WEIGHTED ESTIMATES FOR THE $\bar{\partial}$-NEUMANN PROBLEM ON INTERSECTIONS OF STRICTLY PSEUDOCONVEX DOMAINS IN $\mathbb{C}^2$

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Abstract. We obtain weighted estimates for the $\bar{\partial}$-Neumann operator on intersections of two smooth strictly pseudoconvex domains in $\mathbb{C}^2$. The regularity estimates are described with the use of Sobolev norms with weights which are powers of the defining functions of the two domains.

1. Introduction

This purpose of this work is to study the $\bar{\partial}$-Neumann problem on transversal intersections of strictly pseudoconvex domains. Let $\Omega \subset \mathbb{C}^n$ be the intersection of $m$ smoothly bounded strictly pseudoconvex domains, $\Omega_1, \ldots, \Omega_m$, which intersect (real) transversely, that is for all $1 \leq i_1 < \cdots < i_l \leq m$, we have
\[ d\rho_{i_1} \wedge \cdots \wedge d\rho_{i_l} \neq 0 \]
on $\bigcap_{j=1}^l \partial\Omega_{i_j} \cap \partial\Omega$, where $\rho_j$ is a defining function for $\Omega_j$, or
\[ \Omega_j := \{ z \in \mathbb{C}^n | \rho_j < 0 \} \]
for $j = 1, \ldots, m$. Note in particular, we consider $m \leq n$. This definition is modeled on Range and Siu’s description of piecewise smooth domains [18].

We will be concerned with regularity estimates, as measured by Sobolev norms, of the $\bar{\partial}$-Neumann operator, $N$, defined as the inverse to the $\Box = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$ operator. We should not expect to prove regularity of the solution, even if the given data form is smooth up to the boundary, because of the singularities on the boundary of the domain. Singularities of solutions to elliptic equations (the interior equations of the $\bar{\partial}$-Neumann problem are elliptic) are well known in the situation of singular domains [10, 15]. However, limited regularity, up to a fixed order have been shown by Michel and Shaw in [16]:

**Theorem 1.1.** Let $\Omega \subset \subset \mathbb{C}^n$ be a piecewise smooth strictly pseudoconvex domain. The $\bar{\partial}$-Neumann operator is bounded as a map of $(p,q)$-forms with $L^2$ coefficients, $N : L^2_{p,q}(\Omega) \to L^2_{p,q}(\Omega)$, for $0 \leq p \leq n$, $1 \leq q \leq n-1$. Furthermore, we have the following estimates:
\[ \|Nf\|_{W^{1/2}_{p,q}(\Omega)} \lesssim \|f\|_{L^2_{p,q}(\Omega)} \]
for $f \in L^2_{p,q}(\Omega)$.

We note that the symbol $\lesssim$ means $\leq c$ where $c$ is independent of the functions (or forms) being estimated.

In [9], Lieb and the current author considered singularities of another kind (arising on Henkin-Leiterer domains), and obtained $L^p$ estimates of the $\bar{\partial}$-Neumann
operator with the use of weights which vanish at the singularity of the domains. We use that idea of vanishing weights in the current paper.

We first define the weighted Sobolev norms on half-spaces as in [8]. Let $\mathbb{H}_J^{2n}$ denote the half-plane

$$\mathbb{H}_J^{2n} = \{(x, \rho_j) \in \mathbb{R}^{2n} | \rho_j < 0\},$$

and for multi-index, $I = (i_1, \ldots, i_j)$,

$$\mathbb{H}_I^{2n} := \bigcap_{i \in I} H_i^{2n}.$$  

We use a non-standard notation to write powers of the $\rho_i$ functions using the index notation to stress that the powers (over each $\rho_i$) will be the same. We write

$$\rho_i^{k \times l} := \rho_i^k \rho_i^{k+1} \cdots \rho_i^{k\cdot l}.$$

**Definition 1.2.** Let $I$ be a multi-index, and $J$ a subset of $I$. For $\alpha \in \mathbb{R}$, and $s,k$ integers $\geq 0$ we have the spaces

$$W^{\alpha,s}(\mathbb{H}_I^{2n}, \rho_J, k) = \{ f \in W^{\alpha}(\mathbb{H}_I^{2n}) | |f|^{(sk-rk) \times |J|} f \in W^{\alpha + s-r}(\mathbb{H}_I^{2n}) \text{ for each } 0 \leq r \leq s \}$$

with norm

$$\|f\|_{W^{\alpha,s}(\mathbb{H}_I^{2n}, \rho_J, k)} = \sum_{0 \leq r \leq s} \|\rho_J^{(sk-rk) \times |J|} f\|_{W^{\alpha+s-r}(\mathbb{H}_I^{2n})}.$$  

In the case $J = I$ we will simply write $W^{\alpha,s}(\mathbb{H}_I^{2n}, \rho, k) := W^{\alpha,s}(\mathbb{H}_I^{2n}, \rho_I, k)$.

In dealing with boundary values (restriction to $\rho_j = 0$ for some $j$), a notation to deal with a missing index will be useful. To indicate a missing index (from the multi-index, $I$, which is to be known), we use $\rho_{ij} := \rho_i \cdots \rho_{ij-1} \rho_{ij+1} \cdots \rho_i^k$ and

$$\rho_i^{k \times (l-1)} := \rho_i^k \cdots \rho_{ij-1}^k \rho_{ij+1}^k \cdots \rho_i^k.$$  

Then a weighted Sobolev space can be defined on $\mathbb{H}_I^{2n-1} := \bigcap_{i \in I} \mathbb{H}_i^{2n} | \rho_j = 0$, with norm

$$\|f\|_{W^{\alpha,s}(\mathbb{H}_I^{2n-1}, \rho_{ij}, k)} = \sum_{0 \leq r \leq s} \|\rho_{ij}^{(sk-rk) \times (l-1)} f\|_{W^{\alpha+s-r}(\mathbb{H}_I^{2n-1})}.$$  

We can generalize the above spaces to general intersection domains (of smooth domains) by localizing and using a coordinate system including the $\rho_j$ functions.

Our main result is the following

**Main Theorem 1.** Let $\Omega \subset \subset \mathbb{C}^2$ be an intersection of two smoothly bounded strictly pseudoconvex domains. Let $f \in W_{(0,1)}^{\alpha,2}(\Omega, \rho_j, 2)$ for $j = 1, 2$. Let $N$ be solution operator to the $\bar \partial$-Neumann problem. Then

$$\sum_j \|Nf\|_{W_{(0,1)}^{1/2,s}(\Omega, \rho, 2)} \lesssim \sum_j \|f\|_{W_{(0,1)}^{\alpha,s}(\Omega, \rho_j, 2)}.$$  

We use a representation of the solution to the $\bar \partial$-Neumann problem as a sum of a solution to a homogeneous Dirichlet problem (with the use of Green’s operator) and the solution to an inhomogeneous Dirichlet problem (with the use of a Poisson operator) [4]. Weighted estimates on the boundary will be obtained by reducing the boundary conditions for the $\bar \partial$-Neumann problem to an equation, which to highest
order is just the Kohn boundary problem, for \( \square_b \). This study of the boundary condition is based on our earlier work in [7], and is carried out in Section 7. In particular the Dirichlet to Neumann operator (DNO), a boundary value operator expressing the boundary values of normal derivatives in terms of the given boundary values, plays an important role, and weighted estimates for the DNO are derived in Section 5 which may be of interest in its own right. Estimates for the boundary solution are obtained in Section 8 using standard integration by parts techniques.

In order to use these boundary estimates to conclude estimates for the solution operator, the application of the Poisson operator to the boundary solution is studied. The Poisson operator is represented as a sum of pseudodifferential operators and combinations of pseudodifferential operators with restriction to boundary operators. Weighted estimates for such resulting operators used in Sections 4 and 5 are taken from [8]. In a similar manner weighted estimates for the solution to a homogeneous Dirichlet problem are worked out in Section 6.

For the sake of simplicity, we work with *generic corners* following the terminology of [3] in which

\[
\partial\rho_{i_1} \wedge \cdots \wedge \partial\rho_{i_m} \neq 0
\]

at points where \( \partial\Omega_{i_1}, \ldots, \partial\Omega_{i_m} \) intersect. Although we set up much of the work in \( \mathbb{C}^n \), our results are restricted to \( n = 2 \), mainly in order to make use of the vanishing of a problematic term, which has worse weighted regularity mapping properties than those used to obtain our Main Theorem. We refer the reader to the simplification in Section 7 to see where the problematic term can just be ignored.

2. Notation for operators on intersection domains

We will use the index notation, \( \hat{i} \) to refer to an index, \( i \), which is to be omitted in whatever variables, respectively, operators, are being considered. Thus, for instance, we use \( \text{F.T.}_j \) to denote the partial Fourier Transform in all variables other than the \( j^{th} \) variable. Similarly,

\[
x_j := (x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n).
\]

In specifying a value for a particular coordinate, we mark that value with the index. Thus we write

\[
\left. \left( \text{F.T.}_j g \right) \right|_{x_j = 0} = \tilde{g}(\xi_1, \ldots, \xi_{j-1}, 0, \xi_{j+1}, \ldots, \xi_n).
\]

This should not be confused with \( \tilde{g}(\xi_1, \ldots, \xi_{j-1}, 0, \xi_{j+1}, \ldots, \xi_n) = \tilde{g}(\xi) \bigg|_{\xi_j = 0} \). With slight abuse of notation we will rearrange the order of the arguments and write the variable fixed to a specific value first; we write \( \tilde{g}(0_j, \xi_j) \) in place of \( \tilde{g}(\xi_1, \ldots, \xi_{j-1}, 0_j, \xi_{j+1}, \ldots, \xi_n) \). Note that above we use \( \tilde{g} \) to refer to both the partial and full Fourier Transforms of the function \( g \), the particular transform being clear from context.

Boundary values naturally arise in the Fourier representation of differential equations (on domains with boundary) and it will be useful to have a notation representing the restriction to a given boundary. We will use the operator \( R_j \) to denote the restriction of a function (or form) to \( \partial\Omega_j \):

\[
R_j \phi = \phi \bigg|_{\rho_j = 0}.
\]
We borrow notation from [7] on pseudodifferential operators. In particular, we write \( \Psi^k(\Omega) \) to denote the class of pseudodifferential operators of order \( k \) on \( \Omega \), and we reserve the notation \( \Psi^k \) to indicate an operator belonging to class \( \Psi^k(\Omega) \). Often the meaning of \( \Psi^k \) will change from one line to the next. We also use \( \Psi^k_b \) to denote an operator belonging to class \( \Psi^k(\partial \Omega) \). With slight abuse of notation, we will also use \( \Psi^k_b \) to denote an operator belonging to class \( \Psi^k(\partial \Omega_j) \) for any particular boundary \( \partial \Omega_j \).

As is customary, we write \( \Psi^{-\infty} \) to denote a smoothing operator:

\[
\Psi^{-\infty} : W^s(\Omega) \to W^{-\infty}(\Omega)
\]

for any \( s \).

In writing a pseudodifferential operator (on \( \mathbb{R}^n \)) applied to a distribution supported on a domain, we shall use the convention that the distribution will be considered to be extended by zero to all of \( \mathbb{R}^n \). Thus, for instance, if \( \varphi \in L^2(\mathbb{H}^2) \), where

\[
\mathbb{H}^2 := \{ (x_1, x_2) \in \mathbb{R}^2 : x_2 < 0 \},
\]

and \( A \in \Psi^2(\mathbb{R}^2) \), we write \( A\varphi \) to mean

\[
A\varphi := \frac{1}{(2\pi)^2} \int \sigma(A)\hat{\varphi}(\xi_1, \xi_2)e^{ix\xi}d\xi_1d\xi_2,
\]

where \( \varphi^{E_2} \) refers to extension by zero over \( x_2 > 0 \).

In general if we have a distribution, \( \varphi \), defined on \( \Omega = \bigcap_j \Omega_j \), we will use the notation \( \varphi^{E_k} \) to denote the extension by zero of \( \varphi \) to \( \bigcap_j \neq k \Omega_j \). Similar extensions will be used for boundary distributions. For example, in the case \( \Omega = \Omega_1 \cap \Omega_2 \), consider \( \varphi \in L^2(\partial \Omega_1 \cap \partial \Omega) \). We denote \( \varphi^{E_2} \) to be the distribution supported on all of \( \partial \Omega_1 \) defined by an extension by zero. From Theorem 1.4.2.4 in [10] we have the following results concerning extension by zero: for \( \Omega = \bigcap_j \Omega_j \) and \( 0 \leq s \leq 1/2 \), if \( g \in W^s(\Omega) \) then \( g^{E_\Omega} \in W^s(\bigcap_j \neq k \Omega_j) \).

Operators mapping distributions supported on one boundary, \( \partial \Omega_j \), to a distribution supported on another boundary, \( \partial \Omega_k \), also arise in the pseudodifferential analysis of operators on intersection domains. On the one hand, Fourier Transforms of derivatives lead to boundary value terms, while restrictions to the various boundaries of the intersection domain produce new terms supported on the respective boundaries. It will be necessary to study the Sobolev mapping properties of such operators.

As an example consider the intersection of half-spaces, \( \mathbb{H}^2_{1,2} \), defined by

\[
\mathbb{H}^2_{1,2} := \{ (x_1, x_2) \in \mathbb{R}^2 : x_1, x_2 < 0 \},
\]

and a function, \( h(x_1, x_2) \), in a Sobolev space, \( W^s(\mathbb{H}^2_{1,2}) \), for some \( s \geq 0 \). A Fourier Transform of \( \partial_{x_1}h \) would lead to partial transforms on \( x_1 = 0 \):

\[
\widehat{\partial_{x_1}h} = FT_1 h(0_1, \xi_2) + i\xi_1 \hat{h}.
\]

It is the first term on the right, in combination with pseudodifferential operators and restrictions, which we will now discuss.

We illustrate with an example. Let \( \chi'(\xi_2) \in C^\infty(\mathbb{R}) \) be such that \( \chi'(\xi_2) \equiv 0 \) in a neighborhood of the origin. We look at the operator which sends \( h|_{x_1=0} \) to a

we use the notation \( E \) for elliptic symbols in class \( s \) function supported on \( x \) (2.1) to be made precise shortly) with the property

\[
\gamma \geq 1, \text{ for operators whose symbols satisfy the above conditions. As we will see this condition applied again, we refer to a pseudodifferential operator of order } -k, \Psi^{-k}, \text{ which can be written for any } N \geq k \text{ in the form }
\]

\[
\Psi^{-k} = A_{-k} + \Psi^{-N}
\]

Generalizing the above result to apply to transversal intersections of domains, we use the notation \( \mathcal{E}^{j,k}_{-\alpha,\gamma} \), where \( -\alpha - 1/2 \in \mathbb{N} \), for \( \alpha \geq 1/2 \), for certain operators (to be made precise shortly) with the property

\[
\mathcal{E}^{j,k}_{-\alpha,\gamma} : W^s(\partial \Omega_j) \to W^{s+\alpha}(\partial \Omega_k)
\]

valid for \( s \geq \gamma \). In the case \( \gamma = 0 \) we will use the notation

\[
\mathcal{E}^{j,k}_{-\alpha} := \mathcal{E}^{j,k}_{-\alpha,0}.
\]

To make precise which operators are to be included in the \( \mathcal{E}^{j,k}_{-\alpha,\gamma} \) operators, we start with \( \gamma = 0 \) and include in \( \mathcal{E}^{j,k}_{-\alpha} \) any operators which for any \( N \), can be written in the form

\[
\mathcal{E}^{j,k}_{-\alpha} = R_k \circ A_{-\alpha - 1/2} + R_k \circ \Psi^{-N},
\]

where \( A_{-\alpha - 1/2} \in \Psi^{-\alpha - 1/2}(\Omega) \) is such that its symbol, \( \sigma(A_{-\alpha - 1/2})(x, \rho, \xi, \eta) \) has the property that it is meromorphic in the \( \eta \) variables with poles in \( \eta_j \) which are elliptic symbols in class \( S^1(\Omega) \) (restricted to \( \eta_j = 0 \)). We shall reserve the notation, \( A_{-k} \) for \( k \geq 1 \), for operators whose symbols satisfy the above conditions. As we will see this condition applied again, we refer to a pseudodifferential operator of order \(-k\), \( \Psi^{-k} \), which can be written for any \( N \geq k \) in the form
as decomposable.

Note that in the case \( j = k \) above, we have included in \( E_{-\alpha}^{j} \) (for \( \alpha \in \mathbb{N} \)) terms of the form

\[
E_{-\alpha}^{j} = R_{j} \circ \Psi^{-\alpha-1} \circ R_{j} = \Psi_{b_{j}}^{-\alpha}
\]

for \( \alpha \geq 1 \), where we use the subscript \( b_{j} \) to specify a pseudodifferential operator on the boundary \( \partial \Omega_{j} \). The last line is due to a restriction property of pseudodifferential operators as given in Lemma 2.7 of [8].

Lastly, we will include compositions of such boundary operators, and here the \( \gamma \) value will be of importance. We write

\[
E_{-\alpha,\alpha_{1}}^{l} = E_{-\alpha}^{l} \circ E_{-\alpha_{2}}^{j} \quad \alpha_{1}, \alpha_{2} \geq 1/2, \quad \alpha = \alpha_{1} + \alpha_{2}
\]

for \( l \neq j \).

We will also write \( E_{-\alpha}^{j} = E_{-\alpha}^{l} \) for any \( \alpha \leq \beta \), and \( E_{-\beta,\gamma}^{j} = E_{-\alpha}^{j} \) for \( \alpha = \beta - \gamma \).

We refer to [8] for estimates regarding the \( E_{-\alpha}^{j} \) operators. From Corollary 4.7 in [8], we have

**Theorem 2.1.** Let \(-1/2 \leq \delta \leq 1/2\), and \( g_{b_{j}} \in W^{\delta,s}(\partial \Omega \cap \partial \Omega_{j}, \rho_{j}, \lambda) \). Then for \( \beta \geq \gamma \) with \( \beta - \alpha \leq \delta \), we have

\[
\| E_{-\alpha,\gamma}^{l} g_{b_{j}} \|_{W^{\beta,s}(\partial \Omega \cap \partial \Omega_{j}, \rho_{j}, \lambda)} \lesssim \| g_{b_{j}} \|_{W^{\beta-s,s}(\partial \Omega \cap \partial \Omega_{j}, \rho_{j}, \lambda)}.
\]

To illustrate the importance of the \( \gamma \) value, we consider a composition of two \( E_{-1/2}^{l/2} \) operators: \( E_{-1/2}^{l/2} \circ E_{-1/2}^{l/2} \) cannot be written as \( E_{-1/2}^{l/2} \) due to the condition in Theorem 2.1 that the Sobolev norm on the left-hand side of the estimates be \( \geq 0 \). The condition in two applications of the Theorem to obtain estimates for \( E_{-1/2}^{l/2} \circ E_{-1/2}^{l/2} \) will however be satisfied if we try to estimate Sobolev 1/2 estimates. Thus, we can write

\[
E_{-1/2}^{l/2} \circ E_{-1/2}^{l/2} = E_{-1,1/2}^{l/2}.
\]

Most such operators in this article will involve a \( \gamma \) value of zero, but in a few places a higher order will be needed due to various compositions of operators.

We now give some results concerning interior estimates involving decomposable operators. Suppose that the operator \( A \in \Psi^{-k} \) is decomposable. Then we have the following theorem concerning weighted estimates from [8]:

**Theorem 2.2.** (Theorems 4.3 and 4.4 in [8]) Let \( A_{-\alpha} \) be decomposable operator (of order \( -\alpha \leq -1 \)). Then, for \( 1/2 \leq \beta \leq \alpha + 1/2 \), and \( g_{b_{j}} \in W^{\max\{\beta-\alpha,0\},s}(\partial \Omega \cap \partial \Omega_{j}, \rho_{j}, k) \),

\[
\| A_{-\alpha} g_{b_{j}} \|_{W^{\beta-1/2,s}(\Omega, \rho, k)} \lesssim \| g_{b_{j}} \|_{W^{\beta,s}(\partial \Omega \cap \partial \Omega_{j}, \rho_{j}, k)}.
\]

For pseudodifferential operators acting on a distribution supported on the interior of \( \Omega \) (as opposed to the boundary as in the above theorems), we have

**Theorem 2.3** (Theorem 4.6 in [8]). Let \( A_{-\alpha} \in \Psi^{-\alpha}(\Omega) \) for \( \alpha \geq 0 \). For \( g \in W^{0,s}(\Omega, \rho_{j}, k) \)

\[
\| A_{-\alpha} g \|_{W^{\alpha,s}(\Omega, \rho_{j}, k)} \lesssim \| g \|_{W^{0,s}(\Omega, \rho_{j}, k)}.
\]
We will later consider operators which are matrices composed of operators of the various above types. Suppose, \( M \) is an \( n \times n \) matrix operator and \( u \) is a vector, 
\[
\begin{bmatrix}
  u_1 \\
  \vdots \\
  u_n
\end{bmatrix}
\]
with \( u_j \in H_j \) for some (weighted Sobolev) space denoted \( H_j \). Then we write
\[
M : H_1 + \cdots + H_n \to H'_1 + \cdots + H'_n,
\]
where the \( H'_j \) denote some (weighted Sobolev) spaces, to mean the \( j^{th} \) component of \( Mu \), or \( (Mu)_j \), satisfies
\[
\|(Mu)_j\|_{H'_j} \lesssim \sum_k \|u_k\|_{H_k}.
\]
In the case the \( H_j \) are all the same, we will omit the summation signs.

Furthermore, regarding pseudodifferential operators, we will use the notation \( \Psi^\alpha_{\epsilon} \) to refer to pseudodifferential operators with small operator norm, by which we mean, given some sufficiently small neighborhood \( U \), we have
\[
\|(\Psi^\alpha_{\epsilon} v)\|_s \leq \epsilon \|v\|_{s+\alpha}
\]
for all \( s \geq 0 \) and for all \( v \) with support in \( U \).

In Sections 7 and 8 we discuss the boundary equations related to the \( \bar{\partial} \)-Neumann problem, and in obtaining estimates for the solutions to given boundary equations, we isolate a particular problematic direction in which to obtain a gain of regularity (in a weighted sense). This is a normal phenomenon in the analysis in the theory of the \( \bar{\partial} \)-Neumann problem in which certain operators behave as elliptic operators with the exception of their behavior in one particular microlocal region. To describe this problematic region we recall a microlocal decomposition as given in \([6, 13, 14, 17]\). We describe the situation in \( \mathbb{R}^3 \) (considered as the boundary of a half-plane in \( \mathbb{R}^4 \)).

We choose a smooth partition of the two dimensional unit sphere \( |\xi| = 1 \) with functions \( \psi^+_k \), \( \psi^0_k \), and \( \psi^-_k \), with dependence on a parameter \( k \), in such a way that \( \psi^+_k \) has support in \( \xi_3 > k \sqrt{\xi_1^2 + \xi_2^2} \) and is equal to 1 when \( \xi_3 > (k + 1) \sqrt{\xi_1^2 + \xi_2^2} \). \( \psi^-_k \) is defined symmetrically, so that \( \psi^-_k (\xi_1, \xi_2, -\xi_3) = \psi^+_k (\xi_1, \xi_2, \xi_3) \), and finally, \( \psi^+_k + \psi^0_k + \psi^-_k = 1 \) on the unit sphere.

The functions are then extended to all of \( \mathbb{R}^3 \) in the following way. First, they are extended radially (so they are symbols of zero order pseudodifferential operators) to everywhere outside a neighborhood of the origin. A cutoff equivalently equal to 1 in a neighborhood of the origin is then included in (an extension of) the \( \psi^0_k \) function so that on \( \mathbb{R}^3 \) we have a smooth partition of unity from three order 0 symbols. We refer the reader to the above mentioned papers for more details of the decomposition.

3. Setup of the \( \bar{\partial} \)-Neumann Problem

While our final results are stated in the case of \( n = 2 \), we can set up the \( \bar{\partial} \)-Neumann problem on intersection domains in \( \mathbb{C}^n \) for \( n \geq 2 \). As in the Introduction, we set \( \Omega = \bigcap_{i=1}^m \Omega_i \) where the \( \Omega_i \subset \mathbb{C}^n \) are smoothly bounded strictly pseudoconvex domains intersecting real transversely.
The operator, $\Box$, is defined according to

$$\Box = \bar{\partial}\partial^* + \partial^*\bar{\partial},$$

and for $f$ a $(0, q)$-form with components in $L^2(\Omega)$, written $f \in L^2_{(0, q)}(\Omega)$, the $\bar{\partial}$-Neumann problem is the boundary value problem:

(3.1) \hspace{1cm} $\Box u = f$ \hspace{1cm} in $\Omega$

with the boundary conditions

(3.2) \hspace{1cm} $\bar{\partial}u|\bar{\partial}p_j = 0,$
\hspace{1cm} $u|\bar{\partial}p_j = 0,$

on $p_j = 0$, for $j = 1, \ldots, m$.

We work in a neighborhood of a given point, $p \in \partial\Omega$ at which all the domains intersect; $p \in \bigcap_{i \in I} \partial\Omega_i$ for $I = \{1, \ldots, m\}$. We will further assume that at the point $p$ we have

$$\partial p_1 \wedge \cdots \wedge \partial p_m \neq 0.$$ Such is the case for so-called Bell domains (see [2]), also called domains with generic corners in [3]. The same procedure can be carried out for points at which a subset of domains intersect with obvious modifications.

We work with a metric so that $\omega_1 = \partial p_1, \ldots, \omega_m = \partial p_m$ make up part of an orthonormal frame of $(1, 0)$-forms in a neighborhood of $p$. Let $L_1, \ldots, L_m$ be dual to $\omega_1, \ldots, \omega_m$, respectively. In local coordinates we have the following representations of the vector fields:

(3.3) \hspace{1cm} $L_j = \frac{1}{\sqrt{2}} \frac{\partial}{\partial \rho_j} + iT_j \hspace{1cm} 1 \leq j \leq m$,  

where $T_j$ is tangential to $\partial\Omega_j$, and in local coordinates will be written

$$T_j = \frac{\partial}{\partial x_j} + O(\rho_j) \hspace{1cm} 1 \leq j \leq m.$$ 

We use the convention as in [4] that the holomorphic vector fields transverse to the boundary are written with positive imaginary part.

