Sequential Scaled Sparse Factor Regression

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ABSTRACT
Large-scale association analysis between multivariate responses and predictors is of great practical importance, as exemplified by modern business applications including social media marketing and crisis management. Despite the rapid methodological advances, how to obtain scalable estimators with free tuning of the regularization parameters remains unclear under general noise covariance structures. In this article, we develop a new methodology called sequential scaled sparse factor regression (SESS) based on a new viewpoint that the problem of recovering a jointly low-rank and sparse regression coefficient matrix can be decomposed into several univariate response sparse regressions through regular eigenvalue decomposition. It combines the strengths of sequential estimation and scaled sparse regression, thus sharing the scalability and the tuning free property for sparsity parameters inherited from the two approaches. The stepwise convex formulation, sequential factor regression framework, and tuning insensitiveness make SESS highly scalable for big data applications. Comprehensive theoretical justifications with new insights into high-dimensional multi-response regressions are also provided. We demonstrate the scalability and effectiveness of the proposed method by simulation studies and stock short interest data analysis.

1. Introduction

Highly developed technologies and devices have brought in massive datasets in various fields ranging from health and bioinformatics to marketing and economics. In these big data applications, modeling complex dependence structures of multivariate outcomes using observed features is of great importance since it reveals domain knowledge behind the data. For instance, inferring the influence networks from user activities has wide applications in social media marketing (Gomez-Rodriguez, Leskovec, and Krause 2012) and crisis management (Starbird and Palen 2012). By representing the dependency between the outcomes and the predictors through a jointly low-rank and sparse structure, thus alleviating the curse of dimensionality and facilitating model interpretability, sparse reduced-rank regression has gained increasing popularity in large-scale association analyses.

Depending on how the regression coefficient matrix is recovered, sparse reduced-rank regression can generally be grouped into two classes. One is directly estimating the regression coefficient matrix via different kinds of regularization (Anderson 1951; Izenman 1975; Yuan et al. 2007; Bunea, She, and Wegkamp 2011; Candès and Plan 2011; Giraud 2011; Negahban and Wainwright 2011; Bunea, She, and Wegkamp 2012; Chen and Huang 2012; Chen, Dong, and Chan 2013; Lian, Feng, and Zhao 2015; Liu, Wang, and Zhao 2015; Goh, Dey, and Chen 2017; Fan, Gong, and Zhu 2019), where $L_1$ and nuclear norm penalizations are convex relaxations popularly employed to enforce sparse and low-rank structures, respectively. The other is to recover the coefficient matrix from a latent factor point of view by combining the estimated sparse singular vectors based on singular value decomposition (Chen, Chan, and Stenseth 2012; Mishra, Dey, and Chen 2017; Uematsu et al. 2019; Zheng et al. 2019). Compared with the former class, this type of methods generally enjoy lower computational cost and can be efficiently parallelized in various computing devices. In particular, the sequential estimation procedures proposed in Mishra, Dey, and Chen (2017) and Zheng et al. (2019) demonstrate scalability in large-scale applications by decomposing the estimation of the entire coefficient matrix into unit rank matrix recovery problems. They are guaranteed to stop in a few steps under low-rank structures. Nevertheless, these sequential approaches need to tune the optimal sparsity parameter in each step, which can vary between different layers and account for major computational cost when tuned over a wide range of potential values by either cross-validation or some information criterion.

To further enhance the scalability, it is of urgent need to develop methodology which enjoys free tuning of the regularization parameters for large-scale association analyses. For high-dimensional univariate response sparse regression, tuning free methods have been proposed in Belloni, Chernozhukov, and Wang (2011) and Sun and Zhang (2012), where the universal regularization parameter controlling the sparsity was shown to be independent of the noise level. It was achieved through obtaining an equilibrium between iteratively estimating the noise level via the mean residual square and scaling the penalty in proportion to the estimated noise level. However, it
is much more difficult to develop tuning insensitive methods for large-scale multi-response regression since the population covariance matrix of the noise vector can adopt general high-dimensional structures and its sample estimate is usually not invertible to formulate a joint estimation procedure. When the noise covariance matrix is diagonal, meaning that the noises related to different responses are uncorrelated, Liu, Wang, and Zhao (2015) proposed the calibrated multivariate regression to attain tuning insensitivity for either nuclear norm or sparsity penalization. For general structures of the noise covariance matrix or jointly low-rank and sparse coefficient matrix, to the best of our knowledge, there is no existing work which enjoys tuning free property for the sparsity parameters as estimating an invertible noise covariance matrix needs extra penalization in high dimensions.

In this article, we develop a new methodology for high-dimensional multi-response regression called sequential scaled sparse factor regression (SESS), which combines the strengths of sequential estimation and scaled sparse regression, thus sharing the scalability and the tuning free property for sparsity parameters inherited from the two approaches. The main contributions of this article are as follows. First of all, we rigorously prove that the problem of recovering a jointly low-rank and sparse regression coefficient matrix can be decomposed into several univariate response sparse regressions through regular eigenvalue decomposition, which provides a new viewpoint on large-scale association analyses. Compared with the sparse eigenvalue problem, regular eigenvalue decomposition is convex and guaranteed to converge. Second, based on the new viewpoint, the proposed approach SESS adopts a universal sparsity parameter in the subsequent univariate response regressions, which is among the first attempts to achieve tuning insensitive jointly low-rank and sparse estimation under general noise covariance structures. Accompanied with a simple BIC-type information criterion for identifying the true rank, whose choices are discrete and thus demonstrating significant gaps between the correct one and the other candidates, SESS is tuning insensitive in both sparsity and rank. The stepwise convex formulation, sequential factor regression framework, and tuning insensitiveness make SESS highly scalable for big data applications. Last but not least, we provide comprehensive theoretical justifications on the effectiveness of the suggested methodology including consistency in estimation, prediction, and rank selection under mild and interpretable conditions, which reveal new insights into high-dimensional multi-response regression.

The rest of the article is organized as follows. Section 2 presents the model setting and our new methodology. We establish asymptotic properties of the proposed method in Section 3. In Section 4, we verify the theoretical results empirically through simulation examples. An application to the stock short interest data is provided in Section 5. Section 6 concludes with extensions and possible future work. All technical details are relegated to the supplementary materials.

2. Sequential Scaled Sparse Factor Regression

2.1. Model Setting

Consider the following multi-response regression model in the fixed design setting

\[ Y = XC^* + E, \]

where \( Y = (y_1, \ldots, y_n)^T \) denotes an \( n \times q \) matrix with \( q \) responses, \( X = (x_1, \ldots, x_n)^T \) is an \( n \times p \) design matrix with \( p \) predictors, \( C^* \) is an unknown \( p \times q \) coefficient matrix, and \( E = (e_1, \ldots, e_n)^T \) is an \( n \times q \) random error matrix with each row vector \( e_i \) independent and identically distributed (iid) as \( N(0, \Sigma^1) \). The columns of \( X \) are standardized to have a common \( L_2\)-norm \( \sqrt{n} \). Both dimensions \( p \) and \( q \) are allowed to diverge non-polynomially with the sample size \( n \) and \( C^* \) is assumed to be jointly low-rank and sparse (in rows), entailing the selection of significant predictors.

