Symmetry energy effects on isovector properties of neutron rich nuclei with a density functional approach

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Abstract

We employ a variational method to study the effect of the symmetry energy on the neutron skin thickness and the symmetry energy coefficients of various neutron rich nuclei. We concentrate our interest on $^{208}$Pb, $^{124}$Sn, $^{90}$Zr, and $^{48}$Ca, although the method can be applied in the totality of medium and heavy neutron rich nuclei. Our approach has the advantage that the isospin asymmetry function $\alpha(r)$, which is the key quantity to calculate isovector properties of various nuclei, is directly related with the symmetry energy as a consequence of the variational principle. Moreover, the Coulomb interaction is included in a self-consistent way and its effects can be separated easily from the nucleon-nucleon interaction. We confirm, both qualitatively and quantitatively, the strong dependence of the symmetry energy on the various isovector properties for the relevant nuclei, using possible constraints between the slope and the value of the symmetry energy at the saturation density.

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I. INTRODUCTION

The nuclear symmetry energy (SE) is the basic regulator of the isospin properties of the neutron rich nuclei [1–8]. It is expected to affect the neutron skin thickness, the coefficient of the asymmetry energy in Bethe-Weizsacker formula, e.t.c. In addition, the density dependence of the SE is the main ingredient of the equation of state of neutron rich nuclear matter. Actually there is a variety of neutron star properties which are sensitive to SE, that is the maximum mass value and the corresponding radius, the onset of the direct Urca process, the crust-core transition density and pressure e.t.c. [2, 9].

Recently, there is an extended theoretical [10–34, 36–57, 57–62] and experimental [63–73] interest to constrain the slope of the symmetry energy $L$ close to the value of the saturation density $\rho_0$ of nuclear matter. Both, theoretical and experimental efforts are focused on the study of a possible correlation of $L$ with various nuclear properties including the neutron skin thickness, the dipole polarizability and the pygmy dipole resonance of various neutron rich nuclei as well as the analysis of heavy ion collision data. Additionally, isobaric analog states, nuclei mass formula data and also neutron star observation data are also elaborated.

However, the experimental data for the SE still remain limited and only for low values of density ($\rho < \rho_0$) are accurately constrained. From the theoretical point of view there is an effort to constrain the trend of SE, even for low values of density, from finite nuclei properties and to extrapolate in a way to densities related to neutron stars equation of state (up to $\simeq 5\rho_0$). In any case, the constraints of $L$ or in general the density dependence of SE, even for low values of $\rho$, are very important for astrophysical applications. For example the transition density and pressure between the crust and the core in a neutron star are expected to lie close to the half values of the saturation density $\rho_0$ and consequently similar to the finite nuclei interior densities [2–5].

The structure of a heavy nucleus is a result of the interplay between the strong short range nuclear forces and long range Coulomb interaction. However, in order to exhibit the isovector character of nuclear forces, we have to focus mainly on heavy and additional neutron rich nuclei. Furthermore it is well known that the energy density formalism is able to reproduce properties of finite nuclei including mainly the bulk properties, namely the binding energy as well as the size and shape of the mass and charge distributions [50, 74, 78]. In view of
the above statements, in the present work we employ a variational approach on a suitably constructed energy density functional. Special attention is devoted to the contribution of the nuclear symmetry energy and the self-consistent treatment of the Coulomb interaction. The symmetry energy is suitably parameterized. Actually the present approach can be easily extended to include more complicated expressions for the symmetry energy as well as for the energy of the symmetric nuclear matter.

The key quantity of the present study is the isospin asymmetry function \( \alpha(r) = (\rho_n(r) - \rho_p(r))/\rho(r) \) (where \( \rho_n, \rho_p \) and \( \rho = \rho_n + \rho_p \) are the neutron, proton and total number densities respectively). The method has the advantage that the asymmetry function \( \alpha(r) \) is directly related with the symmetry energy as a consequence of the variational principle. It is expected that the various isovector properties of nuclei (neutron skin thickness, symmetry energy coefficient e.t.c.) depend on the trend of the symmetry energy for densities close to the interior of the nucleus. The motivation of the present work is twofold. Firstly we tried to construct a self-consistent and easily applicable density functional method to study the effect of the symmetry energy on the isovector structure properties of medium and heavy neutron rich nuclei. Secondly, our aim is, if it is possible, to combine our theoretical estimation with the relevant experimental or empirical data in order to suggest constraints on the density dependence of the symmetry energy for densities close to those of the interior of finite nuclei.

The article is organized as follows. In Sec. II we review the density functional method and the variational approach employed for calculating the bulk properties of various neutron rich nuclei. The results are presented and discussed in Sec. III, while Sec. IV summarizes the present study.

