PARTITION MODELS, PERMUTATIONS OF INFINITE SETS WITHOUT FIXED POINTS, AND WEAK FORMS OF AC

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Abstract. We study new relations of the following statements with weak choice principles in ZF (Zermelo–Fraenkel set theory without the Axiom of Choice (AC)) and ZFA (ZF with the axiom of extensionality weakened to allow the existence of atoms).

- There does not exist an infinite Hausdorff space $X$ such that every infinite subset of $X$ contains an infinite compact subset.
- For every set $X$ there is a set $A$ such that there exists a choice function on the collection $[A]^2$ of two-element subsets of $A$ and satisfying $|X| \leq |2^X|$.
- If a field has an algebraic closure then it is unique up to isomorphism.
- For every infinite set $X$, there exists a permutation of $X$ without fixed points.
- vDCP (Van Douwen’s Choice Principle).
- Any infinite locally finite connected graph has a spanning subgraph omitting $K_{2,n}$ for any $2 \leq n \leq \omega$.
- Any infinite locally finite connected graph has a spanning $m$-bush for any even integer $m \geq 4$.

We also study the new status of different weak choice principles in the finite partition model (a type of permutation model of ZFA $+ \neg$AC) introduced by Benjamin Baker Bruce in 2016.

1. Introduction and Abbreviations

1.1. Forms 269, 233, and 304. We study the status of certain weak choice principles in a recent permutation model constructed by Halbeisen and Tachtsis in [12, Theorem 8].

1.1.1. Necessary weak choice forms. Let $X$ and $Y$ be sets. We write $|X| \leq |Y|$ if there is an injection $f : X \to Y$.

- [S] Form 269: For every set $X$ there is a set $A$ such that there exists a choice function on the collection $[A]^2$ of two-element subsets of $A$ and satisfying $|X| \leq |2^A|$.
- [S] Form 233: If a field has an algebraic closure then it is unique up to isomorphism.
- [S] Form 304: There does not exist an infinite Hausdorff space $X$ such that every infinite subset of $X$ contains an infinite compact subset.
- AC$^\text{LO}$ [S, Form 202]: Every linearly ordered family of non-empty sets has a choice function.
- LW [S, Form 90]: Every linearly ordered set can be well-ordered.
- AC$^\text{WO}$ [S, Form 40]: Every well-orderable set of non-empty sets has a choice function.
- AC$^\text{c}_n$ for each $n \in \omega$, $n \geq 2$ [S, Form 342(n)]: Every infinite family $A$ of $n$-element sets has a partial choice function, i.e., $A$ has an infinite subfamily $B$ with a choice function.
- The Chain/Antichain Principle, CAC [S, Form 217]: Every infinite partially ordered set (poset) has an infinite chain or an infinite antichain.
- [S, Form 64]: There are no amorphous sets (An infinite set $X$ is amorphous if $X$ cannot be written as a disjoint union of two infinite subsets).

1.1.2. Results. Pincus proved that [S, Form 233] holds in the basic Fraenkel model (cf. [S, Note 41]). It is also known that in the basic Fraenkel model [S, Form 269] fails, whereas [S, Form 304] holds (cf. [S, Notes 91, 116]). Fix a natural number $2 \leq n \in \omega$. Halbeisen–Tachtsis [12, Theorem 8] constructed a permutation model (we denote by $\mathcal{N}_n^{HT}$) where $\text{AC}^\text{c}_n$ fails but CAC holds. We prove the following (cf. Theorem 3.1, Theorem 3.3):

1. In ZFA, AC$^\text{LO}$ does not imply Form 269.
2. Form 269 fails in $\mathcal{N}_n^{HT}$ whereas Form 233 and Form 304 hold in $\mathcal{N}_n^{HT}$. Consequently, for any integer $n \geq 2$, Form 233 (similarly Form 304) neither implies $\text{AC}^\text{c}_n$ nor implies ‘There are no amorphous sets’ in ZFA.

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1.2. Partition models and permutations of infinite sets. We study the failure of certain weak choice principles in the finite partition model introduced by Bruce in [4].

1.2.1. Necessary weak choice forms and abbreviations. A set $X$ is almost even if there is a permutation $f$ of $X$ without fixed points and such that $f^2 = \text{id}_X$.

- AC$\alpha$ for each $n \in \omega, n \geq 2$ [8 Form 61]: Every family of $n$-element sets has a choice function.
- DF = $F$ [8 Form 9]: Every Dedekind-finite set is finite (A set $X$ is called Dedekind-finite if $\aleph_0 \not\leq |X|$ i.e., if there is no one-to-one function $f : \omega \to X$. Otherwise, $X$ is called Dedekind-infinite).
- W$\aleph_0$ (cf. [13 Chapter 8]): For every $X$, either $|X| \leq \aleph_0$ or $|X| \geq \aleph_0$. We recall that W$\aleph_0$ is equivalent to DF = $F$ in ZF.
- DC$\alpha$, where $\alpha$ is the ordinal such that $\kappa = \aleph_\alpha$ [8 Form 87(\alpha)]: Let $S$ be a non-empty set and let $R$ be a binary relation such that for every $\beta < \kappa$ and every $\beta$-sequence $s = (s_i)_{i<\beta}$ of elements of $S$ there exists $y \in S$ such that $sRy$. Then there is a function $f : \kappa \to S$ such that for every $\beta < \kappa$, $(f \restriction \beta)Rf(\beta)$. We note that DC$\aleph_0$ is a reformulation of DC (the principle of Dependent Choices [8 Form 43]). We denote by DC$\subset \lambda$ the assertion ($\forall \eta < \lambda$)DC$\eta$.
- UT(WO,WO,WO) [8 Form 231]: The union of a well-ordered collection of well-orderable sets is well-orderable.
- ($\forall \alpha$)UT($\aleph_\alpha, \aleph_\alpha, \aleph_\alpha$) [8 Form 23]: For every ordinal $\alpha$, if $A$ and every member of $A$ has cardinality $\aleph_\alpha$, then $|\bigcup A| = \aleph_\alpha$.
- The Axiom of Multiple Choice, MC [8 Form 67]: Every family $\mathcal{A}$ of non-empty sets has a multiple choice function, i.e., there is a function $f$ with domain $\mathcal{A}$ such that for every $A \in \mathcal{A}$, $f(A)$ is a non-empty finite subset of $A$.
- $\leq \aleph_0$-MC (cf. [10 section 1]): For any family $\{A_i : i \in I\}$ of non-empty sets, there is a function $F$ with domain $I$ such that for all $i \in I$, $F(i)$ is a non-empty countable (i.e., finite or countably infinite) subset of $A_i$.
- [8 Form 3]: For every infinite set $X$, the sets $X$ and $X \times \{0, 1\}$ are equipotent (i.e., there exists a bijection $f : X \to X \times \{0, 1\}$).
- ISAE (cf. [18 section 2]): Every infinite set is almost even (i.e., for every infinite set $X$, $f(X)$ is a non-empty set without fixed points and such that $f^2 = \text{id}_X$).
- EPWFP (cf. [18 section 2]): For every infinite set $X$, there exists a permutation of $X$ without fixed points.
- For a set $A$, Sym($A$) and $\text{FSym}(A)$ denote the set of all permutations of $A$ and the set of all $\phi \in \text{Sym}(A)$ such that $\{x \in A : \phi(x) = x\}$ is finite, respectively (cf. [18 section 2]).
- For a set $A$ of size at least $\aleph_\alpha$, $\aleph_\alpha \text{Sym}(A)$ denote the set of all $\phi \in \text{Sym}(A)$ such that $\{x \in A : \phi(x) = x\}$ has cardinality at least $\aleph_\alpha$.
- MA($\kappa$) for a well-ordered cardinal $\kappa$ (cf. [22 section 1]): If $(P, <)$ is a nonempty, c.c.c. quasi order and if $\mathcal{D}$ is a family of $\leq \kappa$ dense sets in $P$, then there is a filter $\mathcal{F}$ of $P$ such that $\mathcal{F} \cap \mathcal{D} = \emptyset$ for all $D \in \mathcal{D}$.

1.2.2. Results. Bruce [4] constructed the finite partition model $\mathcal{V}_p$, which is a variant of the basic Fraenkel model (labeled as Model $\mathcal{N}_1$ in [8]). Many, but not all, properties of $\mathcal{N}_1$ transfer to $\mathcal{V}_p$. In particular, Bruce proved that the set of atoms has no amorphous subset in $\mathcal{V}_p$ unlike in $\mathcal{N}_1$, whereas UT(WO,WO,WO), $\neg$AC$_2$, and $\neg$(DF = $F$) hold in $\mathcal{V}_p$ as in $\mathcal{N}_1$. At the end of the paper, Bruce asked which other choice principles hold in $\mathcal{V}_p$ (cf. [4 section 5]).

We study the status of some weak choice principles in $\mathcal{V}_p$. We also study the status of some weak choice principles in a variant of $\mathcal{V}_p$ mentioned in [4 section 5]. In particular, let $\mathcal{A}$ be an uncountable set of atoms, let $\mathcal{G}$ be the group of all permutations of $\mathcal{A}$, and let the supports be countable partitions of $\mathcal{A}$. We call the corresponding permutation model $\mathcal{V}_p^\mathcal{G}$. At the end of the paper, Bruce asked about the status of different weak choice forms in $\mathcal{V}_p^\mathcal{G}$. Fix any integer $n \geq 2$. We prove the following (cf. Theorem 4.5, Proposition 4.7, Theorem 4.8):

1. $\mathcal{W}_{\aleph_{n+1}}$ implies ‘for any set $X$ of size $\aleph_{n+1}$, Sym($X$) $\neq \aleph_\alpha \text{Sym}(X)$’ in ZF.
2. If $X \in \{\text{EPWFP}, \text{MA}(\aleph_\alpha), \text{AC}_n, \text{MC} \leq \aleph_\alpha \text{-MC}\}$, then $X$ fails in $\mathcal{V}_p$.
3. If $X \in \{\text{EPWFP}, \text{AC}_n, \text{W}_{\aleph_0}\}$, then $X$ fails in $\mathcal{V}_p^\mathcal{G}$.

1.3. Van Douwen’s Choice Principle in permutation models. Howard, Saveliev, and Tachtsis [10 p.175] gave an argument to prove that Van Douwen’s Choice Principle (vDCP) holds in the basic Fraenkel model. We modify the argument slightly to prove that vDCP holds in two recently constructed permutation models (cf. section 5).

