Group analysis approach for finding reciprocal transformations for the
two-dimensional stationary gasdynamics

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Abstract

Equivalence transformations play one of the important roles in continuum mechanics. These transformations reduce the original equations to simpler forms. One of the classes of nonlocal equivalence transformations is the class of reciprocal transformations. Despite the long history of applications of such transformations in continuum mechanics, there is no method of obtaining them. Recently such a method was proposed by the second author of the present paper. The method uses group analysis approach and it consists of similar steps as for finding an equivalence group of transformations. The new method provides a systematic tool for finding classes of reciprocal transformations (group of reciprocal transformations). As an illustration, the method was applied to the one-dimensional gas dynamics equations, and new reciprocal transformations were found. Similar to the classical group analysis this approach can be also applied for finding all reciprocal transformations (not only composing a group) of studied equations. The present paper provides this algorithm. As an illustration the method is applied to the two-dimensional stationary gas dynamics equations. Equivalence group, reciprocal equivalence group and completeness of all discrete reciprocal transformations are presented in the paper.

Keywords: Reciprocal transformations, equivalence group of transformations, Lie group of transformations
Subject Classification (MSC 2010): 35C99, 76W05

1. Introduction

In a study of lift and drag aspects in two-dimensional homentropic irrotational gasdynamics Bateman \textsuperscript{1} established invariance of the governing system under a novel multi-parameter class of relations which have come to be known as reciprocal transformations. The latter are typically associated with conservation laws admitted by a system. These transformations leave invariant the governing equations, up to the equation of state. In the group analysis method such kind of transformations are called equivalence transformations. In nonlinear continuum mechanics, reciprocal transformations have likewise proved to have diverse physical applications\textsuperscript{2}. The preceding attests to the importance of reciprocal transformations in physical applications. De-

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\textsuperscript{1}See, for example in the books \textsuperscript{2, 3}.

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spite on many applications, to the best of our knowledge, there are no systematic derivation of reciprocal transformations.

A link between a one-parameter subclass of infinitesimal reciprocal-type transformations in gasdynamics and the Lie group approach was established in \[4\], where it was shown that the transformations found in \[5\] can compose a Lie group of transformations. This idea was also applied in \[6\] for relativistic gas dynamics equations, where a connection between one-parameter subclasses found recently in \[7, 8\] and a Lie group procedure was shown.

1.1. Equivalence transformations

A nondegenerate change of the dependent and independent variables, which transfers any system of differential equation of the given class

\[F^k(x, u, p, \phi) = 0, \quad (k = 1, 2, \ldots, s)\]  \hspace{1cm} (1.1)

to the system of equations of the same class is called an equivalence transformation. Here \(x\) is the vector of the independent variables, \(u\) is the vector of the dependent variables, \(p\) are derivatives, and the functions \(\phi : V \rightarrow \mathbb{R}^t\) are arbitrary elements of system (1.1). For example, for the gas dynamics equations \(\phi\) is the function related with the state equation.

1.1.1. Equivalence point transformations

The problem of finding an equivalence transformation consists of the construction a transformation of the variables \((x, u, \phi)\) that preserves the equations changing only their representative \(\phi = \phi(x, u)\). For this purpose there are several methods. One of these methods is the direct solution of the equations determining such transformations (see for example \[9\]). Despite its complexity this method gives a complete set of the equivalence point transformations \[10\]. The determining equations become simpler for the equivalence transformations composing a Lie group \[11\], which is called an equivalence group. Notice that using equivalence group and the method proposed in \[12, 13\], all equivalence transformations can be found \[14\].

1.1.2. Reciprocal transformations

For reciprocal transformations the change of the independent variables differs from point transformations: this change is defined by differentials. A link between a one-parameter subclass of infinitesimal reciprocal-type transformations in gasdynamics and the Lie group approach was established in \[4\]. The results of \[4\] led to the idea to use the algorithms developed in the group analysis method \[11\] for finding reciprocal transformations. This idea was realised in \[15\], where reciprocal transformations and equivalence transformations of the one-dimensional gas dynamics equations were compared. The reciprocal transformations obtained by this way we call group of reciprocal transformations.

In the present paper, for finding all reciprocal transformations of systems of differential equations we combined the method of finding reciprocal group of transformations \[15\] with the automorphism-based algebraic method \[12, 13\]. This combination allows finding all reciprocal transformations (not only composing a group). For an illustration the method is applied to the two-dimensional stationary gas dynamics equations. Equivalence group, group of reciprocal transformations, and all discrete reciprocal transformations are presented in the paper. It is proven that the reciprocal transformations found in \[19\] compose a complete set of reciprocal transformations, up to equivalence transformations.

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2 See also references therein.

3 See also \[16, 17, 18\] for further development and extensions of the method.
1.2. The two-dimensional gas dynamics equations

The equations are considered in the present paper are

\[ F_1 = (\rho u)_x + (\rho v)_y = 0, \quad F_2 = \rho(uu_x + vv_y) + p_x = 0, \]
\[ F_3 = \rho(uv_x + vu_y) + p_y = 0, \quad F_4 = uS_x + vS_y = 0, \]  
(1.2)

where \( q = (u, v) \) is the gas velocity, while \( p \) is the gas pressure, \( \rho \) is the gas density and \( S \) is the specific entropy. An appropriate state equation must be added to system (1.2)

\[ p = G(\rho, S). \]  
(1.3)

1.3. Organization of the paper

The paper is organized as follows.

The next Section deals with the equivalence group of the two-dimensional stationary gas dynamics equations. In Section 3, the well-known results related with the reciprocal transformations of the two-dimensional stationary gas dynamics equations are presented. Section 4 discusses applications of the first method for seeking reciprocal transformations using the infinitesimal approach. This method uses two conservation laws written in the form of differentials. Section 5 provides a generalization of the first method, where none of the assumptions on the differentials are required. Section 6 is devoted to the application of the results of the previous section for finding all reciprocal transformations for the two-dimensional stationary gas dynamics equations. Summary of the results is also given in this Section. The last Section gives the Conclusions. Some necessary formulas are given in the Appendix.

2. Equivalence point transformations

Equivalence transformations preserve the structure of equations. Consider an equivalence group preserving system (1.2). A generator of a one-parameter group of equivalence transformations is assumed to be in the form \[ X^e = \xi^x \partial_x + \xi^y \partial_y + \zeta^\rho \partial_\rho + \zeta^u \partial_u + \zeta^v \partial_v + \zeta^S \partial_S + \zeta^p \partial_p, \]

where all coefficients of the generator depend on \((x, y, \rho, u, v, S, p)\), and \( p = G(\rho, S) \) is considered as an arbitrary element.

For finding the equivalence group of transformations the infinitesimal criterion \[ \xi^x \partial_x + \xi^y \partial_y + \zeta^\rho \partial_\rho + \zeta^u \partial_u + \zeta^v \partial_v + \zeta^S \partial_S + \zeta^p \partial_p \]

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where all coefficients of the generator depend on \((x, y, \rho, u, v, S, p)\), and \( p = G(\rho, S) \) is considered as an arbitrary element.
where $\psi = e^{\epsilon h}$, $\epsilon$ is the group-parameter, and only changeable variables are presented. This transformation corresponds to the Munk–Prim transformation \[22\], which is also well-known in the gas dynamics. The transformations related to the generator $X_F$ allow one to change $S' = \Psi(S)$, where $\Psi(S)$ is an arbitrary function.

