Infinitely many nonlocal conservation laws for the $ABC$ equation with $A + B + C \neq 0$

I.S. Krasil’chik$^{1,2}$, A. Sergyeyev$^3$, O.I. Morozov$^4$

$^1$ Independent University of Moscow, B. Vlasyevskiy per. 11, 119002 Moscow, Russia  
$^2$ Russian State University for the Humanities, Miusskaya sq. 6, Moscow, GSP-3, 125993, Russia  
$^3$ Mathematical Institute, Silesian University in Opava, Na Rybníčku 1, 74601 Opava, Czech Republic  
$^4$ Faculty of Applied Mathematics, AGH University of Science and Technology, Al. Mickiewicza 30, 30059 Kraków, Poland  

E-mails: josephkra@gmail.com, Artur.Sergyeyev@math.slu.cz, morozov@agh.edu.pl

August 25, 2016

We construct an infinite hierarchy of nonlocal conservation laws for the $ABC$ equation $A u_t u_{xy} + B u_x u_{ty} + C u_y u_{tx} = 0$, where $A, B, C$ are nonzero constants and $A + B + C \neq 0$, using a non-isospectral Lax pair. As a byproduct, we present new coverings for the equation in question. The method of proof of nontriviality of the conservation laws under study is quite general and can be applied to many other integrable multidimensional systems.

Keywords: integrable systems; conservation laws; Lax pairs.  
MSC 2010: 37K05; 37K10.

1 Introduction

Integrable systems are well known to play an important role in modern mathematics, both pure and applied, see e.g. [1, 4, 6, 7, 8, 9, 10, 11, 14, 15, 16, 17, 32, 34] and references therein. Existence of an infinite hierarchy of conservation laws is among the most important features of integrable systems of partial differential equations [1, 4, 11, 32]. It imposes strong constraints on the associated dynamics making it highly regular.

While such a hierarchy of conservation laws can often be extracted from the Lax-type representation of the system under study, cf. e.g. [4, 11] and references therein, in a relatively straightforward manner, rigorous proof of nontriviality and independence of the conservation laws in question is often a tricky matter, especially in the case of integrable systems of partial differential equations in more than two independent variables when the conservation laws under study often happen to be nonlocal, see e.g. [4, 24].

In the present paper we demonstrate how to prove nontriviality and independence of such nonlocal conservation laws at the example of the $ABC$ equation. The procedure presented below is based on the careful examination of the structure of the kernel of total derivatives and is fairly readily generalized to other multidimensional integrable systems.

Recall that the $ABC$ equation has the form

$$A u_t u_{xy} + B u_x u_{ty} + C u_y u_{tx} = 0,$$

where $A, B, C$ are arbitrary nonzero constants (if one of them vanishes, (1) reduces to a first-order PDE).

*Corresponding author

$^1$In the present paper we mean by integrability existence of a nontrivial Lax pair for the system under study, cf. e.g. [1, 11, 39] and references therein for details.
To the best of our knowledge, equation (1) has first appeared in [38] in connection with the study of geometry of Veronese webs (cf. also [25] and references therein). In the same paper the author has established integrability of (1) for the case of $A + B + C = 0$ by presenting the associated Lax pair; note that in [31] a four-dimensional integrable generalization of (1) with $A + B + C = 0$ was found. Later in [27] (cf. also [28, 30, 35, 37] for related results) a recursion operator for (1) with $A + B + C = 0$ was found, and using the method of hydrodynamic reductions it was shown [4] that (1) is also integrable if $A + B + C \neq 0$. Note also that if $A = B = C \neq 0$ then (1) admits [12] a Lagrangian with the density $-Au_x u_y u_t/2$.

Below we assume that $A + B + C \neq 0$ and put $B = -\kappa_1 A$, $C = -\kappa_2 A$, so $\kappa_1 + \kappa_2 - 1 \neq 0$. Then equation (1) takes the form

$$u_{xy} = \frac{\kappa_1 u_x u_y + \kappa_2 u_y u_t}{u_t}. \tag{2}$$

Integrability of (2) is an immediate consequence of the following result which provides a nonlinear covering for (2) and the associated nonisospectral Lax pair (14) and show how it is related to (5).

To make the exposition self-contained, we give a very brief introduction to the theory of differential coverings in Section 2; for more details of this theory we refer the reader to [4, 22, 23, 24] and references therein; the discussion of how certain types of nonlinear coverings are related to integrability can be found e.g. in [11, 29, 39].

**Remark 1.** Equation (1) is integrable by quadratures. Its general solution is of the form $R(s) = \Omega(s + c_2)$, where $\Omega$ is the inverse function for the function

$$R'' = \frac{(2(\kappa_1 + \kappa_2) - 1) R + ((\kappa_1 + \kappa_2 - 1)(2\kappa_1 + \kappa_2 - 1)/((\kappa_1 + \kappa_2 - 1)R + (\kappa_1 + \kappa_2 - 1))}{(2(\kappa_1 + \kappa_2) - 1) R + ((\kappa_1 + \kappa_2 - 1)(2\kappa_1 + \kappa_2 - 1)/((\kappa_1 + \kappa_2 - 1)R + (\kappa_1 + \kappa_2 - 1))} \tag{4}$$

where $s = q_x/u_x$, and the function $R = R(s)$ is a solution to the ODE

$$R'' = \frac{(2(\kappa_1 + \kappa_2) - 1) R + ((\kappa_1 + \kappa_2 - 1)(2\kappa_1 + \kappa_2 - 1)/((\kappa_1 + \kappa_2 - 1)R + (\kappa_1 + \kappa_2 - 1))}{(2(\kappa_1 + \kappa_2) - 1) R + ((\kappa_1 + \kappa_2 - 1)(2\kappa_1 + \kappa_2 - 1)/((\kappa_1 + \kappa_2 - 1)R + (\kappa_1 + \kappa_2 - 1))} \tag{4}$$

and $c_1, c_2$ are arbitrary constants.

