Equivalence of avalanche dynamics in Burridge-Knopoff and Edwards-Wilkinson models

Soumyajyoti Biswas,† Purusattam Ray,‡ and Bikas K. Chakrabarti§

1 TCMP Division, Saha Institute of Nuclear Physics, 1/AF Bidhannagar, Kolkata 700 064 India.
2 Institute of Mathematical Sciences, CIT Campus, Taramani, Chennai-600113, India

A discretized version of the Burridge-Knopoff train model with (non-linear friction force replaced by) random pinning is studied in one and two dimensions. A scale free distribution of avalanches and the Omori law type behaviour for after-shocks are obtained. The avalanche dynamics of this model becomes precisely similar (identical exponent values) to the Edwards-Wilkinson (EW) model of interface propagation. It also allows the complimentary observation of depinning velocity growth with exponent value identical with that for EW model in this train model and Omori law behaviour of after-shock (depinning) avalanches in the EW model.

I. INTRODUCTION

The spring-block system driven over a rough surface have been studied extensively as a model for intermittent dynamics seen in earthquakes [1] (see [2] for a review). With a non-linear velocity dependent friction force and linear springs, these models are known to produce rich dynamics. Particularly, well known statistical laws of earthquakes, such as the Guttenberg-Richter law, which predicts scale free probability distribution for size of avalanches, are reproduced in this simple model.

Among many variants of the spring-block models are the Burridge-Knopoff model, where each block is pulled with a spring and also the train model, where only one block of one side is pulled. In the present study we are interested in the dynamics of train models. The train model was introduced in Ref. [3], where the friction law was taken to be in the usual velocity dependent non-linear form. Its avalanche statistics showed the scale free behaviour as seen in Burridge-Knopoff models. However, in Ref. [4], it was shown that the complex form of friction force is not actually a necessary ingredient for the intermittent dynamics of the model. They considered a random pinning force that depends on the overlap magnitude of the rough surfaces, in place of friction. The non-linearity appears in the form of the threshold introduced by this pinning force. The avalanche statistics still followed a power-law. This marks a major simplification in the formulation of such models.

However, although the magnitude distribution of the avalanches shows the Guttenberg-Richter law like feature as seen in original Burridge-Knopoff models and train models, the other well known statistical feature of earthquake dynamics, namely the Omori law, is neither seen in Burridge-Knopoff or train models nor in the simplified version of them mentioned above. In the present model we used a different definition for avalanche to capture both these laws. In particular, we distinguish each of the subsequent slip events from the primary one. We consider that the each subsequent slip event occurs at different time. This can be justified from the fact that in earthquakes, the aftershocks are essentially a consequence of the primary adjustments (slips) of the existing contact points and not due to further movement (and thereby stress building) of the tectonic plates, which moves in a time scale orders of magnitude slower than at least the initial aftershock event’s time. Using this definition of avalanche we get both the well known statistical laws of earthquakes, namely the Guttenberg-Richter law and Omori law. Furthermore, we see finite size effects in the avalanche statistics and use the finite size scalings to specify the exponent values and establish the universality by observing that disorder in the equilibrium spacings of the chains give the same exponent values as the one with disordered values in pinning forces. We have also extended the model to the more realistic two dimensional form. The exponent values get changed, while the finite size scalings still work. We have also shown that this version of the train model is very precisely equivalent to the interface propagation model of Edwards-Wilkinson’s (EW) [5] (see [6] for detailed discussion on EW model).

