BRST invariance and de Rham-type cohomology of ’t Hooft-Polyakov monopole

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We exploit the ’t Hooft-Polyakov monopole to define closed algebra of the quantum field operators and the BRST charge $Q_{BRST}$. In the first-class configuration of the Dirac quantization, by including the $Q_{BRST}$-exact gauge fixing term and the Faddeev-Popov ghost term, we find the BRST invariant Hamiltonian to investigate the de Rham-type cohomology group structure for the monopole system. The Bogomol’nyi bound is also discussed in terms of the first-class topological charge defined on the extended internal 2-sphere.

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I. INTRODUCTION

The Becchi-Rouet-Stora-Tyutin (BRST) [1] symmetries have been considerably studied in the constrained physical systems on which many interesting physics phenomena in Nature are described. It is well known that the solitons and monopoles [2] are subject to the second-class constraints which can be rigorously treated in the Dirac Hamiltonian quantization scheme [3]. Despite all the past successes [4] of the quantization of the constrained physical systems through the uses of the first-class configuration of the Dirac Hamiltonian formalism and its corresponding BRST mechanism, there still does not exist a comprehensive understanding of the hidden geometry involved in the BRST symmetry invariance of the constrained systems.

Since the Dirac monopole string [5] was proposed to quantize the electric charges of the particles of matter, there has been lots of progress in understanding the properties of the magnetic monopole systems. It was shown in the Wu-Yang monopole theory [6] that the interaction of the Dirac monopole with the electromagnetic field removes the string line singularity inherent in the monopole. Recently the supersymmetric aspects of a charged particle were investigated in the background of these monopoles [7]. Moreover, it was found that the charged particle in the Dirac monopole background possesses the geometric features associated with cocycles [8]. For the unconstrained gauged Yang-Mills theory, the BRST symmetry was also analyzed in terms of the cocycles and chiral anomaly [9].

In this paper, we will introduce the ’t Hooft-Polyakov monopole [10, 11] to yield its BRST charge, de Rham-type cohomology and closed algebra of the quantum field operators. To do this, we will find the first-class Hamiltonian of the monopole, since the ’t Hooft-Polyakov monopole is classified as the second-class system in the Dirac quantization formalism. We will then define the monopole charge in the U(1) subgroup of the SU(2) gauge group in the first-class configuration to investigate the Bogomol’nyi bound on the extended internal 2-sphere. We will next obtain the explicit form of the BRST invariant Hamiltonian and discuss the geometric aspects of the corresponding de Rham-type cohomology.

In Section II, we will consider the ’t Hooft-Polyakov monopole action to construct the closed algebra of its quantum operators and the first-class Hamiltonian. In Section III, in the first-class configuration, we will study the monopole charge defined in the U(1) subgroup of the SU(2) gauge group and the U(1) gauge invariant electromagnetic fields, and discuss the Bogomol’nyi bound of the first-class static energy of the ’t Hooft-Polyakov monopole. By including the ghost fields, we will then construct the BRST invariant ’t Hooft-Polyakov monopole Hamiltonian and define the de Rham-type cohomology group. In Section IV, we will summarize our results with some comments, and in Appendix A we will list the first-class canonical variables.

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II. CLOSED ALGEBRA OF QUANTUM OPERATORS AND FIRST-CLASS HAMILTONIAN

In this section, in the Dirac quantization formalism, we will construct the closed algebra of the quantum operators and the first-class Hamiltonian of the 't Hooft-Polyakov monopole whose Lagrangian is given by [10, 11],

\[
L_0 = \int d^3x \left[ -\frac{1}{4} G_{\mu\nu} G^{\mu\nu} + \frac{1}{2} D_\mu \phi^a D^\mu \phi^a - \frac{1}{4} \lambda (\phi^a \phi^a - F^2)^2 \right],
\]

