Thermal decoherence of long-distance entanglement in spin-1 chains

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Abstract

The thermal entanglement is generated by weakly interacting atoms with an isotropic spin-1 chain. The decoherence of the entanglement is mainly investigated. The effective Hamiltonian is analytically obtained by the approximation method of perturbation. The scaling behavior of the effective coupling is numerically illustrated by the exact diagonalization. It is found out that the decay of the entanglement is slow in the case of non-interacting spins. The long-distance thermal entangled states can be used as the noisy channel for the achievement of the quantum teleportation.

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I. INTRODUCTION

The entanglement plays a key role in the main tasks of quantum information\textsuperscript{1, 2}. In practice, entangled qubits need be accessed individually for measurements. Consequently, they are well separated in space. Recently, the long-distance entanglement\textsuperscript{3, 4} has been attractive in the field of quantum information processing. A selected pair of distant qubits can retain a sizable amount of entanglement at zero temperature if they are weakly coupled to some spin models. Because spin chains can serve as an efficient communication channel for quantum teleportation\textsuperscript{5} and state transfer\textsuperscript{6}, these models are extensively studied. In many schemes\textsuperscript{7, 8}, spin-\(\frac{1}{2}\) Heisenberg chains acted as the medium for the generation of quantum entanglement when the chain is kept at the ground state. It is found out that the long-distance entanglement decreases and vanishes with the length of the gapless spin chains\textsuperscript{9}. As an appealing spin model, the spin-1 chain exhibits the massive and gapped ground state, which can be realized through confining an \(S = 1\) spinor condensate\textsuperscript{10, 11} in optical lattices. The quantum communication in the spin-1 chain has been investigated by\textsuperscript{12}. Here we expect that the long-distance entanglement can also be generated by the spin-1 chains and show the scaling property which is different from spin-\(\frac{1}{2}\) chains. In the realistic optical lattices, the thermal decoherence from the temperature is unavoidable\textsuperscript{13}. Therefore, it is of fundamental importance to study the impacts of the thermal noise on the long-distance entanglement. Using the long-distant entangled state as the channel, we also suggest the standard scheme of quantum teleportation.

In this report, the thermal entanglement between a pair of distant qubits is present when they are weakly coupled to the general isotropic spin-1 chain with bilinear-biquadratic interactions at finite low temperatures. To study the decoherence, the effective Hamiltonian between two distant sites is analytically obtained by the Fröhlich transformation\textsuperscript{14, 15} in Sec. II. The scaling property of the effective coupling is also given by the exact diagonalization method. The effects of the temperature and the relative strength of biquadratic interactions are considered. In Sec. III, we draw on the master equation to investigate the decay of the long-distance entanglement. The protocol of the quantum teleportation is put forward. Finally, a short discussion concludes the paper.
II. THE EFFECTIVE HAMILTONIAN AT FINITE LOW TEMPERATURES

In the optical lattices, a selected pair of two-level atoms $A$ and $B$ can weakly interact with two open ends of a spin-1 chain. At finite low temperatures, the whole quantum system exhibits the thermal equilibrium state. To study the time evolution of quantum states, the total Hamiltonian can be expressed by

$$H = H_0 + H_I = H_q + H_c + H_I,$$

where

$$H_q = \omega (s_A^z + s_B^z),$$

describes the intrinsic Hamiltonian of two distant atoms,

$$H_c = J \sum_{i=1}^{L-1} \left[ \cos \theta (\vec{S}_i \cdot \vec{S}_{i+1}) + \sin \theta (\vec{S}_i \cdot \vec{S}_{i+1})^2 \right],$$

is the Hamiltonian of general isotropic spin-1 chain with the even length $L$ and

