Let $Z_{r,R}$ be the space of continuous functions on the annulus $B_{r,R}$ in $\mathbb{C}^n$ whose $\lambda$-twisted spherical mean, in the set up of the M"etivier group, vanishes over the spheres $S_s(z) \subset B_{r,R}$ with ball $B_r(0) \subseteq B_s(z)$. We characterize the spherical harmonic coefficients of functions in $Z_{r,R}$, eventually, in terms of polynomial growth, by which we infer support theorem. Further, we prove that non-harmonic complex cone and the boundary of a bounded domain are sets of injectivity for the $\lambda$-twisted spherical means.

1. Introduction

In a remarkable result, Helgason proved a support theorem for continuous functions having polynomial growth whose spherical mean vanishes over the spheres surrounding a ball. In other words, let $\mu_s$ be the normalized surface measure on the sphere $S_{n-1}$. If $f$ is a continuous function on $\mathbb{R}^n$, $(n \geq 2)$ such that $|x|^k f(x)$ is bounded for each non-negative integer $k$, then $f$ is supported in the ball $B_r(0)$ if and only if $f * \mu_s(x) = 0, \forall x \in \mathbb{R}^n$ and $\forall s > |x| + r,$ (see [10]).

Later in [6], Epstein and Kleiner generalized the Helgason’s support theorem significantly, by characterizing the space of all continuous functions on $\mathbb{R}^n$, whose spherical mean vanishes over all spheres surrounding a ball, in terms of spherical harmonic coefficients having polynomial growth. This result was first proved by Globevnik [8] in the plane.

Let $H_k$ be the restriction of the space of homogeneous harmonic polynomials of degree $k$ to the unit sphere $S_{n-1}$, and $\{Y^l_k : l = 1, \ldots, d_k\}$ is an orthonormal basis for $H_k$. Then $f \in C(\mathbb{R}^n)$ can be expressed as

$$f(x) = \sum_{k=0}^{\infty} \sum_{l=1}^{d_k} a_{kl}(\rho) \ Y^l_k(\omega),$$

where $x = \rho \omega$ and $\rho = |x|$. In [6], authors had shown that $f * \mu_s(x) = 0$ for all $x \in \mathbb{R}^n$ and $s > |x| + B$ as long as $a_{kl} \in \text{span}\{\rho^{k-n-2i} : i = 0, 1, \ldots, k-1\}$, whenever $\rho > B$. 

Date: August 27, 2021.

2000 Mathematics Subject Classification. Primary 42A38; Secondary 44A35.

Key words and phrases. Convolution, Spherical harmonic, Heisenberg type group.
Consequently, the support theorem is an immediate corollary of the above result [6]. For other related work, we refer to [2,4,16,24,25].

Further, in the article [15], Thangavelu and Narayanan proved an analogue of the Helgason’s support theorem for the twisted spherical mean (TSM) for certain Schwartz class functions on $\mathbb{C}^n$. In [17], authors characterized the space of all continuous functions on $\mathbb{C}^n$ having TSM mean vanishes over the spheres surrounding a ball, and proved an exact analogue of the Helgason’s support theorem for the twisted spherical mean on $\mathbb{C}^n$ ($n \geq 2$). For $n = 1$, authors have proved a stronger result relaxing decay condition.

In Section 3, we consider M´etivier group for proving necessary conditions for a function to be in $Z^*_{r,R}$, the subspace of ceratin smooth functions on $B_{r,R}$. This result imitate a support theorem for type functions. Further, we derive that the non-harmonic complex cones in $\mathbb{C}^n$ are sets of injectivity for the $\lambda$-twisted spherical mean for the class of continuous functions on M´etivier group. For a brief history of work related to sets of injectivity for TSM on the Heisenberg group, we refer to [1,14,19–22].

Finally, in Section 4, we prove analogous results on $H$-type group, which is a special case of M´etivier group. We prove sufficient condition for a function to be in $Z_{r,R}$, and derive the support theorem and Heche-Bochner identity for the $\lambda$-twisted spherical mean. Then we prove that the boundary of a bounded domain is a set of injectivity for $\lambda$-twisted spherical mean.

We would like to mention that the results in the case of M´etivier group is of restrictive nature with those in Heisenberg group due the fact that the symplectic bilinear form appears in the group action of the M´etivier group cannot be made $U(n)$-invariant, in general, due to higher dimensional center of the M´etivier groups. This fact will be reflected as the distinct eigenvalues of the corresponding symplectic matrix $U_\lambda$ that arises in Section 3. However, we prove that the symplectic bilinear form for $H$-type group is similar to that of the Heisenberg group up to an orthogonal transformation.

2. Preliminaries

Let $G$ be connected, simply connected Lie group with real step two nilpotent Lie algebra $\mathfrak{g}$. Then $\mathfrak{g}$ has the orthogonal decomposition $\mathfrak{g} = \mathfrak{b} \oplus \mathfrak{z}$, where $\mathfrak{z}$ is the center of $\mathfrak{g}$. Since $\mathfrak{g}$ is a nilpotent, the exponential map $\exp : \mathfrak{g} \rightarrow G$ is surjective, and hence $G$ can be parameterized by $\mathfrak{g}$, endowed with the exponential coordinates. Now, we can identify $X + T \in \mathfrak{b} \oplus \mathfrak{z}$ with $\exp(X + T)$ and denote it by $(X, T) \in \mathbb{R}^d \times \mathbb{R}^m$. Since $[\mathfrak{b}, \mathfrak{b}] \subseteq \mathfrak{z}$ and $[\mathfrak{b}, [\mathfrak{b}, \mathfrak{b}]] = 0$, by the Baker-Campbell-Hausdorff formula, the group law on $G$ can be expressed as

$$(X, T)(Y, S) = (X + Y, T + S + \frac{1}{2}[X, Y]),$$

where $X, Y \in \mathfrak{b}$ and $T, S \in \mathfrak{z}$. Now, for $\omega \in \mathfrak{z}^*$, consider the skew-symmetric bilinear form $B_\omega$ on $\mathfrak{b}$ by $B_\omega(X, Y) = \omega([X, Y])$. Let $m_\omega$ be the orthogonal
complement of \( r_\omega = \{ X \in b : B_\omega(X, Y) = 0, \ \forall Y \in b \} \) in \( b \). Then \( B_\omega \) is called a non-degenerate bilinear form when \( r_\omega \) is trivial.

In this article, we discuss some special type of step two nilpotent Lie groups.

**Métivier groups:** We say the group \( G \) is Métivier group if \( B_\omega \) is non-degenerate for all non-zero \( \omega \in \mathfrak{z}^* \). In this case, \( d = 2n \), even. Let \( B_1, \ldots, B_{2n} \) and \( Z_1, \ldots, Z_m \) be orthonormal bases for \( b \) and \( j \) respectively. Since \( [b, b] \subseteq j \), there exist scalars \( U_j^{(k)} \) such that

\[
[B_j, B_l] = \sum_{k=1}^{m} U_j^{(k)} Z_k, \quad 1 \leq j, l \leq 2n.
\]

For \( 1 \leq k \leq m \), define \( 2n \times 2n \) skew-symmetric matrices by \( U^{(k)} = (U_{j,l}^{(k)}) \).

