Minimal Obstructions for 1-Immersions and Hardness of 1-Planarity Testing

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Abstract: A graph is 1-planar if it can be drawn on the plane so that each edge is crossed by no more than one other edge (and any pair of crossing edges cross only once). A non-1-planar graph \( G \) is minimal if the graph \( G - e \) is 1-planar for every edge \( e \) of \( G \). We construct two infinite families...
of minimal non-1-planar graphs and show that for every integer $n \geq 63$, there are at least $2^{(n-54)/4}$ nonisomorphic minimal non-1-planar graphs of order $n$. It is also proved that testing 1-planarity is NP-complete. © 2012 Wiley Periodicals, Inc. J. Graph Theory 72: 30–70, 2013

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1. INTRODUCTION

A graph drawn in the plane is 1-immersed in the plane if any edge is crossed by at most one other edge (and any pair of crossing edges cross only once). A graph is 1-planar if it can be 1-immersed into the plane. It is easy to see that if a graph has 1-immersion in which two edges $e$, $f$ with a common end vertex cross, then the drawing of $e$ and $f$ can be changed so that these two edges no longer cross. Consequently, we may assume that adjacent edges are never crossing each other and that no edge is crossing itself. We take this assumption as a part of the definition of 1-immersions since this limits the number of possible cases when discussing 1-immersions.

The notion of 1-immersion of a graph was introduced by Ringel [14] when trying to color the vertices and faces of a plane graph so that adjacent or incident elements receive distinct colors. In the last two decades this class of graphs received additional attention because of its relationship to the family of map graphs, see [7, 8] for further details.

Little is known about 1-planar graphs. Borodin [1, 2] proved that every 1-planar graph is 6-colorable. Some properties of maximal 1-planar graphs are considered in [15]. It was shown in [3] that every 1-planar graph is acyclically 20-colorable. The existence of subgraphs of bounded vertex degrees in 1-planar graphs is investigated in [10]. It is known (see [4–6]) that a 1-planar graph with $n$ vertices has at most $4n - 8$ edges and that this upper bound is tight. In the paper [9], it was observed that the class of 1-planar graphs is not closed under the operation of edge-contraction.

Much less is known about non-1-planar graphs. The basic question is how to recognize 1-planar graphs. This problem is clearly in NP, but it is not clear at all if there is a polynomial time recognition algorithm. We shall answer this question by proving that 1-planarity testing problem is NP-complete.

The recognition problem is closely related to the study of minimal obstructions for 1-planarity. A graph $G$ is said to be a minimal non-1-planar graph (MN-graph, for short) if $G$ is not 1-planar, but $G - e$ is 1-planar for every edge $e$ of $G$. An obvious question is:

How many MN-graphs are there? Is their number finite? If not, can they be characterized?

The answer to the first question is not hard: there are infinitely many. This was first proved in [12]. Here, we present two additional simple arguments implying the same conclusion.

Example 1. Let $G$ be a graph such that $t = \lceil \text{cr}(G)/|E(G)| \rceil - 1 \geq 1$, where $\text{cr}(G)$ denotes the crossing number of $G$. Let $G_t$ be the graph obtained from $G$ by replacing each edge of $G$ by a path of length $t$. Then $|E(G_t)| = t|E(G)| < \text{cr}(G) = \text{cr}(G_t)$. This

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implies that $G_t$ is not 1-planar. However, $G_t$ contains an MN-subgraph $H$. Clearly, $H$ contains at least one subdivided edge of $G$ in its entirety, so $|V(H)| > t$. Since $t$ can be arbitrarily large (see, e.g., the well-known lower bound on $\text{cr}(K_n)$), this shows that there are infinitely many MN-graphs.

Before giving the next example, it is worth noticing that 3-cycles must be embedded in a planar way in every 1-immersion of a graph in the plane.

**Example 2.** Let $K \in \{K_5, K_{3,3}\}$ be one of the Kuratowski graphs. For each edge $xy \in E(K)$, let $L_{xy}$ be a 5-connected triangulation of the plane and $u, v$ be adjacent vertices of $L_{xy}$ whose degree is at least 6. Let $L'_{xy} = L_{xy} - uv$. Now replace each edge $xy$ of $K$ with $L'_{xy}$ by identifying $x$ with $u$ and $y$ with $v$. It is not hard to see that the resulting graph $G$ is not 1-planar (since two of graphs $L'_{xy}$ must “cross each other,” but that is not possible since they come from 5-connected triangulations). Again, one can argue that they contain large MN-graphs.

The paper [12] and the above examples prove the existence of infinitely many MN-graphs but do not give any concrete examples. They provide no information on properties of MN-graphs. Even the most basic question, if there are infinitely many MN-graphs whose minimum degree is at least 3, cannot be answered by considering these constructions. In [12], two specific MN-graphs of orders 7 and 8, respectively, are given. One of them, the graph $K_7 - E(K_3)$, is the unique 7-vertex MN-graph and since all 6-vertex graphs are 1-planar, the graph $K_7 - E(K_3)$ is the MN-graph with the minimum number of vertices. Surprisingly enough, the two MN-graphs in [12] are the only explicit MN-graphs known in the literature.

The main problem when trying to construct 1-planar graphs is that we have no characterization of 1-planar graphs. The set of 1-planar graphs is not closed under taking minors, so 1-planarity cannot be characterized by forbidding some minors.

In the present paper, we construct two explicit infinite families of MN-graphs whose minimum degree is at least 3 and, correspondingly, we give two different approaches how to prove that a graph has no plane 1-immersion.

In Section 2, we construct MN-graphs based on the Kuratowski graph $K_{3,3}$. To obtain them, we replace six edges of $K_{3,3}$ by some special subgraphs. The minimality of these examples is easy to verify, but their non-1-planarity needs long and delicate arguments. Using these MN-graphs, we show that for every integer $n \geq 63$, there are at least $2^{(n-54)/4}$ nonisomorphic minimal non-1-planar graphs of order $n$. In Section 3, we describe a class of 3-connected planar graphs that have no plane 1-immersions with at least one crossing point (PN-graphs, for short). Every PN-graph has a unique plane 1-immersion, namely, its unique plane embedding. Hence, if a 1-planar graph $G$ contains a PN-graph $H$ as a subgraph, then in every plane 1-immersion of $G$ the subgraph $H$ is 1-immersed in the plane in the same way.

Having constructions of PN-graphs, we can construct 1-planar and non-1-planar graphs with some desired properties: 1-planar graphs that have exactly $k > 0$ different plane 1-immersions; MN-graphs, etc.

In Section 4, we construct MN-graphs based on PN-graphs. Each of these MN-graphs $G$ has as a subgraph a PN-graph $H$ and the unique plane 1-immersion of $H$ prevents 1-immersion of the remaining part of $G$ in the plane.

Despite the fact that minimal obstructions for 1-planarity (i.e., the MN-graphs) have diverse structure, and despite the fact that discovering 1-immersions of
specific graphs can be very tricky, it turned out to be a hard problem to establish hardness of 1-planarity testing. In Section 5 we show that 1-planarity testing is NP-complete. The proof is geometric in the sense that the reduction is from 3-colorability of planar graphs (or similarly, from planar 3-satisfiability).

In Section 6, we show how the proof of Theorem 5 can be modified to obtain a proof that $k$-planarity testing for multigraphs is NP-complete. An extended abstract of this paper was published in *Graph Drawing 2008* [13].

2. CHAIN GRAPHS BASED ON $K_{3,3}$

Two cycles of a graph are *adjacent* if they share a common edge. If a graph $G$ is drawn in the plane, then we say that a vertex $x$ lies *inside* (resp. *outside*) an embedded (i.e., nonself-intersecting) cycle $C$, if $x$ lies in the interior (resp. exterior) of $C$, and does not lie on $C$. Having two embedded adjacent cycles $C$ and $C'$, we say that $C$ lies inside (resp. outside) $C'$ if every point of $C$ either lies inside (resp. outside) $C'$ or lies on $C'$. From this point on, by a 1-immersion of a graph we mean a plane 1-immersion. We assume that in 1-immersions, adjacent edges do not cross each other and no edge crosses itself. Thus, every 3-cycle of a 1-immersed graph is embedded in the plane. Hence, given a 3-cycle of a 1-immersed graph, we can speak about its interior and exterior. We say that an embedded cycle *separates* two vertices $x$ and $y$ on the plane, if one of the vertices lies inside and the other one lies outside the cycle. Two edges $e$ and $e'$ of a graph $G$ *separate* vertices $x$ and $y$ of the graph if $x$ and $y$ belong to different connected components of the graph $G - e - e'$.

Throughout the paper, we will deal with 1-immersed graphs. When an immersion of a graph $G$ is clear from the context, we shall identify vertices, edges, cycles, and subgraphs of $G$ with their image in $\mathbb{R}^2$ under the 1-immersion. Then by a face of a 1-immersion of $G$ we mean any connected component of $\mathbb{R}^2 \setminus G$.

By using Möbius transformations combined with homeomorphisms of the plane, it is always possible to exchange the interior and exterior of any embedded cycle and change any face of a given 1-immersion into the outer face of a 1-immersion. Formally, we have the following observation (which we will use without referring to it every time):

(A) Let $C$ be a cycle of a graph $G$. If $G$ has a 1-immersion $\varphi$ in which $C$ is embedded, then $G$ has a 1-immersion $\varphi'$ with the same number of crossings as $\varphi$, in which $C$ is embedded and all vertices of $G$, which lie inside $C$ in $\varphi$, lie outside $C$ in $\varphi'$ and vice versa.

Now we begin describing a family of MN-graphs based on the graph $K_{3,3}$.

By a link $L(x, y)$ connecting two vertices $x$ and $y$ we mean any of the graphs shown in Figure 1 where $\{z, \bar{z}\} = \{x, y\}$. We say that the vertices $x$ and $y$ are incident with the link. The links in Figure 1(A) and (B) are called *A-link* and *B-link*, respectively, and the one in Figure 1(C) is called a *base link*. Every link has a *free cycle*: both 3-cycles in an A-link are its free cycles, while every B-link or base link has exactly one free cycle (the cycle indicated by thick lines in Fig. 1).

By an *A-chain* of length $n \geq 2$, we mean the graph shown in Figure 2(a). By a *B-chain* of length $n \geq 2$, we mean the graph shown in Figure 2(c) and, for $n \geq 3$, every graph obtained from that graph in the following way: for some integers $h_1, h_2, \ldots, h_t$, where $t \geq 1$ and $1 \leq h_1 < h_2 < \cdots < h_t \leq n - 2$, we replace the link at the left of Figure 2(e)
by the link shown at the right, for \( i = 1, 2, \ldots, t \). Note that, by definition, A- and B-chains have length at least 2. We say that the chains in Figure 2(a) and (c) connect the vertices \( v(0) \) and \( v(n) \), which are called the end vertices of the chain. Two chains are adjacent if they share a common end vertex. A- and B-chains will be represented as shown in Figure 2(b) and (d), respectively, where the arrow points to the end vertex incident with the base link. The vertices \( v(0) \), \( v(1) \), \( v(2) \), \ldots, \( v(n) \) are the core vertices of the chains. Every free cycle of a link contains exactly one core vertex. The two edges of a free cycle \( C \) incident to the core vertex are the core-adjacent edges of \( C \). It is easy to see that two edges \( e \) and \( e' \) of a chain separate the end vertices of the chain if and only if the edges are the core-adjacent edges of a free cycle of a link of the chain.

By a subchain of a chain shown in Figure 2(a) and (c), we mean a subgraph of the chain consisting of links incident with \( v(i) \) and \( v(i + 1) \) for all \( i = m, m + 1, \ldots, m' - 1 \) for some \( 0 \leq m < m' \leq n \). We say that the subchain connects the vertices \( v(m) \) and \( v(m') \).

A chain graph is any graph obtained from \( K_{3,3} \) by replacing three of its edges incident with the same vertex by A-chains and three edges incident with another vertex in the same chromatic class by B-chains, where the chains can have arbitrary lengths \( \geq 2 \). These changes are to be made as shown in Figure 3(a). The vertices \( \Omega(1) \), \( \Omega(2) \), and \( \Omega(3) \) are
the base vertices of the chain graph. The edges joining the vertex $\Omega$ to the base vertices are called the $\Omega$-edges.

We will show that every chain graph is an MN-graph.

**Lemma 1.** Let $G$ be a chain graph and $e \in E(G)$. Then $G - e$ is 1-planar.

**Proof.** If $e$ is an $\Omega$-edge, then $G - e$ is planar and hence 1-planar. Suppose now that $e$ is not an $\Omega$-edge. By symmetry, we may assume that $e$ belongs to an A- or B-chain incident to $\Omega(2)$. If $e$ is the “middle” edge of a B-link, then Figure 3(b) shows that the corresponding B-chain can be crossed by an A-chain, and it is easy to see that this can be made into a 1-immersion of $G - e$. In all other cases, 1-immersions are made by crossing the link $L$ whose edge $e$ is deleted with the edges incident with the vertex $\Omega$. The upper row in Figure 3(c) shows the cases when $L$ is a base link. The lower row covers the cases when $L$ is an A-link or a B-link. The edge $e$ is shown in all cases as the dotted edge.

Our next goal is to show that chain graphs are not 1-planar. In what follows, we let $G$ be a chain graph and $\varphi$ a (hypothetical) 1-immersion of $G$.

**Lemma 2.** Let $\varphi$ be a 1-immersion of a chain graph $G$ such that the number of crossings in $\varphi$ is minimal among all 1-immersions of $G$. If $L$ is a link in an A- or B-chain of $G$, then no two edges of $L$ cross each other in $\varphi$.
Proof. The first thing to observe is that whenever edges $ab$ and $cd$ cross, there is a disk $D$ having $a, c, b, d$ on its boundary, and $D$ contains these two edges but no other points of $G$. In 1-immersions with minimum number of crossings, this implies that no other edges between the vertices $a, c, b, d$ are crossed. Similarly, if $L = L(z, \bar{z})$ is a link in a chain, and an edge incident with $z$ crosses an edge incident with $\bar{z}$, the whole link $L$ can be drawn in $D$ without making any crossings. This shows that the only possible cases for a crossing of two edges $e, f$ in $L$ are the following ones, where we take the notation from Figure 1 and let $u$ be the vertex of $L$ that is not labeled in the figure:

(a) $L$ is a B-link and $e = zv, f = uw$.
(b) $L$ is a B-link and $e = \bar{z}v, f = uw$.
(c) $L$ is a base link and $e = zv, f = uw$.
(d) $L$ is a base link and $e = vw, f = \bar{z}u$.
(e) $L$ is a base link and $e = \bar{z}w, f = vu$.

