H-essential Submodules and Homessential Modules

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Abstract
The main goal of this paper is introducing and studying a new concept, which is named H-essential submodules, and we use it to construct another concept called Homessential modules. Several fundamental properties of these concepts are investigated, and other characterizations for each one of them is given. Moreover, many relationships of Homessential modules with other related concepts are studied such as Quasi-Dedekind, Uniform, Prime and Extending modules.

Keywords: Essential submodules, H-essential submodules, Homessential modules.

1. Introduction
Throughout this paper, all rings are commutative with identity and all modules are unitary left R-modules. "A submodule V of a module U is called essential (simply V ≤_e U), if the intersection of V with any non-zero submodule of U is not equal to zero" [1]. In this paper, we introduce a new concept, named Homessential module. This concept needs to define a certain type of submodules named H-essential submodules, where a proper submodule V of U is called H-essential, if for each non-zero homomorphism f ∈ Hom_R(U/V, U); f(U/V) is an essential submodule of U. A module U is called Homessential if every proper submodule of U is H-essential.

In section two; we investigate the main properties of H-essential submodules; we determine the trace of the quotient module over any H-essential submodule, see Proposition (2.8). Also, we show that the intersection of any two H-essential submodules is also H-essential, see Proposition (2.11). Furthermore, another characterization of H-essential submodule is given, see Theorem (2.12), and we show that every rational submodule is H-essential (see 2.13). Moreover, we prove that in the class of polyform module, every essential submodule is H-essential, Corollary (2.15).

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Section three of this paper is devoted to introduce and study the concept of Homessential module, we give another characterization of Homessential module, see Proposition (3.6). Also, the relationships of Homessential module with some other related concepts such as quasi-Dedekind, uniform and prime and extending modules are studied, see the results (3.9), (3.10), (3.12), (3.14) (3.15), (3.16) and (3.17).

2. H-essential Submodules

This section is devoted to studying the main properties of H-essential submodules.

Definition (2.1): Let U be an R-module. A proper submodule V of U is called H-essential, if for each non-zero homomorphism \( f \in \text{Hom}_R \left( \frac{U}{V}, U \right) \); \( f \left( \frac{U}{V} \right) \) is essential submodule of U.

Examples (2.2):
1. Consider the \( Z \)-module \( Z_4 \), the submodule \( \langle 2 \rangle \) of \( Z_4 \) is H-essential, since the non-zero homomorphism in \( \text{Hom}_Z \left( \frac{Z_4}{\langle 2 \rangle}, Z_4 \right) \) is only the inclusion homomorphism. In fact \( f \left( \frac{Z_4}{\langle 2 \rangle} \right) = \langle 2 \rangle \), and \( \langle 2 \rangle \) is essential submodule of \( Z_4 \).
2. In the \( Z \)-module \( Z_6 \), the submodule \( \langle 3 \rangle \) is not H-essential. In fact; there exists a homomorphism \( f: \frac{Z_6}{\langle 3 \rangle} \to Z_6 \) with \( f \left( \frac{Z_6}{\langle 3 \rangle} \right) \) is not essential submodule of \( Z_6 \).

Proposition (2.3): If V is H-essential submodule of U, then \( rU \subseteq U \) for each \((0 \neq) r \in \text{ann}_R(V) \).

Proof: Let \((0 \neq) r \in \text{ann}_R(V) \). Define \( f: \frac{U}{V} \to U \) by \( f(u + V) = ru \) for each \( u + V \in \frac{U}{V} \). It is clear that \( f \) is well defined and homomorphism. Since V is H-essential, then \( f \left( \frac{U}{V} \right) = ru \) is essential submodule of \( U \).

As a consequence of Proposition (2.3) we have the following.

Corollary (2.4): Every non-zero H-essential submodule cannot be faithful.

We need to give the following lemmas.

Lemma (2.5): [1] If A and B are submodules of an R-module U such that \( A \leq B \leq C \). Then \( A \leq U \) if and only if \( A \leq B \) and \( B \leq U \).

