Single-qubit gate teleportation provides a quantum advantage

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Abstract

Gate-teleportation circuits are arguably among the most basic examples of computations believed to provide a quantum computational advantage: In seminal work [23], Terhal and DiVincenzo have shown that these circuits elude simulation by efficient classical algorithms under plausible complexity-theoretic assumptions. Here we consider possibilistic simulation [25], a particularly weak form of this task where the goal is to output any string appearing with non-zero probability in the output distribution of the circuit. We show that even for single-qubit Clifford-gate-teleportation circuits this simulation problem cannot be solved by constant-depth classical circuits with bounded fan-in gates. Our results are unconditional and are obtained by a reduction to the problem of computing the parity, a well-studied problem in classical circuit complexity.

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1 Introduction and main result

A quantum circuit $Q$ on $n$ qubits takes as input a classical bit string $I = (I_1, \ldots, I_m) \in \{0, 1\}^m$. Starting from a product state $|0\rangle^\otimes n$, it applies one- and two-qubit gates that may be classically controlled by constant-size subsets $\{I_j\}_j$ of input bits, and finally measures each qubit in the computational basis to yield an outcome $O \in \{0, 1\}^n$. A prominent example is the single-qubit (Clifford)-gate-teleportation circuit $\text{Telep}_n$ (on $2n$ qubits) shown in Fig. 1, based on the gate-teleportation protocol of Gottesman and Chuang [14]. Here we seek to characterize this circuit’s computational power.

![Telep_n circuit](image)

Figure 1: The gate-teleportation circuit $\text{Telep}_n$. The circuit takes as input an $n$-tuple $(C_0, \ldots, C_{n-1}) \in \mathcal{C}^n$ of single-qubit Clifford group elements. Each Clifford is applied to one half of Bell state. Subsequently, Bell-state measurements are performed on pairs of qubits. The output can be associated with an $n$-tuple $(P_0, \ldots, P_{n-1}) \in \mathcal{P}^n$ of Pauli elements from the set $\mathcal{P} = \{I, X, Y, Z\}$ corresponding to different outcomes of Bell measurements. The circuit has a cyclic structure: the $n$-th Bell measurement is applied on the first and last qubits in the figure. As the single-qubit Clifford group $\mathcal{C} = \mathcal{P} \cdot \{I, H, S, HS, SHS, (HS)^2\}$ consists of $|\mathcal{C}| = 24$ elements, its elements can be encoded into 5-bit strings. Similarly, elements of $\mathcal{P}$ can be encoded as 2-bit strings (pairs of bits). Correspondingly, we will often interpret inputs $(C_0, \ldots, C_{n-1})$ as elements of $(\{0, 1\}^5)^n$ and outputs $(P_0, \ldots, P_{n-1})$ as elements of $(\{0, 1\}^2)^n$.

The behavior of a quantum circuit $Q$ is fully specified by the conditional distribution $p^Q(O|I)$ of the output $O$ given a fixed input $I$. Its computational power is directly tied to the difficulty of sampling from $p^Q(O|I)$ for different inputs $I$, and, more generally, the hardness of computing or approximating properties of this collection of distributions. Corresponding notions of classically simulating $Q$ give rise to various computational problems with more stringent notions of simu-
lation associated with harder computational problems. Arguably one of the weakest notions of simulating $Q$ is that of possibilistic simulation, a term introduced in [25]. The corresponding computational problem, which we denote by $\text{PSIM}[Q]$, asks, for a given input $I$ (an instance of $\text{PSIM}[Q]$), to produce $O \in \{0, 1\}^n$ such that $p^Q(O|I) > 0$, i.e., such that $O$ occurs with non-zero probability in the output distribution of $Q$.

Our main result is that, for the gate-teleportation circuit $\text{Telep}$, even the problem $\text{PSIM}[\text{Telep}]$ is hard for constant-depth classical circuits. To state this, recall that an $\text{NC}^0$ circuit is a constant-depth polynomial-size classical circuit with bounded fan-in gates (where, following standard terminology, we often refer to an infinite circuit family simply as a circuit).

**Theorem 1.1.** The Telep quantum circuit trivially solves $\text{PSIM}[\text{Telep}]$ for every instance. In contrast, any (even non-uniform) $\text{NC}^0$ circuit fails to solve $\text{PSIM}[\text{Telep}]$ on certain instances.

We note that average-case statements for suitable distributions over instances of $\text{PSIM}[\text{Telep}]$ can be derived in a similar manner as e.g., [8, Supplementary material]. Here we focus on the worst case for simplicity.

The study of the difficulty of classically simulating quantum gate teleportation was initiated in seminal work by Terhal and DiVincenzo [23]. It was shown in [23] (see also [1, 12]) that there is no efficient classical algorithm that can sample exactly from the output distribution of a variant of the gate-teleportation circuit unless the polynomial hierarchy collapses. The gate-teleportation circuit used in this argument teleports $n$ qubits at a time and essentially realizes any $n$-qubit quantum circuit with a depth-4 circuit together with post-selection. In particular, depth-4 circuits form a circuit family which, when combined with post-selection, gives the complexity class PostBQP of (uniform) polynomial-size postselected quantum circuits. This leads to the hardness of classically sampling from their output distribution (also called weak simulation in Ref. [24]) under plausible complexity theory assumptions.

This line of reasoning has also been used to show hardness of simulating “commuting quantum computation” as formalized by IQP circuits [9], level-1-quantum approximate optimization (QAOA) circuits [11] and boson sampling circuits [2].

**Proof outline**

We establish Theorem 1.1 by connecting the teleportation simulation problem $\text{PSIM}[\text{Telep}]$ to a celebrated result from classical circuit complexity: the known circuit-depth lower bounds for $\text{AC}^0$ of computing the parity function – the PARITY problem [4, 13, 16, 19–21, 27]. Our argument proceeds through a reduction to an intermediate computational problem which relates $\text{PSIM}[\text{Telep}]$ and PARITY.

Let us briefly describe the intermediate problem and give a high-level overview of our reasoning.

**Quantum advantage against $\text{NC}^0$**

The intermediate problem we use in our analysis is another simulation problem $\text{PSIM}[\text{CliffP}(Q)_L]$ for a two-qubit quantum circuit $\text{CliffP}(Q)_L$. The circuit $\text{CliffP}(Q)_L$ is defined by using an (arbitrary but fixed) function $Q : C^L \rightarrow P$. The circuit $\text{CliffP}(Q)_L$ is illustrated in Fig. 2. It takes as input $L + 1$ single-qubit Clifford gates

$$(C_1, \ldots, C_L, D) \in C^{L+1}$$

and outputs a Pauli $P \in P$, see the caption of Fig. 1 for a definition of these sets. We call $\text{CliffP}(Q)_L$ the Pauli-corrected Clifford circuit with correction function $Q$.

