Enrichments over symmetric Picard categories

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Abstract

Categorical rings were introduced in [JiPi07], which we call 2-rings. In these notes we present basic definitions and results regarding 2-modules. This is work in progress.

1 Introduction

Categorical rings were introduced by M.Jibladze and T.Pirashvili in [JiPi07]. We call those 2-rings. The present set of notes contains basic results about 2-modules and this work is in progress. Section 2 contains technical preliminaries, namely reminders on symmetric Picard categories as well convenient references to previous works. In particular the developments in [Sch08] for symmetric monoidal categories transpose well to symmetric Picard categories and a suitable tensor product can be defined for the latter. Section 3 and 4 treat enrichments over symmetric Picard categories. Those were introduced in [Dup08] and defined by means of multilinear maps. We also define them using the tensor product. Section 5 contains expected examples of 2-rings and 2-enrichments. Section 6 contains basic results regarding categories of \( A \)-modules for a 2-ring \( A \). In particular we show that \( A \)-modules are particular algebras for the endo-2-functor \( A \otimes - \) of the 2-category of symmetric Picard categories. A large appendix contains the more technical developments.

2 Preliminaries

A categorical group structure \((\mathcal{A}, j)\) consists of a monoidal category \( \mathcal{A} \) and an assignment for every object \( a \) of \( \mathcal{A} \) of an object \( a^* \) with an isomorphism \( j_a : I \rightarrow a^* \otimes a \), \( (a^* \) is an inverse of \( a \). We are concerned in this paper with symmetric Picard categories which are the categorical groups \((\mathcal{A}, j)\) for which \( \mathcal{A} \) has a symmetric monoidal structure and its underlying category is a groupoid. \( SPC \) denotes the 2-category with objects symmetric Picard categories, arrows symmetric monoidal functors and 2-cells monoidal natural transformations. There is a forgetful 2-functor \( SPC \rightarrow SMC \) forgetting the group structure where \( SMC \) denotes the 2-category with objects symmetric monoidal categories, arrows symmetric monoidal functors, and monoidal natural transformations as 2-cells. The 2-categorical properties of \( SPC \) are similar to those of \( SMC \), the latter 2-category has been studied in different works in particular in [HyPo02] and in [Sch08]. We refer the reader to this last work for basic notations, and more elaborate results. In this first section, we describe briefly and compare the important properties of \( SMC \) and \( SPC \).

The 2-category \( SMC \) admits an internal hom and the same holds for \( SPC \) which hom is inherited from \( SMC \). The following was mentioned in [Dup08] with a rather concise explanation.

**Lemma 2.1** Given any two objects in \( \mathcal{A} \) and \( \mathcal{B} \) in \( SMC \) with \( \mathcal{B} \) being a symmetric Picard category, the internal hom \([\mathcal{A}, \mathcal{B}]\) in \( SMC \) admits a symmetric Picard structure given pointwise by that of \( \mathcal{B} \).
We present a proof that relies on coherence results for categorical groups from Laplaza [Lap83]. (This work also contains references to earlier works on the topic.) Let us recall these. For a categorical group $A$ with family of isomorphisms $j_a : I \to a^* \otimes a$ for each object $a$ there is a unique way of extending the assignment $a \mapsto a^*$ into a functor $A^{op} \to A$ that makes the $j_a$ natural in $a$. It is an equivalence. We will write it $(\cdot)^*$ and write $f^* : b^* \to a^*$ for the image of any arrow $f : a \to b$ by this functor. There is a coherence theorem stating that any pair of $a$ and $b$ of objects of $A$ there is at most one “canonical” arrow $a \to b$, those canonical arrows being the ones generated in an expected way from the canonical arrows from the monoidal structures and the $j_a$’s. Eventually Laplaza’s paper also provides a combinatorial description of free categorical groups.

Let us recall also the following known facts for any symmetric Picard categories $A$ and $B$. For any symmetric monoidal functor $(F, F^2, F^0) : A \to B$ the component $F^0$ is determined by $F^2$. Actually a monoidal structure on a functor $F : A \to B$ is given by a natural $F^2_{a,b} : F a \otimes F b \to F((a \otimes b) \otimes c)$ satisfying the only axiom that

$$
\begin{align*}
F a \otimes (F b \otimes F c) & \cong F c \\
1 \otimes F^2_{b,c} & \Downarrow F^2_{a,b} \otimes 1 \\
F a \otimes F(b \otimes c) & \cong F(a \otimes b) \otimes F c \\
F^2_{a,b \otimes c} & \Downarrow F^2_{a \otimes b,1} \\
F(a \otimes (b \otimes c)) & \cong F((a \otimes b) \otimes c)
\end{align*}
$$

commutes for any objects $a$, $b$ and $c$ of $A$. Also any natural transformation $\sigma : F \to G : A \to B$ between monoidal functors is monoidal if it satisfies the only axiom that the diagram in $B$

$$
\begin{align*}
F a \otimes F b & \xrightarrow{\sigma_{a \otimes b}} F(a \otimes b) \\
\sigma_{a \otimes b} & \Downarrow \sigma_{a \otimes b} \\
G a \otimes G b & \xrightarrow{G(\otimes)_{a,b}} G(a \otimes b)
\end{align*}
$$

commutes for any objects $a, b$ of $A$.

Let us consider a symmetric Picard category $(A, j)$. Since $A$ is a groupoid, one has a functor $A^{op} \to A$ which is the identity on objects and sends arrows to their inverses. The functor $\text{inv} : A \to A$ is obtained by composing the previous functors and $(-)^*$ above. Let us denote by $!$ the unique canonical arrow between two objects of $A$, when it exists.

**Lemma 2.2** For any symmetric Picard category $(A, j)$, the associated functor $\text{inv}$ admits a symmetric monoidal structure where $\text{inv}^2$ has component $\text{inv}^2_{a,b} : a^* \otimes b^* \to (a \otimes b)^*$ in any $(a, b)$ the composite $a^* \otimes b^* \xrightarrow{\text{inv}^2_{a,b}} b^* \otimes a^* \xrightarrow{1} (a \otimes b)^*$ (which is also $a^* \otimes b^* \xrightarrow{1} (b \otimes a)^* \xrightarrow{1} (a \otimes b)^*$) according to Lemma 7.1 in Appendix).

**PROOF:** See Appendix 7.2.

2.3 For any symmetric Picard category $(A, j)$ one has a monoidal natural isomorphism $I \to \text{inv} \otimes \text{id} : A \to A$ which component in any object $a$ is $j_a : I \to a^* \otimes a$.

**PROOF:** See Appendix 7.4.
Now given objects \( \mathcal{A} \) and \( \mathcal{B} \) in \( \text{SMC} \) with \((\mathcal{B}, j)\) symmetric Picard, a group structure is obtained on the \( \text{hom} \ [\mathcal{A}, \mathcal{B}] \) in \( \text{SMC} \), which is a groupoid, as follows. The strict symmetric monoidal functor \([\mathcal{A}, -] : [\mathcal{B}, \mathcal{B}] \to [[\mathcal{A}, \mathcal{B}], [\mathcal{A}, \mathcal{B}]]\) sends the monoidal transformation \( j : I \to inv \circ id : B \to B \) of \( [\mathcal{B}, \mathcal{B}] \) to a monoidal transformation

\[
I = [\mathcal{A}, I] \rightsquigarrow [\mathcal{A}, inv \circ id] \rightarrowtail [\mathcal{A}, inv \circ id] \rightarrowtail [\mathcal{A}, inv \circ id] : [\mathcal{A}, B] \to [\mathcal{A}, B].
\]

which we define as the \( j \) on \([\mathcal{A}, \mathcal{B}]\). This is to say that \( F^* \) for any symmetric monoidal \( F \) is the composite

\[
\mathcal{A} \xrightarrow{F} \mathcal{B} \xrightarrow{inv} \mathcal{B}.
\]

and the natural isomorphisms \( j_F : I \cong F^* \circ F \) are pointwise \((j_F)_a : I \to (Fa)^\ast \otimes Fa\). Then for any monoidal \( \sigma : F \to G : \mathcal{A} \to \mathcal{B} \) one has that \( \sigma^* : G^* \to F^* \) is pointwise \((\sigma_a)^* : (Ga)^\ast \to (Fa)^\ast \). We will always consider that this is the chosen group structure on \([\mathcal{A}, \mathcal{B}]\) when considered as an object of \( \text{SPC} \). This structure is determined by that of \( \mathcal{B} \).

One has a notion of strictness for arrows in \( \text{SPC} \), which is different from the notion of strictness in \( \text{SMC} \). Let us consider any symmetric Picard categories \( \mathcal{A} \) and \( \mathcal{B} \). For a symmetric monoidal functor \( F : \mathcal{A} \to \mathcal{B} \) one has the natural isomorphism

\[
2.4 \quad F(a^\ast) \cong_a (Fa)^\ast.
\]

defined precisely in Appendix\(\text{[3]}\). A symmetric monoidal functor \( F : \mathcal{A} \to \mathcal{B} \) is called a strict arrow in \( \text{SPC} \) when it preserves strictly the monoidal structure and moreover preserves strictly the isomorphisms \( j \) meaning that the natural isomorphism \( 2.4 \) is an identity or equivalently that for any object \( a \) in \( \mathcal{A} \), \( F \) sends \( a^\ast \) to \( (Fa)^\ast \) and \( j_a : I \to a^\ast \otimes a \to j_{Fa} : I \to (Fa)^\ast \otimes Fa \). We write \( Str\text{SPC} \) for the sub-2-category of \( \text{SPC} \) with same objects, strict arrows and 2-cells inherited from \( \text{SPC} \).

The results from \( \text{[Sch08]} \) regarding \( \text{SMC} \) transpose rather straightforwardly to \( \text{SPC} \) as follows.

Given an arbitrary symmetric Picard category \( \mathcal{C} \), it happens that the isomorphism \( D_{\mathcal{A}, \mathcal{B}, \mathcal{C}} : [\mathcal{A}, [\mathcal{B}, \mathcal{C}]] \to [\mathcal{B}, [\mathcal{A}, \mathcal{C}]] \) defined in chapter 6 is also a strict arrow in \( \text{SPC} \). For any arrow \( F : \mathcal{A} \to \mathcal{B}, \mathcal{C} \) in \( \text{SPC} \), its image \( F^* \) by \( D \), called it “dual”, is strict in \( \text{SPC} \) if and only for any object \( b \) of \( \mathcal{B} \), the arrow \( F^*(b) : \mathcal{A} \to \mathcal{C} \) is strict in \( \text{SPC} \). For any objects \( \mathcal{A}, \mathcal{B} \) and \( \mathcal{C} \) in \( \text{SPC} \), the arrow \( [\mathcal{A}, -]_{\mathcal{B}, \mathcal{C}} : [\mathcal{B}, \mathcal{C}] \to [[\mathcal{A}, \mathcal{B}], [\mathcal{A}, \mathcal{C}]] \) defined in chapter 8 is strict in \( \text{SPC} \).

The hom 2-functor of \( \text{SMC} \) defined in chapter 9 induces by restriction a hom 2-functor \( \text{SPC}^{op} \times \text{SPC} \to \text{SPC} \) for \( \text{SPC} \). The statements regarding the 2-naturality of \( D \) (chapter 10) and the evaluation functors (chapter 11) still hold when replacing formally \( \text{SMC} \) by \( \text{SPC} \). In particular the evaluation functors are strict arrows in \( \text{SPC} \).

Similarly to the case of the 2-category \( \text{SMC} \), one has a tensor product in \( \text{SPC} \). For any symmetric Picard categories \( \mathcal{A} \) and \( \mathcal{B} \), their tensor \( \mathcal{A} \otimes \mathcal{B} \) satisfies the universal property of the existence of a 2-natural isomorphism

\[
2.5 \quad \text{SPC}(\mathcal{A}, [\mathcal{B}, \mathcal{C}]) \cong_c \text{StrSPC}(\mathcal{A} \otimes \mathcal{B}, \mathcal{C})
\]

between 2-functors \( \text{StrSPC} \to \text{Cat} \) in the argument \( \mathcal{C} \). Note that the 2-naturality in question involves only strict morphisms in \( \text{SPC} \). We briefly sketch a description of the above tensor \( \mathcal{A} \otimes \mathcal{B} \) by generator and relations. It is similar to that given with more details in \( \text{Sch08} \) for the tensor
in $SMC$.

We consider a graph $\mathcal{H}$ with vertices the terms of the free $\{I,(-)\}^n,\otimes\}$-algebra over the set $\text{Obj}(A) \times \text{Obj}(B)$, i.e. they are words of the formal language containing all pairs $(a,b)$ — which we write $a \otimes b$ — for objects $a$ of $A$ and $b$ of $B$, the one-symbol word $I$, the words $X^\star$ for any vertex $X$, and $X \otimes Y$ for any vertices $X$ and $Y$. The set of edges of $\mathcal{H}$ consists of:
- The “canonical” edges for the symmetric monoidal structure which are the edges $\gamma_{X} : X \otimes X \to (X \otimes Y) \otimes Z$, $r_X : X \otimes I \to X$, $l_X : I \otimes X \to X$, $s_{X,Y} : X \otimes Y \to Y \otimes X$ for all vertices $X, Y, Z$;
- Edges $\gamma_{X} : I \to X^\star \otimes X$, one for each vertex $X$;
- Edges $\gamma_{a,a'} : (a \otimes b) \otimes (a' \otimes b) \to (a \otimes a') \otimes b$ and $\delta_{a,b,b'} : (a \otimes b) \otimes (a \otimes b') \to a \otimes (b \otimes b')$ indexed by objects $a, a'$ of $A$ and $b, b'$ of $B$;
- Edges $f \otimes f : a \otimes b \to a \otimes b'$ indexed by objects $a$ of $A$ and arrows $f : b \to b'$ of $B$;
- Edges $f \otimes b : a \otimes b \to a' \otimes b$ indexed by objects $b$ of $B$ and arrows $f : a \to a'$ of $A$;
- Edges $X \otimes p : X \otimes Y \to X \otimes Z$ and $p \otimes X : Y \otimes X \to Z \otimes X$ for any vertex $X$ and any edge $p : Y \to Z$;

with the convention that edges above with different names are different.

Let us consider $FG(\mathcal{H})$ the free groupoid on $\mathcal{H}$, i.e. its arrows are mere concatenations of edges of $\mathcal{H}$ and their formal inverses. For any vertex $X$, one has two graph endomorphisms of $\mathcal{H}$, namely $X \otimes -$ and $- \otimes X$ sending respectively an arbitrary edge $f : Y \to Z$ to $X \otimes Y \to X \otimes Z$, resp. $Y \otimes X \to Z \otimes X$. These two extend uniquely to endofunctors of $FG(\mathcal{H})$ and we extend the notation $X \otimes f$ and $f \otimes X$ to denote the images of arrows of $FG(\mathcal{H})$ by these functors.

The tensor $A \otimes B$ is the quotient of $FG(\mathcal{H})$ by the congruence generated by the following relations $\sim$ on its arrows from $2.6$ to $2.20$ below.

For all edges $X \xrightarrow{t} Y$ and $Z \xrightarrow{s} W$ of $\mathcal{H}$,

![Diagram](2.6)

2.7 Relations giving the coherence conditions for $\text{ass}$, $r$, $l$ and $s$ in $A \otimes B$. These are the following.
- For any vertices $X$, $Y$, $Z$ and $T$,

$$
\begin{align*}
X \otimes (Y \otimes (Z \otimes T)) &\xrightarrow{\text{ass}} (X \otimes Y) \otimes (Z \otimes T) \\
1 \otimes \text{ass} &\sim (X \otimes Y) \otimes (Z \otimes T) \\
X \otimes ((Y \otimes Z) \otimes T) &\xrightarrow{\text{ass}} (X \otimes (Y \otimes Z)) \otimes T.
\end{align*}
$$

- For any vertices $X$ and $Y$,

$$
\begin{align*}
X \otimes (I \otimes Y) &\xrightarrow{\text{ass}} (X \otimes I) \otimes Y \\
1 \otimes I &\sim (X \otimes I) \otimes Y.
\end{align*}
$$
2.8 Relations for the naturalities of $ass$, $r$, $l$, and $s$ in $A \otimes B$.

For instance, one has for any edge $f : X \to X'$ of $\mathcal{H}$, and any vertices $Y$ and $Z$,

$$X \otimes (Y \otimes Z) \xrightarrow{ass X,Y,Z} (X \otimes Y) \otimes Z \otimes (X \otimes Y) \xrightarrow{ass} Z \otimes (X \otimes Y).$$

We will not write here the other relations. There are two more for the naturalities of $ass_{X,Y,Z}$ in $Y$ and $Z$, one for that of $t_X$ in $X$, one for that of $r_X$ in $X$ and two for those of $s_{X,Y}$ in $X$ and $Y$.

For any object $a$ in $A$ and any arrows $b \xrightarrow{f} b' \xrightarrow{g} b''$ in $B$,

2.9

$$a \otimes b \xrightarrow{a \otimes (g \circ f)} a \otimes b'' \xrightarrow{a \otimes g} a \otimes b' \xrightarrow{\sim} a \otimes b.$$

For any object $b$ in $B$ and any arrows $a \xrightarrow{f} a' \xrightarrow{g} a''$ in $A$,

2.10

$$a \otimes b \xleftarrow{(g \circ f) \otimes b} a'' \otimes b \xleftarrow{g \otimes b} a' \otimes b \xleftarrow{\sim} a \otimes b.$$

For any objects $a$ in $A$ and $b$ in $B$,

2.11 $a \otimes id_b \sim id_{a \otimes b}$

and

2.12 $id_a \otimes b \sim id_{a \otimes b}$.

where $id_b$, $id_a$ and $id_{a \otimes b}$ above are the identities respectively at $b$ in $B$, at $a$ in $A$ and at $a \otimes b$ in $\mathcal{F}_G(\mathcal{H})$.

For any arrows $f : a \to a'$ in $A$ and $g : b \to b'$ in $B$,
2.13

\[
\begin{array}{ccc}
   a \otimes b & \xrightarrow{f \otimes b} & a' \otimes b \\
   a \otimes g & \sim & a' \otimes g \\
   a \otimes b' & \xrightarrow{f \otimes b} & a' \otimes b'.
\end{array}
\]

2.14 Relations for the “naturalities” of \( \gamma_{a, a', b} \) in \( a, a' \) and \( b \) and \( \delta_{a, b, b'} \) in \( a, b \) and \( b' \). For instance by the relations for the “naturality” of \( \gamma_{a, a', b} \) in \( a \) and any arrow \( g : b \to b' \) in \( B \),

\[
\begin{array}{c}
   (a \otimes b) \otimes (a' \otimes b) \xrightarrow{\gamma_{a, a', b}} (a \otimes a') \otimes b \\
   (1 \otimes g) \otimes (1 \otimes g) \sim (a \otimes b') \otimes (a' \otimes b') \xrightarrow{\gamma_{a, a', b'}} (a \otimes a') \otimes b'.
\end{array}
\]

We will not write explicitly now the five other relations.

For any objects \( a \) in \( A \) and \( b, b', b'' \) in \( B \),

2.15

\[
(a \otimes b) \otimes ((a \otimes b') \otimes (a \otimes b'')) \xrightarrow{\text{ass}} ((a \otimes b) \otimes (a' \otimes b')) \otimes (a \otimes b'')
\]

\[
\begin{array}{c}
   1 \otimes \delta_{a, b', b''} \otimes \delta_{b', b''} \otimes 1 \\
   \delta_{a, b, b'} \otimes 1 \\
   \delta_{a, b, b'} \otimes \delta_{a, b, b''} \otimes \delta_{a, b', b''} \\
   1 \otimes \text{ass}_{a, b', b''} \\
   a \otimes (b \otimes (b' \otimes b'')) \otimes a' \otimes (b \otimes b'') \otimes b''.
\end{array}
\]

For any objects \( a \) in \( A \) and \( b, b' \) in \( B \),

2.16

\[
(a \otimes b) \otimes (a \otimes b') \xrightarrow{\delta_{a, b, b'}} a \otimes (b \otimes b')
\]

\[
\begin{array}{c}
   \delta_{a, b, b'} \otimes \delta_{b, b'} \otimes \delta_{b', b''} \\
   \sim 1 \otimes \delta_{b, b'} \otimes \delta_{b', b''} \\
   a \otimes (b' \otimes b) \otimes a' \otimes (b' \otimes b) \\
   \delta_{a, b', b} \\
   (a \otimes b') \otimes (a \otimes b) \otimes (b' \otimes b).
\end{array}
\]

For any objects \( a, a', a'' \) in \( A \) and \( b \) in \( B \),

2.17

\[
\begin{array}{c}
   (a \otimes b) \otimes ((a' \otimes b) \otimes (a'' \otimes b)) \xrightarrow{\text{ass}} ((a \otimes b) \otimes (a' \otimes b)) \otimes (a'' \otimes b) \\
   1 \otimes \gamma_{a, a', b'} \otimes \gamma_{a, a'', b} \\
   (a \otimes b) \otimes ((a' \otimes a'') \otimes b) \otimes (a' \otimes b) \otimes (a'' \otimes b) \xrightarrow{\gamma_{a, a', a'', b}} ((a \otimes a') \otimes a'' \otimes b) \otimes (a \otimes b).
\end{array}
\]

For any objects \( a, a' \) in \( A \) and \( b \) in \( B \),
2.18
\[
\begin{align*}
(a \otimes b) \otimes (a' \otimes b) & \xrightarrow{\gamma_{a,a',b}} (a \otimes a') \otimes b \\
& \sim \\
(a' \otimes b) \otimes (a \otimes b) & \xrightarrow{\gamma_{a',a,b}} (a' \otimes a) \otimes b.
\end{align*}
\]

For any objects \( a, a' \) in \( \mathcal{A} \) and \( b, b' \) in \( \mathcal{B} \),

2.19
\[
\begin{align*}
((a \otimes b) \otimes (a \otimes b')) \otimes ((a' \otimes b) \otimes (a' \otimes b')) & \xrightarrow{\delta_{a,b',b'} \otimes \delta_{a',b,b'}} ((a \otimes b) \otimes (a' \otimes b)) \otimes ((a \otimes b') \otimes (a' \otimes b')) \\
& \sim \\
(a \otimes (b \otimes b')) \otimes (a' \otimes (b \otimes b')) & \xrightarrow{\gamma_{a,a',b \otimes b'}} ((a \otimes b') \otimes (a' \otimes b')) \\
(a \otimes (b \otimes b')) \otimes (a' \otimes b) & \xrightarrow{\delta_{a,b,b'}} ((a \otimes b') \otimes (a' \otimes b'))
\end{align*}
\]

where the top arrow is the concatenation

\[
\begin{array}{cccc}
\text{ass}_{X \otimes Y \otimes Z} & \text{ass}_{X \otimes Y \otimes 1}^{-1} & (1 \otimes \text{ass}_{Y \otimes Z}) \otimes T & \text{ass}_{X \otimes Z, Y \otimes T} \otimes \text{ass}_{X \otimes Z, Y \otimes T}^{-1}
\end{array}
\]

with the \( \text{ass}^{-1} \) being the formal inverses of edges ass and \( X, Y, Z \) and \( T \) standing respectively for \( a \otimes b, a \otimes b', a' \otimes b \) and \( a' \otimes b' \).