Most importantly, (3) gives rise to a linear nonisospectral Lax pair for (2):

**Corollary 1.** The ABC equation (2) admits a linear nonisospectral Lax pair of the form

$$\Phi_y = (H_1)_y \Phi_x - (H_1)_x \Phi_\zeta, \quad \Phi_t = (H_2)_y \Phi_x - (H_2)_x \Phi_\zeta, \tag{5}$$

where

$$H_1 = \left( \frac{u_y}{u_t} \frac{\kappa_1 s R' + (\kappa_1 + \kappa_2 - 1) R}{\kappa_1 R'} \right)_{s = \zeta/u_x}, \quad H_2 = \left( \frac{u_t}{u_t} \frac{\kappa_1 s R' + (\kappa_1 + \kappa_2 - 1) R}{\kappa_1 R'} \right)_{s = \zeta/u_x}. $$

The coefficients of (5) depend on the variable spectral parameter $\zeta$ in a highly nontrivial fashion thanks to the presence of the function $R$, so our first order of business is to construct a Lax representation with simpler dependence on the parameter. This is done in Section 3 where we present a new covering for (2) and the associated nonisospectral Lax pair (14) and show how it is related to (5). Finally, in Section 4 we construct an infinite hierarchy of nonlocal conservation laws for (2) and prove their nontriviality. Note that the method of the proof is quite general and can be applied to many other integrable multidimensional systems.

---

2See e.g. [11 Subsection 10.3.3] and [13] and references therein for the general construction leading from a covering of the type (3) to a Lax pair of the type (4), and [11 26] and references therein for nonisospectral Lax pairs in general.
2 Differential coverings

We briefly review here the theory of differential coverings over infinitely prolonged differential equations. The reader can find further details and examples in [1, 2, 3, 4].

Let $M$ be a smooth manifold, $\dim M = n$, and $\pi: E \to M$ be a vector bundle, rank $\pi = m$. We consider the bundles of $k$-jets $\pi_k: J^k(\pi) \to M$, $k \geq 0$, together with the natural projections $\pi_{k+1,k}: J^{k+1}(\pi) \to J^k(\pi)$. Then the manifold of infinite jets $J^\infty(\pi)$ is defined as the inverse limit with respect to these projections and the bundles $\pi_{\infty}: J^\infty(\pi) \to M$ and $\pi_{\infty,k}: J^\infty(\pi) \to J^k(\pi)$ are defined as well. For any section $s: M \to E$ of the bundle $\pi$ its infinite jet $j_\infty(s): M \to J^\infty(\pi)$ is a section of $\pi_{\infty}$. One has the embeddings $\pi^*_k: C^\infty(J^k(\pi)) \to C^\infty(J^{k+1}(\pi))$, and we define the algebra of smooth functions on $J^\infty(\pi)$ as $\mathcal{F}(\pi) = \bigcup_{k \geq 0} C^\infty(J^k(\pi))$.

The mean geometric structure on $J^\infty(\pi)$ is the Cartan distribution $\mathcal{C}$: for any point $\theta \in J^\infty(\pi)$ we define the Cartan plane $\mathcal{C}_\theta$ as the tangent plane to the graph of an infinite jet passing through this point. This distribution is formally integrable: if $X$ and $Y$ are vector fields lying in $\mathcal{C}$ then the commutator $[X, Y]$ lies there as well. Every Cartan plane $\mathcal{C}_\theta$ is $n$-dimensional and projects isomorphically to $T_{\pi_{\infty}(\theta)} M$ by the differential of $\pi_{\infty}$. Due to this, any vector field $Z$ on $M$ can be uniquely lifted to a vector field $\mathcal{C}_X$ on $J^\infty(\pi)$. The correspondence $X \mapsto \mathcal{C}_X$ is $C^\infty(M)$-linear and preserves the commutator. In addition, $\pi_{\infty,\mathcal{C}}(\mathcal{C}_X) = X$. In other words, we have a connection which is called the Cartan connection. In the standard local coordinates $x^1, \ldots, x^n$, $u^1, \ldots, u^m$ in $J^\infty(\pi)$, $\sigma$ being symmetric multi-index consisting of the integers $1, \ldots, n$, the Cartan connection is determined by the correspondence

$$\mathcal{C}: \frac{\partial}{\partial x^i} \mapsto D_{x^i} = \frac{\partial}{\partial x^i} + \sum_{\sigma,j} u^j_{\sigma i} \frac{\partial}{\partial u^\sigma_j},$$

where the fields $D_{x^i}$ are called the total derivatives. Differential operators in total derivatives are called $\mathcal{C}$-differential operators.

A differential equation$^3$ of order $k$ is a submanifold in $J^k(\pi)$. Locally, it can be given by the conditions $F^1 = \cdots = F^r = 0$, where $F^j$ are smooth functions on $J^k(\pi)$. Its infinite prolongation is the submanifold (probably, with singularities) $\mathcal{E}$ in $J^\infty(\pi)$ satisfying the conditions $(D_{x^1} \circ \cdots \circ D_{x^n})(F^j) = 0$ for all $j = 1, \ldots, r$, $s \geq 0$, and $1 \leq i_1, \ldots, i_s \leq n$. The Cartan connection can be restricted from the bundle $\pi_{\infty}$ to its subbundle $\pi_{\infty,\mathcal{E}}: \mathcal{E} \to M$ and hence any $\mathcal{C}$-differential operator restricts from $J^\infty(\pi)$ to $M$. On the other hand, the correspondence $[6]$ allows one to lift any linear differential operator on $M$ to a $\mathcal{C}$-differential operator on $\mathcal{E}$. We always assume that $\mathcal{E}$ is differentially connected which means that the only solutions of the system $D_{x^i}(f) = 0$, $i = 1, \ldots, n$, on $\mathcal{E}$ are constants.

