Low Dimensional Euclidean Volume Preserving Embeddings

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Abstract
Let \( P \) be an \( n \)-point subset of Euclidean space and \( d \geq 3 \) be an integer. In this paper we study the following question: What is the smallest (normalized) relative change of the volume of subsets of \( P \) when it is projected into \( \mathbb{R}^d \). We prove that there exists a linear mapping \( f : P \mapsto \mathbb{R}^d \) that relatively preserves the volume of all subsets of size up to \( \lfloor d/2 \rfloor \) within at most a factor of \( O\left(\frac{n^{2/d} \sqrt{\log n \log \log n}}{\log n} \right) \).

Key words: Volume, Embeddings, Dimensionality Reduction, Discrete Geometry, Distortion

1. Introduction

A classical result of Johnson and Lindenstrauss [JL84] states that any \( n \)-point subset of Euclidean space can be projected into \( O(\log n) \) dimensions while preserving the metric structure of the set. A natural question to pose would be what is the smallest distortion of any \( n \)-point subset of Euclidean space when it is projected into (fixed) \( d \) dimensions. This problem was first studied by Matoušek [Mat90], who proved an \( O\left(n^{2/d} \sqrt{\log n / d} \right) \) upper bound on the distortion by projecting the points into \( \mathbb{R}^d \) using a random \( d \)-dimensional subspace. In Section 3 we re-prove Matoušek’s result using the simplified analysis of [DG03, IM98] adapted in this setting, i.e., bounding the distortion having fixed dimension instead of bounding the target dimension having fixed distortion. Although the simplified proof of the above result is well-known and well-understood, we hope that it is not redundant and that it helps the reader to digest the following theorem

**Theorem 1.** Let \( P \) be a \( n \)-point subset of \( \mathbb{R}^N \) and let \( 3 \leq d \leq c_3 \log n \). Then there is a linear mapping \( f : P \mapsto \mathbb{R}^d \) such that

\[
\forall S \subset P, |S| \leq \lfloor d/2 \rfloor \quad \frac{\text{Vol}(f(S))}{\text{Vol}(S)} \leq c_4 n^{2/d} \sqrt{\log n \log \log n},
\]

where \( c_3, c_4 > 0 \) are absolute constants, and \( \text{Vol}(S) \) is the \((|S| - 1)\)-dimensional volume of the convex hull of \( S \).

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Remark: The case where we fix the relative change of the volume of subsets to be arbitrary close to one, and ask what is the minimum dimension of such a mapping was studied in [MZ08]. Notice that if we only require to preserve pairwise distances the best upper bound is $O\left(\frac{n^2}{d} \sqrt{\log n/d}\right)$, see Section 3; therefore our result can be thought of as a generalization of the distance preserving embeddings since it also guarantees distance preservation. Moreover, there exists $n$-point subset of Euclidean space that any embedding onto $\mathbb{R}^d$ has distortion $\Omega\left(\frac{n}{\left\lfloor \frac{d+1}{2} \right\rfloor}\right)$ [Mat90], and thus the above worst-case upper bound cannot be much improved.

2. Preliminaries and Technical Lemmas

We start by defining an (stochastic) ordering between two random variables $X$ and $Y$, but first let’s motivate this definition. Assume that we have upper and lower bounds on the distribution function of $Y$, and also assume that it’s hard to give precise bounds on the distribution function of $X$. Using this notion of ordering, if $X$ “smaller than” $Y$, then we can bound the “complicated” variable $X$ through bounding the “easy” variable $Y$. We use this notion extensively in this paper.

More formally, let $X$ and $Y$ be two random variables, not necessarily on the same probability space. The random variable $X$ is stochastically smaller than the random variable $Y$ when, for every $x \in \mathbb{R}$, the inequality

$$\mathbb{P}(X \leq x) \geq \mathbb{P}(Y \leq x)$$

holds. We denote this by $X \preceq Y$.