II. ENERGY DENSITY FUNCTIONAL AND VARIATIONAL APPROACH

According to the empirical Bethe-Weizsacker formula the binding energy of a finite nucleus with \( A \) nucleons and atomic number \( Z \) is given by

\[
B(A, Z) = -a_V A + a_S A^{2/3} + a_C \frac{Z(Z - 1)}{A^{1/3}} + a_A \frac{(N - Z)^2}{A} + \Delta E_{mic}.
\]  

(1)

The first term corresponds to the volume effect, the second is the surface term, the third one takes into account the Coulomb repulsion of the protons, while the fourth is the symmetry
energy term. Finally, the last term corresponds to other factors including the pairing interaction e.t.c. Using fits of known masses to this equation one can determine the corresponding coefficients $a_V$, $a_S$, $a_C$ and $a_A$.

The energy density functional is a natural extension of the above formula, where now the total energy is a functional of the proton and neutron densities and consists of terms corresponding with those appearing in relation (1). The minimization of the total energy defines the related densities and consequently the contribution of each term separately. In the present work we apply the energy density formalism, where the total energy of finite nuclei is a functional of the total density $\rho(r)$ and the isospin asymmetry function $\alpha(r)$, that is

$$E[\rho(r), \alpha(r)] = \int_\mathcal{V} \mathcal{E}(\rho(r), \alpha(r)) \, d^3r,$$

(2)

where $\mathcal{E}(r)$ is the local energy density. The integration is performed over the total volume $\mathcal{V}$ occupied by the relevant nuclei.

Now we consider the functional

$$E[\rho, \alpha] = \int_\mathcal{V} \left[ \epsilon_{ANM}(\rho(r), \alpha(r)) + F_0|\nabla \rho(r)|^2 + \frac{1}{4}\rho(1 - \alpha)V_c(r) \right] \, d^3r.$$  

(3)

The first ingredient of the functional, $\epsilon_{ANM}(\rho(r), \alpha(r))$, corresponds to the energy density of the asymmetric nuclear matter given by the expression

$$\epsilon_{ANM}(\rho, \alpha) = \epsilon_{SNM}(\rho) + \alpha^2 \rho S(\rho),$$

(4)

where $\epsilon_{SNM}(\rho, \alpha)$ is the energy density of symmetric nuclear matter and $S(\rho)$ is the symmetry energy per particle of nuclear matter.

The second term $F_0|\nabla \rho(r)|^2$ is the gradient term corresponding to the contribution originating from the finite size character of the density distribution with $F_0$ being a parameter in the interval $(66 - 72)$ MeV. In the present work we consider that $F_0 = 70$ MeV.

The third term corresponds to the Coulomb energy density where the Coulomb potential $V_c(r)$ is defined as

$$V_c(r) = \frac{e^2}{2} \int \frac{\rho(r')(1 - \alpha(r'))}{|\mathbf{r} - \mathbf{r}'|} \, d^3r',$$

(5)

and must satisfy also the Poisson equation

$$\nabla^2 V_c(r) = 4\pi e^2 \left( \frac{1}{2}(1 - \alpha(r)) \right) \rho(r).$$

(6)
Eq. (6) is used to check the convergence of the iteration process involved in such a kind of calculations. Finally the density $\rho(r)$ and the asymmetry function $\alpha(r)$ must obey the following constraints

$$\int \rho(r)d^3r = A, \quad \int \alpha(r)\rho(r)d^3r = N - Z.$$ (7)

The functional (3) and the constraints (7) after some algebra are written as

$$E[\rho, \alpha] = 4\pi \int_0^\infty r^2 \left[ \epsilon_{ANM}(\rho(r), \alpha(r)) + F_0 \left( \frac{d\rho}{dr} \right)^2 + \frac{1}{4}\rho(1 - \alpha)V_c(r) \right] dr$$ (8)

and

$$4\pi \int_0^\infty r^2 \rho(r)dr = A, \quad 4\pi \int_0^\infty r^2 \alpha(r)\rho(r)dr = N - Z.$$ (9)

Eqs. (8) and (9) constitute a variational problem with constraints while the Lagrangian density is given by

$$\mathcal{L} = 4\pi r^2 \left[ \epsilon_{ANM}(\rho, \alpha) + F_0 \left( \frac{d\rho}{dr} \right)^2 + \frac{1}{4}\rho(1 - \alpha)V_c(r) \right] - \lambda_1 4\pi r^2 \rho - \lambda_2 4\pi r^2 \alpha \rho,$$ (10)

In Eq. (10) $\lambda_1$ and $\lambda_2$ are the Lagrange multipliers. The two corresponding Euler-Lagrange equations for $\rho(r)$ and $\alpha(r)$ are defined as follow:

$$\frac{\partial \mathcal{L}}{\partial \rho} - \frac{d}{dr} \left( \frac{\partial \mathcal{L}}{\partial \rho'} \right) = 0,$$ (11)

$$\frac{\partial \mathcal{L}}{\partial \alpha} - \frac{d}{dr} \left( \frac{\partial \mathcal{L}}{\partial \alpha'} \right) = 0.$$ (12)

