COVARIANTS OF BINARY FORMS AND NEW IDENTITIES FOR BERNOULLI, EULER AND HERMITE POLYNOMIALS

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ABSTRACT. Using the methods of classical invariant theory a general approach to finding of identities for Bernulli, Euler and Hermite polynomials is proposed.

1. Introduction

The relationship between the group representations theory and special functions is well known, see. [1]. In this paper we establish the relationship between the classical invariant theory and identities for Bernulli, Euler, and Hermite polynomials.

The polynomials of Bernoulli $B_n(x)$, Euler $E_n(x)$ and Hermite $H_n(x)$, $n = 0, 1, 2, \ldots$ are defined by the following generating functions

$$\frac{te^{xt}}{e^t - 1} = \sum_{i=0}^{\infty} B_n(x) \frac{t^n}{n!} e^{xt} - 1 = \sum_{i=0}^{\infty} E_n(x) \frac{t^n}{n!} e^{xt} = \sum_{i=0}^{\infty} H_n(x) \frac{t^n}{n!}.$$

Particularly $B_0(x) = E_0(x) = H_0(x) = 1$. The numbers $B_n = B_n(0)$ are called the Bernoulli numbers and the numbers $E_n = E_n(0)$ are called the Euler numbers. All these types of polynomials are special cases of the Appell polynomials $\mathcal{A} = \{A_n(x)\}$, where $\deg(A_n(x)) = n$ and the polynomials satisfy the identity

$$A'_n(x) = nA_{n-1}(x), n = 0, 1, 2, \ldots.$$

It is clear that the polynomials $\{x^n\}$ are the Appell polynomials also. Denote by $\mathcal{B, E, H}$, the Bernulli, Euler and Hermite polynomials, respectively. Also, put $\mathcal{T} := \{1, x, x^2, \ldots, x^n, \ldots\}$.

We are interested in finding all polynomial identities for the Appell polynomials, i.e. identities of the form

$$F(A_0(x), A_1(x), \ldots, A_n(x)) = 0,$$

where $F$ is some polynomial of $n + 1$ variables.

First, consider the motivating examples. Let

$$\Delta(x) := \begin{vmatrix} B_0(x) & B_1(x) \\ B_1(x) & B_2(x) \end{vmatrix} = B_0(x)B_2(x) - B_1(x)^2.$$

Taking into account (1) we have

$$\Delta(x)' = (B_0(x)B_2(x) - B_1(x)^2)' = B_0(x)'B_2(x) + B_0(x)B_2(x)' - 2B_0(x)B_1(x) =$$

$$= 2B_0(x)B_1(x) - 2B_0(x)B_1(x) = 0$$

Thus $\Delta(x)$ is a constant, and it is clear that this constant is equal to $\Delta(0)$. Substituting the corresponding Bernoulli polynomial, we find the constant and get the following identity

$$B_0(x)B_2(x) - B_1(x)^2 = B_0B_2 - B_1^2 = -\frac{1}{12}.$$

Similarly, $E_0(x)E_2(x) - E_1(x)^2 = -\frac{1}{4}$ and $H_0(x)H_2(x) - H_1(x)^2 = -1$. 
Consider now the following differential operator
\[ D := a_0 \frac{\partial}{\partial a_1} + 2a_1 \frac{\partial}{\partial a_2} + \cdots + na_{n-1} \frac{\partial}{\partial a_n}, \]
which acts on polynomials of the variables \( a_0, a_1, \ldots, a_n \). The action is very similar to (1). Also, it is easy to see that \( D(a_0 a_2 - a_1^2) = 0 \).

Consider the polynomial
\[
\Delta := \begin{vmatrix} a_0 & 3a_1 & 3a_2 & a_3 & 0 \\ 0 & a_0 & 3a_1 & 3a_2 & a_3 \\ 3a_0 & 6a_1 & 3a_2 & 0 & 0 \\ 0 & 3a_0 & 6a_1 & 3a_2 & 0 \\ 0 & 0 & 3a_0 & 6a_1 & 3a_2 \end{vmatrix}
\]
Note that the polynomial \( \Delta \) is, up to a factor, the discriminant of an binary form of degree 3:
\[ a_0 x^3 + 3a_1 x^2 y + 3a_2 xy^2 + a_3 y^3. \]

By applying the determinant derivative rule we get that \( D(\Delta) \) equals to sum of 5 determinants each of them equal to zero. Thus \( D(\Delta) = 0 \). Similarly, for the determinant
\[
\Delta(A) := \begin{vmatrix} A_0(x) & 3A_1(x) & 3A_2(x) & A_3(x) & 0 \\ 0 & A_0(x) & 3A_1(x) & 3A_2(x) & A_3(x) \\ 3A_0(x) & 6A_1(x) & 3A_2(x) & 0 & 0 \\ 0 & 3A_0(x) & 6A_1(x) & 3A_2(x) & 0 \\ 0 & 0 & 3A_0(x) & 6A_1(x) & 3A_2(x) \end{vmatrix}
\]
we obtain \( \Delta(A)' = 0 \). Therefore, for any Appell polynomials \( \{A_n(x)\} \) the identity
\[ \Delta_3(A) = \text{const} \]
holds. By direct calculations we obtain
\[ \Delta_3(B) = \frac{1}{16}, \Delta_3(C) = \frac{27}{16}, \Delta_3(H) = 108. \]

These examples lead to the hypothesis that if a polynomial \( S(a_0, a_1, \ldots, a_n) \) satisfies the condition \( D(S(a_0, a_1, \ldots, a_n)) = 0 \), then the polynomial \( S(A_0(x), A_1(x), \ldots, A_n(x)) \) is a constant, thus determines the identities between the Appell polynomials.