Let $D$ be a disk as discussed above corresponding to the crossing of $e$ and $f$. In cases (b), (d), and (e), the 3-valent vertex $u$ has all neighbors on the boundary of $D$, so the crossing between $e$ and $f$ can be eliminated by moving $u$ inside $D$ onto the other side of the edge $e$ (see Fig. 4(a)).

It remains to consider cases (a) and (c). Observe that the boundary of $D$ contains vertices $z, u, v, w$ in this order and that $u$ and $v$ both have precisely one additional neighbor $\bar{z}$ outside of $D$. Therefore, we can turn $\varphi$ into another 1-immersion of $G$ by swapping $u$ and $v$ and only redraw the edges inside $D$ (see Fig. 4(b)). However, this eliminates the crossing in $D$ and yields a 1-immersion with fewer crossings, a contradiction. \[\blacksquare\]

Lemma 3. Let $\varphi$ be a 1-immersion of a chain graph $G$ such that the number of crossings in $\varphi$ is minimal among all 1-immersions of $G$. If $\Pi$ and $\Pi'$ are nonadjacent A- and B-chains, respectively, then in $\varphi$ the following holds for every 3-cycle $C$ of $\Pi$:

(i) The core vertices of $\Pi'$ either all lie inside or all lie outside $C$.
(ii) If all core vertices of $\Pi'$ lie inside (resp. outside) $C$, then at most one vertex of $\Pi'$ lies outside (resp. inside) $C$.

Proof. First we show (i). If $C$ does not contain the vertex $A$, then every two core vertices of $\Pi'$ are connected by four edge-disjoint paths not passing through the vertices of $C$, hence (i) holds for $C$.

Suppose now that $C$ contains the vertex $A$ and that core vertices of $\Pi'$ lie inside and outside $C$. Then there is a link $L(z, \bar{z})$ of $\Pi'$ such that the vertex $z$ lies inside and the vertex $\bar{z}$ lies outside $C$. We may assume without loss of generality that $\Pi$ and $\Pi'$ are incident
to the base vertices $\Omega(1)$ and $\Omega(2)$, respectively, and (taking (A) into account) that the vertex $z$ (if $z \neq B$) separates the vertices $B$ and $\bar{z}$ in $\Pi'$ (see Fig. 5(a), where in $L(z, \bar{z})$ the dotted line indicates that the link has either edge $e_z$ or $e_{\bar{z}}$; also if $\bar{z} = \Omega(2)$, then the link indicated in Figure 5(a) is a base link).

The 3-cycle $C$ crosses at least two edges of $L(z, \bar{z})$. The vertex $z$ (resp. $\bar{z}$) is connected to each of the vertices $\Omega(1)$ and $\Omega(3)$ (resp. to the vertex $\Omega(2)$) by two edge-disjoint paths not passing through $V(C)$ or through the noncore vertices of $L(z, \bar{z})$. Hence, $\Omega(1)$ and $\Omega(3)$ lie inside $C$ (resp. $\Omega(2)$ lies outside $C$). It follows that the vertex $\Omega$ lies inside $C$ and the edge $(\Omega, \Omega(2))$ is the third edge that crosses $C$. We conclude that $C$ crosses exactly two edges of $L(z, \bar{z})$ and the two edges separate $z$ from $\bar{z}$ in $L(z, \bar{z})$. Thus, the two edges are the core-adjacent edges of the free cycle of $L(z, \bar{z})$. Hence, in $\varphi$, the link $L(z, \bar{z})$ is 1-immersed as shown in Figure 5(b), where the dotted edges indicate alternative possibilities for the position of $z$ (at top) or $\bar{z}$ (at bottom).
Let \( v, \bar{v} \) be the vertices of \( C \) different from \( A \) and let \( x \) be the fourth vertex of the link containing \( C \). The vertex \( x \) is connected to \( \Omega(1) \) by two edge-disjoint paths not passing through the vertices of \( C \), hence \( x \) lies inside \( C \). At most two vertices of \( C \) lie inside the free cycle of \( L(z, \bar{z}) \). Suppose exactly one of \( v \) and \( \bar{v} \) is inside the free cycle. If we are in the case of the bottom of Figure 5(b), then the path \( v\bar{v} \) cannot lie inside \( C \), a contradiction. In the case of the top of Figure 5(b), if the path \( v\bar{v} \) lies inside \( C \), then \( x \) must lie inside a 3-cycle \( Q \) of \( L(z, \bar{z}) \) incident to \( z \), whereas \( A \) lies outside \( Q \), a contradiction, since \( Q \) is not incident to \( B \) and in \( G \) there are two edge-disjoint paths connecting \( x \) to \( A \) and not passing through the vertices \( v, \bar{v}, \) and the vertices of \( Q \).

If either both \( v \) and \( \bar{v} \) or none of them lie inside the free 4-cycle, then in the case of Figure 5(c) (resp. (d)), where we depict the two possible placements of the nonbase link \( L(z, \bar{z}) \), there are two edge-disjoint paths of \( G \) connecting \( A \) and \( \Omega(3) \) (resp. \( A \) and \( \Omega(2) \)) and not passing through \( z \) (resp. \( \bar{z} \)), a contradiction. Reasoning exactly in the same way, we also obtain a contradiction when \( L(z, \bar{z}) \) is a base link.

Now we prove (ii). By (A), we may assume that all core vertices of \( \Pi' \) lie inside \( C \). By inspecting Figure 1, it is easy to check that for every link \( L \), for every set \( W \) of noncore vertices of \( L \) such that \( |W| \geq 2 \), there are at least four edges joining \( W \) with of \( V(L) \setminus W \). Hence, if at least two noncore vertices belonging to the same link of \( \Pi' \) lie outside \( C \), then at least four edges join them with the vertices of \( \Pi' \) lying inside \( C \), a contradiction. Every noncore vertex of \( \Pi' \) has valence at least 3. Hence, if exactly \( n \) \((n \geq 2)\) noncore vertices of \( \Pi' \) lie outside \( C \) and if they all belong to different links, then at least \( 3n \geq 6 \) edges join them with the vertices of \( \Pi' \) lying inside \( C \), a contradiction.$\blacksquare$

Two chains cross if an edge of one crosses an edge of the other.

**Lemma 4.** Let \( \varphi \) be a 1-immersion of a chain graph \( G \) such that the number of crossings in \( \varphi \) is minimal among all 1-immersions of \( G \). If \( \Pi \) and \( \Pi' \) are nonadjacent A- and B-chains, respectively, then \( \Pi \) does not cross \( \Pi' \) in \( \varphi \).

**Proof.** Suppose, for a contradiction, that \( \Pi \) crosses \( \Pi' \). Then an edge of a link \( L(z, \bar{z}) \) of \( \Pi' \) crosses a 3-cycle \( C = xv\bar{v} \) of a link \( L \) of \( \Pi \). Let \( \overline{C} = \bar{xv} \) be a 3-cycle that is adjacent to \( C \) in \( L \). (If \( L \) is not a base link, then \( L = C \cup \overline{C} \) and \( x, \bar{x} \) are the core vertices of \( L \).) By Lemma 3, we may assume that all core vertices of \( \Pi' \) lie outside \( C \) and that exactly one vertex \( u \) of \( \Pi' \) lies inside \( C \). The vertex \( u \) is 3-valent and is not a core vertex. The three edges incident with \( u \) cross all three edges of \( C \), hence \( \overline{C} \) does not cross \( C \). If \( \overline{C} \) lies inside \( C \), then one of the three edges incident with \( u \) crosses \( C \) and \( \overline{C} \), a contradiction. If \( C \) lies inside \( \overline{C} \), then we consider the plane as the complex plane and apply the Möbius transformation \( f(z) = 1/(z-a) \) with the point \( a \) taken inside \( C \) but outside \( \overline{C} \). This yields a 1-immersion of \( G \) such that

(a) \( C \) lies outside \( \overline{C} \), \( \overline{C} \) lies outside \( C \), and exactly one vertex \( u \) of \( \Pi' \) lies inside \( C \).

Therefore, we may assume that in \( \varphi \) we have (a). Since the three edges incident with \( u \) cross all three edges of \( C \), at least two vertices of \( \Pi' \) lie outside \( \overline{C} \), hence, by Lemma 3, all core vertices of \( \Pi' \) lie outside \( C \) and \( \overline{C} \). Since the edge \( v\bar{v} \) in \( C \cap \overline{C} \) is crossed by an edge of \( L(z, \bar{z}) \), also \( \overline{C} \) contains precisely one vertex \( u' \) of \( L(z, \bar{z}) \) and \( u' \) has degree 3 and is not a core vertex.

Adjacent trivalent vertices \( u, u' \) cannot be contained in a base link. Therefore, \( L(z, \bar{z}) \) is not a base link. Let \( L(z, \bar{z}) \) be depicted as shown in Figure 6(a). Because of symmetry,
we may assume that \( u = w \) and \( u' = \varepsilon \), and that the crossings are as shown in Figure 6(b) and (c).

In the case of Figure 6(b), the adjacent vertices \( z \) and \( \overline{w} \) of \( L(z, \overline{z}) \) are separated by the 3-cycle \( \overline{z}w\varepsilon \), whose edges are crossed by three edges different from the edge \( z\overline{w} \), a contradiction.

Consider the case in Figure 6(c). If \( x \) and \( \overline{x} \) are core vertices of \( \Pi \), then they are separated by the 3-cycle \( \overline{z}w\varepsilon \) of \( \Pi' \), a contradiction, since there are four edge-disjoint paths between \( x \) and \( \overline{x} \) that avoid this 3-cycle (the 3-cycle does not contain the vertex \( B \)).

Suppose that \( x \) and \( \overline{x} \) are not two core vertices. This is possible only when \( L \) is a base link. The 3-cycle \( \overline{z}w\varepsilon \) of \( L(z, \overline{z}) \) crosses the three edges joining the vertex \( \overline{v} \) of \( L \) with three vertices \( x, v, \) and \( \overline{x} \) of \( L \). The fifth vertex of \( L \) is adjacent to at least one vertex from \( x, v, \) and \( \overline{x} \), hence it lies outside \( \overline{z}w\varepsilon \) and is not adjacent to \( \overline{v} \). Thus, \( \overline{v} \) has valence 3 in \( L \). If \( \overline{v} \) is a core vertex, then, since the 3-cycle \( \overline{z}w\varepsilon \) does not contain the vertex \( B \), \( \overline{v} \) is connected to one of the vertices \( x, v, \) and \( \overline{x} \) by a path passing through \( B \) and not passing through the vertices of the 3-cycle \( \overline{z}w\varepsilon \), a contradiction. Hence, \( \overline{v} \) is not a core vertex. The link \( L \) has exactly one noncore vertex \( \overline{v} \) of valence 3 and the vertices \( x \) and \( \overline{x} \) are adjacent. Hence, the 3-cycle \( x\overline{x} \) separates two core vertices \( z \) and \( \overline{z} \) of \( \Pi' \), a contradiction, since \( z \) and \( \overline{z} \) are connected by a path not passing through the vertices of the 3-cycle \( x\overline{x} \).

\[ \square \]

**Theorem 1.** Every chain graph is an MN-graph.

**Proof.** Let \( G \) be a chain graph. By Lemma 1, it suffices to prove that \( G \) is not 1-planar. Consider, for a contradiction, a 1-immersion \( \varphi \) of \( G \) and suppose that \( \varphi \) has minimum number of crossings among all 1-immersions of \( G \).

We know by Lemma 4 that nonadjacent chains do not cross each other. In the sequel, we will consider possible ways that the \( \Omega \)-edges cross with one of the chains. Let us first show that such a crossing is inevitable.

**Claim 1.** At least one of the chains contains a link \( L = L(x, y) \) such that every \((x, y)\)-path in \( L \) is crossed by an \( \Omega \)-edge.

**Proof.** Suppose that for every link \( L = L(x, y) \), an \((x, y)\)-path in \( L \) is not crossed by any \( \Omega \)-edge. Then every chain contains a path joining its end vertices that is not crossed by the \( \Omega \)-edges. All six such paths plus the \( \Omega \)-edges form a subgraph of \( G \) that is homeomorphic to \( K_{3,3} \). By Lemma 4, the only crossings between subdivided edges of

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this $K_{3,3}$-subgraph are among adjacent paths. However, it is easy to eliminate crossings between adjacent paths and obtain an embedding of $K_{3,3}$ in the plane. This contradiction completes the proof of the claim.

Let $L = L(x, y)$ be a link in an A- or B-chain $\Pi$ whose $(x, y)$-paths are all crossed by the $\Omega$-edges. We may assume that $L$ is contained in a chain connecting the vertex $\Omega(1)$ with $A$ or $B$ and that $x$ separates $y$ and $\Omega(1)$ in $\Pi$. By Lemma 2, the induced 1-immersion of $L$ is an embedding. The vertex $\Omega$ lies inside a face of $L$ and all $\Omega$-edges that cross $L$ cross the edges of the boundary of the face. Considering the possible embeddings of $L$, it is easy to see that all $(x, y)$-paths are crossed by $\Omega$-edges only in the case when $\Omega$ lies inside a face of $L$ whose boundary contains two core-adjacent edges of a free cycle $C$ of $L$, and two $\Omega$-edges cross the two core-adjacent edges. By (A), we may assume that $\Omega$ lies inside $C$.

If $C$ is a $k$-cycle, $k \in \{3, 4\}$, then $L$ has another cycle $C'$ that shares with $C$ exactly $k - 2$ edges and contains a core vertex not belonging to $C$. If $C$ lies inside $C'$, then we consider the plane as the complex plane and apply the Möbius transformation $f(z) = 1/(z - a)$ with the point $a$ taken inside $C'$ but outside $C$. This yields a 1-immersion of $G$ such that $C$ does not lie inside $C'$ and $\Omega$ lies inside $C$. Hence, we may assume that $C$ does not lie inside $C'$, that $\Omega$ lies inside $C$ and two $\Omega$-edges $h$ and $h'$ cross two core-adjacent edges of $C$. Note that any two among the vertices $A, B, \Omega(2), \Omega(3)$ are joined by four edge-disjoint paths not using any edges in the chain $\Pi$ containing $L$. Therefore, these four vertices of $G$ are all immersed in the same face of $L$.

