Chiral perturbation theory in a magnetic background - finite-temperature effects

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Abstract: We consider chiral perturbation theory for $SU(2)$ at finite temperature $T$ in a constant magnetic background $B$. We compute the thermal mass of the pions and the pion decay constant to leading order in chiral perturbation theory in the presence of the magnetic field. The magnetic field gives rise to a splitting between $M_{\pi^0}$ and $M_{\pi^\pm}$ as well as between $F_{\pi^0}$ and $F_{\pi^\pm}$. We also calculate the free energy and the quark condensate to next-to-leading order in chiral perturbation theory. Both the pion decay constants and the quark condensate are decreasing slower as a function of temperature as compared to the case with vanishing magnetic field. The latter result suggests that the critical temperature $T_c$ for the chiral transition is larger in the presence of a constant magnetic field. The increase of $T_c$ as a function of $B$ is in agreement with most model calculations but in disagreement with recent lattice calculations.

Keywords: Chiral perturbation theory, finite-temperature field theory, chiral transition, magnetic field.
1 Introduction

While the QCD Lagrangian in the limit of zero quark masses has chiral symmetry, the true ground state of QCD does not respect this symmetry. Chiral symmetry is broken spontaneously in the vacuum by quantum effects. Specifically, the $SU(N_f)_L \times SU(N_f)_R$ symmetry of $\mathcal{L}_{\text{QCD}}$ is broken down to $SU(N_f)_V$, and according to Goldstone’s theorem this gives rise to as many massless spin-zero particles as there are broken generators. For $N_f$ fermions, this implies $N_f^2 - 1$ massless excitations and in phenomenological, we have applications $N_f = 2$ or $N_f = 3$. The symmetry breaking is apparent in the low-energy spectrum of QCD from the existence of the three very light pions. The fact that the pions are not strictly massless is due to the fact that chiral symmetry of nature is only approximate. The associated explicit symmetry breaking in QCD is built in by finite quark masses in $\mathcal{L}_{\text{QCD}}$.

Lattice simulations are currently the only way to calculate the properties of QCD from first principles. However, due to the sign problem, one is restricted to small values of the baryon chemical potential. Other approaches involve low-energy effective theories or model calculations that share some of the properties of QCD such as the same pattern of symmetry breaking. Examples are chiral perturbation theory (ChPT), the Nambu-Jona-Lasinio (NJL) model, linear sigma models, and quark-meson (QM) models. At long wavelengths where the relevant degrees of freedom are the
almost massless Goldstone bosons, one would naturally like to have an effective low-energy theory that contains only these degrees of freedom. Such an effective field theory may simplify calculations of the long-distance properties of the system and one can obtain model-independent predictions with a minimum of assumptions.

An example of a low-energy effective theory for QCD is chiral perturbation theory [1–4]. The chiral Lagrangian that describes the (pseudo)Goldstone bosons is uniquely determined by the global symmetries of QCD and the assumption of chiral symmetry breaking. The Lagrangian $L_{\text{eff}}$ consists of a string of operators that involve an increasing number of derivatives or quark mass factors, each multiplied by a low-energy constant (LEC) $l_i$. However, QCD is a confining and strongly interacting theory at low energies. Thus the coefficients $l_i$ of the chiral Lagrangian cannot be calculated directly from QCD. Instead, the coefficients of the effective Lagrangian are fixed by experiments.

Chiral perturbation theory provides a systematic framework at low energies. It is not an expansion in powers of some small coupling constant, but it is a systematic expansion in powers of momenta $p$ where a derivative counts as one power and the quark masses count as two powers [2]. Chiral perturbation theory is a nonrenormalizable quantum field theory in the old sense of the word. This means that a calculation at a given order $n$ in momentum $p$, requires that one adds higher-order operators in order to cancel the divergences that arise in the calculations at order $n$. This implies that one needs more and more couplings $l_i$ as one goes to higher loop orders, and therefore more experiments to determine them. However, this poses no problem, as long as one is content with finite precision. This is is the essence of effective field theory [5–7]. The order $p^4$-divergences in ChPT were determined in Ref. [2], while the order-$p^6$ divergences were calculated in Ref. [4].

The thermodynamics of a pion gas using ChPT was studied in detail in a series of papers 25 years ago [8–10]. The temperature-dependence of the pion mass $M_\pi$ and the pion decay constant $F_\pi$ were calculated to leading order (LO), while the pressure and the temperature dependence of the quark condensate were calculated to next-to-next-to-leading order (NNLO) in chiral perturbation theory. In the chiral limit, there are two scales at finite temperature, namely the pion decay constant and $T$, and chiral perturbation theory is then an expansion in powers of $T^2/F_\pi^2$. In the present paper, we give the calculational details of a number of quantities in the presence of a constant magnetic field $B$. These include $M_\pi^0, M_\pi^\pm, F_\pi^0, F_\pi^\pm$ at leading order and the free energy and the (normalized) quark condensate at next-to-leading order (NLO).

QCD in external magnetic fields has received a lot of attention in recent years. This is not a purely academic question since the properties of QCD in strong magnetic fields $B$ is relevant in several situations. For example, large magnetic, $B \sim 10^{14} – 10^{15}$ Gauss, exist inside magnetars [11, 12]. If the density of the core is sufficiently high, it contains quark matter. In that case, the core may be color superconducting and so it is important to study the effects of external magnetic fields in this phase [13–20]. Similarly, it has been suggested that strong magnetic fields are created in (noncentral) heavy-ion collisions at the Relativistic Heavy-Ion Collider (RHIC) and the Large Hadron Collider (LHC) and that these play an important role [21–23]. In this case, the magnetic field strength has been estimated to be up to $B \sim 10^{19}$ Gauss, which corresponds to $|qB| \approx 6m_\pi^2$, where $|q|$ is the charge of the pion. Even larger fields could be reached due to the effects of event-by-event fluctuations, see for example [24].

This has spurred the interest in studying QCD in external fields. At zero baryon chemical potential
this can be done from first principles using lattice simulations. Recently, lattice simulations in a constant magnetic background $B$ have been carried out \cite{25–27}. At finite $\mu_B$ this is very difficult due to the infamous sign problem. Therefore one often resorts to effective theories that share some of the features of QCD, such as chiral symmetry breaking.

A low-energy effective theory that provides a systematic framework for systematic calculations is chiral perturbation theory. Chiral perturbation theory has been used to study the quark condensate in strong magnetic fields at zero temperature \cite{28–31} and finite temperature \cite{32}. In Ref. \cite{33}, the thermal corrections to $M_\pi^0$ and $F_\pi^0$ were computed, while the quark-hadron phase transition was investigated in Ref. \cite{34}. The effects of external magnetic fields on the chiral transition have been studied in detail using the NJL model \cite{35–44}, the Polyakov-loop extended NJL model \cite{45,46}, the QM model \cite{43,44,47–49}, the linear sigma model \cite{50}, the (P)QM model \cite{51,52}, and the MIT bag model \cite{53}.

In the present paper, we study pions at finite temperature in a constant external magnetic field $B$ using chiral perturbation theory and it is organized as follows. In Sec. II, we briefly discuss ChPT in an external magnetic field. In Sec. III, we calculate the leading correction to the pion mass as well as the pion decay constant. In Sec. IV, we calculate the free energy and the quark condensate to next-to-leading order in chiral perturbation theory. In Sec. V the numerical results are presented and we summarize in Sec. VI. The necessary sum-integrals are listed in Appendix A, while explicit calculations of those sum-integrals are presented in Appendix B.