There are also known two obvious involutions

\[ E_1 : \quad x' = -x, \quad u' = -u, \]
\[ E_2 : \quad y' = -y, \quad v' = -v. \]

3. The Bateman-type reciprocal transformations of 2D stationary gasdynamics

System (1.2) implies the pair of conservation laws

\[
\begin{align*}
(p u v)_x + (p + p u^2)_y &= 0, \\
(p + p u^2)_x + (p u v)_y &= 0.
\end{align*}
\]

(3.1)

Using these conservation laws, new independent variables can be introduced by the formulas

\[
\begin{align*}
\frac{dx'}{\beta_1 - 1} &= [(p + \beta_2 + p u^2)dx - p u vdy], \\
\frac{dy'}{\beta_1 - 1} &= [-p u vdx + (p + \beta_2 + p u^2)dy].
\end{align*}
\]

(3.2)

subject to the requirement that

\[ p + \beta_2 + p q^2 \neq 0. \]

(3.3)

In [19] it was established that the gasdynamic system (1,2) is invariant under the 4-parameter class of the Bateman-type reciprocal transformations

\[
\begin{align*}
u' &= \frac{\beta_1 u}{p + \beta_2}, \quad v' = \frac{\beta_1 v}{p + \beta_2}, \\
p' &= \frac{\beta_4}{p + \beta_2}, \quad \rho' = \frac{\beta_3 \rho (p + \beta_2)}{p + \beta_2 + p q^2}, \\
S' &= F(S).
\end{align*}
\]

(3.4)

This result has its roots in work of Bateman on lift and drag functions in planar irrotational gasdynamics [1].

In [4], a link between a one-parameter subclass of infinitesimal reciprocal-type transformations in gasdynamics and a Lie group approach was established. It was observed that if one sets

\[ \beta_1 = \beta_2 = \beta_4 = \epsilon^{-1}, \quad \beta_3 = 1, \]

(3.5)

then the one-parameter class of reciprocal transformations

\[
\begin{align*}
u' &= \frac{u}{1 + \epsilon p}, \quad v' = \frac{v}{1 + \epsilon p}, \\
p' &= \frac{p}{1 + \epsilon p}, \quad \rho' = \frac{\rho (1 + \epsilon p)}{1 + \epsilon (p + p q^2)}, \\
S' &= F(S)
\end{align*}
\]

(3.6)

together with

\[
\begin{align*}
dx' &= \epsilon [(p + p u^2)dx - p u vdy], \\
dy' &= \epsilon [-p u vdx + (p + p u^2)dy]
\end{align*}
\]

(3.7)

4Short review of the analysis and applications of these transformations can be found in [4].
composes transformations similar to the Lie group of transformations. This observation led to establishing a method [6, 13] for constructing reciprocal transformations by using procedures developed in the group analysis method. According to [15], transformations (3.6), (3.7) are called by the group of reciprocal transformations. Similar to the classical theory of a Lie group of point transformations, it is convenient to present these transformations by the infinitesimal generators $X = F(S)\partial_S$ and $Y = \rho^2 q^2 \partial_\rho + pu\partial_u + pv\partial_v + p^2 \partial_p + (-p + \rho v^2)dx + \rho uv dy) \partial_{dx} + (\rho uv dx - (p + \rho u^2)dy) \partial_{dy}$, where $F(S)$ is an arbitrary function.

For further presentation it is convenient to introduce the following generators

$$X_1 = -v\partial_u + u\partial_v + dy\partial_{dx} + dx\partial_{dy}, \quad X_2 = dx\partial_{dx} + dy\partial_{dy},$$

$$X_3 = \rho^2 q^2 \partial_\rho + pu\partial_u + pv\partial_v + p^2 \partial_p + (-p + \rho v^2)dx + \rho uv dy) \partial_{dx} + (\rho uv dx - (p + \rho u^2)dy) \partial_{dy},$$

$$X_4 = \frac{1}{2} (2p\partial_\rho + u\partial_u + v\partial_v - dx\partial_{dx} - dy\partial_{dy}), \quad X_5 = \partial_p.$$  

One notices that among these generators only the generator $X_3 = Y$ substantially defines reciprocal transformations: the transformations corresponding to other generators are equivalent to the transformations belonging to equivalence group.

4. The first infinitesimal approach

In [23], two methods are given for constructing a group of reciprocal transformations.

The first method consists of two steps. First, using invariance of the conservation laws, one finds the coefficients of the generator of the group of reciprocal transformations related with the differentials. Then, using this coefficients, one defines the prolongation formulas for the group of reciprocal transformations. Applying the prolonged generator to the studied differential equations, one derives determining equations for the coefficients of the infinitesimal generator related with the dependent and independent variables. The general solution of the determining equations gives the set of generators of one-parameter groups of reciprocal transformations. Solving the Lie equations corresponding to these generators, one finds the one-parameter reciprocal transformations. As in the classical group analysis method, the multi-parameter transformations are constructed by the composition of the one-parameter reciprocal transformations.

Equations (3.1) can be rewritten in the form of differentials

$$S_1 = q_{11} ((p + q_{12} + \rho v^2) dx - (\rho uv + q_{13}) dy) = 0,$$

$$S_2 = q_{21} (-(\rho uv + q_{23}) dx + (p + q_{22} + \rho u^2) dy) = 0,$$

where $q_{ij}, (i = 1, 2; j = 1, 2, 3)$ are constant such that $q_{11}q_{21} \neq 0$.

Consider the transformations defined by the generator

$$X = \zeta^\rho \partial_\rho + \zeta^u \partial_u + \zeta^v \partial_v + \zeta^\rho \partial_\rho + \zeta^S \partial_S + \zeta^{dx} \partial_{dx} + \zeta^{dy} \partial_{dy},$$

where the coefficients $\zeta^\rho, \zeta^u, \zeta^v, \zeta^S$ and $\zeta^{dx}$ and $\zeta^{dy}$ are linear forms with respect to the differentials $dx$ and $dy$ with the coefficients $x\zeta^{dx}, y\zeta^{dx}, x\zeta^{dy}$ and $y\zeta^{dy}$ also depending on the variables $(x, y, \rho, u, v, p, S)$:

$$\zeta^{dx} = x\zeta^{dx} dx + y\zeta^{dx} dy, \quad \zeta^{dy} = x\zeta^{dy} dx + y\zeta^{dy} dy.$$
Requiring that equations (4.1) are invariants of the generator $X$:

$$XS_i = 0,$$

one finds that

$$
\zeta^{dy} = \Delta^{-1}((\zeta^u p v (p + q_{12} + \rho v^2) + \zeta^v \rho (p u + q_{12} u - 2 q_{23} v - \rho u v^2)
+ \zeta^v p (p u + q_{12} u - q_{23} v) - \zeta^p (q_{23} + \rho u v)) dx
+ (\zeta^u \rho (-2 p u - 2 q_{12} u + q_{23} v - \rho u v^2) + \zeta^v \rho u (q_{23} + \rho u v)
+ \zeta^v u (-p u - q_{12} u + q_{23} v) - \zeta^p (p + q_{12} + \rho v^2)) dy)
\zeta^{dx} = \Delta^{-1}((\zeta^u p v (q_{13} + \rho u v) + \zeta^v \rho (-2 p v + q_{13} u - 2 q_{22} v - \rho u v^2)
+ \zeta^v p (-p v + q_{13} u - q_{22} v) - \zeta^p (p + q_{22} + \rho u v)) dx
+ (\zeta^u \rho (p v - 2 q_{12} u + q_{22} v - \rho u v^2) + \zeta^v \rho u (q_{22} + \rho u v)
+ \zeta^v u (p v - q_{12} u + q_{22} v) - \zeta^p (q_{13} + \rho v)) dy),
$$

where

$$
\Delta = p^2 + (q_{12} + q_{22}) p + p p (u^2 + v^2) + q_{12} q_{22} - q_{13} q_{23} + \rho (q_{12} u^2 + q_{22} v^2) - (q_{13} + q_{23}) \rho u v.
$$

