\( \delta N \) Formalism in Anisotropic Inflation and Large Anisotropic Bispectrum and Trispectrum

Ali Akbar Abolhasani\(^1\) \textsuperscript{†} Razieh Emami\(^1\) \textsuperscript{‡} Javad T. Firouzjaee\(^{1,2}\) \textsuperscript{§} and Hassan Firouzjahi\(^3\)

\(^{1}\) School of Physics, Institute for Research in Fundamental Sciences (IPM), P. O. Box 19395-5531, Tehran, Iran and 
\(^{2}\) School of Astronomy, Institute for Research in Fundamental Sciences (IPM), P. O. Box 19395-5531, Tehran, Iran 
(Dated: September 24, 2013)

We present a consistent \( \delta N \) formalism for curvature perturbations in anisotropic cosmological backgrounds. We employ our \( \delta N \) formalism to calculate the power spectrum, the bispectrum and the trispectrum in models of anisotropic inflation with the background gauge fields in Bianchi I universe. Our results coincide exactly with the recent results obtained from in-in formalism. To satisfy the observational constraints the anisotropies generated on power spectrum are kept small but large orientation-dependent non-Gaussianities can be generated. We study the Suyama-Yamaguchi inequality for the amplitudes of the bispectrum and the trispectrum in the presence of anisotropic shapes.

I. INTRODUCTION

Recently there have been many interests in anisotropic inflation. This is partially motivated from the cosmological observations indicating some statistical anisotropies on cosmic microwave background (CMB) \(^1\). Although the statistical significance of the violation of statistical isotropy is not high \(^2,3\), but nonetheless the possibility of having statistically anisotropic seed perturbations are intriguing. One useful method to quantify the statistical anisotropy is to write the curvature perturbation power spectrum \( P_{\zeta} \) in Fourier space for mode \( \vec{k} \) as \(^3\) \( P_{\zeta} = P_0(1 + g_s(\vec{k}, \hat{n})^2) \) in which \( \hat{n} \) is the preferred direction in sky. Constraints from CMB and large scale structure indicate that \( |g_s| \lesssim 0.4 \) \(^3,6\).

The best method to generate anisotropic perturbations is to employ gauge fields during inflation. However, due to conformal invariance of \( U(1) \) gauge fields in FRW background, the background gauge field energy density and their perturbations will be quickly diluted during inflation. Therefore, one has to break conformal invariance such that the gauge field energy density will not decay during inflation. One popular method is to consider a time-dependent gauge kinetic coupling such that the \( U(1) \) action has the form \( \Delta L = -\frac{f(\phi)^2}{4} F_{\mu\nu} F^{\mu\nu} \) in which \( \phi \) is the inflaton field and \( F_{\mu\nu} \) is the \( U(1) \) gauge field strength. Furthermore, in order for the gauge field perturbations to be scale-invariant one has to choose \( f(\phi) \propto a^{-2} \) in which \( a(t) \) is the scale factor. These models in the context of anisotropic inflation and magneto-genesis were studied in great details in literature \(^7,42\).

An interesting model of anisotropic inflation was proposed in \(^16\) in which with \( f(\phi) \propto a^{-2} \) the inflationary system admits an attractor solution in which the gauge field energy density, i.e. the electric field energy density, and the metric anisotropy reaches a small but cosmologically observable level. The cosmological perturbations for this model were studied in \(^49\). Similarly, the cosmological perturbation analysis for a different model of anisotropic inflation \(^31\) involving a complex inflaton field charged under the \( U(1) \) gauge field were performed in \(^49\). These analysis are based on standard in-in formalism which proved technically difficult due to anisotropic background. On the other hand, experiences with \( \delta N \) formalism \(^50,57\) in models of inflation with scalar fields showed that \( \delta N \) analysis are technically much easier to handle when calculating the curvature perturbations and their correlations such as power spectrum and bispectrum. This is specially the case for models in which there are not much interactions when the modes of interest leave the horizon and physically interesting effects, such as non-Gaussianities, originate from local type interactions once the modes are outside the horizon. Therefore it will be very helpful to extend the standard \( \delta N \) formalism \(^52,56\) to models of anisotropic backgrounds such as \(^16\). This is one of our main goal in this work. The applications of \( \delta N \) in models with vector fields were also studied in \(^52,59\).

There have been works in the literature employing the conventional \( \delta N \) formalism for the models containing vector...
or gauge fields but the effects of anisotropic background were not taken into account, i.e. the gauge field is treated on
the same footing as the scalar fields in an FRW background. In this work we present a consistent \( \delta N \) formalism for
anisotropic backgrounds such as in \cite{10} in which the background metric is in the form of Bianchi I. After presenting
our \( \delta N \) formalism we calculate the power spectrum and reproduce exactly the results in \cite{45, 48}. We also calculate
the bispectrum which coincides exactly with the results of \cite{18}.

Planck is expected to release its data soon. Any detection or otherwise of primordial non-Gaussianities from
Planck will have significant implications for inflationary model buildings. Simple models of inflation predict almost
scale-invariant and almost Gaussian perturbations. Therefore, any detection of primordial non-Gaussianity will go a
long way to rule out or classify different inflationary scenarios. Non-Gaussianity may take different shapes in different
models, for a review see \cite{60, 61}. In models of inflation based on scalar fields the shapes of Bispectrum and Trispectrum
are statistically isotropic. However, in models of anisotropic inflation, one obtains new shapes which are anisotropic.

As an important consistency condition for single field inflation, a detection of local form bispectrum in the squeezed
limit can rule out all single field models of inflation provided the system reaches the attractor solution \cite{62, 63} so one
can neglect the evolution of curvature perturbations on super-horizon scales and the curvature perturbations have the
initial Bunch-Davies vacuum state \cite{64, 65}. As a different consistency condition, the Suyama-Yamaguchi (SY)
inequality \cite{66, 67, 68, 69} between the amplitude of the Bispectrum in the squeezed limit, \( f_{NL} \), and the amplitude of
the trispectrum in the collapsed limit, \( \tau_{NL} \), are expected to hold generally in models of inflation based on scalar fields.
It is an interesting question to see if the SY inequality holds when the primordial perturbations are not statistically
isotropic. We will study this question in the context of anisotropic inflation.

The rest of the paper is organized as follows. In Section II we present our \( \delta N \) formalism. In Section III we
study a model of anisotropic inflation which provides a non-trivial set up to employ our \( \delta N \) formalism. In Section
IV we present the bispectrum and the trispectrum analysis for the anisotropic inflation model and study the SY
inequality. The conclusion and discussions are given in Section V. We relegate some technical details into Appendices.

**Note added:** While this work was in its final stages the paper \cite{70} appeared which has overlaps in Bispectrum
and Trispectrum analysis in Section IV with this work.

## II. \( \delta N \) Formalism for Anisotropic Backgrounds

In this section we extend the \( \delta N \) formalism \cite{50, 57} to anisotropic backgrounds. First we present the background
fields equations. After presenting the general metric perturbations, we look into the fields equations using a gradient
expansion method, which is an expansion in \( \epsilon \) defined via

\[
\epsilon \equiv \frac{k}{aH},
\]

in which \( k \) represents the wave number in Fourier space. We demonstrate that the separate universe picture works,
that is, in the limit \( \epsilon \ll 1 \) the background fields equations are locally hold inside each homogenized patch. This proof
is valid to all order in perturbation theory.

### A. Background Equations

Our background is the Bianchi I metric with the scale factors \( a_1(t), a_2(t) \) and \( a_3(t) \)

\[
ds^2 = -dt^2 + a_1(t)^2 dx^2 + a_2(t)^2 dy^2 + a_3(t)^2 dz^2.
\]

We adopt the notations used in \cite{71} in which

\[
H_i(t) = \frac{\dot{a}_i}{a_i}, \quad H \equiv \frac{1}{3} \sum_{i=1}^{3} H_i,
\]

in which \( H_i \) is the Hubble expansion rate for the \( i \)-th spatial direction, \( i = 1, 2, 3 \) and a dot indicates the derivative
with respect to \( t \).
The components of background Ricci tensor are

\[ R^0_0 = 3 \dot{H} + \sum_k H_k^2 \]  
\[ R^0_i = 0 \]  
\[ R^i_j = \delta^i_j \left( \dot{H}_i + 3H H_i \right) \].

(2.4)  
(2.5)  
(2.6)

The background Ricci scalar is

\[ R = 6 \dot{H} + 18H^2 - \sum_{k > k'} H_k H_{k'} \].

(2.7)

To solve the Einstein fields equations we have to specify our energy momentum tensor. The general energy momentum tensor \( T_{\mu\nu} \) for an imperfect fluid has the form \[ T_{\mu\nu} = (\rho + p) u_\mu u_\nu + p g_{\mu\nu} + q_\mu u_\nu + u_\mu q_\nu + \pi_{\mu\nu} \] (2.8)

supplemented with the following conditions

\[ q_\mu u^\mu = 0, \quad \pi^\mu_{\mu} = 0, \quad \pi_{\mu\nu} = \pi_{\nu\mu}, \quad \pi_{\mu\nu} u^\nu = 0, \]

where \( u^\mu \) is the fluid’s four-vector velocity, \( \rho \) is the relativistic energy density, \( p \) is the isotropic pressure, \( \pi_{\mu\nu} \) is the trace-free anisotropic pressure (stress) and \( q^\mu \) usually is referred to as “heat conduction”, which is also the energy flux relative to \( u^\mu \).

The special case of a perfect fluid is identified with \( \pi_{\mu\nu} = q^\mu = 0 \) so we recover the standard form of \( T_{\mu\nu} \) for the perfect fluid

\[ T_{\mu\nu} = (\rho + p) u_\mu u_\nu + p g_{\mu\nu} \] (perfect fluid).

(2.9)

For the comoving coordinate associated with the fluid we have

\[ u^\mu = (1, \vec{0}) \quad , \quad u_\mu = (-1, \vec{0}), \]

(2.10)

so the Einstein equations can be read as

\[ 3H^2 \equiv \sum_{i > j} \dot{H}_i \dot{H}_j = \frac{\dot{\rho}}{M_p^2} \]

(2.11)

\[ \dot{T}^0_0 = \ddot{q}_i = 0 \]

(2.12)

\[ M_p^2 \dot{H}_i = -3M_p^2 \dot{H} \dot{H}_i + \frac{1}{2}(\dot{\rho} - \ddot{\rho}) + \dddot{\pi}_i \]

(2.13)

Here we have used the convention that \( \dot{H}_i \) represents the background Hubble expansion rates while \( \dot{\rho}, \ddot{\rho} \) and so on represent the background fluid’s properties. We also defined \( \hat{H} \) as the effective Hubble expansion rate appearing in Friedmann equation, Eq. (2.11). Note, \( \hat{H} \) here should not be confused with the Hubble expansion rate defined for conformal time usually used in literature.

