Status of the Lambda Lattice Scale for the SU(3) Wilson gauge action

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With the emergence of the Yang-Mills gradient flow technique there is renewed interest in the issue of scale setting in lattice gauge theory. Here I compare for the SU(3) Wilson gauge action the non-perturbative lambda scales of Edwards, Heller and Klassen (EHK), Necco and Sommer (NS), both relying on Sommer’s method using the quark potential, with the lambda scale derived by Bazavov, Berg and Velytsky (BBV) from deconfining phase transition data of the Bielefeld group. It turns out that these scales are based on mutually inconsistent data. Nevertheless their over-all agreement is still at a better than ±2% in the β range for which one expects them to apply. Somewhat surprisingly the scale based on the deconfining transition is consistent with the relevant part of the EHK data (baring the smallest β), while the NS scale is not.

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1. Introduction

With the emergence of the Yang-Mills gradient flow technique there is renewed interest into the issue of scale setting in lattice gauge theory. For a review see \[1\]. Therefore, it appears to be worthwhile to analyze the status of the lambda scale for the SU(3) Wilson gauge action from previous literature. Based on the Sommer scale \[2\] there are two publications, a paper by Edwards, Heller and Klassen \[3\] (EHK) and another by Necco and Sommer \[4\] (NS). Independently this lambda scale was later extracted by Bazavov, Berg and Velytsky \[5\] (BBV) using deconfining transition data of the Bielefeld group \[6\]. We compare these scales in the next section, using the definition \( \beta = 6/g^2 \), where \( g \) is the bare coupling of the SU(3) Wilson gauge action. Summary and conclusions follow in the final section 3.

2. Definition and comparison of the scales

Sommer \[2\] proposed to set a hadronic scale \( r_i/a \) through the force \( F(r) \) between static quarks at intermediate distances \( r \) by

\[
 r_i^2 F(r_i) = c_i \quad \text{(Sommer scale)}.
\]

For their SU(3) investigations NS \[4\] use the values

\[
 r_0^2 F(r_0) = 1.65 \quad \text{and} \quad r_c^2 F(r_c) = 0.65. \tag{2.1}
\]

The \( r_0 \) value was suggested in the original paper by Sommer. It is used by NS for their smaller lattices and also by EHK, who employ even larger values for \( c_i \), which we do not discuss here because they make no difference for the final determination of their scale. The \( r_c \) definition is used by NS for their set of large lattices. While a number of choices have to be made when calculating \( r_i/a \) (for details see the EHK and NS papers), estimations of the deconfining transition temperatures \( T_t = 1/[a(\beta_t)N_t] \) are in essence free of ambiguities when one uses maxima of the Polyakov loop susceptibility on \( N^3 \times N_t \) lattices to determine \( \beta_t(N_t) \) for the limit \( N \to \infty \). In particular, when refining the lattice a switch of a reference value, like in (2.1) from \( r_0 \) to \( r_c \), is unwarranted when \( T_t \) is used.

In the following we compile the analytical expressions of the three scales. The EHK scale, the second of Eqs. (4.4) in their paper \[3\] with \( \hat{a} \) defined by their Eq. (4.1), is given by

\[
 [a\Lambda_L]^{EHK} = f_{\Lambda}^{EHK}(\beta) = \lambda^{EHK}(g^2 f_{\Lambda}^{as}(g^2)), \tag{2.2}
\]

and considered to be reliable in the range \( 5.6 \leq \beta \leq 6.5 \). Here \( f^{as}(g^2) \) is the universal two-loop scaling function of SU(3) gauge theory,

\[
 f^{as}(g^2) = \left( b_0 g^2 \right)^{-b_1/(2b_0^2)} e^{-1/(2b_0 g^2)} \quad \text{with} \quad b_0 = \frac{11}{3}, \quad b_1 = \frac{34}{3} \left( \frac{3}{16\pi^2} \right)^2. \tag{2.3}
\]

