Off-diagonal geometric phase for mixed states

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We extend the off-diagonal geometric phase [Phys. Rev. Lett. 85, 3067 (2000)] to mixed quantal states. The nodal structure of this phase in the qubit (two-level) case is compared with that of the diagonal mixed state geometric phase [Phys. Rev. Lett. 85, 2845 (2000)]. Extension to higher dimensional Hilbert spaces is delineated. A physical scenario for the off-diagonal mixed state geometric phase in polarization-entangled two-photon interferometry is proposed.

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The geometric phase discovered by Berry [1] for cyclic adiabatic evolution has led to important insights into the geometry of quantal evolution as well as to several generalizations. Extension to the nonadiabatic cyclic case was given by Aharonov and Anandan [2], who pointed out that the geometric phase is due to the curvature of the quantal state space. Based upon Pancharatnam’s [3] work on interference of light in distinct state of polarization, Samuel and Bhandari [4] provided a general setting for the geometric phase so as to cover noncyclic and nonunitary evolutions. These noncyclic concepts become undefined when the interfering states are orthogonal, which led Manini and Pistolesi [5] to introduce the off-diagonal pure state geometric phase for pure states in adiabatic evolution. This adiabaticity assumption was subsequently removed by Mukunda et al. [6] and the off-diagonal pure state geometric phase was verified by Hasegawa et al. [7] in a neutron experiment.

Another development of the geometric phase has been its extension to the mixed state case. Uhlmann [8] was probably first to address this issue in the context of purification. More recently another mixed state geometric phase was discovered in the experimental context of interferometry by Sjöqvist et al. [9]. It has been pointed out [10] that this latter mixed state geometric phase can be undefined at nodal points in the parameter space where the interference visibility vanishes.

In this Letter, we expand the concept of mixed state geometric phase to the off-diagonal case. This off-diagonal mixed state geometric phase could contain interference information when the “diagonal” phase in [9] is undefined. The off-diagonal mixed state geometric phase reduces to that proposed in [9] in the limit of pure states and it may be verified experimentally as a shift in the interference oscillations in a polarization-entangled two-photon interferometry set up.

The idea behind the off-diagonal pure state geometric phase arises when considering parallel transport generated by the operator $U^I$ of the $j$th eigenstate $|\psi_j\rangle$ of some time-independent Hermitian operator along the path $\Gamma$ in state space to the $k$th eigenstate $|\psi_k\rangle = U^\parallel |\psi_j\rangle$. Then the scalar product $\langle \psi_j | U^\parallel | \psi_j \rangle$ vanishes and the concomitant relative phase becomes undefined. The only phase information left is in the cross scalar product $\langle \psi_k | U^\parallel | \psi_j \rangle$ ($j \neq k$), from which the off-diagonal geometric phase factor $\gamma_{jk}^\Gamma$ can be defined as

$$\gamma_{jk}^\Gamma \equiv \sigma_{jk} \sigma_{kj}$$

where $\sigma_{jk} = \Phi[\langle \psi_j | U^\parallel | \psi_k \rangle]$ with $\Phi[z] = z/|z|$. This quantity is gauge invariant and consequently measurable. Furthermore, it is reparameterization invariant, real-valued, and it is solely a property of the subjacent geometry of state space. In the qubit (two-level) case it can be shown that the off-diagonal geometric phase $\arg \gamma_{jk}^\Gamma$ becomes $\pi$ for any open path $\Gamma$ on the Bloch sphere.

The above can be generalized to $l$ mutually orthogonal states by defining

$$\gamma_{j_1 j_2 \ldots j_l}^{(l)\Gamma} \equiv \sigma_{j_1 j_2} \sigma_{j_2 j_3} \ldots \sigma_{j_l j_1}$$

as any cyclic product of $\sigma$s is gauge invariant. If $l = 1$ this reduces to the diagonal geometric phase factor $\gamma_j^\Gamma$ and if $l = 2$ we obtain the off-diagonal pure state geometric phase. For $l > 2$ more complex phase relations among off-diagonal components of the eigenstates at the end-points of $\Gamma$ can be described. Such phase relations have been analyzed [6, 11] for the deformed microwave resonator experiments in [12].