In Section 3 we show the following: Assume that $C$ is an (even non-uniform) $\text{NC}^0$-circuit solving $\text{PSIM}[\text{Telep}_n]$ (i.e., producing a valid output for all instances). Then, for $L = \Theta(n)$,
there is a function \( Q : \mathbb{C}^L \rightarrow \mathcal{P} \) (that depends on \( C \)) and a non-uniform NC^0-circuit \( C' \) solving PSIM[CliffP(Q)_L]. The construction of \( C' \) makes use of \( C \) in a non-black-box manner and relies on no-signaling arguments from Section 2. We summarize this (with a slight abuse of notation) as

\[
(\text{PSIM}[\text{Telep}] \in \text{NC}^0/\text{poly}) \Rightarrow (\exists Q: \text{PSIM}[\text{CliffP}(Q)] \in \text{NC}^0/\text{poly}),
\]

where \( \text{NC}^0/\text{poly} \) is the class of problems solved by non-uniform families of \( \text{NC}^0 \)-circuits.

Our second reduction heavily borrows from ideas developed in [15] in the context of interactive quantum advantage protocols. By adapting (specializing) these techniques, we show that the ability of possibilistically simulating CliffP(Q)_L – when combined with an appropriate non-uniform AC^0-circuit – implies the ability of computing products of single-qubit Clifford gates. In particular, given a (deterministic) oracle \( O \) for PSIM[CliffP(Q)] (for any fixed function \( Q \)), this immediately yields a non-uniform AC^0 circuit \( A^O \) with oracle access to \( O \) which solves PARITY. Succinctly, we summarize this as

\[
(O \text{ solves } \text{PSIM}[\text{CliffP}(Q)]) \Rightarrow (\text{PARITY} \in (\text{AC}^0/\text{poly})^O),
\]

where \( \text{AC}^0/\text{poly} \) is the class of problems solved by non-uniform families of \( \text{AC}^0 \) circuits. Combining (1) and (2) shows that

\[
(\text{PSIM}[\text{Telep}] \in \text{NC}^0/\text{poly}) \Rightarrow (\text{PARITY} \in \text{AC}^0/\text{poly}),
\]

i.e., an \( \text{NC}^0 \)-circuit for possibilistically simulating the circuit Telep could be used to compute parity in \( \text{AC}^0/\text{poly} \). Thus the existence of such a circuit would contradict the fact that \( \text{PARITY} \not\in \text{AC}^0/\text{poly} \), i.e., the fact that there are no (not even non-uniform) \( \text{AC}^0 \)-circuits computing parity. In conclusion, this shows that there is no \( \text{NC}^0 \)-circuit for the teleportation simulation problem PSIM[Telep].

**Related work**

Let us briefly summarize related work. The first unconditional separation of constant-depth quantum circuits and constant-depth classical circuits with bounded fan-in gates, i.e., QNC^0
and $\text{NC}^0$, was achieved by Bravyi, Gosset, and König \cite{bravyi2018quantum} for the so-called 2D Hidden Linear Function problem. This result was later extended to noisy quantum circuits by Bravyi, Gosset, König, and Tomamichel \cite{bravyi2018almost}. The latter construction uses a simpler underlying problem – the so-called 1D Magic Square Game. Both of these constructions are physically motivated and based on violation of Bell inequalities. In subsequent work Bene Watts, Kothari, Schaeffer, and Tal \cite{bene2018quantum} achieved an even stronger separation, establishing a quantum advantage of constant-depth quantum circuits against classical constant-depth circuits with unbounded fan-in gates, i.e., $\text{AC}^0$ circuits. The problem used in \cite{bene2018quantum} is the so-called Relaxed Parity Halving problem. Similarly to our result, this work establishes a quantum advantage based on classical circuit complexity lower bounds as opposed to the first two \cite{bravyi2018quant,bravyi2018almost} Bell-inequality-based constructions.

In this work we focus on the minimal quantum information-processing resources required for demonstrating a quantum advantage. Let us thus discuss what geometry and interactions (gates) the quantum circuits for these problems feature. Both quantum circuits for the 2D Hidden Linear Function problem and the Relaxed Parity Halving problem can be realized by geometrically 2D-local circuits with classically controlled Clifford gates. The 1D Magic Square Game problem can be realized by a Clifford circuit on a chain of pairs of qubits. This geometric locality allows for a fault-tolerant implementation by a local quantum circuit in a 3D geometry.

Finally, we note that the work of Grier and Schaeffer \cite{grier2018quantum} provides several quantum advantage protocols involving interaction. In this work, the quantum player can be realized by a constant-depth (classically controlled) Clifford circuit which is realized on a chain consisting of qubits, a pair of qubits, or a grid depending on the complexity-theoretic result considered: For example, a chain of qubits is used to show hardness for $\text{AC}^0$ with additional MOD gates, and chain of pairs of qubits is used to show hardness for classical circuits with bounded fan-in gates and logarithmic depth, i.e., $\text{NC}^1$. We note that, while establishing several containments of classical and quantum complexity classes, this does not directly provide a separation between constant-depth quantum and constant-depth classical circuits (in the sense of separating $\text{QNC}^0$ from e.g., $\text{NC}^0$) because interactive protocols are studied (i.e., these results separate suitably defined complexity involving interaction) We emphasize that – although their focus of this work is different from ours – some of the key ideas developed in \cite{grier2018quantum} are central to our derivation.

Let us briefly comment on the relation of our work to \cite{grier2018quantum}, where an oracle $\tilde{R}$ for the simulation of a certain interactive process is considered. The reduction in \cite{grier2018quantum} assumes that the oracle $\tilde{R}$ provides so-called \textit{rewind} access. This type of reduction does not treat $\tilde{R}$ as a black box but allows to rewind the classical simulator to an earlier state of the computation, change variables and continue the simulation. Notice that this is a natural requirement for a classical machine (i.e. for $\text{NC}^0, \text{AC}^0$ or Turing machines it is implemented by the same class) but is generally impossible in a quantum computational model involving non-unitary dynamics such as measurements.

In more detail, the simulated interactive process considered in \cite{grier2018quantum} is conceptually similar to a Bell experiment: In the first stage, given an $L$-tuple $(C_1, \ldots, C_L)$ of Clifford group elements, the stabilizer state $P(C_1, \ldots, C_L, m) (C_L \cdots C_1 | \Psi_0) = | \Phi \rangle$ is prepared by a constant-depth quantum circuit using measurement, where the Pauli "correction" $P(C_1, \ldots, C_L, m)$ is determined by the Cliffords $(C_1, \ldots, C_L)$ and the measurement results $m$. This first stage corresponds to preparing an initial entangled state for a Bell experiment. In the second stage of the process, an input is provided according to which a suitable Pauli measurement is applied to the prepared state. Without loss of generality, the input takes the form of a Clifford unitary $D$, and the executed measurement is the von Neumann measurement using the basis $\{ D | z \}_z$. This second step corresponds to the execution of a Bell measurement on the prepared state. The work \cite{grier2018quantum} considers simulation problems associated with different interactive protocols of this kind for $k$-qubit Clifford circuits, embedded into so-called graph state simulation problems.

