FACTORY RATIONAL VARIETIES WHICH ADMIT OR FAIL TO ADMIT
AN ELLIPTIC $G_m$-ACTION

GENE FREUDENBURG AND TAKANORI NAGAMINE

ABSTRACT. Over a field $k$, we study rational UFDs of finite transcendence degree $n$ over $k$. We classify such UFDs $B$ when $n = 2$, $k$ is algebraically closed, and $B$ admits a positive $\mathbb{Z}$-grading, showing in particular that $B$ is affine over $k$. We also consider the Russell cubic threefold over $\mathbb{C}$, and the Asanuma threefolds over a field of positive characteristic, showing that these threefolds admit no elliptic $G_m$-action. Finally, we show that, if $X$ is an affine $k$-variety and

$$X \times \mathbb{A}_k^n \cong_k \mathbb{A}_k^{n+m}$$

then $X \cong_k \mathbb{A}_k^n$ if and only if $X$ admits an elliptic $G_m$-action.

1. INTRODUCTION

In his 1977 paper \cite{16}, Mori gives a classification of unique factorization domains (UFDs) which are finitely generated over a field $k$ and which admit a positive $\mathbb{Z}$-grading over $k$. Geometrically, these correspond to factorial affine $G_m$-varieties with elliptic (or good) $G_m$-actions; see Section 3. To each such ring $B$, Mori associates a unique natural number $m$ and a subring $B^{(m)}$ derived from the grading such that $B = B^{(m)}[v^{1/e}]$, where $v$ and $e$ are sequences encoding the ramification data for $B$. The subring $B^{(m)}$ is a UFD defined by a semicomplete polarized $k$-variety $(X,L)$; see Remark 7.4. The algebras $B$ are thus classified by certain semicomplete polarized $k$-varieties $(X,L)$ together with ramification data over the corresponding ring $R(X,L)$. Mori gives an explicit description of all such rings in the case dim$_k B = 2$ and $k$ is algebraically closed.

Let $U_k(n)$ denote the set of $k$-isomorphism classes of UFDs containing $k$ and of transcendence degree $n$ over $k$. Define the following subsets of $U_k(n)$, where $B$ indicates a ring represented in the set.

(1) $U_k(n,A) : B$ is affine over $k$
(2) $U_k(n,D) : B$ admits a positive degree function over $k$
(3) $U_k(n,G) : B$ admits a positive $\mathbb{Z}$-grading over $k$
(4) $U_k(n,P) : B \subset k^{[m]}$ for some integer $m \geq n$
(5) $U_k(n,R) : B$ is rational over $k$

Here, $k^{[m]}$ denotes a polynomial ring in $m$ variables over $k$. Of course, there are other categories of interest, such as noetherian, regular or unirational UFDs, but the foregoing list is of primary interest for this paper.

By a result of Eakin (\cite{6}, Lemma B), definition (4) is equivalent to:

$$(4)' U_k(n,P) : B \subset k^{[n]}$$

Note the containments $U_k(n,G) \subset U_k(n,D)$ and $U_k(n,P) \subset U_k(n,D)$. If $[B]$ denotes the isomorphism class of the ring $B$, then the mapping $[B] \to [B^{[1]}]$ gives an inclusion $U_k(n,* ) \subset U_k(n + 1,*)$ for each of these five properties (*). We use the notation $U_k(n,A,R)$ to denote $U_k(n,A) \cap U_k(n,R)$, etc. In this notation, Mori’s paper describes $U_k(n,A,G)$.

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For $n = 1$ and $k$ algebraically closed, it is known that

$$\mathcal{U}_k(1, A) = \left\{ [k[t]_{f(t)}] \mid f(t) \in k[t] \setminus \{0\} \right\}$$

where $k[t] \cong k^{[1]}$ and $k[t]_{f(t)}$ denotes localization, and:

$$\mathcal{U}_k(1, D) = \mathcal{U}_k(1, G) = \mathcal{U}_k(1, P) = \left\{ [k^{[1]}] \right\}$$

See [9], Lemma 2.9 and Lemma 2.12, and Corollary [2] below.

In Section 3 we consider the family $B(k)$ of two-dimensional affine $k$-domains $B$ defined as follows.

There exist $n \in \mathbb{N}$, pairwise relatively prime integers $a > b > c_1 > \ldots > c_n \geq 2$, and distinct $1 = \lambda_1, \ldots, \lambda_n \in k^*$ such that:

$$B = k[x, y, z_1, \ldots, z_n]/(z^n + \lambda_1 y^b + z_1^{c_1})_{0 \leq i \leq n}$$

If $n = 0$, then $B = k[x, y] \cong k^{[2]}$. If $n = 1$, these are known as factorial Pham-Brieskorn surfaces.

The sequence $a, b, c_1, \ldots, c_n$ is the sequence of ramification indices for $B$.

In [16], Theorem 5.1, Mori shows that $B(k) \subset \mathcal{U}_k(2, A, G)$, with equality in the case $k$ is algebraically closed. Mori’s theorem thus shows $\mathcal{U}_k(2, A, G) \subset \mathcal{U}_k(2, R)$ when $k$ is algebraically closed.

One of our main results is Theorem [5, 7] below, which gives a complete description of $\mathcal{U}_k(2, G, R)$ in the case $k$ is algebraically closed, in particular, showing that $B(k) = \mathcal{U}_k(2, G, R)$. Consequently, $\mathcal{U}_k(2, G, R) \subset \mathcal{U}_k(2, A)$ in this case. Combining this with Mori’s result, we conclude that, when $k$ is algebraically closed:

$$\mathcal{U}_k(2, A, G) = \mathcal{U}_k(2, G, R) = \mathcal{U}_k(2, A, G, R) = B(k)$$

We would like to understand the larger set $\mathcal{U}_k(2, D, R)$, starting with the subset $\mathcal{U}_k(2, P)$. Note that, if $B \in \mathcal{U}_k(2, P)$, then $B$ is affine by Zariski’s Theorem [23], and $B$ is rational by Castelnuovo’s Theorem [2]. Therefore, $\mathcal{U}_k(2, P) = \mathcal{U}_k(2, A, P, R)$.

One motivation to study $\mathcal{U}_k(n, P)$ is the fact that, if $A$ is the ring of invariants for a $\mathbb{G}_a$-action on the affine space $\mathbb{A}^{n+1}_k$, then $A \in \mathcal{U}_k(n, P)$, whereas $A \notin \mathcal{U}_k(n, A)$ in general. For $n = 2$, it known that $A \cong k^{[2]}$ when the characteristic of $k$ is zero (Miyanishi’s Theorem [15]), but it is an open question whether this generalizes to all fields. For $n = 3$, if $A$ is the ring of invariants for a $\mathbb{G}_a$-action on $\mathbb{A}^2_k$ and the characteristic of $k$ is zero, then $A \in \mathcal{U}_k(3, P, R)$ (rationality is due to Deveney and Finston [4]), but it is not known if $A \in \mathcal{U}_k(3, A)$. If the $\mathbb{G}_a$-action is homogeneous for a positive $\mathbb{Z}$-grading, then $A \in \mathcal{U}_k(3, G, P, R)$, but even here we do not know if $A$ is affine.

The main tool in our proof of Theorem [5, 7] is the theory of signature sequences, which is developed in Section 4. Signature sequences are defined for any pair $(B, \deg)$, where $B$ is an integral $k$-domain and $\deg$ is a non-negative degree function on $B$, but they have especially strong properties when $B$ is a UFD and $\deg$ is positive. Section 2 introduces certain criteria for a ring to be a UFD, and Section 3 discusses degree functions and gradings.

When $X$ is a smooth affine variety over $\mathbb{C}$, then $X$ is a topological manifold, and the existence of an elliptic $\mathbb{C}^*$-action on $X$ is a strong form of contractibility: In this case, the $\mathbb{C}^*$-action has a unique (attractive) fixed point $x_0 \in X$. If the action is given by $\lambda x$ ($\lambda \in \mathbb{C}^*, x \in X$), then since all the weights of the action are positive integers, restriction to the real interval $t \in (0, 1]$ yields:

$$\lim_{t \to 0^+} (t x) = x_0 \quad \forall x \in X$$

So the requisite contracting homotopy is given by $F : X \times [0, 1] \to X$, where

$$F(x, t) = \begin{cases} t x & (t \neq 0) \\ x_0 & (t = 0) \end{cases}$$

More precisely, the function $B(k) \to \mathcal{U}_k(2, A, G)$ mapping $B$ to $[B]$ is injective, and when $k$ is algebraically closed, it is also surjective.
A well-known theorem of Ramanujam [18] says that a smooth affine surface over $\mathbb{C}$ which is contractible and simply connected at infinity is isomorphic to $\mathbb{C}^2$. This can be used to show that any smooth affine surface over $\mathbb{C}$ with an elliptic $\mathbb{C}^*$-action is isomorphic to $\mathbb{C}^2$; see [8].

In the same paper, Ramanujam showed that any smooth contractible affine variety over $\mathbb{C}$ of dimension $n \geq 3$ is diffeomorphic to $\mathbb{R}^{2n}$, and is therefore either isomorphic to $\mathbb{C}^n$ or an exotic structure on $\mathbb{C}^n$. A well-known example of this phenomenon is the Russell cubic threefold $X$, which is discussed in Section 2. For the coordinate ring $B$ of $X$, it is known that $B \in \mathcal{U}_\mathbb{C}(3, \mathbf{A}, \mathbf{R})$, that $X$ is smooth and contractible, and that $X \not\cong \mathbb{C}^3$. So $X$ is an exotic structure on $\mathbb{C}^3$. In Theorem 6.2 we show that $X$ does not have the stronger form of contractibility imposed by an elliptic $\mathbb{C}^*$-action, i.e., $B \not\in \mathcal{U}_\mathbb{C}(3, \mathbf{G})$.

Similarly, we consider the Asanuma threefolds over a field of positive characteristic, showing that these also do not admit an elliptic $\mathbb{G}_m$-action (Corollary 6.3). This result is a consequence of Theorem 6.4 which highlights the role of elliptic $\mathbb{G}_m$-actions:

For any field $k$ and positive integers $n, m$, let $X$ be an affine $k$-variety such that $X \times \mathbb{A}_k^n \cong_k \mathbb{A}_k^{n+m}$. Then $X \cong \mathbb{A}_k^n$ if and only if $X$ admits an elliptic $\mathbb{G}_m$-action.

