On the scaling of the electroweak interface tension at finite temperature

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Abstract
We determine the interface tension of the finite-temperature electroweak phase transition in a numerical investigation of the SU(2)–Higgs model on a four-dimensional lattice with temporal extension $L_t = 3$. In this simulation the chosen parameters correspond to a Higgs boson mass of about 16 GeV. As a result the interface tension shows only small scaling violations in comparison with previous studies for $L_t = 2$ lattices. We also report on some experiences with autocorrelations in the applied Monte Carlo simulations of two-phase systems.

1 Introduction

According to the ideas of cosmology and elementary particle physics the electroweak phase transition takes place with decreasing temperature between a symmetric phase and a phase with broken symmetry, the Higgs phase. An order parameter of this transition is the vacuum expectation value of the scalar field, which has a non-zero value in the broken phase and vanishes in the high-temperature, symmetry-restored phase. 

The main interest in the electroweak phase transition in the early universe emerges from the question, whether it alone can provide a mechanism for the observed baryon asymmetry within the minimal Standard Model. A necessary condition for this scenario, essentially depending on the Higgs boson mass value $m_H$, is a transition of strong enough first order type, whereas for a weak transition every asymmetry generated at the phase transition would be washed out in the Higgs phase, because at temperatures $T$ larger than the vector boson mass $m_W$ the baryon and lepton number violation in the Standard Model is enhanced.
After the neglection of the U(1)–gauge field and the fermionic sector one is left with the SU(2)–Higgs model containing all important ingredients for these phenomena. The common theoretical tool for its examination is resummed perturbation theory \[4, 5, 6\], which works well for low and intermediate Higgs masses \(m_H < 50 \text{ GeV}\). Besides the known problems caused by the infrared divergencies in the symmetric phase, its validity is not ensured any longer for higher values of \(m_H\), where the strength of the electroweak phase transition is expected to decrease rapidly. This motivates a nonperturbative treatment on space-time lattices in four dimensions as well as in reduced three-dimensional models, see e.g. refs. \[7, 14, 16, 17\] and ref. \[15\] for a more general review of the whole subject.

The present work is a completion of numerical simulations of the four-dimensional SU(2)–Higgs model on the lattice in \[9, 10\], which focussed on Higgs boson masses below 50 GeV. Since this mass range is ruled out by the actual experimental bound of \(m_H \gtrsim 65 \text{ GeV}\), their main goal is to establish suitable methods for extracting physical quantities from the lattice investigations, and to hint at systematic errors in resummed perturbation theory or dimensional reduction approaches.

In this publication we deal with the interface tension \(\sigma\). This quantity plays a prominent rôle in the course of the electroweak phase transition, because its magnitude is a measure for the strength of this transition. The nucleation rate of the Higgs phase in the symmetric phase is given in the thin-wall approximation by

\[
\Gamma_{\text{nucl}} = \Gamma_0 \exp \left( -\frac{16\pi}{3} \frac{\sigma^3}{(\Delta\epsilon)^2 T_c} \eta^{-2} \right). \tag{1}
\]

Here \(T_c\) denotes the critical temperature, \(\Delta\epsilon\) the latent heat, and \(\eta \equiv \frac{T_c - T}{T_c}\) is the so-called supercooling parameter. The prefactor \(\Gamma_0\) can be approximated by \(T^4\) \[3\]. If \(\sigma^3/(\Delta\epsilon)^2 T_c\) is large, signalling a strong first order phase transition, a substantial supercooling has to be expected, which would lead to an additional suppression of the sphaleron rate.

In the following we use a low value of the Higgs boson mass, where the phase transition is quite strong. For the determination of \(\sigma\) the two-coupling method in the scalar hopping parameter is employed. Compared to transfer-matrix techniques and histogram methods, it gives an optimal ratio between desired accuracy and required CPU-time for the SU(2)–Higgs model \[9, 10, 11\].

