On the Bergman Projection and the Lu Qi-Keng Conjecture

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Abstract: On a reasonable class of domains in $\mathbb{C}^n$, we characterize those holomorphic functions which continue analytically past the boundary. Then we give some applications of this result to holomorphic mappings. In addition, some new results about the Lu Qi-Keng conjecture are treated.

1 Introduction

Work of S. R. Bell (see, e.g., [BEL1]) has demonstrated the importance of the Bergman kernel $K$ and Bergman projection $P$ in understanding holomorphic mappings. In particular, Bell’s Condition $R$ has played a central role for many years.

In the present paper we give a characterization of those functions $\varphi$ on a domain $\Omega$ such that $\varphi$ continues analytically past the boundary. Then we give some applications of this result.

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2 Principal Results

In what follows a domain in $\mathbb{C}^n$ is a connected, open set. Now our main result is this:

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2Key Words: pseudoconvex, domain, Bergman kernel, Bergman projection, biholomorphic mapping.
Theorem 2.1 Let $\Omega \subseteq \mathbb{C}^n$ be a bounded, pseudoconvex domain with real analytic boundary. Assume that the $\overline{\partial}$-Neumann problem on $\Omega$ is real analytic hypoelliptic. If $\varphi$ is a holomorphic function on $\Omega$ that continues real analytically across all boundary points of $\Omega$, then we may find a $g \in C^\infty_c(\Omega)$ such that $P^*g = \varphi$.

Remark 2.2 It may be noted that, if $g \in C^\infty_c(\Omega)$, then $P^*g$ automatically continuous analytically across the boundary by the real analytic hypoellipticity of the $\overline{\partial}$-Neumann problem and by Kohn’s projection formulas $P = I - \overline{\partial}^* N \overline{\partial}$. In particular, since the Bergman kernel $K$ is just the Bergman projection of the Dirac delta mass, we see that $K \cdot, \zeta)$ analytically continues across the boundary for $\zeta \in \Omega$ fixed.

Proof of the Theorem: Let $\mathcal{O}(\overline{\Omega})$ denote those functions which are holomorphic on a neighborhood of the closure $\overline{\Omega}$ of the domain $\Omega$. Our job is to show that the Bergman projection $P$ maps $C^\infty_c(\Omega)$ onto $\mathcal{O}(\overline{\Omega})$. This is equivalent to showing that the adjoint mapping (which is also $P$) maps $\mathcal{O}^*(\overline{\Omega})$ univalently into the dual of $C^\infty_c(\Omega)$. The latter is of course just the space of distributions on $\Omega$.

Now let $\lambda$ be an element of $\mathcal{O}^*(\overline{\Omega})$. We need to see that if $\lambda \neq 0$ then $P\lambda \neq 0$. Suppose to the contrary that

$$\langle P\lambda, \psi \rangle = 0$$

for every $\psi \in C^\infty_c(\Omega)$. Then it follows that

$$\langle \lambda, P\psi \rangle = 0$$

for every $\psi \in C^\infty_c(\Omega)$. We may write this last as

$$\lambda \int_{\Omega} K(z, \zeta) \psi(\zeta) \, dV(\zeta) = 0.$$

The last displayed equation may be written as

$$\int_{\Omega} \lambda_z K(z, \zeta) \psi(\zeta) \, dV(\zeta) = 0$$

for all $\psi \in C^\infty_c(\Omega)$. But then it would follow that

$$\int_{\Omega} \lambda_z K(z, \zeta) h(\zeta) \, dV(\zeta) = 0$$
for every locally $L^2$ function $h$ on $\Omega$. Hence, for each fixed $z$,

$$\lambda_z K(z, \cdot) \equiv 0.$$ 

By earlier remarks, this is true even for $z$ in the boundary.

But this would mean that, if $b$ is any element of the Bergman space on $\Omega$, then

$$\lambda b = \int_{\Omega} \lambda_z K(z, \zeta) b(\zeta) \, dV(\zeta) \equiv 0.$$ 

Hence $\lambda$ is the zero functional, which is a contradiction. So the adjoint of $P$ is univalent. Hence $P$ maps $C^\infty_c(\Omega)$ onto $\mathcal{O}(\Omega)$.

\[ \square \]

\textbf{Remark 2.3} It would be incorrect to suppose that if $Pf \in \mathcal{O}(\Omega)$, then $f \in C^\infty_c(\Omega)$. For example, if $1$ denotes the function that is identically 1 on $\Omega$ then $P1 = 1$.

