Mean field type equations on line bundle over a closed Riemann surface

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Abstract

Let \((L, g)\) be a line bundle over a closed Riemann surface \((\Sigma, g)\), \(\Gamma(L)\) be the set of all smooth sections, and \(D : \Gamma(L) \to T^*\Sigma \otimes \Gamma(L)\) be a connection independent of the bundle metric \(g\), where \(T^*\Sigma\) is the cotangent bundle. Suppose that there exists a global unit frame \(\zeta\) on \(\Gamma(L)\). Precisely for any \(\sigma \in \Gamma(L)\), there exists a unique smooth function \(u : \Sigma \to \mathbb{R}\) such that \(\sigma = u\zeta\) with \(|\zeta| \equiv 1\) on \(\Sigma\). For any real number \(\rho\), we define a functional \(J_\rho : W^{1,2}(\Sigma, L) \to \mathbb{R}\) by

\[
J_\rho(\sigma) = \frac{1}{2} \int_\Sigma |D\sigma|^2 dv_g + \frac{\rho}{|\Sigma|} \int_\Sigma (\sigma, \zeta) dv_g - \rho \log \int_\Sigma h e^{\langle \sigma, \zeta \rangle} dv_g,
\]

where \(W^{1,2}(\Sigma, L)\) is a completion of \(\Gamma(L)\) under the usual Sobolev norm, \(|\Sigma|\) is the area of \((\Sigma, g)\), \(h : \Sigma \to \mathbb{R}\) is a strictly positive smooth function and \(\langle \cdot, \cdot \rangle\) is the inner product induced by \(g\). The Euler-Lagrange equations of \(J_\rho\) are called mean field type equations. Write \(\mathcal{H}_0 = \{\sigma \in W^{1,2}(\Sigma, L) : D\sigma = 0\}\) and \(\mathcal{H}_1 = \{\sigma \in W^{1,2}(\Sigma, L) : \int_\Sigma (\sigma, \tau) dv_g = 0, \forall \tau \in \mathcal{H}_0\}\).

Based on the variational method, we prove that \(J_\rho\) has a constraint critical point on the space \(\mathcal{H}_1\) for any \(\rho < 8\pi\); Based on blow-up analysis, we calculate the exact value of \(\inf_{\sigma \in \mathcal{H}_1} J_\rho(\sigma)\), provided that it is not achieved by any \(\sigma \in \mathcal{H}_1\); If we further assume \(D\zeta = 0\), \(J_\rho : W^{1,2}(\Sigma, L) \to \mathbb{R}\) is reduced to a functional related to the classical mean field equation.

Keywords: Mean field equation; blow-up analysis; analysis on line bundle

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1. Introduction

Let \((\Sigma, g)\) be a closed Riemann surface. The well-known mean field equation has aroused the interests of many mathematicians for a long time. It is written as

\[
\Delta_g u = \rho \left( \frac{h e^u}{\int_\Sigma h e^u dv_g} - \frac{1}{|\Sigma|} \right),
\]  

\[\text{(1)}\]
where $\Delta_p$ is the Laplace-Beltrami operator, $\rho$ is a real number, $h : \Sigma \rightarrow \mathbb{R}$ is a function, and $|\Sigma|$ denotes the area of $\Sigma$. This equation appears in the prescribed Gaussian curvature problem in conformal geometry [18, 6, 16, 24] and also in the abelian Chern-Simons-Higgs model in physics [6, 14, 13, 33, 35]. Let $W^{1,2}(\Sigma)$ be the standard Sobolev space with respect to the norm

$$
||u||_{W^{1,2}(\Sigma)} = \left( \int_{\Sigma} (|\nabla u|^2 + |u|^2)dv_g \right)^{1/2},
$$

where $\nabla$ stands for the gradient operator on $(\Sigma, g)$. Note that (1) has a variational structure. In particular, solutions of (1) are critical points of the functional $J_\rho : W^{1,2}(\Sigma) \rightarrow \mathbb{R}$, which is defined by

$$
J_\rho(u) = \frac{1}{2} \int_\Sigma |\nabla u|^2 dv_g + \frac{\rho}{2} \int_\Sigma udv_g - \rho \log \int_\Sigma h e^u dv_g.
$$

The solvability of the mean field equation (1) is closely related to the Trudinger-Moser inequality contributed by Moser [26], Aubin [2], Fontana [17], among others. Such an inequality together with a direct method of variation implies that $J_\rho$ has a critical point when $\rho < 8\pi$. In the critical case $\rho = 8\pi$, under certain geometric and analytic assumptions, a minimizing solution was found by Ding-Jost-Li-Wang [13] via the method of blow-up analysis. This result was extended to various situations, see for examples [22, 33, 28, 35, 30, 38, 39, 31, 32]. For the related Nirenberg problem, the readers are referred to [2, 7, 8, 18, 26, 27, 29] and the references therein. However, the case $\rho > 8\pi$ is much different from $\rho \leq 8\pi$. Using a min-max method, Ding-Jost-Li-Wang [16] obtained that $J_\rho$ has a critical point if $\rho \in (8\pi, 16\pi)$, $h > 0$ and the genus of the surface is nonzero. A topological method was introduced by Djadli-Malchiodi [12] and Djadli [11] to solve this problem. In particular, for any $\rho \in (8k\pi, 8(k + 1)\pi)$, $k \in \mathbb{N}^+$, $J_\rho$ has a critical point with no assumption on the topology of the surface. Along this direction, there had been a lot of work such as [1, 4, 24, 23, 25] and the references therein. This problem still arouses many people’s interest [5, 16, 34, 21].

In the current paper, our aim is to extend the mean field equation from Riemann surface to the line bundle setting. Given a line bundle $(\mathcal{L}, g)$ over a closed Riemann surface $(\Sigma, g)$, $g$ is a bundle metric. Let $\Gamma(\mathcal{L})$ be the set of all smooth sections $\sigma : \Sigma \rightarrow \mathcal{L}$, $T^*\Sigma$ be the cotangent bundle, and $\mathcal{D} : \Gamma(\mathcal{L}) \rightarrow T^*\Sigma \otimes \Gamma(\mathcal{L})$ be a connection. Here and in the sequel we do not assume $g$ is parallel with respect to the connection $\mathcal{D}$. Define a Sobolev space $W^{1,2}(\Sigma, \mathcal{L})$ as a completion of $\Gamma(\mathcal{L})$ under the norm

$$
||\sigma||_{W^{1,2}(\Sigma, \mathcal{L})} = \left( \int_{\Sigma} (|\mathcal{D}\sigma|^2 + |\sigma|^2)dv_g \right)^{1/2},
$$

where both of the metrics on $\Gamma(\mathcal{L})$ and $T^*\Sigma \otimes \Gamma(\mathcal{L})$ are denoted by the same symbol $| \cdot |$. A unit section $\zeta \in \Gamma(\mathcal{L})$ is said to be a global unit frame provided that $|\zeta(x)| = 1$ for all $x \in \Sigma$ and for any fixed $\sigma \in \Gamma(\mathcal{L})$, there exists a unique smooth function $u : \Sigma \rightarrow \mathbb{R}$ such that $\sigma(x) = u(x)\zeta(x)$ for all $x \in \Sigma$. It is easy to see that any global unit frame on $\Gamma(\mathcal{L})$ would be also a global unit frame on $W^{1,2}(\Sigma, \mathcal{L})$. Then we naturally extend the connection $\mathcal{D}$ from $\Gamma(\mathcal{L})$ to $W^{1,2}(\Sigma, \mathcal{L})$. Indeed, there exists a unique $\omega \in \Gamma(T^*\Sigma)$ satisfying $\mathcal{D}\zeta = \omega \otimes \zeta$. As a consequence, for any $\sigma = u\zeta \in W^{1,2}(\Sigma, \mathcal{L})$, we may write by the Leibniz rule

$$
\mathcal{D}\sigma = du \otimes \zeta + u\omega \otimes \zeta.
$$

(3)
where \langle \cdot , \cdot \rangle denotes the inner product induced by \( g \), and \( h : \Sigma \to \mathbb{R} \) is a positive smooth function. It should be noticed that the above functional \( J \) in the case \( \dim H = 0 \) and \( \dim \mathcal{H} \) depends on the global unit frame \( \zeta \). Define two section sets

\[
\mathcal{H}_0 = \{ \sigma \in W^{1,2}(\Sigma, L) : D\sigma = 0 \}
\]

and

\[
\mathcal{H}_1 = \{ \sigma \in W^{1,2}(\Sigma, L) : \int_{\Sigma} \langle \sigma, \tau \rangle dv_g = 0, \ \forall \tau \in \mathcal{H}_0 \}.
\]

Here and throughout this paper we abuse some notations slightly. Both zero section and real number zero are denoted by the same 0. However, readers can recognize them from the context. Obviously \( \mathcal{H}_0 \cap \mathcal{H}_1 = \{0\} \) and

\[
W^{1,2}(\Sigma, L) = \mathcal{H}_0 \oplus \mathcal{H}_1.
\]

In the line bundle setting, we restrict sections on the space \( \mathcal{H}_1 \) mainly because the Poincaré inequality holds there. To write explicitly the Euler-Lagrange equations for minimizers of \( J_\rho \), we also need to understand \( \mathcal{H}_0 \). We will prove later \( \mathcal{H}_0 \) is a finite dimensional linear space, in particular \( \dim \mathcal{H}_0 \leq 1 \). As for the dimension of \( \mathcal{H}_0 \), there are two possibilities: \( \dim \mathcal{H}_0 = 0 \) or \( \dim \mathcal{H}_0 = 1 \). In view of (3) and the definitions of \( D \) and \( \zeta \), for any \( \tau = u\zeta \in \mathcal{H}_0 \), we have

\[
du \otimes \zeta + u\omega \otimes \zeta = (du + u\omega) \otimes \zeta = 0,
\]

or equivalently

\[
du + u\omega = 0 \quad \text{on} \quad \Sigma.
\]

Obviously \( u \equiv 0 \) is a solution of (7). If (7) has no other solution, then \( \mathcal{H}_0 = \{0\} \), and thus \( \dim \mathcal{H}_0 = 0 \); If (7) has a solution \( u \not\equiv 0 \), then \( \dim \mathcal{H}_0 = 1 \). Let us give an example. Assume \( (\Sigma, g) \) is a closed Riemann surface, \( L = \Sigma \times \mathbb{R} \) is a line bundle and \( g \) is the standard Euclidean metric on \( \mathbb{R} \). Take a global unit frame \( \zeta(x) = (x, 1) \) for all \( x \in \Sigma \) and a bundle connection \( D \) such that \( D\zeta = \omega \otimes \zeta = 0 \), which implies \( \omega = 0 \). Obviously \( W^{1,2}(\Sigma, L) \cong W^{1,2}(\Sigma) \), and (7) has only solutions \( u \equiv C \) for all \( C \in \mathbb{R} \). Hence \( \mathcal{H}_0 \cong \mathbb{R} \) and \( \dim \mathcal{H}_0 = 1 \). In the case \( \dim \mathcal{H}_0 = 1 \), we may assume \( \mathcal{H}_0 = \text{span}\{\tau_1\} \), where \( \{\tau_1\} \) is a normal basis, in particular

\[
\int_{\Sigma} (\tau_1, \tau_1) dv_g = 1.
\]

