Uniformly resolvable decompositions of $K_v$
into $P_3$ and $K_3$ graphs

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Abstract

In this paper we consider the uniformly resolvable decompositions of the complete graph $K_v$, or the complete graph minus a 1-factor as appropriate, into subgraphs such that each resolution class contains only blocks isomorphic to the same graph. We completely determine the spectrum for the case in which all the resolution classes are either $P_3$ or $K_3$.

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1 Introduction and Definitions

Given a collection $\mathcal{H}$ of graphs, an $\mathcal{H}$-decomposition of a graph $G$ is a decomposition of the edge set of $G$ into subgraphs isomorphic to the members of $\mathcal{H}$. The copies of $H \in \mathcal{H}$ in the decomposition are called blocks. Such a decomposition is called resolvable if it is possible to partition the blocks into classes $\mathcal{P}_i$ (often referred to as parallel classes) such that every vertex of $G$ appears in exactly one block of each $\mathcal{P}_i$.

A resolvable $\mathcal{H}$-decomposition of $G$ is sometimes also referred to as an $\mathcal{H}$-factorization of $G$, and a class can be called an $\mathcal{H}$-factor of $G$. The case where $\mathcal{H} = K_2$ (a single edge) is known as a 1-factorization; for $G = K_v$, it is well known to exist if and only if $v$ is even. A single class of a 1-factorization, that is a pairing of all vertices, is also known as a 1-factor or perfect matching.

In many cases we wish to place further constraints on the classes. For example, a class is called uniform if every block of the class is isomorphic to the same graph from $\mathcal{H}$. Of particular note is the result of Rees [12] which finds necessary and sufficient conditions for the existence of uniform $\{K_2, K_3\}$-decompositions of $K_v$. Uniformly resolvable decompositions of $K_v$ have also been studied in [4], [5], [8], [10], [11], [14] and [15].

In this paper we study the existence of uniformly resolvable decompositions into paths $P_3$ and cycles $K_3 \cong C_3$ (both having three vertices) for the complete graph $K_v$ and for the complete graph minus a 1-factor, which we denote by $K_v - I$. The existence of resolvable decompositions for each of $P_3$ and $K_3$ was studied separately already long ago:

- There exists a resolvable $K_3$-decomposition of $K_v$ (called Kirkman Triple System, denoted as KTS($v$)) if and only if $v \equiv 3$ (mod 6).
- There exists a resolvable $K_3$-decomposition of $K_v - I$ (called Nearly Kirkman Triple System, denoted as NKTS($v$)) if and only if $v \equiv 0$ (mod 6) and $v > 12$ [13].
- There exists a resolvable $P_3$-decomposition of $K_v$ if and only if $v \equiv 9$ (mod 12) [9].
- There exists a resolvable $P_3$-decomposition of $K_v - I$ if and only if $v \equiv 6$ (mod 12). (This follows from the case $v = 6$ and from the spectrum of KTS($v$) systems.)
Further results on resolvable path decompositions are given in [7]. Let now

- $G = K_v$ for $v$ odd,
- $G = K_v - I$ for $v$ even,

and let

$$URD(v; P, K) := \{(r, s) : \text{there exists a uniformly resolvable decomposition of } G \text{ into } r \text{ classes containing only copies of } P \text{ and } s \text{ classes containing only copies of } K\}.$$ 

For $v \geq 3$, divisible by 3, define $I(v)$ according to the following table, where the first two lines are meant for $v \geq 18$ only:

| $v$         | $I(v)$                                      |
|-------------|---------------------------------------------|
| $0 \pmod{12}$ | $\{ (3x, \frac{v-2}{2} - 2x), x = 0, 1, \ldots, \frac{v-5}{4} \}$ |
| $6 \pmod{12}$ | $\{ (3x, \frac{v-2}{2} - 2x), x = 0, 1, \ldots, \frac{v-1}{4} \}$ |
| $3 \pmod{12}$ | $\{ (3x, \frac{v-1}{2} - 2x), x = 0, 1, \ldots, \frac{v-3}{4} \}$ |
| $9 \pmod{12}$ | $\{ (3x, \frac{v-1}{2} - 2x), x = 0, 1, \ldots, \frac{v-1}{4} \}$ |
| 6           | $\{ (3, 0) \}$                              |
| 12          | $\{ (3, 3), (6, 1) \}$                      |

Table 1: The set $I(v)$.

In this paper we completely solve the spectrum problem for such systems; i.e., characterize the existence of uniformly resolvable decompositions of $K_v$ and $K_v - I$ into $r$ classes of 3-paths and $s$ classes of 3-cycles, by proving the following result:

**Main Theorem.** For every integer $v \geq 3$, divisible by 3, the set $URD(v; P, K)$ is identical to the set $I(v)$ given in Table 1.

