Optimal proportional control applied to direct current motor

D Devia Narváez¹, D Devia Narváez¹, and R Ospina²
¹ Universidad Tecnológica de Pereira, Pereira, Colombia
² Universidad Industrial de Santander, Bucaramanga, Colombia

E-mail: dmdevian@utp.edu.co

Abstract. This article shows the implementation of the proportional optimum control over a singular input, multiple output system, which corresponds to the speed control of a direct current motor. For this study, the controllability and observability of the model is verified to guarantee stabilization with the applied control, the system of state equations is defined and finally the optimal controller is designed with the specified requirements, in such a way as to minimize the time of response to an input signal.

1. Introduction
The design of the classical control system is generally a trial and error process in which several methods of analysis are used iteratively to determine the design parameters of an “acceptable” system. Acceptable performance is generally defined in terms of time and frequency domain criteria, such as rise time, setup time, peak overshoot, gain and phase margin, and bandwidth. However, radically different performance criteria must be met. Complex systems, multiple inputs and multiple outputs required to meet demands of modern technology [1].

For example, the design of a spacecraft attitude control system that minimizes fuel expense is not amenable to solution by classical methods. A new and direct approach to the synthesis of these complex systems, called optimal control theory, has been possible thanks to the development of the digital computer. The objective of the optimal control theory is to determine the control signals that will make a process satisfy the physical constraints and at the same time minimize (or maximize) some performance criteria. For the development of this document, some verifications are carried out on the proposed system (DC motor) and then proceed to design the optimal proportional control for the motor speed [2].

2. Controllability and observability
Here, it is considered a linear and time invariant system (LTI) represented by the equation of states of n states, q entries Equation (1).

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t) \\
y(t) &= Cx(t) + Du(t)
\end{align*}
\]

(1)

where, \(A \in \mathbb{R}^{n \times n}\) and \(B \in \mathbb{R}^{n \times d}\). Because the controllability of a system only relates inputs and states, the output equation given by \(y(t) = Cx(t) + Du(t)\) is irrelevant in that analysis. For the case of observability, it relates the outputs to the state variables, so the analysis is performed only on the output equation [3].
2.1. Controllability
It is said that the equation of state (1) or the pair \((A; B)\) of a system is controllable if for every initial state \(x(0) = x_0\) and every final state \(x_f\), there is an entry that transfers \(x_0\) to \(x_f\) in a finite time. In another case, (1) or \((A; B)\) is said not controllable. This definition only requires that the entry be able to bring the state to any place in the space of states in finite time no matter which path the state follows. It is equivalent then to say that a system is controllable if its controllability matrix is full range, where the controllability matrix is given by [4] in Equation (2):

\[
C = [B \quad AB \quad A^2B \ldots A^{n-1}B]
\] (2)

2.2. Observability
It is said that the equation of state (Equation (1)) of a system is observable at a time \(t_0\) if with the system in an initial state \(x(t_0)\); it is possible to determine this state from the observations of the output \(y(t)\) during a finite interval of time [5]. An analysis similar to that of controllability is performed verifying that the range of the controllability matrix is complete. The matrix can be calculated using the expression presented in Equation (3):

\[
O = \begin{bmatrix}
C \\
CA \\
\vdots \\
CA^{n-1}
\end{bmatrix}
\] (3)

3. Model
Now, the circuit (Figure 1) and the state space system of a DC motor Equation (4) [6] are presented, with which the optimal proportional control design will be performed. The state variables to be studied are the angular velocity in the rotor \(W_0\) and the armature current \(i_a\) and the control signal is the applied voltage \(e_a\) [7].

![DC motor circuit](image)

**Figure 1.** DC motor circuit.

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} = \begin{bmatrix}
0 & 25 \\
-400 & -200
\end{bmatrix} \begin{bmatrix}
x_1(t) \\
x_2(t)
\end{bmatrix} + \begin{bmatrix}
0 \\
400
\end{bmatrix} u(t)
\] (4)

Initially, the aim is to verify the controllability and observability of the proposed system and, from this, to design an optimal proportional controller and a state observer. The controllability matrix for the presented system is given by Equation (5) [8]:

\[
C = [B \quad AB]
\]

\[
C = \begin{bmatrix}
0 & 10000 \\
400 & -80000
\end{bmatrix}.
\] (5)
where the range of the controllability matrix is 2 (full range matrix), so it can be said that the presented system is controllable. Now we want to verify if the system is also observable. The observability matrix of the system is Equation (6):

\[
\mathcal{O} = \begin{bmatrix}
C \\
CA
\end{bmatrix} = \begin{bmatrix}
1 & 0 \\
0 & 25
\end{bmatrix}
\]

(6)

The range of the observability matrix is 2, being also of full range we can conclude that the system is also observable. Figures 2 and Figure 3 show the response presented by the system when a unit step input is applied to it, in which it is observed and determined that the establishment time of the system is 0.1 seconds (Figure 2) and also the behavior of the state variables (speed and armature current) (Figure 3) [9].