Without loss of generality we take the singular boundary point $p$ to be the origin. Then, lastly, for $j = 1, \ldots, n - m$, we write

(3.4) \hspace{1cm} $V_{m+j} = \frac{1}{2} \left( \frac{\partial}{\partial x_{m+2j-1}} - i \frac{\partial}{\partial x_{m+2j}} \right) + \sum_{k=1}^{2n-m} \ell_k^{m+j}(x) \frac{\partial}{\partial x_k} + \sum_k O(\rho_k).$

We use the standard notation of forms with indices, so that

$$\omega_J = \omega_{j_1} \wedge \cdots \wedge \omega_{j_{|J|}}.$$ 

Let $f$ be of the form $f_J\bar{\omega}_J$, for a single index $J$. If we can solve the $\bar{\partial}$-Neumann problem for all $f$ of the form $f = f_J\bar{\omega}_J$, for a single index, $J$, we can solve the problem for any $(0, |J|)$-form. We thus look at

(3.5) \hspace{1cm} $\bar{\partial}\partial^* u + \partial^*\bar{\partial}u = f_J\bar{\omega}_J.$

We want to find $u_K$ with $|K| = |J|$ so that $u = \sum_K u_K\bar{\omega}_K$ gives the solution to problem (3.5) with the boundary conditions in (3.2).
We also write \( \omega \) modulo forms orthogonal to \( \bar{\omega} \) formally, as in \([7]\) we use the notation \( c^j_l \) to be the function which satisfies
\[
(3.6) \quad \bar{\omega}_j = c^j_l \bar{\omega},
\]
modulo forms orthogonal to \( \bar{\omega}_j \). Note that with the above notation \( |J| = |J'| + 1 \). We also write \( d_j \) for functions arising in integration by parts involving the \( L_j \) fields:
\[
(3.7) \quad u_j \bar{\omega}_j = 0, \text{ for } 1 \leq j \leq m \text{ can be expressed by}
\]
\[
\partial u_j \bar{\rho} = 0 \quad \text{for } j \in K,
\]
on \( \partial \Omega_j \).

We also define the (smooth) functions \( c^K_{j,j} \) by
\[
c^K_{j,j} = \bar{\omega}_K |(\omega_K \wedge \bar{\omega}_j).
\]
Note that this definition can be reconciled with that using \([8,9]\), if the subscript, \( K, j \), is thought of as an ordered set, with \( j \) the last entry. This definition eliminates much of the need for permutation sign functions, \( \varepsilon^j_K \), giving the sign of the permutation between ordered indices, \( J \) and \( K \), and defined to be zero if \( J \) and \( K \) do not contain the same indices.

We write, for fixed \( j \),
\[
\bar{\partial} u_j \bar{\rho}_j = \left( \sum_{K \neq j} L_j u_K \bar{\omega}_j \wedge \bar{\omega}_K + \sum_{K \neq j} \Psi_k u_K \bar{\omega}_K \cup (k) \right) \bar{\rho}_j
\]
\[
+ \sum_{j \not\in K} c^K_{j,j} u_K \bar{\omega}_K - \sum_{i \not\in K} \sum_{j \in K} \varepsilon_{(K \cup j)} c^K_{i,j} u_K \bar{\omega}_K \wedge \bar{\omega}_l
\]
where $\Psi^1_{\rho_j}$ refers to first order operators, tangential to $\partial \Omega_j$. Then we can use (3.7) to express the boundary condition $\bar{\partial}u|_{\bar{\partial}\rho_j} = 0$ on $\rho_j = 0$ by

$$L_j u_K + (-1)^{|K|} c_{jK}^K u_K = 0$$

for $j \notin K$, and $1 \leq j \leq m$.

As the operator on the left-hand side of (3.1) is elliptic, we use the Poisson and Green’s operators to describe a general solution and then insert it into the boundary conditions (3.8) to get the properties of the specific solution to the $\bar{\partial}$-Neumann problem. We thus first derive some properties of the Poisson and Green’s operators on the intersection domains.

**4. The Poisson operator on intersection domains**

We will follow the calculus of pseudodifferential operators in describing the Poisson operator (a good reference for pseudodifferential operators on smooth manifolds is [20]; we follow our own work in [8] in the presentation of pseudodifferential operators on domains with boundary). As we will see, the non-smooth boundary presents problems in this approach, and so we will need to define boundary operators, which, although derived from pseudodifferential operators, do not lend themselves to the calculus (for example, due to the $\mathcal{E}^j_{\alpha_\gamma}$ operators which arise below). Nonetheless, the Sobolev mapping properties of such problematic operators can be characterized and suffice to obtain estimates for the final solution.

We consider the homogeneous Dirichlet problem for the operator $2\Box$ on $\Omega$, with prescribed boundary values, $g_b$:

$$2\Box u = 0 \quad \text{in } \Omega$$

$$u = g_b \quad \text{on } \partial \Omega.$$

From [11] (see Section 5 in [11]; see also Theorem 5.1 of [21] and Theorem 1.4.3 of [19]), given $g_b \in W^s(\partial \Omega)$, for $0 \leq s \leq 1$, there exists a unique solution, $u \in W^{s+1/2}(\Omega)$, to the homogeneous Dirichlet problem, such that $u \to g_b$ almost everywhere, where the limits are taken non-tangentially. We call the unique solution $u$ to be the Poisson solution for $g_b$ associated with the domain $\Omega$.

We look at the Poisson solution locally, in a neighborhood of an intersection point on the boundary, which we take to be the origin as in Section 3. At such a point there are several boundaries of different domains, and to distinguish the boundary values, we use an index. Thus,

$$g_{b_j} := g_b|_{\partial \Omega_j}.$$

Our goal in this section is to obtain an expression for the Poisson solution in terms of order $-1$ pseudodifferential operators acting on $g_b$, modulo lower order error terms. The operator and its errors will be expressed in terms of the data boundary function (or form), $g_b$, and the non-tangential limits of the normal derivatives of the unique solution, $\partial_{\rho_j} u|_{\partial \Omega_j}$. The operator mapping $g_b$ to the solution, $u$, is called the Poisson operator.

To keep track of (smooth) error terms, we use the following notation: we write $R^{-\infty}$ to mean $\Psi^{-\infty}$ applied to $u$ or to $g_{b_j} \times \delta(\rho_j)$ or to $\partial_{\rho_j} u|_{\partial \Omega_j} \times \delta(\rho_j)$. Furthermore, related to a single boundary, $\partial \Omega_j$, we use the notation $R_{b_j}^{-\infty}$ to denote $\Psi_{b_j}^{-\infty}$ applied to $g_{b_j}$ or to $\partial_{\rho_j} u|_{\partial \Omega_j}$, and also to denote $R_j$ composed with a term from $R^{-\infty}$:
\( R_j \circ R^{-\infty} = R^{-\infty}_{b_j} \). And finally, we also write \( R^{-\infty} \) to include any term which can be written in the form

\[
\Psi^{-\alpha}\left( R^{-\infty}_{b_j} \times \delta(\rho_j) \right)
\]

for \( \alpha \geq 1 \), where the \( \Psi^{-\alpha} \) is decomposable.

We note that for estimates of the smooth terms, using Theorems 2.2 and 2.3, we have for any \( \alpha, s, k \geq 0 \)

\[
\| R^{-\infty} \|_{W^{\alpha,s}(\Omega, \rho, k)} \lesssim \| u \|_{W^{\alpha,s}(\Omega, \rho, k)} + \sum_j \| g_{b_j} \|_{W^{\alpha,s}(\partial \Omega \cap \partial \Omega_j, \rho, k)} + \sum_j \| \partial \rho_j u \|_{W^{\alpha,s}(\partial \Omega \cap \partial \Omega_j, \rho, k)}
\]

We can estimate boundary values of a term, \( \partial \rho_j u \big|_{\partial \Omega_j} \) by assuming support in a neighborhood of \( \partial \Omega \) intersected with \( \Omega \) and writing

\[
\partial \rho_j u \big|_{\rho_j=0} = \int_{-\infty}^{0} \partial^2 \rho_j u d\rho_j = \int_{-\infty}^{0} \Lambda^2_{\rho_j} u d\rho_j + \Lambda^1_{\rho_j} g_{b_j},
\]

where \( \Lambda^k_{\rho_j} \), \( k = 1, 2 \), is a \( k \)th order tangential (to \( \partial \Omega_j \)) operator. Applying a tangential smoothing operator to both sides and integrating yields

\[
\| \partial \rho_j u \|_{W^{\alpha,s}(\partial \Omega \cap \partial \Omega_j)} \lesssim \| u \|_{L^2(\Omega)} + \sum_j \| g_{b_j} \|_{L^2(\partial \Omega)} \lesssim \| g_{b} \|_{L^2(\partial \Omega)}.
\]

We thus have

\[
\| R^{-\infty} \|_{W^{\alpha,s}(\Omega, \rho, k)} \lesssim \| g_{b} \|_{L^2(\partial \Omega)}.
\]

Similarly, we have

\[
\| R^{-\infty}_{b_j} \|_{W^{\alpha,s}(\partial \Omega \cap \partial \Omega_j, \rho, k)} \lesssim \| g_{b} \|_{L^2(\partial \Omega)}.
\]

To obtain our expression for the Poisson solution, we assume \( u \) is supported in a small neighborhood of the origin in \( \Omega \) (we can multiply the solution with a smooth cutoff function) with boundary values (as non-tangential limits) given by \( g_{b} \) (also with compact support), and study the operator \( 2 \Box \) applied to (the cutoff multiplied by) \( u \).
We use extensions by zero to consider $\Box u$ on all of $\mathbb{R}^n$. Similarly the result of other operators applied to $u$ will be extended by zero when taking Fourier Transforms. Thus, writing $g_b := u|_{\partial Q}$ and $g_{bj} := u_{bj}$, we have

$$
\frac{\partial^2 u}{\partial \rho_j^2}(\eta, \xi) = \int \int \frac{\partial^2 u}{\partial \rho_j^2} e^{-ip\eta_j}d\rho_j e^{-ix\xi_j}d\rho_jdx
$$

$$
= F.T. \left. \frac{\partial u}{\partial \rho_j} \right|_{\rho_j = 0}(0, \eta_j, \xi) + i\eta_j F.T. \left. \frac{\partial u}{\partial \rho_j} \right|_{\rho_j = 0}(0, \eta_j, \xi) - \eta_j^2 \widehat{u}(\eta, \xi)
$$

$$
= \left. \left( \frac{\partial u}{\partial \rho_j} \right) + i\eta_j \left( g_{bj} \times \delta(\rho_j) \right) \right|_{\rho_j = 0} - \eta_j^2 \widehat{u}(\eta, \xi),
$$

$$
\frac{\partial u}{\partial \rho_j}(\eta, \xi) = F.T. \left. \frac{\partial u}{\partial \rho_j} \right|_{\rho_j = 0}(0, \eta_j, \xi) + i\eta_j \widehat{u}(\eta, \xi)
$$

$$
= \left( g_{bj} \times \delta(\rho_j) \right) + i\eta_j \widehat{u}(\eta, \xi).
$$

Above, $u$ is also extended by zero in writing Fourier Transforms (and partial transforms) of $u$. For instance, a zero order pseudodifferential operator acting on $\frac{\partial u}{\partial \rho_j}$ can be written in the form

$$
(4.2) \quad \Psi^1 u + \Psi^0 \left( g_{bj} \times \delta_j \right),
$$

where $\delta_j := \delta(\rho_j)$.

We consider the above expressions in the sense of distributions (see for example [1] for details of the Dirichlet problem in the sense of distributions), and one of our first tasks in obtaining estimates for the solution will be to obtain a formula and estimates in the sense of distributions for the boundary values of the derivative.

We recall the expressions in local coordinates of the vector fields dual to the $\omega_j$ forms to write their symbols according to the convention that $\eta_j$ is the variable dual to $\rho_j$ for $1 \leq j \leq m$, and $\xi_j$ is the variable dual to $x_j$, for $1 \leq j \leq 2n - m$. The symbols of the vector fields $L_j$ and $V_j$ are given by

$$
\sigma(L_j) = \frac{1}{\sqrt{2}} i\eta_j - \xi_j + O(\rho_j) \quad 1 \leq j \leq m
$$

$$
\sigma(V_{m+j}) = \frac{1}{2} (i\xi_{m+2j-1} + \xi_{m+2j}) + O(x) + \sum_k O(\rho_k) \quad 1 \leq j \leq n - m.
$$

We then have as principal symbol of the second order operator in Proposition 3.1

$$
-\sigma_2 \left( L_1 \nabla_1 + \cdots + L_m \nabla_m + V_{m+1} \nabla_{m+1} + \cdots V_n \nabla_n \right)
$$

$$
= \frac{1}{2} (\eta_1^2 + \cdots + \eta_m^2) + \xi_1^2 + \cdots + \xi_m^2 + \frac{1}{4} (\xi_{m+1}^2 + \cdots + \xi_{2n-m}^2)
$$

$$
+ \sum l_{jk}(x) \xi_j \xi_k + \sum_j O(\rho_j),
$$

(4.3)

where $l_{jk}$ is $O(x)$.

We use the vector notation of forms where each component of a $(0, q)$-form corresponds to an entry of an $(n \choose q)$ vector. Symbols and differential operators will accordingly be matrices. Thus, a symbol such as $\sigma_2 \left( L_1 \nabla_1 + \cdots + V_n \nabla_n \right)$ above is a matrix with diagonal entries given by the right hand side of (4.3).
We also use the notation $\frac{\partial}{\partial \rho_j} u$ to denote
\[
\frac{\partial}{\partial \rho_j} \left( \sum_{|K|=q} u_K \bar{\omega}_K \right)
\]
written in vector notation. For shorthand notation, we use $\frac{\partial}{\partial \rho_j} := \frac{\partial}{\partial \rho_j}$, with a similar notation holding for the $x$ coordinates.

With the use of the expressions in local coordinates given in (3.3) and (3.4), we use Proposition 3.1 to write $2\Box u = 0$ in the local form (see also [7])
\[
- \left[ \partial^2_{\rho_1} + \cdots + \partial^2_{\rho_m} + 2 \left( \partial^2_{x_{m+1}} + \cdots + \partial^2_{x_{2n-m}} \right) \right. \\
\left. + \frac{1}{2} \left( \partial^2_{x_{m+1}} + \cdots + \partial^2_{x_{2n-m}} \right) \right] u \\
\left. + \frac{1}{2} \sum_{ij} l_{ij} \partial_{x_i} \partial_{x_j} \right] u \\
+ \sqrt{2} \sum_{j=1}^m S_j \circ (\partial_{\rho_j} u \times \delta_j) + A(u) + \sum_{j=1}^m \rho_j \tau_j u = 0,
\]
where the $S_k$ operators are diagonal zero order operators, arising from the $\partial_\rho$ components of
\[
\sum_{k=1}^m \left( -1 \right)^{|J'\cup\{k\}|} \left( \epsilon_{J'\setminus\{k\}} L_k - \epsilon_{J'\setminus\{k\}} L_k \right) + d_k L_k,
\]
$A$ is a matrix comprised of all first order tangential operators (tangential to all boundaries simultaneously; note that $L_i$ is orthogonal to $\partial_{\rho_j}$ for $i \neq j$ as are the $V_i$ vector fields), and $\tau_j$ are second order operators, arising from the $O(\rho)$ terms in $L_1 L_1 + \cdots + V_n \nabla n$. The relation in (4.4) is to be understood modulo smoothing terms which are arise due to the local cutoffs introduced in order to study the problem locally. Thus (4.4) holds in a small neighborhood contained in the support of $u$, modulo $R^{-\infty}$.

For the purposes of the Poisson operator we will group the $O(\rho_j)$ terms (the last summation on the left-hand side of (4.4)) with the principal second order operator. Then, using the symbols for the vector fields as above, and using the notation $g_b := u|_{\partial\Omega}$, we write (4.4) as
\[
\frac{1}{(2\pi)^{2n}} \int \left( \eta_1^2 + \cdots + \eta_m^2 + 2 \left( \xi_{m+1}^2 + \cdots + \xi_{2n-m}^2 \right) \right) + \frac{1}{2} \left( \xi_{m+1}^2 + \cdots + \xi_{2n-m}^2 \right) \\
+ 2 \sum_{j=1}^m \Omega_j (\eta_j, \xi_k) + \sum_{j=1}^m \left. \left( F.T.j \partial_{\rho_j} u(0_j, \eta_j, \xi) + i \eta_j \hat{g}_b (\eta_j, \xi) \right) \right) e^{ip \cdot \eta \cdot \xi} d\eta d\xi \\
- \frac{1}{(2\pi)^{2n}} \sum_{j=1}^m \left. \left( F.T.j \partial_{\rho_j} u(0_j, \eta_j, \xi) + i \eta_j \hat{g}_b (\eta_j, \xi) \right) \right) e^{ip \cdot \eta \cdot \xi} d\eta d\xi \\
+ \sqrt{2} \sum_{j=1}^m S_j (\partial_{\rho_j} u) + A(u) = 0.
\]
Let us define the symbols
\[
\Xi(x, \rho, \xi) = \left(2 (\xi_1^2 + \cdots + \xi_m^2) + 1/2 (\xi_{m+1}^2 + \cdots + \xi_{2n-m}^2)\right) \\
+ 2 \sum l_{jk} \xi_j \xi_k + \sum_j O(\rho_j)^{1/2},
\]
where the \(O(\rho_j)\) terms come from the second order operator in (4.6). We further use the notation \(\Xi_{bj}(x, \rho_j, \xi) := \Xi(x, \rho, \xi)_{\mid \rho_j = 0}\).

We will also write
\[
\eta^2 = \eta_1^2 + \cdots + \eta_m^2 \\
\eta_j^2 = \eta_1^2 + \cdots + \eta_{j-1}^2 + \eta_{j+1}^2 + \eta_m^2.
\]

For ease of notation we will omit the delta distributions when applying pseudodifferential operators to distributions supported on the boundaries. Thus, for \(\phi_b \in L^2(\partial \Omega)\), as a shorthand notation we will write
\[
\Psi^\alpha \phi_b := \sum_j \Psi^\alpha (\phi_{bj} \times \delta_j), \\
\Psi^\alpha \phi_{bj} := \Psi^\alpha (\phi_{bj} \times \delta_j).
\]

In what follows we will make repeated use of the fact that multiplying an elliptic operator of negative order acting on a distribution supported on a boundary \(\partial \Omega_j\) with a factor \(\rho_j\) yields lower order terms; thus, for instance, with \(\Psi^{-s}\) for \(s > 0\) denoting a generic pseudodifferential operator in the class \(\Psi^{-s}(\Omega)\), we have for \(k \geq 0\)
\[
(4.7) \quad \rho_j^k \Psi^{-s} g_{bj} \equiv \Psi^{-s-k} g_{bj}.
\]

See [8] for details. From the same reference we have the following restriction property of pseudodifferential operators acting on distributions supported on a boundary:
\[
(4.8) \quad R_j \circ \Psi^{-s} u_{bj} = \Psi_{kj}^{-s+1} u_{bj},
\]
for \(s \geq 2\), where \(\Psi^{-s} \in \Psi^{-s}(\Omega)\), \(\Psi_{kj}^{-s} \in \Psi^{-s}(\partial \Omega_j)\), and \(u_{bj}\) is a distribution supported on \(\partial \Omega_j\).

We recall from Theorem 4.1 in [7], the principal operator, denoted by \(\Theta^+_j\), of the Poisson operator on the (smooth) domain \(\Omega_j\) (but in a neighborhood of the origin, in which \(\rho_1, \ldots, \rho_m, x_1, \ldots, x_{2n-m}\) forms a coordinate system) has as symbol
\[
(4.9) \quad \sigma(\Theta^+_j) := \frac{i}{\eta_j + i \sqrt{\eta_j^2 + \Xi_{bj}^2}}.
\]

The corresponding operator maps \(W^s(\partial \Omega_j)\) into \(W^{s+1/2}(\Omega_j)\), modulo operators which lead to errors of more smooth type, i.e. which map \(W^{s_1}(\partial \Omega_j) \rightarrow W^{s_2}(\Omega_j)\) for \(s_2 > s_1 + 1/2\). As we will show, the same operators (for each of the domains) arise in the Poisson operator for the intersection domain.
Applying an inverse of the principal second order elliptic operator on the left-hand side of (4.6) to both sides of (4.6), and recalling that \(S_\eta\), we can integrate over the \(\Omega\), which yields

\[
u = \sum_{j=1}^{m} \frac{1}{(2\pi)^{2n}} \int \frac{\partial_j F.T.\eta u(0_j, \eta_j, \xi)}{\eta^2 + \Xi^2(x, \rho, \xi)} e^{i\rho \cdot \eta} e^{ix \cdot \xi} d\eta d\xi \]

(4.10)

\[+ \Psi^{-3} (\partial^{\rho} u|_{\partial \Omega}) + \Psi^{-2} (g_b) + \Psi^{-1} u.\]

To be precise we can rewrite (4.6) by adding term, \(u\), to both sides and then invert the operator with symbol \(1 + \eta^2 + \Xi^2\) in order to avoid complications with zeros in the denominators of the symbols of inverse operators. If we consider then a resulting term of the form

\[
\int \frac{\hat{h}_{bj}(\eta_j, \xi)}{1 + \eta^2 + \Xi^2(x, \rho, \xi)} e^{i\rho \cdot \eta} e^{ix \cdot \xi} d\eta d\xi,
\]

we can integrate over the \(\eta_j\) variable using the residue at \(\eta_j = i \sqrt{1 + \eta_j^2 + \Xi^2}\). Alternatively, set \(\chi_j(\eta_j, \xi) \in C^\infty_0 (\mathbb{R}^{2n-1})\) such that \(\chi_j \equiv 1\) near the origin, and set \(\chi'_j = 1 - \chi_j\). We could then use an expansion

\[
\frac{1}{1 + \eta^2 + \Xi^2} = \chi_j(\eta_j, \xi) + \frac{\chi'_j(\eta_j, \xi)}{1 + \eta^2 + \Xi^2} + \frac{\chi''_j(\eta_j, \xi)}{(1 + \eta^2 + \Xi^2)^2} + \cdots,
\]

where the remainder terms are symbols in class \(S^{-3}(\Omega)\). The term

\[
(4.11) \int \chi_j(\eta_j, \xi) \frac{\hat{h}_{bj}(\eta_j, \xi)}{1 + \eta^2 + \Xi^2(x, \rho, \xi)} e^{i\rho \cdot \eta} e^{ix \cdot \xi} d\eta d\xi
\]

is smoothing (on \(\partial \Omega_j\)), which can be seen by integrating over the \(\eta_j\) variable, using the residue calculus, while the term

\[
\int \chi'_j(\eta_j, \xi) \frac{\hat{h}_{bj}(\eta_j, \xi)}{\eta^2 + \Xi^2(x, \rho, \xi)} e^{i\rho \cdot \eta} e^{ix \cdot \xi} d\eta d\xi
\]

can also be analyzed using the residue calculus without any resulting singular terms. We will implicitly adopt this approach in what follows, but for simplicity we will omit the \(\chi'_j\) factors. In our use of symbols which are singular at the origin, we can use the above approach to reduce the application of such symbols to distributions which vanish at the singularities.

We now return to (4.10). Expanding \(\Xi^2(x, \rho, \xi)\) in each \(\rho_j\), we get

\[
u = \sum_{j=1}^{m} \frac{1}{(2\pi)^{2n}} \int \frac{F.T.\eta u(0_j, \eta_j, \xi)}{\eta^2 + \Xi^2} e^{i\rho \cdot \eta} e^{ix \cdot \xi} d\eta d\xi \]

\[+ \Psi^{-1} u + \sum \rho_j \Psi^{-2} (\partial^{\rho} u|_{\partial \Omega_j}) + \sum \rho_j \Psi^{-1} (g_{\rho j}) + \Psi^{-3} (\partial^{\rho} u|_{\partial \Omega}) + \Psi^{-2} (g_{\rho}).\]
Now, using the property stated in (4.7) above of \( \rho_j \) multiplied with an elliptic operator, we can solve for \( u \) and get

\[
\begin{align*}
 u &= \sum_{j=1}^{m} \frac{1}{(2\pi)^{2n}} \int \frac{F.T. j \partial_{\rho_j} u(0_j, \eta_j, \xi) + i \eta_j \hat{g}_{jk}(\eta_j, \xi)}{\eta_j^2 + \Xi_{bk}} e^{i \rho_j \eta_j e^{i \xi} d\eta d\xi} \\
&\quad + \Psi^{-3} (\partial_{\rho_j} u|_{\partial\Omega_j}) + \Psi^{-2} (g_{bk}) + \Psi^{-\infty} u.
\end{align*}
\]

(4.12)

Taking limits as \( \rho_k \to 0^+ \), and using (4.8), we get

\[
0 = \frac{1}{(2\pi)^{2n}} \sum_{j \neq k} \int \frac{F.T. j \partial_{\rho_j} u(0_j, \eta_j, \xi) + i \eta_j \hat{g}_{jk}(\eta_j, \xi)}{\eta_j^2 + \Xi_{bk}} e^{i \rho_k \eta_k e^{i \xi} d\eta d\xi}
\]

\[
+ \frac{1}{(2\pi)^{2n}} \sum_{j \neq k} \int \frac{F.T. j \partial_{\rho_j} u(0_j, \eta_j, \xi) + i \eta_j \hat{g}_{jk}(\eta_j, \xi)}{\eta_j^2 + \Xi_{bj}} e^{i \rho_k \eta_k e^{i \xi} d\eta d\xi}
\]

\[
+ \Psi^{-2} (\partial_{\rho_k} u|_{\partial\Omega_k}) + \Psi^{-1} (g_{bk}) + R_{bk}^{-\infty}
\]

\[
+ \sum_{j \neq k} R_k \circ \Psi^{-3} (\partial_{\rho_j} u|_{\partial\Omega_j}) + \sum_{j \neq k} R_k \circ \Psi^{-2} (g_{bj}).
\]

(4.13)

We perform integrations in the \( \eta \) variables in the integrals in (4.12) for \( \rho_k > 0 \) above and then let \( \rho_k \to 0^+ \):

\[
0 = \frac{i}{(2\pi)^{2n-1}} \int \frac{F.T. j \partial_{\rho_j} u(0_j, \eta_j, \xi)}{2i \sqrt{\eta_j^2 + \Xi_{bk}} \hat{g}_{jk}(\eta_j, \xi)} e^{i \rho_k \eta_k e^{i \xi} d\eta d\xi}
\]

\[
+ \frac{i}{(2\pi)^{2n-1}} \sum_{j \neq k} \int \frac{F.T. j \partial_{\rho_j} u(0_j, \eta_j, \xi) + \sqrt{\eta_j^2 + \Xi_{bj}} \hat{g}_{jk}(\eta_j, \xi)}{2i \sqrt{\eta_j^2 + \Xi_{bj}}} e^{i \rho_j \eta_j e^{i \xi} d\eta_j d\xi}
\]

\[
+ \Psi^{-2} (\partial_{\rho_k} u|_{\partial\Omega_k}) + \Psi^{-1} (g_{bk}) + R_{bk}^{-\infty}
\]

\[
+ \sum_{j \neq k} R_k \circ \Psi^{-3} (\partial_{\rho_j} u|_{\partial\Omega_j}) + \sum_{j \neq k} R_k \circ \Psi^{-2} (g_{bj}).
\]

(4.13)

We note that the terms,

\[
\int \frac{F.T. j \partial_{\rho_j} u(0_j, \eta_j, \xi) + \sqrt{\eta_j^2 + \Xi_{bj}} \hat{g}_{jk}(\eta_j, \xi)}{2i \sqrt{\eta_j^2 + \Xi_{bj}}} e^{i \rho_j \eta_j e^{i \xi} d\eta_j d\xi}
\]

can be thought of as mappings from distributions on \( \partial\Omega_j \) to distributions on \( \partial\Omega_k \).

