"Stringy" Coherent States Inspired By Generalized Uncertainty Principle

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Abstract: Coherent States with Fractional Revival property, that explicitly satisfy the Generalized Uncertainty Principle (GUP), have been constructed in the context of Generalized Harmonic Oscillator. The existence of such states is essential in motivating the GUP based phenomenological results present in the literature which otherwise would be of purely academic interest. The effective phase space is Non-Canonical (or Non-Commutative in popular terminology). Our results have a smooth commutative limit, equivalent to Heisenberg Uncertainty Principle. The Fractional Revival time analysis yields an independent bound on the GUP parameter. Using this and similar bounds obtained here, we derive the largest possible value of the (GUP induced) minimum length scale. Mandel parameter analysis shows that the statistics is Sub-Poiseissonian. Correspondence Principle is deformed in an interesting way. Our computational scheme is very simple as it requires only first order corrected energy values and undeformed basis states.

1 Introduction:

In recent years Generalized Uncertainty Principle (GUP) [1,2] has created a lot of excitement because the works [3,4] have established that Quantum Gravity signatures are quite universal and GUP can serve as a (moderate energy) window to the latter. Contrary to previously held belief, it can lead to small but observable effects at energy scales considerably smaller than Planck energy. The GUP is strikingly distinct from Heisenberg Uncertainty Principle (HUP) since GUP unambiguously fixes the smallest size of the dispersion \( (\Delta x) = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} \). On the other hand in HUP either of \( \Delta x \) or \( \Delta p \) can arbitrarily small at the expense of the other. In a way GUP points to a string like elementary excitations instead of point particle. Furthermore GUP is compatible with a modified Non-Commutative (NC) [5,6] symplectic structure that can have non-trivial
effects in all areas of quantum physics.

But the most important player - an actual quantum state obeying GUP - is missing in all the above analysis [1, 3, 4]. These works naively assume the GUP itself and do not address, let alone show, that quantum states obeying GUP and stable under time-evolution can exist. Compare this situation with the conventional case where one can construct Coherent States (CS) for Harmonic Oscillator (HO) that are stable and does saturate the HUP. In the present Letter we have provided this vital missing information. In the framework of Generalized CS (GCS) [7] we have constructed the first order (in $\beta$ - the GUP parameter) corrected GCS that satisfies GUP and shows the essential Fractional Revival behaviour indicating time-stability. For $\beta = 0$ the conventional CS for HO is recovered. Another major result of our work is that we have provided an independent bound on $\beta$ that agrees with [3]. Correspondence Principle is maintained albeit in an interestingly modified way. Lastly we have developed a simple algorithm to show how first order corrected energy values can yield first order corrected wave functions (GCS) endowed with non-trivial features of the perturbation (GUP and NC physics, in the present case). Indeed, the flexibility in GCS construction [7, 8] allows this act which can be generalized in many interesting systems with non-linear (compared to HO) energy dispersion.

GUP is probably unavoidable at high energy scales where Quantum Gravity prevails since it supplies an energy scale $\sim$ Planck energy $\sim$ in an inherent way. GUP was predicted [2] quite long ago from String Theory considerations and the inherent length scale in GUP intuitively resolves [3] the paradoxes in Black Hole physics. But GUP is directly linked with a Non-Canonical (popularly referred to as Non-Commutative (NC)), phase space symplectic structure [5, 6]. NC phase space has pervaded theoretical physics in recent years since a non-trivial change in the fundamental symplectic structure can affect physics in major ways. NC phase space appears both in relativistic and non-relativistic scenarios and the one relevant to GUP belongs to the latter but can be reduced from the former [6].

GCS are best suited for our purpose since one can quickly grasp the non-classical behaviour of the system being studied. Ever since its introduction by Schrödinger in the context of Harmonic Oscillator (HO), CS have played an ever increasing role in diverse areas of physics. Ideal lasers enjoying Poissonian photon number statistics are described by Glauber-Sudarshan CS but to deal with non-linear interactions in a real laser and furthermore in more exotic theoretical frameworks (such as Quantum Group ideas [9] to describe a real laser [10]) various generalizations of the former canonical CS, compatible with non-Poissonian statistics, have been introduced. We shall follow one particular form of Generalized Coherent state (GCS), the so called Gazeau-Klauder CS, proposed in [8, 7] whose central idea is to construct GCS that are stable in time (i.e GCS will remain GCS throughout its evolution). This property is very relevant in experimental context since this can explain the experimentally observed fascinating phenomenon of fractional revival: an initially well localized wave packet disperses due to quantum effects but after sufficient time, depending on the parameters of the
problem, recombines to form the initial state once again (revival) but during the process the state becomes multi-localized thus exhibiting fractional revival. The latter state appears as a collection of sub-wave packets separated in space but all resembling the initial wave packet. These fractional revival have shorter time scales than the revival time. The fractional revivals are a manifestation of the non-linearity in the energy spectrum (compared to the linear spectrum $E_n \sim n$ in HO) which is precisely the reason of introducing GCS that maintain time stability. In NC physics context CS have appeared in [11] in NC extended Black Hole and GCS in [12] in NC Moyal plane, with which we will compare our results.