We proceed, by way of summarizing the approaches, as follows. Let \( \mathbb{K}[a_0, a_1, \ldots, a_n] \) and \( \mathbb{K}[x] \) be the algebras of polynomials over a field \( \mathbb{K} \) of characteristic zero. Let us consider the substitution homomorphism \( \varphi_A : \mathbb{K}[a_0, a_1, \ldots, a_n] \to \mathbb{K}[x] \) defined by \( \varphi_A(a_i) = A_i(x) \). Put
\[ \ker^* \varphi_A := \{ S \in \mathbb{K}[a_0, a_1, \ldots, a_n] \mid \varphi_A(S) \in \mathbb{K} \}. \]
We will prove that any element \( S(a_0, a_1, \ldots, a_n) \) of the subalgebra \( \ker^* \varphi_A \) defines an identity
\[ S(A) = \| S(A) \|, \]
here \( S(A) := S(A_0(x), A_1(x), \ldots, A_n(x)) \) and \( \| S(A) \| = S(A_0(0), A_1(0), \ldots, A_n(0)) \).

Therefore, the problem of describing all polynomial identities for the Appell polynomials is reduced to that of describing the algebra \( \ker^* \varphi_A \). It will be shown in the paper that the algebra \( \ker^* \varphi_A \) is isomorphic to the algebra of covariants of binary form of order \( n \).

This idea can also be applied for finding identities for different types of Appell polynomials. For instance, we have:
\[
\begin{vmatrix}
B_0(x) & E_0(x) & H_0(x) \\
B_1(x) & E_1(x) & H_1(x) \\
B_2(x) & E_2(x) & H_2(x)
\end{vmatrix} =
B_0(x)E_1(x)H_2(x) - B_0(x)H_1(x)E_2(x) - B_1(x)E_0(x)H_2(x) +
B_1(x)H_0(x)E_2(x) + B_2(x)E_0(x)H_1(x) - B_2(x)H_0(x)E_1(x) = \frac{1}{12}.
\]

The problem of describing all such polynomials identities for different Appell polynomials is reduced to that of describing of the algebra of joint covariants for several binary forms. The algebra of covariants of binary form and the algebra of joint covariants for several binary forms were an object of research in the classical invariant theory of the 19th century. In particular, developed efficient methods to find elements of this algebra.

We will deal mainly with the algebra of semi-invariants rather than the algebra covariants. These algebras are isomorphic. But, the algebra of semi-invariants is simpler object for computation.

In this paper we give a brief introduction to the theory of covariants and semi-invariants of binary form on the locally nilpotent derivations language. Based on the classical invariant theory approach the several types of identities for Appell polynomials are constructed.

2. Covariants and semi-invariants of binary forms

Let us recall, that a derivation of a ring \( R \) is an additive map \( D \) satisfying the Leibniz rule:

\[
D(r_1 r_2) = D(r_1)r_2 + r_1D(r_2), \quad r_1, r_2 \in R.
\]

A derivation \( D \) of a ring \( R \) is called locally nilpotent if for every \( r \in R \) there is an \( n \in \mathbb{N} \) such that \( D^n(r) = 0 \). The subring

\[
\ker D := \{ f \in R | D(f) = 0 \},
\]

is called the kernel of the derivation \( D \).

Let us consider the algebra of polynomials \( \mathbb{K}[a_0, a_1, \ldots, a_n] \) over the field \( \mathbb{K} \) of characteristic 0. Define the derivations \( \mathcal{D}, \mathcal{D}^* \) and \( E \) of the algebra \( \mathbb{K}[A_n] \) by

\[
\mathcal{D}(a_i) = ia_{i-1}, \mathcal{D}^*(a_i) = (n - i)a_{i+1}, E(a_i) = (n - 2i)a_i.
\]

Note, that the derivations \( \mathcal{D}, \mathcal{D}^*, E \) define a representation of the Lie algebra \( \mathfrak{sl}_2(\mathbb{K}) \).

Consider the derivations \( \mathcal{D} - Y \frac{\partial}{\partial X} \) and \( \mathcal{D}^* - X \frac{\partial}{\partial Y} \) of the polynomial algebra \( \mathbb{K}[a_0, \ldots, a_n, X, Y] \). Let us recall some concepts of the classical invariant theory.

**Definition 2.1.** The homogeneous polynomial

\[
(2) \quad \alpha(X, Y) := a_0X^n + na_1X^{n-1}Y + \cdots + \binom{n}{i} a_i X^{n-i}Y^i + \cdots + a_nY^n.
\]

is called the generic binary form of order \( n \).
**Definition 2.2.**

1. The algebra \( C_n := \ker(D - Y \frac{\partial}{\partial X}) \bigcap \ker(D^* - X \frac{\partial}{\partial Y}) \) is called the algebra of covariants for the generic binary form (2);
2. The algebra \( S_n := \ker(D) \) is called the algebra of semi-invariants for the generic binary form;
3. The algebra \( I_n := \ker(D) \bigcap \ker(D^*) \) is called the algebra of invariants for the generic binary form;

The elements of the algebras \( C_n, S_n, I_n \) are called the covariants, semi-invariants and invariants of the binary form. The following obvious inclusion holds: \( I_n \subset C_n \cap I_n \subset S_n \). It is well known that these algebras are finitely generated algebras.

