FORMALITY THEOREM FOR LIE BIALGEBRAS AND QUANTIZATION OF COBOUNDARY r-MATRICES

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Abstract. Let \( (\mathfrak{g}, \delta_h) \) be a Lie bialgebra. Let \( (U_h(\mathfrak{g}), \Delta_h) \) a quantization of \( (\mathfrak{g}, \delta_h) \) through Etingof-Kazhdan quantization functor. We prove the existence of a \( L_\infty \)-morphism between the Lie algebra \( C(\mathfrak{g}) = \Lambda(\mathfrak{g}) \) and the tensor algebra \( TU = T(U_h(\mathfrak{g})[-1]) \) with Lie algebra structure given by the Gerstenhaber bracket. When \( (\mathfrak{g}, \delta_h, r) \) is a coboundary Lie bialgebra, we deduce from the formality morphism the existence of a quantization \( R \) of \( r. \)

0. INTRODUCTION

Let \( K \) be a field of characteristic 0 and \( h \) a formal parameter. Let \( (\mathfrak{g}, [-, -], \delta_h) \) be a Lie bialgebra over \( K \) (all our objects will be \( K[[h]] \)-modules). Using Etingof-Kazhdan quantization functor, one can construct a quantization \( (U_h(\mathfrak{g}), \Delta_h) \) of \( (\mathfrak{g}, \delta_h) \). Let us denote \( C(\mathfrak{g}) = S(\mathfrak{g}[-1]) \) the free graded commutative algebra generated by \( \mathfrak{g} \); \( (C(\mathfrak{g}), [-, -], \Lambda, \delta_h) \) is a differential Gerstenhaber algebra. Let us also denote \( TU = T(U_h(\mathfrak{g})[-1]) \) the tensor algebra over \( U_h(\mathfrak{g}) \) (when \( \delta_h = 0, U = U(\mathfrak{g}) \), the enveloping algebra of \( \mathfrak{g} \)). More generally, we denote \( TE = T(E[-1]) \) the free tensor algebra of a graded vector space \( E \) and \( TE = T(E[1]) \) the cofree tensor coalgebra of \( E \). One can see the elements of \( \mathfrak{g} \) as invariant (under left action) vector fields on the manifold \( G \) where \( G \) is a connected group whose Lie algebra is \( \mathfrak{g} \). In that framework, \( C(\mathfrak{g}) \) corresponds to the Gerstenhaber algebra \( T_{poly}^{inv} \) of invariant multivector fields on \( G \) equipped with Schouten bracket. The space \( TU \) corresponds to the space \( D_{poly}^{inv} \) of multidifferential operators. The space \( D_{poly} \) of Hochschild cochains carries a graded differential Lie algebra structure when equipped with the Hochschild coboundary and the Gerstenhaber bracket. Tamarkin proved in [Ta] that the space \( D_{poly} \) carries a \( G_\infty \) structure. In this paper, for general Lie bialgebra case, we prove:

Theorem 0.1. (1) There exists a \( G_\infty \)-structure on \( TU \), whose underlying \( L_\infty \)-structure is the one given by the differential graded Lie structure with deformed Gerstenhaber bracket and coHochschild differential.

(2) There exists \( \varphi \), a \( L_\infty \)-quasi-isomorphism between \( C(\mathfrak{g}) \) and \( TU \) for the corresponding Lie algebra structures, such that the associated morphism of complex \( \varphi^1 \) maps \( v \in C(\mathfrak{g}) \) to its alternation Alt\((v) \in TU \mod h. \)

Definitions of \( G_\infty \) and \( L_\infty \)-structures will be recalled in section 1 as well as the fact that \( G_\infty \)-algebras have canonical underlying \( L_\infty \)-structure. This theorem generalises a result of Calaque ([Ca]) when \( \delta_h = 0 \) and answers to a conjecture of Tamarkin and Tsygan ([TT]).

Suppose now that \( (\mathfrak{g}, r, Z) \) is a (finite-dimensional) coboundary Lie bialgebra over \( K \). This means that \( \mathfrak{g} \) is a Lie bialgebra, the Lie cobracket \( \delta_h \) is the coboundary of an element \( r \in \Lambda^2(\mathfrak{g}) \): \( \delta_h(x) = h[x \otimes 1 + 1 \otimes x, r] \) for any \( x \in \mathfrak{g} \). This condition means that \( Z := CYB(r) \) belongs to \( \Lambda^3(\mathfrak{g})^s \) (here \( CYB \) is the l.h.s. of the classical Yang-Baxter equation). Quasi-triangular and triangular Lie bialgebras are particular cases of this definition.

Let us recall the definition of a quantization of \( (\mathfrak{g}, r, Z) \) in coboundary Hopf algebra ([Dr1]):

Definition 0.2. A algebra \( (U_h(\mathfrak{g}), R) \) is a coboundary quantization of \( (\mathfrak{g}, r, Z) \) if:
(1) \((U_h(\mathfrak{g}), \Delta_h)\) is a quantization of \(\mathfrak{g}\).
(2) \(R\) is an invertible element of \(U_h(\mathfrak{g})^{\otimes 2}\) such that \(R = 1 + hr + O(h^2)\) that satisfies:
(3) \(R^{1.2}(\Delta_h \otimes \text{id})(R) = R^{2.3}(\text{id} \otimes \Delta_h)(R)\),
(4) \((\varepsilon \otimes \varepsilon)(R) = 1\),
(5) \(R^{2.1} = R^{-1}\),
(6) \(R\) twists \(\Delta_h\) into \(\Delta_h^{op}\):

\[R\Delta_h(a)R^{-1} = \Delta_h^{2.1}(a), \quad a \in U_h(\mathfrak{g}),\]

where \(\Delta_h^{2.1} = \Delta_h^{op}\) is the opposite comultiplication.

Here the notation \(R^{i,j}\) corresponds to the coproduct-insertion map and will be recalled at the end of this introduction. Using the formality map of Theorem 0.1, we prove the existence of a quantization of coboundary \(r\)-matrices \(r\).

**Theorem 0.3.** Let \((\mathfrak{g}, r, Z)\) be a coboundary Lie bialgebra. There exists a Hopf algebra \((U_h(\mathfrak{g}), \Delta_h)\) and an element \(R\) in \(U_h(\mathfrak{g})^{\otimes 2}\) satisfying the first four properties of Definition 0.2.

