COEFFICIENT ESTIMATE OF BI-BAZILEVIČ FUNCTION OF COMPLEX ORDER BASED ON QUASI SUBORDINATION INVOLVING SRIVASTAVA-ATTIYA OPERATOR

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Abstract. In this paper, we introduce and investigate a new subclass of the function class $\Sigma$ of bi-univalent functions defined in the open unit disk, which are associated with the Hurwitz-Lerch zeta function, satisfying subordinate conditions. Furthermore, we find estimates on the Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ for functions in this new subclass. Several (known or new) consequences of the results are also pointed out.

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1. Introduction, Definitions and Preliminaries

Let $A$ denote the class of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

which are analytic in the open unit disk

$$\Delta = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}.$$ 

Further, by $S$ we shall denote the class of all functions in $A$ which are univalent in $\Delta$. Some of the important and well-investigated subclasses of the univalent function class $S$ include (for example) the class $S^*(\alpha)$ of starlike functions of order $\alpha$ in $\Delta$ and the class $K(\alpha)$ of convex functions of order $\alpha$ in $\Delta$.

The convolution or Hadamard product of two functions $f, h \in A$ is denoted by $f \ast h$ and is defined as

$$(f \ast h)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n,$$

where $f(z)$ is given by (1.1) and $h(z) = z + \sum_{n=2}^{\infty} b_n z^n$.

We recall a general Hurwitz-Lerch Zeta function $\Phi(z, s, a)$ defined by (see [24])

$$\Phi(z, s, a) := \sum_{k=0}^{\infty} \frac{z^k}{(k+a)^s} \quad (a \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}; s \in \mathbb{C}, \Re(s) > 1 \text{ and } |z| < 1).$$

Several interesting properties and characteristics of the Hurwitz-Lerch zeta function $\Phi(z, s, a)$ can be found in the recent investigations by Choi and Srivastava [5], Ferreira and Lopez [7], Garg et al [10], Lin et al [15] and others.
For the class $\mathcal{A}$, Srivastava and Attiya [23] (see also Raducanu and Srivastava [21] and Prajapat and Goyal [20]) introduced and investigated linear operator:

$$J^b_\mu : \mathcal{A} \rightarrow \mathcal{A}$$

defined in terms of the Hadamard product (or convolution) by

$$J^b_\mu f(z) = (G^\mu_{b,*} f)(z) \quad (z \in \Delta; \quad b \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}; \mu \in \mathbb{C}; \ f \in \mathcal{A}),$$

where, for convenience.

$$G^\mu_{b,*} f(z) = (1 + b)^\mu [\Phi(z, \mu, b) - b^{-\mu}].$$

It is easy to observe from (given earlier by [20], [21], (1.1), (1.4) and (1.5) that

$$J^b_\mu f(z) = z + \sum_{k=2}^{\infty} \Theta_k a_k z^k,$$

where

$$\Theta_k = \left| \left( \frac{1 + b}{k + b} \right)^\mu \right|$$

and (throughout this paper unless otherwise mentioned) the parameters $\mu, b$ are considered as $\mu \in \mathbb{C}$ and $b \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}$.

We note that

- For $\mu = 1$ and $b = \nu (\nu > -1)$ generalized Libera-Bernardi integral operator [22]

$$J^\nu_1 f(z) = \frac{1 + \nu}{z^\nu} \int_0^z t^{\nu-1} f(t) dt$$

$$= z + \sum_{k=2}^{\infty} \left( \frac{\nu + 1}{k + \nu} \right) a_k z^k = \mathcal{L}_\nu f(z).$$

- For $\mu = \sigma (\sigma > 0)$ and $b = 1$, Jung-Kim-Srivastava integral operator [13]

$$J^\sigma_1 f(z) = \frac{2^{\sigma}}{z \Gamma(\sigma)} \int_0^z \left( \log \left( \frac{z}{t} \right) \right)^{\sigma-1} f(t) dt$$

$$= z + \sum_{k=2}^{\infty} \left( \frac{z}{k + 1} \right)^{\sigma} a_k z^k = \mathcal{J}_\sigma f(z)$$

closely related to some multiplier transformations studied by Flett [8].

