A BOUNDARY-SINGULAR TWO-DIMENSIONAL PARTIAL DATA INVERSE PROBLEM

FREDDY J. F. SYMONS

Abstract. We consider uniqueness in an inverse Schrödinger problem in a bounded domain in $\mathbb{R}^2$ given the Dirichlet-to-Neumann map on part of the boundary. On the remaining boundary we impose a new type of singular boundary condition with unknown parameter. Owing to recent results on this class of boundary conditions, we discuss the necessity of an extra point condition to well-define the data for the inverse problem. Our results are two-fold. At a single frequency the inverse problem displays non-uniqueness, since an unknown boundary condition can spoil “seeing” the Schrödinger potential via the Dirichlet-to-Neumann map. On the other hand, taking as input data the Dirichlet-to-Neumann map at every frequency $\lambda \in \mathbb{R}$ for which it is well-defined yields full uniqueness of the potential and all the boundary conditions. We adapt recent methods in related two-dimensional inverse problems and develop new techniques to cope with the singularity in the boundary condition.

Contents

1. Introduction
2. Main results and outline
3. Summary of differential operators with singular boundary conditions
4. Complex geometric optics for an existing approach with a Dirichlet BC
5. Unique continuation and density results
6. A weighted sum of $q$-values for a singular BC
7. Final steps of proof of Theorem 1
8. The interface Dirichlet-to-Neumann operators
9. Negative eigenvalue asymptotics for operators with singular BCs
10. Discussion
Acknowledgements
References

E-mail address: symonsfj@gmail.com

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1. Introduction

Let $\Omega \subset \mathbb{R}^d$ be an open and bounded set with dimension $d \geq 2$, whose boundary $\partial \Omega = \overline{\Omega} \setminus \Omega$ is a connected piecewise $C^1$ manifold. Separate $\partial \Omega$ into two portions $\Gamma \neq \emptyset$ and $\Gamma_c$, each topologically connected, open with respect to the manifold topology, and chosen so that

$$\Gamma \cap \Gamma_c = \emptyset \quad \text{and} \quad \Gamma \cup \Gamma_c = \partial \Omega.$$

Let $q \in L^\infty(\Omega)$ and $f \in C^1(\Gamma_c)$ both be real-valued. Consider for each $\lambda \in \mathbb{C}$ the Dirichlet-to-Neumann (DN) map $\Lambda_{q,f}(\lambda) : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ mapping $g \mapsto -\partial_\nu u \mid_{\Gamma}$ where $\partial_\nu$ denotes the outward directed normal derivative and $u \in L^2(\Omega)$ solves the boundary-value problem (BVP)

$$\begin{cases}
  -\Delta u + qu &= \lambda u \quad \text{in } \Omega, \\
  u + f\partial_\nu u &= 0 \quad \text{on } \Gamma_c, \\
  u &= g \quad \text{on } \Gamma.
\end{cases} \quad (1)$$

The standard inverse Schrödinger problem is to try to recover $q$ from the operator $\Lambda_{q,0}(0)$. In this paper we examine uniqueness in the problem of recovery of both $q$ and $f$ when $f$ is non-zero.

Owing to the results in [28] (see also [4, 3]), in dimension $d = 2$, if $f$ possesses a simple zero (and otherwise has 0 as an isolated value in its range) then the operator underlying (1) is not self-adjoint and its eigenvalues fill $\mathbb{C}$. This means that for every $\lambda \in \mathbb{C}$ the solution of (1) is not uniquely specified so the DN map $\Lambda_{q,f}(\lambda)$ is ill-defined.

One fixes this by imposing a one-parameter point boundary condition (BC) at the zero $x_0$ of $f$. A certain class of these BCs—arising from limit-circle considerations of certain ordinary differential operators (ODOs)—yields self-adjoint restrictions of the operator associated with (1). We denote this BC (see Definition 4 shortly) by

$$\beta_\vartheta[u] = 0 \quad (\vartheta \in (-\pi/2, \pi/2))$$

for any solution $u$ of (1). Here $\vartheta$ is the polar angle at the point $x_0$ and $\beta \in \mathbb{R}$ a parameter. We can then well define the new DN operator $\Lambda_{q,f,\beta}(\lambda) : g \mapsto -\partial_\nu u \mid_{\Gamma}$ for any $g \in H^{1/2}(\Gamma)$ such that $u$ solves

$$\begin{cases}
  (-\Delta + q)u &= \lambda u \quad \text{in } \Omega, \\
  u - f\partial_\nu u &= 0 \quad \text{on } \Gamma_c, \\
  \beta_\vartheta[u] &= 0 \quad \text{at } x_0, \\
  u &= g \quad \text{on } \Gamma.
\end{cases} \quad (2)$$

The inverse problem we examine is thus:

**Inverse Problem 1.** Given the DN map $\Lambda_{q,f,\beta}(\lambda)$ for some or all $\lambda$, recover the potential $q$ everywhere in $\Omega$, the function $f$ on $\Gamma_c$ and the BC parameter $\beta$. 
With a (non-singular) Dirichlet condition on $\Gamma_c$, this class of inverse problem—often in “conductivity form” with DN map $\Lambda_c^\gamma : h \mapsto -\gamma \partial_n v |_{\Gamma} \ (h \in H^{1/2}(\Gamma))$ for the BVP
\[
\begin{aligned}
-\nabla \cdot (\gamma \nabla v) &= 0 \quad \text{in } \Omega, \\
v &= 0 \quad \text{on } \Gamma_c, \\
v &= h \quad \text{on } \Gamma
\end{aligned}
\]
—has seen plentiful attention. Calderón first proposed this problem of electrical prospection in the ‘80s [11] inspired by his engineering work for the Argentine state oil company. Initial analysis was in simplified situations [25, 26], and the first proof of uniqueness of $\gamma \in C^\infty(\Omega)$ in dimension $d \geq 3$ was then given in [34, 35]. Arguably most of the important subsequent activity on uniqueness can be found in the references [1, 7, 8, 9, 10, 12, 16, 22], covering full data in dimension $d \geq 2$ and partial data in dimension $d \geq 3$, and culminating in [2, 11, 12, 22]. For brevity here, we direct to the latter three and [36, Sec. 2.4 & Ch. 4] for a more detailed historical exposition.

In the more relevant case of partial data in two dimensions the only progress lies in [19, 20, 16, 21]. Owing to their greatest simplicity we will focus on adapting the methods in [19]. We also mention that for the singular case (2) in a symmetric half-disc geometry uniqueness of a radially symmetric $q \in L^\infty_{loc}(0, 1]$ at the single frequency $\lambda = 0$, given known $f$ and $\beta = 0$, was proved in [6, Sec. 4].

### 2. Main results and outline

To state our main results precisely we first need to make some definitions.

**Definition 1** (Specially decomposable domain). We call a bounded two-dimensional simply connected open $\Omega$ a specially decomposable domain if it has a $C^\infty$ boundary and can be written as
\[\Omega = \text{int}(\Omega_1 \cup \Omega_0),\]
where $\Omega_1$ is a half-disc of radius 1 whose straight edge—i.e., diameter—is denoted $\Gamma_1$ and is contained in $\partial \Omega$.

**Definition 2** (Boundary accessibility). Letting $\Omega$ be a specially decomposable domain, specify the further boundary decomposition $\partial \Omega = \Gamma \cup \Gamma_c, \Gamma \cap \Gamma_c = \emptyset$ such that the diameter $\Gamma_1 \subset \Gamma_c$, and $\Gamma$ and $\Gamma_c$ are both relatively open. The boundary portions $\Gamma$ and $\Gamma_c$ are called, respectively, accessible and inaccessible.

**Remark.** For convenience the coordinates of $\mathbb{R}^2$ are specified so that $\Gamma_1$ aligns with the $y$-axis, the mid-point of $\Gamma_1$ is 0, and the subdomain $\Omega_1$ is in the right half-plane. Moreover we will denote by $\Gamma_0$ the (possibly disjoint) portion of the boundary given by $\Gamma_c \setminus \Gamma_1$, and by $\Gamma_i$ the interface between $\Omega_0$ and $\Omega_1$, i.e., $\Gamma_i = (\overline{\Omega_1} \cap \overline{\Omega_0}) \setminus \partial \Omega$. 

A BOUNDARY-SINGULAR INVERSE PROBLEM

3
Definition 3 (Singular boundary condition). The Berry–Dennis boundary condition on $\Omega$ is the requirement on a given $u : \Omega \to \mathbb{C}$ that

$$ (u - f \partial_{\nu} u) \mid_{\Gamma_c} = 0, \quad (3) $$

where $f$ is a bounded, real-valued, a.e. absolutely continuous function possessing only one strictly simple zero at the point $0$, and satisfying that for every $x \in \Gamma_c \setminus \{0\}$ and $\varepsilon > 0$ we have $0$ either not in the range of $f \mid_{\Gamma_c \setminus \{x : |x| < \varepsilon\}}$, or an isolated value in this range. For the same reasons as in [28] we need $f$ to be linear on the straight edge $\Gamma_1$: we assume $\exists b > 0$ such that

$$ f(y) = -by \quad (y \in (-1, 1)), \quad (4) $$

enforcing a simple zero of $f$ at $0$. Such an $f$ is called an admissible boundary function.$^1$

From this definition it is clear that such an $f$ may discontinuously become 0 or be identically 0 in a connected subset of $\Gamma_c$.

Definition 4 (Self-adjoint boundary condition). Let $r = |x|, \, \vartheta = \arg(x)$ be the usual polar coordinates about 0. Denoting by $[\cdot, \cdot]$ the Lagrange bracket in $\mathbb{R}^2$, we define an admissible self-adjoint boundary condition, with real parameter $\beta$, by

$$ \beta_{\vartheta}[u] := \lim_{r \to 0} [u, u_0 + \beta v_0] (r, \vartheta) $$

$$ = \lim_{r \to 0} (u(r, \vartheta)r \partial_r (u_0 + \beta v_0)(r, \vartheta) - (r \partial_r u)(r, \vartheta))(u_0 + \beta v_0)(r, \vartheta) $$

$$ = 0, \quad (5) $$

where $u_0$ and $v_0$ are the solutions to

$$ \begin{cases} 
-\Delta u & = 0 \quad \text{in} \quad \Omega_1, \\
 u - f \partial_{\nu} u & = 0 \quad \text{on} \quad \Gamma_1
\end{cases} $$

$^1$In [28] the parameter “$\varepsilon$” is used instead of $b$. We are not interested in its limit approaching 0, so we use the latter notation.
given in (7) and \( f \) is an admissible boundary function.

**Definition 5** (Admissible potential). A function \( q \in L^\infty(\Omega) \) is called an admissible potential if it is locally radially symmetric about 0, i.e., there is a \( \delta > 0 \) such that in the ball \( r < \delta \) we have \( \partial_\vartheta q(r, \vartheta) = 0 \). Without loss of generality we can assume \( q \rest_{\Omega_1} \) is radially symmetric, since we can rescale the coordinates so that \( \delta = |\Gamma_1|/2 = 1 \).

Our main results are as follows:

**Theorem 1** (Conditional uniqueness at one frequency). Let \( \alpha > 0, \beta \in \mathbb{R} \), and \( \Omega \) be specially decomposable.

(i) Suppose \( q_1, q_2 \in C^{2+\alpha}(\overline{\Omega}) \) are admissible potentials with \( q_1 = q_2 \) in a neighbourhood of the boundary \( \partial \Omega \) and \( f \) is an admissible boundary function. If the DN maps are equal at the frequency \( \lambda = 0 \)—i.e., \( \Lambda_{q_1,f,\beta}(0) = \Lambda_{q_2,f,\beta}(0) \)—then \( q_1 = q_2 \) in all of \( \Omega \).

(ii) Conversely, take admissible boundary functions \( f_1 \) and \( f_2 \). Let \( q \in C^{2+\alpha}(\overline{\Omega}) \) be an admissible potential. If \( \Lambda_{q,f_1,\beta_1}(0) = \Lambda_{q,f_2,\beta_2}(0) \) then \( f_1 = f_2 \).