II. MODEL

Consider the one dimensional train model, where an array of blocks are arranged in a discrete lattice. The blocks are connected by Hookean springs (with identical spring constants). The array is being pulled from one side quasistatically, i.e. the block on the extreme right say, is pulled until instability sets in, after which the pulling is paused as long as there are movements of the blocks. Once all the movements stop, the right most block is pulled again and so on. This puts the system in an intermittent motion after some transients. Three forces act on each block (except for the two blocks at the two extreme ends), two forces from two nearest neighbor springs and another due to friction on that site. The major simplification is that we consider friction force as a random pinning force. The value of the friction force may de-
FIG. 1: Schematic diagram for the discrete spring-block model with pinning. The upper figure depicts the case where there is no disorder in the lattice spacings of the chains (IC) and the lower figure has disorder in the lattice spacing of the chains (DC). The directions of elastic force and pinning force (working as friction) are shown for a typical block. The chains are being pulled from one end quasistatically (see text). The pinning forces in the IC case are distributed in a range (typically \([0 : 1]\)) but for the DC case, pinning force has a single value; it only comes into play when two blocks come on top of each other and the randomness comes from the randomness in the positions of the blocks in the both the chains.

We mainly consider two types of chains: One is like the uniform distribution in \([0 : 1]\) of the rough surfaces are often self-affine, we also consider the case when the blocks in the two chains are arranged in the form of a Cantor set of same generation, let us call this self-affine chain (SAC). This differs from DC in the sense that the disorder here are correlated.

A. Avalanche statistics

When the chain is pulled from one side, after some transients, the whole chain will start moving. All the statistics are to be taken once the whole chain starts moving. Particularly, we are interested in the avalanche statistics. We define an avalanche as follows: Just when an instability sets in (a slip event) the pulling is stopped. Successive slips may continue due to the first slip. Then we count the number of slip event on every scan of the lattice (with parallel updating). This number gives the size of the avalanche and the time is increased by unity after each scan of the blocks of the upper chain. This definition differs from the one used in elsewhere \([3]\), where the avalanche size was defined as the total number of slip events, until a stable condition is reached.

III. RESULTS

Here we report the avalanche statistics in train model for different cases. First we consider the simplest i.e., one dimensional model with regular arrangements of the blocks. Here the disorder is present in terms of the pinning forces, which are taken from a uniform distribution between \([0 : 1]\). Fig. 2 shows the avalanche size distributions. The scaling form assumed here (for this and the subsequent versions of this model) is the following

\[
P(s) \sim (s * L^\alpha)^{-\delta} f \left( \frac{s * L^\alpha}{L^z} \right),
\]

where \(P(s)\) is the probability of an avalanche of size \(s\). This implies that scaling the \(x\) axis by a factor \(L^\alpha\) will make the power-law part of all the curves to coincide and the deviations will be due different finite size cut-offs only, which can then be scaled to obtain the dynamical
The avalanche statistics are universal. We consider the case when a one chain is being pulled over another and some blocks are removed randomly from both the chains. Here we keep the value of the pinning forces to be the same (unity) all through. However, a pinning force. Note that the exponent values are in contrast with that obtained before [4] with earlier definition of avalanches with which we still get \( \beta = z - \alpha = 0.80 \pm 0.01 \).

The values of the exponent obtained here are: \( \delta = 0.70 \pm 0.01, \alpha = 0.40 \pm 0.05 \) and \( z = 1.20 \pm 0.06 \). These values do not depend on the distribution of the pinning force. The exponent values are in contrast with that obtained before [4] with earlier definition of avalanches with which we still get \( \delta \approx 1.5 \) in our model with no finite size scalings. Next we intend to study the analogue of Omori law in this context. We first define an upper cut-off (\( s_1 \)) of the avalanche size. The avalanches above this size is to be called a main-shock. Then we also define a lower cut-off (\( s_0 \)) below which we do not measure an avalanche. Then we measure the probability that an avalanche of size \( s_1 \) or above has occurred at time \( t \) after a main-shock. This is found to decay in a power-law with time

\[
n(t) \sim t^{-p}
\]  

when \( t \) is large and the exponent value is not very sensitive with the thresholds (See Fig. 3). The value of the exponent comes out to be \( p = 0.85 \pm 0.05 \).