\section{An Application}

In the paper [ALE], H. Alexander proved the following striking result:

\textbf{Theorem 3.1} Let $\Phi$ be a proper holomorphic mapping of the unit ball $B$ in $\mathbb{C}^n$, $n > 1$, to itself. Then in fact $\Phi$ must be a biholomorphism.

This solved a problem of longstanding, and was a conceptually important result at the time. It contrasts of course with the situation in $\mathbb{C}^1$. Shortly thereafter, W. Rudin [RUD] came up with a much more elementary proof of a more general result. A bit later, S. Bell [BEL2] was able to put these ideas into a more natural context and give a proof that used key ideas from mapping theory. He was also able to generalize the result from the ball to a more general class of domains.

Recall now the Lu Qi-Keng conjecture (see [BOA]). The question is whether the Bergman kernel for a domain $\Omega \subseteq \mathbb{C}^n$ ever vanishes on $\Omega \times \Omega$. Thanks to work of Boas and others, the answer is known to be negative in a number of cases. But the answer is affirmative, for example, on a bounded, homogeneous, complete circular domain. A domain for which the conjecture is true is said to have the \textit{Lu Qi-Keng property}.
Here we generalize Bell’s result and put his proof into a simple setting. The main result is as follows:

**Theorem 3.2** Let $\Omega_1$, $\Omega_2$ be bounded, pseudoconvex domains with real analytic boundary and each having $\mathcal{D}$-Neumann problem that is real analytic hypoelliptic. Also suppose that $\Omega_1$ has the Lu Qi-Keng property. Let $\Phi: \Omega_1 \to \Omega_2$ be a proper holomorphic mapping. Then in fact $\Phi$ is biholomorphic.

In the proof, we shall let $P_j$ denote the Bergman projection on $\Omega_j$. We begin, as in the paper [BEL2], by noting three facts:

(a) If $\varphi \in C^\infty_c(\Omega_1)$ then $P_1 \varphi$ extends to be holomorphic on a neighborhood of $\Omega_1$. This is immediate from the local real analytic hypoellipticity of the $\overline{\partial}$-Neumann operator $N$, because $P_1 = I - \overline{\partial} N \overline{\partial}$.

(b) For each monomial $z^\alpha$, there is a function $\varphi_\alpha \in C^\infty_c(\Omega_2)$ such that $P_2 \varphi_\alpha = z^\alpha$. This is of course a direct application of our Theorem 2.1.

(c) Let $u = \det (\text{Jac} \Phi)$. If $\varphi \in L^2(\Omega_2)$, then $u \cdot (\varphi \circ \Phi) \in L^2(\Omega_1)$ and $P_1 (u \cdot (\varphi \circ \Phi)) = u \cdot ((P_2 \varphi) \circ \Phi)$. This is a standard formula of Bell, for which see [KRA1, Ch. 11].

**Proof of the Theorem:** Since several of the key ideas appear in [BEL2], we merely outline the argument.

Again using Theorem 2.1 above, let $\varphi_\alpha \in C^\infty_c(\Omega_2)$ be such that $P_2 \varphi_\alpha = z^\alpha$. Thus

$$u \Phi^\alpha = u \cdot ((P_2 \varphi_\alpha) \circ \Phi) = P_1 (u \cdot (\varphi_\alpha \circ \Phi)).$$

