Convergence and its Properties on Soft $𝕊_b$-Metric Spaces

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Abstract: The primary motivation behind this article is to consider an alternative type of speculation of metric space is named a soft $𝕊_b$-metric space. The definition of a soft $𝕊_b$-metric space presented. Cauchy sequence, convergent sequence, and contractive sequence in soft $𝕊_b$-metric space with providing a few outcomes on it given. Examples are given to delineate soft $𝕊_b$-metric space.

Keywords: soft set, soft b-metric Space, soft $𝕊$-metric space, soft $𝕊_b$-metric space.

1. Introduction and Preliminaries
Matric space wide region gives an amazing asset to the examination of optimization, approximation theory, variational inequalities etc. Many papers and studies have been written for developing varicose concepts on matric space [1], and vector matric space [2,10]. Idea of concept of soft set was studied by Molodtsov [14]. In [9,10] introduced soft set theory, soft real set, and their properties. The concepts of fixed-point theorem on G-metric space and soft metric space initiated by [4,11,12]. The notion of the concept S-metric space was introduced by [15]. In [3, 20] studied fixed point type in S-metric spaces also [4] gave some results on S-metric spaces. In 2016, Jong Kyu Kim et al. [19] introduced the notion of S-metric and studied some fixed-point theorems for two maps on complete S- metric spaces. Recently Mujeeb, Mohammad and Muhib studied common fixed-point theorems for compatible and weakly compatible mapping in context of S-metric spaces [19]. In the present paper the notion of soft S-metric space which is generalization of S-metric spaces will be introduced. Also, some common soft fixed-point theorems in the soft S-metric spaces with some examples are discussed. Bakhtin [7] studied the concept of b-metric space. In 2018, Aras, Bayramov and Cafarli [6] introduced the soft of S-metric spaces. In 2016, the idea of $𝕊_b$-metric space as the extension of metric space and S-metric space presented by Souayah and Mlaiki [17]. Also, a common fixed-point theorem for four mappings have been presented overall $𝕊_b$-metric space [16]. Moreover, the idea of an $𝕊_b$-metric used to be generalized to the concept of an $𝐴_p$-metric in [18].

In this article, we review some fundamental concepts which needed in this paper. Moreover, we study soft b-metric Spaces, soft $𝕊_b$-metric Spaces and give several important results that related this concept.
1.1. Definition [17]

Let \( J \) and \( \mathcal{U} \) be a nonempty initial universal and parameter set, respectively. Let \( \mathcal{P}(J) \) be the power set of \( J \). A pair \((\mathcal{F}, \mathcal{U})\) is named a soft set on \( J \) such that \( \mathcal{F} \) is a function from \( \mathcal{U} \) to \( \mathcal{P}(J) \).

1.2. Definition [13]

Let \((\mathcal{F}_1, \mathcal{U})\) and \((\mathcal{F}_2, \mathcal{U})\) be two soft sets over \( J \). Then \((\mathcal{F}_1, \mathcal{U})\) is supposed to be a soft subset of \((\mathcal{F}_2, \mathcal{U})\) if \( \mathcal{F}_1(\zeta) \subseteq \mathcal{F}_2(\zeta), \forall \zeta \in \mathcal{U} \). This is denoted by \(((\mathcal{F}_1, \mathcal{U}) \subseteq (\mathcal{F}_2, \mathcal{U})) \). \((\mathcal{F}_1, \mathcal{U})\) is said to be soft equal to \((\mathcal{F}_2, \mathcal{U})\) if \( \mathcal{F}_1(\zeta) = \mathcal{F}_2(\zeta), \forall \zeta \in \mathcal{U} \). This is denoted by \(((\mathcal{F}_1, \mathcal{U}) = (\mathcal{F}_2, \mathcal{U})\).

1.3. Definition: [7]

The supplement of a soft set \((\mathcal{F}, \mathcal{U})\) is defined as \((\mathcal{F}, \mathcal{U})^c = (\mathcal{F}^c, \mathcal{U})\), where \( \mathcal{F}^c : \mathcal{U} \to \mathcal{P}(J) \) is a mapping given by \( \mathcal{F}^c(\zeta) = J \setminus \mathcal{F}(\zeta), \forall \zeta \in \mathcal{U} \).

1.4. Definition [16]

(i) \((\mathcal{F}, \mathcal{U})\) is called a null soft set if \( \mathcal{F}(\zeta) = \emptyset \) for every \( \zeta \in \mathcal{U} \) and symbolized by \( \tilde{\mathcal{F}} \).

(ii) \((\mathcal{F}, \mathcal{U})\) is called an absolute soft set if \( \mathcal{F}(\zeta) = J \), for every \( \zeta \in \mathcal{U} \) and symbolized by \( \check{\mathcal{F}} \).

Obviously, we obtain \((\mathcal{J})^c = \check{\mathcal{F}} \) and \((\check{\mathcal{F}})^c = \mathcal{J}\).

1.5. Definition [9]

Assume \( J \) is a non-empty set and \( \mathcal{U} \) be a non-empty parameter set. Then a mapping \( \varepsilon : \mathcal{U} \to J \) named soft element of \( J \). A soft element \( \varepsilon \) of \( J \) is called belongs to a soft set \( \mathcal{B} \) of \( J \), that is symbolized by \( \varepsilon \in \mathcal{B} \), if \( \varepsilon(\zeta) \in \mathcal{U}(\zeta), \forall \zeta \in \mathcal{U} \). Therefore, a soft set \( \mathcal{B} \) of \( J \) with respect to the index set \( \mathcal{U} \); we obtain \( B(\varepsilon) = \{ \varepsilon(\zeta) : \varepsilon(\zeta) \in \mathcal{U}(\zeta), \forall \zeta \in \mathcal{U} \} \), \( \varepsilon \in \mathcal{U} \).

