ASPECTS of $W_\infty$ SYMMETRY*

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ABSTRACT

We review the structure of $W_\infty$ algebras, their super and topological extensions, and their contractions down to (super) $w_\infty$. Emphasis is put on the field theoretic realisations of these algebras. We also review the structure of $w_\infty$ and $W_\infty$ gravities and comment on various applications of $W_\infty$ symmetry.

* To appear in the proceedings of the 4th Regional Conference on Mathematical Physics, Tehran, Iran, 1990
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1. Introduction

$W_\infty$ algebra is a particular generalization of the Virasoro algebra which contains fields of conformal spins $2, 3, \ldots, \infty$ [1]. Virasoro algebra is a special truncation of $W_\infty$ algebra containing only the spin-2 field: the 2D energy momentum tensor. The importance of Virasoro algebra in string theory is well known. Virasoro algebra also arises as a symmetry group of the KdV (Korteweg-de Vries) hierarchy which, in turn, plays an important role in the study of non-perturbative 2D gravity coupled to minimal conformal matter [2].

It is natural to search for a higher spin generalization of the Virasoro algebra which may lead to a $W$-string theory containing massless higher spin fields in target spacetime [3]. This would mean a dramatic enlargement of the usual Yang-Mills and diffeomorphism symmetries in spacetime. A more immediate application of $W_\infty$ symmetry is based on the fact that it arises as a symmetry group of the KP (Kadomtsev-Petviashvili) hierarchy [4,5], which is expected to be closely associated with 2D gravity coupled to a $c = 1$ matter, i.e. string in one dimension. Furthermore, a certain contraction of $W_\infty$, known as $w_\infty$ [6,7] emerges as a symmetry group in the study of self-dual gravity, which can be formulated as the $sl(\infty)$ Toda theory [8], as well as the natural background for the critical $N = 2$ superstring [9].

In this review, we begin by recalling the field theoretic realisation of the Virasoro algebra. We then outline the generalisation of the construction for $W_\infty$ and $W_{1+\infty}$ algebras [10]. For completeness we also describe the super $W_\infty$ [11] as well as the topological $W_\infty$ algebras [12]. Contraction of super $W_\infty$ down to super $w_\infty$ algebra is discussed next. We then describe the analog of the 2D Lagrangian $L = \frac{1}{2} \sqrt{-hh^j} \partial_i \phi \partial_j \phi$ (in a gauge in which the Weyl symmetry is fixed) which has local $w_\infty$ symmetry, and is referred to as $w_\infty$ gravity. A similar construction of $W_\infty$ gravity [14] is also reviewed. We conclude by summarising some known facts about various applications of $W_\infty$ symmetry.

2. Virasoro Algebra

Let us recall how the Virasoro algebra arises in the context of a 2D conformal field theory of a free complex scalar $\phi$. The normal ordered energy momentum tensor is given by

$$T(z) = -: \partial \phi(z) \partial \phi^*(z):$$

$$\equiv - \lim_{z \to w} \left[ \partial \phi(z) \partial \phi^*(\omega) + \frac{1}{(z-w)^2} \right]. \quad (1)$$

The conformal spin $j$ of a (chiral) field $\Phi(z)$ is defined by

$$T(z) \Phi(w) \sim \frac{j \Phi}{(z-w)^2} + \frac{\partial \Phi}{z-w}. \quad (2)$$

Using the two-point function

$$\phi(z) \phi^*(w) \sim -log(z-w), \quad (3)$$

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a simple computation involving the use of Wick rules and Taylor expansion yields the following well known operator product expansion (OPE):

\[ T(z)T(w) \sim \frac{\partial T}{z-w} + \frac{2T}{(z-w)^2} + \frac{1}{(z-w)^4}. \]  

Comparing with (2), we see that \( T(z) \) indeed has conformal spin 2. In terms of the Fourier modes defined by

\[ L_n = \oint_C \frac{dz}{2\pi i} z^{n+1} T(z), \]  

one deduces the Virasoro algebra with central extension \( c = 2 \):

\[ [L_m, L_n] = (m-n)L_{m+n} + \frac{1}{12}(m^3 - m)\delta_{m+n,0}. \]  

(For a detailed description of this calculation, and of conformal field theories in general, see the excellent review by Ginsparg [15]). The associativity of OPE guarantees that the Jacobi identities are satisfied by this algebra. Replacing the last coefficient \( \frac{1}{12} \) by \( \frac{c}{24} \) one obtains the Virasoro algebra with central extension \( c \). The Jacobi identities are no longer guaranteed, but one checks that they are indeed still satisfied. At this stage, we may treat the Virasoro algebra in its own right, forgetting about the specific realisation which lead to it. Of course, there are many ways of arriving at the Virasoro algebra. We have chosen to describe it through a field theoretic realisation, because this furnishes an intuitive way of understanding higher spin generalizations, as we shall see below.

3. \( W_\infty \) and \( W_{1+\infty} \) Algebras

A generalization of the Virasoro algebra containing a spin-3 generator \( w \) as discovered in [16]. It is a nonlinear algebra since the commutator of two spin-3 generators yields, among other terms, a term which is quadratic in spin-2 generator. This is known as the \( W_3 \) algebra. A further generalization to \( W_N \) algebra where generators with integer spins 2, 3, ..., \( N \) occur was found in ref. [17]. There are various inequivalent ways of taking an \( N \to \infty \) limit of \( W_N \). A very simple one leading to a linear algebra, called \( w_\infty \) algebra was found in ref. [6]. All the nonlinearities of the \( W_N \) are washed away in this algebra, and it is indeed an infinite dimensional Lie algebra. Geometrical interpretation of \( w_\infty \) is that it is the algebra of symplectic-diffeomorphisms of a cylinder [1], as we shall see later.

A characteristic feature of \( w_\infty \) is that the commutator of a spin-i generator with a spin-j generator yields a spin \((i+j-2)\) generator alone, and the only possible central extension is in the Virasoro sector. A nontrivial generalization of it, called the \( W_\infty \) algebra, in which central terms arise in all spin sectors, and in which the commutator of a spin-i generator with a spin-j generator yields generators with spins \((i+j-2), (i+j-4), ...\), down to spin-2, was discovered in ref. [1]. An important feature of this algebra, besides containing the central
charges in all spin sectors, is that it is indeed an infinite dimensional Lie algebra, since all
the non-linearities of the $W_N$ algebra are absent, just as in the case of $w_\infty$. The construction
of this algebra has been discussed in great detail in ref. [1]. Below, we shall motivate it by
considering its field theoretical realisation, as we did for the Virasoro algebra in the previous
section.

A natural way to generalize the construction of the previous section is to introduce
higher derivative analogs of (1) as follows [18]:

$$V_i(z) = \sum_{k=0}^{i} \alpha_k^i: \partial^{k+1}\varphi(z)\partial^{i+1-k}\varphi^*(z):,$$

where $\alpha_k^i$ are for the moment arbitrary constant coefficients. Choosing $\alpha_0^i = -1$, yields an
expression for $V_0^i(z)$ which coincides with the energy momentum tensor $T(z)$. The next
bilinear current contains terms of the form

$$V^1 := \partial \varphi \partial^2 \varphi^*, \quad \partial^2 \varphi \partial \varphi^*:$$

The sum of the two terms can be written as the derivative of the lower spin current $V_0$, but
the difference is an independent combination. Similarly, at the next level we have

$$V^2 := \partial \varphi \partial^3 \varphi^*, \quad \partial^2 \varphi \partial^2 \varphi^*, \quad \partial^3 \varphi \partial \varphi^*:$$

With a little effort one can show that one linear combination can be written as the derivative
the independent $V^1$ and another linear combination can be written as the double derivative of
$V^0$. Thus, one is left with a single independent linear combination at this level. This pattern
continues at higher levels: At level $i$ there are $(i+1)$ possible bilinear terms and $i$ combination
of them can be written as derivatives of lower level terms, while only one combination is
independent. This means that, with any nondegenerate choice of the coefficients $\alpha_k^i$ the
currents $V_i$ are guaranteed to give closed OPE algebra. By inspection we can see that the
OPE product of $V_i$ with $V_j$, will produce a sum of $V^{i+j-k}$, for $k = 2, 4, ..., i+j$ the last
term corresponding to the central extension. Defining the Fourier modes

$$V_{m}^\ell = \oint_C \frac{1}{2\pi i} V_{\ell}(z)z^{m+\ell+1},$$

one finds an algebra of the form $[V_i^m, V_j^n] \sim V^{i+j}_{m+n} + V^{i+j-2}_{m+n} + ..., with appropriate coefficients,
with “spin” decreasing all the way down to zero. (We shall be more precise about the con-
formal spin of the fields $V_i(z)$ below). The last term, which is the central extension, will
turn out to have a specific value due to the specific choice of realisation involving a single
complex scalar. As before, assigning an arbitrary value to the central extension, one finds
that the Jacobi identities are still satisfied by the algebra. Thus, as in the case of Virasoro
algebra, we have an infinite dimensional algebra in its own right, regardless of the particular realisation that lead to it. The algebra obtained in this manner is the \( W_\infty \) algebra \[1\].