Let the $\Omega$-edges $h$ and $h'$ join the vertex $\Omega$ with basic vertices $\Omega(i)$ and $\Omega(j)$, respectively. If the third basic vertex $\Omega(\ell)$ is $\Omega(2)$ or $\Omega(3)$, then $\Omega(2)$ and $\Omega(3)$ lie inside different faces of $L$, a contradiction, hence $\Omega(\ell) = \Omega(1)$.

The vertex $\Omega(1)$ is connected to one of the vertices $A$ and $B$ by two edge-disjoint paths, not passing the vertices of $C$. Hence, if $C$ is a 3-cycle, then $\Omega(1)$ is not inside $C$.

Now the embeddings of possible links $L$ (so that we can join the vertices $\Omega$ and $\Omega(1)$ by an edge not violating the 1-planarity) are shown in Figure 7.

Let us now consider particular cases (a)–(f) of Figure 7.

(a) In this case, $L$ is a base link and $x = \Omega(1)$. Consider two edge-disjoint paths in a chain $\overline{\Pi}$ joining $\Omega(1)$ with the vertex $A$ or $B$, which is not incident with $\Pi$. These paths must cross the edges $e$ and $f$ indicated in the figure. Let $a$ be the edge crossing $e$ and $b$ be the edge crossing $f$. It is easy to see that $a$ and $b$ cannot be both incident with $\Omega(1)$ since $\Omega(1)$ is incident with three edges of the base link in the chain $\overline{\Pi}$. The 1-planarity implies that the edges $a, b$, and the vertex $\Omega(1)$ separate the graph $G$. Therefore, $a$ and $b$ are core-adjacent edges of a link in the chain $\overline{\Pi}$. If the edge $g$ (shown in the figure) is crossed by an edge $c$ of $\overline{\Pi}$, then also $a, c$, and $\Omega(1)$ separate the graph. Thus $a, c$ would be core-adjacent edges in a link in $\overline{\Pi}$ as well, a contradiction. But if $g$ is not crossed and $a$ is not incident with $\Omega(1)$, then the edge $a$ and the vertex $\Omega(1)$ separate $G$, a contradiction. If $a$ is incident with $\Omega(1)$, then $b$ is not (as proved above). Now we get a contradiction by considering the separation of $G$ by the edge $b$ and the vertex $\Omega(1)$.

(b) In this case, $L$ is a B-link, the cycle $C'$ lies inside $C$ and $\Omega(1)$ is inside $C$ but outside $C'$ (see Fig. 7(b)). In the figure, the vertex labeled $x$ is actually $x$ and not $y$ because the vertex is not $B$ and is connected to $\Omega(1)$ by two edge-disjoint paths not passing through the vertices of $C$, and $y$ is connected to the vertex $B$. 

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FIGURE 7. The $\Omega$-edges crossing a link.

(lying outside $C$) by two edge-disjoint paths not passing through the vertices of $C$. Again, consider two edge-disjoint paths in the A-chain $\overline{\Pi}$ joining $\Omega(1)$ with the vertex $A$. Their edges $a$ and $b$ (say) must cross the edges $e$ and $f$, respectively. As in the proof of case (a), we see that the edges $a$, $b$ and the vertex $\Omega(1)$ separate $G$, thus $a$ and $b$ are core-adjacent edges of a free cycle of a link of $\overline{\Pi}$. This free cycle has length 3 and separates $x$ from $\Omega(1)$, a contradiction, since there is a B-subchain from $x$ to $\Omega(1)$ disjoint from the free 3-cycle and not containing the edges of $L$ incident with $x$.

(c) In this case, $L$ is a B-link, the cycle $C'$ lies outside $C$ and $\Omega(1)$ is inside $C$. The vertex $\Omega(1)$ is connected by four edge-disjoint paths with vertices $x$ and $A$ such that the paths do not pass through noncore vertices of $L$, a contradiction.

(d) and (e) In these cases, $L$ is a B-link and $\Omega(1)$ lies inside a 3-cycle of $L$. Two edge-disjoint paths from $\Omega(1)$ to $A$ cross the edges $e$ and $f$ of $L$. One of the paths also crosses the edge $g$. Then the edges crossing $e$ and $g$ separate $G$, a contradiction, since $G$ is 3-edge-connected.

(f) In this case, $L$ is an A-link and the chain $\Pi$ is joining $\Omega(1)$ with the vertex $A$. Again, consider two edge-disjoint paths in the B-chain $\overline{\Pi}$ joining $\Omega(1)$ with $B$. Their edges $a$ and $b$ (say) must cross the edges $e$ and $f$, respectively. As in the proof of case (a), we see that the edges $a$, $b$ and the vertex $\Omega(1)$ separate $G$, thus $a$, $b$ are core-adjacent edges in a link $L' = L(z, z')$ in $\overline{\Pi}$. Let $z$ be the core vertex of $L'$ incident with the edges $a = zp$ and $b = zq$. Note that in $L'$, vertices $p$, $q$ have two common neighbors, a vertex $r$ of degree 3 in $G$ and the core vertex $z'$, and that $z'$ is adjacent to $r$. Inside $L$, we have the subchain $\Pi'$ of the A-chain $\Pi$ connecting $x$ with $\Omega(1)$. Hence, the free cycle of $L'$ containing the edges $a$ and $b$ must have length at least 4 (i.e., $L'$ is not a base link) and
z is immersed outside L, while p, q, r, z' are inside. The subchain Π' has two edge-disjoint paths that are crossed by the paths prq and pz'q. Each of the paths prq and pz'q crosses core-adjacent edges in Π since the crossed edges and Ω(1) separate G. Thus, they enter faces of these links that are bounded by free cycles of the links. However, the edge rz' would need to cross one edge of each of these two free cycles, hence the two free cycles are adjacent, that is, they are the two free cycles of an A-link of Π. Then the vertex z' lies inside one of the two free cycles and the free cycle separates z' from Ω(1), a contradiction, since z' is connected with Ω(1) by a B-subchain.

The following theorem shows how chain graphs can be used to construct exponentially many nonisomorphic MN-graphs of order n.

**Theorem 2.** For every integer \( n \geq 63 \), there are at least \( 2^{(n-54)/4} \) nonisomorphic MN-graphs of order n.

**Proof.** The A-chain of length \( t \) has \( 3t + 2 \) vertices and a B-chain of length \( t \) has \( 4t + 1 \) vertices. Consider a chain graph whose three A-chains have length 2, 2, and \( \ell \geq 2 \), respectively, and whose B-chains have length 2, 3, and \( t \geq 4 \), respectively. The graph has \( 35 + 3\ell + 4t \) vertices. Applying the modification shown in Figure 2(e) to links of the two B-chains of the graph, which have length at least 3, we obtain \( 2^{\ell-1} \) nonisomorphic chain graphs of order \( 35 + 3\ell + 4t \), where \( \ell \geq 2 \) and \( t \geq 4 \). We claim that for every integer \( n \geq 63 \), there are integers \( 2 \leq \ell \leq 5 \) and \( t \geq 4 \) such that \( n = 35 + 3\ell + 4t \). Indeed, if \( n \equiv 0, 1, 2, 3 \pmod{4} \), put \( \ell = 3, 2, 5, 4 \), respectively. If \( n = 35 + 3\ell + 4t \), where \( 2 \leq \ell \leq 5 \), then \( t \geq n/4 - 50/4 \). Hence, there are at least \( 2^{\ell-1/4} \) nonisomorphic chain graphs of order \( n \geq 63 \). Since every chain graph is an MN-graph, the theorem follows.

### 3. PN-GRAPHS

By a proper 1-immersion of a graph, we mean a 1-immersion with at least one crossing point. Let us recall that a PN-graph is a planar graph that does not have proper 1-immersions. In this section, we describe a class of PN-graphs and construct some graphs of the class. They will be used in Section 4 to construct MN-graphs.

For every cycle \( C \) of \( G \), denote by \( N(C) \), the set of all vertices of the graph not belonging to \( C \) but adjacent to \( C \). Two disjoint edges \( vw \) and \( v'w' \) of a graph \( G \) are paired if the four vertices \( v, w, v', w' \) are all four vertices of two adjacent 3-cycles (two cycles are adjacent if they share an edge).

Following Tutte, we call a cycle \( C \) of a graph \( G \) peripheral if it is an induced cycle in \( G \) and \( G - V(C) \) is connected. If \( G \) is 3-connected and planar, then the face boundaries in its (combinatorially unique) embedding in the plane are precisely its peripheral cycles.

**Theorem 3.** Suppose that a 3-connected planar graph \( G \) satisfies the following conditions:

1. **(C1)** Every vertex has degree at least 4 and at most 6.
2. **(C2)** Every edge belongs to at least one 3-cycle.
3. **(C3)** Every 3-cycle is peripheral (in other words, there are no separating 3-cycles).
Every 3-cycle is adjacent to at most one other 3-cycle.

(C5) No vertex belongs to three mutually edge-disjoint 3-cycles.

(C6) Every 4-cycle is either peripheral or is the boundary of two adjacent triangular faces (this means that there are no separating 4-cycles).

(C7) For every 3-cycle $C$, any two vertices of $V(G) \setminus (V(C) \cup N(C))$ are connected by four edge-disjoint paths not passing through the vertices of $C$.

(C8) If an edge $vw$ of a nontriangular peripheral cycle $C$ is paired with an edge $v'w'$ of a nontriangular peripheral cycle $C'$, then:

(i) $C$ and $C'$ have no vertices in common;

(ii) any two vertices $a$ and $a'$ of $C$ and $C'$, respectively, such that $\{a, a'\} \not\subset \{v, w, v', w'\}$ are nonadjacent and are not connected by a path $aba'$ of length 2, where $b$ does not belong to $C$ and $C'$.

(C9) $G$ does not contain the subgraphs shown in Figure 8 (in this figure, 4-valent (resp. 5-valent) vertices of $G$ are encircled (resp. encircled twice) and the two starred vertices can be the same vertex).

Then $G$ has no proper 1-immersion.

**Proof.** Denote by $f$, the unique plane embedding of $G$. Suppose, for a contradiction, that there is a proper 1-immersion $\varphi$ of $G$. Below we consider the 1-immersion and show that then $G$ has a subgraph that is excluded by (C8) and (C9), thereby obtaining a contradiction. In the figures below, the encircled letter $f$ (resp. $\varphi$) at the top left of a figure means that the figure shows a fragment of the plane embedding $f$ (resp. 1-immersion $\varphi$).

**Lemma 5.** In $\varphi$, there is a 3-cycle such that there is a vertex inside and a vertex outside the cycle.

**Proof.** The 1-immersion $\varphi$ has crossing edges $e$ and $e'$. By (C2), the crossing edges belong to different 3-cycles. If the 3-cycles are nonadjacent, then we apply the following obvious observation:

(a) If two nonadjacent 3-cycles $D$ and $D'$ cross each other, then there is a vertex of $D$ inside and outside $D'$.

If $e = xy$ and $e' = x'y'$ belong to adjacent 3-cycles $xyy'$ and $x'yy'$, respectively (see Fig. 9), then, by (C4), there are nontriangular peripheral cycles $C$ and $C'$ containing $e$ and $e'$, respectively. The cycles $C$ and $C'$ intersect at some point $\delta$ different from the intersection point of edges $e$ and $e'$. By (C8)(i), the two cycles do not have a common vertex, hence $\delta$ is the intersection point of two edges. By (C2), these two edges belong...
to some 3-cycles, $D$ and $D'$. Property (C8)(ii) implies that $D$ and $D'$ are nonadjacent
3-cycles. By (a), the proof is complete. ■

Lemma 6. If $C = u_1u_2u_3$ is a 3-cycle such that there is a vertex inside and a vertex
outside $C$, then there is only one vertex inside $C$ or only one vertex outside $C$, and this
vertex belongs to $N(C)$.

**Proof.** By (C7), we may assume that all vertices of $V(G) \setminus (V(C) \cup N(C))$ lie outside
$C$. Then there can be only vertices of $N(C)$ inside $C$. To prove the lemma, it suffices to
show the following:

(a) For every $Q \subseteq N(C)$, $|Q| \geq 2$, at least four edges join vertices of $Q$ to vertices in
$V(G) \setminus (V(C) \cup Q)$.

By (C1), every vertex of $Q$ has valence at least 4. By (C4), every vertex of $N(C)$ is
adjacent to at most two vertices of $C$. We claim that if a vertex $v \in N(C)$ is
adjacent to two vertices $u_1$ and $u_2$ of $C$, then $v$ is not adjacent to other vertices of
$N(C)$. Suppose, for a contradiction, that $v$ is adjacent to a vertex $w \in N(C)$. Then,
by (C4), the vertex $u$ can be adjacent only to $u_3$ and the 4-cycle $vwu_3u_2$ is not
the boundary of two adjacent 3-cycles, hence, by (C6), the 4-cycle is peripheral.
Then, by (C3), any two of the three edges $u_2v$, $u_2u_1$, and $u_2u_3$ are two edges of a
peripheral cycle, hence $u_2$ has valence 3, contrary to (C1).

Now to prove (a), it suffices to prove the following claim:

(b) For every $Q \subseteq N(C)$, $|Q| = 1$ (resp. $|Q| \geq 2$), such that every vertex of $Q$ is
adjacent to exactly one vertex of $C$, at least 2 (resp. 4) edges join vertices of $Q$ to
vertices of $V(G) \setminus (V(C) \cup Q)$.

The claim is obvious for $|Q| \in \{1, 2\}$. For $|Q| = 3$, it suffices to show that the three
vertices of $Q$ are not all pairwise adjacent. Suppose, for a contradiction, that the vertices
$v_1$, $v_2$, and $v_3$ of $Q$ are pairwise adjacent. Then, by (C4), the three vertices of $Q$ are
not adjacent to the same vertex of $C$. Let $v_1$ and $v_2$ be adjacent to the vertices $u_1$ and $u_2$ of $C$, respectively. Then any two of the edges $v_1u_1$, $v_1v_2$, and $v_1v_3$ are two edges of a 3-cycle
(peripheral cycle) or a 4-cycle that is not the boundary of two adjacent 3-cycles, so by
(C6), that 4-cycle is also peripheral. Hence, $v_1$ has valence 3, contrary to (C1).

