Fake gauge invariant theories of gravity with vectorial nonmetricity

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We discuss on cornerstone aspects of gauge symmetry within (classical) gravitational theories over Weyl space, where the gravitational effects are due not only to spacetime curvature, but also to a Weyl gauge vector or vectorial nonmetricity. It is demonstrated that there are well-known theories of gravity which are supposed to be gauge invariant but which are in fact not gauge invariant theories. The lack of gauge invariance is manifest in the equations of motion (EOM) of these theories, since some of the EOMs are not invariant under the Weyl gauge transformations. This is regardless of the manifest gauge invariance of the actions from which the EOMs are derived. We show that, due to required gauge fixing in order to make predictions within the framework of gauge invariant theories of gravity, the resulting physical picture admits a “many-worlds” interpretation which is discussed. In addition to the well-known result that vectorial nonmetricity does not interact with massless fields of the standard model of particles (SMP), in this paper it is also shown that vectorial nonmetricity does not interact with gravitation either. We conclude that, in their simpler form, gauge invariant theories of gravity over Weyl space are not phenomenologically viable.

I. INTRODUCTION

Weyl geometry [1], the theoretical framework where gauge symmetry was introduced for the first time, played an important role in the early search for alternatives of unification of the fundamental interactions [2–13]. It represented an interesting generalization of Riemann geometry where vectorial nonmetricity – see Eq. (8) below – was assumed. Nonetheless, discussions on the occurrence of the so-called “second clock effect” (SCE) [14–25], ruled out as phenomenologically nonviable.

Recently generalized nonmetricity theories, where the covariant derivative of the metric does not vanish [24],

\[ \nabla_\alpha g_{\mu\nu} = -Q_{\alpha\mu\nu}, \]  

(1)

with \( Q_{\alpha\mu\nu} \) – the nonmetricity tensor, have played an interesting role in the search for alternative explanations to fundamental questions of current interest. The recent resurrection of nonmetricity theories is mainly focused in the so-called teleparallel [27–35] and, specially, the symmetric teleparallel theories [36–45] and their cosmological applications [46–52]. However, in the bulk of these papers, gauge invariance is ignored. Only in Ref. [48], where the role of conformal symmetry within symmetric teleparallel framework is discussed, and in Ref. [28], nonmetricity is investigated from the point of view of gauge symmetry. Gauge symmetry and its break down within the framework of Weyl geometry has been recently investigated in [53–56] in connection with model building beyond the standard model of particles (SMP) and inflation.

Gauge symmetry is one of the most important properties of nonmetricity geometry due to covariance of [1] and of the autoparallel and geodesic equations under the following Weyl gauge transformations [26, 48]:

\[ g_{\mu\nu} \rightarrow \Omega^2 g_{\mu\nu}, \quad g^{\mu\nu} \rightarrow \Omega^{-2} g^{\mu\nu}, \]

\[ Q^\alpha_{\mu\nu} \rightarrow Q^\alpha_{\mu\nu} - 2\partial^\alpha \ln \Omega g_{\mu\nu}, \]

(2)

where the positive smooth function \( \Omega = \Omega(x) \) is the conformal factor and the conformal transformation of the metric does not represent a diffeomorphism or, properly, a conformal isometry, i.e., the spacetime coincidences (as well as the spacetime coordinates that label the events,) are not modified by the conformal transformations.

Gauge freedom, an immediate consequence of gauge invariance, represents a challenge from the point of view of its physical and geometrical implications within the gravitational context. In order to understand the reach of the challenge let us make a comparison with \( U(1) \) gauge freedom in electrodynamics. In this case, from the operational point of view, gauge fixing entails a mathematical constraint on the electromagnetic (EM) potential \( A_\mu \) (on its derivatives), allowing elimination of one redundant degree of freedom and simplification of subsequent computations. The physical interpretation of gauge symmetry when the EM field is coupled to matter (for instance to fermions) is nicely discussed in section 6 of [57]. We have an infinite set of possible descriptions \( (A_\mu + \partial_\mu \lambda(x), e^{-i\sigma_\lambda(x)}\bar{\psi}, \psi) \), where \( \psi \) is the fermion’s spinor and \( \lambda(x) \) can be any function. Any two states, picked out by two different choices \( \lambda_1(x) \) and \( \lambda_2(x) \), are to be identified. This is due to the fact that the probability density \( \propto \psi^* \psi \), which carries the relevant information about the quantum state of the fermion, is not affected by phase shifts \( \sim \lambda(x) \). Meanwhile Maxwell’s equations are not affected by vector potential replacement \( A_\mu \rightarrow A_\mu + \partial_\mu \lambda(x) \).

In the case of a gauge invariant theory of gravity, gauge invariance means that the gravitational laws are not affected by conformal transformations of the metric \( g_{\mu\nu} \rightarrow \Omega^2 g_{\mu\nu} \), together with appropriate transformations of the remaining fields.\(^1\) In this paper we shall call these

\(^1\) In the case of spaces with vectorial nonmetricity \( Q_\mu \) these include gauge transformation of nonmetricity \( Q_\mu \rightarrow Q_\mu - 2\partial_\mu \ln \Omega \) (see...
transformations simply as gauge transformations. Contrary to $U(1)$ gauge symmetry in electrodynamics, there is nothing similar to probability density in the present formalism where the gravitational laws are invariant under the gauge transformations. Instead, the metric tensor carries the required information to make measurements of distances. Conformal transformations of the metric link two different metrics, i.e., two different ways of measuring distances in spacetime. Each one of the conformally related metrics leads to different curvature properties encoded in the curvature tensors (Riemann-Christoffel curvature tensor and its contractions.) Hence, a gauge invariant theory of gravity is not a single theory but a conformal equivalence class of them. The additional complication in a gauge invariant theory of gravity is that it describes the spacetime structure over which the remaining fields – including the EM gauge fields – live. In this context gauge fixing is not a trivial question since it amounts to choosing a specific gravitational theory.

The usual EM-inspired interpretation of gauge symmetry within the framework of the gauge invariant gravitational theories that a specific gauge choice brings no physical consequences since the different gauges describe the same physical state, is not appropriate in this case. We need a different perspective on gauge invariance and on what gauge fixing means for gauge invariant theories of gravity. Here we develop an alternative understanding of gauge symmetry in gravitational theories and we discuss about the physical implications of gauge fixing in our approach. Each gauge choice picks out one possible theory of gravity in the conformal equivalence class, with its physical consequences. But not every gauge choice, although representing a potential description of our universe, gives a phenomenologically viable description.2 There should be a way in which we could determine the gauge where we (and the rest of the matter fields in the universe) live in: this would be the one which better describes the existing amount of observational and experimental evidence.

Another aspect of the challenge deals with the compatibility of Weyl gauge symmetry with the SMP. It is a well-known fact that photons and massless fields (radiation) in general do not interact with nonmetricity3 or, in other words; photons and radiation are blind to non-metricity so that these “see” only the pseudo-Riemann structure of spacetime. Hence, before electroweak (EW) symmetry breaking the massless fields of the SMP react only to the curvature of pseudo-Riemann space so that nonmetricity may be ignored. After $SU(2) \times U(1)$ symmetry breaking the SMP particles acquire masses but, then, as widely accepted the existence of masses breaks the gauge symmetry associated with nonmetricity. It seems that there is no room for nonmetricity in the SMP.

Can anyway EW symmetry breaking and Weyl gauge symmetry coexist together? $SU(2) \times U(1)$ symmetry breaking may be associated with the Higgs Lagrangian,

$$\mathcal{L}_H = -\frac{1}{2} |D_\sigma H|^2 - \frac{\lambda}{2} (|H|^2 - v_0^2)^2,$$

(3)

where $v_0$ is the EW mass parameter, $H$ is the Higgs doublet, and we use the following notation: $|H|^2 \equiv H^\dagger H$, $|D_\sigma H|^2 = g^{\mu\nu} (D_\mu H)^\dagger (D_\nu H)$,

$$D_\mu^a H = \left( \partial_\mu + \frac{i}{2} g W^k_\mu \sigma^k + \frac{i}{2} g' B_\mu \right) H,$$

(4)

with $W^k_\mu$ the $SU(2)$ bosons, $B_\mu$ the $U(1)$ boson, $(g, g')$ – gauge couplings and $\sigma^k$ are the Pauli matrices. Under the conformal transformation in Eq. (2), $(H, H^\dagger) \rightarrow \Omega^{-1}(H, H^\dagger)$, so that the Higgs action,

$$S_H = \int d^4 x \sqrt{-g} \mathcal{L}_H,$$

(5)

is not invariant under the (Weyl) gauge transformations (2). Hence, if we expect gauge symmetry to survive EW symmetry breaking, the Lagrangian (3) has to be modified. The required modification amounts to lifting the mass parameter $v_0$ to a point dependent field $v(x)$ such that under (2), $v^2 \rightarrow \Omega^{-2} v^2$. Besides, the EW gauge covariant derivative in Eq. (4) is to be replaced as well: $D_\mu H \rightarrow D_\mu H - Q_\mu/2$, so that, under the gauge transformations (2), $D_\mu H \rightarrow \Omega^{-1} D_\mu H \Rightarrow |D_\sigma H|^2 \rightarrow \Omega^{-4} |D_\sigma H|^2$. Lifting of the mass parameter to a point dependent field $v(x)$ leads to the masses acquired by the particles of the SMP after EW symmetry breaking, being point dependent quantities as well:3 $m = m(x)$. Under (4) the mass $m$ of a given particle transforms like $m \rightarrow \Omega^{-1} m$. Hence, Weyl gauge symmetry may survive after $SU(2) \times U(1)$ symmetry breaking and thus it may play a role in the past, present and future of the cosmic evolution of our universe. This approach, where gauge symmetry survives EW symmetry breaking, differs from other approaches undertaken in the bulk of papers on gauge theories of gravity, where the Weyl gauge symmetry breaks down either through the Higgs procedure or through other alternative mechanisms.

The main goal of the present paper is to discuss about cornerstone aspects of gauge symmetry including the geometrical and phenomenological consequences of this symmetry within gravitational theories. We shall focus in

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2 Although each gravitational theory in the conformal equivalence class has its own set of experimentally measured quantities, there is a set of physical quantities which is shared by all members in the class: the gauge invariants.

3 Point dependent masses which transform under the conformal transformation of the metric as $m \rightarrow \Omega^{-1} m$, are considered by Dicke in his paper [41] and in subsequent papers on conformal transformations in scalar-tensor theories (STT) [61–63].
spaces with vectorial nonmetricity $Q_{\alpha\mu\nu} = Q_{\alpha} g_{\mu\nu}$, since arbitrary nonmetricity brings with it several issues of fundamental character such as ambiguity in the definition of the gauge covariant derivative operators \[23\] and in the determination of the actual role geodesics and autoparallels play within the geometrical structure of generalized Weyl spaces \[43, 44\] (in spaces with arbitrary nonmetricity the autoparallels and the geodesics do not coincide.) It will be demonstrated, in particular, that gauge invariance of given gravitational theory requires that both the action/Lagrangian and the derived EOM must be invariant under the gauge transformations. As it will be shown here, there are situations when the gravitational action/Lagrangian is gauge invariant but the derived equations of motion are not invariant under gauge transformations. In this case we call the resulting theories as “fake gauge invariant” gravitational theories.\(^4\)

Another important aspect of gauge invariant theories of gravity with vectorial nonmetricity is related with the coupling of matter fields and even of gravity itself to the nonmetricity. It is a well-known result that vectorial nonmetricity does not couple to massless fields of the SMP \[8, 9, 13, 23\]. What is surprising is that it does not couple to gravity either. The implication of this result will be discussed here as well. We shall show that gauge invariant theories of gravity which are based in spacetimes with vectorial nonmetricity are phenomenologically ruled out. Vectorial (and arbitrary) nonmetricity can play a role in the quantum gravitational laws but not in the classical laws of gravity.

In this paper we shall discuss about another not well understood aspect of gauge invariance within the framework of gravitational theories: the geometrical and physical implications of gauge freedom. Here we shall trace a parallel between the well-known “many-worlds” interpretation of quantum physics and the physical picture that emerges from gauge symmetry as the result of gauge fixing (gauge choice).

This paper is organized in the following way. Sections \[\text{IV} \text{ and } \text{V}\] are dedicated mainly to expose the required mathematical formalism, while in Secs. \[\text{VI} \text{ through } \text{IX}\] we discuss on the geometrical and physical aspects of a gauge invariant theories of gravity with vectorial nonmetricity. In Sect. \[\text{VI}\] we demonstrate the Lemma \[\text{I}\] which states, basically, that theories which are based in actions which are linear in the curvature scalar of Weyl space, where a kinetic energy density term of the scalar field vanishes, are not gauge invariant theories of gravity. This lemma teaches us that we have to be careful about claiming gauge invariance of given theory only because the gravitational action/Lagrangian is gauge invariant. It should be checked as well invariance of the equations of motion since invariance of the Lagrangian does not automatically warrant invariance of the derived EOM. A generalization of the above lemma to include scalar multiplets, stated as Lemma \[\text{II}\] is demonstrated as well.

The corollary \[\text{III}\] stating that vectorial nonmetricity does not interact with gravitation, is really a surprising result not previously found. This is a corollary of Lemma \[\text{II}\] which is demonstrated in Sect. \[\text{IV}\] while in Sect. \[\text{V}\] it is shown that this result is not modified by the inclusion of quadratic curvature terms. In Sect. \[\text{VII}\] we demonstrate that only gravitational theories based in spaces with gradient nonmetricity, also known as Weyl integrable geometry (WIG) spaces, can have an impact in the classical laws of gravity. A new approach to the physical and geometrical interpretation of gauge freedom within the framework of gauge invariant theories of gravity, which is based on the “many-worlds” interpretation of quantum physics, is exposed in Sect. \[\text{VIII}\]. Our results are discussed in Sect. \[\text{IX}\] through the check of other gauge invariant gravitational theories already existing in the bibliography. Concluding remarks can be found in section \[\text{X}\].

In order for this paper to be self contained an appendix section has been included, where, among other subjects, the fundamentals of the gauge invariant theory of parallel transport and the consequent derivation of the autoparallel equations (also of the geodesics) in Weyl space, are exposed. In particular, in appendix \[\text{C}\] included for completeness of our exposition, it is discussed the issue with the coupling of matter fields to gauge invariant gravity. It is demonstrated that only massless fields (radiation) can be consistently coupled to gauge invariant gravity (this is a well-known result in the bibliography.) The latter result, together with corollary of Lemma 1, lead to conclude that vectorial nonmetricity and, consequently, Weyl geometry, play no role in the description of gauge invariant laws of gravity.

Unless otherwise stated, here we use natural units where $\hbar = c = 1$ and the following signature of the metric is chosen: \([- + + +\]). Greek indices run over spacetime $\alpha, \beta, ..., \mu, ... = 0, 1, 2, 3$, while latin indices $i, j, k... = 1, 2, 3$ run over three-dimensional space. Some times the spatial components of a vector $v^i$ will be represented as three-dimensional vectors $\bar{v}$. Bold-type letters $\mathbf{v}$ will represent four-dimensional (spacetime) vectors instead. Hence, for instance, $\mathbf{v} = \{v^0, \bar{v}\}$. Also useful is the following notation. For arbitrary quantities $V_\mu, U_\mu$ and $R_{\mu\lambda\nu\sigma}$ with tensorial indexes, the symmetrization and anti-symmetrization of two given indexes are defined as,

\begin{align}
V_\mu U_\nu := \frac{1}{2} (V_\mu U_\nu + V_\nu U_\mu), \\
R_{\mu(\lambda\nu)(\rho\sigma)} := \frac{1}{2} (R_{\mu\lambda\nu\sigma} + R_{\nu\lambda\mu\sigma}),
\end{align}

and

\[\text{4}\] Our designation “fake gauge invariance” is not to be confused with a similar designation in Ref. \[70\], where it is used to mean that the Noether current associated with the symmetry identically vanishes.
\[ V_{[\mu U_\nu]} := \frac{1}{2} (V_\mu U_\nu - V_\nu U_\mu), \]
\[ R_{\mu\nu[\lambda\sigma]} := \frac{1}{2} (R_{\mu\nu\lambda\sigma} - R_{\mu\nu\sigma\lambda}), \]  
(7)

respectively. Standard Riemann space, which is characterized by vanishing nonmetricity: \( \nabla_\alpha g_{\mu\nu} = 0 \), is denoted by \( V_4 \).

II. BACKGROUND AND CONVENTIONS

Weyl geometry space, denoted here by \( W_4 \), is defined as the class of four-dimensional (torsionless) manifolds \( \mathcal{M}_4 \) that are paracompact, Hausdorff, connected \( C^\infty \), endowed with a locally Lorentzian metric \( g \) that obeys the vectorial nonmetricity condition:\(^5\)

\[ \nabla_\alpha g_{\mu\nu} = -Q_\alpha g_{\mu\nu}, \]  
(8)

where \( Q_\alpha \) is the Weyl gauge vector and the covariant derivative \( \nabla_\mu \) is defined with respect to the torsion-free affine connection of the manifold:

\[ \Gamma^{\alpha}_{\mu\nu} = \{^{\alpha}_{\mu\nu} \} + L^{\alpha}_{\mu\nu}, \]  
(9)

where

\[ \{^{\alpha}_{\mu\nu} \} := \frac{1}{2} g^{\alpha\lambda} (\partial_\nu g_{\mu\lambda} + \partial_\mu g_{\nu\lambda} - \partial_\lambda g_{\mu\nu}), \]  
(10)

is the Levi-Civita (LC) connection, while

\[ L^{\alpha}_{\mu\nu} := \frac{1}{2} (Q_\mu \delta^{\alpha}_{\nu} + Q_\nu \delta^{\alpha}_{\mu} - Q^\alpha g_{\mu\nu}), \]  
(11)

is the disformation tensor. The Weyl gauge vector \( Q_\alpha \) measures how much the length of given vector varies during parallel transport.