Using our notation from Section 2, we will write the operators defined by

\[
h|_{\rho_j=0} \mapsto \int \frac{h(0_j, \eta_j, \xi)}{\sqrt{\eta_j^2 + \Xi_{bj}}} e^{i \rho_j \eta_j e^{i \xi} d\eta_j d\xi}
\]

as \( E_{-3/2} (h|_{\rho_j=0}) \). We also note the \( \Psi^{-3} \) and \( \Psi^{-2} \) operators stem from a symbol expansion of the inverse operator to the principal operator on the left-hand side of
and so are decomposable. Then, from Section 2 we have

\[(4.14)\]

\[
R_k \circ \Psi^{-3} \left( \partial_{\rho_j} u|_{\partial\Omega_j} \right) = E_{-5/2}^j \left( \partial_{\rho_j} u|_{\partial\Omega_j} \right)
\]

\[
R_k \circ \Psi^{-2} (g_{b_j}) = E_{-3/2}^j (g_{b_j}),
\]

for \( j \neq k \).

We define an operator, \( \Gamma^\sharp \) by the symbol

\[\sigma (\Gamma^\sharp) = \frac{1}{\eta^2 + \Xi^2 (x, \rho, \xi)},\]

and \( \Gamma^\sharp_j = \Gamma^\sharp|_{\rho_j = 0} \). Let us also define the operators \( |D_{b_j}| \) by the symbols

\[\sigma (|D_{b_j}|) = \sqrt{\eta^2 + \Xi^2_{b_j}}.\]

From (4.13) we can now write

\[
0 = \frac{1}{2} \frac{1}{(2\pi)^{2n-1}} \int \frac{F.T. \partial_{\rho_k} u(0_k, \eta_k, \xi) e^{i\rho_k \cdot \eta_k e^{ix \xi} d\eta d\xi}}{\sqrt{\eta^2_k + \Xi^2_{b_k}}} \]

\[
- \frac{1}{2} \frac{1}{(2\pi)^{2n-1}} \int g_{b_k}(\eta_k, \xi) e^{i\rho_k \cdot \eta_k e^{ix \xi} d\eta}
\]

\[+ \sum_{j \neq k} R_k \circ \Gamma^\sharp_j \left( \partial_{\rho_j} u|_{\partial\Omega_j} + |D_{b_j}| g_{b_j} \right)\]

\[+ \sum_{j \neq k} E_{-5/2}^j \left( \partial_{\rho_j} u|_{\partial\Omega_j} \right) + \sum_{j \neq k} E_{-3/2}^j (g_{b_j})\]

\[+ \Psi_{b_k}^{-2} \left( \partial_{\rho_k} u|_{\partial\Omega_k} \right) + \Psi_{b_k}^{-1} (g_{b_k}) + R_{b_k}^{-\infty}.\]

We now solve for \( \partial_{\rho_k} u(0_k, \rho_k, \xi) \) by inverting the operator with symbol \( 1/ \left(2 \sqrt{\eta^2_k + \Xi^2_{b_k}}\right)\).

Note that the \( E_{-\alpha}^j \) terms above are of the form \( R_k \circ A_{-(\alpha+1/2)} \) where \( A_{-(\alpha+1/2)} \) is decomposable. Thus, \( |D_{b_j}| \circ E_{-\alpha}^j = E_{-\alpha+1}^j \) for the \( E_{-\alpha}^j \) above. We have

\[
\partial_{\rho_k} u|_{\rho_k = 0} = |D_{b_k}| g_{b_k} - 2 \sum_{j \neq k} |D_{b_k}| \circ R_k \circ \Gamma^\sharp_j \left( \partial_{\rho_j} u|_{\partial\Omega_j} + |D_{b_j}| g_{b_j} \right)\]

\[+ \sum_{j \neq k} E_{-3/2}^j \left( \partial_{\rho_j} u|_{\partial\Omega_j} \right) + \sum_{j \neq k} E_{-1/2}^j g_{b_j}\]

\[
+ \Psi_{b_k}^{-1} \left( \partial_{\rho_k} u|_{\partial\Omega_k} \right) + \Psi_{b_k}^{0} (g_{b_k}) + R_{b_k}^{-\infty}.\]
We now iterate (4.15) to get
\[
\partial_{p_k} u \Big|_{p_k=0} = |D_{b_k}| g_{b_k} - 4 \sum_{j \neq k} |D_{b_k}| o R_k o \Gamma_j^2 o |D_{b_j}| g_{b_j} \\
+ 8 \sum_{j \neq k} |D_{b_k}| o R_k o \Gamma_j^2 o |D_{b_j}| o R_j o \Gamma_l^1 o |D_{b_l}| g_{b_l} \\
+ \Psi_{b_k} (g_{b_k}) + \sum_j \mathcal{E}_{-1/2}^{jk} g_{b_j} \\
+ \Psi_{b_k}^{-1} \left( \partial_{p_k} u \big|_{\partial \Omega_k} \right) + \sum_j \mathcal{E}_{-3/2}^{jk} \left( \partial_{p_j} u \big|_{\partial \Omega_j} \right) + R^{-\infty}_k. \tag{4.16}
\]

Note that \(2\Gamma_j^2 o |D_{b_j}| o R_j \equiv \Theta_j^+ o R_j\), with \(\Theta_j^+\) defined as in (4.9). We thus have from (4.16)
\[
\partial_{p_k} u \Big|_{p_k=0} = |D_{b_k}| g_{b_k} - 2 \sum_{j \neq k} |D_{b_k}| o R_k o \Theta_j^+ g_{b_j} \\
+ 2 \sum_{j \neq k} |D_{b_k}| o R_k o \Theta_j^+ o \Theta_l^+ g_{b_l} \\
+ \Psi_{b_k} (g_{b_k}) + \sum_j \mathcal{E}_{-1/2}^{jk} g_{b_j} \\
+ \Psi_{b_k}^{-1} \left( \partial_{p_k} u \big|_{\partial \Omega_k} \right) + \sum_j \mathcal{E}_{-3/2}^{jk} \left( \partial_{p_j} u \big|_{\partial \Omega_j} \right) + R^{-\infty}_k. \tag{4.17}
\]

When estimating the term \(\partial_{p_k} u \big|_{p_k=0}\), it suffices to write the first two sums on the right-hand side simply as a summation of terms of the form \(|D_{b_k}| o \mathcal{E}_{-1/2}^{jk} g_{b_j}\):
\[
\partial_{p_k} u \big|_{p_k=0} = |D_{b_k}| g_{b_k} + \sum_j |D_{b_k}| o \mathcal{E}_{-1/2}^{jk} g_{b_j} + \Psi_{b_k}^0 (g_{b_k}) + \sum_j \mathcal{E}_{-1/2}^{jk} g_{b_j} \\
+ \Psi_{b_k}^{-1} \left( \partial_{p_k} u \big|_{\partial \Omega_k} \right) + \sum_j \mathcal{E}_{-3/2}^{jk} \left( \partial_{p_j} u \big|_{\partial \Omega_j} \right) + R^{-\infty}_k. \tag{4.18}
\]

The expression (4.17) for the normal derivatives leads to an expression for the solution, \(u\), in (4.12). Recall from our convention in Section 2 that the boundary operator, \(\Psi_{b_k}^{-1}\), when acting on \(\partial_{p_k} u \big|_{p_k=0}\) above can be written \(\mathcal{E}_{-1}^{b_k}\):
\[
\Psi_{b_k}^{-1} \left( \partial_{p_k} u \big|_{\partial \Omega_k} \right) = \mathcal{E}_{-1}^{b_k} \left( \partial_{p_k} u \big|_{\partial \Omega_k} \right).
\]

From (4.12), we thus have the expression for the Poisson solution as
\[
u = \sum_j \Theta_j^+ g_{b_j} + \sum_{j,k} \Psi^{-1} \circ \mathcal{E}_{-1/2}^{jk} g_{b_k} + \sum_j \Psi^{-2} g_{b_j} \\
+ \sum_{j,k} \Psi^{-2} \circ \mathcal{E}_{-1}^{jk} \left( \partial_{p_j} u \big|_{\partial \Omega_j} \right) + \Psi^{-3} \left( \partial_{p} u \big|_{\partial \Omega} \right) + R^{-\infty}. \tag{4.19}
\]

In addition, (4.18) above gives an expression for \(\partial_{p_j} u \big|_{\partial \Omega_j}\) (recall the boundary values are to be understood as non-tangential limits to the boundary). We note the above relation for future use. We also note that all the \(\Psi^{-2}\) and \(\Psi^{-3}\) operators are decomposable, as they arise from the inverse to the Laplacian.
We can now derive (weighted) estimates for the Poisson solution from (4.12); we show

**Theorem 4.1.** Let $u$ be the solution to the homogeneous Dirichlet problem (4.11) with boundary data $g_b$ satisfying $g_b \in L^2(\partial\Omega \cap \partial\Omega_j, \rho_j, \lambda)$ for some $\lambda \geq 0$ and for all $1 \leq j \leq m$. Then

$$\|u\|_{W^{1,2,\ast}(\Omega, \rho, \lambda)} \lesssim \sum_j \|g_b\|_{W^{0,\ast}(\partial\Omega_j \cap \partial\Omega, \rho_j, \lambda)}.$$ 

**Proof.** Weighted estimates for $\partial_{\rho_j} u |_{\rho_j = 0}$ can be read from (4.18). From Theorem 5.6 of [11] (see also [12], Theorem 5.1 in [21], and Theorem 1.4.3 in [19]) we have that $\partial_{\rho_j} u |_{\partial\Omega} \in L^2(\partial\Omega)$, and the first subgoal of the proof is to extend these estimates for $g_b \in W^{\gamma}(\partial\Omega)$, with $0 \leq \gamma \leq 1$, using (4.18) and Theorem 2.1. We note the $R_k^\ast$ term stems from terms of the form $R_k \circ \Psi^{-\ast}$ (see (4.12)) in addition to any terms of the form (4.11) resulting from our handling of the singularities in the inverse to the Laplacian operator. The former can be estimated by $\|u\|_{W^{1,\ast}(\Omega)}$; while the latter by $\|\partial_{\rho_j} u |_{\partial\Omega}\|_{-\infty} + \|g_b\|_{-\infty}$ We have, for $-1 \leq \beta \leq 0,$

$$\left\|\partial_{\rho_j} u |_{\partial\Omega_j}\right\|_{W^{\beta,\ast}(\partial\Omega_j \cap \partial\Omega, \rho_j, \lambda)} \lesssim \|g_b\|_{W^{1,\ast}(\partial\Omega_j \cap \partial\Omega, \rho_j, \lambda)} + \sum_k \|g_{b_k}\|_{W^{1,\ast}(\partial\Omega_k \cap \partial\Omega, \rho_k, \lambda)} + \|u\|_{-\infty} + \|\partial_{\rho_j} u |_{\partial\Omega}\|_{-\infty} + \|g_b\|_{-\infty}.$$ 

In applying weighted estimates to (4.18) we use

$$\left\|D_{hk} |_{\partial\Omega}\right\|_{W^{\beta,\ast}(\partial\Omega_k \cap \partial\Omega, \rho_k, \lambda)} \sim \|h_{bk}\|_{W^{1,\ast}(\partial\Omega_k \cap \partial\Omega, \rho_k, \lambda)},$$

for a distribution $h_{bk} \in L^2(\partial\Omega \cap \partial\Omega_k, \rho_k, \lambda)$, which follows from the product rule of differentiation. Then a direct application of Theorems 2.1, 2.2, and 2.3 yields the inequality.

Then summing over $j$ and bringing lower order estimates of boundary values of derivatives to the left-hand side yields

$$\sum_j \left\|\partial_{\rho_j} u |_{\partial\Omega_j}\right\|_{W^{\beta,\ast}(\partial\Omega_j \cap \partial\Omega, \rho_j, \lambda)} \lesssim \sum_j \|g_{b_j}\|_{W^{1,\ast}(\partial\Omega_j \cap \partial\Omega, \rho_j, \lambda)} + \|u\|_{-\infty} + \|\partial_{\rho_j} u |_{\partial\Omega}\|_{-\infty} + \|g_b\|_{-\infty}.\tag{4.20}$$

In particular,

$$\left\|\partial_{\rho_j} u |_{\partial\Omega}\right\|_{W^{1,\ast}(\partial\Omega_j \cap \partial\Omega, \rho_j, \lambda)} \lesssim \sum_j \|g_{b_j}\|_{W^{0,\ast}(\partial\Omega_j \cap \partial\Omega, \rho_j, \lambda)}\tag{4.21}.$$ 

If we use (4.18) (solving first for $\partial_{\rho_k} u |_{\rho_k = 0}$) in (4.19), we get the expression

$$u = \sum_j \Psi^{-1} g_{b_j} + \sum_{j,k} \Psi^{-1} \circ \mathcal{E}^{kj}_{-1/2} g_{b_k} + \sum_{j,k} \Psi^{-2} \circ \mathcal{E}^{kj}_{-1} \left(\partial_{\rho_k} u |_{\partial\Omega_k}\right),$$

where $\mathcal{E}^{kj}$ is the right regularized factorization for $\mathcal{E}^{kj}_{-1}$.
modulo smoothing terms. The $\Psi^{-2}$ and $\Psi^{-1}$ operators are decomposable so Theorem 2.2 applies. Using (4.21) and the above expression as well as the estimates of Theorem 4.2, we can conclude the estimates

$$\|u\|_{W^{1/2,s}(\Omega, \rho, \lambda)} \lesssim \sum_j \|g_{b_j}\|_{W^{1,s}(\partial\Omega_j \cap \partial\Omega, \rho, \lambda)}$$

$$+ \sum_j \left\| \partial_{\rho_j} u \right|_{\rho_j=0} \right\|_{W^{-1,s}(\partial\Omega_j \cap \partial\Omega, \rho, \lambda)}$$

$$+ \|u\|_{-\infty} + \left\| \partial_{\rho_j} u \right|_{-\infty} + \|g_{b}\|_{-\infty}$$

$$\lesssim \sum_j \|g_{b_j}\|_{W^{1,s}(\partial\Omega_j \cap \partial\Omega, \rho, \lambda)}.$$  

We remark that higher order estimates, for instance

$$\|u\|_{W^{1,s}(\Omega, \rho, k)} \lesssim \sum_j \|g_{b_j}\|_{W^{1/2,s}(\partial\Omega_j \cap \partial\Omega, \rho, k)},$$

follow by taking weighted Sobolev 1 estimates from the form

$$u = \sum_j \Psi^{-2} \left( \partial_{\rho_j} u \right)_{\partial\Omega_j} + \sum_j \Psi^{-1}(g_{b_j})$$

from (4.12). However it is the estimates with base-level (by which we mean the Sobolev level whereby $s = 0$) equal to $1/2$ of the Poisson solution which we will use for our Main Theorem.

In particular, if the boundary data is in $W^s(\partial\Omega)$, for instance as the restriction of a function in $W^{s+1/2}$ in some neighborhood of $\cup_j \Omega_j$ to each piece of the boundary, then the solution $u$ can be estimated by

**Theorem 4.2.**

$$\|u\|_{W^{1/2,s}(\Omega, \rho)} \lesssim \sum_j \|g_{b_j}\|_{W^{s}(\partial\Omega)}.$$  

Compare the estimates in Theorem 4.2 to the estimates in 11: Sobolev $\alpha$ estimates are concluded in 11, where it is shown $\|u\|_{W^{\alpha}(\Omega)} \lesssim \|g_b\|_{W^{\alpha-1/2}(\partial\Omega)}$ for $1/2 \leq \alpha \leq 3/2$.

We also note for future reference an extension of the estimates for the normal derivatives. Under the assumptions of Theorem 4.1 so that in particular we know that the boundary values of the normal derivatives (defined as non-tangential limits) exist and are in $L^2(\partial\Omega_j)$, for $0 \leq \beta \leq 3/2$, we can take Sobolev $-\beta$ estimates of 4.18:

$$\left\| \partial_{\rho_j} u \right|_{\partial\Omega_j} \right\|_{W^{-\beta,s}(\partial\Omega_j \cap \partial\Omega, \rho_j, \lambda)}$$

$$\lesssim \|g_{b_j}\|_{W^{1-\beta,s}(\partial\Omega_j \cap \partial\Omega, \rho_j, \lambda)} + \sum_k \|g_{b_k}\|_{W^{\max(1/2-\beta, -1/2),s}(\partial\Omega_k \cap \partial\Omega, \rho_k, \lambda)}$$

$$+ s.c. \sum_k \left\| \partial_{\rho_k} u \right|_{\partial\Omega_k} \right\|_{W^{-1/2,s}(\partial\Omega_k \cap \partial\Omega, \rho_k, \lambda)}$$

$$+ \|u\|_{-\infty} + \left\| \partial_{\rho_j} u \right|_{-\infty} + \|g_b\|_{-\infty}.$$
Summing over the boundaries yields, in the same manner as (4.21) above, the estimates
\[
\left\| \partial_{\rho_j} u \right\|_{L^p(\partial\Omega_j \cap \partial\Omega, \rho_j, \lambda)} \lesssim \sum_j \left\| g_{b_j} \right\|_{W^{1-\beta, s}(\partial\Omega_j \cap \partial\Omega, \rho_j, \lambda)},
\]
modulo (estimates of) smoothing terms.

We conclude this section by illustrating how the above analysis can be used to obtain an expression for the Poisson operator. A Poisson operator, \(P\), associated with \(\square\) on \(\Omega\), with prescribed boundary values, \(g_b\), is the solution operator to a homogeneous Dirichlet problem
\[
2\square P(g_b) = 0 \quad \text{in } \Omega
\]
\[
P(g_b) = g_b \quad \text{on } \partial\Omega.
\]
As seen from (4.19) the principal terms in the Poisson operator are
\[
\sum_j \Theta_j^+ \circ R_j.
\]
And an expression for the Poisson operator follows from (4.19),
\[
u = \sum_j \Theta_j^+ g_{b_j} + \sum_{j,k} \Psi^{-1} \circ \mathcal{E}_{-1/2}^k g_{b_k} + \sum_{j,k} \Psi^{-2} g_{b_j}
+ \sum_{j,k} \Psi^{-2} \circ \mathcal{E}_{-1}^{k} \left( \partial_{\rho_k} u \right|_{\partial\Omega_k} ) + \Psi^{-3} \left( \partial_{\rho} u \right|_{\partial\Omega} ) + R^{-\infty},
\]
to any desired degree by inserting local expressions for the normal derivatives, \(\partial_{\rho} u \mid_{\partial\Omega}\), as in (4.18), and iterating.

5. DNO

The Dirichlet to Neumann operator is defined as the operator which maps boundary values of the solution to the homogeneous Dirichlet problem to the boundary values of the (outward) normal derivatives of the solution to the homogeneous Dirichlet problem.

We note as a corollary from our proof of Theorem 4.1, in particular the inequalities given in (4.20) and (4.22), the following estimates for the DNO:

**Theorem 5.1.** Let \(-3/2 \leq \beta \leq 0\). Let \(u\) be the solution to (4.1) with \(g_{b_j} \in W^{1+\beta}(\partial\Omega \cap \partial\Omega_j, \rho_j, k)\) for all \(1 \leq j \leq m\). Then
\[
\left\| \partial_{\rho_j} u \right\|_{W^{\beta, s}(\partial\Omega_j \cap \partial\Omega, \rho_j, \lambda)} \lesssim \left\| g_{b_j} \right\|_{W^{1+\beta, s}(\partial\Omega_j \cap \partial\Omega, \rho_j, \lambda)} + \left\| u \right\|_{-\infty} + \left\| \partial_{\rho} u \right\|_{-\infty} + \left\| g_b \right\|_{-\infty}.
\]

In the case of \(\beta = 0\) and \(s = 0\) we obtain the known estimates on Lipschitz domains:
\[
\left\| \partial_{\rho_j} u \mid_{\partial\Omega_j} \right\|_{L^2(\partial\Omega \cap \partial\Omega_j)} \lesssim \left\| g_b \right\|_{W^{1}(\partial\Omega)},
\]
where the boundary values are to be understood in the sense of non-tangential limits, for each \(j\) (see Theorem 5.1 in [21] and Theorem 1.4.3 in [19]).
We start with a simplification of the expression for the normal derivative along a boundary as in \((4.15)\):

\[
\partial_{\rho_k} u\big|_{\rho_k=0} = |D_{bk}| g_{bk} - 2 \sum_{j \neq k} |D_{bk}| \circ R_k \circ \Gamma_j^2 \left( \partial_{\rho_j} u \big|_{\partial \Omega_j} + |D_{bj}| g_{bj} \right) \\
+ \sum_{j \neq k} \mathcal{E}_{-3/2}^j \left( \partial_{\rho_j} u \big|_{\rho_j=0} \right) + \sum_{j \neq k} \mathcal{E}_{-1/2}^j g_{bj} \\
+ \Psi_{bk}^{-1} \left( \partial_{\rho_k} u \big|_{\partial \Omega_k} \right) + \Psi_{bk}^0 \left( g_{bk} \right) + R_{bk}^{-\infty}.
\]

(5.1)

Our aim in this section is to calculate the zero order term, written as \(\Psi_{bk}^0 \left( g_{bk} \right)\) in the expression (5.1), which is the same zero order term in \((4.18)\). We first include the zero order operators on \(g_{bk}\) coming from \(\Psi_{bk}^{-1} \left( \partial_{\rho_k} u \big|_{\rho_k=0} \right)\) with the \(\Psi_{bk}^0 \left( g_{bk} \right)\) term in \((4.18)\). Let us denote the zero order boundary pseudodifferential operator acting on \(g_{bk}\) by \(\Lambda_{bk}^0\) so that (4.17) now reads

\[
\partial_{\rho_k} u\big|_{\rho_k=0} = |D_{bk}| g_{bk} - 2 \sum_{j \neq k} |D_{bk}| \circ R_k \circ \Theta_j^+ g_{bj} \\
+ 2 \sum_{j \neq k} |D_{bk}| \circ R_k \circ \Theta_j^+ \circ \Theta_j^+ g_{bj} + \Lambda_{bk}^0 \left( g_{bk} \right) + B_k g_b,
\]

(5.2)

where we write \(B_k g_b\) to denote the error terms

\[
B_k g_b := \sum_j \mathcal{E}_{-1/2}^j g_{bj} + \sum_j \mathcal{E}_{-3/2}^j \left( \partial_{\rho_j} u \big|_{\partial \Omega_j} \right) + R_{bk}^{-\infty}.
\]

We note for future reference the form of the \(\mathcal{E}_{-3/2}^j\) operators is given by

\[
\mathcal{E}_{-3/2}^j = R_k \circ \Psi^{-2} \circ \Theta_j^+ + |D_{bk}| \circ R_k \circ \Psi^{-1} \circ \Theta_j^+ \circ \Theta_j^+ g_{bj} + \Lambda_{bk}^0 \left( g_{bk} \right) + B_k g_b,
\]

(5.3)

for \(k \neq l, l \neq q, q \neq j\), modulo lower order operators. This will be useful in Section \[\text{Section}\] with (5.2) in \((4.12)\), we can improve the expression for the Poisson solution in \((4.19)\):

\[
u = \sum_j \Theta_j^+ g_{bj} + \sum_{j,k} \Psi^{-1} \circ \mathcal{E}_{-1/2}^{kj} g_{bk} + \sum_j \Psi^{-2} g_{bj} \\
+ \sum_{j,k} \Psi^{-2} \circ \mathcal{E}_{-3/2}^{kj} \left( \partial_{\rho_k} u \big|_{\partial \Omega_k} \right) + \Psi^{-3} \left( \partial_{\rho_k} u \big|_{\partial \Omega_k} \right) + R^{-\infty}.
\]

(5.4)

If we return to the derivation of (4.17), we see the \(\Lambda_{bk}^0 g_{bk}\) comes from \(i\) \(-2|D_{bk}| \circ \Psi_{bk}^{-2} \circ \partial_{\rho_k} u\big|_{\rho_k=0}\), where the \(\Psi_{bk}^{-2}\) operator itself comes from the restriction to \(\partial \Omega_k\) of the operator of order \(-3\) in the symbol expansion of the inverse to \(\Gamma\), \(ii\) \(-2|D_{bk}| \circ \Psi_{bk}^{-1} \circ \partial_{\rho_k} u\big|_{\rho_k=0}\) applied to \(g_{bk}\), where the \(\Psi_{bk}^{-1}\) operator comes from the restriction to \(\partial \Omega_k\) of the operator of order \(-3\) in the symbol expansion of the inverse to \(\Gamma\) (composed with the operator with symbol \(i\nu_k\)), and \(iii\) \(2|D_{bk}| \circ R_k \circ \Psi^{-1} \circ \Theta_k^+ \circ \partial_{\rho_k} u\big|_{\rho_k=0}\) applied to \(g_{bk}\), where the \(\Psi^{-1}\) operator is the same \(\Psi^{-1}\) operator in \((4.10)\), coming from

\[
\Gamma^{-1} \circ \left( \sqrt{2} \sum_{j=1}^m S_j \circ \left( \partial_{\rho_j} u + A(u) \right) \right),
\]
as well as $-2|D_{bk} \circ \Psi_{bk}^{-1} g_{bk}$ terms from the operator of order $-2$ stemming from $\Gamma^{-1} \circ S_j$ in the expression (5.5):

$$\Gamma^{-1} \circ S_j(\partial \varphi_j u) = \Psi^{-1} u + \Gamma^{-1} \circ S_j(g)$$

using (4.2).

Regarding the terms from cases i) and ii) above we need to look at the symbol expansion of the inverse of the operator $\Gamma$. Recall the second order operator, $\Gamma$, in (4.4):

$$\Gamma := -\left[ \partial_{\rho_1}^2 + \cdots + \partial_{\rho_m}^2 + 2 \left( \partial_{x_{m+1}}^2 + \cdots + \partial_{x_{2n}}^2 \right) \right] + \frac{1}{2} \left( \partial_{x_{m+1}}^2 + \cdots + \partial_{x_{2n}}^2 \right) + 2 \sum_{ij} l_{ij} \partial_{x_i} \partial_{x_j} \right] + \sum_{j=1}^m \rho_j \tau_j. $$

(5.6)

For the second order operator $\sum \rho_j \tau_j$ in $\Gamma$, we write

$$\tau_k = -\sum_{i,j} \tau^{ij}_k \frac{\partial^2}{\partial x_i \partial x_j},$$

and

$$\tau^{ij}_k = \tau^{ij}_k(\rho_k, x),$$

modulo $O(\rho_k)$.