The next step consists of finding the coefficients $\zeta^p, \zeta^u, \zeta^v, \zeta^p$ and $\zeta^S$, satisfying the determining equations

$$(X F_i)_{12} = 0, \ (i = 1, 2, 3, 4). \quad (4.3)$$

Here $X$ is the prolongation of the generator [12] with the prolongation formulas

$$
\zeta^{fx} = D_x \zeta^f - f_x \frac{\partial \zeta^{dx}}{\partial (dx)} - f_y \frac{\partial \zeta^{dy}}{\partial (dy)}, \quad \zeta^{fy} = D_y \zeta^f - f_x \frac{\partial \zeta^{dx}}{\partial (dy)} - f_y \frac{\partial \zeta^{dy}}{\partial (dy)},
$$

where $f = \rho, \ u, \ v, \ p, \ S, \ D_x$ and $D_y$ are operators of the total derivatives with respect to $x$ and $y$

$$
D_x = \partial_x + \partial_\rho \rho_x + \partial_u u_x + \partial_v v_x + \partial_p p_x + \partial_S S_x,
D_y = \partial_y + \partial_\rho \rho_y + \partial_u u_y + \partial_v v_y + \partial_p p_y + \partial_S S_y,
$$

the derivatives $\frac{\partial \zeta^{dx}}{\partial (dx)}, \frac{\partial \zeta^{dy}}{\partial (dx)}$, $\frac{\partial \zeta^{dx}}{\partial (dy)}$ and $\frac{\partial \zeta^{dy}}{\partial (dy)}$ mean the coefficients of the 1-forms $\zeta^{dx}$ and $\zeta^{dt}$. The prolongation formulas [14] are derived by using the invariance of the differentials $df$ during the transformations. Using this invariance, one finds formulas for transformation of derivatives, and differentiating them with respect to the group-parameter and setting it to zero, one obtains the prolongation formulas [14].

Calculations show that for solving the determining equations, it is necessary to consider three cases (a) $\zeta^p = 0$, (b) $\zeta^p q_{13} \neq 0$, and (c) $\zeta^p \neq 0$ and $q_{13} = 0$.

The case $\zeta^p = 0$ gives that $X = X^h_h + X^e_e$, which means that there are no reciprocal transformations for this case.

4.1. Case $\zeta^p q_{13} \neq 0$

In this case one obtains that $q_{23} = -q_{13}, q_{22} = q_{12}$, and the general solution of the determining equations (4.3) gives that

$$X = k (X_3 + 2 q_{12} X_4 + q_{13} X_1 + (q_{12}^2 + q_{13}^2) X_5) + X^h_h + X^e_e,$$
where \( k \) is an arbitrary constant, \( h(S) \) and \( F(S) \) are arbitrary functions. For finding reciprocal transformations the generators \( X^e_k \) and \( X^e_F \) can be excluded. One first notices that \( \zeta^{dx} = -k(q_{11}^{-1}S_1 + 2q_{13}dy) \) and \( \zeta^{dy} = -k(q_{21}^{-1}S_2 - 2q_{13}dx) \). As \( S_1 \) and \( S_2 \) are invariants, then \( dx' = -\epsilon k(q_{11}^{-1}S_1 + 2q_{13}dy) + dx \) and \( dy' = -\epsilon k(q_{21}^{-1}S_2 - 2q_{13}dx) + dy \). One has that

\[
\begin{align*}
\zeta^{dx} &= -(\epsilon q_{13} + 2k_1) dx - (\rho u v - q_{13}) dy + dx, \\
\zeta^{dy} &= -\epsilon k((\rho u v + q_{13}) dx - (p + q_{13} + \rho u^2)dy) + dy.
\end{align*}
\] 

(4.5)

It is more convenient to obtain transformations of the dependent variables in cylindrical coordinates for the velocity, defined by the change

\[
u = R \sin(\theta), \quad v = R \cos(\theta).
\]

The solution of the Lie equations becomes

\[
R' = \frac{q_{13}R(\sqrt{1 + \lambda^2})}{\lambda(p + q_{12})}, \quad \theta' = \theta - \epsilon q_{13}, \quad \rho' = \rho, \quad \lambda(p + q_{12}) - q_{13},
\]

\[
p' = \frac{q_{13}p + \lambda(pq_{12} + q_{12}^2 + q_{13})}{q_{13} - \lambda(p + q_{12})}, \quad S' = S,
\]

where \( \lambda = \tan(\epsilon q_{13}) \).

4.2. Case \( \zeta^p \neq 0 \) and \( q_{13} = 0 \)

One has that \( q_{23} = 0, q_{22} = q_{12} \), and the general solution of the determining equations (4.3) gives that

\[
X = k_2(X_3 + 2q_{12}X_4 + q_{12}^2X_5) + k_1(2X_4 + 2q_{12}X_5 - X_2) + X^e_h + X^e_F,
\]

where \( k_1 \) and \( k_2 \) are arbitrary constants, \( h(S) \) and \( F(S) \) are arbitrary functions. One obtains that \( \zeta^{dx} = -k_2q_{11}^{-1}S_1 - 2k_1 \) \( dx \) and \( \zeta^{dy} = -k_2q_{21}^{-1}S_2 - 2k_1 \) \( dy \). As \( S_1 \) and \( S_2 \) are invariants, then

\[
\begin{align*}
dx &= \epsilon (k_2 (-p + q_{12} + \rho u^2)dx + \rho uvdy) - 2k_1 dx + dx, \\
dy &= \epsilon (k_2 (\rho u vdx - (p + q_{12} + \rho u^2)dy) - 2k_1 dy) + dy.
\end{align*}
\] 

(4.7)

The formulas for changing the dependent variables depend on \( k_1 \). For \( k_1 \neq 0 \), one finds

\[
\begin{align*}
u' &= u, \quad v' = v, \\
\rho' &= \rho, \quad p' = p, \quad S' = S.
\end{align*}
\] 

(4.8)

For \( k_1 = 0 \), one has

\[
\begin{align*}
u' &= \frac{u}{1 - a(p + q_{12})}, \quad v' = \frac{v}{1 - a(p + q_{12})}, \\
\rho' &= \rho, \quad p' = \frac{p - aq_{12}(p + q_{12})}{1 - a(p + q_{12})}, \quad S' = S.
\end{align*}
\] 

(4.9)

where \( a = k_2 \).
5. Application of the second method

In the previous sections the differential forms \( S_1 \) and \( S_2 \) are used for constructing reciprocal transformations, while this section shows that they are not needed. In the second method there is no separation in the two steps like in the first method, as the second method does not use the conservation laws. The prolongation formulas of the generator of the group of reciprocal transformations are constructed by using the general form of the coefficients related with the differentials of the generator of the group of reciprocal transformations. This method is a little bit more complicated, but it can also be applied to systems without knowing their conservation laws.

