ALGEBRAS OF QUOTIENTS OF PATH ALGEBRAS

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Abstract. Leavitt path algebras are shown to be algebras of right quotients of their corresponding path algebras. Using this fact we obtain maximal algebras of right quotients from those (Leavitt) path algebras whose associated graph satisfies that every vertex connects to a line point (equivalently, the Leavitt path algebra has essential socle). We also introduce and characterize the algebraic counterpart of Toeplitz algebras.

1. Introduction and preliminaries

Leavitt path algebras, natural generalizations of the algebras investigated by Leavitt in [20], are algebraic versions of the Cuntz-Krieger algebras of directed graphs described in [24] which, at the same time, generalize Cuntz algebras (whose introduction and study was motivated by questions in physics). The introduction of Leavitt path algebras in [1] and [7] has recently attracted the interest of a significant number of algebraists as well as of analysts working on $C^*$-algebras. As a sample of this, let us mention the notes of the “Workshop on graph algebras” ([10]), held at the University of Málaga (Spain), focused on both, the analytic and the algebraic part of graph algebras, through the history of the subject and recent developments. Although the algebraic results look very similar (but are not exactly the same, as shown in [9] and [6]), they require quite different techniques to be reached. We recommend the reader the paper by Tomforde [26], where the author considers both areas of graph algebras.

Roughly speaking, for a row-finite graph $E$ and a field $K$, the Leavitt path algebra $L_K(E)$ is the path $K$-algebra associated to $E$, modulo some relations (the so called Cuntz-Krieger relations).

One of the main interests of the researchers in Leavitt path algebras (as well as in graph $C^*$-algebras) is to get a structure theory as deep as possible, and the development of theories of algebras of quotients allows to one to obtain a better understanding of this structure. This has been the motivation for the work in this paper.

Take, for example, a finite and acyclic graph $E$; then the Leavitt path algebra $L_K(E)$ is semisimple and artinian, i.e, it is isomorphic to $\bigoplus_{i=1}^t M_{n_i}(K)$ (see [2] Proposition 3.5). Moreover, $L_K(E)$ is the maximal algebra of right quotients of the path algebra $KE$ ([23]). It is known that when the maximal algebra of right quotients of a semiprime algebra $A$, let us call it $Q$, is semisimple and artinian, then $Q$ coincides with the classical algebra of right quotients of $A$, hence it is natural to ask about when the path algebra $KE$ is semiprime. Another question to be considered is if the Leavitt path algebra of an arbitrary graph is an algebra of right quotients of its corresponding path algebra. A partial answer was given in

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Corollary 3.25]: if the graph \( E \) is finite and acyclic, then \( L_K(E) \) is an algebra of right quotients of the path algebra \( KE \).

In this paper we show that the path algebra \( KE \) associated to any graph \( E \) is semiprime if and only if whenever there is a path joining two vertices, there exists another one from the range to the source of the first one (Proposition 2.1). This implies that for \( E \) a finite and acyclic graph the Leavitt path algebra \( L_K(E) \) is not the classical algebra of right quotients of the path algebra \( KE \) (except if there are no edges joining vertices), although it is the classical algebra of right quotients of any semiprime subalgebra containing the path algebra (Proposition 3.1). Moreover, for a general graph \( E \) we obtain that \( L_K(E) \) is an algebra of right quotients of the path algebra \( KE \) (Proposition 2.2), result that extends that of [23].

There exists a notion of order in nonunital rings which was introduced by Fountain and Gould in [13] and extended by Áhn and Márki to one-sided orders (see [3]). These authors developed, some years later, a general theory of Fountain-Gould one-sided orders (see [4] and the references therein). This localization was carried out by considering generalized inverses like group inverses, and agrees with the classical one (where usual inverses are taken) when the ring is semiprime and coincides with its socle ([14, Corollary 3.4]). If a ring \( R \) has a Fountain-Gould right quotient ring \( Q \), then \( Q \) is unique up to isomorphisms (see [16, Theorem 5.9] or [4, Corollary 3]).

It was shown in [4], and later in [14], that the maximal ring of right quotients provides an appropriate framework where to settle these right orders, specially when the ring \( R \) is a Fountain-Gould right order in a semiprime ring coinciding with its socle. On the other hand, since the socle of a Leavitt path algebra has been studied in [8], we have the required tools to establish results on right orders in Leavitt path algebras which satisfy the descending chain condition on principal one-sided ideals (i.e., are semisimple).

We prove that for an acyclic graph \( E \) any semiprime subalgebra \( A \) such that \( KE \subseteq A \subseteq L_K(E) \) is a Fountain-Gould right order in the Leavitt path algebra \( L_K(E) \) (Proposition 3.2) and characterize these algebras \( A \) when the hereditary closure of the set of line points is the set of all vertices, that is, when \( L_K(E) \) coincides with its socle (Theorem 3.5).

In Section 4, we obtain that the socle of a Leavitt path algebra is an essential ideal if and only if every vertex connects to a line point (Theorem 4.3). This allows to reduce the study of maximal algebras of quotients of the path algebra \( KE \) (for \( E \) as before) to the maximal algebras of quotients of acyclic graphs \( E' \) such that \( L_K(E') \) is semisimple, i.e., coincides with its socle (Theorem 4.4). The maximal symmetric algebra of quotients and the maximal algebra of right quotients of path algebras of locally finite graphs without cycles and such that the set of sinks is a maximal antichain have been described in [23, Chapter 3].

Finally, we define the algebraic counterpart of the Toeplitz algebra, as the Leavitt path algebra \( T \) whose graph is the following:

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Similarly to what happens in the analytic case, \( T \) has an essential ideal, which is the socle of \( T \) in our setting, and there exists an exact sequence

\[0 \to Soc(L_K(E)) \to T \to K[x, x^{-1}] \to 0.\]
Moreover, the Toeplitz algebra $T$ is sandwiched as follows:

$$M_\infty(K) \subseteq T \subseteq \text{RCFM}(K),$$

where $M_\infty(K)$ denotes the algebra of matrices of infinite size with only a finite number of nonzero entries.

As shown in the analytic context, $T$ has not a unique representation. It can be described as the Leavitt path algebra $L_K(E)$, for $E$ a graph such that $E^0 = F^0 \cup \{v\}$, $E^1 = F^1 \cup \{e, e_1, \ldots, e_n\}$, where $e$ has range and source $v$, $s(e_i) = v$, $r(e_i) \in F^0$, and for every $f \in F^0$ its range and source are the corresponding as edges of the graph $F$, being $F$ an acyclic graph such that every vertex connects to a line point. All of these results are collected in Theorem 5.3.

