Finite $N$ unitary matrix model

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**Abstract:** We consider one-plaquette unitary matrix model at finite $N$ using exact expression of the partition function for both $SU(N)$ and $U(N)$ groups.

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1 Introduction

The study of matrix models has been important in understanding and gaining deep insights into the structure of gauge theories. Their range of applicability is wide and some examples include – the geometry of random surfaces, two-dimensional quantum gravity, statistical mechanics of spin systems. They also have connections to supersymmetric gauge theories and non-perturbative formulations of superstring theory. A class of matrix models has also been studied in the context of exhibiting properties similar to the four-dimensional theory of strong interactions (QCD). The techniques of strong coupling expansion and lattice methods [1] along with approximate recursion relations [2] were pioneered to understand the properties of QCD. Around the same time, another important direction was pursued by ’t Hooft in finding an expansion parameter, independent of the scale to study QCD by considering $1/N$ expansion [3]. In this setting, the Feynman diagrams drawn in the double-line notation can be organized to follow an expansion in powers of $N^{\chi=2-2g}$ and were argued to be analogous to the topological expansion in string theory. The large $N$ limit was then used to solve the two-dimensional model of QCD in the light-cone gauge which is now also known as ’t Hooft model [4]. This model has many similarities with the four-dimensional theory but is exactly solvable in the planar limit. In higher dimensions, the success has been scarce but it is at least known that the large $N$ limit does not alter a distinctive feature of QCD – ultraviolet freedom, which is obvious from the $N \to \infty$ limit of the two-loop QCD $\beta$-function which has the leading coefficient with the correct sign. Following these developments, the large $N$ limit of gauge theories have been extensively explored for a wide class of gauge theories. In this topological limit, a remarkable simplification occurs, where only the planar diagrams survive, when we keep $\lambda = g_{YM}^2 N$ fixed and take $N \to \infty$. The interplay between the large $N$ limit of gauge theories and string theory has significantly improved our understanding of strongly coupled gauge theories and resulted in the AdS/CFT conjecture. In this work, we consider a unitary matrix model that is equivalent to two-dimensional pure QCD. The two-dimensional models such as this have rich behavior but are trivial since gauge field is non-dynamical (absence of degrees of freedom) and the expectation value of observables can be calculated exactly as done in [5–7]. In this model, the Wilson loop obeys area law for all couplings since the Coulomb potential is always linear and exhibits both ultraviolet freedom and infrared prison. The two-dimensional lattice gauge theories with Wilson action were shown to exhibit third-order phase transition at large $N$. This transition came as a surprise when it was first observed but now we have a better understanding through AdS/CFT correspondence that these phase transitions at large $N$ and finite volume occurs in several holographic models and are related to transitions between different black hole solutions in supergravity [8, 9]. The occurrence of this phase transition in the large $N$ limit signifies that non-analytic behaviour can occur even in the simplest of models which makes it clear that the strong coupling expansions cannot be usually smoothly continued to the limit of weak coupling. This two-dimensional lattice gauge theory with $SU(N)$ or $U(N)$ gauge group is often described in terms of single plaquette action.
over compact unitary group manifold and will hereafter be referred to as the GWW model after the authors of [5, 6]. The unitary matrix models of this kind have also been studied from the holographic point of view [10, 11]. The interest in this model stems from two main reasons, 1) This model admits an exact solution for any $N$ and coupling, 2) This model is closely related to one of the simplest strings (minimal superstring theory) with manageable non-perturbative description (Type 0 strings in one dimension) [12]. In fact, it was shown that unitary matrix models in the double scaling limit ($N \to \infty$ & $\lambda \to \lambda_{\text{critical}}$ with some well-defined relation between $N$ and $|\lambda - \lambda_{\text{critical}}|$) is described by the Painlevé II equation [13].

