On the Inner Radius of Nodal Domains

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Abstract

Let $M$ be a closed Riemannian manifold. We consider the inner radius of a nodal domain for a large eigenvalue $\lambda$. We give upper and lower bounds on the inner radius of the type $C/\lambda^\alpha (\log \lambda)^\beta$. Our proof is based on a local behavior of eigenfunctions discovered by Donnelly and Fefferman and a Poincaré type inequality proved by Maz’ya. Sharp lower bounds are known only in dimension two. We give an account of this case too.

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1 Introduction and Main Results

Let $(M, g)$ be a closed Riemannian manifold of dimension $n$. Let $\Delta$ be the Laplace–Beltrami operator on $M$. Let $0 < \lambda_1 \leq \lambda_2 \leq \ldots$ be the eigenvalues of $\Delta$. Let $\varphi_\lambda$ be an eigenfunction of $\Delta$ with eigenvalue $\lambda$. A nodal domain is a connected component of $\{\varphi_\lambda \neq 0\}$.

We are interested in the asymptotic geometry of the nodal domains. In particular, in this paper we consider the inner radius of nodal domains.

Let $r_\lambda$ be the inner radius of the $\lambda$-nodal domain $U_\lambda$. Let $C_1, C_2, \ldots$ denote constants which depend only on $(M, g)$. We prove

**Theorem 1.** Let $M$ be a closed Riemannian manifold of dimension $n \geq 3$. Then

$$\frac{C_1}{\sqrt{\lambda}} \geq r_\lambda \geq \frac{C_2}{\lambda^{k(n)}(\log \lambda)^{2n-4}},$$

where $k(n) = n^2 - 15n/8 + 1/4$.

In dimension two we have the following sharp bound
Theorem 2. Let $\Sigma$ be a closed Riemannian surface. Then
\[
\frac{C_3}{\sqrt{\lambda}} \geq r_\lambda \geq \frac{C_4}{\sqrt{\lambda}}.
\]

1.1 Upper Bound

We remark that the upper bound is more or less standard and has been used in the literature (e.g. [8]). However, we explain it here also.

We observe that $\lambda = \lambda_1(U_\lambda)$. This is true since the $\lambda$-eigenfunction does not vanish in $U_\lambda$ ([7], ch. I.5). Therefore, the existence of the upper bound in Theorems 1 and 2 follows from the following general upper bound on $\lambda_1$ of domains $\Omega \subseteq M$.

Theorem 3.
\[
\lambda_1(\Omega) \leq \frac{C_5}{\text{inrad}(\Omega)^2}.
\]

The proof of this theorem is given in §5.

1.2 Lower Bound

For the lower bound on the inner radius in dimensions $\geq 3$, we give a proof in §2.1 which is based on a local behavior of eigenfunctions discovered by H. Donnelly and C. Fefferman (Theorem 4). The same proof gives in dimension two the bound $C/\sqrt{\lambda \log \lambda}$.

In order to get rid of the factor $\sqrt{\log \lambda}$ in dimension two, we treat this case separately in §2.2. The proof for this case can basically be found in [10], and we bring it here for the sake of clarity and completeness.

For the dimension two case we also bring a new proof in §3. Moreover, this proof shows that a big inscribed ball can be taken to be with center at a maximal point of the eigenfunction in the nodal domain. This proof is due to F. Nazarov, L. Polterovich and M. Sodin and is based on complex analytic methods.

1.3 A Short Background

Related to the problem discussed in this paper is the problem of estimating the $(n-1)$-Hausdorff measure $H_{n-1}(\lambda)$ of the nodal set, i.e. the set where an eigenfunction vanishes. J. Br"uning and D. Gromes proved in [4] and [3] sharp lower estimates in dimension two. Namely,
they showed $H_1(\lambda) \geq C\sqrt{\lambda}$. An estimate of the constant $C$ is given in [21]. Later, S. T. Yau conjectured that in any dimension $C_1\sqrt{\lambda} \geq H_{n-1}(\lambda) \geq C_2\sqrt{\lambda}$. This was proved in the case of analytic metrics by H. Donnelly and C. Fefferman in [8].