In this Letter we consider a Generalized HO (GHO) whose coordinate and momentum satisfy a Non-Commutative (NC) algebra that induces the GUP. The non-commutative (or GUP) parameter $\beta$ brings in the minimum length scale. At the same time $\beta$ changes the HO spectrum to a non-linear one thus ushering in the GCS. We show that the fractional revival occurs only for non-zero $\beta$ and hence its experimental (non-)observation can clearly put bounds on $\beta$.

## 2 NC phase space, GUP and GHO

We consider the following Non-Commutative (Jacobi identity satisfying) two-dimensional phase space, leading to the GUP,

$$[x, p] = i\hbar(1 + \beta p^2) ; [x, x] = [p, p] = 0 ; \rightarrow \Delta x \Delta p \geq \frac{\hbar}{2}(1 + \beta A(\Delta p)^2).$$

(1)

where $\beta = l_{PL}^2/(2\hbar^2)$, $l_{PL} = G\hbar/c^3 \equiv ($Planck length$)^2$. Throughout the Letter we will keep only $O(\beta)$ corrections. $A$ is a $\beta$-independent numerical constant. The RHS can be of a slightly more general form. Let us exploit the Darboux map [3] (for the exact relativistic map see Ghosh and Pal in [6]),

$$x \equiv X, p = P(1 + \beta P^2),$$

(2)

where $X, P$ are canonical, $[X, P] = i\hbar$, $[X, X] = [P, P] = 0$. This is different from the conventional map where one deforms $x$.

The GHO

$$H = \frac{p^2}{2m} + \frac{m\omega^2x^2}{2}$$

(3)

in the canonical $X, P$ coordinates becomes

$$H = \frac{P^2}{2m}(1 + \beta P^2)^2 + \frac{m\omega^2X^2}{2} \approx \frac{P^2}{2m} + \frac{m\omega^2X^2}{2} + \frac{\beta P^4}{m}.$$
In [3] this $H$ was used to compute $O(\beta)$ corrections for different potentials. We introduce canonical creation-annihilation operators:

$$
a = \sqrt{\frac{m\omega}{2\hbar}}(X + i \frac{P}{m\omega}), \quad a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}(X - i \frac{P}{m\omega}) \to [a, a^\dagger] = 1; [a, a] = [a^\dagger, a^\dagger] = 0. \quad (5)
$$

Our scheme is such that, in our level of approximation, it is enough to use the canonical HO Fock basis:

$$
a \mid 0 \rangle = 0; \quad a \mid n \rangle = \sqrt{n} \mid n - 1 \rangle, \quad a^\dagger \mid n \rangle = \sqrt{n + 1} \mid n + 1 \rangle. \quad (6)
$$

The Hamiltonian turns out to be,

$$
H = \hbar \omega \left( a^\dagger a + \frac{1}{2} \right) + \frac{\beta m \omega^2 \hbar^2}{4} (a - a^\dagger)^4. \quad (7)
$$

Terms with same number of $a$ and $a^\dagger$ will contribute to the energy spectrum,

$$
E_n = \hbar \omega \left[ n(1 + \lambda(1 + n)) + \frac{1}{2}(1 + \frac{\lambda}{2}) \right]; \quad \lambda \equiv (3\beta m \omega)/2.
$$

We drop the unimportant constant part in $E_n$ and use the notation of [7],

$$
\hbar \omega e_n = \hbar \omega n(1 + \lambda(1 + n)). \quad (9)
$$

### 2.1 Generalized Coherent States for GHO

This section constitutes the main body of our work. The Gazeau-Klauder [7] GCS are defined as

$$
\mid J, \gamma \rangle = \frac{1}{N(J)} \sum_{n \geq 0} \frac{J^n e^{-i\gamma \epsilon_n}}{\sqrt{\rho_n}} \mid n \rangle; \quad \rho_0 = 1, \quad \rho_n = \epsilon_1 \epsilon_2 \ldots \epsilon_n. \quad (10)
$$

where $J$ and $\gamma$ are two real numbers parameterizing the coherent state along with the normalization condition $N(J)^2 = \sum_{n \geq 0} \frac{J^n}{\rho_n}$ that can be expressed in terms of modified Bessel Function (see e.g. [7]).

