Generalized Distribution Amplitudes at the Z pole

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We investigate the exclusive two-pion production at the Z-pole through the process $e^+e^- \rightarrow Z \rightarrow \pi\pi\gamma$. In the kinematical region where the invariant mass of the two pions is much smaller than the mass of the Z boson, the process can be factorized into the convolution of a hard coefficient and a soft matrix element, the generalized distribution amplitude of two pions. We calculate the differential cross-section of charged pion pair production and neutral pion pair production and find the former to be much larger than the latter. We show that combining the measurements of charged pion pair production and neutral pion pair production at the Z-pole provides a convenient approach to access the C-odd part of the $2\pi$ distribution amplitudes.

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I. INTRODUCTION

The QCD description of exclusive processes has been successfully extended to meson pair production through two-photon collisions $\gamma^*\gamma \rightarrow M\bar{M}$, in the kinematical region of large virtuality of one photon and of small center of mass energy [1,2,3]. The scattering amplitude of the process can be factorized into a perturbative calculable hard process $\gamma^*\gamma \rightarrow q\bar{q}$ or $\gamma^*\gamma \rightarrow gg$, and non-perturbative matrix elements describing the transition of the two partons into a hadron pair, which are called generalized distribution amplitudes (GDAs) [1], to emphasize their close connection with the distribution amplitudes introduced many years ago in the description of exclusive hard process [4]. The GDAs are related by crossing [2] to the generalized parton distributions (GPDs) [2,6,7], which enter the factorization of deeply virtual Compton scattering and hard exclusive electroproduction (GPDs) [2,6,7], and encode information of the quark orbital angular momentum contributing to the nucleon spin [3]. A systematic investigation of the pions GDAs (2πDAs) has been performed in Refs. [10,11], and the corresponding phenomenology at $e^+e^-$ colliders has been developed [11] in detail. GDAs can also be defined for more complicated systems. The case of three pions has been studied in [12], two baryon production in [13] (although in a different kinematical region), and the case of two vector mesons ($\rho$ mesons) $\gamma^*\gamma \rightarrow \rho\rho$ in [14], which has been measured first by the L3 Collaboration [15] at LEP and the data has been analyzed in [14].

GDAs appear in a variety of hard scattering processes other than the two-photon process. Studies have also been performed on exotic meson production [16], and exclusive heavy mesons decaying to multimesons [17], where a large scale is set by the heavy quark mass. In this paper we consider another process that has not been studied so far, the pion pair production at the Z-pole: $e^+e^- \rightarrow Z \rightarrow \pi\pi\gamma$. In this process the mass of the Z boson provides a natural hard scale. Thus in the kinematical region where the invariant mass of pion pair $W$ is small, the process can be separated into a hard subprocess $Z \rightarrow q\bar{q}\gamma$ and a non-perturbative two-pion distribution amplitude, analogous to $\gamma^*\gamma \rightarrow \pi\pi$. We give a leading order calculation for neutral and charged pion pair production at the Z-pole. Using an available model for the 2πDAs we find that the cross-section for charged pion pair production is much larger than the case of neutral pion pair, which means that the process is dominated by the isovector channel, that is, the contribution of the C-odd part of the 2πDAs. The same one also contributes to the hard exclusive electroproduction of pion pairs [11,18]. Our study shows that combining the measurement of charged pair and neutral pair production at the Z-pole can give information of the C-odd part of the 2πDAs. The Bremsstrahlung process $e^+e^- \rightarrow \gamma^*\gamma \rightarrow \pi^+\pi^\gamma$ is also considered in our analysis.

II. ANALYSIS OF THE PROCESS $Z \rightarrow \pi\pi\gamma$

The underlying process we study is:

$$Z(q) \rightarrow \pi(p_1) + \pi(p_2) + \gamma(q').$$

(1)

