Killing Weights from the Perspective of $t$-Structures

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Dedicated to Alexey Nikolaevich Parshin, whose memories will live on long after he has passed

Abstract—This paper is devoted to morphisms killing weights in a range (as defined by the first author) and to objects without these weights (as essentially defined by J. Wildeshaus) in a triangulated category endowed with a weight structure $w$. We describe several new criteria for morphisms and objects to be of these types. In some of them we use virtual $t$-truncations and a $t$-structure adjacent to $w$. In the case where the latter exists, we prove that a morphism kills weights $m, \ldots, n$ if and only if it factors through an object without these weights; we also construct new families of torsion theories and projective and injective classes. As a consequence, we obtain some “weakly functorial decompositions” of spectra (in the stable homotopy category $\text{SH}$) and a new description of those morphisms that act trivially on the singular cohomology $H^0_{\text{sing}}(-, \Gamma)$ with coefficients in an arbitrary abelian group $\Gamma$.

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INTRODUCTION

This paper is a complement to [6], whose central notion was that of morphism killing weights $m, \ldots, n$ in a triangulated category $\mathcal{C}$ endowed with a weight structure $w$; here $m \leq n \in \mathbb{Z}$. One says that an object $M$ of $\mathcal{C}$ is without these weights if $\text{id}_M$ kills the weights $m, \ldots, n$, and it was proved in [6] that this version of the notion is closely related to the original definition of J. Wildeshaus. One may consider these notions in the important case of $\mathcal{C} = \text{SH}$ and $w = w_{\text{sph}}$ (this is the spherical weight structure on the stable homotopy category); then $M$ is without weights $m, \ldots, n$ whenever its singular homology $H^m_{\text{sing}}(M, \mathbb{Z})$ vanishes in these degrees and $H^{m-1}_{\text{sing}}(M, \mathbb{Z})$ is a free abelian group.

In the present paper we establish some new criteria for a $\mathcal{C}$-morphism to kill weights $m, \ldots, n$ (in Theorem 2.2) and for the absence of these weights in $M \in \text{Obj} \mathcal{C}$ (see Theorem 2.5). In particular, if $w = w_{\text{sph}}$, then a spectrum $X$ is without weights $m, \ldots, n$ if and only if it possesses a cellular tower with $X^{(m-1)} = X^{(n)}$ (these are the corresponding skeleta of $X$; see [11, Ch. 6, Sect. 3]). In contrast to [6], we employ weight filtrations and virtual $t$-truncations of cohomology in some of our criteria.

Moreover, we study the case where $\mathcal{C}$ is also endowed with a $t$-structure $t$ adjacent to $w$; note that $t$ of this sort exists for $w = w_{\text{sph}}$, and if $t$ exists then virtual $t$-truncations of representable functors are represented by the corresponding $t$-truncations. In this case a morphism $g$ kills weights $m, \ldots, n$ if and only if $g$ factors through an object without these weights (see Theorem 3.1). Moreover, Theorem 3.1 gives certain torsion theories and projective and injective classes in $\mathcal{C}$ (these notions were defined in [10] and [9], respectively); this yields some more criteria for killing weights and the absence of weights.

In Theorem 3.5 we discuss the application of these results to the case where $w$ is purely compactly generated; this includes the case $w = w_{\text{sph}}$. In particular, for an SH-morphism $g$ we have

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$H^0_{\text{sing}}(g, \Gamma) = 0$ for every abelian group $\Gamma$ (that is, $g$ annihilates the zeroth singular cohomology with coefficients in $\Gamma$) if and only if $g$ factors through a spectrum $X$ such that all $H^0_{\text{sing}}(X, \Gamma)$ vanish (see item 1 in Remark 3.6). Furthermore, this statement naturally extends to the equivariant stable homotopy category $\text{SH}(G)$ (see item 2 in Remark 3.6).

1. PRELIMINARIES

In this section we recall several relevant definitions and statements. We do not prove anything new.

1.1. Some categorical notation.

- Let $C$ be a category and $X,Y \in \text{Obj} \ C$. Then we will write $C(X,Y)$ for the set of morphisms from $X$ to $Y$ in $C$.
- We say that $X$ is a retract of $Y$ if $\text{id}_X$ can be factored through $Y$.
- A subcategory $P$ of an additive category $C$ is said to be retraction-closed in $C$ if $P$ contains all retracts of its objects in $C$.
- The symbol $C$ below will always denote some triangulated category; usually it is endowed with a weight structure $w$. Moreover, $w$ will always denote a weight structure on $C$, and $t$ is a $t$-structure on $C$ (see items 1 and 2 in Definition 1.2 below). By $A$ we will denote some abelian category.
- For any $A,B,C \in \text{Obj} \ C$ we say that $C$ is an extension of $B$ by $A$ if there exists a distinguished triangle $A \to C \to B \to A[1]$.
- For $X,Y \in \text{Obj} \ C$ we write $X \perp Y$ if $C(X,Y) = \{0\}$. If $D$ and $E$ are classes of objects or subcategories of $C$, then we will write $D \perp E$ if $X \perp Y$ for all $X \in D$ and $Y \in E$. Moreover, we write $D^\perp$ for the class $\{Y \in \text{Obj} \ C : X \perp Y \forall X \in D\}$;
  dually, $^\perp D$ is the class $\{Y \in \text{Obj} \ C : Y \perp X \forall X \in D\}$.
- For an object $M$ of $C$ we will write $H_M$ for the functor $C(-, M)$.