We now show that the roughness of the surfaces in contact may alternatively be incorporated in terms of the disorder in the spacings of the chains. We consider the case when a one chain is being pulled over another and some blocks are removed randomly from both the chains. Here we keep the value of the pinning forces to be the same (unity) all through. However, a pinning value is only effective when two blocks come on top of each other. In Fig. 4 we report the avalanche statistics when a fraction (0.5) of blocks are randomly removed from the upper and lower chains. We do the similar finite size scaling study as we did for IC. The exponent values come out to be very close to those obtained before. We have checked that the values do not depend on the fraction of the blocks removed, or even for different distribution functions for the inter-block spacings (for random removal it will be exponential; we check for uniform, power-law distributions etc.). In this sense, the avalanche statistics are universal.

Again we measure the probability of an event with size

\[
P(s) \sim s^{-\beta}
\]  

for different distribution functions for the inter-block spacings (for random removal it will be exponential; we check for uniform, power-law distributions etc.). In this sense, the avalanche statistics are universal.
s_l or above after time t of a main-shock (an event of magnitude s_u or above). This again shows the power-law behaviour as the one for IC with same exponent value 0.85 ± 0.05. Now we consider the particular case when the disorders in the two chains are self-similar. We take the particular case when the blocks are arranged in the form of Cantor sets of some finite generation. Rough surfaces were modelled with Cantor sets before in the context of earthquakes [5, 6]. That the distribution of friction values becomes non-trivial even though a single pinning value is taken in every contact is also known [9].

The earlier studies are in fact the limiting cases of the present one when the springs are absolutely rigid. The avalanche statistics and event rate distributions are plotted (see Figs. 6,7). The exponent values remain almost the same. Note that while constructing the Cantor set, in each generation we have removed the middle third of the chain. So, while performing the finite size scaling, the system size used was $2^G$, where G is the generation number. Here also we consider a single value for the pinning force as before, the exponent values obtained remain similar to the values obtained for IC and DC case.

Finally we consider the case of higher dimension (in this case, two). The blocks are initially arranged in a regular square lattice ($L \times L$). The system is being pulled from one side (all $L$ blocks) quasistatically. The definition of avalanche etc are the same. Here we are keeping the pinning force distributed (uniformly) but the lattice is regular (IC). The exponent values for the avalanche statistics change drastically. As can be seen from Fig. 8, the exponent values for 2d becomes $\alpha = 0.07 \pm 0.01$, $\delta = 1.15 \pm 0.02$ and $z \approx 1.27 \pm 0.02$. However, the analogue of Omori-law (see Fig. 9) remains almost similar, although the power-law fit is not very good in this case.

FIG. 5: The probability that an avalanche of size $s_l$ or above takes places after a time $t$ of a main-shock (an event of size equal to or higher than $s_u$) is plotted for DC. This shows a power-law dependence and the exponent value is 0.85 ± 0.05.

A. Equivalence with interface depinning

The avalanche statistics of the discrete train model studied here are similar to the statistics of the interface depinning problem. Particularly, this linear elastic model with threshold activated dynamics is formally similar to the Edwards-Wilkinson (EW) equation with quenched randomness [4, 8]. The EW equation for interface depinning
The exponent values are $\delta = 1.15 \pm 0.01$, which is the equivalent of the Guttenberg-Richter law here, and the other scaling exponents are $\alpha = 0.07 \pm 0.01$, $\beta = z - \alpha = 1.20 \pm 0.01$.

![FIG. 8: The avalanche size distributions for train model with the blocks arranged in form of two dimensional lattice, are plotted for different system sizes. The inset shows the data collapse with the finite size scaling form assumed (Eq. 3). The exponent values are $\delta = 0.77 \pm 0.02$, which is the equivalent of the Guttenberg-Richter law here, and the other scaling exponents are $\alpha = 0.40 \pm 0.01$, $\beta = z - \alpha = 0.82 \pm 0.01$.]

![FIG. 9: The probability that an avalanche of size $s_l$ or above takes places after a time $t$ of a main-shock (an event of size equal to or higher than $s_o$) is plotted for two dimensional train model. This shows a power-law dependence and the exponent value is $0.90 \pm 0.05$.]

