Since January 2020 Elsevier has created a COVID-19 resource centre with free information in English and Mandarin on the novel coronavirus COVID-19. The COVID-19 resource centre is hosted on Elsevier Connect, the company's public news and information website.

Elsevier hereby grants permission to make all its COVID-19-related research that is available on the COVID-19 resource centre - including this research content - immediately available in PubMed Central and other publicly funded repositories, such as the WHO COVID database with rights for unrestricted research re-use and analyses in any form or by any means with acknowledgement of the original source. These permissions are granted for free by Elsevier for as long as the COVID-19 resource centre remains active.
Modeling and analysis on the transmission of covid-19 Pandemic in Ethiopia

Haile Habenom\textsuperscript{a}, Mulualem Aychluha, D.L. Suthara\textsuperscript{a}, Qasem Al-Mdallal\textsuperscript{b,*,c}, S.D. Purohit\textsuperscript{c}

\textsuperscript{a} Department of Mathematics, Wollo University, P.O. Box: 1145, Dessie, Ethiopia
\textsuperscript{b} Department of Mathematical Sciences, UAE University, P.O. Box 15551, Al-Ain, United Arab Emirates
\textsuperscript{c} Department of HEAS (Mathematics), Rajasthan Technical University Kota, India

Received 23 July 2021; revised 4 October 2021; accepted 31 October 2021
Available online 06 November 2021

KEYWORDS
COVID-19;
Legendre polynomials;
Caputo fractional derivative;
Fractional modeling;
Stability analysis;
Basic reproduction number;
Legendre spectral collocation method

Abstract The newest infection is a novel coronavirus named COVID-19, that initially appeared in December 2019, in Wuhan, China, and is still challenging to control. The main focus of this paper is to investigate a novel fractional-order mathematical model that explains the behavior of COVID-19 in Ethiopia. Within the proposed model, the entire population is divided into nine groups, each with its own set of parameters and initial values. A nonlinear system of fractional differential equations for the model is represented using Caputo fractional derivative. Legendre spectral collocation method is used to convert this system into an algebraic system of equations. An inexact Newton iterative method is used to solve the model system. The effective reproduction number ($R_0$) is computed by the next-generation matrix approach. Positivity and boundedness, as well as the existence and uniqueness of solution, are all investigated. Both endemic and disease-free equilibrium points, as well as their stability, are carefully studied. We calculated the parameters and starting conditions (ICs) provided for our model using data from the Ethiopian Public Health Institute (EPHI) and the Ethiopian Ministry of Health from 22 June 2020 to 28 February 2021. The model parameters are determined using least squares curve fitting and MATLAB R2020a is used to run numerical results. The basic reproduction number is $R_0 = 1.4575$. For this value, disease free equilibrium point is asymptotically unstable and endemic equilibrium point is asymptotically stable, both locally and globally.

\textcopyright{} 2021 THE AUTHORS. Published by Elsevier BV on behalf of Faculty of Engineering, Alexandria University. This is an open access article under the CC BY license (http://creativecommons.org/licenses/by/4.0).

1. Introduction

COVID-19 (coronavirus disease 2019) is a significant worldwide health hazard that has affected and killed millions of
people worldwide. Wuhan, China, marking the beginning of Coronavirus outbreak, in December 2019 [1,2]. The World Health Organization named the disease a pandemic, on 11 March 2020 [3]. To limit the spread of this dangerous disease, several control measures are being used, including social distance, case isolation, family quarantine, university and school shutdown [4,5]. In reaction to China’s containment methods, other governments have used extensive community quarantines or lockdowns as a form of control. (see details; [6,7]). In Ethiopia, the first COVID-19 case was reported by the Ethiopian Ministry of Health and Ethiopian Public Health Institute on March 13, 2020, in Addis Ababa [8]. The prevention and spread control of COVID-19 in Ethiopia is complicated by a number of reasons. Some of the reasons are poor physical distancing and the use of face mask, late confirmation of COVID-19 cases from the dead body investigation, inadequacy of quarantine cites, religious and cultural activities, insufficient health workers in treatment centre’s, social instability, a lack of public awareness of the infection, mass transportation system, attending of funeral ceremonies, and poor tradition of using electronic transaction [9]. All of these variables allow infectious and noninfectious people to come into contact.

Around the world, several mathematical models have been constructed to better understand the new virus’s transmission dynamics and intervention techniques. For example, in [10–12] COVID-19 dynamics were examined using the SEIR model, which included the environment and social distancing. They found that in the absence of effective control measures, the basic reproduction number \( R_0 = 2.03 \) indicated that the pandemic would persist in the human population. A fractional mathematical model of COVID-19 epidemics in Pakistan with treatment is discussed in [13]. The authors constructed the Atangana-Baleanu type fractional model and numerically solved it. In [14], a fresh dynamical modeling SEIR was proposed with global analysis incorporated to the real data of spreading COVID-19 in Saudi Arabia. The reproduction number and extensive stability analysis were used to present the dynamics of the suggested model. They focused on the best methods, controls, and techniques for significantly lowering the outbreak quickly (see also [15–17]).

Various mathematical models were created by several scholars in their studies. For instance, [18] explored the epidemic dynamics of COVID-19 in Wuhan, China, using the classical SEIR model in both deterministic and stochastic types. According to the authors’ findings, both exposed and infected classes have a significant role in shaping COVID-19 epidemic dynamics in Wuhan. To predict the spread of COVID-19 in USA, the SEIRD epidemic model was developed in [19]. Many more mathematical models have also been produced on the COVID-19 pandemic, we are referring to [20–29]. Mathematical models were used to study the dynamics of several infectious diseases, not alone COVID-19. Infectious diseases, including dengue [30], Ebola [31], HIV/AIDS [32], Hepatitis B [33], cholera [34,35], and Cultural Hereditary transmission [36] have recently been investigated.

The fractional-order derivative (FOD), which is described as an extension of the integer derivative to a non-integer order (arbitrary order) operator, has been used to model a variety of transmission \( [36] \) have recently been investigated. In [14], a fresh dynamical modeling SEIR was proposed with global analysis incorporated to the real data of spreading COVID-19 in Saudi Arabia. The reproduction number is calculated using the sensitivity of basic reproductive number with respect to the key model parameters. The Legendre spectral collocation method is used to convert the fractional system to an algebraic system. Numerical results of the model are given in graphical forms.

**Preliminaries and Notations.** This section covers the Laplace transform (LT) notation, the Mittag-Leffler (M-L) function, the fractional derivative, and a few key Legendre polynomial characteristics.

**Definition 1.1.** The M-L function is characterized as [39]:

\[
E_\psi(w) = \sum_{i=0}^{\infty} \frac{w^i}{\Gamma(\psi i + 1)}, \quad (\psi > 0). \tag{1.1}
\]

The two-parameter M-L function is defined by [40] as:

\[
E_{\delta, \psi}(w) = \sum_{i=0}^{\infty} \frac{w^i}{\Gamma(\delta i + \psi)} \quad (\delta, \psi > 0). \tag{1.2}
\]

From [41], The LT of the function is

\[
\mathcal{L}\left\{t^{\eta-1}E_{\delta, \psi}(\eta t^\delta) \right\} = \left\{ \begin{array}{ll}
\frac{\varphi^{\eta-1}}{\varphi^\delta}, & \text{for } \eta < 0,
\frac{\varphi^{\eta-1}}{\varphi^\delta}, & \text{for } \eta > 0.
\end{array} \right. \tag{1.3}
\]

**Definition 1.2.** The Caputo fractional derivative of order \( \delta \) is defined [42,43] as follows:

\[
D^\delta f(t) = \frac{1}{\Gamma(n - \delta)} \int_0^t \frac{f^{(n)}(\psi)}{(t - \psi)^{n - \delta + 1}} d\psi, \quad (n - 1 \leq \delta < n, n \in \mathbb{N}). \tag{1.4}
\]

For Caputo fractional derivative, we have

\[
D^\delta f(t) = 0, \quad (f \text{ is a constant}) \tag{1.5}
\]

\[
D^n f = \left\{ \begin{array}{ll}
0, & \text{for } n \in \mathbb{N}_0 \text{ and } n < [\delta],
\sum_{k=0}^{n-1} \frac{\varphi^{n-k-1}}{\Gamma(n-k-1)} f^{(k)}(0), & \text{for } n \in \mathbb{N}_0 \text{ and } n \geq [\delta],
\end{array} \right. \tag{1.6}
\]

where \( \mathbb{N}_0 = \{0, 1, 2,...\} \) and the ceiling function \([\delta]\) denote the smallest integer greater than or equal to \( \delta \).