In this paper we call as curvature tensor of \( W_4 \) spacetime, or simply “curvature tensor”, the curvature of the connection, symbolically \( R(\Gamma) \), whose coordinate components are,

\[ R^\alpha_{\sigma\mu\nu} := \partial_\mu \Gamma^{\alpha}_{\nu\sigma} - \partial_\nu \Gamma^{\alpha}_{\mu\sigma} + \Gamma^{\alpha}_{\mu\lambda} \Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\alpha}_{\nu\lambda} \Gamma^{\lambda}_{\mu\sigma}, \]  
(12)

or, if take into account the decomposition [☐]:

\[ R^\alpha_{\sigma\mu\nu} = \tilde{R}^\alpha_{\sigma\mu\nu} + \nabla_\nu L^\alpha_{\sigma\nu} - \nabla_\sigma L^\alpha_{\nu\mu} + L^\alpha_{\mu\lambda} L^\lambda_{\nu\sigma} - L^\alpha_{\nu\lambda} L^\lambda_{\mu\sigma}, \]  
(13)

where \( \tilde{R}^\alpha_{\sigma\mu\nu} \) is the Riemann-Christoffel or LC curvature tensor,

\[ \tilde{R}^\alpha_{\sigma\mu\nu} := \partial_\mu \{^\alpha_{\nu\sigma} \} - \partial_\nu \{^\alpha_{\mu\sigma} \} + \{^\alpha_{\mu\lambda} \} \{^\lambda_{\nu\sigma} \} - \{^\alpha_{\nu\lambda} \} \{^\lambda_{\mu\sigma} \}, \]  
(14)

and \( \nabla_\alpha \) is the LC covariant derivative. Besides, the LC Ricci tensor \( R_{\mu\nu} = \tilde{R}_{\mu\lambda\nu}, \) and LC curvature scalar read:

\[ \tilde{R} = g^{\mu\nu} \tilde{R}_{\mu\nu}, \]  
(15)

respectively. We call \( R(\Gamma) \) as generalized curvature tensor because it is contributed both by LC curvature \( R^\alpha_{\sigma\mu\nu} \), and by nonmetricity through disformation \( L^\alpha_{\mu\nu} \).

The curvature tensor \( R^\alpha_{\sigma\mu\nu} \) has various linearly independent contractions,

\[ R_{\mu\nu} := g^{\lambda\kappa} R_{\lambda\mu\nu\kappa}, \]
\[ \tilde{R}_{\mu\nu} := g^{\lambda\kappa} \tilde{R}_{\mu\lambda\nu\kappa}, \]
\[ R = g^{\mu\nu} R_{\mu\nu} = g^{\mu\nu} \tilde{R}_{\mu\nu}. \]  
(16)

The first two of these amount to,

\[ R_{\mu\nu} = \tilde{R}_{\mu\nu} - \frac{1}{2} \left( Q_\lambda Q^\lambda + \nabla_\lambda Q^\lambda \right) g_{\mu\nu} + \frac{1}{2} Q_\mu Q_\nu \]
\[ - \nabla_\nu Q_\mu + \frac{1}{2} \left( \nabla_\mu Q_\nu - \nabla_\nu Q_\mu \right), \]  
(17)

and to,

\[ \tilde{R}_{\mu\nu} = \tilde{R}_{\mu\nu} - \frac{1}{2} \left( Q_\lambda Q^\lambda + \nabla_\lambda Q^\lambda \right) g_{\mu\nu} + \frac{1}{2} Q_\mu Q_\nu \]
\[ - \frac{1}{2} \left( \nabla_\mu Q_\nu + \nabla_\nu Q_\mu \right), \]  
(18)

respectively. We shall call \( R_{\mu\nu} \) as first Ricci tensor while \( \tilde{R}_{\mu\nu} \) we shall call as second Ricci tensor. Notice that only the second Ricci tensor is symmetric in its indices: \( \tilde{R}_{\mu\nu} = \tilde{R}_{\nu\mu} \). There are two more contractions of the curvature tensor: \( R^\lambda_{\lambda\mu\nu} \) and \( R^\lambda_{\mu\lambda\nu} \). However, the latter one identically vanishes while the former one is a linear combination of the contractions \( R_{\mu\nu} \) and \( \tilde{R}_{\mu\nu} \):

\[ R^\lambda_{\lambda\mu\nu} = 2 \left( R_{\mu\nu} - \tilde{R}_{\mu\nu} \right) = 2 \left( \nabla_\mu Q_\nu - \nabla_\nu Q_\mu \right), \]

From equations [17] and [18] it follows that,
\[ R(\mu\nu) = \tilde{R}_{\mu\nu}, \quad (19) \]
\[ R_{[\mu\nu]} = \tilde{\nabla}_\mu Q_\nu - \tilde{\nabla}_\nu Q_\mu. \quad (20) \]

Besides, for the Einstein’s tensor \( G_{\mu\nu} := R_{\mu\nu} - g_{\mu\nu}R/2 \) we obtain that,
\[ G(\mu\nu) = \tilde{G}_{\mu\nu} \equiv \tilde{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu}R. \quad (21) \]

A. Properties and identities of the curvature in \( \tilde{W}_4 \)

The curvature tensor of Weyl space satisfies the first (cyclic) identity,
\[ R^\alpha_{\mu\nu\sigma} + R^\alpha_{\nu\sigma\mu} + R^\alpha_{\sigma\mu\nu} = 0, \quad (22) \]
while the torsionless connection \( \tilde{\nabla} \) of \( \tilde{W}_4 \) space satisfies the second Bianchi identity \([71]\):
\[ \tilde{\nabla}_\mu R^\alpha_{\nu\sigma\lambda} + \tilde{\nabla}_\nu R^\alpha_{\mu\lambda\sigma} + \tilde{\nabla}_\lambda R^\alpha_{\nu\mu\sigma} = 0. \quad (23) \]

In general the symmetries of the curvature tensor of \( \tilde{W}_4 \) space differ from those of the curvature tensor of Riemann space \( V_4 \) (LC curvature tensor). For instance:
\[ R^\alpha_{\sigma\mu\nu} = -R^\alpha_{\sigma\nu\mu}, \quad (24) \]
\[ R_{\nu\sigma\mu\nu} = -R_{\sigma\nu\mu\nu} + (\partial_\nu Q_\mu - \partial_\mu Q_\nu)g_{\alpha\sigma}. \quad (25) \]

The last equation, known as the third Bianchi identity, in compact form can be written in the following way: \[ \partial_\mu Q_\nu g_{\alpha\sigma} = R_{(\alpha\sigma)\mu\nu}, \quad (26) \]
or, alternatively,
\[ R_{(\alpha\sigma)\mu\nu} = \frac{1}{2} R_{[\mu\nu]}g_{\alpha\sigma}. \quad (27) \]

III. WEYL GAUGE SYMMETRY

Weyl gauge symmetry (WGS) or invariance under local changes of scale, is a manifest symmetry of \( \tilde{W}_4 \) spaces. \[^6\]

\[\text{The geometric laws that define } \tilde{W}_4, \text{ among which is the nonmetricity condition } \[8\], \text{ are invariant under local Weyl rescalings or, also, Weyl gauge transformations. These represent a particular case of } [24] \text{ and amount to simultaneous conformal transformations of the metric and gauge transformations of the nonmetricity vector } Q_\alpha: \]
\[ g_{\mu\nu} \rightarrow \Omega^2 g_{\mu\nu}, \quad Q_\alpha \rightarrow Q_\alpha - 2\partial_\alpha \ln \Omega, \quad (28) \]

respectively. In what follows we shall call the transformations \([23]\) either as Weyl gauge transformations or, simply, as gauge transformations. Under \([23]\):
\[ \{\alpha\}_{\mu\nu} \rightarrow \{\alpha\}_{\mu\nu} + \left( \delta^\alpha_\mu \partial_\nu + \delta^\alpha_\nu \partial_\mu - g_{\mu\nu} \partial^\alpha \right) \ln \Omega, \]
\[ L^\alpha_{\mu\nu} \rightarrow L^\alpha_{\mu\nu} - \left( \delta^\alpha_\mu \partial_\nu + \delta^\alpha_\nu \partial_\mu - g_{\mu\nu} \partial^\alpha \right) \ln \Omega, \quad (29) \]
so that the generalized affine connection \( \tilde{\Gamma} \) is unchanged by the Weyl rescalings:
\[ \tilde{\Gamma}^\alpha_{\mu\nu} \rightarrow \tilde{\Gamma}^\alpha_{\mu\nu}. \quad (30) \]

This means that the curvature tensor \( R^\alpha_{\sigma\mu\nu} \) of Weyl space \( \tilde{W}_4 \) defined in \([12]\) and the related Ricci tensor, \( R_{\mu\nu} \equiv R^\lambda_{\mu\lambda\nu} \), are unchanged as well, \( R^\alpha_{\mu\sigma\nu} \rightarrow R^\alpha_{\mu\sigma\nu} \).
\[ R_{\mu\nu} \rightarrow R_{\mu\nu}, \text{ while the curvature scalar of } \tilde{W}_4 \text{ space transforms as,} \]
\[ R \rightarrow \Omega^{-2} R. \quad (31) \]

It can demonstrated that the Bianchi identities are gauge invariant expressions as well.

Another important quantity through this paper is the traceless second-rank tensor with coordinate components,
\[ Q_{\mu\nu} := 2\tilde{\nabla}_{[\mu} Q_{\nu]} = \partial_\mu Q_\nu - \partial_\nu Q_\mu. \quad (32) \]

Under the gauge transformations \([23]\) it is not transformed, so that it has vanishing weight \( w(Q_{\mu\nu}) = 0 \). The quantity \([32]\) represents that part of the curvature which is due to nonmetricity of \( \tilde{W}_4 \) space. We shall call it as gauge tensor. It is not difficult to demonstrate that,
\[ Q^{\mu\nu} = g^{\mu\lambda} g^{\nu\kappa} Q_{\lambda\kappa} \equiv \tilde{\nabla}^{\mu} Q^\nu - \tilde{\nabla}^{\nu} Q^\mu, \quad (33) \]
so that \( w(Q^{\mu\nu}) = -4 \).

\[\text{\[6\] In Eq. \([23]\) we have taken into account the symmetry of the connection in its second and third indices:} \]
\[ \tilde{\nabla}_\mu Q_\nu - \tilde{\nabla}_\nu Q_\mu = \tilde{\nabla}_\mu Q_\nu - \tilde{\nabla}_\nu Q_\mu = \partial_\mu Q_\nu - \partial_\nu Q_\mu. \]

\[\text{\[7\] In reference \([20]\) an alternative definition of } \tilde{W}_4 \text{ space is given, where the WGS is made evident (definition 2 of the mentioned reference): “A Weyl structure is a differentiable manifold } \mathcal{M} \text{ endowed with a unique torsion-free affine connection } \tilde{\Gamma}_\omega \text{ and a conformally related equivalence class of metric tensors } [g], \text{ such that the non-metricity tensor is given by } \tilde{Q} = \omega \otimes g, \text{ where } \omega \text{ is any 1-form field and } g \text{ is a representative of the class.”} \]
A. Transformation of LC quantities

Here, for completeness, we include the transformation of LC quantities under the conformal transformation in \([28]\). The Ricci tensor transforms in the following way \((\nabla_{\mu} \ln \Omega = \partial_{\mu} \ln \Omega)\):

\[
\hat{R}_{\mu \nu} \to \hat{R}_{\mu \nu} + 2 \left[ \nabla_{\mu} \ln \Omega \nabla_{\nu} \ln \Omega - g_{\mu \nu} (\nabla \ln \Omega)^2 \right] - 2 \nabla_{\mu} \nabla_{\nu} \ln \Omega - g_{\mu \nu} \nabla^2 \ln \Omega, \tag{34}
\]

where we use the following notation, \(\nabla^2 \equiv g^{\mu \nu} \nabla_{\mu} \nabla_{\nu}, \nabla^{(\chi^2)} \equiv g^{\mu \nu} \nabla_{\mu} \chi \nabla_{\nu} \chi, \) etc. The LC curvature scalar transforms like

\[
\hat{R} \to \Omega^{-2} \left[ \hat{R} - 6 (\partial \ln \Omega)^2 - 6 \nabla^2 \ln \Omega \right], \tag{35}
\]

where \((\partial \ln \Omega)^2 = (\nabla \ln \Omega)^2\). Hence, for the Einstein’s tensor we get that, under the conformal transformation of the metric,

\[
\hat{G}_{\mu \nu} \to \hat{G}_{\mu \nu} + 2 \partial_{\mu} \ln \Omega \partial_{\nu} \ln \Omega + g_{\mu \nu} (\partial \ln \Omega)^2 - 2 \left( \nabla_{\mu} \nabla_{\nu} - g_{\mu \nu} \nabla^2 \right) \ln \Omega. \tag{36}
\]

Under the gauge transformations \([28]\) the LC covariant derivative of the nonmetricity vector transforms in the following way

\[
\nabla_{\mu} Q_{\nu} \to \nabla_{\mu} Q_{\nu} - 2 \nabla_{\mu} \nabla_{\nu} \ln \Omega - 2 \left[ \nabla_{\mu} (\partial_{\nu} \ln \Omega - \frac{1}{2} g_{\mu \nu} \partial^2 \ln \Omega) \right] + 4 \left[ \partial_{\mu} \ln \Omega \partial_{\nu} \ln \Omega - \frac{1}{2} g_{\mu \nu} (\partial \ln \Omega)^2 \right]. \tag{37}
\]

Hence,

\[
\nabla_{\mu} Q_{\nu} \to \Omega^{-2} \left[ \nabla_{\mu} Q_{\nu} + 2 Q_{\lambda} \partial_{\lambda} \ln \Omega - 4 (\partial \ln \Omega)^2 - 2 \nabla^2 \ln \Omega \right]. \tag{38}
\]

For a scalar field \(\psi\) which under \([28]\) transforms like \(\psi \to \Omega^w \psi\), we have that,

\[
(\partial \psi)^2 \to \Omega^{2w} \left[ (\partial \psi)^2 + 2 w \psi \partial_{\mu} \partial_{\nu} \psi \partial^2 \ln \Omega + w^2 \psi^2 (\partial \ln \Omega)^2 \right], \tag{39}
\]

where we recall that \((\partial \psi)^2 = (\nabla \psi)^2\), while

\[
\frac{\nabla_{\mu} \nabla_{\nu} \psi}{\psi} \to \frac{\nabla_{\mu} \nabla_{\nu} \psi}{\psi} + g_{\mu \nu} \partial_{\lambda} \psi \partial^\lambda \ln \Omega + w \psi \partial_{\mu} \partial_{\nu} \psi \partial_{\lambda} \ln \Omega \nonumber \\
+ (w - 1) \left( \frac{\partial_{\mu} \psi}{\psi} \partial_{\nu} \ln \Omega + \frac{\partial_{\nu} \psi}{\psi} \partial_{\mu} \ln \Omega \right) + w (w - 1) \partial_{\mu} \ln \Omega \partial_{\nu} \ln \Omega + w g_{\mu \nu} (\partial \ln \Omega)^2 \nonumber \\
w \psi \nabla_{\mu} \nabla_{\nu} \ln \Omega, \tag{40}
\]

and

\[
\frac{\hat{\nabla}^2 \psi}{\psi} \to \Omega^{-2} \left[ \frac{\hat{\nabla}^2 \psi}{\psi} + 2 (w + 1) \frac{\partial_{\lambda} \psi}{\psi} \partial^\lambda \ln \Omega \right. \right. \nonumber \\
\left. + w (w + 2) \partial_{\mu} \ln \Omega \partial_{\nu} \ln \Omega + \hat{\nabla}^2 \ln \Omega \right]. \tag{41}
\]

As an example, let us consider a scalar field \(\phi\) whose weight is \(w(\phi) = -1\), hence, under \([28]\): \(\phi^2 \to \Omega^{-2} \phi^2\), so that the following transformation property takes place:

\[
\frac{\hat{\nabla}^2 \phi^2}{\phi^2} \to \Omega^{-2} \left[ \frac{\hat{\nabla}^2 \phi^2}{\phi^2} - 2 \hat{\nabla}^2 \ln \Omega \right. \nonumber \\
\left. - 2 \partial_{\mu} \partial_{\nu} \partial_{\lambda} \ln \Omega \right], \tag{42}
\]

or

\[
\nabla^2 \phi^2 \to \Omega^{-4} \left[ \frac{\nabla^2 \phi^2}{\phi^2} - 2 \phi^2 \nabla^2 \ln \Omega \right. \nonumber \\
\left. - 2 \partial_{\mu} \partial_{\nu} \partial_{\lambda} \ln \Omega \right]. \tag{43}
\]

B. Useful relationships

Here we shall write in explicit form the transformation of several quantities under the gauge transformations. These shall be useful for our demonstrations in the subsequent sections. For compactness we use the following definitions:

\[
\hat{\Pi}_{\mu \nu} \equiv \nabla_{\mu} \nabla_{\nu} - g_{\mu \nu} \nabla^2, \tag{44}
\]

\[
\hat{T}^{(Q)}_{\mu \nu} = \frac{3}{2} \left( Q_{\mu \nu} - \frac{1}{2} g_{\mu \nu} Q_{\chi} \chi^2 \right), \tag{45}
\]

\[
\hat{T}^{(\nabla)Q}_{\mu \nu} = \frac{3}{2} \left[ (\nabla_{\mu} Q_{\nu}) - \frac{1}{2} g_{\mu \nu} (\nabla \chi)^2 \right], \tag{46}
\]

\[
\hat{T}^{(\phi)}_{\mu \nu} \equiv \frac{6}{\phi^2} \left[ \partial_{\mu} \partial_{\nu} \phi - \frac{1}{2} g_{\mu \nu} (\partial \phi)^2 \right], \tag{47}
\]

besides, \(\hat{\Pi} \equiv g_{\mu \nu} \hat{\Pi}_{\mu \nu}, \hat{T}^{(Q)} \equiv g_{\mu \nu} \hat{T}^{(Q)}_{\mu \nu}, \) etc.