We start the preliminaries by recalling the definitions of path algebra and Leavitt path algebra. A (directed) graph $E = (E^0, E^1, r, s)$ consists of two countable sets $E^0, E^1$ and maps $r, s : E^1 \to E^0$. The elements of $E^0$ are called vertices and the elements of $E^1$ edges. If $s^{-1}(v)$ is a finite set for every $v \in E^0$, then the graph is called row-finite. Throughout this paper we will be concerned only with row-finite graphs. If $E^0$ is finite then, by the row-finite hypothesis, $E^1$ must necessarily be finite as well; in this case we simply say that $E$ is finite. A vertex which emits no edges is called a source. A path $\mu$ in a graph $E$ is a sequence of edges $\mu = e_1 \ldots e_n$ such that $r(e_i) = s(e_{i+1})$ for $i = 1, \ldots, n - 1$. In this case, $s(\mu) := s(e_1)$ is the source of $\mu$, $r(\mu) := r(e_n)$ is the range of $\mu$, and $n$ is the length of $\mu$, i.e. $l(\mu) = n$. We denote by $\mu^0$ the set of its vertices, that is: $\mu^0 = \{s(e_1), r(e_i) : i = 1, \ldots, n\}$.

Now let $K$ be a field and let $KE$ denote the $K$-vector space which has as a basis the set of paths. It is possible to define an algebra structure on $KE$ as follows: for any two paths $\mu = e_1 \ldots e_m, \nu = f_1 \ldots f_n$, we define $\mu \nu$ as zero if $r(\mu) \neq s(\nu)$ and as $e_1 \ldots e_m f_1 \ldots f_n$ otherwise. This $K$-algebra is called the path algebra of $E$ over $K$.

For a field $K$ and a row-finite graph $E$, the Leavitt path $K$-algebra $L_K(E)$ is defined as the $K$-algebra generated by a set $\{v \mid v \in E^0\}$ of pairwise orthogonal idempotents, together with a set of variables $\{e, e^* \mid e \in E^1\}$, which satisfy the following relations:

1. $s(e)e = e = e r(e)$ for all $e \in E^1$.
2. $r(e)e^* = e^* s(e) = e^*$ for all $e \in E^1$.
3. $e^* e' = \delta_{e,e'} r(e)$ for all $e, e' \in E^1$.
4. $v = \sum_{\{e \in E^1 \mid s(e) = v\}} ee^*$ for every $v \in E^0$ that emits edges.

Relations (3) and (4) are called of Cuntz-Krieger.

The elements of $E^1$ are called (real) edges, while for $e \in E^1$ we call $e^*$ a ghost edge. The set $\{e^* \mid e \in E^1\}$ will be denoted by $(E^1)^*$. We let $r(e^*)$ denote $s(e)$, and we let $s(e^*)$ denote $r(e)$. If $\mu = e_1 \ldots e_n$ is a path, then we denote by $\mu^*$ the element $e_n^* \ldots e_1^*$ of $L_K(E)$.

There exists a natural inclusion of the path algebra $KE$ into the Leavitt path algebra $L_K(E)$ sending vertices to vertices and edges to edges. We will use this monomorphism without any explicit mention to it.

It is shown in [1] that $L_K(E)$ is a $\mathbb{Z}$-graded $K$-algebra, spanned as a $K$-vector space by $\{pq^* \mid p, q \text{ are paths in } E\}$. In particular, for each $n \in \mathbb{Z}$, the degree $n$ component $L_K(E)_n$ is spanned by elements of the form $pq^*$ where $l(p) - l(q) = n$.

The set of homogeneous elements is $\bigcup_{n \in \mathbb{Z}} L_K(E)_n$, and an element of $L_K(E)_n$ is said to be $n$-homogeneous or homogeneous of degree $n$. 

Note that the natural monomorphism from the path algebra $KE$ into the Leavitt path algebra $L_K(E)$ is graded, hence $KE$ is a $\mathbb{Z}$-graded subalgebra of $L_K(E)$.

An easy result which will be used later is the following one.

**Lemma 1.1.** Any set of different paths is $K$-linearly independent.

*Proof.* Consider a graph $E$ and let $\mu_1, \ldots, \mu_n$ be different paths. Write $\sum_i k_i \mu_i = 0$, for $k_i \in K$. Applying that $L_K(E)$ is $\mathbb{Z}$-graded we may suppose that all the paths have the same length. Since $\mu_j' \mu_i = \delta_{ij} r(\mu_j)$ then $0 = \sum_i k_i \mu_j' \mu_i = k_j r(\mu_j)$; this implies $k_j = 0$. \hfill \Box

2. ALGEBRAS OF RIGHT QUOTIENTS

We first study the semiprimeness of the path algebra associated to a graph $E$. Recall that an algebra $A$ is said to be *semiprime* if it has no nonzero ideals of zero square, equivalently, if $aAa = 0$ for $a \in A$ implies $a = 0$ (an algebra $A$ that satisfies this last condition is called in the literature *nondegenerate*).

**Proposition 2.1.** For a graph $E$ and a field $K$ the path algebra $KE$ is semiprime if and only if for every path $\mu$ there exists a path $\mu'$ such that $s(\mu') = r(\mu)$ and $r(\mu') = s(\mu)$.

*Proof.* Suppose first that $KE$ is semiprime. Given a path $\mu$, since $\mu(KE) \mu \neq 0$, there exists a path $\nu \in KE$ such that $\nu \mu \mu \neq 0$. This means that $s(\nu) = r(\mu)$ and $r(\nu) = s(\mu)$.

Now, let us prove the converse. Note that by [21 Proposition II.1.4 (1)], a $\mathbb{Z}$-graded algebra is semiprime if and only if it is graded semiprime. Hence, and taking into account that being graded semiprime and graded nondegenerate are equivalent, it suffices to show that if $x$ is any nonzero homogeneous element of $KE$, then $x(KE)x \neq 0$. Write $x = \sum_{i=1}^n k_i \alpha_i$, with $0 \neq k_i \in K$ and $\alpha_1, \ldots, \alpha_n$ different paths of the same degree (i.e. of the same length). Denote the source and range of $\alpha_1$ by $u_1$ and $v_1$, respectively. Then, by (3), $\alpha_1^* x = k_1 \alpha_1^* \alpha_1 = k_1 v_1$.