2.20  **Expansions of all relations above by iterations of \( X \otimes - \) and \( - \otimes X \) for all vertices \( X \).**

Which means precisely that the set of relations \( \sim \) is the smallest set of relations on arrows of \( \mathcal{F}(\mathcal{H}) \) containing the previous relations \( (2.6) \) to \( (2.19) \) and satisfying the closure properties that for any relation \( f \sim g : Y \rightarrow Z \) that it contains and any vertex \( X \), it contains also the relations

\[
X \otimes f \sim X \otimes g : X \otimes Y \rightarrow X \otimes Z
\]

and

\[
f \otimes X \sim g \otimes X : Y \otimes X \rightarrow Z \otimes X.
\]

The proofs that the above category \( \mathcal{A} \otimes \mathcal{B} \) is a well defined symmetric Picard category and that it satisfies the universal property \( (2.5) \) are similar to those in [Sch08] for the well definition and universal property of the tensor product in SMC. We will therefore not replicate them. For any objects \( \mathcal{A}, \mathcal{B} \) and \( \mathcal{C} \) in \( \text{SPC} \), one obtains an adjunction

\[
\text{En} : [\mathcal{A} \otimes \mathcal{B}, \mathcal{C}] \rightarrow [\mathcal{A}, [\mathcal{B}, \mathcal{C}]]
\]

in the 2-category \( \text{SPC} \) (in this case it is an equivalence) with \( \text{Rn} \circ \text{En} = 1 \) and where the arrows \( \text{Rn}_{A,B,C} \) are strict. The isomorphism \( (2.5) \) becomes 2-natural in \( \mathcal{A} \) and \( \mathcal{B} \) for a unique tensor 2-functor \( \text{SPC} \times \text{SPC} \rightarrow \text{SPC} \).

There exists a free symmetric Picard category on one generator, which we shall write \( I \), that differs obviously from the “unit” for SMC written \( I \) and defined in [Sch08]-chapter 18, but that has a very similar presentation by generators and relation. The only differences are the following. Its set objects is now the free \( \{ I, \otimes, (-)^* \} \)-algebra over one generator \( * \). It is a quotient of free groupoid generated by the graph containing the usual canonical edges for the symmetric monoidal structures.
which lies in \( A \star \tau \) and \( A \star A \)\( A \star A \star \tau \) \( A \star A \star A \). The relations on this free groupoid defining \( I \) are just those for the naturalities of the collections \( \text{ass}, r, I \), and \( s \), those expressing the coherence axioms for the symmetric monoidal structure, and relations expressing the bifunctoriality of \( - \otimes - : I \times I \to I \). The universal property defining \( I \) is that for any symmetric Picard category \( \mathcal{A} \), there exits a unique strict arrow \( v : I \to [\mathcal{A}, \mathcal{A}] \) such that \( v(*) \) is the identity arrow at \( A \to A \) with its strict structure in \( SPC \). One obtains with similar proofs, similar results. Namely:

**2.21** For any symmetric Picard category \( \mathcal{A} \), the dual \( v^* : I \to [\mathcal{I}, \mathcal{A}] \) of \( v : I \to [\mathcal{A}, \mathcal{A}] \) has right adjoint in \( SPC \) the evaluation at \( * \) functor \( ev_* : [\mathcal{I}, \mathcal{A}] \to \mathcal{A} \).

From this one can exhibit a kind of “symmetric monoidal closed 2-structure” on \( SPC \) in the same way as done in [Sch08] chapters 19, 20 and 21 for \( SMC \). Namely one can define the canonical arrows \( A'_{A,B,C} : (A \otimes B) \otimes C \to A \otimes (B \otimes C) \), \( R'_A : A \to A \otimes I \), \( L'_A : A \to I \otimes A \) and \( S_{A,B} : A \otimes B \to B \otimes A \) in this case with respective inverse equivalences \( A_{A,B,C}, R_A, L_A \) and \( S_{B,A} \) and satisfying the (strict) coherence axioms given in chapter 20. Eventually this 2-categorical structure induces a symmetric monoidal closed structure on \( SPC/\sim \) where \( \sim \) denotes the congruence generated by the 2-cells of \( SPC \).

### 3 SPC-categories, SPC-functors and SPC-natural transformations

\( SPC \)-categories, \( SPC \)-functors and \( SPC \)-natural transformations have been considered by M.Dupont in his thesis [Dup08]. They are respectively bicategories with homs in \( SPC \), and pseudo-functors and pseudo-natural transformations with linear components, i.e. with arrows and 2-cells in the 2-category \( SPC \) rather than in \( \text{Cat} \). As such they obviously form a 2-category with a forgetful 2-functor \( SPC \to \text{Cat} \) (see [Sch08] chapter 6).

We start by recalling these notions which are actually slight (enriched) variations of the usual notions of bicategories, pseudo-functors and pseudo-natural transformations.

By a \( n \)-linear natural transformation \( \sigma \) between \( n \)-linear maps \( F \) and \( G \), written \( \sigma : F \to G : A_1 \times \ldots A_n \to B \) we will mean a 2-cell of \([A_1, [A_2, \ldots, [A_n, B]]] \ldots\]. Multi-linear maps and multi-linear natural transformation compose in an evident way, which can be justified by the 2-natural isomorphism \( D_{A,B,C} : [A, [B, C]] \cong [B, [A, C]] \) and the existence of a forgetful 2-functor \( SPC \to \text{Cat} \) (see [Sch08] chapter 6).

An *enrichment* \( (\mathcal{A}, c, \cdot, \alpha, \rho, \lambda) \) over \( SPC \) also named an *\( SPC \)-category* and which we might sometimes denote simply by \( \mathcal{A} \), consists of the following data:

- \( \mathcal{A} \) small set with elements \( x, y, z, \ldots \) called the objects of \( \mathcal{A} \).
- A map \( \mathcal{A} \) sending any pair \( x,y \) of objects to an object \( \mathcal{A}(x,y) \) of \( SPC \) sometimes also written \( \mathcal{A}_{x,y} \) for convenience and called the hom of \( x \) and \( y \).
- A collection of bilinear maps \( c_{x,y,z} : \mathcal{A}_{y,z} \times \mathcal{A}_{x,y} \to \mathcal{A}_{x,z} \), the composition maps indexed by objects \( x, y, z \) of \( \mathcal{A} \) and we write \( g \circ f \) for \( c_{x,y,z}(g,f) \) for any objects \( g \) of \( \mathcal{A}_{y,z} \) and \( f \) of \( \mathcal{A}_{x,y} \) and \( \tau \circ \sigma \) for \( c_{x,y,z}(\tau, \sigma) \) for any arrows \( \tau \) of \( \mathcal{A}_{y,z} \) and \( \sigma \) of \( \mathcal{A}_{x,y} \).
- A collection of objects \( 1_x \) of \( \mathcal{A}_{x,x} \) indexed by objects \( x \) of \( \mathcal{A} \);
- Collections of natural transformations \( \alpha_{x,y,z,t} \), which are *trilinear*, indexed by objects \( x,y,z,t \), and \( \rho_{x,y} \) and \( \lambda_{x,y} \) both *linear*, as follows:

**3.1**

\( (\alpha_{x,y,z,t})_{h,g,f} : h \circ (g \circ f) \to h \circ (g \circ f) \)

which lies in \( \mathcal{A}_{x,t} \) for objects \( h \) in \( \mathcal{A}_{z,t} \), \( g \) in \( \mathcal{A}_{y,z} \), and \( f \) in \( \mathcal{A}_{x,y} \).
3.2 \[(\rho_{x,y})_f : f \to f \circ 1_z\]

which lies in \(A_{x,y}\) for objects \(f\) in \(A_{x,y}\):

3.3 \[(\lambda_{x,y})_f : f \to 1_y \circ f\]

which lies in \(A_{x,y}\) for objects \(f\) in \(A_{x,y}\); and those are subjects to the coherence axioms 3.4 and 3.5 below.

3.4 For any objects \(x, y, t, u\) of \(A\), any objects \(f\) of \(A_{x,y}\), \(g\) of \(A_{y,z}\), \(h\) of \(A_{z,t}\) and \(k\) of \(A_{t,u}\), the diagram in \(A_{x,u}\)

\[
\begin{array}{ccc}
  k \circ (h \circ (g \circ f)) & & (k \circ h) \circ (g \circ f) \\
  k \circ ((h \circ g) \circ f) & & (k \circ h) \circ (h \circ g) \circ f \\
  (\alpha_\circ_{x,y,z,u})_{k,h,g,f} & & (\alpha_\circ_{x,y,z,u})_{k,h,g,f}
\end{array}
\]

commutes.

3.5 For any objects \(x, y, z\) in \(A\) and any objects \(f\) of \(A(x,y)\) and \(g\) of \(A(y,z)\) the diagram in \(A(x,y)\)

\[
\begin{array}{cc}
g \circ f & \quad (\rho_{x,y})_g \circ f \\
(g \circ (1_y \circ f)) & \quad (g \circ 1_y) \circ f \\
(\alpha_{x,y,y,z})_{g,1_y,f} & \quad (\alpha_{x,y,y,z})_{g,1_y,f}
\end{array}
\]

commutes.

For any SPC-category \(A\), its underlying bicategory is denoted \(A^0\).

Given two arbitrary SPC-categories \(A\) and \(B\), a SPC-functor \(F : A \to B\) consists of the following data.
- A map \(F\) sending objects of \(A\) to objects of \(B\);
- Arrows \(F_{x,y} : A_{x,y} \to B_{F_x,F_y}\) in SPC for each pair of objects \(x, y\) in \(A\).
- A collection of bilinear natural transformations \(F_{x,y,z}^2\) indexed by objects \(x, y, z\) of \(A\) with components

\[
(F_{x,y,z}^2)_{g,f} : F_{x,z}(g) \circ F_{x,y}(f) \to F_{x,z}(g \circ f)
\]

in \(B_{F_x,F_z}\) for objects \(g\) in \(A_{y,z}\) and \(f\) in \(A_{x,y}\).
- A collection of arrows \(F_x^0 : 1_{F_x} \to F_{x,x}(1_x)\) in \(B_{F_x,F_x}\) indexed by objects \(x\) of \(A\).

Those are the subjects to the coherence axioms 3.6, 3.7 and 3.8 below.

3.6 For any objects \(x, y, z, t\) of \(A\), and any objects \(f\) of \(A_{x,y}\), \(g\) of \(A_{y,z}\) and \(h\) of \(A_{z,t}\) the diagram in \(B(F_x,F_t)\)

\[
\begin{array}{ccc}
  F_{z,t} \circ h \circ (F_{y,z}g \circ F_{x,y}f) & \quad (\alpha_{F_{x,y},F_{t,f}}^{F_{y,z}g}) \circ F_{z,t}h \circ F_{y,z}g \circ F_{x,y}f \\
  1 \circ (F_{x,y,z,t}^2)_{g,f} & \quad (F_{x,y,z,t}^2)_{g,f} \circ 1 \\
  F_{z,t}h \circ F_{x,z}(g \circ f) & \quad F_{y,t}(h \circ g) \circ F_{x,y}f \\
  (F_{x,y,z,t}^2)_{h,g,f} & \quad (F_{x,y,z,t}^2)_{h,g,f}
\end{array}
\]

\[
\begin{array}{ccc}
  F_{x,t}(h \circ (g \circ f)) & \quad F_{x,t}(h \circ (g \circ f)) \\
  F_{x,t}(h \circ (g \circ f)) & \quad F_{x,t}(h \circ (g \circ f)) \\
  (\alpha_{x,y,z,t})_{h,f,g} & \quad (\alpha_{x,y,z,t})_{h,f,g}
\end{array}
\]
commutes.

3.7 For any objects $x, y$ of $A$ and any object $f$ of $A_{x,y}$, the diagram in $B_{F_x,F_y}$

\[
\begin{array}{ccc}
Ff & \overset{(\rho_{F_x,F_y})_{F_f}}{\longrightarrow} & Ff \circ 1_{F_x} \\
\downarrow F(\rho_{x,y}) & & \downarrow 1_{F_x}^0 \\
F(f \circ 1_x) & \overset{(F_x^{2},y)_{F_f}}{\longrightarrow} & Ff \circ F(1_x)
\end{array}
\]

commutes.

3.8 For any objects $x, y$ of $A$ and any object $f$ of $A_{x,y}$, the diagram in $B_{F_x,F_y}$

\[
\begin{array}{ccc}
Ff & \overset{(\lambda_{F_x,F_y})_{F_f}}{\longrightarrow} & 1_{F_y} \circ Ff \\
\downarrow F(\lambda_{x,y}) & & \downarrow F^0_y \circ 1 \\
F(1_y \circ f) & \overset{(F_y^{2},y)_{1_y,f}}{\longrightarrow} & F(1_y) \circ Ff
\end{array}
\]

commutes.

Given two $SPC$-functors $F, G : A \to B$ a $SPC$-natural transformation $(\sigma, \kappa)$ consists in a family of arrows $\sigma_x$ of $B_{F_x,G_x}$ indexed by objects $x$ of $A$ together with a collection of linear natural transformations $\kappa_{x,y}$ indexed by objects $x$ and $y$ as follows

\[(\kappa_{x,y})_f : Gf \circ \sigma_x \to \sigma_y \circ Ff\]

lies in $B_{F_x,G_y}$ for objects $f$ of $A_{x,y}$, and these satisfy the coherence axioms 3.9 and 3.10 below.

3.9 For any objects $f$ in $A_{x,y}$ and $g$ in $A_{y,z}$, the diagram in $B_{F_x,G_z}$

\[
\begin{array}{ccc}
(Gg \circ Gf) \circ \sigma_z & \overset{(Gg \circ Gf)^2_{y,f} \circ \sigma_z}{\longrightarrow} & G(g \circ f) \circ \sigma_z \\
\downarrow \alpha_{F,G,G,G,y,z} & & \downarrow \alpha_{F,G,G,G,y,z}^* \circ \sigma_z \\
Gg \circ (Gf \circ \sigma_z) & \overset{\sigma_z \circ (Fg \circ Ff)}{\longrightarrow} & \sigma_z \circ (Fg \circ Ff)
\end{array}
\]

commutes.

3.10 For any object $x$ of $A$, the diagram in $B_{F_x,G_x}$

\[
\begin{array}{ccc}
\sigma_x & \overset{(\rho_{F_x,G_x})_{\sigma_x}}{\longrightarrow} & \sigma_x \circ 1_{F_x} \\
\downarrow \sigma_x \circ F^0_x & & \downarrow 1_{G_x} \circ \sigma_x \\
\sigma_x \circ F(1_x) & \overset{\sigma_x \circ (\kappa_{x,x})_{1_x}}{\longrightarrow} & G(1_x) \circ \sigma_x
\end{array}
\]

commutes.
We want to give alternative definitions of SPC-categories, SPC-functors and SPC-natural transformations by means of commuting diagrams in SPC in a first instance without using the tensor. They are obtained by replacing multilinear maps and multilinear natural transformations from the previous definition by corresponding arrows and 2-cells in SPC. This yields the following.

One can define a SPC-category as a collection of objects $\mathcal{A}$, with homs $A_{x,y}$ in SPC as before, with collections of arrows

- $A(x,-)_{y,z} : A_{y,z} \to [A_{x,y},A_{x,z}]$ in SPC, indexed by objects $x, y, z$ of $\mathcal{A}$ with dual $A(-,y) : A_{x,y} \to [A_{y,z},A_{x,z}]$ written $A(-,y)$;
- $u_x : I \to A_{x,z}$ indexed by $x$, which are strict;

and collections of 2-cells:

- $\alpha'_{x,y,z,t}$ in SPC, indexed by objects $x, y, z$ and $t$ of SPC as follows

3.11

- $\rho'_{x,y}$ and $\lambda'_{x,y}$ indexed by objects $x$ and $y$ respectively as follows

3.12 $\rho'_{x,y}$ :

3.13 $\lambda'_{x,y}$ :

those satisfying coherence axioms 3.14 and 3.15 below
3.14 The 2-cells in SPC

\[
\begin{align*}
A_{x,-} &\rightarrow [A_{x,-}, A_{x,-}] \\
A_{y,-} &\rightarrow [A_{y,-}, A_{y,-}] \\
A_{z,-} &\rightarrow [A_{z,-}, A_{z,-}] \\
[1] &\rightarrow [1, 1] \\
\end{align*}
\]

are equal.

3.15 The 2-cell

\[
\begin{align*}
A_{y,z} &\rightarrow id \\
A_{y,z} &\rightarrow [A_{x,y}, A_{z,-}] \\
A_{y,-} &\rightarrow [A_{y,-}, A_{y,-}] \\
\end{align*}
\]
is equal to

\[
\begin{array}{cccccc}
A_{y,z} & \overset{id}{\to} & A_{y,z} & \overset{id}{\to} & A_{y,z} & \overset{id}{\to} & A_{y,z} \\
A(x,-) & \overset{=}{\to} & A(x,-) & \overset{=}{\to} & A(x,-) & \overset{=}{\to} & A(x,-) \\
[A_{x,y},A_{x,z}] & \overset{\alpha_y}{\to} & [A_{x,y},A_{x,z}] & \overset{\alpha_x}{\to} & [A_{x,y},A_{x,z}] & \overset{\alpha_{x,x}}{\to} & [A_{x,y},A_{x,z}] \\
\end{array}
\]

Note: Equality (I) in the first of the pastings above is established in Lemma 11.2 in Appendix. The equality (II) results straightforwardly from Lemma 11.3 given below.

Let us justify the equivalence with the previous definition of SPC-category. For objects \( x, y \) and \( z \) the arrows of SPC

\[ A(x,-) : A(y,z) \to [A(x,y),A(x,z)] \]

correspond to the bilinear

\[ c_{x,y,z} : A(y,z) \times A(x,y) \to A(x,z) \]

and for objects \( x \), the objects \( 1_x \) in \( A_{x,x} \) correspond to strict the strict arrows \( u_x : I \to A_{x,x} \). The trilinear 2-cells \( \alpha_{x,y,z,t} \) correspond to the 2-cells \( \alpha_{x,y,z} \) and the linear natural transformations \( \rho_{x,y} \) and \( \lambda_{x,y} \) correspond respectively to 2-cells \( \rho^x_{x,y} \) and \( \lambda^x_{x,y} \) in SPC.

For data as above, since the forgetful functors \( SPC(X,Y) \to \text{Cat}(X,Y) \) are faithful, the equalities of 2-cells of Axiom 3.1.4 given below are equivalent to the equality of natural transformations of Axioms 3.1 and similarly Axiom 3.1.5 given below and 3.5 are equivalent.

Given two SPC-categories, a SPC-functor \( A \to B \) consists of a map \( F \) sending objects of \( A \) to objects of \( B \), with a collection of arrows in SPC \( F_{x,y} : A_{x,y} \to B_{F_x,F_y} \) indexed by objects \( x \) and \( y \) of \( A \) and collections of 2-cells in SPC:

- \( F^{\ge}x_{y,z} \) indexed by objects \( x, y, z \) and as follows

3.16

\[
\begin{array}{ccc}
A_{y,z} & \overset{F_{x,z}}{\to} & B_{F_y,F_z} \\
A(x,-) & \overset{\rho^x_{x,y,z}}{\to} & [B_{F_x,F_y},B_{F_x,F_z}] \\
[A_{x,y},A_{x,z}] & \overset{[1,F_{x,z}]}{\to} & [A_{x,y},B_{F_x,F_z}] \\
\end{array}
\]

- \( F^0x \) indexed by objects \( x \) as follows
Those satisfy the coherence conditions 3.18, 3.19 and 3.20 below.

3.18 The 2-cells in SPC

\[
\begin{array}{c}
\text{3.18 The 2-cells in SPC}
\end{array}
\]

\[
\begin{array}{c}
\text{and}
\end{array}
\]

are equal.
3.19 The 2-cells

and

are equal.

3.20 The 2-cells in SPC

and

are equal.

One can also define the SPC-natural transformations, in a similar way. For this purpose, we need some notation.

Let \( \mathcal{A} \) be an arbitrary SPC-category. To give a strict arrow \( \mathcal{I} \rightarrow \mathcal{A}(x, y) \) is equivalent to give an arrow \( x \rightarrow y \) of the underlying bicategory \( \mathcal{A}^0 \) and we might confuse the two. We therefore define for any object \( z \) of \( \mathcal{A} \) and any arrow \( f : x \rightarrow y \) of \( \mathcal{A}^0 \) the arrows
3.21 \( A(f, 1) \) as the composite arrow in SPC

\[
\begin{array}{c}
A_{x,y} \xrightarrow{A(x,-)} [A_{x,y}, A_{x,z}] \\
\xrightarrow{[F,1]} [I, A_{x,z}] \\
\xrightarrow{ev_*} A_{x,z}
\end{array}
\]

and

3.22 \( A(1, f) \) as

\[
\begin{array}{c}
A_{z,x} \xrightarrow{A(-,y)} [A_{x,y}, A_{z,y}] \\
\xrightarrow{[F,1]} [I, A_{z,y}] \\
\xrightarrow{ev_*} A_{z,y}
\end{array}
\]

Given any arrows \( x \xrightarrow{f} y \xrightarrow{g} z \xrightarrow{h} t \) in \( A^0 \), we define the two cells

3.23 \( c^1_{f,y,z} \)

\[
\begin{array}{c}
A_{z,t} \xrightarrow{A(y,-)} [A_{y,z}, A_{y,t}] \\
\xrightarrow{A(x,-)} [A_{x,z}, A_{x,t}] \\
\xrightarrow{[1, A(f, 1)]} [A_{y,z}, A_{x,t}]
\end{array}
\]

3.24 \( c^2_{x,g,t} \)

\[
\begin{array}{c}
A_{z,t} \xrightarrow{A(x,-)} [A_{z,x}, A_{x,t}] \\
\xrightarrow{A(g,1)} [A_{z,y}, A_{x,t}] \\
\xrightarrow{[A(1, g), 1]} [A_{z,y}, A_{x,t}]
\end{array}
\]

and eventually

3.25 \( c^3_{x,y,h} \)

\[
\begin{array}{c}
A_{y,z} \xrightarrow{A(1,h)} A_{y,t} \\
\xrightarrow{A(x,-)} [A_{x,y}, A_{x,t}] \\
\xrightarrow{[A(1, h)]} [A_{y,y}, A_{x,t}]
\end{array}
\]

which are obtained from the trilinear natural transformation 3.1

\((\alpha_{x,y,z,t})_{h,g,f} : h \circ (g \circ f) \rightarrow (h \circ g) \circ f\)

by fixing one of its argument. For \( c^1 \), \( c^2 \) and \( c^3 \) fix respectively \( f, g \) and \( h \).