We find easily that

$$\frac{\partial \mathcal{L}}{\partial \rho} = 4\pi r^2 \left[ \frac{\partial \epsilon_{SNM}(\rho)}{\partial \rho} + \alpha^2 \left( S(\rho) + \rho \frac{\partial S(\rho)}{\partial \rho} \right) + \frac{1}{4}(1 - \alpha)V_c(r) - \lambda_1 - \lambda_2 \alpha \right],$$ (13)

$$\frac{\partial \mathcal{L}}{\partial \rho'} = 8\pi r^2 F_0 \rho',$$ (14)

$$\frac{d}{dr} \left( \frac{\partial \mathcal{L}}{\partial \rho'} \right) = 8\pi F_0 r^2 \rho'' + 16\pi F_0 \rho' r.$$ (15)

Also we have

$$\frac{\partial \mathcal{L}}{\partial \alpha} = 4\pi r^2 \left[ 2\alpha \rho S(\rho) - \frac{1}{4}\rho V_c(r) - \lambda_2 \rho \right],$$ (16)

$$\left( \frac{\partial \mathcal{L}}{\partial \alpha'} \right) = 0.$$ (17)
The first Euler-Lagrange equation gives
\[ \rho'' + \frac{2\rho'}{r} - \frac{1}{2F_0} \left[ \frac{\partial \epsilon_{SNM}(\rho)}{\partial \rho} \right] + \alpha^2 \left( S(\rho) + \rho \frac{\partial S(\rho)}{\partial \rho} \right) + \frac{1}{4} (1 - \alpha) V_c(r) - \lambda_1 - \lambda_2 \alpha = 0 \] (18)
and the second one
\[ \alpha(r) = \frac{V_c(r)}{8S(\rho)} + \frac{\lambda_2}{2S(\rho)} = \frac{1}{8S(\rho)} \left( V_c(r) + 4\lambda_2 \right). \] (19)

The asymmetry function \( \alpha(r) \) obeys the constraints \( 0 \leq \alpha(r) \leq 1 \). However, the expression (19) does not ensure the above constraints, since for high values of \( r \) (low values of \( \rho(r) \)) and consequently \( S(\rho) \) \( \alpha(r) \) increases very fast and there is a cut-off radius, \( r_c \) where \( \alpha(r_c) = 1 \) and also \( \alpha(r \geq r_c) \geq 1 \).

In order to overcome this unphysical behavior of \( \alpha(r) \) we use the assumption
\[ \alpha(r) = \begin{cases} 
\frac{1}{8S(\rho)} \left( V_c(r) + 4\lambda_2 \right), & r \leq r_c \\
1, & r \geq r_c.
\end{cases} \] (20)

Accordingly the proton and neutron density distributions take the form
\[ \rho_p(r) = \begin{cases} 
\frac{1}{2} \rho(r) (1 - \alpha(r)), & r \leq r_c \\
0, & r \geq r_c.
\end{cases} \] (21)
\[ \rho_n(r) = \begin{cases} 
\frac{1}{2} \rho(r) (1 + \alpha(r)), & r \leq r_c \\
\rho(r), & r \geq r_c.
\end{cases} \] (22)

The Lagrange multiplier \( \lambda_2 \) is found from the normalization condition
\[ \int_V \alpha(r) \rho(r) d^3r = N - Z, \] (23)
where the integration is performed over the total volume occupied by the specific nucleus considering that \( \alpha(r) \) is given by (20). After a straightforward algebra we get
\[ \lambda_2 = \frac{2}{\left( \int_{V_c} \rho(r) d^3r - \frac{e^2}{8} \int_{V_c} \frac{V_c(r) \rho(r)}{S(\rho)} d^3r - 2Z \right) \left( \int_{V_c} \frac{\rho(r)}{S(\rho)} d^3r \right)^{-1}}, \] (24)
where $V_c$ is the part of the spherical volume of the nucleus for the radius $r_c$. The cut-off radius $r_c$, which reflects the combined effect of the symmetry energy and Coulomb energy on the asymmetry function $\alpha(r)$ according to expression (19), easily can be determined by solving the equation

$$\alpha(r_c) = 0.$$  \hspace{1cm} (25)

Finally from the equations (20), (24) and (25) we see that the asymmetry function $\alpha(r)$ is

$$\alpha(r) = \frac{1}{S(\rho)} \left( \frac{V_c(r) - V_c(r_c)}{8} + S(\rho_c) \right), \quad r \leq r_c,$$  \hspace{1cm} (26)

and $\alpha(r) = 1$ for $r \geq r_c$. In Eq. (26) one can see clearly exhibited the interplay between the long-range Coulomb interaction and the short-range isovector part of the nuclear forces. Actually $S(\rho)$ affects $\alpha(r)$ in a twofold manner: a) directly via the term $S(\rho_c)/S(\rho)$ and b) indirectly since the $V_c(r)$ according to Eq. (5) is a functional of $\alpha(r)$. In the simplified case where $V_c(r)$ is excluded, the asymmetry function is given by the simple formula

$$\alpha(r) = \frac{S(\rho_c)}{S(\rho)}.$$  \hspace{1cm} (27)