If \( \sigma \) is a minimizer of \( J_\rho \) on \( \mathcal{H}_1 \), then it satisfies

\[
\Delta_L \sigma = \rho \frac{he^{(\sigma, \zeta)} \zeta}{\int_{\Sigma} he^{(\sigma, \zeta)} dv_g} - \rho \frac{\zeta}{|\Sigma|} \zeta
\]

in the case \( \dim \mathcal{H}_0 = 0 \), and

\[
\left\{ \begin{array}{l}
\Delta_L \sigma = \rho \frac{he^{(\sigma, \zeta)} \zeta}{\int_{\Sigma} he^{(\sigma, \zeta)} dv_g} - \rho \frac{\zeta}{|\Sigma|} \zeta - \lambda_1 \tau_1 \\
\lambda_1 = \rho \frac{\int_{\Sigma} h \zeta (\tau, \tau) dv_g}{3} - \rho \frac{\int_{\Sigma} \langle \zeta, \tau_1 \rangle dv_g}{3}
\end{array} \right.
\]

in the case \( \dim \mathcal{H}_0 = 1 \).
Theorem 1. Let $H_0 = D^*D$ be the bundle Laplace-Beltrami operator, and $D^*$ is the dual operator of $D$, which is defined by

$$\int_{\Sigma} (D^*D\sigma, \phi)dv_g = \int_{\Sigma} (D\sigma, D\phi)dv_g, \quad \forall \phi \in W^{1,2}(\Sigma, L).$$

Let us explain the geometric meaning of the equation (9). Since $H_0$ is a subset of $L^2(\Sigma, L)$, and $W^{1,2}(\Sigma, L)$ is dense in $L^2(\Sigma, L)$ under the usual $L^2$ norm, we have $L^2(\Sigma, L) = H_0 \oplus \overline{H_1}$, where $\overline{H_1}$ is a closure of $H_1$ in $L^2(\Sigma, L)$, $i = 1, 2$. This is an orthogonal decomposition of $L^2(\Sigma, L)$. There exists a natural projection $P : L^2(\Sigma, L) \to L^2(\Sigma, L)$, which maps a $\xi \in L^2(\Sigma, L)$ to

$$P(\xi) = \xi - \left( \int_{\Sigma} (\xi, \tau)dv_g \right) \tau.$$

In this framework, the equation (9) can be written as

$$\Delta_\xi = P \left\{ \rho \left( \frac{h^{\sigma(\xi)}}{\int_{\Sigma} h^{\sigma(\xi)}dv_g} - \frac{1}{|\Sigma|} \right) \xi \right\}.$$

Concerning critical points of $J_\rho$, we have the following:

**Theorem 1.** Let $(\mathcal{L}, g)$ be a line bundle over a closed Riemann surface $(\Sigma, g)$. $D$ be its bundle connection. Suppose there exists a global unit frame $\xi$ on $\Gamma(\mathcal{L})$. Let $J_\rho$ and $H_1$ be defined as in (2) and (4) respectively. Then for any $\rho < 8\pi$, $J_\rho$ has a minimizer $\sigma_\rho$ on $H_1$.

The proof of Theorem 1 is based on a method of variation. It should be remarked that in this theorem, the infimum is taken over all sections on $H_1$. In general, the infimum is not necessarily attained on the whole space $W^{1,2}(\Sigma, L)$.

**Theorem 2.** Let $(\mathcal{L}, g)$ be a line bundle over a closed Riemann surface $(\Sigma, g)$. $D$ be its bundle connection. Suppose there exists a global unit frame $\xi$ on $\Gamma(\mathcal{L})$. Let $J_\rho$, $H_0$ and $H_1$ be defined as in (2), (3) and (4) respectively. Suppose that $\inf_{\sigma \in H_1} J_{8\pi}(\sigma)$ is not attained on $H_1$. Then we have the following two assertions:

(i) If $\dim H_0 = 0$, then there exist some $p_0 \in \Sigma$ and a Green section $G_0 = G_{p_0, \xi}$ with

$$\Delta_\xi G_0 = 8\pi \left( \delta_{p_0} - \frac{1}{|\Sigma|} \right) \xi$$

such that

$$\inf_{\sigma \in H_1} J_{8\pi}(\sigma) = -8\pi - 4\pi A_{p_0} - 8\pi \log \pi - 8\pi \log h(p_0) + \frac{4\pi}{|\Sigma|} \int_{\Sigma} G_{p_0}dv_g, \quad (10)$$

where $A_{p_0} = \lim_{\rho \to 0}(G_{p_0}(x) + 4 \log d_\rho(x, p_0))$ is a constant, $\Delta_\xi = D^*D$ is the Laplace-Beltrami operator on line bundle, $D^*$ is the dual operator with respect to $D$;

(ii) If $\dim H_0 = 1$ and $\{\tau_1\}$ is a normal basis of $H_0$, then there exist some $p \in \Sigma$ and a Green section $G = G_p, \xi$ with

$$\left\{ \begin{array}{l}
\Delta_\xi G = 8\pi \left( \delta_p - \frac{1}{|\Sigma|} \right) \xi - \lambda_1 \tau_1 \\
\lambda_1 = 8\pi \left( \langle \xi, \tau_1 \rangle(p) - \frac{1}{|\Sigma|} \int_{\Sigma} \langle \xi, \tau_1 \rangle dv_g \right) \end{array} \right. \quad (11)$$

in the case $\dim H_0 = 1$, where $\Delta_\xi = D^*D$ is the bundle Laplace-Beltrami operator, and $D^*$ is the dual operator of $D$, which is defined by

$$\int_{\Sigma} (D^*D\sigma, \phi)dv_g = \int_{\Sigma} (D\sigma, D\phi)dv_g, \quad \forall \phi \in W^{1,2}(\Sigma, L).$$
such that

\[
\inf_{\sigma \in \mathcal{H}_1} \mathcal{J}_{\text{se}}(\sigma) = -8\pi - 4\pi A_p - 8\pi \log \pi - 8\pi \log h(p) + \frac{4\pi}{|\Sigma|} \int G_p dv_g,
\]

where \( A_p = \lim_{\rho \to p}(G_p(x) + 4 \log d(x, p)) \) is a constant.

Indeed, Theorem 2 gives a sufficient condition such that \( \mathcal{J}_{\text{se}} \) has a minimizer on \( \mathcal{H}_1 \). For the proof of Theorem 2 we modify the method of blow-up analysis used by Ding-Jost-Li-Wang \( \cite{13} \) (see also \( \cite{19, 34, 39} \)). Several technical difficulties need to be overcome. Since \( \mathcal{J}_\sigma : W^{1,2}(\Sigma, L) \to \mathbb{R} \) has no translation invariance, the method in \( \cite{13} \) cannot be directly applied here. We overcome this difficulty by employing a technique from \( \cite{39} \). As we shall see later, we need much more analysis on neck domains than in \( \cite{13, 19, 34, 39} \). Moreover, we need to construct a sequence of sections \( \sigma_k \) satisfying \( \mathcal{J}_{\text{se}}(\sigma_k) \) converges to inf_{\sigma \in \mathcal{H}_1} \mathcal{J}_{\text{se}}(\sigma). \) This is also different from \( \cite{13, 19, 34} \).

In the case \( \zeta \) is a global unit frame on \( \Gamma(\mathcal{L}) \), if we further assume \( \mathcal{D}\zeta = 0 \), then we shall prove later that \( \mathcal{H}_0 = \text{span}(\zeta) \). Hence dim\(\mathcal{H}_0 = 1 \), \( \tau_1 = |\Sigma|^{-1/2} \zeta \), and \( \text{(9)} \) would reduce to

\[
\Delta_L \zeta = \rho \left( \frac{h e^{\langle \sigma, \zeta \rangle}}{\int_{\Sigma} h e^{\langle \sigma, \zeta \rangle} dv_g} - \frac{1}{|\Sigma|} \right) \zeta.
\]

Also we shall show \( \Delta_L \zeta = (\Delta_L u) \zeta \) for any \( \sigma = u \zeta \) in this case, and thus \( \text{(13)} \) is essentially the mean field equation \( \text{(1)} \). As a consequence, our third result says

**Theorem 3.** In addition to the assumptions in Theorem 7 we further require that \( \mathcal{D}\zeta = 0 \). Then we have the following three assertions:

(i) When \( \rho < 8\pi \), \( \text{(13)} \) has a solution;
(ii) When \( \rho = 8\pi \), if the Ding-Jost-Li-Wang condition is satisfied, then \( \text{(13)} \) has a solution;
(iii) When \( \rho \in (8k\pi, 8(k + 1)\pi) \), \( \forall k \in \mathbb{N} \), \( \text{(13)} \) has a solution.

Here the **Ding-Jost-Li-Wang condition** means certain assumptions on \( h \) and \( (\Sigma, g) \) proposed by Ding-Jost-Li-Wang \( \cite{13} \), Theorem 1.2. For the proof of Theorem 3 we only need to transform the problem from line bundle setting to Riemann surface setting, and then use the existing results of Ding-Jost-Li-Wang \( \cite{13} \) and Djadli \( \cite{11} \).

The remaining part of this paper is organized as follows: In section 2, we give some preliminaries; In Section 3, by a direct method of variation, we prove Theorem 1. In Section 4, we use the method of blow-up analysis to prove Theorem 2. In Section 5, we transform Theorem 3 to results of Ding-Jost-Li-Wang \( \cite{13} \) and Djadli \( \cite{11} \). Throughout this paper, we do not distinguish sequence and subsequence, often denote various constants by the same \( C \), and represent metrics on different bundles by the same \( | \cdot | \). We always denote a geodesic ball centered at \( x \in \Sigma \) with radius \( r \) by \( B_r(x) = \{ q \in \Sigma : d_q(x) < r \} \), and a Euclidean ball centered at 0 with radius \( r \) by \( \mathbb{B}_r = \{ z \in \mathbb{R}^2 : |z| < r \} \).

**2. Preliminaries**

In this section, we present some useful facts which will be used frequently in the following. First, a distributional decomposition of \( \Delta_L \zeta \) will be given. Next, we show the Poincaré inequality.
holds on the space $\mathcal{H}_1$. Finally, we derive several properties of the spaces $\mathcal{H}_0$ and $\mathcal{H}_1$, which are defined by (5) and (6) respectively.

