A generic nonaligned Josephson junction in the presence of an external magnetic field is theoretically considered and an unusual flux-dependent current-phase relation (CPR) is revealed. We explain the origin of the anomalous CPR via the orbital motion of quasiparticles within a two-dimensional quasiclassical Keldysh-Usadel framework. In particular, it is demonstrated that nonaligned Josephson junctions can be utilized to obtain a ground-state other than 0 and π, corresponding to a so-called \( \varphi \)-junction, which is tunable via the external magnetic flux. Furthermore, we show that the standard Fraunhofer central peak of the critical supercurrent may be inverted into a local minimum solely due to geometrical factors in planar junctions. This yields good consistency with a recent experimental measurement displaying such type of puzzling feature [R. S. Keizer et al., Nature 439, 825 (2006)].

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FIG. 1: Diagram of considered setups in this Letter. An external magnetic field \( \mathbf{H} \) (not shown) is applied to the junction in the \( z \) direction perpendicular to the junction plane. The junction lengths and widths are \( d \) and \( W \), respectively. A): The planar Josephson junction that has experimentally been studied in e.g. Ref. 11. The widths of the superconducting leads are assumed to be \( W_{1L} \) and \( W-(W_{1L}+W_{2L}) \). B): The usual stacked geometry of a Josephson junction with displaced superconducting leads. The superconducting leads’ sizes are \( W_{1L} \) and \( W_{2R} \) at the top and bottom of junction, respectively. C): Qualitative view of the quasiparticles’ orbital motion inside the normal strip subjected to an external magnetic field, which is used to describe the origin of the addressed unusual CPR.

utilized a quasiclassical Keldysh-Usadel technique with the numerical approach in Ref. 10 and solved exactly the resultant linearized equations of motion for the Green’s function. As our first main result, we unveil that the origin of the unexpected interference pattern in the experiment of Ref. 11 lies within the geometry of the setup. In this way, the absence of the standard Fraunhofer pattern, which has not been clearly understood, is resolved. In addition to this, we demonstrate as our second main result that the CPR in non-aligned junctions takes on a very unusual feature: it becomes shifted by a term proportional to the external flux \( \Phi \), namely \( I(\varphi, \Phi) = I_0(\Phi) \sin(\varphi + \Theta(\Phi)) \) where \( \varphi \) is the superconducting phase difference and \( \Theta \) is a geometry-dependent function. Our investigations reveal that the well-known sinusoidal supercurrent and consequently the Fraunhofer pattern manifest only in a specific situations. This result is explained in terms of the motion of the quasiparticles stemming from the orbital effect. An interesting consequence of the external mag-
conditions for studying junctions. In our Josephson system, we employ the Kupriyanov-Lukichev boundary conditions at N/S interfaces [18] and control the leakage of superconductive correlations into the normal strip using an interface parameter \( \zeta \):

\[
\zeta (\hat{G}(x, y, z) \partial \hat{G}(x, y, z)) \cdot \hat{n} = [\hat{G}_{BCS}(\varphi), \hat{G}(x, y, z)],
\]

in which \( \hat{n} \) is a unit vector denoting the perpendicular direction to an interface and \( \varphi \) is the bulk superconducting macroscopic phase. We define \( \zeta = R_B / R_F \) as the ratio between the resistance of the barrier region and the resistance in the normal sandwiched strip. The bulk solution for the retarded Green’s function in a s-wave superconductor is given by [16]

\[
\begin{align*}
\frac{g^R_{BCS}}{g^R} &= \cosh \vartheta(\varepsilon) \\
\frac{f^R_{BCS}}{f^R} &= e^{i\varphi} \sinh \vartheta(\varepsilon)
\end{align*}
\]

in which \( \vartheta(\varepsilon) = \arctanh(|\Delta|/\varepsilon) \). For a weak proximity effect (\( \zeta \gg 1 \)), the normal and anomalous Green’s functions can be approximated by \( g^R \simeq 1 \) and \( |f^R| \ll 1 \), respectively. The current density vector is expressed via the Keldysh block as

\[
\mathbf{J}(\mathbf{R}) = J_0 \cdot \int d\varepsilon \text{Tr}\{\rho_3 (\hat{G}(x, y, z) [\hat{\partial}, \hat{G}(x, y, z)] )\}^K.
\]