Similar to Mishra, Dey, and Chen (2017) and Zheng et al. (2019), we will recover the coefficient matrix \( C^* \) from a latent factor point of view to facilitate sequential estimation. Specifically, based on the SVD representation of \( n^{-1/2}XC^* \), we have

\[
\frac{1}{\sqrt{n}}XC^* = \left( \frac{1}{\sqrt{n}}XU^* \right)D_0V_0^T,
\]

satisfying

\[
\frac{1}{\sqrt{n}}XU^* = \left( \frac{1}{\sqrt{n}}XU^* \right)^T = V_0^TV_0 = 1_r,
\]

where \( C^* = U^*D_0^*V_0^T \in \mathbb{R}^{p \times q} \), \( U^* = (u_1^*, \ldots, u_r^*) \in \mathbb{R}^{p \times r} \), \( D_0^* = \text{diag}(d_1^*, \ldots, d_r^*) \in \mathbb{R}^{r \times r} \), \( V_0^* \in \mathbb{R}^{q \times r} \), and \( r^* \) is the rank of \( C^* \) that allows to be divergent. In the population level, \( V_0^* \) can be identified as the right singular vectors of \( XC^* \). Then \( U^* \) can be obtained from \( VC^*V_0^* \) after rescaling the columns even if \( p \) is larger than \( n \), which gives the feasibility of the above decomposition.

It is worth pointing out that the nonzero singular values in \( D_0^* \) can diverge with the dimensionality \( q \) and their magnitudes can be as large as \( n^{-1/2} |XC^*|_2 \), which is around the order of \( \sqrt{q} \) when each component of the \( n \) by \( q \) noiseless response matrix \( Y^* = XC^* \) is around a constant level. To ensure the identifiability of the left singular vectors \( u_k^* \), we assume that \( u_k^* \) are sparse (inherited from the row sparsity of \( C^* \)) and \( u_k^* \perp \text{Ker}(P) \) with \( \text{Ker}(P) \) the null space of the Gram matrix \( P = n^{-1}XX^T \). For the right singular vectors, we do not impose any sparsity constraint and will discuss in Section 2.2 how to obtain sparse estimates if the population ones are indeed sparse.

For ease of presentation, we set \( V^* = V_0^*D_0^* = (v_1^*, \ldots, v_{r^*}) \in \mathbb{R}^{q \times r^*} \) so that the right singular vectors absorb the singular values and are no longer of unit length. It yields

\[
C^* = U^*V^*T = \sum_{k=1}^{r^*} u_k^*v_k^* = \sum_{k=1}^{r^*} C_k^* = C_1^* \cup C_2^* \cup \cdots \cup C_{r^*}^*,
\]

where \( C_k^* = u_k^*v_k^*T \) is the \( k \)th layer unit rank matrix of \( C^* \). Here the singular vectors are sorted by the magnitudes of the singular values of \( XC^* \), consistent with the contribution to the prediction of \( Y \). Generally speaking, the decomposition in the form of \( C^* = U^*V^*T \) is not unique without orthogonality constraints. But decomposition (3) is the special one that gives \( r^* \) uncorrelated latent factors \( Xu_k^* \) in view of the orthogonality of \( n^{-1/2}UXC^* \) in (2). Each latent factor is a linear combination of a small subset of the predictors due to the sparsity of \( u_k^* \). Our
goal is to scalably and accurately estimate the singular vectors $\mathbf{u}_k^*$ and $\mathbf{v}_k^*$, as well as the true rank $r^*$, so that the latent factors and their impacts can be recovered.

### 2.2. Scalable Estimation by SESS

The proposed method SESS is motivated by the fact that in the noiseless case $\mathbf{Y}^* = \mathbf{X}C^*$ with $C^*$ adopting decomposition (3), the latent factors $\mathbf{Z}_k^* = \mathbf{Xu}_k^*$, $1 \leq k \leq r^*$, are the top-$r^*$ eigenvectors of the following eigenvalue problem

$$(nq)^{-1}\mathbf{Y}^*\mathbf{Y}^{*T}\mathbf{Z} = \lambda \mathbf{Z}.$$  

The corresponding eigenvalues

$$\lambda_k = (nq)^{-1}||\mathbf{XC}_k^*||_F^2 = q^{-1}d_k^2$$  

are typically around the constant level based on the discussion on the magnitudes of $d_k^*$ after (2). Moreover, it can be verified that the right singular vectors $\mathbf{v}_1^*, \ldots, \mathbf{v}_r^*$ satisfy the following intrinsic relationship with $\mathbf{u}_1^*, \ldots, \mathbf{u}_r^*$,

$$\mathbf{v}_k^* = \frac{1}{\mathbf{u}_k^T\mathbf{X}^T\mathbf{X}_k^*}\mathbf{Y}^{*T}\mathbf{X}_k^* = \frac{1}{n}\mathbf{Y}^{*T}\mathbf{X}_k^* = \frac{1}{n}\mathbf{Y}^{*T}\mathbf{Z}_k^*. \quad (5)$$

Therefore, with data matrix $(\mathbf{X}, \mathbf{Y})$, we propose to recover the left singular vectors $\mathbf{u}_k^*$ sequentially in two steps. The first step is to solve the regular eigenvalue problem

$$(nq)^{-1}\mathbf{Y}\mathbf{Y}^{T}\mathbf{Z} = \lambda \mathbf{Z} \quad (6)$$

and get the estimated latent factors $\hat{\mathbf{Z}}_k$ with $||\hat{\mathbf{Z}}_k||_2 = \sqrt{n}$ as well as the corresponding eigenvalues $\hat{\lambda}_k$. Then various kinds of regularization methods can be applied to recover the singular vectors $\mathbf{u}_k^*$ (see Tibshirani 1996; Fan and Li 2001; Zou 2006; Candès and Tao 2007; Fan, Feng, and Wu 2009; Fan and Lv 2011; Fan, Fan, and Barut 2014; Fan and Lv 2014; Yu and Feng 2014; Weng, Feng, and Qiao 2019, among many others).

To facilitate the theoretical analysis, here we use the popularly used Lasso (Tibshirani 1996) to obtain the sparse left singular vectors $\hat{\mathbf{u}}_k$ by solving

$$\hat{\mathbf{u}}_k = \arg\min_{\mathbf{u}} \left\{ \frac{1}{2n}||\hat{\mathbf{Z}}_k - \mathbf{X}\mathbf{u}||_2^2 + \omega_k||\mathbf{u}||_1 \right\},$$

where $\omega_k$ is a regularization parameter controlling sparsity and needs to be tuned by cross-validation or certain information criterion. In big data applications, we can further save the tuning of the regularization parameter through the following scaled version of the Lasso (Sun and Zhang 2012)

$$(\hat{\mathbf{u}}_k, \hat{\sigma}_k) = \arg\min_{\mathbf{u}, \sigma} \left\{ \frac{1}{2n}||\hat{\mathbf{Z}}_k - \mathbf{X}\mathbf{u}||_2^2 + \frac{\sigma}{2} + \omega_0||\mathbf{u}||_1 \right\},$$

where $\omega_0$ is a universal regularization parameter to be specified later. Note that $\hat{\sigma}_k$ here is no longer an estimate of the standard error but used to adjust for the standard error $||\hat{\mathbf{Z}}_k - \mathbf{X}\mathbf{u}||_2/\sqrt{n}$ in the $k$th layer. The solution $\hat{\mathbf{u}}_k$ of the scaled Lasso is the same as that of the Lasso with $\omega_k = \hat{\sigma}_k\omega_0$.

