COMPLEXITY OF SHADOWS & TRAVERSING FLOWS IN TERMS OF THE SIMPLICIAL VOLUME

GABRIEL KATZ

Abstract. We combine Gromov’s amenable localization technique with the Poincaré duality to study the traversally generic vector flows on smooth compact manifolds $X$ with boundary. Such flows generate well-understood stratifications of $X$ by the trajectories that are tangent to the boundary in a particular canonical fashion. Specifically, we get lower estimates of the numbers of connected components of these flow-generated strata of any given codimension. These universal bounds are basically expressed in terms of the normed homology of the fundamental groups $\pi_1(D(X))$, where $D(X)$ denotes the double of $X$. The norm here is the Gromov simplicial semi-norm in homology. It turns out that some close relatives of the normed spaces $H_* (D(X); \mathbb{R})$ form obstructions to the existence of $k$-convex traversally generic vector flows on $X$.

1. Introduction

This paper is a direct extension of [AK]. As the latter article, it draws its inspiration from the paper of Gromov [Gr1] where, among other things, the machinery of amenable localization has been developed. Here we combine the amenable localization with the Poincaré duality to study the traversally generic vector flows (see [K2] or Section 2 for the definition) on smooth compact manifolds $X$ with boundary.

The rest of the paper is divided in three sections. In Section 2, we introduce the notion of a traversally generic vector field and describe its basic properties, needed for Section 4. These properties have been studied in a series of papers [K], [K1]—[K4], and [AK]. They are presented here for the reader’s convenience.

In Section 3, we study maps from compact PL-manifolds $X$ with boundary onto special compact CW-complexes $K$, dim$(K) = \text{dim}(X) - 1$. The local topology (the types of singularities) of $K$ is prescribed a priori; it is independent of $X$. By definition, the fibers of $F : X \to K$ are PL-homeomorphic to closed segments or to singletons. We call such maps $F$ the shadows of $X$. This setting is mimicking the maps $\Gamma : X \to T(v)$, generated by traversally generic fields $v$ on smooth manifolds $X$ with boundary. Here $T(v)$ denotes the space of $v$-trajectories, and $\Gamma$ takes each point $x$ to the trajectory $\gamma_x$ through $x$.

The target spaces $K$ of shadows $F$ come equipped with a natural stratification, defined by the local topology of the singular loci. With the help of $F$, that stratification induces stratifications in $X$. We introduce the $j$-th complexity of $F$ as the number of connected components of the strata of the fixed dimension $j$ (thus the word “complexity” in the title of this paper).
The main results of Section 3 are Theorem 3.1, named “the amenable localization of the Poincaré duality”, and Theorem 3.2. The latter links the singularities of shadows $F : X \to K$ with the “reduced homology” $H^*_\Delta(D(X); \mathbb{R})$, where $D(X)$ is the double of $X$, and the spaces $H^*_\Delta(D(X); \mathbb{R})$ are close relatives of the normed homology spaces $H^*_\Delta(D(X); \mathbb{R})$. The norm here is the Gromov simplicial semi-norm in homology $\text{Gr}$. More accurately, in Theorem 3.2, the $j$-th complexity of any shadow $F$ is estimated from below by the ranks of the groups $H^*_\Delta_{n+1-j}(D(X); \mathbb{R})$, where $n+1 = \dim(X)$. We stress that our results are vacuous when $\Pi = \pi_1(D(X))$ is an amenable group.

In Section 4, we apply the results from Section 3 to the shadows, produced by the traversally generic vector fields. The applications deliver two main results, Theorem 4.2 and Theorem 4.5. The $v$-flow canonically generates some well-understood stratifications of the spaces $\mathcal{T}(v)$, $X$, and $D(X)$ (see [K2]). As in the category of shadows, these stratifications lead to few competing notions of complexity for traversally generic flows. In Theorem 4.5, we get lower estimates of the numbers of connected components of these flow-generated strata of any given dimension. The estimates are universal for the given homotopy type of $D(X)$ and any traversing field on $X$. Again, these universal bounds are expressed in terms of the normed homology of $D(X)$ or $X/\partial X$. It turns out that the normed spaces $H^*\Delta_{n+1-j}(D(X); \mathbb{R})$ and $H^*\Delta_{n+1-j}(X/\partial X; \mathbb{R})$ form obstructions to the existence of the globally $j$-convex (see Definition 2.2) traversally generic vector flows on a given $X$!

2. Basics of Traversally Generic Vector Fields

We start with presenting few basic definitions and facts related to the traversally generic vector fields.

Let $X$ be a compact connected smooth $(n+1)$-dimensional manifold with boundary. A vector field $v$ is called traversing if each $v$-trajectory is either a closed interval with both ends residing in $\partial X$, or a singleton also residing in $\partial X$ (see [K1] for the details). In particular, a traversing field does not vanish in $X$. In fact, $v$ is traversing if and only if $v \neq 0$ and $v$ is of the gradient type (see [K1]).

For traversing fields $v$, the trajectory space $\mathcal{T}(v)$ is homology equivalent to $X$ (Theorem 5.1, [K3]).

We denote by $\mathcal{V}_{\text{trav}}(X)$ the space of traversing fields on $X$.

In this paper, we consider an important subclass of traversing fields which we call traversally generic (see formula (2.4) and Definition 3.2 from [K2]).

For a traversally generic field $v$, the trajectory space $\mathcal{T}(v)$ is stratified by closed subspaces, labeled by the elements $\omega$ of an universal poset $\Omega^\bullet_{(n)}$ which depends only on $\dim(X) = n+1$ (see [K3], Section 2, for the definition and properties of $\Omega^\bullet_{(n)}$). The elements $\omega \in \Omega^\bullet_{(n)}$ correspond to combinatorial patterns that describe the way in which $v$-trajectories $\gamma \subset X$ intersect the boundary $\partial_1 X := \partial X$. Each intersection point $a \in \gamma \cap \partial_1 X$ acquires a well-defined multiplicity $m(a)$, a natural number that reflects the order of tangency of $\gamma$ to $\partial_1 X$ at $a$ (see [K1] and Definition 2.1 for the expanded definition of $m(a)$). So $\gamma \cap \partial_1 X$ can be viewed as a divisor $D_\gamma$ on $\gamma$, an ordered set of points in $\gamma$ with their
multiplicities. Then $\omega$ is just the ordered sequence of multiplicities $\{m(a)\}_{a \in \gamma \cap \partial_1 X}$, the order being prescribed by $v$.

The support of the divisor $D_\gamma$ is either a singleton $a$, in which case $m(a) \equiv 0 \mod 2$, or the minimum and maximum points of sup $D_\gamma$ have odd multiplicities, and the rest of the points have even multiplicities.

![Figure 1. Traversally generic “vertical” field $v$ on the surface $X$, its stratified trajectory space $T(v)$, and the obvious map $\Gamma : X \to T(v)$](image)

Let

$$m(\gamma) := \sum_{a \in \gamma \cap \partial_1 X} m(a) \quad \text{and} \quad m'(\gamma) := \sum_{a \in \gamma \cap \partial_1 X} (m(a) - 1).$$

Similarly, for $\omega := (\omega_1, \omega_2, \ldots, \omega_i, \ldots)$ we introduce the norm and the reduced norm of $\omega$ by the formulas:

$$|\omega| := \sum_i \omega_i \quad \text{and} \quad |\omega'| := \sum_i (\omega_i - 1).$$

Let $\partial_j X := \partial_j X(v)$ denote the locus of points $a \in \partial_1 X$ such that the multiplicity of the $v$-trajectory $\gamma_a$ through $a$ at $a$ is greater than or equal to $j$. This locus has a description in terms of an auxiliary function $z : \hat{X} \to \mathbb{R}$ which satisfies the following three properties:

$$0 \text{ is a regular value of } z,$$
• $z^{-1}(0) = \partial_1 X$, and
• $z^{-1}((-\infty, 0]) = X$.

In terms of $z$, the locus $\partial_j X = \{ z = 0, L_vz = 0, \ldots, L_v^{(j-1)}z = 0 \}$, where $L_v^{(k)}$ stands for the $k$-th iteration of the Lie derivative operator $L_v$ in the direction of $v$ (see [K2]).

The pure stratum $\partial_j X^0 \subset \partial_j X$ is defined by the additional constraint $L_v^{(j)}z \neq 0$.

**Definition 2.1** The multiplicity $m(a)$, where $a \in \partial X$, is the index $j$ such that $a \in \partial_j X^0$.

The characteristic property of traversally generic fields is that they admit special flow-adjusted coordinate systems, in which the boundary is given by quite special polynomial equations (see formula (2.4)) and the trajectories are parallel to one of the preferred coordinates' axis (see [K2], Lemma 3.4). For a traversally generic $v$ on a $(n + 1)$-dimensional $X$, the vicinity $U \subset \hat{X}$ of each $v$-trajectory $\gamma$ of the combinatorial type $\omega$ has a special coordinate system

$$(u, \vec{x}, \vec{y}) : U \to \mathbb{R} \times \mathbb{R}^{\omega'_i} \times \mathbb{R}^{n-|\omega|'}.$$ 

By Lemma 3.4 and formula (3.17) from [K2], in these coordinates, the boundary $\partial_1 X$ is given by the polynomial equation:

$$(2.4) \quad P(u, \vec{x}) := \prod_i [(u - \alpha_i)^{\omega_i} + \sum_{l=0}^{\omega_l-2} x_{i,l}(u - \alpha_i)^l] = 0$$

of an even degree $|\omega|$ in $u$. Here $\vec{x} = \{x_{i,l}\}_{i,l}$, and the numbers $\{\alpha_i\}_i$ are the distinct real roots (of the multiplicities $\{\omega_i\}_i$) of the polynomial $P(u, \vec{0})$, ordered so that $\alpha_i < \alpha_{i+1}$ for all $i$.

At the same time, $X$ is given by the polynomial inequality $\{P(u, \vec{x}) \leq 0\}$. Each $v$-trajectory in $U$ is produced by freezing all the coordinates $\vec{x}, \vec{y}$, while letting $u$ to be free.

We denote by $X(v, \omega)$ the union of $v$-trajectories whose divisors are of a given combinatorial type $\omega \in \Omega_{\{\omega\}}^\bullet$. Its closure $\cup_{\omega' \preceq \omega} X(v, \omega')$ is denoted by $X(v, \omega_{\preceq \bullet})$.

Each pure stratum $\mathcal{T}(v, \omega) \subset \mathcal{T}(v)$ is an open smooth manifold and, as such, has a “conventional” tangent bundle.

We denote by $\mathcal{V}^\dagger(X)$ the space of traversally generic fields on $X$. It turns out that $\mathcal{V}^\dagger(X)$ is an open and dense (in the $C^\infty$-topology) subspace of $\mathcal{V}_{\text{trav}}(X)$ (see [K2], Theorem 3.5).

**Definition 2.2.** We say that a traversing field $v$ on $X$ is globally $k$-convex if $m'(\gamma) < k$ for any $v$-trajectory $\gamma$. 

3. Shadows of Manifolds with Boundary and their Complexity

The notion and properties of shadows (see Definition 3.1), the main subject of this section, are inspired by the maps $\Gamma : X \rightarrow \mathcal{T}(v)$, where the field $v$ is traversally generic.

We are going to pick a fixed and carefully chosen class of compact $n$-dimensional CW-complexes $K$ whose local topological structure is prescribed.

Let $X$ be a compact PL-manifold of dimension $n + 1$ with boundary.

We will consider a variety of surjective maps $\{F : X \rightarrow K\}$ with the $F$-fibers being a particular type of contractible 1-dimensional complexes (segments and singletons). We think of such CW-complexes $K = F(X)$ as “shadows” of the given manifold $X$. We consider singularities in $K$ of particular types $\{K(\omega)\}_\omega$ and intend to count the cardinalities $\#\pi_0(K(\omega))$. We view this count of connected components of the strata $K(\omega)$ as measuring the complexity of the surjection $F$. Then we minimize these numbers over all $F$’s to get various notions of complexity for the given manifold $X$.

Let us start with a quite general setting. Let $\mathcal{S}$ be a poset equipped with two maps: a map $\mu : \mathcal{S} \rightarrow \mathbb{Z}_+$ and an order-preserving map $\mu' : \mathcal{S} \rightarrow \mathbb{Z}_+$. By definition, for each $\omega \in \mathcal{S}$, $\mu'(\omega) < \mu(\omega)$ for each $n \in \mathbb{Z}_+$, we assume that the poset $\mathcal{S}_n = (\mu')^{-1}([0, n])$ is finite.

With each element $\omega \in \mathcal{S}_n$ we associate a model compact CW-complex $T_\omega$ of dimension $\mu'(\omega)$ and a model compact PL-manifold $E_\omega$ of dimension $\mu'(\omega) + 1$. They are linked by a PL-map $p_\omega : E_\omega \rightarrow T_\omega$ whose fibers are closed intervals or singletons. In what follows, we will list additional properties of the two collections, $\mathcal{E} = (E_\omega)_\omega \in \mathcal{S}$ and $\mathcal{T} = (T_\omega)_\omega \in \mathcal{S}$ (exhibiting topologically distinct $T_\omega$’s). We will do it in a recursive fashion.

Consider a set $\text{Cosp}(\mathcal{T}, n)$ of $n$-dimensional compact CW-complexes $K$ such that each point $y \in K$ has a neighborhood which is PL-homeomorphic to the product $T_\omega \times D^{n-\mu'(\omega)}$ for some $\omega \in \mathcal{S}$, where $\mu'(\omega) \leq n$, and $D^{n-\mu'(\omega)}$ denotes the standard ball.

We require that each model space $T_\omega$ topologically will be a cone over a space $S_\omega$ that belongs to the set $\text{Cosp}(\mathcal{T}, n - 1)$.

By definition, $\text{Cosp}(\mathcal{T}, 1)$ consists of finite graphs whose vertices are of valencies 1 and 3 only.

We denote by $K(\omega)$ the set of points in $K$ whose neighborhoods are modeled after the space $T_\omega \times D^{n-\mu'(\omega)}$. It follows that each $K(\omega) \subset K$ is a locally closed PL-manifold.

Let us also consider a filtration

$$K = K_0 \supset K_1 \supset \cdots \supset K_{-n}$$

of $K$ by the closed subcomplexes

$$(3.1) \quad K_{-j} = \bigcup_{\{\omega \in \mathcal{S}_n | \mu'(\omega) \geq j\}} K(\omega)$$

of dimensions $n - j$. Note that $K_{-j} = \emptyset$ implies $K_{-(j+1)} = \emptyset$.

$^1$“Cosp” is an abbreviation of “cospine”.
$^2$“normal” to the $\omega$-labeled stratum $K(\omega)$ in $K$
Definition 3.1. Let $X$ be a compact connected PL-manifold of dimension $n + 1$. We assume that $\partial X \neq \emptyset$. Consider the set $\text{Shad}(X, E \Rightarrow T)$ of surjective PL-maps $F : X \to K$ such that:

- $K$ is a compact CW-complex of the type from $\text{Cosp}(T, n)$,
- each fiber of $F$ is PL-homeomorphic to a closed interval $I$ or to a singleton $pt$, where $\partial I$ and $pt$ reside in $\partial X$,
- $F^\partial \overset{\text{def}}{=} F| : \partial X \to K$ is a surjective map with finite fibers,
- for each $\omega \in S$, the map $F| : F^{-1}(K(\omega)) \to K(\omega)$ is a trivial fibration with an orientable manifold base, and the map $F^\partial| : (F^\partial)^{-1}(K(\omega)) \to K(\omega)$ is a trivial covering with the fiber of cardinality $\mu(\omega) - \mu'(\omega)$, where $\partial I$ and $pt$ reside in $\partial X$.

