}
\]
Since \(\nabla \cdot \Theta = 0\) holds in \(E_T\) by (3.55) and \(\Theta \cdot n = 0\) is true in \(E_T\) as well, one can observe for all \((x, t) \in E_T\) that
\[
\nabla \cdot \overrightarrow{Q}^0(x, t) = 0, \quad \overrightarrow{Q}^0(x, t) \cdot n(x) = 0.
\]
Furthermore, the profile \(-z^0\) also satisfies the flushing property (3.11). Indeed, for \(z^0\) from Lemma 3.5 one can see this from the proof of Lemma 3.5 and the arguments of [14], while for \(z^0\) defined in (3.12) one notices that particles simply travel around the annulus in the opposite direction. In the general case, one can observe this statement from the constructions in [24]. Another option to ensure this property is to choose \(z^0\) with \(T\) replaced by \(T/2\), followed by gluing to it a reversed version starting from \(t = T/2\). Therefore, let \(\overrightarrow{Q}^0\) be the unique solution to
\[
\begin{aligned}
\partial_t \overrightarrow{Q}^0 - (z^0 \cdot \nabla) \overrightarrow{Q}^0 + (\overrightarrow{Q}^0 \cdot \nabla) z^0 &= \eta^0 \quad \text{in } E_T, \\
\overrightarrow{Q}^0(\cdot, 0) &= \overrightarrow{Q}^0(\cdot, T) \quad \text{in } E.
\end{aligned}
\]
with control $\eta^0 \in C^\infty(\overline{E} \times [0, T]; \mathbb{R}^N)$ chosen such that

$$\overline{Q}^0(\cdot, T) = 0,$$

while ensuring the properties

$$\forall (x, t) \in \mathcal{E}_T : \nabla \cdot \overline{Q}^0(x, t) = 0, \quad \forall (x, t) \in \Sigma_T : \overline{Q}^0(x, t) \cdot n(x) = 0,$$

together with

$$\forall (x, t) \in \mathcal{E}_T : \nabla \cdot \eta^0(x, t) = 0, \quad \forall (x, t) \in \Sigma_T : \eta^0(x, t) \cdot n(x) = 0.$$

Finding $\eta^0$ supported in $\overline{E} \setminus \overline{\mathcal{O}}$ is done analogously to controlling (3.19) in the proof of Lemma 3.8, noting that (3.64) and (3.19) are in principle the same equation. As a result, we can define in $\mathcal{E}_T$ the vector field

$$Q^0(x, t) := \overline{Q}^0(x, t) - \overline{Q}^0(x, T - t),$$

which solves the first equation in (3.62) with control $\eta^0$. In particular, one has the desired initial and terminal conditions

$$Q^0(\cdot, 0) = 0, \quad Q^0(\cdot, T) = 0.$$

The next task consists of determining $\eta^1$. Hereto, one may repeat the same arguments as for finding $\eta^0$, but now with the known source term $-4\Phi - 2(\lambda^+ - \lambda^-)Q^0$. Due to the cascade structure of (3.62), in this way all controls $(\eta^j)_{j \in \{1, \ldots, r\}}$ are obtained as desired. Finally, $\eta = \mu^+ - \mu^-$ is constructed via (3.61) and obeys (3.53).

**Step 3. Removing the assumption (3.56).** Without assuming (3.56), the equation (3.57) for $W$ might not be correct in $(\overline{E} \setminus \mathcal{V}) \times \mathbb{R}_+ \times \mathbb{R}$, since (3.33) contains the terms $\partial_x v^\pm$. However, for small $d^* \in (0, d)$ it can be shown that the control $\eta$ obtained in the previous step already provides

$$\overline{v} = 0 \text{ in } (\overline{E} \setminus \mathcal{V}) \times \mathbb{R}_+ \times \mathbb{R}_+ \quad (3.65)$$

for the solution to

$$\begin{aligned}
\partial_t \overline{v} - v_2 \partial_2 \overline{v} + (z^0 \cdot \nabla) \overline{v} - (\overline{v} \cdot \nabla) z^0 &= \eta \quad \text{in } \overline{E} \times \mathbb{R}_+ \times \mathbb{R}_+, \\
\partial_z \overline{v}(x, t, 0) &= g(x, t), \quad x \in \overline{E}, t \in \mathbb{R}_+, \\
\overline{v}(x, t, z) &\rightarrow 0, \quad z \rightarrow +\infty, \quad x \in \overline{E}, t \in \mathbb{R}_+, \\
\overline{v}(x, 0, z) &= 0, \quad z \in \overline{E}.
\end{aligned} \quad (3.66)$$

Because the assumptions (3.52) together with (3.65) imply

$$[(z^0 \cdot \nabla) \overline{v} - (\overline{v} \cdot \nabla) z^0]_{\text{tan}} = (z^0 \cdot \nabla) \overline{v} - (\overline{v} \cdot \nabla) z^0, \quad \int_z \partial_z \overline{v} = 0$$

in all of $\overline{E} \times \mathbb{R}_+ \times \mathbb{R}_+$, it follows that $\overline{v}$ satisfies a version of (3.66) where the first line is replaced by

$$\partial_t \overline{v} - v_2 \partial_2 \overline{v} + [(z^0 \cdot \nabla) \overline{v} - (\overline{v} \cdot \nabla) z^0]_{\text{tan}} + \int_z \partial_z \overline{v} = \eta \quad \text{in } \overline{E} \times \mathbb{R}_+ \times \mathbb{R}_+, \quad (3.66)$$

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which corresponds to the equation for $v^+ - v^-$ derived from (3.33). As a result, after verifying (3.65) for some $d^* \in (0, d)$, we know in retrospect that the equation (3.57) for $W$ is correct even without (3.56).

In order to show (3.65), the ideas from [17, Section 3.4] are applied. For any $q \in (0, d)$, consider the tube $\overline{\mathcal{T}}_q = \{ 0 \leq \varphi_{E} \leq q \}$ and define its maximal distance of influence during the time interval $[0, T]$ under the flow $\mathcal{Z}^0$ via

$$ \mathcal{J}(q) \coloneqq \max_{x,s \in [0, T]} \varphi_{E}(\mathcal{Z}^0(x, s, t)) \geq q. $$

The controls $\eta^k$ in (3.62) only act where pollution caused by $\mathfrak{C}, \eta'$, or $Q^{2l-2}$ is transported by the flow $\mathcal{Z}^0$. Thus, the control $\eta^0$ in (3.62) is supported in $\mathcal{V}^{(\mathcal{J}(d^*))}$ and its action travels at most into $\mathcal{V}^{(\mathcal{J}(d^*))}$. Consequently, the effects of the control $\eta^0$ are propagated into $\mathcal{V}^{(\bar{\mathcal{J}})}$. Since the $x$-support of $\mathfrak{C}$ is contained in $\mathcal{V}^{(\bar{d})}$, by the definition of $\chi_{\partial \mathcal{E}}$, and in view of (3.54), maintaining the $x$-support of $\overline{\mathcal{T}}$ within $\mathcal{V}$ is achieved by adjusting the support of $\chi_{\partial \mathcal{E}}$. Indeed, $\mathcal{J}$ is continuous and one has $\mathcal{J}(0) = 0$ because $\mathcal{Z}^0$ is tangential to $\partial \mathcal{E}$, which yields the existence of $d^*_0 \in (0, d)$ with $\mathcal{J}^{(\bar{d})}(d^*_0) < d$. Then, every choice $d^* \in (0, d^*_0)$ is suitable. \[ \square \]

The last part of the proof of Lemma 3.18 shows why one can take $d^* \in (0, d)$ in the definition of $\chi_{\partial \mathcal{E}}$ such that $|n| = 1$ in the $x$-support of $v^+ - v^-$. A similar argument ensures such a property for $v^+ + v^-$ or the general situation of Lemma 3.17. This has been carried out in [17, Section 3.4] and is summarized as follows.

**Lemma 3.19.** Let $\mu^\pm$ be obtained by Lemma 3.17 or Lemma 3.18. There exists $d_0 \in (0, d)$ for which any choice $d^* \in (0, d_0)$ guarantees that the $x$-supports of $v^\pm$ are included in $\mathcal{V}$.

Now, on the unbounded time interval $t \geq T$ either Lemma 3.17 or Lemma 3.18 is employed with $k = 4, p = 5, s = 3$ and $r = 6$ in order to fix $\mu^\pm$. Subsequently, in order to fix $\chi_{\partial \mathcal{E}}$ in (3.31), any $d^* \in (0, d_0)$ is selected, where $d_0$ is the number determined in Lemma 3.19.

**Remark 3.20.** When $M_2 = 0$ in (1.4), then, as seen in Lemma 3.15, the first case of Lemma 3.18 can be applied. The assumptions of Theorem 1.2 b) provide (3.13), which allows to employ the third part of Lemma 3.18.

### 3.4.3 Properties of the boundary layers and technical profiles

We continue by collecting several properties of the boundary layers and the related technical profiles given in (3.7). Due to the fast variable scaling for the boundary layer profiles $v^\pm, w^\pm$ and $q^\pm$, several estimates will profit from a gain of order $O(\epsilon^{1/4})$ as stated below.

**Lemma 3.21** ([29, Lemma 3]). There exists a constant $C > 0$ such that, for all $\epsilon > 0$ and functions $h = h(x, z)$ in $L^2_\mathcal{E}(\mathbb{R}^+, \mathcal{H}^1_\mathcal{E}(\mathcal{E}))$ with $\cup_{z \in \mathbb{R}^+} \text{supp}(h(\cdot, z)) \subseteq \mathcal{V}$, it holds

$$ \| h \left( \cdot, \frac{\varphi_{E}(\cdot)}{\sqrt{\epsilon}} \right) \|_{L^2_\mathcal{E}(\mathcal{E})} \leq C \varepsilon^{\frac{1}{4}} \| h \|_{L^2_\mathcal{E}(\mathbb{R}^+, \mathcal{H}^1_\mathcal{E}(\mathcal{E}))}. $$
We proceed with three lemmas regarding several attributes of \( q^\pm, \mathbf{w}^\pm \) and \( \theta^{\pm, \epsilon} \).

**Lemma 3.22.** The functions \( q^\pm \) determined by (3.38) satisfy \( \text{supp}(q^\pm(\cdot, \cdot, z)) \subseteq \text{supp}(z^0) \) for all \( z \in \mathbb{R}_+ \). In addition, there exists a constant \( C > 0 \) independent of \( \epsilon > 0 \) such that one has the estimate

\[
\| \nabla q^\pm \|_{L^2(E)} \leq \epsilon^{\frac{1}{4}} C \sum_{\alpha \in \{+,-\}} \| \varphi^\alpha(\cdot, t, \cdot) \|_{H_{E}^{1,3,0}}. \tag{3.67}
\]

**Proof.** One utilizes the properties of \( q^\pm \) observed from (3.38), the fast variable scaling from Lemma 3.21 and integration by parts. Indeed, for \( C > 0 \) and small \( \eta \in (0, 1) \), both independent of \( \epsilon > 0 \), one has

\[
\| \nabla q^\pm \|_{L^2(E)} \leq \epsilon^{\frac{1}{4}} C \| \nabla q^\pm \|_{L^2(\mathbb{R}_+, H^1(E))} \leq \epsilon^{\frac{1}{4}} C \sum_{|\alpha| \leq 1} \int_{\mathbb{R}_+} \int_{E} |\partial_x^{\alpha} \nabla q^\pm|^2 \partial_z dxdz
\]

\[
= \epsilon^{\frac{1}{4}} C(\eta) \sum_{|\alpha| \leq 1} \int_{\mathbb{R}_+} \int_{E} (1 + z^2) |\partial_x^{\alpha} \nabla \partial_z q^\pm|^2 dxdz
\]

\[
+ \epsilon^{\frac{1}{4}} \eta \sum_{|\alpha| \leq 1} \int_{\mathbb{R}_+} \int_{E} |\partial_x^{\alpha} \nabla q^\pm|^2 dxdz,
\]

which implies that

\[
\| \nabla q^\pm \|_{L^2(E)} \leq \epsilon^{\frac{1}{4}} C_1 \sum_{|\alpha| \leq 1} \int_{\mathbb{R}_+} \int_{E} (1 + z^2) |\partial_x^{\alpha} \nabla \partial_z q^\pm|^2 dxdz
\]

\[
\leq \epsilon^{\frac{1}{4}} C_2 \sum_{\alpha \in \{+,-\}} \| \varphi^\alpha(\cdot, t, \cdot) \|_{H_{E}^{1,3,0}}^2
\]

for two constants \( C_1, C_2 \), by using (3.38). \( \square \)

**Lemma 3.23.** For all \( k, m, s \in \mathbb{N} \), the profiles \( \mathbf{w}^\pm \) determined in (3.39) satisfy

\[
\| \mathbf{w}^\pm(\cdot, t, \cdot) \|_{H_{E}^{k,m,s}} \leq C \sum_{\alpha \in \{+,-\}} \| \varphi^\alpha(\cdot, t, \cdot) \|_{H_{E}^{k+1,m+1,s+1, \max \{1,s-1\}}}, \tag{3.68}
\]

\[
\| \partial_z^2 \mathbf{w}^\pm \|_{L^2(E)} \leq \epsilon^{\frac{1}{4}} C \sum_{\alpha \in \{+,-\}} \| \varphi^\alpha(\cdot, t, \cdot) \|_{H_{E}^{1,2,1}}, \tag{3.69}
\]

\[
\| \partial_z \mathbf{w}^\pm \|_{L^2(E)} \leq \epsilon^{\frac{1}{4}} C \sum_{\alpha \in \{+,-\}} \| \varphi^\alpha(\cdot, t, \cdot) \|_{H_{E}^{2,3,3}} + C \| \mu^\pm(\cdot, t, \cdot) \|_{H_{E}^{1,2,1}}. \tag{3.70}
\]

**Proof.** One can show (3.68) by separately estimating the tangential and normal parts

\[
\mathbf{w}_T^\pm \mathbf{x}(x, t, z) := -2 e^{\frac{-s}{2}} \mathbf{N}^\pm(\mathbf{v}^+, \mathbf{v}^-)(x, t, 0),
\]

\[
\mathbf{w}_N^\pm \mathbf{x}(x, t, z) := -n(x) \int_{z}^{\pm \infty} \nabla \cdot \mathbf{v}^\pm(x, t, s) ds.
\]

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For instance, integration by parts yields
\[
\|\omega^\pm_N(\cdot, t, z)\|^2_{H^m_{E,0}} 
\leq C \sum_{\beta \leq m} \int \int_{E, R_+} \partial_\zeta (z + z^{2k+1}) \left| \int \partial^\beta_x (\nabla \cdot \nu^\pm)(x, t, s) \, ds \right|^2 \, dz \, dx 
\]
\[
\leq C \sum_{\beta \leq m} \int \int_{E, R_+} (z + z^{2k+1}) \partial_\zeta \partial^\beta_x (\nu^\pm)(x, t, s) \left( \int \partial^\beta_x (\nabla \cdot \nu^\pm)(x, t, s) \, ds \right) \, dz \, dx.
\]

