A Forward-Backward Splitting Method for Monotone Inclusions Without Cocoercivity

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Abstract

In this work, we propose a simple modification of the forward-backward splitting method for finding a zero in the sum of two monotone operators. Our method converges under the same assumptions as Tseng’s forward-backward-forward method, namely, it does not require cocoercivity of the single-valued operator. Moreover, each iteration only requires one forward evaluation rather than two as is the case for Tseng’s method. Variants of the method incorporating a linesearch, an inertial term, or a structured three operator inclusion are also discussed.

Keywords. forward-backward algorithm · Tseng’s method · operator splitting

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1 Introduction

In this work, we propose an algorithm for finding a zero in the sum of two monotone operators in a (real) Hilbert space $H$. Specifically, we consider the monotone inclusion problem

$$\text{find } x \in H \text{ such that } 0 \in (A + B)(x),$$

(1)

where $A : H \rightrightarrows H$ and $B : H \to H$ are (maximally) monotone operators with $B$ (locally) Lipschitz continuous. Inclusions of the form specified by (1) arise in numerous problems of fundamental importance in mathematical optimization, either directly or through an appropriate reformulation. In what follows, we provide some motivating examples.

Convex minimization: Consider the minimization problem

$$\min_{x \in H} f(x) + g(x),$$

where $f : H \to (-\infty, +\infty]$ is proper, lower semicontinuous (lsc) and convex and $g : H \to \mathbb{R}$ is convex with (locally) Lipschitz continuous gradient denoted $\nabla g$. The solutions to this minimization problem are precisely the points $x \in H$ which satisfy the first order optimality condition:

$$0 \in (\partial f + \nabla g)(x),$$

(2)

where $\partial f$ denotes the subdifferential of $f$. Clearly (2) is of the form specified by (1).

General monotone inclusions: Consider the inclusion problem

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find $x \in \mathcal{H}$ such that $0 \in (A + K^* B K)(x)$, \hspace{1cm} (3)

where $A, B : \mathcal{H} \rightrightarrows \mathcal{H}$ are maximally monotone operators and $K : \mathcal{H} \to \mathcal{H}$ is a linear, bounded operator with adjoint $K^*$. As was observed in [6, 7], solving (3) can be equivalently cast as the following monotone inclusion posed in the product space:

find $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathcal{H} \times \mathcal{H}$ such that $\begin{pmatrix} 0 \\ 0 \end{pmatrix} \in \left( \begin{bmatrix} A & 0 \\ 0 & B^{-1} \end{bmatrix} + \begin{bmatrix} 0 & K^* \\ -K & 0 \end{bmatrix} \right) \begin{pmatrix} x \\ y \end{pmatrix}$.

\hspace{1cm} (4)

Notice that the first operator in (4) is maximally monotone whereas the second is bounded and linear (in particular, it is Lipschitz continuous with full domain). Consequently, (4) is also of the form specified by (1).

Another variant of (1) is the three operator inclusion

find $x \in \mathcal{H}$ such that $0 \in (A + B + C)(x)$, \hspace{1cm} (5)

where the operator $A$ and $B$ are as before and $C : \mathcal{H} \to \mathcal{H}$ is maximally monotone and $\beta$-cocoercive. Problems with this structure have been studied in [8, 10].

**Saddle point problems and variational inequalities:** Many convex optimization problems can be formulated as the **saddle point problem**

$$\min_{x \in \mathcal{H}} \max_{y \in \mathcal{H}} g(x) + \Phi(x, y) - f(y),$$

where $f, g : \mathcal{H} \to (-\infty, +\infty]$ are proper, lsc, convex functions and $\Phi : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ is a smooth convex-concave function. Problems of this form naturally arise in machine learning, statistics, etc., where the dual (maximization) problem comes from either dualizing the constraints in the primal problem or from using the Fenchel–Legendre transform to leverage a nonsmooth composite part. Through its first-order optimality condition, the saddle point problem (6) can expressed as the monotone inclusion

find $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathcal{H} \times \mathcal{H}$ such that $\begin{pmatrix} 0 \\ 0 \end{pmatrix} \in \left( \begin{bmatrix} \partial g(x) \\ \partial f(y) \end{bmatrix} + \begin{bmatrix} \nabla_x \Phi(x, y) \\ -\nabla_y \Phi(x, y) \end{bmatrix} \right)$.

\hspace{1cm} (7)

which is of the form specified by (1). By using the definitions of the respective subdifferentials, (7) can also be expressed in terms of the **variational inequality (VI):** find $z^* = (x^*, y^*)^\top \in \mathcal{H} \times \mathcal{H}$ such that

$$\langle B(z^*), z - z^* \rangle + g(x) - g(x^*) - f(y) + f(y^*) \geq 0 \quad \forall z = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathcal{H} \times \mathcal{H},$$

where $B(x, y) := (\nabla_x \Phi(x, y), -\nabla_y \Phi(x, y))^\top$.

**Splitting algorithms** are a class of methods which can be used to solve (1) by only invoking each operator individually rather than their sum directly. The individual steps within each iteration of these methods can be divided into two categories: **forward evaluations** in which the value of a single-valued operator is computed, and **backward evaluations** in which the **resolvent** of an operator computed. Recall that the resolvent of an operator $A$ is given by $J_A := (I + A)^{-1}$ where $I : \mathcal{H} \to \mathcal{H}$ denotes the identity operator.

When the resolvents of both of the involved operators can be easily computed, there are various algorithms in the literature which are suitable for solving (1) with $B$ not necessarily single-valued. The best known example of such an algorithm is the Douglas–Rachford method [13, 22]. In practice, however, it is usually not the case that both resolvents can be easily
computed and thus in order to efficiently deal with realistic problems, it is often necessary to impose further structure on the operators in (1). Splitting methods which do not require computation of two resolvents are therefore of practical interest.

The best-known splitting method for solving the inclusion (1) when $B$ is single-valued is the forward-backward method, called so because each iteration combines one forward evaluation of $B$ with one backward evaluation of $A$. More precisely, the method generates a sequence according to

$$x_{k+1} = J_{\lambda A}(x_k - \lambda B(x_k)) \quad \forall k \in \mathbb{N},$$

and converges weakly to a solution provided that the operator $B : \mathcal{H} \to \mathcal{H}$ is $1/L$-cocoercive and $\lambda \in (0, 2/L)$. Recall that $B : \mathcal{H} \to \mathcal{H}$ is $\beta$-cocoercive if

$$\langle x - y, B(x) - B(y) \rangle \geq \beta \|B(x) - B(y)\|^2 \quad \forall x, y \in \mathcal{H}.$$

Cocoercivity of an operator is a strictly strong property than Lipschitz continuity and hence can be difficult to satisfy for general monotone inclusions. For instance, apart from the trivial case when $K = 0$, the skew-symmetric operator in (4) is never cocoercive. In order to relax this condition, Tseng [23] proposed a modification of the forward-backward algorithm, known as the Tseng’s method or the forward-backward-forward method, which only requires Lipschitzness of $B$ at the expense of an additional forward evaluation. Applied to (1), Tseng’s method generates sequences according to

$$\begin{cases} y_k &= J_{\lambda A}(x_k - \lambda B(x_k)) \\ x_{k+1} &= y_k - \lambda B(y_k) + \lambda B(x_k) \end{cases} \quad \forall k \in \mathbb{N},$$

and converges weakly provided $B$ is $L$-Lipschitz and $\lambda \in (0, 1/L)$.

