Rotating Intersecting M-Branes

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Abstract

We present intersecting p-brane solutions of eleven-dimensional supergravity (M-branes) which upon toroidal compactification reduce to non-extreme rotating black holes. We identify harmonic functions, associated with each M-brane, and non-extremality functions, specifying a deviation from the BPS limit. These functions are modified due to the angular momentum parameters, which specify the rotation along the transverse directions of the M-branes. We spell out the intersection rules for the eleven-dimensional space-time metric for intersecting (up to three) rotating M-brane configurations (and a boost along the common intersecting direction).

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I. INTRODUCTION

Intersecting, BPS-saturated $p$-brane configurations in string theory and eleven-dimensional supergravity became subjects of intense research since such configurations reduce, upon (toroidal) dimensional compactification, to BPS saturated black holes with non-singular horizons (with non-zero Bekenstein-Hawking (BH) entropy). Within the framework of eleven-dimensional supergravity, such configurations correspond to the intersecting two-branes and five-branes of eleven-dimensional $M$-theory \[1,2\] and they are of special interest since the $D = 10$ backgrounds with NS-NS and R-R charges appear on an equal footing when viewed from eleven dimensions. This approach may shed light on the structure of non-extreme black holes from the point of view of $M$-theory, and, in particular, clarify the origin of their BH entropy.

A discussion of intersections of certain BPS-saturated $M$-branes along with a proposal for intersection rules was first given in \[3\]. A generalization to a number of different harmonic functions specifying intersecting BPS-saturated $M$-branes, which led to a better understanding of these solutions and a construction of new intersecting $p$-brane solutions in $D \leq 11$, was presented in \[4\] (see also related works in Refs. \[2,5–12\]). Specific configurations of that type reduce to the BPS-saturated black holes with regular horizons in $D = 5$ \[4\] and $D = 4$ \[2\], whose properties are determined by three and four charges (associated with the corresponding harmonic functions), respectively. The generalization to intersecting non-extreme $M$-brane configurations, which upon dimensional reduction become non-extreme static black holes, was given in \[13\] (see also \[14\]).

The purpose of the present paper is to spell out the structure of the intersecting non-extreme rotating $M$-brane configurations, whose rotational parameters are associated with the transverse directions. Upon toroidal compactifications, these solutions become the known non-extreme rotating black holes (of the toroidally compactified Type II theory). In particular, we identify the relevant functions that specify the intersecting $M$-brane configurations and concentrate on the role of the rotational parameters. The structure of such configurations may shed light on the role of angular momenta in the structure of non-extreme rotating black holes from the point of view of $M$-theory, and it may shed light on the origin of their contribution to the BH entropy.

In the following, we shall discuss the procedure to construct the non-extreme rotating $M$-brane solutions. We further identify the intersecting configurations of two membrane(s)/five-brane(s) (and also include a boost along the common intersecting direction), and then further generalize the procedure to intersecting three and four rotating $M$-branes. The resulting non-extreme rotating solutions have the eleven-dimensional metric and the field strength of the three-form field specified by a number of independent parameters: the “non-extremality” parameter, representing the deviation from the BPS-saturated limit, the “boost” parameters, specifying charges carried by the configuration, and the angular momentum parameters, specifying the rotational degrees of freedom in the transverse directions. The important information on the eleven-dimensional metric associated with each brane configuration is now encoded in the modified harmonic functions, which are specified not only by the charges but also by the rotational parameters. We also concentrate on the structure of such configurations when the non-extremality parameter is taken to approach zero, i.e., when the configurations correspond to the BPS-saturated limit.
The paper is structured in the following way. In Section II, we obtain rotating, single $M$-brane solutions by uplifting the rotating solutions in lower dimensions. Based upon the observation made from the structure of the rotating single $M$-brane solutions, we infer the general algorithm for constructing the rotating intersecting $M$-brane solutions in Section III. In Section IV, we check the general algorithm against the explicit rotating, intersecting $M$-brane solutions which are obtained by uplifting multicharged rotating solutions in lower dimensions. In Section V, we discuss the BPS limits of the explicit (intersecting) $M$-brane solutions derived in the previous sections. The conclusions are given in Section VI.

II. SINGLE ROTATING $M$-BRANE CONFIGURATION

Before we discuss explicit forms of the intersecting $M$-brane solutions, we fix our notations for the space-time coordinates. For an intersecting $M$-brane solution, the eleven-dimensional space-time coordinates are divided into the “overall world-volume coordinates” $(t, y_1, \ldots, y_{p-r})$, which are (time+internal)-coordinates common to all the constituent $M$-branes, the “relative transverse coordinates” $(y_{p-r+1}, \ldots, y_p)$, which are internal to some of the $M$-branes, and the “overall transverse coordinates” $(x_1, \ldots, x_{10-p})$, which are transverse to all of the $M$-branes. Then, the rotating intersecting $M$-brane solutions depend on the eleven-dimensional space-time coordinates only through $(r, \theta, \psi_i)$, where $r \equiv (x_1^2 + \cdots + x_{10-p}^2)^{1/2}$ and $(\theta, \psi_i)$ are the rests of the angular-coordinates of the transverse space other than the ones $\phi_i$ associated with the rotational planes (see below for definitions).

While non-extreme static solutions $[14,13]$ are specified by the harmonic functions (associated with the electric/magnetic charges) and the non-extremality function (associated with the mass $m$ of the Schwarzschild solution) that depend on the charges and the non-extremality parameter, the rotating non-extreme solutions are specified in addition by non-zero angular momentum parameters $l_i$, which enter the solution in terms of the following form of harmonic functions $g_i$ ($i = 1, \ldots, [D-1]/2)$:

$$g_i \equiv 1 + \frac{l_i^2}{r^2},$$

which appear in the metric also in combinations:

$$G_D \equiv \begin{cases} \alpha^2 + \sum_{i=1}^{[D-1]/2} \mu_i^2 g_i^{-1} & \text{for even } D \\ \sum_{i=1}^{[D-1]/2} \mu_i^2 g_i^{-1} & \text{for odd } D \end{cases},$$

and

$$f_D^{-1} = G_D \prod_{i=1}^{[D-1]/2} g_i,$$

where $\alpha$ and $\mu_i$ are defined as:

- Even dimensions $D$:

$$\mu_1 \equiv \sin \theta, \quad \mu_2 \equiv \cos \theta \sin \psi_1, \quad \cdots, \quad \mu_{D-2} \equiv \cos \theta \cos \psi_1 \cdots \cos \psi_{D-6} \sin \psi_{D-4},$$

$$\alpha \equiv \cos \theta \cos \psi_1 \cdots \cos \psi_{D-4},$$
Odd dimensions $D$:

$$
\begin{align*}
\mu_1 &\equiv \sin \theta, \quad \mu_2 \equiv \cos \theta \sin \psi_1, \quad \ldots, \quad \mu_{D-1} \equiv \cos \theta \cos \psi_1 \cdots \cos \psi_{D-3} \sin \psi_{D-2}, \\
\mu_{D-1} &\equiv \cos \theta \cos \psi_1 \cdots \cos \psi_{D-2}.
\end{align*}
$$

(5)

Here, $D - 1$ specifies the number of spatial transverse directions associated with the rotational degrees of freedom. Note, that when the corresponding 11-dimensional solutions are compactified down to $D$ space-time dimensions the role of parameters $l_i$ is that of the angular momentum parameters of the Kerr-Neumann black hole solutions. The function $f_D$ also modifies the harmonic functions associated with the charge sources of the static $M$-brane solutions. When the rotational parameters $l_i$ become zero, $g_i$ and $G_D$ reduce to one and, thereby, the modified harmonic functions and the non-extremality functions of the rotating (intersecting) $M$-branes reduce to those of static (intersecting) $M$-branes.

Non-extreme rotating membrane and five-brane solutions of the eleven-dimensional supergravity can be obtained by uplifting the nine ($D = 9$) and six ($D = 6$) dimensional charged, rotating black hole solutions, whose charges arise from one two-form $U(1)$ gauge field of the NS-NS sector of toroidally compactified Type II string. These solutions were constructed (in terms of fields of the toroidally compactified heterotic string) in Refs. [15,16]. Following the standard procedure discussed in Ref. [17], we obtain the following form of “rotating” two- and five-brane solutions.

The eleven-dimensional metric of the rotating membrane solution can be cast in the following form:

$$
\begin{align*}
\delta_{11}^2 &= T^{-1/3} \left[ T(-f dt^2 + dy_1^2 + dy_2^2) + f^{-1} dr^2 \\
&+ r^2 \left\{ \left( 1 + \frac{l_2^2 \cos^2 \theta}{r^2} + \frac{t_2^2 \sin^2 \psi_1 + t_3^2 \cos^2 \psi_1 \sin^2 \psi_2 + t_4^2 \cos^2 \psi_1 \cos^2 \psi_2}{r^2} \right) \sin^2 \theta \right\} d\theta^2 \\
&+ \left( 1 + \frac{l_2^2 \cos^2 \psi_1}{r^2} + \frac{t_3^2 \sin^2 \psi_1 \sin^2 \psi_2}{r^2} \right) \cos^2 \theta d\psi_1^2 + \left( 1 + \frac{l_3^2 \cos^2 \psi_2}{r^2} \right) \cos^2 \theta \cos^2 \psi_1 d\psi_2^2 \\
&- 2 \frac{l_2^2 - l_3^2 \sin^2 \psi_2}{r^2} \frac{\cos \theta \cos \psi_1 \sin \psi_1 \cos \psi_2 \sin \psi_2 d\psi_1 d\psi_2}{r^2} \\
&- 2 \frac{l_2^2 - l_3^2 \sin^2 \psi_2}{r^2} \frac{\cos \theta \sin \theta \cos \psi_1 \sin \psi_1 d\theta d\psi_1}{r^2} \\
&- 2 \frac{l_3^2}{r^2} \cos \theta \cos \psi_1 \sin \psi_2 \sin \psi_2 d\theta d\psi_2 \\
&- \frac{4ml_i^2}{r^6} \frac{\mu_i^2}{r^6} f_D T dt d\phi_i + \frac{4ml_i^2 \mu_i^2}{r^6} f_D T d\phi_i d\phi_j \\
&+ \mu_i^2 \left( g_i + f_D T \frac{2ml_i^2 \mu_i^2}{r^8} \right) \left( d\phi_i^2 \right).
\end{align*}
$$

(6)

where the modified “harmonic function” $T^{-1}$ associated with the electric charge $Q \sim 2m \cosh \delta_e \sinh \delta_e$ source is of the form:

$$
T^{-1} \equiv 1 + f_D \left( \frac{2m \sinh^2 \delta_e}{r^{D-3}} \right),
$$

(7)
and the non-extremality functions $f$ and $f'$, appearing respectively in front of $dt^2$ and $dr^2$ terms, are of the form:

$$f \equiv 1 - f_D \left( \frac{2m}{r^{D-3}} \right),$$

$$f' \equiv G_D^{-1} - f_D \left( \frac{2m}{r^{D-3}} \right).$$

Note that in the BPS-saturated limit, i.e., the limit in which $m \to 0$ with $me^{2\delta_e}$ a finite constant, these non-extremality functions reduce respectively to $f = 1$ and $f' = G_D^{-1}$. The non-zero components of the three-form field are of the following form:

$$B_t^{(11)} = 2ml_1 \cosh \delta_e \sinh \delta_e \sinh \delta_e, 
B_\phi^{(11)} = \frac{2ml_2 \sin^2 \theta}{r^2} \cos \theta \sin \psi \sin \psi d\theta d\psi,
Q = 3m \Omega_n \cosh \delta_e \sinh \delta_e,$$

(11)

where $V_2$ is the volume of the 2-dimensional space internal to the $M$ 2-brane, $G_N$ is the eleven-dimensional Newton’s constant, and $\Omega_n$ denotes the area of the unit n-sphere.

