Time-reversal and the Bessel equation

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Abstract. The system of two damped/amplified oscillator equations is of widespread interest in the study of many physical problems and phenomena, from inflationary models of the Universe to thermal field theories, in condensed matter physics as well in high energy physics, and also in neuroscience. In this report we review the equivalence, in a suitable parametrization, between such a system of equations and the Bessel equations. In this connection, we discuss the breakdown of loop-antiloop symmetry, its relation with time-reversal symmetry and the mechanism of group contraction. Euclidean algebras such as \(e(2)\) and \(e(3)\) are also discussed in relation with Virasoro-like algebra.

1. Introduction

Eugene Wigner has remarked [1] that "the role which is common to all special functions is to be matrix elements of representations of the simplest Lie groups". Such a statement could constitute by itself a good motivation to study special functions such as, for example, Bessel functions, whose relation with time reversal symmetry and group contraction we are going to review in this paper. Further motivations, however, come from the specific interest to many physical problems of the so-called Bateman system of the couple of damped/amplified oscillators [2], which, as we will see, is equivalent, in a specific parametrization, to Bessel equations. We will also see that the Euclidean groups \(E(2)\) and \(E(3)\), whose representations are given in terms of planar and spherical Bessel functions, respectively, are obtained by the mechanism of group contraction [3]. Such a mechanism is involved in the breakdown of time-reversal symmetry and of the loop-antiloop symmetry around a preferred axis, which in turn is related to loop-algebras, such as the Virasoro-like algebra.

In this paper we report results whose details can be found in ref. [4]). They are object of renewed interest in these days, since according to Freeman’s observations the distribution functions of impulse responses of cortex to electric shocks resemble Bessel functions [5]. The cortex dynamics can be then studied by considering the equivalence of spherical Bessel equation, in a given parametrization, to two oscillator equations, one damped and one amplified oscillator. The study of such a couple of equations, which are at the basis of the formulation of the dissipative many-body model of brain [6, 7, 8], reveals the structure of the root loci of poles and zeros of solutions of Bessel equations [5]. Therefore, it seems to us interesting to discuss in this paper the mathematics underlying the relation between Bessel function and the system of damped/amplified oscillators.
Following thus ref. [4] let me recall that Bessel functions describe solutions with different Pontryagin number in the punctured plane $\mathbb{R}^2/(0)$ [9]. The elements of the homotopy group, $\Pi_n$, are there represented by differential operators acting on analytic functions:

$$\Pi_n \equiv \frac{\partial^n}{\partial z^n}, \quad n \in \mathbb{N}, \quad (1)$$

$n$ is the loop number around the hole and $\Pi_n \cdot \Pi_m = \Pi_{n+m}$. There are two different kinds of behavior, corresponding to the two different functions $\varphi_m(z)$ and $\psi_m(z)$:

$$\frac{\partial^n}{z\partial z^n} \varphi_m(z) = (-)^m \varphi_{m+n}(z), \quad \varphi_m(z) = \frac{j_m(z)}{z^m} \quad (2)$$

and

$$\frac{\partial^n}{z\partial z^n} \psi_m(z) = \psi_{m-n}(z), \quad \psi_m(z) = z^m j_m(z) \quad (3)$$

respectively. On $\varphi$, $\Pi_n$ acts in counter-clockwise way while on $\psi$ it acts in clockwise way. $j_m(z)$ is the planar Bessel function (Bessel function of integer order). Eqs. (2) and (3) are indeed the first and second kind Bessel functions and their linear combinations (the Hankel functions).

For the planar Bessel functions, the “raising” and “lowering” functions coincide for $m = 0$ (no loops). This is not true with the spherical Bessel functions.

In the following we will exploit these topological properties of the $\varphi$ and $\psi$ functions. In our presentation we will closely follow ref. [4]. We will start considering the spherical Bessel equation in Section 2. Group contraction and loop algebras are discussed in Section 3 and 4, respectively. Section 5 is devoted to conclusions.

