PSEUDO-ROTATIONS VS. ROTATIONS

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Abstract. Continuing the study of Hamiltonian pseudo-rotations of projective spaces, we focus on the conjecture that the fixed point data (actions and the eigenvalues of the linearization) of a pseudo-rotation exactly matches that data for a suitable unique true rotation even though the two maps can have very different dynamics. We prove this conjecture in several instances, e.g., for strongly non-degenerate pseudo-rotations of \( \mathbb{C}P^2 \). The question is closely related to the properties of the action and index spectra of pseudo-rotations. The main new ingredient of the proofs is, however, purely combinatorial and of independent interest. This is the index divisibility theorem connecting the divisibility properties of the Conley–Zehnder index sequence for the iterates of a map with the properties of its spectrum.

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1. Introduction and main results

1.1. Introduction. In this paper we continue the study of Hamiltonian pseudo-rotations of \( \mathbb{C}P^n \) started in [GG18a] and focus on the conjecture that the fixed
point data (the actions and the eigenvalues of the linearization) of a pseudo-rotation exactly matches that data for a suitable unique true rotation. We show that this is indeed the case in many instances, e.g., for strongly non-degenerate pseudo-rotations of $\mathbb{C}P^2$. (Even in dimension two, this is ultimately a highly non-trivial fact, although in this case it is a rather simple consequence of known results; see Section 3.2.) As a consequence, while the two maps can have very different dynamics, it is unlikely that they can be distinguished by Floer theoretical and even more generally by symplectic topological methods.

In the context of this paper, a *pseudo-rotation* of $\mathbb{C}P^n$ is a Hamiltonian diffeomorphism of $\mathbb{C}P^n$ with exactly $n+1$ periodic points; this is the minimal possible number of periodic points by the Arnold conjecture, $[Fl, Fo, FW]$. The periodic points are then necessarily the fixed points. The term comes from low-dimensional dynamics where it is used for maps of $S^2$ (or the closed disk $D^2$) with exactly two (or one) periodic points, which are then also the fixed points. More generally, one should think of a pseudo-rotation as a Hamiltonian diffeomorphism with finite and minimal possible number of periodic points. We refer the reader to $[GG18a, GG18b]$ for a discussion of various ways to make this definition precise for other symplectic manifolds and some other related questions. Here we only mention that many, probably most, symplectic manifolds do not admit Hamiltonian diffeomorphisms with finitely many periodic orbits (and in particular pseudo-rotations) by the Conley conjecture; see $[GG15, GG17]$ and references therein.

Among pseudo-rotations of $\mathbb{C}P^n$ are *true rotations* (or just rotations for brevity) with finitely many periodic points (a generic condition in this class), where by a true rotation we understand a Hamiltonian diffeomorphism arising from the action of an element of $SU(n)$ on $\mathbb{C}P^n$. Once $\mathbb{C}P^n$ is identified with the quotient of the unit sphere $S^{2n+1} \subset \mathbb{C}^{n+1}$, a true rotation is the time-one map generated by a quadratic Hamiltonian $Q = \sum a_i |z_i|^2$, for a suitable choice of linear coordinates $z_i$. A rotation is a pseudo-rotation, i.e., it has exactly $n+1$ periodic points, if and only if $a_i - a_j \not\in \mathbb{Q}$ for $i \neq j$. Otherwise, it has infinitely many periodic orbits.

However, not every pseudo-rotation is conjugate to a true rotation. Indeed, true rotations have simple, essentially trivial dynamics. This is in general not the case for pseudo-rotations, and pseudo-rotations occupy a special place among low-dimensional dynamical systems. In $[AK]$, Anosov and Katok constructed area preserving diffeomorphisms $\varphi$ of $S^2$ with exactly three ergodic invariant measures, the area form and the two fixed points, by introducing what is now known as the conjugation method; see also $[FK]$ and references therein. Such a diffeomorphism $\varphi$ is automatically a pseudo-rotation. Indeed, $\varphi$ is area preserving and hence Hamiltonian, and $\varphi$ has exactly two periodic orbits, which are its fixed points. Furthermore, $\varphi$ is ergodic, necessarily has dense orbits, and thus is not conjugate to a true rotation. As a consequence, the products ($S^2)^n$ also admit pseudo-rotations which are not conjugate to true rotations. It is believed that, at least for $n = 2$, the conjugation method can also be applied to construct dynamically interesting pseudo-rotations of $\mathbb{C}P^n$. Such a construction is yet to be worked out in detail and we refer the reader to $[GG18a]$ for a discussion of the problem.

The study of the dynamics of pseudo-rotations in dimension two by symplectic topological methods (finite energy foliations) was initiated in $[Br15a, Br15b, BH]$. In $[GG18a]$, Floer theory and the results from $[GG09, GG14, GK]$ were utilized to investigate the dynamics of pseudo-rotations in higher dimensions.
1.2. Results. The main goal of the paper is to compare the numerical invariants of periodic orbits (the action, the mean index and Floquet multipliers) for rotations and pseudo-rotations. Hypothetically, for every pseudo-rotation there exists exactly one “matching” true rotation with exactly the same numerical invariants. (In particular, as a consequence of this conjecture, every periodic orbit of a pseudo-rotation is elliptic and non-degenerate.) For $n = 1$, this is an easy consequence of rather standard, although highly non-trivial, results; see Section 3.2. We prove the conjecture for strongly non-degenerate pseudo-rotations of $\mathbb{CP}^2$ and, under some additional assumptions, in all dimensions. Note also that this conjecture is consistent with the general expectation that every pseudo-rotation is obtained from a true rotation by a variant of the conjugation method; cf. [Br15a, FK].

1.2.1. Rotations and pseudo-rotations. To state the results more precisely we need to recall several facts about pseudo-rotations. These facts are essentially of symplectic topological nature and go back to [GG09, GK]. (A more detailed review is given in Section 3 and the relevant definitions are recalled in Section 2.) Here and throughout the paper $\mathbb{CP}^n$ is equipped with the standard, up to a factor, Fubini–Study symplectic structure $\omega$.

**Definition 1.1.** A pseudo-rotation of $\mathbb{CP}^n$ is a Hamiltonian diffeomorphism of $\mathbb{CP}^n$ with exactly $n + 1$ periodic points.

Most of our results concern strongly non-degenerate pseudo-rotations $\varphi$. (Recall that $\varphi$ is called strongly non-degenerate if all iterates $\varphi^k$, $k \in \mathbb{N}$, are non-degenerate.) However, it is useful and illuminating to discuss the key facts in a more general setting.

Let $\varphi = \varphi_H$ be a pseudo-rotation of $\mathbb{CP}^n$ generated by a time-dependent Hamiltonian $H$ viewed as an element of the universal covering $\tilde{\text{Ham}}(\mathbb{CP}^n)$ of the group $\text{Ham}(\mathbb{CP}^n)$ of Hamiltonian diffeomorphisms. We associate two spectra to $\varphi$. One is the standard action spectrum $\tilde{S}(H)$ comprising the actions of capped one-periodic orbits of the Hamiltonian flow $\varphi_H$. For instance, when $\varphi_H$ is a true rotation and $H = \sum a_i |z_i|^2$, the action spectrum $S(H)$ is the union of the sets $a_i + \lambda \mathbb{Z}$ where $\lambda$ is the integral of $\omega$ over $\mathbb{CP}^1$. The second spectrum is the mean index spectrum $S_{\text{ind}}(\varphi)$ formed by the mean indices of capped one-periodic orbits. Furthermore, every point in each of the spectra is marked or labelled by an integer. For a non-degenerate pseudo-rotation $\varphi$, a point in either of the spectra is marked by $l \in \mathbb{Z}$ when the Conley–Zehnder index of the corresponding orbit is $2l - \nu$. Thus we have the marked spectra $\tilde{S}(H)$ and $\tilde{S}_{\text{ind}}(\varphi)$; these are functions $n + 2\mathbb{Z} \to \mathbb{R}$, for all integers of the same parity as $n$ occur as indices. This construction extends to the degenerate pseudo-rotations; see Section 3.1.

The first key fact we need is that for a pseudo-rotation $\varphi = \varphi_H$ the two spectra agree up to a factor and a shift:

$$\tilde{S}(H) = \frac{\lambda}{2(n + 1)} \tilde{S}_{\text{ind}}(\varphi) + \text{const},$$

where $\text{const}$ depends on $H$. Thus by adding a constant to $H$ we can ensure that

$$\tilde{S}(H) = \frac{\lambda}{2(n + 1)} \tilde{S}_{\text{ind}}(\varphi).$$

When $\varphi_H$ is a rotation and $H$ is a quadratic form, this is the condition that $\sum a_i = 0$. Let us denote the points labelled by $0, \ldots, n$ in $\tilde{S}_{\text{ind}}(\varphi)$ by $\Delta_0, \ldots, \Delta_n$. One
can show that $\sum \Delta_i = 0$ when $\varphi$ is a true rotation; see Lemma 5.1. In general, let us call a pseudo-rotation meeting this requirement balanced; see Definition 5.3. It was conjectured in [GK] that every pseudo-rotation is balanced. In Section 5.2 we prove this conjecture for strongly non-degenerate pseudo-rotations of $\mathbb{CP}^2$:

**Theorem 1.2.** Every strongly non-degenerate pseudo-rotation of $\mathbb{CP}^2$ is balanced and all its fixed points are elliptic.

For a rotation $\varphi$ the spectrum $S_{nd}(\varphi)$ completely determines the map $\varphi$ as an element of $\text{Ham}(\mathbb{CP}^n)$; see Section 5.1. Then, as is easy to see, for a balanced strongly non-degenerate pseudo-rotation $\varphi$, there exists a unique rotation $R_{\varphi}$, called the matching rotation, such that $S_{nd}(\varphi) = S_{nd}(R_{\varphi})$ and, moreover, $\dot{S}_{nd}(\varphi) = \dot{S}_{nd}(R_{\varphi})$; see Section 5.2. Thus we have a one-to-one correspondence between the capped one-periodic orbits of $\varphi$ and $R_{\varphi}$, where the corresponding orbits $\tilde{x}_i$ of $\varphi$ and $\tilde{y}_i$ of $R_{\varphi}$ have equal Conley–Zehnder indices, mean indices and, up to a shift, actions. It turns out that under some additional conditions the corresponding orbits $x_i$ and $y_i$ have equal spectra $\sigma(x_i)$ and $\sigma(y_i)$ of the linearized return maps. For instance, we have the following result proved in Section 5.2:

**Theorem 1.3.** Let $\varphi$ be a balanced strongly non-degenerate pseudo-rotation of $\mathbb{CP}^n$ and let $R_{\varphi}$ be its matching rotation. Assume that for every one-periodic orbit $y_i$ of $R_{\varphi}$ all unit eigenvalues at $y_i$ (i.e., the elements of $\sigma(y_i)$) are distinct and $\sigma(y_i) \cap \sigma(y_j) = \emptyset$ for any pair $i \neq j$. Then $\sigma(x_i) = \sigma(y_i)$.

As a consequence, under the conditions of the theorem, the fixed points of $\varphi$ are elliptic and all iterates $\varphi^k$ are balanced. Moreover, for $\mathbb{CP}^2$, the assertion of the theorem holds for any strongly non-degenerate pseudo-rotation without additional assumptions on the spectra $\sigma(y_i)$; see Theorem 5.7 and Corollary 5.8.

1.2.2. **Index Analysis.** The key new ingredient in the proofs of these results is essentially combinatorial. This is Theorem 4.1 (Index Divisibility Theorem) relating the behavior of the Conley–Zehnder index under iterations and the spectrum of a linear map. Although this result plays a purely technical role in the paper, the theorem and its proof are of independent interest. Referring the reader to Sections 4 and 6 for details, here we only briefly outline the underlying idea.

Consider an element $\Phi \in \tilde{Sp}(n)$ which we require to be strongly non-degenerate (i.e., all iterates $\Phi^k$, $k \in \mathbb{N}$, are non-degenerate). Then we have the sequence of the Conley–Zehnder indices $\mu_k := \mu(\Phi^k)$ defined; see, e.g., [Lo, SZ]. We denote by $\mu'_k$ be the derivative or the index jump sequence: $\mu'_k := \mu_{k+1} - \mu_k$. Furthermore, let us decompose $\Phi$ as the product of a loop $\phi$ and the direct sum of three short paths: an elliptic path $\Phi_e$, a positive hyperbolic path $\Phi_h$, and a negative hyperbolic path $\Phi_-$. The requirement that the paths are short makes this decomposition unique up to homotopy. Denote by $\sigma_+(\Phi)$ the part of the spectrum of $\Phi_e$ lying in the upper half plane and, for $\lambda \in \sigma_+(\Phi)$, let $\text{sgn}_N(\Phi)$ be its signature. Furthermore, set $\text{loop}(\Phi)$ to be the mean index of $\phi$ and $\text{mult}_{-1}$ to be half of the dimension of the domain of $\Phi_-$.  

