Laser generated wake fields as a new diagnostic tool for magnetized plasmas

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In the presence of an external magnetic field, wake fields generated by a short laser pulse can propagate out of the plasma, and thereby provide information about the electron density profile. A method for reconstructing the density profile from a measured wake field spectrum is proposed and a numerical example is given. Finally, we compare our proposal with existing plasma density diagnostic techniques.

As is well-known, a short laser pulse propagating in an underdense unmagnetized plasma can excite a wake field of plasma oscillations. This has interesting applications to plasma based particle accelerators and is naturally of importance for the general understanding of laser-plasma interactions. If the plasma is magnetized and the external magnetic field non-parallel to the direction of propagation of the laser pulse, the wake field becomes partially electromagnetic and thereby obtains a nonzero group velocity.

In the present paper we study wake field generation in an inhomogeneous magnetized plasma. In particular we address the question to what extent the wake field can propagate out of the plasma, and thereby provide information about the plasma parameters. The most interesting case is that of a strongly magnetized plasma (i.e. when the electron cyclotron frequency is larger than the plasma frequency), for which almost all of the wake field energy - except that generated in a narrow low density region - may propagate out of the plasma. Since the wake field also in the magnetized case initially has the frequency equal to the local plasma frequency, this provides a way of extracting information about the electron density profile. It turns out that even though wave overtaking - for example when a higher frequency part of the wake field passes a lower frequency part - may occur, the density profile can still be reconstructed by integrating the ray equations of geometric optics backwards. This is more straightforward for a static background density profile, and a numerical example is provided for this case, where the predicted spectrum of the wake field corresponding to an assumed density profile is shown, and a reconstructed profile is calculated. Finally, the method is compared with two of the existing plasma density diagnostic techniques, namely interferometry and reflectometry.

We consider a high frequency laser pulse with frequency \( \omega_H \) propagating in a cold, weakly inhomogeneous magnetized plasma. We assume the ordering \( \omega_H \gg \omega_p, \omega_e \), where \( \omega_p \) and \( \omega_e \equiv |qB_0|/m \) are the plasma and electron cyclotron frequency respectively, \( q \) and \( m \) are the electron charge and mass, and \( B_0 = B_0\hat{x} \) is the (constant) external magnetic field. We let the laser pulse propagate perpendicularly to the external magnetic field.

The ponderomotive force of the laser pulse will generate a "low frequency" wake field mode (which is the low frequency branch of the extraordinary mode, or plasma oscillations modified by the magnetic field, depending on the choice of terminology) during its path through the plasma. The generation mechanism is most efficient if the pulse has a duration of the order of the inverse plasma frequency or shorter. In principle, the laser pulse will broaden due to ordinary dispersion, decrease its energy and frequency due to the interaction with the wake field, etc. These and other effects have been considered in homogeneous plasmas by for example Ref. \[3\]. We will focus on the spectral properties of the wake field, however, and for this purpose it turns out that we can forget about the details of the laser pulse. Basically the effect of the laser field is to provide a well localized ponderomotive source term in the governing equations for the wake field, travelling with almost the speed of light in vacuum.

The wake field quantities are denoted by index \( L \). We introduce the corresponding vector and scalar potentials \( \mathbf{A}_L(z,t) \) and \( \phi_L(z,t) \), using Coulomb gauge, and the electron density is written \( n = n_0(z,t) + n_L(z,t) \), where \( n_0 \) is the unperturbed density, assumed to vary on longer space and time scales than the wake field. Furthermore, the electron velocity \( \mathbf{v} \) is divided into its high- and low frequency part, and we denote the low frequency contributions perpendicular and parallel to the direction of propagation with \( \mathbf{v}_{L\perp} \) and \( \mathbf{v}_{Lz} \) respectively. The ponderomotive force of the laser pulse induces longitudinal wake field motion, which couple to motion in the \( \hat{y} \) -direction through the Lorentz-force, but there is no wake field motion in the direction of the external magnetic field, and accordingly we put \( \mathbf{A}_L = A_L\hat{y} \) and \( \mathbf{v}_{L\perp} = v_{L\perp}\hat{y} \). Linearizing in the low frequency variables, and neglecting derivatives acting on \( n_0 \), we obtain the following set of equations governing the wake field generation

