Consimilarity and quaternion matrix equations

\[ AX - \hat{X}B = C, \quad X - A\hat{X}B = C^* \]

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Abstract

L. Huang [Consimilarity of quaternion matrices and complex matrices, Linear Algebra Appl. 331 (2001) 21–30] gave a canonical form of a quaternion matrix with respect to consimilarity transformations \( A \mapsto S^{-1}AS \) in which \( S \) is a nonsingular quaternion matrix and \( h \mapsto \hat{h} := a - bi + cj - dk \) \((a, b, c, d \in \mathbb{R})\).

We give an analogous canonical form of a quaternion matrix with respect to consimilarity transformations \( A \mapsto S^{-1}AS \) in which \( h \mapsto \hat{h} \) is an arbitrary involutive automorphism of the skew field of quaternions. We apply the obtained canonical form to the quaternion matrix equations \( AX - \hat{X}B = C \) and \( X - A\hat{X}B = C^* \).

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1 Introduction

We give a canonical form of a square quaternion matrix with respect to consimilarity transformations \( A \mapsto \hat{S}^{-1}AS \) in which \( S \) is a nonsingular quaternion matrix and \( \alpha \mapsto \hat{\alpha} \) is a fixed involutive automorphism of the skew
field of quaternions \( \mathbb{H} \), and apply it to the quaternion matrix equations

\[ AX - \hat{X}B = C \quad \text{and} \quad X - A\hat{X}B = C. \]

A canonical form of a square complex matrix \( A \) with respect to consimilarity transformations

\[ A \mapsto \bar{S}^{-1}AS \quad (S \in \mathbb{C}^{n \times n} \text{ is nonsingular}) \tag{1} \]

was given by Hong and Horn [4] (see also [5, Theorem 4.6.12]): \( A \) is consimilar to a direct sum, uniquely determined up to permutation of direct summands, of matrices of the following two types:

\[ J_k(a) \ (a \in \mathbb{R}, \ a \geq 0), \quad J_k(a + bi) \begin{bmatrix} 0 & I_k \\ J_k(a + bi) & 0 \end{bmatrix} \quad (a, b \in \mathbb{R}, \ a < 0 \text{ if } b = 0) \tag{2} \]

in which

\[ J_k(\lambda) := \begin{bmatrix} \lambda & 1 & \cdots & 0 \\ \lambda & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 0 & \cdots & \cdots & \lambda \end{bmatrix} \tag{3} \]

is the \( k \times k \) Jordan block with eigenvalue \( \lambda \).

Huang [7] defined consimilarity transformations of square quaternion matrices

\[ A \mapsto \tilde{S}^{-1}AS \quad (S \in \mathbb{H}^{n \times n} \text{ is nonsingular}) \tag{4} \]

via the involution

\[ h = a + bi + cj + dk \mapsto \tilde{h} := -jhj = a - bi + cj - dk \quad (a, b, c, d \in \mathbb{R}); \tag{5} \]

on the skew field of quaternions \( \mathbb{H} \); we suggest to call (4) \( j \)-consimilarity transformations since \( \tilde{S} = -jSj \). (The main reason why consimilarity transformations of quaternion matrices are not defined via the quaternion conjugation \( h \mapsto \bar{h} = a - bi - cj - dk \) is that the transformations \( A \mapsto \bar{S}^TAS \) are not transitive.)

Huang [7, Theorem 1] noticed that two quaternion matrices

\[ A \text{ and } B \text{ are } j\text{-consimilar} \iff jA \text{ and } jB \text{ are similar}, \tag{6} \]

which holds since \( \tilde{S}^{-1}AS = -jS^{-1}j \cdot AS = B \) if and only if \( S^{-1}jAS = jB \). This statement admitted him to deduce a canonical form of quaternion matrices
under \(j\)-consimilarity from the canonical form for similarity. Huang’s canonical form given in [7, Theorem 3] is a direct sum, uniquely determined up to permutation of summands, of Jordan blocks

\[ J_k(a + bj) \quad (a, b \in \mathbb{R}, \ a \geq 0). \quad (7) \]

The definition of consimilarity transformations via (5) looks special. We show in Lemma 1 that all consimilarity transformations defined via involutive automorphisms of \(\mathbb{H}\) are transformed from one to each other by resclections of the set of orthogonal imaginary units \(i, j, k\). Thus, we can use an involutive automorphism for which the formulas of the theory of consimilarity transformations are simpler.

