New Tribonacci recurrence relations and addition formulas

Kunle Adegoke¹, Adenike Olatinwo² and Winning Oyekanmi³

¹ Department of Physics and Engineering Physics, Obafemi Awolowo University
220005 Ile-Ife, Nigeria
e-mail: adegoke00@gmail.com

² Department of Physics and Engineering Physics, Obafemi Awolowo University
220005 Ile-Ife, Nigeria
e-mail: solakunle711@yahoo.com

³ Department of Physics and Engineering Physics, Obafemi Awolowo University
220005 Ile-Ife, Nigeria
e-mail: winningteam90@yahoo.com

Received: 13 March 2020 Revised: 25 September 2020 Accepted: 4 October 2020

Abstract: Only one three-term recurrence relation, namely, \( W_r = 2W_{r-1} - W_{r-4} \), is known for the generalized Tribonacci numbers, \( W_r, r \in \mathbb{Z} \), defined by \( W_r = W_{r-1} + W_{r-2} + W_{r-3} \) and \( W_{-r} = W_{-r+3} - W_{-r+2} - W_{-r+1} \), where \( W_0, W_1 \) and \( W_2 \) are given, arbitrary integers, not all zero. Also, only one four-term addition formula is known for these numbers, which is \( W_{r+s} = T_{s-1}W_{r-1} + (T_{s-1} + T_{s-2})W_r + T_sW_{r+1} \), where \( (T_r)_{r \in \mathbb{Z}} \) is the Tribonacci sequence, a special case of the generalized Tribonacci sequence, with \( W_0 = T_0 = 0 \) and \( W_1 = W_2 = T_1 = T_2 = 1 \). In this paper we discover three new three-term recurrence relations and two identities from which a plethora of new addition formulas for the generalized Tribonacci numbers may be discovered. We obtain a simple relation connecting the Tribonacci numbers and the Tribonacci–Lucas numbers. Finally, we derive quadratic and cubic recurrence relations for the generalized Tribonacci numbers.

Keywords: Tribonacci number, Tribonacci–Lucas number, Recurrence relation.

2010 Mathematics Subject Classification: 11B39, 11B37.

1 Introduction

For \( r \geq 3 \), we define the generalized Tribonacci numbers \( W_r \) by the third order recurrence relation:
where $W_0$, $W_1$ and $W_2$ are arbitrary integers. By writing $W_{r-1} = W_{r-2} + W_{r-3} + W_{r-4}$ and subtracting this from relation (1), we see that $W_r$ also obeys the useful three-term recurrence

$$W_r = 2W_{r-1} - W_{r-4}.$$  \hspace{1cm} (2)

Extension of the definition of the generalized Tribonacci numbers to negative subscripts is provided by writing identity (2) as $W_{r+4} = 2W_{r+3} - W_r$; so that

$$W_{-r} = 2W_{-r+3} - W_{-r+4}.$$ \hspace{1cm} (3)

Well known examples of $W_r$ are the Tribonacci sequence, $(T_r)$, for which $W = T$, $W_0 = T_0 = 0$, $W_1 = W_2 = T_1 = T_2 = 1$ and the Tribonacci–Lucas sequence, $(K_r)$, for which $W = K$, $W_0 = K_0 = 3$, $W_1 = K_1 = 1$, $W_2 = K_2 = 3$. Table 1 shows the first few Tribonacci and Tribonacci–Lucas numbers for $-20 \leq r \leq 24$.

| $r$  | $-20$ | $-19$ | $-18$ | $-17$ | $-16$ | $-15$ | $-14$ | $-13$ | $-12$ |
|------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| $T_r$ | -56   | 159   | -103  | 0     | 56    | -47   | 9     | 18    | -20   |
| $K_r$ | 795   | -571  | 47    | 271   | -253  | 65    | 83    | -105  | 43    |

| $r$  | $-11$ | $-10$ | $-9$  | $-8$  | $-7$  | $-6$  | $-5$  | $-4$  | $-3$  |
|------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| $T_r$ | 7     | 5     | -8    | 4     | 1     | -3    | 2     | 0     | -1    |
| $K_r$ | 21    | -41   | 23    | 3     | -15   | 11    | -1    | -5    | 5     |

