Supersymmetric Spin Glass

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Abstract

The evidently supersymmetric four-dimensional Wess-Zumino model with quenched disorder is considered at the one-loop level. The infrared fixed points of a beta-function form the moduli space $\mathcal{M} = RP^2$ where two types of phases were found: with and without replica symmetry. While the former phase possesses only a trivial fixed point, this point become unstable in the latter phase which may be interpreted as a spin glass phase.
1 Introduction

There is a great deal of field theory models describing a system in quenched random fields or coupling constants ([1], [2], [3], etc.). In solid state physics such models naturally arise the corresponding pure systems whenever impurities are introduced. It is interesting to extend randomness to other well-studied field theories, just as, for example in [3], disorder was implemented into minimal conformal models. As shown in [4] and subsequent papers stochastic equations as well as the field theories in presence of random external sources often prove to possess some hidden supersymmetry. Kurchan [5] indorsed this result for spin glass dynamics. Because supersymmetry handle perturbative corrections, such random theories are especially interesting. This is what we will do in this paper.

On the other hand in field theories with apparent space-time supersymmetry superpotential is holomorphic function not only of fields but also of coupling constants [6]. Therefore couplings and fields enter potential on equal footing, so that it seems very natural to introduce random (gaussian) distribution of some couplings in the Lagrangian. But the power of supersymmetry is so strong that superpotential gets no quantum corrections [3], [7], i.e. provided that the coupling has no dynamical D-terms integrating it over solves the problem.

In Section 2 we formulate four-dimensional supersymmetric Wess-Zumino theory in random field. In the context of replica method infrared fixed points of one-loop $\beta$-functions are found in Section 3. Analysis of these fixed points suggests two phases on the moduli space $\mathcal{M} = \mathbb{RP}^2$. Numerical evaluation of the most general expressions is eventuated in the phase diagram which is illustrated by two simple examples of the fourth section. Section 5 is devoted to discussions and conclusions.

2 Wess-Zumino model perturbed by randomness

From above arguments it follows that the SUSY analog of a theory with disorder must contain dynamical terms for the random field. In the present paper we consider a four-dimensional Wess-Zumino model that is the supersymmetric counterpart of $\varphi^4$-model (the both theories are defined in the same critical dimension and the scalar potential after integrating auxiliary field in the former model is actually $\varphi^4$). Since, according to [7], Wess-Zumino theory is defined only as a low-energy field theory, we will study the Wilsonian effective action by integrating fast modes with momentum $\Lambda' < P < \Lambda$.

Thereby, let us define a chiral superfield $\Phi = \varphi + \theta \psi + \theta^2 F$ and a random superfield $H$. In this notations the original action is\footnote{For the sake of simplicity the mass terms are omitted.}:

$$S = \int d^4x d^2\theta d^2\bar{\theta}(g\Phi^\dagger\Phi - \Phi^+H - H^+\Phi + \frac{1}{u}H^+H) +$$
\[
\frac{1}{3!} \int d^4 x d^2 \theta (\lambda'_1 \Phi H^2 + \lambda'_2 \Phi^2 H + \lambda'_3 \Phi^3 + \lambda'_4 H^3) + h.c. \tag{1}
\]

This action admits the following treatment. It may be obtained (for the certain set of parameters) from the usual Wess-Zumino action by the replacement \( \Phi \to \Phi + H \), as one usually does in summation over local extremes \([2]\).

One of the most powerful methods to deal with random fields is the replica trick \([1]\), which we will use here to solve this "toy" model. It reduces to introducing \( n \) copies (replicas) of our system, integrating \( H \) field out, then solving \( n \)-replica problem and taking \( n = 0 \) at the end of calculations. After replication the action \([1]\) takes the form:

\[
S = \int d^4 x d^2 \theta [\sum_{a=1}^n (g \Phi^+_a \Phi_a - \Phi^+_a H - H^+ \Phi_a) + \frac{1}{u} H^+ H] + \frac{1}{3!} \int d^4 x d^2 \theta [\sum_{a=1}^n (\lambda'_1 \Phi_a H^2 + \lambda'_2 \Phi^2_a H + \lambda'_3 \Phi^3_a + \lambda'_4 H^3)] + h.c. \tag{2}
\]

As will be shown later the model depends only on the relative values of lambdas, so that one can put them small enough to determine \( H \) field from the saddle point equation on D-term only:

\[
H = u \sum_{a=1}^n \Phi_a \quad \text{and} \quad H^+ = u \sum_{a=1}^n \Phi^+_a \tag{3}
\]

