Physical problem in interpreting classical energy of soliton of sine Gordon equation

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Abstract. The classical energy of soliton of sine Gordon equation was evaluated by using the Hamiltonian density. We classify the condition at which the energy will be either convergent or divergent. For the convergent energy, we also impose the condition at which the energy is real. We justify that there is a satisfied condition to obtain real energy.

1. Introduction
Soliton or solitary wave is a classical nonlinear wave solution where the propagating pulse is stable. If realized, this wave can be generated by combining the dispersive and nonlinear media. The interesting phenomenon is related to the stable pulse that can be interpreted as a particle. So, we deal with the particle-wave dualism in the classical wave solution. Contrary to the quantum case, this interpretation is more realistic due to a localized wave property in soliton solution.

One of the famous nonlinear wave equations is the sine Gordon (SG) equation which is a nonlinear relativistic wave equation. This equation can be found in the discussion of the nonlinear optics [1,2]. In addition, some papers reported that the SG equation has analytical solitary wave/soliton solution [3,4], where the pulse is localized. Therefore, it is a good example to discuss classical energy as well as the interpretation of its energy.

The energy of soliton can represent a mass possessed by the solitary wave/soliton. The energy may be useful to analyze the transfer heat of the electromagnetic wave in the nonlinear optics. Previous studies utilized this equation to investigate the optical soliton in fiber optics using some models [5-7]. Meanwhile, since the SG equation is a nonlinear form of Klein-Gordon equation, we can deduce a similar interpretation of energy in the SG equation.

In this manuscript, we intend to calculate the classical energy by means of the Hamiltonian density and imposing the localized condition of the soliton solution. We found the imposed conditions to make real energy. These conditions are very different with Ref. [8] which give different interpretations. In section 2, we will briefly review the Hamiltonian density by giving the ansatz form of the Lagrangian density. We will calculate the classical energy, impose the suitable condition in section 3, and criticize the interpretations in Alhidayatuddiniiyah [8]. The conclusion will be given in the last section.

2. Methods
In this section, we briefly review the properties of SG equation. The general form of the SG equation representing a nonlinear wave is given by
\( \frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2} - \frac{\partial^2 \psi}{\partial x^2} + \sin \psi = 0, \) \hspace{1cm} (1)

where \( c \) is the speed of light. Equation 1 can be considered as a nonlinear version of a relativistic wave, i.e., the Klein-Gordon equation, with \( \sin \psi \) is a nonlinear term. The analytical soliton solution of Eq. (1), which is obtained by the Bäcklund transformation, can be written as

\[ \psi(x, t) = 4 \arctan(A e^{2/(kx-\omega t)}), \] \hspace{1cm} (2)

where \( A \) is an arbitrary constant while \( \gamma \) is so-called the Lorentz contraction \( \gamma = 1/\sqrt{1-(v^2/c^2)} \). This interprets that the sine Gordon equation represents the dynamic motion of a relativistic particle.

For further discussion, if we define \( u = kx - \omega t \), we find two different solutions of Eq. (2), as shown in Fig. 1.

**Figure 1.** Profiles of soliton of sine Gordon equation for positive \( A \) (a) and negative \( A \) (b).

From Fig. 1, the kink and anti-kink solutions are achieved by setting the positive \( A \) and negative \( A \), respectively. We also conclude that the kink and anti-kink have constant values at \( u \rightarrow \pm \infty \).
Note that the solution of SG equation in Eq. (2) can also be obtained by employing some methods as stated in some references [9-12]. Akgül et al. used the kernel method with some conditions to analyze the solution [9] while Ding et al. utilized the Jacobian elliptic function to find the solution by discretizing the SG equation [10]. Meanwhile, Duan et al. introduced the lattice Boltzmann approach to compute the numerical solution of SG equation [11]. At the same time, Ablowitz et al. introduced the boundary conditions to find the solution [12].

