ON A LOMONACO-KAUFFMAN CONJECTURE

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Abstract. Samuel J. Lomonaco Jr and Louis H. Kauffman conjectured that tame knot theory and knot mosaic theory are equivalent. We give a proof of the Lomonaco-Kauffman conjecture.

1. Introduction

In [3], Samuel J. Lomonaco Jr and Louis H. Kauffman created a formal system \((\mathcal{K}, \mathcal{A})\) consisting of a graded set \(\mathcal{K}\) of symbol strings, called knot mosaics, and a graded subgroup \(\mathcal{A}\), called the knot mosaic ambient group, of the group of all permutations of the set of knot mosaics \(\mathcal{K}\), and they conjectured that the formal system \((\mathcal{K}, \mathcal{A})\) fully captures the entire structure of tame knot theory.

**Conjecture 1.1.** Let \(k_1\) and \(k_2\) be two tame knots (or links), and let \(K_1\) and \(K_2\) be two arbitrary chosen mosaic representatives of \(k_1\) and \(k_2\), respectively. Then \(k_1\) and \(k_2\) are of the same knot type if and only if the representative mosaics \(K_1\) and \(K_2\) are of the same knot mosaic type. In other words, knot mosaic type is a complete invariant of tame knots.

The goal of this paper is to prove the conjecture.

**Theorem 1.2** (Main Theorem). The Lomonaco-Kauffman conjecture is true.

The paper is organized as follows. In Section 2 3 4 we review some notations related to knot mosaic. In Section 5 we recall the definition of the grid diagram. In Section 6 7 we will see a relationship between knot mosaic and grid diagram. In Section 8 we define \(5\times\) zoom map. Section 9 is devoted to proof of the Lomonaco-Kauffman conjecture. In Appendix, we give a list of \(\mathcal{K}(n)/\mathcal{A}(n)\) \((n \leq 4)\). Using this, we can know the mosaic number of some knots.

Acknowledgment. I would like to thank many people for their help with this paper, especially, to Professor Osamu Saeki for his encouragement.

2. Knot Mosaic

Let \(T^{(u)}\) denote the set of the following 11 symbols called (unoriented) tiles. We denote these tiles respectively by the following symbols \(T_0\ , T_1\ , T_2\ , T_3\ , T_4\ , T_5\ , T_6\ , T_7\ , T_8\ , T_9\ , T_{10}\).

**Definition 2.1.** Let \(n\) be a positive integer. We define an (unoriented) \(n\)-mosaic as an \(n \times n\) matrix \(M = (M_{ij}) = (T_{k(i,j)})\) of (unoriented) tiles with rows and columns indexed from 0 to \(n - 1\). We denote the set of \(n\)-mosaics by \(\mathcal{M}^{(n)}\).
Definition 2.2. A connection point of a tile is a midpoint of an edge which is also the endpoint of a curve drawn on a tile.

(1): 0 connection point

(2): 2 connection points

(3): 4 connection points

Definition 2.3. A tile in a mosaic is said to be suitably connected if all its connection points touch the connection points of contiguous tiles.

Definition 2.4. An (unoriented) knot $n$-mosaic is a mosaic in which all tiles are suitably connected. We let $\mathbb{K}^{(n)}$ denote the subset of $\mathbb{M}^{(n)}$ of all knot $n$-mosaics. For example,

$$
\begin{pmatrix}
T_0 & T_2 & T_1 & T_0 \\
T_2 & T_9 & T_{10} & T_1 \\
T_6 & T_3 & T_9 & T_4 \\
T_3 & T_5 & T_4 & T_0
\end{pmatrix} = \begin{pmatrix}
0 & 2 & 1 & 0 \\
2 & 9 & 10 & 1 \\
6 & 3 & 9 & 4 \\
3 & 5 & 4 & 0
\end{pmatrix} \in \mathbb{K}^{(4)}
$$

3. Mosaic moves

Definition 3.1. Let $k$ and $n$ be positive integers such that $k \leq n$. A $k$-mosaic $N$ is said to be a $k$-submosaic of an $n$-mosaic $M$ if it is a $k \times k$ submatrix of $M$. The $k$-submosaic $N$ is said to be at location $(i, j)$ in the $n$-mosaic $M$ if the top left entry of $N$ lies in row $i$ and column $j$ of $M$. Let $M^{(k;i,j)}$ denote the $k$-submosaic of $M$ at location $(i, j)$.