1.3.1. Necessary weak choice forms and abbreviations.

- UT(\aleph_\alpha, \aleph_\alpha, \text{cuf}) [9 Form 420]: Every countable union of countable sets is a cuf set (A set $X$ is called a cuf set if $X$ is expressible as a countable union of finite sets).
• \( \text{M(IC, DI)} \) (cf. [15]): Every infinite compact metrizable space is Dedekind-infinite.
• \( \text{MC}(\aleph_0, \aleph_0) \) [8] Form 350]: Every denumerable family of denumerable sets has a multiple choice function.
• Van Douwen’s Choice Principle, \( \text{vDCP} \): Every family \( X = \{ (x_i, \leq_i) : i \in I \} \) of linearly ordered sets isomorphic with \( (\mathbb{Z}, \leq) \) (\( \leq \) is the usual ordering on \( \mathbb{Z} \)) has a choice function.

1.3.2. Results. Howard and Tachtsis [11] Theorem 3.4] proved that the statement \( \text{LW} \land \neg \text{MC}(\aleph_0, \aleph_0) \) has a permutation model, say \( \mathcal{M} \). The authors of [5] proof of Theorem 3.3] constructed a permutation model \( \mathcal{N} \) where \( \text{UT}(\aleph_0, \aleph_0, \text{cuf}) \) holds. Keremedis, Tachtsis, and Wajch [15] Theorem 13] proved that \( \text{LW} \) holds in \( \mathcal{M} \). We prove the following (cf. Proposition 5.1):

(1) \( \text{vDCP} \) holds in \( \mathcal{N} \) and \( \mathcal{M} \).

1.4. Spanning subgraphs. Fix any \( 2 < n \in \omega \) and any even integer \( 4 \leq m \in \omega \). Delhommé–Morillon [9] Corollary 1, Remark 1] proved that \( \text{AC} \) is equivalent to ‘Every bipartite connected graph has a spanning subgraph omitting \( K_{n,n} \)’, as well as ‘Every connected graph admits a spanning m-bush’. We study new relations between variants of the above statements and weak forms of \( \text{AC} \).

1.4.1. Necessary weak choice forms and abbreviations.

• Let \( n \in \omega \backslash \{0, 1\} \). \( \text{AC}^0_{\leq n} \): Every denumerable family of non-empty sets, each with at most \( n \) elements, has a choice function.
• \( \text{AC}^0_{\leq n} \) [8] Form 10]: Every denumerable family of non-empty finite sets has a choice function.
• \( \text{AC}^0_{\leq \omega} \) [8] Form 165]: Every well-orderable family of non-empty well-orderable sets has a choice function.
• \( n \in \omega \backslash \{0, 1\} \). \( \text{AC}^0_{\leq n} \): Every well-orderable family of non-empty sets, each with at most \( n \) elements, has a choice function.

Fix any \( 2 < k, n \in \omega \) and any even integer \( 4 \leq m \in \omega \). We introduce the following abbreviations.

• \( \text{Q}^m_{\text{lf}, c} \): Any infinite locally finite connected graph has a spanning subgraph omitting \( K_{2,n} \).
• \( \text{Q}^m_{\text{lf}, c} \): Any infinite locally well-orderable connected graph has a spanning subgraph omitting \( K_{k,n} \).
• \( \text{P}^m_{\text{lf}, c} \): Any infinite locally finite connected graph has a spanning m-bush.

Let \( G \) be a graph. We denote by \( P_G \), the class of those infinite graphs whose only components are isomorphic with \( G \). For any graph \( G_1 = (V_{G_1}, E_{G_1}) \in P_G \), we construct a graph \( G_2 = (V_{G_2}, E_{G_2}) \) as follows: Pick a \( t \notin V_{G_1} \). Let \( V_{G_2} = \{ t \} \cup V_{G_1}, E_{G_1} \subseteq E_{G_2} \) and for each \( x \in V_{G_1} \), let \( \{ t, x \} \in E_{G_2} \). We denote by \( P'_G \), the class of graphs of the form \( G_2 \).

1.4.2. Results. Fix \( 2 < n, k \in \omega \) and \( 2 \leq p, q < \omega \). We prove the following in \( \text{ZF} \) (cf. Proposition 6.5 and Proposition 6.6):

(1) \( \text{AC}^0_{\leq n} \) \( \land \text{Q}^m_{\text{lf}, c} \) is equivalent to \( \text{AC}^0_{\leq m} \).
(2) \( \text{UT}(\text{WO}, \text{WO}, \text{WO}) \) implies \( \text{AC}^0_{\leq n} + \text{Q}^m_{\text{lf}, c} \) and the later implies \( \text{AC}^0_{\text{WO}} \).
(3) \( \text{P}^m_{\text{lf}, c} \) is equivalent to \( \text{AC}^0_{\leq m} \) for any even integer \( m \geq 4 \).
(4) \( \text{AC}^0_{\leq 2} \) implies ‘Every graph from the class \( P'_{K_k} \) has a spanning tree’.
(5) \( \text{AC} \) implies ‘Every graph from the class \( P'_{K_k} \) has a spanning tree’.
(6) \( \text{AC} + \text{Q}_{m+1} \) implies ‘Every graph from the class \( P'_{K_{p,q}} \) has a spanning tree’.

2. Basics

Definition 2.1. (Topological definitions) Let \( X = (X, \tau) \) be a topological space. We say \( X \) is Baire if for every countable family \( \mathcal{O} = \{ O_n : n \in \omega \} \) of dense open subsets of \( X \), \( \bigcap \mathcal{O} \) is nonempty and dense. We say \( X \) is compact if for every \( U \subseteq X \) such that \( \bigcup U = X \) there is a finite subset \( V \subseteq U \) such that \( \bigcup V = X \). The space \( X \) is called a Hausdorff (or \( T_2 \))-space if any two distinct points in \( X \) can be separated by disjoint open sets, i.e. if \( x \) and \( y \) are distinct points of \( X \), then there exist disjoint open sets \( U_x \) and \( U_y \) such that \( x \in U_x \) and \( y \in U_y \).

Definition 2.2. (Algebraic definitions) A permutation on a set \( Y \) is a one-to-one correspondence from \( Y \) to itself. The set of all permutations on \( Y \), with operation defined to be the composition of mappings, is the symmetric group of \( Y \), denoted by \( \text{Sym}(Y) \). Let \( X \) be a finite set. Fix \( r \leq |X| \). A permutation \( \sigma \in \text{Sym}(X) \) is a cycle of length \( r \) if there are distinct elements \( i_1, \ldots, i_r \in X \) such that \( \sigma(i_1) = i_2, \sigma(i_2) = i_3, \ldots, \sigma(i_r) = i_1 \) and \( \sigma(i) = i \) for all \( i \in X \backslash \{i_1, \ldots, i_r \} \). In this case we write \( \sigma = (i_1, i_2) \). A cycle of length 2 is called a transposition. We recall that \( (i_1, i_2) = (i_1, i_2)(i_1, i_2)(i_1, i_2) \ldots \). So, every permutation can be written as a product of transpositions. A permutation \( \sigma \in \text{Sym}(X) \) is an even permutation if it can be written as the product
of an even number of transpositions; otherwise, it is an odd permutation. The alternating group of $X$, denoted by $Alt(X)$, is the group of all even permutations in $Sym(X)$. If $G$ is a group and $X$ is a set, an action of $G$ on $X$ is a group homomorphism $F: G \to Sym(X)$. If a group $G$ acts on a set $X$, we say $Orb_G(x) = \{gx : g \in G\}$ is the orbit of $x \in X$ under the action of $G$. We recall that different orbits of the action are disjoint and form a partition of $X$, i.e., $X = \bigcup\{Orb_G(x) : x \in X\}$. Let $\{G_i : i \in I\}$ be an indexed collection of groups. Define the following set.

$$\prod_{i \in I} G_i = \left\{ f : I \to \bigcup_{i \in I} G_i \left| \forall i \in I \right. f(i) \in G_i, f(i) = 1_{G_i}, \text{for all but finitely many } i \right\}. \tag{1}$$

The weak direct product of the groups $\{G_i : i \in I\}$ is the set $\prod_{i \in I} G_i$ with the operation of component-wise multiplicative defined for all $f, g \in \prod_{i \in I} G_i$ by $(fg)(i) = f(i)g(i)$ for all $i \in I$. A field $K$ is algebraically closed if every non-constant polynomial in $K[x]$ has a root in $K$.

**Definition 2.3. (Combinatorial definitions)** The degree of a vertex $v \in V_G$ of a graph $G = (V_G, E_G)$ is the number of edges emerging from $v$. A graph $G = (V_G, E_G)$ is locally finite if every vertex of $G$ has a finite degree.

We say that a graph $G = (V_G, E_G)$ is locally well-orderable if for every $v \in V_G$, the set of neighbors of $v$ is well-orderable. Given a non-negative integer $n$, a path of length $n$ in the graph $G = (V_G, E_G)$ is a one-to-one finite sequence $(x_i)_{0 \leq i < n}$ of vertices such that for each $i < n$, $\{x_i, x_{i+1}\} \in E_G$; such a path joins $x_0$ to $x_n$. The graph $G$ is connected if any two vertices are joined by a path of finite length. For each integer $n \geq 3$, an $n$-cycle of $G$ is a path $(x_i)_{0 \leq i < n}$ such that $\{x_{n-1}, x_0\} \in E_G$ and an $n$-bush is any connected graph with no $n$-cycles. We denote by $K_n$ the complete graph on $n$ vertices. We denote by $C_n$ the circuit of length $n$. A forest is a graph with no cycles and a tree is a connected forest. A spanning subgraph $H = (V_H, E_H)$ of $G = (V_G, E_G)$ is a subgraph that contains all the vertices of $G$ i.e., $V_H = V_G$. A complete bipartite graph is a graph $G = (V_G, E_G)$ whose vertex set $V_G$ can be partitioned into two subsets $V_1$ and $V_2$ such that no edge has both endpoints in the same subset, and every possible edge that could connect vertices in different subsets is a part of the graph. A complete bipartite graph with partitions of size $|V_1| = m$ and $|V_2| = n$, is denoted by $K_{m,n}$ for any natural number $m, n$. Let $(P, \leq)$ be a partially ordered set or a poset. A subset $D \subseteq P$ is called a chain if $(D, \leq)$ is linearly ordered. A subset $A \subseteq P$ is called an antichain if no two elements of $A$ are comparable under $\leq$. The size of the largest antichain of the poset $(P, \leq)$ is known as its width. A subset $C \subseteq P$ is called cofinal in $P$ if for every $x \in P$ there is an element $c \in C$ such that $x \leq c$.