By the hypothesis, there exists a path $\alpha_1'$ such that $s(\alpha_1') = v_1$ and $r(\alpha_1') = u_1$. Observe that $\alpha_1' x \neq 0$; otherwise $0 = (\alpha_1')^* \alpha_1' x = u_1 x$, a contradiction since a set of different paths is always linearly independent over $K$ (Lemma 1.1) and $\alpha_1 = u_1 \alpha_1 \neq 0$. Therefore $0 \neq k_1 \alpha_1' x = k_1 v_1 \alpha_1' x = \alpha_1^* \alpha_1' x \in \alpha_1^* x(KE)x$. \hfill \Box

If $a \in L_K(E)$ and $d \in \mathbb{Z}^+$, then we say that $a$ is *representable as an element of degree $d$ in real (respectively ghost) edges* in case $a$ can be written as a sum of monomials from the spanning set $\{pq^* \mid p, q \text{ are paths in } E\}$, in such a way that $d$ is the maximum length of a path $p$ (respectively $q$) which appears in such monomials. Note that an element of $L_K(E)$ may be represented as an element of different degrees in real (respectively ghost) edges.

The $K$-linear extension of the assignment $pq^* \mapsto qp^*$ (for $p, q$ paths in $E$) yields an involution on $L_K(E)$, which we denote simply as $^*$. Clearly $(L_K(E)_n)^* = L_K(E)_{-n}$ for all $n \in \mathbb{Z}$.

Let $R \subseteq Q$ be rings. Recall that $Q$ is said to be an *algebra of right quotients of $R$* if given $p, q \in Q$, with $p \neq 0$, there exists an element $r \in R$ such that $pr \neq 0$ and $qr \in R$.

**Proposition 2.2.** For any graph $E$ and any field $K$ the Leavitt path algebra $L_K(E)$ is an algebra of right quotients of the path algebra $KE$.

*Proof.* Consider $x, y \in L_K(E)$, with $x \neq 0$. Apply the reasoning in the first paragraph of the proof of [8 Proposition 3.1] to find a path $\mu$ such that $0 \neq x\mu \in KE$. If $y\mu \in KE$ we have finished the proof. If $y\mu \notin KE$, write $y\mu$ as a sum of monomials of the form $ke_1 \ldots e_r f_1^* \ldots f_s^*$
(it is possible by [1] Lemma 1.5), with \( k \in K \) and \( e_1, \ldots, e_r, f_1, \ldots, f_s \in E^1 \), being this expression minimal in ghost edges, and denote by \( m \) the maximum natural number of ghost edges appearing in \( y\mu \).

Note that \( r(\mu) \) is not a sink; otherwise, for any \( f \in E^1 \), \( f^*r(\mu) = 0 \), hence \( y\mu \subseteq KE \), but this is not our case. Therefore, there exists \( h_1 \in E^1 \) such that \( s(h_1) = r(\mu) \). Then \( x\mu h_1 \neq 0 \) because \( x\mu \in KE \setminus \{0\} \) and by Lemma 1.1. If \( y\mu h_1 \in KE \), our proof is complete. Otherwise, repeat this reasoning. This process must stop in at most \( m \) steps. \( \square \)

3. Fountain-Gould and Moore-Penrose right orders

Localization consists of assigning inverses to certain elements. This procedure can be carried out by taking inverses (when the algebra has an identity) or, more generally, by considering generalized inverses like Moore-Penrose or group inverses, in which case the existence of the identity element plays no role at all. There is a well-developed theory of rings of quotients when this kind of inverses is taken. We are speaking about Fountain-Gould orders and Moore-Penrose orders.

For a graph \( E \) having a not finite number of vertices, the Leavitt path algebra \( L_K(E) \) is not unital, hence it cannot be a maximal algebra of quotients of any of its subalgebras. However, we will show that it is a Fountain-Gould right algebra of quotients of any of its semiprime subalgebras containing the path algebra \( KE \) when the graph \( E \) is acyclic (a Moore-Penrose right order if we take into account the involution).

Moreover, we will describe in this section the structure of semiprime subalgebras of \( L_K(E) \) containing the path algebra \( KE \), for \( E \) an acyclic graph such that the set of line points is “dense” in the sense that the saturated closure of \( P_l(E) \) is the whole set of vertices of \( E \).

**Proposition 3.1.** Let \( E \) be a finite and acyclic graph and let \( A \) be a semiprime algebra such that \( KE \subseteq A \subseteq L_K(E) \). Then \( L_K(E) = Q_{\max}^r(KE) = Q_{\text{id}}^r(A) \).

**Proof.** Apply [22] Proposition 3.4] to the opposite graph of \( E \) (the one obtained from \( E \) by changing ranges to sources and sources to ranges) and [2] Proposition 3.5) to show that \( Q_{\max}^r(KE) = L_K(E) \). The fact of \( L_K(E) \) be semiprime and artinian and an algebra of right quotients of \( A \) (because it is an algebra of right quotients of \( KE \), by Proposition 2.2) implies, by [14] Corollary 3.4], that it is the classical algebra of right quotients of \( A \). \( \square \)

Let \( a \) be an element in a ring \( R \). We say that \( b \in R \) is the group inverse of \( a \) if the following conditions hold: \( aba = a, \ bab = b, \ ab = ba \).

It is easy to see that \( a \) has a group inverse \( b \) in \( R \) if and only if there exists a unique idempotent \( e \) \( (e = ab) \) in \( R \) such that \( a \) is invertible in the ring \( eRe \) (with inverse \( b \)), hence the group inverse is unique and \( a \) is said to be locally invertible. Denote by \( a^\sharp \) the group inverse of \( a \).

An element \( a \in R \) is called square cancellable if \( a^2x = a^2y \) implies \( ax = ay \) and \( xa^2 = ya^2 \) implies \( xa = ya \), for \( x, y \in R \cup \{1\} \) (for \( x = 1 \) or \( y = 1 \) this means that \( a^2 = a^2y \) or \( a^2x = a^2 \) implies \( a = ay \) or \( ax = a \), respectively, and analogously for the right hand side). Denote by \( S(R) \) the sets of all square cancellable elements of \( R \).

Recall that a subring \( R \) of a ring \( Q \) is a Fountain-Gould right order in \( Q \) if:

1. Every element of \( S(R) \) has a group inverse in \( Q \) and
2. every element \( q \in Q \) can be written in the form \( q = ab^\sharp \), where \( b \in S(R) \) and \( a \in R \).
Now we extend Proposition 3.1 to the non-necessarily finite case.