The Coulomb potential, given by Eq. (5) due to the discontinuity behavior of the proton density distribution (Eq. (21)) is decomposed in two parts as follow

$$V_c^A(r) = 2\pi e^2 \left[ \frac{1}{r} \int_0^r (1 - \alpha(r')) \rho(r') r'^2 dr' + \int_r^{r_c} (1 - \alpha(r')) \rho(r') r'^2 dr' \right], \quad r \leq r_c$$  \hspace{1cm} (28)

$$V_c^B(r) = 2\pi e^2 \frac{r}{r_c} \int_0^{r_c} (1 - \alpha(r')) \rho(r') r'^2 dr', \quad r \geq r_c.$$  \hspace{1cm} (29)

Actually one has to solve self-consistently Eqs. (18) and (19) with the corresponding constraints (9). In the present work, in order to avoid the complication due to the differential equation (18) we employ a variational method where use is made of an appropriate trial function for $\rho(r)$. This method, as pointed out by Brueckner et. al. \[79, 83\], provides a convenient tool in seeking approximate solution for heavy nuclei. There is a variety of trial density distribution functions suitably parameterized to describe light, medium and heavy nuclei. In the present study we consider the trial function given by the Fermi type formula

$$\rho(r) = \frac{n_0}{1 + \exp[(r - d)/w]}.$$  \hspace{1cm} (30)
In addition, for the basic ingredients of the energy functional we consider a model where the energy of the symmetric nuclear matter is given by

\[ \epsilon_{SNM}(\rho) = \rho T_0 \left( a u^{2/3} - b u + c u^{5/3} \right), \quad u = \rho / \rho_0, \]

(31)

where \( T_0 = 37.0206 \text{ MeV} \) and \( \rho_0 = 0.16144 \text{ fm}^{-3} \) (the saturation density). The corresponding constants are: \( a = -0.08203, b = 0.97342 \) and \( c = 0.61687 \).

The symmetry energy \( S(\rho) \) can be suitably expanded around the saturation density \( \rho_0 \) as follows

\[ S(\rho) = S(\rho_0) + L \delta + \frac{K_{sym}}{2!} \delta^2 + O(\delta^3), \]

(32)

where \( S(\rho_0) \) is the value of the symmetry energy at the saturation density and \( \delta = \frac{\rho - \rho_0}{3\rho_0} \). The coefficient \( L = 3\rho_0 \frac{dS(\rho)}{d\rho} \bigg|_{\rho = \rho_0} \) is related with the slope of the symmetry energy at \( \rho_0 \), while the coefficient \( K_{sym} \) is given by \( K_{sym} = 9\rho_0^2 \frac{d^2S(\rho)}{d\rho^2} \bigg|_{\rho = \rho_0} \).

There are various suggested expressions for the symmetry energy in the literature. Here we employ the simple parameterization

\[ S(\rho) = S(\rho_0) \left( \frac{\rho}{\rho_0} \right)^\gamma = J u^\gamma, \quad S(\rho_0) = J. \]

(33)

Obviously, in this case the parameter \( \gamma \) is related with both the slope \( L \) and \( J \) by the expression

\[ \gamma = \frac{L}{3J}. \]

(34)

It is worth pointing out that for \( \gamma < 1 \) a smaller value of \( \gamma \) gives a stiffer \( S(\rho) \) for \( \rho < \rho_0 \) while, for \( \rho > \rho_0 \) the higher the value of \( \gamma \) the stiffer is \( S(\rho) \). Finally, the symmetry energy density \( s(\rho) \) is given by

\[ s(\rho) = \rho J u^\gamma. \]

(35)

Now the total energy density of the asymmetric nuclear matter is

\[ \epsilon_{ANM}(\rho, \alpha) = \rho T_0 \left( a u^{2/3} - b u + c u^{5/3} \right) + \alpha^2 \rho J u^\gamma. \]

(36)

For each specific set of the Fermi type distribution parameters \( n_0, d, \) and \( w \) and a given symmetry energy \( S(\rho) \), we calculate the asymmetry density \( \alpha(r) \) and the total energy of the specific nucleus. The set of the density distribution parameters is adjusted in order to find
the corresponding minimum value of the total energy given now by the integrals
\[
E[\rho(r); \gamma] = 4\pi \int_{0}^{r_c} r^2 \left( \epsilon_{ANM}(\rho(r), \alpha(r)) + F_0 \left( \frac{d\rho}{dr} \right)^2 + \frac{1}{4} \rho(r)(1 - \alpha(r))V_c(r) \right) dr \tag{37}
\]
\[
+ 4\pi \int_{r_c}^{\infty} r^2 \left( \epsilon_{ANM}(\rho(r), 1) + F_0 \left( \frac{d\rho}{dr} \right)^2 \right) dr.
\]
After finding the density \( \rho(r) \) and asymmetry function \( \alpha(r) \) which minimizes the total energy, all the relevant quantities are easily calculated.