**2.1 Permutation models.** In this subsection, we provide a brief description of the construction of Fraenkel-Mostowski permutation models of ZFA from $\textbf{[13]}$ Chapter 4]. Let $M$ be a model of ZFA + $\textbf{AC}$ where $\textbf{A}$ is a set of atoms or urelements. Let $G$ be a group of permutations of $A$. A set $F_1$ of subgroups of $G$ is a normal filter on $G$ if for all subgroups $H, K$ of $G$, the following holds.

1. $G \in F_1$,
2. If $H \in F_1$ and $H \subseteq K$, then $K \in F_1$,
3. If $H \in F_1$ and $K \in F_1$, then $H \cap K \in F_1$,
4. If $\pi \in G$ and $H \in F_1$, then $\pi H \pi^{-1} \in F_1$,
5. For each $a \in A$, $\{\pi \in G : \pi(a) = a\} \in F_1$.

Let $F$ be a normal filter of subgroups of $G$. For $x \in M$, we say $\text{sym}_G(x) = \{g \in G : g(x) = x\}$ and $\text{fix}_G(x) = \{\phi \in G : \forall y \in x, \phi(y) = y\}$. We say $x$ is symmetric if $\text{sym}_G(x) \in F$ and $x$ is hereditarily symmetric if $x$ is symmetric and each element of the transitive closure of $x$ is symmetric. We define the permutation model $\textbf{N}$ with respect to $G$ and $F$, to be the class of all hereditarily symmetric objects. It is well-known that $\textbf{N}$ is a model of ZFA (cf. $\textbf{[13]}$ Theorem 4.1]). A family $I_1$ of subsets of $A$ is a normal ideal if the following holds.

1. $\emptyset \in I_1$,
2. If $E \in I_1$ and $F \subseteq E$, then $F \in I_1$,
3. If $E, F \in I_1$, then $E \cup F \in I_1$,
4. If $\pi \in G$ and $E \in I_1$, then $\pi[E] \in I_1$,
5. For each $a \in A$, $\{a\} \in I_1$.

If $I \subseteq P(A)$ is a normal ideal, then the set $\{\text{fix}_G(E) : E \in I\}$ generates a normal filter (say $F_I$) over $G$. Let $\textbf{N}$ be the permutation model determined by $M$, $G$, and $F_I$. We say $E \in I$ is a support of a set $\sigma \in \textbf{N}$ if $\text{fix}_G(E) \subseteq \text{sym}_G(\sigma)$.

**Lemma 2.4.** The following hold:
(1) An element $x$ of $\mathcal{N}$ is well-orderable in $\mathcal{N}$ if and only if $\text{fix}_\mathcal{G}(x) \in F_\mathcal{Z}$ (cf. [12] Equation (4.2), p.47). Thus, an element $x$ of $\mathcal{N}$ is well-orderable in $\mathcal{N}$ if $\text{fix}_\mathcal{G}(E) \subseteq \text{fix}_\mathcal{G}(x)$.

(2) Let $\mathcal{G}$ be a group of permutations of a set of atoms $A$ and let $\mathcal{I}$ be a normal ideal of supports. Let $\mathcal{V}$ be the permutation model given by $\mathcal{G}$ and $\mathcal{I}$. Then for all $\pi \in \mathcal{G}$ and all $x \in \mathcal{V}$ such that $E$ is a support of $x$, $\text{sym}_\mathcal{G}(\pi x) = \pi \text{sym}_\mathcal{G}(x)^{-1}$ and $\text{fix}_\mathcal{G}(\pi E) = \pi \text{fix}_\mathcal{G}(E)^{-1}$ (cf. [13] proof of Lemma 4.4]).

A pure set in a model $M$ of ZFA is a set with no atoms in its transitive closure. The kernel is the class of all pure sets of $M$. In this paper,

- Fix an integer $n \geq 2$. We denote by $\mathcal{N}_n^{\text{HF}}$ the permutation model constructed in [12] Theorem 8.
- We denote by $\mathcal{N}_1^{\text{HF}}$ the basic Faenkel model (cf. [8]).
- We denote by $\mathcal{N}_2^{\text{HF}}$ the following variant of the basic Faenkel model: Let $A$ be a set of atoms of size $\aleph_1$, $\mathcal{G}$ be the group of all permutations of $A$, and the supports are countable subsets of $A$ (cf. [8]).
- We denote by $\mathcal{N}_2^{\text{HF}}$ the following variant of the basic Faenkel model: Let $A$ be a set of atoms of size $\aleph_1$, $\mathcal{G}$ be the group of all permutations of $A$, and the supports are subsets of $A$ with cardinality less than $\aleph_1$ (cf. [8]).
- We denote by $\mathcal{V}_p$ the finite partition model constructed in [13].
- We denote by $\mathcal{N}_0^{\text{HF}}$ the countable partition model mentioned in [13] section 5.
- We denote by $\mathcal{N}_0$ Lévy’s permutation model (cf. [8]).

3. Form 269, Form 233, and Form 304

We recall that $\text{AC}^{\text{LO}}$ implies $\text{LW}$ in ZFA and that the implication is not reversible in ZFA (cf. [8]).

**Theorem 3.1.** $\text{AC}^{\text{LO}}$ does not imply Form 269 in ZFA. So, neither LW nor $\text{AC}^{\text{WO}}$ implies Form 269 in ZFA.

**Proof.** We present two known models.

**First model:** Fix a successor aleph $\aleph_{n+1}$. In the proof of [13] Theorem 8.9, Jech proved that $\text{AC}^{\text{WO}}$ holds in the permutation model $\mathcal{N}_2^{\text{HF}}(\aleph_{n+1})$.

We recall a variant of $\mathcal{N}_2^{\text{HF}}(\aleph_{n+1})$ from [13] Theorem 3.5(i)]. Let $\mathcal{N}$ be the permutation model, in which $A$ is a set of atoms of size $\aleph_1$, $\mathcal{G}$ be the group of all permutations of $A$ which move at most $\aleph_n$ atoms, and the supports are subsets of $A$ with cardinality less than $\aleph_{n+1}$. Tachtsis [13] Theorem 3.5(i)] proved that $\mathcal{N} = \mathcal{N}_2^{\text{HF}}(\aleph_{n+1})$ and $\text{AC}^{\text{LO}}$ holds in $\mathcal{N}$. We slightly modify the arguments of [8] Note 91 to prove that Form 269 fails in $\mathcal{N}$. We show that for any set $X$ in $\mathcal{N}$ if the set $|X|^2$ of two-element subsets of $X$ has a choice function, then $X$ is well-orderable in $\mathcal{N}$. Assume that $X$ is a set such a set and let $E$ be a support of $X$ and of a choice function $f$ on $|X|^2$. In order to show that $X$ is well-orderable in $\mathcal{N}$, it is enough to prove that $\text{fix}_\mathcal{G}(E) \subseteq \text{fix}_\mathcal{G}(X)$ (cf. Lemma 2.4(1)). Assume $\text{fix}_\mathcal{G}(E) \nsubseteq \text{fix}_\mathcal{G}(X)$, then there is a $y \in X$ and a $\phi \in \text{fix}_\mathcal{G}(E)$ with $\phi(y) \neq y$. Under such assumptions, Tachtsis constructed a permutation $\psi \in \text{fix}_\mathcal{G}(E)$ such that $\psi(y) \neq y$ but $\psi^2(y) = y$ (cf. the proof of LW in $\mathcal{N}$ from [13] Theorem 3.5(i)). This contradicts our choice of $E$ as a support for a choice function on $|X|^2$ since $\psi$ fixes $\{\psi(y), y\}$ but moves both of its elements. So Form 269 fails in $\mathcal{N}$.

**Second model:** We consider the permutation model $\mathcal{N}$ given in the proof of [19] Theorem 4.7 where $\text{AC}^{\text{LO}}$ and therefore LW hold. Following the above arguments and the arguments in [10] claim 4.10, we can see that Form 269 fails in $\mathcal{N}$. 

We recall a result of Pincus, which we need in order to prove Theorem 3.3.

**Lemma 3.2.** (Pincus; [8] Note 41]) If $\mathcal{K}$ is an algebraically closed field, if $\pi$ is a non-trivial automorphism of $\mathcal{K}$ satisfying $\pi^2 = \text{id}_\mathcal{K}$, and if $i \in \mathcal{K}$ is a square root of $-1$, then $\pi(i) = -i \neq i$.

**Theorem 3.3.** Fix any $2 \leq n \in \omega$. There is a model $\mathcal{M}$ of ZFA where $\text{AC}_n^\omega$ and the statement ‘there are no amorphous sets’ fail. Moreover, the following hold in $\mathcal{M}$:

1. Form 269 fails.
2. Form 233 holds.
3. Form 304 holds.

**Proof.** We consider the permutation model constructed by Halbeisen–Tachtsis [13] Theorem 8] where for arbitrary integer $n \geq 2$, $\text{AC}_n^\omega$ fails. We fix an arbitrary integer $n \geq 2$ and recall the model constructed in the proof of [13] Theorem 8]. We start with a model $M$ of ZFA + AC where $A$ is a countably infinite set of atoms written as a disjoint union $\bigcup\{A_i : i \in \omega\}$ where for each $i \in \omega$, $A_i = \{a_{i1}, a_{i2}, ..., a_{in}\}$ and $|A_i| = n$. The group $\mathcal{G}$ is defined in [13] in a way so that if $\eta \in \mathcal{G}$, then $\eta$ only moves finitely many atoms and for all $i \in \omega$, $\eta(A_i) = A_k$ for
some $k \in \omega$. Let $\mathcal{F}$ be the filter of subgroups of $\mathcal{G}$ generated by $\{\text{fix}_\mathcal{G}(E) : E \in [A]^{<\omega}\}$. We denote by $\mathcal{N}_{HT}^1(n)$ the Fraenkel–Mostowski permutation model determined by $M$, $\mathcal{G}$, and $\mathcal{F}$. Following point 1 in the proof of [12, Theorem 8], both $A$ and $\mathcal{A} = \{A_i\}_{i \in \omega}$ are amorphous in $\mathcal{N}_{HT}^1(n)$. If $X$ is a set in $\mathcal{N}_{HT}^1(n)$, then without loss of generality we may assume that $E = \bigcup_{i=0}^{m} A_i$ is a support of $X$ for some $m \in \omega$.

**claim 3.4.** Suppose $X$ is not a well-ordered set in $\mathcal{N}_{HT}^1(n)$, and let $E = \bigcup_{i=0}^{m} A_i$ be a support of $X$ for some $m \in \omega$. Then there is a $t \in X$ with support $F \supseteq E$ and a permutation $\delta \in \text{fix}_\mathcal{G}(E)$ such that $\delta^2$ is the identity, and $\delta(t) \neq t$.