To write $\Delta_L$ explicitly, we recall $\zeta$ is a global unit frame on $W^{1,2}(\Sigma, L)$ and $D\zeta = \omega \otimes \zeta$. Let $d^*: W^{1,2}(\Sigma, T^*\Sigma) \to W^{1,2}(\Sigma)$ be defined by

$$
\int_{\Sigma} (d^* \varsigma) u dv_g = \int_{\Sigma} (\varsigma, du) dv_g
$$

for all $\varsigma \in W^{1,2}(\Sigma, T^*\Sigma)$ and all $u \in W^{1,2}(\Sigma)$. In a coordinate system $\{x^\alpha\}_{\alpha=1}^2$, if $\varsigma = \varsigma^\alpha d x^\alpha$, then one can easily compute

$$
d^* \varsigma = -\frac{1}{\sqrt{g}} \frac{\partial}{\partial x^\alpha} \left( g^{\alpha\beta} \sqrt{g} \varsigma^\beta \right),
$$

(14)

where $(g^{\alpha\beta})$ is the inverse matrix of $(g_{\alpha\beta})$ and $\sqrt{g} = \sqrt{\det(g_{\alpha\beta})}$. Obviously the Laplace-Beltrami operator $\Delta_g$ is locally written as

$$
\Delta_g u = d^* u = -\frac{1}{\sqrt{g}} \frac{\partial}{\partial x^\alpha} \left( g^{\alpha\beta} \sqrt{g} \frac{\partial u}{\partial x^\beta} \right).
$$

(15)

A straightforward calculation gives

**Lemma 4.** For any $\sigma = u\zeta \in W^{1,2}(\Sigma, L)$, there is a distributional decomposition

$$
\Delta_L \sigma = (\Delta_g u + (|\omega|^2 + d^* \omega) u) \zeta.
$$

**Proof.** Let $\sigma = u\zeta \in W^{1,2}(\Sigma, L)$. Take any $\phi = v\zeta \in \Gamma(L)$. Note that $|\zeta| = 1$. In view of (14) and (15), one has

$$
\int_{\Sigma} (\sigma, \Delta_L \phi) dv_g = \int_{\Sigma} (D\sigma, D\phi) dv_g
\quad = \int_{\Sigma} (du \otimes \zeta + u\omega \otimes \zeta, dv \otimes \xi + v\omega \otimes \zeta) dv_g
\quad = \int_{\Sigma} \left( d^* du + (|\omega|^2 + d^* \omega) u \right) dv_g
\quad = \int_{\Sigma} \left( (\Delta_g u + (|\omega|^2 + d^* \omega) u) \zeta, \phi \right) dv_g.
$$

This is exactly the desired result. \(\square\)

**Lemma 5.** Any section in $\mathcal{H}_0$ is smooth on $\Sigma$. Moreover, $\mathcal{H}_0$ is a finite dimensional linear space, in particular $\dim \mathcal{H}_0 \leq 1$.

**Proof.** Taking any section $\tau$ in $\mathcal{H}_0$, since for all $\varphi \in \Gamma(L)$,

$$
0 = \int_{\Sigma} (D\tau, D\varphi) dv_g = \int_{\Sigma} (\tau, D^* D\varphi) dv_g = \int_{\Sigma} (\tau, \Delta_L \varphi) dv_g,
$$

we know that $\Delta_L \tau = 0$ in the distributional sense. Let $\tau = \phi \zeta$. By Lemma 4, $\phi$ is a distributional solution of

$$
\Delta_g \phi + (|\omega|^2 + d^* \omega) \phi = 0.
$$

(16)
Noting \( \mathbf{\phi} \in W^{1,2}(\Sigma) \), we have by repeatedly applying elliptic estimate (\( \mathbb{F} \), Theorem 3.54) to (15) that \( \mathbf{\phi} \in W^{1,2}(\Sigma) \) for all \( \ell \geq 2 \). Then the Sobolev embedding theorem leads to \( \phi \in C^\infty(\Sigma) \). Hence, we obtain \( \tau \in \Gamma(L) \), which confirms the first assertion.

For any \( \sigma \in \mathcal{H}_0 \), there is a unique \( u \in C^\infty(\Sigma) \) such that \( \sigma = u \zeta \) and \( D\sigma = 0 \). This is equivalent to the equation

\[
du + u\omega = 0.
\]

(17)

Obviously \( u \equiv 0 \) is a solution of (17). If (17) has a unique solution \( u \equiv 0 \), then \( \dim\mathcal{H}_0 = 0 \).

In the following, we consider the case \( \mathcal{H}_0 \neq \{0\} \). We first claim that if \( u \) is a solution of (17) and \( u(x_0) = 0 \) for some \( x_0 \in \Sigma \), then \( u \equiv 0 \) on \( \Sigma \). To see this, for any \( x \in \Sigma \) and \( x \neq x_0 \), we take a geodesic line \( \gamma : [0,1] \to \Sigma \) satisfying \( \gamma(0) = x_0 \) and \( \gamma(1) = x \). Set \( y(t) = u(\gamma(t)) \) and \( a(t) = \omega(\gamma(t))(\gamma'(t)) \) for \( t \in [0,1] \), where \( \gamma'(t) \) denotes the tangent vector field along the geodesic line \( \gamma(t) \). It follows from (17) that

\[
\begin{align*}
\frac{d}{dt}y(t) &= -a(t)y(t) \\
y(0) &= 0.
\end{align*}
\]

(18)

Applying the existence and uniqueness theorem of the linear ODE to (18), we obtain \( y(t) \equiv 0 \) for all \( t \in [0,1] \). Thus \( u(x) = u(\gamma(1)) = y(1) = 0 \). Since \( x \) is an arbitrary point on \( \Sigma \), \( u \) must be identically zero, and our first claim follows. In other words, for any \( \sigma \in \mathcal{H}_0 \), either \( \sigma(x) \equiv 0 \) for all \( x \in \Sigma \) or \( \sigma(x) \neq 0 \) for all \( x \in \Sigma \). Our second claim is that for any two nonzero sections \( \sigma = u\zeta \in \mathcal{H}_0 \) and \( \tau = v\xi \in \mathcal{H}_0 \), there exists a constant \( c \) such that \( \sigma = ct \). Obviously the first claim implies that \( u(x) \neq 0 \) and \( v(x) \neq 0 \) for all \( x \in \Sigma \). This together with (17) gives

\[
\frac{du}{u} = \frac{dv}{v} \quad \text{on} \quad \Sigma.
\]

(19)

Clearly (19) leads to \( \log |u/v| \equiv C \) for some constant \( C \), and there exists some constant \( c \) such that \( u(x) = cv(x) \) for all \( x \in \Sigma \). This confirms the second claim, and implies \( \dim\mathcal{H}_0 = 1 \).

\[\square\]

By Lemma 5 \( \dim\mathcal{H}_0 \leq 1 \). If \( \dim\mathcal{H}_0 = 0 \), then \( \mathcal{H}_0 = \{0\} \); If \( \dim\mathcal{H}_0 = 1 \), then we may assume \( \mathcal{H}_0 = \text{span}\{\tau_1\} \), where \( \{\tau_1\} \) is a normal basis on \( \mathcal{H}_0 \). We next show that the Poincaré inequality holds on the space \( \mathcal{H}_1 \).

**Lemma 6.** There exists a constant \( C \) depending only on \( (\Sigma, g) \) and \( (L, g) \) such that

\[
\int_\Sigma |\sigma|^2 \, dv_g \leq C \int_\Sigma |D\sigma|^2 \, dv_g
\]

for all \( \sigma \in \mathcal{H}_1 \).

**Proof.** Suppose not. For any \( k \in \mathbb{N}^+ \), there would be a section \( \sigma_k \in \mathcal{H}_1 \) satisfying

\[
\int_\Sigma |\sigma_k|^2 \, dv_g = 1
\]

(20)

and

\[
\int_\Sigma |D\sigma_k|^2 \, dv_g \leq \frac{1}{k}.
\]

(21)
Thus \( \{\sigma_k\} \) is bounded in \( W^{1,2}(\Sigma, \mathcal{L}) \). With no loss of generality, we find some \( \sigma_0 \in W^{1,2}(\Sigma, \mathcal{L}) \) such that \( \sigma_k \) converges to \( \sigma_0 \) weakly in \( W^{1,2}(\Sigma, \mathcal{L}) \), strongly in \( L^2(\Sigma, \mathcal{L}) \), and almost everywhere in \( \Sigma \). This implies \( \sigma_0 \in \mathcal{H}_1 \). Moreover (20) and (21) lead to
\[
\int_{\Sigma} |\sigma_0|^2 d\nu_g = 1
\]
and
\[
\int_{\Sigma} |D\sigma_0|^2 d\nu_g = 0.
\]

By (23), we have \( \sigma_0 \in \mathcal{H}_0 \). Hence \( \sigma_0 \in \mathcal{H}_0 \cap \mathcal{H}_1 = \{0\} \), which contradicts (22). In this way, we get the desired result. \( \square \)

3. The subcritical case

In this section, we prove Theorem 1. Since the proof in the case \( \dim \mathcal{H}_0 = 0 \) is a minor modification of that in the case \( \dim \mathcal{H}_0 = 1 \), we only prove the theorem under the assumption \( \dim \mathcal{H}_0 = 1 \) to interested readers. To begin with, we pay attention to the Trudinger-Moser inequality on line bundle. Though a more general inequality was already established in [20], we will give a simple proof in our setting. Also this can be compared with [36].

**Lemma 7.** There exists some \( \alpha_0 > 0 \) such that
\[
\sup_{\sigma \in \mathcal{H}_1, \int |D\sigma|^2 d\nu_g \leq \alpha} \int_{\Sigma} e^{4\pi|\sigma|^2} d\nu_g < +\infty. \tag{24}
\]

**Proof.** Since \( \zeta \) is a global unit frame on \( W^{1,2}(\Sigma, \mathcal{L}) \), for any \( \sigma \in W^{1,2}(\Sigma, \mathcal{L}) \), there must be a unique function \( u \in W^{1,2}(\Sigma) \) such that \( \sigma = u \zeta \) and \( |\sigma| = |u| \). If \( \sigma \in \mathcal{H}_1 \) satisfies \( \int \Sigma |D\sigma|^2 d\nu_g \leq 1 \), then Lemma 6 implies \( \int \Sigma |\sigma|^2 d\nu_g \leq C \), and thus \( \int \Sigma u^2 d\nu_g \leq C \). This together with the formula (3) leads to
\[
\int_{\Sigma} |D\sigma|^2 d\nu_g \leq 2 \int_{\Sigma} |D\sigma|^2 d\nu_g + 2 \int_{\Sigma} u^2 |\sigma|^2 d\nu_g \leq C.
\]
It then follows that \( \|u\|_{W^{1,2}(\Sigma)} \leq C_0 \) for some constant \( C_0 \). By the classical Trudinger-Moser inequality \[26, 17\], we have
\[
\int_{\Sigma} e^{4\pi|\sigma|^2} d\nu_g \leq S_{4\pi} = \sup_{\|u\|_{W^{1,2}(\Sigma)} \leq 1} \int_{\Sigma} e^{4\pi u^2} d\nu_g \leq C \sup_{\|\sigma\|_{W^{1,2}(\Sigma)} \leq 1, \|D\sigma\|_{L^2(\Sigma)} = 0} \int_{\Sigma} e^{8\pi|\sigma|^2} d\nu_g.
\]
Equivalently
\[
\int_{\Sigma} e^{4\pi|\sigma|^2} d\nu_g \leq C.
\]
One can see that any \( \alpha_0 \in (0, 4\pi/C_0^2] \) satisfies (24), as desired. \( \square \)

Define \( a_* = \sup(\alpha : \sup_{\sigma \in \mathcal{H}_1, \int |D\sigma|^2 d\nu_g \leq \alpha} \int_{\Sigma} e^{4\pi|\sigma|^2} d\nu_g < +\infty) \). The following embedding is analogous to the usual one for \( W^{1,2}(\Sigma) \).
Lemma 8. For any $\sigma \in W^{1,2}(\Sigma, \mathcal{L})$, there holds

$$\int_\Sigma e^{\rho \sigma^2} dv_\Sigma < +\infty, \quad \forall p > 1.$$  

Proof. Let $\sigma \in W^{1,2}(\Sigma, \mathcal{L})$ be fixed. There exists a unique function $u$ such that $\sigma = u\zeta$. In view of (3), we have $u \in W^{1,2}(\Sigma)$. Then the results of [24, 17] imply

$$\int_\Sigma e^{\rho \sigma^2} dv_\Sigma = \int_\Sigma e^{\rho u^2} dv_\Sigma < +\infty$$

for any $p > 1$. \hfill \Box

Now we are in a position to estimate $\alpha_*$. 

