Several decades ago, John McKay suggested a correspondence between nodes of the affine $E_8$ Dynkin diagram and certain conjugacy classes in the Monster group. Thanks to Monstrous Moonshine, this correspondence can be recast as an assignment of discrete subgroups of $PSL_2(\mathbb{R})$ to nodes of the affine $E_8$ Dynkin diagram. The purpose of this article is to give an explanation for this latter correspondence using elementary properties of the group $PSL_2(\mathbb{R})$. We also obtain a super analogue of McKay’s observation, in which conjugacy classes of the Monster are replaced by conjugacy classes of Conway’s group — the automorphism group of the Leech lattice.

1 Introduction

Several decades ago, John McKay suggested the following assignment of conjugacy classes in the Monster group to nodes of the affine $E_8$ Dynkin diagram (c.f. [Con85, §14]).

\[
\begin{array}{cccccc}
\otimes & 2A & 3A & 4A & 5A & 6A \\
\downarrow & 3C & & & & \\
& & 4B & & & \\
& & & 2B & & \\
\end{array}
\] (1.1)

This has become known as McKay’s Monstrous $E_8$ observation. Since elements of the Monster group determine principal moduli for genus zero subgroups of $PSL_2(\mathbb{R})$ via Monstrous Moonshine (c.f. [CN79, Bor92]), the assignment (1.1) entails a correspondence between nodes of the affine $E_8$
Dynamkin diagram and certain discrete subgroups of $PSL_2(\mathbb{R})$ that are commensurable with $PSL_2(\mathbb{Z})$.

\[ \begin{array}{c}
1 & 2+ & 3+ & 4+ & 5+ & 6+ \\
\end{array} \]

\[ \begin{array}{c}
\uparrow \\
\downarrow \\
3 \parallel 3 \\
\end{array} \]

\[ \begin{array}{c}
\uparrow \\
\downarrow \\
4 \parallel 2+ \\
\end{array} \]

(1.2)

In this article we furnish a prescription for recovering this latter correspondence (1.2), that is given solely in terms of elementary properties of the group $PSL_2(\mathbb{R})$.

For the purposes of this article, we will say that a subgroup $\Gamma < PSL_2(\mathbb{R})$ is arithmetic if it is commensurable with $PSL_2(\mathbb{Z})$ in the strong sense; viz. if the intersection of $\Gamma$ with $PSL_2(\mathbb{Z})$ has finite index in each of $\Gamma$ and $PSL_2(\mathbb{Z})$.

In §2 we set up a general context for studying arithmetic subgroups of $PSL_2(\mathbb{R})$. The approach here is closely modeled on that described in [Con96]. In §3.2 we furnish arithmetic conditions that are satisfied only by the arithmetic groups appearing in (1.2). Then in §3.3 we show how the arithmetic properties of these groups can be used to recover also the edges of the affine $E_8$ Dynkin diagram (1.2).

When considering the correspondence (1.2), it is hard not to be reminded of the McKay correspondence, which may be regarded as giving a prescription for recovering the extended Dynkin diagrams from representations of finite subgroups of $SU(2)$ (c.f. [McK80]). In particular, the McKay correspondence indicates how to recover the affine $E_8$ Dynkin diagram from the category of representations of the binary icosahedral group, $2.\text{Alt}_5 < SU(2)$. Our prescription for recovering the edges of (1.2) is reminiscent of this procedure.

In §4 we demonstrate how to obtain a kind of “super analogue” of McKay’s Monstrous $E_8$ observation, in which conjugacy classes of Conway’s group $Co_0$ take on the rôle played by conjugacy classes of the Monster in (1.1).

2 Arithmetic groups

Let us agree to say that $\Gamma < PSL_2(\mathbb{R})$ is an arithmetic subgroup of $PSL_2(\mathbb{R})$ if the intersection $PSL_2(\mathbb{Z}) \cap \Gamma$ has finite index both in $PSL_2(\mathbb{Z})$ and in $\Gamma$. In order to study the arithmetic subgroups of $PSL_2(\mathbb{R})$ we will adopt the approach of Conway [Con96], whereby one analyzes these groups in terms of their actions on lattices. We review this approach in the present section. Actually, our exposition will employ a slightly different language to that of [Con96]; the story is nonetheless the same. As a small supplement to the ideas of [Con96], we include, in §2.10, a description of how to identify the cusps of certain arithmetic subgroups of $PSL_2(\mathbb{R})$, in terms of such a group’s action on lattices.

\footnote{The notation in (1.2) for subgroups of $PSL_2(\mathbb{R})$ follows [CN79], [CMS04]; we write $n+$ for $\Gamma_0(n)+$, and $n\parallel h+$ for $\Gamma_0(n\parallel h)+$, for example.}
2.1 Orientation

Let \( \mathbf{k} \) be an ordered field. Write \( \mathbf{k}^\times \) for the multiplicative group of non-zero elements of \( \mathbf{k} \), and write \( \mathbf{k}^+ \) for its subgroup of \textit{positive elements}. Note that an ordered field is necessarily of characteristic 0, so there is a natural embedding \( \mathbb{Q} \hookrightarrow \mathbf{k} \).

Let \( V \) be a vector space of dimension \( n \) over \( \mathbf{k} \). We may consider the \( n \)-th exterior power \( \wedge^n V \). This vector space has a distinguished point; viz. the origin, and we may consider the \( n \)-th exterior power \( \wedge^n V \). Let \( V \) be a vector space over \( \mathbf{k} \). We may consider the complement \((\wedge^n V) \times = \wedge^n V \setminus \{0\} \). The set \((\wedge^n V) \times \) is naturally a \( \mathbf{k}^\times \)-torsor. Upon picking any non-zero element, \( w \) say, of \( \wedge^n V \), we arrive at a decomposition of \((\wedge^n V) \times \) into two disjoint \( \mathbf{k}^+ \)-torsors:

\[
\wedge^n V \times = \mathbf{k}^+ \cdot w \cup \mathbf{k}^+ \cdot (-w).
\]

We define an \textit{orientation} for \( V \) to be a choice of \( \mathbf{k}^+ \)-torsor in \( \wedge^n V \). An \textit{oriented vector space over} \( \mathbf{k} \) is a pair \((V, W)\) where \( V \) is a vector space over \( \mathbf{k} \), and \( W \) is a \( \mathbf{k}^+ \)-torsor in \( \wedge^n V \).

Consider the canonical linear map \( \Delta : V^n \to \wedge^n V \) given by

\[
(v_1, \ldots, v_n) \mapsto \Delta(v_1, \ldots, v_n) := v_1 \wedge \cdots \wedge v_n.
\]  

(2.1)

The set \( \{v_1, \ldots, v_n\} \) underlying the \( n \)-tuple \((v_1, \ldots, v_n)\) is a linearly independent subset of \( V \) if and only if \( \Delta(v_1, \ldots, v_n) \neq 0 \). We say that an \( n \)-tuple \((v_1, \ldots, v_n)\) is an \textit{ordered basis} for \( V \) if \( \Delta(v_1, \ldots, v_n) \neq 0 \). We write \( \mathcal{B} \) for the set of ordered bases for \( V \).

\[
\mathcal{B} := \Delta^{-1}((\wedge^n V) \times) \subset V^n
\]  

(2.2)

For \((V, W)\) an oriented vector space, we say that an ordered set \((v_1, \ldots, v_n)\) \( \in V^n \) is an \textit{oriented basis} for \( V \) if \( \Delta(v_1, \ldots, v_n) \in W \). Let us write \( \mathcal{B}^+ \) for the set of oriented bases in \( V \).

\[
\mathcal{B}^+ := \Delta^{-1}(W) \subset V^n
\]  

(2.3)

Let \( \text{End}(V) \) denote the \( \mathbf{k} \)-algebra of \( \mathbf{k} \)-linear transformations of \( V \). We may regard \( \text{End}(V) \) as acting on \( V \) from the right.

\[
V \times \text{End}(V) \to V
\]

\[
(v, A) \mapsto v \cdot A
\]  

(2.4)

This action extends naturally to an (right) action of \( \text{End}(V) \) on \( V^n \).

\[
(v_1, \ldots, v_n) \cdot A := (v_1 \cdot A, \ldots, v_n \cdot A)
\]  

(2.5)

As is usual, we take \( GL(V) \) to be the complement of \( \det^{-1}(0) \) in \( \text{End}(V) \), where \( \det : \text{End}(V) \to \mathbf{k}^\times \) is the unique map such that

\[
\Delta(v \cdot A) = \det(A) \Delta(v)
\]  

(2.6)

for all \( v \in V^n \) and \( A \in \text{End}(V) \). We take \( SL(V) \) to be the kernel of the restriction \( \det : GL(V) \to \mathbf{k}^\times \), and we set \( GL^+(V) \) to be the preimage of \( \mathbf{k}^+ \) under this map.

\[
GL(V) = \det^{-1}(\mathbf{k}^\times)
\]  

(2.7)

\[
GL^+(V) = \det^{-1}(\mathbf{k}^+)
\]  

(2.8)

\[
SL(V) = \det^{-1}(1)
\]  

(2.9)
Observe that the sets $\mathcal{B}$ and $\mathcal{B}^+$ are naturally torsors for $GL(V)$ and $GL^+(V)$, respectively.

For any choice of ordered basis $\mathbf{v} = (v_1, \ldots, v_n) \in \mathcal{B}$ there is a corresponding isomorphism $\phi_\mathbf{v} : GL(V) \to GL_n(\mathfrak{k})$ given by setting $\phi_\mathbf{v}(A)$ to be the matrix $(a^i_j) \in GL_n(\mathfrak{k})$ such that $v_j \cdot A = \sum_i v_i a^i_j$ for $1 \leq j \leq n$. Thus we obtain a right action of $GL_n(\mathfrak{k})$ on $V^n$ subject to a choice of ordered basis $\mathbf{v} \in \mathcal{B}$. Observe that there is also a canonical (left) action of $GL_n(\mathfrak{k})$ on $V^n$ given by setting

$$
(a^i_j) \cdot (v_1, \ldots, v_n) := (v'_1, \ldots, v'_n)
$$

where $v'_j = \sum_i a^i_j v_i$. Furthermore, this action preserves the subset $\mathcal{B} \subset V^n$; indeed, $GL_n(\mathfrak{k})$ acts simply transitively on $\mathcal{B}$, so that $\mathcal{B}$ is naturally a (left) torsor for $GL_n(\mathfrak{k})$. The set $\mathcal{B}^+$, consisting of oriented bases for $V$, is naturally a (left) torsor for $GL_n^+(\mathfrak{k})$.

### 2.2 Lattices

Let $V = (V, W)$ be an oriented vector space of dimension $n$ over $\mathfrak{k}$, as in \secref{2.1}. Write $\mathcal{B}^+$ for the subset of $V^n$ consisting of oriented bases for $V$.

A **lattice** in $V$ is an additive subgroup, $L$ say, of $V$, such that $L$ is equivalent to $\mathbb{Z}^n$ as a $\mathbb{Z}$-module, and such that $\text{Span}_\mathfrak{k} L$ (the $\mathfrak{k}$-linear span of $L$) is $V$. Let us write $\mathcal{L}$ for the set of all lattices in $V$.

Suppose that $L \in \mathcal{L}$, and let $\{v_1, \ldots, v_n\} \subset V$ be a set of generators for $L$ as a $\mathbb{Z}$-module, so that

$$
L = \mathbb{Z} v_1 + \cdots + \mathbb{Z} v_n.
$$

Then $\{v_1, \ldots, v_n\}$ is a linearly independent subset of $V$, for if $\sum a_i v_i = 0$ for some $a_i \in \mathfrak{k}$, then the $\mathfrak{k}$-linear span of $L$ is vector space of dimension less than $n$, contradicting the property that $\text{Span}_\mathfrak{k} L = V$. Conversely, if $\{v_1, \ldots, v_n\}$ is a linearly independent subset of $V$ then the additive subgroup of $V$ generated by the $v_i$ is a lattice in $V$. We conclude that there is a natural surjective map from the set of bases for $V$ to the set of lattices in $V$.

$$
\{v_1, \ldots, v_n\} \mapsto \mathbb{Z} v_1 + \cdots + \mathbb{Z} v_n
$$

Consider the (pre)composition of this map with the natural map $\mathcal{B}^+ \to \mathcal{P}(V)$, sending an $n$-tuple in $\mathcal{B}^+$ to it’s underlying set. Since any basis for $V$ can be made an oriented basis by equipping it with a suitable ordering, we conclude that this composition furnishes a (natural) surjective map from $\mathcal{B}^+$ to the set of lattices in $V$.

$$
\mathcal{B}^+ \to \mathcal{L}
$$

$$
(v_1, \ldots, v_n) \mapsto \mathbb{Z} v_1 + \cdots + \mathbb{Z} v_n
$$

We would like to know when two oriented bases for $V$ determine the same lattice. Recall that $\mathcal{B}^+$ is naturally a $GL_n^+(\mathfrak{k})$ torsor (c.f. \secref{2.1}). If

$$
\mathbb{Z} v_1 + \cdots + \mathbb{Z} v_n = \mathbb{Z} v'_1 + \cdots + \mathbb{Z} v'_n
$$

then there are some integers $m^i_j$ and $n^i_j$ such that $v'_j = \sum_i m^i_j v_i$ and $v_j = \sum_i n^i_j v'_i$ for $1 \leq j \leq n$, and hence the matrices $(m^i_j)$ and $(n^i_j)$ are mutually inverse. In other words, if two oriented bases
Arithmetic groups and the affine $E_8$ Dynkin diagram

determine the same lattice then they are related, via left-multiplication, by an element of $SL_n(\mathbb{Z})$. Conversely, if $A \in SL_n(\mathbb{Z})$ and $(v_i) \in \mathcal{B}^+$, then we have the equality (2.14) once we set $(v'_i) = (A \cdot v_i)$.

We have shown that the set $\mathcal{L}$ is naturally isomorphic to the orbit space $SL_n(\mathbb{Z}) \backslash \mathcal{B}^+$, where $\mathcal{B}^+$ is naturally a torsor for $GL_n^+(k)$. Given a choice of oriented basis $v \in \mathcal{B}^+$, we obtain an identification of $\mathcal{B}^+$ with $GL_n^+(k)$

$$GL_n^+(k) \leftrightarrow \mathcal{B}^+$$

$$A \leftrightarrow A \cdot v$$

and hence, an identification of $\mathcal{L}$ with $SL_n(\mathbb{Z}) \backslash GL_n^+(k)$.

