DIFFEOMORPHISMS WITH BANACH SPACE DOMAINS

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1. Introduction

The basic element of the following arguments is a \( C^1 \) mapping \( f : X \to Y \), with \( X, Y \) Banach spaces, and with derivative everywhere invertible:

\[
f'(x) \in \text{Isom}(X; Y) \quad \forall x \in X.
\]  

(1.1)

So \( f \) is a local diffeomorphism at every point by the Inverse Function Theorem.

The aim of this paper is to find a sufficient condition for \( f \) to be injective, and so a global diffeomorphism \( X \to f(X) \) (Theorem 2.1), and a sufficient condition for \( f \) to be bijective and so a global diffeomorphism onto \( Y \) (Theorem 3.1). This last condition is also necessary in the particular case \( X = Y = \mathbb{R}^n \).

In these theorems the key role is played by nonnegative auxiliary scalar coercive functions, that is, continuous mappings \( k : X \to \mathbb{R}_+ \) with \( k(x) \to +\infty \) as \( \|x\| \to +\infty \). As far as I know the use of such auxiliary functions in these questions is new. We find some first corollaries. The author hopes that suitable auxiliary functions, adapted to particular problems, may lead to new consequences.

In order to briefly discuss the results, and some of their relations with the literature, let us consider the case where \( X \) is a Hilbert space with scalar product \( \langle \cdot, \cdot \rangle \), and \( k \in C^1 \). This simplifies the formulas a little. However, our results will be formulated and proved for Banach spaces, where we will ask for \( k \) to be locally Lipschitz continuous and to have the right directional derivatives only; these last assumptions are not made to quibble about this matter, but are related to the nondifferentiability of the map \( x \mapsto \|x\|^2 \) in general Banach spaces.

In this paper we use the hypothesis that the operator norm \( \|f'(x)^{-1}\| \) is bounded on bounded sets, i.e.

\[
\sup_{\|x\| \leq r} \|f'(x)^{-1}\| < +\infty, \quad \forall r : 0 < r < +\infty,
\]  

(1.2)

Theorem 2.1 (in the particular case we mentioned above) says that the local diffeomorphism \( f \) is injective if (1.2) holds and there exist a point \( x_0 \in X \), and a coercive function \( k \in C^1(X; \mathbb{R}_+) \), such that

\[
\sup_{x \in X} k'(x)F(x) < +\infty, \quad \text{with} \quad F(x) := -f'(x)^{-1}(f(x) - f(x_0)).
\]  

(1.3)
By a suitable choice of \( k \), the following result is achieved (Corollary 2.2): the local diffeomorphism \( f \) is injective if (1.2) holds and there exist points \( x_0, x_1 \in X \), and non-negative real numbers \( a, b, c \), such that

\[
(x - x_1) \cdot F(x) \leq a + b\|x - x_1\|^2 + c\|f(x) - f(x_0)\|^2, \quad \forall x \in X. \tag{1.4}
\]

To prove Theorem 2.1, the condition in (1.3) is used in connection with the following auxiliary o.d.e. (where \( F \) is as in (1.3))

\[
\dot{x} = F(x). \tag{1.5}
\]

The equation (1.5) plays an important role also in [Zampieri, 1990] to prove the following sufficient condition of invertibility: the restriction of the local diffeomorphism \( f : \Omega = \Omega^o \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^n \) to the ball \( \{ x : |x - x_0| \leq r \} \subset \Omega \), is one-to-one if

\[
(x - x_0) \cdot F(x) \leq 0, \quad \forall x : |x - x_0| = r. \tag{1.6}
\]

More generally, that paper suggests to estimate regions contained in the ‘basin of attraction’ of \( x_0 \) for (1.5) to obtain domains of invertibility of \( f \) around \( x_0 \). The results of [Zampieri, 1990] are generalized to Banach spaces by Gianluca Gorni in [Gorni, 1990].

