Three-quark ground-state potential in the SU(3) lattice QCD

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With the smearing technique, the three-quark (3Q) ground-state potential $V_{3Q}$ is numerically extracted in the SU(3)\textsubscript{c} lattice QCD Monte Carlo simulation with $12^3 \times 24$ and $\beta = 5.7$ at the quenched level. With accuracy better than a few \%, $V_{3Q}$ is well described by a sum of a constant $C_{3Q}$, the two-body Coulomb part $-A_{3Q} \sum_{i<j} \frac{1}{|r_i - r_j|}$ and the three-body linear confinement part $\sigma_{3Q} L_{\text{min}}$, where $L_{\text{min}}$ denotes the minimal length of the color flux tube linking the three quarks. By comparing with the Q-$\bar{Q}$ potential, we find a universal feature of the string tension as $\sigma_{3Q} \approx \sigma_{Q \bar{Q}}$ and the one-gluon-exchange result for the Coulomb coefficient as $A_{3Q} \approx \frac{1}{2} A_{Q \bar{Q}}$. All our results including the constant term are consistent with the requirement on the diquark limit in the lattice regularization.

1. The 3Q Wilson Loop and the 3Q Ground-State Potential in QCD

Similar to the relevant role of the Q-$\bar{Q}$ potential for meson properties, the three-quark (3Q) potential \cite{2,3} is directly responsible to the structure and properties of baryons. In spite of the importance of the 3Q potential in the hadron physics, there were only a few preliminary lattice-QCD studies for the 3Q potential done in 80’s \cite{4,5}.

The 3Q ground-state potential $V_{3Q}$ can be measured in a gauge-invariant manner using the 3Q Wilson loop $W_{3Q}$ as shown in Fig.1,

$$V_{3Q} = - \lim_{T \to \infty} \frac{1}{T} \ln \langle W_{3Q} \rangle, \quad W_{3Q} \equiv \frac{1}{3!} \varepsilon_{abc} \varepsilon_{a'b'c'} U_1^{aa'} U_2^{bb'} U_3^{cc'}, \quad U_k \equiv P e^{ig \int_{V_k} dx^\mu A_\mu (x)}, (1)$$

similar to the derivation of the Q-$\bar{Q}$ potential from the Wilson loop. In principle, $V_{3Q}$ can be obtained in the large $T$ limit, however, the practical lattice-QCD calculation of $\langle W_{3Q} \rangle$ becomes severe for large $T$, because $\langle W_{3Q} \rangle$ decreases exponentially with $T$.

Physically, the true ground state of the 3Q system is expected to consist of three flux tubes, and the 3Q state expressed by the three strings generally includes many excited-state components such as flux-tube vibrational modes. Therefore, for the accurate measurement of the 3Q ground-state potential $V_{3Q}$, the ground-state enhancement or the excitation-component reduction by the smearing technique \cite{1,6,7} is practically indispensable. (This smearing was not applied in Refs. \cite{4,5}.)

In this paper, we study the 3Q ground-state potential $V_{3Q}$ using the ground-state enhancement by the gauge-covariant smearing technique for the link-variable in the SU(3)\textsubscript{c} lattice QCD with the standard action with $\beta = 5.7$ ($a \simeq 0.19\text{fm}$) and $12^3 \times 24$ at the quenched level \cite{1}. We consider 16 patterns of the 3Q configuration where the three

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quarks are put on \((i, 0, 0), (0, j, 0)\) and \((0, 0, k)\) in \(\mathbb{R}^3\) with \(0 \leq i, j, k \leq 3\) in the lattice unit. The junction point \(O\), which does not affect \(V_{3Q}\), is set at the origin \((0,0,0)\) in \(\mathbb{R}^3\).

### 2. The Smearing Technique and the Ground-State Enhancement

The smearing technique is actually successful for accurate measurements of the Q-\(\bar{Q}\) potential in the lattice QCD \([7]\). The standard smearing for link-variables is performed by the iterative replacement of the spatial link-variable \(U_i(s)\) \((i = 1, 2, 3)\) by the obscured link-variable \(\tilde{U}_i(s) \in \text{SU}(3)_c\) \([8]\) which maximizes

\[
\text{Re } \text{tr}\left\{ \tilde{U}_i^\dagger(s)[\alpha U_i(s) + \sum_{j \neq i, \pm} U_{\pm j}(s)U_i(s \pm \hat{j})U_{\pm j}(s + \hat{i})]\right\}, \quad U_{-\mu}(s) \equiv U_{-\mu}^\dagger(s - \hat{\mu}), \tag{2}
\]

with a real smearing parameter \(\alpha\). The \(n\)-th smeared link-variable \(U_{\mu}^{(n)}(s)\) is defined as

\[
U_i^{(n)}(s) \equiv \tilde{U}_i^{(n-1)}(s) \quad (i = 1, 2, 3), \quad U_4^{(n)}(s) \equiv U_4(s), \quad U_\mu^{(0)}(s) \equiv U_\mu(s). \tag{3}
\]