Under the gauge transformations \([28]\) the above quantities transform in the following way:

\[
\frac{\hat{\Pi}_{\mu \nu} \phi^2}{\phi^2} \to \hat{\Pi}_{\mu \nu} \phi^2 - 2 \hat{\Pi}_{\mu \nu} \ln \Omega \nonumber \\
- \frac{12}{\phi} \left[ \partial_{\mu} \phi \partial_{\nu} \ln \Omega - \frac{1}{2} g_{\mu \nu} \partial_{\lambda} \phi \partial^\lambda \ln \Omega \right] + 8 \partial_{\mu} \ln \Omega \partial_{\nu} \ln \Omega - 2 g_{\mu \nu} (\partial \ln \Omega)^2, \tag{48}
\]

\[
\hat{T}^{(Q)}_{\mu \nu} \to \hat{T}^{(Q)}_{\mu \nu} - 6 \left[ Q_{\mu \nu} \ln \Omega - \frac{1}{2} g_{\mu \nu} Q_{\chi} \chi^2 \right] \nonumber \\
+ 6 \partial_{\mu} \ln \Omega \partial_{\nu} \ln \Omega - 3 g_{\mu \nu} (\partial \ln \Omega)^2, \tag{49}
\]

\[
\hat{T}^{(\nabla)Q}_{\mu \nu} \to \hat{T}^{(\nabla)Q}_{\mu \nu} - 6 \left[ \nabla_{\mu} Q_{\nu} \ln \Omega - \frac{1}{2} g_{\mu \nu} (\nabla \chi)^2 \right] \nonumber \\
+ 6 \nabla_{\mu} \ln \Omega \nabla_{\nu} \ln \Omega - 3 g_{\mu \nu} (\partial \ln \Omega)^2, \tag{50}
\]

\[
\hat{T}^{(\phi)}_{\mu \nu} \to \hat{T}^{(\phi)}_{\mu \nu} - 6 \left[ \partial_{\mu} \partial_{\nu} \phi \ln \Omega - \frac{1}{2} g_{\mu \nu} (\partial \phi)^2 \right] \nonumber \\
+ 6 \partial_{\mu} \ln \Omega \partial_{\nu} \ln \Omega - 3 g_{\mu \nu} (\partial \ln \Omega)^2, \tag{51}
\]

\[
\hat{\Pi} \to \hat{\Pi} - 6 \left[ \partial_{\mu} \phi \ln \Omega \right] + 6 \partial_{\mu} \ln \Omega \partial_{\nu} \ln \Omega - 3 g_{\mu \nu} (\partial \ln \Omega)^2, \tag{52}
\]
\[
\dot{T}^{(\nabla Q)}_{\mu\nu} \rightarrow \dot{T}^{(\nabla Q)}_{\mu\nu} - 6 \left[ \nabla_\mu \nabla_\nu \ln \Omega - \frac{1}{2} g_{\mu\nu} \nabla^2 \ln \Omega \right] 
- 6 Q_{(\mu,\partial,\nu)} \ln \Omega + 12 \partial_\mu \ln \Omega \partial_\nu \ln \Omega,
\] (46)
and
\[
\mathcal{T}^{(\phi)}_{\mu\nu} \rightarrow \mathcal{T}^{(\phi)}_{\mu\nu} - \frac{12}{\phi} \left[ \partial_\mu \phi \partial_\nu \ln \Omega - \frac{1}{2} g_{\mu\nu} \partial_\lambda \phi \partial_\lambda \ln \Omega \right] 
+ 6 \partial_\mu \ln \Omega \partial_\nu \ln \Omega - 3g_{\mu\nu} (\partial \ln \Omega)^2,
\] (47)
respectively.

Several interesting results follow from the above transformation properties. Form instance, we have that under \(g_{\mu\nu} = g_{\mu\lambda} g_{\nu\lambda} \),
\[
- \frac{\dot{\Pi}_{\mu\nu}}{\phi^2} + \mathcal{T}^{(\phi)}_{\mu\nu} \rightarrow - \frac{\dot{\Pi}_{\mu\nu}}{\phi^2} + \mathcal{T}^{(\phi)}_{\mu\nu} + 2\dot{\Pi}_{\mu\nu} \ln \Omega 
- 2\partial_\mu \ln \Omega \partial_\nu \ln \Omega - g_{\mu\nu} (\partial \ln \Omega)^2,
\] so that, if take a look at Eq. \(50\), which can be written as:
\[
\dot{G}_{\mu\nu} \rightarrow \dot{G}_{\mu\nu} - 2\dot{\Pi}_{\mu\nu} \ln \Omega 
+ 2\partial_\mu \ln \Omega \partial_\nu \ln \Omega + g_{\mu\nu} (\partial \ln \Omega)^2,
\]
it is evident that the combination:
\[
\dot{\Sigma}_{\mu\nu} \equiv \dot{G}_{\mu\nu} - \frac{\dot{\Pi}_{\mu\nu}}{\phi^2} + \mathcal{T}^{(\phi)}_{\mu\nu},
\] (48)
or, in explicit form:
\[
\dot{\Sigma}_{\mu\nu} \equiv \dot{G}_{\mu\nu} + 6 \left[ \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} g_{\mu\nu} (\partial \phi)^2 \right] 
- \frac{1}{\phi^2} \left( \nabla_\mu \nabla_\nu g_{\mu\nu} \nabla^2 \phi \right)^2,
\] (49)
is invariant under the gauge transformations \(28\). It is evident that under the gauge transformations the trace:
\[
\dot{\Sigma} \equiv g^{\mu\nu} \dot{\Sigma}_{\mu\nu},
\]
transforms like: \(\dot{\Sigma} \rightarrow \Omega^{-2} \dot{\Sigma}\).

IV. A LEMMA ON GAUGE INVARIANCE

Let us consider the simplest gauge invariant gravitational action in \(W_4\) space, which is linear in the curvature scalar of Weyl space \(R\):
\[
S_{\text{lin}} = \frac{1}{2} \int_{\mathcal{M}_4 \in W_4} d^4 x \sqrt{-g} \phi^2 R,
\] (53)
where \(\phi\) is a scalar field with conformal weight \(w(\phi) = -1\). In what follows, for compactness and simplicity, we omit explicit writing of the integration domain.

\[8\] Here it is evident that the ellipsis stand for the trace of the ellipsis in the above equation which, under \(\Phi_{\mu\nu} = g_{\mu\nu} \Phi_{\mu\nu} = \Phi_{\mu\nu} - \Phi_{\mu\nu} \nabla^2 \Phi_{\mu\nu}, \) respectively.
The problem with the theory based on \((53)\) is that, although the action \(S_{\text{lin}}\) is invariant under the gauge transformations \((28)\), the derived equations of motion are not gauge invariant since the scalar field does not have a kinetic energy density term. This result can be stated in the form of the following lemma:

Lemma 1 The Weyl gauge invariant gravitational action \((53)\), which is linear in the curvature scalar \(R\) of Weyl space \(W\), and which lacks a kinetic energy term of the scalar field \(\phi\), leads to equations of motion which are not gauge invariant.

Demonstration. In order to show that the action \((53)\) and the derived equations of motion do not share gauge invariance, let us rewrite the curvature scalar of \(W\) space through its decomposition in LC (Riemannian) quantities. For this purpose it is enough to take the trace of either \((17)\) or \((18)\):

\[
R = \hat{R} - \frac{3}{2} Q_{\mu} Q^\mu - 3 \nabla_\mu Q^\mu,
\]

where, as before, the quantities with a hat are defined in terms of the LC connection \((10)\). The action \((53)\) can be rewritten in the following way,

\[
S_{\text{lin}} = \frac{1}{2} \int d^4x \sqrt{-g} \phi^2 \left( \hat{R} - \frac{3}{2} Q_{\mu} Q^\mu - 3 \nabla_\mu Q^\mu \right). (55)
\]

The Einstein’s EOM which are derived from \((55)\) by varying with respect to the metric read:

\[
\hat{E}_{\mu\nu} \equiv \hat{G}_{\mu\nu} - \frac{1}{\phi^2} \left( \nabla_\mu \nabla_\nu - g_{\mu\nu} \nabla^2 \right) \phi^2 - \frac{3}{2} \left( Q_{\mu} Q_\nu - \frac{1}{2} g_{\mu\nu} Q_\lambda Q^\lambda \right) - 3 \left( \nabla_{(\mu} Q_{\nu)} - \frac{1}{2} g_{\mu\nu} \nabla_\lambda Q^\lambda \right) = 0, (56)
\]

where \(\hat{G}_{\mu\nu} \equiv \hat{R}_{\mu\nu} - g_{\mu\nu} \hat{R}/2\) is the LC Einstein’s tensor of \(V_4\) space. Meanwhile, variation with respect to the scalar field \(\phi\) leads to the following constraint:\(^9\)

\[
\hat{\Sigma} = -\hat{R} + \frac{3}{2} Q_{\mu} Q^\mu + 3 \nabla_\mu Q^\mu = 0. (57)
\]

While this constraint is obviously gauge invariant, as shown in subsection \(11\) by brute force it can be shown that the EOM \((56)\) is not invariant under the gauge transformations \((28)\). Actually, according to equations \((56), (40), (41), (44), (45), (46)\) and \((51)\), under the gauge transformations \((28)\) the quantity

\[
\hat{E}_{\mu\nu} \equiv \hat{G}_{\mu\nu} - \frac{1}{\phi^2} \left( \nabla_\mu \nabla_\nu - g_{\mu\nu} \nabla^2 \right) \phi^2 - \frac{3}{2} \left( Q_{\mu} Q_\nu - \frac{1}{2} g_{\mu\nu} Q_\lambda Q^\lambda \right) - 3 \left( \nabla_{(\mu} Q_{\nu)} - \frac{1}{2} g_{\mu\nu} \nabla_\lambda Q^\lambda \right) = 0, (58)
\]

transforms like,

\[
\hat{E}_{\mu\nu} \rightarrow \hat{E}_{\mu\nu} + 6 \left( \nabla_\mu \nabla_\nu - g_{\mu\nu} \nabla^2 \right) \ln \Omega + \frac{6}{\phi^2} \left( \frac{1}{2} g_{\mu\nu} \partial_\alpha \partial^\alpha \ln \Omega \right) + 6 \left( Q_{(\mu} \partial_{\nu)} - \frac{1}{4} g_{\mu\nu} Q_\lambda \partial^\lambda \ln \Omega \right) - 24 \left( \partial_\mu \ln \Omega \partial_\nu \ln \Omega - \frac{1}{4} g_{\mu\nu} \left( \partial \ln \Omega \right)^2 \right), (59)
\]

while its trace \(\hat{E} \equiv g_{\mu^\nu} \hat{E}_{\mu\nu} = \hat{\Sigma} + 3 \nabla^2 \phi^2 / \phi^2\), transforms like:

\[
\hat{E} \rightarrow \Omega^{-2} \left( \hat{E} - 6 \frac{\partial_\mu \phi^2}{\phi^2} \partial^\mu \ln \Omega - 6 \nabla^2 \ln \Omega \right). (60)
\]

If take into account that under \((28)\): \(\hat{\Sigma} \rightarrow \Omega^{-2} \hat{\Sigma}\), then the above equation leads to \((42)\). The transformation property of \(\hat{E}_{\mu\nu}\) exposed in Eq. \((59)\) demonstrates that the EOM \((56)\) is not invariant under the gauge transformations. Q.E.D.

An alternative (much more easy) way to show that the EOM \((56)\) is not gauge invariant, is by taking its trace: \(\hat{E} = 0\), which can be written as,

\[
\hat{\Sigma} = -\hat{R} + \frac{3}{2} Q_{\mu} Q^\mu + 3 \nabla_\mu Q^\mu = 3 \nabla^2 \phi^2 / \phi^2, (61)
\]

and by further comparing \((61)\) with the constraint \((57)\) we get a dynamical equation for the scalar field \(\phi\):

\[
\nabla^2 \phi^2 = 0. (62)
\]

This is obviously not gauge invariant, since, according to \((42)\):

\[
\nabla^2 \phi^2 = 0 \rightarrow \nabla^2 \phi^2 = 2 \phi^2 \nabla^2 \ln \Omega + 2 \partial_\mu \phi^2 \partial^\mu \ln \Omega.
\]

A. Complex scalar

Lemma \(11\) can be formulated in an alternative way if go to more general gauge invariant Lagrangian, where

\(^9\) Variation with respect to vectorial nonmetricity is not considered since the corresponding EOM is irrelevant to the present demonstration. As a matter of fact, in order to demonstrate the present lemma, it is enough to demonstrate that one of the derived EOMs is not gauge invariant.
the Lagrangian in (63): \( \mathcal{L}_{\text{lin}} = \sqrt{-g} \phi^2 R \), is modified by considering a complex or multicomponent scalar field \( \phi \) with conformal weight \( w(\phi) = -1 \). We also consider a kinetic energy density term for this complex scalar. For generality we shall consider a coupling parameter \( \beta \), so that the resulting gravitational Lagrangian reads:

\[
\mathcal{L}_{\text{grav}} = \frac{\sqrt{-g}}{2} \left[ \frac{1}{2} |\phi|^2 R + |\partial^* \phi|^2 \right], \tag{63}
\]

where we have introduced the following notation: \( |\phi|^2 \equiv \phi^\dagger \phi \), \( |\partial^* \phi|^2 \equiv g_{\mu\nu} (\partial^{\mu} \phi \partial^{\nu} \phi^\dagger) \) and the gauge derivative

\[
\partial^* \phi = \left( \partial_{\mu} - \frac{1}{2} g_{\mu\nu} \right) \phi,
\]

is defined in Eq. (A1) of appendix A. The Lagrangian in (63) is manifestly gauge invariant. The question is: would the derived EOM be gauge invariant as well? The answer to the above question can be stated in the form of the following generalization of Lemma 1.

**Lemma 2** The Weyl gauge invariant gravitational Lagrangian, which is linear in the curvature scalar \( R \) of Weyl space \( W_4 \), and contains a kinetic energy term of the scalar field \( \phi \), leads to equations of motion which are gauge invariant only if the coupling parameter \( \beta = 1/3 \).

**Demonstration.** Let us to start by substituting the decomposition of the generalized curvature scalar \( R \) in terms of LC quantities: \( R = \hat{R} - 3 Q_\Lambda Q^\Lambda /2 - 3 \hat{\nabla}_\lambda Q^\lambda \), in (63) and, also, by writing the gauge derivative explicitly. Up to a gradient \(-\hat{\nabla}_\lambda (|h|^2 Q^\lambda) /2 \), which does not modify the EOM, we get:

\[
\mathcal{L}_{\text{grav}} = \frac{\sqrt{-g}}{2} \left[ \frac{1}{2} |\phi|^2 \hat{R} + |\partial \phi|^2 + \frac{1 - 3 \beta}{4} |\phi|^2 Q_\Lambda Q^\Lambda + \frac{1 - 3 \beta}{2} |\phi|^2 \hat{\nabla}_\lambda Q^\lambda \right]. \tag{64}
\]

Variation of this Lagrangian with respect to the metric yields,

\[
\beta |\phi|^2 \hat{G}_{\mu\nu} - \beta \left( \hat{\nabla}_\mu \hat{\nabla}_\nu - g_{\mu\nu} \hat{\nabla}^2 \right) |\phi|^2 \\
+ 2 \left[ (\partial_{(\mu} \phi^\dagger \partial_{\nu)} \phi - \frac{1}{2} g_{\mu\nu} |\partial \phi|^2 \right] \\
+ \frac{1 - 3 \beta}{2} |\phi|^2 \left( Q_\mu Q^\nu - \frac{1}{2} g_{\mu\nu} Q_\Lambda Q^\Lambda \right) \\
+ (1 - 3 \beta) |\phi|^2 \left[ \hat{\nabla}_{(\mu} Q_{\nu)} - \frac{1}{2} g_{\mu\nu} \hat{\nabla}_\lambda Q^\lambda \right] = 0, \tag{65}
\]

while variation with respect to \( \phi \) and to \( \phi^\dagger \) leads to the following EOM for the Higgs and its transposed conjugated field:

\[
\beta \phi \hat{R} - 2 \hat{\nabla}^2 \phi + \frac{1 - 3 \beta}{2} \phi Q_\Lambda Q^\Lambda \\
+ (1 - 3 \beta) \phi \hat{\nabla}_\lambda Q^\lambda = 0, \tag{66}
\]

\[
\beta \phi \hat{R} - 2 \hat{\nabla}^2 \phi^\dagger + \frac{1 - 3 \beta}{2} \phi^\dagger Q_\Lambda Q^\Lambda \\
+ (1 - 3 \beta) \phi^\dagger \hat{\nabla}_\lambda Q^\lambda = 0, \tag{66}
\]

respectively. If we multiply the first equation above by \( \phi^\dagger \) from the left and the second one by \( \phi \) from the right, and if we take into account that \((\hat{\nabla}^2 \phi^\dagger) \phi + \phi^\dagger \hat{\nabla}^2 \phi = \hat{\nabla}^2 |\phi|^2 - 2 |\partial \phi|^2 \), combining the resulting equations we get that:

\[
\beta |\phi|^2 \hat{R} + 2 |\partial \phi|^2 - \hat{\nabla}^2 |\phi|^2 + \frac{1 - 3 \beta}{2} |\phi|^2 Q_\Lambda Q^\Lambda \\
+ (1 - 3 \beta) |\phi|^2 \hat{\nabla}_\lambda Q^\lambda = 0. \tag{67}
\]