Each singleton soft set (a soft set \((\mathcal{F}, \mathcal{U})\) for which \( \mathcal{F}(\zeta) \) is a singleton set, \( \forall \zeta \in \mathcal{U} \)) is related to a soft element via basically recognizing the singleton set through the element that has been contained \( \forall e \in \mathcal{U} \).

1.6. Definition [9]

The set of real numbers is \( \mathbb{R} \) and \( \mathcal{B}(\mathbb{R}) \) is the family of each non-empty bounded subsets of \( \mathbb{R} \) and \( \mathcal{U} \) be a set of parameters. Thus, the function \( \mathcal{F} : \mathcal{U} \to \mathcal{B}(\mathbb{R}) \) is said to be a soft real set and symbolized by \((\mathcal{F}, \mathcal{U})\) and \( \mathcal{R}(\mathcal{U}) \) symbolizes the set of every soft real sets. Furthermore, \( \mathcal{R}(\mathcal{U})^* \) symbolizes the set of every positive soft real sets \((\mathcal{F}, \mathcal{U})\) is said to be a positive soft real set if \( \mathcal{F}(\zeta) \) is a subset of the set of positive real numbers for every \( \zeta \in \mathcal{U} \).

1.7. Definition [10]

A soft metric \( \mathcal{M} \) on \( \mathcal{I} \) is a mapping \( \mathcal{M} : \mathcal{S}\mathcal{E}(\mathcal{V}) \times \mathcal{S}\mathcal{E}(\mathcal{V}) \to \mathcal{R}(\mathcal{U})^* \) such that \( \mathcal{M} \) fulfills the following conditions for all \( \sigma, \zeta, \lambda \in \mathcal{S}\mathcal{E}(\mathcal{V}) \):

(M1) \( \mathcal{M}(\sigma, \zeta) \geq 0 \)

(M2) \( \mathcal{M}(\sigma, \zeta) = 0 \) if and only if \( \sigma = \zeta \)

(M3) \( \mathcal{M}(\sigma, \zeta) = \mathcal{M}(\zeta, \sigma) \)

(M4) \( \mathcal{M}(\sigma, \zeta) \leq \mathcal{M}(\sigma, \lambda) + \mathcal{M}(\lambda, \zeta) \)
The soft set \( \mathcal{J} \) with a soft metric \( \mathcal{M} \) on \( \mathcal{J} \) is called a soft metric space and is symbolized by \( (\mathcal{J}, \mathcal{M}, \mathcal{A}) \).

1.8. Definition [6]

Assume that \( \mathcal{J} \) and \( \mathcal{A} \) is a non-empty set and set of parameters, respectively. A mapping \( \mathcal{S}: \mathcal{S}(\mathcal{V}) \times \mathcal{S}(\mathcal{V}) \times \mathcal{S}(\mathcal{V}) \to \mathbb{R}(\mathcal{A})^* \) is said to be a soft $\mathcal{S}$-metric on $\mathcal{V}$ if $\mathcal{S}$ satisfies the accompanying conditions $\forall \sigma, \beta, \lambda \in \mathcal{S}(\mathcal{V})$:

1. (1) $\mathcal{S}(\sigma, \beta, \lambda) \geq 0$;
2. (2) $\mathcal{S}(\sigma, \beta, \lambda) = 0$ if and only if $\sigma = \beta = \lambda$;
3. (3) $\mathcal{S}(\sigma, \beta, \lambda) \leq \mathcal{S}(\sigma, \beta, \lambda) + \mathcal{S}(\beta, \beta, \beta) + \mathcal{S}(\lambda, \lambda, \lambda)$.

A triple \( (\mathcal{V}, \mathcal{S}, \mathcal{A}) \) is called a soft $\mathcal{S}$-metric space when $\mathcal{V}$ is a soft set and $\mathcal{S}$ is a soft $\mathcal{S}$-metric over $\mathcal{V}$.

1.9. Example

Assume that $\mathcal{V}$ is absolute soft set. Describe a mapping $\mathcal{S}: \mathcal{S}(\mathcal{V}) \times \mathcal{S}(\mathcal{V}) \times \mathcal{S}(\mathcal{V}) \to \mathbb{R}(\mathcal{A})^*$ through

$$\mathcal{S}(\sigma, \beta, \lambda) = \begin{cases} 0 & \sigma = \beta = \lambda \\ 0.5 & \text{otherwise} \end{cases}$$

For all $\sigma, \beta, \lambda \in \mathcal{S}(\mathcal{V})$. Then \( (\mathcal{V}, \mathcal{S}, \mathcal{A}) \) is a soft $\mathcal{S}$-metric space.

