Dirac Quantization and Fractional Magnetoelectric Effect on Interacting Topological Insulators

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We use Dirac quantization of flux to study fractional charges and axion angles $\theta$ in interacting topological insulators with gapless surface modes protected by time-reversal symmetry. In interacting topological insulators, there are two types of fractional axion angle due to conventional odd and nontrivial even flux quantization at the boundary. On even flux quantization in a gapped time reversal invariant system, we show that there is a halved quarter fractional quantum Hall effect on the surface with Hall conductance of $\frac{e^2}{4h}$ with $p,q$ odd integers. The gapless surface modes can be characterized by a nontrivial $\mathbb{Z}_2$ anomaly emerged from the even flux quantization. It is suggested that the electron can be regarded as a bound state of fractionally charged quarks confined by a nonabelian color gauge field on the Dirac quantization of complex spinor fields.

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Recently there has been paid a great attention to topologically nontrivial states of quantum matter on time-reversal invariant topological insulators (TI) [1,2]. As a theory of 3D TI theories [3,7], topological field theories(TFTs) have been developed in the low energy limit on the various dimensions of TIs [8]. In the formulation of a noninteracting TFT on $(3 + 1)$D, special quantum effects can be induced from the interior of 3D TIs through the couplings between electric and magnetic fields on the Dirac quantization condition of charges and fluxes, similar to the coupling of an axion particle to ordinary electric and magnetic fields. The theory of TIs is described by the effective action

$$S_\theta(E, B) = K_\theta \frac{e^2}{2\pi} \int d^3xdE \cdot B$$

where $K_\theta = \frac{\theta}{2\pi}$, E and B stand for the electromagnetic fields [8,11]. On the condition of normalization all physical quantities become preserved under shifts of $\theta$ by multiples of $2\pi$. Because $E \cdot B$ gets odd under the T operation, there are $0$ or $\pi \mod 2\pi$ in the values allowed by T reversal symmetry. Furthermore, the time(T)-invariant fractional topological states have been suggested in $(4 + 1)$D [12]. Fractional states can, in general, be appeared in terms of strong interactions in $(3 + 1)$D. Hence it is very important to construct a general theory of fractional TIs on a $(3 + 1)$D spin manifold in the presence of strong interactions.

In this brief report, we exploit Dirac quantization to investigate fractional charges and axion angles $\theta$ under the construction of general theory for interacting TIs with gapless surface modes protected by time-reversal symmetry. For interacting TIs, we report three crucial results. First, there exist two kinds of fractional axion angle due to the conventional odd and nontrivial even flux quantization at the boundary. Secondly, when making the even flux quantization required in a gapped time reversal invariant system, we show that there is a halved quarter fractional quantum Hall effect(FQHEs) on the surface with Hall conductance of $\sigma_y^H = \frac{p}{4q} \pi$ with $p,q$ odd integers. Thus it is proposed that the Dirac quantization of even flux leads to the fractional bulk topological quantum number for non-integer, rational multiples such that $K_\theta = \frac{1}{4} \frac{p}{q}$ with $p,q$ odd integers on the $(3 + 1)$D spin manifold of fractional TIs. And finally we claim that the system can have degenerate ground states on a closed topologically nontrivial space- and time-reversal protected gapless surface states which are characterized by $\mathbb{Z}_2$ anomaly of $(-1)^{\omega(S)}$ with $\omega(S)$ mod 2 of a 2-cocycle $\Sigma$, i.e., 2-cycle of a closed form, caused by even flux boundary. In a time reversal (TR) symmetric TI of quantum magnets, there can be two topological objects of fractionalized charge $\frac{1}{2}$ emerged from an interacting TI with a band gap because of the even flux quantization which is twice a 1-cycle flux quantization. Very recently, Maciejko and his companies have suggested the possibility of TR invariant fractional topological insulators for fermions in strong coupled $SU(N)$ gauge theory [13]. In the large $N$ limit, this theoretical construction can have a serious problem due to a TR symmetry broken spontaneously because $SU(N)$ gauge theory can not preserve time reversal symmetry [14]. But within our theoretical framework of interacting TIs, fractional topological insulators can be realized in the absence of TR symmetry breaking caused by strong interacting $SU(N)$ gauge theory. The current theory can only give rise to fractional $\frac{1}{2}$ in a gapped time reversal invariant system of bosons or fermions provided that the system also takes deconfined fractional excitations and associated degenerate ground state on topologically nontrivial spaces. We show the above findings theoretically as followings.
In order to investigate the fractional TI on the basis of a more systematical approach given by Maciejko and his companies, we take into account the projective construction of FQH states for a composite electron on the surface of an emerged spin manifold in \((3 + 1)D\). The electron is decomposed into \(N_f^c\) different flavors of fractionally charged and fermionic partons with \(N_f^c\) partons of each flavor \(f = 1, \ldots, N_f\). This decomposition should obey two fundamental constraints:

\[ N_1^c + \cdots + N_f^c = \text{odd}, \quad N_f^c q_1 + \cdots + N_f^c q_N = e. \]  \(\text{(2)}\)

The first constraint of Eq. \(\text{(2)}\) means that the total number of partons per electron has to be odd because the electron preserves the fermion statistics. The other constraint is that the total charge of the partons has to sum up to the electron charge \(e\) when \(q_f < e\) becomes the fractional charge for partons of flavor \(f\).

Provided that the partons get recombined together to represent the physical electrons, we can construct an interacting many-body wavefunction as a new topological state of electrons emerged in \((3 + 1)D\). The total electron wavefunction is expressed by a product of parton ground state wavefunctions \([15]\)

\[ \prod_{f=1}^{N_f} \psi_f(r_f) = \prod_{f=1}^{N_f} \Psi_f(r_f) \]  \(\text{(3)}\)

Here \(\psi_f(r_f)\) stands for the parton ground state wavefunction given by a Slater determinant which describes the ground state of a noninteracting TI Hamiltonian, and \(r_f, s_i\), \(i = 1, \ldots, N\), the position and spin coordinates of the partons.

To be more specific, let us consider an effective field theory of fractional TI on a diamond lattice of the \(SU(N)\) electrons. The Hamiltonian is given by

\[ H = \sum_{ab} \{ C_{aa}^f a_{ab}^f e^{i \theta A_{ab}^f} C_{ab} + H.C. \} + H_{int}(C^f, C), \]  \(\text{(4)}\)

where \(a, b\) indicate site indices, \(\alpha, \beta\) stand for internal degrees of freedom, \(a_{ab}\) denotes the Hamiltonian matrix, \(A_{ab} = \int r_a dr \cdot A\) with \(A\), the \(U(1)\) electromagnetic vector potential, and \(H_{int}\) means an interaction Hamiltonian between electrons. \(C_{aa}\) is the electron operator decomposed as

\[ C_{aa} = \prod_{f=1}^{N_f} \psi_{f}^j(r_a) \cdots \psi_{N_f}^j(r_a) \]  \(\text{(5)}\)

with obeying constraint rules Eq. \(\text{(2)}\). Here \(\psi_{f}^j(r_a), j = 1, \ldots, N_f^c\) are quark operators with \(N_f^c\) partons of each flavor \(f\). The projection onto the physical Hilbert space can be realized by including \(SU(N_f^c)\) gauge transformation.

Now we take into account the Dirac quantization \([18]\) of fermions generated by complex spinor fields through a two-cycle \(\Sigma\) in the sense of antisymmetric \(N_f^c\) partons, i.e., the odd-number constraint of \(N_f^c\), of a composite electron on \(M_4\). The extension of Dirac quantization to any 2-cycle was described by O. Alvarez \([13, 20]\). Let us study the Dirac quantization by following the Alvarez’s extension. Under the Dirac quantization, the antisymmetric parton wavefunctions can be represented by a nonabelian color gauge field \(SU(N_f^c)\) with an interacting constant \(g\) on \(M_4\) which is covered by a finite number of neighborhoods \(U_i, i = 1, \ldots, N\). In each neighborhood, more structures should be considered on the representation of internal symmetries. In addition to the \(U(1)\) connection or gauge potential, there must be an oriented frame of vierbein \(V_i\) and complex spinor fields of antisymmetric partons \(\{\Psi_{1i}\}^{N_f^c}\) with \(N_f^c\) only odd. These symmetries cannot be independent of choices made in the neighborhood \(U_i\). As choices of degrees of freedom, there can be local \(U(1) \times SU(N_f^c)\) gauge transformations \((\chi, \lambda)\)