Meanwhile, the trace of the Einstein’s equation (66) yields,

\[
\beta |\phi|^2 \hat{R} + 2 |\partial \phi|^2 - 3 \hat{\nabla}^2 |\phi|^2 + \frac{1 - 3 \beta}{2} |\phi|^2 Q_\Lambda Q^\Lambda \\
+ (1 - 3 \beta) |\phi|^2 \hat{\nabla}_\lambda Q^\lambda = 0. \tag{68}
\]

Comparing equations (67) and (68) leads to the following equation:

\[
(1 - 3 \beta) \hat{\nabla}^2 |\phi|^2 = 0. \tag{69}
\]

For \( \beta \neq 1/3 \) this equation is not invariant under the gauge transformations (25) as it was shown for the similar equation in (42) (see (43)). This means that the EOM (63) itself is not gauge invariant. On the contrary, for \( \beta = 1/3 \), Eq. (69) is gauge invariant: the EOM derived from the gravitational Lagrangian reads,

\[
\hat{\mathcal{G}}_{\mu\nu} \equiv \hat{\mathcal{G}}_{\mu\nu} - \frac{\hat{\Pi}_{\mu\nu} |\phi|^2}{|\phi|^2} \\
+ \frac{6}{|\phi|^2} \left[ \partial_{(\mu} \phi^\dagger \partial_{\nu)} \phi - \frac{1}{2} g_{\mu\nu} |\phi|^2 \right] = 0, \tag{70}
\]

where we have taken into account the definition of the operator \( \hat{\Pi}_{\mu\nu} \) given in Eq. (43). If realize that

\[
\frac{1}{|\phi|^2} \partial_{(\mu} |\phi|^2 \partial_{\nu)} \ln \Omega = \frac{1}{|\phi|^2} \partial_{(\mu} \phi^\dagger \partial_{\nu)} \ln \Omega \\
+ \frac{1}{\phi} \partial_{(\mu} \phi \partial_{\nu)} \ln \Omega,
\]

it is not difficult to show that, under the gauge transformations (25), the following transformation equation takes place:
\[ T^{(\varphi, \varphi')} \rightarrow T^{(\varphi, \varphi')} = -\frac{6}{|\varphi|^2} \left[ \partial_\mu (|\varphi|^2 \partial_\nu) \ln \Omega - \frac{1}{2} g_{\mu \nu} \partial_\lambda (|\varphi|^2 \partial^\lambda \ln \Omega) \right] + 6 \left[ \partial_\mu \ln \Omega \partial_\nu \ln \Omega - \frac{1}{2} g_{\mu \nu} (\partial \ln \Omega)^2 \right], \quad (71) \]

where, for compactness of writing, we have defined:

\[ T^{(\varphi, \varphi')} := \frac{6}{|\varphi|^2} \left[ \partial_\mu (|\varphi|^2 \partial_\nu) \ln \Omega - \frac{1}{2} g_{\mu \nu} \partial_\lambda (|\varphi|^2 \partial^\lambda \ln \Omega) \right]. \]

On the other hand, by using the results in subsection III B it is not difficult to show that, under (28):

\[ \hat{G}_{\mu \nu} - \frac{\hat{\Pi}_{\mu \nu} |\varphi|^2}{|\varphi|^2} \rightarrow \hat{G}_{\mu \nu} - \frac{\hat{\Pi}_{\mu \nu} |\varphi|^2}{|\varphi|^2} + 6 \left[ \partial_\mu (|\varphi|^2 \partial_\nu) \ln \Omega - \frac{1}{2} g_{\mu \nu} \partial_\lambda (|\varphi|^2 \partial^\lambda \ln \Omega) \right] - 6 \left[ \partial_\mu \ln \Omega \partial_\nu \ln \Omega - \frac{1}{2} g_{\mu \nu} (\partial \ln \Omega)^2 \right]. \quad (72) \]

Given the expressions (71) and (72), it follows that the quantity \( \hat{C}_{\mu \nu} \) is invariant under the gauge transformations (28). In consequence the EOM (70) is gauge invariant as well. Q.E.D

The above result shows that the theory based in the gravitational Lagrangian (53) is gauge invariant (means that not only the Lagrangian but also the derived EOM are gauge invariant) only for the choice \( \beta = 1/3 \), in which case Eq. (69) becomes an identity. This result is general and for a real scalar field \( \varphi^\dagger = \varphi \) it includes theories of the kind,

\[ S_{\text{grav}} = \frac{1}{2} \int d^4 x \sqrt{-g} \left[ \frac{\phi^2}{6} \dot{\hat{R}} + (\partial \phi)^2 \right]. \]

Hence, not only the absence of kinetic energy density term for the gauge scalar, as in (53), but also the presence of a kinetic term, as in (63), without the appropriate value of the coupling parameter \( \beta = 1/3 \), both lead to the given theory being not really gauge invariant since the resulting EOM are not invariant under the gauge transformations (28).

V. NONMETRICITY DOES NOT INTERACT WITH GRAVITATION

As shown above, in order to make the theory (53) gauge invariant, which means that both the action and the derived EOM are invariant under the gauge transformations (28), it is necessary to add a kinetic energy density term for the scalar field \( \phi \), with the appropriate proportion of the terms \( \dot{\phi}^2 R \) and \( (\partial^* \phi)^2 \), so that the gauge invariance of the action (53) is preserved while allowing for invariance of the derived EOM under (28). The resulting action reads,

\[ S_g = \frac{1}{2} \int d^4 x \sqrt{-g} \left[ \frac{\phi^2}{6} \hat{R} + (\partial^* \phi)^2 \right]. \quad (73) \]

This action and the derived EOM, are fully equivalent to the well-known gauge invariant Riemann geometry-based action for a scalar field conformally coupled to the curvature (11, 58, 69, 72),

\[ S_g = \frac{1}{2} \int d^4 x \sqrt{-g} \left[ \frac{\phi^2}{6} \hat{R} + (\partial \phi)^2 \right]. \quad (74) \]

This equivalence can be re-stated in the form of the following lemma:

**Lemma 3.** Up to a harmless boundary term, the Weyl gauge invariant gravitational actions (73) over Weyl geometry space \( W_4 \) and (74) over Riemann space \( V_4 \), may be identified.

**Demonstration.** Let us first to show that the EOM derived from (74) are gauge invariant equations (this is well-known result in the bibliography.) Actually, by varying (74) with respect to the metric and to the scalar field we get,

\[ \dot{\mathcal{E}}_{\mu \nu} = \mathcal{G}_{\mu \nu} + \frac{6}{\phi^2} \left[ \partial_\mu \phi^* \partial_\nu \phi - \frac{1}{2} g_{\mu \nu} (\partial \phi)^2 \right] - \frac{1}{\phi^2} \left( \tilde{\nabla}_\mu \tilde{\nabla}_\nu - g_{\mu \nu} \tilde{\nabla}^2 \right) \phi^2 = 0, \quad (75) \]

\[ \tilde{\nabla}^2 \phi - \frac{1}{6} \hat{R} \phi = 0 \quad \Leftrightarrow \quad \dot{\mathcal{E}} \equiv -\hat{R} - \frac{6}{\phi^2} (\partial \phi)^2 + 3 \frac{\tilde{\nabla}^2 \phi^2}{\phi^2} = 0, \quad (76) \]

respectively, where \( \dot{\mathcal{E}} = \mathcal{E}^\mu_{\mu} \) is the trace of \( \mathcal{E}_{\mu \nu} \). As shown in Sect. III (subsection III B), invariance of equations (75) and (70) under (28) is due to the transformations properties of \( \dot{\mathcal{E}}_{\mu \nu} \) and \( \dot{\mathcal{E}} \), under the gauge transformations (28),

\[ \dot{\mathcal{E}}_{\mu \nu} \rightarrow \dot{\mathcal{E}}_{\mu \nu}, \quad \dot{\mathcal{E}} \rightarrow \Omega^{-2} \dot{\mathcal{E}}, \]

hence, \( \dot{\mathcal{E}}_{\mu \nu} = 0 \rightarrow \dot{\mathcal{E}}_{\mu \nu} = 0; \quad \dot{\mathcal{E}} = 0 \rightarrow \dot{\mathcal{E}} = 0. \)

Now we shall demonstrate that the actions (74) and (73) are one and the same action. The required demonstration is a matter of simple algebra. Let us substitute \( R \) from (74) and \( \partial^* \phi = (\partial_\mu - Q_\mu/2) \phi \) into (73). We get:
\[ S_g = \int d^4x \sqrt{-g} \left[ \frac{\phi^2}{6} \left( \frac{3}{2} Q_{\mu}^\mu - 3 \nabla_{\mu} Q^\mu \right) + (\partial \phi)^2 - \frac{1}{2} \partial_{\mu} \phi^2 Q^\mu + \frac{\phi^2}{4} Q_{\mu} Q^\mu \right] \]

\[ = \int d^4x \sqrt{-g} \left[ \frac{\phi^2}{6} \left( \frac{3}{2} \hat{R} + (\partial \phi)^2 \right) - \frac{1}{2} \left( \partial_{\mu} \phi^2 Q^\mu + \phi^2 \nabla_{\mu} Q^\mu \right) \right]. \]

Hence, if in the above action integral omit the divergence,

\[ \nabla_{\mu} (\phi^2 Q^\mu) = \partial_{\mu} \phi^2 Q^\mu + \phi^2 \nabla_{\mu} Q^\mu, \]

which does not contribute to the equations of motion, one obtains the action \((73)\), Q.E.D.

Notice that in the above identified actions the kinetic energy density of the scalar field enters with the wrong sign. However, as we shall show below (see Sect. VIII), thanks to gauge symmetry this does not entail any problem since the ghost field \(\phi\) can be gauged away. The main difference between the above equivalent actions is that, while \(S_g\) in \((73)\) is based on background space of Weyl geometric structure \(\hat{W}_4\), the underlying background space associated with \(S_g\) in \((74)\) is Riemann geometry space \(V_4\) instead.

The following corollary of Lemma 3 takes place:

**Corollary 3.1** The Weyl geometric structure of \(\hat{W}_4\) space is irrelevant for the description of gauge invariant gravitational laws.

The above demonstrated Lemma 3 entails that, if require gauge symmetry to be an underlying symmetry of the theory, the nonmetricity \(Q_{\mu}\) does not couple to the background geometry. I. e. \(Q_{\mu}\) plays no role in the description of the gauge invariant laws of gravity when the action is linear in the curvature scalar. Hence one may safely replace Weyl geometry background space \(\hat{W}_4\) by Riemann space \(V_4\). In consequence, we may describe the gravitational phenomena without the need for a Weyl vector (nonmetricity) field.

We underline that, as shown above, the precise combination of the terms within integrals \((73)\) and its equivalent \((74)\) is required if the derived EOM are to be gauge invariant equations. This illustrates the lack of freedom left to us by Nature in order to choose a gauge invariant theory of gravity whose action is linear in the curvature scalar.

**VI. HIGHER CURVATURE TERMS**

Other pieces have to be included in the action \((73)\) in order to get rid of high energy gravitational effects. Higher order curvature terms are required by renormalization of (a would be) quantum gravity \([75, 76]\). Besides, quadratic (and also cubic) corrections arise as countermoments at one loop in gravity coupled to matter \([77]\) and at two loops in pure gravity \([78]\) and also in the effective string gravitational action \([79, 80]\). Here, in addition to the linear action Eq. \((53)\), we shall consider quadratic contributions to the curvature of two types. These comprise the most relevant cases where ghosts can be avoided.

**A. Quadratic contributions of type 1**

Let us consider the following action which contains quadratic curvature contributions:

\[ S_{\text{quad}}^{\text{type 1}} = \frac{\alpha^2}{c} \int d^4x \sqrt{-g} \left\{ a R_{\mu\nu} R^{\mu\nu} + b R_{\lambda\kappa\mu\nu} R^{\lambda\kappa\mu\nu} \right\}, \quad (77) \]

where \(a, b\) and \(c\) are real constants such that \(a + b = c\) and \(\alpha^2\) is a dimensionless coupling constant. According to the third Bianchi identity in the form of Eq. \((27)\) we have that,

\[ R_{\lambda\kappa\mu\nu} R^{\lambda\kappa\mu\nu} = R_{\mu\nu} R^{\mu\nu} = Q_{\mu\nu} Q^{\mu\nu} , \quad (78) \]

where the nonmetricity strength tensor \(Q_{\mu\nu}\) is defined in Eq. \((32)\). Hence the action \((77)\) can be written in the following way:

\[ S_Q = \alpha^2 \int d^4x \sqrt{-g} Q_{\mu\nu} Q^{\mu\nu}, \quad (79) \]

Of course, this piece of action is gauge invariant as well. Equation \((79)\) shows that the precise form of the quadratic terms in the action Eq. \((77)\) leads to second-order equations of motion, hence, to the absence of Ostrogradski ghosts \([82]\).

Variation of \((79)\) with respect to the metric yields,

\[ \delta_g S_Q = -\frac{1}{2} \int d^4x \sqrt{-g} g^{\mu\nu} T_{\mu\nu}^{(Q)}, \]

where,

\[ T_{\mu\nu}^{(Q)} = -4 \alpha^2 \left( Q_{\mu}^\lambda Q_{\nu\lambda} - \frac{1}{4} g_{\mu\nu} Q_{\lambda\sigma} Q^\lambda\sigma \right), \quad (80) \]

is the nonmetricity stress-energy tensor (SET) while variation with respect to the nonmetricity vector \(Q_{\mu}\) leads to,

\[ \delta_Q S_Q = -4 \alpha^2 \int d^4x \sqrt{-g} Q^\mu \nabla^\nu Q_{\nu\mu} = 0 \]

\[ \Rightarrow \nabla^\nu Q_{\nu\mu} = 0. \quad (81) \]
It can be shown that under the gauge transformations $\nabla^{\nu} Q_{\mu} \rightarrow \Omega^{-2} \nabla^{\nu} Q_{\mu}$, so that Eq. (81) is gauge invariant. In the same fashion, under (28) the nonmetricity SET (80) transforms like $T_{\mu \nu}^{(Q)} \rightarrow \Omega^{-2} T_{\mu \nu}^{(Q)}$, so that its contribution to the RHS of (79),

$$\hat{g}_{\mu \nu} = \frac{6}{\phi^2} T_{\mu \nu}^{(Q)},$$

preserves the gauge symmetry since, under (28):

$$\hat{g}_{\mu \nu} \rightarrow \hat{g}_{\mu \nu}, \quad \frac{6}{\phi^2} T_{\mu \nu}^{(Q)} \rightarrow \frac{6}{\phi^2} T_{\mu \nu}^{(Q)}.$$  

Quadratic contributions of type 1 do not introduce ghosts and lead to second order EOM, however, in this case the nonmetricity should be viewed as a vector field in Riemannian background space, nonmetricity should be viewed as a vector field in Riemannian background space, nonmetricity should be viewed as a vector field in Riemannian background space. In the understanding that the action (79) is to be added with action (74), this means that in this theory nonmetricity is a non-geometric radiation field which propagates in $V_4$ spacetime.

**B. Quadratic contributions of type 2**

Another more general type of quadratic contributions is given by the following action:

$$S_{\text{quad}}^{\text{type 2}} = \int d^4 x \sqrt{-g} \left[ \alpha_1 R^2 + \alpha_2 R_{\lambda \sigma \mu \nu} R^{\lambda \sigma \mu \nu} + \alpha_3 \left( R_{\mu \nu} + \hat{R}_{\mu \nu} \right) \left( R^{\mu \nu} + \hat{R}^{\mu \nu} \right) \right],$$

where $\alpha_1$, $\alpha_2$ and $\alpha_3$ are constants and, as defined in section 11, $R_{\mu \nu}$ and $\hat{R}_{\mu \nu}$ are the two independent contractions of the generalized curvature tensor $R^{\alpha \beta \gamma \delta \mu \nu}$ defined in equations (11), (14) and (15), respectively.

Although in principle this theory can have ghosts due to quadratic curvature terms, there are particular cases which are free of ghosts, or at least of the spin-two ghost field with negative square mass, which renders the related quantum theory nonunitary. As a matter of fact, the action (82) can be written in a simplified way if take into account the Gauss-Bonnet (GB) gauge invariant of $W_4$ space,

$$G = R^2 - \left( R_{\mu \nu} + \hat{R}_{\mu \nu} \right) \left( R^{\mu \nu} + \hat{R}^{\mu \nu} \right) + R_{\lambda \sigma \mu \nu} R^{\lambda \sigma \mu \nu},$$

which is also a topological invariant. Taking into account the GB topological invariant, which amounts to a total derivative that does not affect the EOM, the action (82) can be written in the following way:

$$S_{\text{quad}}^{\text{type 2}} = \int d^4 x \sqrt{-g} \left[ (\alpha_1 + \alpha_3) R^2 + (\alpha_2 + \alpha_3) R_{\lambda \sigma \mu \nu} R^{\lambda \sigma \mu \nu} \right].$$

Hence, for the choice of parameters $\alpha_3 = -\alpha_2$, the resulting theory can be free of ghosts.