Definition 3.2. Let $k$ and $n$ be positive integers such that $k \leq n$. For any two $k$-mosaics $N$ and $N'$, we define a $k$-move at location $(i, j)$ on the set of $n$-mosaics $\mathbb{M}^{(n)}$, denoted by

$$
N \xleftarrow{(i,j)} N',
$$
as the map from $\mathbb{M}^{(n)}$ to $\mathbb{M}^{(n)}$ defined by

$$
\left(N \xleftarrow{(i,j)} N'\right)(M) = \begin{cases} 
M & \text{if } M^{(k;i,j)} = N \\
M^{(k;i,j)} & \text{if } M^{(k;i,j)} = N' \\
M^{(k;i,j)} & \text{otherwise}
\end{cases}
$$
Proposition 3.3. Each $k$-move $N \xrightarrow{(i,j)} N'$ is a permutation of $\mathbb{M}^{(n)}$. In fact, it is a permutation which is the product of disjoint transpositions.

Definition 3.4. As an analog to the planar isotopy moves for standard knot diagrams, we define for mosaics the 11 mosaic planar isotopy moves given below:
Definition 3.5. As an analog to the Reidemeister moves for standard knot diagrams, we create for mosaics the **mosaic Reidemeister moves**.

The **mosaic Reidemeister 1 moves** are the following:

and similar moves (see [3] for detail).

the **mosaic Reidemeister 2 moves** are given below:

the **mosaic Reidemeister 3 moves** are given below:

The planar isotopy moves and the Reidemeister moves lie in the permutation group of the set of mosaics. It easily follows that the planar isotopy moves and the Reidemeister moves also lie in the group of all permutations of the set of knot mosaics $\mathbb{K}^{(n)}$. Hence, we can make the following definition:

**Definition 3.6.** We define the **(knot mosaic) ambient group** $\mathbb{A}(n)$ as the group of all permutations of the set of knot $n$-mosaics $\mathbb{K}^{(n)}$ generated by the mosaic planar isotopy and the mosaic Reidemeister moves.

4. Knot mosaic type

We define the **mosaic injection**

$$\tau : \mathbb{M}^{(n)} \to \mathbb{M}^{(n+1)}$$

$$M^{(n)} \mapsto M^{(n+1)}$$
as

\[
M_{ij}^{(n+1)} = \begin{cases} 
M_{ij}^{(n)} & \text{if } 0 \leq i, j < n \\
\text{otherwise} & 
\end{cases}
\]

Thus,

\[
M^{(n)} \rightarrow \iota \rightarrow M^{(n+1)}
\]

**Definition 4.1.** Two \(n\)-mosaics \(M\) and \(N\) are said to be of the same knot \(n\)-type, written

\[M \sim_n N,
\]

provided there is an element of the ambient isotopy group \(A(n)\) which transforms \(M\) into \(N\).

**Definition 4.2.** An \(m\)-mosaic \(M\) and an \(n\)-mosaic \(N\) are said to be of the same knot mosaic type, written

\[M \sim N,
\]

provided there exists a non-negative integer \(\ell\) such that, if \(m \leq n\), then

\[\iota^{\ell+n-m} M \sim_{\ell+n} \iota^{\ell} N,
\]

or if \(m > n\), then

\[\iota^{\ell} M \sim_{\ell+m} \iota^{\ell+m-n} N,
\]

where, for each non-negative integer \(p\), \(\iota^p\) denotes the \(p\)-fold composition \(\underbrace{\iota \circ \iota \circ \cdots \circ \iota}_p\).

**Example 4.3.** Let

\[K_1 = \quad \text{and} \quad K_2 =
\]

, then \(K_1 \not\sim_3 K_2\) but \(K_1 \sim K_2 (\iota K_1 \sim_4 \iota K_2)\).
5. Grid diagram

Let \( K \) be an oriented knot in \( S^3 \). Choose a knot diagram \( D \) for \( K \) such that

- \( D \) is composed entirely of horizontal and vertical segments,
- no two horizontal segments of \( D \) have the same \( y \)-coordinate, and no two vertical segments of \( D \) have the same \( x \)-coordinate, and
- at each crossing, the vertical segment crosses over the horizontal segment.