From now on we will identify $\mathcal{L}$ with the orbit space $SL_n(\mathbb{Z}) \backslash \mathcal{B}^+$

### 2.3 Projective lattices

Let $V$, $\mathcal{B}^+$, and $\mathcal{L}$ be as is [2.2]. For simplicity of exposition, let us assume that $n = \dim V$ is even — the case that $n$ is odd requires one to occasionally replace $k^\times$ with $k^\times$.

There is a natural embedding (of groups) $k^\times \hookrightarrow GL_n^+(k)$. The image of $k^\times$ under this map is central (indeed, this $k^\times$ is the center of $GL_n^+(k)$), so there is a natural action of $k^\times$ on the space of lattices, $\mathcal{L} = SL_n(\mathbb{Z}) \backslash \mathcal{B}^+$.

$$\alpha \cdot SL_n(\mathbb{Z})(v_1, \ldots, v_n) := SL_n(\mathbb{Z})(\alpha v_1, \ldots, \alpha v_n)$$

We wish to consider the quotient space $P \mathcal{L} = k^\times \backslash \mathcal{L}$. That is, we would like to regard two lattices as equivalent if one is a non-zero scalar multiple of the other; we call the elements of $P \mathcal{L}$ the projective lattices in $V$.

If $X$ is a set equipped with an action of $k^\times$, we write $PX$ for the orbit space $k^\times \backslash X$. We write $x \mapsto [x]$ for the natural map $X \to PX$.

Just as $\mathcal{L}$ is naturally identified with the orbit space $SL_n(\mathbb{Z}) \backslash \mathcal{B}^+$. The set $P \mathcal{L}$ is naturally identified with the orbit space $PSL_n(\mathbb{Z}) \backslash PB^+$, where

$$PSL_n(\mathbb{Z}) = SL_n(\mathbb{Z})/(k^\times \cap \mathbb{Z}),$$

$$PB^+ = k^\times \backslash \mathcal{B}^+.$$ (2.17) (2.18)

We call $PB^+$ the set of projective oriented bases for $V$. It is naturally a torsor for $PGL_n^+(k) = GL_n^+(k)/k^\times$, so that after choosing an element $[v] \in PB^+$ we obtain an identification of $PGL_n^+(k)$ with $PB^+$ by setting

$$PGL_n^+(k) \leftrightarrow PB^+$$

$$[A] \leftrightarrow [A] \cdot [v] := [Av],$$

and hence, also, an identification of $PSL_n(\mathbb{Z}) \backslash PGL_n^+(k)$ with $P \mathcal{L}$. 
2.4 Commensurable lattices

Let $V$, $\mathcal{B}^+$, and $\mathcal{L}$ be as in §2.2. Let us chose a lattice $L_1 \in \mathcal{L}$, and let $\mathbf{v}_1 \in \mathcal{B}^+$ satisfy $L_1 = \text{SL}_n(\mathbb{Z})\mathbf{v}_1$. We say that a lattice $L$ in $V$ is commensurable with $L_1$ if the intersection $L \cap L_1$ has finite index both in $L$ and in $L_1$. Let $V_1$ be the $\mathbb{Q}$-linear span of $L_1$ in $V$, so that $V_1$ is a vector space of dimension $n$ over $\mathbb{Q}$ that is contained in $V$, and the $\mathbf{k}$-linear span of $V_1$ is $V$.

The following result is straightforward.

**Proposition 2.1.** The lattices in $V$ that are commensurable with $L_1$ are exactly the additive subgroups of $V_1$ that are equivalent to $\mathbb{Z}^n$ as $\mathbb{Z}$-modules.

The rational vector space $V_1$, determined by $L_1$, inherits an orientation from $V$ (in the sense of §2.1) which we denote by $W_1$. We write $\mathcal{B}_1^+$ for the set of oriented bases for $V_1$ (c.f. §2.2), and we write $\mathcal{L}_1$ for the orbit space $\text{SL}_n(\mathbb{Z}) \backslash \mathcal{B}_1^+$, which is naturally identified with the set of lattices in $V_1$ — in other words, by Proposition 2.1 $\mathcal{L}_1$ is the subset of $\mathcal{L}$ consisting of the lattices in $V$ that are commensurable with $L_1$. Our choice $\mathbf{v}_1 \in \mathcal{B}_1^+$ allows us to construct an identification of $\mathcal{L}_1$ with $\text{SL}_n(\mathbb{Z}) \backslash \text{GL}_n^+(\mathbb{Q})$ (c.f. §2.2).

We will require to distinguish the lattices in $V_1$ only up to their orbits under $\mathbb{Q}^\times$. Similar to §2.3 we write $P\mathcal{L}_1$ for the orbit space $\mathbb{Q}^\times \backslash \mathcal{L}_1$. Then $P\mathcal{L}_1$ is naturally identified with $\text{PSL}_n(\mathbb{Z}) \backslash P\mathcal{B}_1^+$ for $P\mathcal{B}_1^+ = \mathbb{Q}^\times \backslash \mathcal{B}_1^+$. The choice $\mathbf{v}_1 \in \mathcal{B}_1^+$ allows us to identify $P\mathcal{B}_1^+$ with $\text{PSL}_n(\mathbb{Z}) \backslash \text{PGL}_n^+(\mathbb{Q})$ (c.f. §2.3).

We will continue to write $x \mapsto [x]$ for the projection maps $X \rightarrow P\mathcal{X}$. It should be clear from the context whether we are taking orbits with respect to actions by $\mathbb{Q}^\times$ or $\mathbf{k}^\times$.

2.5 Hyperdistance

We continue in the setting of §2.4. In particular, we continue to assume that $n = \dim V$ is even.

There is a canonically defined integer valued function on $M_n(\mathbb{Q})$, which we call the **rational projective determinant**, and denote $P\text{det}$, which is defined as follows. If $A = (a_{ij}) \in M_n(\mathbb{Q})$ is non-zero, then there is a smallest positive rational number $\alpha_A \in \mathbb{Q}^+$ such that $\alpha_A a_{ij} \in \mathbb{Z}$ for all $1 \leq i, j \leq n$; explicitly, if we write each $a_{ij}$ as a quotient $a_{ij} = b_{ij}/c_{ij}$ for some $b_{ij}, c_{ij} \in \mathbb{Z}$, then $\alpha_A = \text{gcd}(c_{ij})/\text{lcm}(b_{ij})$. We set

$$P\text{det}(A) := \begin{cases} \text{det}(\alpha_A A) = \alpha_A^n \text{det}(A), & \text{if } A \neq 0; \\ 0, & \text{if } A = 0. \end{cases} \quad (2.20)$$

Evidently, $P\text{det}(\alpha A) = P\text{det}(A)$ for any $\alpha \in \mathbb{Q}^\times$ (so long as $n$ is even), so that the projective determinant induces a well defined function,

$$P\text{det} : PM_n(\mathbb{Q}) \rightarrow \mathbb{Z}, \quad (2.21)$$

which we also call the **rational projective determinant**, where $PM_n(\mathbb{Q}) = \mathbb{Q}^\times \backslash M_n(\mathbb{Q})$ is the monoid of **rational projective matrices**.

**Lemma 2.2.** If $A \in \text{SL}_n(\mathbb{Z})$ then $P\text{det}(AX) = P\text{det}(X) = P\text{det}(XA)$ for any $X \in M_n(\mathbb{Q})$. 

Proof. We have \( P\det(AX) = \alpha''_AX \det(AX) \) by definition, and \( \det(AX) = \det(X) \) for \( A \in SL_n(\mathbb{Z}) \), so it suffices to show that \( \alpha''_AX = \alpha_X \) for \( A \in SL_n(\mathbb{Z}) \). Since both \( A \) and its inverse have integer entries, \( \alpha''_AX \) belongs to \( M_n(\mathbb{Z}) \) if and only if \( \alpha_X \) does, for any \( \alpha \in \mathbb{Q}^+ \). Thus the sets \( \{ \alpha \in \mathbb{Q}^+ \mid \alpha''_AX \in M_n(\mathbb{Z}) \} \) and \( \{ \alpha \in \mathbb{Q}^+ \mid \alpha_X \in M_n(\mathbb{Z}) \} \) coincide, and thus their minimal elements coincide. \( \square \)

We will be most interested in the restriction of \( P\det \) to the group \( PGL^+_n(\mathbb{Q}) \), where it takes only positive integer values. The following result is an immediate consequence of Lemma 2.2.

**Proposition 2.3.** The projective determinant induces a well defined positive integer valued function on the orbit space \( PSL_n(\mathbb{Z}) \backslash PGL^+_n(\mathbb{Q}) \).

\[
P\det : PSL_n(\mathbb{Z}) \backslash PGL^+_n(\mathbb{Q}) \to \mathbb{Z}_{>0} \tag{2.22}
\]

This function is invariant under the natural right action of \( PSL_n(\mathbb{Z}) \).

In particular then, the projective determinant furnishes us with a natural method for comparing projective lattices in \( V_1 \). For suppose given a pair of projective lattices \( L, L' \in P\mathcal{L}_1 \). We regard \( P\mathcal{L}_1 \) as identified with \( PSL_n(\mathbb{Z}) \backslash PB^+_1 \), so that we have

\[
L = PSL_n(\mathbb{Z}) \cdot [v], \quad L' = PSL_n(\mathbb{Z}) \cdot [v'], \tag{2.23}
\]

for some \([v], [v'] \in PB^+_1 \). The set \( PB^+_1 \) is a \( PGL^+_n(\mathbb{Q}) \)-torsor, so there is a unique \( g \in PGL^+_n(\mathbb{Q}) \) such that \( g \cdot [v] = [v'] \). We define a positive integer \( \delta(L, L') \) by setting

\[
\delta(L, L') = P\det(g). \tag{2.24}
\]

**Proposition 2.4.** The function \( \delta : P\mathcal{L} \times P\mathcal{L} \to \mathbb{Z}_{>0} \) given by (2.24) is well-defined.

Proof. We should check that the value of \( \delta(L, L') \) does not depend upon the choice of representatives (viz. \([v]\) and \([v']\)), for the orbits \( L \) and \( L' \). So let \( L, L' \in P\mathcal{L}_1 \) be as in (2.23), with \( g \in PGL^+_n(\mathbb{Q}) \) satisfying \( g \cdot [v] = [v'] \), and suppose that \( L = PSL_n(\mathbb{Z}) \cdot [w] \) and \( L' = PSL_n(\mathbb{Z}) \cdot [w'] \). Then there are some \( h, h' \in PSL_n(\mathbb{Z}) \) such that \([w] = h \cdot [v]\) and \([w'] = h' \cdot [v']\), and \( h'gh^{-1} \) is the unique element of \( PGL^+_n(\mathbb{Q}) \) satisfying \((h'gh^{-1}) \cdot [w] = [w']\). We require to show that \( P\det(g) = P\det(h'gh^{-1}) \), but this is guaranteed by Proposition 2.3. \( \square \)

In the case that \( n = 2 \) the function \( \delta \) acquires a special property.

**Proposition 2.5.** For \( n = 2 \), the function \( \delta : P\mathcal{L} \times P\mathcal{L} \to \mathbb{Z}_{>0} \) is symmetric.

Proof. If \( X \in M_n(\mathbb{Z}) \) and \( X \) is invertible, then \( \det(AX^{-1}) \in M_n(\mathbb{Z}) \). Applying this same rule with \( \det(AX^{-1}) \) in place of \( X \), we see that

\[
\frac{\det(AX)^n}{\det(AX)} \frac{X}{\det(AX)} \in M_n(\mathbb{Z}) \tag{2.25}
\]

Thus, in the special case that \( n = 2 \), we have that an invertible matrix \( X \) lies in \( M_n(\mathbb{Z}) \) if and only if \( \det(AX^{-1}) \) does. Consequently, for any \( A \in GL^2_2(\mathbb{Q}) \) and any \( \alpha \in \mathbb{Q}^+ \) we have that \( \alpha A \) belongs to
$M_2(\mathbb{Z})$ if and only if $\alpha \det(A)A^{-1}$ does, so that if $\alpha_A$ is the minimal positive rational for which $\alpha_A A$ belongs to $M_2(\mathbb{Z})$, then $\alpha_{A^{-1}} = \det(A)\alpha_A$ is the minimal positive rational that has this property for $A^{-1}$. It is easy to compute now that $P\det(A) = \alpha_2^2 \det(A)$ and $P\det(A^{-1}) = \alpha_2^{-1} \det(A^{-1})$ coincide. This verifies the claim, since if $\delta(L, L') = P\det(g)$ then $\delta(L', L) = P\det(g^{-1})$. \hfill $\square$

Remark. It is easy to find $g \in PGL_n^+(\mathbb{Q})$ such that $P\det(g) \neq P\det(g^{-1})$ when $n > 2$.

Following Conway [Con96], we call the function $\delta$, when defined on pairs of projective lattices in a rational vector space of dimension 2, hyperdistance. The logarithm of $\delta$ is in fact a metric on $PL$ for $n = 2$.

From now on we will restrict attention to the case that $n = \dim V = 2$.

2.6 Names

We continue in the setting of 2.5. In particular, we assume that $n = \dim V = 2$, and we retain our choice of lattice $L_1 \in L$, and generating set $v_1 \in B_1^+$. We regard the set $PL_1$, of projective lattices in $V_1$, as identified with the coset space $PSL_2(\mathbb{Z}) \backslash PGL_2^+(\mathbb{Q})$ via this choice.

\[
PSL_2(\mathbb{Z}) \backslash PGL_2^+(\mathbb{Q}) \leftrightarrow PL_1
\]
\[
PSL_2(\mathbb{Z})[A] \leftrightarrow PSL_2(\mathbb{Z}) \cdot [Av_1]
\]  (2.26)

Of course, $PGL_2^+(\mathbb{Q})$ acts naturally, from the right, on $PSL_2(\mathbb{Z}) \backslash PGL_2^+(\mathbb{Q})$, so the identification (2.26) also allows us to define a right action of $PGL_2^+(\mathbb{Q})$ on $PL_1$. Evidently, hyperdistance (c.f. 2.5) is invariant with respect to this $PGL_2^+(\mathbb{Q})$-action.