In section 1, we recall definitions of \(G_\infty\) and \(L_\infty\)-structures. \(G_\infty\)-structures on a vector space \(A\) are defined on the cofree Lie coalgebra \(S(\mathcal{C}E[1])\), where \(\mathcal{C}E\) is the quotient of the cofree tensor coalgebra \(TE\) of a vector space \(E\) by the image of the shuffles. We also recall the existence of two exact functors: \(L-G_\infty, \eta \rightarrow C(\mathfrak{h})[1]\) between the categories of differential Lie bialgebras and of differential Gerstenhaber algebras, viewed as \(G_\infty\)-algebras and \(L-G_\infty\), \(\mathcal{C}E \rightarrow E\) between the categories of differential Lie bialgebras \(\mathcal{C}E\) and of \(G_\infty\)-algebras.

In section 2, we recall Drinfeld duality between QUE (Quantum Universal Enveloping) and QFSH (Quantum Formal Series Hopf) algebras. In particular we recall the existence of functors \((\cdot)^\vee:\) QUE \(\rightarrow\) QFSH and \((\cdot)^\natural:\) QFSH \(\rightarrow\) QUE. We then recall Etingof-Kazhdan quantization/dequantization functors.

In section 3, we prove the existence of a bialgebra structure on \(\mathcal{C}TTU\) coming from brace operations. Moreover, we prove that \(\mathcal{C}TTU\) is a QFSH algebra deforming the shuffle/cofree (for product/coproduct) structure on \(\mathcal{C}TTU\). Then using Etingof-Kazhdan dequantization functor \(DQ\), we show that the QUE algebra \((\mathcal{C}TTU)^\vee\) is sent to \(\mathcal{C}TTU\). So \(TTU\) has a \(G_\infty\)-structure. This structure reduces to a differential Lie algebra structure with Gerstenhaber bracket and coHochschild differential defined using the coproduct \(\Delta_h\).

In section 4, we prove the existence of a bialgebra quasi-isomorphism \(\varphi_{\text{alg}}: U \rightarrow (\mathcal{C}TTU)^\vee\) and of a Lie bialgebra quasi-isomorphism \(\varphi_{\text{Lie}}: \mathcal{C}TTU \rightarrow \mathcal{C}TC(\mathcal{C}E)\) (for every vector space \(E\)).

In section 5, we recall the existence of an inverse map for any \(L_\infty\)-quasi-isomorphism between \(L_\infty\)-algebras.

In section 6, we deduce the existence of a \(L_\infty\)-quasi-isomorphism between \(C(\mathfrak{g})\) and \(TU\). The proof can be summarised in the following diagram:

\[
\begin{array}{cccc}
\mathcal{C}TC(\mathcal{C}TTU) & C(\mathcal{C}TTU)[1] & C(\mathcal{C}TTU^\vee)[1] & \mathcal{C}TTU^\vee \relax \\
\uparrow \varphi_{\text{Lie}} & \uparrow \varphi_{G_\infty} & \uparrow \varphi_{\text{Ger}_\infty} & \uparrow \varphi_{\text{Lie}} \relax \\
\mathcal{C}TTU & TU^\vee[1] & C(\mathfrak{g})[1] & \mathfrak{g} \relax \\
\end{array}
\]

Thus the composition of \(\varphi_{\text{Ger}_\infty}\) with the inverse of \(\varphi_{G_\infty}\) gives the wanted quasi-isomorphism. From this, we prove Theorem 0.3.
In the last section, we make some remarks on possible applications and related open questions. In particular, we prove that the coHochschild complex \((TU', b_{HH})\) is quasi-isomorphic to the complex \((C(g), \delta_0)\).

**Notations.** We use the standard notation for the coproduct-insertion maps: we say that an ordered subset is a pair of a finite set \(S\) and a bijection \(\{1, \ldots, |S|\} \to S\). For \(I_1, \ldots, I_m\) disjoint ordered subsets of \(\{1, \ldots, n\}\), \((U, \Delta)\) a Hopf algebra and \(a \in U^\otimes m\), we define

\[
a^{I_1, \ldots, I_m} = \sigma_{I_1, \ldots, I_m} \circ (\Delta^{(|I_1|)} \otimes \cdots \otimes \Delta^{(|I_m|)})(a),
\]

with \(\Delta^{(1)} = \text{id}\), \(\Delta^{(2)} = \Delta\), \(\Delta^{(n+1)} = (\text{id}^{\otimes n-1} \otimes \Delta) \circ \Delta^{(n)}\), and \(\sigma_{I_1, \ldots, I_m} : U^\otimes \sum \{I_i\} \to U^\otimes n\) is the morphism corresponding to the map \(\{1, \ldots, \sum I_i\} \to \{1, \ldots, n\}\) taking \((1, \ldots, |I_1|)\) to \(I_1\), \((|I_1| + 1, \ldots, |I_1| + |I_2|)\) to \(I_2\), etc. When \(U\) is cocommutative, this definition depends only on the sets underlying \(I_1, \ldots, I_m\).

Until the end of this paper, although we will often omit to mention it, we will always deal with graded structures.

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### 1. \(G_\infty\)-algebras

**1.1. Definitions.** Let us recall definitions of \(L_\infty\)-algebras and \(L_\infty\)-morphisms. We have denoted \(TA = T(A[-1])\) the free tensor algebra of \(A\) which, equipped with the shuffle coproduct, is a bialgebra. We also have denoted \(C(A) = S(A[-1])\) the free graded commutative algebra generated by \(A[-1]\), seen as a quotient of \(TA\). The shuffle coproduct is still well defined on \(C(A)\) which becomes a cofree cocommutative coalgebra on \(A[-1]\). We also denote \(\Lambda A = S(A[1])\), the analogous graded commutative algebra generated by \(A[1]\) (in particular, for \(A_1, A_2 \in A\), \(A_1 A_2\) stands for the corresponding quotient of \(A_1[1] \otimes A_2[1]\) in \(\Lambda A\)). We will use the notations \(T^n A, \Lambda^n A\) and \(C^n (A)\) for the elements of degree \(n\).