In one direction, the condition $X \times \mathbb{C}^m \cong \mathbb{C}^{n+m}$ ensures that $X$ is smooth, affine and contractible, but does not imply the stronger condition $\mathbb{C}[X] \in \mathcal{U}_\mathbb{C}(n, \mathbf{A}, \mathbf{G})$. In the other direction, if $B \in \mathcal{U}_\mathbb{C}(n, \mathbf{A}, \mathbf{G})$ and $X = \text{Spec}(B)$ is smooth, then either $X \cong \mathbb{C}^n$ or $X$ is an exotic structure on $\mathbb{C}^n$. In Section 7 we conjecture the following characterization of affine space:

Let $k$ be an algebraically closed field, and let $X$ be a factorial rational affine $k$-variety of dimension $n$. If $X$ is smooth and admits an elliptic $\mathbb{G}_m$-action, then $X \cong_k \mathbb{A}_k^n$.

The conjecture is true for $n = 1$ and $n = 2$.

Preliminaries. For the integral domain $B$ and integer $n \geq 0$, $B^n$ is the group of units of $B$ and $B^{[n]}$ is the polynomial ring in $n$ variables over $B$. If $K$ is a field, then $K^{[n]}$ denotes the field of fractions of $K^n$. For a ground field $k$, affine space $n$-space over $k$ is denoted by $\mathbb{A}_k^n$; $\mathbb{G}_a$ is the additive group of $k$, and $\mathbb{G}_m$ the multiplicative group of $k^*$. If $B$ is a $k$-algebra, the Makar-Limanov invariant $ML(B)$ of $B$ is the intersection of all invariant rings of $\mathbb{G}_a$-actions on $B$, and the Derksen invariant $D(B)$ of $B$ is the subring generated by invariants of non-trivial $\mathbb{G}_a$-actions. $B$ is rigid if $ML(B) = B$, and stably rigid if $ML(B^{[n]}) = B$ for every $n \geq 0$. See [9] for details.

2. Criteria for a Ring to be a UFD

Let $A$ be an integral domain. It is well-known that, if $A$ is a UFD, then every localization of $A$ is a UFD. A partial converse is given by Nagata in [17], Lemma 2.

Theorem 2.1. (Nagata’s Criterion) Let $A$ be a noetherian integral domain and $S \subset A \setminus \{0\}$ a multiplicatively closed set generated by a set of prime elements of $A$. Then $A$ is a UFD if and only if $S^{-1}A$ is a UFD.

The main purpose of this section is to introduce two additional criteria for a ring to be a UFD.

2.1. Integral Extensions. The following result generalizes Samuel [21], Theorem 8.1.

Theorem 2.2. Let $A = \bigoplus_{c \in \mathbb{Z}} A_c$ be a $\mathbb{Z}$-graded integral domain which is finitely generated as an $A_0$-algebra, and let $F \in A_\omega \setminus \{0\}$, $\omega \in \mathbb{Z}$. Define $B = A[Z]/(Z^c - F)$, where $A[Z] = A^{[1]}$, $c \in \mathbb{N}$ and $\text{gcd}(c, \omega) = 1$.

(a) $B$ is an integral domain and $\text{frac}(A) \cong_K \text{frac}(B)$, where $K = \text{frac}(A_0)$.

(b) If $F$ is prime in $A$, then $z$ is prime in $B$, where $z = \pi(Z)$ for the surjection $\pi : A[Z] \to B$.

(c) If $A$ is noetherian and $F$ is prime in $A$, then $A$ is a UFD if and only if $B$ is a UFD.

Proof. Let $x_1, \ldots, x_n \in A$ be such that $A = A_0[x_1, \ldots, x_n]$, and let $F = P(x_1, \ldots, x_n)$ for $P \in A_0^{[n]}$. Set $\omega_i = \deg x_i$, $1 \leq i \leq n$. 

3
Consider first the case $c = d\omega \pm 1$ for some $d \in \mathbb{N}$. Define an $A_0$-automorphism $\varphi$ of $A[Z, Z^{-1}]$ by
\[x'_i = \varphi(x_i) = Z^{-d\omega}x_i \quad (1 \leq i \leq n) \quad \text{and} \quad \varphi(Z) = Z\]
and define $\tilde{F} = \varphi(F) = P(x'_1, \ldots, x'_n)$. Set $A' = \varphi(A) = A_0[x'_1, \ldots, x'_n]$. We have:
\[
\begin{align*}
Z^c - F &= Z^c - P(Z^{d\omega}x'_1, \ldots, Z^{d\omega}x'_n) \\
&= Z^c - Z^{d\omega}P(x'_1, \ldots, x'_n) \\
&= Z^{d\omega}(Z^{\pm 1} - \tilde{F})
\end{align*}
\]
Therefore:
\[
B[z^{-1}] = A[Z, Z^{-1}]/(Z^c - F) = A'[Z, Z^{-1}]/(Z^{\pm 1} - \tilde{F}) = A'[	ilde{F}, \tilde{F}^{-1}]
\]
It follows that $B$ is an integral domain and $\text{frac}(B) = \text{frac}(A') \cong_K \text{frac}(A)$. So statement (a) holds in this case.

In general, there exist $j, d \in \mathbb{Z}$ with $j \geq 0$ such that $jc = d\omega \pm 1$. Consider the ring
\[
R := A[T]/(T^{jc} - F) = A[t]
\]
where $A[T] = A^{[1]}$ and $t = p(T)$ for the canonical surjection $p : A[T] \to R$. By what was shown above, $R$ is an integral domain and $\text{frac}(A) \cong_K \text{frac}(R)$.

For the subring $A[t^j] \subset R$ we have:
\[
A[t^j] = A[Z]/(Z^c - F) = B = A[z] \quad \Rightarrow \quad R = B[S]/(S^j - z) = B[t]
\]
where $B[S] = B^{[1]}$. Therefore, $B$ is an integral domain. Let $\psi$ be an $A$-automorphism of the localization $B[S, S^{-1}]$ defined by:
\[
\tilde{z} = \psi(z) = S^{1-j}z \quad \text{and} \quad \psi(S) = S
\]
Set $B' = \psi(B) = A[\tilde{z}]$. Then $B' \cong_A B$ and $B[S, S^{-1}] = B'[S, S^{-1}]$. In addition:
\[
S^j - z = S^j - S^{j-1}\tilde{z} = S^{j-1}(S - \tilde{z})
\]
Therefore:
\[
R[t^{-1}] = B[t, t^{-1}] = B[S, S^{-1}]/(S^j - z) = B'[S, S^{-1}]/(S - \tilde{z}) = B'[\tilde{z}, \tilde{z}^{-1}]
\]
Consequently, $\text{frac}(A) \cong_K \text{frac}(R) = \text{frac}(B') \cong_K \text{frac}(B)$. This completes the proof for part (a).

For part (b), assume that $F$ is prime in $A$. Since $B/zB \cong A/F A$, $z$ is prime in $B$.

For part (c), assume that $A$ is noetherian and $F$ is prime in $A$. Since $z$ is a prime element of $B$, $\tilde{z}$ is a prime element of $B'$.

Consider first the case $c = d\omega \pm 1$ for some $d \in \mathbb{N}$. If $A$ is a UFD, then $A'$ is a UFD, as is $A'[	ilde{F}, \tilde{F}^{-1}] = B[z^{-1}]$. Since $z \in B$ is prime, it follows by Nagata’s criterion that $B$ is a UFD. Conversely, assume that $B$ is a UFD. Then the localization $B[z^{-1}] = A'[	ilde{F}, \tilde{F}^{-1}]$ is a UFD. Since $\tilde{F}$ is prime in $A'$, it follows by Nagata’s criterion that $A'$, hence $A$, is a UFD. So statement (c) holds in this case.

In general, assume $j, d \in \mathbb{Z}$, $j \geq 0$, are such that $jc = d\omega \pm 1$. For the ring $R$ as above, we have shown that $t$ is prime in $R$, and that $R$ is a UFD if and only if $A$ is a UFD.

If $B$ is a UFD, then $B'$ is a UFD, as is $B'[	ilde{z}, \tilde{z}^{-1}] = R[t^{-1}]$. Since $t \in R$ is prime, it follows by Nagata’s criterion that $R$ is a UFD.

Conversely, assume that $R$ is a UFD. Then the localization $R[t^{-1}] = B'[	ilde{z}, \tilde{z}^{-1}]$ is a UFD. Since $\tilde{z}$ is prime in $B'$, it follows by Nagata’s criterion that $B'$, hence $B$, is a UFD.

We have thus shown: $A$ is a UFD if and only if $R$ is a UFD if and only if $B$ is a UFD. So statement (c) is true in the general case. \(\square\)

Note that, although $\text{frac}(A) \cong_K \text{frac}(B)$ in the theorem above, the inclusion $A \subset B$ is not birational if $c \geq 2$. 

4
Let \( \mathfrak{g} \) be the \( \mathbb{Z} \)-grading of \( A \) in Theorem 2.2. Extend the \( \mathbb{Z} \)-grading \( c \mathfrak{g} \) of \( A \) to a \( \mathbb{Z} \)-grading \( \mathfrak{g}' \) of \( A[Z] \) by letting \( Z \) be homogeneous and:

\[
\deg_{\mathfrak{g}'} Z = \deg_{\mathfrak{g}} f = \omega
\]

Then \( Z^c - F \) is homogeneous and the quotient \( B \) has the \( \mathbb{Z} \)-grading induced by \( \mathfrak{g}' \).

### 2.2. Affine Modifications of UFDs

If \( A \) is an integral domain, \( I \subset A \) is an ideal, and \( f \in I \) is nonzero, then the **affine modification** of \( A \) along \( f \) with center \( I \) is the subring of the localization \( A_f \) defined by:

\[
A[f^{-1}I] = A[a/f \mid a \in I]
\]

The reader is referred to [14] for the theory of affine modifications.

The following result generalizes Nagata [17], Theorem 1.

**Theorem 2.3.** Let \( A \) be a noetherian UFD, \( I \subset A \) an ideal, and \( f \in I \). Assume that there exist \( a_1, \ldots, a_n \in A \) such that:

1. \( I = (f, a_1, \ldots, a_n) \)
2. \( \gcd(f, a_1, \ldots, a_n) = 1 \)
3. \( (p, a_1, \ldots, a_n) \) is a prime ideal of \( A \) for every prime divisor \( p \in A \) of \( f \)

Then \( A[f^{-1}I] \) is a UFD. Moreover, any \( \mathbb{Z} \)-grading of \( A \) for which \( f, a_1, \ldots, a_n \) are homogeneous extends to a \( \mathbb{Z} \)-grading of \( A[f^{-1}I] \).