For $|Q| \geq 4$, it suffices to show that no vertex of $Q$ is adjacent to three other vertices in
$Q$. Suppose, for a contradiction, that $v \in Q$ is adjacent to $w_1, w_2, w_3 \in Q$. If $v$ is adjacent
to $u_1$, then the edge $vu_1$ belongs to three cycles $D_1, D_2$, and $D_3$ such that for $i = 1, 2, 3$,
the cycle $D_i$ contains edges $vu_1$ and $vw_i$, has length 3 or 4, and if $D_i$ has length 4, then
$D_i$ is not the boundary of two adjacent 3-cycles. By (C3) and (C6), these three cycles are
peripheral. This contradiction completes the proof of (b). ■

![Figure 9. Crossing two adjacent 3-cycles.](https://doi.org/10.1002/jgt)
Suppose that a vertex $h$ belongs to two adjacent 3-cycles $hvw$ and $hvw'$. Since $\deg(h) \geq 4$, $h$ is adjacent to a vertex $u \not\in \{v, w, w'\}$. By (C2), the edge $hu$ belongs to a 3-cycle $huu'$. By (C4), $u' \not\in \{v, w, w', u\}$. Hence, we have the following:

(B) If an edge $e$ is contained in two 3-cycles of $G$, then both end vertices of $e$ have valence at least 5.

In the remainder of the proof of Theorem 3, we will show that any two crossing edges of the proper 1-immersion $\varphi$ belong to a subgraph that is excluded by (C8) and (C9).

By Lemma 5, there is a 3-cycle $C = xyz$ such that there is a vertex inside and a vertex outside $C$. By Lemma 6, there is only one vertex $v$ inside $C$ and $v$ is adjacent to $x$.

Now we show that there is a 3-cycle $B = vuv$ disjoint from $C$. Let $x, a_1, a_2, \ldots, a_t$ ($t \geq 3$) be all vertices adjacent to $v$. Suppose there is a 3-cycle $D = va_ib$, where $b \in \{x, y, z\}$. If $b \in \{y, z\}$, then the 3-cycle $xvb$ is adjacent to two 3-cycles $C$ and $D$, contrary to (C4). Hence, $D = va_ix$. By (C4), at most two vertices of $\{a_1, a_2, \ldots, a_t\}$ are adjacent to $x$. Hence, there is a vertex $a_j$ such that a 3-cycle $B$ containing the edge $va_j$ is disjoint from $C$.

Since there is only one vertex $v$ inside $C$, exactly two edges of $B$ cross edges of $C$. First, suppose that $B$ separates $x$ from $y$ and $z$ (Fig. 10(a)). By (C2), the edge $xv$ belongs to a 3-cycle $R = xva$. If $a \not\in \{y, z, u, w\}$, then two of the vertices $y, z, u, w$ lie inside $R$ and the other two vertices lie outside $R$ (see Fig. 10(b)), contrary to Lemma 6. So, we may assume, without loss of generality, that $a = z$ (Fig. 10(c)). Then the vertex $x$ belongs to two adjacent 3-cycles, $C$ and $vxz$, hence, by (B), $\deg(x) \geq 5$. By (C4), $x$ is not adjacent to $u$ or $w$. Since $x$ is the only vertex inside $B$, $x$ has valence at most 4, a contradiction. Hence, $B$ cannot separate $x$ from $y$ and $z$.

Now suppose that $B$ separates the vertices $y$ and $z$, and, without loss of generality, let $z$ lie inside $B$ (Fig. 11(a)). If $z$ is adjacent to $v$, then, by (B), $z$ has valence at least 5, hence $z$ is adjacent to a vertex $b \in \{u, w\}$. But then the 3-cycle $xzv$ is adjacent to two 3-cycles,
FIGURE 12. The edge \(uw\) belongs to two 3-cycles.

C and \(vbz\), contrary to (C4). If \(z\) is adjacent to \(u\) and \(w\), then the edge \(xz\) belongs to three peripheral cycles, \(C\), \(xvuz\), and \(xvwz\), a contradiction, since every edge belongs to at most two peripheral cycles. Hence, since \(z\) is the only vertex inside \(B\), \(z\) has valence 4, \(z\) is not adjacent to \(v\) and is adjacent to exactly one of the vertices \(u\) and \(w\). The vertices \(u\) and \(w\) are not symmetric in Figure 11(a), so we have to consider two cases.

Case 1. The vertex \(z\) is adjacent to \(u\) (Fig. 11(b)).

Consider the vertex \(v\). If \(v\) is adjacent to \(y\), then (see Fig. 11(b)) \(x\) is incident with exactly three peripheral cycles and has valence 3, a contradiction. Hence, \(v\) is not adjacent to \(y\) and since \(v\) is the only vertex inside \(C\), \(v\) has valence 4 (see Fig. 11(c)). By (C2), we obtain a subgraph shown in Figure 11(d). By (C9), at least one of the vertices \(u\) and \(x\), say \(u\), is not 4-valent. Then, by (C5), at least one of the edges \(au\) and \(uw\) belongs to two 3-cycles. Here we have two subcases to consider.

Subcase 1.1. The edge \(uw\) belongs to two 3-cycles \(uwv\) and \(uwb\) (Fig. 12(a)).

Now, \(a\) is the only vertex inside the 3-cycle \(uw\) (Fig. 12(b)). By (C4), \(a\) is nonadjacent to \(w\) and \(b\), hence \(a\) has valence 4. The edges \(yz\) and \(za\) (see Fig. 12(a)) belong to a nontriangular peripheral cycle \(yzac\ldots\). The edge \(ac\) belongs to a 3-cycle \(acd\) and \(b\) is the only vertex inside the 3-cycle \(acd\). The vertex \(b\) is not adjacent to \(c\), since the 4-cycle \(bcua\) cannot be peripheral (see Fig. 12(a)). Since \(b\) has valence at least 4, \(b\) is adjacent to \(d\) and has valence 4. The 4-cycle \(dbua\) is peripheral, so we obtain a subgraph of \(G\) shown in Figure 12(c). Note that, by (C3)–(C6), the vertex at the top of Figure 12(c) is different from all other vertices shown in the figure. This contradicts (C9).

Subcase 1.2. The edge \(au\) belongs to two 3-cycles \(auz\) and \(aub\) (Fig. 13(a)).

Since \(b\) has valence at least 4, \(b\) lies outside the 3-cycle \(auz\) (Fig. 13(b)). The edges \(yz\) and \(za\) (resp. \(wu\) and \(ub\)) belong to a nontriangular peripheral cycle \(C_1\) (resp. \(C_2\)). The cycles \(C_1\) and \(C_2\) are paired. In \(\varphi\), the crossing point of edges \(uw\) and \(az\) is an intersection point of \(C_1\) and \(C_2\). The cycles \(C_1\) and \(C_2\) have at least one other crossing point, denote this intersection point by \(\delta\). By (C8)(i), \(C_1\) and \(C_2\) are vertex disjoints, hence, \(\delta\) is the crossing point of \(C_1\) and an edge \(h_1h_2\) of \(C_2\) (Fig. 13(c)). The edge \(h_1h_2\) is not the edge \(uw\) and belongs to a 3-cycle \(h_1h_2h_3\). If \(h_1\) belongs to \(C_2\), then in the embedding \(f\) the edge \(h_1h_3\) is a chord of the embedded peripheral cycle \(C_2\) and thus \(\{h_1, h_3\}\) is a separating vertex set of \(G\). But \(G\) is 3-connected, a contradiction. Hence, \(h_3\) does not belong to \(C_2\).

Now suppose that \(h_1h_2 \neq bu\). We have \(h_2 \notin \{a, z, b, u\}\). By (C8)(ii), \(h_3 \neq a, a\) is not adjacent to \(h_2\) and \(h_3\), and \(C_1\) does not pass through \(h_3\) (i.e., \(h_3\) does not belong to \(C_1\))
and $C_2$). Hence, a vertex $s$ of $C_1$ lies inside the 3-cycle $h_1h_2h_3$ (see Fig. 13(c)). By (B), $\text{deg}(a) \geq 5$. If $s = a$, then, since $s$ is the only vertex inside the 3-cycle $h_1h_2h_3$, $a$ is adjacent to at least one of $h_2$ and $h_3$, a contradiction. Hence, $s \neq a$. Since $z$ is the only vertex inside the 3-cycle $B$, and the 3-cycle $h_1h_2h_3$ is not $B$ (since $h_1h_2 \neq uw$), we have $s \neq z$. Now, since $s$ has valence at least 4, $s$ is adjacent to at least one of the vertices $h_1$, $h_2$, and $h_3$, contrary to (C8)(ii). Hence, $h_1h_2 = bu$ and $h_3 = a$.

We have $C_1 = yzah \ldots$ and the edge $ah$ belongs to a 3-cycle $ahd$ (see Fig. 13(d) and (e)). Considering Figure 13(e), if there is a vertex inside the 3-cycle $auz$, the vertex has valence at most 3, a contradiction. Hence, no edge crosses the edge $au$. If $h$ lies inside the 3-cycle $aub$, then, by (C4), $h$ is not adjacent to $b$ and $u$, $h$ has valence at most 3, a contradiction. Hence, $h$ lies outside the 3-cycle $aub$ and $b$ is the only vertex inside the 3-cycle $ahd$ (see Fig. 13(e)). Note that the edge $ah$ belongs to $C_1$, hence $C_1$ does not cross both $bu$ and $ab$.

By (C4), $b$ is not adjacent to $d$ and $h$, hence $b$ has valence 4 and belongs to a 3-cycle $btt'$ disjoint from $aub$ (Fig. 13(f)), where $C_2 = wubt \ldots$. Now $d$ is the only vertex inside the 3-cycle $btt'$ (Fig. 13(e)). By (C8)(ii), $d$ is not adjacent to $t$. Since $d$ has valence at least 4, $d$ is adjacent to $t'$ and has valence 4. The 4-cycle $dt'ba$ is peripheral. We obtain
a subgraph of $G$ shown in Figure 13(g), contrary to (C9). The obtained contradiction completes the proof in the Case 1.

**Case 2.** *The vertex $z$ is adjacent to $w$.***

This case is dealt in much the same way as Case 1. Here, we describe only what figures will be in Case 2 instead of the Figures 11–13 in Case 1. We hope that the reader is familiar enough with the proof of Case 1 to supply the missing details himself.

In Figures 11(a) and (d) and 12(a) and (c), interchange the letters $u$ and $w$. In Figures 11(c) and 12(b), replace the edge $uz$ by the edge $zw$. In Figures 13(a), (d), (f), and (g) interchange the letters $u$ and $w$. Figures 13(b), (c), and (e) are replaced by the Figures 14(a), (b), and (c), respectively.

Denote by $A$ the class of all 3-connected plane graphs $G$ satisfying the conditions (C1)–(C9) of Theorem 3. In what follows, we show how to construct some graphs in $A$ and, as an example, we shall give two infinite families of graphs in $A$, one of which will be used in Section 4 to construct MN-graphs.

First, we describe a large family of 3-connected plane graphs satisfying the conditions (C1)–(C6) and (C8) of Theorem 3.

Given a 4-valent vertex $v$ of a 3-connected plane graph, two peripheral cycles $C$ and $C'$ containing $v$ are opposite peripheral cycles at $v$ if $C$ and $C'$ have no edges incident with $v$ in common.

Denote by $H$ the class of all 3-connected (simple) planar graphs $H$ satisfying the following conditions (H1)–(H4):

(H1) Every vertex has valence 3 or 4.
(H2) $H$ has no 3-cycles.
(H3) Every 4-cycle is peripheral.
(H4) For every 4-valent vertex $v$ and for any two opposite peripheral cycles $C$ and $C'$ at $v$, no edge joins a vertex of $C - v$ to a vertex of $C' - v$.

A plane graph $G$ is a *medial extension* of a graph $H \in \mathcal{H}$ if $G$ is obtained from $H$ in the following way. The vertex set of $G$ is the set $\{v(e) : e \in E(H)\}$. The edge set of $G$ is defined as follows. For every 3-valent vertex $v$ of $H$, if $e_1$, $e_2$, $e_3$ are the edges incident with $v$, then in $G$ the vertices $v(e_1)$, $v(e_2)$, and $v(e_3)$ are pairwise adjacent (the three edges of $G$ are said to be *associated with* $v$). For every 4-valent vertex $w$ of $H$, if $(e_1, e_2, e_3, e_4)$ is the cyclic order of edges incident to $w$ around $w$ in the plane, then $G$ contains the edges of the 4-cycle $v(e_1)v(e_2)v(e_3)v(e_4)$, and contains either the edge $v(e_1)v(e_3)$ or the edge $v(e_2)v(e_4)$; these five edges of $G$ are said to be *associated with* the 4-valent vertex of $H$.

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Note that $G$ can be obtained from the medial graph of $H$ by adding a diagonal to every 4-cycle associated with a 4-valent vertex of $H$.

**Lemma 7.** Every medial extension $G$ of any graph $H \in \mathcal{H}$ is a 3-connected planar graph satisfying the conditions (C1)–(C6) and (C8) of Theorem 3.

**Proof.** By the construction of $G$, if $\{v(e), v(e')\}$ is a separating vertex set of $G$, then the graph $H - e - e'$ is disconnected, a contradiction, since $H$ is 3-connected. Hence, $G$ is 3-connected. Every peripheral cycle of $H$ induces a peripheral cycle of $G$. It is easy to see that all peripheral cycles of $G$ that are not induced by the peripheral cycles of $H$ are the 3-cycles formed by the edges associated with the vertices of $H$.

It is easy to see that $G$ satisfies (C1)–(C5). To show that $G$ satisfies (C6), let $J = v(e_1)v(e_2)v(e_3)v(e_4)$ be a 4-cycle of $G$. By the construction of $G$, if vertices $v(e)$ and $v(e')$ of $G$ are adjacent, then the edges $e$ and $e'$ of $H$ are adjacent, too. Since in $H$ no three edges among $e_1, e_2, e_3, e_4$ form a cycle (by (H2)), these four edges either form a 4-cycle (in this case $J$ is peripheral) or are the edges incident to a 4-valent vertex of $H$ (in this case, by the construction of $G$, $J$ is the boundary of two adjacent faces of $G$). Hence, $G$ satisfies (C6).