Lemma 9. $\alpha_* = 4\pi$.

Proof. Step 1. There holds $\alpha_* \geq 4\pi$.

Suppose not. In view of Lemma 7, $0 < \alpha_* < 4\pi$. According to the definition of $\alpha_*$, for any $j \in \mathbb{N}^*$, there would exist some $\sigma_j \in \mathcal{H}_1$ such that $\int_\Sigma |D\sigma_j|^2 dv_\Sigma \leq 1$ and

$$\int_\Sigma e^{(\alpha_* + j^{-1}) \sigma_j^2} dv_\Sigma \to +\infty \quad \text{as} \quad j \to \infty. \quad (25)$$

By Lemma 6, $(\sigma_j)$ is bounded in $W^{1,2}(\Sigma, \mathcal{L})$. With no loss of generality, we assume $\sigma_j$ converges to $\sigma^*$ weakly in $W^{1,2}(\Sigma, \mathcal{L})$, strongly in $L^2(\Sigma, \mathcal{L})$, and almost everywhere in $\Sigma$. Clearly, $\sigma^* \in \mathcal{H}_1$. Moreover, we claim that $\sigma^* = 0$. For otherwise, we have $\int_\Sigma |D\sigma^*|^2 dv_\Sigma > 0$ and

$$\int_\Sigma |D(\sigma_j - \sigma^*)|^2 dv_\Sigma = \int_\Sigma |D\sigma_j|^2 dv_\Sigma - \int_\Sigma |D\sigma^*|^2 dv_\Sigma + o_j(1) \leq 1 - \frac{1}{2} \int_\Sigma |D\sigma^*|^2 dv_\Sigma,$$

provided that $j$ is chosen sufficiently large. Then for any $\epsilon > 0$, the Young inequality leads to

$$|\sigma_j|^2 \leq (1 + \epsilon)|\sigma_j - \sigma^*|^2 + (1 + \epsilon^{-1})|\sigma^*|^2 \leq (1 + \epsilon) \left( 1 - \frac{1}{2} \int_\Sigma |D\sigma^*|^2 dv_\Sigma \right) \frac{|\sigma_j - \sigma^*|^2}{\int_\Sigma |D(\sigma_j - \sigma^*)|^2 dv_\Sigma} + (1 + \epsilon^{-1})|\sigma^*|^2,$$

if $j$ is large enough. Fixing $\epsilon$ with $0 < \epsilon < \frac{1}{8} \int_\Sigma |D\sigma^*|^2 dv_\Sigma$, we obtain

$$(\alpha_* + j^{-1})|\sigma_j|^2 \leq (\alpha_* + j^{-1})(1 + \epsilon) \left( 1 - \frac{1}{2} \int_\Sigma |D\sigma^*|^2 dv_\Sigma \right) \frac{|\sigma_j - \sigma^*|^2}{\int_\Sigma |D(\sigma_j - \sigma^*)|^2 dv_\Sigma} + C(\epsilon)|\sigma^*|^2$$

for sufficiently large $j$. By the definition of $\alpha^*$, Lemma 8 and the Hölder inequality, we have

$$\int_\Sigma e^{(\alpha_* + j^{-1}) \sigma_j^2} dv_\Sigma \leq C.$$
This contradicts (25), and confirms our claim $\sigma^* = 0$.

Write $\sigma_j = u_j \zeta$. From the above, we know that $u_j$ converges to 0 weakly in $W^{1,2}(\Sigma)$, strongly in $L^2(\Sigma)$, and almost everywhere in $\Sigma$. On one hand, there holds for any $\delta > 0$,

$$
\int_\Sigma |D\sigma_j|^2 dv_g = \int_\Sigma |du_j \otimes \zeta + u_j \omega \otimes \zeta|^2 dv_g \\
\geq (1 - \delta) \int_\Sigma |du_j|^2 dv_g + o_j(1).
$$

It follows that for sufficiently large $j$,

$$
\int_\Sigma |du_j|^2 dv_g \leq \frac{1}{1 - 2\delta}.
$$

Choosing $\delta > 0$ small enough such that $(\alpha_* + j^{-1})(1 - 2\delta) < 4\pi - \delta$, and noting that $u_j$ converges to 0 weakly in $W^{1,2}(\Sigma)$, strongly in $L^2(\Sigma)$, we obtain by the classical Trudinger-Moser inequality [26, 17],

$$
\int_\Sigma e^{(\alpha_* + j^{-1})\sigma_j} dv_g = \int_\Sigma e^{(\alpha_* + j^{-1})|u_j|^2} dv_g \leq C.
$$

This contradicts (25), and ends the first step.

Step 2. There holds $\alpha_* \leq 4\pi$.

Take $z \in \Sigma$ and $\delta > 0$. For $k \in \mathbb{N}^*$, we define

$$
u_k = \begin{cases} \\
-\frac{\log k}{4\pi} & \text{in } B_{\delta/\sqrt{k}}(z) \\
\frac{\log k}{4\pi} & \text{in } B_\delta(z) \setminus B_{\delta/\sqrt{k}}(z) \\
0 & \text{in } \Sigma \setminus B_\delta(z)
\end{cases} \quad \text{and} \quad \sigma_k = u_k \zeta - \left(\int_\Sigma \langle u_k \zeta, \tau \rangle dv_g\right) \tau_1,
$$

where $r = r(x) = d_g(x, z)$ and $\{\tau_1\}$ is a normal basis on $H_0$. Obviously $\sigma_k \in H_1$. In view of Lemma 5 there exists some constant $C = \max_{x \in \Sigma} |\tau_1(x)|$. For any $\alpha > 4\pi$, one can check that if $\delta > 0$ is chosen sufficiently small, then

$$
\sup_{\sigma \in H_1, \|\sigma\|_{H^1} \leq 1} \int_\Sigma e^{\alpha |\sigma|^2} dv_g \geq \int_\Sigma \exp \left\{ \alpha \frac{|\tau_1|^2}{\int_\Sigma |D\sigma|^2 dv_g} \right\} dv_g \\
\geq \int_{B_{\delta/\sqrt{k}}(z)} \exp \left\{ \frac{\alpha}{4\pi} (1 + o_{\theta}(1) + o_k(1)) \log k \right\} dv_g \\
= \pi \delta^2 (1 + o_{\theta}(1)) k \frac{1 + o_{\theta}(1) + o_k(1)}{1 - \delta} \quad \rightarrow +\infty \quad \text{as } k \rightarrow \infty.
$$

This ends the second step.

The lemma follows immediately from the above two steps. \qed
Therefore

According to Lemma \( \Lambda_0 > -\infty \). Take a minimizing sequence of sections \( \sigma_k \in \mathcal{H}_1 \) such that \( J_\rho(\sigma_k) \to \Lambda_0 \) as \( k \to \infty \). By Lemmas 7 and 9, we have for any \( \alpha < 4\pi \),

\[
\Lambda_\rho + o_k(1) = \frac{1}{2} \int_\Sigma |D\sigma_k|^2 dv_g + \frac{\rho}{2} \int_\Sigma (\sigma_k, \zeta) dv_g - \rho \log \int_\Sigma h e^{(\alpha, \zeta)} dv_g
\]

\[
\geq \left( \frac{1}{2} - \frac{\rho}{4\alpha} - \epsilon \right) \int_\Sigma |D\sigma_k|^2 dv_g - C(\alpha, \epsilon).
\]

Choosing \( \alpha \) satisfying \( \rho/2 < \alpha < 4\pi \) and sufficiently small \( \epsilon > 0 \), we conclude that \( \{\sigma_k\} \) is bounded in \( W^{1,2}(\Sigma, \mathcal{L}) \). Hence, there exists some \( \sigma_0 \) such that \( \sigma_k \) converges to \( \sigma_0 \) weakly in \( W^{1,2}(\Sigma, \mathcal{L}) \), strongly in \( L^2(\Sigma, \mathcal{L}) \), and almost everywhere in \( \Sigma \). It follows that \( \sigma_0 \in \mathcal{H}_1 \),

\[
\int_\Sigma |D\sigma_0|^2 dv_g \leq \limsup_{k \to \infty} \int_\Sigma |D\sigma_k|^2 dv_g,
\]

that

\[
\int_\Sigma (\sigma_0, \zeta) dv_g = \lim_{k \to \infty} \int_\Sigma (\sigma_k, \zeta) dv_g,
\]

and that

\[
\int_\Sigma h e^{(\alpha, \zeta)} dv_g = \lim_{k \to \infty} \int_\Sigma h e^{(\alpha, \zeta)} dv_g.
\]

Summarizing the above three estimates, we conclude

\[
\Lambda_\rho \leq J_\rho(\sigma_0) \leq \lim_{k \to \infty} J_\rho(\sigma_k) = \Lambda_\rho.
\]

Therefore \( \sigma_0 \) is a minimizer of \( J_\rho \) on \( \mathcal{H}_1 \). This completes the proof of the theorem. \( \square \)

4. Critical case

In this section, we shall prove Theorem 2 by using the blow-up analysis. Since the functional \( J_\rho \) has no translation invariance, and the maximum principle is not available in the line bundle setting, our argument is quite different from that of [13], [19], [34], [35]. Since the consideration of the case \( \dim \mathcal{H}_0 = 0 \) is almost the same as that of the case \( \dim \mathcal{H}_0 = 1 \), we only consider the second case, and leave the first case to interested readers.

4.1. Maximizers for subcritical functionals

As Ding-Jost-Li-wang did in [13], we shall analyze maximizers for subcritical functionals. For any positive integer \( k \), we denote \( \rho_k = 8\pi - 1/k \). By Theorem 1 there exists a section sequence \( \sigma_k \in \mathcal{H}_1 \) satisfying \( J_\rho(\sigma_k) = \inf_{\sigma \in \mathcal{H}_1} J_\rho(\sigma) \). Clearly \( \sigma_k \) satisfies the Euler-Lagrange equation

\[
\begin{aligned}
\Delta_L \sigma_k &= \rho_k \left( \frac{e^{(\alpha, \zeta)}}{\mu_k} - \frac{1}{k} \right) \zeta - \lambda_{1,k} \tau_1 \\
\mu_k &= \int_\Sigma he^{(\alpha, \zeta)} dv_g \\
\lambda_{1,k} &= \rho_k \int_\Sigma \left( \frac{e^{(\alpha, \zeta)}}{\mu_k} - \frac{1}{k} \right) (\zeta, \tau_1) dv_g,
\end{aligned}
\]

(26)

where \( \Delta_L = \mathcal{D}^* \mathcal{D} \) and \( \{\tau_1\} \) is a normal basis on \( \mathcal{H}_0 \).

In view of the importance of the coefficients \( \lambda_{1,k} \) and \( \mu_k \), we have the following
Lemma 10. There exists a constant $C$ depending only on $(\Sigma, g)$ and $(L, \tilde{g})$ such that $|\lambda_{1,k}| \leq C$ for all $k \in \mathbb{N}^*$. Moreover, there holds

$$\liminf_{k \to \infty} \mu_k > 0.$$  

Proof. By Lemma 5, we assume $\mathcal{H}_0 = \text{span}\{\tau_1\}$ and $\tau_1 \in \Gamma(L)$. This together with the facts $|\xi(x)| = 1$, $h(x) > 0$ for all $x \in \Sigma$, and $0 < \rho_k \leq 8\pi$ implies $|\lambda_{1,k}| \leq C$ for all $k \in \mathbb{N}^*$.