Regarding the inner radius of nodal domains, we would like to mention the recent work of B. Xu [23], in which he obtains a sharp lower bound on the inner radius for at least two nodal domains, and the work of V. Maz’ya and M. Shubin [17], in which they give sharp bounds on the inner capacity radius of a nodal domain.

1.4 Acknowledgements

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2 The Lower Bound on the Inner Radius

In this section we prove the existence of the lower bounds on the inner radius given in Theorems 1 and 2.

2.1 Lower Bound in Dimension $\geq 3$

In this section we prove the existence of the lower bound in Theorem 1. The proof also gives a bound in the case where $\dim M = 2$, namely $r_\lambda \geq C/\sqrt{\log \lambda}$, but in the next section we treat this case separately to get rid of the $\sqrt{\log \lambda}$ factor.
Let \( \{ \sigma_i \} \) be a finite cellulation of \( M \) by cubes, such that for each \( i \) we can put a Euclidean metric \( e_i \) on \( \sigma_i \), which satisfies \( e_i/4 \leq g \leq 4e_i \). Let \( r_{\lambda,i} \) be the inner radius of \( U_{\lambda,i} = U_{\lambda} \cap \sigma_i \), and \( r_{\lambda,i,e} \) be the Euclidean inner radius of \( U_{\lambda,i} \). Notice that

\[
    r_{\lambda,i,e} \leq 2r_{\lambda,i} \leq 2r_{\lambda}.
\]

**Step 1.** (See Fig. 1). We consider \( \sigma_i \) as a compact cube in \( \mathbb{R}^n \), with edges parallel to the axes directions. We cover \( \sigma_i \) by non-overlapping small cubes with edges of size \( 4h \), where \( r_{\lambda,i,e} < h < 2r_{\lambda,i,e} \). Let \( Q \) be a copy of one of these small cubes. Let \( Q' \) be a concentric cube with parallel edges of size \( 2h \).

**Step 2.** We note that each copy of \( Q' \) contains a point \( p \in \sigma_i \setminus U_{\lambda} \). Otherwise, we would have \( r_{\lambda,i,e} \geq h \), which would contradict the definition of \( h \).

**Step 3.** Denote by \( \text{hole}(p) \) the connected component of \( Q \setminus (\sigma_i \cap U_{\lambda}) \) which contains \( p \). We claim

\[
    \frac{\text{Vol}_e(\text{hole}(p))}{\text{Vol}_e(Q)} \geq \frac{C_1}{\lambda^{\alpha(n)}(\log \lambda)^{4n}},
\]

where \( \alpha(n) = 2n^2 + n/4 \), and \( \text{Vol}_e \) denotes the Euclidean volume. We will denote the right hand side term of (2) by \( \gamma(\lambda) \).

Indeed, \( \text{hole}(p) \) is a connected component of \( U'_{\lambda} \cap Q \) for some \( \lambda \)-nodal domain \( U'_{\lambda} \). Hence, we can apply the following Local Courant’s Nodal Domain Theorem.
Theorem 4 ([9, 5, 13]). Let $B \subseteq M$ be a fixed ball. Let $B'$ be a concentric ball of half the radius of $B$. Let $U_\lambda$ be a $\lambda$-nodal domain which intersects $B'$. Let $B_\lambda$ be a connected component of $B \cap U_\lambda$. Then

$$\frac{\text{Vol}(B_\lambda)}{\text{Vol}(B)} \geq \frac{C_2}{\lambda^{\alpha(n)}(\log \lambda)^{4n}},$$  

(3)

where $\alpha(n) = 2n^2 + n/4$.

We remark that in our case (3) is true also for the quotient of Euclidean volumes, since the Euclidean metric on $\sigma_i$ is comparable with the metric coming from $M$.