\section{Lattice simulation}

The lattice action of the SU(2)–Higgs model is conventionally \[13\] parametrized as

\[
S[U, \varphi] = \beta \sum_p \left( 1 - \frac{1}{2} \text{Tr} U_p \right) \nonumber \\
+ \sum_{x \in \Omega} \left\{ \frac{1}{2} \text{Tr} (\varphi^+_x \varphi_x) + \lambda \left[ \frac{1}{2} \text{Tr} (\varphi^+_x \varphi_x) - 1 \right]^2 - \kappa \sum_{\mu=1}^4 \text{Tr} (\varphi^+_{x+\mu} U_{x,\mu} \varphi_x) \right\} \tag{2}
\]
in terms of the SU(2)–link variables $U_{x,\mu}$ and the site variables $\varphi_x$, which are complex $2 \otimes 2$ matrices, representing the gauge and scalar degrees of freedom, respectively. $U_p = U_{x,\mu} U_{x+\hat{\mu},\nu} U_{x,\nu}^+ U_{x,\nu}^+$ is an elementary plaquette, and we often decompose the Higgs field as $\varphi_x = \rho_x \alpha_x$ with $\rho_x \in \mathbb{R}_{>0}$ and $\alpha_x \in \text{SU}(2)$. If not stated otherwise, the lattice constant $a$ is assumed to be $a = 1$, and the lattice volume is denoted by $\Omega$. The identifications $g^2 = 4/\beta$, $m_0^2 = (1-2\lambda)/\kappa - 8$ and $\lambda_0 = \lambda/4\kappa^2$ relate the lattice parameters $\beta$, $\kappa$ and $\lambda$ to the bare gauge coupling, scalar mass and quartic coupling of the corresponding continuum theory.

A suitable observable for the interface tension is the density of the $\varphi$–link operator $L_{\varphi;x\mu}$, which is conjugate — in the thermodynamic sense — to the hopping parameter $\kappa$ and is itself an order parameter of the phase transition:

$$L_{\varphi} \equiv \frac{1}{4\Omega} \sum_{x \in \Omega} \sum_{\mu=1}^{4} L_{\varphi;x\mu}, \quad L_{\varphi;x\mu} \equiv \frac{1}{2} \text{Tr} \left( \varphi_{x+\hat{\mu}}^+ U_{x,\mu} \varphi_x \right).$$ (3)

When simulating finite-temperature field theory, one utilizes lattices with spacelike extensions much larger than the temporal extension: $L_s \gg L_t$, $s \in \{x, y, z\}$. The physical temperature is given by the timelike lattice extension via $T = 1/aL_t$, and the approach to the scaling region of the model in the continuum limit is realized as $a \to 0$ with $T$ fixed, i.e. as $L_t \to \infty$. This limit goes along the lines of constant physics, on which renormalized couplings and masses are held fixed and only the lattice constant $a$ is varying.

In this spirit we extend the measurement of $\sigma$ at $a^{-1} = 2T_c$ in ref. [10] to $a^{-1} = 3T_c$ with the purpose of gaining control over possible lattice artifacts. The parameters in the former analysis were $\beta = 8.0$ and $\lambda = 0.0001$, leading to a renormalized gauge coupling of $g_R^2 \approx 0.56$ and a Higgs mass of $m_H \approx 16$ GeV. The physical mass scale is set by the vector boson mass value $m_W = 80$ GeV at $T = 0$ in lattice units. A two-coupling simulation on a lattice of size $2 \times 16^2 \times 128$ resulted in a phase transition point at $\kappa_c = 0.12830(5)$ and a finite-volume estimator for the interface tension of $\hat{\sigma}/T_c^3 = 0.84(16)$.

When passing over to smaller lattice spacings, we have to scale all lattice extensions accordingly in order to keep the physical volume constant. In the case of low Higgs mass this is possible with an acceptable demand of computer resources. With increasing $m_H$ the situation becomes worse, because the phase transition weakens and thus one needs larger physical volumes to obtain a stable two-phase situation. Therefore we choose a lattice of size $3 \times 24^2 \times 192$ with the changed parameters $\beta = 8.15$ and $\lambda = 0.00011$. They have been obtained by an integration of the one-loop perturbative renormalization group equations with the transition point of the $L_t = 2$ lattice as a starting value [11]. As shown in ref. [12], a sufficiently precise estimate for the critical point in $\kappa_c$ is only available by numerical methods.