**Definition 1.1.** A vector space \(A\) is endowed with a \(L_\infty\)-algebra (Lie algebra “up to homotopy”) structure if there are degree one linear maps \(d^1, \ldots, d^n : \Lambda^k A \to A[1]\) such that the associated coderivations (extended with respect to the cofree cocommutative structure on \(\Lambda A\))

\[
d : \Lambda A \to \Lambda A, \text{ satisfy } d \circ d = 0 \text{ where } d \text{ is the coderivation}
\]

\[
d = d^1 + d^1 1 + \cdots + d^1 1 + \cdots.
\]

In particular, a differential Lie algebra \((A, b, [-, -])\) is a \(L_\infty\)-algebra with structure maps \(d^1 = b[1]\), \(d^{1,1} = [-, -][1]\) and \(d^{1,\ldots,1} : \Lambda^k A \to A[1]\) are 0 for \(k \geq 3\). One can now define the generalisation of Lie algebra morphisms:

**Definition 1.2.** A \(L_\infty\)-morphism between two \(L_\infty\)-algebras \((A_1, d_1 = d^1_1 + \cdots)\) and \((A_2, d_2 = d^1_2 + \cdots)\) is a morphism of codifferential cofree coalgebras, of degree 0,

\[
\varphi : (\Lambda A_1, d_1) \to (\Lambda A_2, d_2).
\]

In particular \(\varphi \circ d_1 = d_2 \circ \varphi\). As \(\varphi\) is a morphism of cofree cocommutative coalgebras, \(\varphi\) is determined by its image on the cogenerators, i.e., by its components: \(\varphi^{1,\ldots,1} : \Lambda^k A_1 \to A_2[1]\).

Let us denote \(\mathcal{T}(E)\) the cofree tensor coalgebra of \(E\) (with coproduct \(\Delta\)). Equipped with the shuffle product \(\bullet\) (defined on the cogenerators \(\mathcal{T}(E) \otimes \mathcal{T}(E) \to E\) as \(pr \otimes \varepsilon + \varepsilon \otimes pr\), where \(pr : \mathcal{T}(E) \to E\) is the projection and \(\varepsilon\) is the counit), it is a bialgebra. Let \(\mathcal{T}(E)^{\bullet}\) be
the augmentation ideal. We denote \( ^nT(E) = T(E)\) the quotient by the shuffles. It has a graded cofree Lie coalgebra structure (with coproduct \( \Delta = \Delta^\prime - \Delta^\prime\)). Then \( S(\tau T(E)[1]) \) has a structure of cofree coGerstenhaber algebra (i.e. equipped with cofree coLie and cofree cocommutative coproducts satisfying compatibility condition). We use the notation \( ^nT(E) \) for the elements of degree \( n \).

**Definition 1.3.** A vector space \( E \) is endowed with a \( G_\infty \)-algebra (Gerstenhaber algebra “up to homotopy”) structure if there are degree one linear maps \( d^{p_1\ldots p_k} : \tau \tau T(E)[1] \to E[p_k] \) \( \Lambda \cdots \Lambda \tau T(E)[1] \to E[2] \) such that the associated coderivations (extended with respect to the cofree coGerstenhaber structure on \( \tau T(E) \)) \( \Delta : \tau T(E) \to \tau T(E) \) satisfies \( d \circ d = 0 \) where \( d \) is the coderivation

\[
d = d_1 + d_1^{1,1} + \cdots + d_1^{p_1\ldots p_k} + \cdots .
\]

In particular we have

**Remark 1.4.** If \( (E, b, [-,-], \wedge) \) is a differential Gerstenhaber algebra, then \( E[1] \) is a \( G_\infty \)-algebra with structure maps \( d^1 = b[1], d^{1,1} = [-,-][1], d^2 = \wedge[1] \) and other \( d^{p_1\ldots p_k} : \tau \tau T(E)[1] \to E[2] \) are 0.

One can finally define the generalisation of Gerstenhaber algebra morphisms:

**Definition 1.5.** A \( G_\infty \)-morphism between two \( G_\infty \)-algebras \( (E_1, d_1 = d_1^1 + d_1^2 + \cdots ) \) and \( (E_2, d_2 = d_2^1 + d_2^2 + \cdots ) \) is a morphism of differential coGerstenhaber coalgebras, of degree 0,

\[
\varphi : (\tau \tau T(E_1), d_1) \to (\tau \tau T(E_2), d_2).
\]

In particular \( \varphi \circ d_1 = d_2 \circ \varphi \). As \( \varphi \) is a morphism of cofree coGerstenhaber coalgebras, \( \varphi \) is determined by its image on the cogenerators, i.e., by its components: \( \varphi^{p_1\ldots p_k} : \tau \tau T(E_1) \to \tau \tau T(E_2) \).

1.2. **Functors** \( L-G \) and \( L-G_\infty \). Let \( (\h, \delta, d) \) be a differential Lie bialgebra. Let \( C(\h) = S(\h[-1]) \) be the free graded commutative algebra generated by \( \h \). Recall from the previous subsection that \( C(\h) \) is a cofree coalgebra and that coderivations \( C(\h) \to C(\h) \) are defined by their images in \( \h \). Thus, one easily checks that the coderivation \( [-,-] \) extending the Lie bracket (with degree shifted by one) defines a Lie algebra structure on \( C(\h) \) and that \( (C(\h)[-1], [-,-], \cdot, \wedge) \) is a coLie algebra, where \( \cdot \) is the commutative product:

\[
(\alpha, \beta) \mapsto \alpha \wedge \beta = \alpha \Lambda \beta \text{ on } C(\h).
\]

Moreover, one can extend maps \( d : \h \to \h \) and \( \delta : \h \to S^2(\h[-1]) \) on the free commutative algebra \( C(\h) \) so that \( (C(\h)[-1], [-,-], \cdot, \wedge, d + \delta) \) is a differential Gerstenhaber algebra. In fact, the differential \( \delta \) is the Chevalley Eilenberg differential: the space \( C(\h) = S^*(\h[-1]) \) is isomorphic to the standard complex \( (\Lambda^*(\h))[1] \) and \( \delta \) is simply the differential given by the underlying Lie algebra structure of \( \h \). Moreover any morphism \( f : \h_1 \to \h_2 \) can be extended into a morphism \( C(f) : C(\h_1) \to C(\h_2) \) of free commutative algebras thanks to the inclusion \( \h_2 \subset C(\h_2) \). This morphism is easily seen to be a differential Gerstenhaber algebra morphism. Thus, we have defined a functor \( L-G \) from differential Lie bialgebras to differential Gerstenhaber algebras which sends \( \h \) to \( C(\h) \). This functor is exact. As the differential \( \delta \) and \( d \) anticommute by construction, the complex \( (C(\h), d, \delta) \) is a (first quadrant) bicomplex. Hence a quasi-isomorphism \( \h_1 \to \h_2 \) induces a quasi-isomorphism \( C(\h_1) \to C(\h_2) \). Then, thanks to Remark 1.4, we get a functor \( L-G : G_\infty : \h \to C(\h)[1] \) from differential Lie bialgebras to \( G_\infty \)-algebras.