In this work, we introduce and analyze a new method for solving (1) which converges under the same assumptions as Tseng’s method, but whose implementation requires only one forward evaluation per iteration instead of two. For a fixed stepsize $\lambda > 0$, the proposed scheme can be simply described as

$$x_{k+1} = J_{\lambda A}\left(x_k - 2\lambda B(x_k) + \lambda B(x_{k-1})\right) \quad \forall k \in \mathbb{N},$$

and converges weakly if $B$ is $L$-Lipschitz and the stepsize is chosen to satisfy $\lambda < \frac{1}{2L}$. We refer to this scheme as the forward-reflected-backward method. It is worth noting that the analysis of our method is entirely different than existing schemes, and hence is of interest in its own right. In particular, it does not fit within the usual Krasnoselskii–Mann framework. Moreover, to the best of the authors’ knowledge, Tseng’s forward-backward-forward algorithm is the only other known algorithm which can solve inclusion in the form of (1) without cocoercivity. Our proposed method therefore constitutes the first and only known alternative.

We also remark that our method is of particular interest in the setting of the saddle point problem (7). Indeed, one of the first splitting techniques for solving (6) is the famous Arrow–Hurwicz algorithm [3] which suffers from the shortcoming of requiring strict assumptions to ensure convergence. This was remedied in late 70’s when various modification of the algorithm were proposed [2, 12, 20] which turned out to applicable not only to saddle point problems, but also to more general variational inequalities. Note also that the simplest case of (6) occurs when $\Phi$ is a bilinear form and gives rise to the popular primal-dual algorithm, first analyzed by Chambolle & Pock [9]. In a recent preprint [11], a variant of this algorithm, which can be applied when $\Phi$ is not necessarily bilinear, was considered. Such an extension is a significant improvement as it provides an approach to the saddle point problem that is different from variational inequality methods. An interesting common feature of the methods
in [9, 11, 16] as well as the one presented here is that their respective iterations include a “reflection term” in which the value of an operator at the previous point is subtracted from twice its value at the current point.

The remainder of this paper is organized as follows. In Section 2, we introduce our method and prove its convergence (Theorem 2.4). In Section 2.1, this result is refined to show that convergence is linear whenever one of the operators is strongly monotone. In Section 3, we show how to incorporate a linesearch procedure into the method (Theorem 3.4). In Section 4, we consider an inertial version of the method (Theorem 4.3) and finally, in Section 5, we propose a variant which solves the three operator inclusion (5).

2 Forward-reflected-backward splitting

Recall that a set-valued operator $A : \mathcal{H} \rightrightarrows \mathcal{H}$ is monotone if

$$\langle x - y, u - v \rangle \geq 0 \quad \forall (x, u), (y, v) \in \text{gra} A,$$

where $\text{gra} A = \{(x, y) \in \mathcal{H} \times \mathcal{H} : y \in A(x)\}$ denotes the graph of $A$. A monotone operator is maximally monotone if its graph is not properly contained in the graph of any other monotone operator. The resolvent of a maximally monotone operator $A : \mathcal{H} \rightarrow \mathcal{H}$, defined by $J_A := (I + A)^{-1}$, is an everywhere single-valued operator [4]. A single-valued operator $B : \mathcal{H} \rightarrow \mathcal{H}$ is $L$-Lipschitz if $\|B(x) - B(y)\| \leq L \|x - y\|$ for all $x, y \in \mathcal{H}$.

In this section, we consider the problem of finding a point $x \in \mathcal{H}$ such that

$$0 \in (A + B)(x), \quad (11)$$

where $A : \mathcal{H} \rightrightarrows \mathcal{H}$ is maximal monotone, and $B : \mathcal{H} \rightarrow \mathcal{H}$ is monotone and $L$-Lipschitz. Given initial points $x_0, x_{-1} \in \mathcal{H}$, we consider the scheme

$$x_{k+1} = J_{\lambda_k A}(x_k - \lambda_k B(x_k) - \lambda_{k-1}(B(x_k) - B(x_{k-1})) \quad \forall k \in \mathbb{N}, \quad (12)$$

where $(\lambda_k)_{k \in \mathbb{N}} \subseteq \mathbb{R}_+$ is a sequence of step-sizes. Note that, each iteration of this scheme requires one forward evaluation and one backward evaluation. Using the definition of the resolvent $J_{\lambda_k A} = (I + \lambda_k A)^{-1}$, (12) can be equivalently expressed as the inclusion

$$x_{k+1} - x_k + \lambda_k B(x_k) + \lambda_{k-1}(B(x_k) - B(x_{k-1})) \in -\lambda_k A(x_{k+1}) \quad \forall k \in \mathbb{N}. \quad (13)$$

Before turning our attention to the convergence analysis of this method, we first note some special cases in which it recovers known methods.

Remark 2.1 (Special cases of (12)). We consider three cases in which the proposed algorithm reduces or is equivalent to known methods. For simplicity, we only consider the fixed step-size case (i.e., $\exists \lambda > 0$ such that $\lambda_k = \lambda$ for all $k \in \mathbb{N}$). In this case, (12) can be expressed compactly as

$$x_{k+1} = J_{\lambda A}(x_k - 2\lambda B(x_k) + \lambda B(x_{k-1})). \quad (14)$$

(a) If $B = 0$ then (14) simplifies to the proximal point algorithm [21], that is, (14) becomes

$$x_{k+1} = J_{\lambda A}(x_k) \quad \forall k \in \mathbb{N}. \quad$$

(b) If $A = N_C$ and $B$ is an affine operator then (14) can be expressed as

$$x_{k+1} = P_C(x_k - \lambda B(2x_k - x_{k-1})) \quad \forall k \in \mathbb{N}, \quad (15)$$

which coincides with the projected reflected gradient method [15] for VIs.
(c) If $A = N_H = 0$ then the projected reflected gradient method (15) becomes

$$x_{k+1} = x_k - B(2x_k - x_{k-1}) \quad \forall k \in \mathbb{N}.$$ 

Under the change of variables $\overline{x}_k = 2x_k - x_{k-1}$, this becomes

$$\overline{x}_{k+1} = \overline{x}_k - 2\lambda B(\overline{x}_k) + (x_{k-1} - x_k) = \overline{x}_k - 2\lambda B(\overline{x}_k) + \lambda B(\overline{x}_{k-1}) \quad \forall k \in \mathbb{N},$$

which is precisely (12) with $A = 0$. Alternatively, (14) can be expressed as the two step recursion

$$
y_{k+1} = y_k - \lambda B(x_k)
$$

$$x_{k+1} = y_{k+1} - \lambda B(x_k).$$

(16)

This is exactly Popov’s algorithm [20] for unconstrained VIs. In this sense, the three methods coincide in this case up to a change of variable (but not in general).

Before establishing convergence of the method, we require some preparatory lemmas.

**Lemma 2.2.** Let $(z_k)_{n \in \mathbb{N}} \subseteq H$ be a bounded sequence and suppose $\lim_{k \to \infty} \|z_k - z\|$ exists whenever $z$ is a cluster point of $(z_k)_{k \in \mathbb{N}}$. Then $(z_k)_{n \in \mathbb{N}}$ is weakly convergent.