The following eleven-dimensional metric of the rotating $M$-five-brane solution can be obtained in an analogous way by uplifting the magnetically charged rotating solution in six-dimensions:

$$\begin{align*}
\frac{M_{ADM}}{V_2} &= \frac{3m \Omega_7}{8 \pi G_N} (2 \cosh^2 \delta_e + 1), \quad \frac{J_i}{V_2} = \frac{\Omega_7}{4 \pi G_N} m l_i \cosh \delta_e, \quad Q = \frac{3m \Omega_7}{4 \pi G_N} \cosh \delta_e \sinh \delta_e, \\
\end{align*}$$

Here, the modified harmonic function $F^{-1}$ associated with the magnetic charge $P \sim 2m \cosh \delta_m \sinh \delta_m$ source is of the form:
\[ F^{-1} \equiv 1 + f_D \left( \frac{2m \sinh^2 \delta_m}{r^{D-3}} \right), \quad (13) \]

where \( D = 6 \) (therefore, \( f_D^{-1} = g_1 \cos \theta + g_2 \sin \theta \) in Eqs. (12) and (13)) and the modified non-extremality functions are defined in Eqs. (8) and (9) with \( D = 6 \). The non-zero components of the three-form field are of the following form:

\[
\begin{align*}
B_{\phi_1 \phi_2 \psi}^{(11)} &= -2mg_1 f_D \cosh \delta_m \sinh \delta_m \cos^2 \theta, \\
B_{\phi_1 \psi}^{(11)} &= \frac{2ml_2 f_D \sinh \delta_m \sin^2 \theta}{r^2}, \\
B_{\phi_2 \psi}^{(11)} &= \frac{2ml_1 f_D \sinh \delta_m \cos^2 \theta}{r^2}.
\end{align*}
\quad (14)
\]

The ADM mass density \( M_{ADM}/V_5 \) and the angular momentum densities \( J_i/V_5 \) per unit \( M \) 5-brane volume, and the magnetic charge \( P \) are of the following form:

\[
\begin{align*}
\frac{M_{ADM}}{V_5} &= \frac{3m \Omega_4}{8\pi G_N} (\cosh^2 \delta_m + 2), \\
\frac{J_i}{V_5} &= \frac{\Omega_4}{4\pi G_N} ml_i \cosh \delta_m, \\
P &= \frac{3m \Omega_4}{8\pi G_N} \cosh \delta_m \sinh \delta_m,
\end{align*}
\quad (15)
\]

where \( V_5 \) is the volume of the 5-dimensional space internal to the \( M \) 5-brane.

### III. INTERSECTION RULES

Based on the structure obtained in the case of a single rotating membrane and five-brane, we can infer the rules for constructing the intersecting rotating \( M \)-brane solutions. Such rules can be checked by uplifting the corresponding charged, rotating black brane solutions in lower dimensions. Analogously to the case of intersecting non-extreme static \( M \)-brane solutions, one can construct the intersecting non-extreme rotating \( M \)-brane solutions by modifying the flat eleven-dimensional metric, following the intersection rules (at least for the overall conformal factor and the internal part of the metric).

The eleven-dimensional space-time, internal to the intersecting rotating \( M \)-brane configuration, is specified by a “harmonic function” for each constituent \( M \)-brane (associated with the charge source) and the “non-extremality functions” (associated with the Schwarzschild mass), which are now modified by the functions \( G_D \) and \( f_D \) (associated with the rotational parameters), defined in Eqs. (2) and (3). The space transverse to the \( M \)-brane reflects the axial symmetry that involves the charge sources as well as the rotational parameters. Note that such a non-extreme rotating configuration does not have an interpretation as an intersection of separate non-extreme rotating \( M \)-branes, but rather as a bound state solution with a common non-extremality parameter and common rotational parameters associated with the common transverse spatial directions.

In the following, we shall specify the algorithm which would yield the intersecting non-extreme rotating \( M \)-brane solutions \(^1\). (These intersection rules can be checked for specific

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\(^1\)An analogous type of construction applies also to intersecting p-brane solutions in ten dimensions.
cases in which the explicit dimensionally reduced solutions correspond to non-extreme rotating multi-charged black hole solutions which have been explicitly constructed in [15,16].

We shall summarize the algorithm in the following two steps. First, we shall identify the forms of the non-extremality functions and the structure of the transverse part due to the non-zero rotational parameters. We shall also identify the modified harmonic functions, associated with each $M$-brane source. As a step two, we shall spell out the modifications of the eleven-dimensional space-time due to both non-extremality functions and non-zero $M$-brane sources.

Step I

• First, one makes the following replacements in the $D$-dimensional (transverse) space-time parts of the metric:

$$dt^2 \rightarrow f dt^2, \quad dx_n dx_n \rightarrow f'^{-1} dr^2 + r^2 d\Omega^2_{D-2},$$

(16)

where $f$ and $f'$ are the non-extremality functions defined as:

$$f = 1 - f_D \frac{2m}{r^{D-3}}, \quad f' = G_D^{-1} - f_D \frac{2m}{r^{D-3}}.$$  

(17)

with $g_i$, $G_D$ and $f_D$ defined in Eqs. (1), (2) and (3), respectively. $d\Omega^2_{D-2}$ is the angular parts of the rotating solution in $D$ space-time dimensions. When all the charges of the (constituent) $M$-brane(s) are zero, $d\Omega^2_{D-2}$ takes the following form:

$$d\Omega^2_{D-2} = \left(1 + \frac{l_i^2 \cos^2 \theta}{r^2} + \frac{K_1 \sin^2 \theta}{r^2}\right) d\theta^2$$

$$+ \left(1 + \frac{l_{i+1}^2 \cos^2 \psi_i}{r^2} + \frac{K_{i+1} \sin^2 \psi_i}{r^2}\right) \cos^2 \theta \cos^2 \psi_1 \cdots \cos^2 \psi_{i-1} d\psi_i^2$$

$$- 2 \frac{l_i^2 - K_i}{r^2} \cos^2 \theta \cos^2 \psi_1 \cdots \cos^2 \psi_{i-1} \cos \psi_i \sin \psi_i \cdots \cos^2 \psi_{j-1} \cos \psi_j \sin \psi_j d\psi_i d\psi_j$$

$$- 2 \frac{l_{i+1}^2 - K_{i+1}}{r^2} \cos \theta \sin \theta \cos^2 \psi_1 \cdots \cos^2 \psi_{i-1} \cos \psi_i \sin \psi_i d\theta d\psi_i$$

$$- f_D \frac{4m l_i^2 \mu_i^2}{r^{D-1}} dtd\phi_i + \mu_i^2 \left(g_i + f_D \frac{2m l_i^2 \mu_i^2}{r^{D-1}}\right) d\phi_i^2$$

$$+ f_D \frac{4m l_j^2 \mu_j^2 \mu_j^2}{r^{D-1}} d\phi_i d\phi_j,$$  

(18)

where $K_i$ in Eq. (18) are defined, in general, for a $(D-1)$-dimensional transverse space as:

- Even dimensions $D$:

$$K_i \equiv l_{i+1}^2 \sin^2 \psi_i + \cdots + l_{D-3}^2 \cos^2 \psi_i \cdots \cos^2 \psi_{D-6} \sin^2 \psi_{D-4}.$$  

(19)
Odd dimensions $D$:

$$K_i \equiv l_{i+1}^2 \sin^2 \psi_i + \cdots + l_{D-3}^2 \cos^2 \psi_i \cdots \cos^2 \psi_{D-5} \sin^2 \psi_{D-5} + \frac{l_{D-1}^2}{2} \cos^2 \psi_i \cdots \cos^2 \psi_{D-5}. \quad (20)$$

Here $i, j$ in $\psi_i$ and $\phi_i$ run from 1 to $[\frac{D-4}{2}]$ and from 1 to $[\frac{D-1}{2}]$, respectively. When charges carried by intersecting $M$-branes are turned on, the above expression (18) for $d\Omega_{D-2}^2$ gets modified in the $(\phi_i, \phi_j)$ and $(t, \phi_i)$ components.

Note $d\Omega_{D-2}^2$ in Eq. (18) corresponds to the angular part of the Kerr solution in $D$ space-time dimensions [18] and reduces to the infinitesimal length element of $S^{D-2}$ when all the angular momentum parameters $l_i$ are zero.

- Each constituent membrane and five-brane are respectively specified by the following form of modified harmonic functions:

$$T_n^{-1} = 1 + f_D \frac{2m \sinh^2 \delta_n}{r^{D-3}},$$

$$F_n^{-1} = 1 + f_D \frac{2m \sinh^2 \gamma_n}{r^{D-3}}. \quad (21)$$

These harmonic functions modify the overall conformal factor of the eleven-dimensional metric, as well as each metric component of the space-time internal to the configuration (see below).

- In the case when the $M$-brane solution has a null isometry, i.e., intersecting branes have a common string along some direction $y$, one can add a momentum along $y$ by applying the following $SO(1, 1)$-boost coordinate transformation among the coordinates $t$ and $y$:

$$t' = \cosh \delta_{kk} t - \sinh \delta_{kk} y, \quad y' = - \sinh \delta_{kk} t + \cosh \delta_{kk} y \quad (22)$$

to the non-extreme rotating background.

Then, the $(t, t)$- and $(y, y)$-components of the metric are modified as:

$$- f dt^2 + dy^2 \rightarrow - f dt'^2 + dy'^2$$

$$= -dt^2 + dy^2 + f_D \frac{2m}{r^{D-3}}(\cosh \delta_{kk} dt - \sinh \delta_{kk} dy)^2$$

$$= -K^{-1} dt^2 + K \hat{dy}^2, \quad \hat{dy} \equiv dy + [K^{t-1} - 1] dt, \quad (23)$$

where

\[ \text{This procedure is analogous to the one used in introducing a boost along the common intersection of non-extreme static } M\text{-brane configurations [13].} \]
\[ K = 1 + f_D \frac{2m \sinh^2 \delta_{kk}}{r^{D-3}}, \]
\[ K' = 1 - f_D \frac{2m \sinh^2 \delta_{kk}}{r^{D-3}} K^{-1}. \]  
(24)

In addition, \((\vec{y}_a, \phi_i)\) components (corresponding to the induced magnetic field due to non-zero angular momenta) of the metric are induced and \((t, \phi_i)\) components get modified.