Finally, the energy conservation equation \( u_\mu \nabla_\nu T^{\mu\nu} = 0 \) results in

\[ - u_\mu \nabla_\nu T^{\mu\nu} = 3H (\dot{\rho} + \ddot{\rho}) + \dot{H}_i \dddot{\pi}_i \delta^i_j = 0. \]

(2.14)

in which again we have \( \dddot{H} = \sum_i \dot{H}_i / 3. \)

Note that, in this model, the Hubble parameter appearing in \((0,0)\) component of Einstein equation, \( \hat{H} \), and the Hubble parameter appearing in continuity equation, \( \dot{H}_i \) are not equal. The difference between them is given by

\[ \frac{\dddot{H}^2 - \dot{H}^2}{H^2} = \frac{1}{6} \sum_i (\dot{H} - \dot{\bar{H}}_i)^2. \]

(2.15)

As a result \( \dddot{H} > \dot{H} \).
1. Example: U(1) gauge fields in an expanding background

As an example of non-perfect fluid with anisotropic pressure and heat conduction, consider the standard U(1) gauge field theory in an expanding background. This theory will be the base of anisotropic inflation in next section. The action is

\[ L_{em} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \]

in which \( F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu} \) is the field strength associated with the U(1) gauge field \( A_{\mu} \).

The electric field, \( E_\mu \), and the magnetic field, \( H_\mu \), are given by,

\[ E_\mu = F_{\mu\nu} u^\nu \quad (2.16) \]

and

\[ H_\rho = \frac{1}{2} \eta_{\rho\mu\sigma} u^\mu F^{\nu\sigma} . \quad (2.17) \]

The electromagnetic energy-momentum tensor, \( T^{\mu\nu}_{em} \), is

\[ T^{\mu\nu}_{em} = F_{\sigma\nu} F^{\mu\sigma} - \frac{1}{4} g^{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} . \quad (2.18) \]

For an observer comoving with the fluid \( T^{\mu\nu}_{em} \) can be written as

\[ T^{\mu\nu}_{em} = \frac{1}{2} \left( E^2 + H^2 \right) u^\mu u^\nu + \frac{1}{6} \left( E^2 + H^2 \right) h^{\mu\nu} + 2u^{(\mu} \eta^{\nu)}_{\alpha\beta} u_\alpha E_\beta H_\gamma + \pi^{\mu\nu}, \quad (2.19) \]

where \( \eta^{\mu\nu\alpha\beta} \) is the four-dimensional totally antisymmetric volume element \( (\eta_{0123} = \sqrt{-\det g}) \), \( h_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu \) is the projection matrix, \( E^2 = E_\mu E^\mu \) and \( H^2 = H_\mu H^\mu \), respectively, are the magnitudes of the electric and the magnetic fields and \( \pi^{\mu\nu} \) is a traceless and space-like symmetric tensor given by

\[ \pi^{\mu\nu}_{em} = \frac{1}{3} \left( E^2 + H^2 \right) h^{\mu\nu} - E^{\mu} E^{\nu} - H^{\mu} H^{\nu}. \quad (2.20) \]

Eq. \( (2.19) \) can be compared with the energy momentum tensor for a generic imperfect fluid defined in Eq. \( (2.8) \) which yields

\[ \rho_{em} = \frac{1}{2} \left( E^2 + H^2 \right), \quad (2.21) \]

\[ p_{em} = \frac{1}{6} \left( E^2 + H^2 \right), \quad (2.22) \]

\[ q^{\mu}_{em} = \eta^{\mu\nu\alpha\beta} u_\nu E_\alpha H_\beta, \quad (2.23) \]

\[ \pi^{\mu\nu} = \pi^{\mu\nu}_{em}. \quad (2.24) \]

B. Perturbations

Let us now consider the fields equations with perturbations. In our \( \delta N \) analysis we adopt the notation used in \[56\]. The order of spatial derivative or the so-called gradient expansion is denoted by \( \epsilon = k/aH \) while the order of smallness of perturbations are denoted by \( \delta \). In principle, one has to consider different gradient expansion parameters \( \epsilon_i \) for different directions \( \epsilon_i = k/a_i H_i \). However, to simplify the analysis we assume \( \epsilon_i \sim \epsilon \) so there is no hierarchy for gradient expansions along different directions.

We use the standard ADM formalism for the metric decomposition as follows

\[ ds^2 = -dN^2 + \gamma_{ij} \left( dx^i + \beta^i dt \right) \left( dx^j + \beta^j dt \right), \quad (2.25) \]

in which \( N \) is the lapse function, \( \beta_i \) are the shift vectors, and \( \gamma_{ij} \) represent the spatial three-dimensional metric. The spatial indices \( i = 1, 2, 3 \) are raised or lowered by the spatial metric \( \gamma_{ij} \). Furthermore, we decompose the spatial metric as follows

\[ \gamma_{ij} = a_i(t) a_j(t) e^{\psi_i(x,t)+\psi_j(x,t)} \tilde{\gamma}_{ij}, \quad (2.26) \]
where \( a_i(t) \) is the average scale factor for the \( i \)-th spatial direction and \( \psi_i(\mathbf{x}, t) \) are equivalent to curvature perturbation \( \psi \) in the isotropic limit. In linear perturbation theory \( \beta^i, \psi_i \) and \( \tilde{\gamma}_{ij} \) are small perturbations at the order \( \mathcal{O}(\delta) \) with \( \delta \ll 1 \). But in our analysis below, we do not use the assumption that \( \delta \ll 1 \) so our analysis are valid to all orders in perturbation theory.

Note that in general Bianchi Type-I model we considered here there is no spatial symmetry so all physical degrees of freedom are in the form of scalar perturbations, encoded in \( N, \beta_i, \psi_i \) and \( \tilde{\gamma}_{ij}, i \neq j \) and there is no vector or tensor perturbations.

An important step in dealing with the gradient expansion ordering of Einstein equations is the order of the shift \( \beta^i \). We note that at the background level \( \beta^i = 0 \). As a result one expects that the background metric should be valid globally in the limit \( \epsilon \to 0 \) and, as employed in [52], one can assume
\[
\beta^i = \mathcal{O}(\epsilon) . \tag{2.27}
\]

The ordering of \( \beta^i \) in Eq. (2.27) was also obtained in [50] with the assumption that the anisotropic pressure is first order in gradient expansion. We look into ordering of \( \beta^i \) more rigorously in Appendix C and verify Eq. (2.27). Furthermore, as we demonstrated in Appendix C, it can be shown that the non-diagonal spatial metric components, \( \gamma_{ij} \), to all orders in perturbations theory are also at the first order of gradient expansion
\[
\gamma_{ij} = \mathcal{O}(\epsilon) . \tag{2.28}
\]

Now we have all the necessary materials for performing the gradient expansion analysis for the Einstein equations. Here we emphasis that the following expansions are valid to the first order of gradient expansion \( \epsilon \) but to all orders of perturbations \( \delta \).

The \((0,0)\) component of perturbed Einstein tensor is
\[
G^0_0 = \frac{-1}{N^2} \sum_{i>j} (\ddot{H}_i + \ddot{\psi}_i)(\ddot{H}_j + \ddot{\psi}_j) + \mathcal{O}(\epsilon^2) \tag{2.29}
\]

Combining Eq. (2.29) with the background \((0,0)\) component equation, Eq. (2.11), yields
\[
\frac{M_p^2}{N^2} \sum_{i>j} (\ddot{H}_i + \ddot{\psi}_i)(\ddot{H}_j + \ddot{\psi}_j) = \rho(\mathbf{x}, t) + \mathcal{O}(\epsilon^2) \tag{2.30}
\]

Locally, as a function of \((\mathbf{x}, t)\), the above equation takes the form
\[
3M_p^2 \mathcal{H}^2(\mathbf{x}, t) = \rho(\mathbf{x}, t) + \mathcal{O}(\epsilon^2) , \tag{2.31}
\]

in which
\[
\mathcal{H}^2(\mathbf{x}, t) \equiv \frac{1}{3} \sum_{i>j} H_i(\mathbf{x}, t)H_j(\mathbf{x}, t) \tag{2.32}
\]

with the following generalization of local Hubble expansion parameter \( H_i(\mathbf{x}, t) \)
\[
H_i(\mathbf{x}, t) \equiv \frac{\ddot{H}_i + \ddot{\psi}_i(\mathbf{x}, t)}{N} . \tag{2.33}
\]

As a result one can readily associate the average local Hubble expansion rate \( H(\mathbf{x}, t) \) as
\[
H(\mathbf{x}, t) \equiv \frac{1}{3} \sum_i H_i(\mathbf{x}, t) = \frac{\ddot{H}(t) + \frac{4}{3} \sum_i \ddot{\psi}_i(\mathbf{x}, t)}{N} \tag{2.34}
\]

in which the background average Hubble expansion rate \( \bar{H} \) is \( \ddot{H} = \sum_i \ddot{H}_i/3 \).

Now we look at the energy conservation equation in its contracted form \( u_{\mu} \nabla_{\nu} T^{\mu\nu} = 0 \). At the background level the energy conservation equation is given by Eq. (2.14). Defining the fluid’s proper time \( \tau \) via \( \frac{d}{d\tau} = u^\mu \nabla_\mu \simeq \frac{1}{N} \frac{d}{dt} + \mathcal{O}(\epsilon^2) \), the perturbed energy conservation equation is
\[
\frac{d\rho(\mathbf{x}, t)}{d\tau} + 3H(\mathbf{x}, t)(\rho(\mathbf{x}, t) + p(\mathbf{x}, t)) + \left[ -u_\mu \frac{d}{d\tau} q^\mu + \nabla_\mu q^\mu - u_\mu \nabla_\nu q^{\nu} \right] = \mathcal{O}(\epsilon^2) , \tag{2.35}
\]
in which $H(x,t)$ is the average local Hubble expansion rate defined in Eq. (2.34). By using Eq. (B20) and (B18) the above equation takes the following simple local form

$$\frac{d\rho(x,t)}{d\tau} + 3H(x,t)\left(\rho(x,t) + p(x,t)\right) + \sum \pi^i_i(x,t)H_i(x,t) = O(\epsilon^2).$$

in which $\pi^i_i(x,t) = \bar{\pi}^i_i + \delta\pi^i_i(x,t)$ to all orders in perturbations.

So again we conclude that our separate universe recipe works and it is enough to replace any background function $f(t)$ by its local form $f(x,t)$ and also using new local directional Hubble parameters $H_i(x,t)$. Our prescription will be satisfactory if we can also check the $(i = j)$ components of Einstein equations which are identical to the dynamical equations of $\pi^i_i$. The diagonal spatial components of Ricci tensor can be read as

$$R^i_i = \frac{dH_i(x,t)}{d\tau} + 3H(x,t)H_i(x,t) + O(\epsilon^2),$$

so the $(i = j)$ components of Einstein equation simply modifies the corresponding background equation, Eq. (2.13), as follows (for the off-diagonal components of Einstein equation see Appendix C)

$$M_p^2 \frac{dH_i(x,t)}{d\tau} = -3M_p^2\dot{H}(x,t)H_i(x,t) + \frac{1}{2} \left(\rho(x,t) - p(x,t)\right) + \pi^i_i(x,t)$$

Now we have a complete set of local fields equations, Eq. (2.31), Eq. (2.36) and Eq. (2.38), mimicking the corresponding background equations, Eq. (2.11), Eq. (2.14) and Eq. (2.13), with the local Hubble parameters $H_i(x,t)$ defined in Eq. (2.33). We emphasize again that this set of equations are valid to all order in perturbations $\delta$ but to the first order of gradient expansion $\epsilon$.