It remains to show that $G$ satisfies (C8). Let $C$ and $C'$ be nontrivial peripheral cycles of $G$ such that an edge $a$ of $C$ is paired with an edge $a'$ of $C'$. Then, by the construction of $G$, the peripheral cycles $C$ and $C'$ are induced by peripheral cycles $\overline{C}$ and $\overline{C}'$ of $H$, respectively, which are opposite at some 4-valent vertex $u$. If $C$ and $C'$ have a common vertex $v(e)$, then $\overline{C}$ and $\overline{C}'$ have a common edge $e$, hence $H$ has a separating vertex set $\{u, w\}$, where $w$ is a vertex incident to $e$, a contradiction, since $H$ is 3-connected. Now suppose that $G$ has an edge joining a vertex $v(e)$ of $C$ to a vertex $v(e')$ of $C'$ such that at least one of the vertices $v(e)$ and $v(e')$ is not incident to $a$ and $a'$. Then the edges $e$ and $e'$ are incident to the same vertex $w$ of $H$ and the cycles $\overline{C}$ and $\overline{C}'$ pass through $w$. It follows that $\{u, w\}$ is a separating vertex set of $H$, a contradiction. Next suppose that $G$ has a path $v(e)v(b)v(e')$ connecting a vertex $v(e)$ of $C$ to a vertex $v(e')$ of $C'$ such that $v(b)$ does not belong to $C$ and $C'$, and at least one of the vertices $v(e)$ and $v(e')$ is not incident to $a$ or $a'$. If in $H$ the edges $e$, $b$, and $e'$ are incident to the same vertex $w$, then $\{u, w\}$ is a separating vertex set of $H$, a contradiction. If in $H$ the edges $a$ and $b$ (resp. $b$ and $e'$) are incident to a vertex $w$ (resp. $w'$) such that $w \neq w'$, then the edge $b$ joins the vertex $w$ of $\overline{C}$ with the vertex $w'$ of $\overline{C}'$, contrary to (H4). Hence, $G$ satisfies (C8). The proof is complete.

There are medial extensions of graphs in $\mathcal{H}$ that do not satisfy conditions (C7) and (C9). In the sequel we shall describe a way to verify the conditions (C7), and henceforth give examples of graphs satisfying (C1)–(C9). To show that a medial extension $G$ of $H \in \mathcal{H}$ satisfies (C7), it is convenient to proceed in the following way. Subdivide every edge $e$ of $H$ by a two-valent vertex $v(e)$ of $G$. We obtain a graph $H'$ whose vertex set is $V(H) \cup V(G)$ where the vertices of $V(G)$ are all 2-valent vertices of $H$. We will consider paths of $G$ associated with paths of $H'$ connecting 2-valent vertices.

Two paths $P$ and $P'$ of $H'$ are $H$-disjoint if $P \cap P' \cap V(H) = \emptyset$, that is, $P \cap P' \subseteq V(G)$.

Consider a path $P = v(e_1)w_1 v(e_2)w_2 v(e_3) \ldots w_{n-1}v(e_n)$ in $H'$, where $w_1, w_2, \ldots, w_{n-1}$ are the $H$-vertices on $P$. It is easy to see that the edges of $G$ associated with the vertices $w_1, w_2, \ldots, w_{n-1}$ of $H$ contain two edge-disjoint paths connecting in $G$ the vertices $v(e_1)$ and $v(e_n)$ (see Fig. 15); any two such paths in $G$ are said to be associated with the path
FIGURE 15. Two paths of $G$ associated with a path of $\overline{H}$.

$P$ of $\overline{H}$. Since $H$ has no multiple edges, every edge of $G$ is associated with exactly one vertex of $H$. Hence, if $P$ and $P'$ are $H$-disjoint paths in $\overline{H}$, each of which is connecting 2-valent vertices, then every path in $G$ associated with $P$ is edge-disjoint from every path in $G$ associated with $P'$. As a consequence, we have the following conclusion:

(C) If $\overline{H}$ has a cycle containing 2-valent vertices $v(e)$ and $v(e')$, then $G$ has four edge-disjoint paths connecting $v(e)$ and $v(e')$.

The fact that a path in $H$ gives rise to two edge-disjoint paths in $G$ (paths associated with the path of $H$) can be used to check the property (C7) of $G$.

For a 3-cycle $C$ of $G$, a path of $H$ is $C$-independent if the path does not contain vertices of $C$. When checking (C7) for a medial extension $G$ of $H \in \mathcal{H}$, given a 3-cycle $C$ of $G$ and two 2-valent vertices $x, y \in V(G) \setminus (V(C) \cup N(C)) \subset V(\overline{H})$, four $C$-independent edge-disjoint paths $P_1, P_2, P_3,$ and $P_4$ of $G$ connecting $x$ and $y$ in $G$ will be represented in some subsequent figures (see, e.g., Fig. 17) in the following way. The edges of the paths incident to vertices of $N(C)$ are depicted as dashed edges joining 2-valent vertices, the dashed edges are not edges of $H$ (see, e.g., Figure 17(b), where the edges of $H$ are given as solid lines). All other edges of the paths are represented by paths in $H$. If $X$ is a subpath of $P_i$ such that $X$ is associated with a path $\overline{X}$ in $\overline{H}$, then $X$ is represented by a dashed line passing near the edges of $\overline{X}$ in $\overline{H}$. If $X_i$ and $X_j$ are subpaths of $P_i$ and $P_j$ (where possibly $i = j$), respectively, such that $X_i$ and $X_j$ are edge-disjoint paths associated with a path $\overline{X}$ of $\overline{H}$, then $X_i$ and $X_j$ are represented by two (parallel) dashed lines passing near the edges of $\overline{X}$ in $\overline{H}$. Using these conventions, the reader will be able to check that the depicted dashed paths and edges in the figure of $\overline{H}$ represent four $C$-independent edge-disjoint paths of $G$ connecting $x$ and $y$.

Now we describe some graphs in $\mathcal{A}$. Let us recall that graphs in $\mathcal{A}$ are precisely those 3-connected planar graphs that satisfy conditions (C1)–(C9). To simplify the arguments, we construct graphs with lots of symmetries so that, for example, to check the condition (C7) we will have to consider only two 3-cycles of a graph.

For $n \geq 6$, let $H_n \in \mathcal{H}$ be the Cartesian product of the path $P_3$ of length 2 and the cycle $C_n$ of length $n$ (see 16(a)). Let $G_n$ be the medial extension of $H_n$ shown in Figure 16(b).

**Lemma 8.** Each graph $G_n$, $n \geq 6$, is a PN-graph.

**Proof.** We show that $G_n$ satisfies (C1)–(C9) for every $n \geq 6$.

By Lemma 7, $G_n$ satisfies (C1)–(C6) and (C8). Every 4-gonal face of $G_n$ is incident with a 6-valent vertex and $G_n$ has no 5-valent vertices, hence $G_n$ satisfies (C9).

Now we show that $G_n$ satisfies (C7). Consider a fragment of $\overline{H}_n$ shown in Figure 17(a) (in the figure we introduce notation for some vertices and also depict in dashed lines...
FIGURE 16. The graph $H_n \in \mathcal{H}$ and its extension $G_n$.

FIGURE 17. Verifying (C7) for the graph $G_n$.

some edges of $G_n$). Because of the symmetries of $G_n$, it suffices to consider the following two cases for a 3-cycle $C$, when checking (C7):

Case 1. $V(C) = \{6, 9, 10\}$.

Then $N(C) = \{2, 3, 5, 7, 8, 11, 12, 13, 14\}$. If we delete from $\overline{H_n}$ the vertices 1, 2, …, 14, then the obtained graph has only one connected component $U$ with a vertex in $H_n$, and $U$ is 2-connected. Hence, any two 2-valent vertices $x$ and $y$ of $U$ belong to a cycle in $U$ and then, by (C), $G_n$ has four $C$-independent edge-disjoint paths connecting $x$ and $y$. Figure 17(b) shows four $C$-independent edge-disjoint paths in $G_n$ connecting vertices 1 and 4. If we delete from $\overline{H_n}$ the vertices $\{1, 2, \ldots, 18\} \setminus \{1, 4\}$, then in the resulting nontrivial connected component, for every vertex $x \in V(G_n) \setminus \{1, 2, \ldots, 18\}$, there is a path $P$ connecting the vertices 1 and 4, and passing through $x$; combining two edge-disjoint paths of $G_n$ associated with $P$ and the two edge-disjoint paths connecting 1 and 4, shown in Figure 17(b), we obtain four $C$-independent edge-disjoint paths connecting the vertices 4 and $x$ (and, analogously, for the vertices 1 and $x$). Now, because of the symmetries of $\overline{H_n}$, it remains to show that there are four $C$-independent edge-disjoint
paths connecting the vertex 4 with each of the vertices 15, 16, 17, 18; Figure 17(c) shows the paths (since \( n \geq 6 \)).

**Case 2.** \( V(C) = \{2, 3, 6\} \).

We have \( N(C) = \{1, 4, 5, 7, 9, 10\} \). If we delete from \( H'_n \) the vertices 1, 2, \ldots, 7 and 9, 10, 13, then the obtained graph has only one connected component \( U \) with at least two vertices and \( U \) is 2-connected. Hence any two vertices \( x \) and \( y \) in \( H'_n \) that are both in \( U \) belong to a cycle of \( U \) and then, by (C), \( G_n \) has four \( C \)-independent edge-disjoint paths connecting \( x \) and \( y \). It remains to show that for every vertex \( x \) of \( H'_n \) belonging to \( U \), there are four \( C \)-independent edge-disjoint paths connecting \( x \) and the vertex 13. These four paths are shown in Figures 17(d)–(f), depending on the choice of \( x \). We conclude that \( G_n \) satisfies (C1)–(C9), hence \( G_n \) is a PN-graph for every \( n \geq 6 \).

Figure 18 gives another example of an extension \( G'_n \) of a graph \( H'_n \in \mathcal{H} \). By Lemma 7, \( G'_n \) satisfies (C1)–(C6) and (C8). Since \( G'_n \) has no 4-cycles, it satisfies (C9). Using the symmetry of \( H'_n \), one can easily check that for every 3-cycle \( C \) of \( G'_n \), if we delete from \( H'_n \) the vertices \( V(C) \cup N(C) \) of \( G'_n \), then the obtained graph has only one connected component \( U \) with at least two vertices and \( U \) is 2-connected. Then, by (C), any two vertices of \( G'_n \) in \( U \) are connected by four \( C \)-independent edge-disjoint paths. Hence, \( G'_n \) satisfies (C7) and is a PN-graph.

### 4. MN-GRAPHS BASED ON PN-GRAPHS

In this section, we construct MN-graphs based on the PN-graphs \( G_n \) described in Section 3.

For \( m \geq 2 \), denote by \( S_m \), the graph shown in Figure 19. The graph has \( m + 1 \) disjoint cycles of length \( 12m - 2 \) labeled by \( B_0, B_1, \ldots, B_m \) as shown in the figure. The vertices of \( B_0 \) are called the **central vertices** of \( S_m \) and are labeled by \( 1, 2, \ldots, 12m - 2 \) (see Fig. 19). For every central vertex \( x \in \{1, 2, \ldots, 12m - 2\} \), denote by \( x^* \) its “opposite” vertex \( x + (6m - 1) \) if \( x \in \{1, 2, \ldots, 6m - 1\} \) and the vertex \( x - (6m - 1) \) if \( x \in \{6m, 6m + 1, \ldots, 12m - 2\} \). In \( S_m \), any pair \( \{x, x^*\} \) of central vertices is connected by a **central path** \( P(x, x^*) \) of length \( 6m - 3 \) with \( 6m - 4 \) two-valent vertices. There are exactly \( 6m - 1 \) central paths.
For any integers $m \geq 4$ and $n \geq 0$, denote by $\Phi_m(n)$ the set of all $(12m - 2)$-tuples $(n_1, n_2, \ldots, n_{12m-2})$ of nonnegative integers such that $n_1 + n_2 + \cdots + n_{12m-2} = n$. For every $\lambda \in \Phi_m(n)$, denote by $S_m(\lambda)$ the graph obtained from $S_m$ by replacing, for every central vertex $x \in \{1, 2, \ldots, 12m - 2\}$, the eight edges marked by short crossings in Figure 20(a) by $8(1 + n_x)$ new edges marked by crossings in Figure 20(b) (the value $x + 1$ in that figure is to be considered modulo $12m - 2$). The graph $S_m(\lambda)$ has $m - 2$ $(12m - 2)$-cycles $B_0, B_1, \ldots, B_{m-3}$ and three $(12m - 2 + n)$-cycles $B_{m-2}, B_{m-1}, B_m$.

We want to show that for every $m \geq 4$ and for every $\lambda \in \Phi_m(n)$, $n \geq 0$, the graph $S_m(\lambda)$ is an MN-graph.
Lemma 9. For every \( m \geq 4, n \geq 0, \) and \( \lambda \in \Phi_m(n) \), the graph \( S_m(\lambda) - e \) is 1-planar for every edge \( e \).

**Proof.** If we delete an edge of a central path, then the remaining \( 6m - 2 \) central paths, each with \( 6m - 3 \) edges, can be 1-immersed inside \( B_0 \) in Figure 19. If we delete one of the edges shown in Figure 21(a) by a thick line, then the central path \( P(x,x^*) \) can be drawn outside \( B_0 \) with \( 6m - 3 \) crossing points as shown in the figure and then the remaining \( 6m - 2 \) central paths can be 1-immersed inside \( B_0 \). If we delete one of the two edges depicted in Figure 21(a) by a dotted line, then Figure 21(b) shows how to place the central vertex \( x \) so that the path \( P(x,x^*) \) can be drawn outside \( B_0 \) with \( 6m - 3 \) crossing points (analogously to Fig. 21(a)) and then the remaining \( 6m - 2 \) central paths can be 1-immersed inside \( B_0 \). This exhibits all possibilities for the edge \( e \) (up to symmetries of \( S_m \)) and henceforth completes the proof. ■

Given a 1-immersion of a graph \( G \) and an embedded cycle \( C \), we say that \( G \) lies inside (resp. outside) \( C \), if the exterior (resp. interior) of \( C \) does not contain vertices and edges of \( G \).

Denote by \( J_{m-2} \), the graph obtained from the graph \( S_m \) in Figure 19 by deleting the 2-valent vertices of all central paths and all vertices lying outside the cycle \( B_{m-2} \).

Lemma 10. For every \( m \geq 4, J_{m-2} \) is a PN-graph.