**Theorem 1.4** (Index Divisibility). Fix $l \in \mathbb{N}$. The derivative sequence $\mu'_k$ is divisible by $2l$ if and only if the following two conditions are satisfied:

(i) $2l \mid (\text{loop}(\Phi) + \text{mult}_{-1}(\Phi))$,

(ii) $l \mid \text{sgn}_N(\Phi)$ for all $\lambda \in \sigma_+(\Phi)$. 


One consequence of the index divisibility theorem (Corollary 4.2) is that each of the sequences \( \mu_k \) and \( \mu'_k \) completely determines the spectrum \( \sigma_+ (\Phi) \) together with signatures except for the eigenvalues with zero signature and that the jump sequence determines the index sequence. The proof of the theorem, given in Section 6, is geometrical and relies on the properties of a certain cycle associated with \( \Phi \) in the torus \( T^r \), where \( r \) is the number of distinct elements in \( \sigma_+ (\Phi) \).

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2. Preliminaries

The goal of this section is to set notation and conventions, give a brief review of Hamiltonian Floer homology and several other notions from symplectic geometry used in the paper. The reader may consider consulting this section only as necessary.

2.1. Conventions and notation. Throughout the paper the underlying symplectic manifold \((M^{2m}, \omega)\) will be \(\mathbb{CP}^n\) equipped with the standard Fubini–Study symplectic form \(\omega\). It will be convenient however to vary the normalization of this form and so we set \(\lambda = \langle [\omega], \mathbb{CP}^1 \rangle\). In other words, \(\lambda\) is the positive generator of \(\langle [\omega], \pi_2 (M) \rangle \subset \mathbb{R}\), the rationality constant. For the standard Fubini–Study normalization \(\lambda = \pi\). Recall also that the minimal Chern number \(N\), i.e., the positive generator of \(\langle c_1 (TM), \pi_2 (M) \rangle\), is \(n + 1\) for \(\mathbb{CP}^n\).

All Hamiltonians \(H\) are assumed to be \(k\)-periodic in time, i.e., \(H: S^1_k \times M \to \mathbb{R}\), where \(S^1_k = \mathbb{R}/k\mathbb{Z}\) and \(k \in \mathbb{N}\). When the period \(k\) is not specified, it is equal to one and \(S^1 = S^1_1 = \mathbb{R}/\mathbb{Z}\). We set \(H_t = H(t, \cdot)\) for \(t \in S^1_k\). The Hamiltonian vector field \(X_H\) of \(H\) is defined by \(i_{X_H} \omega = -dH\). The (time-dependent) flow of \(X_H\) is denoted by \(\varphi_H^t\) and its time-one map by \(\varphi_H\). Such time-one maps are referred to as Hamiltonian diffeomorphisms. A one-periodic Hamiltonian \(H\) can always be treated as \(k\)-periodic, which we will then denote by \(H^{2k}\) and, abusing terminology, call \(H^{2k}\) the \(k\)th iteration of \(H\). Throughout the paper, all Hamiltonian diffeomorphisms are viewed as elements of the universal covering \(\widetilde{\text{Ham}}(\mathbb{CP}^n)\) of the group \(\text{Ham}(\mathbb{CP}^n)\) of Hamiltonian diffeomorphisms.

Let \(x: S^1_k \to M\) be a contractible loop. A capping of \(x\) is an equivalence class of maps \(A: D^2 \to M\) such that \(A |_{S^1} = x\). Two cappings \(A\) and \(A'\) of \(x\) are equivalent if the integrals of \(\omega\) (and hence of \(c_1 (TM)\)) over the sphere obtained by attaching \(A\) to \(A'\) are equal to zero. A capped closed curve \(\bar{x}\) is, by definition, a closed curve \(x\) equipped with an equivalence class of cappings. In what follows, the presence of capping is always indicated by a bar.

The action of a Hamiltonian \(H\) on a capped closed curve \(\bar{x} = (x, A)\) is

\[
A_H (\bar{x}) = - \int_A \omega + \int_{S^1} H_t (x(t)) \, dt.
\]

The space of capped closed curves is a covering space of the space of contractible loops, and the critical points of \(A_H\) on this space are exactly the capped one-periodic orbits of \(X_H\). The action spectrum \(S(H)\) of \(H\) is the set of critical values of \(A_H\). This is a zero measure set; see, e.g., [HZ].
These definitions extend to $k$-periodic orbits and Hamiltonians in an obvious way. Clearly, the action functional is homogeneous with respect to iteration:

$$A_{H^{(k)}}(\bar{x}^k) = kA_H(\bar{x}),$$

where $\bar{x}^k$ is the $k$th iteration of the capped orbit $\bar{x}$. We denote the set of $k$-periodic orbits of $H$ by $\mathcal{P}_k(H)$. The set of all periodic orbits will be denoted by $\mathcal{P}(H)$. An un-iterated periodic orbit is called simple.

A periodic orbit $x$ of $H$ is said to be non-degenerate if all its one-periodic orbits are non-degenerate and its index $\mu$ does not need to take a completion here. The Floer differential "counts" the gradient trajectories of the action functional and the resulting homology, the Floer homology $HF(H)$ of all its one-periodic orbits are non-degenerate and $H$ is strongly non-degenerate if all periodic orbits of $H$ (of all periods) are non-degenerate.

Let $\bar{x} = (x, A)$ be a non-degenerate capped periodic orbit. The Conley–Zehnder index $\mu(\bar{x}) \in \mathbb{Z}$ is defined, up to a sign, as in [Sa, SZ]. In this paper, we normalize $\mu$ so that $\mu(\bar{x}) = n$ when $x$ is a non-degenerate maximum (with trivial capping) of an autonomous Hamiltonian with small Hessian. The mean index $\bar{\mu}(\bar{x}) \in \mathbb{R}$ measures, roughly speaking, the total angle swept by certain unit eigenvalues of the linearized flow $D\varphi_H|_{\bar{x}}$ with respect to the trivialization associated with the capping; see [Lo, SZ]. The mean index is defined even when $x$ is degenerate and depends continuously on $H$ and $\bar{x}$ in the obvious sense. Furthermore,

$$|\bar{\mu}(\bar{x}) - \mu(\bar{x})| \leq n \tag{2.1}$$

and the inequality is strict when $x$ is non-degenerate or even weakly non-degenerate, i.e., at least one of the eigenvalues is different from one, [SZ]. The mean index is homogeneous with respect to iteration: $\bar{\mu}(\bar{x}^k) = k\bar{\mu}(\bar{x})$.

The index $\mu(x)$ of an un-capped orbit $x$ is well defined as an element of $\mathbb{Z}/2\mathbb{N}$. Likewise, the mean index $\bar{\mu}(x)$ is well defined as an element of $\mathbb{R}/2\mathbb{N}$.

2.2. Floer homology and spectral invariants. In this subsection, we very briefly discuss, mainly to set notation, spectral invariants and Floer homology. We refer the reader to, e.g., [GG09, MS, Sa, SZ] for detailed accounts and additional references.

Fix a ground field $\mathbb{F}$. For a non-degenerate Hamiltonian $H$ on $M$ we denote the filtered Floer complex of $H$ by $\mathcal{CF}^{(a, b)}_{m}(H)$, where $-\infty \leq a < b < \infty$ and $a$ and $b$ are not in $S(H)$. As a vector space, $\mathcal{CF}^{(a, b)}_{m}(H)$ is formed by finite linear combinations

$$\sigma = \sum_{\bar{x} \in \mathcal{P}(H)} \sigma_{\bar{x}} \bar{x},$$

where $\sigma_{\bar{x}} \in \mathbb{F}$ and $\mu(\bar{x}) = m$ and $a < A_H(\bar{x}) < b$. (Since $M$ is monotone we do not need to take a completion here.) The Floer differential "counts" the $L^2$-anti-gradient trajectories of the action functional and the resulting homology, the filtered Floer homology of $H$, is denoted by $HF^{(a, b)}(H)$ and by $HF_*(H)$ when $(a, b) = (-\infty, \infty)$. The degree of a class $\alpha \in HF^{(a, b)}_{m}(H)$ is denoted by $|\alpha|$.

The definition of the Floer homology extends by continuity to all, not necessarily non-degenerate, Hamiltonians. Namely, let $H$ be an arbitrary (one-periodic in time) Hamiltonian on $M$ and let the end-points $a$ and $b$ of the action interval be outside $S(H)$. By definition, we set

$$HF^{(a, b)}_*(H) = HF^{(a, b)}_*(\hat{H}),$$
where \( \tilde{H} \) is a non-degenerate, small perturbation of \( H \). It is easy to see that the right-hand side is independent of \( \tilde{H} \) once \( \tilde{H} \) is sufficiently close to \( H \).

The notion of \textit{local Floer homology} goes back to the original work of Floer and it has been revisited a number of times since then. Here we only briefly recall the definition following mainly [Gi, GG09, GG10] where the reader can find a much more thorough discussion and further references.

Let \( x \) be an isolated one-periodic orbit of a Hamiltonian \( H : S^1 \times M \to \mathbb{R} \). The local Floer homology \( HF(x) \) is the homology of the Floer complex generated by the orbits \( x \) splits into under a \( C^2 \)-small non-degenerate perturbation of \( H \) near \( x \). This homology is well-defined, i.e., independent of the perturbation. The homology \( HF(x) \) is only relatively graded and to fix an absolute grading one can pick a trivialization of \( TM \) along \( x \). This can be done by using, for instance, a capping of \( x \) and in this case we write \( HF_x(x) \). For instance, if \( x \) is non-degenerate and \( \mu_{cz}(\tilde{x}) = m \), we have \( HF_x(\tilde{x}) = \mathbb{F} \) when \( l = m \) and \( HF_x(\tilde{x}) = 0 \) otherwise. This construction is local: it requires \( H \) to be defined only in a neighborhood of \( x \).

The total Floer homology is independent of the Hamiltonian and, up to a shift of the grading and the effect of recapping, is isomorphic to the homology of \( M \). More precisely, we have

\[
HF_x(H) \cong H_{\ast+n}(M; \mathbb{F}) \otimes \Lambda
\]
as graded \( \Lambda \)-modules, where \( \Lambda \) is a suitably defined Novikov ring. For instance, let \( H \) be a Hamiltonian on \( \mathbb{C}P^n \). Then \( HF_n(H) = \mathbb{F} \) when \( n \) has the same parity as \( \Lambda \) and \( HF_m(H) = 0 \) otherwise. To see this, one can just take a non-degenerate quadratic Hamiltonian as \( H \) and observe that all fixed points of \( \varphi_H \) are elliptic and hence their indices have the same parity as \( n \). (In particular, the Floer differential vanishes.)

The machinery of Hamiltonian \textit{spectral invariants} was developed in its present Floer–theoretic form in [Oh, Sc], although the first versions of the theory go back to [HZ, Vi]. Action carriers were introduced in [GG09] and then studied in [CGG, GG12].

Let \( H \) be a Hamiltonian on a closed monotone (or even rational) symplectic manifold \( M^{2n} \). The \textit{spectral invariant} or \textit{action selector} \( c_\alpha \) associated with a class \( \alpha \in HF_x(H) \) is defined as

\[
c_\alpha(H) = \inf\{a \in \mathbb{R} \setminus S(H) \mid \alpha \in \text{im}(i^a)\} = \inf\{a \in \mathbb{R} \setminus S(H) \mid j^a(\alpha) = 0\},
\]
where \( i^a : HF_{(-\infty,a]}(H) \to HF_x(H) \) and \( j^a : HF_x(H) \to HF^{(a,\infty]}(H) \) are the natural “inclusion” and “quotient” maps. It is easy to see that \( c_\alpha(H) > -\infty \) when \( \alpha \neq 0 \) and one can show that \( c_\alpha(H) \in S(H) \). In other words, there exists a capped one-periodic orbit \( \tilde{x} \) of \( H \) such that \( c_\alpha(H) = A_H(\tilde{x}) \). As an immediate consequence of the definition,

\[
c_\alpha(H + a(t)) = c_\alpha(H) + \int_{S^1} a(t) \, dt,
\]
where \( a : S^1 \to \mathbb{R} \).