**Remark.** This immediately implies that if the DN maps \( \Lambda_{q_j,f_j,\beta_j}(0) \) are equal and \( q_1 = q_2 \) in a neighbourhood of \( \partial \Omega \) then \( q_1 = q_2 \) everywhere if and only if \( f_1 = f_2 \) everywhere, i.e., the BC may cloak the potential.

**Theorem 2** (Uniqueness at all frequencies). Let \( \Omega \) be specially decomposable, \( \alpha > 0, q_1, q_2 \in C^{2+\alpha}(\overline{\Omega}) \) admissible potentials that are equal in some neighbourhood of the boundary \( \partial \Omega \), \( f_1 \) and \( f_2 \) admissible boundary functions supported in the straight edge \( \Gamma_1 \), and \( \beta_1, \beta_2 \in \mathbb{R} \) admissible self-adjoint BCs at 0.

If \( \Lambda_{q_1,f_1,\beta_1}(\lambda) = \Lambda_{q_2,f_2,\beta_2}(\lambda) \) at every \( \lambda \in \mathbb{R} \) for which both DN maps are defined then \( q_1 = q_2, f_1 = f_2 \) and \( \beta_1 = \beta_2 \).

We will prove Theorem 1 by adapting various existing approaches to the situation with our singular BC. The proof of Theorem 2 will then apply this conditional uniqueness after extraction of the parameters \( b \) (equivalently \( f \)) and \( \beta \) from the following result.

**Theorem 3** (Negative eigenvalue asymptotics). Let \( \Omega \) be specially decomposable, \( q \) an admissible potential, \( f \) an admissible boundary function that is supported in \( \Gamma_1 \), and \( \beta \in \mathbb{R} \) parameterise a self-adjoint BC at 0. Then the self-adjoint operator \( T \) underlying (2) has discrete spectrum accumulating only at \( \pm \infty \), and its negative eigenvalues possess the asymptotic expansion

\[
\lambda_n = -e^{-2b(\vartheta_0 + \tan^{-1} \beta)} e^{-2n\pi b (1 + o(1))} \quad (n \to -\infty).
\]

We define \( T \) rigorously in (18). Here \( \vartheta_0 \) is a calculable constant (defined explicitly above equation (49)).

We prove Theorem 3 in Section 9 but we can apply it immediately with Theorem 1 and prove Theorem 2.
Proof of Theorem 2. Note that the map \( \Lambda_{q_1,f_1,\beta_1}(\lambda) = \Lambda_{q_2,f_2,\beta_2}(\lambda) \) is an operator-valued Herglotz function of \( \lambda \) (see the proof of Lemma 5). Since we know its behaviour on the real line, we know where its poles lie, \textit{ergo} we know where the eigenvalues \( \lambda_n \) of \( T \) are. In particular, by Theorem 3 we may deduce that, as \( n \to -\infty \), we have

\[
-\frac{\log(-\lambda_n)}{2n\pi} \to b_1 = b_2 =: b,
\]

which determines \( f_1 = f_2 \) completely on \( \Gamma_1 \), where the latter are supported. The constant \( \vartheta_0 \) is fixed and may in principle be calculated (see Lemma 7), so in turn we may calculate from Theorem 3 that, as \( n \to -\infty \),

\[
-\frac{\log(-\lambda_n) + 2b(n\pi + \vartheta_0)}{2b} \to \tan^{-1}(\beta_1) = \tan^{-1}(\beta_2) =: \tan^{-1}(\beta).
\]

Finally, we apply Theorem 1 to the triples \((q_1,f,\beta)\) and \((q_2,f,\beta)\), with equality of the DN maps for any fixed \( \lambda \in \mathbb{R} \), to deduce that \( q_1 = q_2 \) in \( \Omega \). \( \square \)

The remaining paper is devoted to proving Theorems 1 and 3; we structure it as follows. It will be useful to recapitulate the ideas of [28], which we do in Section 3. Then we describe in Section 4 the approach in [19], since we will adapt the methods. In the next three sections we will prove Theorem 1. Our proof utilises unique continuation and density arguments—adapted from [5] and developed in Section 5—applied alongside our version of the weighted sum of values of \( q_1 \) [19, Prop. 4.1], proved in Section 6. Subsequently in Section 7 we conclude the proof of Theorem 1 on conditional uniqueness. In Section 8 we develop some results on the interface DN maps on \( \Gamma_1 \). We apply these in Section 9 to prove the negative eigenvalue asymptotics of Theorem 3. We offer a short discussion in Section 10.

3. Summary of differential operators with singular boundary conditions

When considering inverse problems for partial differential operators (PDOs) it is important for the BVPs to generate self-adjoint operators. This owes to the requirement of well-definedness for the resulting DN map. If the operator is not self-adjoint then the solution to (1) might not exist or be non-unique, leaving the DN map ill-defined. If, as desired, the operator is self-adjoint, then it defines a unitary evolution group (see, e.g., [32]), and consequently—except at points in the spectrum—the associated BVP has a unique solution, non-zero for an inhomogeneous boundary condition. This well defines a DN map. Physically this is realised by the quantity of BCS. Too many, e.g., simultaneous Dirichlet and Neumann conditions, yield only a symmetric operator. Too few and one loses even symmetry.

The recent [4, 28] explored self-adjointness of second-order differential operators in bounded two-dimensional domains, with BCS singular at discrete points. In [28] \( \Omega \) had the Glazman decomposition of Definitions 1 and 2.
Upon considering the operator
\[ D(T_0) = \{ u \in L^2(\Omega) \mid \Delta u \in L^2(\Omega), (u + by\partial_y u) |_{\Gamma_1} = 0 = u |_{\partial\Omega \setminus \Gamma_1} \}, \]
\[ T_0 u = -\Delta u, \quad (6) \]
they showed that despite its seemingly “complete” set of BCs, it is not symmetric. Its adjoint is symmetric and has a one-dimensional deficiency space.

Thus one may specify self-adjoint restrictions of \( T_0 \) by imposing an extra BC at 0. It turns out that any linear combination of the functions
\[ u_0(r, \vartheta) = e^{-\vartheta/b} \sin \left( \log \left( \frac{r}{b} \right) \right), \quad v_0(r, \vartheta) = e^{-\vartheta/b} \cos \left( \log \left( \frac{r}{b} \right) \right) \quad (7) \]
can be used to specify a BC via a (two-dimensional) Lagrange bracket. For example in [28] the BC at 0 is the requirement on functions \( u \) that
\[ [u, u_0](r, \vartheta) := r \left( u \partial_r u_0 - (\partial_r u) u_0 \right) (r, \vartheta) \to 0 \quad (r \to 0). \]

In general one may take any \( \beta \in \mathbb{R} \) and then require
\[ [u, u_0 + \beta v_0](r, \vartheta) \to 0 \quad (r \to 0). \quad (8) \]
The above considerations are the motivation for Definitions 3 and 4.

We now briefly rewrite the key points of [28] incorporating a real-valued radial Schrödinger potential \( q(r) \in L^\infty(0,1;dr) \).

This will explain the following corollary to [28] and provide us with some useful tools.

**Proposition 1** (Marletta–Rozenblum, 2009). Consider the operator
\[ D(L) := \{ u \in L^2(\Omega_1) \mid \Delta u \in L^2(\Omega_1), u |_{\Gamma_1} = 0 = (u - by\partial_y u) |_{\Gamma_1} \}, \]
\[ Lu := (-\Delta + q)u. \]

There exist on \( L^2(0,1;rdr) \) ordinary differential operators \( L_n \) \((n \geq 0) \) regular at 1 for which \( L \) is (equivalent to) the direct sum of operators \( \oplus_{n=0}^\infty L_n \). The \( L_n \) \((n \geq 1) \) are all limit-point at 0 and self-adjoint; \( L_0 \) is limit-circle at 0. The self-adjoint restrictions \( L_0' \) of \( L_0 \) are generated by a one-dimensional Lagrange bracket BC with real parameter \( \beta \). On \( L^2(\Omega_1) \) the PDO
\[ L' := L_0' \oplus \bigoplus_{n=1}^\infty L_n \quad (9) \]
is self-adjoint, restricts \( L \), and has domain
\[ D(L') = \{ u \in L^2(\Omega_1) \mid \Delta u \in L^2(\Omega_1), u |_{\Gamma_1} = 0 = (u - b\partial_y u) |_{\Gamma_1} = \beta_0[u] \}. \quad (10) \]

This is proved for \( q = 0 \) in [28], and since multiplication by \( q \in L^\infty(0,1) \) is a bounded self-adjoint operator on both \( L^2(\Omega_1) \) and \( L^2(0,1;rdr) \) it holds for \( q \neq 0 \) via [23, Thm. V.4.3]. For most of the rest of this section we explain [9], since the underlying tools will be useful.
In polar coordinates the eigenvalue problem for \( L - \lambda \) is

\[
\begin{aligned}
&\left\{ -\left( \frac{1}{r} \partial_r (r \partial_r) - \frac{1}{r^2} \partial_{\vartheta}^2 \right) u(\cdot ; \lambda) + q(\cdot |) u(\cdot ; \lambda) = \lambda u(\cdot ; \lambda) \right. & \quad \text{in } \Omega_1, \\
&\left. u(\cdot ; \lambda) = 0 \right. & \quad \text{on } \Gamma_1, \\
&\left. (b \partial_{\vartheta} u(\cdot ; \lambda) + u(\cdot ; \lambda)) |_{\vartheta = \pm \frac{\pi}{2}} = 0 \right. & \quad \text{on } \Gamma_1.
\end{aligned}
\]

(11)

Separating the variables we find two ordinary differential problems. Firstly the angular problem

\[
\begin{aligned}
&\left\{ -\Theta''(\vartheta) = \mu \Theta(\vartheta) \quad (\vartheta \in (-\frac{\pi}{2}, \frac{\pi}{2})), \\
b \Theta'(\vartheta) + \Theta(\vartheta) = 0 \quad (\vartheta = \pm \frac{\pi}{2}),
\end{aligned}
\]

is easily calculated to possess eigenvalues and eigenfunctions

\[
\mu_0 = -\frac{1}{b^2}; \quad \mu_n = n^2 \quad (n \geq 1); \\
\Theta_n(\vartheta) = \begin{cases} 
\begin{aligned}
e^{-\vartheta/b} & (n = 0), \\
\cos(n \vartheta) - (nb)^{-1} \sin(n \vartheta) & (n \text{ even}), \\
\cos(n \vartheta) + nb \sin(n \vartheta) & (n \text{ odd}).
\end{aligned}
\end{cases}
\]

Replacing the separation-of-variables parameter \( \mu \) by \( \mu_n \) then yields the ordinary differential system

\[
\left\{ \begin{aligned}
\frac{1}{r} (r R_n'(r; \lambda))' + q(r) R_n(r; \lambda) + \frac{\mu_n}{r^2} R_n(r; \lambda) & = \lambda R_n(r; \lambda) \quad (r \in (0, 1)), \\
R_n(1; \lambda) & = 0.
\end{aligned} \right.
\]

(12)

Since \( R(r) \Theta(\vartheta) \in L^2(\Omega_1; rdrd\vartheta) \) if and only if \( R \in L^2(0, 1; rdr) \) and \( \Theta \in L^2(-\frac{\pi}{2}, \frac{\pi}{2}; d\vartheta) \), we see that the natural Hilbert space over which to consider the radial differential system is the weighted space \( L^2(0, 1; rdr) \). The solutions \( R_n \) satisfy the following result.

