On a formula of Thompson and McEnteggert for the adjugate matrix

Kenier Castillo\textsuperscript{a,1}, Ion Zaballa\textsuperscript{b,2},

\textsuperscript{a} CMUC, Department of Mathematics, University of Coimbra, 3001-501 Coimbra, Portugal.
\textsuperscript{b} Departamento de Matemática Aplicada y EIO. Universidad del País Vasco (UPV/EHU).
Apdo. 644. 48080 Bilbao. Spain.

Abstract

For an eigenvalue $\lambda_0$ of a Hermitian matrix $A$, the formula of Thompson and McEnteggert gives an explicit expression of the adjugate of $\lambda_0I - A$, $\text{Adj}(\lambda_0I - A)$, in terms of eigenvectors of $A$ for $\lambda_0$ and all its eigenvalues. In this paper Thompson-McEnteggert’s formula is generalized to include any matrix with entries in an arbitrary field. In addition, for any nonsingular matrix $A$, a formula for the elementary divisors of $\text{Adj}(A)$ is provided in terms of those of $A$. Finally, a generalization of the eigenvalue-eigenvector identity and two applications of the Thompson-McEnteggert’s formula are presented.

Keywords: Adjugate, eigenvalues, eigenvectors, elementary divisors, rank-one matrices.

2000 MSC: 15A18, 15A15

1. Introduction

Let $\mathcal{R}$ be a commutative ring with identity. Following [10, Ch. 30], for a polynomial $p(\lambda) = \sum_{k=0}^{n} p_k \lambda^k \in \mathcal{R}[\lambda]$ its derivative is $p'(\lambda) = \sum_{k=1}^{n} k p_k \lambda^{k-1}$. Recall that if $X \in \mathcal{R}^{n \times n}$ is a square matrix of order $n$ with entries in $\mathcal{R}$ and $M_{ij}(X)$ is the minor obtained from $X$ by deleting the $i$th row and $j$th column then the adjugate of $X$, $\text{Adj}(X)$, is the matrix whose $(i,j)$ entry is $(-1)^{i+j} M_{ji}(X)$; that is, $\text{Adj}(X) = \left[ (-1)^{i+j} M_{ji}(X) \right]_{1 \leq i, j \leq n}$.

Formula (1) below, from now on TM formula, was proved, with $w = v$ and the normalization $w^*v = 1$, for a Hermitian matrix $A \in \mathbb{C}^{n \times n}$ by Thompson and McEnteggert (see [33, pp. 212-213]). Inspection of the proof shows that the formula also holds for normal matrices over $\mathbb{C}$ (see [28]). With the same
arguments we can go further. Recently, Denton, Parke, Tao, and Zhang pointed out that the TM formula has an extension to a non-normal matrix $A \in \mathbb{R}^{n \times n}$, so long as it is diagonalizable (see [12 Rem. 4]). Even more, as shown in Remark 5 of [12] it holds for matrices over commutative rings (see [17] for an informal proof). A more detailed proof of this result will be given in Section 2. However, for matrices over fields (or over integral domains) with repeated eigenvalues, (1) does not provide meaningful information (see Remark 2.4). We will exhibit in Section 2 a generalization of the TM formula which holds for matrices over arbitrary fields with repeated eigenvalues. This new TM formula will be used to generalize the so-called eigenvector-eigenvalue identity (see (20)) for non-diagonalizable matrices over arbitrary fields. In addition we will provide a complete characterization of the similarity invariants of $\text{Adj}(A)$ in terms of those of $A$, generalizing a result about the eigenvalues and the minimal polynomial in [18]. Then in Section 3 two additional consequences of the TM formula will be analysed.

2. The TM formula and its generalization

Let $A \in \mathbb{R}^{n \times n}$ be a square matrix of order $n$ with entries in $\mathbb{R}$. An element $\lambda_0 \in \mathbb{R}$ is said to be an eigenvalue of $A$ if $Ax = \lambda_0 x$ for some nonzero vector $x \in \mathbb{R}^{n \times 1}$ ([7 Def. 17.1]). This vector is said to be a right eigenvector of $A$ for (or associated with) $\lambda_0$. The left eigenvectors of $A$ for $\lambda_0$ are the right eigenvectors for $\lambda_0$ of $A^T$, the transpose of $A$, or, if $\mathbb{R} = \mathbb{C}$ is the field of complex numbers, of $A^*$, the conjugate transpose of $A$. That is to say, $y \in \mathbb{R}^{n \times 1}$ is a left eigenvector of $A$ for $\lambda_0$ if $y^T A = \lambda_0 y^T$ (or $y^* A = \lambda_0 y^*$ if $\mathbb{R} = \mathbb{C}$). The characteristic polynomial of $A$ is $p_A(\lambda) = \det(\lambda I_n - A)$ and $\lambda_0$ is an eigenvalue of $A$ if and only if $p_A(\lambda_0) \in \mathbb{Z}(\mathbb{R})$, where $\mathbb{Z}(\mathbb{R})$ is the set of zero divisors of $\mathbb{R}$ ([7 Lem. 17.2]).

The following result, in a slightly different form, was proved by D. Grinberg in [17].

**Theorem 2.1.** Let $A \in \mathbb{R}^{n \times n}$ and let $\lambda_0 \in \mathbb{R}$ be an eigenvalue of $A$. Let $v, w \in \mathbb{R}^{n \times 1}$ be a right and a left eigenvector, respectively, of $A$ for $\lambda_0$. Then

$$w^T v \text{Adj}(\lambda_0 I_n - A) = p_A'(\lambda_0)vw^T.$$  \hspace{1cm} (1)

The proof in [17] is based on the Lemma 2.2 below which is interesting in its own right. According to McCoy’s theorem ([12 Th. 5.3]) there is a non-zero vector $x \in \mathbb{R}^{n \times 1}$ such that $Ax = 0$ if and only if $\text{rk}(A) < n$, where $\text{rk}(A)$ is the (McCoy) rank of $A$ ([7 Def. 4.10]). In other words, 0 is an eigenvalue of $A$ if and only if $\text{rk}(A) < n$. Note that $\text{rk}(A) = \text{rk}(A^T)$.

**Lemma 2.2.** Let $A \in \mathbb{R}^{n \times n}$ be a matrix such that $\text{rk}(A) < n$ and let $w \in \mathbb{R}^{n \times 1}$ be a left eigenvector of $A$ for the eigenvalue 0. For $j = 1, \ldots, n$, let $(\text{Adj} A)_j$ be the $j$th column of $\text{Adj} A$. Then, for all $i, j = 1, \ldots, n$,

$$w_i (\text{Adj} A)_j = w_j (\text{Adj} A)_i,$$  \hspace{1cm} (2)

where $w = [w_1 \ w_2 \ \cdots \ w_n]^T$.  

2
This is Lemma 3 of [17]. The author himself considers the proof to be informal. So a detailed proof of Lemma 2.2 following Grinberg’s ideas is given next for reader’s convenience.

Proof of Lemma 2.2. Let us take \(i, j \in \{1, \ldots, n\}\) and assume that \(i \neq j\); otherwise, there is nothing to prove. We assume also, without lost of generality, that \(i < j\). Let \(w = [w_1 \ w_2 \ \cdots \ w_n]^T\) and, for \(k = 1, \ldots, n\), let \(a_k\) be the \(k\)th row of \(A\). Define \(B \in \mathbb{R}^{n \times n}\) to be the matrix whose \(k\)th row, \(b_k\), is equal to \(a_k\) if \(k \neq i, j\) and \(b_k = w_k a_k\) if \(k = i, j\). A simple computation shows that \(w_i (\text{Adj } A)_j = (\text{Adj } B)_j\) and \(w_j (\text{Adj } A)_i = (\text{Adj } B)_i\). We claim that \((\text{Adj } B)_j = (\text{Adj } B)_i\). This would prove the lemma.