Formally \( c^1_{f,z,t} \) is the composite

\[
\begin{array}{c}
A_{z,t} \xrightarrow{\alpha^e_{x,y,z,t}} [A_{y,z}, [A_{x,y}, A_{x,t}]] \\
\xrightarrow{[1, [f, 1]]} [A_{y,z}, [I, A_{x,t}]] \\
\xrightarrow{[1, ev_*]} [A_{y,z}, A_{x,t}]
\end{array}
\]

(see 3.26 in Appendix), \( c^2_{x,g,t} \) it is the composite

\[
\begin{array}{c}
A_{z,t} \xrightarrow{\alpha^e_{x,y,z,t}} [A_{y,z}, [A_{x,y}, A_{x,t}]] \\
\xrightarrow{[g, 1]} [I, [A_{x,y}, A_{x,t}]] \\
\xrightarrow{ev_*} [A_{x,y}, A_{x,t}]
\end{array}
\]

and \( c^3_{x,y,h} \) is the image by \( ev_* \) of the composite

\[
\begin{array}{c}
\mathcal{I} \xrightarrow{h} A_{z,t} \xrightarrow{\alpha^e_{x,y,z,t}} [A_{y,z}, [A_{x,y}, A_{x,t}]]
\end{array}
\]
By definition all 2-cells \( c_1, c_2, \) and \( c_3 \) are identities when \( A \) is strict.

One has also for any objects \( x \) and \( y \) of \( A \) the 2-cell in \( SPC \)

\[ 3.26 \quad r'_{x,y} : \]

\[
\begin{array}{ccc}
I & \xrightarrow{u_x} & A_{x,x} \\
& \searrow & \searrow \rho_{x,y} \\
& [A_{x,y}, A_{x,y}] & \xrightarrow{A(-,-)}
\end{array}
\]

which according to Remarks 7.10 is determined by its value in \( \ast \) which is the linear natural transformation

\[ \rho_{x,y} : f \circ 1_x \to f \]

of 3.24 and corresponds by the bijection 7.16/7.18 in Appendix to the 2-cell \( \rho'_{x,y} \)

\[
\begin{array}{ccc}
A_{x,y} & \xrightarrow{A(-,-)} & [A_{x,x}, A_{x,y}] \\
& \downarrow & \downarrow \\
& [u_x,1] & \xrightarrow{\rho'_{x,y}} [I, A_{x,y}]
\end{array}
\]

Similarly for any objects \( x \) and \( y \) of \( A \) one has the 2-cell in \( SPC \)

\[ 3.27 \quad l'_{x,y} : \]

\[
\begin{array}{ccc}
I & \xrightarrow{u_y} & A_{y,y} \\
& \searrow & \searrow \lambda_{x,y} \\
& [A_{x,y}, A_{x,y}] & \xrightarrow{A(-,-)}
\end{array}
\]

that corresponds to the linear natural transformation

\[ (\lambda_{x,y})_f : 1_y \circ f \to f \]

of 3.3 and which corresponds by the bijection 7.16/7.18 in Appendix to the 2-cell \( \lambda'_{x,y} \)

\[
\begin{array}{ccc}
A_{x,y} & \xrightarrow{\lambda_{x,y}} & [A_{y,y}, A_{x,y}] \\
& \downarrow & \downarrow \\
& [u_y,1] & \xrightarrow{\lambda'_{x,y}} [I, A_{x,y}]
\end{array}
\]

Given any strict arrow \( \tilde{f} : I \to A_{x,y} \), with corresponding arrow \( f : x \to y \) in \( A^0 \), one has the 2-cell in \( SPC \)
3.28 $u^1 f$ :

\[
\begin{array}{ccc}
I & \overset{u_y}{\longrightarrow} & A_{y,y} \\
\downarrow f \downarrow \quad & & \quad A(f,1) \\
[ A_{x,y}, A_{x,y} ] & \rightarrow &
\end{array}
\]

which corresponds to the 2-cell $\rho_f : f \rightarrow 1_y \circ f : x \rightarrow y$ of $A^0$. It is the pasting

\[
\begin{array}{ccc}
I & \overset{u_y}{\longrightarrow} & A_{y,y} \\
\downarrow f \downarrow \quad & & \quad A(f,1) \\
[ A_{x,y}, A_{x,y} ] & \rightarrow &
\end{array}
\]

where the bottom identity 2-cell above is established in Lemma 7.11 in Appendix. Similarly one has the 2-cell in SPC

3.29 $u^2 f$ :

\[
\begin{array}{ccc}
I & \overset{u_x}{\longrightarrow} & A_{x,x} \\
\downarrow f \downarrow \quad & & \quad A(1,f) \\
[ A_{x,y}, A_{x,y} ] & \rightarrow &
\end{array}
\]

that corresponds the 2-cell

\[
(\lambda_{x,y})_f : f \rightarrow f \circ 1_x
\]

and is the pasting

\[
\begin{array}{ccc}
I & \overset{u_x}{\longrightarrow} & A_{x,x} \\
\downarrow f \downarrow \quad & & \quad A(1,f) \\
[ A_{x,y}, A_{x,y} ] & \rightarrow &
\end{array}
\]

Given two SPC-functors $F, G : A \rightarrow B$, a SPC-natural transformation $(\sigma, \kappa) : F \rightarrow G : A \rightarrow B$ consists of a collection of strict arrows $\sigma_x : I \rightarrow B(Fx,Gx)$ (or 1-cells $\sigma_x : Fx \rightarrow Gx$ in $B_0$),
indexed by objects $x$ of $\mathcal{A}$ together with a collection of 2-cells $\kappa_{x,y}$ in $\text{SPC}$ for objects $x,y$ of $\mathcal{A}$ as follows

**3.30**

\[
\begin{align*}
A_{x,y} & \xrightarrow{F_{x,y}} B_{F_x,F_y} \\
G_{x,y} & \xrightarrow{B(1,\sigma_y)} B_{G_x,G_y} \\
B_{G_x,G_y} & \xrightarrow{B(\sigma_x,1)} B_{F_x,G_y}
\end{align*}
\]

and that satisfies the two coherence conditions **3.31** and **3.32** and below.

**3.31** For any object $x$, $y$ and $z$ in $\mathcal{A}$, the 2-cells $\Xi_1$, $\Xi_2$, $\Xi_3$, $\Xi_4$, $\Xi_5$, $\Xi_6$ and $\Xi_7$ below satisfy the equality

\[
\Xi_2 \circ \Xi_1 = \Xi_8 \circ (\Xi_7)^{-1} \circ \Xi_6 \circ \Xi_5 \circ \Xi_4 \circ (\Xi_3)^{-1}.
\]

$\Xi_1$ is

\[
\begin{align*}
A_{y,z} & \xrightarrow{G_{y,z}} B_{G_y,G_z} \\
A(y,\cdot) & \xrightarrow{G_2^{(G_z)-1}} B_{G_x,G_y} \\
[A_{x,y},A_{x,z}] & \xrightarrow{[1,G_x]} [A_{x,y},B_{G_x,G_z}]
\end{align*}
\]

$\Xi_2$ is

\[
\begin{align*}
A_{y,z} & \xrightarrow{A(x,-)} [A_{x,y},A_{x,z}] \\
[1,F_x] & \xrightarrow{[1,\kappa_x]} [A_{x,y},B_{F_x,G_z}]
\end{align*}
\]

$\Xi_3$ is

\[
\begin{align*}
B_{G_x,G_y} & \xrightarrow{B(G_z,-)} [1,B(\sigma_x,1)] \\
B_{G_x,G_y} & \xrightarrow{c^1_{B(\sigma_x,1),y,z}} [B_{G_x,G_y},B_{F_x,G_z}]
\end{align*}
\]

$\Xi_4$ is

\[
\begin{align*}
A_{y,z} & \xrightarrow{G_{y,z}} B_{G_y,G_z} \\
B(F_x,-) & \xrightarrow{[B(\sigma_x,1),1]} [B_{G_x,G_y},B_{F_x,G_z}]
\end{align*}
\]
For any object $x$ of $A$, the 2-cells
and

![Diagram](image)

are equal.

4 SPC-categories via the tensor

In this section we give definitions of SPC-categories and SPC-functors that rely on the tensor product in SPC.

A SPC-category \((\mathcal{A}, u, c, \alpha, \rho, \lambda)\) consists of the following data:
- As before: a small set of objects with a map sending any pair \(x, y\) of objects to an object \(A_{x, y}\) of SPC;
- Collections of strict morphisms \(u_x : I \to A_{x, x}\) and \(c_{x, y, z} : A_{y, z} \otimes A_{x, y} \to A_{x, z}\) indexed by objects of \(\mathcal{A}\) with collections of 2-cells \(\alpha_{x, y, z, t} : A_{y, z} \otimes A_{x, y} \to A_{x, z}\), \(\rho_{x, y} : 1 \otimes u_x \to A_{x, x}\), and \(\lambda_{x, y} : u_y \otimes 1 \to A_{y, y}\) in SPC indexed by objects of \(\mathcal{A}\) and as follows

4.1

\[
\begin{array}{c}
(A_{z, t} \otimes A_{y, z}) \otimes A_{x, y} \\
\downarrow c_{y, z, t} \otimes 1 \\
A_{y, t} \otimes A_{x, y} \\
\downarrow c_{x, y, t} \\
A_{x, t} \\
\end{array}
\quad
\begin{array}{c}
A'_{z, t} \otimes (A_{y, z} \otimes A_{x, y}) \\
\downarrow \alpha_{x, y, z, t} \otimes 1 \\
A_{z, t} \otimes A_{x, z} \\
\downarrow c_{x, z, t} \\
A_{x, t} \\
\end{array}
\]

4.2

\[
\begin{array}{c}
A_{x, y} \\
\downarrow \rho_{x, y} \\
A_{x, y} \otimes I \\
\downarrow 1 \otimes u_x \\
A_{x, y}
\end{array}
\]

and

4.3

\[
\begin{array}{c}
A_{x, y} \\
\downarrow \lambda_{x, y} \\
A_{x, x}
\end{array}
\quad
\begin{array}{c}
A_{x, y} \\
\downarrow u_y \otimes 1 \\
A_{y, y} \otimes A_{x, y}
\end{array}
\]
Those satisfy the coherence Axioms 4.4 and 4.5 below.

4.4 For any objects \(x, y, z, t\) and \(u\) of \(A\), the 2-cells

\[
((A_{t,u}A_{z,t})A_{y,z})A_{x,y} \xrightarrow{A'} (A_{t,u}A_{z,t})(A_{y,z}A_{x,y}) \xrightarrow{id} (A_{t,u}A_{z,t})(A_{y,z}A_{x,y}) \xrightarrow{A'} A_{t,u}(A_{z,t}(A_{y,z}A_{x,y}))
\]

are equal. Note that the domains of the above 2-cells are equal since \(A' \circ A' = A' \circ A'\) by Lemma [Sch08]-19.10.

4.5 For any objects \(x, y, z\) of \(A\), the 2-cells

\[
\Xi_1 = (A_{t,u}A_{z,t})A_{y,z} \xrightarrow{\lambda} (A_{t,u}A_{z,t})(A_{y,z}A_{x,y}) \xrightarrow{id} (A_{t,u}A_{z,t})(A_{y,z}A_{x,y}) \xrightarrow{\lambda} A_{t,u}(A_{z,t}(A_{y,z}A_{x,y}))
\]

and

\[
((A_{t,u}A_{z,t})A_{y,z})A_{x,y} \xrightarrow{A' \otimes 1} (A_{t,u}(A_{z,t}A_{y,z}))A_{x,y} \xrightarrow{A'} A_{t,u}(A_{z,t}(A_{y,z}A_{x,y})) \xrightarrow{1 \otimes A' A'} A_{t,u}(A_{z,t}(A_{y,z}A_{x,y}))
\]
\[ \Xi_3 = \]

\[
\begin{array}{ccc}
A_{y,z} \otimes A_{x,y} & \xrightarrow{R \otimes 1} & A_{y,z} \otimes A_{x,y} \\
1 \otimes L & & 1 \otimes (u_y \otimes 1)
\end{array}
\]

\[
\begin{array}{ccc}
A_{y,z} \otimes (I \otimes A_{x,y}) & \xrightarrow{1 \otimes (u_y \otimes 1)} & (A_{y,y} \otimes A_{x,y}) \\
\end{array}
\]

satisfy the equality \( \Xi_1 = \Xi_3 \ast \Xi_2 \)

Let us justify the equivalence of the definitions of \( SPC \)-categories. We define the following bijective correspondence between data involved the definitions. Arrows \( A(x, -) \) and \( c_{x,y,z} \) correspond via the adjunction \[2.5\] By Lemma \[Sch08\]-19.6 one has a bijective correspondence between 2-cells of the kind \( \alpha_{x,y,z,t} \) and 2-cells \( \alpha'_{x,y,z,t} \), the later being images by \( Rn \circ Rn \) of the first ones. The codomains of the 2-cells \( \rho_{x,y} \) and \( \rho'_{x,y} \) are equal by Lemma \[7.4\] and these 2-cells correspond when are equal. The codomains of the 2-cells \( \lambda_{x,y} \) and \( \lambda'_{x,y} \) are equal by Lemma \[7.8\] and these 2-cells correspond when they are equal. For such corresponding data, the proofs of the equivalence of Axioms \[4.4\] and \[3.14\] rely on the adjunction \[2.5\]. The 2-cells of Axiom \[4.4\] have images by \( Rn \circ Rn \circ Rn \) the two 2-cells of Axioms \[3.14\] and their common domain is a strict arrow with strict images by \( Rn \) and \( Rn \circ Rn \). Computation details are in Appendix in \[7.27\]. The proof of the equivalence of Axioms \[4.4\] and Axioms \[5.13\] is similar. The 2-cells \( \Xi_1 \) of Axiom \[4.4\] has a strict domain and its image by \( Rn \) is \( \rho' \) whereas the 2-cell \( \Xi_3 \circ \Xi_2 \) has image by \( Rn \) the second 2-cell of Axiom \[3.15\] Computation details are in Appendix in \[7.28\].

We have an alternative definition for the \( SPC \)-functors with the tensor in \( SPC \).

Given two arbitrary \( SPC \)-categories \( A \) and \( B \), a \( SPC \)-functor \( F : A \to B \) consists of the following data:
- A map \( F \) sending objects of \( A \) to objects of \( B \);
- For any objects \( x, y \) of \( A \), and arrow \( F_{x,y} : A(x, y) \to B(Fx, Fy) \) in \( SPC \);
- Collections of 2-cells of \( SPC \): the \( F_{x,y} \), indexed by pair of objects \( x, y \) of \( A \) and the \( F^0_{x} \), indexed by objects \( x \) of \( A \), as follows

\[ 4.6 \]

\[
\begin{array}{ccc}
A_{y,z} \otimes A_{x,y} & \xrightarrow{F_{x,y} \otimes F_{x,y}} & B_{Fy,Fx} \otimes B_{Fx,Fy} \\
A_{x,z} & \xrightarrow{F_{x,z}} & B_{Fx,Fx} \\
\end{array}
\]

and

\[
\begin{array}{ccc}
I & \xrightarrow{u_x} & A_{x,x} \\
F_{x,x} & \xrightarrow{u_{Fx}} & B_{Fx,Fx} \\
\end{array}
\]

and that satisfy the coherence conditions \[4.7\], \[4.8\] and \[4.9\] below.
4.7 For any objects \(x,y,z,t\) of \(A\), the 2-cells

\[
\begin{align*}
(A_x,t \otimes A_y,z) \\ \otimes A_x,y
\end{align*}
\]

are equal.

4.8 For any objects \(x,y\) of \(A\), the 2-cells

\[
\begin{align*}
A_x,y \\ \otimes \Xi
\end{align*}
\]

are equal.
4.9 For any objects $x, y$ of $A$, the 2-cells

\[
\begin{array}{c}
A_{x,y} \\
\uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \upar
Definition 5.1 A 2-ring consists of a symmetric Picard category \((\mathcal{A}, j)\), where \(\mathcal{A}\) is denoted additively \((\mathcal{A}, +, 0, \text{ass}, r, l, s)\), together with a functor \(\mathcal{A} \times \mathcal{A} \to \mathcal{A}\), denoted by a multiplication \(\cdot\), an object \(1\) of \(\mathcal{A}\) and natural isomorphisms

\[
\tilde{\alpha}_{a,b,c} : (a.b).c \to a.(b.c),
\]

\[
\tilde{\rho}_a : a.1 \to a,
\]

\[
\tilde{\lambda}_a : 1.a \to a,
\]

\[
\varpi_{b,b'} : a.b + a.b' \to a.(b + b'),
\]

\[
\overline{b_a, a'} : a.b + a'.b \to (a + a').b,
\]

such that the data \((\mathcal{A}, ., 1, \tilde{\alpha}, \tilde{\rho}, \tilde{\lambda})\) defines a monoidal structure on \(\mathcal{A}\) and the diagrams below from 5.2 to 5.11 commute for all possible objects of \(\mathcal{A}\).
Note that in the definition given in [JiPi07] inverses of maps $\alpha_{a,b,c}$ and $\beta_{a,b,c}$ rather than the maps themselves are considered and diagrams 5.2 and 5.4 are replaced by...
5.12

\[ a.((b + b') + (c + c')) + a.(b + b') + (a.c + a.c') \]
\[ a.(b + c) + (b' + c') \]
\[ a.(b + c) + a.(b' + c') \]

and similarly diagrams 5.3 and 5.5 are replaced by

5.13

\[ (a + a').c + (b + b').c \]
\[ ((a + a') + (b + b')).c \]
\[ (a.c + a'.c) + (b.c + b'.c) \]

The definitions here and in [JiPi07] are indeed equivalent (To see this use for instance Lemma [Sch08]-7.1.)

Definition 5.14 A morphism of categorical ring \( A \to B \) consists of a functor \( H : A \to B \) with a symmetric monoidal structure between the symmetric categorical groups

\[ H_+ : (A, +, 0, ass, r, l, s) \to (B, +, 0, ass, r, l, s) \]

and a monoidal structure between the monoidal categories

\[ H_\times : (A, 1, \alpha, \beta, \rho, \lambda) \to (B, 1, \alpha, \beta, \rho, \lambda) \]

such that the following diagrams

5.15
and

\[ H(a), H(b) + H(a'), H(b) \xrightarrow{H(a) + H(a')} H(a + a'). h(b) \]

\[ H(a), H(b) + H(a'), H(b) \xrightarrow{H(a) + H(a')} H(a + a'). h(b) \]

\[ H(a + a'), H(b) \xrightarrow{H(a + a')} H((a + a')). h(b) \]

commute for all possible objects involved.

The following is a crucial example of SPC-category.

**Proposition 5.17** The 2-category SPC gets strictly enriched over itself, i.e. it admits a strict enriched structure as follows. The hom map sends any pair \(A, B\) of objects to \([A, B]\), the composition maps are the \([A, -]_{B,C} : [B, C] \to [[A, B], [A, C]]\) and the unit arrows \(u_A\) are the \(v : I \to [A, A]\).

PROOF: See [7.37] in Appendix.

Let us make the following remark about the terminology. If \(SPC'\) denotes just for the purpose of this explanation the enriched structure of \(SPC\) over itself then for any \(A\) in \(SPC\) and any 1-cell \(F : B \to C\) in \(SPC\), one has that:

- \(SPC'(A, -)_{B,C}\) is \([A, -]_{B,C} : [B, C] \to [[A, B], [A, C]]\);
- \(SPC'(-, C)_{A,B}\) is \([- , C]_{A,B} : [A, B] \to [[B, C], [A, C]]\);
- \(SPC'(1, F)\) is \([1, F] : [A, B] \to [A, C]\);
- \(SPC'(F, 1)\) is \([F, 1] : [B, C] \to [A, C]\).

The first two points results from the definitions. The other two points are the following lemma proved in Appendix [7.39]

**Lemma 5.18** For any \(A\) and any arrows \(F : B \to C\) and \(\tilde{F} : I \to [B, C]\) strict with \(ev_*(\tilde{F}) = F\) the diagrams in \(SPC\)

\[ [C, A] \xrightarrow{[F, A]} [B, A] \]

\[ [C, A] \xrightarrow{[F, A]} [B, A] \]

\[ [C, A] \xrightarrow{1 \otimes \tilde{F}} [C, A] \otimes [B, C] \]

\[ [A, B] \xrightarrow{[A, F]} [A, C] \]

\[ [A, B] \xrightarrow{[A, F]} [A, C] \]

\[ I \otimes [A, B] \xrightarrow{F \otimes 1} [B, C] \otimes [A, B] \]

\[ I \otimes [A, B] \xrightarrow{F \otimes 1} [B, C] \otimes [A, B] \]

both commute.

Given any \(SPC\)-category \(A\), 2-rings are obtained by restriction of \(A\) to anyone of its points. The particular case of the strict enriched structure on \(SPC\) yields a strict 2-ring structure on \([A, A]\) for any Picard category \(A\).

Another important example of 2-ring is provided by the unit \(I\) of \(SPC\).
Proposition 5.19 The unit $\mathcal{I}$ of SPC admits a strict 2-ring structure with multiplication given by $L:\mathcal{I} \otimes \mathcal{I} \to \mathcal{I}$ (or $v:\mathcal{I} \to [\mathcal{I}, \mathcal{I}]$) and unit the identity at $\mathcal{I}$.

PROOF: See Appendix 7.

6 Modules and their morphisms

Any 2-ring $\mathcal{A}$ yields a category $\mathcal{A}$-mod of $\mathcal{A}$-modules and their morphisms. Formally $\mathcal{A}$-mod is the category of $\mathcal{SPC}$-functors $\mathcal{A} \to \mathcal{SPC}$ and $\mathcal{SPC}$-natural transformations between them. In this section we present alternative descriptions of $\mathcal{A}$-modules and their morphisms. In particular we show that the category $\mathcal{A}$-mod is isomorphic to a category of $T\mathcal{A} \otimes -$ over $\mathcal{SPC}$. Eventually we prove in Proposition 6.36 that the category $\mathcal{I}-mod$ of modules over the unit 2-ring $\mathcal{I}$ is equivalent to $\mathcal{SPC}$.

In this section $\mathcal{A}$ stand for a 2-ring with multiplication $c':\mathcal{A} \to [\mathcal{A}, \mathcal{A}]/c:\mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ with unit $u:\mathcal{I} \to \mathcal{A}$ and coherence 2-cell $\alpha'/\alpha$ 3.11/4.1, $\rho/\rho'$ 1.2 5.12, and $\lambda/\lambda'$ 3.3 5.13.

Considering an arbitrary $\mathcal{SPC}$-functor $F:\mathcal{A} \to \mathcal{SPC}$, let us write $\mathcal{M}$ for the object $\mathcal{F}(\bullet)$ of $\mathcal{SPC}$ image by $\mathcal{F}$ of the unique point $\bullet$ of $\mathcal{A}$, $\phi'$ for the arrow unique component $F_{\bullet,\bullet}:\mathcal{A} \to [\mathcal{M}, \mathcal{M}]$ of $\mathcal{F}$, $\beta'$ for the 2-cell $F^{\mathcal{I}}_{\bullet,\bullet}$, and $\gamma'$ for the 2-cell $F^0$. Then one obtains the following first definition of $\mathcal{A}$-modules by rewriting the data 3.16 and 3.17 and Axioms 3.18, 3.19 and 3.20 with these new notations.

