Weighted Hardy inequalities with infinitely many sharp missing terms

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Abstract

The main purpose of this article is to improve classical weighted Hardy type inequalities by adding infinitely many sharp new missing terms. The key of proof is to construct a family of positive solutions for ordinary differential equations involving slowly increasing functions in their coefficients.

1. Introduction

It is known that the following weighted Sobolev inequalities

\[ \int_\Omega |\nabla u(x)|^p |x|^\alpha dx \geq C(p,q,\alpha,\beta,n) \left( \int_\Omega |u(x)|^q |x|^\beta dx \right)^{p/q} \]

hold for any \( u \in W^{1,p}_{\alpha,0}(\Omega) \), where \( \Omega \) is a bounded domain of \( \mathbb{R}^n \) containing the origin, \( n \in \mathbb{N} \) and \( p,q,\alpha,\beta \in \mathbb{R} \) are satisfying the relations \( 1 \leq p < \infty, 0 \leq 1/p - 1/q = (1 - \alpha + \beta)/n < 1/p, -n/q < \beta \leq \alpha \). Here \( C(p,q,\alpha,\beta,n) > 0 \) is the best constant depending only on \( p,q,\alpha,\beta \) and \( n \). In special case \( p = q \),

\[ \int_\Omega |\nabla u(x)|^p |x|^\alpha dx \geq \Lambda_{p,\alpha,n} \int_\Omega |u(x)|^p |x|^{(\alpha-1)p} dx \]

are called weighted Hardy inequalities or weighted Hardy-Sobolev inequalities, where the best constant \( \Lambda_{p,\alpha,n} = [(n-p)/p + \alpha]^p \) in the noncritical case \( \alpha \neq 1 - n/p \) is given by the infimum of

\[ \int_\Omega |\nabla u(x)|^p |x|^\alpha dx / \int_\Omega |u(x)|^p |x|^{(\alpha-1)p} dx \]

for \( u \in W^{1,p}_{\alpha,0}(\Omega) \setminus \{0\} \). However, there exists no extremal function in \( W^{1,p}_{\alpha,0}(\Omega) \) which attains the infimum of this quantity. Roughly speaking, the candidates of the extremals are too singular at the origin to belong to the the admissible class \( W^{1,p}_{\alpha,0}(\Omega) \).

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Hence it is natural to consider that there exist “missing terms” in the right-hand side of weighted Hardy inequalities. In view of this, we shall investigate the weighted Hardy inequalities and improve them by finding out missing terms.

In the case $\alpha = 0$ and $p > 1$, several results are known for the existence of missing terms involving the logarithmic type of weights, see for example [1], [5], etc. In [2], it was shown for $p > 1$ that there exist missing terms in the right-hand side of weighted Hardy inequalities such as, in the noncritical case

$$\int_{\Omega} |\nabla u(x)|^p |x|^\alpha dx \geq A_{p,\alpha,n} \int_{\Omega} |u(x)|^p |x|^{\alpha-1} dx + C_n \int_{\Omega} |u(x)|^p |x|^{\alpha-1} \left( A_1(|x|) \right)^{-2} dx,$$

where $A_1(r) = \log(R/r)$ ($0 < r < R = \sup_{x \in \Omega} |x|$) and the constant $C_n > 0$ depends only on $n$.

When $p = 2$ particularly, it was proved in [6] that there exist any finitely many sharp missing terms for $\alpha = 0$ and $n \geq 2$. Moreover, in [7], the improved weighted Hardy inequalities with infinitely many sharp missing terms are shown for $\alpha \geq 1 - n/2$ and $n \geq 3$ as the following

$$\int_{\Omega} |\nabla u(x)|^2 |x|^{2\alpha} dx \geq A_{2,\alpha,n} \int_{\Omega} |u(x)|^2 |x|^{2(\alpha-1)} dx + \frac{1}{4} \sum_{k=1}^{\infty} \int_{\Omega} |u(x)|^2 |x|^{2(\alpha-1)} \left( A_1(|x|) A_2(|x|) \cdots A_k(|x|) \right)^{-2} dx,$$

where $\lim_{r \to 0} \left( A_k(r)/\log^k(1/r) \right) = 1$ for all $k \in \mathbb{N}$ and $\log^k$ denotes $k$th iterated logarithm.

In this paper, we show that the weighted Hardy inequalities can be improved furthermore by adding infinitely many sharp new missing terms for any $\alpha \in \mathbb{R}$ and $n \in \mathbb{N}$. The key of proof is to construct a family of positive solutions for ordinary differential equations involving slowly increasing functions in their coefficients, which will be produced in [4]. Our idea of slowly increasing functions is established in [3].

In sections 2 and 3, we state the main results and give their proofs, respectively.

Let $a > 1$ be fixed. Let

$$F(u) = F_a(u) = a - \log a + \log u = a + \log \frac{u}{a} = a + \int_a^u \frac{1}{t} dt \quad (u \geq a) \quad (1.1)$$

and, for $k \in \mathbb{N}$,

$$F^0(u) = u, \quad F^k(u) = F(F^{k-1}(u)) \quad (u \geq a).$$

Further let

$$\tilde{F}(u) = a \prod_{k=0}^{\infty} \frac{F^k(u)}{a}, \quad (1.2)$$

and

$$\phi(u) = a + \int_a^u \frac{1}{F(t)} dt \quad (u \geq a) \quad (1.3)$$
and, for $k \in \mathbb{N}$,

$$\phi^0(u) = u, \quad \phi^k(u) = \phi(\phi^{k-1}(u)) \quad (u \geq a).$$

**Definition 1.1.** Let $k \in \mathbb{N}$. Let

\begin{align*}
A_k^{(0)}(r) &= F^k(ar), \\
A_k^{(1)}(r) &= \phi(ar), \\
A_k^{(1)}(r) &= F^k(\phi(ar)) = F^k(A_0^{(1)}(r)) \quad (r \geq 1)
\end{align*}

and

$$B^{(0)}(r) = \frac{\bar{F}(ar)}{ar} = \prod_{k=1}^{\infty} \frac{F^k(ar)}{a} = \prod_{k=1}^{\infty} \frac{A_k^{(0)}(r)}{a} \quad (r \geq 1).$$

We note that the following relations

$$\lim_{r \to \infty} \frac{A_k^{(0)}(r)}{\log^k r} = 1, \quad \lim_{r \to \infty} \frac{A_k^{(0)}(r)}{B^{(0)}(r)} = 0,$$

$$\lim_{r \to \infty} \frac{A_0^{(1)}(r)}{A_k^{(0)}(r)} = 0, \quad \lim_{r \to \infty} \frac{A_k^{(1)}(r)}{\log^k A_0^{(1)}(r)} = 1$$

hold for any $k \in \mathbb{N}$.

**Theorem 1.1.** Let $\Omega$ be a bounded domain of $\mathbb{R}^n$ and $0 \in \Omega$. Let $R = R_\Omega = \sup_{x \in \Omega} |x|$. Let $\alpha \in \mathbb{R}$. For $u \in C_0^1(\Omega \setminus \{0\})$, the following inequality holds.

$$\int_{\Omega} |\nabla u(x)|^2 |x|^{2\alpha} dx \geq \left( \alpha + \frac{n-2}{2} \right)^2 \int_{\Omega} (u(x))^2 |x|^{2\alpha-2} dx$$

$$+ \frac{1}{4} \int_{\Omega} (u(x))^2 |x|^{2\alpha-2} \sum_{j=1}^{\infty} \left( \prod_{k=1}^{j} A_k^{(0)} \left( \frac{R}{|x|} \right) \right)^{-2} dx$$

$$+ \frac{1}{4} \int_{\Omega} (u(x))^2 |x|^{2\alpha-2} \left( B^{(0)}(\frac{R}{|x|}) A_0^{(1)}(\frac{R}{|x|}) \right)^{-2} dx. \quad (1.7)$$

Moreover, for $u \in C_0^1(\Omega \setminus \{0\})$, the following inequality holds.

$$\int_{\Omega} |\nabla u(x)|^2 |x|^{2\alpha} dx \geq \left( \alpha + \frac{n-2}{2} \right)^2 \int_{\Omega} (u(x))^2 |x|^{2\alpha-2} dx$$

$$+ \frac{1}{4} \int_{\Omega} (u(x))^2 |x|^{2\alpha-2} \sum_{j=1}^{\infty} \left( \prod_{k=1}^{j} A_k^{(0)} \left( \frac{R}{|x|} \right) \right)^{-2} dx$$

$$+ \frac{1}{4} \int_{\Omega} (u(x))^2 |x|^{2\alpha-2} \sum_{j=0}^{\infty} \left( B^{(0)}(\frac{R}{|x|}) \prod_{k=0}^{j} A_k^{(1)} \left( \frac{R}{|x|} \right) \right)^{-2} dx. \quad (1.8)$$
**Definition 1.2.** Let \( m, k \in \mathbb{N} \). Let
\[
A^{(m)}_0(r) = \phi^m(ar), \quad A^{(m)}_k(r) = F^k(\phi^m(ar)) = F^k(A^{(m)}_0(r)) \quad (r \geq 1)
\]
and
\[
B^{(m)}_k(r) = F(\phi^m(ar)) = a \prod_{k=0}^{\infty} \frac{F^k(\phi^m(ar))}{a} = a \prod_{k=0}^{\infty} \frac{A^{(m)}_k(r)}{a} \quad (r \geq 1).
\]

We note that the following relations
\[
\lim_{r \to \infty} \frac{A^{(l)}_k(r)}{\log^k A^{(l)}_0(r)} = 1, \quad \lim_{r \to \infty} \frac{A^{(l)}_k(r)}{B^{(l)}(r)} = 0, \quad \lim_{r \to \infty} \frac{A^{(l+1)}_k(r)}{A^{(l)}_k(r)} = 0
\]
hold for any \( l, k \in \mathbb{N} \).