The GWW model and several of its modifications have been well-studied using various techniques in the $N \to \infty$ limit. The finite $N$ limit has been surprisingly less explored but recently attracted some attention [14]. Even in these explorations, the gauge group considered was $U(N)$ since it is a little easier to handle and in the planar limit, which is mostly of interest, there is no distinction with $SU(N)$ . However, at finite $N$, they have different behaviour and independent studies of both $G = SU(N)$ and $U(N)$ gauge groups are desirable and to our knowledge no treatment of this unitary matrix model at finite $N$ for $SU(N)$ gauge group has been done yet. This paper aims to fill this gap in the literature. We derive an expression for the partition function considering $SU(N)$ gauge group and study observables in the finite $N$ limit. In case the qualitative behaviour is not severely altered by addition of matter fields i.e. presuming this phase transition is not turned into a smooth crossover, then these studies may be useful in understanding black hole solutions and stringy corrections by considering the finite $N$ limit of matrix models [15] since the transition at $\lambda_{\text{critical}} = 2$ is supposed to separate the supergravity regime from the perturbative gauge theory regime. There are only a few matrix models where the finite $N$ regime is exactly solvable for any coupling and this is one such model. In this respect and various others, a finite $N$ study of these class of matrix models would be useful in probing the quantum effects of gravity using holography.

We briefly outline the structure of the paper. In Section 2, we write down the exact expression of the partition function for special unitary matrix model at any coupling and $N$ in terms of determinant of a Toeplitz matrix\(^1\) and sketch the well-known phase structure of this model at large $N$. In Section 3, we numerically calculate the corrections to the planar result for free energy in both the phases and show that it is simpler to deduce the instanton corrections in the strong coupling phase since the $1/N$ (or genus) expansion terminates at genus-zero for $U(N)$ unitary model. However, this is not the case for the $SU(N)$ model which has contributions to the free energy at small $N$ in both the phases. We also provide results for the Wilson loop in the $SU(N)$ matrix model using our exact expression and show that in the large $N$ limit, they converge to the known results for the $U(N)$ unitary model. We end with a brief mention of some open questions which can be addressed in the context of $SU(N)$ one-plaquette matrix model and other related unitary matrix models.

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\(^1\)Named after German mathematician Otto Toeplitz (1881 - 1940), each descending diagonal from left to right in this matrix is constant and matrix element $(j,k)$ depends on $j - k$
2 Partition function and observables

The central object in the study of matrix models is the partition function which is determined by integration over some compact group manifold. However, there exists only a handful of models which can be solved exactly \([16]\) and most of these proceeds by reduction of \(N^2\) matrix degrees of freedom to \(N\) by exploiting some symmetry in the large \(N\) limit. The simplest among all such matrix models is the well-studied one-matrix model. For a nice review for unitary matrix models, we refer the reader to \([17]\). The lattice Wilson action in two dimensions is equivalent to one-plaquette model by considering the integration over the unitary group. This has also been studied using character expansion\(^2\) which is discussed in \([18, 19]\). It was shown \([2]\) that in two dimensions, the expansion in terms of characters constitute the recursion relations through which one can exactly solve the model. For the large \(N\) analysis, the saddle point methods \([20]\) are often used as was done in \([6]\). However, the saddle point methods are not useful in extracting sub-leading orders in the \(1/N\) expansion for which the method of orthogonal polynomials is usually used as was done in \([7]\). The general partition function can be schematically written as:

\[
Z = \frac{1}{\text{Vol}(G)} \int \prod_{\text{links } l} \mathcal{D}U_l \prod_{\Box} Z_{\Box}, \tag{2.1}
\]

where \(\Box\) denotes the plaquette on whose perimeter we take the ordered product of the link matrices \(U\) of size \(N \times N\). In order to compute expectation values in the two-dimensional model, one has to make a choice between one-plaquette and heat kernel actions. The partition function of the two-dimensional Yang-Mills theory based on the heat kernel action is written in terms of sum over all irreducible representations of the unitary gauge group. We will use the one-plaquette action in this work similar to \([5, 6]\) which can be expressed as\(^3\):

\[
S(U) = \frac{N}{\lambda} \left[ \text{Tr} \left( \prod_{\Box} U \right) + \text{Tr} \left( \prod_{\Box} U^\dagger \right) \right], \tag{2.2}
\]

where \(\prod_{\Box} U\) is the product of links around a plaquette and shorthand for \(U_\mu(n)U_\nu(n + \hat{\mu})U^\dagger_\mu(n + \hat{\mu} + \hat{\nu})U^\dagger_\nu(n + \hat{\nu})\). The convention used is such that \(U_\mu(n)\) denotes a link starting from site \(n\) and going in \(\mu\)-direction.

