A one dimensional analysis of singularities and turbulence for the stochastic Burgers equation in $d$-dimensions

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Abstract

The inviscid limit of the stochastic Burgers equation, with body forces white noise in time, is discussed in terms of the level surfaces of the minimising Hamilton-Jacobi function, the classical mechanical caustic and the Maxwell set and their algebraic pre-images under the classical mechanical flow map. The problem is analysed in terms of a reduced (one dimensional) action function. We give an explicit expression for an algebraic surface containing the Maxwell set and caustic in the polynomial case. Those parts of the caustic and Maxwell set which are singular are characterised. We demonstrate how the geometry of the caustic, level surfaces and Maxwell set can change infinitely rapidly causing turbulent behaviour which is stochastic in nature, and we determine its intermittence in terms of the recurrent behaviour of two processes.

1 Introduction

Burgers equation has been used in studying turbulence and in modelling the large scale structure of the universe [1, 9, 28], as well as to obtain detailed asymptotics for stochastic Schrödinger and heat equations [10, 11, 29, 30, 31, 32]. It has also played a part in Arnol’d’s work on caustics and Maslov’s works in semiclassical quantum mechanics [3, 4, 20, 21].

We consider the stochastic viscous Burgers equation for the velocity field
$v^\mu(x,t) \in \mathbb{R}^d$, where $x \in \mathbb{R}^d$ and $t > 0$,

$$\frac{\partial v^\mu}{\partial t} + (v^\mu \cdot \nabla) v^\mu = \frac{\mu^2}{2} \Delta v^\mu - \nabla V(x) - \epsilon \nabla k_t(x) \dot{W}_t, \quad v^\mu(x,0) = \nabla S_0(x) + \mathcal{O}(\mu^2).$$

Here $\dot{W}_t$ denotes white noise and $\mu^2$ is the coefficient of viscosity which we assume to be small. We are interested in the advent of discontinuities in the inviscid limit of the Burgers fluid velocity $v^0(x,t)$ where $v^\mu(x,t) \to v^0(x,t)$ as $\mu \to 0$.

Using the Hopf-Cole transformation $v^\mu(x,t) = -\frac{\mu^2}{2} \nabla \ln u^\mu(x,t)$, the Burgers equation becomes the Stratonovich heat equation,

$$\frac{\partial u^\mu}{\partial t} = \frac{\mu^2}{2} \Delta u^\mu + \mu^{-2} V(x) u^\mu + \frac{\epsilon}{\mu^2} k_t(x) u^\mu \circ \dot{W}_t, \quad u^\mu(x,0) = \exp \left( -\frac{S_0(x)}{\mu^2} \right) T_0(x),$$

where the convergence factor $T_0$ is related to the initial Burgers fluid density $\mathcal{I}^4$.

Now let,

$$A[X] := \frac{1}{2} \int_0^t \dot{X}^2(s) \, ds - \int_0^t V(X(s)) \, ds - \epsilon \int_0^t k_s(X(s)) \, dW_s, \quad (1)$$

and select a path $X$ which minimises $A[X]$. This requires,

$$d\dot{X}(s) + \nabla V(X(s)) \, ds + \epsilon \nabla k_s(X(s)) \, dW_s = 0. \quad (2)$$

We then define the stochastic action $A(X(0), x, t) := \inf_X \{A[X] : X(t) = x\}$.

Setting,

$$A(X(0), x, t) := S_0(X(0)) + A(X(0), x, t),$$

and then minimising $A$ over $X(0)$, gives $\dot{X}(0) = \nabla S_0(X(0))$. Moreover, it follows that,

$$S_t(x) := \inf_{X(0)} \{A(X(0), x, t)\},$$

is the minimal solution of the Hamilton-Jacobi equation,

$$dS_t + \left( \frac{1}{2} |\nabla S_t|^2 + V(x) \right) \, dt + \epsilon k_t(x) \, dW_t = 0, \quad S_{t=0}(x) = S_0(x). \quad (3)$$

Following the work of Donsker, Freidlin et al $\mathcal{I}^2$, $-\mu^2 \ln u^\mu(x,t) \to S_t(x)$ as $\mu \to 0$. This gives the inviscid limit of the minimal entropy solution of Burgers equation as $v^0(x,t) = \nabla S_t(x)$ $\mathcal{I}$.

**Definition 1.1.** The stochastic wavefront at time $t$ is defined to be the set,

$$\mathcal{W}_t = \{x : S_t(x) = 0\}.$$
For small $\mu$ and fixed $t$, $u^\mu(x,t)$ switches continuously from being exponentially large to small as $x$ crosses the wavefront $\mathcal{W}_t$. However, $u^\mu$ and $v^\mu$ can also switch discontinuously.

Define the classical flow map $\Phi_s: \mathbb{R}^d \rightarrow \mathbb{R}^d$ by,

$$
d\dot{\Phi}_s + \nabla V(\Phi_s) \, ds + \epsilon \nabla k_s(\Phi_s) \, dW_s = 0, \quad \Phi_0 = \text{id}, \quad \dot{\Phi}_0 = \nabla S_0.
$$

Since $X(t) = x$ it follows that $X(s) = \Phi_s(\Phi_t^{-1}(x))$, where the pre-image $x_0(x,t) = \Phi_t^{-1}(x)$ is not necessarily unique.

Given some regularity and boundedness, the global inverse function theorem gives a caustic time $T(\omega)$ such that for $0 < t < T(\omega)$, $\Phi_t$ is a random diffeomorphism; before the caustic time $v^0(x,t) = \dot{\Phi}_t(\Phi_t^{-1}(x))$ is the inviscid limit of a classical solution of the Burgers equation with probability one.

The method of characteristics suggests that discontinuities in $v^0(x,t)$ are associated with the non-uniqueness of the real pre-image $x_0(x,t)$. When this occurs, the classical flow map $\Phi_t$ focusses an infinitesimal volume of points $dx_0$ into a zero volume $dX(t)$.

**Definition 1.2.** The caustic at time $t$ is defined to be the set,

$$
C_t = \left\{ x : \det \left( \frac{\partial X(t)}{\partial x_0} \right) = 0 \right\}.
$$

Assume that $x$ has $n$ real pre-images,

$$
\Phi_t^{-1}\{x\} = \{x_0(1)(x,t), x_0(2)(x,t), \ldots, x_0(n)(x,t)\},
$$

where each $x_0(i)(x,t) \in \mathbb{R}^d$. Then the Feynman-Kac formula and Laplace’s method in infinite dimensions give for a non-degenerate critical point,

$$
u^\mu(x,t) = \sum_{i=1}^n \theta_i \exp \left( -\frac{S_0^i(x,t)}{\mu^2} \right), \tag{4}
$$

where $S_0^i(x,t) := S_0(x_0(i)(x,t)) + A(x_0(i)(x,t), x,t)$, and $\theta_i$ is an asymptotic series in $\mu^2$. An asymptotic series in $\mu^2$ can also be found for $v^\mu(x,t)$.

Note that $S_t(x) = \min\{S_0^i(x,t) : i = 1, 2, \ldots, n\}$.

**Definition 1.3.** The Hamilton-Jacobi level surface is the set,

$$
H^c_t = \left\{ x : S_0^i(x,t) = c \text{ for some } i \right\}.
$$

The zero level surface $H^0_t$ includes the wavefront $\mathcal{W}_t$. 

3
As $\mu \to 0$, the dominant term in the expansion comes from the minimising $x_0(i)(x,t)$ which we denote $\bar{x}_0(x,t)$. Assuming $\bar{x}_0(x,t)$ is unique, we obtain the inviscid limit of the Burgers fluid velocity as $v^0(x,t) = \dot{\Phi}_t(\bar{x}_0(x,t))$.