**Proposition 3.2.** Let $E$ be an acyclic graph and let $A$ be a semiprime algebra such that $KE \subseteq A \subseteq L_K(E)$. Then $A$ is a Fountain-Gould right order in $L_K(E)$.

**Proof.** Use [7 Lemma 2.2] to find a family $\{E_n\}$ of finite subgraphs of $E$ such that $L_K(E) = \lim_{n \to \infty} L_K(E_n)$. Since $E$ is acyclic, this can be said about $E_n$. We show first that if $a$ is a semiregular element of $A$, then it is locally invertible in $L_K(E)$. Indeed, let $m$ be in $\mathbb{N}$ such that $a \in L_K(E_m)$. The element $a$ is semiregular in $L_K(E_m)$; moreover, by [2 Proposition 3.5], $L_K(E_m)$ is a semisimple and artinian algebra, which implies that $a$ is locally invertible in $L_K(E_m)$ (by [13 Proposition 2.6]), hence it is locally invertible in $L_K(E)$.

Now, take $x \in L_K(E)$, and let $r$ be in $\mathbb{N}$ such that $x \in L_K(E_r)$. By Proposition 3.1, $L_K(E_r)$ is the classical algebra of right quotients of $A \cap L_K(E_r)$, so we may write $x = ab^\sharp$, with $a, b \in A \cap L_K(E_r)$, where $b^\sharp$ denotes the inverse of $b$ in $L_K(E_r)$. Then $b^\sharp$ is just the generalized inverse of $b$ in $L_K(E)$, and our claim has been proved. □

**Remark 3.3.** It is not possible to eliminate the hypothesis of semiprimeness in the previous results (Propositions 3.1 and 3.2). Consider, for example, the following graph:

\[ \bullet \rightarrow \bullet \]

Then $\mathcal{M}_2(K) \cong L_K(E) = Q_{\text{max}}(KE)$ would imply, applying [14 Theorem 4.6], that $KE$ is semiprime, which is false by virtue of Proposition 2.1.

There is another way of localizing when the ring has an involution. In this case, it is possible to consider Moore-Penrose inverses instead of group inverses and *-square cancellable elements instead of the square-cancellable ones. Rings of quotients with respect to Moore-Penrose inverses were introduced in a recent paper of the author ([25]), where it is shown among others that these two “local” approaches are equivalent when the ring of quotients (in any of the two senses) is semiprime and coincides with its socle. A general theory of this kind of localization is developed in [7]. Since the Leavitt path algebra has an involution $\ast$, we may consider $(A, \ast)$ as an algebra with involution such that $KE \subseteq A \subseteq L_K(E)$, and then rewrite the previous result as follows.

**Proposition 3.4.** Let $E$ be an acyclic graph and let $(A, \ast)$ be a semiprime algebra with involution such that $KE \subseteq A \subseteq L_K(E)$. Then $A$ is a Moore-Penrose right $\ast$-order in $L_K(E)$.

**Proof.** Use Proposition 3.2, the Main Theorem of [25 and 5 Theorem 2.1]. □

We define a relation $\geq$ on $E^0$ by setting $v \geq w$ if there is a path $\mu \in E^*$ with $s(\mu) = v$ and $r(\mu) = w$. In this case we will say that $v$ connects to the vertex $w$.

Some consequences can be derived from the results in [14] concerning subalgebras of $L_K(E)$ containing the path algebra $KE$ for an acyclic graph $E$ such that every vertex connects to a line point. First, recall the notion of line point (it appears in [8]).

A subset $H$ of $E^0$ is called hereditary if $v \geq w$ and $v \in H$ imply $w \in H$. A hereditary set is saturated if every vertex which feeds into $H$ and only into $H$ is again in $H$, that is, if $s^{-1}(v) \neq \emptyset$ and $r(s^{-1}(v)) \subseteq H$ imply $v \in H$.

The set $T(v) = \{ v \in E^0 \mid v \geq w \}$ is the tree of $v$, and it is the smallest hereditary subset of $E^0$ containing $v$. We extend this definition for an arbitrary set $X \subseteq E^0$ by $T(X) = \bigcup_{v \in X} T(v)$. The hereditary saturated closure of a set $X$ is defined as the smallest hereditary and saturated
subset of $E^0$ containing $X$. It is shown in [7] that the hereditary saturated closure of a set $X$ is $\overline{X} = \bigcup_{n=0}^{\infty} \Lambda_n(X)$, where
\begin{align*}
\Lambda_0(X) &= T(X), \\
\Lambda_n(X) &= \{y \in E^0 \mid s^{-1}(y) \neq \emptyset \text{ and } r(s^{-1}(y)) \subseteq \Lambda_{n-1}(X)\} \cup \Lambda_{n-1}(X), \text{ for } n \geq 1.
\end{align*}

If $\mu$ is a path in a graph $E$ such that $s(\mu) = r(\mu)$ and $s(e_i) \neq s(e_j)$, for every $i \neq j$, then $\mu$ is called a cycle. We say that a vertex $v$ in $E^0$ is a bifurcation (or that there is a bifurcation at $v$) if $s^{-1}(v)$ has at least two elements. A vertex $u$ in $E^0$ will be called a line point if there are neither bifurcations nor cycles at any vertex $w \in T(u)$, where $T(v) = \{w \in E^0 \mid v \leq w\}$ is the tree of $v$. We will denote by $P_i(E)$ the set of all line points in $E^0$. We say that a path $\mu$ contains no bifurcations if the set $\mu^0 \setminus \{r(\mu)\}$ contains no bifurcations, that is, if none of the vertices of the path $\mu$, except perhaps $r(\mu)$, is a bifurcation.

The notion of line point is central in the study of the socle of a Leavitt path algebra (see [8]): for a graph $E$ the algebra $L_K(E)$ has a nonzero socle (equivalently, it has nonzero minimal left (right) ideals) if and only if it has line points. In fact, the socle of $L_K(E)$ is generated, as an ideal, by the line points in $E$.

Every ring $R$ which is left nonsingular and such that every element $a \in R$ has finite left Goldie dimension (the left Goldie dimension of the left ideal generated by $a$ is finite) will be called a left local Goldie ring (equivalently, by [19, 5, (7.5)], $R$ satisfies the ascending chain condition on the left annihilators of the form $\text{lan}(a)$, with $a \in R$, and every element $a \in R$ has finite left Goldie dimension). If additionally $R$ has finite left (global) dimension, then $R$ will be called a left Goldie ring.

