CHARACTERISING THE BIG PIECES OF LIPSCHITZ GRAPHS
PROPERTY USING PROJECTIONS

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ABSTRACT. We characterise the "big pieces of Lipschitz graphs" condition in the plane in terms of projections. Roughly speaking, we prove that if a large subset of a 1-Ahlfors-David regular set $E \subset \mathbb{R}^2$ has plenty of projections in $L^2$, then a large part of $E$ is contained on a single Lipschitz graph. This relates to a question of G. David and S. Semmes.

1. INTRODUCTION AND STATEMENT OF RESULTS

The purpose of this paper is to characterise the "big pieces of Lipschitz graphs" (BPLG) condition in the plane in terms of projections. We begin with some definitions, and then formulate the characterisation. After that, we will discuss the context of the result and provide an outline of the proof.

We are only concerned with 1-Ahlfors-David regular sets:

**Definition 1.1 (1-ADR).** A set $E \subset \mathbb{R}^2$ is 1-Ahlfors-David regular (1-ADR) if there exist constants $0 < c \leq C < \infty$ such that $cr \leq H^1(E \cap B(x,r)) \leq Cr$ for all points $x \in E$ and radii $r \in (0, \text{diam}(E)]$.

**Definition 1.2 (BPLG).** A 1-ADR set $E \subset \mathbb{R}^2$ has big pieces of Lipschitz graphs, if there exist constants $M < \infty$ and $\delta > 0$ with the following property: for every $x \in E$, and every radius $r \in (0, \text{diam}(E)]$, one can find a Lipschitz graph $\Gamma_{x,r}$ (over any 1-dimensional subspace) with Lipschitz constant at most $M$, such that $H^1(E \cap \Gamma_{x,r} \cap B(x,r)) \geq \delta r$.

**Definition 1.3 (Projections).** If $\theta \in S^1$, then $\pi_\theta(x) := x \cdot \theta$, $x \in \mathbb{R}^2$, is (up to an isometry) the orthogonal projection in $\mathbb{R}^2$ onto the line spanned by $\theta$. We occasionally also use the notation $\pi_\theta$ for $\theta \in [-\pi, \pi)$. Then we simply mean $\pi_\theta(x) := \pi_{(\cos \theta, \sin \theta)}(x)$.

The next proposition gives an easy necessary condition for a set to have BPLG:

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Proposition 1.4. Assume that \( E \subset \mathbb{R}^2 \) has BPLG. Then, there exist constants \( \rho > 0 \) and \( C < \infty \) (depending only on \( M \) and \( \delta \) in the definition of BPLG) such that the following holds: for every \( x \in E \) and every radius \( r \in (0, \text{diam}(E)) \), there exists a vector \( \theta_{x,r} \in S^1 \) and a subset \( E_{x,r} \subset E \cap B(x, r) \) with the properties that \( \mathcal{H}^1(E_{x,r}) \geq r/C \), and
\[
\| \pi_{\theta_{x,r}} \mathcal{H}^1 |_{E_{x,r}} \|_{L^\infty} \leq C
\]
for every \( \theta \in B(\theta_{x,r}, \rho) \cap S^1 \).

Proof. Take \( E_{x,r} := E \cap \Gamma_{x,r} \cap B(x, r) \), and let \( \theta_{x,r} \) be the direction of the line, over which \( \Gamma_{x,r} \) is a Lipschitz graph. It is then easy to verify the \( L^\infty \)-condition for all \( \theta \in B(\theta_{x,r}, \rho) \cap S^1 \) for some \( \rho \sim 1/M \).

Remark 1.5. We write \( A \lesssim_p B \) if \( A \leq CB \) for some constant \( C > 0 \) depending only on the parameter \( p \); the notation \( A \lesssim B \) means that the constant \( C \) is absolute, or depends only on parameters, which can be regarded as "fixed" in the situation. The two-sided inequality \( B \lesssim_p A \lesssim_p B \) is abbreviated to \( A \sim_p B \).

The main result of the paper asserts that the necessary condition for BPLG in Proposition 1.4 is also sufficient, and the uniform \( L^\infty \)-bound for the projections can even be relaxed to an averaged bound for the \( L^2 \)-norms:

Theorem 1.6. Let \( E \subset \mathbb{R}^2 \) be a 1-Ahlfors-David regular set. Suppose \( \kappa > 0 \) and \( C < \infty \) are constants such that for every \( x \in E \) and \( r \in (0, \text{diam}(E)) \) there exists a \( \mathcal{H}^1 \)-measurable subset \( E_{x,r} \subset E \cap B(x, r) \) with the following properties:

1. \( \mathcal{H}^1(E_{x,r}) \geq \kappa r \), and
2. there exists \( \theta_{x,r} \in S^1 \) such that \( \pi_{\theta_{x,r}} \mathcal{H}^1 |_{E_{x,r}} \in L^2 \) for a.e. \( \theta \in B(\theta_{x,r}, \kappa) \cap S^1 \), and
\[
\int_{B(\theta_{x,r}, \kappa) \cap S^1} \| \pi_{\theta_{x,r}} \mathcal{H}^1 |_{E_{x,r}} \|_2^2 \, d\mathcal{H}^1(\theta) \leq Cr.
\]

Then \( E \) has BPLG.