The photon emission allows for a small invariance mass to be transferred to the pion pair. Denoting by $q$ and $q'$ the momenta of the Z boson and final state photon, respectively, we use $p_1$ and $p_2$ to denote the momenta of the two-pion mesons, and define the sum of the two-pion momenta as $P$. The lowest order diagrams of pion pair production at the Z-pole are shown in Fig. 1. We choose the pion pair center of mass frame (shown in Fig. 2a) as the reference frame, and define the Z boson momentum direction as the z axis. In this frame we have

$$q = \frac{Q}{\sqrt{2}}v + \frac{Q}{\sqrt{2}}v', \quad q' = \frac{Q^2 - W^2}{\sqrt{2}Q}v',$$

$$P = \frac{Q}{\sqrt{2}}v + \frac{W^2}{\sqrt{2}Q}v',$$

$$p_1'^\mu = \frac{\zeta Q}{\sqrt{2}}v'^\mu + \frac{(1 - \zeta)W^2}{\sqrt{2}Q}v'^\mu + \frac{\Delta_1^\mu}{2},$$

$$p_2'^\mu = \frac{(1 - \zeta)Q}{\sqrt{2}}v'^\mu + \frac{W^2}{\sqrt{2}Q}v'^\mu - \frac{\Delta_2^\mu}{2},$$

(2)
respectively. Here \( v \) and \( v' \) are two lightlike vectors which satisfy: \( v^2 = v'^2 = 0 \), \( v \cdot v' = 1 \), \( W \) is the invariant mass of the pion pair and \( q^2 = Q^2 \). The skewness parameter \( \zeta \) is the momentum fraction of plus momentum carried by \( \pi(p_1) \) with respect to the pion pair:

\[
\zeta = \frac{p_1^+}{P^+} = \frac{1 + \beta \cos \theta}{2}, \quad \beta = \sqrt{1 - 4m^2/W^2}.
\]  

The hadronic tensor of the process is

\[
iT^\mu\nu = -\int d^4x e^{-iq.x} \langle \pi(p_1)\pi(p_2)|T[J_{NC}^\mu(x)J_{EM}^\nu(0)|0\rangle
\]

\[
= \frac{P^+}{2\pi} \int dz H_{\alpha\beta}^{\mu\nu}(z, q, q') \int dx e^{-izP^+ x^-} \times \langle \pi(p_1)\pi(p_2)|\psi_\beta(x^-)\psi_\alpha(0)|0\rangle,
\]

which is expressed as a convolution of a hard coefficient function and a soft correlation function, \( z = k^+/P^+ \) is the momentum fraction of the quark with respect to the hadronic system. \( J_{NC}^\mu \) and \( J_{EM}^\nu \) are the neutral current and electromagnetic current respectively. The light-cone coordinates used here are defined as \( a^\perp = a \cdot v', a^- = a \cdot v \). In the above equation the gauge-link ensuring the gauge invariance of the definition is implied.

The hard coefficient function can be calculated from the subprocess \( Z \to q\bar{q}\gamma \) directly:

\[
H^{\mu\nu} = \frac{i e g}{2 \cos \theta W} \left[ V^n \frac{k^n - q^n}{(k - q)^2 + i\epsilon} + \gamma^\nu \frac{q^n - k^n}{(k - q)^2 + i\epsilon} \right]
\]

\[
= \frac{i e g}{2\sqrt{2} \cos \theta W} \left[ V^n (z - 1)Q^n - Q^n' \gamma^\nu \right]
\]

\[
+ \gamma^\nu \frac{zQ^n + Q^n'}{zQ^2} V^n,\]

where \( g = e/\sin \theta W \) is the weak coupling constant, \( V^n = \gamma^\nu (c^q_V - \gamma^\nu c^q_A) \) is the Z-boson-quark vertex, with \( q \) denoting the quark flavor. The vector and axial-vector coupling to the Z boson are given by:

\[
c^q_V = T^q_3 - 2Q^q \sin^2 \theta W, \quad c^q_A = T^q_3.
\]

Here \( T^q_3 = +1/2 \) for \( q = u \) and \( -1/2 \) for \( q = d, s \), \( Q^q \) is the electric charge of quark \( q \) in units of the electron charge.