1.2. On various torsion theories and projective classes. Our central definition is the following one.

**Definition 1.1.** A pair $s$ of classes $\mathcal{LO}, \mathcal{RO} \subset \text{Obj} \ C$ is said to be a torsion theory (on $C$) if $\mathcal{LO}^\perp = \mathcal{RO}, \mathcal{LO} = ^\perp \mathcal{RO}$, and for every $M \in \text{Obj} \ C$ there exists a distinguished triangle

$$L_sM \xrightarrow{a_M} M \xrightarrow{n_M} R_sM \to L_sM[1]$$

(1.1)

such that $L_sM \in \mathcal{LO}$ and $R_sM \in \mathcal{RO}$. We will call any triangle of this form an $s$-decomposition of $M$.

We also need a collection of related definitions.

**Definition 1.2.** Let $s$ be a torsion theory and $n \in \mathbb{Z}$.

1. We will say that $s$ is weighted if $\mathcal{LO} \subset \mathcal{LO}[1]$. In this case we call the pair $w = (\mathcal{LO}, \mathcal{RO}[-1])$ a weight structure and say that $s$ is associated with $w$. We write $C_{w \leq 0}$ and $C_{w \geq 0}$ for $\mathcal{LO}$ and $\mathcal{RO}[-1]$, respectively; furthermore, we set $C_{w \leq n} = C_{w \leq 0}[n]$ and $C_{w \geq n} = C_{w \geq 0}[n]$.

2. If $\mathcal{LO} \subset \mathcal{LO}[-1]$ then we call the pair $t = (C_{t \leq 0} = \mathcal{RO}[1], C_{t \geq 0} = \mathcal{LO})$ a $t$-structure and say that $s$ is associated with $t$. We set $C_{t \leq n} = C_{t \leq 0}[n]$ and $C_{t \geq n} = C_{t \geq 0}[n]$. Moreover, if $m \leq n$ then we set $C_{[m,n]} = C_{[m,n]} \cap C_{[t \geq m]}$.

3. We say that a weight structure $w$ and a $t$-structure $t$ (as above) are adjacent if $C_{w \geq 0} = C_{t \geq 0}$.

4. We will say (following [14, Definition 3.1]) that $s$ is generated by $P \subset \text{Obj} \ C$ if $P^\perp = \mathcal{RO}$.
**Remark 1.3.** 1. Our definition of torsion theory actually follows [14, Definition 3.2] (where torsion theories were called complete Hom-orthogonal pairs) and is somewhat different from Definition 2.2 of [10], from which our term comes from. However, these two definitions are well known to be equivalent (see assertions 2 and 9 of [8, Proposition 2.4] as well as item 1 of [8, Remark 2.5]).

2. Our definitions of weight structures and t-structures are equivalent to those given in [6] and [5] (see [8, Proposition 3.2]). Note however that in [5] the so-called homological conventions for weight structures and t-structures were used, and they differ from the original ("cohomological") conventions introduced in [1] and [2] (cf. item 4 of Remark 1.2.4 and item 3 of Remark 2.2.3 in [5]).

3. Dually to item 3 in Definition 1.2, one can also consider the case where \( C_{w>0} = C_{t<0} \) (cf. item 6 of Definition 2.2.2 in [5], where both types of this relation between \( w \) and \( t \) are treated). However, we will not need this dual setting in the present paper.

Let us recall some more definitions.

**Definition 1.4.** Assume \( \mathcal{P} \subset \text{Obj} \mathcal{C} \) and \( \mathcal{J} \subset \text{Mor} \mathcal{C} \).

1. We will say that a \( \mathcal{C} \)-morphism \( h \) is \( \mathcal{P} \)-null (\( \mathcal{P} \)-conull) whenever for all \( M \in \mathcal{P} \) we have \( \mathcal{C}(M, h) = 0 \) (respectively, \( \mathcal{C}(h, M) = 0 \)).

2. The pair \( (\mathcal{P}, \mathcal{J}) \) is called a projective class whenever the following conditions are fulfilled:
   (i) \( \mathcal{J} \) is the class of all \( \mathcal{P} \)-null morphisms;
   (ii) for \( M \in \text{Obj} \mathcal{C} \) the functor \( \mathcal{C}(M, -) \) annihilates all elements of \( \mathcal{J} \) if and only if \( M \in \mathcal{P} \);
   (iii) for every \( M \in \text{Obj} \mathcal{C} \) there exists a distinguished triangle
   \[
   PM \xrightarrow{i_M} M \xrightarrow{j_M} IM \to PM[1]
   \]
   such that \( PM \in \mathcal{P} \) and \( j_M \in \mathcal{J} \).

3. We say that \( (\mathcal{P}, \mathcal{J}) \) is an injective class if it becomes a projective class in the category \( \mathcal{C}^{\text{op}} \).

**1.3. On weight range and virtual t-truncations.** Let us recall some properties of truncations with respect to weight structures and t-structures. Recall that by \( w \) and \( t \) we always denote a weight structure and a t-structure on \( \mathcal{C} \), respectively.

**Proposition 1.5.** Let \( M, X, Y \in \text{Obj} \mathcal{C} \).

1. The decomposition triangle (1.1) is determined by \( M \) functorially up to a canonical isomorphism if \( s \) is associated with \( t \) (see item 2 in Definition 1.2).

   Consequently, if \( n \in \mathbb{Z} \) and \( N = M[n] \), then both \( t_{\leq n-1}N = R_sM[n] \) and \( t_{\geq n}N = L_sM[n] \) (we will call these objects t-truncations of \( N \)) are functorially determined by \( N \).