It follows from \(w^T A = 0\) that \(\sum_{k=1}^{n} w_k a_k = 0\) and so
\[
\sum_{k=1, k \neq i, j}^{n} w_k b_k = 0.
\]

Let
\[
P = \begin{bmatrix}
1 & \cdots & 1 \\
\cdot & \ddots & \cdot \\
-1 & \cdots & -1 \\
1 & \cdots & 0 \\
-1 & \cdots & 1 \\
1 & \cdots & 1 \\
\cdot & \ddots & \cdot \\
1 & \cdots & 1
\end{bmatrix}
\]

This matrix is invertible in \(\mathbb{R}\) (its determinant is 1) and by (3),
\[
\tilde{B} = PB = [b_1^T \ \cdots \ b_{i-1}^T \ b_j^T \ b_{i+1}^T \ \cdots \ b_{j-1}^T \ b_i^T \ b_{j+1}^T \ \cdots \ b_n^T]^T.
\]

Then, \(\text{Adj}(\tilde{B}) = \text{Adj}(B) \text{Adj}(P)\) and, since \(P\) is invertible, \(\text{Adj}(P) = (\det P) P^{-1} = P^{-1}\). Hence \(\text{Adj}(B) = \text{Adj}(\tilde{B}) P\) and for \(k = 1, \ldots, n\)
\[
(\text{Adj } B)_{ki} = \sum_{\ell=1}^{n} (\text{Adj } \tilde{B})_{k\ell} P_{\ell i}.
\]

But in the \(i\)th column of \(P\) the only nonzero entry is \(-1\) in position \((i, i)\). Therefore, \((\text{Adj } B)_{ki} = -(\text{Adj } \tilde{B})_{ki}\). Now, taking into account that \(\tilde{B}\) is the

\[\text{(3)}\]

Grinberg’s permission was granted to include the proofs of this Lemma and Theorem 2.1
matrix $B$ with rows $i$th and $j$th interchanged and recalling that $M_{ij}(X)$ is the minor of $X$ obtained by deleting the $i$th row and $j$th column of $X$, we get

$$(\text{Adj } B)_{ki} = -(\text{Adj } \tilde{B})_{ki} = (-1)^{k+i+1}M_{ik}(\tilde{B}) = (-1)^{k+i+1}(-1)^{j-i-1}M_{jk}(B) = (-1)^{k+i+1}M_{ik}(\tilde{B}) = (\text{Adj } B)_{kj},$$

as claimed.

There is a “row version” of Lemma 2.2 which can be proved along the same lines.

**Lemma 2.3.** Let $A \in \mathbb{R}^{n \times n}$ be a matrix such that $\text{rk}(A) < n$ and let $v \in \mathbb{R}^{n \times 1}$ be a right eigenvector of $A$ for the eigenvalue 0. For $j = 1, \ldots, n$ let $(\text{Adj } A)^j$ be the $j$th row of $(\text{Adj } A)$. Then, for all $i, j = 1, \ldots, n$,

$$v_i (\text{Adj } A)^j = v_j (\text{Adj } A)^i,$$

where $v = [v_1 \ v_2 \ \cdots \ v_n]^T$.

The proof of Theorem 2.1 which follows is very much that of Grinberg in [17]. It is included for completion and reader’s convenience.

**Proof of Theorem 2.1.** Let $B = \lambda_0 I_n - A$ and $p_B(\lambda) = \det(\lambda I_n - B)$ its characteristic polynomial. Then $p_B(\lambda) = \lambda^n + \sum_{k=1}^n (-1)^k c_k \lambda^{n-k}$ where, for $k = 0, \ldots, n$,

$$c_k = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \det B(i_1 : i_k, i_1 : i_k),$$

and $B(i_1 : i_k, i_1 : i_k) = [b_{ij}, i_1 \leq j, \ell \leq k]$ is the principal submatrix of $B$ formed by the rows and columns $i_1, \ldots, i_k$. In particular, $c_{n-1} = \sum_{j=1}^n M_{jj}(B)$ where $M_{jj}(B)$ is the principal minor of $B$ obtained by deleting the $j$th row and column. Thus $p_B'(0) = (-1)^{n-1} \sum_{j=1}^n M_{jj}(B)$.

On the other hand, $\det(\lambda I_n - B) = \det(\lambda I_n - \lambda_0 I_n + A) = (-1)^n \det((\lambda_0 - \lambda) I_n - A) = (-1)^n p_A(\lambda_0 - \lambda)$. It follows from the definition of derivative of a polynomial that

$$p_A'(\lambda_0) = (-1)^{n+1}p_B'(0) = \sum_{j=1}^n M_{jj}(\lambda_0 I_n - A).$$

Hence, proving (1) is equivalent to proving

$$w^T v \text{Adj}(B) = \sum_{j=1}^n M_{jj}(B)v w^T$$

where $B = \lambda_0 I_n - A$. It follows from $Av = \lambda_0 v$ and $w^T A = \lambda_0 w^T$ that $Bv = 0$ and $w^T B = 0$, respectively. So we can apply to $B$ properties (2) and (4).
It follows from (2) that \( w_k(\text{Adj} \ B)_{ij} = w_j(\text{Adj} \ B)_{ik} \) for all \( i, j, k \in \{1, \ldots, n\} \). Then \( v_k w_k(\text{Adj} \ B)_{ij} = w_j v_k(\text{Adj} \ B)_{ik} \) and from (1), \( v_k(\text{Adj} \ B)_{ik} = v_i(\text{Adj} \ B)_{kk} \). Hence,
\[
v_k w_k(\text{Adj} \ B)_{ij} = v_i w_j(\text{Adj} \ B)_{kk}, \quad i, j, k = 1, \ldots, n.
\]
Adding on \( k \) and taking into account that \( (\text{Adj} \ B)_{kk} = M_{kk}(B) \), we get
\[
w^T v(\text{Adj} \ B)_{ij} = \sum_{k=1}^n M_{kk}(B)v_i w_j, \quad i, j = 1, \ldots, n.
\]
This is equivalent to (5) and the theorem follows.

\[\square\]

**Remark 2.4.** Assume that \( \mathcal{R} \) is an integral domain and note that in this case \( \text{rk}(A) = \text{rank}(A) \); i.e., the McCoy rank and the usual rank coincide. It is an interesting consequence of (1) that \( w^T v = 0 \) implies \( p'_A(\lambda_0) = 0 \). The converse is not true in general. For example, if \( A = \lambda_0 I_2 \) then \( v = [1 \ 0]^T \) satisfies both \( A v = \lambda_0 v \) and \( v^T A = \lambda_0 v^T \), but \( v^T v = 1 \) and \( p'_A(\lambda_0) = 0 \). However, if \( p'_A(\lambda_0) = 0 \) and \( \text{rank}(\lambda_0 I_n - A) = n - 1 \) then, necessarily, \( w^T v = 0 \) because \( \text{adj}(\lambda_0 I_n - A) \) is not the zero matrix. In particular, if \( \mathbb{F} \) is a field of characteristic zero (see [16, Ch. 30]) then it follows from (1) that if \( w^T v = 0 \) then \( \lambda_0 \) is an eigenvalue of algebraic multiplicity at least 2. On the other hand, it is easily checked that if \( \lambda_0 \) is an eigenvalue of algebraic multiplicity bigger that 1 and geometric multiplicity 1 then \( w^T v = 0 \) for any right and left eigenvectors, \( v \) and \( w \) respectively, of \( A \) for \( \lambda_0 \). This is the case, for example, of \( A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \). For this matrix, the TM formula (1) does not provide any substantial information about \( \text{adj}(\lambda_0 I_n - A) \) because, in this case, \( w^T v = 0 \) and \( p'_A(\lambda_0) = 0 \). Thus, the TM formula (1) is relevant for matrices with simple eigenvalues.