\[
\begin{align*}
-\mu_0 q n_0 v_{L\perp} + \left[ e^{-2}\partial_z^2 - \partial_{Lz}^2 \right] A_L &= 0 \\
e^{-2}\partial_z \partial_{Lz} \phi_L - \mu_0 q n_0 v_{Lz} &= 0 \\
\partial_t v_{L\perp} - \omega_e v_{Lz} + \frac{q}{m} \partial_t A_L &= 0 \\
\partial_t v_{Lz} + \frac{q}{m} \partial_z \phi_L + \omega_e v_{L\perp} &= -\frac{q^2}{2m^2} \partial_z |A_H|^2 \\
\partial_t n_L + n_0 \partial_z v_{Lz} &= 0
\end{align*}
\]
Next we write this on matrix form $Au = b$, where $u = (\phi_L, n_L, v_L, v_{L+}, A_L)$, $A$ is an operator matrix and $b$ is the (ponderomotive) source vector. Since derivatives on $n_0$ are neglected, we can simplify the system (1)-(5) by a simple matrix inversion $u = A^{-1}b$ treating the operators in $A$ like constant coefficients. We introduce the group velocity of the laser pulse $v_{gH} \equiv \partial_t \omega$ and eliminate different operators in the denominators everywhere by “multiplication”. By using $\partial_t \equiv -v_{gH} \partial_z \approx -c \partial_z$ for derivatives acting on the source vector, and focusing on the first component of $u$, we finally obtain after one space and one time integration

$$[\partial_t^2 + (\omega_h^2 + \omega_p^2) \partial_z^2 - (\partial_t^2 + \omega_h^2) \partial_z^2 + \omega_p^2] \phi_L = \frac{q \omega_p^4 |A_H^2|}{2m}$$

where $\omega_h^2 = \omega_p^2 + \omega_p^2$ and we recognize the operator acting on $\phi_L$ as the wave operator for the extraordinary mode. Although derivatives on $n_0$ have been neglected in the derivation of Eq. (1), it should be emphasized that the equations for wake field generation are more complicated here than for a homogeneous plasma. This is because in the inhomogeneous case, the evolution of the wake field is in general not quasi-static in a frame moving with the group velocity of the laser pulse, and thus $\partial_t \approx -v_{gH} \partial_z \approx -c \partial_z$ does not hold for derivatives acting on the wake field.

It is useful to divide the study of the wake field properties into its excitation and its propagation phase. The excitation of one additional wavelength of the wake field takes place during a distance of the order of $2\pi c/\omega_p$, and as a basic assumption of ours -- the variations of $n_0$ is negligible on this length scale. Thus as far as the excitation process is concerned, the plasma can essentially be treated as homogeneous. The solution for the wake field can thus be obtained from previous authors [2]. Changing to comoving coordinates $\xi = z - v_{gH} t$, $\tau = t$ the result for the wake field potential can be written

$$\phi_L = \phi_{L0} \sin[k_p(\xi - \xi_0)]$$

where $k_p$ is the wake field wavenumber $k_p = \omega_p/v_{gH}$, $\xi_0$ is the (constant) position of the (short) laser pulse, and $\phi_{L0} = (q \omega_p^4/v_{gH}) \int_A |A_H|^2d\xi$. The important result here, for our purposes, is the determination of the initial value of the wake field wave number $k_p = \omega_p/v_{gH}$, which corresponds to an initial frequency $\omega_p$ (in the laboratory frame), that will vary with the position of generation.

Next we make the ansatz of geometric optics. Inspecting the wave operator in Eq. (1), we note that there is a resonance at $\omega^2 = \omega_h^2 = \omega_p^2 + \omega_p^2$ and cut-offs at $\omega_L \equiv \frac{1}{2}[\omega_h + (\omega_p^2 + 4 \omega_p^2)^{1/2}]$ and $\omega_\ell \equiv \frac{1}{2}[\omega_h - (\omega_p^2 + 4 \omega_p^2)^{1/2}]$. The dispersion relation for the wake field potential can be written

$$[\partial_t^2 + (\omega_h^2 + \omega_p^2) \partial_z^2 - (\partial_t^2 + \omega_h^2) \partial_z^2 + \omega_p^2] \phi_L = \frac{q \omega_p^4 |A_H^2|}{2m}$$

where $d/dt \equiv \partial_t + v_{gH} \partial_z$. Specifically, for a time-independent medium, the right hand side of the last equation is zero and the wake field propagates, with the local wake field group velocity $v_g \equiv \partial_t \omega$ with unchanged frequency.

It is natural to first consider the density profile reconstruction from a wake field in the case of a quasi-static plasma. The proposed technique can be applied not only to truly static density profiles but to the large class of phenomena varying on a time scale significantly larger than the time of propagation of the wake field through the plasma slab. Then, sequential laser pulses gives a sequence of wake fields spectra from which a “movie” of the density profile can be obtained.