**Notation.** In the constructive parts of the proof we shall use the following notation, where $a_1, a_2, a_3$ may mean any three distinct vertices:

- $(a_1, a_2, a_3)$ denotes the 3-cycle $K_3$ having vertex set $\{a_1, a_2, a_3\}$ and edge set $\{\{a_1, a_2\}, \{a_2, a_3\}, \{a_3, a_1\}\}$;
- $(a_1; a_2, a_3)$ denotes the path $P_3$ having vertex set $\{a_1, a_2, a_3\}$ and edge set $\{\{a_1, a_2\}, \{a_1, a_3\}\}$.
2 Preliminaries and necessary conditions

In this section we introduce some useful definitions and give necessary conditions for
the existence of a uniformly resolvable decomposition of $K_v$ into $P_3$ and $K_3$ graphs.
For missing terms or results that are not explicitly explained in the paper, the
reader is referred to [2] and its online updates. Evidently, for a uniformly resolvable
decomposition of $K_v$ into $P_3$ and $K_3$ graphs to exist, $v$ must be a multiple of 3. A
(resolvable) $H$-decomposition of the complete multipartite graph with $u$ parts each
of size $g$ is known as a (resolvable) group divisible design $H$-(R)GDD; the parts of
size $g$ are called the groups of the design. When $H = K_n$, we call it an
$n$-(R)GDD.

A 3-RGDD of type $g^u$ exists if and only if $g(u - 1)$ is even and $gu \equiv 0 \pmod{3}$,
except when $(g, u) \in \{(2, 6), (2, 3), (6, 3)\}$ [13]. One can see, in particular, that a
3-RGDD of type $2^u$ is a Nearly Kirkman Triple System (NKTS($2^u$)); we mentioned
its spectrum in the Introduction.

Lemma 2.1. Let $v \equiv 3 \pmod{6}$. If $(r, s) \in URD(v; P_3, K_3)$ then $(r, s) \in I(v)$.

Proof. For $v$ odd, we have $G = K_v$. Let $D$ be a decomposition of $K_v$ into $r$ classes
of $P_3$ and $s$ classes of $K_3$ graphs. Counting the edges of $K_v$ that appear in $D$ we obtain

$$\frac{v}{3} \cdot (2r + 3s) = \frac{v(v - 1)}{2},$$

and hence that

$$2r + 3s = \frac{3}{2} (v - 1).$$

This equation implies that $2r \equiv \frac{3}{2} (v - 1) \pmod{3}$ and $3s \equiv \frac{3}{2} (v - 1) \pmod{2}$.

Then we obtain

- $r \equiv 0 \pmod{3}$ and $s \equiv 1 \pmod{2}$ for $v \equiv 3 \pmod{12}$,
- $r \equiv 0 \pmod{3}$ and $s \equiv 0 \pmod{2}$ for $v \equiv 9 \pmod{12}$.

In either case, introducing the notation $x = r/3$, the equation (1) determines that
$s = \frac{v - 3x}{2} - 2x$ must hold. Since $r$ and $s$ cannot be negative, and $x$ is an integer, the
value of $x$ has to be in the range as given in the definition of $I(v)$.

Lemma 2.2. Let $v \equiv 0 \pmod{6}$. If $(r, s) \in URD(v; P_3, K_3)$ then $(r, s) \in I(v)$.
Proof. For \( v \) even, we have \( G = K_v - I \). The argument is similar to the one for \( v \) odd. Let \( D \) be a decomposition of \( K_v - I \) into \( r \) classes of \( P_3 \) and \( s \) classes of 3-cycles. Counting the edges of \( K_v \) that appear in \( D \) we obtain

\[
\frac{v}{3} \cdot (2r + 3s) = \frac{v(v - 2)}{2},
\]

and hence that

\[
2r + 3s = \frac{3}{2} (v - 2). \tag{2}
\]

This equation implies that \( 2r \equiv \frac{3}{2} (v - 2) \pmod{3} \) and \( 3s \equiv \frac{3}{2} (v - 2) \pmod{2} \).

Then we obtain

- \( r \equiv 0 \pmod{3} \) and \( s \equiv 1 \pmod{2} \) for \( v \equiv 0 \pmod{12} \),
- \( r \equiv 0 \pmod{3} \) and \( s \equiv 0 \pmod{2} \) for \( v \equiv 6 \pmod{12} \).

In either case, denoting \( x = r/3 \), the equation (2) yields \( s = \frac{v-1}{2} - 2x \). Since \( r \) and \( s \) cannot be negative, and \( x \) is an integer, the value of \( x \) has to be in the range as given in the definition of \( I(v) \).

3 Small cases

Here we handle the two exceptional cases, namely \( v = 6 \) and \( v = 12 \), for which the set \( I(v) \) is slightly more restricted than for larger \( v \).