For the second assertion, we suppose not. Then up to a subsequence, $\mu_k \to 0$ as $k \to \infty$. One easily sees from the choice of $\sigma_k$ that

$$J_{\rho_k}(\sigma_k) = \frac{1}{2} \int_{\Sigma} |D\sigma_k|^2 dv_g + \frac{\rho_k}{|\Sigma|} \int_{\Sigma} (\sigma_k, \xi) dv_g - \rho_k \log \mu_k$$

$$= \inf_{\sigma \in H} J_{\rho_k}(\sigma)$$

$$\leq J_{\rho_k}(0)$$

$$\leq 8\pi \log \left| \int_{\Sigma} h dv_g \right|,$$  

(27)

and from the Poincaré inequality that

$$\left| \int_{\Sigma} (\sigma_k, \xi) dv_g \right| \leq \int_{\Sigma} |\sigma_k| dv_g$$

$$\leq C \left( \int_{\Sigma} |D\sigma_k|^2 dv_g \right)^{1/2}$$

$$\leq \frac{1}{4} \int_{\Sigma} |D\sigma_k|^2 dv_g + C.$$  

(28)

Combining (27) and (28), one has $\rho_k \log \mu_k \geq -C$ for some positive constant $C$. This contradicts $\mu_k$ converges to 0, and ends the proof of the lemma.

□

Lemma 11. There holds

$$\liminf_{k \to \infty} \inf_{\sigma \in H_1} J_{\rho_k}(\sigma) = \inf_{\sigma \in H_1} J_{8\pi}(\sigma).$$  

(29)

Proof. Since the proof is completely analogous to that of ([39], Lemma 2.3), we omit the details here. □

Lemma 12. If $\mu_k$ is bounded, then $\sigma_k$ is bounded in $W^{1,2}(\Sigma, L)$ and $J_{8\pi}$ has a minimizer on $H_1$.

Proof. If $\mu_k$ is bounded, then (27) gives

$$\int_{\Sigma} |D\sigma_k|^2 dv_g \leq C.$$  

This implies that $\sigma_k$ is bounded in $W^{1,2}(\Sigma, L)$, and thus $\sigma_k$ converges to some $\sigma_0$ weakly in $W^{1,2}(\Sigma, L)$, strongly in $L^2(\Sigma, L)$, and almost everywhere in $\Sigma$. This together with (29) leads to that $\sigma_0 \in H_1$ and $J_{8\pi}(\sigma_0) = \inf_{\sigma \in H_1} J_{8\pi}(\sigma)$. □

From now on in this section, we write $\sigma_k = u_k \xi$ and

$$c_k = \max_{x \in \Sigma} u_k = u_k(x_k).$$  

(30)
Lemma 13. If $c_k$ is bounded from above, then $J_{8\pi}$ has a minimizer on $H_1$.

Proof. In view of Lemma 10, multiplying both sides of (26) by $\sigma_k$ and integrating by parts, one estimates by the Hölder inequality

$$
\int_\Sigma |\nabla \sigma_k|^2 d\nu_g = \rho_k \left( \int_\Sigma \left[ \frac{\rho_k}{\mu_k} - \frac{1}{\mu_k} \right] (\zeta, \sigma_k) d\nu_g - \lambda_1 \int_\Sigma \langle \tau_1, \sigma_k \rangle d\nu_g \right) \leq \rho_k \left( \int_\Sigma |\nabla \sigma_k|^2 d\nu_g \right)^{1/2} + C \left( \int_\Sigma |\sigma_k|^2 d\nu_g \right)^{1/2}.
$$

This implies that $\sigma_k$ is bounded in $W^{1,2}(\Sigma, L)$. Arguing similarly as in the proof of Lemma 12, one concludes the lemma.

If $J_{8\pi}$ has no minimizer on $H_1$, then it follows from Lemmas 12 and 13 that $\mu_k \to +\infty, c_k \to +\infty$.

Let $x_k$ be as in (30). With no loss of generality, we assume

$$
x_k \to p \quad \text{as} \quad k \to \infty.
$$

Take a sequence of isothermal coordinate systems $(U, \phi_k; \{y_1, y_2\})$ near $x_k$ with $\phi_k(x_k) = 0$ such that $\phi_k^{-1}(\mathbb{R}_{2a}) \subset U$ and the metric $g$ is represented by

$$
g_k(y) = \exp(f_k(y))(dy_1^2 + dy_2^2),
$$

where $f_k : \phi_k(U) \to \mathbb{R}$ is a smooth function satisfying $f_k(0) = 0$, $|d^j f_k| \leq C$, $j = 1, 2$,

$$
\exp(f_k(y)) = 1 + O(|y|) = 1 + O(d_k(\phi_k^{-1}(y), x_k)),
$$

$C^{-1}|y| \leq d_k(\phi_k^{-1}(y), x_k) \leq C|y|$ for some constant $C$ independent of $k$. Moreover

$$
|d^2 \Delta_{g_0}| = \exp(-f_k) |d^2 \Delta_{g_2}|, \quad \Delta_{g_0} = -\exp(-f_k) \Delta_{g_2}.
$$

Indeed, such a sequence of isothermal coordinates exist. For its explicit proof, we refer the readers to ([37], Lemma 3.1).

4.2. Blow-up analysis

To analyze the asymptotic behavior of $\sigma_k$, we set

$$
r_k = \frac{\sqrt{\mu_k}}{\sqrt{\rho_k h(p)}} e^{-\frac{c_k^2}{4}}.
$$

Recalling (31), we have an analog of ([39], Lemma 2.7), namely

Lemma 14. For any $\gamma < 1/2$, there holds $r_k^2 e^{r_k} \to 0$ as $k \to \infty$. In particular for any $q > 0$, there holds $r_k e^{r_k} \to 0$ as $k \to \infty$. 

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Proof. Recall $h > 0$ on $\Sigma$. Multiplying both sides of (26) by $\sigma_k$, integrating by parts and using the Poincaré inequality (see Lemma 6 above), we have

$$\int_{\Sigma} |\partial \sigma_k|^2 dv_g = \mu_k \int_{\Sigma} \left( \frac{he^{u_k}}{\mu_k} - \frac{1}{2} \right)(\sigma_k, \zeta) dv_g - \lambda_{1, k} \int_{\Sigma} (\tau_1, \sigma_k) dv_g$$

$$\leq 8\pi c_k \int_{\Sigma} \frac{he^{u_k}}{\mu_k} dv_g + C \int_{\Sigma} |\sigma_k|^2 dv_g$$

$$\leq 8\pi c_k + C \left( \int_{\Sigma} |\partial \sigma_k|^2 dv_g \right)^{1/2}.$$  \hfill (34)

Let $0 < \epsilon < 1$ be a fixed number to be determined later. We conclude from (34) that for all sufficiently large $k$,

$$\int_{\Sigma} |\partial \sigma_k|^2 dv_g \leq 8\pi (1 + \epsilon) c_k.$$  \hfill (35)

This together with Lemma 9 gives

$$\int_{\Sigma} h e^{(\sigma_k, \zeta)} dv_g \leq \left( \max h \right) \int_{\Sigma} \exp \left( \frac{4\pi (1 - \epsilon)|\sigma_k|^2}{\int_{\Sigma} |\partial \sigma_k|^2 dv_g} + \frac{C}{16(1 - \epsilon)} \right) dv_g$$

$$\leq C(\epsilon) \exp \left( \frac{1 + \epsilon}{2(1 - \epsilon) c_k} \right).$$

It follows that

$$r_k^2 = \frac{\int_{\Sigma} h e^{(\sigma_k, \zeta)} dv_g}{\mu_h(p)} e^{-\epsilon r_k} \leq C \exp \left( -\frac{1 - 3\epsilon}{2(1 - \epsilon) c_k} \right).$$

For any $0 < \gamma < 1/2$, we can take $\epsilon > 0$ sufficiently small such that $(1 - 3\epsilon)/(2 - 2\epsilon) > \gamma$. As a consequence, we obtain $r_k^2 e^{-r_k} \to 0$ as $k \to \infty$. \hfill \Box

Recall $\sigma_k = u_k \zeta$. In view of Lemma 4 the Euler-Lagrange equation (26) is transformed into

$$\Delta_{\Sigma} u_k = \rho_k \left( \frac{he^{u_k}}{\mu_k} - \frac{1}{2} \right) - \lambda_{1, k} (\tau_1, \zeta) - (|\omega|^2 + d^* \omega) u_k.$$  \hfill (36)

For simplicity, in the isothermal coordinate systems $(U, \phi_k; \{y_1, y_2\})$, we sometimes denote $u \circ \phi_k^{-1}$ by $\tilde{u}$ for a function $u$ in the sequel. Define $\psi_k(y) = c_k^{-1} \tilde{u}(r_k y)$ and $\varphi_k(y) = \tilde{u}(r_k y) - c_k$ for all $y \in \Omega_k = \{ y \in \mathbb{R}^2 : r_k y \in B_1 \}$. For the convergence of $\psi_k$ and $\varphi_k$, one has

Lemma 15. There hold up to a subsequence, $\psi_k \to 1$ and $\varphi_k \to \varphi = -2 \log(1 + |y|^2/8)$ in $C^1_{\text{loc}}(\mathbb{R}^2)$ as $k \to \infty$.

Proof. By (33), (36) and the definition of $g_{k, x}$, we calculate

$$\Delta_{\Sigma} \psi_k(y) = h(\phi_k^{-1}(r_k y)) e^{u_k(y)} - \frac{r_k^2 \mu_k}{\Sigma c_k} - c_k^{-1} r_k^2 \lambda_{1, k} (\tau_1, \zeta)(\phi_k^{-1}(r_k y))$$

$$- c_k^{-1} r_k^2 (|\omega|^2 + d^* \omega) u_k(\phi_k^{-1}(r_k y)),$$  \hfill (37)

where $\Delta_{\Sigma} \psi_k(y) = -h(\psi_k(y)) \Delta_{\Sigma} \psi_k(y)$. Since for any $R > 0$,

$$\int_{\mathbb{R}^2} \left[ r_k^2 (|\omega|^2 + d^* \omega) u_k(\phi_k^{-1}(r_k y)) \right]^2 dy \leq C r_k^2 \int_{\Sigma} |\sigma_k|^2 dv_g \leq C r_k^2 c_k,$$
we have that $\Delta_{\mathbb{R}^2}\psi_k(y)$ converges to 0 locally uniformly in $\mathbb{R}^2$. Note also that $\psi_k(y) \leq 1$ for all $y \in \Omega_k$. Applying elliptic estimates to (37), we conclude that $\psi_k \to \psi$ in $C^1_{\text{loc}}(\mathbb{R}^2)$, where $\psi$ is a harmonic function with $\psi(y) \leq \psi(0) = 1$. Then the Liouville theorem implies $\psi \equiv 1$ in $\mathbb{R}^2$.