Let us consider the quadratic action (82) with the choice of the free constants which leads to (in principle) ghost free theory: $\alpha_3 = -\alpha_2$, so that we are left with the following quadratic action:

$$\tilde{S}_g = \int d^4 x \sqrt{-\tilde{g}} \hat{R}^2,$$

where $\hat{\beta} = \alpha_1 - \alpha_2$ is a free constant which measures the strength of the quadratic contribution to the curvature effects. Variation of (84) with respect to the metric yields,

$$\delta g \tilde{S}_g = -\frac{1}{2} \int d^4 x \sqrt{-g} \delta g^{\mu \nu} T^{\text{eff}}_{\mu \nu},$$

where we have defined the following effective SET of the quadratic curvature contributions,

$$T^{\text{eff}}_{\mu \nu} = -4\hat{\beta} R \left\{ \hat{R}_{\mu \nu} - \frac{1}{4} g_{\mu \nu} \hat{R} \right\} - \frac{3}{2} \left( Q_{\mu \nu} - \frac{1}{4} g_{\mu \nu} Q \right) - 3 \left( \hat{\nabla}_{(\mu} Q_{\nu)} - \frac{1}{4} g_{\mu \nu} \hat{\nabla} Q \right) - \frac{1}{R} \left( \hat{\nabla}_{\mu} \hat{\nabla}_{\nu} - g_{\mu \nu} \hat{\nabla}^2 \right) R \right\}. $$

Meanwhile, variation of the action (84) with respect to the nonmetricity $Q_{\mu}$ reads,

$$\delta Q_g \tilde{S}_g = -6\hat{\beta} \int d^4 x \sqrt{-g} \left( R Q_{\nu} \delta Q^{\nu} + R \hat{\nabla}_{\mu} \delta Q^{\mu} \right),$$

where we took into account that

$$\delta Q R^2 = 2 R \delta Q R = 2 R \delta Q \left( \hat{R} - \frac{3}{2} Q_{\mu} Q^{\mu} - 3 \hat{\nabla}_{\mu} Q^{\mu} \right).$$

Then, variation (85) leads to the EOM

$$Q_{\mu} = \frac{\hat{\nabla}_{\mu} \hat{R}}{R} = \frac{\partial_{\mu} R}{R}.$$  

This equation fixes the nonmetricity to be a gradient so that the underlying geometric structure of background space is Weyl integrable geometry.

As in the linear case, the theory based on action $S = S_g + \tilde{S}_g$:

$$S = \int d^4 x \sqrt{-g} \left[ \frac{\phi^2}{12} R + \frac{1}{2} (\partial^* \phi)^2 + \hat{\beta} R^2 \right].$$
with EOM,

\[ \hat{\mathcal{E}}_{\mu\nu} = \frac{6}{\phi^2} T_{\mu\nu}^{\text{eff}}, \]  

(90)

plus condition (88), is a false gauge invariant theory. Actually, although the action (89) is manifestly gauge invariant, the derived EOM (90) is not invariant under (28). In a gauge invariant EOM the effective SET (86) should transform as \( T_{\mu\nu}^{\text{eff}} \rightarrow \Omega^{-2} T_{\mu\nu}^{\text{eff}} \) under (28), so that

\[ \frac{6}{\phi^2} T_{\mu\nu}^{\text{eff}} \rightarrow \frac{6}{\phi^2} T_{\mu\nu}^{\text{eff}}, \]  

(91)

is not transformed, since the LHS of Eq. (91); the tensor \( \hat{\mathcal{E}}_{\mu\nu} \), that is defined in Eq. (18) (same as Eq. (49)), is not transformed by the gauge transformations either. However this is not the case. In order to show this, it is easier to deal with the SET trace \( T^{\text{eff}} = g^{\mu\nu}T_{\mu\nu}^{\text{eff}} \), multiplied by the inverse of the gravitational coupling:

\[ \frac{6}{\phi^2} T^{\text{eff}} = -\frac{72\beta}{\phi^2} \hat{\Phi}^2 R. \]  

(91)

In a gauge invariant EOM the quantity (91) should transform like

\[ \frac{6}{\phi^2} T^{\text{eff}} \rightarrow \Omega^{-2} \frac{6}{\phi^2} T^{\text{eff}}, \]

since in the trace of (91):

\[ \hat{\mathcal{E}} = \frac{6}{\phi^2} T^{\text{eff}}, \]  

(92)

the LHS of (92) transforms like \( \hat{\mathcal{E}} \rightarrow \Omega^{-2} \hat{\mathcal{E}} \) under the gauge transformations (28). However, if take into account Eq. (12), the quantity (91) transforms in the following way:

\[ \frac{6}{\phi^2} T^{\text{eff}} \rightarrow \Omega^{-2} \left[ \frac{6}{\phi^2} T^{\text{eff}} + \frac{144 \beta}{\phi^2} (\partial_\lambda R \partial^\lambda \ln \Omega + R \hat{\nabla}^2 \ln \Omega) \right]. \]  

(93)

This means that equations where the SET trace is implied, as in (92), are not gauge invariant. Of course the same is true of the EOM (90) itself.

Although quadratic contributions of type 2 can carry ghosts problems, in the specific form (83) it can be free of ghosts. Unfortunately, in spite that the action (84) is manifestly gauge invariant, the derived Einstein’s EOM (90) is not invariant under the gauge transformations (28), so that the resulting theory is not actually gauge invariant. This might be due to the absence of kinetic terms of the form \( R_{\mu\nu}R^{\mu\nu} \) or of the form \((\partial R)^2/R\) (or their appropriate combination.) We have to recall that the addition of such terms may lead to ghosts or to non-unitarity or both (76, 77). In addition, contribution of type 2 constraints the background space to have Weyl integrable geometric structure, i. e. it is characterized by gradient nonmetricity (see Eq. (88)), \( \nabla_\alpha g_{\mu\nu} = -\partial_\alpha \ln R g_{\mu\nu} \).

VII. GRADIENT NONMETRICITY

In section (10) it was shown that, although in Weyl space \( W_4 \) the action (53) is invariant under the gauge transformations (28), the derived EOM are not gauge invariant. It is due to the fact that the decomposition \( \phi^2 R = \phi^2 (R - 3Q_\mu Q^\mu /2 - 3\hat{\nabla}_\mu Q^\mu) \) does not contain a kinetic energy density term for the scalar field \( \phi \). However, in Weyl integrable space, which we denote as \( W_4^{\text{int}} \), since the nonmetricity vector amounts to a gradient of a scalar field, we have an opportunity to improve the above issue. This can be done by lifting the gauge scalar field \( \phi \) to the category of a geometric field. In other words, we assume that the nonmetricity of WIG space \( W_4^{\text{int}} \) is given by,

\[ \nabla_\alpha g_{\mu\nu} = -2 \frac{\partial_\alpha \phi}{\phi} g_{\mu\nu} , \]  

(94)

i. e. the nonmetricity vector in Eq. (8) is a gradient \( Q_\mu = 2\partial_\mu \phi /\phi \). Under this assumption we have that \( \phi^2 R = \phi^2 \hat{R} - 6\phi \hat{\nabla}^2 \phi \) or, equivalently:

\[ \phi^2 R = \phi^2 \hat{R} + 6(\partial \phi)^2 - 6\hat{\nabla}^\mu (\phi \partial_\mu \phi) , \]  

(95)

where the last term in the RHS, within an action integral amounts to a boundary term that can be omitted. The action of gauge invariant gravity in \( W_4^{\text{int}} \) space reads

\[ S_{g}^{\text{wig}} = \frac{1}{2} \int d^4 x \sqrt{-g} \left[ \frac{\phi^2}{6} \hat{R} \right] \]

\[ = \frac{1}{2} \int d^4 x \sqrt{-g} \left[ \frac{\phi^2}{6} \hat{R} + (\partial \phi)^2 \right] . \]  

(96)

The most interesting property of the above action is that matter fields, whether massless or with the mass, can be consistently coupled to gravity without breaking the gauge symmetry. Consider the gauge invariant action over WIG space \( W_4^{\text{int}} \):

\[ S_{\text{tot}}^{\text{wig}} = \int d^4 x \sqrt{-g} \left[ \frac{\phi^2}{12} \hat{R} + \frac{1}{2} (\partial \phi)^2 + \mathcal{L}_\chi \right] , \]  

(97)

where \( \mathcal{L}_\chi \) is the Lagrangian of the matter fields collectively denoted by \( \chi \). Consistent coupling of arbitrary matter fields is possible thanks to the property that in
WIG space variation of the metric is not independent of variation of the geometric scalar field \( \phi \), since due to gradient nonmetricity law \( \delta g_{\mu\nu} = -2\frac{\delta \phi}{\phi} g_{\mu\nu}, \) one has that (see, for instance, Eq. (3) of Ref. \[74\])

\[
\delta g_{\mu\nu} = -2\frac{\delta \phi}{\phi} g_{\mu\nu}, \quad \delta g^{\mu\nu} = 2\frac{\delta \phi}{\phi} g^{\mu\nu}.
\]

This means, for instance, that variation of the overall Lagrangian in \( \mathcal{L}_{\text{tot}} \),

\[
\mathcal{L}_{\text{tot}} = \sqrt{-g} \left[ \frac{\phi^2}{12} \hat{R} + \frac{1}{2}(\partial \phi)^2 + \mathcal{L}_\chi \right],
\]

\[
\delta \mathcal{L}_{\text{tot}} = \frac{\partial \mathcal{L}_{\text{tot}}}{\partial g^{\mu\nu}} \delta g^{\mu\nu} = 2 \frac{\partial \mathcal{L}_{\text{tot}}}{\partial g^{\mu\nu}} g^{\mu\nu} \frac{\delta \phi}{\phi}.
\]

Hence, since

\[
\frac{\partial \mathcal{L}_{\text{tot}}}{\partial g^{\mu\nu}} = \sqrt{-g} \left[ \frac{\phi^2}{6} G_{\mu\nu} - T_{\mu\nu}^{(\chi)} \right]
\]

\[
= \sqrt{-g} \left[ \frac{\phi^2}{6} \hat{G}_{\mu\nu} + \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} g_{\mu\nu} (\partial \phi)^2 \right]
\]

\[
- \frac{1}{6} \left( \nabla_\mu \nabla_\nu - g_{\mu\nu} \nabla^2 \right) \phi^2 - T_{\mu\nu}^{(\chi)}, \tag{99}
\]

variation of the action \( \mathcal{L}_{\text{tot}} \) with respect to the metric yields the Einstein’s EOM,

\[
G_{\mu\nu} = \frac{6}{\phi^2} T_{\mu\nu}^{(\chi)} \iff \hat{G}_{\mu\nu} = -\frac{6}{\phi^2} \left[ \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} g_{\mu\nu} (\partial \phi)^2 \right]
\]

\[
+ \frac{1}{\phi^2} \left( \nabla_\mu \nabla_\nu - g_{\mu\nu} \nabla^2 \right) \phi^2 + \frac{6}{\phi^2} T_{\mu\nu}^{(\chi)}, \tag{101}
\]

where we have taken into account that the Einstein’s tensor of \( \hat{W}_4 \) space can be written in terms of LC (Riemannian) quantities according to

\[
G_{\mu\nu} = \hat{G}_{\mu\nu} + \frac{6}{\phi^2} \left[ \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} g_{\mu\nu} (\partial \phi)^2 \right]
\]

\[
- \frac{1}{\phi^2} \left( \nabla_\mu \nabla_\nu - g_{\mu\nu} \nabla^2 \right) \phi^2.
\]

Meanwhile, variation of \( \mathcal{L}_{\text{tot}} \) with respect to the geometric scalar field \( \phi \) leads to:

\[
-\hat{R} = -\hat{R} - 6 \frac{(\partial \phi)^2}{\phi^2} + 3 \frac{\nabla^2 \phi^2}{\phi^2} = \frac{6}{\phi^2} T^{(\chi)}, \tag{102}
\]

which coincides with the trace of the Einstein’s EOM \( \hat{R} \). In consequence, the geometric gauge scalar \( \phi \) is not a dynamical field: it can be chosen at will. Different choices lead to different gauges. We shall discuss the physical consequences of the theory \( \hat{R} \) in a forthcoming publication.

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**VIII. GAUGE FREEDOM: THE MANY-WORLDS INTERPRETATION**

In this section, in addition to the gravitational action \( S_{\text{grav}} \) – which is equivalent to \( S_{\text{rad}} \) – we shall consider a radiation piece of action \( S_{\text{rad}} = \int d^3 x \sqrt{-g} \mathcal{L}_{\text{rad}} \), where \( \mathcal{L}_{\text{rad}} \) is the Lagrangian for radiation. In consequence in the RHS of the Einstein’s EOM \( \hat{G}_{\mu\nu} = 0 \) a term \( 6 T_{\mu\nu}^{(\chi)} / \phi^2 \) is to be added, where the trace of the stress-energy tensor of radiation \( T_{\mu\nu}^{(\text{rad})} \equiv g_{\mu\nu} T_{\mu\nu}^{(\chi)} \) vanishes. The reason why we consider radiation exclusively is because only radiation can be consistently coupled in this theory.

Due to vanishing SET trace in the present theory, the gravitational coupling coincides with the measured Newton’s constant \( G_N = 6/8\pi \phi^2 \). The procedure that leads to computation of the measured gravitational constant is based on the weak-field limit of the theory.\(^{10}\) Since for trace-free matter there is no contribution from the coupling parameter, which is the factor of the kinetic energy term of the BD scalar field, we get a sourceless wave equation for the perturbation of the scalar field \( \nabla^2 \phi = 0 \). For this reason the measured Newton’s constant coincides with the gravitational coupling in the present gauge invariant theory. Another way to understand this is by noting that the gravitational coupling is associated with the tensor part of the gravitational interactions, which is the only type of gravitational interaction in this theory since the scalar field is a gauge field.

In the remainder of this section we shall put forth an alternative interpretation of gauge invariance in gravitation, but first let us discuss on the mathematical and physical implications of gauge invariance. Invariance under the gauge transformations \( \phi \) of the theory based in action

\[
S_{\text{tot}} = \int d^3 x \sqrt{-g} \left[ \frac{\phi^2}{12} \hat{R} + \frac{1}{2}(\partial \phi)^2 \right] + S_{\text{rad}}, \tag{103}
\]

and of the derived EOMs:

\[
\hat{\mathcal{E}}_{\mu\nu} = \frac{6}{\phi^2} T_{\mu\nu}^{\text{rad}}, \quad \hat{\mathcal{E}} = 0,
\]

or, if consider the definitions of \( \hat{\mathcal{E}}_{\mu\nu} \) in Eq. \( \text{(49)} \) and of its trace \( \hat{\mathcal{E}} \equiv g_{\mu\nu} \hat{\mathcal{E}}_{\mu\nu} \):

\[
\hat{\mathcal{G}}_{\mu\nu} + \frac{6}{\phi^2} \left[ \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} g_{\mu\nu} (\partial \phi)^2 \right]
\]

\[
- \frac{1}{\phi^2} \left( \nabla_\mu \nabla_\nu - g_{\mu\nu} \nabla^2 \right) \phi^2 = \frac{6}{\phi^2} T_{\mu\nu}^{\text{rad}},
\]

\[
\nabla^2 \phi^2 - 2(\partial \phi)^2 - \frac{2}{\phi^2} \hat{\mathcal{E}} = 0, \tag{104}
\]

---

\(^{10}\) For Brans-Dicke theory this procedure is explained, for instance, in chapter 2.3 of \[53\] (see also section 3.3 of \[83\]).
means that, in addition to the four degrees of freedom to make diffeomorphisms, there is an additional degree of freedom to make conformal transformations. This is reflected in that the scalar field EOM in Eq. (104) is not an independent equation since it coincides with the trace of the Einstein’s EOM (recall that the SET of radiation is traceless.) This means that the scalar field $\phi$ (or any one of the other field degrees of freedom) may be fixed at will. In other words, $\phi$ can be set equal to any nonvanishing (continuous) function $\phi = \phi(t, \vec{x})$, or to any constant we want $\phi = \phi_0$, without conflict with the EOMs (104).

Gauge symmetry of the theory (103), (104) means that, instead of just one given theory we have a whole conformal equivalence class of theories or “gauges.” Any choice of the scalar function $\phi$ – also called as gauge choice – leads to a specific representation of the gravitational laws, i. e. to a specific theory. A clear indication that a specific choice $\phi$ picks out a theory of gravity, which is distinguished from any other theory in the conformal equivalence class, is the fact that the measured gravitational constant $G_N = 3\phi^{-2}/4\pi$, so that the gauge choice is subject to experimental check.

Any given gauge is related with any other possible gauge in the conformal equivalence class through Weyl rescalings (28). Consider two different gauges of the theory (103) plus derived EOM (104),

$$\mathcal{G}_i : \{\mathcal{M}_4, g_{\mu\nu}(i), \phi_{(i)}\}, \mathcal{M}_4 \in V_4,$$

$$\mathcal{G}_j : \{\mathcal{M}_4, g_{\mu\nu}(j), \phi_{(j)}\}, \mathcal{M}_4 \in V_4. \quad (105)$$

The fact that only Riemannian manifolds $\mathcal{M}_4 \in V_4$ are considered, is a consequence of our finding that, if require gauge symmetry to be an underlying symmetry of our theoretical framework, vectorial nonmetricity $Q_{\mu}$ does not couple to the background geometry. Hence, even if start with a Weyl manifold $\mathcal{M}_4 \in \tilde{W}_4$, which is distinguished from $V_4$ precisely by $Q_{\mu}$, we can dispense with the nonmetricity field so that we can safely make the replacement: $\tilde{W}_4 \rightarrow V_4$. The arbitrary gauges $\mathcal{G}_i$ and $\mathcal{G}_j$ are linked by the gauge transformations (28):

$$g_{\mu\nu}^{(i)} \rightarrow \Omega^2 g_{\mu\nu}^{(j)}, \phi^{(i)} \rightarrow \Omega^{-1} \phi^{(j)}. \quad (106)$$

A. General relativity gauge

If in the action (103) make the following replacement $\phi \rightarrow \sqrt{6}M_{\text{pl}}$, where $M_{\text{pl}}$ is the Planck mass, one obtains

$$S_{\text{tot}} = \frac{1}{2} \int d^4x \sqrt{-g}M_{\text{pl}}^2 R + S_{\text{rad}}. \quad (107)$$

This is just the GR action over Riemann $V_4$ space. Alternatively, the action (107) can be obtained from (103) through the gauge transformations (108) with the following choice: $\phi^{(i)} = \phi$, $\phi^{(j)} = \sqrt{6}M_{\text{pl}}$, i. e. through

$$g_{\mu\nu} \rightarrow \Omega^2 g_{\mu\nu}, \phi \rightarrow \Omega^{-1}\sqrt{6}M_{\text{pl}}. \quad (108)$$

The obtained representation $\mathcal{G}_i = \mathcal{G}_{\text{GR}}$ is called as GR gauge. In this specific gauge the manifest gauge symmetry of the theory (103) plus derived EOM, is lost. The inverse of the gauge transformation (108) that links the GR gauge with any other arbitrary gauge $\mathcal{G} : \{M_4, g_{\mu\nu}, \phi\}$, transforms the GR action (107) back into (103).