Hence, for arbitrary \( \eta > 0 \) one has
\[
\|\omega^\pm_N(\cdot, t, z)\|^2_{H^m_{E,0}} 
\leq C(\eta) \sum_{\beta \leq m+1} \int \int_{E, R_+} (1 + z^{2k+2}) \left| \partial^\beta_x \nu^\pm(x, t, z) \right|^2 \, dz \, dx 
\]
\[
+ \eta \sum_{\beta \leq m} \int \int_{E, R_+} \partial_\zeta (z + z^{2k+1}) \left| \int \partial^\beta_x (\nabla \cdot \nu^\pm)(x, t, s) \, ds \right|^2 \, dz \, dx.
\]

Thus, by choosing small \( \eta > 0 \) and a new constant \( C = C(\eta) > 0 \), one obtains
\[
\|\omega^\pm_N(\cdot, t, z)\|^2_{H^m_{E,0}} \leq C\|\nu^\pm(\cdot, t, \cdot)\|_{H^m_{E,0}}.
\]

The other aspects of (3.68) are along the same lines, noting that estimating \( \omega^+_I \) costs one regularity level in \( z \) due the application of a trace theorem. The estimate (3.69) follows from a combination of Lemma 3.21, the above idea for showing (3.68) and the identity
\[
\partial^2_z \omega^\pm = (\nabla \cdot \partial_\zeta \nu^\pm)n - 2 e^{-z} \left[ \mathcal{N}^\pm(\nu^+, \nu^-) \right]_{z=0}.
\]

Regarding (3.70), the starting point is to derive from (3.39) the representation
\[
\partial_t \omega^\pm = \partial_t \tilde{\omega}^\pm n - 2 e^{-z} \left[ \mathcal{N}^\pm(\partial_t \nu^+, \partial_t \nu^-) \right]_{z=0},
\]
into which one can subsequently insert the equation (3.33) and proceed as before. □

**Lemma 3.24.** The Neumann problems (3.41) are well-posed and all solutions \( \theta^{\pm, \epsilon} \) obey for \( l \in \{0, 1, 2\} \) the estimates
\[
\|\theta^{\pm, \epsilon}(\cdot, t)\|_{H^l_{E,0}} \leq e^{\frac{l}{2} - \frac{l}{4}} C\|\omega^\pm(\cdot, t, \cdot)\|_{H^{2l,1}_{E,0}} + C \sum_{\square \in \{+,-\}} \|\nu^{\square}(\cdot, t, \cdot)\|_{H^{1+l,0}_{E,0}}, \quad (3.71)
\]

while in the case when \( \omega^+ - \omega^- \cdot n = 0 \) at \( \partial \mathcal{E} \), one additionally has
\[
\|\theta^{\pm, \epsilon}(\cdot, t) - \theta^{-, \epsilon}(\cdot, t)\|_{H^2_{E,0}} \leq e^{\frac{l}{4} C} \sum_{\square \in \{+,-\}} \|\nu^{\square}(\cdot, t, \cdot)\|_{H^{3,1}_{E,0}}, \quad (3.72)
\]

Furthermore, for all \( t \in [0, T/\epsilon] \), one has
\[
\|\Delta \nabla \theta^{\pm, \epsilon}(\cdot, t)\|_{L^2_{E,0}} \leq e^{\frac{l}{2} - \frac{l}{4}} C \sum_{\square \in \{+,-\}} \|\nu^{\square}(\cdot, t, \cdot)\|_{H^{2,2}_{E,0}}, \quad (3.73)
\]

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Proof. In (3.41), there is no coupling between + and − superscribed functions. Thus, the well-posedness of (3.41) together with (3.71) and (3.73) can be established by analysis similar to [17, Equations (4.29), (4.31)–(4.33) and (4.58)]. In particular, by employing (2.4), (3.37), (3.39), and (3.40), one can verify the necessary compatibility conditions for (3.41) via

\[
\int_{\partial E} \mathbf{w}^\pm(x, t, 0) \cdot \mathbf{n}(x) \, dS(x) \\
= \int_{\partial E} \|\mathbf{w}^+\|_e(x, t) \cdot \mathbf{n}(x) \, dS(x) = \int_{\partial E} \nabla \cdot \|\mathbf{w}^+\|_e(x, t) \, dx \\
= \int_{E} \left( \|\nabla \cdot \mathbf{w}^+\|_e - \frac{1}{\sqrt{\epsilon}} \mathbf{n} \cdot \|\partial_t \mathbf{w}^+\|_e \right)(x, t) \, dx \\
= \int_{E} \left( \|\nabla \cdot \mathbf{w}^+\|_e - \frac{1}{\sqrt{\epsilon}} \nabla \cdot \|\mathbf{v}^\pm\|_e - \frac{1}{\epsilon} \mathbf{n} \cdot \|\partial_t \mathbf{v}^\pm\|_e \right)(x, t) \, dx \\
= \int_{E} \|\nabla \cdot \mathbf{w}^+\|_e(x, t) \, dx.
\]

It remains to show the estimate (3.72) when \((\mathbf{w}^+ - \mathbf{w}^-) \cdot \mathbf{n} = 0\) at \(\partial E\). By elliptic regularity for weak solutions to the Laplace equation (3.41), Lemma 3.21, and (3.68) one obtains

\[
\|((\theta^+ - \theta^-))(\cdot, t)\|_{H^2(E)} \leq C \|\nabla \cdot (\mathbf{w}^+ - \mathbf{w}^-)(\cdot, t, \cdot)\|_e \|\mathbf{l}^2_{\mathcal{E}}(E) \leq \epsilon^{-\frac{1}{2}} C \sum_{\square \in \{+,-\}} \|\mathbf{v}^\square\|_{H_{\mathcal{E}}^{1,1,1}}.
\]

Finally, several properties of the remainder terms \(r^{\pm,\epsilon}\) in the ansatz (3.7) are summarized.

**Lemma 3.25.** The remainder terms \(r^{\pm,\epsilon}\) given in (3.7) satisfy the conditions

\[
\begin{align*}
\mathbf{r}^{\pm,\epsilon}(\cdot, 0) &= \mathbf{0} \quad &\text{in } \mathcal{E}, \\
\nabla \cdot \mathbf{r}^{\pm,\epsilon} &= 0 \quad &\text{in } \mathcal{E} \times \mathbb{R}_+, \\
\mathbf{r}^{\pm,\epsilon} \cdot \mathbf{n} &= 0 \quad &\text{on } \partial \mathcal{E} \times \mathbb{R}_+, \\
\mathbf{N}^{\pm}(\mathbf{r}^{+,\epsilon}, \mathbf{r}^{-,\epsilon}) &= \mathbf{g}^{\pm,\epsilon} \quad &\text{on } \partial \mathcal{E} \times \mathbb{R}_+,
\end{align*}
\]

(3.74)

wherein

\[
\mathbf{g}^{\pm,\epsilon} := \mathbf{N}^{\pm}(\mathbf{z}^{+1}, \mathbf{z}^{-1}) - \mathbf{N}^{\pm}(\nabla \theta^{+,\epsilon}, \nabla \theta^{-,\epsilon}) - \mathbf{N}^{\pm}(\mathbf{w}^+, \mathbf{w}^-)_{|_{\epsilon=0}}.
\]

Moreover, for a constant \(C > 0\) independent of \(\epsilon > 0\), the boundary data \(\mathbf{g}^{\pm,\epsilon}\) can be estimated by

\[
\|\mathbf{g}^{\pm,\epsilon}\|_{L^2((0,T);H^1(\mathcal{E}))} \leq C \sum_{\square \in \{+,-\}} \left( \|\mathbf{z}^{\square,1}\|_{L^2((0,T);H^2(\mathcal{E}))}^2 + \epsilon^{-\frac{1}{2}} \|\mathbf{v}^\square\|_{L^2((0,T);H_{\mathcal{E}}^{1,4,2})}^2 \right).
\]

(3.75)
Proof. The assertions in (3.74) are a consequence of the definitions for the functions in (3.7)–(3.8) studied in Subsections 3.2, 3.3 and 3.4. Particularly, in view of Remark 3.13, Lemma 3.19, the relations in (3.40), and the boundary conditions (3.34), one has

\[ 2\mathcal{N}^\pm(z^0, z^0) = [\partial_z v^\pm]_{|z=0}, \quad 2\mathcal{N}^\pm(v^+, v^-)_{|z=0} = [\partial_z w^\pm]_{|z=0}. \]

\[ \square \]

3.5 Remainder estimates

In this subsection, it will be shown that \( \|z^{\pm, \epsilon}(\cdot, T/\epsilon)\|_{L^2(E)} = O(\epsilon^{9/8}) \) as \( \epsilon \to 0 \). Here, many estimates are similar to those of the Navier-Stokes problem studied in [17]. Therefore, emphasis is put on the parts that are specific for this MHD problem not readily available in the here employed form.

3.5.1 Equations satisfied by the remainders

In order to derive the equations satisfied by the remainders \( r^{\pm, \epsilon} \), the definitions of \( q^\pm \) given in (3.38) are employed for rewriting (3.33) in the form

\[ \partial_t v^\pm - \partial_{zz}(\lambda^\pm v^\pm + \lambda^\mp v^-) + (z^0 \cdot \nabla) v^\pm + (v^\mp \cdot \nabla) z^0 + z^0 \partial_z v^\pm - n \partial_z q^\pm = \mu^\pm. \quad (3.76) \]

Then, (3.7)–(3.8) are inserted into (3.5) while using (2.4), (3.76) and Lemma 3.25. Since the terms \( \|(v^\mp \cdot n) \partial_z v^\pm\|_E \) and \( \|(v^\mp \cdot n) \partial_z w^\pm\|_E \) vanish, the remainders \( r^{\pm, \epsilon} \) satisfy the following problem

\[
\begin{align*}
\partial_t r^{\pm, \epsilon} - \epsilon \Delta (\lambda^\pm r^{\pm, \epsilon} + \lambda^\mp r^{-, \epsilon}) + (z^{\pm, \epsilon} \cdot \nabla) r^{\pm, \epsilon} + \nabla \pi^{\pm, \epsilon} &= \Theta^{\pm, \epsilon} \quad \text{in } \mathcal{E}_{T/\epsilon}, \\
\nabla \cdot r^{\pm, \epsilon} &= 0 \\
r^{\pm, \epsilon} \cdot n &= 0 \\
\mathcal{N}^\pm(r^{+, \epsilon}, r^{-, \epsilon}) &= \mathbf{g}^{\pm, \epsilon} \\
r^{\pm, \epsilon}(\cdot, 0) &= 0
\end{align*}
\]

where the source terms \( \Theta^{\pm, \epsilon} \) are given by

\[ \Theta^{\pm, \epsilon} := \|h^{\pm, \epsilon} - A^{\epsilon, \pm} r^{\mp, \epsilon}\|_E, \]

which contain the amplification terms \( A^{\epsilon, \pm} r^{\mp, \epsilon} \) being defined via

\[ A^{\epsilon, \pm} r^{\mp, \epsilon} := (r^{\mp, \epsilon} \cdot \nabla) \left( z^0 + \sqrt{\epsilon} v^\pm + \epsilon z^{\pm, 1} + \epsilon \nabla \theta^{\pm, \epsilon} + \epsilon w^\pm \right) - (r^{\mp, \epsilon} \cdot n) \left( \partial_z v^\pm + \sqrt{\epsilon} \partial_z w^\pm \right), \]

and the remaining terms

\[ h^{\pm, \epsilon} := \lambda^\pm h^{\pm, \epsilon, 1} + \lambda^\mp h^{-, \epsilon, 1} + h^{\pm, \epsilon, 2} - \partial_z w^\pm - \nabla q^\pm, \]

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with
\[ h^{\pm,\epsilon,1} := \left( \partial_t^2 w^\pm + \Delta \varphi \partial_t v^\pm - 2(n \cdot \nabla) \partial_t v^\pm \right) + \epsilon \left( \Delta w^\pm + \Delta z^{\pm,1} + \Delta \nabla \theta^{\pm,\epsilon} \right) \]
\[ + \sqrt{\epsilon} \left( \Delta v^\pm + \Delta \varphi \partial_t w^\pm - 2(n \cdot \nabla) \partial_t w^\pm \right), \]
and
\[ h^{\pm,\epsilon,2} \]
\[ := - \left( \left[ \varphi^{\pm} + \sqrt{\epsilon} \left( w^{\pm} + z^{\pm,1} + \nabla \theta^{\pm,\epsilon} \right) \right] \cdot \nabla \right) \left[ \varphi^{\pm} + \sqrt{\epsilon} \left( w^{\pm} + z^{\pm,1} + \nabla \theta^{\pm,\epsilon} \right) \right] \]
\[ - (z^0 \cdot \nabla) w^\pm - (w^\pm \cdot \nabla) z^0 + \left( \left[ w^{\pm} + z^{\pm,1} + \nabla \theta^{\pm,\epsilon} \right] \cdot n \right) \partial_t (v^\pm + \sqrt{\epsilon} w^\pm) \]
\[ + \tilde{s}_{\pm,\epsilon} - \left( (z^0 \cdot \nabla) \nabla \theta^{\pm,\epsilon} + \left( \nabla \theta^{\pm,\epsilon} \cdot \nabla \right) z^0 - \nabla (z^0 \cdot \nabla \theta^{\pm,\epsilon}) \right) + \frac{1}{\sqrt{\epsilon}} z^0 \cdot n \partial_t w^\pm. \]

Before deriving energy estimates for \( r^{\pm,\epsilon} \), several asymptotic properties of the right-hand side terms in (3.77) are summarized.