Step II

- The modified harmonic functions \((21)\) rescale the overall conformal factor of the eleven-dimensional metric by factors \(T_n^{-1/3}\) and \(F_m^{-2/3}\), for the membrane and five-brane sources \(^3\), respectively, and therefore the conformal factor of the intersecting \(M\)-brane solution has the form:

\[ ds^2_{11} = \left( \prod_n T_n \right)^{-1/3} \left( \prod_m F_m \right)^{-2/3} [\cdots], \]  
(25)

where the terms in the brace \([\cdots]\) are the rest of the metric that has to be constructed following along the lines spelled out below.

- The \((t, t)\)-component in \([\cdots]\) of Eq. \((25)\) is modified in the following way:

\[ f(r, \theta, \psi_i) dt^2 \rightarrow \left( \prod_n T_n \right)\left( \prod_m F_m \right) f(r, \theta, \psi_i) dt^2, \]  
(26)

whereas the \((r, r)\)-component in \([\cdots]\) remains of the form \(f'^{-1}(r, \theta, \psi_i) dr^2\).

- There is a factor of \(T_n [F_n]\) for the \((y_a, y_a)\)-components (in \([\cdots]\) of Eq. \((25)\)), i.e., the components of the internal part of the membrane [five-brane] solution, for each constituent membrane [five-brane]. Therefore, the internal components are modified as:

\[
\begin{align*}
dy_1 dy_1 + dy_2 dy_2 & \rightarrow T(dy_1 dy_1 + dy_2 dy_2), \quad \text{for a single membrane,} \\
dy_1 dy_1 + \cdots + dy_5 dy_5 & \rightarrow F(dy_1 dy_1 + \cdots + dy_5 dy_5), \quad \text{for a single fivebrane,}
\end{align*}
\]  
(27)

and for an internal coordinate \(y_c\) where membranes and fivebranes with harmonic functions \(T_m\) and \(F_n\) intersect:

\[ dy_c dy_c \rightarrow \left( \prod_m T_m \right)\left( \prod_n F_n \right) dy_c dy_c. \]  
(28)

\(^3\)This structure is the same as in the case of static intersecting BPS-saturated \([3, 4]\) and non-extreme \([14, 13]\) M-branes.
Note that p-branes of the same type can intersect over a \((p-2)\)-brane \(^4\), i.e., membranes \((p = 2)\) can intersect over a 0-brane and five-branes \((p = 5)\) can intersect over a threebrane. A membrane and a five-brane intersect over a string.

In addition, one can also incorporate the boost transformation (22) along the common string along the lines specified at the end of Step I.

- The angular component \(d\Omega^2_{D-2}\) in the transverse parts has information on rotations of (intersecting) \(M\)-brane(s). Prior to the knowledge of general construction rules for \(d\Omega^2_{D-2}\), the expressions for \(d\Omega^2_{D-2}\) can be inferred from the angular components of the \(D\)-dimensional charged, rotating black brane solutions with the same corresponding charge configurations. When all the angular momentum parameters \(l_i\) are zero, \(d\Omega^2_{D-2}\) is given by the infinitesimal length element of (“flat”) \(S^{D-2}\) even when \(M\)-brane charges are non-zero. However, once the angular momenta are turned on, the expression for \(d\Omega^2_{D-2}\) is altered due to the non-zero \(M\)-brane source charges and becomes increasingly complicated with the increase in the number of \(M\)-brane sources.

We were able to find general rules for constructing the transverse part of intersecting two and three \(M\)-brane solutions with non-zero \([\frac{D-1}{2}]\) angular momentum parameters of the \(D\)-dimensional transverse space-time \(^5\). In these cases, the expressions for \(d\Omega^2_{D-2}\) are given as follows:

(1) Configurations with intersecting two \(M\)-branes:

\[
\begin{align*}
    d\Omega^2_{D-2} &= \left(1 + \frac{l_1^2 \cos^2 \theta}{r^2} + \frac{K_1 \sin^2 \theta}{r^2}\right) d\theta^2 \\
    &+ \left(1 + \frac{l_{i+1}^2 \cos^2 \psi_i}{r^2} + \frac{K_{i+1} \sin^2 \psi_i}{r^2}\right) \cos^2 \theta \cos^2 \psi_1 \cdots \cos^2 \psi_{i-1} d\psi_i^2 \\
    &- 2 \frac{l_j^2 - K_j}{r^2} \cos^2 \theta \cos^2 \psi_1 \cdots \cos^2 \psi_{i-1} \cos \psi_i \sin \psi_i \cdots \cos^2 \psi_{j-1} \cos \psi_j \sin \psi_j \sin \psi_{i-1} \cos \psi_j \sin \psi_{i-1} \cos \psi_j \sin \psi_{i-1} \cos \psi_j d\psi_i d\psi_j \\
    &- 2 \frac{l_{i+1}^2 - K_{i+1}}{r^2} \cos \theta \sin \theta \cos^2 \psi_1 \cdots \cos^2 \psi_{i-1} \cos \psi_i \sin \psi_i \cdots \cos^2 \psi_{j-1} \cos \psi_j \sin \psi_j d\theta d\psi_i \\
    &- \frac{4ml_i \mu_i^2}{r^{D-1}} \cosh \delta_1 \cosh \delta_2 f_D H_1 H_2 dt d\phi_i \\
    &+ \mu_i^2 \left[ g_i + f_D H_1 H_2 \frac{2ml_i^2 \mu_i^2}{r^{D-1}} \left(1 - f_D \frac{2m \sin^2 \delta_1 \sin^2 \delta_2}{r^{D-3}}\right) \right] d\phi_i^2
\end{align*}
\]

\(^4\)These rules were first described in Ref. [3] by using the space-time dimensionality argument of the \(M\)-brane solutions in Ref. [19] and can be checked by explicit embeddings of lower dimensional black hole solutions into eleven-dimensions. These rules are also compatible with the intersection rules for D-branes \([20]\) of \(D = 10\) Type II string theory.

\(^5\)In some of these cases, the corresponding rotating solutions in the compactified lower-dimensional theories are known. On the other hand, we do not have a general algorithm for the transverse part of the intersecting four \(M\)-branes.
where $K_i$’s are defined in Eqs. (19) and (20) and the modified harmonic functions $H_n$, specifying each constituent $M$-brane (with electric charges $Q_n$), are given by

$$H_n^{-1} = 1 + f_D \frac{2m \sinh^2 \delta_n}{r^{D-3}}, \quad (n = 1, 2).$$

(2) Configurations with intersecting three $M$-branes and $D - 1 = 4$ transverse spatial dimensions (i.e., a configuration corresponding to three membranes intersecting at a point):

$$
\begin{align*}
\text{d} \Omega_3^2 &= (g_1 \cos^2 \theta + g_2 \sin^2 \theta) \text{d} \theta^2 + \frac{4m \cos^2 \theta \sin^2 \theta}{r^4} \left[ l_1 l_2 \left( 1 - f_D \frac{2m}{r^2} \sum_{n<m} \sinh^2 \delta_n \sinh^2 \delta_m \right) \right] \text{f}_D H_1 H_2 H_3 \text{d} \phi_1 \text{d} \phi_2 \\
&+ \frac{2m}{r^2} \left( l_1^2 + l_2^2 \right) \prod_i \cosh \delta_i - 2l_1 l_2 \prod_i \sinh \delta_i \right) \text{f}_D H_1 H_2 H_3 \text{d} \phi_1 \\
&- \frac{4m \sin^2 \theta}{r^4} \left( l_1 \prod_i \cosh \delta_i - f_l^2 \prod_i \sinh \delta_i \right) \text{f}_D H_1 H_2 H_3 \text{d} \phi_1 \\
&- \frac{4m \cos^2 \theta}{r^4} \left( l_2 \prod_i \cosh \delta_i - f_l^1 \prod_i \sinh \delta_i \right) \text{f}_D H_1 H_2 H_3 \text{d} \phi_2 \\
&+ \sin^2 \theta \left[ f_D^{-1} + \left( \frac{l_2^2 - l_1^2}{r^2} \right) \left( f_D^{-1} + \sum_n \frac{2m \sinh^2 \delta_n}{r^2} \right) - \frac{2ml_1^2}{r^4} \right] \text{f}_D H_1 H_2 H_3 \\
&+ \frac{4m^2}{r^6} \left( l_2^2 \sum_{m<n} \sinh^2 \delta_m \sinh^2 \delta_n - 2l_1 l_2 \prod_n \cosh \delta_n \sinh \delta_n \right) \text{f}_D^2 H_1 H_2 H_3 \text{d} \phi_1 \\
&+ \cos^2 \theta \left[ f_D^{-1} + \left( \frac{l_2^2 - l_1^2}{r^2} \right) \left( f_D^{-1} + \sum_n \frac{2m \sinh^2 \delta_n}{r^2} \right) - \frac{2ml_2^2}{r^4} \right] \text{f}_D H_1 H_2 H_3 \\
&+ \frac{4m^2}{r^6} \left( l_1^2 \sum_{m<n} \sinh^2 \delta_m \sinh^2 \delta_n - 2l_1 l_2 \prod_n \cosh \delta_n \sinh \delta_n \right) \text{f}_D^2 H_1 H_2 H_3 \text{d} \phi_2,
\end{align*}
$$

where the modified harmonic functions associated with the charge sources $Q_n$ are given by

$$H_n^{-1} = 1 + f_D \frac{2m \sinh^2 \delta_n}{r^2}, \quad (n = 1, 2, 3)$$

with $f_D = g_1 \cos^2 \theta + g_2 \sin^2 \theta$.

- For the non-zero components $B_{MNP}^{(11)}$ of the eleven-dimensional three-form field $B$, which can be obtained by uplifting the $D$-dimensional expressions for two-form $U(1)$ fields and the two-form field, we were able to obtain a general rule for the case of intersecting membranes. That is, for intersecting membranes (intersecting at a point) each with electric charges $Q_n$ and internal coordinates $y_{a_n}$ and $y_{b_n}$:
\[ B_{\tau n, b n}^{(11)} = \frac{2m \cosh \delta_n \sinh \delta_n}{r^{D-3}} f_D T_n, \]
\[ B_{\phi, a_n, b_n}^{(11)} = -\frac{2ml_i \mu_i^2 \sinh \delta_n}{r^{D-3}} f_D T_n. \] (33)

For other types of intersecting M-brane configurations discussed in the following sections, we have not found a general pattern for the three-form fields, and, therefore, we simply refer to the following sections for the expressions in these cases.

We can now illustrate the above algorithm from several examples of explicit eleven-dimensional solutions uplifted from the known lower-dimensional non-extreme rotating black hole solutions. Specifically, the single rotating membrane and five-brane described in the previous section are of the form satisfying these rules specific for \( D = 9 \) and \( D = 6 \).

In the following section, we turn to the case of intersecting M-branes and explicitly check the algorithm by uplifting the corresponding multi-charged, non-extreme, rotating solutions to the eleven dimensions.