2. Moreover, if \( m \leq n \), then \( t_{\leq n}(C_{t\geq m}) = C_{[m, n]} \).

3. Assume that \( s \) is associated with \( w \) (see item 1 in Definition 1.2). For every \( n \in \mathbb{Z} \) and \( N = M[n] \) we set \( w_{\leq n}N = L_sM[n] \) and \( w_{\geq n+1}N = R_sM[n] \). Then for any \( m \leq l \in \mathbb{Z} \) any \( g \in \mathcal{C}(X, Y) \) can be extended to a morphism of the corresponding distinguished triangles (cf. item 1 in Remark 1.6 below):

   \[
   \begin{array}{ccc}
   w_{\leq m}X & \xrightarrow{g} & w_{\geq m+1}X \\
   \downarrow & & \downarrow \\
   w_{\leq l}Y & \xrightarrow{g} & w_{\geq l+1}Y
   \end{array}
   \]

Moreover, if \( m < l \), then this extension is unique provided that the rows are fixed.

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1Recall also that D. Pauksztello has introduced weight structures independently (in [13]); he called them co-t-structures.
Proof. 1. This is well known; see [1, Proposition 1.3.3] (yet pay attention to item 2 in Remark 1.3).

2. By [1, Proposition 1.3.5], \( t_{\leq n}(C_{t\geq m}) \) lies in \( C[m, n] \). Next, Proposition 1.3.3 of [1] implies that the restriction of the functor (see assertion 1) \( t_{\leq n} \) to \( C_{t\leq n} \) is the identity functor. Consequently, \( C[m, n] = t_{\leq n}(C[m, n]) \subset t_{\leq n}(C_{t\geq m}) \).

3. This is essentially assertions 1 and 2 of [2, Lemma 1.5.1] (yet pay attention to item 2 in Remark 1.3). \( \square \)

Remark 1.6. 1. The upper row in (1.3) is called an \( m \)-weight decomposition of \( X \); we will say that \( w_{\leq m}X \) and \( w_{\geq m+1}X \) are weight truncations of \( X \). It is easily seen that this triangle is not canonically determined by \( (w, X) \). Yet it will be convenient for us to use the indicated notation below. Moreover, when we write arrows of the type \( w_{\leq m}X \to X \) or \( X \to w_{\geq m+1}X \), we will always assume that they come from some \( m \)-weight decomposition of \( X \).

2. Assertion 3 of Proposition 1.5 says that \( m \)-weight decompositions are “weakly functorial,” that is, a morphism \( g \) between objects extends to a morphism between any choices of their \( m \)-weight decompositions. According to assertion 8 of [8, Proposition 2.4], this property is fulfilled for \( s \)-decomposition triangles (see (1.1)) as well.

Now we pass to weight filtrations and virtual \( t \)-truncations.

Definition 1.7. Let \( H \) be a contravariant functor from \( C \) into \( A \) (where \( A \) is an abelian category), \( m \leq n \in \mathbb{Z} \), and \( M \in \text{Obj}_C \).

1. We define the weight filtration on \( H(M) \) as
   \[
   W^m(H)(M) = \text{Im}(H(w_{\geq m}M) \to H(M));
   \]
   here we take an arbitrary choice of \( w_{\geq m}M \) and use the convention described in Remark 1.6.

2. We define the correspondence (cf. assertion 1 of Proposition 1.8 below) \( \tau_{\leq m}(H) \) as
   \[
   M \mapsto \text{Im}(H(w_{\leq m+1}M) \to H(w_{\leq m}M));
   \]
   here we make arbitrary choices of \( w_{\leq m}M \) and \( w_{\leq m+1}M \), and take \( X = Y = M \) and \( g = \text{id}_M \) in (1.3).

3. If \( H \) is cohomological (that is, it converts distinguished triangles into long exact sequences), we will say that it is of weight range \( \geq m \) if it annihilates \( C_{w\leq m-1} \). We will say that \( H \) is of weight range \( [m, n] \) if it annihilates \( C_{w\geq n+1} \) as well.

Proposition 1.8. In the notation of the previous definition the following statements are valid.

1. The objects \( W^mH(M) \) and \( \tau_{\leq m}(H) \) are \( C \)-functorial in \( M \) (for every \( m \); in particular, they essentially do not depend on the choices of the corresponding weight decompositions of \( M \)).

2. If \( H \) is cohomological, then the functor \( \tau_{\leq m}(H) \) is also cohomological.

3. The functor \( H_M = C(-, M) \) is of weight range \( \geq m \) if and only if \( M \in C_{w\geq m} \).

4. If \( H \) is of weight range \( \geq m \), then \( \tau_{\leq n}(H) \) is of weight range \( [m, n] \).

5. If \( H \) is of weight range \( [m, n] \), then the morphism \( H(w_{\geq m}M) \to H(M) \) is surjective and the morphism \( H(M) \to H(w_{\leq n}M) \) is injective (here we make arbitrary choices of the corresponding weight decompositions of \( M \) and apply \( H \) to their connecting morphisms).

6. Assume that there exists a \( t \)-structure \( t \) adjacent to \( w \). Then the functor \( \tau_{\leq m}(H_M) \) is represented by \( t_{\leq m}M \).

Proof. Assertions 1, 2, and 6 were established in [2]: see assertion 2 of Proposition 2.1.2, assertions I and III.2 of Proposition 2.5.1, and assertion 8 of Theorem 4.4.2 there (and mind the difference in notation).

Assertion 3 is an immediate consequence of our definitions.

Assertions 4 and 5 are given by assertions 8 and 12 of [3, Proposition 2.1.4]. \( \square \)
2. ON MORPHISMS KILLING WEIGHTS AND OBJECTS WITHOUT WEIGHTS IN A RANGE

In this section we recall the central definitions of [6]. We also prove some new criteria for killing weights in terms of virtual $t$-truncations and weight filtrations (see Theorem 2.2) and for the absence of weights $m, \ldots, n$ in objects (see Theorem 2.5).