\[ \{\Psi_{1i}\}^{N_f^c} \rightarrow \{e^{i\alpha_1 + i\alpha_4} \chi \Psi_{1i}\}^{N_f^c}, \]

\[ A_i \rightarrow A_i + d\chi, \quad a_i \rightarrow a_i + d\lambda, \]  \(\text{(6)}\)

and \(SO(4)\) local transformations \([20]\)

\[ V_i \rightarrow R V_i, \quad \{\Psi_{1i}\}^{N_f^c} \rightarrow \{S(R)\Psi_{1i}\}^{N_f^c}, \]  \(\text{(7)}\)

where \(R \in SO(4)\). It is easy to see that there can be a sign ambiguity from the lift of \(R \rightarrow \pm S(R)\) since the quotient of the spin group \(Spin(4)\) by \(Z_2\) is isomorphic to \(SO(4)\). For a double overlap on two contiguous neighborhoods, \(U_i \cap U_j \neq 0\), one should take transition functions associated with transformation groups

\[ A_i \rightarrow A_j + d\chi_{ij}, \quad a_i \rightarrow a_j + d\lambda_{ij}, \quad V_i \rightarrow R_{ij} V_j, \]

\[ \{\Psi_{1i}\}^{N_f^c} \rightarrow \{S(R_{ij}) e^{i\alpha_{ij}} e^{i\lambda_{ij}} \Psi_{1j}\}^{N_f^c}. \]  \(\text{(8)}\)

In a triple overlap region, \(U_i \cap U_j \cap U_k \neq 0\), which is supposed to be contractible, we can have consistency conditions

\[ R_{ij} R_{jk} R_{ki} = I, \quad S(ijk) \equiv S(R_{ij}) S(R_{jk}) S(R_{ki}) = \pm I. \]  \(\text{(9)}\)

It is noted that the above equations have identity elements of \(SO(4)\) and \(Spin(4)\). Consequently, under \(U(1) \times SU(N_f^c)\) gauge transformations and in the spinor representations of antisymmetric \(N_f^c\) partons with \(N_f^c\) odd, we can obtain

\[ \{\Psi_{1i}\}^{N_f^c} \rightarrow \{e^{i\alpha_{ij} + i\alpha_{jk} + i\alpha_{ki}} e^{i\lambda_{ij} + i\lambda_{jk} + i\lambda_{ki}} S(R_{ij}) S(R_{jk}) S(R_{ki}) \Psi_{1i}\}^{N_f^c}. \]  \(\text{(10)}\)

Up to a sign, a crucial point is to take a lift from \(SO(4)\) to \(Spin(4)\) in the right hand side of Eq. \(\text{(10)}\). We cannot determine the sign when it is dependent on the choices made in the transformation groups of Eq. \(\text{(10)}\). In the
sense of different overlaps, the signs of Eq. (10) cannot be totally independent. Thus the spinor consistency condition enables us to obtain
\[ e^{iq_iC_{ijk}e^{i\theta D_{ijk}} = S(ijk)}, \] (11)
in the model of total $U(1) \times SU(N_1^c)$ gauge fields. Here $C_{ijk}$ and $D_{ijk}$ satisfy the self-consistency relations
\[ C_{ijk} \equiv \chi_{ij}(r) + \chi_{jk}(r) + \chi_{ki}(r) \in Z/q_iN_1^c, \]
\[ D_{ijk} \equiv \lambda_{ij} + \lambda_{jk} + \lambda_{ki} \in Z/gN_1^c. \] (12)

Therefore, by $U(1) \times SU(N_1^c)$ gauge theory through a two-cycle $\Sigma$ which is a 2D-manifold without boundary, the Dirac quantization of flux gives rise to
\[ \exp(2\pi i \int_{\Sigma} (q_1F + gG)) = (-1)^{\omega(\Sigma)}, \] (13)
where $G$ is the $SU(N_1^c)$ field strength, with $N_1^c$ odd. Here the sign is determined by the finite product over triple overlaps
\[ (-1)^{\omega(\Sigma)} = \prod_{U_i \cap U_j \cap \Sigma \neq 0} S(ijk). \] (14)