1.10. Example

Suppose that \( (V, d) \) is an ordinary metric on $V$ and $\mathcal{A} \subseteq \mathbb{R}$. The mapping \( \mathcal{S}: \mathcal{S}(\mathcal{V}) \times \mathcal{S}(\mathcal{V}) \times \mathcal{S}(\mathcal{V}) \to \mathbb{R}(\mathcal{A})^* \) defined by $\mathcal{S}(\sigma, \beta, \lambda) = |\sigma - \beta| + |\beta - \lambda| + d(\sigma, \beta) + d(\beta, \lambda)$ is a soft $\mathcal{S}$-metric on $\mathcal{S}(\mathcal{V})$.

1.11. Remark

Every soft $\mathcal{S}$-metric space is a family of parameterized $\mathcal{S}$-metric space.

2. Soft $\mathcal{S}_b$-Metric Space

This section contains a definition of soft $\mathcal{B}$-metric and soft $\mathcal{S}_b$-metric space. Studying and proving for some necessary results showed in this section as well.

2.1. Definition

A mapping $\mathcal{M}_b: \mathcal{S}(\mathcal{V}) \times \mathcal{S}(\mathcal{V}) \to \mathbb{R}(\mathcal{A})^*$ is called a soft $\mathcal{B}$-metric on $\mathcal{J}$ if $\mathcal{M}_b$ fulfills the following conditions such that $\mathcal{B} \geq 1$:

1. (M1) $\mathcal{M}_b(\sigma, \beta) \geq 0$
2. (M2) $\mathcal{M}_b(\sigma, \beta) = 0$ if and only if $\sigma = \beta$
3. (M3) $\mathcal{M}_b(\sigma, \beta) = \mathcal{M}_b(\beta, \sigma)$
4. (M4) $\mathcal{M}_b(\sigma, \beta) \leq b[\mathcal{M}_b(\sigma, \lambda) + \mathcal{M}_b(\lambda, \beta)]$ for all $\sigma, \beta, \lambda \in \mathcal{S}(\mathcal{V})$.

The triple $(\mathcal{J}, \mathcal{M}_b, \mathcal{A})$ is named a soft $\mathcal{B}$-metric space.
2.2. Definition

Let $V$ and $\mathcal{A}$ be a nonempty and set of parameters, respectively with $\mathcal{B} \subseteq \mathbb{R}$. A function $\mathcal{S}_b: \mathcal{S}(\mathbb{V}) \times \mathcal{S}(\mathbb{V}) \times \mathcal{S}(\mathbb{V}) \to \mathbb{R}(\mathcal{A})$ is called a soft $\mathcal{S}_b$-metric on $\mathcal{S}(\mathbb{V})$ if it satisfies the following conditions:

1. $\mathcal{S}_b(\sigma, \varnothing, \lambda) \geq 0$;
2. $\mathcal{S}_b(\sigma, \varnothing, \lambda) = 0$ if and only if $\sigma = \varnothing = \lambda$;
3. $\mathcal{S}_b(\sigma, \varnothing, \lambda) \leq \mathcal{B}[\mathcal{S}_b(\sigma, \sigma, \mathcal{I}) + \mathcal{S}_b(\varnothing, \varnothing, \mathcal{I}) + \mathcal{S}_b(\lambda, \lambda, \mathcal{I})]$. 

A triple $(\mathbb{V}, \mathcal{S}_b, \mathcal{A})$ is called as a soft $\mathcal{S}_b$-metric space when $\mathbb{V}$ is a soft set and $\mathcal{S}_b$ is a soft $\mathcal{S}_b$-metric of $\mathbb{V}$.

We notice that soft $\mathcal{S}_b$-metric space is the generalizations for $S$-metric space then each soft $S$-metric is soft $\mathcal{S}_b$-metric space whenever $\mathcal{B} = 1$.

2.3. Example

Let $V$ be normed space and let $\mathcal{A}$ be a non-empty set of parameters. Describe $\mathcal{S}: \mathcal{S}(\mathbb{V}) \times \mathcal{S}(\mathbb{V}) \times \mathcal{S}(\mathbb{V}) \to \mathbb{R}(\mathcal{A})^*$ through $\mathcal{S}_b(\sigma, \mathcal{A}, \lambda) = |\sigma + \lambda - 2\sigma| + ||\sigma - \lambda|| + ||\sigma + \lambda - 2\sigma|| + ||\sigma - \lambda||$. Then $\mathcal{S}_b$ is soft $\mathcal{S}_b$-metric space on $\mathcal{S}(\mathbb{V})$.

2.4. Example

Let $\mathcal{A}$ be a non-empty set of parameters. Describe $\mathcal{S}: \mathcal{S}(\mathbb{V}) \times \mathcal{S}(\mathbb{V}) \times \mathcal{S}(\mathbb{V}) \to \mathbb{R}(\mathcal{A})^*$ through $\mathcal{S}_b(\sigma, \mathcal{A}, \lambda) = (|\sigma + \lambda - 2\sigma| + ||\sigma - \lambda||)^2$. Then $\mathcal{S}_b$ is soft $\mathcal{S}_b$-metric space on $\mathcal{S}(\mathbb{V})$.

2.5. Proposition

If $(\mathbb{V}, \mathcal{S}_b, \mathcal{A})$ soft $\mathcal{S}_b$-metric space. Then $\mathcal{S}_b(\sigma, \mathcal{A}, \lambda) \leq \mathcal{B}\mathcal{S}_b(\mathcal{A}, \mathcal{A}, \sigma)$ and $\mathcal{S}_b(\mathcal{A}, \mathcal{A}, \sigma) \leq \mathcal{B}\mathcal{S}_b(\sigma, \mathcal{A}, \mathcal{A})$.