Next we recall known results about the Chi-square distribution and also give bounds on its cumulative distribution function. If $X_i, i = 1, \ldots, d$ be independent, identically distributed normal random variables, then the random variable $\chi^2_d = \sum_{i=1}^d X_i^2$ is a Chi-square random variable with $d$ degrees of freedom. Notice that the expected value of $\chi^2_d$ is $d$. It is well known [Fel71, Chapter II, p. 47] that the Chi-square distribution is a special case of the Gamma distribution and its cumulative distribution function is given by

$$\mathbb{P}(\chi^2_d \leq t) = \frac{\gamma(d/2, t/2)}{\Gamma(d/2)},$$

where $\Gamma(x)$ is the Gamma function, $\gamma(a,x) = \int_0^x t^{a-1} e^{-t} dt$ and $\Gamma(a,x) = \int_x^\infty t^{a-1} e^{-t} dt$ is the lower and upper incomplete Gamma function, respectively. Next we present some bounds on the Gamma and incomplete Gamma functions that we use in Sections 3 and 4. We start by presenting the following bound on the Gamma function, see for instance [CD05, Lemmas 2.5, 2.6, 2.7] and [WW63, p.253].

**Lemma 1** (Stirling Bound on Gamma Function). If $\Gamma(a) = \int_0^\infty e^{-t} t^{a-1} dt$, where $a > 0$, then

$$\sqrt{2\pi} a^{a+1/2} e^{-a} < \Gamma(a+1) < \sqrt{2\pi} a^{a+1/2} e^{-a} + \frac{1}{a},$$

Next we upper bound $\gamma(a,x)$. Note that $\gamma(a,x) = \int_0^x t^{a-1} e^{-t} dt \leq \int_0^x t^{a-1} dt$, hence

$$\gamma(a,x) \leq x^a/a.$$
Now for the upper incomplete gamma, we have the following bound.

**Lemma 2.** If \( \Gamma(a, x) = \int_x^\infty e^{-t}t^{a-1} \, dt \) where \( x > 2(a+1) \), then

\[
\Gamma(a, x) < 2 \exp(-x)x^{a+1}.
\]

**(5)**

**Proof.** In [CD05, Lemma 2.6] set \( \alpha = 1 \) and \( d = 2 \).

It is well-known [FB95, pp. 220–235] that the volume that is spanned by the convex hull of a \( k \)-point subset of \( \mathbb{R}^N \) along with the origin is equal to \( \sqrt{\det(P^\top P)/k!} \), where \( P \) is the \( k \times N \) matrix that contains the points as columns. The following lemma gives a connection between the volume of the convex hull of \( k \) points and the determinant of a specific matrix that is constructed using these points.

**Lemma 3.** Let \( \mathcal{P} = \{p_1, p_2, \ldots, p_k\} \) be an \( k \)-point subset of \( \mathbb{R}^N \) in general position and let \( f : \mathbb{R}^N \mapsto \mathbb{R}^d \) be a linear mapping. Let \( P := [p_2 - p_1, p_3 - p_1, \ldots, p_k - p_1] \) be an \( N \times (k-1) \) matrix. Then

\[
\frac{\text{Vol}(f(\mathcal{P}))}{\text{Vol}(\mathcal{P})} = \left( \frac{\det((FP)^\top FP)}{\det(P^\top P)} \right)^{1/2},
\]

where \( F \) is the \( d \times N \) matrix that corresponds to \( f \).

**(6)**

**Proof.** By a translation of the point-set \( \mathcal{P} \), i.e., identifying \( p_1 \) with the origin, it follows that \( \text{Vol}(\mathcal{P}) = \sqrt{\det(P^\top P)/k!} \), since the volume is translation invariant, and similarly \( \text{Vol}(f(\mathcal{P})) = \sqrt{\det((FP)^\top FP)/k!} \). Since \( \mathcal{P} \) is in general position, it follows that

\[
\frac{\text{Vol}(f(\mathcal{P}))}{\text{Vol}(\mathcal{P})} = \left( \frac{\det((FP)^\top FP)}{\det(P^\top P)} \right)^{1/2}.
\]

**Now, let’s consider the above lemma in the setting where \( f \) is a random linear mapping. More specifically, let \( F \) be a Gaussian matrix, i.e., a matrix whose entries are i.i.d. Gaussian \( \mathcal{N}(0, 1) \). First observe that the fraction of the volumes is a random variable. Surprisingly enough, as the following lemma states, the fraction of the volumes in this setting is independent of \( \mathcal{P} \). This can be thought of as a generalization of the 2-stability property of inner products with Gaussian random vectors to matrix multiplication with Gaussian matrices.

