The Use of Analyticity in the $\pi\pi$ and $K\bar{K}$ Coupled Channel System

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Abstract

Studies on the IJ=00 $\pi\pi$ and $K\bar{K}$ coupled–channel system are made using newly derived dispersion relations between the phase shifts and poles and cuts. It is found that the $\sigma$ resonance must be introduced to explain the experimental phase shifts, after evaluating the cut contribution. The effects of nearby branch point singularities to the determination of the $f_0(980)$ resonance are also carefully clarified.

The revived interests in the $\sigma$ particle has stimulated many investigations in recent years [1]. The difficulty in answering the question whether there exists the $\sigma$ particle comes from the fact that the $\sigma$ particle, if exist, must be a very broad resonance as indicated by the broad enhancement of the experimental phase shifts below 1GeV in the IJ=00 channel. However it is difficult to distinguish the contribution of a broad resonance from the contribution of the left hand cut both theoretically and experimentally. In our point of view it is difficult to recognize the existence of the $\sigma$ meson before one can seriously estimate the left hand cut effects, though the $\sigma$ resonance can be naturally generated from many dynamical models producing unitarized partial wave amplitudes fit appropriately to the experimental data.

In two recent papers [2, 3] the present authors devoted to the study on the influence of left hand cuts to the determination of the $\sigma$ meson. For the purpose of separating the cut contribution from the pole contribution to the $S$ matrix we developed a dispersion relation [4] which reads,

$$\sin(2\delta_\pi) = \rho \left( \sum_i \frac{\text{Res}[\mathcal{F}(z_i^{II})]}{s - z_i^{II}} + \frac{1}{\pi} \int_{-\infty}^{0} \frac{\text{Im}_L \mathcal{F}}{s' - s} ds' + \frac{1}{\pi} \int_{4m_K^2}^{\infty} \frac{\text{Im}_R \mathcal{F}}{s' - s} ds' \right) , \quad (1)$$

where $\delta_\pi$ is the $\pi\pi$ phase shift in the single channel unitarity region, and $z_i^{II}$ denotes the resonance pole position on the complex $s$ plane. The function $\mathcal{F}$ is the analytic continuation of twice of the real part of the $\pi\pi$ scattering $T$ matrix (defined in the physical region) on the complex $s$ plane, or in short-hand notation, $\mathcal{F} = 2\text{Re}_R T_{\pi\pi}$ in the physical region, and $\rho$ is the kinematic factor, $\rho(s) = \sqrt{1 - 4m_\pi^2/s}$. Other

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1Summary of talks given by Z.X. at International Conference on Flavor Physics 2001, Zhangjiajie, Hunan, May 30th, 2001; given by H.Z. at BES annual meeting, Jixian, Tianjin, June 11th, 2001 and at Eurodaphne workshop on Nonperturbative Methods in Chiral Theories, Valencia, Spain, June 28th, 2001.

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3Assuming however no bound state exists.
relations which are helpful in understanding Eq. (1) are,
\[
\sin(2\delta_\pi) = \rho \mathcal{F}, \quad \text{Res} [\mathcal{F}(z_i)] = \frac{i}{2\rho(z_i) S'(z_i)}.
\]

The right hand integral appeared on the \textit{r.h.s.} of Eq. (1) is absent in the single channel approximation. In the \pi\pi, \bar{K}K coupled channel case one has
\[
\text{Im}_R \mathcal{F} = \frac{1}{\eta - \eta} \cos(2\delta_\pi),
\]
and hence the right hand integral in Eq. (1) can be evaluated using available experimental data at higher energies. However its contribution is found to be very small even though the right hand integral develops a cusp structure below the \bar{K}K threshold. In Eq. (3) the definition of \delta_\pi and \eta comes from the standard parametrization of the \(S\) matrix in the coupled channel unitarity region,
\[
S = \left( \frac{\eta e^{2i\delta_\pi}}{i \sqrt{1 - \eta^2 e^{i(\delta_\pi + \delta_K)}}}, \frac{i \sqrt{1 - \eta^2 e^{i(\delta_\pi + \delta_K)}}}{\eta e^{2i\delta_K}} \right).
\]

It is important to realize that the \pi\pi phase shift appeared in Eq. (4) and Eq. (3) is defined in the coupled channel region and is different from the \delta_\pi appeared in Eq. (1). Before going to the coupled channel case let us focus on Eq. (1) for the purpose of clarifying the essential characters of the \(\sigma\) resonance in a simpler way. Our analysis revealed that neglecting the effects of the \bar{K}K threshold and the \(f_0(980)\) narrow resonance does no harm to the qualitative understanding of the \(\sigma\) meson \[4\].