**Lemma 3.26.** For all \( t \in [0, T/\epsilon] \), one has
\[ \| A^{\pm,\epsilon} \cdot r^{\pm,\epsilon} \|_{L^1(E \times (0,t))} \leq C \left( \| r^{\pm,\epsilon} \|^2_{L^2(E \times (0,t))} + \| r^{-,\epsilon} \|^2_{L^2(E \times (0,t))} \right) \]
with a constant \( C > 0 \) independent of \( t, \epsilon \) and \( r^{\pm,\epsilon} \). Furthermore, as \( \epsilon \to 0 \) one has
\[ \| g^{\pm,\epsilon} \|^2_{L^2((0,T/\epsilon);H^1(E))} = O(\epsilon^{-\frac{1}{2}}), \quad \| h^{\pm,\epsilon} \|^2_{L^1((0,T/\epsilon);L^2(E))} = O(\epsilon^{\frac{1}{2}}). \]

**Proof.** Throughout, it will be used that \( \mu^\pm \) have been fixed in Section 3.4.2 either by Lemma 3.17 (or Lemma 3.18) applied with \( k = 4, p = 5, s = 3 \) and \( r = 6 \). This provides bounds for \( v^\pm \) in \( L^1((0,T/\epsilon);H^k(E)) \) which are uniform in \( \epsilon \), since for \( \lambda > 0 \) and \( a \in (1, +\infty) \) one has the convergence of the integral
\[ \int_0^\infty \left( \ln(2 + \lambda s) \right)^a \frac{1}{2 + \lambda s} \, ds \leq C = C(\lambda, a) < +\infty. \]

In order to show (3.78), let \( A^{\pm,\epsilon} \) denote the functions
\[ A^{\pm,\epsilon} := \nabla \left( z^0 + \sqrt{\epsilon} v^\pm + z^{\pm,1} + \epsilon \nabla \theta^{\pm,\epsilon} + \epsilon w^\pm \right) - (\partial_t v^\pm + \sqrt{\epsilon} \partial_t w^\pm) n^\top. \]

From Lemma 3.8 one knows that \( z^{\pm,1} \) are bounded in \( L^\infty((0,T);H^3(E)) \) as long as the initial data satisfy \( z^0 \in H^3(E) \cap H(E) \). Hence, combining Sobolev embeddings with Lemma 3.24 allows to infer
\[ \| A^{\pm,\epsilon} \|_{L^\infty(E \times (0,T/\epsilon))} \]
\[ \leq C \sum_{\Delta \epsilon \{+, -\}} \left( \| z^0 \|_{L^\infty((0,T);H^3(E))} + \| z^{\pm,1} \|_{L^\infty((0,T);H^3(E))} + \| \theta^{\pm,\epsilon} \|_{L^\infty((0,T);H^3(E))} \right). \]
Moreover, by Lemma 3.17 or Lemma 3.18 one finds a constant $C > 0$ independent of $\varepsilon$ with
\[ \| u^\pm \|_{L^\infty((0,T);H^{2,5,1}_E)} \leq C, \]
which eventually implies (3.78).

The bounds for $g^{\pm,\varepsilon}$ in (3.79) follow from (3.75) and Lemma 3.17 (or Lemma 3.17) such that it remains to establish the estimates for $h^{\pm,\varepsilon}$ in (3.79). We begin with the terms
\[ a^\pm := \tilde{\zeta}^{\pm,\varepsilon} - \left( (z^0 \cdot \nabla) \nabla \theta^{\pm,\varepsilon} + (\nabla \theta^{\mp,\varepsilon} \cdot \nabla) z^0 - \nabla (z^0 \cdot \nabla \theta^{\pm,\varepsilon}) \right). \]

According to the definition of $\tilde{\zeta}^{\pm,\varepsilon}$ in Section 3.4.1, the terms $a^\pm$ vanish in the case where $\nabla \times z^0 = 0$ and $(w^+ - w^-)_{|z=0} \cdot n \neq 0$. When $(w^+ - w^-)_{|z=0} \cdot n = 0$ is satisfied, the terms $a^\pm$ assume the form
\[ a^\pm = - (\nabla (\theta^{\mp,\varepsilon} - \theta^{\pm,\varepsilon}) \cdot \nabla) z^0. \]

Concerning the latter case, the estimate (3.72) implies
\[ \| (\nabla (\theta^{\mp,\varepsilon} - \theta^{\pm,\varepsilon}) \cdot \nabla) z^0 \|_{L^1((0,T/\varepsilon);L^2(E))} \leq \varepsilon \frac{1}{4} C \sum_{\Delta \in \{+,-\}} \| v^{\Delta} \|_{L^1((0,T);H^{1,3,1}_E)} \]
and an invocation of Lemma 3.17 or Lemma 3.18 yields
\[ \| a^\pm \|_{L^1((0,T/\varepsilon);L^2(E))} = O(\varepsilon^{\frac{1}{2}}). \]

In order to treat the terms which appear with a factor $\varepsilon^{-1/2}$ in $h^{\pm,\varepsilon}$, we resort to a trick similar to [29, Equation (69)]. Indeed, the definitions for $w^\pm$ in (3.39) provide
\[ \left\| \frac{(z^0 \cdot n)}{\sqrt{\varepsilon}} \partial_\varepsilon w^\pm \right\|_{L^1((0,T/\varepsilon);L^2(E))} = - \left\| z \left( (\nabla \cdot v^\pm) n + 2 e^{-\varepsilon^2} (N^\pm (v^+, v^-)_{|z=0}) \right) \|_{L^1((0,T/\varepsilon);L^2(E))} \right\| \]
which due to $f$ being bounded leads to
\[ \left\| \left[ \frac{(z^0 \cdot n)}{\sqrt{\varepsilon}} \partial_\varepsilon w^\pm \right] \right\|_{L^1((0,T/\varepsilon);L^2(E))} \leq C \left\| z \left( (\nabla \cdot v^\pm) n + 2 e^{-\varepsilon^2} (N^\pm (v^+, v^-)_{|z=0}) \right) \right\|_{L^1((0,T/\varepsilon);L^2(E))} \leq I_1^\pm + I_2^\pm, \]

wherein
\[ I_1^\pm := C \left\| z (\nabla \cdot v^\pm) n \right\|_{L^1((0,T/\varepsilon);L^2(E))}, \]
\[ I_2^\pm := C \left\| z e^{-\varepsilon^2} (N^\pm (v^+, v^-)_{|z=0}) \right\|_{L^1((0,T/\varepsilon);L^2(E))}. \]

Since $z (\nabla \cdot v^\pm) n \in L^2_\varepsilon (\mathbb{R}; H^1_E(E))$, Lemma 3.21 can be applied to $I_1^\pm$ and, by similar analysis for $I_2^\pm$, there exists a constant $C > 0$ independent of $\varepsilon$ such that
\[ I_1^\pm + I_2^\pm \leq \varepsilon^{\frac{1}{2}} C \sum_{\Delta \in \{+,-\}} \| v^{\Delta} \|_{L^1((0,T);H^{2,0}_E)}. \]
As a result, Lemma 3.17 (respectively Lemma 3.18) allows to infer
\[ \| \left( \frac{z^0 \cdot n}{\sqrt{\varepsilon}} \right) \partial_x w^{\varepsilon} \|_{L^1((0,T/\varepsilon) ; L^2({\mathcal{E}}))} = O(\varepsilon^{1/2}). \]

The remaining terms contained in \( h^{+,-} \) and \( h^{-,-} \) behave as in the Navier-Stokes case treated in [17, Section 4.4]. Carrying out these details particularly involves the estimates (3.67), (3.69), (3.70) and (3.73). In particular, for estimating \( \partial_t w^{\pm} \), several norms of \( \mu^{\pm} \) enter through (3.70) with \( O(\varepsilon^{1/4}) \) coefficients. Since \( \mu^{\pm} \) are supported in \([0,T]\) and smooth, one has \( O(\varepsilon^{1/4}) \) bounds for these terms. \( \square \)

### 3.5.2 Energy estimates

In this subsection, the bound (3.6) is obtained as a consequence of the next proposition.

**Proposition 3.27.** The functions \( r^{\pm,-} \) determined from (3.7) by means of Subsections 3.2, 3.3 and 3.4 satisfy
\[ \| r^{\pm,-} \|_{L^2((0,T/\varepsilon) ; L^2({\mathcal{E}}))} + \varepsilon \| r^{\pm,-} \|_{L^2((0,T/\varepsilon) ; H^1({\mathcal{E}}))} = O(\varepsilon^{1/2}). \]

**Proof.** The idea is to multiply the first line in (3.77) by \( r^{\pm,-} \) respectively and to integrate over \( \mathcal{E} \times (0,t) \) for \( t \in (0,T/\varepsilon) \), which however is not justified. Indeed, the regularity provided by \( z^{\pm,-} , r^{\pm,-} \in \mathcal{X}_{T/\varepsilon} \) does not guarantee the convergence of the integrals
\[ \int_0^t \int_{\mathcal{E}} (z^{\pm,-}(x,s) \cdot \nabla) r^{\pm,-}(x,s) \cdot r^{\pm,-}(x,s) \, dx \, ds. \]

Since (3.7)–(3.8) provide \( \varepsilon r^{\pm,-} = z^{\pm,-} - s^{\pm,-} \) with \( s^{\pm,-} \) bounded in \( L^\infty((0,T/\varepsilon) ; H^3({\mathcal{E}})) \) and \( \partial_t s^{\pm,-} \in L^2((0,T/\varepsilon) ; L^2({\mathcal{E}})) \), this issue can be transshipped by using the strong energy inequality as explained in [29, Page 167-168].

**Step 1. Idea.** For the sake of completeness, let us sketch the above mentioned idea of using the strong energy inequality for obtaining the desired energy estimates, hereby assuming for simplicity the case from Section 2.4.1 where \( \sigma^0 = 0 \). First, from (3.77) and (3.5), one observes that \( s^{\pm,-} \) satisfy a weak formulation for the problem
\[
\begin{align*}
\partial_t s^{\pm,-} - \varepsilon \Delta (\lambda^{\pm} s^{\pm,+} + \lambda^{-} s^{-,-}) + (z^{\pm,-} \cdot \nabla) s^{\pm,-} + \nabla o^{\pm,-} &= \Xi^{\pm,-} & & & & \text{ in } \mathcal{E}_{T/\varepsilon}, \\
\n \nabla \cdot s^{\pm,-} &= 0 & & & & \text{ in } \mathcal{E}_{T/\varepsilon}, \\
 s^{\pm,-} \cdot n &= 0 & & & & \text{ on } \Sigma_{T/\varepsilon}, \\
 (\nabla \times s^{\pm,-}) \times n &= \rho^{\pm} (s^{\pm,+}, s^{-,-}) - \varepsilon g^{\pm,-} & & & & \text{ on } \Sigma_{T/\varepsilon}, \\
 s^{\pm,-}(\cdot, 0) &= \varepsilon z^{\pm,-}_0 & & & & \text{ in } \mathcal{E},
\end{align*}
\]

where
\[ o^{\pm,-} := \rho^{\pm,\pm,-} - \varepsilon \pi^{\pm,-}, \quad \Xi^{\pm,-} := \xi^{\pm,-} - \varepsilon [h^{\pm,-} - A^{\pm,-} r^{\pm,-}] \|_{L^2}. \]

Multiplying the equations (3.81) with \( z^{\pm,-} - s^{\pm,-} \in \mathcal{X}_{T/\varepsilon} \), integrating over \( \mathcal{E} \times (0,t) \), and summing up the results, leads to
\[ \sum_{\delta \in \{+,-\}} \int_0^t \int_{\mathcal{E}} \partial_t s^{\delta,-}(x) \cdot (z^{\delta,-} - s^{\delta,-}) \, dx \, dt. \]
\[
+ \sum_{\Delta \in \{+,-\}} \epsilon \lambda^+ \int_0^t \int_E \nabla \times s^{\square, \epsilon} \cdot \nabla \times (z^{\Delta, \epsilon} - s^{\square, \epsilon}) \, dx \, dt \\
+ \epsilon \lambda^- \sum_{\Delta \in \{+,-\}} \int_0^t \int_E \nabla \times s^{\square, \epsilon} \cdot \nabla \times (z^{\Delta, \epsilon} - s^{\square, \epsilon}) \, dx \, dt \\
+ \sum_{\Delta \in \{+,-\}} \int_0^t \int_E (z^{\square, \epsilon} \cdot \nabla) s^{\Delta, \epsilon} \cdot z^{\Delta, \epsilon} \, dx \, dt 
\]

\[
= \sum_{\Delta \in \{+,-\}} \int_0^t \int_E (\xi^{\Delta, \epsilon} - \epsilon \| h^{\Delta, \epsilon} \|_e + \epsilon \| A^{\Delta, \epsilon} r^{\square, \epsilon} \|_e) \cdot (z^{\Delta, \epsilon} - s^{\square, \epsilon}) \, dx \, dt \\
+ \epsilon \lambda^+ \sum_{\Delta \in \{+,-\}} \int_0^t \int_{\partial E} (\rho^{\square}(s^{+, \epsilon}, s^{-, \epsilon}) - \epsilon \rho^{\square, \epsilon}) \cdot (z^{\Delta, \epsilon} - s^{\square, \epsilon}) \, ds \, dt \\
+ \epsilon \lambda^- \sum_{\Delta \in \{+,-\}} \int_0^t \int_{\partial E} (\rho^{\circ}(s^{+, \epsilon}, s^{-, \epsilon}) - \epsilon \rho^{\circ, \epsilon}) \cdot (z^{\Delta, \epsilon} - s^{\square, \epsilon}) \, ds \, dt.
\]

In (3.82), the following cancellations, which are due to integration by parts and justified by the regularity of \( s^{\pm, \epsilon} \), have been taken into account:

\[
\sum_{\Delta \in \{+,-\}} \int_0^t \int_E (z^{\square, \epsilon} \cdot \nabla) s^{\Delta, \epsilon} \cdot z^{\Delta, \epsilon} \, dx \, dt = 0.
\]

Second, taking the test function \( s^{\pm, \epsilon} \) in the weak formulation for \( z^{\pm, \epsilon} \), which can be justified by the regularity of \( s^{\pm, \epsilon} \), yields

\[
\sum_{\Delta \in \{+,-\}} \int_E z^{\square, \epsilon}(x, t) \cdot s^{\square, \epsilon}(x, t) \, dx - \sum_{\Delta \in \{+,-\}} \int_E z^{\square, \epsilon}(x, 0) \cdot s^{\square, \epsilon}(x, 0) \, dx \\
- \sum_{\Delta \in \{+,-\}} \int_0^t \int_E z^{\square, \epsilon} \cdot \partial_t s^{\square, \epsilon} \, dx \, dt + \epsilon \lambda^+ \sum_{\Delta \in \{+,-\}} \int_0^t \int_E \nabla \times z^{\square, \epsilon} \cdot \nabla \times s^{\square, \epsilon} \, dx \, dt \\
+ \sum_{\Delta \in \{+,-\}} \int_0^t \int_E (\epsilon \lambda^- \nabla \times z^{\square, \epsilon} \cdot \nabla \times s^{\Delta, \epsilon} + (z^{\square, \epsilon} \cdot \nabla) z^{\Delta, \epsilon} \cdot s^{\Delta, \epsilon}) \, dx \, dt \\
= \sum_{\Delta \in \{+,-\}} \left( \int_0^t \int_E z^{\square, \epsilon} \cdot s^{\square, \epsilon} \, dx \, dt + \epsilon \lambda^+ \int_0^t \int_{\partial E} \rho^{\square, \epsilon}(z^{+, \epsilon}, z^{-, \epsilon}) \cdot s^{\square, \epsilon} \, ds \, dt \right) \\
+ \epsilon \lambda^- \sum_{\Delta \in \{+,-\}} \int_0^t \int_{\partial E} \rho^{\circ, \epsilon}(z^{+, \epsilon}, z^{-, \epsilon}) \cdot s^{\Delta, \epsilon} \, ds \, dt.
\]
Third, multiplying the energy inequality (2.17) with $\varepsilon^2$, followed by evaluation at $s = 0$ and $et$ for $t \in [0, T/\varepsilon]$, while performing the change of variables $r \to \varepsilon r$, one has

$$
\frac{1}{2} \sum_{\xi \in \{+,-\}} \|z^{\xi,\varepsilon}(\cdot, t)\|_{L^2(E)}^2 + \varepsilon \lambda^+ \sum_{\xi \in \{+,-\}} \int_0^t \int_E \nabla \times z^{\xi,\varepsilon} \cdot \nabla \times z^{\xi,\varepsilon} \, dx \, dr + \varepsilon \lambda^- \sum_{(\Delta, o) \in \{(+,-),(+-,)+\}} \int_0^t \int_E \nabla \times z^{\Delta,\varepsilon} \cdot \nabla \times z^{\Delta,\varepsilon} \, dx \, dr
$$

$$
\leq \frac{1}{2} \sum_{\xi \in \{+,-\}} \|z^{\xi,\varepsilon}(\cdot, 0)\|_{L^2(E)}^2 + \varepsilon \lambda^+ \sum_{\xi \in \{+,-\}} \int_0^t \int_{\partial E} \rho^\Delta (z^{+\varepsilon,\varepsilon}, z^{-\varepsilon,\varepsilon}) \cdot z^{\xi,\varepsilon} \, dS \, dr + \varepsilon \lambda^- \sum_{(\Delta, o) \in \{(+,-),(+-,)+\}} \int_0^t \int_{\partial E} \rho^\Delta (z^{+\varepsilon,\varepsilon}, z^{-\varepsilon,\varepsilon}) \cdot z^{\Delta,\varepsilon} \, dS \, dr + \sum_{\xi \in \{+,-\}} \int_0^t \int_E \xi^{\xi,\varepsilon} \cdot z^{\xi,\varepsilon} \, dx \, dr.
$$