$$H_I = J_p (\vec{s}_A \cdot \vec{S}_1 + \vec{S}_N \cdot \vec{s}_B).$$

denotes the weak interaction between two distant atoms and open ends of the chain. Here $\vec{s}_A(B) = \frac{1}{2} \sigma_A(B)$ and $\vec{S}_i$ refer to the spin operators of distant atoms and the $i$th site of the chain respectively. The parameter $\omega$ describes the transition energy from the ground state to the excited one for each atom, and $J \cos \theta \sin \theta$ gives the strength of the bilinear(biquadratic) coupling. As is well known, the energy property of the spin model is determined by the angle $\theta$ \cite{16}. In the context, the biquadratic coupling for $|\theta| < \tan^{-1}\frac{1}{3}$ need be so weak that the general ground state of $H_c$ is a total singlet $|\phi_0\rangle$ with the energy $\epsilon_0$ and the first excited ones are the degenerate triplet states $|\phi^{\lambda=0,1,2}_1\rangle$ with $\epsilon_1$. Here the energy gap $\Delta = \epsilon_1 - \epsilon_0$ is the famous Haldane gap. In general, the thermal equilibrium state $\rho_c(T) = \sum_i \frac{e^{-\epsilon_i/T}}{Z} |\phi_i\rangle \langle \phi_i|$ where $\epsilon_i$ is the $i$-th eigenvalue of $H_c$ and $|\phi_i\rangle$ is the corresponding eigenstate. When the low temperatures satisfy $kT < \Delta$, the components of the ground state and first excited ones become dominant in the thermal equilibrium state. For lower temperatures, this assumption of considering just these states in the thermal fluctuations is more reliable. The approximate expression of the thermal state can be given by $\rho_c(T) \simeq \frac{e^{-\epsilon_0/T}}{Z} (|\phi_0\rangle \langle \phi_0| + e^{-\Delta/T} \sum_{\lambda} |\phi^{\lambda}_1\rangle \langle \phi^{\lambda}_1|)$ where $Z \simeq e^{-\epsilon_0/T} + 3e^{-\epsilon_1/T}$ is the partition function. For the convenience, the Plank constant $\hbar$ and the Boltzman constant $k$ are assumed to be one.
In general, the Fröhlich transformation \[14, 15\] is widely used in condensed matter physics. Recently, this method has been applied to the regime of quantum information processing \[17\]. As a second-order perturbation \[14, 15\], the effective Hamiltonian of the whole system is

\[ H_{\text{eff}} \approx H_0 + \frac{1}{2} [\hat{S}, H_I] \]

where the anti-Hermitian operator \( \hat{S} \) satisfies the relation of \([H_0, \hat{S}] = H_I \) and the elements of this matrix are given by \( \langle \phi^m_i | \hat{S} | \phi^n_j \rangle = \frac{\langle \phi^m_i | H_I | \phi^n_j \rangle}{\epsilon_i - \epsilon_j}, (i \neq j) \) and the diagonal ones are zero for \( m = n, i = j \) \[17\]. Here \( |\phi^m_i\rangle (m = 0, 1, \cdots, d_i - 1) \) is the energy state of \( H_c \) with the corresponding energy \( \epsilon_i \) and \( d_i \) is the degree of degeneracy. In the case with \( J_p \ll J \) at lower temperature, the spin-1 chain is at the state of \( \rho_c \) and then the effective Hamiltonian between two distant atoms is obtained by

\[ H_{\text{eff}}^{A,B} = \text{Tr}_c \left\{ H_0 \rho_c + \frac{1}{2} [\hat{S}, H_I] \rho_c \right\}. \tag{5} \]

where \( \text{Tr}_c \) denotes the trace over the complete energy space of \( H_c \). To simplify the calculation, we assume that the parameter \( \langle \phi^m_k | S^\alpha_i | \phi^n_l \rangle = \tau_{i,\alpha}^{km,ln} = \tau_{i,\alpha} \) where the spin operator \( S^\alpha = S^\pm, S^z \). Due to the invariant symmetry, it is found out that \( \sum_{k\neq l,m,n} \tau_{i,\alpha} \tau_{j,\beta}^* / (\epsilon_k - \epsilon_l) = \Omega_{i,j}^{l,\alpha} \delta_{\alpha,\beta} \) for \( l = 0, 1 \). Here the sum is always zero if \( \alpha \neq \beta \) and the values \( \Omega_{i,j}^{l,\pm} = 2 \Omega_{i,j}^{l,z} \) are real. As a consequence, the effective Hamiltonian can be simplified by the isotropic Heisenberg one

\[ H_{\text{eff}}^{A,B} = J_{\text{eff}} \vec{s}_A \cdot \vec{s}_B + H_q + C. \tag{6} \]