(2.1)

where \( A_\mu^a \) (\( a = 1, 2, 3 \)) and \( \phi^a \) are the SU(2) non-abelian gauge fields and the real scalar Higgs fields, respectively. The field strengths of \( A_\mu^a \) and the covariant derivatives of \( \phi^a \) are defined as

\[
G_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g \epsilon^{abc} A_\mu^b A_\nu^c,
\]

\[
D_\mu \phi^a = \partial_\mu \phi^a + g \epsilon^{abc} A_\mu^b \phi^c.
\]

(2.2)

Using the Lagrangian (2.1), we readily obtain the equations of motion for the fields \((\phi^a, A_\mu^a)\) [10, 11]

\[
D_\mu G^{\mu\nu} - g \epsilon^{abc} (D^\nu \phi^b) \phi^c = 0,
\]

\[
D_\mu D^\mu \phi^a + \lambda (\phi^a \phi^a - F^2) \phi^a = 0.
\]

(2.3)

The canonical momenta \((\pi^a, \pi_\mu^a)\) conjugate to \((\phi^a, A_\mu^a)\) are given by

\[
\pi^a_\mu = D_\mu \phi^a,
\]

\[
\pi_i^a = G^a_{0i},
\]

\[
\pi_0^a = 0.
\]

(2.4) \(\) (2.5) \(\) (2.6)

We also obtain the canonical momentum \(\pi_\lambda\) conjugate to the real scalar multiplier field \(\lambda\) given by

\[
\pi_\lambda = 0.
\]

(2.7)

Exploiting the above momenta and the Legendre transformation, we obtain the Hamiltonian

\[
H = \int d^3x \left[ \frac{1}{2} \pi^a \dot{x}^a + \frac{1}{2} \pi_i^a \dot{\pi}_i^a + \frac{1}{2} D_i \phi^a D^i \phi^a + \frac{1}{4} G^a_{i\mu} G^{a\mu} + \frac{1}{4} \lambda (\phi^a \phi^a - F^2)^2 \right],
\]

(2.8)

where we have used the gauge condition \( A_0^a = 0 \) [10, 11]. The canonical variables are subject to the non-vanishing Poisson brackets

\[
\{ \phi^a(x), \pi^b(y) \} = \delta^{ab} \delta^3(x - y),
\]

\[
\{ A_\mu^a(x), \pi_i^b(y) \} = \delta^{ab} \delta_{\mu i} \delta^3(x - y),
\]

\[
\{ \lambda(x), \pi_\lambda(y) \} = \delta^3(x - y).
\]

(2.9)

By implementing the Dirac quantization scheme [3], we find that our Hamiltonian system is subject to the following second-class constraints

\[
\Omega_1 = \phi^a \phi^a - F^2 \approx 0,
\]

\[
\Omega_2 = \phi^a \pi^a \approx 0.
\]

(2.10) \(\) (2.11)

Here one notes that, in fact, the time evolution of the identity (2.7) yields the constraint (2.10). Moreover, the identities (2.6) and (2.7) are easily shown to be the trivial first-class constraints decoupled from our system of interest. With \( \epsilon^{12} = -\epsilon^{21} = 1 \) this second-class constraint algebra is given by

\[
\Delta_{kk'}(x, y) = \{ \Omega_k(x), \Omega_{k'}(y) \} = \epsilon^{kk'} \phi^a \phi^a \delta^3(x - y).
\]

(2.12)

Next, we consider the Dirac brackets defined as

\[
\{ A(x), B(y) \}_D = \{ A(x), B(y) \} - \int d^3z \int d^3z' \{ A(x), \Omega_k(z) \} \Delta_{kk'}^{-1}(z, z') \{ \Omega_k(z'), B(y) \},
\]