Also we calculate
\[
\Delta_{\mathbb{R}^2}\varphi_k(y) = \frac{h(\phi_k^{-1}(r_k y))}{h(p)} e^{\varphi_k(y)} \left( -r_k^2 \phi_k^{-1}(r_k y) \right)
\]
Similarly as above, we use elliptic estimates to conclude that $\varphi_k \to \varphi$ in $C^1_{\text{loc}}(\mathbb{R}^2)$, where $\varphi$ is a solution of
\[
\left\{ -\Delta_{\mathbb{R}^2}\varphi = e^{\varphi} \text{ in } \mathbb{R}^2 \\
\int_{\mathbb{R}^2} e^{\varphi(y)} dy < \infty.
\right.
\]
It then follows from a classification theorem of Chen-Li [9] that
\[
\int_{B_{k}} \psi \, d\nu_{\varphi_k} \equiv \int_{B_{k}} \varphi \, d\nu_\psi.
\]
We calculate
\[
\int_{B_{k}} \psi \, d\nu_{\varphi_k} = \int_{B_{k}} \varphi \, d\nu_\psi.
\]

Lemma 16. $\mu_k^{-1} h e^{\mu_k} \, dv_{\varphi_k}$ converges to the dirac measure $\delta_p$ in the sense of measure.

**Proof.** In view of (33) and Lemma 15 it is easy to get that
\[
\rho_k \int_{\phi_k^{-1}(B_{k})} \mu_k^{-1} h e^{\mu_k} \, dv_{\varphi_k} = (1 + o_k(1)) \int_{B_{k}} e^{\varphi_k(y)} dy
\]
This together with (38) gives
\[
\lim_{R \to \infty} \lim_{k \to \infty} \int_{\phi_k^{-1}(B_{k})} \mu_k^{-1} h e^{\mu_k} \, dv_{\varphi_k} = 1.
\]
Since $h(x) > 0$ for all $x \in \Sigma$ and $\int_{\Sigma} \mu_k^{-1} h e^{\mu_k} \, dv_{\varphi_k} = 1$, we obtain
\[
\lim_{R \to \infty} \lim_{k \to \infty} \int_{\Sigma \phi_k^{-1}(B_{k})} \mu_k^{-1} h e^{\mu_k} \, dv_{\varphi_k} = 0.
\]
Note also $x_k \to p$ as $k \to \infty$. It follows from (39) and (40) that
\[
\lim_{k \to \infty} \int_{\Sigma} \mu_k^{-1} h e^{\mu_k} \, dv_{\varphi_k} = \eta(p), \quad \forall \eta \in C^0(\Sigma).
\]
This ends the proof of the lemma.\]

Lemma 17. $\sigma_k$ converges to a Green section $G$ weakly in $W^{1,q}(\Sigma, \mathcal{L})$, strongly in $L^r(\Sigma, \mathcal{L})$ for all $r < \frac{2q}{2q-1}$, $1 < q < 2$, and in $C^1_{\text{loc}}(\Sigma \setminus \{p\}, \mathcal{L})$, where $G$ satisfies
\[
\begin{cases}
\Delta_{\mathbb{R}^2} G = 8\pi \left( \delta_p - \frac{1}{|\Sigma|} \right) \zeta - \lambda_1 \tau_1 \\
\lambda_1 = 8\pi \left( \zeta, \tau_1 \right) (p) - \frac{1}{|\Sigma|} \int_{\Sigma} \left( \zeta, \tau_1 \right) dv_{\varphi_k}
\end{cases}
\]
in the distributional sense.
Proof. Denote a section sequence
\[ f_k = p_k \left( \frac{b e^{\sigma_k}}{\mu_k} - \frac{1}{|x|^2} \right) \zeta + A_k \tau_1. \]

We first prove that
\[ \|\sigma_k\|_{L^2(\Sigma, L)} = \left( \int_{\Sigma} |\sigma_k|^2 d\nu_x \right)^{1/2} \leq C. \]

Suppose on the contrary \( \|\sigma_k\|_{L^2(\Sigma, L)} \to \infty \) as \( k \to \infty \). In view of \([20]\), we have
\[ \Delta_{L} \tilde{\sigma}_k = f_k, \]

where \( \tilde{\sigma}_k = \sigma_k/\|\sigma_k\|_{L^2(\Sigma, L)} \) and \( f_k = f_k/\|\sigma_k\|_{L^2(\Sigma, L)} \). Since \( \|\tilde{\sigma}_k\|_{L^2(\Sigma, L)} = 1 \) and \( \|f_k\|_{L^2(\Sigma, L)} = o_k(1) \), we conclude from Proposition 7.1 in \([20]\) that \( \tilde{\sigma}_k \) is bounded in \( W^{1,q}(\Sigma, L) \) for any \( 1 < q < 2 \). We assume with no loss of generality \( \tilde{\sigma}_k \) converges to \( \tilde{\sigma} \) weakly in \( W^{1,q}(\Sigma, L) \), and \( \tilde{\sigma} \) is a distributional solution of
\[ \Delta_{L} \tilde{\sigma} = 0. \] (42)

Moreover \( \tilde{\sigma} \) belongs to \( \mathcal{H}_t \) since \( \tilde{\sigma}_k \in \mathcal{H}_t \). Testing \((43)\) with \( \tilde{\sigma} \) and using the Poincaré inequality (Lemma 6), we conclude \( \|\tilde{\sigma}\|_{L^2(\Sigma, L)} = 0 \), which contradicts \( \|\tilde{\sigma}\|_{L^2(\Sigma, L)} = 1 \). Hence \( \sigma_k \) must be bounded in \( L^2(\Sigma, L) \). Again applying Proposition 7.1 in \([20]\) to \( \Delta_{L} \sigma_k = f_k \), we have that \( \sigma_k \) is bounded in \( W^{1,q}(\Sigma, L) \) for any \( 1 < q < 2 \). Then the weak compactness of \( W^{1,q}(\Sigma, L) \) and the compact embedding \( W^{1,q}(\Sigma, L) \hookrightarrow L^r(\Sigma, L) \), \( \forall r < 2q/(2-q) \), lead to the existence of some section \( G \) such that \( \sigma_k \) converges to \( G \) weakly in \( W^{1,q}(\Sigma, L) \) and strongly in \( L^r(\Sigma, L) \), \( \forall r < 2q/(2-q) \), \( 1 < q < 2 \). By \([20]\) and Lemma 16, \( G \) is a distributional solution of \((41)\). While the convergence in \( C_{b,\text{loc}}(\Sigma \setminus \{p\}, L) \) follows from elliptic estimates on \([20]\). □

From Lemmas \([19]\) and \([17]\), we know that there exists a unique \( G_p \in W^{1,q}(\Sigma) \), \( \forall 1 < q < 2 \), such that \( G = G_p \zeta \) and in the distributional sense
\[ \Delta_{g} G_p + (|\omega|^2 + d^* \omega) G_p = 8\pi \left( \delta_p - \frac{1}{|x|^2} \right) - A_1 \langle \tau_1, \zeta \rangle. \] (43)

Let \( r = r(x) = d_{\text{geod}}(x, p) \) be the geodesic distance between \( x \) and \( p \). Since \( \Delta_{g}(G_p + 4 \log r) \) belongs to \( L^q(\Sigma) \), the elliptic estimate implies that \( G_p + 4 \log r \) belongs to \( C^1(\Sigma) \). This together with the Sobolev embedding theorem leads to a local decomposition
\[ G_p(x) = -4 \log r + A_p + \eta(x), \] (44)

where \( A_p \) is a constant, \( \eta \in C^1(\Sigma) \) satisfies \( \eta(p) = 0 \). As a consequence, the Green section \( G \) can be written as
\[ G(x) = \left( -4 \log r + A_p + \eta(x) \right) \zeta(x). \]
4.3. Lower bound estimate

We now turn our attention to the lower bound estimate of $J_{B_r}$. Firstly we calculate for any fixed $R > 0$,

$$
\int_{\phi_k^{-1}(B_{Rz})} |\mathcal{D}r_k|^2 dv_g = \int_{\phi_k^{-1}(B_{Rz})} |du_k \otimes \zeta + u_k \omega \otimes \zeta|^2 dv_g
$$

$$= \int_{\phi_k^{-1}(B_{Rz})} |du_k|^2 dv_g + 2 \int_{\phi_k^{-1}(B_{Rz})} u_k(du_k, \omega) dv_g
$$

$$+ \int_{\phi_k^{-1}(B_{Rz})} |u_k|^2 |\omega|^2 dv_g, \quad (45)
$$

where $\phi_k : U \to \mathbb{R}^2$ is the sequence of isothermal coordinate systems constructed before. By Lemma 15 we have $\bar{u}_k(z) = c_k - 2 \log(1 + |z|^2/(8r_k^2))$ for all $z \in \mathbb{B}_{Rz}$. Hence

$$
\int_{\phi_k^{-1}(B_{Rz})} |du_k|^2 dv_g = \int_{\mathbb{R}^2} |d\phi(y)|^2 dy
$$

$$= \int_{\mathbb{R}^2} |d\phi(y)|^2 dy + o_k(1)
$$

$$= 16\pi \log(1 + \frac{R^2}{8}) - 16\pi + o_k(1) + o_R(1), \quad (46)
$$

and by Lemma 14

$$
\int_{\phi_k^{-1}(B_{Rz})} |u_k|^2 |\omega|^2 dv_g \leq C(1 + o_k(1))R^2 \delta_k^2 c_k^2 = o_k(1). \quad (47)
$$

As a consequence we have by the Hölder inequality

$$\int_{\phi_k^{-1}(B_{Rz})} u_k(du_k, \omega) dv_g = o_k(1). \quad (48)
$$

Inserting (46)-(48) into (45), we obtain

$$
\int_{\phi_k^{-1}(B_{Rz})} |\mathcal{D}r_k|^2 dv_g = 16\pi \log \left(1 + \frac{R^2}{8}\right) - 16\pi + o_k(1) + o_R(1). \quad (49)
$$

Secondly it follows from Lemma 17 that

$$
\int_{\Sigma \phi_k^{-1}(B_{Rz})} |\mathcal{D}r_k|^2 dv_g = \int_{\Sigma \phi_k^{-1}(B_{Rz})} |\mathcal{D}G|^2 dv_g + o_k(1)
$$

$$= \int_{\Sigma \phi_k^{-1}(B_{Rz})} |dG|^2 dv_g + 2 \int_{\Sigma \phi_k^{-1}(B_{Rz})} G_p(dG_p, \omega) dv_g
$$

$$+ \int_{\Sigma \phi_k^{-1}(B_{Rz})} |G_p|^2 |\omega|^2 dv_g \quad (50)
$$

In view of (32), we conclude $d_k(x_k, \phi_k^{-1}(\mathbb{B}_0)) = (1 + O(\delta))\delta$. Thus there exists some constant $c_1 > 0$ such that $B_{(1-c_1)\delta}(x_k) \subset \phi_k^{-1}(\mathbb{B}_0) \subset B_{(1+c_1)\delta}(x_k)$, provided that $k$ is chosen sufficiently large. By (44), we have

$$
\int_{\phi_k^{-1}(B_{Rz})} G_p(dG_p, \omega) dv_g = o(1) \quad (51)
$$

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where \( o(1) \to 0 \) as \( k \to \infty \) first, and then \( \delta \to 0 \). Inserting (51) and (52) into (50), we have by integration by parts and (43) that

\[
\int_{B_1(\partial \mathbb{B}_\delta)} |D\sigma_k|^2 \, dv_{g} = \int_{\Sigma(B_{1+(\Delta +1)k}(x_k))} |dG_{p}|^2 \, dv_{g} + \int_{\Sigma} (dG_{p}, \omega) \, dv_{g} + \int_{\Sigma} G_{p}^2 |\omega|^2 \, dv_{g} + o(1)
\]

(52)

The infimum is attained by a harmonic function \( \tilde{\sigma} \).