Spectral invariants have several important properties. For instance, the function \( c_\alpha \) is homotopy invariant: \( c_\alpha(H) = c_\alpha(K) \) when \( \varphi_H = \varphi_K \) in \( \text{Ham}(M) \) and \( H \) and \( K \) have the same mean value. Furthermore, it is sub-additive, monotone and Lipschitz in the \( C^0 \)-topology as a function of \( H \).
When $H$ is non-degenerate, the action selector can also be evaluated as

$$c_{\alpha}(H) = \inf_{|\sigma| = \alpha} c_{\sigma}(H),$$

where we set

$$c_{\sigma}(H) = \max \{ A_{H}(\bar{x}) \mid \sigma_{x} \neq 0 \} \text{ for } \sigma = \sum \sigma_{x} \bar{x} \in \text{CF}_{|\alpha|}(H).$$  \hspace{1cm} (2.2)

The infimum here is obviously attained, since $M$ is rational and thus $S(H)$ is closed. Hence there exists a cycle $\sigma = \sum \sigma_{x} \bar{x} \in \text{CF}_{|\alpha|}(H)$, representing the class $\alpha$, such that $c_{\alpha}(H) = A_{H}(\bar{x})$ for an orbit $\bar{x}$ entering $\sigma$. In other words, $\bar{x}$ maximizes the action on $\sigma$ and the cycle $\sigma$ minimizes the action over all cycles in the homology class $\alpha$. Such an orbit $\bar{x}$ is called a carrier of the action selector. This is a stronger requirement than just that $c_{\alpha}(H) = A_{H}(\bar{x})$ and $|\hat{\mu}(\bar{x}) - |\alpha|| \leq n$.

As consequence of the definition of the carrier and continuity of the action and the mean index, we have

$$c_{\alpha}(H) = A_{H}(\bar{x}) \text{ and } |\hat{\mu}(\bar{x}) - |\alpha|| \leq n,$$

and the inequality is strict when $x$ is weakly non-degenerate. Furthermore, a carrier $\bar{x}$ for $c_{\alpha}$ is in some sense homologically essential. Namely, $\text{HF}_{|\alpha|}(\bar{x}) \neq 0$, provided that all one-periodic orbits of $H$ are isolated; cf. ][GG12, Lemma 3.2].

3. Background results on pseudo-rotations

3.1. Action and index spectra. In this section, we briefly recall several symplectic topological results on pseudo-rotations of projective spaces, essential for our purposes. A much more detailed treatment can be found in ][GG18a, Sect. 3.

Let $\varphi = \varphi_{H}$ be a pseudo-rotation of $\mathbb{C}\mathbb{P}^{n}$, which we do not assume to be non-degenerate. Denote by $\alpha_{l}$ the generator in $\text{HF}_{2l-n}(H) = \mathbb{F}$, $l \in \mathbb{Z}$, and let $\bar{x}_{l} \in \mathcal{P}_{1}(H)$ be an action carrier for $\alpha_{l}$. We will write $c_{l} := c_{\alpha_{l}}$. Then, in particular,

$$c_{l}(H) = A_{H}(\bar{x}_{l}) \text{ and } \text{HF}_{2l-n}(\bar{x}_{l}) \neq 0,$$

and hence $\text{HF}(x) \neq 0$ for all $x \in \mathcal{P}(H)$.

**Theorem 3.1** (Lusternik–Schnirelmann Inequalities, [GG09]). For every $l \in \mathbb{Z}$ the action carrier $\bar{x}_{l}$ is unique and the resulting map

$$Z \to \mathcal{P}_{1}(H), \quad l \mapsto \bar{x}_{l}$$

is a bijection. Furthermore, the map

$$Z \to S(H), \quad l \mapsto c_{l}(H) = A_{H}(\bar{x}_{l})$$

is strictly monotone, i.e., $l > l'$ if and only if $A_{H}(\bar{x}_{l}) > A_{H}(\bar{x}_{l'})$. 
One important consequence of the theorem is that distinct capped one-periodic orbits of $\varphi_H$ necessarily have different actions.

The proof of Theorem 3.1 relies on a version of Lusternik–Schnirelmann theory for action selectors. The theorem allows us to extend the notion of the Conley–Zehnder index to capped one-periodic orbits $\bar{x}_l$ of degenerate pseudo-rotations by setting $\mu(\bar{x}) = 2l - n$ for $\bar{x} = \bar{x}_l$. We will call $\mu(\bar{x})$ the LS-index. When $\bar{x}$ is non-degenerate this is just the ordinary Conley–Zehnder index. However, it has many of the expected properties of the Conley–Zehnder index, e.g., $HF(\mu(\bar{x}) \neq 0$ and, when $n = 1$, $HF(\bar{x})$ is concentrated in only one degree which is $\mu(\bar{x})$, [GG10].

With this notion in mind, Theorem 3.1 can be rephrased as that the ordering of $\bar{P}_1(H)$ by the LS-index agrees with that by the action.

**Theorem 3.2** (Action–Index Resonance Relations, [GG09]). For every $\bar{x} \in \bar{P}_1(H)$, we have

$$A_H(\bar{x}) = \frac{\lambda}{2(n+1)} \hat{\mu}(\bar{x}) + \text{const},$$

where const is independent of $x$.

This theorem has been extended to some other symplectic manifolds and a broader class of Hamiltonian diffeomorphisms; see [CGG]. It also has an analog for Reeb flows, [GG16, Sect. 6.1.2].

The marked action spectrum $\hat{S}(H)$ is, by definition, the bijection

$$\hat{S} : \mathbb{Z} \xrightarrow{(3.1)} \bar{P}_1(H) \xrightarrow{(3.2)} S(H),$$

i.e., $\hat{S}(H)$ is simply the spectrum $S(H)$ with its points labelled by $\mathbb{Z}$ (essentially the indices) or, equivalently, by $\bar{P}_1(H)$. In a similar vein, the marked index spectrum $\hat{S}_{ind}(\varphi)$ is the map

$$\hat{S}_{ind} : \mathbb{Z} \xrightarrow{(3.1)} \bar{P}_1(H) \rightarrow S_{ind}(\varphi),$$

where $\varphi = \varphi_H$, which is also a bijection, and

$$S_{ind}(\varphi) = \{ \hat{\mu}(\bar{x}) \mid \bar{x} \in \bar{P}_1(H) \}$$

is the mean index spectrum of $H$ and the second arrow is the map $\bar{x} \mapsto \hat{\mu}(\bar{x})$. Then (3.3) can be rephrased as

$$\hat{S}(H) = \frac{\lambda}{2(n+1)} \hat{S}_{ind}(\varphi) + \text{const},$$

i.e., the action spectrum and the index spectrum agree up to a factor and a shift. The factor can be made equal 1 by scaling $\omega$, and the shift can be made zero by adding a constant to $H$. Then

$$\hat{S}(H) = \hat{S}_{ind}(\varphi).$$

Finally, we have a different type of resonance relations involving only the indices. To state the result, recall that for an un-capped one-periodic orbit $x \in P_1(H)$ the mean index is well defined modulo $2(n+1) = 2N$, i.e., $\hat{\mu}(x) \in S^1_{2N} = \mathbb{R}/2(n+1)\mathbb{Z}$. Let $x_0, \ldots, x_n$ be the fixed points of a pseudo-rotation $\varphi_H$ of $\mathbb{C}\mathbb{P}^n$. Then, as was shown in [GK], for some non-zero vector $\bar{r} = (r_0, \ldots, r_n) \in \mathbb{Z}^{n+1}$, we have

$$\sum r_i \hat{\mu}(x_i) = 0 \text{ in } \mathbb{R}/2(n+1)\mathbb{Z}. \quad (3.5)$$
In other words, the closed subgroup $\Gamma \subset \mathbb{T}^{n+1}$ topologically generated by the mean index vector

$$\bar{\Delta} = \bar{\Delta}(\varphi_H) := (\hat{\mu}(x_0), \ldots, \hat{\mu}(x_n)) \in \mathbb{T}^{n+1} = \mathbb{R}^{n+1}/2(n+1)\mathbb{Z}^{n+1}$$  \hspace{1cm} (3.6)

has positive codimension. Moreover, the codimension is equal to the number of linearly independent resonances, i.e., the rank of the subgroup $\mathcal{R} \subset \mathbb{Z}^{n+1}$ formed by all resonances $\bar{r}$; see [GK].

Clearly, $\mathcal{R}$ depends on $\varphi_H$. However, conjecturally, the resonance relation

$$\sum \hat{\mu}(x_i) = 0 \text{ in } \mathbb{R}/2(n+1)\mathbb{Z}$$  \hspace{1cm} (3.7)

is universal, i.e., satisfied for all pseudo-rotations. (Up to a factor this is the only possible universal resonance relation; for any other relation breaks down for a suitably chosen rotation; see Section 5.1.) For $\mathbb{CP}^1$ this conjecture is known to hold; see the discussion in Section 3.2 below. Theorem 1.2 establishes this conjecture for strongly non-degenerate pseudo-rotations of $\mathbb{CP}^2$.

### 3.2. Pseudo-rotations of $S^2$.

To illustrate our approach, we will use now the results quoted in Section 3.1 to study pseudo-rotations in dimension two.

**Proposition 3.3.** Every pseudo-rotation $\varphi$ of $S^2$ is strongly non-degenerate and its fixed points are elliptic. Furthermore, in the notation from Section 3.1,

$$\hat{\mu}(\bar{x}_0) + \hat{\mu}(\bar{x}_1) = 0.$$  \hspace{1cm} (3.8)

The resonance relation (3.8) asserts, roughly speaking, that $D\varphi$ rotates the tangent spaces at the fixed points by the same angle but in opposite directions. In particular, (3.7) is satisfied and there exists a unique (up to conjugation) true rotation $R_\varphi$ of $S^2$ such that $\bar{S}(\varphi) = \bar{S}(R_\varphi)$ and $\bar{S}_{\text{ind}}(\varphi) = \bar{S}_{\text{ind}}(R_\varphi)$. The first part of the proposition is a standard, although ultimately highly non-trivial, result in two-dimensional dynamics (see [Fr]) and (3.8) readily follows from the Poincaré–Birkhoff theorem, [Br15a, Appendix A.2]; see also [CKRTZ] for a different approach based on (3.5). The proof of this part given below is taken from [GG18a].

**Proof.** Arguing by contradiction, assume that $\varphi$ is a pseudo-rotation of $S^2$ and some iterate of $\varphi$ is degenerate or that one of its fixed points is hyperbolic. In dimension two, a hyperbolic or degenerate fixed point necessarily has integral mean index. Hence, replacing $\varphi$ by a sufficiently large iterate if necessary and using (3.5), we may assume that both fixed points $x_0$ and $x_1$ have mean index equal to zero modulo 4 = 2(n + 1). Let us scale the symplectic structure and adjust the Hamiltonian so that $\bar{S}(H) = \bar{S}_{\text{ind}}(\varphi)$. Then, by (3.3), for suitable cappings of $x_0$ and $x_1$ these orbits have equal actions, which is impossible by Theorem 3.1.

Let us now turn to (3.8). By Theorem 3.1, for any iteration $k$ the orbits $x_0^k$ and $x_1^k$ with any cappings have distinct Conley–Zehnder indices. In particular,

$$\mu(\bar{x}_1^k) - \mu(\bar{x}_0^k) \equiv 2 \pmod{4 = 2(n+1)}.$$  \hspace{1cm} (3.9)

Note also that $-2 < \hat{\mu}(\bar{x}_0) < 0$ and $2 > \hat{\mu}(\bar{x}_1) > 0$ since $\mu(\bar{x}_0) = -1$ and $\mu(\bar{x}_1) = 1$.