**Example 2.1** It is easy to check that the generic form \( \alpha(X, Y) \) is a covariant and its leading coefficient \( a_0 \) (in the ordering \( X > Y \)) is a semi-invariant for the binary form of order \( n \). Also, the element \( a_0a_2 - a_1^2 \) is an invariant of the binary form of degree 2.

Let

\[ \varkappa : C_n \to S_n, \]

be the \( \mathbb{K} \)-linear map that takes each homogeneous covariant to its leading coefficient.

The following theorem holds

**Theorem 2.1** ([2], [3]). The map \( \varkappa \) is the homomorphism of the algebras \( C_n \) and \( S_n \).

The inverse map \( \varkappa^{-1} : S_n \to C_n \), can be defined as follows:

\[ \varkappa^{-1}(s) = \sum_{i=0}^{\text{ord}(s)} \frac{(D^*)^i(s)}{i!} X^{\text{ord}(s) - i} Y^i, \]

where \( \text{ord}(s) = \max\{k | (D^*)^k(s) \neq 0\} \).

The integer number \( \text{ord}(s) \) is called the order of the semi-invariant \( s \). The degree of covariant with respect to the variables \( X, Y \) is called the order of the covariant.

Similarly, we can define the algebras of covariants, semi-invariants and invariants of several generic binary forms. Let us consider the following three generic binary forms of order \( n \):

\[ \beta(X, Y) := b_0X^n + nb_1X^{n-1}Y + \cdots + \binom{m}{i} b_iX^{n-i}Y^i + \cdots + b_nY^n, \]
\[ \gamma(X, Y) := c_0X^n + nc_1X^{n-1}Y + \cdots + \binom{m}{i} c_iX^{n-i}Y^i + \cdots + c_nY^n, \]
\[ \delta(X, Y) := d_0X^n + nd_1X^{n-1}Y + \cdots + \binom{n}{i} d_iX^{n-i}Y^i + \cdots + d_nY^n, \]

Extend the derivations \( D, D^* \) to the polynomial algebra

\[ \mathbb{K}[a_0, \ldots, a_n, b_0, \ldots, b_n, c_0, \ldots, c_n, d_0, \ldots, d_n], \]

by \( D(b_i) = ib_{i-1}, D^*(b_i) = (n - i)b_{i+1}, D(c_i) = ic_{i-1}, D^*(c_i) = (n - i)c_{i+1} \) and \( D(d_i) = id_{i-1}, D^*(d_i) = (n - i)d_{i+1} \).

Then the subalgebra \( \ker(D - Y \frac{\partial}{\partial X}) \bigcap \ker(D^* - X \frac{\partial}{\partial Y}) \) of \( \mathbb{K}[a_0, \ldots, d_n, X, Y] \) is called the algebra of joint covariants of the forms \( \alpha(X, Y), \beta(X, Y), \gamma(X, Y) \) and \( \delta(X, Y) \). The algebras of joint semi-invariants and joint invariants can similarly be defined.

The main computational tool of the classical invariant theory is the transvectant.
Definition 2.3. The $r$-th transvectant of two covariants $f, g$ of orders $n$ and $m$ is called the following covariant

$$(f, g)^r = \sum_{i=0}^{r} (-1)^i \binom{r}{i} \frac{\partial^r f}{\partial X^{r-i} \partial Y^i} \frac{\partial^r g}{\partial X^i \partial Y^{r-i}}, r \leq \min(n, m).$$

For instance, the transvectants $(f, g)^1 \ (f, f)^2$ are equal to the Jacobian $J(f, g)$ and to the Hessian Hes$(f)$.

It is well known, see [3], that each covariant can be represented by transvectants.

Computationally, the semi-invariants is much simple objects that the covariants. To generate semi-invariants in [4] we introduced the semi-transvectant as an analogue of the transvectant.

Definition 2.4. The semi-invariant

$$[p, q]^r := \nabla((\nabla^{-1}(p), \nabla^{-1}(q))^r), r \leq \min(\text{ord}(p), \text{ord}(q)).$$

is called the $r$-th semi-transvectant of the semi-invariants $p, q \in \mathbb{K}[a_0, \ldots, d_n]$

The formula holds

$$(3) \quad [p, q]^r = \sum_{i=0}^{r} (-1)^i \binom{r}{i} \frac{(\nabla^*)^i(p)}{[\text{ord}(p)]^i} \frac{(\nabla^*)^{r-i}(q)}{[\text{ord}(q)]^{r-i}},$$

where $[m]_i = m(m-1)\ldots(m-(i-1))$ is the falling factorial.

Directly from the definition we get the following properties

$$[p, q]^0 = pq,$$

$$[f, g]^k = (-1)^k[g, f]^k,$$

so $[f, f]^k = 0$ if $k$ is odd.

Example 2.2

$$[p, q]^1 := [p, q] = p\frac{\nabla^*(q)}{\text{ord}(q)} - q\frac{\nabla^*(p)}{\text{ord}(p)}, \text{ the semi-jacobian of } p \text{ and } q,$$

$$[p, q]^2 = p\frac{(\nabla^*)^2(q)}{[\text{ord}(q)]^2} - 2\frac{\nabla^*(p)}{[\text{ord}(p)]} \frac{\nabla^*(q)}{[\text{ord}(q)]} + q\frac{(\nabla^*)^2(p)}{[\text{ord}(p)]^2}, \text{ the semi-hessian of } p \text{ and } q.$$

Up to a constant factor, the semi-hessian of the semi-covariant $a_0$ equals

$$\frac{1}{2}[a_0, a_0]^2 = a_0a_2 - a_1^2 = \begin{vmatrix} a_0 & a_1 \\ a_1 & a_2 \end{vmatrix},$$

Definition 2.5.