In our Monte Carlo simulations we use an optimized combination of heatbath and over-relaxation algorithms. In particular, the introduction of the simultaneous overrelaxation of all four cartesian components of the Higgs field [13], instead of the Higgs field length-overrelaxation as proposed in [8], has substantially reduced the integrated autocorrelation...
time \( \tau_{\text{int}} \). In this way we gained factors of \( 3 - 10 \) compared to the values previously found in [11]. This \( \tau_{\text{int}} \)-behaviour is reflected in table 1 and figure 1 for the case of a two-phase simulation, where as typical examples the autocorrelation functions \( \Gamma(t) \) of the \( \phi \)-link operators (3) in both phases and their difference \( \Delta L_{\phi} \) are considered. The observation that this difference has a significantly lower \( \tau_{\text{int}} \) will become relevant in the next section.

| heatbath | overrelaxation | \( \tau_{\text{int}} \) in sweeps |
|----------|----------------|-------------------------------|
| \( U_{x,\mu} \) \( \phi_x \) | \( L_{\phi}^{(1)} \) | 20(6) | 9.4(3.3) | 9.2(3.1) |
| \( U_{x,\mu} \) \( \rho_x \) | \( L_{\phi}^{(2)} \) | 6.3(1.1) | 2.2(4) | 2.0(4) |

Table 1: Autocorrelation times for a 2-\( \kappa \) simulation. Each updating sweep consists of a sequence of different algorithms as given by the numbers in the left part of the table.

![Figure 1: Normalized autocorrelation functions for a 2-\( \kappa \) simulation with \( \rho \)-overrelaxation (left) and \( \phi \)-overrelaxation (right) in the updating sequence.](image)

3 Two-coupling method

The calculation of the phase transition point and the interface tension is done by a modified version of the two-coupling method [19]. Since this method has been used in the SU(2)–Higgs model before, see refs. [9, 10, 11], we only sketch its main idea here. One takes a periodic
lattice with one elongated spacelike direction, i.e. \(L_z \gg L_x = L_y \gg L_t\), and divides the corresponding lattice volume into two halves with different scalar hopping parameters
\[
\kappa = (\kappa_1, \kappa_2) \equiv (\kappa_1 < \kappa_c \text{ for } z \leq L_z/2, \kappa_2 > \kappa_c \text{ for } z > L_z/2) .
\] (4)

Hence the lower half is forced in the symmetric phase and the upper one in the Higgs phase. If in the foregoing inequality the transverse directions \(L_x\) and \(L_y\) are sufficiently large, the system resides in a mixed-phases state and gives rise to an interface pair at the phase boundary perpendicular to the \(z\)–direction.

In a first step one initializes a two-phase situation by a simulation with \(\kappa\)–values far away from the transition point, whose location can roughly be determined by hysteresis runs. Subsequently, the distance in \(\kappa\) is more and more diminished, and the smallest \(\kappa\)–interval, for which the system still resists to turn over into one single phase, gives lower and upper bounds for the critical \(\kappa\). On the \(3 \times 24^2 \times 192\) lattice we obtain the estimate
\[
\kappa_c = 0.128110(3) .
\] (5)