5.1. Group of reciprocal transformations of (1,2)

Consider the generator

\[
X = \zeta^p \partial_p + \zeta^v \partial_v + \zeta^p \partial_p + \zeta^S \partial_S + \zeta^{dt} \partial_{dt} + \zeta^{dx} \partial_{dx},
\]

where the coefficients \( \zeta^p, \zeta^v, \zeta^p \) and \( \zeta^S \) depend on the variables \((x, y, \rho, u, v, p, S)\), \( \zeta^{dx} \) and \( \zeta^{dy} \) are linear 1-forms with respect to \( dx \) and \( dy \) with the coefficients also depending on the variables \((x, y, \rho, u, v, p, S)\):

\[
\zeta^{dx} = x \zeta^{dx} dx + y \zeta^{dx} dy, \quad \zeta^{dy} = x \zeta^{dy} dx + y \zeta^{dy} dy.
\]

Requiring that equations (1.2) compose an invariant manifold of the generator \( X \), one derives the determining equations

\[
\left( \rho \frac{\partial \zeta^u}{\partial y} + \rho \frac{\partial \zeta^v}{\partial y} + \frac{\partial \zeta^u}{\partial x} + u \zeta^p + u \zeta^p + v \zeta^p + v \zeta^p \right)_{1.2} = 0,
\]

\[
\left( \rho u \frac{\partial \zeta^u}{\partial x} + \rho u \frac{\partial \zeta^u}{\partial x} + \rho v \frac{\partial \zeta^u}{\partial x} + u u \zeta^p + u u \zeta^p + v \zeta^p + v \zeta^p \right)_{1.2} = 0,
\]

\[
\left( \rho u \frac{\partial \zeta^u}{\partial x} + \rho u \frac{\partial \zeta^u}{\partial x} + \rho v \frac{\partial \zeta^u}{\partial x} + u u \zeta^p + u u \zeta^p + v \zeta^p + v \zeta^p \right)_{1.2} = 0,
\]

\[
\left( S_x \zeta^u + S_y \zeta^v + u \zeta^S + v \zeta^S \right)_{1.2} = 0,
\]

where the sign \( 1.2 \) has the usual meaning of considering the relations in parenthesis on the manifold defined by equations (1.2), where for \( \zeta^f_x \) and \( \zeta^f_y \), \((f = \rho, u, v, p, S)\) one has to apply the prolongation formulas (4.4).

Beside equations (5.2) one also has to satisfy the equations \( D_x (y \zeta^{dx}) - D_y (x \zeta^{dx}) = 0 \) and \( D_x (x \zeta^{dy}) - D_y (y \zeta^{dy}) = 0 \), which are

\[
-x \zeta^{dx} S_y - x \zeta^{dx} p_y - x \zeta^{dx} \rho_y - x \zeta^{dx} u_y - x \zeta^{dx} v_y - x \zeta^{dx} + y \zeta^{dx} S_x + y \zeta^{dx} p_x + y \zeta^{dx} \rho_x + y \zeta^{dx} u_x + y \zeta^{dx} v_x + y \zeta^{dx} = 0,
\]

\[
-x \zeta^{dy} S_y - x \zeta^{dy} p_y - x \zeta^{dy} \rho_y - x \zeta^{dy} u_y - x \zeta^{dy} v_y - x \zeta^{dy} + y \zeta^{dy} S_x + y \zeta^{dy} p_x + y \zeta^{dy} \rho_x + y \zeta^{dy} u_x + y \zeta^{dy} v_x + y \zeta^{dy} = 0.
\]

The method of solving the determining equations (5.2), (5.3) is similar as in the classical group analysis method [11]. Calculations show that, solving the determining equations (5.2), (5.3), one obtains that

\[
X = \sum_{i=2}^{5} k_i X_i + X^e_h + X^e_F,
\]
where and \( h(S) \) and \( F(S) \) are arbitrary functions, \( k_i, (i = 1, 2, \ldots, 5) \) are arbitrary constants, and the generators \( X_i, (i = 1, 2, \ldots, 5) \) are defined by formulas (3.8):

\[
    X_1 = -v\partial_u + u\partial_v - dy\partial_{dx} + dx\partial_{dy}, \quad X_2 = dx\partial_{dx} + dy\partial_{dy},
    
    X_3 = \rho^2 q^2 \partial_{\rho} + pu\partial_u + pv\partial_v + p^2 \partial_p + \left( - (p + \rho v^2) dx + \rho uv dy \right) \partial_{dx} + \left( \rho uv dx - (p + \rho u^2) dy \right) \partial_{dy},
    
    X_4 = \frac{1}{2} \left( 2p \partial_p + u\partial_u + v\partial_v - dx\partial_{dx} - dy\partial_{dy} \right), \quad X_5 = \partial_p.
\]

5.2. Algebraic properties of the Lie algebra \( L_{rt} = \{ X_1, X_2, X_3, X_4, X_5, X_h, X_F \} \)

The commutator table of \( L_{rt} \) is

|       | \( X_1 \) | \( X_2 \) | \( X_3 \) | \( X_4 \) | \( X_5 \) | \( X_h \) | \( X_F \) |
|-------|--------|--------|--------|--------|--------|--------|--------|
| \( X_1 \) | 0      | 0      | 0      | 0      | 0      | 0      | 0      |
| \( X_2 \) | 0      | 0      | 0      | 0      | 0      | 0      | 0      |
| \( X_3 \) | 0      | 0      | 0      | \(-X_3\) | \(-X_4\) | 0      | 0      |
| \( X_4 \) | 0      | 0      | \( X_3 \) | 0      | \(-X_5\) | 0      | 0      |
| \( X_5 \) | 0      | 0      | \( X_4 \) | \( X_5 \) | 0      | 0      | 0      |
| \( X_h \) | 0      | 0      | 0      | 0      | 0      | \(-X_{h1}\) | 0      |
| \( X_F \) | 0      | 0      | 0      | 0      | 0      | \( X_{h1} \) | 0      |

where \( h_1 = h'F \). From the commutator table, one concludes that the generators \( X_1 \) and \( X_2 \) compose the center, the generators \( X_3, X_4 \) and \( X_5 \) compose an ideal, the derivatives of \( L_{rt} \) are

\[ L_{rt}' = [L_{rt}, L_{rt}] = \{ X_3, X_4, X_5, X_h \}, \quad L_{rt}'' = [L_{rt}', L_{rt}'] = \{ X_3, X_4, X_5 \}. \]

6. Complete set of reciprocal transformations

For finding discrete reciprocal transformations, it is adapted the method proposed in [12] for finding discrete symmetries, and extended in [14] for equivalence transformations. The main idea of the method consists of using invariant sets of a group of reciprocal transformations.

Let be given a system of differential equations \( (S) \), and \( L \) be its (maximal) Lie algebra of reciprocal transformations, where \( Aut(L) \) is the automorphism group. Any reciprocal transformation \( T \) provides an automorphism \( T^* \in Aut(L) \) of the Lie algebra \( L \). The transformation \( T^* \) is defined by the change of a generator under the transformation \( T \). Notice that the change of the generator of a group of reciprocal transformations under a reciprocal transformation follows the same formulas as the change of the generator of a group of point transformations under a point transformation [11]. The property that \( T^* \in Aut(L) \) follows from commutativeness of the operation of commutation of generators and the operation of the change of the variables.

For further study we use the following notations\(^5\). A megaideal \( \tau \) of a Lie algebra \( L \) is a vector subspace of \( L \), which is invariant under any mapping from the automorphism group \( Aut(L) \). That is, for any megaideal \( \tau \) of \( L \), and any transformation \( A \) from \( Aut(L) \) one has that \( A\tau = \tau \).

Some megaideals of \( L \) can be computed without knowing \( Aut(L) \). In particular, the center and the derivative of a Lie algebra are its megaideals [14]. Hence, \( L' = [L, L] \) is a megaideal.

\(^5\)See details in [14] and references therein.
We also use the property that a megaideal τ of a megaideal τ1 of L is also a megaideal of L. In particular, $L'' = (L')'$ is also a megaideal [14]. Thus, one obtains the megaideals of $L_{rt}$:

$L'_{rt} = \{X_3, X_4, X_5, X_h\}$, $L''_{rt} = \{X_3, X_4, X_5\}$, $C_1 = \{X_1\}$, $C_2 = \{X_2\}$.