We will now introduce names for the elements of the coset space

\[
PSL_2(\mathbb{Z}) \backslash PGL_2^+(\mathbb{Q}),
\]  (2.27)

and hence also for the projective lattices in $V_1$. Consider the natural map from $GL_2^+(\mathbb{Q})$ to $PSL_2(\mathbb{Z}) \backslash PGL_2^+(\mathbb{Q})$. Our strategy (following [Con96]) is to try and define a canonical preimage in $GL_2^+(\mathbb{Q})$ for each coset $PSL_2(\mathbb{Z}) g$ in the image of this map.

Let $g \in PGL_2^+(\mathbb{Q})$. We will write

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix}
\]  (2.28)

for the canonical map $M_2(\mathbb{Q}) \to PM_2(\mathbb{Q})$. Then

\[
g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}
\]  (2.29)

for some $a, b, c, d \in \mathbb{Q}$ such that $ad - bc > 0$. Choose integers $s, t \in \mathbb{Z}$ such that $sa + tc = 0$, and observe that then $sb + td \neq 0$, by linear independence of the columns of an invertible matrix. We may assume that $s, t$ have no common factors, and thus there are integers $m, n \in \mathbb{Z}$ such that $mt - sn = 1$. In other words, there is an $h \in PSL_2(\mathbb{Z})$ such that

\[
h \cdot g = g' = \begin{pmatrix} a' & b' \\ 0 & d' \end{pmatrix}
\]  (2.30)
for some $a', b', d' \in \mathbb{Q}$, with $d' \neq 0$. We have
\[
\begin{bmatrix}
a' & b' \\
0 & d'
\end{bmatrix} = \begin{bmatrix}
a'' & b'' \\
0 & 1
\end{bmatrix}
\] (2.31)
for $a'' = a'/d' > 0$ and $b'' = b'/d'$. There is a unique integer, $N$ say, such that $0 \leq b'' + N < 1$. Left-multiplying by
\[
\begin{bmatrix}
1 & N \\
0 & 1
\end{bmatrix},
\] (2.32)
we arrive at an element $g'''$ of $PSL_2(\mathbb{Z})g$ of the form
\[
g''' = \begin{bmatrix}
a''' & b''' \\
0 & 1
\end{bmatrix}
\] (2.33)
where $a''' > 0$ and $0 \leq b''' < 1$. Let us write $\mathcal{M}$ for the set of matrices in $M_2(\mathbb{Q})$ of the form
\[
\begin{pmatrix}
M & b \\
0 & 1
\end{pmatrix}
\] (2.34)
where $M > 0$ and $0 \leq b < 1$. We have established the following result.

Proposition 2.6. The assignment
\[
\begin{pmatrix}
M & b \\
0 & 1
\end{pmatrix} \mapsto PSL_2(\mathbb{Z}) \begin{bmatrix}
M & b \\
0 & 1
\end{bmatrix}
\] (2.35)
defines a bijective correspondence between the elements of $\mathcal{M}$ and the elements of the coset space $PSL_2(\mathbb{Z}) \setminus PGL_2^+(\mathbb{Q})$.

Given $M \in \mathbb{Q}^+$ and $b \in \mathbb{Q} \cap [0, 1)$, let us write $g_{M,b}$ for the projective matrix
\[
g_{M,b} := \begin{bmatrix}
M & b \\
0 & 1
\end{bmatrix}.
\] (2.36)
Let us write $L_{M,b}$ for the corresponding projective lattice with respect to the identification (2.26); i.e. we set
\[
L_{M,b} := PSL_2(\mathbb{Z})g_{M,b}.
\] (2.37)
We will typically abbreviate $g_{M,0}$ to $g_M$, and $L_{M,0}$ to $L_M$. Observe that, under this notational convention, we have
\[
L_1 = L_{1,0} = PSL_2(\mathbb{Z})g_1 = PSL_2(\mathbb{Z}) \leftrightarrow PSL_2(\mathbb{Z})[v_1]
\] (2.38)
in agreement with our notation $L_1$ for the lattice we picked in §2.4.
Observe that to compute the hyperdistance \( \delta(L_{M,b}, L_1) \), from an arbitrary lattice \( L_{M,b} \) to the distinguished lattice \( L_1 \), is the same as computing the projective determinant of the matrix \( g_{M,b} \).

More generally, computing the hyperdistance between \( L_{M',b'} \) and \( L_{M,b} \) is the same as computing the projective determinant of \( g_{M',b'}g_{M,b}^{-1} \).

\[
(g_{M',b'}g_{M,b}^{-1}) \cdot (g_{M,b} \cdot [v_0]) = g_{M',b'} \cdot [v_0] \tag{2.39}
\]

\[
\delta(L_{M',b'}, L_{M,b}) = P\det(g_{M',b'}g_{M,b}^{-1}) \tag{2.40}
\]

Our names \( L_{M,b} \) for the projective lattices in \( V_1 \) are not particularly canonical. We could, for example, have chosen to seek coset representatives for \( PSL_2(Z) \setminus PSL_2^+ (Q) \) with vanishing top-right entry, unital top-left entry, positive bottom-right entry, and bottom-left entry in \( Q \cap [0,1) \). That is, we could have sought representatives in the form \( \bar{g}_{b,M} \) where

\[
\bar{g}_{b,M} = \begin{bmatrix}
1 & 0 & 0 \\
0 & b & M \\
0 & 0 & 1
\end{bmatrix}.
\tag{2.41}
\]

As it turns out, the matrices \( \bar{g}_{b,M} \), for \( b \in Q \cap [0,1) \) and \( M \in Q^+ \), also furnish a complete and irredundant list of coset representatives for \( PSL_2(Z) \) in \( PSL_2(Q) \). We will write \( \bar{L}_{b,M} \) for the projective lattice corresponding to \( PSL_2(Z)\bar{g}_{b,M} \). Of course, every projective lattice \( L_{M,b} \) can be written as \( \bar{L}_{b,M'} \) for some \( b', M' \). The correspondence is such that

\[
L_{M,0} = \bar{L}_{0,1/M}, \quad L_{M,f/g} = \bar{L}_{f'/g,1/g^2M},
\tag{2.42}
\]

where, for \( 0 < f < g \) and \( \gcd \{f,g\} = 1 \), the integer \( f' \) is uniquely determined by the conditions that \( 0 < f' < g \) and \( ff' \equiv 1 \pmod{g} \).

When \( L_{M,b} = \bar{L}_{b,M'} \) we (following [Con96]) call \( \bar{L}_{b,M'} \) the reverse name for \( L_{M,b} \). The reverse names for projective lattices will be useful in \( \S 2.11 \).

### 2.7 Stabilizers

Let us continue with the notation and conventions of \( \S 2.6 \). In particular, we consider the \( Q^\times \)-orbits of lattices in \( V \) that are commensurable with our distinguished lattice \( L_1 \) — the set \( PL_1 \), of projective lattices in \( V_1 \) — and these orbits are in natural correspondence with the \( PSL_2(Z) \setminus PGL_2^+ (Q) \). Each projective lattice \( L \in PL_1 \) arises as \( L = L_{M,b} \) for some \( M \in Q^+ \) and \( b \in Q \cap [0,1) \).

From now on, let us write \( G \) for the group \( PGL_2^+ (Q) \) regarded as a group with right-action on the space \( PL_1 \) of projective lattices in \( V_1 \).

\[
PL_1 \leftrightarrow PSL_2(Z) \setminus PGL_2^+ (Q) \cap PGL_2^+ (Q) =: G
\tag{2.43}
\]

\[
PSL_2(Z) \cdot (g \cdot [v_1]) \leftrightarrow PSL_2(Z)g
\]

For any \( L \in PL_1 \), we may ask for the subgroup of \( G \) that fixes \( L \); i.e. the group \( \text{Fix}_G(L) \). We will write \( G_L \) for \( \text{Fix}_G(L) \), and \( G_{M,b} \) for \( G_L \) if \( L = L_{M,b} \). We may also consider the group that fixes several lattices,

\[
G_{(L^1,\ldots,L^k)} = \bigcap_i G_{L^i} = \{ g \in G \mid L_i \cdot g = L_i, \ 1 \leq i \leq k \} ,
\tag{2.44}
\]
or the group that stabilizes a set of lattices,

\[ G_{\{L_1, \ldots, L_k\}} = \left\{ g \in G \mid \{L^1 \cdot g, \ldots, L^k \cdot g\} = \{L^1, \ldots, L^k\} \right\}. \tag{2.45} \]

Let us compute some examples of the groups \( G_{\{L, \ldots\}}, \) \( G_{\{L, \ldots\}}. \) Note that, by definition,

\[ (\text{PSL}_2(\mathbb{Z}) \cdot [v_1]) \cdot g = \text{PSL}_2(\mathbb{Z})g \cdot [v_1] \tag{2.46} \]

for \( g \in G. \) The identity (2.46) makes it clear that the group \( G_1 = \text{Fix}_G(L_1) \) is none other than the familiar modular group, \( \text{PSL}_2(\mathbb{Z}). \) Evidently, \( G \) acts transitively on \( P\mathcal{L}_1, \) so the lattice \( L_M, \) for \( M \in \mathbb{Q}^+, \) is fixed by a conjugate of \( G_1 \cong \text{PSL}_2(\mathbb{Z}); \) viz.

\[ G_M = g_M^{-1}G_1g_M = \left\{ \begin{bmatrix} a & b/M \\ cM & d \end{bmatrix} \mid a, b, c, d \in \mathbb{Z}, \ ad - bc = 1 \right\}, \tag{2.47} \]

Thus we see that the intersection \( G_{\{1,N\}} = G_1 \cap G_N, \) for \( N \) a positive integer, is the Hecke group of level \( N, \) usually denoted \( \Gamma_0(N). \)

For \( N \) a prime power, \( N = p^a \) say, the group \( G_{\{1,N\}} \) that preserves the set \( \{L_1, L_N\} \) is the group obtained from \( G_{\{1,N\}} \) by adjoining a Fricke involution:

\[ G_{\{1,N\}} = \left\langle G_{\{1,N\}}, \begin{bmatrix} 0 & -1 \\ N & 0 \end{bmatrix} \right\rangle. \tag{2.48} \]

This is the group denoted \( \Gamma_0(N) + N \) (or just \( \Gamma_0(N) + \), since \( N \) is a prime power) in [CN79]. Notice that the subgroup of \( G \) stabilizing the set \( \{L_1, L_p^a\} \) is the same as the subgroup of \( G \) stabilizing the set \( \{L_1, L_p, \ldots, L_p^a\}, \) containing the lattices associated to all powers of \( p \) that divide \( p^a, \) and this latter set is precisely the set of lattices \( L \) for which the product of hyperdistances \( \delta(L_1, L)\delta(L, L_p^a) \) coincides with the hyperdistance between \( L_1 \) and \( L_{p^a}. \) Given \( L' \) and \( L'' \) in \( P\mathcal{L}_1, \) let us write \( (L', L'')^+ \) for the set of lattices \( L \) satisfying \( \delta(L', L)\delta(L, L'') = \delta(L', L''). \)

\[ (L', L'')^+ = \{ L \in P\mathcal{L}_1 \mid \delta(L', L)\delta(L, L'') = \delta(L', L'') \}. \tag{2.49} \]

Following [Con96], we call the set \( (L', L'')^+ \) the \( (L', L'')\)-thread. The sets \((1,6)^+ \) and \((1,4)^+ \) appear in (2.50).

\[ \begin{array}{c}
2 \\
3 \quad 1 \\
6 \\
\end{array} \quad \begin{array}{c}
\quad 1 \\
2 \\
\quad 4 \\
\end{array} \tag{2.50} \]

In terms of threads, we have \( G_{\{1,N\}} = G_{\{1,N\}}^+ \) when \( N \) is a prime power. More generally, for arbitrary \( N \in \mathbb{Z}_{>0}, \) the group \( G_{\{1,N\}}^+ \) is the group obtained by adjoining to \( G_{\{1,N\}}, \) the sets \( W_e = W_e(N), \) where \( e \in \mathbb{Z}_{>0} \) is a divisor of \( N, \) and

\[ W_e = \left\{ \begin{bmatrix} ae & b \\ cN & de \end{bmatrix} \mid a, b, c, d \in \mathbb{Z}, \ ade^2 - bcN = e \right\}. \tag{2.51} \]
Note that $ade^2 - bcN = e$ implies $\gcd\{e, N/e\} = 1$, so the set $W_e$ is non-empty only when $e$ is an exact divisor of $N$. The sets $W_e$ are exactly the cosets of $G_{(1,N)} = W_1$ in $G_{(1,N)+}$. (We will sometimes write $W_e$ for a certain element in $W_e(N)$ — the validity of the statement in which this $W_e$ appears should be independent of the choice that is made.)

Suppose given $h, n \in \mathbb{Z}_{>0}$ such that $h|n$. Then

$$G_{(h,n)} = \left\{ \begin{bmatrix} a & b/h \\ cn & d \end{bmatrix} \mid a, b, c, d \in \mathbb{Z}, \ ad - bcn/h = 1 \right\}. \quad (2.52)$$

This group is conjugate to $G_{(1,n/h)}$ since $G_{(h,n)} = g_h^{-1}G_{(1,n/h)}g_h$ (c.f. (2.50)). The group $G_{(h,n)+}$ is evidently obtained from $G_{(1,n/h)+}$ via conjugation by $g_h$, and so $G_{(h,n)+}$ consists of $G_{(h,n)}$ together with the (other co)sets $g_h^{-1}W_e(n/h)g_h$ for $e$ an exact divisor of $n/h$. The significance of the groups $G_{(h,n)+}$ is demonstrated by the following result.

**Theorem 2.7** (Atkin–Lehner). For $N$ a positive integer, the normalizer of $G_{(1,N)}$ in $\text{PSL}_2(\mathbb{R})$ is the group $G_{(h,n)+}$, where $n = N/h$, and $h$ is the largest divisor of $24$ such that $h^2|N$.

**Remark.** A beautiful proof of the Atkin–Lehner Theorem appears in [Con96].

Recall that the action of $G$ on $\mathcal{P}L_1$ preserves hyperdistance (c.f. (2.245). It follows that for any $L \in \mathcal{P}L_1$, the group $G_L$ acts by permutations on the set $HC_N(L)$, of lattices at hyperdistance $N$ from $L$. We call $HC_N(L)$ the hypercircle of hyperradius $N$ about $L$.

$$HC_N(L) = \{ L' \in \mathcal{P}L_1 \mid \delta(L, L') = N \} \quad (2.53)$$

In fact, the action of $G_L$ on $HC_N(L)$ is transitive.