We assume that the generated wake field spectrum is measured immediately outside the plasma boundary. In consistence with the geometric optics approximation we will treat the measured data as a weakly time dependent spectrum with well defined sharp (quasi-monochromatic) peaks. The data is not necessarily monochromatic at a given time, however. A part of the wake field generated at an earlier time may overtake some part of the wake field generated at a later time, leading to a spectrum with multiple sharp peaks. Generally, we can express the data as a set of distinct frequencies measured at different times, in which case a sharp curve can be recognized, see Fig.1(a). Due to cut-offs and/or resonances, the curve may be discontinuous, the algorithm below only works until the first jump.

Before presenting the reconstruction algorithm we need to introduce some notations. As mentioned, the measurement results in a function of time, in general multi-valued, describing a path $\omega(t)$ as illustrated in Fig.1(a). Let the path be discretized into consecutive points numbered $i = 0, 1, \ldots, n, i = 0$ being the first measured point. Let also the plasma, having length $L$, be discretized into cells, numbered $j = 1, 2, \ldots, N$, of length $\Delta z = L/N$ so that the plasma has the plasma frequency $\omega_{p(j)}$ at $(j-1) \Delta z \leq z \leq j \Delta z$. In order to distinguish the two dis-
cretizations we will use upper indices for quantities discretized according to the spectrum-path discretization, e.g. $t^{(i)}$, and lower indices for quantities discretized according to the plasma discretization, e.g. $\omega_{p(j)}$. In particular we will let $z_{(j)} \equiv (j - 1)\Delta z$ whereas $z^{(i)}$ is the “exact” point in the plasma where the wake field corresponding to the $i$-th point in the spectrum was generated. Furthermore, it is convenient to introduce $\Delta z^{(i)}$, which is defined in Fig.1(b).

The following algorithm can be used to retrace the measured frequency spectrum and thereby reconstruct the electron density profile: We assume that the points $0, 1, 2, ..., i - 1$ (grey dots in Fig.1(b)) have already been retraced so that the plasma frequency of the cells $j + 1, j + 2, ..., N$ (black) is known and the plasma frequency of cell $j$ (grey) has yet to be determined. Consider now the retracing of the next spectrum-point $i$, corresponding to the frequency $\omega^{(i)}$ detected at time $t^{(i)}$. Given the plasma frequency of the already reconstructed cells, the path of spectrum point $i$ is known up to cell $j$, where the plasma frequency and thereby the group velocity is yet to be determined. Since spectrum point $i$ at least belongs in the neighborhood of cell $j$, a first guess – correct to lowest order – is that $\omega_{p(j)} = \omega^{(i)}$. Making this guess, the path of spectrum point $i$ can be reconstructed a bit further by comparing the detection time, $t^{(i)}$, with the generation time of this particular wake field, and using the ray equations (1). In particular it can be decided whether this point belongs to cell $j$ or cell $j - 1$. If it belongs to cell $j$, the algorithm assigns the calculated position to point $i$ and proceeds to spectrum point $i + 1$. If it belongs to $j - 1$, the cell $j$ is filled with spectrum points and a definite plasma frequency is given to this cell, before the algorithm continues in the same manner with reconstructing cell $j - 1$ and the path of spectrum point $i + 1$. For density variations on a faster time-scale than $c/L$, the above algorithm does not work. Preliminary work suggests that it can be generalized to a more rapidly varying regime, however, but it would require multiple laser pulses propagating in the plasma slab simultaneously.

In order to demonstrate the method we numerically calculate a wake field spectrum using the ray equations (1) starting from an assumed density profile, see Fig.2(a). In the reconstruction algorithm this spectrum is then treated as experimental data. In this case we consider a plasma magnetized such that $\omega_c = 1.1 \times \omega_{p,\text{max}}$, where $\omega_{p,\text{max}}$ is the maximum value of the plasma frequency. For simplicity we normalize such that $L = 1, c = 1$, and we let the laser pulse enter the plasma at $t = 0$ and exit at $t = 1$. The spectrum is obtained by treating the wake field as a collection of point particles – jumping into existence as the laser pulse propagates through the plasma – each moving with the local group velocity and the initial wavenumber and frequency determined by Eq. (3). Since the density profile is known, in contrast to the case of re-tracing the spectrum, it is straightforward to propagate these “particles” by means of the ray equations (1).