As an important feature, this smearing procedure keeps the gauge covariance of the smeared link-variable \(U_{\mu}^{(n)}(s)\) properly. In fact, the gauge-transformation property of \(U_{\mu}^{(n)}(s)\) is just the same as that of the original link-variable \(U_\mu(s)\), and therefore the gauge invariance of \(O(U_{\mu}^{(n)}(s))\) is ensured whenever \(O(U_\mu(s))\) is gauge invariant.

Since the smeared link-variable \(U_{\mu}^{(n)}(s)\) includes a spatial extension, the “line” expressed with \(U_{\mu}^{(n)}(s)\) physically corresponds to a “flux tube” with a spatial extension. Therefore, if a suitable smearing is done, the “line” of the smeared link-variable is expected to be close to the ground-state flux tube. Here, the overlap between the ground-state operator and the 3Q-state operator at \(t = 0\) in the 3Q Wilson loop \(W_{3Q}\) is estimated by

\[
C_0 \equiv \langle W_{3Q}(T) \rangle^{T+1}/\langle W_{3Q}(T + 1) \rangle^T \quad \in [0, 1]. \tag{4}
\]

To get the ground-state-dominant 3Q system, we investigate the ground-state overlap \(C_0\) for the 3Q Wilson loop composed of the smeared link-variable \(U_{\mu}^{(n)}(s)\) in the lattice QCD, and we adopt the smearing with \(\alpha = 2.3\) and the iteration number \(n = 12\), which largely enhances \(C_0\) for all of the 3Q configurations in consideration as shown in Fig.2.

### 3. Theoretical Consideration for the 3Q Ground-State Potential

In this section, we consider the potential form of \(V_{3Q}\) based on QCD. In the short-distance limit, the perturbative QCD is applicable and the Coulomb-type potential appears as the one-gluon-exchange (OGE) result. In the long-distance limit at the quenched level, the flux-tube picture would be applicable from the argument of the strong-coupling QCD \([3,8]\), and hence a linear-type confinement potential is expected to appear. Actually, the Q-Q potential \(V_{QQ}\) is well reproduced by the Coulomb-plus-linear potential in the lattice QCD. Then, we conjecture that the 3Q ground-state potential \(V_{3Q}\) is also expressed by a sum of the short-distance OGE result and the long-distance flux-tube result as

\[
V_{3Q} = -A_{3Q} \sum_{i<j} \frac{1}{|\mathbf{r}_i - \mathbf{r}_j|} + \sigma_{3Q} L_{\text{min}} + C_{3Q}, \quad V_{QQ}(r) = -\frac{A_{QQ}}{r} + \sigma_{QQ} r + C_{QQ}, \tag{5}
\]

where \(L_{\text{min}}\) denotes the minimal length of the total flux tubes linking the three quarks.
Next, we consider the diquark limit, where the 3Q system becomes equivalent to a Q-Q̄ system, which leads to a physical requirement on the relation between $V_{3Q}$ and $V_{QQ}$. Here, the constant term is to be considered carefully, because there appears a divergence from the Coulomb term in $V_{3Q}$ as $A_{3Q} \to -\infty$ in the continuum diquark limit. In the lattice regularization, this ultraviolet divergence is regularized to be a finite constant with the lattice spacing $a$ as $A_{3Q} \to \frac{A_{3Q}}{\omega a}$, where $\omega$ is a dimensionless constant satisfying $0 < \omega < 1$ and $\omega \sim 1$. Thus, we get the diquark-limit requirement on the lattice,

$$
\sigma_{3Q} = \sigma_{QQ}, \quad A_{3Q} = \frac{1}{2} A_{QQ}, \quad C_{3Q} - C_{QQ} = \frac{A_{3Q}}{\omega a} \quad (> 0).
$$

(6)

4. Lattice QCD Results for the 3Q Ground-State Potential

We measure the 3Q ground-state potential $V_{3Q}$ from 210 gauge configurations in the SU(3)$_c$ lattice QCD using the smearing technique, and compare the lattice data with the theoretical form of Eq.(5). Table 1 shows the best fit coefficients in $V_{3Q}$ and $V_{QQ}$ in Eq.(5). Table 2 shows the comparison between the lattice QCD data $V_{3Q}^{\text{latt}}$ and the fitting function $V_{3Q}^{\text{fit}}$ in Eq.(5) with the three coefficients listed in Table 1.