In Section 3 we turn our attention to a property stronger than one-to-oneness, namely bijectivity. Several authors have dealt with bijectivity of local diffeomorphisms. We refer the reader to [Berger, 1977], [Ortega & Rheinboldt, 1970], [Plastock, 1974], [Prodi & Ambrosetti, 1973], and to [Radulescu & Radulescu, 1980] for clear discussions on these topics, and some applications to Differential Equations.

The auxiliary functions \( k \) are the ‘common denominator’ of the present paper since they are also used in Section 3 where we shall find a sufficient condition (Theorem 3.1) for bijectivity of local diffeomorphisms between Banach spaces. Theorem 3.1 says that the local diffeomorphism \( f : X \rightarrow Y \) is a global diffeomorphism onto \( Y \) if (1.2) holds and there exists a coercive function \( k \in C^1(X; \mathbb{R}_+) \) such that

\[
\sup_{x \in X} \|k'(x) \circ f'(x)^{-1}\| < +\infty. \tag{1.7}
\]

In \( \mathbb{R}^n \) the ‘if’ becomes ‘if and only if’ (and condition (1.2) is always satisfied). As before, our result actually says much more, and we now present the case \( k \in C^1 \) just for the sake of simplicity.

From Theorem 3.1 we first obtain a new proof of the celebrated Theorem of Hadamard generalized by Levy, Meyer, and Plastock to Banach spaces (Corollary 3.2 below, and [Berger, 1977] Section 5.1). This Theorem is based on a condition on \( \|f'(x)^{-1}\| \) which is assumed bounded or of slow growth in \( \|x\| \) (roughly at most linear).

Moreover, from Theorem 3.1 we deduce another known result: \( f \) is a surjective diffeomorphism if (1.1) and (1.2) hold, and \( f \) is coercive, i.e.

\[
\|f(x)\| \rightarrow +\infty \quad \text{as} \quad \|x\| \rightarrow +\infty \tag{1.8}
\]
As is well known, the local homeomorphism \( f : X \to Y \) is bijective if and only if it is proper, i.e. \( f^{-1}(C) \) is compact for any compact \( C \). This last sentence is an important Theorem (see [Berger, 1977] Theorem 5.14) which, for \( X = Y = \mathbb{R}^n \), was glimpsed by Hadamard (see [Hadamard, 1906] Section 17, and [Hadamard, 1968]). Later it was clarified and generalized in [Caccioppoli, 1932], and in [Banach & Mazur, 1934].

In the particular case of local diffeomorphisms \( \mathbb{R}^n \to \mathbb{R}^n \), the framework of our Section 3, represents a unified point of view on the two celebrated Theorems we mentioned above, the former being based on a condition on the growth of \( \|f'(x)^{-1}\| \) (assumed roughly at most linear), and the latter based on properness of \( f \), which in this case is equivalent to coerciveness. Our Section 4 is devoted to the important case of \( \mathbb{R}^n \).

The mapping \( x \mapsto \arctan x, \ x \in \mathbb{R} \), can be proved to be one-to-one by Corollary 2.2 and it is not surjective. A less trivial example \( f : \mathbb{R}^2 \to \mathbb{R}^2 \) is shown in Section 5. The criterion which includes (1.6) cannot be applied to prove the injectivity of that \( f \).

Hopefully, suitable auxiliary functions can lead to new results of global bijectivity, as well as of plain injectivity, in particular in the realm of Differential Equations.

2. Proving injectivity of local diffeomorphisms

We are going to use nonnegative auxiliary scalar coercive functions, that is continuous mappings \( k : X \to \mathbb{R}_+ \) with \( k(x) \to +\infty \) as \( \|x\| \to +\infty \). We also need the directional right derivatives, i.e. for any \( x, v \in X \) we assume the existence of \( D^+_v k(x) := \lim_{s \to 0^+} \frac{k(x+sv)-k(x)}{s} \). Moreover, we need \( k \) to be locally Lipschitz continuous.

