Generating Functions of Nestohedra and Applications

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Abstract

We examine the family of nestohedra resulting from the complete bipartite graph through the medium of a generating function and demonstrate some of their combinatorial invariants.

1 Introduction

In [3] a ring of simple polytopes $\mathcal{P}$ was introduced, with a formal addition akin to disjoint union, and direct product as multiplication. This ring inherits all the machinery that is normally associated with polytopes. In particular, the $f$-, $h$- and $\gamma$-vectors are defined on the generators of this ring. These vectors are extended, again in [3], to polynomials with the entries of the vectors as coefficients, known as the $f$-, $h$- and $\gamma$-polynomials, which we will favour here. Also introduced in [3] was a derivation, $d$, which produces the disjoint union of the facets of a polytope. In the same paper the concept of a generating function for a family of polytopes was developed together with calculations of the $f$- and $h$-polynomials of some well-known families. In this paper we are going to describe the combinatorial invariants of some additional families of simple polytopes. The particular polytopes involved in these examples are constructed as in [1], so an overview of this construction will also be included in this paper. The main results of this paper will be describing the invariants of the family of polytopes resulting from the complete bipartite graphs and also the method used to obtain them.

2 The Basics

A family of polytopes is a collection of polytopes that share some defining property. A family has at least one representative in each dimension. It is an indexed by a set $J$. For example, we have the family, $I = \{I^n\}_{n \in \mathbb{N}}$, consisting of all cubes, and the family, $\Delta = \{\Delta^n\}_{n \in \mathbb{N}}$, consisting of all simplices.

Definition 2.1. For a family of polytopes $\Psi$, with indexing set $J$, we define the generating function as the formal power series

$$\Psi(x) := \sum_{j \in J} s(j) \ P^n_j \ x^{n+q}$$

in $\mathcal{P} \otimes \mathbb{Q}[x]$. In this series, the parameters $s(j) \in \mathbb{Q}$ and $q \in \mathbb{N}$ are chosen appropriately for the family $\Psi$ in question, to simplify later equations.

For certain families of polytopes we can choose a function $s(i)$ which depends directly on the index of the polytope. We can then choose a $q$ which removes later correcting factors from the differential equations, allowing the generating function to be studied independently from the individual polytopes. All the families studied in this paper will have this desirable property. The choice of $s(P^n)$ and $q$ depends on some subgroup of the group of symmetries of the polytopes and will become more apparent with some examples.

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Example 1. The first example to consider is $I$, the family of cubes. The symmetry group of $I^n$, without reflections, has order $n!$, so we choose $s(I^n) = 1/n!$. We then choose $q$ to be 0, giving $\sum I^n x^n/n!$ as the generating function. We make this choice so that when the generating function is differentiated with respect to $x$ we get a simple expression.

Example 2. Another pertinent example is that of $\Delta$, the family of simplices. In this case, the symmetry group of $\Delta^n$ has order $(n + 1)!$, and hence we choose $s(\Delta^n) = 1/(n + 1)!$. In this case, we set $q = 1$, giving $\sum \delta^n x^{n+1}/(n + 1)!$ as the generating function.

We can extend the machinery that exists on $P$ by linearity to $P \otimes \mathbb{Q}[[x]]$; in particular we will make use of the derivative $d$, the $f$-polynomial, $h$-polynomial and $\gamma$-polynomial, all from [3]. Their extensions are, for a family $\Psi = \{P_j\}_{j \in J}$, respectively,

$$d \Psi(x) = \sum_{j \in J} s(j) d(P_j^n) x^{n+q} \in P \otimes \mathbb{Q}[[x]],$$

$$\Psi_f(\alpha, t, x) = \sum_{j \in J} s(j) f(P_j^n)(\alpha, t) x^{n+q} \in \mathbb{Q}[\alpha, t][[x]],$$

$$\Psi_h(\alpha, t, x) = \sum_{j \in J} s(j) h(P_j^n)(\alpha, t) x^{n+q} \in \mathbb{Q}[\alpha, t][[x]],$$

$$\Psi_\gamma(\tau, z) = \sum_{j \in J} s(j) \gamma(P_j^n)(\tau) z^{n+q} \in \mathbb{Q}[\tau][[x]].$$

In particular, we can restate the identity from Theorem 1 in [3] as

$$(d \Psi)_f(\alpha, t, x) = \frac{\partial}{\partial t} \Psi_f(\alpha, t, x).$$

Similarly, the coordinate changes which relate the $h$-polynomial to the $f$- and $\gamma$-polynomials are unaffected by the extension to generating functions. Setting $a = \alpha + t$, $b = \alpha t$, $\tau = \frac{b}{a}$ and $z = ax$, the coordinate changes are

$$\Psi_f(\alpha, t, x) = \Psi_h(\alpha - t, t, x)$$

$$a^\theta \Psi_\gamma(\tau, z) = \Psi_h(\alpha, t, x).$$

Now we overview the construction of Nestohedra from [1].

Definition 2.2. A building set, $B$, is a set of subsets of $[n + 1] := \{1, \ldots, n + 1\}$, the set consisting of the first $n + 1$ integers, such that

1. if $S_1, S_2 \in B$ such that $S_1 \cap S_2 \neq \emptyset$ then $S_1 \cup S_2 \in B$,
2. the set $\{i\} \in B$ for all $i \in [n + 1]$.

A building set is called connected if $[n + 1] \in B$.

For a graph, $\Gamma$, on $n + 1$ nodes any numbering of then nodes produces a building set $B(\Gamma)$. This building set consists of all non-empty subsets, $I \subset [n + 1]$, such that the induced graph $\Gamma|_I$ is connected. A building set constructed from a graph will be called a graphical building set. A connected graph will produce a connected building set.

Definition 2.3. For a building set $B$, the nestohedron, $P_B$, is the Minkowski sum

$$P_B = \sum_{S \in B} \Delta_S$$

where $\Delta_S := \text{ConvexHull}\{e_i| i \in S\}$ and $e_i$ is the tip of the standard unit basis vector.
Note that \[1\] proves that nestohedra are always simple and that all graphical nestohedra are flag. We can also generalise a result of \[1\] to give us an expression for the differential of a nestohedron.