**Step 2. Conclusion.** By subtracting from (3.84) the equations (3.82) and (3.83), while considering for the general case $\sigma^0 \neq 0$ the identity

$$
\int_0^t \int_E (z^{0,\varepsilon} \cdot \nabla) z^{\Delta,\varepsilon} \cdot s^{\Delta,\varepsilon} \, dx \, dr = - \int_0^t \int_E (z^{0,\varepsilon} \cdot \nabla) s^{\Delta,\varepsilon} \cdot z^{\Delta,\varepsilon} \, dx \, dr - \int_0^t \int_E \sigma^0 s^{\Delta,\varepsilon} \cdot z^{\Delta,\varepsilon} \, dx \, dr,
$$

where $(\Delta, o) \in \{(+,-), (-,+), (++,+,+)\}$, one arrives at the inequality

$$
\frac{1}{2} \sum_{\xi \in \{+,-\}} \|r^{\xi,\varepsilon}(\cdot, t)\|_{L^2(E)}^2 + \varepsilon \lambda^+ \sum_{\xi \in \{+,-\}} \int_0^t \int_E \nabla \times r^{\xi,\varepsilon} \cdot \nabla \times r^{\xi,\varepsilon} \, dx \, dr + \varepsilon \lambda^- \sum_{(\Delta, o) \in \{(+,-), (-,+), (++,+,+)\}} \int_0^t \int_{\partial E} (\|h^{\Delta,\varepsilon}\|_\varepsilon - \|A^{\Delta,\varepsilon} r^{0,\varepsilon}\|_\varepsilon) \cdot r^{\Delta,\varepsilon} \, dS \, dr
$$

$$
+ \varepsilon \lambda^+ \sum_{\xi \in \{+,-\}} \int_0^t \int_{\partial E} f^{\xi} \, dS \, dr + \varepsilon \lambda^- \sum_{(\Delta, o) \in \{(+,-), (-,+), (++,+,+)\}} \int_0^t \int_{\partial E} f^{\Delta,\varepsilon} \, dS \, dr + \frac{1}{2} \sum_{\xi \in \{+,-\}} \int_0^t \int_E \sigma^0 |r^{\xi,\varepsilon}|^2 \, dx \, dr,
$$

with

$$
f^{\Delta,\varepsilon} := (\rho^\Delta (r^{+\varepsilon,\varepsilon}, r^{-\varepsilon,\varepsilon}) + g^{\Delta,\varepsilon}) \cdot r^{\Delta,\varepsilon}.
$$
Since $\nabla \cdot r^{\pm,\varepsilon} = 0$ in $\mathcal{E}$ and $r^{\pm,\varepsilon} \cdot n = 0$ on $\partial \mathcal{E}$, the inequality (2.1) provides
\[
\|r^{\pm,\varepsilon}\|_{H^1(\mathcal{E})} \leq C \left( \|\nabla \times r^{\pm,\varepsilon}\|_{L^2(\mathcal{E})} + \|r^{\pm,\varepsilon}\|_{L^2(\mathcal{E})} \right),
\]
which in turn yields
\[
\frac{1}{2} \sum_{\sigma \in \{+,-\}} \|r^{\pm,\varepsilon}(\sigma, t)\|^2_{L^2(\mathcal{E})} + \varepsilon \sum_{\sigma \in \{+,-\}} (\lambda^+ \Box \lambda^-) \int_0^t \|\left( r^{\pm,\varepsilon} \Box r^{-\varepsilon}(\cdot, r) \right)\|^2_{H^1(\mathcal{E})} \, dr
\leq \sum_{(\lambda, \sigma) \in \{(+,+)\}} \int_0^t \int_{\partial \mathcal{E}} \left( |\partial_{\mathcal{E}} A^{\pm,\varepsilon} \cdot r^{\pm,\varepsilon}(\cdot, r)| - \|\nabla A^{\pm,\varepsilon} \cdot r^{\pm,\varepsilon}(\cdot, r)\| \right) \, dS dr
+ \varepsilon \lambda^+ \sum_{\sigma \in \{+,-\}} \int_0^t \int_{\partial \mathcal{E}} J^{\pm,\varepsilon} \, dS dr + \varepsilon \lambda^- \sum_{(\lambda, \sigma) \in \{(+,+)\}} \int_0^t \int_{\partial \mathcal{E}} J^{\pm,\varepsilon} \, dS dr
+ \varepsilon C \sum_{\sigma \in \{+,-\}} \lambda^+ \Box \lambda^- \int_0^t \|r^{\pm,\varepsilon}(\cdot, r)\|^2_{L^2(\mathcal{E})} \, dr,
\]
where $C > 0$ depends on $\lambda^\pm$ and the fixed quantity $\max_{(x,s) \in \overline{\mathcal{E}} \times [0,T]} |\sigma^0(x, s)|$. The boundary integrals containing $J^{\pm,\varepsilon}$ are treated by applying for $f, h \in H^1(\mathcal{E})$ and $\eta > 0$ the estimate
\[
\left| \int_{\partial \mathcal{E}} f \cdot h \, dS \right| \leq C(\eta) (\|f\|^2_{L^2(\mathcal{E})} + \|h\|^2_{L^2(\mathcal{E})}) + \eta (\|f\|^2_{H^1(\mathcal{E})} + \|h\|^2_{H^1(\mathcal{E})}).
\]
Thus, for $t \in [0, T/\varepsilon]$ and arbitrary $\eta > 0$ one has
\[
\left| \int_{\partial \mathcal{E}} J^{\pm,\varepsilon}(\lambda, x, t) \, dS(x) \right| \
\leq \eta \left( \|g^{\pm,\varepsilon}(\cdot, t)\|^2_{H^1(\mathcal{E})} + \|r^{\pm,\varepsilon}(\cdot, t)\|^2_{H^1(\mathcal{E})} + \|r^{-\varepsilon}(\cdot, t)\|^2_{H^1(\mathcal{E})} \right) + C(\eta) \left( \|g^{\pm,\varepsilon}(\cdot, t)\|^2_{L^2(\mathcal{E})} + \|r^{\pm,\varepsilon}(\cdot, t)\|^2_{L^2(\mathcal{E})} + \|r^{-\varepsilon}(\cdot, t)\|^2_{L^2(\mathcal{E})} \right).
\]
Consequently, by selecting $\eta > 0$ small, employing Lemma 3.26, and applying Grönwall’s inequality in (3.85), one arrives at (3.80).

\begin{corollary}
The functions $z^{\pm,\varepsilon}$ fixed in the beginning of Section 3 satisfy
\[
\|z^{\pm,\varepsilon}(\cdot, T/\varepsilon)\|_{L^2(\mathcal{E})} = O(\varepsilon^{3/2}).
\]
\end{corollary}

\begin{proof}
We again use that $\mu^\pm$ has been fixed via Lemma 3.17 (or Lemma 3.18) with $r = 6$ and $k = 4$, while noting that $\lim_{a \to +\infty} a^{-1/2} \ln(a) = 0$. Therefore, by combining (3.80) with (3.7), Lemma 3.21 and Lemma 3.23, one arrives at
\[
\|z^{\pm,\varepsilon}(\cdot, T/\varepsilon)\|_{L^2(\mathcal{E})} \leq \sqrt{\varepsilon} \|\nabla \psi^{\pm,\varepsilon}(\cdot, T/\varepsilon)\|_{L^2(\mathcal{E})} + \varepsilon \|\nabla w^{\pm,\varepsilon}(\cdot, T/\varepsilon)\|_{L^2(\mathcal{E})} + \|r^{\pm,\varepsilon}(\cdot, T/\varepsilon)\|_{L^2(\mathcal{E})}
\leq \varepsilon^{3/2} \varepsilon^{-\frac{1}{2}} \|\psi^{\pm,\varepsilon}(\cdot, T/\varepsilon)\|_{H^{1.0}_E} + \varepsilon^{3/2} \|w^{\pm,\varepsilon}(\cdot, T/\varepsilon)\|_{H^{0.2,0}_E} + O(\varepsilon^{3/2})
= O(\varepsilon^{3/2}).
\]
\end{proof}
3.6 Controlling towards arbitrary smooth states

Let \( \overline{z}_1^\pm \in C^\infty(\overline{E}) \cap H(E) \) be arbitrarily fixed. The foregoing arguments for reaching the zero state can be modified in order to approach \( \overline{z}^\pm_1 \) approximately. The idea is similar to that for controlling towards smooth trajectories as described in [17, Section 5] for the Navier-Stokes equations. First, the ansatz (3.7) is modified such that for \( z^{\pm,\varepsilon} \) on the time interval \([0, T]\) one chooses an expansion of the form

\[
z^{\pm,\varepsilon} = z^0 + \sqrt{\varepsilon} \left[ \theta^\pm \right]_\varepsilon + \varepsilon \nabla \theta^{\pm,\varepsilon} + \varepsilon \left[ w^\pm \right]_\varepsilon + \varepsilon r^{\pm,\varepsilon},
\]

while on \([T, T/\varepsilon]\) it is assumed that

\[
z^{\pm,\varepsilon} = \sqrt{\varepsilon} \left[ \theta^\pm \right]_\varepsilon + \varepsilon \nabla \theta^{\pm,\varepsilon} + \varepsilon \left[ w^\pm \right]_\varepsilon + \varepsilon r^{\pm,\varepsilon}.
\]

In (3.86), the profiles \( \overline{z}^{\pm,1} \) are bounded in \( L^\infty((0, T); H^1(E)) \) and resolve together with controls \( \overline{\xi}^{\pm,1} \in \mathcal{U}_T \), which obey for all \( t \in (0, T) \) the conditions

\[
\text{supp}(\overline{\xi}^{\pm,1}(\cdot, t)) \subseteq \overline{E} \setminus \overline{\Omega},
\]

the linear controllability problem

\[
\begin{cases}
\partial_t \overline{z}^{\pm,1} + (\overline{z}^{\pm,1} \cdot \nabla) z^0 + (z^0 \cdot \nabla) \overline{z}^{\pm,1} + \nabla p^{\pm,1} = \overline{\xi}^{\pm,1} + (\lambda^{\pm} + \lambda^{\mp}) \Delta z^0 & \text{in } E_T, \\
\nabla \cdot \overline{z}^{\pm,1} = 0 & \text{in } E_T, \\
\overline{z}^{\pm,1} \cdot n = 0 & \text{on } \Sigma_T, \\
\overline{z}^{\pm,1}(\cdot, 0) = z^0_0 & \text{in } E, \\
\overline{z}^{\pm,1}(\cdot, T) = \overline{z}^{\pm}_1 & \text{in } E.
\end{cases}
\]

(3.88)

Such controls \( \overline{\xi}^{\pm,1} \) can be constructed as in Lemma 3.8, see also step 2) in the proof of Lemma 3.18. In particular, all bounds for \( \overline{z}^{\pm,1} \) and \( \overline{\xi}^{\pm,1} \) are independent of \( \varepsilon > 0 \), since this parameter does not appear in (3.88).

Because \( \overline{z}^{\pm,1}_1 \) are smooth and independent of time, by analysis similar to Section 3.5 one can infer the remainder estimates

\[
\|r^{\pm,\varepsilon}(\cdot, T/\varepsilon)\| = O(\varepsilon^4).
\]

As a result, the rescaled functions \( z^{\pm,\varepsilon}(x, t) := \varepsilon^{-1} z^{\pm,\varepsilon}(x, \varepsilon^{-1} t) \) satisfy

\[
\|z^{\pm,\varepsilon}(x, T) - \overline{z}^{\pm,1}_1\|_{L^2(E)} = O(\varepsilon^4).
\]

4 Conclusion of the main results

In order to relax the assumption \( u_0, B_0 \in H^3(E) \cap W(E) \) introduced in Section 3, we connect initial data from \( H(E) \) by a weak controlled trajectory to a state which belongs to \( H^3 \cap W(E) \). This is stated in Lemma 4.1 below, and a proof of this argument, which is a modification of [17, Lemma 9], will be outlined in Appendix B.
Lemma 4.1. Consider the problem (2.7), when \( N = 3 \) assume that \( M_1 \) and \( L_2 \) are symmetric, \( L_1 = M_2 = 0 \), and \( \Omega \) is simply-connected. For any given \( T^* > 0 \) and \( \mathbf{u}_0, B_0 \in H(\mathcal{E}) \), there exists a smooth function \( C_{T^*} > 0 \) with \( C_{T^*}(0) = 0 \) such that a weak Leray-Hopf solution \( (\mathbf{u}, B) \in \mathfrak{X}_T^2 \) to (2.7) obeys for a time \( t_{reg} \in [0, T^*] \) the estimate

\[
\|\mathbf{u}(\cdot, t_{reg})\|_{H^1(\mathcal{E})} + \|B(\cdot, t_{reg})\|_{H^1(\mathcal{E})} \leq C_{T^*} \left( \|\mathbf{u}_0\|_{L^2(\mathcal{E})} + \|B_0\|_{L^2(\mathcal{E})} \right).
\]

Let the control time \( T_{ctrl} > 0 \), the initial states \( \mathbf{u}_0, B_0 \in L^2(\Omega) \), the terminal states \( \mathbf{u}_1, B_1 \in L^2(\Omega) \), and any \( \delta > 0 \) be arbitrarily fixed. Now, Theorem 1.2 is obtained by means of the following steps:

1) The physical domain \( \Omega \) is extended to \( \mathcal{E} \), as explained in Section 2, and the weak formulation given in Section 2.4.1 is chosen. In the case b) of Theorem 1.2, the extension is explicitly given by \( \mathcal{E} = A_i^3 \) as discussed in Section 3.2.

