0/1 Polytopes with Quadratic Chvátal Rank

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Abstract. For a polytope \( P \), the Chvátal closure \( P' \subseteq P \) is obtained by simultaneously strengthening all feasible inequalities \( cx \leq \beta \) (with integral \( c \)) to \( cx \leq \lfloor \beta \rfloor \). The number of iterations of this procedure that are needed until the integral hull of \( P \) is reached is called the Chvátal rank. If \( P \subseteq [0,1]^n \), then it is known that \( O(n^2 \log n) \) iterations always suffice (Eisenbrand and Schulz (1999)) and at least \((1 + 1/e - o(1))n \) iterations are sometimes needed (Pokutta and Stauffer (2011)), leaving a huge gap between lower and upper bounds.

We prove that there is a polytope contained in the 0/1 cube that has Chvátal rank \( \Omega(n^2) \), closing the gap up to a logarithmic factor. In fact, even a superlinear lower bound was mentioned as an open problem by several authors. Our choice of \( P \) is the convex hull of a semi-random Knapsack polytope and a single fractional vertex. The main technical ingredient is linking the Chvátal rank to simultaneous Diophantine approximations w.r.t. the \( \|\cdot\|_1 \) norm of the normal vector defining \( P \).

1 Introduction

Gomory-Chvátal cuts are among the most important classes of cutting planes used to derive the integral hull of polyhedra. The fundamental idea to derive such cuts is that if an inequality \( cx \leq \beta \) is valid for a polytope \( P \) (that is, \( cx \leq \beta \) holds for every \( x \in P \)) and \( c \in \mathbb{Z}^n \), then \( cx \leq \lfloor \beta \rfloor \) is valid for the integral hull \( P_I := \text{conv}(P \cap \mathbb{Z}^n) \). Formally, for a polytope \( P \subseteq \mathbb{R}^n \) and a vector \( c \in \mathbb{Z}^n \),

\[
GC_P(c) := \{ x \in \mathbb{R}^n \mid cx \leq \lfloor \max\{cy \mid y \in P\} \rfloor \}
\]

is the Gomory-Chvátal Cut that is induced by vector \( c \) (for polytope \( P \)). Furthermore,

\[
P' := \bigcap_{c \in \mathbb{Z}^n} GC_P(c)
\]

is the Gomory-Chvátal closure of \( P \). Let \( P^{(i)} := (P^{(i-1)})' \) (and \( P^{(0)} = P \)) be the \( i \)th Gomory-Chvátal closure of \( P \). The Chvátal rank \( \text{rk}(P) \) is the smallest number such that \( P^{(\text{rk}(P))} = P_I \).

* Supported by the Alexander von Humboldt Foundation within the Feodor Lynen program, by ONR grant N00014-11-1-0053 and by NSF contract CCF-0829878.
It is well-known that the Chvátal rank is always finite, but can be arbitrarily large already for 2 dimensional polytopes. However, if we restrict our attention to polytopes $P$ contained in the 0/1 cube the situation becomes much different, and the Chvátal rank can be bounded by a function in $n$. In particular, Bockmayr, Eisenbrand, Hartmann and Schulz [BEHS99] provided the first polynomial upper bound of $\text{rk}(P) \leq O(n^3 \log n)$. Later, Eisenbrand and Schulz [ES99, ES03] proved that $\text{rk}(P) \leq O(n^2 \log n)$, which is still the best known upper bound. Note that if $P \subseteq [0, 1]^n$ and $P \cap \{0, 1\}^n = \emptyset$, then even $\text{rk}(P) \leq n$ (and this is tight if and only if $P$ intersects all the edges of the cube [PS11a]). Already [CCH89] provided lower bounds on the rank for the polytopes corresponding to natural problems like stable-set, set-covering, set-partitioning, knapsack, maxcut and ATSP (however, none of the bounds exceeded $n$). The paper of Eisenbrand and Schulz [ES99, ES03] also provides a lower bound $\text{rk}(P) > (1 + \varepsilon)n$ for a tiny constant $\varepsilon > 0$, which has been quite recently improved by Pokutta and Stauffer [PS11b] to $(1 + \frac{1}{e} - o(1))n$. However, as the authors of [PS11a] state, there is still a very large gap between the best known upper and lower bound. In particular, the question whether there is any superlinear lower bound on the rank of a polytope in the 0/1 cube is open since many years (see e.g. Ziegler [Zie00]).

In this paper, we prove that there is a polytope contained in the 0/1 cube that has Chvátal rank $\Omega(n^2)$, closing the gap up to a logarithmic factor. Specifically, our main result is:

**Theorem 1.** For every $n$, there exists a vector $c \in \{0, \ldots, 2^{n/16}\}^n$ such that the polytope

$$P = \text{conv}\left\{ x \in \{0, 1\}^n : \sum_{i=1}^n c_i x_i \leq \frac{\|c\|_1}{2} \right\} \cup \left\{ (\frac{3}{4}, \ldots, \frac{3}{4}) \right\} \subseteq [0, 1]^n$$

has Chvátal rank $\Omega(n^2)$.

Here $\|c\|_1 := \sum_{i=1}^n |c_i|$ and $\|c\|_\infty := \max_{i=1,\ldots,n} |c_i|$.

### 1.1 Related Work

There is a large amount of results on structural properties of the CG closure. Already Schrijver [Sch80] could prove that the closure of a rational polyhedron is again described by finitely many inequalities. Dadush, Dey and Vielma [DDV11a] showed that $K'$ is a polytope for all compact and strictly convex sets $K \subseteq \mathbb{R}^n$. Later, Dunkel and Schulz [DS10] could prove the same if $K$ is an irrational polytope, while in parallel again Dadush, Dey and Vielma [DDV11b] showed that this holds in fact for any compact convex set.

In the last years, automatic procedures that strengthen existing relaxations became more and more popular in theoretical computer science. Singh and Talwar [ST10] showed that few CG rounds reduce the integrality gap for $k$-uniform hypergraph matchings. However, to obtain approximation algorithms researchers rely more on *Lift-and-Project Methods* such as the hierarchies of Balas, Ceria, Cornuéjols [BCC93]; Lovász, Schrijver [LS91]; Sherali, Adams [SA90] or Lasserre [Las01a, Las01b]. One can optimize over the $t$th level in time $n^{O(t)}$.  