**Theorem 2.4 \([5]\).** For a nestohedron, \(P_B\), on a connected building set \(B\), we have:

\[
d(P_B) = \sum_{S \in B/\{n+1\}} P_{B|S} \times P_{B-S}
\]

where \(B|S\) is the building set consisting of those sets in \(B\) which are subsets of \(S\) and \(B-S\) is the building set consisting of sets in \(B\) with the elements of \(S\) removed.

**Proof.** We are looking at the facets of a Minkowski sum of faces of the standard simplex, \(\Delta^n\), which includes the standard simplex as one of the summands. All the summands will be simplices of dimension \(m \leq n\) and all their faces will also be lower dimensional simplices. The standard definition of the \(m\)-dimensional simplex is

\[
\Delta^m = \{ x = (x_1, \ldots, x_m) : 0 \leq x_i \leq 1, \sum_{i=1}^m x_i = 1 \},
\]

and the faces of \(\Delta^m\) are subsets where some set of coordinates are minimised.

A facet of the sum will have a contribution from each summand. This contribution will be a face of the summand, frequently the entire summand. Each face is defined by the set of coordinates on which it is minimised. If these sets are not restrictions of some set \(C\) the result is not a facet of the sum, it is either in the interior of the polytope or a face of lower dimension.

The facet, as the Minkowski sum of these faces can be split up into the Minkowski sum of two other polytopes, \(X\) and \(Y\). These are both Minkowski sums of faces, \(X\) the parts of those faces in \(\text{span}\{e_i\}_{i \in C}\) and \(Y\) the parts of those faces in \(\text{span}\{e_i\}_{i \notin C}\). Since \(A\) and \(B\) are orthogonal, we have that \(X+Y = X \times Y\), the direct product.

We now examine the faces in two types, those where \(\sum_{i \in C} x_i = 0\) and those where \(\sum_{i \in C} x_i = 1\). It is easy to show that there are no other possibilities. A face, \(\Delta_S\), of the first type contribute 0 to \(X\) and \(\Delta_{S-C}\) to \(Y\). A face, \(\Delta_S\), of the second type contribute \(\Delta_S\) to \(X\) and 0 to \(Y\).

From the above we can clearly see that \(X\) is precisely \(P_{B|S}\) and \(Y\) is \(P_{B-S}\). The product produces a facet precisely when both sums contain the highest possible dimension simplices. This will only occur when \(\Delta_S\) and \(\Delta^n\) are present, which is when \(S\) and \([n+1]\) are distinct and contained in \(B\). So for a connected building set there is a facet for each element of \(B\) apart from \([n+1]\), and it is \(P_{B|S} \times P_{B-S}\).

For a non-connected building set we can split it up into the product of its connected components and use lemma 2.4 result on each component. This result can be restated in terms of graphs for graphical building sets as

**Corollary 2.5.** For a connected graph \(\Gamma\) on \(n+1\) nodes, we have

\[
d(P(\Gamma)) = \sum_* P(\Gamma_G) \times P(\bar{\Gamma}_{G^c})
\]

where

1. \(G \subseteq \{1, \ldots, n+1\}\).
2. \(\Gamma_G\) is the subgraph of \(\Gamma\) with vertex set \(G\).
3. \(\bar{\Gamma}_{G^c}\) is the graph with vertex set \(\{1, \ldots, n+1\} - G\) and arcs between two vertices, \(i\) and \(j\), if they are path connected in \(\Gamma_G \cup \{i,j\}\).
4. * runs over all \(G\) such that \(\Gamma_G\) is connected.
Of particular interest to us is the fact that nestohedra naturally form families. We can also combine theorem 2.4 and equation (2.1) to give an easy way to calculate $\Psi_f(\alpha, t, x)$ for a family $\Psi$. This method was employed in [3] to give results about two families which we shall use here. These families are $Pe = \{Pe^n\}_{n \in \mathbb{N}}$, the family of permutahedra, which is a family of graphical nestohedra where $Pe^n$ is generated from the complete graph on $n + 1$ nodes, and $St = \{St^n\}_{n \in \mathbb{N}}$, the family of stellohedra, which is a family of graphical nestohedra where $St^n$ is generated from the star graph on $n + 1$ nodes. These families have generating functions which we chose to be:

$$Pe(x) = \sum_{n=0}^{\infty} Pe^n \frac{x^{n+1}}{(n+1)!}$$

$$St(x) = \sum_{n=0}^{\infty} St^n \frac{x^n}{n!}.$$

Theorem 2.4 gives formulas for $d$ of the individual nestohedra to be:

$$d(Pe^n) = \sum_{i+j=n-1} \binom{n+1}{i+1} Pe^i \times Pe^j$$

$$d(St^n) = n.St^{n-1} + \sum_{i=0}^{n-1} \binom{n}{i} St^i \times Pe^{n-i-1}.$$

Combining these two gives us that:

$$dPe(x) = \sum_{n=0}^{\infty} \left( \sum_{i+j=n-1} \binom{n+1}{i+1} Pe^i \times Pe^j \right) \frac{x^{n+1}}{(n+1)!} = Pe(x)^2$$

$$dSt(x) = \sum_{n=0}^{\infty} \left( n.St^{n-1} + \sum_{i=0}^{n-1} \binom{n}{i} St^i \times Pe^{n-i-1} \right) \frac{x^n}{n!} = (x + Pe(x)) St(x).$$

Passing to the $f$-polynomial and using equation (2.1) gives us partial differential equations:

$$\frac{\partial}{\partial t} Pe_f(\alpha, t, x) = Pe_f^2(\alpha, t, x)$$

$$\frac{\partial}{\partial t} St_f(\alpha, t, x) = (x + Pe_f(\alpha, t, x)) St_f(\alpha, t, x).$$

Solving these partial differential equations yields:

$$Pe_f(\alpha, t, x) = \frac{e^{\alpha x} - 1}{\alpha - t(e^{\alpha x} - 1)}$$

$$St_f(\alpha, t, x) = e^{(\alpha + t)x} \frac{\alpha}{\alpha - t(e^{\alpha x} - 1)}.$$

### 3 The Method

One of the applications of generating functions we will demonstrate in this paper is to show that these families of nestohedra, which as we have noted are flag, satisfy the Gal conjecture. In this section we will develop the methods which allow us to make these calculations.