Expanding the above equation we arrive at:

\[
H^{\mu\nu} = \frac{i e g}{2 \sqrt{2} \cos \theta W Q} \left\{ c^q_V S^{\mu\nu\alpha\beta} \frac{1 - 2z}{z(1 - z)} \right. \\
+ c^q_A e^{\mu\nu\alpha\beta} \frac{i}{z(1 - z)} v'_\alpha \gamma^\beta + 2ie q^{\mu\nu\alpha\beta} v'_\alpha \gamma^\beta
\]

\[
- 2ie q^{\mu\nu\alpha\beta} v_\alpha \gamma^5 \gamma^\beta + \left[ c^q_A S^{\mu\nu\alpha\beta} \frac{2z - 1}{z(1 - z)} \right.
\]

\[
- \left. c^q_V e^{\mu\nu\alpha\beta} \frac{i}{z(1 - z)} v'_\alpha \gamma^5 \gamma^\beta \right\},
\]

where we have used the identities:

\[
\gamma^\mu \gamma^\alpha \gamma^\nu = S^{\mu\nu\alpha\beta} \gamma^\beta + i e^{\mu\nu\alpha\beta} \gamma^\beta, \quad \gamma^\mu \gamma^\alpha \gamma^\nu \gamma_5 = S^{\mu\nu\alpha\beta} \gamma^5 - i e^{\mu\nu\alpha\beta} \gamma^5, \quad S^{\mu\nu\alpha\beta} = (g^{\mu\nu} g^{\alpha\beta} + g^{\mu\nu} g^{\alpha\beta} - g^{\mu\nu} g^{\alpha\beta}).
\]

Convolting \( H^{\mu\nu} \) with the soft matrix element, and keeping the leading twist contribution (the high-twist contributions are irrelevant at the Z-pole), we get the hadronic tensor:

\[
T^{\mu\nu} = -\frac{g}{4 \cos \theta W} \sum_q e e_q \left\{ c^q_V g^{\mu\nu} \int dz \frac{2z - 1}{z(1 - z)} \Phi_q(z, \zeta, W^2) \right. \\
- \left. i e^A q^{\mu\nu} \int dz \frac{1}{z(1 - z)} \Phi_q(z, \zeta, W^2) \right\}.
\]
The function $\Phi_q(z, \zeta, W^2)$ is the generalized distribution amplitude defined as $[1]$

$$\Phi_q(z, \zeta, W^2) = \int \frac{dx}{2\pi} e^{iz(x^+ x^-)} \times \langle p_1(p_1) p_2 | \psi(x^-) \gamma^+ \psi(0) | 0 \rangle. \quad (11)$$

Note that because the final hadrons are two pions (pseudoscalar mesons), the third and fourth lines of $[3]$ vanish by parity invariance of the strong interactions.

Before further discussion it is useful to recall some symmetry properties of $2\pi$ DAs. Charge conjugation invariance of the strong interactions implies $[1]$

$$\Phi_q^\pi(z, \zeta, W^2) = -\Phi_q^\pi(1 - z, 1 - \zeta, W^2). \quad (12)$$

Following the notation in $[10]$ the $C$-even and $C$-odd parts (In $[11]$ they are also called isoscalar and isovector parts respectively) of $2\pi$ DA can be defined as

$$\Phi_q^\pi(z, \zeta, W^2) = \frac{1}{2} [\Phi_q^+(z, \zeta, W^2) \pm \Phi_q^-(z, 1 - \zeta, W^2)], \quad (13)$$

where the superscript $\pi$ represents $\pi^0 \pi^0$ or $\pi^+ \pi^-$. Then

$$\Phi_q^{\pi \pi}(z, \zeta, W^2) = \Phi_q^+(z, \zeta, W^2) + \Phi_q^-(z, \zeta, W^2). \quad (14)$$

The properties of $\Phi_q^\pi(z, \zeta, W^2)$ under the interchange $z \to 1 - z$ and $\zeta \to 1 - \zeta$ can be easily derived from $[11]$

$$\Phi_q^+(z, \zeta, W^2) = -\Phi_q^+(1 - z, \zeta, W^2),$$

$$\Phi_q^-(z, \zeta, W^2) = \Phi_q^-(1 - z, \zeta, W^2), \quad (15)$$

The $\zeta \to 1 - \zeta$ exchange means the interchange of the two pions, thus in the case in which the final two pions are $\pi^0 \pi^0$ we get

$$\Phi_q^{\pi^0 \pi^0}(z, \zeta, W^2) = \Phi_q^{\pi^0 \pi^0}(1 - z, 1 - \zeta, W^2), \quad (17)$$

Therefore there is only a $C$-even part for $\Phi_q^{\pi^0 \pi^0}(z, \zeta, W^2)$. This is true because the $\pi^0 \pi^0$ state is a $C$-even state.