Huang [7] and other authors (see, for example, [10, 13, 16, 17, 20]) study the \(j\)-consimilarity transformations \((4)\). These transformations act on complex matrices as the consimilarity transformations \((1)\). We suggest to define consimilarity transformations of quaternion matrices extending the similarity transformations of complex matrices. For this purpose, we define \(i\)-consimilarity transformations \(\hat{S}^{-1}AS\) of quaternion matrices via the automorphism

\[ h = a + bi + cj + dk \rightarrow \hat{h} := -ih\bar{i} = a + bi - cj - dk \quad (a, b, c, d \in \mathbb{R}). \]

Our goal is to show that \(i\)-consimilarity transformations are more convenient for use than \(j\)-consimilarity transformations \((4)\):

- Most properties of \(j\)-consimilarity transformations of quaternion matrices are closer to properties of similarity transformations of complex matrices than to properties of consimilarity transformations of complex matrices (see \((6)\) and compare the canonical forms given in \((2)\) and \((7)\)).
- If \(A = U + Vj\), in which \(U, V \in \mathbb{H}^{n \times n}\), then \(\hat{A} = U - Vj\) (compare with \(\bar{A} = \bar{U} + \bar{V}j\) in \([20]\)), which admits us to study similarity and \(i\)-consimilarity transformations of quaternion matrices simultaneously in Sections 2 and 3.
- The canonical form in Theorem 3 of a quaternion matrix for \(i\)-consimilarity transformations \(\hat{S}^{-1}AS\) is a complex matrix (compare with \((7)\)); in Section 3 we apply it to the quaternion matrix equations \(AX - XB = C\) and \(X - A\hat{X}B = C\), reducing them to complex matrix equations.
2 A canonical form for $i$-consimilarity

Consimilarity transformations come from the theory of semilinear operators (which is presented, for example, in Jacobson’s book [8, Chapter 3, Section 12]). It is worth mentioning that the fundamental theorem of projective geometry is formulated in terms of semilinear maps between vector spaces.

Let $V$ be a right vector space over a field or skew field $F$ with a fixed automorphism $\alpha \mapsto \hat{\alpha}$ on $F$. A map $A : V \to V$ is a semilinear operator if

$$A(v + w) = Av + Aw, \quad A(\alpha v) = (A\alpha)\hat{\alpha}$$

for all $v, w \in V$ and $\alpha \in F$. If $A$ is the matrix of $A$ in some basis and $[v]$ is the coordinate vector of $v \in V$, then $[Av] = A[v]$. By transfer to other bases the matrix $A$ is reduced by transformations

$$A \mapsto C^T A \hat{C} \quad (C \text{ is the change of basis matrix}). \quad (8)$$

For the most important types of automorphisms $\alpha \mapsto \hat{\alpha}$, Jacobson [8, Chapter 3, Theorem 34] reduced the problem of classifying matrices with respect to transformations (8) to the problem of classifying matrices with respect to similarity.

All automorphisms $h \mapsto \hat{h}$ on $H$ that we consider are involutive; that is, $\hat{\hat{h}} = h$ for all $h \in H$.

**Lemma 1.** If $h \mapsto \hat{h}$ is an involutive automorphism of $H$, then either it is identical, or the set of orthogonal imaginary units $i, j, k$ can be chosen such that

$$\hat{h} = h^i := i^{-1} hi = -ih = a + bi - cj - dk \quad (9)$$

for each $h = a + bi + cj + dk \in H \ (a, b, c, d \in \mathbb{R})$.

