GEOMETRIC CONFIGURATION OF Riemannian Submanifolds of Arbitrary Codimension

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Abstract. In this paper we study a geometric configuration of submanifolds of arbitrary codimension in an ambient Riemannian space. We obtain relations between the geometry of a \( q \)-codimension submanifold \( M^n \) along its boundary and the geometry of the boundary \( \Sigma^{n-1} \) of \( M^n \) as an hypersurface of a \( q \)-codimensional submanifold \( P^n \) in an ambient space \( \overline{M}^{n+q} \). As a consequence of these geometric relations we get that the ellipticity of the generalized Newton transformations implies the transversality of \( M^n \) and \( P^n \) in \( P^n \) is totally geodesic in \( \overline{M}^{n+q} \).

1. Introduction

Let \( \overline{M}^{n+q} \) be \( n+q \)-dimensional connected and orientable Riemannian manifold with metric \( \langle \cdot, \cdot \rangle \) and Levi-Civita connection \( \nabla \). Denote by \( P^n \) an oriented connected \( n \)- submanifold of \( \overline{M}^{n+q} \) and consider \( \Sigma^{n-1} \) an \( n-1 \)-compact hypersurface of \( P^n \). If \( \Psi : M^n \to \overline{M}^{n+q} \) is an oriented connected and compact submanifold of \( \overline{M}^{n+q} \) with boundary \( \partial M \). \( M^n \) will be said submanifold of \( \overline{M}^{n+q} \) with boundary \( \Sigma^{n-1} \) if the restriction of \( \Psi \) to \( \partial M \) is a diffeomorphism onto \( \Sigma^{n-1} \). A natural question would be: How can one describe the geometry of \( M^n \) along its boundary \( \partial M \) with respect to the geometry of the inclusion \( P^n \subset \overline{M}^{n+q} \)? A partial answer to this question is given by the following formula, obtained in this paper, which holds along the boundary \( \partial M \): for any multi-index \( u = (u_1, \ldots, u_q) \) with length \( |u| \leq n-1 \),

\[
\langle T_u \nu, \nu \rangle = \frac{1}{n-1-|u|} \sum_{l \leq u} \left( \frac{n-1-|l|}{|u|-l} \right) \rho^{l} \mu^{u-l} \sigma_{|u|}(A_{\Sigma}).
\]

where \( (T_u)_u \) stands for the family of the generalized Newton transformations introduced in [4] associated to the matrix \( A = (A_1, \ldots, A_q) \); \( (A_{\alpha})_{\alpha \in \{1, \ldots, q\}} \) is a system of matrices of the shape operators corresponding to a normal basis to the manifold \( M^n \) and \( \tilde{\sigma}_{u} = \tilde{\sigma}_{u}(A_1|_{\Sigma}, \ldots, A_q|_{\Sigma}) \) are the coefficients of the

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Newton polynomial $P_\tilde{A} : \mathbb{R}^q \to \mathbb{R}$ defined by

$$P_\tilde{A}(t) = \sum_{|u| \leq n-1} \tilde{\sigma}_u t^u$$

where $\tilde{A} = (A_1|\Sigma, \ldots, A_q|\Sigma)$, $A_\alpha|\Sigma$ is the restriction of $A_\alpha$ to $\Sigma^{n-1}$ and $\sigma_r(A_\Sigma)$ is the symmetric function coefficient of $t^r$ in the characteristic polynomial of the matrix $A_\Sigma$.

In [2] Alías and Malacarne considered the above geometric configuration to the case of hypersurfaces where $\Sigma^{n-1}$ is an $(n-1)$-dimensional compact submanifold contained in an hyperplane $\Pi$ of $\mathbb{R}^{n+1}$ and $M^n$ stands for a smooth compact, connected and oriented manifold with boundary $\partial M^n$. Moreover $M^n$ is an hypersurface of $\mathbb{R}^{n+1}$ with boundary $\Sigma^{n-1}$ in the sense that there exists $\psi : M^n \to R^{n+1}$ an oriented hypersurface immersed in $\mathbb{R}^{n+1}$ such that the restriction of $\psi$ to the boundary $\partial M^n$ is a diffeomorphism onto $\Sigma^{n-1}$. They showed that along the boundary $\partial M^n$, for every $1 \leq r \leq n-1$:

$$\langle T_r \nu, \nu \rangle = (-1)^r s_r \langle a, \nu \rangle^r$$

where $\nu$ stands for the outward pointing unit conormal vector field along $\partial M^n$ while $T_r$ denotes the classical Newton transformation, $a \in R^{n+1}$ such that $\Pi = a^\perp$ and $s_r$ is the $r$-th symmetric function of the principal curvatures of $\Sigma^{n-1}$ with respect to $\nu$. In [3] Alías, de Lira and Malacarne studied the question in the context of an $(n+1)$-dimensional connected oriented ambient Riemannian manifold $\overline{M}^{n+1}$, they established that along the boundary $\partial M^n$, for every $1 \leq r \leq n-1$:

$$\langle T_r \nu, \nu \rangle = (-1)^r s_r \langle \xi, \nu \rangle^r$$

where $T_r$, $\nu$, $s_r$, are as in relation \[1.2\] where $P^n \subset \overline{M}^{n+1}$ is an embedded totally geodesic submanifold instead of the hyperplane $\Pi$ and $\xi$ is a unitary normal vector field to $P^n$. Relation \[1.3\] shows that the ellipticity of the Newton transformation $T_r$, for some $1 \leq r \leq n-1$ on $M^n$, implies the transversality of the hypersurfaces $M^n$ and $P^n$ along their boundary. Formula \[1.3\] was also obtained, in \[1.4\], by the two first authors in context of pseudo-Riemannian spaces. Moreover we deduce from relation \[1.1\] that the ellipticity of the generalized Newton transformation $T_u$ implies the transversality of the $q$-codimension submanifolds $M^n$ and $P^n$ in case where $P^n$ is totally geodesic submanifold of $\overline{M}^{n+q}$.

2. Preliminaries

In this section, we will recall some properties of the generalized Newton transformations and we will show how our method works.

2.1. Generalized Newton Transformations. Let $E$ be an $n$-dimensional real vector space and $End(E)$ be the vector space of endomorphisms of $E$. Denote by $\mathbb{N}$ the set of nonnegative integers and let $\mathbb{N}^q$ be the one of multi-index $u = (u_1, \ldots, u_q)$ with $u_j \in \mathbb{N}$. The length $|u|$ of $u$ is given by
\(|u| = u_1 + \ldots + u_q\). \(\text{End}^q \(E\)\) stands for the vector space \(\text{End}(E) \times \ldots \times \text{End}(E)\) \(q\)-times. For \(A = (A_1, \ldots, A_q) \in \text{End}^q \(E\), \(t = (t_1, \ldots, t_q) \in \mathbb{R}^q\) and \(u \in \mathbb{N}^q\), we set

\[
\begin{align*}
  tA &= t_1 A_1 + \ldots + t_q A_q \\
  t^u &= t_1^{u_1} \ldots t_q^{u_q}.
\end{align*}
\]

For \(\alpha \in \{1, \ldots, q\}\) we define (see [4]) the musical functions \(\alpha^\flat : \mathbb{N}^q \rightarrow \mathbb{N}^q\) and \(\alpha^\sharp : \mathbb{N}^q \rightarrow \mathbb{N}^q\) by

\[
\begin{align*}
  \alpha^\flat(i_1, \ldots, i_q) &= (i_1, \ldots, i_{\alpha-1}, i_\alpha - 1, i_{\alpha+1}, \ldots, i_q) \\
  \alpha^\sharp(i_1, \ldots, i_q) &= (i_1, \ldots, i_{\alpha-1}, i_\alpha + 1, i_{\alpha+1}, \ldots, i_q)
\end{align*}
\]

It is clear that \(\alpha^\flat\) is the inverse map of \(\alpha^\sharp\).