(2.13)
We also find the closed algebra

\[
\{\phi^a(x), \pi^b(y)\} = i \left( \delta^{ab} - \frac{\phi^a \phi^b}{\phi^c \phi^c} \right) \delta^3(x - y),
\]

\[
\{\pi^a(x), \pi^b(y)\} = \frac{i}{\phi^c \phi^c} \left( \phi^b \pi^a - \phi^a \pi^b \right) \delta^3(x - y),
\]

\[
[A^a_\mu(x), \pi^b_\nu(y)] = i \delta^{ab} \delta_{\mu\nu} \delta^3(x - y),
\]

\[
[\lambda(x), \pi_\lambda(y)] = i \delta^3(x - y),
\]

(2.14)

where the canonical quantum operators are given by

\[
\pi^a = -i \left( \delta^{ab} - \frac{\phi^a \phi^b}{\phi^c \phi^c} \right) \partial_{\phi^b},
\]

\[
\pi^a_\mu = -i \partial_{A^a_\mu}, \quad \pi_\lambda = -i \partial_{\lambda}.
\]

(2.15)

We also find the closed algebra

\[
[S^a, S^b] = \epsilon^{abc} S^c,
\]

\[
[S^a, T^b] = \epsilon^{abc} T^c,
\]

\[
[T^a, T^b] = 0,
\]

(2.16)

where

\[
S^a = \int d^3x \, i \epsilon^{abc} \pi^b \phi^c,
\]

\[
T^a = \int d^3x \, i \phi^a.
\]

(2.17)

Following the Hamiltonian quantization scheme for constrained systems \([3, 4, 12, 13, 14]\), we proceed to convert the second-class constraints \(\Omega_i = 0 \ (i = 1, 2)\) into the first-class ones. For this we introduce two canonically conjugate St"uckelberg fields \((\theta, \pi_\theta)\) with Poisson bracket

\[
\{\theta(x), \pi_\theta(y)\} = \delta^3(x - y).
\]

(2.18)

The strongly involutive first-class constraints \(\tilde{\Omega}_i\) are constructed as a power series of the St"uckelberg fields to yield

\[
\tilde{\Omega}_1 = \Omega_1 + 2\theta,
\]

\[
\tilde{\Omega}_2 = \Omega_2 - \phi^a \phi^b \pi_\theta,
\]

(2.19)

and their commutator is given by

\[
\{\tilde{\Omega}_i, \tilde{\Omega}_j\} = 0.
\]

(2.20)

In general, following the Dirac quantization scheme, we can construct the first-class constraints satisfying the Lie algebra

\[
\{\tilde{\Omega}_i, \tilde{\Omega}_j\} = C^k_{ij} \tilde{\Omega}_k.
\]

(2.21)

Since the first-class constraints are strongly zero to yield \(\{\tilde{\Omega}_i, \tilde{\Omega}_j\}|_{\text{phy}} = 0\) from (2.21), one does not have any difficulties in construction of the quantum commutators and in quantization of the given monopole system. In that sense, one has degrees of freedom in taking a set of the first-class constraints. For instance, the first-class constraints \(\tilde{\Omega}_i\) in (2.19) are a specific choice with \(C^k_{ij} = 0\). In fact, the sets of the first-class constraints form an equivalent family governed by the SO(2) group \([15]\).

Next, after some tedious algebra, we construct the first-class Hamiltonian of (2.8) in terms of the original fields

\[
\tilde{H} = \int d^3x \left[ \frac{1}{2} (\pi^a - \phi^a \pi_\theta)(\pi^a - \phi^a \pi_\theta) \frac{\phi^c \phi^c}{\phi^c \phi^c} + \frac{1}{2} \pi^a_\mu \pi^a_\mu + \frac{1}{2} D_\mu \phi^a D^a \phi_\mu \phi^c \phi^c + \frac{2\theta}{\phi^c \phi^c} + \frac{1}{4} G^a_i C^a_{ij} \right]
\]

\[+ \frac{1}{4} \lambda (\phi^a \phi^a - F^2 + 2\theta)^2. \]

(2.22)
We note that this Hamiltonian is strongly involutive with the first-class constraints,
\[ \{ \tilde{\Omega}_i, \tilde{H} \} = 0. \] (2.23)