**Proof.** By the Skolem–Noether theorem ([18, Theorem 2.1] or [11, Theorem 2.41]), every automorphism of $H$ is of the form

$$q \mapsto q^\sigma := \sigma^{-1} q \sigma \quad \text{for some fixed nonzero } \sigma \in H.$$

This automorphism is involutive if and only if $\sigma^{-2} q \sigma^2 = q$ for all $q \in H$, if and only if $\sigma^2 \in \mathbb{R}$. Multiplying $\sigma$ by a positive number (which does not change the automorphism), we get $\sigma^2 = \pm 1$. Write $\sigma = a + b \tau$, in which $a, b \in \mathbb{R}, \tau \notin \mathbb{R}$, and $|\tau| = 1$. Since $\tau^2 = -1$, $\sigma^2 = (a^2 - b^2) + 2ab\tau = \pm 1$. Hence $ab = 0$. 

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If $\sigma^2 = 1$, then $b = 0$, $a^2 = 1$, $\sigma = a = \pm 1$, and the automorphism $q \mapsto q^\sigma$ is the identity.

Let $\sigma^2 = -1$. Then $a = 0$, $b^2 = 1$, $b = \pm 1$, and so $\sigma = \pm \tau$. Recall that the space of pure quaternions can be identified with the vector space $\mathbb{R}^3$; the product of pure quaternions $u$ and $v$ can be represented in the form $uv = (u, v) + u \times v$ in which $(u, v)$ is the usual inner (scalar) product and $u \times v$ is the vector cross product. Reselecting the imaginary units $i, j, k$, we set $i := \tau$, take as $j$ any pure quaternion such that $|j| = 1$ and $(i, j) = 0$ (i.e, $j$ is any imaginary unit that is perpendicular to $i$), put $k := ij$, and obtain the automorphism (9).

The automorphism (9) can be written in the form

$$h = u + vj \mapsto h^i = u - vj \quad (u, v \in \mathbb{C}).$$

Thus, $h^\sigma = u + \sigma^2 v$ for $\sigma \in \{1, i\}$ and all $h \in \mathbb{H}$. This admits us to study similarity and consimilarity transformations simultaneously using the following definition.

Let $\sigma \in \{1, i\}$. By $\sigma$-consimilarity transformations of quaternion matrices we mean the transformations

$$A \mapsto S^{-\sigma}AS := (\sigma^{-1}S^{-1}\sigma)AS = \begin{cases} S^{-1}AS & \text{if } \sigma = 1, \\ -iS^{-1}iAS & \text{if } \sigma = i, \end{cases}$$

in which $S$ is nonsingular. It suffices to study $\sigma$-consimilarity transformations since by Lemma [1] each involutive automorphism of $\mathbb{H}$ has the form $h \mapsto h^{\sigma} = \sigma^{-1}h\sigma$ ($\sigma \in \{1, i\}$) in suitable $i, j, k$.

**Lemma 2** (cf. [7, Theorem 1]). The following statements are equivalent for $A, B \in \mathbb{H}^{n \times n}$:

(i) $A$ and $B$ are $i$-consimilar;

(ii) $iA$ and $iB$ are similar;

(iii) $iA$ and $Bi$ are similar;

(iv) $Ai$ and $Bi$ are similar.

**Proof.** (i) $\iff$ (ii) since $S^{-i}AS = -iS^{-1}i \cdot AS = B \iff S^{-1}iAS = iB$.

(ii) $\iff$ (iii) since $i^{-1}iBi = Bi$.

(iii) $\iff$ (iv) since $i^{-1}iAi = Ai$. \qed
Theorem 3 (cf. [7, Theorem 3]). Each square quaternion matrix $A$ is $\sigma$-consimilar ($\sigma \in \{1, i\}$) to a complex matrix that is a direct sum, uniquely determined up to permutation of summands, of Jordan blocks $J_k(a + bi),\quad a, b \in \mathbb{R},
\begin{cases}
  b \geq 0 & \text{if } \sigma = 1, \\
  a \geq 0 & \text{if } \sigma = i.
\end{cases}$

Proof. Wiegmann [19, Theorem 1] proved (see also Zhang’s survey [21, Theorem 6.4]) that each square quaternion matrix $A$ is similar to a direct sum, uniquely determined up to permutation of summands, of Jordan blocks $J_k(a + bi),\quad a, b \in \mathbb{R}, b \geq 0$. Using Lemma 2, we get the desired Jordan canonical form of quaternion matrices for $i$-consimilarity.