**Proof.** Let \( B = A[f^{-1}I] = A[a_1/f, \ldots, a_n/f] \) and \( A[Z_1, \ldots, Z_n] = A^n \). Since \( \gcd(f, a_i) = 1 \) for each \( i \), the ring

\[
A[Z_1, \ldots, Z_n]/(fZ_1 - a_1, \ldots, fZ_n - a_n)
\]

is an integral domain isomorphic to \( B \). Let \( p \in A \) be a prime divisor of \( f \). Then:

\[
B/pB \cong A[Z_1, \ldots, Z_n]/(fZ_1 - a_1, \ldots, fZ_n - a_n, p)
\]

\[
\cong A/(a_1, \ldots, a_n, p)[Z_1, \ldots, Z_n]
\]

\[
\cong A/(a_1, \ldots, a_n, p)[a^n]
\]

Since \( a_1 A + \cdots + a_n A + pA \) is a prime ideal of \( A \), \( pB \) is a prime ideal of \( B \).

Let \( S \subset A \) be the multiplicatively closed set generated by the prime divisors of \( f \). We have:

\[
B = A[a_1/f, \ldots, a_n/f] \subset S^{-1}A = S^{-1}B
\]

Since \( A \) is a UFD, \( S^{-1}A = S^{-1}B \) is a UFD. By Nagata’s criterion, \( B \) is a UFD.

Assume that \( A \) has \( \mathbb{Z} \)-grading \( \mathfrak{g} \). Extend \( \mathfrak{g} \) to a \( \mathbb{Z} \)-grading \( \mathfrak{g}' \) of \( A[Z_1, \ldots, Z_n] \) by letting \( Z_i \) be homogeneous with:

\[
\deg_{\mathfrak{g}'} Z_i = \deg_{\mathfrak{g}} a_i - \deg_{\mathfrak{g}} f
\]

Then \( (fZ_1 - a_1, \ldots, fZ_n - a_n) \) is a homogeneous ideal, and the quotient \( A[f^{-1}I] \) has the \( \mathbb{Z} \)-grading induced by \( \mathfrak{g}' \). \( \square \)

### 2.3. An Application

The following lemma generalizes Lemma 2 in [8].

**Lemma 2.4.** Let \( A_0 \) be an integral domain. Given the integer \( n \geq 0 \), let

\[
R_n = A_0[z_0, \ldots, z_n] \cong A_0^{[n+1]}
\]

and let \( a_1, \ldots, a_n, b_1, \ldots, b_n \) be positive integers such that \( \gcd(a_i, b_1 \cdots b_i) = 1 \) for each \( i \). The ideal

\[
I_n = (z_1^{a_1} + z_0^{b_1}, \ldots, z_n^{a_n} + z_{n-1}^{b_n})
\]

is a prime ideal of \( R_n \).
Proof. We proceed by induction on \( n \), the case \( n = 0 \) being clear: \( I_0 = \{0\} \).

Assume, for some \( n \geq 1 \), that \( I_{n-1} \) is a prime ideal of \( R_{n-1} \). Define a \( \mathbb{Z} \)-grading of \( R_{n-1} \) over \( A_0 \) for which \( z_i \) is homogeneous of degree \( b_i \cdots b_{a_i+1} \cdots a_{n-1} \), \( 0 \leq i \leq n-1 \). Then the quotient ring \( A := R_{n-1}/I_{n-1} \) is a \( \mathbb{Z} \)-graded integral domain which is finitely generated over \( A_0 \).

Let \( F \in A \) be the image of \( z_m^{b_m} \), noting that \( \deg F = b_1 \cdots b_n \). By hypothesis, \( \gcd(a_m, \deg F) = 1 \). Therefore, by Theorem 2.2(a), the ring \( A[Z]/(Z^n - F) \cong R_n/I_n \) is an integral domain.

It follows by induction that \( I_n \) is a prime ideal of \( R_n \) for each integer \( n \geq 0 \). \( \Box \)

Theorem 2.5. Let \( K \) be a noetherian UFD. Given the integer \( n \geq 0 \), let \( K[z_0, \ldots, z_{n+1}] = K^{[n+2]} \) and let \( a_1, \ldots, a_n, b_1, \ldots, b_n \) be positive integers such that \( \gcd(a_i, b_i) = 1 \) for each \( i \). Given nonzero \( f \in K \), the ring

\[
A_n := K[z_0, \ldots, z_{n+1}]/(f z_{i+1}^{a_i} + z_i^{b_i})_{0 \leq i \leq n}
\]

is a UFD whose field of fractions equals \( \text{frac}(K[z_0, z_1]) \cong (\text{frac} K)^{(2)} \).

Proof. If \( f \in K^* \), then \( A_n \cong K^{[2]} \) is a UFD. So assume that \( f \) is not a unit of \( K \).

We proceed by induction on \( n \), the case \( n = 0 \) being clear. Note that each ring \( A_m \) is noetherian, \( 0 \leq m \leq n \). Given \( m \geq 1 \), assume that \( A_{m-1} \) is a UFD. Let \( p \in K \) be a prime divisor of \( f \). Then

\[
A_{m-1}/(p, z_{m+1}^{a_m} + z_{m-1}^{b_m}) \cong (K/pK)[Z_0, \ldots, Z_m]/(Z_1^{a_1} + Z_0^{b_0}, \ldots, Z_m^{a_m} + Z_{m-1}^{b_{m-1}})
\]

where \( (K/pK)[Z_0, \ldots, Z_m] \cong (K/pK)^{[m+1]} \). By Lemma 2.4, \( (p, z_{m+1}^{a_m} + z_{m-1}^{b_m}) \) is a prime ideal of \( A_{m-1} \). Define the ideal \( I \subset A_{m-1} \) by \( I = (f, z_{m+1}^{a_m} + z_{m-1}^{b_m}) \). Since \( A_m \cong A_{m-1}/I \), it follows by Theorem 2.3 that \( A_m \) is a UFD.

Therefore, by induction \( A_n \) is a UFD. Since affine modifications preserve quotient fields, we see that \( \text{frac}(A_n) = \text{frac}(A_0) = \text{frac}(K[z_0, z_1]) \cong (\text{frac} K)^{(2)} \). \( \Box \)

Rings of the type described in this theorem are considered in Section 6 where \( K = k^{[1]} \) for a field \( k \).

3. Degree Functions, \( G \)-Gradings and \( \mathbb{G}_m \)-Actions

An abelian group \( G \) is totally ordered if \( G \) has a total order \( \leq \) which is translation invariant:

For all \( x, y, z \in G \), \( x + z \leq y + z \) implies \( x \leq y \).

3.1. Degree Functions. Assume that \( G \) is a totally ordered abelian group, and that \( B \) is an integral domain with degree function \( \deg : B \to G \cup \{-\infty\} \). We say the \( \deg \) has values in \( G \). The induced filtration is

\[
B = \bigcup_{g \in G} F_g
\]

where the sets \( F_g = \{ b \in B \mid \deg b \leq g \} \) are the associated degree modules. The associated degree submodules are:

\[
V_g = \{ f \in B \mid \deg f < g \} \subset F_g
\]

Note that \( \deg \) can be extended to \( K = \text{frac}(B) \) by letting \( \deg(f/g) = \deg f - \deg g \) for \( f, g \in B \), \( g \neq 0 \). Note also that, if \( B \) is a field, then \( \deg \) is a degree function on \( B \) if and only if \( (-\deg) \) is a valuation of \( B \).

Definition 3.1. \( \deg \) is non-negative if \( V_0 = \{0\} \).

Proposition 3.2. With the assumptions and notation above:

(a) \( F_0 \) is a subring of \( B \) which is integrally closed in \( B \).
(b) \( F_g \) is an ideal of \( F_0 \) for each \( g \leq 0 \).
(c) \( F_g \) is an \( F_0 \)-module for each \( g \in G \), and \( V_g \) is a submodule.
(d) If \( \deg \) is non-negative, then \( F_0 \) is factorially closed in \( B \) and \( B^* \subset F_0 \).
(e) If \( \deg \) is non-negative and \( B \) is a UFD, then \( F_0 \) is a UFD.
(f) If $B$ is a normal ring, then $F_0$ is a normal ring.
(g) If $B$ is a field, then $F_0$ is a valuation ring of $B$ and $\text{frac}(F_0) = B$.

Proof. Extend deg to $K = \text{frac}(B)$ and let $V = \{ f \in K \mid \deg f \leq 0 \}$. Then $V$ is a valuation ring of $K$, and $F_0 = V \cap B$. This proves parts (a), (f) and (g). Proofs for statements (b)-(e) are left to the reader. □

3.2. $k$-Algebras. Suppose that $B$ is an integral $k$-domain for a ground field $k$. deg is a degree function over $k$ if $\text{deg}(k^*) = \{0\}$. Hereafter, any degree function on $B$ is assumed to be over $k$ when $k$ is the ground field. In this case, each degree module $F_g$ is a $k$-vector space, and the associated degree submodule $V_g$ is a subspace of $F_g$. Let $W_g$ be a complementary subspace, that is:

$$F_g = V_g \oplus W_g$$

Then $\deg b = g$ for every nonzero $b \in W_g$.

Definition 3.3. Let $\deg$ be a degree function on $B$ with values in $G$.

(1) $\deg$ is positive if it is non-negative and $F_0 = k$.
(2) $\deg$ is of finite type if $\dim_k F_g < \infty$ for each $g \in G$.

Note that these properties are preserved under restriction: If $A \subset B$ is a $k$-subalgebra, then the degree function $\deg|_A$ on $A$ is non-negative (respectively, positive, of finite type) if $\deg$ is non-negative (respectively, positive, of finite type).

Lemma 3.4. If $\deg$ is of finite type, then $\deg$ is non-negative.

Proof. Given $f \in F_g$ for $g < 0$, we have:

$$fk[f] \subset F_g \implies \dim_k fk[f] < \infty \implies \dim_k k[f] < \infty$$

We conclude that $k[f]$ is a field. If $f \neq 0$, then $f \in k[f]^*$. But then

$$f^{-1} \in k[f] \subset F_0 \quad \text{and} \quad \deg f^{-1} > 0$$

which is a contradiction. Therefore, $f = 0$. □

3.3. $G$-Gradings. Let $B$ be an integral $k$-domain and $G$ an abelian group (not necessarily torsion free). Let $g$ be a $G$-grading of $B$ over $k$:

$$B = \bigoplus_{g \in G} B_g, \ k \subset B_0$$

If $G$ is torsion free, then any choice of total order on $G$ gives a degree function $\deg_g$ on $B$. In this case, given $f \in B$, $\bar{f}$ will denote the highest-degree homogeneous summand of $f$.

Definition 3.5. Under the above hypotheses:

(1) $g$ is non-negative if $B_g = \{0\}$ for $g < 0$.
(2) $g$ is positive if it is non-negative and $B_0 = k$.
(3) $g$ is of finite type if $\dim_k B_g < \infty$ for each $g \in G$.