The LT of Caputo fractional derivative of order \( \delta \) is provided by:

\[
\mathcal{L}\left\{D^\delta f(t)\right\} = s^\delta F(s) - \sum_{k=0}^{n-1} s^{n-k-1} f^{(k)}(0), \quad (n \geq \delta > n - 1 \in \mathbb{N}). \tag{1.7}
\]
with $F(s)$ is the Laplace transform of $f(t)$.

**Legendre polynomials:**

On the interval $[-1, 1]$, Legendre polynomials are defined and can be generated with the aid of the following recurrence formula [44]:

$$L_{m+1}(x) = \frac{2m+1}{m+1} xL_m(x) - \frac{m}{m+1} L_{m-1}(x), \quad m = 1, 2, \ldots, \quad (1.8)$$

with starting polynomials $L_0(x) = 1$ and $L_1(x) = x$. To apply these polynomials to the interval $t \in [0, 1]$, we utilize the so-called shifted Legendre polynomials (SLP), which involve changing the variable $x = 2t - 1$. Let the SLP $L_m(2t - 1)$ be denoted by $P_m(t)$, then it is possible to get them as follows:

$$P_m(t) = \frac{(2m+1)(2t-1)}{m+1} P_{m-1}(t) - \frac{m}{m+1} P_{m-2}(t), \quad m = 1, 2, \ldots, \quad (1.9)$$

with starting polynomials $P_0(t) = 1$ and $P_1(t) = 2t - 1$. The analytic form of SLP $P_m(t)$ of degree $m$ given by:

$$P_m(t) = \sum_{k=0}^{m} (-1)^{m+k} \frac{(m+k)!}{(m-k)!k!} t^k. \quad (1.10)$$

In terms of the SLP, a function $y(t)$ that is square integral in [0, 1] may be represented as:

$$y(t) = \sum_{i=0}^{\infty} v_i P_i(t), \quad (1.11)$$

where the coefficients $v_i$ are given by

$$v_i = (2i+1) \int_0^1 y(t)P_i(t)dt, \quad i = 1, 2, \ldots. \quad (1.12)$$

Only the first $n + 1$-terms of the SLP are considered in practice. After then, there is

$$y_n(t) = \sum_{i=0}^{n} v_i P_i(t). \quad (1.13)$$

**Lemma 1.** Let $y_n(t)$ be an approximated function in terms of SLP which is defined in (1.13) and suppose that $\delta > 0$, then

$$D^\delta y_n(t) \approx \sum_{i=0}^{n} \sum_{|\beta|=\delta} v_{\beta} \gamma^{(\beta)} \hat{\gamma}^{(\beta)} t^{|\beta|}, \quad (1.14)$$

where $\gamma^{(\beta)}$ is shown by

$$\gamma^{(\beta)}_{\beta} = (-1)^{i+k} \frac{(k+i)!}{(1+k-\delta)(i-k)!k!}. \quad (1.15)$$

2. Model formulation for COVID-19 pandemic

In this article, a mathematical model for COVID-19 transmission dynamics is created using a compartmental method comprising nine groups known as S-susceptible (individuals who are free to the disease can become infected by coming into touch with infectious people), E-exposed (individuals exposed to the Coronavirus disease but have not yet become infection and cannot transmit the infection to susceptible individuals), A- asymptomatically infected (having Covid-19 but showing no symptoms at all). The nature of asymptomatic people is the main reason why Covid-19 turned from an epidemic (concentrated in one area) to a pandemic (global spread). L-lockdown (individuals ordered to stay at home or prevented from entering or leaving a restricted area during Coronavirus pandemic), F-funeral (individuals attending funeral ceremonies during the Coronavirus pandemic), I-symptomatically infected (individuals infected by the Coronavirus who are capable of transmitting the infection to any susceptible individuals), Q-quarantined (quarantine of infected individuals out of a certain area to reduce Coronavirus transmission including home-based isolation and care (HBIC)), H-hospitalized (Coronavirus infected individuals who are placed in hospital for treatment, care, or observation) and R-recovered (individuals recovered from COVID-19 disease). The total population at a time $t$ is

$$N(t) = S(t) + E(t) + A(t) + L(t) + F(t) + I(t) + Q(t) + H(t) + R(t). \quad (2.1)$$

The natural death rate $\mu$ and the death rate $\omega$ caused by COVID-19 are the same in all subgroups. Natural death or disease-induced death will lead some individuals to be removed from the population. Individuals are recruited into susceptible classes at a constant rate $\nu$ and get infected by Coronavirus infectious individuals with the force of infection $\beta(vA + L + F + I)S/N$. The exposed cases acquire population from Coronavirus infection at a rate $\beta(vA + L + F + I)S/N$. Exposed population progress to symptomatically infected when individuals show symptoms of coronavirus and tested positive at a rate $\theta$ and the remaining exposed individuals will be removed at the natural death rate $\mu$ and enter the asymptomatic class at a rate $\delta(1 - \theta)$. Asymptomatic infected individuals develop symptoms and become symptomatically infected at a rate $\lambda$, enter a lockdown class at a rate $\zeta(1 - \lambda)$, attending the funeral ceremony at a rate $\tau(1 - \xi)(1 - \lambda)$ and recovered from COVID-19 disease at a rate $\tau(1 - \tau)(1 - \xi)(1 - \lambda)$, while the remaining individuals died naturally or as a result of COVID-19 disease at a rate of $\mu + \omega$. Lockdown individuals participate in funerals during the pandemic time at a rate of $\epsilon$, and they join the recovery class at a rate of $\rho(1 - \epsilon)$, with the remaining individuals from this class being removed from the population by natural death or Coronavirus induced death. Individuals who are symptomatically infected are quarantined at rate of $\eta$ and transfer into treatment centers at a rate of $\varsigma(1 - \eta)$. The parameter $\pi$ represents a rate at which quarantined infected individuals are moved to hospitals (treatment centers) for better care and recover at a rate of $\sigma(1 - \pi)$. At the rates of $\phi$ and $\gamma(1 - \phi)$, infected individuals identified from funeral participants are quarantined and recovered, respectively. $\psi$ is the recovery rate of hospitalized individuals, $\beta$ is the transmission rate (contact rate), and $\nu$ is the transmission rate from asymptotically infected to the susceptible individuals. Infectious individuals will be removed from the population at a rate of $\mu + \omega$, and non-infectious individuals might be removed at rate of $\mu$ because of natural death and will not be assigned to any class. This model assumes that recovered individuals may not go back to the susceptible class (no reinfection). Positive values are assumed for model parameters. All susceptible individuals have an equal chance of being infected.
where $\Lambda = \frac{\kappa + \lambda + \mu + \rho}{\mu}$.

In this model, the transmission dynamics of COVID-19 can be expressed through subsequent fractional order differential equations (FODEs) in Caputo sense:

$$\begin{align*}
D^\alpha S(t) &= \frac{\partial}{\partial t} - \Lambda S - \mu S \\
D^\alpha E(t) &= \Lambda S - (\theta + \delta)(1 - \theta) + \mu E \\
D^\alpha A(t) &= \delta(1 - \theta) E - (\lambda + (\xi + \gamma)(1 - \xi) + \zeta(1 - \tau)(1 - \zeta))(1 - \lambda) + \mu + \omega) A \\
D^\alpha L(t) &= \xi(1 - \lambda) A - (\delta + \rho(1 - \varepsilon) + \mu + \omega)L \\
D^\alpha F(t) &= \tau(1 - \zeta)(1 - \lambda) A + \varepsilon L - (\phi + \gamma(1 - \phi) + \mu + \omega) F \\
D^\alpha I(t) &= \theta E + \lambda A - (\eta + \varphi(1 - \eta) + \mu + \omega) I \\
D^\alpha Q(t) &= \eta I + \phi F - (\pi + \sigma(1 - \pi) + \mu + \omega) Q \\
D^\alpha H(t) &= \phi(1 - \eta) I + \pi Q - (\psi + \mu + \omega) H \\
D^\alpha R(t) &= \zeta(1 - \tau)(1 - \zeta)(1 - \lambda) A + \rho(1 - \varepsilon) L + \sigma(1 - \pi) Q + \psi H + \gamma(1 - \phi) F - \mu R
\end{align*}$$

with the positive starting conditions:

$$(S(0), E(0), A(0), L(0), F(0), I(0), Q(0), H(0), R(0)). \tag{2.3}$$

2.1. Existence and uniqueness of solutions

We simplify the proposed model (2.2) in the following form:

$$\begin{align*}
D^\alpha S(t) &= f_1(t, z) = \frac{\partial}{\partial t} - \Lambda S - \mu S \\
D^\alpha E(t) &= f_2(t, z) = \Lambda S - (\theta + \delta)(1 - \theta) + \mu E \\
D^\alpha A(t) &= f_3(t, z) = \delta(1 - \theta) E - (\lambda + (\xi + \gamma)(1 - \xi) + \zeta(1 - \tau)(1 - \zeta))(1 - \lambda) + \mu + \omega) A \\
D^\alpha L(t) &= f_4(t, z) = \xi(1 - \lambda) A - (\delta + \rho(1 - \varepsilon) + \mu + \omega)L \\
D^\alpha F(t) &= f_5(t, z) = \tau(1 - \zeta)(1 - \lambda) A + \varepsilon L - (\phi + \gamma(1 - \phi) + \mu + \omega) F \\
D^\alpha I(t) &= f_6(t, z) = \theta E + \lambda A - (\eta + \varphi(1 - \eta) + \mu + \omega) I \\
D^\alpha Q(t) &= f_7(t, z) = \eta I + \phi F - (\pi + \sigma(1 - \pi) + \mu + \omega) Q \\
D^\alpha H(t) &= f_8(t, z) = \phi(1 - \eta) I + \pi Q - (\psi + \mu + \omega) H \\
D^\alpha R(t) &= f_9(t, z) = \zeta(1 - \tau)(1 - \zeta)(1 - \lambda) A + \rho(1 - \varepsilon) L + \sigma(1 - \pi) Q + \psi H + \gamma(1 - \phi) F - \mu R
\end{align*}$$

where $z$ is the vector presented as $(S, E, A, L, F, I, Q, H, R)$. Thus, model (2.4) has the following form

$$\begin{align*}
D^\alpha y(t) &= \mathbf{F}(t, y(t)), \\
y(0) &= y_0 \geq 0
\end{align*} \tag{2.5}$$

under the condition that

$$\begin{align*}
y(t) &= (t, z)^T, \\
y(0) &= (t, z_0)^T, \\
\mathbf{F}(t, y(t)) &= (f_i(t, z))^T, \quad i = 1, 2, \ldots, 9
\end{align*} \tag{2.6}$$

Problem (2.5) has an integral representation is given by

$$y(t) - y(0) = \frac{1}{\Gamma(\alpha)} \int_0^t \mathbf{F}(\tau, y(t - \tau)^{\alpha-1} d\tau \tag{2.7}$$

Theorem 2.1. For every kernel $f_i$, in (2.4), there exist $\psi_i$, $i = 1, 2, \ldots, 9$ such that

$$\|f_i(t, z) - f_i(t, z_1)\| \leq \psi_i \|z(t) - z_1(t)\| \tag{2.9}$$

are contractions for $0 \leq \psi_i < 1, \ i = 1, 2, \ldots, 9$.

Proof. For variable $S$, we have

$$\|f_i(t, S) - f_i(t, S_1)\| = \|\theta - \Lambda S - \mu S - (\theta - \Lambda S_1 - \mu S_1)\|$$

$$= \|(A + \mu)(S - S_1)\|$$

$$\leq \psi_1 \|S_1 - S\|$$

where $\psi_1 = \frac{2(\theta_0 + \xi_0 + \gamma_0 + \zeta_0)}{N} + \mu$. For a Banach space $\mathcal{S} = \mathbb{C}([0, s]; \mathbb{R})$ of a continuous function with the norm defined by

$$\|y\|_{\mathcal{S}} = \sup_{t \in \mathcal{S}} |y(t)|, \tag{2.8}$$
As a result, \( f_i(t, S) \) fulfills the Lipschitz condition with Lipschitz constant \( \psi_i \). Furthermore, if \( 0 \leq \psi_i < 1 \), then \( f_i(t, S) \) is a contraction. Also, we can show existence of Lipschitz constants \( \psi_i \), and a contraction for \( f_i(t, z) \), \( i = 2, 3, \ldots, 9 \). Now for \( t = t_k, k = 1, 2, \ldots, \) defined the following recursive form of (2.7):

\[
    z_k(t) - z_{k-1}(t) = \frac{1}{\Gamma(z)} \int_0^t f_i(t, z_{k-1})(t - \vartheta)^{z-1}d\vartheta
\]

The difference between successive terms in the above equation and taking the norm on both sides of the resulting equation, we get:

\[
    \|z_k(t) - z_{k-1}(t)\| = \frac{1}{\Gamma(z)} \int_0^t \|f_i(t, z_{k-1}) - f_i(t, z_{k-2})\| \|(t - \vartheta)^{z-1}\|d\vartheta
\]

Taking \( z = S \), (2.11) can also be reduced to the following form:

\[
    \|S_k(t) - S_{k-1}(t)\| = \frac{1}{\Gamma(z)} \int_0^t \|f_i(t, S_{k-1}) - f_i(t, S_{k-2})\| \|(t - \vartheta)^{z-1}\|d\vartheta
\]

As a result, we have:

\[
    \|S_k(t) - S_{k-1}(t)\| \leq \|S_{k-1} - S_{k-2}(t)\| \frac{t^z}{\Gamma(z + 1)}
\]

In the same way, for the rest of expressions, (2.11) can be reduced to the following form:

\[
    \|z_k(t) - z_{k-1}(t)\| \leq \frac{t^z}{\Gamma(z + 1)} \|z_{k-1}(t) - z_{k-2}(t)\|, \quad i = 2, 3, \ldots, 9
\]

\[\square\]

**Theorem 2.2.** The fractional model given in (2.2) has a solution if we can find \( \bar{\Omega} \) satisfying the inequality

\[
    \frac{\bar{\Omega}^z}{\Gamma(z + 1)} \psi_i < 1, \quad i = 1, 2, 3, \ldots, 9.
\]

**Proof.** From (2.12) and (2.13) we have:

\[
    \|z_k(t)\| \leq \|z(0)\| \left( \frac{t^z}{\Gamma(z + 1)} \right)^k \psi_i
\]

**Theorem 2.1** confirms the existence of solution, and we must show that \( S(t), E(t), A(t), L(t), F(t), I(t), Q(t), H(t), R(t) \) are solutions of model (2.2).

Assume that the following condition is met:

\[
    z(t) - z(0) = z_k(t) - w_k(t)
\]

From (2.16) we obtain:

\[
    \|w_k(t)\| = \left( \int_0^t \frac{t^z}{\Gamma(z + 1)} \right)^{k+1} \left( \frac{t^z}{\Gamma(z + 1)} \right) \psi_i |z_k - z_{k-1}|
\]

Repeating the process recursively leads to:

\[
    \|w_k(t)\| \leq \left( \int_0^t \frac{t^z}{\Gamma(z + 1)} \right)^{k+1} \psi_i |z_k - z_{k-1}|^i, \quad i = 1, 2, 3, \ldots, 9.
\]

As \( k \to \infty \) we see that \( \|w_k(t)\| \to 0 \) for

\[
    \frac{\bar{\Omega}^z}{\Gamma(z + 1)} \psi_i < 1, \quad i = 1, 2, 3, \ldots, 9.
\]

The combination of Theorems 2.1 and 2.2 show model (2.2) has solution by fixed point theorem [45]. \[\square\]

**Theorem 2.3 (Uniqueness of solution).** A fractional model (2.2) has unique solution, provided that

\[
    \left( \frac{\bar{\Omega}^z}{\Gamma(z + 1)} \right) \psi_i < 1, \quad i = 1, 2, 3, \ldots, 9.
\]

**Proof.** Let us assume that \( S(t), E(t), A(t), L(t), F(t), I(t), Q(t), H(t), R(t) \) are solutions to model Eq. (2.2). Then

\[
    S(t) = S_1(t) + \frac{1}{\Gamma(z)} \int_0^t f_1(t, S_i) (t - \vartheta)^{z-1}d\vartheta
\]

Taking the norm both sides, we get:

\[
    \|S(t) - S_1(t)\| \leq \left( \frac{t^z}{\Gamma(z + 1)} \right) \psi_i \|S - S_1\|.
\]

Since \( 1 - \psi_i \frac{t^z}{\Gamma(z + 1)} > 0 \), we obtain \( \|S(t) - S_1(t)\| = 0 \). Thus, we have \( S(t) = S_1(t) \). Similarly, we can show that \( z_i(t) = z_{i-1}(t), \quad i = 2, 3, \ldots, 9 \) \[\square\]

2.2. Solution positivity and boundedness

The system (2.2) depicts the human population, it is important to demonstrate that all results of the system with nonnegative beginning values will stay positive for all \( t > 0 \) and are bound. The subsequent lemma and theorem will establish this.
Lemma 2. The feasible region

\[
\Psi = \left\{ S(t), E(t), A(t), L(t), F(t), I(t), Q(t), H(t), R(t) \in \mathbb{R}^9 : N(t) \leq N(0) + \frac{\vartheta}{\mu} \right\},
\]

(2.23)

with positive initial conditions in (2.3), is a positive invariant for system (2.2), where \( N(0) \) denotes the whole population’s starting values.