**Theorem 1.2.** Let \( \Omega \) be a bounded domain of \( \mathbb{R}^n \) and \( 0 \in \Omega \). Let \( R = R_\Omega = \sup_{x \in \Omega} |x| \). Let \( \alpha \in \mathbb{R} \) and \( m \in \mathbb{N} \). For \( u \in C_0^1(\Omega \setminus \{0\}) \), the following inequality holds.
\[
\int_{\Omega} |\nabla u(x)|^2 |x|^{2\alpha} dx \geq \left( \alpha + \frac{n-2}{2} \right)^2 \int_{\Omega} (u(x))^2 |x|^{2\alpha - 2} dx
\]
\[
+ \frac{1}{4} \int_{\Omega} (u(x))^2 |x|^{2\alpha - 2} \sum_{j=1}^{\infty} \left( \prod_{k=1}^{j} \frac{A^{(0)}_k(R)}{|x|} \right)^{-2} dx
\]
\[
+ \frac{1}{4} \int_{\Omega} (u(x))^2 |x|^{2\alpha - 2} \sum_{j=1}^{\infty} \sum_{\nu=1}^{\infty} \left( \prod_{i=0}^{\nu-1} B^{(l)}(R) \prod_{k=0}^{j} \frac{A^{(\nu)}_k(R)}{|x|} \right)^{-2} dx.
\]

Moreover, all the constants \( \alpha + (n-2)/2 \) and \( 1/4 \) are best.

**2. Main results**

From now on, let \( \Omega \) be a bounded domain of \( \mathbb{R}^n \) and \( 0 \in \Omega \), or an exterior domain of \( \mathbb{R}^n \) such that \( \overline{\Omega} = \mathbb{R}^n \setminus \Omega \) is a bounded domain of \( \mathbb{R}^n \) and \( 0 \in \overline{\Omega} \). We define
\( R = R_\Omega \) by \( R = \sup_{x \in \Omega} |x| \) for the bounded domain \( \Omega \), and by \( R = \inf_{x \in \Omega} |x| = d(0, \Omega) \) for the exterior domain \( \Omega \), respectively.

**Definition 2.1.** Let \( m, j \in \mathbb{N} \cup \{0\} \) and \( m + j \geq 1 \). For \( r \geq 1 \), we define \( S^{(m)}_j(r) \) by
\[
S^{(m)}_{j+1}(r) = \frac{1}{2} \sum_{i=1}^{j} \left( \prod_{k=1}^{i} A^{(m)}_k(r) \right)^{-2} \quad (j \geq 1),
\]
\[
S^{(m)}_0(r) = \sum_{i=1}^{\infty} \left( \prod_{k=1}^{i} A^{(0)}_k(r) \right)^{-2}.
\]
and, for $m \geq 1$,
\[
S_{j+1}^{(m)}(r) = S_0^{(m)}(r) + \sum_{i=0}^{j} \left( \prod_{k=0}^{m-1} B^{(l)}(r) \prod_{k=0}^{i} A_k^{(m)}(r) \right)^{-2} ,
\]
(2.4)
\[
S_0^{(m+1)}(r) = \sum_{i=1}^{\infty} \left( \prod_{k=1}^{i} A_k^{(0)}(r) \right)^{-2} + \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \left( \prod_{k=0}^{m-1} B^{(l)}(r) \prod_{k=0}^{j} A_k^{(0)}(r) \right)^{-2} .
\]
(2.5)

For $r \geq 1$, we define $U_j^{(m)}$ by
\[
U_1^{(0)}(r) = r^{1/2} ,
\]
(2.6)
\[
U_{j+1}^{(0)}(r) = \left( r \prod_{k=1}^{j} \frac{A_k^{(0)}(r)}{a} \right)^{1/2} (j \geq 1) ,
\]
(2.7)
\[
U_0^{(1)}(r) = \left( r \prod_{k=1}^{\infty} \frac{A_k^{(0)}(r)}{a} \right)^{1/2} = \left( r B^{(0)}(r) \right)^{1/2} ,
\]
(2.8)
\[
U_{j+1}^{(1)}(r) = \left( r B^{(0)}(r) \prod_{k=0}^{j} \frac{A_k^{(1)}(r)}{a} \right)^{1/2} ,
\]
(2.9)
and, for $m \geq 2$,
\[
U_0^{(m)}(r) = \left( r B^{(0)}(r) \prod_{l=1}^{m-1} \frac{B^{(l)}(r)}{a} \right)^{1/2} ,
\]
(2.10)
\[
U_{j+1}^{(m)}(r) = \left( r B^{(0)}(r) \prod_{l=1}^{m-1} \frac{B^{(l)}(r)}{a} \prod_{k=0}^{j} \frac{A_k^{(m)}(r)}{a} \right)^{1/2} .
\]
(2.11)

**Definition 2.2.** Let $m, j \in \mathbb{N} \cup \{0\}$ and $m + j \geq 1$. Let $\alpha \in \mathbb{R}$. We define $I_{\alpha,j}^{(m)}(u : \Omega)$ as follows. If $\Omega$ is a bounded domain of $\mathbb{R}^n$ and $0 \in \Omega$, we set
\[
I_{\alpha,j}^{(m)}(u : \Omega) = \int_{\Omega} |\nabla u(x)|^2 |x|^{2\alpha} dx - \left( \alpha + \frac{n - 2}{2} \right)^2 \int_{\Omega} (u(x))^2 |x|^{2\alpha - 2} dx
\]
\[ - \frac{1}{4} \int_{\Omega} (u(x))^2 |x|^{2\alpha - 2} S_j^{(m)} \left( \frac{R}{|x|} \right) dx \]  
(2.12)
for $u \in C^1_0(\Omega \setminus \{0\})$, where $R = \sup_{x \in \Omega} |x|$. If $\Omega$ is an exterior domain of $\mathbb{R}^n$ such that $(\Omega^c)$ is a bounded domain of $\mathbb{R}^n$ and $0 \in (\Omega^c)$, we set
\[
I_{\alpha,j}^{(m)}(u : \Omega) = \int_{\Omega} |\nabla u(x)|^2 |x|^{2\alpha} dx - \left( \alpha + \frac{n - 2}{2} \right)^2 \int_{\Omega} (u(x))^2 |x|^{2\alpha - 2} dx
\]
\[ - \frac{1}{4} \int_{\Omega} (u(x))^2 |x|^{2\alpha - 2} S_j^{(m)} \left( \frac{|x|}{R} \right) dx \]  
(2.13)
for $u \in C^1_0(\Omega)$, where $R = \inf_{x \in \Omega} |x|$.
Theorem 2.1. Let \( \Omega \) be a bounded domain of \( \mathbb{R}^n \) and \( 0 \in \Omega \). Let \( R = \sup_{x \in \Omega} |x| \). Let \( \alpha \in \mathbb{R} \) and let \( m, j \in \mathbb{N} \cup \{0\} \) with \( m + j \geq 1 \). For \( u \in C^0_0(\Omega \setminus \{0\}) \), the following equality holds.

\[
I^{(m)}_{\alpha, j}(u : \Omega) = \int_{\Omega} |\nabla u(x)|^{\alpha} |x|^{-\alpha(n-3)/2} U^{(m)}_j \left( \frac{|x|}{R} \right) \left( \frac{R}{|x|} \right)^{1-\alpha} \left( U^{(m)}_j \left( \frac{|x|}{R} \right) \right) ^2 |x|^{3-n} \, dx.
\]

(2.14)

Theorem 2.2. Let \( \Omega \) be a bounded domain of \( \mathbb{R}^n \) and \( 0 \in \Omega \). Let \( R = \sup_{x \in \Omega} |x| \). Let \( \alpha \in \mathbb{R} \) and let \( m, j \in \mathbb{N} \cup \{0\} \) with \( m + j \geq 1 \). For \( u \in C^0_0(\Omega \setminus \{0\}) \), the following inequality holds.

\[
\int_{\Omega} |\nabla u(x)|^2 |x|^{2\alpha} \, dx \geq (\alpha + \frac{n-2}{2})^2 \int_{\Omega} (u(x))^2 |x|^{2\alpha - 2} \, dx + \frac{1}{4} \int_{\Omega} (u(x))^2 |x|^{2\alpha - 2} S^{(m)}_j \left( \frac{|x|}{R} \right) \, dx.
\]

(2.15)

Moreover, all the constants \((\alpha + (n-2)/2)^2 \) and \(1/4\) are best.

We note that Theorems 1.1 and 1.2 are corresponding to the cases \( m = 2, j = 0 \) and \( m = m + 1, j = 0 \) in Theorem 2.2, respectively.