A $\mathcal{A}$-module $\mathcal{M} = (\mathcal{M}, \phi', \beta', \gamma')$ consists of the following data in $\mathcal{SPC}$: an object $\mathcal{M}$, with an arrow $\phi':\mathcal{A} \to [\mathcal{M}, \mathcal{M}]$, called its action, and two 2-cells $\beta'$ and $\gamma'$ as follows

6.1

6.2

and those satisfy the coherence axioms 6.3 6.4 and 6.5 below.
6.3 The 2-cells

and

are equal.

6.4 The 2-cells in SPC

and

are equal.
6.5 The 2-cells in SPC

and

are equal.

We shall also denote by $\gamma''$ for the 2-cell

6.6

that corresponds to $\gamma'$ via the bijection 7.16/7.17.

Consider now a $SPC$-natural transformation between presheaves with domain a one point category $A$

$$(\sigma, \kappa) : F \to G : A \to SPC.$$  

Let us write $\mathcal{M} = (M, \phi', \beta', \gamma'')$ and $\mathcal{N} = (N, \psi', \beta', \gamma'')$ for the two modules corresponding respectively to $F$ and $G$. The $SPC$-natural transformation $\sigma$ has a unique component at the unique object $\star$ of $A$, which is a strict arrow $I \to [M,N]$ in $SPC$ or equivalently an arrow $H : \mathcal{M} \to \mathcal{N}$. The collection $\kappa$ consists of a unique 2-cell

6.7

that corresponds to $\gamma'$ via the bijection 7.16/7.17.
which we name $\delta'$. We obtain therefore the following definition of morphism of $A$-modules by rewriting Axioms 3.31 and 3.32 with these new notations.

A morphism of $A$-module $(H, \delta') : (M, \varphi', \beta', \gamma') \to (N, \psi', \beta', \gamma')$ consists of an arrow $H : M \to N$ in $SPC$ with a 2-cell $\delta'$ as in 6.7 those satisfying Axioms 6.8 and 6.9 below.

6.8 The 2-cells $\Xi_1, \Xi_2, \Xi_3, \Xi_4, \Xi_5, \Xi_6, \Xi_7$ and $\Xi_8$ in $SPC$ below satisfy the equality

$$\Xi_2 \circ \Xi_1 = \Xi_8 \circ \Xi_7 \circ \Xi_6 \circ \Xi_5 \circ \Xi_4 \circ \Xi_3.$$

$\Xi_1$ is

$$\begin{array}{c}
\mathcal{A} \\
\downarrow \psi' \\
\mathcal{A}, \mathcal{A} \quad [\mathcal{N}, \mathcal{N}] \\
\downarrow \beta' \\
\mathcal{A}, \mathcal{N} \quad [\mathcal{N}, \mathcal{N}] \\
\downarrow \psi' \\
\mathcal{A}, \mathcal{M} \quad [\mathcal{M}, \mathcal{N}] \\
\end{array}$$

$\Xi_2$ is

$$\begin{array}{c}
\mathcal{A} \\
\downarrow \psi' \\
\mathcal{A}, \mathcal{A} \\
\downarrow \beta' \\
\mathcal{A}, \mathcal{M} \\
\end{array}$$

$\Xi_3$ is the identity

$$\begin{array}{c}
\mathcal{A} \\
\downarrow \psi' \\
\mathcal{A}, \mathcal{N} \\
\downarrow \beta' \\
\mathcal{A}, \mathcal{N} \\
\downarrow \psi' \\
\mathcal{A}, \mathcal{N} \\
\end{array}$$

$\Xi_4$ is

$$\begin{array}{c}
\mathcal{A} \\
\downarrow \psi' \\
\mathcal{A}, \mathcal{N} \\
\downarrow \beta' \\
\mathcal{A}, \mathcal{N} \\
\end{array}$$
Ξ₅ is the identity 2-cell

\[
A \xrightarrow{\psi'} [N, N] = ([M, N], [M, N]) \quad ([M, -]) \quad ([1, H], 1)
\]

Ξ₆ is

\[
\begin{array}{c}
A \\
\vdash \psi' \quad \phi' \quad [N, N] \\
\downarrow \quad \downarrow \quad [1, H, 1] \\
[M, N] \\
\end{array}
\]

\[
([M, N], [M, N]) \xrightarrow{[M, -]} ([M, M], [M, N]) \xrightarrow{[\psi', 1]} [A, [M, N]]
\]

Ξ₇ is the identity 2-cell

\[
\begin{array}{c}
A \\
\vdash \phi' \quad \phi' \quad [M, M] \\
\downarrow \quad \downarrow \quad [1, H] \\
[M, M] \\
\end{array}
\]

\[
([M, M], [M, M]) \xrightarrow{[M, -]} ([M, M], [M, N]) \xrightarrow{[\phi', 1]} [A, [M, N]]
\]

and Ξ₈ is

\[
\begin{array}{c}
A \\
\vdash \phi' \quad [M, M] \\
\downarrow \quad \downarrow \quad [1, H] \\
[M, M] \\
\end{array}
\]

\[
([M, M], [M, M]) \xrightarrow{[\phi', 1]} [A, [M, M]] \xrightarrow{[1, 1, H]} [A, [M, N]]
\]

6.9 The 2-cells in SPC
and

\[ H \xrightarrow{=} [\mathcal{N}, \mathcal{N}] \xrightarrow{[H,1]} [\mathcal{M}, \mathcal{M}] \]

\[ \xrightarrow{id} \quad \xrightarrow{\gamma} \quad \xrightarrow{\psi'} \quad \xrightarrow{\delta'} \]

\[ \mathcal{I} \xrightarrow{u} \mathcal{A} \xrightarrow{\varphi'} [\mathcal{M}, \mathcal{M}] \]

are equal.

From these first definitions of \( \mathcal{A} \)-modules and \( \mathcal{A} \)-module morphisms, one obtains immediately the following simple other definitions involving multilinear maps and multilinear natural transformations.

A \( \mathcal{A} \)-module \((\mathcal{M}, \varphi, \beta, \gamma)\) consists of a bilinear map \( \varphi : \mathcal{A} \times \mathcal{M} \to \mathcal{M} \), which we write as a multiplication \( \varphi(a,m) = a.m \), with two natural transformations, \( \beta \) which is trilinear, and \( \gamma \) which is linear, as follows.

6.10

\[ \beta : a_1.(a_2.m) \to (a_1.a_2).m \]

lies in \( \mathcal{M} \), for \( a_1, a_2 \) objects of \( \mathcal{A} \) and \( m \) object of \( \mathcal{M} \).

6.11

\[ \gamma : m \to 1_{\mathcal{A}}.m \]

lies in \( \mathcal{M} \) for \( m \) object of \( \mathcal{M} \).

Those satisfy the coherence conditions 6.12, 6.13 and 6.14 below.

6.12 For any objects \( a_1, a_2, a_3 \) of \( \mathcal{A} \) and \( m \) of \( \mathcal{M} \) the diagram in \( \mathcal{M} \)

\[ \xymatrix{ a_1.(a_2.(a_3.m)) \ar[r]^{\beta} & (a_1.a_2).(a_3.m) \ar[r]^{\beta} & (a_1.a_2.a_3).m } \]

commutes.

6.13 For any objects \( a \) of \( \mathcal{A} \) and \( m \) of \( \mathcal{M} \) the diagram in \( \mathcal{M} \)

\[ \xymatrix{ a.m \ar[r]^{a.\gamma} \ar[dr]_{\rho_a.m} & a.(1_{\mathcal{A}}.m) \ar[dr]_{\beta} \ar[r] & (a.1_{\mathcal{A}}).m } \]

commutes.
6.14 For any objects \( a \) of \( \mathcal{A} \) and \( m \) of \( \mathcal{M} \) the diagram in \( \mathcal{M} \)
\[
\begin{array}{ccc}
  a.m & \xrightarrow{\gamma_{a.m}} & 1_{\mathcal{A}}(a.m) \\
    & \searrow & \\
    & \downarrow & \\
(1_{\mathcal{A}}a).m & \xrightarrow{\beta_{1_{\mathcal{A}}a.m}} & \\
\end{array}
\]
commutes.

A \( \mathcal{A} \)-module morphism \((H, \delta) : \mathcal{M} \rightarrow \mathcal{N}\) consists of an arrow \( H : \mathcal{M} \rightarrow \mathcal{N} \) in \( \mathcal{SPC} \) with a bilinear natural transformation
\[
6.15 \quad \delta_{a,m} : a.H(m) \rightarrow H(a.m)
\]
which lies in \( \mathcal{N} \) for a object of \( \mathcal{A} \) and \( m \) object of \( \mathcal{M} \)
which satisfy the Axioms 6.16 and 6.17 below.

6.16 For any object \( m \) of \( \mathcal{M} \) the following diagram in \( \mathcal{N} \)
\[
\begin{array}{ccc}
  Hm & \xrightarrow{H(\gamma_m)} & H(*)m \\
    & \searrow & \\
    & \downarrow & \\
\star.Hm & \xrightarrow{\delta_{\star,m}} & H(\star.m)
\end{array}
\]
commutes.

6.17 For any objects \( a_1, a_2 \) in \( \mathcal{A} \) and \( m \) in \( \mathcal{M} \), the following diagram in \( \mathcal{N} \)
\[
\begin{array}{ccc}
  a_1.(a_2.Hm) & \xrightarrow{\beta_{a_1.a_2.m}} & (a_1,a_2).Hm \\
  a_1.a_2.m & \searrow & \\
  a_1.H(a_2.m) & \xrightarrow{\delta_{a_1.a_2.m}} & H((a_1,a_2).m)
\end{array}
\]
commutes.

Note that Axiom 6.17 is obtained from Axiom 6.9 by evaluation at the generator \( \star \) since the component in \( \star \) of \( u^2_H \) is an identity. By Remark 7.10 Axioms 6.9 and 6.17 are equivalent.

Eventually we give definitions of \( \mathcal{A} \)-modules and their morphisms using the tensor in \( \mathcal{SPC} \). This will show that in which sense \( \mathcal{A} \)-modules occur as algebras for the doctrine \( \mathcal{A} \otimes - \) over \( \mathcal{SPC} \).

A \( \mathcal{A} \)-module \((\mathcal{M}, \varphi, \beta, \gamma)\) consists of a strict arrow \( \varphi : \mathcal{A} \otimes \mathcal{M} \rightarrow \mathcal{M} \) in \( \mathcal{SPC} \) in with 2-cells \( \beta \) and \( \gamma \) in \( \mathcal{SPC} \) as follows
6.18

\[(A \otimes A) \otimes M \xrightarrow{A'} A \otimes (A \otimes M)\]
\[\xrightarrow{\beta} A \otimes M \xrightarrow{\varphi} M\]

and

6.19

\[M \xrightarrow{L'} I \otimes M\]
\[\xrightarrow{id} \xrightarrow{\gamma} \xrightarrow{\psi} A \otimes M\]

satisfying the coherence conditions 6.20, 6.21, and 6.22 below.

6.20 The 2-cells

\[
\begin{array}{c}
((A \otimes A) \otimes A) \otimes M \\
\xrightarrow{A'} (A \otimes A) \otimes (A \otimes M) \\
\xrightarrow{id} (A \otimes A) \otimes (A \otimes M) \\
\xrightarrow{A'} A \otimes (A \otimes M) \\
\xrightarrow{1 \otimes \varphi} M \\
\end{array}
\]

\[
\begin{array}{c}
((A \otimes A) \otimes A) \otimes M \\
\xrightarrow{1 \otimes \varphi} A \otimes (A \otimes M) \\
\xrightarrow{1 \otimes \psi} A \otimes (A \otimes M) \\
\xrightarrow{1 \otimes \psi} (A \otimes A) \otimes M \\
\xrightarrow{1 \otimes A'} A \otimes (A \otimes M) \\
\xrightarrow{1 \otimes \varphi} (A \otimes A) \otimes M \\
\end{array}
\]

and

\[
\begin{array}{c}
((A \otimes A) \otimes A) \otimes M \\
\xrightarrow{A' \otimes 1} (A \otimes A) \otimes (A \otimes M) \\
\xrightarrow{1 \otimes A'} A \otimes (A \otimes M) \\
\xrightarrow{1 \otimes \beta} (A \otimes A) \otimes M \\
\xrightarrow{1 \otimes \psi} A \otimes (A \otimes M) \\
\xrightarrow{1 \otimes \psi} (A \otimes A) \otimes M \\
\xrightarrow{1 \otimes A'} A \otimes (A \otimes M) \\
\xrightarrow{1 \otimes \varphi} (A \otimes A) \otimes M \\
\end{array}
\]

are equal.

6.21 The 2-cells \(\Xi_1 =\)

\[
\begin{array}{c}
A \otimes M \\
\xrightarrow{id} A \otimes M \\
\xrightarrow{\psi} M \\
\end{array}
\]

\[
\begin{array}{c}
(A \otimes I) \otimes M \\
\xrightarrow{(1 \otimes u) \otimes 1} (A \otimes A) \otimes M \\
\end{array}
\]
The 2-cells \( \Xi_1 = \) and \( \Xi_3 = \) satisfy the equality \( \Xi_1 = \Xi_3 \circ \Xi_2 \).

6.22 The 2-cells \( \Xi_1 = \) satisfy the equality \( \Xi_1 = \Xi_4 \circ (\Xi_3)^{-1} \circ \Xi_2 \).

With the previous definition of \( A \)-modules, a morphism of \( A \)-modules \( (H,\delta) : (M,\varphi,\beta,\gamma) \to (N,\psi,\beta,\gamma) \) consists of an arrow \( H : M \to N \) with a 2-cell \( \delta \) in \( SPC \).
that satisfy Axioms 6.24 and 6.25 below.

6.24 The 2-cells

\[
\begin{array}{c}
A(M) \to A(N) \\
\downarrow \downarrow \\
M \to N
\end{array}
\]

are equal.

6.25 The 2-cells

\[
\begin{array}{c}
\downarrow \downarrow \\
A(M) \to A(N) \\
\downarrow \downarrow \\
M \to N
\end{array}
\]

are equal.

To justify these new definitions let us consider again an arbitrary SPC-functor \( F : A \to SPC \) which defines an \( A \)-module \( M \) with multiplication \( \varphi' : A \to [M, M] \) and 2-cells \( \beta' \) as in 6.1 and \( \gamma' \) as in 6.2. The multiplication corresponds by adjunction 2.5 to a strict arrow \( \varphi : A \otimes M \to M \).
According to the adjunction 2.5, Lemmas 7.21 and [Sch08]-19.6, the map $R_n \circ R_n$ defines a bijection between the sets of 2-cells of the following kinds

\[
\begin{array}{c}
(AA)M \\
\downarrow c \otimes 1 \\
AM \\
\downarrow \varphi \\
\end{array} \quad \begin{array}{c}
A' \\
\downarrow 1 \otimes \varphi \\
A(AM) \\
\end{array}
\]

and

\[
\begin{array}{c}
A \\
\downarrow \varphi' \\
[M, M] \\
\downarrow [\varphi', 1] \\
\end{array} \quad \begin{array}{c}
\varphi' \\
\downarrow [\varphi', 1] \\
[A, A] \\
\downarrow c \\
\end{array} \quad \begin{array}{c}
\varphi \\
\downarrow [c, 1] \\
[A, [M, M]] \\
\downarrow [1, \varphi'] \\
\end{array}
\]

and one has a 2-cell $\beta$ corresponding to $\beta'/F^{r_2}_{* *}$ via the above bijection. Note then that $\beta$ has image by $R_n$ the 2-cell $F^{r_2}_{* *}$, which we also write $\beta''$. One obtains a 2-cell $\gamma$ as in 6.19 that is equal to the 2-cell $\gamma''$ according Lemma 7.8.

We are going to check that the points 6.26, 6.27, 6.28, 6.29, 6.30 and 6.31 below hold for a SPC-functor $F$ and the related data as above. Since the arrows domains of the 2-cells of the equalities of Axioms 6.20, 6.21 and 6.22 are strict, it will result from the adjunction 2.5 that these axioms are equivalent respectively to Axioms 4.7, 4.8 and 4.9 for the SPC-functor $F$.

**6.26** The 2-cell

\[
\begin{array}{c}
(A \otimes A) \otimes A \\
\downarrow (\varphi' \otimes \varphi') \otimes \varphi' \\
\end{array} \quad \begin{array}{c}
A \otimes A \\
\downarrow c \otimes 1 \\
\end{array} \quad \begin{array}{c}
A' \\
\downarrow c \otimes 1 \\
\end{array} \quad \begin{array}{c}
A \otimes A \\
\downarrow e \\
\end{array} \quad \begin{array}{c}
A \\
\downarrow c \\
\end{array}
\]

is the image by $R_n$ of the first of the 2-cells of Axiom 6.20.

PROOF: See 7.41 in Appendix.

**6.27** The 2-cell

\[
\begin{array}{c}
(AA)A \\
\downarrow (\varphi' \otimes \varphi'') \otimes \varphi \\
\end{array} \quad \begin{array}{c}
AA \\
\downarrow 1 \otimes c \\
\end{array} \quad \begin{array}{c}
A^{(AA)} \\
\downarrow 1 \otimes c \\
\end{array} \quad \begin{array}{c}
A \\
\downarrow id \\
\end{array}
\]

is the image by $R_n$ of the second 2-cell of Axiom 6.21.
6.28 The 2-cell

\[
\begin{array}{cccccccc}
A & \xrightarrow{id} & A & \xrightarrow{\psi'} & [M, M] \\
\downarrow{R'} & & \downarrow{c} & & \\
A \otimes I & \xrightarrow{1 \otimes u} & A \otimes A & \\
\end{array}
\]

is the image by \( R_n \) of the 2-cell \( \Xi_1 \) of Axiom 6.21.

PROOF: Immediate.

6.29 The 2-cell

\[
\begin{array}{cccccccc}
A & \xrightarrow{\psi'} & [M, M] \\
\downarrow{R'} & = & \downarrow{r'} & \\
A \otimes I & \xrightarrow{1 \otimes \psi'} & [M, M] \otimes I & \\
\downarrow{1 \otimes u} & = & \downarrow{1 \otimes v} & \xrightarrow{id} & \\
A \otimes A & \xrightarrow{\psi \otimes \phi'} & [M, M] \otimes [M, M] & \\
\downarrow{c} & & \downarrow{1 \otimes \phi} & & \\
A & \xrightarrow{\psi'} & [M, M] & \\
\end{array}
\]

is the image by \( R_n \) of the 2-cell \( \Xi_3 \circ \Xi_2 \) of Axiom 6.21.

PROOF: See 7.44 in Appendix.

6.30 The 2-cell

\[
\begin{array}{cccccccc}
A & \xrightarrow{id} & A & \xrightarrow{\psi'} & [M, M] \\
\downarrow{L'} & & \downarrow{\phi} & & \\
I \otimes A & \xrightarrow{u \otimes 1} & A \otimes A & \\
\end{array}
\]

is the image by \( R_n \) of the 2-cell \( \Xi_1 \) of Axiom 6.22.

PROOF: Immediate.

6.31 The 2-cell

\[
\begin{array}{cccccccc}
A & \xrightarrow{\psi'} & [M, M] \\
\downarrow{L'} & = & \downarrow{r'} & \\
I \otimes A & \xrightarrow{1 \otimes \psi'} & I \otimes [M, M] & \\
\downarrow{u \otimes 1} & = & \downarrow{u \otimes 1} & \xrightarrow{id} & \\
A \otimes A & \xrightarrow{\phi \otimes \phi'} & [M, M] \otimes [M, M] & \\
\downarrow{c} & & \downarrow{1 \otimes \phi} & & \\
A & \xrightarrow{\psi'} & [M, M] & \\
\end{array}
\]

is the image by \( R_n \) of the 2-cell \( \Xi_4 \circ (\Xi_3)^{-1} \circ \Xi_2 \) of Axiom 6.22.
Let us consider now two modules $M$ and $N$ with respective multiplications denoted by $\phi : A \otimes M \to M$ and $\psi : A \otimes N \to N$, with sets of coherence two-cells for both written $\beta/\beta'/\beta''$ and $\gamma/\gamma'/\gamma''$, and an arrow $H : M \to N$ in SPC. A 2-cell $\delta$ as in 6.23 corresponds by the adjunction 2.5 to a 2-cell $\delta'$. For the above related data the equivalence of Axioms 6.24 and 6.8 results from the two following points 6.32 and 6.33 below whereas the equivalence 6.25 and 6.9 follows from Remark 7.10 and points 6.34 and 6.35.

6.32 The first of the 2-cell of 6.24 has image by $R_n \circ R_n$ the pasting $\Xi_2 \circ \Xi_1$ of 6.8, it has a strict domain which has a strict image by $R_n$.

PROOF: See 7.46 in Appendix.

6.33 The image by $R_n \circ R_n$ of the second 2-cell of Axiom 6.24 is the pasting $\Xi_7 \circ \Xi_6 \circ \Xi_5 \circ \Xi_4 \circ \Xi_3 \circ \Xi_2 \circ \Xi_1$ of 6.8.

PROOF: See 7.47 in Appendix.

6.34 The first 2-cell of Axiom 6.9 has image by $ev_\star$ the first 2-cell of Axiom 6.25 and has a strict domain.

PROOF: See 7.48 in Appendix.

6.35 The second 2-cell of Axiom 6.9 has image by $ev_\star$, the second 2-cell of Axiom 6.25.

PROOF: See 7.49 in Appendix.

For any 2-ring $A$, a $A$-module $M$ is said strict when the corresponding $SPC$-presheaf is strict which is to say that its action $\varphi' : A \to [M, M]$ is strict and the 2-cells $\beta'$ and $\gamma'$ are identities. A few remarks are in order. Consider any $A$-module $M$. If its 2-cell $\gamma'$ is an identity then certainly its 2-cell $\gamma''$ is also an identity. Conversely if the action $\varphi'$ is strict then for any $m$ in $M$, $\varphi'^* (m) : A \to M$ is strict, and the component $\epsilon_{\varphi'^* (m) ou}$ is an identity and from this fact one has that if $\gamma''$ is strict then also is $\gamma'$. If the 2-cells $\beta$ are identities then certainly are the 2-cells $\beta'$. Conversely if $A$ is a strict 2-ring and the 2-cells $\beta'$ are identities then the 2-cells $\beta$ also are.

**Proposition 6.36** The forgetful functor $I - Mod \to SPC$ is part of an equivalence of categories. Its equivalence inverse factors as

$$SPC \overset{\cong}{\longrightarrow} I - Mod^s \overset{inc}{\longrightarrow} I - Mod$$

where the left functor is an isomorphism between $SPC$ and the full sub-category $I - Mod^s$ of $I - Mod$ generated by the strict modules and $inc$ is the inclusion functor.