2. The Bessel equation and the damped/amplified oscillator system

The spherical Bessel equation of order $n$ is:

$$\eta^2 J_{n;\eta} + 2\eta J_{n;\eta} + [\eta^2 - n(n+1)] J_n = 0, \quad (5)$$

where $n$ is zero or an integer, $n = 0, \pm 1, \pm 2, \ldots$, the labels “; $\eta$” and “; $n\eta$” denote first and second order derivatives, respectively. The solutions of Eq. (5) constitute a complete set of (parametric) decaying functions [10], the so called spherical Bessel functions, and can be expressed in terms of the first and second kind Bessel functions and their linear combinations (the Hankel functions).

Eq. (5) is invariant under the transformation $n \to -(n+1)$: $J_n$ and $J_{-(n+1)}$ are both solutions of the same equation. In other words, $J_n$ and $J_{-(n+1)}$ are degenerate solutions corresponding to the same eigenvalue $n(n+1)$ of the operator $\eta^2 \frac{\partial^2}{\partial \eta^2} + 2\eta \frac{d}{d\eta} + \eta^2$.

Here we show that by means of suitable transformations, Eq. (5) turns out to be equivalent to a set of two equations representing a couple of damped/amplified parametric oscillators. This is true also for the planar Bessel equation and will be discussed in the following.

Perform the change of variables in Eq. (5): $\eta \to \eta \equiv cx$ with $x \equiv e^{-t/\alpha}$, with arbitrary parameters $\epsilon$ and $\alpha$. The new variable $t$ may be thought to denote the time variable. By using $w_{n,l} \equiv J_n \cdot (x)^{-l}$, Eq. (5) then becomes:

$$\ddot{w}_{n,l} - \frac{2l + 1}{\alpha} \dot{w}_{n,l} + \left[ \frac{l(l+1) - n(n+1)}{\alpha^2} + \left( \frac{\epsilon}{\alpha} \right)^2 e^{-2\alpha} \right] w_{n,l} = 0. \quad (6)$$
\( \dot{w} \) denotes derivative with respect to time \( t \). Chosing \( l(l+1) = n(n+1) \) the degeneracy between the solutions \( J_n \) and \( J_{-(n+1)} \) is removed and thus a partition is induced between the two solution sectors \( \{ J_n \} \) and \( \{ J_{-(n+1)} \} \). Correspondingly, two different sets of equations are obtained \[4\], one for \( w_{n,l} \) and the other one for \( w_{-(n+1),l} \), respectively. Moreover, each set is composed by two different equations, one for \( l = -(n+1) \) and the other one for \( l = n \). The symmetry under the transformation \( n \to -(n+1) \) gets broken.

Choose then arbitrary parameters \( \alpha \) and \( \epsilon \) to be \( n \)-dependent: \( \alpha \to \alpha_n \) and \( \epsilon \to \epsilon_n \). Thus we use \( \eta \to \eta_n \equiv \epsilon_n x_n \) with \( x_n \equiv e^{-t/\alpha_n} \). Perform such a choice in such a way that \( \frac{2n+1}{\alpha_n} \equiv \mathcal{L} \) and \( \frac{\epsilon_n}{\alpha_n} \equiv \omega_0 \) depend neither on \( n \), nor on time. By setting \( u_n \equiv w_{n,-(n+1)} \), and \( v_n \equiv w_{n,n} \), the couple of equations for the damped/amplified parametric oscillators is finally obtained:

\[
\begin{align*}
\ddot{u}_n + \mathcal{L} \dot{u}_n + \omega^2_n(t)u_n &= 0, \\
\ddot{v}_n - \mathcal{L} \dot{v}_n + \omega^2_n(t)v_n &= 0. 
\end{align*}
\]

The frequency \( \omega_n(t) \) is \( \omega_n(t) = \omega_0 e^{-\frac{t}{2\alpha_n}} \). We see that \( \omega_n(t) \) approaches to the time-independent value \( \omega_0 \) for \( n \to \infty \): the frequency time-dependence is thus ”graded” by the order \( n \) of the original Bessel equation. \( \mathcal{L} \) and \( \omega_0 \), which may be arbitrarily chosen, are characteristic parameters of the oscillator system.

