On the $(-1)$-curve conjecture of Friedman and Morgan.

Rogier Brussee*

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Abstract

We will prove that every differentiably embedded sphere with self-intersection $-1$ in a simply connected algebraic surface with $p_g > 0$ is homologous to an algebraic class. If the surface has a minimal model with Picard number 1 or $|K_{\text{min}}|$ contains a smooth irreducible curve of genus at least 2, and $p_g$ is even or $K_{\text{min}}^2 \not\equiv 7$ (mod 8), then every such sphere is homologous to a $(-1)$-curve, as conjectured by Friedman and Morgan.

1 Introduction.

It is now well known that the deformation type of an algebraic surface is determined by its oriented diffeomorphism type up to a finite number of choices [FM2, F-M, theorem S.2]. It is therefore natural to ask if a deformation invariant is in fact an invariant of the underlying oriented differentiable manifold. For example, Van de Ven conjectured that this is true for the Kodaira dimension [O-V, F-M, P-T]. In this paper we study whether the deformation invariant decomposition

$$H_2(X) = H_2(X_{\text{min}}) \oplus \mathbf{Z}E_i,$$  \hspace{1cm} (1)

in the homology of the minimal model and the span of the $(-1)$-curves is invariant under orientation preserving diffeomorphisms (cf. [FM1, conj. 2])

A $(-1)$-curve on a complex surface is a smooth holomorphically embedded 2-sphere with self-intersection $-1$. A $(-1)$-curve can be blown down to obtain a

* Math. inst. Oxford university, 24-29 St. Giles, OX1 3LB Oxford UK, e-mail: brussee@maths.oxford.ac.uk, fax:+44-865273583
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new smooth complex surface. Successively contracting all \((-1)\)-curves gives the minimal model \(X_{\text{min}}\) which is unique if \(p_g > 0\). More generally we will call the total transform of a \((-1)\)-curve on some intermediate blow-down a \((-1)\)-curve as well. It is in this sense that the decomposition \(\square\) is a deformation invariant.

A \((-1)\)-sphere on a 4-manifold is a smooth differentiably embedded 2 sphere with self-intersection \(-1\). A classical \((-1)\)-curve is obviously a \((-1)\)-sphere, a reducible one can be deformed to a \((-1)\)-sphere by smoothing out the double points. Moreover if two \((-1)\)-curves are orthogonal, they can be deformed in disjoint \((-1)\)-spheres. Friedman and Morgan conjectured that if a surface has a unique minimal model, then modulo homological equivalence the relation between its \((-1)\)-spheres and its \((-1)\)-curves is the strongest possible.

**Conjecture 1.1.** (Friedman and Morgan \[FM1, conj. 2,3 prop. 4\]) Let \(X\) be a simply connected algebraic surface with Kodaira dimension \(\kappa \geq 0\). Then every \((-1)\)-sphere is homologous to a \((-1)\)-curve up to orientation. In particular the decomposition \(\square\) is invariant under orientation preserving diffeomorphisms.

(I have slightly reformulated the conjecture, and added the simply connectedness hypothesis). Now, \(X\) contains \(n\) disjoint \((-1)\)-spheres if and only if there is a differentiable connected sum decomposition \(X \cong_{\text{diff}} X' \# n\mathbb{P}\). The decomposition \(\square\) can be thought of as being induced by this special connected sum decomposition. Friedman and Morgan also made a conjecture about more general connected sum decompositions, which would imply conjecture \(\square\) above.

**Conjecture 1.2.** (Friedman and Morgan \[FM1, conj. 9\]) Let \(X\) be a simply connected algebraic surface with \(\kappa \geq 0\). Suppose \(X\) admits a connected sum decomposition \(X \cong_{\text{diff}} X' \# N\) for a negative definite manifold \(N\), then \(H_2(N, \mathbb{Z})\) is generated by \((-1)\)-curves.