**Theorem 2.1.** The mapping \( f \in C^1(X; Y) \) is a global diffeomorphism \( X \to f(X) \) if (i) \( f'(x) \in \text{Isom}(X; Y) \) \( \forall x \in X \), (ii)\n
\[
\sup_{\|x\| \leq r} \|f'(x)^{-1}\| < +\infty, \quad \forall r : 0 < r < +\infty.
\]  

(2.1)

and (iii) there exist a point \( x_0 \in X \) and a locally Lipschitzian coercive function \( k : X \to \mathbb{R}_+ \), which admits all \( D^+_v k(x) \), such that

\[
\sup\{D^+_v k(x) : v = -f'(x)^{-1}(f(x) - f(x_0)), \ x \in X\} < +\infty.
\]

(2.2)

Moreover, under these hypotheses, the range \( f(X) \) is star shaped around \( f(x_0) \), i.e.

\[
y \in f(X), \ s \in [0, 1] \implies f(x_0) + s(y - f(x_0)) \in f(X).
\]

(2.3)

For \( k \in C^1 \) formula (2.2) becomes:

\[
\sup_{x \in X} (-k'(x)f'(x)^{-1}(f(x) - f(x_0))) < +\infty.
\]

(2.4)
Proof. Consider the Cauchy problem
\[ \dot{x}(t) = F(x(t)), \quad x(0) = \bar{x}, \quad \text{with} \quad F(x) := -f'(x)^{-1}(f(x) - f(x_0)). \] (2.5)

We have that:
(a) there is local existence, uniqueness, and continuous dependence on the initial conditions for (2.5),
(b) if \( \mathcal{A} \) is the ‘basin of attraction’ of \( x_0 \), i.e. the set of all the points \( \bar{x} \in X \) such that the maximal solution \( t \mapsto x(t, \bar{x}) \) (with \( x(0, \bar{x}) = \bar{x} \)) is defined for all \( t \geq 0 \) and \( x(t, \bar{x}) \rightarrow x_0 \) as \( t \rightarrow +\infty \), then \( \mathcal{A} \) is open and the restriction \( f|\mathcal{A} \) is one-to-one, and
(c) if \( \bar{x} \in \partial \mathcal{A} \), the boundary of \( \mathcal{A} \), then the maximal solution \( t \mapsto x(t, \bar{x}) \) cannot be global in the future, i.e. it cannot be defined for all positive values of \( t \).

A detailed proof of these facts can be found in the Lemmas of [Gorni, 1990]; here let us just remind the following fundamental facts. We can check at once that, for any \((t, \bar{x})\),
\[ f(x(t, \bar{x})) - f(x_0) = e^{-t} (f(\bar{x}) - f(x_0)). \] (2.6)

This permits to prove (a) and (b). To have (c) we use (b) to say that \( f|\mathcal{A} : \mathcal{A} \rightarrow f(\mathcal{A}) \) is a homeomorphism. This implies the existence of \( d > 0 \) such that \( \|f(x) - f(x_0)\| \geq d \) for all \( x \in \partial \mathcal{A} \). Finally, (2.6) gives \( t \leq \ln(\|f(\bar{x}) - f(x_0)\|/d) \) for \( \bar{x} \in \partial \mathcal{A} \) and \( t \) in the domain of definition of the maximal solution \( t \mapsto x(t, \bar{x}) \).

Now, let us show that, under our hypotheses, the solutions to (2.5) are all global in the future. This implies that the boundary \( \partial \mathcal{A} \) is empty (see (c) above). Thus \( \mathcal{A} = X \) and \( f \) is one-to-one (see (b) above).

Assume that \( t \mapsto x(t) \) is a solution to (2.5) (so in particular \( x(0) = \bar{x} \)) which is defined on \([0, b]\), with \( 0 < b < +\infty \). We are going to prove that it can be extended to \( \mathbb{R}_+ \).