It was found that for this model in the \(N \to \infty\) limit with fixed \(\lambda = g_{\text{YM}}^2 N\), one observes a discontinuity in the third derivative of the free energy corresponding to a third-order phase transition. This means that one loses the analytic structure for even simple actions in the large \(N\) limit and the continuation from the weak coupling to strong coupling is non-trivial. This transition is not like the usual phase transitions in statistical mechanics which occur in the infinite volume limit. Here, it occurs in a finite volume (single plaquette) for an infinite rank gauge group.

\(^2\)It is useful to sometimes think of character expansion as Fourier expansion on a compact group manifold

\(^3\)Some authors use \(g\) to denote \(1/\lambda\) or \(2/\lambda\). This distinction is clear from the coupling where phase transition occurs.
We denote the full partition function of the model with $Z$ and since in two dimensions, we can treat all the plaquettes as independent, we deal with just a single plaquette \[Z = Z^{1/N_s N_t}, \] where $N_s N_t$ are the number of nodes in the lattice and equals the number of plaquettes. It is given by:

$$Z = \int_{U(N) \text{ or } SU(N)} D U \exp \left[ \frac{N}{\lambda} \text{Tr} \left( U^\dagger + U \right) \right]. \quad (2.3)$$

It is known that the partition function for the $U(N)$ matrix model given above can be written in terms of Toeplitz determinant\footnote{The determinant of infinite size Toeplitz matrix has a well-defined asymptotic behaviour limit due to a theorem by Szegő.} given by [7, 22]:

$$Z(N, \lambda) = \text{Det} \left[ I_{j-k} \left( \frac{2N}{\lambda} \right) \right]_{j,k=1\ldots N}, \quad (2.4)$$

where $I_{\nu}(x)$ is the modified Bessel function of first kind or Bessel function of imaginary argument of order $\nu$. Note that the argument of the Bessel function is twice the prefactor of action in (2.3). The appearance of partition function in terms of determinant has deep connections to the notion of integrability and special differential equations. For instance, the determinant of Toeplitz matrices also play a role in the context of the Ising model, for a partial set of references, see [23, 24]. Instead of working with the partition function given in (2.3), one can also consider more general unitary matrix model with source arbitrary $N \times N$ matrix $A$ as was considered in [25]. The action for this model is given by:

$$Z = \int D U \exp \text{Tr} \left[ U A^\dagger + A U^\dagger \right], \quad (2.5)$$

where $U$ is product of four ordered links. The exact form of $Z$ was derived in the large $N$ limit and it was shown that the strong and weak coupling regimes in this model are characterized by $(1/N)\text{Tr}(A^\dagger A)^{-1/2}$. One gets back the usual GWW model by setting $A = \mathbb{I}/\lambda$. The model in (2.5) was related to a specific $N \times N$ Hermitian matrix model following some parameter tuning in [26] and an exact solution was found. In this work we will only consider (2.3) and derive the exact expression for the partition function when the integration is over $SU(N)$ group in (2.3).

The analysis for $SU(N)$ is similar to the one for $U(N)$ except that now we have the constraint that the product of eigenvalues should satisfy $\prod_{j=1}^N e^{i \alpha_j} = 1$. We start with the one-plaquette partition function:

$$Z = \int_{SU(N)} D U \exp \left[ \frac{N}{\lambda} \left( \text{Tr} \prod_{\text{single } \square} U + \text{Tr} \prod_{\text{single } \square} U^\dagger \right) \right], \quad (2.6)$$

where the measure is given by:

$$D U = \frac{1}{N!} \prod_{j=1}^N \sum_q \delta \left( \sum_{m=1}^N \alpha_m - 2q\pi \right) \frac{d \alpha_j}{2\pi} \prod_{j<k} \left( e^{i \alpha_j} - e^{i \alpha_k} \right) \left( e^{-i \alpha_j} - e^{-i \alpha_k} \right), \quad (2.7)$$

The determinant of infinite size Toeplitz matrix has a well-defined asymptotic behaviour limit due to a theorem by Szegő.
and the constraint can be written as:

\[ \frac{1}{2\pi} \sum_{p=-\infty}^{\infty} e^{ip\sum \alpha_m}. \]  

(2.8)