The following Lemma can be easily shown.

Lemma (2.6): Let \( \{V_i\}_{i \in I} \) be a family of essential submodules of an R-module U. Then \( \sum_{i \in I} V_i \subseteq U \).

Proof. It’s obvious from Lemma (2.5).

"Recall that a ring \( R \) is called Noetherian if every ideal of \( R \) is finitely generated" [2, P.55].

Proposition (2.7): Let R be a Noetherian ring, and U be an R-module. If V is H-essential submodule of U, then \( \text{ann}_R(V)U \leq U \).

Proof: Since R is a Noetherian ring, then \( \text{ann}_R(V) \) is finitely generated ideal of R. That is \( \text{ann}_R(V) = (r_1, r_2, \ldots, r_n) \) for some \( r_j \in \text{ann}_R(V) \). This implies that \( \text{ann}_R(V)U = \sum_{j=1}^n r_jU \). By Proposition (2.3), \( r_jU \leq U \) for each \( j=1, 2, \ldots, n \), and by Lemma (2.6), \( (\text{ann}_R(V))U \leq U \).

"For R-modules U and V, the trace of an R-module U is defined by \( T_V(U) = \sum_{\varphi} \varphi(U) \) where \( \varphi \in \text{Hom}_R(U, V) \) ([2], p.27)". If V=R, then the trace of U on R is denoted by \( T(U) \).

Proposition (2.8): Let U be an R-module and T(U) is the trace of U on R. If V is an H-essential submodule, then either \( T(\frac{U}{V}) \) is zero or \( T(U/V)U \) is essential submodule of U.

Proof: Suppose that \( T(\frac{U}{V}) \neq 0 \) and \((0 \neq) \varphi \in \text{Hom}_R(\frac{U}{V}, R) \). For each \( u \in U \), one can define \( f_u: R \to U \) by \( f_u(r) = ru \); for all \( r \in R \). It is clear that \( f_u \) is well defined and homomorphism. So \( (f_u \circ \varphi) \in \text{Hom}_R(\frac{U}{V}, U) \). Since V is H-essential submodule of U, then \( f_u \circ \varphi \mid_\frac{U}{V} \leq U \) for each \( u \in U \). This implies that \( \varphi \mid_\frac{U}{V} \leq U \) for each \( u \in U \). Since \( T(\frac{U}{V})U = (\sum_{\varphi} \varphi(\frac{U}{V}))U \), we have \( \varphi(\frac{U}{V})U \leq (\sum_{\varphi} \varphi(\frac{U}{V}))U = T(\frac{U}{V})U \leq U \). By Lemma (2.6), we get \( T(\frac{U}{V})U \leq U \).

Proposition (2.9): Let U be an R-module. If V is an H-essential submodule of U with \( \text{ann}_R(V) \neq 0 \), then \( \text{ann}_R(U) \nsubseteq \text{ann}_R(V) \).

Proof: It is clear that \( \text{ann}_R(U) \nsubseteq \text{ann}_R(V) \). Now let \((0 \neq) r \in \text{ann}_R(V) \), by proposition (2.3), \( rU \leq U \), hence \( rU \neq 0 \), that is \( r \notin \text{ann}_R(U) \). Therefore \( \text{ann}_R(U) \nsubseteq \text{ann}_R(V) \).

Examples (2.10):
1. From Example (2.2)(1), the submodule \( \langle 2 \rangle \) is H-essential of \( Z_4 \). So by Proposition (2.9), \( \text{ann}_Z(\langle 2 \rangle) \nsubseteq \text{ann}_Z(\langle 2 \rangle) \). In fact, \( \text{ann}_Z(\langle 2 \rangle) = 2Z \), and \( \text{ann}_Z(\langle 2 \rangle) = 4Z \).
2. Consider the $\mathbb{Z}$-module $\mathbb{Q}$, $\mathbb{Z}$ is H-essential submodule of $\mathbb{Q}$ (as we will see in Example 2.14), but $\text{ann}_\mathbb{Q}(\mathbb{Q})= \text{ann}_\mathbb{Z}(\mathbb{Z})$ since $\text{ann}_\mathbb{Z}(\mathbb{Z}) = 0$.