We use the expansion

$$\sigma(\Gamma^{-1}) = \frac{1}{\eta^2 + \Xi^2} - \sum_j \partial_{\xi_j} (\eta^2 + \Xi^2) \partial_{x_j} (\eta^2 + \Xi^2) + \sum_j \partial_{\eta_j} (\eta^2 + \Xi^2) \partial_{\rho_j} (\eta^2 + \Xi^2),$$

(5.7)

modulo lower order symbols. We again remind the reader the above expansion is just formal. To avoid the singularities arising at $\eta = \xi = 0$, we could work instead with the operator $\Gamma + I$ and use cutoffs in the expansion (5.7); see the discussion following Equation 4.10.

Recall that, for given $1 \leq j \leq m$, we denote by $\eta_j$ the dual to the tangential (with respect to $\partial \Omega_j$) coordinates $\rho_k$ for $1 \leq k \leq m$. Thus, by $|\eta_j|$ we mean

$$|\eta_j| = \sqrt{\sum_{1 \leq k \leq m} \eta^2_k}. $$

Similarly,

$$|\xi_j| = \sqrt{\sum_{1 \leq k \leq 2n-m} \xi^2_k},$$

for $1 \leq j \leq m$. We also define a notation which gives importance to the vector fields $V_j$ for $m+1 \leq j \leq n$:

$$|\xi_V| = \sqrt{\xi^2_{m+1} + \cdots + \xi^2_{2n-m}}. $$

We extend to $\mathbb{R}^{2n-1}$ the microlocal neighborhoods described in Section 2 for each boundary, $\partial \Omega_j$. Namely, $\psi_{N,bj}$ will be defined in analogy with $\psi^{-}_N$ with support in
the region

\[ \xi_j < -N \sqrt{\eta_j^2 + \xi_j} \]

for \( 1 \leq j \leq m \).

We note

\[
\partial_{x_j}(\eta^2 + \Xi^2) = \partial_{x_j} \Xi^2 = O(x, \rho)O(\xi^2 + \eta^2) + O(\xi V)O(\xi, \eta),
\]

for any \( 1 \leq j \leq 2n - m \) (see Section 3 above), and

\[
\partial_{\rho_j}(\eta^2 + \Xi^2) = \partial_{\rho_j} \Xi^2 = \sum_{k,l} \tau_{jl} \xi_k \xi_l + O(\rho_j),
\]

for \( 1 \leq j \leq m \).

We thus have

\[
\partial_{\xi_j} \Xi^2 \partial_{x_j} \Xi^2 = O(x, \rho) + O\left(\frac{|\xi_j|}{(\eta^2 + \Xi^2)^{3/2}}\right),
\]

while

\[
\partial_{\eta_j} \Xi^2 \partial_{\rho_j} \Xi^2 = 2\tau_{jl} \frac{\eta_j \xi_k \xi_l}{(\eta^2 + \Xi^2)^{3/2}} + O(\rho_j),
\]

which is all we will need to know of this operator. The contribution of the last symbol to the operators written as \( \Psi^{-3} \) in (4.10) is given by

\[
- \frac{i}{(2\pi)^{2n}} \sum_{j,k,l} \int \tau_{jl} \frac{2\eta_j \xi_k \xi_l}{(\eta^2 + \Xi^2)^{3/2}} F.T. \partial_{\rho_j} u(0, \eta_j, \xi) e^{i\rho \eta_j e^{ix} d\eta d\xi},
\]

modulo the \( O(\rho) \) terms, and modulo terms with symbols of order

\[
O\left(\frac{|\eta_j|}{(\eta^2 + \Xi^2)^2}\right)
\]

acting on \( \partial_{\rho_j} u |_{\partial \Omega_j} \). Note that such terms lead to operators with arbitrarily small norm in microlocal neighborhoods defined by the support of \( \psi_{N,bj} \) for large \( N \). Upon integrating with respect to \( \eta_j \), the integrals of the summation term in (5.9) are \( O(\rho_j) \). On the other hand restricting to a boundary \( \Omega_k \) for \( k \neq j \) would lead to \( \mathcal{E}_{-5/2}^{jk} (\partial_{\rho_j} u |_{\rho_j = 0}) \) terms (see (4.14)).

The operator associated with the symbol of order -3 in (5.7) contributes (upon composition with the operators \( 2|D_{\xi}| \)) to the \( \Psi_{bk}^{-1} (\partial_{\rho_k} u |_{\rho_k = 0}) \) in (5.1). Denote this \( \Psi_{bk}^{-1} \) operator by \( A_{bk}^{-1} \). To handle error terms from the order -3 symbol in (5.7), when used as operators, we use the notation \( \Psi_{\xi}^\alpha \) introduced in (4.2) to refer to pseudodifferential operators with small operator norm. We work in a microlocal neighborhood, with respect to \( \partial \Omega_k \), that is with symbols with support in the support of \( \psi_{N,bk} \) with large \( N \). In particular, \( |\xi_k| \ll \sqrt{\xi^2 + \eta_k^2} \), and so for example, a symbol, given by

\[
O\left(\frac{\xi_k^2 + \eta_k^2}{(\xi^2 + \eta_k^2)^2}\right),
\]
of an operator on \( \partial \Omega_k \), will be denoted \( \Psi_{\varepsilon, bk}^0 \). Symbols which are \( O(x, \rho) \) will also be included in \( \Psi_{\varepsilon, bk}^0 \) as we can restrict to a small neighborhood of the point on the boundary under consideration.

We have

\[
A_{bk}^{-1} = -4i \sum_{j,l} |D_{bk}| \circ R_k \circ Op \left( \tau_k^{ijl} \frac{\eta_k \xi_j \xi_l}{(\eta^2 + \Xi_{bk}^2)^3} \right) \circ R_k + \Psi_{\varepsilon, bk}^{-1}
\]

For the terms \(-2|D_{bk}| \circ \Psi_{bk}^{-1}\) arising in case ii) above, we note the \( \Psi_{bk}^{-1} \) operator is just the restriction to the boundary of the operator in \( \Psi^{-2}(g_{bk}) \), from (4.10). Let us denote this operator of order -2 by \( A^{-2} \). As stated earlier, the \( A^{-2} \) operator itself is just the operator of order -3 given by the symbol expansion of the inverse to \( \Gamma \) with symbol as in (5.7), composed with the operator with symbol \( i\eta_k \).

We note that

\[
\frac{i}{(2\pi)^n} \sum_{\alpha, j, l} \tau_\alpha^{ijl} 2\eta_k \xi_j \xi_l (\eta^2 + \Xi_{bk}^2)^3 i\eta_k \hat{g}_{bk}(\eta_k, \xi) e^{i\rho \eta} e^{ix \xi} d\eta d\xi
\]

\[
= -\frac{i}{(2\pi)^n} \sum_{\alpha, j, l} \tau_\alpha^{ijl} \frac{2\eta_k \xi_j \xi_l}{(\eta^2 + \Xi_{bk}^2)^3} i\eta_k \hat{g}_{bk}(\eta_k, \xi) e^{i\rho \eta} e^{ix \xi} d\eta d\xi + O(\rho_k),
\]

using that

\[
\int \frac{\eta_k}{(\eta^2 + \Xi_{bk}^2)^3} \hat{g}_{bk}(\eta_k, \xi) e^{i\rho \eta} e^{ix \xi} d\eta d\xi = O(\rho_k)
\]
as above.

We thus have

\[
R_k \circ A^{-2} g_{bk} = -\frac{i}{(2\pi)^n} R_k \circ \sum_{\alpha, j, l} \tau_\alpha^{ijl} \frac{2\eta_k \xi_j \xi_l}{(\eta^2 + \Xi_{bk}^2)^3} i\eta_k \hat{g}_{bk}(\eta_k, \xi) e^{i\rho \eta} e^{ix \xi} d\eta d\xi
\]

\[
= \frac{2}{(2\pi)^n} R_k \circ \sum_{ij} \tau_{ij}^{kl} \frac{\eta_k^2 \xi_j \xi_l}{(\eta^2 + \Xi_{bk}^2)^3} \hat{g}_{bk}(\eta_k, \xi) e^{i\rho \eta} e^{ix \xi} d\eta d\xi
\]

\[
= \frac{1}{(2\pi)^n} \frac{1}{8} \int \tau_{kk}^{\eta_k^2} (\eta_k^2 + \Xi_{bk}^2)^{3/2} \hat{g}_{bk}(\eta_k, \xi) e^{i\rho \eta} e^{ix \xi} d\eta d\xi,
\]

modulo terms (as in case i) above) which are \( \Psi_{\varepsilon, bk}^{-1} \) in the microlocal neighborhood, with respect to \( \partial \Omega_k \), in which \( |\xi_k| \ll \sqrt{\xi^2 + \eta_k^2} \).

Thus the term in \(-2|D_{bk}| \circ \Psi_{bk}^{-1}(g_{bk})\) stemming from case ii) in a microlocal neighborhood defined by the support of \( \psi_{\Omega_{\varepsilon, bk}} \) can be written as

\[
-2|D_{bk}| \circ \Psi_{bk}^{-1} g_{bk}
\]

\[
= -2|D_{bk}| \circ Op \left( \frac{1}{8} \tau_{kk}^{\eta_k^2} (\eta_k^2 + \Xi_{bk}^2)^{3/2} \right) (g_{bk})
\]

(5.10)

\[
= -\frac{1}{4} \frac{1}{(2\pi)^n} \int \tau_{kk}^{\xi_k^2} (\eta_k^2 + \Xi_{bk}^2) \hat{g}_{bk}(\eta_k, \xi) e^{i\rho \eta} e^{ix \xi} d\eta d\xi,
\]

modulo \( \Psi_{\varepsilon, bk}^0 g_b \) as well as lower order terms.
We now handle case iii) and the terms from
\[ \Gamma^{-1} \circ \left( \sqrt{2} \sum_{j=1}^{m} S_j \circ (\partial_{\rho_j} u) + A(u) \right). \]

We first look at \( \Gamma^{-1} \circ S_j \circ (\partial_{\rho_j} u) \). Let the symbol of \( S_j \) be given by
\[ \sigma(S_j) = s_j(\rho, x). \]
We will also use the notation
\[ s_{0j}(\rho_j, x) = s_j(0, \rho_j, x), \]
and, in the case \( j = k \), simply
\[ s_{0j}(\rho_j, x) = s_j(0, \rho_j, x). \]

Then, modulo lower order terms, we have
\[ \Gamma^{-1} \circ S_j \circ (\partial_{\rho_j} u) = \frac{1}{(2\pi)^n} \int s_j(\rho, x) \frac{\hat{g}_{bj} + i n_j \hat{\mu}}{\eta^2 + \Xi^2} e^{i\rho \cdot \eta} e^{ix\xi} d\eta d\xi. \]

The integral involving \( g_{bj} \) can be calculated by integrating with respect to \( \eta_j \):
\[ \int s_j(\rho, x) \frac{\hat{g}_{bj}}{\eta^2 + \Xi^2} e^{i\rho \cdot \eta} e^{ix\xi} d\eta d\xi = 2\pi \int s_{0j}(\rho_j, x) \frac{\hat{g}_{bj}}{2\eta_j^2 + \Xi_j^2} e^{i\rho_j \cdot \eta_j} e^{ix\xi} d\eta d\xi, \]
modulo lower order terms. Restricting to \( \partial \Omega_k \) and applying \( 2\sqrt{2} D_{bj} \) yields a term
\[ (5.11) \sqrt{2} Op(s_{0k}(\rho_k, x)) \]
which is to be included in the \( \Lambda^0_b \) operator.

For the integral involving \( u \) we use the expression for the Poisson solution in [5.3]. We have
\[ \int s_j(\rho, x) \frac{in_j \hat{\mu}}{\eta^2 + \Xi^2} e^{i\rho \cdot \eta} e^{ix\xi} d\eta d\xi = - \sum_l \int s_{0lj}(\rho_l, x) \frac{\eta_j}{\eta_l^2 + \Xi_l^2} \frac{1}{\sqrt{\eta_l^2 + \Xi_l^2}} \hat{g}_{bl} e^{i\rho_l \cdot \eta_l} e^{ix\xi} d\eta d\xi + \sum_{j,l} \Psi^{-2} \circ \mathcal{E}^{l j}_{-1/2} g_{bl} + \Psi^{-3} g_b \]
\[ (5.12) + \sum_{j,l} \Psi^{-3} \circ \mathcal{E}^{l j}_{-3/2} (\partial_{\rho_l} u) \big|_{\partial \Omega_l} + \Psi^{-4} (\partial_{\rho_l} u) \big|_{\partial \Omega} + R^{-\infty}. \]

We will restrict the above relation to the boundary, \( \partial \Omega_k \), and we analyze the terms in the first summation on the right according to the cases of \( l = k \) or \( l \neq k \), and according to \( j = k \) or \( j \neq k \). In the case \( l \neq k \) restricting to \( \partial \Omega_k \) yields a term \( \mathcal{E}^{lk}_{-3/2} g_{bl} \).
In the case \( l = k \), and \( j \neq k \), we have

\[
-R_k \circ \int s_{k,j}(\rho_k, x) \frac{\eta_j}{\eta^2 + \frac{\xi^2}{\eta^2}} \frac{1}{\eta^2 + \frac{\xi^2}{\eta^2}} g_{jk} e^{i\rho \cdot \eta} e^{i\xi \cdot \xi} d\eta d\xi
\]

\[
= \frac{2\pi i}{4} \sum_{k \neq j} \int s_{k,j}(\rho_k, x) \frac{\eta_j}{\eta^2 + \frac{\xi^2}{\eta^2}} g_{jk} e^{i\rho \cdot \eta} e^{i\xi \cdot \xi} d\eta d\xi
\]

\[
= \Psi_{e,bk}^{-1} g_{bk},
\]

where the symbol of the \( \Psi_{e}^{-1} \) operator is of the form

\[
O \left( \frac{|\eta_j|}{\eta^2 + \frac{\xi^2}{\eta^2}} \right)
\]

and thus can be made arbitrarily small in the support of \( \psi_{N,bk} \) (for large \( N \)).

Finally, in the case \( l = k \) and \( j = k \), we have

\[
-R_k \circ \int s_{k,k}(\rho_k, x) \frac{\eta_k}{\eta^2 + \frac{\xi^2}{\eta^2}} \frac{1}{\eta^2 + \frac{\xi^2}{\eta^2}} g_{kk} e^{i\rho \cdot \eta} e^{i\xi \cdot \xi} d\eta d\xi
\]

\[
= \frac{2\pi}{4} \int s_{0,k}(\rho_k, x) \frac{\eta_k}{\eta^2 + \frac{\xi^2}{\eta^2}} g_{kk} e^{i\rho \cdot \eta} e^{i\xi \cdot \xi} d\eta d\xi.
\]

We can now restrict (5.14) to \( \partial \Omega_k \) and write

\[
R_k \circ \sum_j s_j(\rho, x) \frac{i\eta \bar{\eta}}{\eta^2 + \frac{\xi^2}{\eta^2}} e^{i\rho \cdot \eta} e^{i\xi \cdot \xi} d\eta d\xi
\]

\[
= \left( -\frac{2\pi}{4} \right) \int s_{0,k}(\rho_k, x) \frac{\eta_k}{\eta^2 + \frac{\xi^2}{\eta^2}} g_{kk} e^{i\rho \cdot \eta} e^{i\xi \cdot \xi} d\eta d\xi + \Psi_{e,bk}^{-1} g_{bk}
\]

\[
+ \sum_{j,l} R_k \circ \Psi^{-2} \circ \xi_{-1/2} g_{lk} + R_k \circ \Psi^{-3} g_{lk}
\]

\[
+ \sum_{j,l} R_k \circ \Psi^{-2} \circ \xi_{-3/2} \big( \partial_{\rho} u \big|_{\partial \Omega_k} \big)
\]

\[
+ R_k \circ \Psi^{-4} \big( \partial_{\rho} u \big|_{\partial \Omega} \big) + R_{bk}^{-\infty}.
\]

Applying \( 2\sqrt{2}|D_k| \) yields other terms

\[
(5.13)
\]

\[
-\frac{\sqrt{2}}{2} \text{Op}(s_{0,k}) + \Psi_{e,bk}^0
\]

to be added to \( \Lambda_k^0 \).

We also note that the error terms arising from \( 2\sqrt{2}|D_{bk}| \Gamma^{-1} \circ S_j \circ \big( \partial_{\rho} u \big) \) are of the form

\[
\sum_l \xi_{-1/2} g_{lk} + \Psi_{bk}^{-1} g_{bk} + \sum_l \xi_{-3/2} \big( \partial_{\rho} u \big|_{\partial \Omega} \big) + R_{bk}^{-\infty}
\]

and are thus already included in the formula (5.12).

Putting (5.11) and (5.13) together, we see the terms \( 2\sqrt{2}|D_{bk}| \circ R_k \circ \Gamma^{-1} \circ S_j \circ \partial_{\rho} u \), yield a

\[
(5.14)
\]

\[
\frac{\sqrt{2}}{2} \frac{1}{(2\pi)^{n-1}} \int s_{0,k}(\rho_k, x) \frac{\eta_k}{\eta^2 + \frac{\xi^2}{\eta^2}} g_{kk} e^{i\rho \cdot \eta} e^{i\xi \cdot \xi} d\eta d\xi,
\]
which is to be included in the $A_b^0$ operator.

We next look at $\Gamma^{-1} \circ A(u)$. $A$ is a first order differential operator (tangential to all boundaries $\partial \Omega_j$ for $1 \leq j \leq m$). Denote the symbol of $A$ by

$$\sigma(A) = \alpha(\rho, x, \xi, \eta) = \sum_0 \alpha_j(\rho, x)\xi_j.$$  

In analogy with the symbol $s_{0j}$ we define

$$\alpha_{0k}(\rho_k^j, x) := \alpha(0_k^j, \rho_k^j, x)$$

$$\alpha_{0k,j}(\rho_k^j, x) := \alpha_j(0_k^j, \rho_k^j, x).$$

We again use the expression in (4.19) for $u$ in (5.2).

(5.16) $1$

In analogy with the symbol $s$ which is to be included in the $\Lambda_0^j$ to highest order, i.e. modulo terms of the form (5.15).

We use the notation $\cdots$ in the following representation to indicate terms which upon being operated by $|D_{bk} \circ R_k$ lead to terms of the form

(5.15) $\Psi_{bk}^{-1} g_{bk} + E^{jk}_{-1/2}(g_{bj}) + E^{jk}_{-3/2} (\partial_{\rho_j} u |_{\rho_j=0}) + R_{bk}^{-\infty}$

in (5.2).

We have

$$\Gamma^{-1} \circ A(u) = \frac{1}{(2\pi)^n} \int \alpha(\rho, x, \xi) \frac{\hat{u}}{\eta^2 + \Xi^2} e^{ip \cdot \eta} e^{i \xi \cdot \xi} d\eta d\xi + \Psi^{-2} u$$

$$= \frac{1}{(2\pi)^n} \sum_0 \int \alpha_{0k}(\rho_k^j, x, \xi) \frac{1}{\eta^2 + \Xi_j^2} \frac{\hat{g}_{bj}}{\eta_j + i \sqrt{\eta^2_j + \Xi^2_j}} e^{ip \cdot \eta} e^{i \xi \cdot \xi} d\eta d\xi + \cdots$$

$$= \frac{1}{4} \frac{1}{(2\pi)^{n-1}} \sum_0 \int \alpha_{0k,j}(\rho_k^j, x, \xi) \frac{\hat{g}_{bj}}{\eta_j^2 + \Xi^2_j} e^{ip \cdot \eta} e^{i \xi \cdot \xi} d\eta d\xi + \cdots.$$

Then, the terms $2|D_{bk} \circ R_k \circ \Gamma^{-1} \circ A$ yield

(5.16) $\frac{1}{2} \frac{1}{(2\pi)^{n-1}} \int \alpha_{0k}(\rho_k^j, x, \xi) \frac{\hat{g}_{bk} e^{ip \cdot \eta} e^{i \xi \cdot \xi} d\eta d\xi}

$$\sqrt{\eta_k^2 + \Xi^2_{bk}}$$

to highest order, i.e. modulo terms of the form (5.15).

We are now ready to put together the $A_{0k}^b$ operator according to the terms from cases $i)$, $ii)$ and $iii)$ above. From (5.10), (5.14), and (5.16) above we have

$$A_{0k}^b g_{bk} = -\frac{1}{4} \frac{1}{(2\pi)^{n-1}} \int \tau_{kk} \frac{\eta_k^2}{\eta_k^2 + \Xi^2_{bk}} \hat{g}_{bk}(\eta_k, \xi) e^{ip \cdot \eta} e^{i \xi \cdot \xi} d\eta d\xi$$

$$+ \sqrt{2} \frac{1}{2} \frac{1}{(2\pi)^{n-1}} \int s_{0k}(\rho_k^j, x) \hat{g}_{bk} e^{ip \cdot \eta} e^{i \xi \cdot \xi} d\eta d\xi$$

(5.17) $\frac{1}{2} \frac{1}{(2\pi)^{n-1}} \int \alpha_{0k,j}(\rho_k^j, x, \xi) \frac{\hat{g}_{bk} e^{ip \cdot \eta} e^{i \xi \cdot \xi} d\eta d\xi}

$$\sqrt{\eta_j^2 + \Xi^2_j}$$

modulo $\Psi_{\varepsilon, bk}^{-1} g_b$ (from cases $i$) and $ii$)). Recall that

$$\|\Psi_{\varepsilon, bk}^{-1} g_b\|_{W^{\gamma', \infty}(\Gamma_{\Omega_j \cap \partial \Omega_l, \rho_k, \lambda})} \lesssim_{s,c} \|g_b\|_{W^{\gamma, \infty}(\Omega_{\rho_k, \rho_l, \lambda})}.$$  

We also note the error terms of the form

$$C_{k} g_{bk} = \Psi_{bk}^{-1} g_{bk} + \sum_j E_{-1/2}^{jk} g_{bj} + \sum_j E_{-3/2}^{jk} \left( \partial_{\rho_j} u |_{\rho_j=0} \right) + R_{bk}^{-\infty}$$
in the $B_k g_b$ terms in (6.2) and those resulting from the above expansions. The $E_{-3/2}^{jk}$ terms remain in the form (5.3) with additional terms of the form

$$|D_k| \circ R_k \circ \Psi^{-2} \circ E_{-3/2}^{ij}.$$ 

We thus have from (5.2)

**Theorem 5.2.** Let $N_k^-$ be the DNO operator mapping the boundary values of the homogeneous Dirichlet problem (4.1) to the boundary values on $\partial \Omega_k \cap \partial \Omega$ of the outward normal derivative of the solution. Then

$$N_k^- g_b = |D_k| g_b k - 2 \sum_{j \neq k} |D_{bk}| \circ R_k \circ \Theta_j^+ g_{bj}$$

$$+ 2 \sum_{j \neq k} |D_{bk}| \circ R_k \circ \Theta_j^+ \circ R_j \circ \Theta_j^+ g_{bj} + \Lambda^0_{bk} (g_{bk}) + C_k g_b,$$

where in the microlocal support of a cutoff, $\psi^N_{N, bk}$, we have modulo operators of the form $\psi^{\alpha, \gamma}_{\epsilon, \beta}$ for large $N$,

$$\Lambda^0_{bk} = -\frac{1}{4} Op \left( \sum_{k} \frac{\xi_k^2}{\eta_k^2 + \varepsilon_k^2} \right) + \frac{\sqrt{2}}{2} Op (s_{0k} (\rho_k, \xi)) + \frac{1}{2} Op \left( \frac{\alpha_{0k} (x, \rho_k, \xi)}{\sqrt{\eta_k^2 + \varepsilon_k^2}} \right),$$

and, for $0 \leq \gamma \leq 1$ and $\lambda \geq 0$,

$$\|C_k g_b\|_{W^{\gamma, \lambda} (\partial \Omega_k \cap \partial \Omega, \rho_k, \lambda)} \lesssim \sum_{j} \|g_{bj}\|_{W^{\gamma-1/2, \lambda} (\partial \Omega_k \cap \partial \Omega, \rho_j, \lambda)} + \|u\|_{-\infty} + \|\partial_{\rho} u\|_{-\infty} + \|g_b\|_{-\infty}.$$

6. Green’s operator

The Green’s operator for the $\partial \Omega$ operator is defined as the solution operator, $G$, to

$$2 \Box G(g) = g \quad \text{in } \Omega$$

$$G(g) = 0 \quad \text{on } \partial \Omega.$$

As we did with the Poisson operator, we will find an expression for the Green’s operator, modulo some smoothing terms. For this purpose, we use again the notation $R^{-\infty}$ to refer to smoothing terms, but in this section $R^{-\infty}$ will mean $\Psi^{-\infty}$ applied to $u = G(g)$, or $\Psi^{-\infty}$ applied to the boundary terms $\partial_{\rho} u |_{\partial \Omega}$. Furthermore, $R^{-\infty}$ will denote $\Psi^{-\infty}$ applied to $\partial_{\rho} u |_{\partial \Omega_k}$ and to denote terms described by $R_k \circ R^{-\infty}$.