Determining the structure constants is by no means an easy task. One guiding consideration is to work in a basis in which the central term is diagonal in spin, i.e. nonvanishing only between any pair of equal spin generators. It turns out that, this requirement fixes the coefficients \( \alpha_{im} \) up to an overall \( i \)-dependent scalings. With a particular choice of such an overall scaling these coefficients are given by \[18\]

\[
\alpha_{ik} = \left( -1 \right)^{k+1} 2^{-i-1} (i+2)! \left( \begin{array}{c} i+1 \\ k \end{array} \right) \left( \begin{array}{c} i+1 \\ k+1 \end{array} \right),
\]

and the \( W_\infty \) algebra takes the form \[1\]

\[
[V^i_m, V^j_n] = \sum_{\ell \geq 0} g^{ij}_{2\ell}(m, n) V^{i+j-2\ell}_{m+n} + c_i(m) \delta^{ij} \delta_{m+n,0},
\]

where

\[
c_i(m) = c \frac{2^{2i-3}!(i+2)!}{(2i+1)!!(2i+3)!!} \prod_{k=-i-1}^{i+1} (m+k)
\]

Although it is helpful to examine the OPE products of the currents \( V^i(z) \) to determine the structure constants \( g^{ij}_{2\ell} \), it is more advantageous to directly impose the Jacob identities on the algebra (10). In fact, that is what was originally done in ref. \[1\], where it was found that

\[
g^{ij}_{\ell}(m, n) = \frac{1}{2(\ell+1)!} \phi^{ij}_{\ell} N^{ij}_{\ell}(m, n),
\]

where

\[
\phi^{ij}_{\ell} = 4F_3 \left[ \begin{array}{c} -\frac{1}{2}, \frac{3}{2}, -\ell - \frac{1}{2}, -\ell \\ -i - \frac{1}{2}, -j - \frac{1}{2}, i + j - 2\ell + \frac{5}{2} + 1 \\ \end{array} \right],
\]

\[
N^{ij}_{\ell}(m, n) = \sum_{k=0}^{\ell+1} (-1)^k \left( \begin{array}{c} \ell + 1 \\ k \end{array} \right) [i+1+m]_{\ell+1-k} [i+1-m]_{k} [j+1+n]_{k} [j+1-n]_{\ell+1-k}.
\]

and \( [a]_n \equiv a!/(a-n)! \), while the hypergeometric function \( 4F_3(z) \) is defined by

\[
4F_3 \left[ \begin{array}{c} a_1, a_2, a_3, a_4 \\ b_1, b_2, b_3 \end{array} ; z \right] = \sum_{n=0}^{\infty} \frac{(a_1)_n(a_2)_n(a_3)_n(a_4)_n}{(b_1)_n(b_2)_n(b_3)_n} \frac{z^n}{n!},
\]

where \( (a)_n \equiv (a+n-1)!/(a-1)! \). The sum terminates if any one of the parameters \( a_1, ..., a_4 \) is zero or a negative integer. The function \( N^{ij}_{\ell}(m, n) \) are related to the SL(2,R) Clebsch-Gordan coefficients, while \( \phi^{ij}_{\ell} \) are formally related to Wigner 6-j symbols \[1\].
It is worth mentioning that while \( V^0 \) and \( V^1 \) correspond to true primary fields of conformal spin 2 and 3, respectively, \( V^i \) for \( i \geq 2 \) correspond to quasi-primary fields of spin \( i+2 \) since lower spin terms occur in the commutator \([V_m^i, V_n^j]\) in that case. The choice of basis is relevant to this fact. In another basis, \( V^i(z) \) may not be even quasi-primary. Also, one can define primary fields for all \( i \geq 2 \) as well, but then the algebra becomes nonlinear [19].

The \( W_\infty \) algebra can be enlarged to contain a spin-1 generator. The resulting algebra is called \( W_{1+\infty} \) algebra, and has the following form [10]

\[
[V^i_m, V^j_n] = \sum_{\ell=0}^{\infty} \tilde{g}^{ij}_{2\ell}(m, n) V^{i+j-2\ell}_{m+n} + \tilde{c}_i(m) \delta^{ij} \delta_{m+n,0},
\]

(15)

where

\[
\tilde{c}_i(m) = c \frac{2^{2i-3}(i+1)!^2}{(2i+1)!(2i+3)!} \prod_{k=-i-1}^{i+1} (m+k)
\]

(16)
and the structure constants are

\[
\tilde{g}^{ij}_{\ell}(m, n) = \frac{1}{2(\ell+1)!} \tilde{\phi}^{ij}_{\ell} N^{ij}_{\ell}(m, n),
\]

(17)

with \( N^{ij}_{\ell}(m, n) \) given as before, and

\[
\tilde{\phi}^{ij}_{\ell} = 4F3 \left[ \begin{array}{cccc}
-\frac{1}{2}, & \frac{1}{2}, & -\ell - \frac{1}{2}, & -\ell \\
-i - \frac{1}{2}, & -j - \frac{1}{2}, & i + j - 2\ell + \frac{5}{2} : 1
\end{array} \right].
\]

(18)

There exists a redefinition of the generators, involving finite number of terms at each spin level, which enables one to truncate the \( W_{1+\infty} \) algebra down to the \( W_\infty \) subalgebra [20]. Another interesting subalgebra of the \( W_{1+\infty} \) algebra, denoted by \( W^+_{1+\infty} \), is obtained by restricting the Fourier modes of the generators \( V^i_m \) to lie in the range \( m \geq -\ell - 1 \).

Currents obeying the \( W_{1+\infty} \) algebra can be constructed in terms of a complex fermion \( \psi(z) \) with the propagator

\[
\psi^*(z) \psi(w) \sim \frac{1}{z-w},
\]

(19)
as follows [21,22,11]

\[
\tilde{V}^i(z) = \sum_{k=0}^{i+1} \beta^i_k : \partial^k \psi^* \partial^{i+1-k} \psi :,
\]

(20)
where the coefficients \( \beta^i_k \) are given by

\[
\beta^i_k = (-1)^k \frac{(i+1)(k+1)}{(i+2)(i-k+1)} \alpha^i_k,
\]

(21)
with $\alpha^i_k$ as defined in (9). The currents $\tilde{V}^i(z)$ have (quasi-) conformal spins $s = i + 2$.