**Lemma 2.3.** Let $x \in (A + B)^{-1}(0)$ and let $(x_k)_{k \in \mathbb{N}}$ be given by (12). Let $\varepsilon > 0$ and suppose $(\lambda_k) \subseteq \left[\varepsilon, \frac{1-2\varepsilon}{2L}\right]$. Then, for all $k \in \mathbb{N}$, we have

$$\|x_{k+1} - x\|^2 + 2\lambda_k \langle B(x_{k+1}) - B(x), x - x_{k+1} \rangle + \left(\frac{1}{\lambda^2} + \varepsilon\right) \|x_{k+1} - x\|^2 
\leq \|x_k - x\|^2 + 2\lambda_k \langle B(x_k) - B(x_{k-1}), x - x_k \rangle + \frac{1}{\lambda} \|x_k - x_{k-1}\|^2. \quad (17)$$

**Proof.** Combining (13) with the monotonicity of $A$ gives

$$0 \leq \langle x_{k+1} - x_k + \lambda_k B(x_k) + \lambda_{k-1}(B(x_k) - B(x_{k-1})) - \lambda_k B(x), x - x_{k+1} \rangle,$$

which we rewrite as

$$0 \leq \langle x_{k+1} - x_k, x - x_{k+1} \rangle + \lambda_k \langle B(x_k) - B(x), x - x_{k+1} \rangle + \lambda_{k-1}(B(x_k) - B(x_{k-1}), x - x_k) + \lambda_k \langle B(x_k) - B(x_{k-1}), x - x_{k-1} \rangle. \quad (18)$$

Using the Law of Cosines, the first term can be expressed as

$$\langle x_{k+1} - x_k, x - x_{k+1} \rangle = \frac{1}{2} (\|x_k - x\|^2 - \|x_{k+1} - x_k\|^2 - \|x_{k+1} - x\|^2).$$

Using the monotonicity of $B$, the second term can be estimated as

$$\langle B(x_k) - B(x), x - x_{k+1} \rangle \leq -\langle B(x_{k+1}) - B(x), x - x_{k+1} \rangle$$

and finally, using Lipschitzness of $B$, the fourth term can estimated as

$$\langle B(x_k) - B(x_{k-1}), x - x_k \rangle \leq L \|x_k - x_{k-1}\| \|x_k - x_{k+1}\|$$

$$\leq \frac{L}{2} (\|x_k - x_{k-1}\|^2 + \|x_k - x_{k+1}\|^2). \quad (20)$$

Thus, altogether, (18) implies that

$$0 \leq \|x_k - x\|^2 - \|x_{k+1} - x_k\|^2 - \|x_{k+1} - x\|^2 - 2\lambda_k \langle B(x_{k+1}) - B(x), x - x_{k+1} \rangle$$

$$+ 2\lambda_{k-1}(B(x_k) - B(x_{k-1}), x - x_k) + \lambda_{k-1} L \left(\|x_k - x_{k-1}\|^2 + \|x_k - x_{k+1}\|^2\right),$$

$$\|x_{k+1} - x_{k-1}\|^2 \leq 2\lambda_{k-1} \|x_k - x_{k-1}\|^2 + \lambda_{k-1} L \left(\|x_k - x_{k-1}\|^2 + \|x_k - x_{k+1}\|^2\right),$$

$$\|x_{k+1} - x_{k-1}\|^2 \leq 2\lambda_{k-1} \lambda_{k-1} \|x_k - x_{k-1}\|^2 + \lambda_{k-1} \lambda_{k-1} L \left(\|x_k - x_{k-1}\|^2 + \|x_k - x_{k+1}\|^2\right).$$
Proof. Let \( \lambda \) be a weak cluster point of the bounded sequence \((x_k)_{k \in \mathbb{N}}\) such that \((x_k)_{k \in \mathbb{N}}\) converges weakly to a point \( p \in \mathcal{H} \) such that \( 0 \in A(p) + B(p) \).

Then \((x_k)_{k \in \mathbb{N}}\) is maximally monotone, and suppose that \( A \) is maximally monotone, let \( B : \mathcal{H} \to \mathcal{H} \) be monotone and \( L \)-Lipschitz, and suppose \((x_k)_{k \in \mathbb{N}}\) is convergent weakly to a point \( x = x_0 \) and that \( \lambda = \lambda_0 \), by taking the limit along a subsequence of \((x_k)_{k \in \mathbb{N}}\) and noting the proof remains true without these assumptions.

We are now ready for the first main result regarding convergence of the proposed method. In order to keep the exposition as clear as possible, we assume \( x_{-1} = x_0 \) and \( \lambda_{-1} = \lambda_0 \), but note the proof remains true without these assumptions.

**Theorem 2.4.** Let \( A : \mathcal{H} \rightrightarrows \mathcal{H} \) be maximally monotone, let \( B : \mathcal{H} \to \mathcal{H} \) be monotone and \( L \)-Lipschitz, and suppose that \((A + B)^{-1}(0) \neq \emptyset\). Let \( \varepsilon > 0 \) and suppose \( (\lambda_k) \subseteq [\varepsilon, \frac{1-\varepsilon}{2}] \).

Given \( x_0 \in \mathcal{H} \), define the sequence \((x_k)_{k \in \mathbb{N}}\) according to

\[
x_{k+1} = J_{\lambda_k A}(x_k - \lambda_k B(x_k) - \lambda_{k-1}(B(x_k) - B(x_{k-1}))) \quad \forall k \in \mathbb{N}.
\]

which telescopes to yield

\[
\|x_{k+1} - x\|^2 + 2\lambda_k \langle B(x_{k+1}) - B(x_k), x - x_{k+1}\rangle + \left(\frac{1}{2} + \varepsilon\right) \|x_{k+1} - x_k\|^2 \leq \|x_k - x\|^2 + 2\lambda_{k-1} \langle B(x_k) - B(x_{k-1}), x - x_k\rangle + \frac{1}{2} \|x_k - x_{k-1}\|^2,
\]

which telescopes to yield

\[
\|x_{k+1} - x\|^2 + 2\lambda_k \langle B(x_{k+1}) - B(x_k), x - x_{k+1}\rangle + \frac{1}{2}\|x_{k+1} - x_k\|^2 + \varepsilon \sum_{i=0}^{k} \|x_i - x_{i+1}\|^2 \leq \|x_0 - x\|^2.
\]

Using the Lipschitzness of \( B \), we can estimate

\[
2\lambda_k \langle B(x_{k+1}) - B(x_k), x - x_{k+1}\rangle \geq -2\lambda_k L \|x_{k+1} - x_k\| \|x - x_{k+1}\| \geq -\lambda_k L (\|x_{k+1} - x_k\|^2 + \|x - x_{k+1}\|^2).
\]

Since \( \lambda_k L \leq (1 - 2\varepsilon)/2 < 1/2, \) substituting the previous equation back into (22) gives

\[
\frac{1}{2}\|x_{k+1} - x\|^2 + \varepsilon \sum_{i=0}^{k} \|x_i - x_{i+1}\|^2 \leq \|x_0 - x\|^2,
\]

from which we deduce that \((x_k)_{k \in \mathbb{N}}\) is bounded and that \( \|x_k - x_{k+1}\| \to 0 \).