Note that by theorems of Donaldson (\[D-K, th. 1.3.1, 9.3.4, 10.1.1\]), \(N\) has automatically a standard negative definite intersection form if \(p_g(X) > 0\). Conjecture \(\square\) (and hence conjecture \(\square\)) has been proved for blow-ups of simply connected surfaces with \(p_g > 0\) and big monodromy (like elliptic surfaces or complete intersections), and simply connected surfaces with \(p_g > 0\) whose minimal model admits a spin structure (i.e. \(K_{\text{min}} \equiv 0(2)\)) \[FM, cor. 4.5.4\]. Conjecture \(\square\) has been proved for the Dolgachev surfaces (i.e. \(\kappa = p_g = 0\)).

For minimal surfaces, conjecture \(\square\) would imply strong minimality. A 4-manifold is called strongly minimal if for every diffeomorphism \(X \# N_1 \cong_{\text{diff}} Y \# N_2\) with \(N_i \cong n_i\mathbb{P}\), we have \(H_2(N_2) \subset H_2(N_1)\) (c.f. \[FM, def. IV.4.6\]). Conjecture \(\square\) would also imply that the canonical class of the minimal model \(K_{\text{min}}\) is invariant mod 2 under orientation preserving diffeomorphisms. Conjecture
1.2 would imply that a minimal surface is irreducible i.e. for every decomposition $X \cong X' \# N$, say $N$ is homeomorphic to $S^4$, thereby avoiding the Poincaré conjecture.

In this paper we will show that under the stronger assumption $p_g > 0$, $(-1)$-spheres must give rise, if not to $(-1)$-curves then in any case to special algebraic 1-cycles. Indeed, the main theorem 1.3 below leaves little room for $(-1)$-spheres not homologous to a $(-1)$-curve. Furthermore we will reduce a similar statement for general connected sum decompositions, to a technical problem in gauge theory.

To state the theorem we need some notation. Let $N_1(X)_{\mathbb{Z}} \subset H_2(X, \mathbb{Z})$ be the preferred subgroup of algebraic classes i.e. the subgroup generated by algebraic curves. Its rank $\rho$ is the Picard number. The effective cone $\text{NE}(X) \subset N_1(X)_{\mathbb{Q}}$ is the cone spanned by positive rational multiples of algebraic curves. The subcone $\text{NE}(X_{\text{min}}) = \text{NE}(X) \cap H_2(X_{\text{min}}, \mathbb{Q})$ is the cone spanned by the pullbacks of rational curves on the minimal model, i.e. the effective cone in $H_2(X_{\text{min}})$. Finally we note that since $N_1(X)_{\mathbb{Q}}$ is a finite dimensional vector space, the closure of the effective cone $\text{NE}(X)$ is well defined.

**Theorem 1.3.** Let $X$ be a simply connected algebraic surface with $p_g > 0$ and let $K$ be its canonical divisor. Then for every $(-1)$-sphere in $X$, there is an orientation such that $e$ is either represented by a $(-1)$-curve or $e \in \text{NE}(X_{\text{min}})$, depending on whether $K \cdot e$ is negative or positive respectively.

Note that $K \cdot e \neq 0$ since $K \cdot e \equiv e^2 \pmod{2}$. I have no examples where $K \cdot e$ is positive (i.e. a counter-example to conjecture 1.1) but without further assumptions neither can I exclude this case.

**Corollary 1.4.** In addition to the assumptions of the theorem suppose that the minimal model $X_{\text{min}}$ has Picard number 1 or that the linear system $|K_{\text{min}}|$ contains a smooth irreducible curve of genus at least two, and that $p_g$ is even or $K_{\text{min}}^2 \equiv 7 \pmod{8}$, then every $(-1)$-sphere is homologous to a $(-1)$-curve (i.e. conjecture 1.1 is true for $X$).

we will use this corollary to prove conjecture 1.1 for blow-ups of Horikawa surfaces with $K_{\text{min}}^2$ even and zero-sets of general sections in sufficiently ample $n-2$-bundles on $n$-folds with $\rho = 1$ generalising Friedman and Morgan’s result for complete intersections in $\mathbb{P}^n$.

The proof of theorem 1.3 is based on two very general properties of the SO(3) Donaldson-Kotschick invariant $\phi_k$. Kotschick observed that it follows from the invariance properties of the $\phi_k$ polynomial, that it is divisible by the Poincaré dual of a $(-1)$-sphere. On the other hand, using Morgan’s algebro geometric description of the Donaldson polynomials we show that $\phi_k$ has pure Hodge type
for $k \gg 0$. The theorem and the corollary then follow by using Donaldson’s and O’Grady’s non-triviality results.