Rewind access to an oracle $\tilde{R}$ simulating such an interactive process means that oracle queries to $\tilde{R}$ can be repeated in such a way that it yields an identical output $m$ for the first step. In
other words, the classical machine can be rewound to the beginning of the second stage. In contrast, reproducing this behavior using a quantum protocol would require post-selecting on fixed measurement outcomes occurring with exponentially small probability. The ability of a classical simulator to rewind is at the root of the hardness results of [15] concerning interactive quantum advantage.

Our work removes the need for interaction – we do not require two distinct stages but consider a relation problem. We also do not need to assume rewind access to the classical oracle ˜R. This is achieved by exploiting the internal structure of the classical simulator: We consider oracles realized by circuits and provide a non-black-box reduction. As we discuss below (see Section 3.2), however, our arguments heavily rely on some of the techniques of [15] developed in the interactive scenario.

Outlook

Due to the simplicity of the circuit Telep, our work provides a starting point for a fault-tolerant 3D-local quantum advantage scheme based on the techniques of [7]. A natural open question is whether the Telep-circuit also provides a quantum advantage over AC0, with a similar separation as the (up to the best of our knowledge) strongest known result [26]. More generally, it is unclear whether or not quantum advantage schemes against AC0 are feasible based on geometrically 1D-local quantum circuits only.

2 Locality properties of classical circuits

In this section we identify certain no-signaling properties every NC0 circuit has. These are crucial in our non-black-box reduction from the simulation problem PSIM[CliffP(Q)] to the problem PSIM[Telep] in Section 3.

In Sections 2.1 and 2.2 we define lightcones and locality of functions and NC0 circuits. In Section 2.3 we use combinatorial arguments to prove no-signaling properties of NC0.

2.1 Locality of functions

In the following, we consider functions

\[ f : \{(0,1)^k\}^n \rightarrow \{(0,1)^r\}^s \]

\[ I \mapsto f(I) = (f_1(I), \ldots, f_s(I)) \]

(3)

taking an “input”-n-tuple I of k-bit strings to an “output”-s-tuple f(I) of r-bit strings. We often write I = (I1, \ldots, In) for the input variables and O = (O1, \ldots, Os) for the output variables, where Om = f_m(I) for m \in \{1, \ldots, s\}. We call each of the k-bit-strings making up an input I a block. That is, the j-th block of the input consists of the k binary variables I_j = (I_j(1), \ldots, I_j(k)) of the argument, where j \in \{1, \ldots, n\}. Similarly, the m-th block of the output is associated with O_m = (O_m(1), \ldots, O_m(r)) for m \in \{1, \ldots, s\}.

While a function of the form (3) can be regarded as a function

\[ f : \{0,1\}^N \rightarrow \{0,1\}^M \]

(4)

with N = k \cdot n and M = r \cdot s, i.e., as a function taking N-bit strings to M-bit strings, the block-structure (3) will also be important in the following. To distinguish between the two interpretations of a function f of the form (3), we refer to (4) as the function f with the block-structure ignored. Note also that for k = r = 1, there is no difference between these two
notions. In particular, the following definitions apply to any function \( f : \{0, 1\}^N \to \{0, 1\}^M \) taking \( N \)-bit strings to \( M \)-bit strings, for any \( N, M \in \mathbb{N} \).

For a function \( f \) of the form (3) we say that the \( k \)-th output block \( O_k \) is influenced by the \( j \)-th input \( I_j \) if and only if the output block \( O_k \) depends non-trivially on the value of the \( j \)-th input block, i.e., if there is a choice of \( (I_1, \ldots, I_{j-1}, I_{j+1}, \ldots, I_n) \in (\{0, 1\}^k)^{n-1} \) and \( J, J' \in \{0, 1\}^k \) such that

\[
 f_k(I_1, \ldots, I_{j-1}, J, I_{j+1}, \ldots, I_n) \neq f_k(I_1, \ldots, I_{j-1}, J', I_{j+1}, \ldots, I_n) .
\]

In the following, we will often identify input- and output-blocks with their indices. Correspondingly, we define the forward lightcone \( \mathcal{L}^+ (j) \) of the \( j \)-th input block \( I_j \) as the set of all output blocks that are influenced by it, i.e.,

\[
 \mathcal{L}^+ (j) := \{ k \in [s] \mid O_k \text{ is influenced by } I_j \} .
\]

Similarly, the backward lightcone \( \mathcal{L}^- (k) \) of the \( k \)-th output block \( O_k \) is the set of all input blocks that influence it,

\[
 \mathcal{L}^- (k) := \{ j \in [n] \mid O_k \text{ is influenced by } I_j \} .
\]

A function \( f \) is said to be \( \ell \)-local if the size of all backward lightcones is bounded by \( \ell \), i.e., if

\[
 |\mathcal{L}^- (k)| \leq \ell \quad \text{for all } k \in [s] .
\]

Lightcones naturally extend to a subset \( \mathcal{J} \) of either input and output blocks: for \( \mathcal{J}_{in} \subset [n] \) and \( \mathcal{J}_{out} \subset [s] \), we set

\[
 \mathcal{L}^+ (\mathcal{J}_{in}) = \bigcup_{j \in \mathcal{J}_{in}} \mathcal{L}^+ (j), \quad \text{and} \quad \mathcal{L}^- (\mathcal{J}_{out}) = \bigcup_{j \in \mathcal{J}_{out}} \mathcal{L}^- (j) .
\]

For a subset \( \{ I_j \}_{j \in \mathcal{J}} \) of input blocks (specified by a subset \( \mathcal{J} = \{ j_1, \ldots, j_\ell \} \subset [n] \)), we say that an output block \( O_k \) is determined by \( \{ I_j \}_{j \in \mathcal{J}} \) if it is only a function of the corresponding input variables, i.e., if there is a function \( g : (\{0, 1\}^\ell) \to \{0, 1\}^r \) such that

\[
 f_k(I) = g(I_{j_1}, \ldots, I_{j_\ell}) \quad \text{for all } I = (I_1, \ldots, I_n) \in (\{0, 1\}^k)^n .
\]