Thirdly we estimate the energy of \( \tilde{\sigma} \) on neck domains \( \phi_k^{-1}(\mathbb{B}_\delta) \setminus \phi_k^{-1}(\mathbb{B}_{R_k}) \). Since \( u_k \) converges to \( G_p \) weakly in \( W^{1,q}(\Sigma), 1 < q < 2 \), and strongly in \( L^2(\Sigma), \phi_k^{-1}(\mathbb{B}_\delta) \subset B_{2\delta}(p) \) for sufficiently large \( k \), \( \int_{B_{2\delta}(p)} G_{p}^2 \, dv_{g} = o_\delta(1) \), and \( \int_{B_{2\delta}(p)} G_{p} (dG_{p}, \omega) \, dv_{g} = o_\delta(1) \), there must be

\[
\int_{\phi_k^{-1}(\mathbb{B}_\delta) \setminus \phi_k^{-1}(\mathbb{B}_{R_k})} u_k^2 |\omega|^2 \, dv_{g} \leq C \int_{B_{2\delta}(p)} u_k^2 \, dv_{g} = o(1)
\]

and

\[
\int_{\phi_k^{-1}(\mathbb{B}_\delta) \setminus \phi_k^{-1}(\mathbb{B}_{R_k})} u_k (du_k, \omega) \, dv_{g} = o(1)
\]

Therefore

\[
\int_{\phi_k^{-1}(\mathbb{B}_\delta) \setminus \phi_k^{-1}(\mathbb{B}_{R_k})} |D\sigma_k|^2 \, dv_{g} = \int_{\phi_k^{-1}(\mathbb{B}_\delta) \setminus \phi_k^{-1}(\mathbb{B}_{R_k})} |d\tilde{u}_k|^2 \, dv_{g} + o(1)
\]

\[
= \int_{\phi_k^{-1}(\mathbb{B}_\delta) \setminus \phi_k^{-1}(\mathbb{B}_{R_k})} |d\tilde{u}_k|^2 \, dy + o(1)
\]

(53)

\[
\geq \int_{\phi_k^{-1}(\mathbb{B}_\delta) \setminus \phi_k^{-1}(\mathbb{B}_{R_k})} |d\tilde{u}_k|^2 \, dy + o(1),
\]

where \( \tilde{u}_k \) is \( \max(\min(\tilde{u}_k, a_k), b_k) \), \( a_k = \inf_{\mathbb{B}_\delta \setminus \mathbb{B}_{R_k}} \tilde{u}_k \), and \( b_k = \sup_{\mathbb{B}_\delta \setminus \mathbb{B}_{R_k}} \tilde{u}_k \). Let us now modify an argument of Wang-Liu [34] (see also Li-Li [19] for the Toda system). It is easy to see that the infimum

\[
\inf_{w|_{\mathbb{B}_\delta \setminus \mathbb{B}_{R_k}}=a_k, w|_{\mathbb{B}_{R_k}}=b_k} \int_{\mathbb{B}_\delta \setminus \mathbb{B}_{R_k}} |dw|^2 \, dy
\]

(55)

is attained by a harmonic function

\[
H(y) = \frac{a_k - b_k}{\log(R_k) - \log(\delta)} \log |y| + \frac{a_k \log \delta - b_k \log(R_k)}{-\log(R_k) + \log \delta}, \quad y \in \mathbb{B}_\delta \setminus \mathbb{B}_{R_k}.
\]

As a consequence

\[
\int_{\mathbb{B}_\delta \setminus \mathbb{B}_{R_k}} |dH|^2 \, dy = \frac{2\pi(a_k - b_k)^2}{\log(\delta) - \log(R_k)}.
\]

(56)
Combining (54)-(56), we get by (33)$$\gamma$$ leads to a Taylor expansion of the final term in (57), namely

$$\int_{\phi^{-1}(\mathbb{R}_b)}\rho|dx|^2 |x| \geq \frac{2\pi(a_0 - b_0)^2}{\log \delta - \log(Rr_k)} = \frac{4\pi(c_k + a_k - c_k - b_k)^2}{c_k - \log \mu_k + 2\log \delta - 2\log R + \log \rho_k + \log \delta}.$$  (57)

In view of Lemmas [14, 17] and the local representation (64), we derive

$$a_k - c_k - b_k = -2 \log \left(1 + \frac{R^2}{8}\right) + 4 \log \delta - A_p + o_k(1).$$  (58)

It follows from Lemma [14] that $$c_k - \log \mu_k \geq \gamma c_k - \log \rho_k - \log h(p)$$ for any $$0 < \gamma < 1/2$$, in particular $$\gamma = (c_k - \log \mu_k)/c_k \leq 1$$, provided that $$k$$ is sufficiently large. This together with (58) leads to a Taylor expansion of the final term in (57), namely

$$\frac{2\pi(a_0 - b_0)^2}{\log \delta - \log(Rr_k)} = \frac{4\pi c_k^2}{c_k - \log \mu_k} \left(1 + \frac{2(a_k - c_k - b_k)}{c_k} + \frac{(a_k - c_k - b_k)^2}{c_k^2}\right) \times \left(1 - \frac{2\log \delta - 2\log R + \log \rho_k + \log h(p)}{c_k - \log \mu_k} + O\left(\frac{1}{(c_k - \log \mu_k)^2}\right)\right)

= \frac{4\pi c_k^2}{c_k - \log \mu_k} + \frac{8\pi(a_k - c_k - b_k)c_k}{c_k - \log \mu_k} - \frac{4\pi c_k^2}{(c_k - \log \mu_k)^2} \left(2\log \delta - 2\log R + \log \rho_k + \log h(p)\right) + o_k(1).$$  (59)

Also we calculate

$$\frac{4\pi c_k^2}{c_k - \log \mu_k} = 4\pi(c_k - \log \mu_k)\left(1 + \frac{\log \mu_k}{c_k - \log \mu_k}\right)^2 = 4\pi(c_k - \log \mu_k)\left(1 - \frac{\log \mu_k}{c_k - \log \mu_k}\right)^2 + 16\pi \log \mu_k.$$  (60)

Combining (57), (60) and (64), we obtain

$$\int_{\phi^{-1}(\mathbb{R}_b)}|D\sigma|^{2}d\nu_{\delta} \geq 4\pi(c_k - \log \mu_k)\left(1 - \frac{\log \mu_k}{c_k - \log \mu_k}\right)^2 + 16\pi \log \mu_k

+ \frac{8\pi(a_k - c_k - b_k)c_k}{c_k - \log \mu_k} - \frac{4\pi c_k^2}{(c_k - \log \mu_k)^2} \left(2\log \delta - 2\log R + \log \rho_k + \log h(p)\right) + o(1).$$  (61)
In view of (49), (53), (61) and $\rho_k \leq 8\pi$, there are the following estimates

\[
\mathcal{J}_{\rho_k}(\sigma_k) = \frac{1}{2} \int_{\Sigma} |D\sigma_k|^2 \, dv_x + \frac{\rho_k}{c_k} \int_{\Sigma} u_k \, dv_x - \rho_k \log \mu_k
\]

\[
\geq 2\pi(c_k - \log \mu_k) \left(1 - \frac{\log \mu_k}{c_k - \log \mu_k}\right)^2 + \frac{8\pi}{c_k} \int_{\Sigma} G_{\rho_k} \, dv_x + \frac{4\pi(a_k - c_k - b_k) c_k}{c_k - \log \mu_k}
\]

\[-\frac{2\pi c_k^2}{(c_k - \log \mu_k)^2} (2 \log \delta - 2 \log R + \log \rho_k + \log h(p))
\]

\[-16\pi \log \delta + 4\pi A_p \int_{\Sigma} G_{\rho} \, dv_x + 8\pi \log \left(1 + \frac{R^2}{8}\right) - 8\pi + o(1),
\] (62)

where $o(1) \to 0$ as $k \to \infty$ first, $R \to \infty$ next, and $\delta \to 0$ finally. Noting (58) and $c_k - \log \mu_k \leq \eta$ for any $\eta < 1/2$ and sufficiently large $k$, we have up to a subsequence

\[
\frac{\log \mu_k}{c_k - \log \mu_k} = 1 + o_k(1).
\] (63)

For otherwise, the right-hand side of (62) tends to $+\infty$ as $k \to \infty$. This implies $\mathcal{J}_{\rho_k}(\sigma_k) \to +\infty$ as $k \to \infty$, contradicting (27). Note that (63) implies

\[
\frac{c_k}{c_k - \log \mu_k} = 2 + o_k(1).
\] (64)

Discarding the first term of the right-hand side of (62) and passing to the limit $k \to \infty$ first, then $R \to \infty$, $\delta \to 0$ finally, we have by using (58), (64) and Lemma 11 that

\[
\inf_{\sigma_k \in \mathcal{H}} \mathcal{J}_{\rho_k}(\sigma) \geq \frac{4\pi}{c_k} \int_{\Sigma} G_{\rho} \, dv_x - 4\pi A_p - 8\pi \log \pi - 8\pi - 8\pi \log h(p).
\] (65)

We shall prove that (65) is in fact an equality. It suffices to construct a sequence of sections $Q_k \in \mathcal{H}$ such that $\mathcal{J}_{\rho_k}(Q_k)$ converges to the right-hand side of (65). To this end, we set

\[
q_k(x) = \begin{cases} 
\kappa - 2 \log \left(1 + \frac{\rho_k}{8}\right), & x \in B_{R/4}(p) \\
G_{\rho}(x) - g(x)\eta(x), & x \in B_{2R/4}(p) \setminus B_{R/4}(p) \\
G_{\rho}(x), & x \in \Sigma \setminus B_{2R/4}(p),
\end{cases}
\] (66)

where $\eta$ is given as in (44), $g \in C^1_0(B_{2R/4}(p))$ satisfies $g(x) \equiv 1$ for all $x \in B_{R/4}(p)$, $|d^k(x)| \leq 4k/R$ for all $x \in B_{2R/4}(p)$,

\[
c = 2 \log(1 + R^2/4) - 4 \log R + 4 \log k + A_p,
\]

and $R = R(k)$ satisfies $R \to +\infty$ and $R \log R / k \to 0$ as $k \to \infty$. One easily checks that $q_k$ belongs to $W^{1,2}(\Sigma)$ and

\[
\int_{\Sigma} q_k^2 \, dv_x = \int_{\Sigma} G_{\rho}^2 \, dv_x + o_k(1).
\] (67)

Define

\[
Q_k = q_k \xi - \int_{\Sigma} (q_k \xi, \tau) \, dv_{x} \tau_1,
\] (68)
where \( \{ \tau_1 \} \) is a normal basis of \( \mathcal{H}_0 \). Obviously \( Q_k \in \mathcal{H}_1 \). In view of (67) and \( D\tau_1 = 0 \), we have

\[
\int_{\Sigma} |DQ_k|^2 d\nu_g = \int_{\Sigma} |dq_k \otimes \zeta + q_k \omega \otimes \zeta^2| d\nu_g
\]

\[
= \int_{\Sigma} \left( |dq_k|^2 + q_k^2 (|\omega|^2 + d^* \omega) \right) d\nu_g
\]

\[
= \int_{\Sigma} |dq_k|^2 d\nu_g + \int_{\Sigma} G_p^2 (|\omega|^2 + d^* \omega) d\nu_g + o_k(1). \quad (69)
\]

By the definition of \( q_k \) (see (66) above), we get

\[
\int_{B_{\nu_k}(p)} |dq_k|^2 d\nu_g = 16\pi \log \left( 1 + \frac{R^2}{8} \right) - 16\pi + o_k(1). \quad (70)
\]