**Step 4.** We let $\tilde{\varphi}_\lambda = \chi(U_\lambda)\varphi_\lambda$, where $\chi(U_\lambda)$ is the characteristic function of $U_\lambda$, and similarly, $\tilde{\varphi}_{\lambda,i} = \chi(U_\lambda \cap \sigma_i)\varphi_\lambda$. Then we have the inequality

$$\int_Q |\tilde{\varphi}_{\lambda,i}|^2 \, d(vol) \leq \beta(\lambda) h^2 \int_Q |\nabla \tilde{\varphi}_{\lambda,i}|^2 \, d(vol),$$  

(4)

where

$$\beta(\lambda) = \begin{cases} C_3 \log(1/\gamma(\lambda)), & n = 2, \\ C_4/(\gamma(\lambda))^{(n-2)/n}, & n \geq 3. \end{cases}$$

**Proof.** Observe that $\tilde{\varphi}_{\lambda,i}$ vanishes on hole($p$). We will use the following Poincaré type inequality due to Maz’ya. We discuss it in §4.1. A general version of this inequality with weights instead of Lebesgue measure is proved in [9].

**Theorem 5.** Let $Q \subset \mathbb{R}^n$ be a cube whose edge is of length $a$. Let $0 < \gamma < 1$. Then,

$$\int_Q |u|^2 \, d(vol) \leq \beta a^2 \int_Q |\nabla u|^2 \, d(vol)$$

for all Lipschitz functions $u$ on $Q$, which vanish on a set of measure $\geq \gamma a^n$, and where

$$\beta = \begin{cases} C_5 \log(1/\gamma), & n = 2, \\ C_6/\gamma^{(n-2)/n}, & n \geq 3. \end{cases}$$

From (2) and Theorem 5 applied to $\tilde{\varphi}_{\lambda,i}$, it follows

$$\int_Q |\tilde{\varphi}_{\lambda,i}|^2 \, d(vol_e) \leq \beta(\lambda) h^2 \int_Q |\nabla_e \tilde{\varphi}_{\lambda,i}|^2 \, d(vol_e).$$

Since the metric on $\sigma_i$ is comparable to the Euclidean metric, we have also inequality [1].
**Step 5.**

\[ \int_{\sigma_i} |\tilde{\varphi}_\lambda|^2 \, d(vol) \leq 16\beta(\lambda)r_\lambda^2 \int_{\sigma_i} |\nabla \tilde{\varphi}_\lambda|^2 \, d(vol). \]  

(5)

This is obtained by summing up inequalities (4) over all cubes \( Q \) which cover \( \sigma_i \), and recalling that \( h < 2r_{\lambda_{i,e}} \leq 4r_\lambda \).

**Step 6.** We sum up (5) over all cubical cells \( \sigma_i \) to obtain a global inequality.

\[ \int_{U_\lambda} |\varphi_\lambda|^2 \, d(vol) = \int_M |\tilde{\varphi}_\lambda|^2 \, d(vol) = \sum_i \int_{\sigma_i} |\tilde{\varphi}_\lambda|^2 \, d(vol) \]

\[ \leq 16\beta(\lambda)r_\lambda^2 \sum_i \int_{\sigma_i} |\nabla \tilde{\varphi}_\lambda|^2 \, d(vol) = 16\beta(\lambda)r_\lambda^2 \int_M |\nabla \tilde{\varphi}_\lambda|^2 \, d(vol) \]

\[ = 16\beta(\lambda)r_\lambda^2 \int_{U_\lambda} |\nabla \varphi_\lambda|^2 \, d(vol) \]  

(6)

**Step 7.**

\[ r_\lambda \geq \begin{cases} 
 C_7/\sqrt{\lambda \log \lambda} & , \quad n = 2, \\
 C_8/\lambda^{n^2-15n/8+1/4}(\log \lambda)^2n^{-4} & , \quad n \geq 3.
\end{cases} \]

Indeed, by (6)

\[ \lambda = \frac{\int_{U_\lambda} |\nabla \varphi_\lambda|^2 \, d(vol)}{\int_{U_\lambda} |\varphi_\lambda|^2 \, d(vol)} \geq \frac{1}{16\beta(\lambda)r_\lambda^2}. \]

Thus,

\[ r_\lambda \geq \frac{1}{4\sqrt{\lambda \beta(\lambda)}} = \begin{cases} 
 C_9/\sqrt{\lambda \log(1/\gamma(\lambda))} & , \quad n = 2, \\
 C_{10}/(\lambda^{n-2})/2n/\sqrt{\lambda} & , \quad n \geq 3.
\end{cases} \]

\[ \geq \begin{cases} 
 C_{11}/\sqrt{\lambda \log \lambda} & , \quad n = 2, \\
 C_{12}/(\lambda^{n^2-15n/8+1/4}(\log \lambda)^2n^{-4}) & , \quad n \geq 3.
\end{cases} \]

2.2 Lower Bound in Dimension = 2

We prove the existence of the lower bound on the inner radius in Theorem 2. The arguments below can basically be found in [10], Chapter 7.
We begin the proof of Theorem 2 with Step 1 and Step 2 of §2.1. We proceed as follows.