**Proposition (2.11):** Let $A$ and $B$ be H-essential submodules of an R-module $U$, then $A \cap B$ is an H-essential submodule.

**Proof:** If $A \cap B = 0$, then we are done. Assume that $A \cap B \neq 0$ and $0 \neq f \in \text{Hom}_R(\frac{A}{A \cap B}, U)$. It is clear that $\text{Hom}_R(\frac{A}{A \cap B}, U) \leq \text{Hom}_R(\frac{A}{A}, U) + \text{Hom}_R(\frac{B}{A \cap B}, U) [3]$. So for all $f \in \text{Hom}_R(\frac{A}{A \cap B}, U), f = h + k$ where $h \in \text{Hom}_R(\frac{A}{A}, U)$ and $k \in \text{Hom}_R(\frac{B}{A \cap B}, U)$ with $h(\frac{A}{A}) \leq U$ and $k(\frac{B}{A \cap B}) \leq U$ implies $h(\frac{A}{A}) + k(\frac{B}{A \cap B}) \leq U$. Hence $f(\frac{A}{A \cap B}) \leq U$, hence we are done.

**Theorem (2.12):** Let $U$ be an R-module, and $0 \neq V \leq U$. Then the following statements are equivalent:

1. $V$ is H-essential submodule.
2. For each $f \in \text{End}_R(U)$ such that $V \subseteq \ker f : f(U) \leq \mathfrak{e} U$.

**Proof:** (1) $\Rightarrow$ (2) Let $f \in \text{End}_R(U)$ and $V \leq \ker f$. Define $g : \frac{U}{V} \to U$ by $g(u + V) = f(u)$. It is clear that $g$ is well defined and homomorphism. By assumption, $g(\frac{U}{V}) \leq \mathfrak{e} U$. But by definition of $g$, $g(\frac{U}{V}) = f(U)$, thus $f(U) \leq \mathfrak{e} U$.

(2) $\Rightarrow$ (1) Let $f \in \text{Hom}_R(\frac{U}{V}, U)$, and consider the natural epimorphism $\pi : U \to \frac{U}{V}$, then $f \circ \pi \in \text{End}_R(U)$. But $V \leq \ker (f \circ \pi)$, so by assumption $f \circ \pi (U) \leq \mathfrak{e} U$. This implies that $f(\frac{U}{V}) \leq \mathfrak{e} U$. That is $V$ is H-essential submodule of $U$.

"Recall that a submodule $V$ of an R-module $U$ is called rational if $\text{Hom}_R(\frac{U}{V}, E(U)) = 0$, where $E(U)$ is the injective hull of $U" [2, P.12].

**Proposition (2.13):** Every non-zero rational submodule is H-essential.

**Proof:** Let $V$ be a non-zero submodule of an R-module $U$. It is clear that $\text{Hom}_R(\frac{U}{V}, U) \leq \text{Hom}_R(\frac{U}{V}, E(U))$, and since $V$ is rational submodule, then $\text{Hom}_R(\frac{U}{V}, E(U)) = 0$. This implies that $\text{Hom}_R(\frac{U}{V}, U) = 0$, that is $V$ is H-essential. In fact there is no non-zero homomorphism $f \in \text{Hom}_R(\frac{U}{V}, U)$ with $f(\frac{U}{V}) \leq \mathfrak{e} U$.

**Example (2.14):** Consider the Z-module $\mathbb{Q}$, where $\mathbb{Q}$ is the set of all rational numbers. Since $\mathbb{Z}$ is a rational submodule of $\mathbb{Q}$ hence by Proposition (2.13), $\mathbb{Z}$ is H-essential submodule of $\mathbb{Q}$.