\[\square\]

Our next goal is to provide a generalization of the TM formula (1) which is meaningful for nondiagonalizable matrices over fields. We will use the following notation: \( \mathbb{F} \) will denote an arbitrary field. If \( A \in \mathbb{F}^{n \times n} \) then \( p_1(\lambda), \ldots, p_r(\lambda) \) will be its (possibly repeated) elementary divisors in \( \mathbb{F} \) ([15, Ch. VI, Sec. 3]). These are powers of monic irreducible polynomials of \( \mathbb{F}[\lambda] \) (the ring of polynomials with coefficients in \( \mathbb{F} \)). We will assume that for \( j = 1, \ldots, r \),

\[
p_j(\lambda) = \lambda^{d_j} + a_{j1}\lambda^{d_j-1} + a_{j2}\lambda^{d_j-2} + \cdots + a_{jd_j-1}\lambda + a_{jd_j}.
\]

Let \( \Delta_A \) denote the determinant of \( A \) and \( \Lambda(A) \) the set of eigenvalues (the spectrum) of \( A \) in, perhaps, an extension field, \( \overline{\mathbb{F}} \), of \( \mathbb{F} \). Thus \( \lambda_0 \in \Lambda(A) \) if and only if it is a root in \( \overline{\mathbb{F}} \) of \( p_j(\lambda) \) for some \( j \in \{1, 2, \ldots, r\} \). In particular, \( p_A(\lambda) = \prod_{j=1}^r p_j(\lambda) \) is the characteristic polynomial of \( A \).

Item (ii) of the following theorem is an elementary result that is included for completion.

**Theorem 2.5.** With the above notation:
(i) If $0 \notin \Lambda(A)$ then the elementary divisors of $\text{Adj}(A)$ are $q_1(\lambda), \ldots, q_r(\lambda)$ where for $j = 1, \ldots, r,$

$$q_j(\lambda) = \lambda^{d_j} + \sum_{d_j \leq i \leq d_j-1} \frac{a_{jd_j-i}}{a_{jd_j}} \lambda^{d_j-i} + \cdots + \frac{\Delta A}{a_{jd_j}} \lambda + \frac{\Delta A}{a_{jd_j}} \frac{1}{a_{jd_j}}. \quad (6)$$

(ii) If $0 \in \Lambda(A)$ and there are two indices $i, k \in \{1, \ldots, r\}, i \neq k,$ such that $p_i(0) = p_k(0) = 0$ then $\text{Adj}(A) = 0.$

(iii) If $0 \in \Lambda(A), p_k(0) = 0$ for only one value $k \in \{1, \ldots, r\}$ and $u, v \in \mathbb{F}^{n \times 1}$ are arbitrary right and left eigenvectors of $A,$ respectively, for the eigenvalue $0,$ then $v^T A^{d_k-1} u \neq 0$ and

$$\text{Adj}(A) = \frac{(-1)^{n-1}}{d_k!} p_A^{(d_k)}(0) \frac{uv^T}{v^T A^{d_k-1} u}, \quad (7)$$

where $p_A^{(d_k)}(\lambda)$ is the $d_k$-th derivative of $p_A(\lambda).$

**Proof.** For $j = 1, \ldots, r,$ let the companion matrix of $p_j(\lambda)$ be

$$C_j = \begin{bmatrix}
0 & 0 & \cdots & 0 & -a_{jd_j} \\
1 & 0 & \cdots & 0 & -a_{jd_j-1} \\
0 & 1 & \cdots & 0 & -a_{jd_j-2} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & -a_{jd_1} 
\end{bmatrix}. \quad (8)$$

Then (see [15 Ch. VI, Sec. 6]) there is an invertible matrix $S \in \mathbb{F}^{n \times n}$ such that

$$C = S^{-1} AS = \bigoplus_{j=1}^r C_j. \quad (9)$$

An explicit computation shows that

$$\text{Adj}(C_j) = (-1)^{d_j} \begin{bmatrix}
-a_{jd_j-1} & a_{jd_j} & 0 & \cdots & 0 \\
-a_{jd_j-2} & 0 & a_{jd_j} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
-a_{k1} & 0 & 0 & \cdots & a_{jd_j} \\
-1 & 0 & 0 & \cdots & 0
\end{bmatrix}. \quad (10)$$

Bearing in mind that $\det C_j = (-1)^{d_j} a_{jd_j},$ we obtain $\text{Adj}(C) = \bigoplus_{j=1}^r L_j$ where, for $j = 1, \ldots, r,$

$$L_j = (-1)^n \prod_{i=1, i \neq j}^r a_{id_i} \begin{bmatrix}
-a_{jd_j-1} & a_{jd_j} & 0 & \cdots & 0 \\
-a_{jd_j-2} & 0 & a_{jd_j} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
-a_{k1} & 0 & 0 & \cdots & a_{jd_j} \\
-1 & 0 & 0 & \cdots & 0
\end{bmatrix}. \quad (10)$$
Therefore, from (9) we get

$$\text{Adj}(A) = S \left( \bigoplus_{j=1}^{r} L_j \right) S^{-1}.$$  \hspace{1cm} (11)

(i) Assume that $0 \not\in \Lambda(A)$. This means that $a_{jd_j} \neq 0$ for all $j = 1, \ldots, r$ and we can write

$$L_j = \det A \begin{bmatrix} -a_{jd_j-1} & 1 & 0 & \cdots & 0 \\ -a_{jd_j} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_{jd_1} & 0 & 0 & \cdots & 1 \\ -\frac{1}{a_{jd_j}} & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Taking into account the definition of $q_j(\lambda)$ of (6),

$$\text{det}(\lambda I_{d_j} - L_j) = \Delta^d_A \left( \frac{\lambda^{d_j}}{\Delta^d_A} + \frac{a_{jd_j-1}}{a_{jd_j}} \frac{\lambda^{d_j-1}}{\Delta^d_A} + \cdots + \frac{a_{j1} \lambda}{a_{jd_j} \Delta_A} + \frac{1}{a_{jd_j}} \right) = q_j(\lambda).$$

Let us see that $q_j(\lambda)$ is a power of an irreducible polynomial in $F[\lambda]$. In fact, put

$$s_j(\lambda) = \lambda^{d_j} p_j \left( \frac{1}{\lambda} \right) = a_{jd_j} \lambda^{d_j} + a_{jd_j-1} \lambda^{d_j-1} + \cdots + a_{j1} \lambda + 1.$$

This polynomial is sometimes called the \textit{reversal polynomial} of $p_j(\lambda)$ (see, for example, [22]). Since $p_j(\lambda)$ is an elementary divisor of $A$ in $F$, it is a power of an irreducible polynomial of $F[\lambda]$. By [1] Lemma 4.4, $s_j(\lambda)$ is also a power of an irreducible polynomial. Now, it is not difficult to see that $q_j(\lambda) = \frac{1}{a_{jd_j}} s_j \left( \frac{\lambda}{\Delta_A} \right)$ is a power of an irreducible polynomial too.