**Step 3’**. If hole($p$) does not touch $\partial Q$

\[
\frac{\text{Area}_e(\text{hole}(p))}{\text{Area}_e(Q)} \geq C_1,
\]

where $\text{Area}_e$ denotes the Euclidean area.

**Proof.** We recall the Faber-Krahn inequality in $\mathbb{R}^n$.

**Theorem 6.** Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain. Then $\lambda_1(\Omega) \geq C_2/\text{Vol}(\Omega)^{2/n}$

We apply Theorem 6 with $\Omega = \text{hole}(p)$. We emphasize that $\lambda_1(\text{hole}(p), g) \geq C_3\lambda_1(\text{hole}(p), e)$, since the two metrics are comparable.

Thus, we obtain

\[
\lambda = \lambda_1(\text{hole}(p), g) \geq \frac{C_4}{\text{Area}_e(\text{hole}(p))},
\]

or, written differently, $\text{Area}_e(\text{hole}(p)) \geq C_4/\lambda$. On the other hand, $\text{Area}_e(Q) = (4h)^2 \leq 64r_\lambda^2 \leq 64C_5/\lambda$, where the last inequality is the upper bound on the inner radius in Theorem 2. So take $C_1 = C_4/(64C_5)$.

**Step 4’** (part a). There exists an edge of $Q$, on which the orthogonal projection of hole($p$) is of Euclidean size $\geq \gamma \cdot 4h$, where $0 < \gamma < 1$ is independent of $\lambda$.

Let us denote by $|\text{pr}(\text{hole}(p))|$ the maximal size of the projections of hole($p$) on one of the edges of $Q$. If hole($p$) touches $\partial Q$, then $|\text{pr}(\text{hole}(p))| \geq 4h/4 = h$, and we can take $\gamma = 1/4$. Otherwise, by Step 3’

\[
|\text{pr}(\text{hole}(p))| \geq \frac{\text{Area}_e(\text{hole}(p))}{\sqrt{C_1(4h)^2}} = \frac{\sqrt{C_1h}}{4\sqrt{C_1}}.
\]

So, we can take $\gamma = \sqrt{C_1}$.

**Step 4’** (part b).

\[
\int_Q |\tilde{\varphi}_{\lambda,i}|^2 \text{dvol}_e \leq C_6 h^2 \int_Q |\nabla \tilde{\varphi}_{\lambda,i}|^2 \text{dvol}_e. \tag{7}
\]
Notice that $\tilde{\psi}_{\lambda,i}$ vanishes on hole($p$). Hence, Step 4' (part a) permits us to apply the following Poincaré type inequality to $\tilde{\psi}_{\lambda,i}$. Its proof is given in [12]. An inequality in the same spirit can be found in [22].

**Theorem 7** ([10], ch. 7). Let $Q \subseteq \mathbb{R}^2$ be a cube whose edge is of length $a$. Let $u$ be a Lipschitz function on $Q$ which vanishes on a curve whose projection on one of the edges is of size $\geq \gamma a$. Then

$$\int_Q |u|^2 \, dx \leq C(\gamma)a^2 \int_Q |\nabla u|^2 \, dx.$$  

**Steps 5'–7'**. To conclude we continue in the same way as in Steps 5–7 of §2.1.

### 3 A New Proof in Dimension Two

This section is due to L. Polterovich, M. Sodin and F. Nazarov. In dimension two we give a proof based on the harmonic measure and the fact due to Nadirashvili that an eigenfunction on the scale comparable to the wavelength is almost harmonic in a sense to be defined below. This proof also gives information about the location of a big ball inscribed in the nodal domain $U_{\lambda}$. Namely, we show that if $\phi_{\lambda}(x_0) = \max_{U_{\lambda}} |\phi_{\lambda}|$, then one can find a ball of radius $C/\sqrt{\lambda}$ centered at $x_0$ and inscribed in $U_{\lambda}$.

