Solutions of massive gravity theories in constant scalar invariant geometries

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Abstract

We solve massive gravity field equations in the framework of locally homogeneous and vanishing scalar invariant (VSI) Lorentzian spacetimes, which in three dimensions are the building blocks of constant scalar invariant (CSI) spacetimes. At first, we provide an exhaustive list of all Lorentzian three-dimensional homogeneous spaces and then we determine the Petrov type of the relevant curvature tensors. Among these geometries we determine for which values of their structure constants they are solutions of the field equations of massive gravity theories with a cosmological constant. The homogeneous solutions obtained are all of various Petrov types: I_C, I_R, II, III, D_t, D_s, N, O; the VSI geometries which we found are of Petrov type III. The Petrov types II and III are new explicit CSI space-time solutions of these types. We also examine the conditions under which the obtained anti-de Sitter solutions are free of tachyonic massive graviton modes.

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(Some figures may appear in colour only in the online journal)

1. Introduction

It is known that, in three dimensions, Einstein gravity theory does not possess any local physical degrees of freedom. However, Deser et al proposed a modification of the theory [1], consisting of a Chern–Simons term [2, 3] added to the usual Einstein–Hilbert Lagrangian. The latter is known as topological massive gravity (TMG). This is a chiral three-dimensional gravity theory which contains massive spin-2 excitations that mediate finite-range interactions. Moreover, this modification provides an interesting framework for a unitary quantum theory of gravity.

On the other hand, Bergshoeff et al [8, 9] have recently obtained another type of massive gravity theory (new massive gravity (NMG)): a parity preserving theory that describes (on a Minkowski background) the propagation of a massive positive energy spin-2 field, but now of

1 Its status concerning renormalizability seems not yet definitely established [4–7]. We thank S Deser for a comment on this point.
both helicities; ±2. This theory possesses all the virtues of the TMG, so it may be considered as another consistent candidate of a theory of 3D quantum gravity. These authors have also considered the ‘merging’ of both theories, producing a general massive gravity (GMG) theory, which involves two spin-2 helicity states with different masses (parity-violating) and, as the previous ones, can also be extended by adding a cosmological constant.

Let us mention that solutions of the NMG theory, including AdS 3 black holes, were constructed in [10]. In addition, aspects of the gauge/gravity duality in this theory were found to be in agreement with the holographic studies of TMG theory [11, 12].

This work is motivated by the one of Chow et al [13], who provided a review of a large set of solutions of TMG (with a cosmological constant). In their paper, these authors described a three-dimensional variant of Petrov classification and showed that all the solutions that were found in the literature at this time were of Petrov types D or N, corresponding to locally squashed AdS 3 or AdS pp-wave metrics. Moreover, they proved that all Petrov type D solutions of TMG actually are biaxially squashed AdS 3 metrics. In a companion paper [14], these authors also obtained new solutions of the topologically massive gravity equations by considering Kundt metrics [15, 16] (see also chapters 28 and 31 of [17]). The TMG solutions belonging to this class of metrics are generically of Petrov type II, but there are some special cases of Petrov types D, III, N and O as well. Let us note that a classification of the homogeneous solutions of TMG equations (without a cosmological constant) was given by Ortiz [18] and was recently generalized for a non-vanishing cosmological constant by Moutsopoulos [19].

On the other hand, Coley et al have proved [20] that a Lorentzian three-dimensional spacetime, on which all scalars built out of the curvature tensor are constant (constant scalar invariant (CSI) spaces), can be constructed by means of fibering and warping, from locally homogeneous spaces and a subclass of CSI spaces, namely the vanishing scalar invariant (VSI) spaces.

The purpose of this work is to obtain all scalar invariant geometries (i.e. locally homogenous and VSI Lorentzian spacetimes) that solve the massive gravity (MG) theory equations with a cosmological constant and to classify them according to their Petrov types.

The paper is organized as follows. In section 2, we summarize the various field equations of the MG theories. We start section 3 by revisiting all the three-dimensional homogeneous spaces with Lorentzian signature and determine the Petrov types of the relevant curvature tensors (listed in the appendix), which will prove to be useful for solving the equations and identifying the solutions. We then proceed in section 4 to solve the equations of the MG theories on homogeneous spaces. Finally, in section 5 we study MG equations on VSI spaces.

2. Massive gravity theories

This section is devoted to a short review of the various MG theories in three dimensions.

They all consist by adding extra pieces to the usual Einstein–Hilbert action:

$$S_{EH} = \frac{1}{\kappa^2} \int \sqrt{|g|} (R - 2\Lambda) \, d^3 x, \quad g := \det(g_{\mu\nu}),$$

where $\Lambda$ is the cosmological constant and $\kappa$ is the gravitational coupling with mass dimension $[\kappa] = -\frac{1}{2}$. In what follows, we adopt the mostly plus expression of the metric and define the curvature tensors so that the curvature of the Euclidean round sphere, equipped with its positive definite metric, has a positive curvature\(^2\) and fixes the spacetime orientation by adopting for the tensor density $\varepsilon^{123} = +1$ (so $g^{123} = g^{-1}$).

\(^2\) In others words, according to the conventions: $R^\mu_{\nu\rho\sigma} = \partial_\rho \Gamma^\mu_{\nu\sigma} + \cdots; R_{\nu\rho} = R^\mu_{\nu\rho\mu}$. 

2.1. Topologically massive gravity

TMG is obtained by adding a gravitational Chern–Simons term to the Einstein–Hilbert action

$$S_{TMG} := S_{EH} + \frac{1}{\mu \kappa^2} S_{CS}, \quad S_{CS} = \frac{1}{2} \int g \varepsilon^{\lambda\mu\nu} \Gamma_{\mu\nu}^\rho \left( \partial_\mu \Gamma_{\rho\nu}^\sigma + \frac{2}{3} \Gamma_{\mu\nu}^\sigma \Gamma_{\rho\nu}^\tau \right) \, d^3x,$$

which is expressed through Christoffel symbols of the spacetime metric $g_{\mu\nu}$, while $\mu$ is a new coupling constant with mass dimension 1.

The classical equations of motion read

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} + \frac{1}{\mu} C_{\mu\nu} = 0, \quad C_{\mu\nu} := \varepsilon^{\rho\mu\nu} \nabla^\rho \left( R_{\rho\sigma} - \frac{1}{4} \delta_{\rho\sigma} R \right),$$

where $C_{\mu\nu}$ is the Cotton–York tensor: a symmetric, traceless and divergenceless tensor. An immediate consequence of the equations of motion is that the traceless part of the Ricci tensor and the Cotton–York are proportional

$$S_{\mu\nu} + \frac{1}{\mu} C_{\mu\nu} = 0, \quad S_{\mu\nu} := R_{\mu\nu} - \frac{1}{3} g_{\mu\nu} R,$$

and accordingly of the same Petrov type.

2.2. New massive gravity

This theory is defined by adding a quadratic curvature term [8] to the Einstein–Hilbert action

$$S_{NMG} := S_{EH} - \frac{1}{\xi \kappa^2} S_{QC}, \quad S_{QC} = \int \sqrt{|g|} \varepsilon_{\mu\nu\rho} \nabla^\rho \left( R_{\rho\sigma} - \frac{1}{4} \delta_{\rho\sigma} R \right) \, d^3x,$$

where $\xi$ is a coupling constant with mass dimension 2. The corresponding equations of motion read

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} - \frac{1}{2\xi} K_{\mu\nu} = 0,$$

$$K_{\mu\nu} = 2\nabla^2 R_{\mu\nu} - \frac{1}{2} \nabla_{\mu} R_{\nu\tau} - \frac{9}{2} R_{\nu\tau} R_{\mu\sigma} - 8R_{\mu}^\tau R_{\nu\tau} + g_{\mu\nu} \left( 3R_{\kappa\lambda} R^{\kappa\lambda} - \frac{13}{8} R^2 \right),$$

where $K_{\mu\nu}$ is a symmetric and divergenceless tensor. Similar to equation (2.4) we see that the traceless parts of the Ricci tensor and $K_{\mu\nu}$ tensor are proportional

$$S_{\mu\nu} = \frac{1}{2\xi} \bar{K}_{\mu\nu}, \quad \bar{K}_{\mu\nu} := K_{\mu\nu} - \frac{1}{3} g_{\mu\nu} K,$$
in [8] and [21], respectively. In particular, it was shown in [21] that there are two massive spin-2 modes, which are stable if
\[ \xi (\xi + 4\mu^2) \geq 2\Lambda\mu^2. \] (2.10)

Thus, for pure NMG, when \( \mu \) goes to infinity, an AdS\(_3\) space is exempt of tachyonic graviton if \( \xi \geq \frac{1}{2} \). In what follows, we shall not restrict the sign of the coupling constants, but unless for exact AdS\(_3\) geometries, just provide some plots of their signs according to the parameters appearing in the expressions of the metrics we obtain. Indeed, for an asymptotically anti-de Sitter space (\( \Lambda < 0 \)), condition (2.10) implies that for \( \xi > 0 \), the graviton has no tachyonic massive mode.

3. Homogeneous geometries

In this section, we review the various expressions of the structure constants of the isometry groups characterizing the three-dimensional locally homogenous spacetimes which are going to be used in section 4.1 for solving the MG field equations.

The strategy we adopt to obtain all homogeneous space metrics, that are solutions of the MG field equations, was the one advocated a long time ago by Ozsváth [22] (see also [18, 19]). Let us briefly recall its principle. We consider metrics on homogeneous three-dimensional spaces invariant under the action of a locally simply transitive isometry group. The action being simply transitive implies that locally such spaces can be identified with the groups acting on them. Suppose that we choose the left action of \( \mathcal{G} \) on itself. The vectors tangent to the orbits of the one-parameter subgroups of \( \mathcal{G} \) constitute right invariant vector fields obeying the relations
\[ [\xi_\alpha, \xi_\beta] = -C^\gamma_{\alpha\beta} \xi_\gamma, \] (3.1)
where \( C^\gamma_{\alpha\beta} \) are the structure constants of the Lie algebra of \( \mathcal{G} \). The group being simply transitive also implies that at each point these vectors \( \xi_\alpha \) constitute a local frame. Their dual 1-form \( \theta^\alpha \) (such that \( \theta^\alpha(\xi_\beta) = \delta^\alpha_\beta \)) define the right invariant coframes. Metric tensors whose components with respect to these coframes are constants, also are right invariant and the generators \( \tilde{\xi}_\alpha \) associated to the right action of \( \mathcal{G} \) are their Killing vector fields. In the framework of three-dimensional groups, it is well known how to implement the action of \( GL(3, \mathbb{R}) \) and writing \( C^\gamma_{\alpha\beta} \) in a canonical form according to the Bianchi classification [17]. However, solving the gravitational field equations in a Lorentzian invariant theory appears to be much easier by setting the metric in a canonical form:
\[ (\eta_{\alpha\beta}) = \text{diag}(-1, 1, 1), \] (3.2)
but starting from arbitrary structure constants of the Lie algebra and fixing them by \( \text{Iso}(1, 2) \) transformations. In this framework, all geometrical quantities expressed in the invariant coframe are algebraic functions of the structure constants. For instance, defining \( C_{\alpha\beta\gamma} := C^\delta_{\alpha\beta} \eta_{\delta\gamma} \), we obtain the connection coefficients, the Ricci tensor and its covariant derivative:
\[ \omega_{\alpha\beta\gamma} = \frac{1}{2}(C_{\beta\gamma\alpha} - C_{\alpha\gamma\beta} - C_{\alpha\beta\gamma}), \quad \omega^\delta_{\alpha\beta} = \omega^{\gamma\delta}_{\alpha\beta}, \quad \omega^\delta_{\gamma\beta} = \omega^\delta_{\alpha\beta} - \omega^{\gamma\delta}_{\alpha\beta}, \quad R_{\alpha\beta} = \text{Ricci}(\alpha\beta), \quad \nabla_{\gamma} R_{\alpha\beta} = -R_{\delta\beta} \omega^\delta_{\alpha\gamma} - R_{\delta\alpha} \omega^\delta_{\beta\gamma}. \] (3.3)

Accordingly, all the field equations, we shall encounter in section 4, become algebraic equations. But first we shall review the classification of the structure constants, then put them into the field equations and solve the resulting algebraic equations.
3.1. Bianchi classification revisited

As it was mentioned above, the Lie algebras of the (right) invariant fields (given by equation (3.1)) whose structure constants satisfy the Jacobi identity \( C^\kappa_\alpha_\kappa C^\kappa_\beta_\gamma \epsilon^{\kappa_\alpha_\beta_\gamma} = 0 \) allocate into two classes: the unimodular Lie algebras, such that \( C^\kappa_\alpha_\kappa = 0 \) and the non-unimodular ones such that \( \frac{1}{2}C^\kappa_\alpha_\kappa = k_\alpha \neq 0 \). For the three-dimensional real algebras (first classified by Bianchi), their structure constants can be parametrized in terms of the vector components \( k_\alpha \) and a symmetric tensor density\(^3\) \( n^{\alpha_\beta} \) (see [17] and references therein),

\[
C^\alpha_\beta_\gamma = \epsilon_\beta_\gamma_\zeta n^{\kappa_\alpha_\kappa} + k_\beta_\alpha_\kappa \delta^\alpha_\kappa - k_\gamma_\alpha_\kappa \delta^\alpha_\beta,
\]

while the Jacobi identity reduces to the condition \( k_\alpha_\kappa n^{\alpha_\beta} = 0 \). Thus, the Iso\((1, 2)\) classification of the structure constants reduces to the classification of symmetric tensor densities that annihilate a vector, a problem that was solved in [23] (see also [24, 25]).