On the other hand, motivated by (5), after getting $\hat{\mathbf{Z}}_k$, the right singular vectors $\mathbf{v}_k^*$ can be estimated as

$$\hat{\mathbf{v}}_k = n^{-1}\mathbf{Y}^{T}\hat{\mathbf{Z}}_k.$$  

We will show in Section 3 that the convergence rates of $\hat{\mathbf{v}}_k$ are basically the same as that of $\hat{\mathbf{Z}}_k$ after adjusting for the corresponding scales. Then based on the estimation consistency, a simple entry-wise thresholding will yield sparse right singular vectors to facilitate the selection of response variables if $\mathbf{v}_k^*$ are indeed sparse. However, since the sparse structure of $\mathbf{v}_k^*$ can reduce the magnitude of the singular value $d_k^*$ in view of $d_k^* = n^{-1/2}||\mathbf{XC}_k^*||_F$, if $d_k^*$ becomes not to large compared with the noise $||\mathbf{E}||_2$, better accuracy would be achieved by directly estimating the coefficient matrix via some co-sparsity inducing penalty (Mishra, Dey, and Chen 2017; Uematsu et al. 2019).

Finally, since the true rank $r^*$ is unknown in practice, we will estimate $\hat{\mathbf{Z}}_k$ sequentially until the $k$th eigenvalue $\hat{\lambda}_k$ of (6) is no larger than certain tolerance level $\mu$, which controls the maximum rank $r$. A simple tuning procedure based on $\hat{\mathbf{Z}}_k$ will be provided in Section 3 to identify the optimal rank $\hat{r}$. Then we have unit rank matrices $\hat{\mathbf{C}}_k = \hat{\mathbf{u}}_k\hat{\mathbf{v}}_k^T$ as the estimates of $\mathbf{C}_k^*$ and the estimated regression coefficient matrix $\hat{\mathbf{C}}$ is defined as

$$\hat{\mathbf{C}} = \sum_{k=1}^{\hat{r}} \hat{\mathbf{C}}_k = \sum_{k=1}^{\hat{r}} \hat{\mathbf{u}}_k\hat{\mathbf{v}}_k^T.$$  

The implementation of SESS is summarized in Algorithm 1.

\begin{algorithm}
\begin{algorithmic}
\State Input: $\mathbf{Y} \in \mathbb{R}^{n \times q}$, $\mathbf{X} \in \mathbb{R}^{n \times p}$, and termination parameter $\mu$.
\State $j \leftarrow 1$, $\hat{\mathbf{Y}}_j \leftarrow 0$
\Repeat
\State $(\hat{\mathbf{Z}}_j, \hat{\lambda}_j) \leftarrow j$th eigenvector and eigenvalue of $(nq)^{-1}\mathbf{Y}\mathbf{Y}^{T}\mathbf{Z} = \lambda \mathbf{Z}$.
\If{$\hat{\lambda}_j > \mu$}
\State $\hat{\mathbf{v}}_j \leftarrow n^{-1}\mathbf{Y}^{T}\hat{\mathbf{Z}}_j$
\State $\hat{\mathbf{Y}}_j \leftarrow \hat{\mathbf{Y}}_j + \hat{\mathbf{Z}}_j\hat{\mathbf{v}}_j^T$
\State $j \leftarrow j + 1$
\EndIf
\Until{tune the optimal rank $\hat{r}$ by information criterion (9)}
\State $k \leftarrow 1$
\Repeat
\State $(\hat{\mathbf{u}}_k, \hat{\sigma}_k) \leftarrow \arg\min_{\mathbf{u}, \sigma} \left\{ \frac{1}{2n}||\hat{\mathbf{Z}}_k - \mathbf{X}\mathbf{u}||_2^2 \right. + \frac{\sigma}{2} + \omega_0||\mathbf{u}||_1 \}$
\State $\hat{\mathbf{C}}_k \leftarrow \hat{\mathbf{u}}_k\hat{\mathbf{v}}_k^T$
\State $k \leftarrow k + 1$
\Until{$k > \hat{r}$}
\State output $\hat{\mathbf{C}} = \sum_{k=1}^{\hat{r}} \hat{\mathbf{C}}_k$
\end{algorithmic}
\caption{Algorithm SESS.}
\end{algorithm}

Although SESS adopts a similar sequential estimation framework as those in Mishra, Dey, and Chen (2017) and Zheng et al. (2019), it enjoys two significant advantages. First of all, the optimal sparsity parameter depends on the noise level and varies from layer to layer in Mishra, Dey, and Chen (2017) and Zheng et al. (2019). By contrast, SESS converts the multivariate regression problem into several univariate response regressions, thus making it possible to adopt a universal sparsity parameter $\omega_0$ and substantially improve the computational efficiency. Second, the major sparse estimation technique such
as sparse eigenvalue decomposition used in Zheng et al. (2019) is a nonconvex optimization problem and may not guarantee convergence under general settings (Ma 2013). But the regular eigenvalue problem and the scaled Lasso estimation render SESS a stepwise convex formulation with guaranteed numerical stability.

Nevertheless, we need to address two main issues before claiming the success of SESS. The first one is that when solving the regular eigenvalue problem (6), we lose the space constraint on Z as it may not fall into the column space of X. Then how much price in accuracy we pay to trade for the computational efficiency is unknown. The other issue is that the relationship between the estimated latent factors Ẑ_k and their population counterparts Xu^*_k can be different from the standard high-dimensional linear regression. Therefore, whether regularization methods such as the Lasso or the scaled Lasso apply and how to choose the corresponding regularization parameters deserve careful investigation. We will provide comprehensive theoretical guarantees with new insights in the next section.

3. Theoretical Properties

This section will present the theoretical properties of the proposed method SESS. First of all, we list the following technical conditions and discuss their relevance.

**Condition 1.** For some positive constant d_k, the top-r^* population eigenvalues λ_k in (4) satisfy λ_k − λ_{k+1} ≥ d_k, k = 1, . . . , r^*.

**Condition 2.** The eigenvalues of the population covariance matrix Σ for the random error vector are bounded from above and below by positive constants γ_u and γ_l, respectively.

**Condition 3.** There exists positive constants U and V such that the lengths of the left and right population singular vectors satisfy ||u^*_k||_2 ≤ U and ||v^*_k||_2/√q ≤ V for any k, k = 1, . . . , r^*.

Condition 1 is imposed to ensure the identifiability of the latent factors with d_k, the minimum separation between successive nonzero eigenvalues. Similar identifiability assumptions can be found in Fan, Liao, and Wang (2016), Wang and Fan (2017), and Uematsu et al. (2019). Condition 2 allows for general covariance structure of the random error vector as long as its eigenvalues are bounded. The upper bound controls the noise level while the lower bound is only needed in Theorem 2 to facilitate rank selection. It is weaker than the independence assumption on the error vector imposed in Bunea, She, and Wegkamp (2012) and Liu, Wang, and Zhao (2015). Condition 3 puts a mild assumption on the lengths of the singular vectors. The constant upper bound on ||u^*_k||_2 is reasonable due to the sparsity of u^*_k and can be consistent with the aforementioned scale of ||u^*_k||_2 = √n since the design matrix X can typically satisfy some sparse eigenvalue assumption (Candès and Tao 2005; Uematsu et al. 2019; Zheng et al. 2019).