The separate Universe approach discussion is now complete. The $\delta N$ formalism is also at hand noting that from the equations above one has

$$N_i(x,t_1,t_2) \equiv \int_{t_1}^{t_2} H_i(x,t)Ndt = \int_{t_1}^{t_2} \bar{H}_i dt + \int_{t_1}^{t_2} \dot{\psi}_i dt$$

So one readily finds

$$N_i(x,t_1,t_2) - \bar{N}_i(t) = \psi_i(t_2) - \psi_i(t_1)$$

Now defining the average expansion by

$$N(x,t_1,t_2) = \frac{1}{3} \sum_i N_i(x,t_1,t_2) = \int_{t_1}^{t_2} H(x,t)Ndt$$

one obtains

$$\delta N(x,t_1,t_2) = N(x,t_1,t_2) - \bar{N}(t) = \psi(t_2) - \psi(t_1)$$

in which $\psi(x,t)$ is defined as the average of $\psi$

$$\psi(x,t) \equiv \frac{1}{3} \sum_i \psi_i(x,t).$$

We are interested in curvature perturbation on surface of constant energy density. As was demonstrated in Appendix A the curvature perturbation $\zeta$ defined in Eq. (A9) via

$$-\zeta = \psi - \frac{H}{\dot{\rho}} \delta \rho,$$

is gauge invariant. But this definition just works to the first order in perturbations $\delta$. The definition of $\zeta$ to all orders of perturbation theory can be found in [52]. However, as it is shown below, we calculate $\delta N$ on the surface of uniform energy density so the definition of $\zeta$ to nonlinear orders is irrelevant for our purpose.

The relation between $\zeta$ and $\delta N$ therefore is

$$\zeta(x,t) = \delta N(x,t_i,t_f),$$
in which the initial surface is a flat surface $\psi = 0$ and the final surface should be a uniform energy density surface $\delta \rho = 0$.

Here a comment is in order. The diagonal components of the anisotropic pressure, $\delta \pi_{ii}$ (no sum over $i$), are non-zero at the background level so their perturbations are expected to play some roles in the curvature perturbation analysis. However, the non-diagonal spatial components of anisotropic pressure and the heat conduction terms are absent at the background level so their perturbations will dilute quickly. The diagonal anisotropic pressure plays two different roles in the curvature perturbation analysis, a direct effect and an indirect effect. The direct effect can be seen from the continuity equation, Eq. (2.36), in which $\delta \pi$ contributes to the Hubble expansion rate. This effect, by using Eq. (2.46), can be quantified as follows

$$N(x, t_i, t_f) = \int_{t_i}^{t_f} H(x, t) dt = -\frac{1}{3} \int_{t_i}^{t_f} dt \frac{\dot{\rho}(x, t)}{\rho(x, t) + p(x, t)} - \frac{1}{3} \int_{t_i}^{t_f} dt Q(x, t),$$

in which $Q(x, t)$ is defined as

$$Q(x, t) = \frac{\mathcal{N}}{\rho + p} \left[ -u_{\mu} \frac{d}{dt} q^\mu + \nabla_{\mu} \dot{q}^\mu - u_{\mu} \nabla_{\nu} \pi^{\mu\nu} \right] = \frac{\mathcal{N}(x, t)}{\rho(x, t) + p(x, t)} \sum_i H_i(x, t) \pi_{ii}(x, t) + O(\epsilon^2).$$

The above equation shows that the diagonal anisotropic pressure components $\delta \pi_{ii}$ contributes to $\delta N$ through their effect on continuity equation as captured by the term containing $Q$ in Eq. (2.46).

The indirect effect of anisotropic pressure is more subtle and sometimes can be more important than the contribution from the term containing $Q$ above. This effect can be understood as the back-reactions of fields responsible for anisotropic pressure on the dynamics of other background fields such as the inflaton field. The $\delta N$ formalism automatically includes this indirect effect. We will see this effect in next section in application of our $\delta N$ formalism for models of anisotropic inflation.

### III. ANISOTROPIC INFLATION

In this section we present the model of anisotropic inflation with a $U(1)$ gauge field originally presented in [45] which provide a non-trivial setup to employ our $\delta N$ formalism.

The action is given by

$$S = \int d^4x \sqrt{-g} \left[ \frac{M_p^2}{2} R - \frac{1}{2} \partial_{\mu} \phi \partial^\mu \phi - \frac{f^2(\phi)}{4} F_{\mu\nu} F^{\mu\nu} - V(\phi) \right]$$

in which $\phi$ is the inflaton field and $F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu}$ is the field strength associated with the $U(1)$ gauge field $A_{\mu}$.

To employ the $\delta N$ formalism, as usual we need to have a good control of the background dynamics. We assume that the gauge field has a non-zero classical value along the $x$-direction so $A_{\mu} = (0, A_x(t), 0, 0)$. As a result, the background space-time is in the form of Bianchi I Universe with the metric

$$ds^2 = -dt^2 + e^{2\sigma(t)} \left( e^{-4\sigma(t)} dx^2 + e^{2\sigma(t)} (dy^2 + dz^2) \right) = -dt^2 + a(t)^2 dx^2 + b(t)^2 (dy^2 + dz^2).$$

In this view $H \equiv \dot{a}/a$ and $H_b \equiv \dot{b}/b$ are the expansion rates along the spatial directions $x$ and $y$ and $\dot{\sigma}/H \equiv (H_b - H_a)/H$ is a measure of anisotropic expansion.
A. The background dynamics

The fields equations are given by

\[ \partial_t \left( f^2(\phi)e^{\alpha + 4\sigma} \dot{A}_z \right) = 0 \] (3.3)

\[ \ddot{\phi} + 3\dot{\alpha}\dot{\phi} + V_\phi - f(\phi)f_{,\phi}(\dot{A}_z^2 e^{-2\alpha + 4\sigma}) = 0 \] (3.4)

\[ \frac{1}{2}\ddot{\phi}^2 + V(\phi) + \frac{1}{2}f^2(\phi)\dot{A}_z^2 e^{-2\alpha + 4\sigma} = 3M_p^2 (\alpha^2 - \dot{\phi}^2) \] (3.5)

\[ V(\phi) + \frac{1}{6}f^2(\phi)\dot{A}_z^2 e^{-2\alpha + 4\sigma} = M_p^2 (\dot{\alpha} + 3\dot{\phi}^2) \] (3.6)

\[ \frac{1}{3}f^2(\phi)\dot{A}_z^2 e^{-2\alpha + 4\sigma} = M_p^2 (3\dot{\alpha}\dot{\sigma} + \ddot{\sigma}) \] (3.7)

in which a dot indicates derivative with respect to \( t \).

The equation of motion for \( A_z \) (the Maxwell equation) is easily solved as

\[ \dot{A}_z = f(\phi)^{-2}e^{-\alpha(t) - 4\sigma(t)}p_A, \] (3.8)

where \( p_A \) is a constant of integration.

We are interested in the small anisotropy limit, \( |\dot{\sigma}/H| \ll 1 \), so the background expansion is mainly supported by the isotropic potential term as in conventional models of inflation. In order for the anisotropy to be small, we demand that \( R \ll 1 \) in which

\[ R \equiv \frac{\dot{A}_z^2 f(\phi)^2 e^{-2\alpha}}{2V}. \] (3.9)

In this view \( R \) measures the ratio of the electric field energy density, \( \rho_{em} \), associated with the gauge field to the total potential energy density. Therefore, to have small anisotropies, we require \( \rho_{em} \ll V \).

Although the anisotropy is small, \( R \ll 1 \), so the Hubble expansion rate in modified Friedmann equation (3.5) is mainly dominated by the isotropic potential term, but the back-reactions of the gauge field on the inflaton field induce an effective mass for the inflaton as given by the last term in Eq. (3.4). This in turn will affect the dynamics of the inflaton field. As shown in [16] with the appropriate form of \( f(\phi) \) the the system reaches an attractor solution in which \( R \) reaches a subdominant but nearly constant value. For \( R \) to be constant, we need \( f(\phi) \propto a^n \) with \( n \simeq -2 \).

Indeed, the background expansion is given by

\[ a \propto \exp \left[ -\int d\phi \frac{V}{V_\phi} \right]. \] (3.10)

So if one chooses

\[ f \propto \exp \left[ -n \int d\phi \frac{V}{V_\phi} \right] \] (3.11)

this yields \( f \propto a^n \). The exact form of \( f \) therefore depends on \( V(\phi) \). For the chaotic potential used in [16] we have

\[ V = \frac{1}{2}m^2 \phi^2 \quad \rightarrow \quad f(\phi) = \exp \left( \frac{c\phi^2}{2M_p^2} \right) \] (3.12)

with \( c \) a constant very close to unity. In our discussion below we take the form of \( f \), in terms of \( a = e^{\alpha} \), to be

\[ f = \left( \frac{a}{a_f} \right)^{-2c} \simeq \left( \frac{\eta}{\eta_e} \right)^{2c}, \] (3.13)

in which \( a_e \) and \( \eta_e \) represent the value of the scale factor and the conformal time at the end of inflation.

As shown in [16] the system reaches the attractor solution in which \( R \) is given by

\[ R = \frac{c - 1}{2c} \epsilon_H = \frac{1}{2} I \epsilon_H, \] (3.14)
where we have defined $I \equiv \frac{c}{\epsilon} - 1$ and $\epsilon_H \equiv \dot{H}/H$ is the slow-roll parameter. Combined with the definition of $R$ in Eq. (3.9) we obtain

$$A^2 f^2 e^{-2\alpha} = I \epsilon_H V.$$  

(3.15)

As we shall see below, this equation will be the key equation to find $\delta N$ in terms of $\delta \phi$ and $\delta \dot{A}$.

Furthermore the anisotropy in expansion is given by

$$\dot{\sigma} \dot{\alpha} \simeq I \epsilon_H H^3.$$  

(3.16)

During the attractor phase the inflaton evolution is given by

$$M_P^{-2} d\phi/d\alpha \simeq -\frac{V}{V} \phi + \frac{c-1}{c} \frac{V}{V}.$$  

(3.17)

Interestingly, this means that the back-reactions of the gauge field on the inflation field change the effective mass of the inflaton field as given by the second term above.

Using Eq. (3.12) in Eq. (3.17) results in the following equation

$$\phi_e^2 - \phi^2 = 4M_P^2 \alpha (1 - I)$$  

(3.18)

in which $\phi_e$ is the value of $\phi$ at the end of inflation. We choose the convention such that $\alpha_e = 0$, so during inflation $\alpha < 0$. Eq. (3.18) clearly shows the effect of the gauge field back-reactions on the evolution of the inflaton field. The fact that the evolution of the inflaton field is affected by the gauge field, as given by the correction factor $(1 - I)$ in Eq. (3.18) is the key to calculate $\delta N$ in the presence of gauge field. In passing, we comment that in the previous applications of $\delta N$ in the literature for models with the gauge fields, this important effect is not taken into account. In other words, $\delta N$ in these papers have been written with treating $\delta A_\mu$ in the same footing as $\delta \phi$ in an FRW background without taking into account the back-reactions of the gauge field in the evolution of inflaton field and in the dynamics of the anisotropic background.