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2 The $\phi_k$ polynomials.

We will need SO(3) Donaldson polynomials $q_{L,k,\Omega}$ and in particular the $\phi_k$ invariant introduced by Kotschick [Kot]. Let $X$ be a simply connected 4-manifold with odd $b_+$ $\geq 3$. The polynomial $q_{L,k,\Omega}$ on $H_2(X)$ corresponds to the moduli space $\mathcal{M}^{\text{asd}}(L,k)$ of ASD SO(3)-connections on the SO(3)-bundle $P_k$ with $w_2(P_k) \equiv L \pmod{2}$, and $p_1(P_k) = -4k$, oriented by the choice of the lift $L$ of $w_2(P_k)$, and an orientation $\Omega$ of a maximal positive subspace in $H^2(X, \mathbb{R})$ [D-K, §9.2]. We choose $\Omega$ once and for all (e.g. using a complex structure if present), and we will suppress it in the notation. $q_{L,k}$ has degree $d = 4k - \frac{3}{2}(1 + b_+)$. Note that the SO(3) bundle $P_k$ exists if and only if $p_1 \equiv K^2 \pmod{4}$, and that $k \in \frac{1}{4}\mathbb{Z}$. To define $\phi_k(X)$ we lift the second Stiefel Whitney class $w_2(X)$ of the manifold to an integral class $K$. For complex surfaces, the canonical divisor is such a lift. Now define $\phi_k = q_{K,k}$. $\phi_k(X)$ is invariant under orientation preserving diffeomorphisms up to sign.

Now suppose that $X$ has a decomposition $X \cong X’ \# N$ for a negative definite manifold $N$, necessarily with standard intersection form. Then we have a decomposition $H_2(X) = H_2(X’) \oplus H_2(N)$. Choose generators $e_1, \ldots, e_n$ of $H_2(N)$, such that $K \cdot e_i \equiv -1 \pmod{4}$. This fixes the generators up to permutation. By Poincaré duality we can consider the generators $e_i$ as linear forms on $H_2(X)$. Any polynomial $Q$ on $H_2(X)$ can be uniquely written as a polynomial in the dual classes of $e_i$ with polynomials on $H_2(X’)$ as coefficients (the $e$-expansion).

Definition 2.1. A 4-manifold is said to have a good connected sum decomposition $X \cong X’ \# N$ if for every generator $e_i$ of $H_2(N)$, $e_i$ divides $q_{L,k}(X)$ for all $L$ with $L \cdot e_i$ odd and all $k \gg 0$. 

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Proposition 2.2. (Kotschick [Kot, prop. 8.1]) A connected sum decomposition $X \cong X' \# n\mathbb{P}^2$ is a good connected sum decomposition.

Proof (sketch). For notational simplicity we only prove divisibility for the $\phi_k$ invariant. The reflection $R_e$ in the hyperplane defined by a generator $e$ of $H_2(n\mathbb{P}^2)$ can be represented by an orientation preserving diffeomorphism, for example id#C-conjugation (c.f. [FM2, prop 2.4]). Then it follows from the general invariance properties of the SO(3)-polynomials [D-K, 9.2.2] that $R_e^* \phi_k(X) = -\phi_k(X)$. Hence $\phi_k(X)$ is an odd polynomial in the dual class of $e$, in particular $\phi_k$ is divisible by $e$. 

One should expect that any connected sum decomposition is good. This is because the coefficients $q_{L,k,N,I}$ of the $e$-expansion have the same invariance properties as $q_{L,k}$ under orientation preserving diffeomorphisms of $X'$. Conjecturally, these invariants depend only on the homotopy type of $N$, (c.f. [F-M, conjecture above lemma 4.5.6]) and so the argument for $N = n\mathbb{P}^2$ would give the divisibility by the generators in general. A naive gauge theoretic analysis seems to confirm this conjecture, but some technical difficulties remain to be overcome. In any case we will state and prove our results for surfaces admitting a good connected sum decomposition.