Here the constant of \( C \) is irrelevant to the long distant entanglement. The effective Heisenberg coupling \( J_{\text{eff}} = -2P_{\text{exc}}^\alpha / Z (\Omega_{1,L}^{0,z} + e^{-\Delta/T} \Omega_{1,L}^{1,z}) \) is closely dependent on the energy property of \( H_0 \). By means of the exact diagonalization method, the scaling property of \( J_{\text{eff}} \) at finite low temperatures is demonstrated in Fig. 1. It is shown that the values increase almost exponentially and arrive at a steady one with the length of the chain. According to \[9\], the effective coupling \( J_{\text{eff}} \) is mainly determined by the singlet-triplet gap of the whole system \( H \). From the numerical results of \[18\], the gap of \( H \) for \( L \sim 20 \) is almost the steady one. Therefore, the values of \( J_{\text{eff}} \) saturate rapidly with the increase of the length. This means that the effective coupling can be obtained at finite low temperatures when distant sites are taken infinitely far away. Notice that the parameter \( \Omega_{i,j}^{l,\alpha} \) must be calculated by all of eigenvectors of \( H_c \) and \( \rho_c \) is approximately expressed in the singlet-triplet subspace. For a simplest example of \( L = 2 \), the Hamiltonian of \( H_c \) can be expanded by

\[ H_c = \sum_{\lambda=0}^{2} \epsilon_\lambda \hat{P}_\lambda \]
where the projectors $\hat{P}_\lambda = \sum_{S_{\text{tot}}^z} |S_{\text{tot}} = \lambda, S_{\text{tot}}^z\rangle\langle S_{\text{tot}} = \lambda, S_{\text{tot}}^z|$ and $S_{\text{tot}}^z = -\lambda, \cdots, \lambda$. For very small $|\theta| < \tan^{-1} \frac{1}{3}$, the energy spectrum is given by the ground energy $\epsilon_0 = -2J(\cos \theta - 2 \sin \theta)$, the first excited one $\epsilon_1 = -J(\cos \theta - \sin \theta)$ and the second $\epsilon_2 = J(\cos \theta + \sin \theta)$. Thus the effective coupling is analytically written by

$$J_{\text{eff}} = \frac{e^{-\epsilon_0/T}}{3Z} \left( \frac{4J_p^2 - 4J_p^2 e^{-\Delta/T}}{\epsilon_1 - \epsilon_0} - \frac{5J_p^2 e^{-\Delta/T}}{\epsilon_2 - \epsilon_1} \right). \quad (8)$$

It is necessary to consider the effects of the temperatures and relative strength of biquadratic coupling $\theta$ on the effective coupling. From Fig. 2, it is seen that the values $J_{\text{eff}}$ are increased by the slight increase of $\theta$. For the even length of the chain, the parameters $\Omega_{1,\ell}^\alpha(l = 0, 1)$ are negative. In accordance with Eq.(8), the values of $J_{\text{eff}}$ can be enhanced slightly because the bigger angle $\theta$ leads to the smaller energy gap $\Delta$. For the low temperature, the effective coupling is mainly determined by the first item of $\frac{2J_p^2 e^{-\epsilon_0/T}}{Z} |\Omega_{1,\ell}^{0,2}| = |\Omega_{1,\ell}^{0,1}| = \frac{1+3e^{-\Delta/T}}{1+3e^{-\Delta/T}}$ which is decreased with increasing the temperature.

### III. DECOHERENCE OF ENTANGLEMENT IN THERMAL NOISE

The state of two distant atoms $\rho^{A,B}$ can be gained by tracing out the variables of the chain from the thermal state of the whole system. However, if the temperature $kT \ll \Delta$, we do not expect real excitations of the spin chain to be present. Only the subspace of the states described by $H_{\text{eff}}^{A,B}$ will be populated and then we can calculate the correlations between two atoms using $\rho^{A,B} = e^{-H_{\text{eff}}^{A,B}/T}/Z_q$ where $Z_q = \text{Tr}[e^{-H_{\text{eff}}^{A,B}/T}]$. When two distant sites are simultaneously coupled to the chain, the thermal state $\rho^{A,B}$ can be generated. In accordance with $|\|19, 20, 21, 22, 23\|$ the concurrence of $\rho^{A,B}$ can be written by $C = \frac{1}{Z_q} \max\{0, e^{3J_{\text{eff}}/4T} - 3e^{-J_{\text{eff}}/4T}\}$. Therefore, thermal entanglement exists if the effective coupling satisfies $\frac{J_{\text{eff}}}{T} > \ln 3$. From the point of view of practice, the local operations concerning two distant entangled atoms are needed. It is reasonable to assume that two atoms are coupled with its local thermal reservoirs $E_A, E_B$. According to $[24]$, the two independent reservoirs can lead to the local decoherence of entanglement. Suppose that the initial state at $t = 0$ is $\rho_{\text{tot}} = \rho^{A,B} \otimes (|0_{E_A}0_{E_B}\rangle\langle 0_{E_A}0_{E_B}|)$ where $|0_{E_A}0_{E_B}\rangle$ denotes the vacuum state of the two local reservoirs. The evolution of quantum state between atoms $A$ and $B$ is given by the master equation