Let us compute $x$ and $p$ dispersion for the GCS $\mid J, \gamma \rangle$ (note that the physical variables are $x, p$ and not the canonical $X, P$):

$$
(\Delta x)^2 = \langle x^2 \rangle - (\langle x \rangle)^2; \quad (\Delta p)^2 = \langle p^2 \rangle - (\langle p \rangle)^2. \quad (11)
$$

A straightforward calculation reveals (see Appendix for a brief outline of the computation)

$$
\langle x \rangle = \sqrt{\frac{\hbar}{2m\omega}} < a + a^\dagger >
$$
\[ = \sqrt{\frac{2\hbar J}{m\omega}} \{ \cos \gamma - \lambda \{(1 + \frac{J}{2})\cos \gamma + 2(1 + J)\gamma \sin \gamma \} \}, \quad (12) \]

and in a similar way,

\[ < x^2 > = \frac{\hbar}{2m\omega} < \sigma^2 + \sigma^1 \sigma^1 + \sigma^1 \sigma^1 > \]

\[ = \frac{\hbar}{2m\omega} [1 + 2J + 2J \cos(2\gamma) - 2\lambda(2 + J)(\cos(2\gamma) + (6 + 4J)\gamma \sin(2\gamma))]. \quad (13) \]

This leads to the \( x \)-dispersion,

\[ (\Delta x)^2 = \frac{\hbar}{2m\omega} [1 - \lambda J \{ \cos(2\gamma) + 4\gamma \sin(2\gamma) \}]. \quad (14) \]

We compute analogous expressions for \( p \),

\[ < p > = -i \sqrt{\frac{m\omega}{2} \sigma^z} \frac{\hbar}{2} \frac{\sigma^x}{\sigma^y} \frac{\sigma^y}{\sigma^z} \]

\[ = \sqrt{2m\omega J}[\sin \gamma + \lambda \{ -\frac{J}{2} \sin \gamma + \frac{J}{3} \sin(3\gamma) - 2(1 + J)\gamma \cos \gamma \}]. \quad (15) \]

\[ < p^2 > = \frac{m\omega}{2} \sigma^z \frac{\hbar}{2} \left( \sigma^x \sigma^y \sigma^z \right) \]

\[ = \frac{m\omega}{2} [1 + 2J - 2J \cos(2\gamma) + \lambda \{ 2 + 4J + 2J^2 + (3J - \frac{10}{3} J^2) \cos(2\gamma) + \frac{4}{3} J^2 \cos(4\gamma) + 2J(6 + 4J)\gamma \sin(2\gamma) \}]. \quad (16) \]

and subsequently obtain the \( p \)-dispersion,

\[ (\Delta p)^2 = \frac{m\omega}{2} \sigma^z \frac{\hbar}{2} \left( \sigma^x \sigma^y \sigma^z \right) \]

\[ = \frac{m\omega}{2} [1 + 2J - 2J \cos(2\gamma) + \lambda \{ 2 + 4J - 3J \cos(2\gamma) + 4\gamma \sin(2\gamma) \}]. \quad (17) \]

Hence the product of the dispersions is,

\[ (\Delta x)^2 (\Delta p)^2 = \frac{\hbar^2}{4} [1 + 2\lambda \{ 1 + 2J - 2J \cos(2\gamma) \}]. \quad (18) \]

The energy of the GCS turns out to be,

\[ < E > = \frac{1}{2m}(< p^2 > + m^2 \omega^2 < x^2 >) = \frac{\hbar \omega}{2} [1 + 2J + \lambda \{ 1 - 4J(1 + \frac{2}{3} J) \cos(2\gamma) - \frac{2}{3} J^2 \cos(4\gamma) \}]. \quad (19) \]