![FIG. 10: The avalanche size distributions for one dimensional EW model are plotted for different system sizes. The inset shows the data collapse with the finite size scaling form assumed (Eq. 3). The exponent values are $\delta = 0.77 \pm 0.02$, which is the equivalent of the Guttenberg-Richter law here, and the other scaling exponents are $\alpha = 0.40 \pm 0.01$, $\beta = z - \alpha = 0.82 \pm 0.01$.]

![FIG. 11: The probability that an avalanche of size $s_l$ or above takes places after a time $t$ of a main-shock (an event of size equal to or higher than $s_o$) is plotted for one dimensional EW model. This shows a power-law dependence and the exponent value is $0.80 \pm 0.05$.]

The equation shows a depinning transition in the sense that if one measures the steady state velocity of the interface for different values of external force, one gets it in the form

$$v_{sat} \sim (f_{ext} - f_c)^\theta \quad \text{when} \quad f_{ext} > f_c$$

$$v_{sat} = 0 \quad \text{otherwise.} \quad (5)$$

where $\theta$ is the velocity depinning exponent. Depinning transition in bead spring model was studied before [10]. However, that model was not discretised and the driving was applied on each bead, where Omori law like features cannot be observed. Apart from this steady state exponent $\theta$, one can also study the avalanche dynamics in this model when it is driven quasistatically at the boundaries.
That the boundary driven interface dynamics and train model are in the same universality class was conjectured before [11] (see also [12]). Here we show that the discrete version of the train model is exactly the boundary driven interface model.

The avalanche statistics, with avalanches defined as before, in the interface problem is same as in the case of train model (in 1d). In Fig. 10, we plot the avalanche size distribution and its finite size scaling and get same exponents. Also we measure the rate of events after a large shock and get similar statistics (see Fig. 11). As mentioned before, we also measure the steady state order parameter exponent $\theta$ both in train model and EW model. The results are shown in Fig. 12. The proximity in the values of the exponents suggest that these are manifestation of the same model. Furthermore, we study EW model in 2d to compare with train model in 2d. Here we drive the 2d surface again from the boundaries. The avalanche statistics and rate of activities are plotted in Figs. 13, 14. Again these are similar to the values obtained in train model. Therefore we see that the steady state and dynamical behaviours of the discrete train model with random pinning are same as that of EW model. In both cases the systems are boundary driven and are of linear elastic nature with random pinning.

FIG. 13: The avalanche size distributions for two dimensional EW model (driven at the boundaries) are plotted for different system sizes. The inset shows the data collapse with the finite size scaling form assumed (Eq. 3). The exponent values are $\delta = 1.15 \pm 0.01$, which is the equivalent of the Guttenberg-Richter law here, and the other scaling exponents are $\alpha = 0.05 \pm 0.01$, $\beta = z - \alpha = 1.20 \pm 0.01$.

FIG. 14: The probability that an avalanche of size $s_l$ or above takes places after a time $t$ of a main-shock (an event of size equal to or higher than $s_u$) is plotted for two dimensional EW model. This shows a power-law dependence and the exponent value is $0.80 \pm 0.02$. 

FIG. 12: The steady state velocities as a function of external force for train model (top) and EW model (bottom) in one dimension. In both cases the velocity increases in a power-law beyond a critical point. The exponent are $0.29 \pm 0.01$ and $0.30 \pm 0.01$ for train model and boundary driven EW model respectively.
B. Interface propagation and fluctuation in bulk

As mentioned above, the drive in 2d EW model (and also in 2d train model) is applied along one side. Therefore one may ask how the disturbances propagate through the bulk. Considering the case of 2d EW model (of course everything can be translated to the train model in 2d as well), we study mainly two aspects of this disturbance propagation: (a) Since the drive is applied only along a side, in the early time dynamics, there will be a ‘front’ similar to the propagation of interface through quenched disordered medium. This is a rough line with an average velocity of propagation that decays in a power-law. In Fig. 15 the front velocity and the r.m.s. fluctuation of the front are plotted with time. While the velocity decays as \( v_f(t) \sim t^{-0.5} \), the front width does not show appreciable dependence with time, making it similar to a logarithmic growth. (b) We also measure the fluctuation of the system along the direction perpendicular to the drive. As one goes away from the line which is driven, the widths of each line first increase, reaches a maximum and then decreases to zero in the region which are yet to experience the drive. Since the front velocity scales as \( t^{-0.5} \), the displacement (i.e., the maximum length upto which disturbance has propagated and one gets finite roughness) scales as \( t^{0.5} \). One can therefore scale the displacement axis by \( t^{0.5} \) and make the end-points coincide (see Fig. 16). One then finds a symmetric curve, i.e. the fluctuation in the bulk is maximum at the halfway of the range upto which it has propagated.