| $r$  | $-2$  | $-1$  | $0$   | $1$   | $2$   | $3$   | $4$   | $5$   | $6$   |
|------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| $T_r$ | 1     | 0     | 0     | 1     | 1     | 2     | 4     | 7     | 13    |
| $K_r$ | -1    | -1    | 3     | 1     | 3     | 7     | 11    | 21    | 39    |

| $r$  | $7$   | $8$   | $9$   | $10$  | $11$  | $12$  | $13$  | $14$  | $15$  |
|------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| $T_r$ | 24    | 44    | 81    | 149   | 274   | 504   | 927   | 1705  | 3136  |
| $K_r$ | 71    | 131   | 241   | 443   | 815   | 1499  | 2757  | 5071  | 9327  |

| $r$  | $16$  | $17$  | $18$  | $19$  | $20$  | $21$  | $22$  | $23$  | $24$  |
|------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| $T_r$ | 5768  | 10609 | 19513 | 35890 | 66012 | 121415 | 223317 | 410744 | 755476 |
| $K_r$ | 17155 | 31553 | 58035 | 106743 | 196331 | 361109 | 664183 | 1221623 | 2246915 |

Table 1. The first few Tribonacci and Tribonacci–Lucas numbers.

The following references contain useful information on the properties of the Tribonacci numbers and related: [1, 3, 5, 6, 8, 9]. Various congruence properties of Tribonacci sequences are discussed in the recent paper by Atanassova [2].

Among other interesting results, we found the following three-term recurrence relations which are presumably new: $W_{r-16} = -103W_r + 56W_{r+1}$, $2W_{r-17} = 9W_r - 103W_{r-4}$ and $W_{r-14} = 9W_{r+2} - 56W_{r-1}$.
We also found a simple relation linking the Tribonacci numbers and the Tribonacci–Lucas numbers:

\[ K_{r-2} = 5T_{r-1} - T_{r+1}. \]  

### 2 Linear recurrence relations and addition formulas

**Theorem 2.1.** The following identity holds for integers \(a, b, c, d\) and \(e:\)

\[
(T_{a-c}T_{c-b}T_{b-a} + T_{b-c}T_{c-a}T_{a-b})W_{d+e} \\
= (T_{b-c}T_{c-a}T_{e-b} - T_{b-c}T_{c-b}T_{e-a} + T_{c-b}T_{b-a}T_{e-c})W_{d+a} \\
+ (T_{a-c}T_{c-b}T_{e-a} - T_{a-c}T_{c-a}T_{e-b} + T_{c-a}T_{a-b}T_{e-c})W_{d+b} \\
+ (T_{b-c}T_{a-b}T_{e-a} - T_{a-b}T_{b-a}T_{e-c} + T_{a-c}T_{b-a}T_{e-b})W_{d+c}
\]

**Proof.** We seek to express a generalized Tribonacci number as a linear combination of three Tribonacci numbers. Let

\[ W_{d+e} = f_1T_{e-a} + f_2T_{e-b} + f_3T_{e-c}, \]  

where \(a, b, c, d\) and \(e\) are arbitrary integers and the coefficients \(f_1, f_2\) and \(f_3\) are to be determined. Setting \(e = a, e = b\) and \(e = c\), in turn, we obtain three simultaneous equations:

\[
W_{d+a} = f_2T_{a-b} + f_3T_{a-c}, \quad W_{d+b} = f_1T_{b-a} + f_3T_{b-c}, \quad W_{d+c} = f_1T_{c-a} + f_2T_{c-b}.
\]

The identity of Theorem 2.1 is established by solving these equations for \(f_1, f_2\) and \(f_3\) and substituting the solutions into identity (5). \(\square\)

**Corollary 2.1.1.** The following identities hold for integers \(r\) and \(s:\)

\[
4W_{r+s} = 2T_{s-4}W_{r-4} + (T_{s+4} - 7T_s)W_r + 4T_sW_{r+1};
\]

\[
4W_{r+s} = 2T_{s-4}W_{r-1} + 4(T_{s+1} - 7T_s)W_r + T_sW_{r+4};
\]

\[
W_{r+s} = T_{s-1}W_{r-1} + (T_{s+1} - T_s)W_r + T_sW_{r+1};
\]

\[
4W_{r+s} = T_{s+4}W_{r-4} + (T_{s+4} - 11T_s)W_r + T_sW_{r+4};
\]

\[
W_{r+s} = (T_{s+1} - 2T_s - T_{s-2})W_{r-1} + (T_{s+1} - 2T_s)W_r + T_sW_{r+2}.
\]