2) By Lemma 4.1, there exists \( T_1 \in (0, T_{ctrl}/4) \) such that a weak Leray-Hopf solution \( (\mathbf{u}, B) \) to (2.7) with initial data \( (\mathbf{u}_0, B_0) \), zero forces \( \xi = \eta = 0 \) obeys \( \mathbf{u}(\cdot, T_1), B(\cdot, T_1) \in H^3(\mathcal{E}) \cap H(\mathcal{E}) \).

3) By a density argument, one can select states \( \overline{\mathbf{u}}_1, \overline{B}_1 \in C^\infty(\overline{\mathcal{E}}; \mathbb{R}^N) \cap W(\mathcal{E}) \) with

\[
\|\overline{\mathbf{u}}_1 - \mathbf{u}_1\|_{L^2(\Omega)} + \|\overline{B}_1 - B_1\|_{L^2(\Omega)} < \delta/2.
\]

4) The arguments in Section 3 are carried out with \( T = T_{ctrl} - T_1 \), initial data \( \mathbf{u}(\cdot, T_1), B(\cdot, T_1) \) and target states \( \overline{\mathbf{u}}_1, \overline{B}_1 \). This provides controls \( \xi, \eta \in \mathcal{U}_T \) such that all corresponding weak Leray-Hopf solutions \( (\overline{\mathbf{u}}, \overline{B}) \) to (2.7) with initial data \( \mathbf{u}(\cdot, T_1) \) and \( B(\cdot, T_1) \) satisfy

\[
\|\overline{\mathbf{u}}(\cdot, T_{ctrl} - T_1) - \overline{\mathbf{u}}_1\|_{L^2(\mathcal{E})} + \|\overline{B}(\cdot, T_{ctrl} - T_1) - \overline{B}_1\|_{L^2(\mathcal{E})} < \delta/2.
\]

5) At \( t = T_1 \), a weak Leray-Hopf solution \( (\overline{\mathbf{u}}, \overline{B}) \) chosen via step 4 is glued to a weak Leray-Hopf solution \( (\mathbf{u}, B) \) from step 2. After renaming, one obtains a weak Leray-Hopf solution \( (\mathbf{u}, B) \) to (2.7), defined on the whole time interval \([0, T_{ctrl}]\), which starts from the initial data \((\mathbf{u}_0, B_0)\) and satisfies

\[
\|\mathbf{u}(\cdot, T_{ctrl}) - \mathbf{u}_1\|_{L^2(\Omega)} + \|B(\cdot, T_{ctrl}) - B_1\|_{L^2(\Omega)} < \delta.
\]

It remains to conclude Theorem 1.6. In a similar way as before, this is achieved by means of the following steps:

1) The physical domain \( \Omega \) is extended to \( \mathcal{E} \) as described in Section 2, but now the weak formulation in Section 2.4.3 is considered. When Lemma 4.1 cannot be applied, the extended initial data are chosen with \( \mathbf{u}_0, B_0 \in W(\mathcal{E}) \cap H^3(\mathcal{E}) \). Otherwise, in order to reach a divergence-free state, one takes zero forces \( \xi = \eta = 0 \) and defines \( \sigma^\pm(x, t) := \beta(t)(\nabla \cdot z_\pm^\alpha)(x) \), with \( \beta \in C^\infty(\mathbb{R}) \) obeying \( \beta(0) = 1 \) and \( \beta(t) = 0 \) for all \( t \geq \widehat{T} := T_{ctrl}/8 \). A corresponding weak solution on \([0, \widehat{T}]\) is denoted by \((\overline{z}^+, \overline{z}^+)\) and it follows that \( \overline{z}^+(\overline{T}) \in H(\mathcal{E}) \).
2) Any weak Leray-Hopf solution \((z^+, z^-)\) to (2.18) with \(\sigma^z = 0\) and zero forces \(\xi = \eta = 0\) also obeys by means of the transformation
\[
(u, B) = \frac{1}{2} \left( z^+ + z^-, \frac{1}{\sqrt{\mu}}(z^+ - z^-) \right)
\]
the weak form introduced for (2.7), and vice versa. Therefore, either one can take \(T_1 = 0\), or Lemma 4.1 provides a time \(T_1 \in [\hat{T}, T_{\text{ctrl}}/4]\) such that a weak Leray-Hopf solution \((z^+, z^-)\) to (2.18) with initial data \(\widehat{z}^+\) and zero forces \(\xi = \eta = 0\) obeys \(z^z(\cdot, T_1) \in H^3(E) \cap H(\mathcal{E})\).

3) As before, by density one can choose regular states \(\overline{z}_1^\pm \in C_0^\infty(\overline{E}; \mathbb{R}^N) \cap H(\mathcal{E})\) with
\[
||\overline{z}_1^\pm - (u_1 \pm \sqrt{\mu}B_1)||_{L^2(\Omega)} < \delta/4.
\]

4) Now, Section 3 is applied with \(T = T_{\text{ctrl}} - T_1\), initial data \(z^\pm(\cdot, T_1)\), and target states \(\overline{z}_1^\pm\). As a result, there are controls \(\xi, \eta \in \mathcal{U}_T\) such that all corresponding weak Leray-Hopf solutions \((\overline{z}^+, \overline{z}^-)\) to (2.18) satisfy
\[
||\overline{z}^+ - T_{\text{ctrl}} - T_1 - \overline{z}_1^\pm ||_{L^2(\mathcal{E})} < \delta/4.
\]

5) By a gluing argument, one obtains a weak Leray-Hopf solution \((z^+, z^-)\) to (2.18) on \([0, T_{\text{ctrl}}]\), starting from the initial data \(z_0^\pm\) and satisfying
\[
||z^+ - (u_1 \pm \sqrt{\mu}B_1)||_{L^2(\Omega)} < \delta/2.
\]

### A Boundary layer estimates

In this appendix, the \textit{a priori} estimates for proving Lemma 3.12 are outlined. Since a more general context will not influence the subsequent arguments, we can take arbitrary \(z^{0,0} \in C^\infty(\overline{E} \times [0, T]; \mathbb{R}^N)\) with \(z^{0,0} \cdot n = 0\) along \(\partial \mathcal{E}\) and \(\text{supp}(z^{0,0}(x, \cdot)) \subset (0, T)\) for all \(x \in \overline{E}\). However, of interest is here the case where \(z^{0,0} = z^{+,0}\). We then consider in \(E \times (0, T) \times \mathbb{R}^+\) the equations
\[
\partial_t v^\pm - \partial_{zz}(\lambda^\pm v^\pm + \lambda^\pm v^-) + \left[(z^{0,0} \cdot \nabla) v^\pm + (v^\top \cdot \nabla) z^{0,0}\right]_{\text{tan}} + \tau^\pm \partial_z v^\pm = 0 \tag{A.1}
\]

together with the initial and boundary conditions
\[
\begin{aligned}
\left\{ \begin{array}{l}
\partial_z v^\pm(x, t, 0) - [\partial_z v^\pm(x, t, 0) \cdot n(x)] \hspace{0.5cm} n(x) = g^\pm(x, t), \quad x \in \overline{E}, t \in (0, T), \\
v^\pm(x, t, 0) \cdot n(x) = 0, \\
v^\pm(x, t, z) \rightarrow 0, \hspace{0.5cm} \text{as} \hspace{0.5cm} z \rightarrow +\infty, \\
v^\pm(x, 0, z) = 0,
\end{array} \right.
\end{aligned} \tag{A.2}
\]

While also more general data could be chosen, here we take the cutoff \(\chi_{\partial \mathcal{E}}\) defined in (3.31) and consider
\[
f^\pm(x, t) = -\frac{z^{0,0}(x, t) \cdot n(x)}{\varphi_{\mathcal{E}}(x)}, \quad g^\pm(x, t) = 2\chi_{\partial \mathcal{E}} \mathcal{N}^\pm(z^{+,0}, z^{-0})(x, t).
\]
Multiplying in (A.1) with \( n \), one may similarly to [29, Section 5] establish energy estimates which imply for all \((x, t, z) \in \overline{E} \times [0, T] \times \mathbb{R}_+\) that
\[
[\nu^+(x, t, z) \pm \nu^-(x, t, z)] \cdot n(x) = 0.
\]

The goal consists now of showing the following lemma.

**Lemma A.1.** For any choice of \(k, m_1, m_2, m_3 \in \mathbb{N}_0\), there exists a constant
\[
C = C(E, \lambda^\pm, k, m_1, m_2, m_3, T, z^\pm, 0, M^\pm, L^\pm) > 0
\]
(A.3)
such that every smooth solution \((\nu^+, \nu^-)\) to (A.1) and (A.2) obeys
\[
\|\nu^+\|_{W^{m_2, \infty}((0,T),H^{k,m_1,m_3}_{L})} + \|\nu^-\|_{H^{m_2,0}((0,T),H^{k,m_1,m_3+1}_{L})} \leq C,
\]
(A.4)
with \( C = 0 \) when \( g^\pm = 0 \).

**Proof.** The idea for proving Lemma A.1 is based on [29]. All constants \( C > 0 \) which appear during the estimates can depend on \( E, \lambda^\pm, k, m_1, m_2, m_3, T, z^\pm, 0 \), and \( \rho^\pm \).

**Step 1. Estimates for \( \partial_x^\alpha \partial_z^\gamma \nu^\pm \).** We take in (A.1) the partial derivatives \( \partial_x^\alpha \partial_z^\gamma \) for \( \gamma \in \mathbb{N}_0 \) and \( \alpha \in \mathbb{N}_0^N \). As a result,
\[
\partial_t (\partial_x^\alpha \partial_z^\gamma \nu^\pm) = \partial_{zz} \partial_x^\alpha \partial_z^\gamma (\lambda^\pm \nu^+ + \lambda^\mp \nu^-) - z \partial_x^\alpha \sum_{l=0}^{\gamma} \binom{\gamma}{l} \partial_x^l \partial_z^{\gamma-l} \partial_z \nu^\pm
\]
(A.5)

\[
- \partial_x^\alpha \sum_{l=0}^{\gamma} \binom{\gamma}{l} \left[ (\partial_x^l z^{\pm,0} \cdot \nabla) \partial_z^{\gamma-l} \nu^\pm + (\partial_x^l \nu^\mp \cdot \nabla) \partial_z^{\gamma-l} z^{\pm,0} \right]_{\text{tan}}.
\]

Furthermore, multiplying (A.5) for arbitrary \( k \in \mathbb{N} \) with \((1 + z^{2k})\partial_x^\alpha \partial_z^\gamma \nu^\pm \) and integrating in \((x, z)\) over \( \overline{E} \times \mathbb{R}_+ \) yields
\[
\frac{1}{2} \frac{d}{dt} \int_{\overline{E}} \int_{\mathbb{R}_+} (1 + z^{2k}) |\partial_x^\alpha \partial_z^\gamma \nu^\pm(x, t, z)|^2 \, dz \, dx = I_1^k(t) - I_2^k(t) - I_3^{+,a}(t) - I_3^{+,b}(t)
\]
with the right-hand side being given by
\[
I_1^k := \int_{\overline{E}} \int_{\mathbb{R}_+} (1 + z^{2k}) \partial_{zz} \partial_x^\alpha \partial_z^\gamma (\lambda^\pm \nu^+ + \lambda^\mp \nu^-) \cdot \partial_x^\alpha \partial_z^\gamma \nu^\pm \, dz \, dx,
\]
\[
I_2^k := \int_{\overline{E}} \int_{\mathbb{R}_+} (z + z^{2k+1}) \partial_x^\alpha \sum_{l=0}^{\gamma} \binom{\gamma}{l} \partial_x^l \partial_z^{\gamma-l} \partial_z \nu^\pm \cdot \partial_x^\alpha \partial_z^\gamma \nu^\pm \, dz \, dx,
\]
(A.6)
\[
I_3^{+,a} := \int_{\overline{E}} \int_{\mathbb{R}_+} (1 + z^{2k}) \partial_x^\alpha \sum_{l=0}^{\gamma} \binom{\gamma}{l} \left[ (\partial_x^l z^{\pm,0} \cdot \nabla) \partial_z^{\gamma-l} \nu^\pm \right]_{\text{tan}} \cdot \partial_x^\alpha \partial_z^\gamma \nu^\pm \, dz \, dx,
\]
\[
I_3^{+,b} := \int_{\overline{E}} \int_{\mathbb{R}_+} (1 + z^{2k}) \partial_x^\alpha \sum_{l=0}^{\gamma} \binom{\gamma}{l} \left[ (\partial_x^l \nu^\mp \cdot \nabla) \partial_z^{\gamma-l} z^{\pm,0} \right]_{\text{tan}} \cdot \partial_x^\alpha \partial_z^\gamma \nu^\pm \, dz \, dx.
\]
We focus now on the situations where $\gamma > 0$ and $|\alpha| > 0$. For the terms $I_{11}^\pm$, integration by parts in $z$ leads to

\[
I_{11}^\pm = - \int_E \int_{\mathbb{R}_+} (1 + z^{-2k})(\lambda^+ \partial_z \partial_x^\alpha \partial_t^\gamma v^+ + \lambda^- \partial_z \partial_x^\alpha \partial_t^\gamma v^-) \cdot \partial_z \partial_x^\alpha \partial_t^\gamma v^\pm \, dz \, dx
\]

\[
-2k \int_E \int_{\mathbb{R}_+} z^{2k-1}(\lambda^+ \partial_z \partial_x^\alpha \partial_t^\gamma v^+ + \lambda^- \partial_z \partial_x^\alpha \partial_t^\gamma v^-) \cdot \partial_x^\alpha \partial_t^\gamma v^\pm \, dz \, dx
\]

\[
- \int_E (\lambda^\pm \partial_z \partial_x^\alpha \partial_t^\gamma v^+ + \lambda^- \partial_z \partial_x^\alpha \partial_t^\gamma v^-) (x, t, 0) \cdot \partial_x^\alpha \partial_t^\gamma v^\pm (x, t, 0) \, dx
\]

(A.7)

By means of Young’s inequality and the identities $2v^\pm = (v^+ + v^-) \pm (v^+ - v^-)$, one obtains

\[
|I_{11}^\pm| \leq \sum_{\square \in \{+,-\}} \frac{\lambda^\pm \lambda^-}{8} \int_E \int_{\mathbb{R}_+} (1 + z^{-2k})|\partial_z \partial_x^\alpha \partial_t^\gamma (v^+ \square v^-)|^2 \, dz \, dx
\]

\[
+ C \int_E \int_{\mathbb{R}_+} (1 + z^{-2k})|\partial_x^\alpha \partial_t^\gamma v^\pm|^2 \, dz \, dx. \quad \text{(A.8)}
\]