The generalized Newton transformation (GNT in brief) is a system of endomorphisms \(T_u = T_u \(A\), \(u \in \mathbb{N}^q\), that satisfies the following recursive relations

\[
\begin{align*}
  T_0 &= I & \text{where } 0 = (0, \ldots, 0), \\
  T_u &= \sigma_u I - \sum_{\alpha} A_\alpha T_{\alpha^\flat(u)} & \text{where } |u| > 1 \\
  &= \sigma_u I - \sum_{\alpha} T_{\alpha^\flat(u)} A_\alpha
\end{align*}
\]

where \(\sigma_u\) are the coefficients of the Newton polynomial \(P_A : \mathbb{R}^q \rightarrow \mathbb{R}\) of \(A\), given by

\[
P_A(t) = \det \(I + tA\) = \sum_{|u| \leq n} \sigma_u t^u
\]

\(\sigma_u = \sigma_u \(A_1, \ldots, A_q\)\) depends only on \(A = (A_1, \ldots, A_q)\) and \(I\) is the identity map on \(E\).

### 2.2. The method.

We will describe how our method works.

#### 2.2.1. Hypersurfaces’ case.

Let \(M^n\) be a \(n\)-submanifold of codimension one in \(\overline{M}^{n+1}\) of boundary \(\partial M\). Assume the boundary \(\Sigma^{n-1} = \partial M\) is a codimension one in \(P^n \subset \overline{M}^{n+1}\). Then we have the inclusions

\[
\Sigma^{n-1} \subset M^n \subset \overline{M}^{n+1}, \quad \Sigma^{n-1} \subset P^n \subset \overline{M}^{n+1}.
\]

Denote the corresponding shape operators, respectively, by

\(A_\Sigma, A_P, A_{\Sigma,P}, A\).

In our consideration we will need only \(A_\Sigma, A_P, A\). More precisely we will use

\(A_\Sigma, A_P|_\Sigma, A\).

First two are represented by square matrices of dimension \(n - 1\) whereas the last one by a square matrix of dimension \(n\). The intrinsic geometry of \(\Sigma^{n-1}\)
in \( M^n \) is coded in the pair \((A_\Sigma, A_P|\Sigma)\) and the geometry of \( M^n \subset \overline{M}^{n+1} \) is given by \( A \). Therefore we will use the following Newton Transformation and the generalized Newton Transformations

\[
T_{(k,l)} = T_{(k,l)}(A_\Sigma, A_P|\Sigma) \quad \text{and} \quad T_r = T_r(A)
\]

and corresponding symmetric functions

\[
\sigma_{(k,l)} = \sigma_{(k,l)}(A_\Sigma, A_P|\Sigma) \quad \text{and} \quad \sigma_r = \sigma_r(A).
\]

The goal is to show that

\[
\langle T_r \nu, \nu \rangle = \sum_{k+l=r} \sigma_{(k,l)}
\]

where \( \nu \) is the unit normal vector to \( \Sigma^{n-1} \) in \( M^n \).

The only geometric considerations involved are the ones which lead to the formulas

\[
\langle N, \nu \rangle = \langle \xi, N \rangle, \quad \langle \eta, N \rangle = -\langle \xi, \nu \rangle
\]

and

\[
\langle Ae_i, e_j \rangle = -\langle A_\Sigma e_i, e_j \rangle \langle \xi, \nu \rangle + \langle A_P e_i, e_j \rangle \langle \xi, N \rangle
\]

where \( N \) is unit normal vector with the respect to inclusion \( M^n \subset \overline{M}^{n+1} \), \( \xi \) unit normal vector with respect to \( P^n \subset \overline{M}^{n+1} \) and \( \eta \) is the unit normal vector of \( \Sigma^{n-1} \subset P^n \) and \((e_1, ..., e_{n-1})\) is a local orthonormal basis of \( T\Sigma^{n-1} \), we may assume that this basis consists of eigenvectors of \( A_\Sigma \) i.e. \( A_\Sigma e_i = \tau_i e_i \).

In other words

\[
A|_\Sigma = -\langle \xi, \nu \rangle A_\Sigma + \langle \xi, N \rangle A_P|_\Sigma.
\]

Assuming \( P \) is totally umbilical in \( \overline{M}^{n+1} \), we have \( A_P|_\Sigma = \lambda I_{T\Sigma^{n-1}} \). Hence

\[
A|_\Sigma = -\langle \xi, \nu \rangle A_\Sigma + \lambda \langle \xi, N \rangle I_{T\Sigma^{n-1}}.
\]

Denote by \( \tilde{A} \) the matrix of \( A \) with the respect of the basis \((e_1, ..., e_{n-1}, \nu)\) and by \( A \) the matrix of \( A|_\Sigma \). Then

\[
\tilde{A} = \begin{pmatrix} A & B \\ B^\top & c \end{pmatrix}, \quad \text{where} \quad B = \begin{pmatrix} \langle A\nu, e_1 \rangle \\ \vdots \\ \langle A\nu, e_{n-1} \rangle \end{pmatrix} \quad \text{and} \quad c = \langle A\nu, \nu \rangle.
\]

Let us compare symmetric functions of \( \tilde{A} \) with symmetric functions of \( A \).

We have

\[
P_{\tilde{A}}(t) = \det \left( \begin{array}{cc} I_{n-1} + tA & tB \\ tB^\top & 1 + tc \end{array} \right) = (1 + tc - t^2B^\top(I_{n-1} + tA)^{-1}B)\det(I_{n-1} + tA)
\]

\[
= f(t)P_A(t),
\]

where \( f(t) = 1 + tc - t^2B^\top(I_{n-1} + tA)^{-1}B \). Recall that

\[
P_{\tilde{A}}(t) = \sum_{j=0}^{n} (-1)^j \sigma_j(\tilde{A}) t^j.
\]
Hence
\[
(-1)^r r! \sigma_r(\bar{A}) = \frac{d^r}{dt^r} P_{\bar{A}}(0) = \sum_{j=0}^{r} \binom{r}{j} \frac{d^{r-j}}{dt^{r-j}} P_{\bar{A}}^{(r-j)}(0) f^{(j)}(0)
\]
\[= \sum_{j=0}^{r} \binom{r}{j} (-1)^{r-j} (r-j)! \sigma_{r-j}(A) f^{(j)}(0). \]

It is not hard to see that
\[f(0) = 1, \quad f'(0) = -c, \quad \text{and} \quad f^{(j)}(0) = -j! B^\top A^{j-2}B \text{ for } j \geq 2.\]

Therefore
\[
(2.3) \quad \sigma_r(\bar{A}) = \sigma_r(A) + c \sigma_{r-1}(A) - \sum_{j=2}^{r} (-1)^j \left( B^\top A^{j-2}B \right) \sigma_{r-j}(A). \]