IV. INTERSECTING ROTATING M-BRANES

A. Intersection of two M-branes

Prior to the knowledge of the algorithms described in the previous section, one can construct the intersecting M-brane solutions with two charges by uplifting the lower-dimensional black hole solutions carrying two charges of two-form \( U(1) \) gauge fields associated with the NS-NS or RR sector of toroidally compactified Type II string [15,16].

The eleven-dimensional metric \( G_{MN}^{(11)} \) is related to the \( D \)-dimensional (Einstein-frame) metric \( g_{\mu \nu} \) in the following way. First, we construct the ten-dimensional metric \( \hat{G}_{\hat{\mu} \hat{\nu}}^{(10)} \) following the standard Kaluza-Klein Ansatz given by:

\[ \hat{G}_{\hat{\mu} \hat{\nu}}^{(10)} = \begin{pmatrix} e^{\varphi} g_{\mu \nu} + G_{mn} A_{\mu}^{(1)m} A_{\nu}^{(1)n} & A_{\mu}^{(1)m} G_{mn} \nonumber \\ A_{\nu}^{(1)n} G_{mn} & G_{mn} \end{pmatrix}, \] (34)

where \( \varphi \equiv \hat{\Phi} - \frac{1}{2} \ln \det G_{mn} \) is the \( D \)-dimensional metric (this definition of \( \varphi \) is used to construct the ten-dimensional dilaton \( \hat{\Phi} \) from the \( D \)-dimensional solutions for \( \varphi \) and \( G_{mn} \)) and \( a \equiv \frac{2}{D-2} \). Then, we construct the eleven-dimensional space-time metric from the ten-dimensional metric and dilaton using the Kaluza-Klein Ansatz corresponding to the compactification from the eleven dimensions to the ten dimensions:

\[ G_{MN}^{(11)} = \begin{pmatrix} e^{\hat{\Phi}} \hat{G}_{\hat{\mu} \hat{\nu}}^{(10)} & e^{\hat{\Phi}} B_M \\ \nonumber \\ e^{\hat{\Phi}} B_N & e^{\hat{\Phi}} \end{pmatrix}, \] (35)

where \( B_M \) is the ten-dimensional \( U(1) \) field in the RR-sector of type-IIA string. Also, the ten-dimensional two-form field \( B_{\hat{\mu} \hat{\nu}}^{(10)} \) in the NS-NS sector and three-form field \( B_{\hat{\mu} \hat{\nu} \hat{\rho}}^{(10)} \) in the RR sector are defined from the eleven-dimensional three-form field \( B_{MNP}^{(11)} \) as \( B_{MNP}^{(11)} = (B_{\mu \nu}^{(10)}, B_{\mu \nu \rho}^{(11)} \equiv B_{\mu \nu}^{(10)}) \).
In order to obtain all the possible intersecting two $M$-brane solutions, we begin with the intersecting $M$-brane solution whose lower-dimensional counterpart is explicitly known. The only such an intersecting $M$-brane solution available at this point is the rotating $M$-brane solution corresponding to intersection of membrane and fivebrane intersecting at a string $(2 \perp 5)$. Such a solution can be obtained by uplifting five-dimensional rotating black hole solution $(D = 5)$ [13] carrying the electric charge of the NS-NS two-form $U(1)$ gauge field and the electric charge associated with the NS-NS two-form antisymmetric tensor (or the “magnetic” charge of the corresponding uplifted eleven-dimensional three-form field) of the toroidally compactified heterotic string [15]. The eleven-dimensional metric is of the form:

$$ds^2 = F^{-2/3}T^{-1/3}\left[FT(-f dt^2 + dy_1^2) + F(dy_2^2 + dy_3^2 + dy_4^2 + dy_5^2) + Tdy_6^2 + f'^{-1}dr^2 + r^2d\Omega_3^2\right]$$

(36)

where the modified harmonic functions corresponding to membrane charge source $Q \sim 2m \cosh \delta_e \sinh \delta_e$ and five-brane charge source $P \sim 2m \sinh \delta_m \cosh \delta_m$ are respectively given by

$$F^{-1} \equiv 1 + f_D \frac{2m \sinh^2 \delta_m}{r^2},$$

$$T^{-1} \equiv 1 + f_D \frac{2m \sinh^2 \delta_e}{r^2},$$

(37)

with $f_D^{-1} = g_1 \cos^2 \theta + g_2 \sin^2 \theta$. Here, the angular component $d\Omega_3^2$ is defined in Eq. (29), and the non-extremality functions $f$ and $f'$ are respectively given by Eqs. (8) and (4) with $D = 5$. Non-zero components of the eleven-dimensional three-form field are given by

$$B_{\phi_1\phi_6}^{(11)} = \frac{2m \cosh \delta_e \sinh \delta_e}{r^2} f_D T,$$

$$B_{\phi_i \phi_6}^{(11)} = -2m \cos^2 \theta \cosh \delta_m \sinh \delta_m \left(g_1 + \frac{2m f_D \sinh^2 \delta_e}{r^2}\right) T,$$

$$B_{\phi_1 \phi_6}^{(11)} = \frac{2ml_2 \sin^2 \theta \cosh \delta_e \sinh \delta_m}{r^2} f_D T,$$

$$B_{\phi_1 \phi_6}^{(11)} = -\frac{2ml_1 \sin^2 \theta \sinh \delta_e \cosh \delta_m}{r^2} f_D T,$$

$$B_{\phi_2 \phi_6}^{(11)} = -\frac{2ml_2 \cos^2 \theta \sinh \delta_e \cosh \delta_m}{r^2} f_D T.$$  

(38)

The ADM mass density $M_{ADM}/V_6$ and the angular momentum densities $J_i/V_6$ per unit intersecting $M$ 2-brane and $M$ 5-brane volume, the electric charge $Q$, and magnetic charge $P$ are of the following form:

$$\frac{M_{ADM}}{V_6} = \frac{3m \Omega_3}{8\pi G_N} (2 \cosh^2 \delta_e + \cosh^2 \delta_m), \quad \frac{J_i}{V_6} = \frac{\Omega_3}{4\pi G_N} ml_i \cosh \delta_e \cosh \delta_m,$$

$$Q = \frac{3m \Omega_3}{4\pi G_N} \cosh \delta_e \sinh \delta_e, \quad P = \frac{3m \Omega_3}{8\pi G_N} \cosh \delta_m \sinh \delta_m,$$

(39)

where $V_6$ is the volume of the 6-dimensional space internal to the intersecting $M$ 2-brane $M$ 5-brane.

The rest of the intersecting two $M$-brane solutions, i.e., two membranes intersecting at a point $(2 \perp 2)$ and two five-branes intersecting at a three-brane $(5 \perp 5)$, can be obtained by
imposing the $T$-duality transformation of the toroidally compactified type-II string on the above intersecting membrane and five-brane solution $(2 \perp 5)$ compactified on a torus. The $T$-duality transformations of the toroidally compactified type-II string transform type-IIA [type-IIB] (intersecting) $M$-brane solutions into type-IIB [type-IIA] (intersecting) $M$-brane solutions, while dualizing the overall world-volume directions and the overall transverse directions into each other, and dualizing the relative transverse directions internal to one $M$-brane to the ones internal to the other $M$-brane. For the type-II string with the coordinate $x$ compactified on a circle, the relevant parts of the $T$-duality transformations necessary in constructing the eleven-dimensional metric of the intersecting two $M$-branes are given by

$$
\hat{G}_{xx}^{(10)} \to \hat{G}_{xx}^{(10)\prime} = \frac{1}{G_{xx}^{(10)}}, \quad \hat{G}_{\mu\nu}^{(10)} \to \hat{G}_{\mu\nu}^{(10)} \quad (\mu, \nu \neq x), \quad e^{2\phi} \to e^{2\phi'} = \frac{e^{2\hat{\phi}}}{\hat{G}_{xx}^{(10)\prime}}. \quad (40)
$$

In general, only $T$-duality transformations that decrease the dimensionality of the overall transverse directions guarantee that the $T$-dualized solutions satisfy the Euler-Lagrange equations, since the other types of $T$-duality transformations introduce additional transverse spatial coordinates that the transformed solutions are required to depend on. But since there is only a limited number of available known explicit solution (i.e., only $2 \perp 5$ configuration) from which we have to construct other types of intersecting two $M$-brane solutions through the $T$-duality transformations, we shall also use the types of $T$-duality transformations that increase the number of dimensions of the overall transverse directions. Namely, we shall assume that the $T$-dualized solutions depend on the new introduced transverse spatial coordinate, say $x_{10-p}$, as well, through the radial coordinate $r = (x_1^2 + \cdots + x_{10-p}^2)^{1/2}$ of the transverse space, rather than becoming delocalized [22].

The intersecting two membrane solution $(2 \perp 2)$ in eleven dimensions is obtained as follows. One first compactifies the coordinate $y_5$ of the solution (36) on a circle to obtain the type-IIA string solution where membrane and four-brane intersect at the coordinate $y_1$. Then, one imposes $T$-duality transformation along the coordinate $y_1$ compactified on a circle, resulting in an intersecting $p$-brane solution in type-IIB string where the string and the three-brane intersect over a zero-brane. One transforms this type-IIB solution into a type-IIA solution by imposing a $T$-duality transformation along the coordinate $y_4$ compactified on a circle, resulting in a two membranes intersecting at a point. Finally, one uplifts this type-IIA solution to eleven dimensions to obtain the intersecting two membrane solution.