By the hypothesis (iii) there exists a (nonnegative) function \( k \) which satisﬁes (2.2). Let us deﬁne \( \alpha : [0, b) \rightarrow \mathbb{R}_+, t \mapsto k(x(t)) \). Since (by hypothesis) \( k \) admits all \( D^+_x k(x) \) and it is locally Lipschitzian, and \( t \mapsto x(t) \in C^1 \), then the map \( \alpha \) admits the right derivatives and we have \( D^+\alpha(t) = D^+_x k(x(t)) \). This is checked at once by showing in particular that
\[ \lim_{s \rightarrow 0^+} \frac{k(x(t + s)) - k(x(t) + s\dot{x}(t))}{s} = 0. \] (2.7)

Formulas (2.2) and (2.5) give \( D^+\alpha(t) \leq c \) where \( c > 0 \) is the absolute value of the ‘sup’ in (2.2). Therefore, by a standard argument that we show below, we have that
\[ 0 \leq k(x(t)) =: \alpha(t) \leq \alpha(0) + bc. \] (2.8)

So \( x(t) \in k^{-1}([0, \alpha(0) + bc]) \). This last set is contained in a ball, say \( \|x\| \leq \hat{r} \), since \( k : X \rightarrow \mathbb{R}_+ \) is coercive (\( k(x) \rightarrow +\infty \) as \( \|x\| \rightarrow +\infty \)). Thus
\[ \|x(t)\| \leq \hat{r} \quad \forall t \in [0, b). \] (2.9)

One of our hypotheses is (2.1), so we can define \( a := \sup_{\|x\| \leq \hat{r}} \|f'(x)^{-1}\| < +\infty \). Moreover the map \( t \mapsto \|f(x(t)) - f(x_0)\| \) is decreasing (see (2.6)). From (2.9) and (2.5) we then have
Thus $t \mapsto x(t)$ is Lipschitzian, and has a limit as $t \to b$. So it can be extended in the future, and the maximal solution is defined for all positive $t$.

In the particular case where $k \in C^1$, formula (2.2) gives (2.4) at once.

Finally, let $\tilde{y} \in f(X)$, and $s \in (0,1]$. We define $\tilde{x} = f^{-1}(\tilde{y})$ (we just proved that $f$ is 1-1), and consider the maximal solution $t \mapsto x(t, \tilde{x})$. Since $X = \mathcal{A}$, we have that this is globally defined in the future. So we can consider $t := \ln(1/s)$ and (2.6) proves (2.3) for $s \neq 0$. The case $s = 0$ is trivial.

All we are left to prove is formula (2.8) from $D^+\alpha(t) \leq c$. We fix any $\epsilon > 0$. It is enough to prove that $\alpha(t) - \alpha(0) \leq (c + \epsilon)t$. The set where this holds is an interval $I_{\epsilon}$. We argue by contradiction and assume that $\hat{b} := \sup I_{\epsilon} < b$. Then, by the continuity of $\alpha$, we have that $\hat{b} \in I_{\epsilon}$. The existence of $D^+\alpha(\hat{b})$ implies that

$$\frac{\alpha(t) - \alpha(\hat{b})}{t - \hat{b}} \leq D^+\alpha(\hat{b}) + \epsilon \leq c + \epsilon$$

for $\hat{b} < t < b$, with $t$ near $\hat{b}$. So, for such values of $t$, we have (remind that $\hat{b} \in I_{\epsilon}$)

$$\alpha(t) - \alpha(0) = (\alpha(t) - \alpha(\hat{b})) + (\alpha(\hat{b}) - \alpha(0)) \leq (c + \epsilon)(t - \hat{b}) + (c + \epsilon)\hat{b} = (c + \epsilon)t. \quad \square$$

Corollary 2.2. The mapping $f \in C^1(X; Y)$ is a global diffeomorphism $X \to f(X)$ if conditions (i) and (ii) in Theorem 2.1 are satisfied, and there exist points $x_0, x_1 \in X$, and nonnegative real numbers $a, b, c$, such that

$$D^+_{f(x)} g(x) \leq a + b\|x - x_1\|^2 + c\|f(x) - f(x_0)\|^2, \quad \forall x \in X,$$

where $F(x) := -f'(x)^{-1}(f(x) - f(x_0))$, and $g(x) := \|x - x_1\|^2$. \hfill (2.10)