Finally let us account for the flux through $\Sigma_i$ with non-trivial boundary $\partial\Sigma_i$, $\forall i = 1, 2, \ldots, N$. Then in the representation of complex spinor fields for fermions generated by antisymmetric partons, the spinor field consistency in Eq. (11) leads to the boundary Dirac quantization condition
\[ \exp(2\pi i \int_{\Sigma_i} (q_1F + gG)) = \exp(2\pi i \int_{\partial\Sigma_i} (q_1A + ga)) \times \prod_{U_i \cap U_j \cap \Sigma \neq 0} S(ijk), \forall i = 1, 2, \ldots, N. \] (15)

The problem in question is that the two factors of Eq. (15) cannot be independent of the choice of neighborhoods although the neighborhoods can be independently chosen in the right hand side. Let us consider Dirac flux quantization at the conventional odd boundary of a 4D manifold. In the viewpoint of conventional flux quantization at the odd boundary, i.e., $\partial\Sigma_i = \gamma_i$ which means a loop or a 1-cycle, it is argued that adding a neighborhood to the interior of $\Sigma$ cannot affect the first factor as well as the second factor. Thus the boundary Dirac quantization yields to
\[ \exp(2\pi i \int_{\Sigma_i} (q_1F + gG)) = \exp(2\pi i \phi \left( \frac{e}{N_1^c}A + ga \right)), \forall i = 1, 2, \ldots, N \] (16)

Therefore we can find $q_1 = \frac{\phi}{N_1^c}$ due to a conventional odd Dirac quantization of flux of antisymmetric $N_1^c$ partons at the boundary, i.e., $\partial\Sigma_i = \gamma_i$ on a 4D manifold.

Although any better way is not known for understanding Stokes’ theorem in the current context, the sign problem in Eq. (15) will, however, be resolved if the boundary of $\Sigma_i$ become even such as $\partial\Sigma_i = 2\gamma_i, \forall i = 1, 2, \ldots, N$. Hence the even flux quantization leads to a form \[ (-1)^{\omega(\Sigma)} \]
\[ = (-1)^{\omega(\Sigma)} (q_1F + gG)). \] (17)

If the two factors of Eq. (15) become independent of choices of neighborhoods, they can be well defined since the second factor is well-behaved due to the extra number 2 in the exponent of Eq. (17). Therefore we can obtain $q_1 = \frac{\phi}{2N_1^c}$ from the even flux quantization at the boundary, i.e., $\partial\Sigma = 2\gamma$ on a 4D manifold with spin structures. It follows that $\omega(\Sigma)$ becomes well-defined mod 2 on the $\Sigma$ with even flux boundary condition. This remarkable result shows that the system can have degenerate ground states on a closed topologically nontrivial space-time-reversal-protected gapless surface states which are characterized by $(-1)^{\omega(\Sigma)}$ at the even flux boundary. There can be two topological objects of fractionalized charge $\frac{1}{2}$ emerged from the even flux quantization.

So far we have described the Dirac quantization of flux with the total $U(1) \times SU(N_1^c)$ quark field strength $q_1F + gG$ on the complex spinor representations of antisymmetric parton wavefunctions. The interactions yield the quarks to condensing at low energies into a noninteracting T-invariant TI state with axion angle $\theta$. It has been assumed that $M_4$ has spin structures, and odd and even boundaries. In particular, we have to account for the three main results of boundary flux quantization into the topological term of the effective field theory. In the topological term $\frac{\theta_1}{\pi} F \cdot E \cdot B = \frac{\pi}{24} F_{\mu\nu} \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}$ for noninteracting TIs, the $U(1)$ electron field strength is replaced by the $U(1) \times SU(N_1^c)$ one. We can construct a partition function
\[ Z = C(-1)^{\omega(\Sigma)} \exp(i \int_{M_4} d^2x dt L_{\text{eff}}(F,G)). \] (18)

The effective Lagrangian is given by $L_{\text{eff}}(F,G) = L_0 + L_{\text{top}}$ where $L_0 = -\frac{1}{4} G_{\mu\nu} G^{\mu\nu}$ is the kinetic Yang-Mills (YM) Lagrangian. The second topological term of $L_{\text{eff}}(F,G)$ is expressed in terms of
\[ L_{\text{top}} = \frac{\theta_1}{32\pi^2} \epsilon^{\mu\nu\rho\sigma} \text{Tr} [(q_1F_{\mu\nu} + gG_{\mu\nu})(q_1F_{\rho\sigma} + gG_{\rho\sigma})] = \partial_\mu \epsilon^{\mu\nu\rho\sigma} \left( \frac{\theta_0 \epsilon^2}{8\pi^2} A_\nu \partial_\rho A_\sigma + \frac{\theta_1 g^2}{8\pi^2} (a_\mu \partial_\rho a_\sigma + \frac{2}{3} g a_\mu a_\rho a_\sigma) \right), \] (19)