1. The homogeneous polynomial $F$ is called isobaric if it is an eigenvector of the operator $E$, i.e. $E(F) = \omega(F)F, \omega(F) \in \mathbb{Z}$.

2. The corresponding eigenvalue $\omega(F)$ is called the weight of the isobaric polynomial $F$.

The following theorem holds

Theorem 2.2.

1. $\omega(a_0^{k_0}a_1^{k_1}\cdots a_n^{k_n}) = n(k_0 + k_1 + \cdots + k_n) - 2(k_1 + k_2 + \cdots + k_n),$

2. if $s$ is a homogeneous isobaric semi-invariant then $\text{ord}(s) = \omega(s),$

3. if $p, q$ are homogeneous isobaric semi-invariants then $\omega([p, q]^i) = \omega(p) + \omega(q) - 2i.$
Throughout this paper the semi-invariant will mean an isobaric homogeneous semi-invariant.

**Definition 2.6.** A semi-invariant $S$ of the binary form of order $n$ is called proper if $\frac{\partial S}{\partial a_n} \neq 0$.

In other words, a semi-invariant is the proper one of the binary form of degree $n$ if it is not a semi-invariant of a binary form of smaller order.

**Problem.** Find all irreducible proper semi-invariants of the binary form of order $n$.

### 3. The main theorems

The following theorem is crucial for the constructions of identities for the Appell polynomials.

**Theorem 3.1.** Let $\varphi_A : \mathbb{K}[a_0, a_1, \ldots, a_n] \to \mathbb{K}[x]$ be the substitution homomorphism

$$\varphi_A(a_i) = A_i(x).$$

Then

$$\ker^* \varphi_A = S_n.$$

**Proof.** First we shall show that the homomorphism $\varphi_A$ commutes with the derivative operator $\frac{d}{dx}$, i.e.

$$\varphi_A(\mathcal{D}(h(a_0, a_1, \ldots, a_n))) = \frac{d}{dx}(\varphi_A(h(a_0, a_1, \ldots, a_n))),$$

for all $h(a_0, a_1, \ldots, a_n) \in \mathbb{K}[a_0, a_1, \ldots, a_n]$. The proof is by induction over the degree of the polynomial $h(a_0, a_1, \ldots, a_n)$.

Show that the statement holds for all polynomials of degree 1:

$$\varphi_A(\mathcal{D}(a_i)) = \varphi_A(ia_{i-1}) = iA_{i-1}(x) = \frac{d}{dx}A_i(x) = \frac{d}{dx}\varphi_A(a_i).$$

Assume that it holds for all polynomials $f \in \mathbb{K}[a_0, a_1, \ldots, a_n]$ , deg($f$) $\leq k$.

$$\varphi_A(\mathcal{D}(f)) = \frac{d}{dx}\varphi_A(f).$$

Then for all $i$ we have

$$\varphi_A(\mathcal{D}(a_if)) = \varphi_A(\mathcal{D}(a_i)f) = \varphi_A(\mathcal{D}(a_i))\varphi_A(f) + \varphi_A(a_i)\varphi_A(\mathcal{D}(f)) =$$

$$= \frac{d}{dx}\varphi_A(a_i)\varphi_A(f) + \varphi_A(a_i)\frac{d}{dx}\varphi_A(f) = \frac{d}{dx}(\varphi_A(a_i)\varphi_A(f)) = \frac{d}{dx}(\varphi_A(a_if)).$$

The linearity of the derivations $\mathcal{D}$, $\frac{d}{dx}$ and the linearity of the homomorphism $\varphi_A$ yield that the statement holds for all polynomials of the degree $k + 1$.

Thus, by induction the $\varphi_A$ commutes with the derivative $\frac{d}{dx}$.

Show that $S_n \subset \ker^* \varphi_A$. For $h(a_0, a_1, \ldots, a_n) \in S_n$ we have

$$\frac{d}{dx}(h(A_0(x), \ldots, A_n(x))) = \mathcal{D}\varphi_A(h(A_0(x), \ldots, A_n(x))) = \mathcal{D}(h(a_0, \ldots, a_n)) = 0.$$ 

Therefore, $h(A_0(x), \ldots, A_n(x))$ is a constant as claimed.

On the contrary, assume $g(A_0(x), \ldots, A_n(x)) \in \mathbb{K}$. Then

$$\mathcal{D}(g(a_0, \ldots, a_n)) = \frac{d}{dx}g(a_0(x), \ldots, A_n(x)) = 0.$$

Thus $g(a_0, \ldots, a_n) \in S_n$ and $S_n = \ker^* \varphi_A$. \qed
So, any semi-invariant \( S(a_0, \ldots, a_n) \) defines the identity

\[
S(A) = \|S(A)\|
\]

for the Appell polynomials \( \{A_n(x)\} \).

**Definition 3.1.** For the semi-invariant \( S(a_0, \ldots, a_n) \) the number \( \|S(A)\| \) is said to be the norm of the semi-invariant with respect to the Appell polynomials \( A \).