Now we turn back to Eq. $[10]$. The hard coefficient in the first term inside the brackets is antisymmetric under $z \to 1 - z$, while the second one is symmetric under $z \to 1 - z$. Then by virtue of $[13]$ we have

$$T^{\mu \nu} = -\frac{g}{4\cos \theta_W} \sum_q e_q \left \{ e^q \gamma_\perp^{\mu \nu} \int_0^1 dz \frac{2z - 1}{z(1 - z)} \Phi_q^+(z, \zeta, W^2) \right \} \times \Phi_q^+(z, \zeta, W^2) - i e_A^q \gamma_\perp^{\mu \nu} \int_0^1 dz \frac{1}{z(1 - z)} \Phi_q^-(z, \zeta, W^2). \quad (18)$$

The first term in $[18]$ probes the $C$-even part of the $2\pi$ DAs, which is the same as that in the two-photon process except for the different coupling. Here the main difference with the two-photon process, which can only project out the $C$-even part of the $2\pi$ DAs, is an additional term, the second term in $[18]$, which can probe the $C$-odd part of $2\pi$ DAs.

The amplitude of $Z \to \pi \pi \gamma$ can be calculated from

$$A_{i,j}^{Z \to \pi \pi \gamma} = \epsilon_{ij} \epsilon_3^{(i)} T^{\alpha \beta}(z, \zeta, W^2), \quad (19)$$

where $\epsilon$ and $\epsilon'$ are the polarization vectors of $Z$ boson and real photon, respectively. In our frame these have the form:

$$\epsilon_\mu = \left( \frac{0, 1.92, -i}{\sqrt{2}}, 0 \right), \quad (20)$$

Similar to the two-photon process, in leading twist only the transverse polarization contributes, and the polarizations of the $Z$ boson and the photon must be the same. This follows from angular momentum conservation in the collinear approximation.

### III. CROSS-SECTION AND NUMERICAL RESULT

The pion pair production at the $Z$-pole can be realized in the $e^+e^-$ annihilation process $e^+e^- \to Z \to \pi \pi \gamma$. It is interesting to mention that except for the fact that the process occurs at the $Z$-pole, this process is related by crossing to the virtual Compton process $e\pi \to e\pi \gamma$, as well as the process $e\gamma \to e\pi$.

With the polarizations of the final photon summed the amplitude of the process is calculated from

$$A_{e^+e^- \to Z \to \pi \pi \gamma} = \frac{g}{2\cos \theta_W} \sum_{i,j} \int \frac{1}{q^2 - M_Z^2 - i\Gamma_Z M_Z} A_{i,j}^{Z \to \pi \pi \gamma}. \quad (21)$$

Here $V^{\mu} = \gamma^{\mu}(c_V^{\perp} - \gamma_{5} c_A^{\perp})$ is the $Z$-boson-lepton vertex, and $c_{V}^{\perp}$ and $c_{A}^{\perp}$ have the same form as shown in $[21]$ and $[42]$, where $T_{\perp}^{A}$ for the electron is $-1/2$.

The differential cross-section for the process $e^+e^- \to Z \to \pi \pi \gamma$ is expressed as

$$d\sigma_{e^+e^- \to Z \to \pi \pi \gamma} = \frac{1}{2S_{ee}(2\pi)^3} \frac{d^3p_1}{2p_1^3} \frac{d^3p_2}{2p_2^3} \frac{d^3q'}{2q'^3} \times |A_{e^+e^- \to Z \to \pi \pi \gamma}|^2 \delta^4(q - p_1 - p_2 - q'), \quad (22)$$

where $S_{ee} = (l + l')^2 = Q^2$ is the center of mass energy squared of the lepton pair. We then rearrange the kinematics of the phase space to $[19]$

$$d\sigma_{e^+e^- \to Z \to \pi \pi \gamma} = \frac{1}{16(2\pi)^6} \frac{d^3S_{ee}}{S_{ee}^3} |A_{e^+e^- \to Z \to \pi \pi \gamma}|^2 \times |\mathbf{p}^*||\mathbf{q}||dWd\Omega^*d\Omega', \quad (23)$$

$$\mathbf{p} = \frac{q + p_1 - p_2}{2}, \quad \mathbf{q} = \frac{p_1 - p_2}{2}. \quad (24)$$
in which \((|p^*|, \Omega^*)\) is the momentum of pion 1 in the c.m. frame of the pion pair, and \(\Omega^*\) is the angle of the photon in the rest frame of the \(Z\) boson (that is, the c.m. frame of \(e^+e^-\)). \(|p^*|\) and \(|q'|\) are given by

\[
|p^*_1| = \frac{1}{2}\sqrt{W^2 - 4m^2} = \frac{\beta W}{2},
\]

\[
|q'| = \frac{Q^2 - W^2}{2Q}.
\]