In particular, let $\ell_\mathcal{E}$ denote the restriction of the linearization operator

$$\ell_\mathcal{E} = \left( \sum_{\alpha} \frac{\partial F^\alpha}{\partial u^\sigma_{\alpha}} D_{u^\sigma_{\alpha}} \right), \quad \alpha = 1, \ldots, r, \quad \beta = 1, \ldots, m,$$

to $\mathcal{E}$. Then the solutions of the equation $\ell_\mathcal{E}(\varphi) = 0$ are identified with symmetries of $\mathcal{E}$, i.e., with the evolutionary vector fields $\sum_{\alpha,j} \frac{\partial F^\alpha}{\partial u^\sigma_{\alpha}} \frac{\partial}{\partial u^\sigma_j}$ that preserve the Cartan distribution on $\mathcal{E}$. Dually, the solutions of $\ell_\mathcal{E}^*(\psi) = 0$ are called cosymmetries, where $\ell_\mathcal{E}^*$ is formally adjoint to $\ell_\mathcal{E}$. The lift $d_h$ of the de Rham differential gives rise to the horizontal de Rham complex on $\mathcal{E}$; $d_h$-closed $(n-1)$-forms are conservation laws of $\mathcal{E}$ and $d_h$-exact forms are trivial conservation laws. To any conservation law $\omega$ one can associate its generating section (or the characteristic) $\psi_\omega$ which is a cosymmetry.

A morphism of equations is a smooth map $\tau: \bar{\mathcal{E}} \to \mathcal{E}$ which takes the Cartan distribution $\bar{\mathcal{C}}$ on $\bar{\mathcal{E}}$ to that on $\mathcal{E}$. A morphism $\tau$ is a (differential) covering if its differential maps the Cartan plane $\mathcal{C}_\theta$ to $\mathcal{C}_{\tau(\bar{\theta})}$ isomorphically for any $\bar{\theta} \in \bar{\mathcal{E}}$. In other words, for any vector field $Z$ on $M$ the field $\bar{\mathcal{C}}_Z$ projects to $\mathcal{C}_Z$. Locally, this means that the total derivatives on $\bar{\mathcal{E}}$ are

$$\bar{D}_{x^i} = D_{x^i} + X_i, \quad i = 1, \ldots, n,$$

$^3$By a slight abuse of terminology, we speak of a differential equation even though it could actually be a system of differential equations.
where $D_{x^i}$ are the total derivatives on $\mathcal{E}$, and

$$D_{x^i}(X_j) - D_{x^j}(X_i) + [X_i, X_j] = 0, \quad 1 \leq i < j \leq n,$$

$X_i$ being $\tau$-vertical vector fields on $\tilde{\mathcal{E}}$. A covering $\tau: \tilde{\mathcal{E}} \to \mathcal{E}$ over a differentially connected equation is called irreducible if the covering equation $\tilde{\mathcal{E}}$ is differentially connected as well.

Symmetries, cosymmetries, conservation laws of the covering equation $\tilde{\mathcal{E}}$ are nonlocal symmetries, etc., of $\mathcal{E}$. Local objects depend on formal solutions of $\mathcal{E}$ and their partial derivatives; roughly speaking, nonlocal ones may depend on integrals of these solutions.

For example, the relations

$$\tilde{D}_x = D_x + \frac{wu}{2}, \quad \tilde{D}_t = D_t + \frac{w}{2} \left( \frac{u^2}{2} + u_x \right)$$

define a covering of the Burgers equation $\mathcal{E} = \{u_t = uu_x + u_{xx}\}$ by the heat equation $\tilde{\mathcal{E}} = \{w_t = w_{xx}\}$.

Thus, $w$ is related to $u$ by the formulas

$$w_x = \frac{wu}{2}, \quad w_t = \frac{w}{2} \left( \frac{u^2}{2} + u_x \right).$$

Note that system (9) is compatible by virtue of the Burgers equation.

The form $\omega = w \, dx + w_x \, dt$ is a local conservation law of $\tilde{\mathcal{E}}$, and its pullback to $\mathcal{E}$ gives a nonlocal conservation law for $\mathcal{E}$. The corresponding nonlocal conserved density on $\mathcal{E}$, i.e., $w$, defined by (9), can be informally thought of as $\int \exp(\frac{u^2}{2}) \, dx$.

Going back to the general theory, note that any $C$-differential operator $\Delta$ on $\mathcal{E}$ can be lifted to a $C$-differential operator $\tilde{\Delta}$ on $\tilde{\mathcal{E}}$ using equations (7). In particular, this can be done with the linearization operator $\ell_\mathcal{E}$ and its adjoint. Solutions of the equations

$$\tilde{\ell}_\mathcal{E}(\varphi) = 0, \quad \tilde{\ell}^*_\mathcal{E}(\psi) = 0$$

are called nonlocal shadows of symmetries and cosymmetries, respectively.

We employ the theory of coverings to establish nontriviality and independence of nonlocal conservation laws constructed in Section 4.

3 New coverings and nonisospectral Lax pair

We can readily write down a new covering for (2) expressed solely in terms of the variable $r = R(s)$:

**Proposition 2.** The ABC equation (2) has a covering defined by the system

$$
\begin{align*}
\rho_t &= \frac{r((\kappa_1 + \kappa_2 - 1)(r + \kappa_1 + \kappa_2 - 1)u_{tx} - u_tr_x)}{u_x\kappa_1((\kappa_1 + \kappa_2 - 1)}, \\
\rho_y &= \frac{r((\kappa_1 + \kappa_2 - 1)(r + \kappa_1 + \kappa_2 - 1)u_{xy} - \kappa_2 u_yr_x)}{u_x\kappa_1(r + \kappa_1 + \kappa_2 - 1)}.
\end{align*}
$$

Recall that a symmetry shadow (resp. a cosymmetry) for (2) is a solution of linearized (resp. adjoint linearized) version of (2), see Section 2 and [4, 22, 23, 24] for further details.