**Conjecture 3.1** (Gal '05). For any flag simple polytope, $P$, the $\gamma$-polynomial $\gamma(P)$ has non-negative coefficients
Inspired by this conjecture we shall make two definitions. Firstly we shall fix a grading on the ring \( \mathbb{Q}[\alpha, t][[x]] \) where \( \deg(1) = 0 \), \( \deg(\alpha) = \deg(t) = -2 \) and \( \deg(x) = 2 \). Then,

**Definition 3.2.** A Gal series in \( \mathbb{Z}[\alpha, t][[x]] \) is an element \( \psi(\alpha, t, x) \), such that,

1. \( \psi(\alpha, t, x) = \psi(t, \alpha, x) = \hat{\psi}(a, b, x) \).
2. \( \psi \) is homogenous under the above grading.
3. \( \hat{\psi}(a, b, x) \) has all non-negative coefficients.

A family of nestohedra, \( \Psi \), satisfies the Gal conjecture precisely when \( \Psi_h(\alpha, t, x) \) is a Gal series. Note that by the nature of the \( h \)-polynomial, \( \Psi_h(\alpha, t, x) \) satisfies the fist two conditions for any family of simple polytopes. To show that \( \Psi_h(\alpha, t, x) \) is a Gal series we will employ the following lemma.

**Lemma 3.3.** Let \( \Psi \) be a family of polytopes, indexed by some set \( J \). If \( \Psi_h(\alpha, t, x) \) is such that:

- \( \frac{\partial \Psi_h(\alpha, t, x)}{\partial x} \mid_{x=0} \) is a Gal series.
- \( \frac{\partial \Psi_h(\alpha, t, x)}{\partial x} \) is a homogeneous polynomial

\[
F(a, b, \Psi_h(\alpha, t, x), S_{1,h}(\alpha, t, x), \ldots, S_{k,h}(\alpha, t, x)) \in \mathbb{Z}[\alpha, t][[x]]
\]

with non-negative coefficients, where the \( S_{i,h}(x) \), for \( i = 1, \ldots, k \), are Gal series.

Then \( \Psi_h(\alpha, t, x) \) is a Gal series and the polytopes in \( \Psi \) satisfy the Gal conjecture.

**Proof.** Let \( \frac{\partial \Psi_h(\alpha, t, x, y)}{\partial x} = F(a, b, \Psi_h(\alpha, t, x, y), S_{i,h}(\alpha, t, x)) \)

then by applying the standard substitutions that give the \( \gamma \)-polynomial we have that

\[
\frac{\partial \Psi(\tau, z)}{\partial z} = F(1, \tau, \Psi(\tau, z), S_{\gamma}(\tau, z)).
\] (3.1)

We have

\[
\Psi(\tau, z) = \sum_{j \in J} s(j) \gamma(\Psi^n_j) z^{n+q}
\]

\[
S_{\gamma}(\tau, z) = \sum_{n=1}^{\infty} S^n_{\gamma}(\alpha, t) z^n
\]

\[
\frac{\partial \Psi(\tau, z)}{\partial z} = \sum_{j \in J} (n+q) s(j) \gamma(\Psi^n_j) z^{n+q-1}
\]

where \( s(\Psi^n) \) are known and positive and \( S^n_{\gamma}(\alpha, t) \) is a homogeneous symmetric polynomial.

Examining equation (3.1) term by term in \( z \) gives an identity for \( \gamma(\Psi^n) \) expressed as a polynomial with non-negative coefficients in \( \tau, \gamma(\Psi^m) \) for \( m < n \) and \( S^m_{\gamma} \), \( i = 1, \ldots, k \), for \( m \leq n \). Thus since \( S^m_{\gamma} \) has non-negative coefficients of \( \tau \), if \( \gamma(\Psi^m) \) has all non-negative coefficients of \( \tau \), for all \( m < n \), then \( \gamma(\Psi^n) \) has all non-negative co-efficient of \( \tau \).

Since \( \frac{\partial \Psi_h}{\partial x} \mid_{x=0} = \gamma(\Psi^1) \), thus, by induction, \( \gamma(\Psi^n) \) has non-negative coefficients for all \( n \). Consequently \( \Psi_{\gamma}(\tau, z) = \Psi_h(a, b, x) \) has all non-negative coefficients and \( \Psi_h(\alpha, t, x, y) \) meets the third property for being a Gal series. Since it is related to \( \Psi(x, y) \) by the \( h \)-polynomial, it automatically meets the other two conditions and so is a Gal series.
The Families

Also in this paper we will calculate the $f$- and $h$-polynomials and demonstrate that the Gal conjecture holds for another family of graphical nestohedra. The family we will consider will be those nestohedra generated by the complete bipartite graphs $K_{m,n}$. However to make these calculations we will have to study some other families first. In this section we will define these families.

The graphs producing these families will take the form of the join of two graphs.

Definition 4.1. For two graphs $X^m$ and $Y^n$ we obtain the join, $X + Y = \Gamma_{X,m,Y,n}$, from $X \cup Y$ by adding in all edges between $X$ and $Y$.

We shall denote the resultant nestohedra as $P_{X,m,Y,n}$. The complete bipartite graph, $K_{m,n}$, is the join of the graph on $m$ nodes with no edges and the graph on $n$ nodes with no edges, so we will denote $K_{m,n}$ by $\Gamma_{m,n}$, the individual nestohedra by $P_{\Gamma,m,n}$ and $P_{\Gamma}$ will represent the entire family. The other families we shall look at will be generated by $\Gamma_{\nabla,m,n}$, $\Gamma_{\nabla,m,\Gamma,n}$ and $\Gamma_{\nabla,\Gamma,m,n}$ where $\nabla$ represents the complete graph. However $\Gamma_{\nabla,m,\Gamma,n}$ is the complete graph on $m + n$ nodes, so $P_{\nabla,m,\Gamma,n} = P_{\Gamma,m,n}$ and $\Gamma_{\nabla,\Gamma,m,n} = \Gamma_{\Gamma,n,\nabla,m}$. As such we only need to do the calculations for the families $P_{\nabla,\Gamma}$ and $P_{\Gamma,\nabla}$.