These properties are preserved under restriction to graded subalgebras: If $A \subset B$ is a graded $k$-subalgebra, then the induced grading

$$A = \bigoplus_{g \in G} A_g, \ A_g = B \cap B_g$$

of $A$ is non-negative (respectively, positive, of finite type) if $g$ is non-negative (respectively, positive, of finite type).

Note also that, if $G$ is totally ordered, and if $g$ is non-negative (respectively, positive), then $\deg_g$ is non-negative (respectively, positive). Thus, for any $k$-subalgebra $A$ of $B$, if $g$ is non-negative (respectively, positive), then $\deg_g$ restricts to a non-negative (respectively, positive) degree function on $A$. 7
However, it can happen that \( g \) is of finite type, while \( \deg_g \) is not. For example, if \( B = k[x, x^{-1}] \), the ring of Laurent polynomials with the standard \( \mathbb{Z} \)-grading, then the grading is of finite type, but the associated degree function is not non-negative, and therefore not of finite type. However, if \( g \) is non-negative and of finite type, then \( \deg_g \) is of finite type.

**Lemma 3.6.** If \( g \) is positive, then \( B^* = k^* \).

**Proof.** Assume \( g \) is positive, and let \( u \in B^* \). If \( \deg_g u > 0 \), then \( \deg_g (u^{-1}) < 0 \), which is impossible, since \( \deg b < 0 \) implies \( b = 0 \). Therefore, \( u \in \mathcal{F}_0 = B_0 = k \). \( \square \)

### 3.4. \( \mathbb{G}_m \)-Actions

Assume that \( B \) is an affine \( k \)-domain and set \( X = \text{Spec}(B) \). Let

\[
\rho(g) : \mathbb{G}_m \to \text{Aut}_k(X)
\]

be the \( \mathbb{G}_m \)-action of \( X \) induced by the nonzero \( \mathbb{Z} \)-grading \( g \) of \( B \). Recall the following definitions.

1. \( \rho(g) \) is **effective** if \( \gcd\{\deg_g f \mid f \in B \setminus \{0\}\} = 1 \).
2. \( \rho(g) \) is **elliptic** if either \( g \) or \(-g\) is positive.
3. \( \rho(g) \) is **parabolic** if either \( g \) or \(-g\) is non-negative, but not positive.
4. \( \rho(g) \) is **hyperbolic** if it is neither elliptic nor parabolic.
5. \( \rho(g) \) is **good** if it is both elliptic and effective.

Note that any \( \mathbb{G}_m \)-action on \( X \) is of the form \( \rho(cg) \) for some \( \mathbb{Z} \)-grading \( g \) where \( \rho(g) \) is effective. Two \( \mathbb{G}_m \)-actions \( \sigma, \tau \) on \( X \) are **equivalent** if there exists a \( \mathbb{Z} \)-grading \( g \) of \( B \) and nonzero \( c, d \in \mathbb{Z} \) such that \( \sigma = \rho(cg) \) and \( \tau = \rho(dg) \). In particular, the equivalence class of any nontrivial \( \mathbb{G}_m \)-action contains exactly two effective members, being of the form \( \rho(\pm g) \).

The following result is needed in Section 3 and is due to Flenner and Zaidenberg; see [7], Theorem 3.3.

**Theorem 3.7.** Let \( X \) be a normal affine surface over \( \mathbb{C} \). If \( \mathbb{C}[X] \) is rigid and \( X \not\cong \mathbb{C}^* \times \mathbb{C}^* \), then all \( \mathbb{C}^* \)-actions on \( X \) are equivalent.

### 4. Signature Sequences for Non-Negative Degree Functions

#### 4.1. Definition and Basic Properties

Let \( k \) be a field, \( B \) an integral \( k \)-domain, \( G \) a totally ordered abelian group, and \( \deg : B \to G \cup \{-\infty\} \) a non-negative degree function with filtration:

\[
B = \bigcup_{g \in G} \mathcal{F}_g
\]

**Definition 4.1.** A signature sequence \( \vec{h} = \{h_i\}_{i \in I} \) for \((B, \deg)\) is a sequence \( h_i \in B \) indexed by an interval \( 0 \in I \subset \mathbb{N} \) such that:

1. \( h_0 = 1 \)
2. For each \( n \in I \) with \( n \geq 1 \), \( h_n \in \mathcal{F}_{d_n} \setminus k[h_1, \ldots, h_{n-1}] \) where:

\[
d_n = \min\{g \in G \mid \mathcal{F}_g \not\subseteq k[h_1, \ldots, h_{n-1}]\}
\]

The length of \( \vec{h} \) is \( |\vec{h}| \), and \( \vec{h} \) is **finite** or **infinite** depending on \( |\vec{h}| \). \( \vec{h} \) is **complete** if \( B = k[\vec{h}] \).

Note that the degree sequence \( \{d_n\} \subset G \) has \( d_n \leq d_{n+1} \). In addition, for \( n \leq |\vec{h}| \), the subsequence \( \{h_0, \ldots, h_n\} \) is a signature sequence.

In case the degree function is of the form \( \deg_g \) for some \( G \)-grading \( g \) of \( B \), we say that \( \vec{h} \) is a **homogeneous** signature sequence if each \( h_n \) is homogeneous.

By Lemma 3.3 if a degree function \( \deg \) on \( B \) is of finite type, then it is non-negative. So signature sequences can be formed for any pair \((B, \deg)\) for which \( \deg \) is of finite type.

**Lemma 4.2.** If \( \deg \) is a degree function on \( B \) of finite type, then \((B, \deg)\) admits a complete signature sequence. If \( B \) is of finite type over \( k \), then \((B, \deg)\) admits a complete signature sequence which is finite.
Proof. There are two cases to consider.

Case 1: There exists a complete finite signature sequence for \((B, \deg)\).

Case 2: There is no complete finite signature sequence for \((B, \deg)\). In this case, any finite signature sequence \(\{h_0, \ldots, h_n\}\) can be extended, that is,

\[
d := \min \{g \in G \mid F_g \not\subset k[h_1, \ldots, h_n]\}\]

exists, and we can choose \(h_{n+1} \in F_d \setminus k[h_1, \ldots, h_n]\). By induction, there exists an infinite signature sequence \(\vec{h}\). Since \(\dim_k F_g < \infty\) for each \(g\), it follows that, given \(g \in G\):

\[
F_g \subset k[h_1, \ldots, h_n] \quad \text{for} \quad n > 0
\]

Therefore, \(B = k[\vec{h}]\) and \(\vec{h}\) is complete. \(\square\)

Let \(\vec{h}\) be a signature sequence of length \(L\) for the pair \((B, \deg)\), with degree sequence \(d_i\). Define subgroups \(H_i, H \subset G\) by:

\[
H_i = (d_1, \ldots, d_i), 1 \leq i \leq L, \quad \text{and} \quad H = \deg(B \setminus \{0\})
\]

**Proposition 4.3.** Let \(\vec{h}\) be a signature sequence for \((B, \deg)\) of length at least \(n\).

(a) If \(b \in B\) and \(\deg b < d_n\), then \(b \in k[h_1, \ldots, h_{n-1}]\).

(b) Given \(g \in H_{n-1}\), write \(F_g = V_g \oplus W_g\). If \(g \leq d_n\), then:

\[
W_g \cap k[h_1, \ldots, h_{n-1}] \neq \{0\}
\]

**Proof.** Assume \(f \in B \setminus k[h_1, \ldots, h_{n-1}]\), and set \(g = \deg f\). Then:

\[
f \in F_g \setminus k[h_1, \ldots, h_{n-1}] \implies g \geq d_n
\]

This proves part (a).

For part (b), since \(g \in H_{n-1}\), there exist \(c_1, \ldots, c_{n-1} \in \mathbb{N}\) such that \(\deg(h_1^{c_1} \cdots h_{n-1}^{c_{n-1}}) = g\). Therefore, there exist \(v \in V_g\) and nonzero \(w \in W_g\) such that \(h_1^{c_1} \cdots h_{n-1}^{c_{n-1}} = v + w\). Consequently:

\[
w - h_1^{c_1} \cdots h_{n-1}^{c_{n-1}} = v \in V_g \implies \deg(w - h_1^{c_1} \cdots h_{n-1}^{c_{n-1}}) < g \leq d_n
\]

Part (a) implies \(w - h_1^{c_1} \cdots h_{n-1}^{c_{n-1}} \in k[h_1, \ldots, h_{n-1}]\), so \(w \in k[h_1, \ldots, h_{n-1}]\). This proves part (b). \(\square\)

**Corollary 4.4.** Let \(\vec{h}\) be a signature sequence for \((B, \deg)\) of length at least \(n\). If \(\deg\) is positive, then \(h_n + b\) is irreducible in \(B\) whenever \(\deg b < \deg h_n\).

**Proof.** Assume that \(h_n + b = uv\) for \(u, v \in B\). If \(\deg u < d_n\) and \(\deg v < d_n\), then by Proposition \(4.3(a)\) we have:

\[
u, v, b \in k[h_1, \ldots, h_{n-1}] \implies uv = h_n + b \in k[h_1, \ldots, h_{n-1}] \implies h_n \in k[h_1, \ldots, h_{n-1}]
\]

a contradiction. Therefore, either \(\deg u \geq d_n\) or \(\deg v \geq d_n\). Assume that \(\deg u \geq d_n\). Then:

\[
d_n \leq d_n + \deg v \leq \deg u + \deg v = \deg(h_n + b) = d_n \implies \deg v = 0 \implies v \in k^*
\]

Likewise, \(u \in k^*\) if \(\deg v \geq d_n\). \(\square\)

Note that, if \(\vec{h}\) is a homogeneous signature sequence for a positive \(G\)-grading, this corollary implies that any \(\beta \in B\) with \(\beta = h_n\) is irreducible and \(\beta - h_n \in k[h_1, \ldots, h_{n-1}]\), where \(\beta\) denotes the highest degree homogeneous summand of \(\beta\).
4.2. Signature Sequences in UFDs. In this section, assume that the field $k$ is algebraically closed.

**Theorem 4.5.** Let $B$ be a UFD over $k$ with a positive degree function $\text{deg}$. Assume that $\vec{h}$ is a signature sequence for $(B, \text{deg})$. Given $h_n \in \vec{h}$ and $b \in B$ with $\text{deg} b < \text{deg} h_n$, $k[h_n + b]$ is factorially closed in $B$. Consequently, $h_n + b$ is a prime element of $B$.

**Proof.** We may assume $n \geq 1$. Suppose that $uv \in k[h_n + b]$ for $u, v \in B \setminus k$, and let $h_n + b - \lambda$ be a divisor of $uv$, where $\lambda \in k$. By Corollary 4.4, $h_n + b - \lambda$ is irreducible in $B$, and therefore prime in $B$. It follows that every prime factor of $u$ (respectively, $v$) is of the form $h_n + b - \lambda$ for some $\lambda \in k$. Therefore, $u \in k[h_n + b]$ (respectively, $v \in k[h_n + b]$).