We now discuss the context and history of the result. The motivation stems from a question of G. David and S. Semmes in [2, 4]: the authors ask whether BPLG could be characterised by a property called big projections in plenty of directions (BPPD). Quoting the \( \mathbb{R}^2 \)-version of the definition in [4], a set \( E \subset \mathbb{R}^2 \) has BPPD, if there exists a constant \( \delta > 0 \) with the following property: for every \( x \in E \) and \( r \in (0, \text{diam}(E)) \), there is a vector \( \theta_{x,r} \in S^1 \) such that
\[
|\pi_{\theta}(E \cap B(x, r))| \geq \delta r
\]
for all \( \theta \in B(\theta_{x,r}, \delta) \cap S^1 \). David and Semmes write that it seems to be a hard problem to decide whether this property should imply BPLG and, indeed, the question remains open to date.

To see how the BPPD hypothesis is connected with our assumptions, suppose that a set \( E' \subset E \cap B(x, r) \) satisfies \( \mathcal{H}^1(E') \gtrsim r \), \( \pi_{\theta} \mathcal{H}^1 |_{E'} \in L^2 \) and \( \| \pi_{\theta} \mathcal{H}^1 |_{E'} \|_2 \lesssim r \) for some \( \theta \in S^1 \). Then,
\[
r^2 \lesssim \| \pi_{\theta} \mathcal{H}^1 |_{E'} \|_2^2 \leq |\pi_{\theta}(E')| \cdot \| \pi_{\theta} \mathcal{H}^1 |_{E'} \|_2 \lesssim |\pi_{\theta}(E \cap B(x, r))| \cdot r,
\]
which implies (1.7) for this particular $\theta$. Therefore, for those $\theta$ in Theorem 1.6 such that $\|\pi_{\theta}^*\mathcal{H}^1|_{E_{\theta}}\|_2 \lesssim r$, our hypothesis is strictly stronger than (1.7); on the other hand, the "averaged" hypothesis in (2) is more relaxed than the uniform requirement of BPPD.

We should also mention that, in [4], the condition BPLG was already characterised by a combination of BPPD and an extra hypothesis called the the weak geometric lemma – an additional regularity assumption not directly connected with projections. To the best of our knowledge, our result is the first to characterise BPLG using projections, and projections only.

We now discuss briefly, how the question relates to uniformly rectifiable sets:

**Definition 1.8 (1-UR).** A closed set $E \subset \mathbb{R}^2$ is 1-uniformly rectifiable (1-UR) if it is 1-ADR, and there exist constants $\delta > 0$ and $M < \infty$ with the following property: for every $x \in E$ and $r \in (0, \text{diam}(E))]$ there is a Lipschitz mapping $g: [-r, r] \to \mathbb{R}^2$ such that $\text{Lip}(g) \leq M$ and

$$\mathcal{H}^1(E \cap B(x, r) \cap g([-r, r])) \geq \delta r.$$  

For the basics of 1-UR sets, we refer to the monographs [2, 3] of David and Semmes. Obviously, if a set $E$ has BPLG, then it is 1-UR, so Theorem 1.6 gives a sufficient condition for $E$ to be 1-UR. However, an unpublished example of T. Hrycak shows that BPLG is a strictly stronger condition than 1-UR, even in the plane. In fact, Hrycak’s construction produces, for any given $\epsilon > 0$, a 1-ADR set $E \subset \mathbb{R}^2$, which has length one, which is contained in a single 1-ADR curve of length at most ten, and which has the – fairly surprising – property that $|\pi_\theta(E)| \leq \epsilon$ for every $\theta \in S^1$. In the presence of such an example, it seems plausible that there is simply no natural characterisation of 1-UR sets in terms of projections – and so characterising BPLG instead is the "right question" to ask.

Let us finally mention the recent deep geometric result by J. Azzam and R. Schul [1], which says that 1-UR = (BP)$^2$LG (that is, 1-UR sets contain big pieces of sets which have BPLG).

1.1. **Outline of the proof.** After a suitable translation, scaling and rotation, the proof of Theorem 1.6 reduces to verifying the following statement:

**Theorem 1.9.** Let $E_0 \subset \mathbb{R}^2$ be a 1-ADR set, and assume that $E_1 \subset E_0 \cap B(0, 1)$ is a $\mathcal{H}^1$-measurable subset satisfying the following two properties:

(i) $\mathcal{H}^1(E_1) =: \kappa > 0$, and

(ii) there exists a constant $\theta_0 > 0$ such that $\pi_{\theta}^*\mathcal{H}^1|_{E_1} \in L^2$ for almost every $\theta \in [-\theta_0, \theta_0]$, and

$$\int_{-\theta_0}^{\theta_0} \|\pi_{\theta}^*\mathcal{H}^1|_{E_1}\|_2^2 \, d\theta =: C < \infty.$$  

Then, there exists a Lipschitz-graph

$$\Gamma := \Gamma(f) := \{(t, f(t)) : t \in [-1, 1]\}$$  

such that
\[ \text{Lip}(f) \lesssim_{\kappa, \theta_0, C} 1 \quad \text{and} \quad \mathcal{H}^1(E_1 \cap \Gamma) \gtrsim_{\kappa, \theta_0, C} 1. \]

The proof divides into one main lemma and one main proposition. To formulate these, we introduce notation for vertical one-sided cones.