Note that, since keeping \( \mathcal{L} \) independent of \( n \) implies that \( \alpha_{-(n+1)} = -\alpha_n \), the transformation \( n \to -(n+1) \) leads to solutions \( J_{-(n+1)} \) with frequencies exponentially increasing in time. These solutions can be obtained from the ones of Eqs. (7) by time-reversal \( t \to -t \) and exchanging \( u \) with \( v \) (referred to as ”charge conjugation”). In the large limit, \( \omega_n \to \omega_0 \), \( u_n \) and \( v_n \) are reciprocally time-reversed and the two sectors \( \{ J_i \} \), \( i = n, -(n+1) \) are mapped one into the other one in that limit.

Note also that \( w_{n,n} \) and \( w_{n,-(n+1)} \) are ”harmonically conjugate” functions in the sense that they may be represented as \( \frac{1}{\sqrt{2}} w_{n,-(n+1)}(t) = e^{-\frac{\epsilon t}{\alpha_n}} \) and \( \omega_n(t) = \frac{1}{\sqrt{2}} r_n(t) e^{\frac{\epsilon t}{2}} \), with \( r_n(t) \) solution of the single parametric oscillator

\[
\ddot{r}_n + \Omega_n^2(t)r_n = 0.
\]

The common frequency \( \Omega_n \) is:

\[
\Omega_n(t) = \left( \omega_n^2(t) - \frac{\mathcal{L}^2}{4} \right)^{\frac{1}{2}}.
\]

In ref. [4] it has been shown also that the Bessel-like equation

\[
J_{\eta n} + \frac{\alpha}{\eta} J_n + \left( 1 - \frac{\beta^2}{\eta^2} \right) J = 0,
\]

where \( \alpha = 1 \) or 2 and \( \beta \) is an arbitrary real constant, may represent parametric oscillators with constant damping and with different functional choices for the frequency. Vice-versa, from the equation of the damped parametric oscillator, by use of convenient transformations, one can always obtain the Bessel-like equation (10) [4]. In this report we do not discuss more on this point.

Summarizing, under convenient parametrization, the spherical Bessel equation is represented by a two-fold hierarchy of couples of parametric oscillators with time dependent frequency graded by \( n \) and by \( -(n+1) \) and constant damping/amplification given by \( \mathcal{L} \). The transition \( n \to -(n+1) \) corresponds to time-reversal \( t \to -t \) combined with ”charge conjugation” \( u \to v \). The breakdown of the \( n \to -(n+1) \) symmetry of the spherical Bessel equation corresponds to the breakdown of time-reversal symmetry. In the manifold of the the spherical Bessel functions
{J_i}, i = n, - (n + 1) the arrow of time emerges, corresponding to the arrow of time in each of the damped or amplified oscillator equations.

In the following Section such a feature will be seen to have its root in the E(3) group, whose representations can be indeed constructed by means of the spherical Bessel functions. The breakdown of the n → - (n + 1) symmetry there appears as the breakdown of the loop-antiloop symmetry around a preferred direction in the 4D-space.

Finally, we remark that in expanding geometry (inflationary) models of the Universe the first of Eqs. (7), with n = 1 and L denoting the Hubble constant, is commonly used [11].

3. Group contraction

As already mentioned, some of the features studied in the previous Section are found to be rooted in the Euclidean groups, whose representations can be constructed in terms of Bessel functions [1, 10].

We consider the Euclidean groups in two and in three dimensions, E(2) and E(3), respectively, and their relation with the Laplace equations [1]. We start by considering E(2).

3.1. The planar Bessel equations and E(2)

E(2) is the group of the translations T(\vec{v}) in the plane by the vector \vec{v} (\vec{v} ≡ (a, b)) and the rotations R(θ) of the plane around the origin by the angle θ. The Lie algebra is given in terms of the two translation generators \( P_a, P_b \) and of the rotation generator \( M \):

\[
[P_a, P_b] = 0, \quad [P_a, M] = -P_b, \quad [P_b, M] = P_a. \tag{11}
\]

\( P^2 = P_a^2 + P_b^2 = P_+ P_- \) with \( P_± = P_a ± iP_b \), which has non-positive eigenvalue, \( -p^2 \), is the invariant operator of E(2).