Let us now compute the dispersion of the number operator \( N = a^\dagger a : \)

\[ < N > = \sigma^1 \sigma^1 = J \{ 1 - \lambda(2 + J) \}, \quad < N^2 > = J(1 + J)(1 - 2\lambda(1 + J)) \rightarrow (\Delta N)^2 = J(1 - 2\lambda). \quad (20) \]
2.2 Fractional Revival

This section deals with the essential stability criteria of our GUP satisfying GCS. To discuss revival structure of the coherent states, let us re-write the energy eigenvalues as

\[ E_n = an^2 + bn, \quad a = \lambda \hbar \omega, \quad b = \hbar \omega (1 + \lambda). \tag{21} \]

Then the coherent state at a time \( t \) is given by

\[ |J, \gamma, t \rangle = \frac{1}{N(J)} \sum_{n=0}^{\infty} \frac{J^{n/2} e^{-i(\gamma e_n + E_n t / \hbar)}}{\sqrt{\rho_n}} |n \rangle \tag{22} \]

Next we consider the autocorrelation function defined by

\[ A(t) = \langle J, \gamma, t | J, \gamma, 0 \rangle \tag{23} \]

Thus squared modulus of the autocorrelation function \( |A(t)|^2 \) gives the probability of how far the coherent state resembles its original form. So, if at time \( t = t_R \), the coherent state regains its form at time \( t = 0 \), then \( |A(t_R)|^2 = 1 \) and \( t_R \) is called the period of full revival. Now from the relations (21) it follows that time of full revival is given by

\[ t_R = \frac{2\pi \hbar}{a} \tag{24} \]

provided the following relation is satisfied

\[ \beta m \hbar = \frac{1}{r - 1}, \quad r = integer \geq 2, \tag{25} \]

where we assume \( \beta \) to be positive. From the above relation we find that for the coherent state to exhibit full revival, the frequency has to be chosen suitably. Furthermore, it is known that for suitably fine tuned coupling the coherent states exhibit fractional revivals with periods \( t_{fr} = \frac{q}{r} t_R \) where \( q/r \) is a non reducible fraction of integers. In fact, from Fig 1 it can be seen that there are quarterly revivals at \( t = t_R/4, 3t_R/4 \). It may be of interest to note that these fractional revivals do not exist in the \( \beta \to 0 \) limit and are essentially a consequence of the presence of a minimal length.

3 Discussion

Below we present our results and their implications.

1. **GUP and Dynamics:** First we explicitly show that the GCS satisfy GUP. From [18] we recover the
generalized uncertainty principle (with the original parameter $\beta$ restored):

$$\langle \Delta x \rangle \langle \Delta p \rangle = \frac{\hbar}{2} \left[ 1 + 3\beta \left( \frac{m\hbar\omega}{2} \right) \{ 1 + 2J - 2J\cos(2\gamma) \} \right].$$ \hspace{1cm} (26)

Notice that, quite remarkably, the $\beta$-correction term in r.h.s. is identical to the $\lambda$-independent part of $\langle p^2 \rangle$ as obtained previously in (2.1), $\langle p^2 \rangle = \frac{m\hbar\omega}{2} \{(1 + 2J - 2J\cos(2\gamma)) + O(\lambda)\}$. Hence to $O(\lambda)$ we have shown that

$$\langle \Delta x \rangle \langle \Delta p \rangle = \frac{h}{2} (1 + 3\beta \langle p^2 \rangle).$$ \hspace{1cm} (27)

Indeed this is not completely the GUP (1) since it has $(\Delta p)^2$ on the r.h.s. But from (2.1) and (2.1) we might argue that for the $\lambda$-independent terms. $\sqrt{\langle p^2 \rangle} \gg \langle p \rangle$. Thus at very high momentum regime, where the correction terms become significant $\langle p^2 \rangle \gg \langle p \rangle^2 \rightarrow (\Delta p)^2 \approx \langle p^2 \rangle$ and we roughly recover the GUP (1) from (27). However, since from (17) $(\Delta p)^2 = \frac{m\hbar\omega}{2} (1 + O(\lambda))$, we can also interpret (27) as

$$\langle \Delta x \rangle \langle \Delta p \rangle = \frac{h}{2} (1 + \beta_{eff} (\Delta p)^2),$$ \hspace{1cm} (28)

where $\beta_{eff} = 3(1 + 2J)\beta$ with the oscillatory part averaged out. In this respect $\beta_{eff}$ loses partly its universality status since it becomes $J$-dependent but $J$ being positive $\beta_{eff} \geq \beta$. Since $J$ is related to the energy of the coherent state, this identification will eventually mean that particles having different energies will observe slightly different minimum length scale with the lower bound being determined by $\beta$. However, this observation may not appear too unexpected if we recall that in the Doubly Special Relativity scenario [6], (which induces the Non-Commutative phase space and GUP (1) of the present form in certain limiting condition), the generalized spacetime transformations become explicitly dependent on the energy and momentum of the particle whose coordinates are being transformed and so it is never possible to isolate the kinematics (spacetime transformations) from the dynamical content (particle energy and momentum). It seems that similar type of behaviour reappears for a realistic coherent state obeying the GUP that we have constructed.