**Proposition 3.** The ABC equation (2) has a shadow of nonlocal symmetry in the covering (10) with the characteristic

$$U = \int (r + \kappa_1 + \kappa_2 - 1)^{(\frac{1}{\kappa_1+\kappa_2-1})} \exp(\frac{(1-\kappa_1)}{\kappa_1+\kappa_2-1}) dr.$$
Proposition 4. The ABC equation \((2)\) has a nonlocal cosymmetry with the characteristic

\[
\gamma = u_t \int (r + \kappa_1 + \kappa_2 - 1)^{-\frac{2\kappa_2}{\kappa_1 + \kappa_2}} r^{\frac{2\kappa_2}{\kappa_1 + \kappa_2 - 1}} \, dr.
\] (12)

It can be shown that the shadow \((11)\) cannot be lifted to a nonlocal symmetry for the ABC equation \((2)\) in the covering \((10)\).

Now pass from \((10)\) to a slightly different (but equivalent) covering by putting \(w = ru_x^{-(\kappa_1 + \kappa_2 - 1)/\kappa_1}\). Then we have

\[
w_t = -\frac{u_x^{(\kappa_2 - \kappa_1 - 1)/\kappa_1} w (\kappa_1 u_x u_t w_x + (\kappa_1 + \kappa_2 - 1) w (u_t u_{xx} - \kappa_1 u_x u_{xt}))}{\kappa_1^2 (\kappa_1 + \kappa_2 - 1)},
\]

\[
w_y = -\frac{\kappa_2 u_x^{(\kappa_2 - \kappa_1 - 1)/\kappa_1} w u_y (\kappa_1 u_x u_t w_x + (\kappa_1 + \kappa_2 - 1) w u_{xx})}{\kappa_1^2 (u_x^{(\kappa_1 + \kappa_2 - 1)/\kappa_1} w + \kappa_1 + \kappa_2 - 1)}.
\]

Moreover, a similar change of variables in \((5)\) leads to a significantly simpler Lax pair for \((2)\):

Proposition 5. The ABC equation \((2)\) admits a nonisospectral Lax pair of the form

\[
\Psi_y = \frac{\kappa_2 \lambda u_x^{(\kappa_2 - \kappa_1 - 1)/\kappa_1} u_y (-\kappa_1 u_x \Psi_x + (\kappa_1 + \kappa_2 - 1) \lambda u_{xx} \Psi_\lambda)}{\kappa_1^2 (u_x^{(\kappa_1 + \kappa_2 - 1)/\kappa_1} \lambda + \kappa_1 + \kappa_2 - 1)},
\]

\[
\Psi_t = \frac{\lambda u_x^{(\kappa_2 - \kappa_1 - 1)/\kappa_1} (-\kappa_1 u_x u_t \Psi_x + (\kappa_1 + \kappa_2 - 1) \lambda (u_t u_{xx} - \kappa_1 u_x u_{xt}) \Psi_\lambda)}{\kappa_1^2 (\kappa_1 + \kappa_2 - 1)},
\]

which is related to \((3)\) by the transformation \(\Phi = \Psi, \lambda = u_x^{-(\kappa_1 + \kappa_2 - 1)/\kappa_1} R(\zeta/u_x)\).

Note that \((14)\) can be obtained from \((13)\) via the so-called Pavlov eversion, see \([33]\) Section 2 and \([13] [21]\) for details on the latter.

In stark contrast with \((5)\), the variable spectral parameter \(\lambda\) enters \((14)\) rationally. This enables us to construct an infinite sequence of conservation laws from \((14)\) using a formal Taylor expansion for \(\Psi\) with respect to \(\lambda\) in the fashion outlined below.

4 Nonlocal conservation laws

Substituting into \((14)\) a formal expansion \(\Psi = \sum_{j=\infty}^{\infty} \psi_j \lambda^j\) yields the equations

\[
(\psi_j)_y = \frac{\kappa_2 u_x^{(\kappa_2 - \kappa_1 - 1)/\kappa_1} u_y (-\kappa_1 u_x (\psi_{j-1})_x + (\kappa_1 + \kappa_2 - 1) (j - 1) u_{xx} \psi_{j-1})}{\kappa_1^2 (\kappa_1 + \kappa_2 - 1)} - \frac{u_x^{(\kappa_1 + \kappa_2 - 1)/\kappa_1} (\psi_{j-1})_y}{(\kappa_1 + \kappa_2 - 1)},
\]

\[
(\psi_j)_t = \frac{u_x^{(\kappa_2 - \kappa_1 - 1)/\kappa_1} (-\kappa_1 u_x u_t (\psi_{j-1})_x + (\kappa_1 + \kappa_2 - 1) (j - 1) (u_t u_{xx} - \kappa_1 u_x u_{xt}) \psi_{j-1})}{\kappa_1^2 (\kappa_1 + \kappa_2 - 1)}
\]

for \(j \in \mathbb{Z}\).

However, the covering over \((2)\) defined by \((15)\) for \(j \in \mathbb{Z}\) is pretty much intractable, and there appears to be no way to extract from it any reasonably simple conservation laws for \((2)\).