**Lemma 4.** Let \( \mathcal{P} = \{p_1, p_2, \ldots, p_k\} \) be an \( k \)-point subset of \( \mathbb{R}^N \) in general position. And let \( f : \mathbb{R}^N \mapsto \mathbb{R}^d \) be a random Gaussian linear mapping. Then

\[
\left( \frac{\text{Vol}(f(\mathcal{P}))}{\text{Vol}(\mathcal{P})} \right)^2 \sim \prod_{i=1}^{k-1} \chi^2_{d-i+1}.
\]

**(7)**
Proof. It is a simple consequence of \cite[Lemma 3]{MZ08} and the above lemma.

Remark 1. For $k = 2$ in Lemma \ref{lem:indep} we get $\|f(p_1) - f(p_2)\|^2 / \|p_1 - p_2\|^2 \sim \chi_d^2$.

Equation \ref{eq:indep} gives the distribution of the fraction of the volume as a product of independent random variables. However, in general it’s difficult to deal with such a product, and so we employ the following theorem that sandwiches this product with a single Chi-square distributions.

\begin{theorem}[Theorem 4, \cite{Gor89}] Let $u_i := \chi_{d-i+1}^2$ be independent Chi-square random variables for $i = 1, 2, \ldots, s$. Then the following holds for every $s \geq 1$,
\begin{equation}
\chi_{s(d-s+1)+\frac{s(s-1)}{2}}^2 \leq s \left( \prod_{i=1}^{s} u_i \right)^{1/s} \leq \chi_{s(d-s+1)}^2.
\end{equation}

We now have enough tools at our disposal to prove Theorem \ref{thm:indep}.

3. Distance Distortion

In this section we prove the following

\begin{theorem}
Let $\mathcal{P}$ be a $n$-point subset of $\mathbb{R}^N$ and let $3 \leq d \leq c_1 \log n$, where $c_1$ is a positive constant. Then there exists a linear mapping $f : \mathcal{P} \rightarrow \mathbb{R}^d$ with (distance) distortion $\text{dist}(f) = O\left(n^{2/d} \sqrt{\log n/d}\right)$, i.e., there exists an absolute constant $c > 0$ such that
\[ \forall x, y \in \mathcal{P}, \quad \|x - y\| \leq \|f(x) - f(y)\| \leq c n^{2/d} \sqrt{\log n / d} \|x - y\|. \]
\end{theorem}

\begin{proof}
Similarly as in \cite{Mat90}, Consider the random linear map $f : \mathbb{R}^N \rightarrow \mathbb{R}^d$, $f(x) := R \cdot x$ where $R$ is an $d \times N$ random Gaussian matrix. Using linearity of $f$ and Remark \ref{rem:indep} it follows that $\|f(x) - f(y)\|^2 / \|x - y\|^2 \sim \chi_d^2$ for any $x, y \in \mathcal{P}$. Our goal is to show that $\chi_d^2$ is sufficiently concentrated. More specifically, it suffices to show that $\chi_d^2$ doesn’t fall outside an interval $[a, b]$ for some $a, b \in \mathbb{R}$ with constant probability. This aims to upper bound the probabilities $\Pr[\chi_d^2 \leq a^2]$ and $\Pr[\chi_d^2 \geq b^2]$.

The elements of $\mathcal{P}$ determine at most $\binom{n}{2}$ distinct direction vectors. Applying union bound over all pairs of $\mathcal{P}$ gives that if
\[ \binom{n}{2} \left( \Pr\left( \chi_d^2 \leq a^2 \right) + \Pr\left( \chi_d^2 \geq b^2 \right) \right) < 1, \]
then there exists $f$ that expands every distance in $\mathcal{P}$ by at most $b$ times and contracts at least $a$ times, so \text{dist}(f) \leq b/a. Our goal therefore is to specify $a, b$ in terms of $d$ and $n$ such that Inequality \ref{eq:indep} holds. To do so, we first bound $\Gamma(d/2)$ from below, which will be used later. By Lemma \ref{lem:indep} we have that $\Gamma(d/2) \geq e^{-d/2}(d-2)^{d-1/2}/2^{d/2}$. Now, we will bound $a, b$ separately. We find $a$ such that $\binom{n}{2} \Pr\left( \chi_d^2 \leq a^2 \right) < 1/2$. Using Equation \ref{eq:indep} and the previous analysis we require that $\frac{n^2}{2} e^{-d/2}(d-2)^{d-1/2} < 1/2$, which holds for all $d \geq 3$ if we set $a = c_2 \sqrt{d/n^2/d}$, where $c_2 > 0$ is an
absolute constant. Similarly, we will find $b$ such that $\binom{n}{2} \mathbb{P}(\chi_{d}^{2} \geq b^{2}) < 1/2$. Using Lemma 2 and assume for the moment that $b^{2} > 2d - 2$, we have that