Let us now define the functor \( L-G_\infty \). Consider the category \( CFDB \) of differential Lie bialgebras which are cofree as a Lie coalgebra. In other words we are interested in cofree Lie coalgebra \( ^nT(E) \) on a graded vector space \( E \) together with a differential \( \ell \) and a cobracket \( L \) on
that makes it a differential Lie bialgebra. As \(c^\ell(T(E))\) is cofree, the differential is uniquely determined by its restriction to cogenerators \(p^r: c^\ell p^r(E) \to E\). Similarly, the Lie bracket is uniquely determined by maps \(L^{p_1,p_2}: c^\ell p^1(E) \Lambda^\ell T^{p_2}(E) \to E\). Now, this data determines on \(E\) a structure of \(G_\infty\)-algebra with structure maps given by \(d^{p_1 \cdots p_k}: c^\ell p^1 \Lambda^\ell \cdots \Lambda^\ell p_k(E) \to E\) with \(d^{p_1 \cdots p_k} = 0\) for \(k > 2\) and \(d^{p_1,p_2} = L^{p_1,p_2} = dp = lp\) (with degrees shifted by one). In fact, according to Definition 1.3, a \(G_\infty\)-structure on \(E\) is given by a differential \(d\) on \(\Lambda^\ell T(E)\) which as a space is isomorphic to the standard Chevalley-Eilenberg complex of the differential Lie algebra \((c^\ell(E), \ell, L)\). The differential defined above is simply the Chevalley-Eilenberg differential. In particular \(d\) is the sum \(d = d_1 + d_2 = \sum_{p_1 \geq 1} \|p\|\) and \(d_2 = \sum_{p_1 \geq 2} 1^p\) and \((\Lambda^\ell T(E), d^1, d^2)\) is a bicomplex. Moreover, a morphism \(\varphi: T(E)_1 \to T(E)_2\) of differential Lie bialgebras is determined by its restriction to cogenerator of the cofree Lie coalgebra structure, that is to say by maps: \(\varphi^0: T^0(E)_1 \to T^0(E)_2\). It determines a \(G_\infty\)-morphism \(E_1 \to E_2\) (with the \(G_\infty\)-structures defined above) defined by maps \(\varphi^0: T^n(E)_1 \to T^n(E)_2\), other being 0. This is simply the functoriality of the Chevalley-Eilenberg complex. Thus we have defined a functor from CFDLB to the category of \(G_\infty\)-algebras. This functor is exact. A quasi-isomorphism of differential Lie bialgebras \((c^\ell(E_1),\ell_1) \to (c^\ell(E_2),\ell_2)\) induces a quasi-isomorphism \((\Lambda^\ell T(E_1),d^1_1) \to (\Lambda^\ell T(E_2),d^1_2)\), hence, as \((\Lambda^\ell T(E),d^1,d^2)\) is a (first quadrant) bicomplex, a quasi-isomorphism \((\widetilde{\Lambda}^\ell T(E_1),d_1) \to (\Lambda^\ell T(E_2),d_2)\).

Until the end of the paper, we will use the notations \(TE\) for \(T(E[-1])\) and \(c^\ell TE\) for \(T(E[1])\).

2. Etingof-Kazhdan functors

2.1. Duality of QUE and QFSH algebras. We recall some facts from [Dr1] (proofs can be found in [Gav]). Let us denote by QUE the category of quantized universal enveloping (QUE) algebras and by QFSH the category of quantized formal series Hopf (QFSH) algebras. We denote by QUE\(_{fd}\) and QFSH\(_{fd}\) the subcategories corresponding to finite dimensional Lie bialgebras.

We have contravariant functors QUE\(_{fd}\) \(\to\) QFSH\(_{fd}\), \(U \mapsto U^*\) and QFSH\(_{fd}\) \(\to\) QUE\(_{id}\), \(\mathcal{O} \mapsto \mathcal{O}^\vee\). These functors are inverse to each other. \(U^*\) is the full topological dual of \(U\), i.e., the space of all continuous (for the \(h\)-adic topology) \(\mathbb{K}[[h]]\)-linear maps \(U \to \mathbb{K}[[h]]\). \(\mathcal{O}^\vee\) is the space of continuous \(\mathbb{K}[[h]]\)-linear forms \(O \to \mathbb{K}[[h]]\), where \(O\) is equipped with the \(\mathcal{M}\)-adic topology (here \(\mathcal{M} \subset \mathcal{O}\) is the maximal ideal).

We also have covariant functors QUE \(\to\) QFSH, \(U \mapsto U^V\) and QFSH \(\to\) QUE, \(\mathcal{O} \mapsto \mathcal{O}^\vee\). These functors are also inverse to each other.

\(U^V\) is a subalgebra of \(U\) defined as follows: for any ordered subset \(\Sigma = \{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n\}\) with \(i_1 < \cdots < i_k\), define the morphism \(j_{\Sigma}: U^\otimes k \to U\otimes n\) by \(j_{\Sigma}(a_1 \otimes \cdots \otimes a_k) := b_1 \otimes \cdots \otimes b_n\) with \(b_i := 1\) if \(i \notin \Sigma\) and \(b_{i_m} := a_m\) for \(1 \leq m \leq k\); then set \(\Delta_{\Sigma} := j_{\Sigma} \circ \Delta^{(k)}\) and \(\Delta_0 := \Delta^{(0)}\), and

\[
\delta_{\Sigma} := \sum_{\Sigma' \subset \Sigma} (-1)^{n-|\Sigma'|} \Delta_{\Sigma'}, \quad \delta_0 := \epsilon .
\]

We shall also use the notation \(\delta^{(n)} := \delta_{\{1,2,\ldots,n\}}\), \(\delta^{(0)} := \delta_0\), and the useful formula

\[
\delta^{(n)} = (id_0 - \epsilon)^\otimes n \circ \Delta^{(n)} .
\]

Finally, we define

\[
U^V := \{ a \in U \mid \delta^{(n)}(a) \in \hbar^n U^\otimes n \} \quad (\subseteq U)
\]

and endow it with the induced topology.