**Proposition 2** (Marletta–Rozenblum, 2009, p. 4). Let \( \lambda \in \mathbb{C} \) and \( q = 0 \). For each \( n \geq 1 \) a solution \( R_n(\cdot ; \lambda) \) of the differential equation in (12) is in \( L^2(0, 1; rdr) \) if and only if it is a constant multiple of the Bessel function

\[
J_n(\sqrt{\lambda} r) \quad (r \in (0, 1)).
\]

The function \( R_0(\cdot ; \lambda) \) solves the full BVP (12) in \( L^2(0, 1; rdr) \) with \( n = 0 \) if and only if it is a constant multiple of

\[
Y_{i/b}(\sqrt{\lambda}) J_{i/b}(\sqrt{\lambda} r) - J_{i/b}(\sqrt{\lambda}) Y_{i/b}(\sqrt{\lambda} r) \quad (r \in (0, 1)).
\]

(13)

**Remark.** Even if \( q \neq 0 \), for each \( n \geq 1 \) and \( \lambda \in \mathbb{C} \) the solution \( R_n(r; \lambda) \) is asymptotically equal to a constant multiple of \( J_n(\sqrt{\lambda} r) \) as \( r \to 0 \). The same result fails to hold for \( R_0 \) and (13), as both have zeros accumulating at 0 which are not, in general, the same sequence.
We may now define explicitly the operators \( L_n \) in Proposition 11. For \( n \geq 0 \)
\[
D(L_n) := \left\{ \varphi \in L^2(0, 1; rdr) \left| \frac{1}{r} (r \varphi'(r))' - \frac{\mu_n}{r^2} \varphi(r) \in L^2(0, 1; rdr), \varphi(1) = 0 \right. \right\},
\]
\[
L_n \varphi(r) := -\frac{1}{r} (r \varphi'(r))' + q(r) \varphi(r) + \frac{\mu_n}{r^2} \varphi(r) \quad (r \in (0, 1)).
\]
By the orthogonality in \( L^2\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \) of \( \Theta_n \), the operator \( L \) is equal to the orthogonal sum \( \oplus_{n=0}^{\infty} L_n \). As remarked in [28, pp. 4–5], for every \( n \geq 1 \) the operator \( L_n \) is of limit-point type at 0, whilst \( L_0 \) is of limit-circle type at 0. This means that a self-adjoint restriction of \( L_0 \) may be found by imposing a BC at 0. Assuming 0 is not in the spectrum of \( L_0 \), this BC may be written [30] as a Lagrange bracket with a linear combination of
\[
u_0(r) = \sin\left(\frac{\log(r)}{b}\right)\quad \text{and} \quad v_0(r) = \cos\left(\frac{\log(r)}{b}\right),
\]
which are linearly independent in the kernel of \( L_0 - q \); see (7). For technical reasons we always need \( u_0 \), so we use the combination \( u_0 + \beta v_0 \) for a fixed \( \beta \in \mathbb{R} \). Applied to a function \( \varphi \), the Lagrange bracket BC requires that as \( r \to 0 \) we have
\[
[\varphi, u_0 + \beta v_0](r) := \varphi(r)(u_0 + \beta v_0)'(r) - r \varphi'(r)(u_0 + \beta v_0)(r) \to 0.
\]
Abusing notation we define \( \beta[\varphi] := [\varphi, u_0 + \beta v_0](0^+) \). Thus, our self-adjoint restriction of \( L_0 \) is
\[
D(L'_0) = \left\{ \varphi \in L^2(0, 1; rdr) \left| \frac{1}{r} (r \varphi'(r))' - \frac{\mu_n}{r^2} \varphi(r) \in L^2(0, 1; rdr), \beta[\varphi] = 0 = \varphi(1) \right. \right\},
\]
\[
L'_0 \varphi(r) = -\frac{1}{r} (r \varphi'(r))' + q(r) \varphi(r) + \frac{\mu_n}{r^2} \varphi(r) \quad (r \in (0, 1)),
\]
meaning that the orthogonal sum
\[
L' := L'_0 \oplus \bigoplus_{n=1}^{\infty} L_n
\]
is a self-adjoint restriction of \( L \).
Applying the asymptotics of Proposition 2 we see that the BC (15) when applied to the \( \Theta_0 \)-component of the solution \( u \) of (11) is equivalent to the two-dimensional
\[
[u, u_0 + \beta v_0](0^+, \vartheta) = 0 \quad (\vartheta \in (-\pi/2, \pi/2)).
\]
With the same abuse of notation we are now justified in defining
\[
\beta_{\vartheta}[u] = [u, u_0 + \beta v_0](0^+, \vartheta),
\]
which yields the BC in [28] and proves that the self-adjoint operator \( L' \) has domain
\[
D(L') = \{ u \in L^2(\Omega_1) \mid \Delta u \in L^2(\Omega_1), u \mid_{\Gamma_1} = 0 = (u + b y \partial_\nu u) \mid_{\Gamma_1} = \beta_{\vartheta}[u] \}.
\]
We observe the following corollary to [28, Sec. 4].
Proposition 3. The operator $L'$ has discrete spectrum accumulating exactly at $\pm \infty$.

Proof. It is known \[28\] Sec. 4] that $\sigma(L' - q)$ comprises simple eigenvalues accumulating at $\pm \infty$. Now observe that as a simple consequence of the spectral theorem \[32\] Sec. VIII.3] $L' - q$ has compact resolvent. Then \[23\] Thms. IV.3.17 & V.4.3] imply that $L'$ also has compact resolvent and is self-adjoint. The last two results combined mean that the spectrum of $L'$ accumulates nowhere except possibly at $\pm \infty$. Combining this fact with \[23\] Thm. V.4.10], which here implies

$$\sup_{\lambda \in \sigma(L')} \text{dist}(\lambda, \sigma(L' - q)) \leq \|q\|_{L^\infty(\Omega)},$$

we obtain sequences of eigenvalues of $L'$ that accumulate at both $\pm \infty$.

\[ \square \]

Remark. One cannot guarantee solely from this argument that no new eigenvalues are introduced by adding $q$, which otherwise would alleviate the need for Theorem 3.

By the reasoning in \[28\] Sec. 5] one may then define the self-adjoint operator

$$D(T) := \{ u \in L^2(\Omega) \mid \Delta u \in L^2(\Omega), (u - f \partial \nu u) |_{\Gamma_c} = u |_{\Gamma} = \beta_0[u] = 0 \},$$

$$Tu := (-\Delta + q)u,$$

also possessing purely discrete spectrum accumulating only at $\pm \infty$, and which is a restriction of the operator underlying $\Pi$.

We finish the section with an important fact:

Resolvent Hypothesis. Without a loss of generality, when considering the question of uniqueness outlined in Section 2 we may assume that both of the operators $T_1$ and $T_2$—corresponding respectively to the triples $(q_1, f_1, \beta_1)$ and $(q_2, f_2, \beta_2)$—have 0 in their resolvent set. To see why, suppose either (or both) has 0 in their spectrum. Then we may simply shift their spectra, which are discrete, by adding the same sufficiently small constant to both $q_1$ and $q_2$ so that the resulting operators now both have 0 in their resolvent set. The question of uniqueness is left unchanged by the shift. We will assume the hypothesis holds for the rest of the paper.

4. Complex geometric optics for an existing approach with a Dirichlet bc

To prove Theorem 1 we will adapt some existing methods. To reduce technicality we will use \[19\] Thm. 1.1] instead of the more powerful results in \[16\ 21\]. As previously mentioned, all such results employ a Dirichlet bc on the inaccessible $\Gamma_c$.

The main ingredients in the proof of \[19\ Thm. 1.1] are Lemma 1 of this section (merely summarised from \[19\]) and \[19\ Prop. 4.1]. We will explain the former now. Let any $x = (x_1, x_2) \in \mathbb{R}^2$ be represented by the complex number $z = x_1 + ix_2$. Notating the partial derivative with respect to $x_j$ as $\partial_j$, define the complex derivatives
\( \partial_z = (\partial_1 - i \partial_2)/2 \) and \( \partial_\varphi = (\partial_1 + i \partial_2)/2 \). Note that \( \Phi(z) \ (z \in \Omega) \) is holomorphic if and only if the Cauchy–Riemann equation \( \partial_\varphi \Phi(z) = 0 \ (z \in \Omega) \) is satisfied.

**Definition 6.** We say that a holomorphic \( \Phi = \varphi + i \psi \) on \( \Omega \), with \( \varphi \) and \( \psi \) real-valued and possessing a continuous extension to \( \overline{\Omega} \), is an admissible phase function if the following criteria are met:

(i) its set of critical points \( \mathcal{H} := \{ z \in \overline{\Omega} \mid \partial_z \Phi(z) = 0 \} \) does not intersect \( \Gamma \);
(ii) its critical points are non-degenerate, i.e., \( \partial_\varphi^2 \Phi(z) \neq 0 \) (\( z \in \mathcal{H} \));
(iii) its imaginary part \( \psi \) vanishes on \( \Gamma_c \).

**Remark.** The function \( \Phi \) will be crucial in constructing the complex geometric optics (CGO) solutions. The critical points \( \mathcal{H} \) are finite, by holomorphicity of \( \Phi \).

Now define the primitive operators \( \partial_{\tau}^{-1} \) and \( \partial_{\varphi}^{-1} \) by

\[
\partial_{\tau}^{-1} g(z) := -\frac{1}{\pi} \int_{\Omega} \frac{g(\xi_1 + i \xi_2)}{\xi_1 + i \xi_2 - z} d\xi_2 d\xi_1 =: \overline{\partial_{\varphi}^{-1} g(z)}.
\]

**Lemma 1** (Imanuvilov–Uhlmann–Yamamoto, 2010, Sec. 3). Let \( \alpha > 0 \) and \( q_1 \in C^{2+\alpha}(\overline{\Omega}) \), and take any admissible phase function \( \Phi \). Then, for each \( \tau > 0 \), there is a solution to

\[
\begin{cases}
(-\Delta + q_1) u &= 0 \quad \text{in} \quad \Omega, \\
u &= 0 \quad \text{on} \quad \Gamma_c
\end{cases}
\]

given by

\[
u_1(z; \tau) = e^{\tau \Phi(z)} (a(z) + a_0(z)/\tau) + e^{\tau \overline{\Phi(z)}} \left( a(z) + a_1(z)/\tau \right) + e^{\tau (\Phi + \overline{\Phi})(z)/2} \tilde{v}_1(x; \tau),
\]

where the following conditions hold.

(i) The amplitude function \( a(\cdot) \in C^2(\overline{\Omega}) \) is non-trivial, holomorphic on \( \Omega \), its real part vanishes on \( \Gamma_c \) and \( a = \partial_\varphi a = 0 \) in \( \mathcal{H} \cap \partial \Omega \); such an \( a(\cdot) \) is called admissible.

(ii) The remainder \( \| \tilde{v}_1(\cdot ; \tau) \|_{L^2(\Omega)} = o(1/\tau) \) as \( \tau \to +\infty \).

(iii) The functions \( a_0 \) and \( a_1 \) are holomorphic and satisfy the \( \tau \)-independent BC

\[
(a_0 + a_1) \big|_{\Gamma_c} = \frac{M_1}{4 \partial_\varphi \Phi} + \frac{M_3}{4 \partial_\varphi \Phi}.
\]

The functions \( \tilde{M}_1 := \partial_{\tau}^{-1} (a q_1) - M_1 \) and \( \tilde{M}_3(z) := \partial_{\tau}^{-1} (a(z) q_1(z)) - M_3(z) \), where \( M_1 \) and \( M_3 \) are any polynomials satisfying, for \( j = 0, 1 \) and \( 2 \),

\[
\begin{align}
\partial_\varphi^j (\partial_{\tau}^{-1} (a(z) q_1(z)) - M_1(z)) &= 0, \\
\partial_\varphi^j (\partial_{\tau}^{-1} (a(z) q_1(z)) - M_3(z)) &= 0.
\end{align}
\]
Moreover for any \( q_2 \in C^{2+\alpha}(\overline{\Omega}) \) and the same \( \Phi \) and \( \tau \) we can find the same amplitude function \( a(\cdot) \) so that
\[
v_2(x;\tau) = e^{-\tau \Phi(z)}(a(z) + b_0(z)/\tau) + e^{-\tau \Phi(z)} \left( a(z) + b_1(z)/\tau \right) + e^{-\tau (\Phi + \overline{\Phi})/2} \tilde{v}_2(x;\tau)
\] solves \((-\Delta + q_2)u = 0, u \mid_{\Gamma_c} = 0, \) with
\[
(iv) \| \tilde{v}_2(\cdot;\tau) \|_{L^2(\Omega)} = o(1/\tau) \text{ as } \tau \to +\infty, \text{ and } \\
(v) \text{ holomorphic } b_0 \text{ and } b_1 \text{ satisfying }
\]
\[
(b_0 + b_1) \mid_{\Gamma_c} = -\frac{\tilde{M}_2}{4z^2 \Phi} - \frac{\tilde{M}_4}{4z^2 \Phi}.
\] (24)

Here \( \tilde{M}_2 := \partial_z^{-1}(aq_2) - M_2 \) and \( \tilde{M}_4 := \partial_z^{-1}(a(z)q_2(z)) - M_4(\overline{\Phi}) \), where \( M_2 \) and \( M_4 \) are any polynomials satisfying, for \( j = 0, 1 \) and \( 2 \),
\[
\partial^j_z(\partial_z^{-1}(a(z)q_2(x)) - M_2(z)) = 0, \quad (25) \\
\partial^j_z(\partial_z^{-1}(\overline{\Phi}(z)q_2(x)) - M_4(\overline{\Phi})) = 0. \quad (26)
\]

5. Unique continuation and density results

To utilise the arguments of [19] we will prove density, in the full space of solutions of the differential equation in the \( \text{BVP} \) (1), of solutions that also satisfy the bcs (3) and (5). To do this we will adapt certain results of [5]. We note that these results are not entirely new to [5], being related to early Runge-type theorems, e.g., [33].