When we consider the time evolution of the constraint algebra, as determined by computing the Poisson brackets of the constraints with the Hamiltonian (2.22), we readily see from the Poisson bracket \( \{ \Omega_1, H \} = 0 \) that we need to improve the Hamiltonian into the following, equivalent first-class Hamiltonian,
\[ \tilde{H}' = \tilde{H} + \int d^3 x \, \pi_\theta \tilde{\Omega}_2. \] (2.24)

In fact, this improved Hamiltonian generates the constraint algebra
\[ \{ \tilde{\Omega}_1, \tilde{H}' \} = 2 \tilde{\Omega}_2, \]
\[ \{ \tilde{\Omega}_2, \tilde{H}' \} = 0. \] (2.25)

Since the Hamiltonians \( \tilde{H} \) and \( \tilde{H}' \) only differ by a term which vanishes on the constraint surface, they lead to an equivalent dynamics on the constraint surface. Finally, the first-class canonical variables are explicitly constructed and listed in Appendix A.

### III. MONOPOLE CHARGE, BRST SYMMETRY AND DE RHAM-TYPE COHOMOLOGY

In this section, we will investigate the de Rham-type cohomology group structure for the ’t Hooft-Polyakov monopole system, after constructing its first-class monopole charge and BRST charge. To do this, we first revisit the original ’t Hooft-Polyakov monopole Lagrangian in (2.1) to consider the monopole charge which is defined in the U(1) subgroup of the SU(2) gauge group. The U(1) gauge invariant electromagnetic fields \( F_{\mu\nu} \) are defined as \([10, 11]\)
\[ F_{\mu\nu} = \bar{\phi}^a G^a_{\mu\nu} - \frac{1}{g} \epsilon^{abc} \bar{\phi}^a D_\mu \bar{\phi}^b D_\nu \bar{\phi}^c, \] (3.1)
and the topological current \( k^\mu \) is also defined as \([16]\)
\[ k^\mu = -\frac{1}{8\pi} \epsilon^{\mu\nu\rho\sigma} \epsilon^{abc} \partial_\nu \bar{\phi}^a \partial_\rho \bar{\phi}^b \partial_\sigma \bar{\phi}^c, \] (3.2)
where the rescaled real scalar Higgs fields are given by
\[ \bar{\phi}^a = \frac{\phi^a}{(\phi^c \phi^c)^{1/2}}. \] (3.3)

Exploiting the conformal map condition
\[ \bar{\phi}^a D_\mu \bar{\phi}^a = \bar{\phi}^a \partial_\mu \bar{\phi}^a = 0, \] (3.4)
one readily checks that the dual equations of motion for the electromagnetic fields \( F_{\mu\nu} \) in (3.1) yield
\[ \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} \partial_\nu F_{\rho\sigma} = \frac{4\pi}{g} k^\mu, \] (3.5)
from which the magnetic monopole charge \( m \) is given by
\[ m = \frac{1}{g} \int d^3 x \, k^0 = \frac{1}{g} Q_{top}. \] (3.6)
Here \( Q_{top} \) is the topological charge to be discussed later.