The differential cross-section of this process in the \(\theta\) distribution from the interference (dashed line) between the \(C\)-odd and \(C\)-even parts of the \(2\pi DA\).

The amplitude square is

\[
|A_{e^+e^-\to Z\to \pi\pi\gamma}|^2 = \frac{e^2a_w^2Q^2}{(Q^2 - M_Z^2)^2 + \Gamma_Z^2M_Z^2}
\]

\[
\times \{ (c_{V}^2 + c_{A}^2)I_1(y)\left[ |V(\cos \theta, W^2)|^2 + |A(\cos \theta, W^2)|^2 \right] + 4c_{V}^2c_{A}I_2(y) \text{Im}\{ V^*(\cos \theta, W^2)A(\cos \theta, W^2) \}\}.
\]

The symbols in the above equation are given by

\[
a_w = \frac{e^2}{4\sin^2 \theta_W \cos^2 \theta_W},
\]

\[
I_1(y) = \left( \frac{1}{2} - y + y^2 \right),
\]

\[
I_2(y) = (1 - 2y),
\]

\[
V(\cos \theta, W^2) = \sum q e_q^2 c_q \int_0^1 dz \frac{2z - 1}{z(1 - z)}
\]

\[
\times \Phi^+(z, \zeta(\cos \theta), W^2),
\]

\[
A(\cos \theta, W^2) = \sum q e_q^2 c_q \int_0^1 dz \frac{1}{z(1 - z)}
\]

\[
\times \Phi^-(z, \zeta(\cos \theta), W^2),
\]

\[
\text{where } y = l \cdot q' / q \cdot q'. \text{ In the c.m. frame of the lepton pair (see Fig. 2b) there is the relation } y = (1 + \cos \theta')/2, \text{ where } \theta' \text{ is the angle of the photon with respect to the momentum of the incoming positron. Thus } y \text{ is in the interval } 0 < y < 1.
\]

Then we write the differential cross-section as

\[
\frac{d\sigma^{e^+e^-\to Z\to \pi\pi\gamma}}{dW^2 d\cos \theta d\phi dy} = \frac{e^2a_w^2\beta(Q^2 - W^2)}{128(2\pi)^4((Q^2 - M_Z^2)^2 + \Gamma_Z^2M_Z^2)Q^2}
\]

\[
\times \{ I_1(y)(c_{V}^2 + c_{A}^2)\left[ |V(\cos \theta, W^2)|^2 + |A(\cos \theta, W^2)|^2 \right] + 4I_2(y)c_{V}^2c_{A} \text{Im}\{ V^*(\cos \theta, W^2)A(\cos \theta, W^2) \}\}.
\]

The three terms in the above equation give the contribution from the \(C\)-even channel, the \(C\)-odd channel and the interference of the two channels, respectively.

In the case of the charged pion pair production, however, there is also contribution from a Bremsstrahlung type process that \(e^- e^+ \to \gamma^* \gamma \to e^+ e^- \pi^+ \pi^- \), where a time-like photon produced in a QED process decays into the pion pair. The differential cross-section of this process in the kinematics that we consider \((Q^2 = M_Z^2 \gg W^2)\) has the form

\[
\frac{d\sigma^B}{dW^2 d\cos \theta d\phi dy} = \frac{e^6\beta(Q^2 - W^2)}{128(2\pi)^4Q^2W^2}
\]

\[
\times \frac{2\beta^2I_1(y)}{y(1 - y)} |F_\pi(W^2)|^2 \sin^2 \theta.
\]
To complete the analysis, we give the the interference term between the production at the \( Z \)-pole and the Bremsstrahlung process:

\[
\frac{d\sigma}{dW^2d\cos\theta d\phi dy} = \frac{e^4a_u\beta(Q^2-W^2)}{128(2\pi)^4Q^4\Gamma_Z W^2} \times \left\{ \begin{array}{l}
\frac{2\beta^2c_1I_1(y)}{\sqrt{y(1-y)}} \text{Re}\{F_\pi(W^2)^*V\} \sin\theta \cos\phi \\
-\frac{2\beta^2c_1I_1(y)}{\sqrt{y(1-y)}} \text{Re}\{F_\pi(W^2)^*A\} \sin\theta \cos\phi \\
-\frac{\beta^2c_1I_2(y)}{\sqrt{y(1-y)}} \frac{Q^2}{\sqrt{y(1-y)} Q^2-W^2} \text{Im}\{F_\pi(W^2)^*A\} \sin\theta \sin\phi \\
+\frac{2\beta^2c_1I_1(y)}{\sqrt{y(1-y)}} \text{Im}\{F_\pi(W^2)^*V\} \sin\theta \sin\phi \end{array} \right\}.
\]

(34)

In Eq. (34) and (35) the \( W/Q \) suppressed terms are not kept, since they are irrelevant at the \( Z \)-pole energy scale.

We now use a simple model for the \( 2\pi \) DAs to estimate the cross-section of pion pair production at the \( Z \)-pole through Eq. (32). In the following we use \( \Phi^q_{\pi^+\pi^-}(z,\zeta,W^2) \) to represent the \( C \)-even/odd part of \( \Phi^q_{\pi^+\pi^-}(z,\zeta,W^2) \). Isospin invariance implies

\[
\Phi^u_{\pi^+\pi^-}(z,\zeta,W^2) = \Phi^+_{\pi^+}(z,\zeta,W^2),
\]

(36)

Diehl et al. gave a simple model for \( \Phi^+_{\pi^+} \) based on the Legendre polynomials expansion of the GDAs, as follows

\[
\Phi^+_{\pi^+}(z,\zeta(W^2) = 10z(1-z)(2z-1)R_\pi \times \left[ -\frac{3-\beta^2}{2} e^{i\delta^u(W^2)} + \beta^2 e^{i\delta^u(W^2)} P_2(\cos\theta) \right],
\]

(37)

in which \( \delta^u(W^2) \) and \( \delta^u(W^2) \) are the \( S \) wave and \( D \) wave phase shifts of elastic \( \pi\pi \) scattering, respectively. The analysis for these phase shifts is available, for example, in Ref. [20]. This model is valid in the regime \( W < 1 \) GeV, and we restrict our calculation to this regime. For \( \Phi^u_{\pi^+} \) we use the asymptotic form at large \( Q^2 \), corresponding to the \( P \) wave contribution [11]:

\[
\Phi^u_{\pi^+}(z,\zeta(W^2) = 6z(1-z)\beta P_1(\cos\theta)F_\pi(W^2),
\]

(38)

where \( F_\pi(W^2) \) is the time-like pion form factor, for which we use the parametrization \( N = 1 \) given in Ref. [21].

In Fig. 3 we show the \( W \) dependence of the differential cross-section for the neutral pion and charged pion pair production using the \( 2\pi \) DAs given in (37) and (38). We also show the contribution from the interference term of the \( C \)-odd and \( C \)-even parts of the \( 2\pi DA \) to the charged pion production at the \( Z \)-pole. The corresponding \( y \) dependence and \( \theta \) dependence of the differential cross-section are given in Fig. 4 and Fig. 5 respectively. The figures show that the cross-section of the charged pion pair production is much larger than that of the neutral pion pair, nearly \( 1 \sim 2 \) order of magnitude in the region \( 0.4 \) GeV \( < W < 0.9 \) GeV. The contribution of the interference term is significant compared with the cross-section of the neutral pion pair production, but still several times smaller than that of the charged pion production. One can conclude that most of the contribution to the charged pion production comes from \( \Phi^u_{\pi^+} \). This is expected from the large magnitude of the time-like form factor of the pion and the different \( z \) dependence of the \( \Phi^+_{\pi^+} \) given in (37) and (38).