Substituting it back into \([2]\) yields:

\[
S = \sum_{a,b=1}^n \int d^4 x d^2 \theta [\lambda_1 \Phi_a \Phi_b] + \frac{1}{3!} \int d^4 x d^2 \theta [\sum_{a,b,c=1}^n \lambda_1 \Phi_a \Phi_b \Phi_c + \sum_{a,b=1}^n \lambda_3 \Phi^2_a \Phi_b + \sum_{a=1}^n \lambda_3 \Phi^3_a] + h.c. \tag{4}
\]

where \( g_{aa} = g + 3u, \ g_{a \neq b} = 3u \) and three types of vertexes \( \lambda_1 = \lambda'_1 u^2 + \lambda'_4 u^3, \ \lambda_2 = \lambda'_2 u, \ \lambda_3 = \lambda'_3 \) band differently replica indices. It is the action \([1]\) that we are going to study.

### 3 Fixed points of \( \beta \)-functions

Renormalisation group (RG) equations for \( g_{ab} \) easily follow from the one-loop diagram for the pure Wess-Zumino theory \([2]\):

\[
\frac{d g_{ab}}{d \ln \Lambda} = \frac{1}{288 \pi^2} \left( 9 \lambda_3^2 g_{ab}^2 + 2 \lambda_2^2 \sum_{c,d=1}^n [(g_{ac} + g_{bc}) g_{cd} + g_{ac} g_{bd}] + 3 \lambda_2 \lambda_3 \left[ \sum_{c=1}^n (g_{ac}^2 + g_{bc}^2) + 2 g_{ab} \sum_{c=1}^n (g_{ac} + g_{bc}) \right] + 9 \lambda_1 \lambda_3 \sum_{c,d=1}^n (g_{ac} g_{ad} + g_{bc} g_{bd}) \right) \tag{5}
\]

Taking into account the possible replica symmetry breaking we take the Parisi ansatz for \( g_{ab} \)[1]: off-diagonal part of \( g_{ab} \) is parametrized by internal function \( g(x) \) defined on a unite interval \( x \in [0, 1] \) and diagonal part is \( g_{aa} = \tilde{g} \). Replica-symmetric case is obtained by putting \( g(x) = g = \text{constant} \). Algebra of Parisi matrices \( a = (\tilde{a}, a(x)) \) is defined by

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the multiplication rule [1]:
\[
\mathbf{c} = \mathbf{a} \mathbf{b} : \quad \mathbf{c} = \mathbf{a} \mathbf{b} - \int_0^1 dx a(x) b(x)
\]
\[
c(x) = b(x)[a - \int_0^1 dx a(y)] + a(x)[b - \int_0^1 dx b(y)] - \int_0^x dy (a(x) - a(y))(b(x) - b(y))
\]

By means of this rule we get sums over replica indices that appear in (5) in the \( n \to 0 \) limit:
\[
\sum_{b=1}^n g_{ac} \to \bar{g} - \bar{g} \quad \sum_{c,d=1}^n g_{ac} g_{cd} \to (\bar{g} - \bar{g})^2 \quad \sum_{b=1}^n g_{ac}^2 \to \bar{g}^2 - \bar{g}^2
\]
where
\[
\bar{g} = \int_0^1 dx g(x) \quad \text{and} \quad \bar{g}^2 = \int_0^1 dx g^2(x)
\]

A question arises, as usual in spin glass theory, to find infrared (IR) fixed points of (5) which determine the dynamics of the system:
\[
\frac{3}{2} \lambda_2^2 \bar{g}^2 + (\lambda_2^2 + 3 \lambda_1 \lambda_3)(\bar{g} - \bar{g})^2 + \lambda_2 \lambda_3[2\bar{g}(\bar{g} - \bar{g}) + \bar{g}^2 - \bar{g}^2] = 0
\]
\[
\frac{3}{2} \lambda_3^2 \bar{g}^2(x) + (\lambda_2^2 + 3 \lambda_1 \lambda_3)(\bar{g} - \bar{g})^2 + \lambda_2 \lambda_3[2g(x)(\bar{g} - \bar{g}) + \bar{g}^2 - \bar{g}^2] = 0
\]

For example, the \( \lambda_2^2 \)-term is produced by the two non-vanishing (with the number of replicas) diagrams shown on Fig.1.