Due to decoherence effects or improper preparation procedures it is more realistic to talk about mixed states in quantum mechanics. To cover such situations the concept of diagonal mixed state geometric phase was introduced in [13] by considering Mach-Zehnder interferometry with a nondegenerate mixed internal input state $\rho$. This phase arises naturally as the shift of the interference oscillations determined by

$$\gamma_{\rho} = \Phi[\text{Tr}(U^\parallel \rho)]$$
with the unitarity $U^\parallel$ parallel transporting each eigenstate of $\rho$ in one arm of the interferometer. The geometric phase factor $\gamma_\rho$ is a property of the subjacent geometry of state space and reduces to that of the standard geometric phase in the limit of pure states. It becomes undefined at its nodal points where the visibility factor $|\text{Tr}(U^\parallel \rho)|$ vanishes.

To generalize the above to the off-diagonal mixed state case we have to find an appropriate notion of “maximal orthogonality” between unitarily connected density matrices. One approach would be to take the density matrices $\rho$ and $\rho' = U \rho U^\dagger$ as maximally orthogonal if their Bures fidelity $F_B[\rho, \rho'] = \frac{1}{2} \left(\text{Tr} \sqrt{\sqrt{\rho} \rho' \sqrt{\rho}}\right)^2$ is at infimum, as this would ensure maximal distinguishability between $\rho$ and $\rho'$ \footnote{For nondegenerate density matrices as $\rho = \sum_{k=1}^N \lambda_k |\psi_k\rangle \langle \psi_k|$ and $\rho' \equiv \rho^\perp = \sum_{k=1}^N \lambda_k |\psi_k^\perp\rangle \langle \psi_k^\perp|$, we demonstrate below that it can be assigned an operational meaning in terms of a purification lift that can be experimentally tested using two-particle interferometry.}

Using the above concept of quasi-orthogonality, we now define the off-diagonal mixed state geometric phase factor $\gamma_{\rho \rho^\perp}$ for nondegenerate density matrices as

$$\gamma_{\rho \rho^\perp} \equiv \Phi \left[ \text{Tr}(U^\parallel \sqrt{\rho} U^\parallel \sqrt{\rho^\perp}) \right],$$

where the unitarity $U^\parallel$ parallel transports each eigenstate $|\psi_k\rangle$ of $\rho$. This definition can be seen as a natural extension of \footnote{The Bures fidelity $F_B[\rho, \rho^\perp] = 1 - r^2$ vanishes for pure states and the off-diagonal geometric phase becomes undefined only for cyclic evolution, where the diagonal geometric phase is well-defined. In the mixed state case $0 < r < 1$ the diagonal geometric phase becomes undefined only for rotations that flip the Bloch vector, corresponding to $\eta = 0$. For such rotations $|\text{Tr}(U^\parallel \sqrt{\rho} U^\parallel \sqrt{\rho^\perp})| = 1$ and the off-diagonal mixed state geometric phase is well-defined and equals the pure state.}

with $|\langle \psi_k | \psi_k^\perp \rangle| = 0$ for each $k = 1, \ldots, N$, $N$ being the dimension of the Hilbert space. In the qubit case, this notion is equivalent to solving the minimization problem for the Bures fidelity, but in higher dimensional cases it is straightforward to find examples where the two approaches differ.

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We may extend the off-diagonal mixed state geometric phase to $l \leq N$ mutually quasi-orthogonal density matrices $\rho_{j_k}$, $k = 1, 2, \ldots, l$, yielding the expression

$$\gamma_{\rho_{j_1} \rho_{j_2} \ldots \rho_{j_l}} \equiv \Phi \left[ \text{Tr}(U^\parallel \sqrt{\rho_{j_1}} U^\parallel \sqrt{\rho_{j_2}} \ldots U^\parallel \sqrt{\rho_{j_l}}) \right],$$

which is gauge invariant and independent of cyclic permutations of the indexes $j_1, j_2, \ldots, j_l$. It reduces to the diagonal mixed state geometric phase $\arg \text{Tr}(U^\parallel \rho)$ for $l = 1$ and to the off-diagonal mixed state geometric phase $\arg \text{Tr}(U^\parallel \sqrt{\rho} U^\parallel \sqrt{\rho^\perp})$ for $l = 2$. In the limit of pure states it is equivalent to Eq. \footnote{The Bures fidelity $F_B[\rho, \rho^\perp] = 1 - r^2$ vanishes for pure states and the off-diagonal geometric phase becomes undefined only for cyclic evolution, where the diagonal geometric phase is well-defined. In the mixed state case $0 < r < 1$ the diagonal geometric phase becomes undefined only for rotations that flip the Bloch vector, corresponding to $\eta = 0$. For such rotations $|\text{Tr}(U^\parallel \sqrt{\rho} U^\parallel \sqrt{\rho^\perp})| = 1$ and the off-diagonal mixed state geometric phase is well-defined and equals the pure state.}.