Proof. The dynamics of the total population can be gained by adding each equation of the model (2.2), given by

\[
D^0N(t) = \vartheta - \mu N - \omega(A + L + F + I + Q + H) \leq \vartheta - \mu N.
\]

(2.24)

Applying the Laplace transform on (2.24) and the idea of (1.3) we get:

\[
N(t) \leq E_{a,1}(-\mu t^2)N(0) + \frac{\vartheta}{\mu}(1 - E_{a,1}(-\mu t^2)).
\]

(2.25)

Since, \( 0 \leq E_{a,1}(-\mu t^2) \leq 1 \), we have \( N(t) \leq N(0) + \frac{\vartheta}{\mu} \) and hence, the region \( \Psi \) is a positively invariant set for the system (2.2). \( \Box \)

Theorem 2.4. If the initial solutions satisfy \( S(0) \geq 0, E(0) \geq 0, A(0) \geq 0, L(0) \geq 0, F(0) \geq 0, I(0) \geq 0, Q(0) \geq 0, H(0) \geq 0, R(0) \geq 0 \), then the solutions \( S(t), E(t), A(t), L(t), F(t), I(t), Q(t), H(t), R(t) \) of the system (2.2) are positive \( \forall t \geq 0 \).

Proof. Starting with the first equation of the system (2.2)

\[
D^0S(t) = \vartheta - \nu A(t) - \mu S(t) \geq -(\lambda + \mu)S(t),
\]

(2.26)

where

\[
\Lambda = \beta(vA(t) + L(t) + F(t) + I(t)) / N.
\]

From Lemma 2, \( vA(t) + L(t) + F(t) + I(t) \) is bounded by a constant \( \Lambda_0 \), we have

\[
\delta_1 = \vartheta + \delta(1 - \theta) + \mu,
\]

\[
\delta_2 = \lambda + (\xi + \tau(1 - \xi) + \zeta(1 - \tau) - (1 - \tau)(1 - \xi))(1 - \delta) + \mu + \omega,
\]

\[
\delta_3 = \varepsilon + \rho(1 - e) + \mu + \omega,
\]

\[
\delta_4 = \phi + \gamma(1 - \phi) + \mu + \omega,
\]

\[
\delta_5 = \eta + \phi(1 - \eta) + \mu + \omega
\]

(2.31)

\[
D^0S(t) \geq -\gamma S(t),
\]

(2.27)

where \( \gamma \) is a constant equal to \( \frac{\delta_6}{\delta_5} + \mu \). Applying the LT in (2.27), we arrive at

\[
S(t) \geq E_{a,1}(-\mu t^2)S(0).
\]

(2.28)

Since \( E_{a,1}(-\mu t^2) \geq 0 \) and \( S(0) \geq 0 \), then the solution \( S(t) \) is positive. Following the above process, from the remaining equations of (2.2), we can readily demonstrate model’s other state variables remain positive \( \forall t \geq 0 \). \( \Box \)

2.3. Stability analysis

2.3.1. The Disease-Free Equilibrium (DFE) point

The disease-free equilibrium is attained when there are no infections in the population. The infectious compartments have all been reset to zero. Also, in our model equation, we set the fractional derivatives of non-infectious compartments to zero. As a result, the disease-free equilibrium point is

\[
\mathcal{E}_0 = (S_0, E_0, A_0, L_0, F_0, I_0, Q_0, H_0, R_0) = \left( \frac{\vartheta}{\mu}, 0, 0, 0, 0, 0, 0, 0 \right).
\]

(2.29)

2.3.2. The basic reproduction number

When a typical infective enters a susceptible individual, the average number of secondary infections is defined as the basic reproduction number. The next-generation matrix technique [46] can be used to calculate this value. We pick epidemiologically valid matrices and algorithms to generate the expression. The spectral radius of the next generation matrix provides the required effective reproduction number.

\[
\mathcal{R}_0 = \beta \delta_1(1 - \theta) \left( 1 - \delta \right) \left( \frac{\zeta}{\delta_1} + \frac{\tau(1 - \xi)}{\delta_3} + \frac{\zeta(1 - \tau)}{\delta_5} \right) + \lambda + \frac{\mu}{\delta_5} + \nu + \frac{\beta \theta}{\delta_2 \delta_5}
\]

(2.30)

where,

\[
\delta_1 = \theta + \delta(1 - \theta) + \mu,
\]

\[
\delta_2 = \lambda + (\xi + \tau(1 - \xi) + \zeta(1 - \tau) - (1 - \tau)(1 - \xi))(1 - \delta) + \mu + \omega,
\]

\[
\delta_3 = \varepsilon + \rho(1 - e) + \mu + \omega,
\]

\[
\delta_4 = \phi + \gamma(1 - \phi) + \mu + \omega,
\]

\[
\delta_5 = \eta + \phi(1 - \eta) + \mu + \omega
\]

Theorem 2.5. If \( \mathcal{R}_0 < 1 \), then the disease-free equilibrium point in Eq. (2.29) is locally asymptotically stable and unstable for \( \mathcal{R}_0 > 1 \).
Proof. To determine the local stability of the disease-free equilibrium, we examine the behavior of our model population near this equilibrium solution. Now, the Jacobian matrix of (2.2) at $\mathcal{E}_0$ is:

$$J_{\mathcal{E}_0} = \begin{bmatrix}
-\mu & 0 & -\beta v & -\beta \\
0 & -\delta_1 & 0 & \beta v \\
0 & \delta(1 - \theta) & -\delta_2 & 0 \\
0 & 0 & \zeta(1 - \lambda) & -\delta_3 \\
0 & 0 & \tau(1 - \zeta)(1 - \lambda) & \rho(1 - \epsilon) \\
-\beta & -\beta & 0 & 0 & 0 \\
\beta & \beta & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
-\delta_4 & 0 & 0 & 0 & 0 \\
0 & -\delta_5 & 0 & 0 & 0 \\
0 & \eta & -\delta_6 & 0 & 0 \\
0 & \sigma(1 - \eta) & \pi & -\delta_7 & 0 \\
\gamma(1 - \phi) & 0 & \sigma(1 - \pi) & \psi & -\mu
\end{bmatrix}$$

In this Jacobian matrix, four of the eigenvalues are negative, that is $m_1 = m_2 = -\mu, m_3 = -\delta_6,$ and $m_4 = -\delta_7.$ The remaining eigenvalues can be obtained from the characteristic equation:

$$m^5 + d_1m^4 + d_2m^3 + d_3m^2 + d_4m + d_5 = 0,$$

where

$$d_1 = \delta_1 + \delta_3 + \delta_5 + \delta_4 + \delta_4,$$

$$d_2 = (\delta_3 + \delta_4)(\delta_1 + \delta_2 + \delta_5) + \delta_5\delta_3 + \delta_2\delta_5 + \frac{\delta_2\delta_1(1 - \theta)}{\delta_1} + \frac{\delta_3\delta_1(1 - \theta)}{\delta_1},$$

$$d_3 = \delta_2\delta_5\delta_3 + \delta_1(\delta_2\delta_3 + \delta_2\delta_5 + \delta_2\delta_5 + \delta_5\delta_3 + \delta_5\delta_3 + \delta_5\delta_3),$$

$$d_4 = \delta_2\delta_5\delta_3 + \delta_1(\delta_2\delta_3 + \delta_2\delta_5 + \delta_2\delta_5 + \delta_5\delta_3 + \delta_5\delta_3 + \delta_5\delta_3),$$

and

$$E = \frac{\delta_1(1 - \theta)^2}{\delta_2}.$$

In the above expressions, the coefficient $d_i$ is positive when $\Re_0 < 1,$ and all the other coefficients are positive. Further, the Routh-Hurwitz criteria for fifth-order polynomials are $d_i > 0,$ for $i = 1, 2, 3, 4, 5,$ and $(d_1d_4 - d_2d_3) + (d_2d_5 - d_3d_1) > d_1d_4 + d_2d_3$ can be easily satisfied by using the above coefficients. So, the model (2.2) at $\mathcal{E}_0$ is locally asymptotically stable if $\Re_0 < 1$ and unstable if $\Re_0 > 1.$