These equations have two remarkable properties: they are homogeneous in \( \lambda \) and \( g \), i.e. depend only on the squares of the both. Such dependence on \( \lambda \) tells us that zeroes of beta-functions [1]-[4] do not depend on the values of the couplings themselves, but only on their mutual ratios, so that the moduli space of the theory is \( \mathbb{R}P^2 \) instead of \( \mathbb{R}^3 = \{\lambda_1, \lambda_2, \lambda_3\} \). Therefore, without loss of generality, we may put couplings very small keeping their ratios fixed. In this limit the results that we are going to obtain are exact. Moreover, in what follows we will assume \( \lambda_3 \neq 0 \), so that we can choose it to be \( \lambda_3 = 1 \) and denote \( \lambda_2 = \lambda \) and \( \lambda_1 = \mu \) (affined map). The special case of \( \lambda_3 = 0 \) will be studied in the first example of Section 4.

\[\text{The points where } \beta \text{-functions vanish.}\]
\[\text{If } \lambda_3 \neq 1 \text{ then the right parameters are } \lambda = \frac{\lambda_2}{\lambda_3} \text{ and } \mu = \frac{\lambda_1}{\lambda_3}.\]
Quadratic dependence on $g$ in (10) means that for each set of general characteristics, such as $\bar{g}$, $g^2$ and $\tilde{g}$, there are only two possible values $g_{1,2}$ (if any) which the function $g(x)$ can take in a IR-fixed point. Moreover, the same must be true for $\tilde{g}$ because formally it also satisfies similar equation (9). We are free to choose $\tilde{g} = g_1$, for instance. Let us denote the measure of points on a unite interval of $x$ where $g(x) = g_1$ as $1 - x_0$ and the measure of points where $g(x) = g_2$ as $x_0$. For example, it may be a stepwise distribution:

$$g(x) = \begin{cases} g_1, & x_0 < x < 1 \\ g_2, & 0 < x < x_0 \end{cases}$$

Thus we have two equations (9)-(10) on three quantities $g_1$, $g_2$ and $x_0$ depending on them. If $g_1$ and $g_2$ are not simultaneously equal to zero then, actually, we have only two unknowns: $x_0$ and the ratio $p = \frac{g_2}{g_1}$. In this notations (9)-(10) may be rewritten as:

$$\left\{ \begin{array}{l} 1 + \left( \frac{2}{3} \lambda^2 + 2 \mu \right) x_0^2 (1-p)^2 + \frac{2}{3} x_0 \lambda \left[ 2(1-p) + (1-p^2) \right] = 0 \\ p^2 + \left( \frac{2}{3} \lambda^2 + 2 \mu \right) \tilde{g}_0^2 (1-p)^2 + \frac{2}{3} \tilde{g}_0 \lambda \left[ 2p(1-p) + (1-p^2) \right] = 0 \end{array} \right.$$  

(12)

determining both $p$ and $x_0$, and, consequently, the phase of the system.

Curiously enough, for a given solution $p$ and $x_0$ we get the whole set of RG-fixed points $\{\tilde{g}, g(x)\}$ differing in arbitrary factor. Of course, this degeneracy will be removed by higher loop corrections, so that particular value of the fixed point will be determined by the full perturbative expansion. At the one-loop approximation, the explicit data $(\tilde{g}, g(x))$ in the fixed point may be determined by the initial conditions $g$ and $u$.

If for some set of couplings there is no solution to (12) except the trivial one $\tilde{g} = g(x) = 0$, we will refer to this point on the phase space $\{\lambda, \mu\} \in \mathcal{M} = \mathbb{R}P^2$ as a replica-symmetric point and will denote the corresponding phase "RS". Otherwise, replica symmetry is broken with $x_0$ being the solution of (12), and the corresponding phase "RSB" looks like a spin glass system.