**Definition 1.10 (Cones).** Given \(0 < \alpha \leq \pi\) and \(x \in \mathbb{R}^2\), let \(X(x, \alpha)\) be the one-sided closed cone, which is centred at \(x\), points upwards and has opening angle \(\alpha\) in the sense that if \(x = (x_1, x_2) \in \mathbb{R}^2\), then
\[ X(x, \alpha) := \{ y = (y_1, y_2) : y_2 \geq x_2 \text{ and } \pi_\theta(x) = \pi_\theta(y) \text{ for some } \theta \in [-\frac{\alpha}{2}, \frac{\alpha}{2}] \}. \]

Given such a cone \(X(x, \alpha)\) and two radii \(0 < r < R < \infty\), we write
\[ X(x, \alpha, R, r) := X(x, \alpha) \cap [B(x, R) \setminus U(x, r)], \]
where \(B(x, \delta)\) and \(U(x, \delta)\) are, respectively, the closed and open balls of radius \(\delta > 0\) centred at \(x\). Note that \(X(x, \alpha, R, r)\) is a closed set for \(0 < r < R < \infty\).

Our main lemma reads as follows:

**Lemma 1.11.** Assume that \(E_0\) and \(E_1\) satisfy the hypotheses of Theorem 1.9. Then, there exists a natural number \(M \in \mathbb{N}\) (depending only on the constants \(\kappa, \theta_0, C\)) and a \(\mathcal{H}^1\)-measurable subset \(E_2 \subset E_1\) with the following properties: \(\mathcal{H}^1(E_2) \sim_{\kappa, \theta_0, C} 1\), and if \(x \in E_2\), then
\[ X(x, \theta_0, 2^{-j}, 2^{-j-1}) \cap E_2 \neq \emptyset \]
for at most \(M\) scales \(2^{-j}\), \(j \in \mathbb{Z}\).

This will be useful in combination with the main proposition below:

**Proposition 1.12.** Assume that a \(\mathcal{H}^1\)-measurable set \(E_2 \subset \mathbb{R}^2\) satisfies the following conditions:
\begin{enumerate}
  \item \(\mathcal{H}^1(E_2) \sim 1\), and \(E_2 \subset E_0 \cap B(0, 1)\) for some 1-ADR set \(E_0\), and
  \item for some \(\alpha \in (0, \pi/4]\), \(M \in \mathbb{N}\), and for every point \(x \in E_0\), there are at most \(M\) scales \(2^{-j}\) such that
\[ X(x, \alpha, 2^{-j}, 2^{-j-1}) \cap E_2 \neq \emptyset. \]
\end{enumerate}

Then, if \(M \geq 1\), there exists a compact subset \(E_3 \subset E_2\) with \(\mathcal{H}^1(E_3) \sim 1\), which satisfies (a) and (b) with \(\alpha\) replaced by \(\alpha/2\) and \(M\) replaced by \(M - 1\).

Naturally, the idea is to iterate this proposition until \(M = 0\), because it is well-known that any set \(E\) satisfying (b) with \(M = 0\) is contained in a Lipschitz graph with Lipschitz constant \(\lesssim 1/\alpha\), see for example [5, Lemma 15.13]. Thus, Lemma 1.11 and Proposition 1.12 combined give Theorem 1.9. The rest of the paper is devoted to proving Lemma 1.11 and Proposition 1.12.
2. Proof of the main lemma

We start by proving an easy but very useful auxiliary lemma:

**Lemma 2.1.** Let $E_0$ be a 1-AD-regular set with $\mathcal{H}^1(E_0) \gtrsim 1$, let $E_1 \subset E_0 \cap B(0, 1)$ be a $\mathcal{H}^1$-measurable subset, and let

$$E_{1, \epsilon} := \{x \in E_1 : \mathcal{H}^1(E_1 \cap B(x, r_x)) \leq \epsilon r_x \text{ for some radius } 0 < r_x \leq 1\}.$$

Then $\mathcal{H}^1(E_{1, \epsilon}) \lesssim \epsilon$, where the implicit constants only depend on the 1-AD-regularity constants of $E_0$. In particular, if $E_1 \subset E_{1, \epsilon}$, then $\mathcal{H}^1(E_1) \lesssim \epsilon$.

**Proof.** The set $E_{1, \epsilon}$ is covered by the balls $B(x, r_x/5)$, $x \in E_{1, \epsilon}$, so the $5r$-covering lemma can be used to extract a disjoint subcollection $\{B(x_i, r_x/5)\}_{i \in \mathbb{N}}$ with the property that the balls $\{B(x_i, r_x)\}_{i \in \mathbb{N}}$ cover $E_{1, \epsilon}$. Let $C_0 > 5$ be so large that $r_x/C_0 \leq d(E_0)$. Now, we have that

$$\mathcal{H}^1(E_{1, \epsilon}) \leq \sum_{i \in \mathbb{N}} \mathcal{H}^1(E_1 \cap B(x_i, r_x)) \leq C_0 \sum_{i \in \mathbb{N}} r_x,$$

$$\lesssim \epsilon \sum_{i \in \mathbb{N}} \mathcal{H}^1(E_0 \cap B(x_i, r_x/C_0))$$

$$\leq \epsilon \cdot \mathcal{H}^1(E_0 \cap B(0, 2)) \lesssim \epsilon,$$

as claimed. \hfill \Box

Of course, the lemma cannot be used to conclude that the set $E_1 \setminus E_{1, \epsilon}$ is 1-ADR, but it is still somewhat more regular than $E_1$, and and this will be useful in the following proofs.