Let \( \bar{p} \) be a weak cluster point of the bounded sequence \((x_k)_{k \in \mathbb{N}}\). From (13),

\[
\frac{1}{\lambda_{k-1}} \left( x_{k-1} - x_{k} + \lambda_{k-1} (B(x_k) - B(x_{k-1})) + \lambda_{k-2} (B(x_{k-2}) - B(x_{k-1})) \right) \in (A + B)(x_k).
\]

Since \( A + B \) is maximally monotone [4, Corollaries 24.4(i) & 20.25], its graph is demiclosed (\( i.e., \) sequentially closed in the weak-strong topology on \( \mathcal{H} \times \mathcal{H} \)) [4, Proposition 20.33]. Thus, by taking the limit along a subsequence of \((x_k)_{k \in \mathbb{N}}\) which converges to \( \overline{p} \) in (24) and noting
that $\lambda_k \geq \delta$ for all $k \in \mathbb{N}$, we deduce that $0 \in (A + B)(\overline{x})$. To show that $(x_k)_{n \in \mathbb{N}}$ is weakly convergent, first note that, by combining (21) and (23), we deduce existence of the limit

$$\lim_{k \to \infty} \left( \|x_k - \overline{x}\|^2 + 2\lambda_{k-1} \langle B(x_k) - B(x_{k-1}), \overline{x} - x_k \rangle + \frac{1}{2}\|x_k - x_{k-1}\|^2 \right).$$

Since $(x_k)_{k \in \mathbb{N}}$ is bounded, $\|x_k - x_{k+1}\| \to 0$, and $B$ is continuous, it then follows that

$$\lim_{k \to \infty} \left( \|x_k - \overline{x}\|^2 + 2\lambda_{k-1} \langle B(x_k) - B(x_{k-1}), \overline{x} - x_k \rangle + \frac{1}{2}\|x_k - x_{k-1}\|^2 \right) = \lim_{k \to \infty} \|x_k - \overline{x}\|^2.$$

Since the cluster point $\overline{x}$ of $(x_k)_{k \in \mathbb{N}}$ was chosen arbitrarily, the sequence $(x_k)_{k \in \mathbb{N}}$ is weakly convergent by Lemma 2.2 and the proof is complete.

As an immediately consequence of Theorem 2.4, we obtain the following corollary when the stepsize sequence $(\lambda_k)_{k \in \mathbb{N}}$ is constant.

**Corollary 2.5.** Let $A : \mathcal{H} \rightrightarrows \mathcal{H}$ be maximally monotone, let $B : \mathcal{H} \to \mathcal{H}$ be monotone and $L$-Lipschitz, and suppose that $(A + B)^{-1}(0) \neq \emptyset$. Choose $\lambda \in (0, \frac{1}{2L})$. Given $x_0 \in \mathcal{H}$, define the sequence $(x_k)_{k \in \mathbb{N}}$ according to

$$x_{k+1} = J_{\lambda_k A} \left( x_k - 2\lambda B(x_k) + \lambda B(x_{k-1}) \right) \quad \forall k \in \mathbb{N}.$$

Then $(x_k)_{k \in \mathbb{N}}$ converges weakly to a point $\overline{x} \in \mathcal{H}$ such that $0 \in A(\overline{x}) + B(\overline{x}).$

**Remark 2.6.** In practice, it can be desirable to analyze an algorithm with respect to an auxiliary metric to encourage faster convergence. This is done by considering the metric induced by the inner product $\langle \cdot, \cdot \rangle_M$ corresponding to a symmetric positive definite operator $M : \mathcal{H} \to \mathcal{H}$. In the case of saddle point problems for example, choosing the operator $M$ to be a diagonal scaling matrix gives different weights to primal and dual variables. To keep our presentation as simple and as clear as possible, we present our analysis only for the case when $M = I$. Nevertheless, the more general case can be easily obtained through a straightforward modification of the proof. In particular, instead of (12), we can consider the iteration

$$x_{k+1} = J^M_{\lambda_k A} \left( x_k - M^{-1} [\lambda_k B(x_k) + \lambda_{k-1} (B(x_k) - B(x_{k-1}))] \right) \quad \forall k \in \mathbb{N},$$

where $J^M_A = (I + M^{-1} A)^{-1}$ denotes the generalized resolvent of $A$.

### 2.1 Linear Convergence

In this section, we establish $R$-linear of the sequence generated by the forward-reflected-backward method when $A$ is strongly monotone. Recall that $A : \mathcal{H} \to \mathcal{H}$ is $m$-strongly monotone if $m > 0$ and

$$\langle x - y, u - v \rangle \geq m \|x - y\|^2 \quad \forall (x, u), (y, v) \in \text{gra} A.$$

Strong monotonicity is a standard assumption for proving linear convergence of first order methods. We also note that there is no loss of generality in assuming that $A$ is strongly monotone. For if $B$ is $m$-strongly monotone, we can always augment the operators by the identity, i.e., $A + B = (A + mI) + (B - mI)$, without destroying monotonicity and Lipschitz continuity. Notice this does not complicate computing the resolvent of $(A + mI)$, as we have $J_{A+mI} x = J_{\frac{1}{1+m}} A x$ for all $x \in \mathcal{H}$.

Before establishing linear convergence of our proposed method, we shall require the following preparatory lemma.
Lemma 2.7. Let \((a_k)_{k \in \mathbb{N}}\) and \((b_k)_{k \in \mathbb{N}}\) be two nonnegative sequences of real numbers. Suppose there exist constants \(\alpha > 1\) and \(\delta > 0\) such that
\[
\alpha a_{k+1} + b_{k+1} \leq a_k + b_k \quad \text{and} \quad \delta b_k \leq a_k \quad \forall k \in \mathbb{N}.
\] (25)
Then \((a_k)_{k \in \mathbb{N}}\) and \((b_k)_{k \in \mathbb{N}}\) converge to zero with R-linear rate.

Proof. Let \(\beta \in (1, \alpha)\). Then, since \(\delta b_{k+1} \leq a_{k+1}\), we have
\[
\alpha a_{k+1} = \beta a_{k+1} + (\alpha - \beta)a_{k+1} \geq \beta a_{k+1} + \delta (\alpha - \beta)b_{k+1}.
\]
Set \(r := \min\{\beta, 1 + \delta (\alpha - \beta)\} > 1\). Combining the previous inequality with (25) gives
\[
r(a_{k+1} + b_{k+1}) \leq \beta a_{k+1} + (1 + \delta (\alpha - \beta)) b_{k+1}
\]
\[
= [\beta a_{k+1} + \delta (\alpha - \beta) b_{k+1}] + b_{k+1}
\]
\[
\leq \alpha a_{k+1} + b_{k+1} \leq a_k + b_k.
\]
This implies that
\[
a_{k+1} + b_{k+1} \leq \frac{1}{r^k}(a_0 + b_0) \quad \forall k \in \mathbb{N},
\]
from which the claim follows.

Theorem 2.8. Let \(A : \mathcal{H} \rightrightarrows \mathcal{H}\) be maximally monotone and \(m\)-strongly monotone, let \(B : \mathcal{H} \to \mathcal{H}\) monotone and L-Lipschitz, and suppose \((A + B)^{-1}(0) \neq \emptyset\). Let \(\lambda \in (0, \frac{1}{2L})\).
Given \(x_0 \in \mathcal{H}\), define the sequence \((x_k)_{k \in \mathbb{N}}\) according to
\[
x_{k+1} = J_{\lambda A}(x_k - 2\lambda B(x_k) + \lambda B(x_{k-1})) \quad \forall k \in \mathbb{N}.
\]
Then \((x_k)_{k \in \mathbb{N}}\) converges R-linearly to the unique point \(\pi \in \mathcal{H}\) satisfying \(0 \in A(\pi) + B(\pi)\).