In connection with our discussion in previous section the energy density, pressure, momentum density and stress associated with the electro-magnetic field are given by

$$\rho_{em} = \frac{1}{2} (E^2 + B^2) = \frac{3}{2} I \epsilon_H H^2,$$  

(3.19)

$$p_{em} = \frac{1}{6} (E^2 + B^2) = \frac{1}{2} I \epsilon_H H^2,$$  

(3.20)

$$q_{em}^i = \eta^{ijk} u_j E_k B_q = 0,$$  

(3.21)

(3.22)

and

$$\pi^\nu_{\mu} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -2I \epsilon_H H^2 & 0 & 0 \\ 0 & 0 & I \epsilon_H H^2 & 0 \\ 0 & 0 & 0 & I \epsilon_H H^2 \end{bmatrix}.$$  

(3.23)

Plugging the value of $\rho_{em}$ into definition of $R$ and using the attractor value Eq. (3.14) we obtains $\rho_{em} \simeq RV$ as advertised before. Also Eq. (3.19) indicates that $E = \sqrt{3I \epsilon_H H}$ and $B = 0$.

### B. $\delta N$ in anisotropic inflation

Our goal here is to calculate the curvature perturbations in this model by employing our $\delta N$ formalism. As we argued before the contribution of the gauge field into the Hubble expansion rate and total energy density is sub-dominant. This means that the surface of end of inflation is controlled only by the inflaton field. However, the gauge field plays an important role in Klein-Gordon equation and in the evolution of the inflaton field as can be seen in Eq.(3.18).

Perturbing Eq.(3.18) we have,

$$2\phi \delta \phi = -4\delta N + 4N \delta I.$$  

(3.24)
As a result
\[ \delta N = -\frac{1}{2} \phi \dot{\phi} + N \delta I. \] (3.25)

Note that, in order to connect to the standard notation we made the replacement \( \alpha \rightarrow N \) so \( \delta \alpha = \delta N \) from now on. Also note that \( R \) is related to \( I \) by Eq. (3.14) so by \( \delta I \), we actually mean \( \delta (R/\epsilon) \). Since it is easier to work with \( \delta I \) than \( \delta R \), we use \( \delta I \) from now on.

The first term in Eq. (3.25) is the contribution of the inflaton field, while the second term is due to the back-reaction of the gauge field on the inflaton dynamics. Also by perturbing Eq. (3.15) we have
\[ \epsilon_H \delta I = -12R \delta N + 4R \frac{\delta A_x}{A_x} \] (3.26)

Now combining Eq. (3.25) and Eq. (3.26) we have
\[ \left( 1 + \frac{12RN}{\epsilon_H} \right) \delta N = -\frac{\phi}{2M_p^2} \dot{\phi} + \frac{4RN}{\epsilon_H} \frac{\delta A_x}{A_x} \] (3.27)

Now using Eq. (3.14) we have \( RN/\epsilon_H \sim IN \). As we shall see, we require \( NI \ll 1 \) in order not to produce too much anisotropy in power spectrum so we can neglect the second term in the left hand side of Eq. (3.27) and
\[ \delta N \simeq -\frac{\phi}{2M_p^2} \dot{\phi} + 2IN \frac{\delta A_x}{A_x}. \] (3.28)

This is our result for \( \delta N \) to linear order in terms of \( \delta \phi \) and \( \delta A \). Interestingly, since in this model the leading contribution into the anisotropic power spectrum comes from the electric field instead of the magnetic field, we see that in \( \delta N \) only \( \delta \dot{A} \) and not \( \delta A \) appears. This should be compared with the conventional models of \( \delta N \) involving scalar fields \( \phi_I \) in which \( \delta \phi_I \) and not \( \partial_\eta \delta \phi_I \) appears. This is because for light scalar fields \( \delta \phi_I \) become negligible on super-horizon scales once the attractor solution has been reached.

To calculate the power spectrum and the higher order correlations, we have to know the behavior of \( \frac{\delta A}{A} \) outside the horizon. For this purpose, we have to solve the mode function for \( \delta A_i \) with the initial Bunch-Davies vacuum deep inside the horizon. As shown in [48], the canonically normalized gauge field quantum fluctuations are given by
\[ \delta A_i = \sum_{\lambda = \pm} \int \frac{d^3k}{(2\pi)^{3/2}} e^{ik\cdot x} \epsilon_\lambda(k) \hat{V}_i/k, \] (3.29)

in which
\[ \hat{V} = a_\lambda(-\vec{k})V_\lambda(k) + a_\lambda^\dagger(-\vec{k})V_\lambda^*(k). \] (3.30)

Here \( a_\lambda(-\vec{k}) \) and \( a_\lambda^\dagger(-\vec{k}) \) represent the annihilation and the creation operators and \( \epsilon_\lambda \) for \( \lambda = \pm \) represents the circular polarization with the properties \( \vec{k} \cdot \epsilon_\pm(k) = 0, \vec{k} \times \epsilon_\pm(k) = \mp ik \epsilon_\pm(k), \epsilon_\lambda(-\vec{k}) = \epsilon_\lambda^*(\vec{k}) \), normalized via \( \epsilon_\lambda(k) \cdot \epsilon_\lambda^*(k) = \delta_{\lambda\lambda'} \) and
\[ \sum_{\lambda} \epsilon_{\lambda,i}(\vec{k}) \epsilon_{\lambda,j}^*(\vec{k}) = \delta_{ij} - k_i k_j. \] (3.31)

The mode functions satisfy the evolution equation
\[ V_\lambda''(k) + \left( k^2 - \frac{f''}{f} \right) V_\lambda(k) = 0, \] (3.32)

where the prime denotes the derivative with respect to conformal time \( d\eta = dt/a(t) \). For \( f \) given in Eq. (3.13) the normalized gauge field mode function is the same as that of a massless scalar field in dS space with
\[ V_\lambda(k) \simeq \frac{1 + ik\eta}{\sqrt{2k^{3/2}\eta}} e^{-ik\eta}. \] (3.33)
Using this form of the wave function and the attractor solution Eq. (3.15), one can easily show that on super-horizon scales
\[
\frac{\delta \hat{A}}{A} = \sum_{\lambda} \hat{\epsilon}_{\lambda} \frac{\sqrt{3} H}{\sqrt{2} t c_H k^3} \quad (k > aH)
\] (3.34)

In particular, we see that on super-horizon scale, \( \delta A / \dot{A} \) is a constant.

Now we are in the position to calculate the total effect of the gauge field in curvature perturbation \( \zeta \). From Eq. (2.46) and Eq. (3.28) we see that there are two different terms that encode the contributions of the gauge field in \( \zeta \). Eq. (3.28) encodes the indirect effects of the gauge field on \( \delta N \) originating from its back-reaction on inflaton field dynamics. However, the direct contribution of the gauge field in \( \delta N \) is encoded in term \( Q \) in Eq. (2.46). As we shall prove below, the contribution from the \( Q \) term in the curvature perturbation is negligible and the leading contribution of the gauge field in curvature perturbation is from its back-reaction effects in Eq. (3.28).

Calculating \( Q \) from Eq. (2.47) yields (note that in this model \( q \) is proportional to the product of the electric and magnetic fields and since in this model the magnetic field is zero therefore there is no correction from \( q \))
\[
Q = \frac{1}{(\rho + p)} \left( H_a \pi_1^1 + 2 H_b \pi_2^2 \right)
\]
\[
= \frac{2 H^2}{(\rho + p)} \left( H_b - H_a \right) I \epsilon_H
\]
\[
= I^2 H \epsilon_H \ ,
\] (3.35)

where we have used \( (\rho + p) \simeq \dot{\phi}^2 = 2 H^2 \epsilon_H \) and \( H_b - H_a = H I \epsilon_H \) from Eq. (3.16).

Perturbing Eq. (3.35), we have
\[
\delta Q = 2 IH \epsilon_H \delta I
\]
\[
= 2IH \left( -12 R \delta N + 4 R \frac{\delta A_x}{A} \right) .
\] (3.36)

Now integrating Eq. (3.36) over \( t \) we have,
\[
\int_{t_1}^{t_2} \delta Q dt = \int_{t_1}^{t_2} 2IH \left( -12 R \delta N + 4 R \frac{\delta A_x}{A} \right) dt
\]
\[
= 2IN \left( -12 R \delta N + 4 R \frac{\delta A_x}{A} \right)
\] (3.37)

To perform this integral, we have assumed that \( H \) and \( R \) are nearly constant in the slow-roll approximation. Furthermore, \( \delta A_x / A \) is also nearly constant as can be seen from Eq. (3.34).

Eq. (3.37) indicates that the contribution of \( Q \) in \( \delta N \) is at the order of \( INR \sim NI^2 \). However, as we shall see, in order not to produce too much anisotropy we require \( IN^2 < 1 \) so \( NI^2 \ll 1 \) and we can safely neglect the contribution of \( Q \) in \( \delta N \).

In conclusion, the only contribution of the gauge field in curvature perturbations comes in \( \delta N \) as given by the second term in Eq. (3.28). As a result we have
\[
\zeta = \delta N
\]
\[
= -\frac{\dot{\phi}}{2M_p^2} \delta \phi + 2IN \frac{\delta A_x}{A} .
\] (3.38)

We are interested in curvature perturbation power spectrum \( P_\zeta \) defined via
\[
\langle \hat{\zeta}_{\vec{k}_1} \hat{\zeta}_{\vec{k}_2} \rangle = (2\pi)^3 \delta^3(\vec{k}_1 + \vec{k}_2) P_\zeta(\vec{k}_1) \ , \quad P_\zeta(\vec{k}) = \frac{k^3}{2\pi^2} \mathcal{P}_\zeta(\vec{k}) .
\] (3.39)

We decompose the power spectrum into the isotropic part, \( \mathcal{P}_0 \), coming from the \( \delta \phi \) contribution in Eq. (3.38) and the anisotropic power spectrum, \( \Delta \mathcal{P} \), coming from \( \delta \hat{A} \) in Eq. (3.38)
\[
\mathcal{P}_\zeta = \mathcal{P}_0 + \Delta \mathcal{P} .
\] (3.40)
As usual the isotropic power spectrum is given by
\[
P_0 = \frac{H^2}{8\pi^2 M_p^2 \epsilon_H}.
\] (3.41)

To calculate the anisotropic power spectrum we note that \(\delta \phi\) and \(\delta \dot{A}\) are mutually uncorrelated so \(\langle \delta \phi \delta \dot{A} \rangle_* = 0\). As a result
\[
\Delta P = \frac{k^3}{2\pi^2} 4 I^2 N^2 \left( \frac{\delta \dot{A}_x(k_1) \delta \dot{A}_x(k_2)}{A_x} \right)
\] (3.42)
in which the angle \(\theta\) is defined via \(\cos \theta = \hat{n} \cdot \hat{k}\). Now comparing this with the anisotropy factor \(g_*\) defined via
\[
P_\zeta(\vec{k}) = P_0 \left( 1 + g_*(\hat{k}, \hat{n})^2 \right).
\] (3.43)
we obtained
\[
g_* = -24 I N^2
\] (3.44)

Very interestingly this is the result obtained in [45, 48] using the standard in-in formalism. The advantage of using \(\delta N\) formalism is that we only needed to use the background attractor solutions with the information about \(\delta \dot{A}\) at the time of horizon crossing. This should be compared with the tedious analysis employed in [45, 48, 49] using in-in formalism to calculate \(\Delta P\). Physically, one expects that the \(\delta N\) method to be applicable in this model. The reason is that all the dynamics between the inflaton field and the gauge field are in the form of local interactions and the dynamics of modes are trivial inside the horizon and at the time of horizon crossing.