3 Pureness of the Donaldson polynomials.

Now we come to the algebraic geometric part of the proof. The Hodge structure on $H^2(X, \mathbb{Z})$ induces a natural Hodge structure on $S^dH^2(X)$. Let

$$S^dH^2(X) \hookrightarrow H^{2d}(X \times \cdots \times X)$$

be the natural injection in the cohomology of the $d$-fold product of $X$. Then $j$ is a map of Hodge structures. Hence a polynomial $q \in S^dH^2(X)$ is of pure Hodge type $(d, d)$ if and only if $j(q)$ is pure of Hodge type $(d, d)$. Now clearly a sufficient condition for $j(q)$ to be of type $(d, d)$ is that it is represented by an algebraic cycle. We will prove the rather natural statement that those Donaldson polynomials that can be computed completely by algebraic geometry give rise to algebraic cycles. However to make this statement precise requires serious work (as so often in mathematics). Fortunately almost all of the work has already been done by J. Morgan [Mor].

Proposition 3.1. Let $X$ be a simply connected algebraic surface with $p_g > 0$. Then if $L \in NS(X)$, there is constant $k_0 > 0$ such that all Donaldson polynomials $q_{L,k}$ with $k > k_0$ and the integer $\frac{1}{2}(L^2 - L \cdot K) - \frac{1}{4}(L^2 + 4k)$ odd are
represented by algebraic cycles. In particular these polynomials are of Hodge type \((d, d)\), where \(d = \deg(q_{L, k}) = 4k - 3(1 + p_g)\).

**Proof.** First suppose that \(L \equiv 0 \pmod{2}\), then \(q_{L, k}\) is up to sign just the SU(2) polynomial \(q_k\). Now the lemma follows directly from recent results of Morgan \([\text{Mor}]\). He shows that for odd \(k \gg 0\), \(q_k\) can be computed as follows.

Let \(\overline{M}^G_k = \overline{M}^G_k(H, 0, k)\) be the closure of the moduli space of \(H\)-slope stable bundles in the moduli space of Gieseker \(H\)-stable sheaves with \(c_1 = 0\), \(c_2 = k\). For odd \(c_2\), there exists a universal sheaf \(\xi\) on \(\overline{M}^G_k\) (cf. \([\text{Mu}, \text{Remark A7}]\) and \([\text{OG}, \text{prop. 2.2}]\)) which determines a correspondence

\[
\nu: H_2(X) \rightarrow H^2(\overline{M}^G_k)
\]

\[
\Sigma \rightarrow c_2(\xi)/\Sigma.
\]

Then if \(H\) is \(k\)-generic (in a sense to be made more precise below) and \(k \gg 0\), we have \([\text{Mor}, \text{theorem 1}]\)

\[
q_k(\Sigma) = \left\langle (\Sigma)^d, [\overline{M}^G_k] \right\rangle. \tag{2}
\]

Now since the universal sheaf \(\xi\) is algebraic, it actually determines Chow cohomology classes \(c_2(\xi) \in A^2(X \times \overline{M}^G_k)\) (cf. \([\text{Ful}, \text{Definition 17.3}]\)). Consider the diagram

\[
\begin{array}{ccc}
X^d \times \overline{M}^G_k & \xrightarrow{\pi_{X^d}} & (X \times \overline{M}^G_k).
\end{array}
\]

Then by equation (2), the algebraic cycle

\[
j(q_k) = \int_{[\overline{M}^G_k]} \pi_{X^d}^* c_2(\xi) \cdots \pi_{X^d}^* c_2(\xi) \in A^d(X^d) \cong A_d(X^d),
\]

represents the image of the Donaldson polynomial \(q_k\) on the level of Chow groups. (Integration over the fibre \(\int_{[\overline{M}^G_k]}\) is defined formally as the composition

\[
A^i(X^d \times \overline{M}^G_k) \xrightarrow{[\pi_{X^d}]} A^{i-d}(X^d \times \overline{M}^G_k) \rightarrow A^d(X^d) \xrightarrow{\pi_{X^d}} A^{i-d}(X^d),
\]

where \([\pi_{X^d}]\) is the orientation class of the flat map \(\pi_{X^d}\) (cf. \([\text{Ful}, \text{section 17.4}]\)). This proves the lemma if \(L \equiv 0 \pmod{2}\).