The following lemmata achieve unique continuation and the required density. Define \( \Omega^* := \overline{\Omega} \setminus \{0\} \).

**Lemma 2** (Unique continuation principles for a Schrödinger-type equation). Let \( \Omega, \) \( q \) and \( f \) be admissible and \( \Omega^* \subset \Omega \), with \( \Omega^* \) non-empty, bounded, connected and open, such that \( \partial \Omega^* \subset C^2 \) and \( \Omega \setminus \overline{\Omega^*} \) is connected.

(i) If \( u \in H^2_{\text{loc}}(\Omega^*) \) satisfies \((-\Delta + q)u = 0 \) in \( \Omega \), and there is a ball \( B \) with \( \overline{B} \subset \Omega \) and \( u \mid_{\partial B} = 0 \), then \( u = 0 \). 

(ii) If \( u \in H^2_{\text{loc}}(\Omega^* \setminus \Omega') \), \((-\Delta + q)u \mid_{\Omega^* \setminus \overline{\Omega'}} = 0 \) and \( u \mid_{\Gamma} = 0 = \partial_{\nu} u \mid_{\Gamma} \), then \( u \mid_{\Omega \setminus \Omega'} = 0 \).

**Proof.** Part (i) is standard, e.g., [27], Cor. 1.1. Part (ii) follows extending \( u \) by 0 through \( \Gamma \) and applying (i). \( \square \)

**Lemma 3.** Under the hypotheses of Lemma 2 define the sets
\[
K := \{ v \in H^2_{\text{loc}}(\Omega^*) \mid (-\Delta + q)v \mid_{\Omega} = 0 \}, \\
\tilde{K} := \{ g \in K \mid (g - f \partial_n g) \mid_{\Gamma_c} = 0 = \beta_0[g] \}.
\]

Then \( \tilde{K} \) is dense in \( K \) under the topology induced by \( \| \cdot \|_{L^2(\Omega^*)} \).
Proof. We adapt the proofs of [5, Prop. 5.1-2]. For any measurable $A \subset \mathbb{R}^d$ denote by $\langle \cdot, \cdot \rangle_A$ the inner product on $L^2(A)$. Let $v \in K$ such that $\langle g, v \rangle_{\Omega'} = 0$ for every $g \in \tilde{K}$; we aim to show $v = 0$. By the Resolvent Hypothesis we may uniquely define $w \in D(T)$ to solve $Tw = \chi_{\Omega'} \pi$, where $\chi_A(x) = 1$ $(x \in A)$; $0$ $(x \notin A)$.

Now we make some technical definitions (see Fig. 2): the sub-domain $\Omega_2 \subset \Omega \setminus \overline{\Omega}'$ is taken to have boundary $\partial \Omega_2 = \Gamma \cup \Gamma' \cup \tilde{\Gamma} \in C^{2,1}$ such that $\Gamma$, $\Gamma'$ and $\tilde{\Gamma}$ are all relatively open and disjoint. Here $\tilde{\Gamma}$ continuously extends $\Gamma$ in $\partial \Omega_2 \cap \partial \Omega$ at both its endpoints, $\Gamma'$ is entirely contained in $\Omega \setminus \overline{\Omega}$, and $0 \notin \partial \Omega_2$. This means $\Omega_2$ is a neighbourhood of $\Gamma$, and its complement is separated from $\Gamma$, i.e., $\Omega \setminus \Omega_2 \cap \Gamma = \emptyset$.

Take a smooth cut-off function $\mu$ on $\Omega_2$ to be 1 in a neighbourhood of $\Gamma$ and 0 in a neighbourhood of $\Gamma'$. Assume without loss of generality that the level curves of $\mu$ are orthogonal to $\tilde{\Gamma}$.

Consider $g \in \tilde{K} \subset H^2(\Omega_2)$. We wish to make a decomposition of $g$ into two parts: $g_0 \in D(T)$, and $g_1 \in H^2(\Omega_2)$ which is supported in $\Omega_2$ and satisfies

$$
(g_1 - f \partial_\nu g_1) \mid_{\Gamma} = (g - f \partial_\nu g) \mid_{\Gamma}.
$$

(27)

Firstly extend $f$ to a bounded, a.e. continuous function in the interior of $\Omega$. By the trace theorem [37, Thm. 8.7] $(g - f \partial_\nu g) \mid_{\Gamma}$ can be extended by 0 to $\tilde{F}_2 := (g - f \partial_\nu g) \mid_{\partial \Omega} \in H^{1/2}(\partial \Omega)$. Take any $F_2 \in H^{1/2}(\partial \Omega_2)$ which agrees with $\tilde{F}_2$ on $\Gamma \cup \tilde{\Gamma}$ and is 0 on $\Gamma'$. Define $F_1 = \mu g \mid_{\partial \Omega_2} \in H^{3/2}(\partial \Omega_2)$.

The inverse trace theorem [37, Thm. 8.8] guarantees the existence of $g_1 \in H^2(\Omega_2)$ such that $g_1 \mid_{\partial \Omega_2} = F_1 \in H^{3/2}(\partial \Omega)$. Furthermore, since

$$
(g_1 - F_2)/f \mid_{\partial \Omega_2} = (\mu g - F_2)/f
$$

$$
= \begin{cases} 
\partial_\nu g &\in H^{1/2}(\Gamma), \text{ on } \Gamma, \\
mug/f &\in H^{3/2}(\tilde{\Gamma}), \text{ on } \tilde{\Gamma}, \\
0 &\in H^{1/2}(\Gamma'), \text{ on } \Gamma'
\end{cases}
$$
is clearly in $H^{1/2}(\partial \Omega_2)$, we can choose this $g_1$ to satisfy $\partial_v g_1 \big|_{\partial \Omega_2} = (g_1 - F_2)/f \big|_{\partial \Omega_2} \in H^{1/2}(\partial \Omega_2)$. This ensures that $\partial_v g_1 \big|_{\Gamma} = 0$; since $g_1 \big|_{\Gamma} = F_1 \big|_{\Gamma} = 0$ we may extend this $g_1$ by 0 into $\Omega$, and note that by construction (27) holds.

Now define $g_0 = g - g_1$. By checking the BCs on $\Gamma, \tilde{\Gamma}$ and $\Gamma_c \setminus \tilde{\Gamma}$, we see that $g_0 \in D(T)$.

Therefore, for any $g \in \tilde{K}$,

$$0 = \langle g, v \rangle_{L^2(\Omega)} = \langle g_0, T\overline{w} \rangle_{L^2(\Omega)} + \langle g_1, T\overline{w} \rangle_{L^2(\Omega)}$$

$$= \langle Tg_0, \overline{w} \rangle_{L^2(\Omega)} + \int_{\Omega_2} g_1(-\Delta + q)w$$

$$= \langle (-\Delta + q)g, \overline{w} \rangle_{L^2(\Omega)} + \int_{\Gamma,\tilde{\Gamma},\Gamma_c} (-g_1 \partial_v w + w \partial_v g_1) = -\int_{\Gamma} g \partial_v w, \quad (28)$$

where we used the self-adjointness of $T$, Green’s formula, and the fact $g_1, w \in H^2(\Omega_2)$. The final equality was achieved by noting that $w - f \partial_v w = 0 = g - f \partial_v g$ on $\tilde{\Gamma}$, $g = \partial_v g = 0$ on $\Gamma'$, and $w \in D(T)$ means that $w = 0$ on $\Gamma$.

Now observe that the $L^2(\Gamma)$-closure of $\{g \big|_{\Gamma} \mid g \in \tilde{K}\}$ is precisely $L^2(\Gamma)$. This is because for any given basis $\psi_n$ of $L^2(\Gamma)$ we can solve

$$\begin{cases}
(-\Delta + q)g &= 0 \text{ in } \Omega, \\
g - f \partial_v g &= 0 \text{ on } \Gamma_c, \\
\beta g &= 0 \text{ at } 0, \\
g &= \psi_n \text{ on } \Gamma,
\end{cases}$$

as $0$ is in the resolvent set of $T$. Thus, from (28), we see $\partial_v w \big|_{\Gamma} = 0$. Since $w \big|_{\Gamma} = 0$, the unique continuation in Lemma 4(ii) implies, from $(-\Delta + q)w \big|_{\Omega \setminus \tilde{\Gamma}} = 0$, that $w \big|_{\Omega \setminus \tilde{\Gamma}} = 0$.

In particular $w \big|_{\partial \Gamma'} = \partial_v w \big|_{\partial \Gamma'} = 0$, and so

$$\langle v, v \rangle_{L^2(\Omega')} = \int_{\Omega'} v(-\Delta + q)w$$

$$= \int_{\Omega'} w(-\Delta + q)v + \int_{\partial \Gamma'} (-v \partial_v w + w \partial_v v) = 0.$$

By the unique continuation in Lemma 2(i) we deduce that $v \big|_{\Omega} = 0$. \hfill \Box

6. A weighted sum of $q$-values for a singular BC

In this section we will prove an analogue to [19, Prop. 4.1] in the case of the DN map $\Lambda_{q,f,\beta}(0)$ from Inverse Problem 1 as opposed to the more standard DN map $\Lambda_{q,0}(0)$ used in, e.g., [19]. We will use the cgo solutions from Lemma 1 by density we can “approach” such solutions with those satisfying the singular BC, thanks to Lemma 3.

As in Lemmata 2 and 3, we consider $\Omega' \subset \Omega$ to be non-empty, bounded, open and connected subsets of $\mathbb{R}^2$ with $\Omega \setminus \overline{\Omega'}$ connected and $\partial \Omega' \in C^2$. However we are forced,
for the same technical reasons as in [19], to accept the restriction \( \partial \Omega \in C^\infty \), included as a requirement in our class of admissible domains.

**Proposition 4.** Suppose we have an admissible phase function \( \Phi \) (see Definition 6) and functions \( a, a_0, a_1, b_0, b_1 \) satisfying (6) (from Lemma 1, 20) and (24), where \( M_1, M_2, M_3, M_4 \) satisfy (21), (22), (25) and (26). Denote by \( H_\Phi \) the Hessian matrix of \( \Phi \). Let \( q_1, q_2 \in C^{2+\alpha}(\Omega) \) for some \( \alpha > 0 \), set \( q = q_1 - q_2 \). Suppose the DN maps with the same \( f \) and \( \beta \) are equal at \( \lambda = 0 \), so \( \Lambda_{q_1,f,\beta}(0) = \Lambda_{q_2,f,\beta}(0) \). Then, for any \( \tau > 0 \),

\[
\sum_{z \in H} |a(z)|^2 \cos(2\tau \text{Im}[\Phi(z)]) = \frac{1}{8\pi} \int_\Omega \left[ \left( \frac{\partial H_\Phi(z)}{\partial z} - 4(a_0 + b_0) \right) a + \left( \frac{\partial H_\Phi(z)}{\partial z} - 4(a_1 + b_1) \right) \frac{1}{2} q \right] q. \tag{29}
\]

To prove Proposition 4 we will need an integration-by-parts formula. Since elements of \( D(T) \) are not necessarily in \( H^2(\Omega) \)—in fact, in polar coordinates \( \sin(b^{-1}\log r) + \beta \cos(b^{-1}\log r) \) may be \( C^2 \)-extended to an element of \( D(L') \), and this linear combination clearly fails to be in \( H^2 \) at 0—we see that Green’s formula cannot be applied. We circumvent this in the following way.