**Proof of Lemma 1.11.** In what follows, the constants $\kappa, \theta_0$ and $C$ from the statement of Lemma 1.11 will be treated as "fixed" in the sense that "$\lesssim_{\kappa, \theta_0, C}$" is abbreviated to "$\lesssim$". We begin by applying the previous lemma twice. First, with $\epsilon \sim \mathcal{H}^1(E_1)$, remove $E_{1, \epsilon}$ from $E_1$: thus, for a suitable $\epsilon \sim 1$, the set $E' := E_1 \setminus E_{1, \epsilon}$ satisfies $\mathcal{H}^1(E') \sim 1$ and has the property that if $x \in E'$ and $0 < r \leq 1$, then

$$\mathcal{H}^1(E_1 \cap B(x, r)) \sim r. \tag{2.2}$$

Then, we apply the lemma again to $E'$, this time with $\epsilon' \sim \mathcal{H}^1(E')$, to the effect that the set $E := E' \setminus E'_{\epsilon'}$ still satisfies $\mathcal{H}^1(E) \sim 1$, and if $x \in E$, $0 < r \leq 1$, then

$$\mathcal{H}^1(E' \cap B(x, r)) \sim r. \tag{2.3}$$

Next, let $E^M$ consist of those $x \in E$ for which there are at least $M$ scales $2^{-j}$, $j \in \mathbb{Z}$, such that

$$X(x, \theta_0, 2^{-j}, 2^{-j-1}) \cap E \neq \emptyset. \tag{2.4}$$

In a moment, we will show that $\mathcal{H}^1(E^M) \lesssim 1/M$; this completes the proof, because then, for a large enough $M$, the set $E_2 = E \setminus E^M$ will be the kind of set we were looking for.

We start with a preliminary reduction. Let $C \geq 1$ be a large absolute constant. Observe that for every $x \in E^M$, there is a constant $\delta_x > 0$ such that there are at
least $M$ scales $2^{-j} \geq C\delta$ satisfying (2.4). If $E^{M,\delta} = \{x \in E^M : \delta_x \geq \delta\}$, we can choose $\delta > 0$ so small that $\mathcal{H}^1(E^{M,\delta}) \geq \mathcal{H}^1(E^M)/2$. In particular, it suffices to show that $\mathcal{H}^1(E^{M,\delta}) \lesssim 1/M$, where the implicit constant does not depend on $\delta$. This is what we will do, but in order to avoid obscuring the notation any further, we assume that $E^M = E^{M,\delta}$; note that (2.2) and (2.3) are obviously unaffected by the passage from $E^M$ to $E^{M,\delta}$.

With $\delta$ as in the previous paragraph, let $F^M$ and $F'$ be maximal $\delta$-separated sets inside $E^M$ and $E'$, respectively; since $E^M \subset E'$, we can also arrange so that $F^M \subset F'$. We wish to find lower and upper bounds on the amount of triples $(x, y, \theta)$, where $x, y \in F'$, $\theta \in [-\theta_0, \theta_0]$, and $|\pi_\theta(x) - \pi_\theta(y)| \leq \delta$. The notation $\theta \in [-\theta_0, \theta_0]$ refers to those angles $\theta$ such that the line $\pi_\theta(\mathbb{R}^2)$ forms an angle $\theta$ with the $x$-axis. The idea is that (ii) will give us an upper bound for such triples, whereas a lower bound can be obtained, via (2.4), by choosing $x \in F^M \subset F'$ and $y \in F'$.

We start with the lower bound. Note that

$$\#F^M \gtrsim \mathcal{H}^1(E^M)\delta^{-1},$$  \hfill (2.5)

because $E^M$ is covered by the balls $B(x, 2\delta)$, $x \in F^M$, and the AD-regularity of $E_0 \supset E^M$ implies that $\mathcal{H}^1(B(x, 2\delta) \cap E^M) \lesssim \delta$. Pick $x \in F^M$, and let $2^{-j} \geq C\delta$ be one of the scales such that (2.4) holds, and choose a point $y_j \in X(x, \theta_0, 2^{-j}, 2^{-j-1}) \cap E$.