To delineate the nodal structure of the mixed state geometric phases in Eqs. \footnote{The Bures fidelity $F_B[\rho, \rho^\perp] = 1 - r^2$ vanishes for pure states and the off-diagonal geometric phase becomes undefined only for cyclic evolution, where the diagonal geometric phase is well-defined. In the mixed state case $0 < r < 1$ the diagonal geometric phase becomes undefined only for rotations that flip the Bloch vector, corresponding to $\eta = 0$. For such rotations $|\text{Tr}(U^\parallel \sqrt{\rho} U^\parallel \sqrt{\rho^\perp})| = 1$ and the off-diagonal mixed state geometric phase is well-defined and equals the pure state.} and \footnote{The Bures fidelity $F_B[\rho, \rho^\perp] = 1 - r^2$ vanishes for pure states and the off-diagonal geometric phase becomes undefined only for cyclic evolution, where the diagonal geometric phase is well-defined. In the mixed state case $0 < r < 1$ the diagonal geometric phase becomes undefined only for rotations that flip the Bloch vector, corresponding to $\eta = 0$. For such rotations $|\text{Tr}(U^\parallel \sqrt{\rho} U^\parallel \sqrt{\rho^\perp})| = 1$ and the off-diagonal mixed state geometric phase is well-defined and equals the pure state.} let us first consider the qubit case with $\rho = \frac{1}{2}(1 + r \sigma_z)$, where $r \neq 0$ is the length of the Bloch vector and $\sigma_z$ is the standard Pauli operator in the $|\psi_1\rangle, |\psi_2\rangle$ basis. Using the definition of quasi-orthogonality yields $\rho^\perp = \frac{1}{2}(1 - r \sigma_z)$. Putting these density matrices into Eq. \footnote{The Bures fidelity $F_B[\rho, \rho^\perp] = 1 - r^2$ vanishes for pure states and the off-diagonal geometric phase becomes undefined only for cyclic evolution, where the diagonal geometric phase is well-defined. In the mixed state case $0 < r < 1$ the diagonal geometric phase becomes undefined only for rotations that flip the Bloch vector, corresponding to $\eta = 0$. For such rotations $|\text{Tr}(U^\parallel \sqrt{\rho} U^\parallel \sqrt{\rho^\perp})| = 1$ and the off-diagonal mixed state geometric phase is well-defined and equals the pure state.} we obtain

$$\text{Tr}(U^\parallel \sqrt{\rho} U^\parallel \sqrt{\rho^\perp}) = \eta^2 \sqrt{F_B[\rho, \rho^\perp] \cos \Omega + (1 - \eta^2) \gamma_{12}}. \quad (7)$$

Here, $\eta = |\langle \psi_1 | \parallel \rho \parallel \psi_1 \rangle|$ is the pure state visibility and $\Omega$ is the solid angle enclosed by the path $\Gamma$ and the shortest geodesic connecting its end-points on the Bloch sphere. Similarly, the expression in the diagonal case becomes

$$\text{Tr}(U^\parallel \rho) = \eta \sqrt{\cos^2 \frac{\Omega}{2} + r^2 \sin^2 \frac{\Omega}{2}} \times \exp \left( -i \arctan \left[ r \tan \frac{\Omega}{2} \right] \right). \quad (8)$$

FIG. 1: Nodal surfaces of the off-diagonal mixed state geometric phase for a qubit. For Bures fidelity $F_B > 0$ (mixed states), there are nodes also for paths with pure state visibility $\eta \neq 1$ at various solid angles $\Omega$. The Bures fidelity $F_B[\rho, \rho^\perp] = 1 - r^2$ vanishes for pure states and the off-diagonal geometric phase becomes undefined only for cyclic evolution, where the diagonal geometric phase is well-defined. In the mixed state case $0 < r < 1$ the diagonal geometric phase becomes undefined only for rotations that flip the Bloch vector, corresponding to $\eta = 0$. For such rotations $|\text{Tr}(U^\parallel \sqrt{\rho} U^\parallel \sqrt{\rho^\perp})| = 1$ and the off-diagonal mixed state geometric phase is well-defined and equals the pure state.
value \( \arg \gamma_{12} \) = \( \pi \). This shows in the qubit case that the diagonal and off-diagonal mixed state geometric phases never become undefined simultaneously.