Every knot admits such a diagram (Starting with a knot diagram \( D \), one approximates \( D \) using horizontal and vertical segments, so that crossings are always vertical over horizontal). The only data in such a diagram are the endpoints of the segments, which we record by placing \( X \)'s and \( O \)'s at these endpoints, alternately around the knot, and so that the knot is oriented from \( X \) to \( O \) along vertical segments. The data of the knot is entirely encoded in these \( X \)'s and \( O \)'s, which we can see as sitting in the middle of squares on a piece of graph paper. Notice that no two \( X \)'s (respectively \( O \)'s) lie on the same horizontal or vertical line (Perturb the result so that no segments lie on the same horizontal or vertical line).
Definition 5.1. Let $X = \{X_i\}_{i=1}^{\infty}$ and $O = \{O_i\}_{i=1}^{\infty}$ denote the set of $X$’s and $O$’s, respectively. Up to isotopy of the knot (and renumbering of the $X_i$), we may assume that the coordinates of $X_i$ are $(i - \frac{1}{2}, \sigma_X(i) - \frac{1}{2})$ for some permutation $\sigma_X \in S_N$. Then (after renumbering), the coordinates of $O_i$ are $(i - \frac{1}{2}, \sigma_O(i) - \frac{1}{2})$ for some permutation $\sigma_O \in S_N$. The data $(\mathbb{R}^2, X, O)$ is a planar grid diagram for the knot $K$.

Of course, different grid diagrams can lead to the same link. However, this is controlled by the following theorem:

Theorem 5.2. \cite{2} Any two grid diagrams which describe the same link can be connected by a finite sequence of the following elementary moves:

- **Cyclic permutation** cyclic permutation of the columns (resp. rows);
- **Commutation** commutation of two adjacent columns (resp. rows) under the condition that all the decorations of one of the two commuting columns (resp. rows) are strictly above (resp. strictly on the right of) the decorations of the other one, where a decoration is an element of $O \cup X$;
- **Stabilization/Destabilization** addition (resp. removal) of one column and one row by replacing (resp. substituting) locally a decorated square by (resp. to) a $(2 \times 2)$-grid containing three decorations in such a way that it remains globally a grid diagram.

6. GRID DIAGRAM AS A KNOT MOSAIC DIAGRAM

We can regard a grid diagram as a knot mosaic diagram using following map:

Using this map, we identify the set of $(n \times n)$-grid diagrams with a subset of $\mathbb{K}^{(n)}$ generated by $\{T_0, T_1, T_2, T_3, T_4, T_5, T_6, T_{10}\}$ denoted by $\mathbb{G}^{(n)}$. 
7. ELEMENTARY MOVES AS MOSAIC MOVES

In this section, we will check elementary moves can be represented by mosaic moves under identification of previous section.

- **Cyclic permutation**
  
  In [4], P. S. Ozsváth, Z. Szábo, and D. P. Thurston proved the following fact, so it is sufficiently to check Commutation and Stabilization/Destabilization.

  **Lemma 7.1.** A cyclic permutation is equivalent to a sequence of commutations and (de)stabilizations.

- **Commutation**

  ![Commutation Diagram]
• Stabilization/Destabilization

As an immediate corollary of previous and this sections, we get the following proposition.

Proposition 7.2. \((G, E) \subset (K, A)\), where \(E\) is the group of all elementary moves of \(G\).
8. 5× ZOOM

Definition 8.1. We define the zoom with ratio of 5× as

\[ z_{5\times} : M^{(n)} \rightarrow M^{(5n)} \]

\[ M^{(n)} \rightarrow M^{(5n)} \]
For any positive integer $n$ and arbitrary knot $n$-mosaic $K \in \mathbb{K}^{(n)}$, $z_{5 \times}(K) \sim_{5n} i^{\ln}K$. Thus, $z_{5 \times}(K) \sim \bar{K}$. 

**Example 8.2.**
Proof. Under mosaic planar isotopy moves,

\[
z_5x(T_7) = z_5x(\begin{array}{c} \includegraphics[width=1cm]{figure1} \end{array}) = \begin{array}{c} \includegraphics[width=5cm]{figure2} \end{array} \sim \begin{array}{c} \includegraphics[width=5cm]{figure3} \end{array}.
\]

\[
z_5x(T_8) = z_5x(\begin{array}{c} \includegraphics[width=1cm]{figure4} \end{array}) = \begin{array}{c} \includegraphics[width=5cm]{figure5} \end{array} \sim \begin{array}{c} \includegraphics[width=5cm]{figure6} \end{array}.
\]

\[
z_5x(T_9) = z_5x(\begin{array}{c} \includegraphics[width=1cm]{figure7} \end{array}) = \begin{array}{c} \includegraphics[width=5cm]{figure8} \end{array} \sim \begin{array}{c} \includegraphics[width=5cm]{figure9} \end{array}.
\]

After this arrangement, we can contract segments. As parallel translation can be realized by mosaic moves, we have required result.