"Recall that an R-module $U$ is called polyform if every essential submodule is rational" [4]. By proposition (2.13), we conclude the following.

**Corollary (2.15):** If $U$ is polyform module, then every essential submodule of $U$ is H-essential.

3. **Homessential Module**

In this section, we introduce the class of Homessential module.

**Definition (3.1):** An R-module $U$ is called Homessential module, if every proper submodule of $U$ is H-essential. A ring $R$ is called Homessential, if $R$ is Homessential R-module.

**Examples (3.2):**

1. The zero R-module is a Homessential module, since $(0)$ has no non-zero H-essential submodule $V$ of $(0)$ such that $f(V)$ is not essential submodule of $(0)$.
2. $\mathbb{Z}_4$ is a Homessential module, since the only proper submodule of $\mathbb{Z}_4$ is $\langle 2 \rangle$, and $\langle 2 \rangle$ is H-essential submodule see (example 2.2)(1).
3. The Z-module $\mathbb{Z}_{p^{\infty}}$ is Homessential module. In fact, it's uniform module [5], then the result follows Remark (5.3)(4).

**Remarks (3.3):**

1. Every simple module is a Homessential module.
2. A semisimple module is not Homessential module.
3. Every integral domain $R$ is Homessential R-module.
4. Every uniform module is Homessential module, where an R-module $U$ is called uniform if every non-zero submodule is essential in $U$ [2].
5. Every monomorph module is Homessential, where a module $U$ is called monomorph if every non-zero submodule is rational [5], in fact this follows directly from Proposition (2.13).
Proof: 3. Let be an integral domain, and be a non-zero ideal of . We can easily show that in an integral domain every non-zero ideal is an essential ideal. This implies that is singular -module, hence, it is torsion. On the other hand, is torsion free -module, therefore , that is is Homessential module. In particular, is Homessential.

4. It is obvious. □

Proposition (3.4): A direct summand of a Homessential module is Homessential.
Proof: Let be a Homessential module, where and be -submodules of , and be a submodule of . Let . Consider the following sequence of homomorphism:

Note that , and hence . Since is Homessential, then by Theorem (2.12), . This implies that , and so that . □

Proposition (3.5): Let be an -module. Then is Homessential -module if and only if is Homessential -module for each ideal .

Proof: Assume that is Homessential -module. Since , then . Then . This implies that . The converse is in similar way. □

The following theorem gives another characterization of Homessential module.

Proposition (3.6): Let be an -module, and be a non-zero submodule of . Then the following statements are equivalent:

1. is a Homessential module.
2. For each with , .

Proof: (1) ⇒ (2) Let be a Homessential module, and with , by assumption . By Theorem (2.12), .

(2) ⇒ (1) Let be a non-zero proper submodule of , and . Note that , where : . Since , then . By Theorem (2.12), is an -essential submodule, hence is Homessential module. □

"Recall that a submodule of a module is said to be closed if has no proper essential submodule in " [1, p.18]

Proposition (3.7): Let be a Homessential module. If is a semisimple module, then for every , is an epimorphism.

Proof: Let be a proper submodule of and . By the definition of Homessential module, . But is semisimple, so that is closed submodule of [7]. This implies that , hence is an epimorphism. □

"Recall that an -module is said to be quasi-Dedekind, if for every non-zero homomorphism , then is Homessential." [8]

Proposition (3.8): Let be a quasi-Dedekind module with for , then is Homessential.

Proof: Put , since is a maximal ideal, then is field, and clearly is quasi-Dedekind -module, therefore is free -module. This implies that , and . Since is indecomposable [8, Rem (1.3), p.24], then is as -module. Thus is simple, and by Remarks (3.3)(1), is Homessential. □

Corollary (3.9): If is a finitely generated quasi-Dedekind module, then is Homessential.