On the other way, \(\mathcal{O}^\vee\) is the \(h\)-adic completion of \(\sum_{k \geq 0} \hbar^{-k} \mathcal{M}^k \subset \mathcal{O}[1/h]\).
We also have canonical isomorphisms \((U')^\circ \simeq (U^*)^\circ\) and \((O^\vee)^* \simeq (O^\circ)'\).

2.2. The functor \(DQ\). In [GH], a generalisation of Etingof-Kazhdan theorem ([EK]) was proven in an appendix by Enriquez and Etingof:

**Theorem 2.1.** We have an equivalence of categories

\[DQ_\Phi : DGQUE \to DGLBA_h\]

from the category of differential graded quantized universal enveloping super-algebras to that of differential graded Lie super-bialgebras such that if \(U \in \text{Ob}(DGQUE)\) and \(a = DQ(U)\), then \(U/hU = U(a/ha)\), where \(U\) is the universal algebra functor; taking a differential graded Lie super-algebra to a differential graded super-Hopf algebra.

Here \(\Phi\) is a Drinfeld associator. We will use any of these functor and denote it \(DQ\).

3. Bialgebra structure on \(\mathcal{T} T U\)

3.1. Brace structures. Here, we will define a structure on \(\mathcal{T} T U\). Following [GV, GJ, Ka], it is a brace structure (although usually brace structures are defined on the space of Hochschild cochains of an algebra) but we will not recall the definition of brace structure as we will only need the result of the following subsection. Let us define those structures for a general Hopf algebra. More precisely, let \((U, \Delta, \times)\) be a Hopf algebra (in our case \(U\) will be the Etingof-Kazhdan quantization \(U_h(g)\) of the Lie bialgebra \(g\)). We will define a brace structure on the cofree tensor coalgebra \(\mathcal{T} T U\) of the free tensor algebra \(T(U[-1])\). To distinguish the two tensor products, we denote \(\otimes\) the tensor product on \(TU\) and \(\boxtimes\) the tensor product on \(\mathcal{T} T U\).

**Definition 3.1.** We define a brace structure on \(\mathcal{T} T U\) by extending the following maps given on the cogenerators of the cofree coalgebra \(\mathcal{T} T U\):

1. \(B^0 = 0\),
2. \(B^1 = b_{cH}\) (the coHochschild coboundary on \(TU\)),
3. \(B^2 : \alpha \boxtimes \beta \mapsto \alpha \otimes \beta\),
4. \(B^n = 0\) for \(n > 2\),
5. \(B^{0,1} = B^{1,0} = \text{id}\),
6. \(B^{0,n} = B^{n,0} = 0\) for \(n > 1\),
7. \(B^{1,n} : (\alpha, \beta_1 \boxtimes \cdots \boxtimes \beta_n) \mapsto \sum_{\substack{0 \leq i_1, \ldots, i_m + k_m \leq n \\text{ s.t. } i + k_i \leq i_{i+1}}} (-1)^\varepsilon \alpha^{1-i_1-1-i_2-k_1, \ldots, i_m+1-i_m-k_m, \ldots, n} \otimes \beta_1 \otimes \cdots \otimes \beta_n \otimes 1_{i_1+k_1=i_1} \otimes \cdots \otimes 1_{i_m+k_m=i_m} \otimes 1_{n-(i_m+k_m)},\)
   where \(k_s = \lfloor \beta_s \rfloor, n = \lfloor \alpha \rfloor + \sum k_s - m\) and \(\varepsilon = \sum k_s - 1\)
8. \(B^{k,l} = 0\) for \(k > 1\).

Note that, when \(U = U(g)\), the enveloping algebra of a Lie algebra \(g\), \(T(U[-1])\) can been seen as the space of invariant multidifferential operators over the Lie group corresponding to \(g\) and in that case, our definition coincides with those of [GV, GJ, Ka].
3.2. From Hopf algebra $U$ to Bialgebra structure on $\mathcal{TT}U$. Still following [GV, GJ, Ka], we have:

**Theorem 3.2.** The brace structure of Definition 3.1 defines a differential bialgebra structure on the cofree tensor coalgebra $\mathcal{TT}U$, with product $\ast$ extending $\sum B^{p_1,p_2}$ and differential $d$ extending $\sum B^p$.

**Proof.** To prove the associativity of $\ast$, one has to check the following equation for $\alpha, \beta_1, \ldots, \beta_t, \gamma_1, \ldots, \gamma_l \in TU$:

$$
\sum_{0 \leq i_1 \leq \cdots \leq i_l \leq m} (-1)^{\varepsilon} \{\alpha\} \{\gamma_1, \ldots, \gamma_{i_1}, \{\beta_1\} \{\gamma_{i_1+1}, \ldots, \gamma_{i_2}, \{\beta_2\} \{\gamma_{i_2+1}, \ldots, \gamma_{i_3}, \ldots, \gamma_{i_l}, \{\beta_l\} \{\gamma_{i_l+1}, \ldots, \gamma_m\} = \{\{\alpha\} \{\beta_1, \ldots, \beta_l\}\} \{\gamma_1, \ldots, \gamma_m\},
$$

where $\varepsilon = \sum_{p=1}^{t} (|\beta_p| - 1) \sum_{q=1}^{p} (|\gamma_q| - 1)$ and $\{\alpha\} \{\beta_1, \ldots, \beta_l\}$ is $B^{1, l}(\alpha, \beta_1 \boxtimes \cdots \boxtimes \beta_l)$. This equation is a consequence of the associativity, coassociativity and compatibility of $\times$ and $\Delta$:

$$(\Delta_h^k \otimes \text{id}^\otimes \otimes \Delta_h^m)(\alpha) = (\Delta_h^k \otimes \text{id}^\otimes \otimes \Delta_h^{m}) \circ (\text{id}^\otimes \otimes \Delta_h^{m})(\alpha)$$

and

$$(\Delta_h^k \otimes \text{id}^\otimes \otimes \Delta_h^{n+p-2})(\Delta_h^n \otimes \text{id}^\otimes \otimes \alpha) \times (\beta \otimes 1^\otimes p^{-1}) = (\Delta_h^{n+k-1} \otimes \text{id}^\otimes \otimes \alpha) \times (\Delta_h^k \otimes \text{id}^\otimes \otimes \alpha).$$

For $\alpha, \beta \in TU$ of degree $p$ and $n$.