Let $D_p \subseteq \Sigma$ be a metric disk centered at $p$. Let $f$ be a function defined on $D$. Let $\mathbb{D}$ denote the unit disk in $\mathbb{C}$.

**Definition 8.** We say that $f$ is $(K, \delta)$-quasiharmonic if there exists a $K$-quasiconformal homeomorphism $h : D \rightarrow \mathbb{D}$, a harmonic function $u$ on $\mathbb{D}$, and a function $v$ on $\mathbb{D}$ with $1 - \delta \leq v \leq 1$, such that

$$f = (v \cdot u) \circ h.$$  

(8)

**Remark.** We will assume without loss of generality that $h(p) = 0$.

**Theorem 9** ([18] [19]). There exist $K, \varepsilon, \delta > 0$ such that for every eigenvalue $\lambda$ and disk $D \subseteq \Sigma$ of radius $\leq \varepsilon/\sqrt{\lambda}$, $\varphi_{\lambda}|_D$ is $(K, \delta)$-quasiharmonic.

We now choose a preferred system of conformal coordinates on $(\Sigma, g)$.  

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Lemma 10. There exist positive constants $q_+, q_-, \rho$ such that for each point $p \in M$, there exists a disk $D_{p,\rho}$ centered at $p$ of radius $\rho$, a conformal map $\Psi_p : \mathbb{D} \to D_{p,\rho}$ with $\Psi_p(0) = p$, and a positive function $q(z)$ on $\mathbb{D}$ such that

$$\Psi_p^*(g) = q(z)|dz|^2,$$

with $q_- < q < q_+$. Let us take a point $p$, where $|\varphi_\lambda|$ admits its maximum on $U_\lambda$. Let $R = \varepsilon / \sqrt{\lambda q_+}$. Let $D_{p,R\sqrt{q_+}} \subseteq D_{p,\rho}$ be a disk of radius $R\sqrt{q_+}$ centered at $p$.

We now take the functions $u, v$ defined on $\mathbb{D}$ which correspond to $\varphi_\lambda|_{D_{p,R\sqrt{q_+}}}$ in Theorem 9. We observe that

$$\varphi_\lambda(p) = u(0)v(0) \geq u(z)v(z) \geq u(z)(1-\delta),$$

for all $z \in \mathbb{D}$. Hence

$$u(0) \geq (1-\delta) \max_{\mathbb{D}} u. \tag{9}$$

Now we apply the harmonic measure technic. Let $U_\lambda^0 \subseteq \mathbb{D}$ be the connected component of $\{u > 0\}$, which contains 0. Let $E = \mathbb{D} \setminus U_\lambda^0$. Let $\omega$ be the harmonic measure of $E$ in $\mathbb{D}$. $\omega$ is a bounded harmonic function on $U_\lambda^0$, which tends to 1 on $\partial U_\lambda^0 \cap \text{Int}(\mathbb{D})$ and to 0 on the interior points of $\partial U_\lambda^0 \cap \partial \mathbb{D}$. Let $r_0 = \inf\{|z| : z \in E\}$.

By the Beurling-Nevanlinna theorem ([2], sec. 3-3),

$$\omega(0) \geq 1 - C_1 \sqrt{r_0}. \tag{10}$$

By the majorization principle

$$u(0) / \max u \leq 1 - \omega(0). \tag{11}$$

Combining inequalities (9), (10) and (11) gives us

$$r_0 \geq C_2. \tag{12}$$

In the final step we apply a distortion theorem proved by Mori for quasiconformal maps. Denote by $\mathbb{D}_r \subseteq \mathbb{C}$ the disk $\{|z| < r\}$. Observe that

$$\Psi_p(\mathbb{D}_R) \subseteq D_{p,R\sqrt{q_+}}.$$

Hence, we can compose

$$\tilde{h} = h \circ \Psi_p : \mathbb{D}_R \to \mathbb{D}.\tag{9}$$
is a $K$-quasiconformal map. By Mori’s Theorem ([1], Ch. III.C) it is $\frac{1}{K}$-H"older. Moreover, it satisfies an inequality