Then, if $c = c_0 > 0$ is a suitable small constant, we have

$$B(y_j, c2^{-j}) \subset X(x, 2\theta_0, 2^{-j+1}, 2^{-j-2}),$$

and $c, C$ can be chosen so that $c2^{-j} \geq cC\delta \geq 10\delta$ (see Figure 1). Since $y_j \in E$, we...
infer from (2.3) that
\[ \mathcal{H}^1(E' \cap B(y_j, c2^{-j_1})) \gtrsim 2^{-j}, \]
and since \( E' \cap B(y_j, c2^{-j_1}) \subset \bigcup_{w \in F' \cap B(y_j, c2^{-j})} B(w, 2\delta) \) we see that
\[
\#(F' \cap B(y_j, c2^{-j})) \gtrsim \delta^{-1} 2^{-j}, \tag{2.6}
\]
again by the AD-regularity of \( E_0 \). Now, it is a simple geometric fact that if \( x, y \in \mathbb{R}^2 \) and \( |x - y| \geq \delta \), then
\[ \mathcal{H}^1(\{ \theta \in S^1 : |\pi_\theta(x) - \pi_\theta(y)| \leq \delta \}) \gtrsim \frac{\delta}{|x - y|}. \]
Indeed, the set of the left hand side contains an arc of length \( \gtrsim \delta/|x - y| \) around each point \( \theta \in S^1 \) such that \( \pi_\theta(x) = \pi_\theta(y) \). Moreover, assuming that \( y \in X(x, 2\theta_0) \) (which means by definition that \( \pi_\theta(x) = \pi_\theta(y) \) for some \( \theta \in [-\theta_0, \theta_0] \)), we can improve this to
\[ \mathcal{H}^1(\{ \theta \in [-\theta_0, \theta_0] : |\pi_\theta(x) - \pi_\theta(y)| \leq \delta \}) \gtrsim \frac{\delta}{|x - y|}. \tag{2.7} \]
Of course, the implicit constants here depend on \( \theta_0 \). Applying (2.7) to each point \( y \in F' \cap B(y_j, c2^{-j}) \), and recalling (2.6), we obtain
\[
\sum_{y \in F' \cap B(y_j, c2^{-j})} \mathcal{H}^1(\{ \theta \in [-\theta_0, \theta_0] : |\pi_\theta(x) - \pi_\theta(y)| \leq \delta \}) \gtrsim 1.
\]
Next, observe that by varying the scale \( 2^{-j} \) we can – by the definition of \( x \in F^M \subset E^M \) – obtain \( \gtrsim M \) disjoint balls of the form \( B(y_j, c2^{-j}), j \in \mathbb{Z} \), and the estimate above can be repeated for every such ball to the effect that
\[
\sum_{y \in F' \cap X(x, 2\theta_0)} \mathcal{H}^1(\{ \theta \in [-\theta_0, \theta_0] : |\pi_\theta(x) - \pi_\theta(y)| \leq \delta \}) \gtrsim M.
\]
Finally, summing over \( x \in F^M \) and recalling (2.5), we obtain
\[
\sum_{x, y \in F'} \mathcal{H}^1(\{ \theta \in [-\theta_0, \theta_0] : |\pi_\theta(x) - \pi_\theta(y)| \leq \delta \}) \gtrsim \frac{\mathcal{H}^1(E^M) M}{\delta}. \tag{2.8}
\]
Next, we aim for an upper bound for the left hand side of (2.8). For \( \theta \in [-\theta_0, \theta_0] \) fixed, define a function \( f_\theta : \mathbb{R} \to \mathbb{R} \) by
\[ f_\theta(t) := \sum_{x \in F'} 1_{B(\pi_\theta(x), \delta)}(t). \]
Then,
\[
\int_{\mathbb{R}} f_\theta(t)^2 \, dt = \sum_{x, y \in F'} \int_{\mathbb{R}} 1_{B(\pi_\theta(x), \delta) \cap B(\pi_\theta(y), \delta)}(t) \, dt
\geq \delta \cdot \#\{ (x, y) \in F' \times F' : |\pi_\theta(x) - \pi_\theta(y)| \leq \delta \},
\]
so that, after exchanging the order of summation and integration on the left hand side of (2.8),
\[ \mathcal{H}^1(E^M)M \lesssim \delta \cdot \sum_{x,y \in F'} \mathcal{H}^1(\{ \theta \in [-\theta_0, \theta_0] : |\pi_\theta(x) - \pi_\theta(y)| \leq \delta \}) \leq \int_{-\theta_0}^{\theta_0} \int_{\mathbb{R}} f_\theta(t)^2 \, dt \, d\theta. \]

Finally, recall that if \( x \in F' \subset E' \), then \( \mathcal{H}^1(E_1 \cap B(x, \delta)) \sim \delta \) by (2.2). This gives
\[ f_\theta(t) \lesssim \delta^{-1} \sum_{x \in F'} \mathcal{H}^1(E_1 \cap B(x, \delta)) \]
\[ \lesssim \delta^{-1} \mathcal{H}^1(E_1 \cap \pi_\theta^{-1}[t - 2\delta, t + 2\delta]) \]
\[ \lesssim M(\pi_\theta \mathcal{H}^1|_{E_1})(t), \]
where \( M(\pi_\theta \mathcal{H}^1|_{E_1}) \) denotes the Hardy-Littlewood maximal function of \( \pi_\theta \mathcal{H}^1|_{E_1} \). By (ii) and the previous considerations, we infer that
\[ \mathcal{H}^1(E^M)M \lesssim \int_{-\theta_0}^{\theta_0} \| M(\pi_\theta \mathcal{H}^1|_{E_1}) \|^2_2 \, d\theta \lesssim \int_{-\theta_0}^{\theta_0} \| M(\pi_\theta \mathcal{H}^1|_{E_1}) \|^2_2 \, d\theta \lesssim 1, \]
so that \( \mathcal{H}^1(E^M) \lesssim 1/M \). This completes the proof. \( \square \)

3. PROOF OF THE MAIN PROPOSITION

Let us recall the statement (with minor changes in the numbering of the sets involved):

**Proposition 3.1.** Assume that a \( \mathcal{H}^1 \)-measurable set \( E_1 \subset \mathbb{R}^2 \) satisfies the following conditions:

(a) \( \mathcal{H}^1(E_1) \sim 1 \), and \( E_1 \subset E_0 \cap B(0,1) \) for some 1-AD-regular set \( E_0 \subset B(0,1) \), and

(b) for some \( \alpha \in (0, \pi/4) \), \( M \in \mathbb{N} \), and for every point \( x \in E_1 \), there are at most \( M \) scales \( 2^{-j} \) such that
\[ X(x, \alpha, 2^{-j}, 2^{-j-1}) \cap E_1 \neq \emptyset. \]

Then, if \( M \geq 1 \), there exists a compact subset \( E_2 \subset E_1 \) with \( \mathcal{H}^1(E_2) \sim 1 \), which satisfies (a) and (b) with \( \alpha \) replaced by \( \alpha/2 \) and \( M \) replaced by \( M - 1 \).