Chow et al [13] suggested us to classify solutions of TMG according to the Segre classification of the traceless Ricci tensor \( S^{\alpha_\beta} \), which, in their framework, is equivalent to the Petrov classification of the Cotton–York tensor but not necessarily in the case of NMG. Hereafter, we shall present the various canonical forms of the Lie algebras obtained using Iso\((1, 2)\) transformations. We also determine the Segre–Petrov types of the traceless Ricci, Cotton–York and \( \hat{K}^{\alpha_\beta} \) tensors obtained from their invariant coframes and the canonical form (3.2) of the metric. On a practical level, to obtain this classification it is not necessary to know explicitly the eigenvalues of the tensor, when they all are different. So, we have just to compute the discriminant of its characteristic equation. If it is positive, then two eigenvalues are complex conjugate and one is real: it corresponds to the case \( I_C \) (see [13] for the notations). If it is negative, the three eigenvalues are real and distinct, corresponding to the case \( I_R \). It is only when it is zero, in which case we know that at least one eigenvalue is double, that further analysis is required to determine the Jordan form of the tensor. In other words, it is the degeneracy of the eigenvalues that greatly facilitates the analysis. In this case, when the characteristic polynomial reduces to its cubic term \( P(\lambda) = \lambda^3 \), the tensor is of Petrov type \( O, N \) or \( III \); otherwise if the three roots are equal, but nonvanishing, the tensor is of Petrov type \( D_s, D_t \) or \( II \).

3.2. Unimodular Lie algebras

Four types of normal forms are possible.

- Type I

\[
\left( n^{\alpha_\beta} \right) = \begin{pmatrix} a & 0 & 0 \\ 0 & b & c \\ 0 & c & b \end{pmatrix},
\]

(3.5)

this form is equivalent to diag \((a, b + c, b - c)\), but the latter appears to be more easy to handle for solving the field equations. By changing the orientation (in which case \( \mu \) is changing sign), we may always assume that if \( a \) is non-zero it is positive, otherwise that \( b > 0 \) if \( b \) is non-zero, etc.

Obviously, this structure tensor density may correspond to any of the unimodular Bianchi types \((I, II, VI_0, VII_0, VIII, IX)\) of real Lie algebras (see for instance Ortiz [18]).

\(^3\)In what follows, we shall call them respectively structure vector and structure tensor density.
The traceless part of the Ricci tensor is
\[
(S_{\alpha \beta}) = \begin{pmatrix}
\frac{1}{2}(a^2 + ab - 2c^2) & 0 & 0 \\
0 & \frac{1}{2}(a^2 + ab - 2c^2) & -(a + 2b)c \\
0 & -(a + 2b)c & \frac{1}{2}(a^2 + ab - 2c^2)
\end{pmatrix}.
\] (3.6)

To determine its Petrov type, we have to obtain the Jordan form of the matrix of components $S_{\alpha \beta}$. Fortunately, as already mentioned, we will not have to handle the explicit solutions of the third-degree characteristic polynomial
\[
P(\lambda) := \det \left[ \lambda \delta_{\alpha \beta} - S_{\alpha \beta} \right]
\] (3.7)
in the general case. The traceless condition implies that this cubic polynomial will always be of the form $P(\lambda) = \lambda^3 + p \lambda + q$. Its discriminant, which is defined as $\Delta := \frac{p^2}{4} + \frac{q^2}{27}$, partially fixes the number and the nature of its different roots. If $\Delta > 0$, one root is real and the two others are complex conjugate; if $\Delta < 0$, the three roots are real and distinct; if $\Delta = 0$, at least two roots are equal. Accordingly, when $\Delta > 0$ the traceless tensor will be of Petrov type $I_C$ and when $\Delta < 0$ of Petrov type $I_B$. When $\Delta = 0$, the tensor is of special Petrov type. It is of type $II$ or $D$ when $p \neq 0$, and of type $III$, $N$ or $O$ when $P(\lambda) = \lambda^3$. Then, its precise determination will need more investigation, but things are greatly facilitated because at least one of the eigenvalues of the tensor is degenerate. For example, in the case of the tensor (3.6), we obtain
\[
\Delta_S = -\frac{1}{3} \left( a + 2b \right)^2 c^2 \left( (a + b)^2 - c^2 \right)^2 \left( a^2 - 4c^2 \right)^2,
\] (3.8)
which generically is negative and thus defines a tensor of type $I_C$; exceptions occur when it vanishes. The results of this analysis, both for the $S_{\alpha \beta}$ with $C_{\alpha \beta}$ and $\bar{K}_{\alpha \beta}$ tensors, are summarized in tables A1–A6. Thus, solutions of TMG or NMG can be easily found by requiring matching of the Petrov classifications of $S_{\alpha \beta}$ with $C_{\alpha \beta}$ or $\bar{K}_{\alpha \beta}$ respectively. However, this is not a priori the case for GMG.

- **Type II**
\[
(n_{II}^{\alpha \beta}) = \begin{pmatrix}
v + a & v & 0 \\
v & v - a & 0 \\
0 & 0 & b
\end{pmatrix}, \quad v = \pm 1,
\] (3.9)
which does not admit timelike eigenvector but a double null vector.

Diagonalizing $n_{II}$ with a $GL(3, \mathbb{R})$ transformation, we see that if $a = b = 0$, it corresponds to a Lie algebras of Bianchi type $II$, if $b = 0$ but $a \neq 0$ or $a = 0$ and $vb < 0$ to the ones of Bianchi type $VI_0$, if $a = 0$ and $vb > 0$ to Bianchi type $VII_0$ and otherwise to Bianchi type $VIII$.

- **Type III**
\[
(n_{III}^{\alpha \beta}) = \begin{pmatrix}
a & 1 & 0 \\
1 & -a & 1 \\
0 & 1 & -a
\end{pmatrix},
\] (3.10)
which admits a triple null vector.

This structure constant density may only correspond to Lie algebras of Bianchi type $VI_0$ if $a = 0$ or $VIII$ if $a \neq 0$.

- **Type IV**
\[
(n_{IV}^{\alpha \beta}) = \begin{pmatrix}
0 & v & 0 \\
v & a & 0 \\
0 & 0 & b
\end{pmatrix},
\] (3.11)
where $a^2 < 4v^2$ and that has only one simple spacelike eigenvector, but no timelike or null eigenvector.
Here again, only the structure constants of Lie algebras of Bianchi types VI\text{0} if \( b = 0 \) or VIII if \( b \neq 0 \) are available.

### 3.3. Non-unimodular Lie algebras

In the case of non-unimodular Lie algebras, we have to consider three possibilities: the vector \( k\alpha \) is timelike, spacelike or null; four type of normal forms will occur.

- **Timelike** \( k\alpha \): we choose the frame such that \( k\alpha = (k, 0, 0) \). The Jacobi identity implies that \( n^\alpha{}_{\beta\gamma} \) is a spacelike symmetric tensor that can be diagonalized by a rotation in the \([1, 2]\) plane. But again this normal form turns out not to be the most suitable one for solving the field equations, and we prefer to use the following form:

  \[
  (n^\alpha{}_{\beta\gamma})_{T} = \begin{pmatrix}
  0 & 0 & 0 \\
  0 & a & b \\
  0 & b & a
  \end{pmatrix}
  \]  
  \( \text{(type } T \text{)} \). \hspace{1cm} (3.12)

  Obviously, all non-unimodular Lie algebras may lead to this form of the structure tensor density. More precisely, we have Lie algebras of Bianchi type V if \( a = b = 0 \), of type IV if \( a = \pm b \neq 0 \), and otherwise of types III, VI\text{0} for \( |b| > |a| \) or VIII\text{0} for \( |a| > |b| \) with \( h = k^2/(a^2 - b^2) \).

- **Spacelike** \( k\alpha \): we choose the frame such that \( k\alpha = (0, 0, k) \). Then, the structure tensor density may take three different canonical forms. If it admits a timelike (and thus a spacelike) eigenvector

  \[
  (n^\alpha{}_{\beta\gamma})_{SI} = \begin{pmatrix}
  a & 0 & 0 \\
  0 & b & 0 \\
  0 & 0 & 0
  \end{pmatrix}
  \]  
  \( \text{(type } SI \text{)} \), \hspace{1cm} (3.13)

  then it may correspond to any type B Bianchi space, namely: Bianchi type V if \( a = b = 0 \), type IV if \( a \) or \( b \) non-zero and otherwise types III, VI\text{0} for \( ab < 0 \) or VIII\text{0} for \( ab > 0 \) with \( h = k^2/(a b) \). If it has a double null eigenvector, then

  \[
  (n^\alpha{}_{\beta\gamma})_{SII} = \begin{pmatrix}
  v + a & v & 0 \\
  v & v - a & 0 \\
  0 & 0 & 0
  \end{pmatrix}
  \]  
  \( v = \pm 1 \) \( \text{(type } SII \text{)} \), \hspace{1cm} (3.14)

  and it corresponds to Lie algebras of Bianchi type IV if \( a = 0 \), or otherwise Lie algebras of Bianchi types III and VIII\text{0} with \( h = -k^2/a^2 \). If the structure tensor density does not have any other eigenvector, it can be put in the form

  \[
  (n^\alpha{}_{\beta\gamma})_{SIII} = \begin{pmatrix}
  0 & v & 0 \\
  v & a & 0 \\
  0 & 0 & 0
  \end{pmatrix}
  \]  
  \( 4 v^2 > a^2 \) \hspace{1cm} (type SIII) \hspace{1cm} (3.15)

  and corresponds to Bianchi types III and VIII\text{0} with \( h = -k^2/v^2 \).

- **Lightlike** \( k\alpha \): here, without loss of generality, we may assume \( k\alpha = (1, 1, 0) \). Using the Jacobi identity, we obtain the expression

  \[
  (n^\alpha{}_{\beta\gamma}) = \begin{pmatrix}
  a & -a & b \\
  -a & a & -b \\
  b & -b & c
  \end{pmatrix}
  \]  
  \hspace{1cm} (3.16)
corresponding to Bianchi type $V$ if $a = b = c = 0$, type $IV$ if $b^2 = ac$ and Bianchi types $III$, $VI_s$ for $b^2 > ac$ or $VII_s$ for $b^2 < ac$ with $h = 1/(ac - b^2)$ in the other cases. Without loss of generality, it can still be simplified by performing an appropriate null rotation around $k_o$ that leads to

\[
(n^{\alpha\beta}_{L3}) = \begin{pmatrix} a & -a & 0 \\ -a & a & 0 \\ 0 & 0 & c \end{pmatrix} \quad \text{(type $L$)} \quad \text{or} \quad (n^{\alpha\beta}_{L3'}) = \begin{pmatrix} 0 & 0 & b \\ 0 & 0 & -b \\ b & -b & 0 \end{pmatrix} \quad \text{(type $L'$).}
\]

(3.17)

4. Solutions of the field equations

4.1. Simply transitive groups

In this section, we solve the MG field equations, which were presented in section 2, in terms of homogeneous spaces of section 3.

As already mentioned, in the framework of Bianchi spaces, the field equations reduce to algebraic equations. To solve these equations, we first obtain the link between the cosmological and coupling constants $\Lambda$ and $\mu$ (resp. $\xi$) and the structure constant parameters. Then, we insert them into the field equations and discuss the remaining constraints that have to be satisfied.

We shall provide some details in the first case, whereas for the other ones we shall just display the solutions of the equations.

- Unimodular Lie algebras

Type $I$

TMG: using expression (3.5) of the tensor density defining the structure constants, the TMG field equations reduce to three independent equations:

\[
2a^2(a + b) + 8bc^2 + (a^2 - 4c^2 + 4\Lambda)\mu = 0, \quad (4.1)
\]

\[
\frac{a^2}{4} + ab + c^2 - 3\Lambda = 0, \quad (4.2)
\]

\[
c(-a^2 + 4ab + 8b^2 + 4c^2 - 2(a + 2b)\mu) = 0. \quad (4.3)
\]

First, let us assume $c = 0$, so that equation (4.3) is trivially satisfied. We deduce from the other two that

\[
\Lambda = \frac{1}{12}a(a + 4b), \quad (4.4)
\]

\[
\mu = -\frac{3}{2}a, \quad (4.5)
\]

from which we obtain the Petrov type $D_t$ solution

\[
a = -\frac{2}{3}\mu, \quad b = \frac{\mu^2 - 27\Lambda}{6\mu}, \quad c = 0. \quad (4.6)
\]

To isolate the value (4.5) of $\mu$ in equation (4.1) we have assumed that $a^2 + 4\Lambda \neq 0$. If $a^2 + 4\Lambda = 0$, then the parameter $\mu$ remains undetermined and the field equations are satisfied if the cosmological constant is negative and

\[
a = -b = 2\sqrt{-\Lambda},
\]

(4.7)

or if $\Lambda = 0$, for $a = 0$ and $b$ arbitrary. Of course, these last two solutions correspond to conformally flat spacetimes (Petrov type $O$). More precisely, if $a = 0$ the solution is flat; otherwise it is AdS$_3$ when $a = -b \neq 0$. 
If we assume \( c \neq 0 \), then the values of \( \Lambda \) and \( \mu \) provided by the first two field equations (4.1) and (4.2) become

\[
\Lambda = \frac{1}{12} (a^2 + 4ab + 4c^2),
\]

\[
\mu = -\frac{3}{2} \left( \frac{a^2(a + b) + 4bc^2}{a(a + b) - 2c^2} \right).
\]

Inserting these values into the third nontrivial field equation (4.3) we obtain

\[
((a + b)^2 - c^2)(a^2 + 4ab + 4c^2) = 0,
\]

where we have assumed that \( \mu \neq 0 \). This leads to two solutions (changing the sign of \( c \) corresponds to a reflexion of \( \theta^1 \) or \( \theta^2 \) and thus to change in the sign of \( \mu \)) of Petrov type \( D_r \):

\[
a = \frac{27\Lambda - \mu^2}{6\mu}, \quad b = \frac{5\mu^2 - 27\Lambda}{12\mu}, \quad c = \pm(a + b) = \pm \frac{\mu^2 + 9\Lambda}{4\mu}
\]

and a third one of Petrov type \( I^3 \), nonflat, but with a vanishing cosmological constant, and

\[
b = \frac{1}{2} (\mu + a), \quad c^2 = -\frac{1}{4} a(3a + 2\mu),
\]

which restricts the parameter \( a \) to belong to the interval\(^4\): \( a \in \left[ 0, \frac{-2}{3}\mu \right] \).