Lemma 3.1. \( URD(6; P_3, K_3) = \{(3,0)\} \).

Proof. The case \( r = 0 \) would correspond to an NKTS(6), which does not exist [13]. On the other hand, for \( r = 3 \) and \( s = 0 \) we can take the groups to be \{0,1\}, \{2,3\}, \{4,5\} and the three classes \{(0;2,4), (1;3,5)\}, \{(2;4,1), (3;5,0)\}, (4;1,3), \{(5;2,0)\}.

Lemma 3.2. \( URD(12; P_3, K_3) = \{(3,3), (6,1)\} \).

Proof. The case \( r = 0 \) would correspond to an NKTS(12), which does not exist [13]. For the other two cases, the following systems prove the assertion:
Moreover, the edges of $B$ of $G$ each vertex appears in precisely one block of $Q$. $Q$ represent the intersection structure of classes of $Q$.

Proof. Let $q \equiv 0 \pmod{3}, v \geq 9$. The union of any two edge-disjoint parallel classes of $3$-cycles of $K_v$ can be decomposed into three parallel classes of $P_3$.

Lemma 4.1. Let $v \equiv 0 \pmod{3}, v \geq 9$. The union of any two edge-disjoint parallel classes of $3$-cycles of $K_v$ can be decomposed into three parallel classes of $P_3$.

Proof. Let $Q' = \{q'_1, \ldots, q'_{v/3}\}$ and $Q'' = \{q''_1, \ldots, q''_{v/3}\}$ be two edge-disjoint parallel classes of $K_3$, whose union composes the edge set of graph $G$ on $v$ vertices. We represent the intersection structure of $Q'$ and $Q''$ with a bipartite graph $B$ with vertex bipartition $X' \cup X''$, where $|X'| = |X''| = v/3$ and each vertex $x'_i \in X'$ and $x''_j \in X''$ for $1 \leq i, j \leq v/3$ corresponds to a block $q'_i \in Q'$ and $q''_j \in Q''$, respectively. Vertex $x'_i$ is connected to vertex $x''_j$ by an edge of $B$ if their corresponding blocks $q'_i$ and $q''_j$ have a vertex in common.

Every block of $Q'$ ($Q''$) meets exactly three distinct blocks of $Q''$ ($Q'$) because each vertex appears in precisely one block of $Q'$ and also of $Q''$, and no vertex pair of $G$ is contained in blocks of both classes. Thus, $B$ is a 3-regular bipartite graph. Moreover, the edges of $B$ are in one-to-one correspondence with the vertices of $G$, and $G$ is the line graph of $B$. We are going to define three edge decompositions of $B$, each of them being the union of $v/3$ mutually edge-disjoint copies of $P_3$ starting

4 Constructions for general $v$

The key tool in this section is the following important lemma. At the end of the paper we give some related information in the “Historical remarks and acknowledgements”.

\[ (3, 3) \in URD(12; P_3, K_3): \]
\[
\{(1; 6, a), (8; 0, 2), (3; 4, 9), (7; 5, b)\}, \{(4; 7, 1), (5; 2, b), (6; 8, 3), (9; 0, a)\}, \\
\{(0; 4, 5), (a; 6, 8), (b; 1, 3), (2; 7, 9)\}; \{(1; 2, 3), (4; 5, 6), (7; 8, 9), (0; a, b)\}, \\
\{(1; 5, 9), (4; 8, b), (3; 7, a), (2; 6, 0)\}; \{(1; 7, 0), (2; 4, a), (3; 5, 8), (6; 9, b)\}; \\
I = \{(1; 8), (2; b), (3; 0), (4; 9), (5; a), (6; 7)\}. \]

\[ (6, 1) \in URD(12; P_3, K_3): \]
\[
\{(1; 4, 7), (5; 8, 0), (9; 2, b), (a; 3, 6)\}, \{(2; 6, 8), (4; 9, a), (7; 3, 0), (b; 1, 5)\}, \\
\{(0; 4, 2), (3; 5, 9), (6; 7, b), (8; 1, a)\}, \{(1; 5, 6), (4; 8, 7), (9; 0, a), (b; 3, 2)\}, \\
\{(2; 4, 5), (6; 9, 8), (7; a, b), (0; 1, 3)\}; \{(3; 4, 6), (5; 7, 9), (8; 0, b), (a; 1, 2)\}; \\
I = \{(1; 9), (2; 7), (3; 8), (4; b), (5; a), (6; 0)\}. \]

\[ \Box \]

4 Constructions for general $v$

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Lemma 4.1. Let $v \equiv 0 \pmod{3}, v \geq 9$. The union of any two edge-disjoint parallel classes of $3$-cycles of $K_v$ can be decomposed into three parallel classes of $P_3$.