**Proof.** To derive identity (6), set \(a = r - 4, b = r, c = r + 1, d = 0\) and \(e = r + s\) in the identity of Theorem 2.1. The proof of (7)—(10) is similar, (see Table 2). \(\square\)

| Identity | \(a\) | \(b\) | \(c\) | \(d\) | \(e\) |
|----------|------|------|------|------|------|
| (6)      | \(r - 4\) | \(r\) | \(r + 1\) | 0    | \(r + s\) |
| (7)      | \(r + 4\) | \(r\) | \(r - 1\) | 0    | \(r + s\) |
| (8)      | \(r - 1\) | \(r\) | \(r + 1\) | 0    | \(r + s\) |

| Identity | \(a\) | \(b\) | \(c\) | \(d\) | \(e\) |
|----------|------|------|------|------|------|
| (9)      | \(r + 4\) | \(r\) | \(r - 4\) | 0    | \(r + s\) |
| (10)     | \(r + 2\) | \(r\) | \(r - 1\) | 0    | \(r + s\) |

Table 2. Appropriate substitutions in the identity of Theorem 2.1 to obtain identities (6)—(10) of Corollary 2.1.1.
Note that identity (8) can be written in the familiar form

\[ W_{r+s} = T_{s-1}W_{r-1} + (T_{s-1} + T_{s-2})W_r + T_sW_{r+1} \, . \]  
(11)

Evaluating identity (6) at \( s = -3 \), \( s = -16 \) and at \( s = -17 \), in turn, we find the following three-term recurrence relations for the generalized Tribonacci numbers:

\[ W_{r-3} = 2W_r - W_{r+1} \, , \]  
(12)
\[ W_{r-16} = -103W_r + 56W_{r+1} \]  
(13)
and

\[ 2W_{r-17} = 9W_r - 103W_{r-4} \, . \]  
(14)

Evaluating identity (10) at \( s = -14 \) produces yet another three-term recurrence relation for the generalized Tribonacci numbers:

\[ W_{r-14} = 9W_{r+2} - 56W_{r-1} \, . \]  
(15)

Note that by interchanging \( r \) and \( s \) in each case and making use of the defining recurrence relation for \( W \) and \( T \), the identities (6) – (10) can also be written

\[ 4W_{r+s} = 2W_{s-4}T_{r-4} + (W_{s+4} - 7W_s)T_r + 4W_sT_{r+1} \, , \]  
(16)
\[ 4W_{r+s} = 2W_{s-4}T_{r-1} + (4W_{s+1} - 7W_s)T_r + W_sT_{r+4} \, , \]  
(17)
\[ W_{r+s} = W_{s-1}T_{r-1} + (W_{s+1} - W_s)T_r + W_sT_{r+1} \, , \]  
(18)
\[ 4W_{r+s} = W_{s-4}T_{r-4} + (W_{s+4} - 11W_s)T_r + W_sT_{r+4} \, , \]  
(19)
\[ W_{r+s} = (W_{s+1} - 2W_s - W_{s-2})T_{r-1} + (W_{s+1} - 2W_s)T_r + W_sT_{r+2} \, . \]  
(20)

Evaluating identity (18) at \( s = -2 \) with \( W = K \) gives

\[ K_{r-2} = 5T_{r-1} - T_{r+1} \, , \]  
(21)
a three-term relation connecting the Tribonacci numbers and the Tribonacci–Lucas numbers.

Relations similar to (4) and (21) are also derived by Frontczak [4] and Komatsu [7].