2.1. On morphisms killing weights: Definitions and new criteria.

Definition 2.1. Let $m \leq n \in \mathbb{Z}$. We say that a morphism $g \in \mathcal{C}(M, N)$ in a weighted category kills weights $m, \ldots, n$ whenever $g$ satisfies the following equivalent conditions (see [6, Proposition 2.1.1]).

1. There exists a choice of $w \leq n M$ and $w \leq m - 1 N$ along with the corresponding connecting morphisms (see item 1 in Remark 1.6) and a morphism $h$ making the following square commutative:

$$
\begin{array}{ccc}
  w_{\leq n} M & \xrightarrow{x} & M \\
  \downarrow h & & \downarrow g \\
  w_{\leq m-1} N & \longrightarrow & N
\end{array}
$$

(2.1)

2. There exists a choice of $w \leq n M$ and $w \geq m N$ such that the corresponding composed morphism $w_{\leq n} M \xrightarrow{x} M \xrightarrow{g} N \xrightarrow{y} w_{\geq m} N$ is zero.

3. Any choice of an $n$-decomposition triangle for $M$ and of an $(m-1)$-decomposition triangle for $N$ can be completed to a diagram of the form

$$
\begin{array}{ccc}
  w_{\leq n} M & \xrightarrow{x} & M & \longrightarrow & w_{\geq n+1} M \\
  \downarrow h & & \downarrow g & & \downarrow \\
  w_{\leq m-1} N & \longrightarrow & N & \xrightarrow{y} & w_{\geq m} N
\end{array}
$$

(2.2)

We will write $\text{Mor}_{[m,n]}^{[\mathcal{C}]}$ for the class of $\mathcal{C}$-morphisms that kill weights $m, \ldots, n$.

Now we relate these conditions to weight ranges of functors (see Definition 1.7).

Theorem 2.2. Adopt the notation of Definition 2.1. Then the following conditions are equivalent:

1. $g$ kills weights $m, \ldots, n$;

2. $H(g)$ sends $W^m(H)(N)$ inside $W^{n+1}(H)(M)$ for every contravariant functor $H: \mathcal{C} \rightarrow \mathcal{A}$;

3. $H(g)$ sends $W^m(H_I)(N)$ inside $W^{n+1}(H_I)(M)$ for all $I \in \mathcal{C}_{w \geq m}$;

4. $H(g) = 0$ if $H$ is an arbitrary cohomological functor $(\mathcal{C} \rightarrow \mathcal{A})$ of weight range $[m,n]$;

5. $H(g) = 0$ for $H = \tau_{\leq n}(H_I)$ whenever $I \in \mathcal{C}_{w \geq m}$;

6. $H(g) = 0$, where $H = \tau_{\leq n}(H_{I_0})$ and $I_0$ is some fixed choice of $w_{\geq m} N$.

Proof. Condition (3) is just a particular case of condition (2), and condition (6) is a particular case of condition (5). Next, condition (4) implies condition (5) by assertions 3 and 4 of Proposition 1.8.

Now assume that $g$ kills weights $m, \ldots, n$. Then we have a commutative diagram

$$
\begin{array}{ccc}
  M & \longrightarrow & w_{\geq n+1} M \\
  \downarrow g & & \downarrow j \\
  N & \xrightarrow{y} & w_{\geq m} N
\end{array}
$$

(2.3)
(it does not matter here whether we fix some choices of the rows or not; see condition 3 in Definition 2.1). Applying $H$ to this diagram, we obtain condition (2).

Next we fix some choice of the rows of (2.3) and take $I = w_{\geq m}N$. Assume that $g$ satisfies condition (3); then the morphism $y \circ g$ belongs to the image of $\mathcal{C}(w_{\geq n+1}M, w_{\geq m}N)$ in $\mathcal{C}(M, w_{\geq m}N)$. Thus there exists a choice of $y$ that makes (2.3) commutative; hence $g$ kills the weights $m, \ldots, n$ (see condition 1 in Definition 2.1).

It remains to deduce condition (4) from condition (1), and deduce the latter from condition (6).

Assume that $g$ kills weights $m, \ldots, n$. If $H$ is a (cohomological) functor of weight range $[m, n]$, then the morphism $H(y): H(w_{\geq m}N) \to H(N)$ is surjective and the morphism $H(x): H(M) \to H(w_{\leq n}M)$ is injective (for any choices of the corresponding weight decompositions); see assertion 5 of Proposition 1.8. Since the composed morphism $a: w_{\leq n}M \to w_{\geq m}N$ is zero (see condition 3 in Definition 2.1; thus $H(a) = 0$ as well), we obtain condition (4).

Now assume that condition (6) is fulfilled. Consider the element $r$ of the group $\tau_{\leq n}(H_{I_0})(N) = \text{Im}(\mathcal{C}(w_{\leq n+1}N, I_0) \to \mathcal{C}(w_{\leq n}N, I_0))$ obtained by composing the corresponding connecting morphisms (recall that $I_0 = w_{\geq m}N$). Since $r$ vanishes in $\tau_{\leq n}(H_{I_0})(M) \subset \mathcal{C}(w_{\leq n}M, I_0)$, the composed morphism $a \in \mathcal{C}(w_{\leq n}M, w_{\geq m}N)$ is zero. Thus we obtain condition (1) (see condition 2 in Definition 2.1). □

**Remark 2.3.** In [6] much attention was paid to morphisms *killing weight $m$*, that is, to the case $n = m$ of Definition 2.1 (cf. item 1 in Remark 3.6 below). Recall that functors of weight range $[0, 0]$ are the (cohomological) *pure* ones in the sense of [6, Definition 2.4.1], and they can be expressed in terms of *weight complexes* (see [6, Sect. 1.3] and assertion 3 of [6, Theorem 2.4.2]). This yields a relation between weight complexes and killing weight $m$ (see assertion 1 of [6, Theorem 2.3.1]).