An analytic function $f(z)$ is quasi-subordinate to an analytic function $h(z)$, in the open unit disk if there exist analytic functions $\phi$ and $w$, with $w(0) = 0$ such that $|\phi(z)| < 1, |w(z)| < 1$ and $f(z) = \phi(z) h[w(z)]$. Then we write $f(z) \prec_\phi h(z)$. If $\phi(z) = 1$, then the quasi-subordination reduces to the subordination. Also, if $w(z) = z$ then $f(z) = \phi(z) h(z)$ and in this case we say that $f(z)$ is majorized by $h(z)$ and it is written as $f(z) \ll h(z)$ in $\Delta$. Hence it is obvious that quasi-subordination is the generalization of subordination as well as majorization. It is unfortunate that the concept quasi-subordination is so far an underlying concept in the area of complex function theory although it deserves much attention as it unifies the concept of both subordination and majorization. Throughout this paper it is assumed that $\phi$ is analytic in $\Delta$ with $\phi(0) = 1$ and let

$$\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots, \quad (B_1 > 0).$$

(1.8)
also let
\[ \psi(z) = C_0 + C_1 z + C_2 z^2 + C_3 z^3 + \cdots, \quad (|\psi(z)| < 1) \quad z \in \Delta \] (1.9)

2. Bi-Univalent function Class \( \Sigma \)

It is well known that every function \( f \in S \) has an inverse \( f^{-1} \), defined by
\[ f^{-1}(f(z)) = z \quad (z \in \Delta) \]
and
\[ f(f^{-1}(w)) = w \quad \left( |w| < r_0(f); \quad r_0(f) \geq \frac{1}{4} \right), \]
where
\[ g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3) w^3 - (5a_2^3 - 5a_2 a_3 + a_4) w^4 + \cdots \] (2.1)

A function \( f \in A \) is said to be bi-univalent in \( \Delta \) if both \( f(z) \) and \( f^{-1}(z) \) are univalent in \( \Delta \). Let \( \Sigma \) denote the class of bi-univalent functions in \( \Delta \) given by (1.1).

Recently there has been triggering interest to study bi-univalent function class \( \Sigma \) and obtained non-sharp coefficient estimates on the first two coefficients \(|a_2|\) and \(|a_3|\) of (1.1). But the coefficient problem for each of the following Taylor-Maclaurin coefficients:
\[ |a_n| \quad (n \in \mathbb{N} \setminus \{1, 2, 3\}; \quad \mathbb{N} := \{1, 2, 3, \ldots \} \]
is still an open problem (see [2, 3, 4, 11, 13, 18, 26]). Many researchers (see [9, 11, 16, 17, 25]) have recently introduced and investigated several interesting subclasses of the bi-univalent function class \( \Sigma \) and they have found non-sharp estimates on the first two Taylor-Maclaurin coefficients \(|a_2|\) and \(|a_3|\).

Several authors have discussed various subfamilies of Bazilevič functions of type \( \lambda \) from various perspective. They discussed it from the perspective of convexity, inclusion theorem, radii of starlikeness and convexity boundary rotational problem, subordination just to mention few. The most amazing thing is that, it is difficult to see any of this authors discussing the coefficient inequalities, and coefficient bounds of these subfamilies of Bazilevič function most especially when the parameter \( \lambda \) is greater than 1 (\( \lambda \in \mathbb{R} \)). Motivated by the earlier work of Deniz[6] in the present paper we introduce new families of Bazilevič functions of complex order [12] of the function class \( \Sigma \), involving Hurwitz-Lerch zeta function, and find estimates on the coefficients \(|a_2|\) and \(|a_3|\) for functions in the new subclasses of function class \( \Sigma \). Several related classes are also considered, and connection to earlier known results are made.