The Heisenberg equation of motion $\langle \dot{A} \rangle = \frac{i}{\hbar} \langle [H, A] \rangle$ yields,

$$\langle \dot{x} \rangle = \sqrt{\frac{2\hbar\omega J}{m}} \left[ -\sin\gamma + \lambda \{(1 + J/2)\sin\gamma - 2\gamma\cos\gamma - \frac{4J}{3}\sin(3\gamma)\} \right].$$ \hspace{1cm} (29)

Comparison with (2.1) shows a mismatch in numerical factors. For $\lambda = 0$ the exact Correspondence Principle is recovered. Again below are the Newton’s equations in operator form with numerical mismatch:

$$\langle \dot{p} \rangle = \sqrt{\frac{\hbar\omega^3}{2}} \langle [a^\dagger a, (a - a^\dagger)] - \frac{\lambda}{3} [a^\dagger a, (a - a^\dagger)^3] \rangle,$$ \hspace{1cm} (30)
Finally we provide dynamics of the GCS,

\[ <\ddot{x}> = \sqrt{\frac{\hbar \omega^3}{2m}} < [a^\dagger a, (a - a^\dagger)] - \frac{4\lambda}{3} [a^\dagger a, (a - a^\dagger)^3] >. \]  

(31)

Hence the centre of the wave packet does follow a HO path but with an effective reduced frequency. As we have advertized earlier, Correspondence Principle is satisfied but with a twist. It is curious to note the charged planar HO in a perpendicular constant magnetic field also undergoes a reduction in the effective frequency. Furthermore the condition \( (1 - \frac{8\lambda}{3}) \geq 0 \) produces same order of magnitude bound on the GUP parameter \( \beta_0 \) as discussed below.

2. Minimum Length Scale: In [3] the authors consider Landau levels in a GUP corrected scenario and comparing with (Scanning Tunnelling Microscope) experimental data come up with a bound \( \beta_0 \leq 10^{50} \) for the dimensionless parameter \( \beta_0 = (M_{Planck}c)^2 \beta \) and \( \omega_c \sim 10^3 \) GHz (\( \omega_c \) being the cyclotron frequency of an electron in 10 \( T \) magnetic field). This is relevant in our context as well since we also use undeformed HO states in constructing the GCS. But this can have interesting consequences as we now show. The GUP (1) yields the lower bound for \( \Delta x \geq \hbar \sqrt{A \beta} \) (the minimum length scale) that in the present case reduces to

\[ \Delta x \geq \hbar \sqrt{3(1 + 2J)\beta} = \frac{\hbar \sqrt{3(1 + 2J)\beta_0}}{M_{Planck}c}. \]  

(33)

Taking the largest allowed \( \beta = 10^{50} \) [3] and for an electron-mass particle in our HO with \( \omega = \omega_c = 10^3GHz \) we derive the largest allowed value for the minimum length scale to be \( \approx 10^{-9} \) meter. This value can be considerably smaller depending upon \( \beta \). It is conceivable to suggest that these states might be experimentally observed.

3. Non-Poission Statistics: With the help of (20) we determine the Mandel parameter \( Q = (\Delta n)^2 / < n > - 1 \) that is a measure of the deviation from \( Q = 0 \), the Poissonian distribution. \( Q \geq 0 (Q \leq 0) \) are termed as Super-Poissionian (Sub-Poissionian) statistics. In the present case

\[ Q = -\lambda(4 + J) \]  

(34)

showing that the statistics is Sub-Poissionian. This can be compared with the constant NC space GCS of [12] that enjoys Poissionian statistics.

4. Revival Time Induced Bound: Rigorous estimates of the upper bound on \( \beta_0 < 10^{36} \), taking quantum field theoretic (Lamb shift) measurements in to account, have already been provided in [3]. In this section our
aim is modest since we want only to ensure that the properties of the GCS constructed here can give certain bounds on $\beta_0$ which are compatible with [3], albeit much weaker.