Let us now move to symmetric functions of two matrices. Notice first that
\[
\sigma_r(\bar{A} + \lambda I_n) = \sum_{j=0}^{r} \binom{n-j}{r-j} \lambda^{r-j} \sigma_j(\bar{A}). \]

Indeed, \( P_{\bar{A} + \lambda I_n}(t) = (1 + t(a_1 + \lambda))... (1 + t(a_n + \lambda)) \) if \( a_1, ..., a_n \) are the eigenvalues of \( \bar{A} \). Notice moreover that
\[
P_{\bar{A}, \lambda I_n}(t) = \det \left( I_n + t\bar{A} + s\lambda I_n \right)
\]
\[= (1 + ta_1 + s\lambda)...(1 + ta_n + s\lambda). \]

Thus
\[
(2.4) \quad \sigma_{(k,l)}(\bar{A}, \lambda I_n) = \binom{n-k}{l} \lambda^l \sigma_k(\bar{A}). \]

Hence
\[
(2.5) \quad \sigma_r(\bar{A} + \lambda I_n) = \sum_{j=0}^{r} \sigma_{(j,r-j)}(\bar{A}, \lambda I_n) \]

We need to show, by \((2.5)\) and \((2.1)\), that
\[
(2.6) \quad \langle T_r \nu, \nu \rangle = \bar{\sigma}_r(A). \]

By the recurrence formula for \( T_r \) we have
\[
\langle T_r \nu, \nu \rangle = \sigma_r(\bar{A}) - \langle T_{r-1} \nu, \bar{A} \nu \rangle
\]
\[= \sigma_r(\bar{A}) - c \langle T_{r-1} \nu, \nu \rangle - \sum_{i=1}^{n-1} \langle T_{r-1} \nu, e_i \rangle \langle \bar{A} \nu, e_i \rangle. \]

We will show that
\[
(2.7) \quad \sum_{i=1}^{n-1} \langle T_k \nu, e_i \rangle \langle \bar{A} \nu, e_i \rangle = - \sum_{j=2}^{k} (-1)^j \left( B^\top A^{j-2}B \right) \sigma_{k-j}(A). \]
Indeed, assuming \((e_i)\) is a basis consisting of eigenvectors with eigenvalues \((\tau_i)\) by induction we have

\[
\sum_{i=1}^{n-1} \langle T_k \nu, e_i \rangle \langle \tilde{A} \nu, e_i \rangle = -b_i^2 \sum_{j=2}^{k} (-j) \sigma_k \tau_j^{j-2} \tau_i^{j-2} \sigma_k \tau_j (A)
\]

where \(B^T = (b_i)_i\), which proves (2.7). Thus by induction and (2.3), we get

\[
\langle T_r \nu, \nu \rangle = \sigma_r (\tilde{A}) - \alpha \sigma_{r-1} (A) + \sum_{j=2}^{r} (-1)^j \left( B^T A^{j-2} B \right) \sigma_{r-j} (A) = \tilde{\sigma}_r (A).
\]

2.2.2. General case. We generalize considerations to any codimension \(q\). We generalize equations (2.1) and (2.6). Let \(P^n\) and \(M^n\) be of codimension \(q\) in \(\overline{M}^{n+q}\). We assume that \(P^n\) is totally umbilical in \(\overline{M}^{n+1}\). As before \(\Sigma^{n-1} \subset P^n\) is a boundary of \(M^n\). Then we have the following shape operators \(A_{\Sigma}, A_{\xi_1}, \ldots, A_{\xi_q}\), \(A^{N_1}, \ldots, A^{N_q}\), corresponding to inclusions \(\Sigma^{n-1} \subset P^n, P^n \subset \overline{M}^{n+q}\), and \(M^n \subset \overline{M}^{n+q}\), where \((\xi_1, \ldots, \xi_q)\) are orthogonal to \(P^n\) and \((N_1, \ldots, N_q)\) are orthogonal to \(M^n\). Let

\[
T_u = T_u (A^{N_1}, \ldots, A^{N_q}), \quad T_v = T_v \left( A_{\Sigma}, A_{\xi_1} |_{\Sigma}, \ldots, A_{\xi_q} |_{\Sigma} \right)
\]

and

\[
\tilde{T}_u = \tilde{T}_u (A^{N_1} |_{\Sigma}, \ldots, A^{N_q} |_{\Sigma})
\]

where \(u\) is of length \(q\) and \(v\) of length \(q+1\). We hope that the following relations hold

(2.8)

\[
\langle T_u \nu, \nu \rangle = \tilde{\sigma}_u
\]

and

\[
\tilde{\sigma}_u = \sum_{|v|=|u|, v \text{ not increasing}} c_v \sigma_v
\]

where \(c_v\) is a coefficient independent of \((A^{N_1} |_{\Sigma}, \ldots, A^{N_q} |_{\Sigma})\). Both of them imply

\[
\langle T_u \nu, \nu \rangle = \sum_{|v|=|u|, v \text{ not increasing}} c_v \sigma_v.
\]

The first equality is a generalization of (2.6), whereas the second one of (2.5). Moreover, the first one is purely algebraic and the second one uses correspondence between \(A_{\Sigma}, A_{\xi_1} |_{\Sigma}, \ldots, A_{\xi_q} |_{\Sigma}\) and \(A^{N_1} |_{\Sigma}, \ldots, A^{N_q} |_{\Sigma}\) analogous to the relation (2.2) in the codimension one case.

Let us be more precise. First notice that
\[ \nabla_{e_i} e_j = \sum_{k=1}^{n-1} \langle \nabla_{e_i} e_j, e_k \rangle e_k + \langle \nabla_{e_i} e_j, \nu \rangle \nu + \sum_{\alpha=1}^{q} \langle \nabla_{e_i} e_j, N_\alpha \rangle N_\alpha \]
\[ = \sum_{k=1}^{n-1} \langle \nabla_{e_i} e_j, e_k \rangle e_k + \langle \nabla_{e_i} e_j, \nu \rangle \nu + \sum_{\alpha=1}^{q} \langle A^\alpha e_i, e_j \rangle N_\alpha \]

and
\[ \nabla_{e_i} e_j = \sum_{k=1}^{n-1} \langle \nabla_{e_i} e_j, e_k \rangle e_k + \langle \nabla_{e_i} e_j, \eta \rangle \eta + \sum_{\alpha=1}^{q} \langle \nabla_{e_i} e_j, \xi_\alpha \rangle \xi_\alpha \]
\[ = \sum_{k=1}^{n-1} \langle \nabla_{e_i} e_j, e_k \rangle e_k + \langle A^\eta e_i, e_j \rangle \eta + \sum_{\alpha=1}^{q} \langle A^\xi_\alpha e_i, e_j \rangle \xi_\alpha \]

Thus
\[ \langle \nabla_{e_i} e_j, \nu \rangle \nu + \sum_{\alpha=1}^{q} \langle A^\alpha e_i, e_j \rangle N_\alpha = \langle A^\eta e_i, e_j \rangle \eta + \sum_{\alpha=1}^{q} \langle A^\xi_\alpha e_i, e_j \rangle \xi_\alpha \]