There are two things to be mentioned here, firstly the ‘front’ may have overhangs. To deal with it, we are considering the surface which can be formed by over-estimating the overhangs (see Fig. 16). One can estimate the effect of over-hangs in other ways, and it may differ substantially only near a dynamical transition (depinning), which we are not studying here. Secondly, while one can study the time dependence by quasistatically increasing the displacement at one end and wait for the system to come to a halt (that makes one time step), one can also make the entire displacement at one go and one time step consists of one scan of the lattice. These two approaches are not same in terms of avalanche statistics, but in fluctuation in the bulk and velocity of the front, these two turns out to be same. Hence we followed the later (in this section), which is computationally cheaper.

IV. SUMMARY AND DISCUSSION

We first study a discretised version of the train model of earthquake dynamics. The non-linear velocity weakening friction term is replaced by (random) pinning force threshold and the movements of the blocks are discretised, in the sense that one block can move one lattice constant at a given time. This allows us to model the system as two chains; one dragged over the other by pulling the upper chain from one end. The blocks in the upper chain, which is being dragged, are connected by linear springs with identical spring constants, while the blocks in the lower chain are fixed in position (see Fig. 1). The (friction) pinning force is non-zero only in places where two blocks come on top of one another. When both the chains are intact (IC model; blocks are placed at equal distances initially), whenever one block comes on top of another, a random pinning force is drawn from a uniform distribution in \([0 : 1]\). In the case (DC model) both the chains are disordered, i.e., some blocks are randomly removed from both the chains, the pinning (friction) force is taken to be simply a constant (unity) whenever one block comes over another. We make the connection that this version of the train model is precisely same as the EW model for interface propagation.
through quenched disordered medium. This is apparent from Eq. (2), when the EW front is driven by its two ends in the transverse direction (perpendicular to the initial position of the chain). Both of these models have also been studied in two dimensions, where in train model the system is driven along the plane of the surface, and for EW model it is driven perpendicular to the surface. We obtained equivalent results. That these two models may be in the same universality class, was conjectured before (see e.g., [11]), but here we show that the dynamics in these two models can be translated to be precisely the same with all its qualitative features intact and identical complimentary features: Power law of growth of depinning velocity in the train model and Omori law in EW model.