$$|\tilde{h}(z_1) - \tilde{h}(z_2)| \leq M \left(\frac{|z_1 - z_2|}{R}\right)^{1/K}, \quad (13)$$

with $M$ depending only on $K$. Inequalities (12) and (13) imply that

$$\frac{\text{dist}(p, \partial(U_\lambda \cap D_{p,R}))}{R} \geq \left(\frac{C_2}{M}\right)^K \sqrt{q^-}. \quad (14)$$

Hence,

$$\text{inrad}(U_\lambda) \geq \left(\frac{C_2}{M}\right)^K \sqrt{q^-}R = C_3/\sqrt{\lambda}, \quad (15)$$
as desired.

4 A Review of Poincaré Type Inequalities

We give an overview of several Poincaré type inequalities. In particular, we prove Theorem 5 and Theorem 7.

4.1 Poincaré Inequality and Capacity

Theorem 5 is a direct corollary of the following two inequalities proved by Maz’ya.

**Theorem 11** ([14], §10.1.2 in [15]). Let $Q \subseteq \mathbb{R}^n$ be a cube whose edge is of length $a$. Let $F \subseteq Q$. Then

$$\int_Q |u|^2 \, d(\text{vol}) \leq \frac{C_1 a^n}{\text{cap}(F, 2Q)} \int_Q |\nabla u|^2 \, d(\text{vol})$$

for all Lipschitz functions $u$ on $Q$ which vanish on $F$.

A few remarks:

(a) $2Q$ denotes a cube concentric with $Q$, with parallel edges of size twice as large.
(b) If $\Omega \subseteq \mathbb{R}^n$ is an open set, and $\overline{F} \subseteq \Omega$, then $\text{cap}(F, \Omega)$ denotes the $L^2$-capacity of $F$ in $\Omega$, namely
\[
\text{cap}(F, \Omega) = \inf_{u \in \mathcal{F}} \left\{ \int_{\Omega} |\nabla u|^2 \, dx \right\},
\]
where $\mathcal{F} = \{ u \in C^\infty(\Omega), \ u \equiv 1 \text{ on } F, \ \text{supp}(u) \subseteq \Omega \}$.

(c) By Rademacher’s Theorem ([24]), a Lipschitz function is differentiable almost everywhere, and thus the right hand side has a meaning.

(d) A generalization of the inequality to a body which is starlike with respect to a ball is proved in [16].

The next theorem is a capacity–volume inequality.

**Theorem 12** (§2.2.3 in [15]).
\[
\text{cap}(F, \Omega) \geq \left\{ \begin{array}{ll}
C_2 / \log(\text{Area}(\Omega) / \text{Area}(F)) & , \ n = 2, \\
C_3 / (\text{Vol}(F)^{(n-2)/n} - \text{Vol}(\Omega)^{(n-2)/n}) & , \ n \geq 3.
\end{array} \right.
\]

In particular, for $n \geq 3$ we have
\[
\text{cap}(F, \Omega) \geq C_3 \text{Vol}(F)^{(n-2)/n}.
\]

4.2 A Poincaré Inequality in Dimension Two

In this section we prove Theorem 7. The proof can be found in chapter 7 of [10]. We bring it here for the sake of clarity.

**Proof.** Let the coordinates be such that $Q = \{0 \leq x_1, x_2 \leq a\}$. Let the given edge be $Q \cap \{x_1 = 0\}$, and let $pr$ denote the projection from $Q$ onto this edge. Set $E = pr^{-1}(pr(\text{hole}(p)))$. We claim
\[
\int_{E} |u|^2 \, dx \leq a^2 \int_{Q} |\nabla u|^2 \, dx. \tag{16}
\]

Indeed, let $E_t := E \cap \{x_2 = t\}$.

We recall the following Poincaré type inequality in dimension one whose proof is given below.

**Lemma 13.**
\[
\int_{a}^{b} |u|^2 \, dx \leq |b - a|^2 \int_{a}^{b} |u'|^2 \, dx \tag{17}
\]
for all Lipschitz functions $u$ on $[a, b]$ which vanish at a point of $[a, b]$.  