* NMG: the strategy is the same, but the equations are a little bit more cumbersome. The field equations lead to

\[
-21a^4 - 20a^3b + 80abc^3 + 256b^2c^2 + 80c^4 + 8x (a^2 + 2ab - 4\Lambda) = 0,
\]

\[
-63a^4 - 80a^3b + 8a^2 (-2b^2 + 3c^2 + 2\xi^3) + 16 (16b^2c^2 + 5c^4 - 4c^2\xi + 4\Lambda \xi^3) = 0,
\]

\[
c (5a^3 - 2a^2b + 40ab^2 - 8\xi (a + 2b) + 20a^2c^2 + 64b^3 + 104b^2c^2) = 0.
\]

In this case, there are also three different types of solutions.

If we assume \( c = 0 \), then we obtain

\[
\Lambda = \frac{a(21a^2 + 72ab + 16b^2)}{8(21a + 4b)},
\]

\[
\xi = \frac{a}{8} (21a + 4b),
\]

which leads to a Petrov type \( D_r \) solution:

\[
a^2 = \frac{16}{21} (3\xi \pm \sqrt{\xi (7\Lambda + 5\xi)}),
\]

\[
b^2 = \frac{42\Lambda \xi \pm \sqrt{3(21\Lambda - 17\xi)\sqrt{\xi (7\Lambda + 5\xi)} - 66\xi^2}}{4(\Lambda - \xi)}.
\]

We find more convenient to discuss this solution by introducing the parameter \( x = b/a \) in terms of which we may rewrite equations (4.16) and (4.17) as

\[
\Lambda = \frac{a^2}{8} \frac{21 + 72x + 16x^2}{21 + 4x}, \quad \xi = \frac{a^2}{8} (21 + 4x).
\]

\(^4\) When we write an interval as \( [x_0, x_1] \), we do not assume \( x_0 \leq x_1 \), but consider the union of the sets \( [x|x_0 \leq x \leq x_1] \cup [x|x_1 \leq x \leq x_0] \) even, unless \( x_0 = x_1 \), one of these two sets is always empty. This convention allows us to avoid tedious (but elementary) discussion about signs.
In order to have $\xi > 0$ we need, in addition to $a \neq 0$, to impose that $x > -21/4$, which implies that $\Lambda > 0$ if $-(2\sqrt{15} + 9)/4 > x > -21/4$ or $x > (2\sqrt{15} - 9)/4$.

If we do not assume $c = 0$, the first two equations (4.13) and (4.14) lead to much more complicated expressions

$$\Lambda = \frac{2a^2c^2(15a^2 + 28a^2b + 12b^2) + a^2(a^2b)(21a^2 + 72ab + 16b^2) - 16b^2c^2(15a + 32b) - 160b^2}{8(a^2b)(21a^2 + 4b^2) - 2c^2(3a^2 + 10ab + 64b^2) - 40c^2}$$

(4.21)

$$\xi = \frac{a^2(2a + b)(21a + 4b) - 2c^2(3a^2 + 10ab + 64b^2) - 40c^2}{16(a^2 + b^2 - 2c^2)}.$$  

(4.22)

Inserting these into the field equation (4.15) we find

$$a(a + b)^2(a^2 + 2ab - 4b^2) = (a^3 + 10a^2b + 12ab^2 + 8b^3)c^2 + 8bc^4 = 0$$

(4.23)

whose solutions are $c^2 = (a + b)^2 = a^2(1 + x)^2$ and $c^2 = a(a^2 + 2ab - 4b^2)/8b = a^2(1 + 2x - 4x^2)/8x$.

From the first solution, which is Petrov type $D_5$, we obtain

$$\xi = \frac{a^2}{16}(1 + 2x)(25 + 42x), \quad \Lambda = \frac{a^2}{8}(1 + 2x)(84x^2 + 228x + 109).$$

(4.24)

which satisfy the requirement $\xi > 0$ if $x \notin [-25/42, -1/2]$. We can immediately see that this restriction is compatible with both signs of the cosmological constant: $\Lambda < 0$ if $x \in (-8\sqrt{15} + 57)/42, (8\sqrt{15} - 57)/42 \cup -25/42, -1/2]$; $\Lambda = 0$ at the boundary of these intervals, excepted at $x = -25/42$ where it diverges; otherwise $\Lambda > 0$ if $x \notin [-8\sqrt{15} + 57)/42, (8\sqrt{15} - 57)/42 \cup -25/42, -1/2]$. Thus, $\Lambda$ is bounded from below (by approximatively $-0.3a^2$), but it can be chosen arbitrarily positive.

For the second solution $c^2 = a^2(1 + 2x - 4x^2)/8x$, things are a little bit more involved. We obtain

$$\xi = \frac{a^2}{32x}(64x^3 - 44x^2 + 36x + 5), \quad \Lambda = \frac{5a^2}{16x}(1 + 2x)^2(64x^3 - 44x^2 + 36x + 5).$$

(4.25)

Thus, $\xi$ will be positive only if $x \notin \{x_0, 0\}$, where $x_0 \simeq -0.119$ is the single real root of the cubic polynomial $64x^3 - 44x^2 + 36x + 5$. For $x \in \{x_0, 0\}$, we always have $\Lambda < 0$; for $x \notin \{x_0, 0\}$, we have $\Lambda > 0$, a decreasing function of $x$ which varies from $+\infty$ to $5a^2/64$.

* GMG: the combination of the two theories introduces three constants that are related to the components of the structure tensor density (3.5) by

$$\Lambda = -\frac{5(a^2 + 4a b + 4c^3)^3}{8(a^2(3a^2 - 40ab - 112b^2) - 8(9a^2 + 20ab - 16b^2)c^2 - 80c^2))}$$

(4.26)

$$\mu = \frac{a^2(-3a^2 + 40ab + 112b^2) + 8(9a^2 + 20ab - 16b^2)c^2 + 80c^4}{16(a^3 + 2a^2b - 4ab^2 - 8bc^2)}$$

(4.27)

$$\xi = \frac{a^2(-3a^2 + 40ab + 112b^2) + 8(9a^2 + 20ab - 16b^2)c^2 + 80c^4}{8(a^2 + 4ab + 4c^2)}.$$  

(4.28)

Inverting this system as well as discussing in general the positivity of $\xi$ is not very illuminating. So, we have plotted the region in the $(x = b/a, y = c/a)$ plane of the sign of $\xi$ in figure 1. Let us also remark that $\Lambda \xi = (5/64)(a^2 + 4ab + 4c^2)^2 > 0$.

However, using the special values of the structure tensor density we can easily see from condition (2.10) that there are Petrov type $O$ solutions, with arbitrary value of $\mu$ and negative values of $\Lambda = -(45/184)a^2$ and $\xi = -(23/8)a^2$ that are stable for
\( \mu^2 \leq (12167/16208) \ a^2 \). There are also Petrov types \( D_r \) and \( D_s \) solutions that allow positive values of \( \xi \). As for a structure tensor density of type \( I \), we have as many independent parameters in the solution as there are physical constants of the problem. Thus, we may expect that for some range of values a finite number of solutions are always defined. Of course, when the number of geometrical parameters will be less than 3, the solutions, if any, will exist only for special values of the physical constants.

Type \( II^* \) TMG: three different solutions occur. One of Petrov type \( N \), with a negative cosmological constant

\[
\Lambda = -\frac{\mu^2}{9}, \quad a = -\frac{2\mu}{3}, \quad b = \frac{2\mu}{3};
\]

and two solutions, with a vanishing cosmological constant, of Petrov types \( N \) and \( II \) respectively,

\[
\Lambda = 0, \quad a = -\frac{\mu}{2}, \quad b = 0; \quad (4.30)
\]

\[
\Lambda = 0, \quad a = -\frac{\mu}{6}, \quad b = \frac{2\mu}{3}. \quad (4.31)
\]

Let us mention that trivial flat solutions also occur as solutions of the field equations, for instance, we find a solution with \( a = b = 0 \), corresponding to the flat space\(^6\); we shall no

---

\(^5\) Generically, the Jacobian \( (J) \) of the transformation defined by the equations \( (4.26)-(4.28) \) is:

\[
J = 40 \ a \ c \ \frac{(a+b)^2-c^2)(c^2-b^2)(a^2+4ab+4c^2)}{(a^2+2ab-4ab^2-8bc^2)^3}.
\]

\(^6\) This emphasizes the fact that Bianchi \( I \) is always a solution, and it simply corresponds to a flat space, but flat space may also appear as a Bianchi \( II \) model. In the same way, anti-de Sitter space can be seen as a Bianchi \( III \) space, but, locally, also as a Bianchi \( III \) model.
longer insist on such solutions. ∗ NMG: we also obtain three types of solutions; two of Petrov type $N$:

$$
\begin{align*}
 b = 0, & \quad a^2 = \frac{1}{4} \xi, & \Lambda = 0 \\
 b = -a, & \quad a^2 = \frac{8}{17} \xi, & \Lambda = -\frac{135}{229} \xi
\end{align*}
$$

and a third one of Petrov type $II$:

$$
\begin{align*}
 b = -a, & \quad a^2 = \frac{8}{17} \xi, & \Lambda = -\frac{135}{229} \xi
\end{align*}
$$

Let us note that the cosmological constant as well as the coupling constant are independent of $\nu$ (see equation (3.9)), but the Riemann curvature tensor is $\nu$ dependent.

∗ GMG: we obtain solutions of Petrov type $II$:

$$
\begin{align*}
 \Lambda &= \frac{5b(4a + b)^3}{8(-112 a^2 - 40ab + 3b^2)}, \\
 \mu &= \frac{b(-112 a^2 - 40ab + 3b^2)}{16(-4a^2 + 2 ab + b^2)}, \\
 \xi &= \frac{b(112 a^2 + 40ab - 3b^2)}{8(4a + b)},
\end{align*}
$$

where $\Lambda \geq 0$. To have $\xi > 0$, $b$ must be in the intervals $]a (20 + 4\sqrt{46})/3, 0[\cup ]a (20 - 4\sqrt{46})/3, -4 a[$.

Solutions of Petrov type $N$ also occur in the cases where $b = 0$

$$
\begin{align*}
 \Lambda &= 0, & \xi &= \frac{4 a^2 \mu}{2 a + \mu},
\end{align*}
$$

or when $b = -a$

$$
\begin{align*}
 \Lambda &= -a^2 \frac{70 \mu + 3 a}{272 \mu}, & \xi &= \frac{17 a^2 \mu}{4(3a + 2\mu)}.
\end{align*}
$$

Type $III$

∗ TMG: the condition $\mu \neq 0$ is incompatible with the field equations.

∗ NMG: there is no solution with $\xi \neq 0$.

∗ GMG: there is no solution with $\xi > 0$ but a special one of Petrov type $III$ with $\xi \leq 0$, namely

$$
\begin{align*}
 \mu &= -\frac{60}{89} a, & \Lambda &= -\frac{45}{143} a^2 \leq 0, & \xi &= -\frac{23}{17} a^2 \leq 0.
\end{align*}
$$

Type $IV$

∗ TMG: taking into account the condition $4 \nu^2 > a^2$, we obtain as the only (real) solution

$$
\begin{align*}
 a &= \frac{\mu}{2} \mp \nu, & b &= \frac{\mu}{2} \pm \nu, & \Lambda &= 0
\end{align*}
$$

of Petrov type $I_C$, and subject to the condition: $\nu \notin [\mp \mu/2, \pm \mu/6]$.

∗ NMG: once $\Lambda$ and $\xi$ are expressed in terms of $a, b$ and $\nu$, the field equations are satisfied if

$$
\nu^2 = b(a - b) \quad \text{or} \quad \nu^2 = \frac{(a^2 - b^2)(a - b)}{4a}.
$$

7 We shall make it more explicit in section 4.2.
It is easy to verify that the first one is disallowed by the condition $4 \nu^2 > a^2$, whereas for the second one, we may parametrize again the solutions as follows:

\begin{align}
  b &= x a, \\
  \nu^2 &= \frac{a^2}{4} (1 - x^2)(1 + x), \\
  \Lambda &= \frac{5 a^2}{8} \frac{(1 - x)^4 x^2}{(8 + 11 x + 18 x^2 - 5 x^3)}, \\
  \xi &= \frac{a^2}{8} (8 + 11 x + 18 x^2 - 5 x^3).
\end{align}

The condition $4 \nu^2 > a^2$ implies that $x > (1 + \sqrt{5})/2$ or $0 > x > (1 - \sqrt{5})/2$, while the positivity of $\xi$ requires $x < x_0$, where $x_0$ is the single real root of the cubic polynomial $8 + 11 x + 18 x^2 - 5 x^3$, i.e. $x_0 = (18 + \sqrt{12987} - 60\sqrt{14370} + \sqrt{3(4329 + 20\sqrt{14370})})/15$, $x_0 \approx 4.21$, and ensures that $\Lambda > 0$. This solution is of Petrov type $T$, both with respect to the classification of the Ricci and the Cotton–York tensors.