On the other hand, virtual $t$-truncations and weight filtrations are not mentioned in [6].

### 2.2. On objects without weights $m, \ldots, n$

Now we pass to a class of objects that is important for this paper.

**Definition 2.4.** We say that an object $M$ of a weighted category $\mathcal{C}$ is *without weights $m, \ldots, n$* (for $m \leq n \in \mathbb{Z}$) whenever $\text{id}_M$ kills these weights.

We write $\mathcal{C}_{w \notin [m, n]}$ for the class of objects of this sort.

**Theorem 2.5.** I. Under the assumptions of Definition 2.4, the following conditions are equivalent:

1. $M$ is without weights $m, \ldots, n$;
2. $M$ is a retract of some $\tilde{M} \in \text{Obj} \mathcal{C}$ that is an extension of an element of $\mathcal{C}_{w_{\geq n+1}}$ by an element of $\mathcal{C}_{w_{\leq m-1}}$;
3. $H(M) = 0$ if $H$ is of weight range $[m, n]$;
4. $H(M) = \{0\}$ for $H = \tau_{\leq n}(H_I)$ whenever $I \in \mathcal{C}_{w_{\geq m}}$;
5. $H(M) = \{0\}$, where $H = \tau_{\leq n}(H_{I_0})$ and $I_0$ is a fixed choice of $w_{\geq m}M$.

II. Moreover, if $\mathcal{C}$ is idempotent complete, that is, if all idempotent endomorphisms split in it (cf. [12, Proposition 1.6.8]), then the above conditions are also equivalent to the following ones:

1. $M$ itself is an extension of an element of $\mathcal{C}_{w_{\geq n+1}}$ by an element of $\mathcal{C}_{w_{\leq m-1}}$;
2. there exist isomorphic choices of the objects $w_{\leq m-1}M$ and $w_{\leq n}M$ (see item 1 of Remark 1.6);
3. there exists a choice of $w_{\leq m-1}M$ and $w_{\leq n}M$ such that the corresponding connecting morphism defined as in item 2 of Definition 1.7 is an isomorphism.

**Proof.** I. Theorem 2.2 implies the equivalence of conditions (1) and (3)–(5) of part I. Moreover, Theorem 2.2 (see condition (4) in it) implies that condition (1) of part I follows from condition (2) of part I.

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Now assume that $M$ is without weights $m, \ldots, n$. Then, combining assertions 6 and 10 of \cite[Theorem 2.2.1]{6} with assertion 2 of \cite[Proposition 3.1.2]{6}, we obtain the existence of a triangulated category $\mathcal{C}' \supset \mathcal{C}$ along with a $\mathcal{C}'$-distinguished triangle

$$L'M \to M \to R'M \to L'M[1]$$

(2.4)

such that $L'M$ is a retract of an element $\tilde{L}M$ of $\mathcal{C}_{w \leq m-1}$ and $R'M$ is a retract of $\tilde{R}M \in \mathcal{C}_{w \geq n+1}$. Since $\mathcal{C}'$ is triangulated, there exist objects $L''M$ and $R''M$ of $\mathcal{C}'$ such that $LM \oplus L''M \cong LM$ and $R'M \oplus R''M \cong \tilde{R}M$. Thus we can add (2.4) with (say) a split distinguished triangle

$$L''M \to M'' = L''M \oplus R''M \to R''M \to L''M[1]$$

to obtain the triangle $\tilde{L}M \to M \oplus M'' \to \tilde{R}M \to \tilde{L}M[1]$. Since $\tilde{L}M$ and $\tilde{R}M$ are objects of $\mathcal{C}$, $M \oplus M''$ is $\mathcal{C}'$-isomorphic to an object of $\mathcal{C}$. Thus one can take $\tilde{M} \cong M \oplus M''$ and find that condition (1) of part I implies condition (2) of part I.

II. Assertions 9 and 10 of \cite[Theorem 2.2.1]{6} state that conditions (1) of part I and (1) of part II are equivalent if $\mathcal{C}$ is weight-Karoubian, that is, if the category $Hw = \mathcal{C}_{w \leq 0} \cap \mathcal{C}_{w \geq 0}$ is idempotent complete. Now, this assumption is valid whenever $\mathcal{C}$ is idempotent complete itself (see the simple assertion 7 of \cite[Proposition 1.2.4]{6}).

Next, if there exists a distinguished triangle

$$L \to M \to R \to L[1]$$

(2.5)

with $L \in \mathcal{C}_{w \leq m-1}$ and $N \in \mathcal{C}_{w \geq n+1}$, then one can take $w_{m-1}M = w_nM = L$. Consequently, condition (1) of part II implies condition (3) of part II. Moreover, condition (2) of part II is just a weaker version of condition (3) of part II.

Lastly, if condition (2) of part II is fulfilled, then for the corresponding $n$-weight decomposition $w_{n-1}M \to M \to w_nM$ the object $w_{n-1}M$ is simultaneously a choice of $w_{m-1}M$. Hence (this choice of) $w_{n-1}M$ belongs to $\mathcal{C}_{w \leq m-1}$, and this decomposition triangle satisfies the assumptions for (2.5). □

**Remark 2.6.** 1. Originally, objects without weights were defined by J. Wildeshaus. His definition \cite[Definition 1.10]{15} coincides with condition (1) of part II of Theorem 2.5. Thus his definition is equivalent to ours whenever $\mathcal{C}$ is weight-Karoubian (see the proof of part II of Theorem 2.5). Yet this equivalence fails in general (see \cite[Sect. 3.3]{6}).