$$\mathbb{P}(\chi_{d}^{2} \geq b^{2}) \leq \frac{e^{-b^{2}/2} (b^{2}/2)^{d/2 - 1}}{\Gamma(d/2)} \leq \frac{b^{d-2} e^{-b^{2}/2 - d/2}}{(d-2)^{(d-1)/2}}.$$ 

It suffices to show that $\ln \left( n^{2} b^{d-2} e^{-b^{2}/2 - d/2} \right) / (d-2)^{(d-1)/2} )$ is negative for large enough $n$. Indeed,

$$\ln \left( n^{2} b^{d-2} e^{-b^{2}/2 - d/2} \right) \leq 2 \ln n + (d-2) \ln b - b^{2}/2 - d/2 - \frac{d-1}{2} \ln(d-2).$$

Note that if $d > d'$ then $\mathbb{P}(\chi_{d'}^{2} \geq b^{2}) \leq \mathbb{P}(\chi_{d}^{2} \geq b^{2})$. Thus we can assume that $d = c_{1} \log n$, since if we can bound it, then we can bound it for all fixed $d < c_{1} \log n$. Define $g(b,n) = 2 \ln n + (d-2) \ln b - b^{2}/2 - d/2 - \frac{d-1}{2} \ln(d-2)$. We want to show that $g(b,n) < 0$ for large enough $n$. By choosing $b = 5 c_{1} \sqrt{\ln n}$, and recall that $d = c_{1} \log n$ hence $b^{2} > 2d - 2$, we conclude that $\lim_{n \to \infty} g(5 \sqrt{\ln n}, n) = -\infty$ as desired. Hence, we can choose $a, b$ functions of $n$ such that $b/a = 5 c_{1} \sqrt{\ln n} = c n^{2/d} \sqrt{\ln n/d}$. 

4. Proof of Main Theorem

Our goal is to find a mapping $f : \mathcal{P} \to \mathbb{R}^{d}$ such that

$$\forall S \subset \mathcal{P}, |S| \leq k \quad 1 \leq \left( \frac{\text{Vol}(f(S))}{\text{Vol}(S)} \right)^{1/t} \leq D,$$

where $D$ is the volume distortion of the mapping. We will see in the analysis below that we can set $k = \lfloor d/2 \rfloor$ and $D = O(n^{2/d} \sqrt{\log n \log \log n})$. We can assume w.l.o.g. that the input points are in general position, i.e., every subset of size up to $k$ is affinely independent. If not, both the original points and projected points will span zero volume.

Similarly with Section 3, we take a random $f$ using a Gaussian random matrix and show that it satisfies (10) with constant probability. To do so, we first bound the probability that a fixed subset “contracts” its’ volume by more than a factor $a$.

**Lemma 5.** Fix any subset $S \subset \mathcal{P}$ of size $|S| = s + 1$ with $1 \leq s < k$. Then

$$\mathbb{P} \left( \left( \frac{\text{Vol}(f(S))}{\text{Vol}(S)} \right)^{1/t} \leq a \right) \leq \frac{(esa^{2})^{t/2}}{t(t-2)^{(t-1)/2}},$$

where $t = s(d-s+1)$.

**Proof.** Using Lemma 4 we know that the above probability is equal to $\mathbb{P} \left( \left( \prod_{i=1}^{s} \chi_{d-i+1}^{2} \right)^{1/s} \leq a^{2} \right)$. Using Theorem 2 we can bound the above probability of product of Chi-square random variables.
with a single Chi-square. More specifically, using the stochastic ordering we have the following inequality
\[ \mathbb{P} \left( \left( \prod_{i=1}^{s} \chi_{d-i+1}^{2} \right)^{1/s} \leq a^2 \right) \leq \mathbb{P} \left( \chi_{d-s+1}^{2} \leq s \cdot a^2 \right) \]
for every \( 1 \leq s < k \). Now, we have a single Chi-square random variable and thus we can bound it from above, the same way as we did in Section 3 using Lemma (1) and Equation (4). It follows that
\[ \mathbb{P} \left( \chi_{i}^{2} \leq s \cdot a^2 \right) = \frac{\Gamma(\frac{t+2}{2}, \frac{sa^2}{2})}{\Gamma(t/2)} \leq \frac{(esa^2)^{t/2}}{t(s-1)^{s-1/2}}. \]

Similarly, we bound the probability that a fixed subset “expands” its volume by more than a factor \( b \).