If the minimising pre-image $\bar{x}_0(x,t)$ suddenly changes value between two pre-images $x_0(i)(x,t)$ and $x_0(j)(x,t)$, a jump discontinuity will also occur in the inviscid limit of the Burgers fluid velocity. There are two distinct ways in which the minimiser can change; either two pre-images coalesce and disappear (become complex), or the minimiser switches between two pre-images at the same action value. The first of these occurs as $x$ crosses the caustic and when the minimiser disappears the caustic is said to be cool. The second occurs as $x$ crosses the Maxwell set and again, when the minimiser is involved the Maxwell set is said to be cool.

**Definition 1.4.** The Maxwell set is given by,

$$M_t = \{ x : \exists x_0, \bar{x}_0 \in \mathbb{R}^d \text{ s.t. } x = \Phi_t(x_0) = \Phi_t(\bar{x}_0), x_0 \neq \bar{x}_0 \text{ and } A(x_0, x, t) = A(\bar{x}_0, x, t) \}.$$

**Example 1.5 (The generic Cusp).** Let $V(x,y) = 0$, $k_t(x,y) = 0$ and $S_0(x_0, y_0) = x_0^2y_0/2$. This initial condition leads to the *generic Cusp*, a semicubical parabolic caustic shown in Figure 1. The caustic $C_t$ (long dash) is given by,

$$x_t(x_0) = t^2x_0^3, \quad y_t(x_0) = \frac{3}{2}t^2x_0^2 - \frac{1}{t}.$$

The zero level surface $H_t^0$ (solid line) is,

$$x_{(t,0)}(x_0) = \frac{x_0}{2} \left( 1 \pm \sqrt{1 - t^2x_0^2} \right), \quad y_{(t,0)}(x_0) = \frac{1}{2t} \left( t^2x_0^2 - 1 \pm \sqrt{1 - t^2x_0^2} \right),$$

and the Maxwell set $M_t$ (short dash) is $x = 0$ for $y > -1/t$.

**Notation:** Throughout this paper $x, x_0, x_t$ etc will denote vectors, where normally $x = \Phi_t(x_0)$. Cartesian coordinates of these will be indicated using a sub/superscript where relevant; thus $x = (x_1, x_2, \ldots, x_d)$, $x_0 = (x_0^1, x_0^2, \ldots, x_0^d)$ etc. The only exception will be in discussions of explicit examples in two and three dimensions when we will use $(x, y)$ and $(x_0, y_0)$ etc to denote the vectors.

## 2 Some background

We begin by summarising some of the geometrical results established by Davies, Truman and Zhao (DTZ) [6, 7, 8] and presenting some minor generalisations of their results [22, 25]. Following equation (1), let the stochastic
Figure 1: Cusp and Tricorn.

action be defined,

\[
A(x_0, p_0, t) = \frac{1}{2} \int_0^t \dot{X}(s)^2 \, ds - \int_0^t \left[ V(X(s)) \, ds + \epsilon k_s(X(s)) \, dW_s \right],
\]

where \( X(s) = X(s, x_0, p_0) \in \mathbb{R}^d \) and,

\[
d\dot{X}(s) = -\nabla V(X(s)) \, ds - \epsilon \nabla k_s(X(s)) \, dW_s, \quad X(0) = x_0, \quad \dot{X}(0) = p_0,
\]

for \( s \in [0, t] \) with \( x_0, p_0 \in \mathbb{R}^d \). We assume \( X(s) \) is \( \mathcal{F}_s \) measurable and unique.

**Lemma 2.1.** Assume \( S_0, V \in C^2 \) and \( k_t \in C^{2,0} \), \( \nabla V, \nabla k_t \) Lipschitz with Hessians \( \nabla^2 V, \nabla^2 k_t \) and all second derivatives with respect to space variables of \( V \) and \( k_t \) bounded. Then for \( p_0 \), possibly \( x_0 \) dependent,

\[
\frac{\partial A}{\partial x_0^\alpha}(x_0, p_0, t) = \dot{X}(t) \cdot \frac{\partial X(t)}{\partial x_0^\alpha} - X_\alpha(0), \quad \alpha = 1, 2, \ldots, d.
\]

Methods of Kolokoltsov et al. \cite{13, 19} guarantee that for small \( t \) the map \( p_0 \mapsto X(t, x_0, p_0) \) is onto for all \( x_0 \). Therefore, we can define,

\[
A(x_0, x, t) = A(x_0, p_0, t)|_{p_0 = p_0(x_0, x, t)},
\]

where \( p_0 = p_0(x_0, x, t) \) is the random minimiser (assumed unique) of \( A(x_0, p_0, t) \) when \( X(t, x_0, p_0) = x \). The stochastic action corresponding to the initial momentum \( \nabla S_0(x_0) \) is then \( A(x_0, x, t) := A(x_0, x, t) + S_0(x_0) \).

**Theorem 2.2.** If \( \Phi_t \) is the stochastic flow map, then \( \Phi_t(x_0) = x \) is equivalent to,

\[
\frac{\partial}{\partial x_0^\alpha}[A(x_0, x, t)] = 0, \quad \alpha = 1, 2, \ldots, d.
\]
The Hamilton-Jacobi level surface $H^c_t$ is obtained by eliminating $x_0$ between,

$$A(x_0, x, t) = c \quad \text{and} \quad \frac{\partial A}{\partial x_0}(x_0, x, t) = 0, \quad \alpha = 1, 2, \ldots, d.$$ 

Alternatively, if we eliminate $x$ to give an expression in $x_0$, we have the pre-level surface $\Phi_t^{-1}H^c_t$. Similarly the caustic $C_t$ (and pre-caustic $\Phi_t^{-1}C_t$) are obtained by eliminating $x_0$ (or $x$) between,

$$\det \left( \frac{\partial^2 A}{\partial x_0^\alpha \partial x_0^\beta}(x_0, x, t) \right)_{\alpha, \beta = 1, 2, \ldots, d} = 0 \quad \text{and} \quad \frac{\partial A}{\partial x_0^\alpha}(x_0, x, t) = 0 \quad \alpha = 1, 2, \ldots, d.$$ 

These pre-images are calculated algebraically which are not necessarily the topological inverse images of the surfaces $C_t$ and $H^c_t$ under $\Phi_t$.

Assume that $A(x_0, x, t)$ is $C^4$ in space variables with $\det \left( \frac{\partial^2 A}{\partial x_0^\alpha \partial x_0^\beta} \right) \neq 0$.

**Definition 2.3.** A curve $x = x(\gamma)$, $\gamma \in N(\gamma_0, \delta)$, is said to have a generalised cusp at $\gamma = \gamma_0$, $\gamma$ being an intrinsic variable such as arc length, if $\frac{dx}{d\gamma}(\gamma_0) = 0$.

**Lemma 2.4.** Let $\Phi_t$ denote the flow map and let $\Phi_t^{-1} \Gamma_t$ and $\Gamma_t$ be some surfaces where if $x_0 \in \Phi_t^{-1} \Gamma_t$ then $x = \Phi_t(x_0) \in \Gamma_t$. Then $\Phi_t$ is a differentiable map from $\Phi_t^{-1} \Gamma_t$ to $\Gamma_t$ with Frechet derivative,

$$D\Phi_t(x_0) = \left( -\frac{\partial^2 A}{\partial x_0 \partial x_0}(x_0, x, t) \right)^{-1} \left( \frac{\partial^2 A}{\partial x^2_0}(x_0, x, t) \right).$$

**Lemma 2.5.** Let $x_0(s)$ be any two dimensional intrinsically parameterised curve, and define $x(s) = \Phi_t(x_0(s))$. Let $e_0$ denote the zero eigenvector of $\left( \frac{\partial^2 A}{\partial x_0 \partial x_0} \right)$ and assume that $\ker \left( \frac{\partial^2 A}{\partial x_0 \partial x_0} \right) = \langle e_0 \rangle$. Then, there is a generalised cusp on $x(s)$ when $s = \sigma$ if and only if either:

1. there is a generalised cusp on $x_0(s)$ when $s = \sigma$; or,
2. $x_0(\sigma)$ is on the pre-caustic and the tangent $\frac{dx_0}{ds}(s)$ at $s = \sigma$ is parallel to $e_0$.