The dynamics of the train model follow Eqs. (1) and (2). The pulling of the upper chain is stopped whenever a slip event occurs. An avalanche is defined as the number of displacements in one scan of the (entire) upper chain (parallel updates are made). Time is increased by unity after each scan. Further pulling is resumed only when all the blocks have relaxed. This makes an avalanche distinguishable from all the previously occurring avalanches and allows for the study of aftershock statistics. The avalanche statistics changes significantly (see sec. III). In both the cases (IC and DC), the avalanche size distribution exponent becomes $\delta = 0.70 \pm 0.01$ and $\alpha = 0.40 \pm 0.05$, $z = 1.20 \pm 0.06$. The universality of the avalanche statistics is clear: these exponent values remain unchained irrespective of the details of the models (IC or DC or values of spring constants etc.). Next we study the after-shock behaviour (Omori law) of the avalanches. We first search for upper size ($s_u$) of the avalanche and choose a lower cut-off $s_l$. An avalanche of size greater than $s_u$ is then taken as the main-shock. We calculate the probability of having an aftershock of size $s_l$ or above after time $t$ of the main-shock. This follows a scale free behavior (see Eq. (4)) with exponent $0.85 \pm 0.05$, analogous to Omori law. This Omori law was not detected earlier in any spring-block type model. Like the EW model for depinning, we measure the depinned velocity for the train model (see Fig. 12). The depinning exponent value ($0.29 \pm 0.01$) is the same as the EW depinning exponent value. The exponent values mentioned above changes significantly in two dimensions (see Fig. 5), though remain universal (independent of spring constant, pinning force distribution etc.). The value of the avalanche exponent becomes $\delta = 1.15 \pm 0.01$; the other finite size scaling exponents are $\alpha = 0.07 \pm 0.01, z = 1.27 \pm 0.02$. The Omori law exponent value ($0.90 \pm 0.05$), however, remains close to one dimensional value. The fluctuation propagation through the bulk in the 2d EW model was also studied. It is found that when the disturbance (of pulling the system by one end line) has not reached the other end, the fluctuation is maximum just half-way of the total lattice constant propagated (see Fig. 15). The propagation front has a power-law decay in velocity, with exponent value $0.50 \pm 0.01$ (see Fig. 16), while the r.m.s. fluctuation grows only logarithmically.

In conclusion, a discretized version of the Burridge-Knopoff train model with (non-linear friction force replaced by) random pinning is studied in one and two dimensions. With suitable definitions of avalanches, a scale free distribution of avalanche and the Omori law type behaviour for after-shocks are obtained. With this simplification, the avalanche dynamics of the model becomes precisely similar (identical exponent values) to the Edwards-Wilkinson model of interface propagation. It also allows the complimentary observation of depinning velocity growth with exponent value identical with that for Edwards-Wilkinson model in train model and Omori law behaviour of after-shock (depinning) avalanches in the Edwards-Wilkinson model. The observations of universalities of avalanche dynamics and interface depinning dynamics should shed new light in the existing conjectures regarding the statistical similarity in the dynamics of interface and fracture [17], interface and earthquake [11] and the recent experimental observations regarding the dynamics of fracture and earthquakes [13].

---

[1] R. Burridge, L. Knopoff, Bull. Seismol. Soc. A. 57, 341 (1967).
[2] H. Kawamura, T. Hatano, N. Kato, S. Biswas, B. K. Chakrabarti, Rev. Mod. Phys. 84, 839 (2012).
[3] M. Vieira, Phys. Rev. A 46. 6288 (1992).
[4] C. V. Chianca, J. S. S. Martins, P. M. C. de Oliveira, Eur. Phys. J. B 68, 549 (2009).
[5] S. F. Edwards, D. R. Wilkinson, Proc. R. Soc. A 381, 17 (1982).
[6] Fractal concepts in surface growth, A.-L. Barabási, H. E. Stanley, Cambridge University Press, Cambridge (1995).
[7] B. K. Chakrabarti, R. B. Stinchcombe, Physica A 270, 27 (1999); P. Bhattacharyya, Physica A 348, 199 (2005).
[8] Modelling critical and catastrophic phenomena in geoscience, P. Bhattacharyya, B. K. Chakrabarti (Eds.), Springer-Verlag, Heidelberg (2006).
[9] J. A. Eriksen, S. Biswas, B. K. Chakrabarti, Phys. Rev. E 82, 041124 (2010).
[10] D. Cule, T. Hwa, Phys. Rev. Lett. 77, 278 (1996).
[11] M. Paczuski, S. Boettcher, Phys. Rev. Lett 77, 111 (1996).
[12] A. Malthe-Sørenssen, Phys. Rev. E 59, 4169 (1999).
[13] N. J. Zhou, B. Zheng, Phys. Rev. E 82, 031139 (2010).
[14] S. Ramanathan and D. S. Fisher, Phys. Rev. B 58, 6026 (1998).
[15] J. Baró, A. Corral, X. Illa, A. Planes, E. K. H. Salije, W. Schranz, D. E. Solo-Parra, E. Vives, Phys. Rev. Lett. 110, 088702 (2013).