As mentioned in [45, 48], in order to satisfy the observational bound \(|g_*| < 0.3\), and taking \(N = 60\) to solve the horizon and the flatness problems, we require \(I < 10^{-6}\). As a result \(c\) is very close to unity and \(R \ll 1\) as advertised before.

As emphasized in [48] if one allows \(N\) to be too large then the accumulative anisotropies produced from the IR modes can become too large. Therefore, the total number of e-foldings should not be too large in this model.

## IV. BISPECTRUM AND TRISPECTRUM

In this section we calculate the bispectrum and the trispectrum in the anisotropic inflation model studied in previous section using our \(\delta N\) method and compare the results with the corresponding results in [48] and [70] obtained from the standard in-in formalism. As we shall see the results for bispectrum and the trispectrum are in exact agreements.

As usual, in order to calculate the bispectrum and the trispectrum in \(\delta N\) formalism, we have to expand \(\delta N\) to higher orders in perturbations. The expansion of \(\delta N\) to linear order is given in previous section in Eq. (3.38). Here we generalize it to second order. To this purpose, we perturb the attractor solution Eq. (3.15) to second order in fields perturbations
\[
\delta I = \frac{2f_\phi}{f} \delta \phi + \frac{2\delta \dot{A}_x}{A_x} + \left[ \left( \frac{f_\phi}{f} \right)^2 + \frac{f_{\phi \phi}}{f} \right] \delta \phi^2 + \left( \frac{\delta \dot{A}}{A} \right)^2 + \frac{4f_\phi}{f} \frac{\delta \dot{A}_x}{A} \delta \phi
\] (4.1)

This formula gives a relation between \(\delta I, \delta N\) and different powers of \(\delta \phi\) and \(\delta \dot{A}\). On the other hand, Eq. (3.25) from the perturbation of the evolution of \(\phi(\alpha)\) gives a relation between \(\delta I, \delta N\) and \(\delta \phi\) which is valid to all orders in \(\delta I\) and \(\delta N\). Now plugging back Eq. (4.1) into Eq. (3.25) and keeping the leading corrections from \(I \ll 1\) we obtain
\[
\delta N = N_\phi \delta \phi + N_{\dot{A}} \delta \dot{A} + \frac{N_{\phi \phi}}{2} \delta \phi^2 + \frac{N_{\phi \dot{A}}}{2} \delta \dot{A}^2 + N_{\phi \dot{A}} \delta \phi \delta \dot{A}_x
\] (4.2)
in which to leading order in $I$

$$N_\phi \simeq -\frac{\phi}{2M_p^2}, \quad N_{\phi\phi} \simeq \frac{2f_{\phi}^2}{f^2} + \frac{2f_{\phi\phi}}{f} + \frac{\phi^2}{M_p^2} + \frac{4\phi}{M_p^2} \frac{f_{\phi}}{f} \quad (4.3)$$

and

$$N_{\phi A} \simeq \frac{2IN}{A}, \quad N_{\phi A} \simeq \frac{2IN}{A^2}, \quad N_{\phi A} \phi A \simeq \frac{4IN f_\phi}{A} \quad (4.4)$$

Having calculated $\delta N$ to second order in Eq. (4.2), we can calculate the bispectrum $B_\zeta(\vec{k}_1, \vec{k}_2, \vec{k}_3)$ defined via

$$\langle \zeta(\vec{k}_1)\zeta(\vec{k}_2)\zeta(\vec{k}_3) \rangle \equiv \langle 2\pi \rangle^3 \delta^3(\vec{k}_1 + \vec{k}_2 + \vec{k}_3) B_\zeta(\vec{k}_1, \vec{k}_2, \vec{k}_3) \quad (4.5)$$

There are three contributions into bispectrum: (a): $N_{\phi\phi} N_{\phi\phi} \langle \delta \phi^4 \rangle$, (b): $N_{\phi A} N_{\phi A} N_{\phi A} \langle \delta \phi^2 \delta \hat{A}^2 \rangle$ and (c): $N_{\phi \hat{A}} N_{\phi \hat{A}} N_{\phi \hat{A}} \langle \delta \phi^2 \delta \hat{A}^2 \rangle$. The term (a) is the one expected from scalar field theory and is very small. The term (b) is purely from the gauge field while the term (c) is from the mixing of inflaton and the gauge field. One expects that the contribution of term (c) to be sub-leading as compared to the contribution of the term (b). Indeed, a direct analysis shows that the ratio of (b) to (c) is $N$ so for $N \sim 60$ one can safely neglect the contribution from the term (c). In conclusion, the leading contribution to the bispectrum comes from $\langle \delta \hat{A}^2 \rangle$ and

$$\langle \zeta(\vec{k}_1)\zeta(\vec{k}_2)\zeta(\vec{k}_3) \rangle \simeq \frac{1}{2} N_{\phi A} N_{\phi A} N_{\phi A} \int \frac{d^3p}{(2\pi)^3} \langle \delta \hat{A}_\vec{k}_1 \delta \hat{A}_\vec{k}_2 \delta \hat{A}_\vec{k}_3 \rangle + 2\text{perm.} \quad (4.6)$$

in which $N(k_i)$ represents the time when the mode $k_i$ leaves the horizon. Now in the Coulomb gauge $A_0 = 0$, the gauge field perturbations $\delta A_i(\vec{k})$ are given by

$$\left. \frac{\delta \hat{A}_i}{A} \right|_{k_0} = \sum_{\lambda} \tilde{e}_\lambda \frac{\sqrt{3}H}{\sqrt{2I\epsilon_H k^3}} \quad (4.7)$$

Plugging these in Eq. (4.6) yields

$$\langle \zeta(\vec{k}_1)\zeta(\vec{k}_2)\zeta(\vec{k}_3) \rangle \simeq 288IN(k_1)N(k_2)N(k_3) \left( C(\vec{k}_1, \vec{k}_2) P_\zeta(k_1) P_\zeta(k_2) + 2\text{perm.} \right) (2\pi)^3 \delta^3(\vec{k}_1 + \vec{k}_2 + \vec{k}_3) \quad (4.8)$$

in which the momentum shape function $C(\vec{k}_1, \vec{k}_2)$ is defined via

$$C(\vec{k}_1, \vec{k}_2) \equiv \left( 1 - (\vec{k}_1, \vec{n})^2 - (\vec{k}_2, \vec{n})^2 + (\vec{k}_1, \vec{n})(\vec{k}_2, \vec{n})(\vec{k}_1, \vec{k}_2) \right) \quad (4.9)$$

To obtain Eq. (4.8) we have used $P_\zeta(k_1) = \frac{H^2}{4\epsilon_H M_p^2}$ for the isotropic power spectrum and

$$\left( \frac{\delta \hat{A}_i(\vec{k}_1)}{A} \frac{\delta \hat{A}_j(\vec{k}_2)}{A} \right) = \frac{3H^2}{2I\epsilon_H k^3 M_p^2} \left( \delta_{ij} - \vec{k}_1\vec{k}_1 \right) (2\pi)^3 \delta^3(\vec{k}_1 + \vec{k}_2) \quad (4.10)$$

Using Eq. (4.8), one can calculate the bispectrum as

$$B_\zeta(\vec{k}_1, \vec{k}_2, \vec{k}_3) = 288IN(k_1)N(k_2)N(k_3) \left( C(\vec{k}_1, \vec{k}_2) P_\zeta(k_1) P_\zeta(k_2) + 2\text{perm.} \right) \quad (4.11)$$

This completes our results for the bispectrum. As expected, the shape of the bispectrum is anisotropic. Very interestingly our formula Eq. (4.8) and Eq. (4.11) agree exactly with the result of 18 obtained using the standard in-in formalism.

To calculate $f_{NL}$ we go to the squeezed limit $k_1 \ll k_2 \simeq k_3$ in which

$$f_{NL}(\vec{k}_1, \vec{k}_2, \vec{k}_3) = \lim_{k_1 \to 0} \frac{5}{12} \frac{B_\zeta(\vec{k}_1, \vec{k}_2, \vec{k}_3)}{P_\zeta(k_1) P_\zeta(k_2)} \quad (4.12)$$
In the collapsed limit \(\vec{k}\) in trispectrum. Therefore to leading order we have
\[ (4.13) \]

\[ f_{NL} = 240IN(k_1)N(k_2)^2C(\vec{k}_1, \vec{k}_2) \quad (k_1 \ll k_2 \simeq k_3) \]
\[ (4.14) \]

Taking \(N \sim 60\) and \(|g_*| \sim 0.1\) and neglecting the orientation-dependence in \(f_{NL}\) this leads to large non-Gaussianity \(f_{NL} \sim 60\).

Now we are in the position to calculate the trispectrum of our model. The trispectrum is defined via
\[ (4.15) \]

\[ \langle \zeta(k_1)\zeta(k_2)\zeta(k_3)\zeta(k_4) \rangle = (2\pi)^3\delta^3 \left( \vec{k}_1 + \vec{k}_2 + \vec{k}_3 + \vec{k}_4 \right) T_\zeta(\vec{k}_1, \vec{k}_2, \vec{k}_3, \vec{k}_4). \]

In the collapsed limit \(\vec{k}_1 + \vec{k}_3 = \vec{k}_2 + \vec{k}_4 = 0\) we can calculate the parameter \(\tau_{NL}\) via
\[ (4.16) \]

\[ \tau_{NL}(\vec{k}_i) = \lim_{\vec{k}_i + \vec{k}_3 \to 0} \frac{1}{4} \frac{T_\zeta(\vec{k}_1, \vec{k}_2, \vec{k}_3, \vec{k}_4)}{P_\zeta(\vec{k}_1)P_\zeta(\vec{k}_2)P_\zeta(\vec{k}_3)} \]