Besides the above assumptions, we also need to characterize the model identifiability about the sparse left singular vectors by restricting the correlations between the significant predictors and the noise ones. Recall that the Gram matrix P = n^{-1}X^TX. Given ξ ≥ 0 and S ⊂ {1, . . . , p}, the sign-restricted cone invertibility factors introduced in Ye and Zhang (2010) are defined as

\[ F_m(ξ, S) = \inf \left\{ |S|^{1/m} ||P_u||_\infty / ||u||_m : u \in G_-(ξ, S) \right\} \]

for positive integer m with the sign-restricted cone G_-(ξ, S) = {u : ||u||_1 ≤ ξ ||u||_1 ≠ 0, u^TXu ≤ 0, ∀j \notin S}. As pointed out in Sun and Zhang (2012), the bounded sign-restricted cone invertibility factor assumption can be slightly weaker than the parallel condition on the compatibility factor or the restricted eigenvalue (Bickel, Ritov, and Tsybakov 2009). So we put it as follows to ensure the identifiability of S_k = supp(u^*_k), which are the supports of the left population singular vectors.

**Condition 4.** For certain positive constants ξ, F_1, and F_2, the sign-restricted cone invertibility factors F_1(ξ, S_k) ≥ F_1 and F_2(ξ, S_k) ≥ F_2 for any k, k = 1, . . . , r^*.

Since both Ẑ_k and −Ẑ_k are solutions of the regular eigenvalue problem (6), we assume that Ẑ_k takes the correct direction to facilitate the theoretical analysis. That is, the angle between Ẑ_k and Z^*_k is no more than a right angle. Otherwise, we can change Ẑ_k to −Ẑ_k to satisfy that. Now we are ready to show the main results.

**Proposition 1 (Consistency of latent factors).** Under Conditions 1 and 2, for sufficiently large n, the following statement holds with probability at least 1 − exp(−n^2q^2) for any k, 1 ≤ k ≤ r^*,

\[ \frac{1}{\sqrt{n}} ||ž_k - Xu^*_k||_2^2 \leq \frac{4q \sqrt{λ_k} }{d_k} \left( \sqrt{q} + \sqrt{nq} \right) \]

Proposition 1 provides a uniform convergence rate around the order of (√n + √q)/√mq for all the latent factors relative to the top-r^* singular values with significant probability. The numerator indicates the magnitude of ||E||_2, which is the largest singular value of the error matrix, while the denominator corresponds to the order of the top-r^* singular values ||X||_F. In Bunea, She, and Wegkamp (2012), it was shown by random matrix theory that ||E||_2 is of the magnitude (√n + √q) for independent entries and we generalize this result to allow for correlated random errors. Our convergence rate is as fast as n^{-1/2} + q^{-1/2} based on the current signal strength √mq as we do not impose any sparsity assumption on the right singular vectors v^*_k. In view of the technical argument, the consistency of estimated latent factors can still be guaranteed as long as the signal strengths ||X||_F are above the noise level. Moreover, the eigenvalue λ_k and the eigengap d_k here are assumed to be around the constant level for simplicity under the low rank structure, and the convergence rates still hold even if they become divergent.

The results of Proposition 1 are the bases of our two-step procedure for estimating the left singular vectors in SESS, which show that the latent factors can be consistently recovered from a regular eigenvalue problem even if we do not force the solutions to lie in the column space of the design matrix X. The underlying reason is that the directions achieving the maximum variations remain close to the population ones when the perturbation is relatively small compared with the signals of the factors. Moreover, the penalized regression in the second step is indeed a
relaxed projection of \( \tilde{Z}_k \) to the column space of \( X \), which further alleviates the issue of lacking subspace constraint in the first step. Then the proposed two-step procedure substantially simplifies the computational complexity compared with the nonconvex generalized sparse eigenvalue problem in Zheng et al. (2019), making it possible to decompose the original multi-response regression problem into several univariate response regressions. Extra benefits on tuning the sparsity parameter and the rank will be demonstrated through the subsequent theorems.

Note that a random vector \( w = (w_1, \ldots, w_q)^T \in \mathbb{R}^q \) is said to be sub-Gaussian distributed if there exists some positive constant \( k \) such that the marginal random variable \( \mathbf{m}^T w \) satisfies \( \mathbb{P}(\mathbf{m}^T w > t) \leq \exp(-t^2/k^2) \) for any \( t > 0 \) and any unit length vector \( \mathbf{m} \in \mathbb{R}^q \). Its second moment matrix is defined as \( \mathbb{E}(w^T w) \). Since the Gaussian assumption of the random error vector is not essential for our method, the following corollary generalizes the results of Proposition 1 to sub-Gaussian errors. It guarantees that the subsequent theoretical results can also hold for sub-Gaussian errors after some constant adjustment by applying the same technical arguments.

Corollary 1. Suppose that the rows of the random error matrix \( E \) in (1) are independent sub-Gaussian vectors with a common second moment matrix \( \Sigma^* \), whose eigenvalues are bounded from above. Then under Condition 1, for sufficiently large \( n \) and some constant \( \gamma_u \), it holds that with probability at least \( 1 - \exp(-\frac{n}{4}) \) for any \( k, 1 \leq k \leq r^* \),

\[
\frac{1}{\sqrt{n}} ||\hat{\mathbf{u}}_k - \mathbf{u}_k^*||_2 \leq \frac{4\sqrt{\gamma_y} \lambda_{\min}(\Sigma^*)}{d_k} \left( \sqrt{n} + \sqrt{q} \right).
\]

With the estimated latent factors \( \hat{Z}_k \), our second step is to recover the sparse left singular vectors through penalized regressions. However, this problem can be somewhat between model selection and sparse recovery (Candes and Tao 2005; 2006; Lv and Fan 2009) since the residual vector converges to zero when regressing \( \hat{Z}_k \) on \( X \) with the true coefficient vector \( \mathbf{u}_k^* \). As the solution of the Lasso is the same as that of the scaled Lasso with \( \omega_k = 1/\lambda_{\min}(\Sigma^*) \), the following theorem guarantees the estimation accuracy of SESS.

Theorem 1 (Consistency of sequential estimation). Suppose that Conditions 1–4 hold and \( \omega_k = \tilde{C} \lambda_{\min}(\Sigma^*)^{(\frac{1}{k} + \frac{3}{4})} \) for any constant \( \tilde{C} = \frac{4\sqrt{\gamma_y} \lambda_{\min}(\Sigma^*)}{d_k} \). Then for sufficiently large \( n \), with probability at least \( 1 - \exp(-\frac{n}{4}) \), the following statements hold simultaneously for any \( k, 1 \leq k \leq r^* \),

\[
||\hat{\mathbf{u}}_k - \mathbf{u}_k^*||_2 \leq C_u \sqrt{s} \left( \sqrt{n} + \sqrt{q} \right),
\]

\[
\frac{1}{\sqrt{n}} ||\hat{X}_k - \mathbf{u}_k^*||_2 \leq \tilde{C}_u \sqrt{s} \left( \sqrt{n} + \sqrt{q} \right),
\]

\[
\frac{1}{\sqrt{q}} ||\hat{v}_k - v_k^*||_2 \leq C_v \sqrt{s} \left( \sqrt{n} + \sqrt{q} \right),
\]

\[
\frac{1}{\sqrt{q}} \|\hat{C}_k - C_k^*\|_F \leq (VC_u + UC_v) \sqrt{s} \left( \sqrt{n} + \sqrt{q} \right),
\]

\[
||\hat{Z}_k - \mathbf{X} C_k^*\|_F \leq (V C_u + C_v) \sqrt{s} \left( \sqrt{n} + \sqrt{q} \right),
\]

\[
||\mathbf{X} \hat{C}_k - \mathbf{X} C_k^*\|_F \leq (V C_u + C_v) \sqrt{s} \left( \sqrt{n} + \sqrt{q} \right),
\]

where \( s = \max_{k=1}^{s^*} ||\mathbf{u}_k||_2 \) is the maximum sparsity level, \( C_u = 2\tilde{C} \xi/(\xi - 1) F_2 \), \( \tilde{C}_u = 2\tilde{C} \xi/(\xi - 1) F_1 \), and \( C_v = (4\lambda_1/d_k + 2)\gamma_{u} \) are positive constants.