### 2.2 Locality of circuits

Note that for circuit \( C : \{0, 1\}^N \to \{0, 1\}^M \) of depth \( d \), the maximal fan-in \( \nu \) of the gates in the circuit sets an upper bound on the size of all forward lightcones: we have

\[
 |\mathcal{L}^+ (j)| \leq \nu^d \quad \text{for all } j \in [N] .
\]

In particular, a depth-\( d \) circuit with gates of fan-in upper bounded by \( \nu \) is \( \ell \)-local with

\[
 \ell \leq \nu^d .
\]

Similar considerations apply to every “block” function \( C : (\{0, 1\}^k)^n \to (\{0, 1\}^r)^s \) that is realized by a circuit \( C \). If each gate has fan-in upper bounded by \( \nu \) (i.e., has \( \nu \) bits as output), then each input bit can influence at most \( \nu^d \) output bits. Thus each input bit can influence at most \( \nu^d \) blocks. Since every input block consists of \( k \) bits, it follows that

\[
 |\mathcal{L}^+ (j)| \leq k \nu^d \quad \text{for all } j \in [n] ,
\]

(5) i.e., the circuit is \( \ell \)-(block)-local with \( \ell \leq k \nu^d \).

In the following, we often consider families of functions

\[
 \{ f_n : (\{0, 1\}^k)^n \to (\{0, 1\}^r)^{\nu(n)} \}_{n \in \mathbb{N}}
\]

where \( k, r \) are constant and \( \nu(n) = \text{poly}(n) \) is polynomial in \( n \). Eq. (5) implies that if such a function family is represented by an \( \text{NC}^0 \)-circuit, then it has constant locality, i.e., it is \( O(1) \)-local.
We establish no-signaling properties for a specific class of NC\(^0\)-circuits that have the same number of input and output blocks. This is exactly the type of circuits we want to analyze for the PSIM [Telep] problem. Let C: \((\{0,1\}^{k})^n \rightarrow (\{0,1\})^n\) be a function. That is, C takes as input \((I_1, \ldots, I_n) \in (\{0,1\}^k)^n\) and outputs \((O_1, \ldots, O_n) \in (\{0,1\})^n\). We assume that the circuit C has (block-)locality \(\ell\).

Let \(L \in [n]\). We say that \(m \in [n]\) is \(L\)-good if and only if
\[
\mathcal{L}^{-}\left(\left\{ m \right\}\right) \cap [L] = \emptyset.
\]
That is, an input index \(m \in [n]\) is \(L\)-good if and only if the input block \(I_m\) does not influence the first \(L\) output blocks \((O_1, \ldots, O_L)\). We illustrate the \(L\)-good property in Figure 3a. Let us denote by \(\text{Good}(L) \subset [n]\) the subset of \(m \in [n]\) such that \(m\) is \(L\)-good. We also set \(\text{Bad}(L) := [n] \setminus \text{Good}(L)\).

In a similar fashion, let us say that the \(m\)-th input block \(I_m\) has limited signaling if and only if
\[
|\mathcal{L}^{-}(m)| \leq 8\ell ,
\]
i.e., if and only if \(I_m\) influences at most \(8\ell\) output blocks. (The choice of factor 8 is for convenience only.) We depict the limited signaling property in Figure 3b. Let us denote the set of limited-signaling input indices by \(\text{LimitedSignaling} \subset [n]\).

Our goal in this section is to show that for a suitably chosen value of \(L\), there are indices \(m \in [n]\) such that
\begin{enumerate}
  \item \(m\) is \(L\)-good,
\end{enumerate}
(ii) $m$ has the limited signaling property, and

(iii) $m > n/2$ (i.e., $m$ belongs to the “second half” of input indices).

To establish this statement, we first show a lower bound on the number of indices $m \in [n]$ satisfying (ii) and (iii). The following notation will be convenient: Consider a subset $\mathcal{G} \subset [n]$ of input block indices. We will often partition such a set as

\[
\mathcal{G} = \mathcal{G}_\leq \cup \mathcal{G}_> \quad \text{where} \quad \mathcal{G}_\leq = \{ m \in \mathcal{G} \mid m \leq n/2 \} \quad \text{and} \quad \mathcal{G}_> = \{ m \in \mathcal{G} \mid m > n/2 \} .
\]

**Lemma 2.1.** Set

\[
L := \left\lfloor \frac{n}{4\ell} \right\rfloor .
\]

Then

\[
|\text{Good}(L)_>| \geq \frac{n}{2} \cdot \frac{1}{2} .
\]

**Proof.** By definition of the set $\text{Bad}(L)_>$, we have

\[
\mathcal{L}^\rightarrow(m) \cap [L] \neq \emptyset \quad \text{for every} \quad m \in \text{Bad}(L)_> .
\]  

By the definition of the forward and backward lightcones, condition (6) is equivalent to

\[
m \in \mathcal{L}^\rightarrow([L]) \quad \text{for every} \quad m \in \text{Bad}(L)_> ,
\]

that is,

\[
\text{Bad}(L)_> \subset \mathcal{L}^\rightarrow([L]) .
\]

It follows that

\[
|\text{Bad}(L)_>| \leq |\mathcal{L}^\rightarrow([L])| .
\]  

By the locality-$\ell$ assumption on $C$, we have

\[
|\mathcal{L}^\rightarrow(j)| \leq \ell \quad \text{for all} \quad j \in [n]
\]

and thus

\[
|\mathcal{L}^\rightarrow([L])| \leq L\ell .
\]  

Combining (8) with (7) yields

\[
|\text{Bad}(L)_>| \leq L\ell .
\]

or equivalently

\[
|\text{Good}(L)_>| \geq \lfloor n/2 \rfloor - L\ell \geq n/4
\]

by the definition of $L$. 

We next derive a lower bound on the number of indices $m \in [n]$ satisfying (ii) and (iii). The statement and proof of the following result are analogous to [8, Claim 5].
Lemma 2.2. We have

$$|\text{LimitedSignaling}_>| \geq \frac{3n}{8} = \frac{n}{2} \cdot \frac{3}{4}.$$  

Proof. Consider a bipartite graph $G = (V_I \cup V_O, E)$ with two sets of vertices

$$V_I = \{v_i \mid i \in [n]_>\}$$
$$V_O = \{w_o \mid o \in [n]\},$$

where $(v_i, w_o) \in V_I \times V_O$ are connected by an edge if and only if

$$i \in \mathcal{L}^{-}(o). \tag{9}$$

Observe that (9) is equivalent to

$$o \in \mathcal{L}^{-}(i). \tag{10}$$

Using (9), the set of edges of this graph can be upper bounded by

$$|E| \leq \sum_{o=1}^{n} |\mathcal{L}^{-}(o)| \leq n\ell. \tag{11}$$

On the other hand, using the definition of the forward lightcone and (10), we have that for any $m \in \{\lfloor n/2 + 1 \rfloor, \ldots, n\} \setminus \text{LimitedSignaling}$, the subset $\{w_o \mid o \in \mathcal{L}^{-}(m)\} \subset V_O$ has more than $8\ell$ neighbors in $V_I$. This gives the lower bound

$$|E| \geq \{\lfloor n/2 + 1 \rfloor, \ldots, n\} \setminus \text{LimitedSignaling}_> | \cdot 8\ell \tag{12}$$

$$= (\lfloor n/2 \rfloor - |\text{LimitedSignaling}_>|) \cdot 8\ell.$$  

Combining (12) with (11) gives the claim. \qed

Finally, we can combine both properties to show that there is a large number of indices $m \in [n]$ with $m > n/2$ (property (iii)) that are $L$-good (cf. (i)) and have the limited-signaling property (cf. (ii)).