$$\dot{\rho}(t) = -i[H_{\text{eff}}, \rho] + \tilde{L}(\rho), \quad (9)$$
where the Lindbald operator

\[
\hat{L}(\rho) = \sum_{i=A,B} (\bar{n}_i + 1) \Gamma_i \left( 2\sigma_i^- \rho \sigma_i^+ - \rho \sigma_i^+ \sigma_i^- + \sigma_i^+ \sigma_i^- \rho \right) + \bar{n}_i \Gamma_i \left( 2\sigma_i^+ \rho \sigma_i^- - \rho \sigma_i^- \sigma_i^+ + \sigma_i^- \sigma_i^+ \rho \right).
\] (10)

Here \( \bar{n}_i = \bar{n} \) is the mean number of the thermal reservoir and \( \Gamma_i = \Gamma \) signifies the rate of spontaneous emission for each atom.

If one of two weak couplings \( J_p \) is turned off after the preparation of the long-distance entanglement, the effective Hamiltonian of two atoms is obtained by \( H_{eff} = H_q + C_{eff} \) which means there is no mutual interaction between atoms. In this case, the evolution of \( \rho(t) \) can be described by a completely positive trace-preserving map \([25]\). For a general two-qubit mixed state \( \rho(0) = \sum_{kl,nn} a_{mn,kl} |kl\rangle\langle mn | \), the evolved state in time can be written by

\[
\rho(t) = \sum_{kl,nn} \sum_{j,j'} a_{mn,kl} (K_{Aj} |k\rangle_A \langle m| K_{Aj}^\dagger) \otimes (K_{Bj'} |l\rangle_B \langle p| K_{Bj'}^\dagger) \text{ where the Kraus operators}
\]

\[
K_{i0} = \sqrt{\frac{\bar{n} + 1}{2\bar{n} + 1}} (|g\rangle_i \langle g| + \sqrt{1-p}|e\rangle_i \langle e|), \quad K_{i1} = \sqrt{\frac{(\bar{n} + 1)p}{2\bar{n} + 1}} |g\rangle_i \langle e|, \quad K_{i2} = \sqrt{\frac{\bar{n}}{2\bar{n} + 1}} (\sqrt{1-p}|g\rangle_i \langle g| + |e\rangle_i \langle e|) \text{ and } K_{i3} = \sqrt{\frac{2p}{2\bar{n} + 1}} |e\rangle_i \langle g|. \]

Here \( |g(e)\rangle_i \) is the ground(excited) state of atoms \( i = A, B \) and \( p(t) = 1 - e^{-\Gamma(2\bar{n} + 1)t} \) means the probability of the atom exchanging a quantum with the reservoir. The density matrix of the quantum state at any time is expanded in the Hilbert space of \( \{ |gg\rangle_{AB}, |ge\rangle_{AB}, |eg\rangle_{AB}, |ee\rangle_{AB} \} \)

\[
\rho(t) = \frac{1}{Z_q} \begin{pmatrix}
    u & 0 & 0 & 0 \\
    0 & x & y & 0 \\
    0 & y & x & 0 \\
    0 & 0 & 0 & v
\end{pmatrix},
\] (11)

The elements of \( \rho(t) \) are expressed by \( u = (1-a)^2 e^{-\frac{3J_{eff}}{4} - \omega}/T + a^2 e^{-\frac{3J_{eff}}{4} + \omega}/T + a(1-a) (e^{-\frac{3J_{eff}}{4}} + e^{3J_{eff}/4}) \), \( v = (1-a)^2 e^{-\frac{3J_{eff}}{4} + \omega}/T + a^2 e^{-\frac{3J_{eff}}{4} - \omega}/T + a(1-a) (e^{-\frac{3J_{eff}}{4}} + e^{3J_{eff}/4}) \), \( x = a(1-a) (e^{-\frac{3J_{eff}}{4} + \omega}/T + e^{-\frac{3J_{eff}}{4} - \omega}/T) + \frac{1}{2} [(1-a)^2 + a^2] (e^{-\frac{3J_{eff}}{4}} + e^{3J_{eff}/4}) \) and \( y = \frac{1}{2} p (e^{-\frac{J_{eff}}{4}} - e^{\frac{3J_{eff}}{4}}) \) where \( a = \frac{bp}{2\bar{n} + 1} \). The concurrence \([19, 20, 21, 22, 23]\) is used to evaluate the long distant entanglement

\[
C = \frac{2}{Z_q} \max\{0, |y| - \sqrt{uv}\}.
\] (12)