Hence
\[ \langle A^\alpha e_i, e_j \rangle = \langle \eta, N_\alpha \rangle \langle A^\eta(e_i), e_j \rangle + \sum_{\beta=1}^{q} \langle \xi_\beta, N_\alpha \rangle \langle A^\xi_\beta e_i, e_j \rangle \]
\[ = \langle \eta, N_\alpha \rangle \langle A^\eta(e_i), e_j \rangle + \sum_{\beta=1}^{q} \langle \xi_\beta, N_\alpha \rangle \langle A^\xi_\beta e_i, e_j \rangle \]

Assuming that \( P^n \) is totally umbilical, i.e. \( A^\xi_\alpha = \lambda_\alpha I_n \), where \( I_n \) denotes the identity map of the tangent space \( T_p P^n \), we get
\[ A^\alpha |_\Sigma = \langle \eta, N_\alpha \rangle A^\eta + \sum_{\beta=1}^{q} \langle \xi_\beta, N_\alpha \rangle \lambda_\beta I_{n-1}. \]

Notice that it can be written in the form
\[ (2.9) \quad A^\alpha |_\Sigma = \langle \eta, N_\alpha \rangle A^\eta + \langle V, N_\alpha \rangle I_{n-1}, \quad \text{where} \quad V = \sum_{\beta=1}^{q} \lambda_\beta \xi_\beta. \]

Moreover one can show that
\[ \langle \eta, V \rangle = \det (\langle \xi_\alpha, N_\beta \rangle)_{\alpha,\beta} \quad \text{and} \quad \langle \eta, N_\alpha \rangle = -\det C_\alpha \]
where \( C_\alpha \) is a matrix obtained from \( C = (\langle \xi_\alpha, N_\beta \rangle)_{\alpha,\beta} \) by replacing the \( \alpha \)-th column by \( \langle \xi_\alpha, \nu \rangle \).

Hence we have
\[ \tilde{T}_u = \tilde{T}_u (\rho_1 A^\eta + \mu_1 I_{n-1}, \ldots, \rho_q A^\eta + \mu_q I_{n-1}), \quad \text{where} \quad \rho_\alpha = \langle \eta, N_\alpha \rangle, \ \mu_\alpha = \langle V, N_\alpha \rangle. \]
Now if \( \{ e_1, \ldots, e_{n-1} \} \) is a basis of eigenvectors of the shape operator \( A_\Sigma \), then, for every \( i \in \{ 1, \ldots, n-1 \} \),
\[
A_\Sigma(e_i) = \tau_i e_i,
\]
where \( \tau_i \) are the corresponding eigenvalues.

By relation (2.9) we get
\[
\langle A_\alpha(e_i), e_j \rangle = \left( \langle \eta, N_\alpha \rangle \tau_i + \langle V, N_\alpha \rangle \right) \delta_{ij} = \gamma_{i,\alpha} \delta_{ij},
\]
where \( \gamma_{i,\alpha} = \langle \eta, N_\alpha \rangle \tau_i + \langle V, N_\alpha \rangle \).

Thus the matrix associated to the shape operator \( A_\alpha \) with respect to the
basis \( \{ e_1, \ldots, e_{n-1}, \nu \} \) is given by
\[
A_\alpha = \begin{pmatrix}
\gamma_{1,\alpha} & 0 & \ldots & 0 & \langle A_\alpha \nu, e_1 \rangle \\
0 & \gamma_{2,\alpha} & \ldots & 0 & \vdots \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \ldots & \ldots & \gamma_{n-1,\alpha} & \langle A_\alpha \nu, e_{n-1} \rangle \\
\langle A_\alpha \nu, e_1 \rangle & \ldots & \ldots & \langle A_\alpha \nu, e_{n-1} \rangle & \langle A_\alpha \nu, \nu \rangle
\end{pmatrix}.
\]

3. Algebraic Formulas

Let us prove the algebraic relation (2.8), to do so, we put
\[
\rho_\alpha = \langle \eta, N_\alpha \rangle, \quad \mu_\alpha = \langle V, N_\alpha \rangle.
\]
Moreover, we will write \( v \leq u \) for multi–indices \( v, u \in \mathbb{N}^q \) if the difference \( u - v \in \mathbb{N}^q \). For a multi–index \( u = (u_1, \ldots, u_q) \in \mathbb{N}^q \) of length \( |u| = n \), we set
\[
\binom{n}{u} = \frac{n!}{u! u_1! \ldots u_q!}.
\]

**Proposition 1.** Let \( \hat{A} = (A_1|\Sigma, \ldots, A_q|\Sigma) = (\rho_1 A_\Sigma + \mu_1 I, \ldots, \rho_q A_\Sigma + \mu_q I) \). Put \( \tilde{\sigma}_u = \sigma_u(\hat{A}) \). Then for every multi–index \( u \in \mathbb{N}^q \) we have
\[
(3.1) \quad \tilde{\sigma}_u = \frac{1}{(n-1-|u|)!} \sum_{l \leq u} \binom{|u|}{l} \rho^l \mu^{u-l} \left( \frac{n-1-|l|}{u-l} \right) \sigma_{|l|} (A_\Sigma).
\]

**Proof.** First, by applying the following relation (see \cite{6})
\[
\sigma_u(aA_1, \ldots, A_q) = a^{u_1} \sigma_u(A_1, \ldots, A_q)
\]
we obtain
\[
(3.2) \quad \tilde{\sigma}_u(\rho_1 A_\Sigma + \mu_1 I, \ldots, \rho_q A_\Sigma + \mu_q I) = \rho^u \tilde{\sigma}_u(A_\Sigma + \theta_1 I, \ldots, A_\Sigma + \theta_q I)
\]
where
\[
\theta_\alpha = \frac{\mu_\alpha}{\rho_\alpha} \quad \text{and} \quad \rho^u = \rho_1^{u_1} \ldots \rho_q^{u_q}.
\]
Consider the GNT \( T_u = T_u(\hat{A}) \), with
\[
\hat{A} = (A_\Sigma + \theta_1 I, \ldots, A_\Sigma + \theta_q I).
\]
The characteristic polynomial associate to \( \hat{A} \) is then
\[
P_{\hat{A}}(t) = \sum_{|u| \leq n-1} \hat{\sigma}_u t^u
\]
Moreover, we have
\[
(3.3) \quad P_{\hat{A}}(t) = \det \left( I + \sum_{\alpha=1}^q t_\alpha (A_\Sigma + \theta_\alpha) I \right)
\]
\[
(3.4) \quad = \prod_{j=1}^{n-1} (1 + \tau_j (t_1 + \ldots + t_q) + t_1 \theta_1 + \ldots + t_q \theta_q).
\]
By expanding the linear factorization of a monic polynomial we obtain
\[
P_{\hat{A}}(t) = \sum_{j=0}^{n-1} (1 + t_1 \theta_1 + \ldots + t_q \theta_q)^{n-1-j} (t_1 + \ldots + t_q)^j \sigma_j (A_\Sigma)
\]
and by the multinomial theorem we get
\[
P_{\hat{A}}(t) = \sum_{j=0}^{n-1} \left( \sum_{|u| \leq n-1} \sum_{|v| \leq n-1} \sum_{|l| |l| = j} \frac{1}{(n-1-j-k)!} \left( \begin{array}{c} j \\ l \end{array} \right) \left( \begin{array}{c} n-1-j \\ v \end{array} \right) \sigma_j (A_\Sigma) \theta_v^u t_v^j \right).
\]
where \( l = (l_1, \ldots, l_q), v = (v_1, \ldots, v_q), \theta = (\theta_1, \ldots, \theta_q) \) and \( t = (t_1, \ldots, t_q) \). Let \( u = v + l \) so \( j = |u| - |v| \) and
\[
P_{\hat{A}}(t) = \sum_{|u| \leq n-1} \sum_{1 \leq u \leq n-1} \left( \frac{1}{(n-1-|u|)!} \left( \begin{array}{c} |l| \\ l \end{array} \right) \left( \begin{array}{c} n-1-|l| \\ u-l \end{array} \right) \theta_v^u \sigma_l (A_\Sigma) t^u \right).
\]
Thus the coefficient before \( t^u \) is given by
\[
\tilde{\sigma}_u (A_\Sigma + \theta_1 I, \ldots, A_\Sigma + \theta_q I) = \frac{1}{(n-1-|u|)!} \sum_{l \leq u} \left( \begin{array}{c} |l| \\ l \end{array} \right) \theta_v^u \sigma_l (A_\Sigma).
\]
Consequently
\[
\tilde{\sigma}_u (\rho_1 A_\Sigma + \mu_1 I, \ldots, \rho_q A_\Sigma + \mu_q I) = \left( \begin{array}{c} 1 \\ (n-1-|u|)! \end{array} \right) \sum_{l \leq u} \left( \begin{array}{c} |l| \\ l \end{array} \right) \rho_l \mu_l t^u \sigma_l (A_\Sigma).
\]
On the other hand
Denote by \( \bar{A} = (A_\Sigma, \mu_1 I, \ldots, \mu_q I) \) and \( \tilde{\sigma}_{(k,v)} = \sigma_{(k,v)}(\bar{A}) \). We have
\[
\sigma_{(k,v)}(\bar{A}) = \mu^v \sigma_{(k,v)}(A_\Sigma, I, \ldots, I).
\]
Now, we use the following formula (see [6])