**Example 3.1** Let \( \Gamma(a_0, a_1, a_2) = \frac{1}{2}[a_0, a_0]^2 \) be the semi-hessian. Then

\[
\begin{align*}
\Gamma(B) &= B_0(x)B_2(x) - B_1(x)^2 = \frac{1}{6} + x^2 - x - \left(x - \frac{1}{2}\right)^2 = -\frac{1}{12}, \\
\Gamma(E) &= E_0(x)E_2(x) - E_1(x)^2 = x^2 - x - \left(x - \frac{1}{2}\right)^2 = -\frac{1}{4}, \\
\Gamma(H) &= H_0(x)H_2(x) - H_1(x)^2 = -1 + x^2 - x^2 = -1, \\
\Gamma(T) &= 0.
\end{align*}
\]

**Theorem 3.2.** The semi-invariant \( S(a_0, a_1, \ldots, a_n) \) determines the identity \( S(1, 1, \ldots, 1) = 0 \).

**Proof.** It easy to see that for a homogeneous isobaric polynomial \( S(a_0, a_1, \ldots, a_n) \) we have

\[
S(T) = S(1, x, x^2, \ldots, x^n) = x^mS(1, 1, \ldots, 1),
\]

for some integer \( m \).

Therefore, \( \|S(T)\| = 0 \). From another hand, the identity \( S(T) = \|S(T)\| \), implies

\[
x^mS(1, 1, \ldots, 1) = 0,
\]

for all \( x \). Thus \( S(1, 1, \ldots, 1) = 0 \). \qed

For the algebra of joint semi-invariants one can easily formulate and prove similar theorems.

4. **Identities for unique Appel sequence**

To describe identities for Appell polynomials of the same type let us describe the low degree proper semi-invariants for the finary form \( \alpha(X, Y) \). The formula \( [B_0, B_0]^i \) generates semi-invariants of degree 2, namely \( [a_0, a_0]^i, i = 0, \ldots, n \). Therefore, a proper semi-invariant of the degree 2 is the semi-transvectant

\[
[a_0, a_0]^n = \sum_{i=0}^n (-1)^i \binom{n}{i} a_i a_{n-i}.
\]

Denote it by \( Dv_n(a_0) \) and its image \( \varphi_A(Dv_n(a_0)) \) denote by \( Dv_n(A) \)

\[
Dv_n(A) := \sum_{i=0}^n (-1)^i \binom{n}{i} A_i(x)A_{n-i}(x).
\]

It is easy to check that the variable \( a_i \) can be expressed by \( a_0 \) as follows \( a_i = \frac{(D^*)^i(a_0)}{[n]_i} \). So, for simplicity of notation we write \( Dv_n(a_0) \) instead of \( Dv_n(a_0, a_1, \ldots, a_n) \).

**Example 4.1.** For the Bernoulli polynomials we have

\[
Dv_n(B) = \sum_{i=0}^n (-1)^i \binom{n}{i} B_i(x)B_{n-i}(x).
\]
Now
\[ \|Dv_n(B)\| = \sum_{i=0}^{n} (-1)^i \binom{n}{i} B_i(0) B_{n-i}(0) = \sum_{i=0}^{n} (-1)^i \binom{n}{i} B_i B_{n-i}. \]

From other hand, by direct calculation one can show [5] that
\[ \sum_{i=0}^{n} (-1)^i \binom{n}{i} B_i B_{n-i} = (1 - n) B_n. \]

Therefore we obtain the identity for the Bernoulli polynomials
\[ \sum_{i=0}^{n} (-1)^i \binom{n}{i} B_i(x) B_{n-i}(x) = (1 - n) B_n. \]

For the Euler polynomials we propose the conjecture
Conjecture. \[ \sum_{i=0}^{n} (-1)^i \binom{n}{i} E_i(x) E_{n-i}(x) = -2E_{n+1}. \]

Theorem 3.2 implies the well-known binomial identity \[ \sum_{i=1}^{n} (-1)^i \binom{n}{i} = 0. \]

In the paper [4] we found another proper semi-invariant of degree 2:
\[ W_n(a_0) := \sum_{i=1}^{n} (-1)^i \binom{n}{i} a_{n-i} a_i^2. \]

To construct a proper semi-invariant of degree 3 use the semi-hessian \[ \frac{1}{2} [a_0, a_0]^2 = \begin{bmatrix} a_0 & a_1 \\ a_1 & a_2 \end{bmatrix} \]
the semi-invariant \( a_0 \). Denote \( \text{Tr}_n(a_0) := [a_0, \frac{1}{2}[a_0, a_0]^2]^n \).

**Theorem 4.1.** For \( n \geq 4 \) the formula holds
\[ \text{Tr}_n(a_0) := \sum_{i=0}^{n} \sum_{j=0}^{i} \frac{(-1)^i}{[2n-4]^j} \binom{n}{i} \binom{i}{j} a_{n-i} \begin{bmatrix} [n]_j a_j \\ [n-1]_j a_{j+1} \\ [n-2]_{i-j} a_{i-j+2} \end{bmatrix}. \]

**Proof.** Since the semi-hessian has the weight \( 2n - 4 \), then the semi-transvectant
\[ [a_0, \frac{1}{2}[a_0, a_0]^2]^n \]
has the order \( n + 2n - 4 - 2n = n - 4 \). Thus it well-defined for \( n \geq 4 \). We have
\[ (D^*)^i(a_k) = [n - k]_i a_{i+k}. \]