Moreover, in view of (66), we have by integration by parts

\[
\int_{\Sigma \backslash B_{\nu_k}(p)} |dq_k|^2 d\nu_g = \int_{\Sigma \backslash B_{\nu_k}(p)} |dG_p|^2 d\nu_g + \int_{B_{\nu_k}(p) \backslash B_{\nu_k}(p)} |d(\eta)|^2 d\nu_g - 2 \int_{B_{\nu_k}(p) \backslash B_{\nu_k}(p)} (dG_p, d(\eta)) d\nu_g
\]

\[
= \int_{\Sigma \backslash B_{\nu_k}(p)} \frac{G_p \partial G_p}{\partial \nu} d\nu_g + \int_{B_{\nu_k}(p) \backslash B_{\nu_k}(p)} G_p \Delta d\nu_g
\]

\[
+ \int_{B_{\nu_k}(p) \backslash B_{\nu_k}(p)} |d(\eta)|^2 d\nu_g - 2 \int_{B_{\nu_k}(p) \backslash B_{\nu_k}(p)} \eta \Delta d\nu_g + 2 \int_{\partial B_{\nu_k}(p)} \eta \partial G_p \partial \nu_g. \quad (71)
\]

By (44) and (20), Lemma 7.2, we have

\[
- \int_{\partial B_{\nu_k}(p)} G_p \frac{\partial G_p}{\partial \nu} ds_g = -32\pi \log R + 32\pi \log k + 8\pi A_p + o_k(1). \quad (72)
\]

Noting that

\[
\int_{\Sigma} G_p (\tau_1, \zeta) d\nu_g = \int_{\Sigma} (\tau_1, G) d\nu_g = 0, \quad (73)
\]

we also have by using (43) and (44),

\[
\int_{\Sigma \backslash B_{\nu_k}(p)} G_p \Delta d\nu_g = \frac{8\pi}{|\Sigma|} \int_{\Sigma} G_p d\nu_g - \int_{\Sigma} (|\omega|^2 + d^* \omega) G_p^2 d\nu_g + o_k(1). \quad (74)
\]

Using again (43), (44) and (20), Lemma 7.2, we know that all the last three terms on the righthand side of (71) are infinitesimals as \( k \to \infty \). In view of these infinitesimals, (71) and (72), we have by combining (69), (70) and (74) that

\[
\int_{\Sigma} |DQ_k|^2 d\nu_g = 32\pi \log k - 16\pi \log 8 - 16\pi + 8\pi A_p - \frac{8\pi}{|\Sigma|} \int_{\Sigma} G_p d\nu_g + o_k(1). \quad (75)
\]
Next it is obvious to see
\[ \frac{8\pi}{|\Sigma|} \int_{\Sigma} (Q_k, \zeta) \, dv_g = \frac{8\pi}{|\Sigma|} \int_{\Sigma} G_{\rho} \, dv_g + o_k(1). \]

We are now computing the integral \( \int_{\Sigma} h e^{(Q_k, \zeta)} \, dv_g \). Fixing some small \( \delta > 0 \), we write
\[
\int_{\Sigma} h e^{(Q_k, \zeta)} \, dv_g = h(p) \int_{B_{R_0}(p)} e^{(Q_k, \zeta)} \, dv_g + \int_{B_{R_0}(p)} (h - h(p)) e^{(Q_k, \zeta)} \, dv_g + \int_{B_{R_0}(p)} h e^{(Q_k, \zeta)} \, dv_g + \int_{\Sigma \setminus B_{R_0}(p)} h e^{(Q_k, \zeta)} \, dv_g. \tag{76}
\]
Recalling (75), there holds \( \int_{\Sigma} (\xi_0, \tau_1) \, dv_g = o_k(1) \). This together with (65) and (68) gives
\[
h(p) \int_{B_{R_0}(p)} e^{(Q_k, \zeta)} \, dv_g = h(p) \int_{B_{R_0}(p)} e^{(Q_k, \zeta) + o_k(1)} \, dv_g = (1 + o_k(1)) 8\pi h(p) e^{-2 \log 8 + 2 \log k + A_p}.
\]
While three integrals \( \int_{B_{R_0}(p)} (h - h(p)) e^{(Q_k, \zeta)} \, dv_g, \int_{B_{R_0}(p)} h e^{(Q_k, \zeta)} \, dv_g \) and \( \int_{\Sigma \setminus B_{R_0}(p)} h e^{(Q_k, \zeta)} \, dv_g \) are all \( o_k(1)k^2 \). These estimates together with (76) lead to
\[
\log \int_{\Sigma} h e^{(Q_k, \zeta)} \, dv_g = -4 \log 8 + \log(h(p)) + 2 \log k + A_p + o_k(1). \tag{77}
\]
Combining (75) and (77), we obtain
\[
J_{8k}(Q_k) = \frac{1}{2} \int_{\Sigma} |\nabla Q_k|^2 \, dv_g + \frac{8\pi}{|\Sigma|} \int_{\Sigma} (Q_k, \zeta) \, dv_g - 8 \log \int_{\Sigma} h e^{(Q_k, \zeta)} \, dv_g = -8 \log \pi - 8 \log h(p) - 4\pi A_p + \frac{4\pi}{|\Sigma|} \int_{\Sigma} G_{\rho} \, dv_g + o_k(1).
\]
This immediately leads to
\[
\inf_{\sigma \in \mathcal{H}_1} J_{8k}(\sigma) \leq -8 \log \pi - 8 \log h(p) - 4\pi A_p + \frac{4\pi}{|\Sigma|} \int_{\Sigma} G_{\rho} \, dv_g, \tag{78}
\]
which together with (65) implies (12).

### 4.4. Completion of the proof of Theorem 2

It is just a summary of the previous subsections. Our purpose is to prove that if \( J_{8k} \) has no minimizer on \( \mathcal{H}_1 \), that is,
\[
J_{8k}(\sigma_0) \neq \inf_{\sigma \in \mathcal{H}_1} J_{8k}(\sigma), \quad \text{for all} \quad \sigma_0 \in W^{1,2}(\Sigma, \mathcal{L}),
\]
then \( \inf_{\sigma \in \mathcal{H}_1} J_{8k}(\sigma) \) has an exact value as in (13). First of all, let \( \rho_k = 8\pi - 1/k \) for any \( k \in \mathbb{N}^+ \). By Theorem 1, there exists a sequence of sections \( \sigma_k = u_k \xi \in W^{1,2}(\Sigma, \mathcal{L}) \) such that
\[
J_{\rho_k}(\sigma_k) = \inf_{\sigma \in \mathcal{H}_1} J_{\rho}(\sigma). \tag{22}
\]
By Lemma [11], $J_{p_k}(σ_k) \to \inf_{σ \in \mathcal{H}_1} J_{8r}(σ)$ as $k \to \infty$. Let $c_k = \max_Σ u_k = u_k(x_k)$. Then Lemma [13] leads to $c_k \to +\infty$ as $k \to \infty$. In view of Lemma [17], there exists a unique Green section $G = G_pζ$ such that $σ_k$ converges to $G$ weakly in $W^{1,q}(Σ, L)$ for any $1 < q < 2$, strongly in $L'(Σ, L)$ for $r < 2q/(2 - q)$, $1 < q < 2$, and in $C^1_c(Σ \setminus \{p\}, L)$. Locally $G_p$ is written as $G_p(x) = -4 log r + A_p + o(1)$, where $o(1) \to 0$ as $r = d_p(x, p) \to 0$. Finally we derive a lower bound of $J_{8r}$ on $\mathcal{H}_1$. Also we construct a sequence of sections $Q_k \in \mathcal{H}_1$ indicating that the infimum of $J_{8r}$ on $\mathcal{H}_1$ satisfies (78). Combining (65) and (78), we obtain (12). This completes the proof of the theorem.

5. Further assumption on the frame $ζ$

We now turn to Theorem 3. A key observation is as follows.

Lemma 18. Suppose that $Dζ = 0$. Then $\dim \mathcal{H}_0 = 1$, in particular $\mathcal{H}_0 = \text{span}[ζ]$. Moreover if we set $σ = uζ$, then $Δ_L σ = (Δ_L u)ζ$.

Proof. Since $ζ$ is a global unit frame on $W^{1,2}(Σ, L)$, for any $τ \in H_0$, there exists a unique function $v$ such that $τ = νζ$. Clearly

$$0 = Dτ = dv \otimes ζ + νDζ = dv \otimes ζ.$$  

This leads to $dv = 0$, and thus $ν \equiv C$ for some constant $C$. Hence $τ = Cζ$ and $H_0 = \text{span}[ζ]$.

In view of Lemma 4 for any $σ = uζ$, we have

$$Δ_L σ = (Δ_L u + (|ω|^2 + d^*ω)u)ζ,$$

where $ω$ satisfies $Dζ = ω \otimes ζ$. Obviously $ω = 0$ since $Dζ = 0$. Hence $Δ_L σ = (Δ_L u)ζ$. □

Proof of Theorem 3. Suppose $Dζ = 0$. If $σ = uζ \in H_1$, then we have $⟨σ, ζ⟩ = u$ and

$$\int_Σ |Dτ|^2dvg = \int_Σ |du|^2dv_ζ.$$  

In view of the definitions of functionals $J_ρ$ and $J_ρ$ (see (12) and (4) above), there holds

$$J_ρ(σ) = J_ρ(u).$$  

(79)

Define a function space

$$H_1 = \left\{ u \in W^{1,2}(Σ) : \int_Σ u dv_ζ = 0 \right\}.$$

(i) $ρ < 8π$.

By Theorem 1 there exists some $σ_ρ \in H_1$ such that $J_ρ(σ_ρ) = \inf_{σ \in H_1} J_ρ(σ)$. A straightforward calculation shows $σ_ρ$ satisfies the Euler-Lagrange equation (9). In view of Lemma 18 one can easily check that (9) reduces to (13).

(ii) $ρ = 8π$.

Note that

$$\inf_{σ \in H_1} J_{8r}(σ) = \inf_{u \in H_1} J_{8r}(u).$$
Under the Ding-Jost-Li-Wang condition (13), Theorem 1.2, there exists some $u_0 \in H_1$ such that

$$J_{8\pi}(u_0) = \inf_{u \in H_1} J_{8\pi}(u).$$

Let $\sigma_0 = u_0\zeta$. Then $\sigma_0 \in H_1$ satisfies

$$J_{8\pi}(\sigma_0) = \inf_{\sigma \in H_1} J_{8\pi}(\sigma).$$

Clearly $\sigma_0$ satisfies the Euler-Lagrange equation (13).

(iii) $\rho \in (8k\pi, 8(k+1)\pi)$, $\forall k \in \mathbb{N}^*$. It was proved by Djadli [11] that for any $\rho \in (8k\pi, 8(k+1)\pi)$, $k \in \mathbb{N}^*$, $J_{\rho}$ has a critical point $u_\rho \in W^{1,2}(\Sigma)$. Let $\sigma_\rho = u_\rho\zeta$. In view of (79), we conclude that $\sigma_\rho$ is a critical point of $J_{\rho}$ on the Sobolev space $W^{1,2}(\Sigma, L_2)$. Obviously $\sigma_\rho$ is a solution of the mean field equation (13).

Combining the above three assertions, we finish the proof of the theorem. \qed

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Competing interests

On behalf of all authors, the corresponding author states that there is no conflict of interest.

Data availability

Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

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