Notice also that the nonzero structure constants for $L''_{rt}$ are

$c_{34}^3 = -1$, $c_{35}^4 = -1$, $c_{45}^5 = -1$.

Consider an invertible transformation $T$ defined by the relations

$\rho' = R(\rho, u, v, p, S)$, $u' = U(\rho, u, v, p, S)$, $v' = V(\rho, u, v, p, S)$,

$p' = P(\rho, u, v, p, S)$, $S' = H(\rho, u, v, p, S)$,

$dx' = x f^dx(\rho, u, v, p, S) dx + y f^dy(\rho, u, v, p, S) dy$,

$dy' = x f^dy(\rho, u, v, p, S) dx + y f^dx(\rho, u, v, p, S) dy$.

The transformation $T$ is a reciprocal transformation if

$-R Ud'y' + RV dx' = 0$, $(P + RV^2)dx' - RUV dy' = 0$,

$RUV dx' - (RU^2 + P)dy' = 0$, $RHU dy' - RVH dx' = 0$.

The transformation $T_*$ acts on a generator $X \in L_{rt}$

$X = \zeta^\rho \partial_\rho + \zeta^v \partial_v + \zeta^p \partial_p + \zeta^S \partial_S + \zeta^t \partial_t + \zeta^x \partial_x$, (6.3)

as follows

$T_*X = \zeta'^\rho \partial_\rho' + \zeta'^v \partial_v' + \zeta'^p \partial_p' + \zeta'^S \partial_S' + \zeta'^t \partial_t' + \zeta'^x \partial_x'$, (6.4)

where

$\zeta'^\rho = X \rho'$, $\zeta'^v = X v'$, $\zeta'^p = X p'$, $\zeta'^S = X S'$, $\zeta'^t = X (dt')$, $\zeta'^x = X (dx')$.

One of the sets of equations for the invertible transformation $T$ to be a reciprocal transformation is defined by the conditions that $dx'$ and $dy'$ are differentials:

$D_y(x f^dx) = D_x(y f^dx)$, $D_y(x f^dy) = D_x(y f^dy)$.

(6.5)

Analysis of these equations is similar to the analysis of determining equations. As these equations have to be satisfied for any solution of equations [12], then substituting the main derivatives of equations (1.2) into (6.3), and splitting them with respect to parametrical derivatives, one derives the set of equations, which we also call determining equations.

6.1. Use of the megaideal $L''_{rt}$

Let $X$ belong to the megaideal $L''_{rt} = \{X_3, X_4, X_5\}$. As $T_* \in Aut(L''_{rt})$, then $T_*X \in L''_{rt}$, which means that there exist constants such that

$T_*X = a_3 X_3' + a_4 X_4' + a_5 X_5'$,

10
where prime in $X'_i$ means that it is the generator $X_i$ with primed variables:

$$
X'_3 = \rho'^2 + \rho'u\partial_{u'} + p'\rho'\partial_{\rho'} + \rho'v\partial_{v'} + p'\partial_{p'}
$$

$$
+ (p' + \rho'v^2)dx' + p'u'\partial_{dx'} + (\rho'u'\partial_{dx'} - (p' + \rho'u^2)dy')\partial_{dy'},
$$

$$
X'_4 = \frac{1}{2}(2p'\partial_{p'} + u'\partial_{u'} + v'\partial_{v'} - dx'\partial_{dx'} - dy'\partial_{dy'}),
$$

$$
X'_5 = \partial_{\rho'}.
$$

In particular,

$$
T^*X_i = a_{3i}X'_3 + a_{4i}X'_4 + a_{5i}X'_5, \quad (i = 3, 4, 5),
$$

(6.6)

where the matrix

$$
A = \begin{pmatrix} 
  a_{33} & a_{34} & a_{35} \\
  a_{43} & a_{44} & a_{45} \\
  a_{53} & a_{54} & a_{55}
\end{pmatrix}
$$

is nonsingular.

The constants $a_{ij}, \quad (i, j = 3, 4, 5)$ have to satisfy the following relations. As $T^* \in \text{Aut}(L''_{\text{rt}})$, then

$$
[T^*X_i, T^*X_j] = T*[X_i, X_j] = \sum_{k=3}^{5} c_{ij}^k T^*X_k, \quad (i, j = 3, 4, 5),
$$

which lead to the conditions

$$
\sum_{k=3}^{5} \sum_{s=3}^{5} \sum_{n=3}^{5} a_{ik}a_{js}c_{ks}^n X'_n = \sum_{k=3}^{5} \sum_{n=3}^{5} c_{ij}^k a_{nk}X'_n, \quad (i, j = 3, 4, 5)
$$

or

$$
\sum_{k=3}^{5} \sum_{s=3}^{5} a_{ik}a_{js}c_{ks}^n = \sum_{k=3}^{5} c_{ij}^k a_{nk}, \quad (i, j, n = 3, 4, 5) \quad (6.7)
$$

Calculations show that equations (6.6) define the derivatives with respect $\rho, u$ and $p$ of all unknown functions (6.1). The solution of equations (6.6) and the representation of equations (6.7) are given in the Appendix. Analysis of equations (6.7) leads to the study of two cases (a) $a_{35} = 0$, and (b) $a_{35} \neq 0$.

All calculations were performed using symbolic manipulation system Reduce [24].

6.2. Case $a_{35} = 0$

In this case, by virtue of $\det A \neq 0$, one derives from equations (6.7) that $a_{33} \neq 0$ and

$$
a_{34} = 0, \quad a_{43} = a_{54}a_{33}, \quad a_{44} = 1, \quad a_{45} = 0, \quad a_{53} = \frac{1}{2}a_{54}a_{33}, \quad a_{55} = a_{33}^{-1}. \quad (6.8)
$$

6.2.1. Using the megaideal $\{X_1\}$

The generator $X_1$ is a center of the Lie algebra $L$. As any center is a megaideal, then there exists constant $a_{11}$ such that

$$
T^*X_1 = a_{11}X'_1.
$$

(6.9)

Calculations give that the latter equation defines derivatives with respect $v$ of all unknown functions:
\[ R_v = 0, \quad U_v = \frac{U v - V a_{11} u}{u^2 + v^2}, \quad V_v = \frac{U a_{11} u + V v}{u^2 + v^2}, \quad P_v = \frac{2 P a_{33} v + 2 v (a_{54} a_{33} - p)}{a_{33} (u^2 + v^2)}, \quad H_v = 0, \]

(6.10)

\[ x f^d x = -u \frac{y f^d x + x f^d y a_{11}}{u^2 + v^2}, \quad y f^d x = u \frac{x f^d x - y f^d y a_{11}}{u^2 + v^2}, \]

\[ x f^d y = u \frac{x f^d x a_{11} - y f^d y}{u^2 + v^2}, \quad y f^d y = u \frac{y f^d x a_{11} + x f^d y}{u^2 + v^2}. \]

The remaining equations are equations (6.5).