For instance, we have, for any $\alpha \geq 0$,

$$\|R_{bk}^{-\infty}\|_{W^{\alpha, \lambda} (\partial \Omega_k \cap \partial \Omega, \rho_k, \lambda)} \lesssim \|R_k \circ \Psi^{-\infty} u\|_{W^{\alpha, \lambda} (\partial \Omega_k)} + \|R_k \circ \Psi^{-\infty} (\partial_{\rho} u |_{\partial \Omega})\|_{W^{\alpha, \lambda} (\partial \Omega_k)}$$

$$+ \|\partial_{\rho} u |_{\partial \Omega_k}\|_{W^{-\infty} (\partial \Omega_k)} \lesssim \|\Psi^{-\infty} u\|_{W^{\alpha + 1/2, \lambda} (\Omega_k)} + \|\Psi^{-\infty} (\partial_{\rho} u |_{\partial \Omega})\|_{W^{\alpha + 1/2, \lambda} (\Omega_k)}$$

$$+ \|\partial_{\rho} u |_{\partial \Omega_k}\|_{W^{-\infty} (\partial \Omega_k)} \lesssim \|u\|_{L^2 (\Omega)} + \|\partial_{\rho} u |_{\partial \Omega}\|_{W^{-\infty} (\partial \Omega)}.$$ (6.1)
where $\epsilon \geq 0$ is chosen so that $\alpha + \epsilon > 0$, and thus so that the Trace Theorem applies. The $L^2$ norm of $u$ can be bounded by $\|g\|_{L^2(\Omega)}$ (see Theorem 5 of [11]). To obtain estimates for the boundary values of the normal derivative, we argue as in Section 4.

\[
\partial_{\rho_j} u \big|_{\rho_j=0} = \int_{-\infty}^{0} \partial_{\rho_j}^2 u d\rho_j
\]

(6.2)

\[
= \int_{-\infty}^{0} \Lambda_i^2 u d\rho_j + \int_{-\infty}^{0} \Lambda_i^1 g d\rho_j,
\]

where $\Lambda_i^j$ is an operator of order $i$ tangential to $\partial \Omega_j$. Then applying a smoothing tangential operator yields

\[
\|\partial_{\rho} u \big|_{\partial \Omega}\|_{W^{-\infty}(\partial \Omega)} \lesssim \|u\|_{L^2(\Omega)} + \|g\|_{L^2(\Omega)} + \|g\|_{L^2(\Omega)}.
\]

Putting all this together yields

\[
\|R_{bk}^{-\infty} \|_{W^{-\infty}(\partial \Omega_k \cap \partial \Omega, \lambda)} \lesssim \|u\|_{L^2(\Omega)} + \|g\|_{L^2(\Omega)} + \|g\|_{L^2(\Omega)}
\]

for any $\lambda \geq 0$.

Using the notation of Section 4 with the $\Gamma$ operator defined as in (5.6), for the solution using Green’s operator, we write the interior equation as

\[
\Gamma u + \sqrt{2} S \partial_{\rho} u + A u = g,
\]

where $\sqrt{2} S \partial_{\rho} u$ is the short-hand notation for the sum of terms

\[
\sqrt{2} S \partial_{\rho} u := \sqrt{2} \sum_{j=1}^{m} S_j \circ (\partial_{\rho_j} u)
\]

as in (4.6), and with boundary conditions $u = 0$ on $\rho_i = 0$, $i = 1, \ldots, m$.

Applying the operator, $\Gamma^2$, with symbol $\frac{1}{\eta + 2\sigma}$ we get, in a similar manner to (4.10) above,

\[
u = \sum_j \Gamma^2 \left( \partial_{\rho_j} u \right) \big|_{\rho_j=0} + \Psi^{-3} (\partial_{\rho_j} u \big|_{\partial \Omega}) + \Gamma^2 g + \Psi^{-3} g + \Psi^{-1} u.
\]

Solving for $u$ yields

\[
u = \sum_j \Gamma^2 \left( \partial_{\rho_j} u \right) \big|_{\rho_j=0} + \Psi^{-3} (\partial_{\rho_j} u \big|_{\partial \Omega}) + \Gamma^2 g + \Psi^{-3} g + \Gamma^2 g
\]

(6.3)

We now obtain an expression for $\partial_{\rho_k} u \big|_{\rho_k=0}$. We use

\[
R_k \circ \partial_{\rho_k} \circ \sum_j \Gamma^2 \left( \partial_{\rho_j} u \right) \big|_{\rho_j=0} = \frac{1}{2} \partial_{\rho_k} u \big|_{\rho_k=0} + \sum_{j \neq k} \mathcal{E}^{ik}_{-1/2} \left( \partial_{\rho_j} u \right) \big|_{\rho_j=0}
\]
and

\[
\partial_{\rho_k} \Gamma^g = \frac{1}{(2\pi)^n} \int \frac{i\eta_k}{\eta_k^2 + \left(\eta_k^2 + \Xi^2\right)} \tilde{g}(\eta, \xi)e^{i\rho_k \eta} \mathcal{E}^{j\eta_k} d\xi d\eta
\]

\[
= \frac{1}{(2\pi)^n} \int \frac{i\eta_k}{\eta_k^2 + \left(\eta_k^2 + \Xi^2\right)} \int_{-\infty}^{0} \tilde{g}(t_k, \eta_k, \xi)e^{it_k \eta} \mathcal{E}^{j\eta_k} dt_k d\eta
\]

\[
= \frac{1}{(2\pi)^n} \int \frac{i\eta_k}{\eta_k^2 + \left(\eta_k^2 + \Xi^2\right)} \tilde{g}(t_k, \eta_k, \xi)e^{i(\rho_k - t_k) \eta} \mathcal{E}^{j\eta_k} dt_k d\eta
\]

\[
= \frac{1}{(2\pi)^n} \int \frac{1}{2} \left( \int_{-\infty}^{0} \text{sgn}(t_k - \rho_k) \tilde{g}(t_k, \eta_k, \xi) \times e^{-|\rho_k - t_k| \sqrt{\eta_k^2 + \Xi^2}} dt_k \mathcal{E}^{j\eta_k} d\eta_k. \right)
\]

which, in the limit \( \rho_k \to 0 \), tends to

\[
(6.4) \quad - \frac{1}{(2\pi)^n} \int \tilde{g}(t_k, \eta_k, \xi)e^{i\sqrt{\eta_k^2 + \Xi^2}} dt_k \mathcal{E}^{j\eta_k} d\eta_k.
\]

Let \( \Theta_j^- \in \Psi^{-1}(\Omega) \) be the operator with symbol

\[
\sigma(\Theta_j^-) = \frac{i}{\eta_j - i/\sqrt{\eta_j^2 + \Xi^2}}.
\]

We can then rewrite the term in (6.4) as \( \frac{1}{2} R_k \circ \Theta_j^- g \), and

\[
R_k \circ \partial_{\rho_k} \circ \Gamma^g = \frac{1}{2} R_k \circ \Theta_j^- g.
\]

Returning to (6.3) and applying \( R_k \circ \partial_{\rho_k} \) yields

\[
\partial_{\rho_k} u|_{\rho_k=0} = \frac{1}{2} \partial_{\rho_k} u|_{\rho_k=0} + \frac{1}{2} R_k \circ \Theta_k^- g + \sum_{j \neq k} \mathcal{E}^{j\eta_k} \left( \partial_{\rho_j} u|_{\rho_j=0} \right)
\]

\[
+ \Psi_{bk}^{-1}(\partial_{\rho_k} u|_{\rho_k=0}) + R_k \circ \Psi^{-2} g + R_{bk}^{-\infty}.
\]

Thus

\[
(6.5) \quad \partial_{\rho_k} u|_{\rho_k=0} = R_k \circ \Theta_k^- g + \sum_{j \neq k} \mathcal{E}^{j\eta_k} \left( \partial_{\rho_j} u|_{\rho_j=0} \right) + R_k \circ \Psi^{-2} g + R_{bk}^{-\infty}.
\]

Our aim is to provide (weighted) estimates for \( G(g) \). To deduce these with the help of (6.3), we need estimates for boundary values of the normal derivatives, and we start with estimating the term \( \Theta_j^- g \) in (6.5) (see also Theorem 2.3 whose proof can be read from that below). In the following Theorems (and Corollaries), we will assume the data function (form) satisfies

\[
g \in W^{0,s}(\Omega, \rho_j, \lambda)
\]

for some \( s, \lambda \geq 0 \) and for all \( j = 1, \ldots m \). We start with

**Theorem 6.1.**

\[
\|\Theta_j^- g\|_{W^{1,s}(\Omega, \rho_j, \lambda)} \lesssim \|g\|_{W^{0,s}(\Omega, \rho_j, \lambda)}.
\]
Proof. We recall the convention that $\Theta_i^{-} g$ for $g$ defined on $\Omega$ refers to $\Theta_i^{-} g^{E_i}$, where the superscript $E_i$ denotes extensions (defined locally) by zero across $\rho_i = 0$ for $i \in I := \{1, \ldots, m\}$.

Since $\Theta_j^{-} g = \Psi^{-1} g^{E_j}$, we have
\[
\|\Theta_j^{-} g\|_{W^{1, r}(\Omega_j, \lambda)} \lesssim \sum_{r \leq s} \left\| \rho_j^{r\lambda \times (m-1)} \Psi^{-1} g^{E_j} \right\|_{W^{1+r}(\mathbb{R}^n)},
\]
whereas for each $r \leq s$ it holds that
\[
\left\| \rho_j^{r\lambda \times (m-1)} \Psi^{-1} g^{E_j} \right\|_{W^{1+r}(\mathbb{R}^n)} \lesssim \sum_{t \leq r} \left\| \Psi^{-1-(r-t)} \rho_j^{t\lambda \times (m-1)} g^{E_j} \right\|_{W^{1+r}(\mathbb{R}^n)} \lesssim \|g^{E_j}\|_{W^{0, r}(\mathbb{R}^n, \rho_j, \lambda)}.
\]

Summing over $r \leq s$ yields
\[
\|\Theta_j^{-} g\|_{W^{1, r}(\Omega_j, \rho_j, \lambda)} \lesssim \|g^{E_j}\|_{W^{0, r}(\mathbb{R}^n, \rho_j, \lambda)} \lesssim \|g\|_{W^{0, r}(\Omega_j, \rho_j, \lambda)}.
\]

As a corollary of the Sobolev Trace Theorem (applied to each smooth domain $\Omega_j$) we obtain

**Corollary 6.2.**
\[
\|R_j \circ \Theta_j^{-} g\|_{W^{1/2, r}(\partial \Omega_j \cap \partial \Omega_j, \rho_j, \lambda)} \lesssim \|g\|_{W^{0, r}(\Omega_j, \rho_j, \lambda)}.
\]

**Proof.** For $\varphi$, a function defined on $\mathbb{R}^n$, let $Z_{\Omega_j}(\varphi)$ denote the extension from $\Omega$ to $\Omega_j$ defined by
\[
Z_{\Omega_j}(\varphi) := \begin{cases} 
\varphi & \text{in } \Omega \\
0 & \text{in } \Omega_j \setminus \Omega.
\end{cases}
\]

We then have
\[
\|R_j \circ \Theta_j^{-} g\|_{W^{1/2, r}(\partial \Omega_j \cap \partial \Omega_j, \rho_j, \lambda)} \approx \sum_{r \leq s} \|R_j \circ Z_{\Omega_j} \left( \rho_j^{r\lambda \times (m-1)} \Theta_j^{-} g \right)\|_{W^{1/2+r}(\partial \Omega_j)} \lesssim \sum_{r \leq s} \left\| Z_{\Omega_j} \left( \rho_j^{r\lambda \times (m-1)} \Theta_j^{-} g \right) \right\|_{W^{1+r}(\Omega_j)}.
\]

(6.6)

Furthermore, since
\[
\rho_j^{r\lambda \times (m-1)} \Theta_j^{-} g \in W^{1+r}(\Omega)
\]
by Theorem 6.1, we have
\[
Z_{\Omega_j} \left( \rho_j^{r\lambda \times (m-1)} \Theta_j^{-} g \right) \in W^{1+r}(\Omega_j)
\]
with
\[
\left\| Z_{\Omega_j} \left( \rho_j^{r\lambda \times (m-1)} \Theta_j^{-} g \right) \right\|_{W^{1+r}(\Omega_j)} \lesssim \left\| \rho_j^{r\lambda \times (m-1)} \Theta_j^{-} g \right\|_{W^{1+r}(\Omega)}
\]
by Lemma 4.2 in [8].

Inserting these last estimates into (6.6) gives
\[
\|R_j \circ \Theta_j^{-} g\|_{W^{1/2, r}(\partial \Omega_j \cap \partial \Omega_j, \rho_j, \lambda)} \lesssim \sum_{r \leq s} \left\| \rho_j^{r\lambda \times (m-1)} \Theta_j^{-} g \right\|_{W^{1+r}(\Omega)} \lesssim \|\Theta_j^{-} g\|_{W^{1, r}(\Omega_j, \rho_j, \lambda)}.
\]
The Corollary now follows from another application of Theorem 6.1.

□

A similar proof can be used to estimate the terms which appear as \( R_j \circ \Psi^{-2}g \) in \((6.5)\) above. We obtain

\[
\| R_j \circ \Psi^{-2}g \|_{W^{1/2,\infty}(\partial \Omega_j \cap \partial \Omega, \rho, \lambda)} \lesssim \| g \|_{W^{0,\infty}(\Omega, \rho, \lambda)}.
\]

We can now establish estimates for the boundary values of normal derivatives of the Green’s solution.

**Corollary 6.3.**

\[
\sum_j \| \partial_{\rho_j} u \|_{\rho_j=0} \| W^{1/2,\infty}(\partial \Omega_j \cap \partial \Omega, \rho, \lambda) \lesssim \sum_j \| g \|_{W^{0,\infty}(\Omega, \rho, \lambda)}.
\]

**Proof.** First, we note that with the assumption that \( g \in L^2(\Omega) \), we have \( \partial_{\rho_j} u \rvert_{\partial \Omega} \in W^{-1}(\partial \Omega) \) by \((6.2)\) and interior estimates for for the solution to the inhomogeneous Dirichlet problem \((11)\). Using a bootstrapping argument, we can use \((6.5)\) to show \( \partial_{\rho_j} u \rvert_{\partial \Omega} \in W^{1/2}(\partial \Omega) \). Applications of \((6.5)\) can then be used to get weighted estimates. First, to show \( \rho_j \partial_{\rho_j} u \rvert_{\partial \Omega_j} \in W^{1}(\partial \Omega_j \cap \partial \Omega) \), and with this, that \( \rho_j \partial_{\rho_j} u \rvert_{\partial \Omega_j} \in W^{3/2}(\partial \Omega_j \cap \partial \Omega) \). Then multiplication with \( \rho_j^2 \) can be used to show first \( \rho_j^2 \partial_{\rho_j} u \rvert_{\partial \Omega_j} \in W^{2}(\partial \Omega_j \cap \partial \Omega) \), and then \( \rho_j^2 \partial_{\rho_j} u \rvert_{\partial \Omega_j} \in W^{5/2}(\partial \Omega_j \cap \partial \Omega) \), and so forth.

For the estimates, we take weighted estimates of \((6.5)\), applying the estimates for \( R_j \circ \Theta_j^{-1}g \) and \( R_j \circ \Psi^{-2}g \) above:

\[
\| \partial_{\rho_j} u \|_{\rho_j=0} \| W^{1/2,\infty}(\partial \Omega_j \cap \partial \Omega, \rho, \lambda) \lesssim \| g \|_{W^{0,\infty}(\Omega, \rho, \lambda)} + \sum_k \| \partial_{\rho_k} u \|_{\rho_k=0} \| W^{0,\infty}(\partial \Omega_k \cap \partial \Omega, \rho, \lambda)\).
\]

We used the estimates in \((6.1)\) to estimate the smooth terms, \( R_{bj} \) from \((6.5)\). Summing over all boundaries \( \partial \Omega_j \) and using

\[
\| \partial_{\rho_k} u \|_{\rho_k=0} \| W^{0,\infty}(\partial \Omega_k \cap \partial \Omega, \rho, \lambda) \lesssim \quad \text{s.c.} \| \partial_{\rho_k} v \|_{\rho_k=0} \| W^{1/2,\infty}(\partial \Omega_k \cap \partial \Omega, \rho, \lambda) + \| \partial_{\rho_k} u \|_{\rho_k=0} \| W^{-\infty}(\partial \Omega) \|
\]

to bring the last sum on the right to the left hand side yields

\[
\sum_j \| \partial_{\rho_j} u \|_{\rho_j=0} \| W^{1/2,\infty}(\partial \Omega_j \cap \partial \Omega, \rho, \lambda) \lesssim \sum_j \| g \|_{W^{0,\infty}(\Omega, \rho, \lambda)}.
\]

As a corollary we can prove the

**Theorem 6.4.** For any \( k \),

\[
\| G(g) \|_{W^{2,\infty}(\Omega, \rho_k, \lambda)} \lesssim \| g \|_{W^{0,\infty}(\Omega, \rho_k, \lambda)},
\]

for \( s \geq 0 \).

**Proof.** As above, let \( u = G(g) \). Estimates come by taking \( W^{2,s} \) norms of the terms on the right-hand side of \((6.3)\):

\[
u = \sum_j \Psi^{-2} \left( \partial_{\rho_j} u \rvert_{\rho_j=0} \right) + \Psi^{-2} g + R^{-\infty}.
\]
The first $\Psi^{-2}$ operator is decomposable, arising as the inverse to the Laplacian. Therefore the estimates of Theorem 2.2 can be applied to the term $\Psi^{-2} \left( \partial_{\rho_j} u \big|_{\rho_j=0} \right)$. Estimates for $\Psi^{-2} g$ follow as in Theorem 6.1, and those for $R^{-\infty}$ follow as in (6.1).

Thus in the case of intersections of smooth domains we obtain a (weighted) gain of two derivatives. This is also the case with Lipschitz domains however with Lipschitz domains the level of Sobolev norms for the solution is restricted between $W^{1/2}$ and $W^{3/2}$ (Theorem 5.11).

7. Boundary equations

We now return to the conditions in (3.8): for $j \notin K$,

$$L_j u_K + (-1)^{\left| \mathcal{K} \right|} c_{jK}^K u_K = 0$$

on $\rho_j = 0$ for $1 \leq j \leq m$.

We write $u = G(2f) + P(u_b)$. From Section 5 we have

$$\partial_{\rho_j} P_K(u_b)_{\rho_j=0} = |D_{bj}| u_{K, bj} + E u_b + \Lambda_{bj}^0 u_{K, bj},$$

$E$ is given by the $|D_{bj}| \circ \sum_k \mathcal{E}_{-1/2}^{kj} u_{K, bk} + C_j u_b$ terms in Theorem 5.2 and $\Lambda_{bj}^0$ is also as in Theorem 5.2.

On the boundary $\partial \Omega_j$ we use the notation $X_{k, bj}$ to denote the complex tangential (to $\Omega_j$) vector fields: for $1 \leq j \leq m$, and $k \neq j$, we set

$$X_{k, bj} = \begin{cases} L_k_{\rho_j=0} \quad &\text{if } 1 \leq k \leq m \\ V_k_{\rho_j=0} \quad &\text{if } m + 1 \leq k \leq n. \end{cases}$$

$\Lambda_{bj}^0$ is a matrix of pseudodifferential operators, and we write $\sigma(\Lambda_{bj}^0)_{K, K}$ to refer to the symbol in the $(K, K)^{th}$ entry of the matrix symbol $\sigma(\Lambda_{bj}^0)$.

We will use $| \cdot |_{L_j}$ to denote the Levi-norm length of a vector field: the Levi norm is given by Levi metric, which is defined by

$$ds_j^2 = \sum \frac{\partial^2 \rho_j}{\partial \bar{z}_k \bar{z}_l} d z_k d \bar{z}_l,$$

and the norm of a vector field,

$$Z = \sum \gamma_j \frac{\partial}{\partial z_j}$$

with respect to this metric will be written as

$$|Z|_{L_j} := \sum \frac{\partial^2 \rho_j}{\partial \bar{z}_k \bar{z}_l} \gamma_k \bar{\gamma}_l.$$

We can use Proposition 5.1 of [7], which relates the symbols of the $S_j$, $A$, and $\tau_j$ operators of (4.4) with the terms $c_{j K}^K$ as well as the Levi-norms (on $\partial \Omega_j$) of
tangential vectors, to find the symbol of $\Lambda^0_{bj}$. On the diagonal of $\Lambda^0_{bj}$ we have

\[
\sigma(\Lambda^0_{bj})_{K,K} = -(-1)^{|K|} \sqrt{2e_j^{K\bar{k}}} + \sum_{k \notin K} |X_{k,bj}|_{\xi_j}^2 - \sum_{k \in K} |X_{k,bj}|_{\xi_j}^2
\]

\[
+ O(x, \rho_j) + O \left( \frac{\eta_j^2 + \xi_j^2}{\eta_j^2 + \varepsilon_{bj}^2} \right)
\]

in a microlocal neighborhood defined by the set product of a neighborhood of the origin with the support of the $\psi_{N,bj}$ symbol defined in Section 5. There will also be some entries off the diagonal for $\sigma(\Lambda^0_{bj})$, which arise from the contribution of the first order operators, $\sum_{k \notin K} j K_j \cdot (\bar{W}_j, W_k)[u_{K,j}k \bar{\omega}_K, \ (j \in K)$ in Proposition 3.1 $(ii)$ to the terms off the diagonal in the symbol $\sigma(A) = \alpha(p, x, \xi)$. We note for now that the commutators, $[\bar{W}_j, W_k]$, are tangential with respect to $\partial \Omega_j$.

Thus, for $1 \leq j \leq m$, we can write

\[
\frac{1}{\sqrt{2}} (\Lambda^0_{bj})_{K,K} + (-1)^{|K|} e_j^{K}_{jK} = \left( \frac{1}{\sqrt{2}} \sum_{k \notin K} |X_{k,bj}|_{\xi_j}^2 - \frac{1}{\sqrt{2}} \sum_{k \in K} |X_{k,bj}|_{\xi_j}^2 + O(x, \rho_j) + O \left( \frac{\eta_j^2 + \xi_j^2}{\eta_j^2 + \varepsilon_{bj}^2} \right) \right)
\]

on $\rho_j = 0$ in a microlocal neighborhood in which $\xi_j \ll -\sqrt{\eta_j^2 + \xi_j^2}$.

From this point forward, we work with $(0,1)$-forms in $\mathbb{C}^2$. We set $n = m = 2$, and look at resulting simplifications in the boundary equations. We first deal with the non-diagonal tangential operators contained in $\Lambda^0_{bj}$. Without loss of generality we work on the particular boundary $\partial \Omega_1$. We write

\[
u = u_1 \bar{\omega}_1 + u_2 \bar{\omega}_2.
\]

On $\partial \Omega_1$, $(5.7)$ leads to $u_1|_{\partial \Omega_1} = 0$. We will also use the notation $u_{bj} = u_{1,bj} \bar{\omega}_1 + u_{2,bj} \bar{\omega}_2$. From $\nu = P(u_b) + G(2f)$ we write

\[
u_j = P_j(u_b) + G_j(2f)
\]

for $j = 1, 2$.

Turning to $(5.8)$ on $\partial \Omega_1$, we examine in more detail the normal derivative,

\[
R_1 \circ T_1 u_2 = R_1 \circ T_1 (P_2(u_b) + G_2(2f))
\]

\[
= \frac{1}{\sqrt{2}} (N^{-1}_1 u_b) - i T_1 u_{2,b1} + \sqrt{2} \partial_{\rho_1} G_2 f,
\]

where $N^{-1}_1$ is given as in Theorem 5.2, and we use a subscript around the $N^{-1}_1 u_b$ term to denote which component (of the vector result) we are taking. Thus

\[
(N^{-1}_1 u_b)_2 := N^{-1}_1 u_b | \bar{\omega}_2.
\]

Taking the $\bar{\omega}_2$ components, we obtain

\[
R_1 \circ (N^{-1}_1 u_b)_2 = |D_{b1}| u_{2,b1} + 2 |D_{b1}| \circ R_1 \circ \Theta_{b2}^{\bar{z}} \circ R_2 \circ \Theta_{b1}^{\bar{z}} u_{2,b1} + (\Lambda^0_{b1} u_b)_2 + R_1 \circ (C_{b} u_b)_2,
\]
and $C_1 u_b$ can be estimated as in the theorem (assuming a microlocal neighborhood in which $|\xi| \gg \sqrt{\eta_2^2 + \xi_2^2}$). For convenience, we group the second term on the right with the last error term, and write

$$ Eu_b = 2|D_{b_1}| \circ R_1 \circ \Theta_2^+ \circ R_2 \circ \Theta_1^+ u_b + R_1 \circ C_1 u_b. $$

so that

$$(Eu_b)_2 = 2|D_{b_1}| \circ R_1 \circ \Theta_2^+ \circ R_2 \circ \Theta_1^+ u_{b,2} + R_1 \circ (C_1 u_b)_2.$$

Furthermore, as mentioned above, the terms off the diagonal of the operator $\Lambda_0^{b_1}$ are tangential with respect to $\partial \Omega_1$, and so

$$\Lambda_0^{b_1} u_{b_1} = \left[ (\Lambda_0^{b_1})_{1,1} u_{b_1} + (\Lambda_0^{b_1})_{1,2} u_{b_2} \right] \tilde{\omega}_1$$

$$+ \left[ (\Lambda_0^{b_1})_{2,1} u_{b_1} + (\Lambda_0^{b_1})_{2,2} u_{b_2} \right] \tilde{\omega}_2$$

$$= \left[ (\Lambda_0^{b_1})_{1,1} u_{b,1} \right] \tilde{\omega}_1 + \left[ (\Lambda_0^{b_1})_{2,2} u_{b,1} \right] \tilde{\omega}_2.$$  

Hence,

$$(\Lambda_0^{b_1} u_{b_1})_2 = (\Lambda_0^{b_1})_{2,2} u_{b,2}.$$  

We let $\zeta(x, \rho_2)$ have compact support in a neighborhood of 0 on $\partial \Omega_1$ which provides a coordinate as in Section 3 with $\zeta \equiv 1$ near 0. Dividing Fourier space into regions as described in Section 3 with symbols $\psi_k^+$ with support in $\xi_1 > k \sqrt{\eta_2^2 + \xi_2^2}$, $\psi_k^-$ with support in $\xi_1 < -k \sqrt{\eta_2^2 + \xi_2^2}$, and $\psi_k^0$ defined by $\psi_k^+ + \psi_k^0 + \psi_k^- = 1$, we write

$$u_b = u_b^+ + u_b^0 + u_b^-,$$

in a small neighborhood of $0 \in \partial \Omega_1$, where $u_b^+ := \Psi_k^+ u_b$, $\Psi_k^+$ being the operator with symbol

$$\sigma(\Psi_k^+) = \zeta(x, \rho_2)\psi_k^+ (\xi, \eta_2),$$

and with $u_b^0$ and $u_b^-$ defined similarly, with the same cutoff, $\zeta(x, \rho_2)$, in base space.

We apply the operators $\Psi_k^+$, respectively $\Psi_k^0$, to both sides of the boundary condition (on $\partial \Omega_1$)

$$\mathbf{T}_1 u_2 - c_{12} u_2 = 0$$

to write

$$(7.4) \left( \frac{1}{\sqrt{2}} |D_{b_1}| - iT_1^0 \right) u_{b,1}^+ + \Psi_0^{b_1} u_b + E_2^+ u_b = -\sqrt{2}R_1 \circ \Psi_k^+ \circ \partial_{\rho_1} \circ G_2 f,$$

where $T_1^0 = R_1 \circ T_1$, and $E_2^+ u_b := \Psi_k^+ \circ (Eu_b)_2$, $E$ being defined above in (7.2).