PROOF: One has a forgetful 2-functor $I - Mod \to SPC$ and the result follows then from Lemmas 6.37, 6.38 and 6.40 below.

**Lemma 6.37** Any object $A$ of $SPC$ admits a unique strict $I$-module structure, its multiplication is given by the arrow $\varphi' = v : I \to [A, A]$ (or equivalently $\varphi = L_A : I \otimes A \to A$).
PROOF: If $A$ has a strict $I$-module structure with multiplication $\varphi' = v : I \to [A, A]$ the 2-cell $\gamma''$ in this case being an identity one has that the composite

$$A \xrightarrow{\varphi'^*} [I, A] \xrightarrow{ev} A$$

in SPC is necessarily the identity at $A$. Actually there is a unique arrow $f : A \to [I, A]$ in SPC with strict images in $[I, A]$ – or equivalently such that $f^*$ is strict – and such that the composite

$$A \xrightarrow{f} [I, A] \xrightarrow{ev} A$$

is the identity and this arrow is $v^*$.

Now to establish that $\varphi' = v : I \to [A, A]$ gives a strict $I$-module structure on $A$, it remains to check that one has in this case an identity 2-cell $\beta'$ which is the commutativity of the external diagram in the pasting below

$\begin{array}{ccc}
I & \overset{v}{\rightarrow} & [A, A] \\
& \underset{\varphi'}{\searrow} & \downarrow \quad [A, -] \\
& & \downarrow \quad [[A, A], [A, A]] \\
[I, I] & \overset{[1, v]}{\rightarrow} & [I, [A, A]]
\end{array}$

where the top right diagram commutes according to Lemma [Sch08]-18.5 and the bottom left also does according to Lemma [Sch08]-18.8.

**Lemma 6.38** For any $I$-module $A$ and any arrow $H : A \to B$ in SPC there is a unique $I$-module morphism from $A$ to the strict $I$-module structure on the symmetric Picard category $B$ which underlying map in SPC is $H$. If the multiplication of $A$ is given by $\varphi' : I \to [A, A]$ this morphism has 2-cell $\delta'$ as in 6.17

$\begin{array}{ccc}
I & \overset{v}{\rightarrow} & [B, B] \\
& \overset{\varphi'}{\searrow} & \downarrow \quad [H, 1] \\
[A, A] & \overset{[1, H]}{\rightarrow} & [A, B],
\end{array}$

which is determined by its value in $*$ by Remark [7.10] and such that

$$(\delta'_*)_a = \ast H a \xrightarrow{id} H a \xrightarrow{H(\varphi')} H(\varphi'(\ast)(a)).$$

PROOF: Let us write $t \times a$ for $\varphi'(t)(a)$ for any objects $t$ of $I$ and $a$ of $A$.

The coherence Axiom 6.16 for the pair $H$ and $\delta'$, with corresponding bilinear $\hat{\delta}$ as in 6.15 amounts to the commutativity of the diagram in $B$

$\begin{array}{ccc}
H a & \xrightarrow{H(\varphi')} & H(\ast \times a) \\
\downarrow \quad \hat{\delta}_*, a & & \downarrow \quad \ast H a.
\end{array}$
That the arrow $H : \mathcal{A} \to \mathcal{B}$ in $SPC$ together with the 2-cell $\delta$ defined by the condition above satisfies Axiom 6.17 amounts to the commutativity of the diagram in $\mathcal{B}$

6.39

\[
\begin{array}{c}
t_1.(t_2.Ha) \\
\downarrow \delta_{t_2,a} \\
t_1.H(t_2 \times a) \\
\end{array}
\begin{array}{c}
\to id \\
\to H(t_1 \times (t_2 \times a)) \\
\to H((t_1.t_2) \times a) \\
\end{array}
\begin{array}{c}
(t_1.t_2).Ha \\
\downarrow \delta_{t_1.t_2,a} \\
\end{array}
\]

for all objects $t_1, t_2$ in $\mathcal{I}$ and $a$ in $\mathcal{A}$.

We prove this last point by induction on the structure of the objects $t_1$ in $\mathcal{I}$ for arbitrary objects $t_2$ and $a$.

For $t_1 = I$ diagram 6.39 is the external diagram in the pasting

\[
\begin{array}{c}
I \\
\downarrow id \\
\end{array}
\begin{array}{c}
\overset{id}{\to} H(I) \\
\overset{H^o}{\to} H(\beta) \\
\overset{\delta}{\to} \delta_{t,a} \\
\end{array}
\begin{array}{c}
I \\
\downarrow H(I) \\
\downarrow H(\beta) \\
\end{array}
\begin{array}{c}
\overset{H}{\to} H(I \times a) \\
\overset{H}{\to} H(I \times (t_2 \times a)) \\
\overset{H}{\to} H((I \times a) \times (t_2 \times a)) \\
\end{array}
\]

in which all diagrams commute and in particular the bottom-right triangle since the natural transformation $\beta : - \times (t_2 \times a) \to (-.t_2) \times a : \mathcal{I} \to \mathcal{A}$ is monoidal.

For $t_1 = \ast$ diagram 6.39 is the external diagram in the pasting

\[
\begin{array}{c}
\ast.(t_2.Ha) \\
\downarrow \delta_{t_2,a} \\
\ast.H(t_2 \times a) \\
\downarrow id \\
\end{array}
\begin{array}{c}
\overset{id}{\to} (\ast.t_2).Ha \\
\overset{\delta}{\to} (\ast.t_2.a) \\
\overset{id}{\to} H((\ast.t_2) \times a) \\
\end{array}
\begin{array}{c}
(\ast.t_2.a) \\
\end{array}
\begin{array}{c}
\overset{H}{\to} H((\ast \times (t_2 \times a)) \\
\end{array}
\]

where the bottom diagram commutes by the coherence Axiom 6.14 for the $\mathcal{I}$-module $\mathcal{A}$.
For \( t_1 = t'_1 \otimes t''_1 \) diagram [3.39] is the external diagram in the pasting

\[
(t'_1 \otimes t''_1).H(2.H(a)) \xrightarrow{id} ((t'_1 \otimes t''_1).2).H(a)
\]

where the middle diagram is commutative if the diagram [3.39] commutes for the values \( t_1 = t'_1 \) and \( t_1 = t''_1 \) and the bottom diagram commutes since the natural transformation \( \beta_{-t_2.a} : - \times (t_2 \times a) \rightarrow (-t_2) \times a : \mathcal{I} \rightarrow \mathcal{A} \) is monoidal.

For \( t_1 = t^* \), the diagram [3.9] is

\[
t^*.H(2.H(a)) \xrightarrow{id} (t^*.t_2).H(a)
\]

Note that according to Lemmas [3.7] and [3.6] the arrow \( \delta_{t_1.a} \) is

\[
t^*.H(a) \xrightarrow{(t.H(a))} H(t \times a) \xrightarrow{\sim} H((t \times a)^*) \xrightarrow{H(t)} H(t^* \times a).
\]

Therefore the left-bottom leg rewrites

1. \( t^*.H(2.H(a)) \xrightarrow{t^*.\delta_{t_2.a}} t^*.H(t_2 \times a) \xrightarrow{\delta_{t_2.a} \times a} H(t \times (t_2 \times a)) \xrightarrow{H(t \times (t_2 \times a))} H((t \times (t_2 \times a))^*) \)
2. \( t^*.H(2.H(a)) \xrightarrow{id} (t^*.H(a)) \xrightarrow{\delta_{t_2.a} \times a} (t^*.H(t_2 \times a)) \xrightarrow{H(t^* \times (t_2 \times a))} H((t \times (t_2 \times a))^*) \)
3. \( t^*.H(2.H(a)) \xrightarrow{id} (t^*.H(a)) \xrightarrow{\delta_{t_2.a} \times a} (t^*.H(t_2 \times a)) \xrightarrow{H(t \times (t_2 \times a))} H((t \times (t_2 \times a))^*) \)
4. \( t^*.H(2.H(a)) \xrightarrow{id} (t^*.H(a)) \xrightarrow{\delta_{t_2.a} \times a} (t^*.H(t_2 \times a)) \xrightarrow{H(t \times (t_2 \times a))} H((t \times (t_2 \times a))^*) \)
In the derivation above arrows 2. and 3. are equal due to Lemma 6.39 and arrows 3. and 4. are equal due to the naturality of the isomorphism [2,4].

The top-right leg of the diagram above rewrites

1. \( t^\ast.(t_2.Ha) \xrightarrow{id} (t^\ast.t_2).Ha \xrightarrow{\delta(t_2)_a} H((t^\ast.t_2) \times a) \)
2. \( t^\ast.(t_2.Ha) \xrightarrow{id} (t^\ast.t_2).Ha \xrightarrow{id} (t_2^\ast).Ha \xrightarrow{\delta(t^\ast.t_2)\ast a} H((t.t_2)^\ast \times a) \xrightarrow{id} H(((t^\ast.t_2) \times a)) \)
3. \( t^\ast.(t_2.Ha) \xrightarrow{id} (t_2^\ast).Ha \xrightarrow{id} ((t_2.t).Ha) \xrightarrow{(\delta(t^\ast.t_2)\ast a)^\ast} H((t.t_2)^\ast \times a) \xrightarrow{\cong} H((((t^\ast.t_2) \times a)^\ast)) \)

Therefore the two legs above are equal when diagram 6.39 commutes for \( t_1 = t \).

**Lemma 6.40** For any \( I \)-module \( A \) the unique \( I \)-module morphism from \( A \) to the strict \( I \)-module structure on the symmetric Picard category \( A \) and which underlying map is the identity map \( 1_A \) at \( A \) in \( SPC \) – given by Lemma 6.39 – is invertible in \( I \)-mod.

**PROOF:** Let \( \varphi' : I \to [A, A] \) denote the multiplication of \( A \), and \( t \times a \) stand for \( \varphi(t)(a) \) for any objects \( t \) of \( I \) and \( a \) of \( A \). The 2-cell \( \delta' \) given by 6.38 for the morphism from \( A \) to the strict \( I \)-module on \( A \) in \( SPC \), is in this case of the form \( \bigtriangleup \xrightarrow{\varphi'} \Psi [A, A] \). Its inverse is therefore of the expected form to be part of a module morphism from the strict \( I \)-module on \( A \) to the module \( A \) with multiplication \( \varphi' \).

Recall that the coherence Axiom 6.10 for the pair \( (1_A, \delta') \) as a morphism from \( A \) with multiplication \( \varphi' \) to the strict \( I \)-module on \( A \) is the commutativity of the diagram in \( A \)

\[ \begin{array}{ccc}
\ast \times a & \xrightarrow{\varphi'} & \ast \times a \\
\downarrow{id} & & \downarrow{id} \\
\ast \times a & \xrightarrow{\delta'} & \ast \times a \\
\end{array} \]

for any object \( a \) where the bilinear \( \delta' \) corresponds to \( \delta' \), whereas Axiom 6.17 amounts to the commutation of the diagram in \( A \)

\[ \begin{array}{ccc}
t_1.(t_2.a) & \xrightarrow{id} & (t_1.t_2).a \\
t_1.(t_2,a) \downarrow{\varphi'} & & \downarrow{\varphi'} \\
t_1.(t_2 \times a) & \xrightarrow{\delta'} & (1_A, \delta') \\
\end{array} \]

for all objects \( t_1, t_2 \) in \( I \) and \( a \) in \( A \).
Axiom 6.16 for the pair \((1_A, \delta'^{-1})\) amounts the commutation for any object \(a\) in \(A\) of the diagram

\[
\begin{array}{c}
\Delta_{-1, a} \\
\downarrow \\
\star \times a \\
\end{array}
\]

\[
\begin{array}{ccc}
a & \overset{id}{\longrightarrow} & \star \times a \\
\downarrow & & \downarrow \\
\gamma_a & \downarrow & \Delta_{-1, a} \\
\end{array}
\]

which does commute since diagram 6.41 does.

Axiom 6.17 for the pair \((1_A, \delta^{-1})\) is the commutation for any objects \(t_1, t_2\) in \(\mathcal{I}\) and \(a\) in \(A\) of the diagram

\[
\begin{array}{ccc}
t_1 \times (t_2 \times a) & \overset{\beta_{t_1, t_2, a}}{\longrightarrow} & (t_1, t_2) \times a \\
\downarrow & & \downarrow \\
t_1 \times (t_2, a) & \overset{\Delta_{t_1, t_2, a}}{\longrightarrow} & t_1, (t_2, a) \overset{id}{\longrightarrow} (t_1, t_2).a
\end{array}
\]

which is equivalent to the commutation of the external diagram in the pasting

\[
\begin{array}{ccc}
t_1, (t_2, a) & \overset{id}{\longrightarrow} & (t_1, t_2).a \\
\downarrow & & \downarrow \\
t_1 \times (t_2, a) & \overset{\Delta_{t_1, t_2, a}}{\longrightarrow} & (t_1, t_2) \times a \\
\downarrow & & \downarrow \\
t_1 \times (t_2, a) & \overset{\Delta_{t_1, t_2, a}}{\longrightarrow} & (t_1, t_2) \times a \\
\downarrow & & \downarrow \\
t_1 \times (t_2, a) & \overset{\Delta_{t_1, t_2, a}}{\longrightarrow} & (t_1, t_2) \times a
\end{array}
\]

in which the left diagram commutes by naturality of \(\delta_{t_1} : v(t_1) \to \varphi(t_1) : \mathcal{I} \to A\) and the right diagram is diagram 6.42.

7 Appendix

This section contains various technical developments.

Section 2

Lemma 7.1 In any symmetric Picard category the diagram

\[
\begin{array}{ccc}
a^* \otimes b^* & \overset{1}{\longrightarrow} & (b \otimes a)^* \\
\downarrow & & \downarrow \\
b^* \otimes a^* & \overset{1}{\longrightarrow} & (a \otimes b)^*
\end{array}
\]

commutes for any objects \(a\) and \(b\).
PROOF: By definition the canonical arrow $a^* \otimes b^* \to (b \otimes a)^*$ is the only arrow $f$ making the diagram

\[
\begin{array}{c}
(a^* \otimes b^*) \otimes (b \otimes a) \\
\downarrow \cong \\
(a^* \otimes ((b^* \otimes b) \otimes a)) \\
\downarrow 1 \otimes (j \otimes 1) \\
(a^* \otimes (I \otimes a)) \\
\downarrow \cong \\
(a^* \otimes a)
\end{array}
\quad f \otimes 1
\begin{array}{c}
(b \otimes a)^* \otimes (b \otimes a) \\
\downarrow j \\
\end{array}
\]

commute (see [Lap83]-p.310). In the following pasting all diagrams commute,

\[
\begin{array}{c}
(a^* \otimes b^*) \otimes (b \otimes a) \\
\downarrow 1 \otimes s \\
(a^* \otimes (a \otimes b)) \\
\downarrow \cong \\
(a^* \otimes (I \otimes a)) \\
\downarrow 1 \otimes r \\
\end{array}
\begin{array}{c}
(b^* \otimes ((a^* \otimes a) \otimes b)) \\
\downarrow 1 \otimes (j \otimes 1) \\
(b^* \otimes (I \otimes b)) \\
\downarrow j \\
\end{array}
\]

and by the coherence theorem for symmetric monoidal categories the two left legs of the two diagrams above are equal.

\[\square\]

7.2 Proof of Lemma 2.2

PROOF: That the functor $\text{inv} : A \to A$ is monoidal result from points 7.3 that it is symmetric amounts to Lemma 7.1.

7.3 In any symmetric Picard category the diagram

\[
\begin{array}{c}
a^* \otimes (b^* \otimes c^*) \xrightarrow{\text{ass}} (a^* \otimes b^*) \otimes c^*
\end{array}
\begin{array}{c}
\downarrow 1 \otimes s \\
a^* \otimes (c^* \otimes b^*) \\
\downarrow s \otimes 1 \\
(b^* \otimes a^*) \otimes c^*
\end{array}
\begin{array}{c}
\downarrow 1 \otimes 1 \\
(a^* \otimes (b \otimes c)^*) \\
\downarrow 1 \\
(b \otimes c)^* \otimes a^*
\end{array}
\begin{array}{c}
\downarrow s \\
(c^* \otimes (a \otimes b)^*) \\
\downarrow 1 \\
((a \otimes b) \otimes c)^*
\end{array}
\]

commutes for any objects $a$, $b$ and $c$. 

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PROOF: By the naturality of $s$, the lemma is equivalent to the commutation of the external diagram in the pasting

$$
\begin{array}{c}
\xymatrix{
a^* \otimes (b^* \otimes c^*) \ar[r]^{ass} & (a^* \otimes b^*) \otimes c^* \\
1 \otimes s \ar[u] & s \otimes 1 \ar[u] \\
(a^* \otimes (c^* \otimes b^*)) \ar[r]^s & (b^* \otimes a^*) \otimes c^* \\
1 \otimes 1 \ar[u] & 1 \otimes 1 \ar[u] \\
(b \otimes c^*) \otimes a^* \ar[r]^{ass} & c^* \otimes (b \otimes a^*) \\
1 \ar[u] & 1 \ar[u] \\
(a \otimes (b \otimes c))^* \ar[r]^{ass} & ((a \otimes b) \otimes c)^*
\end{array}
$$

where the top diagram commutes according to the coherence for the symmetric monoidal structure and the bottom theorem commutes according to the coherence theorem for the group structure.

7.4 Proof of Lemma 2.3

PROOF: The collection $j$ is natural by definition of the functor $inv$. That it is monoidal amounts to the commutation of the diagram

$$
\begin{array}{c}
\xymatrix{I \ar[r]^{j_{a \otimes b}} & (a \otimes b)^* \otimes (a \otimes b) \\
I \otimes I \ar[r]_{j_{a \otimes b}} & (a^* \otimes a) \otimes (b^* \otimes b)
\end{array}
$$

for any objects $a$ and $b$ of $\mathcal{A}$, which holds by definition of the arrow $!: I : (a \otimes b)^* \to b^* \otimes a^*$ see [Lap83] p.310.

7.5 Definition of the isomorphism 2.4
The isomorphism 2.4 is defined pointwise in a as the only arrow in \( B \) making the diagram commute.

**Lemma 7.6** Given arrows \( A \xrightarrow{F} B \xrightarrow{G} C \) in SPC the diagram in \( C \)

\[
\begin{array}{ccc}
(FG(a))^* & \cong & F(G(a))^* \\
\cong & \cong & \\
\cong & \cong & \\
\end{array}
\]

where all the \( \cong \) are of type 2.4 is commutative.

**PROOF:** Consider the pasting of commutative diagrams below where all the \( \cong \) are of type 2.4

**Lemma 7.7** For any monoidal transformation \( \sigma : F \to G : \mathcal{A} \to \mathcal{B} \) where \( \mathcal{A} \) and \( \mathcal{B} \) are objects of SPC, for any object \( a \) of \( \mathcal{A} \), the diagram in \( \mathcal{B} \)

\[
\begin{array}{ccc}
(Fa)^* & \cong & (Ga)^* \\
\cong & \cong & \\
\cong & \cong & \\
\end{array}
\]

where the \( \cong \) denote isomorphisms of type 2.4 is commutative.
PROOF: Consider the pasting of diagrams in $\mathcal{B}$

\[ \begin{array}{ccc}
I & \xrightarrow{J \mathcal{F}_a} & (Fa)^{\bullet} \otimes Fa \\
F^0 & \xrightarrow{id} & \otimes 1 \\
F_{a^a} & \xrightarrow{\sigma_a \otimes \sigma_a} & (Ga)^{\bullet} \otimes Ga \\
G^0 & \xrightarrow{\sigma_a \otimes \sigma_a} & \otimes 1 \\
G_{a^a} & \xrightarrow{\sigma_a \otimes \sigma_a} & \otimes 1 \\
\end{array} \]

where the $\cong$ denote isomorphisms of type 2.4. All diagrams above commute apart from the one consisting of the four dotted arrows. Since all arrows are invertible this last diagram commutes. The result follows since tensoring with $Fa$ is an equivalence $\mathcal{B} \rightarrow \mathcal{B}$.

A few computational lemmas for $\mathcal{SPC}$.
We present here a couple of results not stated in [Sch08].

Lemma 7.8 The 2-cells in $\mathcal{SPC}$

\[ A \xrightarrow{L'} I \otimes A \xrightarrow{U \otimes 1} B \otimes A \xrightarrow{\tau} C \]

and

\[ A \xrightarrow{R_n(\tau)^*} [B, C] \xrightarrow{[U, 1]} [I, C] \xrightarrow{ev_*} C \]

are equal for any 2-cell $\tau : B \otimes A \rightarrow C$ and any arrow $U : I \rightarrow B$.

PROOF: The 2-cell $\tau \circ (U \otimes 1) \circ L'$ above rewrites successively

1. $A \xrightarrow{\eta^*} [I, I \otimes A] \xrightarrow{ev_*} I \otimes A \xrightarrow{U \otimes 1} B \otimes A \xrightarrow{\tau} C$
2. $A \xrightarrow{\eta^*} [I, I \otimes A] \xrightarrow{[1, U \otimes 1]} [I, B \otimes A] \xrightarrow{[1, \tau]} [I, C] \xrightarrow{ev_*} C$
3. $A \xrightarrow{\eta^*} [B, B \otimes A] \xrightarrow{[U, 1]} [I, B \otimes A] \xrightarrow{[1, \tau]} [I, C] \xrightarrow{ev_*} C$
4. $A \xrightarrow{\eta^*} [B, B \otimes A] \xrightarrow{[U, 1]} [I, C] \xrightarrow{ev_*} C$
5. $A \xrightarrow{(R_n(\tau))^*} [B, C] \xrightarrow{[U, 1]} [I, C] \xrightarrow{ev_*} C$.

Lemma 7.9 The 2-cells in $\mathcal{SPC}$

\[ A \xrightarrow{K'} A \otimes I \xrightarrow{1 \otimes U} A \otimes B \xrightarrow{\tau} C \]

and

\[ A \xrightarrow{R_n(\tau)} [B, C] \xrightarrow{[U, 1]} [I, C] \xrightarrow{ev_*} C \]

are equal for any 2-cell $\tau : A \otimes B \rightarrow C$ and any arrow $U : I \rightarrow B$. 