Moreover (2.3) holds. If $X$ is a Hilbert space then formula (2.10) becomes:

$$-2(x - x_1) \cdot f'(x)^{-1}(f(x) - f(x_0)) \leq a + b\|x - x_1\|^2 + c\|f(x) - f(x_0)\|^2, \quad \forall x \quad (2.11)$$

Proof. We can assume that $a \geq b > 2$ (if this is false, we may use $a + b + 3$ and $b + 3$ instead of $a$ and $b$ respectively). Consider the following auxiliary function

$$k(x) := \ln h(x) \quad \text{with} \quad h(x) := \frac{a}{b} + \|x - x_1\|^2 + \frac{c}{b - 2}\|f(x) - f(x_0)\|^2. \quad (2.12)$$

This is trivially coercive. Moreover, it is locally Lipschitzian and it admits all $D^+_x k(x)$. We have

$$h(x) D^+_F(x) k(x) = D^+_F g(x) + 2\frac{c}{b - 2}\|f(x) - f(x_0)\|\lim_{s \to 0^+} \frac{\|f(x + sF(x)) - f(x_0)\| - \|f(x) - f(x_0)\|}{s},$$

where $D^+_F g(x) \leq a + b\|x - x_1\|^2 + c\|f(x) - f(x_0)\|^2, \quad \forall x \in X$. \hfill (2.13)
where $D^+_F(x) g(x)$ and the other limit exist as one checks by using the convexity of the norm. So (2.10) gives
\[
h(x) \cdot D^+_F(x) k(x) \leq a + b\|x - x_1\|^2 + c\|f(x) - f(x_0)\|^2 + 2\frac{c}{b - 2} \|f(x) - f(x_0)\| \|f'(x)F(x)\| = bh(x).
\]
This shows that (2.2) holds. Finally Theorem 2.1 gives Corollary 2.2. \qed

3. Proving bijectivity of local diffeomorphisms

**Theorem 3.1.** The mapping $f \in C^1(X; Y)$ is a global diffeomorphism onto $Y$ if (i) $f'(x) \in \text{Isom}(X; Y) \quad \forall x \in X$, (ii)
\[
\sup_{\|x\| \leq r} \|f'(x)^{-1}\| < +\infty, \quad \forall r : 0 < r < +\infty,
\]
and (iii) there exists a locally Lipschitzian coercive function $k : X \to \mathbb{R}_+$ which admits all $D^+_v k(x)$, and it is such that
\[
\sup\{D^+_v k(x) : v = f'(x)^{-1}u, \ x \in X, \ u \in Y, \ \|u\| = 1\} < +\infty.
\]
For $k \in C^1$ this last formula is equivalent to: $\sup_{x \in X} \|k'(x) \circ f'(x)^{-1}\| < +\infty$.

**Proof.** Consider the Cauchy problem (2.5), that is
\[
\dot{x}(t) = F(x(t)), \quad x(0) = \bar{x}, \quad \text{with} \quad F(x) := -f'(x)^{-1}(f(x) - f(x_0)) \quad (3.3)
\]
where $\bar{x}, x_0$ are any distinct points. Its maximal solution $t \mapsto x(t, \bar{x})$ satisfies (2.6), i.e.
\[
f(x(t, \bar{x})) - f(x_0) = e^{-t} \left(f(\bar{x}) - f(x_0)\right). \quad (3.4)
\]
Since
\[
t \mapsto \|f(x(t, \bar{x})) - f(x_0)\|
\]
is bounded whenever $t$ ranges on a bounded interval, we may just repeat some arguments of the proof of Theorem 2.1 to have the global existence in the future of the solution to our Cauchy problem. In these arguments we consider the derivative $D^+_v k(x)$ with
\[
v = -f'(x)^{-1}u, \quad u = \frac{f(\bar{x}) - f(x_0)}{\|f(\bar{x}) - f(x_0)\|}
\]
(see (3.3) and remind (3.4)). The global existence in the future implies the injectivity of $f$ as we saw in the proof of Theorem 2.1.
Our actual hypothesis (3.2), unlike the one of Theorem 2.1, permits to say that the solution to the Cauchy problem (3.3) is global in the past too. Indeed, we just need to consider the opposite vector field, namely the differential equation
\[ \dot{z}(t) = -F(z(t)), \quad \text{with} \quad F(x) = -f'(x)^{-1}(f(x) - f(x_0)), \]  
(3.5)
whose solution starting at \( \bar{x} \) is \( z(t) = x(-t, \bar{x}) \). Now the map \( t \mapsto \|f(z(t)) - f(x_0)\| \) increases, but what we need is boundedness on bounded intervals only.