**Lemma 4.** Take admissible \( \Omega \) and open, connected subset \( \Omega' \subset \Omega \). Suppose \( \Omega \setminus \overline{\Omega'} \) is connected and \( \partial \Omega' \in C^2 \). For \( j = 1, 2 \) let \( q_j \) be admissible potentials with \( (q_1 - q_2) |_{\partial \Omega} = 0 \) and \( f_j \) admissible boundary functions. Suppose \( f_1 \) and \( f_2 \) are identical on \( \Gamma_1 \), i.e., \( b_1 = b_2 \). Choose an admissible self-adjoint \( \mathcal{BC} \beta \). Let \( u_j \) be any respective solutions to

\[
\begin{cases}
(-\Delta + q_j) u_j = 0 & \text{in } \Omega, \\
u_j - f_j \partial_\nu u_j = 0 & \text{on } \Gamma_c, \\
\beta_\theta [u_j] = 0 & \text{at } 0.
\end{cases} \tag{30}
\]

Then, if \( \Lambda_{q_1,f_1,\beta}(0) = \Lambda_{q_2,f_2,\beta}(0) \), we have

\[
\int_\Omega (q_1 - q_2) u_1 u_2 = \int_{\partial \Omega} (u_2 \partial_\nu u_1 - u_1 \partial_\nu u_2). \tag{31}
\]

**Proof.** Take a \( \delta \)-radius half-disc \( \Omega_\delta = \delta \Omega_1 \subset \Omega_1 \) for some \( 0 < \delta < 1 \) and define \( \Omega_{0,\delta} = \Omega \setminus \overline{\Omega_\delta} \). Without losing generality we may assume \( \overline{\Omega_1} \cap \overline{\Omega'} = \emptyset \). Set \( \Gamma_\delta = \delta \Gamma_1 \) and \( \Gamma_{1,\delta} \) to be, respectively, the straight and semi-circular parts of \( \partial \Omega_\delta \). Then
where we applied identity of the \( q_j \) outside \( \Omega' \), identity of \( f_j \) on \( \Gamma_1 \), and equality of the DN maps to eliminate various terms, and Green’s formula over \( \Omega_{0,\delta} \) to achieve the second line. Thus the lemma follows if the second integral on the right-hand side of (32) converges to 0 as \( \delta \to 0 \).

Observe that

\[
\int_{\Gamma_{1,\delta}} (u_1 \partial_\nu u_2 - u_2 \partial_\nu u_1) = \int_{-\pi/2}^{\pi/2} \delta(u_1 \partial_r u_2 - u_2 \partial_r u_1)(\delta, \vartheta)d\vartheta = \int_{-\pi/2}^{\pi/2} [u_1, u_2](\delta, \vartheta)d\vartheta.
\]

Moreover, by expanding the following determinant and calculating \([v_0, u_0](r, \vartheta) = b^{-1}e^{-2\vartheta/b}\), one can easily see that

\[
[u_1, u_2](r, \vartheta) = be^{2\vartheta/b} \begin{vmatrix} [u_1, v_0] & [u_1, u_0] \\ [u_2, v_0] & [u_2, u_0] \end{vmatrix} (r, \vartheta).
\]

Now apply \([u_j, u_0 + \beta v_0](r, \vartheta) \to 0 \) as \( r \to 0 \) to see that the columns in the right-hand side of (33) become collinear as \( r \to 0 \). The lemma follows.

\[\square\]

**Remark.** The identity (33) is usually written for solutions of ordinary differential equations; see, e.g., [15, (2.8-9)].

**Proof of Proposition 4.** We consider all solutions \( u_j \) \((j = 1, 2)\) of (30) with \( f_j = f \), \( \beta_j = \beta \). Define \( \Lambda = \Lambda_{q_1, f, \beta}(0) = \Lambda_{q_2, f, \beta}(0) \), \( h_1 = u_1 \big| \Gamma \) and \( h_2 = u_2 \big| \Gamma \). Also define
\[ u_2 \in L^2(\Omega) \text{ by} \]
\[
\begin{cases}
(\Delta - q_1)u_2 &= 0 \quad \text{in } \Omega, \\
u_2 - f \partial_\nu \tilde{u}_2 &= 0 \quad \text{in } \Gamma_c, \\
\beta_0[\tilde{u}_2] &= 0 \quad \text{at } \partial \Omega, \\
u_2 &= h_2 \quad \text{on } \Gamma,
\end{cases}
\]
(34)
i.e., with potential \( q_1 \) but bc \( h_2 = u_2 |_\Gamma \) on \( \Gamma \). By Lemma [4] we see
\[
\int_\Omega (q_1 - q_2)u_1u_2 = \int_\partial \Omega (u_2 \partial_\nu u_1 - u_1 \partial_\nu u_2)
= \int_\Gamma (h_1 \Lambda h_2 - h_2 \Lambda h_1)
= \int_\partial \Omega (\tilde{u}_2 \partial_\nu u_1 - u_1 \partial_\nu \tilde{u}_2)
= \int_\Omega (q_1 - q_2)u_1 u_2 = 0.
\]
(35)
Note the hypothesis \( q_1 - q_2 = 0 \) outside \( \Omega' \). Clearly \( v_j \in K \)—see (19) and (23)—and \( u_j \in \tilde{K} \), so using Lemma [3] we deduce from (31) that
\[
\int_\Omega (q_1 - q_2)v_1(\cdot ; \tau)v_2(\cdot ; \tau) = 0 \quad (\tau > 0).
\]
We have now arrived at precisely [19, Eq. (4.3)]. From here on our proof exactly follows that of [19, Prop. 4.1]. □

7. Final steps of proof of Theorem

The proof of Theorem [4] is now straight-forward:

Proof of Theorem [4]. To prove that \( \Lambda_{q_1,f,\beta}(0) = \Lambda_{q_2,f,\beta}(0) \) and \( (q_1 - q_2) |_{\Omega' \setminus \Omega} = 0 \) implies \( q_1 = q_2 \) everywhere, simply apply all but one of the steps in the proof of [19 Thm. 1.1], replacing [19 Prop. 4.1] with our Proposition [4].

Now for the other claim, namely that \( \Lambda_{q,f,\beta}(0) = \Lambda_{q,f,\beta}(0) \) implies \( f_1 = f_2 \). Let \( g \in H^{1/2}(\Gamma) \), and choose functions \( u_1 \) and \( u_2 \) solving
\[
\begin{cases}
(\Delta + q)u_j &= 0 \quad \text{in } \Omega, \\
u_j - f_j \partial_\nu u_j &= 0 \quad \text{on } \Gamma_c, \\
\beta_0[u_j] &= 0 \quad \text{at } \partial \Omega, \\
u_j &= g \quad \text{on } \Gamma.
\end{cases}
\]
By hypothesis \( \partial_\nu u_1 = \partial_\nu u_2 = -\Lambda g \). Then \( u_j \in H^2_{\text{loc}}(\Omega^*) \), so with the definition \( u = u_1 - u_2 \) we see
\[
\begin{cases}
(\Delta + q)u &= 0 \quad \text{in } \Omega, \\
u = 0 = \partial_\nu u \quad \text{on } \Gamma.
\end{cases}
\]
The unique continuation principle Lemma 2(ii) immediately implies \( u = 0 \) in \( \Omega \), whence \( u_1 = u_2 \) in \( \Omega \) so \( \partial_\nu u_1 = \partial_\nu u_2 \) on \( \Gamma_c \). Thus, along \( \Gamma_c \), \( f_1 = u_1/\partial_\nu u_1 = u_2/\partial_\nu u_2 = f_2 \).

8. The Interface Dirichlet-to-Neumann Operators

To prove the asymptotics of Theorem 3 we will find upper and lower asymptotic bounds on the difference between the counting functions of the negative eigenvalues for, respectively, \( T_0 \) and \( L' \) (see (9) and (18) for the operator definitions). To achieve the upper bound we will consider a pencil of interface Dirichlet-to-Neumann operators on \( \Gamma_i \). In this section we will develop these operators, establishing results for our proof of Theorem 3. To ensure sign-definiteness of one of these \( \Lambda_i \) operators we need the admissible boundary function \( f \) to be supported in \( \Gamma_1 \), justifying the corresponding requirement in both Theorems 2 and 3.

These interface Dirichlet-to-Neumann operators are defined as follows: for \( j = 0, 1 \),

\[
\Lambda_j(\lambda) : H^{1/2}(\Gamma_i) \ni h \mapsto -\partial_\nu w_j \in H^{-1/2}(\Gamma_i),
\]

where \( w_j \) solve the boundary-value problems

\[
\begin{cases}
-\Delta w_0 = \lambda w_0 & \text{in } \Omega_0, \\
w_0 = 0 & \text{on } \Gamma_0 \cup \Gamma, \\
w_0 = h & \text{on } \Gamma_i,
\end{cases}
\]

and

\[
\begin{cases}
-\Delta w_1 = \lambda w_1 & \text{in } \Omega_1, \\
w_1 = f \partial_\nu w_1 & \text{on } \Gamma_1, \\
\beta_\theta[w_1] = 0 & \text{at } 0, \\
w_1 = h & \text{on } \Gamma_i,
\end{cases}
\]

and \( \partial_\nu \) denotes the outward directed normal derivative for the subdomain \( \Omega_j \).

**Remark.** It is clear that a real number \( \lambda \) is an eigenvalue for \( T_0 \) if and only if the pencil of operators \( \Lambda_1(\lambda) + \Lambda_0(\lambda) \) has a non-trivial kernel \( K(\lambda) \), since any function in this kernel will correspond to a pair \((w_0, w_1)\) solving, respectively, (36) and (37), for which \( \partial_{\nu_0} w_0 = -\partial_{\nu_1} w_1 \) on \( \Gamma_i \).

To conduct our analysis we utilise the normalised \( L^2(-\pi/2, \pi/2) \)-basis \( \Theta_n \) on \( \Gamma_i \) (see Section 3), defined by

\[
\Theta_n(\vartheta) = \begin{cases} 
k_0 e^{-\vartheta/b} & (n = 0), \\
_k(n b \cos(n \vartheta) - \sin(n \vartheta)) & (n \text{ even}), \\
_k(n \cos(n \vartheta) - nb \sin(n \vartheta)) & (n \text{ odd}),
\end{cases}
\]

where the \( k_n \) are chosen so that \( \|\Theta_n\|_{L^2(-\pi/2, \pi/2)} = 1 \). Since the sufficiently negative eigenvalues of \( T_0 \) can only arise from the presence of the singular \( \text{BC} \) (regular \( \text{BCs} \) would yield a spectrum that is bounded below), any function \( h = \sum_{n=0}^{\infty} h_n \Theta_n \in K(\lambda) \) for \( \lambda \ll 0 \) is either identically 0 or has non-trivial zeroth component. It follows that
for \( \lambda \ll 0 \) such that \( \mathcal{K}(\lambda) \neq \{0\} \) we may normalise the non-trivial kernel element \( h \) so that \( h_0 = 1 \).