Next, we return to the first-class physical system described in the previous section. In this configuration, the first-class topological current is given by
\[ \tilde{k}^\mu = -\frac{1}{8\pi} \epsilon^{\mu\nu\rho\sigma} \epsilon^{abc} \partial_\nu \bar{\phi}^a \partial_\rho \bar{\phi}^b \partial_\sigma \bar{\phi}^c \left( \frac{\phi^c \phi^c + 2\theta}{\bar{\phi}^a \bar{\phi}^a} \right)^{3/2}. \] (3.7)
Here we have used the first-class rescaled fields \( \tilde{\phi}^a \) defined as
\[
\tilde{\phi}^a = \phi^a \left( \frac{\delta^c \phi_c + 2\theta}{\phi^c \phi^c} \right)^{1/2},
\]
which satisfies
\[
\tilde{\phi}^a \tilde{\phi}^a - 1 = 0. \tag{3.9}
\]
Exploiting the antisymmetric property of \( \epsilon^{\mu
u\rho\sigma} \) in (3.2) and (3.7), one readily check the following divergence
\[
\partial_\mu \tilde{k}^\mu = 0, \tag{3.10}
\]
as in the second-class case: \( \partial_\mu k^\mu = 0 \). Moreover, the first-class magnetic monopole charge \( \tilde{m} \) is given by
\[
\tilde{m} = \frac{1}{g} \tilde{Q}_{\text{top}}, \tag{3.11}
\]
where the first-class topological charge \( \tilde{Q}_{\text{top}} \) is given by
\[
\tilde{Q}_{\text{top}} = \frac{1}{4\pi} \int_{\tilde{S}^2_{\text{int}}} d\tilde{A}^{\text{int}}_{a} \tilde{\phi}^a, \tag{3.12}
\]
where \( \tilde{A}^{\text{int}} \) is the surface of a unit 2-sphere \( \tilde{S}^2_{\text{int}} \). Here one notes that \( \tilde{Q}_{\text{top}} \) yields the winding number in the map: \( S^2_{\text{ph}/y} \rightarrow \tilde{S}^2_{\text{int}} \), where \( S^2_{\text{ph}/y} \) and \( \tilde{S}^2_{\text{int}} \) are the 2-sphere compactified at infinity in the physical coordinate space and the other 2-sphere of unit radius in the extended internal space of \( \tilde{\phi}^a \) satisfying the first-class constraint (3.9), respectively, associated with the homotopy group \( \pi_2(\tilde{S}^2) = Z \). In the second-class configuration, the topological charge \( Q_{\text{top}} \) is described by the winding number in the map: \( S^2_{\text{ph}/y} \rightarrow S^2_{\text{int}} \), where \( S^2_{\text{int}} \) is the 2-sphere of unit radius in the internal space of \( \tilde{\phi}^a \) with \( \tilde{\phi}^a \tilde{\phi}^a - 1 \approx 0 \). The static conserved energy \( E \) of the 't Hooft-Polyakov monopole in the first-class configuration, corresponding to the static limit of the first-class Hamiltonian \( \tilde{H} \) in (2.22) with \( \lambda = 0 \), is now given in terms of the \( \tilde{Q}_{\text{top}} \)
\[
E = \int d^3x \frac{1}{4} \left( \tilde{G}^a_{ij} - \epsilon_{ijk} \tilde{D}^k \tilde{\phi}^a \right)^2 + \frac{4\pi F}{g} \tilde{Q}_{\text{top}}, \tag{3.13}
\]
where the first-class variables \( \tilde{G}^a_{ij} \) and \( \tilde{D}^k \tilde{\phi}^a \) are obtainable from Appendix A. For a given \( \tilde{Q}_{\text{top}} \)-sector, the static energy \( E \) has the Bogomol’nyi lower bound \( \frac{4\pi F}{g} \tilde{Q}_{\text{top}} \) when the variables satisfy the condition
\[
\tilde{G}^a_{ij} = \epsilon_{ijk} \tilde{D}^k \tilde{\phi}^a. \tag{3.14}
\]

Now, in order to investigate the de Rham-type cohomology group structure for the 't Hooft-Polyakov monopole system, we proceed to implement the covariant Batalin-Fradkin-Vilkovisky formalism [17]. We start by the construction of the nilpotent BRST operator, by introducing two canonical sets of ghost number \( \eta = 1 \) field and ghost number \( \eta = -1 \) field \( (C^i, P_i), (\bar{P}^i, \bar{C}) \) and the ghost number \( \eta = 0 \) auxiliary fields \( (N^i, B_i) \), which satisfy the (anti)commutators,
\[
\{C^i(x), \bar{P}_j(y)\} = \{P^i(x), \bar{C}_j(y)\} = \{N^i(x), B_j(y)\} = \delta^i_j \delta^3(x - y) \quad (i = 1, 2). \tag{3.15}
\]
Here the super-Poisson bracket is defined as
\[
\{A, B\} = \frac{\delta A}{\delta q}_r \frac{\delta B}{\delta p}_l - (-1)^{\eta_A \eta_B} \frac{\delta B}{\delta q}_r \frac{\delta A}{\delta p}_l, \tag{3.16}
\]
where \( \eta_A \) is the ghost number in \( A \) and the subscript \( r \) and \( l \) denote right and left derivatives, respectively.