The explicit form of this rotating $M$-brane solution is then given as follows:

$$
\begin{align*}
\text{ds}^2 &= T_1^{-1/3}T_2^{-1/3} \left[ -T_1 T_2 f dt^2 + T_1 (dy_1^2 + dy_2^2) + T_2 (dy_3^2 + dy_4^2) \
&\quad + f^{-1} dr^2 + r^2 \left\{ (1 + \frac{l_1^2 \cos^2 \theta}{r^2}) + \frac{(l_2^2 \sin^2 \psi + l_3^2 \cos^2 \psi) \sin^2 \theta}{r^2} \right\} d\theta^2 \
&\quad + (1 + \frac{l_2^2 \cos^2 \psi}{r^2}) \cos^2 \theta d\psi^2 - 2 \frac{l_2^2}{r^2} \cos \theta \sin \theta \cos \psi \sin \psi d\theta d\psi \right].
\end{align*}
$$

\[6\] The complete $T$-duality transformation of the type-II string compactified on a torus can be found in Ref. [21].
\[\begin{align*}
&+ \frac{4ml\,l_j\mu^2 \mu^2_j}{r^4} (1 - f_D \frac{2m \sinh^2 \delta_1 \sinh^2 \delta_2}{r^4}) f_D T_1 T_2 d\phi_i d\phi_j \\
&- \frac{4ml \mu^2 \cosh \delta_1 \cosh \delta_2}{r^4} f_D T_1 T_2 dt d\phi_i \\
&+ \mu_i^{2} \left[ g_i + fT_1 T_2 \frac{2ml^2 \mu^2_i}{r^6} (1 - f_D \frac{2m \sinh^2 \delta_1 \sinh^2 \delta_2}{r^4})] d\phi_i^{2}\right],
\end{align*}\]

where the modified harmonic functions \( T_n \) associated with electric charges \( Q_n \) are given by

\[ T_n^{-1} = 1 + f_D \frac{2m \sinh^2 \delta_n}{r^4}, \quad n = 1, 2 \]

and \( f \) and \( f' \) are given in Eqs. (8) and (9) with \( D = 7 \). The non-zero components of the eleven-dimensional three-form field are given by:

\[\begin{align*}
B_{y_1 y_2}^{(11)} &= \frac{2m \cosh \delta_1 \sinh \delta_1}{r^4} f_D T_1, \\
B_{\phi y_1 y_2}^{(11)} &= -\frac{2ml \mu^2 \sinh \delta_1}{r^4} f_D T_1 \\
B_{y_3 y_4}^{(11)} &= \frac{2m \cosh \delta_2 \sinh \delta_2}{r^4} f_D T_2, \\
B_{\phi y_3 y_4}^{(11)} &= -\frac{2ml \mu^2 \sinh \delta_2}{r^4} f_D T_2.
\end{align*}\]

The ADM mass density \( M_{ADM}/V_4 \) and the angular momentum densities \( J_i/V_4 \) per unit intersecting two \( M \) 2-brane volume, and the electric charges \( Q_n \) \((n = 1, 2)\) are of the following form:

\[\begin{align*}
\frac{M_{ADM}}{V_4} &= \frac{3m \Omega_5}{8\pi G_N} (2 \cosh^2 \delta_1 + 2 \cosh^2 \delta_2 - 1), \\
\frac{J_i}{V_4} &= \frac{\Omega_5}{4\pi G_N} ml \cosh \delta_1 \cosh \delta_2, \\
Q_n &= \frac{3m \Omega_5}{4\pi G_N} \cosh \delta_n \sinh \delta_n.
\end{align*}\]

where \( V_4 \) is the volume of the 4-dimensional space internal to the intersecting two \( M \) 2-brane.

The intersecting \( M \)-brane solution where two five-branes intersect over a three-brane are obtained as follows. First, one compactifies the coordinate \( \phi_2 \) of solution (11) (with \( l_2 = 0 \)) on a circle to obtain intersecting two membranes. Then, one imposes the \( T \)-duality transformation along the coordinate \( y_1 \) compactified on a circle to obtain type-IIB solution of two three-branes intersecting at a string (along the coordinate \( y_1 \)). This solution is transformed into type-IIA solution of two four-branes intersecting at a membrane (along the coordinates \( y_1 \) and \( y_5 \)), by applying the \( T \)-duality transformation along the coordinate \( y_5 \) compactified on a circle. Finally, one uplifts this solution to eleven-dimensions to obtain two five-branes intersecting at a three-brane.

The final form of this rotating \( M \)-brane configuration is given by:

\[\begin{align*}
\frac{m}{r^3} (g + f_D F_1 F_2 \frac{2ml^2 \sin^2 \theta}{r^3} (1 - f_D \frac{2m \sinh^2 \delta_1 \sinh^2 \delta_2}{r^4})] d\phi^2 \right) \right].
\end{align*}\]

\[\begin{align*}
 ds^2 &= F_1^{-2/3} F_2^{-2/3} \left[ F_1 F_2 (-f dt^2 + dy_1^2 + dy_2^2 + dy_3^2) + F_1 (dy_4^2 + dy_5^2) + F_2 (dy_6^2 + dy_7^2) \\
&+ f'^{-1} dr^2 + r^2 \left\{ d\theta^2 - \frac{4ml \sin^2 \theta \cosh \delta_1 \cosh \delta_2}{r} f_D F_1 F_2 dt d\phi \\
&+ \sin^2 \theta (g + f_D F_1 F_2 \frac{2ml^2 \sin^2 \theta}{r^3} (1 - f_D \frac{2m \sinh^2 \delta_1 \sinh^2 \delta_2}{r^4})] d\phi^2 \right) \right],
\end{align*}\]

\[\begin{align*}
 ds^2 &= F_1^{-2/3} F_2^{-2/3} \left[ F_1 F_2 (-f dt^2 + dy_1^2 + dy_2^2 + dy_3^2) + F_1 (dy_4^2 + dy_5^2) + F_2 (dy_6^2 + dy_7^2) \\
&+ f'^{-1} dr^2 + r^2 \left\{ d\theta^2 - \frac{4ml \sin^2 \theta \cosh \delta_1 \cosh \delta_2}{r} f_D F_1 F_2 dt d\phi \\
&+ \sin^2 \theta (g + f_D F_1 F_2 \frac{2ml^2 \sin^2 \theta}{r^3} (1 - f_D \frac{2m \sinh^2 \delta_1 \sinh^2 \delta_2}{r^4})] d\phi^2 \right) \right],
\end{align*}\]
where the modified harmonic functions specified by magnetic charges $P_n$ are given by

$$F_n^{-1} = 1 + f_D \frac{2m \sinh^2 \delta_n}{r}, \quad n = 1, 2,$$

(46)

with $f_D^{-1} = g \cos^2 \theta + \sin^2 \theta$. The non-zero components of the three-form field are given by

$$B^{(11)}_{t\delta} = \frac{2mtf_D \sinh \delta_1 \cos \theta}{r^2}, \quad B^{(11)}_{\phi\delta} = -2mgf_D \cosh \delta_1 \sinh \delta_1 \cos \theta,$$

$$B^{(11)}_{t\phi} = \frac{2mtf_D \sinh \delta_2 \cos \theta}{r^2}, \quad B^{(11)}_{\phi\phi} = -2mgf_D \cosh \delta_2 \sinh \delta_2 \cos \theta.$$

(47)

Here, in the above $g = 1 + \frac{l^2}{r^2}$. The ADM mass density $M_{ADM}/V_7$ and the angular momentum densities $J_i/V_7$ per unit intersecting two $M$-brane volume, and the magnetic charges $P_n$ ($n = 1, 2$) are of the following form:

$$\frac{M_{ADM}}{V_7} = \frac{3m\Omega_2}{8\pi G_N} (\cosh^2 \delta_1 + \cosh^2 \delta_2 + 1),$$

$$\frac{J_i}{V_7} = \frac{4\Omega_2}{4\pi G_N} ml_i \cosh \delta_1 \cosh \delta_2,$$

$$P_n = \frac{3m\Omega_2}{8\pi G_N} \cosh \delta_n \sinh \delta_n,$$

(48)

where $V_7$ is the volume of the 7-dimensional space internal to the intersecting two $M$-branes.

One can check that the above $M$-brane solutions for intersecting two $M$-branes ($2 \perp 5$, $2 \perp 2$ and $5 \perp 5$) satisfy the general intersection rules discussed in the previous section.

**B. Intersection of three $M$-branes**

The case of intersecting three $M$-branes can be obtained by uplifting the five-dimensional rotating black hole solution constructed in Ref. [15]. Namely, when the five-dimensional three-charged rotating black hole solution in Ref. [15] is uplifted to transverse space-time components of the eleven dimensions assuming that all the three charges are originated from the eleven-dimensional three-form field, the $(t,t)$-component of the metric has a suggestive form as a product of three modified harmonic functions, with each modified harmonic function corresponding to each non-zero $M$-brane source charge, in agreement with

---

7Note, one of these three charges of the solution in Ref. [15] is the electric charge of the Kaluza-Klein $U(1)$ gauge field (i.e, the electric charge associated with the $SO(1,1)$ boost along the isometry direction). However, this assumption can be justified by the fact that the solution in [15] is the generating solution for the most general solution and, therefore, can be related to the rotating solution with all the three charges associated with the eleven-dimensional three-form field through $U$-duality (of the five-dimensional type-II string theory), which do not change the Einstein frame space-time.
the intersection rules discussed in the previous section. Therefore, we shall take the angular component \(d\Omega_3^2\) of the intersecting three \(M\)-brane with the transverse spatial dimension \(D - 1 = 4\) to be the corresponding angular coordinate components of the three-charged five-dimensional rotating black hole solution constructed in Ref. [15].

The \(M\)-brane solution where three membranes intersect over a zero brane \((2 \perp 2 \perp 2)\) has \(D - 1 = 4\) transverse spatial dimensions. Such a configuration should have the form following the general intersection rules discussed in section III with the angular component \(d\Omega_3^2\) given by the corresponding angular components of the five-dimensional, rotating, three-charged black hole obtained in Ref. [15]. The explicit form of the rotating \(M\)-brane solution where three membranes intersect \((2 \perp 2 \perp 2)\) is then given by:

\[
ds_{11}^2 = (T_1T_2T_3)^{-1/3} \left[ -T_1T_2T_3 f dt^2 + T_1(dy_1^2 + dy_2^2) + T_2(dy_3^2 + dy_4^2) + T_3(dy_5^2 + dy_6^2) \right]
\]

where the modified harmonic functions corresponding to the electric charge \(Q_n \sim 2m \cosh \delta_n \sinh \delta_n\) sources are given by

\[
T_n^{-1} = 1 + f_D \frac{2m \sinh^2 \delta_n}{r^2}, \quad n = 1, 2, 3,
\]

with \(f_D^{-1} = g_1 \cos^2 \theta + g_2 \sin^2 \theta\). Here, the angular component \(d\Omega_3^2\) is given by Eq. (31).

The non-zero components of the harmonic functions are of the following form:

\[
B_{t12}^{(11)} = \frac{2m \cosh \delta_1 \sinh \delta_1}{r^2} f_D T_1, \quad B_{\phi 12}^{(11)} = -\frac{2m \mu_1^2 \sinh \delta_1}{r^2} f_D T_1,
\]

\[
B_{t34}^{(11)} = \frac{2m \cosh \delta_2 \sinh \delta_2}{r^2} f_D T_2, \quad B_{\phi 34}^{(11)} = -\frac{2m \mu_1^2 \sinh \delta_2}{r^2} f_D T_2,
\]

\[
B_{t56}^{(11)} = \frac{2m \cosh \delta_3 \sinh \delta_3}{r^2} f_D T_3, \quad B_{\phi 56}^{(11)} = -\frac{2m \mu_2 \sinh \delta_3}{r^2} f_D T_3,
\]

where \(\mu_1 = \sin \theta\) and \(\mu_2 = \cos \theta\).