The simplified version of Eq. (1), i.e., without the right hand integral and the \(f_0^{II}(980)\) resonance looks like the following,
\[
\sin(2\delta_\pi(s)) = \rho \left( \sum_{z_i=\pi^+,\pi^-} \frac{i/2\rho(z_i)}{S'(z_i)(s - z_i)} + a + \frac{s - m_\pi^2/2}{\pi} \int_{-\infty}^0 \frac{\text{Im}_L \mathcal{F}}{(s' - m_\pi^2/2)(s' - s)} ds' \right).
\]

In Eq. (5) we have on the \textit{l.h.s.} the experimental data of \(\sin(2\delta_\pi)\) truncated at certain scale below the \bar{K}K threshold in order to exclude the threshold and the narrow \(f_0^{II}\) effects. On the \textit{r.h.s.} we have the \(\sigma\) pole term put by hand hence one has to demonstrate that such a term is really needed. However in order to make such a demonstration possible one has to be able to estimate the left hand integral in a reliable way, at least in the qualitative sense. The integral is once subtracted at the point which happens to be the Adler zero of the \(IJ=00\) lowest order chiral amplitude. The subtraction constant, \(a\), is not fixed by dispersion theory. The way we estimate the left hand integral is to use the 1–loop chiral perturbation theory \((\text{CHPT})\) result on \(\text{Im}_L \mathcal{F}\) and truncate the integral at certain scale \(\Lambda\) above which CHPT results become no longer trustworthy. Implicit in our approximation is the assumption that physics above the scale \(\Lambda\) does not influence the low energy physics at qualitative level. One may argue that the major contribution from high energies is already included in the subtraction constant which has to be determined by the fit, rather than the theoretical calculation here. We also make use of the \([1,1]\) Padé approximant of CHPT to estimate the left hand integral, as inspired by a series
of recent efforts [3]. Of course, the use of the unitarized amplitude automatically regularize the high energy contribution to the left hand integral. We bear in mind that none of these estimates is perfect in the eye of a perfectionist.

Having estimated the left hand integral contribution to the physical observable, \( \sin(2\delta_\pi) \), it becomes possible now to address the problem whether the \( \sigma \) resonance is really needed. The situation is nicely summarized in fig. 1 which is a modified version of fig. 6 in Ref. [2], that is in fig. 1 the effect of the kinematic factor \( \rho \) is removed. It is clearly shown in fig. 1 that the experimental data exhibit a very broad peak or an enhancement which can not be explained by the dynamical cut effects: the latter is concave irrespective of the different choice of the cutoff parameter. Actually, the derivative of the cut contribution to \( \mathcal{F} \) with respect to \( s \) take the following form,

\[
\frac{d}{ds} \mathcal{F}|_{\text{cut}} = \frac{1}{\pi} \int_{-\Lambda^2}^{0} \frac{\text{Im} L \mathcal{F} \left( s' - s \right)}{(s' - s)^2} ds' , \quad \frac{d^2}{ds^2} \mathcal{F}|_{\text{cut}} = \frac{2}{\pi} \int_{-\Lambda^2}^{0} \frac{\text{Im} L \mathcal{F} \left( s' - s \right)}{(s' - s)^3} ds' .
\]

Since both the CHPT prediction and the [1,1] Padé prediction on the sign of \( \text{Im} L \mathcal{F} \) are always negative within a reasonable range of the \( \Lambda \) parameter, it is clear from the above expressions that the cut contribution to \( \mathcal{F} \) must be concave.

![Figure 1: A typical fit of 5 parameters (4 resonance + 1 subtraction constant) in the I=J=0 channel, with \( \Lambda_{\chi PT} = 0.7 \text{GeV} \). Different estimates on the left–hand integral are also plotted: The dotted line corresponds to \( \Lambda_{\chi PT} = 0.6 \text{GeV} \), the dashed line corresponds to \( \Lambda_{\chi PT} = 0.7 \text{GeV} \), the dot–dashed line corresponds to \( \Lambda_{\chi PT} = 0.85 \text{GeV} \) and the solid line corresponds to the Padé solution.](image)

Therefore it becomes unavoidable to call for the \( \sigma \) resonance which turns over the curve in fig. 1. In the fit by using Eq. [3], the derivative of the \( S \) matrix at
the pole position is taken as a free parameter. The uncertainty of our fit result at quantitative level mainly comes from the uncertainty in the estimation of the left hand cut contribution. However, one can still manage to determine the location of the $\sigma$ pole within a reasonable range, by varying the $\Lambda$ parameter. It is worth noticing that the inclusion of the left hand cut drives the $\sigma$ pole moving towards left on the complex $s$ plane [6], though the effect is not very strong. In the fit we find that the recent $K_{e4}$ data from the E865 collaboration [7] is crucial in reducing the magnitude of the scattering length parameter towards the chiral result [8]. But we point out here that the global fit still favors a somewhat larger value of the scattering length parameter $a_0^0$.