**Proof**

$\mathcal{S}_b(\sigma, \mathcal{A}, \lambda) \leq \mathcal{B}[2\mathcal{S}_b(\sigma, \mathcal{A}, \sigma) + \mathcal{S}_b(\mathcal{A}, \mathcal{A}, \sigma)] = \mathcal{B}\mathcal{S}_b(\mathcal{A}, \mathcal{A}, \sigma)$

and also

$\mathcal{S}_b(\mathcal{A}, \mathcal{A}, \sigma) \leq \mathcal{B}[2\mathcal{S}_b(\mathcal{A}, \mathcal{A}, \mathcal{A}) + \mathcal{S}_b(\sigma, \mathcal{A}, \mathcal{A})] = \mathcal{B}\mathcal{S}_b(\sigma, \mathcal{A}, \mathcal{A})$.

2.6. Proposition

If soft $\mathcal{S}_b$-metric space, so the next conditions are satisfies:

1. $\mathcal{S}_b(\sigma, \mathcal{A}, \lambda) \leq \mathcal{B}^2[\mathcal{S}_b(\sigma, \mathcal{A}, \sigma) + \mathcal{S}_b(\sigma, \lambda)]$
2. $\mathcal{S}_b(\sigma, \mathcal{A}, \mathcal{A}) \leq 2\mathcal{B}^2\mathcal{S}_b(\sigma, \sigma, \mathcal{A})$
3. $\mathcal{S}_b(\sigma, \sigma, \mathcal{A}) \leq 2\mathcal{B}\mathcal{S}_b(\sigma, \sigma, \mathcal{A}) + \mathcal{B}^2\mathcal{S}_b(\sigma, \sigma, \lambda)$
Proof

1)
\[ \mathbb{S}_b(\sigma, \lambda) \leq \| b \mathbb{S}_b(\sigma, \sigma, \sigma) + \mathbb{S}_b(\lambda, \lambda, \sigma) \| \leq \| b \mathbb{S}_b(\sigma, \sigma, \sigma) + \mathbb{S}_b(\lambda, \lambda, \sigma) \| \leq \| b^2 \mathbb{S}_b(\sigma, \sigma, \sigma) + \mathbb{S}_b(\lambda, \lambda, \sigma) \| \]

2)
\[ \mathbb{S}_b(\sigma, \lambda) \leq \| b \mathbb{S}_b(\sigma, \sigma, \sigma) + \mathbb{S}_b(\lambda, \lambda, \sigma) \| \leq 2b \mathbb{S}_b(\lambda, \lambda, \sigma) \]

3)
\[ \mathbb{S}_b(\sigma, \sigma, \lambda) \leq \| b \mathbb{S}_b(\sigma, \sigma, \sigma) + \mathbb{S}_b(\lambda, \lambda, \sigma) \| \leq 2b \mathbb{S}_b(\lambda, \lambda, \sigma) \]

2.7. Proposition

Let \((\mathcal{V}, \mathbb{S}_b, \mathcal{U})\) be soft \(\mathbb{S}_b\)-metric space such that \(\mathbb{S}_b\) is symmetric where \(b \geq 1\).

Define the function \(M_b: \mathcal{J}(\mathcal{V}) \times \mathcal{J}(\mathcal{V}) \to [0, \infty)\) by \(M_b(\sigma, \lambda) = \| b \mathbb{S}_b(\lambda, \lambda, \sigma) \|\) for all \(\sigma, \lambda \in \mathcal{J}(\mathcal{V})\). Then \(M_b\) is soft \(b\)-metric on \(\mathcal{J}(\mathcal{V})\).

Proof

The first, second and third conditions are easily to be proved. Let us focus then to satisfy the last condition.

Hence,
\[ M_b(\sigma, \lambda) \leq \| b \mathbb{S}_b(\lambda, \lambda, \sigma) \| \leq \| b \| [2 \mathbb{S}_b(\lambda, \lambda, \sigma) + \mathbb{S}_b(\lambda, \lambda, \sigma)] \leq \| b \| [2 \mathbb{S}_b(\lambda, \lambda, \sigma) + \mathbb{S}_b(\lambda, \lambda, \sigma)] \leq \frac{3}{2} \| b \| [M_b(\lambda, \lambda) + M_b(\sigma, \lambda)] \]

Thus \(d\) is soft \(b\)-metric on \(\mathcal{V}\).

2.8. Definition

Suppose that \((\mathcal{V}, \mathbb{S}_b, \mathcal{U})\) is soft \(\mathbb{S}_b\)-metric space, let \(\sigma \in \mathcal{J}(\mathcal{V})\) and let \(q > 0\), we define a soft \(\mathbb{S}_b\)-open ball through the center \(\sigma\) and radius \(q\) as follows:

\[ \mathcal{B}_{\mathbb{S}_b}(\sigma, q) = \{ \lambda \in \mathcal{J}(\mathcal{V}) : \mathbb{S}_b(\lambda, \lambda, \sigma) < q \} \]

Also, we can define a soft \(\mathbb{S}_b\)-closed ball with the center \(\sigma\) and radius \(q\) by:

\[ \mathcal{B}_{\mathbb{S}_b}[\sigma, q] = \{ \lambda \in \mathcal{J}(\mathcal{V}) : \mathbb{S}_b(\lambda, \lambda, \sigma) \leq q \} \]