Although GR itself is clearly not a gauge invariant theory of gravity, in the present framework it is no more than one of the infinitely many equivalent gauges in the conformal equivalence class

$$\mathcal{K}_{\text{conf}} : \{\mathcal{G}_0 \equiv \mathcal{G}_{\text{GR}}, \mathcal{G}_1, \mathcal{G}_2, ..., \mathcal{G}_k, ..., \mathcal{G}_N \mid k \in \mathbb{N}\}, \quad (109)$$

where $N \rightarrow \infty$. Hence, since gauge invariance is the underlying symmetry behind the class $\mathcal{K}_{\text{conf}}$, GR is part of a bigger gauge invariant theory.

B. Many-worlds interpretation

Despite obvious differences, there is a certain resemblance between the present picture and the “many-worlds” interpretation of quantum physics \[86–95\], since different gauges represent different theories, yielding different potential descriptions of the gravitational laws.

Let us illustrate the above statement in the simplest case: the GR gauge. Since, in order to fix the GR gauge, the choice of a constant value of the scalar field $\phi$ is arbitrary, depending of the chosen constant value of the scalar field, one has (in principle) an infinite set of GR copies with different values of the Planck mass, of the masses of the SMP fields and of the cosmological constant, among others. Recall that there are certain constants, such as the Planck constant $\hbar$, the speed of light $c$, the electron charge $e$, etc. which are not affected by the gauge transformations (28), so that these are the same constants in all gauges. Hence the GR gauge is a subclass within the conformal class $\mathcal{K}_{\text{conf}}$, which comprises an infinite number of GR copies:

$$\mathcal{G}_{\text{GR}} = \{\mathcal{G}_{0}^{k}, \mathcal{G}_{1}^{k}, \mathcal{G}_{2}^{k}, ..., \mathcal{G}_{N}^{k} \}, \quad (110)$$

where $k = 0, 1, 2, 3, ..., N \ (N \rightarrow \infty)$ and the general element of the GR gauge can be expressed as

$$\mathcal{G}_{\text{GR}}^{k} = \{M_4, g_{\mu\nu}, \phi_{0k}\}, \mathcal{M}_4 \in V_4. \quad (111)$$

The different constants $\phi_{0k} \in \mathbb{R}$ generate different sets of physical constants: $\{M_{\text{pl},k}^2, \nu_k, ..., \}$, where $M_{\text{pl},k} = \phi_{0k}/\sqrt{6}$ is the Planck mass in the $k$-th element of the GR gauge, $\nu_k = \nu_0\phi_{0k}$ is the corresponding mass parameter of the SMP ($\nu_0$ is a dimensionless constant) and the
ellipsis stand for other possible physical constants which are sensible to the choice of gauge such as, for instance, the cosmological constant. Actually, if in the Lagrangian one replaces the Higgs self-interacting potential by

\[ V(|H|) \propto \left( |H|^2 - \nu_0^2 \phi_0^2 \right)^2, \]

the mass parameter that appears in the expressions for the masses acquired by the different fields after EW symmetry breaking is \( m_{\nu, k} \propto \nu_0 \phi_0 k \), where \( m_{\nu, k} \) is the mass of given SMP field \( \chi \) in the \( k \)-th representation or gauge. Besides, if in (74) add a gauge invariant term \( \sqrt{-g} \lambda \phi^4 / 48 \), where \( \lambda \) is a dimensionless coupling constant,

\[ S_g = \frac{1}{2} \int d^4x \sqrt{-g} \left[ \frac{\phi^2}{6} \hat{R} + (\phi)^2 - \frac{\lambda}{4!} \phi^4 \right], \]

then in the \( k \)-th element of the GR gauge the following cosmological constant arises: \( \Lambda_k = 3 \lambda M_{pl,k}^2 / 4 \). As we have explained, the GR gauge consists of \( N \) copies of GR theory, where each copy has its own set of physical constants:

\[ \{ M_{pl,k}, \nu_k, \Lambda_k, h, c, e, \ldots \}. \]

This classic gravitational version of the many-worlds interpretation of quantum physics is interesting because it provides a different perspective on the relation between theory and experiment. Usually experiment is useful in order to corroborate the theoretical predictions made on the basis of given theoretical framework. In the present case – a gauge invariant theory of gravity – experiment allows to determine which one of the infinitely many gauges is the one which better describes our Universe through associating experimental values to the constants of Nature.

We underline that the above discussed illustration is no more than a very simple example which is not intended to be a potential model of our Universe. Among an infinity of possibilities where \( \phi \) can be any spacetime function, we have chosen a specific class: the class consisting of different GR copies comprised in the GR gauge, where the scalar field is constant. Besides, up to now we have considered that only radiation is coupled to gravity. It is a well-known fact that coupling matter fields with nonvanishing mass in this theory is not possible (see appendix C): The only way to couple timelike fields to gravitation in this framework is by breaking gauge symmetry.

**IX. DISCUSSION**

In this section we shall discuss our previous results:

1. There are theories of gravity which carry fake gauge symmetry since, although the action of the theory is invariant under the gauge transformations (28), the derived equations of motion are not invariant under these transformations.

2. Gauge invariant theories of gravity with vectorial nonmetricity are trivial gravitational theories, i.e. nonmetricity plays no role so that the gravitational effects may be associated with the curvature of Riemann space \( V_4 \) exclusively, and we shall confront them with other known results in the bibliography. But first, let us make a comment on the second result above.

Triviality of vectorial nonmetricity in gauge invariant theories of gravity can be explained in the following way. According to corollary 3.1 in these theories nonmetricity does not couple to gravity so that it must be ignored (see appendix C for a similar demonstration for the massless fields of the SMP.) Only massless fields and radiation can be coupled to gravity if gauge symmetry is to be preserved. But massless fields and radiation follow null geodesics of Riemann space where nonmetricity plays no role. Hence, in a theory of gravity and of the SMP with \( SU(3), SU(2) \times U(1) \) and gauge symmetries, only the Riemann structure of background space matters. Meanwhile, after breakdown of these symmetries, the gauge symmetric structure of Weyl space with vectorial nonmetricity is not required at all.

**A. Fake gauge invariant theories of gravity**

We have shown in section [IV] that the theory based in (53) is not actually gauge invariant even if the action itself is invariant under (28). This happens because either a kinetic energy density term for the scalar field \( \phi \) is lacking or it appears with the incorrect numeric factor. Theories of gravity with this property fall within the class of “fake gauge invariant” theories. In this class belong, for instance, the theory of spacetime structure at short distances explored in [7], the gauge invariant gravity theory with the inclusion of the fields of the SMP explored in [8, 9], which will be analyzed separately, and also the theoretical framework developed in Refs. [53–56]. All of these are theories built over Weyl geometric background spaces \( W_4 \), which are characterized by vectorial nonmetricity \( \nabla_{\mu} g_{\mu \nu} = -Q_\mu g_{\mu \nu} \).

In Ref. [7] the following gravitational Lagrangian is investigated (in this bibliographic reference the Weyl vector is represented by \( b_\mu = -Q_\mu \) and difference in the signs is due to a different choice of the metric signature):

\[ \mathcal{L}_{sh} = \sqrt{-g} \left[ \frac{c}{2} \phi^2 R + \frac{1}{2} (\partial^\mu \phi)^2 - \frac{1}{4g^2} Q^2 + \cdots \right], \tag{112} \]

where, as defined in Eq. (32), \( Q_{\mu \nu} = \partial_\mu Q_\nu - \partial_\nu Q_\mu \) and we have introduced the shorthand notation \( Q^2 = Q_{\mu \nu} Q^{\mu \nu} \), while \( c \) and \( g^2 \) are free constants. In the above equation...
the ellipsis stand for terms quadratic in the curvature that are dropped since they induce unphysical poles in the graviton propagator [75] and, besides, they do not contribute to the low-energy phenomenology. This theory is intended to understand the short-distance behavior of gravity. The problem is that, although the Lagrangian (112) is gauge invariant, as shown in section [115], the derived EOM are not gauge invariant, unless \( c = 1/6 \). But, in this latter case, after applying the procedure undertaken in [76], the particle spectrum differs from the one obtained in this reference. In particular, after gauge symmetry breaking the Weyl vector \( Q_\mu \), remains massless, contrary to [7] where the mass of the Weyl vector field \( \phi \) is intended to understand the short-distance behavior of gravity. Theory of gravity, which includes the derived equations of motion, is not gauge invariant as incorrectly assumed in [53-56]. But, in this case, the gauge vector \( Q_\mu \) disappears from the Lagrangian (115), the resulting Lagrangian for massless fermions and bosons, respectively, while:

\[
\mathcal{L}_{\text{ferm}} = \sqrt{-g} \left[ \frac{\beta}{2} |h|^2 R + |D^\mu h|^2 \right],
\]

where \( \beta \) is a coupling constant and \( h \) is the (isodoublet) Higgs field. In this Lagrangian we have introduced the following notation: \( |h|^2 \equiv \bar{h} h \), \( |D^\mu h|^2 \equiv g^{\mu\nu} (D^\mu h)(D^\nu h) \) and

\[
D^\mu h \equiv \left( \partial^\mu + i g W^i_\mu \sigma^i + i g' B_\mu \right) h,
\]

where, as before, \( W^i_\mu, B_\mu \) are the \( SU(2) \) and \( U(1) \) gauge bosons, respectively, \( \sigma^i \) are the Pauli matrices and the gauge derivative – Eq. (A1) in appendix A – amounts to \( \partial^\mu h = (\partial_\mu - Q_\mu / 2) h \).

Through using the decomposition of the generalized curvature scalar \( R \) into its Riemannian components [54]:

\[ R = \tilde{R} - 3Q_\Lambda Q^\Lambda / 2 - 3\nabla_\lambda Q^\lambda \]

and explicitly developing the Higgs field kinetic energy density term, the Lagrangian (116) can be written in the following way:

\[
\mathcal{L}_{\text{ferm}} = \sqrt{-g} \left[ \frac{\beta}{2} |h|^2 R + |\partial h|^2 + \frac{1}{2} - \frac{3\beta}{4} |h|^2 Q_\Lambda Q^\Lambda \right.
\]

\[ + \left. \frac{1}{2} |h|^2 \nabla_\lambda Q^\lambda \right],
\]

where we have omitted the terms containing the \( SU(2) \) and \( U(1) \) gauge bosons, which are not transformed by the gauge transformations (28). In section [54] it has been demonstrated that the Einstein’s EOM derived from this Lagrangian (same as Lagrangian (114) of that section if we make the replacements \( \varphi \to h \), \( \varphi^\dagger \to h^\dagger \)) are gauge invariant only if \( \beta = 1/3 \). Hence, it is confirmed that the only possible gauge invariant combination is given by the unit \( |h|^2 \tilde{R} + 6|\partial h|^2 \). But, in this case, the gauge vector \( Q_\mu \) disappears from the Lagrangian density, i. e. the nonmetricity can be safely ignored from start. The same conclusion applies to the Lagrangians \( \mathcal{L}_{\text{ferm}} \) and \( \mathcal{L}_{\text{bos}} \) as shown in appendix C.

B. A critique to nonmetricity as dark matter in the Universe

In reference [8] (see also the subsequent [9]) the author proposed a gauge invariant gravitational theory with the inclusion of \( SU(2) \times U(1) \) symmetric EW interactions in Weyl space \( \tilde{W}_4 \) with vectorial nonmetricity \( Q_\mu \) (represented by the vector \( S_\mu \) in the mentioned reference.) The overall Lagrangian of this model can be written in the following way (our notation slightly differs from the one undertaken in [8, 9]):

\[
\mathcal{L}_{\text{tot}} = \mathcal{L}_{\tilde{W}-h} + \mathcal{L}_{\text{ferm}} + \mathcal{L}_{\text{bos}},
\]

where \( \mathcal{L}_{\text{ferm}} \) given by Eq. (C7) and \( \mathcal{L}_{\text{bos}} \) given by Eq. (C10), are the \( SU(2) \times U(1) \) and Weyl gauge symmetric Lagrangians for massless fermions and bosons, respectively.
gauge symmetry plays a role. In conclusion the Weyl gauge vector $Q_μ$ can not describe the dark matter in the universe as claimed in [8, 9].

Besides, if as argued in [8], the coupling constant $β^{-1} \approx 10^{-32}$ in order to explain the weakness of the gravitational interactions, then $β \neq 1/3$ so that the theory developed in the mentioned bibliographic references also belongs in the class of the “fake gauge invariant” theories of gravity and of the SMP. In this case break down of gauge symmetry can not happen because the resulting theory is not gauge invariant.

C. Quadratic gauge invariant theory of gravity

In the bibliographic references [110–113] a conformal invariant quadratic theory of gravity, which operates in Riemann space $V_4$, has been proposed to solve several problems, including a possible explanation to the dark matter issue. The proposed conformal invariant action reads,

$$S_W = \alpha \int d^4x \sqrt{-g} \hat{C}^2 = 2\alpha \int d^4x \sqrt{-g} \left[ \hat{R}_{\mu\nu} \hat{R}^{\mu\nu} - \frac{1}{3} \hat{R}^2 \right],$$

(118)

where $\alpha$ is a dimensionless constant, $\hat{C}^2 \equiv \hat{C}_{\mu\lambda\nu\sigma} \hat{C}^{\mu\lambda\nu\sigma}$, $\hat{C}_{\mu\lambda\nu\sigma}$ is the Weyl tensor of Riemann space and we have used the Gauss-Bonnet invariant to trade the $\hat{C}^2$-term by the terms within square brackets in (118).

This theory has many problems. For instance, it does not have the Einstein-Hilbert (low-curvature) limit. Besides, the derived equations of motion are traceless, so that only traceless matter fields can be consistently coupled. In a cosmological context, for instance, this theory could describe the radiation dominated epoch of the cosmic evolution exclusively. But the matter dominated stage, where the formation of cosmic structure happens and where the dark matter and the dark energy play the most important part, requires of a different theory that should replace (118). Hence, dark matter can not be explained in the present setup as incorrectly claimed in [111–113]. Yet, the most serious drawback of this theory is related with its perturbation spectrum. According to Appendix A (see also [114]), to lowest order, fluctuations of the metric average flat space in a theory with quadratic action of the form:

$$S_{st} = \frac{1}{2} \int d^4x \sqrt{-g} \left[ \hat{R} + \alpha \hat{R}^2 + \beta \hat{R}_{\mu\nu} \hat{R}^{\mu\nu} + \gamma \hat{R}_{\lambda\mu\sigma\tau} \hat{R}^{\lambda\mu\sigma\tau} \right],$$

(119)

where $\alpha$, $\beta$ and $\gamma$ are dimensionless constants, which can be brought to the form:

$$S_{st} = \frac{1}{2} \int d^4x \sqrt{-g} \left[ \hat{R} + \frac{1}{6m_0^2} \hat{R}^2 - \frac{1}{2m_0^2} \hat{C}^2 \right],$$

(120)

in addition to the graviton, the perturbations spectrum contains a scalar field with square mass $m_0^2 = 6\alpha + 2\beta + 2\gamma$ and a spin-two field with square mass $m_2^2 = -\beta - 4\gamma$. This quadratic theory is renormalizable but nonunitary. The theory (118), in contrast, contains only the ghost-like spin-two field and has no graviton in its spectrum. This rules out this theory as a phenomenologically viable description of low curvature gravitational phenomena.

D. Observational signatures of gauge invariant theory with vectorial nonmetricity

Despite of the demonstrated fact that there is not much room for vectorial nonmetricity to have an impact in the classical laws of gravitation, in this subsection we shall assume that indeed nonmetricity plays a role in the classical description of gravity. No matter whether this assumption is correct or not, here we shall look for the phenomenological consequences of such a possibility.

In general relativity when two identical clocks, initially synchronized, are parallel transported along different paths, a certain loss of synchronization arises that is called as the “first clock effect.” In Weyl spacetimes $W_4$, where the vectorial nonmetricity condition (8) is satisfied, an additional effect arises: the two clocks not only have lost their initial synchronization, but, they go at different rates. It is known as the “second clock effect” [3, 10, 13, 19, 22, 23]. The SCE causes a serious observational issue: an unobserved broadening of spectral lines. This issue was enough to reject the original Weyl’s gauge invariant gravitational theory and its related geometrical framework [1, 13]. In spite of this, several recent papers have put into discussion the occurrence of the second clock effect [19, 22, 24, 25].

The mathematical basis for the SCE is given by (E6) of Appendix B:

$$m(x, \mathcal{C}) = m_0 \exp \left( \frac{1}{2} \int_0^{\infty} Q_\mu w^\mu d\tau \right),$$

(121)

where $\mathcal{C}$ is the parallel transport path joining the origin $x_0 = \{0, \vec{0}\}$ with an arbitrary point with coordinates $x^\mu$: $x = \{t, \vec{x}\}$, $w^\mu = dx^\mu/d\tau$ is the four-velocity, $\tau$ is an affine parameter along the path and the integration constant $m_0 = m(0)$.