By the determinant derivative rule we have
\[ (D^*)^i \left( \begin{bmatrix} a_0 & a_1 \\ a_1 & a_2 \end{bmatrix} \right) = \sum_{j=0}^{i} \binom{i}{j} (D^*)^j(a_0) (D^*)^{i-j}(a_1) = \]
\[ = \sum_{j=0}^{i} \binom{i}{j} \begin{bmatrix} [n]_j a_j \\ [n-1]_j a_{j+1} \\ [n-2]_{i-j} a_{i-j+2} \end{bmatrix}. \]
By (3) we get

\[ [a_0, [a_0, a_0]^2]^n = \sum_{i=0}^{n} \frac{(-1)^i}{n-i}[2n-4]^i \left( \begin{array}{c} n \\ i \end{array} \right) (\mathcal{D}^*)^{n-i}(a_0)(\mathcal{D}^*)^i \left( \begin{array}{cc} a_0 & a_1 \\ a_1 & a_2 \end{array} \right) = \]

\[ = \sum_{i=0}^{n} \frac{(-1)^i}{[2n-4]^i} \left( \begin{array}{c} n \\ i \end{array} \right) \sum_{j=0}^{i} \left( \begin{array}{c} i \\ j \end{array} \right) a_{n-i} \left[ \begin{array}{cc} [n]_j a_j \\ [n-1]_j a_{j+1} \end{array} \right] \left[ \begin{array}{cc} [n-1]_{i-j} a_{i-j+1} \\ [n-2]_{i-j} a_{i-j+2} \end{array} \right]. \]

As above, the direct calculations the \( n \)-th semi-transvectant \((n \geq 4)\) of the two semi-hessians yields the semi-invariant of degree 4:

\[
\text{Ch}_n(a_0) := \left( \frac{1}{2} [a_0, a_0]^2 ; \frac{1}{2} [a_0, a_0]^2 \right)^n = \sum_{i=0}^{n} \sum_{j=0}^{i} \sum_{k=0}^{n-i} \frac{(-1)^i (n_i)(n_j)(n_k)}{[2n-4]^i} A_{i,j,k},
\]

where

\[
A_{i,j,k} := \left| \begin{array}{cccc} [n]_k a_k & [n-1]_{i-k} a_{n-i-k+1} \\ [n-1]_k a_{k+1} & [n-2]_{i-k} a_{n-i-k+2} \end{array} \right| \left| \begin{array}{cccc} [n]_j a_j & [n-1]_{i-j} a_{i-j+1} \\ [n-1]_j a_{j+1} & [n-2]_{i-j} a_{i-j+2} \end{array} \right|.
\]

Now, let us consider the discriminant of a binary form. The discriminant is a well known invariant which can be defined as the \((2n-1) \times (2n-1)\) determinant of the Sylvester matrix of the binary form \(\alpha(X,Y) : \)

\[
\text{Discr}_n(a_0) := \left| \begin{array}{cccccc} a_0 & na_1 & \cdots & a_n & 0 & \cdots & 0 \\ 0 & a_0 & \cdots & na_{n-1} & a_n & 0 & \cdots & 0 \\ \vdots & \hdots & \ddots & \vdots & \hdots & \ddots & \vdots & \vdots \\ 0 & na_0 & \cdots & na_{n-1} & 0 & \cdots & 0 \end{array} \right| 
\]

The corresponding identities has the form \(\text{Discr}_n(A) = ||\text{Discr}_n(A)||. \)

**Example 4.2** Let \(A = B, n = 3.\) Simplifying the identity

\[ \text{Discr}_3(B) = ||\text{Discr}_3(B)||, \]

we obtain

\[ -27 B_3(x)^2 B_0(x)^2 + 162 B_3(x) B_0(x) B_1(x) B_2(x) + 81 B_2(x)^2 B_1(x)^2 - 108 B_2(x)^3 B_0(x) - 108 B_1(x)^3 B_3(x) = \frac{1}{16}. \]

**Conjecture.** \(||\text{Discr}_n(H)|| = \prod_{k=1}^{n} k^k.\)

Thus, we get the five types of identities for the Appell polynomials.
Theorem 4.2. Let $\mathcal{A} = \{A_n(x)\}$ be the Appell polynomials. Then the following identities hold

\[(4) \quad Dv_n(\mathcal{A}) = \|Dv_n(\mathcal{A})\|,\]
\[(5) \quad Tr_n(\mathcal{A}) = \|Tr_n(\mathcal{A})\|,\]
\[(6) \quad Ch_n(\mathcal{A}) = \|Ch_n(\mathcal{A})\|,\]
\[(7) \quad Discr_n(\mathcal{A}) = \|Discr_n(\mathcal{A})\|,\]
\[(8) \quad W_n(\mathcal{A}) = \|W_n(\mathcal{A})\|.\]

By applying Theorem 3.3 to the above identities we derive the following binomial identities:

\[(9) \quad \sum_{i=0}^{n} \sum_{j=0}^{i} \frac{(-1)^i}{[2n-4]^i} \binom{n}{i} \binom{i}{j} \mid [n]_j \mid [n-1]_j \mid [n-2]_j = 0,\]
\[(10) \quad \sum_{i=0}^{n} \sum_{j=0}^{n-i} \frac{(-1)^i}{[2n-4]^i} \binom{n-i}{j} \binom{n-j}{k} \mid [n]_k \mid [n-1]_k \mid [n-2]_k = 0.\]

5. Joint identities

Let us find joint proper semi-invariants of the binary forms $\alpha(X, Y)$ and $\beta(X, Y)$. First of all we consider the $n$-th semi-transvectant of the semi-invariants $a_0$ and $b_0$:

\[Dv_n(a_0, b_0) := [a_0, b_0]^n = \sum_{i=0}^{n} (-1)^i \binom{n}{i} a_i b_{n-i}.\]