- $\{\sigma_i(\Delta)\}$ intersect $K$. Put $\|c\|_{l_1} = \sum_i |r_i|$. We define the simplicial semi-norm of $h$ by the formula:

$$\|h\|_\Delta = \inf_c \{\|c\|_{l_1}\}.$$
This construction defines a semi-norm \( \| \sim \|_{\Delta} \) on the vector space \( H_k(X,Y; \mathbb{R}) \). The norm is monotone decreasing under continuous maps of pairs of spaces:

\[
\|h\|_{\Delta} \geq \|f_*(h)\|_{\Delta}
\]

for any \( h \in H_k(X_1,Y_1; \mathbb{Z}) \) and a continuous map \( f : (X_1,Y_1) \to (X_2,Y_2) \).

Moreover, if \( f : X_1 \to X_2 \) is a continuous map such that \( f_* : \pi_1(X_1) \to \pi_1(X_2) \) is an isomorphism of the fundamental groups, then \( \|f_*(h)\|_{\Delta} = \|h\|_{\Delta} \) for any \( h \in H_k(X_1; \mathbb{R}) \).

If \( M \) is any closed, oriented hyperbolic manifold, then

\[
\text{Vol}(M) = c(n) \cdot \|[M]\|_{\Delta},
\]

where \([M]\) denotes the fundamental class of \( M \) and \( c(n) \) is an universal positive constant (this is the Proportionality Theorem, page 11 of [Gr]). For this reason, the simplicial norm of the fundamental class \([X]\) is often called the simplicial volume.

Gromov’s Localization Lemma 3.1 below relies on the notion of the stratified simplicial semi-norm, available for stratified spaces \( X \) and pairs \( X \supset Y \) of stratified spaces.

We consider stratified spaces such that if a stratum \( S \) intersects the closure \( \overline{S}' \) of another stratum \( S' \), then \( S \subseteq \overline{S}' \). In this case we write \( S \preceq S' \). If neither \( S \preceq S' \) nor \( S' \preceq S \), then we say the two strata are incomparable.

Recall that the corank of a stratum \( Y_\omega \) in a \( \mathcal{S} \)-stratification of a given space \( Y \) is the maximal length \( k \) of a filtration \( Y_\omega \subset \overline{Y}_{\omega_1} \subset \cdots \subset \overline{Y}_{\omega_k} \) by the distinct strata whose closures contain \( Y_\omega \).

**Definition 3.4.** Let \( X \) be a \( \mathcal{S} \)-stratified topological space. Given a real homology class \( h \in H_k(X; \mathbb{R}) \), consider all singular cycles \( c = \sum_i r_i \sigma_i \) that represent \( h \), where \( r_i \in \mathbb{R} \) and \( \sigma_i : \Delta^k \to X \) are singular simplicies that are consistent with the stratification \( \mathcal{S} \), in the following sense:

- We require that for each simplex \( \sigma_i \) of \( c \), the image of the interior of each face (of any dimension) must be contained in one stratum. We call this the **cellular** condition.
- The (ord) condition states that the image of each simplex of \( c \) must be contained in a totally ordered chain of strata; that is, the simplex does not intersect any two incomparable strata.
- The (int) condition states that for each simplex of \( c \), if the boundary of a face (of any dimension) maps into a stratum \( S \), then the whole face maps into \( S \).
- For technical reasons (involving the Amenable Reduction Lemma in [AK]), we require that if two vertices of a simplex \( \sigma_i \) map to the same point \( v \in X \), then the edge between them must be constant at \( v \). We call this the **loop** condition.

\[\text{Gr}\].
Figure 2. Examples of a singular 2-simplex in relation to a stratification of the plane by a single stratum of codimension 2, three strata of codimension 1, and two strata of codimension 0. Diagrams (a), (b), (c) are consistent with the four bullet list from Definition 2.4; diagram (d) violates the second bullet, diagram (e) violates the third bullet, diagram (f) violates the first bullet.

We define the $S$-stratified simplicial semi-norm of $h$ by the formula:

$$\|h\|_S^\Delta = \inf_c \{ \|c\|_{l_1} \},$$

where $c$ runs over all the cycles $c = \sum_i r_i \sigma_i$ that represent $h$, subject to the four properties above.

Similar definition of $\|h\|_S^\Delta$ is available for relative homology classes $h \in H_k(X,Y;\mathbb{R})$, where the inclusion $Y \subset X$ is a $S$-stratified map.

**Definition 3.5.** A discrete group $G$ is called amenable if for every finite subset $S \subset G$ and every $\epsilon > 0$, there exists a finite non-empty set $A \subset G$ such that the proportion of cardinalities

$$\frac{|((g \cdot A) \cup A) \setminus ((g \cdot A) \cap A)|}{|A|} < \epsilon$$

for all $g \in S$.

Finally, we are in position to state the pivotal Gromov’s Localization Lemma from [Gr1], page 772. Its proof there is a bit rough; a detailed proof can be found in [AK].

**Lemma 3.1 (Gromov’s Localization Lemma).** Let $X$ be a closed $(n+1)$-manifold with stratification $S$ consisting of finitely many connected locally closed submanifolds. Pick
an integer \( j \) from the interval \([0, n + 1]\). Let \( Z \) be a space with the contractible universal cover\(^4\), and let \( \alpha : X \to Z \) be a continuous map such that the \( \alpha \)-image of the fundamental group of each stratum of codimension less than \( j \) is an amenable subgroup of \( \pi_1(Z) \).

Let \( X_{< j} \subseteq X \) denote the union of strata with codimension at least \( j \), and let \( U \) be a neighborhood of \( X_{< j} \) in \( X \). Then the \( \alpha \)-image of every \( j \)-dimensional homology class \( h \in H_j(X) \) satisfies the upper bound

\[
\| \alpha_\ast(h) \|_\Delta \leq h_U \|_\Delta^n,
\]

where \( h_U \in H_j(U, \partial U) \) denotes the restriction of \( h \) to \( U \), obtained via the composite homomorphism

\[
H_j(X) \to H_j(X, X \setminus U) \to H_j(U, \partial U),
\]

where the last map is the excision isomorphism. \( \bullet \)

Let \( X \) be a compact oriented manifold with boundary. Let \( D(X) \) denote its double \( X \cup_{\partial X} X \). The orientation on \( X \) extends to an orientation on its double. Then there is an orientation-reversing involution \( \tau : D(X) \to D(X) \) whose orbit space is \( X \) and whose fixed point set \( D(X)^\tau = \partial X \).

Let \( S \) be a stratification of \( X \) by strata \( S \) such that the sets \( \{ S \cap \partial X \}_{S \in S} \) forms a stratification \( S^\Delta \) of the boundary \( \partial X \). Then \( S \) gives rise to a stratification \( DS \) of the double \( D(X) \) so that any stratum \( S \in DS \) either belongs to \( \partial X \) (in which case \( S \in S^\partial \)) or to \( D(X) \setminus \partial X \). Conversely, any \( \tau \)-invariant stratification \( DS \) of the double induces a stratification \( S \) of \( X \).

**Lemma 3.2.** Let \( X \) be a connected compact and oriented \((n + 1)\)-manifold with boundary. Let \( \beta : D(X) \to X/\partial X \) be the degree 1 map which collapses the second copy \( \tau(X) \subseteq D(X) \) of \( X \) to the point \( * = \partial X/\partial X \in X/\partial X \), and let \( \alpha : (X/\partial X, *) \to (X/\partial X, *) \) be the obvious map of pairs.

Then, for any \( h \in H_j(X, \partial X) \), we get

\[
2\| h \|_\Delta \geq \| h - \tau_\ast(h) \|_\Delta \geq \| \alpha_\ast(h) \|_\Delta.
\]

In particular, for the fundamental class \( h = [X, \partial X] \in H_{n+1}(X, \partial X; \mathbb{R}) \), the class \( h - \tau_\ast(h) = [D(X)] \) is fundamental; so we get

\[
2\| [X, \partial X] \|_\Delta \geq \| [D(X)] \|_\Delta \geq \| [X/\partial X] \|_\Delta.
\]

**Proof.** If \( c = \sum_i r_i \sigma_i \) is a relative cycle with the \( l_1 \)-norm \( \| c \| \) being \( \epsilon \)-close to \( \| h \|_\Delta \), then the chain \( \sum_i r_i \sigma_i - \sum_i r_i \tau(\sigma_i) \) is an absolute cycle in \( D(X) \) whose \( l_1 \)-norm is \( 2\| c \| \) at most. Thus \( 2\| h \|_\Delta \geq \| h - \tau_\ast(h) \|_\Delta \).

On the other hand, consider the degree 1 map \( \beta : D(X) \to X/\partial X \). By [Gr], the simplicial volume does not increase under continuous maps. Therefore, observing that \( \alpha_\ast(h) = \beta_\ast(h - \tau_\ast(h)) \), we get \( 2\| h \|_\Delta \geq \| \alpha_\ast(h) \|_\Delta \).

\( \square \)

\( ^4\)that is, a \( K(\pi, 1) \)-space
Next, we are going to associate few useful differential chain complexes and their homology groups with shadows of manifolds $X$ with boundary. Eventually they will become useful instruments in our investigations of various notions of complexity of traversing flows.

Let $F \in \text{Shad}(X, E \Rightarrow T)$ be a shadow (see Definition 3.1) of a compact orientable PL-manifold $X$ of dimension $n+1$. Put $K = \text{def } F(X)$. We consider the finite $S_n$-stratification $\{K(\omega)\}_{\omega \in S_n}$ of $K$ by the model (normal to the strata) spaces $\{T_\omega\}$ of dimensions $\{\mu'(\omega)\}$. We denote by $X(F, \omega)$ the $F$-preimage of the pure stratum $K(\omega)$. These sets form a stratification $S_F(X)$ of $X$.

The CW-complex $K \in \text{Cosp}(T, n)$ comes equipped with a filtration

$$K = \text{def } K_0 \supset K_1 \supset \cdots \supset K_n$$

which has been introduced in (2.1) and employs the more refined $S_n$-stratification.

Let $\mathcal{A}$ be an abelian coefficient system on $K$. As a default, $\mathcal{A} = \mathbb{R}$, the trivial coefficient system with the real numbers for a stalk.

For each $j \in [0, n]$, consider the relative homology groups

$$\{\mathcal{C}_j \mathcal{U}(K) := H_j(K_{n+j}, K_{n+j-1}; \mathcal{A})\}_j$$

associated with the filtration. Note that $\dim(K_{n+j}) = j$, so $\mathcal{C}_j \mathcal{U}(K)$ is the top homology of the quotient $K_{n+j}/K_{n+j-1}$.

These homology groups can be organized into a differential complex

$$\mathcal{C}_n \mathcal{U}(K) = \text{def } \{0 \rightarrow \mathcal{C}_n \mathcal{U}(K) \xrightarrow{\partial} \mathcal{C}_{n-1} \mathcal{U}(K) \xrightarrow{\partial} \cdots \xrightarrow{\partial} \mathcal{C}_0 \mathcal{U}(K) \rightarrow 0\},$$

where the differentials $\{\partial_j\}$ are the boundary homomorphisms from the long exact homology sequences of the triples

$$\{K_{n+j} \supset K_{n+j-1} \supset K_{n+j-2}\}_j.$$

Since the pure strata $K^o_{n+j} = \text{def } K_{n+j} \setminus K_{n+j-1}$ are open manifolds of dimension $j$, the $\mathcal{A}$-modules $\{\mathcal{C}_j \mathcal{U}(K)\}_j$ are free and finitely generated, the number of generators being the number of connected components of the locus $K^o_{n+j}$.

We call the homology groups $H_n \mathcal{U}(K)$, associated with the differential complex (3.4), the $\mathcal{U}$-homology of $K$.

In fact, the $\mathcal{U}$-homology is an ingredient of the homology spectral sequence associated with the filtration $\{K_{-j}\}_j$ and converging to the regular homology $H_*(K) \approx H_*(X)$ for any shadow $K = F(X)$. Therefore the is a canonical homomorphism $A^\mathcal{U}_*: H^\mathcal{U}_*(K) \rightarrow H_*(X)$.

These spectral sequence instruments and their applications to traversally generic fields will be developed in the paper to follow.

**Definition 3.5.** With any filtered CW-complex $K \in \text{Shad}(T, n)$ and its filtration as in (2.1), we associate the ordered collection of ranks

$$\{c_j \mathcal{U}(K) = \text{def } \text{rk}_\mathcal{A}(\mathcal{C}_j \mathcal{U}(K))\}_{0 \leq j \leq n},$$

where the groups $\{\mathcal{C}_j \mathcal{U}(K)\}_j$ were introduced in (3.3).

\[\text{in our notations, we suppress the dependance of these homology groups on the coefficients } \mathcal{A}.\]
We call \( c_j^{\text{shad}}(K) \) the \( j \)-th \( \mathcal{U} \)-complexity of \( K \).

**Definition 3.6.** Let \( X \) be a compact connected PL-manifold with boundary, \( \dim(X) = n + 1 \). Consider the variety of maps \( F : X \to K \) from the set \( \text{Shad}(X, E \Rightarrow T) \) as in Definition 3.1.

Each \( F \) produces the sequence of \( \mathcal{U} \)-complexities:

\[
c(F) = \{ c_0^{\text{shad}}(F(X)), c_1^{\text{shad}}(F(X)), \ldots, c_n^{\text{shad}}(F(X)) \}
\]

Consider the *lexicographical minima*

\[
c^{\text{shad}}(X, E \Rightarrow T) = \text{lex. min}_{\{ F \in \text{Shad}(X, E \Rightarrow T) \}} c(F(X))
\]

We call them the *lexicographic shadow complexity* of \( X \).

**Remark 3.1.** Of course, the definitions above rely on the set \( \text{Shad}(X, E \Rightarrow T) \) being nonempty, a nontrivial fact which requires a carefully designed poset \( S \) and a system of models \( \{ p_\omega : E_\omega \to T_\omega \}_{\omega \in S} \) (as the ones introduced in \([K2]\)).

**Remark 3.2.** If a compact connected PL-manifold \( X \) of dimension \( n + 1 \) is globally \( k \)-convex (see Definition 3.2), then evidently \( c_j^{\text{shad}}(X, E \Rightarrow T) = 0 \) for all \( j \leq n - k \).

Our next goal is to find some lower estimates of the complexities \( \{ c_j^{\text{shad}}(X, E \Rightarrow T) \}_j \) in terms of the algebraic topology of \( X \).

Let us consider a refinement \( S^*_F(X) \) of the stratification \( S_F(X) \) of \( X \), formed by the connected components of the sets:

\[
\{ X^\circ(F, \omega) = X(F, \omega) \setminus (\partial X \cap X(F, \omega)), \ X^\partial(F, \omega) = \partial X \cap X(F, \omega) \}_{\omega \in S}.
\]

See Fig. 1 which shows an example of such a stratification \( S^*_F(X) \) on a surface.

For technical reasons we are interested in the double \( D(X) = X \cup \partial X \) of \( X \), a closed manifold equipped with the orientation-reversing involution \( \tau \). The double acquires the \( \tau \)-equivariant stratification \( S_F(D(X)) \) whose pure strata are:

\[
\{ X^\circ(F, \omega), \ \tau(X^\circ(F, \omega)), \ X^\partial(\omega) \}_{\omega}.
\]

and its refinement \( S^*_F(D(X)) \), formed by the connected components of the strata from \( S_F(D(X)) \).

The stratifications \( S^*_F(X) \) and \( S^*_F(D(X)) \) give rise to the filtrations:

\[
X = X^F_0 \supset X^F_1 \supset \cdots \supset X^F_{-(n+1)},
\]

\[
D(X) = D(X)^F_0 \supset D(X)^F_1 \supset \cdots \supset D(X)^F_{-(n+1)}
\]

by the union of strata of a fixed codimension, each pure stratum being an open manifold.