Similar notation is used to write $u_{b,2}^0$. We use the notation $u_{b,2}^0$ to denote either $u_{b,2}^0$ or $u_{b,2}^0$. The first order operator, $\frac{1}{\sqrt{2}} |D_{b_1}| - iT_1^0$ is elliptic in the regions of support $\psi_k^+$ and $\psi_k^0$, and so leads to (weighted) estimates with a gain of a derivative:

$$||u_{b,1}^+||_{W^{1,r}(\partial \Omega_1, \rho_2, \lambda)}$$

$$\leq ||u_{b,1}||_{W^{0,r}(\partial \Omega_1, \rho_2, \lambda)} + ||u_{b,2}||_{W^{0,r}(\partial \Omega_2, \rho_1, \lambda)}$$

$$+ ||R_1 \circ \partial_{\rho_1} \circ G_2 f||_{W^{0,r}(\partial \Omega_2, \rho_2, \lambda)} + ||E u_b||_{W^{0,r}(\partial \Omega_1, \rho_2, \lambda)}$$

$$(7.5) \leq ||u_{b,1}||_{W^{0,r}(\partial \Omega_1, \rho_2, \lambda)} + ||u_{b,2}||_{W^{0,r}(\partial \Omega_2, \rho_1, \lambda)} + ||f||_{W^{0,r}(\Omega, \rho_2, \lambda)},$$

where we use Corollary 3.3 in the last step. In fact, the same reasoning shows for operators $\Psi_k^+$ and $\Psi_k^0$, defined by symbols $\psi_{k_1}$ and $\psi_{k_2}$, respectively, with
k_1 < k < k_2$, and with the properties $\psi_k^+ \equiv 1$ on the support of $\psi_{k_2}^+$, and similarly, $\psi_k^0 \equiv 1$ on the support of $\psi_{k_1}^0$, we have

$$
\| \Psi_{k_1} u_{2, b_1} \|_{W^{1, r} (\partial \Omega_1, \rho_2)} \lesssim \| u_{2, b_1} \|_{W^{0, r} (\partial \Omega_1, \rho_2)} + \| u_{1, b_2} \|_{W^{0, r} (\partial \Omega_2, \rho_1, \lambda)}
$$

+ $\| f \|_{W^{0, r} (\Omega, \rho)}$

(7.6) $$
\| \Psi_{k_2} u_{2, b_1} \|_{W^{1, r} (\partial \Omega_1, \rho_2)} \lesssim \| u_{2, b_1} \|_{W^{0, r} (\partial \Omega_1, \rho_2)} + \| u_{1, b_2} \|_{W^{0, r} (\partial \Omega_2, \rho_1, \lambda)}
$$

+ $\| f \|_{W^{0, r} (\Omega, \rho)}$

We now write $u_{2, b_1} = u_{2, b_1}^+ + u_{2, b_1}^0 + u_{2, b_1}^-$ in the boundary condition

$$
\bar{L}_1 u_2 - c_{12}^2 u_2 = 0
$$

and use

$$
R_1 \circ (\bar{T}_1 - c_{12}^2) u_2^- = R_1 \circ \Psi_k^- \circ (\bar{T}_1 - c_{12}^2) u_2 + R_1 \circ \left( \left[ (\bar{T}_1 - c_{12}^2), \Psi_k^- \right] u_2 \right)
$$

$$
= \left[ (\bar{T}_1 - c_{12}^2), \Psi_k^- \right] u_{2, b_1}.
$$

If we let $\varphi (x, \rho_2) \in C^\infty (\partial \Omega_1)$ be such that the support of $\varphi$ is contained in the region where $\zeta \equiv 1$, then the support of the symbol, $\varphi (x, \rho_2) \sigma \left( \left[ (\bar{T}_1 - c_{12}^2), \Psi_k^- \right] \right)$, is contained in the region where $\sigma (\Psi_{k_2}^0) \equiv 1$ for some $k_2 > k$. Let us fix a notation for such an operator. Let $\Psi_{b_1, \varphi}$ denote an operator of order $r$ on $\partial \Omega_1$ whose symbol has support in the region where $\sigma (\Psi_{b_2}^0) \equiv 1$.

For such an operator, we can use (7.6) to conclude

$$
\| \Psi_{b_1, \varphi}^0 \|_{W^{1, r} (\partial \Omega_1, \rho_2, \lambda)} \lesssim \| u_{2, b_1} \|_{W^{0, r} (\partial \Omega_1, \rho_2, \lambda)} + \| u_{1, b_2} \|_{W^{0, r} (\partial \Omega_2, \rho_1, \lambda)}
$$

+ $\| f \|_{W^{0, r} (\Omega, \rho)}$

We write in a similar manner to (7.4) the boundary condition in the microlocal region determined by $\Psi_k^-:

(7.7) $$
\left( \frac{1}{\sqrt{2}} | D_{b_1} | - i T_1^0 \right) u_{2, b_1}^- + \frac{1}{\sqrt{2}} \left( \sum_{k \not\in K} | X_{k, b_1} |^2_{L_1} - \sum_{k \in K} | X_{k, b_1} |^2_{L_1} \right) u_{2, b_1}^-
$$

$$
+ \Psi_{b_1, \varphi}^0 u_{2, b_1} + \Psi_{\varepsilon, b_1}^0 u_b + E_2^- u_b = -\sqrt{2} R_1 \circ \Psi_k^- \circ \partial_{\rho_1} \circ G_2 f,
$$

(in the support of a cutoff, $\varphi$, as outlined above), where we use the notation, $\Psi_{\varepsilon, b_1}$ as outlined in (7.2), the $\varepsilon$ signifying the property

$$
\| \Psi_{\varepsilon, b_1} \phi \|_{W^r (\mathbb{R}^3)} \leq s.c. \| \phi \|_{W^r (\mathbb{R}^3)}
$$

where the constant of inequality, written as "s.c.", can be made arbitrarily small by choosing an appropriately large constant, $k$, for the functions, $\psi_k^+$, $\psi_k^0$, and $\psi_k^-$, used to divide Fourier space above. Recall that the sums of norms of vector fields on the left-hand side of (7.7) come from the zero order term of the DNO as in (7.1). For these norms, we use the notation $L_1$ to denote the Levi form is used with respect to the defining function, $\rho_1$, on $\partial \Omega_1$.

From the estimates in (7.6), it suffices to consider (7.7), and get estimates for $u_b^-$.

On $\partial \Omega_1$, the vector field, $X_{2, b_1}$ is given by $L_{2, b_1}$, while $X_{1, b_1}$ does not exist. Thus

$$
\sum_{k \not\in K} | X_{k, b_1} |^2_{L_1} - \sum_{k \in K} | X_{k, b_1} |^2_{L_1} = -| L_{2, b_1} |^2_{L_1}.
$$
The strict pseudoconvexity condition gives
\[ |L_{2,b1}|^2_{L^1} = \beta > 0 \]
on \partial \Omega_1. The boundary condition, (7.7), on \( \partial \Omega_1 \) thus becomes
\[
\begin{aligned}
(7.8) \quad & \left( \frac{1}{\sqrt{2}} D_{b1} \right) \left| D_{b1} \right| - i T^0_1 \right) u_{2,b1} - \frac{\beta}{\sqrt{2}} u_{2,b1} \\
& + \Psi_{b1,\psi}^0 u_{2,b1} + \Psi_{\epsilon,b}^0 u_{2,b1}^2 + E_{b} u_b = -\sqrt{2} R_1 \circ \psi_{b} \circ \partial_{\rho_1} G_2 f.
\end{aligned}
\]

The boundary condition on \( \partial \Omega_2 \) is of course symmetric, so it suffices to obtain estimates for \( u_2 \) on \( \partial \Omega_1 \).

We now apply \( \frac{1}{\sqrt{2}} D_{b1} \mid | T^0_1 \rangle \) to both sides of (7.8) above. We use
\[
(7.9) \quad \left( \frac{1}{\sqrt{2}} D_{b1} \mid + i T^0_1 \right) \left( \frac{1}{\sqrt{2}} D_{b1} \mid - i T^0_1 \right) = \left( \frac{1}{2} D_{b1}^2 + (T^0_1)^2 \right) + \Psi_{b1}^1,
\]
where \( \Psi_{b1}^1 \) is given by \( i \sqrt{2} \mid T^0_1 \mid | D_{b1} \rangle \). The symbol of this first order term has the property
\[
\sigma_1 \left( \mid T^0_1 \mid | D_{b1} \rangle \right) = O \left( \sqrt{\eta_2^2 + \xi_2^2} \right) + O(x, \rho_2)
\]
from (5.8) (see also Proposition 3.4 in [4] or Section 6 of [7]). We thus write the first order operator in (7.9) above as \( \Psi_{\epsilon,b1}^1 \).

We also note that the operator, \( \frac{1}{2} D_{b1}^2 + (T^0_1)^2 \), is given by
\[
\left( \frac{1}{2} D_{b1}^2 + (T^0_1)^2 \right) = L_{2,b1} \bar{L}_{2,b1} + i \sqrt{2} L_{2,b1} \mid T^0_1 \rangle + \Psi_{\epsilon,b1}^1
\]
(see the discussion in [7]).

After applying \( \frac{1}{\sqrt{2}} D_{b1} \mid + i T^0_1 \) to both sides of (7.8) above, we thus have
\[
\begin{aligned}
L_{2,b1} \bar{L}_{2,b1} u_{2,b1} + i \sqrt{2} \beta T^0_1 u_{2,b1} - \beta \left( \frac{1}{\sqrt{2}} D_{b1} \mid + i T^0_1 \right) u_{2,b1} \\
& - \sqrt{2} R_1 \circ \left( \frac{1}{\sqrt{2}} D_{b1} \mid + i T^0_1 \right) \circ \psi_{b} \circ \partial_{\rho_1} G_2 f \\
& + \Psi_{b1,\psi}^1 u_{2,b1} + \Psi_{\epsilon,b1}^1 u_{2,b1}^2 + E_{b} u_b = -\sqrt{2} R_1 \circ \psi_{b} \circ \partial_{\rho_1} G_2 f \\
(7.10)
& + \Psi_{b1,\psi}^1 u_{2,b1} + \Psi_{\epsilon,b1}^1 u_{2,b1}^2 + \Psi_{b1}^0 u_{2,b1} + \Psi_{b1}^1 E_{b} u_b.
\end{aligned}
\]

Expanding the (symbols of the) last two terms on the left hand side of (7.10) for large (negative) \( \xi_1 \), we see the terms cancel, modulo operators of small operator norm:
\[
L_{2,b1} \bar{L}_{2,b1} u_{2,b1} - \sqrt{2} R_1 \circ \left( \frac{1}{\sqrt{2}} D_{b1} \mid + i T^0_1 \right) \circ \psi_{b} \circ \partial_{\rho_1} G_2 f \\
+ \Psi_{b1,\psi}^1 u_{2,b1} + \Psi_{\epsilon,b1}^1 u_{2,b1}^2 + \Psi_{b1}^0 u_{2,b1} + \Psi_{b1}^1 E_{b} u_b.
\]

8. A priori weighted \( L^2 \) boundary estimates

To show estimates for the boundary solution, \( u_b \), we start with the property
\[
(8.1) \quad u_{k,j} \in L^2 \left( \partial \Omega_j \cap \partial \Omega \right)
\]
for \( k = 1, 2 \).

In the weighted estimates we start with the base case, \( s = 0 \). From [16] (Theorem 3.1), we have the solution, \( u \), to the \( \bar{\partial} \)-Neumann problem is in \( W^{1/2}(\Omega) \) as are \( \bar{\partial} u \).
and $\bar{\partial}^* u$. We now follow the proof in Lemma 5.2.3 of \[5\] to show the boundary estimates in \[8.1\].

We write

$$\|u_b\|_{L^2(\partial \Omega)} := \|u_{b1}\|_{L^2(\partial \Omega_1 \cap \partial \Omega)} + \|u_{b2}\|_{L^2(\partial \Omega_2 \cap \partial \Omega)}.$$We will also use the short-hand estimates in \(8.1\).

We let $\Lambda_{\partial j}^{1/2} \in \Psi^{1/2}(\partial \Omega_j \cap \partial \Omega)$ for $j = 1, 2$ denote the operator with symbol

$$\sigma(\Lambda_{\partial j}^{1/2}) = (1 + \xi^2 + \eta^2)^{1/2},$$

where $k \neq j$, and $\xi^2 = \xi_1^2 + \xi_2^2$, i.e. $\sigma(\Lambda_{\partial j}^{1/2}) \simeq 1 + \sqrt{\eta^2 + \Xi_j^2}$. We recall the superscript $E_k$ meaning extension by zero across $\rho_k = 0$ as in Section \[2\] and we note

$$\|\Lambda_{\partial j}^{1/2} u_{E_k}\|_{L^2(\partial \Omega_j)} \simeq \|u_{E_k}\|_{W^{1/2}(\partial \Omega_j)}$$

(8.2)

by the Extension Theorem, Theorem 1.4.2.4 in \[10\].

For each $u_{E_k}$ we write as in \[5\]

$$\|u_{E_k}\|_{L^2(\partial \Omega_j \cap \partial \Omega)} = \|u_{E_k}\|_{L^2(\partial \Omega_j)}$$

(8.3)

Note that we have

$$\partial_{\rho_j} u_{E_k} = (\partial_{\rho_j} u)^{E_k},$$

and from (8.2),

$$\|\Lambda_{\partial j}^{1/2} u_{E_k}\|_{L^2(\Omega_j)} \simeq \left\| (\Lambda_{\partial j}^{1/2} u)^{E_k} \right\|_{L^2(\Omega_j)}$$

so that (8.3) shows $\|u_{E_k}\|_{L^2(\partial \Omega_j \cap \partial \Omega)}$ is bounded by a sum of terms of the form

$$\left\| (\Lambda_{\partial j}^{1/2} u)^{E_k} \right\|_{L^2(\Omega_j)} + \|\Lambda_{\partial j}^{-1/2}(\partial_{\rho_j} u_{E_k})\|_{L^2(\Omega_j)}$$

Now write the operator, $\partial_{\rho_j}$, as a combination of (components) of $\bar{\partial}$, $\bar{\partial}^*$, and tangential (to $\partial \Omega_j$) operators:

$$\|\Lambda_{\partial j}^{1/2}(\partial_{\rho_j} u_{E_k})\|_{L^2(\Omega_j)} \lesssim \|\Lambda_{\partial j}^{-1/2}(\bar{\partial} u_{E_k})\|_{L^2(\Omega_j)} + \|\Lambda_{\partial j}^{-1/2}(\bar{\partial}^* u_{E_k})\|_{L^2(\Omega_j)} + \|\Lambda_{\partial j}^{1/2} u_{E_k}\|_{L^2(\Omega_j)}$$

$$\lesssim \|\bar{\partial} u\|_{L^2(\Omega_j)} + \|\bar{\partial}^* u\|_{L^2(\Omega_j)} + \|u\|_{W^{1/2}(\Omega_j)}.$$Combining this with (8.3) as well as Theorem 3.1 of \[10\], we get $u_b \in L^2(\partial \Omega)$, with estimates

$$\|u_b\|_{L^2(\partial \Omega)} \lesssim \|f\|_{L^2(\Omega)}.$$(8.4)

We will concentrate on the more difficult estimates in the microlocal region defined by $\psi_k$; that is we assume $-i \sigma(T_1) = -k \sqrt{\sigma(T_1)^2 + |\sigma(\partial_{\rho_j})|^2}$. We will also just consider the boundary, $\partial \Omega_1$, the results being analogous on $\partial \Omega_2$. We will
thus drop the subscripts, writing \( L_{b_1} \) to mean \( L_{2,b_1} \) in (7.11). From (7.11), the boundary equation reads

\[
(8.5) \quad L_{b_1} \nabla_{b_1} u_{2,b_1} = g + \Psi_{\text{c},b_1}^1 u_{2,b_1} + \Psi_{\text{c},b_1}^0 u_{2,b_1} + \Psi_{\text{b}_1}^1 u_{2,b_1} + \Psi_{\text{b}_1}^0 E_{2}^- u_b,
\]

where \( g \) is of the form \( R_1 \circ \Psi_{\text{b}_1}^1 \circ \partial_{\rho_1} \circ G_2 f \).

We first start with the assumption that \( u_b \in C^\infty(\partial \Omega) \) so that integration by parts can be performed. This is not a necessary assumption as the use of regularizing operators could be used from the beginning (see [14]). We separate the regularizing argument in this paper, mainly out of aesthetic concerns, but also due to the more complicated regularizing of terms involving the \( E_{2}^- \) operator in (8.5). The assumption of smooth forms will be removed in Section 9.

Using an argument of Kohn [14], we write

\[
T_1 = \frac{i}{\lambda} (L_{b_1} \nabla_{b_1} - \nabla_{b_1} L_{b_1})
\]

modulo \( L_{b_1} \) and \( \nabla_{b_1} \), where \( \lambda > 0 \) on \( \partial \Omega_1 \). Sobolev 1/2 estimates follow as in the smooth case: let \( \varphi \in C^\infty(\partial \Omega_1) \) with support away from \( \rho_2 \geq 0 \), but still contained in the region where \( \zeta \equiv 1 \), where \( \zeta \) is the cutoff defining the operator \( \Psi_{\zeta} \) as in (7.3). Then we have by integration by parts

\[
\| \varphi u_{2,b_1} \|_{1/2}^2 \lesssim \left| T_1 \varphi u_{2,b_1}, \varphi u_{2,b_1} \right| \lesssim \| L_{b_1} \varphi u_{2,b_1} \|^2 + \| \nabla_{b_1} \varphi u_{2,b_1} \|^2 \lesssim \| \varphi L_{b_1} v \|^2 + \| \varphi \nabla_{b_1} u_{2,b_1} \|^2 + \| u_{2,b_1} \|^2.
\]

The \( L^2 \)-norms are with respect to \( \partial \Omega_1 \), and we write \( \| \cdot \|_{1/2} \) as a shorthand for \( \| \cdot \|_{W_{1/2}(\partial \Omega_1)} \). In the case we need to specify one boundary norm over another, we will write explicitly the domain on which the norms are calculated; otherwise the boundary, \( \partial \Omega_1 \), is to be the default.

We also have

\[
\| \varphi L_{b_1} u_{2,b_1} \|^2 + \| \varphi \nabla_{b_1} u_{2,b_1} \|^2 \lesssim - (\varphi \nabla_{b_1} L_{b_1} u_{2,b_1}, \varphi u_{2,b_1}) + O \left( \| \varphi L_{b_1} u_{2,b_1} \| \| u_{2,b_1} \| \right) \]

(8.7)

For the term \( \nabla_{b_1} L_{b_1} u_{2,b_1} \), we write (in the support of \( \varphi \))

\[
\nabla_{b_1} L_{b_1} u_{2,b_1} = L_{b_1} \nabla_{b_1} u_{2,b_1} + \nabla_{b_1} L_{b_1} u_{2,b_1}
\]

(8.8)

In the support of \( \psi_{\zeta} \), the symbol of \( -iT_1 \) is negative, hence we can write

\[
- (\varphi i \lambda T_1 \varphi u_{2,b_1}, \varphi u_{2,b_1}) \lesssim \| u_{2,b_1} \|^2,
\]

which also follows from Gårding’s inequality.
Using this in (8.7) above, we have
\[
\|\varphi_{L1}u_{2,\bar{b}1}\|^2 + \|\varphi_{\cal T_{b}1}u_{2,\bar{b}1}\|^2 \lesssim \\
2(|\varphi_{\Psi_{b1}}(\varphi_{u_{2,\bar{b}1}}) + |(\varphi_{\Psi_{b1}1}u_{2,\bar{b}1}, \varphi_{u_{2,\bar{b}1}})| + |(\varphi_{\Psi_{b1}1}u_{b}^-, \varphi_{u_{2,\bar{b}1}})| \\
+ \left|\left(\varphi_{\Psi_{b1}} E^- u_{b}, u_{2,\bar{b}1}\right)\right| + \|u_{2,\bar{b}1}\|^2 \\
+ O\left(\left(\|\varphi_{\cal T_{b}1}u_{2,\bar{b}1}\| + \|\varphi_{L1}u_{2,\bar{b}1}\|\right)\|u_{2,\bar{b}1}\|\right),
\]

Estimates of the terms on the right-side then yield estimates for \(L_{b1}\) and \(\cal T_{b1}\) applied to \(u_{2,\bar{b}1}\) which in turn yield 1/2-estimates for \(u_{2,\bar{b}1}\) (when combined with estimates for \(u_{1,\bar{b}2}\)). We apply this approach, but with weights, in order to obtain estimates near a singularity.

We now look at estimates as in (8.6) but without the assumption of support away from the boundary singularities. For higher order estimates, we work with the operator, \(\Lambda\), with symbol
\[
|\sigma(\Lambda)| \approx \sqrt{|\sigma(T_1)|^2 + |\sigma(T_2)|^2 + |\sigma(\partial_{\rho_2})|^2},
\]
on \(\partial\Omega_1\). So that integration by parts can still be used, we multiply by factors of \(\rho_2\) (recall we work with estimates on \(\partial\Omega_1\)); in (8.3) we let the cutoff, \(\varphi\), have support in a neighborhood of the singularity, assumed to be at 0, and introduce a factor, \(\rho_2^{2\alpha}\) for some integer \(2\alpha \geq 1\), and we also replace \(u_{2,\bar{b}1}\) with \(\Lambda^\alpha u_{2,\bar{b}1}\):

\[
(8.9) \quad \|\rho_2^{2\alpha+1}\Lambda^\alpha u_{2,\bar{b}1}\|^2 \lesssim \\
\|\rho_2^{2\alpha+1}L_{b1}\Lambda^\alpha u_{2,\bar{b}1}\|^2 + \|\rho_2^{2\alpha+1}\cal T_{b1}\Lambda^\alpha u_{2,\bar{b}1}\|^2 + \|\rho_2^{2\alpha}\Lambda^\alpha u_{2,\bar{b}1}\|^2.
\]

The vanishing of the boundary terms arising in the integration by parts occurs due to our assumption that \(u_b \in C^\infty(\partial\Omega)\). We omit writing the cutoffs \(\varphi\); they could also be understood to be part of the \(\Lambda\) operators.

For integer \(\alpha\), we use
\[
\|\rho_2^{2\alpha}\Lambda^\alpha u_{2,\bar{b}1}\|^2 \approx \sum_{l \leq \alpha} \|\Psi_{b1}^{\alpha-l}\rho_2^{2\alpha-2l}u_{2,\bar{b}1}\|^2 \lesssim \sum_{l \leq \alpha} \|\rho_2^{2\alpha-2l}u_{2,\bar{b}1}\|^2_{\alpha-l},
\]
whereas for \(\alpha\) of the form \(\alpha = (2k + 1)/2\), we use
\[
\|\rho_2^{2k+1}\Lambda^{k+1/2}u_{2,\bar{b}1}\|^2 \approx \sum_{l \leq k} \left\|\rho_2^{2k-2l}u_{2,\bar{b}1}\right\|^2 \\
\lesssim \sum_{l \leq k} \|\rho_2^{2k-2l}u_{2,\bar{b}1}\|^2_{k-l+1/2} + \|u_{2,\bar{b}1}\|^2 \\
\lesssim \sum_{l \leq [\alpha]} \|\rho_2^{2\alpha-2l}u_{2,\bar{b}1}\|^2_{\alpha-l} + \|u_{2,\bar{b}1}\|^2_{\alpha-l}.
\]

In each case we write
\[
(8.10) \quad \|\rho_2^{2\alpha}\Lambda^\alpha u_{2,\bar{b}1}\|^2 \approx \sum_{l \leq [\alpha]} \|\rho_2^{2\alpha-2l}u_{2,\bar{b}1}\|^2_{\alpha-l} + \|u_{2,\bar{b}1}\|^2_{\alpha-l}.
\]

We consider the first term on the right-hand side of (8.9), and integrate by parts (again, for the time being, assuming the vanishing of the arising boundary integrals.
at $\rho_2 = 0$:
\[
(p_2^{2\alpha+1} L_{b_1} A^\alpha u_{-2,b_1}, p_2^{2\alpha+1} L_{b_1} A^\alpha u_{-2,b_1}) \\
= - (p_2^{2\alpha+1} T_{b_1} L_{b_1} A^\alpha u_{-2,b_1}, p_2^{2\alpha+1} L_{b_1} A^\alpha u_{-2,b_1}) \\
+ O \left( \|p_2^{2\alpha+1} L_{b_1} A^\alpha u_{-2,b_1}\| \right) \\
\lesssim - (p_2^{2\alpha+1} T_{b_1} L_{b_1} A^\alpha u_{-2,b_1}, p_2^{2\alpha+1} L_{b_1} A^\alpha u_{-2,b_1}) \\
+ s.c. \|p_2^{2\alpha+1} L_{b_1} A^\alpha u_{-2,b_1}\|^2 + \|p_2^{2\alpha} A^\alpha u_{-2,b_1}\|^2,
\]
or
\[
\|p_2^{2\alpha+1} L_{b_1} A^\alpha u_{-2,b_1}\|^2 \lesssim - (p_2^{2\alpha+1} T_{b_1} L_{b_1} A^\alpha u_{-2,b_1}, p_2^{2\alpha+1} L_{b_1} A^\alpha u_{-2,b_1}) + \|p_2^{2\alpha} A^\alpha u_{-2,b_1}\|^2.
\]

We now commute the $A^\alpha$ operator through the $L$ derivatives and use Gårding’s inequality:
\[
-(p_2^{2\alpha+1} T_{b_1} L_{b_1} A^\alpha u_{-2,b_1}, p_2^{2\alpha+1} L_{b_1} A^\alpha u_{-2,b_1}) \\
\lesssim - (p_2^{2\alpha+1} T_{b_1} L_{b_1} A^\alpha u_{-2,b_1}, p_2^{2\alpha+1} L_{b_1} A^\alpha u_{-2,b_1}) \\
\lesssim \left| (p_2^{2\alpha+1} A^\alpha L_{b_1} u_{-2,b_1}, p_2^{2\alpha+1} L_{b_1} A^\alpha u_{-2,b_1}) \right| + s.c. \|p_2^{2\alpha+1} L_{b_1} \Psi_{b_1} A^\alpha u_{-2,b_1}\|^2 \\
+ s.c. \|p_2^{2\alpha+1} \Psi_{b_1} A^\alpha u_{-2,b_1}\|^2 + \|p_2^{2\alpha+1} L_{b_1} A^\alpha u_{-2,b_1}\|^2.
\]

The next relation can be derived in a similar manner as we did (8.10):
\[
(\rho_2^{2\alpha+1} \Psi_{b_1} A^\alpha u_{-2,b_1}, \rho_2^{2\alpha+1} \Psi_{b_1} A^\alpha u_{-2,b_1}) = \sum_{l \leq |\alpha|} \Psi_{b_1}^{\alpha-l} \rho_2^{2\alpha+1-2l} w + \Psi_{b_1}^{1/2} \rho_2 w + \Psi_{b_1}^{1} w
\]
and in particular, when $w = L_{b_1} u_{-2,b_1}$,
\[
(\rho_2^{2\alpha+1} \Psi_{b_1}^{\alpha} L_{b_1} u_{-2,b_1}) = \sum_{l \leq |\alpha|} \Psi_{b_1}^{\alpha-l} \rho_2^{2\alpha+1-2l} L_{b_1} u_{-2,b_1} + \Psi_{b_1}^{1/2} \rho_2 u_{-2,b_1} + \Psi_{b_1}^{0} u_{-2,b_1}
\]
with similar inequalities in which $T_{b_1}$ replaces the $L_{b_1}$ derivative.