Proof. Let \(x \in (A + B)^{-1}(0)\). Using the strong monotonicity of \(A\) (in place of monotonicity), inequality (18) in the proof of Lemma 2.3 can be improved to
\[
m\lambda \|x_{k+1} - x\|^2 \leq \langle x_{k+1} - x, x - x_{k+1} \rangle + \lambda \langle B(x_k) - B(x), x - x_{k+1} \rangle
\]
\[
+ \lambda \langle B(x_k) - B(x_{k-1}), x - x_k \rangle + \lambda \langle B(x_k) - B(x_{k-1}), x - x_{k+1} \rangle.
\]
Propagating this improvement through the proof of Lemma 2.3 gives
\[
(1 + 2m\lambda) \|x_{k+1} - x\|^2 + 2\lambda \langle B(x_{k+1}) - B(x_k), x - x_{k+1} \rangle + (1 - \lambda L) \|x_{k+1} - x\|^2
\]
\[
\leq \|x_k - x\|^2 + 2\lambda \langle B(x_k) - B(x_{k-1}), x - x_k \rangle + \frac{1}{2} \|x_k - x_{k-1}\|^2. \quad (26)
\]
By denoting \(\alpha := 1 + 4m\lambda > 1\), (26) implies
\[
\alpha a_{k+1} + b_{k+1} \leq a_k + b_k, \quad (27)
\]
where the nonnegative sequences \((a_k)_{k \in \mathbb{N}}\) and \((b_k)_{k \in \mathbb{N}}\) are given by
\[
a_k := \frac{1}{2}\|x_k - x\|^2 \geq 0,
\]
\[
b_k := \frac{1}{2}\|x_k - x\|^2 + 2\lambda \langle B(x_k) - B(x_{k-1}), x - x_k \rangle + \frac{1}{2} \|x_k - x_{k-1}\|^2.
\]
Using Lipschitzness of \(B\), we have
\[
b_k \geq \frac{1}{2}\|x_k - x\|^2 - 2\lambda L \|x_k - x_{k-1}\| \|x_k - x\| + \frac{1}{2} \|x_k - x_{k-1}\|^2
\]
\[
\geq \frac{1}{2}\|x_k - x\|^2 - \lambda L \left(\|x_k - x_{k-1}\|^2 + \|x_k - x\|^2\right) + \frac{1}{2} \|x_k - x_{k-1}\|^2 \geq 0.
\]
The conditions of Lemma 2.7 are thus satisfied, hence \(x_k \to x\) with \(R\)-linear rate. Since \(x\) was chosen arbitrarily from \((A + B)^{-1}(0)\), it must be unique.

\[\square\]
3 Forward-reflected-backward splitting with linesearch

The algorithm presented in the previous section required information about the single-valued operator’s Lipschitz constant in order to select an appropriate stepsize. In practice, this requirement is undesirable for several reasons. Firstly, obtaining the Lipschitz constant (or an estimation) is usually non-trivial and often a computationally expensive problem itself. Secondly, as a global constant, the (global) Lipschitz constant can often lead to an over-conservative stepsize scheme although local properties (around the current iterate) may permit the use of larger stepsizes and ultimately lead to faster convergence. Finally, when the single-valued operator is not Lipschitz continuous, any fixed stepsize scheme based on Lipschitz continuity will potentially fail to converge.

To address these shortcomings, most known methods can incorporate an additional procedure called *linesearch* (or *backtracking*) which is run in each iteration. It is worth noting however, that in the more restrictive context of variational inequalities, the method proposed in [16] overcomes the aforementioned difficulties without resorting to a linesearch procedure.

In what follows, we show that the forward-reflected-backward method with such a linesearch procedure converges whenever the single-valued operator is *locally Lipschitz*.

**Algorithm 1** The forward-reflected backward method with linesearch.

**Initialization:** Choose $x_0 \in \mathcal{H}$, $x_{-1} = x_0$, $\lambda_0 > 0$, $\delta \in (0, 1)$, and $\sigma \in (0, 1)$.

**Iteration:** Having $x_k$, $\lambda_{k-1}$, and $B(x_{k-1})$, choose $\rho \in \{1, \sigma^{-1}\}$ and compute

$$x_{k+1} := J_{\lambda A}(x_k - \lambda_k B(x_k) - \lambda_{k-1}(B(x_k) - B(x_{k-1}))),$$

where $\lambda_k = \rho \lambda_{k-1} \sigma^i$ with $i$ being the smallest nonnegative integer satisfying

$$\lambda_k \|B(x_{k+1}) - B(x_k)\| \leq \frac{\delta}{2} \|x_{k+1} - x_k\|. \quad (29)$$

**Remark 3.1.** The parameter $\rho$ in Algorithm 1 has been introduced to allow for greater flexibility in the choice of possible step sizes. Indeed, there are two possible scenarios for the value of $\lambda_k$ in the first iteration of the linesearch procedure (i.e., when $i = 0$): either $\rho = \sigma^{-1}$ and $\lambda_k = \sigma^{-1} \lambda_{k-1} > \lambda_{k-1}$, or $\rho = 1$ and $\lambda_k = \lambda_{k-1}$. The former, more aggressive scenario allows for possibility of larger stepsizes at the price of a potential increase in the number of linesearch iterations.

The following lemma shows that the linesearch procedure described in Algorithm 1 is well-defined as long as the operator $B$ is locally Lipschitz continuous.

**Lemma 3.2.** Suppose $B : \mathcal{H} \to \mathcal{H}$ is locally Lipschitz. Then the linesearch procedure in (28)–(29) always terminates. i.e., $\lambda_k$ is well defined.

**Proof.** Denote $x_{k+1}(\lambda) := J_{\lambda A}(x_k - \lambda B(x_k) - \lambda_{k-1}(B(x_k) - B(x_{k-1})))$. From [4, Theorem 23.47], we have that $J_{\lambda A}(x_{k+1}(0)) \to P_{\text{dom } A}(x_{k+1}(0))$ as $\lambda \searrow 0$ which, together with the nonexpansivity of $J_{\lambda A}$, yields

$$\|x_{k+1}(\lambda) - P_{\text{dom } A}(x_{k+1}(0))\| \leq \|x_{k+1}(\lambda) - J_{\lambda A}(x_{k+1}(0))\| + \|J_{\lambda A}(x_{k+1}(0)) - P_{\text{dom } A}(x_{k+1}(0))\| \leq \lambda \|B(x_k)\| + \|J_{\lambda A}(x_{k+1}(0)) - P_{\text{dom } A}(x_{k+1}(0))\|.$$

By taking the limit as $\lambda \searrow 0$, we deduce that $x_{k+1}(\lambda) \to P_{\text{dom } A}(x_{k+1}(0))$. 


Now, by way of a contradiction, suppose that the linesearch procedure in Algorithm 1 fails to terminate at $k$-th iteration. Then, for all $\lambda = \rho \lambda_{k-1} \sigma^i$ with $i = 0, 1, \ldots$, we have

$$\rho \lambda_{k-1} \sigma^i \|B(x_{k+1}^i(\lambda)) - B(x_k)\| > \frac{\delta}{2} \|x_{k+1}^i(\lambda) - x_k\|. \quad (30)$$

On one hand, taking the limit as $i \to \infty$ in (28) gives $P_{\text{dom}A}(x_{k+1}(0)) = x_k$. On the other hand, since $B$ is locally Lipschitz at $x_k$ there exists an $L > 0$ such that for $i$ is sufficiently large, we have

$$\rho \lambda_{k-1} \sigma^i \|B(x_{k+1}^i(\lambda)) - B(x_k)\| > \frac{\delta}{2} \|x_{k+1}^i(\lambda) - x_k\| \geq \frac{\delta L}{2} \|B(x_{k+1}^i(\lambda)) - B(x_k)\|.$$

Diving both sides by $\|B(x_{k+1}^i(\lambda)) - B(x_k)\|$ gives $\delta L/2 < \rho \lambda_{k-1} \sigma^i$. Since $\sigma^i \to 0$ as $i \to \infty$, this inequality gives a contradiction which completes the proof. \qed

The next lemma is a direct extension of Lemma 2.3.