2 Chiral perturbation theory

As explained in the introduction, chiral perturbation theory is a low-energy effective field theory that can be used to systematically calculate physical quantities as a power series in momentum $p$. The effective Lagrangian is given by an infinite string of operators involving an increasing number of derivatives or quark masses. Schematically, we can write

$$\mathcal{L}_{\text{eff}} = \mathcal{L}^{(2)} + \mathcal{L}^{(4)} + \mathcal{L}^{(6)} + \ldots$$  \hspace{1cm} (2.1)

where the superscript indicates the powers of momentum. In Euclidean space, the leading term is given by

$$\mathcal{L}^{(2)} = \frac{1}{4} F^2 \text{Tr} \left[ (D_\mu U) \Gamma (D_\mu U) - M^2 \text{Tr}(U + U^\dagger) \right].$$  \hspace{1cm} (2.2)

The first term is the Lagrangian of the nonlinear sigma model and the second term is the kinetic term for the gt. Here $U = \exp[i\tau^a \pi^a/F]$ is a unitary $SU(2)$ matrix, where $\pi^a$ are the pion fields and $\tau^a$ are the Pauli spin matrices. The low-energy constants $M$ and $F$ are the tree-level values for the pion mass $M_\pi$ and the pion decay constant $F_\pi$, respectively. Moreover $D_\mu$ is the covariant derivative which replaces the partial derivative:

$$D_\mu = \partial_\mu - i[U, v_\mu] - i \{U, a_\mu\},$$  \hspace{1cm} (2.3)
where $v_\mu$ is an external vector field and $a_\mu$ is an external axial vector field. The next-to-leading order terms in the low-energy expansion are \cite{2, 10, 31} \footnote{The last two operators of Eq. (2.4) are so-called contact terms. The term $-h_1 M^4$ contributes to the vacuum energy. Loop corrections to the vacuum energy are divergent and renormalizing the couplings such as $h_1$ eliminates (some of) the divergences and is needed to render the vacuum energy finite. We will not need them in the following since we are subtracting the divergences in the vacuum diagrams that appear for $B = 0$. The remaining divergences are then eliminated by renormalizing the low-energy couplings $\bar{l}_i$.}

$$
\mathcal{L}^{(4)} = -\frac{1}{4} l_4 \text{Tr} \left[ (D_\mu U)^\dagger (D_\mu U) \right]^2 - \frac{1}{4} l_2 \text{Tr} \left[ (D_\mu U)^\dagger (D_\nu U) \right] \text{Tr} \left[ (D_\mu U)^\dagger (D_\nu U) \right] \\
+ \frac{1}{8} l_4 M^2 \text{Tr} \left[ (D_\mu U)^\dagger (D_\nu U) \right] \text{Tr}(U + U^\dagger) - \frac{1}{16} (l_3 + l_4) M^4 \text{Tr}((U + U^\dagger)^2 - l_5 \text{Tr}[U^\dagger F^R_{\mu\nu} UF^L_{\mu\nu}] \\
- \frac{1}{2} i l_6 \text{Tr}[F^R_{\mu\nu}(D_\mu U)(D_\nu U)^\dagger + F^L_{\mu\nu}(D_\mu U)(D_\nu U)] - h_1 M^4 + h_2 \text{Tr}[F^L_{\mu\nu}F^L_{\mu\nu}] .
$$

(2.4)

Here $F^L_{\mu\nu} = \partial_\mu F^l_{\nu} - \partial_\nu F^l_{\mu} - i[F^l_{\mu\nu}, F^l_{\nu\rho}]$, where $l = L, R$, and $F^R_{\mu\nu} = v_\mu + a_\mu$ and $F^L_{\mu\nu} = v_\mu - a_\mu$.

The Lagrangian $\mathcal{L}^{(6)}$ is very complicated as it contains more than 50 terms for $SU(2)$ \cite{4}. However, only one term is relevant for the calculations \cite{29–31}, namely

$$
\mathcal{L}^{(6),\text{relevant}} = -4 c_{34} M^2 (q F_{\mu\nu})^2 .
$$

(2.5)

We consider in the following a constant external magnetic field so $v_\mu$ is an Abelian gauge field, $v_\mu = \frac{1}{2} q A_\mu \tau^3$, and $a_\mu = 0$. Here $q$ is the electric charge of the pion and we choose the four-vector potential to be $A_\mu = B \delta_{\mu 2} x_1$. By expanding the Lagrangian Eq. (2.2) to fourth order in the pion field $\pi$, we obtain

$$
\mathcal{L}^{(2)} = -F^2 M^2 + \frac{1}{2} \left( \partial_\mu \pi^0 \right)^2 + \frac{1}{2} M^2 (\pi^0)^2 + \left( \partial_\mu + i q A_\mu \right) \pi^+(\partial_\mu - i q A_\mu) \pi^- + M^2 \pi^+ \pi^- \\
- \frac{M^2}{24 F^2} \left[ (\pi^0)^2 + 2 \pi^+ \pi^- \right]^2 + \frac{1}{6 F^2} \left\{ 2 \pi^0 [\partial_\mu \pi^0] [\partial_\mu (\pi^+ \pi^-)] - 2 \pi^+ \pi^-(\partial_\mu \pi^0)^2 \\
- 2 \left[ (\pi^0)^2 + 2 \pi^+ \pi^- \right] (\partial_\mu \pi^+) (\partial_\mu \pi^-) + [\partial_\mu (\pi^+ \pi^-)]^2 \right\} ,
$$

(2.6)

where we have defined the complex pion fields as $\pi_\pm = \frac{1}{\sqrt{2}}(\pi_1 \pm i \pi_2)$ In the same manner, we can expand $\mathcal{L}^{(4)}$:

$$
\mathcal{L}^{(4)} = \frac{1}{4} F^2 \mu^2 + \frac{2 l_5}{F^2} (q F_{\mu\nu})^2 \pi^+ \pi^- + \frac{2 i l_6}{F^2} q F_{\mu\nu} \left[ (\partial_\mu \pi^0)(\partial_\nu \pi^+) + i q A_\mu \partial_\nu (\pi^+ \pi^-) \right] + (l_3 + l_4) \frac{M^4}{F^2} (\pi^0)^2 \\
+ 2(l_3 + l_4) \frac{M^4}{F^2} \pi^+ \pi^- + l_4 \frac{M^2}{F^2} (\partial_\mu \pi^0)^2 + 2 l_4 \frac{M^2}{F^2} (\partial_\mu + i q A_\mu) \pi^+(\partial_\mu - i q A_\mu) \pi^- .
$$

(2.7)

We note in passing that our expression (2.6) for the truncated Lagrangian $\mathcal{L}^{(2)}$ differs from the expression found in Refs. \cite{29, 30} since they use a different parametrization for the unitary matrix $U$, namely the Weinberg parametrization. However, we obtain the same expressions for physical quantities independent of parametrization \cite{54}. This is simply the statement that physical quantities are independent of the coordinate system used.
At this point it is appropriate to briefly discuss the number of Goldstone modes in the presence of an external electromagnetic field. Due to the different electric charges of the $u$ and $d$ quarks, flavor symmetry is broken in an external electromagnetic field. In particular, the axial $SU(2)_A$-symmetry is broken down to a $U(1)_A$-symmetry, which corresponds to a rotation of the $u$ and $d$-quarks with opposite angles. The formation of a quark condensate breaks this Abelian symmetry which gives rise to a Goldstone boson - the neutral pion $\pi^0$ [28]. The charged pions are thus no longer Goldstone modes. In fact, the presence of the external field allows for an effective mass term even in the chiral limit when $M = 0$, namely the first two terms in Eq. (2.7).

The chiral Lagrangian comes with a number of undetermined parameters or low-energy constants (LECs). These parameters can be determined by experiments. However, loop corrections involve renormalization of these parameters and the physical quantities are no longer equal to the parameters of the chiral Lagrangian. For example, $F$ and $M$ are no longer the measured pion decay constant and the physical pion mass, respectively. The relation between the bare and renormalized parameters has been found in [2] and can be expressed as 2

$$ l_i = -\frac{\gamma_i}{2(4\pi)^2} \left[ \frac{1}{\epsilon} + 1 - \tilde{l}_i \right], \quad (2.8) $$

where $\gamma_i$ are coefficients that are tabulated in [2] and $\tilde{l}_i$ are scale-independent parameters, i.e. they are the renormalized running couplings evaluated at the renormalization scale $\Lambda = M$.