To preform these calculations we need to have generating functions for these families. We define these generating functions to be;

$$P_{\nabla,\Gamma}(x,y) = \sum_{k=0}^\infty \sum_{l=0}^\infty P_{\nabla,k,l} \frac{x^k y^l}{k! l!}$$

$$P_{\Gamma,\nabla}(x,y) = \sum_{k=0}^\infty \sum_{l=0}^\infty P_{\nabla,k,l} \frac{x^k y^l}{k! l!}$$

These generating functions have two variables $x$ and $y$, rather than just $x$. However, the addition of an extra variable is consistent with the motivation behind definition 2.1 and all the machinery used above extends by linearity to the two variable case.

5 The First Family, $P_{\nabla,1,\Gamma} = St^*$

We will now apply our method to $\Gamma_{\nabla,1,\Gamma}(x,y)$. We notice that this family of graphs is featured in [3] as the stellohedron $St(x)$, and we have that

$$St(x) = \sum_{n \geq 0} St^n \frac{x^n}{n!}$$

$$dSt(x) = (x + Pe(x))St(x)$$

$$St_f(\alpha,t,x) = e^{(\alpha t)x} \frac{\alpha}{\alpha - t(e^{\alpha x} - 1)}$$

which were obtained using the same method employed here.

However, this generating function does not match our general generating function for the join of a complete graph and an empty graph. Fortunately, the difference in the generating functions is a constant factor of $y$. So we can extend the above results to

$$P_{\nabla,1,\Gamma}(\alpha,t,x,y) = e^{(\alpha t)x} \frac{\alpha y}{\alpha - t(e^{\alpha x} - 1)}$$

We know from [1] that each individual stellohedron satisfies the Gal conjecture since it is generated by a chordal graph; that is one with no induced cycles of length 4 or more. However, we would like to show that our method works independently of this result, so we shall apply it to the series generated by the stellohedra.

Theorem 5.1. The series $P_{h,\nabla,1,\Gamma}(\alpha,t,x,y)$ is a Gal series.
Proof. We will use 3.3 for this. We begin by calculating the partial derivative of the $h$-polynomial with respect to $x$

$$P_{h,\nabla^{1}::}(\alpha, t, x, y) = e^{(\alpha + t)x}$$

$$\frac{\partial}{\partial x} P_{h,\nabla^{1}::}(\alpha, t, x, y) = \frac{(\alpha + t)y}{\alpha e^{\alpha x} - te^{\alpha x}}$$

since $Pe_{h}(\alpha, t, x)$ is a Gal series, $P_{h,\nabla^{1}::}(\alpha, t, x, y)$ fits the conditions of 3.3 if and only if $P_{h,\nabla^{1}::}(\alpha, 0, x, y)$ has non-negative coefficients.

$$P_{h,\nabla^{1}::}(\alpha, 0, x, y) = \sum_n \alpha^n \frac{x^n}{n!} = ye^{\alpha x},$$

does have non-negative coefficients so $P_{h,\nabla^{1}::}(\alpha, t, x, y)$ is a Gal series.

Thus each individual stellohedron satisfies the Gal conjecture.

6 The Second Family, $P_{\nabla^{i}::}$

In this section we wish to extend our calculations to show that those stellohedra generated by all possible graphs $\Gamma_{\nabla^{i}::, j}$ satisfy the Gal conjecture. Unlike in the previous section we must find the $f$-polynomial of this family explicitly. We start by calculating a formula for the derivative of the family.

Lemma 6.1. The formula for $d$ of the series $P_{\nabla^{i}::}$ is $dP_{\nabla^{i}::}(x, y) = P_{\nabla^{i}::}(x, y) (y + Pe(x + y)).$

Proof. We also have from 2.4 that

$$dP_{\nabla^{i}::} = tP_{\nabla^{i}::, t} + \sum_{i=1}^{s} \sum_{j=0}^{t} \binom{s}{i} \binom{t}{j} P_{\nabla^{i}::, j} \times Pe^{n-i-j}.$$
\[
\sum_{n=0}^{\infty} \sum_{k+l=n+1, k \geq 1} \sum_{i=1}^{k} \sum_{j=0}^{l} \binom{k}{i} \binom{l}{j} P_{\gamma, i \cdots j} Pe^{n-i-j} x^k y^j \frac{1}{(k)! (l)!}
\]
\[
= \sum_{n=0}^{\infty} \sum_{k+l=n+1, k \geq 1} \sum_{i=1}^{k} \sum_{j=0}^{l} \frac{k!}{(k-i)! i!} \frac{l!}{j!(l-j)!} P_{\gamma, i \cdots j} Pe^{n-i-j} x^k y^j \frac{1}{(k)! (l)!}
\]
\[
= \sum_{k=1}^{\infty} \sum_{l=0}^{\infty} \sum_{i=1}^{k} \sum_{j=0}^{l} \frac{1}{i! g! j! h!} P_{\gamma, i \cdots j} Pe^{g+h-1} x^k y^j
\]
\[
= \sum_{k=1}^{\infty} \sum_{l=0}^{\infty} \sum_{i=1}^{k} \sum_{j=0}^{l} \frac{1}{i! g! j! h!} P_{\gamma, i \cdots j} Pe^{g+h-1} x^i y^j y^h
\]
\[
= \sum_{k=1}^{\infty} \sum_{l=0}^{\infty} \sum_{i=1}^{k} \sum_{j=0}^{l} \sum_{g=0}^{\infty} \sum_{h=0}^{\infty} P_{\gamma, i \cdots j} Pe^{g+h-1} x^i y^j y^h \frac{1}{i! g! j! h!}
\]
\[
= \left( \sum_{i=1}^{\infty} \sum_{j=0}^{\infty} P_{\gamma, i \cdots j} x^i y^j \right) \left( \sum_{g=0}^{\infty} \sum_{h=0}^{\infty} Pe^{g+h-1} \frac{x^g y^h}{g! h!} \right)
\]
\[
= P_{\gamma, \gamma}(x, y) Pe(x + y).
\]

Recombining this we get
\[
dP_{\gamma, \gamma}(x, y) = P_{\gamma, \gamma}(x, y) (y + Pe(x + y)).
\]

Now that we have a formula for the derivative, we use it along with equation 2.1 to calculate the \(f\)-polynomial of this family.