3 Quaternion matrix equations $AX - \hat{X}B = C$ and $X - A\hat{X}B = C$

In this section, we consider the quaternion matrix equations

\begin{align*}
AX - \hat{X}B &= C, \\
X - A\hat{X}B &= C,
\end{align*}

in which $A \in \mathbb{H}^{m \times m}, B \in \mathbb{H}^{n \times n}, C \in \mathbb{H}^{m \times n}$ and $h \mapsto \hat{h}$ is an arbitrary involutive automorphism of $\mathbb{H}$. These equations for the automorphism (5) are studied in [10, 15, 16, 17, 20] mainly by means of the replacement of $p \times q$ quaternion matrices with the corresponding $2p \times 2q$ complex matrices or $4p \times 4q$ real matrices. We use the consimilarity canonical form from Theorem 3 (in a similar way, Bevis, Hall, and Hartwig [1] studied the complex matrix equation $A\hat{X} - XB = C$ using the consimilarity canonical form of complex matrices).

By Lemma 1 we can suppose that $\hat{h} = h^\sigma$ for some $\sigma \in \{1, i\}$ and all $h \in \mathbb{H}$, and get the equations

\begin{align*}
AX - X^\sigma B &= C, \\
X - AX^\sigma B &= C \quad (\sigma \in \{1, i\}).
\end{align*}

(10)

For all nonsingular $S \in \mathbb{H}^{m \times m}$ and $R \in \mathbb{H}^{n \times n}$, these equations are equivalent to

\begin{align*}
S^{-\sigma} AS^{-1} XR - S^{-\sigma} X^\sigma R^\sigma R^{-\sigma} BR &= S^{-\sigma} CR, \\
S^{-\sigma} XR - S^{-\sigma} ASS^{-1} X^\sigma R^\sigma R^{-\sigma} BR &= S^{-\sigma} CR.
\end{align*}

Taking $S$ and $R$ such that $S^{-\sigma} AS$ and $R^{-\sigma} BR$ are the complex canonical forms of $A$ and $B$ determined by Theorem 3, we obtain the matrix equations that are considered in the following simple theorem and its corollaries.
Theorem 4. Let the quaternion matrix equations (10) be given by complex matrices $A$ and $B$ and a quaternion matrix $C = C_1 + C_2j$ ($C_1, C_2 \in \mathbb{C}^{m \times n}$). Then the sets of solutions of the equations (10) consist of all matrices $X = X_1 + X_2j$ in which $X_1$ and $X_2$ are complex matrices satisfying

\begin{align}
AX_1 - X_1B &= C_1, \\
AX_2 - \sigma^2X_2B &= C_2
\end{align}

(11)

for the first equation in (10) and, respectively,

\begin{align}
X_1 - AX_1B &= C_1, \\
X_2 - \sigma^2AX_2B &= C_2
\end{align}

(12)

for the second equation in (10).

Proof. Write $X = X_1 + X_2j$ and $C = C_1 + C_2j$ ($X_1, X_2, C_1, C_2 \in \mathbb{C}^{m \times n}$). Then $X^i = X_1 - X_2j$, $X^\sigma = X_1 + \sigma^2X_2j$, and the equations (10) take the form

\begin{align}
A(X_1 + X_2j) - (X_1 + \sigma^2X_2j)B &= C_1 + C_2j, \\
(X_1 + X_2j) - A(X_1 + \sigma^2X_2j)B &= C_1 + C_2j.
\end{align}

(13)

Since $A$ and $B$ are complex matrices and $jB = \bar{B}j$, the quaternion matrix equations (13) are partitioned into two pairs of complex matrix equations (11) and (12).

The theory of complex matrix equations $AX - XB = C$ and $X - AXB = C$ was developed in [2, Chapter 18], [3, Chapter VIII], [6, Section 4.4], [14, Sections 12.3 and 12.5], [9, 12, 13], and in many other books and articles.