□

Using this Lemma, we have the following:

**Proposition 8.4.** Let \(k\) be tame knot (or link) and let \(K\) be arbitrary chosen mosaic representative of \(k\), then there is a grid diagram \(G\) of \(k\) such that \(K \sim G\).

**Proof.** \(z_5x(K)\) has no tiles like \(T_7, T_8, T_9\). Using mosaic planar isotopy moves (and mosaic injection \(i\), if necessary), we can perturb the result so that no segments lie on the same horizontal or vertical line. Thus, we get a grid diagram of \(k\). □

### 9. Proof of main theorem

Let \(k_1\) and \(k_2\) be two tame knots (or links), let \(K_1\) and \(K_2\) be two arbitrary chosen mosaic representatives of \(k_1\) and \(k_2\), respectively, and let \(G_1\) and \(G_2\) be two arbitrary chosen grid diagrams of \(k_1\) and \(k_2\), respectively, then the following conditions are equivalent.

1. \(k_1\) and \(k_2\) are of the same knot type.
2. \(K_1\) and \(K_2\) are of the same knot mosaic type.
3. \(G_1\) and \(G_2\) can be connected by a finite sequence of the elementary moves.
Proof of main theorem.

(1) ⇒ (3): Theorem 5.2

(3) ⇒ (2): By Proposition 7.2, assumption means \( G_1 \sim G_2 \). By Proposition 8.4, \( \exists G(K_i) \in G \) s.t. \( K_i \sim G(K_i) \) \((i = 1, 2)\). Here, \( G(K_i) \) and \( G_i \) are grid diagrams of the same knot \( k_i \), so, by using Theorem 5.2 and Proposition 7.2, \( G(K_i) \sim G_i \). From these facts, result follows.

(2) ⇒ (1): By the forgetting map of the square tiling of knot mosaic, \( K_i \) can naturally be identified with a diagram of the knot \( k_i \), and mosaic moves can be regarded as ambient istopies.

\[ \blacksquare \]

10. ★ Appendix ★ A list of \( \mathbb{K}^{(n)}/\mathbb{A}(n) \) \((n \leq 4)\) and mosaic number of some knots

Theorem 10.1.

(1)

\[ \mathbb{K}^{(1)}/\mathbb{A}(1) = \{ \}\]

(2)

\[ \mathbb{K}^{(2)}/\mathbb{A}(2) = \{ \}

(3)

\[ \mathbb{K}^{(3)}/\mathbb{A}(3) = \{ \}

(4)

\[ \mathbb{K}^{(4)}/\mathbb{A}(4) = \{ \}

Definition 10.2. Define the mosaic number \( m(k) \) of a knot \( k \) as the smallest integer \( n \) for which \( k \) is representable as a knot \( n \)-mosaic. For example, \( m(\text{unknot}) = 2 \) and \( m(\text{trefoil}) = 4 \).

As a corollary of Theorem 10.1, we can get some knots with mosaic number 5.
Corollary 10.3.

\[ m(4_1) = m(5_1) = m(5_2) = m(6_2) = m(7_4) = 5 \]

Proof.
These knots did not appear in the list of Theorem 10.1, so we have required results.

$\square$

**Conjecture 10.4.** $m(6_3) = 6$

### References

[1] Benjamin Audoux, *Généralisation de l’homologie d’Heegaard-Floer aux entrelacs singuliers & Raffinement de l’homologie de Khovanov aux entrelacs restreints*, preprint, (2008) arXiv:0803.4478

[2] I. Dynnikov, *Arc-presentations of links: monotonic simplification*, Fund. Math, 190 (2006), 29–76.

[3] Samuel J. Lomonaco Jr and Louis H. Kauffman, *Quantum Knots and Mosaics*, preprint, (2008) arXiv:0805.0339

[4] P. S. Ozsváth, Z. Szábo, and D. P. Thurston, *Legendrian knots, transverse knots and combinatorial Floer homology*, Geom. Topol, 12 (2008), no. 2, 941–980.

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