Theorem 2.3. Let \( \Omega \) be an exterior domain of \( \mathbb{R}^n \) such that \( \overline{\Omega}^c \) is a bounded domain of \( \mathbb{R}^n \) and \( 0 \in \overline{\Omega}^c \). Let \( R = \inf_{x \in \Omega} |x| \). Let \( \alpha \in \mathbb{R} \) and let \( m, j \in \mathbb{N} \cup \{0\} \) with \( m + j \geq 1 \). For \( u \in C^0_0(\Omega) \), the following equality holds.

\[
I^{(m)}_{\alpha, j}(u : \Omega) = \int_{\Omega} |\nabla u(x)|^{\alpha} |x|^{-\alpha(n-1)/2} U^{(m)}_j \left( \frac{|x|}{R} \right) \left( \frac{R}{|x|} \right)^{1-\alpha} \left( U^{(m)}_j \left( \frac{|x|}{R} \right) \right) ^2 |x|^{1-n} \, dx.
\]

(2.16)

Theorem 2.4. Let \( \Omega \) be an exterior domain of \( \mathbb{R}^n \) such that \( \overline{\Omega}^c \) is a bounded domain of \( \mathbb{R}^n \) and \( 0 \in \overline{\Omega}^c \). Let \( R = \inf_{x \in \Omega} |x| \). Let \( \alpha \in \mathbb{R} \) and let \( m, j \in \mathbb{N} \cup \{0\} \) with \( m + j \geq 1 \). For \( u \in C^0_0(\Omega) \), the following inequality holds.

\[
\int_{\Omega} |\nabla u(x)|^2 |x|^{2\alpha} \, dx \geq (\alpha + \frac{n-2}{2})^2 \int_{\Omega} (u(x))^2 |x|^{2\alpha - 2} \, dx + \frac{1}{4} \int_{\Omega} (u(x))^2 |x|^{2\alpha - 2} S^{(m)}_j \left( \frac{|x|}{R} \right) \, dx.
\]

(2.17)

Moreover, all the constants \((\alpha + (n-2)/2)^2 \) and \(1/4\) are best.

Definition 2.3. Let \( \alpha \in \mathbb{R} \) and let \( m, j \in \mathbb{N} \cup \{0\} \) with \( m + j \geq 1 \). Let \( X \) be a nonempty open subset of \( \Omega \). We define the norm \( \| \cdot \|_{W^{(m)}_{\alpha, j}(X ; \Omega)} \) by

\[
\| u \|_{W^{(m)}_{\alpha, j}(X ; \Omega)} = \left( I^{(m)}_{\alpha, j}(u : \Omega) \right)^{1/2}.
\]

(2.18)
for $u \in C_0^1(\Omega \setminus \{0\})$. In particular, we use a simple notation $\|\|_{W^{(m)}_{\alpha,j}(\Omega)} = \|\|_{W^{(m)}_{\alpha,j}(\Omega;\Omega)}$.

By embedding $C_0^1(\Omega \setminus \{0\}) \subset C_0^1(\Omega \setminus \{0\})$, from Theorems 2.1 and 2.3 it follows that $\|\|_{W^{(m)}_{\alpha,j}(\Omega)}$ is the norm of $C_0^1(\Omega \setminus \{0\})$.

**Definition 2.4.** Let $\alpha \in \mathbb{R}$ and let $m, j \in \mathbb{N} \cup \{0\}$ with $m + j \geq 1$. Let $X$ be a nonempty open subset of $\Omega$. $W^{(m)}_{\alpha,j}(X : \Omega)$ denotes the completion of $C_0^1(\Omega \setminus \{0\})$ with respect to the norm $\|\|_{W^{(m)}_{\alpha,j}(X;\Omega)}$. In particular, we use a simple notation $W^{(m)}_{\alpha,j}(\Omega) = W^{(m)}_{\alpha,j}(\Omega : \Omega)$.

**Proposition 2.1.** Let $\alpha \in \mathbb{R}$. For each $m, j \in \mathbb{N} \cup \{0\}$ with $m + j \geq 1$, the following relations hold.

\[
W^{(m)}_{\alpha,j}(\Omega) \not\subset W^{(m)}_{\alpha,j+1}(\Omega),
\]
\[
W^{(m)}_{\alpha,j}(\Omega) \not\subset W^{(m+1)}_{\alpha,0}(\Omega).
\]

3. Proofs

We provide the notations for the proofs of main results. For $x \in \mathbb{R}^n \setminus \{0\}$, $\tilde{x}$ denotes the inversion of $x$, that is, $\tilde{x} = x/|x|^2$. For $\emptyset \neq X \subset \mathbb{R}^n$, we define $\tilde{X}$ by $\tilde{X} = \{\tilde{x} : x \in X \setminus \{0\}\}$. Let $\mathbb{B}_\rho^n = \{x \in \mathbb{R}^n : |x| < \rho\}$ for $\rho > 0$. For $\emptyset \neq X \subset \mathbb{R}^n$ and a function $f$ defined on $X$, we define a function $\tilde{f}$ on $\tilde{X}$ by

\[
\tilde{f}(x) = f(\tilde{x}) \quad (x \in \tilde{X}).
\]

We see that

- $\tilde{x} = x$, $|\tilde{x}| = |x|^{-1}$ $(x \in \mathbb{R}^n \setminus \{0\})$,
- $\tilde{X} = X \setminus \{0\}$ $(\emptyset \neq X \subset \mathbb{R}^n)$,
- $\tilde{f}(\tilde{x}) = f(x)$ $(x \in \tilde{X})$.

$\mathbb{B}^n_\rho = \{x \in \mathbb{R}^n : |x| > \rho^{-1}\} = \left(\mathbb{B}^n_{\rho^{-1}}\right)^c$ ($\rho > 0$).

For a function $f$ and $\rho > 0$, we define $f_\rho$ by

\[
f_\rho(x) = f(\rho x).
\]

Now we will give the proofs in order of Theorems 2.1, 2.3, 2.4, 2.2 and Proposition 2.1.

**Proof of Theorem 2.1.** Changing the variable from $x \in \Omega \setminus \{0\}$ to $y \in \tilde{\Omega}$ by $y = \tilde{x}$, in view of

\[
|x| = |\tilde{y}| = |y|^{-1}, \quad dx = d\tilde{y} = |y|^{-2n}dy, \quad |\nabla_x u(x)| = |\nabla_{\tilde{y}} u(\tilde{y})| = |y|^2|\nabla_{\tilde{y}} \tilde{u}(y)|,
\]

(3.1)
we see that \( \tilde{u} \in C_0^1(\tilde{\Omega}) \) and
\[
\int_{\Omega} |\nabla u(x)|^2|x|^{2\alpha}dx - \left( \alpha + \frac{n-2}{2} \right)^2 \int_{\Omega} (u(x))^2|x|^{2\alpha-2}dx \\
- \frac{1}{4} \int_{\Omega} (u(x))^2|x|^{2\alpha-2}S_j^{(m)} \left( \frac{R}{|x|} \right) dx
\]
\[
= \int_{\widetilde{\Omega}} |\nabla \tilde{u}(y)|^2|y|^{2(2-n-\alpha)}dy - \left( 2 - n - \alpha + \frac{n-2}{2} \right)^2 \int_{\Omega} (\tilde{u}(y))^2|y|^{2(2-n-\alpha)-2}dy \\
- \frac{1}{4} \int_{\widetilde{\Omega}} (\tilde{u}(y))^2|y|^{2(2-n-\alpha)-2}S_j^{(m)} \left( \frac{|y|}{R-1} \right) dy,
\]
which implies the equality
\[
I^{(m)}_{\alpha,j}(u : \Omega) = I^{(m)}_{2-n-\alpha,j}(\tilde{u} : \tilde{\Omega}) \tag{3.2}
\]
because of \( \inf_{y \in \widetilde{\Omega}} |y| = \inf_{x \in \Omega \setminus \{0\}} |x|^{-1} = \left( \sup_{x \in \Omega \setminus \{0\}} |x| \right)^{-1} = R^{-1} \) and \( \tilde{\Omega}^c \supset \mathbb{B}_R^\alpha \). Since there exists a \( \rho \) such that \( 0 < \rho \leq R \) and \( \mathbb{B}_\rho^\alpha \subset \Omega \subset \mathbb{B}_{\rho+1}^\alpha \), we have \( \mathbb{B}_{\rho-1}^\alpha \supset \tilde{\Omega} \supset \mathbb{B}_\rho^\alpha = (\mathbb{R}_R^\alpha)^c \), that is, \( \mathbb{B}_{\rho-1}^\alpha \supset \tilde{\Omega}^c \). Hence we apply Theorem 2.3 for \( \tilde{\Omega} \) so that
\[
I^{(m)}_{2-n-\alpha,j}(\tilde{u} : \tilde{\Omega}) \\
= \int_{\tilde{\Omega}} \left| \nabla \tilde{u}(y) \left( |y|^{-2-n-\alpha-(n-1)/2}U_j^{(m)} \left( \frac{|y|}{R-1} \right) \right) \right|^2 \left( U_j^{(m)} \left( \frac{|y|}{R-1} \right) \right)^2 |y|^{1-n}dy \\
= \int_{\Omega} \left| \nabla_x u(x) \left( |x|^{2-n-\alpha+(n-1)/2}U_j^{(m)} \left( \frac{R}{|x|} \right) \right) \right|^2 \left( U_j^{(m)} \left( \frac{R}{|x|} \right) \right)^2 |x|^{n-1} \\
\cdot |x|^{-2n}dx \tag{3.3}
\]
by using the relations (3.1). Consequently (2.14) follows from (3.2) and (3.3).