We note that $u \cdot (\varphi_\alpha \circ \Phi)$ is a function in $C^\infty_c(\Omega_1)$ just because $\Phi$ is a proper mapping. Thus Fact (a) implies that $u \Phi^\alpha$ extends to be holomorphic in a neighborhood of $\overline{\Omega_1}$. Now let $z \in \partial \Omega_1$. We have that $u \cdot \Phi^\alpha$ belongs to the ring of germs of holomorphic functions at $z$ for all multi-indices $\alpha$, including $\alpha = (0,0,\ldots,0)$. Because this ring is a unique factorization domain, we may decompose each of the functions $u \cdot \Phi^\alpha$ into a product of powers of irreducible elements of the ring. We take the special case $\alpha = (1,0,0,\ldots,0)$. A simple analysis of the decomposition into irreducible elements (see [BEL2]) shows that $\Phi_1$ (the first component of $\Phi$) extends to be holomorphic in a neighborhood of $z$. Likewise, the other components of $\Phi$ extend to be holomorphic in a neighborhood of $z$. 
Finally we must show that Φ is unbranched. For this we use the Lu Qi-Keng hypothesis and the standard mapping formula for the Bergman kernel. Namely, we know that

\[ K_1(z, \zeta) = \det (\text{Jac}_C \Phi)(z) \cdot K_2(\Phi(z), \Phi(\zeta)) \cdot \det (\text{Jac}_C \Phi)(\zeta). \]

Now \( K_1 \) does not vanish on \( \Omega \times \Omega \), and a simple application of Hurwitz’s theorem allows us to conclude then that \( K_1 \) does not vanish on \( \partial \Omega \times \Omega \) (of course \( K \) is the Bergman projection of the Dirac delta mass, so it analytically continues across the boundary). But then we can conclude that \( \det (\text{Jac}_C \Phi)(z) \) does not vanish. Therefore \( \Phi \) does not branch, so it must be biholomorphic.

\[ \square \]

4 On the Lu Qi-Keng Conjecture

In the paper [JPDA], D’Angelo proved the Lu Qi-Keng conjecture for domains of the form

\[ \Omega_{1,m} = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^{2m} < 1\} \]

where \( m \) is a positive integer. He did so by producing an explicit formula for the Bergman kernel.

We also note that the paper [BFS] treats domains of the form

\[ \{(z_1, z_2, \ldots, z_n) \in \mathbb{C}^n : |z_1|^{2/p_1} + |z_2|^{2/p_2} + \cdots + |z_n|^{2/p_n} < 1\} \]

for the \( p_j \) positive integers. That paper finds domains for which the Lu Qi-Keng conjecture fails.

It has been an open problem to decide the Lu Qi-Keng conjecture for domains of the form

\[ \Omega_{m_1, m_2, \ldots, m_n} = \{(z_1, z_2, \ldots, z_n) \in \mathbb{C}^n : |z_1|^{2m_1} + |z_2|^{2m_2} + \cdots + |z_n|^{2m_n} < 1\}, \]

where \( m_1, m_2, \ldots, m_n \) are positive integers. We do so affirmatively in the present section.

To keep notation simple, we restrict attention to dimension two. So we concentrate on a domain

\[ \Omega_{m,n} = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^{2m} + |z_2|^{2n} < 1\}, \]
for $m, n$ positive integers. Seeking a contradiction, we suppose that, for $j$ a large positive integer, the domain

$$\Omega_{jm,jn} = \{(z_1, z_2) : |z_1|^{2mj} + |z_2|^{2nj} < 1\}$$

fails the Lu Qi-Keng property. Let $K_j$ be the Bergman kernel for this last domain, and suppose that $K_j(z, \zeta) = 0$. Applying a rotation $e^{i\theta}$ in the $z_1$ variable, and using the usual transformation formula for the Bergman kernel (see [KRA1, §1.4]), we see that $K_j(e^{i\theta}z_1, z_2, \ldots, z_n, e^{i\theta}\zeta_1, \zeta_2, \ldots, \zeta_n) = 0$ for all $0 \leq \theta < 2\pi$. Now applying the sub-mean value property for subharmonic functions to $|K_j(e^{i\theta}z_1, z_2, \ldots, z_n, e^{i\theta}\zeta_1, \zeta_2, \ldots, \zeta_n)|$, we conclude that $K_j(0, z_2, \ldots, z_n, 0, \zeta_2, \ldots, \zeta_n) = 0$. We may repeat this argument in the $z_2, z_3 \ldots, z_n$ variables to conclude that $K_j(0, 0) = 0$.