Note that this result means that every term $h_i$ of the signature sequence $\vec{h}$ is prime.

**Corollary 4.6.** Let $B$ be a UFD over $k$ with a positive degree function $\text{deg}$. Assume that $\vec{h}$ is a signature sequence for $(B, \text{deg})$. Given $h_m, h_n \in \vec{h}$ with $0 < m < n$, $k[h_m, h_n] \cong k[2]$. 

**Proof.** By Theorem 4.5, $k[h_m]$ is factorially closed in $B$, hence algebraically closed in $B$. Since $h_n \notin k[h_m]$, it follows that $h_n$ is transcendental over $k[h_m]$. Since $m > 0$, $k[h_m] \cong k[1]$. Therefore, $k[h_m, h_n] \cong k[2]$.

**Corollary 4.7.** $U_k(1, G) = U_k(1, D) = \{k[1]\}$

**Proof.** Let $B \in U_k(1, D)$ and let $\text{deg}$ be a positive degree function on $B$. Since $\text{tr.deg}_{k} B = 1$, there exists a signature sequence $\vec{h}$ for $(B, \text{deg})$ of length at least one. By Theorem 4.5, $k[h_1]$ is factorially closed in $B$, hence algebraically closed in $B$. Since $\text{tr.deg}_{k} B = 1$, this means $B = k[h_1] \cong k[1]$. We thus have:

$$\{k[1]\} \subset U_k(1, G) \subset U_k(1, D) \subset \{k[1]\}$$

5. Rational UFDs of Transcendence Degree Two

Let $B(k)$ be the family of rings defined in (1) above. The main goal of this section is to prove the following classification.

**Theorem 5.1.** $B(k) \subset U_k(2, G, R)$. If $k$ is algebraically closed, then $B(k) = U_k(2, G, R)$.

**Proof.** Assume that $k$ is algebraically closed and $B \in U_k(2, G, R)$. Let $g$ be a positive $\mathbb{Z}$-grading of $B$, given by $B = \bigoplus_{i \in \mathbb{Z}} B_i$. If $B \cong k[2]$, then $B \in B(k)$. So assume that $B \not\cong k[2]$.

Since $\text{tr.deg}_{k} B = 2$ and $B \not\cong k[2]$, there exists a homogeneous signature sequence $\vec{h}$ of $(B, \text{deg}_g)$ of length at least three. Let $f = h_1$ and $g = h_2$. Set $K = \text{frac}(B) \cong k[2]$ and let $K_0 \subset K$ be the subfield:

$$K_0 = \{u/v \in K \mid u, v \in B_t, v \neq 0, i \in \mathbb{N}\}$$

By [9], Proposition 1.1(c), $K_0$ is algebraically closed in $K$. Since $k \neq K_0$ and $K_0 \neq K$, we conclude that $\text{tr.deg}_{k} K_0 = 1$. Since $\text{frac}(B) = k[2]$, Lüroth’s Theorem implies that $K_0 = k(\zeta) = k[1]$ for some $\zeta \in K_0$. Let $\zeta = u/v$ for $u, v \in B_t$ with $\text{gcd}(u, v) = 1$ for positive $t \in \mathbb{Z}$.

Let $a \geq 1$ be such that $d_1 = \text{deg} f = bd$ and $d_2 = \text{deg} g = ad$ for $d = \text{gcd}(d_1, d_2)$ and $\text{gcd}(a, b) = 1$. Then there exist standard homogeneous $F, G \in k[X, Y] = k[2]$ of the same degree $r$ such that:

$$\text{gcd}(F(X, Y), G(X, Y)) = 1 \quad \text{and} \quad f^a G(u, v) = g^b F(u, v)$$

Let $L_i, M_j \in k[X, Y]$, $1 \leq i, j \leq r$, be linear forms such that $F = L_1 \cdots L_r$ and $G = M_1 \cdots M_r$. If $\lambda \in B$ is prime and $F(u, v), G(u, v) \in \lambda B$, then $L_i(u, v), M_j(u, v) \in \lambda B$ for some $i, j$. Since $L_i$ and $M_j$ are linearly independent, $u, v \in \lambda B$, which implies $\lambda \in k^*$. Therefore, $\text{gcd}_{2}(F(u, v), G(u, v)) = 1$. It follows that $f$ is the only prime divisor of $L_i(u, v)$ and $g$ is the only prime divisor of $M_1(u, v)$. If
L_1(u,v) = f^{a_1} \text{ and } M_1(u,v) = g^{b_1} \text{ for } a_1 \leq a \text{ and } b_1 \leq b, \text{ then } \deg f^{a_1} = \deg g^{b_1} \implies a_1 = a \text{ and } b_1 = b. \text{ Since } (L_1, M_1) \in GL_2(k), \text{ it follows that } K_0 = k(f^a/g^b).

Let } \xi \in B \text{ be homogeneous and irreducible, where } \xi \in B_i \text{ for positive } l \in \mathbb{Z}. \text{ Assume } \xi B \neq gB \text{ and } \xi B \neq fB. \text{ We have } \xi^d/\xi^l \in K_0. \text{ Reasoning as above, we conclude that there exists a linear form } N \in k[X,Y] \text{ and positive } e \in \mathbb{Z} \text{ such that:}

\[ \xi^e = N(f^a,g^b) \]

Moreover, } \gcd(e,ab) = 1: \text{ Assume that } s_0 \in \mathbb{Z} \text{ is a prime dividing } e \text{ and } ab. \text{ Since } \gcd(a,b) = 1, \text{ either } a \in s_0\mathbb{Z} \text{ or } b \in s_0\mathbb{Z}. \text{ Suppose that } e = s_0c_0 \text{ and } a = s_0a_0 \text{ for integers } c_0, a_0. \text{ Furthermore, set } s_1 = \gcd(e_0, a_0), e_0 = s_1e_1 \text{ and } a_0 = s_1a_1 \text{ for integers } e_1, a_1. \text{ Then } e = ec_1 \text{ and } a = sa_1, \text{ where } s = s_0s_1. \text{ Since } \gcd(e_1, a_1) = 1, \text{ the equation above then yields } g^{b_1} = \lambda \xi^{c_1} + \mu f^{a_1} \text{ for } b_1 \leq b \text{ and } \lambda, \mu \in k^*. \text{ Since } \deg g^{b_1} = \deg f^{a_1}, \text{ we must have } a_1 = a \text{ and } b_1 = b. \text{ But then } s = 1, \text{ which is impossible. Therefore, no such prime } s_0 \text{ exists.}

Suppose that } a = b. \text{ Then } a = b = 1 \text{ and } d_1 = d_2 = d. \text{ By equation (2), } e \deg \xi = d \text{ for all homogeneous primes } \xi \in B. \text{ If } e > 1, \text{ this implies } \deg \xi < d = d_1, \text{ a contradiction. So } e = 1 \text{ and } \xi \in k[f,g]. \text{ But then } B = k[f,g], \text{ also a contradiction. Therefore, } a > b.

According to equation (2), there exist integers } c_n \geq 2 \text{ such that } h_n^{c_n} \in \langle f^a, g^b \rangle. \text{ By Theorem 4.1, } h_n \text{ is prime for each } n \geq 1. \text{ Therefore, } \deg h_n \text{ divides } abd \text{ for all } n \geq 3, \text{ which implies the sequence of degrees } d_n \text{ is bounded.}

Suppose, for some pair } m \neq n, \text{ that } c_m = tp \text{ and } c_n = tq \text{ for } t = \gcd(c_m, c_n) \text{ and nonzero } p, q \in \mathbb{N}. \text{ Being powers of distinct primes, we see that } h_m^{c_m} \text{ and } h_n^{c_n} \text{ are } k \text{-linearly independent. Therefore, from equation (2) it follows that}

\[ f^a, g^b \in \langle h_m^{c_m}, h_n^{c_n} \rangle = \langle h_m^{tp}, h_n^{tq} \rangle \implies f^{a'} \in \langle h_m^p, h_n^q \rangle \]

for some } a' \leq a \text{ and } b' \leq b. \text{ But then:}

\[ a' \deg f = b' \deg g \implies a' = a \text{ and } b' = b \implies t = 1 \]

Therefore, } \gcd(c_m, c_n) = 1 \text{ for all pairs } m \neq n. \text{ In particular, this means } d_{n+1} \neq d_n \text{ for all } n \geq 0. \text{ So } d_n \text{ is a strictly increasing sequence, and } c_n, n \geq 3, \text{ is a strictly decreasing sequence. Since } d_{n} \text{ is also bounded, we conclude } d_{n}' \text{ is finite. Consequently, } (B, \deg g) \text{ admits a finite complete homogeneous signature sequence } \tilde{h}, \text{ and if the length of } \tilde{h} \text{ is } n, \text{ then } B = k[f,g, h_3, \ldots, h_n].

By re-scaling equations from (2), we may assume that } f^a + \kappa_i y^b + h_i^{c_i} = 0, \text{ where } \kappa_i \in k^*, \text{ } 3 \leq i \leq n. \text{ Replacing } y \text{ with } \kappa_i^{1/b} y, \text{ we may assume } \kappa_3 = 1. \text{ By linear independence, we see that } \kappa_i \neq \kappa_j \text{ if } i \neq j. \text{ We may thus write}

\[ B \cong_k k[x,y,z_3, \ldots, z_n]/(x^a + \kappa_i y^b + z_i^{c_i})_{3 \leq i \leq n} \quad (\kappa_i \in k^*, \kappa_i \neq \kappa_j \text{ if } i \neq j) \]

where } a > b > c_3 > \cdots > c_n \geq 2 \text{ are pairwise relatively prime integers. Therefore, } B \in B(k).

Conversely, for any field } k, \text{ suppose that } B \in B(k) \text{ has the form (1), and consider subrings } R_i = k[x,y,z_3, \ldots, z_i], \text{ for } 2 \leq i \leq n. \text{ For } 3 \leq i \leq n, \text{ define } F_i = x^a + \lambda_i y^b. \text{ We see that } R_2 = k[x,y] \cong k^2 \text{ is a rational UFD with the } Z \text{-grading for which } \deg x = b \text{ and } \deg y = a, \text{ and that } x^a + \mu y^b \text{ is irreducible and homogeneous in } R_2 \text{ for every } \mu \in k^*.