**Theorem 3.5.** Let $E$ be an acyclic graph such that $\overline{P_i(E)} = E^0$. Then, for every semiprime algebra $A$ such that $KE \subseteq A \subseteq L_K(E)$ we have:

(i) $A$ is a left local Goldie ring

(ii) $A$ is prime if and only if the only hereditary and saturated subsets of $E^0$ are $\emptyset$ and $E^0$.

(iii) $A$ has finite left Goldie dimension if and only if $E$ is finite.

**Proof.** (i). The condition $\overline{P_i(E)} = E^0$ implies that $L_K(E)$ coincides with its socle ([8, Theorem 4.2]). Moreover, $L_K(E)$ is an algebra of right quotients of $A$ because it is an algebra of right quotients of $KE$ (Proposition 2.2). Now, (i) follows by [14, Proposition 3.5 (i)]. The statement (ii) can be obtained applying [14, Proposition 3.5 (1)] and [11, Theorem 3.11]. Finally, (iii) follows from [14, Proposition 3.5 (2)] and the obvious fact that $L_K(E)$ artinian implies $E$ finite.

4. **Maximal algebras of quotients**

A Leavitt path algebra has essential socle if and only if every vertex of $E^0$ connects to a line point. This characterization is the key tool to compute maximal algebras of right quotients of $L_K(E)$, for $E$ satisfying the above condition.

An edge $e$ is an exit for a path $\mu = e_1 \ldots e_n$ if there exists $i$ such that $s(e) = s(e_i)$ and $e \neq e_i$.

**Proposition 4.1.** For any graph $E$ the Leavitt path algebra $L_K(E)$ is nonsingular.
Proof. Suppose that the left singular ideal \( Z_1(L_K(E)) \) contains a nonzero element \( x \). By [8 Proposition 3.1] there exist \( \gamma, \mu \in L_K(E) \) such that \( 0 \neq \gamma x \mu = kv \) for a certain vertex \( v \) or \( 0 \neq \gamma x \mu \in wL_K(E)w \cong K[t, t^{-1}] \), where \( w \) is a vertex for which there is a cycle without exits based at it. Since the left singular ideal is an ideal, we have in the first case that it contains the vertex \( v \), which is not possible because \( Z_1(L_K(E)) \) does not have idempotents. In the second case, \( 0 \neq \gamma x \mu \in Z_1(L_K(E)) \) implies, using [14 GS, proposition 2.1 (viii)] that the local algebra of \( L_K(E) \) at \( \gamma x \mu \), i.e. \( L_K(E)_{\gamma x \mu} \), is left nonsingular. But \( L_K(E)_{\gamma x \mu} \) of \( (wL_K(E)w)_{\gamma x \mu} \), that is, there exists a nonzero element \( u \in K[t, t^{-1}] \) satisfying that \( K[t, t^{-1}]u \) is left nonsingular. This implies, by [14 Proposition 2.1 (vii)], that \( u \) lies in the left singular ideal of \( K[t, t^{-1}] \), a contradiction since \( K[t, t^{-1}] \) is a nonsingular algebra.

The right nonsingularity of \( L_K(E) \) can be proved analogously. \( \square \)

Remark 4.2. The previous result implies that for any graph \( E \) the essential left (respectively right/bi-) modules over the Leavitt path algebra \( L_K(E) \) coincide with the dense left (respectively right/bi-) modules (use [15 Lemma 2.24 (b)]). Similar results can be obtained if we consider the graded notions.

Theorem 4.3. Let \( E \) be a graph. Then \( \text{Soc}(L_K(E)) \) is an essential ideal of \( L_K(E) \) if and only if every vertex connects to a line point.

Proof. Suppose first that every vertex connects to a line point. Let \( x \) be a nonzero element in \( L_K(E) \). By [8 Proposition 3.1] there exist \( v \in E^0 \) and \( \gamma, \mu \in L_K(E) \) such that \( 0 \neq \gamma x \mu = kv \in Kv \), or there exists a cycle \( c \) without exits, and \( w \in c^0 \), such that \( 0 \neq \gamma x \mu \in wL_K(E)w \cong K[t, t^{-1}] \). In the first case, since every vertex connects to a line point, there exist \( u \in P_1(E) \) and a path \( \alpha \in E^* \) satisfying \( s(\alpha) = v \) and \( r(\alpha) = u \). Then \( u = \alpha^* \alpha = \alpha^* v \alpha = k^{-1} \alpha^* x \mu \alpha \). The second case cannot happen because since \( w \) connects to an element of \( P_1(E) \) and the vertices of any cycle are not in \( P_1(E) \) (by the very definition), \( w \) connects to a vertex which is not in \( c^0 \), hence \( c \) has an exit. This shows that \( \text{Soc}(L_K(E)) \) is an essential ideal of \( L_K(E) \).

Now, suppose the socle is an essential ideal of \( L_K(E) \). We remark that every vertex of \( P_1(E) \) connects to a line point because \( P_1(E) = \bigcup_{n=0}^{\infty} \Lambda_n(P_1(E)) \) and since \( P_1(E) \) is a hereditary subset of \( E^0 \), every \( \Lambda_n(P_1(E)) \) is hereditary again. Pick a vertex \( v \) in \( E^0 \). Apply that \( \text{Soc}(L_K(E)) \) is essential as a left ideal, and [8 Theorem 4.2], to find \( \alpha, \alpha_i, \beta_i \in L_K(E) \) and \( u_i \in P_1(E) \), for \( i = 1, \ldots, r \), such that \( 0 \neq \alpha v = \sum_i \alpha_i u_i \beta_i = \sum_i \alpha_i u_i \beta_i v \), hence for some \( i \) we have \( 0 \neq u_i \beta_i v \). Suppose for a moment that only (1) can happen in [8 Proposition 3.1]. Then, there exist: an element \( \mu \in KE, k \in K \) and \( w \in E^0 \) such that \( 0 \neq u_i \beta_i v \mu = kw \). Note that \( w \in P_1(E) \) because \( u_i \in \text{Soc}(L_K(E)) = I(P_1(E)) \) ([8 Theorem 4.2]) and by [9 Lemma 2.1]. If \( \gamma \) is a path connecting \( w \) to a line point \( u \), then \( 0 \neq \gamma = w \gamma u = k^{-1} u_i \beta_i v \mu \gamma u \). In particular, \( v \mu \gamma u \neq 0 \). Take into account that \( \mu \gamma \in KE \), hence for some path \( \lambda \) appearing in the summands of \( \mu \gamma, 0 \neq v \lambda u \), that is, \( \lambda \) is a path that joins \( v \) to \( u \).