![Figure 2. Step response.](image)

![Figure 3. States variables.](image)

For the description of the quadratic linear regulatory system (LRQ), the algebraic equation of Ricatti (ARE) is presented below Equation (7).

\[
A^TP + PA - PBR^{-1}B^TP + Q = 0
\]

(7)

This equation has a unique positive semidefinite solution \( P = P^T \geq 0 \), which provides stability to the matrix \( A_{LC} = A - BR^{-1}B^TP \), corresponding to the original dynamic system \( \dot{x} = Ax + Bu \), if the system is controllable and observable. Here, we obtain a positive definite symmetric matrix \( P \) is obtained, where the gain of the regulator \( K \) is obtained by Equation (8) [10]:

\[
K = R^{-1}B^TP
\]

(8)

Where, \( Q \) is taken as the identity matrix and \( R = 0.5 \), which correspond to design parameters for the optimal proportional controller. Figure 4 shows the coefficients obtained from the Ricatti Equation (9) [11].

\[
P = \begin{bmatrix}
0.0524 & 0.0012 \\
0.0012 & 0.0027
\end{bmatrix} \Rightarrow K = R^{-1}B^TP = \begin{bmatrix}
0.1049 & 0.0025
\end{bmatrix}
\]

(9)
Figure 4. Ricatti coefficients.

Vector presented below Equations (10) to Equation (15):

\[
A_p = A - B \cdot K = \begin{bmatrix} -0.1049 & 24.9975 \\ -400 & -200 \end{bmatrix} \tag{10}
\]

\[
B_p = B \cdot K(1) = \begin{bmatrix} 0.1049 \\ 0 \end{bmatrix} \tag{11}
\]

\[
C_p = C = [1 \ 0] \tag{12}
\]

\[
D_p = D = [0] \tag{13}
\]

\[
\begin{bmatrix} \dot{w} \\ \dot{y} \end{bmatrix}_p = \begin{bmatrix} -0.1049 & 24.9975 \\ -400 & -200 \end{bmatrix} \begin{bmatrix} w \\ y \end{bmatrix}_p + \begin{bmatrix} 0.1049 \\ 0 \end{bmatrix} e_a 
+ \begin{bmatrix} 0 \\ e_d \end{bmatrix}_u(t) \tag{14}
\]

\[
\begin{align*}
\dot{x}_p(t) &= A_p x(t) + B_p u(t) \\
y_p(t) &= C_p x(t) + D_p u(t)
\end{align*} \tag{15}
\]

Finally, a state space representation is obtained for the system controlled from the gain values obtained K and in Figure 5 the results for the behavior of the state variables and the input-output error of the system are shown [12].
4. Conclusion
The optimal control represents an alternative form of control, in which it is possible to define a behavior for certain variables of a system. We can focus the work of a controller to minimize the energy consumption of the actuators, to accelerate the convergence to a point of equilibrium, so that the variation of the error is more or less rapid, etc. Everything depends on the specifications imposed by the controller designer or the circumstances under which the control system is located.

The solution of the Ricatti algebraic equation may not be unique. However, it can be forced to be if it is guaranteed that the matrix $P$ is positive defined and this is achieved if the system is controllable and observable. Which means that both the controllability and observability matrices are full range.

References
[1] A Rantzer 2000 Piecewise linear quadratic optimal control IEEE transactions on automatic control 45 629
[2] C Ramos 2012 Diseño de controladores basados en técnicas de control óptimo lqr + iyh2 para un prototipo del péndulo invertido sobre ruedas Revista Politécnica 8 45
[3] J Willems 1971 Least squares stationary optimal control and the algebraic Riccati equation IEEE Transactions on Automatic Control 16 621
[4] D Atherton 1999 PID controller tuning Computing and control engineering journal 10 44
[5] K Narendra 1993 Control of nonlinear dynamical systems using neural networks: controllability and stabilization IEEE Transactions on Neural Networks 4 192
[6] H Purnawan 2017 Design of linear quadratic regulator (LQR) control system for flight stability of LSU-05 Journal of Physics: Conference Series 890 012056
[7] N Praboo 2013 Simulation work on fractional order pi control strategy for control of DC motor based on stability boundary Locus method International Journal of Engineering Trends and Technology 4 3403
[8] V Shrivastval 2014 Performance analysis of speed control of direct current (dc) motor using traditional tuning controller International Journal of Emerging Technology and Advanced Engineering 4 119
[9] J Kanieski 2016 robust adaptive controller combined with a linear quadratic regulator based on Kalman filtering IEEE Trans. Autom. Control 6 1373
[10] R Dwivedi 2015 PID conventional controller and LQR optimal controller for speed analysis of dc motor: a comparative study International Research Journal of Engineering and Technology 2 1
[11] H Alasooly 2011 Control of DC motor using different controller strategies global Journal of Technology and Optimization 2 21
[12] H Ang 2005 PID Control system analysis, design, and technology IEEE Transactions on Control System Technology 13 559