By using the representation of the modified Bessel function as:

\[ I_{k-j-p}\left(\frac{2N}{\lambda}\right) = I_{j-k+p}\left(\frac{2N}{\lambda}\right) = \frac{1}{2\pi} \int_{0}^{2\pi} e^{\frac{2N}{\lambda} \cos \alpha} e^{i(j-k+p)\alpha} d\alpha, \]

(2.9)

where, \( e^{i\alpha} \) are the eigenvalues of \( U \) and (2.9) fills the corresponding \( j \)th and \( k \)th element of the matrix \( M_{\alpha} = I_{j-k+\alpha}(\frac{2N}{\lambda}) \) and using (2.6) through (2.9), we obtain the partition function for \( SU(N) \) matrix model as:

\[ Z(N, \lambda) = \sum_{p=-\infty}^{\infty} \text{Det} \left[ I_{j-k+p}\left(\frac{2N}{\lambda}\right) \right]_{j,k=1}^{N}. \]

(2.10)

In contrast to the expression for the exact partition function for \( U(N) \) matrix model i.e. (2.4), there is an additional sum over the index \( p \) from the constraint in (2.7). In this paper, we study different observables using this partition function. However, for practical purposes of computation using Mathematica, we simply replace the \( \infty \) in (2.10) by a large number. We checked that \( |p| \leq 15 \) suffices i.e.,

\[ Z(N, \lambda) = \sum_{p=-15, \neq 0}^{15} \text{Det} \left[ I_{j-k+p}\left(\frac{2N}{\lambda}\right) \right] + \text{Det} \left[ I_{j-k}\left(\frac{2N}{\lambda}\right) \right]. \]

(2.11)

This partition function enables us to evaluate free energy which can be written as a sum over genus expansion for this model:

\[ F(N, \lambda) = \sum_{g \in \mathbb{N}} F_g N^{2-2g} + \mathcal{O}(e^{-N}), \]

(2.12)

where \( \mathbb{N} \) denotes the set of non-negative integers. The exact result for the leading coefficient, \( F_0 \), in the planar limit is given by:

\[ F_0 = \begin{cases} \frac{1}{\lambda^2} & \lambda \geq 2 \\ \frac{1}{\lambda^2} - \frac{1}{\lambda} \ln \frac{\lambda}{2} + 3 \frac{3}{4} & \lambda < 2 \end{cases}. \]

(2.13)

Another important observable in unitary matrix models and the one we consider are the Wilson loops. The finite \( N \) analysis of Wilson loops in various representations for \( U(N) \) gauge theory was recently done in [14]. An expression for the expectation value of normalized winding Wilson loops defined as:

\[ \mathcal{W}_k(N, \lambda) = \frac{1}{N} \langle \text{Tr} \left( \prod_{\mathcal{C}} U \right)^k \rangle, \]

(2.14)
was given in terms of $\text{Tr}(\mathcal{M}_k/\mathcal{M}_0)$ with $k \in \mathbb{Z}^+$ denoting the winding number and the expectation value is computed over a closed contour $\mathcal{C}$. A similar expression for the $\text{SU}(N)$ case is yet unknown. Note that like the partition function, we can write $W_{k,\mathcal{C}}(N,\lambda)$ as $W_k(N,\lambda)^{N_sN_t}$, where the contour is of time extent $aN_t$ and spatial extent $aN_s$. The single winding ($k = 1$) Wilson loop is related to the derivative of the free energy and is given by:

$$W_1(N,\lambda) = \frac{-\lambda^2}{2N^2} \frac{\partial \ln Z}{\partial \lambda} = \begin{cases} \frac{1}{\lambda} & \lambda \geq 2 \\ 1 - \frac{\lambda}{4} & \lambda \leq 2 \end{cases}.$$  \hspace{1cm} (2.15)

For this $\text{U}(N)$ matrix model as mentioned above, there is another equivalent definition of $W_1$ given by:

$$W_1(N,\lambda) = \text{Tr} \left( \frac{\mathcal{M}_1}{\mathcal{M}_0} \right),$$ \hspace{1cm} (2.16)

where $\mathcal{M}_\alpha$ is defined below (2.9). We will present results for the free energy and Wilson loops in Section 3 for the $\text{SU}(N)$ and $\text{U}(N)$ models.