2.3.3. Endemic equilibrium

Adjusting the model equation to zero and solve simultaneously, we get the endemic equilibrium point $EE = (S^*, E^*, A^*, L^*, F^*, P^*, Q^*, H^*, R^*),$ where

$$S^* = \frac{\beta}{\mu}E^*, \quad A^* = \frac{\delta(1 - \theta)}{\delta_2}E^*, \quad L^* = \frac{\delta\zeta(1 - \theta)(1 - \lambda)}{\delta_2\delta_3}E^*, \quad F^* = \left(\tau(1 - \zeta) + \frac{\delta\zeta}{\delta_3}\right)\frac{\delta(1 - \theta)(1 - \lambda)}{\delta_2\delta_3}E^*,$$

$$P^* = \left(\delta + \frac{\delta(1 - \theta)}{\delta_2}\right)\frac{1}{\delta_3}E^*,$$

$$Q^* = \left(\frac{\delta\theta}{\delta_2} + \frac{\lambda\eta}{\delta_3}\right)\frac{\delta(1 - \theta)(1 - \lambda)}{\delta_2\delta_3}E^* + \frac{\delta\eta\zeta(1 - \theta)(1 - \lambda)}{\delta_2\delta_3}E^*,$$

and

$$H^* = \frac{\delta\theta\eta}{\delta_3} + \frac{\delta\eta\zeta(1 - \theta)(1 - \lambda)}{\delta_2\delta_3}E^*.$$

2.4. Global stability analysis of equilibrium points

Using the Lyapunov function method, we showed the global asymptotic stability of both equilibrium points in this subsection.

i. Global stability of disease-free equilibrium point

COVID-19 can be controlled in Ethiopia when $\Re_0 < 1$ if the initial sizes of the cases in each of compartment or society are very close to the disease free equilibrium point; the initial size is in the DFE’s basin of attraction.

$$\mathcal{E}_0 = \left(\frac{\beta}{\beta}, \frac{\delta}{\delta}, 0, 0, 0, 0, 0, 0, 0\right),$$

as previously demonstrated by the local asymptotic stability of the disease free equilibrium. It is incredibly important to verify that the elmi-
nation of COVID-19 is independent of the starting size of the number of cases in society by showing that the DFE is globally asymptotically stable if \( R_0 < 1 \). The following theorem is used to explore this situation.

**Theorem 2.6.** If \( R_0 \leq 1 \), then the disease-free equilibrium point \( \delta_0 \) in (2.29) is globally asymptotically stable on a positively invariant region \( \Psi \).

**Proof.** Consider the Lyapunov function

\[
\mathcal{F}(t) = c_1 E(t) + c_2 A(t) + c_3 L(t) + c_4 F(t) + c_5 I(t).
\]

Differentiate the Lyapunov function \( \mathcal{F}(t) \) with respect to time \( t \) using Caputo fractional derivative of order \( \alpha \), we get

\[
D^\alpha \mathcal{F}(t) = c_1 D^\alpha E + c_2 D^\alpha A + c_3 D^\alpha L + c_4 D^\alpha F + c_5 D^\alpha I.
\]

Substituting the appropriate values from (2.2) and using (2.31), we get

\[
D^\alpha \mathcal{F}(t) = c_1 \left( \frac{\beta(t)(1 - \frac{A}{S})}{S - \delta_1} + c_2 \delta(1 - \theta)E - \delta_2 A \
+ c_3 (\xi(1 - \lambda)A - \delta_3 L) 
+ c_4 (\tau(1 - \xi)(1 - \lambda)A + \varepsilon L - \delta_4 F) 
+ c_5 (\theta E + \lambda A - \delta_1 I) \right)
\]

Since, \( S \leq S_0 \), we have

\[
D^\alpha \mathcal{F}(t) \leq c_1 \beta(t)(1 - \frac{A}{S}) + c_2 \delta(1 - \theta)E - c_3 \delta_2 A 
+ c_4 \xi(1 - \lambda)A - c_5 \delta_3 L 
+ c_4 \tau(1 - \xi)(1 - \lambda)A + c_6 \varepsilon L - c_5 \delta_4 F 
+ c_5 \theta E + c_5 \lambda A - c_5 \delta_1 I
\]

Choose \( c_1 = \frac{c_2(1 - \theta) + c_3 \xi(1 - \lambda) + c_4 \varepsilon + c_5 \theta}{c_1} \), \( c_2 = \frac{c_2 \delta}{c_1} \), \( c_3 = \frac{c_5 \xi}{c_1} \), and \( c_4 = \frac{c_6}{c_1} \). Then

\[
D^\alpha \mathcal{F}(t) \leq \left( \frac{c_2(1 - \theta) + c_3 \xi(1 - \lambda) + c_4 \varepsilon + c_5 \theta}{c_1} \right) c_1 \delta_1 E 
= \left( \frac{c_2(1 - \theta) + c_3 \xi(1 - \lambda) + c_4 \varepsilon + c_5 \theta}{c_1} - 1 \right) c_1 \delta_1 E 
= \left( \frac{c_2(1 - \theta) + c_3 \xi(1 - \lambda) + c_4 \varepsilon + c_5 \theta}{c_1} - 1 \right) c_1 \delta_1 E 
= (9R_0 - 1)c_1 \delta_1 E 
\leq 0 \text{ for } R_0 \leq 1
\]

All the model parameters are positive; it follows that \( D^\alpha \mathcal{F}(t) \leq 0 \) for \( R_0 \leq 1 \) and \( D^\alpha \mathcal{F}(t) = 0 \) if and only if, \( E = 0 \). Thus, by LaSalle’s invariance principle the disease-free equilibrium point \( \delta_0 \) is globally asymptotically stable in a positively invariant region if \( R_0 \leq 1 \). \( \square \)

**Theorem 2.7.** If \( R_0 > 1 \), then endemic equilibrium point given by

\[
EE = (S^*, E^*, A^*, L^*, F^*, \Gamma^*, Q^*, H^*, R^*)
\]

is globally asymptotically stable in the region \( \Psi \).

**Proof.** Assume that the effective reproduction number \( R_0 > 1 \), which indicates that the endemic equilibrium point exists. We’ll look at the Lyapunov function defined by

\[
\mathcal{H}(t) = (S - S^*)\ln \frac{S}{S^*} + (E - E^*)\ln \frac{E}{E^*} + (A - A^*)\ln \frac{A}{A^*} 
+ (L - L^*)\ln \frac{L}{L^*} + (F - F^*)\ln \frac{F}{F^*} + (\Gamma - \Gamma^*)\ln \frac{\Gamma}{\Gamma^*} 
+ (Q - Q^*)\ln \frac{Q}{Q^*} + (H - H^*)\ln \frac{H}{H^*} + (R - R^*)\ln \frac{R}{R^*}.
\]

Then, time derivative of the Lyapunov function in Caputo sense and following [47], we get

\[
D^\alpha \mathcal{H}(t) = \left( 1 - \frac{S}{S^*} \right) D^\alpha S^* + \left( 1 - \frac{E}{E^*} \right) D^\alpha E^* + \left( 1 - \frac{A}{A^*} \right) D^\alpha A^* 
+ \left( 1 - \frac{L}{L^*} \right) D^\alpha L^* + \left( 1 - \frac{F}{F^*} \right) D^\alpha F^* + \left( 1 - \frac{\Gamma}{\Gamma^*} \right) D^\alpha \Gamma^* 
+ \left( 1 - \frac{Q}{Q^*} \right) D^\alpha Q^* + \left( 1 - \frac{H}{H^*} \right) D^\alpha H^* + \left( 1 - \frac{R}{R^*} \right) D^\alpha R^*.
\]