The off-diagonal mixed state geometric phase in the qubit case has a nontrivial nodal structure that arises due to the nonvanishing Bures fidelity. This can be seen by putting the left-hand side of Eq. (10) to zero and solving for \( \eta^2 \) yielding

\[
\eta^2 = (1 + \sqrt{\mathcal{F}_B[\rho, \rho^+] \cos \Omega})^{-1},
\]

which has solutions at \( \eta \neq 1 \) for \( \cos \Omega > 0 \) and \( \mathcal{F}_B[\rho, \rho^+] > 0 \). Thus, the off-diagonal mixed state geometric phase factor may change sign across the nodal surfaces in the parameter space \( \{\mathcal{F}_B[\rho, \rho^+], \eta, \Omega\} \) defined by the solutions of Eq. (9), as shown in Fig. 1. Thus, the corresponding off-diagonal mixed state geometric phase can take both values 0 and \( \pi \), contrary to the corresponding pure state phase, which can only be \( \pi \).

In the maximally mixed state case (\( r = 0 \)), the density matrix is degenerate and the geometric phases are undefined since there is no direction in space singled out. Still, there is a unique notion of relative phase in this case with a nontrivial nodal structure discussed in [10]. The degeneracy matrix is degenerate and the geometric phases are undefined since there is no direction in space singled out.

Next, let us generalize to arbitrary Hilbert space dimensions \( N \). We take the set

\[
\begin{align*}
\rho_1 &= \lambda_1 |\psi_1\rangle \langle \psi_1| + \lambda_2 |\psi_2\rangle \langle \psi_2| + \ldots + \lambda_N |\psi_N\rangle \langle \psi_N|, \\
\rho_2 &= \lambda_1 |\psi_2\rangle \langle \psi_2| + \lambda_2 |\psi_3\rangle \langle \psi_3| + \ldots + \lambda_N |\psi_1\rangle \langle \psi_1|, \\
\rho_N &= \lambda_1 |\psi_N\rangle \langle \psi_N| + \lambda_2 |\psi_1\rangle \langle \psi_1| + \ldots + \lambda_N |\psi_{N-1}\rangle \langle \psi_{N-1}|.
\end{align*}
\]

of mutually quasi-orthogonal nondegenerate density matrices and consider parallel transporting unitarities that permute the eigenstates \( |\psi_i\rangle, \ldots, |\psi_N\rangle \). By appropriate labeling of the eigenstates, such operators can always be decomposed into the direct sum

\[
\mathcal{D}^{(i)}_{\rho_1 \cdots \rho_{N-i}} = \sum_{k=m+1}^{N} (U^{(k)}_{kk})^\dagger \sqrt{\lambda_{k_1} \cdots \lambda_{k_l}}
\]

where the \( \mathcal{D}^{(i)}_{\rho_1 \cdots \rho_{N-i}} \)'s can be written as sums of terms of the form \( \sqrt{\lambda_{a_1} \ldots \lambda_{a_l}} \). For other \( l \), the \( \mathcal{P}^{(i)}_{\rho_1 \cdots \rho_{N-i}} \) vanish as there is \( K \times m \) steps needed to connect \( \sqrt{\rho_{ij}} \) and \( \sqrt{\rho_{ji}} \) with \( u_p \). In the extreme case where all \( N \) eigenstates are permuted, only \( \mathcal{P}^{(N)}_{\rho_1 \cdots \rho_{N-1}} \) may be nonvanishing. In particular, we have the \( \lambda \)-independent expression

\[
\mathcal{P}^{(N)}_{\rho_1 \cdots \rho_N} = (-1)^{N-1} \det \mathcal{U}^\|,
\]

with \( \mathcal{U}^\| \) the matrix elements of \( u_p \) in the eigenbasis of the \( \rho \)'s. As the density matrices are nondegenerate, it follows that all \( \lambda_{k_n} \)'s are different in each term on the right-hand side of Eq. (11), and \( \mathcal{P}^{(l)}_{\rho_1 \cdots \rho_{N-l}} \) must vanish if \( l > \text{rank of the } \rho \)'s.

Let us revisit the qubit (\( N = 2 \)) case using the above general theory. If \( m = 0 \), both \( \gamma^{(1)}_{\rho_1} \) and \( \gamma^{(1)}_{\rho_2} \) exist. Moreover, we have

\[
\mathcal{P}^{(2)}_{\rho_1 \rho_2} = -1,
\]

in agreement with Eq. (7) for \( \eta = 0 \).