Higher perturbative and non-perturbative corrections are parametrized by

\[
 \lambda^{EHK}(g^2) = (1 + a_1 \hat{a}^2 + a_2 \hat{a}^4)/a_0 \quad \text{with} \quad \hat{a} = \hat{a}(g^2) = f^{as}(g^2)/f^{as}(1) \tag{2.4}
\]

and the coefficients are given by \( a_0 = 0.01596, a_1 = 0.2106, a_2 = 0.05492 \). Up to the free over-all constant the asymptotic scale \( f^{as}(g^2) \) is approached for \( \beta \to \infty \). In contrast to that NS present their
scale in form of a polynomial fit, Eq. (2.6) in their paper [4], which is supposed to be valid in the region $5.7 \leq \beta \leq 6.92$:

$$[a\Lambda_L]^{NS} = f^{NS}_\lambda(\beta) = \exp[-1.6804 - 1.7331 (\beta - 6) + 0.7849 (\beta - 6)^2 - 0.4428 (\beta - 6)^3].$$

The BBV scale, Eq. (19) in their paper [5], is given by

$$[a\Lambda_L]^{BBV} = f^{BBV}_\lambda(\beta) = 10 \times \lambda^{BBV}(g^2) f^{as}(g^2),$$

where $f^{as}$ is again the asymptotic scaling function (2.3) and higher perturbative and non-perturbative corrections are parametrized by

$$\lambda^{BBV}(g^2) = 1 + e^{\ln a_1} e^{-a_2/g^2} + a_3 g^2 + a_4 g^4$$

with the coefficients $\ln a_1 = 18.08596$, $a_2 = 19.48099$, $a_3 = -0.03772473$, $a_4 = 0.5089052$. As the EHK scale, the BBV scale approaches up to a constant factor $f^{as}(g^2)$ for $\beta \to \infty$.

In table 1 data are compiled on which the scales rely. As usual error bars are given in parenthesis and apply to the last digits. The EHK data are from table 4 of their paper [3], which includes also results from other groups. Thus several data point exists at some $\beta$, which are here combined into one estimate per $\beta$ value. Their $\beta = 5.54$ data point is omitted, because it is not used for the determination of their $r_0/a$ scale (2.2). The NS data are from table 1 of their paper [4]. The Bielefeld data are from table 2 of their paper [6]. Together with other information from this paper they were used to extract the BBV scale (2.6). The Bielefeld $\beta_t$ estimates are in good agreement with a by-product of some recent work [7] (BW). As BW data are not used for the calculation of the BBV scale, we do not list their values but give only the likelihoods $Q$ (for the definition see, e.g., Ref. [8]) that the differences between the corresponding Bielefeld and BW estimates are due to chance (last column of table 1). The statistical errors for estimates of deconfining transition transition temperatures are in $\beta_t$ with $N_t$ fixed. To allow for comparison with the statistical accuracy of the Sommer method, we attach to $(aT_t)^{-1}$ error bars by means of the equation

$$\triangle (aT_t)^{-1} = \frac{N_t}{f^{BBV}_\lambda(\beta_t)} \left[ f^{BBV}_\lambda(\beta_t) - f^{BBV}_\lambda(\beta_t - \triangle \beta_t) \right].$$

1To get convenient constants in the upcoming table 2 our definition (2.6) differs by a factor 10 from the one in [4].
For each of the four data sets of table 2 we perform the best one-parameter fit of the form

\[ c/f_\lambda(\beta) \]

(2.9)
to each of the three scales. The results for the twelve constants are compiled in table 2. Even more interesting than the constants are the thus obtained goodness of fit values \( Q \), which are given in table 3. We see that the EHK \( r_0 \) data are only consistent with the EHK scale, similarly the NS \( r_0 \) data are only consistent with the NS scale and the Bielefeld data only with the BBV scale. The NS \( r_c \) data from large lattices are rather inaccurate. They are consistent with the NS and BBV scales and almost consistent with the EHK scale.

There is a twist with respect to the small lattice data. For EHK we find in table 2 a \( r_0/a \) estimate at \( \beta = 5.6 \), which was used in the determination of their scale. We may not expect universal scaling at such a small \( \beta \) value. Leaving the \( \beta = 5.6 \) data point out, we obtain the \( Q \) values of the column EHK \( r_0 - 1 \) of table 3. The EHK data are then consistent with the BBV scale, but still in disagreement with the NS scale. The Bielefeld data remain inconsistent with the EHK scale, because the \( \beta = 5.6 \) data point is used in deriving the EHK scale.