* GMG: the generic solution is of Petrov type $I_C$ with

\begin{align}
  \Lambda &= -\frac{5((a-b)^2 - 4\nu^2)^3}{8[(a-b)^2(3a^2 + 26ab + 3b^2) + 8\nu^2(a-b)(a-b) - 80\nu^4]}, \\
  \mu &= \frac{(a-b)^2(3a^2 + 26ab + 3b^2) + 8\nu^2(a-b)(a-b) - 80\nu^4}{16[(a-b)^2(a+b) - 4\nu^2]}, \\
  \xi &= -\frac{(a-b)^2(3a^2 + 26ab + 3b^2) + 8\nu^2(a-b)(a-b) - 80\nu^4}{8[(a-b)^2 - 4\nu^2]}
\end{align}

where $\Lambda \xi = (5/64)((a-b)^2 - 4\nu^2)^2 > 0$. We have plotted in figure 2 the zero and singular curves of $\Lambda, \mu$ and $\xi$ on the $(x = a/\nu, y = b/\nu)$ plane.

We also found a Petrov type $D_s$ solution:

\begin{align}
  a = b = -\frac{\xi}{2\mu}, \quad \nu^2 = -\frac{2}{5} \xi > 0, \quad \text{with:} \quad \Lambda = \frac{1}{5} \xi.
\end{align}

* Non-unimodular Lie Algebras

Type $T$

* TMG: we obtained two solutions, only defined for a positive cosmological constant. The first one is of Petrov $O$ and locally a dS$_3$ space,

\begin{align}
  b &= 0, \quad k^2 = \Lambda,
\end{align}

while the second one is of Petrov type $D_s$

\begin{align}
  a = \frac{\mu}{3}, \quad k^2 &= \frac{3}{4} \Lambda - \frac{1}{36} \mu^2, \quad b^2 = \frac{3}{4} \Lambda + \frac{1}{12} \mu^2, \quad \Lambda > \mu^2/27.
\end{align}

* NMG: three different solutions are available. The first one is of Petrov type $I_6$ (or Petrov type $D_s$ when $\xi = 2\Lambda$)

\begin{align}
  a = 0, \quad b^2 = \frac{1}{8}(3\xi + \Lambda), \quad k^2 = \frac{1}{8}(5\Lambda - \xi)
\end{align}

which of course requires that $\Lambda > \xi/5$.

The second one is of Petrov type $O$ and locally a dS$_3$ space:

\begin{align}
  b &= 0, \quad k^2 = 2(\xi \pm \sqrt{\xi}(\xi - \Lambda)).
\end{align}
Note that \( \xi \notin [0, \Lambda] \), and \( k^2 \geq 0 \) requires the plus sign in front of the square root unless \( \xi \geq \lambda \geq 0 \) in which case both signs are accepted.

The third one is of Petrov type \( D_\ast \):

\[
\begin{align*}
&b^2 = a^2 + k^2, \\
&a^2 = \frac{6 \xi - \sqrt{\xi (15 \xi + 21 \Lambda)}}{21}, \\
&k^2 = \frac{\sqrt{\xi (15 \xi + 21 \Lambda)} \pm 4 \xi}{4} \quad \text{(4.53)}
\end{align*}
\]

which are real for the upper signs if \( \xi > 0 \) and for the lower ones if \( \xi \in [0, 21 \Lambda] \) or \( \xi \in [\Lambda, 0] \) depending on whether \( \Lambda \) is positive or negative respectively. Note that for \( \xi = \Lambda \) the Cotton–York tensor is vanishing, while the traceless Ricci and \( \tilde{K}_{\alpha \beta} \) tensors remain of Petrov type \( D_{\ast} \).

* GMG: generically we obtain solutions of Petrov type \( I_R \):

\[
\begin{align*}
&\Lambda = \frac{3 k^2 - 14 b^2 k^2 + 16 a^2 k^2 - 5 b^2}{2(k^2 + 8 a^2 - 5 b^2)}, \quad \text{(4.54)} \\
&\mu = \frac{k^2 + 8 a^2 - 5 b^2}{8 a}, \quad \text{(4.55)} \\
&\xi = -k^2 - 8 a^2 + 5 b^2. \quad \text{(4.56)}
\end{align*}
\]

The singular and zero curves of \( \Lambda \), \( \mu \) and \( \xi \) in the \( (x = a/k, y = b/k) \) plane are depicted in figure 3. Let us mention that in the region where \( \xi > 0 \) (the interior of the hyperbola), we have also that \( \Lambda > 0 \).
Figure 3. Graphical representation, in the framework of non-unimodular group spaces of type $T_4$, of the zero and singular curves of the cosmological constant $\Lambda_1$ and of the coupling constants $\xi$ and $\mu$ as a function of the rescaled parameters $x = a/k$ and $y = b/k$ ($\Lambda = \infty$ but $\xi = 0$ and $\mu = 0$ on the (dashed) hyperbola, $\Lambda = 0$ on the black fourth-order algebraic (solid) curve, $\mu = \infty$ on the $y$ axis). For each region of the plane delimited by these curves, we indicate the sign of $\Lambda_1$, $\mu$ and $\xi$.

There are also solutions of Petrov type $D_s$, with $k^2 = b^2 - a^2$, $\mu$ arbitrary and

$$\Lambda = \frac{3a^3 \xi - 35a^6 \mu - 8a^3 b^2 + 40a^2 b^2 \mu - 16ab^4 + 16b^4 \mu}{2\mu(17a^2 + 4b^2)}, \quad \xi = \frac{17a^2 + 4b^2}{2(\mu - 3a)}.$$  

(4.57)

We plot in figure 4 the zero curve of $\Lambda$ and the singular straight line of $\xi$ in the $(x = a/\mu, y = b/\mu)$ plane.

Finally, solutions of Petrov type $O$ are obtained for arbitrary values of the coupling constant $\mu$ and $\xi$. These are dS$_3$ (or flat) spaces but with

$$\Lambda = \frac{4\xi - k^2}{4\xi}.$$  

(4.58)

Let us notice that for $a = 0$, the solutions (4.53) and (4.57) correspond to conformally flat geometries with $\Lambda = \xi \geq 0$, like $\mathbb{R} \times \text{dS}_2$ or $\mathbb{R} \times \text{AdS}_2$ respectively, the latter having been considered by Clement [10].

Type $SI$

* TMG: two nontrivial solutions occur. The first one is of Petrov type $O$ and locally an AdS$_3$ space with $\Lambda < 0$,

$$b = -a, \quad k^2 = -\Lambda.$$  

(4.59)

The second one, with $\mu^2/27 > \Lambda \geq -\mu^2/9$, is of Petrov type $D$:

$$a = \frac{-2\mu \pm \sqrt{27\Lambda + 3\mu^2}}{6}, \quad b = \frac{2\mu \pm \sqrt{27\Lambda + 3\mu^2}}{6}, \quad k^2 = -ab = \frac{\mu^2 - 27\Lambda}{36}.$$  

(4.60)
More precisely, with the upper sign (+), the solution is of Petrov type $D_1$ (resp. $D_2$) for $\mu > 0$ (resp. $\mu < 0$); with the lower sign (−), it is the converse: $D_2$ (resp. $D_1$) for $\mu > 0$ (resp. $\mu < 0$).

* NMG: here we obtain $\xi = \frac{1}{8} [5 (a + b)^2 + 16 (a^2 + b^2) + 36 k^2] > 0$ and the remaining field equations are satisfied in three cases. If

$$a = -b, \quad k^2 = 2 (-\xi \pm \sqrt{\xi (\xi - \Lambda)})$$

the minus sign requires that $0 > \Lambda \geq \xi$, whereas for the plus sign: $\xi > 0$ and $\Lambda < 0$ or $\xi < 0$ and $\Lambda \geq \xi$. It is a Petrov type $O$ solution, corresponding to an AdS$_3$ space, which is stable when $\xi > 0$.

We also obtain $k^2 = -ab$, in which case, parametrizing the solution as before by $b = xa$, we have

$$\xi = \frac{a^2}{8} [8(x^2 + 1) + 13(x - 1)^2] > 0 \quad \text{and} \quad \Lambda = \frac{a^4}{64\xi} (21 x^4 + 204 x^3$$

$$-194 x^2 + 204 x + 21).$$

The positivity of $k^2$ requires that $x < 0$. So the maximum of $\Lambda/\xi$ is reached at $x = 0$ or $x = -\infty$ where the ratio tends to $1/21$. In the limit $x = 0$, we obtain $\Lambda = 0 < \xi / 21 = a^2/8$.

As the quartic polynomial $21 x^4 + 204 x^3 - 194 x^2 + 204 x + 21$ has only two (negative) real roots: $x_1 \approx -0.1$ and $x_2 \approx -10.7$, $\Lambda$ is negative for the value of $x$ between these two roots. It reaches its minimum at $x \approx -1$. The coupling constant $\xi$ increases monotonically as $x$ decreases, when $x$ goes to $-\infty$, where both $\Lambda$ and $\xi$ diverge. This solution is of Petrov type $D_1$ if $x \in [ -1, 0 ]$, Petrov type $O$ when $x = -1$ and Petrov type $D_2$ if $x \in (-\infty, -1)$. There also is a third solution of Petrov type $I_C$ (unless $a = b = 0$)

$$a = b = \pm \frac{\sqrt{\Lambda + 3\xi / 2}}{2\sqrt{2}}, \quad k^2 = \frac{\xi - 10\Lambda}{16}.$$ (4.63)

which according to the sign of $\Lambda$ requires $\xi > 10\Lambda \geq 0$ or $\xi > -\frac{2}{3} \Lambda \geq 0$ to be defined.
* GMG: generically, we obtain a Petrov type I solution

$$\Lambda = -\frac{5(a+b)^2 - 8(a^2 - 30ab + b^2)k^2 + 48k^4}{8(3a^2 - 26ab + 3b^2 - 4k^2)},$$

$$\mu = -\frac{3a^2 + 26ab - 3b^2 + 4k^2}{16(a-b)},$$

$$\xi = \frac{1}{8}(-3a^2 + 26ab - 3b^2 + 4k^2).$$

(4.64)

(4.65)

(4.66)

We also recover a Petrov type O solution, namely an AdS3 space, when

$$b = -a, \quad \Lambda = -k^2 - \frac{4\xi}{4k}.$$  

(4.67)

The absence of the tachyonic massive mode (2.10) reads

$$\xi(28^2(\xi + 4\mu^2) + k^2\mu^2(4\xi + k^2)) \geq 0.$$  

(4.68)

Accordingly, at least for large positive or negative values of $\xi$ the solutions will not contain tachyons.

We also obtain special solutions for $k^2 = -ab$, in which case we parametrize the solutions as before by $b = x a$ with $x < 0$:

$$\Lambda = -a^2\left(\frac{-28x^2 + 14x^3 + 55x^4 + 21x^5}{16(21(1+x^2) - 26x)^2}\mu\right),$$

$$\xi = \frac{a^2\mu}{4}\left(\frac{21(1+x^2) - 26x}{3(1-x)a + 2\mu}\right).$$

(4.69)

These solutions are of Petrov type $D_i$ if $x \notin \pm 1$, Petrov type $O$ when $x = -1$ and Petrov type $D_i$ if $x \notin \pm \infty$, $-1$.

Type $III$

* TMG: we easily obtain a Petrov type N solution

$$k = \pm \sqrt{-\Lambda}, \quad a = \frac{\mp \sqrt{-\Lambda} - \mu}{2};$$

we also recover a Petrov type O solution, namely an AdS3 space when

$$a = -k, \quad \Lambda = -k^2. $$

(4.70)

(4.71)

* NMG: here, we obtain a Petrov type N solution. If

$$\Lambda = -\frac{k^2(4a + 3k)(4a + k)}{2(8a^2 + 8ak + k^2)}, \quad \xi = \frac{1}{2}(8a^2 + 8ak + k^2)$$

(4.72)

the field equations are satisfied. Accordingly, imposing $\xi > 0$, we obtain solutions for all values of $a$ and $k$ such that $k \notin [-4a - 2\sqrt{2}|a|, -4a + 2\sqrt{2}|a|]$. All these solutions have a negative cosmological constant. Let us also note that, here again, the value of the parameter $\nu$ does not play any role in the parametrization of the solutions, but only appears in the curvature. For $a = -k$, we reobtain a Petrov type O solution, a stable AdS3 geometry. For $a = -k/2$, we have also a conformally flat geometry, solving the field equations, but for a negative value of $\xi = -2a^2$.

* GMG: here, we obtain a Petrov type N solution

$$\Lambda = -\frac{16a^2\mu + k^2(k + 3\mu) + 2ak(k + 8\mu)}{2(8a^2 + 8ak + k^2)\mu}k^2, \quad \xi = \frac{8a^2 + 8ak + k^2}{2(2a + k + \mu)}.$$  

(4.73)
Moreover, there also exists a solution of Petrov $O$ type for $a = -k$, which is locally an AdS$_3$ space with $\Lambda = -\frac{k^2 + 4\xi k}{4\xi}$, and whose absence of tachyonic mode is also provided by equation (4.68).

**Type S III**

* TMG: we obtain a solution of Petrov type $D$:

\[
a = \frac{2\mu}{3}, \quad k^2 = \nu^2 = \frac{\mu^2 - 27\Lambda}{36}, \quad \mu^2 < -9\Lambda, \tag{4.74}
\]

which is of Petrov type $D_s$ (resp. $D_t$) for $k = \nu$ (resp. $k = -\nu$).

* NMG: the following set of solutions occurs.

First, a Petrov type $I$ solution:

\[
a = 0, \quad k^2 = \frac{\xi - 5\Lambda}{8}, \quad \nu^2 = \frac{3\xi + \Lambda}{8}, \tag{4.75}
\]

which implies that $\Lambda < -3\xi$.

Secondly, we have also solutions of Petrov type $D$:

\[
k = \pm \nu, \quad a^2 = \frac{4}{27} (6\xi + \sqrt{3\xi (5\xi + 7\Lambda)}), \quad \nu^2 = \frac{1}{4} (4\xi + \sqrt{3\xi (5\xi + 7\Lambda)}), \tag{4.76}
\]

which requires, taking into account the condition $4\nu^2 > a^2$ and assuming $\xi > 0$, that $\Lambda \geq -5\xi/7$, otherwise $2\Lambda \leq \xi < 0$.