Let us base the physical analysis of the SCE on the functioning of an atomic clock which, for definiteness and simplicity, we shall assume is made of hydrogen atoms. The principle of operation of such an atomic clock is based on atomic physics: it measures the electromagnetic signal that electrons in the hydrogen atoms emit when they change energy levels. For instance, the energy
of each energy level in the hydrogen atom, labeled by \( n \), is given by: 
\[ E_n = -\frac{\alpha^2}{2n^2}, \]
where \( m \) is the mass of the electron and \( \alpha \approx 1/137 \) is the fine-structure constant. Any changes in the mass \( m \) over spacetime will cause changes in the energy levels and, consequently, in the energy of the atomic transitions
\[ \nu_{ij} = |E_{n_j} - E_{n_i}| = \frac{m\alpha^2}{2} \left( \frac{1}{n_j^2} - \frac{1}{n_i^2} \right). \]

Hence, the functioning of atomic clocks will be affected by the variation of masses over spacetime, according to equation (121).

Let us consider a collection of identical hydrogen atoms that are parallel transported along neighboring paths from the origin \( x_0 = (0, 0) \) to a given point \( x \). Let us take the larger difference arising between the masses of any two atoms in the collection at \( x \): \( \Delta m = m(x) - \bar{m}(x) \). Then, according to (121) one gets the following gauge invariant ratio:
\[ \frac{\Delta \nu_{ij}}{\nu_{ij}} = \frac{\Delta m}{m} = 1 - \exp \left( \frac{1}{2} \right) \left( \int_{\mathcal{C}} Q_{\mu} dx^\mu - \int_{\mathcal{C}} Q_{\mu} dx^\mu \right) \]
where \( \Delta \nu_{ij} \) quantifies the broadening of a given spectral line. The spectral line is sharp: \( \Delta \nu_{ij} = 0 \), only if either \( \mathcal{C} = \mathcal{C}_0 \), or if \( Q_{\mu} = \partial_{\mu} \varphi \), where \( \varphi \) is the Weyl gauge scalar. In this last case \( \int_{\mathcal{C}} \partial_{\mu} \varphi dx^\mu = \varphi(x) - \varphi(0) \), independent of the path joining the starting and final points. WIG is the resulting geometric structure. Hence, only for gradient nonmetricity the second clock effect does not arise. The SCE alone is sufficient to reject theories with vectorial nonmetricity as phenomenologically nonviable descriptions of our classical world.

It has been stated [8] that vanishing of the quantity \( Q_{\mu\nu} = 2\partial_{[\mu}Q_{\nu]} = 0 \), is the necessary and sufficient condition for the given vector (four-momentum in particular) to return to itself after parallel transport in a closed trajectory in \( W_4 \) space. Actually, due to the Stokes theorem:
\[ \oint_{\mathcal{C}} Q_{\mu} dx^\mu = \frac{1}{2} \int_{S} Q_{\mu\nu} dx^\mu \wedge dx^\nu, \]

where \( Q_{\mu\nu} \) is defined by Eq. (32) and \( \mathcal{C} \) is the boundary of the oriented surface \( S \), if \( Q_{\mu\nu} = 0 \), then \( \oint_{\mathcal{C}} Q_{\mu} dx^\mu = 0 \). Nevertheless, the demonstration that the SCE does not arise based in the argument that Eq. (123) vanishes when \( \partial_{\mu}Q_{\nu} = 0 \) [19, 24], would be incorrect in general [23].

As a matter of fact, the above conclusion would require the given body – in our case an hydrogen atomic clock – to be submitted to parallel transport in a closed timelike trajectory \( \mathcal{C} \) in spacetime. In general, closed timelike curves (CTCs) in spacetime carry causality issues and timelike worldlines of observers with clocks – aimed at the check of the SCE – are not the exception.

In this regard we should differentiate the timelike worldlines \( \mathcal{C}_{\text{open}} \) with coordinates \( x^\mu(\xi) \), where \( \xi \) is an affine parameter along the worldline, which start and end up at a same spatial point (different values of the time coordinate):
\[ x^0(\xi_{\text{start}}) \neq x^0(\xi_{\text{end}}), \]
\[ x^i(\xi_{\text{start}}) = x^i(\xi_{\text{end}}) \Rightarrow x^\mu(\xi_{\text{start}}) \neq x^\mu(\xi_{\text{end}}), \]

from those worldlines \( \mathcal{C}_{\text{closed}} \), which start and end up at the same spacetime point:
\[ x^\mu(\xi_{\text{start}}) = x^\mu(\xi_{\text{end}}) \Rightarrow x^i(\xi_{\text{start}}) = x^i(\xi_{\text{end}}). \]

While timelike worldlines of type \( \mathcal{C}_{\text{open}} \) can be associated with real classical motions, timelike worldlines of type \( \mathcal{C}_{\text{closed}} \) are CTCs which are plagued by causality issues as long as a CTC represents time travel [97–102].

Closed curves of this latter type are the ones that are considered in Eq. (123).

### E. Quantum considerations

The above causality argument that is based in the existence of CTCs, may not be correct in the quantum domain. Hence, the impact of vector nonmetricity in quantum gravitational phenomena may not be underestimated. Following a reasoning line similar to that of the pioneering work [2], one can make the replacement \( Q_{\mu} \rightarrow 2iq_{\mu} \) in Eq. (123), where \( i \) is the imaginary unit. Then one gets that the following quantization requirement,
\[ \oint_{\mathcal{C}} q_{\mu} dx^\mu = \frac{1}{2} \int_{S} f_{\mu\nu} dx^\mu \wedge dx^\nu = 2\pi n, \]

where \( f_{\mu\nu} \equiv \partial_{\mu}q_{\nu} - \partial_{\nu}q_{\mu} \) is the field strength associated with the quantum nonmetricity vector \( q_{\mu} \) and \( n \) is an integer, avoids the occurrence of the SCE. Hence, the phenomenological viability of vectorial nonmetricity in the quantum world amounts to quantization of the flux of the vector \( q_{\mu} \) through the surface \( S \) in spacetime. This is similar to the phenomenon of magnetic flux quantization.

Similar argument may be applied to arbitrary nonmetricity \( Q_{\alpha\mu\nu} \),
\[ Q_{\alpha\mu\nu} \rightarrow iq_{\alpha\mu\nu}, \]
where the above field strength $f_{\mu\nu}$ is to be replaced as it follows:

$$f_{\mu\nu} \rightarrow (\nabla_{\alpha} q_{\mu\alpha\beta} - \nabla_{\nu} q_{\mu\alpha\beta}) u^\alpha u^\beta.$$ 

Hence, the impact of arbitrary nonmetricity in quantum gravitational phenomena may not be excluded.

\section{Conclusion}

Nonmetricity theories have received renewed interest within the gravitational community due to the cosmological applications of the symmetric teleparallel theories of gravity \cite{32, 34, 36, 37}. However, one of the most important properties of nonmetricity geometry: gauge invariance, has not been investigated with the same interest. It has been known since the first theory of this kind was published by Weyl \cite{1, 13}, that gauge symmetry is the distinctive feature of nonmetricity. This is not modified by considering the teleparallel condition that $R_{\mu\nu\rho\sigma} = 0$.

The argument many authors make to justify not considering gauge symmetry is that one has the freedom to choose the action of the theory, which is the one that defines the underlying symmetries. Others base their work on the dynamical equivalence existing between GR and (pure nonmetricity) symmetric teleparallel equivalent of GR (STEGR) \cite{45}. Then, since GR is not a Weyl gauge invariant theory, the STEGR should not possess gauge symmetry. The question is, why a theory where gravity has geometric nature should not respect the underlying symmetry of the geometrical background? Weyl answered to this question by choosing an action for his theory which satisfied gauge symmetry.

In this paper we have followed a similar approach by looking for gravitational theories whose action possesses the same symmetries of the $W_4$ geometric background. We have considered also quadratic curvature contributions. Our results can be summarized as it follows:

1. There are theories of gravity whose action is gauge invariant but the derived equations of motion are not invariant under the gauge transformations. These “fake gauge invariant” theories do not really possess gauge symmetry.

2. Gauge invariant theories of gravity over Weyl space $W_4$ which are linear in the curvature scalar, are the same as gauge invariant theories of gravity over Riemann space $V_4$. In other words, vectorial nonmetricity does not couple to gravitation if gauge symmetry is required. Addition of quadratic curvature terms of types \cite{77} and \cite{82} does not modify this result.

3. Only massless fields and radiation, which follow null geodesics of Riemann space, can be consistently coupled to the above gauge invariant theory of gravity.

The last two results above can be comprised in the following statement: In a gauge invariant theoretical framework gravitation and massless fields do not interact with vectorial nonmetricity, so that we may dispense with Weyl space $W_4$.

These results complete a consistent gauge invariant picture, where nonmetricity may be ignored and massless fields of the SMP interact with the LC curvature of Riemann space exclusively. This consistency can be demonstrated by noting that, under the gauge transformations \cite{28}, $\nabla_{\alpha} q_{\mu\nu} \rightarrow \Omega^2 \nabla_{\alpha} q_{\mu\nu}$, so that the Riemann metricity condition $\nabla_{\alpha} q_{\mu\nu} = 0$ is not modified by the gauge transformations. Besides, as shown in appendix D of Ref. \cite{90} (see the appendix \cite{B} below,) the null geodesics of Riemann space $V_4$ are gauge invariant as well. This means that gauge invariance is compatible with Riemann geometry as long as only massless fields and radiation propagate in $V_4$.

In a hypothetical situation where vectorial nonmetricity had a certain impact in the classical gravitational interactions, due to the SCE this theory was phenomenologically ruled out. Notwithstanding, it may have led its footprints in the quantum era. As we have shown, there is only a possibility for gradient nonmetricity to play a role in the classical description of the gravitational interactions. In a separate publication we investigate this possibility and its phenomenological consequences.

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\section*{Appendix A: Gauge symmetry and parallel transport}

Parallel transport consistent with gauge symmetry is required to define gauge covariant differentiation of vectors and tensors in generalized Weyl spaces. Below we shall expose the theory of gauge invariant parallel transport, which is cornerstone to discuss, among others, on the second clock effect. Although our exposition contains new elements not previously considered, to a great extent it is based in the work of references \cite{3, 6}.

\begin{footnotesize}
\begin{itemize}
\item[$12$] Particles with mass break gauge invariance in Riemann $V_4$ space since timelike geodesics are not gauge invariant equations. Actually, under \cite{28}, the timelike geodesic equation in $V_4$ space:
\begin{equation}
\frac{d^2 x^\alpha}{ds^2} + \{\frac{\alpha}{\mu}\} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} = 0,
\end{equation}
transforms into,
\begin{equation}
\frac{d^2 x^\alpha}{ds^2} + \{\frac{\alpha}{\mu}\} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} = h^{\alpha\mu} \partial_\mu \ln \Omega,
\end{equation}
where, as defined in Eq. \cite{B3} of the appendix, $h^{\alpha\mu}$ is the orthogonal projection tensor. The RHS of the above equation may be understood as a fifth force.
\end{itemize}
\end{footnotesize}
1. Gauge derivative operators

In order to make the gauge symmetry compatible with well-known derivation rules and with the inclusion of fields into $W_4$, it is necessary to introduce the Weyl gauge derivative operators in a way that is equivalent to the one appearing in [3, 4, 6]. Let $T$ be a $(p, q)$-tensor in $\tilde{W}_4$, with coordinate components $T_{\beta_1\beta_2\cdots\beta_q}^{\alpha_1\alpha_2\cdots\alpha_p}$, and with conformal weight $w(T) = w$, so that under (28): $T \rightarrow \Omega^w T$. Then, the Weyl gauge differential of the tensor, its Weyl gauge derivative and Weyl gauge covariant derivative, respectively, are defined as it follows:

$$
\begin{align*}
&d^*T := dT + \frac{w}{2} Q \, dx^\lambda T, \\
&\partial^*_\alpha T := \partial_\alpha T + \frac{w}{2} Q_\alpha T, \\
&\nabla^*_\alpha := \nabla_\alpha + \frac{w}{2} Q_\alpha,
\end{align*}
$$

(1A)

where $d^*T = d^\mu \partial^*_\mu T$. These definitions warrant that the gauge differential, the gauge derivative and the gauge covariant derivative, transform like the geometrical object itself, i.e., under the gauge transformations (28):

$$
\begin{align*}
&d^*T \rightarrow \Omega^w d^*T, \quad \partial^*_\alpha T \rightarrow \Omega^w \partial^*_\alpha T, \\
&\nabla^*_\alpha T \rightarrow \Omega^w \nabla^*_\alpha T, \quad \nabla^*_\alpha \Omega \rightarrow \Omega^w \nabla^*_\alpha \Omega.
\end{align*}
$$

2. Parallel transport in $\tilde{W}_4$ space

Let $C$ be a curve in $\tilde{W}_4$ that is parametrized by the affine parameter $\xi$. I.e., $C$ has coordinates $x^\mu(\xi)$. We can define the gauge covariant derivative along the path $x^\mu(\xi)$ to be given by the following operator:

$$
\frac{D^*}{d\xi} := \frac{dx^\mu}{d\xi} \nabla^*_\mu,
$$

(2A)

where the gauge covariant derivative $\nabla^*_\mu$ is given by (A1). Then, the parallel transport of given tensor $T$ with coordinate components $T_{\beta_1\beta_2\cdots\beta_q}^{\alpha_1\alpha_2\cdots\alpha_p}$, along the path $x^\mu(\xi)$, is defined by the following requirement (this definition coincides with the one in [3, 4]):

$$
\frac{D^*T}{d\xi} := \frac{dx^\mu}{d\xi} \nabla^*_\mu T \Rightarrow D^* \quad \Rightarrow D^*_{\beta_1\beta_2\cdots\beta_q}^{\alpha_1\alpha_2\cdots\alpha_p} = 0.
$$

(A3)

Appendix B: Gauge symmetry, autoparallels and geodesics

In general autoparallels — “straightest curves” of the geometry — do not coincide with the geodesics, which are the “shortest curves” [3, 4, 5, 6]. There goes a discussion on whether autoparallels or geodesics describe the motion of test particles in spaces $W_4$ with generalized nonmetricity $Q_{\alpha\nu\mu}$ [3, 4, 6]. However, in Weyl space $\tilde{W}_4$, autoparallels and geodesics coincide as in GR. Anyway, geodesics and autoparallels can be associated exclusively with the motion of spinless point particles. Spinor fields like the fermions obey the Dirac equation in curved background, while extended spinning test bodies obey the Mathisson-Papapetrou-Dixon equations [105–108].

1. Auto-parallels

In Weyl space $\tilde{W}_4$ the “timelike” autoparallels are those curves along which the gauge covariant derivative of the tangent four-velocity vector $u$, vanishes (for a concise account of gauge derivative operators and gauge invariant parallel transport see appendix [A]) Here $u^\mu = dx^\mu / d\tau$ are the coordinate components of $u$ and, as long as this does not cause loss of generality, we chose the proper time $\tau$ to be the affine parameter along the autoparallel curve. The conformal weight of the four-velocity vector $w(u) = -1$. Autoparallel curves satisfy:

$$
\frac{D^* u}{d\tau} = u^\mu \nabla^*_\mu u = 0,
$$

(B1)

or, in explicit form, in terms of the arc-length $d\tau \rightarrow ds$ and of the LC connection:

$$
\frac{d^2 x^\alpha}{ds^2} + \{\alpha \beta\} \frac{dx^\beta}{ds} \frac{dx^\nu}{ds} - \frac{1}{2} Q_{\mu \nu} h^\nu = 0,
$$

(B2)

where

$$
h^\mu\nu := g^\mu\nu + u^\mu u^\nu - g^\mu\nu \frac{dx^\mu}{ds} \frac{dx^\nu}{ds},
$$

(B3)

is the orthogonal projection tensor, which projects any vector or tensor onto the hypersurface orthogonal to the four-velocity vector $u^\mu = dx^\mu / d\tau$.

The parallel transport law is obeyed by any vector including the four-momentum vector $p = mu$, where $m$ is the mass of the point particle. Since under (28) the point mass transforms like $m \rightarrow \Omega^{-1} m$, it has a conformal weight $w(m) = -1$. Consequently, the weight of the four-momentum $w(p) = -2$. Hence, from the law of parallel transport of the four-momentum

$$
\frac{D^* p}{d\tau} = u^\mu \nabla^*_\mu p = u^\mu (\nabla^*_\mu m) u + mu^\nu \nabla^*_\nu u = 0,
$$

(B4)

it follows that,

$$
u^\mu \nabla^*_\mu m = 0 \Rightarrow \delta m - \frac{m}{2} Q_\mu dx^\mu = 0.
$$

(B5)

Here we use variation instead of differentiation to underline that, in general, $\delta m$ is not a perfect differential.
Integration of equation \( E \) along the parallel transport path \( \gamma \), joining the origin \( x = \{0\} \) with the point with coordinates \( x = \{x^\mu\} \) and parametrized by some affine parameter \( \tau \), yields

\[
m(x, \gamma) = m_0 \exp \left( \frac{1}{2} \int_0^\tau Q_\mu dx^\mu \right),
\]

where \( m_0 \) is an integration constant that we can identify with the value of the parameter \( m \) evaluated at the origin \( m_0 = m(0) \). Eq. \( E \) is the basis of the second clock effect. It says that the mass \( m \) of a point particle at some point \( x^\mu \) depends not only on the point but also on the path joining this point with the origin.