One can then notice that the map $d$ is the commutator, with respect to the product $\ast$, of the element $1 \otimes 1 \in TU \subset \mathcal{TT}U$. Thus $d$ is compatible with the multiplication. Finally, $d$ is a differential as $[1 \otimes 1, 1 \otimes 1]_\ast = 0$. \hfill $\Box$

**Remark 3.3.** Note that we have only used here the bialgebra structure of $U$. So one would get a similar result when replacing $U$ with $U'$ (see section 2).

We have now:

**Proposition 3.4.** The algebra $\mathcal{TT}U'$ is a QFSHA.

**Proof.** This is known when $U = U(hg)$ ([Ta, TT] or [GH]). Proof can be done in the same way for any QUE algebra $U_h(g)$: one considers the dual bialgebra $\mathcal{TT}U'^\ast$. It is a free algebra and so a QUE algebra: as a $\mathbb{K}[[h]]$-module, it is isomorphic to the enveloping algebra of the corresponding free Lie algebra $\mathcal{TT}U'^\ast$. So $\mathcal{TT}U'$ is a QFSH algebra \hfill $\Box$

**Remark 3.5.** In this proof, we have shown that the corresponding differential Lie bialgebra to $\mathcal{TT}U'^\ast$ through Etingof-Kazhdan dequantization functor $\mathcal{D}$ is isomorphic to $\mathcal{TT}U'$ as a $\mathbb{K}[[h]]$-module.

Thus $TU'$ is equipped with a $G_\infty$-structure (see section 1).

Finally, one easily checks that

**Proposition 3.6.** The underlying differential Lie bialgebra structure on $TU'$ corresponding to the differential Lie bialgebra structure on $\mathcal{TT}U'$ is given by Gerstenhaber bracket

$$[\alpha, \beta]_G = B^{1,1}(\alpha, \beta) - (-1)^{|\alpha|-1}|\beta|-1\}B^{1,1}(\beta, \alpha)$$

and coHochschild differential

$$b_{CH}(\alpha) = [1 \otimes 1, \alpha]_G.$$
4. Bialgebra quasi-isomorphisms

4.1. A bialgebra quasi-isomorphism $\varphi_{\text{alg}} : U \to (\mathcal{TTU})'$. Let us first define a bialgebra quasi-isomorphism $\varphi_{\text{alg}} : U' \to \mathcal{TTU}'$ between the bialgebra $(U', \Delta_h, \times)$ (in our case $U = U_h(\mathfrak{g})$ the Etingof-Kazhdan quantization of $\mathfrak{g}$) and the bialgebra $(\mathcal{TTU}', \Delta, \ast)$ whose structure was described in the previous section. The definition of $U'$ was given in section 2. As $\mathcal{TTU}'$ is a cofree coalgebra, as a coalgebra map $\varphi_{\text{alg}}$ is uniquely determined by its restriction $U \to \mathcal{T U}'$ to cogenerators of $\mathcal{TTU}'$. We define

$$\varphi_{\text{alg}} = \text{inc} - \varepsilon 1,$$

where inc is actually the inclusion $U' \subset \mathcal{T U}'$. Equivalently, following [Ta2], let us consider $M$ the augmentation ideal of $U'$ and denote

$$\delta^{(2)} : M \to M \otimes M,$$

$$x \mapsto \Delta_h(x) - (1 \otimes x + x \otimes 1).$$

We still use the notation $TM$ for $T(M[-1])$ and $\mathcal{TT}M$ for $\mathcal{T}(TM[1])$. One now defines

$$\delta : M \to T(M),$$

$$x \mapsto x + \sum_{k \geq 2} \delta^{(k)}(x),$$

where $\delta^{(k)}$ is the $k$-1-th iterate of $\delta^{(2)}$ (we will set $\delta^{(1)} = \text{inc}$). This map is well defined thanks to the definition of $U'$. Now inclusion $i : U'[-1] \to T(U'[-1])$ defines, after prolongation on the cofree tensor coalgebra, a map $Ti : T(U'[-1])[1] \to \mathcal{TTU}'$. We have $\varphi_{\text{alg}} = Ti \circ \delta$. We can then take into account the unit and define the map $\varphi_{\text{alg}}$ on $U' : \mathbb{K} \oplus M \to \mathbb{K} \oplus \mathcal{TT}M$.

As every maps are coalgebra morphisms, to check that $\varphi_{\text{alg}}$ is an algebra morphism, one only has to check that the following diagram commutes after canonical projection $Pr : \mathcal{TTU}' \to \mathcal{T}U'[1]$:

$$\begin{array}{ccc}
U' \otimes U' & \xrightarrow{\varphi_{\text{alg}} \otimes \varphi_{\text{alg}}} & \mathcal{TTU}' \otimes \mathcal{TTU}' \\
\downarrow \times & & \downarrow \star \\
U' & \xrightarrow{\varphi_{\text{alg}}} & \mathcal{TTU}' \xrightarrow{Pr} \mathcal{T}U'[1]
\end{array}$$

Let us check the commutation for $\mu \otimes \eta \in M \otimes M$. Commutation is obvious if one of the elements $\mu$ or $\eta$ is in $\mathbb{K}$, so let us consider the image of $\mu \otimes \eta \in M \otimes M$ through the two paths of this diagram. Using the notations of section 3, we have $\delta \alpha = \sum_n \delta^{(n)} \alpha = \sum \alpha_{n_1} \otimes \cdots \otimes \alpha_{n_k}$ and $\delta \beta = \sum \beta_{m_1} \otimes \cdots \otimes \beta_{m_l}$ with $\alpha_{n_i}, \beta_{m_j} \in U$. Note that $B^{1,n}(\alpha_{n_1}, \beta_{m_1})$ is 0 for $n > 1$ (otherwise) and that $B^{p>1,q} = 0$. So we get

$$Pr(\varphi_{\text{alg}}(\alpha) \ast \varphi_{\text{alg}}(\beta)) = Pr(\alpha \times \varphi_{\text{alg}}(\beta)) = \alpha \times \beta,$$

which is the commutation property.