2. In the case where there exists a $t$-structure $t$ adjacent to $w$ (on $\mathcal{C}$), some more descriptions of $\text{Mor}_{[\text{part}]}^w \mathcal{C}$ and $\mathcal{C}_{w \notin [m,n]}$ are provided by Theorem 3.1 below.

Finally, we make a simple nice observation.

**Lemma 2.7.** Assume a $\mathcal{C}$-morphism $h$ factors through some $M \in \mathcal{C}_{w \notin [m,n]}$ (for $m \leq n \in \mathbb{Z}$). Then $h$ kills the weights $m, \ldots, n$.

**Proof.** Our assumption means that $h = a \circ \text{id}_M \circ b$ for some $\mathcal{C}$-morphisms $a$ and $b$. Applying assertion 3 of \cite[Theorem 2.2.1]{6}, we find that $h$ kills the weights $m, \ldots, n$ indeed. □

**Remark 2.8.** Assertion 3 of Theorem 3.1 below gives the converse implication in the case where a $t$-structure adjacent to $w$ exists. The authors suspect that this equivalence fails if this additional assumption is omitted. In particular, it would be interesting to take $\mathcal{C}$ to be the subcategory of finite spectra in SH (cf. Theorem 3.5 below and assertion 1 of \cite[Theorem 4.2.1]{4}).

3. ON THE RELATION TO THE ADJACENT $t$-STRUCTURE

In Subsection 3.1 we construct some “new” torsion theories and injective classes on $\mathcal{C}$ whenever it is endowed with adjacent (weight structure) $w$ and ($t$-structure) $t$.

In Subsection 3.2 we consider the so-called purely compactly generated examples of this setting. In particular, we discuss the application of our results to the case $\mathcal{C} = \text{SH}$. 
3.1. Some new torsion theories and injective classes.

**Theorem 3.1.** Assume that $\mathcal{C}$ is endowed with a weight structure $w$ and an adjacent $t$-structure $t$, and $m \leq n \in \mathbb{Z}$.

1. Then the pair $s = (\mathcal{C}_{w\leq [m,n]}, \mathcal{C}_{[m,n]})$ is a torsion theory.

2. The pair $(\mathcal{C}_{w\leq [m,n]}, \mathcal{J}_{[m,n]})$ is a projective class, where $\mathcal{J}_{[m,n]}$ is the class of those morphisms that factor through $\mathcal{C}_{[m,n]}$ (cf. Lemma 2.7).

3. The pair $(\mathcal{C}_{[m,n]}, \text{Mor}_{[m,n]} \mathcal{C})$ is an injective class, and $\text{Mor}_{[m,n]} \mathcal{C}$ consists of those morphisms that factor through $\mathcal{C}_{w\leq [m,n]}$.

**Proof.** 1. According to assertion 9 of [8, Proposition 2.4], it suffices to verify that the classes $\mathcal{C}_{w\leq [m,n]}$ and $\mathcal{C}_{[m,n]}$ are retraction-closed in $\mathcal{C}$: $\mathcal{C}_{w\leq [m,n]} \perp \mathcal{C}_{[m,n]}$, and for every $M \in \text{Obj} \mathcal{C}$ there exists an $s$-decomposition (1.1).

The class $\mathcal{C}_{[m,n]}$ is retraction-closed in $\mathcal{C}$ since it equals the intersection of the classes $\mathcal{C}_{t\leq n}$ and $\mathcal{C}_{t\geq m}$, whereas the equalities $\mathcal{LO}^{t \leq n} = \mathcal{RO}^{t \leq n}$ and $\mathcal{LO}^{t \geq m} = \mathcal{RO}^{t \geq m}$ (for the torsion theory $s^t$ associated with $t$) imply that these two classes are retraction-closed in $\mathcal{C}$. Moreover, it follows from Lemma 2.7 that $\mathcal{C}_{w\leq [m,n]}$ is retraction-closed in $\mathcal{C}$ as well.

Next, the orthogonality conditions in Definition 1.1 (applied to the torsion theories associated with $w$ and $t$) imply that for $N \in \mathcal{C}_{[m,n]}$ the functor $H_N = \mathcal{C}(-, N)$ is of weight range $[m,n]$. Hence $\mathcal{C}_{w\leq [m,n]} \perp \mathcal{C}_{[m,n]}$ by condition (3) of part 1 in Theorem 2.5.

It remains to verify the existence of an $s$-decomposition for an arbitrary $M \in \text{Obj} \mathcal{C}$.

We fix some $w_{\geq m}M$, denote $t_{\leq n}(w_{\geq m}M)$ by $RM$, and complete the corresponding composed morphism $h \in \mathcal{C}(M, RM)$ to a triangle $LM \to M \to RM \to LM[1]$. Then $LM$ is an extension of $t_{n+1}(w_{\geq m}M)$ by $w_{\leq m}M$ (by the octahedron axiom of triangulated categories). Since $t_{n+1}(w_{\geq m}M) \in \mathcal{C}_{w_{\geq m}+1}$ (by the definition of adjacent structures), $LM$ is without weights $m, \ldots, n$ (see condition (2) of part 1 in Theorem 2.5). Finally, since $w_{\geq m}M \in \mathcal{C}_{t\geq m}$, we have $RM \in \mathcal{C}_{[m,n]}$ by assertion 2 of Proposition 1.5.

2. The statement immediately follows from (the simple) Lemma 3.2 below.

3. Applying Lemma 3.2 to the category $\mathcal{C}^{\text{op}}$, we find that $(\mathcal{C}_{[m,n]}, \mathcal{J}'_{[m,n]})$ is an injective class (in $\mathcal{C}$), where $\mathcal{J}'_{[m,n]}$ is the class of $\mathcal{C}_{[m,n]}$-conull morphisms. Moreover, $\mathcal{J}'_{[m,n]}$ coincides with the class of morphisms that factor through $\mathcal{C}_{w\leq [m,n]}$.