Theorem 1 presents a uniform convergence rate of the order \( \sqrt{s} (\sqrt{n} + \sqrt{q})/\sqrt{m} \) for the left singular vectors and the unit rank matrices corresponding to the top-\( r^* \) singular values with significant probability. Compared with the uniform convergence rate in Proposition 1 and that of the right singular vectors, there is an extra term \( \sqrt{s} \) reflecting the price we pay for estimating the nonzero entries in \( \mathbf{u}_k^* \). Interestingly, here we do not observe a \( \log p \) term that typically exists in high-dimensional regression problems. The \( \log p \) term is used to be induced by the penalization parameter of a magnitude no smaller than the maximum spurious correlation \( ||n^{-1}X^T e||_{\infty} \) (\( e \) denotes a general random error vector) to suppress the noise variables and exclude them from the selected model. By contrast, as the columns of \( X \) are standardized to have a common \( L_2 \)-norm \( \sqrt{n} \), the corresponding maximum spurious correlation in our setup would be

\[
n^{-1} ||X^T(\hat{Z}_k - \mathbf{u}_k^*)||_{\infty} \leq n^{-1/2} ||\hat{Z}_k - \mathbf{u}_k^*||_2 = O_p(\sqrt{n}/\sqrt{mq}),
\]

with the true coefficient vector \( \mathbf{u}_k^* \), which can be better than our corresponding rate \( n^{-1/2} \sqrt{mq} \) when \( \sqrt{n} > \sqrt{q} \). Another advantage to out-of-sample prediction, which contains an extra term \( \sqrt{s} \) since then we can only use the regression coefficient matrix \( \mathbf{X} C_k^* \) instead of the estimated latent factor \( \hat{Z}_k \). Another advantage of SESS is that it only requires the tolerated sparsity level for model identification in Condition 4 be larger than the number of nonzero components in each \( \mathbf{u}_k^* \) instead of that of the whole regression coefficient matrix \( \mathbf{C}_k^* \), which alleviates the correlation constraints on the design matrix \( \mathbf{X} \).
Furthermore, the specific choice of \( \omega_k \) is derived from the requirement that \( \omega_k (\xi - 1)/(\xi + 1) \geq ||X^T(\hat{Z}_k - \hat{X}_u)^T)||_\infty/n \), similarly as in Ye and Zhang (2010) and Sun and Zhang (2012). Based on (8), it suffices to guarantee that
\[
\left( \frac{\xi - 1}{\xi + 1} \right) \omega_k \geq \tilde{C} \left( \frac{\sqrt{n} + \sqrt{q}}{\sqrt{m}} \right) \geq \frac{||\hat{Z}_k - \hat{X}_u||_2}{\sqrt{n}}
\]
which yields the choice of \( \tilde{C} \) in Theorem 1. Moreover, since \( \omega_0 = \omega_k/\tilde{C} \) and \( \delta_k = ||\hat{Z}_k - \hat{X}_u||_2/\sqrt{n} \) should be close to \( ||\hat{Z}_k - \hat{X}_u||_2/\sqrt{n} \), we suggest a universal regularization parameter \( \omega_0 \) around the constant level. This is different from the choice of \( \sqrt{2}(\log p)/n \) in Sun and Zhang (2012) for model selection. In our numerical studies, setting \( \omega_0 = 1 \) gives satisfactory finite sample performance.

Based on the results of Theorem 1, the regression coefficient matrix \( C^* \) can be accurately recovered once the rank is correctly identified. After decomposing the multi-response regression into univariate response regressions, the optimal rank can be tuned separately from the sparsity parameters in SESS. Moreover, since the true rank corresponds to the underlying number of latent factors, we propose the following BIC-type information criterion based on the estimated latent factors \( \hat{Z}_k \) and their factor loadings \( \hat{v}_k = n^{-1}Y^T\hat{Z}_k \).

**Theorem 2 (Consistency of rank recovery).** Suppose that Conditions 1–3 hold, \( r^* \log n/n^{1/2} = o(1), \) \( r \left( \frac{\sqrt{n} + \sqrt{q}}{\sqrt{m}} \right)^2 = o(1), \) and \( \sqrt{\log n}/\sqrt{n} = o(1). \) Then for sufficiently large \( n \), the following information criterion
\[
\tilde{C}(k) = \sqrt{n} \log \mathcal{L}(k) + k \log n,
\]
where \( \mathcal{L}(k) = (nq)^{-1}||Y - \hat{Y}_k||_F^2 \) with \( \hat{Y}_k = \sum_{j=1}^{k} \hat{Z}_j \hat{v}_j^T \), attains its minimum value when \( k = r^* \) with probability at least \( 1 - 2 \exp(-\frac{r^2}{2}) \).

As pointed out in Fan and Tang (2013), some power of the logarithmic factor of dimensionality is usually needed in the model complexity penalty to consistently identify the true model in high dimensions. But a BIC-type information criterion still applies here due to the separation of tuning procedures for the rank and the sparsity parameter. Compared with tuning both parameters via a GIC-type information criterion in Zheng et al. (2019), information criterion (9) enjoys much lower computational cost, as well as better statistical accuracy since the estimation of \( \hat{Z}_k \) bypasses the high-dimensional predictors so that their estimation error bounds do not involve \( s \) or \( \log p \). When the dimensionality \( q \) is less than \( n \), we can replace \( n \) with \( q \) in (9) so that the true rank can still be identified with significant probability by a similar technical argument.

## 4. Simulation Studies

In this section, we use simulated data to investigate the finite sample performance of SESS and compare it with four other methods: column-wise Lasso (Lasso), reduced rank regression (RRR), rank constrained group Lasso (RCGL), and sequential co-sparse factor regression (SeCURE). Lasso and RRR are two classical methods which generate sparse and low-rank estimates, respectively. RCGL yields a jointly row-sparse and low-rank estimate that achieves the optimal prediction error rate (Bunea, She, and Wegkamp 2012), while SeCURE sequentially estimates the sparse unit rank matrices to enjoy co-sparse structures in both left and right singular vectors (Mishra, Dey, and Chen 2017).

These methods were implemented as follows. Lasso was implemented by R package “lars” with the sparsity parameters tuned by BIC. RRR was implemented using R package “rrpack” with the rank tuned by the criterion of joint rank and row selection (JRRS) proposed in Bunea, She, and Wegkamp (2012). RCGL was implemented by R package “rrpack” with the sparsity parameter and the rank tuned by JRRS. SeCURE was implemented using R package “secure” and tuned by BIC for both sparsity parameters and the rank. By contrast, the proposed method SESS used a universal sparsity parameter \( \omega_0 = 1 \) and its rank was chosen by the BIC-type information criterion (9).