**Proof.** The graph \( J_{m-2} \) contains \( m - 3 \) subgraphs \( L_1, L_2, \ldots, L_{m-3} \) isomorphic to the PN-graph \( G_{12m-2} \) such that for \( i = 1, 2, \ldots, m - 1 \), the graph \( L_i \) contains the cycles \( B_{i-1}, B_i, \) and \( B_{i+1} \). Consider an arbitrary 1-immersion \( \varphi \) of \( J_{m-2} \). Suppose that in the plane embedding of the PN-graph \( L_1 \) in \( \varphi \), the cycle \( B_2 \) is the boundary cycle of the outer \( (12m - 2) \)-gonal face of the embedding. Then the embedding of \( L_1 \) determines an
embedding of the subgraph of \( L_2 \) bounded by the cycles \( B_1 \) and \( B_2 \). Since \( L_2 \) is a PN-graph, the subgraph of \( L_2 \) bounded by \( B_2 \) and \( B_3 \) lies outside the cycle \( B_2 \). Reasoning similarly, we obtain that for \( i = 3, 4, \ldots, m - 3 \), the subgraph of the PN-graph \( L_i \) bounded by \( B_i \) and \( B_{i+1} \) lies outside \( B_i \). As a result, \( \varphi \) is a plane embedding of \( J_{m-2} \), hence \( J_{m-2} \) is a PN-graph.

Denote by \( \overline{S}_m(\lambda) \) the graph obtained from \( S_m(\lambda) \), where \( m \geq 4 \) and \( \lambda \in \Phi_m(n) \), by deleting the 2-valent vertices of all central paths.

**Lemma 11.** For every \( m \geq 4 \), \( n \geq 0 \) and \( \lambda \in \Phi_m(n) \), \( \overline{S}_m(\lambda) \) is a PN-graph.

**Proof.** The graph \( \overline{S}_m(\lambda) \) contains a subgraph \( G \) isomorphic to the PN-graph \( G_{12m-2+n} \) and contains a subgraph \( G' \) homeomorphic to the PN-graph \( J_{m-2} \). The graph \( G \) contains the cycles \( B_{m-2} \), \( B_{m-1} \), and \( B_m \) of \( \overline{S}_m(\lambda) \), and the graph \( G' \) contains the cycles \( B_0, B_1, \ldots, B_{m-2} \) of \( \overline{S}_m(\lambda) \) and is obtained from \( J_{m-2} \) by subdividing the edges of the cycle \( B_{m-2} \) (by using \( n \) 2-valent vertices in total).

Consider, for a contradiction, a proper 1-immersion \( \varphi \) of \( \overline{S}_m(\lambda) \). In \( \varphi \), the graph \( G \) has a plane embedding and we shall investigate in which faces of the embedding of \( G \) lie the vertices of \( G' \). We shall show that they all lie in the face of \( G \) bounded by the (subdivided) cycle \( B_{m-2} \).

In the graph \( \overline{S}_m(\lambda) \), the cycles \( B_{m-2} \) and \( B_i, i \in \{0, 1, \ldots, m-3\} \) are connected by \( 24m - 4 \) edge-disjoint paths. This implies that no 3- or 4-gonal face of \( G \) contains all vertices of \( B_i \) in its interior.

Any two vertices of \( B_i \) are connected by six edge-disjoint paths in \( G' - B_{m-2} \). Therefore

(a) No 3- or 4-gonal face of \( G \) contains any vertex of the cycles \( B_i, i = 0, 1, \ldots, m - 3 \), in its interior.

Suppose that a vertex \( v \) of \( G' \) does not belong to the cycles \( B_i, i = 1, 2, \ldots, m - 2 \), and lies inside a 3- or 4-gonal face \( F \) of \( G \). By construction of \( G' \), the vertex \( v \) is adjacent to two vertices \( w \) and \( w' \) of some \( B_j, j \in \{0, 1, \ldots, m - 3\} \). By (a), \( w \) and \( w' \) do not lie inside \( F \), hence they lie, respectively, in faces \( F_1 \) and \( F_2 \) of \( G \) adjacent to \( F \). However, at least one of \( F_1 \) and \( F_2 \) is 3- or 4-gonal, contrary to (a). Therefore, no 3- or 4-gonal face of \( G \) contains any vertex of \( G' - B_{m-2} \). If there is a vertex of \( G' - B_{m-2} \) outside \( B_m \), then \( G' \) has two adjacent vertices such that one of them is either a vertex of \( B_{m-2} \) or lies inside \( B_{m-2} \), and the other vertex lies outside \( B_m \), a contradiction, since the edge joining the vertices crosses at least five edges of \( G \). This implies that all vertices of \( G' - B_{m-2} \) lie inside the face of \( G \) bounded by \( B_{m-2} \). Hence, \( G' \) lies inside \( B_{m-2} \) and has a proper 1-immersion in \( \varphi \). If in this 1-immersion of \( G' \) we ignore the 2-valent vertices on the cycle \( B_{m-2} \) of \( \overline{S}_m(\lambda) \), then we obtain a proper 1-immersion of the PN-graph \( J_{m-2} \), a contradiction. ■

By the paths of \( \overline{S}_m(\lambda) \) associated with any central vertex \( x \), we mean the two paths shown in Figure 22; one of them is depicted in thick line and the other in dashed line. Every edge of \( \overline{S}_m(\lambda) \) not belonging to the cycles \( B_0, B_1, \ldots, B_m \) is assigned a type \( t \in \{1, 2, \ldots, 2m\} \) as shown in Figure 22 such that for \( i = 1, 2, \ldots, m \), the edges of type \( 2i - 1 \) and \( 2i \) are all edges lying between the cycles \( B_{i-1} \) and \( B_i \), and the edges of type \( 2i - 1 \) (resp. \( 2i \)) are incident to vertices of \( B_{i-1} \) (resp. \( B_i \)).

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Suppose that there is a 1-immersion \( \varphi \) of \( S_m(\lambda) \). By Lemma 11, \( \overline{S}_m(\lambda) \) is a PN-graph. Thus, \( \varphi \) induces an embedding of this graph. We shall assume that the outer face \( F_0 \) of this embedding is bounded by the cycle \( B_m \). We shall first show that \( F_0 \) is also a face of \( \varphi \). To prove this, it suffices to see that no central path can enter \( F_0 \).

Any central vertex \( x \) is separated from \( F_0 \) by \( 3m-1 \) edge-disjoint cycles: \( m \) cycles \( B_1, B_2, \ldots, B_m \) and \( 2m-1 \) cycles \( C_2, C_3, \ldots, C_{2m} \), where the cycle \( C_i \) consists of all edges of type \( i \) (\( i = 2, 3, \ldots, 2m \)). The central path \( P = P(x, x^*) \) can have at most \( 6m-3 \) crossing points, hence, all \( 6m-1 \) central paths cannot be 1-immersed inside \( B_0 \). Then there is a central path that crosses some edges of \( \overline{S}_m(\lambda) \), a contradiction. Hence, in \( \varphi \) any central path either lies inside \( B_0 \) or crosses some edges of \( \overline{S}_m(\lambda) \) but does not lie entirely between \( B_0 \) and \( B_m \).

The main goal of this section is to show that \( S_m(\lambda) \) has no 1-immersions (see Theorem 4 in the sequel). Roughly speaking, the main idea of the proof is as follows. Suppose, for a contradiction, that \( S_m(\lambda) \) has a 1-immersion. Every central path can have at most \( 6m-3 \) crossing points, hence, all \( 6m-1 \) central paths cannot be 1-immersed inside \( B_0 \). Then there is a central path that crosses some edges of \( \overline{S}_m(\lambda) \). Let \( P \) be a central path with maximum number of such crossings. Since \( P \) can have at most \( 6m-3 \) crossing points, some of the other \( 6m-2 \) central paths do not cross \( P \) and have to “go around” \( P \) and, in doing so, one of the paths has to cross more edges of \( \overline{S}_m(\lambda) \) than \( P \) does, a contradiction. Before proving Theorem 4, we need some definitions and preliminary Lemmas 12 and 13.

Consider a 1-immersion of \( S_m(\lambda) \) (if it exists). If a central path \( P = P(x, x^*) \) does not lie inside \( B_0 \), consider the sequence \( \delta_1, \delta_2, \ldots, \delta_r \) (\( r \geq 2 \)), where \( \delta_1 = x \) and \( \delta_r = x^* \), obtained by listing the intersection points of the path and \( B_0 \) when traversing the path from the vertex \( x \) to the vertex \( x^* \) (here \( \delta_2, \delta_3, \ldots, \delta_{r-1} \) are crossing points). By a piece of \( P \), we
mean the segment of $P$ from $\delta_i$ to $\delta_{i+1}$ for some $i \in \{1, 2, \ldots, r-1\}$; denote the piece by $P(\delta_i, \delta_{i+1})$. A piece of $P$ with an end point $x$ or $x^*$ is called an end piece of $P$ at the vertex $x$ or $x^*$, respectively. An outer piece of $P$ is every piece of $P$ that is immersed outside $B_0$. Clearly, either $P(\delta_1, \delta_2)$, $P(\delta_3, \delta_4)$, $P(\delta_5, \delta_6)$, \ldots or $P(\delta_2, \delta_3)$, $P(\delta_4, \delta_5)$, $P(\delta_6, \delta_7)$, \ldots are all outer pieces of $P$. The end points $\delta$ and $\delta'$ of an outer piece $\Pi$ of $P$ partition $B_0$ into two curves $A$ and $A'$ such that the curve $A$ lies inside the closed curve consisting of $\Pi$ and $A'$ (see Fig. 23). The central vertices belonging to $A$ and different from $\delta$ and $\delta'$ are said to be bypassed by $\Pi$ and $P$ (cf. Fig. 23).

**Lemma 12.** If $P(x, x^*)$ bypasses neither a central vertex $y$ nor its opposite vertex $y^*$, and $P(y, y^*)$ bypasses neither $x$ nor $x^*$, then $P(x, x^*)$ crosses $P(y, y^*)$.

**Proof.** Suppose, for a contradiction, that $P(x, x^*)$ does not cross $P(y, y^*)$. For every outer piece of the two paths, we can replace a curve of a path containing the piece by a new curve lying inside $B_0$ so that the path $P(x, x^*)$ (resp. $P(y, y^*)$) becomes a new path $P'(x, x^*)$ (resp. $P'(y, y^*)$) connecting the vertices $x$ and $x^*$ (resp. $y$ and $y^*$) such that the two new paths lie inside $B_0$ and do not cross each other, a contradiction. How the replacements can be done is shown in Figure 24, where the new curves are depicted in thick line. (Note that in Fig. 24(b), since $P(y, y^*)$ does not bypass $x$, the depicted pieces bypassing $x$ belong to $P(x, x^*)$.)

By a type of an outer piece of a central path, we mean the maximal type of an edge of $\bar{S}_m(\lambda)$ crossed by the path.

For an outer piece $\Pi$ of a central path $P(x, x^*)$, denote by $b(\Pi)$ the number of central vertices bypassed by $\Pi$, and by $\Delta(\Pi)$ the number of intersection points of $\Pi$ and $\bar{S}_m(\lambda)$, including the crossings at the end points of $\Pi$ (except if an end point is $x$ or $x^*$).
Lemma 13. If $\Pi$ is an outer piece of type $t$ of a central path $P$, then
\[ \Delta(\Pi) - b(\Pi) \geq 2t - \tau, \]
where $\tau = 1$ if $\Pi$ is an end piece and $\tau = 0$ otherwise.

**Proof.** The piece $\Pi$ crossing an edge of type $t$ has a point separated from the interior of $B_0$ by $\left\lceil \frac{t+1}{2} \right\rceil + t$ edge-disjoint cycles: $B_0, B_1, \ldots, B_{\left\lceil \frac{t+1}{2} \right\rceil - 1}$, and the cycles $C_1, C_2, \ldots, C_t$, where the cycle $C_i$ consists of all edges of type $i$ ($i = 1, 2, \ldots, t$). Thus, $\Pi$ crosses each of these cycles twice, except that for an end piece, we may miss one crossing with $C_1$. The piece $\Pi$ bypasses $b(\Pi)$ central vertices, hence $\Pi$ crosses $2b(\Pi)$ paths associated with those $b(\Pi)$ vertices. Hence, we obtain two inequalities,
\[ \Delta(\Pi) \geq 2 \left\lceil \frac{t+1}{2} \right\rceil + 2t - \tau \tag{1} \]
and
\[ \Delta(\Pi) \geq 2 \left\lceil \frac{t+1}{2} \right\rceil + 2b(\Pi) - \tau. \tag{2} \]
Now add (1) and (2), apply $2 \left\lceil \frac{t+1}{2} \right\rceil \geq t$, then divide by 2 and rearrange to get the result. ■

By the *type* of a central path not lying (entirely) inside $B_0$, we mean the maximal type of the outer pieces of the path. If $t$ different central paths bypass a central vertex $x$, then all the paths cross edges of the same path $T$ associated with $x$ and since the edges of $T$ have pairwise different types, we obtain that one of the central paths crosses an edge of type at least $t$. Hence we have

(D) If $t$ different central paths bypass the same central vertex, then one of the paths has type at least $t$.

**Theorem 4.** For every $m \geq 4$ and $\lambda \in \Phi_m(n)$, the graph $S_m(\lambda)$ is not 1-planar.

**Proof.** Consider, for a contradiction, a 1-immersion $\varphi$ of $S_m(\lambda)$ and a path $P = P(x, x^*)$ of maximal type $t > 0$.

As above, let $\Delta(P)$ be the number of crossing points of $P$ and $\overline{S}_m(\lambda)$, and let $b(P)$ be the number of distinct central vertices bypassed by $P$ and different from $x$ and $x^*$.

There are at least $6m - 2 - b(P)$ different pairs $\{y, y^*\}$ ($\{y, y^*\} \neq \{x, x^*\}$) of central vertices such that $P$ does not bypass $y$ and $y^*$; denote by $\mathcal{P}$ the set of the corresponding (at least $6m - 2 - b(P)$) paths $P(y, y^*)$. If $P(x, x^*)$ does not bypass $y$ and $y^*$, then $P(y, y^*)$ either does not bypass $x$ and $x^*$ (in this case $P(y, y^*)$ crosses $P(x, x^*)$ by Lemma 12) or bypasses at least one of the vertices $x$ and $x^*$. Hence, we have
\[ 6m - 2 - b(P) \leq \beta + \gamma + \epsilon, \tag{3} \]
where $\beta$ is the number of paths of $\mathcal{P}$ that cross $P$ and do not bypass $x$ or $x^*$; $\gamma$ is the number of paths of $\mathcal{P}$ that bypass $x$ or $x^*$ and do not cross $P$; $\epsilon$ is the number of paths of $\mathcal{P}$ that cross $P$ and bypass $x$ or $x^*$. We are interested in the number $\gamma + \epsilon$ of paths of $\mathcal{P}$ that bypass $x$ or $x^*$.