Proof. Let $Q' = \{q'_1, \ldots, q'_{v/3}\}$ and $Q'' = \{q''_1, \ldots, q''_{v/3}\}$ be two edge-disjoint parallel classes of $K_3$, whose union composes the edge set of graph $G$ on $v$ vertices. We represent the intersection structure of $Q'$ and $Q''$ with a bipartite graph $B$ with vertex bipartition $X' \cup X''$, where $|X'| = |X''| = v/3$ and each vertex $x'_i \in X'$ and $x''_j \in X''$ for $1 \leq i, j \leq v/3$ corresponds to a block $q'_i \in Q'$ and $q''_j \in Q''$, respectively. Vertex $x'_i$ is connected to vertex $x''_j$ by an edge of $B$ if their corresponding blocks $q'_i$ and $q''_j$ have a vertex in common.

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\[ 1^{\text{In fact, } B \text{ is the hypergraph-theoretic dual of the 2-regular 3-uniform hypergraph whose hyperedges are the triples in } Q' \cup Q''}. \]

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in $X'$ and ending in $X''$, in such a way that each intersecting edge-pair of $B$ occurs together in precisely one of those $3 \times v/3$ copies of $P_4$. Since $G$ is the line graph of $B$, this will yield the three parallel classes of $P_3$ as required.

It follows from the König-Hall theorem \cite{1} that the edge set of $B$ can be decomposed into three edge-disjoint perfect matchings; we view this as a proper 3-edge-coloring with three colors, say colors $a$, $b$, and $c$. We define

- $\mathcal{P}_{abc} = \{\text{paths } P_4 \text{ in } B, \text{ starting in } X', \text{ whose color sequence is } (a, b, c) \text{ in this order}\}$.

This $\mathcal{P}_{abc}$ is well-defined and yields an edge decomposition of $B$ indeed, because each color class is a perfect matching. We define $\mathcal{P}_{bca}$ and $\mathcal{P}_{cab}$ analogously, replacing the sequence $(a, b, c)$ with $(b, c, a)$ and $(c, a, b)$, respectively.

It is easy to verify that the three edge decompositions $\mathcal{P}_{abc}$, $\mathcal{P}_{bca}$, $\mathcal{P}_{cab}$ of $B$ satisfy the requirements. For example, if an edge $e_a$ of color $a$ meets an edge $e_c$ of color $c$ in $B$, then they are consecutive in one $P_4$ of $\mathcal{P}_{bca}$ if $e_a \cap e_c \in X'$ or in one $P_4$ of $\mathcal{P}_{cab}$ if $e_a \cap e_c \in X''$ (and they are not consecutive in any other $P_4$ of $\mathcal{P}_{abc} \cup \mathcal{P}_{bca} \cup \mathcal{P}_{cab}$).

Lemma 4.2. For every $v \equiv 3 \pmod{6}$, $I(v) \subseteq URD(v; P_3, K_3)$.

Proof. Let $R_1, R_2, \ldots, R_{v-1}$ be the parallel classes of a resolvable KTS($v$). Define

$$S_i = R_{2i+1} \cup R_{2i+2}, \quad i = 0, 1, \ldots, \frac{v-7}{4}$$

for $v \equiv 3 \pmod{12}$, and

$$T_i = R_{2i+1} \cup R_{2i+2}, \quad i = 0, 1, \ldots, \frac{v-5}{4}$$

for $v \equiv 9 \pmod{12}$.

By Lemma 4.1 we know that each $S_i$ and each $T_i$ can be decomposed into three parallel classes of $P_3$. Thus, in order to generate a member $(r, s) = (3x, \frac{v+1}{2} - 2x)$ of $I(v)$, we apply the lemma to $(S_0, S_1, \ldots, S_{x-1})$ or to $(T_0, T_1, \ldots, T_{x-1})$, depending on the residue of $v$ modulo 12. The range given above for $i$ covers the entire range of $x$ in $I(v)$.

Lemma 4.3. For every $v \equiv 0 \pmod{6} \geq 18$, $I(v) \subseteq URD(v; P_3, K_3)$.

Proof. Start with a A 3-RGDD of type $2^{v/3}$ \cite{13}. This gives that $K_v - I$ can be decomposed into $\frac{v}{3} - 1$ parallel classes of triples. Now the result can be easily obtained by using an argument similar to the proof of Lemma 4.2. \qed
5 Conclusion

We are now in a position to prove the main result of the paper.

**Theorem 5.1.** For every $v \equiv 0 \pmod{3}$, we have \( URD(v; P_3, K_3) = I(v) \).

**Proof.** Necessity follows from Lemmas 2.1 and 2.2. Sufficiency follows from Lemmas 3.1, 3.2, 4.2 and 4.3. This completes the proof. \( \Box \)

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