Finally, we also obtain

\[
k = \pm \nu, \quad a^2 = \frac{4}{27} (6\xi - \sqrt{3\xi (5\xi + 7\Lambda)}), \quad \nu^2 = \frac{1}{4} (4\xi - \sqrt{3\xi (5\xi + 7\Lambda)}), \tag{4.77}
\]

which requires that $0 > -35\xi/289 > \Lambda \geq -5\xi/7$. These solutions are of Petrov type $D_s$ (resp. $D_t$) for $k = \nu$ (resp. $k = -\nu$).

* GMG: here, we obtain solutions of Petrov type $I$

\[
\Lambda = \frac{-5a^4 - 8a^2(5\nu^2 - k^2) + 16(3k^4 - 14k^2\nu^2 - 5\nu^4)}{8(3a^2 - 4k^2 + 20\nu^2)}, \tag{4.78}
\]

\[
\mu = \frac{3a^2 - 4k^2 + 20\nu^2}{16a}, \tag{4.79}
\]

\[
\xi = \frac{1}{8} (-3a^2 + 4(k^2 - 5\nu^2)). \tag{4.80}
\]

We also find solutions, with $k = \pm \nu$, of Petrov type $D$

\[
\Lambda = \frac{-21a^4 - 42a^2\mu - 160a^3\nu^2 + 576a^2\mu \nu^2 + 256\mu \nu^2 - 512\mu \nu^4}{336\mu a^2 - 256\mu \nu^2}, \tag{4.81}
\]

\[
\xi = \frac{-21a^2\mu + 16\mu \nu^2}{4(3a^2 - 2\mu)}, \tag{4.82}
\]

which are of Petrov type $D_s$ (resp. $D_t$) for $k = \nu$ (resp. $k = -\nu$).
Type L:

* TMG: there is one solution of Petrov type $D_s$, defined only for special values of $\mu$ and $\Lambda$ (positive):

$$a = -\frac{1}{c}, \quad \mu = \pm 3\sqrt{3} \Lambda, \quad c = \pm 2\sqrt{3} \Lambda. \quad (4.83)$$

* NMG: here also, only one solution of Petrov type $D_s$ is obtained, for special values of $\xi$ and $\Lambda$ (positive):

$$a = -\frac{1}{c}, \quad \xi = 21 \Lambda, \quad c = \pm 2\sqrt{2} \Lambda. \quad (4.84)$$

* GMG: there is a solution of Petrov type $D_s$:

$$a = -\frac{1}{c}, \quad \xi = \frac{21 \mu}{a(12 + 8a \mu)}, \quad \Lambda = \frac{1 + 2a \mu}{16a^2 \mu}. \quad (4.85)$$

We also found a class of solutions for arbitrary $a$ which are of Petrov type $II$:

$$\Lambda = -\frac{5}{24} c^2, \quad \mu = \frac{3}{16} c, \quad \xi = -\frac{3}{8} c^2. \quad (4.86)$$

Type $L'$: the solutions we obtain are flat spaces.

* TMG, NMG and GMG: the only solution is the flat space, obtained with

$$b = 0 \quad \text{or} \quad b = -1 \quad \text{and} \quad \Lambda = 0. \quad (4.87)$$

### 4.2. Coordinate representations of the metrics

Having at our disposal all the homogeneous solutions of MG theories in a formal way, the purpose of this subsection is to illustrate how to express them in terms of the invariant forms that define the coordinate system.

This can be achieved in two ways. Either algebraically by determining the $GL(3, \mathbb{R})$ matrix which transforms the canonical expression of the structure vector and tensor density in the forms we use or by a direct integration of the Cartan equations (3.1) defining the invariant vector fields from which we deduce the expression of their dual basis.

For illustrative purpose, we now sketch both approaches. The first one is performed in the framework of the GMG solution (4.38) obtained from the unimodular type $III$ structure tensor density $^8$ (3.10); the second one for the non-unimodular $SII$ solutions, built from the structure tensor density (3.14).

To diagonalize and put into the standard form

$$n_{(\alpha\beta)\,III} = \text{diag.}(−1, 1, 1) \quad (4.88)$$

the structure tensor density (3.10) we just have to determine the eigenvalues and the appropriately rescaled eigenvectors of the matrix $n^{\alpha\beta}_{III}$. The eigenvalues are given by the three real, distinct (and nonvanishing) roots $\lambda^{(p)}$ of the polynomial

$$\lambda^3 + a \lambda^2 - (a^2 + 2) \lambda - a^3 = 0. \quad (4.89)$$

For positive values of $a$, we always have two eigenvalues negative and one positive ; it is the converse for negative $a$.

The corresponding eigenvectors can be chosen proportional to

$$u^{(p)}_a = ((a + \lambda^{(p)})^2 - 1, \quad (a + \lambda^{(p)}), \quad 1). \quad (4.90)$$

$^8$ Assuming $a \neq 0$. 

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They may be used to diagonalize $\sigma_{\alpha\beta}^\mu$ and lead, at an intermediate step, to the diagonal matrix 

$$
\tilde{\sigma}^{\mu \nu} = -\lambda^{(p)} \frac{u^{(p)} \cdot u^{(q)}}{(\lambda^{(1)} - \lambda^{(2)})(\lambda^{(2)} - \lambda^{(3)})(\lambda^{(3)} - \lambda^{(1)})} = \delta^{\mu \nu} \lambda^{(p)} \frac{4a^2 + 1)(\lambda^{(p)})^2 + 4a(2a^2 + 1) \lambda^{(p)} + (4a^4 - a^2 + 2)}{2\sqrt[3]{16a^4 + 13a^2 + 8}} =: \tilde{\sigma}^{\mu \nu} \delta^{\mu \nu}.
$$

(4.91)

To obtain the standard expression of the structure constant density tensor we still have to rescale this matrix. Assuming that we ordered the roots such that $\lambda^{(1)} > \lambda^{(2)} > \lambda^{(3)}$ when $a > 0$ or $\lambda^{(1)} < \lambda^{(2)} < \lambda^{(3)}$ when $a < 0$, this is achieved thanks to the transformation defined by the diagonal matrix $L$ of components:

$$
L^{\mu \nu} := -\text{sgn}(a) \frac{\lambda^{(p)}}{2|\tilde{\sigma}^{\mu \nu}|} \sqrt[3]{16a^4 + 13a^2 + 8} =: \delta^{\mu \nu} L_{\alpha \beta}^{(p)}.
$$

(4.92)

Finally, the metric components with respect to the usual right invariant 1-forms on the Bianchi VIII group:

$$
\theta^1 = dt - \sinh x dy, \quad \theta^2 = \cos t dx - \sin t \cosh x dy, \quad \theta^3 = \sin t dx + \cos t \cosh x dy
$$

(4.93)

are given by the matrix product

$$
g_{\mu \nu} \equiv \Lambda_{\mu}^{\alpha} \Lambda_{\nu}^{\beta} \eta_{\alpha \beta}
$$

(4.94)

with $(\Lambda^{-1})_{\mu}^{\alpha} = L^{(p)}_{\mu \alpha}$. Let us emphasize that this metric will not be diagonal and cannot be diagonalized by an Iso$(1, 2)$ transformation (being itself also of ’type II’), but put into the form:

$$
dx^2 = a^{-2}(-\theta^1)^2 + (\theta^2)^2 + (\theta^3)^2 + \gamma (\theta^1 + \theta^3) \theta^2,
$$

(4.95)

where $\gamma \neq 0$ is an arbitrary parameter.

Similarly, solutions ((4.31), (4.34) and (4.35)) can be written, in terms of the 1-forms (4.93), as:

$$
dx^2 = \delta_{\alpha \beta}^{II} \theta^\alpha \theta^\beta,
$$

(4.96)

where the matrix of component ($\delta_{\alpha \beta}^{II}$) is numerically equal to the matrix ($\delta_{\mu \nu}^{\alpha \beta}$) that was introduced in equation (3.9); we also demand the constraint $b a^2 = 1$ upon its elements.

Furthermore, the solution (4.86) can be written,

$$
dx^2 = |c| (\theta^1)^2 - \frac{2 \theta^2 \theta^3}{\sqrt[3]{c}}
$$

(4.97)

in terms of the 1-forms defining Bianchi spaces $VI_h$ or $VII_h$.

$$
\theta^1 = e^{-h z} (\cosh z dx + \sinh z dy), \quad \theta^2 = e^{-h z} (\sinh z dx + \cosh z dy), \quad \theta^3 = dz
$$

(4.98)

$$
\theta^1 = e^{-h z} (\sin z dx + \cos z dy), \quad \theta^2 = e^{-h z} (\cos z dx - \sin z dy), \quad \theta^3 = dz.
$$

(4.99)

Let us now illustrate the analytical approach. The $III$ structure constants correspond to a group admitting an Abelian two-dimensional subgroup. The corresponding Cartan equations are obtained from the structure tensor density (3.14) and the spacelike structure vector $k_\alpha = (0, 0, k)$. They read

$$
[\xi_1, \xi_2] = 0, \quad [\xi_1, \xi_3] = (k + \nu) \xi_1 + (\nu - a) \xi_2, \quad [\xi_2, \xi_3] = -(a + \nu) \xi_1 + (k - \nu) \xi_2.
$$

(4.100)

9 In other words, the solution (4.38) cannot be obtained from a diagonal ansatz like 

$$
dx^2 = A(\theta^1)^2 + B(\theta^2)^2 + C(\theta^3)^2,
$$

which always leads to metrics of Petrov type $D$. 
Thanks to the presence of the Abelian subgroup, the integration of these equations is immediate (see [27] for a discussion of this problem in the framework of the Bianchi III group) and, after a specific choice of the integration constants, leads to
\[
\xi_1 = \partial_x, \quad \xi_2 = \partial_y, \quad \xi_3 = \partial_z + r(x, y) \partial_x + s(x, y) \partial_y,
\] (4.101)
where the functions \( r(x, y) \) and \( s(x, y) \) are linear:
\[
r(x, y) = (k + v) x - (a + v) y \quad \text{and} \quad s(x, y) = (v - a) x + (k - v) y.
\] (4.102)

We immediately obtain the dual right invariant 1-forms that define the metric
\[
\theta^1 = dx - r(x, y) dz, \quad \theta^2 = dy - s(x, y) dz, \quad \theta^3 = dz.
\] (4.103)

The metric reads
\[
dx^2 = -(dx - r(x, y) dz)^2 + (dy - s(x, y) dz)^2 + dz^2
\] (4.104)
and solves the MG equations when the algebraic conditions (4.70), (4.72) and (4.73) are satisfied.

The metric (4.104) has an explicit one-parameter (hereafter labelled as \( \lambda_3 \)) isometry subgroup:
\[
z \mapsto z + \lambda_3,
\] (4.105)
but hides the two-parameter Abelian isometry subgroup given by
\[
x \mapsto x + p(z), \quad y \mapsto y + q(z), \quad z \mapsto z,
\] (4.106)
where \( p(z) \) and \( q(z) \) are the solutions (depending on two arbitrary constants denoted hereafter \( \lambda_1 \) and \( \lambda_2 \): the group parameters) of the differential system:
\[
p'(z) = (k + v) p(z) - (a + v) q(z),
\] (4.107)
\[
q'(z) = (v - a) p(z) + (k - v) q(z),
\] (4.108)
whose solution reads
\[
p(z) = \lambda_1 e^{kz} (a \cosh(az) + v \sinh(az)) - \lambda_2 e^{kz} (v + a) \sinh(az)
\] (4.109)
\[
:= \lambda_1 p_1(z) + \lambda_2 p_2(z),
\] (4.110)
\[
q(z) = \lambda_1 e^{kz} (v - a) \sinh(az) + \lambda_2 e^{kz} (a \cosh(az) - v \sinh(az))
\] (4.111)
\[
:= \lambda_1 q_1(z) + \lambda_2 q_2(z).
\] (4.112)

Now it is immediate to write the most general left invariant vector, a Killing vector of the metric (4.104) depending on the three parameters \( \lambda_1, \lambda_2 \) and \( \lambda_3 \) introduced respectively in equations (4.110) or (4.112) and (4.105):
\[
\tilde{\xi} = \lambda_3 \partial_z + p(z) \partial_x + q(z) \partial_y.
\] (4.113)
To make explicit the action of the Abelian two-dimensional subgroup we have to use the group parameters as coordinates. Assuming \( a \neq 0 \), we introduce new coordinates \( X \) and \( Y \) which are defined by
\[
x = p_1(z) X + p_2(z) Y,
\] (4.114)
\[ y = q_1(z) X + q_2(z) Y, \quad (4.115) \]

and we obtain
\[ \theta^1 = p_1(z) \, dX + p_2(z) \, dY \]
\[ = \frac{e^{\ell z}}{2} \left[ e^{az} (v + a) (dX + dY) + e^{-az} ((a - v) \, dX + (a + v) \, dY) \right], \quad (4.116) \]

\[ \theta^2 = q_1(z) \, dX + q_2(z) \, dY \]
\[ = \frac{e^{\ell z}}{2} \left[ e^{az} (v - a) (dX - dY) + e^{-az} ((a - v) \, dX + (a + v) \, dY) \right]. \quad (4.117) \]

A last coordinate transformation
\[ X \mapsto \frac{\sigma \, v \, v + k^2 (v + a) \, u}{2 \, k^2 \, a \, \sqrt{|a|}}, \quad Y \mapsto \frac{\sigma \, v \, v + k^2 (v - a) \, u}{2 \, k^2 \, a \, \sqrt{|a|}}, \quad z \mapsto -\frac{1}{k} \ln \zeta, \quad (4.118) \]

where \( \sigma := -\text{sgn}(a) \, v \), provides the usual expression of the so-called pp-wave AdS [26, 19] metric or the null warped AdS metric:
\[ ds^2 = \frac{1}{k^2} \, \frac{du \, dv + d\zeta^2}{\zeta^2} + \sigma \, \frac{d\xi^2}{\zeta^{2(\alpha/k + \nu)}}. \quad (4.119) \]

Obviously, the coordinate system \((u, v, \zeta)\) (or equivalently \((x, y, z)\)) defines a local chart but does not cover the whole manifold.