Corollary 5. Let us apply Theorem 4 to the quaternion matrix equation $AX - X^\sigma B = C$ with $\sigma \in \{1, i\}$. Let

\[ A = J_{k_1}(\lambda_1) \oplus \cdots \oplus J_{k_p}(\lambda_p), \quad B = J_{l_1}(\mu_1) \oplus \cdots \oplus J_{l_q}(\mu_q) \quad \text{(all } \lambda_i, \mu_j \in \mathbb{C}) \]

(14)

be two complex Jordan matrices (for instance, canonical forms from Theorem 3 of quaternion matrices with respect to $\sigma$-consimilarity). Write

\[ M_\sigma := \{\lambda_1, \ldots, \lambda_p\} \cap \{\mu_1, \ldots, \mu_q, \sigma^2\hat{\mu}_1, \ldots, \sigma^2\hat{\mu}_q\}. \]

(15)

(a) The following statements hold:

- If $M_\sigma = \emptyset$, then $AX - X^\sigma B = C$ has a unique solution.
If $M_\sigma \neq \emptyset$, then two cases may arise: either $AX - X^\sigma B = C$ has no solutions, or the set of its solutions is infinite and consists of all matrices $X_0 + Y$ in which $X_0$ is a fixed particular solution of $AX - X^\sigma B = C$ and $Y$ runs over all solutions of $AY - Y^\sigma B = 0$.

(b) The set of solutions of $AY - Y^\sigma B = 0$ is described as follows. Let us partition $Y$ into blocks in accordance with the partitions of $A$ and $B$:

$$Y = [Y_{\alpha\beta}]_{\alpha=1,1=1}^p_q, \quad Y_{\alpha\beta} \text{ is } k_\alpha \times l_\beta. \quad (16)$$

Then the set of solutions of $AY - Y^\sigma B = 0$ consists of all quaternion matrices of the form $U + Vj$, in which

- $U = [U_{\alpha\beta}]_{\alpha=1,1=1}^p_q$ and $V = [V_{\alpha\beta}]_{\alpha=1,1=1}^p_q$ are complex matrices that are partitioned conformally to $Y$,
- $U_{\alpha\beta} = 0$ if $\lambda_\alpha \neq \mu_\beta$,
- $V_{\alpha\beta} = 0$ if $\lambda_\alpha \neq \sigma^2 \overline{\mu}_\beta$,
- the other $U_{\alpha\beta}$ and $V_{\alpha\beta}$ have the form

$$\begin{bmatrix}
a & b & \cdots & d \\
a & \ddots & & \\
\vdots & & \ddots & \\
0 & \cdots & & a
\end{bmatrix} \quad \text{or} \quad \begin{bmatrix}
a & b & \cdots & d \\
a & \ddots & & \\
\vdots & \ddots & & \\
0 & \cdots & & a
\end{bmatrix} \quad (a, b, \ldots, d \in \mathbb{C})$$

if the number of rows is less than or equal to the number of columns or, respectively, the number of rows is greater than the number of columns (we write the off-diagonal units in all Jordan blocks over the diagonal; see (3)).

Proof. The statement (a) follows from Theorem 4 since by [3, Chapter VIII, §3] a complex matrix equation $MX - XN = P$ with square matrices $M$ and $N$ has a unique complex solution if and only if $M$ and $N$ do not have common eigenvalues; otherwise it is contradictory or it has an infinite number of solutions.

The statement (b) follows from Theorem 4 and from the description in [3, Chapter VIII, §1] of the set of all complex matrices $Y$ such that $MY = YN$ in which $M$ and $N$ are complex Jordan matrices. \qed
Example 3.1. Let us consider the quaternion matrix equation

\[
\begin{bmatrix}
0 & 0 \\
0 & i
\end{bmatrix} X - X^\sigma \begin{bmatrix}
i & 1 \\
0 & i
\end{bmatrix} = \begin{bmatrix}
-k & j \\
0 & 0
\end{bmatrix} \quad (\sigma \in \{1, i\}).
\]  

(17)

Its set (15) is \{i\}. Its particular solution is

\[
\begin{bmatrix}
-j & 0 \\
0 & 0
\end{bmatrix} \text{ if } \sigma = 1, \quad \begin{bmatrix}
j & 0 \\
0 & j
\end{bmatrix} \text{ if } \sigma = i.
\]