**Proof of Theorem 2.3.** Since \( \Omega \subset (\mathbb{B}_R^\alpha)^c \) by the assumption, we have \( C_0^1(\Omega) \subset C_0^1((\mathbb{R}_R^\alpha)^c) \) and
\[
I^{(m)}_{\alpha,j}(u : \Omega) = I^{(m)}_{\alpha,j}(u : (\mathbb{B}_R^\alpha)^c) \quad \text{for} \quad u \in C_0^1(\Omega). \tag{3.4}
\]
If \( u \in C_0^1((\mathbb{B}_R^\alpha)^c) \), then \( u_R \in C_0^1((\mathbb{B}_R^\alpha)^c) \) and
\[
I^{(m)}_{\alpha,j}(u : (\mathbb{B}_R^\alpha)^c) = R^{2\alpha+n-2}I^{(m)}_{\alpha,j}(u_R : (\mathbb{B}_R^\alpha)^c). \tag{3.5}
\]
Therefore we have
\[
I^{(m)}_{\alpha,j}(u : \Omega) = R^{2\alpha+n-2}I^{(m)}_{\alpha,j}(u_R : (\mathbb{B}_R^\alpha)^c) \quad \text{for} \quad u \in C_0^1(\Omega). \tag{3.6}
\]
Hence (2.16) follows from (3.6) and the next proposition. \( \square \)
Proposition 3.1. Let $\alpha \in \mathbb{R}$ and let $m, j \in \mathbb{N} \cup \{0\}$ with $m + j \geq 1$. For $u \in C^1_0((\mathbb{R}^n)^c)$, the following equality holds.

\[
I_{\alpha,j}^{(m)}(u : (\mathbb{R}^n)^c) = \int_{(\mathbb{R}^n)^c} \nabla \left( u(x) \left( |x|^{-\alpha-(n-1)/2} U_j^{(m)}(|x|) \right)^{-1} \right) \left( U_j^{(m)}(|x|) \right)^2 |x|^{1-n} dx. \tag{3.7}
\]

To prove Proposition 3.1, we prepare the following two propositions.

Proposition 3.2. For each $m, j \in \mathbb{N} \cup \{0\}$ with $m + j \geq 1$, the equality

\[
- \left( U_j^{(m)}(r) \right)'' = \frac{1}{4} r^{-2} \left( 1 + S_j^{(m)}(r) \right) U_j^{(m)}(r) \quad (r \geq 1) \tag{3.8}
\]

holds.

The proof of Proposition 3.2 will be given in [4].

Proposition 3.3. Let $\alpha \in \mathbb{R}$ and let $m, j \in \mathbb{N} \cup \{0\}$ with $m + j \geq 1$. For $u \in C^1_0((1, \infty))$, the following equality holds.

\[
\int_1^\infty (u'(r))^2 r^{2\alpha} dr = \left( \alpha - \frac{1}{2} \right)^2 \int_1^\infty (u(r))^2 r^{2\alpha-2} dr + \frac{1}{4} \int_1^\infty (u(r))^2 r^{2\alpha-2} S_j^{(m)}(r) dr + \int_1^\infty \left\{ \left( u(r) \left( r^{-\alpha} U_j^{(m)}(r) \right)^{-1} \right)' \right\}^2 \left( U_j^{(m)}(r) \right)^2 dr. \tag{3.9}
\]

Proof. Let $T \in C^2((1, \infty))$ and $T(r) > 0 \ (r > 1)$. For $u \in C^1_0((1, \infty))$, by

\[
\{(u(r)T(r))'\}^2 = (u'(r)T(r) + u(r)T'(r))^2 = (u'(r))^2 (T(r))^2 + \{(u(r))^2 T(r)T'(r)\}' - (u(r))^2 T(r)T''(r) = (u'(r))^2 (T(r))^2 - (u(r)T(r))^2 (T(r))^{-1} T''(r) + \{(u(r)T(r))^2 (\log T(r))'\}'. \tag{3.10}
\]

and

\[
(u'(r))^2 = \left( \{u(r)(T(r))^{-1} T(r)\}' \right)^2 = \left( \{u(r)(T(r))^{-1} \}' \right)^2 (T(r))^2 - (u(r))^2 (T(r))^{-1} T''(r) + \{(u(r))^2 (\log T(r))'\}', \tag{3.11}
\]

\[
= (u'(r))^2 (T(r))^2 - (u(r))^2 (T(r))^{-1} T''(r) + \{(u(r))^2 (\log T(r))'\}'. \tag{3.12}
\]
we have the equalities

\[
\int_1^\infty (u'(r))^2 (T(r))^2 \, dr = \int_1^\infty \{ (u(r) T(r))' \}^2 \, dr + \int_1^\infty (u(r) T(r))^2 (T(r))^{-1} T''(r) \, dr
\]

and

\[
\int_1^\infty (u'(r))^2 \, dr = \int_1^\infty (u(r) (T(r))^{-1})^2 (T(r))^2 \, dr - \int_1^\infty (u(r))^2 (T(r))^{-1} T''(r) \, dr,
\]

respectively. Applying (3.12) with \(T(r) = r^\alpha\), we see that

\[
\int_1^\infty (u'(r))^2 r^{2\alpha} \, dr = \int_1^\infty \{ (u(r) r^\alpha) \}'^2 \, dr + \int_1^\infty (u(r) r^\alpha)^2 r^{-\alpha} (r^\alpha)'' \, dr
\]

\[
= \int_1^\infty \{ (u(r) r^\alpha) \}'^2 \, dr + \alpha(\alpha - 1) \int_1^\infty (u(r))^2 r^{2\alpha - 2} \, dr.
\]

In addition, by using (3.13) replaced \(u(r)\) and \(T(r)\) by \(u(r) r^\alpha\) and \(U_j^{(m)}(r)\) respectively, we get

\[
\int_1^\infty \{ (u(r) r^\alpha) \}'^2 \, dr = \int_1^\infty \left\{ \left( u(r) r^\alpha \left( U_j^{(m)}(r) \right)^{-1} \right)' \right\}^2 \left( U_j^{(m)}(r) \right)^2 \, dr
\]

\[
- \int_1^\infty (u(r) r^\alpha)^2 \left( U_j^{(m)}(r) \right)^{-1} \left( U_j^{(m)}(r) \right)'' \, dr
\]

\[
= \int_1^\infty \left\{ \left( u(r) \left( r^{-\alpha} U_j^{(m)}(r) \right)^{-1} \right)' \right\}^2 \left( U_j^{(m)}(r) \right)^2 \, dr
\]

\[
+ \frac{1}{4} \int_1^\infty (u(r))^2 r^{2\alpha - 2} \left( 1 + S_j^{(m)}(r) \right) \, dr
\]

by virtue of (3.8). Combining (3.14) and (3.15), we obtain the desired equality (3.9).

**Proof of Proposition 3.1.** Case \(n = 1\) : Note that \((\mathbb{R}^1_+)^c = (-\infty, -1) \cup (1, \infty)\). For \(u \in C_0^1((\mathbb{R}^1_+)^c)\), we define \(u_+, u_- \in C_0^1((1, \infty))\) by

\[
u_+(x) = u(x), \quad u_-(x) = u(-x) \quad (x > 1).
\]
Since
\[ I^{(m)}_{\alpha,j}(u : (\mathbb{M}^1)} = \int_1^\infty (u'_+(x))^2 x^{2\alpha} dx - \left( \alpha - \frac{1}{2} \right)^2 \int_1^\infty (u_+(x))^2 x^{2\alpha-2} dx \\
- \frac{1}{4} \int_1^\infty (u_+(x))^2 x^{2\alpha-2} S_j^{(m)}(x) dx \\
+ \int_1^\infty (u'_-(x))^2 x^{2\alpha} dx - \left( \alpha - \frac{1}{2} \right)^2 \int_1^\infty (u_-(x))^2 x^{2\alpha-2} dx \\
- \frac{1}{4} \int_1^\infty (u_-(x))^2 x^{2\alpha-2} S_j^{(m)}(x) dx, \]
from the equality (3.9) it follows that
\[ I^{(m)}_{\alpha,j}(u : (\mathbb{M}^1)} = \int_1^\infty \left\{ \left( u_+(x) \left( x^{-\alpha} U_j^{(m)}(x) \right)^{-1} \right) \right\} \left( U_j^{(m)}(x) \right)^2 dx \\
+ \int_1^\infty \left\{ \left( u_-(x) \left( x^{-\alpha} U_j^{(m)}(x) \right)^{-1} \right) \right\} \left( U_j^{(m)}(x) \right)^2 dx \\
= \int_{(\mathbb{M})} \left\{ \left( u(x) \left( |x|^{-\alpha} U_j^{(m)}(|x|) \right)^{-1} \right) \right\} \left( U_j^{(m)}(|x|) \right)^2 dx, \]
which implies (3.7).