**Lemma 6.** Fix any subset \( S \subset \mathcal{P} \) of size \( |S| = s + 1 \) with \( 1 \leq s < k \). If \( sb^2 > 2l + 4 \), then
\[ \mathbb{P} \left( \left( \frac{\text{Vol}(f(S))}{\text{Vol}(S)} \right)^{s} \geq b \right) \leq e^{-\frac{a^2}{2} \left( \frac{(sb^2)^{1/2+1}}{(l-2)(l-1)/2} \right)} \]
where \( l = s(d - s + 1) + \frac{(s-1)(s-2)}{2} \).

**Proof.** As in the previous lemma the above probability is equal to \( \mathbb{P} \left( \left( \prod_{i=1}^{s} \chi_{d-i+1}^{2} \right)^{1/s} \geq b^2 \right) \), and again using Theorem (2) it follows that
\[ \mathbb{P} \left( \left( \prod_{i=1}^{s} \chi_{d-i+1}^{2} \right)^{1/s} \geq b^2 \right) \leq \mathbb{P} \left( \chi_{s(d-s+1) + \frac{(s-1)(s-2)}{2}}^{2} \geq s \cdot b^2 \right) := E_{d,s}. \]
Using Lemmas (1, 2) it follows that \( \mathbb{P} \left( \chi_{i}^{2} \geq s \cdot b^2 \right) = \frac{\Gamma(\frac{t+2,\frac{sb^2}{2})}{\Gamma(t/2)} \leq \frac{(esa^2)^{t/2}}{t(s-1)^{s-1/2}}. \)

Notice that if \( d' > d \), then \( E_{d,s} \leq E_{d',s} \) from the stochastic ordering of the Chi-square distribution. Now we are ready to apply union bound. Our goal is to find \( a \) such that with probability at least \( 1/2 \), our embedding does not contract volumes of subsets of size up to \( k \) by a factor \( a \).

By union bounding over all sets of fixed size \( i, 1 \leq i \leq k \), we want to find \( a \) such that
\[ \binom{n}{i+1} \frac{(ea^2)^{i/2}}{t_{i}(t_{i}-2)^{(n-1)/2} < \frac{1}{2k},} \]
where \( t_{i} = i(d - i + 1) \). Note that if we sum over all different size of subsets \( (i = 1, \ldots, k) \) we get that the failure probability is at most \( 1/2 \). It suffices to show that \( \ln \left( \frac{2k \binom{n}{i+1}}{t_{i}(t_{i}-2)^{(n-1)/2} e^{a^2/2}} \right) \) is negative for large enough \( n \) and for every \( 1 \leq i \leq k \) and \( d \geq 3 \), or equivalently the following is negative
\[ \ln 2 + \ln k + (i+1) \ln n + t_{i} \ln a + (t_{i}/2 - i) \ln i + (t_{i}/2 + i) \ln t_{i} - \left( \frac{t_{i}-1}{2} \right) \ln(t_{i}-2). \]
Let’s group the terms of the right hand size and bound them individually. It is not hard to see that

$$(t_i/2 - i) \ln i - \left(\frac{i-1}{2}\right) \ln (t_i - 2) < 0$$

and $\ln k - \ln t_i \leq 0$ since $k \leq d \leq t_i$ and $t_i = i(d - i + 1)$, when $i = 1, \ldots, k$ and for $d \geq 3$. Hence, it suffices to show that

$$\ln 2 + (i + 1) \ln n + t_i \ln a + (t_i/2 + i) < 0.$$ 

Set $a = c_e n^{-\gamma}$, for some positive $\gamma$ that will be specified shortly and $c_e$ a sufficient small positive constant. Recall that we want the above inequality to hold for every $1 \leq i \leq k$. We can choose $c_e$ smaller than $e^{-1}$ and take care of the $t_i/2 + i + \ln 2$ term. Let’s now focus on the dominate term $(i + 1) \ln n$. It follows that the above quantity is negative if $\gamma \geq \frac{i+1}{i(d-i+1)}$, for all $i = 1, \ldots, k$.