In the \( W \) range which Fig. 2 covers, the contributions from the Bremsstrahlung process and its interference with the production at the \( Z \)-pole are not suppressed, due to the factor \( W^2 \) and \( W \) in the denominator of Eq. (38) and Eq. (34). The differential cross-sections contributed by these two terms at \( Q = M_Z \) are depicted in Fig. 6 where the interference contribution has been scaled by a factor 10. In Fig. 7 we also give a comparison between Bremsstrahlung with the production at the \( Z \)-pole which is scaled by a factor 2\( \pi \). The numerical results show that the contribution from the Bremsstrahlung process is several times larger than that from the production at the \( Z \)-pole (note that in Fig. 3 and 4 the azimuthal angle \( \phi \) has been integrated over). The interference term given in (35) is about one order of magnitude less than the Bremsstrahlung process.
The large contribution of the C-odd channel production at the Z-pole compared to the production at the C-even channel provides an opportunity to obtain information on $\Phi_q^-$. The same part also contributes in hard electroproduction [11, 18] and charm or B meson decay [17].

However, in these cases the GDAs convolute with non-perturbative GPDs or meson distribution amplitudes. In the pion pair production case at the Z-pole since there are no hadronic states in the initial state, GDAs only convolute with the perturbative calculable hard coefficients, thus $\Phi_q^-$ can be accessed more cleanly in this case.

We now discuss how to obtain the contribution coming from $\Phi_q^-$ in $e^+e^-$ annihilation process at the Z-pole. First one confront the large contribution from the Bremsstrahlung process which need to be carefully treated. Because the modulus of $F_q$ has been well measured, the contribution from Bremsstrahlung can be predicted quite well by Eq. [33]. The interference between the Bremsstrahlung process and the production at the Z-pole is eliminated by integrating over azimuthal angle $\phi$. Therefore the Z-pole contribution can be separated out. Again notice that the interference term in Eq. [39] vanishes after integrating over $y$ (or alternatively, integrating over $\theta$), therefore the difference of the differential cross-sections of $\pi^+\pi^-$ and $\pi^0\pi^0$ production at the Z-pole (after integrating over $y$) is

$$\frac{d\sigma^{e^+e^-\rightarrow Z\rightarrow \pi^+\pi^-}}{dW^2d\cos\theta} - \frac{d\sigma^{e^+e^-\rightarrow Z\rightarrow \pi^0\pi^0}}{dW^2d\cos\theta} = \frac{e^2a_0^2(Q^2 - W^2)(c_1^2 + c_3^2)|A(\cos\theta,W^2)|^2}{384(2\pi)^3((Q^2 - M_Z^2)^2 + \Gamma_Z^2M_Z^2)Q^2}.$$  (39)

The above result depends only on the moment of $\Phi_q^-$. Thus combining the measurement of charged pair production and neutral pair production at the Z-pole, the moment of $\Phi_q^-$ is obtained. Fig. 4 shows the contribution of $\Phi_q^-$ to the charged pair production which is calculated from Eq. [33]. Due to the relative small cross-section,
the extraction of $\Phi_{\gamma}$ is difficult through the pion pair production at the $Z$-pole based on the LEP-I data \cite{22}, while it is feasible at the possible Giga-$Z$ option \cite{23} of the planned International Linear Collider, at which the integrated luminosity can reach 100 fb$^{-1}$ in about one yr of running.

**IV. SUMMARY**

We have shown that pion pairs with small invariance mass, produced at the $Z$-pole through $e^+e^- \rightarrow Z \rightarrow \pi\pi\gamma$, can be factorized into a hard scattering coefficient convoluted with the non-perturbative two-pion distribution amplitudes. We calculated the cross-sections for the production of a charged pion pair and of a neutral pion pair in leading order. The comparison of these two cases show that in the regime $0.4 \text{ GeV} < W < 0.9 \text{ GeV}$ the cross-section of the charged pair production is much larger than that of neutral pair production, showing that the process is dominated by the contribution from the $C$-odd part of the $2\pi$DAs. By virtue of isospin symmetry, one can combine the measurements of the charged pair and neutral pair production at the $Z$-pole to access the $C$-odd part of the $2\pi$DAs cleanly. The pion pair production at the $Z$-pole at $e^+e^-$ facilities can open opportunities to obtain valuable information on the $2\pi$DAs, especially the $C$-odd part of $2\pi$DAs, and can be viewed as the useful complement to the two-photon process.

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