**Lemma 6.2.** We have that \(P_{f, \gamma, \gamma}(x, y) = e^{(\alpha + t)y} \frac{\eta(x)}{1 - t\eta(x+y)}\).

**Proof.** We can the pass to the face polynomial, we set \(\eta(z) = \frac{e^{\alpha z} - 1}{\alpha}\) and we get
\[
\frac{\partial}{\partial t} P_{f, \gamma, \gamma}(x, y) = P_{f, \gamma, \gamma}(x, y) (y + Pe(f(x + y))
\]
\[
= P_{f, \gamma, \gamma}(x, y) \left( y - \frac{\partial}{\partial t} \ln(1 - t\eta(x + y)) \right)
\]
\[
\frac{\partial}{\partial t} \ln(P_{f, \gamma, \gamma}(x, y)) = y - \frac{\partial}{\partial t} \ln(1 - t\eta(x + y))
\]
\[
P_{f, \gamma, \gamma}(x, y) = e^{\text{ust}} \frac{1}{1 - t\eta(x+y)}.
\]
Looking at \( t = 0 \) we have initial conditions \( P_{f,\nabla}:(\alpha, 0, x, y) = e^{\alpha y}\eta(x) \) and so

\[
P_{f,\nabla}:(x, y) = e^{(\alpha+t)y} \frac{\eta(x)}{1-t\eta(x+y)}.
\]

Letting \( y = 0 \) we have that \( P_{f,\nabla}:(x, 0) = \frac{\eta(x)}{1-t\eta(x)} = P_e(x) \) as expected, since \( P_{\nabla,\nabla}:(0) = P_e^\ast \).

Now that we have the \( f \)-polynomial of this family, we can repeat theorem 3.1 for this family to show that it too consists of Gal polytopes.

**Theorem 6.3.** \( P_{h,\nabla}:(\alpha, t, x, y) \) is a Gal series.

**Proof.** First we calculate the \( h \) polynomial and we have

\[
P_{h,\nabla}:(x, y) = P_{f,\nabla}:(\alpha-t, t, x, y) = e^{\alpha y}e^{ty} - e^{tx}.
\]

So, we calculate the partial derivative or the \( h \)-polynomial with respect to \( x \),

\[
\frac{\partial}{\partial x} P_{h,\nabla}:(\alpha, t, x) = e^{(\alpha+t)y} \frac{e^{\alpha x} - e^{tx}}{\alpha e^{t(x+y)} - te^{\alpha(x+y)}}
\]

\[
\frac{\partial}{\partial y} \frac{\alpha e^{\alpha x} - te^{tx}}{\alpha e^{t(x+y)} - te^{\alpha(x+y)}} = \frac{-(\alpha e^{\alpha x} - te^{tx})(ate^{t(x+y)} - e^{\alpha}(x+y))}{(\alpha e^{t(x+y)} - te^{\alpha(x+y)})^2}
\]

\[
= \frac{\alpha t e h(\alpha, t, x + y)}{\alpha e^{\alpha x} - te^{tx}} \phi_{h}(\alpha, t, x, y)
\]

where \( \phi_{h}(\alpha, t, x, y) = \frac{\alpha e^{\alpha x} - te^{tx}}{\alpha e^{t(x+y)} - te^{\alpha(x+y)}} \). By 3.3 \( P_{h,\nabla}:(\alpha, t, x, y) \) is Gal if \( \phi_{h}(\alpha, t, x, y) \) is Gal. We now apply lemma 3.3 to \( \phi_{h}(\alpha, t, x, y) \) differentiating with respect to \( y \) rather than \( x \).

\[
= \frac{\alpha t e h(\alpha, t, x + y) \phi_{h}(\alpha, t, x, y)}{\alpha e^{\alpha x} - te^{tx}}
\]

and we have that

\[
\frac{\partial \phi_{h}(\alpha, t, x, y)}{\partial y} \bigg|_{y=0} = \frac{\alpha e^{\alpha x} - te^{tx}}{\alpha e^{tx} - te^{\alpha x}} = 1 + (\alpha + t)pe h(\alpha, t, x)
\]

which is Gal. So by repeated application of the lemma, \( \phi_{h} \) is Gal and so is \( P_{h,\nabla}:(\alpha, t, x, y) \).

## 7 The Final Family, \( \Gamma_{\nabla,\nabla} \)

With these preliminaries over, we can finally move on to demonstrate that the Gal conjecture holds for the family \( P_{\nabla,\nabla} \), which is generated by a family of graphs which in general is non-chordal. We will follow the same steps as used in section 6. So we start by finding a formula for the derivative of \( P_{\nabla,\nabla} \).

**Lemma 7.1.** We shall show that \( dP_{\nabla,\nabla}:(x, y) \) is

\[
xP_{\nabla,\nabla}:(y, x) + yP_{\nabla,\nabla}:(y, x) + P_{\nabla,\nabla}:(x, y)Pe(x + y) - (x + y)Pe(x + y).
\]
Proof. By expanding on the work of N. Erokhovets in [6] we have that, for $s, t \geq 2$,

$$dP_{s,t} = sP_{t-1,1} + tP_{s-1,1} + \sum_{i=1}^{s-1} \sum_{j=1}^{t-1} \binom{s}{i} \binom{t}{j} P_{i,j} \times Pe^{t-i-j-1} + \sum_{i=1}^{s-1} P_{i,t} \times Pe^{s-i-1} + \sum_{j=1}^{t-1} P_{s,j} \times Pe^{t-j-1}.$$ 

when either $s < 2$ or $t < 2$ there are only two possible outcomes. Since, if $s = 0$ then we must have $t = 1$ for the graph to be connected and vice versa, we must have either $s = 1$ or $t = 1$ or both. Here we notice that $P_{1,k} = P_{k,1} = St^k$.