Therefore, $\mu(\bar{x}_1^{k+1}) - \mu(\bar{x}_0^k)$ is either 0 or 2 and $\mu(\bar{x}_0^{k+1}) - \mu(\bar{x}_0^k)$ is either 0 or -2. Combining these facts it is not hard to see that

$$\mu(\bar{x}_1^{k+1}) - \mu(\bar{x}_1^k) = -(\mu(\bar{x}_0^{k+1}) - \mu(\bar{x}_0^k)).$$
for all $k \in \mathbb{N}$. (Otherwise, at the first moment when one eigenvalue changes and the other does not, (3.9) is violated.) As a consequence,

$$\mu(\bar{x}_k) = -\mu(\bar{x}_0)$$

for all $k \in \mathbb{N}$. Since $\hat{\mu}(\bar{x}_i) = \lim_{k \to \infty} \mu(\bar{x}_k)/k$, we conclude that $\hat{\mu}(\bar{x}_1) = -\hat{\mu}(\bar{x}_0)$. □

4. INDEX DIVISIBILITY

In this section we state the main index theory result used in our comparison of true rotations and pseudo-rotations. Let $\Phi \in \tilde{Sp}(2m)$ be a path parametrized by $[0,1]$, starting at the identity and taken up to homotopy with fixed endpoints. Denote by $\sigma(\Phi)$ the unit spectrum (the collection of unit eigenvalues) of $\Phi(1)$. Throughout this section we assume that $\Phi$ is strongly non-degenerate, i.e., all iterates $\Phi^k$ are non-degenerate or, equivalently, that $\sigma(\Phi)$ does not contain any root of unity. We denote the part of $\sigma(\Phi)$ lying in the upper half circle by $\sigma^+(\Phi)$ and refer to it as the positive unit spectrum.

Our goal is to associate to $\Phi$ several numerical invariants which determine its index behavior under iterations. For the sake of simplicity let us first assume that all unit eigenvalues of $\Phi$ are semi-simple although not necessarily distinct. Let us write $\Phi$ as a product of a loop $\phi$ with the direct sum $\Phi_h \oplus \Phi_{-h} \oplus \Phi_e$. Here $\Phi_h$ is hyperbolic with complex or positive real eigenvalues and zero mean index. The second term $\Phi_{-h}$ is hyperbolic with negative real eigenvalues. As an element of the universal covering it is specified by that its mean index is equal to half of the dimension of its domain or equivalently by that it is connected to $I$ by the counterclockwise rotation by $\pi$ and a hyperbolic transformation. We set $\text{mult}_{-1}(\Phi) := \hat{\mu}(\Phi_{-h})$ and $\text{loop}(\Phi) := \hat{\mu}(\phi)$.

The remaining term $\Phi_e$ is elliptic. Up to conjugation, it decomposes as a direct sum of “short” rotations $R_{\theta}$ in an angle $\pi \theta \in (-\pi, \pi)$. Thus, in particular, $\sigma(\Phi) = \sigma(\phi_e)$ is the collection of eigenvalues $\exp(\pm \pi \sqrt{-1} \theta)$. For a rotation $R_\theta$ the eigenvalue $\exp(\pi \sqrt{-1} \theta)$ is said to be of the first (Krein) type. In other words, for a counter-clock-wise rotation we pick up the eigenvalue in the upper half-plane as the first Krein type, and the eigenvalue in the lower half-plane for a clock-wise rotation.

The signature $\text{sgn}_\lambda(\Phi)$ of $\lambda \in \sigma_+(\Phi)$ is by definition the difference $p - q$, where $p$ is the number of times $\lambda$ enters $\sigma(\Phi)$ as the first type eigenvalue and $q$ is the number of times $\bar{\lambda}$ occurs as the first type eigenvalue. (This is indeed the Krein signature of the corresponding complex eigenspaces. It is convenient to define $\text{sgn}_\lambda(\Phi)$ for all $\lambda \in S^1$ by setting $\text{sgn}_\lambda(\Phi) := \text{sgn}_\lambda(\Phi)$ (no negative sign!) when $\lambda \in \sigma(\Phi) \setminus \sigma_+(\Phi)$ and $\text{sgn}_\lambda(\Phi) = 0$ when $\lambda \notin \sigma(\Phi)$. Thus we can think of $\lambda \mapsto \text{sgn}_\lambda(\Phi)$ as a function on $S^1$ which depends on $\Phi$.)

It is not hard to extend these definitions to transformations $\Phi$ with not necessarily semi-simple eigenvalues. To this end, we may, for instance, connect $\Phi$ to a transformation $\Phi'$ with all unit eigenvalues semi-simple and the same spectrum as $\Phi$ via a family of isospectral transformations, i.e., a family of transformations with constant spectrum. We set $\text{mult}_{-1}(\Phi) := \text{mult}_{-1}(\Phi')$ and $\text{loop}(\Phi) := \text{loop}(\Phi')$ and $\text{sgn}_\lambda(\Phi) := \text{sgn}_\lambda(\Phi')$. Then it is easy to show that these invariants are independent of the choice of $\Phi'$. 

Furthermore, it is clear that these invariants are additive with respect to direct sum and that \( \text{loop} + \text{mult} - 1 \) and \( sgn_\lambda \) change sign when the transformation is replaced by its inverse.

Essentially by definition,

\[
\mu(\Phi) = \text{loop}(\Phi) + \text{mult} - 1(\Phi) + \sum_{\lambda \in \sigma_+(\Phi)} sgn_\lambda(\Phi)
\]

and

\[
\hat{\mu}(\Phi) = \text{loop}(\Phi) + \text{mult} - 1(\Phi) + \sum_{\lambda \in \sigma_+(\Phi)} sgn_\lambda(\Phi)\theta,
\]

where \( \lambda = \exp(\pi\sqrt{-1})^\theta \) with \( \theta \in (0, 1) \).

Our key combinatorial result is the following theorem proved in Section 6.

**Theorem 4.1 (Index Divisibility).** Fix \( l \in \mathbb{N} \). The following two conditions are equivalent:

(a) \( 2l | (\mu(\Phi^{k+1}) - \mu(\Phi^k)) \) for all \( k \in \mathbb{N} \);

(b) the following two divisibility requirements are met:

(i) \( 2l | (\text{loop}(\Phi) + \text{mult} - 1(\Phi)) \),

(ii) \( l | sgn_\lambda(\Phi) \) for all \( \lambda \in \sigma(\Phi) \).

This is Theorem 1.4 from the introduction. Note that Condition (a) is satisfied whenever \( 2l | \mu(\Phi^k) \) for all \( k \in \mathbb{N} \). The latter requirement is stronger than and not equivalent to (a) or (b) as simple examples show. (By (4.1) and (6.1), what follows from (a) or (b) is only that \( l | \mu(\Phi^k) \).) However, we can infer from the theorem that the sequence of indices or the sequence of index jumps determines the eigenvalues with signature, except for the zero signature eigenvalues:

**Corollary 4.2.** Let \( \Phi \) and \( \Psi \) be strongly non-degenerate. Then the following three conditions are equivalent:

(a) \( \mu(\Phi^k) = \mu(\Psi^k) \) for all \( k \in \mathbb{N} \);

(b) \( \mu(\Phi^{k+1}) - \mu(\Phi^k) = \mu(\Psi^{k+1}) - \mu(\Psi^k) \) for all \( k \in \mathbb{N} \);

(c) \( sgn_\lambda(\Phi) = sgn_\lambda(\Psi) \) for all \( \lambda \) and \( \text{loop}(\Phi) + \text{mult} - 1(\Phi) = \text{loop}(\Psi) + \text{mult} - 1(\Psi) \).

**Remark 4.3.** In general, neither the index sequence \( \mu(\Phi^k) \) nor the jump sequence \( \mu(\Phi^{k+1}) - \mu(\Phi^k) \) completely determines \( \sigma(\Phi) \). For instance, the zero signature eigenvalues are not detected in Theorem 4.1 and Corollary 4.2.

**Proof of Corollary 4.2.** Clearly, (a) implies (b). Assume that (b) holds. Then

\[
\mu((\Phi \oplus \Psi^{-1})^{k+1}) - \mu((\Phi \oplus \Psi^{-1})^k) = 0
\]

and thus, by Theorem 4.1 applied to this transformation,

\[
sgn_\lambda((\Phi \oplus \Psi^{-1})^k) = 0
\]

for all \( \lambda \) and

\[
\text{loop}((\Phi \oplus \Psi^{-1})^k) + \text{mult} - 1((\Phi \oplus \Psi^{-1})^k) = 0.
\]

Here we are using the fact that 0 is the only integer divisible by infinitely many integers. Now, by additivity, we see that the signatures and \( \text{loop} + \text{mult} - 1 \) for \( \Phi \) and \( \Psi \) are equal.

To prove that (c) implies (a) we need to show that \( \text{loop}(\Phi) + \text{mult} - 1(\Phi) \) and the signatures for \( \Phi \) determine these invariants for all iterations \( \Phi^k \). This is easy to prove directly. Alternatively, by the theorem, the \( \text{loop}(\Phi) + \text{mult} - 1(\Phi) \) and the
signatures for $\Phi$ determine the jump sequence $\mu(\Phi^{k+1}) - \mu(\Phi^k)$ and also, by (4.1), the initial condition $\mu(\Phi)$. Hence they also determine the sequence $\mu(\Phi^k)$. \hfill \Box

5. Pseudo-rotations vs. rotations

5.1. True rotations. Consider a true rotation of $\mathbb{C}P^n$, i.e., a Hamiltonian diffeomorphism $\phi_Q$ of $\mathbb{C}P^n$ generated by a quadratic Hamiltonian

$$Q(z) = \sum_{i=0}^{n} a_i |z_i|^2,$$

(5.1)

where we have identified $\mathbb{C}P^n$ with the quotient of the unit sphere $S^{2n+1} \subset \mathbb{C}^{n+1}$ and renormalized the standard symplectic form $\omega$ on $\mathbb{C}P^n$ so that

$$\int_{\mathbb{C}P^n} \omega = 1.$$

(For the standard normalization this integral is $\pi$.)

Most of the time it will be convenient to order the eigenvalues $a_i$ of $Q$ so that

$$a_0 \leq \ldots \leq a_n.$$  

(5.2)

Furthermore, since the Hamiltonian $\sum |z_i|^2$ reduces to a constant Hamiltonian on $\mathbb{C}P^n$, we can assume without loss of generality that

$$\sum a_i = 0,$$

(5.3)

which is equivalent to that $Q$ is normalized, i.e.,

$$\int_{\mathbb{C}P^n} Q \omega^n = 0.$$

Finally, $\phi_Q$ is non-degenerate if and only if $a_i - a_j \not\in \mathbb{Z}$ and strongly non-degenerate if and only if $a_i - a_j \not\in \mathbb{Q}$. Among the periodic orbits of $\phi_Q$ are the coordinate axes $x_0, \ldots, x_n$, all of which are one-periodic, and these are the only periodic orbits when $\phi_Q$ is strongly non-degenerate. Thus $\phi_Q$ is then a pseudo-rotation. We will assume this to be the case from now on unless specifically stated otherwise.

The first type eigenvalues of $D\phi_Q$ at $x_i$ are $\exp\left(2\sqrt{-1}(a_i - a_j)\right)$ where $j \neq i$. Viewing $\phi_Q$ as an element in the universal covering $\tilde{\text{Ham}}(\mathbb{C}P^n, \omega)$ generated by the flow of $Q$, we have the mean index of $\phi_Q$ at $x_i$ defined once $x_i$ is equipped with a capping. The orbit $x_i$ is constant and, in particular, it can be given a trivial capping. We denote the resulting trivially capped orbit by $\hat{x}_i$. It is easy to see that

$$\hat{\mu}(\hat{x}_i) = 2 \sum_{j \neq i} (a_i - a_j) = -2 \sum_j a_j + 2(n+1)a_i = 2(n+1)a_i,$$

(5.4)

where in the last equality we used (5.3). With $a_i$ arranged in an increasing order as in (5.2), the Conley–Zehnder index of $\hat{x}_i$ is $-n+2i$ when $Q$ is small. Without the latter requirement, $\mu(\hat{x}_i)$ can be any integer of the same parity as $n$. The action of $\hat{x}_i$ is $a_i$, and hence we have

$$S(Q) = \prod_i (a_i + \mathbb{Z}),$$

(5.5)

where as above we assumed $\phi_Q$ to be non-degenerate.

For our purposes, however, it is more useful to cap $x_i$ so that its Conley–Zehnder index is in the range from $-n$ to $n$ even when $Q$ is large. There exists exactly one such capping of $x_i$ and, as in Section 3.1, we denote the resulting capped orbit by
\[ \phi \] is an isomorphism. Hence the loop can evaluate the mean index at any of the fixed points. The composition \( \tilde{\square} \) in the mean index of a loop modulo \( \mu \) for all \( i \) values. (The argument goes back to Weinstein, \([\text{Sc}]\).)