Then the group law for the Métivier group can be expressed as

\[
(x, t).J(\tau, \xi) = \begin{pmatrix}
x_i + \xi_i, \quad i = 1, \ldots, 2n \\
t_j + \tau_j + \frac{1}{2} \langle x, U^{(j)} \xi \rangle, \quad j = 1, \ldots, m
\end{pmatrix},
\]

where \( x, \xi \in \mathbb{R}^{2n} \) and \( t, \tau \in \mathbb{R}^m \). For \( x = (x_1, \ldots, x_n, y_1, \ldots, y_n) \in \mathbb{R}^{2n} \), write \( z = (x_1 + iy_1, \ldots, x_n + iy_n) = (z_1, \ldots, z_n) \) and say, \( z \) be the complexification of \( x \). Let \( z, w \in \mathbb{C}^n \) be the complexification of \( x, \xi \in \mathbb{R}^{2n} \). If we fix the notation \( U^{(j)}w \) for the complexification of \( U^{(j)} \xi \), then (2.1) can be simplified to

\[
(z, t).J(w, \tau) = \begin{pmatrix}
t_j + \tau_j + \frac{1}{2} \text{Re} \left( z \cdot U^{(j)}w \right), \quad j = 1, \ldots, m
\end{pmatrix}.
\]

**H-type groups:** Suppose \( g \) is endowed with an inner product \( \langle \cdot, \cdot \rangle \) such that for each \( Z \in \mathfrak{z} \), the map \( J_Z : b \rightarrow b \) defined by \( \langle J_Z(X), Y \rangle = \langle Z, [X, Y] \rangle \) for \( X, Y \in b \), satisfies \( J_Z^2 = -I \). We say that \( g \) is H-type if \( J_Z^2 = -|Z|^2 I \), whenever \( Z \in \mathfrak{z} \). Hence it follows that \( J_ZJ_{Z'} + J_{Z'}J_Z = -2\langle Z, Z' \rangle I \) for all \( Z, Z' \in \mathfrak{z} \), where \( I \) denotes the identity mapping. A connected, simply connected Lie group \( G \) with H-type Lie algebra is called Heisenberg type (or H-type) group. The H-type groups, introduced by A. Kaplan [11], are examples of Métivier group. However, there are Métivier groups that differ from the H-type groups. For more details, see [12][13].

**Theorem 2.1.** [3] Let \( G \) be connected, simply connected Lie group with real step two nilpotent Lie algebra \( g \). Then \( G \) is a H-type group if and only if \( G \) is isomorphic to \( \mathbb{R}^{2n+m} \) with the group law (2.1) and the matrices \( U^{(1)}, \ldots, U^{(m)} \) satisfy the following conditions:

(a) \( U^{(j)} \) is skew-symmetric \( 2n \times 2n \) orthogonal matrix, for \( j = 1, \ldots, m \).

(b) \( U^{(j)}U^{(l)} + U^{(l)}U^{(j)} = 0 \) for all \( j, l = 1, \ldots, m \) with \( j \neq l \).

For \( \lambda \in \mathfrak{z} \setminus \{0\} \), it follows from Theorem 2.1 that \( \sum_{j=1}^{m} \lambda_j U^{(j)} = |\lambda| V \), where \( V \) is an orthogonal matrix. This fact will enable us to deduce that \( A \)-twisted spherical mean on H-type group is similar to the \( |\lambda| \)-twisted spherical mean on the Heisenberg group.
Let \( \mu_s \) be the normalized surface measure on the set \( \{(z,0): |z| = s\} \subset G \). Then the partial spherical means of a function \( F \in L^1_{\text{loc}}(G) \) can be defined by

\[
F \ast \mu_s(z, t) = \int_{|w|=s} F((z, t)(-w, 0)) \, d\mu_s(w).
\]

Let

\[
F^\lambda(z) = \int_{\mathbb{R}^m} F(z, t)e^{i\lambda t} \, dt,
\]

be the inverse Fourier transform of \( F \) in the \( t \) variable. Then

\[
(F \ast \mu_s)^\lambda(z) = \int_{|w|=s} F^\lambda(z - w)e^{\frac{i}{2}\sum_{j=1}^m \lambda_j \Re(z \cdot \overline{U_j(w)})} \, d\mu_s(w).
\]

Define the \( \lambda \)-twisted spherical means of \( f \in L^1(\mathbb{C}^n) \) by

\[
f \times \lambda \mu_s(z) = \int_{|w|=s} f(z - w) e^{\frac{i}{2}\sum_{j=1}^m \lambda_j \Re(z \cdot \overline{U_j(w)})} \, d\mu_s(w).
\]

From (2.2) we get \( (F \ast \mu_s)^\lambda = F^\lambda \times \lambda \mu_s \). Thus, the partial spherical mean \( F \ast \mu_s \) on the Métivier group \( G \) can be thought of as the \( \lambda \)-twisted spherical mean \( F^\lambda \times \lambda \mu_s \). Note that \( \lambda \)-twisted spherical mean (2.5) is the complexification of the mean

\[
f \times \lambda \mu_s(x) = \int_{|\xi|=s} f(x - \xi) e^{\frac{i}{2}\sum_{j=1}^m \lambda_j (x \cdot \overline{U_j(\xi)})} \, d\mu_s(\xi).
\]

**Definition 2.2.** Let \( B_{r,R} = \{z \in \mathbb{C}^n : r < |z| < R\} \) be an open annulus in \( \mathbb{C}^n \), where \( 0 \leq r < R \leq \infty \). Let \( L_{r,R} \) be the space of all continuous functions \( f \) on \( B_{r,R} \) such that \( f \times \lambda \mu_s(z) = 0 \) on the spheres \( S_s(z) \subset B_{r,R} \) and the ball \( B_r(0) \subseteq B_s(z) \).

Let \( Z_{r,R}^\infty \) be space of all smooth functions in \( Z_{r,R} \). Consider a smooth non-negative radial function \( \phi \) on \( \mathbb{C}^n \), supported in \( B_1(0) \) and \( \int_{\mathbb{C}^n} \phi = 1 \). When \( \epsilon > 0 \), write \( \phi_\epsilon(z) = \epsilon^{-2n} \phi(\frac{z}{\epsilon}) \). For \( f \in Z_{r,R}^\infty \), define \( S_\epsilon(f) \) by

\[
S_\epsilon(f)(z) = \int_{\mathbb{C}^n} f(z - w)\phi_\epsilon(w)e^{\frac{i}{2}\sum_{j=1}^m \lambda_j \Re(z \cdot \overline{U_j(w)})} \, dw.
\]

Then we can deduce that \( S_\epsilon(f) \in Z_{r+\epsilon,R-\epsilon}^\infty \). Since \( \text{supp} \, \phi_\epsilon \subseteq B_\epsilon(0) \), and

\[
S_\epsilon(f)(z) - f(z) = \int_{|w| \leq \epsilon} \phi_\epsilon(w)e^{\frac{i}{2}\sum_{j=1}^m \lambda_j \Re(z \cdot \overline{U_j(w)})}(f(z - w) - f(z)) \, dw
\]

\[+ \int_{|w| \leq \epsilon} \left( e^{\frac{i}{2}\sum_{j=1}^m \lambda_j \Re(z \cdot \overline{U_j(w)})} - 1 \right) f(z)\phi_\epsilon(w) \, dw,
\]

together with \( f \) is continuous, letting \( \epsilon \) goes to 0, it follows that \( S_\epsilon(f) \) converges to \( f \) locally uniformly. Thus, without loss of generality, we can assume the functions in \( Z_{r,R} \) are smooth.

Since \( Z_{r,R} \) is closed under small translation, it follow that \( Z_{r,R} \) will be invariant under the action of appropriate vector fields on \( G \).
and \((x, t) = (x_1, \ldots, x_n, y_1, \ldots, y_m, t_1, \ldots, t_m) \in \mathbb{R}^{2n} \times \mathbb{R}^m\). In fact, they generate a basis of the Lie algebra of the Métivier group \(G\). Given \(U^{(s)}\)'s are skew-symmetry, we obtain the following commutation relations

\[
[X_i, X_j] = \sum_{k=1}^{m} U_{i,j}^{(k)} \frac{\partial}{\partial t_k}, \quad [Y_i, Y_j] = \sum_{k=1}^{m} U_{i,j}^{(k)} \frac{\partial}{\partial t_k}, \quad \text{for } i, j = 1, \ldots, n.
\]

Since \(U^{(1)}, \ldots, U^{(m)}\) are linearly independent, the dimension of the space spanned by \(\{(U_{i,j}^{(1)}, \ldots, U_{i,j}^{(m)}) : i, j = 1, \ldots, n\}\) will be \(m\).