**Theorem 2.2.** The following identity holds for integers \( a, b, c, d \) and \( e \):

\[
\{K_{b-a-1}K_{a-b-1} + K_{b-a-1}K_{c-b-1}K_{a-c} + K_{c-a-1}K_{b-c-1}K_{a-b-1} + K_{a-c-1}K_{b-c-1}K_{a-b-1} + K_{c-a-1}K_{a-c-1} + K_{b-c-1}K_{b-c-1} - 1\}W_{d+e} \\
= \{(1 - K_{c-b-1}K_{b-c-1})K_{e-a} + (K_{b-a-1} + K_{c-a-1}K_{b-c-1})K_{e-b} + (K_{c-a-1} + K_{b-a-1}K_{c-b-1})K_{e-c}\}W_{d+a-1} \\
+ \{(1 - K_{c-a-1}K_{a-c-1})K_{e-b} + (K_{a-b-1} + K_{c-b-1}K_{a-c-1})K_{e-a} + (K_{c-b-1} + K_{a-b-1}K_{c-a-1})K_{e-c}\}W_{d+b-1} \\
+ \{(1 - K_{a-c-1}K_{a-b-1})K_{e-c} + (K_{b-c-1} + K_{b-a-1}K_{a-c-1})K_{e-b} + (K_{a-c-1} + K_{b-c-1}K_{a-b-1})K_{e-a}\}W_{d+c-1} \, .
\]
Proof. We wish to express a generalized Tribonacci number as a linear combination of three Tribonacci–Lucas numbers. Let
\[
W_{d+e} = f_1K_{e-a} + f_2K_{e-b} + f_3K_{e-c}, \tag{22}
\]
where \(a, b, c, d\) and \(e\) are arbitrary integers and the coefficients \(f_1, f_2, f_3\) are to be determined. Setting \(e = a - 1, e = b - 1\) and \(e = c - 1\), in turn, we obtain three simultaneous equations:
\[
W_{d+a-1} = -f_1 + f_2K_{a-b} + f_3K_{a-c-1}, \quad W_{d+b-1} = f_1K_{b-a-1} - f_2 + f_3K_{b-c-1},
\]
\[
W_{d+c-1} = f_1K_{c-a-1} + f_2K_{e-b-1} - f_3.
\]
The identity of Theorem 2.2 is obtained by solving these equations for \(f_1, f_2\) and \(f_3\) and substituting the solutions into identity (22).

\[\square\]

Corollary 2.2.1. The following identities hold for integers \(r\) and \(s\):

\[
44W_{r+s} = (9K_{s+3} - K_{s+5} + 2K_{s+1})W_{r-6}
+ (9K_{s+1} + K_{s+5} + 2K_{s+3})W_{r-4} + (K_{s+3} + 6K_{s+5} - K_{s+1})W_{r-2}, \tag{23}
\]
\[
88W_{r+s} = (8K_{s+3} - 30K_{s-1} - 2K_{s-2})W_r
+ (K_{s+3} - K_{s-1} + 8K_{s-2})W_{r-4} + (3K_{s+3} + 19K_{s-1} + 2K_{s-2})W_{r+1}, \tag{24}
\]
\[
22W_{r+s} = (5K_{s-2} + K_{s-1} + 2K_s)W_{r-1}
+ (K_{s-2} - 2K_{s-1} + 7K_s)W_r + (2K_{s-2} + 7K_{s-1} + 3K_s)W_{r+1}, \tag{25}
\]
\[
5060W_{r+s} = (-264K_{s-3} + K_{s+9} - 1079K_{s-1})W_{r-10}
+ (-4279K_{s-3} + 21K_{s+9} - 21394K_{s-1})W_r
+ (6325K_{s-1} + 1265K_{s-3})W_{r+2}, \tag{26}
\]
and
\[
5060W_{r+s} = (-264K_{s-11} + 1265K_{s+1} - 4279K_{s-1})W_{r-2}
+ (-1079K_{s-11} + 6325K_{s+1} - 21394K_{s-1})W_r
+ (21K_{s-1} + K_{s-11})W_{r+10}. \tag{27}
\]

Identities (23) – (27) can also be written as follows.

\[
44W_{r+s} = (9W_{s+3} - W_{s+5} + 2W_{s+1})K_{r-6}
+ (9W_{s+1} + W_{s+5} + 2W_{s+3})K_{r-4} + (W_{s+3} + 6W_{s+5} - W_{s+1})K_{r-2}, \tag{28}
\]
\[
88W_{r+s} = (8W_{s+3} - 30W_{s-1} - 2W_{s-2})K_r
+ (W_{s+3} - W_{s-1} + 8W_{s-2})K_{r-4} + (3W_{s+3} + 19W_{s-1} + 2W_{s-2})K_{r+1}, \tag{29}
\]
\[
22W_{r+s} = (5W_{s-2} + W_{s-1} + 2W_s)K_{r-1}
+ (W_{s-2} - 2W_{s-1} + 7W_s)K_r + (2W_{s-2} + 7W_{s-1} + 3W_s)K_{r+1}, \tag{30}
\]
\[
5060W_{r+s} = (-264W_{s-3} + W_{s+9} - 1079W_{s-1})K_{r-10}
+ (-4279W_{s-3} + 21W_{s+9} - 21394W_{s-1})K_r
+ (6325W_{s-1} + 1265W_{s-3})K_{r+2}. \tag{31}
\]
\[ 5060W_{r+s} = (-264W_{s-11} + 1265W_{s+1} - 4279W_{s-1})K_{r-2} + (-1079W_{s-11} + 6325W_{s+1} - 21394W_{s-1})K_r + (21W_{s-1} + W_{s-11})K_{r+10}. \] 