As a further illustration, let us work out the \( N = 3 \) case in detail. For \( m = 0 \), all the \( \gamma^{(1)}_\lambda \)'s are well-defined. The dependence upon the rank of the density matrices is visible for higher \( l \), namely

\[
\begin{align*}
\mathcal{D}^{(2)}_{\rho_1 \rho_2} &= \sqrt{\lambda_1 \lambda_3 (U_{11}^\|)^2 + \sqrt{\lambda_1 \lambda_2} (U_{22}^\|)^2} + \sqrt{\lambda_2 \lambda_3} (U_{33}^\|)^2, \\
\mathcal{D}^{(3)}_{\rho_1 \rho_2 \rho_3} &= \mathcal{D}^{(3)}_{\rho_1 \rho_2 \rho_3} = \sqrt{\lambda_1 \lambda_2 \lambda_3} (U_{33}^\|)^3, \\
\mathcal{P}^{(2)}_{\rho_1 \rho_2} &= U_{12}^\| U_{21}^\| (\lambda_1 + \sqrt{\lambda_2 \lambda_3}),
\end{align*}
\]

where we have used \( (-1)^{m-1} \det u_p = U_{12}^\| U_{21}^\| \). The remaining \( \mathcal{D}^{(1)}_{\rho_2}, \mathcal{D}^{(1)}_{\rho_3}, \mathcal{D}^{(2)}_{\rho_2 \rho_3}, \mathcal{D}^{(2)}_{\rho_3 \rho_2}, \mathcal{P}^{(2)}_{\rho_2 \rho_3}, \mathcal{P}^{(2)}_{\rho_3 \rho_2} \) are given...
by appropriate permutations of the \( \lambda \)'s. For \( m = 3 \) (full permutation) the only possible contributions are

\[
\rho^{(3)}_{\rho_1,\rho_2,\rho_3} = \frac{1}{3}, \\
\rho^{(3)}_{\rho_1,\rho_2,\rho_2} = 3 \sqrt{\lambda_1 \lambda_2 \lambda_3},
\]

where the latter requires full rank to be nonvanishing and we have used \((-1)^{N-1} \det U = +1 \) for \( N = 3 \).

Let us now turn to the issue how to measure \( \gamma^{(2)}_{\rho,\rho^\perp} \). In general, the procedure to achieve this is based upon lifting \( \rho \) and \( \rho^\perp \) to the pure states \( |\Psi_{sa}\rangle \) and \( |\Psi_{sa}^\perp\rangle \), respectively, by attaching an ancilla system \( a \) in such a way that \( \rho = \text{Tr}_a |\Psi_{sa}\rangle \langle \Psi_{sa}| \) and \( \rho^\perp = \text{Tr}_a |\Psi_{sa}^\perp\rangle \langle \Psi_{sa}^\perp| \). This purification can be performed experimentally in the qubit case by using a Franson type interferometer set up and polarization-entangled photon pairs (system and ancilla photon) \( \frac{\Psi_{sa}}{\sqrt{2}} \), see Fig. 2. A source of this type that produces photons in the horizontal-vertical (h-v) basis has been demonstrated in \[\frac{\Psi_{sa}}{\sqrt{2}}\]. Such photon pairs are sent to the interferometer in the polarization-entangled state \( |\Psi_{sa}\rangle = \sqrt{\frac{1}{2}} (|h\rangle \otimes |h\rangle + |r\rangle \otimes |v\rangle) \) and in the short arms the polarization is flipped so as to obtain \( |\Psi_{sa}^\perp\rangle = \sqrt{\frac{1}{2}} (|h\rangle \otimes |h\rangle + |r\rangle \otimes |v\rangle) \), where \( r \) is the degree of polarization in single-photon measurement. By an appropriate choice of unitarity \( V \), the off-diagonal phase \( \gamma^{(2)}_{\rho,\rho^\perp} \) can be measured as a shift of the interference pattern obtained by varying the \( U(1) \) phase \( \chi \) in this two-photon scenario. For simplicity, one may consider parallel transporting unitaries that rotates linear polarization into linear polarization. With \( \beta \) the polarization angle of the photons with respect to the horizontal axis, this amounts to \( U = \exp \left[ -\beta (|h\rangle \langle v| - |v\rangle \langle h|) \right] \), and the desired output intensity detected in coincidence is