Now, for \(1 \leq j \leq n\), define

\[
Z_j = \frac{1}{2}(X_j - iY_j)
\]

\[
= \frac{\partial}{\partial z_j} + \frac{1}{4} \sum_{k=1}^{m} \sum_{l=1}^{n} \left\{x_l \left(U_{i,j}^{(k)} - iU_{i,n+j}^{(k)} \right) + y_l \left(U_{n+i,j}^{(k)} - iU_{n+i,n+j}^{(k)} \right) \right\} \frac{\partial}{\partial t_k}.
\]

\[
\bar{Z}_j = \frac{1}{2}(X_j + iY_j)
\]

\[
= \frac{\partial}{\partial \bar{z}_j} + \frac{1}{4} \sum_{k=1}^{m} \sum_{l=1}^{n} \left\{x_l \left(U_{i,j}^{(k)} + iU_{i,n+j}^{(k)} \right) + y_l \left(U_{n+i,j}^{(k)} + iU_{n+i,n+j}^{(k)} \right) \right\} \frac{\partial}{\partial t_k}.
\]

Consider the function \(F\) on \(G = \mathbb{C}^n \times \mathbb{R}^m\) of type \(F(z, t) = e^{\lambda t} f(z)\), where \(\lambda \in \mathfrak{g} \setminus \{0\}\). Then the vector fields \(Z_j\) and \(\bar{Z}_j\) reduce to

\[
Z_j^\lambda = \frac{\partial}{\partial z_j} + \frac{1}{4} \sum_{l=1}^{n} \left\{(\beta_l^\lambda + i\alpha_l^\lambda) z_l + (-\beta_l^\lambda + i\alpha_l^\lambda) \bar{z}_l \right\}
\]

\[
(2.7) = \frac{\partial}{\partial z_j} + \frac{1}{4} \nu_j \bar{z}_j + \frac{1}{4} \sum_{l=1}^{n} (\eta_l z_l + \nu_l \bar{z}_l),
\]

\[
\bar{Z}_j^\lambda = \frac{\partial}{\partial \bar{z}_j} + \frac{1}{4} \sum_{l=1}^{n} \left\{(\bar{\beta}_l^\lambda + i\bar{\alpha}_l^\lambda) z_l + (-\bar{\beta}_l^\lambda + i\bar{\alpha}_l^\lambda) \bar{z}_l \right\}
\]

\[
(2.8) = \frac{\partial}{\partial \bar{z}_j} - \frac{1}{4} \bar{\nu}_j z_j - \frac{1}{4} \sum_{l=1}^{n} (\bar{\eta}_l z_l + \bar{\nu}_l \bar{z}_l),
\]
since \( \eta_j = 0 \), where we denote

\[
\alpha_t^\lambda = \frac{1}{2} \sum_{k=1}^{m} \lambda_k \left( U_{i,j}^{(k)} - i U_{i,n+j}^{(k)} \right), \quad \beta_t^\lambda = \frac{1}{2} \sum_{k=1}^{m} \lambda_k \left( U_{n+i,j}^{(k)} - i U_{n+i,n+j}^{(k)} \right)
\]

and \( \eta_l = \beta_t^\lambda + i \alpha_t^\lambda \), \( \nu_l = -\beta_t^\lambda + i \alpha_t^\lambda \) for \( 1 \leq l \leq n \).

The differential operators \( Z_j^\lambda \) and \( \bar{Z}_j^\lambda \) play a role of left-invariant vector fields for \( \lambda \)-twisted convolution on \( \mathbb{C}^n \). That is,

\[
Z_j^\lambda(f \times_\lambda \mu_s) = Z_j^\lambda f \times_\lambda \mu_s \quad \text{and} \quad \bar{Z}_j^\lambda(f \times_\lambda \mu_s) = \bar{Z}_j^\lambda f \times_\lambda \mu_s.
\]

As an effect, if \( f \in Z_{r,R} \), then \( Z_j^\lambda f \) and \( \bar{Z}_j^\lambda f \) both are in \( Z_{r,R} \).

2.1. Bi-graded spherical harmonics. We require the bi-graded spherical harmonic expansion of continuous function on \( \mathbb{C}^n \). See [5, 9, 18, 23] for details.

For \( p, q \in \mathbb{Z}_+ \), the set of all non-negative integers, let \( P_{p,q} \) denote the space of all polynomials \( P \) in \( z \) and \( \bar{z} \) of the form

\[
P(z) = \sum_{|\alpha|=p} \sum_{|\beta|=q} c_{\alpha\beta} z^\alpha \bar{z}^\beta.
\]

Write \( H_{p,q} = \{ P \in P_{p,q} : \Delta P = 0 \} \), where \( \Delta \) stands for the Laplacian on \( \mathbb{C}^n \). The elements of \( H_{p,q} \) restricted to the unit sphere \( S^{2n-1} \) are called bi-graded spherical harmonics. Now, we identify \( H_{p,q} \) as the space of bi-graded spherical harmonics on \( S^{2n-1} \). Let \( \{ Y_{j}^{p,q} : 1 \leq j \leq d_{p,q} \} \) be an orthonormal basis of \( H_{p,q} \). By Peter-Weyl theorem the set \( \{ Y_{j}^{p,q} : 1 \leq j \leq d_{p,q}, \; p, q \in \mathbb{Z}_+ \} \) forms an orthonormal basis for \( L^2(S^{2n-1}) \), and hence a continuous function \( f \) on \( \mathbb{C}^n \) can be expressed as

\[
f(\rho, \omega) = \sum_{p,q} \sum_{j=1}^{d_{p,q}} a_j^{p,q}(\rho) \; Y_j^{p,q}(\omega),
\]

where \( \rho > 0 \), \( \omega \in S^{2n-1} \), and \( a_j^{p,q} \) are called the spherical harmonic coefficients of \( f \). The \( (p, q)^{th} \) projection of \( f \) is given by

\[
\Pi_{p,q}(f)(\rho, \omega) = \sum_{j=1}^{d_{p,q}} a_j^{p,q}(\rho) \; Y_j^{p,q}(\omega).
\]

We need the following lemma to decompose a homogeneous polynomial into homogeneous harmonic polynomials.

**Lemma 2.3.** [23] Every \( P \in H_{p,q} \) can be uniquely expressed as \( P(z) = P_0(z) + |z|^2 P_1(z) + \cdots + |z|^{2l} P_l(z) \), where \( P_k \in H_{p-k,q-k} \) and \( l \leq \min(p, q) \).

**Corollary 2.4.** [17] Let \( P \in H_{p,q} \). Then it follows that

\[
\bar{z}_j P(z) = P_0(z) + \gamma_{p,q} |z|^2 \frac{\partial P}{\partial \bar{z}_j}, \quad z_j P(z) = P_0'(z) + \gamma_{p,q} |z|^2 \frac{\partial P}{\partial z_j},
\]

where \( \gamma_{p,q} = \frac{1}{(n+p+q-1)} \), \( P_0 \in H_{p+1,q} \) and \( P_0' \in H_{p+1,q} \).
3. Results on Métivier groups

3.1. Characterization of certain continuous functions. As we know that the $\lambda$-twisted spherical mean of the Métivier group is not $U(n)$-invariant, we require to modify the space $Z_{r,R}$ appropriately.

We first recall the following fact from Geller [7]. The operator analog of a bi-graded harmonic polynomial can be identical to the polynomial itself. Now for our purpose, we assume $P_{j}^{p,q}(z) = |z|^{p+q}Y_{j}^{p,q}(\frac{z}{|z|})$ contains the term $z^{\alpha}z^{\beta}$, for some multi-index $\alpha, \beta \in \mathbb{Z}^+_{+}$ with $|\alpha| = p$ and $|\beta| = q$. By abuse of notation, we denote

$$ P_{j}^{p,q}(Z) = Z^{\alpha} \bar{Z}^{\beta}, $$

where $Z^{\alpha} = (Z_1)^{\alpha_1} \cdots (Z_n)^{\alpha_n}$ and $\bar{Z}^{\beta} = (\bar{Z}_1)^{\beta_1} \cdots (\bar{Z}_n)^{\beta_n}$. Let $a_{j}^{p,q}(\rho) = \rho^{-(p+q)a_{j}^{p,q}}$, where $a_{j}^{p,q}$ as appears in (2.9). Let $Z_{r,R}^{*}$ be the space of smooth functions $f$ on $B_{r,R}$ satisfying the conditions

$$ \Pi_{0,0} \left( P_{j}^{p,q}(Z) \left( \Pi_{0,0} \left( P_{j}^{p,0}(Z) \left( a_{j}^{p,q} P_{j}^{p,q} \right) \right) \right) \right) \in Z_{r,R} $$

for all $p, q \in \mathbb{Z}_{+}$ and $1 \leq j \leq d_{p,q}$.