As a consequence, $q_1(\lambda), q_2(\lambda), \ldots, q_r(\lambda)$ are the elementary divisors of $\text{Adj}(C) = \bigoplus_{j=1}^{r} L_j$. Since this and $\text{Adj}(A)$ are similar matrices (cf. (11)), $q_1(\lambda), q_2(\lambda), \ldots, q_r(\lambda)$ are the elementary divisors of $\text{Adj}(A)$. This proves (i).

(ii) If $p_i(0) = p_j(0) = 0$ for $i \neq j$, then $\text{rank}(A) = \text{rank}(C) \leq n - 2$. Hence all minors of $A$ of order $n - 1$ are equal to zero and so $\text{Adj}(A) = 0$.

(iii) Assume now that there is only one index $k \in \{1, \ldots, r\}$ such that $a_{kd_k} = 0$. Then $p_k(\lambda) = \lambda^{d_k}$ because it is a power of an irreducible polynomial. Thus
\( a_{kj} = 0 \) for \( j = 1, \ldots, d_k \) and by (8) and (10), \( C_k = \begin{bmatrix} 0 & 0 \\ t_{ke_{k-1}} & 0 \end{bmatrix} \) and

\[
\begin{align*}
L_k &= (-1)^{n-1} \left\{ \prod_{j=1, j \neq k}^r a_{jd_j} 0 \right\}^{0} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \end{bmatrix} \\
 &= (-1)^{n-1} \left\{ \prod_{j=1, j \neq k}^r a_{jd_j} e_{d_k} e_{d_k}^T \right\},
\end{align*}
\]

respectively. Also, it follows from \( a_{kd_k} = 0 \) that \( L_j = 0 \) for \( j = 1, \ldots, r, j \neq k \).

Recall now that \( S^{-1}AS = C = \bigoplus_{j=1}^r C_j \) and split \( S \) and \( S^{-1} \) accordingly:

\[
S = \begin{bmatrix} S_1 & S_2 & \ldots & S_r \end{bmatrix}, \quad S^{-1} = \begin{bmatrix} T_1 \\ T_2 \\ \vdots \\ T_r \end{bmatrix}
\]

with \( S_j \in \mathbb{F}^{n \times d_j} \) and \( T_j \in \mathbb{F}^{d_j \times n}, j = 1, \ldots, r \). Then

\[
AS_k = S_k C_k, \quad T_k A = C_k T_k.
\]

For \( i = 1, \ldots, d_k \) let \( s_{ki} \) and \( t_{ki}^T \) be the \( i \)-th column and row of \( S_k \) and \( T_k \), respectively:

\[
S_k = \begin{bmatrix} s_{k1} & s_{k2} & \cdots & s_{kd_k} \end{bmatrix}, \quad T_k = \begin{bmatrix} t_{k1}^T \\ t_{k2}^T \\ \vdots \\ t_{kd_k}^T \end{bmatrix}
\]

Bearing in mind that \( \text{Adj}(A) = S(\bigoplus_{j=1}^r L_j)S^{-1} \) (cf. (11)), the representation of \( L_k \) as a rank-one matrix of (12) and that \( L_j = 0 \) for \( j \neq k \), we get

\[
\text{Adj}(A) = S_k L_k T_k = (-1)^{n-1} \left\{ \prod_{j=1, j \neq k}^r a_{jd_j} \right\} s_{kd_k} t_{k1}^T.
\]

Now, it follows from (13) that

\[
\begin{align*}
s_{kj} &= As_{kj-1}, & t_{kJ-1}^T &= t_{kJ}^T A, & j = 2, 3, \ldots, d_k, \\
As_{kd_k} &= 0, & t_{k1}^T A &= 0.
\end{align*}
\]

Henceforth, \( s_{kd_k} \) and \( t_{k1}^T \) are right and left eigenvectors of \( A \) for the eigenvalue 0. Also, \( J_k = < s_{k1}, As_{k1}, \ldots, A^{d_k-1}s_{k1} > \) is a cyclic \( A \)-invariant subspace with \( s_{k1} \) as generating vector. Similarly, \( J_k = < t_{kd_k}, A^T t_{kd_k}, \ldots, \)
In other words, and it is plain that singular matrices for any choice of \( u,v \) and left eigenvectors for which such a normalization may fail to hold. Since \( \ker A \) and \( \ker A^T \) span the cyclic subspaces \( J_1 \) and \( J_2 \). Specifically, \( T_k S_k = I_{dn} \). However, we are looking for a more general representation in terms of arbitrary right and left eigenvectors for which such a normalization may fail to hold.

Let us assume that \( u,v \in F^{n \times 1} \) are arbitrary right and left eigenvectors of \( A \) for the eigenvalue 0. Then \( Au = 0 \) and \( v^T A = 0 \) and since \( \ker A = \ker A^T = 1 \), there are nonzero scalars \( \alpha_1, \beta_1 \in F \) such that \( u = \alpha_1 s_{kd_k} \) and \( v = \beta_1 t_{k1} \). Put \( u_{dk} = u, v_1 = v \) and for \( j = 1, 2, \ldots, d_k - 1 \) define

\[
\begin{align*}
  u_{dk-j} &= \alpha_{j+1}s_{kd_k} + \alpha_j s_{kd_k-1} + \cdots + \alpha_1 s_{kd_k-j} \\
v_{j+1} &= \beta_{j+1}t_{k1} + \beta_j t_{k2} + \cdots + \beta_1 t_{kj+1}
\end{align*}
\]

with \( \alpha_2, \ldots, \alpha_{d_k}, \beta_2, \ldots, \beta_{d_k} \in F \) arbitrary scalars. Using these scalars we define the following triangular matrices

\[
X = \begin{bmatrix}
  \alpha_1 & \alpha_2 & \alpha_3 & \cdots & \alpha_d \\
  \alpha_2 & \alpha_3 & \alpha_4 & \cdots & \alpha_d \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  \alpha_{d-2} & \alpha_{d-1} & \alpha_d \\
  \alpha_d & \alpha_{d-1} & \alpha_{d-2} & \cdots & \alpha_1
\end{bmatrix}, \quad
Y = \begin{bmatrix}
  \beta_1 & \beta_2 & \beta_3 & \cdots & \beta_d \\
  \beta_2 & \beta_3 & \beta_4 & \cdots & \beta_d \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  \beta_{d-2} & \beta_{d-1} & \beta_d \\
  \beta_d & \beta_{d-1} & \beta_{d-2} & \cdots & \beta_1
\end{bmatrix}
\]

It is plain that \( [u_1 \ u_2 \ \cdots \ u_{dk}] = [s_{k1} \ s_{k2} \ \cdots \ s_{kd_k}] X \) and also \( [v_1 \ v_2 \ \cdots \ v_{dk}] = [t_{k1} \ t_{k2} \ \cdots \ t_{kd_k}] Y \). Since \( X \) and \( Y \) are non-singular matrices for any choice of \( \alpha_2, \ldots, \alpha_{d_k}, \beta_2, \ldots, \beta_{d_k} \) (because \( \alpha_1 \neq 0 \) and \( \beta_1 \neq 0 \)), we conclude that \( \mathcal{J}_k \equiv u_1, u_2, \ldots, u_{d_k} > \) and \( \mathcal{J}_k \equiv v_1, v_2, \ldots, v_{d_k} > \).