Analogously to (3.3), we consider the homology and cohomology groups with coefficients in \( \mathcal{A} \):

\[
\{ C_j^j(D(X), F) = H_j(D(X)^F_{j-n-1}, D(X)^F_{j-n}; \mathcal{A}) \}_j,
\]

\[
C_j^j(D(X), F) = H^j(D(X)^F_{j-n-1}, D(X)^F_{j-n}; \mathcal{A})_j.
\]
They can be organized into a differential complex:

\[ (3.8) \quad C^j_\natural(D(X), F) =_{\text{def}} \{ 0 \to C^i_{n+1}(D(X), F) \overset{\partial_{n+1}}{\to} C^i_n(D(X), F) \overset{\partial_n}{\to} \cdots \overset{\partial_1}{\to} C^i_0(D(X), F) \to 0 \}, \]

where the differentials \( \{ \partial_j \} \) are the boundary homomorphisms from the long exact homology sequences of triples

\[ \{ D(X)^{E_j} \supset D(X)^{E_{j-1}} \supset D(X)^{E_{j-2}} \} \jmath. \]

Similarly, we introduce the dual differential complex

\[ (3.9) \quad C^\natural(D(X), F) =_{\text{def}} \{ 0 \leftarrow C^\natural_{n+1}(D(X), F) \overset{\partial_{n+1}}{\leftarrow} C^\natural_n(D(X), F) \overset{\partial_n}{\leftarrow} \cdots \overset{\partial_1}{\leftarrow} C^\natural_0(D(X), F) \leftarrow 0 \} \]

When \( \mathcal{A} = \mathbb{Z} \), since \( D(X)^{F_{j-n-1}} \setminus D(X)^{F_{j-n}} \) is an open orientable \( j \)-manifold, by \cite{Hat}, Corollary 3.28, the torsion

\[ \text{tor}\{ H_{j-1}(D(X)^{F_{j-n-1}}, D(X)^{F_{j-n}}; \mathbb{Z}) \} = 0 \]

which results in the natural isomorphism

\[ C^\natural_j(D(X), F) \approx \text{Hom}_\mathbb{Z}(C^j_\natural(D(X), F), \mathbb{Z}). \]

In the notations to follow, we drop the dependence of the constructions on the choice of the coefficient group \( \mathcal{A} \), but presume that \( \mathcal{A} = \mathbb{Z} \) or \( \mathbb{R} \).

So the complexes \( C^j_\natural(D(X), F) \) and \( C^\natural_j(D(X), F) \) are comprised of free \( \mathcal{A} \)-modules whose generators \( \{ [\sigma] \} \) are in 1-to-1 correspondence with the strata \( \sigma \in S^*_F(D(X)) \) (see (3.6))

We denote by \( Z^j_\natural(D(X), F) \) the kernels of the differentials \( \partial_j \) and by \( B^j_\natural(D(X), F) \) the images of the differentials \( \partial_{j+1} \) from (3.8)

We form the \( \mathcal{U} \)-homology

\[ H^\mathcal{U}_j(D(X), F) := Z^\mathcal{U}_j(D(X), F)/B^\mathcal{U}_j(D(X), F) \]

of the double \( D(X) \), associated with its stratification \( S^*_F(D(X)) \).

Since each cycle \( \zeta \in Z^\mathcal{U}_j(D(X), F) \) is a common singular cycle in \( D(X) \) (indeed, by (3.7) and (3.8), the boundary of the \( j \)-chain \( \zeta \) is mapped in the subcomplex of \( D(X) \) of dimension \( j - 2 \)), there is a canonical homomorphism

\[ A^\mathcal{U}_\natural : H^\mathcal{U}_\natural(D(X), F) \to H_\natural(D(X)). \]

Since the differential complex \( (3.9) \) is the dual of the differential complex \( (3.8) \), the natural paring \( C^{n-k}_\natural(D(X), F) \otimes C^{n-k}_\natural(D(X), F) \to \mathbb{R} \) produces an isomorphism

\[ \Phi : C^{n-k}_\natural(D(X), F)/B^{n-k}_\natural(D(X), F) \approx (Z^{\mathcal{U}}_{n-k}(D(X), F))^*, \]

where \( (Z^{\mathcal{U}}_{n-k}(D(X), F))^* \) denotes the dual space of \( Z^{\mathcal{U}}_{n-k}(D(X), F) \).

One of the crucial questions is to understand how the differential complexes \( C^\mathcal{U}_\natural(F(X)) \) in (3.4) and \( C^\mathcal{U}_\natural(D(X), F) \) in (3.8) vary, as \( F \) runs over the set of shadows \( \text{Shad}(X, E \Rightarrow T) \).
Such understanding would allow to treat the appropriate equivalence class of a particular differential complex $C^n_i(F(X))$ as an invariant of $X$. We will devote another article to such investigations.

The next localization construction is central to the rest of this section. Using the Poincaré duality $\mathcal{D}$ on the closed oriented manifold $D(X)$, for each $k \in [-1,n]$, we introduce the localization transfer map

$$L_{k+1}^F : H_{k+1}(D(X)) \cong D H^n_{-k}(D(X)) \to H^n_{-k}(D(X)_{-k+1}^F),$$

where $i : D(X)_{-k+1}^F \subset D(X)$ is the embedding.

Because the locus $D(X)_{-k+1}^F$ is a $(n-k)$-dimensional $CW$-complex, the top cohomology group $H^n_{-k}(D(X)_{-k+1}^F)$ can be identified with the quotient group

$$C^n_{-k}(D(X), F)/\mathcal{B}^{n-k}_{-k}(D(X), F)$$

from the complex in (3.8). This fact follows from the diagram chase in the two long exact cohomology sequences of the pairs

$$(D(X)_{-k+1}^F, D(X)^F_{-(k+2)}), \quad (D(X)_{-k+1}^F, D(X)_{-(k+3)}) \to 0,$$

The first pair gives rise to the fragment of the exact sequence

$$\to H^n_{-k-1}(D(X)_{-(k+2)}) \delta \to C^n_{-k}(D(X), F) \to H^n_{-k}(D(X)_{-k+1}^F) \to 0,$$

while the second pair to the fragment

$$\to C^n_{-k-1}(D(X), F) \to H^n_{-k-1}(D(X)_{-(k+2)}^F) \to 0,$$

so that the composition $j^* \circ \delta$ coincides with the boundary map

$$\delta^n_{-k-1} : C^n_{-k-1}(D(X), F) \to C^n_{-k}(D(X), F)$$

from the differential complex $C^n_{-k}(D(X), F)$. Therefore

$$H^n_{-k}(D(X)_{-k+1}^F) \approx C^n_{-k}(D(X), F)/\mathcal{B}^{n-k}_{-k}(D(X), F).$$

Similar diagram chase in homology leads to an isomorphism

$$H^n_{-k}(D(X)_{-k+1}^F) \approx \mathbb{Z}^{n-k}_{-k}(D(X), F).$$

As a result, for each shadow $F$, we get the transfer homomorphism

$$L_{k+1}^{F,0} : H_{k+1}(D(X)) \cong D H^n_{-k}(D(X)) \to C^n_{-k}(D(X), F)/\mathcal{B}^{n-k}_{-k}(D(X), F).$$

Our next goal is to introduce a norm $\|\cdot\|$ in the target space

$$C^n_{-k}(D(X), F)/\mathcal{B}^{n-k}_{-k}(D(X), F)$$

and to show that

$$\|h\|_{\Delta} \leq const(n) \cdot \|L_{k+1}^{F,0}(h)\|_{\mathcal{B}}$$

for any $h \in H_{k+1}(D(X))$ and some universal constant $const(n) > 0$ which depends only on the list of model spaces $\{E_\omega \to T_\omega\}_\omega$ as in the last bullet of Definition 3.1.
Consider the $l_1$-norm $\| \sim \|_{l_1}$ on the space $C_{\beta}^{k-}(D(X), F)$ in the unitary basis $\{[\sigma]^i\}$, labeled by the strata $\sigma \in S_{\rho}^k(D(X))$ of dimension $n-k$, and dual to the preferred basis $\{[\sigma]\}$ of $C_{\beta}^{n-k}(D(X), F)$. Then the norm $\|\sim\|_{l_3}$ on $C_{\beta}^{k-}(D(X), F)/B_{\beta}^{n-k}(D(X), F)$ is by definition the quotient norm induced by the norm $\| \sim \|_{l_1}$ on the space $C_{\beta}^{n-k}(D(X), F)$.

We may regard the maximal proportion

$$\sup_{\{h \in H_{k+1}(D(X)); \|h\|_\Delta \neq 0\}} \{\|[\mathcal{L}_{k+1}^{F,\beta}(h)]_{l_3}/\|h\|_\Delta\}\}
$$

as the norm of the operator $\mathcal{L}_{k+1}^{F,\beta}$. So the previous inequality (see also (3.11)) testifies that these norms admit a positive lower bound, universal for all $X$ and $F$.

**Theorem 3.1 (Amenable Localization of the Poincaré Duality).**

Let $X$ be a compact connected $(n+1)$-dimensional PL-manifold with a nonempty boundary.

Assume that for each connected component of the boundary $\partial X$, the image of its fundamental group in $\pi_1(D(X))$ is an amenable group.

Then, there exists an universal constant $\Theta > 0$ such that, for any shadow $F \in \text{Shad}(X, E \Rightarrow T)$, the space $C_{\beta}^{k-}(D(X), F)/B_{\beta}^{n-k}(D(X), F)$ admits a norm $\| \|_{l_3}$ so that

$$(3.11) \quad \|h\|_\Delta \leq \Theta \cdot \|[\mathcal{L}_{k+1}^{F,\beta}(h)]_{l_3}\|
$$

for any $h \in H_{k+1}(D(X))$. Here the operator $\mathcal{L}_{k+1}^{F,\beta}$ is introduced in (3.10), and the constant $\Theta$ depends only on $n$, the poset $S$, and the list of model maps $\{E_\omega \to T_\omega\}_{\omega \in S}$ in the way that is described in formula (3.16).

**Proof.** Let $U$ be a regular neighborhood of the set $D(X)^{F,\beta}_{-(k+1)}$ in $D(X)$. So the PL-manifold $U$ has a homotopy type of a $(n-k)$-dimensional CW-complex $D(X)^{F,\beta}_{-(k+1)}$.

Let $\Pi = \text{def} \; \pi_1(D(X))$. In order to apply Localization Lemma [3.1] to the neighborhood $U$, the homology class $h \in H_{k+1}(D(X); Z)$, and the classifying map $\beta : D(X) \to K(\Pi, 1)$, we need to check that, for each stratum $\sigma \in S_{\rho}^k(D(X))$ of $D(X)$, the subgroup $\beta_\beta \pi_1(\sigma)$ of $\Pi$ is an amenable group. This is true if $\sigma \subseteq \partial X$ since for each connected component of $\partial X$, the image of its fundamental group in $\Pi$ is amenable, and every subgroup of an amenable group is amenable. Otherwise, $\sigma$ is contained in some $X^c(\omega)$ or in some $\tau(X^c(\omega))$.

By the fourth bullet of Definition 3.1, this preimage $X^c(\beta, \omega)$ is a trivial oriented bundle $F : X^c(\beta, \omega) \to K(\omega)$ whose fiber is a disjointed union of open intervals. Thus $F : \sigma \to F(\sigma) \subset K(\omega)$ is a trivial fibration whose fiber is an open interval. As a result, $F_* : \pi_1(\sigma) \to \pi_1(F(\sigma))$ is an isomorphism.

On the other hand, the covering $F : X^c(\beta, \omega) \to K(\omega)$ is trivial and therefore admits a section $\rho : K(\omega) \to X^c(\beta, \omega)$ such that its image intersects with the closure $\text{cl}(\sigma)$ of $\sigma$ in $X$. Put $\bar{\sigma} = \text{def} \; \sigma \cup \rho(F(\sigma))$. The map $F : \bar{\sigma} \to F(\sigma)$ is a trivial fibration with a semi-open interval for the fiber. Therefore the imbedding $j : \sigma \subset \bar{\sigma}$ is a homotopy equivalence, and so is the obvious imbedding $q : \rho(F(\sigma)) \subset \bar{\sigma}$. Hence $j_* : \pi_1(\sigma) \to \pi_1(\bar{\sigma})$ and $q_* : \pi_1(\rho(F(\sigma))) \to \pi_1(\bar{\sigma})$ are isomorphisms. Since $\rho(F(\sigma)) \subset \partial X$ and, for every choice of the base point $x_* \in \partial X$, the $\beta_*$-image of each fundamental group $\pi_1(\partial X, x_*)$ in
Π is amenable, so is the $\beta_\ast$-image of $\pi_1(\rho(F(\sigma)))$—a subgroup of an amenable group is amenable. By the previous arguments, the $\beta_\ast$-image of $\pi_1(\sigma)$ is amenable.

Now, applying localization Lemma 3.1 gives the inequality:

$$\|h\|_\Delta \leq \|h_U\|^S_{\Delta(D(U))},$$

where $h_U$ denotes the restriction of the absolute homology class $h$ to $(U, \partial U)$.

![Figure 3](image.png)

**Figure 3.** The cycle $h \in H_{k+1}(D(X))$, its transversal intersections with the strata $\sigma \subset D(X)^F_{(k+1)}$, and the normal disks $D^{k+1}_\sigma$ which represent the localization $h_U$.

For each $(n - k)$-dimensional stratum $\sigma \in S^*_F(D(X))$, consider an oriented disk

$$(D^{k+1}_\sigma, \partial D^{k+1}_\sigma) \subset (U, \partial U),$$

normal to the open manifold $\sigma \subset U$ at its typical point. Taking a smaller regular neighborhood $U$ of the subcomplex $D(X)^F_{(k+1)} \subset D(X)$ if necessarily, we can arrange for $\{D^{k+1}_\sigma\}_\sigma$ to be disjointed, so that each disk $D^{k+1}_\sigma$ hits its stratum $\sigma$ transversally at a singleton and misses the rest of the strata. Note that the relative integral homology classes $\{[D^{k+1}_\sigma]\}_\sigma$ may be dependent in $H_{k+1}(U, \partial U)$.

We claim that any element $h_U \in H_{k+1}(U, \partial U)$ can be written as a linear combination of relative cycles $\{[D^{k+1}_\sigma]\} \in H_{k+1}(U, \partial U)_\sigma$:

$$h_U = \sum_{\sigma \in S^*_F(D(X)), \dim \sigma = n-k} r_\sigma \cdot [D^{k+1}_\sigma].$$
In fact, $H_{k+1}(U, \partial U)$ can be recovered from the free $\mathbb{Z}$-module, generated by the elements \{[$D^{k+1}_\sigma$]$\}_{\sigma}$, by factoring the module by the appropriate relations.

To justify the presentation in (3.13), we notice that, since $D(X)^F_{-(k+1)}$ is a deformation retract of $U$, any $(n-k)$-dimensional homology class in $U$ is represented by a cycle $z$ in $D(X)^E_{-(k+1)}$, a combination $\sum_\sigma n_\sigma \cdot \sigma$ of the top strata of the $(n-k)$-dimensional CW-complex $D(X)^E_{-(k+1)}$. The algebraic intersection of $z \circ D^{k+1}_\sigma = n_\sigma$, since $\sigma \circ D^{k+1}_\sigma = 1$ and $\sigma' \circ D^{k+1}_\sigma = 0$ for any $\sigma' \neq \sigma$ by the very choice of the normal disks $D^{k+1}_\sigma$'s. For the $\mathbb{R}$-coefficients, by the Poincaré duality, any $(k+1)$-dimensional homology class $\hat{h} \in H_{k+1}(U, \partial U)$ is determined by its algebraic intersections with the cycles $z \in H_{n-k}(U) \approx H_{n-k}(D(X)^E_{-(k+1)})$. Therefore any $\hat{h}$ is a linear combination of \{[$D^{k+1}_\sigma$]$\}_{\sigma}$.

We introduce the norm of $h_U$ by the formula

$$
[h_U] = \inf\left\{\text{representations of } h_U \mid \sum_{\sigma \in S^*_\sigma(D(X)), \dim \sigma = n-k} |r_\sigma|\right\}
$$

the minimum being taken over all representations of $h_U$ as in (3.13).