Now, we fix some notations for our convenience. Denote $D_{j} = \rho \frac{\partial}{\partial \rho} + \nu_{j} \rho^{2}$ and $\tilde{D}_{j} = \rho \frac{\partial}{\partial \rho} - \nu_{j} \rho^{2}$, where $\nu_{j}$ is defined in (2.3). For multi-index $\alpha, \beta$, define

$$ D^{\alpha} = \prod_{i_{1}=1}^{\alpha_{1}} (\kappa_{i_{1},i_{1}} D_{1}+2) \cdots \prod_{i_{n}=1}^{\alpha_{n}} (\kappa_{n,i_{n}} D_{n}+2) \text{ and } \tilde{D}^{\beta} = \prod_{j_{1}=1}^{\beta_{1}} (\tilde{\kappa}_{j_{1},j_{1}} \tilde{D}_{1}+2) \cdots \prod_{j_{n}=1}^{\beta_{n}} (\tilde{\kappa}_{n,j_{n}} \tilde{D}_{n}+2), $$

where $\kappa_{i_{1},i_{1}}, \tilde{\kappa}_{j_{1},j_{1}} \in \{\gamma_{p',q'} = \frac{1}{(n+p'+q'-1)} : 0 \leq p' \leq p, 0 \leq q' \leq q\}$.

In order to prove the result for the functions in $Z_{r,R}^{*}$, it would be enough to consider the following theorem.

**Theorem 3.1.** Let $f(z) = \tilde{a}(\rho) P_{p,q}(z)$, where $\rho = |z|$ and $P_{p,q} \in H_{p,q}$. Then a necessary condition for $f \in Z_{r,R}^{*}$ is that $\tilde{a}$ satisfies the ODE

$$ \left( \sum_{|\beta|+k=q} d_{\beta,k} \rho^{2k} \tilde{D}^{\beta} \right) \left( \sum_{|\alpha|+l=p} c_{\alpha,l} \rho^{2l} D^{\alpha} \right) \tilde{a} = 0 $$

for some scalars $c_{\alpha,l}, d_{\beta,k} \in \mathbb{C}$.

In particular, if $P_{p,q}(z) = z_{1}^{l_{1}} z_{2}^{l_{2}}$ for some $1 \leq l_{1}, l_{2} \leq n$, then there exist $A_{i}, B_{k} \in \mathbb{C}$ such that

$$ \tilde{a}(\rho) = \sum_{i=0}^{p} A_{i} e^{-\frac{\nu_{i}}{r^{2}}} \rho^{2(\nu_{i}+n-i)} + \sum_{k=0}^{q} B_{k} e^{-\frac{\nu_{k}}{r^{2}}} \rho^{2(\nu_{k}+n-k)}, $$

where $r < \rho < R$ and $A_{0} = B_{0} = 0$. 

Proof. For \( p = q = 0 \), we have \( \tilde{a}(\rho) = \tilde{a} \times \mu_\rho(0) = 0 \), whenever \( r < \rho < R \). To proceed the other cases, we need to apply the operator \( Z_j^\lambda \) to \( f \).

\[
Z_j^\lambda f = \frac{\partial f}{\partial z_j} + \frac{1}{4} \nu_j \tilde{z}_j f + \frac{1}{4} \sum_{l=1}^{n} (\eta_l z_l + \nu_l \tilde{z}_l) f.
\]

Given that \( f = \tilde{a} P \), the above equation will take the form

\[
(3.2) \quad Z_j^\lambda f = \frac{1}{2\rho^2} (D_j \tilde{a}) \tilde{z}_j P(z) + \tilde{a} \frac{\partial P}{\partial z_j} + \frac{1}{4} \sum_{l\neq j} (\eta_l z_l + \nu_l \tilde{z}_l) \tilde{a} P(z).
\]

Substituting the values of \( \tilde{z}_j P(z) \) and \( z_j P(z) \) from Corollary [2.4] we have

\[
Z_j^\lambda f = \frac{1}{2\rho^2} D_j \tilde{a} \left( P_0 + \gamma_{p,q} |z|^2 \frac{\partial P}{\partial z_j} \right) + \tilde{a} \frac{\partial P}{\partial z_j} + \frac{1}{4} \sum_{l\neq j} \eta_l \tilde{a} \left( P_0 + \gamma_{p,q} |z|^2 \frac{\partial P}{\partial z_l} \right) + \nu_l \tilde{a} \left( P_0 + \gamma_{p,q} |z|^2 \frac{\partial P}{\partial z_l} \right).
\]

After rearranging the terms, we get

\[
Z_j^\lambda f = \frac{1}{2\rho^2} D_j \tilde{a} P_0 + \frac{1}{4} \sum_{l\neq j} \nu_l \tilde{a} P_0 + \frac{1}{4} \sum_{l\neq j} \eta_l \tilde{a} P_0
\]

\[
+ \frac{1}{2} (\gamma_{p,q} D_j + 2) \tilde{a} \frac{\partial P}{\partial z_j} + \frac{1}{4} \rho^2 \gamma_{p,q} \sum_{l\neq j} \left( \eta_l \frac{\partial P}{\partial z_l} + \nu_l \frac{\partial P}{\partial z_l} \right) \tilde{a}.
\]

Now, the projection \( \Pi_{p-1,q} \) of \( Z_j^\lambda f \) is given by

\[
\Pi_{p-1,q} Z_j^\lambda f = \frac{1}{2} (\gamma_{p,q} D_j + 2) \tilde{a} \frac{\partial P}{\partial z_j} + \frac{1}{4} \rho^2 \gamma_{p,q} \sum_{l\neq j} \frac{\partial P}{\partial z_l} \tilde{a}.
\]

If \( p = 1 \) and \( q = 0 \), then \( \frac{\partial P}{\partial z_j} \) is a non-zero constant for some \( j \), say \( \zeta_j \). Thus,

\[
(3.3) \quad \Pi_{0,0} Z_j^\lambda f = \left\{ \frac{1}{2n} \left( \rho \frac{\partial}{\partial \rho} + \frac{\nu_j}{2} \rho^2 \right) + 1 \right\} \tilde{a} \zeta_j + \frac{1}{4} \sum_{l\neq j} \rho^2 \frac{\nu_l}{n} \tilde{a} \zeta_l
\]

\[
= \zeta_j \left\{ \frac{1}{2n} \left( \rho \frac{\partial}{\partial \rho} + \frac{\nu_j}{2} + \sum_{l\neq j} \frac{\nu_l \zeta_l}{2 \zeta_j} \rho^2 \right) + 1 \right\} \tilde{a}
\]

\[
= \zeta_j \left\{ \frac{1}{2n} \left( \rho \frac{\partial}{\partial \rho} + \frac{d_0}{2} \rho^2 \right) + 1 \right\} \tilde{a}(\rho),
\]

where \( d_1 = (\nu_j + \frac{\sum_{l\neq j} \nu_l \zeta_l}{\zeta_j}) \). By definition of \( Z_{r,R}^*; \Pi_{0,0}(Z_j^\lambda f) \in Z_{r,R} \). Evaluating \( \lambda \)-twisted spherical mean of \( \Pi_{0,0}(Z_j^\lambda f) \) at \( z = 0 \), we get

\[
\left\{ \frac{1}{2n} \left( \rho \frac{\partial}{\partial \rho} + \frac{d_1}{2} \rho^2 \right) + 1 \right\} \tilde{a}(\rho) = 0.
\]
By replacing \( \tilde{a}(\rho) = e^{-\frac{d}{4}\rho^2} \tilde{a}'(\rho) \) in the above equation, we get

\[
e^{-\frac{d}{4}\rho^2} \left\{ \frac{1}{2n} \rho \frac{\partial}{\partial \rho} + 1 \right\} \tilde{a}'(\rho) = 0.
\]

Thus, for \( p = 1, q = 0 \) we infer that

\[
\tilde{a}(\rho) = A_1 e^{-\frac{d}{4}\rho^2} \rho^{-2n}.
\]

However, for the case \( q = 0 \) and \( p \geq 2 \), it would be difficult to solve the ODE. For instance, consider \( P(z) = z_{1z2} \). Then, after applying \( Z_1 Z_2 \) to \( \tilde{a}P \), we get

\[
\{(\gamma_{1,0} D_1 + 2)(\gamma_{2,0} D_2 + 2) + c_1 c_2 \rho^4 \} \tilde{a} = 0,
\]

which we yet to solve.