One possibility is to calculate the symmetry energy coefficient \( a_A \), defined in Bethe-Weizsacker formula via the local density approximation. In this approach \( a_A \) is defined by the integral
\[
a_A = \frac{A}{(N - Z)^2} \int \rho(r)S(\rho)\alpha^2(r)d^3r. \tag{38}
\]
Definition (38) shows explicitly the direct strong dependence of \( a_A \) on the symmetry energy \( S(\rho) \) and the asymmetry function \( \alpha(r) \). Actually, according to the present study, the total integral is split in two parts as follows
\[
a_A = \frac{A}{(N - Z)^2} \left( \int_{V_c} \rho(r)S(\rho)\alpha^2(r)d^3r + \int_{V > V_c} \rho(r)S(\rho)d^3r \right). \tag{39}
\]
One of the most important quantities concerning the isovector character of the nuclear forces is the neutron skin thickness defined as
\[
R_{skin} = R_n - R_p, \tag{40}
\]
with \( R_n \) and \( R_p \) the neutron and proton radii respectively defined as
\[
R_n = \left( \frac{1}{N} \int r^2 \rho_n(r)d^3r \right)^{1/2}, \quad R_p = \left( \frac{1}{Z} \int r^2 \rho_p(r)d^3r \right)^{1/2}. \tag{41}
\]
In the framework of the present approach they are given respectively by the expressions
\[
R_n = \left[ \frac{1}{N} \left( \int_{V_c} r^2 \frac{1}{2}\rho(r) (1 + \alpha(r)) d^3r + \int_{V > V_c} r^2 \rho(r)d^3r \right) \right]^{1/2} \tag{42}
\]
and
\[
R_p = \left( \frac{1}{Z} \int_{V_c} r^2 \frac{1}{2}\rho(r) (1 - \alpha(r)) d^3r \right)^{1/2}. \tag{43}
\]
Actually, \( R_{skin} \) is not directly dependent on \( S(\rho) \), compared to the case of \( a_A \), but indirectly via the dependence of \( \alpha(r) \). However, recent studies conjecture that \( R_{skin} \) is a strong indicator of the isospin character of the nuclear interaction expected to be strongly correlated with the symmetry energy slope \( L \) and the value \( J \) or in general with the values of the symmetry energy close to the saturation density.
III. RESULTS AND DISCUSSION

We employ a variational approach to study the effect of the symmetry energy on isovector properties of various medium and heavy nuclei. The method, even its simplicity, has the advantage that the dependence of the asymmetry function $\alpha(r)$ on the symmetry energy, and the Coulomb potential, are introduced explicitly. More specifically, the total energy density of the nucleus consists of the nuclear and Coulomb contributions. The nuclear term consists of two parts i.e. the symmetric nuclear matter and also the asymmetry energy one. In order to be able to study the effects of the symmetry energy we parameterized suitably the related expression. We study the dependence of the neutron skin thickness $R_{\text{skin}}$ and the asymmetry coefficient $a_A$ on the slope $L$ of the SE at the saturation density $\rho_0$. In the present method the asymmetry function $\alpha(r)$ is treated as a variational function and the total density $\rho(r)$ as a trial function. For each $\rho(r)$ the corresponding $\alpha(r)$ and the total energy are found. The process continues up to find the function $\rho$ which minimizes the total energy. All the relevant quantities, which are functionals of $\rho(r)$, $\alpha(r)$ and $S(\rho)$, are easily calculated.

The outline of our approach is the following: We start from the general relation $R = r_0 A^{1/3}$ which gives an averaged estimate of the nuclear radius and accordingly we consider a Fermi form for the total density distribution $\rho(r)$ \[85\]. Afterwards, for a fixed $\rho(r)$ the asymmetry function is rearranged accordingly so that the total energy of the nucleus is the lowest one.

It is worth to point out, following the discussion by Brueckner et.al. \[80\] that the energy density functional \[3\] breaks down at the edge of the nucleus for two reasons. Firstly, the Thomas-Fermi approximation, which is the basis of the present work, fails for low densities. Secondly at the edge of the nucleus the asymmetry function $\alpha(r)$ tends to unity and the potential contribution to the total energy functional is not accurate.

