Irreducibility of the Hilbert scheme of smooth curves in $\mathbb{P}^4$ of degree $g + 2$ and genus $g$

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Abstract. We denote by $\mathcal{H}_{d,g,r}$ the Hilbert scheme of smooth curves, which is the union of components whose general point corresponds to a smooth irreducible and non-degenerate curve of degree $d$ and genus $g$ in $\mathbb{P}^r$. In this note, we show that any non-empty $\mathcal{H}_{g+2,g,4}$ is irreducible without any restriction on the genus $g$. Our result augments the irreducibility result obtained earlier by Hristo Iliev (2006), in which several low genus $g \leq 10$ cases have been left untreated.

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1. An overview, preliminaries and basic set-up

Given non-negative integers $d$, $g$ and $r \geq 3$, let $\mathcal{H}_{d,g,r}$ be the Hilbert scheme of smooth curves parametrizing smooth irreducible and non-degenerate curves of degree $d$ and genus $g$ in $\mathbb{P}^r$.

After Severi asserted that $\mathcal{H}_{d,g,r}$ is irreducible for $d \geq g + r$ in [14, Anhang G, p. 368] with an incomplete proof, the irreducibility of $\mathcal{H}_{d,g,r}$ has been studied by several authors. Most noteworthy result regarding the irreducibility of $\mathcal{H}_{d,g,r}$ is due to L. Ein, who proved Severi’s claim for $& r = 4$ (& for $r = 3$ as well); cf. [7, Theorem 7] & [6, Theorem 4], see also [11, Theorem 1.5], [13, Proposition 2.1 & Proposition 3.2], [9, Theorem 3.1], [10, Theorem 2.1] for a different proof and several extensions concerning the irreducibility of $\mathcal{H}_{d,g,3}$ in the range $d \geq g$.

For smooth curves in $\mathbb{P}^4$, the irreducibility of $\mathcal{H}_{g+3,g,4}$ ($g \geq 5$) and $\mathcal{H}_{g+2,g,4}$ ($g \geq 11$) has been pushed forward by Hristo Iliev [9, Theorem 3.2] beyond the range $d \geq g + 4$ which has been known by a work of L. Ein [7, Theorem 7]. However, the irreducibility of $\mathcal{H}_{g+2,g,4}$ (and the non-emptyness as well) for lower genus $g \leq 10$ has been left unsettled. In this article, we...
show that $H_{g+2,g,4}$ is non-empty and irreducible for $7 \leq g \leq 10$ which in turn implies (together with [9, Theorem 3.2(b)]) that any non-empty $H_{g+2,g,4}$ is irreducible without any restriction on the genus $g$; cf. Theorem 2.1, Corollary 2.2 and Remark 2.5.

Before proceeding, we recall several related results which are rather well-known; cf. [1] and [2]. Let $M_g$ be the moduli space of smooth curves of genus $g$. For any given isomorphism class $[C] \in M_g$ corresponding to a smooth irreducible curve $C$, there exist a neighborhood $U \subset M_g$ of the class $[C]$ and a smooth connected variety $M$ which is a finite ramified covering $h : M \rightarrow U$, as well as varieties $C$, $W_d$ and $G_d$ proper over $M$ with the following properties:

1. $\xi : C \rightarrow M$ is a universal curve, i.e. for every $p \in M$, $\xi^{-1}(p)$ is a smooth curve of genus $g$ whose isomorphism class is $h(p)$.
2. $W_d$ parametrizes the pairs $(p, L)$ where $L$ is a line bundle of degree $d$ and $h^0(L) \geq r + 1$ on $\xi^{-1}(p)$,
3. $G_d$ parametrizes the couples $(p, D)$, where $D$ is possibly an incomplete linear series of degree $d$ and dimension $r$ on $\xi^{-1}(p)$ - which is usually denoted by $g^r_d$.

Let $\tilde{G}$ be the union of components of $G_d$ whose general element $(p, D)$ corresponds to a very ample linear series $D$ on the curve $C = \xi^{-1}(p)$. Note that an open subset of $H_{d,g,r}$ consisting of points corresponding to smooth irreducible and non-degenerate curves is a $\mathbb{PGL}(r+1)$-bundle over an open subset of $\tilde{G}$. Hence the irreducibility of $\tilde{G}$ guarantees the irreducibility of $H_{d,g,r}$. We also make a note of the following well-known facts regarding the scheme $G_d$; cf. [1], [4, Chapt. 21, §3, 5, 6, 11, 12] and [8, §2.a, p. 67].

**Proposition 1.1.** For non-negative integers $d$, $g$ and $r$, let $\rho(d,g,r) := g - (r+1)(g-d+r)$ be the Brill-Noether number.