**Lemma 5.1.** We have
\[ \sum \mu(\tilde{x}_i) = 0. \]

**Proof.** The total recapping from the orbits \( \tilde{x}_i \) to the orbits \( x_i \) is
\[ \sum \mu(\tilde{x}_i) - \sum \mu(\tilde{x}_i) = \sum \mu(x_i) - \sum \mu(x_i), \]
where we identified \( \pi_2(\mathbb{CP}^n) \) with \( 2(n+1)\mathbb{Z} \) via \( 2c_1(T\mathbb{CP}^n) \). By (5.3) and (5.4), we have
\[ \sum \mu(\tilde{x}_i) = 0, \]
and hence
\[ \sum \mu(\tilde{x}_i) = \sum \mu(x_i) - \sum \mu(x_i), \]
The first sum on the right-hand side is zero. For, after if necessary rearranging the terms,
\[ \sum \mu(\tilde{x}_i) = -n + (-n+2) + \cdots + (n-2) + n = 0. \]
We also have
\[ \sum \mu(x_i) = 0. \]

To see this, note that this sum is the Conley–Zehnder index of the path
\[ \Phi(t) = \bigoplus D\varphi^t|_{x_i}, \quad t \in [0,1], \]
in \( \text{Sp} \left(2n(n+1)\right) \). The first type eigenvalues of \( \Phi \) break down into complex conjugate pairs \( \exp(2\sqrt{-1}(a_i - a_j)) \) where \( j \neq i \) and the eigenvalues within each pair have the same multiplicity. Hence \( \mu(\Phi) = 0 \), which concludes the proof of the lemma. \( \square \)

A rotation \( \varphi_Q \), viewed as an element of \( \widetilde{\text{Ham}}(\mathbb{CP}^n, \omega) \) and not necessarily non-degenerate, lies in \( \text{SU}(n+1) = \tilde{\text{PU}}(n+1) \) and conversely every element of \( \text{SU}(n+1) \) can be generated by a diagonal quadratic Hamiltonian \( Q \) as in (5.1) for a suitable choice of coordinates. Let us next examine the question when \( \varphi_Q \) is trivial as an element of this universal covering, i.e., \( \varphi_Q = \text{id} \) in \( \widetilde{\text{Ham}}(\mathbb{CP}^n, \omega) \).

Clearly, \( \varphi_Q \) is trivial if and only if it is a contractible loop. For this, first of all, the path \( \varphi_Q^t, t \in [0,1], \) must be a loop, which is equivalent to that \( a_i - a_j \in \mathbb{Z} \) for all \( i \) and \( j \). Next, this loop must be contractible, i.e., it must represent the zero class in \( \pi_1(\tilde{\text{Ham}}(\mathbb{CP}^n, \omega)) \). Note that we have the “Maslov index” homomorphism \( \pi_1(\tilde{\text{Ham}}(\mathbb{CP}^n, \omega)) \to \mathbb{Z}_{n+1} = \mathbb{Z}/(n+1)\mathbb{Z} \) given by the evaluation of one-half of the mean index of a loop modulo \( n+1 \) on any orbit. In particular, for the loop \( \varphi_Q \) we can evaluate the mean index at any of the fixed points. The composition
\[ \mathbb{Z}_{n+1} = \pi_1(\tilde{\text{PU}}(n+1)) \to \pi_1(\tilde{\text{Ham}}(\mathbb{CP}^n, \omega)) \to \mathbb{Z}_{n+1} \]
is an isomorphism. Hence the loop \( \varphi_Q \), which actually lies in \( \tilde{\text{PU}}(n+1) \), is trivial in \( \tilde{\text{Ham}}(\mathbb{CP}^n, \omega) \) if and only if it is trivial in \( \tilde{\text{PU}}(n+1) \) and if and only if \( a_i - a_j \in \mathbb{Z} \) for all \( i \) and \( j \) and \( (n+1)a_i \in \mathbb{Z} \) for one eigenvalue \( a_i \) or equivalently for all eigenvalues. (The argument goes back to Weinstein, \([\text{We}]\), and comprises a linear algebra counterpart of the Seidel representation, \([\text{Sc}]\). This observation, of course, readily translates into a criterion in terms of the eigenvalues for two simultaneously diagonalizable quadratic Hamiltonians to generate the same rotation. For our purposes, however, the following condition expressed via action spectra is more useful.
Lemma 5.2. Two non-degenerate rotations $\varphi_Q$ and $\varphi_{Q'}$ with simultaneously diagonalizable $Q$ and $Q'$ are equal as elements of $\tilde{\text{Ham}}(\mathbb{C}P^n, \omega)$ if and only if they have the same action spectrum:

$$S(Q) = S(Q').$$

Proof. The action spectrum is completely determined by an element of $\tilde{\text{Ham}}(\mathbb{C}P^n)$ and hence we only need to show that two rotations with equal action spectra are the same.

Let $a_i$ and $a'_i$ be the eigenvalues of $Q$ and, respectively, $Q'$ normalized to satisfy (5.2) and (5.3). By (5.5), we have

$$\prod_i a_i + Z = \prod_i a'_i + Z =: S.$$

Our goal is to show that $a_i = a'_i$ for all $i$. The actions $a_i$ or $a'_i$ for $i = 0, \ldots, n$ are $n + 1$ consequent points in $S$ with sum equal to zero. (This follows from the fact that the ordering of $S$ by the Conley–Zehnder index agrees with the ordering of $S$ by the action, i.e., as a subset of $\mathbb{R}$; see Theorem 3.1.) There is at most one way to pick up such $n + 1$ consequent points, and hence $a_i = a'_i$. \hfill \Box

5.2. Pseudo-rotations. Consider a non-degenerate pseudo-rotation $\varphi$ of $\mathbb{C}P^n$ and let, as in Section 3.1, $\bar{x}_0, \ldots, \bar{x}_n$ be its fixed points uniquely capped so that $|\mu(\bar{x}_i)| \leq n$. Furthermore, it will often be convenient to order the fixed points by requiring the Conley–Zehnder index (or equivalently the action) to increase

$$\mu(\bar{x}_0) = -n, \mu(\bar{x}_1) = -n + 2, \ldots, \mu(\bar{x}_n) = n.$$

Definition 5.3. A non-degenerate pseudo-rotation $\varphi$ is balanced if

$$\sum_i \hat{\mu}(\bar{x}_i) = 0.$$  \hspace{1cm} (5.6)

Example 5.4. By Lemma 5.1, every strongly non-degenerate (true) rotation is balanced.

Remark 5.5. Replacing the Conley–Zehnder index by the LS-index (see Section 3.1) we can extend this definition to all, not necessarily non-degenerate, pseudo-rotations.

Note that the condition that $\varphi$ is balanced does not automatically imply that all iterates $\varphi^k$ are balanced. However, as is easy to see, these iterates are balanced modulo $2(n + 1)$, i.e., (3.7) which is a slightly weaker condition holds. Hypothetically, every pseudo-rotation is balanced at least under suitable non-degeneracy conditions. Then (3.7) is also satisfied and thus (3.7) would indeed be a universal resonance relation; see Section 3. By Proposition 3.3 or the Poincaré–Birkhoff theorem (cf. [Br15a, Appendix A.2]), every pseudo-rotation of $S^2$ is balanced. Furthermore, this conjecture, which in a somewhat different form is already stated in [GK], is supported by the following result (Theorem 1.2 from the introduction):

Theorem 5.6. Every strongly non-degenerate pseudo-rotation of $\mathbb{C}P^2$ is balanced and all its fixed points are elliptic.

Proof. Let $\varphi: \mathbb{C}P^2 \to \mathbb{C}P^2$ be a strongly non-degenerate pseudo-rotation. As above, we denote its capped fixed points with Conley–Zehnder indices $-2, 0$ and $2$ by, respectively, $\bar{x}_0, \bar{x}_1$ and $\bar{x}_2$. 


To see that all three fixed points of $\varphi$ are necessarily elliptic, assume otherwise. Then, since the fixed points have even indices and the dimension is four, one (or more) fixed point is hyperbolic. However, in this case, $\varphi$ must have infinitely many periodic orbits; see [GG14] or [GG18a].

The key to the proof of the theorem is the fact that by Theorem 3.1 for all iterations $k \in \mathbb{N}$ the orbits $x^k_i$ have distinct even indices modulo $2(n + 1) = 6$. (Recall that the indices modulo $2(n + 1)$ are well-defined without capping.)

As in the proof of Lemma 5.1, consider

$$\Phi(t) = \bigoplus_i D\varphi^t|_{\bar{x}_i}, \quad t \in [0, 1],$$

which we will treat as an element of $\tilde{\text{Sp}}(2n(n + 1)) = \tilde{\text{Sp}}(12)$. Note that here, as the notation indicates, we use the capping of $\bar{x}_i$ to turn $D\varphi^t|x_i$ into an element of $\tilde{\text{Sp}}(2n)$. Clearly,

$$\mu(\Phi) = \sum \mu(\bar{x}_i) = 0 \quad (5.7)$$

and

$$\mu(\Phi^k) = \sum \mu(\bar{x}^k_i) = 0 \quad \text{mod} \ 2(n + 1) = 6$$

due to the fact that for every $k$ the indices $\mu(\bar{x}^k_i)$ assume only the values $-2, 0$ and $2$ modulo $6$, and are all distinct.

Furthermore, $\text{mult}_{-1}(\Phi) = 0$ since the orbits $x_i$, and hence the transformation $\Phi$, are elliptic. Applying Theorem 4.1 to $\Phi$, we see that $6 \mid \text{loop}(\Phi)$ and $3 \mid \text{sgn}_\lambda(\Phi)$ for all $\lambda \in \sigma_+(\Phi)$. Since $\sigma_+(\Phi)$ comprises exactly six eigenvalues counting with multiplicity, there are now three possibilities:

Case 0: there are three eigenvalues (not necessarily distinct) in $\sigma_+(\Phi)$ and all eigenvalues have multiplicity 2 and signature 0;

Case 1: all eigenvalues in $\sigma_+(\Phi)$ are equal and the signature is $\pm 6$;

Case 2: there are two distinct eigenvalues in $\sigma_+(\Phi)$ and their signatures are $\pm 3$.

In Case 0, we also have $\text{loop}(\Phi) = 0$ due to (4.1) and (5.7). Therefore, by (4.2),

$$\sum \hat{\mu}(\bar{x}_i) = \hat{\mu}(\Phi) = \text{loop}(\Phi) = 0,$$

and the proof is finished.

To rule out Case 1, first observe that by passing to an iteration of $\varphi$ we may assume that the only eigenvalue in $\sigma_+(\Phi)$ is arbitrarily close to 1. (It suffices to ensure that the distance in $S^1$ from the eigenvalue to 1 is less than $\pi/6$.) Then it is easy to see that the iterated orbits $x^6_i$ have all index 2 modulo 6 if the signature is 6 and index -2 modulo 6 if the signature is -6. Either case contradicts the fact that the indices are distinct modulo 6.

The goal of the rest of the proof is to rule out Case 2. Denote the distinct eigenvalues in $\sigma_+(\Phi)$ by $\lambda$ and $\eta$. By passing to an iteration we may again assume that $\lambda$ and $\eta$ are arbitrarily close to 1. When these eigenvalues have the same signature, either 3 or -3, we arrive at a contradiction exactly as in Case 1. Namely, then all three points $x^6_i$ have the same index modulo 6. (The index is 2 if the signature is positive and -2 if the signature is negative.)

Thus the remaining case is when one eigenvalue, $\lambda$, has signature 3 and the other, $\eta$, has signature -3. The fixed points of $\varphi$ have Floquet multipliers, i.e., the first type eigenvalues, $(\bar{\eta}, \bar{\eta}), (\bar{\eta}, \lambda)$ and $(\lambda, \lambda)$. Let us denote the fixed points, in exactly
that order, by \(y_0, y_1\) and \(y_2\). Note that this labeling of the fixed points might differ from \(x_i\) which are labelled according to their indices.