From the above realization of $W_{1+\infty}$ in terms of fermionic field $\psi(z)$, one can obtain a bosonic realisation of it in terms of a single scalar field $\phi(z)$, through the bosonization formulae

\begin{align*}
\psi(z) = : e^{\phi(z)} : \\
\psi^*(z) = : e^{-\phi(z)} : 
\end{align*}

The result is \[23\]

\[\tilde{V}^i(z) = \sum_{\ell=0}^{i+1} (-1)^{\ell} \frac{([i+1]!\ell)^2}{(i-\ell+1)! [2i]\ell} \partial^{\ell} P^{(i-\ell+1)}(z) \]

where

\[P^{(k)}(z) \equiv e^{-\phi(z)} \partial^k e^{\phi(z)} : \]

First few examples are

\begin{align*}
\tilde{V}^{-1} &= \partial \phi \\
\tilde{V}^0 &= \frac{1}{2} : (\partial \phi)^2 : \\
\tilde{V}^1 &= \frac{1}{3} : (\partial \phi)^3 : \\
\tilde{V}^2 &= \frac{1}{4} : (\partial \phi)^4 : -\frac{3}{20} : (\partial^2 \phi)^2 : + \frac{1}{10} : \partial \phi \partial^3 \phi : 
\end{align*}

4. Super-$W_\infty$ Algebra

There exists an $N = 2$ supersymmetric extension of $W_\infty$ algebra \cite{11}. More precisely, it is the superextension of the bosonic algebra $W_\infty \oplus W_{1+\infty}$, in which the following supercurrents are introduced \cite{11}

\[G^\alpha = \sum_{k=0}^{\alpha} \gamma^i_k : \partial^{\alpha+1-k} \phi^* \partial^k \psi^i : , \]

where

\[\gamma^i_k = (-1)^k \frac{2(k+1)}{(i+2)} \alpha^i_k , \]

with $\alpha^i_k$ as defined in (9). The currents $G^\alpha$ have (quasi-) conformal spins $s = \alpha + \frac{3}{2}$.

One finds that the currents (7), (20) and (26) obey the super-$W_\infty$ algebra, which, in terms
of the Fourier modes of the generators, is given by (10), (15) and the following relations [11]

\[
\{ \bar{G}_r^\alpha, G_s^\beta \} = \sum_{\ell=0}^{\infty} \left( b^{\alpha \beta}_\ell (r, s) V^{\alpha+\beta-\ell}_{r+s} + \tilde{b}^{\alpha \beta}_\ell (r, s) \bar{V}^{\alpha+\beta-\ell}_{r+s} \right) + \hat{c}_\alpha (r) \delta^{\alpha \beta} \delta_{r+s,0},
\]

\[
[V^i_m, G_r^\alpha] = \sum_{\ell=0}^{\infty} a^i_\ell (m, r) G^{\alpha+i-\ell+1}_{m+r},
\]

\[
[\bar{V}^i_m, G_r^\alpha] = \sum_{\ell=0}^{\infty} \tilde{a}^i_\ell (m, r) G^{\alpha+i-\ell+1}_{m+r},
\]

\[
[V^i_m, \bar{G}_r^\alpha] = \sum_{\ell=0}^{\infty} (-1)^{\ell+1} a^i_\ell (m, r) \bar{G}^{\alpha+i-\ell+1}_{m+r},
\]

\[
[\bar{V}^i_m, \bar{G}_r^\alpha] = \sum_{\ell=0}^{\infty} (-1)^{\ell+1} \tilde{a}^i_\ell (m, r) \bar{G}^{\alpha+i-\ell+1}_{m+r}.
\]

Explicit expressions for \( \hat{c}_\alpha (r) \) and the structure constants \( \tilde{g}, \ldots, \tilde{a} \) can be found in ref. [11]. The spectral flow in the above algebra, as well as its various truncations including the \( N = 1 \) supersymmetric truncation is also discussed in ref. [11].

One can choose other types of higher derivative bilinear currents giving rise to a super-\( W_\infty \) algebra which isomorphic to the above one. For example, one can choose \( \bar{V}^i(z) \sim \psi^* \partial^{i+1} \psi \). Although this gives rise to central terms between generators of different spin, this realisation may nonetheless have some advantages in dealing with certain problems [5,24].

The redefinition of the basis elements in \( W_{1+\infty} \) algebra has been discussed in ref. [20], where it was shown that a one parameter family of isomorphic algebras can be obtained in this way. One application of basis redefinition is to show that the spin-1 generator can be consistently truncated, as a consequence of which one is left with \( W_\infty \) algebra. Recently, these issues have been treated in great detail in ref. [25], where a one parameter family of isomorphic \( W_\infty \) algebras, denoted by super \( W_\infty(\lambda) \), were also obtained. The construction of ref. [25] uses an infinite set of differential operators, and enables them to compute explicitly the structure constants of the algebra for any value of the parameter \( \lambda \). These generators are of the form [25]

\[
V^i_\lambda (\Omega^{(i)}) = \sum_{k=0}^{2i+2} A^k(i, \lambda) (D^{2i+2-k} \Omega^{(i)}) D^k,
\]

where \( i = 0, \frac{1}{2}, 1, \ldots \), \( \Omega^{(i)}(z, \theta) \) are the generating superfunctions, \( D = \frac{\partial}{\partial \theta} - \theta \partial \) and the coefficients \( A^k(i, \lambda) \) can be found in ref. [25].

A geometric interpretation of \( W_{1+\infty} \) is that it is the algebra of smooth differential operators on a circle [20]. It can also be viewed as the universal enveloping algebra of the \( U(1) \) Kac-Moody algebra generated by \( \bar{V}^{-1}_m \equiv j_m \) modulo the ideal \( j_m j_n - j_{m+n} = 0 \) [20]. Similarly, \( W_\infty \) can be viewed as the universal enveloping algebra of the Virasoro algebra.
adding new generators of a given spin would correspond to a (quasi) primary conformal field [1].

In fact, those are precisely the one obtains an infinite dimensional algebra in which only positive spin generators occur. In fact, those are precisely the\[ W_\infty \] and \[ W_{1+\infty} \] algebras, respectively. Dramatic simplifications occur in the structure constants of \( U_L(SL(2,R)) \) algebra moded by the ideal \( C_2 + \frac{3}{10} = 0 \). However, this algebra, associated with the so called symplecton, singleton or metaleptic representation of \( SL(2,R) \), does not admit extension to a \( W \)-like algebra in which generators of a given spin would correspond to a (quasi) primary conformal field [1].

It would be interesting to apply the above considerations to the case of universal enveloping algebras of any Lie (super) algebra. In this context, see the interesting work of Fradkin and Linetsky [26]. See also ref. [27], where an \( SU(N) \) structure is introduced by adding new generators \( V_{n}^{i,a} \) where the index \( a = 1, ..., p^2 - 1 \) labels the adjoint representation of \( SU(p) \). The generator \( V_{n}^{-1,a} \) obeys the affine \( \hat{su}(p) \) algebra with level \( k \). The \( N = 2 \) supersymmetric version of this algebra, contains the generators \( V_{n}^{i}, \bar{V}_{n}^{i}, V_{n}^{i,a}, G_{r}^{\alpha,A} \) and \( G_{\bar{r}}^{\alpha,A} \), where the index \( A = 1, ..., p \) labels the fundamental representation of \( SU(p) \). In the bosonic case, \( V^{i} \) and \( \bar{V}_{i,a} \) can be combined to represent the \( p^2 \) dimensional representation of \( U(p) \), denoted by \( V^{i,r} \), \( (r = 0, 1, ..., p^2 - 1) \), and the currents (7) are now replaced by

\[
V^{i,r}(z) = \sum_{A,B=1}^{p} \sum_{k=0}^{i} \alpha_{k}^{i} T_{AB}^{r} : \partial^{k+1} \varphi^{A}(z) \partial^{i+1-k} \varphi^{B}(z) :, \quad (30)
\]

where \( T_{AB}^{r} \) are the generators of \( U(p) \) in the fundamental representation. The algebra generated by these currents is called \( W_{\infty}^{p} \), and it can be viewed as the large \( N \) limit of the Grassmannian coset model \( G_{N}(p) = SU(p+1)_{N}/SU(p)_{N} \otimes U(1) \) [28]. The large \( p \) limit of \( W_{\infty}^{p} \) takes the form [28]

\[
[V_{i,m}^{\vec{k}}, V_{n}^{j,\vec{\ell}}] = [(j+1)m - (i+1)n]V_{m+n}^{i+j,\vec{k}+\vec{\ell}} + \vec{k} \times \vec{\ell} \cdot V_{m+n}^{i+j+1,1,\vec{k}+\vec{\ell}}, \quad (31)
\]

where \( \vec{k} = (k_1, k_2) \) and \( \vec{\ell} = (\ell_1, \ell_2) \). One can show that this algebra describes symplectic diffeomorphisms in four dimensions [28].