The trispectrum \(\langle \zeta(\vec{k}_1)\zeta(\vec{k}_2)\zeta(\vec{k}_3)\zeta(\vec{k}_4) \rangle\) have 6 contributions in the forms of
\[
N_{\phi,\phi}^2N_{\phi,\phi}^2\langle \delta\phi^6 \rangle, \quad N_{\phi,\phi}^2N_{\phi,\phi}^2\langle \delta\phi^4\delta\dot{A}^4 \rangle, \quad N_{\phi,\phi}N_{\phi,\phi}N_{\phi,\phi}N_{\phi,\phi}(\delta\phi^2\delta\dot{A}^4), \quad N_{\phi,\phi}N_{\phi,\phi}N_{\phi,\phi}N_{\phi,\phi}(\delta\phi^4\delta\dot{A}^2)
\]
and \(N_{\phi,\phi}^2N_{\phi,\phi}^2\langle \delta\dot{A}^4 \rangle\). As in the case of bispectrum, one can easily check that the last term has the dominant contribution in trispectrum. Therefore to leading order we have
\[ (4.17) \]

\[ \langle \zeta(\vec{k}_1)\zeta(\vec{k}_2)\zeta(\vec{k}_3)\zeta(\vec{k}_4) \rangle \simeq N_{\phi,\phi}(k_1)N_{\phi,\phi}(k_2)N_{\phi,\phi}(k_3)N_{\phi,\phi}(k_4) \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3q}{(2\pi)^3} \left\langle \delta\dot{A}_x(\vec{k}_3)\delta\dot{A}_y(\vec{k}_4)\delta\dot{A}_z(\vec{q})\delta\dot{A}_r(\vec{p})\delta\dot{A}_s(\vec{k}_2 - \vec{p}) \right\rangle + \text{perm.} \]
\[ = 3456IN(k_1)N(k_2)N(k_3)N(k_4) \left( D(\vec{k}_3, \vec{k}_4, \vec{k}_1 + \vec{k}_3)P(\vec{k}_3)P(\vec{k}_4)P(\vec{k}_1 + \vec{k}_3) + 11\text{perm.} \right) (2\pi)^3\delta^3 \left( \vec{k}_1 + \vec{k}_2 + \vec{k}_3 + \vec{k}_4 \right), \]

in which \(D(\vec{k}_3, \vec{k}_4, \vec{k}_1 + \vec{k}_3)\) refers to the trispectrum’s shape function and is given by
\[ (4.18) \]

\[
D(\vec{k}_3, \vec{k}_4, \vec{k}_1 + \vec{k}_3) = 1 - (\vec{k}_4, \vec{n})^2 - (\vec{k}_3, \vec{n})^2 - (\vec{k}_1 + \vec{k}_3, \vec{n})^2 + (\vec{k}_3, \vec{n})(\vec{k}_4, \vec{n})(\vec{k}_3, \vec{k}_4) + (\vec{k}_4, \vec{n})(\vec{k}_1 + \vec{k}_3, \vec{n})(\vec{k}_1 + \vec{k}_3, \vec{k}_4) + (\vec{k}_3, \vec{n})(\vec{k}_1 + \vec{k}_3, \vec{n})(\vec{k}_1 + \vec{k}_3, \vec{k}_3) - (\vec{k}_3, \vec{n})(\vec{k}_1, \vec{n})(\vec{k}_1 + \vec{k}_3, \vec{k}_3)(\vec{k}_1 + \vec{k}_3, \vec{k}_4).
\]

Comparing Eq. (4.14) with the definition of trispectrum we obtain
\[ (4.19) \]

\[ T_\zeta(\vec{k}_1, \vec{k}_2, \vec{k}_3, \vec{k}_4) = 3456IN(k_1)N(k_2)N(k_3)N(k_4) \left( D(\vec{k}_3, \vec{k}_4, \vec{k}_1 + \vec{k}_3)P_\zeta(\vec{k}_3)P_\zeta(\vec{k}_4)P_\zeta(\vec{k}_1 + \vec{k}_3) + 11\text{perm.} \right). \]

Now going to the collapsed limit \(\vec{k}_1 + \vec{k}_3 = \vec{k}_2 + \vec{k}_4 = 0\) and using the definition of \(\tau_{NL}\) given in Eq. (4.16) we obtain
\[ (4.20) \]

\[ \tau_{NL}(k_1, k_2, k_3, k_4) \simeq 3456IN(k_3)^2N(k_4)^2D(\vec{k}_3, \vec{k}_4, \vec{k}_1 + \vec{k}_3). \]

As in the case of bispectrum the trispectrum is anisotropic so \(\tau_{NL}\) has direction-dependence. Comparing our trispectrum with the results of [70] obtained from in-in formalism, we find the exact agreement between these two results.

Now we are in the position to check the SY inequality between \(f_{NL}\) and \(\tau_{NL}\) which states
\[ (4.21) \]

\[ \tau_{NL} \geq \left( \frac{6}{5}f_{NL} \right)^2. \]

The importance of SY inequality as a tool to rule out inflationary scenarios was studied in [61, 67]. The SY inequality presented in the form of Eq. (4.21) is for the models in which \(f_{NL}\) and \(\tau_{NL}\) are either scale-invariant or have the same scale-dependence. In our case, see also [74, 75], we have complicated shape-dependent for \(f_{NL}\) and \(\tau_{NL}\) so the
original SY inequality as given in Eq. (4.21) is not applicable. Instead, in [69] a general integral representation of SY is proved which states
\[ \int d^3q_1 d^3q_2 \tau_{NL}(\vec{q}_1, \vec{k} - \vec{q}_1, \vec{q}_2 - \vec{k}) P_\zeta(q_1)P_\zeta(q_2) \geq \left( \int d^3q \frac{6}{5} f_{NL}(\vec{q}, -\vec{q} - \vec{k})P_\zeta(q) \right)^2, \] (4.22)
in which \( k \to 0 \). As demonstrated in [70] this integral form of SY inequality does hold in our model if we assume \( g_* < 1 \). Indeed, the condition \( g_* < 1 \) is necessary from the observational constraints on the power spectrum and the consistency of our starting assumption that the anisotropic power spectrum is sub-leading so \( \Delta P < P_0 \).

To see qualitatively that the SY inequality in its simple form, Eq. (4.21), does hold in our model, let us neglect the direction-dependence in \( f_{NL} \) and \( \tau_{NL} \) coming from \( D(\vec{k}^3, \vec{k}^4, \vec{k}_1 + \vec{k}_3) \) and \( C(\vec{k}_1, \vec{k}_2) \). As a result
\[ \frac{\tau_{NL}(k_1, k_2, k_3, k_4)}{(\frac{f_{NL}(k_3)}{f_{NL}(k_3)})^2} \simeq \frac{1}{g_*} \quad (g_* < 1) \] (4.23)
in which we have taken \( N(k_i) = N \) and \( g_* \simeq -24IN^2 \) from Eq. (3.43). Demanding that \( g_* < 1 \) from the cosmological observations and also from the consistency of our analysis we conclude that \( \tau_{NL} > \frac{6}{5}(f_{NL})^2 \) so the SY inequality does hold.

V. CONCLUSION AND DISCUSSIONS

In this work we have presented the consistent \( \delta N \) formalism in anisotropic backgrounds. We have demonstrated that the separate universe approach works. In each homogenized patch the local continuity equation and the local Friedmann equation hold which have the same form as the corresponding background equations. We note that the Hubble expansion rate appearing in continuity equation, \( H(x, t) \), and the Hubble expansion rate appearing in Friedmann equation, \( \mathcal{H}(x, t) \), are different.

The anisotropic pressure has two different effects in \( \delta N \) analysis. One is the direct effect encoded by the term containing \( Q \) in Eq. (2.47). The second effect is indirect and comes from the back-reactions of the source of anisotropic pressure on the dynamics of other fields, such as the inflaton field. We have calculated these two effects in model of anisotropic inflation containing a \( U(1) \) gauge field. We have shown that the second effect, i.e. the back-reaction effect, is much larger than the direct effect coming from the \( Q \) term in \( \delta N \) formula. In previous works in the literature, this back-reaction effect during inflation was not taken into account. The gauge field contribution \( \delta A_i \) is added trivially as a non-interacting field during inflation.

Taking into account the back-reaction of gauge field on inflaton dynamics, we have calculated \( \delta N \) to linear and to second order in perturbations. We have demonstrated that our \( \delta N \) formalism exactly reproduces the power spectrum and the bispectrum results obtained in previous works using standard in-in formalism. This is a non-trivial verification of the validity of our \( \delta N \) analysis. The advantage in using \( \delta N \) formalism is that all we need to know to calculate the power spectrum and higher order correlations is the background dynamics and the profile of gauge field fluctuations on super-horizon scales. This method seems to be considerably simpler than the standard in-in formalism.

We also calculated the bispectrum and the trispectrum in anisotropic inflation model. The bispectrum and the trispectrum are both orientation-dependent and scale-dependent. As a result the SY inequality in its simple form is not applicable. However, a generalization of the SY inequality in its integrated form indeed holds.

Large amount of non-Gaussianity with \( f_{NL} \sim 60 \) can easily be generated in this model. It is an interesting exercise to compare the predictions of this model on CMB and large scale structure formation along [76, 77].

Acknowledgments

We would like to thank Shant Baghram, Paolo Creminelli, Eiichiro Komatsu, Karim Malik, Mohammad Hossein Namjoo, Marco Peloso and Misao Sasaki for useful discussions and comments. R. E. would like to thank ICTP for hospitality where this work was in its final stage. We also thank the anonymous referee for the insightful comments about the ordering of \( \delta N \) which were helpful to improve our presentations.

Appendix A: Metric perturbations and Gauge invariant perturbations

In this Appendix we specify the properties of metric transformations under the general coordinate transformation and construct the gauge invariant curvature perturbations \( \zeta \) to linear order in perturbation theory \( \delta \).
The background metric is given in Eq. (2.2). The most general form of the scalar perturbations for the Bianchi I metric is introduced in Eqs. (2.25) and (2.26). For the later convenience we introduce new variables \( \tilde{\beta}_i \) and \( \tilde{\gamma}^{ij} \) as

\[
\partial_i \tilde{\beta}_i \equiv \beta^i, \quad \partial_i \partial_j \tilde{\gamma}^{ij} \equiv \gamma^{ij},
\]  
with no sum on repeated indices.

Consider the general coordinate transformation

\[
x^\mu \to x^\mu + \xi^\mu, \quad \xi^\mu = \left( \xi^0, \partial_i \hat{\xi}^i \right)
\]  

in which \( \xi^0 \) and \( \xi^i = \partial_i \hat{\xi}^i \) for \( i = 1, 2, 3 \) are scalars. Under the coordinate transformation Eq. (A2) we have

\[
\delta g_{\mu\nu} \to \delta g_{\mu\nu} - \bar{g}_{\mu\nu,\kappa} \xi^\kappa - \bar{g}_{\alpha\mu} \partial_\nu \xi^\alpha - \bar{g}_{\alpha\mu} \partial_\nu \xi^\alpha
\]

in which \( \bar{g}_{\alpha\mu} \) is the background Bianchi metric given in Eq. (2.2). More explicitly, one can check that

\[
A \to A - \partial_\xi \hat{\xi}^0 \quad \text{(A4)}
\]

\[
\tilde{\beta}^i \to \tilde{\beta}^i - \frac{1}{a_i^2} \xi^0 - \partial_i \hat{\xi}^i \quad \text{(A5)}
\]

\[
\psi_i \to \psi_i - H \xi^0 - 2 \partial_i \hat{\xi}^i \quad \text{(A6)}
\]

\[
\tilde{\gamma}^{ij} \to \tilde{\gamma}^{ij} - \frac{\partial_i \hat{\xi}^i}{a_j} - \frac{\partial_j \hat{\xi}^j}{a_i} \quad \text{(A7)}
\]

in which \( N \equiv 1 + A \).