Applying (3.12) to any presentation of $h_U$ as in (3.13), we get

$$
\|h\|_{\Delta} \leq \sum_{\sigma \in S^*_\sigma(D(X)), \dim \sigma = n-k} |r_\sigma| \cdot \|[[D^{k+1}_\sigma]]_{\Delta}^{S^*_\sigma(D(U))}.
$$

Thus,

$$
\|h\|_{\Delta} \leq \Theta \cdot \sum_{\sigma \in S^*_\sigma(D(X)), \dim \sigma = n-k} |r_\sigma|,
$$

where

$$
\Theta = \max_{\sigma \in S^*_\sigma(D(X)), \dim \sigma = n-k} \left\{[[D^{k+1}_\sigma]]_{\Delta}^{S^*_\sigma(D(U))}\right\}.
$$

Here we stratify the normal disk $D^{k+1}_\sigma$ by intersecting it with the $S^*_\sigma(D(X))$-stratification in the ambient space $D(X)$. Employing Definition 3.1, the normal to $\sigma$ disk $D^{k+1}_\sigma$ can be viewed as a subspace of the model double space $D(E_{\omega_\sigma})$, stratified with the help of the model map $p_{\omega_\sigma} : E_{\omega_\sigma} \rightarrow T_{\omega_\sigma}$. The normal disk acquires its stratification from the ambient space $D(E_{\omega_\sigma})$. Hence, as a stratified topological space, $D^{k+1}_\sigma$ depends only on the position of the stratum $\sigma$ in the canonical stratification of $D(E_{\omega_\sigma})$. That position is determined by the appropriate combinatorial data, provided by formula (2.4). As a result, the stratified simplicial norms $[[D^{k+1}_\sigma]]_{\Delta}^{S^*_\sigma(D(U))}$ take only a finite set of values, which depend only on the list of model maps \{${p_\omega : E_\omega \rightarrow T_\omega}$\}. So the constant $\Theta$ in (3.16) is universal for any $X$ and its shadow $F \in \text{Shad}(X, \mathbb{E} \Rightarrow T)$.

Therefore, in line with formula (3.12) and the definition in (3.14), we get

$$
\|h\|_{\Delta} \leq \Theta \cdot [h_U]
$$

where $\Theta > 0$ is universal for $X$ and $h$. 

Next, we are going to reinterpret the norm \([h_U]\) in terms of the differential complexes \(\mathcal{C}_k^+ (D(X), F)\) and \(\mathcal{C}_k^- (D(X), F)\) to make it “more computable”.

Let \(\{[\sigma]^*\}_\sigma\) be the basis of \(\mathcal{C}_0^{n-k} (D(X), F)\), dual to the basis \(\{[\sigma]\}_\sigma\) in \(\mathcal{C}_n^{n-k} (D(X), F)\). The \([\sim]\|^{-}\)-norm of a class
\[
\eta^* \in \mathcal{C}_0^{n-k} (D(X), F)/\mathcal{B}_0^{n-k} (D(X), F)
\]
is the quotient norm, induced by the \(l_1\)-norm on \(\mathcal{C}_0^{n-k} (D(X), F)\); it is defined by the formula (which resembles the formula in (3.14))
\[
\||\eta^*\||^{1}_{\sim} = \inf_{\{\zeta^\sim \equiv \eta^* \text{ mod } \mathcal{B}_0^{n-k} (D(X), F)\}} \left\{ \sum_{\sigma \in S^*_{\sim} (D(X))} \dim \sigma = j \right\},
\]
where
\[
\zeta^* = \sum_{\sigma \in S^*_{\sim} (D(X))} \sigma \cdot \sigma^*.
\]

Recall that in (3.13) we have considered an epimorphism \(A : \mathcal{C}_k^+ \to H_{k+1} (U, \partial U)\), where \(\mathcal{C}_k^+\) denotes the free module over \(\mathbb{R}\) (or over \(\mathbb{Z}\)), generated by the normal relative disks \(\{D^k_{\sigma}\}_\sigma\). Put \(R^+_k = \ker (A)\), so that
\[
\mathcal{C}_k^+ / R^+_k \approx H_{k+1} (U, \partial U).
\]

On the other hand, the Poincaré duality \(D_U\) produces an isomorphism
\[
B : H_{k+1} (U, \partial U) \approx D_U^* \Rightarrow \mathcal{C}_k^+ / R^+_k \approx H_{n-k} (D(X), F)/\mathcal{B}_0^{n-k} (D(X), F).
\]
The composition \(B \circ A\) takes each generator \(D^k_{\sigma}\) to the class of \([\sigma]^*\) and identifies the space of relations \(R^+_k\) with the space \(\mathcal{B}_0^{n-k} (D(X), F)\).

Let \(D_U (h_U)\) denotes the Poincaré dual in \(U\) of the relative homology class \(h_U\). Examining the definitions of the norm in (3.14) and of the norm \([\sim]\|^{1}_{\sim}\) and tracing the nature of the Poincaré duality (in terms of the intersections of relative and absolute cycles of complementary dimensions in \(U\)), we see that the norm \([h_U]\) in (3.14) coincides with the norm \([D_U (h_U)]^{1}_{\sim}\). where
\[
D_U (h_U) \subset H_{n-k} (D(X), F) \approx \mathcal{C}_0^{n-k} (D(X), F)/\mathcal{B}_0^{n-k} (D(X), F).
\]

Let \(\{[\sigma]^*\}_\sigma\) be the basis of \(\mathcal{C}_0^{n-k} (D(X), F)\), dual to the basis \(\{[\sigma]\}_\sigma\) in \(\mathcal{C}_n^{n-k} (D(X), F)\). The \([\sim]\|^{-}\)-norm of a class
\[
\eta^* \in \mathcal{C}_0^{n-k} (D(X), F)/\mathcal{B}_0^{n-k} (D(X), F)
\]
gets a general form that is often encountered in the study of simplicial complexes. The unit balls in the \(l_1\)-norms on the spaces \(\mathcal{C}_0^+ (D(X), F)\) are convex closures of the vectors \(\{\pm [\sigma]^*\}_\sigma\), where \(\sigma\) are strata of dimension \(j\). 

---

\(^6\)The unit balls in the \(l_1\)-norms on the spaces \(\mathcal{C}_0^+ (D(X), F)\) are convex closures of the vectors \(\{\pm [\sigma]^*\}_\sigma\), where \(\sigma\) are strata of dimension \(j\).
is the quotient norm, induced by the $l_1$-norm on $C_{ij}^{n-k}(D(X), F)$; it is defined by the formula (which resembles the formula in (3.14))

$$||[\eta^*]|\|^i_0 = \sup_{\eta^* \in \Omega \mod B_0^{n-k}(D(X), F)} \left\{ \sum_{\sigma \in S^*_n(D(X)), \dim \sigma = j} |r_{\sigma}| \right\},$$

where

$$\xi^*_i = \sum_{\sigma \in S^*_n(D(X)), \dim \sigma = j} r_{\sigma} \cdot [\sigma]^*_i$$

Recall that in (3.13) we have considered an epimorphism $A : C^i_{k+1} \to H_{k+1}(U, \partial U)$, where $C^i_{k+1}$ denotes the free module over $\mathbb{R}$ (or over $\mathbb{Z}$), generated by the normal relative disks $\{D^{k+1}_{\sigma}\}_\sigma$. Put $R^i_{k+1} = \text{def} \ker(A)$, so that

$$C^i_{k+1}/R^i_{k+1} \approx H_{k+1}(U, \partial U).$$

On the other hand, the Poincaré duality $D_U$ produces an isomorphism

$$B : H^{n-k}(U, \partial U) \cong D_U H^{n-k}(U, \partial U) \cong C^{n-k}_0(D(X), F)/B^{n-k}_0(D(X), F).$$

The composition $B \circ A$ takes each generator $D^{k+1}_{\sigma} \in C^1_{k+1}$ to the class of $[\sigma]^*$ and identifies the space of relations $R^i_{k+1}$ with the space $B^{n-k}_0(D(X), F)$.

Let $D_U(h_U)$ denotes the Poincaré dual in $U$ of the relative homology class $h_U$. Examining the definitions of the norm in (3.14) and of the norm $||[\sim]|\|^i_0$ and tracing the nature of the Poincaré duality (in terms of the intersections of relative and absolute cycles of complementary dimensions in $U$), we see that the norm $||h_U||$ in (3.14) coincides with the norm $||[D_U(h_U)]||_0$, where

$$D_U(h_U) \in H^{n-k}(D(X)^{F_{(k+1)}}(k+1)) \approx C^{n-k}_0(D(X), F)/B^{n-k}_0(D(X), F).$$

Therefore, combining this observation with (3.17), we get the desired inequality:

(3.18) $$\|h\|_\Delta \leq \Theta \cdot ||L^{F_{(k+1)}}_{k+1}(h)||_U.$$ 

For any shadow $F$, Theorem 3.1 has a “more geometric” interpretation in terms of counting the intersections of $(k+1)$-cycles in $D(X)$ with the locus $D(X)^{F_{(k+1)}}_{(k+1)}$ of the complementary dimension. Fig. 3 illustrates the idea of the arguments below.

**Corollary 3.1.** Let a homology class $h \in H_{k+1}(D(X); \mathbb{Z})$ be realized my a singular closed oriented PL-manifold $f : M \to D(X)$, $\dim(M) = k + 1$. Under the hypotheses of Theorem 3.1, the number of intersections of $f(M)$ with the locus $D(X)^{F_{(k+1)}}_{(k+1)}$ is greater than or equal to $\Theta^{-1} \cdot \|h\|_\Delta$, provided that $f$ is in general position with the subcomplex $D(X)^{F_{(k+1)}}_{(k+1)} \subset D(X)$.

\footnote{The unit balls in the $l_1$-norms on the spaces $C_{ij}(D(X), F)$ are convex closures of the vectors $\{\pm[\sigma]^*_i\}_\sigma$, where $\sigma$ are strata of dimension $j$.}
Proof. Recall that any integral homology class \( h \in H_{k+1}(D(X);\mathbb{Z}) \) can be realized by a 
\((2m + 1)\)-multiple of a singular oriented manifold \( f : M \to D(X) \), where the integer \( m \) depends only on \( X \) and \( h \) [CF].

By a small perturbation, we can assume that \( f(M) \) intersects transversally only with the pure \((n-k)\)-dimensional strata in \( D(X)^F_{\sim(k+1)} \) from the poset \( S^*_F(D(X)) \).

Consider the transversal intersections from the set \( f(M) \cap D(X)^F_{\sim(k+1)} \). Then, for a sufficiently narrow regular neighborhood \( U \) of \( D(X)^F_{\sim(k+1)} \), the localized class \( h_U \) has a representation

\[
h_U = \sum_{x_\sigma \in f(M) \cap D(X)^F_{\sim(k+1)}} r_\sigma \cdot [D^{k+1}_{x_\sigma}],
\]

where the relative disk \( (D^{k+1}_{x_\sigma}, \partial D^{k+1}_{x_\sigma}) \subset (U, \partial U) \cap f(M) \) and \( r_\sigma = \pm 1 \). By (3.18), the norm

\[
\|h_U\| \leq \sum_{x_\sigma \in f(M) \cap D(X)^F_{\sim(k+1)}} 1 = \#(f(M) \cap D(X)^F_{\sim(k+1)}).
\]

Again, by (3.18), we get \( \|h\|_\Delta \leq \Theta \cdot \#(f(M) \cap D(X)^F_{\sim(k+1)}) \). Thus

\[
\#(f(M) \cap D(X)^F_{\sim(k+1)}) \geq \Theta^{-1} \cdot \|h\|_\Delta.
\]

where the positive constant \( \Theta^{-1} \) depends only on \( n \), the poset \( S \), and the list of model maps \( E \Rightarrow T \) in the way that is described in formula (3.16).

\( \square \)

Question 3.1. Can one probe faithfully the shape of the “unit ball” in the semi-norm \( H_{\sim}\Delta \) on \( H_{k+1}(D(X)) \) by counting the cardinalities

\[
\#(f(M) \cap D(X)^F_{\sim(k+1)})
\]

for various shadows \( F \) and singular manifolds \( f : M \to D(X) \)?

The next corollary is a weaker version of Corollary 3.1, but with no amenability assumptions about the image of \( \pi_1(\partial X, pt) \to \pi_1(D(X)) \) (since these groups automatically have trivial images in \( \pi_1(X/\partial X) \)). Its proof is similar.

Corollary 3.2. Let a homology class \( h \in H_{k+1}(X, \partial X;\mathbb{Z}) \) be realized my a singular compact oriented PL-manifold \( f : (M, \partial M) \to (X, \partial X) \), \( \dim(M) = k + 1 \). Then, the cardinality of the intersections \( f(M) \cap X^F_{\sim(k+1)} \) is greater than or equal to \( \Theta^{-1} \cdot \|h\|_\Delta \), provided that \( f \) is in general position with the subcomplex \( X^F_{\sim(k+1)} \subset X \).

For a topological space \( Z \) and an integer \( j \geq 0 \), consider the subspace/subgroup

\[
K_j^{\Delta=0}(Z) \subset H_j(Z),
\]

formed by the homology classes \( h \in H_j(Z) \) whose simplicial semi-norm \( \|h\|_\Delta = 0 \). Then \( \| \sim \|_\Delta \) becomes a norm on the quotient space

\[
H^\Delta_j(Z) = \text{def} H_j(Z)/K_j^{\Delta=0}(Z).
\]
Note that if $Z$ admits a continuous self map $\phi : Z \to Z$ whose action $\phi_* : H_j(Z) \to H_j(Z)$ on homology has an eigenelement $h$ such that $\phi_*(h) = \lambda \cdot h$ for a scalar $\lambda$, subject to $|\lambda| > 1$, then

$$\|h\|_\Delta \geq \|\phi_*(h)\|_\Delta = |\lambda| \cdot \|h\|_\Delta.$$  

Thus $\|h\|_\Delta = 0$; so such an element $h \in H_j(Z)$ dies in the quotient $H_j^\Delta(Z)$.

In fact, the construction of $H_j^\Delta(\sim)$ always produces a trivial result: for any $Z$, $H_j^\Delta(Z) = 0$. However, if $Z$ is a closed surface of genus $g > 1$, then $H^\Delta_2(Z) \neq 0$. Moreover, when $Z$ is a product of many closed surfaces of genera which exceed 1, then $H_j^\Delta(Z) \neq 0$ is rich.

If $f : Z \to W$ is a continuous map of topological spaces, then $\|h\|_\Delta \geq \|f_*(h)\|_\Delta$. Therefore, $f$ induces a continuous linear map of normed spaces $f_* : H_j^\Delta(Z) \to H_j^\Delta(W)$ whose operator norm is $\leq 1$. It is not difficult to verify that when $f$ is a homotopy equivalence, then this map $f_*$ is an isometry (in the simplicial norm) between the normed spaces $H_j^\Delta(Z)$ and $H_j^\Delta(W)$. In particular, the shape of the unit ball in $H_j^\Delta(Z)$ is an invariant of the homotopy type of $Z$.

We can take this observation one step further. The Mapping Theorem from [Gr], section 3.1, implies that the classifying map $f : Z \to K(\pi_1(Z), 1)$ induces an isometry $f_* : H_j^\Delta(Z) \to H_j^\Delta(K(\pi_1(Z), 1))$ (so that $f_*$ is a monomorphism).

It turns out that formula (3.18) gives a nontrivial lower bound for the number of strata $\sigma \in S^*_F(D(X))$ of dimension $n - k$ for any shadow $F \in \text{Shad}(X, E \Rightarrow T)$. This universal boundary is constructed in terms of the group $H_{k+1}(D(X))$.

The next theorem is also an expression of the amenable localization, coupled with the Poincaré duality. It can be viewed as a version of Morse Inequalities, where a Morse function $f : X \to \mathbb{R}$ is replaced by a shadow $F$, the $f$-critical points are replaced by the $F$-induced strata in the double $D(X)$, and the homology of $X$ by the “homology” $H^\Delta(D(X))$, enhanced with a simplicial norm. It shows, in particular, that the groups $H_{k+1}(D(X))$ provide lower bounds of the ranks of the groups $C_{(k+1)}^n(D(X), F)$ from the differential complex in (3.9).