In the present calculations, we need $\gamma_3 = -\frac{1}{2}$, $\gamma_4 = 2$, $\gamma_5 = -\frac{1}{6}$, and $\gamma_6 = -\frac{1}{3}$ [2, 3]. The coupling $c_{34}$ in $\mathcal{L}^{(6)}$ can be renormalized as in [4] by using

$$ c_{34} = \frac{1}{F^2} \left[ \tilde{c}_{34} - \frac{1}{2(4\pi)^2\epsilon} \left( \tilde{l}_5 - \frac{1}{2}\tilde{l}_6 \right) \right]. \quad (2.9) $$

We close this section by briefly discussing the bare propagators of the pions in a magnetic background. The classical solutions to the Klein-Gordon equation in a constant magnetic field $B$ are well known and the dispersion relation is given by

$$ (E^{\pm}_{m,p_z})^2 = p_z^2 + M^2 + (2m + 1)|qB|, \quad (2.10) $$

where $m = 0, 1, 2, \ldots$ denotes the $m$th Landau level, $q$ is the electric charge of the pion, and $p_z$ is the spatial momentum in the $z$-direction. The subscript $\pm$ denotes $\pi^{\pm}$ and we note that the dispersion relations for the two charged pions are identical. The imaginary-time propagator for a charged pions is then $\Delta_{\pi^\pm}(P_0, p) = 1/(P_0^2 + p_z^2 + M^2 + (2m + 1)|qB|)$, while for neutral pions it is given by the usual $\Delta_{\pi^0}(P_0, p) = 1/(P_0^2 + p_z^2 + M^2)$, where $P_0 = 2n\pi T$ is the $n$th Matsubara frequency.

3 Pion masses and pion decay constants

In this section, we discuss the thermal pion masses and the pion decay constants. In order to calculate the thermal pion masses, we consider the inverse propagators at one-loop. For example, the inverse propagator for the neutral pion can be written as

$$ \Delta_{\pi^0}^{-1}(P_0, p) = P_0^2 + M^2 + \Sigma_{m_0}(P_0, p), \quad (3.1) $$

\[\text{In contrast to ref. [2], we have introduced the renormalization scale } \Lambda \text{ in the definition of the sum-integrals instead of as a part of the relation between the bare and renormalized parameters. See Appendix A for details.}\]
where the self-energy is given by

\[ \Sigma_{\pi_0}(P_0, p) = [A_0 - \Delta Z_0]P^2 + [B_0 - \Delta Z_0]M^2, \]  

(3.2)

where \( \Delta Z_0 \) is the wavefunction renormalization counterterm for \( \pi^0 \) and \( \pi^\pm \), respectively. The coefficients \( A_0 \) and \( B_0 \) are given by

\[ A_0 = -\frac{2}{3F^2} \int P \frac{1}{P^2 + P_B^2} \frac{1}{2} + 2M^2 F \langle \bar{I}_4 \rangle, \]  

(3.3)

\[ B_0 = \frac{1}{6F^2} \left[ 2 \int P \frac{1}{P^2 + P_B^2} \frac{1}{2} - 3 \int P \frac{1}{P^2 + P_B^2} + 2M^2 F \langle \bar{I}_3 + \bar{I}_4 \rangle. \]  

(3.4)

The sum-integrals \( \sum_P \) and \( \sum_P^B \) are defined in Appendix A. The counterterms \( \Delta Z_0 \) is chosen such that the self-energy \( \Sigma_{\pi_0} \) is independent of the momentum of the pions [55]. This implies that \( \Delta Z_0 = A_0 \). The propagator then reduces to

\[ \Delta_{\pi_0}^{-1}(P_0, p) = P^2 + M^2 [1 - A_0 + B_0]. \]  

(3.5)

Inserting Eqs. (3.3) and (3.4) into Eq. (3.5), we can then read off the mass \( m_{\pi_0} \) from the propagator and we find

\[ M_{\pi_0}^2 = M^2 + \frac{M^2}{F^2} \int P \frac{1}{P^2 + P_B^2} \frac{1}{2} - \frac{1}{2} M^2 F \langle \bar{I}_4 \rangle + 2\bar{I}_3 \frac{M^4}{F^2}. \]  

(3.6)

The mass \( M_{\pi^\pm} \) of the charged pion can be found in a similar way

\[ M_{\pi^\pm}^2 = M^2 + \frac{M^2}{2F^2} \int P \frac{1}{P^2 + M^2} + 2\bar{I}_3 \frac{M^4}{F^2} + \frac{2}{F^2} (2l_5 - l_6)(qB)^2. \]  

(3.7)

Renormalization of \( l_5 \) and \( l_6 \) are carried out according to Eq. (2.8), which shows that we can replace \( 2l_5 - l_6 \) by \( (l_6 - l_5)/6(4\pi)^2 \) [29, 30]. After renormalization, one obtains

\[ M_{\pi_0}^2 = M^2 \left[ 1 - \frac{1}{(4\pi)^2F^2} \left( I_B(M) + \frac{1}{2} J_1(\beta M) T^2 - J_1^B(\beta M) |qB| \right) \right], \]  

(3.8)

\[ M_{\pi^\pm}^2 = M^2 \left[ 1 + \frac{1}{2(4\pi)^2F^2} J_1(\beta M) T^2 \right] + \frac{(qB)^2}{3(4\pi)^2F^2} (l_6 - l_5), \]  

(3.9)

where the function \( I_B(M) \) is defined by

\[ I_B(M) = M^2 \log \frac{M^2}{2|qB|} - M^2 - 2\zeta(1,0)(0, \frac{1}{2} + x) |qB| \]

\[ = M^2 \log \frac{M^2}{2|qB|} - M^2 - 2\log[\Gamma(0, \frac{1}{2} + x)]|qB| + \log(2\pi)|qB|, \]  

(3.10)

and \( x = \frac{M^2}{2|qB|} \). In particular, note that \( I_B(0) = |qB| \log 2 \). The thermal integrals \( J_1(\beta M) \) and \( J_1^B(\beta M) \) are defined in Appendix A, and evaluated at \( \epsilon = 0 \). Moreover, \( \zeta(1,0)(b,y) = \frac{1}{b^2} \zeta(x - \frac{1}{2}) \).
A current of Eq. (3.12) of the charged pion is no longer a Goldstone mode in the presence of an electromagnetic field. F constant for the time component and one for the spatial component of A, respectively.

The result (3.8) is in agreement with the calculations of Ref. [33]. In the limit B → 0, Eqs. (3.8)–(3.9) reduce to the result of Ref. [8]. We notice that temperature dependence of the charged pion mass Mπ± is the same as for vanishing magnetic field since the only loop correction to the mass involves a neutral pion, cf. Eq. (3.7). The only difference is a temperature-independent constant proportional to (qB)². This constant survives the chiral limit, M → 0, which shows that the charged pion is no longer a Goldstone mode in the presence of an electromagnetic field.

We next discuss the pion decay constant Fπ0 for the neutral pion. The components of the axial current Aₜμ are given by [54, 55]

\[ A_\mu = -F\partial_\mu\pi^0 + \frac{2}{3F} \left[ 2\pi^+\pi^-\partial_\mu\pi^0 - \pi^0\partial_\mu(\pi^+\pi^-) \right] - F^2 F_4\partial_\mu\pi^0, \]

In order to calculate the pion decay constant, we need to evaluate the matrix element \( F_{\pi^0} = \langle 0 | A^0_\mu | \pi^0 \rangle \). The matrix elements are proportional to \( ip_\mu \). Moreover, in the leading term \( -F\partial_\mu\pi^0 \) of Eq. (3.12), we must take into account wavefunction renormalization of the pion field given by Eqs. (3.3). A straightforward calculation of the matrix element then gives

\[ F_{\pi^0} = F \left[ 1 - \frac{F}{2\pi^2} \int_0^B \frac{1}{p^2 + M^2} \int_0^B \frac{1}{p^2 + M^2 + M_B^2} + \frac{M^2}{F^2 F_4} \right]. \]

In a similar way we can calculate the pion decay constant for the charged pion, \( F_{\pi^±} \), and one finds

\[ F_{\pi^±} = F \left[ 1 - \frac{F}{2\pi^2} \int_0^B \frac{1}{p^2 + M^2} \int_0^B \frac{1}{p^2 + M^2 + M_B^2} + \frac{M^2}{F^2 F_4} \right]. \]