**Proposition 2.6.** The normal to $\Phi_t^{-1}H^c_t$ is,

$$n(x_0) = -\left( \frac{\partial^2 A}{\partial x_0 \partial x_0}(x_0, x, t) \right) \left( \frac{\partial^2 A}{\partial x_0 \partial x} \right)^{-1} \dot{X}(t, x_0, \nabla S_0(x_0)).$$
**Corollary 2.7.** In two dimensions, let \( \Phi^{-1}H^c_t \) meet \( \Phi^{-1}C_t \) at \( x_0 \) where \( n(x_0) \neq 0 \) and \( \ker \left( \frac{\partial^2 A}{(\partial x_0)^2} \right) = \langle e_0 \rangle \). Then the tangent to \( \Phi^{-1}H^c_t \) at \( x_0 \) is parallel to \( e_0 \).

**Proposition 2.8.** In two dimensions, assume that \( n(x_0) \neq 0 \) where \( x_0 \in \Phi^{-1}H^c_t \), so that \( \Phi^{-1}H^c_t \) does not have a generalised cusp at \( x_0 \). Then \( H^c_t \) can only have a generalised cusp at \( \Phi_t(x_0) \) if \( \Phi_t(x_0) \in C_t \). Moreover, if \( x = \Phi_t(x_0) \in \Phi_t \{ \Phi_t^{-1} C_t \cap H^{-1} \} \) then \( H^c_t \) will have a generalised cusp.

**Example 2.9** (The generic Cusp). Figure 2 shows that a point lying on three level surfaces has three distinct real pre-images each on a separate pre-level surface. A cusp only occurs on the corresponding level surface when the pre-level surface intersects the pre-caustic. Thus, a level surface only has a cusp on the caustic, but it does not have to be cusped when it meets the caustic.

![Figure 2](image.png)

Figure 2: (a) The pre-level surface (solid line) and pre-caustic (dashed), (b) the level surface (solid line) and caustic (dashed), both for the generic Cusp with \( c > 0 \).

**Theorem 2.10.** Let,

\[
x \in \text{Cusp}(H^c_t) = \left\{ x \in \Phi_t \left( \Phi_t^{-1} C_t \cap \Phi_t^{-1} H^c_t \right), x = \Phi_t(x_0), n(x_0) \neq 0 \right\}.
\]

Then in three dimensions in the stochastic case, with probability one, \( T_x \) the tangent space to the level surface at \( x \) is at most one dimensional.
3  A one dimensional analysis

In this section we outline a one dimensional analysis first described by Reynolds, Truman and Williams (RTW) [34].

Definition 3.1. The \( d \)-dimensional flow map \( \Phi_t \) is globally reducible if for any \( x = (x_1, x_2, \ldots, x_d) \) and \( x_0 = (x_0^1, x_0^2, \ldots, x_0^d) \) where \( x = \Phi_t(x_0) \), it is possible to write each coordinate \( x_0^\alpha \) as a function of the lower coordinates. That is,

\[
x = \Phi_t(x_0) \Rightarrow x_0^\alpha = x_0^\alpha(x, x_0^1, x_0^2, \ldots, x_0^{\alpha-1}, t) \text{ for } \alpha = d, d-1, \ldots, 2. \quad (5)
\]

Therefore, using Theorem 2.2, the flow map is globally reducible if we can find a chain of \( C^2 \) functions \( x_0^d, x_0^{d-1}, \ldots, x_0^2 \) such that,

\[
x_0^d = x_0^d(x, x_0^1, x_0^2, \ldots, x_0^{d-1}, t) \iff \frac{\partial A}{\partial x_0^d}(x_0, x, t) = 0,
\]

\[
x_0^{d-1} = x_0^{d-1}(x, x_0^1, x_0^2, \ldots, x_0^{d-2}, t) \iff \frac{\partial A}{\partial x_0^{d-1}}(x_0^1, x_0^2, \ldots, x_0^{d-1}, x, t) = 0,
\]

\[
\vdots
\]

\[
x_0^2 = x_0^2(x, x_0^1, t) \iff \frac{\partial A}{\partial x_0^2}(x_0^1, x_0^2, x_0^3(x, x_0^1, x_0^2, t), \ldots, x_0^d(x, \ldots, x_0^d), x, t) = 0,
\]

where \( x_0^\ell(\ldots) \) is the expression only involving \( x_0^1 \) and \( x_0^2 \) gained by substituting each of the functions \( x_0^3, \ldots, x_0^{d-1} \) repeatedly into \( x_0^d(x, x_0^1, x_0^2, \ldots, x_0^{d-1}, t) \). This requires that no roots are repeated to ensure that none of the second derivatives of \( A \) vanish. We assume also that there is a favoured ordering of coordinates and a corresponding decomposition of \( \Phi_t \) which allows the non-uniqueness to be reduced to the level of the \( x_0^1 \) coordinate. This assumption appears to be quite restrictive. However, local reducibility at \( x \) follows from the implicit function theorem and some mild assumptions on the derivatives of \( A \).

Definition 3.2. If \( \Phi_t \) is globally reducible then the reduced action function is the univariate function gained from evaluating the action with equations (5),

\[
f(x, t)(x_0^1) := f(x_0^1, x, t) = A(x_0^1, x_0^2(x, x_0^1, t), x_0^3(x, \ldots, x_0^d), \ldots, x, t).
\]
Lemma 3.3. If \( \Phi_t \) is globally reducible, modulo the above assumptions,
\[
\left| \det \left( \frac{\partial^2 A}{(\partial x_0)^2}(x_0, x, t) \right) \right|_{x_0=(x_0^1,...,x_0^d(\ldots))} = \prod_{\alpha=1}^{d} \left[ \left( \frac{\partial}{\partial x_0^\alpha} \right)^2 A(x_0^1, \ldots, x_0^\alpha, x_0^\alpha+1(\ldots), \ldots, x_0^d(\ldots), x, t) \right]_{x_0^\alpha=x_0^\alpha(\ldots) \cdots x_0^2=x_0^2(\ldots)}
\]
where the first term is \( f''_{(x,t)}(x_0^1) \) and the last \( d-1 \) terms are non zero.

Theorem 3.4. Let the classical mechanical flow map \( \Phi_t \) be globally reducible. Then:
1. \( f'_{(x,t)}(x_0^1) = 0 \) and the equations \( \Xi \) \( \Leftrightarrow x = \Phi_t(x_0) \),
2. \( f'_{(x,t)}(x_0^1) = f''_{(x,t)}(x_0^1) = 0 \) and the equations \( \Xi' \)
\( \Leftrightarrow x = \Phi_t(x_0) \) is such that the number of real solutions \( x_0 \) changes.

4 Analysis of the caustic

We begin by parameterising the caustic \( 0 = \det (D\Phi_t(x_0)) \) from Definition 1.2. This equation only involves \( x_0 \) and \( t \), and is therefore the pre-caustic. We use this to parameterise the pre-caustic as,
\[
x_0^1 = \lambda_1, \quad x_0^2 = \lambda_2, \ldots, \quad x_0^{d-1} = \lambda_{d-1} \quad \text{and} \quad x_0^d = x_0^d(\lambda_1, \lambda_2, \ldots, \lambda_{d-1}).
\]
The parameters are restricted to be real so that only real pre-images are considered.

Definition 4.1. For any \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{d-1}) \in \mathbb{R}^{d-1} \) the pre-parameterisation of the caustic is given by \( x_t(\lambda) := \Phi_t(\lambda, x_0^d(\lambda)) \).