Consider the subgroup \(\Gamma\) topologically generated by the point \((\lambda, \bar{\eta}) \in S^1 \times S^1 = T^2\). Let \((\zeta, \bar{\zeta})\) be the point closest to 1 in the intersection of \(\Gamma\) with the diagonal \(\Delta\) in \(T^2\). Clearly, \(\Gamma \cap \Delta\) is a cyclic subgroup of \(\Delta\) and hence \(\zeta = \exp (2\pi \sqrt{-1}/q)\) for some \(q \in \mathbb{N}\). (Arguing as in Case 1 one can show that \(q \leq 12\), but we do not need this fact.)

Since the positive semi-orbit is dense in the group, \(\Gamma\) is the closure of the set \(\{(\lambda^k, \bar{\eta}^k) \mid k \in \mathbb{N}\}\). By passing to an iteration \(\varphi^k\) and re-denoting \(\lambda^k\) by \(\lambda\) and \(\eta^k\) by \(\eta\), we can assume that \(\lambda\) and \(\bar{\eta}\) are arbitrarily close to \(\zeta\). (It is enough to ensure that the distance from \(\lambda\) and \(\bar{\eta}\) to \(\zeta\) is less than \(\pi/6q\).)

Next, we claim that \(\zeta\) lies in the short arc connecting \(\lambda\) and \(\bar{\eta}\). Indeed, assume otherwise: \(\lambda\) and \(\bar{\eta}\) are on the same side of \(\zeta\). Then after iterating \(q\) times, we see that \(\lambda^q\) and \(\eta^q\) have the same signature and after passing again to the 6th iteration of the resulting map, the three fixed points of \(\varphi^{6q}\) have the same index modulo 6, which is impossible.

Thus \(\zeta\) is between \(\bar{\eta}\) and \(\lambda\) and, in what follows, we will assume that \(\bar{\eta}\) is closer to 1. (The other case is handled similarly.) There are six ways to assign indices -2, 0 and 2 modulo 6 to the fixed points \(y_i\). For instance, all three points can have zero loop part and then \(\mu(y_0) = -2, \mu(y_1) = 0\) and \(\mu(y_2) = 2\) modulo 6; or, in a different scenario, \(y_0\) can have loop part 2 and index 0, \(y_1\) can have loop part -2 and index -2, and \(y_2\) can have loop part 0 and index 2. Then, as a direct calculation shows, in each of the six cases after \(\varphi\) is iterated \(3\eta\) times all three points \(\bar{y}_i^{3\eta}\) have the same index 2 (or -2 depending on the index assignment) modulo 6, which is again impossible.

One important feature of balanced pseudo-rotations \(\varphi = \varphi_H\) is that the mean-index spectrum \(S_{\text{ind}}(\varphi)\) (or equivalently the action spectrum) completely determines the marked spectrum \(\hat{S}_{\text{ind}}(\varphi)\); see Section 3.1. This is an immediate consequence of the observation that \(S_{\text{ind}}(\varphi)\) contains at most one collection of \(n + 1\) consecutive points \(a_0, \ldots, a_n\) with \(\sum a_i = 0\). These points are then assigned indices \(-n, -n + 2, \ldots, n\) and the rest of the spectrum is labelled accordingly.

Let \(\varphi = \varphi_H\) be a balanced strongly non-degenerate pseudo-rotation of \(\mathbb{CP}^n\) and let, as above, \(\bar{x}_0, \ldots, \bar{x}_n\) be its fixed points capped so that \(\mu(\bar{x}_i) = 2i - n\). Then there exists a unique true rotation \(R_{\varphi}\), called the matching rotation, such that

\[
S_{\text{ind}}(R_{\varphi}) = S_{\text{ind}}(\varphi)
\]

or, equivalently,

\[
S(Q) = S(H)
\]

up to a shift, where \(Q\) is the quadratic form generating \(R_{\varphi}\). Then, since for both \(R_{\varphi}\) and \(\varphi\) the index spectrum determines the marked spectrum,

\[
\hat{S}_{\text{ind}}(R_{\varphi}) = \hat{S}_{\text{ind}}(\varphi).
\]

The rotation \(R_{\varphi}\) can be given, for instance, by the Hamiltonian

\[
Q(z) = \sum \mu(\bar{x}_i)|z_i|^2.
\]

The uniqueness follows from Lemma 5.2. Denoting by \(\bar{y}_i\) the capped fixed points of \(R_{\varphi}\) with cappings again chosen so that \(\mu(\bar{y}_i) = 2i - n\), for all \(i = 0, \ldots, n\) we have

\[
\hat{\mu}(\bar{y}_i) = \hat{\mu}(\bar{x}_i).
\]
Our next two results show that in many instances the points $x_i$ and $y_i$, have, roughly speaking, the same Floquet multipliers. To state the results it is convenient to introduce the notion of the decorated spectrum $\hat{\sigma}(\Phi)$ of an element $\Phi \in \check{S}p(2n)$. Namely, this is the collection of pairs $(\lambda, \text{sgn}_\lambda(\Phi))$ where $\lambda \in \sigma_+(\Phi)$ and $\text{sgn}_\lambda(\Phi) \neq 0$. (Thus the eigenvalues with zero signature do not register in the decorated spectrum, and Corollary 4.2 can be rephrased as that the sequence of indices $\mu(\Phi^k)$ determines $\hat{\sigma}(\Phi)$.) For a capped one-periodic orbit $\bar{x}$, we set $\hat{\sigma}(\bar{x}) := \hat{\sigma}(D\varphi^1|_{\bar{x}})$, where as always we used the capping of $\bar{x}$ to turn the linearized flow into an element of $\check{S}p(2n)$. Likewise, set $\text{loop}(\bar{x}) := \text{loop}(D\varphi^1|_{\bar{x}})$.

**Theorem 5.7.** Let $\varphi$ be a strongly non-degenerate pseudo-rotation of $\mathbb{C}P^n$ such that all pseudo-rotations $\varphi^k$, $k \in \mathbb{N}$, are balanced and let $R_\varphi$ be its matching rotation. Then, in the above notation, for all $i = 0, \ldots, n$

$$\hat{\sigma}(\bar{x}_i) = \hat{\sigma}(\bar{y}_i) \text{ and } \text{loop}(\bar{x}_i) = \text{loop}(\bar{y}_i). \quad (5.8)$$

This theorem together with Theorem 5.6 yields

**Corollary 5.8.** Let $\varphi$ be a strongly non-degenerate pseudo-rotation of $\mathbb{C}P^2$ and let $R_\varphi$ be its matching rotation. Then (5.8) holds.

**Proof of Theorem 5.7.** First, observe that

$$S_{\text{ind}}(\varphi^k) = S_{\text{ind}}(R_\varphi^k) \quad (5.9)$$

since the left-hand side is determined by $S_{\text{ind}}(\varphi)$ and the right-hand side by $S_{\text{ind}}(R_\varphi)$. (Note that this does not automatically imply that $\varphi^k$ is balanced but only that it is “balanced modulo $2(n + 1)$”. ) Combining this with the condition that the iterates $\varphi^k$ are balanced, we see that

$$S_{\text{ind}}(\varphi^k) = S_{\text{ind}}(R_\varphi^k). \quad (5.10)$$

As a consequence, $\mu(x_k^i) = \mu(y_k^i)$ for all $k \in \mathbb{N}$, and (5.8) now follows from Corollary 4.2. (For the equality $\text{loop}(\bar{x}_i) = \text{loop}(\bar{y}_i)$ we also need to use the fact that $-1$ is neither in $\sigma(x_i)$ nor in $\sigma(y_i)$ due to non-degeneracy.)

The requirement that all iterations $\varphi^k$ are balanced can be eliminated if we assume that all Floquet multipliers are distinct. Namely, we have the following result.

**Theorem 5.9.** Let $\varphi$ be a balanced strongly non-degenerate pseudo-rotation and let $R_\varphi$ be its matching rotation. Assume that for every one-periodic orbit $y_i$ of $R_\varphi$ all unit eigenvalues at $y_i$ (i.e., the elements of $\sigma(y_i)$) are distinct and $\sigma(y_i) \cap \sigma(y_j) = \emptyset$ for any pair $i \neq j$. Then (5.8) holds and, in particular, $\sigma(x_i) = \sigma(y_i)$.

This is Theorem 1.3 from the introduction.

**Remark 5.10.** The spectra $\sigma(y_i)$ are determined by the collection of the mean indices $\hat{\mu}(y_i) = \hat{\mu}(x_j)$; see Section 5.1. Namely, the assumption of the theorem can be explicitly rephrased as that all eigenvalues $\exp\left(2\pi \sqrt{-1}(\hat{\mu}(x_i) - \hat{\mu}(x_j))/2(n + 1)\right)$, where $i \neq j$, are distinct.

**Corollary 5.11.** Under the conditions of the theorem, the fixed points of $\varphi$ are elliptic and all iterates $\varphi^k$ are balanced.
Proof of Theorem 5.9. As in the proof of Theorem 5.7, (5.9) holds. However, we do not know if the equality extends to the marked spectra, i.e., if (5.10) holds, for \( k \geq 2 \). The markings of the spectra by indices may differ by a shift of degree
\[
\sigma_i \sim \mu(\bar{x}_i) - \mu(\bar{y}_i)
\]
which is independent of \( i \). Note that \( s(1) = 0 \) since both \( \varphi \) and \( R_\varphi \) are balanced. Denote by \( \Phi_i \) the linearized flow at \( \bar{x}_i \) and by \( \Psi_i \) the linearized flow at \( \bar{y}_i \). Then
\[
\mu((\Phi_i + \Psi_i^{-1})^k) = s(k) = \mu((\Phi_j + \Psi_j^{-1})^k)
\]
for any \( i \) and \( j \), and, by Corollary (4.2), we have
\[
\hat{s}((\Phi_i + \Psi_i^{-1})^k) = \hat{s}((\Phi_j + \Psi_j^{-1})^k).
\]
To prove the theorem, it is enough to show that these decorated spectra are empty and that \( \text{loop}(\Phi_i + \Psi_i^{-1}) = 0 \). In fact, we only need to do this for just one of them. For then \( s(k) = 0 \) and, by Corollary (4.2), the same is true for all \( i = 0, \ldots, n \).

Let us first prove that the decorated spectra are empty. For the sake of brevity, set \( \Gamma_i = \hat{s}((\Phi_i + \Psi_i^{-1})^k) \), and let \( A_i \subset \Gamma_i \) be the subset comprising all pairs \((a, m) \in \Gamma_i \) where \( a \in \sigma_+(\Phi_i) \) but \( a \notin \sigma_+(\Psi_i) \). Furthermore, we automatically have \( m = \text{sgn}_a(\Phi_i) \neq 0 \) since \((a, m) \in A_i \). Our first goal is to show that at least two of the sets \( A_i \) are empty.

To this end, denote by \( B_i \) the complement to \( A_i \) in \( \Gamma_i \). The set \( B_i \) consists of pairs \((a, m) \in \Gamma_i \) with \( a \in \sigma_+(\Phi_i) \). In other words, \((a, m) \in B_i \) if and only if \( a \in \sigma_+(\Psi_i) \) and \( m = \text{sgn}_a(\Phi_i) - \text{sgn}_a(\Psi_i) \neq 0 \).

We claim that \( |B_i| < n \) for all \( i \). (Henceforth, \(| \cdot | \) stands for the cardinality.) Without loss of generality we may set \( i = 0 \) and note that \( |B_0| \leq |\sigma_+(\Phi_0)| \leq n \).

It suffices to show that \( |B_0| \neq n \). Arguing by contradiction assume the contrary: \( |B_0| = n \). Then, as is easy to see, we have a one-to-one map \( \hat{s}(\Psi_0) \cong \sigma_+(\Psi_0) \to \Gamma_j \) sending a point \( a \) to the unique pair \((a, m) \in \hat{s}(\Phi_j + \Psi_j^{-1}) \). (This map need not preserve the signature and its image may overlap with \( B_0 \).) Then for every \( l \) such that \( 0 < l \leq n \), we have \(|B_l| = |\hat{s}(\Psi_0)| = n \) by (5.11) and the assumption that \( \sigma(y_l) \cap \sigma(y) = \emptyset \). The set formed by the first components of the points in \( B_l \) is exactly the spectrum \( \sigma_+(\Psi_0) \). Hence, we also have \(|A_l| = n \) since \( \sigma(y_l) \cap \sigma(y) = \emptyset \). Without loss of generality we may assume that \( n \geq 2 \). (Otherwise, the assertion of the theorem is obvious.) Applying (5.11) (with \( i = 1 \) and \( j = 2 \)) we see that \( A_1 = A_2 \) and thus \( \sigma_+(y_1) = \sigma_+(y_2) \), which contradicts the assumptions of the theorem.