**Theorem 3.2.** Let $X$ be a compact connected $(n+1)$-dimensional PL-manifold with a nonempty boundary. Assume that, for each connected component of the boundary $\partial X$, the image of its fundamental group in $\pi_1(D(X))$ is an amenable group.

Then, for any shadow $F \in \text{Shad}(X, E \Rightarrow T)$,

$$(3.20) \quad \text{rk}(\text{im}(L_{k+1}^{F,1})) \geq \text{rk}(H_{k+1}^\Delta(D(X))).$$

As a result, $\text{rk}(C_{(k+1)}^n(D(X), F))$—the number of strata $\sigma \in S^*_F(D(X))$ of dimension $(n - k)$—is greater than or equal to $\text{rk}(H_{k+1}^\Delta(D(X)))$.

**Proof.** If $h \in H_{k+1}(D(X))$ belongs to the kernel of the homomorphism $L_{k+1}^{F,1}$, then evidently $\|L_{k+1}^{F,1}(h)\|_{\Omega} = \|h\|_{\Omega} = 0$. By the inequality (3.17), $\|h\|_\Delta = 0$. Therefore, $h \in K^\Delta_{k+1} = \text{def} K^{\Delta_{k+1}}_{k+1}$. In other words, $\ker(L_{k+1}^{F,1}) \subset K^\Delta_{k+1}$. Hence, $\text{rk}(H_{k+1}(D(X))/K^\Delta_{k+1}) \leq \text{rk}(H_{k+1}(D(X))/\ker(L_{k+1}^{F,1})) = \text{rk}(\text{im}(L_{k+1}^{F,1})).$
So we get
\[
\text{rk}(H_{k+1}^\Delta(D(X))) \leq \text{rk}(C_{n-k}^n(D(X), F)/\mathcal{B}_{n-k}^n(D(X), F)) \leq \text{rk}(C_{n-k}^n(D(X), F)),
\]
the later rank is the number of strata \( \sigma \) of dimension \( n-k \) in the stratification \( \mathcal{S}^\bullet_k(D(X)) \). \( \square \)

**Remark 3.1.** Note that the best lower estimate that formula (3.20) could provide equals
\[
\text{rk}(H_{k+1}^\Delta(K(\Pi, 1))) \geq \text{rk}(H_{k+1}^\Delta(D(X))),
\]
where \( \Pi = \pi_1(D(X)) \). \( \square \)

In one interesting case that deals with the strata of maximal codimension, Theorem 3.1 and Corollary 3.1 can be made more explicit.

**Theorem 3.3.** Let \( X \) be a compact connected and oriented \((n+1)\)-dimensional PL-manifold with a nonempty boundary. Assume that, for each connected component of the boundary \( \partial X \), the image of its fundamental group in \( \pi_1(D(X)) \) is an amenable group.

Then there is a \( X \)-independent universal constant \( \theta > 0 \) such that, for any shadow \( F \in \text{Shad}(X, \mathbb{E} \Rightarrow \mathbb{T}) \), the cardinality of the set \( F(X)_{-n} \) satisfies the inequality
\[
\#(F(X)_{-n}) \geq \theta \cdot \| [D(X)] \| \Delta.
\]

**Proof.** The argument is based on the proof of Theorem 3.1. Note that the fundamental class \([D(X)] \in H_{n+1}(D(X))\), being restricted to the regular neighborhood
\[
U = \text{def} \prod_{\{ \sigma \in \mathcal{S}^\bullet_k(D(X)) | \mu'(\omega_\sigma) = n \}} D^{n+1}_\sigma
\]
of the locus \( D(X)^F_{-(n+1)} \), equals the sum
\[
\sum_{\{ \sigma \in \mathcal{S}^\bullet_k(D(X)) | \mu'(\omega_\sigma) = n \}} [D^{n+1}_\sigma].
\]
Therefore, by (3.14), \( \| [D(X)]_U \| = \#(D(X)^F_{-(n+1)}) \). Employing inequality (3.18), we get
\[
\| [D(X)]_\Delta \| \leq \Theta \cdot \#(D(X)^F_{-(n+1)}) \leq \Theta \cdot \kappa \cdot \#(F(X)_{-n}),
\]
where
\[
\kappa = \text{def} \max_{\{ \omega | \mu'(\omega) = n \}} \left( \mu(\omega) - \mu'(\omega) \right)
\]
is the maximal cardinality of the fibers of the map
\[
F : D(X)^F_{-(n+1)} = F^{-1}(F(X)_{-n}) \cap \partial X \rightarrow F(X)_{-n}.
\]
Choosing \( \theta = (\Theta \cdot \kappa)^{-1} \) completes the argument. \( \square \)

The next implication of Theorem 3.2 reveals the groups \( H_{k+1}^\Delta(D(X)) \) and \( H_{k+1}^\Delta(X/\partial X) \) as obstructions to the existence of globally \( k \)-convex shadows of \( X \).
Corollary 3.3. Let $X$ be a compact connected $(n + 1)$-dimensional PL-manifold with a nonempty boundary. Assume that, for each connected component of the boundary $\partial X$, the image of its fundamental group in $\pi_1(D(X))$ is an amenable group.

If $X$ is globally $k$-convex, then the simplicial semi-norm is trivial on $H_{k+1}(D(X))$, and thus on $H_{k+1}(X/\partial X)$.

In particular, if an oriented $X$ admits a shadow $F \in \Shad(X, E \Rightarrow T)$ such that $F(X)_{-n} = \emptyset$, then $\|D(X)\|_\Delta = 0$, and thus $\|X/\partial X\|_\Delta = 0$.

Proof. Note that $F(X)_{-k} = \emptyset$ implies that $D(X)^{F}_{-(k+1)} = \emptyset$, since the $(n - k)$-dimensional strata from $S^*_D(D(X))$ are contributed only by the the strata of $F(X)$ of dimensions $(n - k)$ and $(n - k - 1)$. The exception to this rule is provided by the 0-dimensional strata from $S^*_D(D(X))$: they are contributed by the points from $F(X)_{-n}$ only.

For any $h \in H_{k+1}(D(X))$, we get $\|h\|_\Delta \geq \|g_*(h)\|_\Delta$, where $g : D(X) \to X/\partial X$ is the map that collapses the second copy of $X$ to a point. Thus, if the simplicial norm is nontrivial on $H_{k+1}(X/\partial X)$, it is nontrivial on $H_{k+1}(D(X))$. Hence $H^A_{k+1}(X/\partial X) \neq 0$ implies that $H^A_{k+1}(D(X)) \neq 0$.

Definition 3.7. Let $X$ be a compact connected $(n + 1)$-dimensional PL-manifold with a nonempty boundary, and $A$ denotes an abelian group or a field. For any shadow $F \in \Shad(X, E \Rightarrow T)$ and each $j \in [0, n + 1]$, let

$$\Sigma c^j_{\text{shad}}(X, F) = \text{def} \ \text{rk}_A(C^j_{\text{shad}}(X, F)).$$

We call this integer the $j$-th suspension shad-complexity of the shadow $F$.

Let

$$\Sigma c^j_{\text{shad}}(X) = \text{def} \ \min_{F \in \Shad(X, E \Rightarrow T)} \Sigma c^j_{\text{shad}}(X, F).$$

We call $\Sigma c^j_{\text{shad}}(X)$ the $j$-th suspension shadow complexity of $X$.

These numbers can be organized into a sequence

$$\Sigma c_{\text{shad}}(X) = \text{def} \ (\Sigma c^0_{\text{shad}}(X), \Sigma c^1_{\text{shad}}(X), \ldots, \Sigma c^{n+1}_{\text{shad}}(X)).$$

Note that this optimal sequence may not be realizable by a single shadow! To avoid this fundamental difficulty, we will need a more manageable version of the previous definition.

Definition 3.8. Let $X$ be a compact connected $(n + 1)$-dimensional PL-manifold with a nonempty boundary. For any shadow $F \in \Shad(X, E \Rightarrow T)$, form the sequence

$$\Sigma c^j_{\text{shad}}(X, F) = \text{def} \ (\Sigma c^0_{\text{shad}}(X, F), \Sigma c^1_{\text{shad}}(X, F), \ldots, \Sigma c^{n+1}_{\text{shad}}(X, F)).$$

and take the lexicographic minimum

$$\Sigma c^j_{\text{shad}}(X) = \text{def} \ \text{lex.min}_{F \in \Shad(X, E \Rightarrow T)} \Sigma c^j_{\text{shad}}(X, F).$$

We denote by $\Sigma c^j_{\text{shad}}(X)$ the $(j + 1)$ component of the vector $\Sigma c^j_{\text{shad}}(X)$ and call it the $j$-th lexicographic suspension shadow complexity of $X$.

Remark 3.2. By its very definition, the lexicographically optimal sequence of complexities is delivered by some shadow $F$!
Evidently, for each \( j \),
\[
\Sigma c_{\text{shad}}^{\leq j}(X) \geq \Sigma c_{\text{shad}}^j(X).
\]

Theorem 3.4 has an immediate implication.

**Corollary 3.4.** Let \( X \) be a compact connected \((n+1)\)-dimensional PL-manifold with a nonempty boundary. Assume that, for each connected component of the boundary \( \partial X \), the image of its fundamental group in \( \pi_1(D(X)) \) is an amenable group.

Then, for any \( k \) from the interval \([-1, n]\), the suspension shadow complexity of \( X \) satisfies the inequality
\[
\Sigma c_{\text{shad}}^{n-k}(X) \geq \text{rk}(H_{k+1}^\Delta(D(X))).
\]

Dropping the assumption about amenability of the images of the fundamental groups \( \pi_1(\partial X, pt) \) in \( \pi_1(D(X)) \), we get a weaker inequality
\[
\Sigma c_{\text{shad}}^{n-k}(X) \geq \text{rk}(H_{k+1}(X/\partial X)).
\]

Let us consider the sequence
\[
\text{rk}(H_{n+1}^\Delta(D(X))) = \text{def} \left( \text{rk}(H_1^\Delta(D(X))), \text{rk}(H_2^\Delta(D(X))), \ldots, \text{rk}(H_0^\Delta(D(X))) \right).
\]

Then Corollary 3.4 can be expressed in its compressed form as
\[
\Sigma c_{\text{shad}}^{\leq}(X) \geq \Sigma c_{\text{shad}}(X) \geq \text{rk}(H_0^\Delta(D(X))),
\]
where the vectorial inequality is understood as the inequality among all the corresponding components of the participating vectors.

Let \( X \) be a compact connected \((n+1)\)-dimensional PL-manifold with a nonempty boundary, and \( F : X \to K \) its shadow. Recall that, by Definition 3.1, the cardinality of the fiber \( F(x, \omega) \cap \partial X \to K(\omega) \) depends only on \( \omega \in S \): it is the difference \( \mu(\omega) - \mu'(\omega) \).

**Definition 3.9.** Consider a filtered CW-complex \( K \in \text{Shad}(T, n) \) and its filtration as in (3.1). Put
\[
\kappa_j(n) = \text{def} \max_{\{\omega | \mu'(\omega) = n-j\}} (\mu(\omega) - \mu'(\omega)).
\]

The weighted \( j \)-th complexity of \( K \) is defined by the formula
\[
\sharp_{\leq o}^j(K) = \text{def} \kappa_j(n) \cdot c_{oj}^j(K).
\]

The next proposition helps to link the suspension complexities \( \Sigma c_{oj}^j(X, F) \) to the weighted ones \( \sharp_{oj}^j(F(X)) \) and thus motivates Definition 3.9.

**Corollary 3.5.** Let \( X \) be a compact connected \((n+1)\)-dimensional PL-manifold with a nonempty boundary. Assume that, for each connected component of the boundary \( \partial X \), the image of its fundamental group in \( \pi_1(D(X)) \) is an amenable group.

Then, for any shadow \( F \in \text{Shad}(X, E \Rightarrow T) \) and each \( k \in [-1, n] \),
\[
\sharp_{oj}^{n-k}(F(X)) + 2 \cdot \sharp_{oj}^{n-k-1}(F(X)) \geq \text{rk}(H_{k+1}^\Delta(D(X))).
\]
Proof. Let $K = F(X)$ be a shadow of $X$. Any connected component of $K(\omega)$ of dimension $n - \mu'(\omega)$ (by Definition 3.1) gives rise to $\mu(\omega) - \mu'(\omega)$ strata $\sigma \in S^*_F(D(X))$ of dimension $n - \mu'(\omega)$ and to $2(\mu(\omega) - \mu'(\omega) - 1)$ strata $\sigma \in S^*_F(D(X))$ of dimension $n + 1 - \mu'(\omega)$. Therefore the number of strata $\sigma \in S^*_F(D(X))$ of dimension $j = n - k$ is given by the formula:

$$\sum_{\omega \mid \mu'(\omega) = k} (\mu(\omega) - k) \cdot \#(\pi_0(K(\omega))) + \sum_{\omega \mid \mu'(\omega) = k+1} 2(\mu(\omega) - k - 1) \cdot \#(\pi_0(K(\omega)))$$

By the definition of $\{\kappa_j(n)\}_j$, the latter number is smaller than or equal to

$$\kappa_{n-k}(n) \cdot \sum_{\omega \mid \mu'(\omega) = k} \#(\pi_0(K(\omega))) + 2\kappa_{n-k-1}(n) \cdot \sum_{\omega \mid \mu'(\omega) = k+1} \#(\pi_0(K(\omega))),$$

where the first sum is the number of strata in $F(X)$ of dimension $n - k$, and the second sum is the number of strata of dimension $n - k - 1$. Thus the previous formula is an upper bound of the number of components in $S^*_F(D(X))$ of dimension $n - k$.

Applying Theorem 3.4, we prove the formula (3.24). □

Corollary [3.5] has the following two immediate implications that reveal the non-triviality of the group/space $H^\Delta_{k+1}(D(X)) \subset H^\Delta_{k+1}(\Pi)$ as an obstruction to the existence of shadows $F$ with no strata in $F(X)$ of codimensions grater than or equal to $k$.

**Corollary 3.6.** Under the hypotheses of Corollary [3.5], if $H^\Delta_{k+1}(D(X)) \neq 0$, then either $c_{n-k}^\mu(F(X)) \neq 0$ or $c_{n-k-1}^\mu(F(X)) \neq 0$ for any shadow $F$.

**Corollary 3.7.** Let $X$ be a compact connected $(n + 1)$-dimensional PL-manifold with a nonempty boundary. Assume that, for each connected component of the boundary $\partial X$, the image of its fundamental group in $\pi_1(D(X))$ is an amenable group.

Then, for each $k \in [0, n + 1]$,

$$\tau_{c_{n-k}^{\text{shad}}(X, E \Rightarrow T)} + 2 \cdot \tau_{c_{n-k-1}^{\text{shad}}(X, E \Rightarrow T)} \geq \text{rk}(H^\Delta_{k+1}(D(X))).$$

4. **Complexity of Traversing Flows**

Now let us examine applications of these results about the complexities of shadows to the traversing flows on smooth manifolds with boundary. In fact, the entire general setting in Section 3 was designed with these applications in mind. As we will show next, any proposition about shadows of PL-manifolds with boundary and the simplicial semi-norms has an analogue for the smooth manifolds with boundary that carry a traversally generic vector field. In short, the traversally generic fields are a good source of shadows.

We will apply Theorem [3.1], Theorem [3.2] and their corollaries to traversing vector fields $v$. In this setting, the poset $S = \Omega^\bullet$, $\mu := | \sim |$, $\mu' := | \sim' |$, the role of shadows is played by the obvious maps $\Gamma : X \to T(v)$ which belong to an appropriate set $\text{Shad}(X, E \Rightarrow T)$.
Theorem 4.1. Let \( X \) be a smooth compact connected and oriented \((n+1)\)-manifold with boundary. Any traversally generic vector field \( v \) gives rise to a shadow \( \Gamma : X \rightarrow T(v) \) in the sense of Definition 3.1. The model projections \( \{ p_\omega : E_\omega \rightarrow T_\omega \}_{\omega \in \Omega_{(n)}^*} \) are described in [K3], Theorems 7.4 and 7.5., utilizing special coordinates as in (2.4).