On square integrable functions \( f \), the group transformations are represented by the action of the operator \( D(\vec{v}, \theta) \) [1]:

\[
D(\vec{v}, 0)f(\phi) = e^{i\vec{v} \cdot \vec{r}} f(\phi),
\]

\[
D(0, \theta)f(\phi) = f(\phi - \theta). \tag{12}
\]

Assume as basis functions the complete set of normalized eigenfunctions of the rotation subgroup \( \{f_n(\phi)\} \). \( f_n = (2\pi)^{-\frac{1}{4}} i^{-n} e^{in\phi} \), \( D(0, \theta)f_n(\phi) = e^{in\theta} f_n(\phi) \). The representation of E(2) in terms of the planar Bessel functions \( j_m \) [1] is expressed in the form:

\[
D(\vec{v}, \theta)f_n = \sum m \Delta(\vec{v}, \theta)_{mn} f_m,
\]

\[
\Delta(\vec{v}, \theta)_{mn} = (-1)^{m-n} e^{-im\beta} j_{m-n}(pr)e^{in(\beta-\theta)}. \tag{13}
\]

(\( r, \beta \)) are the polar coordinates of \( \vec{v} \) and \( j_m \) is the planar Bessel function of order m.

In order to study the \( P^2 \) eigenvalue equation we consider the 3D-Laplace equation in an isotropic and homogeneous 3D-space. In the cylindrical coordinates we have:

\[
\left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial x_3^2} \right) \psi = 0. \tag{14}
\]

We are interested in solutions of the type:

\[
\psi(r, \theta, x_3) = \varphi(r, \theta) \cdot \sigma(x_3), \quad \frac{\partial^2}{\partial x_3^2} \sigma(x_3) \equiv p^2 \sigma(x_3). \tag{15}
\]

For \( p \) positive, by selecting, for positive \( x_3 \), the solution: \( \sigma = e^{-x_3 p} \) and for negative \( x_3 \) the solution: \( \sigma = e^{x_3 p} \), one may obtain square integrable eigenfunctions.
Clearly, the choice of the cylindrical coordinates, instead of the spherical one, breaks the symmetry of 3D-spatial rotation group $SO(3)$: when cylindrical coordinates are chosen, $x_3$ is differently treated with respect to the two remaining coordinates and this singles out a privileged axis for rotations. The resulting symmetry group is $E(2)$, the group contraction of $SO(3)$. Indeed, the Laplace equation (14) reduces to the eigenvalue equation for $P^2$ (in polar coordinates), i.e. the 2D-Helmholtz equation:

$$P_+ P_- \varphi = \left( \frac{\partial^2}{\partial r^2} + \frac{\partial}{r \partial r} + \frac{\partial^2}{r^2 \partial \theta^2} \right) \varphi = -\lambda^2 \varphi. \tag{16}$$

Assuming $\varphi(r, \theta) = f_n(r) \cdot e^{i n \theta}$, solving this equation gives $f_n(r) = j_n(pr)$, with $j_n(p r)$ solution of the planar Bessel equation of order $n$:

$$j_n;+1 + \frac{1}{\eta} j_n;+ \left[ 1 - \frac{n^2}{\eta^2} \right] j_n = 0, \tag{17}$$

with $\eta = pr$. Positive/negative $n$ values correspond to positive/negative rotations (loop/antiloop) around the $x_3$ axis, namely to different orientations of the $x_3$ axis and thus to different solutions [10]. A similar situation occurs in the case of spherical Bessel functions. Thus, in a natural way we are led to consider topological properties of Bessel functions related with loop operators and loop-algebras, according to our comments in the Introduction.

We remark that although the solutions are not symmetric under the reversal of the $x_3$ axis (i.e. under time-reversal in our choice $x_3 \equiv t$), Eq. (17) is invariant under the $n \rightarrow -n$ exchange. Such an invariance reflects indeed the existence of the two sets, $D^n$ and $D^{-n-1}$, of the $SO(3)$ independent representations [12].

As in the case of the spherical Bessel equation (5) in Section 2, the planar Bessel equation (17), by setting $w_{n,l}(\eta) = (\eta)^{-l}J_n(\eta)$, $\eta = e^{-t/\alpha}$, and by introducing the mirror parameter $l = \pm n$, gives the couple of damped/amplified harmonic oscillators:

$$l = -n : \quad \ddot{w}_{n,-n} + \frac{2n}{\alpha} \omega \dot{w}_{n,-n} + \left( \frac{\lambda}{\alpha} \right)^2 e^{-2t/\alpha} \dot{w}_{n,-n} = 0, \quad \ddot{w}_{n,-n} = 0,$$

$$l = n : \quad \ddot{w}_{n,n} - \frac{2n}{\alpha} \dot{w}_{n,n} + \left( \frac{\lambda}{\alpha} \right)^2 e^{-2t/\alpha} \dot{w}_{n,n} = 0, \quad \ddot{w}_{n,n} = 0, \tag{18}$$

where the value of $l$ is connected with the choice of negative/positive eigenvalues of the rotation generator $M$.