3. Results and discussions
The Lagrangian and Hamiltonian densities of the SG equation are respectively given by Alhidayatuddiniyah [8]

\[ L = -\frac{1}{2c^2} \left( \frac{\partial \psi}{\partial t} \right)^2 + \frac{1}{2} \left( \frac{\partial \psi}{\partial x} \right)^2 - \cos \psi + D, \]  
(3)

\[ H = -\frac{1}{2c^2} \left( \frac{\partial \psi}{\partial t} \right)^2 - \frac{1}{2} \left( \frac{\partial \psi}{\partial x} \right)^2 + \cos \psi - D, \]  
(4)

where \( D \) is a constant. Therefore, the classical energy of the SG equation then can be expressed by Alhidayatuddiniyah [8] and Prayitno et al. [13]

\[ E = \int_{-\infty}^{\infty} H(x, t) \, dx = \int_{-\infty}^{\infty} \left[ -\frac{1}{2c^2} \left( \frac{\partial \psi}{\partial t} \right)^2 - \frac{1}{2} \left( \frac{\partial \psi}{\partial x} \right)^2 + \cos \psi - D \right] \, dx. \]  
(5)

According to Alhidayatuddiniyah [8], considering the dependence of \( D \) on \( E \), \( D = -2 \), \( D = 0 \), and \( D = 2 \) yield divergent \( E \). Meanwhile, \( D = -1 \) and \( D = 1 \) yield the negative \( E \) and positive \( E \), respectively. Here, we would like to give objections regarding the interpretations in Alhidayatuddiniyah [8]. In Alhidayatuddiniyah [8], they claimed that the real energy only holds for \( D = 1 \), and not \( D = -1 \). Our different interpretations will be given as follows

- For \( D = 1 \), the calculated energy yields

\[ E = \frac{\left( 1 + 8k^2 \gamma^2 - 8 \frac{\omega^2 \gamma^2}{c^2} \right)}{ky}. \]  
(6)

Imposing the condition \( E > 0 \), we find two possibilities

1) if the soliton is the massive object with the mass \( m > 0 \) \((v << c)\), then it should maintain

the condition \( 1 + 8k^2 \gamma^2 > 8 \frac{\omega^2 \gamma^2}{c^2} \), leading to a real energy.

2) if the soliton is the massless object with the mass \( m = 0 \) \((v = c)\), then it should get the

energy \( E = 1/k \), thus the energy is also real.

- For \( D = -1 \), the calculated energy yields

\[ E = -\frac{\left( 1 + 8k^2 \gamma^2 - 8 \frac{\omega^2 \gamma^2}{c^2} \right)}{ky}. \]  
(7)

Imposing the condition \( E > 0 \), we find also two possibilities

1) if the soliton is the massive object with the mass \( m > 0 \) \((v << c)\), then it should maintain

the condition \( 1 + 8k^2 \gamma^2 < 8 \frac{\omega^2 \gamma^2}{c^2} \), which still give a real energy.
2) if the soliton is the massless object with the mass \( m = 0 \) (\( v = c \)), then it should get the energy \( E = -1/k \), which cannot give a real energy.

Based on our results, the interpretations are slightly different with Alhidayatuddiniah [8], which only hold for \( D = 1 \). Their interpretation is also incorrect for the divergent \( E \), which gives the delocalized property of soliton. In our opinion, the divergent \( E \) leads to unrealistic soliton as a particle. Therefore, only \( D = 1 \) for all conditions and \( D = -1 \) for \( m > 0 \) can interpret the soliton as a particle.

Compared to Wazwaz [14], which includes the time-dependent coefficient, we do not need an initial condition as long as \( D \) is appropriate. In addition, since the SG equation can be considered as a nonlinear version of Klein-Gordon equation, we can make the same interpretation of the positive and negative energies, as shown in Eqs. (6) and (7). In quantum field theory, the positive and negative energies correspond to the particle and anti-particle, respectively. This interpretation may be brought in this nonlinear version. Also, the appearance of kink and anti-kink solutions may also be related to the positive and negative energies. Although the SG equation is widely considered as a nonlinear version of the Klein-Gordon equation, another nonlinear form of the Klein-Gordon equation may appear. Prayitno [15] provided the model of the nonlinear Klein-Gordon equation, which described a realistic quantum particle.

4. Conclusions
We found the conditions, at which the energy of SG equation can be real, which gives the slight different interpretations with Alhidayatuddiniah [8]. For \( D = 1 \) in all conditions, the soliton can be interpreted as either massive object or massless object. Meanwhile, \( D = -1 \) for \( m > 0 \) interprets the soliton as a massive object.

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