The ADM mass density \(M_{ADM}/V_6\) and the angular momentum densities \(J_i/V_6\) per unit intersecting three \(M\)-brane volume, and the electric charges \(Q_n\) \((n = 1, 2, 3)\) are of the following form:

\[
\frac{M_{ADM}}{V_6} = \frac{3m\Omega_3}{8\pi G_N} \left( 2 \cosh^2 \delta_1 + 2 \cosh^2 \delta_2 + 2 \cosh^2 \delta_3 - 3 \right),
\]

\[
\frac{J_1}{V_6} = \frac{\Omega_3}{4\pi G_N} m (l_1 \prod_i \cosh \delta_i - l_2 \prod_i \sinh \delta_i),
\]

\[
\frac{J_2}{V_6} = \frac{\Omega_3}{4\pi G_N} m (l_2 \prod_i \cosh \delta_i - l_1 \prod_i \sinh \delta_i),
\]

\[
Q_n = \frac{3m\Omega_3}{4\pi G_N} \cosh \delta_n \sinh \delta_n,
\]

where \(V_6\) is the volume of the 6-dimensional space internal to the intersecting three \(M\)-branes.
C. Intersection of four M-branes

As for the intersecting four M-brane solutions, there is no available lower dimensional rotating charged solution to which the intersecting four M-brane configuration would be directly related via dimensional reduction. The explicit general rotating black hole solution with four non-zero charges, obtained in Ref. [24], carries two electric charges of the Kaluza-Klein $U(1)$ and two-form $U(1)$ gauge fields and two magnetic charges of the Kaluza-Klein $U(1)$ and two-form $U(1)$ gauge fields. In principle, by applying $U$-duality transformations (e.g., $O(5,5)$ transformations of Type II theory in six-dimensions [25,26]) on this known solution one should be able to relate the transformed solution to the intersecting configuration of rotating four M-branes (i.e., $2 \perp 2 \perp 5 \perp 5$) or the intersecting configuration of rotating three M-branes with a boost along the common intersection direction (i.e., $5 \perp 5 \perp 5 + \text{boost}$).

We postpone such a study and, following the intersection rules of Section III, we simply present a general structure for the intersection of two membranes and two fivebranes  ($2 \perp 2 \perp 5 \perp 5$), which becomes, after a dimensional reduction, a four-dimensional rotating black hole with four-charges. Such an intersecting M-brane solution has the following structure:

$$dS^2_{11} = (T_1 T_2)^{-1/3} (F_1 F_2)^{-2/3} \left[ -T_1 T_2 F_1 F_2 f dt^2 + F_1 (T_1 dy_1^2 + T_2 dy_2^2) + F_2 (T_1 dy_3^2 + T_2 dy_4^2) + F_1 F_2 (dy_5^2 + dy_6^2 + dy_7^2) + f^{-1} dr^2 + r^2 d\Omega_2^2 \right]$$

(53)

where the “modified” harmonic functions $T_i$ and $F_i$ associated, respectively, with the electric charges $Q_i \sim 2m \cosh \delta e_i \sinh \delta e_i$ and the magnetic charges $P_i \sim 2m \cosh \delta p_i \sinh \delta p_i$, and the non-extremality functions are given by

$$T_i^{-1} \equiv 1 + f_D \frac{2m \sinh^2 \delta e_i}{r}, \quad F_i^{-1} \equiv 1 + f_D \frac{2m \sinh^2 \delta p_i}{r}, \quad i = 1, 2,$$

$$f \equiv 1 - f_D \frac{2m}{r}, \quad f' \equiv f_D \left( 1 + \frac{l^2}{r^2} - \frac{2m}{r} \right),$$

(54)

where $f_D^{-1} = 1 + \frac{l^2 \cos^2 \theta}{r^2}$. Here, the explicit expression for $d\Omega_2^2$ is not known yet.

V. INTERSECTING M-BRANE SOLUTIONS WITH A BOOST

Prior to the knowledge of general M-brane construction rules, rotating M-brane solutions with a boost along a null isometry direction are obtained by uplifting lower-dimensional rotating charged solutions carrying Kaluza-Klein electric charge. Such a Kaluza-Klein electric charge can be viewed as a boost along the internal isometry direction (or the internal momentum) from the viewpoint of the lower-dimensional black hole solutions, and when it is uplifted to ten dimensions it corresponds to the momentum of the string along the compactified direction.

For example, the following intersecting five-brane and membrane with a boost along the intersection coordinate $y_1$ is obtained by uplifting the general class of five-dimensional rotating charged black hole solution in heterotic string [13] to eleven dimensions:
\[ ds^2 = F^{-2/3}T^{-1/3} \left[ FT(-K^{-1} f dt^2 + K \hat{dy}_1^2) + F(dy_2^2 + \cdots + dy_5^2) + T dy_6^2 \right] \\
\quad + \frac{4m \sin^2 \theta}{r^2} (l_1 \sinh \delta_{kk} \cosh \delta_e \cosh \delta_m - l_2 \cosh \delta_{kk} \sinh \delta_e \sinh \delta_m) f_D TF d\phi_1 \hat{dy}_1 \\
\quad + \frac{4m \cos^2 \theta}{r^2} (l_2 \sinh \delta_{kk} \cosh \delta_e \cosh \delta_m - l_1 \cosh \delta_{kk} \sinh \delta_e \sinh \delta_m) f_D TF d\phi_2 \hat{dy}_1 \\
\quad + f'^{-1} dr^2 + r^2 \left\{ (g_1 \cos^2 \theta + g_2 \sin^2 \theta) d\theta^2 + \frac{4ml_1 l_2 \cos^2 \theta \sin^2 \theta}{r^4} \\
\quad \times (1 - f_D^2 \frac{2m \sinh^2 \delta_m \sin^2 \delta_e}{r^2}) f_D FT d\phi_1 d\phi_2 \\
\quad - \frac{4m \sin^2 \theta}{r^4} (l_1 \cosh \delta_e \cosh \delta_m \cosh \delta_{kk} - l_2 \sinh \delta_e \sinh \delta_m \sinh \delta_{kk}) K^{-1} FT dt d\phi_1 \\
\quad - \frac{4m \cos^2 \theta}{r^2} (l_2 \cosh \delta_e \cosh \delta_m \cosh \delta_{kk} - l_1 \sinh \delta_e \sinh \delta_m \sinh \delta_{kk}) K^{-1} FT dt d\phi_2 \\
\quad + \left[ g_1 + f_D FT 2ml_1 l_2 \sin^2 \theta \left(1 - \frac{2m \sinh^2 \delta_m \sin^2 \delta_e}{r^2}\right) \right] \sin^2 \theta d\phi_1^2 \\
\quad + \left[ g_2 + f_D FT 2ml_1^2 \cos^2 \theta \left(1 - \frac{2m \sinh^2 \delta_m \sin^2 \delta_e}{r^2}\right) \right] \cos^2 \theta d\phi_2^2 \right\} \right] \
\]

where the modified harmonic functions \( F, T \) and \( K \) associated with the electric \( Q \sim 2m \cosh \delta_e \sinh \delta_k \), magnetic \( P \sim 2m \cosh \delta_m \sinh \delta_m \) and boost \( Q_{kk} \sim 2m \cosh \delta_{kk} \sinh \delta_{kk} \) charge sources are given by

\[
F^{-1} \equiv 1 + f_D \frac{2m \sinh^2 \delta_m}{r^2}, \\
T^{-1} \equiv 1 + f_D \frac{2m \sinh^2 \delta_e}{r^2}, \\
K \equiv 1 + f_D \frac{2m \sinh^2 \delta_{kk}}{r^2},
\]

and the modified infinitesimal length element \( \hat{dy}_1 \) of the intersection coordinate \( y_1 \) is given by

\[
\hat{dy}_1 \equiv dy_1 + (K'^{-1} - 1) dt, \quad K'^{-1} \equiv 1 - \frac{2m \sinh \delta_{kk} \cosh \delta_{kk}}{r^2} f_D^{-1} K^{-1}.
\]

The non-zero components of the three-form fields are given by

\[
B_{1t6}^{(11)} = \frac{2m \cosh \delta_e \sinh \delta_e}{r^2} f_D T, \\
B_{\phi_1 \phi_6}^{(11)} = -2m \cos^2 \theta \cosh \delta_m \sinh \delta_m \left( g_1 + \frac{2mf_D \sinh^2 \delta_e}{r^2} \right) T, \\
B_{t6\phi_1}^{(11)} = \frac{2m \sin^2 \theta (l_2 \cosh \delta_e \sinh \delta_m \cosh \delta_{kk} - l_1 \sinh \delta_e \cosh \delta_m \sinh \delta_{kk})}{r^2} f_D T, \\
B_{t2\phi_6}^{(11)} = \frac{2m \cos^2 \theta (l_1 \cosh \delta_e \sinh \delta_m \cosh \delta_{kk} - l_2 \sinh \delta_e \cosh \delta_m \sinh \delta_{kk})}{r^2} f_D T, \\
B_{\phi_1 t6}^{(11)} = \frac{2m \sin^2 \theta (l_2 \cosh \delta_e \sinh \delta_m \sinh \delta_{kk} - l_1 \sinh \delta_e \cosh \delta_m \sinh \delta_{kk})}{r^2} f_D T,
\]

\text{19}
The ADM mass density \( M_{\text{ADM}}/V_6 \) and the angular momentum densities \( J_i/V_6 \) per unit intersecting \( M \) 2-brane and \( M \) 5-brane volume, the electric charge \( Q \), magnetic charge \( P \), and the Kaluza-Klein electric charge \( Q_{kk} \) are of the following form:

\[
M_{\text{ADM}} = \frac{3m\Omega_3}{8\pi G_N} (2 \cosh^2 \delta_e + \cosh^2 \delta_m + 3 \cosh^2 \delta_{kk} - 3),
\]

\[
J_1 = \frac{\Omega_3}{4\pi G_N} m (l_1 \cosh \delta_e \cosh \delta_m \cosh \delta_{kk} - l_2 \sinh \delta_e \sinh \delta_m \sinh \delta_{kk}),
\]

\[
J_2 = \frac{\Omega_3}{4\pi G_N} m (l_2 \cosh \delta_e \cosh \delta_m \cosh \delta_{kk} - l_1 \sinh \delta_e \sinh \delta_m \sinh \delta_{kk}),
\]

\[
Q = \frac{3m\Omega_3}{4\pi G_N} \cosh \delta_e \sinh \delta_e, \quad P = \frac{3m\Omega_3}{8\pi G_N} \cosh \delta_m \sinh \delta_m,
\]

\[
Q_{kk} = \frac{9m\Omega_3}{8\pi G_N} \cosh \delta_{kk} \sinh \delta_{kk},
\]

(59)

where \( V_6 \) is the volume of the 6-dimensional space internal to the intersecting \( M \) 2-brane and \( M \) 5-brane.

Note that when \( \delta_e = 0 \) (and, therefore, \( T^{-1} = 1 \)) \( \delta_m = 0 \) (and, therefore, \( F^{-1} = 1 \)), the solution (53) reduces to a special case of a rotating five-brane with a boost [a rotating membrane with a boost].

It would be of interest to find the explicit expression for intersecting three fivebranes with a \( SO(1,1) \) boost along the common string \((5 \perp 5 \perp 5 + \text{boost})\). Such a solution becomes the four-dimensional rotating black hole with four charges, after the compactification, and is related to the four-dimensional rotating solution constructed in Ref. [24] through \( U \)-duality transformations of the toroidally compactified, six-dimensional type-II string.