There are six performance measures in total for evaluating different methods. The first three measures are: the normalized prediction error (PE) \( ||Y_{test} - X_{test}\hat{C}||_F/||Y_{test}||_F \) based on an independent test sample of size 10,000, the normalized estimation error (EE) \( ||\hat{C} - C||_F/||C||_F \), and the rank recovery error (RE) ||rank(\( \hat{C} \)) - rank(C)||. The fourth and fifth measures are the false positive rate (FPR) and the false negative rate (FNR) suggested in Mishra, Dey, and Chen (2017) to evaluate the results of variable selection for different layers, obtained by comparing the sparsity pattern of \( (\hat{u}_1, \ldots, \hat{u}_s) \) to that of \( (u_1^*, \ldots, u_s^*) \). These two measures do not apply to Lasso, RRR, and RCGL since they do not recover the latent factors. The last one is the averaged CPU time for obtaining the corresponding estimate on a PC with 16 GB RAM and Intel Core i7-8700 CPU (3.20 GHz).

### 4.1. Simulation Example 1

We generated 100 datasets from multivariate regression model (1) with similar setups as those in Zheng et al. (2019). For each dataset, the rows of \( X \) were sampled as iid copies from \( N(0, \Sigma_X) \) with \( \Sigma_X = (0.5|s|)_{p \times p} \). Similarly, the rows of \( E \) are iid from \( N(0, \gamma \Sigma_E) \) with \( \Sigma_E = (0.5|s|)_{q \times q} \) and \( \gamma = 0.1 \). The parameter matrix \( C^* \) was generated as follows. First, we created a matrix \( C \) with 90 nonzero entries and each nonzero entry was drawn independently from \( N(0, 1) \). Second, based on the singular value decomposition of \( C = USV^T \), we replaced the first \( r \) diagonal components of \( S \) by 100, 99, \ldots, 101 - \( r \) and others by 0, yielding a jointly sparse and low-rank coefficient matrix \( C^* \) with around 50 nonzero entries. We considered two different settings with \( (n, q, r) = (100, 200, 3) \) and \( (200, 300, 10) \), respectively. For both settings, the dimensionality \( p \) can vary in \( (800, 1500, 2000) \).

Table 1 summarizes the results of the performance measures except the CPU time. It is clear that the performance of SESS is among the best in terms of either prediction and estimation accuracies or variable selection under various settings. Although the computational efficiency of Lasso and RRR is good in view of Figure 1, the Lasso cannot recover and use the low rank structure, which in turn lowers its estimation and prediction accuracies, while the RRR suffers from the curse of dimen-
Table 1. Means and standard errors (in parentheses) of different performance measures in Section 4.1.

| $p$ | Method | $PE \times 10^{-2}$ | $EE \times 10^{-2}$ | $RE$ | $FNR$ | $FPR$ |
|-----|--------|-------------------|-------------------|------|-------|-------|
| 800 | Lasso  | 5.20 (0.01)       | 4.36 (0.02)       | 87.90 (1.12) | –     | –     |
|     | RRR    | 29.87 (0.01)      | 27.37 (0.01)      | 0 (0) | –     | –     |
|     | SESS   | 2.66 (0.00)       | 0.78 (0.01)       | 0 (0) | 0 (0) | 0.06 (0.01) |
|     | RCGL   | 2.71 (0.00)       | 1.34 (0.01)       | 0 (0) | –     | –     |
|     | SeCURE | 3.99 (0.02)       | 2.61 (0.08)       | 0 (0) | 0.13 (0.08) | 1.85 (0.25) |
| 1500| Lasso  | 5.20 (0.01)       | 4.37 (0.02)       | 88.56 (1.25) | –     | –     |
|     | RRR    | 28.13 (0.03)      | 26.65 (0.03)      | 0.02 (0.04) | –     | –     |
|     | SESS   | 2.65 (0.00)       | 0.74 (0.00)       | 0 (0) | 0 (0) | 0.06 (0.01) |
|     | RCGL   | 2.71 (0.00)       | 1.59 (0.01)       | 0 (0) | –     | –     |
|     | SeCURE | 3.14 (0.03)       | 2.79 (0.06)       | 0 (0) | 0.06 (0.01) | 0.21 (0.01) |
| 2000| Lasso  | 5.21 (0.01)       | 4.41 (0.02)       | 89.00 (2.97) | –     | –     |
|     | RRR    | 28.29 (0.03)      | 27.21 (0.04)      | 0.08 (0.02) | –     | –     |
|     | SESS   | 2.66 (0.00)       | 0.79 (0.01)       | 0 (0) | 0 (0) | 0.05 (0.01) |
|     | RCGL   | 2.74 (0.00)       | 1.25 (0.01)       | 0 (0) | –     | –     |
|     | SeCURE | 3.12 (0.05)       | 3.11 (0.05)       | 0 (0) | 0.06 (0.01) | 0.11 (0.01) |

Figure 1. CPU times of different methods.

sionality regardless of the correct identification of the rank. By contrast, SESS, RCGL, and SeCURE perform much better since they take advantage of the jointly low-rank and sparse structure.

Nevertheless, it can be seen from Figure 1 that SESS enjoys tremendous computational advantages over the other two comparable methods by increasing the speed for tens to hundreds of times, benefiting from its sequential formulation and tuning free property for the sparsity parameter. Specifically, when the dimensionality is 2000 and the true rank equals to 10, both RCGL and SeCURE need more than 2 hr to obtain the estimated
coefficient matrix, while SESS costs less than 2 min in the same device. Furthermore, in view of Figure 2, there are significant gaps between the BIC values of the true rank and other candidate ranks for SESS no matter how the dimensionality varies, which is due to the discrete nature of the rank. It makes SESS fairly easy to identify the correct one as it does not need to tune the sparsity level at the same time.

4.2. Simulation Example 2

In this second example, we generated 100 datasets and adopted the model setup similar to that in Mishra, Dey, and Chen (2017) with \((n, p, q, r) = (400, 500, 200, 3)\). Specifically, the true regression coefficient matrix \(C^* = \sum_{j=1}^{r^*} d_j^* v_j^* v_j^T\) with the rank \(r^* = 3\), \(d_1^* = 60\), \(d_2^* = 30\), and \(d_3^* = 10\). The singular vectors were created as follows. We first generated \(\tilde{u}_1 = [\text{unif}(\mathcal{A}_u, s_1), \text{rep}(0, p - s_1)]^T\), \(\tilde{u}_2 = [\text{rep}(0, 5), \text{unif}(\mathcal{A}_u, s_2), \text{rep}(0, p - 5 - s_2)]^T\), and \(\tilde{u}_3 = [\text{rep}(0, 11), \text{unif}(\mathcal{A}_u, s_3), \text{rep}(0, p - 11 - s_3)]^T\), where \(\mathcal{A}_u = [1, -1]\). The \(s_k\) denotes a \(k\)-dimensional vector whose entries are iid uniformly distributed on set \(\mathcal{A}\), and \(\text{rep}(\alpha, k)\) denotes a \(k\)-dimensional vector whose entries are all equal to \(\alpha\). Then we normalized them to have unit length, so that \(u_j^* = \tilde{u}_j/\|\tilde{u}_j\|_2, j = 1, 2, 3\). Note that \(s_k\) indicates the number of nonzero components in \(u_k^*\) and can be various as displayed in Table 2. Similarly, we got \(\tilde{v}_1 = [\text{unif}(\mathcal{A}_v, 5), \text{rep}(0, q - 5)]^T\), \(\tilde{v}_2 = [\text{rep}(0, 5), \text{unif}(\mathcal{A}_v, 5), \text{rep}(0, q - 10)]^T\), \(\tilde{v}_3 = [\text{rep}(0, 10), \text{unif}(\mathcal{A}_v, 5), \text{rep}(0, q - 15)]^T\), and \(\mathcal{A}_v = [-1, -0.3] \cup [0.3, 1]\), and \(v_j^* = \tilde{v}_j/\|\tilde{v}_j\|_2\) for \(j = 1, 2, 3\).