So the solution to (3.3) is defined on the whole \( \mathbb{R} \) and \( f \) maps it to the half-line \( \{ y \in Y : y = f(x_0) + \xi (f(\bar{x}) - f(x_0)), \ 0 < \xi < +\infty \} \) (see (3.4)). This implies the surjectivity of \( f \) since \( f(\bar{x}) \) ranges in a full neighbourhood of \( f(x_0) \) (\( f(x_0) \) excluded).

The following theorem is known. We find it again as a consequence of Theorem 3.1.

**Corollary 3.2.** Let \( f \in C^1(X;Y) \), \( f'(x) \in \text{Isom}(X;Y) \ \forall x \in X \). Then \( f \) is a global diffeomorphism onto \( Y \) if there exists a continuous map \( \omega : \mathbb{R}_+ \to \mathbb{R}_+ \setminus \{0\} \) such that
\[ \int_0^{+\infty} \frac{1}{\omega(s)} \, ds = +\infty, \quad \|f'(x)^{-1}\| \leq \omega(\|x\|). \]  
(3.6)

In particular this holds if, for some \( a, b \in \mathbb{R}_+ \), we have
\[ \|f'(x)^{-1}\| \leq a + b\|x\|. \]  
(3.7)

**Proof.** Define
\[ k : X \to \mathbb{R}_+, \ x \mapsto \int_0^{\|x\|} \frac{1}{\omega(s)} \, ds. \]  
(3.8)

By the first condition in (3.6) we have that this function is coercive. Moreover it is locally Lipschitz continuous and it admits all the derivatives \( D^+_{v} k(x) \). For \( v = f'(x)^{-1}u, \ x \in X, \ u \in Y, \|u\| = 1 \), we have (see (3.6))
\[ D^+_{v} k(x) = \frac{1}{\omega(\|x\|)} \lim_{s \to 0^+} \frac{\|x + sv\| - \|x\|}{s} \leq \frac{1}{\|f'(x)^{-1}\|} \|v\| \leq 1. \]

Where the limit exists as one verifies by means of the convexity of the norm. So (3.2) is satisfied, (3.1) holds by (3.6), and Theorem 3.1 gives Corollary 3.2.

\[ \square \]
As we said in Section 1, Corollary 3.2 was discovered by Hadamard in $\mathbb{R}^n$. For Banach spaces it was proved in [Levy, 1920] under condition (3.7) with $b = 0$; [Meyer, 1968] demonstrated that (3.7) is sufficient, and finally [Plastock, 1974] gave a proof for the general statement.

Now, we are going to see another known consequence of Theorem 3.1.

**Corollary 3.3.** The mapping $f \in C^1(X; Y)$ is a global surjective diffeomorphism if (i) it is coercive, i.e. $\|f(x)\| \to +\infty$ as $\|x\| \to +\infty$, (ii) $f'(x) \in \text{Isom}(X; Y)$ $\forall x \in X$, and (iii) the condition in (3.1) is satisfied.