In the basis \( \Theta_n \) on \( \Gamma_1 \) the map \( \Lambda_1(\lambda) \) takes the form of an infinite diagonal matrix, represented in the block partitioned form

\[
\begin{pmatrix}
  m_0(\lambda) & 0^T \\
  0 & M(\lambda)
\end{pmatrix}.
\]

Here \( m_0(\lambda) := -\varphi'_0(1;\lambda)/\varphi_0(1;\lambda) \) (\( \varphi_0 \neq 0 \)) is the Weyl–Titchmarsh \( m \)-function for the \( L^2(0,1) \)-limit-circle ordinary differential problem

\[
\begin{aligned}
-\frac{1}{r} \frac{d}{dr} \left(r \frac{d\varphi_0}{dr}(r;\lambda)\right) - \frac{1}{b^2 r^2} \varphi_0(r;\lambda) &= \lambda \varphi_0(r;\lambda) \quad (r \in (0,1)), \\

r \left(\varphi_0 \partial_r (u_0 + \beta v_0) - (\partial_r \varphi_0)(u_0 + \beta v_0)\right)(r;\lambda) &\to 0 \quad (r \to 0),
\end{aligned}
\]

the infinite column-vector of zeros is denoted by \( 0 \), and \( M(\lambda) \) is the diagonal submatrix whose \( n \)-th diagonal term (\( n = 1, 2, 3, \ldots \)) is the \( L^2(0,1) \)-limit-point \( m \)-function \( m_n(\lambda) := -\varphi'_n(1;\lambda)/\varphi_n(1;\lambda) \) (\( \varphi_n \neq 0 \)) for the ODE

\[
-\frac{1}{r} \frac{d}{dr} \left(r \frac{d\varphi_n}{dr}(r;\lambda)\right) + \frac{n^2}{r^2} \varphi_n(r;\lambda) = \lambda \varphi_n(r;\lambda) \quad (r \in (0,1)).
\]

In the same basis, \( \Lambda_0(\lambda) \) lacks this diagonal structure, although it is symmetric. We may nevertheless use the basis to partition it the same way, labelling it

\[
\begin{pmatrix}
  a(\lambda) & b(\lambda)^T \\
  b(\lambda) & C(\lambda)
\end{pmatrix}.
\]

Lemma 5.

(i) The pencil of DN maps \( \Lambda_1 + \Lambda_0 \) is a holomorphic, operator-valued function on \( \mathbb{C} \setminus \mathbb{R} \).

(ii) The derivative of the pencil is a compact operator on the Sobolev space \( H^k(\Gamma_1) \) for any \( k \geq 1/2 \).

(iii) If the admissible boundary function \( f \) is supported in \( \Gamma_1 \) then the quadratic form \( \langle (\Lambda_1 + \Lambda_0)(\lambda)h, h \rangle_{L^2(\Gamma_1)} \) has imaginary part of the same sign as \( \text{Im}(\lambda) \).

Remark. This makes \( \Lambda_1 + \Lambda_0 \) an operator-valued Herglotz function.

Proof. (i) and (ii). It suffices to show holomorphicity of any DN map, since the sum of any two will also have the property. Let \( \tilde{\Omega} \subset \mathbb{R}^2 \) be bounded and simply connected with piecewise \( C^2 \) boundary \( \partial \tilde{\Omega} \), and suppose \( \tilde{\Gamma} \) is a connected subset of \( \partial \tilde{\Omega} \), whilst \( \tilde{\Gamma}_c = \partial \tilde{\Omega} \setminus \tilde{\Gamma} \) is its complement. Denote by \( S \) the self-adjoint operator \(-\Delta\) with homogeneous Dirichlet conditions on \( \tilde{\Gamma} \) and any BC \( \text{SA}[\cdot] = 0 \) on \( \tilde{\Gamma}_c \) that suffices to make \( S \) self-adjoint. Then define the DN operator \( \tilde{\Lambda}(\lambda) : h \mapsto \partial_\nu w \big|_{\tilde{\Gamma}} \), where

\[
\begin{aligned}
-\Delta w &= \lambda w \quad \text{in} \quad \tilde{\Omega}, \\
\text{SA}[w] &= 0 \quad \text{on} \quad \tilde{\Gamma}_c, \\
w &= h \quad \text{on} \quad \tilde{\Gamma}.
\end{aligned}
\]

Proof
If $h \in H^k(\tilde{\Gamma})$ for some $k \geq 1/2$ then we may choose a $w_0 \in L^2(\tilde{\Omega})$ taking the value $h$ on $\tilde{\Gamma}$. Then if $\text{SA}[w_0] = 0$ and $\Delta w_0 \in L^2(\tilde{\Omega})$, we see that the solution to the boundary-value problem \((\Omega)\) is given by
\[
w = (1 - (S - \lambda)^{-1}(-\Delta - \lambda))w_0.
\]
Hence, in terms of the trace maps
\[
D(\gamma_0) = \{ v \in L^2(\tilde{\Omega}) \mid \Delta v \in L^2(\tilde{\Omega}), \text{SA}[v] = 0 \}, \quad \gamma_0 \mid_{C^0(\tilde{\Omega})}: v \mapsto v \mid_{\tilde{\Gamma}},
\]
\[
D(\gamma_1) = \{ v \in L^2(\tilde{\Omega}) \mid \text{SA}[v] = 0 \}, \quad \gamma_1 \mid_{C^1(\tilde{\Omega})}: v \mapsto \partial_\nu v \mid_{\tilde{\Gamma}},
\]
we see that $\tilde{\Lambda}(\lambda) = \gamma_1(1 - (S - \lambda)^{-1}(-\Delta - \lambda))\gamma_0^{-1}$, where by $\gamma_0^{-1}$ we mean any right-inverse of $\gamma_0$. Thus we find, via the resolvent formula \([32, \text{Thm. VIII.2}]\), that
\[
\frac{\tilde{\Lambda}(\lambda) - \tilde{\Lambda}(\mu)}{\lambda - \mu} = \gamma_1 \frac{(S - \lambda)^{-1}(\Delta + \lambda) - (S - \mu)^{-1}(\Delta + \mu)}{\lambda - \mu} \gamma_0^{-1}
\]
\[
= \gamma_1 \frac{\lambda(S - \lambda)^{-1} - \mu(S - \mu)^{-1} + ((S - \lambda)^{-1} - (S - \mu)^{-1})(\Delta)}{\lambda - \mu} \gamma_0^{-1}
\]
\[
= \gamma_1 (S - \lambda)^{-1} (1 + \mu(S - \mu)^{-1} + (S - \mu)^{-1}(\Delta)) \gamma_0^{-1}
\]
is a smoothing operator of order $-1$ on the scale of Sobolev spaces on $\tilde{\Gamma}$, since it is a product (from right to left) of operators with order $1/2, 0, -2$ and $1/2$. By Sobolev embedding \([14, \text{Thm. V.}4.18]\) this Newton quotient is compact, and in particular is bounded. As $\mu \to \lambda$, its norm limit is the compact operator
\[
\tilde{\Lambda}'(\lambda) = \gamma_1 (S - \lambda)^{-1} (1 - (S - \lambda)^{-1}(-\Delta - \lambda)) \gamma_0^{-1}.
\]
Of course, this limit is only defined for $\lambda$ in the resolvent set of $S$, which owing to $S = S^*$ contains $C \setminus R$.

(iii). To establish the Herglotz property, we need to examine the imaginary parts of both $\langle \Lambda_1(\lambda)h, h \rangle_{\Gamma_1}$ and $\langle \Lambda_0(\lambda)h, h \rangle_{\Gamma_1}$. The latter involves straightforward integration by parts. Using Green’s formula and the solution $w_0$ to \((\Omega)\), and in particular the homogeneous Dirichlet BC on $\Gamma \cup \Gamma_0$ (since $f \mid_{\Gamma_0} = 0$), we see
\[
\langle \Lambda_0(\lambda)h, h \rangle_{\Gamma_1} = \int_{\Gamma_1} (-\partial_\nu w_0)\overline{w_0} = -\int_{\Omega_0} ((\Delta w_0)\overline{w_0} + |\nabla w_0|^2)
\]
\[
= \lambda \int_{\Omega_0} |w_0|^2 - \int_{\Omega_0} |\nabla w_0|^2.
\]
This clearly has imaginary part of the same sign as that of $\lambda$.

On the other hand, integration by parts fails for the solutions of \((\Omega)\), since, e.g., the solutions $u_0$ and $v_0$ defined in \((\Omega)\) are not in $H^2(\Omega)$. Instead, we decompose the solution as in the proof of Lemma \([11]\) effectively treating $-\lambda$ and $-\bar{\lambda}$ as, respectively,
the potentials $q_1$ and $q_2$ (we no longer have equality in a neighbourhood of the boundary). Recall the radial solutions $R_n$ for \([12]\); we assume they have unit $L^2$-norm. Then letting $w_1$ solve \([37]\), there are real constants $c_n$ ($n = 0, 1, 2, \ldots$) so that $W_n(r, \vartheta) = c_n R_n(r) \Theta_n(\vartheta)$ form the terms in a series expansion: $w_1 = \sum_{n=0}^{\infty} W_n =: W + W$. The term $W$ arises from the regular part of the problem, and is in $H^2(\Omega_1)$ \([28]\), and moreover all the $W_n$ are pairwise orthogonal in $L^2(\Omega_1)$. Hence, by Green’s formula,

\[
(\lambda - \overline{\lambda}) \int_{\Omega_1} w_1 \overline{w_1} = (\lambda - \overline{\lambda}) \int_{\Omega_1} W_0 \overline{W_0} - \int_{\Omega_1} (\Delta W) \overline{W} - W \Delta \overline{W}
\]

\[
= c_0^2 (\lambda - \overline{\lambda}) \int_0^1 r dr |R_0(r)|^2 \int_{-\pi/2}^{\pi/2} d\vartheta |\Theta_0(\vartheta)|^2 \\
- \int_{\partial \Omega_1} (\partial_r W) \overline{\partial_r W} - W \partial_r \overline{W}
\]

\[
= c_0^2 (\lambda - \overline{\lambda}) \int_0^1 r dr |R_0(r)|^2 + \int_{\Gamma_1} (W \partial_r \overline{W} - (\partial_r W) \overline{W}). \tag{43}
\]

Without loss of generality we scale $c_0$ to be 1, set $0 < \delta < 1$ and examine

\[
(\lambda - \overline{\lambda}) \int_\delta^1 r dr \ R_0(r) \overline{R_0(r)} = R_0(1) \overline{R_0(1)} - R_0(1) \overline{R_0(1)} - \delta (R_0(\delta) \overline{R_0(\delta)} - R_0(\delta) \overline{R_0(\delta)}) \\
= \int_{\Gamma_1} (W_0 \partial_\nu \overline{W_0} - (\partial_\nu W_0) \overline{W_0}) - [R_0, \overline{R_0}] (\delta). \tag{44}
\]

Clearly the lemma will follow if we can show that $[R_0, \overline{R_0}] (\delta)$ vanishes as $\delta \rightarrow 0$, since by dominated convergence the left-hand side of \((44)\) tends to $(\lambda - \overline{\lambda}) \int_0^1 r |R_0(r)|^2 dr$. Combining this with \((43)\) yields

\[
\text{Im}(\lambda) \int_{\Omega_1} |w_1|^2 = \text{Im} \left( \langle \Lambda_1(\lambda) h, h \rangle_{L^2(\Gamma_1)} \right).
\]

Similarly to Lemma \[44\] we relate the boundary behaviour of $R_0$ to that of the solutions $u_0$ and $v_0$ for $\lambda = 0$ by applying the elementary identity

\[
[R_0, \overline{R_0}] = -b \begin{vmatrix} [R_0, u_0] & [R_0, v_0] \\ [R_0, u_0] & [R_0, v_0] \end{vmatrix}, \tag{45}
\]

since $u_0$ and $-b v_0$ form a fundamental system satisfying $[u_0, v_0] = -b^{-1}$. Thus, since both $R_0$ and $\overline{R_0}$ in its place satisfy $[R_0, u_0 + \beta v_0] (0^+) = 0$, we see that the columns in the right-hand side of \((45)\) become collinear as its argument approaches 0. \[\square\]

Lemma 6. Let the admissible boundary function $f$ be supported in $\Gamma_1$, and let $\lambda$ be less than the infima of the spectra of each $L_j$ ($j = 1, 2, 3, \ldots$; see Section \[8\]) and of the Laplace operator in $\Omega_0$ with homogeneous Dirichlet BCs. If $z \in \mathbb{C}_c := \mathbb{C} \setminus [0, +\infty)$
then both \((M + C)(\lambda)\) and \((M + C)(\lambda + z)\) are invertible matrices, and satisfy
\[
[(M + C)(\lambda + z)]^{-1} = \left( 1 + [(M + C)(\lambda)]^{-1} \int_{[\lambda, \lambda + z]} (M + C)' \right)^{-1} [(M + C)(\lambda)]^{-1}.
\]

Proof. Lemma 3 implies that \(M + C\) is differentiable anywhere in \(\lambda + \mathbb{C}_c\), and its derivative is compact. By the fundamental theorem of calculus (for operator-valued analytic functions), \((M + C)(\lambda + z) - (M + C)(\lambda) = \int_{[\lambda, \lambda + z]} (M + C)'\). Hence, the lemma will follow from showing that \((M + C)(\lambda)\) and subsequently \(1 + [(M + C)(\lambda)]^{-1} \int_{[\lambda, \lambda + z]} (M + C)'\) are invertible. The first will be achieved by checking the definiteness of the sign of \((M + C)(\lambda)\), the second is a consequence of the analytic Fredholm theorem [32, p. 201].