In Fig. 1, the symmetry energy versus the total density is plotted, according to Eq. (33) for various values of the slope parameter $L$. It is noted that lower values of $L$ correspond to higher values of $S(\rho)$. This behavior of $S(\rho)$ is well reflected on the values of the total binding energy $E_{\text{tot}}$ and the asymmetry function $\alpha(r)$. More precisely, higher values of $L$ lead to lower contribution of the $S(\rho)$ on the total binding energy and consequently the nucleons become more bound. For example in Table I are presented the results (concerning
the total binding energy $E_{\text{tot}}$, the proton $R_p$ and neutron $R_n$ rms radii, the neutron skin $R_{\text{skin}}$ and the asymmetry coefficient $a_A$) for the nucleus $^{208}\text{Pb}$ for the case $J = 30$ MeV and for $10 \text{ MeV} \leq L \leq 100 \text{ MeV}$. In addition, since the values of the density distribution inside the nucleus are lower than the value of the saturation density $\rho_0 = 0.16 \text{ fm}^{-3}$, it is concluded that the isovector properties of nuclei are related with the trend of the symmetry energy in the region $0 < \rho < \rho_0$ and vice-versa, that is the experimental isovector measurements give information for the lower part of the SE.

In Fig. 2 we plot the density distributions (total, proton and neutron) as well as the corresponding asymmetry function $\alpha(r)$ for various values of $L$ and two nuclei ($^{208}\text{Pb}$ and $^{48}\text{Ca}$.) The softness symmetry energy (higher values of $L$) shift the neutron distribution to the outer part of the nucleus, while at the same time it concentrates deeper the protons. This is clearly reflected both on the corresponding values of $R_p$ and $R_n$ as well as on the neutron skin $R_{\text{skin}}$ (see also table I). The effects of the symmetry energy is even more pronounced on the trend of the asymmetry function. Higher values of $L$ shift the cut-off radius $r_c$ to even more lower values increasing dramatically the neutron skin and forming a kind of neutron halo inside the nucleus. It is obvious from the above analysis that $S(\rho)$ and consequently, according to expression (26), the asymmetry function $\alpha(r)$ acts as a regulator on the proton and neutron distributions in order to minimize, in every case, the total energy of the nucleus. On the other hand, and accordingly (see also expression (26)) the Coulomb potential $V_c(r)$ acts inversely, compared to $S(\rho)$ and its main effect is to shift the proton distribution to the outer part of the nucleus. Actually, the interplay between the long range coulomb forces, the nuclear forces and mainly the isovector part of nuclear forces is responsible for the creation of the neutron skin thickness. However, although the Coulomb contribution is well defined, the contribution of the symmetry energy still remains an open problem even for low values of densities.

Fig. 3 displays the neutron skin $R_{\text{skin}}$ as a function of $L$ for various values of $J$ for the four studied neutron rich nuclei ($^{208}\text{Pb}$, $^{124}\text{Sn}$, $^{90}\text{Zr}$ and $^{48}\text{Ca}$). The most striking feature is, in all cases, the strong dependence of $R_{\text{skin}}$ on $L$. For a comparison, we include for the case of $^{208}\text{Pb}$ an approximate linear dependence

$$R_{\text{skin}}(\text{fm}) = 0.101 + 0.00147 \, L \, (\text{MeV}),$$  \hspace{1cm} (44)

established by Centelles et. al., using a wide range of non-relativistic and relativistic
models. It is obvious that relation (44) supports a softer dependence of $R_{\text{skin}}$ on $L$ compared to the present study. However we note that we present a systematic study of the effects of $L$ on $R_{\text{skin}}$ and in a large range of values of $L$ without trying to reproduce for example the experimental value of the binding energy or the charge radius of the specific nucleus. Even in this case, we found that the intersection between our results and the results compatible with (44) corresponds to values of binding energy very close to the experimental for the specific nuclei.

Very recently the Lead Radius Experiment (PREX) at the Jefferson Laboratory has provided the first model-independent evidence for the existence of a neutron-rich skin in $^{208}\text{Pb}$ \cite{66,67}. The determined neutron skin was $R_{\text{skin}} = 0.33^{+0.16}_{-0.18}$fm. However such a large error is not enough to constrain the various nuclear models. In addition the large determined neutron skin (compared to previous experimental measurements) creates a new open problem concerning the correlation between the nuclear equation of state of nuclear matter and the density functional theory in finite nuclei (see for a pertinent discussion in \cite{34,57}).

In Fig. 4 we display the coefficients $a_A$ as a function of $L$, for various values of $J$. It is obvious that $a_A$ is a decreasing function of $L$. Actually, for specific pairs of values of $N$ and $Z$, and according to Bethe-Weizsacker formula (1), a softer $S(\rho)$ (high values of $L$) leads to a lower value of $\alpha(r)$ (a property directly connected with the contribution of the symmetry energy to the total energy). Obviously, $a_A$ exhibits a mass depended $A$ behavior.