6.2.2. Analysis of equations (6.5)

Splitting these equations with respect to parametric derivatives of equations (1.2), one obtains the determining equations

\[ a^2_{11} = 1, \quad y f^d x = -x f^d y a_{11}, \quad x f^d x = y f^d y a_{11}, \]

(6.11)

\[ x f^d y a_{11} (p + \rho u^2 - a_{33} (P + RV^2 + a_{54})) + y f^d y (a_{11} \rho uv - RUV a_{33}) = 0, \]

\[ x f^d y (a_{11} \rho uv + RUV a_{33}) + y f^d y a_{11} (p + \rho u^2 - a_{33} (P + RV^2 + a_{54})) = 0, \]

(6.12)

\[ -x f^d y a_{11} (RV a_{33} + a_{11} \rho uv) + y f^d y (p + \rho u^2 - a_{33} (P + RU^2 + a_{54})) = 0, \]

\[ -x f^d y a_{11} (p + \rho v^2 - a_{33} (P + RU^2 + a_{54})) + y f^d y (a_{11} \rho uv - RUV a_{33}) = 0, \]

Considering equations (6.12) as a system of linear algebraic equations with respect to \( x f^d y \) and \( y f^d y \), and because \((x f^d y)^2 + (y f^d y)^2 \neq 0\), one derives that

\[ P = \frac{1}{2} \left( \frac{2 p + \rho (u^2 + v^2)}{a_{33}} - R (U^2 + V^2 + 2a_{54}) \right), \quad R = \mu \rho \frac{u^2 + v^2}{a_{33} (U^2 + V^2)}, \]

(6.13)

where \( \mu^2 = 1 \), and equations (6.12) are reduced to the equations

\[ \mu \left( x f^d y (V^2 - U^2) + 2 y f^d y U V a_{11} \right) (u^2 + v^2) + (-2 y f^d y u v + x f^d y (v^2 - u^2)) (V^2 + U^2) = 0, \]

(6.14)

\[ (-2 x f^d y u v + y f^d y (u^2 - v^2)) (U^2 + V^2) + \mu \left( y f^d y (V^2 - U^2) - 2 x f^d y U V a_{11} \right) (u^2 + v^2) = 0. \]

Integrating equations (6.6)\(^6\) and (6.10) for \( U \) and \( V \), one derives that

\[ U = \rho^{(1-\mu)/2} (\varphi_1 u + \varphi_2 v), \quad V = a_{11} \rho^{(1-\mu)/2} (\varphi_1 v - \varphi_2 u), \]

(6.15)

where \( \varphi_1(S) \) and \( \varphi_2(S) \) are arbitrary functions such that \( \varphi_1^2 + \varphi_2^2 \neq 0 \). Equations (6.13) become

\[ P = \frac{p}{a_{33}} - a_{54} + \frac{(1 - \mu)}{2 a_{33}} \rho (u^2 + v^2), \quad R = \frac{\mu \rho^\mu}{a_{33} (\varphi_1^2 + \varphi_2^2)}. \]

\(^6\)Presented in Appendix.
Equations (6.11) give
\[ dx' = a_{11}(y f dy dx - x f dy dy), \quad dy' = x f dy dx + y f dy dy. \] (6.17)

Further analysis of finding the coefficients \( x f dy \) and \( y f dy \) depends on \( \mu \).
If \( \mu = 1 \), then equations (6.14) provide that
\[ x f dy \varphi_1 + y f dy \varphi_2 = 0. \]
In symmetric form one can represent a solution of the latter equation as
\[ y f dy = \varphi_1 \psi^{-1}, \quad x f dy = -\varphi_2 \psi^{-1}, \]
where it can it is also obtained that \( \psi(S) \) is an arbitrary function. Equations (6.5) require that
\[ \varphi_1 = \alpha \psi, \quad \varphi_2 = \beta \psi, \]
where \( \alpha \) and \( \beta \) are constant such that \( \alpha^2 + \beta^2 \neq 0 \).

Thus, one obtains that the transformation can be written in the form
\[ P = \frac{p}{a_{33}} - a_{54}, \quad R = \frac{\rho}{a_{33} \psi^2(\alpha^2 + \beta^2)}, \quad U = \psi(\alpha u + \beta v), \quad V = a_{11} \psi(\alpha v - \beta u), \quad H = F(S), \]
\[ dx' = a_{11}(\alpha dx + \beta dy), \quad dy' = -\beta dx + \alpha dy. \] (6.18)

As the coefficients \( x f dx, y f dx, x f dy \) and \( y f dy \) are constant, then the transformation (6.18) is equivalent to an equivalence transformation: in particular, composition of (2.2) with the rotation.

If \( \mu = -1 \), then (6.15), (6.16) become
\[ P = \frac{p}{a_{33}} - a_{54} + \frac{1}{a_{33}} \rho (u^2 + v^2), \quad R = -\frac{1}{a_{33} \rho (\varphi_1^2 + \varphi_2^2)}, \]
\[ U = \rho (\varphi_1 u + \varphi_2 v), \quad V = a_{11} \rho (\varphi_1 v - \varphi_2 u). \] (6.19)

Equations (6.14) give that
\[ x f dy \varphi_2 - y f dy \varphi_1 = 0, \]
In symmetric form one can represent a solution of the latter equation as \( y f dy = \varphi_2 \psi^{-1}, \quad x f dy = \varphi_1 \psi^{-1} \). Equations (6.5) require that \( \varphi_1 = \alpha \psi, \quad \varphi_2 = \beta \psi \). Equations (6.2) in this case are only satisfied if the original solution is isentropic and irrotational. Hence, the case \( \mu = -1 \) also does not provide a reciprocal transformation.

6.3. Case \( a_{35} \neq 0 \)

For this case
\[ a_{33} = \frac{a_{34}^2}{2a_{35}}, \quad a_{43} = \frac{a_{34}(a_{45}a_{34} - 2a_{35})}{2a_{35}^2}, \quad a_{44} = \frac{a_{34}a_{34} - 2a_{35}}{a_{35}}, \]
\[ a_{53} = \frac{a_{35}^2 a_{34}^2 - 4a_{45}a_{35}a_{34} + 4a_{35}^2}{4a_{35}^3}, \quad a_{54} = \frac{a_{34}(a_{45}a_{34} - 2a_{35})}{2a_{35}}, \quad a_{55} = \frac{a_{35}^2}{2a_{35}}, \]
and analysis of the equations defining the reciprocal transformations is similar to the case \( a_{35} = 0 \), but more cumbersome. Because of their cumbersomeness, we only describe the main steps of the finding of the reciprocal transformations.
Using the megaideal \( \{X_1\} \), one finds the derivatives \( R_v, U_v, V_v, P_v, x f_v^{dx}, y f_v^{dx}, x f_v^{dy}, \) and \( y f_v^{dy} \). After that equations (6.5) give the relation \( \nu x f_S^{dx} + y f_S^{dx} = 0 \), and an algebraic system of ten homogeneous linear equations with respect to \( x f^{dx}, y f^{dx}, x f^{dy}, \) and \( y f^{dy} \). As \( (x f^{dx})(y f^{dy}) - (y f^{dx})(x f^{dy}) \neq 0 \), then the rank \( r \) of the matrix with respect to these variables satisfies the inequality \( r \leq 3 \). From the analysis of the minors of latter system of linear homogeneous equations one finds \( P \) and \( R \). Integrating the overdetermined system of equations for the functions \( U \) and \( V \), one funds them. Substituting all the expressions of \( R, U, V, \) and \( P \) into a linear system for \( x f^{dx}, y f^{dx}, x f^{dy}, \) and \( y f^{dy} \), one finds the reciprocal transformations

\[
\begin{align*}
R &= \frac{2 \rho(p - g)}{\psi^2 a_{35}(\alpha^2 + \beta^2)(p + \rho q^2 - g)}, \quad P = -\frac{a_{45}}{a_{35}} - \frac{2}{a_{35}(p - g)}, \\
U &= \frac{\psi(\alpha v + \beta u)}{p - g}, \quad V = a_{11} \frac{\psi(-\alpha u + \beta v)}{p - g}, \quad H = F, \\
dx' &= k ((\alpha \rho uv - \beta(p + \rho v^2 - g)) \, dx + (-\alpha(p + \rho u^2 - g) + \beta \rho uv) \, dy), \\
dy' &= k a_{11} ((\alpha(p + \rho v^2 - g) + \beta \rho uv) \, dx - (\alpha \rho uv + \beta(p + \rho u^2 - g)) \, dy),
\end{align*}
\]

where \( g = a_{34}a_{35}^{-1}, a_{11}^2 = 1, \alpha, \beta, \) and \( k \) are constant, \( \psi(S) \) and \( F(S) \) are arbitrary functions. Recall that \( a_{34}, \) and \( a_{45} \) are arbitrary constants, and \( a_{35} \neq 0 \).