We now use (8.3), noting that the $\varphi$ cutoff allows us to use an operator $\Psi_{b_1,\psi}$ as before, in the first term on the right of (8.11):
\[
\left| (p_2^{2\alpha+1} T_{b_1} L_{b_1} A^\alpha u_{-2,b_1}, p_2^{2\alpha+1} L_{b_1} A^\alpha u_{-2,b_1}) \right| \\
\lesssim \left| (p_2^{2\alpha+1} A^\alpha g, p_2^{2\alpha+1} A^\alpha u_{-2,b_1}) \right| + \left| (p_2^{2\alpha+1} A^\alpha \Psi_{b_1,\psi} u_{-2,b_1}, p_2^{2\alpha+1} L_{b_1} A^\alpha u_{-2,b_1}) \right| \\
+ \left| (p_2^{2\alpha+1} A^\alpha \Psi_{b_1,\psi} u_{-2,b_1}, p_2^{2\alpha+1} L_{b_1} A^\alpha u_{-2,b_1}) \right| \\
+ \left| (p_2^{2\alpha+1} A^\alpha \Psi_{b_1,\psi} u_{-2,b_1}, p_2^{2\alpha+1} L_{b_1} A^\alpha u_{-2,b_1}) \right| \\
+ s.c. \|p_2^{2\alpha+1} \Psi_{b_1}^{\alpha} L_{b_1} u_{-2,b_1}\|^2 + s.c. \|p_2^{2\alpha+1} \Psi_{b_1}^{\alpha} L_{b_1} u_{-2,b_1}\|^2 + \|p_2^{2\alpha+1} \Psi_{b_1}^{\alpha} u_{-2,b_1}\|^2.
\]

Using (a cruder form of (8.12))
\[
\rho_2^{2\alpha+1} \Psi_{b_1}^{\alpha} = \sum_{l \leq |\alpha|} \Psi_{b_1}^{\alpha-l} \rho_2^{2\alpha+1-2l} + \Psi_{b_1}^{1}
\]
and
\[
\rho_2^{2\alpha+1} \Psi_{b_1,\psi}^{\alpha+1} = \Psi_{b_1,\psi}^{\alpha+1} \rho_2^{2\alpha+1} + \sum_{l \leq |\alpha|} \Psi_{b_1}^{\alpha-l+1} \rho_2^{2\alpha+1-2l} + \Psi_{b_1}^{0}
\]
in the third term on the right-hand side of \((8.14)\) we obtain
\[
\left| (\rho^2_{\alpha} \Lambda_{\alpha}^1 \Psi^1_{b,1} u_{2,b,1}, \rho^2_{\alpha} \Lambda_{\alpha}^1 u_{2,b,1}) \right| \lesssim s.c.|\rho^2_{\alpha} \Lambda_{\alpha}^1 u_{2,b,1}|^2 \\
+ \sum_{1 \leq l \leq [\alpha]} \left| \rho^2_{\alpha} \Lambda_{\alpha}^1 u_{2,b,1} \right|^2.
\]
Similarly, the second term on the right-hand side of \((8.14)\) can be estimated by
\[
\left| (\rho^2_{\alpha} \Lambda_{\alpha}^1 \Psi^1_{b,1} u_{2,b,1}, \rho^2_{\alpha} \Lambda_{\alpha}^1 u_{2,b,1}) \right| \lesssim \sum_{l \leq [\alpha]} \left| \rho^2_{\alpha} \Lambda_{\alpha}^1 u_{2,b,1} \right|^2 + \sum_{l \leq [\alpha]} \left| \rho^2_{\alpha} \Lambda_{\alpha}^1 u_{2,b,1} \right|^2.
\]
For the fourth term on the right of \((8.14)\) we distinguish the two cases for each of the terms composing \(E^-_{\alpha} u_b\); \(\alpha\) is of the form \(i\) \(\alpha = k + 1/2\), or \(ii\) \(\alpha = k\), for \(k\) an integer. We recall that
\[
E^-_{\alpha} u_b = 2\Psi^- \circ D_{b,1} \circ R_1 \circ \Theta_{2}^+ \circ R_2 \circ \Theta_{1}^+ u_{2,b,1} \\
+ \Psi_{b,1}^{-1} u_{2,b,1} + \mathcal{E}_{-1/2}^1 u_{b,2} + \mathcal{E}_{-3/2}^1 \left( \partial_{\rho^2_{\alpha}} \Psi_{2} u_b \right)_{\partial_{\rho^2_{\alpha}}} + R_{b,1}^{-\infty}.
\]
Consider the term \(|D_{b,1} \circ R_1 \circ \Theta_{2}^+ \circ R_2 \circ \Theta_{1}^+ u_{2,b,1}\), which we will write as \(\mathcal{E}_{0}^{11} u_b\). In case \(i\) we estimate
\[
\left| (\rho^2_{\alpha} \Lambda_{\alpha}^1 \Psi^1_{b,1} e_{0}^{11} u_{b}, \rho^2_{\alpha} \Lambda_{\alpha}^1 e_{0}^{11} u_{b,1}) \right| \lesssim \sum_{0 \leq l \leq [\alpha] + 1} \left| \rho^2_{\alpha} e_{0}^{11} u_{b} \right|_{1/2}^{1/2} + \left| \rho^2_{\alpha} e_{0}^{11} u_{b} \right|^{2} \\
+ \sum_{0 \leq l \leq [\alpha]} \left| \rho^2_{\alpha} e_{0}^{11} u_{b,1} \right|_{1/2}^{1/2} + \left| \rho^2_{\alpha} e_{0}^{11} u_{b,1} \right|^{2}.
\]
Note that
\[
\rho^2_{\alpha} |\partial_{\rho^2_{\alpha}} e_{0}^{11} u_{b} | \circ R_1 \circ \Theta_{2}^+ \circ R_2 \circ \Theta_{1}^+ u_{2,b,1} = \mathcal{E}_{-1/2}^1 u_{2,b,1}
\]
by \((4.7)\).
In case \(i\) for the first term on the right of the inequality we have
\[
\sum_{l \leq k + 1} \left| \rho^2_{\alpha} e_{0}^{11} u_{b} \right|_{1/2}^{1/2} + \left| \rho^2_{\alpha} e_{0}^{11} u_{b} \right|^{2} \lesssim \left| \rho^2_{\alpha} e_{0}^{11} u_{b} \right|_{W^{1/2,k+1} (\partial\Omega_1 \cap \partial\Omega_2)} \\
\lesssim \left| e_{0}^{11} u_{b} \right|_{W^{1/2,k+1} (\partial\Omega_1 \cap \partial\Omega_2)} \\
\lesssim \left| u_{b,1} \right|_{W^{1/2,k+1} (\partial\Omega_1 \cap \partial\Omega_2)} \\
\lesssim s.c. \sum_{l \leq [\alpha]} \left| \rho^2_{\alpha} e_{0}^{11} u_{b,1} \right|_{1/2}^{1/2} + s.c. \left| u_{b,1} \right|^{2}.
\]
In case \(ii\) we estimate
\[
\left| (\rho^2_{\alpha} \Lambda_{\alpha}^1 \Psi^1_{b,1} e_{0}^{11} u_{b}, \rho^2_{\alpha} \Lambda_{\alpha}^1 e_{0}^{11} u_{b,1}) \right| \lesssim \sum_{0 \leq l \leq [\alpha]} \left| \rho^2_{\alpha} e_{0}^{11} u_{b} \right|_{1/2}^{1/2} + \left| \rho^2_{\alpha} e_{0}^{11} u_{b} \right|^{2} \\
+ s.c. \sum_{0 \leq l \leq [\alpha]} \left| \rho^2_{\alpha} e_{0}^{11} u_{b,1} \right|_{1/2}^{1/2} + \left| \rho^2_{\alpha} e_{0}^{11} u_{b,1} \right|^{2}.
\]
The first term on the right of the inequality can be estimated (in case \( ii \)) by

\[
\sum_{l \leq k} \left\| \rho^{2k-2l+1} \mathcal{E}^{11}_0 u_b \right\|_{1/2+k-l} \lesssim \left\| \rho_2 \mathcal{E}^{11}_0 u_b \right\|_{W^{1/2,k}(\partial \Omega_1 \cap \partial \Omega, \rho_2, 2)}
\]

\[
\lesssim \left\| \mathcal{E}^{11}_{-1,1/2} u_b \right\|_{W^{1/2,k}(\partial \Omega_1 \cap \partial \Omega, \rho_2, 2)}
\]

\[
\lesssim \left\| u_b \right\|_{W^{-1/2,k}(\partial \Omega_1 \cap \partial \Omega, \rho_2, 2)}
\]

\[
\lesssim \sum_{l \leq [\alpha]} \left\| \rho^{2\alpha-2l} u_b \right\|_{\alpha-l}.
\]

It remains to write estimates for the \( \mathcal{E}^{21}_\beta \) terms in the \( E^{-}_{2} \) operator. We will go through the estimates for the \( \mathcal{E}^{21}_{-3/2} \partial \rho_2 P_2(u_b) \big|_{\partial \Omega_2} \) term, the remaining estimates for the terms involving \( \mathcal{E}^{21}_{-1/2} u_b \) being proved similarly.

We write

\[
\left| (\rho_2^{2\alpha+1} \Lambda^{\alpha} \Psi_{\beta}^{1} \mathcal{E}^{21}_{-3/2} \partial \rho_2 P_2(u_b), \rho_2^{2\alpha+1} \Lambda^{\alpha} u_{2, b1} ) \right|
\]

\[
\lesssim \sum_{0 \leq l \leq [\alpha]} \left\| \rho_2^{2\alpha+1-2l} \mathcal{E}^{21}_{-3/2} \partial \rho_2 P_2(u_b) \right\|_{1/2+\alpha-l}^2 + \left\| \mathcal{E}^{21}_{-3/2} \partial \rho_2 P_2(u_b) \right\|^2
\]

(8.15) \quad + \text{s.c.} \sum_{0 \leq l \leq [\alpha]} \left\| \rho_2^{2\alpha+1-2l} u_{2, b1} \right\|_{1/2+\alpha-l}^2 + \left\| u_{2, b1} \right\|^2.

The first term on the right of the inequality can be estimated (in case \( i \)) by

\[
\sum_{l \leq k+1} \left\| \rho_2^{2k-2l+2} \mathcal{E}^{21}_{-3/2} \partial \rho_2 P_2(u_b) \big|_{\partial \Omega_2} \right\|_{k+1-l}
\]

\[
\lesssim \sum_{l \leq k} \left\| \rho_2^{2k-2l+2} \mathcal{E}^{21}_{-3/2} \partial \rho_2 P_2(u_b) \big|_{\partial \Omega_2} \right\|_{k+1-l} + \left\| \mathcal{E}^{21}_{-3/2} \partial \rho_2 P_2(u_b) \big|_{\partial \Omega_2} \right\|
\]

\[
\lesssim \sum_{l \leq k} \left\| \rho_2^{2k-2l+1} \mathcal{E}^{21}_{-3/2} \partial \rho_2 P_2(u_b) \big|_{\partial \Omega_2} \right\|_{k-l}
\]

\[
+ \sum_{l \leq k} \left\| \rho_2^{2k-2l+2} \mathcal{E}^{21}_{-1/2} \partial \rho_2 P_2(u_b) \big|_{\partial \Omega_2} \right\|_{k-l} + \left\| \mathcal{E}^{21}_{-3/2} \partial \rho_2 P_2(u_b) \big|_{\partial \Omega_2} \right\|
\]

Note that the \( \mathcal{E}^{21}_{-1/2} \) operator is of the form

\[
|D_1| \circ R_1 \circ \Psi^{-2} + |D_1|^2 \circ R_1 \circ \Psi^{-1} \circ R_2 \circ \Psi^{-1} \circ R_1 \circ \Psi^{-2},
\]

by (5.3), and hence with this \( \mathcal{E}^{21}_{-1/2} \) operator we have

\[
\rho_2 \mathcal{E}^{21}_{-1/2} \partial \rho_2 P_2(u_b) \big|_{\partial \Omega_2} = \mathcal{E}^{21}_{-3/2} \partial \rho_2 P_2(u_b) \big|_{\partial \Omega_2}.
\]
With this property, we continue the estimates:

\[
\sum_{l \leq k} \left\| \rho_2^{2k-2l+2} \mathcal{E}_{-3/2}^{21} \partial_{\rho_2} P_2(u_b) \right\|_{\partial \Omega_2}^{k+1-l} \leq \sum_{l \leq k} \left\| \rho_2^{2k-2l+1} \mathcal{E}_{-3/2}^{21} \partial_{\rho_2} P_2(u_b) \right\|_{\partial \Omega_2}^{k-l} + \text{s.c.} \| u_b \|
\]

\[
\lesssim s.c. \| u_b \|_{W^{0,k}((\partial \Omega_1 \cap \partial \Omega_2,2)} + \text{s.c.} \| u_b \|
\]

\[
\lesssim s.c. \sum_{l \leq [\alpha]} \| \rho_2^{2\alpha-2l} u_b \|_{W^{0,l}((\partial \Omega)} + \text{s.c.} \| u_b \|,
\]

modulo estimates of smooth terms,

\[
\| P(u_b) \|_{-\infty} + \left\| \partial_{\rho} P(u_b) \right\|_{\partial \Omega_1} \|_{-\infty} + \| u_b \|_{-\infty} \lesssim \| u_b \|_{L^2(\partial \Omega)}.
\]

On the other hand, in case \( ii \) we can estimate the first term on right of (8.15) by

\[
\sum_{0 \leq l \leq k} \left\| \rho_2^{2k+1-2l} \mathcal{E}_{-3/2}^{21} \partial_{\rho_2} P_2(u_b) \right\|_{\partial \Omega_2}^{2/1+2-l} \lesssim s.c. \| \mathcal{E}_{-3/2}^{21} \partial_{\rho_2} P_2(u_b) \|_{\partial \Omega_2}^{W^{1/2,k}((\partial \Omega_1 \cap \partial \Omega_2,2)}
\]

\[
\lesssim s.c. \| \partial_{\rho_2} P_2(u_b) \|_{\partial \Omega_2}^{W^{0,k}((\partial \Omega_2,2)}
\]

\[
\lesssim s.c. \| u_b \|_{W^{0,k}((\partial \Omega,2)}
\]

\[
\lesssim \sum_{l \leq [\alpha]} \| \rho_2^{2\alpha-2l} u_b \|_{W^{0,l}((\partial \Omega)}.
\]

In fact, the estimates for the \( \mathcal{E}_{-3/2}^{21} \partial_{\rho_2} P_2(u_b) \|_{\partial \Omega_2} \) could be improved by taking into account the vanishing of \( u_2 \) along \( \partial \Omega_2 \), but the weaker estimates we have above suffice for our purposes.

Putting this together, we have

\[
\| \rho_2^{2\alpha+1} L_{b_1} \Lambda^\alpha u_{-,-2b_1} \|^2 \lesssim \| \rho_2^{2\alpha+1} \Lambda^\alpha y \|_{-1/2}^2 + s.c. \| \rho_2^{2\alpha+1} u_{-,-2b_1} \|_{1/2+\alpha}^2 + \sum_{1 \leq l \leq [\alpha]} \| \rho_2^{2\alpha+1-2l} u_{-,-2b_1} \|_{1/2+\alpha-l} \| u_{-,-2b_1} \|^2
\]

\[
+ \sum_{l \leq [\alpha]} \| \rho_2^{2\alpha+1-2l} u_{-,-2b_1} \|^2_{\alpha+1-l} + \| u_{-,-2b_1} \|^2 + s.c. \| \rho_2^{2\alpha+1} L_{b_1} \Psi_0 u_{-,-2b_1} \|^2 + s.c. \| \rho_2^{2\alpha+1} \Psi_0 u_{-,-2b_1} \|^2 + \sum_{l \leq [\alpha]} \| \rho_2^{2\alpha-2l} u_b \|^2_{W^{0,-l}((\partial \Omega)} + \| u_b \|^2.
\]
Hence, using (8.10), (8.12), and (8.13), we have

\[ \| \rho_2^{2\alpha+1} L_{b_1} \Lambda^\alpha u_{2,b_1}^- \|^2 \lesssim \]
\[ \sum_{l \leq [\alpha]} \| \rho_2^{2\alpha-2l+1} g \|_{\alpha-l-1/2}^2 + \| \Psi_b^{-1/2} g \|_{2-1/2}^2 + s.c. \| \rho_2^{2\alpha+1} u_{2,b_1}^- \|_{1/2+\alpha}^2 \]
\[ + \sum_{l \leq [\alpha]} \| \rho_2^{2\alpha+1-2l} u_{2,b_1}^- \|_{1+2\alpha-l}^2 + \sum_{l \leq [\alpha]} \| \rho_2^{2\alpha-2l'} u_b \|_{W^{\alpha-1}(\partial \Omega)}^2 \]
\[ + \sum_{l \leq [\alpha]} \| \rho_2^{2\alpha+1-2l} u_{2,b_1}^0 \|_{\alpha+1-l}^2 + \| u_b \|_{\alpha}^2 \]
\[ (8.16) \]
\[ + s.c. \sum_{l \leq [\alpha]} \| \rho_2^{2\alpha-2l+1} L_{b_1} u_{2,b_1}^- \|_{\alpha-l}^2 + s.c. \sum_{l \leq [\alpha]} \| \rho_2^{2\alpha-2l+1} L_{b_1} u_{2,b_1}^- \|_{\alpha-l}^2. \]

Similarly, for any $\Psi_b^{\alpha}$, we can estimate $\| \rho_2^{2\alpha+1} L_{b_1} \Psi_b^{\alpha} u_{2,b_1}^- \|^2$ by the right-hand side of (8.16). And since

\[ \| \rho_2^{2\alpha+1} L_{b_1} u_{2,b_1}^- \|_{\alpha-l}^2 \lesssim \sum_{l \leq [\alpha]} \| \rho_2^{2\alpha-2l+1} L_{b_1} \Psi_b^{\alpha-l} u_{2,b_1}^- \|_{2}^2 + \| \rho_2 u_{2,b_1}^- \|_{1/2}^2 \]
\[ + \sum_{l \leq [\alpha]} \| \rho_2^{2\alpha-2l+1} \Psi_b^{\alpha-l} u_{2,b_1}^- \|_{2}^2 + \| u_{2,b_1}^- \|_{2}^2 \]
\[ \lesssim \sum_{l \leq [\alpha]} \| \rho_2^{2\alpha-2l+1} L_{b_1} \Psi_b^{\alpha-l} u_{2,b_1}^- \|_{2}^2 + \| \rho_2 u_{2,b_1}^- \|_{1/2}^2 \]
\[ + \sum_{l \leq [\alpha]} \| \rho_2^{2\alpha-2l+1} u_{2,b_1}^- \|_{\alpha-l}^2 + \| u_{2,b_1}^- \|_{2}^2, \]

we have

\[ \sum_{l \leq [\alpha]} \| \rho_2^{2\alpha-2l+1} L_{b_1} u_{2,b_1}^- \|_{\alpha-l}^2 \lesssim \]
\[ \sum_{l \leq [\alpha]} \| \rho_2^{2\alpha-2l+1} g \|_{\alpha-l-1/2}^2 + \| \Psi_b^{-1} g \|_{2}^2 + s.c. \| \rho_2^{2\alpha+1} u_{2,b_1}^- \|_{1/2+\alpha}^2 \]
\[ + \sum_{l \leq [\alpha]} \| \rho_2^{2\alpha+1-2l} u_{2,b_1}^- \|_{1/2+\alpha-l}^2 + \sum_{l \leq [\alpha]} \| \rho_2^{2\alpha-2l'} u_b \|_{W^{\alpha-1}(\partial \Omega)}^2 \]
\[ + \sum_{l \leq [\alpha]} \| \rho_2^{2\alpha+1-2l} u_{2,b_1}^0 \|_{\alpha+1-l}^2 + \| u_b \|_{\alpha}^2 \]
\[ (8.17) \]
\[ + s.c. \sum_{l \leq [\alpha]} \| \rho_2^{2\alpha-2l+1} L_{b_1} u_{2,b_1}^- \|_{\alpha-l}^2 + s.c. \sum_{l \leq [\alpha]} \| \rho_2^{2\alpha-2l+1} L_{b_1} u_b \|_{\alpha-l}^2. \]

We similarly have that

\[ \sum_{l \leq [\alpha]} \| \rho_2^{2\alpha-2l+1} L_{b_1} u_{2,b_1}^- \|_{\alpha-l}^2 \]

is bounded by the right-hand side of (8.17).
In particular, from (8.9), we have
\[
\|\rho_2^{2a+1}\Psi_{b_1}^\alpha u_{2,b_1}\|^2 \lesssim \sum_{l \leq \lfloor \alpha \rfloor} \|\rho_2^{2a-2l+1}g\|_{\alpha-l-1/2}^2 + \|\Psi_{b_1}^{-1}g\|^2 + s.c. \|\rho_2^{2a+1}u_{2,b_1}\|^{2/3}_{1/2+a}
\] (8.18)
\[
\sum_{l \leq \lfloor \alpha \rfloor} \|\rho_2^{2a+1-2l}u_{2,b_1}\|_{1/2+a-1}^2 + \sum_{l \leq \lfloor \alpha \rfloor} \|\rho_2^{2a-2l}u_{b_0}\|^2_{W^{1/2}(\partial\Omega)}
\]
\[
+ \sum_{l \leq \lfloor \alpha \rfloor} \|\rho_2^{2a+1-2l}u_{2,b_1}\|_{\alpha+1-l}^2 + \|u_b\|^2.
\]
Together with (8.4) to estimate \(\|u_{2,b_1}\|_{1/2}^2\), this yields

In analogy with (8.12), we write
\[
\Psi_{b_1}^\alpha \rho_2^{2a+1} = \sum_{l \leq \lfloor \alpha \rfloor} \rho_2^{2a-2l+1}\Psi_{b_1}^{\alpha-l} + \Psi_{b_1}^{-1}
\]
so that we have
\[
\|\rho_2^{2a+1}u_{2,b_1}\|_{\alpha+1/2}^2 \lesssim \sum_{l \leq \lfloor \alpha \rfloor} \|\rho_2^{2a-2l+1}\Psi_{b_1}^{\alpha-l} u_{2,b_1}\|_{1/2}^2 + \|u_{2,b_1}\|^2.
\] Then using (8.18) for the terms in the summation on the right yields
\[
\|\rho_2^{2a+1}u_{2,b_1}\|_{\alpha+1/2}^2 \lesssim \sum_{l \leq \lfloor \alpha \rfloor} \|\rho_2^{2a-2l+1}g\|_{\alpha-l-1/2}^2 + \|\Psi_{b_1}^{-1}g\|^2 + s.c. \|\rho_2^{2a+1}u_{2,b_1}\|_{1/2+a}^2
\] (8.19)
\[
+ \sum_{l \leq \lfloor \alpha \rfloor} \|\rho_2^{2a+1-2l}u_{2,b_1}\|_{1/2+a-1}^2 + \sum_{l \leq \lfloor \alpha \rfloor} \|\rho_2^{2a-2l}u_{b_0}\|^2_{W^{1/2}(\partial\Omega)}
\]
\[
+ \sum_{l \leq \lfloor \alpha \rfloor} \|\rho_2^{2a+1-2l}u_{2,b_1}\|_{\alpha+1-l}^2 + \|u_b\|^2.
\]

The estimates will be obtained by induction. To illustrate the process we calculate the first few estimates, starting with \(\|\rho_2 u_{2,b_1}\|_{1/2}^2\). We continue to refrain from writing the boundary over which the norms are taken in the cases where it is clear from the boundary distribution. Thus, for instance we write \(\|\rho_2 u_{2,b_1}\|_{1/2}^2\) to mean the \(W^{1/2}(\partial\Omega_1 \cap \partial\Omega)\) norm, and \(\|\rho_1 u_{1,b_2}\|_{1/2}^2\) to mean the \(W^{1/2}(\partial\Omega_2 \cap \partial\Omega)\) norm.