The interface tension is defined as the free energy \(F\) per unit area of the walls separating the two phases. As derived in ref. [10], its lattice version
\[
a^3 \sigma = \frac{1}{2L_x L_y L_t} \left\{ F(\kappa_1, \kappa_2) - \frac{1}{2} F(\kappa_1, \kappa_1) - \frac{1}{2} F(\kappa_2, \kappa_2) \right\}
\] (6)
is related to the expectation values \(L_{\varphi}^{(1)}(\kappa_1, \kappa_2)\) and \(L_{\varphi}^{(2)}(\kappa_1, \kappa_2)\) of \(L_{\varphi}\) in each phase by
\[
a^3 \sigma = \frac{1}{2} \lim_{\kappa_2 \rightarrow \kappa_c} \lim_{\kappa_1 \rightarrow \kappa_c} \left\{ (\kappa_1 - \kappa_2) \lim_{L_z \rightarrow \infty} L_z \cdot \Delta L_{\varphi}(\kappa_1, \kappa_2) \right\} ,
\] (7)
where \(\Delta L_{\varphi}(\kappa_1, \kappa_2)\) denotes their difference \(L_{\varphi}^{(2)}(\kappa_1, \kappa_2) - L_{\varphi}^{(1)}(\kappa_1, \kappa_2)\). With the \((N + 2)\)–parametric Laurent ansatz
\[
L_{\varphi}^{(i)}(\kappa_1, \kappa_2) = -\frac{c_i}{\kappa_i - \kappa_c} + \sum_{j=0}^N \gamma_i^{(j)}(\kappa_i - \kappa_c)^j + \mathcal{O}\left(|\kappa_i - \kappa_c|^{N+1}\right), \quad i = 1, 2 ,
\] (8)
this leads to the finite-volume estimator for the interface tension
\[
a^3 \hat{\sigma} = L_z (c_1 + c_2) .
\] (9)
The inverse-linear term in eq. (8) is motivated by the fact that for \(\Delta \kappa \equiv |\kappa_i - \kappa_c| \ll 1\) the free energy change between the two phases behaves as \(\Delta F \simeq \mathcal{O}(\Delta \kappa)\). The probability \(p\) for an interface being at \(z_0 > z\) is essentially given by \(\exp\{-\text{const} \cdot \Delta \kappa(z - z_0)\}\) and therefore \(\int dz \, p \simeq 1/\Delta \kappa\).

Thus one proceeds in a similar way as for the \(\kappa_c\)–determination, but due to practical limitations on \(L_z\), one has to prevent the interfaces from touching each other by a choice
of large enough $\kappa$–intervals. Figure 2 displays typical two-phase structures from the 2–$\kappa$ method. Note that the plateaus become narrower for smaller $\kappa$–intervals, especially in the broken phase, but the phases are still clearly developed, and the interfaces continue to exist.

The results of our simulations are presented in table 2. Owing to correlations between $L^{(1)}_\varphi$ and $L^{(2)}_\varphi$, the statistical errors on $\Delta L_{\varphi}$ are usually smaller than those of $L^{(2)}_\varphi$. This was already suggested by the autocorrelations in figure 1 and is caused by shifts of the interfaces during the simulation, which cancel out in $\Delta L_{\varphi}$ to some extent.

| $\kappa_1$ | $\kappa_2$ | sweeps | $L^{(1)}_\varphi$ | $L^{(2)}_\varphi$ | $\Delta L_{\varphi}$ |
|-----------|-----------|--------|----------------|----------------|----------------|
| 0.12776   | 0.12846   | 5000   | 1.5161(24)      | 37.8216(70)    | 36.3055(67)    |
| 0.12781   | 0.12841   | 5000   | 1.5149(26)      | 34.7287(68)    | 33.2138(60)    |
| 0.12786   | 0.12836   | 5000   | 1.5285(29)      | 31.5085(77)    | 29.9799(76)    |
| 0.12791   | 0.12831   | 10000  | 1.5398(19)      | 28.1144(55)    | 26.5746(57)    |
| 0.12796   | 0.12826   | 10000  | 1.5616(30)      | 24.5264(76)    | 22.9648(71)    |
| 0.12801   | 0.12821   | 20000  | 1.6019(32)      | 20.5877(66)    | 18.9858(59)    |

Table 2: Results for $L^{(1)}_\varphi$, $L^{(2)}_\varphi$ and $\Delta L_{\varphi}$ on a $3 \times 24^2 \times 192$ lattice. The errors are obtained by binning.

In order to give a reliable estimate for the statistical error when fitting these $\varphi$–link averages to the functions in eq. (8), we take their correlations into account by a bootstrap analysis.
The characteristic ingredient is to calculate secondary quantities from bootstrap sub-
samples randomly taken with repetition from the original measurements, which itself can be
interpreted as the empirical probability distribution of the observables under consideration.
The errors of the fit parameters are extracted as half of the central 68.3%–interval of their
distributions from fits performed on these sample averages. A more extensive description of
this method in the same context is contained in ref. [11].