5. Topological \( W_{\infty} \) Algebra

The \( N = 2 \) super-\( W_{\infty} \) algebra described above can be twisted to give a topological algebra which we call \( W_{\infty}^{top} \) [12]. The idea is to identify one of the fermionic generators as
the nilpotent BRST charge \( Q \), and to define bosonic generators which can be written in the form \( \hat{V}_m^i = \{ Q, \text{something} \} \). This is the higher-spin generalization of the property that holds for the energy-momentum tensor of a topological field theory. A suitable candidate for the BRST charge is

\[
Q = -\bar{G}_0 \frac{1}{2}.
\]  

(32)

Inspired by the twisting of the \( N = 2 \) super-Virasoro algebra, we then define the generators of \( W_\infty^{\text{top}} \) to be \( G_{m+\frac{1}{2}}^i \) and define \( \hat{V}_m^i \) as follows

\[
\hat{V}_m^i = -\{Q, G_{m+\frac{1}{2}}^i \}.
\]  

(33)

It can be easily shown that these generators form a closed algebra. Finding the structure constants requires more work. Towards that end, from (33) we first find

\[
\hat{V}_m^i = V_m^i + \tilde{V}_m^i - 2i(m+i+1)\frac{2i}{2i+1}V_m^{i-1} + \frac{(2i+2)(m+i+1)}{2i+1}V_m^{i-1}. \]  

(34)

With respect to the new energy-momentum tensor \( \hat{V}_0^0(z) \) the (quasi)conformal spin of the fields \( \hat{V}_i^0(z) \) is shifted up by a half. After some algebra, we then find that the \( W_\infty^{\text{top}} \) algebra takes the form [12]

\[
\left[ \hat{V}_m^i, \hat{V}_n^j \right] = \sum_{\ell \geq 0} \hat{g}^{ij}_\ell (m, n) \hat{V}_{m+n}^{i+j-\ell},
\]

\[
\left[ \hat{V}_m^i, G_{n+\frac{1}{2}}^j \right] = \sum_{\ell \geq 0} \hat{g}^{ij}_\ell (m, n) G_{m+n+\frac{1}{2}}^{i+j-\ell},
\]  

(35)

\[
\{ G_{m+\frac{1}{2}}^i, G_{n+\frac{1}{2}}^j \} = 0,
\]

where

\[
\hat{g}^{ij}_\ell (m, n) = a^{ij}_{\ell+1} (m, n + \frac{1}{2}) + \tilde{a}^{ij}_{\ell+1} (m, n + \frac{1}{2}) - \frac{2i(m+i+1)}{2i+1}a^{i-1,j}_{\ell} (m, n + \frac{1}{2}) + \frac{(2i+2)(m+i+1)}{2i+1}\tilde{a}^{i-1,j}_{\ell} (m, n + \frac{1}{2}).
\]  

(36)

Note that the structure constants for \( \left[ \hat{V}_m^i, G_{n+\frac{1}{2}}^j \right] \) are the same as those for \( \left[ \hat{V}_m^i, \hat{V}_n^j \right] \), and that the algebra is centerless. Furthermore, one can show that \( \hat{V}_m^i \)'s generate the diagonal subalgebra in \( W_\infty \times W_{1+\infty} \) with vanishing central charge. A field theoretic realisation of \( W_\infty^{\text{top}} \) is given in ref. [12].

6. Contraction of \( W_\infty \) Down to \( W_\infty^{1+\infty} \)

There exists an interesting contraction of all the \( W \) algebras discussed above. Consider for simplicity \( W_{1+\infty} \) algebra. A rescaling of the form \( \hat{V}_m^i \to q^{-i}v_m^i \) followed by the limit \( q \to 0 \) yields the following remarkably simple result:

\[
[v_m^i, v_n^j] = [(j+1)m-(i+1)n]v_m^{i+j} + \frac{c}{12} (m^3 - m) \delta^{i,0} \delta^{j,0} \delta_{m+n,0}.
\]  

(37)
where $v_m^i$ now represent the Fourier modes of a true primary field of conformal spin $(i+2)$, and $i \geq -1$ and $-\infty < m < \infty$. This is known as the $w_\infty$ algebra. Note that all the lower spin terms on the right hand side are now absent, and that all the central charges have also disappeared except in the Virasoro sector. The above algebra has the nice geometric interpretation \[6\] as the symplectic diffeomorphisms of a cylinder [1]. (In two dimensions symplectic diffeomorphisms coincide with are area-preserving diffeomorphisms). In general, the symplectic diffeomorphisms of a $2n$ dimensional symplectic manifold are those which leave the symplectic structure (a non-degenerate 2-form) invariant, and they are easily shown to be generated by vector fields of the form $\xi^a = \Omega^{ab} \partial_b \Lambda$, where $\Omega^{ab}$ are the components of the inverse of the symplectic 2-form, and $\Lambda$ is an arbitrary function. The generator of the symplectic-diffeomorphisms can be written as

$$v_\Lambda = \xi^a(\Lambda) \partial_a = \Omega^{ab} \partial_b \Lambda \partial_a$$

These generators obey the algebra $[v_{A_1}, v_{A_2}] = v_{A_3}$, where the composition parameter $\Lambda_3$ is given by the Poisson bracket $\Lambda_3 = \Omega^{ab} \partial_b \Lambda_1 \partial_a \Lambda_2$. Let us now consider the case of a cylinder and define a complete set of functions on the cylinder $S^1 \times \mathbb{R}$ with coordinates $0 \leq x \leq 2\pi$ and $-\infty \leq y \leq \infty$ as follows

$$u_{\ell m} = -ie^{imx}y^{\ell+1}$$

Expanding $\Lambda(x, y) = \sum_{\ell,m} \Lambda_{\ell m} u_{\ell m}$, and defining the Fourier modes of the generators as $v_\Lambda = \sum_{\ell,m} \Lambda_{\ell m} v_{\ell m}$, one finds that

$$v_{\ell m} = \Omega^{ab} \partial_b u_{\ell m} \partial_a$$

It can be easily shown that these generators obey the centerless part of the $w_\infty$ algebra (37). In ref. [29], on quite general grounds, it was found that the symplectic-algebra of a Riemann surface of genus $g$ admitted a $2g$ parameter central extension. Since the cylinder has genus one, we expect that the algebra admits a one parameter central extension. From the formula in ref. [29], one can check that it indeed has the form given in (37). We next consider the contraction of the $N = 2$ super $W_\infty$ algebra down to the corresponding super $w_\infty$ algebra. To this end we perform the rescalings

$$V_{m}^\ell \to q^{-\ell}(v_{m}^\ell - \frac{1}{2}q^{-1}J_{m}^\ell)/2,$$

$$\tilde{V}_{m}^\ell \to q^{-\ell}(v_{m}^\ell + \frac{1}{2}q^{-1}J_{m}^\ell)/2,$$

$$G^\alpha \to q^{-\ell}G^\alpha \sqrt{2},$$

$$\tilde{G}^\alpha \to q^{-\ell}\tilde{G}^\alpha \sqrt{2}.$$
Taking the limit \( q \to 0 \) in the \( N = 2 \) super \( W_\infty \) algebra given in (10), (15) and (28) yields the result [30]\(^\dagger\)

\[
[v^i_m, v^j_n] = [(j + 1)m - (i + 1)n]v^{i+j} + \frac{c}{8}(m^3 - m)\delta^{i,0}\delta^{j,0}\delta_{m+n,0},
\]

\[
[v^i_m, J^{j-1}_n] = [jm - (i + 1)n]J^{i+j-1}_{m+n},
\]

\[
\{G^\alpha_r, G^\beta_s\} = 2v^{\alpha+\beta}_{r+s} - 2[(\beta + \frac{1}{2})r - (\alpha + \frac{1}{2})s]J^{\alpha+\beta-1}_{r+s} + \frac{c}{2}(r^2 - \frac{1}{4})\delta^{i,0}\delta^{j,0}\delta_{r+s,0},
\]

\[
[v^i_m, G^\alpha_r] = [(\alpha + \frac{1}{2})m - (i + 1)r]G^{\alpha+i}_{m+r},
\]

\[
[v^i_m, G^\alpha_r] = [(\alpha + \frac{1}{2})m - (i + 1)r]G^{\alpha+i}_{m+r},
\]

\[
[J^{i-1}_m, G^\alpha_r] = G^{i+\alpha}_{m+r},
\]

\[
[J^{i-1}_m, G^\alpha_r] = -G^{\alpha+i}_{m+r},
\]

\[
[J^{i-1}_m, J^{j-1}_n] = \frac{c}{2}m\delta^{i,0}\delta^{j,0}\delta_{m+n,0}
\]

In fact, this is the algebra of symplectic diffeomorphisms on a \((2,2)\) superplane, i.e. a plane of two bosonic and two fermionic dimensions [30]. The \( N = 1 \) super \( w_\infty \) algebra can be obtained from the above algebra by truncation, or directly as an algebra of the symplectic diffeomorphisms of a \((2,1)\) superplane [30,31]. For \( i = j = \alpha = \beta = 0 \), the algebra (42) reduces to the well known \( N = 2 \) superconformal algebra.