From (8.19), we have
\[
\|\rho_2 u_{2,b_1}\|_{1/2}^2 \lesssim \|g\|_{1/2}^2 + \|u_b\|^2 + \|\rho_2 u_{2,b_1}\|_{1/2}^2
\] (8.20)
\[
\lesssim \|f\|_{L^2(\Omega)}^2,
\]
where we use the estimates in (7.6) for the \(u_{2,b_1}\) term. To estimate \(g = R_1 \circ \Psi_{b_1} \circ \partial_{\nu_1} G_2(f)\), we use Corollary 6.3
\[
\|R_1 \circ \Psi_{b_1} \circ \partial_{\nu_1} G_2(f)\|_{1/2} \lesssim \|\partial_{\nu_1} G_2(f)\|_{W^{1/2}(\partial\Omega_1 \cap \partial\Omega)} \lesssim \|f\|_{L^2(\Omega)}.
\]
Together with (8.4) to estimate \(\|u_b\|_{L^2(\partial\Omega)}\) in (8.20), this yields
\[
\|\rho_2 u_{2,b_1}\|_{1/2}^2 \lesssim \|f\|_{L^2(\Omega)}^2.
\]
Combining this with weighted $1/2$ estimates for $u^\pm_{2,b_1}$ and $u^0_{2,b_1}$ as well as the analogous estimates for $u_{1,b_2}$ we get

$$\|\rho^2 u_{2,b_1}\|_{1/2}^2 + \|\rho_1 u_{1,b_2}\|_{1/2}^2 \lesssim \|f\|_{1}^2,$$

Next, we calculate from (8.19)

$$\|\rho^2 u_{2,b_1}\|_{1/2}^2 \lesssim \|\rho^2 g\|^2_{1/2} + \|g\|_{-1}^2 + \|\rho^2 u_{2,b_1}\|_{1/2}^2 + \|\rho_1 u_{1,b_2}\|_{1/2}^2
+ \|\rho^2 u_{2,b_1}\|_{3/2}^2 + \|u^0_{2,b_1}\|_{1/2}^2 + \|u_b\|^2.$$

To estimate $\|\rho^2 g\|^2_{1/2}$, we use

$$\|\rho^2 \Psi^1 \circ \partial_{\rho_1} \circ G_2(f)\| \lesssim \|\rho^2 \circ \partial_{\rho_1} \circ G_2(f)\|_1 + \|\partial_{\rho_1} \circ G_2(f)\|
\lesssim \|f\|_{W^{0,1}(\Omega_{\rho_2,2})},$$

by Corollary 6.3. We also use, from (7.26),

$$\|\rho^2 u^0_{2,b_1}\|_{3/2}^2 + \|u^0_{2,b_1}\|_{1/2}^2 \lesssim s.c.\|u^0_{2,b_1}\|_{W^{0,1}(\partial \Omega_{1}\cap \partial \Omega_{\rho_2,2})} + \|u_b\|^2
\lesssim s.c.\|u_{2,b_1}\|_{W^{0,1}(\partial \Omega_{1}\cap \partial \Omega_{\rho_2,2})} + s.c.\|u_{1,b_2}\|_{W^{0,1}(\partial \Omega_{2}\cap \partial \Omega_{\rho_2,2})}
+ \|f\|_{W^{0,1}(\Omega_{\rho_2,2})}^2.$$

Combining these estimates with those for $\rho^2 u^+_{2,b_1}$ and $\rho^2 u^0_{2,b_1}$, as well as estimates for $u_{1,b_2}$, we have

$$\|\rho^2 u_{2,b_1}\|_{1/2}^2 \lesssim \|f\|_{W^{0,1}(\Omega_{\rho_2,2})}^2,$$

and

$$\|\rho^2 u_{2,b_1}\|_{1/2}^2 + \|\rho^2 u_{1,b_2}\|_{1}^2 \lesssim \|f\|_{W^{0,1}(\Omega_{\rho_2,2})}^2.$$

Next we can calculate $\|\rho^2 u_{2,b_1}\|_{3/2}^2$ using (8.19) (with $\alpha = 1$).

$$\|\rho^3 u^0_{2,b_1}\|_{3/2}^2 \lesssim \|\rho^2 g\|_{1/2}^2 + \|\rho^2 g\|_{3/2}^2 + \|g\|_{-1}^2
+ \|\rho^2 u_{2,b_1}\|_{1}^2 + \|\rho^2 u_{1,b_2}\|_{1}^2
+ \|\rho^2 u_{2,b_1}\|_{3/2}^2 + \|\rho^2 u_{1,b_2}\|_{1}^2
+ \|\rho^2 u_{2,b_1}\|_{1}^2 + \|\rho^2 u_{1,b_2}\|_{1}^2
+ \|\rho^2 u_{2,b_1}\|_{3/2}^2 + \|\rho^2 u_{1,b_2}\|_{1}^2 + \|u_b\|^2.$$

The terms involving $u^0_{2,b_1}$ can be handled as before with

$$\|\rho^3 u^0_{2,b_1}\|_{3/2}^2 + \|\rho^2 u^0_{2,b_1}\|_{1}^2 \lesssim s.c.\|u^0_{2,b_1}\|_{W^{0,1}(\partial \Omega_{1}\cap \partial \Omega_{\rho_2,2})} + s.c.\|u_{1,b_2}\|_{W^{0,1}(\partial \Omega_{2}\cap \partial \Omega_{\rho_2,2})}
+ \|f\|_{W^{0,1}(\Omega_{\rho_2,2})}^2.$$

The only remaining term we need to estimate is the $\|\rho^2 g\|_{1/2}^2$ term. This follows as above:

$$\|\rho^2 \Psi^1 \circ \partial_{\rho_1} \circ G_2(f)\|_{1/2} \lesssim \|\rho^2 \circ \partial_{\rho_1} \circ G_2(f)\|_{3/2} + \|\partial_{\rho_1} \circ G_2(f)\|_{1/2}
\lesssim \|R_{1} \circ \partial_{\rho_1} \circ G_2(f)\|_{W^{1,2,1}(\partial \Omega_{1}\cap \partial \Omega_{\rho_2,2})}
\lesssim \|f\|_{W^{0,1}(\Omega_{\rho_2,2})}.$$

We thus have

$$\|\rho^2 u_{2,b_1}\|_{1/2}^2 \lesssim \|f\|_{W^{0,1}(\Omega_{\rho_2,2})}^2,$$

which can be combined with the other weighted $3/2$ estimates in the usual way.
For a last illustrative step, we estimate $\|\rho_2^2 u_{2,61}\|_2^2$, for which we use (8.19) (with $\alpha = 3/2$):

$$\|\rho_2^2 u_{2,61}\|_2^2 \lesssim \|\rho_2^2 g\|_1^2 + \|\rho_2^2 g\|_2^2 + \|g\|_2^2 \quad + \|\rho_3^2 u_{1,62}\|_3^2 / 2 + \|\rho_3^2 u_{2,61}\|_3^2 / 2 + \|\rho_2^2 u_{1,62}\|_1^2$$

$$+ \|\rho_2^2 u_{2,61}\|_1^2 + \|\rho_1 u_{1,62}\|_2^2 \quad + \|\rho_2^2 u_{2,61}\|_2^2$$

$$+ s.c. \|u_{2,61}\|_{W^{1,2}(\partial \Omega_1 \cap \partial \Omega_2, \rho_2, 2)}^2 + \|u_b\|_2^2.$$

We note

$$\|\rho_2 g\|_1 \lesssim \|\rho_2^2 \partial_{\rho_1} \circ G_2(f)\|_2 + \|\rho_2^2 \partial_{\rho_1} \circ G_2(f)\|_1 + \|\partial_{\rho_1} \circ G_2(f)\|$$

$$\lesssim \|R_1 \circ \partial_{\rho_1} \circ G_2(f)\|_{W^{0,2}(\partial \Omega_1 \cap \partial \Omega_2, \rho_2, 2)}$$

$$\lesssim \|f\|_{W^{0,2}(\Omega, \rho_2, 2)}.$$

and, from (7.6),

$$\|u_{2,61}\|_{W^{1,2}(\partial \Omega_1 \cap \partial \Omega_2, \rho_2, 2)} \lesssim \|u_{1,62}\|_{W^{1,2}(\partial \Omega_2, \rho_2, 2)}$$

$$\quad \quad \quad + \|f\|_{W^{0,2}(\Omega, \rho_2, 2)}$$

to conclude

$$\|u_{2,61}\|_{W^{0,2}(\partial \Omega_1 \cap \partial \Omega_2, \rho_2, 2)} + \|u_{1,62}\|_{W^{0,2}(\partial \Omega_2, \rho_2, 2)} \lesssim \|f\|_{W^{0,2}(\Omega, \rho_2, 2)} + \|f\|_{W^{0,2}(\Omega, \rho_1, 2)}.$$

Higher order (weighted) estimates are calculated with repeated application of (8.19) and the already obtained lower order estimates.

We put together the estimates for the boundary solution in the following

**Theorem 8.1.** Assume $u_b$ is a smooth form on $\partial \Omega$ satisfying (8.5) with analogous expression for other microlocal regions. Then we have the estimates

$$\sum_j \|u_b\|_{W^{0,\alpha}(\partial \Omega_1 \cap \partial \Omega_2, \rho_2, 2)} \lesssim \sum_j \|f\|_{W^{0,\alpha}(\Omega, \rho_1, 2)}.$$

9. REGULARIZING OPERATORS

Here we sketch the argument validating the regularity assumption in Section 8 we used to integrate by parts. We adopt the regularizing operators used in [13]. We let $\zeta(x, \rho_2)$ be a cutoff function with support in a neighborhood of the origin intersected with $\partial \Omega_1$ (i.e. a neighborhood of the boundary singularity in which we are working). For $\delta > 0$ we write $P^\delta_{-}$ to denote an operator in $\Psi^0(\partial \Omega_1)$ whose symbol, $p^\delta_{-}(x, \rho_2, \eta_2, \xi) = \sigma(P^\delta_{-})$ is of the form

$$p^\delta_{-} = c_\delta \left( \sqrt{\eta_2^2 + \xi^2} \right) \zeta(x, \rho_2) \psi_\delta^{-}(\eta_2, \xi),$$

where $c_\delta(t) \in C^0_0(\mathbb{R}_+)$ with $c_\delta(t) \equiv 1$ for $t \in [0, \delta]$ and $c_\delta(t) \equiv 0$ for $t > \delta + 1$.

Furthermore, the symbol estimates, realizing $p^\delta_{-}$ as a function in class $S^0(\partial \Omega_1 \times \mathbb{R}^3)$, are independent of the parameter $\delta$. We also write $P^\delta_{-}$ for operators which dominate $P^\delta_{-}$, by which we mean that the symbol, $p^\delta_{-}$ can be written

$$p^\delta_{-} = c_\delta' \left( \sqrt{\eta_2^2 + \xi^2} \right) \zeta'(x, \rho_2) \psi_\delta^{-}(\eta_2, \xi),$$
where \( \delta' > \delta, \zeta' \in C^\infty(\partial \Omega_1) \) with support near the origin and \( \zeta' \equiv 1 \) on the support of \( \zeta \), and \( k' < k \).

We note \( \mathcal{P}_\delta^- u_{2,b_1} \in C^\infty(\partial \Omega_1) \) and hence the integration by parts leading to (5.3) is valid for \( \mathcal{P}_\delta^- u_{2,b_1} \). We recall that we consider \( \Lambda^\alpha \) to be an operator with symbol

\[
\sigma(\Lambda^\alpha) = \phi(x, \rho_2) \left( 1 + \xi^2 + \eta_2^2 \right)^{\alpha/2},
\]

where \( \phi \) has support where \( \zeta \equiv 1 \).

\[
\| \rho_2^{2\alpha+1} \Lambda^\alpha \mathcal{P}_\delta^- u_{2,b_1} \|_{1/2}^2 \lesssim \| (T_1 \rho_2^{2\alpha+1} \Lambda^\alpha \mathcal{P}_\delta^- u_{2,b_1}, \rho_2^{2\alpha+1} \Lambda^\alpha \mathcal{P}_\delta^- u_{2,b_1}) \|_2
\]

\[
\lesssim \| L_{b_1} \rho_2^{2\alpha+1} \Lambda^\alpha \mathcal{P}_\delta^- u_{2,b_1} \|_2 + \| \mathcal{T}_{b_1} \rho_2^{2\alpha+1} \Lambda^\alpha \mathcal{P}_\delta^- u_{2,b_1} \|_2
\]

\[
+ \| \rho_2^{2\alpha+1} \Lambda^\alpha \mathcal{P}_\delta^- u_{2,b_1} \|_2^2.
\]

In estimating the first two terms on the right, we follow our previous analysis to write

\[
(\rho_2^{2\alpha+1} L_{b_1} \Lambda^\alpha \mathcal{P}_\delta^- u_{2,b_1}, \rho_2^{2\alpha+1} L_{b_1} \Lambda^\alpha \mathcal{P}_\delta^- u_{2,b_1}) \lesssim - (\rho_2^{2\alpha+1} \mathcal{T}_{b_1} L_{b_1} \Lambda^\alpha \mathcal{P}_\delta^- u_{2,b_1}, \rho_2^{2\alpha+1} \Lambda^\alpha \mathcal{P}_\delta^- u_{2,b_1})
\]

\[
+ s.c. \| \rho_2^{2\alpha+1} L_{b_1} \Lambda^\alpha \mathcal{P}_\delta^- u_{2,b_1} \|_2 + \| \rho_2^{2\alpha+1} \Lambda^\alpha \mathcal{P}_\delta^- u_{2,b_1} \|_2^2.
\]

We can commute the two \( L \) derivatives past the \( \Lambda^\alpha \) operator as before, but we need then an expression for the resulting operator applied to \( \mathcal{P}_\delta^- u_{2,b_1} \). For this, we need a replacement for (5.3):

\[
L_{b_1} \mathcal{T}_{b_1} \mathcal{P}_\delta^- u_{2,b_1} = \mathcal{P}_\delta^- g + \Psi_{b_1,\psi,\phi}^1 \mathcal{P}_\delta^- u_{2,b_1} + \Psi_{b_1,\phi}^1 \mathcal{P}_\delta^- u_{2,b_1} + \Psi_{b_1,\phi}^1 \mathcal{E}_\phi u_b
\]

\[
+ \Psi_{b_1}^0 \mathcal{P}_\delta^- L_{b_1} u_{2,b_1} + \Psi_{b_1}^0 \mathcal{P}_\delta^- \mathcal{T}_{b_1} u_{2,b_1} + \Psi_{b_1}^0 \mathcal{P}_\delta^- u_{2,b_1}.
\]

Again, (5.2) holds in the support of \( \phi \). That the support of \( \phi \) is contained in \( \zeta \equiv 1 \) gives rise to the \( \Psi_{b_1,\phi,\psi}^1 \) operator.

We can then estimate the term in (9.1) by

\[
(\rho_2^{2\alpha+1} L_{b_1} \Lambda^\alpha \mathcal{P}_\delta^- u_{2,b_1}, \rho_2^{2\alpha+1} L_{b_1} \Lambda^\alpha \mathcal{P}_\delta^- u_{2,b_1}) \lesssim - (\rho_2^{2\alpha+1} \Lambda^\alpha L_{b_1} \mathcal{T}_{b_1} \mathcal{P}_\delta^- u_{2,b_1}, \rho_2^{2\alpha+1} \Lambda^\alpha \mathcal{P}_\delta^- u_{2,b_1})
\]

\[
+ s.c. \| \rho_2^{2\alpha+1} L_{b_1} \Lambda^\alpha \mathcal{P}_\delta^- u_{2,b_1} \|_2 + s.c. \| \rho_2^{2\alpha+1} \mathcal{T}_{b_1} \Lambda^\alpha \mathcal{P}_\delta^- u_{2,b_1} \|_2 + \| \rho_2^{2\alpha+1} \Lambda^\alpha \mathcal{P}_\delta^- u_{2,b_1} \|_2^2.
\]
again using Gårding’s inequality to handle the commutation term, $[\mathcal{T}_{b_1}, L_{b_1}]$, and using (0.2) then gives for the first term on the right

$$
\langle \rho_2^{2\alpha+1} \Lambda^\alpha L_{b_1} \bar{L}_{b_1} \mathcal{P}_\delta^- u_{2,b_1}, \rho_2^{2\alpha+1} \Lambda^\alpha \mathcal{P}_\delta^- u_{2,b_1} \rangle
\lesssim \langle \rho_2^{2\alpha+1} \Lambda^\alpha \mathcal{P}_\delta^- g, \rho_2^{2\alpha+1} \Lambda^\alpha \mathcal{P}_\delta^- u_{2,b_1} \rangle
+ \langle \rho_2^{2\alpha+1} \Lambda^\alpha \Psi_{b_1,v_0}^1 \mathcal{P}_\delta^- u_{2,b_1}, \rho_2^{2\alpha+1} \Lambda^\alpha \mathcal{P}_\delta^- u_{2,b_1} \rangle
+ \langle \rho_2^{2\alpha+1} \Lambda^\alpha \Psi_{b_1}^1 \mathcal{P}_\delta^- u_{2,b_1}, \rho_2^{2\alpha+1} \Lambda^\alpha \mathcal{P}_\delta^- u_{2,b_1} \rangle
+ \langle \rho_2^{2\alpha+1} \Lambda^\alpha \Psi_{b_1}^1 E_2 u_b, \rho_2^{2\alpha+1} \Lambda^\alpha \mathcal{P}_\delta^- u_{2,b_1} \rangle
+ \langle \rho_2^{2\alpha+1} \Lambda^\alpha \mathcal{P}_\delta^- L_{b_1} u_{2,b_1}, \rho_2^{2\alpha+1} \Lambda^\alpha \mathcal{P}_\delta^- u_{2,b_1} \rangle
+ \langle \rho_2^{2\alpha+1} \Lambda^\alpha \mathcal{P}_\delta^- \bar{L}_{b_1} u_{2,b_1}, \rho_2^{2\alpha+1} \Lambda^\alpha \mathcal{P}_\delta^- u_{2,b_1} \rangle
+ \| \rho_2^{2\alpha} \Lambda^\alpha \mathcal{P}_\delta^{-1} u_{2,b_1} \|^2.
$$

From Section 8 the first three terms on the right can be bounded by

$$
\sum_{l \leq [\alpha]} \| \rho_2^{2\alpha-2l+1} \mathcal{P}_\delta^- g \|^2 \alpha-l/2 + \| \mathcal{P}_\delta^- g \|^2 + s.c. \| \rho_2^{2\alpha+1} \mathcal{P}_\delta^- u_{2,b_1} \|^2_{1/2+\alpha}
+ \sum_{l \leq [\alpha]} \| \rho_2^{2\alpha+1-2l} \mathcal{P}_\delta^- u_{2,b_1} \|^2_{\alpha+1-l} + \| \mathcal{P}_\delta^- u_{2,b_1} \|^2 \nonumber\quad
+ \sum_{1 \leq l \leq [\alpha]} \| \rho_2^{2\alpha+1-2l} \mathcal{P}_\delta^- u_{2,b_1} \|^2_{1/2+\alpha-l} + \| \mathcal{P}_\delta^- u_{2,b_1} \|^2.
$$

We note that in the second and third to last terms, the $\mathcal{P}_\delta^-'$ and $\mathcal{P}_\delta^-$ can be switched, the error involving only lower order terms:

$$
\langle \rho_2^{2\alpha+1} \Lambda^\alpha \mathcal{P}_\delta^- \bar{L}_{b_1} u_{2,b_1}, \rho_2^{2\alpha+1} \Lambda^\alpha \mathcal{P}_\delta^- u_{2,b_1} \rangle
\lesssim \langle \rho_2^{2\alpha+1} \Lambda^\alpha \mathcal{P}_\delta^- \bar{L}_{b_1} u_{2,b_1}, \rho_2^{2\alpha+1} \Lambda^\alpha \mathcal{P}_\delta^- u_{2,b_1} \rangle
+ \sum_{1 \leq l \leq [\alpha]} \| \rho_2^{2\alpha-2l-1} \Psi_b^{a-l} \mathcal{P}_\delta^- L_{b_1} u_{2,b_1} \|^2
+ \sum_{1 \leq l \leq [\alpha]} \| \rho_2^{2\alpha-2l-1} \Psi_b^{a-l} \mathcal{P}_\delta^- \bar{L}_{b_1} u_{2,b_1} \|^2
+ s.c. \| \rho_2^{2\alpha+1} \Lambda^\alpha \mathcal{P}_\delta^- u_{2,b_1} \|^2 + \| u_b \|^2,
$$

where $\mathcal{P}_\delta^-$ dominates $\mathcal{P}_\delta^-$ (similar relations hold for $\bar{L}_{b_1}$), and where we use

$$
\rho_2^{2\alpha+1} \mathcal{P}_\delta^- \Psi_b^a = \mathcal{P}_\delta^- \rho_2^{2\alpha+1} \Psi_b^a + \sum_{1 \leq l \leq [\alpha]} \rho_2^{2\alpha+1-2l} \Psi_b^{a-l} \mathcal{P}_\delta^- + \Psi_b^{-1}.
$$

To handle the terms with $\Psi_b^1 E_2 u_b$ we argue as in Section 8 to reduce to lower order terms:

$$
\langle \rho_2^{2\alpha+1} \Lambda^\alpha \Psi_b^1 E_2 u_b, \rho_2^{2\alpha+1} \Lambda^\alpha \mathcal{P}_\delta^- u_{2,b_1} \rangle
\lesssim \sum_{l \leq [\alpha]} \| \rho_2^{2\alpha-2l} u_b \|_{\alpha-l} + \| u_b \| + s.c. \| \rho_2^{2\alpha+1} \mathcal{P}_\delta^- u_{2,b_1} \|^2_{1/2+\alpha}
+ \sum_{1 \leq l \leq [\alpha]} \| \rho_2^{2\alpha+1-2l} \mathcal{P}_\delta^- u_{2,b_1} \|^2_{1/2+\alpha-l}.
$$
Putting this together yields
\[
\langle \rho_2^{2n+1} \Lambda^\alpha L_{b1} \mathbf{P}_\delta^{-} u_{2,b1}, \rho_2^{2n+1} \Lambda^\alpha \mathbf{P}_\delta^{-} u_{2,b1} \rangle
\]
\[
\lesssim \sum_{l \leq \lceil \alpha \rceil} \| \rho_2^{2n-2l+1} \mathbf{P}_\delta^{-} g \|^2_{\alpha-l-1/2} + \| \mathbf{P}_\delta^{-} g \|^2_{\alpha-l-1/2} + \frac{\| \mathbf{P}_\delta^{-} u_{2,b1} \|^2_{1/2+\alpha}}{s.c.} + \sum_{l \leq \lceil \alpha \rceil} \| \rho_2^{2n+1} \mathbf{P}_\delta^{-} u_{2,b1} \|^2_{1/2+\alpha-l} \]
\[
+ \frac{\| \rho_2^{2n+1} \mathbf{P}_\delta^{-} u_{2,b1} \|^2_{1/2+\alpha-l}}{s.c.} + \sum_{l \leq \lceil \alpha \rceil} \| \rho_2^{2n+1} \mathbf{P}_\delta^{-} u_{2,b1} \|^2_{1/2+\alpha-l} + \frac{\| \rho_2^{2n+1} \mathbf{P}_\delta^{-} u_{2,b1} \|^2_{1/2+\alpha-l}}{s.c.} + \sum_{l \leq \lceil \alpha \rceil} \| \rho_2^{2n+1} \mathbf{P}_\delta^{-} u_{2,b1} \|^2_{1/2+\alpha-l} \]
\[
(9.4)
\]

Genuine estimates can be obtained from the following argument. From (9.3), and the corresponding estimates for its analogue with the \( \bar{L}_{b1} \) operator replacing \( L_{b1} \), and from (9.3), we can obtain estimates for
\[
(9.5) \quad \| \rho_2^{2n+1} L_{b1} \Lambda^\alpha \mathbf{P}_\delta^{-} u_{2,b1} \|^2_{1/2+\alpha} + \| \rho_2^{2n+1} \bar{L}_{b1} \Lambda^\alpha \mathbf{P}_\delta^{-} u_{2,b1} \|^2_{1/2+\alpha}
\]
in terms of a s.c.\( \| \rho_2^{2n+1} \mathbf{P}_\delta^{-} u_{2,b1} \|^2_{1/2+\alpha} \) plus terms of (weighted) lower order of \( u_b \) as well as weighted lower order terms of \( L_{b1} u_{2,b1} \) and \( \bar{L}_{b1} u_{2,b1} \). The higher order norms involving \( u_{2,b1}^0 \) can be handled using (7.0):
\[
\sum_{l \leq \lceil \alpha \rceil} \| \rho_2^{2n+1-2l} \mathbf{P}_\delta^{-} u_{2,b1}^0 \|^2_{1/2+\alpha-l} \lesssim \sum_{l \leq \lceil \alpha \rceil} \| \rho_2^{2n-2l} u_b \|^2_{1/2+\alpha-l} + \| f \|_{W^{0,1}(\Omega_{\rho_1,2})}^2
\]

An induction argument thus gives an estimate for \( \| \rho_2^{2n+1} \mathbf{P}_\delta^{-} u_{2,b1} \|^2_{1/2+\alpha} \) which can then be used to deduce estimates for (9.5). Letting \( \delta \rightarrow \infty \) and combining estimates for \( u_{2,b1}^0 \) and \( u_{2,b1}^0 \) yields estimates for \( \| \rho_2^{2n+1} u_{2,b1} \|^2_{1/2+\alpha} \), as well as
\[
\| \rho_2^{2n+1} L_{b1} \Lambda^\alpha u_{2,b1} \|^2_{1/2+\alpha} + \| \rho_2^{2n+1} \bar{L}_{b1} \Lambda^\alpha u_{2,b1} \|^2_{1/2+\alpha}.
\]
All these estimates can in turn be used in the next induction step to estimate \( \| \rho_2^{2n+2} \mathbf{P}_\delta^{-} u_{2,b1} \|^2_{1/2+\alpha} \) and so forth.

Recalling the notation \( g = R_1 \circ \Psi_0^{-1} \circ \partial_{\rho_1} G_2(f) \) we have

**Theorem 9.1.** Let \( \Omega \subset \mathbb{C}^2 \) be the piecewise smooth intersection domain, \( \Omega = \Omega_1 \cap \Omega_2 \) with generic corners. Let \( u = u_1 \bar{\omega}_1 + u_2 \bar{\omega}_2 \) be the solution to the \( \bar{\partial} \)-Neumann problem (3.1) with boundary conditions (3.2). Let \( u_{b,j} = u|_{\partial \Omega_j \cap \partial \Omega_1} \). Then
\[
\sum_j \| u_{b,j} \|_{W^{0,\alpha}(\partial \Omega_j \cap \partial \Omega_1, \rho_1, 2)} \lesssim \sum_j \| f \|_{W^{0,\alpha}(\Omega_2, \rho_1, 2)}.
\]
10. Estimates of solution operator

By definition of the Poisson and Green operators, the solution to (3.1) with boundary conditions (3.2) can be written

\[ u = G(2f) + P(u_b). \]

From the weighted estimates for Green’s operator (see Theorem 6.4), we have

\[ \sum_j \left\| G(2f) \right\|_{W^{2,s}(\Omega, \rho_j, 2)} \lesssim \sum_j \left\| f \right\|_{W^{0,s}(\Omega, \rho_j, 2)}. \]

In addition, we have the weighted estimates for the Poisson operator from Theorem 4.1:

\[ \left\| P(u_b) \right\|_{W^{1/2,s}(\Omega, \rho, 2)} \lesssim \sum_j \left\| u_{b_j} \right\|_{W^{0,s}(\partial \Omega_j \cap \partial \Omega, \rho_k, 2)} \lesssim \sum_j \left\| f \right\|_{W^{0,s}(\Omega, \rho_j, 2)}. \]

We thus have estimates for the \( \bar{\partial} \)-Neumann problem:

**Theorem 10.1.** Let \( \Omega \) be as in Theorem 9.1. Let \( f \in W^{0,s}(\Omega, \rho_j, 2) \) for \( j = 1, 2 \). Let \( N \) be solution operator to the \( \bar{\partial} \)-Neumann problem (3.1), (3.2). Then

\[ \sum_j \left\| Nf \right\|_{W^{1/2,s}(\Omega, \rho, 2)} \lesssim \sum_j \left\| f \right\|_{W^{0,s}(\Omega, \rho_j, 2)}. \]

When \( s = 0 \) this is the same result of Michel and Shaw in [10] (Theorem 1.2). If we also suppose that the data form \( f \in W^s(\Omega) \). Then since

\[ \sum_j \left\| f \right\|_{W^{0,s}(\Omega, \rho_j, 2)} \lesssim \left\| f \right\|_{W^s(\Omega)}, \]

Theorem 10.1 implies

\[ \left\| Nf \right\|_{W^{1/2,s}(\Omega, \rho, 2)} \lesssim \left\| f \right\|_{W^s(\Omega)}. \]

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