We may thus summarize our discussion and results up to this point in the following way: the breakdown of the rotational symmetry of $SO(3)$, which leads to its group contraction $E(2)$, has been shown to introduce a crucial difference in the double choice of the $x_3$ axis orientation, namely the loss of loop-antiloop symmetry. This results, in turn, in the difference between the planar Bessel functions or order $+n$ and the ones of order $-n$, in terms of which the $E(2)$ representations can be built. Then, we have shown that the planar Bessel equation (17) can be cast, by a convenient re-parametrization, into the set of Eqs. (18) for the damped/amplified harmonic oscillators: the mirror index $\pm n$ of the Bessel functions is associated to the couple of damped/amplified harmonic oscillators: it is a time-mirror index$^1$.

The physical content of the contraction of $SO(3)$ to $E(2)$ can be geometrically depicted as the projection of the sphere, corresponding to $SO(3)$, on the plane tangent to one of the poles. The radius $\rho$ of the sphere acts as a ”scale”: the $E(2)$ translations in the tangent plane are ”good” approximations of rotations in the limit $\rho \rightarrow \infty$, namely for distances much smaller than

$^1$ Identifying $x_3$ with the time $t$, $p$ acquires the dimensions of an energy over an action; see, e.g., the eigenfunctions $\sigma = e^{\pm x_3 p}$. 

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$\rho$. The physical meaning of this is that the $SO(3)$ contraction to $E(2)$ manifests itself in local observations. However, in local observations the orientation of the $x_3$ axis is "locked", which means that the symmetry of the solutions under the $n \rightarrow -n$ exchange is lost (breakdown of the loop-antiloop symmetry). Specifying the direction of the $x_3$ axis, namely, choosing one of the two possible forms for $\sigma$ in Eq. (15), produces topologically inequivalent configurations [9].

Finally note that the "spinor" representation of $SO(3)$, which involves half-integer $n$ and uses a representation in terms of the Pauli matrices $\sigma_i$, $i = 1, 2, 3$, maps the planar Bessel equation into the harmonic oscillator equation. In fact, for $n = \pm\frac{1}{2}$ and with $w = \sqrt{\eta} j_{\pm\frac{1}{2}}$, Eq. (17) reduces to the harmonic oscillator equation

$$\frac{d^2 w(r)}{dr^2} + p^2 w(r) = 0, \quad r = \frac{\eta}{p},$$

with frequency $p$ and $r$ playing the role of time. The oscillator so obtained may be thought as a possible classical analogue of a Fermi oscillator based on a 'rotation system' [13].

3.2. The spherical Bessel equation and $E(3)$

$E(3)$ is the group contraction of $SO(4)$. The algebra $e(3)$ has six translation and rotation generators $P_i$ and $M_i$, $i = 1, 2, 3$, respectively, with commutation relations:

$$[P_i, P_j] = 0, \quad [M_i, M_j] = \epsilon_{ijk} M_k, \quad [P_i, M_j] = \epsilon_{ijk} P_k;$$

(20)

The $SO(3)$ subgroup generated by the $M_i$'s is left unchanged in the contraction process. The algebra has two invariants, $P^2 = \Sigma P_i^2$ and $\Sigma P_i \cdot M_i$.