Next, observe that, by the hypotheses of the theorem, the sets \( A_i \) are disjoint for all \( i \). Fix \( j \). By (5.11), each set \( A_i \) with \( i \neq j \) is mapped one-to-one into \( B_j \) and the images are also disjoint. Since \(|B_j| < n \) and the number of the sets \( A_i \) with \( i \neq j \) is \( n \) we see that one of the sets \( A_i \) must be empty. Moreover, since this is true for every \( j \), there must be at least two such sets.

Thus we may assume without loss of generality that \( A_0 = \emptyset \) and \( A_1 = \emptyset \). Then \( \sigma_+(x_0) = \sigma_+(y_0) \) and, similarly, \( \sigma_+(x_1) = \sigma_+(y_1) \). Furthermore, by (5.11), \( B_0 = B_1 \). Assume that this set is non-empty and denote by \( X \subset \sigma_+(x_0) \cap \sigma_+(x_1) \) be the set of the first components of \( B_0 = B_1 \). Clearly, we also have \( X \neq \emptyset \). Hence, \( \sigma_+(y_0) \cap \sigma_+(y_1) \subset X \neq \emptyset \) which is impossible. Therefore, \( B_0 = \emptyset \) and \( \hat{s}(\Phi_0 + \Psi_0^{-1}) = \emptyset \).

Since the decorated spectra are empty, we have \( \text{loop}(\Phi_i + \Psi_i^{-1}) = s(1) = 0 \), which concludes the proof of the theorem. \( \square \)
6. Proof of the Index Divisibility Theorem

Our goal in this section is to prove the index divisibility theorem, i.e., Theorem 4.1 (or, Theorem 1.4 from the introduction). Throughout the proof we keep the notation and conventions from Section 4. Furthermore, set \( \mu_k := \mu(\Phi^k) \) and \( \mu'_k = \mu_{k+1} - \mu_k \). These sequences are additive with respect to the direct sum. When it is essential to emphasize the role of the map \( \Phi \), we will write \( \mu(\Phi) \) and \( \mu'_k(\Phi) \). For every \( \lambda \in \sigma_+(\Phi) \), we define the logarithmic eigenvalue \( \theta \in (0, 1) \) by

\[
\lambda = \exp(\pi \sqrt{-1} \theta).
\]

Since \( \Phi \) is strongly non-degenerate, \( \theta \not\in \mathbb{Q} \).

Essentially by the definition of the Conley–Zehnder index (see, e.g., [SZ]),

\[
\mu'_k = \text{loop}(\Phi) + \text{mult}_{-1}(\Phi) + 2 \sum_{\lambda \in \sigma_+(\Phi)} a_\lambda(k) \text{sgn}_\lambda(\Phi), \tag{6.1}
\]

where

\[
a_\lambda(k) = \begin{cases} 
0 & \text{when } \lfloor (k+1)\theta/2 \rfloor = \lfloor k\theta/2 \rfloor, \\
1 & \text{when } \lfloor (k+1)\theta/2 \rfloor = \lfloor k\theta/2 \rfloor + 1.
\end{cases} \tag{6.2}
\]

In the latter case we say that the eigenvalue \( \lambda \) jumps at \( k \).

The implication (b) \( \Rightarrow \) (a) is an immediate consequence of (6.1). The rest of this section comprises the proof of the main assertion of the theorem, the implication (a) \( \Rightarrow \) (b): the fact that (i) and (ii) hold whenever \( 2l | \mu'_k(\Phi) \) for all \( k \in \mathbb{N} \).

First, note that by (6.1) and (6.2) we have

\[
\mu'_1(\Phi) = \text{loop}(\Phi) + \text{mult}_{-1}(\Phi)
\]

which proves (i) and, as a consequence, the assertion in the case where the map \( \Phi(1) \) has no elliptic part, i.e., \( \Phi_e = 0 \) in the decomposition \( \Phi = \phi \circ (\Phi_h \oplus \Phi_{-h} \oplus \Phi_e) \) from Section 4. This also shows that the assertion holds for \( \Phi \) if and only if it holds for \( \Phi_{h} \). Thus in what follows we can assume that \( \Phi \) has no hyperbolic and loop parts, i.e., \( \Phi = \Phi_e \), and, in particular,

\[
\text{loop}(\Phi) + \text{mult}_{-1}(\Phi) = 0.
\]

Then the iterations \( \Phi^k \) also have no hyperbolic part, although the loop part of the iterated map may be (and usually is) non-trivial.

The map \( \Phi \) decomposes into the direct sum of maps \( \Phi(\lambda) \) with eigenvalues \( \lambda \) and \( \bar{\lambda} \), where \( \lambda \in \sigma_+(\Phi) \):

\[
\Phi = \bigoplus_{\lambda \in \sigma_+(\Phi)} \Phi(\lambda).
\]

Clearly, \( \text{sgn}_\lambda(\Phi(\lambda)) = \text{sgn}_{\lambda}(\Phi) \) and \( \text{sgn}_{\lambda'}(\Phi(\lambda)) = 0 \) when \( \lambda' \neq \lambda \).

The assertion holds for \( \Phi \) and one of the maps \( \Phi(\lambda) \) if and only if it holds for \( \Phi(\lambda) \) and the sum \( \Psi := \bigoplus_{\lambda' \neq \lambda} \Phi(\lambda') \) of the remaining maps. The idea of the proof is to show that

\[
l | \text{sgn}_\lambda(\Phi) \tag{6.3}
\]

for some eigenvalue \( \lambda \). Once (6.3) is established, \( l | \mu'_k(\Phi(\lambda)) \) because (b) implies (a). Since

\[
\mu'_k(\Phi) = \mu'_k(\Psi) + \mu'_k(\Phi(\lambda)),
\]

where as above \( \Phi = \Psi \oplus \Phi(\lambda) \), to prove the assertion for \( \Phi \) it is enough to establish it for \( \Psi \). Now the result follows by induction on dimension.
Turning to the actual proof, we first need to introduce some terminology and notation. Let us order the positive unit spectrum $\sigma_+(\Phi)$ in an arbitrary way:

$$\sigma_+(\Phi) = \{\lambda_1, \ldots, \lambda_r\},$$

and consider the eigenvalue “vector”

$$\vec{\lambda}(\Phi) = (\lambda_1, \ldots, \lambda_r) \in T^r := (S^1)^r.$$ (6.4)

Denote by $\Gamma(\Phi)$ or just $\Gamma$ the closure of the positive orbit

$$\Lambda = \{\vec{\lambda}_k \mid k \in \mathbb{N}\} \subset T^r.$$

This is a subgroup of $T^r$. Let $\Gamma_0$ be the connected component of the identity of $\Gamma$.

Denoting the coordinates on $T^r$ by $(z_1, \ldots, z_r)$, consider the codimension one sub-tori $T^r_{i-1}$ given by the condition $z_i = 1$. It is easy to see from the requirement that $\Phi$ is strongly non-degenerate (i.e., all $\theta_i \notin \mathbb{Q}$) that the group $\Gamma$ intersects the submanifolds $T^r_{i-1}$ transversely. We co-orient $T^r_{i-1}$ via the positive (clockwise) orientation of $z_i$. Furthermore, let us call a formal linear combination of closed co-oriented submanifolds of $T^r$ with integer coefficients a cycle. For a codimension-one cycle $T$ and an oriented path $\eta$ in $T^r$ with end-points outside $T$ the intersection index $\langle \eta, T \rangle \in \mathbb{Z}$ is defined in an obvious way.

To $\Phi$, we associate the cycle

$$T = \sum T_i, \text{ where } T_i = \text{sgn}_{\lambda_i}(\Phi)T^r_{i-1},$$

which we call the index cycle of $\Phi$.

Finally, consider the oriented path

$$A = \{e^{\pi \sqrt{-1} \theta_1 t}, \ldots, e^{\pi \sqrt{-1} \theta_r t} \mid t \in [0, 1]\},$$

which we will refer to as the generating arc. Then the arc $\vec{\lambda}^k A$, where we use multiplicative notation for the group operation in $T^r$, connects $\vec{\lambda}^k$ to $\vec{\lambda}^{k+1}$. The points $\vec{\lambda}^k$, $k \in \mathbb{N}$, are never in $T$ due to the strong non-degeneracy of $\Phi$ and thus the intersection index $\langle \vec{\lambda}^k A, T \rangle \in \mathbb{Z}$ is defined. Then (6.1) is simply the fact that

$$2\langle \vec{\lambda}^k A, T \rangle = \mu_k,$$ (6.5)

where we have assumed that $\Phi$ does not have the loop and hyperbolic parts. The individual terms in (6.1) can be interpreted in a similar vein. Namely, with $a_{\lambda_i}(k)$ defined by (6.2), we have

$$\langle \vec{\lambda}^k A, T_i \rangle = a_{\lambda_i}(k)\text{sgn}_{\lambda_i}(\Phi).$$

The proof of the theorem hinges on two lemmas.

**Lemma 6.1.** For every oriented path $\eta$ in $\Gamma$ with end points outside $T$, the intersection index $\langle \eta, T \rangle$ is divisible by $l$:

$$l \mid \langle \eta, T \rangle.$$ (6.6)

**Remark 6.2.** Observe that this lemma must hold if the theorem is true: the cycles $T^r_{i-1}$ enter $T$ with coefficients divisible by $l$. Moreover, (6.6) must be satisfied for all paths $\eta$ in $T^r$, but not just in $\Gamma$, with end points outside $T$.

Postponing the proof of the lemma, set $C_i = \Gamma \cap T^r_{i-1}$. As has been pointed out above, this intersection is transverse. (Note that the subgroup $C_i$ can have several connected components even when $\Gamma$ is connected.)
Lemma 6.3. There exists $i$ such that $C_i$ is not entirely contained in the union of the subgroups $C_j$ with $j \neq i$:

$$C_i \not\subset \bigcup_{j \neq i} C_j. \quad (6.7)$$

Remark 6.4. This lemma depends only on the assumptions that the components $\lambda_i$ of $\vec{\lambda}$ are distinct, in contrast with the eigenvalues of $\Phi$, and that $\theta_i \in (0, 1)$ for all $i$. Note also that, as will be clear from the proof, while those $i$ for which (6.7) holds can be explicitly described, (6.7) does not need to be satisfied for all $i$. In fact, we can have $C_i \subset C_j$ for some $i \neq j$.

The theorem readily follows from these two lemmas. Namely, let $i$ be as in Lemma 6.3. Pick a short path $\eta$ in $\Gamma$ transverse to $C_i$, intersecting $C_i$ at one point and not intersecting any $C_j$ with $j \neq i$. Such a path exists since the complement to the union of $C_j$, $j \neq i$, in $C_i$ is non-empty.

Then, by Lemma 6.1,

$$l \mid \langle \eta, T \rangle = \langle \eta, T_i \rangle = \pm \text{sgn}_{\lambda_i}(\Phi).$$

Splitting off $\Phi(\lambda_i)$ as described above, we reduce the dimension and the theorem follows by induction. To complete the proof, it remains to prove the lemmas.

Proof of Lemma 6.1. Let us equip $\Gamma$ with the metric induced by the standard flat metric on $\mathbb{T}^r$. It suffices to prove the lemma for short geodesics $\xi$ (of length less than some $\epsilon > 0$ to be specified later) connecting points of the orbit $\Lambda$. Indeed, any path $\eta$ can be arbitrarily well approximated by broken geodesics with segments $\xi_q$ of this type. Thus $\langle \eta, T \rangle = \sum \langle \xi_q, T \rangle$, where every term on the right is divisible by $l$. (In fact, proving the lemma for such short geodesics $\xi$ would be sufficient for our purposes.)