After renormalization, we find

\[ F_{\pi^0} = F_\pi \left[ 1 + \frac{1}{(4\pi)^2 F^2} \left( J_B(M) - J_B^R(\beta M)qB \right) \right], \]

\[ F_{\pi^±} = F_\pi \left[ 1 + \frac{1}{2(4\pi)^2 F^2} \left( J_B(M) - J_1(\beta M)T^2 - J_B^R(\beta M)qB \right) \right], \]

where the thermal integrals \( J_1(\beta M) \) and \( J_B^R(\beta M) \) are evaluated at \( \epsilon = 0 \), and where the pion decay constant \( F_\pi \) in the vacuum is

\[ F_\pi = \left[ 1 + \frac{M^2}{(4\pi)^2 F^2 F_4} \right]. \]

The result (3.15) agrees with the calculations of Agasian and Shushpanov [33]. In the limit B → 0, \( F_{\pi^0} = F_{\pi^±} \) and they reduce to the result of [9] as they should. At zero magnetic field, the decay constants are identical \( F_{\pi^0} = F_{\pi^±} \) but there are two pion decay constants at finite temperature; one for the time component and one for the spatial component of \( A_\mu \). The difference between them is an order-\( p^3 \) effect [56] and thus the calculation of the difference is beyond the scope of this paper.
4 Free energy and quark condensate

In this section, we compute the free energy and the quark condensate to NLO in chiral perturbation theory. We are interested in the contributions to the free energy that are due to a nonzero magnetic field and finite temperature. We therefore write the contribution to the free energy at the $n$th loop order, $F_n$, as a sum of three different terms

$$F_n = F_{n}^{\text{vac}} + F_{n}^{B} + F_{n}^{T}, \quad (4.1)$$

where $F_{n}^{\text{vac}}$ is the contribution in the vacuum, i.e. for $B = T = 0$, $F_{n}^{B}$ is the zero-temperature contribution due to a finite magnetic field, and $F_{n}^{T}$ is the finite-temperature contribution. The last term also depends on $B$. The strategy is to isolate the term $F_{n}^{\text{vac}}$ and subtract it from Eq. (4.1). This term contains ultraviolet divergences which are removed by renormalization of the low-energy constants of the chiral Lagrangian in the usual way and the renormalized expression for $F_{n}^{\text{vac}}$ represents the contribution to the vacuum energy of the theory from the $n$th loop order. The term $F_{n}^{B}$ generally contains additional ultraviolet divergences as well and it is rendered finite by the same renormalization procedure. The term $F_{n}^{T}$ is finite.

The free energy at tree level is given by the first term in Eqs. (2.6) and (2.7):

$$F_0 = \frac{1}{2} B^2 - F^2 M^2. \quad (4.2)$$

The first term in Eq. (4.2) is needed when we renormalize the one-loop contribution to the free energy in the next subsection. The second term is independent of $B$, so we neglect it in the following.

4.1 One-loop free energy

The one-loop contribution $F_1$ to the free energy can be written as the sum of the contributions from the charged and neutral pions, and tree-level terms from $\mathcal{L}^{(4)}$

$$F_1 = F_{\pi^0} + F_{\pi^+} + F_{\pi^-} + \text{tree level terms from } \mathcal{L}^{(4)}, \quad (4.3)$$

where

$$F_{\pi^0} = \frac{1}{2} \sum_{P} \log \left[ P^2 + M_{\pi}^2 \right], \quad (4.4)$$

$$F_{\pi^\pm} = \frac{1}{2} \sum_{P} B \log \left[ P_0^2 + p_z^2 + M_{\pi}^2 \right], \quad (4.5)$$

The tree-level terms are necessary in the renormalization of the vacuum energy, but according to our strategy we do not need to consider these since they are independent of the magnetic field. Using the expressions for the various sum-integrals in Appendix A, we obtain

$$F_1 = \frac{1}{2(4\pi)^2} \left( \frac{\Lambda^2}{2|qB|} \right)^\varepsilon \left[ \left(\frac{(qB)^2}{3} - M^4 \right) \left(\frac{1}{\varepsilon} + 1 \right) + 8\zeta^{(1,0)}(-1, \frac{1}{2} + x)(qB)^2 - 2J_0^2(\beta M)|qB|T^2 \right] - \frac{1}{4(4\pi)^2} \left( \frac{\Lambda^2}{M^2} \right)^\varepsilon \left[ \left(\frac{1}{\varepsilon} + \frac{3}{2} \right) M^4 + 2J_0(\beta M)T^4 \right]. \quad (4.6)$$
The divergence that depends on the external magnetic field $B$ is removed by wavefunction renormalization, i.e. by the replacement

$$B^2 \rightarrow B^2 \left[ 1 - \frac{q^2}{6(4\pi)^2 \epsilon} \right],$$

in the tree-level term $\frac{1}{2}B^2$ in Eq. (4.2). Subtracting the expressions for the diagrams for $T = B = 0$, we remove the remaining divergences and obtain a finite result. One finds

$$\mathcal{F}_1^B = \frac{M^4}{4(4\pi)^2} \left[ 1 - 2 \log \frac{M^2}{2|qB|} \right] + \frac{4(qB)^2}{(4\pi)^2} \zeta(1,0)(-1, \frac{1}{2} + x) + \frac{(qB)^2}{6(4\pi)^2} \log \frac{\Lambda^2}{2|qB|},$$

$$\mathcal{F}_1^T = -\frac{1}{2(4\pi)^2} \left[ J_0(\beta M)T^4 + 2J_0^B(\beta M)|qB|T^2 \right].$$

4.2 Two-loop free energy

Figure 1. Feynman diagrams contributing to the free energy at order $p^4$.

The order-$p^4$ diagrams that contribute to the free energy are shown in Fig. 1. Dotted lines denote a neutral pion and solid lines denote a charged pion. Wiggly line denote the background field $|qB|$. The two-loop contribution $\mathcal{F}_2$ to the free energy can be written as

$$\mathcal{F}_2 = \mathcal{F}_{2a} + \mathcal{F}_{2b} + \mathcal{F}_{2c} + \mathcal{F}_{2d} + \mathcal{F}_{2e},$$

where the expressions for the diagrams are

$$\mathcal{F}_{2a} = -\frac{1}{8} \frac{M^2}{F^2} \left( \int Pf \int P^2 + M^2 \right)^2,$$

$$\mathcal{F}_{2b} = \frac{1}{8} \frac{M^2}{F^2} \int Pf \int P^2 + M^2 \int Q \frac{1}{Q^2 + M^2},$$

$$\mathcal{F}_{2c} = 0,$$

$$\mathcal{F}_{2d} = l_3 \frac{M^4}{F^2} \sum_{F} \frac{1}{P^2 + M^2} + 2l_3 \frac{M^4}{F^2} \sum_{P} \frac{1}{P^2 + M_B^2},$$

$$\mathcal{F}_{2e} = 2 \frac{(qB)^2}{F^2} \sum_{P} (2l_5 - l_6) \frac{1}{P^2 + M_B^2},$$

$$\mathcal{F}_{2f} = -8c_{34}(qB)^2 M^2.$$

---

It was shown in Refs. [29, 31] that the diagram $\mathcal{F}_{2c}$ vanishes at $T = 0$. It also vanishes at finite temperature.
Using the expressions for the sum-integrals in Appendix A, we obtain

\[
\mathcal{F}_{2a} = -\frac{1}{8(4\pi)^4} \frac{M^2}{F^2} \left( \frac{\Lambda^2}{M^2} \right)^2 \left[ \left( \frac{1}{\epsilon^2} + \frac{2}{\epsilon} + \frac{\pi^2}{6} + 3 \right) M^4 - 2 \left( \frac{1}{\epsilon} + 1 \right) J_1(\beta M) M^2 T^2 \\
+ J_1^B(\beta M) T^4 + \mathcal{O}(\epsilon) \right],
\]