In addition, for \( j = 1, 2, \ldots, d_k - 1 \)

\[
Au_{dk-j} = \alpha_{j+1}As_{kd_k} + \alpha_j As_{kd_k-1} + \cdots + \alpha_1 As_{kd_k-j} = \alpha_j s_{kd_k} + \alpha_{j-1}s_{kd_k-1} + \cdots + \alpha_1 As_{kd_k-j+1} = u_{dk-j+1},
\]

and

\[
v_j^T A = \beta_j t_{k1} A + \beta_{j-1} t_{k2} A + \cdots + \beta_1 t_{kj} A = \beta_j t_{k1} A + \beta_{j-1} t_{k2} A + \cdots + \beta_1 t_{kj-1} = v_{j-1}^T.
\]

In other words, \( u_1 \) and \( v_{dk} \) are generating vectors of \( \mathcal{J}_k \) and \( \mathcal{J}_k \) and \( u = u_{dk} = A^{-1} u_1 \) and \( v = v_{dk} = v_1 = A^T u_{dk} \) are the given right and left eigenvectors of \( A \) for the eigenvalue 0. Now, it follows from \( u = \alpha_1 s_{kd_k} \), \( v = \beta_1 t_{k1} \) and \([14]\) that

\[
\text{Adj}(A) = (-1)^{n-1} \left( \prod_{j=1, j \neq k}^r a_{jd_j} \right) \frac{uv^T}{\alpha_1 \beta_1}.
\]

Since \( T_k S_k = I_{dn} \),

\[
\begin{bmatrix}
v_1^T \\
v_2^T \\
\vdots \\
v_{dk}^T
\end{bmatrix}
\begin{bmatrix}
u_1 \\
u_2 \\
\vdots \\
u_{dk}
\end{bmatrix} = Y^T T_k S_k X = Y^T X.
\]
But $Y^TX$ is a lower triangular matrix whose diagonal elements are all equal to $\alpha_1 \beta_1$. Thus, for $j = 1, \ldots, d_k$, $\alpha_1 \beta_1 = v_j^T u_j = v_j^T A^{d_k-1} u_{d_k} = v^T A^{d_k-1} u$. Since $\alpha_1 \neq 0$ and $\beta_1 \neq 0$, $v^T A^{d_k-1} u \neq 0$ as claimed. Now, from (15)

$$\text{Adj}(A) = (−1)^{n−1} \left( \prod_{j=1,j\neq k}^r a_{jd_j} \right) \frac{uv^T}{v^T A^{d_k-1} u}. \quad (16)$$

Finally, $p_A(\lambda) = \prod_{j=1}^r p_j(\lambda) = \lambda^{d_k} \prod_{j=1,j\neq k}^r p_j(\lambda)$. Therefore, $p_A^{(d_k)}(0) = d_k! \prod_{j=1,j\neq k}^r p_j(0) = d_k! \prod_{j=1,j\neq k}^r a_{jd_j}$ and (7) follows from (16).

□

As a first consequence of Theorem 2.5 we present a generalization of the formula for the eigenvalues of the adjugate matrix (see [18]).

**Corollary 2.6.** Let $A \in \mathbb{F}^{n \times n}$ be a nonsingular matrix. Let $\lambda_0 \in \Lambda(A)$ and let $m_1 \geq \ldots \geq m_s$ be its partial multiplicities (i.e., the sizes of the Jordan blocks associated to $\lambda_0$ in any Jordan form of $A$ in, perhaps, a extension field $\overline{\mathbb{F}}$. Then $\Delta_A^{m_i}$ is an eigenvalue of $\text{Adj}(A)$ with $m_1 \geq \ldots \geq m_s$ as partial multiplicities.

**Proof.** The elementary divisors of $A$ for the eigenvalue $\lambda_0$ in $\overline{\mathbb{F}}(\lambda)$ are $(\lambda - \lambda_0)^{m_1}, \ldots, (\lambda - \lambda_0)^{m_s}$. Then, it follows from item (i) of Theorem 2.5 (see [6]) that $(\lambda - \Delta_A^{m_i}) \cdots (\lambda - \Delta_A^{m_s})$ are the corresponding elementary divisors of $\text{Adj}(A)$.

□

**Corollary 2.7.** Let $A \in \mathbb{F}^{n \times n}$, $\lambda_0 \in \Lambda(A) \cap \mathbb{F}$ and let $m_1 \geq \ldots \geq m_s$ be its partial multiplicities. Let $u, v \in \mathbb{F}^{n \times 1}$ be arbitrary right and left eigenvectors of $A$ for $\lambda_0$. Then

$$\text{Adj}(\lambda_0 I_n - A) = \left( -\delta_{j_1} \right)^{m-1} \frac{\left(\lambda_0 - \Delta_A^{m_i}\right)}{m!} \frac{\left(\lambda_0 - \Delta_A^{m_k}\right)^{m-1} u v^T}{v^T \left(\lambda_0 I_n - A\right)^{-1} u} \quad (17)$$

where $m$ is the algebraic multiplicity of $\lambda_0$ and $\delta_{j_1}$ is the Kronecker delta.

**Proof.** Put $B = \lambda_0 I_n - A$. Then $0 \in \Lambda(B)$, $u$ and $v$ are right and left eigenvectors of $B$ for the eigenvalue $0$ and $m_1 \geq \ldots \geq m_s$ are the partial multiplicities of this eigenvalue. By Theorem 2.5, $\text{Adj}(\lambda_0 I_n - A) = \text{Adj}(B) \neq 0$ if and only if $s = 1$. In this case,

$$\text{Adj}(\lambda_0 I_n - A) = \text{Adj}(B) = \frac{(-1)^{n-1}}{m!} \frac{p_B^{(m)}(0) u v^T}{v^T B^{-1} u}.$$ 

Therefore (17) follows from the fact that $p_B^{(m)}(0) = (-1)^{n+m} p_A^{(m)}(\lambda_0)$ (see the proof of Theorem 2.1).

□
The following result is an immediate consequence of Corollary 2.7

**Corollary 2.8.** Let \( A \in \mathbb{F}^{n \times n} \) and let \( \Lambda(A) = \{ \lambda_1, \ldots, \lambda_s \} \) be its spectrum. Assume that \( \Lambda(A) \subset \mathbb{F} \) and let \( m_j \) and \( g_j \) be the algebraic and geometric multiplicities of \( A \) for the eigenvalue \( \lambda_j, j = 1, \ldots, s \). Fix \( k \in \{ 1, \ldots, s \} \) and let \( u_k \) and \( v_k \) be right and left eigenvectors of \( A \) for \( \lambda_k \). Then

\[
\text{Adj}(\lambda_k I - A) = (-\delta_{1g_k})^{m_k-1} \prod_{j=1, j\neq k}^{s} (\lambda_k - \lambda_j)^{m_j} \frac{u_k v_k^T}{v_k^T A^{m_k-1} u_k}.
\]  
(18)

The TM formula (1) can be used to provide an easy proof of the so-called eigenvector-eigenvalue identity (see [12, Sec. 2.1]). In fact, under the hypothesis of Theorem 2.1 it follows from (1) that \( w^T v \text{Adj}(\lambda_0 I_n - A)]_{jj} = p'_A(\lambda_0) v_j w_j, j = 1, \ldots, n \) (see [12, Rem 5]). Hence, recalling that for \( j = 1, \ldots, n \), \( M_{jj} \) is the principal minor of \( \Lambda(A) \), the following identity follows from (18) for the non-repeated eigenvalues \( \mu_j \), \( j = 1, \ldots, n \) (see [12, Thm. 1]).