In case \(L \not\equiv 0 \pmod{2}\), the results of Morgan carry over virtually unchanged, in fact the corresponding results are rather easier. To be more precise, for a Hodge metric \(g_H\), the moduli space of irreducible ASD SO(3)-connections
with \( w_2 \equiv L \pmod{2} \) and \(-p_1 = 4k\) can be identified with the moduli space \( \mathcal{M}_k \) of \( H \)-slope stable bundles with \( c_1 = L \) and \( 4c_2 - c_1^2 = 4k \). For \( k \gg 0 \), the closure of the moduli space of \( H \)-stable bundles in the moduli space of Gieseker stable sheaves \( \mathcal{M}_k^G \) has the proper complex dimension \( d = 4k - 3(1 + p_g) \) and is generically smooth. Moreover \( \mathcal{M}_k^G \) carries a universal sheaf \( \xi \) [O’G, prop. 2.2], and the class \( c_2(\xi) - \frac{1}{16} c_1^2(\xi) \) defines a \( \nu \) correspondence and Chow cohomology classes just as in the discussion above. Finally, we choose a polarisation \( H \) which is \( k \)-generic in the sense that \( H \cdot (L - 2N) \neq 0 \) for all \( N \in \text{NS}(X) \) with \(-4k \leq (L - 2N)^2 < 0\) (i.e. \( H \) is not on a wall). Then since \( L \neq 0 \) \pmod{2} every Gieseker \( H \)-semistable sheaf is actually slope \( H \)-stable. Now all of the discussion in [Mor] to relate the Gieseker and the Uhlenbeck compactification, and the \( \mu \) and \( \nu \) correspondence as far as it is concerned with slope stable sheaves and bundles carries over. 

\( \square \)

Remark 3.2. The pureness of the polynomials \( q_{L,k} \) is also suggested by a differential geometric argument, which seems to be the point of view taken by Tyurin [Tyu, §4.22]. The complex structure on \( X \) induces the complex structure \( T_{10}^* \mathbb{B}_X^* = \{ a \in A^{10}(\text{End}_0(V)), \ d^* a = 0 \} \) on the space of irreducible connections modulo gauge \( \mathbb{B}_X^* \). The space of irreducible ASD connections with respect to a Kähler metric is then an analytic subspace. Now the explicit formulas in [D-K, Proposition 5.2.18] for the forms representing \( \mu(\text{Pd}(\omega)) \) for a harmonic form \( \omega \in H^2(X, \mathbb{C}) \), show that \( \mu \) preserves the Hodge structure. If we write formally

\[
\phi_k(\text{Pd}(\omega_1), \ldots, \text{Pd}(\omega_d)) = \int_{\mathcal{M}_k^{\text{red}}} \mu(\text{Pd}(\omega_1)) \cdots \mu(\text{Pd}(\omega_d)),
\]

then it is clear that \( \phi_k(X) \neq 0 \) only if the total Hodge type of \( \omega_1, \ldots, \omega_d \) is \((d, d)\). The (probably inessential) problem is that it is not \textit{a priori} obvious whether integrating the form representatives over the non compact manifold \( \mathcal{M}_k^{\text{red}} \) gives a valid way of computing \( \phi_k \).

### 4 Proof of theorem 1.3.

We can now give proofs of the results stated in the introduction. The main theorem 1.3 is the special case of theorem 4.1 below for \( N = n\mathbb{P}^2 \) (see definition 2.1 for the definition of good connected sum decomposition).
Theorem 4.1. Let $X$ be a simply connected algebraic surface with $p_g > 0$. Let $X_{\text{min}}$ be its minimal model and let $K$ be its canonical divisor. Suppose $X$ admits a good smooth connected sum decomposition $X \overset{\text{def}}{=} X' \# N$ for a negative definite manifold $N$. Then $H_2(N, \mathbb{Z})$ is generated by classes $e_1, \ldots, e_n$, with $e_i^2 = -1$ such that either $e_i$ is represented by a $(-1)$-curve, or $e_i \in \overline{NE}(X_{\text{min}})$ depending on whether $K \cdot e_i$ is negative or positive respectively.