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PROOF: The 2-cell \( \tau \circ (1 \otimes U) \circ R' \) above rewrites successively

1. \[ A \xrightarrow{\eta} [I, A \otimes I] \xrightarrow{ev_*} A \otimes I \xrightarrow{1 \otimes U} A \otimes B \xrightarrow{\tau} C \]
2. \[ A \xrightarrow{\eta} [I, A \otimes I] \xrightarrow{[1, 1 \otimes U]} [I, I \otimes B] \xrightarrow{[1, \tau]} [I, C] \xrightarrow{ev_*} C \]
3. \[ A \xrightarrow{\eta} [B, A \otimes B] \xrightarrow{[U, 1]} [I, A \otimes B] \xrightarrow{[1, \tau]} [I, C] \xrightarrow{ev_*} C \]
4. \[ A \xrightarrow{\eta} [B, A \otimes B] \xrightarrow{[1, \tau]} [A, C] \xrightarrow{[U, 1]} [I, C] \xrightarrow{ev_*} C \]
5. \[ A \xrightarrow{Rn(\tau)} [B, C] \xrightarrow{[U, 1]} [I, C] \xrightarrow{ev_*} C \]

where in the above derivation arrows 1. and 2. are equal according to Corollary \( \text{(Sch08)-11.2} \). □

Remark 7.10 Any 2-cell \( \sigma : F \to G : I \to A \) with \( F \) strict in SPC is fully determined by its component in \( * \) since \( F \) being strict the component at \( F \) of counit of the adjunction \( \epsilon_F : v^* \circ ev_*(F) \to F \) is an identity and by naturality of \( \epsilon \) one has the commutation of

\[
v^*(F(*)) \xrightarrow{v^*(\sigma_*)} v^*(G(*))
\]

in SPC.

Lemma 7.11 For any strict arrow \( F : I \to A \) the diagram in SPC

\[
\begin{array}{ccc}
I & \xrightarrow{v} & [A, A] \\
\downarrow{F} & & \downarrow{[F, 1]} \\
A & \xrightarrow{ev_*} & [I, A]
\end{array}
\]

commutes.

PROOF: The arrow

\[
I \xrightarrow{v} [A, A] \xrightarrow{[F, 1]} [I, A]
\]

has dual

\[
I \xrightarrow{F} A \xrightarrow{v^*} [I, A].
\]

Since the arrow \( F \) is strict it is equal to \( v^* \circ ev_*(F) \) and the result follows then from Lemma \( \text{(Sch08)-11.9} \). □

Remark 7.12 Since the units of the adjunctions \( v^* - \Downarrow ev_* \) are identities in SPC, one has a bijection for any \( A \) and \( B \) between sets of 2-cells in SPC of the following kind

\[
\begin{array}{ccc}
F & \xrightarrow{v^*} & B \\
\downarrow{G} & & \downarrow{[I, B]} \\
A & \xrightarrow{G^*} & [I, B]
\end{array}
\]

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and

7.14

\[
\begin{array}{ccc}
A & \xrightarrow{F} & B \\
\downarrow{G} & & \downarrow{ev_*} \\
[I, A] & & \\
\end{array}
\]

sending any 2-cell \(\Xi\) of the type of the kind 7.13 to

\[
\begin{array}{ccc}
B & \xrightarrow{v^*} & \downarrow{ev_*} \\
\downarrow{F} & & \\
A & \xrightarrow{G} & [I, B] \\
\end{array}
\]

with inverse sending any 2-cell \(\Xi'\) of type of the kind 7.14 to

\[
\begin{array}{ccc}
A & \xrightarrow{F} & B \\
\downarrow{G} & & \downarrow{ev_*} \\
[I, A] & & \xrightarrow{id} A \\
\end{array}
\]

We shall describe for any arrow \(\varphi : A \otimes M \to M\) and \(U : I \to A\) of \(SPC\) some bijections between sets of 2-cells of the following kind:

7.15

\[
\begin{array}{ccc}
I M & \xrightarrow{U \otimes 1} & A M \\
\downarrow{L} & & \downarrow{\varphi} \\
M & & \\
\end{array}
\]

7.16

\[
\begin{array}{ccc}
I & \xrightarrow{U} & A \\
\downarrow{v} & & \downarrow{Rn(\varphi)} \\
[M, M] & & \\
\end{array}
\]

7.17

\[
\begin{array}{ccc}
M & \xrightarrow{(Rn(\varphi))^*} & [A, M] \\
\downarrow{v^*} & & \downarrow{[U, 1]} \\
[I, M] & & \\
\end{array}
\]

53
and

7.19

as follows.
- Since $L$ is the image by $En$ of $v : I \to [M, M]$ and the image by $Rn$ of

$$\begin{array}{c}
IM \\
\downarrow \\
A \text{ev}^\star \\
\downarrow \\
[M, M]
\end{array}$$

is

$$\begin{array}{c}
I \\
\downarrow \\
A \text{Rn}(\varphi) \\
\downarrow \\
[M, M]
\end{array}$$

the maps $Rn/En$ define the bijection (and its inverse) between sets of 2-cells $7.15$ and $7.16$.
- 2-cells $7.16$ and $7.17$ correspond by duality.
- The bijection between sets of 2-cells $7.17$ and $7.18$ sends any 2-cell $\Xi : v^* \to [j, 1] \circ (Rn(\varphi))^*$ to

$$\begin{array}{c}
M \\
\downarrow \\
[1, M]
\end{array}$$

which is as expected a 2-cell

$$id \to ev_* \circ [U, 1] \circ (Rn(\varphi))^*$$

according to the adjunction $2.21$. Its inverse sends any 2-cell $\Xi : 1 \to ev_* \circ [U, 1] \circ (Rn(\varphi))^*$ to the pasting

$$\begin{array}{c}
M \\
\downarrow \\
[1, M]
\end{array}$$

- Eventually 2-cells $7.18$ and $7.19$ are the same since their codomains arrows respectively $ev_* \circ [U, 1] \circ Rn(\varphi)^*$ and $\varphi \circ (U \otimes 1) \circ L'$ are equal according to Lemma $7.8$. 

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Lemma 7.20  The above bijection between sets of 2-cells \([7.15][7.19]\) sends any \(\Xi : L \to \phi \circ (U \otimes 1) : \mathcal{LM} \to \mathcal{M}\) to the pasting

![Diagram](image)

PROOF: According to Lemma \([7.8]\) the above 2-cell \(A \xrightarrow{L'} \mathcal{M} \xrightarrow{\Xi} \mathcal{A} \xrightarrow{\phi} \mathcal{M}\) is

\[ A \xrightarrow{Rn(\Xi)^{\vee}} [\mathcal{I}, \mathcal{A}] \xrightarrow{ev^{\vee}} \mathcal{A}. \]

Consider then the image of the above 2-cell by the bijection \([7.18] \to [7.17]\) it is

\[ A \xrightarrow{v^{\vee}} [\mathcal{I}, \mathcal{A}] \xrightarrow{id} [\mathcal{I}, \mathcal{A}] \xrightarrow{\epsilon} [\mathcal{I}, \mathcal{A}] \]

which is just \((Rn(\bar{\lambda}))^{\vee} : A \to [\mathcal{I}, \mathcal{A}]\) since \(A \xrightarrow{v^{\vee}} [\mathcal{I}, \mathcal{A}] \xrightarrow{\epsilon} \mathcal{A}\) is an identity 2-cell, which dual has image by \(En\) the 2-cell \(\bar{\lambda}\).

Easy computation also gives the following.

Remark 7.21  For any arrows \(F : A \to [\mathcal{B}, \mathcal{X}]\) and \(G : \mathcal{X} \to [\mathcal{C}, \mathcal{D}]\) of \(SPC\) the arrow

\[(A \otimes B) \otimes \mathcal{C} \xrightarrow{En(F) \otimes 1} \mathcal{X} \otimes \mathcal{C} \xrightarrow{En(G)} \mathcal{D}\]

has image by \(Rn\)

\[A \otimes B \xrightarrow{En(F)} \mathcal{X} \xrightarrow{G} [\mathcal{C}, \mathcal{D}]\]

and has image by \(Rn \circ Rn\) the arrow

\[A \xrightarrow{F} [\mathcal{B}, \mathcal{X}] \xrightarrow{[1, G]} [\mathcal{B}, [\mathcal{C}, \mathcal{D}]].\]

Sections 3 and 4.

7.22  Proof of Equality (I) in second pasting of Axiom \([3.15]\).

PROOF: To check the equality (I) of arrows, consider the derivation of equal composite arrows in \(SPC\) for any arrows \(d : A \to [\mathcal{A}, \mathcal{A}]\) and \(j : \mathcal{I} \to \mathcal{A}\).

1. \([A, \mathcal{B}] \xrightarrow{[A, -]} [\mathcal{A}, [\mathcal{A}, \mathcal{B}]] \xrightarrow{[d, 1]} [A, [A, B]] \xrightarrow{[j, 1]} [\mathcal{I}, [A, B]] \xrightarrow{ev_{\vee}} [A, B]
2. \([A, \mathcal{B}] \xrightarrow{[A, -]} [\mathcal{A}, [\mathcal{A}, \mathcal{B}]] \xrightarrow{[d, 1]} [A, [A, B]] \xrightarrow{ev_{\vee}(\ast)} [A, B]
3. \([A, \mathcal{B}] \xrightarrow{[A, -]} [\mathcal{A}, [\mathcal{A}, \mathcal{B}]] \xrightarrow{[d', 1]} [A, [A, B]] \xrightarrow{D} [A, [A, B]] \xrightarrow{ev_{\vee}(\ast)} [A, B]
The codomain of \( c^1_{f,y,z} \) rewrites
\[
\begin{align*}
&\mathcal{A}_{x,y} \xrightarrow{A(x,y)} \mathcal{A}_{x,y} \xrightarrow{A(x',y), A_{x',z}} \mathcal{A}_{x,y} \xrightarrow{A(x',y), A_{x',z}} [A_{x,y} \xrightarrow{[1,f]} [A_{x,y}, A_{x',z}]] \xrightarrow{[1,e_{v_{f,x}}]} [A_{x,y}, A_{x',z}] \\
&\mathcal{A}_{x,y} \xrightarrow{A(x,y)} \mathcal{A}_{x,y} \xrightarrow{A(x',y), A_{x',z}} \mathcal{A}_{x,y} \xrightarrow{A(x',y), A_{x',z}} [A_{x,y} \xrightarrow{[1,f]} [A_{x,y}, A_{x',z}]] \xrightarrow{[1,e_{v_{f,x}}]} [A_{x,y}, A_{x',z}]
\end{align*}
\]

Remark 7.24 For any arrows \( f : \mathcal{A} \to \mathcal{B} \) and \( \tilde{f} : \mathcal{I} \to [\mathcal{A}, \mathcal{B}] \) such that \( e_{v_{f}}(\tilde{f}) = f \) and any object \( D \) in \( SPC \), all the diagrams in the pasting below commute
\[
\begin{array}{cccc}
& [\mathcal{A}, \mathcal{B}] & [D, \mathcal{A}] & [D, \mathcal{B}] \\
& [f, \mathcal{B}] \xrightarrow{e_{v_{f}}} & [f, 1] & \xrightarrow{[f, 1]} [f, 1] \xrightarrow{[f, 1]} [f, 1]
\end{array}
\]
The top left one commutes according to Corollary [Sch08]-11.6 and the bottom right one does according to the 2-naturality of the collection of arrows \( q \).

**7.25 Definition of the 2-cell** \( c^2_{x,y'; y', z} \).

Observe that the image by \( \text{ev}_* : SPC(I, [A_{x,y'}, A_{x,y}]) \to SPC(A_{x,y'}, A_{x,y}) \) of

\[
\begin{array}{c}
I \\ {\xrightarrow{g}} \\
A_{y', y} \xrightarrow{A(z,-)} [A_{x,y'}, A_{x,y}]
\end{array}
\]

is equal to \( A(1, g) : A_{x,y'} \to A_{x,y} \). Therefore according to Remark [24] the domain of this 2-cell which is

\[
\begin{array}{c}
A_{y,z} \xrightarrow{A(x,-)} [A_{x,y}, A_{x,z}] \\
\xrightarrow{[A_{x,y'}, A_{x,y}]} [A_{x,y'}, A_{x,y}] \\
\xrightarrow{A(z,-)} [A_{x,y'}, A_{x,y'}] \\
\xrightarrow{g} [I, [A_{x,y'}, A_{x,z}]] \\
\xrightarrow{\text{ev}_*} [A_{x,y'}, A_{x,z}]
\end{array}
\]

is equal to

\[
A_{y,z} \xrightarrow{A(x,-)} [A_{x,y}, A_{x,z}] \xrightarrow{[A(1,g), 1]} [A_{x,y'}, A_{x,z}]
\]

The codomain of the 2-cell \( c^2_{x,y'; y', z} \) rewrites successively

1. \( A_{y,z} \xrightarrow{A(y',-)} [A_{y', y}, A_{y', z}] \xrightarrow{\{A(1, x), 1, 1\}} [A_{y', y}, A_{y', z}] \xrightarrow{[g, 1]} [I, [A_{x,y'}, A_{x,z}]] \xrightarrow{\text{ev}_*} [A_{x,y'}, A_{x,z}]
\)
2. \( A_{y,z} \xrightarrow{A(y',-)} [A_{y', y}, A_{y', z}] \xrightarrow{[g, 1]} [I, [A_{x,y'}, A_{x,z}]] \xrightarrow{[1, A(x,-)]} [I, [A_{x,y'}, A_{x,z}]] \xrightarrow{\text{ev}_*} [A_{x,y'}, A_{x,z}]
\)
3. \( A_{y,z} \xrightarrow{A(y',-)} [A_{y', y}, A_{y', z}] \xrightarrow{[g, 1]} [I, [A_{x,y'}, A_{x,z}]] \xrightarrow{\text{ev}_*} A_{y', z} \xrightarrow{A(z,-)} [A_{x,y'}, A_{x,z}]
\)
4. \( A_{y,z} \xrightarrow{A(y,1)} A_{y', z} \xrightarrow{A(x,-)} [A_{x,y'}, A_{x,z}]
\)

where in the above derivation arrows 2. and 3. are equal due to Lemma [Sch08]-11.2.

**7.26 Definition of the 2-cell** \( c^3_{x,y, h; z' \to z'} \).

The 2-cell

\[
\begin{array}{c}
I \\ {\xrightarrow{h}} \\
A_{z', z'} \xrightarrow{A(x,-)} [A_{x,z}, A_{x,z}] \xrightarrow{[A_{x,y}, A_{x,z}]} [A_{x,y}, A_{x,z}] \xrightarrow{[A(x,-), h]} [A_{y,z}, A_{x,y}, A_{x,z}]
\end{array}
\]

has image by \( \text{ev}_* \)

\[
A_{y,z} \xrightarrow{A(x,-)} [A_{x,y}, A_{x,z}] \xrightarrow{[1, A(1, h)]} [A_{x,y}, A_{x,z}]
\]

which is the domain of \( c^3_{x,y, h; z' \to z'} \).

The 2-cell

\[
\begin{array}{c}
I \\ {\xrightarrow{h}} \\
A_{z', z'} \xrightarrow{A(y,-)} [A_{y,z}, A_{y,z}] \xrightarrow{[A(1, x), -]} [A_{y,z}, A_{x,y}, A_{x,z}]
\end{array}
\]

has image by \( \text{ev}_* \)

\[
A_{y,z} \xrightarrow{A(1, h)} A_{y,z} \xrightarrow{A(x,-)} [A_{x,y}, A_{x,z}]
\]

which is the codomain of \( c^3_{x,y, h; z' \to z'} \).

**7.27 Proof of the equivalence of Axioms [4.4][3.14]**
The first of the 2-cell of Axiom 4.4 decomposes as the product \( \Xi_2 \circ \Xi_1 \) where \( \Xi_1 \) is

\[
((\{A_{x,u}A_{z,v}\}A_{y,z})A_{x,u}) \overset{\alpha'}{\longrightarrow} (\{A_{x,u}A_{z,v}\}A_{y,z})A_{x,u} \quad \overset{1 \otimes c_{x,y,z}}{\longrightarrow} (\{A_{x,u}A_{z,v}\}A_{y,z})A_{x,u}
\]

and \( \Xi_2 \) is

\[
((\{A_{x,u}A_{z,v}\}A_{y,z})A_{x,u}) \overset{(c_{x,u}u \otimes 1)^\dagger}{\longrightarrow} (\{A_{x,u}A_{z,v}\}A_{y,z})A_{x,u}
\]

The 2-cell \( \Xi_1 \) has a strict domain which image by \( Rn \) is strict since the arrows \( A', Rn(A') \) and \( c \) are strict. According to Lemma [Schütz, 19.6], the 2-cell \( \Xi_1 \) has image by \( Rn \circ Rn \)

\[
A_{x,u} \overset{Rn(\alpha'_{x,y,z},u)}{\longrightarrow} [A_{z,v}, A_{x,u}, \{A_{x,y}, A_{z,v}\}] \overset{Rn(\alpha'_{x,y,z},u)}{\longrightarrow} [A_{x,y}, A_{z,v}, A_{x,u}].
\]

This 2-cell has again strict domain and its image by \( Rn \) is the 2-cell

\[
A_{x,u} \overset{\alpha'}{\longrightarrow} [A_{z,v}, A_{y,z}, A_{x,u}] \overset{1 \otimes c_{x,y,z}}{\longrightarrow} [A_{z,v}, [A_{x,y}, A_{z,v}, A_{x,u}]] \overset{1 \otimes \alpha'_{x,y,z}}{\longrightarrow} [A_{z,v}, [A_{x,y}, [A_{z,v}, A_{x,u}]]].
\]

The image by \( Rn \) of \( \Xi_3 \) is

\[
A_{x,u} \overset{\alpha'_{x,y,z}}{\longrightarrow} A_{x,u} \overset{\alpha'_{x,y,z}}{\longrightarrow} A_{x,u} \overset{1 \otimes c_{x,y,z}}{\longrightarrow} [A_{x,y}, A_{z,v}, A_{x,u}].
\]

The second 2-cell from Axiom 4.4 decomposes as \( \Xi_3 \circ \Xi_4 \circ \Xi_3 \) where \( \Xi_3 \) is

\[
((\{A_{x,u}A_{z,v}\}A_{y,z})A_{x,u}) \overset{\alpha'_{1 \otimes c_{x,y,z}}}{\longrightarrow} (\{A_{x,u}A_{z,v}\}A_{y,z})A_{x,u} \quad \overset{1 \otimes c_{x,y,z}}{\longrightarrow} (\{A_{x,u}A_{z,v}\}A_{y,z})A_{x,u}
\]

\( \Xi_4 \) is

\[
((\{A_{x,u}A_{z,v}\}A_{y,z})A_{x,u}) \overset{\alpha'_{1 \otimes c_{x,y,z}}}{\longrightarrow} (\{A_{x,u}A_{z,v}\}A_{y,z})A_{x,u} \quad \overset{\alpha'_{1 \otimes c_{x,y,z}}}{\longrightarrow} (\{A_{x,u}A_{z,v}\}A_{y,z})A_{x,u}
\]

and \( \Xi_5 \) is

\[
((\{A_{x,u}A_{z,v}\}A_{y,z})A_{x,u}) \overset{\alpha'_{1 \otimes c_{x,y,z}}}{\longrightarrow} (\{A_{x,u}A_{z,v}\}A_{y,z})A_{x,u}
\]

The image by \( Rn \circ Rn \circ Rn \) of the 2-cell \( \Xi_3 \) is

\[
A_{x,u} \overset{Rn(\alpha'_{x,y,z})}{\longrightarrow} [A_{x,y}, A_{z,v}, A_{x,u}] \overset{1 \otimes \alpha'_{x,y,z}}{\longrightarrow} [A_{x,y}, [A_{z,v}, A_{x,u}]] \overset{1 \otimes \alpha'_{x,y,z}}{\longrightarrow} [A_{x,y}, [A_{z,v}, [A_{x,u}]]]
\]

which rewrites

1. \( A_{x,u} \overset{Rn(\alpha'_{x,y,z})}{\longrightarrow} [A_{x,y}, A_{z,v}, A_{x,u}] \quad \overset{1 \otimes \alpha'_{x,y,z}}{\longrightarrow} [A_{x,y}, [A_{z,v}, A_{x,u}]] \)

2. \( A_{x,u} \overset{Rn(\alpha'_{x,y,z})}{\longrightarrow} [A_{x,y}, A_{z,v}, A_{x,u}] \quad \overset{1 \otimes \alpha'_{x,y,z}}{\longrightarrow} [A_{x,y}, [A_{z,v}, A_{x,u}]] \)

3. \( A_{x,u} \overset{Rn(\alpha'_{x,y,z})}{\longrightarrow} [A_{x,y}, A_{z,v}, A_{x,u}] \quad \overset{1 \otimes \alpha'_{x,y,z}}{\longrightarrow} [A_{x,y}, [A_{z,v}, A_{x,u}]] \)

4. \( A_{x,u} \overset{Rn(\alpha'_{x,y,z})}{\longrightarrow} [A_{x,y}, A_{z,v}, A_{x,u}] \quad \overset{1 \otimes \alpha'_{x,y,z}}{\longrightarrow} [A_{x,y}, [A_{z,v}, A_{x,u}]] \)

5. \( A_{x,u} \overset{Rn(\alpha'_{x,y,z})}{\longrightarrow} [A_{x,y}, A_{z,v}, A_{x,u}] \quad \overset{1 \otimes \alpha'_{x,y,z}}{\longrightarrow} [A_{x,y}, [A_{z,v}, A_{x,u}]] \)

6. \( A_{x,u} \overset{Rn(\alpha'_{x,y,z})}{\longrightarrow} [A_{x,y}, A_{z,v}, A_{x,u}] \quad \overset{1 \otimes \alpha'_{x,y,z}}{\longrightarrow} [A_{x,y}, [A_{z,v}, A_{x,u}]] \)

7. \( A_{x,u} \overset{Rn(\alpha'_{x,y,z})}{\longrightarrow} [A_{x,y}, A_{z,v}, A_{x,u}] \quad \overset{1 \otimes \alpha'_{x,y,z}}{\longrightarrow} [A_{x,y}, [A_{z,v}, A_{x,u}]] \)

8. \( A_{x,u} \overset{Rn(\alpha'_{x,y,z})}{\longrightarrow} [A_{x,y}, A_{z,v}, A_{x,u}] \quad \overset{1 \otimes \alpha'_{x,y,z}}{\longrightarrow} [A_{x,y}, [A_{z,v}, A_{x,u}]] \)

9. \( A_{x,u} \overset{Rn(\alpha'_{x,y,z})}{\longrightarrow} [A_{x,y}, A_{z,v}, A_{x,u}] \quad \overset{1 \otimes \alpha'_{x,y,z}}{\longrightarrow} [A_{x,y}, [A_{z,v}, A_{x,u}]] \)

10. \( A_{x,u} \overset{Rn(\alpha'_{x,y,z})}{\longrightarrow} [A_{x,y}, A_{z,v}, A_{x,u}] \quad \overset{1 \otimes \alpha'_{x,y,z}}{\longrightarrow} [A_{x,y}, [A_{z,v}, A_{x,u}]] \)

This rewrites
The image by $R_n$ of the 2-cell $\Xi_4$ is

$$
(A_{t,u}A_{z,t})_{A_{y,z}} \xrightarrow{\alpha_{y,z,t}} A_{t,u}A_{z,t}A_{y,z} \xrightarrow{\eta} A_{t,u}A_{z,u} \xrightarrow{Rn} [A_{x,y}, A_{x,u}]
$$

which according to Lemma [Sch08]–19.6 as image by $R_n \circ R_n$ the 2-cell

$$
A_{t,u} \xrightarrow{\alpha'_{y,z,t,u}} [A_{y,t}, [A_{x,y}, A_{x,u}]] \xrightarrow{\alpha_{y,z,t}} [[A_{y,z}, A_{y,t}], [A_{y,z}, [A_{x,y}, A_{x,u}]]] \xrightarrow{[A(y,-), 1]} [A_{z,t}, [A_{y,z}, [A_{x,y}, A_{x,u}]]].
$$

The image by $R_n \circ R_n \circ R_n$ of the 2-cell $\Xi_5$ is

$$
A_{t,u} \xrightarrow{\alpha'_{y,z,t,u}} [A_{z,t}, [A_{y,z}, A_{y,u}]] \xrightarrow{[A(-,-), 1]} [A_{z,t}, [A_{y,z}, [A_{x,y}, A_{x,u}]]]
$$

7.28 Proof of the equivalence of Axioms 4.3 and 5.1

PROOF: It is easy to check that the image by $R_n$ of the 2-cell $\Xi_1$ of Axiom 4.3 is $A(x,-)_{y,z} \ast \rho'_{y,z}$ the first 2-cell of the Axiom 5.13

The image by $R_n$ of the 2-cell $\Xi_2$ is

$$
A_{y,z} \xrightarrow{A(x,-)} [A_{x,y}, A_{x,z}] \xrightarrow{id} [A_{x,y}, A_{x,z}] \xrightarrow{[\alpha_{x,z}, 1]} [A_{x,y}, A_{x,z}] \xrightarrow{[\lambda y, 1]} [A_{x,y}, A_{x,z}] \xrightarrow{[\mu y, 1], 1]} [[A_{y,z}, A_{y,x}]], [A_{x,z}]]
$$

The 2-cell $\Xi_3$, namely

$$
A_{y,z} \otimes A_{x,y} \xrightarrow{R' \otimes 1} (A_{y,z} \otimes T) \otimes A_{x,y} \xrightarrow{(1 \otimes \eta_x) \otimes 1} A_{y,z} \otimes A_{y,y} \xrightarrow{\alpha_{y,y,z}} A_{x,z}
$$

has image by $R_n$ the 2-cell

$$
A_{y,z} \xrightarrow{R'} A_{y,z} \xrightarrow{1 \otimes \eta_y} A_{y,z} \xrightarrow{Rn(\alpha_{y,y,z})} [A_{x,y}, A_{x,z}]
$$

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which is according to Lemma 7.9

\[ A_{y,z} \xrightarrow{\alpha_{y,z}} [A_{y,y}, [A_{z,y}, A_{x,z}]] \xrightarrow{[u_y,1]} [I, [A_{x,y}, A_{x,z}]] \xrightarrow{[v_x,1]} [A_{x,y}, A_{x,z}] \]

7.29 Equivalence of Axioms 3.18 and 7.18

First let us remark that the two 2-cells of Axiom 3.18 have the same domain. This results from the following sequence of equal arrows.