For a constant $C_0 > 0$ which vanishes when $g^\pm = 0$, one can infer

\[
|I_{12}^\pm| \leq \left| \int_E \partial_x^\alpha \partial_t^\gamma (\lambda^\pm g^+ + \lambda^- g^-) (x, t) \cdot \left[ \int_{\mathbb{R}_+} \partial_z \partial_x^\alpha \partial_t^\gamma v^\pm (x, t, z) \, dz \right] \, dx \right|
\]

\[
\leq \int_E \int_{\mathbb{R}_+} \left| (1 + z^{-2k})^{-\frac{1}{2}} \partial_x^\alpha \partial_t^\gamma (\lambda^\pm g^+ + \lambda^- g^-) (x, t) \cdot (1 + z^{-2k})^{\frac{1}{2}} \partial_z \partial_x^\alpha \partial_t^\gamma v^\pm (x, t, z) \right| \, dz \, dx
\]

\[
\leq \sum_{\square \in \{+,-\}} \frac{\lambda^\pm \lambda^-}{8} \int_E \int_{\mathbb{R}_+} (1 + z^{-2k})|\partial_z \partial_x^\alpha \partial_t^\gamma (v^+ \square v^-)|^2 \, dz \, dx + C_0.
\]

(A.9)

Thus, after collecting (A.7)–(A.9), one obtains the bound

\[
\sum_{\square \in \{+,-\}} \left( |I_{11}^\pm| + \frac{\lambda^\pm \lambda^-}{4} \int_E \int_{\mathbb{R}_+} (1 + z^{-2k})|\partial_z \partial_x^\alpha \partial_t^\gamma (v^+ \square v^-)|^2 \, dz \, dx \right)
\]

\[
\leq C \sum_{\square \in \{+,-\}} \|\partial_t^\gamma v^\pm\|^2_{L^2(0,T;\mathbb{R})} + C_0. \quad \text{(A.10)}
\]
Concerning $I_2^\pm$, expanding the derivatives leads to

$$
|I_2^\pm| \leq \left| \sum_{l=0}^{\gamma} \sum_{0 < \kappa \leq \alpha} \left( \gamma \right) \left( \alpha \right) \int_{E \subset \mathbb{R}^n} \left( z + z^{2k+1} \right) \partial_x^\kappa \partial_{y_l}^\pm \partial_x^\alpha - \kappa \partial_{y_l}^\gamma \cdot \partial_x^\kappa \partial_{y_l}^\gamma \cdot \partial_{y_l}^\pm \partial_x^\alpha \partial_{y_l}^\gamma \partial_{y_l}^\pm \, dz \, dx \right|
$$

$$
+ \left| \sum_{l=0}^{\gamma} \left( \gamma \right) \int_{E \subset \mathbb{R}^n} \left( z + z^{2k+1} \right) \partial_{y_l}^\gamma \partial_{y_l}^\pm \partial_x^\alpha \partial_{y_l}^\gamma \partial_{y_l}^\pm \partial_x^\alpha \partial_{y_l}^\gamma \partial_{y_l}^\pm \, dz \, dx \right|

$$

which implies

$$
|I_2^\pm| \leq C \gamma \sum_{l=0}^{\gamma} \sum_{0 < \kappa \leq \alpha} \int_{E \subset \mathbb{R}^n} \left( z + z^{2k+1} \right) \| \partial_x^\kappa \partial_{y_l}^\pm \|_{L^2(E)} \| \partial_x^\alpha - \kappa \partial_{y_l}^\gamma \partial_{y_l}^\pm \|_{L^2(E)} \| \partial_x^\alpha \partial_{y_l}^\gamma \partial_{y_l}^\pm \|_{L^2(E)} \, dz
$$

$$
+ C \gamma \sum_{l=0}^{\gamma} \int_{E \subset \mathbb{R}^n} \left( z + z^{2k+1} \right) \| \partial_x^\alpha \partial_{y_l}^\gamma \partial_{y_l}^\pm \partial_x^\alpha \partial_{y_l}^\gamma \partial_{y_l}^\pm \|_{L^2(E)} \, dz
$$

$$
+ C \gamma \int_{E \subset \mathbb{R}^n} \left( 1 + (2k + 1)z^{2k} \right) \| \partial_x^\alpha \partial_{y_l}^\gamma \partial_{y_l}^\pm \|_{L^2(E)}^2 \, dz.
$$

(A.11)

Hence, the interpolation inequality $\| \cdot \|_{L^2(E)} \leq C\| \cdot \|_{L^1(E)}^{1/2} \| \cdot \|_{H^1(E)}^{1/2}$ and (A.11) yield for arbitrary $\eta > 0$ that

$$
|I_2^\pm| \leq C \gamma \sum_{l=0}^{\gamma} \sum_{0 < \kappa \leq \alpha} \int_{E \subset \mathbb{R}^n} \left\{ \left( 1 + z^{2k+1} \right) \| \partial_x^\kappa \partial_{y_l}^\pm \|_{L^2(E)} \| \partial_x^\alpha - \kappa \partial_{y_l}^\gamma \partial_{y_l}^\pm \|_{L^2(E)} \right\}

$$

$$
\times \left( 1 + z^{2k+1} \right) \| \partial_x^\alpha \partial_{y_l}^\gamma \partial_{y_l}^\pm \partial_x^\alpha \partial_{y_l}^\gamma \partial_{y_l}^\pm \|_{L^2(E)}^2 \right) \, dz
$$

$$
+ \eta \sum_{l=1}^{\gamma} \| \partial_{y_l}^\gamma \partial_{y_l}^\pm \|_{k+1,|\alpha|,1,E}^2 + C(\eta) \| \partial_{y_l}^\gamma \partial_{y_l}^\pm \|_{k,|\alpha|,0,E}^2 + C \| \partial_{y_l}^\gamma \partial_{y_l}^\pm \|_{k,|\alpha|,0,E}^2
$$

$$
\leq \sum_{l=0}^{\gamma} \sum_{0 < \kappa \leq \alpha} C \| \partial_{y_l}^\alpha \partial_{y_l}^\pm \|_{k+2,|\alpha|,1,E}^2 \| \partial_{y_l}^\gamma \partial_{y_l}^\pm \|_{k,|\alpha|,1,E}^2 \| \partial_{y_l}^\gamma \partial_{y_l}^\pm \|_{k,|\alpha|,0,E}^2
$$

$$
+ \eta \sum_{l=1}^{\gamma} \| \partial_{y_l}^\gamma \partial_{y_l}^\pm \|_{k+1,|\alpha|,1,E}^2 + C(\eta) \| \partial_{y_l}^\gamma \partial_{y_l}^\pm \|_{k,|\alpha|,0,E}^2.
$$

(A.12)

Above in (A.12), we used the elementary inequality $z + z^{2k+1} \leq (1 + z^k)(1 + z^{k+1})$. Then, the bounds for $I_2^\pm$ are concluded by

$$
|I_2^\pm| \leq \sum_{l=0}^{\gamma} \sum_{0 < \kappa \leq \alpha} \left( C(\eta) \left( \| \partial_{y_l}^\alpha \partial_{y_l}^\pm \|_{k+2,|\alpha|,1,E}^2 + \| \partial_{y_l}^\gamma \partial_{y_l}^\pm \|_{k,|\alpha|,0,E}^2 \right) + \eta \| \partial_{y_l}^\gamma \partial_{y_l}^\pm \|_{k,|\alpha|,1,E}^2 \right)
$$

$$
+ \eta \sum_{l=1}^{\gamma} \| \partial_{y_l}^\gamma \partial_{y_l}^\pm \|_{k+1,|\alpha|,1,E}^2 + C(\eta) \| \partial_{y_l}^\gamma \partial_{y_l}^\pm \|_{k,|\alpha|,0,E}^2.
$$

(A.13)
It remains to treat $I^{\pm,a}_3$ and $I^{\pm,b}_3$. For a vector field $\mathbf{z}$ on $\mathcal{E}$, we denote by $D^m(\mathbf{z})$ arbitrary linear combinations of components of $\mathbf{z}$ and derivatives of such, which are taken in $x$ and are of order $\leq m$, while the coefficients can depend on $n$. Given a multi-index $\vec{\alpha}$ with $|\vec{\alpha}| = \vec{m} \in \mathbb{N}$, the relations $v^\pm \cdot n = 0$ imply
\[ n \cdot \partial_{x}^\iota \partial_{x}^\alpha \mathbf{v}^\pm = D^{\vec{m} - 1}(\partial_{x}^\iota \mathbf{v}^\pm) = \partial_{x}^\iota D^{\vec{m} - 1}(\mathbf{v}^\pm). \tag{A.14} \]
Therefore, in view of (A.14) and the definition of the tangential part $[\cdot]_{\text{tan}}$, one may write
\[
I^{\pm,a}_3 = \sum_{l=0}^{y} \left( \gamma \right) \int_{E} \int_{\mathbb{R}^n} (1 + z^{2l}) D^{[\alpha]}(\partial_{x}^\iota z^\pm) D^{[\alpha]}(\partial_{x}^\alpha \mathbf{v}^\pm) \, dz \, dx
\]
\[ + \sum_{l=0}^{y} \left( \gamma \right) \int_{E} \int_{\mathbb{R}^n} (1 + z^{2l}) \left( (\partial_{x}^\iota z^\pm \cdot \nabla) \partial_{x}^\alpha \mathbf{v}^\pm \cdot \mathbf{n} \right) D^{[\alpha]}(\partial_{x}^\iota \mathbf{v}^\pm) \, dz \, dx
\]
\[ + \sum_{l=0}^{y} \left( \gamma \right) \int_{E} \int_{\mathbb{R}^n} (1 + z^{2l}) (\partial_{x}^\iota z^\pm \cdot \nabla) \partial_{x}^\alpha \partial_{x}^\iota \mathbf{v}^\pm \cdot \partial_{x}^\alpha \partial_{x}^\iota \mathbf{v}^\pm dx \, dz, \tag{A.15} \]
and integration by parts implies that the second line is of the same type as the first one, thus it follows that
\[
I^{\pm,a}_3 = \sum_{l=0}^{y} \left( \gamma \right) \int_{E} \int_{\mathbb{R}^n} (1 + z^{2l}) D^{[\alpha]}(\partial_{x}^\iota z^\pm) D^{[\alpha]}(\partial_{x}^\iota \mathbf{v}^\pm) \, dz \, dx
\]
\[ + \sum_{l=1}^{y} \left( \gamma \right) \int_{E} \int_{\mathbb{R}^n} (1 + z^{2l}) (\partial_{x}^\iota z^\pm \cdot \nabla) \partial_{x}^\alpha \partial_{x}^\iota \mathbf{v}^\pm \cdot \partial_{x}^\alpha \partial_{x}^\iota \mathbf{v}^\pm dx \, dz
\]
\[ + \int_{E} \int_{\mathbb{R}^n} (1 + z^{2l}) (z^\pm \cdot \nabla) \partial_{x}^\alpha \partial_{x}^\iota \mathbf{v}^\pm \cdot \partial_{x}^\alpha \partial_{x}^\iota \mathbf{v}^\pm dx \, dz. \]
Due to $z^\pm \cdot \mathbf{n} = 0$ on $\partial \mathcal{E}$, the last integral in (A.15) reads
\[
\int_{E} \int_{\mathbb{R}^n} (1 + z^{2l}) (z^\pm \cdot \nabla) \partial_{x}^\alpha \partial_{x}^\iota \mathbf{v}^\pm \cdot \partial_{x}^\alpha \partial_{x}^\iota \mathbf{v}^\pm dx \, dz
\]
\[ = - \frac{1}{2} \int_{E} \int_{\mathbb{R}^n} (1 + z^{2l}) |\partial_{x}^\alpha \partial_{x}^\iota \mathbf{v}^\pm|^2 (\nabla \cdot z^\pm) \, dz \, dx
\]
such that (A.15) eventually implies the bound
\[
|I^{\pm,a}_3| \leq C \sum_{l=1}^{y} \|\partial_{x}^\iota \mathbf{v}^\pm\|_{k,|\alpha|+1,0,\mathcal{E}}^2 + C \|\partial_{x}^\alpha \mathbf{v}^\pm\|^2_{k,|\alpha|,0,\mathcal{E}}. \tag{A.16} \]
When it comes to $I^{\pm,b}_3$, the corresponding estimates are less demanding compared to those for $I^{\pm,a}_3$ and one finds
\[
|I^{\pm,a}_3| + |I^{\pm,b}_3| \leq C \left( \|\partial_{x}^\iota \mathbf{v}^\pm\|^2_{k,|\alpha|,0,\mathcal{E}} + \|\partial_{x}^\alpha \mathbf{v}^\pm\|^2_{k,|\alpha|,0,\mathcal{E}} \right)
\]
\[ + C \sum_{l=0}^{y-1} \left( \|\partial_{x}^\iota \mathbf{v}^\pm\|^2_{k,|\alpha|+1,0,\mathcal{E}} + \|\partial_{x}^\alpha \mathbf{v}^\pm\|^2_{k,|\alpha|+1,0,\mathcal{E}} \right). \tag{A.17} \]
In order to collect the foregoing estimates, for fixed \( m_1, m_2 \in \mathbb{N}_0 \), we sum in (A.10), (A.13), (A.16), and (A.17) over all \( |\alpha| \leq m_1 \) and \( \gamma \leq m_2 \). Moreover, we denote

\[
\Phi^\pm := C(\eta) \sum_{\gamma=0}^{m_2} \delta^\pm_{0,m_1} \| \partial^\gamma_\nu v^\pm \|_{H^{k+2,\max(m_1-1,0)}_E}^2 + \| \partial^\gamma_\nu v^\pm \|_{H^{k,m_1+1}_E}^2 + \eta \sum_{\gamma=0}^{m_2} \| \partial^\gamma_\nu v^\pm \|_{H^{k,m_1+1}_E}^2,
\]

as a result, one obtains the estimate

\[
\sum_{\Box \in \{+, -\}} \left( \partial_i \sum_{\gamma=0}^{m_2} \| \partial^\gamma_\nu v^\Box \|_{H^{k,m_1+1}_E}^2 + \sum_{\gamma=0}^{m_2} \frac{\lambda^{+} \Box \lambda^{-}}{2} \| \partial^\gamma_\nu (v^+ \Box v^-) \|_{H^{k,m_1+1}_E}^2 \right) \leq 2 (\Phi^+ + \Phi^-). \tag{A.18}
\]

On the right-hand side of (A.18), all terms containing norms of the spaces

\[
H^{k+2,\max(m_1-1,0)}_E, \quad H^{k+1,m_1+1}_E, \quad H^{k,m_1+1}_E
\]

disappear in the respective base cases when \( m_1 = 0 \) or \( m_2 = 0 \). Thus, inductively with respect to \( m_1 \) and \( m_2 \), and by using a Grönwall argument for (A.18) with \( \eta > 0 \) sufficiently small, one can obtain

\[
\partial^\gamma_\nu v^\pm \in L^\infty((0, T); H^{k,m_1+1}_E), \quad \gamma \in \{0, \ldots, m_2\}
\]

and the estimate (A.4) when \( m_3 = 0 \).