For \( p \geq 2 \) and \( q = 0 \), we conclude that

\[
\sum_{|\alpha|+l=p} c_{\alpha,l} \rho^{2l} D^\alpha \tilde{a} = 0.
\]

By similar argument for \( p = 0 \) and \( q \geq 1 \), we get

\[
\sum_{|\beta|+k=q} d_{\beta,k} \rho^{2k} \bar{D}^\beta \tilde{a} = 0.
\]

In general, while \( p, q \geq 1 \), we conclude that

\[
\left( \sum_{|\beta|+k=q} d_{\beta,k} \rho^{2k} \bar{D}^\beta \right) \left( \sum_{|\alpha|+l=p} c_{\alpha,l} \rho^{2l} D^\alpha \right) \tilde{a} = 0.
\]

However, if \( P(z) \) is of the form \( z_{1l1}^p \), then we can express \( \tilde{a} \) explicitly as earlier. For this, first consider the case \( q = 0 \) and \( p \geq 1 \). Since \( \tilde{a} z_{1i}^p \in Z^*_r \), it follows that \( \Pi_{0,0} Z^p_t (\tilde{a}P) \in Z^*_r \). Thus, by evaluating \( \lambda \)-twisted spherical mean of \( \Pi_{0,0} Z^p_t (\tilde{a}P) \) at \( z = 0 \), we get

\[
\prod_{i=1}^p \left( \gamma_{p-(i-1),0} D_{t_i} + 2 \right) \tilde{a} = 0.
\]

This, in turn, implies that

\[
\tilde{a}(\rho) = \sum_{i=1}^p A_i e^{-\frac{\gamma_{p,i}}{4}\rho^2} \rho^{-2(n+p-i)}.
\]

Similarly, for \( p = 0 \) and \( q \geq 1 \), by considering the operator \( \tilde{Z}_{l2}^q \), we can derive that

\[
\tilde{a}(\rho) = \sum_{k=1}^q B_k e^{-\frac{\gamma_{q,k}}{4}\rho^2} \rho^{-2(n+q-k)}.
\]
If \( p, q \geq 1 \), by evaluating \( \lambda \)-twisted spherical mean of \( \Pi_{0,0}Z_{r,0}^{p}(\tilde{a}P) \) at \( z = 0 \), we obtain

\[
\prod_{k=1}^{q} (\gamma_{p,q+(k-1)} \tilde{D}_{l_{2}} + 2) \prod_{i=1}^{p} (\gamma_{p-(i-1),q} D_{l_{1}} + 2) \tilde{a} = 0.
\]

Hence, a solution to the above equation can be expressed as

\[
\tilde{a}(\rho) = \sum_{i=1}^{p} A_{i} e^{-\nu_{i} \rho^{2}} \rho^{-(n+p+q-i)} + \sum_{k=1}^{q} B_{k} e^{-\nu_{k} \rho^{2}} \rho^{-(2(n+p+q-k))}.
\]

This completes the proof. \( \square \)

**Remark 3.2.** In the definition of \( Z_{r,R}^{*} \) we have assumed that, for all \( p, q \in \mathbb{Z}_{+} \) and \( 1 \leq j \leq d_{p,q} \),

\[
(3.4) \quad \Pi_{0,0} \left( \Pi_{0,0} \left( P_{j}^{p,0}(Z) \left( \Pi_{0,0} \left( P_{j}^{p,0}(Z) \left( \tilde{a}_{j}^{p,q} \bar{P}_{j}^{p,q} \right) \right) \right) \right) \right) \times_{\lambda} \mu_{\lambda}(z) = 0
\]

for all \( z \in \mathbb{C}^{n} \) and \( s > 0 \) with \( S_{s}(z) \subseteq B_{r,R} \) and \( B_{s}(0) \subseteq B_{s}(z) \). However, for a proof of Theorem 3.1, it is enough to assume that (3.4) holds for \( z = 0 \), whenever \( r < s < R \). Consequently, sufficient part of Theorem 3.1 at \( z = 0 \), is obviously true.

Further, as compared to the Heisenberg group, it would be a reasonable question to consider \( e^{\pi |z|^{2}} |z|^{-2(n+p+q-1)} P(z) \) to be in \( Z_{r,\infty} \) for appropriate choice of \( c \) and \( i \), where \( P \in H_{p,q} \). In general, the matrix \( \sum_{j=1}^{m} \lambda_{j} U^{(j)} \), arises from the symplectic form, has distinct eigenvalues, makes the difficulty to find out such \( c \). However, in the case of \( H \)-type group, all the eigenvalues are identical, we have such a result in Section 4, Theorem 4.1.

### 3.2. Injectivity and support theorem.

In this section, we simplify the \( \lambda \)-twisted spherical mean on the Métivier group to another mean, which is similar to the TSM on the Heisenberg group. This will ease to prove support theorem for the \( \lambda \)-twisted spherical mean on Métivier group, for the type function. Further, we prove that a non-harmonic complex cone aligned with one of the coordinate axes in \( \mathbb{C}^{n} \) is a set of injectivity for the \( \lambda \)-twisted spherical mean on the Métivier groups.

For \( \lambda \in \mathbb{R} \setminus \{0\} \), the skew symmetric matrix \( V_{\lambda} = \sum_{j=1}^{m} \lambda_{j} U^{(j)} \) is non-singular (see [13]). Let \( u_{1} \pm iv_{1}, \ldots, u_{n} \pm iv_{n} \) be the eigenvectors of \( V_{\lambda} \) with corresponding eigenvalues \( \pm i\mu_{\lambda,1}, \ldots, \pm i\mu_{\lambda,n} \), where \( \mu_{\lambda,1} \geq \cdots \geq \mu_{\lambda,n} > 0 \). Define \( A_{\lambda} = (\sqrt{2} v_{1}, \ldots, \sqrt{2} v_{n}, \sqrt{2} u_{1}, \ldots, \sqrt{2} u_{n}) \). Then \( A_{\lambda} \) is an orthogonal matrix and satisfies \( V_{\lambda} A_{\lambda} = A_{\lambda} \lambda_{\lambda} \), where

\[
(3.5) \quad U_{\lambda} = \begin{pmatrix} 0_{n} & -J_{\lambda} \\ J_{\lambda} & 0_{n} \end{pmatrix}
\]
with $J_{\lambda} = \text{diag}(\mu_{\lambda,1}, \ldots, \mu_{\lambda,n})$ and $0_n$ is zero matrix of order $n$. Thus, in view of (3.5), we have
\[
\sum_{j=1}^{m} \lambda_j \langle x, U^{-j} \xi \rangle = \langle x, V_\lambda \xi \rangle = \langle A_\lambda^l x, U_\lambda A_\lambda^l \xi \rangle,
\]
where $A_\lambda A_\lambda^l = I$. That is,
\[
\sum_{j=1}^{m} \lambda_j \text{Re} (z \cdot U^{-j} w) = \sum_{j=1}^{n} \mu_{\lambda,j} \text{Im} ((z_\lambda)_j \cdot (\bar{w}_\lambda)_j),
\]
where $z_\lambda$ and $w_\lambda$ are complexification of $A_\lambda^l x$ and $A_\lambda^l \xi$ respectively.

Let $f \in L^1(\mathbb{C}^n)$, then define
\[
f_\lambda(z) = f(\tilde{z}_\lambda),
\]
where $z, \tilde{z}_\lambda \in \mathbb{C}^n$ be the complexification of $x, A_\lambda x \in \mathbb{R}^{2n}$ respectively. The following lemma would give a simplification of the $\lambda$-twisted spherical mean on the M"{e}tivier group defined by (2.5).