VI. BPS LIMITS

In this section, we shall discuss BPS limits (i.e., the limit in which the ADM energy of the \( M \)-brane configuration saturates the BPS bound) of rotating intersecting \( M \)-brane solutions. For the case of static, spherically symmetric solutions, the BPS-saturated solutions are a subset of extreme solutions (defined as solutions for which the event horizon and the Cauchy horizon coincide), but not all of the extreme solutions correspond to the BPS-saturated solutions \(^8\). However, when the angular momenta are non-zero, the BPS limit and the extreme limit are not correlated. For a generic rotating charged solution, the extreme limit is reached before the BPS limit is reached. Thus, most of BPS-saturated \( rotating \) solutions have a naked singularity.

For solutions embedded in supergravity theories, the BPS-saturated solutions preserve some of supersymmetries. The BPS limit is of special interest since (in the case of \( N \geq 4 \))

\(^8\)See Refs. [24,27] for some of examples of non-BPS extreme solutions. Also, Ref. [28] discusses systematic criteria for determining the BPS nature of extreme solutions.
supersymmetry protects classical solutions from being modified by quantum corrections, and therefore one can trust their classical form. In particular, those are solutions which have been studied in detail in order to shed light on the connection between the thermodynamic and statistical (microscopic) origins of their entropy.

In the following, we shall obtain the explicit forms of the BPS limits for intersecting rotating $M$-brane solutions discussed in the previous sections. The BPS limit is achieved by taking the boost parameters $\delta$’s to infinity and the non-extremality parameter $m$ to zero in such a way that the charges carried by the intersecting $M$-branes are kept as finite constants. On the other hand, the rotational parameters $l_i$ are kept finite in general, however, in this case they may produce singular BPS solutions upon compactification to lower dimensions (see below).

Note that such explicit solutions might shed light on the understanding of the microscopic entropy of rotating BPS-saturated black hole solutions from the point of the $M$-brane picture.

**A. Singular BPS limit**

For BPS limits, with angular momentum parameters kept as non-zero constants, most of the charged, rotating black holes exhibit a naked singularity. This is also the case for the corresponding BPS limits of rotating intersecting $M$-branes, constructed in the previous sections. Namely, when we take the limit $m \to 0$ with $l_i$ kept as non-zero constants, the event horizon (defined as roots $r = r_H$ of the equation $G^{(11)}_{rr} = 0$) disappears. Thus, a singularity, which the rotating, intersecting $M$-brane solutions may have, becomes a naked singularity. Therefore, in order to reach the regular BPS limit (i.e., the limit in which regular horizon exists in directions transverse to the $M$-brane), one has to take the angular momentum parameters to zero, thus resulting (in most cases) in static charged configurations. Even though the BPS solutions with non-zero rotational parameters, which have a naked singularity, may not have an immediate relevance for the study of their thermodynamic properties, we shall present them anyway. In addition, we shall also present non-singular BPS limits for some of (intersecting) rotating $M$-brane solutions derived in the previous sections.

The singular BPS limit (i.e. the limit in which the space-time transverse to the configuration has a naked singularity) corresponds to a limit with $\delta's \to \infty$ and $m \to 0$ in such way that charges remain as finite constants (i.e. $Q_n \sim Q_n \equiv 2m \sinh \delta_n \cosh \delta_n = 2m(\sinh \delta_n)^2$).

\footnote{In \cite{29}, it was proposed that such a naked singularity of BPS-saturated rotating black holes can be resolved by a generalized dimensional reduction (i.e., the dimensional reduction including massive Kaluza-Klein modes as well) of the corresponding higher-dimensional solution, which has a milder singularity, thereby making it possible to analyze the thermodynamic entropy of such singular solutions.}

\footnote{For the space-time dimensions higher than $D = 5$, the charged rotating black holes with one non-zero angular momentum have regular horizon in the BPS limit \cite{30}.}
while the angular momentum parameters $l_i$ are kept finite. In this case, the non-extremality functions and the modified harmonic functions take the following forms:

$$f \to 1, \quad f' \to G_D^{-1}, \quad H_n \to 1 + f_D \frac{Q_n}{r^{D-3}}. \quad (60)$$

In the following, we shall first obtain the singular BPS saturated solutions of a single membrane and a single five-brane, and then find expressions for singular BPS limits of intersecting two $M$-branes.

- **single $M$-brane**

  The singular BPS limit of a membrane:

  $$ds_{11}^2 = T^{-1/3} \left[ T(-dt^2 + dy_1^2 + dy_2^2) + G_D dr^2 
  + \frac{r^2}{2} \left\{ \left( \frac{l_2^2 \cos^2 \theta + l_3^2 \sin^2 \psi_1 + l_4^2 \cos^2 \psi_2 + l_5^2 \sin^2 \psi_2}{r^2} \right) d\theta^2 
  + \left( \frac{l_2^2 - \frac{l_3^2 \sin^2 \psi_2 - l_4^2 \cos^2 \psi_2}{r^2}}{r^2} \right) \cos \theta d\psi^2 
  - 2 \frac{l_2^2 - l_3^2 \sin^2 \psi_2 - l_4^2 \cos^2 \psi_2}{r^2} \cos \theta \sin \psi \sin \psi d\theta d\phi 
  - 2 \frac{l_2^2 \cos \theta \sin \psi \sin \psi d\theta d\phi}{r^2} \right\} \right], \quad (61)$$

  where the modified harmonic function $T$ associated with the electric charge source $Q$ is of the form:

  $$T^{-1} = 1 + f_D \frac{Q}{r^6}, \quad Q \equiv \frac{m}{2} e^{2\delta_k}. \quad (62)$$

  The non-zero component of the three-form field is given by

  $$B_{112}^{(11)} = T^{-1} \quad (63)$$

  The singular BPS limit of a five-brane:

  $$ds_{11}^2 = F^{-2/3} \left[ F(-dt^2 + dy_1^2 + \cdots + dy_5^2) 
  + G_D dr^2 + \frac{r^3}{2} \left\{ \left( \frac{l_2^2 \cos^2 \theta + l_3^2 \sin^2 \psi}{r^2} \right) d\theta^2 
  + \left( \frac{l_2^2 \cos \theta d\psi^2 - 2 \frac{l_2^2}{r^2} \cos \theta \sin \psi \sin \psi d\theta d\phi}{r^2} \right) \right\} \right], \quad (64)$$

  where the modified harmonic function $F$ associated with the magnetic charge source $P$ is given by

  $$F^{-1} = 1 + f_D \frac{P}{r^3}, \quad P \equiv \frac{m}{2} e^{2\delta_m}. \quad (65)$$
The non-zero component of the three-form field is given by

\[ B_{\phi_1 \phi_2 \psi}^{(11)} = -\mathcal{P} g_1 f_D \cos^2 \theta \]  

(66)

Note, in the BPS limit the angular momenta of single \( M \)-branes become zero (as one can see from the expressions for angular momenta \( J_i \) in Eqs. (11) and (12)), while the metric is non-trivially modified due to non-zero rotational parameters \( l_i \) since we are keeping \( l_i \) in the solutions Eqs. (11) and (12) as non-zero constants.

* Intersecting two \( M \)-branes

The singular BPS limit of an intersecting membrane and five-brane:

\[
\begin{align*}
\frac{ds^2}{F^{-2/3} T^{-1/3}} &= FT(-dt^2 + dy_1^2) + F(dy_2^2 + dy_3^2 + dy_4^2 + T dy_5^2) \\
&+ \mathcal{G}_D dr^2 + r^2 \left\{ (g_1 \cos^2 \theta + g_2 \sin^2 \theta) \text{d}\theta^2 - \frac{2 \mathcal{Q} l_1 l_2 \cos^2 \theta \sin^2 \theta}{r^4} f_D FT d\phi_1 d\phi_2 \\
&- \frac{2 \sqrt{\mathcal{Q}l_1 \sin^2 \theta}}{r^2} f_D FT d\phi_2 - \frac{2 \sqrt{\mathcal{Q}l_2 \cos^2 \theta}}{r^2} f_D FT d\phi_2 \\
&+ \left( g_1 - \frac{\mathcal{Q}l_1^2 \sin^2 \theta}{r^6} f_D^2 FT \right) \sin^2 \theta d\phi_1^2 + \left( g_2 - \frac{\mathcal{Q}l_2^2 \cos^2 \theta}{r^6} f_D^2 FT \right) \cos^2 \theta d\phi_2^2 \right\} \\
\end{align*}
\]

(67)

where the modified harmonic functions \( T \) and \( F \) associated with the electric \( Q \) and magnetic \( P \) charge sources are given by

\[ T^{-1} = 1 + f_D \frac{Q}{r^2}, \quad F^{-1} = 1 + f_D \frac{P}{r^2} \]

(68)

The non-zero components of the three-form field are of the following form:

\[
\begin{align*}
B_{\phi_1 \phi_2 \psi}^{(11)} &= T^{-1}, \quad B_{\phi_1 \phi_2 \phi_3}^{(11)} = -\mathcal{P} \cos^2 \theta \left( g_1 + \frac{Q f_D}{r^2} \right) T, \\
B_{\phi_1 \phi_2 \phi_3}^{(11)} &= l_1 \frac{\sqrt{\mathcal{Q} l_1 \sin^2 \theta}}{r^2} f_D T, \quad B_{\phi_2 \phi_3}^{(11)} = l_1 \frac{\sqrt{\mathcal{Q} l_2 \cos^2 \theta}}{r^2} f_D T, \\
B_{\phi_1 \phi_2 \phi_3}^{(11)} &= -l_1 \frac{\sqrt{\mathcal{Q} l_1 \sin^2 \theta}}{r^2} f_D T, \quad B_{\phi_2 \phi_3}^{(11)} = -l_2 \frac{\sqrt{\mathcal{Q} l_2 \cos^2 \theta}}{r^2} f_D T. \\
\end{align*}
\]

(69)

The singular BPS limit of intersecting two membranes:

\[
\begin{align*}
\frac{ds^2}{T_1^{-1/3} T_2^{-1/3}} &= \left[ -T_1 T_2 dt^2 + T_1 (dy_1^2 + dy_2^2) + T_2 (dy_3^2 + dy_4^2) \\
&+ \mathcal{G}_D dr^2 + r^2 \left\{ 1 + \frac{l_1^2 \cos^2 \psi}{r^2} + \left( \frac{l_1^2 \sin^2 \psi + l_2^2 \cos^2 \psi}{r^2} \right) \sin^2 \theta \right\} d\theta^2 \\
&+ \left( 1 + \frac{l_2^2 \cos^2 \psi}{r^2} \right) \cos^2 \theta d\psi^2 - 2 \frac{l_2^2}{r^2} \cos \theta \sin \theta \cos \psi \sin \psi d\theta d\psi \\
&- 2 \frac{\sqrt{\mathcal{Q}_1 \mathcal{Q}_2 l_1 \mu_1^2}}{r^4} f_D T_1 T_2 d\phi_i + \mu_1^2 \left( g_1 - \frac{\mathcal{Q}_1 \mathcal{Q}_2 l_1^2 \mu_1^2}{r^{10}} f_D T_1 T_2 \right) d\phi_i^2 \\
\end{align*}
\]