Here $\overline{NE}(X_{\text{min}})$ is the closure of the cone spanned by positive rational multiples of algebraic curves on the minimal model.

Proof. Choose a generator $e$ of $H_2(N)$. We first prove that $e$ is homologous to an algebraic cycle. Since $e$ is certainly integral, it suffices by the Lefschetz $(1, 1)$ theorem [G-H, p. 163] to prove that its Poincaré dual is of pure type $(1, 1)$. Choose $k$ sufficiently large as in proposition 3.1 (with $L = K$), and the definition 2.1 of good. Then $\phi_k(X)$ is non trivial and has pure Hodge type $(d, d)$. On the other hand we have $\phi_k = e\psi$. Since the number of Hodge types of $e\psi$ is at least the number of Hodge types of $e$, $e$ has to be of pure type as well. Since $e$ is a real class, it is then of type $(1, 1)$.

To show that for the proper orientation $e$ lies on the closure of the full effective cone $\overline{NE}(X)$, it is enough to show that $e \cdot H \neq 0$ for all ample divisors $H$. In fact, since the closure of the effective cone and the nef cone are in duality [Vil, proposition 2.3], $e \in \pm \overline{NE}(X)$ if and only if $e$ defines a strictly positive or strictly negative form on the ample cone. But since the ample cone is connected, it suffices to show that the form $e$ has no sign change i.e. does not vanish on the ample cone. Since $e$ is a rational class we need to check this only for integral ample classes. Now for a fixed ample divisor $H$ there is a $k_0 = k_0(H)$ such that $\phi_k(X)(H) \neq 0$ for $k > k_0$ [D-K, th. 10.1.1]. Since $e$ divides $\phi_k(X)$, it follows that $e \cdot H \neq 0$.

By the orthogonality result [F-M, th. 4.5.3], for every $(-1)$-sphere $S$ in $X$ we have either $e \cdot S = 0$ or $e = \pm |S|$. Hence for the “effective orientation” of $e$ found above, $e$ is either homologous to a $(-1)$-curve, or $e$ is orthogonal to all $(-1)$-curves, i.e. $e \in H_2(X_{\text{min}})$. In the first case $e \cdot K = -1 < 0$, in the latter case we have $e \cdot K = e \cdot K_{\text{min}} > 0$ for as $p_g$ is positive, $K_{\text{min}}$ is nef, and $K \cdot e \equiv e^2 \equiv 1 \pmod{2}$. Since a divisor on the minimal model is effective if and only if its pullback to $X$ is effective, we have $\overline{NE}(X_{\text{min}}) = \overline{NE}(X) \cap H_2(X_{\text{min}})$, and the result follows.

Theorem 1.1 gives the following technical refinement of corollary 1.4, proving conjecture 1.2 for the pair $(X, N)$ under an additional hypothesis.

Corollary 4.2. In addition to the assumptions of theorem 1.1, suppose $X$ has a deformation $Y$ with a minimal model $Y_{\text{min}}$ such that there are no classes
$C \in \overline{\text{NE}}(\text{Y}_\text{min})$ with $C^2 = -1$ dividing all Donaldson polynomials $q_{L,k}$ with $L \cdot C$ odd and $k \gg 0$. Then $H^2(N)$ is generated by $(-1)$-curves. In particular this is true if

(a) the linear system $|K_{\text{Y}_\text{min}}|$ contains a smooth irreducible curve of genus at least 2, and either $p_g$ is even or $K_{\text{min}}^2 \not\equiv 7 \pmod{8}$, or

(b) the Picard number $\rho(\text{Y}_\text{min}) = 1$.

As mentioned in the introduction this corollary has already been proved without the goodness condition under the assumptions $X_{\text{min}}$ is spin or $X_{\text{min}}$ has big monodromy [F-M, cor 5.4].