**Proposition 2.8.** For any positive integer $N$, the group $G_L = \text{Fix}_G(L)$ acts transitively on the set of lattices that are hyperdistant $N$ from $L$.

We will furnish a proof of Proposition 2.8 in §2.11.

We may consider the subgroup of $G$ that fixes all the lattices in $HC_N(L)$ for given $N \in \mathbb{Z}_{>0}$ and $L \in \mathcal{P}L_1$. When $L$ is the distinguished lattice $L = L_1$, this is just the group $\Gamma(N)$ — the principal congruence group of level $N$.

$$1 \to \Gamma(N) \to \text{PSL}_2(\mathbb{Z}) \to \text{PSL}_2(\mathbb{Z}/N) \to 1 \quad (2.54)$$

Recall that a group $H < G_1 \cong \text{PSL}_2(\mathbb{Z})$ is called a congruence group if $H$ contains $\Gamma(N)$ for some $N$. A congruence group $H$ is said to have level $N$ if $N$ is the smallest positive integer such that $\Gamma(N) < H$. We will apply these definitions also to arithmetic groups: we will say that a subgroup $\Gamma < \text{PSL}_2(\mathbb{R})$ that is commensurable with $G_1$ is a congruence group if $\Gamma$ contains $\Gamma(N)$ for some positive integer $N$, and we will say that such a group $\Gamma$ has level $N$, if $N$ is the minimal positive integer for which $\Gamma(N) < \Gamma$.

**Remark.** See Lemma 2.19 for an explicit description of $HC_N(1)$ in the case that $N$ is a prime power.
2.8 Trees

Let \( p \) be a prime, and consider the set \( (P\mathcal{L}_1)_p \subset P\mathcal{L}_1 \) consisting of (projective) lattices that are hyperdistant a power of \( p \) from \( L_1 \).

\[
(P\mathcal{L}_1)_p = \left\{ L \in P\mathcal{L}_1 \mid \delta(L_1, L) \in p\mathbb{Z} \right\}
\]  

(2.55)

Following Conway \cite{Con96}, we call \( (P\mathcal{L}_1)_p \) the \( p \)-adic tree in \( P\mathcal{L}_1 \).

For \( L, L' \in P\mathcal{L}_1 \) we say \( L \) and \( L' \) are \( p \)-adically equivalent, and write \( L \sim_p L' \), if \( p \nmid \delta(L, L') \). It is easy to check that the \( p \)-adic equivalence class of any projective lattice \( L \in P\mathcal{L}_1 \) has a unique representative in \( (P\mathcal{L}_1)_p \).

**Proposition 2.9.** For any \( L \in P\mathcal{L}_1 \), and any prime \( p \), there is a unique \( L' \in P\mathcal{L}_1 \) such that \( L \sim_p L' \) and \( L' \in (P\mathcal{L}_1)_p \).

Consequently, we obtain a map \( \pi_p : P\mathcal{L}_1 \rightarrow (P\mathcal{L}_1)_p \), called the \( p \)-adic projection, by setting \( \pi_p(L) = L' \) when \( L \) and \( L' \) are as in Proposition 2.9.

As explained in \cite{Con96}, the set \( (P\mathcal{L}_1)_p \) has a natural tree structure; viz. if we regard the elements of \( (P\mathcal{L}_1)_p \) as vertices of a graph, with edges joining just those lattices that are hyperdistant \( p \) from each other, we obtain a graph with the property that there is a unique shortest path between any two vertices. That is, we obtain a tree. The \( p \)-adic tree \( (P\mathcal{L}_1)_p \) has infinitely many nodes, and each node is \( p + 1 \) valent.

A finite subset \( S \subset P\mathcal{L}_1 \) is called a cell if, for each prime \( p \), the subtree of \( (P\mathcal{L}_1)_p \) generated by the set \( \pi_p(S) \subset (P\mathcal{L}_1)_p \) is either a point, or two points joined by an edge.

The methods of \cite{Con96} illustrate the utility of the trees \( (P\mathcal{L}_1)_p \) and the projections \( \pi_p \). The following result is a prime example of this.

**Proposition 2.10** (\cite{Con96}). If \( \Gamma \) is a subgroup of \( PGL_2^+(\mathbb{Q}) \) that is commensurable with \( G_1 \cong PSL_2(\mathbb{Z}) \), then \( \Gamma \) stabilizes a cell in \( P\mathcal{L}_1 \).

It is well known that a subgroup of \( PGL_2^+(\mathbb{R}) \) that is commensurable with \( PSL_2(\mathbb{Z}) \) is contained in \( PGL_2^+(\mathbb{Q}) \) (c.f. Proposition 2.11). For any cell \( S \subset P\mathcal{L}_1 \) we can find lattices \( L \) and \( L' \) in \( S \) that maximize \( \delta(L, L') \). After conjugation by an element of \( PGL_2^+(\mathbb{Q}) \) we may assume that \( L = L_1 \). Since \( G_1 \cong PSL_2(\mathbb{Z}) \) acts transitively on the lattices of any given hyperdistance from \( L_1 \) (c.f. Proposition 2.8), we may conjugate the pair \( (L, L') \) to \( (L_1, L_N) \) for some \( N \). These observations, together with Proposition 2.10, quickly imply the following result, known as Helling’s Theorem.

**Theorem 2.11** (\cite{Hel66}). The maximal arithmetic subgroups of \( PSL_2(\mathbb{R}) \) are the conjugates of \( G_{(1,N)+} \) for square-free \( N \).

2.9 Characters

Let \( N \) be a positive integer, and let \( h \) be the largest divisor of 24 such that \( h^2 \mid N \). Observe that \( G_{(h,N/h)} \) contains \( G_{(1,N)} \). According to \cite{CN79} (c.f. \cite{CMS04}), there is a canonically defined
subgroup of index \( h \) in \( G(h,N/h) \) that contains \( G(1,N) \), and it may be realized as the kernel of a (non-canonically defined) homomorphism

\[
\lambda : G(h,N/h) \to \mathbb{Z}/h.
\]  

(2.56)

We will write \( G^{(h)}_{(h,N/h)} \) for this group that arises as \( \ker(\lambda) \). This group is denoted \( \Gamma_0(n|h) \) in [CN79]. The cases that \( N = 9 \) and \( N = 8 \) will be of particular relevance in §3.

2.9.1 Example: \( N = 9 \).

If \( N = 9 \) then \( h = 3 \) and \( N/h = 3 \). Evidently, \( G_{(3,3)} = G_3 \). Define generators \( x \) and \( y \) for \( G_3/G_{(1,9)} \) (recall from Theorem 2.7 that \( G_{(1,9)} \) is normal in \( G_3 \)) by setting

\[
x = G_{(1,9)}T^{1/3} = G_{(1,9)}\begin{bmatrix} 1 & 1/3 \\ 0 & 1 \end{bmatrix},
\]

(2.57)

\[
y = G_{(1,9)}(T^3)^t = G_{(1,9)}\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}.
\]

(2.58)

A suitable map \( \lambda : G_3/G_{(1,9)} \to \mathbb{Z}/3 \) may be defined by assigning \( \lambda(y) = \sigma \) and \( \lambda(x) = \sigma^{-1} \), where \( \sigma \) is a generator for \( \mathbb{Z}/3 \) (c.f. [CMS04]).

Let us also analyze the situation in terms of the 3-adic tree (c.f. §2.8). The group \( G_3 \) preserves the set \( HC_3(3) \) — the hypercircle of hyperradius 3 about \( L_3 \). There are exactly 4 lattices in \( HC_3(3) \); they appear in the diagram (2.59), which displays the smallest subtree of the 3-adic tree \( (PL_1)_3 \) that contains \( HC_3(3) \).

Observe that \( G_{1,9} \) fixes every element of \( HC_3(3) \). Thus, there is a natural map \( G_3/G_{1,9} \to \text{Sym}(HC_3(3)) \). Diagram (2.59) also displays the 3-cycles in \( \text{Sym}(HC_3(3)) \) generated by (the images in \( \text{Sym}(HC_3(3)) \) of) \( x \) and \( y \). Evidently, the image of \( G_3/G_{1,9} \) in \( \text{Sym}(HC_3(3)) \) is just \( \text{Alt}(HC_3(3)) \) — a copy of the alternating group on 4 letters. This group has order 12. It contains 8 elements of order 3, and 3 elements of order 2. Any non-trivial homomorphism \( \text{Alt}(HC_3(3)) \to \mathbb{Z}/3 \) must be trivial on elements of order 2, and non-trivial on the elements of order 3.

We conclude that \( G_{3}^{(3)} \) consists of all the elements of \( G_3 \) that induce permutations of order 2 (or 1) on the set \( HC_3(3) \), and the permutations of order 2 occurring are just those that arise as a product of 2 disjoint transpositions.

\[
G_{3}^{(3)} = \langle G_{(1,9)}, (T^3)^tT^{1/3}, T^{-1/3}(T^3)^tT^{-1/3} \rangle
\]

(2.60)
The group $G_3^{(3)}$ is denoted $\Gamma_0(3|3)$ in [CN79].

### 2.9.2 Example: $N = 8.$

If $N = 8$ then $h = 2$ and $N/h = 4$. The group $G_{(2,4)}$ contains $G_{(1,8)}$ normally by Theorem 2.7. Consider the set $S = HC_2(2) \cup HC_2(4)$.

![Diagram](2.61) Diagram (2.61) displays the smallest subtree of the 2-adic tree (c.f. §2.8) containing $S$, together with the actions on this tree induced by some elements of $G \cong PGL_2^+(\mathbb{Q})$ (c.f. §2.7). The symbol $W_8$ in (2.61) denotes an (arbitrary) element of $g_2^{-1}W_2(2)g_2$ (c.f. §2.7).

The group $G_{(2,4)}$ acts by permutations on $S$; the subgroup $G_{(1,8)}$ is exactly the subgroup that fixes every element of $S$. We see from (2.61) that $G_{(2,4)}$ is generated by $G_{(1,8)}$ together with $T^{1/2}$ and $(T^4)^t$. The group $G_{(2,4)}$ — the full normalizer of $G_{(1,8)}$ — is obtained by adjoining $W_8$ to $G_{(2,4)}$.

We thus obtain generators $x, y$ for $G_{(2,4)}/G_{(1,8)}$ by setting $x = G_{(1,8)}T^{1/2}$ and $y = G_{(1,8)}(T^4)^t$, say. We obtain a suitable map $\lambda : G_{(2,4)}/G_{(1,8)} \to \mathbb{Z}/2$ by setting $\lambda(x) = \lambda(y) = \sigma$, where $\sigma$ is the non-trivial element of $\mathbb{Z}/2$.

$$G_{(2,4)}^{(2)} = \langle G_{(1,8)}, (T^4)^tT^{1/2} \rangle$$  \hspace{1cm} (2.62)

We can also carry out this story with $G_{(2,4)}^{(2)}$ in place of $G_{(2,4)}$; that is, we can adjoin the set $g_2^{-1}W_2(2)g_2$ to (2.62), just as we adjoin this same set to $G_{(2,4)}$ in order to recover $G_{(2,4)}$ (c.f. §2.51). We set

$$G_{(2,4)}^{(2)} = \langle G_{(1,8)}, T^4, T^{1/2}, W_8 \rangle.$$ \hspace{1cm} (2.63)

Evidently, $G_{(2,4)}^{(2)}$ may be characterized as the kernel of the following composition of natural mappings: $G_{(2,4)} \to \text{Sym}(S) \to \mathbb{Z}/2$. This group is denoted $\Gamma_0(4|2)^+$ in [CN79].

### 2.10 Cusps

From now on we will restrict to the case that $k = \mathbb{R}$. 

---

**Reference**

[CN79]: Cohn, N. P.; Norton, S. P., *Arithmetic groups and the affine $E_8$ Dynkin diagram*.
There is an obvious embedding $PSL_2(\mathbb{R}) \hookrightarrow PGL_2^+(\mathbb{R})$ arising from the embedding of groups $SL_2(\mathbb{R}) \hookrightarrow GL_2^+(\mathbb{R})$. Observe that this map

$$PSL_2(\mathbb{R}) \hookrightarrow PGL_2^+(\mathbb{R}) \quad (2.64)$$

is in fact surjective, so that $PGL_2^+(\mathbb{R}) = PSL_2(\mathbb{R})$. There is also an embedding $GL_2^+(\mathbb{Q}) \hookrightarrow GL_2^+(\mathbb{R})$ coming the the embedding of fields $\mathbb{Q} \hookrightarrow \mathbb{R}$. The map $PGL_2^+(\mathbb{Q}) \to PGL_2^+(\mathbb{R})$ sending a $\mathbb{Q}^\times$-orbit in $GL_2^+(\mathbb{Q})$ to the $\mathbb{R}^\times$-orbit of its image in $GL_2^+(\mathbb{R})$ is readily checked to be an embedding; we conclude that $PGL_2^+(\mathbb{Q})$ embeds naturally in $PSL_2(\mathbb{R})$.

Our choice $v_1 \in B_1^+$ allows us to identify $PV^\times = \mathbb{R}^\times \setminus V^\times$ with the real projective line $\mathbb{P}^1(\mathbb{R}) = \mathbb{R} \cup \{\infty\}$; viz.

$$[\alpha u_1 + \beta v_1] \leftrightarrow \begin{cases} \alpha/\beta, & \text{if } \beta \neq 0; \\ \infty, & \text{if } \beta = 0; \end{cases} \quad (2.65)$$

where $v_1 = (u_1, v_1)$. The resulting left action of $PGL_2(\mathbb{R})$ on $PV^\times$ is given by

$$\left[ \begin{array}{cc} a & b \\ c & d \end{array} \right] \cdot [\alpha u_1 + \beta v_1] = [(a\alpha + b\beta)u_1 + (c\alpha + d\beta)v_1]. \quad (2.66)$$

In terms of the identification $PV^\times \hookrightarrow \mathbb{R} \cup \{\infty\}$ this translates to the familiar prescription

$$\left[ \begin{array}{cc} a & b \\ c & d \end{array} \right] \cdot \alpha = \frac{a\alpha + b}{c\alpha + d}. \quad (2.67)$$

Evidently, it is natural to regard the group $PGL_2^+(\mathbb{R})$ as acting both from the right on projective lattices $PL$, and from the left on the projective line $\mathbb{P}^1(\mathbb{R})$. Similarly, we may regard our group $G \cong PGL_2^+(\mathbb{Q})$ (see (2.71)) as acting both from the right on the projective lattices $PL_1$ in $V_1$, and from the left on the rational projective line $\mathbb{P}^1(\mathbb{Q})$. Also, there is a natural embedding $\mathbb{P}^1(\mathbb{Q}) \hookrightarrow \mathbb{P}^1(\mathbb{R})$.