Theorem 2.3. Let $C$ be a circuit with locality upper bounded by $\ell$. Set

$$L := \left\lfloor \frac{n}{4\ell} \right\rfloor.$$  

Then there is some input index $m \in \{\lfloor n/2 + 1 \rfloor, \ldots, n\}$ that is both $L$-good and has limited signaling, i.e.,

$$\mathcal{L}^{-}(m) \cap [L] = \emptyset$$

and

$$|\mathcal{L}^{-}(m)| \leq 8\ell.$$  

Proof. This follows immediately by combining Lemma 2.1 with Lemma 2.2. Indeed, we have

$$|\text{Good}(L)_>| \geq (n/2) \cdot (1/2)$$

by Lemma 2.1 and

$$|\text{LimitedSignaling}_>| \geq (n/2) \cdot (3/4)$$

by Lemma 2.2. But both $\text{Good}(L)_>$ and $\text{LimitedSignaling}_>$ are subsets of $[n]_>$. thus we must have

$$|\text{Good}(L)_> \cap \text{LimitedSignaling}_>| \geq (n/2) \cdot (1/4).$$

This implies the claim. \qed
3 Quantum advantage against NC\(^0\)

Our goal is to show that PSIM [Telep] \(\not\in NC^0/poly\), i.e., constant-depth quantum circuits have an advantage over (even non-uniform) \(NC^0\) for the possibilistic simulation of the gate-teleportation circuit. Our proof is by contradiction: we show the implication

\[
(PSIM [Telep] \in NC^0/poly) \implies (PARITY \in AC^0/poly).
\]

This implication shows that PSIM [Telep] \(\not\in NC^0/poly\) since we would otherwise get a contradiction to the well-known lower-bound \(PARITY \not\in AC^0/poly\) on the PARITY problem (see e.g. [16]).

In Section 3.1.1, we prove that any \(NC^0\) algorithm for the PSIM [Telep] problem can be used to construct a non-uniform \(NC^0\) algorithm for the PSIM [CliffP(Q)\(_L\)] problem for \(L = \Theta(n)\) and a fixed function \(Q\). In Section 3.2, we follow and specialize the techniques of [15] to obtain an \(AC^0/poly\) algorithm for the possibilistic simulation of the gate-teleportation circuit \(Telep\) to learn some information about the stabilizer group of states of the form \((I \otimes C_L \cdots C_1)\Phi\), where \(\Phi\) is a Bell state. Finally, in Section 3.3, we use this learning problem to solve PARITY. This establishes (13).

Since the teleportation circuit \(Telep\) trivially solves the PSIM [Telep] problem, this completes the proof of Theorem 1.1, i.e., establishes a quantum advantage over \(NC^0\).

3.1 Relating PSIM [Telep] to PSIM [CliffP(Q)]

Here we argue that an \(NC^0\)-circuit \(C\) which possibilistically simulates the gate-teleportation circuit \(Telep_n\) gives rise to a non-uniform \(NC^0\)-circuit (using oracle access to \(C\)) which simulates \(CliffP(Q)\(_L\)\) for \(L = \Theta(n)\) and a suitably chosen function \(Q\). That is, we show the implication

\[
(PSIM [Telep] \in NC^0/poly) \implies (\exists Q: PSIM [CliffP(Q)] \in NC^0/poly).
\]

3.1.1 Motivation

To motivate the construction, recall what the effect of the (concurrent) Bell measurements in a gate-teleportation circuit is: It essentially amounts to the application of a product \(C_n \cdots C_1\) of Clifford group elements to one half of a maximally entangled state, but the resulting state \((I \otimes C_n \cdots C_1)\Phi\) is corrupted by a Pauli error \(\widetilde{Q}\) determined by the measurement outcomes. In standard uses of gate teleportation (such as in error correction), this Pauli error is computed and applied in a correction step. In contrast, the circuit \(Telep_n\) does not incorporate such a correction step. We note that the computation of \(\widetilde{Q}\) from the measurement outcomes is not feasible using, e.g., an \(NC^0\) or an \(AC^0\)-circuit.

Our construction avoids the necessity of computing Pauli corrections such as \(\widetilde{Q}\) by exploiting the fact that the correction function \(Q\) in the definition of the circuit \(CliffP(Q)\(_L\)\) can be chosen arbitrarily (without computational complexity considerations). Very roughly, the circuit \(Telep_n\) emulates the circuit \(CliffP(\widetilde{Q})\(_L\)\) for suitably chosen \(L\) and \(\widetilde{Q}\).

In more detail, let us examine the relationship between the quantum circuits for \(Telep_n\) and \(CliffP(Q)\(_L\)\). The output distribution of \(Telep_n\) is given by

\[
p^{Telep_n}(P_0, \ldots, P_{n-1} | C_0, \ldots, C_{n-1}) = \frac{1}{2^{n-1}} \text{Tr} \left( P_{n-1} C_{n-1} \cdots P_1 C_1 P_0 C_0 \right),
\]

which can be written as

\[
p^{Telep_n}(P_0, \ldots, P_{n-1} | C_0, \ldots, C_{n-1}) = \frac{1}{2^{n-1}} \text{Tr} \left( (P_{n-1} C_{n-1})(P_{n-2} C_{n-2} \cdots P_1 C_1 P_0 C_0) \right)
\]

\[
= \frac{1}{2^{n-1}} \text{Tr} \left( (P_{n-1} C_{n-1}) \widetilde{Q} C_{n-2} \cdots C_0 \right),
\]

(15)
where the Pauli operator

\[ \tilde{Q} = \tilde{Q}(C_0, \ldots, C_{n-2}, P_0, P_1, \ldots, P_{n-2}) \]

is obtained by commuting all the Pauli operators in the product \( P_{n-2}C_{n-2} \cdots P_0C_0 \) to the left, i.e.,

\[ \tilde{Q} \prod_{i=0}^{n-2} C_i = P_{n-2}C_{n-2} \cdots P_0C_0 . \]