Let \(X\) follow the multivariate Gaussian distribution \(N(0, \Sigma_X)\) with \(\Sigma_X = (0.5^{(j-l)})_{p \times p}\) and \(x_1 = U^T x\) for some \(U\) so that \(x_1 \sim N(0, I_p)\). To generate the predictor matrix \(X\), we first created \(X_1 \in \mathbb{R}^{n \times r}\) by drawing \(n\) random samples from \(x_1 \sim N(0, I_p)\). Then based on \(U = (u_1^*, u_2^*, u_3^*)\), we can find a \(U_1^* \in \mathbb{R}^{n \times (p-r)}\) such that \(P = (U^*, U_1^*) \in \mathbb{R}^{n \times p}\) and \(\text{rank}(P) = p\). Let \(x_2 = U_1^T x\) and \(X_2 \in \mathbb{R}^{n \times (p-r)}\) was generated by drawing \(n\) random samples from the conditional distribution of \(x_2\) given \(x_1\). Finally, the predictor matrix \(X = (X_1, X_2)^{p^{-1}}\). Moreover, we generated a non-Gaussian error matrix \(E\) by first creating matrix \(E\) whose components are iid from a scaled \(t\)-distribution with 5 degrees of freedom and unit variance, and then let \(E = \sigma \hat{E} \Sigma_{E}^{1/2}\) with \(\Sigma_{E} = (0.5^{(l-j)})_{q \times q}\). The noise level \(\sigma\) is set so that the signal-to-noise ratio (SNR) defined as \(\text{SNR} = \|d_\sigma^* x u_\sigma^* y_\sigma^* T / \|E\|_F\) equals to 0.75.

The results for different methods are summarized in Table 2. Similar to the first example, the performance of the jointly low-rank and sparse estimates including SESS, RCGL, and SeCURE is better than that of Lasso and RRR in terms of prediction and estimation accuracies and their performance is relatively stable regardless of the increasing number of nonzero components in \(u_k^*\). Among them, SESS enjoys the highest computational efficiency similarly as in Section 4.1. It also demonstrates the effectiveness of jointly low-rank and sparse estimation under some non-Gaussian errors.

5. Application to Stock Short Interest Data

In this section, we will analyze the monthly stock short interest dataset originally studied in Rapach, Ringgenberg, and Zhou (2016), available at Compustat (http://www.hec.unil.ch/agoyal/). The raw dataset reported short interest at the firm-level as the number of shares that were held short in a given firm. As short interest was shown in Rapach, Ringgenberg, and Zhou (2016) to be the strongest predictor of aggregate stock returns, we will analyze the short interest influence networks among the firms and find the most influential firms through following vector autoregression model with the maximal time lag \(L\),

\[ y(t) = \sum_{i=1}^{L} C_i^T y(t - i) + \epsilon(t). \]
Here \( y(t) = [y_1(t), \ldots, y_q(t)]^T \in \mathbb{R}^q \) consists of the short interests of \( q \) firms at time \( t \), \( C_t \in \mathbb{R}^{q \times q} \) are the regression coefficient matrices, and \( \epsilon(t) \in \mathbb{R}^q \) denotes the random noise vector. By setting \( x(t) = (y(t - L)^T, \ldots, y(t - 1)^T)^T \in \mathbb{R}^{Lq} \) and \( C = [C_1, \ldots, C_q]^T \), the model can be rewritten as a multi-response regression model

\[
y(t) = C^T x(t) + \epsilon(t).
\]

After the preprocessing, the dataset consists of short interests of 3269 firms at the month level from January 1973 to December 2013, including 492 months in total. We set the maximal time lag \( L = 5 \) and the results are similar for larger lags. It yields a triple of \( (n, p, q) = (487, 16,345, 3269) \). Since both RCGL and SeCURE are no longer applicable due to the memory constraint in such large-scale data analysis, we report the performance of other methods in Section 4. By treating the first 366 samples as training data, we fit the multi-response regression model and then calculated the averaged \( R^2 \) statistics over \( q \) firms for the significant latent factors and the averaged forecast error \( (nq)^{-1} \| Y - XC \|_2^2 \) based on the remaining 121 testing samples. In view of the results summarized in Table 3, SESS enjoys the lowest prediction error and identifies one significant latent factor. Its out-of-sample averaged \( R^2 \) statistic is as high as 15.20%, demonstrating its importance in forecasting the short interests. Moreover, there are 220 nonzero entries in the significant left singular vector with five entries much larger than others (at least 5 times larger). Correspondingly, the five most influential firms are two investment trust companies (Washington Prime Group Inc. and Invesco) and three resource mining companies (Asanko Gold Inc., O’Kiep Copper, and Mesa Royalty Trust).

Several researches reveal that market frictions and behavioral biases may cause price to deviate from fundamental value (Miller 1977; Hong and Stein 1999) and that short sellers can exploit these situations since they are skilled at processing firm-specific information and information about future aggregate cash flows that is not reflected in current market prices (Diether, Lee, and Werner 2009; Rapach, Ringgenberg, and Zhou 2016). Therefore, by applying our method to analyze the short interest of each company and their influence networks and forecast the behavior of short sellers, investors can make better judgments in response to market frictions to reasonably avoid certain risks.

6. Discussion

In this article, we have developed a new method SESS for high-dimensional multi-response regression, which recovers regression coefficient matrix and latent factors sequentially by converting the original problem into several univariate response regressions. Numerical studies demonstrate the statistical accuracy and high scalability of the proposed method. Our two-step sequential estimation procedure may be extended to deal with data containing measurement errors and outliers or more general model settings such as the generalized linear model, which will be interesting topics for future research.