**Proof.** Since $X$ is not well-ordered, and $E$ is a support of $X$, $\text{fix}_\mathcal{G}(E) \not\subseteq \text{fix}_\mathcal{G}(X)$ by Lemma 2.4(1). So there is a $t \in X$ and a $\psi \in \text{fix}_\mathcal{G}(E)$ such that $\psi(t) \neq t$. Let $F$ be a support of $t$ containing $E$. Without loss of generality, we may assume that $F$ is a union of finitely many $A_i$'s. We slightly modify the arguments of [19, claim 4.10]. Let $W = \{a \in A : \psi(a) \neq a\}$. We note that $W$ is finite since if $\eta \in \mathcal{G}$, then $\eta$ only moves finitely many atoms. Let $U$ be a finite subset of $A$ which is disjoint from $F \cup W$ and such that there exists a bijection $H : tr(U) \to tr((F \cup W) \setminus E)$ (where for a set $x \subseteq A$, $tr(x) = \{i \in \omega : A_i \cap x \neq \emptyset\}$) with the property that if $i \in tr((F \cup W) \setminus E)$ is such that $A_i \subseteq (F \cup W) \setminus E$ then $A_{H^{-1}(i)} \subseteq U$; otherwise if $A_i \not\subseteq (F \cup W) \setminus E$, which means that $A_i \cap F = \emptyset$ and $A_i \not\subseteq W$, then $|W \cap A_i| = |U \cap A_i|$. Let $f : U \to (F \cup W) \setminus E$ be a bijection such that $\forall i \in tr(U), f^{-1} \cap U \cap A_i$ is a one-to-one function from $U \cap A_i$ onto $((F \cup W) \setminus E) \cap A_i$. Let $f' : \bigcup_{i \in tr(U)} A_i \setminus (U \cap A_i) \to \bigcup_{i \in tr(U)} A_{H(i)} \setminus (((F \cup W) \setminus E) \cap A_{H(i)})$ be a bijection such that $\forall i \in tr(U), f' \cap (A_i \setminus (U \cap A_i))$ is a one-to-one function from $A_i \setminus (U \cap A_i)$ onto $A_{H(i)} \setminus (((F \cup W) \setminus E) \cap A_{H(i)})$. Let

$$\delta = \prod_{u \in U} (u, f(u)) \prod_{i \in tr(U)} A_i \setminus (U \cap A_i) \cup \left\{(u, f'(u)) \right\}$$

be a product of disjoint transpositions. It is clear that $\delta$ only moves finitely many atoms, and for all $i \in \omega$, $\delta(A_i) = A_k$ for some $k \in \omega$. Moreover, $\delta \in \text{fix}_\mathcal{G}(E)$, $\delta^2$ is the identity, and $\delta(t) \neq t$ by the arguments in [19, claim 4.10].

**claim 3.5.** In $\mathcal{N}_{HT}^1(n)$, the following hold:

1. Form 269 fails.
2. Form 304 holds.
3. Form 233 holds.

**Proof.** (1). Following claim 3.4 and the arguments in the proof of Theorem 3.1, Form 269 fails in $\mathcal{N}_{HT}^1(n)$.

(2). We modify the arguments of [S, Note 116] to prove that Form 304 holds in $\mathcal{N}_{HT}^1(n)$. Let $X$ be an infinite Hausdorff space in $\mathcal{N}_{HT}^1(n)$, and $E = \bigcup_{i \in K} A_i$ be a support of $X$ and its topology where $K \in [\omega]^{<\omega}$. We show there is an infinite $Y \subseteq X$ in $\mathcal{N}_{HT}^1(n)$ such that $Y$ has no infinite compact subsets in $\mathcal{N}_{HT}^1(n)$. If $X$ is well-orderable, then we can use transfinite induction without using any form of choice to finish the proof. Suppose $X$ is not well-orderable in $\mathcal{N}_{HT}^1(n)$. By Lemma 2.4(1), there is an $x \in X$ and a $\phi \in \text{fix}_\mathcal{G}(E)$ such that $\phi(x) \neq x$. Let $F = \bigcup_{i \in K} A_i$ be a support of $x$ where $K' \in [\omega]^{<\omega}$. Since $E$ is not a support of $x$, $F \cap E = \emptyset$. Without loss of generality assume that $E \subseteq F$. We also assume that $\{A_i : i \in K'\}$ has the fewest possible copies $A_i$ outside $\{A_i : i \in K\}$. Let $i_0 \in K'$ such that $A_{i_0} \cap E = \emptyset$. We define

$$f = \{((x, \psi(A_{i_0})) : \psi \in \text{fix}_\mathcal{G}(F \setminus A_{i_0})\}.$$

Tacthys proved that $f$ is a function with $\text{dom}(f) \subseteq X$ and $\text{ran}(f) = \mathcal{A} \setminus \{A_i : i \in K, i \neq i_0\}$, where $\mathcal{A} = \{A_i : i \in \omega\}$ and $Y = \text{dom}(f)$ is an amorphous function of $X$ (cf. proof of [23, Lemma 2]). Since $\phi(x) \neq x$ and $X$ is an infinite Hausdorff space, we can choose open sets $C$ and $D$ so that $x \in C$, $\phi(x) \in D$ and $C \cap D = \emptyset$. Since $Y$ is amorphous in $\mathcal{N}_{HT}^1(n)$, every subset of $Y$ in the model must be finite or cofinite. Thus if at least one of $Y \cap C$ or $Y \cap D$ is finite, we may assume that $Y \cap C$ is finite. Then we can conclude that

$$C = \{\psi(C) \cap Y : \psi \in \text{fix}_\mathcal{G}(F \setminus A_{i_0})\}$$

is an open cover for $Y$ and each element of $C$ is finite. So there is an infinite $Y \subseteq X$ in $\mathcal{N}_{HT}^1(n)$ such that for any infinite subset $Z$ of $Y$, $C$ is an open cover for $Z$ without a finite subcover.

(3). We follow the arguments due to Pincus from [S, Note 41] and use claim 3.4 to prove that Form 233 holds in $\mathcal{N}_{HT}^1(n)$. For the reader's convenience, we write down the proof. Let $(K, +, \cdot, 0, 1)$ be a field in $\mathcal{N}_{HT}^1(n)$ with finite support $E \subseteq A$ and assume that $K$ is algebraically closed. Without loss of generality assume that $E = \bigcup_{i=0}^{m} A_i$ for some natural number $m \in \omega$. We show that every element of $K$ has support $E$ which implies that $K$ is well-orderable in $\mathcal{N}_{HT}^1(n)$ and therefore the standard proof of the uniqueness of algebraic closures (using $\mathcal{A}C$) is valid in $\mathcal{N}_{HT}^1(n)$. For the sake of contradiction, assume that $x \in K$ does not have support $E$. By claim 3.4, there is a
permutation $\psi \in \text{fix}_G(E)$ such that $\psi(x) \neq x$ and $\psi^2$ is the identity. The permutation $\psi$ induces an automorphism of $(K, +, \cdot, 0, 1)$ and we can therefore apply Lemma 3.2 to conclude that $\psi(i) = -i \neq i$ for some square root $i$ of $-1$ in $K$.

On the other hand, we can follow the arguments from [8] Note 41 to see that for every permutation $\pi$ of $A$ such that $\pi \in \text{fix}_G(E)$, $\pi(i) = i$ for every square root of $i$ of $-1$ in $K$. In particular, fix an $i = \sqrt{-1} \in K$. It is enough to show that $E$ is a support of $i$. We note that $i$ is a solution to the equation $x^2 + 1 = 0$ all of whose coefficients are fixed by any $\eta \in \text{fix}_G(E)$. So if $\eta \in \text{fix}_G(E)$, then $\eta(i)$ is also a solution to $x^2 + 1 = 0$. Suppose $E$ is not a support of $i$. Let $E' = \bigcup_{i=0}^{n+k} A_i$ be a support of $i$ for some natural number $k$ and let $F = E' \setminus E$. Then $F \neq \emptyset$ (since $E$ is not a support of $i$) and $F \cap E = \emptyset$. By applying [23] Lemma 1 (where Tachtsis proved that if $x \in N_1^{HT}(n)$ and $E_1, E_2$ are supports of $x$, then $E_1 \cap E_2$ is a support of $x$), we can see that if $\phi, \phi' \in \text{fix}_G(E)$ and $\phi(F) \cap \phi'(F) = \emptyset$, then $\phi(i) \neq \phi'(i)$. Consequently, we can obtain an infinite set $S = \{\phi_k(i) : k \in \omega\}$ such that $\phi_k \in \text{fix}_G(E)$ and, $\phi_k(i)$ is in $N_1^{HT}(n)$ for every $k \in \omega$, and for all $k, l \in \omega$, if $k \neq l$ then $\phi_k(i) \neq \phi_l(i)$. Thus, the equation $x^2 + 1 = 0$ has infinitely many solutions in $K$, which is a contradiction. Thus, $E$ is a support of $i$. This completes the proof of the third assertion.

\[ \square \]

Remark 3.6. Fix $2 \leq n \in \omega$. For each regular $\aleph_n$ we denote by $\text{CAC}^{\aleph_n}$ the statement “If in a poset all antichains are finite and all chains have size at most $\aleph_n$ and there exists at least one chain with size $\aleph_n$, then the poset has size $\aleph_n$.” In [1], Theorem 4.3, Remark 4.4 we proved that the statement “For every regular $\aleph_n$, $\text{CAC}^{\aleph_n}$ holds in $N_1^{HT}(n)$ and $N_1$.” We present different proofs to show that the statement “For every regular $\aleph_n$, $\text{CAC}^{\aleph_n}$ holds in $N_1^{HT}(n)$ and $N_1$.” First, we recall the following result communicated to us by Tachtsis from [1].

**Lemma 3.7.** (cf. [1], Lemma 4.1, Corollary 4.2) The following hold:

1. The statement “If $(P, \leq)$ is a poset such that $P$ is well-ordered, and if all antichains in $P$ are finite and all chains in $P$ are countable, then $P$ is countable” holds in any Fraenkel-Mostowski model.

2. $\text{UT}(\aleph_n, \aleph_n, \aleph_n)$ implies the statement “If $(P, \leq)$ is a poset such that $P$ is well-ordered, and if all antichains in $P$ are finite and all chains in $P$ have size at most $\aleph_n$ and there exists at least one chain with size $\aleph_n$, then $P$ has size $\aleph_n$” for any regular $\aleph_n$ in ZF.