On the other hand, the output distribution of CliffP\((Q)_L\) is

\[
p^{\text{CliffP}(Q)_L}(P \mid C_1, \ldots, C_L, D) = | \langle \Phi | I_A \otimes (DQ(C_1, \ldots, C_L))(C_L \cdots C_1) | \Phi \rangle |^2
\]

which can be written as

\[
p^{\text{CliffP}(Q)_L}(P \mid C_1, \ldots, C_L, D) = \frac{1}{2} \text{tr} (PDQC_L \cdots C_1) , \quad \text{where} \quad Q = Q(C_1, \ldots, C_L) . \tag{17}
\]

Comparing (15) and (17), setting \( n := L + 1 \) and

\[
C'_j := \begin{cases} 
C_{j+1} & \text{for } j = 0, \ldots, n - 2 \\
D & \text{for } j = n - 1
\end{cases}
\tag{18}
\]

we have the implication

\[
p^{\text{Telep}_n}(P_0, \ldots, P_{n-1} \mid C'_0, \ldots, C'_{n-1}) > 0 \quad \Rightarrow \quad \tilde{p}(P \mid C_1, \ldots, C_L, D) > 0 , \tag{19}
\]

where we have written \( P \) for \( P_{n-1} \), and where the distribution \( \tilde{p} \) is defined as in (17) but with the Pauli operator \( Q(C_1, \ldots, C_L) \) replaced by \( \tilde{Q}(C_1, \ldots, C_L, P_0, \ldots, P_{L-1}) \), that is,

\[
\tilde{p}(P \mid C_1, \ldots, C_L, D) := \frac{1}{2} \text{tr} (PD\tilde{Q}C_L \cdots C_1) \quad \tilde{Q} = \tilde{Q}(C_1, \ldots, C_L, P_0, \ldots, P_{L-1}) . \tag{20}
\]

Eq. (19) and comparing (20) with (17) suggests that a possibilistic simulation of \( \text{Telep}_n \) can be used to possibilistically simulate CliffP\((Q)_L\) for a suitably chosen function \( Q \). However, the fact that \( \tilde{Q} \) also depends on \( P_0, \ldots, P_{L-1} \) needs to be addressed – the expression (20) differs from the output distribution (17) of CliffP\((Q)_L\). This is where no-signaling considerations enter the argument.

### 3.1.2 An algorithm for PSIM [CliffP\((Q)\)]

Suppose \( C : \mathcal{C}^n \to \mathcal{P}^n \) is a deterministic oracle (i.e., a function) that solves PSIM [\( \text{Telep}_n \)]. In this section, we argue that query access to \( C \) allows to solve the problem PSIM [CliffP\((Q)_L\)] for some fixed function \( Q \), assuming that \( C \) has certain locality properties we detail below. Moreover, this property matches exactly what we established in Section 2.3 for NC\(^0\)-circuits. The main idea behind the reduction is identical to that explained in the previous section: We embed instances of PSIM [CliffP\((Q)_L\)] into PSIM [\( \text{Telep}_n \)]. However, we need to deal with some amount of signaling since the \( Q \) in Eq. (20) could be indirectly influenced by \( D \) through the Pauli operators \( (P_0, \ldots, P_{L-1}) \). Therefore, contrary to the choice of embedding (18), our embeddings use longer sequences (i.e., large values of \( n \)), essentially “padding” with identity operators.

To define these embeddings, suppose that there are \( L \) and \( m \) such that \( L \leq m < n \), which satisfy properties illustrated in Figure 4.
We assume the input $C'_m := D$ cannot signal to outputs $P_0, \ldots, P_{L-1}$ and can only signal to a constant $k$ number of outputs with indices from the set $J$. We depict the outputs influenced by the input $C'_m$ in grey.

We embed an instance $(C_1, \ldots, C_L, D) \in C_{L+1}$ of $\text{PSIM}[\text{CliffP}(Q)_L]$ (where $Q : C^L \to \mathcal{P}$ can be any fixed function) into an instance $(C'_0, \ldots, C'_{n-1}) \in C^n$ of $\text{PSIM}[\text{Telep}_n]$ as follows: We set

$$C'_j = \begin{cases} C_{j+1} & \text{for } j = 0, \ldots, L - 1, \\ I & \text{for } j \geq L, j \neq m, \\ D & \text{for } j = m. \end{cases} \quad (21)$$

To examine the result of this embedding, we need to consider the output probabilities for an embedded instance: We have

$$p^{\text{Telep}_n}(P_0, \ldots, P_{n-1} \mid C'_0, \ldots, C'_{n-1}) = \frac{1}{2^{n-1}} \text{tr} \left( \left( \prod_{r=m}^{n-1} P_r \right) D \left( \prod_{s=L}^{m-1} P_s \right) \left( \prod_{t=1}^{L} (P_{t-1} C_t) \right) \right). \quad (22)$$

We display the trace defining the probability $p^{\text{Telep}_n}(P_0, \ldots, P_{n-1} \mid C'_0, \ldots, C'_{n-1})$ as well as the same trace after we inserted variables from the embedding (i.e. the right-hand side of Eq. (22)) in tensor network diagrammatic representation in Figure 5.

We need to relate this to the output probability $p^{\text{CliffP}(Q)_L}(P \mid C_1, \ldots, C_L, D)$ of the circuit $\text{CliffP}(Q)_L$, see Eq. (17). It is easy to see (using commutation relations) that expression (22) can indeed be related to $p^{\text{CliffP}(Q)_L}(P \mid C_1, \ldots, C_L, D)$ for suitable choices of $Q$ and $P$. The less obvious a priori is that, when going from $(P_0, \ldots, P_{n-1})$ to $(Q, P)$, the dependence on the Paulis $(P_{j_1}, \ldots, P_{j_k})$ for a fixed subset $J := \{j_1 < \cdots < j_k\}$ of positions can be limited to $P$ (whereas $Q$ can be chosen not to depend on these Paulis). This is a consequence of the cyclic structure of the circuit $\text{Telep}_n$. We formalize it in the following Lemma 3.1.
(a) Tensor network representation of the trace defining $p_{\text{Telep}}(P_0, \ldots, P_{n-1}|C'_0, \ldots, C'_{n-1})$, from Eq. (14). We depict Clifford unitaries by squares and Pauli operators by circles. The output Pauli operators influenced by the input $C'_m$ are illustrated in grey.

(b) The trace defining the output probability distribution of $\text{Telep}_n$ after we insert the inputs from Eq. (21) (also later in Algorithm 2 after the lines (2)-(8)).

Figure 5: Tensor network diagrammatic representation of (a) the trace that defines $p_{\text{Telep}}(P_0, \ldots, P_{n-1}|C'_0, \ldots, C'_{n-1})$, from Eq. (14), and (b) the same trace after we inserted variables from the embedding (21).