Proof: Since is a finitely generated quasi-Dedekind module, then is uniform [8], and by Remark (3.3)(4), is Homessential. □
"A ring R is called regular (in the sense of Von Neumann) if for each a ∈ R there exists b ∈ R such that a = aba." [2, P.4]. Under certain conditions, Homessential module can be quasi-Dedekind as the following proposition shows.

**Proposition (3.10):** Let U be a Homessential R-module. If $\text{End}_R(U)$ is a regular ring, then U is quasi-Dedekind.

**Proof:** Let $0 \neq f \in \text{End}_R(U)$. We have to show that $\ker f = \{0\}$, otherwise; since U is a Homessential module, so by Proposition (3.6), $f(U) \subseteq \text{ann}_R(U)$. But $\text{End}_R(U)$, then $U = f(U) \oplus \ker f$ [7, Exc. 17(a), P.272]. This implies that $f(U) \cap \ker f = \{0\}$. Since $\ker f \neq \{0\}$, so $f(U)$ is not essential in U, which is a contradiction. □

**Remark (3.11):** The condition "$\text{End}_R(U)$ is regular" cannot be dropped from Proposition (3.10), for example; $\mathbb{Z}_4$ is Homessential Z-module but not quasi Dedekind, since $\text{End}_R(\mathbb{Z}_4) = \mathbb{Z}_4$ which is not regular.

It is known that if U is a semisimple module, then $\text{End}_R(U)$ is regular [9, Cor.(2.22, P.52)]. So by this fact and Proposition (3.10) we have the following.

**Corollary (3.12):** If U is semisimple and Homessential module, then U quasi-Dedekind.

**Remarks (3.13):**
1. The condition "semisimple" in Corollary (3.12) cannot be dropped. For example, the Z-module $\mathbb{Z}_{p_{\infty}}$ is uniform module, so it is Homessential (see Remark (3.3)(4)) but it is not quasi-Dedekind module, since it is not semisimple module.
2. In general, Homessential module need not be prime module; for example: the Z-module $\mathbb{Z}_4$ is Homessential but not prime module.

The following Corollary gives a necessary condition under which Homessential module can be prime.

**Corollary (3.14):** Let U be a Homessential module. If U is semisimple, then:
1. U is a prime module.
2. $\text{ann}_R(U)$ is a prime ideal of R.

**Proof:** 1. The result follows by Corollary (3.12), and [8, Prop.(1.7), p.26].
2. The result follows by Corollary (3.12), and [8, Cor.1.8, p.26]. □

"Recall that an R-module P is said to be projective if for every epimorphism $f: B \twoheadrightarrow C$ and for every homomorphism $g: P \rightarrow C$ there is a homomorphism $h: P \rightarrow B$ with $g = hf"$ [7, Def. (5.3.1) (b), p. 116].

**Corollary (3.15):** Let R be a regular ring and U is a finitely generated projective module. If U is Homessential, then U is a quasi-Dedekind module.

**Proof:** Assume that U is a Homessential module. Since U is a finitely generated projective module over regular ring, then $\text{End}_R(U)$ is regular [7,Exc.17(c), P.272], and by Proposition (3.10), we are done. □

"Recall that an R-module U is called Z-regular, if every cyclic submodule of U is direct summand and projective" [10].

**Corollary (3.16):** Let U be a Z-regular module. If U is a Homessential module, then U is quasi-Dedekind.

**Proof:** Since U is a Z-regular module, then $\text{End}_R(U)$ is regular [10]. But U is Homessential, then by proposition (3.10), U is a quasi-Dedekind. □

"Recall that an R-module U is called extending if every submodule of U is essential in a direct summand of U" [2, p.118], and U is indecomposable, if the only decomposition $U = A \oplus B$ are those in which either $A = \{0\}$ or $B = \{0\}$ [7, p.285].

**Proposition (3.17):** Every non-zero extending and indecomposable module is Homessential.

**Proof:** Since every non-zero extending and indecomposable module is uniform, then the result follows by Remark (3.3)(4). □

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