Fortunately, the situation improves dramatically when we truncate the expansion for \(\Psi\). One natural possibility to do this is to pass from the Laurent expansion for \(\Psi\) to the Taylor one, i.e., to assume that \(\psi_j = 0\) for \(j < 0\). The substitution of \(\Psi = \sum_{j=0}^{\infty} \psi_j \lambda^j\) into \((14)\) yields \((15)\) for \(j = 1, 2, 3, \ldots\), and the equations

\[
(\psi_0)_t = 0, \quad (\psi_0)_y = 0.
\]

(16)
System (15) for \( j = 1, 2, 3, \ldots \) together with equations (16) defines an infinite-dimensional covering over (2) and yields, cf. e.g. [2, 3, 20, 35] and references therein, an infinite sequence of (in general nonlocal) two-component conservation laws for (2) of the form

\[
\mathcal{D}_y \left( \frac{u_x^{(\kappa_2-\kappa_1-1)/\kappa_1}(-\kappa_1u_xu_t(\psi_{j-1})_x + (\kappa_1 + \kappa_2 - 1)(j - 1)(u_tu_{xx} - \kappa_1u_xu_{xt})\psi_{j-1})}{\kappa_1^2(\kappa_1 + \kappa_2 - 1)} \right) = 0,
\]

\[
\mathcal{D}_t \left( \frac{u_x^{(\kappa_2-\kappa_1-1)/\kappa_1}u_y(-\kappa_1u_x(\psi_{j-1})_x + (\kappa_1 + \kappa_2 - 1)(j - 1)u_{xx}\psi_{j-1})}{\kappa_1^2(\kappa_1 + \kappa_2 - 1)} \right) = 0 - \frac{u_x^{(\kappa_1+\kappa_2-2)/\kappa_1}(\psi_{j-1})_y}{(\kappa_1 + \kappa_2 - 1)}
\]

(17)

for \( j = 1, 2, 3, \ldots \). The operators \( \mathcal{D}_y \) and \( \mathcal{D}_t \) denote here the total \( y \)- and \( t \)-derivatives in the covering (15), cf. e.g. [3, 24] and the discussion after Lemma 2 below for details. The nonlocal variables and the associated conservation laws are constructed recursively.

The simplest choice \( \psi_0 = 0 \), and even a more general choice \( \psi_0 = \text{const} \), yields by virtue of (15)

\[
(\psi_1)_t = 0, \quad (\psi_1)_y = 0.
\]

If we now choose \( \psi_1 = 1 \), so \( \Psi = \lambda + \sum_{j=2}^{\infty} \psi_j \lambda^j \), then we have

\[
(\psi_2)_y = \frac{\kappa_2u_x^{(\kappa_2-\kappa_1-1)/\kappa_1}u_yu_{xx}}{\kappa_1^2},
\]

\[
(\psi_2)_t = \frac{u_x^{(\kappa_2-\kappa_1-1)/\kappa_1}(u_tu_{xx} - \kappa_1u_xu_{xt})}{\kappa_1^2},
\]

which gives rise to the first nontrivial conservation law in the sequence (17),

\[
\mathcal{D}_y \left( \frac{u_x^{(\kappa_2-\kappa_1-1)/\kappa_1}(u_tu_{xx} - \kappa_1u_xu_{xt})}{\kappa_1^2} \right) = \mathcal{D}_t \left( \frac{u_x^{(\kappa_2-\kappa_1-1)/\kappa_1}u_yu_{xx}}{\kappa_1^2} \right).
\]

(19)

This conservation law is local but the conservation laws (17) for \( j = 3, 4, \ldots \) will be nonlocal:

**Proposition 6.** The ABC equation (2) has an infinite sequence of nontrivial nonlocal linearly independent conservation laws (17) for \( j = 3, 4, \ldots \), where \( \psi_0 = 0, \psi_1 = 1 \), the nonlocal variable \( \psi_2 \) is defined by (18), and the nonlocal variables \( \psi_j, j = 3, 4, \ldots \), are defined recursively via (15).

**Proof.** We begin with a general construction. Let \( \mathcal{E} \) be a differentially connected equation and suppose that \( \omega = A_1 \ dx^1 \wedge dx^3 \wedge \cdots \wedge dx^n + A_2 \ dx^2 \wedge dx^3 \wedge \cdots \wedge dx^n \) is a nontrivial two-component conservation law on \( \mathcal{E} \), i.e.,

\[
D_x^1(A_1) = D_x^1(A_2),
\]

where \( D_{x^i} \) are the total derivatives on \( \mathcal{E} \). Consider the covering \( \tau_\omega: \mathcal{E}_\omega \to \mathcal{E} \) naturally associated to \( \omega \). This covering contains nonlocal variables \( \psi_\sigma \), where \( \sigma \) is a multi-index whose components take values in the set \( \{3, \ldots, n\} \), that satisfy the defining equations

\[
(\psi_\sigma)_{x^1} = D_\sigma(A_1),
\]

\[
(\psi_\sigma)_{x^2} = D_\sigma(A_2),
\]

\[
(\psi_\sigma)_{x^i} = \psi_{\sigma i},
\]

(20)

where \( i = 3, \ldots, n \).

Thus, infinitely many two-component conservation laws of the form

\[
\omega_\sigma = D_\sigma(A_1) \ dx^1 \wedge dx^3 \wedge \cdots \wedge dx^n + D_\sigma(A_2) \ dx^2 \wedge dx^3 \wedge \cdots \wedge dx^n
\]

arise on \( \mathcal{E} \).

A straightforward generalization of the results proved in [19] is the following
Lemma 1. If the equation $E$ is differentially connected then the conservation laws $\omega_\sigma$ are linearly independent if and only if the only solutions of the system

$$\tilde{D}_{x^1}(f) = \tilde{D}_{x^2}(f) = 0$$

are functions of $x^3, \ldots, x^n$, where $\tilde{D}_{x^i}$ are the total derivatives on $E_\omega$.

From equations (20) it also immediately follows that under the assumptions of Lemma 1 the covering $\tau_\omega$ is irreducible.