**Lemma 3.3.** Let $x \in (A + B)^{-1}(0)$ and let $(x_k)_{k \in \mathbb{N}}$ be generated by Algorithm 1. Then there exist $\varepsilon > 0$ such that, for all $k \in \mathbb{N}$, we have

$$\|x_{k+1} - x\|^2 + 2\lambda_k \langle B(x_{k+1}) - B(x_k), x - x_{k+1} \rangle + \left(\frac{1}{2} + \varepsilon\right) \|x_{k+1} - x_k\|^2$$

$$\leq \|x_k - x\|^2 + 2\lambda_{k-1} \langle B(x_k) - B(x_{k-1}), x - x_k \rangle + \frac{1}{2} \|x_k - x_{k-1}\|^2. \quad (31)$$

**Proof.** The proof is exactly the same as Lemma 2.3 with the only change being that instead of using Lipschitzness of $B$ to deduce the inequality (20), we use (29), which is well-defined due to Lemma 3.2. \qed

**Theorem 3.4.** Let $\mathcal{H}$ be finite dimensional, $A : \mathcal{H} \rightrightarrows \mathcal{H}$ be maximally monotone, $B : \mathcal{H} \to \mathcal{H}$ be monotone and locally Lipschitz continuous, and suppose that $(A + B)^{-1}(0) \neq \emptyset$. Then the sequence $(x_k)_{k \in \mathbb{N}}$ generated by Algorithm 1 converges weakly to a point $\bar{x} \in \mathcal{H}$ such that $0 \in A(\bar{x}) + B(\bar{x})$.

**Proof.** We argue similarly to Theorem 2.4 but using Lemma 3.3 in place of Lemma 2.3, and (29) in place of Lipschitzness of $B$. This yields (23) from which we deduce that $(x_k)_{k \in \mathbb{N}}$ is bounded and $\|x_k - x_{k+1}\| \to 0$. As a locally Lipschitz operator on finite dimensional space, $B$ is Lipschitz on bounded sets. Thus, since $(x_k)_{k \in \mathbb{N}}$ is bounded, there exists a constant $L > 0$ such that

$$\|B(x_{k+1}) - B(x_k)\| \leq L \|x_{k+1} - x_k\| \quad \forall k \in \mathbb{N}. \quad (32)$$

By combining (29) and (32), we see that $(\lambda_k)_{k \in \mathbb{N}}$ is bounded away from zero. The remainder of the proof is the same as Theorem 2.4. \qed

## 4 Inertial forward-reflected-backward splitting

In this section, we consider an inertial variant of the forward-reflected-backward splitting algorithm. Such variants are of interest in practice because they tend to improve performance.

Consider the monotone inclusion

$$\text{find } x \in \mathcal{H} \text{ such that } 0 \in A(x) + B(x),$$

where $A : \mathcal{H} \rightrightarrows \mathcal{H}$ and $B : \mathcal{H} \to \mathcal{H}$ is monotone are maximally monotone operators with $B$ $1/L$-cocoercive. The inertial algorithm is given by

$$x_{k+1} := J_{\lambda A} \left(x_k - 2\lambda B(x_k) + \lambda B(x_{k-1}) + \alpha(x_k - x_{k-1})\right) \quad \forall k \in \mathbb{N}, \quad (33)$$
Thus, altogether, \( \sqrt{c} \) and \( \lambda \in \left( 0, \frac{1 + \alpha}{2\rho} \right) \).

To prove convergence of the scheme, we first prove two lemmas.

**Lemma 4.1.** Suppose \( B : \mathcal{H} \to \mathcal{H} \) \( 1/L \)-cocoercive and \( \rho \in \left[ 0, \frac{1}{2} \right] \). Then the operator \( B' := B - \rho I \) is \( L' \)-Lipschitz where \( L' := L - \rho \).

**Proof.** Let \( x, y \in \mathcal{H} \). By using \( 1/L \)-cocoercivity of \( B \), we deduce

\[
\|(B - \rho I)x - (B - \rho I)y\|^2 = \|Bx - By\|^2 - 2\rho \langle Bx - By, x - y \rangle + \rho^2 \|x - y\|^2 \\
\leq \left( 1 - \frac{2\rho}{L} \right) \|Bx - By\|^2 + \rho^2 \|x - y\|^2 \\
\leq \left( L^2 - \frac{2\rho}{L} L^2 + \rho^2 \right) \|x - y\|^2 = (L - \rho)^2 \|x - y\|^2,
\]

from which the result follows. \( \square \)

In the following lemma, we show that in this section the operator \( B' \) plays the same role as the operator \( B \) in Section 2.

**Lemma 4.2.** Let \( x \in (A + B)^{-1}(0) \) and let \( (x_k)_{k \in \mathbb{N}} \) be given by (33). Let \( \lambda > 0 \) and \( \alpha \geq 0 \) be constants satisfying \( 2\alpha \leq \lambda L < \frac{1 + \alpha}{2} \). Then there exists an \( \varepsilon > 0 \) such that

\[
(1 - \alpha) \|x_{k+1} - x\|^2 + 2\lambda \langle B'(x_{k+1}) - B'(x_k), x - x_{k+1} \rangle + ((\lambda \lambda - \alpha) + \varepsilon) \|x_{k+1} - x_k\|^2 \\
\leq (1 - \alpha) \|x_k - x\|^2 + 2\lambda \langle B'(x_k) - B'(x_{k-1}), x - x_k \rangle + (\lambda \lambda - \alpha) \|x_k - x_{k-1}\|^2,
\]

where \( \rho := \alpha / \lambda, B' := B - \rho I \) and \( L' := L - \rho \).

**Proof.** Combining (33) with the monotonicity of \( A \) gives

\[
0 \leq \langle x_{k+1} - x_k + \lambda B(x_k) + \lambda(B(x_k) - B(x_{k-1})) - \alpha(x_k - x_{k-1}) - \lambda B(x), x - x_{k+1} \rangle,
\]

which can be rewritten as

\[
0 \leq (1 - \alpha) \langle x_{k+1} - x_k, x - x_{k+1} \rangle + \lambda \langle B(x_k) - B(x), x - x_{k+1} \rangle + \rho \langle x_{k+1} - x_k, x - x_{k+1} \rangle \\
+ \lambda \langle B'(x_k) - B'(x_{k-1}), x - x_k \rangle + \lambda \langle B'(x_k) - B'(x_{k-1}), x_k - x_{k+1} \rangle.
\]

Using the monotonicity of \( B \), the second term can be estimated as

\[
\langle B(x_k) - B(x), x - x_{k+1} \rangle \leq -\langle B(x_{k+1}) - B(x_k), x - x_{k+1} \rangle \\
= -\langle B'(x_{k+1}) - B'(x_k), x - x_{k+1} \rangle - \rho \langle x_{k+1} - x_k, x - x_{k+1} \rangle.
\]

By Lemma 4.1, \( B' \) is \( (L - \rho) \)-Lipschitz continuous and, consequently, the fourth term can estimated as

\[
\langle B'(x_k) - B'(x_{k-1}), x - x_{k+1} \rangle \leq (L - \rho) \|x_k - x_{k+1}\| \|x_k - x_{k+1}\| \\
\leq \frac{\lambda L - \alpha}{2\lambda} \left( \|x_k - x_{k-1}\|^2 + \|x_k - x_{k+1}\|^2 \right).
\]

Thus, altogether, (35) implies that

\[
0 \leq (1 - \alpha) \left( \|x_k - x\|^2 - \|x_{k+1} - x_k\|^2 - \|x_{k+1} - x_k\|^2 \right) - 2\lambda \langle B'(x_{k+1}) - B'(x_k), x - x_{k+1} \rangle \\
+ 2\lambda \langle B'(x_k) - B'(x_{k-1}), x - x_{k-1} \rangle + (\lambda L - \alpha) \left( \|x_k - x_{k-1}\|^2 + \|x_k - x_{k+1}\|^2 \right),
\]

which we rearrange to yield

\[
(1 - \alpha) \|x_{k+1} - x\|^2 + 2\lambda \langle B'(x_{k+1}) - B'(x_k), x - x_{k+1} \rangle + (1 - \lambda L) \|x_{k+1} - x_k\|^2 \\
\leq (1 - \alpha) \|x_k - x\|^2 + 2\lambda \langle B'(x_k) - B'(x_{k-1}), x - x_{k-1} \rangle + (\lambda L - \alpha) \|x_k - x_{k-1}\|^2.
\]

The claimed inequality follows with \( \varepsilon := (1 - \lambda L) - (\lambda L - \alpha) = 1 + \alpha - 2\lambda L > 0 \). \( \square \)
The following is our main result regarding convergence of the inertial scheme.