Retracing the spectrum according to the algorithm presented above results in a density profile that can be compared with the one we assumed, see Fig. 2(b). A small numerical error – that can be removed with a finer discretization – can be seen. Note that the entire plasma profile cannot be reconstructed. The left most points in Fig. 2(b) are missing. This is because the wake field generated in this region of low density cannot propagate through the plasma since there is a cut-off prohibiting this. The information of this region is already missing in the wake field spectrum.

Existing techniques for unperturbing plasma density diagnostics include reflectometry and interferometry methods as well as incoherent scattering [8]. It is most natural to compare our proposal with the two former methods: Reflectometry essentially makes use of the reflection of electromagnetic waves against a cut-off occurring at a point where the frequency of the electromagnetic wave equals the local plasma frequency. The plasma frequency as a function of position can be derived from time-of-flight measurements for a suitable spectrum of electromagnetic waves. Applying interferometry, a laser beam is split up into two (or more) beams. One of the beams passes through the plasma and experiences a phase shift relative to the beam propagating in vacuum. This phase shift – obtained by letting the beams interfere – is related to the density averaged along the path through the plasma.

In many respects our new method compare favorably with existing techniques, provided the basic criterion of a strong external magnetic field is fulfilled. Some advantages are: 1) Coaxial profile information: Interferometry can at best provide information about the density profile in a direction perpendicular to the penetrating laser beams (if multiple beams are used) whereas the proposed method reconstructs the density profile along the path of the laser pulse. 2) High spatial resolution: The requirement that the length scales of inhomogeneities must be much larger than the local plasma wave length $\lambda_p \equiv 2\pi v_g/\omega_p$ sets the spatial resolution limit to roughly $10^6n^{-1/2}m$ for a strongly magnetized plasma ($\omega_c \sim \omega_p$), where $n$ is the electron number density in units m$^{-3}$. 3) High temporal resolution: The temporal resolution limit is equal to the time that separates sequential laser pulses probing the dynamics, and thus $t_{\text{res}} \sim \Delta L/c$, where $\Delta L$ is the spatial separation of the laser pulses. For use of the “static” density profile reconstruction scheme $\Delta L$ must be comparable to the width of the plasma column, implying that $t_{\text{res}}$ is not longer than $10^{-8}s$, for typical laboratory plasma dimensions. 4) Density minimum access In our method the density profile can be measured even in the vicinity of a local density minimum, in contrast to reflectometric methods for which only monotonous profiles
can be fully reconstructed. This feature together with points 2) and 3) makes it possible to study the density evolution in detail of various dynamical processes, e.g. wave propagation and instabilities, using our method.

Given a suitable plasma – a regime close to thermonuclear fusion conditions seems appropriate – the equipment necessary to apply the proposed scheme is firstly a laser that can produce powerful and short pulses. Commercial Ti:sapphire lasers typically fulfill these requirements by far. Secondly, the detector need to operate over a broad frequency band and produce spectrums that are time-resolved on (roughly) the time-scale \( \frac{L_{ih}}{c} \), where \( L_{ih} \) is the density inhomogeneity scale length. Similar detector requirements have been fulfilled in the applications of ultra-short pulse reflectometry [4], however, and thus we deduce that it is not necessary to develop new technology for our scheme to work.

There are a numbers of issues not addressed in our density reconstruction scheme. Firstly, what happens if the external magnetic field is not homogeneous? This is simple provided the magnetic field profile is know, since our static reconstruction scheme is essentially unaffected by this change, and we only need to include the corresponding variations in the dispersion relation. Secondly, what happens if the profile to be reconstructed has two- or three-dimensional spatial variations? Unfortunately – unless the extra variations occur on longer spatial scales – this complicates the application a lot, at least from a practical point of view. Two parts of the wake field generated at different positions will follow different orbits in the plasma and exit at different locations. By using multiple lasers and detectors placed at suitable positions, it seems possible that 3-D density profiles can be reconstructed in principle, but to address this question in any detail is beyond the scope of this paper. Finally we point out that the question of the usefulness of our scheme is unlikely to be settled by purely theoretical arguments, and instead we want to encourage experimental investigations of wake field generation in magnetized plasmas. A first interesting step, before complete density profile reconstructions are performed, would be to experimentally verify that the maximum frequency of the wake field coincide with the maximum value of the local plasma frequency.

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I. FIGURE CAPTIONS

Fig. 1 Illustration of the discretization scheme. (a) Spectrum path discretization (b) Plasma slab discretization

Fig. 2 Example of a reconstruction. (a) Predicted wake field spectrum normalized against the maximum frequency (b). Normalized plasma profiles. The solid line marks the assumed profile and the crosses marks the reconstructed profile.
Figure 1
Figure 2