(32)

### 3 Quadratic relations

Our goal in this section is to derive expressions involving only pure squares of generalized Tribonacci numbers. To achieve this we must be able to express the anticipated cross-terms such as \( W_{r-1}W_r \) and \( W_{r-1}W_{r-4} \) as squares of generalized Tribonacci numbers.

Rearranging identity (2) and squaring, we have

\[ 4W_{r-1}W_r = 4W_{r-1}^2 - W_{r-4}^2 + W_r^2, \] 

(33)

\[ 4W_{r-1}W_{r-4} = 4W_{r-1}^2 + W_{r-4}^2 - W_r^2 \] 

(34)

and

\[ 2W_rW_{r-4} = 4W_{r-1}^2 - W_{r-4}^2 - W_r^2. \] 

(35)

Rearranging identity (2) and multiplying through by \( 4W_{r-3} \) to obtain

\[ 8W_{r-1}W_{r-3} = 4W_rW_{r-3} + 4W_{r-4}W_{r-3}, \] 

(36)

and using identities (33) and (34) to resolve the right hand side gives

\[ 8W_{r-1}W_{r-3} = 4W_r^2 + 2W_{r-3}^2 - W_{r+1}^2 + 4W_{r-4}^2 - W_{r-7}^2. \] 

(37)

Multiplying through identity (2) by \( 4W_{r-5} \) to obtain

\[ 4W_rW_{r-5} = 8W_{r-1}W_{r-5} - 4W_{r-4}W_{r-5}, \] 

(38)

which, with the use of (33) and (35), translates to

\[ 4W_rW_{r-5} = 16W_{r-2}^2 - 8W_{r-5}^2 - 4W_{r-1}^2 + W_{r-8}^2 - W_{r-4}^2. \] 

(39)

Rearranging identity (2), shifting index \( r \) and multiplying through by \( 2W_r \) gives

\[ 2W_rW_{r-8} = 4W_rW_{r-5} - 2W_rW_{r-4}, \] 

(40)

from which, using (35) and (39), we get

\[ 2W_rW_{r-8} = 16W_{r-2}^2 - 8W_{r-5}^2 - 8W_{r-1}^2 + W_{r-8}^2 + W_r^2. \] 

(41)

Rearranging identity (13) and squaring, we have

\[ 112W_{r-17}W_r = W_{r-17}^2 + 3136W_r^2 - 10609W_{r-1}^2 \] 

(42)

\[ 11536W_{r-1}W_r = -W_{r-17}^2 + 3136W_r^2 + 10609W_{r-1}^2 \] 

(43)
and
\[ 206W_{r-17}W_{r-1} = -W^2_{r-17} - 10609W^2_{r-1} + 3136W^2_r. \]  
(44)

Rearranging and squaring identity (14), we have
\[ 36W_{r-17}W_r = 4W^2_{r-17} + 81W^2_r - 10609W^2_{r-4}, \]  
(45)
\[ 1854W_rW_{r-4} = 81W^2_r + 10609W^2_{r-4} - 4W^2_{r-17} \]  
(46)

and
\[ 412W_{r-17}W_{r-4} = -4W^2_{r-17} - 10609W^2_{r-4} + 81W^2_r. \]  
(47)

Finally, squaring and rearranging identity (15) produces
\[ 18W_{r-17}W_r = W^2_{r-17} + 81W^2_{r-1} - 3136W^2_{r-4}, \]  
(48)
\[ 1008W_{r-4}W_{r-4} = 81W^2_{r-1} + 3136W^2_{r-4} - W^2_{r-17} \]  
(49)

and
\[ 112W_{r-17}W_{r-4} = -W^2_{r-17} - 3136W^2_{r-4} + 81W^2_{r-1}. \]  
(50)