Definition 2.1. A function \( f \in \Sigma \) given by (1.1) is said to be in the class \( \mathcal{B}_{\Sigma}^\mu(h, \lambda, \phi) \) if the following conditions are satisfied:
\[ \frac{1}{\gamma} \left( \frac{z^{1-\lambda}(\partial^\mu_{\mu} f(z))'}{|\partial^\mu_{\mu} f(z)|^{1-\lambda}} - 1 \right) \prec_{\tilde{q}} (\phi(z) - 1) \] (2.2)
and
\[ \frac{1}{\gamma} \left( \frac{w^{1-\lambda}(\partial^\mu_{\mu} g(w))'}{|\partial^\mu_{\mu} g(w)|^{1-\lambda}} - 1 \right) \prec_{\tilde{q}} (\phi(w) - 1) \] (2.3)
where \( \gamma \in \mathbb{C} \setminus \{0\}; \lambda \geq 0; \quad z, w \in \Delta \) and the function \( g \) is given by (2.1).

On specializing the parameters \( \lambda \) one can define the various new subclasses of \( \Sigma \) associated with Hurwitz-Lerch zeta function as illustrated in the following examples.
Example 1. For $\lambda = 0$ and a function $f \in \Sigma$, given by (1.1) is said to be in the class $S_{\Sigma}^\mu b(\gamma, \phi)$ if the following conditions are satisfied:

$$\frac{1}{\gamma} \left( \frac{z(\partial^b_\mu f(z))'}{\partial^b_\mu f(z)} - 1 \right) \prec \tilde{q} (\phi(z) - 1)$$

(2.4) and

$$\frac{1}{\gamma} \left( \frac{w(\partial^b_\mu g(w))'}{\partial^b_\mu g(w)} - 1 \right) \prec \tilde{q} (\phi(w) - 1)$$

(2.5)

where $\gamma \in \mathbb{C} \setminus \{0\}; z, w \in \Delta$ and the function $g$ is given by (2.1).

Example 2. For $\lambda = 1$ and a function $f \in \Sigma$, given by (1.1) is said to be in the class $H_{\Sigma}^\mu b(\gamma, \phi)$ if the following conditions are satisfied:

$$\frac{1}{\gamma} \left( (\partial^b_\mu f(z))' - 1 \right) \prec \tilde{q} (\phi(z) - 1)$$

(2.6) and

$$\frac{1}{\gamma} \left( (\partial^b_\mu g(w))' - 1 \right) \prec \tilde{q} (\phi(w) - 1)$$

(2.7)

where $\gamma \in \mathbb{C} \setminus \{0\}; z, w \in \Delta$ and the function $g$ is given by (2.1).

It is of interest to note that for $\gamma = 1$ the class $B_{\Sigma}^\mu b(\gamma, \lambda, \phi)$ reduces to the following new subclass $B_{\Sigma}^\mu b(\lambda, \phi)$.

Definition 2.2. A function $f \in \Sigma$ given by (1.1) is said to be in the class $B_{\Sigma}^\mu b(\lambda, \phi)$ if the following conditions are satisfied:

$$\left( \frac{z^{1-\lambda}(\partial^b_\mu f(z))'}{|\partial^b_\mu f(z)|^{1-\lambda}} - 1 \right) \prec \tilde{q} (\phi(z) - 1)$$

(2.8) and

$$\left( \frac{w^{1-\lambda}(\partial^b_\mu g(w))'}{|\partial^b_\mu g(w)|^{1-\lambda}} - 1 \right) \prec \tilde{q} (\phi(w) - 1)$$

(2.9)

where $\lambda \geq 0; z, w \in \Delta$ and the function $g$ is given by (2.1).

For particular values of $\lambda$, we have

Example 3. For $\lambda = 0$ and a function $f \in \Sigma$, given by (1.1) is said to be in the class $B_{\Sigma}^\mu b(0, \phi) \equiv S_{\Sigma}^\mu b(\phi)$ if the following conditions are satisfied:

$$\frac{z(\partial^b_\mu f(z))'}{\partial^b_\mu f(z)} \prec \tilde{q} (\phi(z) - 1)$$

(2.10) and

$$\frac{w(\partial^b_\mu g(w))'}{\partial^b_\mu g(w)} \prec \tilde{q} (\phi(w) - 1)$$

(2.11)

where $z, w \in \Delta$ and the function $g$ is given by (2.1).
Example 4. For \( \lambda = 1 \) and a function \( f \in \Sigma \), given by (1.1) is said to be in the class \( B_{\Sigma}^{\mu,b}(1, \phi) \equiv \mathcal{K}_{\Sigma}^{\mu,b}(\phi) \) if the following conditions are satisfied:
\[
(\partial^b_{\mu} f(z))' \prec_\gamma (\phi(z) - 1)
\]
and
\[
(\partial^b_{\mu} g(w))' \prec_\gamma (\phi(w) - 1)
\]
where \( z, w \in \Delta \) and the function \( g \) is given by (2.1).