**Lemma 3.3.** Let $f \in L^1(\mathbb{C}^n)$ and $f_\lambda$ be as in (3.7). Then $f \times_\lambda \mu_s(\tilde{z}_\lambda) = f_\lambda \times_\lambda \mu_s(z)$, where
\[
f_\lambda \times_\lambda \mu_s(z) = \int_{|w|=s} f_\lambda(z - w) e^{\frac{1}{2} \sum_{j=1}^{n} \mu_{\lambda,j} \text{Im}(z_j \cdot w_j)} \, d\mu_s(w).
\]

**Proof.** In view of (2.6), we can write
\[
f \times_\lambda \mu_s(A_\lambda x) = \int_{|\xi|=s} f(A_\lambda x - \xi) e^{\frac{1}{2} \sum_{j=1}^{n} \mu_{\lambda,j} \text{Im}(z_j \cdot \bar{w}_j)} \, d\mu_s(\xi)
\]
\[
= \int_{|\xi|=s} f_\lambda(x - A_\lambda^l \xi) e^{\frac{1}{2} (z_\lambda \cdot U_\lambda A_\lambda^l \xi)} \, d\mu_s(\xi)
\]
\[
= \int_{|\xi|=s} f_\lambda(x - \xi) e^{\frac{1}{2} (x_\lambda \cdot U_\lambda \xi)} \, d\mu_s(\xi)
\]
\[
= \int_{|w|=s} f_\lambda(z - w) e^{\frac{1}{2} \sum_{j=1}^{n} \mu_{\lambda,j} \text{Im}(z_j \cdot w_j)} \, d\mu_s(w)
\]
\[
= f_\lambda \times_\lambda \mu_s(z).
\]

To deal with the modified $\lambda$-twisted spherical mean $f_\lambda \times_\lambda \mu_s$, defined in (3.8), it is required to study the function $f_\lambda$. In particular, we need to find out those polynomials $P$ such that $P_\lambda \in H_{p,q}$ for some $p, q$. Let $P_\lambda \in H_{p,q}$. By identifying $\mathbb{C}^n$ with $\mathbb{R}^{2n}$, we get $P_\lambda \in H_l$, where $l = p + q$, and $H_l$ is the space of all homogeneous harmonic polynomials of degree $l$ on $\mathbb{R}^{2n}$. Since $P(x) = P_\lambda(A_\lambda^l x)$ and the Laplacian is rotation invariant, we get $P \in H_l$, and hence
\[
P \in \bigoplus_{p' + q' = l} H_{p',q'}.
\]
With the above observation, we define

\[(3.9) \quad H^\lambda_{p,q} = \{ P \in \bigoplus_{p'+q' = p+q} H^\lambda_{p',q'} : P_\lambda \in H_{p,q} \}. \]

Next, we prove a similar result to support theorem for the Métivier groups. Consider the following left-invariant differential operators for the \( \lambda \)-twisted spherical mean \( [3.5, 8] \),

\[
\tilde{Z}_j^\lambda = \frac{\partial}{\partial z_j} - \frac{\mu_{\lambda,j}}{4} \tilde{z}_j \text{ and } \tilde{Z}_j^\lambda = \frac{\partial}{\partial z_j} + \frac{\mu_{\lambda,j}}{4} z_j, \quad j = 1, 2, \ldots, n.
\]

Since \( P_\lambda \in H_{p,q} \), define \( P^\lambda_{p,q}(\tilde{Z}) \) as in \([3.1]\), replacing \( Z \) by \( \tilde{Z} \).

**Theorem 3.4.** Let \( f = \tilde{a}P \), where \( P \in H^\lambda_{p,q} \), be a smooth function on \( \mathbb{C}^n \) and \( |z|^k e^{\frac{\mu_{\lambda,1}|z|^2}{4}} f(z) \) is bounded for each \( k \in \mathbb{Z}_+ \). Then \( \Pi_{0,0} \left( P^\lambda_{p,q}(\tilde{Z})f_\lambda \right) \tilde{x}_\lambda \mu_s(z) = 0 \) for all \( z \in \mathbb{C}^n \) and \( s > r + |z| \) if and only if \( f \) is supported in \( |z| \leq r \).

**Proof.** If \( f = \tilde{a}P \), then \( f_\lambda = aP_\lambda \). Clearly for \( p = q = 0, \tilde{a} = 0 \). Let \( p \geq 1 \), then applying \( \tilde{Z}_j^\lambda \) to \( f_\lambda \), we have

\[
\tilde{Z}_j^\lambda f_\lambda = \frac{1}{2} \left( \frac{1}{\rho} \frac{\partial}{\partial \rho} - \frac{\mu_{\lambda,j}}{2} \right) \tilde{a} \rho P_\lambda + \tilde{a} \frac{\partial P_\lambda}{\partial z_j}.
\]

Since \( P_\lambda \in H_{p,q} \), substituting the value of \( \tilde{Z}_j^\lambda P_\lambda \) from corollary \([2.4]\) we have

\[
\tilde{Z}_j^\lambda f_\lambda = \frac{1}{2} \left( \frac{1}{\rho} \frac{\partial}{\partial \rho} - \frac{\mu_{\lambda,j}}{2} \right) \tilde{a} \rho P_\lambda + \left[ \frac{\gamma_{p,q}}{2} \left( \rho \frac{\partial}{\partial \rho} - \rho^2 \frac{\mu_{\lambda,j}}{2} \right) + 1 \right] \tilde{a} \frac{\partial P_\lambda}{\partial z_j},
\]

where \( \gamma_{p,q} = \frac{1}{(n+p+q-1)} \).

Consider \( q = 0 \) and \( p = 1 \). Then there exists a \( j_0 \) such that \( \frac{\partial P_\lambda}{\partial z_{j_0}} \neq 0 \). Thus, from the given condition that \( \Pi_{0,0}(\tilde{Z}_{j_0}^\lambda f_\lambda) \tilde{x}_\lambda \mu_s(0) = 0 \) for all \( s > r \), we arrived at

\[
\left[ \frac{1}{2n} \left( \rho \frac{\partial}{\partial \rho} - \rho^2 \frac{\mu_{\lambda,j_0}}{2} \right) + 1 \right] \tilde{a} = 0,
\]

whenever \( \rho > r \). This leads to a solution

\[
\tilde{a}(\rho) = A_1 e^{-\frac{\mu_{\lambda,j_0} \rho^2}{4}} \rho^{-2n}.
\]

By an induction argument, for \( q = 0 \) and \( p \geq 1 \), from \( \Pi_{0,0}(\tilde{Z}_\lambda^\mu f_\lambda) \tilde{x}_\lambda \mu_s(0) = 0 \) for all \( s > r \), it follows that

\[
\prod_{i=1}^{p} \left\{ \frac{1}{2(n + p - i)} \left( \rho \frac{\partial}{\partial \rho} - \rho^2 \frac{\mu_{\lambda,j}}{2} \right) + 1 \right\} \tilde{a} = 0.
\]

Solving the above equation we get

\[
\tilde{a}_{p,0}(\rho) = \sum_{i=1}^{p} A_i e^{-\frac{\mu_{\lambda,j}}{4} \rho^2} \rho^{-2(n+i)},
\]
where \( c_i \in \{ \mu_{\lambda,j} : 1 \leq j \leq n \} \). Similar conclusion holds true for \( p = 0, q \geq 1 \).