By this lemma
\[ \int_{E_t} |u|^2 \, dx_1 \leq a^2 \int_{E_t} |\partial_1 u(x_1, t)|^2 \, dx_1. \]

Integrating over \( t \in \text{pr}(\text{hole}(p)) \) gives us (16).

Next we show
\[ \int_Q |u|^2 \, dx \leq C_1 a^2 \int_Q |\nabla u|^2 \, dx. \] (18)

By the mean value theorem \( \exists t_0 \) such that
\[ \int_{E_{t_0}} |u|^2 \, dx_1 \leq \frac{1}{\gamma \cdot a} \int_E |u|^2 \, dx. \] (19)

In addition, we have
\[
|u(x)|^2 \leq 2|u(x_1, t_0)|^2 + 2|u(x) - u(x_1, t_0)|^2 \\
\leq 2|u(x_1, t_0)|^2 + 2 \left( \int_{t_0}^{x_2} |\partial_2 u(x_1, s)| \, ds \right)^2 \\
\leq 2|u(x_1, t_0)|^2 + 2 \cdot a \int_{t_0}^{a} |\partial_2 u(x_1, s)|^2 \, ds.
\]

Integrating the last inequality over \( Q \) gives us
\[ \int_Q |u|^2 \, dx \leq 2 \cdot a \int_{E_{t_0}} |u(x_1, t_0)|^2 \, dx_1 + 2a^2 \int_Q |\partial_2 u|^2 \, dx. \] (20)

Finally, we combine (16), (19) and (20) to get (18).
\[
\int_Q |u|^2 \, dx \leq 2 \cdot a \cdot \frac{1}{\gamma a} \int_E |u|^2 \, dx + 2 \cdot a^2 \int_Q |\nabla u|^2 \, dx \\
\leq C_1 a^2 \int_Q |\nabla u|^2 \, dx.
\]

\[\square\]

### 4.3 A Poincaré Inequality in Dimension One

We prove Lemma 13.
Proof. By scaling, it is enough to prove (17) for the segment \([0, 1]\). Suppose \(u(x_0) = 0\). Since a Lipschitz function is absolutely continuous, we have
\[
|u(x)|^2 = \left| \int_{x_0}^x u'(t) \, dt \right|^2 \leq \int_0^1 |u'(t)|^2 \, dt.
\]
We integrate over \([0, 1]\) to get the desired inequality. □

5 \(\lambda_1\) and Inner Radius

We prove Theorem 3 which relates the inner radius to \(\lambda_1\).

Proof. Let \(\{V_i\}\) be a finite open cover of \(M\), such that for each \(i\) one can put a Euclidean metric \(e_i\) on \(V_i\), which satisfies \(e_i/4 \leq g \leq 4e_i\). Let \(\alpha\) be the Lebesgue number of the covering.

Let \(r = \min(\text{inrad}(\Omega), \alpha)\). Let \(B \subseteq \Omega\) be a ball of radius \(r\). We can assume that \(B \subseteq V_1\). Let \(B_e \subseteq B\) be a Euclidean ball of radius \(r/2\). By monotonicity of \(\lambda_1\), we know that \(\lambda_1(\Omega, g) \leq \lambda_1(B, g) \leq \lambda_1(B_e, g)\), but since the Riemannian metric on \(B_e\) is comparable to the Euclidean metric on it, it follows from the variational principle that
\[
\lambda_1(B_e, g) \leq C_1 \lambda_1(B_e, e_1) = C_2/r^2 \leq C_3/\text{inrad}(\Omega)^2,
\]
where in the last inequality we used the fact that \(\text{inrad}(\Omega) \leq C_4\). □

Remark. We would like to emphasize that in general there is no lower bound on \(\lambda_1\) in terms of the inner radius. However, in dimension two, as pointed out to us by Daniel Grieser and Mikhail Shubin, there exists a lower bound on \(\lambda_1\) in terms of the inner radius and the connectivity of \(\Omega\). This was proved in [11] and [20]. For a more detailed account of the subject one can consult [12].

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