As mentioned in Remark 2.4, if \( \mathbb{F} \) is a field of characteristic zero and \( A \in \mathbb{F}^{n \times n} \) then (19) is meaningful if and only if \( \lambda_0 \) is a simple eigenvalue. If \( \lambda_0 \) is defective and its geometric multiplicity is bigger than 1 then (19) becomes a trivial identity because, in this case, \( \text{Adj}(\lambda_0 I_n - A) = 0 \) (item (ii) of Theorem 2.5) and so \( p_M(\lambda_0) = \det(\lambda_0 I_n - M) = 0 \). However, if \( \lambda_0 \) is defective and its geometric multiplicity is 1, then (17) can be used to obtain a generalization of the eigenvector-eigenvalue identity. In fact, one readily gets from (17):

\[
p_{M_{jj}}(\lambda_0) = \frac{(-\delta_{1g})^{m-1}}{m!} p_A^{(m)}(\lambda_0) \frac{u_j v_j}{v_j^T (\lambda_0 I_n - A)^{m-1} u_j}, \quad j = 1, \ldots, n,
\]  
(21)

where \( m \) and \( g \) are the algebraic and geometric multiplicities of \( \lambda_0 \), respectively. Moreover, if both \( p_A(\lambda) \) and \( p_{M_{jj}}(\lambda) \) split in \( \mathbb{F} \) then, with the notation of Corollary 2.8, the following identity follows from (18) for the non-repeated eigenvalues \( \{ \mu_{j_1}, \ldots, \mu_{j_r} \} \) of \( M_{jj} \) and for \( i = 1, \ldots, s \):

\[
\prod_{k=1}^{r} (\lambda_i - \mu_{jk})^{g_k} = (-\delta_{1g_i})^{m_i-1} \frac{u_{ij} v_{ij}}{v_{ij}^T A^{m_i-1} u_{ij}} \prod_{k=1, k \neq i}^{s} (\lambda_i - \lambda_k)^{m_k}, \quad j = 1, \ldots, n,
\]  
(22)
where \( u_i = [u_{i1} \cdots u_{in}]^T \), \( v_i = [v_{i1} \cdots v_{in}]^T \), and \( q_{jk} \) is the algebraic multiplicity of \( \mu_{jk} \), \( k = 1, \ldots, r \) and \( j = 1, \ldots, n \).

In the following section two additional applications will be presented.

3. Two additional consequences of the TM formula

The well-known formula (23) below gives the derivative of a simple eigenvalue of a matrix depending on a (real or complex) parameter. The investigation about the eigenvalue sensitivity of matrices depending on one or several parameters can be traced back to the work of Jacobi [19]. However a systematic study of the perturbation theory of the eigenvalue problem starts with the books of Rellich (1953), Wilkinson (1965) and Kato (1966), as well as the papers by Lancaster [20], Osborne and Michaelson [27], Fox and Kapoor [14], Crossley and Porter [9] (see also [31] and the references therein). Since then this topic has become classical as evidenced by an extensive literature including books and papers addressed to mathematicians and a broad spectrum of scientist and engineers. In addition to the above early references, a short, and by no means exhaustive, list of books could include [4, p. 463], [24, Ch. 8, Sec. 9], [10, Sec. 4.2] or [21, pp. 134-135].

In proving (23), one first must prove, of course, that the eigenvalues smoothly depend on the parameter. It is also a common practice to prove or assume (see [23], [13, Ch. 11, Th. 2] and the referred books), the existence of eigenvectors which depend smoothly on the parameter. It is worth-remarking that in the proof by Lancaster in [20] only the existence of eigenvectors continuously depending on the parameter is required. We propose a simple and alternative proof of (23) where no assumption is made on the right and left eigenvector functions.

Let \( D_{\epsilon}(\omega_0) \) be the open disc of radius \( \epsilon > 0 \) with center \( \omega_0 \). For the following result \( F \) will be either the field of real numbers \( \mathbb{R} \) or of the complex numbers \( \mathbb{C} \). Recall that \( v \in \mathbb{C}^{n \times 1} \) is a left eigenvector of \( A \in \mathbb{C}^{n \times n} \) for an eigenvalue \( z_0 \) if \( v^*A = z_0v^* \) where \( v^* = \overline{v}^T \) is the transpose conjugate of \( v \). Hence, we will change \( T \) by * to include complex vectors in our discussion.

Proposition 3.1. Let \( A(\omega) \in \mathbb{F}^{n \times n} \) be a square matrix-valued function whose entries are analytic at \( \omega_0 \in \mathbb{C} \). Let \( z_0 \) be a simple eigenvalue of \( A(\omega_0) \). Then there exist \( \epsilon > 0 \) and \( \delta > 0 \) so that \( z : D_{\epsilon}(\omega_0) \rightarrow D_{\delta}(z_0) \) is the unique eigenvalue of \( A(\omega) \) with \( z(\omega) \in D_{\delta}(z_0) \) for each \( \omega \in D_{\epsilon}(\omega_0) \). Moreover, \( z \) is analytic on \( D_{\epsilon}(\omega_0) \) and

\[
\frac{dz(\omega)}{d\omega} = \frac{v(\omega)^*A'(\omega)u(\omega)}{v(\omega)^*u(\omega)},
\]

where, for \( w \in D_{\epsilon}(\omega_0) \), \( u(\omega) \) and \( v(\omega) \) are arbitrary right and left eigenvector, respectively, of \( A \) for \( z(\omega) \).

Proof. Since \( z_0 \) is a simple root of \( p(z, \omega) = \det(zI - A(\omega)) \), by the analytic implicit function theorem, we have, in addition to the first part of the result,
that
\[ z'(\omega) = -\frac{\partial p}{\partial \omega}(z(\omega), \omega) \frac{\partial p}{\partial z}(z(\omega), \omega). \]

By the Jacobi formula for the derivative of the determinant and TM formula \([1]\), we have (note that since \(z(\omega)\) is a simple eigenvalue, \(v(\omega)^*u(\omega) \neq 0\) for any right and left eigenvectors \(u(\omega)\) and \(v(\omega))

\[ \frac{\partial p}{\partial z}(z(\omega), \omega) = \text{tr}(\text{Adj}(z(\omega)I_n - A(\omega))) \]
\[ = p'(z(\omega), \omega) \]
\[ \frac{\partial p}{\partial \omega}(z(\omega), \omega) = -\text{tr}(\text{Adj}(z(\omega)I_n - A(\omega)))A'(\omega)) \]
\[ = -p'(z(\omega), \omega)\frac{v(\omega)^*A'(\omega)u(\omega)}{v(\omega)^*u(\omega)}, \]

and the result follows.

\[ \square \]

**Remark 3.2.** (a) The same conclusion can be drawn in Proposition 3.1 if \(A\) is a complex or real matrix-valued differentiable function of a real variable. In the first case, we would need a non-standard version of the implicit function theorem like the one in \([3\), Theorem 2.4\]. In the second case the standard implicit function theorem is enough.

(b) It is shown in \([2\) that the existence of eigenvectors smoothly depending on the parameter can be easily obtained from the properties of the adjugate matrix. In fact, since \(z(\omega)\) is a simple eigenvalue of \(A(\omega)\) for each \(\omega \in D(\omega_0)\), \(\text{rank}(z(\omega)I_n - A(\omega)) = n - 1\) and so by the TM formula, \(\text{rank} \text{Adj}(z(\omega)I_n - A(\omega)) = 1\) (see Remark 2.4). Now, \(\text{Adj}(z(\omega)I_n - A(\omega)))\) is a differentiable matrix function of \(\omega \in D(\omega_0)\) and \((z(\omega)I_n - A(\omega))(\text{Adj}(z(\omega)I_n - A(\omega))) = (\text{Adj}(z(\omega)I_n - A(\omega)))(z(\omega)I_n - A(\omega)) = \text{det}(z(\omega)I_n - A(\omega))I_n = 0\). Henceforth, all nonzero columns of \(\text{Adj}(z(\omega)I_n - A(\omega))\), which are all proportional, are (right and left) eigenvectors of \(A(\omega)\) for \(z(\omega)\).