Now return to the $ABC$ equation (11). Using the presentation (2) introduce the internal variables

$$u_k = \frac{\partial^k u}{\partial t^k}, \quad v_{k,l} = \frac{\partial^{k+l} u}{\partial t^k \partial x^l}, \quad w_{k,l} = \frac{\partial^{k+l} u}{\partial t^k \partial y^l},$$

where $k \geq 0$, $l \geq 1$. The total derivatives in these coordinates take the form

$$D_t = \frac{\partial}{\partial t} + \sum_k u_{k+1} \frac{\partial}{\partial u_k} + \sum_{k,l} \left( v_{k+l+1} \frac{\partial}{\partial v_{k,l}} + w_{k+l+1} \frac{\partial}{\partial w_{k,l}} \right),$$

$$D_x = \frac{\partial}{\partial x} + \sum_k v_{k,1} \frac{\partial}{\partial u_k} + \sum_{k,l} \left( v_{k,l+1} \frac{\partial}{\partial v_{k,l}} + D_t^k D_y^{l-1} (\Upsilon) \frac{\partial}{\partial w_{k,l}} \right),$$

$$D_y = \frac{\partial}{\partial y} + \sum_k w_{k,1} \frac{\partial}{\partial u_k} + \sum_{k,l} \left( D_t^k D_x^{l-1} (\Upsilon) \frac{\partial}{\partial v_{k,l}} + w_{k,l+1} \frac{\partial}{\partial w_{k,l}} \right),$$

where $\Upsilon$ denotes the right-hand side of (2).

From the above formulas we immediately obtain

Lemma 2. The $ABC$ equation (2) is differentially connected.

Now denote by $Y_k$ the right-hand sides of first equations in systems (15) and (18) and by $T_k$ the right-hand sides of second equations in these systems, $k \geq 2$. Then we have

$$\tilde{D}_y = D_y + \sum_{j \geq 2} \sum_{k \geq 0} \tilde{D}_x^k (Y_j) \frac{\partial}{\partial \psi_{j,k}}, \quad \tilde{D}_t = D_t + \sum_{j \geq 2} \sum_{k \geq 0} \tilde{D}_x^k (T_j) \frac{\partial}{\partial \psi_{j,k}},$$

where $\psi_{j,0} = \psi_j$ and $\psi_{j,k+1} = (\psi_{j,k})_x$.

The subsequent lemma is proved by direct computations using the expressions (18) and (15) for $T_j$:

Lemma 3. The following estimates hold:

$$\tilde{D}_x^k (T_2) = \frac{\nu_{0,1}^\alpha}{\kappa_1^2} (u_{1,v_{0,k+2} - \kappa_1 v_{0,1} v_{1,k+1}}) + o(2,k)$$

and

$$\tilde{D}_x^k (T_j) = (j-1) \frac{\nu_{0,1}^\alpha}{\kappa_1^2} (u_{1,v_{0,k+2} - \kappa_1 v_{0,1} v_{1,k+1}}) \psi_{j-1,0} - \frac{\nu_{0,1}^{\alpha+1} u_1}{\kappa_1 (\kappa_1 + \kappa_2 - 1)} \psi_{j-1,k+1} + o(j,k), \quad j > 2,$$

where $\alpha = (\kappa_2 - \kappa_1 - 1)/\kappa_1$, and $o(j,k)$ is a function that does not depend on the variables $\psi_{s,l}$ for $s > j-1$, $\psi_{j-1,l}$ for $l > k+1$, and $v_{0,l+2}$, $v_{1,l+1}$ for $l > k$.

We are now ready to establish a stronger result:

Lemma 4. The only solutions of the equation

$$\tilde{D}_t (f) = 0$$

are functions of $x$ and $y$.  

\pagebreak
Proof of Lemma 4. Suppose that

\[ f = f(x, y, t, \ldots, \psi_{2,0}, \ldots, \psi_{2,k_2}, \ldots, \psi_{j,0}, \ldots, \psi_{j,k_j}) \]  

(25)

is a solution of equation (24). Let us stress that in addition to \( x, y, t \) and \( \psi_{j,k} \) the function \( f \) is allowed to depend on finitely many internal variables (21) on the \( ABC \) equation.

Now proceed by induction on \( j \).

**Base of induction:** \( j = 2 \). Let \( f = f(x, y, t, \ldots, \psi_{2,0}, \ldots, \psi_{2,k_2}) \). Then from (22) one has

\[ \hat{D}_t(f) + \frac{\theta_0}{\kappa_1^2} \sum_{k=0}^{k_2} ((u_1 v_{0,k+2} - \kappa_1 v_{0,1} v_{1,k+1}) + o(2, k)) \frac{\partial f}{\partial \psi_{2,k}} = 0. \]

We now perform induction on \( k_2 \) and show that \( f \) cannot depend on nonlocal variables \( \psi_{2,0}, \ldots, \psi_{2,k_2} \).

Indeed, for \( k_2 = 0 \) one has

\[ \hat{D}_t(f) + \frac{\theta_0}{\kappa_1^2} ((u_1 v_{0,2} - \kappa_1 v_{0,1} v_{1,1}) + o(2, 0)) \frac{\partial f}{\partial \psi_{2,0}} = 0. \]

But from the structure of the operator \( \hat{D}_t \) it immediately follows that \( f \) is independent of \( v_{0,2} \), and thus \( \partial f / \partial \psi_{2,0} = 0 \); hence the claim holds true. If \( k_2 > 0 \) then for similar reasons \( f \) cannot depend on \( v_{0,k_2+2} \) and consequently \( f = f(x, y, t, \ldots, \psi_{2,0}, \ldots, \psi_{2,k_2-1}) \). However, this form of \( f \) contradicts our initial assumption that \( \partial f / \partial \psi_{2,k_2} \neq 0 \), and hence \( f \) is local.