Case \( n \geq 2 \): For \( x \in (\mathbb{M}^1)^c \), we use the polar coordinates \( x = r \omega \) such that \( r = |x| \in (1, \infty) \) and \( \omega = x/|x| \in \mathbb{S}^{n-1} \), where \( \mathbb{S}^{n-1} = \{ x \in \mathbb{R}^n : |x| = 1 \} \). For \( u \in C^1_0((\mathbb{M}^1)^c) \), we have
\[
\int_{(\mathbb{M})} |\nabla u(x)|^2 |x|^{2\alpha} dx \\
= \int_1^\infty \int_{\mathbb{S}^{n-1}} \left\{ (\partial_r u(r \omega))^2 + \left( r^{-1}(-\Delta_{\mathbb{S}^{n-1}})^{1/2} u(r \omega) \right)^2 \right\} r^{2\alpha+n-1} dr d\omega \\
= \int_{\mathbb{S}^{n-1}} d\omega \int_1^\infty (\partial_r u(r \omega))^2 r^{2\alpha+n-1} dr \\
+ \int_1^\infty \int_{\mathbb{S}^{n-1}} \left( r^{-1}(-\Delta_{\mathbb{S}^{n-1}})^{1/2} u(r \omega) \right)^2 r^{2\alpha+n-1} dr d\omega, \tag{3.16}
\]
where \( \Delta_{\mathbb{S}^{n-1}} \) is the Laplace-Beltrami operator on \( \mathbb{S}^{n-1} \). Applying (3.9), we have
\[
\int_1^\infty (\partial_r u(r \omega))^2 r^{2\alpha+n-1} dr \\
= \left( \alpha + \frac{n-1}{2} - \frac{1}{2} \right)^2 \int_1^\infty (u(r \omega))^2 r^{2\alpha+n-3} dr + \frac{1}{4} \int_1^\infty (u(r \omega))^2 r^{2\alpha+n-3} S_j^{(m)}(r) dr \\
+ \int_1^\infty \left\{ \partial_r \left( u(r \omega) \left( r^{-\alpha-(n-1)/2} U_j^{(m)}(r) \right)^{-1} \right) \right\} \left( U_j^{(m)}(r) \right)^2 dr. \tag{3.17}
\]
Combining (3.16) and (3.17), we obtain

\[
\int_{|x|=1} |\nabla u(x)|^2 |x|^{2\alpha} \, dx
\]

\[
= \left( \alpha + \frac{n-2}{2} \right)^2 \int_{|x|=1} (u(x))^2 |x|^{2\alpha-2} \, dx + \frac{1}{4} \int_{|x|=1} (u(x))^2 |x|^{2\alpha-2} S_j^{(m)} (|x|) \, dx
\]

\[
+ \int_{|x|=1} \left| \nabla \left( |x|^{-\alpha - (n-1)/2} j_u^{(m)} (|x|) \right) \right|^2 \left( j_u^{(m)} (|x|) \right)^2 |x|^{1-n} \, dx,
\]

which implies the intended equality (3.7). That finishes the proof of Proposition 3.1.

Proof of Theorem 2.4. The inequality (2.17) is a direct consequence of Theorem 2.3. We show the sharpness of constants by induction on \( j \) for each fixed \( m \). Suppose that the constant \( 1/4 \) in inequality (2.17) is best till \( j \). For \( u \in C_0^1(\Omega) \), from (2.13) it follows that

\[
I_{\alpha, j}(u : \Omega) = I_{\alpha, j+1}(u : \Omega) + \frac{1}{4} \int_\Omega (u(x))^2 |x|^{2\alpha-2} \left( \prod_{k=1}^j A_k^{(0)} \left( \frac{|x|}{R} \right) \right)^2 \, dx \quad (j \geq 1)
\]

(3.19)

and, for \( m \geq 1 \),

\[
I_{\alpha, j}^{(m)} (u : \Omega) = I_{\alpha, j+1}^{(m)} (u : \Omega) + \frac{1}{4} \int_\Omega (u(x))^2 |x|^{2\alpha-2} \left( \prod_{l=0}^{m-1} B_l^{(0)} \left( \frac{|x|}{R} \right) \prod_{k=0}^j A_k^{(m)} \left( \frac{|x|}{R} \right) \right)^2 \, dx
\]

(3.20)

For \( u \in C_0^1(\Omega) \setminus \{0\} \), we put

\[
J_{\alpha, j+1}^{(0)} (u : \Omega)
\]

\[
= \int_\Omega |\nabla u(x)|^2 |x|^{2\alpha} \, dx - \left( \alpha + \frac{n-2}{2} \right)^2 \int_\Omega (u(x))^2 |x|^{2\alpha-2} \, dx - \frac{1}{4} \int_\Omega (u(x))^2 |x|^{2\alpha-2} S_j^{(m)} (|x|) \, dx
\]

\[
= \frac{I_{\alpha, j}(u : \Omega)}{I_{\alpha, j+1}^{(0)} (u : \Omega) - I_{\alpha, j}^{(0)} (u : \Omega)} \quad (j \geq 1)
\]
and, for $m \geq 1$,

\[
J_{\alpha,j+1}^{(m)}(u : \Omega) = \frac{\int_{\Omega} |\nabla u(x)|^2 |x|^{2\alpha} \, dx - (\alpha + \frac{n-2}{2})^2 \int_{\Omega} (u(x))^2 |x|^{2\alpha-2} \, dx - \frac{1}{4} \int_{\Omega} (u(x))^2 |x|^{2\alpha-2} S_{j+1}^{(m)} \left( \frac{|x|}{R} \right) \, dx}{\int_{\Omega} (u(x))^2 |x|^{2\alpha-2} \left( \prod_{l=0}^{m-1} B(l) \left( \frac{|x|}{R} \right) \prod_{k=0}^{j} A_k^{(m)} \left( \frac{|x|}{R} \right) \right)^2 \, dx}
\]

\[
= \frac{\int_{\Omega} (u(x))^2 |x|^{2\alpha-2} \left( \prod_{l=0}^{m-1} B(l) \left( \frac{|x|}{R} \right) \prod_{k=0}^{j} A_k^{(m)} \left( \frac{|x|}{R} \right) \right)^{-2} \, dx}{\frac{1}{4} \left( 1 + \frac{I_{\alpha,j}^{(m)}(u : \Omega)}{I_{\alpha,j+1}^{(m)}(u : \Omega)} \right)} (j \geq 0).
\]

In view of (3.19), (3.20) and (2.16) we see that

\[
I_{\alpha,j}^{(m)}(u : \Omega) > I_{\alpha,j+1}^{(m)}(u : \Omega) > 0 \quad \text{for} \quad u \in C_0^1(\Omega) \setminus \{0\},
\]

which implies

\[
\inf_{u \in C_0^1(\Omega) \setminus \{0\}} J_{\alpha,j+1}^{(m)}(u : \Omega) \geq \frac{1}{4}.
\]

We show that

\[
\inf_{u \in C_0^1(\Omega) \setminus \{0\}} J_{\alpha,j+1}^{(m)}(u : \Omega) = \frac{1}{4}.
\]

By the assumption, there exists a $\rho \geq R$ such that $(\overline{B}_\rho)^c \subset \Omega \subset (\overline{B}_r)^c$. Then by $C_0^1((\overline{B}_r)^c) \subset C_0^1(\Omega)$ we see that

\[
\inf_{u \in C_0^1((\overline{B}_r)^c) \setminus \{0\}} J_{\alpha,j+1}^{(m)}(u : \Omega) = \inf_{u \in C_0^1((\overline{B}_r)^c) \setminus \{0\}} J_{\alpha,j+1}^{(m)}(u : (\overline{B}_r)^c) 
\]

\[
\leq \inf_{u \in C_0^1((\overline{B}_r)^c) \setminus \{0\}} J_{\alpha,j+1}^{(m)}(u : (\overline{B}_r)^c) 
\]

\[
= \inf_{u \in C_0^1((\overline{B}_r)^c) \setminus \{0\}} J_{\alpha,j+1}^{(m)}(u : (\overline{B}_r)^c) \quad \text{[by (3.5)]}
\]

\[
= \inf_{u \in C_0^1((\overline{B}_r)^c) \setminus \{0\}} J_{\alpha,j+1}^{(m)}(u : (\overline{B}_r)^c). 
\]

Therefore it is enough for (3.21) to show that

\[
\inf_{u \in C_0^1((\overline{B}_r)^c) \setminus \{0\}} J_{\alpha,j+1}^{(m)}(u : (\overline{B}_r)^c) \leq \frac{1}{4}
\]

(3.22)

for $\sigma \geq 1$. By (3.42), which will be shown in the proof of Proposition 2.1 as stated last, there exists a sequence $\{u_\ell\} \subset C_0^1((\overline{B}_r)^c)$ such that

\[
\lim_{\ell \to \infty} I_{\alpha,j+1}^{(m)}(u_\ell : (\overline{B}_r)^c) < \infty, \quad \lim_{\ell \to \infty} I_{\alpha,j}^{(m)}(u_\ell : (\overline{B}_r)^c) = \infty.
\]
Hence it follows that

$$J^{(m)}_{a,j+1}(u^\ell : (\mathbb{R}^n_+)^c) = \frac{1}{4} \left( 1 + \frac{I^{(m)}_{a,j+1}(u^\ell : (\mathbb{R}^n_+)^c)}{I^{(m)}_{a,j}(u^\ell : (\mathbb{R}^n_+)^c) - I^{(m)}_{a,j+1}(u^\ell : (\mathbb{R}^n_+)^c)} \right) \rightarrow \frac{1}{4}$$

as $\ell \to \infty$, which implies (3.22), and so (3.21). This shows that the constant 1/4 in inequality (2.17) is best for $j+1$. Consequently the proof of Theorem 2.4 is over.

**Proof of Theorem 2.2.** Theorem 2.2 follows from Theorem 2.4 by using the inversion formula (3.2).

For the completion of proofs of main results, we only remain to prove Proposition 3.4. Let $\alpha \in \mathbb{R}$ and let $m, j \in \mathbb{N} \cup \{0\}$ with $m+j \geq 1$. For $U \in C^1([1, \infty))$, from (2.16) and (2.18) it follows that

$$\left\| \cdot \right\|_{W^{m,j}_j(\mathbb{R}^n_+)^c} = \int_{\mathbb{R}^n_+} |\nabla(U(|x|))|^2 \left( U^{(m)}_j(|x|) \right)^2 |x|^{1-n} dx = |S^{n-1}| \int_1^\infty (U'(r))^2 \left( U^{(m)}_j(r) \right)^2 dr = |S^{n-1}| \left\| U' U^{(m)}_j \right\|_{L^2([1, \infty))}^2. \quad (3.23)$$

Let $\chi \in C^\infty([0, \infty))$ be a nonincreasing function satisfying

$$\chi(r) = 1 \quad (0 \leq r \leq 1), \quad \chi(r) = 0 \quad (r \geq 2), \quad \| \chi' \|_{L^\infty([0, \infty))} = \sup_{r \geq 0} |\chi'(r)| < \infty.$$

For $\ell \in \mathbb{N}$, we set

$$\varphi^{(m)}_{j,\ell}(r) = \left( 1 - \chi(\ell(r-1)) \right) \chi \left( \frac{A^{(m)}_j(r)}{a\ell} \right) \quad (r \geq 1).$$

Then $\text{supp} \varphi^{(m)}_{j,\ell} \subset [1 + 1/\ell, \left( A^{(m)}_j \right)^{-1}(2a\ell)]$ and $\varphi^{(m)}_{j,\ell} \in C^\infty((1, \infty))$.