Even though this matrix model is one of the simplest it has a wide range of interesting features. In [27], by considering the trans-series solutions of the pre-string equation to obtain instanton corrections, it was deduced that the instanton action vanishes at the critical point i.e. $\lambda = 2$ and it was concluded that the GWW transition is caused by the effect of instantons\(^5\). There exist other examples where a similar phenomenon occurs [28]. The behaviour of the corrections to the planar result due to $1/N$ and instanton have been well-studied. This model also exhibits resurgence behaviour thoroughly explored in [29]. One of the striking features of these studies, which is clearly evident in our results is that the strong coupling phase has no $1/N$ corrections [7] but only $O(e^{-N})$ corrections from the contributions due to instantons. In GWW model, both gapped and ungapped phases have instanton corrections, albeit of different nature. In the ungapped phase, the eigenvalues of the holonomy matrix fill the circle while in the gapped phase they are distributed over some interval. When the eigenvalue distribution is restricted to some domain, it is called a one-cut (or single interval) distribution/solution. In the ungapped phase, the instanton contribution to the free energy can simply be evaluated by subtracting genus-zero contribution from the total free energy. As we will see, this does not hold for the $\text{SU}(N)$ model.

The distribution of eigenvalues, which is one of the central objects in these matrix models, have

\(^5\)Usually, when one thinks of large $N$ limit, it seems that the instanton corrections which goes as $\exp(-A/g_s) \sim \exp(-AN/\lambda)$ are insignificant with $A$ denoting the instanton action. In fact, a more general form is $\exp(-F(\lambda)N)$, where $F$ is some non-negative function of the coupling $\lambda$ and proportional to $A$. When $F(\lambda) = 0$ corresponding to vanishing action, the exponentially suppressed instanton contribution to $1/N$ expansion becomes important. It turns out that for the GWW model this happens exactly at $\lambda_{\text{critical}} = 2$ where the third-order phase transition takes place. Therefore, one also refers to these as instanton driven large $N$ phase transitions and physically relate them to condensation of instantons. We thank M. Mariño for email correspondence regarding the one-plaquette model and his book ’Instantons and Large $N$’ for clarification regarding the instanton contributions.
been studied in both the phases for the $\mathbb{U}(N)$ model and is given by:

$$
\rho(\lambda, \theta) = \begin{cases} 
\frac{2}{\pi \lambda} \cos \left( \frac{\theta}{2} \right) \sqrt{\frac{\lambda}{2} - \sin^2 \frac{\theta}{2}} & \lambda < 2 \\
\frac{1}{2\pi} \left( 1 + \frac{2}{\lambda} \cos \theta \right) & \lambda > 2 
\end{cases}
$$

(2.17)

In the gapped phase, the distribution is only supported on some interval $\theta \in [-a, a]$ while it is uniform in the ungapped phase. For $\text{SU}(N)$ unitary matrix model, there are corrections to the distributions above which was discussed in [30] and will not be further discussed in this work.

$$
\lambda = g^2 N = \frac{2}{\lambda} \rightarrow \infty \quad \lambda \rightarrow 0
$$

$$
F_{g<2} = N^2 \left( F_0 + \frac{F_1}{N^2} + \frac{F_2}{N^4} + \cdots \right) + \mathcal{O}(e^{-N})
$$

$$
F_{g>2}\mathbb{U}(N) = N^2 \left( F_0 + \frac{F_1}{N^2} + \frac{F_2}{N^4} \right) + \mathcal{O}(e^{-N})
$$

Gapped \hspace{1cm} Ungapped

Figure 1. A telegraphic summary describing the phase diagram of the one-plaquette matrix model at large $N$ and the exact expression of the partition function for $\text{SU}(N)$ model as given in the main text.

3 Results & Conclusions

In this section we present the results obtained using (2.11) and (2.4) for $\text{SU}(N)$ and $\mathbb{U}(N)$ groups respectively. We primarily focus on the free energy for different couplings to emphasize behaviours in both phases and at the critical coupling. Our results converge to expected results in (2.13) when we take large $N$, while also probing the finite $N$ coefficients i.e. $F_g$ with $g \neq 0$ according to (2.12). In the weak coupling limit, $\lambda < 2$, the free energy up to genus-two was calculated in [7] and given by:

$$
F(\lambda, N) = F_0 - \frac{1}{N^2} \left( \frac{1}{12} - \ln A - \frac{1}{12} \ln N - \frac{1}{8} \ln(1 - (\lambda/2)) \right) - \frac{1}{N^4} \left( \frac{3\lambda^3}{1024} \left( \frac{1}{1 - (\lambda/2)^2} \right)^3 + \frac{1}{240} \right),
$$