2.9. Example

Let \(\mathcal{U}\) be a non-empty set of parameters. Describe \(\mathcal{S}: \mathcal{J}(\mathcal{V}) \times \mathcal{J}(\mathcal{V}) \times \mathcal{J}(\mathcal{V}) \to \mathbb{R}^{\mathcal{U}}\) through \(\mathcal{S}_b(\sigma, \lambda, \lambda) = (|\lambda + \lambda - 2\sigma| + |\lambda - \lambda|)^2\). We obtain

\[ \mathcal{B}_{\mathbb{S}_b}(0, 3) = \{ \lambda \in \mathcal{J}(\mathcal{V}) : \mathbb{S}_b(\lambda, \lambda, 0) < 3 \} = \left\{ \frac{3}{2}, \frac{3}{2}, \frac{3}{2} \right\} \]

and

\[ \mathcal{B}_{\mathbb{S}_b}(0, 3) = \{ \lambda \in \mathcal{J}(\mathcal{V}) : \mathbb{S}_b(\lambda, \lambda, 0) \leq 3 \} = \left\{ \frac{3}{2}, \frac{3}{2}, \frac{3}{2} \right\} \]
2.10. Definition

Let $\mathcal{V}, \mathcal{F}_b, \mathcal{A}$ be soft $\mathcal{F}_b$-metric space. Then

1) A sequence $\{\sigma_n\}$ in $\mathcal{V}, \mathcal{F}_b, \mathcal{A}$ is said to be convergent and converges to $\mathcal{D}$ if $\mathcal{F}_b(\sigma_n, \sigma_n, \sigma) \to 0$ as $n \to \infty$. That is for each $\delta > 0$, there exists $n_0 \in N$ such that for all $n_0 \geq n$ we have $\mathcal{F}_b(\sigma_n, \sigma_n, \sigma) < \epsilon$.

2) A sequence $\{\sigma_n\}$ in $\mathcal{V}, \mathcal{F}_b, \mathcal{A}$ is said to be Cauchy if for $\epsilon > 0$, there exists $l \in N$ such that $\mathcal{F}_b(\sigma_n, \sigma_n, \sigma_m) < \epsilon$ for each $n, m \geq l$.

Let $\mathcal{V}, \mathcal{F}_b, \mathcal{A}$ be soft $\mathcal{F}_b$-metric space and let $\mathcal{D} \subseteq \mathcal{V}$. $\mathcal{D}$ is called soft open in $\mathcal{V}$ if for all soft element $\sigma$ there exists $\exists \rho > 0$ such that $\mathcal{F}_b(\sigma, \rho) \subseteq \mathcal{D}$.

2.11. Lemma

Let $\mathcal{V}, \mathcal{F}_b, \mathcal{A}$ be soft $\mathcal{F}_b$-metric space. Then every soft open ball is soft open set in $\mathcal{V}, \mathcal{F}_b, \mathcal{A}$.

Proof

Suppose that $\mathcal{F}_b(\sigma, \rho)$ and $\mathcal{F}_b(\mathcal{D}, \rho)$ are soft open ball and soft open set, respectively. Let $\mathcal{D} \subseteq \mathcal{F}_b(\sigma, \rho)$, so $\mathcal{F}_b(\mathcal{D}, \mathcal{D}, \mathcal{D}) \leq \rho$. Put $\mathcal{F}_b(\mathcal{D}, \mathcal{D}, \mathcal{D}) = k$, $\rho_1 = \frac{\epsilon - k}{2\delta}$. We shall show that $\mathcal{F}_b(\mathcal{D}, \mathcal{D}, \mathcal{D}) \subseteq \mathcal{F}_b(\sigma, \rho)$. Assume $a \in \mathcal{F}_b(\mathcal{D}, \mathcal{D}, \mathcal{D})$ such that $\mathcal{F}_b(a, a, \mathcal{D}) \leq \rho_1$. Hence $\mathcal{F}_b(a, a, \mathcal{D}) \leq \rho_1$. Hence $\mathcal{F}_b(a, a, \mathcal{D}) \leq \rho_1$. Therefore, $\mathcal{F}_b(a, a, \mathcal{D}) \leq \rho$, hence $\mathcal{F}_b(a, a, \mathcal{D}) \subseteq \mathcal{F}_b(\sigma, \rho)$.

2.12. Proposition

Let $\mathcal{V}, \mathcal{F}_b, \mathcal{A}$ be soft $\mathcal{F}_b$-metric space and let $\{\sigma_n\}$ be a sequence in $\mathcal{V}, \mathcal{F}_b, \mathcal{A}$ such that $\sigma_n \to \sigma$. Then $\sigma$ is unique.