The path $P$ has at most $6m - 3$ crossing points, hence
\[ 6m - 3 - \Delta(P) \geq \beta + \epsilon \]
and, by (3), we obtain
\[ \gamma \geq 6m - 2 - b(P) - (\beta + \epsilon) \geq \Delta(P) - b(P) + 1, \]
when
\[ \gamma + \epsilon \geq \Delta(P) - b(P) + 1 + \epsilon. \] (4)

Let \( \Pi_1, \Pi_2, \ldots, \Pi_\ell (\ell \geq 1) \) be all outer pieces of \( P \), and \( \Pi_1 \) be of the maximal type \( t \). We have \( \Delta(P) = \sum_{i=1}^\ell \Delta(\Pi_i) \) and \( b(P) \leq \sum_{i=1}^\ell b(\Pi_i) \) (the vertices \( x \) and \( x^\ast \) can be bypassed by \( P \), and some central vertices can be bypassed by \( P \) more than once). By Lemma 13, \( \Delta(\Pi_i) - b(\Pi_i) \geq 0 \) for every \( i = 1, 2, \ldots, \ell \). Hence, by (4) and Lemma 13, we obtain
\[ \gamma + \epsilon \geq \sum_{i=1}^\ell \Delta(\Pi_i) - \sum_{i=1}^\ell b(\Pi_i) + 1 + \epsilon \]
\[ \geq \Delta(\Pi_1) - b(\Pi_1) + 1 + \epsilon \geq (2t + 1) - \tau + \epsilon, \] (5)
where \( \tau = 1 \) if \( \Pi_1 \) is an end piece and \( \tau = 0 \) otherwise. If \( \Pi_1 \) is not an end piece or \( \epsilon \geq 1 \), then, by 5, \( \gamma + \epsilon \geq 2t + 1 \), hence one of the vertices \( x \) and \( x^\ast \) is bypassed by at least \( t + 1 \) paths of \( \mathcal{P} \). Now, by (D), one of the \( t + 1 \) paths has type at least \( t + 1 \), a contradiction. Now suppose that \( \Pi_1 \) is an end piece at the vertex \( x \) and \( \epsilon = 0 \). Then every path of \( \mathcal{P} \) either crosses \( P \) or bypasses \( x \) or \( x^\ast \), and at least \( 2t \) paths of \( \mathcal{P} \) bypass \( x \) or \( x^\ast \). If no one of the \( 2t \) paths bypasses \( x \), then all the \( 2t \) paths bypass \( x^\ast \) and, by (D), one of the paths has type at least \( t + 1 \). If one of the \( 2t \) paths, say, \( P' \), bypasses \( x \), then \( P' \) has an outer piece \( \Pi' \) that bypasses \( x \) and does not cross \( \Pi_1 \) (since \( \epsilon = 0 \), \( P' \) does not cross \( P \)). The piece \( \Pi_1 \) has type \( t \) and is an end piece at \( x \), hence \( \Pi' \) has type at least \( t + 1 \), a contradiction. 

We have shown that every graph \( S_m(\lambda) \), where \( m \geq 4 \) and \( \lambda \in \Phi_m(n) \), is an \( \mathcal{M} \)-graph. These graphs have order \( (5m - 1) (12m - 2) + 5n \). Clearly, graphs \( S_{m_1}(\lambda_1) \) and \( S_{m_2}(\lambda_2) \), where \( \lambda_1 \in \Phi_{m_1}(n_1) \) and \( \lambda_2 \in \Phi_{m_2}(n_2) \), are nonisomorphic for \( m_1 \neq m_2 \) and for \( m_1 = m_2 \) and \( n_1 \neq n_2 \).

Claim 2. For any integers \( m \geq 4 \) and \( n \geq 0 \), there are at least \( \frac{1}{(24m - 4)} \left( \frac{n + 12m - 3}{12m - 3} \right) \) nonisomorphic \( \mathcal{M} \)-graphs \( S_m(\lambda) \), where \( \lambda \in \Phi_m(n) \).

Proof. The automorphism group of \( S_m \) is the dihedral group \( D_{12m - 2} \) of order \( 24m - 4 \). Now the claim follows by recalling a well-known fact that \( |\Phi_m(n)| = \left( \frac{n + 12m - 3}{12m - 3} \right) \).

5. TESTING 1-IMMERSIBILITY IS HARD

In this section, we prove that testing 1-immersibility is NP-hard. This shows that it is extremely unlikely that there exists a nice classification of \( \mathcal{M} \)-graphs.

Theorem 5. It is NP-complete to decide if a given input graph is 1-immersible.

Since 1-immersions can be represented combinatorially, it is clear that 1-immersibility is in NP. To prove its completeness, we shall make a reduction from a known NP-complete problem, that of 3-colorability of planar graphs of maximum degree at most 4 [11].

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The rest of this section is devoted to the proof of Theorem 5. Let $G$ be a given plane graph of maximum degree 4 whose 3-colorability is to be tested. We shall show how to construct, in polynomial time, a related graph $\overline{G}$ such that $\overline{G}$ is 1-immersible if and only if $G$ is 3-colorable. We may assume that $G$ has no vertices of degree less than 3 (since degree 1 and 2 vertices may be deleted without affecting 3-colorability).

To construct $\overline{G}$, we will use as building blocks graphs that have a unique 1-immersion. These building blocks are connected with each other by edges to form a graph that also has a unique 1-immersion. Then we add some additional paths to obtain $\overline{G}$.

We say that a 1-planar graph $G$ has a unique 1-immersion if, whenever two edges $e$ and $f$ cross each other in some 1-immersion, then they cross each other in every 1-immersion of $G$, and second, if $G^*$ is the planar graph obtained from $G$ by replacing each pair of crossing edges $e = ab$ and $f = cd$ by a new vertex of degree 4 joined to $a$, $b$, $c$, $d$, then $G^*$ is 3-connected (and thus has combinatorially unique embedding in the plane – the one obtained from 1-immersions of $G$).

It was proved in [12] that for every $n \geq 6$, the graph with $4n$ vertices and $13n$ edges shown in Figure 25(a) has a unique 1-immersion. (To be precise, the paper [12] considers the graph for even values of $n \geq 6$ only, but one can check that the proof does not depend on whether $n \geq 6$ is even or odd.) We call the graph a U-graph. Figure 25(b) shows a designation of the U-graph used in what follows. In the 1-immersion of the U-graph shown in Figure 25, the vertices $1, 2, 3, \ldots, n - 1, n$, which lie on the boundary of the outer face of the spanning embedding (the boundary is called the outer boundary cycle of the 1-immersed U-graph), are called the boundary vertices of the U-graph in the 1-immersion. If a graph has a U-graph as a subgraph, then the U-graph is called the U-subgraph of the graph.

Take two 1-immersed U-graphs $U_1$ and $U_2$ such that each of them has the outer boundary cycle of length at least 7, and construct the 1-immersed graphs shown in Figure 26(a) and (b), respectively, where by $1, 2, \ldots, 7$ we denote seven consecutive vertices on the outer boundary cycle of each of the 1-immersed graphs. We say that in Figure 26(a) (resp. (b)) the U-graphs $U_1$ and $U_2$ are connected by a (1)-grid (resp. (2)-grid). The vertices labeled $1, 2, \ldots, 7$ are the basic vertices of the grid and for $i = 1, 2, \ldots, 7$, the $h$-path connecting the vertices labeled $i$ of the $(h)$-grid, $h \in \{1, 2\}$, is called the basic path of the grid connecting these vertices. Let us denote the $i$th basic path by $P_i$. The paths $P_{i-1}$ and $P_i$, $i = 2, 3, \ldots, 7$, are neighboring basic paths of the grid.
For two basic paths \( P = P_i \) and \( P' = P_j, 1 \leq i < j \leq 7 \), denote by \( C(P, P') \) the cycle of the graph in Figure 26 consisting of the two paths and of the edges \( (i, i + 1), (i + 1, i + 2), \ldots, (j - 1, j) \) of the two graphs \( U_1 \) and \( U_2 \).

By a \( U \)-supergraph, we mean every graph obtained in the following way. Consider a plane connected graph \( H \). Now, for every vertex \( v \in V(H) \), take a 1-immersed \( U \)-graph \( U(v) \) of order at least \( 28 \cdot \deg(v) \) and for any two adjacent vertices \( u \) and \( w \) of the graph, connect \( U(u) \) and \( U(w) \) by a \((1)\) or \((2)\)-grid as shown in Figure 27 such that any two distinct grids have no basic vertices in common. We obtain a 1-immersed \( U \)-supergraph.

**Theorem 6.** Every \( U \)-supergraph \( M \) has a unique 1-immersion.

**Proof.** It suffices to show the following:

(a) The graph consisting of two \( U \)-graphs connected by an \((h)\)-grid, \( h = 1, 2 \), has a unique 1-immersion.

(b) In every 1-immersion \( \varphi \) of \( M \), the edges of distinct grids do not intersect.

Note that \( M \) contains no subgraph that can be 1-immersed inside the boundary cycle of a 1-immersed \( U \)-subgraph of \( M \) in a 1-immersion of \( M \) as shown in Figure 28 in dashed line. Hence, in every 1-immersion of \( M \), the boundary edges of the \( U \)-subgraphs of \( M \) are not crossed.

We prove (a) and (b) in the following way. We consider a 1-immersed subgraph \( W \) of \( M \) (cf. Fig. 26) consisting of two \( U \)-graphs \( U_1 \) and \( U_2 \) connected by an \((h)\)-grid \( \Gamma, h \in \{1, 2\} \), and we show that in every 1-immersion \( \varphi \) of \( M \), the graph \( W \) has the same 1-immersion and the edges of \( \Gamma \) are not crossed by edges of other grids.
Suppose, for a contradiction, that $U_1$ and $U_2$ are 1-immersed under $\varphi$ as shown in Figure 29(a) (the situation described in the figure arises when $U_1$ is drawn clockwise and $U_2$ counter-clockwise (or vice versa)). Clearly, there are two basic paths $P_i$ and $P_j$ of $\Gamma$, $1 \leq i < j \leq 7$, which do not intersect. Then the cycle shown in Figure 29(a) in thick line is embedded in the plane, a contradiction, since the cycle is crossed by five other basic paths of $\Gamma$, but the cycle has only $2h \leq 4$ edges that can be crossed by other edges. Hence, $U_1$ and $U_2$ are 1-immersed as shown in Figure 26.

Suppose that in $\varphi$, a basic path $P_i$ of $\Gamma$ crosses a basic path $Q$ of some grid of $M$ exactly once. If $Q = P_j$ is a basic path of $\Gamma$, $j \neq i$ (see Fig. 29(b)), then the closed curves $C_1$ and $C_2$ shown in Figure 29(b) by dashed cycles, are embedded in the plane and each of the other five basic paths of $\Gamma$ crosses an edge of $C_1$ or $C_2$, a contradiction, since $C_1$ and $C_2$ have $2h - 2 \leq 2$ edges in total, which can be crossed by other edges. If $Q$ is a basic path of a grid $\Gamma'$ different from $\Gamma$, then there is a basic path $P_j$, $j \neq i$, of $\Gamma$ such that $P_j$ is not crossed by $P_i$ and $Q$. Hence, the cycle $C(P_i, P_j)$ is embedded and $Q$ crosses the edges of the cycle exactly once. Then for every other basic path $Q'$ of $\Gamma'$, the cycle $C(Q, Q')$ crosses $C(P_i, P_j)$ at least twice and $Q'$ crosses $P_i$ or $P_j$ (the edges of different U-subgraphs do not intersect). We have that the seven basic paths of $\Gamma'$ cross $P_i$ and $P_j$, a contradiction. Hence, in $\varphi$, if two basic paths intersect, then they intersect twice. In particular, only basic paths of (2)-grids can intersect.

Now we claim the following:

(E) If in $\varphi$ two neighboring basic paths $P_{i-1}$ and $P_i$ of the (2)-grid $\Gamma$ do not intersect, then the edge $e$ joining the middle vertices of $P_{i-1}$ and $P_i$ lies inside the embedded cycle $C(P_{i-1}, P_i)$.

Indeed, if $e$ lies outside $C(P_{i-1}, P_i)$, then the 4-cycle shown in Figure 29(c) in thick line is crossed by five basic paths $P_r$, $r \neq i - 1, i$, a contradiction.

Suppose that in $\varphi$, a basic path of $\Gamma$ crosses a basic path of a grid $\Gamma'$ twice (i.e., $\Gamma$ and $\Gamma'$ are (2)-grids). Since the number of basic paths of $\Gamma$ is odd (namely, seven), it cannot be that every basic path of $\Gamma$ crosses some other basic path of $\Gamma$ twice. Then there is a basic path of $\Gamma$, which is not crossed by other basic paths of $\Gamma$. Hence, if $\Gamma = \Gamma'$
(resp. $\Gamma \neq \Gamma'$), then there are two neighboring basic paths $P_{i-1}$ and $P_i$ of $\Gamma$ such that one of them, say, $P_{i-1}$, is crossed twice by some basic path $Q$ of $\Gamma$ (resp. $\Gamma'$), and the other basic path $P_i \neq Q$ is not crossed by $Q$. Then the cycle $C(P_{i-1}, P_i)$ is embedded. By (E), the edge $e$ joining the middle vertices of $P_{i-1}$ and $P_i$ lies inside $C(P_{i-1}, P_i)$. Denote by $C_1$ and $C_2$ the two embedded adjacent 4-cycles each of which consists of $e$ and edges of the 6-cycle $C(P_{i-1}, P_i)$. The middle vertex of $Q$ lies outside $C(P_{i-1}, P_i)$ and the two end vertices of $Q$ lie inside $C_1$ and $C_2$, respectively. The end vertices of $Q$ belong to two U-subgraphs connected by $\Gamma'$. Since the edges of the two U-subgraphs do not cross the edges of $C_1$ and $C_2$, we obtain that one of the U-subgraphs lies inside $C_1$ and the other lies inside $C_2$, a contradiction, since the two U-subgraphs are connected by at least four basic paths different from $Q$, $P_{i-1}$, and $P_i$. Hence, no basic path of $\Gamma$ crosses some other basic path twice.