\[
\mathcal{F}_{2b} = \frac{1}{2(4\pi)^4} \frac{M^2}{F^2} \left( \frac{\Lambda^2}{M^2} \right)^\epsilon \left( \frac{\Lambda^2}{2qB} \right)^\epsilon \left[ \left( \frac{1}{\epsilon^2} + \frac{\pi^2}{6} + 1 \right) M^4 - 2 \left( \frac{1}{\epsilon} + 1 \right) \zeta(1,0)(0, \frac{1}{2} + x)|qB| \right.
- \zeta(2,0)(0, \frac{1}{2} + x)|qB|M^2 - \frac{1}{\epsilon} (J_1(\beta M) T^2 + J_1^B(\beta M)|qB|) M^2 - J_1^B(\beta M)|qB|M^2 \\
+ J_1(\beta M) J_1^B(\beta M) T^2 |qB| + 2\zeta(1,0)(0, \frac{1}{2} + x) J_1(\beta M)|qB| T^2 + \mathcal{O}(\epsilon) \left] \right.\),
\]

\[
\mathcal{F}_{2c} = 0 ,
\]

\[
\mathcal{F}_{2d} = -\frac{l_3}{(4\pi)^4} \frac{M^4}{F^2} \left( \frac{\Lambda^2}{M^2} \right)^\epsilon \left[ \frac{1}{\epsilon} + 1 + \left( \frac{\pi^2}{12} + 1 \right) \epsilon \right] M^2 - J_1(\beta M) T^2 + \mathcal{O}(\epsilon^2) \right)
- 2 l_3 \frac{M^4}{(4\pi)^4} \frac{F^2}{2|qB|} \left[ \left( \frac{1}{\epsilon} + \frac{\pi^2}{12} \right) M^2 - 2\zeta(1,0)(0, \frac{1}{2} + x)|qB| \right.
- \zeta(2,0)(0, \frac{1}{2} + x)|qB| - J_1^B(\beta M)|qB| + \mathcal{O}(\epsilon^2) \left] \right. ,
\]

\[
\mathcal{F}_{2e} = \frac{2(qB)^2}{F^2} (2l_5 - l_6) \frac{1}{(4\pi)^2} \left( \frac{\Lambda^2}{2qB} \right)^\epsilon \left[ \frac{1}{\epsilon} M^2 - 2\zeta(1,0)(0, \frac{1}{2} + x)|qB| - J_1^B(\beta M)|qB| \right.
+ \mathcal{O}(\epsilon) \right] .
\]

After having subtracted the expressions for the diagrams for \( T = B = 0 \), we are still left with some simple poles in \( \epsilon \). These remaining divergences are now eliminated by renormalizing the couplings \( l_3 \), \( l_5 \), and \( l_6 \) according to the prescription (2.8). Since the renormalized couplings \( l_i^r \) are evaluated at the scale \( \Lambda^2 = M^2 \), the final result simplifies significantly. After a lengthy calculations, we obtain

\[
\mathcal{F}^B_2 = \frac{(qB)^2}{2(4\pi)^4} \frac{F^2}{F^2} \left( I_B \left[ \frac{M^4}{(qB)^2} \frac{1}{l_3} - \frac{2}{3} (\bar{l}_6 - \bar{l}_5) \right] - 2\bar{d} M^2 \right) ,
\]

\[
\mathcal{F}^T_2 = \frac{M^2}{8(4\pi)^4} \frac{F^2}{F^2} \left[ - J_1^2(\beta M) T^4 + 4 I_B J_1(\beta M) J_1^B(\beta M) T^2 |qB| - 2I_B (J_1(\beta M) T^2 + J_1^B(\beta M)|qB|) M^2 \\
- 4 I_B J_1(\beta M) T^2 \right] + \frac{(qB)^2}{3(4\pi)^4} \frac{F^2}{F^2} \left( \bar{l}_6 - \bar{l}_5 \right) J_1^B(\beta M)|qB| ,
\]

where

\[
\bar{d}(M^2) = 8(4\pi)^4 \frac{c_3}{2} + \frac{1}{3} (\bar{l}_6 - \bar{l}_5) \log \frac{\Lambda^2}{M^2} .
\]

Note in particular that all the divergent terms involving the thermal integrals \( J_1 \) and \( J_1^B \) cancel amongst themselves and so we can evaluate the remaining ones at \( \epsilon = 0 \).

If we express the contributions \( \mathcal{F}^B_1 \) and \( \mathcal{F}^T_1 \) in terms of the physical pion masses \( M_{\pi^0}(0) \) and \( M_{\pi^\pm}(0) \) using Eqs. (3.8) and (3.9) instead of the parameter \( M \), the dependence on \( \bar{l}_3 \), \( l_5 \), and \( l_6 \)
cancels in the sums $\mathcal{F}_{1+2}^B$ and $\mathcal{F}_{1+2}^T$. This yields

$$\mathcal{F}_{1+2}^B = \frac{M_{\pi^\pm}^4(0)}{4(4\pi)^2} \left[ 1 - 2 \log \frac{M_{\pi^\pm}^2(0)}{2|qB|} \right] + \frac{4(qB)^2}{(4\pi)^2} \zeta(1,0)(-1, \frac{1}{2} + x_\pm) + \frac{(qB)^2}{6(4\pi)^2} \log \frac{\Lambda^2}{2|qB|}$$

\[- \frac{(qB)^2}{(4\pi)^2 F^2} \delta(|qB|) M, \]  

(4.26)

$$\mathcal{F}_{1+2}^T = -\frac{1}{2(4\pi)^2} \left[ J_0(\beta M_{\pi^\pm}(0))T^4 + 2J_0^B(\beta M_{\pi^\pm}(0))|qB|^2 T^2 \right] + \frac{M^2}{8(4\pi)^4 F^2} \left[ -J_1^2(\beta M)T^4 + 4J_1(\beta M)J_1^B(\beta M)T^2|qB| - 4I_B J_1(\beta M)T^2 \right],$$

(4.27)

where $x_\pm = M_{\pi^\pm}^2(0)/2|qB|$. The result for the vacuum energy $\mathcal{F}_{1+2}^B$ is in agreement with the calculation of Ref. [31]. Since $M_{\pi^\pm}$ is nonzero in the chiral limit ($M = 0$) and it appears in the thermal integrals $J_0^B$ in Eq. (4.27), the NLO correction to the free energy does not vanish in contrast to the case of zero magnetic field.

We next discuss the quark condensate in the presence of the magnetic field $B$. At finite temperature, the quark condensate is [10]

$$\langle \bar{q}q \rangle = \langle 0|\bar{q}q|0 \rangle \left[ 1 - \frac{c}{F^2} \frac{\partial (\mathcal{F}_B + \mathcal{F}_T)}{\partial M_{\pi}^2} \right],$$

(4.28)

where $c$ is a constant defined by

$$c = -F^2 \frac{\partial M_{\pi}^2(0)}{\partial m} \langle 0|\bar{q}q|0 \rangle^{-1},$$

(4.29)

where $m$ is the quark mass. In the chiral limit, we have $c = 1$ [10]. In this case, the quark condensate reduces to

$$\langle \bar{q}q \rangle = \langle 0|\bar{q}q|0 \rangle \left\{ 1 + \frac{|qB|}{4\pi^2 F^2} I_B(M_{\pi^\pm}(0)) + \frac{(qB)^2}{(4\pi)^4 F^4} \bar{d}(qB) - \frac{1}{2(4\pi)^2 F^2} \left[ J_1(0)T^2 + 2J_1^B(\beta M_{\pi^\pm}(0))|qB| \right] \right\},$$

(4.30)

where we have used the recursion relations for $J_n$ and $J_n^B$. We notice that we in the limit $B \rightarrow 0$ recover the result by Gerber and Leutwyler [10]

$$\langle \bar{q}q \rangle = \langle 0|\bar{q}q|0 \rangle \left[ 1 - \frac{T^2}{8F^2} - \frac{T^4}{384 F^4} \right],$$

(4.31)

where we have used $J_1(0) = 4\pi^2/3$.}

\footnote{Note that the final result of Ref. [31] is expressed in terms of the pion mass $M_{\pi}$ in the vacuum and so the terms involving $\bar{c}$ and $\bar{c}$ are not absorbed in the order $p^4$ term.}
5 High-temperature expansion

In this section, we discuss the expansion of our results in the limit of weak fields, i.e. for $|qB|/T^2 \ll 1$. For simplicity, we restrict our analysis to the chiral limit, $M = 0$.