\[ \sigma(v,k)(B_1,\ldots,B_q,l) = \binom{n-1-|v|}{k} \sigma(B_1,\ldots,B_q) \]

consecutively \( q \)-times and we get

\[ \sigma_{(j,v)}(A_{\Sigma},I,\ldots,I) = \binom{n-1-|v|+v_1}{v_1} \cdots \binom{n-1-|v|+v_q}{v_q} \sigma_j(A_{\Sigma}) \]

Taking \( j = |u| - |v| \), we obtain

\[ = \binom{n-1-|u|+|v|}{v} \sigma_{|u|-|v|}(A_{\Sigma}) \]

and if we let \( l = u - v \), we have

\[ \sigma(|l|,u-l)(A_{\Sigma},I,\ldots,I) = \binom{n-1-|l|}{u-l} \sigma_{|l|}(A_{\Sigma}) \]

So by relations (3.2) and (3.6), we get

\[ (3.7) \quad \tilde{\sigma}_u = \binom{|l|}{l} \sum_{l \leq \alpha} \binom{|l|}{l} \rho^l \mu^u \sigma(|l|,u-l) \sigma_{|l|}(A_{\Sigma}) \]

Notice, that in the proof of the above Proposition, we assumed \( \rho_\alpha \neq 0 \) for all \( \alpha \), this allowed us to defined the constants \( \theta_\alpha \). The assumption \( \rho_\alpha \neq 0 \) is not necessary. Since if there exists \( \alpha \in \{1,\ldots,q\} \) such that \( \rho_\alpha = 0 \). then (see [6])

\[ \tilde{\sigma}_u = \tilde{\sigma}_u(\rho_1 A_{\Sigma} + \mu_1 I,\ldots,\rho_q A_{\Sigma} + \mu_q I) \]

\[ = \tilde{\sigma}_u(\rho_1 A_{\Sigma} + \mu_1 I,\ldots,\rho_{i-1} A_{\Sigma} + \mu_{i-1} I,\rho_i A_{\Sigma} + \mu_i I,\rho_{i+1} A_{\Sigma} + \mu_{i+1} I,\ldots,\rho_q A_{\Sigma} + \mu_q I) \]

\[ = \mu_i^u \tilde{\sigma}_u(\rho_1 A_{\Sigma} + \mu_1 I,\ldots,\rho_i A_{\Sigma} + \mu_i I,\rho_{i+1} A_{\Sigma} + \mu_{i+1} I,\ldots,\rho_q A_{\Sigma} + \mu_q I), \]

where

\[ \tilde{u} = (u_1,\ldots,u_{i-1},u_{i+1},\ldots,u_q) \]

and we may apply the above Proposition to \( \tilde{\sigma}_u \).

**Proposition 2.** Let \( \tilde{A} = (A_1|_{\Sigma},\ldots,A_q|_{\Sigma}) \), where \( A_\alpha|_{\Sigma} = (\rho_\alpha A_{\Sigma} + \mu_\alpha I, \) and \( \overline{A} = (A_{\Sigma},\mu_1 I,\ldots,\mu_q I) \). For any multi-index \( u \in \mathbb{N}^q \), put \( \tilde{\sigma}_u = \sigma_u(\tilde{A}) \) and \( \overline{\sigma}_u = \sigma_u(\overline{A}) \). Then we have

\[ (3.8) \quad \tilde{\sigma}_u = \sum_{l \leq u} \binom{|l|}{l} \rho^l \sigma_{|l|,u-l}(\tilde{A}). \]
Proof. First notice that
\[ \sigma_{(j,v)} = \sigma_{(j,v)}(\tilde{A}) = u^v \sigma_{(j,v)}(A_\Sigma, I, \ldots, I). \]

Using the following formula (see [6])
\[ \sigma_{(v,j)}(B_1, \ldots, B_q, I) = \left( n - 1 - |v| \right) \sigma_v(B_1, \ldots, B_q) \]
consecutively \( q \)-times, we obtain
\[ \sigma_{(j,v)}(A_\Sigma, I, \ldots, I) = \frac{1}{(n - 1 - |v|)!} \left( n - 1 - j \right) \sigma_j(A_\Sigma) \]
or by putting \( l = u - v \)
\[ \sigma_{(|l|, u - l)} = \frac{1}{(n - 1 - |u|)!} \left( n - 1 - |l| \right) \sigma_{|l|}(A_\Sigma) \]
so
\[ \sigma_{(|l|, u - l)} = \frac{\mu^{u - l}}{(n - 1 - |u|)!} \left( n - 1 - |l| \right) \sigma_{|l|}(A_\Sigma) \]

Thus by (3.1) we get (3.8). \( \Box \)

Let us now state the relation between symmetric functions corresponding to the families \((A_\alpha)\) and \((A_\alpha|\Sigma)\). Notice that there are of different sizes.