(3.1)

where, $A = 2^{\frac{7}{4}} \pi \frac{\pi}{\sin^3 \frac{\pi}{2}} \exp \left[ \frac{1}{3} + \frac{2}{3} \int_0^\frac{3}{4} \ln \Gamma(1 + x)dx \right] = 1.2824271291 \cdots$ is the Glaisher-Kinkelin constant and is related to the derivative of zeta function as, $\zeta'(-1) = \frac{1}{12} - \ln A$. A general expression for genus $g$ free energy $F_g(\lambda)$ is also known [7, 13, 14, 27] and can be written as:

$$
F_g(\lambda) = \frac{B_{2g}}{2g(2g - 2)} + \frac{1}{\left( \frac{2}{\lambda} - 1 \right)^{3g-3}} \sum_{n=0}^{g-2} C_n^g \lambda^{-n},
$$

(3.2)
where $B_{2g}$ is the Bernoulli number\(^6\).

A general expression for the $SU(N)$ case is unknown but is expected to have some similarity since the origin of these Bernoulli numbers are in the volume of the $U(N)$ group which is similar to that of $SU(N)$. In Figure 2, we compute the free energy for $\lambda = 4/3$ and show the results for $SU(N)$ and $U(N)$ models.

\[ \chi(M_{g,n}) \] using Harer-Zagier formula and set $n = 0$ we also obtain $\frac{B_{2g}}{2g(2g-2)}$.}

---

\(^{6}\) We note that if we calculate the orbifold Euler characteristic of the moduli space of Riemann surfaces of genus $g$ with $n$ marked points i.e. $\chi(M_{g,n})$ using Harer-Zagier formula and set $n = 0$ we also obtain $\frac{B_{2g}}{2g(2g-2)}$.\]
Figure 3. The dependence of the free energy (normalized by $N^2$) for $SU(N)$ and $U(N)$ model at $\lambda = 2$ on $N$. In the $N \to \infty$ limit, the exact value is $F_0 = -0.25$.

Figure 4. The dependence of the free energy (normalized by $N^2$) on $N$ for $\lambda = 4$ (strong coupling). There are no $1/N$ corrections for the $U(N)$ model while for $SU(N)$ the genus expansion does not terminate at genus-zero.
Finally, in Figure 5, we study the Wilson loop by taking the numerical derivative of the free energy for a range of couplings with $N = 4, 10$ for both $SU(N)$ and $U(N)$ groups. The most distinct feature is that the planar result is approached from different sides for $SU(N)$ and $U(N)$ models (similar to free energy behaviour) and signifies that the $1/N$ corrections to planar values come with opposite sign. It will be interesting to compute the exact expression for the Wilson loop winding around $k$ times as done for the $U(N)$ model in [14].

A related model to the one we studied here is the ‘double trace model’ for which the action is $Tr U Tr U^\dagger$ and can be written in terms of the partition function of GWW model. This model is closely related to a truncated limit of $\mathcal{N} = 4$ SYM and in the double scaling limit exhibits the Hagedorn/deconfinement phase transition. It would be interesting to understand the finite $N$ limit of this model while not restricting to $U(N)$ integral as done in [31].

In this paper, we have given an exact expression for the partition function of $SU(N)$ one-plaquette matrix model valid for all $N$ and couplings and computed exact results for free energy and Wilson loop at finite $N$ for several couplings. We concluded that the $1/N$ corrections to free energy vanish for $U(N)$ matrix model in the ungapped (strongly coupled) phase where the only corrections to the planar result come from the instanton correction, while for $SU(N)$ matrix model, the genus expansion contribution to the free energy does not terminate at genus-zero. Our results suggest that that the finite $N$ behaviour of $SU(N)$ is very different from the $U(N)$ matrix model and deserves further analysis. It would be interesting to understand results for multiply winded Wilson loops and $1/N$ expansion of the free energy for $SU(N)$ along the lines as done for $U(N)$ group.

**Figure 5.** The expectation value of Wilson loop against coupling for $N = 4, 10$ around the critical coupling. The dashed lines (different colours) are the planar limit result in different phases.
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