Proof. Recall that, for a traversally generic \( v \), the trajectory space \( T(v) \) is a Whitney stratified space (see [K4], Theorem 2.2), which implies that \( \Gamma \) is a simplicial map with respect to the appropriate triangulations of the source and target spaces. Moreover, the triangulation of \( X \) can be chosen to be smooth.

By the very definition of a traversing flow, each fiber \( \Gamma^{-1}(\gamma), \gamma \in T(v) \) is either a closed segment, or a singleton.

By Corollary 5.1 from [K3], the map \( \partial_1 X \rightarrow T(v) \), being restricted to the preimage of each proper stratum \( T(v,\omega) \), is a cover with the trivial monodromy and fibers of cardinality

\[
#(\text{sup}(\omega)) = |\omega| - |\omega'| = \mu(\omega) - \mu'(\omega).
\]

Moreover, by Theorem 2.1 from [K4], each pure stratum \( T(v,\omega) \) is an open orientable smooth manifold.

By Lemma 3.4 from [K2], Theorems 5.2 and 5.3 from [K3], each point \( \gamma \in T(v,\omega) \), has a regular neighborhood \( V_\gamma \subset T(v) \), so that \( \Gamma : \Gamma^{-1}(V_\gamma) \rightarrow V_\gamma \) is PL-homeomorphic to the model projection \( p_\omega : E_\omega \rightarrow T_\omega \).

This completes the checklist of bullets from Definition 3.1. \( \square \)

Let \( X \) be a smooth compact connected and oriented \((n+1)\)-manifold with boundary, and \( v \) a traversally generic vector field.

Let us consider a filtration

\[
T(v) = T(v)_0 \supset T(v)_1 \supset \cdots \supset T(v)_n
\]

of the trajectory space \( T(v) \) by the closed subcomplexes

\[
T(v)_{-j} = \text{def} \bigcup_{\omega \in \Omega_{(n)}^*, |\omega'| \geq j} T(v,\omega)
\]

of dimensions \( n - j \).

Let \( \mathcal{A} \) be an abelian coefficient system on \( T(v) \) (equivalently, on \( X \)). As a default, \( \mathcal{A} = \mathbb{R} \). For each \( j \in [0,n] \), consider the relative homology groups

\[
\{ C^V_j(T(v)) := H_j(T(v)_{-n+j}, T(v)_{-n+j-1}; \mathcal{A}) \}_j
\]

associated with the filtration. As in the context of shadows, \( C^V_j(T(v)) \) is the top homology of the quotient \( T(v)_{-n+j}/T(v)_{-n+j-1} \).

These homology groups can be organized into a differential complex

\[
\{ 0 \rightarrow C^V_n(T(v)) \xrightarrow{\partial_n} C^V_{n-1}(T(v)) \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_1} C^V_0(T(v)) \rightarrow 0 \},
\]

8in our notations, we suppress the dependence of these homology groups on the coefficients \( \mathcal{A} \).
where the differentials \( \{ \partial_j \} \) are the boundary homomorphisms from the long exact homology sequences of the triples
\[
\{ T(v)_{-n+j} \supset T(v)_{-n+j-1} \supset T(v)_{-n+j-2} \}_{j}.
\]
The modules \( C^j_\omega(T(v)) \) are free and finitely generated, the number of generators being the number of connected components of the set
\[
T(v)_{-n+j} = \operatorname{def} T(v)_{-n+j} \setminus T(v)_{-n+j-1}.
\]
This observation is valid, since by Theorem 2.1 from [K4], the components of \( T(v)_{-n+j} \) are orientable smooth manifolds.

The differential complex \((4.3)\) gives rise to the \( \mathcal{U} \)-homology groups
\[
H^j_\omega(v) = \operatorname{def} H^j_\omega(T(v))
\]
of the traversally generic field \( v \).

As in the category of shadows, the traversing fields give rise to several notions of complexity.

**Definition 4.1.** With any traversally generic vector field \( v \) on a smooth compact connected and oriented \((n+1)\)-manifold \( X \) with boundary, we associate the ordered collection of ranks
\[
\left\{ tc^j_\omega(v) = \operatorname{def} \operatorname{rk}_A \left( C^j_\omega(T(v)) \right) \right\}_{0 \leq j \leq n},
\]
where the groups \( C^j_\omega(T(v)) \) have been introduced in \((4.2)\).

We call \( tc^j_\omega(v) \) the \( j \)-th traversing complexity ("tc" for short) of the field \( v \).

**Remark 4.1.** Note that \( tc^j_\omega(v) = 0 \) implies that \( T(v)_{-n+j} = \emptyset \). Examining the local models \( \{ p_\omega : E_\omega \to T_\omega \}_\omega \), we notice that if a particular combinatorial type \( \omega' \) is missing in a model, then all the smaller combinatorial types \( \omega'' < \omega' \) (of greater codimensions) are missing as well. Therefore \( T(v)_{-n+j} = \emptyset \) implies that \( \{ T(v)_{-n+j} = \emptyset \}_{k \leq j} \). Hence \( T(v)_{-n+k} = \emptyset \) for all \( k \leq j \). As a result, if the complexity \( tc^j_\omega(v) = 0 \), then the complexities \( tc^k_\omega(v) = 0 \) for all \( k \leq j \).

**Definition 4.2.** Let \( X \) be a compact connected and oriented smooth \((n+1)\)-manifold with boundary. For each traversally generic field \( v \in \mathcal{U}(X) \), we form the sequence of traversing complexities:
\[
\textbf{tc}(v) = \operatorname{def} \left\{ tc^0_\omega(v), tc^1_\omega(v), \ldots, tc^n_\omega(v) \right\}
\]

Consider the lexicographical minimum
\[
\textbf{tc}^{\text{lex}}(X) = \operatorname{def} \operatorname{lex} \min \{ \textbf{tc}(v) \}_{v \in \mathcal{U}(X)}
\]
We call this vector the lexicographical traversing complexity of \( X \).

We denote by \( tc^{\text{lex}}(X) \) the \((j+1)\)-component of the vector \( \textbf{tc}^{\text{lex}}(X) \).

**Remark 4.2.** If a compact connected and oriented smooth manifold \( X \) of dimension \( n+1 \) is globally \( k \)-convex in the sense of Definition 2.2, then evidently \( tc^{\text{lex}}_j(X) = 0 \) for all \( j \leq n - k \).
Remark 4.3. Thanks to Theorem 4.1 traversally generic fields produce a particular kind of shadows from the set $\text{Shad}(X, E \Rightarrow T)$. Therefore,

$$c_{\text{shad}}^{\text{lex}}(X) \leq c_{\text{lex}}^{\text{shad}}(X).$$

As with shadows, we will need “suspending” of these notions and constructions.

We consider the stratification $\Omega(X, v)$ of $X$ by the connected components of the strata of a fixed codimension, each pure stratum being an open manifold.

$\Omega(X, v) = \{X^0(v, \omega) = \text{def} X(v, \omega) \setminus (\partial X \cap X(v, \omega)), X^\partial(v, \omega) = \text{def} \partial X \cap X(v, \omega)\}_{\omega \in \Omega_{*}(v)}.$

and the $\tau$-invariant stratification $\Omega^*(D(X), v)$ of the double $D(X)$, which is induced by $\Omega^*(X, v)$.

The stratifications $\Omega^*(X, v)$ and $\Omega^*(D(X), v)$ give rise to the filtrations:

$$X = \text{def} X^v_0 \supset X^v_1 \supset \cdots \supset X^v_{n+1},$$

$$D(X) = \text{def} D(X)^v_0 \supset D(X)^v_1 \supset \cdots \supset D(X)^v_{n+1}$$

by the union of strata of a fixed codimension, each pure stratum being an open manifold.

Analogously to (3.7), we consider the homology and cohomology groups with coefficients in $\mathcal{A} = \mathbb{R}$ or $\mathcal{A} = \mathbb{Z}$:

$$\{C_j^\Omega(D(X), v) = \text{def} H_j(D(X)_j^{\leq n-1}; \mathcal{A})\},$$

$$\{C_j^\Omega(D(X), v) = \text{def} H_j(D(X)_j^{\leq n-1}; \mathcal{A})\}.$$

As in (3.8) and (3.9), they can be organized into a differential complex:

$$C_{n+1}^\Omega(D(X), v) = \text{def} \{0 \rightarrow C_{n+1}^\Omega(D(X), v) \xrightarrow{\partial_n} C_n^\Omega(D(X), v) \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_1} C_0^\Omega(D(X), v) \rightarrow 0\},$$

(4.7)

where the differentials $\{\partial_j\}$ are the boundary homomorphisms from the long exact homology sequences of triples

$$\{D(X)^v_{j+1} \supset D(X)^v_j \supset D(X)^v_{j-1}\}_j.$$

Similarly, we introduce the dual differential complex

$$C_{n+1}^\Omega(D(X), v) = \text{def} \{0 \leftarrow C_{n+1}^\Omega(D(X), v) \xleftarrow{\delta_n} C_n^\Omega(D(X), v) \xleftarrow{\delta_{n-1}} \cdots \xleftarrow{\delta_1} C_0^\Omega(D(X), v) \leftarrow 0\}$$

(4.8)

These differential complexes produce the suspension $\Omega$-homology $H^\Omega(D(X), v)$ and $\Omega$-cohomology $H^\Omega(D(X), v)$ of the traversally generic $v$-flows on $X$.

Definition 4.3. Let $X$ be a compact connected oriented and smooth $(n+1)$-dimensional manifold with a boundary. For any traversally generic field $v \in \mathcal{V}(X)$ and each $j \in [0, n+1]$, let

$$\Sigma t^j_c(D(X), v) = \text{def} \text{rk}_\mathcal{A}(C_j^\Omega(D(X), v)).$$
We call this integer the $j$-th suspension $\mathcal{D}$-complexity of the field $v$.

**Definition 4.4.** Let $X$ be a compact connected oriented and smooth $(n+1)$-dimensional manifold with a boundary. For any traversally generic field $v \in \mathcal{V}^i(X)$, we form the sequence

$$\Sigma \text{tc}_{ij}^c(X, v) = \text{def} \left( \Sigma \text{tc}_{ij}^0(X, v), \Sigma \text{tc}_{ij}^1(X, v), \ldots, \Sigma \text{tc}_{ij}^{n+1}(X, v) \right),$$

and take the lexicographic minimum

$$\Sigma \text{tc}_{ij}^{\text{lex}}(X) = \text{def} \, \text{lex.min}_{v \in \mathcal{V}^i(X)} \, \Sigma \text{tc}_{ij}^c(X, v).$$

We denote by $\Sigma \text{tc}_{ij}^{\text{lex}, j}(X)$ the $(j + 1)$ component of the vector $\Sigma \text{tc}_{ij}^{\text{lex}}(X)$ and call it the $j$-th lexicographic suspension traversing complexity of $X$.

**Remark 4.4.** By its very definition, the lexicographically optimal sequence of complexities is delivered by some traversally generic vector field!

Evidently,

$$\Sigma \text{tc}_{ij}^{\text{lex}}(X) \geq \Sigma \text{tc}_{ij}^{\text{shad}}(X),$$

where “shad” refers to the shadows, based on the list of model maps $\{E_\omega \to T_\omega \}_{\omega \in \Omega^{(n)}}$, exemplifying the local structure of traversally generic flows.

A priori, both vectors, $\text{tc}_{ij}^{\text{lex}}(X)$ and $\Sigma \text{tc}_{ij}^{\text{lex}}(X)$, are invariants of the smooth topological type of $X$, while $\text{tc}_{ij}^{\text{shad}}(X)$ and $\Sigma \text{tc}_{ij}^{\text{shad}}(X)$ are invariants of the PL-topological type.

Consider the space $\mathcal{V}^i(X)$ of traversally generic vector fields on $X$ and its subspace $\mathcal{V}_\text{fold}^i(X)$, formed by fields for which the multiplicity $m(a) \leq 2$ for any $v$-trajectory $\gamma$ at each point $a \in \gamma \cap \partial X$. Locally, for such fields $v$, $\Gamma : \partial X \to T(v)$ is a folding map.

**Definition 4.5.** Let $X$ be a compact connected oriented and smooth $(n+1)$-dimensional manifold with a boundary.

For any traversally generic field $v \in \mathcal{V}_\text{fold}^i(X)$ we define the lexicographic fold complexity of $X$ by

$$\text{tc}_{\text{fold}^i(X)} = \text{def} \, \text{lex.min}_{v \in \mathcal{V}_\text{fold}^i(X)} \, \text{tc}(v).$$

Clearly, $\text{tc}_{\text{fold}^i(X)} \geq \text{tc}^\text{lex}_{\text{trav}}(X)$, provided $\mathcal{V}_\text{fold}^i(X) \neq \emptyset$. When $X$ is a 3-fold, by Theorem 9.5 in [K],

$$\text{tc}_{\text{fold}^i(X)} = \text{tc}^0_{\text{trav}}(X).$$

Besides the inequality above, we know little about the relation between $\text{tc}_{\text{fold}^i(X)}$ and $\text{tc}^\text{lex}_{\text{trav}}(X)$.

Now let us examine how the marriage of Amenable Localization and Poincaré Duality works in the environment of traversing fields.

For each $k \in [-1, n]$, by composing the Poincaré duality map with the localization to the subset $D(X)^{\nu}_{-(k+1)} \subset D(X)$, we get the localization transfer map

$$\mathcal{L}^{\nu}_{k+1} : H_{k+1}(D(X)) \xrightarrow{\approx} H^{n-k}(D(X)) \xrightarrow{i^*} H^{n-k}(D(X)^{\nu}_{-(k+1)}).$$
whose target can be identified with the quotient
\[ C^{n-k}_0(D(X), v) / B^{n-k}_0(D(X), v), \]
produced by the differential complex in (4.8). We denote by
\[ \mathcal{L}^{v,i,j}_{k+1} : H_{k+1}(D(X)) \to C^{n-k}_0(D(X), v) / B^{n-k}_0(D(X), v) \]
the resulting operator.

Now Theorem 3.1, being combined with Theorem 4.1, delivers

**Theorem 4.2 (Amenable localization of the Poincaré duality for traversing flows).**

Let \( X \) be a compact connected oriented and smooth manifold \( X \) with boundary, \( \dim(X) = n + 1 \). Assume that for each connected component of the boundary \( \partial X \), the image of its fundamental group in \( \pi(X) \) flows).

Then, there exists an universal constant \( \Theta > 0 \) such that, for any traversally generic vector field \( v \), the space \( C^{n-k}_0(D(X), v) / B^{n-k}_0(D(X), v) \) admits a norm \( \| [~] \|_0 \) so that
\[ \| h \|_\Delta \leq \Theta : \| [\mathcal{L}^{v,i,j}_{k+1}(h)] \|_0 \]
for any \( h \in H_{k+1}(D(X)) \). Here the operator \( \mathcal{L}^{v,i,j}_{k+1} \) is introduced in (4.9), and the constant \( \Theta > 0 \) depends only on the list of model maps \( \{ p_\omega : E_\omega \to T_\omega \} \in \Omega_0 \) in the way that is described in formula (3.16).

Theorem 4.1 together with Corollary 3.1 produce

**Corollary 4.1.** Let an integral homology class \( h \in H_{k+1}(D(X)) \) be realized my a singular closed oriented PL-manifold \( f : M \to D(X) \), \( \dim(M) = k + 1 \).

Under the hypotheses of Theorem 4.2, the number of intersections of the cycle \( f(M) \) with the locus \( D(X)^{v}_{-(k+1)} \) is greater than or equal to \( \Theta^{-1} : \| h \|_\Delta \), provided that \( f \) is in general position with the subcomplex \( D(X)^{v}_{-(k+1)} \subset D(X) \).

The following example is a product of the author’s conversations with Larry Guth.