Notice that because of the equivalence transformation corresponding to the involution \( E_2 \) and the rotation, one can assume that \( a_{11} = 1 \) and \( \alpha = 0 \). By virtue of the equivalence transformation (2.2), one can reduce \( \psi \) from formulas (6.21). Introducing the constants \( \beta_i, (i = 1, 2, 3, 4) \):

\[
\beta = \frac{2}{\beta_1 \beta_3}, \quad a_{34} = -\frac{2 \beta_2}{\beta_1^2 \beta_3}, \quad a_{35} = \frac{2}{\beta_1^2 \beta_3}, \quad a_{45} = -\frac{2 \beta_4}{\beta_1^2 \beta_3}, \quad k = -\frac{\beta_3}{2},
\]

formulas (6.21) coincide with (3.2), (3.4).

From the above study one can conclude that for finding all reciprocal transformations it was sufficient to use the megaideal \( L_v' = \{X_3, X_4, X_5\} \) and the center \( \{X_1\} \). The final results obtained can be formulated as follows.

**Theorem 6.1.** The complete set of reciprocal transformations of the two-dimensional stationary gas dynamics equations, considered up to the equivalence transformations corresponding to (2.1) and the involution \( E_2 \), consists of the transformations (3.2), (3.4):

\[
\begin{align*}
u' &= \frac{\beta_1 u}{p + \beta_2}, \quad v' = \frac{\beta_1 v}{p + \beta_2}, \\
p' &= \beta_4 - \frac{\beta_2 \beta_3}{p + \beta_2}, \quad \rho' = \frac{\beta_3 \rho(p + \beta_2)}{p + \beta_2 + \rho q^2}, \quad S' = F(S), \\
dx' &= \beta_1^{-1}[(p + \beta_2 + \rho v^2) \, dx - \rho uv dy], \\
dy' &= \beta_3^{-1}[\rho v dx + (p + \beta_2 + \rho u^2) \, dy].
\end{align*}
\]

**Remark 6.2.** The transformations (6.23), (6.22) can be further simplified by the equivalence transformations corresponding to (2.1). In particular, using the transformation corresponding

\^As \( \alpha^2 + \beta^2 \neq 0 \), one has \( \beta \neq 0 \).
to $X_e^6$, one can assume that $\beta_2 = 0$. Because of the transformation corresponding to $X_e^5$, one can assume that $\beta_1 = 1$. The reciprocal transformations (6.23), (6.22) become

\[
\begin{align*}
    u' &= \frac{u}{p}, \quad v' = \frac{v}{p}, \quad p' = \frac{\tilde{\beta}_3}{p} - \frac{\tilde{\beta}_4}{p}, \quad \rho' = \frac{\tilde{\beta}_3 \rho p}{p + \rho q^2}, \quad S' = F(S),
    \\
    dx' &= (p + \rho v^2)dx - \rho uvdv, \quad dy' = -\rho uvdx + (p + \rho u^2)dy.
\end{align*}
\]

7. Conclusions

Two methods for constructing a group of reciprocal transformations are presented in the paper. These methods are demonstrated by the two-dimensional stationary gas dynamics equations. Both methods use the infinitesimal approach. The first method requires two properties to be satisfied. The first property is that two conservation laws written in the form of differentials are required to be invariant under these transformations. This property gives the representation of the coefficients of the generator corresponding to the differentials. Using these coefficients, the prolongation of the generator is obtained. The second method provides a generalization of the first one, where none of the assumptions about the differentials are required. This method can also be applied to systems without knowing their conservation laws. The solution obtained by the second method allowed us to state the theorem about all reciprocal transformations for the two-dimensional stationary gas dynamics equations.

The proposed methods provide systematic tools for finding reciprocal transformations. The developed approach can be also extended to equations with more than two independent variables.

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Appendix

Equations (6,3) can be written in the forms

\[ R_\rho = (R^2 U^2 (a_{35} p^2 - 2a_{34} p + 2a_{33}) + R^2 V^2 (a_{35} p^2 - 2a_{34} p + 2a_{33}))/(2\rho^2 (u^2 + v^2)), \]

\[ R_u = (-R_v v + R^2 U^2 (-a_{35} p + a_{34}) + R^2 V^2 (-a_{35} p + a_{34}))/u, \]

\[ R_p = R^2 a_{35} (U^2 + V^2)/2, \]

\[ U_\rho = (PU(a_{35} p^2 - 2a_{34} p + 2a_{33}) + U(a_{45} p^2 - 2a_{44} p + 2a_{43}))/(2\rho^2 (u^2 + v^2)), \]

\[ U_u = (-U_v v + PU(-a_{35} p + a_{34}) + U(-a_{45} p + a_{44}))/u, \]

\[ U_p = (PU a_{35} + U a_{45})/2, \]

\[ V_\rho = (PV(a_{35} p^2 - 2a_{34} p + 2a_{33}) + V(a_{45} p^2 - 2a_{44} p + 2a_{43}))/(2\rho^2 (u^2 + v^2)), \]

\[ V_u = (-V_v v + PV(-a_{35} p + a_{34}) + V(-a_{45} p + a_{44}))/u, \]

\[ V_p = (PV a_{35} + V a_{45})/2, \]

\[ P_\rho = (P^2 (a_{35} p^2 - 2a_{34} p + 2a_{33}) + 2P(a_{45} p^2 - 2a_{44} p + 2a_{43}) + 2a_{55} p^2 - 2a_{54} p + 2a_{53}))/(2\rho^2 (u^2 + v^2)), \]

\[ P_u = (-P_v v + P^2 (-a_{35} p + a_{34}) + 2P(-a_{45} p + a_{44}) + 2(-a_{55} p + a_{54}))/u, \]

\[ P_p = (P^2 a_{35} + 2P a_{45} + 2a_{55})/2, \]

\[ H_\rho = 0, \quad H_u = -H_v v/u, \quad H_p = 0, \]

\[ x f^d_p = (x f^d x P(-a_{35} p^2 + 2a_{34} p - 2a_{33}) + x f^d x R^2 V(-a_{35} p^2 + 2a_{34} p - 2a_{33}) + x f^d x (-a_{45} p^2 + 2a_{44} p - 2a_{43}) + 2u f^d x \rho u v )/(2\rho^2 (u^2 + v^2)), \]

\[ x f^d_u = (-x f^d v + x f^d x P(a_{35} p - a_{34}) + x f^d x R^2 V(a_{35} p - a_{34}) + x f^d x (a_{45} p - a_{44} + 1) + x f^d y RUV(-a_{35} p + a_{34}))/u, \]

\[ y f^d_p = (-x f^d x P a_{35} - x f^d x R^2 a_{35} - x f^d x a_{45} + x f^d y RUV a_{35})/2, \]