The BRST operator for our constraint algebra is then simply given by
\[
Q_{\text{BRST}} = \int d^3x \ (C^i \bar{Q}_i + \bar{P}^i B_i). \tag{3.17}
\]
We choose the unitary gauge with
\[
\chi^1 = \Omega_1, \quad \chi^2 = \Omega_2 \tag{3.18}
\]
by selecting the gauge fixing functional
\[ \Psi = \int d^3x \left( \tilde{C}_i \chi^i + \tilde{P}_i N^i \right). \] (3.19)

One can now readily see that \( Q_B \) is nilpotent
\[ Q_B^2 = \{ Q_B, Q_B \} = 0, \] (3.20)
and \( Q_{BRST} \) is the generator of the infinitesimal BRST transformations
\[
\begin{align*}
\delta_{Q_{BRST}} \phi^a &= -c^2 \phi^a, \\
\delta_{Q_{BRST}} A^a_\mu &= 0, \\
\delta_{Q_{BRST}} \theta &= c^2 \phi^a, \\
\delta_{Q_{BRST}} C^i &= 0, \\
\delta_{Q_{BRST}} \pi^i_\mu &= 0, \\
\delta_{Q_{BRST}} N^i &= -\mu^i, \\
\delta_{Q_{BRST}} \lambda &= 0.
\end{align*}
\] (3.21)

Furthermore, the first-class Hamiltonian \( \tilde{H} \) in (2.22) is \( Q_{BRST} \)-closed
\[ \delta_{Q_{BRST}} \tilde{H} = \{ Q_{BRST}, \tilde{H} \} = 0, \] (3.22)
and
\[ \delta_{Q_{BRST}} \{ Q_{BRST}, \Psi \} = \{ Q_{BRST}, \{ Q_{BRST}, \Psi \} \} = 0, \] (3.23)
which follows from the nilpotency of the charge \( Q_{BRST} \). The gauge fixed BRST invariant Hamiltonian is now given by
\[
\begin{align*}
H_{\text{eff}} &= \tilde{H}'' - \{ Q_{BRST}, \Psi \}, \\
\tilde{H}'' &= \tilde{H} + \int d^3x \left( \pi_\theta \Omega_2 - 2c^1 \bar{P}_2 \right),
\end{align*}
\] (3.24)
with \( \tilde{H} \) defined in (2.22). In order to guarantee the BRST invariance of \( H_{\text{eff}} \), we have included in \( H_{\text{eff}} \) of (3.24) the \( Q_{BRST} \)-exact term, and in \( \tilde{H}'' \) of (3.25) the term associated with \( \pi_\theta \) in \( \tilde{H}' \) in (2.24) and the Faddeev-Popov ghost term [18]. In fact, the term \( \{ Q_{BRST}, \Psi \} \) fixes the particular unitary gauge corresponding to the fixed point \( (\theta = 0, \pi_\theta = 0) \) in the gauge degrees of freedom associated with two dimensional manifold described by the internal phase space coordinates \( (\theta, \pi_\theta) \), which physically speaking are two canonically conjugate St"uckelberg fields.