1. The dimension of any component of $G_d$ is at least $3g - 3 + \rho(d,g,r)$ which is denoted by $\lambda(d,g,r)$.
2. $G_d$ is smooth and irreducible of dimension $\lambda(d,g,1)$ if $g > 1$, $d \geq 2$ and $d \leq g + 1$.

We will utilize the following upper bound of the dimension of a certain irreducible component of $W_d$, which was proved and used effectively in [9].

**Proposition 1.2 ([9, Proposition 2.1]).** Let $d$, $g$ and $r \geq 2$ be positive integers such that $d \leq g + r - 2$ and let $W$ be an irreducible component of $W_d$. For a general element $(p, L) \in W$, let $b$ be the degree of the base locus of the line bundle $L = |D|$ on $C = \xi^{-1}(p)$. Assume further that for a general $(p, L) \in W$ the curve $C = \xi^{-1}(p)$ is not hyperelliptic. If the moving part of $L = |D|$ is very ample and $r \geq 3$, then

$$\dim W \leq 3d + g + 1 - 5r - 2b.$$
For notations and conventions, we usually follow those in [3] and [4]; e.g., \( \pi(d, r) \) is the maximal possible arithmetic genus of an irreducible and non-degenerate curve of degree \( d \) in \( \mathbb{P}^r \); \( \pi(d, r) := \left( \frac{m}{2} \right)(r - 1) + m \epsilon \) where \( m = \left\lceil \frac{d - 1}{r - 1} \right\rceil \) and \( d - 1 = m(r - 1) + \epsilon \). Throughout we work over the field of complex numbers.

2. Irreducibility of \( \mathcal{H}_{g+2, g, 4} \)

We first recall that the irreducibility of \( \mathcal{H}_{g+3, g, 4} \) for \( g \geq 5 \) has been shown by Hristo Iliev; [9, Theorem 3.2(a)]. We also remark that this result of Hristo Iliev has been stated with the fullest possible generality; note that \( \pi(g + 3, 4) < g \) for \( g \leq 4 \) and hence \( \mathcal{H}_{g+3, g, 4} = \emptyset \) if \( g \leq 4 \).

In the same vein, one may easily see that \( \mathcal{H}_{g+2, g, 4} = \emptyset \) for \( g \leq 6 \); by the Castelnuovo genus bound, one checks that there is no smooth and non-degenerate curve in \( \mathbb{P}^4 \) of degree \( g + 2 \) and genus \( g \) if \( g \leq 6 \). Therefore in conjunction with the theorem of Hristo Iliev [9, Theorem 3.2(b)], i.e. \( \mathcal{H}_{g+2, g, 4} \) being irreducible for \( g \geq 11 \), we shall assume \( 7 \leq g \leq 10 \) for the rest of this section. The main result of this article is the following theorem.

**Theorem 2.1.** \( \mathcal{H}_{g+2, g, 4} \) is irreducible for any \( g \) with \( 7 \leq g \leq 10 \).

Consequently Theorem 2.1 together with a result of Hristo Iliev [9, Theorem 3.2(b)] readily imply the following statement.

**Corollary 2.2.** Any non-empty \( \mathcal{H}_{g+2, g, 4} \) is irreducible.

**Remark 2.3.** It is worthwhile to note that the genus range \( g \geq 11 \) in [9, Theorem 3.2(b)] is exactly the range where the Brill-Noether number \( \rho(g + 2, g, 4) \) is strictly positive so that there exists a unique component of the Hilbert scheme \( \mathcal{H}_{g+2, g, 4} \) dominating the moduli space \( \mathcal{M}_g \). For curves of genus \( g \) in the range \( 7 \leq g \leq 10 \) – in which case \( \rho(g + 2, g, 4) \leq 0 \) – we will see in Remark 2.5 that \( \mathcal{H}_{g+2, g, 4} \neq \emptyset \) and the unique component of \( \mathcal{H}_{g+2, g, 4} \) is indeed the component which dominates the irreducible locus \( \mathcal{M}^1_{g,g-4} \) in \( \mathcal{M}_g \) consisting of \((g - 4)\)-gonal curves.

The following lemma, which is an intermediate step toward the proof of the irreducibility of \( \mathcal{H}_{g+2, g, 4} \), asserts that a general element in any component of \( \mathcal{H}_{g+2, g, 4} \) corresponds to a smooth curve in \( \mathbb{P}^4 \) which is linearly normal.