Example 5.1 For the Bernoulli and Euler polynomials we have

\[Dv_n(\mathcal{B}, \mathcal{E}) = \sum_{i=0}^{n} (-1)^i \binom{n}{i} B_i(x) E_{n-i}(x).\]

By direct calculations we get

\[\|Dv_1(\mathcal{B}, \mathcal{E})\| = 0, \|Dv_2(\mathcal{B}, \mathcal{E})\| = -\frac{1}{3}, \|Dv_3(\mathcal{B}, \mathcal{E})\| = 0, \|Dv_4(\mathcal{B}, \mathcal{E})\| = \frac{7}{15}.\]

For the general case we propose the conjecture

Conjecture. $\sum_{i=0}^{n} (-1)^i \binom{n}{i} B_i(x) E_{n-i}(x) = -2(2^n - 1)B_{2n}.$

Similarly, the identity

\[Dv_n(\mathcal{B}, \mathcal{T}) = \|Dv_n(\mathcal{B}, \mathcal{T})\|,\]

implies that $Dv_n(\mathcal{B}, \mathcal{T}) = \sum_{i=0}^{n} (-1)^i \binom{n}{i} B_i(x) x^{n-i}$. It follows

\[Dv_n(\mathcal{B}, \mathcal{T})|_{x=0} = \|Dv_n(\mathcal{B}, \mathcal{T})\| = B_n(0) = B_n.\]

After simplification we get the identity for the Bernoulli polynomials

\[B_n(x) = \sum_{i=0}^{n-1} (-1)^{i+1} \binom{n}{i} B_i(x) x^{n-i} + B_n.\]
In the same way we get the identities for the Euler and Hermite polynomials

\[ E_n(x) = \sum_{i=0}^{n-1} (-1)^{i+1} \binom{n}{i} E_i(x) x^{n-i} + E_n, \]

\[ H_n(x) = \sum_{i=0}^{n-1} (-1)^{i+1} \binom{n}{i} H_i(x) x^{n-i} + H_n(0). \]

The proper joint semi-invariants of degree 3 are the following semi-transvectants \([a_0, [a_0, b_0]]^n\) for \(2i \leq n\). By direct calculations for \(i = 1\) we get

\[ \text{Tr}_n(a_0, b_0) := [a_0, [a_0, b_0]]^n = \left[ a_0, \left[ a_0, b_0 \right] \right]^n = \sum_{i=0}^{n} \sum_{j=0}^{i} \frac{(-1)^i}{[2n-2]_i} \binom{n}{i} \binom{i}{j} a_{n-i} \left[ \begin{array}{c} [n]_j a_j \\ [n-1]_j a_{j+1} \\ [n-1]_{i-j} b_{i-j+1} \\ [n]_{i-j} b_{i-j} \end{array} \right], \]

and

\[ \overline{\text{Tr}}_n(a_0, b_0) := [a_0, [a_0, b_0]]^n = [a_0, a_0 b_2 - 2a_1 b_1 + a_2 b_0]^n = \sum_{i=0}^{n} \sum_{j=0}^{i} \frac{(-1)^i}{[2n-2]_i} \binom{n}{i} \binom{i}{j} a_{n-i} A_{i,j}, \]

where

\[ A_{i,j} := [n]_{i-j} a_{i+1} b_{i-2} - 2[n-1]_{i-j} a_{i+1} b_{i-1} + n-2]_{i-j} a_{i+1} b_{i-2}. \]

The well known joint covariant is the resultant of two binary form. The corresponding semi-invariant \(s\text{Res}_n(a_0, b_0)\) has form

\[ s\text{Res}_n(a_0, b_0) := \left[ \begin{array}{cccccc} a_0 & n a_1 & \cdots & a_n & 0 & \cdots & 0 \\ 0 & a_0 & \cdots & n a_{n-1} & a_n & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & a_0 & \binom{n}{2} a_2 & \cdots & a_n \\ b_0 & n b_1 & \cdots & b_n & 0 & \cdots & 0 \\ 0 & b_0 & \cdots & n b_{n-1} & b_n & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & b_0 & b_1 & \binom{n}{2} b_2 & \cdots & b_n \end{array} \right]. \]

**Example 5.2** The semi-resultant of two binary forms of order 2 leads to the following identity for the Bernoulli and Euler polynomials

\[ s\text{Res}_2(B, \mathcal{E}) = ||s\text{Res}_2(B, \mathcal{E})||. \]

We expand the determinants and get

\[ B_2(x)^2 E_0(x)^2 - 2 B_2(x) E_0(x) E_2(x) B_0(x) + E_2(x)^2 B_0(x)^2 - 4 B_1(x) B_2(x) E_1(x) E_0(x) - 4 B_1(x) E_1(x) E_2(x) B_0(x) + 4 E_2(x) B_1(x)^2 E_0(x) + 4 B_0(x) B_2(x) E_1(x)^2 = \frac{1}{36}. \]

Let us find out the joint proper semi-invariants of the binary forms \(\alpha(X, Y), \beta(X, Y)\) and \(\gamma(X, Y)\). Since the semi-jacobian \([b_0, c_0]\) has the weight \(2n-2\), the semi-transvectant \([a_0, [b_0, c_0]]^n\) is well-defined. We have

\[ (\mathcal{D}^*)^i(b_k) = [n-k]_i b_{i+k}, (\mathcal{D}^*)^i(c_k) = [n-k]_i c_{i+k}. \]
Therefore
\[(D^*)^i \left( \begin{array}{cc} b_0 & c_0 \\ b_1 & c_1 \end{array} \right) = \sum_{j=0}^{i} \binom{i}{j} [n]_j [n-1]_{i-j} \left| \begin{array}{cc} b_j & c_j \\ b_{i-j+1} & c_{i-j+1} \end{array} \right| .\]