Substituting the appropriate values from (2.2) for each fractional derivative and using (2.31), we arrive at

\[
D^\alpha \mathcal{H}(t) \leq \left( 1 - \frac{S}{S^*} \right) (\theta - \Lambda S - \mu S) + \left( 1 - \frac{E}{E^*} \right) (\Lambda S - \delta_1 E) 
+ \left( 1 - \frac{A}{A^*} \right) (\delta(1 - \theta)E - \delta_2 A) 
+ \left( 1 - \frac{L}{L^*} \right) (\xi(1 - \lambda)A - \delta_3 L) 
+ \left( 1 - \frac{F}{F^*} \right) (\tau(1 - \xi)(1 - \lambda)A + \varepsilon L - \delta_4 F) 
+ \left( 1 - \frac{\Gamma}{\Gamma^*} \right) (\theta E + \lambda A - \delta_1 I) 
+ \left( 1 - \frac{Q}{Q^*} \right) (\eta I + \phi F - \delta_6 Q) 
+ \left( 1 - \frac{H}{H^*} \right) (\varphi(1 - \eta)I + \pi Q - \delta_7 H) 
+ \left( 1 - \frac{R}{R^*} \right) (\zeta(1 - \tau)(1 - \xi)(1 - \lambda)A + \rho(1 - \varepsilon)L) 
+ \sigma(1 - \pi)Q + \psi H.
\]

\[\square\]
Now, use the Lemma 1 and Eq. (3.1) on the model Eq. (2.2). We obtain equations, on that equations, we collocate at
\(m + 1 - |z|\) points \(t_p(p = 0, 1, \ldots, m - |z|)\) as:

\[
\sum_{i=0}^{m} \sum_{j=|i|}^{m} a_{i,j} x_{j-|i|} t_p^{|i|} = \theta - \frac{\beta}{N} \sum_{i=0}^{m} (v_i + d_i + e_i + f_i) P_i(t_p),
\]

\[
- \mu \sum_{i=0}^{m} a_{i} P_i(t_p)
\]

\[
\sum_{i=0}^{m} \sum_{j=|i|}^{m} b_{i,j} x_{j-|i|} t_p^{|i|} = \frac{\beta}{N} \sum_{i=0}^{m} (v_i + d_i + e_i + f_i) P_i(t_p),
\]

\[
- (\theta + \delta(1 - \xi) + \mu) \sum_{i=0}^{m} b_i P_i(t_p)
\]

Since all the parameters used in the model (2.2) are positive, we have \( D^x \mathcal{M}(t) \leq 0 \) for \( v_1 + v_2 \leq 0 \) and \( D^x \mathcal{M}(t) = 0 \) for \( v_1 + v_2 = 0 \) which in turn implies that \( D^x \mathcal{M}(t) = 0 \) if and only if \( S = S^*, E = E^*, A = A^*, L = L^*, F = F^*, I = I^*, Q = Q^*, H = H^* \), and \( R = R^* \). Thus, by LaSalle’s invariance principle the EEP is globally asymptotically stable. □

Theorem 2.7 shows that, independent of the starting size of infectious people in the community, COVID-19 will establish itself in society when \( R_0 > 1 \).

3. The Legendre spectral collocation method for the solution of COVID-19 model

To apply the Legendre collocation method for the model Eq. (2.2) we first approximate \( S(t), E(t), A(t), L(t), F(t), I(t), Q(t), H(t), R(t) \) as follows:

\[
S(t) = \sum_{i=0}^{m} a_i P_i(t), \quad E(t) = \sum_{i=0}^{m} b_i P_i(t), \quad A(t) = \sum_{i=0}^{m} c_i P_i(t),
\]

\[
L(t) = \sum_{i=0}^{m} d_i P_i(t), \quad F(t) = \sum_{i=0}^{m} e_i P_i(t), \quad I(t) = \sum_{i=0}^{m} f_i P_i(t),
\]

\[
Q(t) = \sum_{i=0}^{m} g_i P_i(t), \quad H(t) = \sum_{i=0}^{m} h_i P_i(t), \quad R(t) = \sum_{i=0}^{m} q_i P_i(t)
\]

\[
\sum_{i=0}^{m} \sum_{j=|i|}^{m} d_{i,j} x_{j-|i|} t_p^{|i|} = \xi(1 - \lambda) \sum_{i=0}^{m} c_i P_i(t_p)
\]

(3.1)
\[
\sum_{i=1}^{m} \sum_{k=1}^{n} \xi_i \phi_{k, i}^{\beta} \rho_{k} = \tau(1 - \xi)(1 - \lambda) \sum_{i=0}^{m} \epsilon_i P_i(t_p) + \epsilon \sum_{i=0}^{m} d_i P_i(t_p)
\]

\[
- (\phi + \gamma(1 - \phi) + \mu + \omega) \sum_{i=0}^{m} c_i P_i(t_p)
\]

\[
- (\eta + \varphi(1 - \eta) + \mu + \omega) \sum_{i=0}^{m} f_i P_i(t_p)
\]

\[
- (\pi + \sigma(1 - \pi) + \mu + \omega) \sum_{i=0}^{m} g_i P_i(t_p)
\]

\[
- (\psi + \mu + \omega) \sum_{i=0}^{m} h_i P_i(t_p)
\]

\[
\sum_{i=1}^{m} \sum_{k=1}^{n} \eta_i \phi_{k, i}^{\beta} \rho_{k} = \xi(1 - \tau)(1 - \xi)(1 - \lambda) \sum_{i=0}^{m} c_i P_i(t_p) + \rho(1 - \varphi) \sum_{i=0}^{m} d_i P_i(t_p)
\]

\[
\psi \sum_{i=0}^{m} h_i P_i(t_p) - \mu \sum_{i=0}^{m} q_i P_i(t_p)
\]

\[
\gamma(1 - \phi) \sum_{i=0}^{m} c_i P_i(t_p) + \sigma(1 - \pi) \sum_{i=0}^{m} g_i P_i(t_p)
\]

\[
\sum_{i=0}^{m} (-1)^i a_i = S(0), \quad \sum_{i=0}^{m} (-1)^i b_i = E(0), \quad \sum_{i=0}^{m} (-1)^i c_i = I(0),
\]

\[
\sum_{i=0}^{m} (-1)^i d_i = L(0), \quad \sum_{i=0}^{m} (-1)^i e_i = F(0),
\]

We utilize roots of the shifted Legendre polynomial \(P_{m+1}(-1)\) to find appropriate collocation points. We can also obtain nine equations by replacing Eq. (3.1) in the beginning conditions (2.3).

Table 1 Summary of model parameters and their descriptions

| Parameter | Value/week | Source | Parameter | Value/week | Source |
|-----------|------------|--------|-----------|------------|--------|
| \(\phi\)  | 32.730     | Calculated | \(\omega\) | 0.000412 | Calculated |
| \(\beta\) | 0.189254   | Fitted   | \(\epsilon\) | 0.000025 | Assumed |
| \(\mu\)   | 0.000287   | Calculated | \(\rho\) | 0.000043 | Fitted |
| \(\theta\) | 0.017893   | Fitted   | \(\phi\) | 0.00156  | Fitted |
| \(\delta\) | 0.000123   | Fitted   | \(\gamma\) | 0.080962 | Fitted |
| \(\lambda\) | 0.003578  | Fitted   | \(\eta\) | 0.187373 | Fitted |
| \(\xi\)   | 0.002056   | Fitted   | \(\sigma\) | 0.00206 | Fitted |
| \(\tau\)  | 0.000124   | Assumed  | \(\varphi\) | 0.011624 | Fitted |
| \(\zeta\) | 0.00001    | Assumed  | \(\psi\) | 0.001    | Assumed |
| \(\psi\)  | 0.011742   | Fitted   | \(\varphi\) | 0.011624 | Fitted |

Fig. 1 The flow rate of the parameters of the model.
\[\sum_{i=0}^{m} (-1)^i f_i = I(0), \quad \sum_{i=0}^{m} (-1)^i g_i = Q(0), \quad \sum_{i=0}^{m} (-1)^i h_i = H(0), \quad \sum_{i=0}^{m} (-1)^i q_i = R(0)\]  
\hspace{1cm} (3.3)

Eq. (3.2), together with the equations of the initial conditions (3.3), give \(9m + 9\) equations that can be solved using Newton’s iterative methods, for the unknowns \(a_i, b_i, c_i, d_i, f_i, g_i, h_i, q_i\), and \(p_i, i = 0, 1, \ldots, m\).