**Proof.** We define the function

$$k(x) := \ln(1 + \|f(x)\|^2). \quad (3.9)$$

This is coercive as well as $f$. Furthermore it admits all $D^+_v k(x)$ and it is locally Lipschitz continuous. For $v = f'(x)^{-1} u$, $x \in X$, $u \in Y$, $\|u\| = 1$, we have

$$D^+_v k(x) = \frac{2\|f(x)\|}{1 + \|f(x)\|^2} \lim_{s \to 0^+} \frac{\|f(x + sv)\| - \|f(x)\|}{s} \leq \frac{2\|f(x)\|}{1 + \|f(x)\|^2} \|f'(x)v\| \leq 1. \quad (3.10)$$

So the condition (3.2) holds and Theorem 3.1 gives Corollary 3.3.

\[\square\]

4. Finite dimension

Of course the case of the Euclidean space is particularly important and relevant applications of global inverse function theorems in finite dimension arise in Numerical Analysis, Network Theory, Economics and other fields (see [Sandberg, 1980]). Let me also mention the Jacobian conjecture for global asymptotic stability. In the plane this conjecture leads to an injectivity problem which is still open (see [Zampieri & Gorni, 1991]).

In $\mathbb{R}^n$ all we have said is simpler. The auxiliary functions $k$ in Theorems 2.1 and 3.1 may be taken $C^1$. Let us see how the previous results can be stated in $\mathbb{R}^n$.

**Theorem 4.1.** Let

$$f : \mathbb{R}^n \to \mathbb{R}^n, \quad f \in C^1, \quad \text{det} f'(x) \neq 0, \quad \forall x \in \mathbb{R}^n. \quad (4.1)$$

Then $f$ is one-to-one if there exists a coercive function $k \in C^1(\mathbb{R}^n; \mathbb{R}_+)$, and a point $x_0 \in \mathbb{R}^n$ such that

$$\sup \{k'(x) F(x) : x \in \mathbb{R}^n\} < +\infty, \quad \text{with} \quad F(x) := -f'(x)^{-1}(f(x) - f(x_0)). \quad (4.2)$$
If one prefers $k'(x) F(x) = \nabla k(x) \cdot F(x)$, i.e. the scalar product of the gradient of $k$ and the vector field $F$. This theorem still gives Corollary 2.2 in the finite dimension, i.e. the following

**Corollary 4.2.** Let $f$ be as in formula (4.1). Then $f$ is one-to-one if there exist points $x_0, x_1 \in \mathbb{R}^n$, and nonnegative real numbers $a, b, c$, such that

$$
(x - x_1) \cdot F(x) \leq a + b|x - x_1|^2 + c|f(x) - f(x_0)|^2, \quad \forall x \in \mathbb{R}^n,
$$

where $F(x)$ is as in (4.2).

Now let us turn our attention to bijectivity. The ‘if’ in Theorem 3.1 and Corollary 3.3 can be substituted by ‘if and only if’. To verify the necessity we just remark that if $f : \mathbb{R}^n \to \mathbb{R}^n$ is a global surjective diffeomorphism then $f'(x) \in \text{Isom}(\mathbb{R}^n; \mathbb{R}^n) \; \forall x \in \mathbb{R}^n$ and $f$ is coercive. Moreover we can just consider the mapping in (3.9), which is coercive as well as $f$, and (3.10) completes the argument.

**Theorem 4.3.** The mapping $f \in C^1(\mathbb{R}^n; \mathbb{R}^n)$ is a (global) diffeomorphism onto $\mathbb{R}^n$ if and only if (i) $\det f'(x) \neq 0$ at every $x \in \mathbb{R}^n$, and (ii) there exists a coercive function $k \in C^1(\mathbb{R}^n; \mathbb{R}_+)$ such that

$$
\sup \{\|k'(x) \circ f'(x)^{-1}\| : x \in \mathbb{R}^n\} < +\infty.
$$

(4.4)

In other words, (4.4) may be written as

$$
\sup \{\|\nabla k(x) \cdot f'(x)^{-1} u\| : x \in \mathbb{R}^n, u \in \mathbb{R}^n, |u| = 1\} < +\infty.
$$

(4.5)

From the last theorem we can easily deduce the following two celebrated results. The proofs above can be easily adapted to $k \in C^1$ (of course the map in (3.8) is not differentiable at $x = 0$ but we can just define $k$ in a different way for $|x| \leq 1$).