Given } i \text{ with } 2 \leq i \leq n - 1, \text{ suppose that } R_i \text{ is a rational UFD with positive } Z \text{-grading } g, \text{ and that } x^a + \mu y^b \text{ is irreducible and homogeneous in } R_i \text{ for every } \mu \in k^*. \text{ Since}

\[ R_{i+1} = R_i[z_{i+1}] = R_i[Z]/(Z^{c_{i+1}} - F_{i+1}) \]

it follows by Theorem 2.2 that } R_{i+1} \text{ is a rational UFD. \text{ Let } } \mu \in k^* \setminus \{\lambda_3, \ldots, \lambda_{i+1}\} \text{ and consider } G := x^a + \mu y^b \in R_{i+1}. \text{ We have:}

\[ R_{i+1}/GR_{i+1} = R[Z]/(Z^{c_{i+1}} - F_{i+1}, G) = (R_i/GR_i)[Z]/(Z^{c_{i+1}} - (\lambda_{i+1} - \mu)y^b) \]

By the inductive hypothesis, } R_i/GR_i \text{ is an integral domain. By Theorem 2.2(a), } R_{i+1}/GR_{i+1} \text{ is an integral domain. Therefore, } G \text{ is irreducible and homogeneous in } R_{i+1}.\]
Finally, we may extend the \( Z \)-grading \( c_{i+1} \) on \( R_i \) to a positive \( Z \)-grading on \( R_{i+1} \) by letting \( z_{i+1} \) be homogeneous of degree equal to \( \deg \, F_i \).

It follows by induction on \( i \) that \( B = R_n \in \mathcal{U}_k(2, \mathbf{G}, \mathbf{R}) \) and that \( x^a + \mu y^b \) is irreducible and homogeneous in \( B \) for every \( \mu \in k^* \setminus \{\lambda_1, \ldots, \lambda_n\} \).

This completes the proof. \( \square \)

Note that the positive \( Z \)-grading on \( B \in \mathcal{B}(k) \) as defined in (1) is given by

\[
\deg(x, y, z_1, \ldots, z_n) = (N/a, N/b, N/c_1, \ldots, N/c_n)
\]

where \( N = abc_1 \cdots c_n \) and \( x, y, z_1, \ldots, z_n \) are homogeneous.

**Corollary 5.2.** \( \mathcal{U}_k(2, \mathbf{G}, \mathbf{R}) = \mathcal{U}_k(2, \mathbf{A}, \mathbf{G}, \mathbf{R}) \)

**Theorem 5.3.** Given \( B \in \mathcal{B}(k) \) as defined in (1), the minimum number of generators of \( B \) over \( k \) is \( n + 2 \).

**Proof.** By hypothesis, we have

\[
B = \mathbb{C}[x, y, z_1, \ldots, z_n]/(x^a + \lambda_i y^b + z_i^{c_i})_{0 \leq i \leq n}
\]

where \( a > b > c_1 > \cdots > c_n \geq 2 \) are pairwise relatively prime integers and \( 1 = \lambda_1, \ldots, \lambda_n \in k^* \) are distinct. Let \( d \) be the minimum number of generators of \( B \) over \( k \). Then clearly \( d \leq n + 2 \). Set \( X = \text{Spec}(B) \subset \mathbb{A}^{n+2} \). For \( 1 \leq i \leq n \), let \( f_i = x^a + \lambda_i y^b + z_i^{c_i} \). Let \( J \) be the Jacobian matrix of \((f_1, \cdots, f_n)\), namely:

\[
J = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z_1} \\ \vdots & \vdots & \vdots \\ \frac{\partial f_i}{\partial x} & \frac{\partial f_i}{\partial y} & \frac{\partial f_i}{\partial z_i} \end{pmatrix}_{1 \leq i, j \leq n}
\]

Then \( J \) is of dimension \((n + 2) \times n \). For a closed point \( p \in X \), we denote \( J(p) \) by the Jacobian matrix at \( p \), that is:

\[
J(p) = \begin{pmatrix} \frac{\partial f_1}{\partial x}(p) & \frac{\partial f_1}{\partial y}(p) & \frac{\partial f_1}{\partial z}(p) \\ \vdots & \vdots & \vdots \\ \frac{\partial f_i}{\partial x}(p) & \frac{\partial f_i}{\partial y}(p) & \frac{\partial f_i}{\partial z_i}(p) \end{pmatrix}_{1 \leq i, j \leq n}
\]

Let \( m_p \) be a maximal ideal of \( B \) corresponding to the origin \( p = (0, \ldots, 0) \in X \). Since \( a > b > c_i \geq 2 \) for each \( i \), we see that \( \text{rank}(J(p)) = 0 \), and we have:

\[
\dim_k(m_p/m_p^2) = (n + 2) + \text{rank}(J(p)) = n + 2
\]

Therefore, the dimension of the tangent space at the origin \( p \) is \( n + 2 \), which implies \( d \geq n + 2 \). \( \square \)

**Theorem 5.4.** Assume that the characteristic of \( k \) equals 0. Given \( B \in \mathcal{B}(k) \), let \( R \) be a UFD such that \( B \subset R \) and \( B \) is factorially closed in \( R \). If \( B \not\cong k[2] \) and \( B \not\cong k[x, y, z]/(x^5 + y^3 + z^2) \), then \( B \subset ML(R) \). In particular, \( B \) is rigid (respectively, stably rigid) in these cases.

**Proof.** Assume that

\[
B = \mathbb{C}[x, y, z_1, \ldots, z_n]/(x^a + \lambda_i y^b + z_i^{c_i})_{0 \leq i \leq n}
\]

where \( a > b > c_1 > \cdots > c_n \geq 2 \) are pairwise relatively prime integers and \( 1 = \lambda_1, \ldots, \lambda_n \in k^* \) are distinct. We may assume \( n \geq 1 \), since otherwise \( B = k[x, y] \). From the proof of Theorem 5.1 we see that \( x, y, z_1, \ldots, z_n \) are distinct elements of \( B \), since \( x = h_1, y = h_2, z_1 = h_3, \ldots, z_n = h_{n+2} \) is a complete signature sequence in \( B \). Since \( B \) is factorially closed in \( R \), it follows that \( x, y, z_1, \ldots, z_n \) are distinct primes in \( R \).

If \( a^{-1} + b^{-1} + c_i^{-1} > 1 \), then it is easy to check that \((a, b, c_i) = (5, 3, 2)\). By the ABC Theorem (\[9\], Thm. 2.48), it follows that \( k[x, y, z_i] \subset ML(R) \) whenever \((a, b, c_i) \neq (5, 3, 2)\). Since \( B \) is algebraic over \( k[x, y, z_i] \) and \( ML(R) \) is algebraically closed in \( R \), it follows that \( B \subset ML(R) \) if \((a, b, c_i) \neq (5, 3, 2)\) for all \( i \).

If \((a, b, c_i) = (5, 3, 2)\) for some \( i \), then \( n = 1 \) and \((a, b, c_1) = (5, 3, 2)\). \( \square \)
Proposition 6.1. From the defining equations for $\phi$, we may assume that $\phi(\xi) = K$. Therefore:

\[ \phi = \lambda_1, \ldots, \lambda_n \in \mathbb{C}^* \] are distinct; and

\[ B' = \mathbb{C}[x', y', z_1', \ldots, z_m']/(x'^{a'} + \lambda_i y'^{b'} + (z'_i)^{c_i})_{0 \leq i \leq m} \]

where $a' > b' > c'_1 > \cdots > c'_m \geq 2$ are pairwise relatively prime integers and $1 = \lambda'_1, \ldots, \lambda'_m \in \mathbb{C}^*$ are distinct. By Theorem 5.5, we must have $m = n$. If $m = n = 2$, then $B, B' \cong \mathbb{C}^2$. So assume that $m = n \geq 3$. By Theorem 5.4 the rings $B$ and $B'$ are rigid; see also Remark 7.1 below.

Assume that $\varphi : B' \to B$ is a $\mathbb{C}$-algebra isomorphism. Let $g$ and $g'$ be the positive $\mathbb{Z}$-gradings of $B$ and $B'$, respectively, as given in (1). In addition, let $\varphi(g')$ be the $\mathbb{Z}$-grading of $B$ induced by $\varphi$. According to Theorem 5.5 there exists $\xi \in \mathbb{Z}$ such that $\varphi(g') = \xi g$. Since $\varphi$ is surjective, we see that $\xi = \pm 1$. If $\xi = -1$, we may compose $\varphi$ with an involution $\alpha$ of $B$ so that $\alpha \varphi(g') = g$. So we may assume that $\xi = 1$, and $\varphi(g') = g$.

From the proof of Theorem 5.4 we have that $\mathcal{Z} = \{x, y, z_1, \ldots, z_n\}$ is a homogeneous signature sequence for $(B, g)$, and that $\mathcal{Z}' = \{x', y', z_1', \ldots, z_n'\}$ is an homogeneous signature sequence for $(B', g') = \varphi(B, g)$. Therefore, $\mathcal{K} := \varphi(\mathcal{Z}')$ is also a homogeneous signature sequence for $(B, g)$.

Write $\mathcal{K} = \{f, g, h_3, \ldots, h_{n+2}\}$, where:

\[ \varphi(x) = f, \varphi(y) = g, \varphi(z_i) = h_{i+2} \quad (1 \leq i \leq n) \]

Let $d_1 = \deg x$, $d_2 = \deg y$ and $d_{i+2} = \deg z_i$ for $i \geq 1$. The proof of Theorem 5.4 shows that the sequence $d_i$ is strictly increasing. Therefore, for each $i$ with $1 \leq i \leq n$:

\[ \dim_{\mathbb{C}} B_{d_i} - \dim_{\mathbb{C}}(B_{d_i} \cap \mathbb{C}[x, y, z_1, \ldots, z_{i-1}]) = 1 \]

If follows that:

(1) There exist $u, v, w_i \in \mathbb{C}^*$ such that $f = ux$, $g = vy$ and $h_{i+2} = w_i z_i$ for $1 \leq i \leq n$

(2) $a = a'$, $b = b'$ and $c_i = c'_i$ for $1 \leq i \leq n$

From the defining equations for $B$ and $B'$ we thus obtain for $1 \leq i \leq n$:

\[ -z_i^{c_i} = x^a + \lambda_i y^b \quad \text{and} \quad -h_i^{c_i} = f^a + \lambda'_i g^b \]

Therefore:

\[ -z_i^{c_i} = \frac{u^a}{w_i^{c_i}}x^a + \frac{v^b}{w_i^{c_i}}\lambda'_i y^b = x^a + \lambda_i y^b \]

Since $x^a$ and $y^b$ are linearly independent over $k$, it follows that $u^a = w_i^{c_i}$ and $w_i^{c_i}\lambda_i = v^b\lambda'_i$ for $1 \leq i \leq n$. Set $\zeta = v^b/u^a \in \mathbb{C}^*$. Then $\lambda_i = \zeta^{c_i}$ for $1 \leq i \leq n$. Since $\lambda_1 = \lambda'_1 = 1$, we see that $\zeta = 1$, hence $\lambda_i = \lambda'_i$ for $1 \leq i \leq n$.