Here, \( J_0 = N_0 e D/4 \) and \( N_0 \) is the density of states at the Fermi level. The flux penetrating the junction is given by \( \Phi = dW H \). We also investigate the spatial variation of pair potential inside the normal region calculated via:

\[
U = U_0 \text{Tr}\{[\hat{\rho}_1 - i\hat{\rho}_2] \} \int d\varepsilon \hat{\tau}_3 \hat{G}^K(x, y, z),
\]

where \( U_0 = -N_0 e / 16 \) [16]. To study the considered Josephson junction we use a collocation finite element numerical method the same as Ref. 10. The components of approximate solution are assumed to be linear combinations of bicubic Hermite basis functions satisfying boundary conditions. Ultimately, the resultant nonsymmetric linear algebraic equations are solved via a Jacobi conjugate-gradient method. For more details, see Ref. 19. All lengths and energies are normalized by the superconducting coherent length \( \xi_S \) and superconducting gap at absolute zero \( \Delta_0 \). The barrier resistance \( \zeta \) is fixed at 7 ensuring the validity of weak proximity regime. Temperature and junction width are \( T = 0.05 T_c \) and \( W = 10 \xi_S \). Units also are considered so that \( \hbar = \kappa_B = 1 \).

Figure 2 illustrates the response of the critical Josephson current in a planar junction to an external magnetic field as shown schematically in Fig. 1 A). Various parameter values have been considered in order to make our analysis as general as possible. To do so, we have considered three scenarios where the superconducting leads have different sizes (first row) and where they have equal sizes with a large (second row) and small (third row) separation distance. Specifically, the third row is relevant with regard to the experiment in Ref. 11 where the size of the electrodes far exceeds the separation distance. As seen, in this case the interference pattern exhibits a local minimum at \( \Phi = 0 \) rather than a maximum as in...
FIG. 3: Critical supercurrent against external magnetic flux and corresponding pair potential spatial maps of usual Josephson junction with displaced superconducting leads including various lead sizes. For the pair potential maps, the superconducting phase difference and external magnetic flux are fixed at $\phi=0$ and $\Phi=4\Phi_0$, respectively. The junction thickness and width are set to $d=2\xi_S$ and $W=10\xi_S$, respectively.

FIG. 4: i) Left column: $W_1=3\xi_S$, $W_2=4\xi_S$, $W_1=3\xi_S$, $W_2=4\xi_S$, $W_1=6\xi_S$, $W_2=4\xi_S$, $W_1=0$, $W_2=4\xi_S$ and finally iii) right column: $W_1=2\xi_S$, $W_2=6\xi_S$. The top panels represent the CPRs for various values of $\Phi/\Phi_0=0, 3.92, 6.28$. The current density spatial maps in the bottom row show the results for $\phi=0$ and $\Phi=4\Phi_0$. The superconducting leads’ sizes are then set equal at $4\xi_S$ for all cases as schematically depicted on top of each column.

the Fraunhofer case, which is fully consistent with the experimental results in Ref. 11. Whereas it was speculated that this minimum might be attributed to a shift in the entire interference curve due to a finite sample magnetization in Ref. 11, it is obvious that this is not the case here since the sandwiched strip is not ferromagnetic. Moreover, such a shift would make the current vs. flux curve manifestly asymmetric (see e.g. Ref. 20), in contrast to the experimental results of Ref. 11 where the central minimum is flanked by two large peaks, similar to our results. Based on this, it seems reasonable to explain the deviation from the standard Fraunhofer pattern as a result originating from the combination of a planar geometry with the size and separation distance of the superconducting electrodes. The latter fact is seen by considering the second row of Fig. 2 where the separation distance is large compared to the superconductors: a Fraunhofer-like pattern emerges, although the decay becomes more monotonic as the thickness $d$ of the normal strip increases. Even columns in both Figs. 2 and 3 show the pair potential where the superconducting phase difference is zero $\phi=0$ and external magnetic flux is set to $\Phi=4\Phi_0$. As seen, the predicted proximity vortices in Refs. 4 and 10 vanish for the planar junction geometry. However, as it will be discussed further below, they reappear in a specific condition of stacked geometry (Fig. 1 B)).

It is instructive to contrast these results with the geometry of Fig. 1 B) where the two superconducting leads are connected to the normal strip at opposite edges. This is reminiscent to the experimentally often used stacked geometry. The order of frames (critical current and corresponding pair potential spatial map) are identical to those in Fig. 3 and various lead sizes and locations are investigated. It is seen that the location and size of both terminals are vital in terms of determining how the critical current responds to the external flux. For instance, our results reveal that only in specific case where the width of the leads’ are sufficiently large and connected to opposite edges precisely in front of each other does one recover a proximity-induced vortex pattern along with the Fraunhofer curve i.e.

$$I(\phi, \Phi) \propto \Phi^{-1} \sin \Phi \sin \phi$$

which is a special case corresponding to the scenario of Ref. 4. The results for the other scenarios in Fig. 3 also show good consistency with previous experimental observations [5].