From (25) we can obtain an upper bound on $\beta_0$ that is independent of the bounds existing in the literature. In this case taking the minimum value of $r = 2$ yields $\beta_0 \leq \frac{(M_{\text{Planck}}c)^2}{m\omega\hbar} \approx 10^{53}$. This is comparable to the one derived in [3] that we have used before and provides an independent consistency check of the entire framework. On the other hand, exploiting the (24) one can get a relation

$$t_R\beta_0 \approx 10^{41},$$

(35)

for the same parameters $m, \omega$. For $\beta_0 \sim 10^{50}$ we find the lower bound $t_R \geq 10^{-9}\text{s}$. However (35) can be used to restrict $\beta_0$ as well. Also from (32) we arrive at a similar bound on $\beta_0$. This is the independent bound on GUP parameter that emerges from our GCS analysis.

**Conclusion and Future Prospects:** Let us put our work in proper perspective. We emphasize that until and unless one can (at least theoretically) establish the existence of stable quantum states obeying GUP, the GUP-based phenomenological results [3, 4] are of purely academic interest. This Letter provided the first explicit example of such states in the form of Generalized Coherent States. Hence our work non only provides this missing link in GUP-based phenomenology, it also motivates and strengthens earlier results [3, 4].

We have successfully constructed GCS that satisfies the GUP with its inherent length scale. Various features of these states are studied. In subsection 1 of Discussion we demonstrate the validity of GUP for the GCS constructed here and in subsection 3 we derive the independent bound revealed by our Revival Time analysis. These results are two of our most important contributions.

A major convenience of our scheme is its computational simplicity: only first order corrected energy together with undeformed Fock basis states are sufficient to generate non-trivial behaviour of the perturbed system. This has been clearly shown in the present Letter.

We recall that originally the idea of a GUP and the associated minimum length scale was mooted in [2] from high energy string scattering. In the present set up we can try to compute scattering cross-sections by using directly the GCS. The form of GUP and the NC phase space are precursors to the relativistic $\kappa$-Minkowski spacetime [6]. It will be interesting if one can construct GCS for the relativistic framework (see [13] for the formalism). Lastly our formalism can be exploited in other physically relevant non-linear oscillator systems [14].

**Appendix:** The results presented in this paper are expressible in closed form mainly because we restrict ourselves to $O(\lambda)$ corrections only. Below we provide a brief outline of the computational scheme.
From the Gazeau-Klauder GCS let us compute $\langle a \rangle$:

$$\langle a \rangle = \langle J, \gamma \vert a \vert J, \gamma \rangle = \frac{1}{N^2(J)} \sum_{n,m \geq 0} \sqrt{n} \frac{J^{(n+m)/2} e^{i \gamma (\epsilon_n - \epsilon_m)}}{\sqrt{\rho_n \rho_m}} \delta_{m,n-1}. \tag{36}$$

Using $\epsilon_{n-1} - \epsilon_n = -(1 + 2n \lambda)$ we find

$$\langle a \rangle = \frac{e^{-i \gamma}}{\sqrt{JN^2}} \left[ \sum_n n J^n e^{-i 2n \lambda \gamma} \sqrt{\rho_n} \right].$$

Using $e^n - 1 - e^n = -(1 + 2n \lambda)$ we find

$$\langle a \rangle = \frac{e^{-i \gamma}}{\sqrt{JN^2}} \left[ \sum_n J^n (1 - i 2n \lambda \gamma)(1 - \frac{1}{2}(1 + n)) \right].$$

Calculations for higher order terms in $a, a^\dagger$ follow in a straightforward manner. We provide another example below.

$$\langle N \rangle = \langle a^\dagger a \rangle = \sum_{m,n \geq 0} \frac{J^{(m+n)/2} e^{i \gamma (\epsilon_m - \epsilon_n)}}{\sqrt{\rho_m \rho_n}} n \delta_{m,n} = \sum_{n \geq 0} \frac{n J^n}{\rho_n}.$$

$$= \sum_{n \geq 1} \frac{J^n}{\rho_{n-1}(1 + \lambda(1 + n))} \approx \sum_{n \geq 0} \frac{J^{n+1}(1 - \lambda(1 + n))}{\rho_n} = J(1 + J)(1 - 2\lambda(1 + J)). \tag{38}$$

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