To solve the corresponding equation in which the right-hand part is zero:

\[
\begin{bmatrix}
0 & 0 \\
0 & i
\end{bmatrix} Y - Y^\sigma \begin{bmatrix}
i & 1 \\
0 & i
\end{bmatrix} = \begin{bmatrix}
0 & 0 \\
0 & 0
\end{bmatrix},
\]  

(18)

we write

\[
Y = \begin{bmatrix}
Y_{11} \\
Y_{21}
\end{bmatrix} = \begin{bmatrix}
U_{11} \\
U_{21}
\end{bmatrix} + j \begin{bmatrix}
V_{11} \\
V_{21}
\end{bmatrix}.
\]

Replacing these blocks by the corresponding pairs of eigenvalues (in the notation of Corollary 5(b)), we get

\[
\begin{bmatrix}
U_{11} \\
U_{21}
\end{bmatrix} \mapsto \begin{bmatrix}
\lambda_1, \mu_1 \\
\lambda_2, \mu_1
\end{bmatrix} = \begin{bmatrix}
0, i \\
i, i
\end{bmatrix}, \quad \begin{bmatrix}
V_{11} \\
V_{21}
\end{bmatrix} \mapsto \begin{bmatrix}
\lambda_1, \delta^2 \bar{\mu}_1 \\
\lambda_2, \delta^2 \bar{\mu}_1
\end{bmatrix} = \begin{bmatrix}
0, -\sigma^2 i \\
i, -\sigma^2 i
\end{bmatrix}.
\]

By Corollary 5, the set of solutions of (18) consists of all matrices

\[
\begin{bmatrix}
0 & 0 \\
0 & *
\end{bmatrix} + \begin{bmatrix}
0 & 0 \\
0 & 0
\end{bmatrix} j \text{ if } \sigma = 1, \quad \begin{bmatrix}
0 & 0 \\
0 & *
\end{bmatrix} + \begin{bmatrix}
0 & 0 \\
0 & 0
\end{bmatrix} j \text{ if } \sigma = i,
\]

in which the stars denote complex numbers, and the set of solutions of (17) consists of all matrices

\[
X = \begin{bmatrix}
-j & 0 \\
0 & 0
\end{bmatrix} + \begin{bmatrix}
0 & 0 \\
0 & c
\end{bmatrix} (c \in \mathbb{C}) \quad \text{if } \sigma = 1,
\]

\[
\begin{bmatrix}
j & 0 \\
0 & 0
\end{bmatrix} + \begin{bmatrix}
0 & 0 \\
0 & 0
\end{bmatrix} h \quad (h \in \mathbb{H}) \quad \text{if } \sigma = i.
\]
Corollary 6. Let us apply Theorem 4 to the quaternion matrix equation $X - AX^\sigma B = C$ in which $\sigma \in \{1, i\}$ and $A, B$ are the matrices \([14]\). Write

$$M_\sigma := \{\lambda_1^{-1}, \ldots, \lambda_p^{-1}\} \cap \{\mu_1, \ldots, \mu_q, \sigma^2 \mu_1, \ldots, \sigma^2 \mu_q\}$$

in which $0^{-1} := \infty$.

The following statements hold:

- If $M_\sigma = \emptyset$, then $X - AX^\sigma B = C$ has a unique solution.
- If $M_\sigma \neq \emptyset$, then two cases may arise: either $X - AX^\sigma B = C$ has no solutions, or the set of its solutions is infinite and consists of all matrices $X_0+Y$ in which $X_0$ is a fixed particular solution of $X - AX^\sigma B = C$ and $Y$ runs over all solutions of $Y - AY^\sigma B = 0$.

Proof. These statements hold since by \([2, \text{Theorem 18.2}]\) a complex matrix equation $X - MXN = P$ with square $M$ and $N$ has a unique complex solution if and only if $\lambda \mu \neq 1$ for each eigenvalue $\lambda$ of $M$ and for each eigenvalue $\mu$ of $N$.

Conclusion: In all the papers that we know, the consimilarity of quaternion matrices is defined via the automorphism $h \mapsto -jhj$. We show that the consimilarity defined via $h \mapsto -ihi$ is more convenient for studying. All consimilarities of quaternion matrices defined via involutive automorphisms are reduced each other by reselecting of orthogonal imaginary units $i, j, k$ in $\mathbb{H}$.

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