**Step 2. Estimates for \( \partial^\alpha_\xi \partial^\gamma_\nu \partial^\beta_\rho v^\pm \).** From the equation (A.2) and the regularity obtained in Step 1, one can estimate the boundary values of \( \partial^\alpha_\xi \partial^\gamma_\nu \partial^\beta_\rho v^\pm \) at \( z = 0 \) in \( L^\infty((0, T); H^{m}(E)) \) for any \( m \in \mathbb{N} \) by a constant \( C > 0 \) of the type stated in (A.3). Therefore, after acting on (A.2) with \( \partial_\zeta \), one obtains the \( W^{m,\infty}((0, T); H^{k,m_1+1}_E) \cap H^{m_2}((0, T); H^{k,m_1+3}_E) \) estimates for \( v^\pm \) by analysis similar to that given in Step 1. The boundary values of \( \partial^\alpha_\xi \partial^\gamma_\nu \partial^\beta_\rho v^\pm \) at \( z = 0 \) are then again bounded via (A.2) and the previous step. After acting with \( \partial^2_\zeta \) on (A.2), one can derive the \( W^{m,\infty}((0, T); H^{k,m_1+3}_E) \cap H^{m_2}((0, T); H^{k,m_1+5}_E) \) estimates for \( v^\pm \). By induction on the order of \( m_3 \), the proof of Lemma A.1 is concluded.

**Remark A.2.** In Section 3, at most \( v^\pm \in W^{m_2,\infty}((0, T); H^{4,5,3}_E) \) is required.
B  Proof of Lemma 4.1

In this appendix, a proof of Lemma 4.1 is outlined. At the level of a priori estimates, we proceed along the lines of [17, Lemma 9] and [10, Lemma 2.1], where the corresponding problems for the Navier-Stokes equations and Boussinesq systems have been considered. While these a priori estimates are valid for \( N \in \{2, 3\} \), they are only justified for sufficiently smooth solutions.

Since the boundary conditions are not symmetric in general, it appears out of reach to employ a standard Galerkin method with a basis consisting of eigenvectors for coupled Stokes type problems with Navier slip-with-friction boundary conditions. However, when such a basis is available, then one can carry out the estimates given in Appendix B.1 below at the level of Galerkin approximations and obtain Lemma 4.1 with \( N = 3 \). Such a basis has been utilized for instance in [26], where the domain is simply-connected, \( M_1 = L_2 = M \) with \( M \) being positive symmetric, and \( M_2 = L_1 = 0 \). As discussed in [43], one also obtains such a basis when the symmetric matrix \( M \) is non-positive. Then, for the uncoupled case where \( L_1 = M_2 = 0 \), we can by the argument from [43] allow also symmetric matrices \( M_1 \) and \( L_2 \) with \( M_1 \neq L_2 \).

Whether general symmetric boundary conditions, where \( M_1, L_2 \) are symmetric and \( v_1 L_1 = v_2 M_2 \top \), would still lead to a suitable basis has to be further investigated. Due to the coupled boundary conditions, one could not simply utilize basis functions as found in [43].

In general two dimensional domains and without imposing restrictions on the boundary conditions, we can take a different approach for justifying the a priori estimates given in Appendix B.1 below. First, the initial data are approximated in \( L^2(E) \) by \( W(E) \cap H^2(E) \) functions satisfying the Navier slip-with-friction boundary conditions. Then, a sequence of strong solutions of the type \( L^\infty(0, T; H^2(E) \cap W(E)) \) which converges to a weak Leray-Hopf solution is constructed. This idea relies on the assumption \( N = 2 \), which guarantees that the sequence of strong solutions is defined on a fixed time interval \([0, T]\). This argument will be provided in [37].

B.1  Estimates for sufficiently regular solutions

In order to write down the estimates for \( u \) and \( B \) simultaneously, the symmetric notations from Section 2.4.2 are employed. If \((u, B) \in X_T \times X_T^\ast\) is a weak Leray-Hopf solution to (2.7), then the functions \( z^\pm = u \pm \sqrt{\mu} B \) obey the energy inequality (2.17) and a corresponding weak formulation for the Elsasser system

\[
\begin{align*}
\partial_t z^\pm - \Delta(\lambda^\pm z^+ + \lambda^\mp z^-) + (z^\pm \cdot \nabla) z^\pm + \nabla p^\pm &= 0 \quad \text{in } E_T^\ast, \\
\nabla \cdot z^\pm &= 0 \\
\nabla \cdot n &= 0 \\
\n(\nabla \times z^\pm) \times n &= \rho^\pm(z^+, z^-) \\
z^\pm(\cdot, 0) &= z_0^\pm
\end{align*}
\]

on \( \Sigma_T \),

on \( \Sigma_T^\ast \),

in \( E \).

To begin with, a priori estimates for a related stationary problem are shown based on known results for the Navier-Stokes equations.
Lemma B.1. Let \( k \in \mathbb{N}_0, f^\pm \in H^k(\mathcal{E}), \) tangential \( b^\pm \in H^{k+1/2}(\partial \mathcal{E}) \) and friction operators \( M^\pm, L^\pm \in C^0(\partial \mathcal{E}; \mathbb{R}^{N \times N}) \) be arbitrarily fixed. Then, every solution \((Z^+, Z^-, P^+, P^-)\) with \( Z^\pm \in H^{k+1}(\mathcal{E}) \) to the coupled Stokes type system:

\[
\begin{cases}
-\Delta(\lambda^+ Z^+ + \lambda^- Z^-) + \nabla P^\pm = f^\pm & \text{in } \mathcal{E}, \\
\nabla \cdot Z^\pm = 0 & \text{in } \mathcal{E}, \\
Z^\pm \cdot n = 0 & \text{on } \partial \mathcal{E}, \\
(\nabla \times Z^\pm) \times n + [M^\pm Z^+ + L^\pm Z^-]_{\text{tan}} = b^\pm & \text{on } \partial \mathcal{E}
\end{cases}
\]

(B.2)

obeys the estimate

\[
\sum_{\sigma \in \{+,-\}} \left( \|Z^\sigma\|_{H^{k+2}(\mathcal{E})} + \|P^\sigma\|_{H^{k+1}(\mathcal{E})} \right) 
\leq C \sum_{\sigma \in \{+,-\}} \left( \|f^\sigma\|_{H^k(\mathcal{E})} + \|b^\sigma\|_{H^{k+1/2}(\partial \mathcal{E})} + \|Z^\sigma\|_{H^{k+1}(\mathcal{E})} \right). \tag{B.3}
\]

Proof. The proof is reduced to known results for Stokes equations under Navier slip-with-friction boundary conditions. Indeed, if we first assume the uncoupled boundary conditions where \( M^\pm = L^\pm = 0, \) then \( U := Z^+ + Z^- \) and \( V := Z^+ - Z^- \) both obey independent Stokes problems under Navier slip-with-friction boundary conditions. Thus, from [25, Pages 90-94] one has for each \( k \geq 0 \) the estimates

\[
\|U\|_{H^{k+2}(\mathcal{E})} + \|P^+ - P^-\|_{H^{k+1}(\mathcal{E})} 
\leq C \left( \|f^+ - f^-\|_{H^k(\mathcal{E})} + \|b^+ - b^-\|_{H^{k+1/2}(\partial \mathcal{E})} + \|U\|_{H^{k+1}(\mathcal{E})} \right)
\]

and

\[
\|V\|_{H^{k+2}(\mathcal{E})} + \|P^+ - P^-\|_{H^{k+1}(\mathcal{E})} 
\leq C \left( \|f^+ - f^-\|_{H^k(\mathcal{E})} + \|b^+ - b^-\|_{H^{k+1/2}(\partial \mathcal{E})} + \|V\|_{H^{k+1}(\mathcal{E})} \right),
\]

which imply (B.3) by means of the triangle inequality. For the general case, we start with \( k = 0 \) and observe that every solution \((Z^+, Z^-, P^+, P^-)\) to (B.2) satisfies

\[
\begin{cases}
-\Delta(\lambda^+ Z^+ + \lambda^- Z^-) + \nabla P^\pm = f^\pm & \text{in } \mathcal{E}, \\
\nabla \cdot Z^\pm = 0 & \text{in } \mathcal{E}, \\
Z^\pm \cdot n = 0 & \text{on } \partial \mathcal{E}, \\
(\nabla \times Z^\pm) \times n = \tilde{b}^\pm & \text{on } \partial \mathcal{E},
\end{cases}
\]

(B.4)

with

\[
\tilde{b}^+ := b^+ - [M^+ Z^+ + L^+ Z^-]_{\text{tan}}, \quad \tilde{b}^- := b^- - [M^- Z^+ + L^- Z^-]_{\text{tan}}.
\]
Since $\tilde{b}^\pm \in H^{1/2}(\partial\mathcal{E})$, after applying to (B.4) the result for uncoupled boundary conditions explained above, one finds
\[
\sum_{\square \in \{+,-\}} \left( \|Z^\square\|_{H^2(\mathcal{E})} + \|P^\square\|_{H^1(\mathcal{E})} \right) \leq C \sum_{\square \in \{+,-\}} \left( \|f^\square\|_{L^2(\mathcal{E})} + \|b^\square\|_{H^{1/2}(\partial\mathcal{E})} + \|Z^\square\|_{H^1(\mathcal{E})} \right).
\]
Inductively, if (B.3) is assumed to be true for any fixed $k \in \mathbb{N}$, one has $\tilde{b}^\pm \in H^{k+3/2}(\partial\mathcal{E})$ and the known estimates for (B.4) lead to (B.3) with $k$ being replaced by $k + 1$. $\square$

**Remark B.2.** The idea of the proof of (B.2), which is to reduce the *a priori* estimates for (B.1) to *a priori* estimates for corresponding Stokes problems with non-homogeneous Navier slip (without friction) boundary conditions, also provides $W^{2,p}(\mathcal{E})$ estimates for $p \in (1, +\infty)$. Indeed, if $Z^\pm \in W^{1,p}(\mathcal{E})$ and $b^\pm \in W^{1-1/p,p}(\partial\mathcal{E})$, then $\tilde{b}^\pm \in W^{1-1/p,p}(\partial\mathcal{E})$ in (B.4) and one can apply [4, Theorem 4.1].

In what follows, the operator $\mathbb{P}$ denotes the Leray projector in $L^2(\mathcal{E})$ onto $H(\mathcal{E})$ and thus, for any selection $h^+, h^- \in H(\mathcal{E}) \cap H^2(\mathcal{E})$ with $N^\pm(h^+, h^-) = 0$, Lemma B.1 provides
\[
\|h^\pm\|_{H^2(\mathcal{E})}^2 \leq C \left( \sum_{\square \in \{+,-\}} \|\mathbb{P}\Delta(h^\pm + \lambda^\pm h^-)\|_{L^2(\mathcal{E})}^2 + \|h^\square\|_{H^1(\mathcal{E})}^2 \right).
\]

The proof of Lemma 4.1 is now completed by means of the following steps.

**Step 1. Basic energy estimates when $t \in (0, T^*/3)$.** We introduce for a positive parameter $\kappa > 0$ the quantity
\[
F^\kappa(t) := \frac{1}{2} \sum_{\square \in \{+,-\}} \left( \frac{d}{dt}\|Z^\square(\cdot, t)\|_{L^2(\mathcal{E})}^2 + \kappa(\lambda^\square \Delta h^-) \|\nabla \times (z^\square \nabla z^-)(\cdot, t)\|_{L^2(\mathcal{E})}^2 \right).
\]
Then, the strong energy inequality (2.17) with $\xi^\pm = 0$ provides
\[
\int_0^t F^1(s) \, ds \leq \sum_{\Delta, \square \in \{(+,-), (-,+), (-,-)\}} \int_0^t \int_{\partial\mathcal{E}} \left( \lambda^\square \rho^\Delta(z^+, z^-) + \lambda^\square \rho^\Delta(z^+, z^-) \cdot z^\partial \, dS \right) \, ds.
\]
Furthermore, by (2.1) and trace estimates, together with $\rho^\pm \in C^\infty(\partial\mathcal{E}; \mathbb{R}^{N \times 2N})$ being fixed, one has for small $\eta > 0$ and $\square, \Delta \in \{+,-\}$ the bound
\[
\int_{\partial\mathcal{E}} \rho^\square(z^+, z^-) \cdot z^\Delta \, dS \leq C \sum_{\square, \Delta \in \{+,-\}} \|z^\partial\|_{H^1(\mathcal{E})} \|z^\square\|_{L^2(\mathcal{E})} \leq \sum_{\square, \Delta \in \{+,-\}} \left( \eta \|\nabla \times z^\square\|_{L^2(\mathcal{E})}^2 + C(\eta) \|z^\square\|_{L^2(\mathcal{E})}^2 \right).
\]
Thus, for $\eta > 0$ sufficiently small it follows
\[\int_0^t F^{1/2}(s) \, ds \leq C(\eta) \int_0^t \left( \|z^+(\cdot, s)\|_{L^2(\mathcal{E})}^2 + \|z^-\|_{L^2(\mathcal{E})}^2 \right) \, ds.\]

As a result, by employing (2.1) similarly as in Section 3.5.2 and further utilizing Grönwall’s inequality, one obtains for $t \in (0, T^*)$ the energy estimate
\[\sum_{\delta \in \{+, -\}} \left( \|z^\delta(t, \cdot)\|^2_{L^2(\mathcal{E})} + \frac{(\lambda^+ \cdot \lambda^-)}{2} \int_0^t \|z^\delta(\cdot, s)\|^2_{H^1(\mathcal{E})} \, ds \right) \leq C \sum_{\delta \in \{+, -\}} \|z^\delta_0\|^2_{L^2(\mathcal{E})}, \quad (B.6)\]

Therefore, by a contradiction argument, there exists $C_1 > 0$ and a possibly small time $t_1 \in [0, T^*/3]$ for which
\[\|z^+(\cdot, t_1)\|_{H^1(\mathcal{E})} \leq \sqrt{\frac{3C_1}{T^*}} \left( \|z^+_0\|^2_{L^2(\mathcal{E})} + \|z^-_0\|^2_{L^2(\mathcal{E})} \right).\]