In order to impose some possible constraints on the values of $L$, we plot in Fig. 5(a) for each of the four nuclei $^{208}\text{Pb}$, $^{124}\text{Sn}$, $^{90}\text{Zr}$, and $^{48}\text{Ca}$ the pairs of $L$ and $J$ consistent with the corresponding empirical values of $a_A$ determined by the formula (13)

$$a_A^{-1} = (a_V)^{-1} + (a_S)^{-1} A^{-1/3}, \quad (45)$$

where we use for the volume and surface coefficients $a_V = 35.5$ MeV and $a_S = 9.9$ MeV respectively. It is seen in Fig. 5(a) that the set $J = 34$ MeV and $L = 65$ MeV reproduces very well the empirical values of $a_A$ for almost all the medium and heavy isotopes.

Two important features of the relation between $L$ and $J$ are useful. First, the inequality $4.13 \leq \Delta L/\Delta J \leq 5.18$ holds approximately. This means that a change of 1 MeV in the value of $J$ results in a corresponding change $4.13 \leq \Delta L \leq 5.18$ MeV. That is according to the present approach the accuracy on the measurements of $a_A$ and $J$ will impose strong constraints on the values of $L$. 

12
Second, since the four almost linear curves are arranged very close with a similar slope, we may conjecture that a possible universal dependence holds between $L$ and $J$ for nuclei at least in the mass region $A=40$-200. That means that the same set of $L$ and $J$, related with nuclear symmetry energy, reproduce in a very good accuracy the symmetry energy coefficient for medium as well as heavy nuclei. Especially for values of $J$ and $L$ in the region $34^{+0.2}_{-0.2}$ MeV and $65^{+1}_{-1}$ MeV respectively the accuracy is high.

In addition, in Fig. 5(b) we present the mass dependence of the coefficient $a_A$ of a several isotopes of Ca, Zr, Sn and Pb, for two sets of $L$ and $J$. For a comparison we include also the formula (45). The first set ($J = 32$ MeV and $L = 70$ MeV) reproduces on the average the binding energies of the corresponding isotopes while the second set ($J = 34$ MeV and $L = 65$ MeV) as we mentioned above, reproduces on the average the empirical values of $a_A$ for the isotopes $^{208}$Pb, $^{124}$Sn, $^{90}$Zr, and $^{48}$Ca.

In Fig. 6 we compare the allowed pairs of $L$ and $J$ constrained from heavy-ion collisions and nuclear structure observable [10] with those found in the present approach. Actually the present results lie inside the intersection area suggested by the measurements of the dipole polarizability $a_D$ as well as those found by heavy-ion collisions experiments. However, they lie outside the interval constrained by the nuclear masses measurements, connected with the binding energy, but only for $J < 33$ MeV. This is due to the fact that constraints between $L$ and $J$ are based on the adjustment of the theoretical to the empirical value of the asymmetry coefficient $a_A$ and not on the corresponding experimental values of the binding energies. It is remarkable that the allowed pairs of $L$ and $J$ consistent with our approach (see the colored lines in Fig. 6) lie at the intersection of the bands originating from other approaches [10].

Fig. 7(a) exhibits the dependence of the coefficient $a_A$ on the asymmetry parameter $I = (N - Z)/A$ for various isotopes, for the cases $J = 32$, $L = 70$ and $J = 34$, $L = 65$. In almost all cases there is a soft dependence of $a_A$ on $I$ but, as expected, a strong dependence on the value of $J$. Similarly, in Fig. 7(b) we indicate the dependence of the neutron skin on the asymmetry parameter $I$. The most characteristic trend is the occurrence of strong and linear dependence of $R_{\text{skin}}$ on $I$ that is

$$R_{\text{skin}} = a + bI$$

where the constants $a$ and $b$ vary in the intervals $-0.02 \leq a \leq 0.045$ and $1.31 \leq b \leq 1.45$. Of course those intervals are strongly dependent on the specific set of values of $L$ and $J$. For
a comparison a similar relation suggested in [69] is presented

\[ R_{\text{skin}} = (1.01 \pm 0.15)I + (-0.04 \pm 0.03), \]  

(47)

although in a number of analyzed cases the statistical errors are rather large.

In any case, additional experimental work is necessary to constrain the neutron skin [10]. In particular the upcoming Lead Radius Experiment II (PREX-II) promises to determine the neutron-skin thickness of \(^{208}\)Pb with a ±0.06 fm accuracy while the Calcium Radius Experiment (CREX) will provide a high precision measurement for the neutron radius of \(^{48}\)Ca with accuracy ±0.02 fm [10]. The above measurements will be a very good test for all the theoretical models including the present variational approach.