**Lemma 3.1.** Let $L \leq m < n$ be integers. Let

$$\mathcal{J} = \{j_1 < \cdots < j_k\}$$

be a subset of $\{L, L+1, \ldots, n-1\}$. Then there are two functions

$$P : \mathcal{P}^k \times \mathcal{C} \to \mathcal{P}$$
$$Q : \mathcal{P}^{n-k} \times \mathcal{C}^L \to \mathcal{P}$$

such that

$$\left| \mathrm{tr} \left( \left( \prod_{r=m}^{n-1} P_r \right) D \left( \prod_{s=L}^{m-1} P_s \right) \left( \prod_{t=1}^{L} (P_{t-1}C_t) \right) \right) \right| = \left| \mathrm{tr} \left( PDQ \prod_{j=1}^{L} C_j \right) \right|$$

for all $(P_0, \ldots, P_{n-1}) \in \mathcal{P}^n$ and $(C_1, \ldots, C_L, D) \in \mathcal{C}^{L+1}$, where

$$P = P \left( (P_j)_{j \in \mathcal{J}} ; D \right)$$
$$Q = Q \left( (P_j)_{j \notin \mathcal{J}} , C_1, \ldots, C_L \right) .$$

Here we write $(P_j)_{j \in \mathcal{J}}$ for $(P_{j_1}, \ldots, P_{j_k})$ (and similarly for $(P_j)_{j \notin \mathcal{J}}$).

We illustrate each step in the proof of Lemma 3.1 in the tensor network diagrammatic representation for clarity. The left-hand side of Eq. (23) is shown in Figure 5b.

**Proof.** Let $(P_0, \ldots, P_{n-1}) \in \mathcal{P}^n$ and $(C_1, \ldots, C_L, D) \in \mathcal{C}^{L+1}$ be arbitrary. Define

$$R := \left( \prod_{r=m}^{n-1} P_r \right) \quad \text{and} \quad S := \left( \prod_{s=L}^{m-1} P_s \right) .$$
Then the quantity of interest is equal to
\[
\text{tr}\left( \left( \prod_{r=m}^{n-1} P_r \right) D \left( \prod_{s=L}^{m-1} P_s \right) \left( \prod_{t=1}^{L} (P_{t-1} C_t) \right) \right) = \text{tr}\left( RDS \left( \prod_{t=1}^{L} (P_{t-1} C_t) \right) \right). \tag{26}
\]
Since Pauli operators either commute or anticommute, we can factor the products of Pauli operators \( R \) and \( S \) into factors \( P_j \) with \( j \in \mathcal{J} \), and factors \( P_j \) with \( j \in \mathbb{Z}_n \setminus \mathcal{J} \). That is, we have
\[
R = \pm Q' P' \quad \text{and} \quad S = \pm P'' Q'', \tag{27}
\]
where
\[
P' := \prod_{r=m}^{n-1} \frac{P_{r}}{P_{r-\delta_{r}}}, \quad P'':= \prod_{s=L}^{m-1} \frac{P_{s}}{P_{s-\delta_{s}}}, \quad Q' := \frac{Q}{Q_{r-1}}, \quad Q'' := \frac{Q}{Q_{s-1}},
\tag{28}
\]
where \( \delta_{r} \in \{0, 1\} \) indicates whether or not \( r \in \mathcal{J} \), that is,
\[
\delta_{r} := \begin{cases} 1 & \text{if } r \in \mathcal{J} \\ 0 & \text{otherwise}. \end{cases}
\]
Inserting the factorization (27) into expression (26) we have
\[
\pm \text{tr}\left( RDS \left( \prod_{t=1}^{L} (P_{t-1} C_t) \right) \right) = \text{tr}\left( (Q' P') D (P'' Q'') \left( \prod_{k=1}^{L} (P_{k-1} C_k) \right) \right)
= \text{tr}\left( Q' P' D P'' Q'' \left( \prod_{j=1}^{L} C_j \right) \right).
\]
Here we commuted the Pauli operators \( P_0, \ldots, P_{L-1} \) in the product \( \prod_{t=1}^{L} (P_{t-1} C_t) \) to the left, combining them into a Pauli operator \( Q'' \in \mathcal{P} \) defined by
\[
Q'' := \left( \prod_{r=1}^{L} (P_{r-1} C_r) \right) \left( \prod_{r=1}^{L} C_r \right)^{-1}.
\]
At this point, we use the cyclicity of the trace to move \( Q' \) to the right of the product in the argument, obtaining
\[
\text{tr}\left( Q' P' D P'' Q'' \left( \prod_{j=1}^{L} C_j \right) \right) = \text{tr}\left( P' D P'' Q'' \left( \prod_{j=1}^{L} C_j \right) Q' \right)
= \text{tr}\left( P' D P'' Q \left( \prod_{j=1}^{L} C_j \right) \right), \tag{29}
\]
where we defined
\[
Q := Q'' \left( \prod_{j=1}^{L} C_j \right) Q' \left( \prod_{j=1}^{L} C_j \right)^{-1}. \tag{30}
\]
Using the cyclicity of the trace to move \( Q' \) to the right (instead of commuting it past \( P' D P'' \)) here ensures that the resulting operator \( Q \) has no dependence on \( D \). Indeed, it is clear from the definition of \( Q', Q'', Q''' \) and (30) that \( Q \) is a function of \( (P_j)_{j \in \mathcal{J}} \) and \( (C_1, \ldots, C_L) \) only as claimed, see Eq. (25). We illustrate these steps (commuting Paulis) in Figure 6.
Figure 6: The use of the cyclic property of trace to commute Pauli operators not influenced by the input $D$ in a way that does not create additional dependence of the resulting Pauli operator on $D$. This figure illustrates the steps from the Eq. (26) to Eq. (29).

We can rewrite (29) as

$$\text{tr} \left( P' D P'' Q \left( \prod_{j=1}^{L} C_j \right) \right) = \text{tr} \left( P D Q \left( \prod_{j=1}^{L} C_j \right) \right)$$

(31)

by defining

$$P := P' D P'' D^{-1}. \quad (32)$$

It is easy, to check using the definition of $P'$ and $P''$ and (32) that $P$ is a function of only $(P_j)_{j \in J}$ and $D$ as claimed (see Eq. (24)). We illustrate the computation of $P$ in Figure 7.

Figure 7: In this step we commute all Pauli operators depending on $D$ to the left of it. This illustrates the Eq. (31) and the step (4) of Algorithm 1.

This proves the lemma since we have shown that (23) is satisfied with these definitions, and we can assume without loss of generality that $P, Q \in P$ (as (23) includes absolute values). The resulting trace is depicted in Figure 8.