Let us show now that $\varphi_{\text{alg}}$ is a quasi-isomorphism of complexes. Recall that the differential $d$ on $\mathcal{TTU}'$ is defined as extension of $B^1 + B^2$ (c.f. definition 3.1). Let us prove the following lemma

**Lemma 4.1.** We have $H.(\mathcal{TT}(U'), d) \simeq U'$ in degree 1 and is 0 for degree $\geq 2$.

**Proof.** The differential $d$ is the sum $d = d^1 + d^2$ where $d^1$ and $d^2$ correspond respectively to the maps $B^1$ and $B^2$. Thus $\mathcal{TTU}'$ has a structure of bicomplex. Let us compute the homology with respect to $d^2$: $(\mathcal{TTU}' \simeq \oplus \mathcal{T^nTU}', d^2)$ is the Hochschild complexe associated to $T U'$ and $\mathcal{T}U'$ is a free associative algebra then

$$H.(\mathcal{TT}(U'), d^2) \simeq H.(\mathcal{T^nTU}', d^2) \simeq U',$$
concentrated in degree 1. So $H_c(TT(U'), d^1 + d^2) \simeq \ker(U', d^1) = U'$ as $d^1(H_c(TU')) \cong 0$. 

Finally, we check that $\varphi_{\text{alg}}$ is a morphism of complexes. As before, it is enough to check on the cogenerators that $\text{Pr}(d(\varphi_{\text{alg}}(\alpha))) = 0$ for $\alpha \in U'$. Still writing $\delta\alpha = \sum \alpha_{n_1} \otimes \cdots \otimes \alpha_{n_k}$ we get

$$\text{Pr}(d(\varphi_{\text{alg}}(\alpha))) = \text{Pr}(d^1(\alpha) + d^2(Ti\delta(2)(\alpha))) = b_{\text{ch}}(\alpha) - \delta^2(\alpha) = 0,$$

where $b_{\text{ch}}$ is the coHochschild codifferential.

Thus we have a bialgebra quasi-isomorphism $\varphi_{\text{alg}} : U' \to ^cTTU'$. Applying to it the Drinfeld functor $(-)^\vee$, we get a bialgebra quasi-isomorphism $\varphi_{\text{alg}} : U \to ^cTTU'^\vee$.

4.2. A Lie bialgebra quasi-isomorphism $\varphi_{\text{Lie}} : ^cTA \to ^cTC(^cTA)$. Let $A$ be a vector space. Suppose now that the cofree Lie coalgebra $^cTA$ has a structure $(^cTA, \delta, [-, -], d)$ of a differential Lie bialgebra. Using the functor $L - G$ (see section 1), one gets a differential Gerstenhaber algebra $(C(^cTA), [-, -], \wedge, d + \delta)$. One can extend the structure maps on the cofree Lie coalgebra $^cTC(^cTA)$ and one gets a differential cofree Lie bialgebra $(^cTC(^cTA), \delta', [-, -], d + \delta + \wedge)$ (we will set $d^1 = d + \delta$ and $d^2 = \wedge$). We can now prove the existence of a differential Lie bialgebra quasi-isomorphism $\varphi_{\text{Lie}} : ^cTA \to ^cTC(^cTA)$. Let us extend the inclusion $\text{inc} : ^cTA \to C(^cTA)$ to a coderivation $\varphi_{\text{Lie}} : ^cTA \to ^cTC(^cTA)$ on the cofree Lie coalgebra. More explicitly, using the Lie cobracket $\delta$ of $^cTA$, one defines

$$\delta : ^cTA \to S(^cTA),
\quad x \mapsto \sum_{k \geq 1} \delta^{(k)}(x),$$

where $\delta^{(k)}$ is the $k - 1$-th iterate of $\delta$ and $\delta^{(1)} = \text{inc}$. Now inclusion $i : ^cTA[-1] \to C(^cTA)$ defines, after prolongation on the cofree tensor Lie coalgebra, a map $Ti : S(^cTA) \to ^cTC(^cTA)$. We have $\varphi_{\text{Lie}} = Ti \circ \delta$. We can reproduce the proof of the previous subsection (all structures are much more simpler) and easily check that $\varphi_{\text{Lie}} : ^cTA \to ^cTC(^cTA)$ is a differential Lie bialgebra quasi-isomorphism. It is obviously a Lie bialgebra morphism. Moreover, for $\alpha \in ^cTA$, we get after projection on the cogenerators,

$$(d^1 + d^2)(\varphi_{\text{alg}}(\alpha)) = d^1(\alpha) + d^2(Ti\delta(\alpha)) = d(\alpha) - \delta(\alpha) + \delta(\alpha) = d(\alpha).$$

The fact that $\varphi_{\text{Lie}}$ is a quasi-isomorphism can be proved as in the previous section.

5. Inversion of formality morphisms

Let us recall Theorem 4.4 of Kontsevich ([Ko]):

**Theorem 5.1.** Let $g_1$ and $g_2$ be two $L_\infty$-algebras and $F$ be a $L_\infty$-morphism from $g_1$ to $g_2$. Assume that $F$ is a quasi-isomorphism. Then there exists an $L_\infty$-morphism from $g_2$ to $g_1$ inducing the inverse isomorphism between associated cohomology of complexes.

**Remark 5.2.** We know from private communications the existence of a similar $G_\infty$-version of this theorem. This result would imply the existence of a corresponding $G_\infty$-morphism in Theorem 0.1.
6. Proof of the main theorems

6.1. Proof of Theorem 0.1. Let \((\mathfrak{g}, \delta_\hbar)\) be a Lie bialgebra. We write \(\delta_\hbar = \hbar \delta_1 + \hbar^2 \delta_2 + \cdots\). Let 
\((U_\hbar(\mathfrak{g}), \Delta_\hbar)\) be the Etingof-Kazhdan canonical quantization of \((\mathfrak{g}, \delta_\hbar)\). We denote \(U = U_\hbar(\mathfrak{g})\) for short. In section 3, we proved the existence of a bialgebra structure \(\varphi\) on \(\mathfrak{cTTU}\) and thanks to section 4, we have a bialgebra quasi-isomorphism \(\varphi_{\text{alg}} : U \to \mathfrak{cTTU}^\vee\). Thanks to Etingof-Kazhdan dequantization functor (see section 2), and the fact that \((\mathfrak{cTTU})^\vee\) is a QUE algebra quantizing \(\mathfrak{cTTU}\) (see section 3), we get a Lie bialgebra quasi-isomorphism \(\varphi_{\text{Lie}} : \mathfrak{g} \to \mathfrak{cTTU}\), a differential Lie bialgebra. Using the exact functor \(L-G-G_\infty\) (see section 1), we get a quasi-isomorphism of differential Gerstenhaber algebras \(\varphi_{\text{Ger}_\infty} : C(\mathfrak{g})[1] \to C(\mathfrak{cTTU})[1]\).