The high level idea of the argument is to write down an explicit algorithm, which refines \( E_1 \) by deleting some points in several stages, but all the time keeps track that not too much is wasted. When the algorithm eventually stops, it will output the desired set \( E_2 \).

**Proof of Proposition 3.1.** Before starting to describe the algorithm, we make two easy reductions: first, without loss of generality, we may assume that if \( x \in E_1 \) and
\[ X(x, \alpha, 2^{-j}, 2^{-j-1}) \cap E_1 \neq \emptyset, \]
then $2^{-j} \geq \delta$ for some small constant $\delta > 0$. Simply, for every $x \in E_1$, there is some $\delta_x > 0$ with this property, and then we can take $\delta > 0$ so small that $\mathcal{H}^1(E_1) \geq \mathcal{H}^1(E_1')/2$, where $E_1' := E_1 \setminus \{ x \in E_1 : \delta_x < \delta \}$. After this, we would proceed with the proof as below, only replacing $E_1$ by $E_1'$. Second, we may assume that $E_1$ is compact; otherwise we can always find a compact subset of $E_1$ (or $E_1'$) with almost the same $\mathcal{H}^1$-measure, and then we can find $E_2$ inside this subset as below.

We now begin to describe the algorithm. The following points (I)–(IV) summarise the key features.

(I) There will be a sequence of compact sets $E_1^0 \supset E_1^1 \supset E_1^2, \ldots$, where $E_1^{k+1}$ is obtained from $E_1^k$ by deleting a certain open set $D^k$.

(II) Thus, there will also be a sequence of deleted sets $D^k \subset E_1^k$, $k \in \{0, 1, \ldots\}$.

(III) There will be a sequence of saved sets $S^k \subset E_1^k \subset E_1$, $k = \{0, 1, \ldots\}$, which are disjoint from each other and all the deleted sets $D^k$, $i \geq k$, satisfy

$$\mathcal{H}^1(S^k) \gtrsim \max\{\mathcal{H}^1(D^k), \delta\},$$

and have the property that if

$$x \in \bigcup_{i \leq k} S^i,$$

then there are at most $M - 1$ scales $2^{-j}$ such that

$$X(x, \alpha/2, 2^{-j}, 2^{-j-1}) \cap \bigcup_{i \leq k} S^i \neq \emptyset.$$

(IV) We describe the structure of the saved sets. Let $E_1^{k,M}$ be the set of points in $E_1^k$ such that there are exactly $M$ scales $2^{-j} \geq \delta$ such that

$$X(x, \alpha/2, 2^{-j}, 2^{-j-1}) \cap E_1^k \neq \emptyset.$$

A point $x \in E_1^k$ is then called $k$-bad, if $x \in E_1^{k,M}$, and furthermore

$$\mathcal{H}^1(B(x, r) \cap E_1^{k,M}) \geq \epsilon r$$

for all radii $0 < r \leq 1$, where $\epsilon \sim \mathcal{H}^1(E_1)$ is a constant to be specified in Stopping condition 3.3 below. Using the compactness of $E_1^k$ and the uniform lower bound for the numbers $2^{-j}$, it is easy to verify that the set of $k$-bad points is compact. Thus, if there are any $k$-bad points to begin with, there exists a (possibly non-unique) $k$-bad point $x_k = (x_1^k, x_2^k)$ with the smallest second coordinate $x_2^k$. The saved set $S^k$ will be defined as $B(x_k, r_k) \cap E_1^{k,M}$ for some suitable radius $r_k \geq \delta$.

Note that if $x$ is $k$-bad and $k \geq 1$, then $x$ is also $(k - 1)$-bad, simply because $E_1^k \subset E_1^{k-1}$ and $E_1^{k,M} \subset E_1^{k-1,M}$. This implies, by the definition of $x_k$, that the second coordinates of the points $x_0, x_1, \ldots, x_k$ form a non-decreasing sequence.

$^1$The sets $S^k$ are also disjoint from the deleted sets $D^i$ with $i < k$, as $S^k \subset E_1^k = E_1 \setminus \bigcup_{i<k} D^i$.Dimensions: 595.3x841.9
Finally, to every set $S_k = B(x_k, r_k) \cap E_{1,M}^{k}$ we associate a somewhat larger set $B_k := B(x_k, 100r_k) \cap E_{1}^k$, which will have the property that if $x \in B_k$, then there are at most $M - 1$ scales $2^{-j}$ such that

$$X(x, \alpha/2, 2^{-j}, 2^{-j-1}) \cap E_{1}^{k+1} \neq \emptyset.$$ 

There will be two different stopping conditions, which bring the algorithm to a halt and output the desired set $E_2$.