Finally, justify that every cycle has an exit. If \( c \) were a cycle without exits, and \( v \) any vertex in \( c^0 \), with similar ideas to that of [11 Proof of Theorem 3.11] it could be shown that \( vL_K(E)v \cong K[x, x^{-1}] \). Apply that the socle of \( L_K(E) \) is essential as a left ideal to find a nonzero \( \alpha \in L_K(E)v \cap \text{Soc}(L_K(E)) \). By the semiprimeness of \( L_K(E) \) ([8 Proposition 1.1]) there exists \( \beta \in L_K(E) \) satisfying \( 0 \neq y := v \beta \alpha v \in vL_K(E)v \cap \text{Soc}(L_K(E)) \). Then, since \( K[x, x^{-1}] \) has no minimal one-sided ideals the local algebra of \( vL_K(E)v \) at \( y \), that is, \( (vL_K(E)v)_y = L_K(E)_y \), is not artinian by [14 Proposition 2.1 (v)] (see [14] for the definition.
of local algebra at an element). But, on the other hand, the same result and \( y \in Soc(L_K(E)) \) imply that \( L_K(E)_y \) is artinian, a contradiction.

The following definitions are particular cases of those appearing in [12, Definition 1.3]:
Let \( E \) be a graph, and let \( \emptyset \neq H \in \mathcal{H}_E \). Define
\[
F_E(H) = \{ \alpha = \alpha_1 \ldots \alpha_n \mid \alpha_i \in E^1, s(\alpha_1) \in E^0 \setminus H, r(\alpha_i) \in E^0 \setminus H \text{ for } i < n, r(\alpha_n) \in H \}.
\]
Denote by \( \overline{F}_E(H) \) another copy of \( F_E(H) \). For \( \alpha \in F_E(H) \), we write \( \overline{\alpha} \) to denote a copy of \( \alpha \) in \( \overline{F}_E(H) \). Then, we define the graph \( \overline{H}E = (\overline{H}E^0, \overline{H}E^1, s', r') \) as follows:
1. \( (\overline{H}E)^0 = H \cup F_E(H) \).
2. \( (\overline{H}E)^1 = \{ e \in E^1 \mid s(e) \in H \} \cup \overline{F}_E(H) \).
3. For every \( e \in E^1 \) with \( s(e) \in H \), \( s'(e) = s(e) \) and \( r'(e) = r(e) \).
4. For every \( \overline{\alpha} \in \overline{F}_E(H) \), \( s'(\overline{\alpha}) = \alpha \) and \( r'(\overline{\alpha}) = r(\alpha) \).

A (semiprime) algebra which coincides with its socle will be called semisimple.

**Theorem 4.4.** Let \( E \) be a graph such that every vertex connects to a line point and denote by \( H \) the saturated closure of \( P(E) \). Then:
\[
Q_v(L_K(E)) \cong Q_v(I(H)) \cong Q_v(L_K(HE)),
\]
where \( Q_v(\_\_ \_ \_) \) is the Martindale symmetric algebra of quotients, the maximal symmetric algebra of quotients, the maximal algebra of left/right quotients or the graded maximal algebra of left/right quotients, and \( \overline{H}E \) is an acyclic graph such that \( \overline{P}(\overline{H}E) = \overline{H}E^0 \), that is, \( L_K(\overline{H}E) \) is a semisimple algebra.

**Proof.** Note that by [8, Theorem 4.2], \( Soc(L_K(E)) = I(H) \) (where \( I(H) \) is the ideal of \( L_K(E) \) generated by the vertices of \( H \); it is a graded ideal of \( L_K(E) \) -see [9, Remark 2.2]-), and by [6, Lemma 1.2], \( I(H) \cong L_K(\overline{H}E) \), where \( \overline{H}E \) is an acyclic graph (see the proof of [8, Theorem 4.6]) and satisfies that the saturated closure of \( P(\overline{H}E) \) is \( (\overline{H}E)^0 \) \( (L_K(\overline{H}E) \) coincides with its socle and we apply [8, Theorem 4.2]). By Theorem [13] and Remark [12] \( I(H) \) is a dense subalgebra of \( L_K(E) \), hence the result follows for the Martindale symmetric, the maximal symmetric and the maximal one-sided algebras of quotients. For the graded case, apply [11, Lemma 2.8].

**Remark 4.5.** The condition “every vertex connects to a line point” is milder than \( \overline{P}(\overline{E}) = E^0 \). For example, the graph

\[
\begin{array}{c}
v_1 \\
\downarrow \\
v_2 \\
\downarrow \\
v_3 \\
\ldots
\end{array}
\]

\[
\begin{array}{c}
u_1 \\
\downarrow \\
u_2 \\
\downarrow \\
u_3 \\
\ldots
\end{array}
\]

satisfies the first condition, although not the second one, as was shown in [8, Example 4.6].

**Corollary 4.6.** Let \( E \) be a graph such that every vertex connects to a line point and denote by \( H \) the saturated closure of \( P(E) \). Then
\[
Q^r_{max}(KE) = Q^r_{max}(L_K(E)) \cong Q^r_{max}(I(H)) \cong Q^r_{max}(L_K(\overline{H}E)),
\]
where \( \overline{H}E \) is an acyclic graph such that \( \overline{P}(\overline{E}) = E^0 \), that is, \( L_K(\overline{H}E) \) is a semisimple algebra.

The same can be said about the maximal graded algebra of right quotients.
Example 4.7. Consider the graph $E$ in Remark 4.5. Since every vertex connects to a line point, by Theorem 4.3, $I(H) = \text{Soc}(L_K(E))$ is an essential ideal of $L_K(E)$. It was shown in [8, Example 4.6] that $\text{Soc}(L_K(E)) = I(H)$, where $H = \{v_n \mid n \in \mathbb{N}\} = P(E)$. Apply Theorem 4.4 to have that $Q_v(L_K(E)) \cong Q_v(\infty \oplus M_n(K))$. Define $e_n$ as the element of $E^1$ having source $u_n$ and range $v_n$, and by $f_n$ the edge whose source is $u_n$ and whose range is $u_{n+1}$, for every $n \in \mathbb{N}$. Then $H_E$ is the graph:

![Graph Diagram]

It is clear that the associated Leavitt path algebra is isomorphic to $\infty \oplus \bigoplus_{n=2}^{\infty} M_n(K)$, therefore $Q_v(L_K(E)) \cong Q_v(\infty \oplus \bigoplus_{n=2}^{\infty} M_n(K)) = \prod_{n=2}^{\infty} M_n(K)$.