Recall that $T$ typically denotes the element of $PSL_2(\mathbb{R}) = PGL_2^+(\mathbb{R})$ represented by the upper-triangular unipotent matrix with $1$ in the top right-hand corner. We convene to write $T^A$, for $A \in \mathbb{R}$, for the element of $PGL_2^+(\mathbb{R})$ represented by the upper-triangular unipotent matrix with $A$ in the top right-hand corner.

$$T^A := \left[ \begin{array}{cc} 1 & A \\ 0 & 1 \end{array} \right] \in PGL_2^+(\mathbb{R}) \quad (2.68)$$

Evidently, $T^A T^B = T^{A + B}$ for $A, B \in \mathbb{R}$, so that the assignment $A \mapsto T^A$ defines an embedding (of groups) $\mathbb{R} \hookrightarrow PGL_2^+(\mathbb{R})$. Observe that the stabilizer of $\infty \in \mathbb{P}^1(\mathbb{R})$ in $PGL_2^+(\mathbb{R})$ (let us write $\text{Fix}(\infty)$ for this group) contains the image of $\mathbb{R}$ under this embedding.

$$\text{Fix}(\infty) = \left\{ \left[ \begin{array}{cc} a & b \\ 0 & 1 \end{array} \right] \mid a, b \in \mathbb{R}, \ a > 0 \right\} \quad (2.69)$$

In fact, the image of $\mathbb{R}$ here is a normal subgroup of $\text{Fix}(\infty)$, and $\text{Fix}(\infty)$ may be described as the split extension $\text{Fix}(\infty) \cong \mathbb{R} \rtimes \mathbb{R}^+$, corresponding to the natural action of (the multiplicative...
group) $\mathbb{R}^+$ on (the additive group) $\mathbb{R}$. Given a subgroup $H < PGL_2^+(\mathbb{R})$ let us write $\text{Fix}_H(\infty)$ for the intersection $\text{Fix}(\infty) \cap H$. We have

$$\text{Fix}_G(\infty) \cong \mathbb{Q} \times \mathbb{Q}^+. \quad (2.70)$$

Observe that $\text{Fix}_G(\infty)$ acts transitively on the subset $\mathbb{P}^1(\mathbb{Q}) \subset \mathbb{P}^1(\mathbb{R})$, and no element of $PGL_1$ is fixed by any non-trivial element of the subgroup $\mathbb{Q}^+ < \text{Fix}_G(\infty)$. Observe also that the intersection $\text{Fix}_{G_1}(\infty) = \text{Fix}_G(\infty) \cap G_1$ is contained in the image (under $A \mapsto T^A$) of $\mathbb{Q}$; indeed, $\text{Fix}_{G_1}(\infty) = \langle T \rangle$. This property is shared by the subgroups of $PSL_2(\mathbb{R})$ that are commensurable with $G_1 \cong PSL_2(\mathbb{Z})$ (c.f. (2.71)).

**Proposition 2.12.** Suppose $H < PSL_2(\mathbb{R})$ is commensurable with $G_1$. Then $\text{Fix}_H(\infty)$ is contained in the image of $\mathbb{Q}$ under $A \mapsto T^A$.

**Proof.** Let $h \in \text{Fix}_H(\infty)$. Choose $a, b \in \mathbb{R}$ such that

$$h = \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}. \quad (2.71)$$

If $H$ is commensurable with $G_1$ then $h^n \in G_1$ for some $n \in \mathbb{Z}$. This implies that $a^n = 1$, and hence $a = 1$, since $a$ must be positive. Given that $a = 1$, we have $nb \in \mathbb{Z}$, so that $b \in \mathbb{Q}$, and $h = T^b$ lies in the image of $\mathbb{Q}$ in $\text{Fix}_G(\infty)$, as required. \qed

Recall that an element $\gamma \in PSL_2(\mathbb{R})$ is called parabolic if there is a unique fixed point for its action on $\mathbb{P}^1(\mathbb{R})$. Equivalently, $\gamma \in PSL_2(\mathbb{R})$ is parabolic if it is not the identity, and if $\gamma = [A]$ for some $A \in SL_2(\mathbb{R})$ with $\text{tr}(A) = 2$. For any subgroup $H < PGL_2^+(\mathbb{R})$ we may ask for the orbits of $H$ on $\mathbb{P}^1(\mathbb{R})$ that are comprised of points that are fixed by some parabolic element of $H$. (If $h \cdot s = s$ for some parabolic $h \in H$ and some $s \in \mathbb{P}^1(\mathbb{R})$ then every $s' \in H \cdot s$ is fixed by some parabolic element of $H$.) Such an orbit for the action of $H$ on $\mathbb{P}^1(\mathbb{R})$ is called a cusp of $H$. We write $C(H)$ for the set of cusps of $H$, and we define

$$C(H) := \bigcup_{H \cdot s \in C(H)} H \cdot s, \quad (2.72)$$

so that $C(H)$ consists of all the points in $\mathbb{P}^1(\mathbb{R})$ which are fixed by some parabolic element of $H$.

**Proposition 2.13.** The assignment $H \mapsto C(H)$ is constant on commensurability classes.

**Proof.** It suffices to show that $C(H) = C(H')$ whenever $H'$ is a subgroup of finite index in $H$. Certainly, $C(H') \subset C(H)$ in this case, so let $s \in C(H)$. Then there is some parabolic $h \in H$ such that $h \cdot s = s$. Since $H'$ has finite index in $H$, there is some $n > 0$ such that $h^n \in H'$, but $h^n$ is parabolic whenever $h$ is (since $A^2 = \text{tr}(A)A - 1$ for $A \in SL_2(\mathbb{R})$), so there is a parabolic element of $H'$ that fixes $s$, and $C(H) \subset C(H')$, as required. \qed

For $H < PGL_2^+(\mathbb{R})$ set $c(H)$ to be the number of cusps of $H$.

$$c(H) := |C(H)| \quad (2.73)$$
It is easy to check that \( C(G) = \mathbb{P}^1(\mathbb{Q}) \), and it is clear that \( G \) acts transitively on \( \mathbb{P}^1(\mathbb{Q}) \), so we have \( c(G) = 1 \). In fact, these statements remain valid with \( G_1 \cong PSL_2(\mathbb{Z}) \) in place of \( G \).

\[
c(G_1) = 1, \quad C(G_1) = \mathbb{P}^1(\mathbb{Q}), \quad C(G_1) = \{ G_1 \cdot \infty \}. \tag{2.74}
\]

By Proposition 2.13, we have \( C(H) = \mathbb{P}^1(\mathbb{Q}) \) for any subgroup of \( PGL^+_2(\mathbb{R}) \) that is commensurable with \( G_1 \). This has the following useful consequence.

**Proposition 2.14.** If \( H < PSL_2(\mathbb{R}) \) is commensurable with \( G_1 \cong PSL_2(\mathbb{Z}) \) then \( H \) is contained in \( G \cong PGL^+_2(\mathbb{Q}) \).

**Proof.** We see from Proposition 2.13 that the action of such a group \( H \) on \( \mathbb{P}^1(\mathbb{R}) \) stabilizes \( \mathbb{P}^1(\mathbb{Q}) \). Any element of \( H \) can be written as a Möbius transformation, and in particular, is determined by its action on 0, 1 and \( \infty \), for example. Since these points are mapped to points of \( \mathbb{P}^1(\mathbb{Q}) \), any element of \( H \) can be represented by a rational matrix. \( \Box \)

Let \( H < G \) such that \( C(H) = \mathbb{P}^1(\mathbb{Q}) \). Then \( c(H^g) = c(H) \) for any \( g \in G \), since \( G \) acts transitively on \( \mathbb{P}^1(\mathbb{Q}) \). This shows, for example, that \( c(G_L) = 1 \) for any \( L \in P\mathcal{L}_1 \).

We may ask what the action of a group on \( P\mathcal{L}_1 \) says about its cusps. The following lemma is useful in this regard.

**Lemma 2.15.** Let \( H \) be a group equipped with a transitive left-action on a set \( L \), and a transitive right-action on a set \( R \). Let \( l \in L \) and \( r \in R \), and write \( H_x \) for the stabilizer of \( x \) in \( H \), for \( x \) in \( L \) or \( R \). Then the \( H_x \)-orbits on \( L \) are in bijective correspondence with the \( H_1 \)-orbits on \( R \).

\[
L/H_r \leftrightarrow H_1 \backslash R \tag{2.75}
\]

**Proof.** Since the \( H \)-actions are transitive, we have bijections \( L \leftrightarrow H_1 \backslash H \) and \( H/H_r \leftrightarrow R \), so both sides of \( (2.75) \) are in bijection with \( H_1 \backslash H/H_r \). \( \Box \)

Note that the correspondence in \( (2.75) \) is given explicitly by

\[
l \cdot hH_r \leftrightarrow H_1h \cdot r. \tag{2.76}
\]

Lemma 2.15 has the following immediate consequence.

**Proposition 2.16.** Let \( H \) be a subgroup of \( G \) with a single cusp, let \( \mathcal{O} \) be an orbit for the action of \( H \) on \( P\mathcal{L}_1 \), and let \( H_0 \) be the subgroup of \( H \) fixing some \( L_0 \in \mathcal{O} \). Then the cusps of \( H_0 \) are in bijective correspondence with the orbits of \( \text{Fix}_H(\infty) \) on \( \mathcal{O} \).

Suppose now that \( H \) is a subgroup of \( PSL_2(\mathbb{R}) \) commensurable with \( G_1 \), so that \( H \) is contained in \( G \cong PGL^+_2(\mathbb{Q}) \), by Proposition 2.14 and suppose that \( H \) has a single cusp (i.e. acts transitively on \( \mathbb{P}^1(\mathbb{Q}) \)). For a subgroup \( H_0 \) of \( H \), we have \( C(H_0) = H_0 \backslash \mathbb{P}^1(\mathbb{Q}) \) for the cusps of \( H_0 \). If \( H_0 \) has finite index in \( H \) then we may define a function

\[
w_H : C(H_0) \to \mathbb{Q}^+ \tag{2.77}
\]

in the following way. For \( H_0 \cdot x \in H_0 \backslash \mathbb{P}^1(\mathbb{Q}) \) let \( g \in H \) such that \( x = g \cdot \infty \), and consider the intersection \( H_0^g \cap \text{Fix}_H(\infty) \). Since \( H \) is commensurable with \( G_1 \), the group \( \text{Fix}_H(\infty) \) is an infinite cyclic group, by Proposition 2.12. Since \( H_0 \) has finite index in \( H \), the intersection \( H_0^g \cap \text{Fix}_H(\infty) \) is also an infinite cyclic group. Define \( w_H(H_0 \cdot x) = A \in \mathbb{Q}^+ \) just when \( T^A \) generates \( H_0^g \cap \text{Fix}_H(\infty) \).
Proposition 2.17. The function $w_H$, of (2.77), is well-defined.

Proof. If $g, g' \in H$ satisfy $g \cdot \infty = g' \cdot \infty = x$ then $g^{-1}g' = T^B$ for some $B \in \mathbb{Q}$, since $\text{Fix}_H(\infty)$ is contained in the image of $\mathbb{Q}$ in $\text{Fix}_G(\infty)$, by Proposition 2.12. We have $H^g_0 = (H^g_0)^{T^B}$, so that $H^0_0 \cap \text{Fix}_H(\infty)$ and $H^0_0 \cap \text{Fix}_H(\infty)$ coincide, since $T^B$ centralizes $\text{Fix}_H(\infty)$. This shows that the value of $w_H(H_0 \cdot x)$ doesn’t depend upon the choice of $g \in H$ mapping $\infty$ to $x$. Suppose that $H_0 \cdot x = H_0 \cdot x'$. Then $x' = h \cdot x$ for some $h \in H_0$, and $g \cdot \infty = x$ implies $hg \cdot \infty = x'$. We have $H^h_0 = H^0_0$, so the value of $w_H$ at the cusp $H_0 \cdot x$ is independent of the choice of representative point $x \in \mathbb{P}^1(\mathbb{Q})$. □

For $H$ and $H_0$ as above, and $C \in \mathcal{C}(H_0)$, we call $w_H(C)$ the width of $C$ relative to $H$. If $H = G_1$ then $w_H$ recovers the usual notion of width for cusps of finite index subgroups of the modular group. Observe that if $H' < \text{PSL}_2(\mathbb{R})$ is commensurable with $G_1$ and contains $H$, then $H_0$ is a subgroup of finite index in $H'$, and the functions $w_{H'}$ and $w_H$ coincide on $\mathcal{C}(H_0)$. Consequently, for a group $H_0 < \text{PSL}_2(\mathbb{R})$ commensurable with $G_1$, to know all the widths of a cusp $H_0 \cdot x \in H_0 \setminus \mathbb{P}^1(\mathbb{Q})$, it suffices to know the values $w_H(H_0 \cdot x)$ for each supergroup $H > H'$ that is maximal subject to being commensurable with $G_1$. The subgroups of $\text{PSL}_2(\mathbb{R})$ that are maximal subject to being commensurable with $G_1$ are determined by Proposition 2.10.

Even though the functions $w_H$ and $w_{H'}$ in general don’t coincide for $H_0 < H \cap H'$, the particular value $w_H(H_0 \cdot \infty)$ is easily checked to be independent of the choice of supergroup $H$. Thus we can speak unambiguously of the width of $H_0$ at $\infty$, to be denoted $w(H_0 \cdot \infty)$, whenever $H_0$ is commensurable with $G_1$.

Proposition 2.18. Suppose $H$ and $H_0$ are as in Proposition 2.10, and suppose that $H$ is commensurable with $G_1$. Then

$$\frac{w_H(H_0h \cdot \infty)}{A} = \#(L_0 : h \text{Fix}_H(\infty)).$$

(2.78)

for all $h \in H$, where $A \in \mathbb{Q}^+$ is the width of $H$ at $\infty$.