7. \( w_\infty \) Gravity

It is natural to look for Lagrangian field theories in which the \( W \) symmetries of the kind we have discussed so far are realised. In the case of Virasoro symmetry, the classic example is the Polyakov type Lagrangian

\[
\mathcal{L} = -\frac{1}{4}\sqrt{-h}h^{ij}\partial_i\phi\partial_j\phi,
\]

where \( h^{ij} \) is the inverse of the worldsheet metric \( h_{ij} \), \((i,j = 0,1)\), \( h = d\text{eth}_{ij} \) and \( \phi \) is a real scalar. This Lagrangian clearly possesses the 2D diffeomorphism and Weyl symmetries. In a conformal gauge the residual symmetry becomes the Virasoro symmetry. Generalizations of this Lagrangian in which \( W \) symmetry is realised, have been constructed, and although they are not purely gauge theories of \( W \) algebras, they have been called \( W \) gravity Lagrangians, motivated by the fact that they are invariant under local \( W \) symmetries. Here, we shall present selected few examples of these Lagrangian field theories.

To generalize (43), it is convenient to first fix its Weyl symmetry so that we are left with diffeomorphism symmetry. A convenient gauge choice in which \( h \) is a perfect square is as follows

\[
h_{+-} = \frac{1}{2}(1 + h_{++}h_{--}),
\]

\(^\dagger\) To obtain this result, it is sufficient to know that \( \beta^{ij} = \bar{g}^{ij} = (j+1)m-(i+1)n \), \( \epsilon^{\alpha\beta} = \bar{\delta}^{\alpha\beta} = 1 \), \( b^{\alpha\beta}(r,s) = -(\alpha\beta)(r,s) = [(\frac{1}{2}r+s) - (\frac{1}{2}r+s)] \), \( a^{\alpha\alpha} = -\bar{a}^{\alpha\alpha} = \frac{1}{4}, \) \( a^{r}(m,r) = \bar{a}^{r}(m,r) = \frac{1}{4}(2m+1) - (2i+2)r \).
where the light-cone coordinate bases defined by $x^\pm = (x^0 \pm x^1)$ is used. In this gauge, the Lagrangian (43) reduces to

$$\mathcal{L} = \frac{1}{2} (1 - h_{++} h_{--})^{-1} (\partial_+ \phi - h_{++} \partial_- \phi)(\partial_- \phi - h_{--} \partial_+ \phi). \quad (45)$$

It turns out to be very useful to rewrite this Lagrangian in the following first order form

$$\mathcal{L} = -\frac{1}{2} \partial_+ \phi \partial_- \phi - J_+ J_- + J_+ \partial_- \phi - J_- \partial_+ \phi - \frac{1}{2} h_{++} J_+^2 - \frac{1}{2} h_{++} J_-^2, \quad (46)$$

where $J_\pm$ are auxiliary fields which obey the field equations

$$J_+ = \partial_+ \phi - h_{++} J_-,$$
$$J_- = \partial_- \phi - h_{--} J_+,$$

These equations define a set of nested covariant derivatives [32]. Solving for $J_\pm$ and substituting into (46) indeed yields (45). Thus the two Lagrangians are classically equivalent, though in principal they may be quantum inequivalent. The action of the Lagrangian (46) is invariant under 2D diffeomorphism transformations, with a general parameter $k_+(x^+, x^-)$, given by [13]

$$\delta \phi = k_+ J_-,$$
$$\delta h_{++} = \partial_+ k_+ - h_{++} \partial_- k_+ + k_+ \partial_- h_{++},$$
$$\delta h_{--} = 0,$$
$$\delta J_- = \partial_-(k_+ J_-),$$
$$\delta J_+ = 0,$$

and 2D diffeomorphisms with parameters $k_-(x^+, x^-)$, which can be obtained from the above transformations by changing $+ \leftrightarrow -$ everywhere.

The $W$ symmetric generalization of (46) is now remarkably simple. With the further generalization to the case in which the fields $\phi$ and $J_\pm$ take their values in the Lie algebra of $SU(N)$ the answer can be written as follows [13]

$$\mathcal{L} = \text{tr} \left( -\frac{1}{2} \partial_+ \phi \partial_- \phi - J_+ J_- + J_+ \partial_- \phi + J_- \partial_+ \phi \right)$$
$$- \sum_{\ell \geq 0} \frac{1}{\ell + 2} A_{++} \ell \text{tr} J_-^\ell J_+^{\ell+2} - \sum_{\ell \geq 0} \frac{1}{\ell + 2} A_{--} \ell \text{tr} J_+^\ell J_-^{\ell+2}. \quad (49)$$

Note that $A_{0+} = h_{++}$ and $A_{0-} = h_{--}$. The equations of motion for the auxiliary fields now reads

$$J_+ = \partial_+ \phi - \sum_{\ell \geq 0} A_{++} \ell J_-^{\ell+1},$$
$$J_- = \partial_- \phi - \sum_{\ell \geq 0} A_{--} \ell J_+^{\ell+1}. \quad (50)$$
The Lagrangian (49) possesses the W-symmetry with parameters $k_{+\ell}(x^+, x^-)$ that generalize (48) as follows

$$\delta \phi = \sum_{\ell \geq -1} k_{+\ell} J_{\ell+1}$$

$$\delta A_{+\ell} = \partial_{+} k_{+\ell} - \sum_{j=0}^{\ell} [(j+1) A_{+j} \partial_{-} k_{+\ell-j} - (\ell-j+1) k_{+\ell-j} \partial_{-} A_{+j}]$$

$$\delta A_{-\ell} = 0$$

$$\delta J_{-} = \sum_{\ell \geq -1} \partial_{-} [k_{+\ell} (J_{-})^{\ell+1}]$$

$$\delta J_{+} = 0,$$

and W transformations with parameters $k_{+\ell}(x^+, x^-)$ which can be obtained from above by the replacement $+ \leftrightarrow -$ everywhere. It is important to note that we must set $k_{-1\pm} = -\frac{1}{N} \sum_{\ell \geq 1} k_{\pm\ell} \text{tr}(J_{\pm})^{\ell+1},$ (52)

to ensure the tracelessness $\delta \phi$ and $\delta J_{\pm}$ in the transformation rules above.

The Lagrangian (49) has also Stueckelberg type shift symmetries which arise due to the fact that for $SU(N)$, only $(N-1)$ Casimirs of the form $\text{tr}(J_{\pm})^{\ell+2}$ are really independent, while the rest can be factorise into products of these Casimirs. There is not a simple way to write down the resulting symmetries for any value of $N$; one must consider each case separately. To see how this works, consider for example, the case of $SU(3)$. It is sufficient to consider, say, the left-handed fields, since all results hold independently also for the right-handed fields. We observe that, for $SU(3)$, the terms containing the left-handed gauge fields in the Lagrangian can be written as (dropping the handedness subscripts)

$$\mathcal{L}(A) = -\frac{1}{2} A_0 \text{tr}J^2 - \frac{1}{3} A_1 \text{tr}J^3 - \sum_{\ell \geq 2} \frac{1}{\ell + 2} A_{\ell} X_{\ell} \text{tr}J^2 + \sum_{\ell \geq 3} \frac{1}{\ell + 2} (A_{\ell} Y_{\ell} \text{tr}J^3,$$ (53)

where $X$ and $Y$ are themselves products of $\text{tr}J^2$ and/or $\text{tr}J^3$ factors. For example, since $\text{tr}J^4 = \frac{1}{2}(\text{tr}J^2)^2$, this implies that $X_2 = \frac{1}{2} \text{tr}J^2$. It is now easy to see that the Lagrangian (49) has the following symmetry

$$\delta A_0 = -\sum_{\ell \geq 2} \frac{2}{\ell + 2} \alpha_{\ell} X_{\ell},$$

$$\delta A_1 = -\sum_{\ell \geq 3} \frac{3}{\ell + 2} \alpha_{\ell} Y_{\ell},$$

$$\delta A_{\ell} = \alpha_{\ell}, \ \ell \geq 2,$$ (54)