In the 4D-space the Laplace equation may be solved by using the position $\psi(x_1, x_2, x_3, x_4) = \varphi(r, \theta, \phi) \cdot \sigma(x_4)$, with $r, \theta, \phi$ spherical coordinates, $\sigma = e^{\pm x_4p}$. The role of time $t$ may be played by $x_4$. The solution is a functions $\varphi = Y_{n,m}(\theta, \phi) \cdot J_n(pr)$, with $Y_{n,m}$ the spherical harmonics and $J_n$ solution of the spherical Bessel equation. The spherical Bessel functions depend on the continuous eigenvalue $p^2$ of $P^2$. Their label $n$ (integer or zero) is related to the discrete eigenvalue $n(n + 1)$ of the rotation operator $M^2$ and classifies the representations $D^n$ of the compact subgroup $SO(3)$ of $E(3)$. The existence of two sets of independent representations [12], $D^n$ and $D^{-n-1}$, is reflected the invariance of Eq. (5) under the transformation $n \rightarrow -(n + 1)$. For $n = 0$, or $n = -1$ Eq. (5) reduces to the harmonic oscillator equation with frequency $p$:

$$\frac{d^2 w(r)}{dr^2} + p^2 w(r) = 0, \quad r = \frac{\eta}{p},$$

(21)

where $w = \eta J_0$ or $\eta J_{-1}$. The harmonic oscillator thus appears to be related to the "ground state" (in the $D^n$ or $D^{-n-1}$ spectra) of the Bessel system. Here, the so-called "true" representation of $SO(3)$, i.e. the one with integer values of $n$ [12], has been used.

The breakdown of the symmetry under the transformation $n \rightarrow -(n + 1)$ is built in in the geometrical structure of the $E(3)$ group: it corresponds to the breakdown of the $x_4$ axis reversal symmetry, or loop-antiloop symmetry, i.e. of time-reversal symmetry when $x_4$ is identified with time. Again, the physical meaning is that the $SO(4)$ contraction to $E(3)$ manifests itself in local observations and the $x_4$ axis orientation then gets "locked", turning in the loss of the symmetry of the solutions under the $n \rightarrow -(n + 1)$.

4. Loop algebras

It is known that infinite-dimensional algebras and the so-called "loop-algebras" can be constructed on some finite-dimensional group as $SU(1,1)$ [14]. The algebra $e(3)$, which, as we have seen, is related to the topological properties of the Bessel functions, is related to a
particular structure of loop-algebras. We focus on the Virasoro algebra, which plays a central role in the conformal field theories.

We will show that a contraction procedure, based on the so called graded contraction method [15, 16], maps the Virasoro algebra into a generalization of the Euclidean algebra $e(3)$.

The commutation relations of the Virasoro algebra $\mathcal{L}$ of central charge $c$ ($c$ commuting with all the $T$'s) are

$$[T_n, T_m] = (n - m)T_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m,0} , \quad m, n \in \mathbb{Z} .$$

(22)

The $\mathbb{Z}_2$-grading of the algebra consists in dividing the set of the $T_n$ generators into an even set $L_0 \equiv \{A_n, c\}$ and an odd set $L_1 \equiv \{B_n\}$, so that $\mathcal{L} = L_0 \oplus L_1$ and

$$A_n = \frac{1}{2}\left(T_{2n} + \frac{c}{8}\delta_{n,0}\right) , \quad B_n = \frac{1}{2}T_{2n+1} ,$$

(23)

$$[L_0, L_0] \subseteq L_0 , \quad [L_0, L_1] \subseteq L_1 , \quad [L_1, L_1] \subseteq L_0 .$$

(24)

The commutation relations of the graded generators are given by [17]

$$[A_n, A_m] = (n - m)A_{n+m} + \frac{2c}{12}(n^3 - n)\delta_{n+m,0} ,$$

(25)

$$[B_n, B_m] = (n - m)A_{n+m+1} + \frac{2c}{12}(n - \frac{1}{2})(n + \frac{1}{2})(n + \frac{3}{2})\delta_{n+m+1,0} ,$$

(26)

$$[A_n, B_m] = (n - m - \frac{1}{2})B_{n+m} .$$

(27)

Eq. (25) shows that $\{A_n, c\}$ is again a Virasoro algebra but with central charge $2c$.