Proof. Observe that the size of the orbit $L_0 \cdot h \text{Fix}_H(\infty)$ is just the smallest $n \in \mathbb{Z}_{>0}$ such that $T^nA \in \text{Fix}_H(L_0 \cdot h)$. Observe also that $T^B$, for $b \in \mathbb{Q}$, belongs to $\text{Fix}_H(L_0 \cdot h)$ if and only if $T^B \in H^0_{h}$. For $T^B \in \text{Fix}_H(L_0 \cdot h)$ if and only if $hT^B = h_0h$ for some $h_0 \in H_0$, and this occurs if and only if $T^B \in h^{-1}H_0h$. Thus if $B = w_H(H_0h \cdot \infty)$, so that $B$ is the smallest positive rational such that $T^B \in H^0_h$, then $B = nA$ where $n$ is $\#(L_0 : h \text{Fix}_H(\infty))$. This completes the proof. □

Consider the case that $H = G_1$ is the modular group, and $\mathcal{O} = HC_N(1)$ is the hypercircle of hyperradius $N$ about $L_1$ for some positive integer $N$ (c.f. (2.7)), and take $L_0 = L_N$, so that $H_0 = G_{(1,N)}$. By Proposition 2.10, the cusps of $G_{(1,N)}$ are in natural correspondence with the orbits of $\text{Fix}_G(\infty) \cap G_1 = \langle T \rangle$ on $HC_N(1)$, and by Proposition 2.13, the width of each cusp is just the cardinality of the corresponding orbit of $\langle T \rangle$.

It turns out that the orbit structure of $\langle T \rangle$ on $HC_N(1)$ is not difficult to describe. We will not give the full analysis here (since none of it would be new), but we will furnish the following first step, which will be of use in 2.14.
Lemma 2.19. Suppose $N = p^n$ for some positive integer $n$. Then $HC_N(1)$ consists of the lattices of the form $p^{n-2a}, k/p^a$ where $a$ and $k$ satisfy one of the following conditions.

1. $a = 0$ and $k = 0$;
2. $0 < a < n$ and $0 < k < p^a$ and $\gcd\{k, p\} = 1$;
3. $a = n$ and $0 \leq k < p^n$.

In other words, we have

\[
HC_{p^n}(1) = \{p^n\} \cup \left\{\frac{p^{n-a}}{p^a} \cdot k \mid 0 < a < n, \ 0 < k < p^a, \ \gcd\{k, p\} = 1\right\} \\
\cup \left\{\frac{1}{p^n}, \frac{k}{p^a} \mid 0 \leq k < p^n\right\}.
\]  

(2.79)

2.11 Orbits

In this section we use Lemma 2.19 to furnish a proof of Proposition 2.8. Since $G \cong \text{PSL}_2^+(\mathbb{Q})$ acts transitively on $\mathcal{P}_1$, we have verified Proposition 2.8 as soon as we show that $G_1 \cong \text{PSL}_2(\mathbb{Z})$ acts transitively on the hypercircles $HC_N(1)$ for all $N$. Actually, a stronger result is true.

Proposition 2.20. If $M$ and $N$ are positive integers with $\gcd\{M, N\} = 1$, then the group $G_{(1,M)}$ acts transitively on $HC_N(1)$.

Proof. Consider first the case that $N = p^n$ is a prime power. Then $M$ is a positive integer such that $p \nmid M$. The group $G_{(1,M)}$ contains both $T$ and $(T^M)^t$.

\[
T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad (T^M)^t = \begin{bmatrix} 1 & 0 \\ M & 1 \end{bmatrix}.
\]

(2.80)

Recall from Lemma 2.19 that $HC_{p^n}(1)$ consists of the lattices $L_{A,b}$ with $A = p^{n-a}/p^a$ and $b = k/p^a$ for some $0 \leq a \leq n$ and some $0 \leq k < p^a$, with $\gcd\{k, p^a\} = 1$ in case $a < n$. Recall also, from [2.6] that every projective lattice $L_{A,f/g}$ (for coprime positive integers $f, g$ with $0 \leq f < g$) has a reverse name $L_{f'/g, 1/g^2A}$, where $f'$ is the unique positive integer less than $g$ such that $ff' \equiv 1$ (mod $g$). It is easy to check then that $L_{A,b}$ belongs to $HC_{p^n}(1)$ just when $L_{b,A}$ does. The precise correspondence is as follows.

\[
L_{p^n,0} = \bar{L}_{0,1/p^n}; \\
L_{p^{n-a}/p^a,k/p^a} = \bar{L}_{k'/p^a,1/p^n}, \quad p \nmid k, \ 0 < a < n; \\
L_{1/p^n,k/p^a} = \bar{L}_{k'/p^a,p^{a-a}/p^a}, \quad p \nmid k, \ 0 < a \leq n; \\
L_{1/p^n,0} = \bar{L}_{0,0/p^n}.
\]

(2.81)

In the above, $k'$ denotes the unique positive integer less than $p^a$ such that $kk' \equiv 1$ (mod $p^a$). The reverse labels $\bar{L}_{b,A}$ are convenient for describing the action of $T^t$. For example, we have $\bar{L}_{b,A} \cdot (T^M)^t = \bar{L}_{b+AM,A}$. In particular,

\[
\bar{L}_{0,1/p^n} \cdot (T^M)^t = \bar{L}_{M/p^n,1/p^n}.
\]

(2.82)
Since $M$ is coprime to $p$, we see that there is some power of $T^i$ in $G_{(1,M)}$ that induces the permutation

$$\bar{L}_{k/p^n,1/p^n} \mapsto \bar{L}_{(k+1)/p^n,1/p^n}$$

(2.83)
on the lattices $\bar{L}_{b,A}$ in $HC_{p^n}(1)$ with $A = 1/p^n$ and $b = k/p^n$ for some $k$ satisfying $0 \leq k < p^n$. Recall that $T \in G_{(1,M)}$ induces a cyclic permutation on the lattices $L_{A,b}$ in $HC_{p^n}(1)$ for the very same $A$ and $b$ — viz. the permutation obtained by removing the bars in $\bar{(2.83)}$. Comparing with $(2.81)$ we see that the $G_{(1,M)}$ orbit containing $L_{p^n,0}$ contains every lattice in $HC_{p^n}(1)$; that is, $G_{(1,M)}$ acts transitively on $HC_{p^n}(1)$ if $p \nmid M$.

More generally, consider the action of $G_{(1,M)}$ on $HC_N(1)$ for $N$ coprime to $M$. Then the elements of $HC_N(1)$ are in natural correspondence with the elements of the cartesian product

$$HC_{p_1^{a_1}}(1) \times \cdots \times HC_{p_k^{a_k}}(1)$$

(2.84)

for $N = p_1^{a_1} \cdots p_k^{a_k}$ a prime decomposition of $N$ (c.f. $(2.81)$). The factors in $(2.84)$ are sets of mutually coprime order acted on transitively by $G_{(1,M)}$. It follows that $G_{(1,M)}$ acts transitively on their product. This completes the proof. 

\[\Box\]

3 Diagrams

In this section we give our prescription for recovering McKay’s Monstrous $E_8$ observation $(1.2)$ — at least, its reformulation in terms of discrete subgroups of $PSL_2(\mathbb{R})$ (c.f. $(1.1)$) — using elementary properties of the group $PSL_2(\mathbb{R})$.

3.1 Setting

We adopt the setting of $(2.10)$ (more particularly, of $(2.10)$, with $k = \mathbb{R}$, so that $V$ is an oriented real vector space of dimension 2, and $v_1 = (u_1,v_1)$ is an ordered basis for $V$, and $V_1 \subset V$ is the rational vector space generated by $\{u_1,v_1\}$. The choice $v_1 \in B^+$ (c.f. $(2.2)$) entails well-defined actions of the groups $PSL_2(\mathbb{R})$, $PGL_2^+(\mathbb{Q})$, and $PSL_2(\mathbb{Z})$ on $V$, and on various objects related to $V$ (c.f. $(2.7)$ $(2.10)$).

3.2 Vertices

If $\Gamma$ is a group and $\Gamma'$ is a finite index subgroup of $\Gamma$, we write $[\Gamma : \Gamma']$ for the index of $\Gamma'$ in $\Gamma$.

There is a well known formula for the index of $G_{(1,N)}$ (a.k.a $\Gamma_0(N)$) in $G_1 \cong PSL_2(\mathbb{Z})$. As demonstrated in $(2.10)$ it is easy to recover this formula by considering the projections $\pi_p(L_N)$. For if $N = p_1^{a_1} \cdots p_k^{a_k}$ is a prime decomposition of $N$, then, by Proposition 2.8 the index of $G_{(1,N)}$ in $G_1$ is the product over $i$ of the number of lattices at hyperdistance $p_i^{a_i}$ from the distinguished lattice $L_1$ (c.f. $(2.6)$), since $\pi_p(L_N) = L_{p_i^{a_i}}$. The cardinality of $HC_{p_i^{a_i}}(L_1)$ is $(p_i + 1)p_i^{a_i-1}$ (by Lemma 2.19 for example). We thus obtain the following expression,

$$[G_1 : G_{(1,N)}] = \prod_i (p_i + 1)p_i^{a_i-1},$$

(3.1)
which easily implies the following lemma.

**Lemma 3.1.** If the index of $G_{1,n}$ in $G_1$ does not exceed 12, then $n$ belongs to the following set.

$$\{1, 2, 3, 4, 5, 6, 7, 8, 9, 11\} \quad (3.2)$$

Given a group $\Gamma < PSL_2(\mathbb{R})$ that is commensurable with $G_1 \cong PSL_2(\mathbb{Z})$, let us write $I_{G_1}^\Gamma$ for the index of $\Gamma \cap G_1$ in $G_1$, and $I_{G_1}^\Gamma$ for the index of $\Gamma \cap G_1$ in $\Gamma$.

\[ \begin{array}{c}
G_1 \\
\downarrow
\end{array} 
\begin{array}{c}
\Gamma
\end{array} 
\begin{array}{c}
\downarrow
\end{array} 
\begin{array}{c}
\Gamma \cap G_1
\end{array} 
\begin{array}{c}
\downarrow
\end{array} 
\begin{array}{c}
I_{G_1}^\Gamma
\end{array} 
\begin{array}{c}
\downarrow
\end{array} 
\begin{array}{c}
I_{G_1}^\Gamma
\end{array} \quad (3.3)

**Proposition 3.2.** Suppose $\Gamma < PSL_2(\mathbb{R})$ satisfies the following conditions.

1. $\Gamma$ is arithmetic;
2. $\Gamma$ has width 1 at $\infty$ (c.f. §2.10);
3. there is some $N$ such that $\Gamma$ contains and normalizes $G_{1,N}$, and the quotient $\Gamma/G_{1,N}$ is a group of exponent 2;
4. $I_{G_1}^\Gamma \leq 12$ and $I_{G_1}^\Gamma / I_{G_1}^\Gamma \leq 3$.

Then $\Gamma$ is one of the groups in $\mathcal{E}$, where

$$\mathcal{E} = \left\{ G_1, G_{1,2}, G_{1,3}, G_{1,4}, G_{1,5}, G_{1,2,3,6}, G_{2,4}^{(2)}, G_3^{(3)} \right\} \quad (3.4)$$

**Remark.** Since $G_{1,N}$ and its normalizer are both commensurable with $G_1$ for any $N$, condition 3 implies condition 1.

**Remark.** We may also write $G_{1,N}^+$ for $G_{1,N}$ when $N \in \{2, 3, 4, 5\}$. We may write $G_{1,6}^+$ for $G_{1,2,3,6}$. The notation in (3.3) has been chosen because it suggests, more strongly, how we can determine the correct valence for each $\Gamma$ in $\mathcal{E}$ as a vertex in the affine $E_8$ Dynkin diagram (cf. §3.3.3).

**Proof.** By the Atkin–Lehner Theorem (Theorem 2.7), the intersection of the normalizer of $G_{1,N}$ with $G_1$ is $G_{1,N/h}$ where $h$ is the largest divisor of 24 such that $h^2 | N$. By condition 3 then, we have $G_{1,N} < \Gamma \cap G_1 < G_{1,N/h}$, for some $N$, with $h$ as in the previous sentence. Equivalently, we have

$$G_{1,hn} < \Gamma \cap G_1 < G_{1,n} \quad (3.5)$$

for some $n$ and some $h$, where $h$ is a divisor of $\gcd\{n, 24\}$ such that neither $4h$ nor $9h$ divide $n$.

By Lemma 3.1, the only possibilities for the $n$ in (3.5) are those in the set (3.2) if the first inequality of condition 4 is to be satisfied; we will consider these 10 cases separately.

Let us agree to write $\Gamma_1$ for the intersection $\Gamma \cap G_1$. 
• Case: \( n = 1 \). The inequality (3.5) reduces to \( G_1 < \Gamma_1 < G_1 \) in this case, so \( \Gamma_1 = G_1 \). The normalizer of \( G_1 \) is \( G_1 \), so \( \Gamma = \Gamma_1 = G_1 \) in this case. Conditions 1 through 4 are satisfied when \( \Gamma = G_1 \).

• Case: \( n = 2 \). If \( n = 2 \) then \( h \in \{1, 2\} \).

If \( h = 1 \) then \( \Gamma_1 = G_{(1,2)} \). The normalizer of \( G_{(1,2)} \) is \( G_{(1,2)} \). The former group has index 2 in the latter, so \( \Gamma \) is one of \( G_{(1,2)} \) or \( G_{(1,2)} \). The conditions 1 through 4 are satisfied in both cases.

If \( h = 2 \) then \( \Gamma \) is assumed to normalize and contain \( G_{(1,4)} \), so that \( G_{(1,4)} < \Gamma_1 < G_{(1,2)} \), and \( \Gamma < G_2 \).

Diagram (3.7) displays the smallest subtree of the 2-adic tree (c.f. §2.8) containing \( H \mathcal{C}_2(2) \), together with elements of \( G_2 \) that give rise to each of the transpositions in \( \text{Sym}(H \mathcal{C}_2(2)) \).
(We may take $W_4 = g^{1/2}Sg^2$.) Our group $\Gamma$ does not contain $T^{1/2}$ by condition $2$ and it
does not contain $(T^2)^l$ since this element lies in $G_1$ but not in $\Gamma_1 = G_{(1,4)}$. The remaining
possibility is that $\Gamma = \langle G_{(1,4)}, W_4 \rangle = G_{\{1,4\}}$. This group satisfies conditions $1$ through $4$.