**Example 4.1.** Consider a fibration \( p : E \to M \) whose base is a closed oriented hyperbolic manifold of dimension \( k + 1 \) and whose fiber \( F \) is a closed manifold. Assume that the \( (n+1) \)-dimensional manifold \( E \) is oriented and that \( p \) admits a section \( s : M \to E \). In the complement to \( s(M) \), pick a smooth codimension zero manifold \( V \) such that \( \{ \pi_1(\partial V, pt) \} \) are amenable groups. Let \( X = E \setminus \text{int}(V) \).

Recall that \( \| [M] \|_\Delta \) is proportional to the hyperbolic volume \( \text{vol}(M) \) with an universal positive proportionality constant \( \Theta \). Since \( s \) is a section, and the simplicial seminorm does not increase under the continuous maps, it follows that the simplicial norm of \( s_*([M]) \in H_{k+1}(X) \) is proportional to \( \text{vol}(M) \) with the same proportionality constant.

For any traversally generic vector field \( v \) on \( X \), consider the locus
\[ X^v_{-(k+1)} \overset{\text{def}}{=} \bigcup_{\{ \omega | \omega \geq k+1 \}} X(v, \omega), \]
where \( X(v, \omega) \) denotes the union of \( v \)-trajectories of the combinatorial type \( \omega \).
We can perturb the section $s$ so that the intersections of the cycle $s(M)$ is transversal to the locus $X^u_{-(k+1)}$.

According to Corollary 4.1 there exits a universal constant $\rho > 0$ so that, for any such $X$ and any traversally generic vector field $v$ on $X$, the transversal intersections of the perturbed cycle $s(M)$ with the locus $X^u_{-(k+1)}$ has $\rho \cdot \text{vol}(M)$ intersections at least.

Since by the choice of $V$, $s(M) \cap \partial V = \emptyset$, it follows that there exist at least $\rho \cdot \text{vol}(M)$ trajectories $\gamma$ of the reduced multiplicity $m'(\gamma) \geq k+1$ that link $\partial X = \partial V$ with the section $s(M) \subseteq \text{int}(X)$.

It worth describing the important special case $h = [D(X)]$ of formula (3.16), being applied to the shadows that are produced by traversally generic flows $\theta$.

**Theorem 4.3.** Let $X$ be a smooth compact connected and oriented $(n+1)$-manifold with boundary. Assume that, for each connected component of the boundary $\partial X$ the image of its fundamental group in $\pi_1(D(X))$ is an amenable group.

Let $v$ be a traversally generic vector field on $X$. Then

$$
\sum_{\omega \in \Omega^* | |\omega'| = n} \theta(\omega) \cdot \#(T(v, \omega)) \geq \|D(X)\|_\Delta,
$$

where the universal constant

$$
\theta(\omega) = \text{def} \left\| \left[ D(E_\omega), D(\delta E_\omega) \right] \right\|_{\Omega^*_{\{n\}}} \Delta,
$$

is the $\Omega^*_{\{n\}}$-stratified simplicial volume of the model pair $(D(E_\omega), D(\delta E_\omega))$.

Although desirable, an exact computation of the universal constants $\{\theta(\omega)\}$ in formula (4.11) is somewhat tricky. We can estimate the number of cells-strata of the top dimension $|\omega'|$ in the model space $T_\omega$. That number can be calculated in terms of the chambers in which the discriminant of each polynomial

$$
\psi_i(u, \bar{x}) = \text{def} (u - i)^{\omega_i} + \sum_{j=0}^{\omega_i-2} x_{i,j} (u - i)^j
$$
divides the space $\mathbb{R}^{\omega_i - 1}$.

For $\omega_i = \text{def} (1 \leq \omega_i \leq 2)$, the model space $T_{\omega_i}$ is formed by attaching $n$ copies of a half-disk $D^n_+$ to a disk $D^n$ along the coordinate hyperplanes which divide $D^n$ into $2^n$ quadrants. So the number $\kappa(\omega_i)$ of $n$-dimensional cells-strata in $T_{\omega_i}$ is $2^n + n$ (see [K3]). Therefore, the pull-back of the $\Omega^*$-stratification in $T_{\omega_i}$ divides the model space $E_{\omega_i}$ into $2^n + n$ chambers-cells of dimension $n + 1$. As a result, the stratified simplicial norm

$$
\left\| [E_\omega, \delta E_\omega] \right\|_{\Omega^*_{\{n\}}} \geq 2^n + n,
$$

$$
\left\| [D(E_\omega), \partial D(E_\omega)] \right\|_{\Omega^*_{\{n\}}} \geq 2^{n+1} + 2n.
$$

It seems that $\kappa(\omega) < \kappa(\omega^*)$ for any minimal $\omega \neq \omega^*$.

---

9Compare the next theorem with the proposition that results by applying Theorem 4.2 to the fundamental class $h = [D(X)]$. 
Of course, it is more desirable to get lower bounds of these stratified simplicial norms...

The following proposition from [AK] is implied by Theorem 3.3 in combination with Theorem 4.1.

**Theorem 4.4. (Alpert, Katz)** Let $X$ be a smooth compact connected and oriented $(n+1)$-manifold with boundary. Assume that, for each connected component of the boundary, the image of its fundamental group in $\pi_1(D(X))$ is an amenable group. Let $v$ be a traversally generic vector field on $X$.

Then there is a $X$-independent universal constant $\theta > 0$ such that, for any $v$, the cardinality of the set $T(v)_{-n}$, formed by the trajectories of the maximal reduced multiplicity $n$, satisfies the inequality

$$\#(T(v)_{-n}) \geq \theta \cdot \|\|D(X)\|_\Delta.$$  

(4.12)

Combining Theorem 3.2 with Theorem 4.1, we get the following proposition.

**Theorem 4.5.** Let $X$ be a smooth compact connected and oriented manifold with boundary, $\dim(X) = n+1$. Assume that, for each connected component of the boundary $\partial X$, the image of its fundamental group in $\pi_1(D(X))$ is an amenable group. Let $v$ be a traversally generic vector field on $X$. Then

$$\text{rk}(\text{im}(L_{k+1}^{n,k+1})) \geq \text{rk}(H^\Delta_{k+1}(D(X))).$$

(4.13)

As a result, $\text{rk}(C_{n-k}^{n-k}(D(X),v))$---the number of $(n-k)$-dimensional strata in the stratification $\Omega^*(D(X),v)$ of the double $D(X)$---is greater than or equal to $\text{rk}(H^\Delta_{k+1}(D(X)))$.

The last claim in Theorem 4.5, together with the definitions of suspension complexities, leads to

**Theorem 4.6.** Let $X$ be a smooth compact connected and oriented manifold with boundary, $\dim(X) = n+1$. Assume that, for each connected component of the boundary $\partial X$, the image of its fundamental group in $\pi_1(D(X))$ is an amenable group. Then

$$\Sigma_{tcl}^{\text{trav}}(X) \geq \Sigma_{cshad}^{\text{trav}}(X) \geq \text{rk}(H^\Delta_*(D(X))),$$

where the vectorial inequality is understood as the inequality among all the the corresponding components of the participating vectors. The vector $\text{rk}(H^\Delta_*(D(X)))$ has been introduced prior to formula (3.23).

•

By the construction of the stratification $\Omega^*(D(X),v)$ of the double $D(X)$ (see Fig. 1 and Fig. 3), we get a formula which connects the combinatorics of $\Omega^*(D(X),v)$ to

\text{see (11.32)}
the combinatorics of the stratification $\Omega^*(\mathcal{T}(v))$ of the trajectory space by the connected components of the strata $\mathcal{T}(v, \omega)$:

$$\# \pi_0(D(X)^v_{\omega}) = \sum_{\{\omega| |\omega'|=j\}} \# \sup(\omega) \cdot \# \pi_0(\mathcal{T}(v, \omega)) + 2 \sum_{\{\omega| |\omega'|=j+1\}} \# \sup(\omega) \cdot \# \pi_0(\mathcal{T}(v, \omega))$$

$$\leq (n + 2) \left( \sum_{\{\omega| |\omega'|=j\}} \# \pi_0(\mathcal{T}(v, \omega)) + 2 \sum_{\{\omega| |\omega'|=j+1\}} \# \pi_0(\mathcal{T}(v, \omega)) \right)$$

$$= (n + 2) \left( \# \pi_0(\mathcal{T}(v, \omega)^v_{\omega}) + 2 \cdot \# \pi_0(\mathcal{T}(v, \omega)^{\omega}_{\omega}(j+1)) \right).$$

Therefore, the suspended complexity of $v$ can be estimated in terms of the $v$-complexities:

$$\Sigma c_{\Omega}^{n+1-j}(X, v) \leq (n + 2) \left( tc_{\Omega}^{n-j}(v) + 2 \cdot tc_{\Omega}^{n-j-1}(v) \right).$$

By Theorem 1.5 and under its hypotheses, for each $k$, we get

$$\text{rk}(H_{k+1}^v(D(X))) \leq (n + 2) \left( tc_{\Omega}^{n-k}(v) + 2 \cdot tc_{\Omega}^{n-k-1}(v) \right).$$

We notice that if $tc_{\Omega}^{n-k}(v) = 0$, then by Remark 4.1, $tc_{\Omega}^{n-k-1}(v) = 0$. Therefore, $tc_{\Omega}^{n-k}(v) = 0$ implies $H_{k+1}^v(D(X)) = 0$.

**Example 4.2.** Let $\{M_i\}_{1 \leq i \leq N}$ be several closed orientable surfaces of genera greater than 1. Consider the connected sum

$$Y \stackrel{\text{def}}{=} (M_1 \times S^1) \# (M_2 \times S^1) \# \ldots \# (M_N \times S^1),$$

and let $Z = Y \setminus B^3$, the complement to a smooth 3-ball. Put $\Sigma(M_i) = \text{def} M_i \times S^1$.

We may assume that the 1-handles $\{H_i \approx S^2 \times D^1\}_{1 \leq i \leq N-1}$, participating in the connected sum construction of $Y$, are attached to the complements to the surfaces $M_i \times \star_i$ in $M_i \times S^1$ and to the complement of $B^3$.

Consider a map $f_j : Z \rightarrow \Sigma(M_j)$ which is a homeomorphism on $M_j \times \star_i$. We construct $f_j$ in stages. First, we map each handle $H_i \subset Y$ to its core $D_i$ so that $Y$ is mapped to the union $W$ of $\Sigma(M_i)$’s to which the 1-cores $D_i'$’s are attached; each core is attached at a point $a_i \in \Sigma(M_i)$ and at a point $b_{i+1} \in \Sigma(M_{i+1})$. Under $f_j$, each of the two 3-disks, $D_a$ and $D_b$, from $D^3 \times \partial D^1$ is mapped to its center $a_i$ or $b_{i+1}$. Finally for each $j \in [2, N-1]$, we map the complement $Y \setminus \left( \Sigma(M_j) \setminus (D^3 \cup D^3) \right)$ to $a_j \coprod b_j$. For $j = 1$ and $j = N$, the constructions of $f_1$ and $f_N$ are similar.

For each $j$, by an isotopy argument, we may assume that the ball $B^3$ belongs to $Y \setminus \Sigma(M_j)$ and then restrict $f_j$ to $Z$.

We claim that the 2-dimensional classes $\{I_*[M] \in H^2_2(\Sigma Z)\}_{i}$, where $I : \coprod_i M_i \rightarrow Z$ is the obvious embedding, are linearly independent. Indeed, assume that some combination $h = \text{def} \sum_i r_i I_*[M]$ has the property $\|h\|_\Delta = 0$. Then $\|(f_j)_* h\|_\Delta = 0$. On the other hand,

$$(f_j)_* h = r_j \cdot (I_j)_* [M_j] \in H_2(\Sigma(M_j)) \approx \mathbb{R},$$
where the imbedding $I_j : M_j \subset \Sigma(M_j)$, being composed with $f_j$, produces the identity map of $M_j$. Therefore by the property of the simplicial semi-norm not to increase under continuous maps $\|I_j\|_\Delta = \|M_j\|_\Delta \neq 0$. Thus $r_j = 0$.

Let $X$ be a compact smooth 3-fold which is homotopy equivalent to $Z$ and has a spherical boundary. Let $v$ be a traversally generic field on $X$. Since $\pi_1(\partial X) = 0$, by Theorem 1.5 and following the arguments that lead to (4.15), we count the 1-dimensional connected strata in the stratification $\Omega^1(D(X), v)$ to get the inequality

$$6 \cdot \# T(v, 1221) + 2 \cdot \# T(v, 13) + 2 \cdot \# T(v, 31) + 3 \cdot \# \pi_0(T(v, 121)) + \# \pi_0(T(v, 2)) \geq \text{rk}\left(H_2^\Delta(D(Z))\right) = 2N.$$ 

Here the coefficients 6, 2, 3, and 1 next to the cardinalities are determined by the cardinalities of the support of the corresponding combinatorial types $\omega = 1221, 13, 31, 12, 2$.

Note that $\|[D(X)]\|_\Delta = 0$. So, for $k + 1 = 3$, Theorem 1.5 provides a trivial estimate for

$$6 \cdot \# T(v, 1221) + 2 \cdot \# T(v, 13) + 2 \cdot \# T(v, 31),$$

the number of 0-dimensional strata in $\Omega^1(D(X), v)$.

Let $\pi = \text{def } \pi_1(X/\partial X)$. Utilizing formula (4.15), Theorems 4.2 and 4.5 expose the groups $H_{k+1}^\Delta(D(X)) \subset H_{k+1}^\Delta(\Pi)$ and $H_{k+1}^\Delta(X/\partial X) \subset H_{k+1}^\Delta(\pi)$ as obstructions to the existence of globally $k$-convex traversing flows on $X$.

**Corollary 4.2.** Let $X$ be a smooth compact connected and oriented $(n + 1)$-manifold with boundary. Assume that, for each connected component of the boundary $\partial X$, the image of its fundamental group in $\pi_1(D(X))$ is an amenable group.

If $X$ admits a globally $k$-convex traversally generic vector field $v$, then the simplicial semi-norm is trivial on $H_{j+1}(D(X))$ and thus on $H_{j+1}(X/\partial X)$ for all $j \geq k$.

In particular, if $X$ admits a traversally generic vector field such that $T(v)_{-n} = 0$, then $\|[D(X)]\|_\Delta = 0$, and thus $\|[X/\partial X]\|_\Delta = 0$.

The next theorem, proven in $[AK]$, should be compared with Theorem 7.5 from $[K]$, its older 3D-relative. In a way, this theorem is the source of motivation for developing the machinery of amenable localization in $[AK]$ and in the present paper. For 3-folds $\mathcal{M}$ with a simply-connected boundary, Theorem 4.7 below is known with $c(2) = 1/\text{Vol}(\Delta^3)$, the inverse of the volume of the ideal tetrahedron in the hyperbolic space (see $[K]$).

**Theorem 4.7 (Alpert, Katz).** Let $M$ be a closed, oriented hyperbolic manifold of dimension $n + 1 \geq 2$. Let $X$ be the space obtained by deleting a domain $U$ from $M$, such that $U$ is contained in a ball $B_n^* \subset M$. Let $v$ be a traversally generic vector field on $X$.

Then the cardinality of the set of $v$-trajectories of the maximal reduced multiplicity $n$ satisfies the inequality

$$\#(T(v)_{-n}) \geq c(n) \cdot \text{Vol}(M),$$

where $c(n) > 0$ is an universal constant, and $\text{Vol}(M)$ denotes the hyperbolic volume of $M$.

\footnote{These are fields with no trajectories with the reduced multiplicity $\geq k$.}
Corollary 4.3. The inequality (4.16) of Theorem 4.7 is valid for any $X$, obtained by deleting a number of $(n+1)$-balls from a closed hyperbolic manifold $M$.

Proof. When the domain $U$ from Theorem 4.7 is a union of $(n+1)$-balls, we can encapsulate them into a single $(n+1)$-ball $B$. Therefore, Theorem 4.7 is applicable to $X = M \setminus U$. □

Let us apply Corollary 4.3 to the landscape of Morse functions on closed hyperbolic manifolds.