This completes the proof of the theorem. \qed

6. Rational UFDs of transcendence degree three

6.1. Certain Affine Modifications of $k[3]$. Let $k[x] = k[1]$ for a field $k$, and let $f = p(x) \in k[x] \setminus \{0\}$. Define the affine $k$-algebra

\[ B_n = k[x][z_0, \ldots, z_{n+1}]/(p(x)z_{i+1} + z_i^{a_i} + z_i^{b_i})_{0 \leq i \leq n} \]

where $a_1, \ldots, a_n, b_1, \ldots, b_n$ are positive integers such that $\gcd(a_i, b_1 \cdots b_i) = 1$ for each $i$. Using $K = k[x]$ in Theorem 5.4 it follows that $B_n \in \mathcal{U}_k(3, A, R)$.

Proposition 6.1. If $p(x) \notin k$ and $a_i, b_i \geq 2$ for all $i$, then the minimum number of generators of $B_n$ over $k$ is $n + 3$. 
Proof. Let $d$ be the minimum number of generators of $B_n$ over $k$. Then clearly $d \leq n + 3$. Set $X = \text{Spec}(B_n) \subset \mathbb{A}^{n+3}$. For $0 \leq i \leq n$, let $f_i = p(x)z_{i+1} + z_i^a + z_{i-1}^b$. Let $J$ be the Jacobian matrix of $(f_0, \ldots, f_n)$, namely:

$$J = \begin{pmatrix} \frac{\partial f_i}{\partial x_i} & \frac{\partial f_i}{\partial z_j} \end{pmatrix}_{0 \leq i \leq n, 0 \leq j \leq n+1}$$

Then $J$ is of a matrix of size $(n + 3) \times (n + 1)$. For $0 \leq i \leq n$ and $0 \leq j \leq n + 1$, we have $\partial f_i / \partial x = p'(x)z_{i+1}$ and:

$$\frac{\partial f_i}{\partial z_j} = \begin{cases} p(x) & (j = i + 1) \\ a_i z_i^{a_i-1} & (j = i) \\ b_i z_i^{b_i-1} & (j = i + 1) \\ 0 & \text{(otherwise)} \end{cases}$$

For a maximal ideal $m$ of $B_n$, we denote $J(m)$ by the Jacobian matrix at $m$, that is,

$$J(m) = \begin{pmatrix} \frac{\partial f_i}{\partial x_i}(m), \frac{\partial f_i}{\partial z_j}(m) \end{pmatrix}_{0 \leq i \leq n, 0 \leq j \leq n+1}$$

where for $g \in B_n$, $g(m)$ means the image of $g$ in $B_n/m$.

Take a prime divisor $q(x) \in k[x]$ of $p(x)$, which is possible since $p(x) \not\equiv 0$. Let $m$ be the maximal ideal of $B_n$ generated by $q(x), z_0, \ldots, z_{n+1}$. Since $a_i, b_i \geq 2$ for each $i$, we see that $\text{rank}(J(m)) = 0$, hence we have:

$$\text{dim}_k(m/m^2) = (n + 3) + \text{rank}(J(m)) = n + 3$$

Therefore, the dimension of the tangent space at $m$ is $n + 3$, which implies $d \geq n + 3$. \hfill \Box

The threefolds listed in (4) are of interest, since some of them occur as the kernel $a$ of locally nilpotent derivation of $k[4]$ when the characteristic of $k$ is $0$. For instance, Example 8.11 and Example 8.15 of [21] give kernels isomorphic to

(5) $B_1 = k[x, z_0, z_1, z_2]/(x^2 z_2 + z_1^2 + z_0^3)$ and $B_2 = k[x, z_0, z_1, z_2, z_3]/(x z_2 + z_1^2 + z_0^3, x z_3 + z_2^2 + z_1^3)$

respectively. $B_1$ has two independent positive $\mathbb{Z}$-gradings $g_1$ and $g_2$, where $x, z_0, z_1, z_2$ are homogeneous with:

$$\deg_{g_1}(x, z_0, z_1, z_2) = (1, 2, 3, 4) \text{ and } \deg_{g_2}(x, z_0, z_1, z_2) = (2, 2, 3, 2)$$

$B_2$ has positive $\mathbb{Z}$-grading $\mathfrak{g}$, where $x, z_0, z_1, z_2, z_3$ are homogeneous with:

$$\deg_{\mathfrak{g}}(x, z_0, z_1, z_2, z_3) = (3, 4, 6, 9, 15)$$

For $n \geq 3$, it is easy to show that $B_n$ admits no positive $\mathbb{Z}$-grading for which $x, z_0, \ldots, z_n$ are homogeneous.

6.2. The Russell Cubic Threefold. The Russell cubic threefold over $k$ is $X = \text{Spec}(B)$, where:

$$B = k[x, y, z, t]/(x + x^2 y + z^2 + t^3) \in \mathcal{U}_k(3, \mathbf{A}, \mathbf{R})$$

$X$ is smooth and admits the hyperbolic $\mathbb{G}_m$-action $\rho(\mathfrak{g})$ induced by the $\mathbb{Z}$-grading $\mathfrak{g}$ of $B$ for which $x, y, z, t$ are homogeneous and $\deg_{\mathfrak{g}}(x, y, z, t) = (6, -6, 3, 2)$.

Assume that $k = \mathbb{C}$. Dubouloz, Moser-Jauslin and Poloni describe the automorphism group $G = \text{Aut}_\mathbb{C}(B)$ in [23] as follows.

It is known that $\text{ML}(B) = \mathbb{C}[x]$ and $\mathcal{D}(B) = \mathbb{C}[x, z, t]$. Thus, any element of $G$ restricts to both $\mathbb{C}[x]$ and $\mathbb{C}[x, z, t] \cong \mathbb{C}[3]$. Define the ideal $I \subset \mathbb{C}[x, z, t]$ by $I = (x^2, z^2 + t^3 + x)$, and define the group:

$$K = \{ \alpha \in \text{Aut}_{\mathbb{C}[x]}\mathbb{C}[x, z, t] \mid \alpha(I) = I, \alpha \equiv 1 \text{ (mod } x) \}$$
Let $\varphi : \mathbb{C}^* \to \text{Aut}_C(K)$ be the restriction of $\rho(\mathfrak{g})$ to $C[x, z, t]$. Then
\begin{equation}
G \cong K \rtimes \mathbb{C}^*
\end{equation}
where the isomorphism is gotten by restricting elements of $G$ to $C[x, z, t]$. As a consequence, every automorphism of $X$ fixes the point $0 \in X$ defined by the maximal ideal $(x, y, z, t)$ of $B$.

**Theorem 6.2.** $B \not\in \mathcal{U}_C(3, G)$

**Proof.** Let $\mathfrak{h}$ be a positive $\mathbb{Z}$-grading of $B$, and let $\psi : \mathbb{C}^* \to G$ be the elliptic $\mathbb{C}^*$-action on $B$ induced by $\mathfrak{h}$. Then $\psi$ restricts to $R := C[x, z, t]$ and is completely determined by its action on $R$.

Note first that $\psi$ fixes $C[x]$, and $x$ must therefore be $\mathfrak{h}$-homogeneous. Since $\mathfrak{h}$ is positive, Lemma [6,3] below implies that there exist $\mathfrak{h}$-homogeneous $Z, T \in R$ with $R = C[x, Z, T]$. Define $\beta \in K$ by $\beta(x, z, t) = (x, Z, T)$, and for $\lambda \in \mathbb{C}^*$, suppose that $\psi(\lambda)(x, Z, T) = (\lambda^p x, \lambda^q Z, \lambda^r T)$ for positive $p, q, r \in \mathbb{Z}$. Given $\lambda \in \mathbb{C}^*$, write
\[
\psi(\lambda) = \kappa_\lambda \varphi(\mu_\lambda), \quad \kappa_\lambda \in K, \quad \mu_\lambda \in \mathbb{C}^*
\]
according to the decomposition of $G$ given in [10]. By [3], Proposition 3.6, there exist $f, g, \alpha \in C[z, t]$ such that:
\[
\kappa_\lambda(z) = z + x(\alpha(z^2 + t^3))t + x^2 f \quad \text{and} \quad \kappa_\lambda(t) = t - x(\alpha(z^2 + t^3))z + x^2 g
\]
In addition $Z = \beta(z)$ and $T = \beta(t)$ also have this form. Therefore:
\[
\psi(\lambda)(x) = \kappa_\lambda \varphi(\mu_\lambda)(x) = \kappa_\lambda(\mu_\lambda^3 x) = \mu_\lambda^3 \kappa_\lambda(x) = \mu_\lambda^3 x
\]
\[
\psi(\lambda)(z) = \kappa_\lambda \varphi(\mu_\lambda)(z) = \kappa_\lambda(\mu_\lambda z) = \mu_\lambda^3 \kappa_\lambda(z) = \mu_\lambda^3 (z + x(\alpha(z^2 + t^3))t + x^2 f)
\]
\[
\psi(\lambda)(t) = \kappa_\lambda \varphi(\mu_\lambda)(t) = \kappa_\lambda(\mu_\lambda^2 t) = \mu_\lambda^3 \kappa_\lambda(t) = \mu_\lambda^3 (t - x(\alpha(z^2 + t^3))z + x^2 g)
\]
Let $V$ be the tangent space to $\text{Spec}(R) \cong \mathbb{C}^3$ at 0, and let $\Psi$ denote the $\mathbb{C}^*$-action on $V$ induced by $\psi|_R$. Since $x(\alpha(z^2 + t^3))z, x(\alpha(z^2 + t^3))t \in (x, z, t)^2$, it follows that:
\[
\Phi(\lambda)(x, Z, T) = (\mu_\lambda^3 x, \mu_\lambda^3 z, \mu_\lambda^3 t) = (\lambda^p x, \lambda^q Z, \lambda^r T)
\]
Therefore, $\psi(\lambda)(x, Z, T) = (\lambda^{6c} x, \lambda^{3c} Z, \lambda^{2c} T)$ for some $c \in \mathbb{Z}$, which implies:
\[
\psi|_R = \beta(\rho(\mathfrak{g})|_R) \beta^{-1} = (\beta \rho(\mathfrak{g}) \beta^{-1})|_R \implies \psi = \beta \rho(\mathfrak{g}) \beta^{-1}
\]
But this is impossible, since $\psi$ is elliptic. Therefore, $B$ admits no positive $\mathbb{Z}$-grading. \qed

The following result, which is due to Daigle, implies that any elliptic $G_m$-action on $A^3_0$ is linearizable; see [3], Proposition 3.42.