In \( \mathcal{W}_4 \) the “null” autoparallels are those curves along which the gauge covariant derivative of the wave vector \( k \) with components \( k^\mu := dx^\mu / d\lambda \) (\( \lambda \) is a parameter along the null autoparallel), vanishes:

\[
\frac{D^* k^\alpha}{d\lambda} = k^\mu \nabla^*_\mu k^\alpha = 0,
\]

or, in explicit form:

\[
\frac{dk^\alpha}{d\lambda} + \{k^\alpha, k^\nu\} k^\nu = 0,
\]

where we took into account that \( (k, k) = g_{\mu\nu} k^\mu k^\nu = 0 \). In other words: photons and radiation in general do not interact with the gauge vector \( Q_\alpha \). It is clearly seen in metric theories of gravity, no matter whether local scale invariant or not, since massless fields follow null geodesics of Riemann space, which are already conformal invariant (see appendix D of \[69\]), so that nonmetricity plays no role.

2. Geodesic equations

The geodesic equations are equations of motion in the sense that these are the result of applying the variational principle of least action. Time-like and null particles follow geodesics. When these are compared with the corresponding auto-parallels one can measure how much the motion paths depart from the straightest curves of the geometry.

In the GR context the action of timelike particles reads \( S = m \int ds \), where \( m \) is the constant mass parameter. In Weyl space \( \mathcal{W}_4 \), since the mass, being the squared length of the four-momentum of the particle, varies in spacetime, then \( m \) cannot be taken out of the action integral. The action integral in \( \mathcal{W}_4 \) reads:

\[
S = \int m ds.
\]

From this action the following equations of motion – geodesic equations – can be derived:

\[
\frac{d^2 x^\alpha}{ds^2} + \{_{\mu\nu}, x^\alpha\} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} - \frac{1}{m} \frac{\delta m}{\delta x^\mu} h^\mu\alpha = 0,
\]

where the non-Riemannian term \( \propto \delta m / m \delta x^\mu \) accounts for the variation of mass during parallel transport. Hence, from Eq. \( E \) it follows that,

\[
\frac{1}{m^2} \frac{\delta m}{\delta x^\mu} = \frac{1}{2} Q_\mu.
\]

This shows that time-like autoparallels and time-like geodesics coincide in \( \mathcal{W}_4 \) space.

In a similar way the null geodesic equations can be derived from the following action:

\[
S_{null} = \frac{1}{2} \int g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu d\xi,
\]

where the dot accounts for derivative with respect to the parameter \( \lambda \) of the path \( x^\mu(\lambda) \) followed by photons (by radiation in general). From \( B \) the GR null geodesic equations are obtained. These coincide with the null autoparallels Eq. \( B \). Hence, the null geodesic equations do not depend of nonmetricity \( Q_\mu \). This means that photons and radiation probe the Riemann affine structure of spacetime, i.e. these interact only with the metric field (with the LC curvature of spacetime.)

Appendix C: Coupling of matter fields to gravity

That the coupling of matter fields with nonvanishing mass, in gauge invariant theories of gravity of type \( \{7\} \), is a difficult task was recognized long time ago in \[89\], for theories with underlying Riemann geometric structure of background space \( V_3 \). In order to expose the above difficulty we shall follow the demonstration in \[89\]. Let us add in action \( \{4\} \) a potential term for the scalar field \( V = V(\phi) \), as well as a Lagrangian \( L_\chi \equiv L(\chi, \partial \chi) \) for the matter fields, collectively denoted here by \( \chi \). We get

\[
S_{tot} = \int d^4 x \sqrt{-g} \left\{ \left[ \frac{\phi^2}{12} \dot{\phi}^2 + \frac{1}{2} (\partial \phi)^2 - V \right] + L_\chi \right\} . \]

The following EOM:
\[ \dot{C}_{\mu\nu} = \frac{6}{\phi^2} \left( \partial_{\mu}\phi \partial_{\nu}\phi - \frac{1}{2} g_{\mu\nu}(\partial_{\phi})^2 \right) - \frac{6V}{\phi^2} g_{\mu\nu} + \frac{1}{\phi^2} \left( \nabla_{\mu} \nabla_{\nu} - g_{\mu\nu} \nabla^2 \right) \phi^2 + \frac{6}{\phi^2} T^{(\chi)}_{\mu\nu}, \] (C2)

\[ \nabla^2 \phi^2 - 2(\partial_{\phi})^2 - \frac{1}{3} R \phi^2 + 2\phi V_{\phi} = 0, \] (C3)

are obtained by varying action (C1) with respect to the metric and to the scalar field, respectively. In the above equations \( V_{\phi} \equiv dV/d\phi \) and

\[ T^{(\chi)}_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta (\sqrt{-g} \mathcal{L}_{\chi})}{\delta g^{\mu\nu}}, \] (C4)

is the stress-energy tensor for the matter degrees of freedom. The trace of (C2) reads,

\[ \nabla^2 \phi^2 - 2(\partial_{\phi})^2 - \frac{1}{3} R \phi^2 + 2\phi V_{\phi} = 2T^{(\chi)}, \] (C5)

where \( T^{(\chi)} = g^{\mu\nu} T^{(\chi)}_{\mu\nu} \) is the trace of the matter SET. If compare equations (C3) and (C5) we obtain the following equation:

\[ \phi V_{\phi} - 4V = -T^{(\chi)}. \] (C6)

Notice that for the only self-interacting potential \( V \) which preserves gauge invariance of the theory (C1): \( V \propto \phi^4 \), the LHS of (C6) vanishes. This means that only massless matter fields for which \( T^{(\chi)} = 0 \) (light and radiation in general,) can be coupled to gauge invariant gravity.

### 1. Massless fields of the SMP

The above results are consistent with other results in the bibliography. In [8,9], for instance, it was shown that massless fields (fermions and gauge bosons previous to EW symmetry breaking,) do not interact with the Weyl vector or vectorial nonmetricity (see also the related discussion in [21].) Here, for completeness, we include a discussion of these results. We shall show that the action of massless SMP fields with \( SU(2) \times U(1) \) and \( SU(3) \) symmetries, in a curved background space \( \tilde{W}_4 \) with gauge symmetry, is fully equivalent to the same action in background Riemann space \( V_4 \) so that the nonmetricity may be ignored.

#### a. \( SU(2) \times U(1) \) fermions

Actually, in \( \tilde{W}_4 \) space the fermion Lagrangian which respects both \( SU(2) \times U(1) \) and Weyl gauge symmetry, reads,

\[ \mathcal{L}_{\text{term}} = \sqrt{-g} \bar{\psi} \mathcal{D}^\mu \psi, \] (C7)

where, for shorthand notation, we have introduced the gauge invariant operator

\[ \mathcal{D}^\mu := \gamma^\mu \left[ D^\mu - \frac{1}{2} \sigma_{ab} e^{\mu}_{\nu} (\nabla^\nu e^a_b) \right]. \] (C8)

In the above equations \((\bar{\psi}, \tilde{\psi})\) are the fermion spinor and its adjoint spinor (both have conformal weight \( w(\bar{\psi}) = w(\tilde{\psi}) = -3/2 \), \( \gamma^a \) are the (flat) Dirac gamma matrices, \( e^a_{\mu\nu} \) are the tetrad fields such that, \( g_{\mu\nu} = \eta_{ab} e^a_{\mu\nu} \) (\( \eta_{ab} \) is the Minkowski metric), so that their conformal weight is \( w(e^a_{\mu\nu}) = 1 \), \( w(e^a_{\mu\nu}) = -1 \). Besides \( \gamma^a = e^a_b \gamma^b \), etc.,

\[ \sigma_{ab} = \frac{1}{2} [\gamma_a, \gamma_b] = \frac{1}{4} (\gamma_a \gamma_b - \gamma_b \gamma_a), \]

are the generators of the Lorentz group in the spin representation [21]. \( \nabla^\mu \) is the gauge covariant derivative (see the corresponding definition in appendix A1 Eq. (A1)) and the gauge \( SU(2) \times U(1) \) derivative,

\[ D^\mu_{\psi} := \left[ \partial^\mu + ig W^i_{\mu} T^i - \frac{i}{2} \gamma^\nu Y B^\nu \right] \psi. \] (C9)

In the latter equation \( \partial^\mu \) is the gauge derivative defined in Eq. (A1) of appendix A1 \( W^i_{\mu} \) and \( B^\mu \) are the \( SU(2) \) and \( U(1) \) bosons, respectively, while \( (g, g^\nu) \) are the gauge couplings, \( Y \) is the hypercharge for \( \psi \) and \( T^i \) are the isospin matrices. It is not difficult to show that [9]:

\[ \mathcal{L}_{\text{term}} = \sqrt{-g} \bar{\psi} \mathcal{D}^\mu \psi = \sqrt{-g} \bar{\psi} \tilde{\mathcal{D}} \psi = \mathcal{L}_{\text{term}}, \]

where \( \tilde{\mathcal{D}} \) stands for the LC (Riemannian) part of the operator \( \mathcal{D}^\mu \), so that the terms with the nonmetricity \( Q_{\mu} \) may be safely ignored. In other words, the fermion field \( \psi \) does not interact with the nonmetricity.

#### b. \( SU(2) \times U(1) \) bosons

A similar result is obtained for massless \( SU(2) \times U(1) \) bosons. Let us consider the Lagrangian density of the \( SU(2) \times U(1) \) gauge bosons in \( \tilde{W}_4 \) background space,

\[ \mathcal{L}_{\text{bos}} = -\frac{\sqrt{-g}}{4} \left( W^i_{\mu\nu} W^{i\mu\nu} + B_{\mu\nu} B^{\mu\nu} \right), \] (C10)

where

\[ B_{\mu\nu} := \nabla_{\mu} B_{\nu} - \nabla_{\nu} B_{\mu}, \]

\[ W^i_{\mu\nu} := \nabla_{\mu} W^i_{\nu} - \nabla_{\nu} W^i_{\mu} - g_{ijk} W^j_{\mu} W^k_{\nu}. \]
Since the conformal weight \( w(B_\mu) = w(W^a_\mu) = 0 \), then in the above equations one may replace \( \nabla^*_\mu \rightarrow \nabla_\mu \). Besides, since the disformation tensor of \( W_4 \) space is symmetric in its lower indices, then one may further replace \( \nabla_\mu \rightarrow \nabla^*_\mu \). We get that, in the Lagrangian density \( \mathcal{L}_{\mathrm{bos}} \),

\[
B_{\mu\nu} := \nabla_\mu B_\nu - \nabla_\nu B_\mu = \tilde{B}_{\mu\nu},
\]

\[
W^i_{\mu\nu} := \nabla_\mu W^i_\nu - \nabla_\nu W^i_\mu - g\epsilon_{ijk}W^j_\mu W^k_\nu = \tilde{W}^i_{\mu\nu}.
\]

Hence \( \mathcal{L}_{\mathrm{bos}} = \tilde{\mathcal{L}}_{\mathrm{bos}} \), so that the nonmetricity tensor may be safely ignored.

e. Massless QCD fields

The SMP over \( \tilde{W}_4 \) space includes the quantum chromodynamics (QCD) \( SU(3) \) symmetric Lagrangian,

\[
\mathcal{L}_{\mathrm{QCD}} = \sqrt{-g} \left( \tilde{Q} \tilde{D}^\rho Q - \frac{1}{4} G^a_{\mu\nu} G^{a\mu\nu} \right), \tag{C11}
\]

where \( \tilde{D}^\rho \equiv \gamma^\mu D^\rho_\mu \), the gauge derivative \( D^\rho_\mu \) is given by Eq. \( \text{(C9)} \) with the addition of the term \(-ig^\rho{}^a X^a G^a_{\mu\nu}/2 \) (\( a = 1,2,\ldots,8 \)), \( Q \) is a triplet of spin 1/2 fermions, \( g^\rho \) is the gauge coupling, \( G^a_{\mu\nu} \) represent the gauge fields ( gluons), \( X^a \) are the eight matrices of \( SU(3) \), which obey \([X^a, X^b] = 2if_{abc}X^c\), \( f_{abc} \) are the structure constants and \( G^a_{\mu\nu} \) are the field strength tensors of the gluons,

\[
G^a_{\mu\nu} = \nabla_\mu G^a_\nu - \nabla_\nu G^a_\mu - g''f_{abc}G^b_\mu G^c_\nu. \tag{C12}
\]

It can be demonstrated that

\[
G^a_{\mu\nu} = \tilde{G}^a_{\mu\nu} = \tilde{\nabla}_\mu G^a_\nu - \tilde{\nabla}_\nu G^a_\mu - g''f_{abc}G^b_\mu G^c_\nu,
\]

and \( \tilde{Q} \tilde{D}^\rho Q = \tilde{Q} \tilde{D}^\rho \tilde{Q} \). Hence, nonmetricity plays no role in \( \text{(C11)} \).

The above results confirm that massless fields of the SMP do not interact with vectorial nonmetricity (see the related discussion in \[21\] where, in addition to nonmetricity, torsion is also considered.) This demonstrates that the action of SMP fields with \( SU(2) \times U(1) \) and \( SU(3) \) symmetries in a curved background space \( \tilde{W}_4 \) with gauge symmetry,

\[
S_{\text{tot}} = \int d^4x\sqrt{-g} \left[ \frac{g^2}{12} R + \frac{1}{2} (\partial^* \phi)^2 - \frac{\lambda}{48} \phi^4 
\right.
\]

\[
\left. + \bar{\psi} i \tilde{D}^\rho \psi - \frac{1}{4} (W^i_{\mu\nu} W^{i\mu\nu} + B_{\mu\nu} B^{\mu\nu}) 
\right]
\]

\[
+ \tilde{Q} \tilde{D}^\rho Q - \frac{1}{4} G^a_{\mu\nu} G^{a\mu\nu} \right], \tag{C13}
\]

is fully equivalent to the same action (same symmetries) in background Riemann space \( V_4 \). Q.E.D.

If the masses of the SMP fields were considered, then the above conclusion were not correct. Take, for instance, a fermion field \( \psi \) with mass \( m_\psi \). In this case the Lagrangian \( \text{(C7)} \) is to be replaced by

\[
\mathcal{L}_{\text{term}} = \sqrt{-\tilde{g}} (i\tilde{D}^\rho - m_\psi) \psi, \tag{C14}
\]

where, according to Eq. \( \text{(B6)} \) in appendix \[B\]

\[
m_\psi(x, C) = m_{0\psi} \exp \left( \frac{1}{2} \int_C Q_\mu u^\mu d\tau \right),
\]

where \( m_{0\psi} \) is an integration constant, \( u^\mu \) are the coordinate components of the tangent to the parallel transport path \( C \) at each point and \( \tau \) is an affine parameter along the trajectory, so that there is a implicit dependence on nonmetricity \( Q_\mu \) through the mass in Eq. \( \text{(C14)} \).

[1] H. Weyl, Annalen Phys. 54, 117 (1917); Math. Z. 2, 384 (1918); Annalen Phys. 59, 101 (1919).
[2] F. London, Z. Physik 42, 375 (1927).
[3] P.A.M. Dirac, Proc. Roy. Soc. Lond. A 333, 403 (1973).
[4] R. Utiyama, Prog. Theor. Phys. 50, 2080 (1973); Prog. Theor. Phys. 53, 565 (1975).
[5] R. Adler, M. Bazin, M. Schiffer, “Introduction to General Relativity,” 2nd Edition (Mc Graw-Hill, 1975).
[6] P. Bouvier, A. Maeder, Astrophys. Space Sci. 54, 497 (1978).
[7] L. Smolin, Nucl. Phys. B 160, 253 (1979).
[8] H. Cheng, Phys. Rev. Lett. 61, 2182 (1988).
[9] H. Cheng, ePrint: math-ph/0407010.
[10] V. Perlick, Class. Quantum Grav. 8, 1369 (1991).
[11] W. Drechsler, A. Tamm, Found. Phys. 29, 1023 (1999).
[12] G. ’t Hooft, Int. J. Mod. Phys. D 24, 1543001 (2015).
[13] S. De Bianchi, C. Kiefer, “One hundred years of gauge theory: Past, present and future perspectives” (Springer, 2020).
[14] P. Teyssandier, R.W. Tucker, Class. Quant. Grav. 13, 145 (1996).
[15] J.T. Wheeler, J. Math. Phys. 39, 299 (1998).
[16] J.A. Spencer, J.T. Wheeler, Int. J. Geom. Meth. Mod. Phys. 8, 273 (2011).
[17] J.T. Wheeler, Gen. Rel. Grav. 50, 80 (2018).
[18] I.P. Lobo, C. Romero, Phys. Lett. B 783, 306 (2018).
[19] J.B. Jiménez, L. Heisenberg, T. Koivisto, Class. Quant. Grav. 37, 195013 (2020).
[20] A. Delhom, I.P. Lobo, G.J. Olmo, C. Romero, Eur. Phys. J. C 80, 084040 (2020).
[21] A. Delhom, Eur. Phys. J. C 80, 728 (2020).
[22] M.P. Hobson, A.N. Lasenby, Phys. Rev. D 102, 084040 (2020).
[23] I. Quiros, Phys. Rev. D 105, 104060 (2022).
(1951).
[107] W.G. Dixon, Proc. Roy. Soc. Lond. A 314, 499 (1970).
[108] R.M. Wald, Phys. Rev. D 6, 406 (1972).
[109] C.M. Will, Living Rev. Rel. 17, 4 (2014).
[110] P.D. Mannheim, D. Kazanas, Astrophys. J. 342, 635 (1989).
[111] P.D. Mannheim, Gen. Rel. Grav. 22, 289 (1990).
[112] P.D. Mannheim, Prog. Part. Nucl. Phys. 56, 340 (2006).
[113] P.D. Mannheim, Found. Phys. 42, 388 (2012).
[114] A. Hindawi, B.A. Ovrut, D. Waldram, Phys. Rev. D 53, 5583 (1996).