Let us now examine the generating function $P_{s,t}(x, y)$, since we have two distinct indices the generating function is of two variables. We have

$$P_{s,t}(x, y) = \sum_{n=0}^{\infty} \sum_{k+l=n+1, l \geq 0} P_{s,t} \cdot \frac{x^k y^l}{k! l!} = \sum_{n=0}^{\infty} \sum_{k+l=n+1, l \geq 2} P_{s,t} \cdot \frac{x^k y^l}{k! l!} + \sum_{n=0}^{\infty} \sum_{k=n, l=1} P_{s,t} \cdot \frac{x^k}{k!} y^l$$

$$+ \sum_{n=0}^{\infty} \sum_{l=n, k=1} P_{s,t} \cdot \frac{x^k}{k!} y^l - P_{s,t}(x, y)$$

$$= \sum_{n=0}^{\infty} \sum_{k=2}^{\infty} P_{s,t} \cdot \frac{x^k}{k!} y^n + \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} P_{s,t} \cdot \frac{x^k}{k!} y^n + \sum_{n=0}^{\infty} \sum_{k=n, l=1} P_{s,t} \cdot \frac{x^k}{k!} y^n - P_{s,t}(x, y)$$

$$= \sum_{n=0}^{\infty} \sum_{k=2}^{\infty} P_{s,t} \cdot \frac{x^k}{k!} y^n + P_{\infty,1}(x, y) + P_{\infty,1}(y, x) - I^1 xy.$$ 

so

$$dP_{s,t}(x, y) = d \sum_{k=2}^{\infty} \sum_{l=2}^{\infty} P_{s,t} \cdot \frac{x^k y^l}{k! l!} + dP_{\infty,1}(x, y) + dP_{\infty,1}(y, x) - (dI^1) xy$$

$$= \sum_{k=2}^{\infty} \sum_{l=2}^{\infty} \left( kP_{s,t} \cdot \frac{x^k y^l}{k! l!} + lP_{s,t} \cdot \frac{x^k y^l}{k! l!} + \sum_{i=1}^{k-1} \sum_{j=1}^{l-1} \binom{k}{i} \binom{l}{j} P_{s,t} \cdot \frac{x^k y^l}{k! l!} \right)$$

$$+ \sum_{k=2}^{\infty} \sum_{l=2}^{\infty} \frac{k}{k! l!} kP_{s,t} \cdot \frac{x^k y^l}{k! l!} + \sum_{k=2}^{\infty} \sum_{l=2}^{\infty} \frac{l}{k! l!} lP_{s,t} \cdot \frac{x^k y^l}{k! l!}$$

$$+ \sum_{k=2}^{\infty} \sum_{l=2}^{\infty} \frac{k}{k! l!} \frac{k-1}{l-1} \sum_{i=1}^{k-1} \sum_{j=1}^{l-1} \binom{k}{i} \binom{l}{j} P_{s,t} \cdot \frac{x^k y^l}{k! l!}$$

$$+ \sum_{k=2}^{\infty} \sum_{l=2}^{\infty} \frac{k}{k! l!} \frac{k-1}{l-1} \sum_{i=1}^{k-1} \sum_{j=1}^{l-1} \binom{k}{i} \binom{l}{j} P_{s,t} \cdot \frac{x^k y^l}{k! l!}$$

$$+ (x + Pe(x))P_{\infty,1}(x, y) + (y + Pe(y))P_{\infty,1}(y, x) - (dI^1) xy.$$
Let us now consider each sum in turn. Taking the first sum, we have that
\[
\sum_{k=2}^{\infty} \sum_{l=2}^{\infty} \frac{x^k}{k!} \frac{y^l}{l!} kP_{k+1,\nu,l} = x \sum_{k=2}^{\infty} \sum_{l=2}^{\infty} \frac{x^{k-1}}{(k-1)!} \frac{y^l}{l!} P_{k+1,\nu,l}
\]
\[
= x \sum_{k=1}^{\infty} \sum_{l=2}^{\infty} \frac{x^k}{k!} \frac{y^l}{l!} P_{k,\nu,l}
\]
\[
= x \sum_{k=0}^{\infty} \sum_{l=1}^{\infty} \frac{x^k}{k!} \frac{y^l}{l!} P_{k,\nu,l}
\]
\[
-x \sum_{k=1}^{\infty} \frac{x^k}{k!} yP_{k+1,\nu,1} - x \sum_{l=1}^{\infty} \frac{y^l}{l!} P_{0,\nu,1}
\]
\[
= xP_{\nu,1}(y,x) - x(P_{\nu,1}(x,y) - y) - x \sum_{l=1}^{\infty} \frac{y^l}{l!}
\]
\[
= xP_{\nu,1}(y,x) - x(P_{\nu,1}(x,y) - y) - xPe(y).
\]

It is clear that the second sum is the same as the first sum with \(x\) and \(y\) reversed. Proceeding to the third sum, setting \(g = k - i\) and \(h = l - j\), we have
\[
\sum_{k=2}^{\infty} \sum_{l=2}^{\infty} \frac{x^k}{k!} \frac{y^l}{l!} \sum_{i=1}^{k-1} \sum_{j=1}^{l-1} \binom{k}{i} \binom{l}{j} P_{i,j} \times Pe^{k-i-j-1}
\]
\[
= \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \sum_{i=2}^{k} \sum_{j=1}^{l} P_{i,j} \times Pe^{k-i-j-1} \frac{x^i}{i!} \frac{x^{k-i}}{(k-i)!} \frac{y^j}{j!} \frac{y^{l-j}}{(l-j)!}
\]
\[
= \sum_{g=1}^{\infty} \sum_{h=1}^{\infty} \sum_{i=1}^{g} \sum_{j=1}^{h} P_{i,j} \times Pe^{g+h-1} \frac{x^i}{i!} \frac{x^g}{g!} \frac{y^j}{j!} \frac{y^h}{h!}
\]
\[
= (P_{\nu,1}(x,y) - (x+y)) \left( Pe(x+y) - \sum_{g=1}^{\infty} \frac{Pe^{g-1}x^g}{g!} - \sum_{h=1}^{\infty} \frac{Pe^{h-1}y^h}{h!} \right)
\]
\[
= (P_{\nu,1}(x,y) - (x+y))(Pe(x+y) - Pe(x) - Pe(y)).
\]

Again we notice the similarities between the fourth and fifth sums. We examine the fourth in detail,
again with $g = k - i$.