1. \([\mathcal{G}_{F_x,F_y} \otimes [\mathcal{G}_{F_x,F_y} \otimes A_{x,y}]] \xrightarrow{[\mathcal{G}_{F_x,F_y}, \mathcal{G}_{F_x,F_y} \otimes A_{x,y}]} [\mathcal{G}_{F_x,F_y}, [A_{x,y}, A_{x,z}]] \xrightarrow{[u_y,1]} [I, [A_{x,y}, A_{x,z}]] \xrightarrow{[v_x,1]} [A_{x,y}, A_{x,z}] \]

2. \([\mathcal{G}_{F_x,F_y} \otimes [\mathcal{G}_{F_x,F_y} \otimes A_{x,y}]] \xrightarrow{[\mathcal{G}_{F_x,F_y}, \mathcal{G}_{F_x,F_y} \otimes A_{x,y}]} [\mathcal{G}_{F_x,F_y}, [A_{x,y}, A_{x,z}]] \xrightarrow{[u_y,1]} [I, [A_{x,y}, A_{x,z}]] \xrightarrow{[v_x,1]} [A_{x,y}, A_{x,z}] \]

3. \([\mathcal{G}_{F_x,F_y} \otimes [\mathcal{G}_{F_x,F_y} \otimes A_{x,y}]] \xrightarrow{[\mathcal{G}_{F_x,F_y}, \mathcal{G}_{F_x,F_y} \otimes A_{x,y}]} [\mathcal{G}_{F_x,F_y}, [A_{x,y}, A_{x,z}]] \xrightarrow{[u_y,1]} [I, [A_{x,y}, A_{x,z}]] \xrightarrow{[v_x,1]} [A_{x,y}, A_{x,z}] \]

4. \([\mathcal{G}_{F_x,F_y} \otimes [\mathcal{G}_{F_x,F_y} \otimes A_{x,y}]] \xrightarrow{[\mathcal{G}_{F_x,F_y}, \mathcal{G}_{F_x,F_y} \otimes A_{x,y}]} [\mathcal{G}_{F_x,F_y}, [A_{x,y}, A_{x,z}]] \xrightarrow{[u_y,1]} [I, [A_{x,y}, A_{x,z}]] \xrightarrow{[v_x,1]} [A_{x,y}, A_{x,z}] \]

In the above derivation arrows 3. and 4. are equal according to Lemma [Sch08]-11.

The 2-cell

\((A_{x,z} \otimes A_{y,z}) \otimes A_{x,y} \xrightarrow{\alpha_{x,z} \otimes \alpha_{x,y}} [B_{F_y,F_z} \otimes B_{F_y,F_z}] \otimes A_{x,y} \xrightarrow{1 \otimes F_{x,y}} (B_{F_y,F_z} \otimes B_{F_y,F_z}) \otimes B_{F_y,F_z} \xrightarrow{\alpha_{y,p} \otimes F_{x,y}} B_{F_x,F_t} \)

has a strict domain and image by \(Rn\)

\(A_{x,t} \otimes A_{y,z} \xrightarrow{F_{x,t} \otimes F_{y,z}} B_{F_x,F_t} \otimes B_{F_y,F_z} \xrightarrow{Rn(\alpha_{x,y} \otimes F_{x,y})} B_{F_x,F_t} \otimes B_{F_y,F_z} \xrightarrow{[F_{y,z},1]} [A_{x,y}, B_{F_x,F_t}] \)

which has a strict domain and image by \(Rn\)

\(A_{x,t} \xrightarrow{F_{x,t}} B_{F_x,F_t} \xrightarrow{\alpha_{x,y} \otimes \alpha_{x,z}} [B_{F_y,F_z} \otimes B_{F_y,F_z}] \xrightarrow{[u_y,1]} [A_{x,y}, B_{F_x,F_t}] \xrightarrow{[v_x,1]} [1, [F_{y,z},1]] \xrightarrow{[A_{y,z}, [A_{x,y}, B_{F_x,F_t}]]} [A_{y,z}, [A_{x,y}, B_{F_x,F_t}]] \]

The 2-cell

\((A_{x,t} \otimes A_{y,z}) \otimes A_{x,y} \xrightarrow{F_{x,t} \otimes F_{y,z}} B_{F_y,F_t} \otimes B_{F_y,F_z} \xrightarrow{1 \otimes F_{x,y}} B_{F_y,F_t} \otimes B_{F_y,F_z} \xrightarrow{e} B_{F_x,F_t} \)

has image by \(Rn\) that rewrites

\(A_{x,t} \otimes A_{y,z} \xrightarrow{F_{x,t} \otimes F_{y,z}} B_{F_y,F_t} \xrightarrow{Rn(e \otimes (1 \otimes F_{x,y}))} [A_{y,z}, B_{F_x,F_t}] \)

\(A_{x,t} \otimes A_{y,z} \xrightarrow{F_{x,t} \otimes F_{y,z}} B_{F_y,F_t} \xrightarrow{[F_{x,y},1]} [B_{F_x,F_t}, B_{F_x,F_t}] \xrightarrow{[F_{x,y},1]} [B_{F_x,F_t}, B_{F_x,F_t}] \)

The image by \(Rn\) of this last arrow is

\(A_{x,t} \xrightarrow{F_{x,t} \otimes F_{y,z}} [A_{y,z}, B_{F_y,F_t}] \xrightarrow{[1, [F_{x,y},1]]} [A_{y,z}, B_{F_y,F_t}] \xrightarrow{[1, [F_{x,y},1]]} [A_{y,z}, B_{F_y,F_t}] \)

The 2-cell

\((A_{x,t} \otimes A_{y,z}) \otimes A_{x,y} \xrightarrow{c \otimes 1} A_{y,t} \otimes A_{x,y} \xrightarrow{F_{x,t},1} B_{F_x,F_t} \)

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has image by \( R_n \) the 2-cell

\[
A_{z,t} \otimes A_{y,z} \xrightarrow{c} A_{y,t} \xrightarrow{F^2} [A_{x,y}, B_{F_x,F_t}]
\]

which has image by \( R_n \)

\[
A_{z,t} \xrightarrow{A'(y,-)} [A_{y,z}, A_{y,t}] \xrightarrow{[1, F^2]} [A_{y,z}, [A_{x,y}, B_{F_x,F_t}]].
\]

The 2-cell

\[
A_{z,t} \otimes B_{F_x,F_t} \xrightarrow{F_{z,t} \otimes 1} B_{F_x,F_t} \otimes B_{F_x,F_t} \xrightarrow{c_{F_x,F_t}} B_{F_x,F_t}
\]

has image by \( R_n \) the 2-cell

\[
A_{z,t} \xrightarrow{F_{z,t}} B_{F_x,F_t} \xrightarrow{B(F_x,-)} [B_{F_x,F_t}, B_{F_x,F_t}]
\]

therefore according to Lemma 7.21 the 2-cell

\[
(A_{z,t} \otimes A_{y,z}) \otimes A_{x,y} \xrightarrow{A' \otimes A_{t} \otimes (A_{y,z} \otimes A_{x,y})^{1 \otimes c_{F_x,F_y}}} A_{z,t} \otimes B_{F_x,F_t} \xrightarrow{F_{z,t} \otimes 1} B_{F_x,F_t} \otimes B_{F_x,F_t} \xrightarrow{c_{F_x,F_t}} B_{F_x,F_t}
\]

has image by \( R_n \circ R_n \) the 2-cell

\[
A_{z,t} \xrightarrow{F_{z,t}} B_{F_x,F_t} \xrightarrow{B(F_x,-)} [B_{F_x,F_t}, B_{F_x,F_t}] \xrightarrow{[A_{x,y}, B_{F_x,F_t}]} [A_{x,y}, [A_{x,y}, B_{F_x,F_t}]].
\]

According to Lemma 7.21 the 2-cell

\[
(A_{z,t} \otimes A_{y,z}) \otimes A_{x,y} \xrightarrow{A' \otimes (A_{y,z} \otimes A_{x,y})^{1 \otimes c_{F_x,F_y}}} A_{z,t} \otimes A_{x,z} \xrightarrow{F^2_{z,x},t} B_{F_x,F_t}
\]

has image by \( R_n \circ R_n \) the 2-cell

\[
A_{z,t} \xrightarrow{F^2_{z,x},t} [A_{x,z}, B_{F_x,F_t}] \xrightarrow{[A_{x,y}, -]} [[A_{x,y}, A_{x,z}], [A_{x,y}, B_{F_x,F_t}]] \xrightarrow{[A(F_{z,x}, -), 1]} [A_{y,z}, [A_{x,y}, B_{F_x,F_t}]].
\]

**7.30 Equivalence of Axioms 4.8 and 3.19**

**PROOF:** According to Lemma 7.9 the arrow

\[
A_{x,y} \xrightarrow{R'} A_{x,y} \otimes I \xrightarrow{1 \otimes u_x} A_{x,y} \otimes A_{x,x} \xrightarrow{F^2_{x,y}} A_{x,y}
\]

is equal to

\[
A_{x,y} \xrightarrow{F^2_{x,y}} [A_{x,x}, B_{F_x,F_y}] \xrightarrow{[u_x, 1]} [I, B_{F_x,F_y}] \xrightarrow{c_{u_x}} B_{F_x,F_y}.
\]

Since the image by \( R_n \) of

\[
A_{x,y} \otimes B_{F_x,F_x} \xrightarrow{F_{x,y} \otimes 1} B_{F_x,F_y} \otimes B_{F_x,F_x} \xrightarrow{c} B_{F_x,F_y}
\]

is

\[
A_{x,y} \xrightarrow{F_{x,y}} B_{F_x,F_y} \xrightarrow{B(F_x,-)} [B_{F_x,F_x}, B_{F_x,F_y}],
\]

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the arrow
\[ A_{x,y} \xrightarrow{F_x} A_{x,y} \otimes I \xrightarrow{F_y \otimes 1} A_{x,y} \otimes B_{F_x,F_x} \xrightarrow{c} B_{F_x,F_y} \]
is equal to
\[ A_{x,y} \xrightarrow{F_x} B_{F_x,F_x} \xrightarrow{B(F_x,-)} [B_{F_x,F_x},B_{F_x,F_y}] \xrightarrow{\{F_0,1\}} [I,B_{F_x,F_y}] \xrightarrow{ev_x} B_{F_x,F_y} \]
according to Lemma 7.9.

7.31 Equivalence of Axioms 4.9 and 3.20.

PROOF: The arrow
\[ B_{F_y,F_y} \otimes A_{x,y} \xrightarrow{1 \otimes F_y} B_{F_y,F_y} \otimes B_{F_x,F_y} \xrightarrow{c} B_{F_x,F_y} \]
has image by \( R_n \)
\[ B_{F_y,F_y} \xrightarrow{B(F_y,-)} [B_{F_x,F_y},B_{F_x,F_y}] \xrightarrow{\{F_0,1\}} [A_{x,y},B_{F_x,F_y}] \]
which has dual
\[ A_{x,y} \xrightarrow{F_x} B_{F_x,F_y} \xrightarrow{B(-,F_y)} [B_{F_y,F_y},B_{F_x,F_y}] \]
Therefore according to Lemma 7.8, the 2-cell
\[ A_{x,y} \xrightarrow{L'} I \otimes A_{x,y} \xrightarrow{F_0 \otimes 1} A_{x,y} \otimes A_{x,y} \xrightarrow{1 \otimes F_y} A_{x,y} \otimes B_{F_x,F_y} \xrightarrow{c} B_{F_x,F_y} \]
is equal to
\[ A_{x,y} \xrightarrow{F_x} B_{F_x,F_y} \xrightarrow{B(-,F_y)} [B_{F_y,F_y},B_{F_x,F_y}] \xrightarrow{\{F_0,1\}} [I,B_{F_x,F_y}] \xrightarrow{ev_x} B_{F_x,F_y} \]
According to Lemma 7.8 the 2-cell
\[ A_{x,y} \xrightarrow{L'} I \otimes A_{x,y} \xrightarrow{u_y \otimes 1} A_{y,y} \otimes A_{x,y} \xrightarrow{F_0 \otimes 1} A_{x,y} \otimes B_{F_x,F_y} \xrightarrow{c} B_{F_x,F_y} \]
is equal to
\[ A_{x,y} \xrightarrow{(F_0^2 \circ \sigma_{a,b})^*} [A_{y,y},A_{x,y}] \xrightarrow{[a_y,1]} [I,A_{x,y}] \xrightarrow{ev} A_{x,y} \]

Section 5.

We will need the following characterization of bilinear natural transformations.

Remark 7.32 For any symmetric monoidal functors \( F, G : A \rightarrow [B, C] \) with respective underlying functors \( F', G' : A \times B \rightarrow C \). Any 2-cell \( \sigma : F \rightarrow G : A \rightarrow [B, C] \) in SPC corresponds to a collection of arrows \( \sigma_{a,b} : F'(a,b) \rightarrow G'(a,b) \) in \( C \), natural in \( a \) and \( b \) and that satisfies the following two conditions:
- For all objects \( a, a', b \) and \( b' \) of \( A \) the following diagram commutes
\[ \begin{array}{ccc}
F'(a,b) + F'(a,b') & \xrightarrow{F_x \circ \sigma_{a,b} + \sigma_{a,b'}} & F'(a, b + b') \\
\downarrow \sigma_{a,b} + \sigma_{a,b'} & & \downarrow \sigma_{a,b + b'} \\
F'(a',b) + F'(a',b') & \xrightarrow{F_x \circ \sigma_{a,b} + \sigma_{a,b'}} & F'(a', b + b')
\end{array} \]
- For all objects \(a, a', b\) and \(b'\) of \(A\) the following diagram commutes

\[
\begin{array}{c}
F'(a, b) + F'(a', b) \\
\downarrow \sigma_{a,b} + \sigma_{a',b} \quad \text{and} \quad \downarrow \sigma_{a+a',b} \\
F'(a, b') + F'(a', b')
\end{array}
\]

\[
F'(a, b) + F'(a', b) \xrightarrow{F^2} F'(a + a', b) \xrightarrow{F'(a + a', b)} F'(a, b')
\]

7.33 Proof that 2-rings in the sense of 5.1 are exactly one point SPC-categories.

PROOF: Let us start with a 2-ring \(A\) as defined in 5.1. Its corresponds to a one-point SPC-category as follows. Since monoidal functors \(I \to A\) are in one-to-one correspondence with objects of \(A\), the object \(1\) correspond to a strict arrow \(u : I \to A\) in SPC. That the multiplication ",", which is already a functor \(A \times A \to A\), defines an arrow \(\varphi' : A \to [A, A]\) in SPC corresponds to the existence of the natural arrows \(\omega_{a,b}\) and \(\tau_{a,a'}\) and the commutation of Diagrams 5.2, 5.3, 5.4, 5.5 and 5.6. Precisely for any object \(a\) of \(A\), the multiplication on the left by \(a\) defines a functor \(a.- : A \to A\). That this one is monoidal corresponds to the existence of the arrows \(\omega_{a,b}\) natural in \(b\) and \(b'\) and such that Diagrams 5.2 commute for all \(b, b'\). That the monoidal \(a.-\) is symmetric corresponds to the commutation of 5.4 for all \(b, b'\). That for any arrow \(f : a \to a'\), the natural transformation \(f.- : a.- \to a'.- : A \to A\) is monoidal for the structures described previously on \(a.-\) and \(a'.-\) corresponds to the naturality in the argument \(a\) of the maps \(\omega_{a,b}\).

The assignments \(a \mapsto a.-\) and \((f : a \to a') \mapsto (f.- : a.- \to a'.-)\) define therefore a functor say \(\varphi' : A \to SPC(A, A)\). The existence of a natural transformation \((a.-) + (a'.-) \to (a+a').-\) corresponds to the existence of arrows \(\tau_{a,a'}\) natural in \(b\). This transformation is monoidal since Diagrams 5.6 commute. Eventually that the collection of these arrows for all \(a\) and \(a'\) defines a symmetric monoidal structure on the above functor \(\varphi'\) results from the the naturality of the collection in the arguments \(a\) and \(a'\) and the commutation Diagrams 5.3 and 5.5.

According to Remark 7.32 a 2-cell \(\alpha'\) in SPC as in 5.11 corresponds to a natural collection of arrows \(\alpha_{a,b,c} : a.(b,c) \to (a,b).c\) in \(A\) such that Diagrams 5.7, 5.8 and 5.9 commute. A 2-cell \(\rho' : 1 \to ev \circ [u, 1] \circ \varphi' : A \to A\) in SPC corresponds to a natural collection of arrows \(\rho_{a} : a.1 \to a\) such that Diagrams 5.10 commute. Similarly a 2-cell \(\lambda' : 1 \to ev \circ [u, 1] \circ \varphi' : A \to A\) amounts to a natural collection of arrows \(\lambda_{a} : 1.a \to a\) such that Diagrams 5.11 commute.

Then the coherence conditions 8.14 and 8.15 for \(\alpha', \rho'\) and \(\lambda'\) above are equivalent to the coherence conditions for the associativity and unit laws of the monoidal category \((A, \cdot, I, \alpha, \rho, \lambda)\). □

7.34 Proof that 2-ring morphisms in the sense of 5.12 are exactly SPC-functors.

PROOF: According to the two observations below the result becomes clear after inspection of Axioms 3.18, 3.19 and 3.20.

Observe first that according to Remark 7.32 a 2-cell
in \( SPC \) is the same thing than a collection of arrows \( \Xi_{a,b} : HA.Hb \to H(a + b) \) natural in \( a \) and \( b \) and satisfying the conditions that for any objects \( a, b, c \) of \( A \), the two diagrams in \( B \) below commute

\[
\begin{align*}
H(a).H(b) + H(a).H(c) & \xrightarrow{H(a)} H(a).(H(b) + H(c)) & H(a).H(b + c) \\
H(a) \Xi_{a,b} + H(a) \Xi_{a,c} & \xrightarrow{H(a)} H(a).H(b + a + c)
\end{align*}
\]

\[
\begin{align*}
H(a).H(c) + H(b).H(c) & \xrightarrow{H(c)} H(a + b).H(c) & H(a + b).H(c) \\
H(a + b) \Xi_{a,b,c} + H(a + b) \Xi_{c,b} & \xrightarrow{H(c)} H((a + b).c)
\end{align*}
\]

where we write the \( \varphi' \) as products. Also according to Remark 7.10 in Appendix any 2-cell

\[
\begin{array}{c}
\text{in } SPC \\
\text{is fully determined by its component at the generator } \star \text{ which is an arrow } 1_B \to H(1_A) \\
\text{if } 1_A \text{ and } 1_B \text{ denote the images } u(\star), \text{ and conversely every such arrow corresponds to a 2-cell as above in this way.}
\end{array}
\]

7.37 Proof of Proposition 5.17.

PROOF: That there are identity 2-cells \( 3.31 \) amounts to the fact that the diagram in \( SPC \)

\[
\begin{array}{c}
\text{commutes for any objects } A, B, C \text{ and } D. \text{ Such a diagram involves only strict arrows in } SPC \text{ and}
\end{array}
\]

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its underlying diagram in $\textbf{Cat}$ is

\[
\begin{tikzcd}
& SPC(C, D) \\
SPC([B, C], [B, D]) \\
SPC([A, C], [A, D]) \\
SPC([B, C], [[A, B], [A, D]]) & SPC([A, C], [A, D]) \\
SPC([1, [A, -]]) & SPC([A, -], 1) \\
SPC([B, C], [[A, B], [A, D]]) \\
\end{tikzcd}
\]

which commutes according to Lemma [Sch08]-9.10.

One has identity 2-cells $\rho'$ since for any objects $A$ and $B$ in $SPC$ the composite
\[
[A, B] \xrightarrow{[A, -]} [[A, A], [A, B]] \xrightarrow{[v, 1]} [I, [A, B]] \xrightarrow{ev_*} [A, B]
\]
which is strict and has underlying functor that is an identity, hence is an identity in $SPC$.

One has identity 2-cells $\lambda'$ since the composite
\[
[A, B] \xrightarrow{[-, B]} [[B, B], [A, B]] \xrightarrow{[v, 1]} [I, [A, B]] \xrightarrow{ev_*} [A, B]
\]
is the identity at $[A, B]$ as shown below. The arrow $[A, B] \xrightarrow{[A, -]} [[A, B], [B, B]] \xrightarrow{[v, 1]} [I, [A, B]]$ is $v^*$ since it has dual
\[
I \xrightarrow{v} [B, B] \xrightarrow{[A, -]} [[A, B], [A, B]]
\]
that is $v$ according to Lemma [Sch08]-18.5. One concludes since $ev_* \circ v^* = 1$.