Proof

Assume $\sigma_n \to \sigma$ and $\Omega_n \to \mathcal{B}$ such that $\mathcal{F}_b(\sigma_n, \sigma, \mathcal{B}) = \epsilon$. Since $\sigma_n \to \sigma$, then for every $\epsilon > 0$, there is $k_1 \in N$ such that $\mathcal{F}_b(\sigma_n, \sigma_n, \sigma) < \frac{\epsilon}{4\delta^2}$, $\forall n \geq k_1$. Also, Since $\sigma_n \to \mathcal{B}$, then for every $\epsilon > 0$, there is $k_2 \in N$ such that $\mathcal{F}_b(\sigma, \sigma, \mathcal{B}) < \frac{\epsilon}{2\delta^2}$, $\forall n \geq k_2$. Put $k = \max\{k_1, k_2\}$, hence for all $n \geq k$, we get

$$
\mathcal{F}_b(\sigma, \sigma, \mathcal{B}) \leq \mathcal{F}_b(\sigma, \sigma, \sigma_n) + \mathcal{F}_b(\mathcal{B}, \mathcal{B}, \sigma_n) \\
\leq \mathcal{F}_b(2\mathcal{F}_b(\sigma_n, \sigma_n, \sigma) + \mathcal{F}_b(\mathcal{B}, \mathcal{B}, \sigma_n)) < 2\delta^2 \frac{\epsilon}{4\delta^2} + \frac{\epsilon}{2\delta^2} = \epsilon
$$

This is contradiction. Then limit point is unique.

2.13. Proposition

Let $\mathcal{V}, \mathcal{F}_b, \mathcal{A}$ be soft $\mathcal{F}_b$-metric space and let $\{\Omega_n\}$ be a sequence in $\mathcal{V}, \mathcal{F}_b, \mathcal{A}$ such that $\sigma_n \to \sigma$. Then $\{\sigma_n\}$ is soft Cauchy sequence in $\mathcal{V}, \mathcal{F}_b, \mathcal{A}$.

Proof

Since $\sigma_n \to \sigma$, then for every $\epsilon > 0$, there is $k_1 \in N$ such that $\mathcal{F}_b(\sigma_n, \sigma_n, \sigma) < \frac{\epsilon}{4\delta}$.
\( \forall n \geq k_1 \). Also, \( \sigma_m \to \sigma \), then for every \( \varepsilon > 0 \), there is \( k_2 \in N \) such that \( \overline{S_b}(\sigma_m, \sigma, \sigma) < \frac{\varepsilon}{2b} \) \( \forall m \geq k_2 \). Put \( k = \max\{k_1, k_2\} \), hence for all \( n, m \geq k \), we obtain
\[
\overline{S_b}(\sigma_n, \sigma_n, \sigma_n) < \delta \left[ 2\overline{S_b}(\sigma_n, \sigma_n, \sigma) + \overline{S_b}(\sigma_m, \sigma_m, \sigma) \right] < 2b \frac{\varepsilon}{4b} + b \frac{\varepsilon}{2b} = \varepsilon
\]
Thus, \( \{\sigma_n\} \) is soft Cauchy sequence in \((\overline{V}, \overline{S_b}, \mathfrak{U})\).

2.14. Proposition

Let \((\overline{V}, \overline{S_b}, \mathfrak{U})\) be soft \(S_b\)-metric space. If \( \sigma_n \to \sigma \) and \( \mathfrak{U}_n \to \mathfrak{U} \) in \((\overline{V}, \overline{S_b}, \mathfrak{U})\), then \( \overline{S_b}(\sigma_n, \sigma_n, \mathfrak{U}_n) \to \overline{S_b}(\sigma, \sigma, \mathfrak{U}) \) in \((\overline{V}, \overline{S_b}, \mathfrak{U})\).

Proof

Let \( \sigma_n \to \sigma \), thus for every \( \varepsilon > 0 \), there is \( k_1 \in N \) such that \( \overline{S_b}(\sigma_n, \sigma_n, \sigma) < \frac{\varepsilon}{4b} \) \( \forall n \geq k_1 \). Also, \( \mathfrak{U}_n \to \mathfrak{U} \), then for every \( \varepsilon > 0 \), there is \( k_2 \in N \) such that \( \overline{S_b}(\mathfrak{U}_n, \mathfrak{U}_n, \mathfrak{U}) < \frac{\varepsilon}{2b} \) \( \forall \mathfrak{U}_n \geq k_2 \). Put \( k = \max\{k_1, k_2\} \), hence for all \( n \geq k \), we obtain
\[
\overline{S_b}(\sigma_n, \sigma_n, \mathfrak{U}_n) \leq b \left[ 2\overline{S_b}(\sigma_n, \sigma_n, \sigma) + \overline{S_b}(\mathfrak{U}_n, \mathfrak{U}_n, \mathfrak{U}) \right] \leq b \left[ 2\overline{S_b}(\sigma_n, \sigma_n, \sigma) + 2b \overline{S_b}(\sigma_n, \sigma_n, \sigma) + b \overline{S_b}(\sigma_n, \sigma_n, \sigma) \right] < 2b \frac{\varepsilon}{4b} + 2b^2 \frac{\varepsilon}{4b^2} + \overline{S_b}(\sigma, \sigma, \mathfrak{U}) < \varepsilon + \overline{S_b}(\sigma, \sigma, \mathfrak{U})
\]
Therefore, \( \overline{S_b}(\sigma_n, \sigma_n, \mathfrak{U}_n) - \overline{S_b}(\sigma, \sigma, \mathfrak{U}) < \varepsilon \). But
\[
\overline{S_b}(\sigma_n, \sigma_n, \mathfrak{U}_n) \leq b \left[ 2\overline{S_b}(\sigma_n, \sigma_n, \sigma) + \overline{S_b}(\mathfrak{U}_n, \mathfrak{U}_n, \mathfrak{U}) \right] \leq b \left[ 2\overline{S_b}(\sigma_n, \sigma_n, \sigma) + 2b \overline{S_b}(\sigma_n, \sigma_n, \sigma) + b \overline{S_b}(\sigma_n, \sigma_n, \sigma) \right] < 2b \frac{\varepsilon}{4b} + 2b^2 \frac{\varepsilon}{4b^2} + \overline{S_b}(\sigma_n, \sigma_n, \mathfrak{U}_n) < \varepsilon + \overline{S_b}(\sigma_n, \sigma_n, \mathfrak{U}_n)
\]
With \( b \geq 1 \). So \( \overline{S_b}(\sigma_n, \sigma_n, \mathfrak{U}_n) - \overline{S_b}(\sigma_n, \sigma_n, \mathfrak{U}_n) < \varepsilon \), consequently \( |\overline{S_b}(\sigma_n, \sigma_n, \mathfrak{U}_n) - \overline{S_b}(\sigma_n, \sigma_n, \mathfrak{U}_n)| < \varepsilon \). Hence \( \overline{S_b}(\sigma_n, \sigma_n, \mathfrak{U}_n) \to \overline{S_b}(\sigma, \sigma, \mathfrak{U}) \).