5. Toeplitz algebras

The Toeplitz algebra is defined (see, for example [18]) as the $C^*$-algebra of continuous functions on the quantum disc. It can be described as the graph algebra associated to the following graph (see, for example, [17]):

![Graph Diagram]

although other descriptions in terms of graph algebras can be given (see [18, Section 3]).

Definition 5.1. Define the (algebraic) Toeplitz algebra as the Leavitt path algebra associated to the graph given above. Denote it by $T$.

Note that the Toeplitz algebra (we will avoid the use of the word “algebraic” because we are in an algebraic context) is a Leavitt path algebra associated to a graph $E$ for which every vertex connects to a line point. Our main concern in this section will be to give a description of the Toeplitz algebra, similar to that given in the analytic context, and to show that if we substitute the line point in the graph for a connected acyclic graph without bifurcations, and the edge connecting the loop to the sink to any (finite) number of edges, then the resulting Leavitt path algebra is again the defined Toeplitz algebra.

We first start by describing those Leavitt path algebras $L_K(E)$ such that there are neither bifurcations nor cycles at any point of $E^0$.

As we have said before, it was shown in [1] Lemma 1.5 that every monomial in $L_K(E)$ is of the form: $kv$, with $k \in K$ and $v \in E^0$, or $ke_1 \cdots e_m f_1^* \cdots f_n^*$ for $k \in K$, $m, n \in \mathbb{N}$, $e_i, f_j \in E^1$. By a reduced expression of a monomial we will understand an expression of the form $e_1 \cdots e_m f_1^* \cdots f_n^*$ with $m + n$ minimal. If this is the case, we will say that $e_1 \cdots e_m f_1^* \cdots f_n^*$ is a reduced monomial.

The following definition can be found in [24, pg. 56]: a walk in a directed graph $E$ is a path in the underlying undirected graph. Formally, a walk $\mu$ is a sequence $\mu = \mu_1 \cdots \mu_n$ with $\mu_i \in E^1 \cup (E^1)^*$ and $s(\mu_i) = r(\mu_{i+1})$ for $1 \leq i < n$. The directed graph $E$ is connected if...
for every two vertices \( v, w \in E^0 \) there is a walk \( \mu = \mu_1 \ldots \mu_n \) with \( v = s(\mu) \) and \( w = r(\mu) \). Intuitively, \( E \) is connected if \( E \) cannot be written as the union of two disjoint subgraphs, or equivalently, \( E \) is connected in case the corresponding undirected graph of \( E \) is so in the usual sense. It is easy to show that if \( E \) is the disjoint union of subgraphs \( \{ E_i \} \), then \( L_K(E) \cong \bigoplus L_K(E_i) \). If all the \( E_i \)'s are connected, then each \( E_i \) will be called a connected component of \( E \).

**Proposition 5.2.** Let \( E \) be an acyclic graph such that there are no bifurcations at any vertex of \( E^0 \) and write \( E = \bigcup_{i \in \mathcal{I}} E_i \), where the \( E_i \)'s are the connected components of \( E \). Then \( L_K(E) \) is isomorphic to \( \bigoplus_{i \in \mathcal{I}} \mathcal{M}_{\alpha_i}(K) \), where \( \alpha_i = \sharp(E_i^0) \), the cardinal of \( E_i^0 \), and \( \alpha_i \in \mathbb{N} \cup \{ \infty \} \).

**Proof.** As we have explained, \( L_K(E) \cong \bigoplus \) \( i \in \mathcal{I} \) \( L_K(E_i) \), hence we may reduce our study to the case of a connected graph \( E \). We have divided the proof into three steps.

**Step 1.** Every monomial in \( L_K(E) \) has a unique reduced expression, that is, for every monomial \( z \in L_K(E) \), there exist two unique paths \( \alpha, \beta \in L_K(E) \), with length(\( \alpha \)) + length(\( \beta \)) minimal such that \( z = \alpha \beta^* \).

Let \( \alpha, \beta, \mu, \nu \) be paths such that \( \alpha \beta^* = \mu \nu^* \), being both reduced expressions. Suppose that the length of \( \alpha \) is strictly less to the length of \( \mu \). Use (3) to show that if \( \alpha = e_1 \ldots e_n \), then \( \mu = e_1 \ldots e_n g_1 \ldots g_r \), for some \( e_i, g_j \in E^1 \), and write \( \beta^* = f_1^* \ldots f_m^* \) and \( \nu^* = h_1^* \ldots h_{m+r}^* \), with \( f_i, h_j \in E^1 \). Then \( e_1 \ldots e_n f_1^* \ldots f_m^* = e_1 \ldots e_n g_1 \ldots g_r h_1^* \ldots h_{m+r}^* \) implies, by (3), \( f_1^* \ldots f_m^* = g_1 \ldots g_r h_1^* \ldots h_{m+r}^* \). Since there are no bifurcations in \( E \), \( (f_m^* \ldots f_1^*) (f_m^* \ldots f_1^*) = s(f_m) \). Using this and (3), after multiplying the expression below by \( f_m \ldots f_1 \) on the left hand side, we obtain:

\[
\sum f_m \ldots f_1 g_1 \ldots g_r h_1^* \ldots h_{m+r}^* = s(f_m).
\]

This means that \( s(f_m) = r(h_{m+r}^*) = s(h_{m+r}^*) \), but there are no bifurcations in \( E \), hence \( f_m = h_{m+r} \). Multiply by \( f_m^* \) on the left hand side and by \( f_m \) on the right hand side. Then

\[
f_m \ldots f_1 g_1 \ldots g_r h_1^* \ldots h_{m+r-1}^* = s(f_m-1).
\]

Proceed again in this form and in \( m + r \) steps we will have obtained \( f_i = h_{r+i} \), for \( i \in \{1, \ldots, m\} \) and \( g_j = h_{r+j+1} \) for \( j \in \{1, \ldots, r\} \). This implies that

\[
\mu \nu^* = e_1 \ldots e_n g_1 \ldots g_r h_1^* \ldots h_{m+r}^* = e_1 \ldots e_n h_1 \ldots h_{m+r} = e_1 \ldots e_n h_{r+1} \ldots h_{m+r},
\]

which is not a reduced expression of \( \mu \nu^* \) since here the number of edges plus the number of ghost edges is \( n + m \), while in the first expression of \( \mu \nu^* \) this sum was \( n + m + 2r \).