We now introduce another new subclass of the function class \( \Sigma \) of complex order \( \gamma \in \mathbb{C} \setminus \{0\} \).

**Definition 2.3.** A function \( f \in \Sigma \) given by (1.1) is said to be in the class \( \mathcal{S}_{\Sigma}^{\mu,b}(\gamma, \lambda, \phi) \) if the following conditions are satisfied:
\[
\frac{1}{\gamma} \left( \frac{z(\partial^b_{\mu} f(z))'}{(1 - \lambda)(\partial^b_{\mu} f(z) + \lambda z(\partial^b_{\mu} f(z))')} - 1 \right) \prec_\gamma (\phi(z) - 1)
\]
and
\[
\frac{1}{\gamma} \left( \frac{w(\partial^b_{\mu} g(w))'}{(1 - \lambda)(\partial^b_{\mu} g(w) + \lambda z(\partial^b_{\mu} g(w))')} - 1 \right) \prec_\gamma (\phi(w) - 1)
\]
where \( \gamma \in \mathbb{C} \setminus \{0\} \), \( 0 \leq \lambda < 1 \), \( z, w \in \Delta \) and the function \( g \) is given by (2.1).

We note that by taking \( \lambda = 0 \) we have
\[
\mathcal{S}_{\Sigma}^{\mu,b}(\gamma, 0, \phi) \equiv \mathcal{S}_{\Sigma}^{\mu,b}(\gamma, \phi)
\]

In the following section we find estimates on the coefficients \(|a_2|\) and \(|a_3|\) for functions in the above-defined subclasses \( B_{\Sigma}^{\mu,b}(\gamma, \lambda, \phi) \) of the function class \( \Sigma \).

In order to derive our main results, we shall need the following lemma:

**Lemma 2.4.** (see [19]) If \( p \in \mathcal{P} \), then \(|p_k| \leq 2\) for each \( k \), where \( \mathcal{P} \) is the family of all functions \( p \) analytic in \( \Delta \) for which \( \Re(p(z)) > 0 \), where \( p(z) = 1 + p_1 z + p_2 z^2 + \cdots \) for \( z \in \Delta \).

### 3. Coefficient Bounds for the Function Class \( B_{\Sigma}^{\mu,b}(\gamma, \lambda, \phi) \)

We begin by finding the estimates on the coefficients \(|a_2|\) and \(|a_3|\) for functions in the class \( B_{\Sigma}^{\mu,b}(\gamma, \lambda, \phi) \).

**Theorem 3.1.** Let the function \( f(z) \) given by (1.1) be in the class \( B_{\Sigma}^{\mu,b}(\gamma, \lambda, \phi) \). Then
\[
|a_2| \leq \frac{|\gamma| |C_0 B_1 \sqrt{2B_1}|}{\sqrt{|\gamma C_0 B_1^2[(\lambda - 1)(\lambda + 2)\Theta_2^2 + 2(\lambda + 2)\Theta_3] - 2(B_2 - B_1)(1 + \lambda)^2\Theta_2^2}}
\]
and
\[
|a_3| \leq \frac{|\gamma||C_0 B_1}{(\lambda + 2)\Theta_3} + \frac{|\gamma||C_1 B_1}{(\lambda + 2)\Theta_3} + \left( \frac{|\gamma||C_0 B_1}{(1 + \lambda)\Theta_2} \right)^2.
\]