It remains to verify that $\mathcal{J}'_{[m,n]} = \text{Mor}_{[m,n]} \mathcal{C}$. According to Theorem 2.2 (see condition (5) in it), $g$ belongs to $\text{Mor}_{[m,n]} \mathcal{C}$ if and only if $H(g) = 0$ whenever $H = \tau_{\leq n}(H_I)$ and $I \in \mathcal{C}_{w_{\geq m}}$. Next, assertion 6 of Proposition 1.8 says that $\tau_{\leq n}(H_I) \cong \mathcal{C}(-, t_{\leq n}I)$. Recalling assertion 2 of Proposition 1.5, we conclude that the class $\text{Mor}_{[m,n]} \mathcal{C}$ equals $\mathcal{J}'_{[m,n]}$ indeed. \qed

**Lemma 3.2.** Let $(\mathcal{LO}, \mathcal{RO})$ be a torsion theory. Then $(\mathcal{LO}, \mathcal{J})$ is a projective class, where $\mathcal{J}$ is the class of $\mathcal{LO}$-null morphisms. Moreover, $\mathcal{J}$ coincides with the class of those morphisms that factor through $\mathcal{RO}$.

**Proof.** This is Proposition 5.2 of [8]. \qed

**Remark 3.3.** 1. In particular, $\text{Mor}_{[0,1]} \mathcal{C}$ is the class of $\mathcal{C}_{[0,1]}$-conull morphisms (cf. assertion 3 of [6, Corollary 4.1.6]). Recall here that in some papers of the first author the class $\mathcal{C}_{[0,1]}$ was denoted by $\mathcal{C}_{t=0}$ (cf. [4, Definition 4.3.1]).

2. There exists one more description of the class $\mathcal{J}_{[m,n]}$ in Theorem 3.1. Indeed, Lemma 3.2 implies that the pairs $(\mathcal{C}_{w_{\geq n}+1}, \mathcal{J}_1)$ and $(\mathcal{C}_{w_{\leq m}+1}, \mathcal{J}_2)$ are projective classes; here $\mathcal{J}_1$ is the class of morphisms that factor through (elements of) $\mathcal{C}_{w_{\geq n}+1} = \mathcal{C}_{t\geq n+1} = \mathcal{C}_{t\geq n}$ and $\mathcal{J}_2$ consists of those morphisms that factor through $\mathcal{C}_{w_{\leq m}+1} = \mathcal{C}_{w_{\leq m}} = \mathcal{C}_{t\leq m}$. Thus Proposition 3.3 of [9] yields that $(\mathcal{C}_{w\leq [m,n]}, \mathcal{J}_{[m,n]})$ equals the product of $(\mathcal{C}_{w_{\geq n}+1}, \mathcal{J}_1)$ and $(\mathcal{C}_{w_{\leq m}+1}, \mathcal{J}_2)$; that is, $\mathcal{J}_{[m,n]} = \mathcal{J}_1 \circ \mathcal{J}_2$ (we take all possible pairwise compositions of this form), and here we use the description of $\mathcal{C}_{w\leq [m,n]}$ provided by condition (2) of part I in Theorem 2.5.
3.2. Examples (including stable homotopy ones). Recall that several general statements on the existence of adjacent weight structures and t-structures are given by [5, Theorems 2.3.4, 2.4.2] and [3, Theorems 3.2.3, 4.1.2, Proposition 4.2.1]. Here we will only discuss the most “explicit” family of examples.

Throughout this subsection we assume that all coproducts are small and $C$ is smashing, that is, closed with respect to (small) coproducts. Note that any $C$ that satisfies this condition is well known to be idempotent complete (cf. part II of Theorem 2.5; see [12, Proposition 1.6.8]).

Definition 3.4. Assume that $P$ is a full subcategory of $C$.

1. An object $M$ of $C$ is said to be compact if the functor $H^M = C(M, -): C \to \text{Ab}$ respects coproducts.

2. We will say that $C$ is compactly generated by $P$ if $P$ is small, the objects of $P$ are compact in $C$, and $C$ equals its own smallest strict triangulated subcategory that is closed with respect to $C$-coproducts and contains $P$.

3. We will say that $P$ is connective in $C$ if $P \perp \bigcup_{i \geq 0} \mathcal{P}[i]$.\(^2\)

Theorem 3.5. 1. Let $P$ be a connective subcategory of $C$ that compactly generates it. Then the following statements are valid.

1. Set $C_{w \leq 0}$ (respectively, $C_{w \geq 0}$) to be the smallest subclass of $\text{Obj} \, C$ that is closed with respect to coproducts and extensions and contains $\text{Obj} \, \mathcal{P}[i]$ for $i \leq 0$ (respectively, for $i \geq 0$). Then $w = (C_{w \leq 0}, C_{w \geq 0})$ is a weight structure.

More generally, $C_{w \geq 0} = (\bigcup_{i \leq 0} \text{Obj} \, C[i])^{\perp}$.

2. There exists a t-structure adjacent to $w$; accordingly, we have $C_{t \geq 0} = C_{w \geq 0}$ and $C_{t \leq 0} = (\bigcup_{i \geq 0} \text{Obj} \, C[i])^{\perp}$.

3. If $m \leq n$ then the corresponding class $C_{w \notin [m, n]}$ is the smallest class of objects of $C$ that is closed with respect to coproducts and extensions and contains the class $C' = \bigcup_{i \leq 0} \text{Obj} \, C[i]$.