**Lemma 2.4.** Let \( G \subset G^4_{g+2} \) be an irreducible component whose general element \((p, D)\) is a very ample linear series \( D \) on the curve \( C = \xi^{-1}(p) \) and assume \( 7 \leq g \leq 10 \). Then

1. \( D \) is complete and \( \dim G = 4g - 13 \).
2. a general element of the component \( \mathcal{W}^V \subset \mathcal{W}^1_{g-4} \) consisting of the residual series (with respect to the canonical series on the corresponding curve) of those elements in \( G \) is a complete pencil.
Proof. By Proposition 1.1, we have
\[ \lambda(g + 2, g, 4) = 3g - 3 + \rho(g + 2, g, 4) = 4g - 13 \leq \dim \mathcal{G}. \]
We set \( r := h^0(C, |D|) - 1 \) for a general \((p, D) \in \mathcal{G}\). Let \( \mathcal{W} \subset \mathcal{W}_{g+2}^r \) be the component containing the image of the natural rational map \( \mathcal{G} \rightarrow \mathcal{W}_{g+2}^r \) with \( \iota(D) = |D| \). Since \( \dim \mathcal{G} \leq \dim \mathcal{W} + \dim \mathbb{G}(4, r) \), it follows by Proposition 1.2 that
\[ \lambda(g + 2, g, 4) = 4g - 13 \leq \dim \mathcal{G} \leq (4g + 7 - 5r) + 5(r - 4) = 4g - 13, \]
hence
\[ \dim \mathcal{G} = 4g - 13 \quad \text{and} \quad \dim \mathcal{W} = 4g + 7 - 5r. \quad (2.1) \]

Let \( \mathcal{W}^c \subset \mathcal{W}_{g-4}^{r-3} \) be the locus consisting of the residual series (with respect to the canonical series on the corresponding curve) of those elements in \( \mathcal{W} \), i.e. \( \mathcal{W}^c = \{(p, \omega_C \otimes L^{-1}) : (p, L) \in \mathcal{W}\} \).

Assume that \( r \geq 5 \) and we will argue by contradiction for each \( g \) with \( 7 \leq g \leq 10 \).

(a) \( g = 7 \): We have \( g_5^2 \) on \( C = \xi^{-1}(p) \) corresponding to an element in \( \mathcal{W}^c \subset \mathcal{W}_{g-4}^{r-3} \), contradicting Clifford’s theorem.

(b) \( g = 8 \): We have \( g_5^2 \) on \( C = \xi^{-1}(p) \) corresponding to an element in \( \mathcal{W}^c \subset \mathcal{W}_{g-3}^{r-4} \), hence \( C \) is hyperelliptic by Clifford’s theorem. However a hyperelliptic curve of genus \( g \) cannot have a very ample \( g_{g+2}^4 \).

(c) \( g = 9 \): We have \( g_5^2 \) on \( C = \xi^{-1}(p) \) and since \( C \) is not a plane quintic, our \( g_5^2 \) has a base point and hence \( C \) is a hyperelliptic curve, which cannot have a very ample \( g_{g+2}^4 \).

(d) \( g = 10 \): In this case, we have either a \( g_5^2 \) (when \( r = 5 \)) or a \( g_5^3 \) (when \( r = 6 \)) on \( C = \xi^{-1}(p) \) corresponding to an element in \( \mathcal{W}^c \subset \mathcal{W}_{g-4}^{r-3} \).

If there were a \( g_5^3 \), \( C \) is an hyperelliptic curve on which there does not exist a very ample \( g_{g+2}^4 \). Therefore \( r = 5 \). Note that our \( g_5^2 \) is base-point-free, for otherwise the same reasoning as in (c) applies. Therefore it follows that \( C \) is either trigonal with a unique trigonal pencil \( g_3^1 \) so that \( 2g_3^1 = g_5^2 \) or a smooth plane sextic with \( |K_C - g_6^3| = |2g_6^2| = g_5^{12} \), where \( g_6^2 \) is the unique very ample net of degree 6. Suppose that \( C \) is a smooth plane sextic. We recall that two smooth plane curves of the same degree \( d \geq 4 \) are isomorphic if and only if they are projectively equivalent. Hence the family of smooth plane curves of degree \( d = 6 \) moves in \( \dim \mathbb{P}H^0(\mathbb{P}^2, \mathcal{O}(d)) = \dim \mathbb{P}\text{GL}(3) = \frac{(d+3)(d+2)}{2} - \dim \mathbb{P}\text{GL}(3) = 27 - 8 = 19 \) dimensional locus \( \mathcal{M} \) in \( \mathcal{M}_g \). In the sequence of natural rational maps \( \mathcal{G} \rightarrow \mathcal{W} \rightarrow \mathcal{W}^c \rightarrow \mathcal{M} \subset \mathcal{M}_g \), we note that the rational map \( \zeta \) which takes a complete linear series to its residual series is clearly birational and the projection map \( \eta \) is also birational by the uniqueness of \( g_5^2 \). Therefore we have \( \dim \mathcal{W} = \dim \mathcal{W}^c = \dim \mathcal{M} = 19 \), contradicting (2.1). If \( C \) is trigonal with \( g_5^2 = 2g_3^1 \), we also have a sequence of rational maps \( \mathcal{G} \rightarrow \mathcal{W} \rightarrow \mathcal{W}^c \rightarrow \mathcal{M} \subset \mathcal{M}_g \), where
\( \mathcal{M}_{g,3}^4 \) is the irreducible locus of trigonal curves of genus \( g \). Again the projection map \( \eta \) is birational since there exists a unique trigonal pencil on any trigonal curve of genus \( g \) when \( g \geq 5 \). Hence \( \dim \mathcal{W} = \dim \mathcal{W}^\vee = \dim \mathcal{M}_{g,3}^4 = 2g + 1 = 21 \), again contradicting (2.1).