Using the best fits to the BBV scale, regardless of good or bad \( Q \) values, Fig. 1 is obtained for the differences between the data and the BBV fit divided by the BBV fit values (relative deviation). Similarly the relative deviations to the scale functions (2.3) belonging to each data set are calculated and shown in the figure. In the same way Figs. 2 and 3 are obtained for the relative deviations from the NS and EHK scales(2). The ratio between the NS data sets \( r_0 \) and \( r_c \) changes when different scales are used. From the constants of table 2 one finds

\[ (r_c/r_0)^{BBV} = 0.11024 (35)/0.21415 (21) = 0.5148 (18), \]

(2.10)
\[ (r_c/r_0)^{NS} = 0.5140 (17)/0.99995 (98) = 0.5140 (18), \]

(2.11)
\[ (r_c/r_0)^{EHK} = 0.5172 (17)/0.99204 (97) = 0.5214 (18). \]

(2.12)

(2)The rather strong downshift of the \( T_c \) data in Fig. 2 comes from the large statistical weight of the \( N_t = 4 \) data point. Similarly the \( \beta = 5.8 \) data point dominates the NS scale. Error bars of such data points should perhaps be adjusted by hand so that more weight goes to the larger lattices.
The first values for \((r_c/r_0)^{BBV}\) and \((r_c/r_0)^{NS}\) are in statistical agreement with one another as well as with the ratio \(r_c/r_0 = 0.5133\) (24), which is given in Eq. (2.5) of the NS paper and used to determine the NS scale. For \((r_c/r_0)^{NS}\) this is obvious in Fig. 2, where the NS scale (i.e., the zero-line) fits both NS data sets well. All other data are in disagreement with this scale. In Fig. 1 the BBV scale fits the EHK \(r_0 - 1\) data\(^3\), the \(T_c\) and the NS \(r_c\) data well and is in disagreement with the

\(^3\)The fit drawn is for the EHK \(r_0\) data. It becomes good for the EHK \(r_0 - 1\) data (omission of the \(\beta = 5.6\) data point implies also small changes for the EHK data coefficients in table 2).
NS \( r_0 \) data and the \( \beta = 5.6 \) EHK value. Due to the slight difference between the ratios (2.10) and (2.11), the NS scale on its \( r_0 \) data should in Fig. 1 be slightly higher than the NS scale on its \( r_c \) data. As this stays within statistical errors, we have just averaged the two curves, but use distinct colors, red for the \( r_0 \) and blue for the \( r_c \) range. Such averaging is not possible when plotting the NS data on the EHK scale, because the ratio (2.12) is incompatible with the other two ratios. It amounts to the difference between the red and blue curves in Fig. 3.

3. Summary and conclusions

Table 3 shows that the three scales (EHK, NS and BBV) are derived from data sets of table 1 which are mutually inconsistent in the range up to \( \beta = 6.4 \), while the NS \( r_c \) data for the range \( 6.57 \leq \beta \leq 6.92 \) are not very restrictive. Nevertheless, in the range \( 5.65 \leq \beta \leq 6.92 \) the relative discrepancy between the scales is never larger than \( \pm 2\% \) as is shown in the upper part of Fig. 4 for ratios of the form \( \text{const} \, f_{EHK}^{\lambda} / f_{BBV}^{\lambda} \) and \( \text{const} \, f_{NS}^{\lambda} / f_{BBV}^{\lambda} \) (the upper abscissa and the right ordinate apply and the constants (2.9) used from table 3 are the same as those for Fig. 1). Note that the previous figures, which exhibit the deviation of the data from one another cover a similar range [-0.03:0.02].

The lower part of Fig. 4 shows that the EHK and BBV scales approach the universal asymptotic scale (2.3) in rather distinct ways, whereas such a parametrization is not attempted by NS (this part of the figure uses a normalization in which all scales agree at \( \beta = 6 \)).

In conclusion, more accurate results from large lattices are desirable, which could come from calculations of the SU(3) deconfining temperature for \( N_t > 12 \). Despite the “time of transition” anticipated in Ref. [1], I was before the symposium not able to find SU(3) scale calculations with the gradient method. One should be careful before jumping to conclusions about a paradigm change.
Figure 4: Ratios with respect to the BBV scale (upper part, top abscissa and right ordinate) and asymptotic behavior of the scales (lower part, bottom abscissa and left ordinate).

For instance, the sensitivity of the gradient method to topological excitations, as for SU(3) studied in [9], could turn into a disadvantage when it comes to accurate scale calculations.

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