**Proof.** Let \( f \in \mathcal{S}_{\Sigma}^{\mu,b}(\gamma, \lambda, \phi) \) and \( g = f^{-1} \). Then there are analytic functions \( u, v : \Delta \rightarrow \Delta \) with \( u(0) = 0 = v(0) \), satisfying
\[
\frac{1}{\gamma} \left( \frac{z^{1-\lambda}(\partial^b_{\mu} f(z))'}{[\partial^b_{\mu} f(z)]^{1-\lambda}} - 1 \right) = \psi(z)[\phi(u(z)) - 1]
\]
and
\[ \frac{1}{\gamma} \left( \frac{w^{1-\lambda}(f_{\mu} g(w))'}{|f_{\mu} g(w)|^{1-\lambda}} - 1 \right) = \psi(w)[\phi(u(w)) - 1]. \] (3.4)

Define the functions \( p(z) \) and \( q(z) \) by
\[ p(z) := \frac{1 + u(z)}{1 - u(z)} = 1 + p_1 z + p_2 z^2 + \cdots \]
and
\[ q(z) := \frac{1 + v(z)}{1 - v(z)} = 1 + q_1 z + q_2 z^2 + \cdots \]
or, equivalently,
\[ u(z) := \frac{p(z) - 1}{p(z) + 1} = \frac{1}{2} \left[ p_1 z + \left( p_2 - \frac{p_1^2}{2} \right) z^2 + \cdots \right] \] (3.5)
\[ v(z) := \frac{q(z) - 1}{q(z) + 1} = \frac{1}{2} \left[ q_1 z + \left( q_2 - \frac{q_1^2}{2} \right) z^2 + \cdots \right]. \] (3.6)

Then \( p(z) \) and \( q(z) \) are analytic in \( \Delta \) with \( p(0) = 1 = q(0) \). Since \( u, v : \Delta \to \Delta \), the functions \( p(z) \) and \( q(z) \) have a positive real part in \( \Delta \), and \( |p_i| \leq 2 \) and \( |q_i| \leq 2 \).

Now,
\[ \psi(z)[\phi(u(z)) - 1] = \frac{1}{2} C_0 B_1 p_1 z + \left[ \frac{1}{2} C_1 B_1 p_1 + \frac{1}{2} C_0 B_1 \left( p_2 - \frac{p_1^2}{2} \right) \right] z^2 + \cdots \] (3.7)
and
\[ \psi(w)[\phi(v(w)) - 1] = \frac{1}{2} C_0 B_1 q_1 w + \left[ \frac{1}{2} C_1 B_1 q_1 + \frac{1}{2} C_0 B_1 \left( q_2 - \frac{q_1^2}{2} \right) \right] w^2 + \cdots \] (3.8)
respectively.

In light of (1.1) - (1.8), from (3.7) and (3.8), it is evident that
\[ \frac{(\lambda + 1)}{\gamma} \Theta_2 a_2 z + \frac{1}{\gamma} \left[ (\lambda + 2) \Theta_3 a_3 + \frac{(\lambda - 1)(\lambda + 2)}{2} \Theta_2^2 a_2^2 \right] z^2 + \cdots \]
\[ = \frac{1}{2} C_0 B_1 p_1 z + \left[ \frac{1}{2} C_1 B_1 p_1 + \frac{1}{2} C_0 B_1 \left( p_2 - \frac{p_1^2}{2} \right) + \frac{C_0 B_2}{4} p_1^2 \right] z^2 + \cdots \]
and
\[ - \frac{(\lambda + 1)}{\gamma} \Theta_2 a_2 w + \frac{1}{\gamma} \left[ -(\lambda + 2) \Theta_3 a_3 + \frac{(\lambda - 1)(\lambda + 2)}{2} \Theta_2^2 + 2(\lambda + 2) \Theta_3 \right] a_2 w^2 + \cdots \]
\[ = \frac{1}{2} C_0 B_1 q_1 w + \left[ \frac{1}{2} C_1 B_1 q_1 + \frac{1}{2} C_0 B_1 \left( q_2 - \frac{q_1^2}{2} \right) + \frac{C_0 B_2}{4} q_1^2 \right] w^2 + \cdots \]
which yields the following relations.
\[ (1 + \lambda) \Theta_2 a_2 = \frac{\gamma}{2} C_0 B_1 p_1 \] (3.9)
\[ \frac{(\lambda - 1)(\lambda + 2)}{2} \Theta_2^2 a_2^2 + (\lambda + 2) \Theta_3 a_3 = \gamma \left[ \frac{1}{2} C_1 B_1 p_1 + \frac{1}{2} C_0 B_1 \left( p_2 - \frac{p_1^2}{2} \right) + \frac{C_0 B_2}{4} p_1^2 \right] \] (3.10)
\[ -(\lambda + 1) \Theta_2 a_2 = \frac{\gamma}{2} C_0 B_1 q_1 \] (3.11)
and
\[
(2(\lambda + 2)\Theta_3 + \frac{(\lambda - 1)(\lambda + 2)}{2} \Theta_2^2) a_2^2 - (\lambda + 2)\Theta_3 a_3 = \gamma \left[ \frac{1}{2} C_1 B_1 q_1 + \frac{1}{2} C_0 B_1 \left( a_2 - \frac{q_1^2}{2} \right) + \frac{C_0 B_2}{4} q_1^2 \right].
\]