\[
\sum_{k=2}^{\infty} \sum_{l=2}^{k-1} \frac{x^k y^l}{k! l!} \sum_{i=1}^{k-1} \left( \begin{array}{c} k \\ i \end{array} \right) P_{i,i}, l \times Pe^{k-i-1} = \sum_{k=2}^{\infty} \sum_{l=2}^{k-1} \sum_{i=1}^{k-1} P_{i,i}, l \times Pe^{k-i-1} \frac{x^{k-i} x^i y^l}{(k-i)! i! l!} \\
= \sum_{g=1}^{\infty} \sum_{i=1}^{\infty} \sum_{l=2}^{g-1} P_{g,i}, l \times Pe^{g-1} \frac{x^g x^i y^l}{g! i! l!} \\
= \left( \sum_{i=1}^{\infty} P_{i,i}, l \frac{x^i y^l}{i! l!} \right) \left( \sum_{g=1}^{\infty} Pe^{g-1} \frac{x^g}{g!} \right) \\
= \left( \sum_{i=1}^{\infty} \sum_{l=0}^{\infty} P_{i,i}, l \frac{x^i y^l}{i! l!} \right) \left( \sum_{g=1}^{\infty} Pe^{g-1} \frac{x^g}{g!} \right) \\
= (P_{i,i}, (x, y) - x - P_{i,1}, (x, y)) (Pe(x)).
\]

We can combine all of these to get

\[
dP_{i,i},(x, y) = \sum_{k=2}^{\infty} \sum_{l=2}^{k-1} \frac{x^k y^l}{k! l!} kP_{i,i}, k, \nabla, l + \sum_{k=2}^{\infty} \sum_{l=2}^{k-1} \frac{x^k y^l}{k! l!} lP_{i,i}, k, l-1 \\
+ \sum_{k=2}^{\infty} \sum_{l=2}^{k-1} \frac{x^k y^l}{k! l!} \sum_{i=1}^{k-1} \sum_{j=1}^{l-1} \left( \begin{array}{c} k \\ i \end{array} \right) \left( \begin{array}{c} l \\ j \end{array} \right) P_{i,i}, i, j \times Pe^{k+i-1-j-1} \\
+ \sum_{k=2}^{\infty} \sum_{l=2}^{k-1} \frac{x^k y^l}{k! l!} \sum_{i=1}^{k-1} \left( \begin{array}{c} k \\ i \end{array} \right) P_{i,i}, i, l \times Pe^{k-i-1} \\
+ \sum_{k=2}^{\infty} \sum_{l=2}^{k-1} \frac{x^k y^l}{k! l!} \sum_{j=1}^{l-1} \left( \begin{array}{c} l \\ j \end{array} \right) P_{i,i}, k, j \times Pe^{l-j-1} \\
+ (x + Pe(x))P_{i,1}, (x, y) + (y + Pe(y))P_{i,1}, (y, x) - d(I^1)xy \\
= xP_{i,i},(y, x) - xP_{i,1},(y, x) - y - xPe(y) + yP_{i,i},(x, y) - yP_{i,1},(y, x) - y - xPe(x) \\
+ (P_{i,i},(x, y) - (x+y)) (Pe(x) - Pe(y)) \\
+ (P_{i,i},(x, y) - x - P_{i,1},(x, y)) (Pe(x)) + (P_{i,1},(x, y) - y - P_{i,1},(x, y)) (Pe(y)) \\
+ (x + Pe(x))P_{i,1},(x, y) + (y + Pe(y))P_{i,1},(y, x) - d(I^1)xy \\
= xP_{i,i},(y, x) + yP_{i,i},(x, y) + P_{i,1},(x, y) Pe(x + y) - (x + y)Pe(x + y).
\]

\[
\square
\]

Now that we have the formula expressing the boundary of this series of polytopes we can, as before, use it to calculate the $f$-polynomial of the series. Solving the appropriate differential equation, as set out below, gives us:

**Lemma 7.2.** We have that, with $\eta(x)$ as before, $P_{f,i,i},(\alpha, t, x, y)$ is

\[
\frac{1}{1 - t\eta(x + y)} \left( e^{(\alpha + t)x} \eta(y) + e^{(\alpha + t)y} \eta(x) + \alpha \eta(y) \eta(x) - e^{\alpha x} \eta(y) - e^{\alpha y} \eta(x) \right) + (x + y).
\]
Proof. By the identity [2.1] we have

\[
\frac{\partial}{\partial t} P_{f,:\cdots}(x, y) = xP_{f,:\cdots}(y, x) + yP_{f,:\cdots}(x, y) + P_{f,:\cdots}(x, y)Pe_f(x + y) - (x + y)Pe_f(x + y)
\]

\[
= xe^{(\alpha+t)x} \frac{\eta(y)}{1 - t\eta(x + y)} + ye^{(\alpha+t)y} \frac{\eta(x)}{1 - t\eta(x + y)} + P_{f,:\cdots}(x, y) \frac{\eta(x + y)}{1 - t\eta(x + y)} - (x + y) \frac{\eta(x + y)}{1 - t\eta(x + y)}.
\]

To solve this we shall start by setting \( \hat{P} = P_{f,:\cdots} - (x + y) \), then we have

\[
\frac{\partial}{\partial t} \hat{P}(x, y) = xe^{(\alpha+t)x} \eta(y) + ye^{(\alpha+t)y} \eta(x) + \hat{P}(x, y)\eta(x + y) \]

If we now set \( \hat{P} = P_1P_2 \) and \( P_1 = \frac{c_1}{1 - t\eta(x + y)} \) then we get, by application of the quotient rule and integrating,

\[
P_2 = e^{(\alpha+t)x} \eta(y) + e^{(\alpha+t)y} \eta(x) + c_2.
\]

Combining all these we have

\[
P_2(\alpha, t, x, y) = e^{(\alpha+t)x} \eta(y) + e^{(\alpha+t)y} \eta(x) + c_2
\]

\[
\hat{P}(\alpha, t, x, y) = \frac{c_1}{1 - t\eta(x + y)} \left( e^{(\alpha+t)x} \eta(y) + e^{(\alpha+t)y} \eta(x) + c_2 \right)
\]

\[
P_{f,:\cdots}(\alpha, t, x, y) = \frac{c_1}{1 - t\eta(x + y)} \left( e^{(\alpha+t)x} \eta(y) + e^{(\alpha+t)y} \eta(x) + c_2 \right) + (x + y).
\]