The second application is related to the problem of characterizing the admissible eigenstructures and, more generally, the similarity orbits of the rank-one updated matrices. There is a vast literature on this problem. A non-exhaustive list of publications is \([22, 29, 34, 26, 6, 25, 8, 5\) and the references therein. It is a consequence of Theorem 2 in \([22\) that if \(\lambda_0\) is an eigenvalue of \(A \in \mathbb{F}^{n \times n}\) with geometric multiplicity 1 and \(\text{rank}(B - A) = 1\) then \(\lambda_0\) may or may not be an eigenvalue of \(B \in \mathbb{F}^{n \times n}\). It is then proved in \([25\), Th. 2.3\] that in the complex case, generically, \(\lambda_0\) is not an eigenvalue of \(B\). That is to say, there is a Zariski open set \(\Omega \subset \mathbb{C}^n \times \mathbb{C}^n\) such that for all \((x, y) \in \Omega\), \(\lambda_0\) is not an eigenvalue of \(A + xy^T\). With the help of the TM formula we can be a little more precise about the set \(\Omega\). Form now on, \(\mathbb{F}\) will be again an arbitrary field.
Proposition 3.3. Let $A \in \mathbb{F}^{n \times n}$ and let $\lambda_0$ be an eigenvalue of $A$ in, perhaps, an extension field $\mathbb{F}$. Assume that the geometric multiplicity of $\lambda_0$ is 1 and its algebraic multiplicity is $m$. Let $u_0, v_0 \in \mathbb{F}^{n \times 1}$ be right and left eigenvectors of $A$ for $\lambda_0$. If $x, y \in \mathbb{F}^{n \times 1}$ then $\lambda_0$ is an eigenvalue of $A + xy^T$ if and only if $y^T u_0 = 0$ or $v_0^T x = 0$.

Proof. Let $B = A + xy^T$. Then $\lambda I_n - A = \lambda I_n - B - xy^T$. Taking into account that $\lambda I_n - B$ is invertible in $\mathbb{F}(s)^{n \times n}$, where $\mathbb{F}(s)$ the field of rational functions, and using the formula of the determinant of updated rank-one matrices, we get

$$p_B(\lambda) = p_A(\lambda) + p_A(\lambda)y^T (\lambda I_n - A)^{-1} x = p_A(\lambda) + y^T \text{Adj}(\lambda I_n - A)x.$$  

In particular,

$$p_B(\lambda_0) = p_A(\lambda_0) + y^T \text{Adj}(\lambda_0 I_n - A)x = y^T \text{Adj}(\lambda_0 I_n - A)x.$$  

(24)

It follows from (17) that (recall that $v_0^T (\lambda_0 I_n - A)^{m-1} u_0 \neq 0$)

$$p_B(\lambda_0) = \frac{(-1)^{m-1}}{m!} p_A^{(m)}(\lambda_0) \frac{y^T u_0 v_0^T x}{v_0^T (\lambda_0 I_n - A)^{m-1} u_0}.$$  

Since $p_A^{(m)}(\lambda_0) \neq 0$, the Proposition follows. \hfill \Box

Remark 3.4. Note that, by (24) and item (ii) of Theorem 2.5 if the geometric multiplicity of $\lambda_0$ as eigenvalue of $A$ is 2 then $\text{Adj}(\lambda_0 I_n - A) = 0$ and so, $\lambda_0$ is necessarily an eigenvalue of $A + xy^T$. This is an easy consequence of the interlacing inequalities of [32 Th. 2]. However, proving that those interlacing inequalities are necessary conditions that the invariant polynomials of $A$ and $A + xy^T$ must satisfy is by no means a trivial matter. \hfill \Box

The eigenvalues of rank-one updated matrices are at the core of the divide and conquer algorithm to compute the eigenvalues of real symmetric or complex hermitian matrices (see, for example, [11 Sec. 5.3.3], [30 Sec. 2.1]). At each step of the algorithm a diagonal matrix $D = D_1 \oplus D_2$ and a vector $u \in \mathbb{C}^{n \times 1}$ are given such that the eigenvalues and eigenvectors of $D + uu^*$ are to be computed. In order the algorithm to run smoothly, it is required, among other things, that the diagonal elements of $D$ are all distinct. Thus, a so-called deflation process must be carried out. This amounts to check at each step the presence of repeated eigenvalues and, if so, remove and save them. The result that follows is related to the problem of detecting repeated eigenvalues but for much more general matrices over arbitrary fields.

Proposition 3.5. Let $A = A_1 \oplus A_2$ with $A_i \in \mathbb{F}^{n_i \times n_i}, i = 1, 2$. Let $x, y \in \mathbb{F}^{n \times 1}$ and split $B = A + xy^T = [B_{ij}]_{ij=1,2}$ into $2 \times 2$ blocks such that $B_{ii} \in \mathbb{F}^{n_i \times n_i}, i = 1, 2$. Assume also that the eigenvalues of $A_1$ and $A_2$ have geometric multiplicity equal to 1 and $\Lambda(A_1) \cap \Lambda(B_{11}) = \Lambda(A_2) \cap \Lambda(B_{22}) = \emptyset$. Then

$$\Lambda(A_1) \cap \Lambda(A_2) = \Lambda(B) \cap \Lambda(A_1) = \Lambda(B) \cap \Lambda(A_2).$$  

14
Proof. - If \( \lambda_0 \in \Lambda(A_1) \cap \Lambda(A_2) \) then \( \lambda_0 \), as eigenvalue of \( A \), has geometric multiplicity 2. By Remark 3.4 \( \lambda_0 \in \Lambda(B) \cap \Lambda(A_1) \cap \Lambda(A_2) \). Assume that \( \lambda_0 \in \Lambda(B) \cap \Lambda(A_1) \) but \( \lambda_0 \not\in \Lambda(A_2) \). Let us see that this assumption leads to a contradiction. Let \( u_0, v_0 \in \mathbb{F}^{n \times 1} \) be a right and a left eigenvectors of \( A_1 \), respectively. Then \( w_0 = \begin{bmatrix} u_0^T & 0 \end{bmatrix}^T \in \mathbb{F}^{n \times 1} \) and \( z_0 = \begin{bmatrix} w_0^T & 0 \end{bmatrix}^T \in \mathbb{F}^{n \times 1} \) are right and left eigenvectors of \( A \), respectively, for \( \lambda_0 \). Since \( \lambda_0 \not\in \Lambda(A_2) \), the geometric multiplicity of \( \lambda_0 \) as eigenvalue of \( A \) is 1. Then, by Proposition 3.3 \( y^T w_0 = 0 \) or \( z_0^T x = 0 \) because \( \lambda_0 \in \Lambda(B) \). Let us assume that \( y^T w_0 = 0 \), on the contrary we would proceed similarly with \( z_0^T x = 0 \). If we put \( y = \begin{bmatrix} y_1^T & y_2^T \end{bmatrix}^T \) and \( x = \begin{bmatrix} x_1^T & x_2^T \end{bmatrix}^T \), with \( x_1, y_1 \in \mathbb{F}^{n \times 1} \), then \( y_1^T u_0 = 0 \) and \( B_{11} = A_{11} + x_1 y_1^T \). It follows from Proposition 3.3 that \( \lambda_0 \in \Lambda(B_{11}) \), contradicting the hypothesis \( \Lambda(A_1) \cap \Lambda(B_{11}) = \emptyset \). That \( \Lambda(B) \cap \Lambda(A_2) \subset \Lambda(A_1) \cap \Lambda(A_2) \) is proved similarly. \( \square \\