**Induction step:** \( j > 2 \). Let \( f \) be of the form (25). Then from the estimate (23) it follows that \( k_{j-1} > k_j \); otherwise \( f \) will be independent of \( \psi_{j,k_j} \). Repeating this argument, we obtain

\[ k_2 > k_3 > \cdots > k_{j-1} > k_j. \]

Using now the estimate (22), we see that the coefficient at \( \partial / \partial \psi_{2,k_2} \) linearly depends on \( v_{0,k_2+2} \), where \( k_2 + 2 \geq k_j + j \). But this dependence is impossible for the reasons similar to those used in the proof of the case \( j = 2 \). Consequently, \( f \) is independent of all \( \psi_{j,k_j}, k = 0, \ldots, k_j \), and we arrive at the situation of the induction hypothesis, which completes the proof of Lemma 4.

From Lemma 4 it now follows that the only solutions of the equation \( \hat{D}_t(f) = \hat{D}_y(f) = 0 \) are functions of \( x \) alone, and the result of Proposition 6 now readily follows from Lemma 1.

5 Closing remarks

In the present paper we have constructed an infinite hierarchy of nonlocal conservation laws (17) for the \( ABC \) equation (2) and proved the nontriviality of those. An interesting byproduct of our work is the change of spectral parameter which simplifies the original nonisospectral Lax pair (5) down to (14).

Let us reiterate that the method of proving nontriviality of the nonlocal conservation laws in question is quite general and can be applied *mutatis mutandis* to many other multidimensional integrable systems.

In addition to being an important integrability attribute *per se*, the conservation laws found in our paper can be employed, by means of the associated potentials \( \psi_j, j = 2, 3, \ldots \), for the construction of nonlocal symmetries, nonlocal cosymmetries and further nonlocal conservation laws for the equation under study, cf., for example, [4, 7, 23, 24] and references therein.

On the other hand, just as the local conservation laws, the nonlocal ones give rise, once we perform a suitable change of independent variables and then rewrite our equation as an evolutionary system, to nonlocal integrals of motion, cf. e.g. [7, 32]. On the more speculative side, perhaps it could have been possible to apply the nonlocal conservation laws for the construction of exact solutions of the equation under study using a suitably adapted version of the method of conservation laws [18].
In closing note that in addition to the hierarchy of nonlocal conservation laws (17), all of which are two-component, the ABC equation (24) also admits *inter alia* a three-component conservation law of the form [5]

\[
D_x ((\kappa_1 - \kappa_2 + 1)u_y u_t) + D_y ((1 - \kappa_1 + \kappa_2)u_x u_t) - D_t ((\kappa_1 + \kappa_2 + 1)u_x u_y) = 0. \tag{26}
\]

This shows, in particular, that the set of conservation laws (19), (17) for (2) is by no means complete.

In fact, acting on the conservation law (26) by appropriately chosen (nonlocal) symmetries, or even by shadows, is likely to yield many more (nonlocal) three-component conservation laws for (2). More broadly, finding non-two-component nonlocal conservation laws for multidimensional integrable systems is an interesting problem on its own right which is, however, beyond the scope of the present paper (cf. also the discussion in [7] and references therein for the systems which are not necessarily integrable).

**Acknowledgments**

The research of AS was supported in part by the Grant Agency of the Czech Republic (GA ČR) under grant P201/12/G028 and by the Ministry of Education, Youth and Sports of the Czech Republic (MSMT ČR) under RVO funding for IČ47813059. The work of ISK was partially supported by the Simons-IUM fellowship. OIM gratefully acknowledges financial support from the Polish Ministry of Science and Higher Education. AS is pleased to thank E.V. Ferapontov and R.O. Popovych for stimulating discussions.

This research was initiated in the course of visits of OIM to Silesian University in Opava and of AS to the AGH University of Science and Technology. The authors thank the universities in question for warm hospitality extended to them.

The authors thank the editor and the anonymous referee for useful suggestions.

**References**

[1] M. Ablowitz, P.A. Clarkson, Solitons, nonlinear evolution equations and inverse scattering, Cambridge University Press, Cambridge, 1991.

[2] H. Baran, I.S. Krasil’shchik, O.I. Morozov, P. Vojčák, Integrability properties of some equations obtained by symmetry reductions, J. Nonlinear Math. Phys. 22 (2015), no. 2, 210–232, [arXiv:1412.6461](http://arxiv.org/abs/1412.6461)

[3] H. Baran, I.S. Krasil’shchik, O.I. Morozov, P. Vojčák, Coverings over Lax integrable equations and their nonlocal symmetries, Theor. Math. Phys., to appear, [arXiv:1507.00897](http://arxiv.org/abs/1507.00897)

[4] A.V. Bocharov et al., Symmetries and Conservation Laws for Differential Equations of Mathematical Physics, AMS, Providence, RI, 1999

[5] P.A. Burovskiy, E.V. Ferapontov, S.P. Tsarev, Second order quasilinear PDEs and conformal structures in projective space, Int. J. Math. 21 (2010), no. 6, 799–841, [arXiv:0802.2626](http://arxiv.org/abs/0802.2626)

[6] F. Calogero, Why are certain nonlinear PDEs both widely applicable and integrable?, in What is integrability?, ed. by V.E. Zakharov, Springer, Berlin, 1991, 1–62.

[7] A. Cheviakov, G. Bluman, Multidimensional partial differential equation systems: nonlocal symmetries, nonlocal conservation laws, exact solutions, J. Math. Phys. 51 (2010), no. 10, paper 103522.

[8] A. Constantin, On the relevance of soliton theory to tsunami modelling, Wave Motion 46 (2009), 420–426.

[9] A. Constantin, B. Kolev, Integrability of invariant metrics on the diffeomorphism group of the circle, J. Nonlinear Sci. 16 (2006), no. 2, 109–122.