### 4.3. Nonsimply transitive groups

Apart from the aforementioned cases in subsection 4.1, there is also a special class of homogeneous spaces: the Kantowsky–Sachs spacetimes [28] that do not admit a simply transitive three-dimensional isometry group. The corresponding one-parameter metric describes homogeneous spaces of the form \( R \times S^2 \), on which acts (multi-transitively) a four-parameter isometry group that does not contain any three-parameter transitive subgroup:
\[ ds^2 = -dt^2 + R^2 (d\theta^2 + \sin^2 \theta \, d\phi^2). \quad (4.120) \]

This geometry is conformally flat, with the traceless Ricci and \( \tilde{K}_{\alpha \beta} \) tensors of Petrov type \( D_1 \). Thus, it can utmost satisfy the NMG (and so the GMG) field equations. It turns out that the latter are satisfied for arbitrary \( \mu \) but \( \Lambda = \xi = \frac{1}{2 \sqrt{r}} \). Let us emphasize that the hyperbolic or the flat version of this metric (with \( \sin \theta \) is replaced by \( \sinh \theta \) or \( \theta \) respectively) admits three-parameter transitive isometry groups. Indeed, the \( SO(3) \) group only admits one-dimensional subgroups, whereas in the framework of the hyperbolic version of the metric, the Lorentz group \( SO(1, 2) \) possesses two-dimensional subgroups. Thus, the latter geometry has already been considered in the previous subsections. The hyperbolic metric is conformally flat, with the traceless Ricci and \( \tilde{K}_{\alpha \beta} \) tensors of Petrov type \( D_r \). It solves the equations of NMG (and GMG) with arbitrary \( \mu \) but \( \Lambda = \xi = -\frac{1}{2 \sqrt{r}} \). Actually, it appears as a special solution of Bianchi type \( III \), obtained from a \( S III \) non-unimodular Lie algebra (3.15) with \( a = 0 \) and \( k = -\nu \), given by equation (4.77) and equations (4.81), (4.82) for NMG and GMG respectively.
5. Vanishing scalar invariant geometries

In this section, we complete the list of Lorentzian spacetimes with CSI geometries solving the MG field equations by examining the solutions provided by VSI geometries.

All the homogeneous geometries considered here above share the common property that all their scalar geometrical objects are constants. Notably, in the framework of Lorentzian geometries, there exist spaces (VSI spaces) which are not locally homogeneous but have all scalar invariants built out of their curvature tensors vanishing. These geometries constitute a subclass of the Kundt geometries considered in [14] and are explicitly known in three dimensions [29]. Their metrics are characterized by the existence of a null geodesic vector field (which in three dimensions implies vanishing of shear and twist). If we exclude flat space, then there are two possible such metrics\(^{10}\) (labelled A1 and B1 in [29]) that can be written as follows

\[
\text{ds}^2 = -2 du \left[ dv + \frac{1}{2} F(u, v, x) du + W(u, v, x) dx \right] + dx^2
\]

with special expressions of the function \( F(u, v, x) \) and \( W(u, v, x) \). Using a null frame \([l, m, n]\) such that \( \varphi^\alpha n_{\alpha} = -1 = -m^\alpha m_{\alpha} \), these metrics lead to Ricci tensors that read

\[
R_{\alpha \beta} = \phi l_{\alpha} l_{\beta} \quad \text{or} \quad R_{\alpha \beta} = \psi l_{\alpha} m_{\beta},
\]

with \( \phi \neq 0 \) and \( \psi \neq 0 \). The first one is of Petrov type \( N \), whereas the second one is of Petrov type \( III \). Of course the functions appearing in the metric (5.1) can be modified by coordinate transformations that preserve their writing:

\[
\begin{align*}
    u &\mapsto \mathcal{U}[\tilde{u}], \\
v &\mapsto \tilde{v} \left[ \frac{\tilde{u}}{\mathcal{U}[\tilde{u}]} \right] + \mathcal{F}[\tilde{u}, \tilde{x}], \\
x &\mapsto \tilde{x} + \mathcal{G}[\tilde{u}].
\end{align*}
\]

Using these transformations it is easy to check that the induced transformations on the functions \((F, W)\) read

\[
\tilde{F} = F \mathcal{U}^2 + 2 \left( \mathcal{F} \mathcal{U} + W \mathcal{G} \mathcal{U} - \tilde{v} \frac{\tilde{u}}{\mathcal{U}} \right) - \mathcal{G}^2, \quad \tilde{W} = W + \mathcal{F}' - \mathcal{G},
\]

where \((F, W)\) are expressed in terms of the \((\tilde{u}, \tilde{v}, \tilde{x})\) coordinates via their \((u, v, x)\) dependence given by (5.3). Dot and prime denote partial derivatives with respect to the new coordinates \(\tilde{u}\) and \(\tilde{x}\). In what follows, we describe the resolution of the MG field equations on VSI spaces. Let us note that for consistency we have to assume a zero cosmological constant, as all scalar invariants of VSI metrics vanish.

We have solved the equations of TMG, NMG and GMG for types A1 and B1. It turns out that the only nontrivial equations are those corresponding the \([u, x]\) and \([u, u]\) components. We start by considering the simplest one, namely the \([u, x]\) equation, plug its solution into the \([u, u]\) one and solve it.

- **Type A1**

  Here, the metric components are a priori of the form

  \[
  F(u, v, x) = v f_1(u, x) + f_0(u, x), \quad W(u, v, x) = w_0(u, x).
  \]

  Before proceeding to the equations of motion we shall first discuss the gauge fixings that are allowed by the coordinate transformations (5.3). At first we note that we can always eliminate \(w_0(u, x)\) with an appropriate choice of \(\mathcal{F}\)

  \[
  \mathcal{F}(\tilde{u}, \tilde{x}) = - \int w_0(\tilde{u}, \tilde{x}) d\tilde{x} + \tilde{G}(\tilde{u}) \tilde{x} + \mathcal{T}(\tilde{u});
  \]

10 In [29], four nonflat expressions of the metric are displayed, but the last two (denoted as D1 and F1) are special cases of the first ones.
thus, we consistently assume that \( w_0(u, x) = 0 \). Using the latter, it easy to check that the residual coordinate transformations induce the transformations

\[
\begin{align*}
\tilde{f}_1 &= f_1 \dot{U} - 2 \frac{\dot{U}}{U}, \\
\tilde{f}_0 &= f_0 U^2 + (\tilde{\xi} \tilde{G} + \mathcal{H}) f_1 \dot{U}^2 + 2 (\tilde{\xi} \tilde{G} + \mathcal{H}) \dot{U} - \dot{G}^2.
\end{align*}
\] (5.7)

Hereafter, we shall present the solution of the field equations in a fixed coordinate system, where we have eliminated as many as possible gauge functions in the expression of the metric.

* TMG: the only nontrivial field equations are \([u, x]\) and \([u, u]\). The first one gives

\[
\partial_x^2 f_1(u, x) + \mu \partial_x f_1(u, x) = 0,
\] (5.8)

whose general solution reads

\[
f_1(u, x) = c_0(u) + c_1(u) e^{-\mu x}.
\] (5.9)

Using (5.7) we can eliminate \( c_0(u) \) and set (locally) \( c_1(u) \) to 1 by choosing \( 2 \dot{U}/\dot{U}^2 = c_0(u) \), \( c_1(u) e^{-\mu \tilde{G}} \dot{U} = 1 \), i.e. in equation (5.7) the \( u = U(\tilde{u}) \) coordinate transformation as the inverse transformation of

\[
\tilde{u} = \frac{1}{c_*} \int e^{-\frac{1}{2} \int c_0(u) \, du}, \quad c_* = \text{const.}
\] (5.10)

and with an appropriate sign for \( c_* \)

\[
G(\tilde{u}) = \frac{1}{\mu} \left( \frac{1}{2} \int c_0(u) \, du + \ln[c_*, c_1(\mathcal{U}(\tilde{u}))] \right).
\] (5.11)

Let us note that if \( c_1(u) = 0 \) the metric is flat. Using the latter solution, the \([u, u]\) equation reduces to

\[
\partial_x^3 f_0(u, x) + \mu \partial_x^2 f_0(u, x) = \frac{\mu}{2} e^{-2\mu x},
\] (5.12)

whose general solution reads

\[
f_0(u, x) = q_0(u) + q_1(u) x + q_2(u) e^{-\mu x} + \frac{e^{-2\mu x}}{8 \mu^2}.
\] (5.13)

Last but not least, we can also eliminate \( q_2(u) \) with the appropriate choice of \( \mathcal{H}(\tilde{u}) = -q_2(\tilde{u}) \).

* NMG: working similarly as in the TMG we obtain

\[
f_1(u, x) = e^{-\sqrt{\tau} x} + c_2(u) e^{\sqrt{\tau} x}
\] (5.14)

and

\[
\begin{align*}
f_0(u, x) &= q_0(u) + q_1(u) x + q_2(u) e^{-\sqrt{\tau} x} + \frac{2\sqrt{\tau} - 5}{2\xi} c_2(u) e^{\sqrt{\tau} x} \\
&\quad + \frac{1}{6\xi} e^{-2\sqrt{\tau} x} + \frac{c_2(u)}{6\xi^2} e^{2\sqrt{\tau} x}.
\end{align*}
\] (5.15)

* GMG: in the same way, we find

\[
f_1(u, x) = \begin{cases} 
\chi_{\pm} x + c_2(u) e^{\lambda_{\pm} x}, & \text{where } \lambda_{\pm} = \frac{\xi \pm \sqrt{\xi (\xi + 4\mu^2)}}{2\mu^2}, \\
c_2(u) x e^{-2\mu x}, & \text{if } \xi = -4\mu^2.
\end{cases}
\] (5.16)
and

\[
f_0(u, x) = q_0(u) + r_0(u, x) + (q_1(u) + r_1(u, x)) x + r_2(u, x) e^{\lambda x}
\]

where

\[
r_0(u, x) = \frac{1 + \mu x}{\xi \mu} j(u, x), \quad r_1(u, x) = -\frac{j(u, x)}{\xi}, \quad r_2(u, x) = \pm \frac{e^{-\xi x} j(u, x)}{\lambda_+ (\lambda_+ - \lambda_-)}.
\]

whereas for \( \xi = -4\mu^2 \)

\[
f_0(u, x) = q_0(u) + r_0(u, x) + (q_1(u) + r_1(u, x)) x + r_2(u, x)
\]

\[
+ (q_3(u) + r_3(u, x)) x e^{-2\mu x},
\]

\[
r_0(u, x) = -\frac{1 + \mu x}{4\mu^2} j(u, x), \quad r_1'(u, x) = \frac{j(u, x)}{4\mu^2}, \quad r_2'(u, x) = \frac{(1 - \mu x) e^{2\mu x} j(u, x)}{4\mu^3},
\]

\[
r_3(u, x) = \frac{e^{2\mu x} j(u, x)}{4\mu^2},
\]

where

\[
j(u, x) := \left( (\partial_x f_1(u, x)) \partial_x + f_1(u, x) \left( \frac{\partial_x^2}{\mu} - \frac{\xi}{2\mu} \partial_x \right) - \frac{\xi}{\mu} \partial_x \partial_y + 2\partial_y \partial_x \right) f_1(u, x).
\]

**Type B1**

The metric components are given by

\[
F(u, v, x) = -\frac{v^2}{x^2} + v f_1(u, x) + f_0(u, x), \quad W(v, u, x) = -\frac{2v}{x} + w_0(u, x).
\]

Applying the coordinate transformations (5.3), in this case we find that the ones preserving the form of \( W \) require \( u = \tilde{u} + u_0, x = \tilde{x} + x_0 \) (where \( u_0 \) and \( x_0 \) are constants), while \( F \) remains an arbitrary function. Fixing it as follows:

\[
\left( \frac{\partial}{\partial x} - \frac{2}{x} \right) F(u, x) = -w_0(u, x) \implies F(u, x) = -x \left( \int \frac{w_0(u, x')}{x'} \, dx' - H(u) \right)
\]

allows us to eliminate \( w_0(u, x) \). Thus, we can set consistently \( w_0(u, x) = 0 \) and obtain

\[
\tilde{F} = -\frac{v^2}{x^2} + \tilde{v} f_1 + \tilde{f}_0,
\]

\[
\tilde{f}_1 = f_1 - 2\tilde{H}, \quad \tilde{f}_0 = f_0 + \tilde{x}^2 (f_1 - 1) + 2\tilde{x}^2 H.
\]

As for type A1, we shall present the solutions of the field equations in a coordinate system fixed by eliminating as many as possible gauge functions in the expression of the metric.