Now we notice that, as $j \to \infty$, the domains $\Omega_{jm,jn}$ converge in the Hausdorff metric on domains to the bidisc $D^2$. By Ramadanov’s theorem (see also [KRA2]), the Bergman kernels on the $\Omega_{jm,jn}$ converge uniformly on compact sets to the Bergman kernel on $D^2$. Hence the Bergman kernel on $D^2$ has zeros. That is a contradiction.

We conclude that, for $j$ large, the Bergman kernel for $\Omega_{jm,jn}$ has no zeros. But now we can apply Bell’s projection formula for the Bergman kernel under a proper holomorphic covering (see, for instance, [BOA]) because $\Omega_{jm,jn}$ covers $\Omega_{j'm,j'n}$ for $j' < j$. And we may conclude that the Bergman kernel for $\Omega_{j'm,j'n}$ has no zeros. Hence the Bergman kernel for $\Omega_{m,n}$ is zero-free for any positive integers $m$ and $n$.

It is easy to see how the proof just presented generalizes to arbitrary $\Omega_{m_1,m_2,\ldots,m_n}$ in any dimension.

### 5 Additional Results

We now have the following result.

**Proposition 5.1** Let $\Omega_1, \Omega_2$ be as in Theorem 2.1. Let $\Phi : \Omega_1 \to \Omega_2$ be biholomorphic. Suppose that $u$ is a function in the Bergman space of $\Omega_2$ that analytically continues past $\partial\Omega_2$. Then $(u \circ \Phi) \cdot \det \text{Jac} \Phi$ analytically continues past $\partial\Omega_1$. 

6
Proof: By our Theorem 2.1, there is a function \( g \in C^\infty_c(\Omega_2) \) such that \( u = P_2 g \). Now we calculate:

\[
\begin{align*}
    u \circ \Phi(z) \cdot \det \text{Jac}_C \Phi(z) &= P_2 g \circ \Phi(z) \cdot \det \text{Jac}_C \Phi(z) \\
    &= \det \text{Jac}_C \Phi(z) \cdot \int_{\Omega_2} K_2(\Phi(z), \zeta) g(\zeta) dV(\zeta) \\
    &= \int_{\Omega_2} K_1(z, \Phi^{-1}(\xi)) g(\zeta) \det \text{Jac}_C \Phi^{-1}(z) \\
    &\quad \cdot \det \text{Jac}_C \Phi(z) \cdot \det \text{Jac}_C \Phi^{-1}(\xi) dV(\zeta) \\
    &= \int_{\Omega_1} K_1(z, \xi) g(\Phi(\xi)) \det \text{Jac}_C \Phi(\xi) dV(\xi) \\
    &= P_1((g \circ \Phi)(\xi) \cdot \det \text{Jac}_C \Phi(\xi))(z).
\end{align*}
\]

Because \( \Phi \) is proper, \( g \circ \Phi \) is \( C^\infty_c \) hence \( (g \circ \Phi) \cdot \det \text{Jac}_C \Phi \) is \( C^\infty_c \). So we see that \( u \circ \Phi(z) \cdot \det \text{Jac}_C \Phi(z) \) is the Bergman projection on \( \Omega_1 \) of a \( C^\infty_c \) function. So it analytically continues past the boundary. \( \square \)

The next result is a consequence of the proof of Theorem 2.1.

**Proposition 5.2** Let \( \Omega_1, \Omega_2 \) be domains as in the hypothesis of Theorem 2.1 and let \( \Phi \) be a biholomorphic mapping of these domains. Then \( \Phi \) and \( \Phi^{-1} \) extend analytically past the boundary of \( \Omega_1 \) and \( \Omega_2 \) respectively.

**Remark 5.3** Theorems 2.1, 3.2, as well as Propositions 5.1, 5.2 apply to domains of the form

\[
\Omega = \{(z_1, z_2, \ldots, z_n) \in \mathbb{C}^n : |z_1|^{2m_1} + |z_2|^{2m_2} + \cdots + |z_n|^{2m_n} < 1\}
\]

for positive integers \( m_1, m_2, \ldots, m_n \).

### 6 Concluding Remarks

Given any function space \( X \) on a domain \( \Omega \), it would be of interest to know which functions have Bergman projection that lies in \( X \). Clearly this set of questions is related to Bell’s Condition \( R \).

We hope to investigate these matters in future work.
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