- **Case: $n = 3$.** If $n = 3$ then $h \in \{1, 3\}$.
  If $h = 1$ then $\Gamma_1 = G_{(1,3)}$ and $\Gamma < G_{\{1,3\}}$. The group $G_{(1,3)}$ has index 2 in $G_{\{1,3\}}$, so either
  $\Gamma = G_{(1,3)}$ or $\Gamma = G_{\{1,3\}}$. In the former case the second inequality of condition $4$ is
  violated. In the latter case conditions $1$ through $4$ are all satisfied.
  If $h = 3$ then $\Gamma$ is assumed to normalize and contain $G_{(1,9)}$, so that $G_{(1,9)} < \Gamma_1 < G_{(1,3)}$, and
  $\Gamma < G_3$.

\[
\begin{array}{c}
G_1 \\
\downarrow 4 \quad \downarrow 4 \\
G_{(1,3)} \quad \Gamma \\
\downarrow 3/c \quad \downarrow b \\
3 \\
\downarrow c \\
G_{(1,9)}
\end{array}
\quad \Gamma_1
\]

The indices of these containments are as in (3.8), for some $a, b, c$. Evidently, $abc = 12$. In
order for $\Gamma/G_{(1,9)}$ to have exponent 2 (c.f. condition $3$), it must be that $3|a$, and in particular, we must have $c = 1$; i.e. $\Gamma_1 = G_{(1,9)}$. In order that the second inequality of condition $4$ be
satisfied, we require that $b \geq 4$. Consequently, $\Gamma$ is a subgroup of index 3 in $G_3$ whose
intersection with $G_1$ is exactly the group $G_{(1,9)}$.

The group $G_3$ acts by permutations on $HC_3(3)$, and $G_{(1,9)}$ is exactly the kernel of the corre-
sponding map $G_3 \rightarrow \text{Sym}(HC_3(3))$ (c.f. $\S 2.9.1$). The image of $G_3$ under this map is a copy of
the alternating group on 4 symbols. The image of $\Gamma$ in $\text{Sym}(HC_3(3))$ must be a subgroup of
order 4 in this $\text{Alt}_4$; such a subgroup is unique — it is the kernel of any non-trivial homomor-
phism $\text{Alt}_4 \rightarrow \mathbb{Z}/3$. We conclude that $\Gamma = G_{3}^{(3)}$ (c.f. $\S 2.9.1$). This group satisfies conditions
$1$ through $4$.

- **Case: $n = 4$.** If $n = 4$ then $h \in \{2, 4\}$.
If $h = 2$ then $G_{(1,8)} < \Gamma_1 < G_{(1,4)}$ and $\Gamma < G_{(2,4)}$.

![Diagram](image)

We have $abc = 8$ for $a, b, c$ as in (3.9). The group $G_{(2,4)}$ does not have width 1 at $\infty$, so $a \geq 2$. On the other hand, by condition 4 we have $I_{\Gamma_1}^{G_{(2,4)}}/I_{\Gamma}^{G_{(2,4)}} = 12/bc \leq 3$, so $bc \geq 4$, so we conclude $a = 2$ and $bc = 4$.

The group $G_{(2,4)}$ acts by permutations on the set $S = HC_2(2) \cup HC_2(4)$, and the image of $G_{(2,4)}$ in $\text{Sym}(S)$ is a group of the shape

$$(\mathbb{Z}/2 \times \mathbb{Z}/2) \rtimes \mathbb{Z}/2$$

(i.e. a Dihedral group of order 8) where the central factors $\mathbb{Z}/2$ are generated by $T^{1/2}$ and $(T^4)^t$, and a non-central $\mathbb{Z}/2$ is generated by the Atkin–Lehner involution $W_8$ (c.f. §2.9.2). The group $G_{(1,8)}$ is the kernel of the natural map $G_{(2,4)} \to \text{Sym}(S)$. Consequently, the image of $\Gamma$ in $\text{Sym}(S)$ is a subgroup of order four in (3.10). By condition 2 it does not contain $T^{1/2}$. By condition 3 it is not cyclic. There is exactly one possibility: $\Gamma = \langle G_{(1,8)}, (T^4)^tT^{1/2}, W_8 \rangle$; i.e. $\Gamma = G_{(2,4)}^{(2)}$ (c.f. §2.9.2). This group satisfies all the required properties.

If $h = 4$ then $G_{(1,16)} < \Gamma_1 < G_{(1,4)}$ and $\Gamma < G_4$.

![Diagram](image)
We have \(abc = 24\) for \(a, b, c\) as in (3.11). By condition 3 we have \(3 \mid a\). By condition 4 we have \(I_{G_1}^{I_G} = 24/bc \leq 3\), so \(bc \geq 8\), and this implies \(a = 3\) and \(bc = 8\).

The diagram in (3.12) shows the part of the 2-adic tree containing \(H_{C_4}(4)\); the group \(G_4\) acts as automorphisms of this tree, and the subgroup \(G_{(1,16)}\) fixes every node. We see from (3.12) that the quotient \(G_4/G_{(1,16)}\) has the structure \((\mathbb{Z}/2 \times \mathbb{Z}/2) \rtimes \text{Sym}_3\), with the elementary abelian subgroup generated by \(T^{1/2}\) and \((T^8)^t\), and the symmetric group acting on it generated by \(T^{1/4}\) and \((T^4)^t\) (c.f. (3.7)). If \(\Gamma\) is a subgroup of \(G_4\) of order 8 containing \(G_{(1,16)}\), then the image of \(\Gamma\) in \(G_4/G_{(1,16)}\) contains the subgroup \((\mathbb{Z}/2 \times \mathbb{Z}/2)\). It follows that \(\Gamma\) contains \(T^{1/2}\), but this violates condition 2.

**Case: \(n = 5\).** If \(n = 5\) then \(h = 1\) and \(\Gamma_1 = G_{(1,5)}\), and \(G_{(1,5)} < \Gamma < G_{(1,5)}\). Similar to the case that \(N = nh = 3\), the only possibility satisfying conditions 1 through 4 is \(\Gamma = G_{(1,5)}\).

**Case: \(n = 6\).** If \(n = 6\) then \(h \in \{1, 2, 3\}\).

If \(h = 1\) then \(\Gamma_1 = G_{(1,6)}\), and \(\Gamma < G_{(1,2,3,6)}\). The group \(G_{(1,6)}\) has index 12 in \(G_1\), and index 4 in \(G_{\{1,2,3,6\}}\), so in order to satisfy condition 4 it must be that \(\Gamma = G_{\{1,2,3,6\}}\). This group satisfies all the required properties.

If \(h = 2\) then \(G_{\{1,12\}} < \Gamma_1 < G_{(1,6)}\), and \(\Gamma < G_{\{2,6\}}\). The group \(G_{(1,6)}\) has index 12 in \(G_1\), so
it must be that \( \Gamma_1 = G_{(1,6)} \), by the first inequality of condition \( \text{I} \).

\[
\begin{array}{c}
G_1 \\
\downarrow_{12} \downarrow \downarrow_{2} \\
G_{(2,6)} \downarrow \downarrow_{b} \\
\quad \Gamma \\
G_{(1,6)} = \Gamma_1
\end{array}
\quad (3.13)
\]

We have \( ab = 4 \) for \( a, b \) as in \((3.13)\). For condition \( \text{II} \) we require \( b \geq 4 \), so that \( \Gamma = G_{(2,6)} \), but this group does not satisfy condition \( \text{II} \). Diagram \((3.14)\) shows the cell \( \{L_2, L_6\} \) together with sufficiently many adjacent nodes (in the 2 and 3-adic trees) so as to allow us to display the action of \( G_{(2,6)}/G_{(1,6)} \) (and verify the indices in \((3.13)\)).

\[
\begin{array}{c}
3 \quad 6 \quad 12 \\
\downarrow \downarrow \downarrow \\
\Gamma_1 \downarrow \downarrow \downarrow \\
\quad \Gamma \\
\quad \Gamma_1
\end{array}
\quad (3.14)
\]

If \( h = 3 \) then \( G_{(1,18)} < \Gamma_1 < G_{(1,6)} \) and \( \Gamma < G_{(3,6)} \). Again, we must have \( \Gamma_1 = G_{(1,6)} \), by condition \( \text{I} \).

\[
\begin{array}{c}
G_1 \\
\downarrow_{12} \downarrow \downarrow_{2} \\
G_{(3,6)} \downarrow \downarrow_{b} \\
\quad \Gamma \\
G_{(1,6)} = \Gamma_1
\end{array}
\quad (3.15)
\]

Considering indices (c.f. \((3.15)\)) we see that \( \Gamma \) must coincide with \( G_{(3,6)} \) if condition \( \text{I} \) is to be satisfied, but this group fails to satisfy condition \( \text{II} \). Diagram \((3.16)\) is the analogue of \((3.14)\).
for $h = 3$.

\begin{equation}
\begin{array}{c}
\hline
\text{2} & \text{6} & \text{18} \\
\hline
\text{1} & \text{3} & \text{9} \\
\hline
\end{array}
\end{equation}

\begin{array}{c}
\text{W}_2 \\
\text{W}_9
\end{array}

- **Case:** $n = 7$. If $n = 7$ then $h = 1$, so we have $\Gamma_1 = G_{(1,7)}$ and $\Gamma < G_{(1,7)}$. Then $G_{(1,7)}$ has index 8 in $G_1$, and index 2 in $G_{\{1,7\}}$, so there is no possibility for $\Gamma$ that satisfies condition 4.

- **Case:** $n = 8$. If $n = 8$ then $h \in \{4, 8\}$. Since $G_{(1,8)}$ has index 12 in $G_1$, we have $\Gamma_1 = G_{(1,8)}$. The group $\Gamma$ is assumed to normalize $G_{(1,8\ell)}$, and the quotient $\Gamma / G_{(1,8\ell)}$ should have exponent 2, by condition 3. Observe that $G_{(1,8)} / G_{(1,8\ell)}$ is a cyclic group of order $h$ — we may take the image of $(T^8)^t$ as a generator. Since $h > 2$, there are no groups in $\mathcal{E}$ that arise for $n = 8$.

- **Case:** $n = 9$. For $n = 9$ we have $h \in \{3, 9\}$. Since $G_{(1,9)}$ has index 12 in $G_1$, we have $\Gamma_1 = G_{(1,9)}$. The group $\Gamma$ is assumed to normalize $G_{(1,9\ell)}$, and, by condition 3 it is required that $\Gamma / G_{(1,9\ell)}$ have exponent 2, but $G_{(1,9\ell)}$ has index $h$ in $\Gamma_1$. We conclude that no elements of $\mathcal{E}$ arise for $n = 9$.

- **Case:** $n = 11$. Similar to the case that $n = 7$, we must have $\Gamma_1 = G_{(1,11)}$ and $G_{(1,11)} < \Gamma < G_{(1,11)}$. But $G_{(1,11)}$ has index 12 in $G_1$, and index 2 in $G_{\{1,11\}}$, so there is no possibility for $\Gamma$ that satisfies condition 4.

We have accounted for all the groups appearing in the set $\mathcal{E}$, and we have verified that no other subgroups of $PSL_2(\mathbb{R})$ satisfy conditions 1 through 4.

\section*{3.3 Edges}

The groups in $\mathcal{E}$ are exactly those that label the vertices in the diagram (1.2). We now seek to reconstruct the edges of the diagram (1.2), by examination of the properties of the groups in $\mathcal{E}$.

\subsection*{3.3.1 Thread}

By condition 3 of Proposition 3.2 each group $\Gamma \in \mathcal{E}$ satisfies

\begin{equation}
G_{(1,N)} < \Gamma < N_G(G_{(1,N)}) = G_{(h,N/h)^+}
\end{equation}

for some positive integer $N$ (where $h$ is the largest divisor of 24 such that $h^2$ divides $N$ — c.f. Theorem 2.7). For $\Gamma \in \mathcal{E}$ let $N_\Gamma$ be the minimal $N$ with this property. (The group $\Gamma = G_{(1,2)}$ satisfies (3.17) for $N = 2$ and $N = 4$.) Now let $a_\Gamma$ be the largest divisor of 24 such that $a_\Gamma^2$ divides
\[ N_{\Gamma}, \text{ and } G_{(a,N/a)} \text{ contains } \Gamma \cap G_{(a,N/a)} \text{ to index } a, \text{ for } a = a_{\Gamma} \text{ and } N = N_{\Gamma}. \]

\[ \begin{array}{c}
\xymatrix{ & G_{(h,N/h)}+ \\
G_{(h,N/h)} \ar[ur] & \Gamma \ar[l] \ar[d] \ar[r] & a \\
G_{(a,N/a)} \ar[u] & \Gamma \cap G_{(a,N/a)} \ar[u] & G_{(1,N)} \ar[l]}
\end{array} \]

(3.18)

**Lemma 3.3.** If \( \Gamma \neq G_{\{1,4\}} \) then \( a_{\Gamma} \) is the largest divisor of 24 such that \( a_{\Gamma}^2 \) divides \( N_{\Gamma} \); that is, \( a = a_{\Gamma} = h \) in (3.18). If \( \Gamma = G_{\{1,4\}} \) then \( a_{\Gamma} = 1 \).

For each \( \Gamma \in \mathcal{E} \) define a subgroup \( \Gamma_0 < \Gamma \) by setting \( \Gamma_0 = \Gamma \cap G_{(a,N/a)} \), where \( a = a_{\Gamma} \) and \( N = N_{\Gamma} \). If we agree to write also \( G_{(1,N)}^{(1)} \) for \( G_{(1,N)} \), and \( G_{(a,a)}^{(a)} \) for \( G_{(a)}^{(a)} \), then we have

\[ \Gamma_0 = G_{(a,N/a)}^{(a)} \]

for \( a = a_{\Gamma} \) and \( N = N_{\Gamma} \), for each \( \Gamma \in \mathcal{E} \).

The groups \( \Gamma_0 \), for \( \Gamma \in \mathcal{E} \), may be recovered in a more geometric way as follows.

Relax for a moment the assumption that \( \Gamma \in \mathcal{E} \), and suppose, more generally, that \( \Gamma \) is an arithmetic subgroup of \( PSL_2(\mathbb{R}) \) such that \( \Gamma \) contains and normalizes some \( G_{(1,N)}^{(1)} \). Then \( \Gamma \) stabilizes the set \((1, N)^+\), and the normalizer of \( \Gamma \) stabilizes some subset of \((1, N)^+\).