Therefore it finally follows that \( r = 4 \) and by (2.1), we have

\[
\dim \mathcal{G} = \dim \mathcal{W} = \dim \mathcal{W}^\vee = 4g - 13. \tag{2.2}
\]

The second statement (2) is obvious from (1).

The irreducibility of \( \mathcal{H}_{g+2,g,4} \) follows easily as an immediate consequence of Lemma 2.4 together with Proposition 1.1(2).

**Proof of Theorem 2.1.** Retaining the same notations as before, let \( \tilde{\mathcal{G}} \) be the union of irreducible components \( \mathcal{G} \) of \( \mathcal{G}_{g+2}^4 \) whose general element corresponds to a pair \( (p, D) \) such that \( D \) is very ample linear series on \( C := \xi^{-1}(p) \). Let \( \tilde{\mathcal{W}}^\vee \) be the union of the components \( \mathcal{W}^\vee \) of \( \mathcal{W}_{g-4}^1 \), where \( \mathcal{W}^\vee \) consists of the residual series of elements in a component \( \mathcal{G} \) of \( \tilde{\mathcal{G}} \). By Lemma 2.4 (or (2.2)),

\[
\dim \mathcal{W}^\vee = \dim \mathcal{G} = 4g - 13 = \lambda(g - 4, g, 1) = \dim \mathcal{G}_{g-4}^1. \tag{2.3}
\]

Since a general element of any component \( \mathcal{W}^\vee \subset \tilde{\mathcal{W}}^\vee \subset \mathcal{W}_{g-4}^1 \) is a complete pencil by Lemma 2.4, there is a natural rational map \( \tilde{\mathcal{W}}^\vee \to \mathcal{G}_{g-4}^1 \) with \( \kappa(|D|) = D \) which is clearly injective on an open subset \( \tilde{\mathcal{W}}^\vee \kappa \) of \( \tilde{\mathcal{W}}^\vee \) consisting of those which are complete pencils. Therefore the rational map \( \kappa \) is dominant by (2.3). We also note that there is another natural rational map \( \mathcal{G}_{g-4}^1 \to \tilde{\mathcal{W}}^\vee \) with \( \iota(D) = |D| \), which is an inverse to \( \kappa \) (wherever it is defined). Therefore it follows that \( \tilde{\mathcal{W}}^\vee \) is birationally equivalent to the irreducible locus \( \mathcal{G}_{g-4}^1 \), hence \( \tilde{\mathcal{W}}^\vee \) is irreducible and so is \( \tilde{\mathcal{G}} \). Since \( \mathcal{H}_{g+2,g,4} \) is a \( \mathbb{P}\text{GL}(5) \)-bundle over an open subset of \( \tilde{\mathcal{G}} \), \( \mathcal{H}_{g+2,g,4} \) is irreducible. \( \square \)

**Remark 2.5.** We finally remark that \( \mathcal{H}_{g+2,g,4} \neq \emptyset \) for \( 7 \leq g \leq 10 \). As was suggested by the proof of Theorem 2.1 one may argue that \( \mathcal{H}_{g+2,g,4} \) dominates (and is a \( \mathbb{P}\text{GL}(5) \)-bundle over) the irreducible locus \( \mathcal{M}_{g,g-4}^4 \) consisting of \( (g-4) \)-gonal curves as follows. Recall that the Clifford index \( e \) of a general \( (e+2) \)-gonal curve of genus \( g \geq 2e+2 \) can only be computed by the unique pencil computing the gonality as long as \( e \neq 0 \), i.e. there does not exist a \( g_{2r+e} \) with \( 2r+e \leq g-1 \), \( r \geq 2 \); cf. [5, Theorem] or [12, Corollary 1]. Therefore on a general \( (g-4) \)-gonal curve \( C \), the residual series of the unique \( g_{g-4}^1 \) is a very ample \( g_{g+2}^4 \); for otherwise there exists a \( g_{g-2}^2 = g_{g-4}^1 \otimes \mathcal{O}_C(p+q) \) for some \( p, q \in C \), computing the Clifford index of a general \( (g-4) \)-gonal curve contradicting the result just mentioned.
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