**Step 2. Higher order \textit{a priori} estimates when} $t \in (t_1, 2T^*/3)$. We apply the Leray projector $P$ in (B.1) and multiply with $P \Delta z^\pm$. Subsequently, the results are added up and integrated over $\mathcal{E}$. Hereto, we denote for $\delta > 0$ the auxiliary function
\[G^\delta(t) := \frac{1}{2} \sum_{\delta \in \{+, -\}} \left( \frac{d}{dt} \|\nabla \times z^\delta(\cdot, t)\|_{L^2(\mathcal{E})}^2 + \delta(\lambda^+ \cdot \lambda^-) \|\Delta(z^\delta \cdot z^-)(\cdot, t)\|_{L^2(\mathcal{E})}^2 \right),\]

and the nonlinear terms
\[J^+(s) := \int_{\mathcal{E}} \left| P(z^+(x, s) \cdot \nabla)z^+(x, s) \cdot P\Delta z^+(x, s) \right| \, dx.\]

**Lemma B.3.** Assume that $M_1, L_2$ are symmetric and $v_1 L_1 = v_2 M_2^\top$. For any small $\eta > 0$, and a constant $C = C(\eta)$ which is reciprocal to $\eta$, it holds
\[\int_{t_1}^t G^{1/2}(s) \, ds \leq \int_{t_1}^t J^+(s) \, ds + \int_{t_1}^t J^-(s) \, ds + \sum_{\delta \in \{+, -\}} \left( \eta \|\nabla \times z^\delta(\cdot, t)\|_{L^2(\mathcal{E})}^2 + C(1, \eta) \|z^\delta_0\|^2_{L^2(\mathcal{E})} \right), \quad (B.7)\]

**Proof.** One has to estimate several boundary integrals of the form
\[\int_{t_1}^t \int_{\partial \mathcal{E}} \partial_t z \cdot \bar{M} \cdot \nabla z \, dS \, ds,

66
with $\mathbf{M}^{\Box, \Box} \in C^\infty(\mathcal{E}^c; \mathbb{R}^{N \times N})$ and $\Box, \Box \in \{+,-\}$. Thanks to the symmetry assumptions, it follows that

$$
\int_{t_1}^t \int_{\partial \mathcal{E}} (v_1 M_1 u \cdot \partial_t u + v_2 L_2 B \cdot \partial_t B) \, dSds \\
\leq \frac{1}{2} \int_{t_1}^t \partial_t \int_{\partial \mathcal{E}} (v_1 M_1 u \cdot u + v_2 L_2 B \cdot B) \, dSds \\
\leq \eta \left( \|u(\cdot, t)\|_{H^1(\mathcal{E})}^2 + \|B(\cdot, t)\|^2_{L^2(\mathcal{E})} \right) + C(\eta) \left( \|u(\cdot, t)\|_{L^2(\mathcal{E})}^2 + \|B(\cdot, t)\|_{L^2(\mathcal{E})}^2 \right) \\
+ C \left( \|u(\cdot, t)\|_{H^1(\mathcal{E})}^2 + \|B(\cdot, t)\|_{H^1(\mathcal{E})}^2 \right),
$$

for arbitrary $\eta > 0$, and similarly one can treat

$$
\int_{t_1}^t \int_{\partial \mathcal{E}} (v_2 M_2 u \cdot \partial_t B + v_1 L_1 B \cdot \partial_t u) \, dSds = \nu_1 \int_{t_1}^t \partial_t \int_{\partial \mathcal{E}} L_1 B \cdot u \, dSds.
$$

Therefore, (B.7) can be inferred from (2.1) and the basic energy estimate (B.6).

**Remark B.4.** When $M_1, M_2, L_1, L_2 \in C^\infty(\mathcal{E}^c; \mathbb{R}^{N \times N})$ are arbitrary, as mentioned in the proof for [17, Lemma 9], one can resort to parallel energy estimates. Hereon, one additionally multiplies in (B.1) with $\partial_t z^\pm$, which, in combination with the estimates that arise from multiplying with $\mathbb{P}\Delta z^\pm$, allows to absorb norms of the form $\|z^\pm\|_Y$, where

$$
Y := H^1((t_1, t); L^2(\mathcal{E})) \cap L^2((t_1, t); H^2(\mathcal{E})).
$$

Thus, similarly to [25, Page 995], one can use interpolation arguments.

It remains to further bound the integrals $J^\pm$ in (B.7). Applying inequalities of Hölder, Gagliardo–Nirenberg and Young, one obtains for any small constant $\eta > 0$ the bound

$$
J^\pm \leq \| (\nabla \cdot \nabla) z^\pm \|_{L^2(\mathcal{E})} \| \mathbb{P}\Delta z^\pm \|_{L^2(\mathcal{E})} \\
\leq C \| z^\pm \|_{H^1(\mathcal{E})} \| \nabla z^\pm \|_{L^2(\mathcal{E})} \| \mathbb{P}\Delta z^\pm \|_{L^2(\mathcal{E})} \\
\leq C(\eta) \| z^\pm \|_{H^1(\mathcal{E})} \| z^\pm \|_{L^2(\mathcal{E})} \| z^\pm \|^2_{H^1(\mathcal{E})} + \eta \| \mathbb{P}\Delta z^\pm \|^2_{L^2(\mathcal{E})} \\
\leq C(\eta) \| z^\pm \|^2_{H^1(\mathcal{E})} \| z^\pm \|^4_{H^1(\mathcal{E})} + \eta \left( \| z^\pm \|^2_{H^1(\mathcal{E})} + \| \mathbb{P}\Delta z^\pm \|^2_{L^2(\mathcal{E})} \right).
$$

Moreover, (2.1) and Young’s inequality with $1/3 + 4/6 = 1$ provides

$$
\| z^\pm \|^2_{H^1(\mathcal{E})} \| z^\pm \|^4_{H^1(\mathcal{E})} \leq C \sum_{\Box \in \{+,-\}} \left( \| z^\Box \|_{L^2(\mathcal{E})} + \| \nabla \times z^\Box \|_{L^2(\mathcal{E})} \right)^6,
$$

while (B.5) allows to infer

$$
\| z^\pm \|^2_{H^1(\mathcal{E})} \leq C \sum_{\Box \in \{+,-\}} \left( \| \mathbb{P}\Delta z^\Box \|^2_{L^2(\mathcal{E})} + \| \nabla \times z^\Box \|^2_{L^2(\mathcal{E})} + \| z^\Box \|^2_{L^2(\mathcal{E})} \right).
$$
Thus, by combining (B.7)–(B.10) and the basic energy estimate (B.6), one obtains
\[
\int_{t_1}^{t} G^{1/2}(s) \, ds \leq C(\eta) \sum_{\delta \in \{+,-\}} \int_{t_1}^{t} \left( \|z^\delta(\cdot, s)\|_{L^2(\mathcal{E})} + \|\nabla \times z^\delta(\cdot, s)\|_{L^2(\mathcal{E})} \right)^6 \, ds \\
+ \sum_{\delta \in \{+,-\}} \int_{t_1}^{t} \left( \eta \|\Delta z^\delta(\cdot, s)\|_{L^2(\mathcal{E})}^2 + C\|\nabla \times z^\delta(\cdot, s)\|_{L^2(\mathcal{E})}^2 \right) \, ds \\
+ \sum_{\delta \in \{+,-\}} \left( \eta \|\nabla \times z^\delta(\cdot, t)\|_{L^2(\mathcal{E})}^2 + C(C_1, \eta) \|z^\delta_0\|_{L^2(\mathcal{E})}^2 \right).
\]

Therefore, for sufficiently small parameters \(\delta_1, \delta_2 \in (0, 1)\), one arrives at
\[
\int_{t_1}^{t} F^{\delta_1}(s) + G^{\delta_2}(s) \, ds \\
\leq C \sum_{\delta \in \{+,-\}} \int_{t_1}^{t} \left( \|z^\delta(\cdot, s)\|_{L^2(\mathcal{E})} + \|\nabla \times z^\delta(\cdot, s)\|_{L^2(\mathcal{E})} \right)^6 \, ds \\
+ C \sum_{\delta \in \{+,-\}} \int_{t_1}^{t} \|\nabla \times z^\delta(\cdot, s)\|_{L^2(\mathcal{E})}^2 \, ds + C(C_1, \delta_1, \delta_2) \sum_{\delta \in \{+,-\}} \|z^\delta_0\|^2_{L^2(\mathcal{E})}. \tag{B.11}
\]

In order to apply Grönwall’s lemma in (B.11), first one utilizes the elementary inequality
\[
\sum_{\delta \in \{+,-\}} \left( \|z^\delta(\cdot, t)\|_{L^2(\mathcal{E})} + \|\nabla \times z^\delta(\cdot, t)\|_{L^2(\mathcal{E})} \right)^2 \\
\leq 2 \sum_{\delta \in \{+,-\}} \left( \|z^\delta(\cdot, t)\|^2_{L^2(\mathcal{E})} + \|\nabla \times z^\delta(\cdot, t)\|^2_{L^2(\mathcal{E})} \right),
\]
such that for \(t > t_1\) and sufficiently small \(c_1 > 0\) one has the estimate
\[
\sum_{\delta \in \{+,-\}} \left( \|z^\delta(\cdot, t)\|_{L^2(\mathcal{E})} + \|\nabla \times z^\delta(\cdot, t)\|_{L^2(\mathcal{E})} \right)^2 + c_1 \sum_{\delta \in \{+,-\}} \int_{t_1}^{t} \|z^\delta(\cdot, s)\|^2_{H^1(\mathcal{E})} \, ds \\
\leq \sum_{\delta \in \{+,-\}} \left( \frac{C(C_1, \eta) \|z^\delta_0\|^2_{L^2(\mathcal{E})}}{C} + C \int_{t_1}^{t} \left( \|z^\delta(\cdot, s)\|_{L^2(\mathcal{E})} + \|\nabla \times z^\delta(\cdot, s)\|_{L^2(\mathcal{E})} \right)^6 \, ds \right). 
\]

Thus, for a generic constant \(C = C(C_1, \eta) > 0\), the function
\[
\Phi(t) := C \sum_{\delta \in \{+,-\}} \left( \|z^\delta_0\|^2_{L^2(\mathcal{E})} + \int_{t_1}^{t} \left( \|z^\delta(\cdot, s)\|_{L^2(\mathcal{E})} + \|\nabla \times z^\delta(\cdot, s)\|_{L^2(\mathcal{E})} \right)^6 \, ds \right),
\]
obey\(\Phi/\Phi^3 \leq C\), where \(\Phi = \frac{d\Phi}{dt}\). Taking \(s_1 > 0\) small enough and integrating the latter differential inequality leads for \(t \in [t_1, t_1 + s_1]\) to
\[
\Phi(t)^2 \leq \frac{C^2 \left( \sum_{\delta \in \{+,-\}} \|z^\delta_0\|^2_{L^2(\mathcal{E})} \right)^2}{1 - 2(t - t_1) C^3 \left( \sum_{\delta \in \{+,-\}} \|z^\delta_0\|^2_{L^2(\mathcal{E})} \right)^2}.
\]
Consequently, for some constant $C_2 > 0$ and all $t \in [t_1, t_1 + s_1]$ one has the estimate

$$\sum_{\partial \in \{+,-\}} \left( \|z^\partial(\cdot, t)\|^2_{\mathcal{H}^1(E)} + c_1 \int_{t_1}^t \|z^\partial(\cdot, s)\|^2_{\mathcal{H}^1(E)} \, ds \right) \leq C_2 \sum_{\partial \in \{+,-\}} \|z^\partial\|^2_{L^2(E)}.$$ 

Therefore, there exists $t_2 \in (t_1, 2T^* / 3)$ and $C_3 > 0$ such that

$$\|z^+(\cdot, t_2)\|_{\mathcal{H}^1(E)} \leq \sqrt{\frac{C_3}{s_1}} \left(\|z_{0+}\|^2_{L^2(E)} + \|z_{0-}\|^2_{L^2(E)}\right)^{1/2}.$$ 

**Step 3. Additional estimates for $\partial_t z^\pm$ when $t \in (t_2, T^*)$.** By taking the derivative $\partial_t$ in (B.1), multiplying the resulting equations with $\partial_t z^\pm$ respectively, and integrating over $E$, one obtains for

$$H(t) := \frac{1}{2} \sum_{\partial \in \{+,-\}} \left( \frac{d}{dt} \|\partial_t z^\partial(\cdot, t)\|^2_{L^2(E)} + (\lambda^+ \Delta \lambda^-) \|\nabla \times (\partial_t z^\partial \partial_t z^-)(\cdot, t)\|^2_{L^2(E)} \right)$$

the estimate

$$H \leq -\int_E (\partial_t z^- \cdot \nabla) z^+ \cdot \partial_t z^+ \, dx - \int_E (\partial_t z^+ \cdot \nabla) z^- \cdot \partial_t z^- \, dx$$

$$+ \sum_{(\lambda, \partial) \in \{(+,-), (-,+)} \int_{\partial E} (\lambda^+ \rho^+ (\partial_t z^+, \partial_t z_-) + \lambda^- \rho^- (\partial_t z^+, \partial_t z_-)) \cdot \partial_t z^\partial \, ds.$$ 

Therefore, considerations similar to the previous steps lead for some constants $C_4 > 0$, $c_2 \in (0, 1)$, a possibly small time $s_2 \in (0, T^*/3)$ and all $t \in [t_2, t_2 + s_2]$ to

$$\|\partial_t z^\pm(\cdot, t)\|^2_{L^2(E)} + c_2 \int_{t_2}^t \|\partial_t z^\pm(\cdot, s)\|_{\mathcal{H}^1(E)} \, ds \leq C_4 \sum_{\partial \in \{+,-\}} \|\partial_t z^\partial(\cdot, t_2)\|^2_{L^2(E)},$$

noting that

$$\|\partial_t z^\pm(\cdot, t_2)\|^2_{L^2(E)} \leq \|(z^\pm \cdot \nabla) z^\pm(\cdot, t_2)\|^2_{L^2(E)} + \|\nabla p^\pm(\cdot, t_2)\|^2_{L^2(E)}$$

$$+ \|\Delta (\lambda^\pm z^\pm + \lambda^\mp z^-)(\cdot, t_2)\|^2_{L^2(E)}$$

$$\leq C \sum_{\partial \in \{+,-\}} \|z^\partial(\cdot, t_2)\|^2_{H^1(E)}.$$ 

Hence, there exists a time $t_3 \in [t_2, t_2 + s_2]$ and a constant $C_5 > 0$ such that

$$\|\partial_t z^\pm(\cdot, t_3)\|_{H^1(E)} \leq \sqrt{\frac{C_5}{s_2}} \sum_{\partial \in \{+,-\}} \|\partial_t z^\partial(\cdot, t_2)\|^2_{L^2(E)}.$$ 

**Step 4. Conclusions.** The proof of Lemma 4.1 is concluded by shifting the time derivative and the nonlinear terms in (B.1) to the right-hand side, followed by multiple applications of Lemma B.1 and Remark B.2. This first provides $L^\infty([t_2, t_2 + s_2]; H^2(E))$ bounds for $z^\pm$ and subsequently $H^3(E)$ bounds for $z^\pm(\cdot, t_3)$.
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