Proof. Since the deformations of a surface are all oriented diffeomorphic, we conclude that if $X$ admits a good connected sum decomposition $X \cong X' \# N$, so does its deformation $Y$. Moreover, the subgroup generated by $(-1)$-curves is stable under deformation by [BPV, IV.3.1]. Hence if $H_2(N) \subset H_2(Y)$ is generated by $(-1)$-curves, so is $H_2(N) \subset H_2(X)$. Thus we can assume $X = Y$.

By definition 2.1 of a good decomposition, it is clear that no generator of $H^2(N)$ can be in $\text{NE}(\text{X}_\text{min})$. Hence by theorem 4.1, $H^2(N)$ is generated by $(-1)$-curves. It remains to see that the extra condition is satisfied in the given special cases.

In case (b), NS$(X_{\text{min}})$ is positive definite. For case (a) we argue by contradiction. Suppose there is a class $C \in \text{NE}(X_{\text{min}})$, $C^2 = -1$ and $C$ divides $q_{C,k}$ for all $k \gg 0$.

Since $C$ is orthogonal to all $(-1)$-curves, the proof of theorem [Do2, th. 4.8], [D-K, th. 9.3.14] gives that

$$q_{C,k}(X)|_{H_2(X_{\text{min}})} = \pm q_{C,k}(X_{\text{min}}).$$

Since $C \in \text{NS}(X_{\text{min}})$, Morgan’s comparison formula [2] and O’Grady’s non triviality result [O’G, cor. 2.4, th. 2.4], give that for every $\omega \in H^0(K_{\text{min}})$, which vanishes on a smooth irreducible curve of genus $g \geq 2$ we have

$$q_{C,k}(X_{\text{min}})(\text{Pd}(\omega + \overline{\omega})) \neq 0$$

if $4k - 3(1 + p_g)$ is even and $k \gg 0$ with $\frac{1}{4}(C^2 - C \cdot K) - \frac{1}{4}(C^2 + 4k)$ odd. (Strictly speaking O’Grady uses a slightly different polynomial defined on $C^1 \subset H_2(X)$, but it is easy to see that on $C^1$, $q_{C,k}$ coincides with his polynomial).

Since $4k \equiv -C^2 \pmod{4}$, and $\langle C, \omega \rangle = 0$ this contradicts the divisibility of $q_{C,k}$ by $C$ if $p_g$ is even. If $p_g$ is odd, the same argument gives a contradiction if there is a polynomial $q_{L,k}$ with $L \in \text{NS}(X_{\text{min}})$, $L \cdot C \equiv 1$, and $L^2 \equiv K_{\text{min}} \cdot L \equiv 0$. This an affine equation for $L \pmod{2}$ in $\text{NS}(X_{\text{min}}) \otimes \mathbb{Z}/2\mathbb{Z}$, so it has a solution if $C \neq K_{\text{min}} \pmod{2}$. But if $C \equiv K_{\text{min}}$, then $C^2 = -1 \equiv K_{\text{min}}^2 \pmod{8}$ contrary to assumption. \qed
Remark 4.3. If $|K_{\text{min}}|$ contains a smooth irreducible curve but $p_g$ is odd and $K_{\text{min}}^2 \equiv 7 \pmod{8}$ the proof above shows that there is up to orientation at most one generator $e_0$ of $H^2(N)$ which is not homologous to a $(-1)$-curve. Hence all $(-2)$-spheres in $H_2(X_{\text{min}})$ are orthogonal to $e_0$, because the reflections they generate are represented by diffeomorphisms. We also get that if $e_0$ exists, $w_2(X)$ is represented by the sum of the generators of $H_2(N)$, hence $X'$ is spin.

Remark 4.4. It follows from results in [Br2] that if $C$ is orthogonal to all $(-1)$-curves, then the divisibility of say $\phi_k(X)$ by $C$ implies the divisibility of $\phi_k(X')$ by $C$ for all blow-downs of $X'$ intermediate between $X$ and $X_{\text{min}}$. Hence the extra condition in corollary 4.2 gets stronger as we blow-up more points.

It is interesting to see how the minimality of a surface plays a role here. For a non minimal surface every curve in $|K|$ is reducible since it contains an exceptional divisor, unless $X$ is a K3-surface blown-up once. Indeed in the non minimal case $q_{E,k}$ is divisible by the exceptional divisor $E$, and so O’Grady’s theorem could not possibly be true. It would be very interesting if O’Grady’s results would be true assuming the existence of an irreducible curve in $|K|$, or stretching things even further, a smooth irreducible curve in a pluricanonical system $|nK|$.