Consequently, \( r = 0 \) and \( e_1 \ldots e_n f_1^* \ldots f_m^* = e_1 \ldots e_n h_1^* \ldots h_m^* \). This implies (by (4) and since there are no bifurcations) \( f_1^* \ldots f_m^* = h_1^* \ldots h_m^* \); by (3), \( f_i = h_i \) for \( i \in \{1, \ldots, m\} \), and so \( \alpha = \mu \) and \( \beta = \nu \).

**Step 2.** The set of all reduced monomials is a basis of \( L_K(E) \) as a \( K \)-vector space. Denote it by \( \mathcal{B} \).

We know from [11, Lemma 1.5] that the expressions \( \{ \alpha \beta^* \} \) generate \( L_K(E) \) as a vector space, hence we only need to check that they are linearly independent.

Suppose we have \( \sum_i \kappa_i \alpha_i \beta_i^* = 0 \), where all the summands are different from zero and \( \alpha_i \beta_i^* \neq \alpha_j \beta_j^* \). Moreover, taking into account the degree, we may suppose that each summand has the same degree.
Let \( \beta_1 \) be with maximal length among the \( \beta_i \)'s. Then \( 0 = \sum_i k_i \alpha_i \beta_i^* \beta_1 \), where at least one of the summands is nonzero (\( k_1 \alpha_1 \beta_1^* \beta_1 = k_1 \alpha_1 \)) and being each summand in only real edges. Let us call \( \gamma_1 = \alpha_1 \) and \( \gamma_i = \alpha_i \beta_i^* \beta_1 \), for \( i \neq 1 \). With this notation the formula below reads \( \sum_i \gamma_i = 0 \). Multiply by \( \gamma_1^* \) on the right hand side and apply that \( \gamma_1 \gamma_1^* = s(\gamma_1) \), because there are no exits. Then multiply by \( \gamma_1^* \) on the left hand side and apply (3). We obtain the following formulas:

\[
-k_1 s(\gamma_1) = \sum_{i \neq 1} k_i \gamma_i \gamma_1^* \\
-k_1 r(\gamma_1) = \sum_{i \neq 1} k_i \gamma_1^* \gamma_i
\]

The first one implies that \( s(\gamma_i) = s(\gamma_1) \) for all \( i \)'s given nonzero terms. The second one that \( r(\gamma_i) = s(\gamma_1) \) for the same \( i \)'s. This implies \( \gamma_i = \gamma_1 \) for these terms. Hence, there exists at least one \( i \) such that \( \alpha_1 = \gamma_1 = \gamma_i = \alpha_i \beta_i^* \beta_1 \), so \( \alpha_1 \beta_1^* = \alpha_i \beta_i^* \), a contradiction.

Step 3. The result.

Let \( u \) and \( v \) be vertices. Since we are considering that the graph \( E \) is connected, there exists a walk \( \mu \) starting at \( v \) and ending at \( w \). By Step 1, \( \mu \) has a reduced expression and it is an element of the basis \( \mathcal{B} \) (Step 2), therefore we may describe \( \mathcal{B} \) as the set of walks \( \mu_{j,k} \), where \( \mu_{j,k} \) is the only element in \( L_K(E) \) such that \( s(\mu_{j,k}) = v_j \) and \( r(\mu_k) = v_k \), for \( v_j, v_k \in E^0 \). Define, for \( \alpha = \sharp(E^0) \) the map \( \varphi : L_K(E) \to \mathcal{M}_\alpha(K) \) as the \( K \)-linear map which acts on the elements of \( \mathcal{B} \) as follows: \( \varphi(\mu_{j,k}) = e_{jk} \), being \( e_{jk} \) the matrix unit having all the entries equal to zero except those in row \( j \) and column \( k \). It is not difficult to show that it is an isomorphism of \( K \)-algebras. \( \square \)

**Theorem 5.3.** Let \( F \) be an acyclic graph such that every vertex connects to a line point, and consider the graph \( E = E(n, F) \) such that \( E^0 = F^0 \cup \{ v \} \), \( E^1 = F^1 \cup \{ e_1, \ldots, e_n \} \), where \( e \) has range and source \( v \), \( s(e_i) = v \), \( r(e_i) \in F^0 \), and for every \( f \in F^0 \), its range and source are the corresponding as edges of the graph \( F \). Then:

(i) \( \text{Soc}(L_K(E)) \) is an essential ideal of \( L_K(E) \).

(ii) There exists an exact sequence:

\[
0 \to \text{Soc}(L_K(E)) \to T \to K[x, x^{-1}] \to 0.
\]

(iii) \( L_K(E) \) is isomorphic to the Toeplitz algebra \( T \).

(iv) There is a subalgebra \( T' \) of \( \text{RCFM}(K) \), isomorphic to \( T \), such that \( \mathcal{M}_\infty(K) \subseteq T' \subseteq \text{RCFM}(K) \).

**Proof.** (i). Since every vertex of \( E^0 \) connects to a line point, we may apply Theorem 4.3 to obtain that the socle of \( L_K(E) \) is an essential ideal.

(ii). Observe that \( F^0 \), which is the saturated closure of the set of line points of \( E \), is an hereditary and saturated subset of \( E^0 \), therefore \( \text{Soc}(L_K(E)) = I(F^0) \) by [8] Theorem 5.2 \]. Moreover, [9] Lemma 2.3 (1) implies that

\[
0 \to \text{Soc}(L_K(E)) \to T \to L_K(E/F^0) \to 0
\]

is an exact sequence, where \( E/F^0 \) is the graph having one vertex and one edge (\( E/F^0 \) is called the quotient graph; see, for example, [7] for its definition). Since \( L_K(E/F^0) \cong K[x, x^{-1}] \), we have proved this item.
(iii). By [6, Lemma 1.2], $I(F^0)$ is isomorphic to $L_K(F_0E)$. Note that $F_0E$ is an acyclic graph without bifurcations at any point and having an infinite number of edges. In the particular case of being $n = 1$ and $F$ the graph having one vertex and no edges, $F_0E$ is

\[
\begin{array}{c}
\bullet \\
\end{array}
\]

By Proposition 5.2, $L_K(F_0E)$ is isomorphic to $\mathcal{M}_\infty(K)$. This fact, jointly with (ii), imply the result.

(iv). It has been proved in (iii) that $I(F^0) \cong L_K(F_0E) \cong \mathcal{M}_\infty(K)$. By Corollary 4.6 $Q^r_{\max}(L_K(E)) \cong Q^r_{\max}(I(F^0)) \cong Q^r_{\max}(\mathcal{M}_\infty) = \text{RCFM}(K)$.

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