We conclude that the basic paths of the grids connecting U-subgraphs do not intersect.

Now it remains to show that if $\Gamma$ is a (2)-grid, then the edges joining the middle vertices of the basic paths of $\Gamma$ are not crossed. Consider any two neighboring basic paths $P_{i-1}$ and $P_i$ of $\Gamma$. The cycle $C(P_{i-1}, P_i)$ is embedded and, by (E), the edge $e$ joining the middle vertices of $P_{i-1}$ and $P_i$ lies inside $C(P_{i-1}, P_i)$. It is easy to see that for every edge $e'$ of $G$ not belonging to U-subgraphs and different from $e$ and the edges of $C(P_{i-1}, P_i)$, in the graph $G - e'$ the end vertices of $e'$ are connected by a path that consists of edges of U-subgraphs and basic paths of grids and which does not pass through the vertices of $C(P_{i-1}, P_i)$. Now, if the edge $e$ is crossed by some other edge $e'$, then $e'$ is not an edge of a U-subgraph, the end vertices of $e'$ lie inside the cycles $C_1$ and $C_2$, respectively (where $C_1$ and $C_2$ are defined as in the preceding paragraph), whose edges are not crossed by edges of U-subgraphs and basic paths, a contradiction.

Therefore, the edges of $M$ do not intersect and that the graph $W$ has a unique 1-immersion. This completes the proof of the theorem. 

Now, given a plane graph $G$ every vertex of which has degree 3 or 4, we construct a graph $\overline{G}$ such that $G$ is 3-colorable if and only if $\overline{G}$ is 1-immersible. To obtain $\overline{G}$, we proceed as follows. First, we construct a subgraph $G^{(1)}$ of $\overline{G}$ such that $G^{(1)}$ has a unique 1-immersion. The graph $G^{(1)}$ is obtained from a U-supergraph $W$ by adding some additional vertices and edges. By inspection of the subsequent figures that illustrate the construction of $G^{(1)}$ and its 1-immersion, the reader will easily identify the additional vertices and edges: they do not belong to U-subgraphs and grids. Then one can easily check that given the 1-immersion of $W$, the additional vertices and edges can be placed in the plane in a unique way to obtain a 1-immersion of $G^{(1)}$, hence $G^{(1)}$ has a unique 1-immersion also. Now, given the unique 1-immersion of $G^{(1)}$, to construct $\overline{G}$ we place some new additional paths “between” 1-immersed U-subgraphs of $G^{(1)}$. Notice that due to circle inversion, one may assume that in a 1-planar drawing of $\overline{G}$ each U-subgraph is drawn in such a way that the outer boundary cycle is containing all other vertices of the U-subgraph inside the region it bounds.

The graph $G^{(1)}$ is obtained from the plane graph $G$ if we replace every face $F$ of the embedding of $G$ by a U-graph $U(F)$ and replace every vertex $v$ by a vertex-block $B(v)$ as shown at the top of Figure 30. At the bottom of Figure 30, we show the designation of a (1)-grid used at the top of the figure and at what follows. The vertex-block $B(v)$ has a unique 1-immersion and is obtained from a U-supergraph by adding some additional vertices and edges. Figure 30 shows schematically the boundary of $B(v)$ Figure 33 shows $B(v)$ in more detail. For a $k$-valent vertex $v$ of $G$, $3 \leq k \leq 4$, the vertex-block $B(v)$ has
3k boundary vertices labeled clockwise as \(a, b, c, a, b, c, \ldots, a, b, c\); these vertices do not belong to U-subgraphs of \(B(v)\). In Figures 30 and 33, we only show the case of a 3-valent vertex \(v\); for a 4-valent vertex, the construction is analogous—there are three more boundary vertices labeled \(a, b, c\), respectively.

We say that vertex-blocks \(B(v)\) and \(B(w)\) are adjacent if \(v\) and \(w\) are adjacent vertices of \(G\).

The graph \(G\) is obtained from \(G^{(1)}\) if we take a collection of additional disjoint paths of length \(\geq 1\) (they are called the pending paths) and identify the end vertices of every path with two vertices, respectively, of \(G^{(1)}\). The graph \(G^{(1)}\) has a unique 1-immersion and the edges of the U-subgraphs of \(G^{(1)}\) cannot be crossed by the pending paths, hence the 1-immersed \(G^{(1)}\) restricts the ways in which the pending paths can be placed in the plane to obtain a 1-immersion of \(G\). Every pending path connects either boundary vertices of adjacent vertex blocks or vertices of the same vertex block.

The graph \(G\) is such that for any two adjacent vertex blocks, there are exactly three pending paths connecting the vertices of the vertex blocks. The paths have length 3 and are shown in Figure 31; we say that these pending paths are incident with the two vertex blocks. Each of the three pending paths connects the boundary vertices labeled by the same letter: \(a, b,\) or \(c\). For \(h \in \{a, b, c\}\), the pending path connecting vertices labeled \(h\) is called the \((h)\)-path connecting the two vertex blocks. In Figure 31 the \((h)\)-path, \(h \in \{a, b, c\}\), is labeled by the letter \(h\).

Denote by \(G^{(2)}\) the graph obtained from \(G^{(1)}\) if we add all triples of pending paths connecting vertex-blocks \(B(v), B(w)\), for all edges \(vw \in E(G)\).

The graph \(G\) also satisfies the properties stated below. The pending paths connecting vertices of the same vertex-block \(B(v)\) are divided into three families called, respectively,
the \( a \), \( b \), and \( c \)-families of \( B(v) \). Given the 1-immersion of \( G^{(1)} \), for every \( h \in \{a, b, c\} \), the \( h \)-family of \( B(v) \) has the following properties:

(i) Every path \( P \) of the \( h \)-family admits exactly two embeddings in the plane such that we obtain a 1-immersion of \( G^{(2)} \cup P \).

(ii) The \( h \)-family consists of paths \( P_1, P_2, \ldots, P_n \) such that the graph \( G^{(2)} \cup P_1 \cup P_2 \cup \ldots \cup P_n \) has exactly two 1-immersions. In the two 1-immersions, every path \( P_i \) uses its two embeddings. In one of the 1-immersions, paths of the \( h \)-family cross all \((h)\)-paths incident with \( B(v) \). In the other 1-immersion, the paths of the \( h \)-family do not cross any \((h)\)-path incident with \( B(v) \).

Figure 32 shows fragments of the two 1-immersions of the union of \( G^{(2)} \) and the pending paths of an \( h \)-family. In the figure, each of the depicted (in thick line) six edges of the family, they are labeled by 1, 2, \ldots, 6, respectively, uses its two embeddings in one of the two 1-immersions.

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If in a 1-immersion of $\overline{G}$, paths of an $h$-family of $B(v)$, $h \in \{a, b, c\}$, cross $(h)$-paths incident with $B(v)$, then we say that the $h$-family of $B(v)$ is activated in the 1-immersion of $\overline{G}$.

Figures 33 and 34 show a vertex-block $B(v)$ and the $h$-families of the vertex-block $(h = a, b, c)$ in the case where $v$ is 3-valent (the generalization for a 4-valent vertex $v$ is straightforward). The pending paths of the three $h$-families are shown by thick lines and the three families are activated. To avoid cluttering a figure, Figure 33 contains a fragment denoted by $R$, which is given in more detail in Figure 34. Recall that the gray areas in Figures 33 and 34 represent U-graphs.

In Figures 33 and 34, we use designations of some fragments of $B(v)$; the designations are given at the left of Figure 35 and the corresponding fragments are given at the right of Figure 35 (i.e., the connections between the gray areas in Figures 33 and 34 consist of seven basic paths). The reader can easily check that for every pending path $P$ of the three families, there are exactly two ways to embed the path so that we obtain a 1-immersion of $G^{(2)} \cup P$. The vertex-block $B(v)$ contains a 2-path connecting the vertices labeled 0 in Figure 34 and a 1-path connecting the vertices labeled 1 in Figure 34; we call the paths the (0)- and (1)-blocking paths, respectively. For every $h \in \{a, b, c\}$, exactly one pending path of the $h$-family of $B(v)$ crosses a blocking path: the pending path has length 33, crosses the (1)-blocking (resp. (0)-blocking) path when the $h$-family is activated (resp. not activated), and the pending path in each of its two embeddings crosses exactly one pending path of each of the other two families. Figure 34 shows the two embeddings of the pending 33-path of the $b$-family (one of them is in thick line, the other, when the family is not activated, is in dashed line). Note that Figure 34 shows something that is
not a 1-immersion, since all three families of paths are activated, and the (1)-blocking path is crossed three times.

Denote by $G^{(2)}$ the union of $G^{(2)}$ and the paths of all three $h$-families of $B(v)$. Now the reader can check that $B(v)$ and the $h$-families of $B(v)$ are constructed in such a way that the following holds:

(F) In every 1-immersion of $G^{(2)}$ (and, hence, of $\bar{G}$) exactly one $h$-family of $B(v)$ is activated, and for each $h \in \{a, b, c\}$, there is a 1-immersion of $G^{(2)}$ in which the $h$-family of $B(v)$ is activated.

By construction of $\bar{G}$, if $\bar{G}$ has a 1-immersion, then in the 1-immersion for every vertex $v$ of $G$, exactly one $h$-family ($h \in \{a, b, c\}$) of $B(v)$ is activated and taking Figure 31 into account we obtain that in the 1-immersion the $h$-families of the vertex blocks adjacent to $B(v)$ are not activated.

Now take a 1-immersion of $\bar{G}$ (if it exists) and assign every vertex $v$ of $G$ a color $h \in \{a, b, c\}$ such that the $h$-family of $B(v)$ is activated in the 1-immersion of $\bar{G}$. We obtain a proper 3-coloring of $G$ with colors $\{a, b, c\}$.

Take a proper 3-coloring of $G$ (if it exists) with colors $\{a, b, c\}$ and for every vertex $v$ of $G$, if $h(v)$ is the color of $v$, take the $h(v)$-family of $B(v)$ to be activated and the other two families not to be activated. By the construction of $\bar{G}$, and the mentioned
6. **k-PLANARITY TESTING FOR MULTIGRAPHS**

A graph drawn in the plane is *k-immersed* in the plane \((k \geq 1)\) if any edge is crossed by at most \(k\) other edges (and any pair of crossing edges cross only once). A graph is *k-planar* if it can be \(k\)-immersed into the plane.

It appears that we can slightly modify the proof of Theorem 5 so as to obtain a proof that \(k\)-planarity testing \((k \geq 2)\) for multigraphs is NP-complete. Below we give only a sketch of the proof, the reader can easily fill in the missing details.

Denote by \(G(k)\), \(\overline{G}(k)\), and \(G^{(1)}(k)\), respectively, the multigraphs obtained from the graphs \(G\), \(\overline{G}\), and \(G^{(1)}\) if we replace every edge by \(k\) parallel edges. For an edge \(e\) of the...
FIGURE 36. Different plane 1-immersions of $G^{(1)}$. 

(a) \hspace{2cm} (b) 

FIGURE 37. Edges of an $h$-family in $\overline{G}(k)$. 

multigraphs denote by $H(e)$ the set consisting of $e$ and all other $k-1$ edges parallel to $e$. Denote by $\varphi$ the unique plane 1-immersion of $G^{(1)}$, and by $\varphi_k$ the plane $k$-immersion of $G^{(1)}(k)$ obtained from $\varphi$ if we replace every edge of $G^{(1)}$ by $k$ parallel edges.

**Lemma 14.** The multigraph $G^{(1)}(k)$, $k \geq 2$, has a unique plane $k$-immersion.

**Proof.** We consider an arbitrary plane $k$-immersion $\psi$ of $G^{(1)}(k)$ and show that $\psi$ is $\varphi_k$.

First, we show that if edges $e_1$ and $e_2$ of $G^{(1)}(k)$ cross in $\psi$, then each edge of $H(e_1)$ intersects every edge of $H(e_2)$. Suppose, for a contradiction, that an edge $e'_2$ of $H(e_2)$ does not intersect $e_1$ (see Fig. 36(a)). Consider the 2-cell $D$ whose boundary consists of the edges $e_2$ and $e'_2$. Since $e_2$ and $e'_2$ can have at most $2k$ crossings in total, there are at most two vertices lying outside $D$ that are adjacent to vertices inside $D$. This means (see Fig. 36(b)) that $G^{(1)}$ has two different plane 1-immersions (the edge of $G^{(1)}$ joining $u$ and $w$ has different positions in the two different plane 1-immersions), a contradiction. Hence, each edge of $H(e_1)$ intersects every edge of $H(e_2)$. Delete $k-1$ edges from every $k$ parallel edges. We obtain a plane 1-immersion of $G^{(1)}$, that is, $\varphi$. Hence, $\psi$ is $\varphi_k$. 

The graph $\overline{G}(k)$ is obtained from $G^{(1)}(k)$ if we add the pending paths of $\overline{G}$ where every edge is replaced by $k$ parallel edges. Now, considering a pending path of an $h$-family, we have (see Fig. 37, where each thick edge represents $k$ parallel edges) that if $e' \in H(e)$, then each of the edges $e$ and $e'$ is already crossed by $k$ edges of $H(e_1)$ and $H(e_2)$, respectively, thus the edges of the pending path incident with the vertex $v$ cannot cross edges $e$ and $e'$, a contradiction. Hence, all edges of $H(e)$ cross either $H(e_1)$ or $H(e_2)$. As
a result, the $h$-families and the other pending paths of $\overline{G}(k)$ “behave” in the same way as in $\overline{G}$. We conclude that $\overline{G}$ has a plane 1-immersion if and only if $\overline{G}(k)$ has a plane $k$-immersion. Since $\overline{G}$ has a plane 1-immersion if and only if $G$ has a proper 3-coloring, we get that $k$-planarity testing for multigraphs is NP-complete.

If we restrict ourselves to simple graphs only, then to have a proof analogous to the proof of Theorem 5 we need simple graphs that have a unique plane $k$-immersion ($k \geq 2$), but the construction of such graphs seems to be nontrivial and does not readily follow from the construction of U-graphs in Section 5.

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