The pre-parameterisation will be intrinsic if \( \ker(D\Phi_t) \) is one dimensional.

Corollary 4.2. Let \( x_t(\lambda) \) denote the pre-parameterisation of the caustic where \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{d-1}) \in \mathbb{R}^{d-1} \). Then \( f'_{(x_t(\lambda),t)}(\lambda_1) = f''_{(x_t(\lambda),t)}(\lambda_1) = 0 \).

Proposition 4.3. Let \( x_t(\lambda) \) denote the pre-parameterisation of the caustic where \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{d-1}) \in \mathbb{R}^{d-1} \). Assume \( f_{(x_t(\lambda),t)}(x_0^1) \in C^{p+1} \) then, in \( d \)-dimensions, if the tangent to the caustic is at most \( (d-p+1) \)-dimensional at \( x_t(\lambda) \),
\[
f'_{(x_t(\lambda),t)}(\tilde{\lambda}_1) = f''_{(x_t(\lambda),t)}(\tilde{\lambda}_1) = \ldots = f^{(p)}_{(x_t(\lambda),t)}(\tilde{\lambda}_1) = 0.
\]
Proof. Follows by repeatedly differentiating $f'''_{(x_{t}(\lambda),t)}(\lambda_1) = 0$, which holds if the tangent space at $x_{t} \lambda$ is $(d-2)$-dimensional [22].

From Corollary 4.2 there is a critical point of inflexion on $f_{(x,t)}(x_{0}^1)$ at $x_{0}^1 = \lambda_1$ when $x = x_{t} \lambda$. Consider an example where for $x$ on one side of the caustic there are four real critical points on $f_{(x,t)}(x_{0}^1) = 0$. Let them be enumerated $x_{0}^1(i)(x,t)$ for $i = 1$ to 4 and denote the minimising critical point $\tilde{x}_{0}^1(x,t)$. Figure 3 illustrates how the minimiser jumps from (a) to (b) as $x$ crosses the caustic. This will cause $u^\mu$ and $v^\mu$ to jump for small $\mu$ and the caustic at such a point is described as being cool.

Before Caustic

\[ x_{0}^1(1) = x_{0}^1(3), \]

Minimiser at $x_{0}^1(2)(x,t) = \tilde{x}_{0}^1(x,t)$.

On Cool Caustic

Two $x_{0}^1$’s coalescing form point of inflexion.

Beyond Caustic

Minimiser jumps.

Figure 3: The graph of $f_{(x,t)}(x_{0}^1)$ as $x$ crosses the caustic.

Definition 4.4. Let $x_{t} \lambda$ be the pre-parameterisation of the caustic. Then $x_{t} \lambda$ is on the cool part of the caustic if $f_{(x_{t} \lambda),t}(\lambda_1) \leq f_{(x_{t} \lambda),t}(x_{0}^1(i)(x_{t} \lambda),t)$ for all $i = 1, 2, \ldots, n$ where $x_{0}^1(i)(x,t)$ denotes an enumeration of all the real roots for $x_{0}^1$ to $f'_{(x,t)}(x_{0}^1) = 0$. If the caustic is not cool it is hot.

Definition 4.5. The pre-normalised reduced action function evaluated on the caustic is given by $F_{\lambda}(x_{0}^1) := f_{(x_{t} \lambda),t}(x_{0}^1) - f_{(x_{t} \lambda),t}(\lambda_1)$.

Assume that $F_{\lambda}(x_{0}^1)$ is a real analytic function in a neighbourhood of $\lambda_1 \in \mathbb{R}$. Then,

\[ F_{\lambda}(x_{0}^1) = (x_{0}^1 - \lambda_1)^3 \tilde{F}(x_{0}^1), \]

where $\tilde{F}$ is real analytic. When the inflexion at $x_{0}^1 = \lambda_1$ is the minimising critical point of $F_{\lambda}$, the caustic will be cool. Therefore, on a hot/cool boundary this inflexion is about to become or cease being the minimiser.

Proposition 4.6. A necessary condition for $x_{t} \lambda \in C_t$ to be on a hot/cool boundary is that either $\tilde{F}(x_{0}^1)$ or $\tilde{G}(x_{0}^1)$ has a repeated root at $x_{0}^1 = r$ where,

\[ \tilde{G}(x_{0}^1) = 3\tilde{F}(x_{0}^1) + (x_{0}^1 - \lambda_1)\tilde{F}'(x_{0}^1). \]
Proof. The minimiser could change when either \( \tilde{F} \) has a repeated root which is the minimiser, or there is a second inflexion at a lower minimising value \( [23] \).

The condition is not sufficient as it includes cases where the minimiser is not about to change (see Figure 4).

\[
\frac{\text{Increasing } \lambda}{\text{Possible hot/cool boundary}} \quad \begin{array}{c|c|c}
\text{Caustic changes hot to cool} & \text{No change in caustic} \\
\hline
\includegraphics[width=0.8\textwidth]{example47.png}
\end{array}
\]

Figure 4: Graphs of \( F_\lambda(x_0) \) as \( \lambda \) varies.

**Example 4.7** (The polynomial swallowtail). Let \( V(x, y) \equiv 0, k_t(x, y) \equiv x \), and \( S_0(x_0, y_0) = x_0^5 + x_0^2y_0 \). This gives global reducibility and \( k_t(x, y) \equiv x \) means that the effect of the noise is to translate \( \epsilon = 0 \) picture through \( \left(-\epsilon \int_t^0 W_s \, ds, 0\right) \). A simple calculation gives,

\[
\tilde{F}(x_0) = 12\lambda^2 - 3\lambda t + 6\lambda x_0 - tx_0 + 2x_0^2, \\
\tilde{G}(x_0) = 15\lambda^2 - 4\lambda t + 10\lambda x_0 - 2tx_0 + 5x_0^2.
\]

\[
k = \left(-\frac{\epsilon^2}{500} - \epsilon \int_t^0 W_s \, ds, \frac{\epsilon^2}{50} - \frac{1}{27}\right)
\]

\[
\psi = \left(-\frac{\epsilon^2(3+8\sqrt{6})}{18000} - \epsilon \int_t^0 W_s \, ds, \frac{\epsilon^2(9-\sqrt{6})}{450} - \frac{1}{27}\right)
\]

Figure 5: Hot and cool parts of the polynomial swallowtail caustic for \( t = 1 \).
Example 4.8 (The three dimensional polynomial swallowtail). Let \( V(x, y) \equiv 0, \ k_t(x, y) \equiv 0 \), and \( S_0(x_0, y_0, z_0) = x_0^7 + x_0^3 y_0 + x_0^2 z_0 \). The functions \( \tilde{F} \) and \( \tilde{G} \) can be easily found, and an exact expression for the boundary extracted [22] this is shown in Figure 6.

![Boundary on the caustic.](image1)

![Hot and cool parts.](image2)

Figure 6: The hot (plain) and cool (mesh) parts of the 3D polynomial swallowtail caustic at time \( t = 1 \).

5 Swallowtail perestroikas

The geometry of a caustic or wavefront can suddenly change with singularities appearing and disappearing [2]. We consider the formation or collapse of a swallowtail using some earlier works of Cayley and Klein. This section provides a summary of results from [23] where all proofs can be found.

We begin by recalling the classification of double points of a two dimensional algebraic curve as acnodes, crunodes and cusps (Figure 7).

![Acnode. Crunode. Cusp.](image3)

Figure 7: The classification of double points.

In Cayley’s work on plane algebraic curves, he describes the possible triple points of a curve [27] by considering the collapse of systems of double points which would lead to the existence of three tangents at a point. The four
possibilities are shown in Figure 8. The systems will collapse to form a triple point with respectively, three real distinct tangents, three real tangents with two coincident, three real tangents all of which are coincident, or one real tangent and two complex tangents. It is the interchange between the last two cases which will lead to the formation of a swallowtail on a curve [15]. This interchange was investigated by Felix Klein [17].