For values $|qB|/T^2 \leq 1$, the sum over Landau levels in $J_n^B(\beta M)$ converges very slowly. It is advantageous to use the Euler-McLaurin summation formula [57] to evaluate $J_n^B$ in this regime [50]. The sum over integers $m$ can be written as

$$
\sum_{m=k}^{l} f(m) = \int_{k}^{l} f(x) \, dx + \frac{1}{2} [f(l) + f(k)] + \sum_{i=1}^{n} \frac{b_{2i}}{(2i)!} [f(2i-1)(l) - f(2i-1)(k)]
$$

$$
+ \int_{k}^{l} B_{2n+1} \{ x \} f(2n+1)(x) \, dx ,
$$

where $B_n$ are the Bernoulli numbers and $\{ x \}$ means the fractional part of $x$. The last term in Eq. (5.1) is the remainder. We will illustrate the use of Eq. (5.1) in the case of $J_1^B$. Setting $n = 2$, $k = 0$, and $l = \infty$, we obtain

$$
J_1^B = 8 \int_{0}^{\infty} dz \left\{ \int_{0}^{\infty} \frac{dk}{E(z, M, T, B, k)} - \frac{1}{2} \frac{1}{E(z, M, T, B, 0)} \right\} + \frac{|qB|}{12T^2} \frac{1}{E(z, M, T, B, 0)} \left\{ \frac{1}{E(z, M, T, B, 0)} + \frac{1}{E(z, M, T, B, 0)} \right\} ,
$$

where we have defined $E(z, M, T, B, k) = \sqrt{z^2 + M^2/T^2 + (2k+1)|qB|/T^2}$. Integrating over $k$ in the first term of Eq. (5.2), we obtain

$$
J_1^B = -\frac{8}{a^2} \int_{0}^{\infty} dz \log \left[ 1 - e^{-\sqrt{z^2+a^2}} \right] + 4 \int_{0}^{\infty} \frac{dz}{\sqrt{z^2+a^2}} - \frac{1}{\sqrt{z^2+a^2}} - 1
$$

$$
+ \frac{2a^2}{3} \int_{0}^{\infty} \frac{dz}{z^2+a^2} \left[ \frac{1}{\sqrt{z^2+a^2}} + \frac{e^{\sqrt{z^2+a^2}}}{e^{\sqrt{z^2+a^2}}-1} \right] + \ldots ,
$$

where $a^2 = |qB|/T^2$. The Euler-McLaurin formula is next used to expand the integrals $J_n^B$ in a Taylor series around $B = 0$. This is particularly simple in the chiral limit where $J_n^B$ only depends on the ratio $|qB|/T^2$, and the series is in powers of $\sqrt{qB}/T$. Integrating the first and the last term in Eq. (5.3) by parts, we can write

$$
J_1^B = \frac{a^2}{a^2} J_1 + J_2 + \frac{a^2}{3} J_3 + \ldots ,
$$

where the argument of $J_n$ is $\beta a$. The expansion of the integrals $J_n(x)$ can be found in e.g. [60] and the leading terms are $J_1 = 4\pi^2/3 - 4\pi a$, $J_2 = 2\pi/a$, and $J_3 = \pi/a^3$. This yields

$$
J_1^B |qB| = \frac{4\pi^2 T^2}{3} \left[ 1 - \frac{5}{4\pi} \sqrt{qB}/T + O \left( \frac{qB}{T^2} \right) \right] .
$$

Inserting this expression into Eqs. (3.15), and (3.16), we obtain in the chiral limit

$$
F_{\nu^0} = F_{\pi} \left[ 1 + \frac{|qB|}{(4\pi)^2 F^2} \log 2 - \frac{T^2}{12 F^2} + 5\sqrt{qB}/48\pi F^2 + \ldots \right] ,
$$

$$
F_{\pi^\pm} = F_{\pi} \left[ 1 + \frac{|qB|}{2(4\pi)^2 F^2} \log 2 - \frac{T^2}{12 F^2} + \frac{5\sqrt{qB}}{96\pi F^2} + \ldots \right] .
$$
In the same manner, we can expand the temperature-dependent part of the quark condensate around $qB = 0$. A straightforward calculation yields

$$
\langle \bar{q}q \rangle = \langle 0 | \bar{q}q | 0 \rangle \left[ 1 - \frac{T^2}{8F^2} + \frac{5\sqrt{qBT}}{48\pi F^2} - \frac{T^4}{384F^4} + \frac{5\sqrt{qBT^3}}{1536\pi F^4} + \frac{T^2 |qB|}{384F^4} \log 2 + \ldots \right].
$$

(5.8)

6 Numerical results and discussion

We need to evaluate the integrals $J^B_0$ and $J^B_1$. By expanding the Bose-Einstein distribution function in $J^B_0$ and $J^B_1$, and integrating over three-momenta, we write it as a double sum involving a modified Bessel function $K_l(x)$ of order $l = 0$:

$$J^B_0(\beta M) = -8 \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \sqrt{\frac{M^2_\pi + (2m + 1)|qB|}{T}} K_1 \left( \frac{n\sqrt{M^2_\pi + (2m + 1)|qB|}}{T} \right),
$$

(6.1)

$$J^B_1(\beta M) = 8 \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} K_0 \left( \frac{n\sqrt{M^2_\pi + (2m + 1)|qB|}}{T} \right).
$$

(6.2)

For values $|qB|/T^2 \geq 1$, the sums converge fast. In the remainder of the paper, we use $|qB| = 5(140\text{MeV})^2$ and so we use the expressions (6.1)–(6.2) in the numerical work.

In Fig. 2, we show the pion masses $M^2_{\pi^0}$ and $M^2_{\pi^\pm}$ normalized to their vacuum valued $M^2_\pi$ as a function of $T$.

![Figure 2. Pion masses $M^2_{\pi^0}$ and $M^2_{\pi^\pm}$ normalized to their vacuum valued $M^2_\pi$ as a function of $T$.](image)

In Fig. 3, we show the pion decay constant $F^0_\pi$ given by Eq. (3.15) at $T = 0$ normalized by its vacuum value $F_\pi = 93$ MeV as a function of $|qB|/F^2_\pi$ in the chiral limit as well as at the physical point. At $T = 0$, the dependence on the magnetic field is governed by the function $I_B(M)$ defined
in Eq. (3.10). In the chiral limit, $F_{\pi^0}$ is a linear function of $|qB|$ since $I_B(0) = |qB| \log 2$. In both cases, we note that pion decays constant is an increasing function of $|qB|$. We do not show $F_B^\pm$ since the functional dependence at $T = 0$ is the same, cf. Eq. (3.16).

![Figure 3. Pion decay constant $F_{\pi^0}$ at $T = 0$ scaled by its vacuum value $F_{\pi}$ as a function of $|qB|/F_{\pi}^2$.](image)

In Fig. 4, we show the normalized pion decay constant $F_{\pi^0}$ in the chiral limit as a function of $T$ for $|qB| = 0$ (dashed line) and $|qB| = 5(140\text{MeV})^2$ (solid line). In vanishing magnetic field, the temperature dependence is given by

$$F_{\pi^0} = F_{\pi} \left( 1 - \frac{T^2}{12F_{\pi}^2} \right).$$

(6.3)

The pion decay constant is larger in a magnetic background. This is partly due to $I_B(M)$ which is an increasing function of $B$ and partly due to the fact that $J_B^2|qB| \leq J_B T^2$ for all temperatures.

In Fig. 5, we show the contribution $\mathcal{F}^B$ to free energy in the chiral limit at one and two loops normalized to $F_{\pi}^4$ as a function of $|qB|/F_{\pi}^2$. We have omitted the term $\frac{(qB)^2}{6(4\pi)^2} \log \frac{\Lambda^2}{2|qB|}$ as customary in the literature. The uncertainty in the coupling $\tilde{d}$ is large and even the sign is undetermined [58]. Varying $\tilde{d}$ would give rise to large bands for the free energy as well as the quark condensate [31]. Since the experimental value is consistent with zero, we will simply use $\bar{d} = 0$ in the remainder of the paper. The two-loop result (4.26) for the free energy then reduces to the one-loop result (4.8) with an effective mass for the charged pion given by $M_{\pi^\pm} = \frac{(qB)^2}{3(4\pi)^2F_{\pi}^2}(\tilde{l}_6 - \tilde{l}_5)$.