Since (12) must be solved by the same $p$, equating the solutions to each equation we get the relation between $x_0$ and $\{\lambda, \mu\} \in \mathcal{M}$. Instead of writing the resulting complicated formulae (partly because it can not be resolved relatively $x_0$), we display it for $x_0 = 1$:

$$\frac{\lambda^2 + 3 \mu + \lambda \pm \sqrt{\frac{3}{2} \lambda^2 - \frac{9}{2} \mu + \frac{3}{2} \lambda}}{\lambda^2 + 3 \mu - \lambda} = \frac{\lambda^2 + 3 \mu - \lambda \pm \sqrt{\frac{3}{2} \lambda^2 - \frac{9}{2} \mu - \frac{3}{2} \lambda}}{\frac{3}{2} + \lambda^2 + 3 \mu - 3 \lambda}$$

(13)

where sings in the both parts are taken independently. Replacing $\lambda \to x_0 \lambda$ and $\mu \to x_0^2 \mu$, we get the equation (13) for arbitrary $x_0$. This expression describes (part of) a curve in $\mathcal{M}$ that separates RS and RSB phases as shown on Fig.2. The shaded region indicate replica-symmetric phase and the unshaded region corresponds to replica symmetry breaking where there is a non-trivial solution to (13), and the trivial point $\tilde{g} = g(x) = 0$ becomes unstable, as will be descanted in the second example of the next section.

4Otherwise we get a trivial replica-symmetric fixed point.
4 Two simple examples

- $\lambda_3 = 0$
  
  In this case the beta-functions \([3]\) become

  $$
  \frac{d\tilde{g}}{d\ln \Lambda} = \frac{1}{48\pi^2} \lambda_2^2 (\tilde{g} - \bar{g})^2
  $$

  $$
  \frac{dg(x)}{d\ln \Lambda} = \frac{1}{48\pi^2} \lambda_2^2 (\tilde{g} - \bar{g})^2
  $$

  These equations may be easily integrated with the result:

  $$
  \tilde{g}_\Lambda = \tilde{g}_{0,\Lambda'} + \frac{A}{48\pi^2} \lambda_2^2 \ln \frac{\Lambda}{\Lambda'}
  $$

  $$
  g_\Lambda(x) = g_{0,\Lambda'}(x) + \frac{A}{48\pi^2} \lambda_2^2 \ln \frac{\Lambda}{\Lambda'}
  $$

  where a constant $A = (\tilde{g} - \bar{g})^2$ is determined by initial conditions and remains unchanged during renormalisation group flow. Since for any $\lambda_2$ the only fixed point is $\tilde{g} = g(x) = 0$, this phase is always replica-symmetric and is not as interesting as others.
• $\lambda_2 = 0 \iff \lambda = 0$

Equations (9)-(10) take the form:

$$\begin{align*}
\tilde{g}^2 + 2\mu(\tilde{g} - \bar{g})^2 &= 0 \\
g^2(x) + 2\mu(\tilde{g} - \bar{g})^2 &= 0
\end{align*}$$

(14)

for which $g_{1,2} = \pm g$ for some $g \neq 0$ in the SG phase. In parametrization (11)

$$\tilde{g} = (g_1 - g_2)x_0 = 2gx_0 \quad \text{and} \quad \tilde{g}^2 = (g_1^2 - g_2^2)x_0 = 2g^2x_0$$

(15)

Substitution it into (14) yields a nontrivial solution:

$$-8\mu x_0^2 = 1 \quad \text{or} \quad x_0 = \frac{1}{\sqrt{-8\mu}}$$

(16)

which exists only for $\mu < -\frac{1}{8}$. It is the range of $\mu$ where the RSB phase can be found. Let us emphasize that exactly for these points in $\mathcal{M}$ the trivial fixed point $\tilde{g} = g(x) = 0$ becomes unstable, for example, with respect to perturbations in $\tilde{g}$. To see this consider $\tilde{g} = \epsilon$:

$$\frac{d\epsilon}{d\ln \Lambda} = \alpha\epsilon^2$$

(17)

where $\alpha < 0$ if (14) is true (i.e. arbitrary small $\epsilon$ increases during the flow to low energies). This simple case illustrates the behavior of general system (12). On the phase diagram it corresponds to $\mu$ axis where the both RS and RSB phases exist.

## 5 Summary

Having started from the (space-time) supersymmetric Wess-Zumino model in a random and quenched background (1) we have found that renormalisation group equations (3) in a fixed point are quadratic homogenous equations in couplings and in $g$. The former property allowed us to take couplings very small as well as to reduce the moduli space to $\mathcal{M} = \mathbb{RP}^2$. There are two types of points (phases) on this moduli space with either broken replica symmetry or not.

Though we have found all IR-fixed points of one-loop $\beta$-function, stability of nontrivial fixed points and analytic RG flow to them remain unexplored. Finally, it is interesting to generalize this analysis to more complex supersymmetric theories as well as to find realistic models whose critical behavior correspond to such theories.

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