Figure 8: Cayley’s triple points.

In Section 3, we restricted the pre-parameter to be real to only consider points with real pre-images. This does not allow there to be any isolated double points. We now allow the parameter to vary throughout the complex plane and consider when this maps to real points. We begin by working with a general curve of the form 

\[ x(\lambda) = (x_1(\lambda), x_2(\lambda)) \]

where each \( x_\alpha(\lambda) \) is real analytic in \( \lambda \in \mathbb{C} \). If \( \text{Im}\{x(a + i\eta)\} = 0 \), it follows that \( x(a + i\eta) = x(a - i\eta) \), so this is a “complex double point” of the curve \( x(\lambda) \).

**Lemma 5.1.** If \( x(\lambda) = (x_1(\lambda), x_2(\lambda)) \) is a real analytic parameterisation of a curve and \( \lambda \) is an intrinsic parameter, then there is a generalised cusp at \( \lambda = \lambda_0 \) if and only if the curves,

\[ 0 = \frac{1}{\eta} \text{Im}\{x_\alpha(a + i\eta)\} \quad \alpha = 1, 2, \]

intersect at \( (\lambda_0, 0) \) in the \( (a, \eta) \) plane.

Now consider a family of parameterised curves \( x_t(\lambda) = (x_1^t(\lambda), x_2^t(\lambda)) \). As \( t \) varies the geometry of the curve can change with swallowtails forming and disappearing.

**Proposition 5.2.** If a swallowtail on the curve \( x_t(\lambda) \) collapses to a point where \( \lambda = \tilde{\lambda} \) when \( t = \tilde{t} \) then,

\[ \frac{dx_i}{d\lambda}(\lambda) = \frac{d^2x_i}{d\lambda^2}(\tilde{\lambda}) = 0. \]
Proposition 5.3. Assume that there exists a neighbourhood of $\tilde{\lambda} \in \mathbb{R}$ such that $\frac{b_{a\lambda}}{a\lambda}(\lambda) \neq 0$ for $t \in (\tilde{t} - \delta, \tilde{t})$ where $\delta > 0$. If a complex double point joins the curve $x_t(\lambda)$ at $\lambda = \tilde{\lambda}$ when $t = \tilde{t}$ then,

$$
\frac{dx_i}{d\lambda}(\tilde{\lambda}) = \frac{d^2x_i}{d\lambda^2}(\tilde{\lambda}) = 0.
$$

These provide a necessary condition for the formation or destruction of a swallowtail, and for complex double points to join or leave the main curve.

Definition 5.4. A family of parameterised curves $x_t(\lambda)$, (where $\lambda$ is some intrinsic parameter) for which,

$$
\frac{dx_i}{d\lambda}(\lambda) = \frac{d^2x_i}{d\lambda^2}(\lambda) = 0
$$

is said to have a point of swallowtail perestroika when $\lambda = \tilde{\lambda}$ and $t = \tilde{t}$.

As with generalised cusps, we have not ruled out further degeneracy at these points. Moreover, as Cayley highlighted, these points are not cusped and are barely distinguishable from an ordinary point of the curve [27].

5.1 The complex caustic in two dimensions

The complex caustic is the complete caustic found by allowing the parameter $\lambda$ in the pre-parameterisation $x_t(\lambda) \in \mathbb{R}^2$ to vary over the complex plane. By considering the complex caustic, we are determining solutions $a = a_t$ and $\eta = \eta_t$ to,

$$
f'_{(x,t)}(a + i\eta) = f''_{(x,t)}(a + i\eta) = 0,
$$

where $x \in \mathbb{R}^2$. We are interested in these points if they join the main caustic at some finite critical time $\tilde{t}$. That is, there exists a finite value $\tilde{t} > 0$ such that $\eta_t \to 0$ as $t \uparrow \tilde{t}$. If this holds then a swallowtail can develop at the critical time $\tilde{t}$.

Theorem 5.5. For a two dimensional caustic, assume that $x_t(\lambda)$ is a real analytic function. If at a time $\tilde{t}$ a swallowtail perestroika occurs on the caustic, then $x = x_{\tilde{t}}(\lambda)$ is a real solution for $x$ to,

$$
f'_{(x,\tilde{t})}(\lambda) = f''_{(x,\tilde{t})}(\lambda) = f'''_{(x,\tilde{t})}(\lambda) = f^{(4)}_{(x,\tilde{t})}(\lambda) = 0,
$$

where $\lambda = a_{\tilde{t}}$. 

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Theorem 5.6. For a two dimensional caustic, assume that \( x_t(\lambda) \) is a real analytic function. If at a time \( \tilde{t} \) there is a real solution for \( x \) to,

\[
    f'_x(x, \tilde{t}) = f''_x(x, \tilde{t}) = f'''_x(x, \tilde{t}) = f^{(4)}_x(x, \tilde{t}) = 0,
\]

and the vectors \( \nabla_x f'_x(x, \tilde{t}) \) and \( \nabla_x f''_x(x, \tilde{t}) \) are linearly independent, then \( x \) is a point of swallowtail perestroika on the caustic.

Example 5.7. Let \( V(x, y) = 0, k_t(x, y) \equiv 0 \) and \( S_0(x_0, y_0) = x_0^5 + x_0^5y_0 \). The caustic has no cusps for times \( t < \tilde{t} \) and two cusps for times \( t > \tilde{t} \) where \( \tilde{t} = 4\sqrt{2} \times 33^{3/4} \times 7^{-7/4} = 2.5854\ldots \).

At the critical time \( \tilde{t} \) the caustic has a point of swallowtail perestroika as shown in Figures 9 and 10. The conjugate pairs of intersections of the curves in Figure 9 are the complex double points. There are five before the critical time and four afterwards. The remaining complex double points do not join the main caustic and so do not influence its behaviour for real times.

![Figure 9: Im\{x_t(a + i\eta)\} = 0 (solid) and Im\{y_t(a + i\eta)\} = 0 (dashed) in (a, \eta) plane.](image)

![Figure 10: Caustic plotted at corresponding times.](image)

5.2 Level surfaces

Unsurprisingly, these phenomena are not restricted to caustics. There is an interplay between the level surfaces and the caustics, characterised by their pre-images.
Proposition 5.8. Assume that in two dimensions at $x_0 \in \Phi^{-1}_t H^c_t \cap \Phi^{-1}_t C_t$ the normal to the pre-level surface $n(x_0) \neq 0$ and the normal to the pre-caustic $\tilde{n}(x_0) \neq 0$ so that the pre-caustic is not cusped at $x_0$. Then $\tilde{n}(x_0)$ is parallel to $n(x_0)$ if and only if there is a generalised cusp on the caustic.

Corollary 5.9. Assume that in two dimensions at $x_0 \in \Phi^{-1}_t H^c_t \cap \Phi^{-1}_t C_t$ the normal to the pre-level surface $n(x_0) \neq 0$. Then at $\Phi_t(x_0)$ there is a point of swallowtail perestroika on the level surface $H^c_t$ if and only if there is a generalised cusp on the caustic $C_t$ at $\Phi_t(x_0)$.

Example 5.10. Let $V(x, y) = 0$, $k_t(x, y) = 0$, and $S_0(x_0, y_0) = x_0^5 + x_0^6 y_0$. Consider the behaviour of the level surfaces through a point inside the caustic swallowtail at a fixed time as the point is moved through a cusp on the caustic. This is illustrated in Figure 11. Part (a) shows all five of the level surfaces through the point demonstrating how three swallowtail level surfaces collapse together at the cusp to form a single level surface with a point of swallowtail perestroika. Parts (b) and (c) show how one of these swallowtails collapses on its own and how its pre-image behaves.