\[
I \propto |\langle e^{-i\chi} |\Psi_{sa}^\perp| + U^\parallel \otimes V |\Psi_{sa}\rangle|^2 \\
\propto 1 + |\text{Tr}(U^\parallel \sqrt{\rho^\parallel \otimes \rho^\perp})| \cos (\chi - \gamma^{(2)}_{\rho,\rho^\perp})
\]

if we choose \( V = \exp \left[ \beta (|h\rangle \langle v| - |v\rangle \langle h|) \right] \). Explicit calculation predicts \( \text{Tr}(\sqrt{\rho^\parallel \otimes \rho^\perp})^\parallel (\beta) \sqrt{\rho^\parallel \otimes U^\parallel (\beta)} = \sqrt{1 - r^2} \cos^2 \beta - \sin^2 \beta \), which can be positive and negative for \( r \neq 1 \) depending upon \( \beta \). Thus, such an experiment would test that the off-diagonal geometric phase is either 0 or \( \pi \) for mixed qubit states.

In conclusion, we have introduced the concept of off-diagonal geometric phase for mixed states. In the qubit case we have demonstrated that the nodal points of the diagonal and off-diagonal mixed state geometric phase never coincide. Extension to cyclic products of density matrices in arbitrary Hilbert space dimensions is shown to further enrich the mixed state phase. We have also proposed a polarization-entangled two-photon experiment that could test the off-diagonal mixed state geometric phase, and in particular check the sign change property across its nodal surfaces.

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[1] M.V. Berry, Proc. Roy. Soc. London Ser. A \textbf{392}, 45 (1984).

[2] Y. Aharonov and J. Anandan, Phys. Rev. Lett. \textbf{58}, 1593 (1987).

[3] S. Pancharatnam, Proc. Indian Acad. Sci. A \textbf{44}, 247 (1956).

[4] J. Samuel and R. Bhandari, Phys. Rev. Lett. \textbf{60}, 2339 (1988).

[5] N. Manini and F. Pistolesi, Phys. Rev. Lett. \textbf{85}, 3067 (2000).

[6] N. Mukunda, Arvind, S. Chaturvedi, and R. Simon, Phys. Rev. A \textbf{65}, 012102 (2002).

[7] Y. Hasegawa, R. Loidl, M. Baron, G. Badurek, and H. Rauch, Phys. Rev. Lett. \textbf{87}, 070401 (2001); Y. Hasegawa, R. Loidl, G. Badurek, M. Baron, N. Manini, F. Pistolesi, and H. Rauch, Phys. Rev. A \textbf{65}, 052111 (2002).

[8] A. Uhlmann, Rep. Math. Phys. \textbf{24}, 229 (1986).

[9] E. Sjöqvist, A.K. Pati, A. Ekert, J.S. Anandan, M. Ericsson, D.K.L. Oi, and V. Vedral, Phys. Rev. Lett. \textbf{85}, 2845 (2000).

[10] R. Bhandari, Phys. Rev. Lett. \textbf{89}, 268901 (2002); J.S. Anandan, E. Sjöqvist, A.K. Pati, A. Ekert, M. Ericsson, D.K.L. Oi, and V. Vedral, Phys. Rev. Lett. \textbf{89}, 268902 (2002).

[11] F. Pistolesi and N. Manini, Phys. Rev. Lett. \textbf{85}, 1585 (2000).

[12] H.-M. Lauber, et al. Phys. Rev. Lett. \textbf{72}, 1004 (1994); D.E. Manolopoulos and M.S. Child, Phys. Rev. Lett. \textbf{82}, 2223 (1999); J. Samuel and A. Dhar, Phys. Rev. Lett. \textbf{87}, 260401 (2001).

[13] D. Bures, Trans. Am. Math. Soc. \textbf{135}, 199 (1969); A. Uhlmann, Rep. Math. Phys. \textbf{9}, 273 (1976).

[14] The Bures fidelity provides the worst case measure of distinguishability for purifications consistent with the pair of density matrices, see: R. Jozsa, J. Mod. Opt. \textbf{41}, 2315 (1994).

[15] J.D. Franson, Phys. Rev. Lett. \textbf{62}, 2205 (1989); B. Hes-
[16] P.G. Kwiat, E. Waks, A.G. White, I. Appelbaum, and P.H. Eberhard, Phys. Rev. A 60, R773 (1999); A.G. White, D. James, P. Eberhard, and P. Kwiat, Phys. Rev. Lett. 83, 3103 (1999).