By (42) we see that \(\Lambda_0(\mu) = 0\) for any \(\mu \leq 0\), from which \(C(\lambda) \leq 0\) follows immediately. Furthermore, the diagonal entries of \(M(\lambda)\) are by definition \(m_n(\lambda)\); see the discussion preceding [32]. Owing to a remark in [28, p. 4], for \(n = 1, 2, 3, \ldots\), we have
\[
m_n(\lambda) = -i \sqrt{-\lambda} J_n'(i \sqrt{-\lambda}) / J_n(i \sqrt{-\lambda}).
\]
We may apply Bessel function properties [13, Eqs. 10.6.2, 10.19.1] to show by an algebraic calculation that with fixed \(\lambda < 0\), as \(n \to \infty\),
\[
m_n(\lambda) = -n(1 + o(1)). \tag{46}
\]
We deduce that \((M + C)(\lambda) < 0\), and its invertibility follows.

Consider, now, the analytic operator-valued function
\[
\mathcal{A}(z) := [(M + C)(\lambda)]^{-1} \int_{[\lambda, \lambda + z]} (M + C)' \quad (z \in \mathbb{C}_c). \tag{47}
\]
We may apply the reasoning that led to equation (41) to see that the integral \(\int_{[\lambda, \lambda + z]} (M + C)' = (M + C)(\lambda + z) - (M + C)(\lambda)\) is the matrix of a compact operator, for any \(z \in \mathbb{C}_c\). Hence, since \([(M + C)(\lambda)]^{-1}\) is bounded, we observe that \(\mathcal{A}(z)\) is compact for every \(z \in \mathbb{C}_c\). Furthermore, if \(z \in \mathbb{C} \setminus \mathbb{R}\) then ker \(((M + C)(\lambda + z)) = \{0\}\), since if this were not the case we would have a non-trivial function on the interface \(\Gamma_1\), meaning (by the Remark on page 118) there would be an eigenfunction for \(T\) with a non-real eigenvalue, which is forbidden by the self-adjointness of \(T\). Indeed, this is also contradictory for any \(z < 0\) since neither \(M\) nor \(C\) can give rise to eigenvalues less than \(\lambda\). Therefore, by the analytic Fredholm theorem [32, p. 201], the following two cases are mutually exhaustive:

(i) \((1 + \mathcal{A}(z))^{-1}\) exists for no \(z \in \mathbb{C}_c\);  (ii) \((1 + \mathcal{A}(z))^{-1}\) exists for every \(z \in \mathbb{C}_c\).

Clearly for any \(z \in \mathbb{C}_c\) we have
\[
(M + C)(\lambda + z) = (M + C)(\lambda) \left( 1 + [(M + C)(\lambda)]^{-1} \int_{[\lambda, \lambda + z]} (M + C)' \right)
\]
\[
= (M + C)(\lambda)(1 + \mathcal{A}(z)),
\]
so if \(z < 0\) then both sides are invertible. This excludes case (i).
9. Negative eigenvalue asymptotics for operators with singular bcs

In this section we prove Theorem 3. Throughout, for convenience, we set Λ = Λ_{q,f,β}.

In the case of symmetric geometry Ω = Ω_1, owing to the poles of Λ being eigenvalues of the operator (9), one can sharpen the proof of [28, Sec. 4] to achieve the following result. A crucial consideration is—as in the proof of Proposition 3—that adding \( q \neq 0 \) to the operator leaves the essential spectrum unchanged and perturbs the discrete spectrum by at most \( \|q\|_{L^\infty(\Omega_1)} \), via [23, Thms. IV.3.17, V.4.3 & .10].

Lemma 7 (Marletta–Rozenblum, 2009). Take the operator \( L' \) defined in (9), and label its eigenvalues by \( \lambda_{1n} \) so that \( \lambda_{1n} - 1 < 0 \leq \lambda_{10} \). Then as \( n \to -\infty \) we have

\[
\lambda_{1n} = -e^{-2b(\vartheta_0 + \tan^{-1} \beta)} e^{-2bn\pi}(1 + o(1)).
\]  

(48)

Here \( \vartheta_0 = \tan^{-1}(A/B) \in (-\pi/2, \pi/2] \) is known, where

\[
A = \lim_{t \to +\infty} e^{-t}w_s(t), \quad B = \lim_{t \to +\infty} e^{-t}w_c(t),
\]  

(49)

and the non-trivial functions \( w_s \) and \( w_c \) satisfy, as \( t \to 0 \),

\[
[w_s(t), t^{1/2} \sin{(b^{-1} \log(t))}] \to 0, \quad [w_c(t), t^{1/2} \cos{(b^{-1} \log(t))}] \to 0
\]  

(50)

and solve \(-w''(t) - (1/4 + 1/b^2)t^{-2}w(t) = -w(t)\) on the half-line \((0, +\infty)\).

We now prove Theorem 3, i.e., for the operator \( T \) in (18) the same asymptotics hold. Our approach shows the counting functions of the eigenvalues asymptotically agree. To avoid ambiguity we enumerate the eigenvalues \( \lambda_n \) and \( \lambda_{1n} \) of, respectively, \( T \) and \( L' \) so that \( \lambda_0 \) and \( \lambda_{10} \) are respectively their smallest non-negative eigenvalues.

Proof of Theorem 3. For discreteness and accumulation points of the spectrum of \( T \) we refer to [28, Sec. 5]. The remainder of the proof is split into two parts, in which we asymptotically bound the counting function for the negative eigenvalues \( \lambda_n \) of \( T \) from, in turn, above and below by that for the negative eigenvalues \( \lambda_{1n} \) of \( L' \). The lower bound will follow from an asymptotic analysis of approximate eigenfunctions of \( T \). To find the upper bound we consider the pencil of DN operators on \( \Gamma_i \) defined in Section 8. As in the proof of Proposition 3 we apply [23, Thms. IV.3.17, V.4.3 & V.4.10] to assume without loss of generality \( q = 0 \).

1. Bound from below. Take any smooth cut-off function \( \mu \) on \( \Omega \) that is supported and radially symmetric in \( \Omega_1 \), and takes value 1 in \( \frac{1}{2}\Omega_1 \) (in particular, \( \partial_\nu \mu \mid_{\Gamma_1} = 0 \)). Note that the partial derivatives of \( \mu \) are supported in the half-annulus \( A := \Omega_1 \setminus \frac{1}{2}\Omega_1 \). Let \( n < 0 \), and choose eigenfunction \( \varphi_n \) for \( L' \) at the eigenvalue \( \lambda_{1n} \), such that \( \|\varphi_n\|_{L^2(\Omega_1)} = 1 \). We will show that these \( \varphi_n \) are pseudo-modes for \( T \), i.e.,

\[
\frac{\|(T - \lambda_{1n})\mu\varphi_n\|_{L^2(\Omega)}}{\|\mu\varphi_n\|_{L^2(\Omega)}} =: \varepsilon_n \to 0 \quad (n \to -\infty).
\]  

(51)
By the spectral theorem, denoting the normalised eigenfunctions of $T$ as $\psi_j$ corresponding to $\lambda_j$ ($j \in \mathbb{Z}$), we may write

$$
\|(T - \lambda_n^1)\varphi_n\|^2 = \sum_{j \in \mathbb{Z}} (\lambda_j - \lambda_n^1)^2 \langle \mu \varphi_n, \psi_j \rangle_{L^2(\Omega)}^2.
$$

Suppose $|\lambda_j - \lambda_n^1| > \varepsilon_n$ for every $j \leq -1$. Then $\varepsilon_n^2 \|\mu \varphi_n\|^2_{L^2(\Omega)} > \varepsilon_n^2 \sum_{j \in \mathbb{Z}} \langle \mu \varphi_n, \psi_j \rangle_{L^2(\Omega)}^2 = \varepsilon_n^2 \|\mu \varphi_n\|^2_{L^2(\Omega)}$, a contradiction. Thus there is a subsequence $\lambda_{j_n}$ satisfying $|\lambda_{j_n} - \lambda_n^1| \leq \varepsilon_n$ ($n \leq -1$). Hence if $\varepsilon_n \to 0$ ($n \to -\infty$) we will have established the lower bound.

Firstly observe that, by choice of $\mu$ and the fact $\varphi_n \in D(L')$, we have

$$
\mu \varphi_n - f \partial_\nu (\mu \varphi_n) = \mu (\varphi_n - f \partial_\nu \varphi_n) - f (\partial_\nu \mu) \varphi_n = 0,
$$

implying $\mu \varphi_n$ is indeed in $D(T)$. Next we calculate that, clearly,

$$
(T - \lambda_n^1)\mu \varphi_n = \left\{ \begin{array}{ll}
0 & \text{in } \Omega \setminus \mathcal{A}, \\
-(\Delta \mu) \varphi_n - 2 \nabla \mu \cdot \nabla \varphi_n & \text{in } \mathcal{A}.
\end{array} \right.
$$

If we can show that $\varphi_n$ and $\nabla \mu \cdot \nabla \varphi_n$ go to 0 uniformly in $\mathcal{A}$ then (51) will follow.

Set $\kappa_n := \sqrt{-\lambda_n^1}$, and define the (more conveniently notated) Hankel functions $H^\pm_{i\pi} = J_{i\pi} \pm iY_{i\pi}$, where $J_z$ and $Y_z$ are the Bessel functions of first and second kind.

Since $\lambda_n^1 < 0$ we know that the eigenfunction $\varphi_n$ for $L'$ comes from the $L_0'$ operator in the decomposition (9). Owing to Proposition 2, $\varphi_n$ is a constant multiple of

$$
\phi_n(r, \vartheta) := e^{-\vartheta/b} \left( H_{i/b}^+(i\kappa_n)H_{i/b}^-(i\kappa_n) - H_{i/b}^-(i\kappa_n)H_{i/b}^+(i\kappa_n) \right) \\
= e^{-\vartheta/b} \mathcal{H}_n(r).
$$

It follows that, to normalise $\phi_n$ asymptotically, we need to know the leading-order behaviour, as $n \to -\infty$, of

$$
\frac{1}{b \sinh \left( \frac{\pi}{b} \right)} \int_{\Omega_1} |\phi_n|^2 = \int_0^1 |\mathcal{H}_n(r)|^2 r dr \\
= \int_0^{\kappa_n} \left| H_{i/b}^+(it)H_{i/b}^-(i\kappa_n) - H_{i/b}^-(it)H_{i/b}^+(i\kappa_n) \right|^2 \frac{t dt}{\kappa_n^2}.
$$

One may easily calculate from Bessel function asymptotics [13, Eq. 10.17.5–6] that, for $x > 0$,

$$
H_{i/b}^\pm(ix) \sim \sqrt{\frac{2}{\pi}} e^{-i(1+1)\pi/4+i\pi/2b}e^{\pm x}x^{-1/2} \quad (x \to +\infty).
$$
Expanding the absolute value in the right-hand side of (52), we see
\[ \kappa_n^2 \int_0^1 \frac{|H_n(r)|^2 r dr}{|H_{i/b}(i\kappa_n)|^2} = \int_0^{\kappa_n} |H_{i/b}(i\kappa_n)|^2 \left\{ 1 + \frac{|H_{i/b}^-(it)|^2}{|H_{i/b}^-(i\kappa_n)|^2} \left( \frac{|H_{i/b}^+(it)|^2}{|H_{i/b}^+(i\kappa_n)|^2} + 2 \text{Re} \left( \frac{H_{i/b}^-(it)}{H_{i/b}^-(i\kappa_n)} \frac{H_{i/b}^+(it)}{H_{i/b}^+(i\kappa_n)} \right) \right) t dt. \]