From (3.9) and (3.11), it follows that
\[
p_1 = -q_1
\]
and
\[
8(\lambda + 1)^2 \Theta_2^2 a_2 = \gamma^2 C_0^2 B_1^2 (p_1^2 + q_1^2).
\]
Adding (3.10) and (3.12), we obtain
\[
[(\lambda - 1)(\lambda + 2)\Theta_2^2 + 2(\lambda + 2)\Theta_3] a_2^2 = \frac{\gamma C_0 B_1}{2} (p_2 + q_2) + \frac{\gamma C_0 (B_2 - B_1) (p_1^2 + q_1^2)}{4}.
\]
Using (3.14) in (3.15), we get
\[
a_2^2 = \frac{\gamma^2 C_0^2 B_1^2 (p_2 + q_2)}{2\gamma C_0 B_1^2 [(\lambda - 1)(\lambda + 2)\Theta_2^2 + 2(\lambda + 2)\Theta_3] - 4(B_2 - B_1)(1 + \lambda)^2 \Theta_2^2}.
\]
Applying Lemma 2.4 for the coefficients \(p_2\) and \(q_2\), we immediately have
\[
|a_2|^2 \leq \frac{2|\gamma||C_0|^2 B_1^3}{(\lambda - 1)(\lambda + 2)\Theta_2^2 + 2(\lambda + 2)\Theta_3 - 2(B_2 - B_1)(1 + \lambda)^2 \Theta_2^2}.
\]
This gives the bound on \(|a_2|\) as asserted in (3.1).

Next, in order to find the bound on \(|a_3|\), by subtracting (3.12) from (3.10), we get
\[
[(\lambda - 1)(\lambda + 2)\Theta_3 a_3 - 2(\lambda + 2)\Theta_3 a_2^2] = \gamma C_1 B_1 (p_2 - q_2) + \frac{\gamma C_1 B_1 (p_1 - q_1)}{2} + \frac{\gamma C_0 (B_2 - B_1) (p_1^2 - q_1^2)}{4}.
\]
Using (3.13) and (3.14) in (3.17), we get
\[
a_3 = \frac{\gamma C_1 B_1 (p_2 - q_2)}{2(\lambda + 2)\Theta_3} + \frac{\gamma B_1 C_1 (p_1 - q_1)}{4(\lambda + 2)\Theta_3} + \frac{\gamma^2 C_0^2 B_1^2 (p_1^2 + q_1^2)}{8(1 + \lambda)^2 \Theta_2^2}.
\]
Applying Lemma 2.4 once again for the coefficients \(p_1, q_1, p_2\) and \(q_2\), we readily get (3.2). This completes the proof of Theorem 3.1.

Putting \(\lambda = 0\) in Theorem 3.1 we have the following corollary.

**Corollary 3.2.** Let the function \(f(z)\) given by (1.1) be in the class \(S^{a,b}_{\Sigma}(\gamma, \phi)\). Then
\[
|a_2| \leq \frac{|\gamma||C_0|B_1 \sqrt{B_1}}{\sqrt{|\gamma| C_0 B_1^2 (2 \Theta_3 - \Theta_2^2) - (B_2 - B_1) \Theta_2^2|}}
\]
and
\[
|a_3| \leq \frac{|\gamma||C_0|B_1}{2 \Theta_3} + \frac{|\gamma||C_1|B_1}{2 \Theta_3} + \left( \frac{|\gamma||C_0|B_1}{\Theta_2} \right)^2.
\]

Putting \(\lambda = 1\) in Theorem 3.1 we have the following corollary.