We quote our final result for two four-parameter fits with $\chi^2$–values of 2.58 and 0.82.
Using (9), we find
\[
\left( \frac{\hat{\sigma}}{T_c^3} \right)_{L_{\varphi}} = 0.764(52 + 47),
\]
whereby $aT_c = 1/L_t = \frac{1}{3}$ in lattice units. The error consists of two parts, coming from a
bootstrap analysis with 10000 iterations and the uncertainty of $\kappa_c$ in eq. (5).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure.png}
\caption{Four-parameter least-squares fits of $L_{\varphi}^{(i)}$, $i = 1, 2$, separately in each phase.}
\end{figure}

At this place some explanations about the number of necessary fit parameters are in
order. In the examinations of $L_t = 2$ lattices [10, 11] a three-parameter fit was sufficient
to give a reliable value for the interface tension; an inclusion of higher order terms gave
no improvement. The situation for $L_t = 3$ seems to be different for the following reason:
Basically, one has to keep in mind that the expansion in eq. (8) is only a phenomenological
ansatz. Furthermore the lattice volume $\Omega$ has increased, and the phase transition, which
in a strict sense is only present in infinite volumes, is more pronounced. Consequently, the
contribution of the higher Laurent coefficients becomes more important for the slopes at
larger $\kappa$–intervals.

In fact, the three-parameter fit has no satisfactory $\chi^2$. A five-parameter fit gives
$\hat{\sigma}/T_c^3 = 0.85(20 + 5)$ and $\chi^2$–values equal to 1.42 and 0.82. This is compatible to [10].
within errors, but fairly sensitive to the number of fitted data points. A more careful inspection of the bootstrap calculations reveals that the last fit parameter is not very significant in such cases, where its bootstrap error is roughly as large as its value; this holds true for the five-parameter fits and also the four-parameter fit in the symmetric phase. So for the sake of completeness we combined the three-parameter fit in the symmetric phase with the four-parameter fit in the broken phase to $\hat{\sigma}/T_c^3 = 0.793$, although we are nevertheless convinced of the four-parameter fit in both phases to lead to the most reasonable result.

The preceding remarks on the $\varphi$–link correlations should have made clear that $\Delta L_\varphi$ is the most natural variable for estimating the interface tension via eq. (7). This requires that the chosen $\kappa$–intervals are symmetric with respect to $\kappa_c$. A four-parameter fit to an ansatz similar to (8) for $\Delta L_\varphi$ yields

$$\left(\frac{\hat{\sigma}}{T_c^3}\right)_{\Delta L_\varphi} = 0.767(53)$$

with $\chi^2 = 0.39$ and is illustrated in figure 4. Here we only quote the statistical error, which now comes from 1000 normally distributed random data, since the different $\Delta L_\varphi$–averages are uncorrelated. Note the perfect agreement of this result, and especially of its error, with the numbers in eq. (10) above.

Figure 4: Four-parameter least-squares fit of $\Delta L_\varphi$ as a function of $\kappa_2 - \kappa_1$. 

\lattice{3\times24\times24\times192, \beta=8.15, \lambda=0.00011}
4 Discussion

We have determined the interface tension of the four-dimensional SU(2)–Higgs model with the two-coupling method. All estimates from acceptable fits with a reasonable number of parameters show a very good consistency.

To our knowledge this is the first attempt to supply information about the scaling of $\sigma$ when going over to a finer lattice. Compared to the $L_t = 2$ results

\[
\left( \frac{\hat{\sigma}}{T_c^3} \right)_{2-\kappa} = 0.84(16), \quad \left( \frac{\sigma}{T_c^3} \right)_{\text{hist}} = 0.83(4) \tag{12}
\]

from refs. [9, 10] — the second number referring to the histogram method — the accuracy of the $2-\kappa$ estimate has been improved to the value $\hat{\sigma}/T_c^3 = 0.76(10)$ in eq. (14). The observed small deviation between $L_t = 2$ and $L_t = 3$ confirms the smallness of scaling violations, as e.g. also found recently for the critical temperature in [12].

Finally, we confront our result with the perturbative estimate [3] up to order $g^4, \lambda_0^2$

\[
\left( \frac{\sigma}{T_c^3} \right)_{\text{pert}} = 0.78(1), \tag{13}
\]

with an error coming from the uncertainties in the renormalized parameters on the lattice. As expected, the agreement with perturbation theory on a quantitative level in this range of the Higgs boson mass is excellent.

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