Thus
\[ [a_0, [b_0, c_0]]^n = \sum_{i=0}^{n} \frac{(-1)^i}{[n]_{n-i} [2n-2]_i} \binom{n}{i} (D^*)^{n-i} (a_0) (D^*)^i \left( \begin{array}{cc} b_0 & c_0 \\ b_1 & c_1 \end{array} \right) = \]
\[ = \sum_{i=0}^{n} \frac{(-1)^i}{[n]_{n-i} [2n-2]_i} \binom{n}{i} [n]_{n-i} a_{n-i} \sum_{j=0}^{i} \binom{i}{j} [n]_j [n-1]_{i-j} \left| \begin{array}{cc} b_j & c_j \\ b_{i-j+1} & c_{i-j+1} \end{array} \right| .\]

It implies
\[ \text{Tr}_n (a_0, b_0, c_0) := [a_0, [b_0, c_0]]^n = \sum_{i=0}^{n} \sum_{j=0}^{i} \frac{(-1)^i}{[n]_{n-i} [2n-2]_i} \binom{n}{i} \binom{i}{j} a_{n-i} \left| \begin{array}{cc} b_j & c_j \\ b_{i-j+1} & c_{i-j+1} \end{array} \right| .\]

Finally, let us find the joint proper semi-invariants of the four binary forms \( \alpha(X, Y), \beta(X, Y), \gamma(X, Y) \) and \( \delta(X, Y) \). It is easy to see that the determinant
\[ \Delta := \left| \begin{array}{ccc} b_0 & c_0 & d_0 \\ b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \end{array} \right| \]
is a semi-invariant with the weight \( 3n - 6 \). Then the semi-teransvectant \([d_0, \Delta]^n\) is well defined for \( n \geq 3 \). As above, we obtain
\[ \text{Ch}_n (a_0, b_0, c_0, d_0) := [d_0, \Delta]^n = \]
\[ = \sum_{i=0}^{n} \frac{(-1)^i}{[3n-6]_i} \binom{n}{i} a_{n-i} \sum_{i_1+i_2+i_3=i} \frac{i!}{i_1! i_2! i_3!} \left| \begin{array}{ccc} [n]_{i_1} & b_{i_1} \\ [n-1]_{i_1} & b_{i_1+1} \\ [n-2]_{i_1} & b_{i_1+2} \end{array} \right| \left| \begin{array}{ccc} [n]_{i_2} & c_{i_2} \\ [n-1]_{i_2} & c_{i_2+1} \\ [n-2]_{i_2} & c_{i_2+2} \end{array} \right| \left| \begin{array}{ccc} [n]_{i_3} & d_{i_3} \\ [n-1]_{i_3} & d_{i_3+1} \\ [n-2]_{i_3} & d_{i_3+2} \end{array} \right| .\]

Therefore we get the following identities for the Appell polynomials of different series \( A_1, A_2, A_3, A_4 \):

**Theorem 5.1.**

(11) \( \text{Dv}_n (A_1, A_2) = \| \text{Dv}_n (A_1, A_2) \|, \)

(12) \( \text{Tr}_n (A_1, A_2) = \| \text{Tr}_n (A_1, A_2) \|, \)

(13) \( \overline{\text{Tr}}_n (A_1, A_2) = \| \overline{\text{Tr}}_n (A_1, A_2) \|, \)

(14) \( \text{Ch}_n (A_1, A_2) = \| \text{Ch}_n (A_1, A_2) \|, \)

(15) \( \text{sRes}_n (A_1, A_2) = \| \text{sRes}_n (A_1, A_2) \|, \)

(16) \( \text{Tr}_n (A_1, A_2, A_3) = \| \text{Tr}_n (A_1, A_2, A_3) \|, \)

(17) \( \text{Ch}_n (A_1, A_2, A_3, A_4) = \| \text{Ch}_n (A_1, A_2, A_3, A_4) \|. \)
By using Theorem 3.3 we get the binomial identities:

\[
\sum_{i=0}^{n} \sum_{j=0}^{i} \frac{(-1)^i}{[2n-2]^i} \binom{n}{i} \binom{i}{j} \left| \begin{array}{cc}
\frac{[n]_j}{[n-1]_j} & \frac{[n]_{i-j}}{[n-1]_{i-j}} \\
\end{array} \right| = 0,
\]

(18)

\[
\sum_{i=0}^{n} \sum_{j=0}^{i} \frac{(-1)^i}{[2n-2]^i} \binom{n}{i} \binom{i}{j} ([n]_i [n - 2]_{i-j} - 2[n - 1]_i [n - 1]_{i-j} + [n - 2]_i [n]_{i-j}) = 0,
\]

(19)

\[
\sum_{i=0}^{n} \binom{n}{i} \sum_{i_1 + i_2 + i_3 = i} \frac{i!}{i_1!i_2!i_3!} \left| \begin{array}{ccc}
\frac{[n]_{i_1}}{[n-1]_{i_1}} & \frac{[n]_{i_2}}{[n-1]_{i_2}} & \frac{[n]_{i_3}}{[n-1]_{i_3}} \\
\end{array} \right| = 0.
\]

(20)

Unfortunately, all efforts to find the norms of the above semi-invariants were unsuccessful.

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