If we apply the gradient expansion approximation \( \partial_i^2 = O(\epsilon^2) \), then \( \zeta \) defined via

\[
-\zeta = \left( \psi_1 + \psi_2 + \psi_3 \right) - \left( H_1 + H_2 + H_3 \right) \frac{\delta \rho}{\bar{\rho}} = \psi - H \frac{\delta \rho}{\bar{\rho}}
\]

is gauge invariant and can be interpreted as the average curvature perturbations in our setup. The definition of \( \zeta \) to all orders of perturbation theory can be found in [52].

Appendix B: Gradient Expansion Ordering of Perturbations

In this section we estimate the ordering of \( \beta^i \) and \( \delta q^\mu \) and calculate the contributions of \( \delta q^\mu \) and \( \delta \pi_{\mu}^\nu \) in the energy conservation equation, Eq. (2.35).

First of all let us check the transverse conditions on the heat flow and anisotropic pressure. By definition one has

\[
u^\mu q_\mu = 0, \quad \nu^\mu \pi_{\mu}^\nu = 0.
\]

The fluid’s 4-velocity can be read as

\[
u^\mu = \left[ \frac{1}{N}, \hat{\xi}^0 \right] + O(\epsilon^2) \quad \text{,} \quad \nu_\mu = \left[ -N, \frac{\beta^i}{N} \right] + O(\epsilon^2).
\]

From the background equations we conclude that \( \bar{q}^\mu \) and \( \bar{\pi}_{\mu}^0 \) are zero. Now using the transverse condition (B1) one concludes that to all order

\[
\delta q_0 = 0 \quad \text{,} \quad \delta \pi_{0}^0 = 0.
\]

This also yields \( \delta q^0 = O(\delta^2) \). For the ordering of \( \delta \pi_{0}^i \) one has

\[
\delta \pi_{0}^i = a_i^2 \delta \pi_i^i + \beta_i \pi_i^i = \beta_i \pi_i^i.
\]

We will use this equation later in order to find the ordering of the gradient expansion of perturbations.
Let us now look at the gradient expansion ordering of $\beta^i$. For this we look at $\delta G^0$, and $\delta G^i_0$ components of Einstein equations. With some efforts one can show that
\begin{equation}
\delta G^0 = \epsilon \mathcal{O}(\delta) + \beta \mathcal{O}(\delta) \tag{B5}
\end{equation}
\begin{equation}
\delta G^i_0 = \beta^i \left( 3 H_i \delta - \sum_i \delta H_i^2 - 3 \dot{H} + \dot{H}_i \right) + \epsilon \mathcal{O}(\delta) + \beta \mathcal{O}(\delta) \tag{B6}
\end{equation}
The easiest way to see this is to adopt the local inertial frame in which $\Gamma^3_{\beta \gamma} = 0$. Therefore, the corresponding Einstein equations yield
\begin{equation}
\delta T^0_i = \epsilon \mathcal{O}(\delta) + \beta \mathcal{O}(\delta) \tag{B7}
\end{equation}
\begin{equation}
\delta T^i_0 = \beta^i \left( \bar{R}^i_0 - \bar{R}^0_0 \right) + \epsilon \mathcal{O}(\delta) + \beta \mathcal{O}(\delta). \tag{B8}
\end{equation}
Similarly, using the spatial components of energy momentum conservation one can put limits on $\delta T^0_i$. The continuity equation $\nabla_i T^i$ to leading order yields
\begin{equation}
(\partial_0 + 3 H - H_i) \delta T^0_i - a^2 H_i \delta T^i_0 = - H_i \beta_i \left( \bar{T}^i_0 - \bar{T}^0_0 \right) + \epsilon \mathcal{O}(\delta) + \beta \mathcal{O}(\delta)
= - H_i \beta_i \left( \bar{R}^i_0 - \bar{R}^0_0 \right) + \epsilon \mathcal{O}(\delta) + \beta \mathcal{O}(\delta). \tag{B9}
\end{equation}
One can show that the expression, $\bar{T}^i_0 - \bar{T}^0_0 = \bar{R}^i_0 - \bar{R}^0_0$ is non-vanishing in general and is of the order of $\dot{H}$.

Plugging Eqs. (B7) and (B8) into continuity equation Eq. (B9) yields
\begin{equation}
(\partial_0 + 3 H - H_i) \delta T^0_i = \epsilon \mathcal{O}(\delta) + \beta \mathcal{O}(\delta). \tag{B10}
\end{equation}
This equation shows that $\delta T^0_i$ has decaying solutions approximately like $1/a^2$. So one can readily deduce that $\delta T^0_i$ should be higher order in gradient expansion as
\begin{equation}
\delta T^0_i = \epsilon^2 \mathcal{O}(\delta) + \epsilon \beta \mathcal{O}(\delta). \tag{B11}
\end{equation}
Before discussing about the consequences of the above equation it is more convenient to rephrase Eq. (B8) as follows
\begin{equation}
- \delta q_i = \beta_i \left( \bar{T}^i_0 - \bar{T}^0_0 \right) + \epsilon \mathcal{O}(\delta) + \beta \mathcal{O}(\delta), \tag{B12}
\end{equation}
Furthermore for $\delta T^0_i$ we have
\begin{equation}
\delta T^0_i = (\bar{\rho} + \bar{p}) \beta_i - \delta q_i + \delta \pi^0_i + \epsilon \mathcal{O}(\delta) + \beta \mathcal{O}(\delta) \tag{B13}
\end{equation}
By using Eq. (B12) to eliminate $\delta q^i$ in favor of $\beta^i$ and Eq. (B14) to express $\delta \pi^0_i$ as a function of $\beta^i$ one can show that the leading order terms of $\beta^i$ cancel each other and one obtains
\begin{equation}
\delta T^0_i = \beta^i \mathcal{O}(\delta) + \epsilon \mathcal{O}(\delta) + \beta \mathcal{O}(\delta). \tag{B14}
\end{equation}
On the other hand, comparing Eq. (B11) with Eq. (B14), one obtains the following result for the ordering of $\beta^i$
\begin{equation}
\beta^i = \mathcal{O}(\epsilon), \tag{B15}
\end{equation}
This also yields
\begin{equation}
\delta q_i = \mathcal{O}(\epsilon) \tag{B16}
\end{equation}
\begin{equation}
\delta \pi^0_i = \mathcal{O}(\epsilon) \tag{B17}
\end{equation}
Now we investigate the contribution of heat flow in continuity equation Eq. (2.35). Using Eq. (B16) and Eq. (B17), one finds that $\delta q^\mu \sim \mathcal{O}(\epsilon)$ and by noting that the background value of $q^\mu$ is also zero, we get
\begin{equation}
- u_\mu \pi^\nu \nabla_\nu q^\mu + \nabla_\mu q^\mu \sim \mathcal{O}(\epsilon^2, \epsilon^4) \tag{B18}
\end{equation}
As a result one can deduce that heat conduction can be ignored in the continuity equation at the first order of perturbations and gradient expansion.

Now it is time to calculate the contribution of anisotropic pressure on the continuity equation
\begin{equation}
u^\mu \nabla_\nu \pi^\nu_{\mu} = \bar{u}^\mu \pi^\nu_{\mu} + \delta \left( u^\mu \partial_\nu \pi^\nu_{\mu} + u^\nu \Gamma^\nu_{\rho \mu} \pi^\rho_{\mu} - u^\mu \Gamma^\rho_{\nu \mu} \pi^\nu_{\rho} \right) \tag{B19}
\end{equation}
Noting that $u^\mu = \left[ 1/N, \tilde{\eta} \right] + \mathcal{O}(\epsilon^2)$ and $\delta \pi^0_0 = 0$ to the all orders of perturbations $\delta$, one has
\begin{equation}
u^\mu \nabla_\nu \pi^\nu_{\mu} = \left[ -1/N \right] H_i (x,t) \pi^i_0 (x,t). \tag{B20}
\end{equation}
Appendix C: \(i \neq j\) Components of Einstein equation

In this Appendix we look into off-diagonal components of spatial Einstein equations \(M_p^2 \delta G^i_j = \delta T^i_j\) for \(i \neq j\). These equations are trivial at the background level. The leading order perturbation equations lead

\[
(1)^{\hat{\eta}}_{ij} + 3H(1)^{\hat{\eta}}_{ij} + \left[3H_iH_j - 3(H^2 + \dot{H}) - 2\sum^3_i H^2 - |\epsilon^{ijk}|\dot{H}_k\right] (1)^{\hat{\eta}}_{ij} = \frac{2}{a_{ij}M^2_p} (1)^{\hat{\eta}}_{ij}, \hspace{1cm} (i \neq j)
\]

in which \(\epsilon^{ijk}\) is the Levi-Civita symbol. As one can see the above equation has decaying solutions. This is due to the fact that the background metric does not admit off-diagonal spatial components. Weinberg has argued that the anisotropic stress for a wide class of theories to be some linear combinations of \(\delta u, \delta p\) and \(\delta \rho\) [28]. We partially extend this assumption and assume that the anisotropic stress can also obtain contributions from gauge fields, \(A_\mu\). So anisotropic stress tensor \(\pi_{ij}\) for \(i \neq j\) can be some linear combination of \(\partial_i \partial_j \rho, \partial_i \partial_j u, \partial_i A_j, \partial_i q_j, u_i u_j, u_i A_j, u_i q_j\) and finally \(A_i A_j\). As \(A_i A_j\) for \(i \neq j\) is forbidden by the background equations, this term does not contribute to the off-diagonal part of \(\pi_{ij}\). These contributions are at least at the first order of gradient expansion \(\epsilon\). So one readily deduces that

\[
\pi_{ij} = \mathcal{O}(\epsilon).
\]

Now, Eq. \((C1)\) can be rephrased as

\[
(1)^{\hat{\eta}}_{ij} + 3H(1)^{\hat{\eta}}_{ij} + m^2 \gamma_{ij} = \mathcal{O}(\epsilon),
\]

with \(m^2 \sim H^2\) so \(\gamma_{ij}\) has decaying solutions scaling approximately as \(a^{3/2}\). This is a consequence of the fact that the background equations do not admit \(\gamma_{ij} \neq 0\) for \(i \neq j\). At the second order in perturbation variables \(\delta\), the homogeneous equation has the same form, but it can be verified that all possible source terms are at least at the first order in gradient expansion \(\epsilon\). This argument can be repeated for all orders of perturbations. This argument leads to the conclusion that in the \(n\)-th order of perturbation theory, the off diagonal spatial part of metric, after the decaying solutions become negligible, are of the first order of gradient expansion. As a result, one deduces

\[
\gamma_{ij} = \mathcal{O}(\epsilon)
\]

This equation is important for gradient expansion of Einstein equations.