Reinstate now the assumption that \( \Gamma \in \mathcal{E} \), and define \( S_{\Gamma} \) to be the largest subset of the \((1, N)^-\)-thread that is stabilized by \( N_{G}(\Gamma) \). For each \( \Gamma \in \mathcal{E} \) except for \( \Gamma = G_{\{1,4\}} \), the normalizer \( N_{G}(\Gamma) \), of \( \Gamma \), is a maximal discrete subgroup of \( PSL_2(\mathbb{R}) \), and therefore, by Proposition \( 2.10 \), stabilizes a unique cell in \( PL_1 \). The set \( S_{\Gamma} \) then recovers this cell. In the case that \( \Gamma = G_{\{1,4\}} \), we have \( N_{G}(\Gamma) = \Gamma \), while the group \( G_{2} \) properly contains \( \Gamma \). We have \( S_{\Gamma} = \{L_1, L_2, L_4\} \) (c.f. (2.50)) for \( \Gamma = G_{\{1,4\}} \).

For \( \Gamma \in \mathcal{E} \) consider the subgroup of \( N_{G}(\Gamma) \) fixing the elements of \( S_{\Gamma} \) point-wise; this group is exactly \( \Gamma_0 \), for each \( \Gamma \in \mathcal{E} \).

The following lemma is easily checked — by inspection of the first two lines of Table 2, for example.

**Lemma 3.4.** For each \( \Gamma \in \mathcal{E} \), the subgroup \( \Gamma_0 \) is contained normally in \( \Gamma \), and the quotient \( \Gamma/\Gamma_0 \) is a group of exponent 2.

The group \( \Gamma_0 \) may be regarded as the group we obtain from \( \Gamma \) by removing its Atkin–Lehner involutions.
3.3.2 Level

Let \( \Gamma \) be an arithmetic subgroup of \( \text{PSL}_2(\mathbb{R}) \), and suppose that \( \Gamma \) is a congruence group (c.f. \( \S 2.7 \)). Then \( \Gamma \) has a well-defined level; viz. the minimal positive integer \( N \) such that \( \Gamma(N) < \Gamma \). Let us write \( \text{lev}(\Gamma) \) for the level of an arithmetic congruence group \( \Gamma \). For \( \Gamma \in \mathcal{E} \) we define the normalized level of \( \Gamma \), to be denoted \( \text{lev}_0(\Gamma) \), by setting

\[
\text{lev}_0(\Gamma) = \frac{\text{lev}(\Gamma)}{a_{\Gamma}} \tag{3.20}
\]

where \( a_{\Gamma} \) is as in \( \S 3.3.1 \). The values \( a_{\Gamma} \) and the normalized levels \( \text{lev}_0(\Gamma) \) are displayed in Table 1.

| \( \Gamma \)       | \( a_{\Gamma} \) | \( \text{lev}_0(\Gamma) \) |
|-------------------|-----------------|-------------------------|
| \( G_1 \)         | 1               | 1                       |
| \( G^{(1,2)} \)   | 1               | 2                       |
| \( G^{(1,3)} \)   | 1               | 3                       |
| \( G^{(1,4)} \)   | 1               | 4                       |
| \( G^{(1,5)} \)   | 1               | 5                       |
| \( G^{(1,2,3,6)} \) | 1             | 6                       |
| \( G^{(3)} \)     | 2               | 3                       |
| \( G^{(2,4)} \)   | 2               | 4                       |

3.3.3 Valency

By Lemma \( \S 3.4 \), the group \( \Gamma_0 \) is normal in \( \Gamma \) for each \( \Gamma \in \mathcal{E} \), and the quotient \( \Gamma/\Gamma_0 \) has exponent 2. For \( \Gamma \in \mathcal{E} \) we define the valency of \( \Gamma \), to be denoted \( \text{val}(\Gamma) \), by setting \( \text{val}(\Gamma) = m + 1 \), where \( m \) satisfies

\[
\Gamma/\Gamma_0 \cong (\mathbb{Z}/2)^m. \tag{3.21}
\]

The value \( \text{val}(\Gamma) \) encodes the number of Atkin–Lehner involutions in \( \Gamma \). The valencies of the groups in \( \mathcal{E} \) are recorded in Table 2.

| \( \Gamma \)       | \( \Gamma_0 \) | \( \text{val}(\Gamma) \) |
|-------------------|---------------|-------------------------|
| \( G_1 \)         | \( G^{(1,2)} \) | 1                       |
| \( G^{(1,3)} \)   | \( G^{(1,3)} \) | 2                       |
| \( G^{(1,4)} \)   | \( G^{(1,4)} \) | 2                       |
| \( G^{(1,5)} \)   | \( G^{(1,5)} \) | 2                       |
| \( G^{(1,2,3,6)} \) | \( G^{(1,6)} \) | 3                       |
| \( G^{(3)} \)     | \( G^{(3)} \)  | 1                       |
| \( G^{(2,4)} \)   | \( G^{(2,4)} \) | 2                       |

3.3.4 Faithfulness

Recall that in the classical McKay Correspondence for the binary icosahedral group \( 2.\text{Alt}_5 \), the nodes in the affine \( E_8 \) Dynkin diagram are labeled by irreducible representations of \( 2.\text{Alt}_5 \). In particular, four of the nodes correspond to faithful representations of \( 2.\text{Alt}_5 \), and the remaining five nodes correspond to representations that factor through \( \text{Alt}_5 \).

With this in mind, we will say that a group \( \Gamma \in \mathcal{E} \) is \emph{faithful} if \( \Gamma \) is “not far” from \( G^{(1,2)} \), in the sense that \( [\Gamma, \Gamma \cap G^{(1,2)}] \leq 2 \). We set \( \mathcal{E}_1 \) to be the subset of groups in \( \mathcal{E} \) that are faithful, and
we set $\mathcal{E}_0 = \mathcal{E} \setminus \mathcal{E}_1$.

\[
\mathcal{E}_1 = \{G_{\{1,2\}}, G_{\{1,4\}}, G_{\{1,2,3,6\}}, G_{\{1,2\}}\}
\]

\[
\mathcal{E}_0 = \{G_1, G_{\{1,3\}}, G_{\{1,5\}}, G_{3}^{(3)}, G_{3}^{(2)}, G_{\{2,4\}}\}
\]

### 3.3.5 Prescription

The following proposition is easily checked.

**Proposition 3.5.** There is a unique graph with vertex set $\mathcal{E}$ satisfying the following properties.

- The valence of $\Gamma \in \mathcal{E}$ is $\text{val}(\Gamma)$.
- The identity $2 \text{lev}_0(\Gamma) = \sum_{\Gamma' \in \text{adj}(\Gamma)} \text{lev}_0(\Gamma')$ holds for all $\Gamma \in \mathcal{E}$, where $\text{adj}(\Gamma)$ denotes the set of vertices that are adjacent to $\Gamma$.
- If $\Gamma \in \mathcal{E}_1$ then $\text{adj}(\Gamma) \subset \mathcal{E}_0$.

The graph whose existence and uniqueness is guaranteed by Proposition 3.5 is displayed in (3.22).

\[
\begin{aligned}
G_{3}^{(3)} & \quad G_{\{1,2\}} - G_{\{1,3\}} - G_{\{1,4\}} - G_{\{1,5\}} - G_{\{1,2,3,6\}} \\
& \quad G_{\{2,4\}} \\
& \quad G_{\{1,2\}}
\end{aligned}
\]

(3.24)

In terms of the more standard notation for the discrete groups of Monstrous Moonshine, this is exactly the diagram (1.2) we sought to recover.

### 4 Super analogue

For each $\Gamma \in \mathcal{E}$, we define an arithmetic group $^s\Gamma < PSL_2(\mathbb{R})$ as follows. Recall that for each $\Gamma \in \mathcal{E}$, the corresponding group $\Gamma_0$ (c.f. §3.3.1) can be written in the form $G_{(a,N/a)}^{(a)}$ for some $a = a_\Gamma$ and $N = N_\Gamma$. Define groups $^s\Gamma_0$ by setting

\[
^s\Gamma_0 = G_{(a,2N/a)}^{(a)}
\]

(4.1)

when $\Gamma_0 = G_{(a,N/a)}^{(a)}$. We will arrive at the group $^s\Gamma$ by adjoining to $^s\Gamma_0$ the appropriate “scalings” of the Atkin–Lehner involutions of $\Gamma$; i.e. scalings of the cosets $\Gamma/\Gamma_0$. More precisely, a coset $W_e(N)$ (c.f. §2.7) in $\Gamma/\Gamma_0$, for $N = N_\Gamma$ and $e$ an exact divisor of $N$, is scaled (i.e. sent) to $W_{2e}(2N)$ or $W_e(2N)$ according as $2|e$ or not.

\[
^sW_e(N) = \begin{cases} 
W_{2e}(2N), & \text{if } 2|e; \\
W_e(2N), & \text{else.}
\end{cases}
\]

(4.2)
This handles the cosets of $\Gamma_0$ in $\Gamma$ in the cases that $\Gamma \neq G^{(2)}_{(2,4)}$. For $\Gamma = G^{(2)}_{(2,4)}$ we apply the same idea, but the notation is slightly more involved: the unique non-trivial coset in $\Gamma/\Gamma_0$ is $g_2^{-1}W_2(2)g_2$ in this case, and we set $2(g_2^{-1}W_2(2)g_2) = g_2^{-1}W_4(4)g_2$. Now we define $^s\Gamma$, for $\Gamma \in \mathcal{E}$, by setting

$$^s\Gamma = \bigcup_{X \in \Gamma/\Gamma_0} ^sX.$$  \hspace{1cm} (4.3)

It is straightforward to check that $^s\Gamma$ is an arithmetic subgroup of $\text{PSL}_2(\mathbb{R})$ for each $\Gamma \in \mathcal{E}$. Even more than this, each group $^s\Gamma$ also appears in Monstrous Moonshine. Define a set $^s\mathcal{E}$ by setting

$$^s\mathcal{E} = \{ ^s\Gamma \mid \Gamma \in \mathcal{E} \}.$$  \hspace{1cm} (4.4)

We may consider the diagram obtained from (1.2) by replacing the labels $\Gamma \in \mathcal{E}$ with the corresponding groups $^s\Gamma \in ^s\mathcal{E}$. In the notation of [CN79] and [CMS04], the labeling of the affine $E_8$ Dynkin diagram thus obtained is displayed in (4.5).

$$
\begin{array}{c}
6|3 \\
2-4+6+6-8-10+10-12+ \\
8|2+ \\
4
\end{array}
$$  \hspace{1cm} (4.5)

Each of the groups in (4.5) is attached to a unique conjugacy class of the Monster group, courtesy of Monstrous Moonshine. Each conjugacy class $nZ$ in $\mathbb{M}$ so obtained has the property that if $g \in nZ$ then $g^{n/2} \in 2B$. Further, each of these conjugacy classes is a “lift” of some conjugacy class in the Conway group $\text{Co}_0$. More precisely, for each group $^s\Gamma$ in $^s\mathcal{E}$ there is a Frame shape $A_1^{a_1}A_2^{a_2}\cdots$ say, such that the eta product

$$\frac{\eta(A_1\tau)^{a_1}\eta(A_2\tau)^{a_2}\cdots}{\eta(2A_1\tau)^{a_1}\eta(2A_2\tau)^{a_2}\cdots}$$  \hspace{1cm} (4.6)

furnishes a principal modulus (a.k.a. hauptmodul) for $^s\Gamma$, and also encodes the eigenvalues of the elements of a unique conjugacy class of $\text{Co}_0$.

$$
\begin{array}{c}
2^8 \\
2^{24}/1^{24} - 3^{12}/1^{12} - 4^8/1^8 - 5^6/1^6 - 2^6/6^6/1^6/3^6 \\
4^{12}/2^{12} \\
1^8/2^8
\end{array}
$$  \hspace{1cm} (4.7)

\footnote{a generalization of the cycle notation for permutations, which encodes the eigenvalues of an orthogonal transformation of finite order that is writable over $\mathbb{Z}$.}
Diagram (4.7) shows the labeling obtained by replacing the groups in (4.5) with the corresponding Frame shapes of $\text{Co}_0$.

Evidently, (4.7) furnishes a reformulation of McKay’s Monstrous $E_8$ observation, in which conjugacy classes of the Monster are replaced by conjugacy classes of $\text{Co}_0$.

It is amusing to observe that the highest root labeling of the affine $E_8$ Dynkin diagram can be read off directly from the Frame shapes in (4.7): consider the maximal $A_i$ in $A_1^{a_1}A_2^{a_2}\cdots$ (i.e. the order of the corresponding class in $\text{Co}_0$). One can also predict the valency of the vertex corresponding to any Frame shape in (4.7): the valency of the vertex labeled $A_1^{a_1}A_2^{a_2}\cdots$ is the number of negative $a_i$ plus 1.

A direct link between the classes in (4.7) and the principal moduli of the groups in (4.5) can be obtained by considering the McKay–Thompson series associated to conjugacy classes of $\text{Co}_0$ via the action of this group on a suitably defined vertex operator superalgebra (c.f. [Dun07]).

Acknowledgement

This article is dedicated to John McKay, with gratitude and respect. The author is grateful to Noam Elkies and Curtis McMullen for helpful conversations

References

[Bor92] Richard E. Borcherds. Monstrous moonshine and monstrous Lie superalgebras. *Invent. Math.*, 109(2):405–444, 1992.

[CMS04] John Conway, John McKay, and Abdellah Sebbar. On the discrete groups of Moonshine. *Proc. Amer. Math. Soc.*, 132:2233–2240, 2004.

[CN79] J. H. Conway and S. P. Norton. Monstrous moonshine. *Bull. London Math. Soc.*, 11(3):308–339, 1979.

[Con85] J. H. Conway. A simple construction for the Fischer-Griess monster group. *Invent. Math.*, 79(3):513–540, 1985.

[Con96] J. H. Conway. Understanding groups like $\Gamma_0(N)$. In *Groups, difference sets, and the Monster (Columbus, OH, 1993)*, volume 4 of *Ohio State Univ. Math. Res. Inst. Publ.*, pages 327–343. de Gruyter, Berlin, 1996.

[Dun07] John F. Duncan. Super-Moonshine for Conway’s largest sporadic group. *Duke Math. J.*, 139(2):255–315, 2007.

[Hel66] Heinz Helling. Bestimmung der Kommensurabilitätsklasse der Hilbertschen Modulgruppe. *Math. Z.*, 92:269–280, 1966.

[McK80] John McKay. Graphs, singularities, and finite groups. In *The Santa Cruz Conference on Finite Groups (Univ. California, Santa Cruz, Calif., 1979)*, volume 37 of *Proc. Sympos. Pure Math.*, pages 183–186. Amer. Math. Soc., Providence, R.I., 1980.