* TMG: the only nontrivial field equations are \([u, x]\) and \([u, v]\). The first one gives

\[
\left( \partial_x^2 + \left( \mu + \frac{1}{x} \right) \partial_x \right) f_1(u, x) = 0,
\]

whose solution reads

\[
f_1(u, x) = c_0(u) + c(u) Ei(-\mu x),
\]

where \( Ei(z) = -PV \int_z^{\infty} e^{-t}/t \, dt \) denotes the exponential integral function. Using (5.22) we can eliminate, for example, \( c_0(u) \) by choosing \( c_0(u) = 2 H(u) \). Using the latter equation, the \([u, u]\) equation reads

\[
\left( \partial_x^3 + \left( \mu - \frac{3}{x} \right) \left( \partial_x^2 - \frac{2}{x} \partial_x + \frac{1}{x^2} \right) \right) f_0(u, x) = \tilde{j}(u, x),
\]

\[
\tilde{j}(u, x) := \frac{e^{-\mu x}}{2x} (c^2(u) Ei(-\mu x) + 2c'(u)),
\]

25
whose solution is given by
\[ f_0(u, x) = x(q_0(u) + r_0(u, x)) + (q_1(u) + r_1(u, x)) x + (q_2(u) + r_2(u, x)) e^{-\mu x}, \]
\[ r_0'(u, x) = -\frac{(1 + \mu x) j(u, x)}{\mu^2 x}, \quad r_1'(u, x) = \frac{j(u, x)}{\mu x}, \quad r_2'(u, x) = \frac{e^{\mu x} j(u, x)}{\mu^2 x}. \]  
(5.26)

* NMG: working similarly as in the TMG we obtain
\[ f_1(u, x) = c_1(u) E\hat{x}(-x\sqrt{\xi}) + c_2(u) E\hat{x}(x\sqrt{\xi}) \]
and
\[ f_0(u, x) = x(q_0(u) + r_0(u, x)) + (q_1(u) + r_1(u, x)) x + (q_2(u) + r_2(u, x)) e^{-x\sqrt{\xi}} + (q_3(u) + r_3(u, x)) e^{x\sqrt{\xi}}, \]
\[ r_0'(u, x) = \frac{j(u, x)}{x}, \quad r_1'(u, x) = -\frac{j(u, x)}{x}, \quad r_2'_{23}(u, x) = \mp \frac{e^{\pm x\sqrt{\xi}} j(u, x)}{2x \xi^{3/2}}, \]
where
\[ j(u, x) := (\partial_x f_i(u, x)) \partial_x + 2\partial_x \partial_x^2 + f_i(u, x) \partial_x^2 \]
(5.29)

* GMG: as in the TMG, we obtain
\[ f_1(u, x) = \begin{cases} 
  c_1(u) E\hat{x}(\lambda_+ x) + c_2(u) E\hat{x}(\lambda_- x), & \text{where } \lambda_{\pm} = \frac{\xi \pm \sqrt{(\xi + 4\mu^2)}}{2\mu}, \\
  c_1(u) e^{-2\mu x} + c_2(u) E\hat{x}(-2\mu x), & \text{if } \xi = -4\mu^2 \end{cases} 
\]
and
\[ f_0(u, x) = x(q_0(u) + r_0(u, x)) + (q_1(u) + r_1(u, x)) x + (q_2(u) + r_2(u, x)) e^{\lambda_+ x} + (q_3(u) + r_3(u, x)) e^{\lambda_- x}, \]
\[ r_0'(u, x) = \frac{1 + \mu x}{\xi \mu x} j(u, x), \quad r_1'(u, x) = -\frac{j(u, x)}{x}, \quad r_2'_{23}(u, x) = \mp \frac{e^{-\mu x} j(u, x)}{\lambda_+ \xi (\lambda_+ - \lambda_-) x}, \]
(5.31)
whereas for \( \xi = -4\mu^2 \)
\[ f_0(u, x) = x(q_0(u) + r_0(u, x)) + (q_1(u) + r_1(u, x)) x + (q_2(u) + r_2(u, x)) e^{-2\mu x} + (q_3(u) + r_3(u, x)) x e^{-2\mu x}, \]
\[ r_0'(u, x) = -\frac{1 + \mu x}{4\mu^3 x} j(u, x), \quad r_1'(u, x) = \frac{j(u, x)}{4\mu^2 x}, \quad r_2'(u, x) = \frac{(1 - \mu x) e^{2\mu x} j(u, x)}{4\mu^3 x}, \]
\[ r_0'(u, x) = -\frac{e^{2\mu x} j(u, x)}{4\mu^2 x}. \]
(5.32)
where
\[ j(u, x) := (\partial_x f_i(u, x)) \partial_x + f_i(u, x) \left( \partial_x^2 - \frac{\xi}{2\mu} \partial_x \right) - \frac{\xi}{\mu} \partial_x \partial_x^2 + 2\partial_x \partial_x^2 \]
(5.33)

6. Discussion and conclusion

To summarize, we have obtained all locally homogeneous spaces, solutions of the TMG, NMG and GMG field equations (with a cosmological constant), at least formally. We have classified them according to the canonical representations of the structure constants obtained by Lorentz transformations, and for anti-de Sitter geometries we have discussed the appearance of a tachyonic massive mode of the graviton. To obtain the explicit expressions of the metrics, we still have to solve an elementary algebraic problem that consists of finding the linear
transformation that maps the expressions of the structure tensor density and the structure vector from which we start on their usual canonical expressions, i.e. express the 1-forms defining our coframe as a linear combination of the canonical ones. We may also directly integrate the expressions of the right invariant forms. Both approaches are explicitly illustrated in subsection 4.2, for solutions with structure tensor densities of types $III$ and $SII$.

We have also determined the Petrov types of the traceless Ricci tensor, the Cotton–York tensor and the traceless $K_{\mu\nu}$ tensor of all Lorentzian three-dimensional homogeneous geometries, which proves to be a useful tool to recognize equivalent solutions: for instance solutions of TMG, which are of Petrov type $D$, are biaxially squashed AdS$_3$ geometries [13].

In brief, we found solutions with structure tensor densities of type $I$ that correspond to all unimodular Bianchi types and allow all values of the cosmological constant; for those structure tensor densities of type $II$ (which could lead to the Bianchi types $II$, $VII_0$, $VIII_0$ and $VIII$), there are solutions of Petrov types $N$ and $II$; in case structure tensor densities of type $III$ (corresponding to Bianchi type $VII_0$ and $VIII$) only the GMG has solutions of Petrov type $III$ with a negative cosmological constant, which was not previously known; for structure tensor densities of type $IV$ (corresponding to Bianchi type $VI_0$ and $VIII$) there are solutions of Petrov type $I_C$ and a solution of Petrov type $D_s$ in the case of GMG.

In the case of non-unimodular homogeneous spaces, the Lorentzian classification of the algebras leads to solutions for structure tensor densities of types $T$ and $SI$ (corresponding to all non-unimodular types Bianchi types $III$, $IV$, $VI_0$ and $VII_0$) that allow both signs of the cosmological constant; for structure tensor densities of type $SII$ (corresponding to Bianchi types $IV$, $III$ and $VI_0$), there are solutions with only negative values of the cosmological constant; for structure tensor densities of type $SIII$ (corresponding to Bianchi types $III$ and $VII_0$), both signs of the cosmological constant are allowed; for structure tensor densities of type $L$ (corresponding to all non-unimodular Bianchi types), there are new solutions of Petrov type $II$ with a positive cosmological constant for TMG and NMG and negative one for the GMG; for structure tensor densities of type $L'$ (corresponding to Bianchi types $III$, $V$, $VI_0$), the only possible solution is flat space.

In addition, we have also obtained the solutions of TMG, NMG and GMG field equation for VSI geometries (which imply a vanishing cosmological constant). The solutions that we found are of Petrov type $III$. After having fixed the coordinate system, they all contain several arbitrary functions of a lightlike coordinate, reflecting the physical degrees of freedom of these solutions.

Regarding the solutions we obtained, let us make two more remarks. First, we recovered, as structure tensor densities types $I$, $SI$ and $SIII$ (and Petrov type $D_s$) the one-parameter family of deformed anti-de Sitter geometries [30] that includes the famous Gödel geometry [31]. On the other hand, each of the VSI solutions, which were given in section 5, contains various arbitrary functions. As it is known, these functions reflect the wave propagating aspect of these solutions, and describe the arbitrariness of the profile of these waves.

Of course, there are numerous known solutions among the ones that we obtained. Nevertheless (to our knowledge) some of them are new. In particular, we found CSI solutions of Petrov types $II$ and $III$ in equations (4.31), (4.34), (4.35), (4.38), (4.86) and the VSI spacetimes in section 5. However, the main goal of our analysis was to make a complete list of all the locally homogeneous and VSI geometries that constitute CSI spacetimes. From a physical perspective such solutions are among the simplest to study, and may also contain fruitful results that deserves further study.

The above results provide a classification of all the solutions of MG theories on CSI geometries. Of course, not all of them can be considered as classical background geometries. For instance, it is well known that on Bianchi IX solutions, due to the compactness of the
space, no global causal structure could be defined. Some of the squashed anti-de Sitter metrics that were obtained may also suffer from causality pathologies [30]. Moreover, we insist on the fact that the solutions that were obtained are often only local solutions, providing geodesically incomplete spaces. Thus, it would be very interesting and instructive to study the perturbative stability and the global structure of all these backgrounds, and to identify all the physically relevant configurations. It also would be newsworthy to check if the Petrov type D solutions of NMG and GMG are biaxially squashed AdS$_3$, like as in the TMG [13].

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Appendix. Petrov classification of homogeneous spaces

In this appendix, we provide the Petrov classification of $S_{\alpha\beta}$, $C_{\alpha\beta}$ and $\hat{K}_{\alpha\beta}$ tensors, for homogeneous spaces of types A and B, corresponding to unimodular and non-unimodular Lie algebras. This classification is presented in tables A1–A6.

Finally, we give the expressions of the discriminants of the $\hat{K}_{\alpha\beta}$ tensor for all the cases:

$$\Delta_K^I = -\frac{1}{110560} ((a + b)^2 - c^2)c^2 \left((5a^3 - 2a^2b + 40ab^2 + 64b^3 + 4(5a + 26b)c^2)^2 \times (21a^3 + 4ac(-6b + 5c) + a^2(4b + 26c) + 8c(8b^2 - 8bc + 5c^2)) \times (21a^3 + a^2(4b - 26c) + 4ac(6b + 5c) - 8c(8b^2 + 8bc + 5c^2))^2 \right) \leq 0, \quad (A.1)$$

$$\Delta_K^{II} = 0, \quad (A.2)$$

$$\Delta_K^{III} = 0, \quad (A.3)$$

$$\Delta_K^{IV} = \frac{1}{442368} (4v^2 - a^2)(-ab + b^2 + v^2)^2((-21a^3 + 15a^2b + ab^2 + 5b^3 + 4(13a - 5b)v^2)^2 \times (105a^6 + 76a^5b + 381a^4b^2 - 160a^3b^3 - 45a^2b^4 + 84ab^5 - 441b^6 - 4(171a^4 + 56a^3b + 486a^2b^2 - 160ab^3 - 41b^4)v^4 + 16(91a^2 - 20ab + 105b^2)v^4 - 1600v^6) \geq 0, \quad (A.4)$$

$$\Delta_K^V = -\frac{63}{27} b^6(a^2 - b^2 + k^2)^2((8a^3 + 13ab^2)^2 + (112a^4 - 74a^2b^2 + 25b^4)k^2 + (49a^2 - 10b^2)k^4 + k^6) \times (64a^4 + (-5b^2 + k^2)^2 + 16a^2(b^2 + 3k^2))^2 \leq 0, \quad (A.5)$$

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### Table A1. Generic Petrov type of the traceless Ricci tensor on homogeneous space of class \( A \) and particular values of the structure tensor density, leading to special Petrov types.

| Bianchi type | Discriminant of the traceless Ricci tensor | Special values | Petrov type | Remarks |
|--------------|-------------------------------------------|----------------|-------------|---------|
| \( I \)     | \( \Delta_S = -\frac{1}{12} (a + 2b)^2 c^2 (a + b)^2 - c^2 (a^2 - 4c^2)^2 \leq 0 \) | \( c = 0, a \neq -b \) | \( I_S \)     | Generic |
|              | \( c = 0, a = -b \)                      |                | \( D_3 \)    |         |
|              | \( c = \pm (a + b) \)                     |                | \( D_3 \)    |         |
|              | \( a = -2b, b \neq c \)                  |                | \( D_3 \)    |         |
|              | \( a = \pm 2c, b \neq \mp c \neq 0 \)   |                | \( D_3 \)    |         |
| \( \mathcal{H} \) | \( \Delta_S = 0 \)                       | \( b(a + b) = 0, a \text{ or } b \neq 0 \) | \( H \)       | Generic |
| \( \mathcal{H} \) |                                            |                | \( N \)       | \( \text{If } a = b = 0, \text{ flat space} \) |
| \( \mathcal{I} \) | \( \Delta_S = 0 \)                       | \( a = 0 \)    | \( \mathcal{I} \) | Generic |
| \( \mathcal{I} \) |                                            |                | \( N \)       |         |
| \( \mathcal{IV} \) | \( \Delta_S = \frac{1}{18}(a - b)^2 (4\nu^2 - a^2)(-ab + b^2 + \nu^2)(-ab + b^2 + 4\nu^2)^2 \geq 0 \) | \( a = b \)   | \( \mathcal{I}_C \) | Generic |
| \( \mathcal{IV} \) |                                            |                | \( D_3 \)    |         |
### Table A2. Generic Petrov type of the traceless Ricci tensor on homogeneous space of class $B$ and particular values of the structure tensor density, leading to special Petrov types.