IV. CONCLUSIONS

In the present work we employ a variational method, in the framework of the local density approximation, in order to study the symmetry energy effects on isovector properties of various neutron rich nuclei. The key quantity of the present approximation is the asymmetry function \(\alpha(r)\) naturally computed by the variational principle. Actually, \(\alpha(r)\) is a functional both of the symmetry energy as well as the Coulomb potential, it contains the interplay between the long-range Coulomb forces and short-range nuclear ones and it defines the density distribution of neutrons and protons. All the calculated properties are studied as a function of the slope of the symmetry energy and the value of the symmetry energy at the nuclear saturation density. Since, the SE even for low values of nuclear matter is uncertain, the above parameterization is necessary. We find that the neutron skin thickness is very sensitive to \(L\) i.e. it increases rapidly with \(L\). This is expected at least in the present approximation, since the main ingredient of the relevant calculated integrals, the function \(\alpha(r)\) approaches unity very rapidly close to the critical value of \(r_c\) (at the surface of the proton distribution). The above characteristic behavior of \(\alpha(r)\) is well reflected on the asymmetry coefficient \(a_A\) which is a decreasing function of \(L\). In the case of \(^{208}\)Pb we compare our results with those originating from additional calculations with different models. We conclude that the present approximation supports a stronger sensitivity of the neutron skin thickness on \(L\). However, constraining the total binding energy to be close to the experimental one we see that our results are very close to the mentioned empirical formula.
Our findings, from the present study, show that the experimental knowledge of the symmetry energy at the saturation density \( J \) will impose, via the values of the symmetry coefficient \( a_A \), strong constraints on \( L \). More specifically, we note that the set \( J = 34, \ L = 65 \) reproduces very well the empirical values of \( a_A \) corresponding to the nuclei under consideration. In any case, further experimental and theoretical work is necessary for a more detailed exploration of the effects of the symmetry energy on the properties of finite nuclei as well as on the neutron star structure.

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TABLE I: The slope parameter $L$ (in MeV), the binding energy $E$ (in MeV), the neutron radius $R_n$ (in fm), the proton radius $R_p$ (in fm), the neutron skin thickness $R_{\text{skin}}$ (in fm) and the asymmetry coefficient $a_A$ (in MeV) calculated for a fixed value $J = 30$ MeV.

| $L$ | $E$     | $R_n$ | $R_p$ | $R_{\text{skin}}$ | $a_A$ |
|-----|---------|-------|-------|-------------------|-------|
| 10  | -1581.36| 5.629 | 5.623 | 0.006             | 28.52 |
| 20  | -1593.44| 5.659 | 5.604 | 0.055             | 26.77 |
| 30  | -1606.35| 5.695 | 5.586 | 0.109             | 24.85 |
| 40  | -1620.28| 5.723 | 5.560 | 0.163             | 22.90 |
| 50  | -1633.62| 5.756 | 5.537 | 0.219             | 20.87 |
| 60  | -1646.40| 5.780 | 5.517 | 0.263             | 19.03 |
| 70  | -1658.36| 5.803 | 5.500 | 0.303             | 17.32 |
| 80  | -1669.35| 5.816 | 5.478 | 0.338             | 15.80 |
| 90  | -1679.37| 5.828 | 5.458 | 0.370             | 14.41 |
| 100 | -1688.46| 5.846 | 5.448 | 0.398             | 13.11 |
FIG. 1: The nuclear symmetry energy $S(\rho)$ as a function of the density $\rho$ for various values of the slope parameter $L$ and the specific value $J = 30$ MeV.
FIG. 2: The density distribution of neutrons, protons and the total one (2(a) and 2(c)) for $^{208}$Pb and $^{48}$Ca for three values of $L$ (figures 2(a) and 2(c)) and the corresponding asymmetry functions $\alpha(r)$ for a variety of values of $L$ (figures 2(b) and 2(d)).
FIG. 3: The neutron skin $R_{\text{skin}}$ as a function of the symmetry energy slope $L$, for various values of $J$. 
FIG. 4: The asymmetry coefficient $a_A$ as a function of the symmetry energy slope $L$ for various values of the parameter $J$. 
FIG. 5: (a) The plot of the pairs $L$ and $J$ which reproduce the empirical value of $a_A$ given by (45) for four nuclei. (b) The asymmetry coefficients $a_A$ as a function of $A$ for the relevant isotopes and for the set $L = 70, J = 32$ and $L = 65, J = 34$. The solid line corresponds to the empirical formula (45) (for more details see text).
FIG. 6: Regions of allowed values of pairs \( J \) and \( L \) (three bands) constrained from heavy-ion collisions (HIC(Sn+Sn) case) and nuclear structure observables (\( a_D \) and nuclear masses) (for more details see Ref. [10]) in comparison with the corresponding results constrained from the present approach. The solid, dashed and dotted arrows indicate constraints related with \( a_D \), nuclear masses and heavy-ion collisions respectively. The four colored lines intersecting at the cross show the dependence of \( L \) on \( J \) according to Fig. 5(a) of our present work. The star and the cross correspond to the set \( L = 70, J = 32 \) and \( L = 65, J = 34 \) respectively (for more details see text).

FIG. 7: (a) The asymmetry coefficient \( a_A \) as a function of the asymmetry parameter \( I \) for various isotopes. (b) The corresponding neutron skin \( R_{\text{skin}} \) dependence on \( I \).