**Theorem 3.1.** The following identities hold for any integer \( r \):
\[ 252W^2_r - 927W^2_{r-1} + 2884W^2_{r-4} - W^2_{r-17} = 0, \]  
(51)
\[ W^2_r - 2W^2_{r-1} - 3W^2_{r-2} - 6W^2_{r-3} + W^2_{r-4} + W^2_{r-6} = 0. \]  
(52)

**Proof.** Eliminating \( W_{r-1}W_r \) between identities (33) and (43) proves identity (51). To prove identity (52), write \( W_r - W_{r-1} = W_{r-2} + W_{r-3} \), square both sides and use the identity (33) to resolve the cross-products \( W_rW_{r-1} \) and \( W_{r-2}W_{r-3} \). \( \square \)

Substituting for \( W^2_{r-17} \) from the identity of Theorem 3.1 into identities (42), (44) and (50), we have the following simpler versions of these identities:
\[ 4W_{r-17}W_r = -412W^2_{r-1} + 103W^2_{r-4} + 121W^2_r, \]  
(53)
\[ W_{r-17}W_{r-1} = -47W^2_{r-1} - 14W^2_{r-4} + 14W^2_r \]  
(54)

and
\[ 4W_{r-17}W_{r-4} = 36W^2_{r-1} - 215W^2_{r-4} - 9W^2_r. \]  
(55)

Next we show how to express the square of a Tribonacci–Lucas number in terms of squares of Tribonacci numbers.

**Theorem 3.2.** The following identity holds for any integer \( r \):
\[ 4K^2_r = 5T^2_{r+5} - 20T^2_{r+4} + 4T^2_{r+3} + 90T^2_{r+1} - 20T^2_r + 5T^2_{r-3}. \]  
(56)

**Proof.** Square identity (21) and use identity (37) to eliminate the cross-term. \( \square \)
Theorem 3.3. The following identities hold for integers \( r \) and \( s \):

\[
16W_{r+s}^2 = -(T_s + T_{s+4})(-T_{s+4} + 2T_{s+1} + 7T_r)W_r^2 + 4T_{s-1}T_sW_{r-7}^2 \\
+ 2T_{s-1}(-T_{s+4} - 9T_s + 2T_{s+1})W_{r-2}^2 - 2T_s(T_{s+4} - 7T_s + 2T_{s-1})W_{r-3}^2 \\
+ 8T_{s-1}(T_s + T_{s+4})W_{r-1}^2 + 2T_s(T_s + T_{s+4})W_{r+1}^2 \\
16W_{r+s}^2 = 4(2T_s - T_{s+1})(7T_s - T_{s-4} - 4T_{s+1})W_r^2 + 4T_{s-4}(2T_s - T_{s+1})W_{r-4}^2 \\
- 4T_{s-4}(-T_{s-4} - 4T_{s+1} + 9T_s)W_{r-1}^2 - 16T_{s-4}T_sW_{r+2}^2 \\
- 4T_s(T_s + T_{s-4} - 4T_{s+1})W_{r+3}^2 + 4T_s(2T_s - T_{s+1})W_{r+4}^2 ,
\]

\[
4W_{r+s}^2 = -2(-T_{s+1} + T_s)(2T_{s+2} - T_{s-1})W_r^2 - T_{s-1}T_sW_{r-5}^2 \\
+ 2T_{s-1}(-T_{s+1} + T_s)W_{r-4}^2 + 2T_s(-T_{s+1} + T_s)W_{r-3}^2 \\
+ 4T_{s-1}T_sW_{r-2}^2 + 2T_{s-1}(2T_{s-1} + 4T_{s+1} - 3T_s)W_{r+1}^2 \\
+ 2T_s(T_s + T_{s+1})W_{r-1}^2 - T_{s-1}T_sW_{r+3}^2 + 4T_{s-1}T_sW_{r+2}^2 .
\]

4 Cubic recurrence relations

Theorem 4.1. The following identity holds for integer \( r \):

\[
W_r^3 - 4W_{r-1}^3 - 9W_{r-2}^3 - 34W_{r-3}^3 + 24W_{r-4}^3 - 2W_{r-5}^3 \\
+ 40W_{r-6}^3 - 14W_{r-7}^3 - W_{r-8}^3 - 2W_{r-9}^3 + W_{r-10}^3 = 0 .
\]