Fix $N \in \{N_1^{HT}(n), N_1\}$. Let $(P, \leq)$ be a poset in $N$ such that all antichains in $P$ are finite and all chains in $P$ have size $\aleph_n$. Let $E \in \mathcal{A}[\langle \omega \rangle]$ be a support of $(P, \leq)$.

**Case (i):** Let $N = N_1^{HT}(n)$. Then for each element $p \in P$, either $\text{Orb}_{\text{fix}_G(E)}(p) = \{\phi(p) : \phi \in \text{fix}_G(E)\} = \{p\}$ or $\text{Orb}_{\text{fix}_G(E)}(p)$ is infinite (cf. [23] Remark 2.2)). Following the arguments of [23] claim 3) we can see that for each $p \in P$, $\text{Orb}_{\text{fix}_G(E)}(p)$ is an anti-chain in $P$. So by assumption, $\text{Orb}_{\text{fix}_G(E)}(p) = \{p\}$. Following the arguments of [23], claim 4) we can see that $0 = \{\text{Orb}_{\text{fix}_G(E)}(p) : p \in P\}$ is a well-ordered partition of $P$. Thus $P$ is also well-orderable. The rest follows from Lemma 3.7(2), since $\text{UT}(\text{WO}, \text{WO}, \text{WO})$ holds in $N$ and $\text{UT}(\text{WO}, \text{WO}, \text{WO})$ implies $\text{UT}(\aleph_n, \aleph_n, \aleph_n)$ in any FM-model (cf. [8] p. 176).

**Case (ii):** Let $N = N_1$. If $P$ is well-orderable, then we are done. Suppose $P$ is not well-orderable. By Lemma 2.4(1), there is a $t \in P$ and a $\pi \in \text{fix}_G(E)$ such that $\pi(t) \neq t$. Let $F \cup \{a\}$ be a support of $t$ where $a \in A\langle E \cup F \rangle$. Under such assumptions, Blass [8] p.389 proved that

$$f = \{(\pi(a), \pi(t)) : \pi \in \text{fix}_G(E \cup F)\}$$

is a bijection from $A\langle E \cup F \rangle$ onto $\text{ran}(f) \subset P$. Now, $\text{ran}(f) = \text{Orb}_{\text{fix}_G(E \cup F)}(t) = \{\pi(t) : \pi \in \text{fix}_G(E \cup F)\}$ is an infinite antichain of $P$ (cf. the proof of [13], Lemma 9.3) in $N$, which contradicts our assumption.

Fix $k, n \in \omega \setminus \{0, 1\}$. Tachtsis [20], Theorem 3.7, Remark 3.8 proved that the statement “If $P$ is a poset with width $k$ while at least one $k$-element subset of $P$ is an antichain, then $P$ can be partitioned into $k$ chains”, abbreviated as $\text{DT}$, holds in $N_1$ and $N_1^{HT}(2)$. Using the above arguments we can give a different proof of $\text{DT}$ in $N_1^{HT}(n)$ and $N_1$ since $\text{DT}$ for well-ordered infinite posets with finite width is provable in ZF.

4. Partition models, weak choice forms, and permutations of infinite sets

We recall some known results, which we need in order to prove Theorem 4.5 and Theorem 4.8.

**Lemma 4.1.** (Kerremeris–Herrlich–Tachtsis; cf. [22], Remark 2.7, [14], Theorem 3.1) The following hold:

1. We note that Tachtsis assumed that a support $E$ has the property that $\forall i \in \omega, A_i \subseteq E$ or $A_i \cap E = \emptyset$ in order to prove [22] Lemma 1.

2. We assume that $\alpha$ is an ordinal and that $\aleph_\alpha$ is the $\alpha$th infinite initial ordinal (where an ordinal $\beta$ is “initial” if $\beta$ is not equipotent with an ordinal $\gamma < \beta$).
(1) $\text{AC}_{\omega_1}^n + \text{MA}(\aleph_0) \rightarrow \text{‘for every infinite set } X, \ 2^X \text{ is Baire’}.$

(2) ‘For every infinite set } X, \ 2^X \text{ is Baire’} \rightarrow \text{‘For every infinite set } X, \ P(X) \text{ is Dedekind-infinite’}.$

**Lemma 4.2.** (Lévy; [17]) MC if and only if every infinite set has a well-ordered partition into non-empty finite sets.

**Lemma 4.3.** (Howard–Saveliev–Tachtsis; [10] Lemma 1.3, Theorem 3.1] The following hold:

(1) $\leq \aleph_0\text{-MC}$ implies “for every infinite set has a well-ordered partition into non-empty countable sets.

(2) $\leq \aleph_0\text{-MC}$ implies “for every infinite set $X$, $P(X)$ is Dedekind-infinite’’, which in turn is equivalent to “for every infinite set $P$ there is a partial ordering $\leq$ on $P$ such that $(P, \leq)$ has a countably infinite disjoint family of cofinal subsets”.

**Lemma 4.4.** (Tachtsis; [18] Theorem 3.1] The following hold:

(1) Each of the following statements implies the one beneath it:

(a) Form 3;

(b) ISAE;

(c) EPWFP;

(d) For every infinite set $X$, $\text{Sym}(X) \neq \text{FSym}(X)$.

(2) DF $= F$ implies “For every infinite set $X$, $\text{Sym}(X) \neq \text{FSym}(X)$”.

4.1. Weak choice forms in the finite partition model. We recall the finite partition model $V_p$ from [4]. In order to describe $V_p$, we start with a model $M$ of $\text{ZFA} + \text{AC}$ where $A$ is a countably infinite set of atoms. Let $\mathcal{G}$ be the group of all permutations of $A$, $S$ be the set of all finite partitions of $A$, and $\mathcal{F} = \{ \mathcal{H} : \mathcal{H} \text{ is a subgroup of } \mathcal{G} \}$.

**Theorem 4.5.** The following hold in $V_p$:

(1) If $X \in \{\text{Form 3}, \text{ISAE}, \text{EPWFP}\}$, then $X$ fails.

(2) $\text{AC}_f$ fails for any integer $n \geq 2$.

(3) $\text{MA}(\aleph_0)$ fails.

(4) If $X \in \{\text{MC}, \leq \aleph_0\text{-MC}\}$, then $X$ fails.

**Proof.** (1). By Lemma 4.4, it is enough to show that $(\text{Sym}(A))^V_p = \text{FSym}(A)$. For the sake of contradiction, assume that $f$ is a permutation of $A$ in $V_p$, which moves infinitely many atoms. Let $P = \{P_j : j \leq k\}$ be a support of $f$ for some $k \in \omega$. Without loss of generality, assume that $P_0, ..., P_n$ are the singleton and tuple blocks for some $n \leq k$. Then there exist $n < i \leq k$ where $a \in P_i$ and $b \in \bigcup P \setminus \{P_0 \cup ... \cup P_n \cup \{a\}\}$ such that $b = f(a)$.

**Case (i):** Let $b \in P_i$. Consider $\phi \in \text{fix}_G(P)$ such that $\phi$ fixes all the atoms in all the blocks other than $P_i$ and $f$ moves every atom in $P_i$ except $b$. Thus, $\phi(b) = b$, $\phi(a) \neq a$, and $\phi(f) = f$ since $P$ is the support of $f$. Thus $(a, b) \in f \implies (\phi(a), \phi(b)) \in \phi(f) \implies (\phi(a), \phi(b)) \in f$. So $f$ is not injective; contradiction.

**Case (ii):** Let $b \notin P_i$. Consider $\phi \in \text{fix}_G(P)$ such that $\phi$ fixes all the atoms in all the blocks other than $P_i$ and $\phi$ moves every atom in $P_i$. Then again we can obtain a contradiction as in Case (i).

(2). Fix any integer $n \geq 2$. We show that the set $[A]^n$ has no choice function in $V_p$. Assume that $f$ is a choice function of $[A]^n$ and let $P$ be a support of $f$. Since $A$ is countably infinite and $P$ is a finite partition of $A$, there is a $p \in P$ such that $|p|$ is infinite. Let $a_1, a_2, ..., a_n \in P$ and $\pi \in \text{fix}_G(P)$ be such that $\pi\{a_1, a_2, ..., a_n\} = a_1$. Without loss of generality, we assume that $\pi\{a_1, a_2, ..., a_n\} = a_1$. Thus, $f(\{a_1, a_2, ..., a_n\}) = a_1$. This contradicts the fact that $f$ is not a function; contradiction.

(3). It is known that $\mathcal{P}(A)$ is Dedekind-finite and $\text{UT}(\text{WO}, \text{WO})$ holds in $V_p$ (cf. [4] Proposition 4.9, Theorem 4.18). So $\text{AC}_{\omega_1}^n$ holds as well. Thus by Lemma 4.1(2), the statement “for every infinite set $X$, $2^X \text{ is Baire’’}” is false in $V_p$. Hence by Lemma 4.1(1), $\text{MA}(\aleph_0)$ is false in $V_p$.

(4). Follows from Lemmas 4.2, 4.3(1) and the fact that $\text{UT}(\text{WO}, \text{WO})$ holds in $V_p$. Alternatively, we can also use Lemma 4.3(2), to see that $\leq \aleph_0\text{-MC}$ fails in $V_p$ since $\mathcal{P}(A)$ is Dedekind-finite in $V_p$. So we may also conclude by Lemma 4.3(2) that the statement “for every infinite set $P$ there is a partial ordering $\leq$ on $P$ such that $(P, \leq)$ has a countably infinite disjoint family of cofinal subsets” fails in $V_p$.

4.2. Weak choice forms in the countable partition model. Let $M$ be a model of $\text{ZFA} + \text{AC}$ where $A$ is an uncountable set of atoms and $\mathcal{G}$ is the group of all permutations of $A$.\[\square\]
Lemma 4.6. Let $S$ be the set of all countable partitions of $A$. Then $\mathcal{F} = \{H : H$ is a subgroup of $\mathcal{G}$, $H \supseteq \text{fix}_P(P)$ for some $P \in S\}$ is a normal filter of subgroups of $\mathcal{G}$.

Proof. We modify the arguments of [4] Lemma 4.1 and verify the clauses 1-5 of a normal filter (cf. section 2.1).

(1) We can see that $G \subseteq F$.

(2) Let $H \in \mathcal{F}$ and $K$ be a subgroup of $\mathcal{G}$ such that $H \subseteq K$. Then there exists $P \in S$ such that $\text{fix}_G(P) \subseteq H$. So, $\text{fix}_G(P) \subseteq K$ and $K \in \mathcal{F}$.