In Fig. 6, we show the contribution $\mathcal{F}^B$ to the free energy at the physical point at one - and two loops normalized to $F_{\pi}^4$ as a function of $|qB|/F_{\pi}^2$. The correction is tiny and the two-loop result starts to deviate from the one-loop result first at very large values of the magnetic field. Again the two-loop result is obtained from the one-loop formula (4.8) using the effective mass $M_{\pi^\pm} = M_{\pi} + \frac{(qB)^2}{3(4\pi)^2F_{\pi}^2}(\tilde{l}_6 - \tilde{l}_5)$ instead of $M_{\pi}$.

In Fig. 7, we show the thermal part $\mathcal{F}^T$ of the free energy as a function of $T$ for $qB = 5(140\text{MeV})^2$ in the chiral limit normalized to the free energy $\mathcal{F}_0 = -\pi^2 T^4/30$ of an ideal massless pion gas. In the chiral limit, the two-loop result (4.27) reduces to the one-loop (4.9) evaluated with the $M_{\pi^\pm} =$
Figure 4. Normalized pion decay constant $F_{\pi 0}$ for and $|qB| = 0$ and $|qB| = 5(140\text{MeV})^2$ as a function of $T$.

### Figure 5. The contribution $F_B^{\mathcal{F}}$ to the one and two-loop free energy scaled by $F_{\pi}^2$ in the chiral limit.

\[(qB)^2(\tilde{l}_6 - \tilde{l}_5)/3(4\pi)^2 F_{\pi}^2\] instead of $M_\pi = 0$. For low temperatures, only the neutral pion contributes to the free energy since the contributions from the charged pions are Boltzmann suppressed. From $T$ around 50 MeV onwards, the normalized free energy increases sharply and approaches the normalized Stefan-Boltzmann result 1 as $T \to \infty$. It has the value of approximately 0.97 for $T = 313$ MeV, i.e. for $T^2/|qB| = 1$. The two-loop result lies below the one-loop result since in this approximation the charged pions are not massless but has an effective mass due to the magnetic field, $M_{\pi \pm}^2 = (qB)^2(\tilde{l}_6 - \tilde{l}_5)/3(4\pi)^2 F_{\pi}^2$.

In Fig. 8, we show the LO and NLO results for the normalized quark condensate in the chiral limit including the $T = 0$ contribution. For comparison, we also show the LO and NLO results for the quark condensate for $B = 0$ given by Eq. (4.31). In Fig. 9, we show the LO and NLO results for the normalized quark condensate in the chiral limit excluding the zero-temperature contributions in
Figure 6. The contribution $\mathcal{F}^B$ to the one and two-loop free energy scaled by $F_4^4\pi$ at the physical point.

Figure 7. One and two-loop thermal part $\mathcal{F}^T$ of the free energy normalized to the free energy $F_0 = -\pi^2T^4/30$ of an ideal massless pion gas.

order to disentangle the effects of the magnetic field at $T = 0$ and the finite-temperature effects. For comparison, we again show the LO and NLO results for the quark condensate for $B = 0$. Figs. 8 and 9 show that the one-loop and two-loop results for the condensate are very close to each other in the entire temperature range. The LO and NLO results for the condensate are closer in a magnetic background than for $B = 0$ and this suggests that chiral perturbation theory converges at least as well for finite $B$.

Figs. 8 and 9 show that the quark condensate for nonzero $B$ goes to zero slower than for vanishing magnetic field and the reason is two-fold. Firstly, there is an enhanced quark condensate at $T = 0$, which to leading order in ChPT is given by the function $I_B(M)$. This is the familiar enhancement of the chiral condensate in a magnetic background. Secondly, there are finite-temperature effects. The
Figure 8. Temperature dependence of the quark condensate including the $T = 0$ contribution normalized to its vacuum value $|qB| = 5(140 \text{ MeV})^2$ at LO and NLO in chiral perturbation theory. For comparison, we show the LO and NLO results for $|qB| = 0$ as well.

Figure 9. Temperature dependence of the quark condensate excluding the $T = 0$ contribution normalized to its vacuum value $|qB| = 5(140 \text{ MeV})^2$ at LO and NLO in chiral perturbation theory. For comparison, we show the LO and NLO results for $|qB| = 0$ as well.

The basic mechanism is that $J_1^B$ is a decreasing function of the magnetic field $B$ and so $J_1T^2 \geq J_1^B|qB|$ for all nonzero $B$.

Comparing the results for the condensate at $B = 0$ and $|qB| = 5(140 \text{ MeV})^2$, we see that its effects are quantitatively large. This is due to a very strong magnetic field. For weaker magnetic fields, the difference between the two sets of curves will be smaller too. The results suggest that the critical temperature $T_c$ for the chiral transition is higher in a magnetic background. However, since ChPT is known to break down for large temperatures we do not make any quantitative statements about
As a function of the magnetic field $B$. In Ref. [34], the authors are using chiral perturbation theory to investigate the quark-hadron phase transition as a function of the magnetic field at the physical point. They compare the pressure of a hot pion gas with that of an ideal quark-gluon plasma with with a subtracted vacuum energy term due to a nonzero gluon condensate. For weak magnetic fields, the transition is first order. The line of first-order transitions ends at critical point $(\sqrt{|qB|}, T) = (600, 104)$ MeV. The critical temperature determined this way is a decreasing function of the magnetic field $B$.

D’Elia et al have recently carried out lattice simulations in a constant magnetic background [25, 26]. They explored various constituent quark masses corresponding to a pion mass of $200 - 480$ MeV and different magnetic fields, up to $|qB| \sim 20 M_\pi^2$ for the lightest quark masses. For these values of the pion mass, they found that there is a slight increase in the critical temperature $T_c$ for the chiral transition. These results have been confirmed by Bali et al [27, 62]. The same group has also carried out lattice simulations for physical values of the pion mass, i.e. $M_\pi = 140$ MeV. Their results which are extrapolated to the continuum limit show that the critical temperature is a decreasing function of the magnetic field [27, 62]. Hence the critical temperature for fixed $|qB|$ as a function of the quark mass is nontrivial. This is in stark contrast to most model calculations that imply an increasing critical temperature as a function of $B$. These include mean-field calculations as well as functional renormalization group calculations that incorporate quantum as well as thermal fluctuations [49, 52]. Since this discrepancy is not understood, more work is needed to resolve this problem.

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A Sum-integrals

In the imaginary-time formalism for thermal field theory, a boson has Euclidean 4-momentum $P = (P_0, \mathbf{p})$ with $P^2 = P_0^2 + \mathbf{p}^2$. The Euclidean energy $P_0$ has discrete values: $P_0 = 2n\pi T$, where $n$ is an integer. Loop diagrams involve sums over $P_0$ and integrals over $\mathbf{p}$. With dimensional regularization, the integral is generalized to $d = 3 - 2\epsilon$ spatial dimensions. We define the dimensionally regularized sum-integral by

$$\sum \int_{P_0, \mathbf{p}} = T \int_{P_0 = 2n\pi T} \int_{\mathbf{p}}, \quad (A.1)$$

where the integral is defined

$$\int_{\mathbf{p}} = \left(\frac{e^{\gamma E}}{4\pi}\right)^\epsilon \int \frac{d^d\mathbf{p}}{(2\pi)^d}, \quad (A.2)$$

and $\Lambda$ is an arbitrary momentum scale. The factor $(e^{\gamma E}/4\pi)^\epsilon$ is introduced so that, after minimal subtraction of the poles in $\epsilon$ due to ultraviolet divergences, $\Lambda$ coincides with the renormalization scale of the $\overline{\text{MS}}$ renormalization scheme.
In this appendix, we explicitly calculate the sum-integrals that we need.