In general, by introducing the BRST operator
\[ Q_{BRST} : \omega_p \rightarrow \omega_{p+1}, \] (3.26)
where \( \omega_p \) is a ghost number \( p \)-form with \( \eta_{\omega_p} = p \), we define the \( p \)-th de Rham-type cohomology group \( H^p(M, R) \) of the manifold \( M \) and the field of real number \( R \) with the following quotient group
\[ H^p(M, R) = \frac{Z^p(M, R)}{B^p(M, R)}. \] (3.27)
Here \( Z^p(M, R) \) are the collection of all \( Q_{BRST} \)-closed ghost number \( p \)-forms \( \omega_p \) for which \( Q_{BRST} \omega_p = 0 \) and \( B^p(M, R) \) are the collection of all \( Q_{BRST} \)-exact ghost number \( p \)-forms \( \omega_p \) for which \( \omega_p = Q_{BRST} \omega_{p-1} \).

For the case of the first-class Hamiltonian in (3.25), the Hamiltonians \( H_{\text{eff}} \) and \( \tilde{H}'' \) are readily shown to be \( Q_{BRST} \)-closed as in the case of \( Q_{BRST} \Psi = \{ Q_{BRST}, \Psi \} \). With these ghost number 0-forms, we define the \( Z^0(M, R) \) with \( M \) being the monopole Hilbert space and \( R \) being the real field. Since \( \Psi \) is the ghost number \(-1\)-form and \( Q_{BRST} \Psi \) is \( Q_{BRST} \)-exact ghost number 0-form, we also define the \( B^0(M, R) \). Moreover, the ghost number 0-form \( H_{\text{eff}} \) is deformed into the other ghost number 0-form \( \tilde{H}'' \). In other words, \( H_{\text{eff}} \) is homologous to \( \tilde{H}'' \) under the BRST transformation \( Q_{BRST}, H_{\text{eff}} \sim \tilde{H}'' \), since \( Q_{BRST} \Psi = \tilde{H}'' - H_{\text{eff}} \). With these \( Z^p(M, R) \) and \( B^0(M, R) \), we define the 0-th de Rham-type cohomology group \( H^0(M, R) \) for the ’t Hooft-Polyakov monopole system.

Finally, after the path integral algebra related to the evaluation of the Legendre transformation of \( H_{\text{eff}} \), we arrive at the manifestly covariant BRST improved Lagrangian
\[ L_{\text{eff}} = L_0 + L_{WZ} + L_{\text{ghost}} \] (3.28)
where \( L_0 \) is given by (2.1) and

\[
L_{WZ} = \int d^3x \left[ \frac{\theta}{\phi^a \phi^a} D_\mu \phi^a D^\mu \phi^a - \lambda \theta (\phi^a \phi^a - F^2 + \theta) - \frac{F^2}{2(\phi^a \phi^a)^2} D_\mu \theta D^\mu \theta \right],
\]

\[
L_{\text{ghost}} = \int d^3x \left[ -\frac{1}{2F^2} (\phi^a \phi^a)^2 (B_2 + 2\bar{\theta} \bar{\varphi} c^2)^2 - \frac{1}{\phi^a \phi^a} D_\mu \theta D^\mu B_2 + D_\mu \bar{\theta} D^\mu \bar{\varphi} c^2 \right].
\]  

(3.29)

The Lagrangian \( L_{\text{eff}} \) in (3.28) can be readily shown to be a covariant form of the ’t Hooft-Polyakov monopole Lagrangian in (2.1). Here, we note that the canonical fields \( (\phi^a, A^a_\mu, \lambda) \) in \( L_{\text{eff}} \) are unconstrained ones and the St"{u}ckelberg field \( \theta \) becomes a nontrivial propagating field. The BRST gauge fixed effective Lagrangian (3.28) is readily shown to be manifestly invariant under the following BRST transformations,

\[
\delta_\epsilon \phi^a = \epsilon \phi^a c^2, \quad \delta_\epsilon A^a_\mu = 0, \\
\delta_\epsilon \lambda = 0, \quad \delta_\epsilon \theta = -\epsilon \phi^a \phi^a c^2, \\
\delta_\epsilon \bar{\varphi} = -\epsilon B_2, \quad \delta_\epsilon c^2 = \delta_\epsilon B_2 = 0,
\]

(3.30)

where \( \epsilon \) is an infinitesimal Grassmann valued parameter.