On the other hand, it is assumed that the two atoms directly interact with each other in the form of the Hamiltonian given by Eq.(6). In this case, the analytical solution of the master equation is tedious. The expression of the density matrix of quantum states is also...
similar to that of Eq.(11). The decoherence of the thermal entanglement in two cases can be illustrated by Fig. 3(a). It is seen that the entanglement of two qubits without mutual interactions is decreased much more slowly than that of two directly interacting qubits. This point demonstrates that the decoherence time for long distant entanglement is so long as to be useful for the implementation of solid-state quantum computation.

The standard teleportation through the mixed states can be regarded as a general depolarising channel [5]. An arbitrary unknown quantum state $|\Psi\rangle = \cos \frac{\theta}{2} |g\rangle + \sin \frac{\theta}{2} e^{i\phi} |e\rangle$, $(0 \leq \theta \leq \pi, 0 \leq \varphi \leq 2\pi)$ is destroyed and its replica state appears at the remote place after applying the Bell measurement and the corresponding local operations. When single-qubit state $\rho_{in} = |\Phi\rangle\langle \Psi|$ is teleported via the noisy channel of $\rho$ like Eq.(12), the output state $\rho_{out}$ is written by

$$\rho_{out} = \sum_i \text{Tr}[E_i \rho] \sigma_i \rho_{in} \sigma_i^i. \quad (13)$$

In the above equation, $i = 0, x, y, z$ and the projectors $E^0 = |\psi^-\rangle\langle \psi^-|$, $E^i = \sigma^i E^0 \sigma^i$ where a Bell state $|\psi^-\rangle = \frac{1}{\sqrt{2}}(|ge\rangle - |eg\rangle)$. The average fidelity of this teleportation is given by

$$F_A = \frac{1}{\sqrt{2}} \int_0^{2\pi} \int_0^{\pi} F \sin \theta \, d\theta \, d\phi = \frac{1}{6} + \frac{3x - 2y}{3Z_q}. \quad (14)$$

According to [1], the fidelity for a pure input state $F = \{\text{Tr}[(\rho_{in})^{1/2} \rho_{out}(\rho_{in})^{1/2}]\}^2 = \text{Tr}[\rho_{out}\rho_{in}]$. The effect of the thermal noise on the average fidelity of the standard teleportation is illustrated by Fig. 3(b). It is shown that the average fidelity of quantum teleportation with thermal decoherence is larger than 2/3 before a certain time. This means that the quantum teleportation via the channel of long-distance entangled state is better than the classic communication in the range of finite time. In the condition of the thermal noise, the quantum teleportation as the channel of the long-distance thermal entangled state is better than that of the thermal entangled state between two qubits interacting directly.

IV. DISCUSSION

The long-distance thermal entanglement can be obtained when two atoms are weakly interacting with the isotropic spin-1 chain at finite low temperatures. For the massively gapped quantum systems, the scaling law for the effective coupling shows the exponential
increase with the length of the spin chain. Under the influence of thermal noise, the entanglement of two distant qubits without mutual interactions is decreased much more slowly. It is demonstrated that the resource of long-distance entanglement can be used for quantum information processing. We suggest the efficient scheme of the standard teleportation via the channel of long-distance entanglement.

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Figure caption

Figure 1
The finite-size scaling behavior of the effective coupling $J_{\text{eff}}/J$ is demonstrated at the low temperature $T = 0.1$. The square data were obtained with the exact diagonalization method when the couplings $J_p/J = 0.1$ and $\theta = 0$. The solid line was the exponential fit curve for the data.

Figure 2
The effective coupling $J_{\text{eff}}/J$ is plotted as functions of the temperature $T$ and the relative strength of the biquadratic coupling $\theta$ when the couplings $J_p/J = 0.1$.

Figure 3
(a). The decoherence of the entanglement is shown. (b). The average fidelity of the standard teleportation is illustrated. The dash line denotes the case that two qubits interact directly. The solid one refers to the case that one of the weak couplings between the distant atoms and the chain is turned off. The length of the chain is infinite and $J_p/J = 0.1$, $\theta = 0$ and $T = 0.01$. The mean number of the thermal reservoir is chosen to be $\bar{n} = 1$ and the rate of spontaneous emission is $\gamma = 0.1$. 