**Corollary 3.3.** Let the function \(f(z)\) given by (1.1) be in the class \(H^{a,b}_{\Sigma}(\gamma, \phi)\). Then
\[
|a_2| \leq \frac{|\gamma||C_0|B_1 \sqrt{B_1}}{\sqrt{3|\gamma| C_0 B_1^2 \Theta_3 - 4(B_2 - B_1) \Theta_2^2|}}
\]
and

\[ |a_3| \leq \frac{\gamma||C_0|B_1}{3\Theta_3} + \frac{\gamma||C_1|B_1}{3\Theta_3} + \left(\frac{\gamma||C_0|B_1}{2\Theta_2}\right)^2. \]  

(3.21)

If \( \partial_\mu^b \) is the identity map, from Corollary 3.2 and 3.3, we get the following corollaries.

**Corollary 3.4.** Let the function \( f(z) \) given by (1.1) be in the class \( S^*_\Sigma(\gamma, \phi) \). Then

\[ |a_2| \leq \frac{\gamma||C_0|B_1\sqrt{B_1}}{\sqrt{\gamma C_0 B_1^2 - (B_2 - B_1)}} \]  

(3.22)

and

\[ |a_3| \leq \frac{\gamma||C_0|B_1}{2} + \frac{\gamma||C_1|B_1}{2} + (\gamma||C_0|B_1)^2. \]  

(3.23)

**Corollary 3.5.** Let the function \( f(z) \) given by (1.1) be in the class \( \mathcal{K}_\Sigma(\gamma, \phi) \). Then

\[ |a_2| \leq \frac{\gamma||C_0|B_1\sqrt{B_1}}{\sqrt{3\gamma C_0 B_1^2 - 4(B_2 - B_1)}} \]  

(3.24)

and

\[ |a_3| \leq \frac{\gamma||C_0|B_1}{3} + \frac{\gamma||C_1|B_1}{3} + \left(\frac{\gamma||C_0|B_1}{2}\right)^2. \]  

(3.25)

4. **Coefficient Bounds for the Function Class \( \mathcal{G}_{\Sigma}^{\mu,b}(\gamma, \lambda, \phi) \)**

**Theorem 4.1.** Let the function \( f(z) \) given by (1.1) be in the class \( \mathcal{G}_{\Sigma}^{\mu,b}(\gamma, \lambda, \phi) \). Then

\[ |a_2| \leq \frac{\gamma||C_0|B_1\sqrt{B_1}}{\sqrt{||\gamma C_0 (\lambda^2 - 1)B_1^2 + (1 - \lambda)^2(B_1 - B_2)|\Theta_2^2 + 2\gamma(1 - \lambda)C_0 B_1^2\Theta_3}} \]  

(4.1)

and

\[ |a_3| \leq \frac{\gamma||C_1|B_1}{2(1 - \lambda)\Theta_3} + \frac{\gamma||C_0|B_1}{2(1 - \lambda)\Theta_3} + \left(\frac{\gamma||C_0|B_1^2}{2(1 - \lambda)\Theta_3}\right). \]  

(4.2)

**Proof.** It follows from (2.14) and (2.15) that

\[ \frac{1}{\gamma} \left( \frac{z(\partial_\mu^b f(z))'}{(1 - \lambda)\partial_\mu^b f(z) + \lambda z(\partial_\mu^b f(z))'} - 1 \right) = \psi(z)[\phi(u(z)) - 1] \]  

(4.3)

and

\[ \frac{1}{\gamma} \left( \frac{w(\partial_\mu^b g(w))'}{(1 - \lambda)\partial_\mu^b g(w) + \lambda z(\partial_\mu^b g(w))'} - 1 \right) = \psi(w)[\phi(v(w)) - 1], \]  

(4.4)

where \( \psi[\phi(u(z)) - 1] \) and \( \psi[\phi(v(w)) - 1] \) are given in (3.7) and (3.8) respectively.