(3) Let $K_1, K_2 \in \mathcal{F}$. Then there exist $P_1, P_2 \in S$ such that $\text{fix}_G(P_1) \subseteq K_1$ and $\text{fix}_G(P_2) \subseteq K_2$. Let $P_1 \land P_2$ denote the coarsest common refinement of $P_1$ and $P_2$, given by $P_1 \land P_2 = \{p \cap q : p \in P_1, q \in P_2, p \cap q \neq \emptyset\}$. Clearly, $\text{fix}_G(P_1 \land P_2) \subseteq \text{fix}_G(P_1) \cap \text{fix}_G(P_2) \subseteq K_1 \cap K_2$. Since the product of two countable sets is countable, $P_1 \land P_2 \in S$. Thus, $K_1 \cap K_2 \in \mathcal{F}$.

(4) Let $\pi \in \mathcal{G}$ and $H \in \mathcal{F}$. Then there exists $P \in S$ such that $\text{fix}_G(P) \subseteq H$. Since $\text{fix}_G(\pi P) = \pi \text{fix}_G(P) \pi^{-1} \subseteq \pi H \pi^{-1}$ by Lemma 2.4(2), it is enough to show $\pi P \in S$. Clearly, $\pi P$ is countable, since $P$ is countable.

Following the arguments of [4] Lemma 4.1(iv] we can see that $\pi P$ is a partition of $A$.

(5) Fix any $a \in A$. Consider any countable partition $P$ of $A$ where $\{a\}$ is a singleton block of $P$. We can see that $\text{fix}_G(P) \subseteq \{\pi \in \mathcal{G} : \pi(a) = a\}$. Thus, $\pi \in \mathcal{G}$: $\pi(a) = a \in \mathcal{F}$.

We call the permutation model (denoted by $\mathcal{V}_p^+$) determined by $M$, $\mathcal{G}$, and $\mathcal{F}$, the countable partition model. Tachtis [13 Theorem 3.1(2)] proved that $\mathcal{D}F = F$ implies "For every infinite set $X$, $\text{Sym}(X) \neq F\text{Sym}(X)$" in $ZF$.

Inspired by that idea we first prove the following.

Proposition 4.7. (ZF) The following hold:

1. $\mathcal{W}_{\aleph_1^+}$ implies "for any set $X$ of size $\leq \aleph_\alpha$, $\text{Sym}(X) \neq \aleph_\alpha \text{Sym}(X)$".

2. $\text{EPWFP}$ implies "for any set $X$ of size $\leq \aleph_\alpha$, $\text{Sym}(X) \neq \aleph_\alpha \text{Sym}(X)$".

Proof. (1). Let $X$ be a set of size $\leq \aleph_\alpha$ and let us assume $\text{Sym}(X) = \aleph_\alpha \text{Sym}(X)$. We prove that there is no injection from $\aleph_{\alpha+1}$ into $X$. Assume there exists such an $f$. Let $\{y_n\}_{n \in \aleph_{\alpha+1}}$ be an enumeration of the elements of $Y = f[\aleph_{\alpha+1}]$. We can use transfinite recursion, without using any form of choice, to construct a bijection $h : Y \to Y$ such that $h(x) \neq x$ for any $x \in Y$. Define $g : X \to X$ as follows: $g(x) = h(x)$ if $x \in Y$, and $g(x) = x$ if $x \notin Y$. Clearly $g \in \text{Sym}(X) \setminus \aleph_\alpha \text{Sym}(X)$, and hence $\text{Sym}(X) \neq \aleph_\alpha \text{Sym}(X)$, a contradiction.

(2). This is straightforward.

(3). The following hold:

1. $\mathcal{N}_{12}(\aleph_1) \subseteq \mathcal{V}_p^+$. (1).

2. If $X \in \{\text{Form 3, ISAE, EPWFP}\}$, then $X$ fails in $\mathcal{V}_p^+$. (2).

3. $\text{AC}_n$ fails in $\mathcal{V}_p^+$ for any integer $n \geq 2$. (3).

4. $\text{A cannot be linearly ordered}$. (4).

5. If $X \in \{\mathcal{W}_{\aleph_1}, \mathcal{DC}_{\aleph_1}\}$, then $X$ fails in $\mathcal{V}_p^+$. (5).

Proof. (1). Let $x \in \mathcal{N}_{12}(\aleph_1)$ with support $E$. So $\text{fix}_G(E) \subseteq \text{sym}_G(x)$. Then $P = \{\{a\}_{a \in E} \cup \{A \setminus E\} \mid A$ is a countable partition of $A$, and $\text{fix}_G(P) = \text{fix}_G(E)$}. Thus $\text{fix}_G(P) \subseteq \text{sym}_G(x)$ and so $x \in \mathcal{V}_p^+$ with support $P$.

(2). Similarly to the proof of $\neg\text{EPWFP}$ in $\mathcal{V}_p$ (cf. the proof of Theorem 4.5(1)), one may verify that if $f$ is a permutation of $A$ in $\mathcal{V}_p^+$, then the set $\{x \in A : f(x) \neq x\}$ has cardinality at most $\aleph_0$. Since $A$ is uncountable, it follows that 'for any uncountable $X$, $\text{Sym}(X) \neq \aleph_\alpha \text{Sym}(X)$' fails in $\mathcal{V}_p^+$. Consequently, if $X \in \{\text{Form 3, ISAE, EPWFP}\}$, then $X$ fails in $\mathcal{V}_p^+$ by Proposition 4.7(2).

(3). Fix any integer $n \geq 2$. Similarly to the proof of Theorem 4.5(2), one may verify that the set $[A]^n$ has no choice function in $\mathcal{V}_p^+$. Consequently, $\text{AC}_n$ fails in $\mathcal{V}_p^+$.

(4). Follows from (3).

(5). We can use the arguments in (2) and Proposition 4.7(1) to show that $\mathcal{W}_{\aleph_1}$ fails in $\mathcal{V}_p^+$. The rest follows from the fact that $\mathcal{DC}_{\aleph_1}$ implies $\mathcal{W}_{\aleph_1}$ in $ZF$ (cf. [13 Theorem 8.1(b)]). However, we write a different argument. In order to show that $\mathcal{W}_{\aleph_1}$ fails in $\mathcal{V}_p^+$, we prove that there is no injection $f$ from $\aleph_1$ into $A$. Assume there exists such an $f$ with support $P$, and let $\pi \in \text{fix}_G(P)$ be such that $\pi$ moves every atom in each non-singleton block of $P$. Since $P$ contains only countably many singletons, $\pi$ fixes only countably many atoms. Fix $n \in \aleph_1$. Since $n$ is in the kernel (the class of all pure sets), we have $\pi(n) = n$. Thus $\pi(f(n)) = f(\pi(n)) = f(n)$. But $f$ is one-to-one, and thus, $\pi$ fixes $\aleph_1$ many values of $f$ in $A$, a contradiction. □
Remark 4.9. Let $M$ be a model of $\text{ZFA} + \text{AC}$ where $A$ is a set of atoms of cardinality $\aleph_{\alpha+1}$. Let $G$ be the group of all permutations of $A$, $S$ be the set of all $\aleph_{\alpha}$-partitions of $A$, and $F = \{H : H$ is a subgroup of $G, H \supseteq \text{fix}(P)$ for some $P \in S\}$ be a normal filter of subgroups of $G$. Let $\mathcal{Y}_p^{\aleph_{\alpha+1}}$ be the permutation model determined by $M$, $G$, and $F$. Following the arguments of Theorem 4.8(1)(2), we can see $N_{12}(\aleph_{\alpha+1}) \subset \mathcal{Y}_p^{\aleph_{\alpha+1}}$ and EPWFP fails in $\mathcal{Y}_p^{\aleph_{\alpha+1}}$.

5. Van Douwen’s Choice Principle in two permutation models

Proposition 5.1. The following hold:

(1) The statement vDCP $\land$ UT($\aleph_0, \aleph_0$, cuf) $\land$ $\neg$M(IC, DI) has a permutation model.

(2) The statement vDCP $\land$ $\neg$MC($\aleph_0, \aleph_0$) has a permutation model.

Proof. (1) We recall the permutation model $\mathcal{N}$ which was constructed in [5] proof of Theorem 3.3 where UT($\aleph_0, \aleph_0$, cuf) holds. In order to describe $\mathcal{N}$, we start with a model $M$ of ZFA + AC with a set $A$ of atoms such that $A$ has a denumerable partition $\{A_i : i \in \omega\}$ into denumerable sets, and for each $i \in \omega$, $A_i$ has a denumerable partition $P_i = \{A_{i,j} : j \in \mathbb{N}\}$ into finite sets such that, for every $j \in \mathbb{N}$, $|A_{i,j}| = j$. Let $G = \{\phi \in \text{Sym}(A) : (\forall i \in \omega)(\phi(A_i) = A_i)\}$ and $\{|x \in A : \phi(x) \neq x\} < \aleph_0$, where Sym(A) is the group of all permutations of $A$. Let $P_i = \{\phi(P_i) : \phi \in G\}$ for each $i \in \omega$ and let $P = \bigcup\{P_i : i \in \omega\}$. Let $F$ be the normal filter of subgroups of $G$ generated by the filter base $\{\text{fix}(E) : E \in [P]^{<\omega}\}$. Then $\mathcal{N}$ is the permutation model determined by $M$, $G$ and $F$. Keremedis, Tachtsis, and Wajch proved that M(IC, DI) fails in $\mathcal{N}$ (cf. [15] proof of Theorem 13(i)). We follow steps (1), (2), and (4) from the proof of [11] Lemma 5.1 to see vDCP holds in $\mathcal{N}$. For the sake of convenience, we write down the proof.

Lemma 5.2. If $(X, \leq)$ is a poset in $\mathcal{N}$, then $X$ can be written as a well-ordered disjoint union $\bigcup\{W_\alpha : \alpha < \kappa\}$ of antichains.

Proof. Let $(X, \leq)$ be a poset in $\mathcal{N}$ and $E \subseteq [P]^{<\omega}$ be a support of $(X, \leq)$. Following the arguments of [20] claim 3.5 we can see that for each $p \in X$, the set $\text{Orb}_{\text{fix}(E)}(p)$ is an antichain in $(X, \leq)$ since every element $\eta \in G$ moves only finitely many atoms. Following the arguments of [20] claim 3.6 we can see that

$$O = \{\text{Orb}_{\text{fix}(E)}(p) : p \in X\}$$

is a well-ordered partition of $X$. 