**Lemma 6.3.** Let $k$ be a field and $R = k[x^r]$ ($r \geq 1$), and let $\mathfrak{g}$ be a positive $\mathbb{Z}$-grading of $R$. If $f_1, \ldots, f_n \in R$ are $\mathfrak{g}$-homogeneous and $R = k[f_1, \ldots, f_n]$, then there is a subset $\{g_1, \ldots, g_r\}$ of $\{f_1, \ldots, f_n\}$ such that $R = k[g_1, \ldots, g_r]$.

**6.3. Asanuma Threefolds.** Let $k$ be a field of characteristic $p > 0$. In [11], Asanuma introduced the rational threefolds
\[
A_m = k[x, y, z, t]/(x^m y + f(z, t))
\]
where $m \geq 1$, $f(z, t) \in k[z, t]$ and $k[z, t]/(f) \cong k_f[k]^{[1]}$ but $k[z, t] \neq k_f[k]^{[1]}$. Segre [22] gives such non-standard line embeddings in $A^2$, for example, defined by
\[
f(z, t) = z^{p^e} + t + t^{p^f}
\]
where $s, e \in \mathbb{Z}^+$ and $p^e$ and $sp$ do not divide each other; see also the Introduction to [11]. Asanuma showed that $A_m^{[1]} \cong k_f[k]^{[4]}$ for each $m \geq 1$. From this, it follows that $A_m \in \mathcal{U}_C(3, A, P, R)$ and that each threefold $\text{Spec}(A_m)$ is smooth.

These rings are considered by N. Gupta in [11,12], showing that, when $m \geq 2$, $ML(A_m) = k[x]$ and $D(A_m) = k[x, z, t]$. So $A_m \not\cong k_f[k]^{[3]}$ when $m \geq 2$, thus providing counterexamples for the cancellation problem for affine spaces in positive characteristic. It is an open problem whether $A_1 \cong k_f[k]^{[3]}$. 

\[\text{Page 15}\]
Theorem 6.4. Let $k$ be a field and $A$ an affine $k$-domain, $n = \dim_k A$. The following conditions are equivalent.

1. $A \in \mathcal{U}_k(n, G)$ and $A^{[m]} \cong_k k^{[n+m]}$ for some $m \in \mathbb{N}$
2. $A \cong_k k^{[n]}$

Proof. The implication (2) implies (1) is clear. For the converse, assume that condition (1) holds, and let $g$ be a positive $\mathbb{Z}$-grading of $A$ over $k$. There exist an integer $s \geq n$ and $g$-homogeneous elements $a_1, \ldots, a_s \in A$ such that $A = k[a_1, \ldots, a_s]$.

Let $B = A[x_1, \ldots, x_m] = A^{[m]}$, and let $I \subset B$ be the ideal $I = x_1 B + \cdots + x_m B$. Extend the $\mathbb{Z}$-grading $2g$ on $A$ to a $\mathbb{Z}$-grading $g'$ of $B$ by letting each $x_i$ be homogeneous of degree one, $1 \leq i \leq m$, and let $B = \bigoplus_{i \in \mathbb{N}} B_i$ be the decomposition of $B$ for $g'$. Then $B_0 = k$ and $B_1 = kx_1 \oplus \cdots \oplus kx_m$.

By Lemma 6.3, there exists a subset $\{g_1, \ldots, g_{n+m}\}$ of $\{a_1, \ldots, a_s, x_1, \ldots, x_m\}$ such that $B = k[g_1, \ldots, g_{n+m}]$. Since $g'$ is positive and $\deg_g a_i \geq 2$ for each $i$, it follows that:

$$\{x_1, \ldots, x_m\} \subset \{g_1, \ldots, g_{n+m}\}$$

By re-indexing the set $\{g_1, \ldots, g_{n+m}\}$, we may assume that $g_i = x_i$, $1 \leq i \leq m$. Therefore, $B = k[x_1, \ldots, x_m, g_{1+m}, \ldots, g_{n+m}]$, which implies:

$$A \cong k/B \cong k[g_{1+m}, \ldots, g_{n+m}]/I \cong_k k^{[n]}$$

\[\square\]

Corollary 6.5. For $m \geq 2$, $A_m \not\in \mathcal{U}_k(3, G)$.

In fact, Gupta found counterexamples to cancellation in positive characteristic for every dimension $n \geq 3$; see [13]. Thus, for each such counterexample $R$, we have $R \not\in \mathcal{U}_k(n, G)$.

7. Conclusion

We conclude with some remarks and a conjecture.

Remark 7.1. The ring $B = k[x, y, z]/(x^5 + y^3 + z^2)$ is also rigid (see [9], Thm. 9.7), but it is not known whether it satisfies the stronger property described in Theorem 5.4.

Remark 7.2. If $k$ is not algebraically closed, then in general, $\mathcal{B}(k) \neq \mathcal{U}_k(2, G, R)$. For example, Theorem 2.2 shows that

$$\mathbb{R}[x, y, z]/(x^2 + y^2 + z^{2k+1})$$

is a rational UFD which admits a positive $\mathbb{Z}$-grading for all integers $k \geq 0$.

Remark 7.3. The set $\mathcal{U}_k(2, A, G, P, R)$ contains more than just $k[x, y]$. For example, it is well-known that the ring

$$B = k[x, y, z]/(x^5 + y^3 + z^2)$$

is the ring of invariants for an action of an icosahedral group on the plane, so $B \subset k[\mathbb{C}]$. For a specific polynomial parametrization, see [19], §2.E. Likewise, Russell [20] gives the subalgebra:

$$B' = k[u^b v^c, v T^e, T] \subset k[u, v] \quad \text{where} \quad T = u^e + v^b, \gcd(b, c) = 1$$

Then $B' \cong k[x, y, z]/(x^{bc+1} + y^b + z^c)$.

Remark 7.4. For $R \in \mathcal{U}_k(n, A, G)$, let $g$ be a positive $\mathbb{Z}$-grading of $R$ given by $R = \bigoplus_{i \in \mathbb{N}} R_i$. Given $a \in \mathbb{N}$, define the homogeneous subalgebra:

$$R^{(a)} = \bigoplus_{i \in \mathbb{N}} R_{ia}$$

Note that $R^{(a)} = R$ and that $R^{(b)} \subset R^{(a)}$ when $a$ divides $b$. In [16], Mori shows that there exists a unique positive integer $m$ with the property:

$$R^{(a)}$$

is a UFD if and only if $a$ divides $m$.
This integer $m$ is called the **index** of the pair $(R, g)$.

Finally, we propose the following characterization of the affine space $A^n_k$.

**Conjecture.** Let $k$ be an algebraically closed field and $n \in \mathbb{Z}$ positive. If $B \in U_k(n, A, G, R)$ and $X = \text{Spec}(B)$ is smooth, then $X \cong_k A^n_k$.

The conjecture is true if $\dim_k X \leq 2$: The case $\dim_k X = 1$ follows by Corollary 4.7, and the case $\dim_k X = 2$ follows by Theorem 5.7.

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**References**

[1] T. Asanuma, *Polynomial fibre rings of algebras over noetherian rings*, Invent. Math. **87** (1987), 101–127.
[2] G. Castelnuovo, *Sulla razionalit`a delle involuzioni piane*, Math. Ann. **44** (1894), 125–155.
[3] D. Daigle and G. Freudenburg, *A note on triangular derivations of $k[X_1, X_2, X_3, X_4]$*, Proc. Amer. Math. Soc. **129** (2001), 657–662.
[4] J. Deveney and D. Finston, *Fields of $G_a$ invariants are ruled*, Canad. Math. Bull. **37** (1994), 37–41.
[5] A. Dubouloz, L. Moser-Jauslin, and P.-M. Poloni, *Inequivalent embeddings of the Koras-Russell cubic 3-fold*, Michigan Math. J. **59** (2010), 679–694.
[6] P. Eakin, *A note on finite dimensional subrings of polynomial rings*, Proc. Amer. Math. Soc. **31** (1972), 75–80.
[7] H. Flenner and M. Zaidenberg, *On the uniqueness of $C^\ast$-actions on affine surfaces*, Contemp. Math. **369** (2005), 97–111.
[8] , *On a result of Miyanishi-Masuda*, Arch. Math. (Basel) **87** (2006), 15–18.
[9] G. Freudenburg, *Algebraic Theory of Locally Nilpotent Derivations*, second ed., Encyclopaedia of Mathematical Sciences, vol. 136, Springer-Verlag, Berlin, Heidelberg, New York, 2017.
[10] R. Ganong, *The pencil of translates of a line in the plane*, RM Proc. Lecture Notes **54** (2011), 57–71.
[11] N. Gupta, *On the cancellation problem for the affine space $A^3$ in characteristic $p$*, Invent. Math. **195** (2014), 279–288.
[12] , *On the family of affine threefolds $z^m y = f(x, z, t)$*, Compos. Math. **150** (2014), 979–998.
[13] , *On Zariski’s cancellation problem in positive characteristic*, Adv. Math. **264** (2014), 296–307.
[14] S. Kaliman and M. Zaidenberg, *Affine modifications and affine hypersurfaces with a very transitive automorphism group*, Transform. Groups **4** (1999), 53–95.
[15] M. Miyanishi, *Normal affine subalgebras of a polynomial ring*, Algebraic and Topological Theories—to the memory of Dr. Tsuchikio Miyata (Tokyo), Kinokuniya, 1985, pp. 37–51.
[16] S. Mori, *Graded factorial domains*, Japan J. Math. **3** (1977), 224–238.
[17] M. Nagata, *A remark on the unique factorization theorem*, J. Math. Soc. Japan **9** (1957), 143–145.
[18] C. P. Ramanujam, *A topological characterization of the affine plane as an algebraic variety*, Ann. of Math. **94** (1971), 69–88.
[19] O. Riemschneider, *Die Invarianten der endlichen Untergruppen von GL(2, C)*, Math. Z. **153** (1977), 37–50.
[20] K. P. Russell, *On affine-ruled rational surfaces*, Math. Ann. **255** (1981), 287–302.
[21] P. Samuel, *Lectures on unique factorization domains*, Tata Institute of Fundamental Research Lectures on Mathematics, vol. 30, Tata Institute of Fundamental Research, Bombay, 1964.
[22] B. Segre, *Corrispondenze di m`obius e trasformazioni cremoniane intere (Italian)*, Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **91** (1956/1957), 3–19.
[23] O. Zariski, *Interpretations algebro-geometriques du quatorzieme probleme de Hilbert*, Bull. Sci. Math. **78** (1954), 155–168.

Department of Mathematics
Western Michigan University
Kalamazoo, Michigan 49008
USA
gene.freudenburg@wmich.edu

Graduate School of Science and Technology
Niigata University
8050 Ikarashininocho, Niigata 950-2181
Japan
t.nagamine14@m.sc.niigata-u.ac.jp