In general, for \( p, q \geq 1 \), \( \hat{a} \) satisfies the ODE

\[
\prod_{k=1}^{q} \left\{ \frac{\gamma_{\rho,q+1-k}}{2} \left( \rho \frac{\partial}{\partial \rho} + \rho^2 d_k \right) + 1 \right\} \prod_{i=1}^{p} \left\{ \frac{\gamma_{\rho+1-i,q}}{2} \left( \rho \frac{\partial}{\partial \rho} - \rho^2 c_i \right) + 1 \right\} \hat{a} = 0,
\]

and be expressed as

\[
\hat{a}_{p,q}(\rho) = \sum_{i=1}^{p} A_i e^{-\frac{i\pi}{2} \rho^3 \rho^{-2(p+q+n-i)}} + \sum_{k=1}^{q} B_k e^{\frac{i\pi}{2} \rho^3 \rho^{-2(p+q+n-k)}}
\]

for all \( \rho > r \), where \( c_i, d_k \in \{ \mu_{\lambda,j} : 1 \leq j \leq n \} \) and \( A_i, B_k \) are constants. Since \( \mu_{\lambda,1} \geq \mu_{\lambda,j} > 0 \) for all \( j \), by the given growth conditions, we infer that \( f_{\lambda}(z) = 0 \) for all \( |z| > r \). Thus, we conclude that \( f \) is supported in \( |z| \leq r \). \( \square \)

A set \( K \subset \mathbb{C}^n (n \geq 2) \), which is closed under complex scaling, is known as a complex cone. Further, a complex cone that does not intersect the zero set of any bi-graded homogeneous harmonic polynomial is called non-harmonic. The zero set of the polynomial \( H(z) = az \bar{z}_2 + |z|^2 \), where \( a \neq 0 \) and \( z \in \mathbb{C}^n \) is a non-harmonic complex cone, (see [22]).

Let \( z \in \mathbb{C}^n \) be the complexification of \( x \in \mathbb{R}^{2n} \), and \( \bar{z}_\lambda \) be the complexification of \( A_\lambda x \). For a complex cone \( K \), define \( K_\lambda = \{ z \in \mathbb{C}^n : \bar{z}_\lambda \in K \} \). Then \( K_\lambda \) is also a complex cone, and \( K \) is non-harmonic if and only if \( K_\lambda \) is non-harmonic.

**Theorem 3.5.** Suppose \( K \) is a non-harmonic complex cone such that \( K_\lambda \) aligned with one of the coordinate axes in \( \mathbb{C}^n (n \geq 2) \). Let \( f \) be a continuous function on \( \mathbb{C}^n \) such that \( f \times_\lambda \mu_\tau(z) = 0 \), for all \( \tau > 0 \) and \( z \in K \). Then \( f = 0 \).

**Proof.** In view of Lemma 3.3, given \( f \times_\lambda \mu_\tau = 0 \) on \( K \) implies \( f_{\lambda,\bar{z}_\lambda} \mu_\tau = 0 \) on \( K_\lambda \). By hypothesis, without loss of generality, we can assume \( z = (z_1, 0, ..., 0) \in K_\lambda \) for all \( z_1 \in \mathbb{C} \). Thus,

\[
\int_{|w| \leq r} f_{\lambda}(z + w)e^{-\frac{i\pi}{2} \sum_{j=1}^{n} \mu_{\lambda,j} \text{Im}(z_j \bar{w}_j)} dw = \int_{0}^{r} f_{\lambda,\bar{z}_\lambda}(z) s^{2n-1} ds = 0
\]

for all \( r > 0 \) and \( z \in K_\lambda \). Applying \( 2 \partial_{z_1} \) to the above equation, we get

\[
\int_{|w| \leq r} \frac{\partial}{\partial w_1} \left( f_{\lambda}(z + w)e^{-\frac{i\pi}{2} \sum_{j=1}^{n} \mu_{\lambda,j} \text{Im}(z_j \bar{w}_j)} \right) dw
\]

\[
- \frac{\mu_{\lambda,1}}{2} \int_{|w| \leq r} w_1 f_{\lambda}(z + w)e^{-\frac{i\pi}{2} \sum_{j=1}^{n} \mu_{\lambda,j} \text{Im}(z_j \bar{w}_j)} ds = 0.
\]

It follows by an application of Green’s theorem that

\[
\int_{|w|=r} \frac{w_1}{r} \left( f_{\lambda}(z + w)e^{-\frac{i\pi}{2} \sum_{j=1}^{n} \mu_{\lambda,j} \text{Im}(z_j \bar{w}_j)} \right) dw
\]

\[
= \frac{\mu_{\lambda,1}}{2} \int_{|w| \leq r} \bar{w}_1 f_{\lambda}(z + w)e^{-\frac{i\pi}{2} \sum_{j=1}^{n} \mu_{\lambda,j} \text{Im}(z_j \bar{w}_j)} dw.
\]
Let \( F(t) = t^{2n-1} g \tilde{x}_\lambda \mu_t(z) \), where \( g(z) = \tilde{z}_1 f_\lambda(z) \). Then we have

\[
F'(r) = \frac{\mu_{\lambda_1}}{2} \int_0^r F(s)ds.
\]

(3.11)

It is easy to see that (3.11) satisfies the ODE

\[
F'(r) = \left( \frac{\mu_{\lambda_1} r}{2} + \frac{1}{r} \right) F(r)
\]

having the general solution

\[
F(r) = \frac{c(z)}{r} e^{\frac{\mu_{\lambda_1} r^2}{4}}.
\]

That is,

\[
r^{2n-2} g \tilde{x}_\lambda \mu_r(z) = c(z) e^{\frac{\mu_{\lambda_1} r^2}{4}}.
\]

Letting \( r \to 0 \), we get \( c(z) = 0 \). Hence \( \tilde{z}_1 f_\lambda \tilde{x}_\lambda \mu_r(z) = 0 \) for all \( r > 0 \) and \( z \in K_\lambda \). By replicating the above procedure, we get \( (P f_\lambda) \tilde{x}_\lambda \mu_r(z) = 0 \) for arbitrary polynomial \( P(z_1, \tilde{z}_1) \). By a similar argument as in (22, Lemma 2.6) we conclude that \( f_\lambda = 0 \) and hence \( f = 0 \).

\[\square\]

**Remark 3.6.** If we consider \( H \)-type group instead of the general Métivier group, then the restriction on the cone to align with one of the coordinates axes could be relaxed.

### 4. Some results on \( H \)-type groups

In this section, we see that the \( \lambda \)-twisted spherical mean on the \( H \)-type group can be related to the \( |\lambda| \)-twisted spherical mean on the Heisenberg group. Although the \( \lambda \)-twisted spherical mean on \( H \)-type group is not \( U(n) \)-invariant, we can prove sufficient condition for a function to be in \( Z_{r,\infty} \), and an analogue of Helgason’s support theorem together with Heche-Bochner identity for the \( H \)-type group. Further, we prove that the boundary of a bounded domain is a set of injectivity for \( \lambda \)-twisted spherical mean on the \( H \)-type group.

We know that for the \( H \)-type groups, \( \mu_{\lambda,j} = |\lambda| \) for all \( j \), due to the fact that \( \sum_{j=1}^m \lambda_j U^{(j)} = |\lambda| V \). Thus (3.6) becomes

\[
\sum_{j=1}^m \lambda_j \text{Re} (z \cdot U^{(j)} w) = |\lambda| \text{Im} (z_\lambda \cdot \tilde{w}_\lambda)
\]

and from Lemma 3.3 we have

\[
f \times_\lambda \mu_s(z) = f_\lambda \tilde{x}_\lambda \mu_s(z_\lambda) = f_\lambda \times |\lambda| \mu_s(z_\lambda),
\]

where \( f_\lambda \times |\lambda| \mu_s \) is the \( |\lambda| \)-twisted spherical mean on the Heisenberg group. Similarly, the \( \lambda \)-twisted convolution on the \( H \)-type group can be related to the twisted convolution on the Heisenberg group by

\[
f \times_\lambda g(z) = f_\lambda \times |\lambda| g_\lambda(z_\lambda).
\]
Next, we present the sufficient condition for functions to be in $Z_{r,\infty}$, which we mentioned in Remark 3.2.

**Theorem 4.1.** Let $P \in H^\lambda_{p,q}$ and $h(z) = \frac{e^{\frac{\lambda |z|^2}{2(n+p+q-i)}} P(z)}{|z|^{2(n+p+q-i)}}$, where $1 \leq i \leq p$ and $H^\lambda_{p,q}$ is defined in (3.3). Then $h \in Z_{r,\infty}$.