11 This may be an example of a violation of classical “no-hair” theorem. Another example can be found in Ref. [3].
\[ - \frac{2Q_1 Q_2 l_i l_j \mu_i^2 \mu_j^2}{r^8} f_D^2 T_1 T_2 d\phi_i d\phi_j \right] , \tag{70} \]

where the modified harmonic functions \( T_n \) associated with the electric charge sources \( Q_n \) are given by

\[ T_n^{-1} = 1 + f_D \frac{Q_n}{r^4}, \quad (n = 1, 2). \tag{71} \]

The non-zero components of the three-form field are given by

\[ B_{11}^{(1)} = T_1^{-1}, \quad B_{34}^{(11)} = T_2^{-1}. \tag{72} \]

The singular BPS limit of intersecting two five-branes:

\[ ds^2 = F_1^{-2/3} F_2^{-2/3} \left[ \begin{array}{c} F_1 F_2 (-dt^2 + dy_1^2 + dy_2^2 + dy_3^2) + F_1 (dy_4^2 + dy_5^2) + F_2 (dy_6^2 + dy_7^2) \\ + G_D dr^2 + r^2 \left\{ d\theta^2 - \frac{2\sqrt{P_1 P_2 l^2 \sin^2 \theta}}{r} f_D F_1 F_2 d\phi d\phi \\
+ \sin^2 \theta \left( g - \frac{P_1 P_2 l^2 \sin^2 \theta}{r^4} f_D^2 F_1 F_2 \right) d\phi^2 \right\} \right], \tag{73} \]

where the modified harmonic functions \( F_n \) associated with the magnetic charge sources \( P_n \) are given by

\[ F_n^{-1} = 1 + f_D \frac{P_n}{r}, \quad (n = 1, 2). \tag{74} \]

The non-zero components of the three-form fields are given by

\[ B_{67}^{(11)} = -P_1 g f_D \cos \theta, \quad B_{45}^{(11)} = -P_2 g f_D \cos \theta. \tag{75} \]

**B. Regular BPS limit**

Again for generic non-extreme, charged, rotating black holes, when the regular BPS limit is reached, the solutions reduce to static solutions. However, for the general five-dimensional rotating black holes with three charges, one can reach the regular BPS limit, i.e. the BPS limit with a regular horizon, with non-zero angular momenta by choosing one linear combination of angular momenta to be non-zero (see e.g., \[22\], \[3\]).

For the intersecting \( M \)-brane solutions with three charges (which is compactified to the above general class of five-dimensional black hole with regular BPS limit) \( Q_k \sim 2m \cosh \delta_k \sinh \delta_k, \quad k = 1, 2, 3 \) (including electric/magnetic charges of the constituent \( M \)-branes, electric charge associated with the \( SO(1,1) \) boost along the isometry direction of the internal coordinate, etc), and \( D - 1 = 4 \) transverse space dimensions, the regular BPS limit is achieved by taking the limit in which \( \delta_k \to \infty \) and \( m, l_i \to 0 \) in such a way that the following combinations remain as constants:
\[ Q_k = \frac{1}{2} me^{2\delta_k}, \quad L_i = \frac{l_i}{\sqrt{m}} \quad (k = 1, 2, 3; \ i = 1, 2). \] (76)

In the regular BPS limit, the non-extremality functions and the Harmonic functions take the following forms:

\[ g_i, G, f, f' \to 1, \quad H_n \to 1 + \frac{Q_n}{r^{D-3}} \], (77)

and \( K_i \) defined in Eqs. (19) and (20) become zero.

The first example of the intersecting \( M \)-brane solution with \( D - 1 = 4 \) transverse spatial dimensions which has the regular BPS limit with non-zero angular momenta corresponds to the intersecting membrane and fivebrane solution with a boost along the \( y_1 \)-directions (Eq. (55)). The explicit form of the BPS limit is given as follows:

\[
\begin{align*}
\frac{d s^2}{F^{-2/3}T^{-1/3}} &= \frac{F T (-K^{-1} dt^2 + K \overline{dy_1}^2)}{T^1 T^2 T^3} + F (dy_2^2 + \cdots + dy_5^2) + T dy_6^2 \\
&+ \frac{J \sin^2 \theta}{r^2} T F d \phi_1 \overline{dy_1} - \frac{J \cos^2 \theta}{r^2} T F d \phi_2 \overline{dy_1} + dr^2 \\
&+ r^2 \left\{ d \theta^2 - \frac{J \sin^2 \theta}{r^4} K^{-1} F T d t d \phi_1 + \frac{J \cos^2 \theta}{r^4} K^{-1} F T d t d \phi_2 + \sin^2 \theta d \phi_1^2 + \cos^2 \theta d \phi_2^2 \right\} ,
\end{align*}
\] (78)

where the harmonic functions \( F, T \) and \( K \) associated with the electric \( Q \), magnetic \( P \) and the boost electric \( Q_{kk} \) charges are given by

\[
\begin{align*}
F^{-1} &= 1 + \frac{P}{r^2}, \quad T^{-1} = 1 + \frac{Q}{r^2}, \quad K = 1 + \frac{Q_{kk}}{r^2},
\end{align*}
\] (79)

and the modified infinitesimal line element \( \overline{dy_1} \) along the boost direction \( y_1 \) is given by

\[
\overline{dy_1} = dy_1 + (K^{-1} - 1) dt.
\] (80)

Here, the angular momenta are defined in terms of the constants in Eq. (76) as

\[
J \equiv J_\phi_1 = -J_\phi_2 = (2Q \overline{P} Q_{kk})^{1/2} (L_1 - L_2)
\] (81)

The non-zero components of the three-form field are given by:

\[
\begin{align*}
B_{t16}^{(11)} &= T^{-1}, & B_{\phi_1 \phi_2 6}^{(11)} &= -P \cos^2 \theta, & B_{t\phi_1 6}^{(11)} &= -\frac{J \sin^2 \theta}{2r^2} T, \\
B_{t\phi_2 6}^{(11)} &= \frac{J \cos^2 \theta}{r^2} T, & B_{\phi_1 16}^{(11)} &= \frac{J \sin^2 \theta}{2r^2} T, & B_{\phi_2 16}^{(11)} &= \frac{J \cos^2 \theta}{2r^2} T.
\end{align*}
\] (82)

The second example corresponds to the intersecting three membrane \( (2 \perp 2 \perp 2) \) solution (49) with the regular BPS limit having the following form:

\[
\begin{align*}
\frac{d s_{11}^2}{(T_1 T_2 T_3)^{-1/3}} &= \left[ -T_1 T_2 T_3 dt^2 + T_1 (dy_1^2 + dy_2^2) + T_2 (dy_3^2 + dy_4^2) + T_3 (dy_5^2 + dy_6^2) \\
&+ dr^2 + r^2 \left\{ d \theta^2 + \frac{J^2}{2r^6} T_1 T_2 T_3 \cos^2 \theta \sin^2 \theta d \phi_1 d \phi_2 \right\}
\end{align*}
\]
\begin{eqnarray}
\frac{2J}{r^4} T_1 T_2 T_3 \sin^2 \theta dt d\phi_1 + \frac{2J}{r^4} T_1 T_2 T_3 \cos^2 \theta dt d\phi_2 \\
+ \sin^2 \theta \left(1 - \frac{J^2}{4r^6} T_1 T_2 T_3 \sin^2 \theta\right) d\phi_1^2 + \cos^2 \theta \left(1 - \frac{J^2}{4r^6} T_1 T_2 T_3 \cos^2 \theta\right) d\phi_1^2 \right)
\end{eqnarray}

where the harmonic functions $T_n$ associated with the electric charge $Q_n$ are given by

$$T_n^{-1} = 1 + \frac{Q_n}{r^2}, \quad n = 1, 2, 3,$$

and the angular momenta are defined in terms of the constants in (76) as

$$J_1 = -J_2 = J = (2Q_1 Q_2 Q_3)^\frac{1}{2}(L_1 - L_2).$$

The non-zero components of the three-form field are given as follows:

$$B_{\alpha \mu \beta}^{(11)} = T_n^{-1}, \quad (n = 1, 2, 3)$$

where $(a_n, b_n) = (1, 2), (3, 4), (5, 6)$ for $n = 1, 2, 3$, respectively.

\section*{VII. CONCLUSIONS}

In this paper, we set out to address a general structure of non-extreme rotating $M$-brane configurations of eleven-dimensional supergravity. Such configurations should be interpreted as bound state solutions of $M$-branes with a common non-extremality parameter and common rotational parameters associated with the transverse spatial directions of the $M$-brane configuration.

An important result is a general algorithm for constructing the overall conformal factor and the internal components of the eleven-dimensional metric for such configurations. We also spelled out the action of a boost along the common intersecting direction. The space-time describing the internal part of such (intersecting) configurations is specified entirely by “harmonic functions” for each constituent $M$-brane (associated with each charge source) and the “non-extremality functions” (associated with the Schwarzschild mass), which are, in contrast to the static case, modified by functions that depend on the the rotational parameters (see Section III). This general algorithm (while inferred from the structure of the solution for a single rotating membrane and a single rotating five-brane) can be checked (and confirmed) against the explicit solutions of intersecting $M$-brane solutions, uplifted from the known rotating, charged black hole solutions in lower dimensions and the $T$-duality related solutions thereof.

On the other hand, the transverse part of the configuration, which reflects the axial symmetry of the solution, involves charge sources as well as the rotational parameters in a more involved manner, and cannot be simply written in terms of modified harmonic functions and non-extremality functions, only \footnote{Note that in the case of static solutions the transverse part has a uniform structure of the form $f^{-1}(r) dr^2 + r^2 d\Omega_{D-1}^2$ with $d\Omega_{D-1}^2$ given by the infinitesimal length element of the unit $(D - 2)$-sphere $S^{D-2}$, independently of the number of non-zero charges.}. Therefore, in contrast to the case of static solutions,
the general algorithm for constructing the transverse part of the metric was obtained only for intersecting two M-branes and for intersecting three membranes with the transverse spatial dimensions $D-1 = 4$. Again this structure can be checked against the corresponding explicit solutions uplifted from known lower-dimensional solutions.

In contrast to the static solutions, the eleven-dimensional three-form field acquires additional non-zero components due to the electric [magnetic] field induced from rotating magnetic [electric] charge sources, thus assuming a complicated structure. The structure of the three-form field was obtained only in the case of intersecting two M-branes and the intersecting membrane and fivebrane with the $SO(1,1)$ boost, when it can be checked against the explicit solutions.

The hope is that structure of such configurations may shed light on the role of angular momentum components of the non-extreme rotating black holes from the point of view of M-theory, and it may ultimately shed light on the origin of their contribution to the entropy of such a system. While the part internal to the configuration is well understood and has a clear interpretation in terms of harmonic functions, as “sources” for each M-brane (and the non-extremality functions), the structure of the transverse part of the metric needs further investigations. In the BPS limit, the structure of the transverse part simplifies significantly and therefore it may be possible to write the general structure in a tractable form.

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