**Stopping condition 3.2.** Assume that the sets $D^0, \ldots, D^k$ and $S^0, \ldots, S^k$ have been defined, and either

$$\sum_{i=0}^{k} \mathcal{H}^1(D^i) \geq \mathcal{H}^1(E_1)/2$$

or

$$\sum_{i=0}^{k} \mathcal{H}^1(S^i) \geq \mathcal{H}^1(E_1)/2.$$ 

In both cases, we set

$$E_2 := \bigcup_{i \leq k} S^i.$$ 

By (III), the set $E_2$ satisfies the requirements of Proposition 3.1, and the proof is complete.

**Stopping condition 3.3.** Assume that the set $E_{1,M}^{k}$ has been defined, and satisfies $\mathcal{H}^1(E_{1,M}^{k}) \geq \mathcal{H}^1(E_1)/2$, and that the set of $k$-bad points, as in (IV), is empty. Thus, for every $x \in E_{1,M}^{k}$, we have

$$\mathcal{H}^1(E_{1,M}^{k} \cap B(x, r_x)) \leq \epsilon r_x$$

for some radius $0 < r_x \leq 1$. Now choose $\epsilon \sim \mathcal{H}^1(E_1) \sim 1$ so small that, using Lemma 2.1, we have $\mathcal{H}^1(E_{1,M}^{k}) \leq \mathcal{H}^1(E_1)/4$. We set

$$E_2 := E_{1,M}^{k} \setminus E_{1,M}^{k}.$$ 

Then, $\mathcal{H}^1(E_2) \geq \mathcal{H}^1(E_{1,M}^{k}) - \mathcal{H}^1(E_{1,M}^{k}) \geq \mathcal{H}^1(E_1)/4$, and for every $x \in E_2$ there are at most $M - 1$ scales $2^{-j}$ such that

$$X(x, \alpha/2, 2^{-j}, 2^{-j-1}) \cap E_2 \neq \emptyset.$$ 

Thus, $E_2$ satisfies the requirements of Proposition 3.1, and the proof is complete.

**Remark 3.4.** Notice that since $\mathcal{H}^1(S^k) \geq \delta$ for every $k$, the first stopping condition will be reached in $\lesssim \delta^{-1}$ steps (unless the second stopping condition was reached before that). In particular, the algorithm terminates and outputs $E_2$ after finitely many steps.
Next, we will explicitly describe how to construct the various sets $E^k_1$, $S^k$ and $D^k$. Define $S^{-1} = \emptyset$, $D^{-1} = \emptyset$ and $E^0_1 := E_1$. Assume that $k \geq 0$ and the sets $E^0_1, \ldots, E^k_1, D^1, \ldots, D^{k-1}$ and $S^1, \ldots, S^{k-1}$ have already been defined, and satisfy the properties listed in (I)–(IV). Assume that the first stopping condition is not satisfied; otherwise the algorithm terminates and the proof is complete. In particular,

$$\mathcal{H}^1(E^k_1) \geq \mathcal{H}^1(E_1) - \sum_{i<k} \mathcal{H}^1(D^i) \geq \mathcal{H}^1(E_1)/2. \quad (3.5)$$

Next, assume that the second stopping condition is not satisfied; because of (3.5), this means that the set of $k$-bad points is non-empty, and – as required by (IV) – we find one of them, $x_k$, with minimal second coordinate. Let $2^{-j_k}$ be one of the $M$ scales such that

$$X(x_k, \alpha/2, 2^{-j_k}, 2^{-j_k-1}) \cap E^k_1 \neq \emptyset.$$ 

Let $r_k := c2^{-j_k}$ for a suitable small absolute constant $c > 0$ to be specified later, and set

$$S^k := B(x_k, r_k) \cap E^k_1, \quad B^k := B(x_k, 100r_k) \cap E^k_1,$$

as required by (IV). Then

$$\mathcal{H}^1(S^k) \gtrsim 2^{-j_k} \geq \delta, \quad (3.6)$$

by the definition of $k$-badness. Furthermore, $S^k$ is disjoint from all the previous sets $S^i, i < k$, and even the larger sets $B^i, i < k$, because $S^k \subset E^k_1$, but if $x \in B^i$, then

$$X(x, \alpha/2, 2^{-m}, 2^{-m-1}) \cap E^k_1 \neq \emptyset$$

can only hold for $M - 1$ scales $2^{-m}$ by (IV). Next, we define the deleted set $D^k$ by

$$D^k := E^k_1 \cap \bigcup_{l \in \{-1,0,1\}} \bigcup_{x \in B^k} X^o(x, \alpha, 2^{-j_k+l}, 2^{-j_k-1+l}).$$

Here $X^o$ stands for the open cone (we want the deleted set to be relatively open in $E^k_1$). Then $D^k$ is contained in a single disc of radius $\lesssim 2^{-j_k}$, so $\mathcal{H}^1(D^k) \lesssim 2^{-j_k}$. Combining this with (3.6), we see that

$$\mathcal{H}^1(S^k) \gtrsim \max\{\mathcal{H}^1(D^k), \delta\},$$

as required in (III).