**Corollary 4.4.** Let $f$ be as in formula (4.1). Then $f$ is a diffeomorphism onto $\mathbb{R}^n$ if there exists a continuous function $\omega : \mathbb{R}_+ \to \mathbb{R}_+ \setminus \{0\}$ such that

$$
\int_0^{+\infty} \frac{1}{\omega(s)} \, ds = +\infty, \quad \|f'(x)^{-1}\| \leq \omega(|x|).
$$

(4.6)

In particular this holds if, for some $a, b \in \mathbb{R}_+$, we have

$$
\|f'(x)^{-1}\| \leq a + b|x|.
$$

(4.7)

Finally we have the following known Theorem.
Corollary 4.5. The mapping \( f \in C^1(\mathbb{R}^n; \mathbb{R}^n) \) is a (global) diffeomorphism onto \( \mathbb{R}^n \) if and only if (i) \( \det f'(x) \neq 0 \) at every \( x \in \mathbb{R}^n \), and (ii) it is coercive, i.e.

\[
|f(x)| \to +\infty, \quad \text{as} \quad |x| \to +\infty.
\] (4.8)

5. A nonsurjective example

Let us give an example \( f : \mathbb{R}^2 \to \mathbb{R}^2 \) which satisfies the condition (4.3) in Corollary 4.2 but which is not onto \( \mathbb{R}^2 \). Moreover, in this example the left hand side of (4.3) assumes values of both signs on every circumference \( |x| = r > 0 \), so the criterion which includes (1.6) cannot be applied to prove the injectivity.

In the sequel \( x = (\xi, \eta)^T \in \mathbb{R}^2 \) and our function is

\[
f : \mathbb{R}^2 \to \mathbb{R}^2, \quad \begin{pmatrix} \xi \\ \eta \end{pmatrix} \mapsto \frac{e^\xi}{\sqrt{1 + \eta^2}} \begin{pmatrix} 1 \\ \eta(1 + \eta^2) \end{pmatrix}.
\] (5.1)

This mapping is not surjective since its first component is positive.

We are going to prove that condition (4.3) for injectivity is satisfied if we choose \( x_0 = x_1 \) at the origin, \( a = b = 1 \), and \( c = 0 \); namely

\[
x \cdot F(x) \leq 1 + |x|^2, \quad \text{where} \quad F(x) = -f'(x)^{-1} \left( f(x) - f(0) \right).
\] (5.2)

We have

\[
f'(x) = \frac{e^\xi}{(1 + \eta^2)^{\frac{3}{2}}} \begin{pmatrix} 1 + \eta^2 & -\eta \\ \eta(1 + \eta^2) & 1 \end{pmatrix},
\] (5.3)

whose determinant nowhere vanishes. Moreover

\[
f'(x)^{-1} = \frac{e^{-\xi}}{\sqrt{1 + \eta^2}} \begin{pmatrix} 1 & \eta \\ -\eta(1 + \eta^2) & 1 + \eta^2 \end{pmatrix},
\] (5.4)

\[
F(x) = \frac{e^{-\xi}}{\sqrt{1 + \eta^2}} \begin{pmatrix} 1 \\ -\eta(1 + \eta^2) \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix}.
\] (5.5)

Therefore the left hand side of the inequality that we are checking is

\[
x \cdot F(x) = \xi \left[ \frac{e^{-\xi}}{\sqrt{1 + \eta^2}} - 1 \right] - \eta^2 e^{-\xi} \sqrt{1 + \eta^2}.
\] (5.6)

If (i) \( \xi \geq 0 \) or (ii) \( \xi < 0 \) and the term between brackets in (5.6) is nonnegative, then \( x \cdot F(x) \leq 0 \) and formula (5.2) holds. Otherwise we have (iii) \( \xi < 0 \) and the term between brackets is negative too (and greater than \(-1\)). In this last case we have

\[
x \cdot F(x) \leq \xi \left[ \frac{e^{-\xi}}{\sqrt{1 + \eta^2}} - 1 \right] \leq |\xi| \leq 1 + \xi^2 \leq 1 + \xi^2 + \eta^2.
\] (5.7)
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