By (53), as \( n \to -\infty \), the term in \{ \} converges pointwise to 1, and moreover for sufficiently large \( n < 0 \)—denoting the greatest such \( n \) by \( n_0 < 0 \)—this term is bounded by 4. By the latter it follows that for \( n \leq n_0 \) the integrand of the right-hand side is bounded by \( 4t|H_{i/b}^+(it)|^2 \), which, owing to (53), is certainly integrable over \((0, \infty)\). Hence dominated convergence applies, yielding, as \( n \to -\infty \),
\[ \int_0^1 |H_n(r)|^2 r dr \sim \left( \int_0^\infty |H_{i/b}^+(it)|^2 t dt \right)^{\kappa_n^{-2}} \left| H_{i/b}^-(i\kappa_n) \right|^2 \]
\[ \sim 2 \int_0^\infty \frac{|H_{i/b}^+(it)|^2 t dt}{\pi e^{\pi/b}} \kappa_n^{-3} e^{2\kappa_n}, \]
from which we find
\[ \int_{\Omega_i} |\phi_n|^2 \sim \frac{2b(1 - e^{-2\pi/b})}{\pi} \int_0^\infty |H_{i/b}^+(it)|^2 t dt \kappa_n^{-3} e^{2\kappa_n} =: \eta^2. \]

According to this definition of \( c_n \) we have—up to sign—the pointwise asymptotics \( \varphi_n \sim c_n^{-1} \phi_n \) as \( n \to -\infty \). More explicitly, we can calculate from (53) that, pointwise, as \( n \to -\infty \), we have
\[ e^{\vartheta/b} \phi_n(r, \vartheta) \sim -\frac{2i}{\pi} r^{-1/2} \kappa_n^{-1} e^{(1-r)\kappa_n} \implies e^{\vartheta/b} \varphi_n(r, \vartheta) \sim -\frac{2i}{\pi \eta} r^{-1/2} \kappa_n^{1/2} e^{-r\kappa_n}, \]
and since within \( \mathcal{A} \) we have \( 1/2 < r < 1 \) it is clear that \( \varphi_n \upharpoonright_{\mathcal{A} \to 0} \) uniformly as \( n \to -\infty \).

Now we examine
\[ \nabla \mu(r, \vartheta) \cdot \nabla \varphi_n(r, \vartheta) = |\nabla \mu(r, \vartheta)||\nabla \varphi_n(r, \vartheta)| \cos \left( \text{arg} \nabla \mu(r, \vartheta) - \text{arg} \nabla \varphi_n(r, \vartheta) \right) \]
\[ = |\partial_r \mu(r, \vartheta) \partial_r \varphi_n(r, \vartheta)| \cos \left( \frac{\partial_{\vartheta} \varphi_n(r, \vartheta)}{r} \right). \]
It is clear from our prior calculations that $\partial_r \varphi_n \sim c_n^{-1} \partial_r \phi_n$. Now recall [13] Eq. 10.6.2 that $\frac{d}{dz} H^\pm(z) = \frac{z}{2} H^\pm(z) - H^\pm_{z+1}(z)$, from which we derive

$$e^{\partial_r/\partial_r \varphi_n(r, \vartheta)} = \frac{ib}{r_b} \left( H^{+}_{i/b}(ir\kappa_n) H^{-}_{i/b}(i\kappa_n) - H^{-}_{i/b}(ir\kappa_n) H^{+}_{i/b}(i\kappa_n) \right) - i\kappa_n \left( H^{+}_{1+i/b}(ir\kappa_n) H^{-}_{i/b}(i\kappa_n) - H^{-}_{1+i/b}(ir\kappa_n) H^{+}_{i/b}(i\kappa_n) \right).$$ (54)

We calculate [13] Eq. 10.17.5-6] that $H^{\pm}_{1+i/b}(ix) \sim \sqrt{\frac{2}{\pi}} e^{-i(1\pm)\pi/4+\pi/2b} e\mp x^{-1/2}$, from which we easily show that the first and second terms in the right-hand side of (54) are asymptotically equivalent respectively to

$$\frac{2}{\pi r_b} e^{(1-r)\kappa_n r^{-1/2} \kappa_n^{-1}} \quad \text{and} \quad -\frac{2}{\pi} e^{(1-r)\kappa_n r^{-1/2}}.$$ (55)

It follows, after substituting these into (54) then dividing by $c_n$, that

$$e^{\partial_r/\partial_r \varphi_n(r, \vartheta)} \sim \frac{-2}{\pi \eta} r^{-1/2} \kappa_n^{3/2} e^{-r\kappa_n} \quad (n \to -\infty),$$ (56)

and therefore $\nabla \mu \cdot \nabla \varphi_n$ must go to 0 uniformly in $\mathcal{A}$. The lower bound on the difference between the counting functions for the negative eigenvalues of $T$ and $L'$ follows.

2. Bound from above. Here we will use the pencil of DN operators on $\Gamma_1$ from the sub-domains either side of the interface, and analyse non-triviality of its kernel, which occurs precisely when the sub-domains either side of the interface, and analyse non-triviality of its kernel, that

$$\frac{d}{dz} H^\pm(z) = \frac{z}{2} H^\pm(z) - H^\pm_{z+1}(z),$$

from which we derive

$$e^{\partial_r/\partial_r \varphi_n(r, \vartheta)} = \frac{ib}{r_b} \left( H^{+}_{i/b}(ir\kappa_n) H^{-}_{i/b}(i\kappa_n) - H^{-}_{i/b}(ir\kappa_n) H^{+}_{i/b}(i\kappa_n) \right) - i\kappa_n \left( H^{+}_{1+i/b}(ir\kappa_n) H^{-}_{i/b}(i\kappa_n) - H^{-}_{1+i/b}(ir\kappa_n) H^{+}_{i/b}(i\kappa_n) \right).$$ (54)

We calculate [13] Eq. 10.17.5-6] that $H^{\pm}_{1+i/b}(ix) \sim \sqrt{\frac{2}{\pi}} e^{-i(1\pm)\pi/4+\pi/2b} e\mp x^{-1/2}$, from which we easily show that the first and second terms in the right-hand side of (54) are asymptotically equivalent respectively to

$$\frac{2}{\pi r_b} e^{(1-r)\kappa_n r^{-1/2} \kappa_n^{-1}} \quad \text{and} \quad -\frac{2}{\pi} e^{(1-r)\kappa_n r^{-1/2}}.$$ (55)

It follows, after substituting these into (54) then dividing by $c_n$, that

$$e^{\partial_r/\partial_r \varphi_n(r, \vartheta)} \sim \frac{-2}{\pi \eta} r^{-1/2} \kappa_n^{3/2} e^{-r\kappa_n} \quad (n \to -\infty),$$ (56)

and therefore $\nabla \mu \cdot \nabla \varphi_n$ must go to 0 uniformly in $\mathcal{A}$. The lower bound on the difference between the counting functions for the negative eigenvalues of $T$ and $L'$ follows.

The normalisation $h_0 = 1$ (from the discussion following the Remark on p. [13]) ensures $\lambda \ll 0$ is an eigenvalue if and only if $\exists \mathbf{h} = (h_1, h_2, \ldots)^T$ with

$$\begin{pmatrix} m_0(\lambda) + a(\lambda) & \mathbf{b}(\lambda)^T \\ \mathbf{b}(\lambda) & M(\lambda) + C(\lambda) \end{pmatrix} \begin{pmatrix} 1 \\ \mathbf{h} \end{pmatrix} = 0.$$ (57)

After expanding the product we find that this can only happen if $\mathbf{b}(\lambda) = -\left( M(\lambda) + C(\lambda) \right) \mathbf{h}$. Owing to Lemma [6] we know $M(\lambda) + C(\lambda)$ is invertible for $\lambda \ll 0$. Define $\mathbf{h}(\lambda) = -\left( M(\lambda) + C(\lambda) \right)^{-1} \mathbf{b}(\lambda)$ and set $h(\lambda) = \left( \frac{1}{\mathbf{h}(\lambda)} \right)$. Substituting $h$ into (57) and then taking a quadratic form we observe that $\lambda \ll 0$ is an eigenvalue for $T$ if the following expression vanishes:

$$\mathcal{E}(\lambda) := \langle \left( \Lambda_1(\lambda) + \Lambda_0(\lambda) \right) h(\lambda), h(\lambda) \rangle_{L^2(\Gamma)} = m_0(\lambda) + a(\lambda) - \mathbf{b}(\lambda)^T \left( M(\lambda) + C(\lambda) \right)^{-1} \mathbf{b}(\lambda).$$

Thanks to Lemma [5] both $\Lambda_j$ are Herglotz in quadratic form, and analytic as operator-valued functions; in particular any sub-block is analytic. Furthermore $\Lambda_0$ arises from a problem on a bounded domain with regular $\text{BCs}$. Therefore we see that $a(\lambda)$, $\mathbf{b}(\lambda)$ and $C(\lambda)$ lack poles when $\lambda$ is sufficiently negative or non-real. For the same $\lambda$ the coefficient $a(\lambda)$ is never 0, $\mathbf{b}(\lambda)$ is not identically the zero vector—though
it could have some zero entries—and $C(\lambda)$ is never null. Moreover the Herglotz property of $\Lambda_1 + \Lambda_0$ ensures that $\text{Im}(\mathcal{E}(\lambda))\text{Im}(\lambda) \geq 0$. The invertibility of $M(\lambda) + C(\lambda)$ for non-real $\lambda$ follows from Lemma [6] and is enough to ensure analyticity of $\mathcal{E}$ away from $\mathbb{R}$, and establish that $\mathcal{E}$ is, like $\Lambda_1 + \Lambda_0$, Herglotz. Now, for $\lambda \ll 0$ $M(\lambda)$ has fixed sign, so we see that the poles of $\mathcal{E}$ and $m_0$ are identical; since poles and zeros of Herglotz functions interlace we see that between any two sufficiently negative poles of $m_0$ there is a zero of $\mathcal{E}$ and hence at most one eigenvalue of $T$. The upper bound follows. \hfill \Box

10. Discussion

A strong motivation for this investigation lies in the question: What sort of bc can we expect on the inaccessible portion $\Gamma_c$ of the boundary $\partial \Omega$? Considering the inverse conductivity problem for (3), a Neumann condition on $\Gamma_c$ corresponds to a “perfect insulator” (no electric potential flux across the boundary), whilst a Dirichlet condition corresponds to a “perfect conductor”. Clearly, at least in problems paralleling most physical scenarios, one cannot expect a pure Dirichlet or Neumann condition.

The only (linear first-order) alternative is a Robin condition with an unknown Dirichlet-to-Neumann ratio $f$. In this situation [4, 28] tell us we need to be wary of points at which $f$ vanishes locally linearly. But as just established, we can exploit this type of singular bc to recover $f$ in a neighbourhood of its zero, and subsequently our self-adjointness bc and the Schrödinger potential.

Our methods do not generalise to further singularities, since it would no longer be clear from which of the $b$-parameters a given negative eigenvalue had arisen.

Nevertheless there are various possible further routes.

- **Generalise the admissible class of $f$.** In the above, $f$ must be linear in a neighbourhood of 0. Given the results suggested in [4, 28, 3], it should be possible to prove that a general $f$ with finitely (possibly countably) many simple zeros yields, from (2), an operator with both symmetric adjoint and self-adjoint restriction, sufficient to generalise Inverse Problem [1].

- **Change the shape of $\text{supp}(f) = \Gamma_1$ (equivalently $\Omega_1$).** Expressing a general $\Gamma_1 \in C^1$ in the form of a perturbation of a straight line is a possible route towards this.

- **Reduce the number of boundary data used.** Inverse Problem [1] requests recovery of the triple $(q, f, \beta)$, which has 3 variables. The Schwartz kernel of the DN map at a fixed $\lambda \in \mathbb{R}$ has 2 variables. Thus it should suffice for full uniqueness to know the DN map at just two points in $\mathbb{R}$—indeed, it should be over-determined.
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