### Supplementary Materials

In the supplementary materials, we give the technical proofs for Proposition 1, Corollary 1 and Theorems 1–2.
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References

Anderson, T. W. (1951), “Estimating Linear Restrictions on Regression Coefficients for Multivariate Normal Distributions,” The Annals of Mathematical Statistics, 22, 327–351. [595]
Belloni, A., Chernozhukov, V., and Wang, L. (2011), “Square-Root Lasso: Pivotal Recovery of Sparse Signals via Conic Programming,” Biometrika, 98, 791–806. [595]
Bickel, P., Ritov, Y., and Tsybakov, A. (2009), “Simultaneous Analysis of Lasso and Dantzig Selector,” The Annals of Statistics, 37, 1705–1732. [598]
Bunea, F., She, Y., and Wiegkamp, M. (2011), “Optimal Selection of Reduced Rank Estimators of High-Dimensional Matrices,” The Annals of Statistics, 39, 1282–1309. [595]
Candes, E. J., and Plan, Y. (2011), “Decoding by Linear Programming,” IEEE Transactions on Information Theory, 57, 2342–2359. [595]
Candes, E. J., and Tao, T. (2005), “Near-Optimal Signal Recovery From Random Projections: Universal Encoding Strategies,” IEEE Transactions on Information Theory, 52, 5406–5425. [599]
Candes, E. J., and Tao, T. (2005), “The Dantzig Selector: Statistical Estimation When p Is Much Larger Than n” (with discussion), The Annals of Statistics, 35, 2313–2404. [597]
Chen, K., Chan, K.-S., and Stenseth, N. C. (2012), “Reduced Rank Stochastic Regression With A Sparse Singular Value Decomposition,” Journal of the Royal Statistical Society, Series B, 74, 203–221. [595]
Chen, K., Dong, H., and Chan, K.-S. (2013), “Reduced Rank Regression via Adaptive Nuclear Norm Penalization,” Biometrika, 100, 901–920. [595]
Chen, L., and Huang, J. Z. (2012), “Sparse Reduced-Rank Regression for Simultaneous Dimension Reduction and Variable Selection,” Journal of the American Statistical Association, 107, 1533–1545. [595]
Diether, K. B., Lee, K.-H., and Werner, I. M. (2009), “Short-Sale Strategies and Return Predictability,” The Review of Financial Studies, 22, 575–607. [603]
Fan, J., Fan, Y. and Barut, E. (2014), “Adaptive Robust Variable Selection,” The Annals of Statistics, 42, 324–351. [597]
Fan, J., Feng, Y. and Wu, Y. (2009), “Network Exploration via the Adaptive Lasso and SCAD Penalties,” The Annals of Applied Statistics, 3, 521–541. [597]
Fan, J., Gong, W., and Zhu, Z. (2019), “Generalized High-Dimensional Trace Regression via Nuclear Norm Regularization,” Journal of Econometrics, 212, 177–202. [595]
Fan, J., and Li, R. (2001), “Variable Selection via Nonconcave Penalized Likelihood and Its Oracle Properties,” Journal of the American Statistical Association, 96, 1348–1360. [597]
Fan, J., Liao, Y., and Wang, W. (2016), “Projected Principal Component Analysis in Factor Models,” The Annals of Statistics, 44, 219–254. [598]
Fan, J., and Lv, J. (2011), “Nonconcave Penalized Likelihood With NP-Dimensionality,” IEEE Transactions on Information Theory, 57, 5467–5484. [597]
Fan, Y., and Tang, C. Y. (2013), “Tuning Parameter Selection in High Dimensional Penalized Likelihood,” Journal of the Royal Statistical Society, Series B, 75, 531–552. [600]
Giraud, C. (2011), “Low Rank Multivariate Regression,” Electronic Journal of Statistics, 5, 773–799. [595]
Goh, G., Dey, D. K., and Chen, K. (2017), “Bayesian Sparse Reduced Rank Multivariate Regression,” Journal of Multivariate Analysis, 157, 14–28. [595]
Gomez-Rodriguez, M., Leskovec, J., and Krause, A. (2012), “Inferring Networks of Diffusion and Influence,” ACM Transactions on Knowledge Discovery From Data (TKDD), 5, Article 21. [595]
Hong, H., and Stein, J. C. (1999), “A Unified Theory of Underreaction, Momentum Trading, and Overreaction in Asset Markets,” The Journal of Finance, 54, 2143–2184. [603]
Izenman, A. J. (1975), “Reduced-Rank Regression for the Multivariate Linear Model,” Journal of Multivariate Analysis, 5, 248–264. [595]
Lian, H., Feng, S., and Zhao, K. (2015), “Parametric and Semiparametric Reduced-Rank Regression With Flexible Sparsity,” Journal of Multivariate Analysis, 136, 163–174. [595]
Liu, H., Wang, L., and Zhao, T. (2015), “Calibrated Multivariate Regression With Application to Neural Semantic Basis Discovery,” Journal of Machine Learning Research, 16, 1579–1606. [595,596,598]
Lv, J., and Fan, Y. (2009), “A Unified Approach to Model Selection and Sparsification Using Regularized Least Squares,” The Annals of Statistics, 37, 3498–3528. [599]
Ma, Z. (2013), “Sparse Principal Component Analysis and Iterative Thresholding,” The Annals of Statistics, 41, 772–801. [598]
Miller, E. M. (1977), “Risk, Uncertainty, and Divergence of Opinion,” The Journal of Finance, 32, 1151–1168. [603]
Mishra, A., Dey, D. K., and Chen, K. (2017), “Sequential Co-Sparse Factor Regression,” Journal of Computational and Graphical Statistics, 26, 814–825. [595,596,597,600,602]
Negahban, S., and Wainwright, M. J. (2011), “Estimation of (Near) Low-Rank Matrices With Noise and High-Dimensional Scaling,” The Annals of Statistics, 39, 1069–1097. [595]
Rapach, D. E., Ringgenberg, M. C., and Zhou, G. (2016), “Short Interest and Aggregate Stock Returns,” Journal of Financial Economics, 121, 46–65. [602,603]
Starbird, K., and Palen, L. (2012), “How Will the Revolution Be Retweeted? Information Diffusion and the 2011 Egyptian Uprising,” in Proceedings of the ACM 2012 Conference on Computer Supported Cooperative Work, pp. 7–16. [595]
Sun, T., and Zhang, C.-H. (2012), “Scaled Sparse Linear Regression,” Biometrika, 99, 879–898. [595,596,597,600]
Tibshirani, R. (1996), “Regression Shrinkage and Selection via the Lasso,” Journal of the Royal Statistical Society, Series B, 58, 267–288. [597]
Uematsu, Y., Fan, Y., Chen, K., Lv, J., and Lin, W. (2019), “SOFAR: Large-Scale Association Network Learning,” IEEE Transactions on Information Theory, 65, 4924–4939. [595,596,597]
Weng, H., Feng, Y., and Qiao, X. (2019), “Regularization After Retention in Ultrahigh Dimensional Linear Regression Models,” Statistica Sinica, 29, 387–407. [597]
Ye, E., and Zhang, C.-H. (2010), “Rate Minimaxity of the Lasso and Dantzig Selector for the εq Loss in εq-Balls,” Journal of Machine Learning Research, 11, 3519–3540. [598,600]
Yi, Y., and Feng, Y. (2014), “Modified Cross-Validation for Lasso Penalized High-Dimensional Linear Models,” Journal of Computational and Graphical Statistics, 23, 1009–1027. [597]
Wang, W., and Fan, J. (2017), “Asymptotics of Empirical Eigen-Structure for High Dimensional Spiked Covariance,” The Annals of Statistics, 45, 1342–1374. [598]
Yuan, M., Ekici, A., Lu, Z., and Monteiro, R. (2007), “Dimension Reduction and Coefficient Estimation in Multivariate Linear Regression,” Journal of the Royal Statistical Society, Series B, 69, 329–346. [595]
Zheng, Z., Bahadori, M. T., Liu, Y., and Lv, J. (2019), “Scalable Interpretable Multi-Response Regression via SEED,” Journal of Machine Learning Research, 20, 1–34. [595,596,597,598,599,600]
Zou, H. (2006), “The Adaptive Lasso and Its Oracle Properties,” Journal of the American Statistical Association, 101, 1418–1429. [597]