[10] A. Constantin, D. Lannes, The hydrodynamical relevance of the Camassa–Holm and Degasperis–Procesi equations, Arch. Ration. Mech. Anal. 192 (2009), no. 1, 165–186.

[11] M. Dunajski, Solitons, Instantons and Twistors, Oxford University Press, Oxford, 2009.

[12] E.V. Ferapontov, K.R. Khushnutdinova, S.P. Tsarev, On a Class of Three-Dimensional Integrable Lagrangians, Commun. Math. Phys. 261 (2006), 225–243, [arXiv:nlin/0407035](http://arxiv.org/abs/nlin/0407035)
13. E.V. Ferapontov, B.S. Kruglikov, Dispersionless integrable systems in 3D and Einstein–Weyl geometry. J. Differential Geom. 97 (2014), no. 2, 215–254, arXiv:1208.2728v3.
14. M.A. Guest, From quantum cohomology to integrable systems, Oxford University Press, Oxford, 2008.
15. F. Hélein, Constant mean curvature surfaces, harmonic maps and integrable systems, Birkhäuser, Basel, 2001.
16. F. Hélein, Four lambda stories, an introduction to completely integrable systems, in Partial differential equations and applications, Soc. Math. France, Paris, 2007, 47–118.
17. A.R. Hodge, M. Mulase, Hitchin integrable systems, deformations of spectral curves, and KP-type equations, in New developments in algebraic geometry, integrable systems and mirror symmetry (RIMS, Kyoto, 2008), Math. Soc. Japan, Tokyo, 2010, 31–77.
18. N.H. Ibragimov, E.D. Avdonina, Nonlinear self-adjointness, conservation laws, and the construction of solutions to partial differential equations using conservation laws, Uspekhi Mat. Nauk 68 (2013), no. 5 (413), 111–146 (in Russian); English translation: Russian Math. Surveys 68 (2013), no. 5, 889–921.
19. I. Krasil’shchik, Integrability in differential coverings, J. Geom. Phys. 87 (2015), 296–304, arXiv:1310.1189
20. I.S. Krasil’shchik, A. Sergiyeyev, Integrability of S-deformable surfaces: conservation laws, Hamiltonian structures and more, J. Geom. Phys. 97 (2015), 266–278, arXiv:1501.07171
21. I.S. Krasil’shchik, A natural geometric construction underlying a class of Lax pairs, Lobachevskii J. Math. 37 (2016), no. 1, 61–66, arXiv:1401.0612
22. I.S. Krasil’shchik, A.M. Vinogradov, Nonlocal symmetries and the theory of coverings: an addendum to Vinogradov’s “Local symmetries and conservation laws” [Acta Appl. Math. 2 (1984), no. 1, 21–78], Acta Appl. Math. 2 (1984), no. 1, 79–96.
23. I.S. Krasil’shchik, A.M. Vinogradov, Nonlocal trends in the geometry of differential equations: symmetries, conservation laws, and Bäcklund transformations, Acta Appl. Math. 15 (1989), no. 1-2, 161–209.
24. J. Krasil’shchik, A.M. Verbovetsky, Geometry of jet spaces and integrable systems, J. Geom. Phys. 61 (2011), 1633–1674, arXiv:1002.0077
25. B. Kruglikov, A. Panasyuk, Veronese webs and nonlinear PDEs, arXiv:1602.07346
26. S.V. Manakov, P.M. Santini, Integrable dispersionless PDEs arising as commutation condition of pairs of vector fields, J. Phys. Conf. Ser. 482 (2014), 012029, arXiv:1312.2740
27. M. Marvan, A. Sergiyeyev, Recursion operators for dispersionless integrable systems in any dimension, Inverse Problems 28 (2012), no. 2, 025011, 12 pp., arXiv:1107.0784
28. M. Marvan, A. Sergiyeyev, Recursion operator for the stationary Nizhnik–Veselov–Novikov equation, J. Phys. A: Math. Gen. 36 (2003), no. 5, L87–L92, arXiv:nlin/0210028
29. O.I. Morozov, Contact integrable extensions of symmetry pseudo-groups and coverings of (2+1) dispersionless integrable equations, J. Geom. Phys. 59 (2009), no. 11, 1461–1475.
30. O.I. Morozov, A recursion operator for the universal hierarchy equation via Cartan’s method of equivalence, Cent. Eur. J. Math. 12 (2014), no. 2, 271–283, arXiv:1205.5748v1.
31. O.I. Morozov, A. Sergiyeyev, The four-dimensional Martínez Alonso–Shabat equation: reductions and nonlocal symmetries, J. Geom. Phys. 85 (2014), 40–45, arXiv:1401.7942
32. P.J. Olver, Applications of Lie Groups to Differential Equations, 2nd ed., Springer, N.Y., 1993.
33. M.V. Pavlov, J.H. Chang, Y.T. Chen, Integrability of the Manakov–Santini hierarchy, arXiv:0910.2400
34. R.O. Popovych, A. Sergiyeyev, Conservation laws and normal forms of evolution equations, Phys. Lett. A 374 (2010), no. 22, 2210–2217, arXiv:1003.1648
35. A. Sergiyeyev, A new class of (3+1)-dimensional integrable systems related to contact geometry, arXiv:1401.2122
36. A. Sergiyeyev, Recursion operators for multidimensional integrable systems, arXiv:1501.01955
37. A. Sergiyeyev, A strange recursion operator demystified, J. Phys. A: Math. Gen. 38 (2005), no. 15, L257–L262, arXiv:nlin/0406032
38. I. Zakharievich, Nonlinear wave equation, nonlinear Riemann problem, and the twistor transform of Veronese webs, arXiv:math-ph/0006001
39. V.E. Zakharov, Dispersionless limit of integrable systems in 2+1 dimensions, in Singular limits of dispersive waves (Lyon, 1991), Plenum, New York, 1994, 165–174.