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where $\alpha_\ell(x^+, x^-)$ is an arbitrary parameter. This symmetry can be used to set $A_{\pm \ell} = 0$ for $\ell \geq 2$. If we also set $k_{\pm \ell} = 0$ for $\ell \geq 2$ then from (51) we see that $A_{\pm \ell}$, $\ell \geq 3$ remain zero for all $k_{\pm 0}$ and $k_{\pm 1}$ transformations, while $A_{\pm 2}$ remains zero provided we make a compensating $\alpha_{\pm 2}$ transformation given by

$$
\alpha_{\pm 2} = 2A_{\pm 1}\partial_\mp k_{\pm 1} - 2k_{\pm 1}\partial_\mp A_{\pm 1}.
$$

(55)

This means that, while the transformation rule for $A_{\pm 1}$ receives no modifications, the transformation rule for $A_{\pm 0}$ does receive a modification given by

$$
(\delta A_{\pm 0})_{\text{extra}} = -\frac{1}{2}\alpha_2 X_2,
$$

(56)

where, as was noted above, we have $X_2 = \frac{1}{2}\text{tr}J^2$. Putting all these together one finds the gauge theory of nonchiral $W_3$, which can be found in ref. [13], which appears to be equivalent to that of ref. [32]. (In the latter reference, the Lagrangian exhibits the spin-2 symmetry in a covariant fashion, whereas in our case all spin symmetries, including spin-2, are treated on equal footing.) Our Lagrangian has additional symmetries, called the $\beta$ and $\gamma$ symmetries, whose origin and role is not quite clear so far.

There exists an interesting chiral truncation of the $W$ gravity theory discussed above. It is achieved by setting $A_{+\ell} = 0$. In that case from (50) we have $J_+ = \partial_+ \phi$ and $J_- = \partial_- \phi - \sum_{\ell \geq 0} A_- \partial_- \text{tr}(\partial_+ \phi)^{\ell+1}$. In this case, it is more convenient to work in second order formalism. Thus, substituting for $J_{\pm}$ into the Lagrangian (49), we obtain [13]

$$
\mathcal{L} = \frac{1}{2}\text{tr}\partial_+ \phi \partial_- \phi - \sum_{\ell \geq 0} \frac{1}{\ell + 2} A_\ell \text{tr}(\partial_+ \phi)^{\ell+2},
$$

(57)

where we have used the notation $A_{-\ell} = A_\ell$. This Lagrangian has the following symmetry [13]

$$
\delta \phi = \sum_{\ell \geq -1} k_\ell (\partial_+ \phi)^{\ell+1}
$$

$$
\delta A_\ell = \partial_- k_\ell - \sum_{j=0}^{\ell+1} [(j+1)A_j \partial_+ k_{\ell-j} - (\ell - j + 1)k_{\ell-j} \partial_- A_j]
$$

(58)

The Lagrangian (57) has also the appropriate Stueckelberg symmetry. Using this symmetry one can obtain [13] the chiral $W_3$ gravity of ref. [33]. Note that the interaction term in this Lagrangian has the form of a gauge field $\times$ conserved current $\frac{1}{\ell + 2}\text{tr}(\partial_+ \phi)^{\ell+2}$. It is important to note that the OPE of these currents do not form a closed algebra, while they do close with respect to Poisson bracket. Hence, the $w_\infty$ symmetry described above is a classical symmetry, as expected.

The nonchiral $w_\infty$ gravity described in (49), (50) and (51) is a light-cone formulation of a covariant $w_\infty$ gravity which needs to be defined. One way to do this is to introduce extra
degrees of freedom at each spin level, removable by newly introduced symmetries, so that the
gauge fields transform as irreducible representations of the 2D Lorentz group. Thus, spin $j$
gauge field can be represented by $\epsilon_{\mu}^{a_1...a_{j-1}}$ which is symmetric and traceless in the
indices $a_1...a_{j-1}$. In the light-cone basis, this means that they are all +’s or -’s. Since, the
index $\mu$ takes the values $\pm$, the gauge fields contain four components at each spin level.
Therefore, passage from the nonchiral theory where the gauge fields have two components
to a covariant theory, one needs to introduce two extra components at each spin level. They
can be removed by the Lorentz and Weyl type symmetries present at each spin level. The
action and transformation rules of covariant $w_\infty$ gravity obtained in this way can be found
in ref. [34]. The same results were already derived in ref. [35] by gauging of the $w_\infty$ algebra.

It should be noted that $w_\infty$ symmetry is realised nonlinearly on the scalar field $\phi$ in
$w_\infty$ gravity theories described above. This situation can be described in terms of a coset
construction which provides a geometrical picture. We have already noted that $w_{1+\infty}$ algebra
can be linearly realised as the algebra of symplectic diffeomorphisms of a cylinder. Let the
coordinates of the cylinder be $(x \equiv x^+, y)$. From (39) we see that $y$ independent functions
are the spin-1 generator $v_m^{-1}$, which is the only generator to be added to $w_\infty$ to obtain
$w_{1+\infty}$. Thus, suppressing the irrelevant $x^-$ dependence, the $y$-independent scalar
field $\phi(x)$ can be considered as parametrising the coset $w_{1+\infty}/w_\infty$. (For simplicity, we
consider the case of a single scalar. The general case follows straightforwardly.) For a
general noninfinitesimal symplectic transformation, the action on a scalar field $f(x, y)$ is
given by

$$f \rightarrow \tilde{f} = e^{Ad_\Lambda}f = f + \{\Lambda, f\} + \frac{1}{2!}\{\Lambda, \{\Lambda, f\}\} + ...,$$

where $\Lambda(x, y)$ is the arbitrary parameter, and the Poisson bracket is defined in $(x, y)$ space.
The coset representative is obtained by setting $\Lambda(x, y) = \phi(x)$. Considering the action of
the group element with an infinitesimal parameter $\lambda$, the change in $\phi$ is found by standard
methods to be

$$\delta \phi = \left( e^{-Ad_\phi} \lambda \right)_{y=0},$$

where setting $y = 0$ amounts to restriction to the coset direction. For the choice $\lambda = k_\ell y^{\ell+1}$,
this transformation rules yields the promised result: $\delta \phi = k_\ell (\partial_+ \phi)^{\ell+1}$. In a similar fashion, it
can be shown that the transformation rule of the scalar in the nonchiral theory can be viewed
as a special symplectic diffeomorphism of a four dimensional manifold with coordinates
$x^+, x^-, y, \bar{y}$ [13].

From the commutation rules (37), it can be seen that the coset space $w_{1+\infty}/w_\infty$ is not
a symmetric space. In particular, the coset generators do not form a representation of the
subalgebra $w_\infty$. Hence, some features of the general theory of nonlinear realisations are going
to be non-standard. Instead, one could start from the symmetric coset space $w_{1+\infty}^+ / Vir^+$
[36], where the $+$ on $w_{1+\infty}$ indicates the restriction of the Fourier modes of the generators
$v_m^\ell$ to $m \geq -\ell - 1$, while the $+$ on Vir denotes the restriction of the Virasoro generators $v_m^0$
to \( m \geq -1 \). Here, the coset generators do form a representation of the subalgebra \( \text{Vir}^+ \), but the coset is infinite dimensional, which means the introduction of infinitely many Goldstone fields. It turns out that, all but those which lie at the “left edge of the wedge”, i.e. \( \phi^\ell_{-\ell-1} \) can be eliminated by the imposition of suitable constraints \([36]\). In particular, the transformation of the scalar \( \phi^{-1}_0 \) turns out to be exactly the nonlinear transformation rule of the scalar \( \phi \) we encountered above in \( w_\infty \) gravity \([36]\). However, now there is an infinite chain of scalar fields, all of which are arising in a geometrical fashion, and this may open up interesting new possibilities with regard to the field theoretic realisation of \( w_\infty \) symmetry.