The $\mathbb{Z}_2$-graded contraction of the algebra (25)-(27) is obtained [17] by putting equal to zero the commutator $[B_n, B_m]$:

$$[B_n, B_m] = 0 ,$$

(28)

In the centerless case ($c = 0$), the $A_0$ and $A_{\pm 1}$ generators close the algebra isomorphic to $so(3) \sim su(2)$ and the set of these three generators and the operators $B_{-\frac{1}{2}}, B_{\frac{1}{2}}$ and $B_{-\frac{3}{2}}$ close the $e(3)$ isomorphic algebra. This is shown by setting:

$$M_+ \equiv A_1 , \quad M_- \equiv A_{-1} , \quad M_3 \equiv iA_0 ,$$

$$P_+ \equiv B_{\frac{1}{2}} , \quad P_- \equiv B_{-\frac{3}{2}} , \quad P_3 \equiv iB_{-\frac{1}{2}} ,$$

(29)

with the $M$s and $P$s generators satisfying indeed the $e(3)$ commutation relations (20).

A general extension of this result is the following: the algebra $\mathcal{E}_n \equiv \{A_0, A_{\pm n}\} \oplus \{B_{-\frac{1}{2}}, B_{\pm n-\frac{1}{2}}\}$ reproduces the $e(3)$ algebra for each integer value of $n$, provided the following positions are satisfied:

$$M_+ \equiv \frac{1}{n}A_n , \quad M_- \equiv \frac{1}{n}A_{-n} , \quad M_3 \equiv \frac{i}{n}A_0 ,$$

$$P_+ \equiv B_{n-\frac{1}{2}} , \quad P_- \equiv B_{n-\frac{3}{2}} , \quad P_3 \equiv iB_{n-\frac{1}{2}} .$$

(30)

We remark that, by choosing $A_{\pm n} = 0$, for non-zero values of $n$, the $e(2)$ algebra can be obtained as a subalgebra of (30).

Summarizing, the extension of the Virasoro algebra by means of its $\mathbb{Z}_2$-grading with the subsequent step of the $\mathbb{Z}_2$-graded contraction leads to a $n$-graded hierarchy of Euclidean algebras. Considering the discussion in the previous Sections, an interesting relation then emerges between the couple of damped/amplified parametric oscillators graded by $n$ and the loop algebras.
5. Conclusions
We have shown that a relation exists between spherical and planar Bessel equations and dissipation/amplification processes. Some topological properties of the Bessel functions in connection with Virasoro-like loop-algebras have been also considered. After a convenient parametrization, the breakdown of the \( n \to -(n + 1) \) \((n \to -n)\) symmetry of the spherical (planar) Bessel equation corresponds to the breakdown of time-reversal symmetry: the \textit{arrow of time} emerges in the manifold of the Bessel functions \( \{ J_i \}, \ i = n, -(n + 1) \) \((i = n, -n)\). Such a breakdown is realized through the mechanism of group contraction leading to the Euclidean groups \( E(3) \) and \( E(2) \), whose representations are constructed by means of the spherical and the planar Bessel functions, respectively.

We thus find that in our parametrization the symmetry of the Bessel equation is not the symmetry of their solutions. In quantum field theory (QFT) the spontaneous breakdown of symmetry occurs when the \textit{continuous} symmetry of the dynamical equations is not the symmetry of the physical vacuum. In the Bessel equation case the \textit{discrete} time-reversal symmetry is broken. However, since breakdown of time-reversal symmetry implies dissipation/amplification phenomena, also continuous time translational symmetry is broken. Moreover, in QFT the effects of spontaneous breakdown of symmetry are solely observable in the contraction limit, namely at the observation scale \( [18] \). In the Bessel equation case, the breakdown of time-reversal symmetry manifests itself at observational level as dissipation/amplification phenomena.

We have considered only the classical sector. The quantization of the damped harmonic oscillator has been extensively studied in a number of papers \([19] - [27]\). The couple of damped/amplified oscillators has been used in inflationary models of the Universe \([11]\), thermal field theories \([21]\), Chern-Simons gauge theory \([26]\), Bloch electrons in metals \([26]\), the dissipative quantum model of brain \([6, 7, 27]\), etc.

The canonical quantization of the simple damped harmonic oscillator requires the “doubling” of the system degrees of freedom by introducing the amplified oscillator, representing the environment, as the time-reversed copy of the damped oscillator \([20, 21, 28]\). Such a doubling is not required at a classical level, however, where it is possible to solve the damped oscillator equation by ignoring the environment. From our discussion it emerges that apparently Bessel equation may represent both the damped and the amplified oscillator as an inseparable “doublet” also at the classical level.

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