**Proposition 3.** Let \( A = (A_1, \ldots, A_q) \) and \( \tilde{A} = (A_1|\Sigma, \ldots, A_q|\Sigma) \). Put \( \sigma_u = \sigma_u(A) \) and \( \tilde{\sigma}_u = \sigma_u(\tilde{A}) \). Then
\[ \sigma_u = \tilde{\sigma}_u + \sum_\alpha C_\alpha \tilde{\sigma}_{\alpha_\Sigma(v)} + \sum_{\alpha^* \beta^* (0) \leq w \leq u, \alpha, \beta} (-1)^{|w|-|v|+1} \left( |u| - |w| \right) B_{\alpha^*}^{\top} \tilde{A}^u - w B_{\beta} \tilde{\sigma}_{\alpha_\beta_\Sigma(w)} \]
where \( B_{\alpha}^{\top} = (\langle A_\alpha v, e_1 \rangle, \ldots, \langle A_\alpha v, e_{n-1} \rangle) \) and \( C_\alpha = \langle A_\alpha v, v \rangle \).

**Proof.** First calculate the characteristic polynomial \( P_\mathcal{A}(t) \) of \( A = (A_1, \ldots, A_q) \). By definition, we have
\[ P_\mathcal{A}(t) = \det(I + \sum_\alpha t_\alpha A_\alpha) \]
\[ = \det \left( I_{n-1} + \sum_\alpha t_\alpha A_\alpha|\Sigma \right) \det \left( C - B^{\top} \left( I_{n-1} + \sum_\alpha t_\alpha A_\alpha|\Sigma \right)^{-1} B \right) \]
where \( B^{\top} \) and \( C \) are respectively given by
\[ B^{\top} = \left( \sum_\alpha t_\alpha \langle A_\alpha v, e_1 \rangle, \ldots, \sum_\alpha t_\alpha \langle A_\alpha v, e_{n-1} \rangle \right) = \sum_\alpha t_\alpha B_{\alpha}^{\top} \]
and
\[ C = 1 + \sum_{\alpha} t_{\alpha}(A_{\alpha}v, v). \]

To simplify the expressions, we put
\[ M = \left( I_{n-1} + \sum_{\alpha} t_{\alpha}A_{\alpha}|\Sigma \right). \]

and
\[ f(t) = \det(C - B^\top M^{-1}B). \]

Moreover it is clear that
\[ B^\top M^{-1}B = \sum_{\alpha,\beta} t_{\alpha}t_{\beta}B_{\alpha}^\top M^{-1}B_{\beta}, \]

therefore we get
\[ f(t) = \left( 1 + \sum_{\alpha} t_{\alpha}C_{\alpha} - \sum_{\alpha,\beta} t_{\alpha}t_{\beta}B_{\alpha}^\top M^{-1}B_{\beta} \right). \]

As it is easily seen that
\[ P_A(t) = f(t)P_{\tilde{A}}(t), \]

where \( P_A(t) \) and \( P_{\tilde{A}}(t) \) are defined by
\[ P_A(t) = \sum_{|u| \leq n} \sigma_u t^u \]

and
\[ P_{\tilde{A}}(t) = \sum_{|u| \leq n-1} \tilde{\sigma}_u t^u. \]

Thus we get
\[ \frac{\partial^u}{\partial t^u} P_A(t) \big|_{t=0} = u!\sigma_u, \quad \frac{\partial^u}{\partial t^u} P_{\tilde{A}}(t) \big|_{t=0} = u!\tilde{\sigma}_u. \]

Similarly we obtain
\[ \frac{\partial^u}{\partial t^u} (P_{\tilde{A}}(t).f(t)) \big|_{t=0} = \sum_{v \leq u} \binom{u}{v} \left( \frac{\partial^v}{\partial t^v} P_{\tilde{A}}(t), \frac{\partial^{u-v}}{\partial t^{u-v}} f(t) \right) \big|_{t=0}, \]

where
\[ \binom{u}{v} = \frac{u!}{v!} = \frac{u_1! \ldots u_1!}{v_1! \ldots v_q!}. \]

Now we will compute \( \frac{\partial^{u-v}}{\partial t^{u-v}} f(t) \). For \( v = u \), it is checked
\[ \frac{\partial^{u-v}}{\partial t^{u-v}} f(t) \big|_{t=0} = f(0, ..., 0) = 1. \]

If \( v = \alpha_\beta(u) \), we have
\[ \frac{\partial^{u-v}}{\partial t^{u-v}} f(t) \big|_{t=0} = \frac{\partial}{\partial t^\alpha} f(t) \big|_{t=0} = C_\alpha \]
and for any \( v \leq \alpha \beta_\gamma(u) \), we obtain

\[
\frac{\partial^{u-v}}{\partial t^{u-v}} f(t) \big|_{t=0} = \frac{\partial^{u-v}}{\partial t^{u-v}} \left( \left( I_{n-1} + \sum_{\alpha} t_\alpha A_\alpha |\Sigma \right)^{-1} \left( 1 + \sum_{\alpha} t_\alpha C_\alpha - \sum_{\alpha,\beta} t_\alpha t_\beta B_\alpha^\top \left( I_{n-1} + \sum_{\alpha} t_\alpha A_\alpha |\Sigma \right)^{-1} B_\beta \right) \right) \big|_{t=0}.
\]

Taking \( |t| < \varepsilon \) for some small enough \( \varepsilon \) we get

\[
\left\| \sum_{\alpha} t_\alpha A_\alpha |\Sigma \right\| \leq 1.
\]

Thus

\[
\sum_{\alpha} t_\alpha A_\alpha |\Sigma \right) \left( \sum_{\alpha} \right)^{-1} = \sum_{k=0}^\infty (-1)^k \left( \sum_{\alpha} t_\alpha A_\alpha |\Sigma \right)^k.
\]

Since the matrices \( A_\alpha |\Sigma \) are diagonal, then they commute. Therefore, we may write

\[
\left( \sum_{\alpha} t_\alpha A_\alpha |\Sigma \right)^k = \sum_{w_1+\ldots+w_q=k} \binom{k}{w} t^w \tilde{A}^w.
\]

Hence

\[
\left( I_{n-1} + \sum_{\alpha} t_\alpha A_\alpha |\Sigma \right)^{-1} = \sum_{k=0}^\infty (-1)^k \left( \sum_{w_1+\ldots+w_q=k} \binom{k}{w} t^w \tilde{A}^w \right).
\]

Which gives

\[
\frac{\partial^{u-v}}{\partial t^{u-v}} f(t) \big|_{t=0} = \frac{\partial^{u-v}}{\partial t^{u-v}} \left( -\sum_{\alpha,\beta} t_\alpha t_\beta B_\alpha^\top \sum_{k=0}^\infty (-1)^k \left( \sum_{w_1+\ldots+w_q=|w|} \binom{|w|}{w} t^w \tilde{A}^w \right) B_\beta \right) \big|_{t=0}
\]

\[
= \sum_{\alpha,\beta} B_\alpha^\top \sum_{k=0}^\infty (-1)^k \left( \sum_{w_1+\ldots+w_q=|w|} \binom{|w|}{w} t^{w+\alpha \beta}(0) \tilde{A}^w \right) B_\beta \big|_{t=0}
\]

\[
= \sum_{\alpha,\beta} \sum_{v \leq \alpha \beta_\gamma(u)} B_\alpha^\top (-1)^{|u|-|v|-2(u-v)!} \left( u - \frac{|v| - 2}{u - \alpha \beta_\gamma(v)} \right) \tilde{A}^{u-\alpha \beta_\gamma(v)} B_\beta \big|_{t=0}
\]