where $\text{Tr}$ stands for the trace in the $N_1^c$ representation of $SU(N_1^c)$, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $G_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu + ig[a_\mu, a_\nu]$ indicate $U(1)$ and $SU(N_1^c)$ field strengths, respectively, and $\theta_1$ is an action angle for $N_f = 1$. It is noted
that the crossed terms such as \( \text{Tr}(F_{\mu\nu}F_{\rho\sigma}) \) become zero owing to the tracelessness of the \( SU(N_f^c) \) gauge field. Under the nonabelian gauge theory, there can possibly be two phases such as confined phases with chiral symmetry breaking and deconfined phases with gapless gauge modes for large flavor cases. In this work, we however focus on the deconfined phases for interacting TIs with topologically protected surface states, and the YM kinetic term does not make a crucial contribution in the large flavor cases of TIs.

The electromagnetic response allows us to have two classes of effective axion angles obtained from conventional odd and nontrivial even boundary Dirac quantization caused by Eq. (16) and Eq. (17), respectively. From the first term of Eq. (19), they are given, respectively, by

\[
\theta_{eff}^{(1)} = \frac{\theta_1}{N_1^f} = 0, \pm \frac{\pi}{N_1^f}, \pm \frac{3\pi}{N_1^f}, \pm \frac{5\pi}{N_1^f}, \cdots, \frac{N_1^c}{2N_1^f} \quad \text{odd,}
\]

\[
\theta_{eff}^{(2)} = \frac{\theta_1}{2N_1^f} = 0, \pm \frac{\pi}{2N_1^f}, \pm \frac{3\pi}{2N_1^f}, \pm \frac{5\pi}{2N_1^f}, \cdots, \frac{N_1^c}{2N_1^f} \quad \text{odd.}
\]

According to the first and second term in Eq. (19), the T-invariance enables us to quantize \( \theta_{eff} \) in integer multiples of \( \pi \) if the minimal electric charge becomes \( e \), and in the current parton model of quarks, \( \theta_1 = \frac{\pi}{N_1^f} \) and \( \frac{e}{2N_1^f} \) due to odd and even flux quantization at the boundary, respectively. But conventional odd flux quantization does not lead to a time reversal protected topological insulator because \( SU(N_1^f) \) gauge theory has a serious TR symmetry breaking in the large \( N_1^f \) limit [14]. On the other hand, owing to \( \mathbb{Z}_2 \) anomaly of \((-1)^{\omega(\Sigma)}\) in Eq. (18) with \( \omega(\Sigma) \mod 2 \) of a 2-cocycle \( \Sigma \) at the boundary, it follows that even flux quantization can yield a topological insulator of TR invariance. Consequently, \( \theta_{eff} \) should have quantized values in odd integer multiples of \( \frac{\pi}{2N_1^f} \).

Let us consider the effective field theory in the multiple flavor values \( N_f^c \geq 1 \). Assume that quarks of flavor \( f \) produces a noninteracting TI with \( \theta_f = \pi \mod 2\pi \). Then after integrating them, the effective theory has a gauge group of \( U(1) \times \prod_{f=1}^{N_f^c} U(N_f^c) \) where \( U(1) \) denotes the overall \( U(1) \) gauge transformation of the electron operator. This gauge group leads to the axion angle

\[
\theta_{eff} = \{ \sum_{f=1}^{N_f^c} N_f^c \}^{-1}. \]

If \( \frac{\pi}{N_f^c} \) becomes odd for each flavor, then there can be \( p \) and \( q \) of odd integers such as \( \theta_{eff} = \pi \frac{p}{2N_f^c} \). The partition function of the gauge fields allows us to have important physical properties for the interacting TI. In general, the surface of the interacting TI has an action domain wall with the electromagnetic action angle which jumps from \( \theta_{eff} \) in the fractional TI to 0 in the vacuum. Hence by even boundary Dirac quantization we can find a halved quarter FQHE on the surface with Hall conductance

\[
\sigma_H = \frac{D}{4q} \epsilon^2 \quad p, q \text{ odd.}
\]