As shown in Table 2, the three-quark ground-state potential $V_{3Q}$ is well described by Eq.(5) with accuracy better than a few %. From Table 1, we find a universal feature of the string tension, $\sigma_{3Q} \simeq \sigma_{QQ}$, as well as the OGE result for the Coulomb coefficient, $A_{3Q} \simeq \frac{1}{2} A_{QQ}$. All the diquark-limit requirements in Eq.(6) are satisfied for $\omega \simeq 0.46$.

Another fitting with the $\Delta$-type flux-tube ansatz [4,5] seems rather worse, because of unacceptably large $\chi^2/N_{DS} \geq 10.9$. However, as an approximation, $V_{3Q}$ seems described by a simple sum of an effective two-body Q-Q potential with a reduced string tension as $\sigma_{QQ} \approx 0.53 \sigma$. This reduction factor can be naturally understood as a geometrical factor rather than the color factor, since the ratio between $L_{\text{min}}$ and the perimeter length $L_P$ satisfies $\frac{1}{2} \leq \frac{L_{\text{min}}}{L_P} \leq \frac{1}{\sqrt{3}}$, which leads to $L_{\text{min}} \sigma = L_P \sigma_{QQ}$ with $\sigma_{QQ} = (0.5 \sim 0.58) \sigma$. 

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Figure 1. The 3Q Wilson loop $W_{3Q}$. The 3Q state is generated at $t = 0$ and is annihilated at $t = T$. The three quarks are spatially fixed in $R^3$ for $0 < t < T$.

Figure 2. The ground-state overlap $C_0$ for the 3Q system with smeared link-variables (upper data) and with unsmeared link-variables (lower data). The horizontal axis denotes $L_{\text{min}}$. For each 3Q configuration, $C_0$ is largely enhanced as $0 < C_0 < 1$ by the smearing.
Table 1
The string tension, the Coulomb coefficient and the constant term for the 3Q ground-state potential $V_{3Q}$ and the Q-\(\bar{Q}\) potential $V_{Q\bar{Q}}$ in the lattice unit with $a \simeq 0.19$ fm.

|      | $\sigma$     | $A$          | $C$         |
|------|--------------|--------------|-------------|
| 3Q   | 0.1528(27)   | 0.1316(62)   | 0.9140(201) |
| Q\(\bar{Q}\)| 0.1629(47)   | 0.2793(116)  | 0.6293(161) |

Table 2
The lattice QCD result for the 3Q ground-state potential $V_{3Q}^{\text{latt}}$ and the fitting function $V_{3Q}^{\text{fit}}$ in Eq.(5) for the 3Q system where the three quarks are put on $(i, 0, 0)$, $(0, j, 0)$ and $(0, 0, k)$ in $\mathbb{R}^3$ in the lattice unit. The error of $V_{3Q}^{\text{latt}}$ denotes the statistical error estimated with the jackknife method. (The systematic error is not included.)

| $(i, j, k)$ | $V_{3Q}^{\text{latt}}$ | $V_{3Q}^{\text{fit}}$ | $V_{3Q}^{\text{latt}} - V_{3Q}^{\text{fit}}$ |
|-------------|------------------------|------------------------|---------------------------------|
| (0, 1, 1)   | 0.8459(36)             | 0.8529                 | 0.0070                          |
| (0, 1, 2)   | 1.0970(40)             | 1.1023                 | 0.0053                          |
| (0, 1, 3)   | 1.2935(39)             | 1.2926                 | $-0.0009$                       |
| (0, 2, 2)   | 1.3164(40)             | 1.3262                 | 0.0098                          |
| (0, 2, 3)   | 1.5032(58)             | 1.5069                 | 0.0037                          |
| (0, 3, 3)   | 1.6741(40)             | 1.6808                 | 0.0067                          |
| (1, 1, 1)   | 1.0231(38)             | 1.0091                 | $-0.0140$                       |
| (1, 1, 2)   | 1.2181(61)             | 1.2145                 | $-0.0036$                       |
| (1, 1, 3)   | 1.4154(49)             | 1.3958                 | $-0.0196$                       |
| (1, 2, 2)   | 1.3870(46)             | 1.3887                 | 0.0017                          |
| (1, 2, 3)   | 1.5588(60)             | 1.5580                 | $-0.0008$                       |
| (1, 3, 3)   | 1.7141(43)             | 1.7195                 | 0.0054                          |
| (2, 2, 2)   | 1.5216(33)             | 1.5230                 | 0.0014                          |
| (2, 2, 3)   | 1.6745(11)             | 1.6755                 | 0.0010                          |
| (2, 3, 3)   | 1.8242(54)             | 1.8169                 | $-0.0073$                       |
| (3, 3, 3)   | 1.9607(92)             | 1.9438                 | $-0.0169$                       |

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