We recall the arguments from the 1st-paragraph of [10] p.175 to give a proof of vDCP in $\mathcal{N}$. Let $A = \{(A_i, \leq_i) : i \in I\}$ be a family as in vDCP. Without loss of generality, we assume that $A$ is pairwise disjoint. Let $R = \bigcup A$. We partially order $R$ by requiring $x < y$ if and only if there exists an index $i \in I$ such that $x, y \in A_i$ and $x <_i y$. By Lemma 5.2, $R$ can be written as a well-ordered disjoint union $\bigcup\{W_\alpha : \alpha < \kappa\}$ of antichains. For each $i \in I$, let $\alpha_i = \min(\alpha \in \kappa : A_i \cap W_\alpha \neq \emptyset)$. Since for all $i \in I$, $A_i$ is linearly ordered, it follows that $A_i \cap W_{\alpha_i}$ is a singleton for each $i \in I$. Consequently, $F = \{(i, \cup(A_i \cap W_{\alpha_i})) : i \in I\}$ is a choice function of $A$. Thus, vDCP holds in $\mathcal{N}$. (2).

We recall the permutation model (say $M$) which was constructed in [11] proof of Theorem 3.4. In order to describe $M$, we start with a model $M$ of ZFA + AC with a denumerable set $A$ of atoms which is written as a disjoint union $\bigcup\{A_n : n \in \omega\}$, where $|A_n| = \aleph_0$ for all $n \in \omega$. For each $n \in \omega$, we let $G_n$ be the group of all permutations of $A_n$ which move only finitely many elements of $A_n$. Let $G$ be the weak direct product of the $G_n$’s for $n \in \omega$. Consequently, every permutation of $A$ in $G$ moves only finitely many atoms. Let $I$ be the normal ideal of subsets of $A$ which is generated by finite unions of $A_n$’s. Let $F$ be the normal filter on $G$ generated by the subgroups $\text{fix}(E)$, $E \in I$. Let $M$ be the Fraenkel–Mostowski model, which is determined by $M$, $G$, and $F$. Howard and Tachtsis [11] proof of Theorem 3.4 proved that $\text{MC}(\aleph_0, \aleph_0)$ fails in $M$. Since every permutation of $A$ in $G$ moves only finitely many atoms, following the arguments in the proof of (1), vDCP holds in $M$. 

Remark 5.3. In every Fraenkel-Mostowski permutation model, CS (Every poset without a maximal element has two disjoint cofinal subsets) implies vDCP (cf. [10] Theorem 3.15(3)). We can see that in the above-mentioned permutation models (i.e., $\mathcal{N}$ and $M$) CS and CWF (Every poset has a cofinal well-founded subset) hold applying Lemma 5.2 and following the methods of [10] Theorem 3.26 and [21] proof of Theorem 10 (ii).

6. Spanning subgraphs and weak choice forms

We recall some known results.

Lemma 6.1. ([ZF; Delhomme–Morillon; [63] Lemma 1]) Given a set $X$ and a set $A$ which is the range of no mapping with domain $X$, consider a mapping $f : A \rightarrow P(X)\setminus\{\emptyset\}$. Then
(1) There are distinct \(a\) and \(b\) in \(A\) such that \(f(a) \cap f(b) \neq \emptyset\).

(2) If the set \(A\) is infinite and well-orderable, then for every positive integer \(p\), there is an \(F \in [A]^p\) such that \(\bigcap f[F] := \bigcap_{a \in F} f(a)\) is non-empty.

**Lemma 6.2.** (Cayley's formula; ZF) The number of spanning trees in \(K_n\) is \(n^{n-2}\) for any \(n \in \omega \setminus \{0, 1, 2\}\).

**Lemma 6.3.** (Soini's formula; ZF) The number of spanning trees in \(K_{m,n}\) is \(n^{m-1}m^{n-1}\) for any \(n, m \in \omega \setminus \{0, 1\}\).

**Lemma 6.4.** (cf. [16] Chapter 30, Problem 5) \(\AC_m\) implies \(\AC_n\) if \(m\) is a multiple of \(n\).

Applying the above lemmas we prove the following propositions.

**Proposition 6.5.** (ZF) The following hold:

1. \(\AC_{\leq n-1}^\omega + Q_{n,k}^n\) is equivalent to \(\AC_{\leq n-1}^\omega\) for any \(2 < n \in \omega\).
2. \(\UT(\WO, \WO, \WO)\) implies \(\AC_{\leq n-1}^\omega + Q_{n,k}^n\) and the later implies \(\AC_{\leq n-1}^\omega\) for any \(2 < n, k \in \omega\).
3. \(P_{n,k}^n\) is equivalent to \(\AC_{\leq n-1}^\omega\) for any even integer \(m \geq 4\).
4. \(Q_{n,k}^n\) fails in \(\aleph_0\) for any \(2 < n \in \omega\).

**Proof.** (1). (\(\Rightarrow\)) Assume \(\AC_{\leq n-1}^\omega\). Fix any \(2 < n \in \omega\). We know that \(\AC_{\leq n-1}^\omega\) implies \(\AC_{\leq n-1}^\omega\) in ZF. More over, \(\AC_{\leq n-1}^\omega\) implies \(Q_{n,k}^n\) in ZF (cf. [3] Theorem 2).

(\(\Leftarrow\)) Fix any \(2 < n \in \omega\). We show that \(\AC_{\leq n-1}^\omega + Q_{n,k}^n\) implies \(\AC_{\leq n-1}^\omega\) in ZF. Let \(A = \{A_i : i \in \omega\} =\) a countably infinite set of non-empty finite sets. Without loss of generality, we assume that \(A\) is disjoint. Let \(A = \bigsqcup_{i \in \omega} A_i\). Consider a countably infinite family \(B = \{B_i, A_i : i \in \omega\}\) of well-ordered sets such that \(|B_i| = |A_i| + k\) for some fixed \(1 \leq k \in \omega\), for each \(i \in \omega\), \(B_i\) is disjoint from \(A\) and the other \(B_j\)'s, and there is no mapping with domain \(A_i\) and range \(B_i\) (cf. the proof of [3] Theorem 1, Remark 6). Let \(B = \bigsqcup_{i \in \omega} B_i\). Consider another countably infinite sequence \(T = \{t_i : i \in \omega\}\) disjoint from \(A\) and \(B\). We construct a graph \(G_1 = (V_{G_1}, E_{G_1})\) (cf. Figure 1).

**Figure 1. The graph \(G_1\)**

Let \(V_{G_1} = A \cup B \cup T\), and

\[E_{G_1} = \{\{t, t_{i+1}\} : i \in \omega\} \cup \{\{t, x\} : i \in \omega, x \in A_i\} \cup \{\{x, y\} : i \in \omega, x \in A_i, y \in B_i\}\]

Clearly, the graph \(G_1 = (V_{G_1}, E_{G_1})\) is connected and locally finite. By assumption, \(G_1\) has a spanning subgraph \(G'_1\) omitting \(K_{2,n}\). For each \(i \in \omega\), let \(f_i : B_i \rightarrow \mathcal{P}(A_i) \setminus \{\emptyset\}\) map each element of \(B_i\) to its neighbourhood in \(G'_1\). We can see that for any two distinct \(e_1\) and \(e_2\) in \(B_i\), \(f_i(e_1) \cap f_i(e_2)\) has at most \(n-1\) elements, hence \(G'_1\) has no \(K_{2,n}\). By Lemma 6.1(1), there are tuples \((e'_1, e'_2) \in B_i \times B_i\) such that \(f_i(e'_1) \cap f_i(e'_2) \neq \emptyset\). Consider the first such tuple \((e'_1, e'_2)\) with respect to the well-ordering on \(B_i \times B_i\). Let \(A'_i = f_i(e'_1) \cap f_i(e'_2)\). By \(\AC_{\leq n-1}^\omega\), we can obtain a choice function of \(A' = \{A'_i : i \in \omega\}\), which is a choice function of \(A\).

(2). For the first implication, we know that \(\UT(\WO, \WO, \WO)\) implies \(\AC_{\leq n-1}^\omega\) as well as the statement ‘Every locally well-orderable connected graph is well-orderable’ in ZF. The rest follows from the fact that every well-ordered connected graph has a spanning tree in ZF.

We show that \(\AC_{\leq n-1}^\omega + Q_{n,k}^n\) implies \(\AC_{\leq n-1}^\omega\). Let \(A = \{A_n : n \in \kappa\}\) be a well-orderable set of non-empty well-orderable sets. Without loss of generality, we assume that \(A\) is disjoint. Let \(A = \bigcup_{i \in \kappa} A_i\). Consider an infinite \(3\), i.e., for each \(i \in \omega\), let \(\{t, t_{i+1}\} \in E_{G_1}\) and \(\{t, x\} \in E_{G_1}\) for every element \(x \in A_i\). Also for each \(i \in \omega\), join each \(x \in A_i\) to every element of \(B_i\).
well-orderable family \((B_i, <_i)_{i \in \kappa}\) of infinite well-orderable sets such that for each \(i \in \kappa\), \(B_i\) is disjoint from \(A\) and the other \(B_j\)'s, and there is no mapping with domain \(A_i\) and range \(B_i\) (cf. the proof of [8] Theorem 1, Remark 6). Let \(B = \bigcup_{i \in \kappa} B_i\). Consider another \(\kappa\)-sequence \(T = \{t_n : n \in \kappa\}\) disjoint from \(A\) and \(B\). We construct a graph \(G_2 = (V_{G_2}, E_{G_2})\) as follows:

\[
V_{G_2} = A \cup B \cup T, \quad \text{and} \quad E_{G_2} = \{\{t_i, t_j\} : i, j \in \kappa \text{ and } i \neq j\} \cup \{\{t_i, x\} : i \in \kappa, x \in A_i\} \cup \{\{x, y\} : i \in \kappa, x \in A_i, y \in B_i\}.
\]

Clearly, the graph \(G_2\) is connected and locally well-orderable. By assumption, \(G_2\) has a spanning subgraph \(G_2'\) omitting \(K_{k,n}\). For each \(i \in \kappa\), let \(f_i : B_i \to \mathcal{P}(A_i)\setminus\{\emptyset\}\) map each element of \(B_i\) to its neighbourhood in \(G_2'\). We can see that for any finite \(k\)-subset \(H_i \subseteq B_i\), \(\bigcap_{i \in H} f_i(e)\) has at most \(n - 1\) elements, since \(G_2'\) has no \(K_{k,n}\). Since each \(B_i\) is infinite and well-orderable, by Lemma 6.1(2), there are tuples \((\epsilon_1, \epsilon_2, \ldots, \epsilon_k) \in B_i^k\) such that \(\bigcap_{i \in \kappa} f_i(\epsilon_i) \neq \emptyset\). Consider the first such tuple \((\epsilon'_1, \epsilon'_2, \ldots, \epsilon'_k)\) with respect to the well-ordering on \(B_i^k\). Let \(A' = \bigcap_{i \in \kappa} f_i(\epsilon'_i)\). By \(\mathcal{AC}_{\leq n-1}\), we can obtain a choice function of \(A' = \{A'_n : n \in \kappa\}\), which is a choice function of \(A\).