Now, equating the coefficients in (4.3) and (4.4), we get

\[ \frac{(1 - \lambda)}{\gamma} \Theta_2 a_2 = \frac{1}{2} C_0 B_1 p_1, \]  

(4.5)

\[ \frac{(\lambda^2 - 1)}{\gamma} \Theta_2 a_2 + \frac{2(1 - \lambda)}{\gamma} \Theta_3 a_3 = \frac{1}{2} C_1 B_1 p_1 + \frac{1}{2} C_0 B_1 \left( p_2 - \frac{p_1^2}{2} \right) + \frac{C_0 B_2}{4} p_1^2, \]  

(4.6)

\[ - \frac{(1 - \lambda)}{\gamma} \Theta_2 a_2 = \frac{1}{2} C_0 B_1 q_1, \]  

(4.7)
and
\[
\frac{(\lambda^2 - 1)}{\gamma} \Theta_{2a_2}^2 + \frac{2(1 - \lambda)}{\gamma} \Theta_3(2a_2^2 - a_3) = \frac{1}{2} C_1 B_1 q_1 + \frac{1}{2} C_0 B_1 \left( q_2 - \frac{q_1^2}{2} \right) + \frac{C_0 B_2}{4} q_1^2.
\] (4.8)

Proceeding as in Theorem 3.1 we get the desired results.

Note that by taking \( \lambda = 0 \) we get the result as in Corollary 3.2

5. CONCLUDING REMARK

For the class of strongly starlike functions, the function \( \phi \) is given by
\[
\phi(z) = \left( \frac{1 + z}{1 - z} \right)^\alpha = 1 + 2\alpha z + 2\alpha^2 z^2 + \cdots \quad (0 < \alpha \leq 1),
\] (5.1)
which gives
\[ B_1 = 2\alpha \quad \text{and} \quad B_2 = 2\alpha^2. \]

On the other hand for \(-1 \leq B \leq A < 1\) if we take
\[
\phi(z) = \frac{1 + Az}{1 + Bz} = 1 + (A - B)z - B(A - B)z^2 + B^2(A - B)z^3 + \cdots.
\] (5.2)
then we have
\[ B_1 = (A - B), \quad B_2 = -B(A - B). \]

By taking, \( A = (1 - 2\beta) \) where \( 0 \leq \beta < 1 \) and \( B = -1 \) in (5.2), we get,
\[
\phi(z) = \frac{1 + (1 - 2\beta)z}{1 - z} = 1 + 2(1 - \beta)z + 2(1 - \beta)z^2 + 2(1 - \beta)z^3 + \cdots.
\] (5.3)
Hence, we have
\[ B_1 = B_2 = 2(1 - \beta). \]

Further, by taking \( \beta = 0 \), in (5.3), we get,
\[
\phi(z) = \frac{1 + z}{1 - z} = 1 + 2z + 2z^2 + 2z^3 + \cdots,
\] (5.4)
Hence,
\[ B_1 = B_2 = B_3 = 2. \]

Various Choices of \( \phi \) as mentioned above and suitably choosing the values of \( B_1 \) and \( B_2 \), we state some interesting results analogous to Theorem 3.1, Theorem 4.1 and the Corollaries 3.2 to 3.5.

It is of interest to note that, if \( \mu = 1, b = \nu (\nu > -1) \) the operator \( J^b_\mu \) turns into Libera-Bernardi integral operator \( L_\nu \) and if \( \mu = \sigma (\sigma > 0), \ b = 1 \) the operator \( J^b_\mu \) turns into Jung-Kim-Srivastava integral operator \( I_\sigma \). So, various other interesting corollaries and consequences of our main results (which are asserted by Theorem 3.1 and 4.1 above) can be derived similarly. If \( J^b_\mu \) is the identity map and \( \psi(z) = 1 \) yields some known results stated in [9, 11, ?, 17, 25]. The details involved may be left as an exercise for the interested reader.
COEFFICIENT ESTIMATE OF BI-BAZILEVIĆ FUNCTION... 10

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