Now we start from the following fundamental result.

**Proposition 3.4.** Let $\alpha \in \mathbb{R}$ and let $m, j \in \mathbb{N} \cup \{0\}$ with $m+j \geq 1$. Let $U \in C^\infty([1, \infty))$ and $U(1) = 0$. If $U^{(m)}_j U' \in L^2([1, \infty))$ and there exists a constant $\beta < 1/2$ such that $\left( A^{(m)}_j \right)^{-\beta} U \in L^\infty([1, \infty))$, then it holds that

$$\left\| \cdot \right\|_{W^{(m)}_{\alpha,j}(\mathbb{R}^n_+)^c} = \left\| \cdot \right\|_{W^{m,j}_j(\mathbb{R}^n_+)^c} \in W^{(m)}_{\alpha,j}(\mathbb{R}^n_+)^c. \quad (3.24)$$
Proof. We show that
\[
C_{\alpha}^{\infty}(\mathbb{B}_{1}^{n}) \ni \varphi_{j,\ell}^{(m)}(\cdot, |) \cdot |^{-\alpha - (n-1)/2}U_{j}^{(m)}(\cdot, |)U(| \cdot |) \rightarrow | \cdot |^{-\alpha - (n-1)/2}U_{j}^{(m)}(\cdot, |)U(| \cdot |) \quad \text{in} \quad W_{\alpha,j}^{(m)}(\mathbb{B}_{1}^{n})
\] (3.25)
as \ell \rightarrow \infty. Using (3.23) and the inequality \((X + Y + Z)^2 \leq 3(X^2 + Y^2 + Z^2)\) for \(X, Y, Z \in \mathbb{R}\), we have the following estimate
\[
\left\| | \cdot |^{-\alpha - (n-1)/2}U_{j}^{(m)}(\cdot, |)U(| \cdot |) - | \cdot |^{-\alpha - (n-1)/2}U_{j}^{(m)}(\cdot, |)U(| \cdot |) \right\|^2_{W_{\alpha,j}^{(m)}(\mathbb{B}_{1}^{n})} \\
= |S_{1}^{n-1}| \int_{1}^{\infty} \left\{ \left( \varphi_{j,\ell}^{(m)}(r) - 1 \right) U(r) \right\}^2 \left( U_{j}^{(m)}(r) \right)^2 dr \\
\leq 3 |S_{1}^{n-1}| \int_{1}^{\infty} \left\{ \left( \varphi_{j,\ell}^{(m)}(r) - 1 \right) \left( U'(r) \right)^2 \left( U_{j}^{(m)}(r) \right)^2 \right\} dr \\
+ \int_{1}^{\infty} \left\{ \left( 1 - \chi(\ell(r-1)) \right) \right\}^2 \left( \chi \left( \frac{A_{j}^{(m)}(r)}{a_{\ell}} \right) \right)^2 \left( U(r) \right)^2 \left( U_{j}^{(m)}(r) \right)^2 dr \\
+ \int_{1}^{\infty} \left\{ \left( 1 - \chi(\ell(r-1)) \right) \right\}^2 \left( \left( \frac{A_{j}^{(m)}(r)}{a_{\ell}} \right) \right)^2 \left( U'(r) \right)^2 \left( U_{j}^{(m)}(r) \right)^2 dr \right\}. \tag{3.26}
\]

[First term in \{ \} part of (3.26)] From the assumption \(U_{j}^{(m)}U' \in L^2([1, \infty))\) it follows that
\[
\int_{1}^{\infty} \left( \varphi_{j,\ell}^{(m)}(r) - 1 \right)^2 \left( U'(r) \right)^2 \left( U_{j}^{(m)}(r) \right)^2 dr \rightarrow 0 \quad \text{as} \quad \ell \rightarrow \infty
\]
by dominated convergence theorem.

[Second term in \{ \} part of (3.26)] Since, for \(1 \leq r \leq 3\),
\[
|U(r)| = |U(r) - U(1)| \leq \| U' \|_{L^\infty([1,3])} (r - 1),
\]
it follows that
\[
\int_{1}^{\infty} \left\{ \left( 1 - \chi(\ell(r-1)) \right) \right\}^2 \left( \chi \left( \frac{A_{j}^{(m)}(r)}{a_{\ell}} \right) \right)^2 \left( U(r) \right)^2 \left( U_{j}^{(m)}(r) \right)^2 dr \\
\leq \ell^2 \int_{1+1/\ell}^{1+2/\ell} \left\{ \chi'(\ell(r-1)) \right\}^2 \left( U'(r) \right)^2 \left( U_{j}^{(m)}(r) \right)^2 dr \\
\leq \ell^2 \| \chi' \|^2_{L^\infty([0,\infty))} \left\| U_{j}^{(m)} \right\|^2_{L^\infty([1,3])} \| U' \|^2_{L^\infty([1,3])} \int_{1+1/\ell}^{1+2/\ell} (r - 1)^2 dr \\
= \ell^2 \| \chi' \|^2_{L^\infty([0,\infty))} \left\| U_{j}^{(m)} \right\|^2_{L^\infty([1,3])} \| U' \|^2_{L^\infty([1,3])} \cdot \frac{7}{3\ell^3} \rightarrow 0 \quad \text{as} \quad \ell \rightarrow \infty.
\]
Proof. Noting that Lemma 3.1. Let

\[
\left( \frac{A_j^{(m)}}{\alpha} \right)' = \left( \frac{F_j^{(a^m}}{\alpha} \right)' = \frac{1}{m+a+j} \left( U_j^{(m)}(r) \right)^{-2},
\]

and so

\[
\left\{ \chi \left( \frac{A_j^{(m)}}{a^\ell} \right) \right\}'^2 \left( U_j^{(m)}(r) \right)^2 = \chi \left( \frac{A_j^{(m)}}{a^\ell} \right)' \left\{ \chi \left( \frac{A_j^{(m)}}{a^\ell} \right) \right\}' \left( U_j^{(m)}(r) \right)^2
\]

\[
= \frac{1}{\ell a^{m+j}} \chi \left( \frac{A_j^{(m)}}{a^\ell} \right)' \left( U_j^{(m)}(r) \right)^2.
\]

Hence from the assumption \( \left( A_j^{(m)}(\cdot) \right)^{-\beta} U \in L^\infty([1, \infty)) \) it follows that

\[
\int_1^\infty \left( 1 - \chi(\ell(r-1)) \right)^2 \left\{ \chi \left( \frac{A_j^{(m)}}{a^\ell} \right) \right\}'^2 \left( U(r) \right)^2 \left( U_j^{(m)}(r) \right)^2 dr
\]

\[
= \frac{1}{\ell a^{m+j}} \int_1^{(A_j^{(m)})^{-1}(2a\ell)} \left( 1 - \chi(\ell(r-1)) \right)^2 \chi \left( \frac{A_j^{(m)}}{a^\ell} \right)' \left( A_j^{(m)}(r) \right) \left( U(r) \right)^2 dr
\]

\[
\leq \frac{\ell^{2\beta-1}}{a^{m+j-2\beta}} \| \chi' \|^2_{L^\infty([1, \infty))} \left\| \left( A_j^{(m)}(\cdot) \right)^{-\beta} U \right\|_{L^\infty([1, \infty))}^2
\]

\[
\cdot \int_1^{(A_j^{(m)})^{-1}(2a\ell)} \left( A_j^{(m)}(r) \right) \left( a^\ell \right) \left( A_j^{(m)}(r) \right)^{2\beta} dr
\]

\[
= \frac{\ell^{2\beta-1}}{a^{m+j-2\beta}} \| \chi' \|^2_{L^\infty([1, \infty))} \left\| \left( A_j^{(m)}(\cdot) \right)^{-\beta} U \right\|_{L^\infty([1, \infty))}^2 \int_1^2 \ell^{2\beta} dt \to 0 \quad \text{as} \quad \ell \to \infty
\]

by \( \beta < 1/2 \).