It remains to check the disjointness of $D_k$ from the previous saved sets $S^i, i < k$, and the latter claim in (III) about $\bigcup_{i \leq k} S^i$, plus the claim about the set $B^k$ at the end of (IV). We begin with the last and easiest task. By assumption, there exists a point

$$z_k \in X(x_k, \alpha/2, 2^{-j_k}, 2^{-j_k-1}) \cap E^k_1.$$ 

Now, if the constant $c$ in $r_k = c2^{-j_k}$ is chosen small enough (depending on $\alpha$), and $x \in B^k \subset B(x_k, 100r_k)$, one can check that

$$z_k \in \bigcup_{l \in \{-1,0,1\}} X(x, \alpha, 2^{-j_k+l}, 2^{-j_k-1+l}).$$
In particular, one of the three scales $2^{-j}k^l$, $l \in \{-1, 0, 1\}$, is among the at most $M$ scales $2^{-m}$ such that $X(x, \alpha, 2^{-m}, 2^{-m} - 1) \cap E^k_1 \neq \emptyset$. Then $D^k$ certainly contains all the points in the intersection $X(x, \alpha/2, 2^{-m}, 2^{-m} - 1) \cap E_1^k$, so

$$X(x, \alpha/2, 2^{-m}, 2^{-m} - 1) \cap E_1^{k+1} = X(x, \alpha/2, 2^{-m}, 2^{-m} - 1) \cap (E_1^k \setminus D^k) = \emptyset.$$ 

Thus, there can only remain at most $M - 1$ scales $2^{-m}$ such that the intersection $X(x, \alpha/2, 2^{-m}, 2^{-m} - 1) \cap E_1^{k+1}$ is non-empty, and this is exactly what is claimed at the end of (IV).

Finally, we establish the remaining claims in (III) by proving that $D^k$ is disjoint from the saved sets $S^i$, $i \leq k$. This, and induction, implies that every saved set $S^i$ is in fact disjoint from all the deleted sets $D^l$, $l \geq 0$, so that in particular

$$\bigcup_{i \leq k} S^i \subset E_1^{k+1}. \quad (3.7)$$

Because for every $x \in \bigcup_{i \leq k} S^i$ there are at most $M - 1$ scales $2^{-j}$ such that

$$X(x, \alpha/2, 2^{-j}, 2^{-j-1}) \cap E_1^{k+1} \neq \emptyset,$$

recalling that the sets $E_1^i$ are nested, we infer from (3.7) that for every $x \in \bigcup_{i \leq k} S^i$ there are also at most $M - 1$ scales $2^{-j}$ such that

$$X(x, \alpha/2, 2^{-j}, 2^{-j-1}) \cap \bigcup_{i \leq k} S^i \neq \emptyset.$$

This is what was claimed at the end of (III).

Now, we fix $i \leq k$, and establish that $D^k$ is disjoint from $S^i$. If $i = k$, this is immediate from the construction (recall that $S^k \subset B(x_k, r_k)$, whereas $D^k$ lies inside the union of certain annuli, all at distance $r_k$ from $x_k$). So, we assume that $i < k$. There are two cases to consider. First, assume that $100r_k \leq r_i$ (see...
Figure 3. The case $100r_k \leq r_i$.

In this case, we simply prove that if $x \in B^k \subset B(x_k, 100r_k) \subset B(x_k, r_i)$, then

$$X(x, \alpha) \cap B(x_i, r_i) = \emptyset, \quad (3.8)$$

which is clearly a stronger statement than $D_k \cap S^i = \emptyset$. Fix $x \in B^k$, and recall that $x_k \notin B^i = B(x_i, 100r_i) \cap E^i_k$ (because $x_k \in S_k$, and $S_k$ is disjoint from $B^i$, as remarked earlier). Because $x_k \in S^k \subset E^i_k \subset E^i_1$, this implies that $x_k \notin B(x_i, 100r_i)$, and hence, by $100r_k \leq r_i$,

$$x \notin B(x_i, 50r_i). \quad (3.9)$$

Now, recall that the second coordinate of $x_k$ is no smaller than the second coordinate of $x_i$ by (IV). So, if we write $x = (x^1, x^2), x_i = (x_i^1, x_i^2)$, we have

$$x^2 \geq x_i^2 - r_i.$$  \hfill (3.10)

It is now easy to check, based on (3.9) and (3.10) and the assumption $\alpha \leq \pi/4$, that (3.8) holds.

Next, assume that $100r_k > r_i$ (see Figure 4). Recall that $r_k = c2^{-jk}$, and $D^k$ is contained in the union of the annuli $X^\circ(x, \alpha, 2^{-jk+l}, 2^{-jk-1+l})$, where $x \in B^k \subset B(x_k, 100r_k)$ and $l \in \{-1, 0, 1\}$. If $x = (x^1, x^2) \in B^k$ is fixed, then by the same argument that gave (3.10), we now have

$$x^2 \geq x_i^2 - 100r_k = x_i^2 - 100c2^{-jk}.$$  \hfill (3.11)

If

$$ y = (y^1, y^2) \in X^\circ(x, \alpha, 2^{-jk+l}, 2^{-jk-1+l}),$$

then, using (3.11) and choosing $c > 0$ small enough,

$$y^2 \geq x^2 + 2^{-jk-10} \geq x_i^2 + 2^{-jk-20} \geq x_i^2 + 1000c2^{-jk} \geq x_i^2 + 10r_i.$$  

In particular, $y$ cannot lie in $B(x_i, r_i) \supset S^i$, and the proof is complete. \hfill \Box

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