\[ y f^d_x = (-2 y f^d x \rho u v + y f^d x P(-a_{35} p^2 + 2a_{34} p - 2a_{33}) + y f^d x R^2 V(-a_{35} p^2 + 2a_{34} p - 2a_{33}) + y f^d x (-a_{45} p^2 + 2a_{44} p - 2a_{43}) + 2u f^d x \rho u v )/(2\rho^2 (u^2 + v^2)), \]

\[ y f^d_u = (-y f^d v + y f^d x P(a_{35} p - a_{34}) + y f^d x R^2 V(a_{35} p - a_{34}) + y f^d x (a_{45} p - a_{44} + 1) + y f^d y RUV(-a_{35} p + a_{34}))/u, \]

\[ y f^d_p = (-y f^d x P a_{35} - y f^d x R^2 a_{35} - y f^d x a_{45} + y f^d y RUV a_{35})/2, \]

\[ x f^d_y = (x f^d x RUV(a_{35} p^2 - 2a_{34} p + 2a_{33}) + x f^d x P(-a_{35} p^2 + 2a_{34} p - 2a_{33}) + x f^d y RUV a_{35} - x f^d y a_{45} - x f^d y a_{35} - x f^d y a_{45})/2, \]

\[ x f^d_u = (-x f^d v + x f^d x RUV(-a_{35} p + a_{34}) + x f^d y P(a_{35} p - a_{34}) + x f^d y RUV a_{35} + x f^d y (a_{45} p - a_{44} + 1))/u, \]

\[ x f^d_p = x f^d y = (x f^d x RUV a_{35} - x f^d y P a_{35} - x f^d y RUV a_{35} - x f^d y a_{45})/2, \]

\[ y f^d_y = (y f^d x RUV(a_{35} p^2 - 2a_{34} p + 2a_{33}) + a_{45} p^2 - 2a_{44} p + 2a_{43}) + 2a_{55} p^2 - 2a_{54} p + 2a_{53} )/(2\rho^2 (u^2 + v^2)), \]

\[ 16 \]
\[ y f_u^d = (-y f_v^d v + y f_x^R U V (-a_{35} p + a_{34}) + y f_y^d P (a_{35} p - a_{34}) + y f_y^d R U^2 (a_{35} p - a_{34})) + y f_y^d (a_{45} p - a_{44} + 1))/u, \]
\[ y f_p^d = (y f_x^R U V a_{35} - y f_y^d P a_{35} - y f_y^d R U^2 a_{35} - y f_y^d a_{45})/2, \]

Equations (6.7) are

\[ a_{44} a_{33} - a_{43} a_{34} - a_{33} = 0, \quad a_{54} a_{33} - a_{53} a_{34} - a_{43} = 0, \quad a_{54} a_{33} - a_{53} a_{44} - a_{53} = 0, \]
\[ a_{45} a_{33} - a_{43} a_{35} - a_{34} = 0, \quad a_{55} a_{33} - a_{53} a_{35} - a_{44} = 0, \quad a_{55} a_{43} - a_{54} - a_{53} a_{45} = 0, \]
\[ a_{45} a_{34} - a_{44} a_{35} - a_{35} = 0, \quad a_{55} a_{34} - a_{54} a_{35} - a_{45} = 0, \quad a_{55} a_{44} - a_{55} - a_{54} a_{45} = 0. \]

References

[1] H. Bateman. The lift and drag functions for an elastic fluid in two-dimensional irrotational flow. *Proc. Nat. Acad. Sci.*, 24A:246–251, 1938.

[2] C. Rogers and W. F. Shadwick. *Bäcklund Transformations and Their Applications*. Academic Press, Mathematics in Science and Engineering Series, New York, 1982.

[3] A. M. Meirmanov, V. V. Pukhnachov, and S. I. Shmarev. *Evolution Equations and Lagrangian Coordinates*. Walter de Gruyter, New York, 1997.

[4] N. H. Ibragimov and C. Rogers. On infinitesimal reciprocal-type transformations in gasdynamics. Lie group connections and nonlinear self-adjointness. *Ufa Mathematical Journal*, 4(4):196–207, 2012.

[5] C. Rogers. Reciprocal relations in non-steady one-dimensional gasdynamics. *Zeit. ang. Math. Phys.*, 19:58–63, 1968.

[6] S. V. Meleshko and C. Rogers. Reciprocal transformations in relativistic gasdynamics. Lie group connections. *Open Communications in Nonlinear Mathematical Physics*, 1:4, 2021.

[7] C. Rogers and T. Ruggeri. On invariance in 1+1-dimensional isentropic relativistic gasdynamics. *Wave Motion*, 94:102527, 2020.

[8] C. Rogers, T. Ruggeri, and W. K. Schief. On relativistic gasdynamics: invariance under a class of reciprocal-type transformations and integrable Heisenberg spin connections. *Proc. R. Soc. A*, 476:20200487, 2020. https://doi.org/10.1098/rspa.2020.0487.

[9] J G Kingston and C Sophocleous. On form-preserving point transformations of partial differential equations. *Journal of Physics A: Mathematical and General*, 31(6):1597–1619, feb 1998.

[10] O. O. Vaneeva, A. G. Johnpillai, R. O. Popovych, and C. Sophocleous. Enhanced group analysis and conservation laws of variable coefficient reaction-diffusion equations with power nonlinearities. *J. Math. Anal. Appl.*, 330:1363–1386, 2007.

[11] L. V. Ovsiiannikov. *Group Analysis of Differential Equations*. Nauka, Moscow, 1978. English translation, Ames, W.F., Ed., published by Academic Press, New York, 1982.
[12] P. E. Hydon. Discrete point symmetries of ordinary differential equations. *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.*, 454:1961–1972, 1998.

[13] P. E. Hydon. How to construct the discrete symmetries of partial differential equations. *Eur. J. Appl. Math.*, 11:515–527, 2000.

[14] E. Dos Santos Cardoso-Bihlo, A. Bihlo, and R. O. Popovych. Algebraic method for finding equivalence groups. *J. Phys. Conf. Ser.*, 621:012001, 2015.

[15] S. V. Meleshko. Methods for constructing reciprocal transformations. *Wave Motion*. Submitted.

[16] E. M. Dos Santos Cardoso-Bihlo and R. O. Popovych. Complete point symmetry group of the barotropic vorticity equation on a rotating sphere. *J. Engrg. Math.*, 82:31–38, 2013.

[17] S. Kontogiorgis, R. O. Popovych, and C. Sophocleous. Enhanced symmetry analysis of two-dimensional burgers system. *Acta Appl. Math.*, 163:91–128, 2019.

[18] A. Bihlo, N. Poltavets, and R. O. Popovych. Lie symmetries of two-dimensional shallow water equations with variable bottom topography. *Chaos*, (073132), 2020. doi: 10.1063/5.0007274.

[19] G. Power and P. Smith. Reciprocal properties of plane gas flows. *J. Math. Mech.*, 10:349–361, 1961.

[20] L. V. Ovsiannikov. *Lectures on Basis of the Gas Dynamics*. Institute of Computer Studies, Moscow-Izhevsk, 2003. 2nd Edition.

[21] S. V. Meleshko. *Methods for constructing exact solutions of partial differential equations*. Mathematical and Analytical Techniques with Applications to Engineering. Springer, New York, 2005.

[22] M. Munk and R. Prim. On the multiplicity of steady gas flows having the same streamline pattern. *Proc. Nat. Acad. Sci.*, 33:137–141, 1947.

[23] S. V. Meleshko. Method for constructing reciprocal transformations. *Communications in Nonlinear Science and Simulation*. submitted.

[24] A. C. Hearn. *REDUCE Users Manual, ver. 3.3*. The Rand Corporation CP 78, Santa Monica, 1987.