**Explicit calculations**

replaced by a sum over Matsubara frequencies $P_0 = 2\pi n T$, a sum over Landau levels $m$, and an integral over momenta in $d - 2 = 1 - 2\epsilon$ dimensions. We then define

$$\sum_{p_0} \int_{p_0} B = \frac{|qB|}{2\pi} \sum_{m=0}^{\infty} T \sum_{P_0=2\pi n T} \int_{p_0},$$

where the integral is defined

$$\int_{p_0} = \left(\frac{\epsilon^{\gamma_E} A^2}{4\pi}\right) \int \frac{d^{d-2} p}{(2\pi)^{d-2}},$$

and where the prefactor $\frac{|qB|}{2\pi}$ takes into account the degeneracy of the Landau levels.

The specific sum-integrals that we need are

$$\sum_{p_0} \log \left[ P^2 + M^2 \right] = -\frac{1}{2(4\pi)^2} \left( \frac{\Lambda^2}{M^2} \right)^\epsilon \left[ \left( \frac{1}{\epsilon^2} + \frac{3}{2} + \frac{21 + \pi^2}{12} \right) M^4 + 2J_0 T^4 + O(\epsilon^2) \right],$$

$$\sum_{p_0} \log \left[ P_0^2 + p_z^2 + M_B^2 \right] = \frac{1}{2(4\pi)^2} \left( \frac{\Lambda^2}{2|qB|} \right)^\epsilon \left[ \left( \frac{(qB)^2}{3} - M^4 \right) \left( \frac{1}{\epsilon^2} + 1 \right) + 8\zeta^{(1,0)}(-1, \frac{1}{2} + x)(qB)^2 \right.$$

$$\left. - 2J_0 |qB| T^2 + O(\epsilon) \right],$$

$$\sum_{p_0} \frac{1}{P^2 + M^2} = -\frac{1}{(4\pi)^2} \left( \frac{\Lambda^2}{M^2} \right)^\epsilon \left\{ \left[ \frac{1}{\epsilon^2} + 1 + \left( \frac{\pi^2}{12} + 1 \right) \right] M^2 - J_1 T^2 + O(\epsilon^2) \right\},$$

$$\sum_{p_0} \frac{1}{P_0^2 + p_z^2 + M_B^2} = -\frac{1}{(4\pi)^2} \left( \frac{\Lambda^2}{2|qB|} \right)^\epsilon \left\{ \left[ \frac{1}{\epsilon^2} + \frac{\pi^2}{12} \right] M^2 - 2\zeta^{(1,0)}(0, \frac{1}{2} + x)|qB| \right.$$

$$\left. - \zeta^{(2,0)}(0, \frac{1}{2} + x)|qB| \epsilon - J_1^2 |qB| + O(\epsilon^2) \right\},$$

where $M_B^2 = M^2 + (2m + 1)|qB|$ and where we have defined the functions

$$J_n(\beta M) = \frac{4\epsilon^{\gamma_E} \Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3}{2} - n - \epsilon\right)} \beta^{1-2n} M^{2\epsilon} \int_0^{\infty} dp \frac{p^{4-2n-2\epsilon}}{\sqrt{p^2 + M^2}} \frac{1}{e^\beta \sqrt{p^2 + M^2} - 1},$$

$$J_n^B(\beta M) = \frac{8\epsilon^{\gamma_E} \Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3}{2} - n - \epsilon\right)} \beta^{2-2n} (2|qB|)^{2\epsilon} \sum_{m=0}^{\infty} \int_0^{\infty} dp \frac{p^{2-2n-2\epsilon}}{\sqrt{p^2 + M_B^2}} \frac{1}{e^\beta \sqrt{p^2 + M_B^2} - 1}.$$

The functions $J_n(x)$ and $J_n^B(x)$ satisfy the recursion relations

$$xJ_n'(x) = 2\epsilon J_n(x) - 2x^2 J_{n+1}(x),$$

$$xJ_n^B(x) = -2x^2 J_{n+1}^B(x).$$

**B Explicit calculations**

In this appendix, we explicitly calculate the sum-integrals that we need.
The first sum-integral we need is given by Eq. (A.5). Summing over Matsubara frequencies we can write

\[
\sum_{\vec{p}} \log [P^2 + M^2] = \int_{\vec{p}} \left\{ \sqrt{p_x^2 + M^2} + 2T \log \left[ 1 - e^{-\beta \sqrt{p_x^2 + M^2}} \right] \right\}.
\]  
(B.1)

The first term in Eq. (B.1) is ultraviolet divergent. Calculating it with dimensional regularization, we obtain

\[
\int_{\vec{p}} \sqrt{p_x^2 + M^2} = -\left( \frac{\epsilon \gamma \Lambda^2}{4\pi} \right) \Gamma \left( \frac{d+1}{2} \right) M^{d+1}.
\]  
(B.2)

The second term in Eq. (B.1) can be expressed in terms of \( J_0 \) defined in Eq. (A.9) using integration by parts. Expanding Eq. (B.2) in powers of \( \epsilon \) through order \( \epsilon \), we obtain Eq. (A.5). We next consider the sum-integral (A.6). Summing over the Matsubara frequencies, we obtain

\[
\sum_{\vec{p}} \log [P_0^2 + p_x^2 + M_B^2] = \frac{|qB|}{2\pi} \sum_{m=0}^{\infty} \int_{p_x} \left\{ \sqrt{p_x^2 + M_B^2} + 2T \log \left[ 1 - e^{-\beta \sqrt{p_x^2 + M_B^2}} \right] \right\}.
\]  
(B.3)

The first integral is ultraviolet divergent and we compute it in dimensional regularization using Eq. (B.2) with \( d = 1 - 2\epsilon \). This yields

\[
\int_{p_x} \sqrt{p_x^2 + M_B^2} = -\frac{1}{4\pi} \left( \frac{\epsilon \gamma \Lambda^2}{M_B^2} \right) \Gamma(-1 + \epsilon) M_B^2.
\]  
(B.4)

Eq. (B.4) shows that the sum over Landau levels \( m \) involves the term \( M_B^{2-2\epsilon} \). This sum is divergent for \( \epsilon = 0 \) and we regulate it using zeta-function regularization. After scaling out a factor of \((2|qB|)^{1-\epsilon}\), this sum can be written as

\[
\sum_{m=0}^{\infty} M_B^{2-2\epsilon} = (2|qB|)^{1-\epsilon} \sum_{m=0}^{\infty} \left[ m + \frac{1}{2} + \frac{M_B^2}{2|qB|} \right]^{1-\epsilon}
\]  
(B.5)

where \( x = \frac{M_B^2}{2|qB|} \) and \( \zeta(q, s) = \sum_{m=0}^{\infty} (q + m)^{-s} \) is the Hurwitz zeta-function. We then find

\[
\frac{|qB|}{2\pi} \sum_{m=0}^{\infty} \int_{p_x} \left\{ \sqrt{p_x^2 + M_B^2} \right\} = -\frac{4}{(4\pi)^2} \left( \frac{\epsilon \gamma \Lambda^2}{2|qB|} \right) \Gamma(-1 + \epsilon) \zeta(-1 + \epsilon, \frac{1}{2} + x) (qB)^2.
\]  
(B.6)

The second term in Eq. (B.3) can be expressed in terms of \( J_0^B \) defined in Eq. (A.10) using integration by parts. Expanding Eq. (B.6) in powers of \( \epsilon \), we obtain Eq. (A.6).

Taking the derivative of Eq. (A.5) with respect to \( M^2 \) and using the recursion relation (A.11) for \( J_n \), one obtains Eq. (A.7).

We next consider Eq. (A.8). It is given by the derivative of Eq. (A.6) with respect to \( M_B^2 \). The temperature-independent part of the sum-integral is given by the derivative of Eq. (B.6) with respect to \( M^2 \). This yields

\[
\frac{|qB|}{2\pi} \sum_{m=0}^{\infty} \int_{p_x} \frac{1}{2\sqrt{p_x^2 + M_B^2}} = \frac{2|qB|}{(4\pi)^2} \left( \frac{\epsilon \gamma \Lambda^2}{2|qB|} \right) \Gamma(\epsilon) \zeta(\epsilon, \frac{1}{2} + x).
\]  
(B.7)

Using the recursion relation (A.12), the temperature-dependent term of Eq. (A.8) is expressed in terms of \( J_1^B \).
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