2.15. Proposition

Let \( \mathcal{M} : \mathcal{S}(\overline{V}) \times \mathcal{S}(\overline{V}) \to \mathbb{R}(\mathfrak{U})^* \) be a soft metric defined by
\[
\overline{S_b}(\sigma, \mathfrak{U}, \lambda) = [\mathcal{M}(\sigma, \mathfrak{U}) + \mathcal{M}(\sigma, \lambda) + \mathcal{M}(\mathfrak{U}, \lambda)]^p
\]
for all \( \sigma, \mathfrak{U}, \lambda \in \mathcal{S}(\overline{V}) \). Then \((\overline{V}, \overline{S_b}, \mathfrak{U})\) is soft \(S_b\)-metric space (i.e. Every soft metric space is soft \(S_b\)-metric space).

Proof

\[
\overline{S_b}(\sigma, \mathfrak{U}, \lambda) = [\mathcal{M}(\sigma, \mathfrak{U}) + \mathcal{M}(\sigma, \lambda) + \mathcal{M}(\mathfrak{U}, \lambda)]^p
\]
\[
\leq [\mathcal{M}(\sigma, \mathfrak{U}) + \mathcal{M}(\sigma, \mathfrak{U}) + \mathcal{M}(\sigma, \lambda) + \mathcal{M}(\mathfrak{U}, \mathfrak{U}) + \mathcal{M}(\mathfrak{U}, \lambda)]^p
\]
\[
= [2\mathcal{M}(\sigma, \mathfrak{U}) + 2\mathcal{M}(\sigma, \lambda) + 2\mathcal{M}(\mathfrak{U}, \mathfrak{U})]^p
\]
\[
\leq 2^{p-1} \left[ 2\mathcal{M}(\sigma, \mathfrak{U}) \right]^p + 2\mathcal{M}(\sigma, \lambda) + 2\mathcal{M}(\mathfrak{U}, \mathfrak{U})]^p
\]
\[
\leq \delta \left[ \overline{S_b}(\sigma, \mathfrak{U}) + \overline{S_b}(\sigma, \lambda) + \overline{S_b}(\mathfrak{U}, \mathfrak{U}) \right]
\]
Then \( \overline{S_b} \) is soft \(S_b\)-metric space, where \( b = 2^{p-1} \).

2.16. Definition

Let \((\overline{V}, \overline{S_b}, \mathfrak{U})\) be soft \(S_b\)-metric space. Sequence \( \{\sigma\} \) is said to be contractive if there is a constant \( h, 0 < h < 1 \). so \( \overline{S_b}(\sigma_{m+3}, \sigma_{m+2}, \sigma_{n+1}) \leq h[\overline{S_b}(\sigma_{m+2}, \sigma_{n+1}, \sigma_n)] \) for every \( n \in N \).
2.17. Proposition

Let \((\mathcal{V}, \mathcal{S}_b, \mathcal{A})\) be soft \(\mathcal{S}_b\)-metric space. Every contractive sequence \(\{\sigma\}\) on \((\mathcal{V}, \mathcal{S}_b, \mathcal{A})\) is Cauchy sequence.

Proof

Take \(\{\sigma\}\) contractive sequence in soft \(\mathcal{S}_b\)-metric space \((\mathcal{V}, \mathcal{S}_b, \mathcal{A})\). According to definition 2.16 there is constant \(h\), \(0 < h < 1\), so \(\mathcal{S}_b(\sigma_{m+3}, \sigma_{m+2}, \sigma_{n+1}) \leq h[\mathcal{S}_b(\sigma_{n+2}, \sigma_{n+1}, \sigma_n)]\) for every \(n \in \mathbb{N}\).