4. Input parameters and assumptions

The parameters are estimated using real-world data from COVID-19 confirmed cases in Ethiopia from 22 June 2020 to 28 February 2021. Data available at [http://www.ephi.gov.et](http://www.ephi.gov.et) and [https://www.worldometers.info](https://www.worldometers.info) [48]. The following parameters are derived from the data: Ethiopia has a total population of roughly \(N = 114.041.946\), and the life expectancy in 2020 is 66.95 years. As a result, the natural death rate is computed appropriately. Further, from 29,424 tests performed in the first week of this study (June 22–28, 2020), 1,157 new confirmed COVID-19 cases in Ethiopia were reported. Of these, 736 were asymptomatic infected cases. In the first week of this work, there were 919 newly recovered COVID-19 cases, 3,160 cases on treatment, 7,079 cases on follow-up, 120 cases on HBIC (Home Based Isolation and Care), and assuming the initial number of funeral cases to be 87. The initial population is assumed \(N(0) = 114,041,946\).

All the graphical depictions sketched in the below sections are based on the parametric values listed on Table 1.

**Sensitivity analysis of \(R_0\):** The sensitivity of \(R_0\) to a parameter \(\varphi\) can be computed using the following formula

\[\frac{\delta R_0}{\delta \varphi} = \frac{\partial R_0}{\partial \varphi}.\]  
\hspace{1cm} (4.1)

The sensitive parameters and their sensitivity index are shown in the table above. On the next figures, we have also shown the effect of the most sensitive factors for \(R_0\) (see Fig. 1).

The forecast of our model generated by the system of nonlinear fractional differential Eqs. (2.2) with real data from the Ethiopian Ministry of Health and the Ethiopian Public Health Institute incorporated for 36 weeks is shown in Fig. 2. Furthermore, the numerical finding is immediately recognizable as being close to the real data (see Fig. 3 and 4).

The graphical interpretations of Figs. 5–13 show the influence of the fractional order \(\alpha\) on each compartment of the model.
Fig. 4  Effects of hospitalization rate of symptomatically infected cases $\varphi$ and rate of lockdown individuals attending the funeral $\varepsilon$ on the basic reproduction number $R_0$.

Fig. 5  Dynamical behavior of the susceptible class over time for different values of $\alpha$.

Fig. 6  Dynamical behavior of the exposed class over time for different values of $\alpha$. 
Fig. 7  Dynamical behavior of the asymptomatic class over time for different values of $\alpha$.

Fig. 8  Dynamical behavior of the lockdown class over time for different values of $\alpha$.

Fig. 9  Dynamical behavior of the funeral class over time for different values of $\alpha$. 
Fig. 10  Dynamical behavior of the symptomatically infected class over time for different values of $a$.

Fig. 11  Dynamical behavior of the quarantined class over time for different values of $a$.

Fig. 12  Dynamical behavior of the hospitalized class over time for different values of $a$. 
Numerical simulations of the behavior of each compartment in relation to the sensitive parameters are shown in the graphs below.

5. Discussion

In this study, a fractional mathematical model for COVID-19 transmission dynamics in Ethiopia was developed, and its many properties, such as existence and uniqueness, local and global stability analysis of equilibrium points were investigated. Based on the real data that was made accessible in [48], the fitted parameters were constructed using least-square curve fitting in Matlab R2020a. The basic reproduction number is then computed to be $R_0 = 1.4575 > 1$, showing that the endemic equilibrium point is both locally and globally asymptotically stable and the infection free equilibrium point is both locally and globally asymptotically unstable.

Table 2 also discusses the sensitivity of the basic reproduction number $R_0$, pointing out that it is more sensitive to the quarantine and hospitalization rates of symptomatically

| Parameter | Sensitivity index | Parameter | Sensitivity index |
|-----------|------------------|-----------|------------------|
| $\beta$   | +1.0000          | $\eta$    | −4.6900          |
| $\gamma$  | $+5.536 \times 10^{-9}$ | $\varphi$ | −3.8543          |
| $\delta$  | +0.1282          | $\varepsilon$ | −1.1788          |
| $\zeta$   | +0.2484          | $\rho$    | −0.1424          |
| $\theta$  | −0.3662          | $\phi$    | −0.0002          |
| $\lambda$ | −0.2022          | $\gamma$  | −0.0046          |
| $\tau$    | −0.0068          | $\zeta$   | −0.0006          |

Based on the real data that was made accessible in [48], the fitted parameters were constructed using least-square curve fitting in Matlab R2020a. The basic reproduction number is then computed to be $R_0 = 1.4575 > 1$, showing that the endemic equilibrium point is both locally and globally asymptotically stable and the infection free equilibrium point is both locally and globally asymptotically unstable.

Table 2 also discusses the sensitivity of the basic reproduction number $R_0$, pointing out that it is more sensitive to the quarantine and hospitalization rates of symptomatically

---

**Fig. 13** Dynamical behavior of the recovered class over time for different values of $\alpha$.

**Fig. 14** Behaviors of susceptible and exposed cases when the value of the parameter $\eta$ is increasing for $\alpha = 0.7$. 
Behaviors of asymptomatic infected and symptomatic infected cases when the value of the parameter $\eta$ is increasing for $\alpha = 0.7$.

Behaviors of quarantined and hospitalized cases when the value of the parameter $\eta$ is increasing for $\alpha = 0.7$.

Behaviors of susceptible and exposed cases when the value of the parameter $\beta$ is decreasing for $\alpha = 0.7$. 

Fig. 15  Behaviors of asymptomatic infected and symptomatic infected cases when the value of the parameter $\eta$ is increasing for $\alpha = 0.7$.

Fig. 16  Behaviors of quarantined and hospitalized cases when the value of the parameter $\eta$ is increasing for $\alpha = 0.7$.

Fig. 17  Behaviors of susceptible and exposed cases when the value of the parameter $\beta$ is decreasing for $\alpha = 0.7$. 


Fig. 18  Behaviors of asymptomatic and symptomatic infected cases when the value of the parameter $\beta$ is decreasing for $\alpha = 0.7$.

Fig. 19  Behaviors of quarantined and hospitalized cases when the value of the parameter $\beta$ is decreasing for $\alpha = 0.7$.

Fig. 20  Behaviors of susceptible and exposed cases when the value of the parameter $\varphi$ is increasing for $\alpha = 0.2$. 
infected people, as well as the lockdown rate of people attending funerals and the transmission rate. $\beta$ has positive impact, which means decreasing this parameter leads to a reduction in the value of $R_0$. $\eta$, $\varphi$ and $\varepsilon$ have negative impacts. That is, increasing the value of these parameters leads to a decrease in $R_0$. When $\beta \leq 0.1299$, $\eta \geq 0.3776$, $\varphi \geq 0.2420$ and $\varepsilon \geq 0.0075$ and the remaining parameters in Table 1 are unchanged, the stable infection free equilibrium point for Eq. (2.2) is obtained. Other parameters change within their suitable range, making all its significant effects, even the value of the reproduction number $R_0$ greater than or less than one (see Figs. 14–22).

Fig. 21  Behaviors of asymptomatic and symptomatic infected cases when the value of the parameter $\varphi$ is increasing for $\alpha = 0.2$.

Fig. 22  Behaviors of quarantined and hospitalized cases when the value of the parameter $\varphi$ is increasing for $\alpha = 0.2$.

Fig. 23  Numerical results of $A(t)$ and $I(t)$ for different values of $R_0$. 
The behavior of approximate solutions of a model Eq. (2.2) for different values of fractional order \( \alpha \) is shown in Figs. 5–13. The fractional order \( \alpha \) has its own impact on a model’s numerical solutions. We observed a significant reduction in the number of infected individuals in a fractional order \( \alpha = 0.5 \). Fig. 23 shows that the highest value of the basic reproduction number \( R_0 \) shows the highest number of asymptomatic and symptomatic cases, and as the number of infectious cases increases, the susceptible population decreases. For \( R_0 = 2.4125 \), we have an evidence that in this simulation, most of the infected individuals are rapidly increasing. This indicates that one infected person will cause more than two new infections, and virus transmission will become uncontrollable. When the basic reproduction number \( R_0 = 0.7519 \) is used, the curves reveal that infected individuals tend to be few. This indicates that the number of newly infected individuals is decreasing, that society is safe from the virus, and that the infection will eventually die out from the Ethiopia.