Having unveiled the origin of the anomalous inverted Fraunhofer response, we now turn to the second main result of this paper: the possibility to generate a $\phi$-junction in an SNS system with an applied magnetic field. In Fig. 4, we provide
the CPR in addition to a spatial map of the current-flow in the normal strip for three represented geometries. In i) the leads are connected opposite to each other, in ii) they are connected antisymmetrically, whereas in iii) they are connected symmetrically in a planar geometry similar to Ref. 11. It is clear that the CPR remains sinusoidal as a function of the superconducting phase difference \( \varphi \) in both i) and ii) independent on the applied flux. However, case iii) is qualitatively different. The generic form of the CPR is now revealed as:

\[
I(\varphi, \Phi) = I_0(\Phi) \sin(\varphi + \Theta(\Phi))
\]  

(7)
in which \( I_0(\Phi) \) and \( \Theta(\Phi) \) are geometry-dependent functions of external magnetic flux as seen in Fig. 4. In fact, Eq. (7) holds for all situations considered in Fig. 2 where we have demonstrated the CPR is never purely sinusoidal. The standard sinusoidal CPR is recovered only for symmetric situations relative the induced orbital motion by the external magnetic field, see Fig. 1 C). This observation has a highly interesting consequence: the anomalous magnetic flux-coupled CPR ensures that the ground-state of the system may be tuned so that the equilibrium phase difference differs from the conventional 0 or \( \pi \) solutions. Instead, a so-called \( \varphi \)-junction may be realized where the ground-state phase difference \( \varphi \) is tunable via the external flux. We therfore arrive at a ground-state with Josephson energy \( E_J(\Phi) = 1 - \cos(\varphi + \Theta(\Phi)) \) which can be controlled by adjusting the applied external magnetic field. The idea of a \( \varphi \)-junction via a superconducting phase difference shift has been considered previously [7] in the context of a non-centrosymmetric normal layer with a Rashba spin-orbit interaction. However, in our setup the external flux is a well-controlled parameter which allows for easy tuning of the ground-state, as opposed to controlling a spin-orbit interaction parameter. Moreover, our finding is different from Ref. 8 where two magnetic junctions, one in 0-state and the other in \( \pi \)-state with different lengths, are connected in parallel and consequently generate an extra cosinusoidal term in addition to negative second harmonic.

What is then the physical origin of this anomalous CPR? The answer to this question may be obtained by investigating the orbital motion of the quasiparticles under the influence of an external magnetic field inside the normal strip, as seen in the current-density maps of Fig. 4 and simple schematic in Fig. 1 C). For zero phase difference \( \varphi = 0 \), the external magnetic field induces a current flow where the orbital paths taken by the quasiparticles move with the same flux in and out of the superconducting regions, in effect no net current flow, only in special geometrical configurations. For instance, both in i) and ii) the current flow between the superconductors in any part of the normal region is seen to have an antisymmetric, and thus cancelling, contribution in a different part of the normal strip at zero phase difference \( \varphi = 0 \). In contrast, this is no longer the case in setup iii): there is a net flow of current induced by the orbital response due to the magnetic flux, even at \( \varphi = 0 \). To elucidate this clearly in the current-flow, one would have to consider the amplitude of the local current as well, but the supercurrent-phase curves nevertheless demonstrate that this interpretation is correct. In essence, this is a geometry-dependent effect since it relies on the positioning of the leads relative the induced current-flow via the applied field. Thus, it gives rise to the unique possibility to alter the standard CPR so that the ground-state of system can be adjusted by tuning the external flux.

To conclude, we have studied the Josephson critical current and its response to an external magnetic flux in experimentally feasible nonaligned junctions. Specifically, a planar geometry similar to a recent experiment [11] is considered and it is demonstrated that the observed suppression at zero flux may stem from the junction geometry rather than any intrinsic magnetization. Moreover, it is shown that a highly unusual supercurrent-phase difference-shift occurs inevitably in a class of nonaligned junctions due to an external magnetic flux. Its precise form is sensitive to the size and location of the superconducting leads. Consequently, this offers a route to a tunable junction ground-state. The physical origin of this effect is traced back to the orbital motion of the quasiparticles in the presence of an external field relative the position of the superconducting leads. As an interesting consequence, this type of Josephson junctions constitute an attainable way of realizing the so-called \( \varphi \)-junction experimentally.

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