Remark 3.6. \ (i) \  Note that, with the notation of the proof of Proposition 3.5, \( B_{11} = A_1 + x_1 y_1^T \) and \( B_{22} = A_2 + x_2 y_2^T \). Then, according to Proposition 3.3 \( \lambda_0 \not\in \Lambda(B_{11}) \) unless \( (y_1^T u_0) (v_0^T x_1) = 0 \). Hence, the hypothesis \( \Lambda(A_1) \cap \Lambda(B_{11}) = \emptyset \) is a generic property, and so is \( \Lambda(A_2) \cap \Lambda(B_{22}) = \emptyset \).

(ii) Consider Proposition 3.5 over \( \mathbb{C} \). If \( A \) and \( B \) are both Hermitian or unitary, then \( \Lambda(B) \setminus \left( \Lambda(A_1) \cap \Lambda(A_2) \right) \) and \( \Lambda(A_1) \cup \left( \Lambda(A_2) \setminus \left( \Lambda(A_1) \cap \Lambda(A_2) \right) \right) \) strictly interlace on the real line or the unit circle, respectively (see, for example, [30 Th. 2.1, Sec. 2]). \( \square 

References

[1] A. Amparan, S. Marcaida, and I. Zaballa. On the structure invariants of proper rational matrices with prescribed finite poles. Linear and Multilinear Algebra, 61(11):1464–1486, 2013.

[2] A. L. Andrew, K.-W. E. Chu, and P. Lancaster. Derivatives of eigenvalues and eigenvectors of matrix functions. SIAM J. Matrix Anal. Appl., 14(4):903–926, 1993.

[3] M. S. Ashbaugh and E. M. Harrell II. Perturbation theory for shape resonances and large barrier potentials. Comm. Math. Phys., 83(2):151–170, 1982.

[4] F. V. Atkinson. Discrete and continuous boundary problems, volume 8 of Mathematics in Science and Engineering. Academic Press, New York-London, 1964.

[5] I. Baragaña. The number of distinct eigenvalues of a regular pencil and of a square matrix after rank perturbation. Linear Algebra Appl., 588:101–121, 2020.
[6] M. A. Beitia, I. de Hoyos, and I. Zaballa. The change of the Jordan structure under one row perturbations. *Linear Algebra Appl.*, 401:119 – 134, 2005.

[7] W. C. Brown. *Matrices over Commutative Rings*. Marcel Dekker Inc., New York, 1993.

[8] R. Bru, R. Cantó, and A. M. Urbano. Eigenstructure of rank one updated matrices. *Linear Algebra Appl.*, 485:372–391, 2015.

[9] T. R. Crossley and B. Porter. Eigenvalue and eigenvector sensitivities in linear system theory. *Int. J. Control*, 10:163–170, 1969.

[10] Hinrichsen D. and Pritchard A. J. *Mathematical System Theory I. Modelling, State Space Analysis, Stability and Robustness*. Springer, Berlin, 2005.

[11] J. W. Demmel. *Applied Numerical Linear Algebra*. SIAM, Philadelphia, 1997.

[12] P. B. Denton, S. J. Parke, T. Tao, and X. Zhang. Eigenvectors from eigenvalues: a survey of a basic identity in linear algebra. *arXiv:1908.03795*, 2020.

[13] L. C. Evans. *Partial differential equations*, volume 19 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, second edition, 2010.

[14] R. L. Fox and M. P. Kapoor. Rate of change of eigenvalues and eigenvectors. *AIAA J.*, 6:2426–2429, 1968.

[15] F. R. Gantmacher. *The Theory of Matrices*. AMS Chelsea Publishing, Providence, Rhode Island, 1988.

[16] R. Godement. *Cours d’algèbre*. Hermann Éditeurs, Paris, 2005.

[17] D. Grinberg. Eigenvectors from eigenvalues: a survey of a basic identity in linear algebra — what’s new. *https://terrytao.wordpress.com/2019/12/03/eigenvectors-from-eigenvalues-a-survey-of-a-basic-identity-in-linear-algebra/#comment-531597*, 2019.

[18] R. D. Hill and E. E. Underwood. On the matrix adjoint (adjugate). *SIAM J. Algebraic Discrete Methods*, 6(4):731–737, 1985.

[19] C. G. J. Jacobi. Über ein leichtes verfahren die in der theorie der säcularstörungen vorkommenden gleichungen numerisch aufzulösen. *J. für die Reine und Angew. Math.*, 1846(30):51–94, 1846.

[20] P. Lancaster. On eigenvalues of matrices dependent on a parameter. *Numer. Math.*, 6:377–387, 1964.
[21] P. D. Lax. *Linear Algebra and its Applications*. Pure and Applied Mathematics (Hoboken). Wiley-Interscience [John Wiley & Sons], Hoboken, NJ, second edition, 2007.

[22] D. S. Mackey, N. Mackey, C. Mehl, and V. Mehrmann. Vector spaces of linearizations for matrix polynomials. *SIAM J. Matrix Anal. Appl.*, 28(4):971–1004, 2006.

[23] J. R. Magnus. On differentiating eigenvalues and eigenvectors. *Econometric Theory*, 1:179–191, 1985.

[24] J. R. Magnus and H. Neudecker. *Matrix Differential Calculus with Applications in Statistics and Econometrics*. John Wiley & Sons, Chichester, 1988.

[25] C. Mehl, V. Mehrmann, A. C. M. Ran, and L. Rodman. Eigenvalue perturbation theory of classes of structured matrices under generic structured rank one perturbations. *Linear Algebra Appl.*, 435(3):687–716, 2011.

[26] J. Moro and F. M. Dopico. Low rank perturbation of Jordan structure. *SIAM J. Matrix Anal. Appl.*, 25(2):495–506, 2003.

[27] M. R. Osborne and S. Michaelson. The numerical solution of eigenvalue problems in which the eigenvalue appears nonlinearly, with an application to differential equations. *Computer J.*, 7:66–71, 1964.

[28] D. S. Scott. How to make the Lanczos algorithm converge slowly. *Math. Comp.*, 33:239–247, 1979.

[29] F. C. Silva. The rank of the difference of matrices with prescribed similarity classes. *Linear and Multilinear Algebra*, 24(1):51–58, 1988.

[30] G. W. Stewart. *Matrix Algorithms, Volume II: Eigensystems*. SIAM, Philadelphia, 2001.

[31] J. G. Su. Multiple eigenvalue sensitivity analysis. *Linear Algebra Appl.*, 137(4):183–211, 1990.

[32] R. C. Thompson. Invariant factors under rank one perturbations. *Canad. J. Math.*, 32(1):240–245, 1980.

[33] R. C. Thompson and P. McEnteggert. Principal submatrices. II: The upper and lower quadratic inequalities. *Linear Algebra Appl.*, 1:211–243, 1968.

[34] I. Zaballa. Pole assignment and additive perturbations of fixed rank. *SIAM J. Matrix Anal. Appl.*, 12(1):16–23, 1991.