| Bianchi type $B$ | Discriminant of the traceless Ricci tensor | Special values | Petrov type | Remarks |
|------------------|-------------------------------------------|----------------|-------------|---------|
| $T$              | $\Delta_T = -\frac{64}{27} b^6 (a^2 + k^2) (a^2 + k^2 - b^2)^2 \leq 0$ | $b = 0$ | $I_8$ | Generic |
|                  | $k^2 = b^2 - a^2$ | $D_4$ | $R^a_\nu = 2 k^2 \delta^a_\nu$ (locally) $dS_3$ |
| $	ext{SI}$      | $\Delta_{SI} = \frac{1}{108} (a + b)^6 (4 k^2 - (a - b)^2) (k^2 + a b)^2$ | $k = \pm (a - b)/2$ | $I$ | $I_C$ if $4 k^2 > (a - b)^2$, $I_S$ if $4 k^2 < (a - b)^2$
|                  | $a = -b$ | $II$ | $a \neq -b$ |
|                  | $k^2 = -a b$ | $O$ | $R^a_\nu = -2 k^2 \delta^a_\nu$ (locally) $AdS_3$
|                  | $a b < 0$; if $|b| > |a|$, type $D_s$, if $|a| > |b|$, type $D_t$ |
| $SH$             | $\Delta_{SH} = 0$ | $k = -a$ | $N$ | Generic |
|                  | $O$ | $R^a_\nu = -2 k^2 \delta^a_\nu$ (locally) $AdS_3$ |
| $SIII$           | $\Delta_{SIII} = -\frac{1}{108} (4 k^2 - a^2) (4 v^2 - a^2) (k^2 - v^2)^2$ | $k = \pm a/2$ | $I$ | $I_S$ if $k^2 > a^2/4$, $k^2 \neq v^2$, $I_C$ if $k^2 < a^2/4$
|                  | $k^2 = v^2$ | $D$ | $D_t$ if $k = v$, $D_t$ if $k = -v$
|                  | $k = \pm a/2$ | $II$ | $II$ if $k = a = 0$
| $L$              | $\Delta_L = 0$ | $b^2 = 1 + a c$ | $I_8$ | Generic |
|                  | $c = 0$ | $D_4$ | $D_s$ |
|                  | If $b = 0$ or $b = -1$, flat space. |
Table A3. Generic Petrov type of the Cotton–York tensor on homogeneous space of class \( A \) and particular values of the structure tensor density, leading to special Petrov types.

| Bianchi type \( A \) | Discriminant of the Cotton–York tensor | Special values | Petrov type | Remarks |
|----------------------|--------------------------------------|----------------|-------------|---------|
| \( I \) | \( \Delta_c = -\frac{1}{1728}c^2((a+b)^2-c^2)^2((3a^2+4c^2)^2-(4ac-8bc)^2)^2 \times (a^2-4ab-4(2b^2+c^2))^2 \leq 0 \) | \( c = 0, a \neq -b \) | \( I_\| \) | Generic |
| | | \( c = 0, a = -b \) | \( D_\| \) | AdS, or flat spaces |
| | | \( c = \pm(a+b) \) | \( O \) | |
| | | \( c = \pm\sqrt[4]{(a-2b)^2-12b^4} \) | \( D_I \) | \( a \notin [2(\sqrt{3}-1)b, 2(\sqrt{3}+1)b] \) |
| | | \( c = \pm(\frac{a}{2}-b \pm \sqrt[4]{(3b^2-(a+b)^2)}) \) | \( D_{iO} \) | \( a \in [\sqrt{3}-1)b, -(\sqrt{3}+1)b[ \) |
| \( II \) | \( \Delta_c = 0 \) | \( b(a+b) = 0, a \) or \( b \neq 0 \) | \( I_\| \) | Generic |
| | | \( a = 0 \) | \( N \) | If \( a = b = 0 \), flat space |
| \( III \) | \( \Delta_c = 0 \) | \( a \) | \( I_\| \) | Generic |
| | | | \( O \) | conformally flat, Ricci of type \( N \) |
| \( IV \) | \( \Delta_c = \frac{1}{972}(4\nu^2-a^2)((a-b)(3a+b)-4\nu^2)(-ab+b^2+v^2)^2 \times (-3a^4-8a^3b+2a^2b^2+9b^4+8(a^2+4ab-b^2)v^2+16v^4)^2 \geq 0 \) | \( 4\nu^2 = (a-b)(3a+b) \) | \( I_\| \) | Generic |
| | | \( a = -b = v \) | \( D_\| \) | conformally flat, Ricci of type \( I_\| \) |
Table A4. Generic Petrov type of the Cotton–York tensor on homogeneous space of class \( B \) and particular values of the structure tensor density, leading to special Petrov types.

| Bianchi type \( B \) | Discriminant of the Cotton–York tensor | Special values | Petrov type | Remarks |
|-----------------------|---------------------------------------|----------------|----------------|--------|
| \( T \)               | \( \Delta_C = -\frac{64}{27}b^2(a^2 - b^2 + k^2)^2 \) × \((4a^2 - b^2 + k^2)^2(2a^2 + b^2)^2 + k^2(5a^2 - 2b^2) + k^4 \) ≤ 0 | \( b = 0 \) \( k^2 = b^2 - a^2 \) \( k^2 = b^2 - 4a^2 \) | \( I_\Sigma \) | Generic |
|                       | \( b = 0 \) \( k^2 = b^2 - a^2 \) \( k^2 = b^2 - 4a^2 \) | \( O \) | \( dS_3 \) |
|                       | \( k^2 = b^2 - a^2 \) \( k^2 = b^2 - 4a^2 \) | \( D_1 \) | \( D_1 \) |
|                       | \( k^2 = b^2 - a^2 \) \( k^2 = b^2 - 4a^2 \) | \( O \) | | |
|                       | \( k^2 = b^2, a = 0 \) | | | Conformally flat, Ricci type \( D_1 \) |
| \( SI \)              | \( \Delta_C = -\frac{1}{16}(a + b)^6(ab + k^2)^2 \) × \((a - 3b)(3a - b) - 4k^2)^2 \) \( (3(a - b)^2 + 4ab)^2 \) | \( a = -b \) \( k^2 = -a b \) | \( I \) | If \( (...) > 0 \) \( I_\Sigma \) and if \( (...) < 0 \) \( I_C \) \( O \) AdS\(_3 \) \( D_1 \) if \( |b| > |a|, D_1 \) if \( |a| > |b| \) \( D_1 \) if \( |b| > |a|, D_1 \) if \( |a| > |b| \) |
|                       | \( 4k^2 = (a - 3b)(3a - b) \) | \( D \) | | \( a \neq [b/3, 3b], D_0 \) if \( a \in [-b, 3 b], D_1 \) if \( a \notin [-b, 3b] \) |
|                       | \( 16k^2 = 3(a - b) \pm \sqrt{(3a + b)(a + 5b)} \) | \( II \) | | \( O \) if \( b = -a \) and \( k^2 = a^2 \) or \( 4a^2 \) |
| \( SII \)             | \( \Delta_C = 0 \) | \( a = -k/2 \) \( k^2 = -a \) | \( N \) | Generic \( O \) Ricci type \( N \) \( O \) AdS\(_3 \) |
| \( SIII \)            | \( \Delta_C = \frac{1}{16}(4a^2 - a^2)(k^2 - v^2)^2 \) × \((4a^2 - 4k^2 + 3a^2)^2 \) \( (9a^2 + 16(k^2 - v^2)^2) \) \( -12a^2(k^2 + 2v^2) \) | \( k^2 = v^2 \) \( 4k^2 = 4v^2 + 3a^2 \) | \( I \) | If \( (...) < 0 \) \( I_\Sigma \) and if \( (...) > 0 \) \( I_C \) \( D \) \( D \) if \( v k > 0, D_1 \) if \( v k < 0 \) |
|                       | \( 16k^2 = 3a \pm \sqrt{16v^2 - 3a^2} \) | \( D_1 \) if \( k = v, D_1 \) if \( k = -v \) \( D_1 \) if \( v k > 0, D_1 \) if \( v k < 0 \) |
| \( L \)               | \( \Delta_C = 0 \) | \( b^2 = 1 + a c \) \( c = 0 \) | \( II \) | | \( D_1 \) | | \( D_1 \) | \( O \) Conformally flat, Ricci type \( N \) |

Symbol definitions:

- \( b, a, k \): Parameters of the structure tensor density.
- \( C \): Cotton–York tensor.
- \( \Delta_C \): Discriminant of the Cotton–York tensor.
- \( I_\Sigma \): Generic Petrov type.
- \( dS_3 \): Ricci type on AdS\(_3 \).
- \( I_C \): Special Petrov types leading to AdS\(_3 \).
- \( D_1 \): Special Petrov types leading to AdS\(_3 \).
- \( N \): Generic Petrov type.
- \( D_0 \): Special Petrov types leading to AdS\(_3 \).

Remarks:

- If \( \Delta_C < 0 \), the space is AdS\(_3 \).
- If \( \Delta_C > 0 \), the space is not AdS\(_3 \).
- If \( \Delta_C = 0 \), the space is conformally flat.

The table provides a comprehensive list of Petrov types and their associated discriminants, special values, and remarks for various classes of homogeneous spaces.
| Bianchi type A | Discriminant of the $\hat{K}_{\alpha\beta}$ tensor | Special values | Petrov type | Remarks |
|---------------|---------------------------------|----------------|------------|---------|
| $I$           | $\Delta_R = \cdots \leq 0$, in (A.1) | $c = 0$, $(4b + 21a)(b + a) \neq 0$ | $I_B$      | Generic |
| $I$           | $\Delta_R = \cdots \leq 0$, in (A.1) | $c = 0$, $(4b + 21a)(b + a) = 0$ | $O$        | for $b = -a$ AdS$_3$ or flat spaces |
| $I$           | $\Delta_R = \cdots \leq 0$, in (A.1) | $c = \pm (a + b)$ | $D_4$      |         |
| $I$           | $\Delta_R = \cdots \leq 0$, in (A.1) | $c = \pm \sqrt{-5a^2 - 40ab^2 - 64b^4}$ | $D_7$      |         |
| $I$           | $\Delta_R = \cdots \leq 0$, in (A.1) | $21a^2 + 4ac(\pm 6b + 5c) + a^2(4b \pm 26c)$ | $D_7$      |         |
| $I$           | $\Delta_R = \cdots \leq 0$, in (A.1) | $\pm 8c(8b^2 \pm 8bc + 5c^2) = 0$ | $D_7$      |         |
| $II$          | $\Delta_R = 0$ | $b(4b + 21a)(b + a) = 0$, $a$ or $b \neq 0$ | $II$       | Generic |
| $II$          | $\Delta_R = 0$ | $b(4b + 21a)(b + a) = 0$, $a$ or $b \neq 0$ | $N$        | If $a = b = 0$, flat space |
| $III$         | $\Delta_R = 0$ | $a = 0$ | $III$     | Generic |
| $III$         | $\Delta_R = 0$ | $a = 0$ | $O$        | conformally flat, Ricci of type $N$ |
| $IV$          | $\Delta_R = \cdots \geq 0$, in (A.4) | $-21a^3 + 15a^2b + ab^2 + 5b^3 + 4(13a - 5b)v^2 = 0$ | $I_C$     | Generic |
| $IV$          | $\Delta_R = \cdots \geq 0$, in (A.4) | $-21a^3 + 15a^2b + ab^2 + 5b^3 + 4(13a - 5b)v^2 = 0$ | $D_7$     |         |
Table A6. Generic Petrov type of the $\hat{K}_{\alpha\beta}$ tensor on homogeneous space of class $B$ and particular values of the structure tensor density, leading to special Petrov types.

| Class | Petrov type | Remarks |
|-------|-------------|---------|
| $T$   | $I_\emptyset$ | Generic |
|       | $O$         | dS$_3$   |
|       | $D_0$       |          |
|       | $D_1$       |          |
| $SI$  | $I$         | If $(...)>0$ and if $(...) < 0$ I$_C$ |
|       | $O$         | AdS$_3$  |
|       | $D_0$       |          |
|       | $D_1$       |          |
|       | $D_2$       |          |
|       | $D_3$       |          |
|       | $D_4$       |          |
| $SII$ | $N$         | Generic |
|       | $O$         | Ricci type $N$ |
|       | $O$         | AdS$_3$  |
| $SIII$| $I$         | If $(...) < 0$ I$_B$ and if $(...) > 0$ I$_C$ |
|       | $D_0$       |          |
|       | $D_1$       |          |
|       | $D_2$       |          |
|       | $D_3$       |          |
|       | $D_4$       |          |

Discriminant of the $\hat{K}_{\alpha\beta}$ tensor:
- $b = 0$
- $k^2 = b^2 - a^2$
- $k^2 = -24a^2 + 5b^2 \pm 16\sqrt{2}|a|\sqrt{a^2 - b^2}$
- $\# \geq 0$, in (A.5)

Special values:
- $a = -b$
- $k^2 = -a b$
- $4k^2 = 19(a - b)^2$
- $-20ab \pm 16|a - b|\sqrt{a^2 - 6ab + b^2}$

$N$ in $k^2 = v^2$:
- $4a = k(-2 \pm \sqrt{2})$
- $a = -k$

$O$ in $k^2 = v^2$:
- $\# \geq 0$, in (A.6)

$H$ in $k^2 = 5v^2$:
- $b^2 = 1 + ac$
- $c = 0$

$O$ Conformally flat, Ricci type $N$
\[ \Delta S^I_{K} = -\frac{1}{4\Delta S} (a + b)^6 (ab + k^2)^2 (105a^4 - 156a^3b + 502a^2b^2 - 156ab^3 + 105b^4 - 8(19a^2 - 88ab + 19b^2)k^2 + 16k^4)(a - b)^2 (21a^2 + 10ab + 21b^2)^2 - 4(63a^4 - 348a^3b + 970a^2b^2 - 348ab^3 + 63b^4)k^2 + 16(39a^2 - 118ab + 39b^2)k^4 - 64k^6), \]  
(A.6)

\[ \Delta S^I_{K} = 0, \]  
(A.7)

\[ \Delta S^II_{K} = \frac{1}{4\Delta S} (4\nu^2 - \nu^2)^3 (k^2 - \nu^2)^2 (105a^4 + 16(k^2 - 5\nu^2)^2 - 8a^2(19k^2 + 33\nu^2))^2 \times (441a^6 - 84\nu^2(3k^2 + 26\nu^2) - 64(k^3 - 5\nu^2)^2 + 16\nu^2(39k^4 - 24k^2\nu^2 + 169\nu^4)), \]  
(A.8)

\[ \Delta L_{K} = 0. \]  
(A.9)

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