5 Examples.

We give two examples of the use of corollary 4.2. The first example follows basically by leafing through [BPV]. For the other we use Noether-Lefschetz theory to reprove and generalise Friedman and Morgan’s result that for complete intersections with $p_g > 0$ conjecture 1.1 is true. Since one approach to Noether-Lefschetz theory is through monodromy groups, it is not surprising that this approach gives results similar to those using big monodromy. However the proof may be interesting because the Noether Lefschetz theorems we will use are proved using the “infinitesimal method”, based on Hodge and deformation theory.

**Proposition 5.1.** Suppose $X$ is a smooth simply connected surface admitting a good connected sum decomposition $X \cong X' \# N$. Then $H_2(N)$ is generated by $(-1)$-curves if

(a) $X$ is the blow-up of a Horikawa surface with $K^2$ even (see [BPV, table 10]),

or

(b) $(Y, O(1))$ is a simply connected projective local complete intersection of dimension $r + 2$ with Picard number $\rho = 1$, $V$ is an ample $r$-bundle, and $X$ is the blow up of the smooth zero locus $X_a$ of a section in $V(a)$ with $a \gg 0$. 

Proof. Case (a). By \([\text{BPV, th. 10.1, remark VII.10.1}]\) and Bertini’s theorem, a Horikawa surface with \(K^2\) even is simply connected and the linear system \(|K|\) contains a smooth curve.

Case (b). For \(r = 0\) the proposition is a special case of corollary \([4.2]\) so we assume \(r > 0\).

Choose \(a\) so large that \(V(a)\) is globally generated. By an application of the vector bundle version of the Lefschetz hyperplane theorem \([S-V], \text{Oko, cor. 22}\), \(X_a\) is simply connected \([G-H, \text{p.158}]\). To prove that \(p_g(X_a) > 0\) consider the sequence

\[
0 \rightarrow I_X \det(V(a)) \otimes O_Y(K_Y) \rightarrow \det(V(a)) \otimes O_Y(K_Y) \rightarrow O_X(K_X) \rightarrow 0.
\]

Now choose \(a\) so large that \(\det V \otimes O_Y(K_Y) \otimes O_Y(ra)\) has a section non-vanishing on \(X\). Finally, if \(a\) is sufficiently large then \(\rho(X_a) = 1\) for the general section by Ein’s generalization of the Noether-Lefschetz theorem in \(\mathbb{P}^3\) to ample vector bundles on projective varieties \([\text{Ein, th. 2.4}]\). \(\square\)

Remark 5.2. Suppose that in case (b), \(Y\) is smooth, and \(V = \bigoplus_{i=1}^r O(d_i)\).

We can then be more precise since there is no need to twist up. It suffices that \(V\) and \(\det(V) \otimes O_Y(K_Y)\) are spanned by global sections, \(H^{1,1}(\det(V)) = H^{1,1}(V \otimes \det(V)) = 0\), and

\[
H^0(V) \otimes H^0(\det(V) \otimes O_Y(K_Y)) \rightarrow H^0(V \otimes \det(V) \otimes O_Y(K_Y))
\]

is surjective (e.g. if the \(d_i \gg 0\) or \(Y = \mathbb{P}^n\), with the exception of \(n = 3\), \(V = O(2)\) or \(O(3)\), and \(n = 4\), \(V = O(2) \oplus O(2)\)). This follows from judicially checking the cohomological conditions \([\text{Spa, lemma 3.2.1,3.2.2,3.2.3}]\) using Kodaira-Nakano vanishing. For \(\mathbb{P}^n\) the statement follows from the classical Noether Lefschetz theorem. Also note that by choosing \(a\) sufficiently large we can make \(K_{X_a}\) very ample, and so we can find a smooth irreducible curve of genus at least 2 in its linear system. Hence case (b) follows directly from corollary \([4.2]\) if \(p_g(X_a)\) or \(K_{X_a}^2\) are even.

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