We show that $l \mid \langle \xi, T \rangle$ in several steps. First consider a path $\alpha$ in $\mathbb{T}^r$ – an iterated arc – obtained by concatenating several adjacent copies of the generating arc $A$:

$$\alpha = \tilde{X}^k A \cup \tilde{X}^{k+1} A \cup \ldots \cup \tilde{X}^m A. \quad (6.8)$$

Then, by (6.5),

$$2 \langle \alpha, T \rangle = \mu'_k + \mu'_{k+1} + \ldots + \mu'_m,$$

and hence

$$l \mid \langle \alpha, T \rangle. \quad (6.9)$$

Next, fix a small neighborhood $V$ of $\tilde{X}$ disjoint from $T$ and set $\epsilon > 0$ to be the radius of $V$, i.e., the supremum of the length of a geodesic in $V$ starting at $\tilde{X}$. Pick a point $\tilde{X}^k \in V \cap \Lambda$ and let $\alpha$ be a path as above connecting $\tilde{X}$ to $\tilde{X}^k$. We complete $\alpha$ to a loop $\gamma$ by concatenating it with a geodesic $\zeta$ in $V$ connecting its end points. Then

$$\langle \gamma, T \rangle = \langle \alpha, T \rangle,$$

since $V$ is disjoint from $T$ and thus $\langle \zeta, T \rangle = 0$. Therefore, by (6.9),

$$l \mid \langle \gamma, T \rangle. \quad (6.10)$$

Moreover, since the intersection index $\langle \gamma, T \rangle$ depends only on the homology class of $\gamma$, the same is true for any loop $\gamma'$ obtained from $\gamma$ by parallel transport in $\mathbb{T}^r$.

Let $\xi$ be a geodesic in $\Gamma$ of length less than $\epsilon$ connecting $\tilde{X}^k$ to $\tilde{X}^{k'}$ for some $k'$, where without loss of generality we can assume that $k < k'$. Let us connect $\tilde{X}^k$ to
\(\tilde{X}^k\) by the iterated arc \(\alpha'\) defined by (6.8) with \(m = k'\). Concatenating \(\alpha'\) with the geodesic \(-\xi\) (the reversed orientation) we obtain a loop \(\gamma'\) in \(T^r\).

Consider the loop \(\gamma = \tilde{X}^{-k+1}\gamma'\). Then \(\langle \gamma', T \rangle = \langle \gamma, T \rangle\) and thus
\[
(\xi, T) = \langle \gamma', T \rangle - \langle \alpha', T \rangle = \langle \gamma, T \rangle - \langle \alpha', T \rangle.
\]

By (6.9) and (6.10), both terms on the right-hand side are divisible by \(l\). Therefore, \(l \mid \langle \xi, T \rangle\), which concludes the proof of the lemma.

**Proof of Lemma 6.3.** The argument is carried out in three steps.

**Step I.** To set the stage for dealing with more complicated situations, let us first consider the case where \(\Gamma\) is connected even though, formally speaking, Steps II and III are logically independent of this case. Then \(\Gamma\) contains a dense one-parameter subgroup
\[
\alpha^t = \{e^{\pi \alpha_i \sqrt{-1} t}, \ldots, e^{\pi \alpha_r \sqrt{-1} t} \mid t \in \mathbb{R}\}.
\]

Observe that all coefficients \(\alpha_i\) are necessarily distinct. Indeed, assume otherwise: e.g., \(\alpha_1 = \alpha_2\). Then, since the one-parameter subgroup is dense in \(\Gamma\), the coordinates \(z_1\) and \(z_2\) agree for all points in \(\Gamma\), which is impossible because \(\lambda_1 \neq \lambda_2\).

Moreover, we claim that the absolute values \(|\alpha_i|\) are also distinct. Indeed, assume by contradiction, assume that, e.g., \(\alpha_1 = -\alpha_2\). Then \(z_1 = z_2\) at every point of \(\Gamma\). However, we have \(\lambda_i = \exp(\pi \sqrt{-1} \theta_i)\), where \(0 < \theta_i < 1\) for all \(i\), and hence \(\lambda_i \neq \lambda_j\) for any \(i\) and \(j\).

Pick \(\alpha_i\) with the largest absolute value and set \(t = 2/\alpha_i\). Then, as is easy to see,
\[
\alpha^t \in C_i, \text{ but } \alpha^t \notin C_j \text{ when } j \neq i.
\]

**Step II.** Next, let us focus on the case where \(\Gamma\) is one-dimensional – this is the key step of the proof. Then the connected component of the identity \(\Gamma_0\) of \(\Gamma\) is a one-parameter subgroup of the form (6.11), where now \(\alpha_i \in \mathbb{Z}\) for all \(i\). However, in contrast with Step I, the map \(t \mapsto \alpha^t\) from \(\mathbb{R}\) to \(\Gamma_0\) is not one-to-one; this map is a group homomorphism with kernel \(2\mathbb{Z}\). Since \(\Gamma\) and hence \(\Gamma_0\) are transverse to \(C_i\), we have \(\alpha_i \neq 0\) for all \(i\). Without loss of generality, we may assume that \(\gcd(\alpha_1, \ldots, \alpha_r) = 1\).

Consider the intersections \(C_i^0 = \Gamma_0 \cap T_i^{-1} \subset C_i\). Thus \(C_i^0\) is the group of the roots of unity in \(\Gamma_0 \cong S^1\) (i.e., just a cyclic subgroup) of order \(|C_i^0| = |\alpha_i|\). We partially order the set of these subgroups by inclusion, which is equivalent to partially ordering the collection \(\{\alpha_i\}\) by divisibility: \(\alpha_i\) is considered to be greater than or equal to \(\alpha_j\) if \(\alpha_j\) divides \(\alpha_i\) or equivalently \(C_j^0 \subset C_i^0\).

**Claim 6.5.** Let \(C_i^0\) be a maximal element among all subgroups \(C_j^0\) with respect to the inclusion partial order. Then (6.7) holds.

This claim establishes Lemma 6.3 when \(\dim \Gamma = 1\). Without loss of generality, we may require that \(i = r\). Thus \(C_r^0\) is not contained in any other subgroup \(C_j^0\) unless \(C_j^0 = C_r^0\) or, equivalently, \(\alpha_r\) does not divide \(\alpha_j\) unless \(|\alpha_r| = |\alpha_j|\). (The reader can assume that \(|\alpha_r| \geq |\alpha_j|\) for all \(j\) – this is sufficient for our purposes.)

Let \(I\) be the collection of indices \(j\) in the range from 0 to \(r - 1\) such that \(|\alpha_j| \neq |\alpha_r|\). (The essential point is that \(|\alpha_j|\) is not divisible by \(|\alpha_r|\) for any \(j \in I\).) In other words, since \(C_i^0\) is maximal, \(C_r^0 \neq C_j^0\) or, equivalently, \(C_r^0 \not\subset C_j^0\) if and only if \(j \in I\). Then
\[
C_r^0 \not\subset \bigcup_{j \in I} C_j^0.
\]
Indeed, a generator \( z \) of \( C^0_j \) is not contained in any subgroup \( C^0_j \), \( j \in I \), for otherwise we would have \( C^0_r \subset C^0_j \).

Next, let \( \Gamma_1 \) be the connected component of \( \Gamma \) containing the topological generator \( \mathbf{\lambda} \) of \( \Gamma \) and let \( C^1_j = \Gamma_1 \cap C_j \). When \( \Gamma \) is connected, we necessarily have \( \Gamma_0 = \Gamma_1 = \Gamma \) and \( C^0_j = C^1_j = C_j \). Note that \( C^1_j \neq \emptyset \) since \( \Gamma_1 \) is homologous to \( \Gamma_0 \) in \( \mathbb{T}^r \), and hence the cardinality of \( C^1_j \) is, up to a sign,

\[
\langle \Gamma_1, \mathbb{T}_j^{-1} \rangle = \langle \Gamma_0, \mathbb{T}_j^{-1} \rangle = \alpha_i \neq 0.
\]

In other words, \( |C^0_j| = |C^1_j| \). The same is true, of course, for other connected components of \( \Gamma \).

The set \( C^1_j \) generates the group \( C_j \) for all \( j = 1, \ldots, r \). This readily follows from the observation that the natural map

\[ C_j/C^0_j \rightarrow \Gamma/\Gamma_0 \]

hits the generator of the cyclic group \( \Gamma/\Gamma_0 \) because \( C^1_j \neq \emptyset \) and thus is onto. (In fact, by construction, this map is then an isomorphism.)

We claim that

\[ C^1_r \not\subset \bigcup_{j \in I} C^1_j. \]  \hspace{1cm} (6.13)

Indeed, assume the contrary:

\[ C^1_r \subset \bigcup_{j \in I} C^1_j. \]

Then, since \( C^1_j \) generates \( C_j \) for all \( j \), we have

\[ C_r \subset \bigcup_{j \in I} C_j. \]

Intersecting this inclusion with \( \Gamma_0 \), we conclude that

\[ C^0_r \subset \bigcup_{j \in I} C^0_j, \]

which is impossible by (6.12).

There are now two possibilities: either \( C^0_r = C^0_j \) for some other subgroup \( C^0_j \) or not, i.e., \( C^0_r \) is strictly maximal. In the former case, there exists \( s \neq r \) such that \( \alpha_s = \pm \alpha_r \). In the latter case \( I = \{1, \ldots, r-1\} \), and the claim follows from (6.12).

It remains to prove the claim when \( C^0_r = C^0_s \) for some \( s < r \). In other words, \( s \neq r \) and \( s \notin I \) and \( C^0_r = C^0_s \) is a maximal subgroup. Let us show that

\[ C^1_r \cap C^1_s = \emptyset. \]  \hspace{1cm} (6.14)

We argue by contradiction. Set

\[ \pi = (\pi_s, \pi_r): \mathbb{T}^r \rightarrow \mathbb{T}^2 \]

to be the projection to the product of the last two coordinate circles \( S^1_s \) and \( S^1_r \). These circles intersect only at the origin in \( \mathbb{T}^2 \). Thus, if \( C^1_s \cap C^1_r \) is non-empty,

\[ \pi(C^1_s \cap C^1_r) \subset \pi(C^1_s) \cap \pi(C^1_r) \subset S^1_s \cap S^1_r = \{(1,1)\} \]

which is exactly the origin in \( \mathbb{T}^2 \). As a consequence, \( \pi(\Gamma_1) \) contains the origin \( \{(1,1)\} \). On the other hand, \( \pi(\Gamma_1) \) is a parallel transport of a subgroup and, to be more precise, of \( \pi(\Gamma_0) \). Hence, then it is also a subgroup and

\[ \pi(\Gamma_1) = \pi(\Gamma_0), \]
and furthermore $\pi(\Gamma_0) = \pi(\Gamma)$ since $\lambda$ is a topological generator.

In particular, $\pi(\lambda) \in \pi(\Gamma_0)$. As a consequence, for some $t \in \mathbb{R}$ we have

$$\left(\lambda_s, \lambda_r\right) = \pi(\lambda) = \left(e^{\pi s \sqrt{-1} t}, e^{\pi r \sqrt{-1} t}\right).$$

This implies that $\lambda_s = \lambda_r$ if $\alpha_s = \alpha_r$, which is impossible since all components of $\lambda$ are distinct. If $\alpha_s = -\alpha_r$, we have $\lambda_s = \lambda_r$ which is also impossible since $\theta_s$ and $\theta_r$ are both in $(0, 1)$. This completes the proof of (6.14).

Now combining (6.13) and (6.14), we see that

$$C^1_r \not\subset \bigcup_{j< r} C^1_j,$$

which concludes the proof of the claim and the proof of the lemma in the case where $\Gamma$ is one-dimensional.

**Step III.** To deal with the general case, we observe that the lemma automatically holds for $\Gamma$ whenever it holds for a closed subgroup $\Gamma'$ of $\Gamma$. (Note that, in general, $\Gamma'$ is not connected even when $\Gamma$ is.)

Indeed, let $k$ be the order of the cyclic group $\Gamma/\Gamma_0$, i.e., $k$ is the smallest positive integer such that $\lambda^k \in \Gamma_0$. The assertion is clear when $\Gamma$ is connected – it suffices to take a point with “rational” coordinates – and thus arbitrarily close to $\lambda^k$ there exists a point $\zeta$ topologically generating a closed one-dimensional subgroup of $\Gamma_0$. The map $z \mapsto z^k$ from $\Gamma_1$ to $\Gamma_0$ is a diffeomorphism and we can set $\lambda^k$ to be the inverse image of $\zeta$.

When $\lambda^k$ is sufficiently close to $\lambda$, its components $\lambda_i^k = e^{\pi \sqrt{-1} \theta_i^k}$ are distinct and all $\theta_i^k$ can be taken close to $\theta_i$ and hence in $(0, 1)$. Therefore, by Step II, the lemma holds for $\Gamma'$ and, as has been pointed out above, it then also holds for $\Gamma$. $\square$

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