Examining the initial conditions we have that

\[
P_{f,:\cdots}(\alpha, 0, x, y) = \sum_{n=0}^{\infty} \sum_{k+l=n+1, k,l > 0} \alpha^n \frac{x^k y^l}{k! l!} + x + y
\]

\[
= \sum_{n=0}^{\infty} \sum_{k+l=n+1, k,l > 0} \alpha^n \frac{x^k y^l}{k! l!} - \sum_{n=0}^{\infty} \sum_{k+l=n+1, k > 0, l = 0} \alpha^n \frac{x^k}{k!} + x + y
\]

\[
= \sum_{n=0}^{\infty} \sum_{k+l=n+1, k,l > 0} \alpha^n x^k \frac{k!}{k! l!} - \sum_{k=1}^{\infty} \alpha^{k+1} \frac{x^k}{k!} + x + y
\]

\[
= P_{f,:\cdots}(\alpha, 0, x, y) - \eta(x) + x + y
\]

\[
e^{\alpha x} \eta(x) - \eta(x) + x + y
\]

\[
= \alpha \eta(y) \eta(x) + x + y
\]

so, setting \( c_1 = 1 \), we have

\[
\alpha \eta(y) \eta(x) + x + y = \frac{1}{1} \left( e^{\alpha x} \eta(y) + e^{\alpha y} \eta(x) + c_2 \right) + (x + y)
\]

\[
c_2 = \alpha \eta(y) \eta(x) - e^{\alpha x} \eta(y) - e^{\alpha y} \eta(x).
\]

Then we have

\[
P_{f,:\cdots}(\alpha, t, x, y) = \frac{1}{1 - t\eta(x + y)} \left( e^{(\alpha+t)x} \eta(y) + e^{(\alpha+t)y} \eta(x) + \alpha \eta(y) \eta(x) - e^{\alpha x} \eta(y) - e^{\alpha y} \eta(x) \right) + (x + y).
\]
With the $f$-polynomial of the family now calculated we can demonstrate that $P_{\gamma,\ldots,\gamma}$ is a family of polytopes that satisfy the Gal conjecture. We will do this in the same way we did for the families $P_{\nu,1,\ldots,\gamma}$ and $P_{\gamma,\ldots,\gamma}$, using lemma [3.3]. Unlike in the previous sections we can no longer use the result from [1] since for $n,m \geq 2$, $\Gamma_{\gamma,\ldots,\gamma,m}$ is not a chordal graph.

**Theorem 7.3.** $P_{h,\ldots,\gamma}(\alpha,t,x,y)$ is a Gal series.

**Proof.** As before we start by calculating the series $P_{h,\ldots,\gamma}(\alpha,t,x,y)$.

$$P_{h,\ldots,\gamma}(\alpha,t,x,y) = P_{f,\ldots,\gamma}(\alpha - t, t, x, y)$$

$$= \frac{1}{1 - t^{e^{(\alpha-t)x}y}} \left( e^{((\alpha-t)+t)x} - 1 \right) + e^{(\alpha-t)x} - 1 \right) + e^{(\alpha-t)y} - 1 \right) + (x + y)$$

$$= \frac{1}{1 - t^{e^{(\alpha-t)x}y}} \left( e^{\alpha y e^{ax} - e^{ax} e^{ty} + e^{ax} e^{ay} e^{tx} e^{ty} - e^{ay} e^{tx} e^{ty} + e^{tx} e^{ty} - e^{ax} e^{ay} + \frac{1}{(\alpha-t)} \left( (x + y) \right) \right)$$

$$= \frac{1}{1 - t^{e^{(\alpha-t)x}y}} \left( e^{\alpha y e^{ax} - e^{ax} e^{ty} + e^{ax} e^{ay} e^{tx} e^{ty} - e^{ay} e^{tx} e^{ty} + e^{tx} e^{ty} - e^{ax} e^{ay} + \frac{1}{(\alpha-t)} \left( (x + y) \right) \right)$$

So by our lemma, $P_{h,\ldots,\gamma}$ is a Gal series if $\phi_h(\alpha,t,x,y) = \frac{\alpha e^{ax} - e^{tx}}{\alpha e^{t(x+y)} - e^{(x+y)}}$ is a Gal series. We showed that this series was Gal in the previous section, so $P_{\gamma,\ldots,\gamma}$ is a Gal series by [3.3].
References

[1] Alexander Postnikov, Victor Reiner and Lauren Williams, “Faces of Generalised Permutohedra” \[\text{arXiv:math/0609184v2 [math.CO]}\].

[2] Buchstaber, Victor M. and Panov, Taras E., \textit{Torus Actions and Their Applications in Topology and Combinatorics} University Lecture Series, Volume 24, American Mathematical Society, 2002.

[3] Buchstaber, Victor M., “Ring of Simple Polytopes and Differential Equations,” \textit{Proceedings of the Steklov Institute of Mathematics}, volume 263, (Pleiades Publishing, Ltd., 2008): 1–25.

[4] Gal., S. R., “Real Root Conjecture Fails for Five and Higher Dimensional Spheres,” \texttt{arXiv:math/0501046v1 [math.CO]}.

[5] Fenn, A. G., “The Face-Polynomial of Nestohedra” in \textit{Differential Equations and Topology: Abstr. Int. Conf. Dedicated to the Centennial Anniversary of L.S. Pontryagin, Moscow, June 17-22, 2008} (Maks Press, Moscow, 2008), p. 435.

[6] Erokhovets, N. Yu., “Permutations of Vertices of Polyhedra and the Nestohedra $K_{m,n}$,” \textit{Proceedings of the Steklov Institute of Mathematics}, volume 266, (Pleiades Publishing, Ltd., 2008): Accepted.

[7] Bollobás, Béla, \textit{Graph Theory, An Introductory Course}, Graduate Texts in Mathematics, Volume 63, Springer-Verlag, 1979.