Consequently (3.25) was shown by the above argumentations consisting of three parts. That completes the proof.

\[ \Box \]

Lemma 3.1. Let \( m, j \in \mathbb{N} \cup \{0\} \) and \( m + j \geq 1 \). Then it holds that

\[
U_j^{(m)} \left\{ \left( U_j^{(m)}(\cdot) \right)^{-1} \right\}'^2 \in L^2([1, \infty)).
\]

Proof. Noting that

\[
\left( U_j^{(m)}(r) \left\{ \left( U_j^{(m)}(r) \right)^{-1} \right\} \right)^2 = \left\{ \left( U_j^{(m)}(r) \right)^{-1} \left( U_j^{(m)}(r) \right)' \right\}^2
\]

\[
= \left( U_j^{(m)}(r) \right)^{-1} \left( U_j^{(m)}(r) \right)' - \left( \log U_j^{(m)}(r) \right)
\]
and
\[
\left( U_j^{(m)}(r) \right)'' < 0
\]
by (3.8), we have
\[
\left( U_j^{(m)}(r) \left\{ \left( U_j^{(m)}(r) \right)^{-1} \right\}' \right)^2 < - \left( \log U_j^{(m)}(r) \right)'' \quad (r \geq 1).
\]
(3.28)
Further from (3.27) it follows that
\[
0 < \left( \log \frac{A_j^{(m)}(r)}{a} \right)' = \left( F^{j+1}(\phi^m(ar)) - a \right)' = \frac{1}{a^{m+j}} \left( U_j^{(m)}(r) \right)^{-2}
\]
\[
< \frac{1}{a^{m+j}} \left( U_j^{(0)}(r) \right)^{-2} = \frac{1}{a^{m+j}} r^{-1} \quad (r > 1),
\]
and so
\[
0 < \left( \log U_j^{(m)}(r) \right)',
\]
\[
< \left( \log U_0^{(m+1)}(r) \right)' = \frac{1}{2} \left( \log r + \log a + \sum_{l=0}^{m} \sum_{k=0}^{\infty} \left( F^{k+1}(\phi^l(ar)) - a \right) \right)',
\]
\[
< \frac{1}{2} \left( r^{-1} + \sum_{l=0}^{m} \sum_{k=0}^{\infty} \frac{1}{a^{l+k}} r^{-1} \right)
\]
\[
< \frac{1}{2} r^{-1} \left( 1 + \left( \frac{a}{a-1} \right)^2 \right) \quad (r > 1),
\]
which implies
\[
\lim_{r \to \infty} \left( \log U_j^{(m)}(r) \right)' = 0.
\]
(3.29)
Hence the assertion follows from (3.28) and (3.29).

**Lemma 3.2.** Let \( m, j \in \mathbb{N} \cup \{0\} \) and \( m + j \geq 1 \). Let \( \beta \in \mathbb{R} \). Then
\[
U_j^{(m)} \left\{ \left( A_j^{(m)}(r) \right)^\beta \right\}' \in L^2([1, \infty))
\]
holds if and only if \( \beta < 1/2 \).

**Proof.** From (3.27) it follows that
\[
\left\{ \left( A_j^{(m)}(r) \right)^\beta \right\}'^2 = \left\{ \beta \left( A_j^{(m)}(r) \right)^{\beta-1} \left( A_j^{(m)}(r) \right)' \right\}^2
\]
\[
= \frac{\beta^2}{a^{m+j-1}} \left( A_j^{(m)}(r) \right)^{2\beta-2} \left( A_j^{(m)}(r) \right)' \left( U_j^{(m)}(r) \right)^{-2}.
\]
(3.30)
Hence we have
\[
\int_1^\infty \left( U_j^{(m)}(r) \right)^2 \left( \left\{ (A_j^{(m)}(r))^\beta \right\} \right)^2 \, dr = \int_1^\infty \frac{\beta^2}{am+j-1} \left( A_j^{(m)}(r) \right)^{2\beta-2} \left( A_j^{(m)}(r) \right)' \, dr \\
= \int_1^\infty \frac{\beta^2}{am+j-1} t^{2\beta-2} \, dt,
\]
which implies the assertion.

Lemma 3.3. Let \( m, j \in \mathbb{N} \cup \{0\} \) and \( m + j \geq 1 \). Let \( \beta \in \mathbb{R} \). Then

\[
U_j^{(m)} \left\{ (A_j^{(m)}(r))^\beta \right\} \in L^2([1, \infty))
\]
holds if and only if \( \beta \leq 0 \).

Proof. Noting that
\[
U_j^{(m)}(r) = U_j^{(m)}(r) \left( \frac{A_j^{(m)}(r)}{a} \right)^{1/2},
\]
from (3.30) it follows that
\[
\int_1^\infty \left( t_j^{(m)}(r) \right)^2 \left( \left\{ (A_j^{(m)}(r))^\beta \right\} \right)^2 \, dr = \int_1^\infty \frac{\beta^2}{am+j} \left( A_j^{(m)}(r) \right)^{2\beta-1} \left( A_j^{(m)}(r) \right)' \, dr \\
= \int_1^\infty \frac{\beta^2}{am+j} t^{2\beta-1} \, dt,
\]
which implies the assertion.

Proposition 3.5. Let \( \alpha \in \mathbb{R} \) and let \( m, j \in \mathbb{N} \cup \{0\} \) with \( m + j \geq 1 \). If \( \beta < 1/2 \), then it holds that

\[
| \cdot |^{-\alpha-(n-1)/2} \left( U_j^{(m)}(| \cdot |) \left( \frac{A_j^{(m)}(| \cdot |)}{a} \right)^\beta - 1 \right) \in W_{\alpha,j}^{r,m}(\mathbb{R}_1^n). \tag{3.31}
\]

In particular, it holds that

\[
| \cdot |^{-\alpha-(n-1)/2} \left( U_j^{(m)}(| \cdot |) - 1 \right) \in W_{\alpha,j}^{r,m}(\mathbb{R}_1^n). \tag{3.32} \text{ [ case } \beta = 0 \text{ ]}
\]

If \( \beta \geq 1/2 \), then it holds that

\[
| \cdot |^{-\alpha-(n-1)/2} \left( U_j^{(m)}(| \cdot |) \left( \frac{A_j^{(m)}(| \cdot |)}{a} \right)^\beta - 1 \right) \notin W_{\alpha,j}^{r,m}(\mathbb{R}_1^n). \tag{3.33}
\]
In particular, it holds that
\[ |\cdot|^{-\alpha-(n-1)/2} \left( U_{j+1}^{(m)} (|\cdot|) - 1 \right) \notin W_{\alpha,j}^{(m)} (\mathbb{B}_1^\infty). \quad [\text{case } \beta = 1/2] \quad (3.34) \]
Moreover, if \( \beta > 1/2 \), then it holds that
\[ |\cdot|^{-\alpha-(n-1)/2} \left( U_{j}^{(m)} (|\cdot|) \left( \frac{A_j^{(m)}(|\cdot|)}{a} \right)^\beta - 1 \right) \notin W_{\alpha,j+1}^{(m)} (\mathbb{B}_1^\infty). \quad (3.35) \]

**Corollary 3.1.** Let \( \alpha \in \mathbb{R} \) and let \( m, j \in \mathbb{N} \cup \{0\} \) with \( m+j \geq 1 \). From \((3.32)\) and \((3.34)\) it follows that
\[ |\cdot|^{-\alpha-(n-1)/2} \left( U_{j+1}^{(m)} (|\cdot|) - 1 \right) \in W_{\alpha,j+1}^{(m)} (\mathbb{B}_1^\infty) \setminus W_{\alpha,j}^{(m)} (\mathbb{B}_1^\infty). \quad (3.36) \]

**Proof of Proposition 3.5.** Suppose \( \beta < 1/2 \). By Lemmas 3.1 and 3.2, we apply Proposition 3.4 to \( U = 1 - \left( U_{j}^{(m)} (\cdot) \right)^{-1} \) and \( U = \left( A_j^{(m)} (\cdot)/a \right)^\beta - 1 \), respectively.

Then we have
\[ |\cdot|^{-\alpha-(n-1)/2} U_j^{(m)} (|\cdot|) \left( 1 - \left( U_j^{(m)} (|\cdot|) \right)^{-1} \right) \in W_{\alpha,j}^{(m)} (\mathbb{B}_1^\infty) \quad (3.37) \]
and
\[ |\cdot|^{-\alpha-(n-1)/2} U_j^{(m)} (|\cdot|) \left( \left( \frac{A_j^{(m)}(|\cdot|)}{a} \right)^\beta - 1 \right) \in W_{\alpha,j}^{(m)} (\mathbb{B}_1^\infty), \]
respectively. Hence it follows that
\[ |\cdot|^{-\alpha-(n-1)/2} \left( U_j^{(m)} (|\cdot|) \left( \frac{A_j^{(m)}(|\cdot|)}{a} \right)^\beta - 1 \right) \in W_{\alpha,j+1}^{(m)} (\mathbb{B}_1^\infty). \]

Suppose \( \beta \geq 1/2 \). By Lemma 3.2 and the equality \((3.23)\), we see that
\[ |\cdot|^{-\alpha-(n-1)/2} U_j^{(m)} (|\cdot|) \left( \left( \frac{A_j^{(m)}(|\cdot|)}{a} \right)^\beta - 1 \right) \notin W_{\alpha,j}^{(m)} (\mathbb{B}_1^\infty), \]
which implies
\[ |\cdot|^{-\alpha-(n-1)/2} \left( U_j^{(m)} (|\cdot|) \left( \frac{A_j^{(m)}(|\cdot|)}{a} \right)^\beta - 1 \right) \notin W_{\alpha,j}^{(m)} (\mathbb{B}_1^\infty), \]
by \((3.37)\). Suppose \( \beta > 1/2 \). Since
\[ U_{j+1}^{(m)} \left\{ \left( A_j^{(m)} (\cdot) \right)^{\beta-1/2} \right\}' \notin L^2 ([1, \infty)) \]
In addition, by (3.27) we see that
\[ | \cdot |^{-\alpha-(n-1)/2} U_{j+1}^{(m)} (| \cdot |) \left( \frac{\hat{A}_j^{(m)}(| \cdot |)}{a} \right)^{\beta-1/2} - 1 \notin W_{\alpha,j+1}^{(m)} (\mathbb{B}_1^n). \]