(3) \((\Rightarrow)\) Fix any even integer \(m = 2(k + 1) \geq 4\). We prove that \(\mathcal{P}_{\lf,c}^{\omega}\) implies \(\mathcal{AC}_{\omega}^{\kappa}\). Let \(A = \{A_i : i \in \omega\}\) be a countably infinite set of non-empty finite sets and \(A = \bigcup_{i \in \omega} A_i\). Let \(T = \{t_i : i \in \omega\}\) be a sequence such that \(t_i\)'s are pair-wise distinct and belong to no \(A_j \times \{1, \ldots, k\}\), and \(R = \{r_i : i \in \omega\}\) be a sequence such that \(r_i\)'s are pair-wise distinct and belong to no \((A_j \times \{1, \ldots, k\}) \cup \{t_j\}\) for any \(i, j \in \omega\). We construct a graph \(G_3 = (V_{G_3}, E_{G_3})\) as follows (cf. Figure 2):

\[
\text{Let } V_{G_3} = (\bigcup_{i \in \omega} (A_i \times \{1, \ldots, k\})) \cup R \cup T, \quad \text{and} \quad E_{G_3} = (\bigcup_{i \in \omega, x \in A_i} \{\{r_i, (x, 1)\}, \{(x, 1), (x, 2)\}, \ldots, \{(x, k), t_i\}\}) \cup \{\{r_i, r_{i+1}\} : i \in \omega\}.
\]

\textbf{Figure 2. The graph } G_3, \text{ where } \{t_1, (x_1, k), \ldots, (x_1, 1), r_1\} \text{ is a path in } \zeta.\]

Clearly, \(G_3\) is locally finite and connected. By assumption, \(G_3\) has a spanning \(m\)-bush \(\zeta\). We can see that \(\zeta\) generates a choice function of \(A\): for each \(i \in I\), there is a unique \(x \in A_i\), say \(x_i\), such that \(\{t_i, (x_i, 1), \ldots, (x_i, 1), r_i\}\) is a path in \(\zeta\).

(\(\Leftarrow\)) Fix any even integer \(m \geq 4\). We prove that \(\mathcal{AC}_{\omega}^{\kappa}\) implies \(\mathcal{P}_{\lf,c}^{\omega}\). We know that \(\mathcal{AC}_{\omega}^{\kappa}\) implies the statement ‘Every infinite locally finite connected graph is countably infinite’ in \(\mathbf{ZF}\). The rest follows from the fact that every well-ordered graph has a spanning tree in \(\mathbf{ZF}\) and any spanning tree is a spanning \(m\)-bush.

(4) In \(\mathcal{N}_6\), \(\mathcal{AC}_{\omega m}\) fails, whereas \(\mathcal{AC}_{\omega}^{\omega m}\) holds for any natural number \(n > 2\) (cf. [13, proof of Theorem 7.11, p.110]). By Proposition 6.5(1), \(\mathcal{Q}_{\lf,c}^{\omega m}\) fails in the model.

We recall the definition of \(P_G^\prime\) from \S 1.4.1.

**Proposition 6.6. (ZF)** Fix any \(2 < k \in \omega\) and any \(2 \leq p, q < \omega\). The following hold:

1. \(\mathcal{AC}_{\omega k}^{\omega} \) implies the statement ‘Every graph from the class \(P_{\omega k}^\prime\) has a spanning tree’.
2. \(\mathcal{AC}_{\omega}^{\kappa} \) implies the statement ‘Every graph from the class \(P_{\omega \kappa}^\prime\) has a spanning tree’.
3. \(\mathcal{AC}_{\omega^p-\omega q}^{\wedge} + \mathcal{AC}_{\omega+q}^{\kappa}\) implies the statement ‘Every graph from the class \(P_{\omega^p-\omega q, \omega+q}^\prime\) has a spanning tree’.
Proof. We prove (1). Let $G_2 = (V_{G_2}, E_{G_2})$ be a graph from the class $P'_{K_k}$. Then there is a $G_1 \in P_{K_k}$ (an infinite graph whose only components are $K_k$) such that $V_{G_2} = V_{G_1} \cup \{t\}$ for some $t \notin V_{G_1}$. Let $\{A_i : i \in I\}$ be the components of $G_1$. By $AC_k$ (which follows from $AC_{k+2}$ (cf. Lemma 6.4)), we choose a sequence of vertices $\{a_i : i \in I\}$ such that $a_i \in A_i$ for all $i \in I$. By Lemma 6.2, the number of spanning trees in $A_i$ is $k^{k-2}$ for any $i \in I$. By $AC_{k+2}$, we choose a sequence $\{s_i : i \in I\}$ such that $s_i = (V_{S_i}, E_{S_i})$ is a spanning tree of $A_i$ for all $i \in I$. We construct a graph $S_{G_2} = (V_{S_{G_2}}, E_{S_{G_2}})$ as follows:

Let $V_{S_{G_2}} = \{t\} \cup \bigcup_{i \in I} V_{S_i}$, and

$E_{S_{G_2}} = \{\{t, a_i\} : i \in I\} \cup \bigcup_{i \in I} E_{S_i}$.

Then the graph $S_{G_2} = (V_{S_{G_2}}, E_{S_{G_2}})$ is a spanning tree of $G_2$. Similarly, we can prove (2) and (3) since the number of spanning trees in $C_k$ is $k$ and the number of spanning trees in $K_{P,q}$ is $p^q - 1$ (cf. Lemma 6.3).

\[ \square \]

7. Synopsis of results, questions, and further studies

7.1. Synopsis of results. Fix any $2 < n, k \in \omega$, any $2 \leq q \in \omega$, and any even integer $m \geq 4$. In Figure 3, known results are depicted with dashed arrows, new results in ZF are mentioned with simple arrows, and new results in ZFA are mentioned with thick dotted arrows.

![Figure 3](image-url)

In the above figure, we summarize the results of this note from sections 3, 6.

Consistency results:

- $N_{HT}^q (\{\}) = \neg$ Form 269 $\land$ Form 304 $\land$ Form 233. Consequently, for any integer $q \geq 2$, neither Form 233 nor Form 304 implies $AC_\omega^\omega$ in ZFA.
- $AC^L \Rightarrow$ Form 269 in ZFA.
- $V P \models$ Form 3 $\land \neg$ ISAE $\land \neg$ EPWFP $\land \neg AC_q \land \neg MA(N_0) \land \neg \leq N_0$-MC.
- $V P^+ \models$ Form 3 $\land \neg$ ISAE $\land \neg$ EPWFP $\land \neg AC_q \land \neg W_{N_0} \land \neg DC_{\bar{N}_0}$.

The statements ‘$\exists$ DCP $\land$ UT$(N_0, N_0, cuf) \land \neg M(\{\{\emptyset\}\})$’ and ‘$\exists$ DCP $\land \neg MC(N_0, N_0)$’ have permutation models.

7.2. Questions, and further studies.

Question 7.1. Which other choice principles hold in $V P$? In particular does CAC, the infinite Ramsey’s Theorem (RT) $\mathbb{Z}$ Form 17], and Form 233 hold in $V P$?

Bruce [4] proved that UT(WO, WO, WO) holds in $V P$.

Question 7.2. Does UT(WO, WO, WO) hold in $V P^+$?

We recall that every symmetric extension (symmetric submodel of a forcing extension where AC can consistently fail) is given by a symmetric system $(\mathcal{P}, G, F)$, where $\mathcal{P}$ is a forcing notion, $G$ is a group of permutations of $\mathcal{P}$, and $F$ is a normal filter of subgroups over $G$. We recall the definition of Feferman–Lévy’s symmetric extension from Dimitriou’s Ph.D. thesis (cf. [2] Chapter 1, section 2).

Forcing notion $\mathcal{P}_1$: Let $\mathcal{P}_1 = \{p : \omega \times \omega \rightarrow K_\omega : (|p| < \omega \text{ and } \forall (n, i) \in \text{dom}(p), p(n, i) < \omega_i)\}$ be a forcing notion ordered by reverse inclusion, i.e., $p \leq q$ if $p \supseteq q$ (We denote by $p : A \rightarrow B$ a partial function from $A$ to $B$).

Group of permutations $G_1$ of $\mathcal{P}_1$: Let $G_1$ be the full permutation group of $\omega$. Extend $G_1$ to an automorphism group of $\mathcal{P}_1$ by letting an $a \in G_1$ act on a $p \in \mathcal{P}_1$ by $a^*(p) = \{(n, a(i), \beta) : (n, i, \beta) \in p\}$. We identify $a^*$ with $a \in G_1$. We can see that this is an automorphism group of $\mathcal{P}_1$.

Normal filter $\mathcal{F}_1$ of subgroups over $G_1$: For every $n \in \omega$ define the following sets:

$E_n = \{p \cap (n \times \omega \times \omega_n) : p \in \mathcal{P}_1\}$, $\text{Fix}_{G_1}(E_n) = \{a \in G_1 : \forall p \in E_n(a(p) = p)\}$. (3)
We can see that $F_1 = \{X \subseteq G_1 : \exists n \in \omega, \operatorname{fix}_{G_1}(E_n) \subseteq X\}$ is a normal filter of subgroups over $G_1$.

Feferman–Lévy’s symmetric extension is the symmetric extension obtained by $(P_1, G_1, F_1)$ where the statement ‘$R_1$ is regular’ [3 Form 34] fails. It is known that the following statements follow from ‘$R_1$ is regular’ in ZF (cf. [1] [2]).

(\ast): If $P$ is a poset such that the underlying set has a well-ordering and if all antichains in $P$ are finite and all chains in $P$ are countable, then $P$ is countable.

(\ast\ast): If $P$ is a poset such that the underlying set has a well-ordering and if all antichains in $P$ are countable and all chains in $P$ are finite, then $P$ is countable.

Question 7.3. Does any of (\ast\ast) and (\ast) is true in Feferman–Lévy’s symmetric extension?

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