Let’s study closer the function $h_d(x) = \frac{x+1}{x(d-x+1)}$. We will show that $h_d(x)$ is convex on the domain $[1, d/2]$ and also increasing in the domain $[d/4, d]$ for any fixed $d \geq 3$. A simple calculation shows that $h_d'(x) > 0$ for $x \in [1, d]$ and $h_d'(x) > 0$ for $x \in [d/4, d]$ (details omitted). Also note that $h_d(1) = h_d(d/2) = 2/d$. By convexity in $[1, d/2]$, we get that $h_d(x) \leq 2/d$ for all $x \in [1, d/2]$.

The above analysis gives a bound on the parameter $k$, i.e., the maximum size of subsets that we can consider. Thus, we get that $k$ should be less than or equal to $\lceil d/2 \rceil$.

To sum up, we have proved that if $a = c_e n^{-\gamma}$, then with probability at least $1/2$ our embedding doesn’t contract the normalized volumes of subsets of size at most $\lceil d/2 \rceil$ by more than a multiplicative factor of $a$.

Next our goal is to find $b$ such that with probability at least $1/2$, $f$ does not expand volumes by more than a factor of $b$. Let $l_i = i(d - i + 1) + \frac{(i-1)(i-2)}{2}$. We apply union bound over all sets of fixed size $i$, $1 \leq i < k$ together with Lemma 5 assuming for the moment that $ib^2 > 4l_i + 8$. We want to find $b$ such that

$$\left(\frac{n}{i+1}\right) e^{-\frac{a^2}{2}} \frac{(ib^2)^l/2+1}{(l_i - 2)^{(l-1)/2}} < \frac{1}{2k}.$$ 

Summing over all different size of subsets we get the desired property with probability at least $1/2$.

It suffices to show that $\ln 2k \left(\frac{n}{i+1}\right) e^{-\frac{a^2}{2}} \frac{(ib^2)^l/2+1}{(l_i - 2)^{(l-1)/2}}$ is negative for every $1 \leq i < k$ and $d \in [3, \log n]$. Similarly with Section 3 we can assume without loss of generality that $d = c_3 \log n$, using the fact that if $d' \leq d$ then $E_{d,s} \leq E_{d,s}$.

Now, since there are at most $\left(\frac{n}{i+1}\right) \leq \left(\frac{\alpha}{i+1}\right)^{i+1}$ subsets of size $i + 1$, it suffices to show that the following quantity is negative,

$$\ln \left(\frac{kn^{i+1} e^{-\frac{a^2}{2}} (ib^2)^l/2+1}{(l_i - 2)^{(l-1)/2}}\right) \leq \ln \left(\frac{kn^{i+1} e^{-\frac{a^2}{2}} (ib^2)^l/2+1}{(l_i - 2)^{(l-1)/2}}\right) <$$

$$(l_i/2 + 1) \ln i + (i + 1) \ln n + l_i \ln b + l_i/2 + 2i + \ln k - \left(\frac{ib^2}{2} + \frac{l_i - 1}{2} \ln li\right).$$
Note that in the last quantity the positive terms are of order $O(id\ln i + i\ln n)$. The negative terms are of order $O(ib^2)$. Recall that $i < d = c_3 \log n$. It is not hard to see that by choosing $b = c_2 \sqrt{\log n \log \log n}$, where $c_2 > 0$ a sufficient large constant, then $ib^2 > 4i_i + 8$ and the above quantity goes to $-\infty$ as $n$ grows for every $1 \leq i < k$.

To sum up, we proved that with probability at least $1/2$, $f$ doesn’t expand normalized volumes of subsets of size at most $\lceil d/2 \rceil$ by more than a multiplicative factor of $b$.

Rescaling $f$ by $a$, we conclude that there exists $a, b$ with $a < b$ such that

$$
\mathbb{P} \left( \forall S \subset P, |S| \leq |d/2|, 1 \leq \left( \frac{\text{Vol}(f(S))}{\text{Vol}(S)} \right)^{1/1} \leq \frac{b}{a} \right) > 0.
$$

This concludes the proof of Theorem I

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