6. Conclusion

The transmission dynamics of the Covid-19 epidemic are studied using a mathematical model developed in this work. In the Caputo sense, the model is presented as a system of fractional order differential equations, with the numerical solution achieved using the Legendre collocation method. The findings of the proposed model are found to be quite close to the real data. The model considers the impact of various control techniques on disease transmission. These strategies have been shown to result in considerable changes to reduce the danger of disease transmission, as simulated above. Ethiopia’s government must take the required steps to make control tactics a must throughout the epidemic. If used properly, control techniques such as lockdown, quarantine, and hospitalization can considerably reduce the disruptive effects of COVID-19 while also protecting the nation. The government may also direct its control strategy toward reducing the \( R_0 \) via the sensitive parameters.

Data Availability

Data from COVID-19 confirmed cases in Ethiopia from 22 June 2020 to 28 February 2021 were utilized to support this investigation.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgement

The authors are grateful to the editor and reviewers for their thorough review and comments, which contributed to improving this article. In addition, the authors would like to acknowledge and express their gratitude to the United Arab Emirates University, Al Ain, UAE for providing financial support with Grant No. 12S086.

References

[1] H. Zhu, L. Wei, P. Niu, The novel coronavirus outbreak in Wuhan, China, Glob. Health Res. Policy 5 (6) (2020).
[2] Centers for Disease Control and Prevention, Outbreak of 2019 Novel Coronavirus (2019-nCoV) in Wuhan, China, 21 January (2020). Available: www.cdc.gov/ncidod/dhfs/coronavirus/2019-ncov/2019-ncov-wuhan-china-update.html.
[3] World Health Organization, WHO Director-General’s opening remarks at the media briefing on COVID-19 on 30 January 2020, Geneva, Switzerland: https://www.who.int/director-general-speeches/detail/who-director-general-opening-remarks-covid-19-situation-updates.
[4] World Health Organization (WHO), Advice for Public, Retrieved from the original on 26 January 2020. Retrieved 10 February 2020.
[5] S.E. Moore, E. Okyere, Controlling the Transmission Dynamics of COVID-19, Univ. Cape Coast 1 (1) (2020) 13–21.
[6] R.M. Anderson, H. Heesterbeek, D. Klinkenberg, T. Hollingsworth, How will country-based mitigation measures influence the course of the COVID-19 epidemic?, Lancet 395 (2020).
[7] J. Cohen, K. Kupferschmidt, The coronavirus seems unstoppable. What should the world do now?, Science 2020 (news). Retrieved 22 March 2020.
[8] World Health Organization, Ethiopia, the first case of COVID-19 confirmed in Ethiopia, 13 March 2020. Available: https://www.afro.who.int/news/first-March-case-covid-19-confirmed-ethiopia.
[9] T. Kejela, Probable Factors Contributing to the Fast Spread of the Novel Coronavirus (COVID-19) in Ethiopia, J. Infect. Dis. Epidemiol. 6 (2020) 169, https://doi.org/10.23937/2474-3658/1510169.
[10] S. Mwalili, M. Kimathi, V. Ojiambo, D. Gathungu, R. Mbogo, SEIR model for COVID-19 dynamics incorporating the environment and social distancing, BMC Res. Notes 13 (2020) 352.
[11] A. Boudaoui, Y. El hadj Moussa, Z. Hammouch, S. Ullah, A fractional-order model describing the dynamics of the novel coronavirus (COVID-19) with nonsingular kernel, Chaos Solitons Fract. 146 (2021) 110859, https://doi.org/10.1016/j.chaos.2021.110859.
[12] K.S. Nisar, S. Ahmad, U. Ullah, K. Shah, H. Alrabiaah, M. Arfan, Mathematical analysis of SIRD model of COVID-19 with Caputo fractional derivative based on real data, Res. Phys. 21 (2021) 103772, https://doi.org/10.1016/j.rinp.2020.103772.
[13] P.A. Naik, M. Yavuz, S. Qureshi, J. Zu, S. Townley, Modeling and analysis of COVID-19 epidemics with treatment in fractional derivatives using real data from Pakistan, Eur. Phys. J. Plus 135 (2020) 795, https://doi.org/10.1140/epjp/s13660-020-00819-5.
[14] H.M. Youssef, N.A. Alghamdi, M.A. Ezzat, A.A. El-Baray, A. M. Shawky, A New Dynamical Modelling SEIR with global analysis applied to the real data of spreading COVID-19 in Saudi Arabia, J. Math. Biol. Sci. Eng. 17 (6) (2020) 7018–7044, https://doi.org/10.3934/mbe.2020362.
[15] A.M. Mishra, R. Agarwal, S.D. Purohit, K. Jangid, Nonlinear Dynamics of SARS-CoV2 Virus: India and Its Government Policy, Math. Model. Soft Comput. Epidemiol. (2020) 291–301.
[16] J. Danane, Z. Hammouch, K. Allali, S. Rashid, J. Singh, A fractional-order model of coronavirus disease 2019 (COVID-19) with governmental action and individual reaction, Math. Methods Appl. Sci. (2021), https://doi.org/10.1002/mma.7759.
[17] A.S. Shaikh, I.N. Shaikh, K.S. Nisar, A mathematical model of COVID-19 using fractional derivative: outbreak in India with dynamics of transmission and control, Adv. Differ. Equ. 73 (2020), https://doi.org/10.1186/s13662-020-02834-3.
[18] D. Olabodey, J. Culpy, A. Fisher, A. Tower, D. Hull-Nye, X. Wang, Deterministic and stochastic models for the epidemic...
A.A. Khan, H.M. Alshehri, T. Abdeljawad, Q.M. Al-Mdallal, H.F. Bozkurt, A. Yousef, T. Abdeljawad, A. Kalinli, Q.M. Al-Mdallal, A. Rachah, D.F.M. Torres, Dynamics and optimal control of COVID-19 model with fractional differential operators with singular and non-singular kernels: Analysis and numerical scheme based on Newton polynomial, Alexandria Eng. J. 60 (2021) 3781–3806.

M.A. Khan, A. Atangana, Modeling the dynamics of novel coronavirus (2019-nCoV) with fractional derivative, Alexandria Eng. J (2020), https://doi.org/10.1016/j.aej.2020.02.033.

I. Ahmed, I.A. Baba, A. Yusuf, P. Kumam, W. Kumam, Analysis of Caputo fractional-order model for COVID-19 with lockdown, Adv. Differ. Equ. 394 (2020).

J. Danane, K. Allali, Z. Hammouch, K.S. Nisar, Mathematical analysis and simulation of a stochastic COVID-19 Lévy jump model with isolation strategy, Res. Phys. 103994 (2021), https://doi.org/10.1016/j.rinp.2021.103994.

K. Logeswari, C. Ravichandran, K.S. Nisar, Mathematical model for spreading of COVID-19 virus with the Mittag-Leffler kernel, Numer. Methods Partial Different. Eqs. (2020), https://doi.org/10.1002/num.22652.

A.M. Mishra, S.D. Purohit, K.M. Owolabi, Y.D. Sharma, A nonlinear epidemiological model considering asymptotic and quarantine classes for SARS CoV-2 virus, Chaos Solitons Fract. 138 (2020) 109953.

S. Ahmad, A. Ullah, Q.M. Al-Mdallal, H. Khan, K. Shah, A. Khan, Fractional order mathematical modeling of COVID-19 transmission, Chaos Solitons Fract. 139 (2020) 110256.

A. Khan, H.M. Alshhehri, T. Abdeljawad, Q.M. Al-Mdallal, H. Khan, Stability analysis of fractional nabla difference COVID-19 model, Res. Phys. 22 (2021) 103888.

T.N. Sudhu, A. Shiﬁq, Q.M. Al-Mdallal, Exponentiated transformation of Gumbel Type-II distribution for modeling COVID-19 data, Alexandria Eng. J. 60 (1) (2021) 671–689.

F. Bozkurt, A. Yousef, T. Abdeljawad, A. Kalinli, Q.M. Al-Mdallal, A Fractional-Order Model of COVID-19 considering the Fear Effect of the Media and Social Networks on the Community, Chaos Solitons Fract. 152 (2021) 111403.

H.S. Rodrigues, M.T.T. Monteiro, D.F.M. Torres, Seasonality effects on dengue: basic reproduction number, sensitivity analysis, and optimal control, Math. Methods Appl. Sci. 39 (16) (2016) 4671–4679.

A. Rachah, D.F.M. Torres, Dynamics and optimal control of Ebola transmission, Math. Comput. Sci. 10 (3) (2016) 331–342.

C.J. Silva, D.F.M. Torres, A SICA compartmental model in epidemiology with application to HIV/AIDS in Cape Verde, Ecol. Complexity 30 (2017) 70–75.