**Proof.** To prove the result, it needs to verify that $h \times_\lambda \mu_s(z) = 0$ for all $z \in \mathbb{C}^n$ and $s > |z| + r$. From (4.1), it is enough to show that $h_\lambda \times_\lambda \mu_s(z) = 0$ for all $z \in \mathbb{C}^n$ and $s > |z| + r$.

Let $\eta = n + p + q$ and consider

$$h_\lambda \times_\lambda \mu_s(z) = \int_{|w| = s} e^{\frac{\lambda |z+w|^2}{2(n+\eta-i)}} e^{-\frac{\lambda}{2} \operatorname{Im}(z-w)} d\mu_s(w).$$

Simplifying the exponential terms, it is enough to show the following integral is zero

$$\int_{|w| = s} e^{\frac{\lambda}{2} \operatorname{Re}(z+w)} P_\lambda(z + w) \frac{d\mu_s(w)}{|z+w|^{2(n-\eta)}}.$$

Again if we expand exponential term the above integral will reduce to

$$\int_{|w| = s} \frac{w^\eta P_\lambda(z+w)}{|z+w|^{2(n-\eta)}} d\mu_s(w).$$

Hence we arrived at the same Euclidean situation, which was proved by the author in (17, Theorem 3.3). Thus, it follows that $h_\lambda \times_\lambda \mu_s(z) = 0$ for all $z \in \mathbb{C}^n$ and $s > |z| + r$. \hfill \Box

Next, we shall prove support theorem for the $\lambda$- twisted spherical mean on the H-type group, for which we need to recall the following support theorem for the TSM on the Heisenberg group.

**Theorem 4.2.** [17] Let $g$ be a continuous function on $\mathbb{C}^n$ such that for each $k \in \mathbb{Z}_+$, $|z|^k e^{\frac{\lambda |z|^2}{2}} g(z)$ is bounded for every $k \in \mathbb{Z}_+$. Then $g$ is supported in $|z| \leq r$ if and only if $g \times_\lambda \mu_s(z) = 0$ for all $z \in \mathbb{C}^n$ and $s > r + |z|$.

Using (4.1) we prove the following support theorem for the $H$-type groups.

**Theorem 4.3.** Suppose $f$ is a continuous function on $\mathbb{C}^n$ such that each of the function $|z|^k e^{\frac{\lambda |z|^2}{2}} f(z)$ is bounded. Then $f$ is supported in $|z| \leq r$ if and only if $f \times_\lambda \mu_s(z) = 0$ for all $z \in \mathbb{C}^n$ and $s > r + |z|$.

**Proof.** We know that $f_\lambda(z_{\lambda}) = f(z)$, where $z$ and $z_{\lambda}$ are the complexification of $x$ and $A_\lambda x$ respectively. Since $|z|^k e^{\frac{\lambda |z|^2}{2}} f(z)$, is bounded, it follows that $|z|^k e^{\frac{\lambda |z|^2}{2}} f_\lambda(z)$ is bounded because $|z| = |z_{\lambda}|$. Now, $f$ is supported in $|z| \leq r$ if and only if $f_\lambda$ is supported in $|z| \leq r$. Hence, from (4.1) and Theorem 4.2 we get the desired result. \hfill \Box
Theorem 4.4. Let \( f \in L^1(C^n) \) be of the form \( f = Pg \), where \( g \) is radial and \( P \in H_{p,q} \). Then for \( \lambda > 0 \),
\[
 f \times_\lambda \varphi_{k,\lambda}^{n-1}(z) = \begin{cases} 
 (2\pi)^{-n} \lambda^{p+q} P(z) g \times_\lambda \varphi_{k-p,\lambda}^{n+p+q-1}(z), & \text{if } k \geq p \\
 0, & \text{otherwise}
\end{cases}
\]
and for \( \lambda < 0 \),
\[
 f \times_\lambda \varphi_{k,\lambda}^{n-1}(z) = \begin{cases} 
 (2\pi)^{-n} |\lambda|^{p+q} P(z) g \times_\lambda \varphi_{k-q,\lambda}^{n+p+q-1}(z), & \text{if } k \geq q \\
 0, & \text{otherwise}
\end{cases}
\]
where convolution on the right-hand side is on \( C^{n+p+q} \).

An analogue of the above Hecke-Bochner identity for \( H \)-type group can be stated as follows.

Theorem 4.5. Let \( f \in L^1(C^n) \) be of the form \( f = gP \) where \( g \) is radial and \( P \in H_{p,q}^\lambda \), where \( H_{p,q}^\lambda \) defined in (2.2). Then for \( \lambda \in \mathbb{R}^m \setminus \{0\} \),
\[
 f \times_\lambda \varphi_{k,\lambda}^{n-1}(z) = (2\pi)^{-n} |\lambda|^{p+q} P(z) g \times_\lambda \varphi_{k-p,\lambda}^{n+p+q-1}(z'),
\]
if \( k \geq p \) and 0 otherwise, where \( z' \in C^{n+p+q} \) be such that \( |z| = |z'| \) and convolution on the right is on \( C^{n+p+q} \).

Proof. Since \( \varphi_{k,\lambda}^{n-1} \) is radial, by (1.2) and Theorem 4.3 we get
\[
 f \times_\lambda \varphi_{k,\lambda}^{n-1}(z) = f_{\lambda} \times_{|\lambda|} \varphi_{k,\lambda}^{n-1}(z_{\lambda}) = (2\pi)^{-n} |\lambda|^{p+q} P_{\lambda}(z_{\lambda}) g \times_{|\lambda|} \varphi_{k-p,\lambda}^{n+p+q-1}(z'_{\lambda}) = (2\pi)^{-n} |\lambda|^{p+q} P(z) g \times_\lambda \varphi_{k-p,\lambda}^{n+p+q-1}(z'),
\]
where \( z_{\lambda}', z'_{\lambda} \in C^{n+p+q} \) such that \( |z_{\lambda}| = |z'_{\lambda}| = |z'| \). \( \Box \)

Next, we deduce an injectivity result for the \( H \)-type groups, which is known for the Heisenberg group.

Theorem 4.6. Let \( \partial \Omega \) be the boundary of a bounded domain \( \Omega \) in \( C^n \). Let \( f \) be such that \( f(z)e^{(\frac{1}{2}+\epsilon)|z|^2} \in L^p(C^n) \), for some \( \epsilon > 0 \) and \( 1 \leq p \leq \infty \). Suppose that \( f \times \mu_s(z) = 0 \) for all \( z \in \partial \Omega \) and \( s > 0 \). Then \( f = 0 \).

Now, we state an analogue of the above result for the \( H \)-type groups.

Theorem 4.7. Let \( \partial \Omega \) be the boundary of a bounded domain \( \Omega \) in \( C^n \). Let \( f \) be such that \( f(z)e^{(\frac{1}{2}+\epsilon)|\lambda|^2} \in L^p(C^n) \), for some \( \epsilon > 0 \) and \( 1 \leq p \leq \infty \). Suppose that \( f \times_\lambda \mu_s(z) = 0 \) for all \( z \in \partial \Omega \) and \( s > 0 \). Then \( f = 0 \).
Proof. From (4.1) we have $f \times \lambda \mu_s(z) = f \times |\lambda| \mu_s(z\lambda)$. Define $\Omega' = \{z\lambda \in \mathbb{C}^n : z \in \Omega\}$. Since the boundary of bounded domain $\Omega'$ is $\partial\Omega' = \{z\lambda : z \in \partial\Omega\}$, by Theorem 4.6 we can conclude that $f\lambda = 0$ and hence $f = 0$. \qed

Concluding remark: We know that in the case of the Métivier groups, the symplectic bilinear form $\sum_{j=1}^{m} \lambda_j \text{Re} (z \cdot U(j)w)$ cannot be made $U(n)$-invariant, because all of $\mu_{\lambda,j}$ need not be identical. Hence we require more assumptions on the functions to prove analogous results as to the Heisenberg group. However, in the case of the $H$-type groups, all $\mu_{\lambda,j}$ are identical, we do not require further assumption to prove the results for $H$-type groups.

Acknowledgements: The first and second authors would like to gratefully acknowledge the support provided by IIT Guwahati, Government of India.

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