According to section 4, we also have a differential Lie bialgebra quasi-isomorphism \(\varphi_{\text{Lie}} : \mathfrak{cTTU} \to \mathfrak{cTC(cTTU)}\). This quasi-isomorphism is sent to a \(G_\infty\)-quasi-isomorphism \(\varphi_{G_\infty} : TU'[1] \to C\mathfrak{cTTU}'[1]\) using the functor \(L-G\) defined in section 1.

Finally, in section 5, we recalled the existence of an inverse map for any \(L_\infty\)-quasi-isomorphism between \(L_\infty\)-algebras so the \(L_\infty\)-restriction of the quasi-isomorphism \(\varphi_{G_\infty}\) is invertible. Then one can define \(\varphi : C(\mathfrak{g}) \to TU\) as the composition of the \(L_\infty\)-restriction of \(\varphi_{\text{Ger}_\infty}\) with the inverse of the \(L_\infty\)-restriction of \(\varphi_{G_\infty}\).

One has to show now that \(\varphi^1\) maps \(v \in C(\mathfrak{g})\) to \(\Alt(v) \in TU\) mod \(\hbar\). Let us replace \(\mathfrak{g}\) with \(h\mathfrak{g}\) in the previous construction. Let \(h\mathfrak{v}\) be an element of \(h\mathfrak{g}\). Let us still call \(hv\) its image in the quantization \(U\). By definition of \(\varphi_{\text{alg}}\), \(\varphi_{\text{alg}}(hv) = hv\) mod \(\hbar^2\) (once again, we still use the same notation for its image in \((\mathfrak{cTTU})^\vee\)). So \(\varphi_{\text{Lie}}(hv) = hv\) mod \(\hbar^2\) (\(hv\) on the right hand side is the image of \(hv\) in \((\mathfrak{cTTU})^\vee\)). Note that here appeared highly non trivial terms in \(O(h^2)\). So \(\varphi_{\text{Ger}_\infty}(h(v_1 \wedge \cdots \wedge hv_n)) = h^n \Alt(v_1 \otimes \cdots \otimes v_n)\) mod \(\hbar^{n+1}\).

Moreover, it is clear by definition, that \(\varphi_{\text{Lie}} : \mathfrak{cTTU} \to \mathfrak{cTC(cTTU)}\) is the identity map id mod \(\hbar\) when restricted to \(TU\), thus so is the corresponding map of complexes \(\varphi_{G_\infty} : TU' \to C(\mathfrak{cTTU}')\). Then \(\varphi^1 : C^n(h\mathfrak{g}) \to TU'\) is \(\Alt\) mod \(\hbar^{n+1}\).

6.2. Proof of Theorem 0.3. Suppose now that \((\mathfrak{g}, \delta_\hbar, [-, -], r, Z)\) is a (finite-dimensional) coboundary Lie bialgebra over \(K\). This means that the Lie cobracket \(\delta_\hbar\) is the coboundary of \(r \in \Lambda^2(\mathfrak{g})\): \(\delta_\hbar(x) = \hbar [x \otimes 1 + 1 \otimes x, r]\) for any \(x \in \mathfrak{g}\). Let \(r' = -hr\). This means that \(r'\) is a Mauer-Cartan element in \(h\mathfrak{g}\) \([r', r'] + \delta(r') = 0\). Let us write the \(L_\infty\)-morphism \(\varphi\) of Theorem 0.1: \(\varphi = \sum_{k \geq 1} \varphi^{1, \ldots, 1}_k : \Lambda^n \mathfrak{g} \to TU'\). Let us define \(R' = 1 \otimes 1 + \sum_{k \geq 1} \frac{1}{k!} (\Lambda^n r')\).

By definition of \(L_\infty\)-morphism, we get
\[
(\Delta_\hbar \otimes \text{id})(R') R'^{1,2} - (\text{id} \otimes \Delta_\hbar)(R') R'^{2,3} = 0.
\]
Let us now define \(R = R'^{-1}\). We have a solution of Theorem 0.3. \(\square\)

7. Concluding remarks

Let us consider the \(L_\infty\)-quasi-isomorphism of Theorem 0.1. It induces a quasi-isomorphism of complexes:

**Theorem 7.1.** Let \((\mathfrak{g}, \delta_\hbar)\) be a Lie bialgebra. Let \((U_\hbar(\mathfrak{g}), \Delta_\hbar)\) be the associated Etingof-Kazhdan quantization. The restriction \(\varphi\) of the \(L_\infty\)-quasi-isomorphism of Theorem 0.1 defines a quasi-isomorphism between the coHochschild complex \((T(U_\hbar(\mathfrak{g})), b_{\text{ch}})\) (with differential associated to the coproduct \(\Delta_\hbar\)) and \((C(\mathfrak{g}), \delta_\hbar)\) the exterior product over the Lie algebra \(\mathfrak{g}\) with differential given by the Lie cobracket.
The theorem generalises well known theorem for $\delta_\hbar = 0$ (see [Dr3], Proposition 2.2).

Let us discuss some possible generalisations:

**Remark 7.2.** The main tool used in that paper is Etingof-Kazhdan quantization/dequantization theorem. Suppose now that one can prove an analogue of this theorem for Lie bialgebroids and Hopf algebroid (which is still a conjecture). Using the same framework, one would get a $L_\infty$-quasi-isomorphism (or even $G_\infty$-quasi-isomorphism, c.f. Remark 5.2) between the exterior power of a any Lie bialgebroid and the tensor product of the associated quantized Hopf algebroid. In the case the Lie algebroid is the algebroid of tensor fields over a manifold $M$, this would give directly a global formality theorem between tensor fields and multidifferential operators.

**Remark 7.3.** To answer completely to “Drinfeld last unsolved problem” ([Dr2]), one should also check the last two conditions of Definition 0.2: $R^{-1} = R^{2,1}$ and $R\Delta_\hbar(-)R^{-1} = \Delta^{2,1}_\hbar(-)$. Those properties are maybe not satisfied for every quantization $U_\hbar(g)$ of $(g, \delta_\hbar)$.

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