7.38 Proof of Proposition 5.19

PROOF: That one has an identity 2-cell $\rho$ amounts to the commutation of the diagram

\[
\begin{tikzcd}
& I \\
[I, I] \\
[I, I] \\
[[I, I], [I, I]] & [I, [I, I]] \\
[[I, I], [I, I]] \\
\end{tikzcd}
\]

This diagram commutes since the arrow
\[
I \xrightarrow{v} [I, I] \xrightarrow{[I, -]} [[I, I], [I, I]] \xrightarrow{[v, 1]} [I, [I, I]]
\]
is
\[
I \xrightarrow{v} [[I, I], [I, I]] \xrightarrow{[v, 1]} [I, [I, I]]
\]
according to Lemma [Sch08]-18.5, which is equal to
\[
I \xrightarrow{v} [I, I] \xrightarrow{[1, v]} [I, [I, I]].
\]
according Lemma [Sch08]-18.8.

One has an identity 2-cell $\rho'$ since the composite $\mathcal{I} \xrightarrow{v} \mathcal{I} \xrightarrow{ev} \mathcal{I}$ is the identity.

One has also an identity 2-cell $\lambda'$ since the composite $\mathcal{I} \xrightarrow{v} \mathcal{I} \xrightarrow{ev} \mathcal{I}$ is an identity according to Lemma [Sch08]-18.4.

7.39 Proof of Lemma 5.18.

PROOF: According to Lemma [Sch08]-11.4, the diagram in $\text{SPC} \left[ \mathcal{I}, \mathcal{I}, [A, B] \right] \xrightarrow{ev} [A, B]$ commutes, therefore also does the diagram $\left[ [A, B], [A, C] \right] \xrightarrow{[F, 1]} [I'[A, C]]$.

From this and according to Corollary [Sch08]-11.6 all diagrams in the pasting below commute and according to Lemma 7.9 the commutation of the external diagram above is equivalent to the commutation of the first diagram of the Lemma.

In the pasting $\left[ [A, B], [C, B] \right] \xrightarrow{[F, 1]} [I, [C, B]]$ the top-left diagram commutes according to Corollary [Sch08]-11.8 and we have already seen that the bottom-left diagram commutes. The commutation of the external diagram above is equivalent to the commutation of the second diagram of the Lemma according to Lemma 7.8.

Section 6.66
7.40 Definition of the 2-cell $\theta$ from Axiom [6.26]

PROOF: The arrows

$$A \otimes M \xrightarrow{L' \otimes 1} (I \otimes A) \otimes M \xrightarrow{A'} I \otimes (A \otimes M)$$

is strict and, as shown below, it has the same image by $Rn$ as the arrow $L' : A \otimes M \rightarrow I \otimes (A \otimes M)$. The 2-cell $\theta$ corresponds then via the adjunction $L \otimes A$ to the identity 2-cell.

The image by $Rn$ of $L'_A$ is

$$A \xrightarrow{\eta} [M, AM] \xrightarrow{[1, \eta]^*} [M, [I, I(AM)]] \xrightarrow{[1, ev_1]} [M, I(AM)].$$

The arrow $A' \circ L' \otimes 1$ above as image by $Rn$ that rewrites

1. $A \xrightarrow{L'} I \xrightarrow{Rn(A')} [M, I(AM)]$
2. $A \xrightarrow{\eta^*} [I, I] \xrightarrow{ev} I \xrightarrow{Rn(A')} [M, I(AM)]$
3. $A \xrightarrow{\eta^*} [I, I] \xrightarrow{[1, Rn(A')]^*} [I, [M, I(AM)]] \xrightarrow{ev} [M, I(AM)]$
4. $A \xrightarrow{Rn(Rn(A'))^*} [I, [M, I(AM)]] \xrightarrow{ev} [M, I(AM)]$
5. $A \xrightarrow{\eta} [M, AM] \xrightarrow{[-, I(AM)]} [[AM, I(AM)], [M, I(AM)]] \xrightarrow{[\eta, 1]} [I, [M, I(AM)]] \xrightarrow{ev} [M, I(AM)]$
6. $A \xrightarrow{\eta} [M, AM] \xrightarrow{[1, \eta]^*} [I, [M, I(AM)]] \xrightarrow{D} [I, [M, I(AM)]] \xrightarrow{ev} [M, I(AM)]$
7. $A \xrightarrow{\eta} [M, AM] \xrightarrow{[1, \eta]^*} [I, [M, I(AM)]] \xrightarrow{[1, ev_1]} [M, I(AM)]$

where in the above derivation the equalities between arrows hold for the following reasons:

- 4. and 5. by definition of $A'$ since its image by $Rn \circ Rn$ is in this case

$$I \xrightarrow{\eta} [M, AM] \xrightarrow{[-, I(AM)]} [[AM, I(AM)], [M, I(AM)]] \xrightarrow{[\eta, 1]} [M, I(AM)]$$

- 5. and 6. by Lemma [Sch08]-10.8;

- 6. and 7. by Lemma [Sch08]-11.9.

7.41 Proof of [6.26]

PROOF: The first of the 2-cells of Axiom [6.20] can be decomposed as the composite $\Xi_2 \circ \Xi_1$ where $\Xi_1$ is

$$(A(AA), A)M \xrightarrow{A'} (AA(AA))M \xrightarrow{1 \otimes \psi} (AA)M \xrightarrow{\beta} M$$

and $\Xi_2$ is

$$(AA)(AAM) \xrightarrow{(e \otimes 1) \otimes 1} (AA)M \xrightarrow{\beta} M.$$ 

The 2-cell of [6.26] decomposes as $\Xi_4 \circ \Xi_3$ where $\Xi_3$ is

$$(A \otimes A) \otimes A \xrightarrow{\beta' \otimes 1} [M, M] \otimes A \xrightarrow{1 \otimes \psi'} [M, M] \otimes [M, M] \xrightarrow{c} [M, M]$$

and $\Xi_4$ is

$$(AA)A \xrightarrow{c \otimes 1} AA \xrightarrow{\beta'} [M, M].$$

The 2-cell $\Xi_3$ has a strict domain and an easy computation gives that its image by $Rn$ is

$$A \otimes A \xrightarrow{\beta'} [M, M] \xrightarrow{[M, \eta]} [[M, M], [M, M]] \xrightarrow{[\eta', 1]} [A, [A, [M, M]]].$$
According to Lemma 7.8 the 2-cell $\Xi_1$ has an image by $Rn$ with a strict domain and has image by $Rn \circ Rn$ the 2-cell

$$A \xrightarrow{\alpha \otimes 1} A \xrightarrow{\beta} \mathcal{M}$$

Therefore the 2-cell $\Xi_1$ has image by $Rn$ the 2-cell $\Xi_2$.

An easy computation shows that the 2-cell $\Xi_2$ has image by $Rn$

$$(A \xrightarrow{\phi} A) \xrightarrow{\otimes 1} \mathcal{M}$$

which is $\Xi_4$.

**7.42 Proof of 6.27**

PROOF: The second 2-cells of Axiom 6.20 can be decomposed as the composite $\Xi_3 \circ \Xi_2 \circ \Xi_1$ where $\Xi_1$ is

$$((A \xrightarrow{A} A) \xrightarrow{\alpha \otimes 1} (A \xrightarrow{\beta} A)) \xrightarrow{\phi} \mathcal{M}$$

$\Xi_2$ is

$$((A \xrightarrow{A} A) \xrightarrow{\alpha \otimes 1} (A \xrightarrow{\beta} A)) \xrightarrow{\phi \otimes 1} \mathcal{M}$$

and $\Xi_3$ is

$$((A \xrightarrow{A} A) \xrightarrow{\alpha \otimes 1} \mathcal{M})$$

The 2-cell of 6.27 decomposes as $\Xi_6 \circ \Xi_5 \circ \Xi_4$ where $\Xi_4$ is the 2-cell $\xi$.

$$A \xrightarrow{\phi} [A, \mathcal{M}]$$

$\Xi_5$ is

$$A \xrightarrow{\phi} [A, \mathcal{M} \otimes \mathcal{M} \otimes \mathcal{M}]$$

and $\Xi_6$ is

$$A \xrightarrow{\phi} [A, \mathcal{M}]$$

The 2-cell $\Xi_1$ has image by $Rn$ the 2-cell

$$A \xrightarrow{\phi} [A, \mathcal{M}]$$

The 2-cell

$$\Xi_7 = A \xrightarrow{Rn(A)} [A, A((A \xrightarrow{\beta} A))] \xrightarrow{[1, 1, \beta]} [A, \mathcal{M}]$$

has a strict domain and according to Lemma 19.8.6 its image by $Rn$ is

$$\Xi_8 = \mathcal{M} \xrightarrow{[1, \phi]} [\mathcal{M}, \mathcal{M}]$$

The 2-cell

$$\Xi_9 = \mathcal{M} \xrightarrow{\phi} [\mathcal{M}, \mathcal{M}]$$

$$\Xi_9 = \mathcal{M} \xrightarrow{\phi} [\mathcal{M}, \mathcal{M}]$$

$$\Xi_9 = \mathcal{M} \xrightarrow{\phi} [\mathcal{M}, \mathcal{M}]$$

$\Xi_9 = \mathcal{M} \xrightarrow{\phi} [\mathcal{M}, \mathcal{M}]$
One the other hand the dual of the second 2-cell of the lemma re writes

\[ \text{PROOF:} \text{The 2-cell from 6.29 decomposes as} \]

\[ (\mathcal{A} \mathcal{A}) A \longrightarrow A (\mathcal{A} \mathcal{A}) \quad 1 \otimes c \quad A A \quad Rn(\beta) \longrightarrow [\mathcal{M}, \mathcal{M}] \]

which is \( \Xi_5 \).

Eventually the image by \( Rn \) of \( \Xi_2 \) is trivially \( \Xi_6 \).

\[ \text{To prove 6.29 we shall use the following lemma.} \]

**Lemma 7.43** For any objects \( A, B, C \) and \( D \), any 2-cell \( C \xrightarrow{\tau} [D, A] \) of SPC and any \( c \in C \), the 2-cells in SPC

\[ [A, B] \xrightarrow{[D, -]} [D, A] \xrightarrow{[D, B]} \xrightarrow{[\tau, 1]} [C, [D, B]] \xrightarrow{ev_c} [D, B] \]

and

\[ [A, B] \xrightarrow{[\tau, 1]} [C, A] \xrightarrow{[\tau, 1]} [D, B] \]

are equal.

**PROOF:** According to Lemma [Sch08]-11.9, The first of the 2-cell rewrites

1. \( [A, B] \xrightarrow{[D, -]} [D, A] \xrightarrow{[D, B]} [C, [D, B]] \xrightarrow{[\tau, 1]} [D, [C, B]] \xrightarrow{ev_c} [D, B] \)

according to Lemma [Sch08]-10.9,

2. \( [A, B] \xrightarrow{[\tau, 1]} [C, A] \xrightarrow{[\tau, 1]} [D, [C, B]] \)

and this last arrow has dual

\[ D \xrightarrow{\tau} [C, A] \xrightarrow{[\tau, B]} [A, B] \xrightarrow{[\tau, 1]} [A, B] \xrightarrow{ev_c} [A, B] \xrightarrow{[\tau, 1]} [B, B] \]

One the other hand the dual of the second 2-cell of the lemma rewrites

1. \( D \xrightarrow{\tau} [C, A] \xrightarrow{[\tau, 1]} [A, B] \xrightarrow{[\tau, 1]} [B, B] \)

2. \( D \xrightarrow{\tau} [C, A] \xrightarrow{[\tau, 1]} [A, B] \xrightarrow{[\tau, 1]} [B, B] \)

3. \( D \xrightarrow{\tau} [C, A] \xrightarrow{[\tau, 1]} [A, B] \xrightarrow{[\tau, 1]} [B, B] \)

where in the above derivation arrows 2. and 3. are equal by Corollary [Sch08]-11.4.

\[ \text{7.44 Proof of 6.29} \]

**PROOF:** The 2-cell from 6.29 decomposes as \( \zeta_2 \circ \zeta_1 \) where \( \zeta_1 = \)

\[ A \xrightarrow{\kappa} A \otimes I \xrightarrow{1 \otimes \gamma} A \otimes [\mathcal{M}, \mathcal{M}] \xrightarrow{\alpha} [\mathcal{M}, \mathcal{M}] \otimes [\mathcal{M}, \mathcal{M}] \xrightarrow{\epsilon} [\mathcal{M}, \mathcal{M}] \]

and \( \zeta_2 = \)

\[ A \xrightarrow{\kappa} A \otimes I \xrightarrow{1 \otimes \gamma} A \otimes A \xrightarrow{\beta} [\mathcal{M}, \mathcal{M}] \]

We show below that the image by \( Rn \) of the 2-cell \( \Xi_2 \) is \( \zeta_1 \) whereas the image by \( Rn \) of \( \Xi_3 \) is \( \zeta_2 \).

The image by \( Rn \) of the 2-cell \( \Xi_2 \) rewrites

...
1. $\mathcal{A} \xrightarrow{\eta} [\mathcal{A}, \mathcal{M}] \xrightarrow{[\gamma,1]} [\mathcal{M}, \mathcal{A} \otimes \mathcal{M}] \xrightarrow{[1,\psi]} [\mathcal{M}, \mathcal{M}]$
2. $\mathcal{A} \xrightarrow{\eta} [\mathcal{M}, \mathcal{A} \otimes \mathcal{M}] \xrightarrow{[1,\psi]} [\mathcal{M}, \mathcal{M}] \xrightarrow{[\gamma,1]} [\mathcal{M}, \mathcal{M}]$
3. $\mathcal{A} \xrightarrow{\phi'} [\mathcal{M}, \mathcal{M}] \xrightarrow{[\gamma,1]} [\mathcal{M}, \mathcal{M}]$

According to Lemma 7.43 and since the 2-cell $\gamma$ is $\ev_* * (\gamma')$, the 2-cell 3. above is

$$\mathcal{A} \xrightarrow{\phi'} [\mathcal{M}, \mathcal{M}] \xrightarrow{[\gamma,1]} [[\mathcal{M}, \mathcal{M}], [\mathcal{M}, \mathcal{M}]] \xrightarrow{[\gamma',1]} [\mathcal{I}, [\mathcal{M}, \mathcal{M}]] \xrightarrow{\ev_*} [\mathcal{M}, \mathcal{M}].$$

This last 2-cell is actually $\zeta_1$. To check this use Lemma 7.21 and the fact that the image by $R\mathcal{N}$ of the arrow

$$\mathcal{A} \otimes [\mathcal{M}, \mathcal{M}] \xrightarrow{\phi' \otimes 1} [\mathcal{M}, \mathcal{M}] \otimes [\mathcal{M}, \mathcal{M}] \xrightarrow{c} [\mathcal{M}, \mathcal{M}]$$

is

$$\mathcal{A} \xrightarrow{\phi'} [\mathcal{M}, \mathcal{M}] \xrightarrow{[\gamma,1]} [[\mathcal{M}, \mathcal{M}], [\mathcal{M}, \mathcal{M}]].$$

The image by $R\mathcal{N}$ of the 2-cell $\Xi_3$ is $\mathcal{A} \xrightarrow{R' \mathcal{N}} \mathcal{A} \otimes \mathcal{I} \xrightarrow{1 \otimes \mathcal{N}} \mathcal{A} \otimes \mathcal{A} \xrightarrow{R\mathcal{N}(\beta)} [\mathcal{M}, \mathcal{M}]$ which is $\zeta_2$. $

\textbf{7.45 Proof of 6.31}

\textbf{PROOF:} The 2-cell $\Xi_2$ rewrites

$$\mathcal{A} \otimes \mathcal{M} \xrightarrow{\phi'} \mathcal{M} \xrightarrow{id} \mathcal{M} \xrightarrow{\gamma''} [\mathcal{A}, \mathcal{M}] \xrightarrow{[\nu,1]} [\mathcal{I}, \mathcal{M}]$$

which image by $R\mathcal{N}$ is the 2-cell

$$\mathcal{A} \xrightarrow{\phi'} [\mathcal{M}, \mathcal{M}] \xrightarrow{id} [\mathcal{M}, \mathcal{M}] \xrightarrow{[1,\gamma''']} [\mathcal{M}, [\mathcal{A}, \mathcal{M}]] \xrightarrow{[1,\nu,1]} [\mathcal{M}, [\mathcal{I}, \mathcal{M}]].$$

On the other hand the 2-cell

$$\mathcal{A} \xrightarrow{L'} \mathcal{I} \otimes \mathcal{A} \xrightarrow{1 \otimes \phi'} \mathcal{I} \otimes [\mathcal{M}, \mathcal{M}] \xrightarrow{\gamma' \otimes 1} [\mathcal{M}, \mathcal{M}] \otimes [\mathcal{M}, \mathcal{M}] \xrightarrow{c} [\mathcal{M}, \mathcal{M}]$$

rewrites

1. $\mathcal{A} \xrightarrow{\phi'} [\mathcal{M}, \mathcal{M}] \xrightarrow{L'} \mathcal{I} \otimes [\mathcal{M}, \mathcal{M}] \xrightarrow{\gamma' \otimes 1} [\mathcal{M}, \mathcal{M}] \otimes [\mathcal{M}, \mathcal{M}] \xrightarrow{c} [\mathcal{M}, \mathcal{M}]$
2. $\mathcal{A} \xrightarrow{\phi'} [\mathcal{M}, \mathcal{M}] \xrightarrow{[\gamma,\zeta]} [[\mathcal{M}, \mathcal{M}], [\mathcal{M}, \mathcal{M}]] \xrightarrow{[\gamma',1]} [\mathcal{I}, [\mathcal{M}, \mathcal{M}]] \xrightarrow{\ev_*} [\mathcal{M}, \mathcal{M}]$
3. $\mathcal{A} \xrightarrow{\phi'} [\mathcal{M}, \mathcal{M}] \xrightarrow{[1,\gamma''']} [\mathcal{M}, [\mathcal{I}, \mathcal{M}]] \xrightarrow{D} [\mathcal{I}, [\mathcal{M}, \mathcal{M}]] \xrightarrow{\ev_*} [\mathcal{M}, \mathcal{M}]$
4. $\mathcal{A} \xrightarrow{\phi'} [\mathcal{M}, [\mathcal{I}, \mathcal{M}]] \xrightarrow{[1,\nu,1]} [\mathcal{M}, [\mathcal{I}, \mathcal{M}]] \xrightarrow{[1,\ev_*]} [\mathcal{M}, \mathcal{M}]$
5. $\mathcal{A} \xrightarrow{\phi'} [\mathcal{M}, \mathcal{M}] \xrightarrow{[1,\gamma''']} [\mathcal{M}, \mathcal{M}]$. 

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In the above derivation the equality between arrows hold for the following reasons:
- 1. and 2. by Lemma \[7.8\],
- 2. and 3. by Lemma \[Sch08\]-10.8(which can be improved to take 2-cells into account),
- 3. and 4. by Lemma \[Sch08\]-11.9.

By definition of the 2-cell $\theta$, the 2-cell $\Xi_3$ has image by $Rn$ an identity 2-cell. Eventually it is rather straightforward that 2-cell $\Xi_4$ has image by $Rn$ since the 2-cell $\beta''$ is $Rn(\beta)$.

**7.46 PROOF of 6.32**

PROOF: The 2-cell

\[
(AA)M \xrightarrow{1 \otimes H} AAN \xrightarrow{\beta} N
\]

has a strict domain

\[
1 \otimes H \quad A' \quad 1 \otimes \psi \quad \psi
\]

and has image by $Rn$

\[
AA \xrightarrow{\beta''} [N, N] \xrightarrow{[H, 1]} [M, N]
\]

which is image by $Rn$ is $\Xi_1$. The 2-cell

\[
(AA)M \xrightarrow{c \otimes 1} AM \xrightarrow{\delta} N
\]

has image by $Rn$ the 2-cell

\[
AA \xrightarrow{c} A \xrightarrow{\delta'} [M, N]
\]

which image by $Rn$ is $\Xi_2$.

**7.47 Proof of 6.33**

PROOF: According to Lemma \[7.21\] the 2-cell

\[
(AA)M \xrightarrow{A'} A(AM) \xrightarrow{1 \otimes \delta} AN \xrightarrow{\psi} N
\]

has image by $Rn \circ Rn$ the 2-cell $\Xi_4$. According to Lemma \[7.21\] the 2-cell

\[
(AA)M \xrightarrow{A'} A(AM) \xrightarrow{1 \otimes \psi} AM \xrightarrow{\delta} N
\]

has image by $Rn \circ Rn$ the 2-cell $\Xi_6$. Eventually the 2-cell

\[
(AA)M \xrightarrow{\beta} M \xrightarrow{\sigma} N
\]

has image by $Rn \circ Rn$ the 2-cell $\Xi_7$.

**7.48 Proof of 6.34**
PROOF: The 2-cell
\[ u^2_H : H \to [1,H] \circ v : I \to \mathcal{M}, \mathcal{N} \]
has image by \( ev_* \), an identity 2-cell and the 2-cell
\[ I \overset{\gamma'}{\longrightarrow} \mathcal{M} \overset{[1,H]}{\longrightarrow} \mathcal{M} \]
has image by \( ev_* \), the 2-cell
\[ \mathcal{M} \overset{\gamma}{\longrightarrow} \mathcal{M} \overset{H}{\longrightarrow} \mathcal{N} \]

\[ 7.49 \text{ Proof of 6.35} \]

PROOF: The arrow \( H : I \to \mathcal{M}, \mathcal{N} \) is strict has image by the functor \( ev_* \), the arrow \( H : \mathcal{M} \to \mathcal{N} \). The 2-cell
\[ I \overset{\gamma'}{\longrightarrow} \mathcal{M} \overset{[1,H]}{\longrightarrow} \mathcal{M} \]
has image by \( ev_* \), the 2-cell
\[ \mathcal{M} \overset{H}{\longrightarrow} \mathcal{N} \overset{\gamma}{\longrightarrow} \mathcal{N} \]

The 2-cell
\[ I \overset{u}{\longrightarrow} \mathcal{M} \overset{\delta'}{\longrightarrow} \mathcal{M} \]
has image by \( ev_* = SMC(1, ev_*) \circ D \) the 2-cell
\[ \mathcal{M} \overset{Rn(\delta)^*}{\longrightarrow} \mathcal{M} \overset{[1,H]}{\longrightarrow} \mathcal{M} \overset{ev_*}{\longrightarrow} \mathcal{N} \]
which is according to Lemma 7.8
\[ \mathcal{M} \overset{L'}{\longrightarrow} \mathcal{M} \overset{u \otimes 1}{\longrightarrow} \mathcal{M} \overset{\delta}{\longrightarrow} \mathcal{M} \]

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