Finally we obtain

\[
\sigma_u = u! \tilde{\sigma}_u + \sum_{\alpha} (\alpha_\beta(u))! \left( \sum_{\alpha} C_\alpha \tilde{\sigma}_\alpha(u) + \sum_{\alpha,\beta} \left( \sum_{v \leq \alpha \beta_\gamma(u)} (-1)^{|u|-|v|-1(u-v)!} \right) \left( u - \frac{|v| - 2}{u - \alpha \beta_\gamma(v)} \right) \tilde{A}^{u-\alpha \beta_\gamma(v)} B_\beta \right) \big|_{t=0}.
\]
or equivalently,
\[
\sigma_u = \tilde{\sigma}_u + \sum_{\alpha} C_\alpha \tilde{\sigma}_{\alpha}(u) + \sum_{\alpha, \beta} \sum_{0 \leq e \leq \alpha, \beta} (-1)^{|v|-|w|-1} \left( \frac{|u| - |v| - 2}{u - \alpha \beta(v)} \right) B_{\alpha} \tilde{A}^{u-a \beta(v)} B_{\beta} \tilde{\sigma}_v.
\]

Hence
\[
\sigma_u = \tilde{\sigma}_u + \sum_{\alpha} C_\alpha \tilde{\sigma}_{\alpha}(u) + \sum_{\alpha, \beta} \sum_{0 \leq w \leq \alpha, \beta} (-1)^{|w|-|v|+1} \left( \frac{|u| - |w|}{u - w} \right) B_{\alpha} \tilde{A}^{u-w} B_{\beta} \tilde{\sigma}_{\alpha, \beta}(w).
\]

\[\square\]

4. Generalized Newton transformation on the boundary

We use the same notations as in previous sections. In this section we give the expression of the GNT \( T_u = T_u(\tilde{A}) \), where \( \tilde{A} = (A_1|\Sigma, \ldots, A_q|\Sigma) \), on the boundary \( \Sigma^{n-1} \) of \( M^n \). Recall that
\[
A_\alpha|\Sigma = \rho_\alpha A_\Sigma + \mu_\alpha I,
\]
where
\[
\rho_\alpha = \langle \eta, N_\alpha \rangle, \quad \mu_\alpha = \langle V, N_\alpha \rangle, \quad V = \sum_{\alpha=1}^q \lambda_\alpha \xi_\alpha.
\]

**Proposition 4.** Let \( \overline{M}^{n+q} \) be an \((n+q)\)-Riemannian manifold and \( P^n \subset \overline{M}^{n+q} \) an oriented totally umbilical \( n \)-submanifold of \( \overline{M}^{n+q} \). Denote by \( \Sigma^{n-1} \subset P^n \) an \((n-1)\)-compact hypersurface of \( P^n \). Let \( \Psi : M^n \rightarrow \overline{M}^{n+q} \) be an oriented connected and compact submanifold of \( \overline{M}^{n+q} \) with boundary \( \Sigma^{n-1} = \Psi(\partial M) \). Then along the boundary \( \partial M \), we have
\[
\langle T_u \nu, \nu \rangle = \tilde{\sigma}_u(A_1|\Sigma, \ldots, A_q|\Sigma).
\]

**Proof.** We make a recursive proof. Assume that \([11]\) holds for any multi–index \( v < u \). We have by the recurrence definition of \( T_u \)
\[
\langle T_u \nu, \nu \rangle = \sigma_u(\nu, \nu) - \sum_{\alpha} \langle A_\alpha T_{\alpha}(u) \nu, \nu \rangle
\]
\[
= \sigma_u - \sum_{\alpha} \langle T_{\alpha}(u) \nu, A_\alpha \nu \rangle
\]
\[
= \sigma_u - \sum_{\alpha} \langle T_{\alpha}(u) \nu, \nu \rangle \langle A_\alpha \nu, \nu \rangle - \sum_{\alpha, i} \langle T_{\alpha}(u) \nu, e_i \rangle \langle A_\alpha e_i, \nu \rangle.
\]

Put
\[
C_\alpha = \langle A_\alpha \nu, \nu \rangle, \quad b_{i, \alpha} = \langle A_\alpha e_i, \nu \rangle
\]
then
\[
\langle T_u \nu, \nu \rangle = \sigma_u - \sum_{\alpha} C_\alpha \tilde{\sigma}_{\alpha}(u) - \sum_{\alpha, i} b_{i, \alpha} \langle T_{\alpha}(u) \nu, e_i \rangle.
\]

Let us compute \( \langle T_u \nu, e_i \rangle \) for any multi–index \( u \in \mathbb{N}^q \). Notice that
\[
A_\alpha e_i = \rho_\alpha A_\Sigma e_i + \mu_\alpha e_i + b_{i, \alpha} \nu
\]
assuming that \( \{e_1, \ldots, e_{n-1}\} \) is an orthonormal basis of \( T_p\Sigma^{n-1} \) consisting of eigenvectors of \( A\Sigma \) (with eigenvalues \( \tau_i \)). Thus
\[
A_\alpha e_i = \gamma_{i,\alpha} e_i + b_{i,\alpha} \nu
\]
where
\[
\gamma_{i,\alpha} = \rho_\alpha \tau_i + \mu_\alpha.
\]
We will show inductively that
\[
(4.2) \quad \langle T_\nu, e_i \rangle = \sum_\alpha \sum_{\alpha^\sharp(0) \leq w \leq u} (-1)^{|u|-|w|+1} \left( \frac{|u| - |w|}{u - w} \right) b_{i,\alpha} \gamma_{i}^{u-w} \tilde{\sigma}_{\alpha}(w)
\]
where
\[
\gamma_i = (\gamma_{i,1}, \ldots, \gamma_{i,q})
\]
Indeed, for \( u = \beta^\sharp(0) \) we have,
\[
\langle T_\nu, e_i \rangle = \sigma_{\beta^\sharp(0)} \langle \nu, e_i \rangle - \sum_\alpha \left( A_\alpha T_{\alpha}(\beta^\sharp(0)) \nu, \nu \right)
\]
\[
= -\sum_\alpha \langle A_\alpha \nu, \nu \rangle
\]
\[
= -b_{i,\beta}
\]
\[
= \sum_\alpha \sum_{\alpha^\sharp(0) \leq w \leq \beta^\sharp(0)} (-1)^{|\beta^\sharp(0)|-|w|+1} \left( \frac{|\beta^\sharp(0)|}{\beta^\sharp(0) - w} \right) b_{i,\alpha} \gamma_{i}^{\beta^\sharp(0)-w} \tilde{\sigma}_{\alpha}(w),
\]
since the sum reduces to one element for \( \alpha = \beta \).