Proof. Setting \( a = r - 8, b = r, c = r - 10, d = 0 \) and \( e = r - s \) with \( s \in \{1, 2, 3, 4, 5, 6, 7, 9\} \) in the identity of Theorem 2.1, the following linear combinations are formed:

\[
W_{r-1} = \frac{37}{68} W_r - \frac{5}{68} W_{r-8} + \frac{1}{17} W_{r-10}, \quad W_{r-2} = \frac{5}{17} W_r + \frac{3}{17} W_{r-8} + \frac{1}{17} W_{r-10}, \\
W_{r-3} = \frac{11}{68} W_r - \frac{7}{68} W_{r-8} - \frac{2}{17} W_{r-10}, \quad W_{r-4} = \frac{3}{34} W_r - \frac{5}{34} W_{r-8} + \frac{2}{17} W_{r-10}, \\
W_{r-5} = \frac{3}{68} W_r + \frac{9}{68} W_{r-8} + \frac{1}{17} W_{r-10}, \quad W_{r-6} = \frac{1}{34} W_r - \frac{13}{34} W_{r-8} - \frac{5}{17} W_{r-10}, \\
W_{r-7} = \frac{1}{68} W_r - \frac{13}{68} W_{r-8} + \frac{6}{17} W_{r-10}, \quad W_{r-9} = \frac{1}{68} W_r - \frac{81}{68} W_{r-8} - \frac{11}{17} W_{r-10} .
\]

The above \( W_{r-i}, i \in \{1, 2, 3, 4, 5, 6, 7, 9\} \) verify the identity of Theorem 4.1. □

Taking the cube in identities (12) and (13) and solving two simultaneous equations, we find

\[
155736 W_r W_{r-1}^2 = -W_{r-17}^3 + 199305 W_{r-1}^3 - 161504 W_{r}^3 + 14112 W_r^3
\]

and

\[
77868 W_{r}^2 W_{r-1} = -148526 W_{r-4}^3 + 27090 W_r W_{r-4}^2 - W_{r-17}^3 + 95481 W_{r-1}^3.
\]

Theorem 4.2. The following identity holds for integer \( r \):

\[
11844 W_r^3 + 3 W_r W_{r-9} + W_{r-17} + 458556 W_{r-6}^3 \\
+ 135548 W_{r-4}^3 - 442179 W_{r-3}^3 - 120204 W_{r-2}^3 - 43569 W_{r-1}^3 = 0 .
\]

Proof. Write the defining recurrence relation of the generalized Tribonacci numbers as \( W_r - W_{r-1} = W_{r-2} + W_{r-3} \), take the cube of both sides and use identities (60) and (61) to remove cross-terms. □

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5 Conclusion

We have derived many new recurrence relations for the generalized Tribonacci sequence. We discovered three new three-term recurrence relations and two identities from which numerous new addition formulas for generalized Tribonacci numbers may be discovered. We obtained a simple relation connecting the Tribonacci numbers and the Tribonacci–Lucas numbers. Finally, we derived quadratic and cubic recurrence relations for the generalized Tribonacci sequence.

Acknowledgements

Thanks are due to the two anonymous referees whose comments helped to improve the presentation.

References

[1] Adegoke, K. (2018). Weighted Tribonacci sums, arXiv:1804.06449[math.CA].

[2] Atanassova, L. (2019). A remark on the Tribonacci sequences, Notes on Number Theory and Discrete Mathematics, 25(3), 138–141.

[3] Feng, J. (2011). More identities on the Tribonacci numbers, Ars Combinatoria, 100, 73–78.

[4] Frontczak, R. (2018). Convolutions for generalized Tribonacci numbers and related results, International Journal of Mathematical Analysis, 12 (7), 307–324.

[5] Frontczak, R. (2018). Sums of Tribonacci and Tribonacci–Lucas numbers, International Journal of Mathematical Analysis, 12(1), 19–24.

[6] Gabai, H. (1970). Generalized Fibonacci k-sequences, The Fibonacci Quarterly, 8(1), 31–38.

[7] Komatsu, T. (2011). On the sum of reciprocal Tribonacci numbers, Ars Combinatoria, 98, 447–459.

[8] Shah, D. V . (2011). Some Tribonacci identities, Mathematics Today, 27, 1–9.

[9] Waddill, M. E., & Sacks, L. (1967). Another generalized Fibonacci sequence, The Fibonacci Quarterly, 5(3), 209–222.