![Figure 11](image-url)

Figure 11: (a) All level surfaces (solid line) through a point as it crosses the caustic (dashed line) at a cusp, (b) one of these level surfaces with its complex double point, and (c) its real pre-image.

6 Maxwell sets

A jump will occur in the inviscid limit of the Burgers velocity field if we cross a point at which there are two different global minimisers $x_0(t)(x, t)$ and $x_0(j)(x, t)$ returning the same value of the action.
In terms of the reduced action function, the Maxwell set corresponds to values of \( x \) for which \( f(x,t)(x^1_0) \) has two critical points at the same height. If this occurs at the minimising value then the Burgers fluid velocity will jump as shown in Figure 12.

### Figure 12: The graph of the reduced action function as \( x \) crosses the Maxwell set.

---

6.1 The Maxwell-Klein set

We begin with the two dimensionals polynomial case by considering the classification of double points of a curve (Figure 7).

**Lemma 6.1.** A point \( x \) is in the Maxwell set if and only if there is a Hamilton-Jacobi level surface with a point of self-intersection (crunode) at \( x \).

**Proof.** Follows from Definition 1.4.

**Definition 6.2.** The Maxwell-Klein set \( B_t \) is the set of points which are non-cusp double points of some Hamilton-Jacobi level surface curve.

It follows from this definition that a point is in the Maxwell-Klein set if it is either a complex double point (acnode) or point of self-intersection (crunode) of some Hamilton-Jacobi level surface. Using the geometric results of DTZ outlined in Section 2, it is easy to calculate this set in the polynomial case as the cusps of the level surfaces sweep out the caustic.

**Theorem 6.3.** Let \( D_t \) be the set of double points of the Hamilton-Jacobi level surfaces, \( C_t \) the caustic set, and \( B_t \) the Maxwell-Klein set. Then, from Cayley and Klein’s classification of double points as crunodes, acnodes, and cusps, by definition \( D_t = C_t \cup B_t \) and the corresponding defining algebraic equations factorise \( D_t = C_t^n \cdot B_t^m \), where \( m, n \) are positive integers.
Proof. Follows from Proposition 2.8 and Lemma 6.1.

\textbf{Theorem 6.4.} Let $\rho_{(t,c)}(x)$ be the resultant,

$$\rho_{(t,c)}(x) = R\left( f_{(x,t)}(\cdot) - c, f'_{(x,t)}(\cdot) \right),$$

where $x = (x_1, x_2)$. Then $x \in D_t$ if and only if for some $c$,

$$\rho_{(t,c)}(x) = \frac{\partial \rho_{(t,c)}}{\partial x_1}(x) = \frac{\partial \rho_{(t,c)}}{\partial x_2}(x) = 0.$$

Further,

$$D_t(x) = \gcd(\rho_1^1(x), \rho_2^1(x)),$$

where $\gcd(\cdot, \cdot)$ denotes the greatest common divisor and $\rho_1^1$ and $\rho_2^1$ are the resultants.

**Proof.** Recall that the equation of the level surface of Hamilton-Jacobi functions is merely the result of eliminating $x_0$ between the equations,

$$f_{(x,t)}(x_0^1) = c \quad \text{and} \quad f'_{(x,t)}(x_0^1) = 0.$$

We form the resultant $\rho_{(t,c)}(x)$ using Sylvester’s formula. The double points of the level surface must satisfy for some $c \in \mathbb{R}$,

$$\rho_{(t,c)}(x) = 0, \quad \frac{\partial \rho_{(t,c)}}{\partial x_1}(x) = 0 \quad \text{and} \quad \frac{\partial \rho_{(t,c)}}{\partial x_2}(x) = 0.$$

Sylvester’s formula proves all three equations are polynomial in $c$. To proceed we eliminate $c$ between pairs of these equations using resultants giving,

$$R\left( \rho_{(t,\cdot)}(x), \frac{\partial \rho_{(t,\cdot)}}{\partial x_1}(x) \right) = \rho_1^1(x) \quad \text{and} \quad R\left( \frac{\partial \rho_{(t,\cdot)}}{\partial x_1}(x), \frac{\partial \rho_{(t,\cdot)}}{\partial x_2}(x) \right) = \rho_2^1(x).$$

Let $D_t = \gcd(\rho_1^1, \rho_2^1)$ be the greatest common divisor of the algebraic $\rho_1^1$ and $\rho_2^1$. Then $D_t(x) = 0$ is the equation of double points.

We now extend this to $d$-dimensions, where the Maxwell-Klein set corresponds to points which satisfy the Maxwell set condition but have both real pre-images (Maxwell) or complex pre-images (Klein).
Theorem 6.5. Let the reduced action function \( f_{(x,t)}(x^1) \) be a polynomial in all space variables. Then the set of all possible discontinuities for a \( d \)-dimensional Burgers fluid velocity field in the inviscid limit is the double discriminant,

\[
D(t) := D_c \left\{ D_\lambda \left( f_{(x,t)}(\lambda) - c \right) \right\} = 0,
\]

where \( D_x(p(x)) \) is the discriminant of the polynomial \( p \) with respect to \( x \).

Proof. By considering the Sylvester matrix of the first discriminant,

\[
D_\lambda \left( f_{(x,t)}(\lambda) - c \right) = K \prod_{i=1}^{m} \left( f_{(x,t)}(x^1(i)(x,t)) - c \right),
\]

where \( x^1(i)(x,t) \) is an enumeration of the real and complex roots \( \lambda \) of \( f'_{(x,t)}(\lambda) = 0 \) and \( K \) is some constant. Then the second discriminant is simply,

\[
D_c \left( D_\lambda \left( f_{(x,t)}(\lambda) - c \right) \right) = K^{2m-2} \prod_{i<j} \left( f_{(x,t)}(x^1(i)(x,t)) - f_{(x,t)}(x^1(j)(x,t)) \right)^2.
\]

Theorem 6.6. The double discriminant \( D(t) \) factorises as,

\[
D(t) = b_0^{2m-2} \cdot (C_t)^3 \cdot (B_t)^2,
\]

where \( B_t = 0 \) is the equation of the Maxwell-Klein set and \( C_t = 0 \) is the equation of the caustic. The expressions \( B_t \) and \( C_t \) are both algebraic in \( x \) and \( t \).

Proof. See [23].

Example 6.7 (The polynomial swallowtail). Let \( V(x,y) = 0 \), \( k_t(x,y) = 0 \) and, \( S_0(x_0,y_0) = x_0^5 + x_0^2y_0 \). The Maxwell-Klein set can be found by factorisation giving,

\[
0 = -675 + 52t^4 - t^8 + 3120t^3x - 224t^7x + 4t^{11}x - 38400t^2x^2 + 1408t^6x^2 \\
+ 128000tx^3 - 5400t^4y + 312t^5y - 4t^9y + 12480t^4xy - 448t^8xy \\
- 7680t^3x^2y - 16200t^2y^2 + 624t^6y^2 - 4t^{10}y^2 + 12480t^5xy^2 \\
- 21600t^3y^3 + 416t^7y^3 - 10800t^3y^4.
\]

Outside of the swallowtail on the caustic there are two real and two complex pre-images whereas inside there are four real and no complex pre-images. Therefore, any part of the Maxwell-Klein set outside of the caustic swallowtail must correspond to Klein double points and any part inside must correspond to the Maxwell set. This is shown in Figure [13].
6.2 The pre-Maxwell set

If the Maxwell set is defined as in Definition 1.4 then the pre-Maxwell set is the set of all the pre-images $x_0$ and $\tilde{x}_0$ which give rise to the Maxwell set.