**Theorem 4.3.** Let \( A : \mathcal{H} \rightrightarrows \mathcal{H} \) be maximally monotone, let \( B : \mathcal{H} \rightarrow \mathcal{H} \) be monotone, \(1/L\)-cocoercive, and suppose that \( (A + B)^{-1}(0) \neq \emptyset\). Let \( \alpha \in [0, \frac{1}{4}) \) and \( \lambda \in (0, \frac{1+\alpha}{2L}) \). Given \( x_0 \in \mathcal{H}, x_{-1} = x_0 \), define the sequence \( (x_k)_{k \in \mathbb{N}} \) according to

\[
x_{k+1} = J_{\lambda A}(x_k - 2\lambda B(x_k) + \lambda B(x_{k-1}) + \alpha(x_k - x_{k-1})) \quad \forall k \in \mathbb{N}.
\]

Then \( (x_k)_{k \in \mathbb{N}} \) converges weakly to a point \( \overline{x} \in \mathcal{H} \) such that \( 0 \in A(\overline{x}) + B(\overline{x}) \).

**Proof.** Without loss of generality, we may and do assume that

\[
2\alpha \leq \lambda L < \frac{1+\alpha}{2}.
\]

To see why this is the case, suppose that \( \lambda L < 2\alpha \). Since \( \alpha < 1/3 \), we always have \( 2\alpha < \frac{1+\alpha}{2} \) and hence there exists \( \bar{L} > L \) such that \( 2\alpha \leq \lambda \bar{L} < \frac{1+\alpha}{2} \). As \( \bar{L} > L \), the operator \( B \) is \(1/\bar{L}\)-cocoercive and thus (37) holds with \( L \) replaced by \( \bar{L} \).

We now proceed with the proof. From Lemma 4.2, we deduce that

\[
(1 - \alpha) \|x_1 - x\|^2 + 2\lambda \langle B'(x_1) - B'(x_0), x - x_1 \rangle + (\lambda L - \alpha) \|x_1 - x_0\|^2 \\
\geq (1 - \alpha) \|x - x_{k+1}\|^2 + 2\lambda \langle B'(x_{k+1}) - B'(x_k), x - x_{k+1} \rangle \\
+ (\lambda L - \alpha) \|x_{k+1} - x_k\|^2 + \varepsilon \sum_{i=1}^{k} \|x_{i+1} - x_i\|^2.
\]

By Lemma 4.1, \( B' \) is \((L - \rho)\)-Lipschitz and thus

\[
2\lambda \langle B'(x_{k+1}) - B'(x_k), x - x_{k+1} \rangle \geq -2\lambda(L - \rho) \|x_{k+1} - x_k\| \|x - x_{k+1}\| \\
\geq -(\lambda L - \alpha) \left( \|x_{k+1} - x_k\|^2 + \|x - x_{k+1}\|^2 \right).
\]

This, together with (38), yields the inequality

\[
(1 - \lambda L) \|x_{k+1} - x\|^2 + \varepsilon \sum_{i=0}^{k} \|x_{i+1} - x_i\|^2 \leq (1 - \alpha) \|x_0 - x\|^2,
\]

which shows that \( (x_k)_{k \in \mathbb{N}} \) is bounded and \( \|x_k - x_{k+1}\| \rightarrow 0 \). The remainder of the proof follows a similar argument to Theorem 2.4. \( \square \)

**Remark 4.4.** By setting \( B = 0 \) in Theorem 4.3, the scheme (33) reduces to the classical inertial proximal algorithm first considered in [1]. It is interesting to note that the proof presented here seems to be simpler, even in this special case. In contrast, all other analyses of first order inertial operator splitting methods [5, 14, 17] use the same technique as in [1].

**Remark 4.5.** Note that the proof of Theorem 4.3 never directly invokes cocoercivity of \( B \) but rather only relies on Lipschitz continuity of the operator \( B' \). Indeed, cocoercivity of \( B \) was only used to obtain the Lipschitz bound for \( B' \) in Lemma 4.1. The conclusion of Theorem 4.3 thus remain true when \( B \) is merely monotone albeit with worse bounds on the constants \( \alpha \) and \( \lambda \).
5 Three operators splitting

In this section, we consider a structured three operator monotone inclusion. Specifically, we consider the inclusion

$$ \text{find } x \in \mathcal{H} \text{ such that } 0 \in (A + B + C)(x), $$

where $A : \mathcal{H} \rightrightarrows \mathcal{H}$ is maximal monotone, $B : \mathcal{H} \to \mathcal{H}$ is monotone and $L_1$-Lipschitz, and $C : \mathcal{H} \to \mathcal{H}$ is monotone and $1/L_2$-cocoercive. This problem could be solved using the two operator splitting algorithm in Section 2 applied to $A$ and $(B + C)$, where we note that $(B + C)$ is $L$-Lipschitz continuous with $L := L_1 + L_2$. Consequently, to apply Theorem 2.4, the stepsize $\lambda$ should satisfy

$$ \lambda < \frac{1}{2L} = \frac{1}{2L_1 + 2L_2}. $$

In this section, we show that this can be improved by exploiting the additional structure in (39). Indeed, we propose a modification which only requires $\lambda > 0$ to satisfy

$$ \lambda < \frac{2}{4L_1 + L_2} = \frac{1}{2L_1 + \frac{3}{2}L_2}. $$

Given initial points $x_0, x_{-1} \in \mathcal{H}$, our modified scheme is given by

$$ x_{k+1} = J_{\lambda A} \left( x_k - 2\lambda B(x_k) + B(x_{k-1}) - \lambda C(x_k) \right) \quad \forall k \in \mathbb{N}. \quad (40) $$

In other words, the algorithm only uses a standard forward step of the operator $C$, as is employed in the forward-backward method (9).

We begin our analysis with the three operator analogue of Lemma 2.3.