Consequently, the torsion theory $(C_{w \notin [m, n]}, C_{[m, n]})$ is generated by $C'$ (see item 4 in Definition 1.2).

II. Take $C = \text{SH}$ (the stable homotopy category) and $P = \{S^0\}$ (where $S^0$ is the sphere spectrum).

1. Then the assumptions of part I are fulfilled. In this case $w$ was denoted by $w_{\text{sph}}$ in [6, Theorem 4.2.4]; moreover, $\text{SH}_{w_{\text{sph}} \geq n+1}$ is the class of $n$-connected spectra, and $\text{SH}_{w_{\text{sph}} \leq m-1}$ consists of $(m - 1)$-skeleta in the sense of [11, Ch. 6, Sect. 3];

2. The class $\text{SH}_{t \leq n}$ (respectively, $\text{SH}_{t \geq m} = \text{SH}_{w_{\text{sph}} \geq m}$) is characterized by the vanishing of the stable homotopy groups $\pi_i(-) = \text{SH}(S^0[i], -)$ for $i > n$ (respectively, for $i < m$). Consequently, $\text{SH}[0, 0]$ consists of Eilenberg–Maclane spectra.

3. $\text{SH}_{w_{\text{sph}} \notin [m, n]}$ consists of those spectra $X$ that possess cellular towers with $X^{(m-1)} \cong X^{(n)}$ (these are the corresponding skeleta of $X$; see [11, Ch. 6, Sect. 3]). Moreover, it is characterized by the vanishing of the singular homology $H_i^{\text{sing}}(X, \mathbb{Z})$ for $m \leq i \leq n$ and the freeness of $H_{m-1}^{\text{sing}}(X, \mathbb{Z})$ (as an abelian group).

Proof. I. Assertions 1 and 2 of part I are contained in [2, Theorem 4.5.2] (see assertion 3 of Theorem 3.2.3 and assertion 1 of Proposition 4.3.3 in [4] for more details).

\(^2\)In earlier texts of the authors, connective subcategories were called negative ones; another related notion is silting.
Let us prove assertion 3. Theorem 2.5 (see condition (2) of part I in it) implies that the class $C$ specified in the assertion contains $C_{w\notin[m,n]}$.

Conversely, $C \subseteq C_{w\notin[m,n]}$ since the latter class contains $\text{Obj}\, \mathcal{P}[i]$ both for all $i < m$ and for all $i > n$, and assertion 1 of Theorem 3.1 implies that it is closed with respect to coproducts and extensions.

Next, recall that $C$-representable functors are cohomological and convert $C$-coproducts into coproducts (in the category of abelian groups). Consequently, $C^{\perp} = C'^{\perp}$. Finally, $C_{[m,n]} = C^{\perp}$ since $(C_{w\notin[m,n]}, C_{[m,n]})$ is a torsion theory (see Definition 1.1).

II. The first part of assertion 1 is a particular case of assertion 1 in [4, Theorem 4.1.1] (cf. item 2 in Remark 3.6 below), which is well known. Next, the description of $w^{\text{sp}}$ in the assertion is contained in assertion 3 of [6, Proposition 4.2.1] and assertion 1 of [6, Theorem 4.2.5].

Assertion 2 of part II immediately follows from assertion 2 of part I.

Let us prove assertion 3 of part II. Assertion 2 of [6, Theorem 4.2.5] essentially says (see assertion 1 of [4, Proposition 1.3.4] for additional details) that for every $k \in \mathbb{Z}$ the possible choices of $X^{(k)}$ are precisely the ones of $\text{SH}_{w^{\text{sph}} \leq k}$. Hence the first description of $\text{SH}_{w^{\text{sph}} \notin[m,n]}$ in our assertion is given by assertion II of Theorem 2.5. The second description is provided by assertion 5 of [6, Theorem 4.2.5].

**Remark 3.6.** 1. Combining assertion 1 of part II of Theorem 3.5 with assertion 1 of Theorem 3.1, we obtain rather curious “decompositions” of objects of SH. Moreover, these decompositions of spectra are weakly functorial in the sense of item 2 of Remark 1.6.

Theorem 3.1 also yields some more descriptions of morphisms killing weights and spectra without weights in a range. Combining some of them with assertion 2 of part II of Theorem 3.5, we obtain the following remarkable statement: for an SH-morphism $g$ we have $H^{0}_{\text{sing}}(g, \Gamma) = 0$ for every abelian group $\Gamma$ (that is, $g$ acts trivially on the singular cohomology with coefficients in $\Gamma$) if and only if $g$ factors through a spectrum $X$ such that all $H^{0}_{\text{sing}}(X, \Gamma)$ vanish. Moreover, one can also describe $\text{SH}_{w^{\text{sph}} \notin[0,0]}$ by means of assertion 3 of part II of Theorem 3.5.

2. Assertion 1 of part II of Theorem 3.5 can be extended to the equivariant stable homotopy $\text{SH}(G)$, where $G$ is a compact Lie group (see [6, Sect. 4.2] and [4, § 4.1]). Recall that the corresponding analog of the functor $H^{0}_{\text{sing}}(-, \Gamma)$ (see the previous part of this remark) is the Bredon cohomology represented by an Eilenberg–MacLane $G$-spectrum (see loc. cit.).

One of the main distinctions of this general case from the case where $G = \{e\}$ (and $\text{SH}(G) = \text{SH}$) is that some of the definitions and results of [11] have never been extended to the equivariant context.

3. The compactness assumption in part I of Theorem 3.5 can be weakened (see assertion 1 of Corollary 2.3.1 and assertion 1 of part II of Corollary 4.1.4 in [7] as well as item 3 of [4, Remark 3.2.4]).

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