**Definition 6.8.** The pre-Maxwell set $\Phi_t^{-1} M_t$ is the set of all points $x_0 \in \mathbb{R}^d$ where there exists $x, \tilde{x}_0 \in \mathbb{R}^d$ such that $x = \Phi_t(x_0)$ and $x = \Phi_t(\tilde{x}_0)$ with $x_0 \neq \tilde{x}_0$ and,

$$A(x_0, x, t) = A(\tilde{x}_0, x, t).$$

With the caustic and level surfaces, each regular point was linked by $\Phi_t^{-1}$ to a single point on the relevant pre-surface. However, every point on the Maxwell set is linked by $\Phi_t^{-1}$ to at least two points on the pre-Maxwell set.

**Theorem 6.9.** The pre-Maxwell set is given by the discriminant $D_{x_0^1} (G(\tilde{x}_0^1)) = 0$ where,

$$G(\tilde{x}_0^1) = \frac{f(\Phi_t(x_0), t)(x_0^1) - f(\Phi_t(x_0), t)(\tilde{x}_0^1)}{(x_0^1 - \tilde{x}_0^1)^2}.$$

**Proof.** From the Definition 6.8 and Theorem 3.4 it follows that the pre-Maxwell set is found by eliminating $x$ and $\tilde{x}_0^1$ between,

$$f(x, t)(x_0^1) = f(x, t)(\tilde{x}_0^1), \quad f'(x, t)(x_0^1) = f'(x, t)(\tilde{x}_0^1) = 0.$$

This surface would include the pre-caustic where $x_0^1 = \tilde{x}_0^1$ and so this repeated root must be eliminated. 

---

**Figure 13:** The caustic and Maxwell-Klein set.
We can use this to pre-parameterise the Maxwell set as has been done with the caustic and level surfaces. By restricting the parameter to be real, we only get the Maxwell set as the Klein points have complex pre-images.

We now summarise the results of [25].

**Lemma 6.10.** Assume that a point \( x \) on the Maxwell set corresponds to exactly two pre-images on the pre-Maxwell set, \( x_0 \) and \( \tilde{x}_0 \). Then the normal to the pre-Maxwell set at \( x_0 \) is to within a scalar multiplier given by,

\[
\mathbf{n}(x_0) = - \left( \frac{\partial^2 A}{\partial x_0^2}(x_0, x, t) \right) \left( \frac{\partial^2 A}{\partial x \partial x_0}(x_0, x, t) \right)^{-1} \left( \dot{X}(t, x_0, \nabla S_0(x_0)) - \dot{X}(t, \tilde{x}_0, \nabla S_0(\tilde{x}_0)) \right).
\]

**Corollary 6.11.** In two dimensions let the pre-Maxwell set meet the pre-caustic at a point \( x_0 \) where \( n \neq 0 \) and

\[
\ker \left( \frac{\partial^2 A}{(\partial x_0)^2}(x_0, \Phi_t(x_0), t) \right) = \langle e_0 \rangle,
\]

where \( e_0 \) is the zero eigenvector. Then the tangent plane to the pre-Maxwell set at \( x_0 \), \( T_{x_0} \) is spanned by \( e_0 \).

**Proposition 6.12.** Assume that in two dimensions at \( x_0 \in \Phi_t^{-1}M_t \) the normal \( n(x_0) \neq 0 \) so that the pre-Maxwell set does not have a generalised cusp at \( x_0 \). Then the Maxwell set can only have a cusp at \( \Phi_t(x_0) \) if \( \Phi_t(x_0) \in C_t \). Moreover, if

\[
x = \Phi_t(x_0) \in \Phi_t \left\{ \Phi_t^{-1} C_t \cap \Phi_t^{-1} M_t \right\},
\]

the Maxwell set will have a generalised cusp at \( x \).

**Corollary 6.13.** In two dimensions, if the pre-Maxwell set intersects the pre-caustic at a point \( x_0 \), so that there is a cusp on the Maxwell set at the corresponding point where it intersects the caustic, then the pre-Maxwell set touches the pre-level surface \( \Phi_t^{-1}H^c \) at the point \( x_0 \). Moreover, if the cusp on the Maxwell set intersects the caustic at a regular point of the caustic, then there will be a cusp on the pre-Maxwell set which also meets the same pre-level surface \( \Phi_t^{-1}H^c \) at another point \( \tilde{x}_0 \).

**Corollary 6.14.** When the pre-Maxwell set touches the pre-caustic and pre-level surface, the Maxwell set intersects a cusp on the caustic.

**Example 6.15** (The polynomial swallowtail). Let \( V(x, y) = 0 \), \( k_t(x, y) = 0 \) and, \( S_0(x_0, y_0) = x_0^5 + x_0^2 y_0 \).
From Proposition 6.12, the cusps on the Maxwell set correspond to the intersections of the pre-curves (points 3 and 6 on Figure 14). But from Corollary 6.13, the cusps on the Maxwell set also correspond to the cusps on the pre-Maxwell set (points 2 and 5 on Figure 14 and also Figure 15). The Maxwell set terminates when it reaches the cusps on the caustic. These points satisfy the condition for a generalised cusp but, instead of appearing cusped, the curve stops and the parameterisation begins again in the sense that it maps back exactly on itself. At such points the pre-surfaces all touch (Figure 15).

These two different forms of cusps correspond to very different geometric behaviours of the level surfaces. Where the Maxwell set stops or cusps corresponds to the disappearance of a point of self-intersection on a level surface. There are two distinct ways in which this can happen. Firstly, the level surface will have a point of swallowtail perestroika when it meets a cusp on the caustic. At such a point only one point of self-intersection will disappear,
and so there will be only one path of the Maxwell set which will terminate at that point. However, when we approach the caustic at a regular point, the level surface must have a cusp but not a swallowtail peresoika. This corresponds to the collapse of the second system of double points in Figure 8. Thus, two different points of self intersection coalesce and so two paths of the Maxwell set must approach the point and produce the cusp (see Figure 16).

Figure 16: The caustic (long dash) Maxwell set (solid line) and level surface (short dash).

7 Some applications to turbulence in two dimensions

7.1 Real turbulence and the $\zeta$ process

Definition 7.1. The turbulent times $t$ are times when the pre-level surface of the minimising Hamilton-Jacobi function touches the pre-caustic. Such times $t$ are zeros of a stochastic process $\zeta^c(\cdot)$, i.e. $\zeta^c(t) = 0$.

These turbulent times are times at which the number of cusps on the corresponding level surface will change. We begin with some minor generalisations of results in RTW 31 and also 23 26.

Proposition 7.2. Assume $\Phi_t$ is globally reducible and that $x_t(\lambda)$ is the pre-parameterisation of a two dimensional caustic. Then the turbulence process at $\lambda$ is given by,

$$\zeta^c(t) = f_{(x_t(\lambda_0), t)}(\lambda_0) - c,$$

where $f_{(x, t)}(x_0)$ is the reduced action evaluated at points $x = x_t(\lambda_0)$ where $x_t(\lambda_0) = \Phi_t(\lambda_0, x^2_0(\lambda_0)) \in C_t$, $\lambda = \lambda_0$ satisfying,

$$\dot{X}_t(\lambda) \cdot \frac{dx_t}{d\lambda}(\lambda) = 0,$$
where $\tilde{X}_t(\lambda) = \tilde{\Phi}_t(\lambda, x_0^2, c(\lambda))$ and $x_t(\lambda_0) \in C^c_t$, the cool part of the caustic.

Hence, there are three kinds of real stochastic turbulence:

1. **Cusped**, where there is a cusp on the caustic,
2. **Zero speed**, where the Burgers fluid velocity is zero,
3. **Orthogonal**, where the Burgers fluid velocity is orthogonal to the caustic.