IV. CONCLUSIONS

In conclusion, we have found the closed algebra of the quantum operators in the the ’t Hooft-Polyakov monopole using the Dirac brackets of the Dirac Hamiltonian formalism. Next, in the first-class configuration, we have constructed the first-class monopole charge in the U(1) subgroup of the SU(2) gauge group to discuss the Bogomol’nyi bound defined on the extended internal 2-sphere. We then have included the \( Q_{\text{BRST}} \)-exact gauge fixing term and the Faddeev-Popov ghost term in the first-class Hamiltonian to define the BRST symmetries and the de Rham-type cohomology group for the monopole system.

The cohomology group structure discussed in this paper is a generic property shared by the constrained physical systems, such as the monopoles, Skyrman model, CP(n) model, O(n) model, and so on [4]. In these models, the fields \( \phi^a \) satisfy the geometric constraint of the typical form \( \phi^a \phi^a - F^2 \approx 0 \). Using the time evolution of this constraint, we construct the second-class constraints of the models, which can be converted into the first-class systems in the Dirac Hamiltonian quantization by introducing the St"{u}ckelberg fields. Next by including the ghost degrees of freedom in the constrained models, we can discuss the BRST symmetries and also define the de Rham-type cohomology group as in this ’t Hooft-Polyakov monopole system. In the future studies, it will be interesting to discuss the generic features of the cocycles involved in the constrained physical systems.

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APPENDIX A: FIRST-CLASS CANONICAL VARIABLES

We construct the first-class canonical variables \( \tilde{\mathcal{F}} = (\tilde{\phi}^a, \tilde{\pi}^a, \tilde{A}^a_\mu, \tilde{\lambda}, \tilde{\pi}_\lambda) \), associated with the original variables \( \mathcal{F} = (\phi^a, \pi^a, A^a_\mu, \pi_\lambda, \lambda) \), in the extended phase space. These variables are obtained as a power series in the St"{u}ckelberg fields \( (\theta, \pi_0) \), by demanding that they are in strong involution with the first-class constraints (2.19),

\[
\{\tilde{\Omega}_i, \tilde{\mathcal{F}}\} = 0.
\]

(A.1)

After some algebra similar to the case of the first-class Hamiltonian, we obtain for the first-class canonical variables

\[
\tilde{\phi}^a = \phi^a \left( \frac{\phi^c \phi^c + 2\theta}{\phi^a \phi^a} \right)^{1/2},
\]

\[
\tilde{\pi}^a = (\pi^a - \phi^a \pi_0) \left( \frac{\phi^c \phi^c}{\phi^a \phi^a} + 2\theta \right)^{1/2},
\]

\[
\tilde{A}^a_\mu = A^a_\mu, \quad \tilde{\pi}_\mu = \pi_\mu, \\
\tilde{\lambda} = \lambda, \quad \tilde{\pi}_\lambda = \pi_\lambda.
\]

(A.2)
Next, we find for the Hamiltonian in (2.22)

$$\tilde{H} = \int d^3 x \left[ \frac{1}{2} \tilde{\pi}^a \tilde{\pi}^a + \frac{1}{2} \tilde{\pi}^a_i \tilde{\pi}^a_i + \frac{1}{2} \tilde{D}_i \tilde{\phi}^a \tilde{D}_i \tilde{\phi}^a + \frac{1}{4} \tilde{G}_{ij} \tilde{G}_{ij} + \frac{1}{4} \lambda (\tilde{\phi}^a \tilde{\phi}^a - F^2)^2 \right], \quad (A.3)$$

where the first-class variables are given above and

$$\tilde{D}_i \tilde{\phi}^a = D_i \phi^a \left( \frac{\phi^c \phi^c + 2 \theta}{\phi^a \phi^a} \right)^{1/2},$$

$$\tilde{G}^a_{ij} = G^a_{ij}. \quad (A.4)$$

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