So, we get
\[
\mathcal{S}_b(\sigma_{m+3}, \sigma_{m+2}, \sigma_{n+1}) \leq h[\mathcal{S}_b(\sigma_{n+2}, \sigma_{n+1}, \sigma_n)] \leq c^2[\mathcal{S}_b(\sigma_{n+2}, \sigma_{n+1}, \sigma_n)] \leq \cdots \cdots \\
\leq c^n[\mathcal{S}_b(\sigma_3, \sigma_2, \sigma_1)]
\]

Choose \(\mathcal{F}_1 = \sigma_{m-1}\), \(\mathcal{F}_2 = \sigma_{m-2}\), \ldots \ldots \(\mathcal{F}_n = \sigma_{n+1}\)

Let
\[
\mathcal{S}_b(\sigma_m, \sigma_m, \sigma_n) \leq q[\mathcal{S}_b(\sigma_m, \sigma_m, \sigma_{nm-1}) + \mathcal{S}_b(\sigma_{m-1}, \sigma_{m-2}, \sigma_{m-3}) + \cdots \cdots + \mathcal{S}_b(\sigma_{n+3}, \sigma_{n+2}, \sigma_{n+1}) + \mathcal{S}_b(\sigma_{n+2}, \sigma_{n+1}, \sigma_n)]
\]
\[
\leq q\left(h^{m-2} + h^{m-3} + \cdots \cdots + h^{n-1}\right)[\mathcal{S}_b(\sigma_3, \sigma_2, \sigma_1)] \leq q\left(h^{n-1}\left(1 - \frac{1}{1 - h}\right)\right)[\mathcal{S}_b(\sigma_3, \sigma_2, \sigma_1)]
\]

Because \(0 < h < 1\), then \(\lim_{n \to \infty} h^n = 0\). Then \(\lim_{n,m \to \infty} \mathcal{S}_b(\sigma_m, \sigma_m, \sigma_n) = 0\). Therefore \(\{\sigma\}\) is a Cauchy sequence.

We can get the next corollary from Proposition 2.17.

2.18. Corollary

Let \((\mathcal{V}, \mathcal{S}_b, \mathcal{A})\) be a complete type-metric space. Every contractive sequence \(\{\sigma\}\) on \((\mathcal{V}, \mathcal{S}_b, \mathcal{A})\) is convergent sequence.

3. Conclusion

The main point of this article is to define a soft \(\mathcal{S}_b\) – metric space and some properties. Further, the convergent sequence, Cauchy sequence and every contractive sequence \(\{\sigma\}\) in soft \(\mathcal{S}_b\) – metric space is a Cauchy sequence and some related theorems have been established too.

4. References

[1] Abd Al-Rahem M, Abood Z, and Mohammad S 2019, Generalizing on \(C^*\)- Algebra Valued A - Metric Space, Int. Math. Forum 14 (1), 17-26.
[2] Abood Z, Abd Al-Rahem M and Abd Al-Hassain Z 2018, A Study of \(\mathcal{F}\) – Convergence in Vector Metric Space, Jour. Of Adv. Research in Dynamical & Control Systems 10 (10), 106 – 11.
[3] Afra J M 2014, Fixed point type theorem in S-Metric spaces, *Mid-E. J. of Sc. R.* **22** (6), 864–869.

[4] Afra J M 2015, Double Contraction in S-Metric space, *Int. J. of Math. Ana.* **9** (3), 117-125.

[5] Ali M, Feng F, Liu X, Min W and Shabir M 2009, On some new operations in soft set theory, *Comput. Math. Appl.* **57** (9), 1547–1564.

[6] Aras C, Bayrmov S and Cafarli V 2018, A Study on Soft S-Metric Spaces, *Communications and Applications* **19** (40), 713-23.

[7] Bakhtin A 1989, The contraction mapping principle in quasi-metric spaces, *Funct. Anal. Unianowsk Gos. Ped. Inst.* **30**, 26–37.

[8] Das S and Samanta S 2013, On Soft metric spaces, *J. Fuzzy Math.* **21** (3), 707–734.

[9] Das S and Samanta S 2012, Soft Real Set, Soft Real Number and Their Properties, *J. Fuzzy Math.* **20** (3) 551-576.

[10] Faydh S, Hassan Z, Abd Al-Rahem M and Abdulameer D 2019, Some Result in 2-Fuzzy Vector Matric Space, *IOP Conf. Series: Materials Science and Engineering* **571**.

[11] Guler A, Yıldırım E and Ozbakır O 2016, A Fixed-Point Theorem on Soft G-metric Spaces, *J. Nonlinear Sci. Appl.* **9**, 885-894.

[12] Jleli M and Samet B 2012, Remarks on G-metric Spaces and Fixed-Point Theorems, *Fixed Point theory Appl.* **210**, 7 pages.

[13] Maji P, Biswas R and Roy A 2003, Soft set theory, *Comput. Math. Appl.* **45**, 555-562.

[14] Molodtsov D 1999, Soft set theory-first results, *Comput. Math. Appl.* **37** (4-5), 19-31.

[15] Sedghi S 2012, A Generalization of Fixed-Point Theorem in S-metric spaces, *Math. V.* **64** (3), 258-66.

[16] Shabir M and Naz M 2011, On soft topological spaces, *Comput. Math. Appl.* **61** (7), 1786-99.

[17] Sedghi S, Gholidahneh A, Došenović T, Esfahani J and Radenović S 2016, Common fixed point of four maps in $S_b$-metric spaces, *J. Linear Topol. Algebra* **5** (2), 93-104.

[18] Souayah N and Mlaiki N 2016, A fixed point theorem in $S_b$-metric space, *J. Math. Computer Sci.* 131-9.

[19] Rahman M, Sarwar M and UR Rahman M 2017, Some common fixed-point theorems on S-metric spaces, *J. fixed point theory* **2** ISSN:2052-5338.

[20] Ughade M, Turkoglu D, Singh S and Daheriy R 2016, Some fixed-point theorems in $A_b$-metric space, *British Journal of Mathematics & Computer Science* **19** (6), 1-24.