We formulate the following

Conjecture 4.1. Let $f : M \to \mathbb{R}$ be a Morse function on a closed manifold $M$ of dimension $(n+1)$, and $v$ its gradient-like field. Assume that $v$ satisfies the Morse-Smale transversality condition ($S$, $S_1$). Then, for all sufficiently small $\epsilon > 0$, the field $v$ on

$$X = \text{def } M \setminus \bigcup_{x \in \Sigma_f} B_{\epsilon}(x)$$

is traversally generic. The combinatorial types of the $v$-trajectories on $X$ are drawn from the list:

$$(11), (121), (1221), \ldots, (1 \underbrace{2 \cdot \ldots \cdot 2}_{n})_1.$$\n
Moreover, there exists a universal constant $c(n) > 0$ such that the number of broken $v$-trajectories on $M$, comprising $(n+1)$ segments, and the number of $n$-tangent $v$-trajectories in $X$ are related by the formula

$$(4.17) \quad \#(T(v, \omega_*)) = c(n) \cdot \#(\text{broken}_{(n+1)}(v)),$$

where $\omega_* = (1 \underbrace{2 \cdot \ldots \cdot 2}_{n})_1$. \hfill *•*

We can show that $c(2) = 4$ and $c(3) = 4$.

Combining Conjecture 4.1 with Corollary 4.3 one could arrive to a lower estimate of $\#(\text{broken}_{(n+1)}(v))$ for Morse-Smale gradient fields $v$ on closed hyperbolic manifolds.

Fortunately, regardless of the validity of the conjecture, $\#(\text{broken}_{(n+1)}(v))$ has a lower boundary, delivered by the normalized hyperbolic volume! The beautiful proposition below has been recently proven by H. Alpert [A]. The proof involves the same circle of amenable localization techniques as in [Gr1], [AK].

Theorem 4.8 (Alpert). Let $f : M \to \mathbb{R}$ be a Morse function on a closed hyperbolic manifold $M$ of dimension $(n+1)$ and $v$ its gradient-like field. Assume that $v$ satisfies the Morse-Smale transversality condition. Then

$$\#(\text{broken}_{(n+1)}(v)) \geq \text{Vol}(M).$$ \hfill *•*

Remark 4.5. To estimate from above the constant $c(n)$ in Theorem 4.7, it will suffice to compute the proportion $\#(T(v) \setminus n)/\|[(D(X)]\|_\Delta$ for some $(n+1)$-dimensional $X$ and some traversally generic field $v$ on it.
To compute the local contributions $\theta(\omega)$ in formula (4.11) requires a detailed understanding of the stratified geometry of the pair $(D(\mathcal{E}_\omega), D(\partial \mathcal{E}_\omega))$, a tricky task. Instead, it seems that computing $\#(T(v)_{-2})/\|D(X)\|_\Delta$ is within a reach when $X$ is a complement to a number of $(n+1)$-balls in a closed (hyperbolic) manifold $M$ comprised of $N$ (ideal) simplices and the vector field $v$ is “adjusted” to the first barycentric subdivision $\beta T$ of the smooth triangulation $F : T \to M$. So it seems that the universal constant admits a supper-exponential upper estimate: $\text{const}(n) \leq 2(n+2)!$. We conjecture that a better exponential estimate

$$\text{const}(n) \leq 2^{n+1} + 2n$$

is valid. The RHS of this inequality is the number of $n$-dimensional chambers-cells in which the stratification divides $D(\mathcal{E}_\omega)$, where $\omega = (1 2 \ldots 2 \underbrace{1}_{n})$.

To illustrate the statements of our theorems in 2D, let us turn to two simple examples.

**Example 4.3.** Consider the constant vertical field $v = \partial_y$ on a standard annulus $X \subset \mathbb{R}^2$. Since $X$ and $D(X)$ each admits self-maps of the degree 2, the simplicial semi-norm $\|X, \partial X\|_\Delta = 0$ and $\|D(X)\|_\Delta = 0$ (see [Gr]). So Theorem 4.4 and Theorem 4.7 say little about the cardinality $\#(T(v)_{-2}) = 4$. On the other hand, $X$ does admit a radial traversing field for which $T(v)_{-2} = \emptyset$.

**Example 4.4.** Take the surface $X \subset \mathbb{R}^2$ and the constant vertical field $v = \partial_y$ on it as in Fig. 1. Since $X$ is a disk with 4 holes, the double $D(X)$ is a closed surface of genus 4. It admits a hyperbolic metric. By [Gr], it follows that

$$\|D(X)\|_\Delta = 2(2 \cdot 4 - 2) = 12.$$

By Lemma 3.2

$$2\|X, \partial X\|_\Delta \geq 12 \geq \|X/\partial X\|_\Delta.$$

Thus $6 \leq \|X, \partial X\|_\Delta$ and $\|X/\partial X\|_\Delta \leq 12$.

At the same time, $\#(T(v)_{-2}) = 12$ (see Fig. 1). As a result, the universal constant in the inequality of Theorem 4.4 gets an upper bound: $\text{const}(1) \leq 2$.

However, the field $v$ in Fig. 1 is not the “optimal” one. To optimize $v$, take the radial field on an annulus $A$. Delete tree convex disks from $A$ to form $X$ and restrict the radial field to $X$. For the restricted field $v$ on $X$, the graph $T(v)$ is trivalent with 6 vertices. So we get $\#(T(v)_{-2}) = 6$. As a result, the universal constant must satisfy the inequality $\text{const}(1) \leq 1$.

We just have internalized the crucial role that non-amenable fundamental groups $\pi_1(X/\partial X)$ and $\pi_1(D(X))$ play in delivering some lower bounds of the traversing complexities $\text{tc}_{\text{trav}}(X)$ and $\Sigma \text{tc}_{\text{trav}}(X)$.

Now we will connect the complexity of the fundamental group $\pi_1(X/\partial X)$ with the number of minimal connected components in the $\Omega^*$-stratification $\{X(v, \omega)\}_\omega$ of $X$. The considerations to follow are rather elementary; in particular, the amenability properties do not play any role here.
Each \( v \)-trajectory \( \gamma \) of the combinatorial type \( \omega \) defines a loop or a bouquet of loops in the quotient space \( X/\partial X \). Similarly, each \( \gamma \) defines its double \( D(\gamma) \subseteq D(X) \). The double \( D(\gamma) \) is a chain of loops (like “\( \infty \)” or “ooo”), the number of loops in the chain being equal to \( \#(\gamma \cap \partial X) - 1 \).

Therefore each \( \gamma \) produces an element \( [\gamma] \in \pi_1(X/\partial X) \) and a subgroup \( [[\gamma]] \) of \( \pi_1(X/\partial X) \), equipped with the ordered set of \( (|\text{sup}(\omega)| - 1) \) generators—the \( v \)-ordered loops of the bouquet \( [\gamma] \). Here \( |\text{sup}(\omega)| \) is the cardinality of the set \( \gamma \cap \partial X \). The element \( [\gamma] \) and the subgroup \( [[\gamma]] \) are constant within each connected component \( X(v, \sigma) \) of the pure stratum \( X(v, \omega) \). This follows from the fact that the finite covering

\[
\Gamma : X(v, \sigma) \cap \partial X \rightarrow \Gamma(X(v, \sigma) \cap \partial X)
\]

is trivial \([\text{K3}], \text{Corollary } 5.1\). Let us denote by \( S^•(v) \) the poset whose elements are the connected components \( X(v, \sigma) \).

Therefore, we get a system of groups \( \{[[\gamma_\sigma]]\}_{\sigma \in S^•(v)} \), linked by homomorphisms

\[
\psi_{\sigma, \sigma'} : [[\gamma_\sigma]] \rightarrow [[\gamma_{\sigma'}]]
\]

for any pair \( \sigma \succ \sigma' \) in \( S^•(v) \).

This construction leads to the following

**Lemma 4.1.** For a traversally generic field \( v \) on \( X \), the subgroups \( \{[[\gamma_\sigma]]\}_{\sigma \in S_{\text{min}}^•(v)} \) generate \( \pi_1(X/\partial X) \).

**Proof.** We need to show that any loop \( \rho : I/\partial I \rightarrow X/\partial X \) through the point \( \ast = \partial X/\partial X \) is a product of loops of the form the groups \( [[\gamma_\sigma]] \), where \( \sigma \) is a minimal element of the poset \( S^•(v) \). Equivalently, it will suffice to show that any path \( (\rho, \partial \rho) : (I, \partial I) \rightarrow (X, \partial X) \) can be homotoped, relatively to the boundary \( \partial X \), to an ordered union several segments

\[
\text{of oriented trajectories } \gamma, \text{ labeled with the elements of the minimal set } S_{\text{min}}^•(v). \text{ Here the segments of trajectories } \gamma \text{ are bounded by two points from } \gamma \cap \partial X \text{ and the orientations of segments are not necessarily the ones induced by } v.
\]

Note that each oriented segment of \( \gamma \) belongs to the subgroup \( [[\gamma]] \).

First, with the help of the \((-v)\)-flow, we homotop \( (\rho, \partial \rho) \) to a path that is realized by several segments of trajectories, intermingled with some paths residing in \( \partial X \).

The rest of argument is an induction based on the order \( \succ \) in \( S^•(v) \). We can replace any segment of \( \gamma \subseteq X(v, \sigma) \) with the corresponding segment of any other trajectory from that stratum. This replacement is possible since \( \Gamma : X(v, \sigma) \cap \partial X \rightarrow T(v, \sigma) \) is a trivial covering. Moreover, we can homotop such segment of \( \gamma \) to a segment of some trajectory \( \gamma' \), residing in any stratum \( X(v, \sigma') \) that belongs to the closure of \( X(v, \sigma) \). In other words, we can replace any segment of \( \gamma \subseteq X(v, \sigma) \) with some segment of \( \gamma' \subseteq X(v, \sigma') \), where \( \sigma' \prec \sigma \). Following these replacements, eventually we will arrive to an ordered finite collection of segments of \( \{\gamma_\sigma \subseteq X(v, \sigma)\}_{\sigma \in S_{\text{min}}^•(v)} \); the collection will represent the relative homotopy class of the path \( \rho \) (some of the the segments of trajectories \( \gamma_\sigma \)'s in this representation may appear with integral multiplicities). \( \square \)
We denote by $c_{\text{gen}}(\pi)$ the minimal number of generators in finite presentations of the group $\pi$.

**Theorem 4.9.** For a traversally generic field $v$ on a connected compact smooth manifold $X$ with boundary,

$$c_{\text{gen}}(\pi_1(X/\partial X)) \leq \sum_{\sigma \in S_{\text{min}}^e(v)} \left( |\sup(\omega(\sigma))| - 1 \right),$$

where $\omega(\sigma)$ is the combinatorial type of a typical $v$-trajectory passing through the connected component labeled by $\sigma$.

**Proof.** By Lemma 4.1, any element $\beta \in \pi_1(X/\partial X)$ produces a word in the alphabet whose letters are the loops, generated by the segments of special trajectories that label the minimal strata of $S_{\text{min}}^e(v)$; each minimal stratum $X(v, \sigma)$, $\sigma \in S_{\text{min}}^e(v)$, contains a unique special trajectory. The number of loops-letters in each trajectory is $|\sup(\omega(\sigma))| - 1$. Therefore the total number of letters in the alphabet is given by the RHS of the formula above. \(\square\)

**Corollary 4.4.** For a traversally generic field $v$ on a connected compact smooth $(n + 1)$-dimensional manifold $X$ with boundary, the number of minimal connected components in $T(v)$ satisfies the inequality:

$$\#(S_{\text{min}}^e(v)) \geq c_{\text{gen}}(\pi_1(X/\partial X))/(n + 1).$$

In particular, if $\partial X$ consists of $m$ components, then

$$\#(S_{\text{min}}^e(v)) \geq (m - 1)/(n + 1).$$

**Proof.** Note that for a traversally generic $v$, $|\sup(\omega(\sigma))| \leq n + 2$, so that the number of segments in which a typical trajectory $\gamma_\sigma$ is divided by $\partial X$ is $n + 1$ at most. Utilizing the arguments from Lemma 4.1, Theorem 4.9 implies the first inequality of the corollary.

To prove the second one, note that $X/\partial X$ admits a continuous map $F$ onto a connected graph $G$ with two vertices, $a$ and $b$, and $m$ edges. Indeed, let $U$ be a collar of $\partial X$ in $X$. We send $X \setminus \text{int}(U)$ to $a \in G$, $\partial X$ to $b$, and each component of the collar $U$ to the corresponding edge. Each basic loop in $G$ lifts against $F$ to a loop in $X/\partial X$. Therefore the exists an epimorphism $\pi_1(X/\partial X) \to F_{m-1}$, where $F_{m-1}$ denotes the free group in $m - 1$ generators. Thus $c_{\text{gen}}(\pi_1(X/\partial X)) \geq m - 1$. \(\square\)

**Remark 4.6.** Note that, by definition, $\#(S_{\text{min}}^e(v)) \geq \#(T(v)_{-n})$. Thus, under the hypotheses of Theorem 4.7 and by that theorem, there exists an universal positive constant such that $\#(S_{\text{min}}^e(v)) \geq \text{const}(n) \cdot \text{Vol}(M)$, where the hyperbolic volume $\text{Vol}(M)$ depends only on $\pi_1(M)$.

If $X$ is obtained from $M$ by deleting a single ball, $\pi_1(M) \approx \pi_1(X/\partial X)$. In such a case, $\#(S_{\text{min}}^e(v)) \geq c_{\text{gen}}(\pi_1(M))/(n + 1)$. So, for the hyperbolic $X = M \setminus D^{n+1}$, both lower bounds for $\#(S_{\text{min}}^e(v))$ are expressed essentially in terms of $\pi_1(M)$. \(\bullet\)

**Acknowledgments.** The author is grateful to Larry Guth and Hannah Alpert for the enjoyable in-depth discussions that have led to this work.
References

[A] Allpert, H., Manuscript.

[AK] Allpert, H., Katz, G., Using Simplicial Volume to Count Multi-tangent Trajectories of Traversing Vector Fields, arXiv:1503.02583v1 [math.DG] (9 Mar 2015).

[CF] Conner, P.E., Floyd, E.E., Differentiable Periodic Maps, Springer-Verlag, 1964.

[Gr] Gromov, M., Volume and bounded cohomology Publ. Math. I.H.E.S., tome 56 (1982), 5-99.

[Gr1] Gromov, M., Singularities, Expanders and Topology of Maps. Part I: Homology versus Volume in the Spaces of Cycles, Geom. Funct. Anal. vol 19 (2009), 743-841.

[Hat] Hatcher, A., Algebraic Topology, Cambridge University Press, 2002.

[K] Katz, G., Convexity of Morse Stratifications and Gradient Spines of 3-Manifolds, JP Journal of Geometry and Topology, v.1 no 1, (2009), 1-119 (math.GT/0611005 v1(31 Oct. 2006)).

[K1] Katz, G., Stratiﬁed Convexity & Concavity of Gradient Flows on Manifolds with Boundary. Applied Mathematics, 2014, 5, 2823-2848, http://www.scirp.org/journal/am (also arXiv:1406.6907v1 [mathGT] (26 June, 2014)).

[K2] Katz, G., Traversally Generic & Versal Flows: Semi-algebraic Models of Tangency to the Boundary arXiv:1406.1345v1 [math.GT] (4 July, 2014).

[K3] Katz, G., The Stratified Spaces of Real Polynomials & Trajectory Spaces of Traversing Flows, arXiv:1407.2898v3 [math.GT] (6 Aug 2014).

[K4] Katz, G., Causal Holography of Traversing Flows, arXiv:1409.0588v1[math.GT] (2 Sep 2014).

[S] Smale, S., Generalized Poincare’s conjecture in dimensions greater than four, Ann. of Math. (2) 74 (1961), 39-106.

[S1] Smale, S., On the structure of manifolds, Amer. J. Math., 84 (1962) 387-399.

5 Bridle Path Circle, Framingham, MA 01701, USA
E-mail address: gabkatz@gmail.com