References

[1] E. Komatsu et al. [WMAP Collaboration], “Seven-Year Wilkinson Microwave Anisotropy Probe (WMAP) Observations: Cosmological Interpretation,” Astrophys. J. Suppl. 192, 18 (2011) [arXiv:1001.4338 [astro-ph.CO]].
[2] D. Hanson, A. Lewis, “Estimators for CMB Statistical Anisotropy,” Phys. Rev. D80, 063004 (2009). [arXiv:0908.0963 [astro-ph.CO]].
[3] D. Hanson, A. Lewis, A. Challinor, “Asymmetric Beams and CMB Statistical Anisotropy,” Phys. Rev. D81, 103003 (2010). [arXiv:1003.0198 [astro-ph.CO]].
[4] L. Ackerman, S. M. Carroll and B. Wise, “Imprints of a Primordial Preferred Direction on the Microwave Background,” Phys. Rev. D 75, 083502 (2007).
[5] N. E. Groeneboom, L. Ackerman, I. K. Wehus and H. K. Eriksen, “Bayesian analysis of an anisotropic universe model: systematics and polarization,” Astrophys. J. 722, 452 (2010) [arXiv:0911.0150 [astro-ph.CO]].
[6] A. R. Pullen and C. M. Hirata, “Non-detection of a statistically anisotropic power spectrum in large-scale structure,” JCAP 1005, 027 (2010) [arXiv:1003.0673 [astro-ph.CO]].
[7] M. S. Turner and L. M. Widrow, “Inflation Produced, Large Scale Magnetic Fields,” Phys. Rev. D 37, 2743 (1988).
[8] B. Ratra, “Cosmological ‘seed’ magnetic field from inflation,” Astrophys. J. 391, L1 (1992).
[9] V. Demozzi, V. Mukhanov, H. Rubinstein, “Magnetic fields from inflation?,” JCAP 0908, 025 (2009) [arXiv:0907.1030 [astro-ph.CO]].
[10] J. Martin and J. Yokoyama, “Generation of Large-Scale Magnetic Fields in Single-Field Inflation,” JCAP 0801, 025 (2008) [arXiv:0711.4307 [astro-ph]].
[11] R. Emami, H. Firouzjahi and M. S. Movahed, “Inflation from Charged Scalar and Primordial Magnetic Fields?,” Phys. Rev. D 81, 083526 (2010) [arXiv:0908.4161 [hep-th]].
nonminimally coupled to the curvature,” Phys. Rev. D 80, 123530 (2009) [arXiv:0909.3524 [astro-ph.CO]].

[38] M. Giovannini, “On the variation of the gauge couplings during inflation,” Phys. Rev. D 64, 061301 (2001) [astro-ph/0012409].

[39] M. Giovannini, “Magnetogenesis, spectator fields and CMB signatures,” Phys. Lett. B 659, 661 (2008) [arXiv:0711.3273 [astro-ph]].

[40] K. E. Kunze, “CMB and matter power spectra from cross correlations of primordial curvature and magnetic fields,” arXiv:1301.6105 [astro-ph.CO].

[41] A. Kandus, K. E. Kunze and C. G. Tsagas, “Primordial magnetogenesis,” Phys. Rept. 505, 1 (2011) [arXiv:1007.3891 [astro-ph.CO]].

[42] T. Kahniashvili, A. Brandenburg, L. Campanelli, B. Ratra and A. G. Tsvjadze, “Evolution of inflation-generated magnetic field through phase transitions,” Phys. Rev. D 86, 103005 (2012) [arXiv:1206.2428 [astro-ph.CO]].

[43] T. R. Dulaney, M. I. Gresham, “Primordial Power Spectra from Anisotropic Inflation,” Phys. Rev. D81, 103532 (2010). [arXiv:1001.2301 [astro-ph.CO]].

[44] A. E. Gumrukcuoglu, B. Himmetoglu, M. Peloso, “Scalar-Scalar, Scalar-Tensor, and Tensor-Tensor Correlators from Anisotropic Inflation,” Phys. Rev. D81, 063528 (2010). [arXiv:1001.4088 [astro-ph.CO]].

[45] M. a. Watanabe, S. Kanno and J. Soda, “The Nature of Primordial Fluctuations from Anisotropic Inflation,” Prog. Theor. Phys. 123, 1041 (2010) [arXiv:1003.0056 [astro-ph.CO]].

[46] H. Funakoshi and K. Yamamoto, “Primordial bispectrum from inflation with background gauge fields,” arXiv:1212.2615 [astro-ph.CO].

[47] K. Yamamoto, “Primordial Fluctuations from Inflation with a Triad of Background Gauge Fields,” Phys. Rev. D 85, 123504 (2012) [arXiv:1203.1071 [astro-ph.CO]].

[48] N. Bartolo, S. Matarrese, M. Peloso and A. Ricciardone, “The anisotropic power spectrum and bispectrum in the $f(\phi)F^2$ mechanism,” arXiv:1210.3297 [astro-ph.CO].

[49] R. Emami and H. Firouzjahi, “Curvature Perturbations in Anisotropic Inflation with Symmetry Breaking,” arXiv:1301.1219 [hep-th].

[50] M. Sasaki, E. D. Stewart, “A General analytic formula for the spectral index of the density perturbations produced during inflation,” Prog. Theor. Phys. 95, 71-78 (1996), astro-ph/9507001.

[51] D. Wands, K. A. Malik, D. H. Lyth et al., “A New approach to the evolution of cosmological perturbations on large scales,” Phys. Rev. D62, 043527 (2000). astro-ph/0003278.

[52] D. H. Lyth, K. A. Malik and M. Sasaki, “A General proof of the conservation of the curvature perturbation,” JCAP 0505, 004 (2005) astro-ph/0411220.

[53] D. H. Lyth, Y. Rodriguez, “The Inflationary prediction for primordial non-Gaussianity,” Phys. Rev. Lett. 95, 121302 (2005) astro-ph/0504045.

[54] A. Naruko, Y. -i. Takamizu and M. Sasaki, “Beyond $\delta N$ formalism,” arXiv:1210.6520 [astro-ph.CO].

[55] A. Naruko, “A General proof of the equivalence between the $\delta N$ and covariant formalisms,” Europhys. Lett. 98, 69001 (2012) [arXiv:1202.1516 [astro-ph.CO]].

[56] N. S. Sugiyama, E. Komatsu and T. Futamase, “The $\delta N$ Formalism,” Phys. Rev. D 87, 023530 (2013) [arXiv:1208.1073 [astro-ph.CO]].

[57] M. Dias, R. H. Ribeiro and D. Seery, “The $\delta N$ formula is the dynamical renormalization group,” arXiv:1210.7800 [astro-ph.CO].

[58] K. Dimopoulos, M. Karcauskaas, D. H. Lyth and Y. Rodriguez, “Statistical anisotropy of the curvature perturbation from vector field perturbations,” JCAP 0905, 013 (2009) [arXiv:0809.1055 [astro-ph]].

[59] C. A. Valenzuela-Toledo, Y. Rodriguez and J. P. Beltran Almeida, “Feynman-like Rules for Calculating $n$-Point Correlators of the Primordial Curvature Perturbation,” JCAP 1110, 020 (2011) [arXiv:1107.3186 [astro-ph.CO]].

[60] X. Chen, “Primordial Non-Gaussianities from Inflation Models,” Adv. Astron. 2010, 638979 (2010) [arXiv:1002.1416 [astro-ph.CO]].

[61] E. Komatsu, “Hunting for Primordial Non-Gaussianity in the Cosmic Microwave Background,” Class. Quant. Grav. 27, 124010 (2010) [arXiv:1003.6097 [astro-ph.CO]].

[62] M. H. Namjoo, H. Firouzjahi and M. Sasaki, “Violation of non-Gaussianity consistency relation in a single field inflationary model,” arXiv:1210.3692 [astro-ph.CO].

[63] X. Chen, H. Firouzjahi, M. H. Namjoo and M. Sasaki, “A Single Field Inflation Model with Large Local Non-Gaussianity,” arXiv:1301.5699 [hep-th].

[64] I. Agullo and L. Parker, “Non-gaussianities and the Stimulated creation of quanta in the inflationary universe,” Phys. Rev. D 83, 063526 (2011) [arXiv:1010.5766 [astro-ph.CO]].

[65] J. Ganc, “Calculating the local-type fNL for slow-roll inflation with a non-vacuum initial state,” Phys. Rev. D 84, 063514 (2011) [arXiv:1104.0244 [astro-ph.CO]].

[66] T. Suyama and M. Yamaguchi, “Non-Gaussianity in the modulated reheating scenario,” Phys. Rev. D 84, 023505 (2011) [arXiv:1101.3630 [astro-ph]].

[67] N. S. Sugiyama, E. Komatsu and T. Futamase, “Non-Gaussianity Consistency Relation for Multi-field Inflation,” Phys. Rev. Lett. 106, 251301 (2011) [arXiv:1101.3630 [gr-qc]].

[68] K. M. Smith, M. LoVerde and M. Zaldarriaga, “A universal bound on N-point correlations from inflation,” Phys. Rev. Lett. 107, 191301 (2011) [arXiv:1105.1805 [astro-ph.CO]].

[69] V. Assassi, D. Baumann and D. Green, “On Soft Limits of Inflationary Correlation Functions,” JCAP 1211, 047 (2012) [arXiv:1204.4207 [hep-th]].
[70] M. Shiraishi, E. Komatsu, M. Peloso and N. Barnaby, “Signatures of anisotropic sources in the squeezed-limit bispectrum of the cosmic microwave background,” arXiv:1302.3056 [astro-ph.CO].

[71] P. G. Miedema, W. A. van Leeuwen “Cosmological perturbations in Bianchi type-I Universe,” Phys. Rev. D 47, 8 (1993).

[72] G. F. R. Ellis and H. van Elst, “Cosmological models: Cargese lectures 1998,” NATO Adv. Study Inst. Ser. C. Math. Phys. Sci. 541, 1 (1999) [gr-qc/9812046].

[73] C. G. Tsagas and J. D. Barrow, “A Gauge invariant analysis of magnetic fields in general relativistic cosmology,” Class. Quant. Grav. 14, 2539 (1997) [gr-qc/9704015].

[74] Y. Rodriguez, J. P. B. Almeida and C. A. Valenzuela-Toledo, “The different varieties of the Suyama-Yamaguchi consistency relation and its violation as a signal of statistical inhomogeneity,” arXiv:1301.5843 [astro-ph.CO].

[75] J. P. Beltran Almeida, Y. Rodriguez and C. A. Valenzuela-Toledo, “The Suyama-Yamaguchi consistency relation in the presence of vector fields,” Mod. Phys. Lett. A 28 (2013) 1350012 [arXiv:1112.6149 [astro-ph.CO]].

[76] M. Shiraishi, S. Yokoyama, K. Ichiki and T. Matsubara, “Scale-dependent bias due to primordial vector field,” arXiv:1301.2778 [astro-ph.CO].

[77] A. R. Pullen and M. Kamionkowski, “Cosmic Microwave Background Statistics for a Direction-Dependent Primordial Power Spectrum,” Phys. Rev. D 76, 103529 (2007) [arXiv:0709.1144 [astro-ph]].

[78] S. Weinberg, Phys. Rev. D 67, 123504 (2003) [astro-ph/0302326].