So far we have described the classical \( w_\infty \) gravity. Recently, quantum \( w_\infty \) gravity was studied, and it was found that infinitely many counterterms are needed in order to remove matter dependent anomalies \([37]\). Remarkably, these counterterms correspond precisely to a renormalisation of the classical \( w_\infty \) currents to quantum \( W_\infty \) currents. For example, the classical spin-4 current \( \frac{1}{4!}(\partial \phi)^4 \) renormalises to the quantum spin-4 current of the form given in (25). In ref. \([37]\), it is further shown that the matter independent gauge anomalies are cancelled by the ghost contributions.

8. \( W_\infty \) Gravity

The gauge field \( \times \) conserved current form of the chiral \( w_\infty \) gravity Lagrangian discussed above suggests a \( W_\infty \) gravity Lagrangian of similar form. To this end, let us consider the following Lagrangian,

\[
\mathcal{L} = \partial_+ \phi^* \partial_- \phi + \sum_{i \geq 0} A_i V^i, \tag{61}
\]

where \( V^i \) now represent the \( W_\infty \) current given in (7). This Lagrangian is indeed \( W_\infty \) symmetric. The transformation rule for the scalar can be derived by considering its Poisson bracket with the current. Equivalently, when appropriate (see below), one can instead use the OPE rules. For example, using OPE, the transformation rule for the scalar field can be obtained as follows

\[
\delta \phi = \oint \frac{dz}{2\pi i} k_i(z) V^i(z) \phi = \sum_{i \geq 0} \sum_{k=0} a_i^k \partial_+^k (k_i \partial_+^{i+1-k} \phi), \tag{62}
\]

where the basic OPE rule (3) has been used. To derive the transformation property of the gauge fields, it is convenient to first derive the transformation of the current \( V^i \) and then demand the invariance of the Lagrangian (61). To this end we need to know the OPE of two currents. This is known to be \([1,10,14]\]

\[
V^i(z)V^j(w) \sim -\sum_{\ell=0}^\infty f_{ij}^{\ell} \partial_z \partial_w \left( \frac{V^i+j-2\ell(w)}{z-w} \right) + \text{central terms}, \tag{63}
\]
where
\[ f_{ij}^k (m, n) = \frac{1}{2(\ell+1)!} \phi_{ij}^k M_{ij}^{\ell+1}(m, n), \]
\[ M_{ij}^{\ell+1}(m, n) \equiv \sum_{k=0}^{\ell+1} (-1)^k \binom{\ell+1}{k} (2i-\ell+2)k[2j+2-\ell]k_n^{\ell+1-k}m^{\ell+1-k}n^k. \]

Note that \( M_{ij}^{\ell+1}(m, n) \) is obtained from \( N_{ij}^{\ell+1}(m, n) \) by picking up the leading component, with total degree \((\ell+1)\) in \( m \) and \( n \).

To derive the classical transformation rule of the current \( V^i \) which is bilinear in scalar field, we must consider only the singular and field dependent terms in the OPE above, since that would correspond to a single contraction of the scalar field, giving results equivalent to those obtainable by using the Poisson bracket. (The central terms which we need to omit correspond to double contraction of the scalar fields.) With this in mind, we obtain
\[ \delta V^i = \oint \frac{dz}{2\pi i} k_i(z)V^i(z)V^j(\omega) = \sum_{\ell \geq 0} \int dz k_j(z) f_{2\ell}^{ij}(\partial_\omega, \partial_z) \left( \delta(z-\omega)V^{i+j-2\ell}(\omega) \right). \]

The invariance of the Lagrangian (61) is now seen to be achieved by letting the gauge fields to transform as
\[ \delta A_i = \partial_{-} k_i + \sum_{\ell \geq 0} \sum_{j=0}^{i+2\ell} f_{2\ell}^{ji,j+2\ell}(\partial_A, \partial_k) A_j k_{i-j+2\ell}, \]
where \( \partial_A \) and \( \partial_k \), which replace \( m \) and \( n \), are the \( \partial_+ \) derivatives acting on \( A \) and \( k \), respectively. Writing \( \delta A_i = \partial_{-} k_i + \hat{\delta} A_i \), we observe that \( \hat{\delta} A_i \) is a co-adjoint transformation of \( A_i \), while \( V^i \) transforms in the adjoint representation of \( W_\infty \); i.e. \( \int \left( \hat{\delta} A_i V^i + A_i \delta V^i \right) = 0 \). Unlike the case of finite-dimensional semi-simple Lie algebras, one must distinguish between the adjoint and co-adjoint representations of \( W_\infty \) since there is no Cartan-Killing metric available to raise or lower indices. One consequence of this is that whereas the transformation (65) of \( V^i \) under \( k_j \) for given \( i \) and \( j \) involves a finite number of terms, the transformation (66) of \( A_i \) under \( k_j \) will involve an infinite number. The simultaneous gauging of both the left-handed and the right-handed \( W_\infty \) symmetries is also possible. The results, which are rather analogous to those of \( w_\infty \) gravity, can be found in [14].

9. Various Applications of \( w_\infty \) and \( W_\infty \) Algebras

So far we have primarily discussed the structure of various \( W_\infty \) algebras, and their gauging. There are various applications of \( w_\infty \) or \( W_\infty \) algebras where these structures are relevant. The ultimate application would be the construction of a \( W \)-string or \( W \)-membrane
theory. Below we shall comment briefly on a selected few topics with regard to the application of W-symmetry.

1. $w_\infty$ arises as a symmetry algebra of $sl(\infty)$ Toda theory, which itself arises from a particular reduction of the 4D self-dual gravity equation \cite{8}. In ref. \cite{8}, these results are obtained from a Wess-Zumino-Witten model by Hamiltonian reduction procedure. It should also be pointed out that self-dual metrics of a spacetime with signature $(2,2)$ are the ones that arise as consistent backgrounds for $N=2$ supersymmetric string theory \cite{9}.

2. It has been shown that the Poisson bracket algebra of an important, integrable hierarchy of nonlinear partial differential equations known as the KP hierarchy, is isomorphic to the $W_{1+\infty}$ algebra \cite{24}. Furthermore, it is known that another $W_{1+\infty}$ algebra, which commutes with the first one arises as the symmetry algebra of the KP hierarchy \cite{4,5}. These facts play an important role in the nonperturbative analysis of $d=1$ string theory.

The KP hierarchy is neatly formulated in terms of the pseudo-differential operator

$$Q = \partial_x + q_0 \partial_x^{-1} + q_1 \partial_x^{-2} + q_2 \partial_x^{-3} + \cdots$$

(67)

where $q_i$, $i = 0, 1, 2, \ldots$ are functions of $x_1 \equiv x$, $x_2 \equiv y$, $x_3 \equiv t$ and the “higher time variables” $x_4, x_5, \ldots$. Defining the Hamiltonians as

$$H_n = \frac{1}{n+1} \int \text{res} \ Q^{n+1},$$

(68)

where the residue is defined to be the coefficient of $\partial^{-1}$, the KP equations are simply the Hamiltonian flows

$$\frac{\partial q_i}{\partial x_n} = \{q_i, H_n\}.$$  

(69)

with respect to the following Poisson bracket

$$\{q_i(x), q_j(x')\} = K_{ij}(x, x')$$

(70)

where \cite{38}

$$K_{ij} = \sum_{\ell=0}^{i} (-1)^{\ell} \left( \binom{i}{\ell} q_{i+j-\ell} \partial_x^{\ell} - \sum_{\ell=0}^{j} \left( \binom{j}{\ell} \partial_x^{\ell} q_{i+j-\ell} \right) \right).$$

(71)

The $n = 1$ equation is content-free, while the $n = 2, 3$ equations together imply the well known KP equation $\partial_x (u_t + 6u u_x + u_{xxx}) = \pm 3u_{yy}$ where $u \equiv q_0$, $u_t \equiv \frac{\partial}{\partial t}$, $u_x \equiv \frac{\partial}{\partial x}$, etc. The flows (69) commute, i.e. $\{H_m, H_n\} = 0$, which shows that the KP hierarchy is integrable. It has been shown that the Poisson bracket algebra (70) is isomorphic to the OPE algebra of fermionic bilinears $\tilde{\psi} \partial^{i-1} \psi$, i.e. the following correspondence exists \cite{24}

$$\{q_i(x), q_j(x')\} \leftrightarrow [\tilde{\psi} \partial^{i-1} \psi, \tilde{\psi} \partial^{j-1} \psi].$$

(72)
On the other hand, the OPE algebra of these bilinears is just the $W_{1+\infty}$ algebra. There exists another $W_{1+\infty}$ symmetry which commutes with this one, and arises as the non-isospectral flow of the KP potentials $q_i$ [4,5]. We refer the reader to [5] for a discussion of how this symmetry can be utilized in the formulation of the matrix model approach to 2D gravity. A basic connection is the identification of the partition function of the 2D gravity with a suitable function of certain KP potentials.