Assume that (4.2) holds for all multi index \( v < u \). Then again by the recursive definition of \( T_u \) we get
\[
\langle T_\nu, e_i \rangle = \sigma_u \langle \nu, e_i \rangle - \sum_\alpha \left( A_\alpha T_{\alpha}(u) \nu, e_i \right)
\]
\[
= -\sum_\alpha \left( A_\alpha T_{\alpha}(u) \nu, e_i \right)
\]
\[
= -\sum_\alpha \left( A_\alpha T_{\alpha}(u) \nu, A_\alpha e_i \right) - \sum_\alpha \left( T_{\alpha}(u) \nu, A_\alpha e_i \right) - \sum_{i,j} \left( T_{\alpha}(u) \nu, e_i \right) \left( A_\alpha e_i, e_j \right)
\]
\[
= -\sum_\alpha \left( T_{\alpha}(u) \nu, e_i \right) \gamma_{i,\alpha} - \sum_\alpha b_{i,\alpha} \tilde{\sigma}_{\alpha}(u)
\]
\[
= \sum_\alpha \gamma_{i,\alpha} \left( \sum_\beta \sum_{\beta^\sharp(0) \leq u \leq \alpha^\sharp(0)} (-1)^{|u|-|\alpha^\sharp(0)|} \left( \frac{|u| - |\alpha^\sharp(0)| - 1}{\alpha^\sharp(0) - w} \right) b_{i,\beta} \gamma_{i}^{\alpha^\sharp(0)-w} \tilde{\sigma}_{\beta}(w) \right)
\]
\[
- \sum_\alpha b_{i,\alpha} \tilde{\sigma}_{\alpha}(u).
\]
Clearly
\[
\gamma_{i,\alpha} \gamma_{i}(u) - w = \gamma_{i}^{u-w}
\]
Notice that taking \( w = u \) we get the last sum. Thus we obtain
\[
\langle T_u, e_i \rangle = \sum_{\alpha, \beta} \left( \sum_{\beta \leq \alpha} (-1)^{|u| - |w| + 1} \left( \frac{|u| - |w| - 1}{\alpha_\beta (u) - w} \right) b_{i, \beta} \gamma_i^{u - w} \sigma_{\beta}(w) \right).
\]

Since
\[
\sum_{\alpha} \left( \frac{|u| - |w| - 1}{\alpha_\beta (u) - w} \right) = \left( \frac{|u| - |w|}{u - w} \right)
\]
we deduce
\[
\langle T_u, e_i \rangle = \sum_{\beta} \left( \sum_{\beta \leq \alpha} (-1)^{|u| - |w| + 1} \left( \frac{|u| - |w|}{u - w} \right) b_{i, \beta} \gamma_i^{u - w} \sigma_{\beta}(w) \right)
\]
which end the proof of (4.2).

Now, we may prove (4.1). First, we have
\[
\sum_{\alpha, i} \langle T_{\alpha}(u), e_i \rangle b_{i, \alpha} = \sum_{\alpha, \beta, i} \left( \sum_{\beta \leq \alpha} (-1)^{|u| - |w|} \left( \frac{|u| - |w| - 1}{\alpha_\beta (u) - w} \right) b_{i, \beta} \gamma_i^{u - w} \sigma_{\beta}(w) \right)
\]
Replacing \( w \) by \( \alpha_\beta \sigma_{\beta}(w) \), we obtain
\[
\sum_{\alpha, i} \langle T_{\alpha}(u), e_i \rangle b_{i, \alpha} = \sum_{\alpha, \beta, i} \left( \sum_{\beta \leq \alpha} (-1)^{|u| - |w| + 1} \left( \frac{|u| - |w|}{u - w} \right) b_{i, \alpha} \gamma_i^{u - w} b_{i, \beta} \sigma_{\alpha_\beta}(w) \right)
\]
Noticing that
\[
\sum_{i} b_{i, \alpha} \gamma_i^{u - w} b_{i, \beta} = B_{\alpha}^T \tilde{A}^{u - w} B_{\beta}
\]
we infer
\[
\sum_{\alpha, i} \langle T_{\alpha}(u), e_i \rangle b_{i, \alpha} = \sum_{\alpha, \beta} \sum_{\beta \leq \alpha} (-1)^{|u| - |w| + 1} \left( \frac{|u| - |w|}{u - w} \right) B_{\alpha}^T \tilde{A}^{u - w} B_{\beta} \sigma_{\alpha_\beta}(w)
\]
Summing up all the above considerations we get
\[
\langle T_u, \nu \rangle = \sigma_u - \sum_{\alpha} \langle T_{\alpha}(u), \nu \rangle \langle A_\alpha \nu, \nu \rangle - \sum_{\alpha, i} \langle T_{\alpha}(u), e_i \rangle \langle A_\alpha e_i, \nu \rangle
\]
\[
= \sigma_u - \sum_{\alpha} C_\alpha \tilde{\sigma}_{\alpha}(u) - \sum_{\alpha, \beta} \sum_{\alpha_\beta \geq \beta} (-1)^{|u| - |w| + 1} \left( \frac{|u| - |w|}{u - w} \right) B_{\alpha}^T \tilde{A}^{u - w} B_{\beta} \tilde{\sigma}_{\alpha_\beta}(w)
\]
or, equivalently,
\[
\sigma_u = \tilde{\sigma}_u + \sum_{\alpha} C_\alpha \tilde{\sigma}_{\alpha}(u) + \sum_{\alpha, \beta} \sum_{\alpha_\beta \geq \beta} (-1)^{|u| - |w| + 1} \left( \frac{|u| - |w|}{u - w} \right) B_{\alpha}^T \tilde{A}^{u - w} B_{\beta} \tilde{\sigma}_{\alpha_\beta}(w)
\]
Applying Proposition 4, we obtain
\[
\langle T_u, \nu \rangle = \tilde{\sigma}_u.
\]
By Proposition (1), we obtain the following expression of $\langle T_u \nu, \nu \rangle$ in terms of symmetric functions of the shape operator $A_{\Sigma}$.

**Corollary 1.** With the conditions of Proposition (4), we have

$$
\langle T_u \nu, \nu \rangle = \frac{1}{n - 1 - |u|} \sum_{l \leq u} \binom{n - 1 - |u|}{|u| - l} \rho^l \mu^{u-l} \sigma_{|u|}(A_{\Sigma}).
$$

The relation (4.3) will be more simple if we suppose that the embedding $P^n \subset M^{n+q}$ is totally geodesic.

**Corollary 2.** With the conditions of Proposition (4) and assuming that $P^n \subset M^{n+q}$ is totally geodesic, then for every multi-index $u$ with length $|u| \leq n - 1$, we have

$$
\langle T_u \nu, \nu \rangle = \rho^u \sigma_{|u|}(A_{\Sigma})
$$

**Proof.** It suffices to use (4.3) with $\mu_{\alpha} = 0$. \[\square\]

5. **Transversality of Submanifolds**

The formula for the generalized Newton transformation implies the relation between transversality of $M^n$ and $P^n$ and ellipticity of $T_u$ provided that $P^n$ is totally geodesic in $M^{n+1}$. This generalizes the result in (2) to any arbitrary codimension.

**Theorem 1.** With the conditions in Corollary (2) the submanifolds $M^n$ and $P^n$ are transversal along $\partial M$ provided that for some multi-index $u$ of length $1 \leq |u| \leq n - 1$, the generalized Newton transformation $T_u$ is positive definite on $M^n$.

**Proof.** Saying that $M^n$ and $P^n$ are not transversal means that there exist $p \in \partial M$ such that for every $\alpha \in \{1, \ldots, q\}$ we have

$$
\rho_u (\eta, N_\alpha) = 0 \quad \text{at } p.
$$

Therefore, if we suppose that for all $p \in M_1^{n+q}$, $T_u$ is positive definite, then by Corollary 7, $\rho^u (p) \neq 0$. Thus

$$
\langle \eta, N_\alpha \rangle \neq 0,
$$

hence $M^n$ and $P^n$ are transversal. \[\square\]

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