In addition, noting that
\[ | \cdot |^{-\alpha-(n-1)/2} \left( U_{j+1}^{(m)} (| \cdot |) - 1 \right) \in W_{\alpha,j+1}^{(m)} (\mathbb{B}_1^n) \]
by (3.32), we have
\[ | \cdot |^{-\alpha-(n-1)/2} \left( U_{j+1}^{(m)} (| \cdot |) - 1 \right) \notin W_{\alpha,j+1}^{(m)} (\mathbb{B}_1^n), \]
which implies (3.35). That finishes the proof of Proposition 3.5. \( \square \)

**Proposition 3.6.** Let \( \alpha \in \mathbb{R} \) and let \( m, j \in \mathbb{N} \cup \{0\} \) with \( m + j \geq 1 \). Then it holds that
\[ | \cdot |^{-\alpha-(n-1)/2} U_0^{(m+1)} (| \cdot |) - 1 \notin W_{\alpha,j}^{(m)} (\mathbb{B}_1^n). \quad (3.38) \]

**Proof.** In view of (3.32) it suffices to show
\[ | \cdot |^{-\alpha-(n-1)/2} U_j^{(m)} (| \cdot |) \left( \frac{U_0^{(m+1)} (| \cdot |)}{U_j^{(m)} (| \cdot |)} - 1 \right) \notin W_{\alpha,j}^{(m)} (\mathbb{B}_1^n). \quad (3.39) \]

From the relation
\[ \frac{U_0^{(m+1)} (r)}{U_j^{(m)} (r)} = \left( \frac{A_j^{(m)} (r)}{a} \right)^{1/2} \frac{U_0^{(m+1)} (r)}{U_j^{(m+1)} (r)} \]
it follows that
\[ \left( \frac{U_0^{(m+1)} (r)}{U_j^{(m)} (r)} \right)' = \left\{ \left( \frac{A_j^{(m)} (r)}{a} \right)^{1/2} \frac{U_0^{(m+1)} (r)}{U_j^{(m+1)} (r)} \right\}' + \left( \frac{A_j^{(m)} (r)}{a} \right)^{1/2} \left( \frac{U_0^{(m+1)} (r)}{U_j^{(m+1)} (r)} \right)' \]
Note that \( U_0^{(m+1)} (r) / U_j^{(m+1)} (r) \) is the increasing and \( U_0^{(m+1)} (r) / U_j^{(m+1)} (r) \geq 1 \) \( (r \geq 1) \).

In addition, by (3.27) we see that
\[ \frac{A_j^{(m)} (r)}{a} \geq 1, \quad \left( \frac{A_j^{(m)} (r)}{a} \right)' > 0, \]
\[ \left\{ \left( \frac{A_j^{(m)} (r)}{a} \right)^{1/2} \right\}' = \frac{1}{2} \left( \frac{A_j^{(m)} (r)}{a} \right)^{-1/2} \left( \frac{A_j^{(m)} (r)}{a} \right)' > 0 \quad (r \geq 1). \]
Therefore we have

\[
\left( \frac{U_0^{m+1}(r)}{U_j^{m}(r)} \right) ' \geq \left\{ \left( \frac{A_j^{(m)}(r)}{a} \right)^{1/2} \right\} ' > 0 \quad (r \geq 1).
\]

Hence from Lemma 3.2 it follows that

\[
U_j^{(m)} \left( \frac{I_0^{(m+1)}}{U_j^{(m)}} \right)' \notin L^2([1, \infty)),
\]

which implies (3.39) by means of the equality (3.23).

Proof of Proposition 2.1. In view of the inversion formula (3.2), it suffices to show (2.19) and (2.20) in the case where \( \Omega \) is an exterior domain. By the assumption of exterior domain \( \Omega \), there exists a \( \rho \geq R \) such that \( (B^n_R)^c \subset \Omega \subset (B^n_R)^c \). Then, from embedding \( C^1_0((B^n_R)^c) \subset C^1_0(\Omega) \subset C^1_0((B^n_R)^c) \) and the relation \( I_{a,j}(u) \Omega = I_{a,j}(u) (B^n_R)^c \) for \( u \in C^1_0(\Omega) \) it follows that \( W^{(m)}_{a,j}(\Omega) = W^{(m)}_{a,j}(\Omega : (B^n_R)^c) \) and

\[
W^{(m)}_{a,j}((B^n_R)^c : (B^n_R)^c) \subset W^{(m)}_{a,j}(\Omega : (B^n_R)^c) \subset W^{(m)}_{a,j}((B^n_R)^c : (B^n_R)^c).
\]

So we have

\[
W^{(m)}_{a,j+1}(\Omega) \Omega W^{(m)}_{a,j}(\Omega) \supset W^{(m)}_{a,j+1}((B^n_R)^c : (B^n_R)^c) \setminus W^{(m)}_{a,j}((B^n_R)^c : (B^n_R)^c)
\]

and

\[
W^{(m+1)}_{a,0}(\Omega) \Omega W^{(m+1)}_{a,0}(\Omega) \supset W^{(m+1)}_{a,0}((B^n_R)^c : (B^n_R)^c) \setminus W^{(m)}_{a,j}((B^n_R)^c : (B^n_R)^c).
\]

Therefore, to see (2.19) and (2.20), we show that

\[
W^{(m)}_{a,j+1}((B^n_R)^c : (B^n_R)^c) \setminus W^{(m)}_{a,j}((B^n_R)^c : (B^n_R)^c) \neq \emptyset
\]

(3.40)

and

\[
W^{(m+1)}_{a,0}((B^n_R)^c : (B^n_R)^c) \setminus W^{(m)}_{a,j}((B^n_R)^c : (B^n_R)^c) \neq \emptyset,
\]

(3.41)

respectively. From the scaling relation (3.5) it follows that \( u \in W^{(m)}_{a,j}((B^n_R)^c : (B^n_R)^c) \) is equivalent to \( u_R \in W^{(m)}_{a,j}((B^n_{\rho R})^c : (B^n_R)^c) \), furthermore \( u \in W^{(m)}_{a,j+1}((B^n_R)^c : (B^n_R)^c) \setminus W^{(m)}_{a,j}((B^n_R)^c : (B^n_R)^c) \) is equivalent to \( u_R \in W^{(m)}_{a,j+1}((B^n_{\rho R})^c : (B^n_R)^c) \setminus W^{(m)}_{a,j}((B^n_R)^c : (B^n_R)^c) \). Hence (3.40) is reduced to

\[
W^{(m)}_{a,j+1}((B^n_R)^c : (B^n_R)^c) \setminus W^{(m)}_{a,j}((B^n_R)^c : (B^n_R)^c) \neq \emptyset
\]

(3.42)
for $\sigma \geq 1$. By Corollary 3.1 we have a radial function

$$U = | \cdot |^{-\alpha-(\alpha-1)/2} \left( U_{j+1}^{(m)} (| \cdot | - 1) \right) \in W^{(m)}_{\alpha,j+1} \left( \left( \mathbb{B}^c_1 \right)^c : \left( \mathbb{B}^c_1 \right)^c \right) \setminus W^{(m)}_{\alpha,j} \left( \left( \mathbb{B}^c_1 \right)^c : \left( \mathbb{B}^c_1 \right)^c \right),$$

which implies (3.42) for $\sigma = 1$. Suppose $\sigma > 1$. Let $\psi \in C_0^\infty \left( \left( \mathbb{B}^c_1 \right)^c \right)$ be a cutoff function satisfying $0 \leq \psi \leq 1$, $\text{supp} \psi \subset \left( \mathbb{B}^c_1 \right)^c$ and $\psi = 1$ on $\left( \mathbb{B}^c_2 \right)^c$. For each $i$, since $\text{supp} \partial \psi / \partial x_i$ is a compact set in $\left( \mathbb{B}^c_1 \right)^c$ by $\text{supp} \partial \psi / \partial x_i \subset \mathbb{B}^c_2 \setminus \mathbb{B}^c_1$, from (3.43) it follows that

$$\psi U \in W^{(m)}_{\alpha,j+1} \left( \left( \mathbb{B}^c_1 \right)^c : \left( \mathbb{B}^c_1 \right)^c \right) \setminus W^{(m)}_{\alpha,j} \left( \left( \mathbb{B}^c_1 \right)^c : \left( \mathbb{B}^c_1 \right)^c \right),$$

which implies (3.42). In similar way, we can show (3.41) by virtue of Proposition 3.6. That concludes the proof of Proposition 2.1.

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**References**

[1] Adimurthi, N. Chaudhuri and M. Ramaswamy, An improved Hardy-Sobolev inequality and its application, Proc. Amer. Math. Soc., Vol.130, No.2 (2001), 489-505.

[2] H. Ando and T. Horiuchi, Missing terms in the weighted Hardy-Sobolev inequalities and its application, Kyoto J. Math., Vol.52, No.4 (2012), 759-796.

[3] H. Ando, T. Horiuchi and E. Nakai, Construction of slowly increasing functions, Sci. Math. Jpn., Vol.75, No.2 (2012), 187–201.

[4] H. Ando, T. Horiuchi and E. Nakai, Some properties of slowly increasing functions, Math. J. Ibaraki Univ., Vol.46 (2014), 37–49.

[5] G. Barbatis, S. Filippas and A. Tertikas, Series expansion for $L^p$ Hardy inequalities, Indiana Univ. Math. J., Vol.52, No.1 (2003), 171–190.

[6] A. Detalla, T. Horiuchi and H. Ando, Sharp remainder terms of Hardy-Sobolev inequalities, Math. J. Ibaraki Univ., Vol.37 (2005), 39–52.

[7] S. Filippas and A. Tertikas, Optimizing improved Hardy inequalities, J. Funct. Anal., Vol.192 (2002), 186-233.