3. Perhaps one of the most important applications of $W$ symmetry is the construction of a possible $W$ string theory. So far, most of the studies on $W$ algebras have primarily aimed at a better understanding of their algebraic structure, and there have been very few attempts to build a $W$ string theory. This, indeed, seems to be a rather nontrivial challenge. In [3], various aspects of a possible $W$ string theory were conjectured, with emphasis on $W_3$. The critical dimension for $W_3$ string was suggested to be $d = 100$. In particular, it was conjectured that massless target space higher spin fields should occur in the spectrum. More recently, the issues of physical states in a $W_N$ string theory with emphasis on $W_3$ has been investigated within the framework of a specific matter coupling to $W$ gravity [39]. Consider $d$ free scalars $X^i$, and the fields $\phi_1, \phi_2$ coming from the $W_3$ gravity sector. The spin-2 generator of $W_3$ algebra is the total energy momentum tensor given by

$$T(z) = -\frac{1}{4}(\partial\phi_1)^2 + i a_1 \partial^2 \phi_1 - \frac{1}{4}(\partial\phi_2)^2 + i a_2 \partial^2 \phi_2 - \frac{1}{2}(\partial \vec{X})^2$$

$$\equiv T_1 + T_2 + T_X \quad (73)$$

where $a_1^2 = (12\alpha_0^2 + c_m)/24$, $c_m$ is the central charge of the matter fields (i.e. $d$, for $d$ free scalars), $a_2^2 = 3\alpha_0/2$ and $\alpha_0 = -49/4$ [39]. The observation made in ref. [39] is that, since $\phi_1$ occurs only through its energy-momentum tensor $T_1$ in the expression for the spin-3 current of pure $W_3$ gravity, a natural way to couple matter fields $X^i$ is to make the replacement $T_1 \rightarrow T_1 + T_X$ in that expression. In this way one arrives at the following spin-3 generator [39]

$$W = \frac{b}{12i}((\partial\phi_2)^3 - 6i a_2 (\partial\phi_2)(\partial^2 \phi_2) + 12\partial \phi_2 (T_1 + T_X) - 12i a_2 \partial (T_1 + T_X), \quad (74)$$

where $b^2 = \frac{16}{22 + 10(1 - 24\alpha_0^2)}$. Ref. [39] then considers the following operator which creates a scalar (tachyon) mode of the string

$$V_T(z) = e^{i\vec{\beta} \cdot \vec{\phi}} e^{i\vec{k} \cdot \vec{X}}. \quad (75)$$

Physical state conditions imposes restrictions on $\vec{\beta}$ and $\vec{k}$. It turns out that a tachyon occurs for $d > \frac{1}{2}$, and thus the lower critical dimension is $d = \frac{1}{2}$ [39]. The upper critical dimension in which a full symmetry between the usual coordinates and the $W_3$ gravity coordinate arises turns out to be $d = \frac{49}{2}$ [39]. The full spectrum and other salient features of an off-critical $W$ string is not known at present.
Then, there is the question of what an $W_\infty$ string theory would be like. Those features which arise in $W_N$ string as a consequence of the nonlinear nature of the $W_N$ algebra may disappear, and new features having to do with the infinite $N$ limit may arise. We would like to conjecture that in a super $W_\infty$ string theory, the target space spectrum will contain an infinite tower of higher spin gauge fields, gauging an infinite dimensional higher spin algebra which contains the usual Poincaré or anti de Sitter superalgebras as a subalgebra. If we assume that the motivations of Fradkin and Vasiliev [40] for considering the anti de Sitter rather than Poincaré based higher spin algebras persist also in a $W_\infty$ string theory, then it is reasonable to try construct the generators of the Fradkin-Vasiliev type higher spin algebra from the $W_\infty$ structures on the worldsheet, and show that they obey the correct algebra at the quantum level.

4. Symplectic diffeomorphisms of a surface also arises in membrane theories as a residual symmetry group in a light-cone gauge [41]. Consider the membrane action

$$ I = \int d\tau d\sigma d\rho \left( \frac{1}{2} \sqrt{-g} g^{ij} \partial_i X^\mu \partial_j X^\nu \eta_{\mu\nu} - \frac{1}{2} \sqrt{-g} \right), \quad (76) $$

where $i = \tau, \sigma, \rho$ labels the coordinates of the membrane world-volume with metric $g_{ij}$, and $X^\mu$, where $\mu = 0, ..., d-1$, are the coordinates of a $d$-dimensional Minkowski spacetime with metric $\eta_{\mu\nu}$. The action is evidently invariant under the reparametrization of the world-volume, $\delta \sigma^i = \xi^i$. Imposing the following gauge conditions [42]

$$ X^+ = \tau, \quad g^{00} = -h^{-1}, \quad (77) $$

where $X^+ = \frac{1}{\sqrt{2}}(X^0 + X^{d-1})$, $h = \det g_{ab}$, and $a = \sigma, \rho$, we are left as a residual symmetry with the symplectic diffeomorphisms of the membrane satisfying the condition $\partial_a \xi^a = 0$. To maintain the above gauges, one finds that, at least locally, $g^{0a}$ must have the form

$$ g^{0a} = -\epsilon^{ab} h^{-1} \partial_b \omega, \quad (78) $$

where $\omega$ is a time-dependent gauge potential which transforms as

$$ \delta \omega = \partial_0 \Lambda + \epsilon^{ab} \partial_b \omega \partial_a \Lambda $$

$$ \equiv \partial_0 \Lambda + \{\omega, \Lambda\} \equiv D_0 \Lambda. \quad (79) $$

Here $\Lambda(\tau, \sigma, \rho)$ is an arbitrary gauge parameter. With the gauge choices (77), and recalling (78), one can show that the membrane action (76) reduces to

$$ I = \int d\tau \int d\sigma d\rho \left[ \frac{1}{2} D_0 X^r D_0 X^s - \frac{1}{4} \{X^r, X^s\} \{X^r, X^s\} \right], \quad r, s = 1, ..., d-2, \quad (80) $$

where $D_0 X^r = \partial_0 X^r + \{\omega, X^r\}$. The action (76) is invariant under

$$ \delta \omega = D_0 \Lambda, \quad \delta X^r = -\{\Lambda, X^r\}. \quad (81) $$
Thus we see that the action (76) is the gauge theory in one dimension (time) of the symplectic diffeomorphisms of the membrane. The role of the usual trace is played by $\int d\sigma d\rho$, and Yang-Mills commutator is replaced by the Poisson bracket $\{,\}$. Typically spherical or toroidal membranes have been considered so far. Perhaps, one should consider a cylindrical membrane, and investigate the role of $w_\infty$ symmetry in the theory. It would be interesting to see if the membrane theory is integrable under any circumstances and if the representation theory of $w_\infty$ could be utilized in analysing the quantum theory. It would also be interesting to determine the quantum fate of $w_\infty$ symmetry and see whether it deforms to $W_\infty$ symmetry in a manner similar to the one encountered in the study of $w_\infty$ gravity [37].

Acknowledgments

I would like to thank E. Bergshoeff, A. Das, C. Pope, X. Shen, and S. Sin for useful discussions. I also would like to thank Professor Abdus Salam, the International Atomic Energy Agency and UNESCO for hospitality at the International Center for Theoretical Physics where this work was completed. This work is partially supported by NSF grant PHY-8907887.
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