In order to discuss the properties of another interesting resonance named $f_0(980)$, it is appropriate to go to the coupled channel region. In such case we can also write down a series of dispersion relation which are similar to Eq. (1),

\[
\begin{align*}
(\eta + \frac{1}{\eta}) \sin(2\delta_\pi) &= \rho_1 \left( \Phi_{11}(s) + \frac{1}{2\pi i} \int_L Disc \left( T\mathbf{C} \right)_{11} dz \right), \\
(\eta + \frac{1}{\eta}) \sin(2\delta_K) &= \rho_2 \left( \Phi_{22}(s) + \frac{1}{2\pi i} \int_L Disc \left( T\mathbf{C} \right)_{22} dz \right), \\
\sqrt{1 - \eta^2} \left( \cos(\delta_\pi + \delta_K) + \frac{1}{\eta} \cos(\delta_\pi - \delta_K) \right) &= \sqrt{\rho_1 \rho_2} \left( \Phi_{12}(s) + \frac{1}{2\pi i} \int_L Disc \left( T\mathbf{C} \right)_{12} dz \right)
\end{align*}
\]

in which the matrix function $T\mathbf{C}$ is the sum of the T matrices defined on different sheets:

\[
T\mathbf{C}(z) \equiv T^I(z) + T^{II}(z) + T^{III}(z) + T^{IV}(z).
\]

In Eq. (7) the matrix function $\Phi$ represents the sum over all possible pole contributions on different sheets. The left hand integrals appeared in Eq. (7) reflects the effect of the left hand cut in $T_{\bar{K}K}$ generated by $t$ channel $2\pi$ exchanges: it starts from $4m_K^2 - 4m_\pi^2$ to $-\infty$. We can immediately draw some important conclusions by comparing Eq. (1) with Eq. (7):

1. The experimental data below the $\bar{K}K$ threshold only contribute to the determination of the second sheet pole, only the data above the second threshold contribute to the determination of the 3rd and/or 4th sheet pole. This simple observation explains the reason why the third sheet pole found from various fits differ so much in the literature whereas the results on the second sheet $f_0^{II}(980)$ narrow resonance agree with each other qualitatively: the second sheet pole is much easier to fix by experimental data. Especially for $f_0^{II}(980)$, it is almost uniquely determined by the data which are very close but below the $\bar{K}K$ threshold.

2. The fit above the $\bar{K}K$ threshold will be polluted by the uncertainty of the left hand cut at $(\infty, 4m_K^2 - 4m_\pi^2)$, but the cut is expected to be smooth.

\footnote{The function $\Phi$ defined in the following differs by a sign to that in Ref. [3].}
function in the absence of nearby narrow 3rd or 4th sheet poles. Especially the cut influence is not important and may be neglected when only discussing the second sheet poles. A typical dynamical assumption made in dynamical models like in the case of using Lippman–Schwinger equation is that one only takes care of the $s$ channel force and neglects the crossed channel forces by assuming them to behave mildly. Similar assumptions occurred in the more phenomenological $K$ matrix fits. In some sense our above analysis justifies the commonly used assumption for neglecting the left hand cut provided that there is no narrow 3rd or 4th sheet pole close to the branch point $4m_K^2 - 4m_{\pi}^2$. The latter condition seems to be indeed satisfied in the $IJ=00 \pi\pi$ and $\bar{K}K$ coupled channel system.

To end the discussion we quote the estimated value of the pole positions of the $\sigma$ and $f_0^{II}(980)$ resonances, which may be a little bit optimistic:

$$
M_\sigma = 478 \sim 500\text{MeV}, \quad \Gamma_\sigma = 480 \sim 550\text{MeV};
$$

$$
M_{f_0^{II}} = 982 \sim 990\text{MeV}, \quad \Gamma_{f_0^{II}} = 35 \sim 37\text{MeV}.
$$

(9)

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