**Lemma 5.1.** Let $x \in (A + B + C)^{-1}(0)$ and let the sequence $(x_k)_{k \in \mathbb{N}}$ be given by (40). Suppose $\lambda \in \left(0, \frac{2}{4L_1 + L_2}\right)$. Then there exists an $\varepsilon > 0$ such that, for all $k \in \mathbb{N}$, we have

$$ \|x_{k+1} - x\|^2 + 2\lambda \langle B(x_{k+1}) - B(x_k), x - x_{k+1} \rangle + (\lambda L_1 + \varepsilon) \|x_{k+1} - x_k\|^2 \\ \leq \|x_k - x\|^2 + 2\lambda \langle B(x_k) - B(x_{k-1}), x - x_k \rangle + \lambda L_1 \|x_k - x_{k-1}\|^2. $$

**Proof.** Since $0 \in (A + B + C)(x)$, we have $-(B + C)(x) \in A(x)$. Combined with the monotonicity of $A$, this gives

$$ 0 \leq \langle x_{k+1} - x_k + \lambda(B + C)(x_k) + \lambda(B(x_k) - B(x_{k-1})) - \lambda(B + C)(x), x - x_{k+1} \rangle, $$

which we rewrite as

$$ 0 \leq \langle x_{k+1} - x_k, x - x_{k+1} \rangle + \lambda \langle B(x_k) - B(x), x - x_{k+1} \rangle + \lambda \langle B(x_k) - B(x_{k-1}), x - x_k \rangle + \lambda \langle B(x_k) - B(x_{k-1}), x - x_{k+1} \rangle \\ + \lambda \langle C(x_k) - C(x), x - x_k \rangle + \lambda \langle C(x_k) - C(x), x - x_{k+1} \rangle. \quad (41) $$

The first through fourth terms can be estimated as in Lemma 2.3. Using $1/L_2$-cocoercivity of $C$, the fifth term can be estimated as

$$ \langle C(x_k) - C(x), x - x_k \rangle \leq -\frac{1}{L_2} \|C(x_k) - C(x)\|^2, $$

and the final term can be estimated as

$$ \langle C(x_k) - C(x), x_k - x_{k+1} \rangle \leq \|C(x_k) - C(x)\| \|x_k - x_{k+1}\| \\ \leq \frac{1}{L_2} \|C(x_k) - C(x)\|^2 + \frac{L_2}{4} \|x_k - x_{k+1}\|^2. $$

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Thus, altogether, (41) implies that

\[
0 \leq \|x_k - x\|^2 - \|x_{k+1} - x\|^2 - \|x_{k+1} - x\|^2 - 2\lambda \langle B(x_{k+1}) - B(x_k), x - x_{k+1} \rangle \\
+ 2\lambda \langle B(x_k) - B(x_{k-1}), x - x_k \rangle + \lambda L_1 \left( \|x_k - x_{k-1}\|^2 + \|x_k - x_{k+1}\|^2 \right) \\
- \frac{2\lambda}{L_2} \|C(x_k) - C(x)\|^2 + \lambda \left( \frac{2}{L_2} \|C(x_k) - C(x)\|^2 + \frac{L_2}{2} \|x_k - x_{k+1}\|^2 \right),
\]

which, on rearranging, gives

\[
\|x_{k+1} - x\|^2 + 2\lambda \langle B(x_{k+1}) - B(x_k), x - x_{k+1} \rangle + \left( 1 - \lambda L_1 - \frac{\lambda L_2}{2} \right) \|x_{k+1} - x\|^2 \\
\leq \|x_k - x\|^2 + 2\lambda \langle B(x_k) - B(x_{k-1}), x - x_k \rangle + \lambda L_1 \|x_k - x_{k-1}\|^2.
\]

The claim inequality follows with \(\varepsilon := \left( 1 - \lambda L_1 - \frac{\lambda L_2}{2} \right) - \lambda L_1 = 1 - 2\lambda L_1 - \frac{\lambda L_2}{2} > 0.\)

The following theorem is our main result regarding convergence of three operator splitting scheme.

**Theorem 5.2.** Let \(A : \mathcal{H} \ni \mathcal{H}\) be maximally monotone, let \(B : \mathcal{H} \to \mathcal{H}\) be monotone and \(L_1\)-Lipschitz, and let \(C : \mathcal{H} \to \mathcal{H}\) be monotone and \(1/L_2\)-cocoercive. Suppose that \((A + B + C)^{-1}(0) \neq \emptyset\) and \(\lambda \in \left( 0, \frac{2}{4L_1 + L_2} \right)\). Given \(x_0 \in \mathcal{H}\), \(x_{-1} = x_0\), define the sequence \((x_k)_{k \in \mathbb{N}}\) according to

\[x_{k+1} = J_{\lambda A} \left( x_k - 2\lambda B(x_k) + \lambda B(x_{k-1}) - \lambda C(x_k) \right) \quad \forall k \in \mathbb{N}.
\]

Then \((x_k)_{k \in \mathbb{N}}\) converges weakly to a point \(\mathbf{p} \in \mathcal{H}\) such that \(0 \in (A + B + C)(\mathbf{p})\).

**Proof.** The proof is more or less the same as Theorem 2.4 but uses Lemma 5.1 in place of Lemma 2.3. The only other thing to check is that

\[
\|x_{k+1} - x\|^2 + 2\lambda \langle B(x_{k+1}) - B(x_k), x - x_{k+1} \rangle + \lambda L_1 \|x_{k+1} - x\|^2
\]

is bounded from below by zero. To see this, observe that

\[
2\lambda \langle B(x_{k+1}) - B(x_k), x - x_{k+1} \rangle \geq -2\lambda L_1 \|x_{k+1} - x\| \|x_{k+1} - x\| \\
\geq -\lambda L_1 \left( \|x_{k+1} - x\|^2 + \|x_{k+1} - x\|^2 \right),
\]

and \(1 - \lambda L_1 > 0\) since \(\lambda < \frac{2}{4L_1 + L_2} < \frac{1}{L_1}\). \(\Box\)

6 Concluding remarks

In this work, we have proposed a modification of the forward-backward algorithm for the finding a zero in the sum of two monotone operators which does not require cocoercivity. To conclude, we outline three possible directions for further research into the method.

**Fixed point interpretations:** As the proof of the forward-reflected-backward method does not conform to the usual Krasnoselskii–Mann framework, it would be interesting to see if the method can be analyzed from the perspective of fixed point theory. To this end, consider the two operators \(M, T : \mathcal{H} \times \mathcal{H} \to \mathcal{H} \times \mathcal{H}\) given by

\[M := \begin{bmatrix} J_{\lambda A} & 0 \\
0 & I \end{bmatrix}, \quad T := \begin{bmatrix} I - 2\lambda B & \lambda I \\
B & 0 \end{bmatrix}.
\]
By introducing the auxiliary variable \( u_{k+1} := B(x_k) \), it is easy to see that (14) may be expressed as the fixed point iteration in \( \mathcal{H} \times \mathcal{H} \) given by

\[
\begin{pmatrix}
x_{k+1} \\
w_{k+1}
\end{pmatrix} = (M \circ T)\begin{pmatrix} x_k \\ u_k \end{pmatrix}.
\]

From the perspective of fixed point theory, it is not clear what properties the operator \( M \circ T \) possesses which can be used to deduce convergence. For instance, although \( M \) is \textit{firmly nonexpansive}, the operator \( T \) need not be. A similar question regarding interpretations of the \textit{golden ratio algorithm}, for which the operator \( M \) is of the same form, was posed in [16].

**Stochastic and coordinate extensions:** In large-scale problems, it is not always possible to fully evaluate the operator \( B \) owing to its high computational cost. Two possibilities for reducing the computational requirements are \textit{stochastic approximations} of \( B(x_k) \) and \textit{block coordinate} variants of the algorithm. Both approaches work by employing low-cost approximation of \( B(x_k) \) in each iteration. It would be interesting to consider stochastic and coordinate extensions of the method proposed here.

**Acceleration schemes:** As explained in Section 1, the forward-reflected-backward method can be specialized to solve a minimization problem involving the sum of two convex functions, one of which is smooth. In 2007, in [18], Nesterov exploited his original idea from [19] to derive accelerated proximal gradient methods that enjoy better complexity rates than the standard forward-backward method. It therefore seems reasonable that the forward-reflected-backward method could be adapted to incorporate a Nesterov-type acceleration.

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