**Proof.** The number of cusps on the relevant pre-level surface is,

$$n_c(t) = \# \{ \lambda \in \mathbb{R} : f(x_t(\lambda), t)(\lambda) = c \},$$

where the roots $\lambda = \lambda_0$ correspond to points in the cool part of the caustic. The pre-surfaces touch when $n_c(t)$ changes, which occurs when,

$$\frac{d}{d\lambda} f(x_t(\lambda), t)(\lambda) = 0.$$

For stochastic turbulence to be intermittent we require that the process $\zeta^c(t)$ is recurrent.

**Proposition 7.3.** Let $V(x, y) = 0$, $k_t(x, y) = x$ and

$$S_0(x_0, y_0) = f(x_0) + g(x_0)y_0,$$

where $f, g, f'$ and $g'$ are zero at $x_0 = a$ but $g''(a) \neq 0$. Then, for orthogonal turbulence at $a$,

$$\zeta^c(t) = -a\epsilon W_t + \epsilon^2 W_t \int_0^t W_s \, ds - \frac{\epsilon^2}{2} \int_0^t W_s^2 \, ds - c.$$

We note the following result of RTW [34].

**Lemma 7.4.** Let $W_t$ be a $BM(\mathbb{R})$ process starting at 0, $c$ any real constant and

$$Y_t = -a\epsilon W_t + \epsilon^2 W_t \int_0^t W_s \, ds - \frac{\epsilon^2}{2} \int_0^t W_s^2 \, ds - c.$$

Then, with probability one, there exists a sequence of times $t_n \nearrow \infty$ such that

$$Y_{t_n} = 0 \quad \text{for every } n.$$

We also note that this can be extended to a $d$-dimensional setting where for a $d$-dimensional Wiener process $W(t)$ the zeta process can be found explicitly [22].
Theorem 7.5. In $d$-dimensions, the zeta process is given by,

$$
\zeta_t = f^0_{(x^0_t(\lambda),t)}(\lambda_1) - \epsilon x^0_t(\lambda) \cdot W(t) + \epsilon^2 W(t) \cdot \int_0^t W(s) \, ds + \frac{\epsilon^2}{2} \int_0^t |W(s)|^2 \, ds
$$

where $f^0_{(x,t)}(x^1_t)$ denotes the deterministic reduced action function, $x^0_t(\lambda)$ denotes the pre-parameterisation of the deterministic caustic and $\lambda$ must satisfy the equation,

$$
\nabla_\lambda \left( f^0_{(x^0_t(\lambda),t)}(\lambda_1) - \epsilon x^0_t(\lambda) \cdot W(t) \right) = 0.
$$

When $\lambda$ is deterministic, the recurrence of this process can be shown using the same argument as for the two dimensional case (further results on recurrence can be found in [24]). Here we recapitulate our belief that cusped turbulence will be the most important. As we have shown, when the cusp on the caustic passes through a level surface, it forces a swallowtail to form on the level surface. The points of self intersection of this swallowtail form the Maxwell set.

7.2 Complex turbulence and the resultant $\eta$ process

We now consider a completely different approach to turbulence. Let $(\lambda, x^2_{0,C}(\lambda))$ denote the parameterisation of the pre-caustic at time $t$. When,

$$
Z_t = \text{Im} \left\{ \Phi_t(a + i\eta, x^2_{0,C}(a + i\eta)) \right\},
$$

is random, the values of $\eta(t)$ for which $Z_t = 0$ will form a stochastic process. The zeros of this new process will correspond to points at which the real pre-caustic touches the complex pre-caustic. The points at which these surfaces touch correspond to swallowtail perestroikas on the caustic. When such a perestroika occurs there is a solution of the equations,

$$
f'_{(x,t)}(\lambda) = f''_{(x,t)}(\lambda) = f'''_{(x,t)}(\lambda) = f^{(4)}_{(x,t)}(\lambda) = 0.
$$

Assuming that $f_{(x,t)}(x^1_t)$ is polynomial in $x^1_t$ we can use the resultant to state explicit conditions for which this holds [23].

Lemma 7.6. Let $g$ and $h$ be polynomials of degrees $m$ and $n$ respectively with no common roots or zeros. Let $f = gh$ be the product polynomial. Then the resultant,

$$
R(f, f') = (-1)^{mn} \left( \frac{m!n!}{N!} \frac{f^{(N)}(0)}{g^{(m)}(0)h^{(n)}(0)} \right)^{N-1} R(g, g')R(h, h')R(g, h)^2,
$$

where $N = m + n$ and $R(g, h) \neq 0$. 

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Since \( f'_{(x_t(\lambda),t)}(x_0^1) \) is a polynomial in \( x_0 \) with real coefficients, its zeros are real or occur in complex conjugate pairs. Of the real roots, \( x_0 = \lambda \) is repeated. So,

\[
f'_{(x_t(\lambda),t)}(x_0^1) = (x_0^1 - \lambda)^2 Q_{(\lambda,t)}(x_0^1) H_{(\lambda,t)}(x_0^1),
\]

where \( Q \) is the product of quadratic factors,

\[
Q_{(\lambda,t)}(x_0^1) = \prod_{i=1}^{q} \left\{ (x_0^1 - a_i^t)^2 + (\eta_i^t)^2 \right\},
\]

and \( H_{(\lambda,t)}(x_0^1) \) the product of real factors corresponding to real zeros. This gives,

\[
f'''_{(x_t(\lambda),t)}(x_0^1)|_{x_0^1 = \lambda} = 2 \prod_{i=1}^{q} \left\{ (\lambda - a_i^t)^2 + (\eta_i^t)^2 \right\} H_{(\lambda,t)}(\lambda).
\]

We now assume that the real roots of \( H \) are distinct as are the complex roots of \( Q \). Denoting \( f'''_{(x_t(\lambda),t)}(x_0^1)|_{x_0^1 = \lambda} \) by \( f'''_{t}(\lambda) \) etc, a simple calculation gives

\[
\left| R_{\lambda}(f'''_{t}(\lambda), f^{(4)}_{t}(\lambda)) \right| =
\]

\[
K_t \prod_{k=1}^{q} (\eta_k^t)^2 \prod_{j \neq k} \left\{ (a_k^t - a_j^t)^4 + 2((\eta_k^t)^2 + (\eta_j^t)^2)(a_k^t - a_j^t)^2 + ((\eta_k^t)^2 - (\eta_j^t)^2)^2 \right\}
\]

\[
\times |R_{\lambda}(H, H')| |R_{\lambda}(Q, H)|^2,
\]

\( K_t \) being a positive constant. Thus, the condition for a swallowtail pere-stroika to occur is that

\[
\rho_\eta(t) := \left| R_{\lambda}(f'''_{t}(\lambda), f^{(4)}_{t}(\lambda)) \right| = 0,
\]

where we call \( \rho_\eta(t) \) the resultant eta process.

When the zeros of \( \rho_\eta(t) \) form a perfect set, swallowtails will spontaneously appear and disappear on the caustic infinitely rapidly. As they do so, the geometry of the cool part of the caustic will rapidly change as the \( \lambda \) shaped sections typical of a swallowtail caustic appear and disappear. Moreover, Maxwell sets will be created and destroyed with each swallowtail that forms and vanishes adding to the turbulent nature of the solution in these regions. We call this ‘complex turbulence’ occurring at the turbulent times which are the zeros of the resultant eta process.
Complex turbulence can be seen as a special case of real turbulence which occurs at specific generalised cusps of the caustic. Recall that when a swallowtail perestroika occurs on a curve, it also satisfies the conditions for having a generalised cusp. Thus, the zeros of the resultant eta process must coincide with some of the zeros of the zeta process for certain forms of cusped turbulence. At points where the complex and real pre-caustic touch, the real pre-caustic and pre-level surface touch in a particular manner (a double touch) since at such a point two swallowtail perestroikas on the level surface have coalesced.

Thus, our separation of complex turbulence from real turbulence can be seen as an alternative form of categorisation to that outlined in Section 7.1 which could be extended to include other perestroikas.

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