Figure 8: The resulting trace defining the probability $p^\text{CliffP}(Q)_L(P | C_1, \ldots, C_L, D)$ and therefore the solution to PSIM $[\text{CliffP}(Q)]$ problem. This illustrates the trace from Eq. (17) to which we arrive at Eq. (31).

In the following, we will use that $P$ only depends on the subsets of Paulis $(P_j)_{j \in J}$ with indices belonging to $J$ and the Clifford $D$ as expressed by Eq. (24). In fact, the Pauli $P$ can
Algorithm 1 Algorithm defining the function $A_{J,m} : \mathcal{P} \times \mathcal{C} \rightarrow \mathcal{P}$. Here $J \subset \mathbb{Z}_m$ is a subset and $m \in \mathbb{Z}_n$.

1: function $A_{J,m}(P_0, \ldots, P_{n-1}, D)$
2: $P' \leftarrow \prod_{p=m}^{n-1} P_p^{\delta_p}$ where $\delta_p = 1$ if $p \in J$ and $\delta_p = 0$ otherwise
3: $P'' \leftarrow \prod_{q=1}^{m-1} P_q^{\delta_q}$
4: $P \leftarrow P'DP''D^{-1}$.
5: Ignore (remove) the global phase to ensure that $P \in \mathcal{P}$.
6: return $P$

Easily be computed: The proof of Lemma 3.1 also shows that the algorithm $A_{J,m}$ (Algorithm 1) computes this function, i.e., we have

$$A_{J,m}(P_0, \ldots, P_{n-1}, D) = P \left( (P_j)_{j \in J}, D \right) \text{ for all } (P_0, \ldots, P_{n-1}) \in \mathcal{P}^n \text{ and } D \in \mathcal{C} . \quad (33)$$

Importantly, the algorithm $A_{J,m}$ can be implemented efficiently.

Lemma 3.2. Let $k$ be a constant, let $L = \text{poly}(n) < n$ and $m$ be such that $L \leq m < n$. Let $J \subset \mathbb{Z}_m$ be of size $|J| = k$. Then Algorithm 1 for computing the function $A_{J,m} : \mathcal{P}^n \times \mathcal{C} \rightarrow \mathcal{P}$ can be implemented by an $\text{NC}^0$-circuit.

Proof. This follows immediately from equations (28) and (32). Since $k$ is constant, the corresponding products can be computed by an $\text{NC}^0$-circuit, see Lemma 3.1 \hfill \Box

Algorithm 2 The algorithm $B_{L,m,J}^C$, which takes as input an $L+1$-tuple $(C_1, \ldots, C_L, D) \in \mathcal{C}^{L+1}$ of Clifford group elements. It uses an oracle $C$ for the PSIM $[\text{Telep}_n]$ problem. For a suitable choice of $(L, m, J)$, the output $P$ of $B_{L,m,J}^C$ is a valid solution of PSIM $[\text{CliffP}(Q)_L]$ for a certain function $Q : \mathcal{C}^L \rightarrow \mathcal{P}$, see Lemma 3.3.

1: function $B_{L,m,J}^C(C_1, \ldots, C_L, D)$
2: for $j = 0, \ldots, L-1$ do
3: \hspace{1cm} $C'_j \leftarrow C_{j+1}$
4: for $j = L, \ldots, n-1$ do
5: \hspace{1cm} if $j \neq m$ then
6: \hspace{1.5cm} $C'_j \leftarrow I$
7: \hspace{1cm} else
8: \hspace{2cm} $C'_j \leftarrow D$
9: \hspace{1cm} $(P_0, \ldots, P_{n-1}) \leftarrow C(C'_0, \ldots, C'_{n-1})$
10: $P \leftarrow A_{J,m}(P_0, \ldots, P_{n-1}, D)$
11: return $P$

Our main result of this section combines Lemma 3.1 and Lemma 3.2 to show that an $\text{NC}^0$ circuit $C$ for PSIM $[\text{Telep}_n]$ can be used to solve PSIM $[\text{CliffP}(Q)_L]$. To arrive at this result, we make a suitable choice of $L$, $m$ and the subset $J \subset \mathbb{Z}_m$ depending on the oracle $C$ in non-black-box manner. More precisely, we use the no-signaling properties of $C$ established in Section 2.3. The resulting reduction is therefore non-uniform: While the algorithm requires oracle access to $C$, it is only correct if $(L, m, J)$ are appropriately chosen. The following theorem formalizes this and shows that a suitable choice always exists.

Lemma 3.3. Let $C : \mathcal{C}^n \rightarrow \mathcal{P}^n$ be a (possibly non-uniform) $\text{NC}^0$ circuit for PSIM $[\text{Telep}_n]$. Then there are $(L, m, J)$ with $L = \Theta(n)$, $L \leq m < n$ and $J \subset \mathbb{Z}_m$, as well as a function $Q : \mathcal{C}^L \rightarrow \mathcal{P}$ such that the algorithm $B_{L,m,J}^C$ (Algorithm 2) has the following properties:
(i) Given an arbitrary input \((C_1, \ldots, C_L, D) \in \mathcal{C}^{L+1}\), the output \(P = \mathcal{B}_{L,m,J}^c(C_1, \ldots, C_L, D)\) is a valid solution of \(\text{PSIM}[\text{CliffP}(Q)]\), associated with the instance \((C_1, \ldots, C_L, D)\).

(ii) The algorithm \(\mathcal{B}_{L,m,J}^c\) can be implemented by an \(\mathcal{NC}^0\)-circuit making one oracle query to \(C\).

**Proof.** Suppose that \(C: \mathcal{C}^n \rightarrow \mathcal{P}^n\) is an \(\mathcal{NC}^0\) circuit. Let us denote input- and output-variables of \(C\) by \((C'_0, \ldots, C'_{n-1})\) and \((P_0, \ldots, P_{n-1})\), respectively, i.e., we write

\[
(P_0, \ldots, P_{n-1}) = C(C'_0, \ldots, C'_{n-1}).
\]

We note that by definition, the algorithm \(\mathcal{B}_{L,m,J}^c\) uses the embedding \((21)\), i.e., for an instance \((C_1, \ldots, C_L, D)\) of \(\text{PSIM}[\text{CliffP}(Q)]\), it calls the oracle \(C\) with \((C'_0, \ldots, C'_{n-1})\) defined by \((21)\).

As a constant-depth circuit with bounded fan-in gates, the circuit \(C\) is \(\ell\)-local with locality bounded by \(\ell = O(1)\), see Eq. \((22)\). Let \(L := [n/(8\ell)]\). Then \(L = \Theta(n)\) by definition, and, as shown in Theorem \((2.3)\) there is an integer \(m \in \{\lfloor n/2 + 1 \rfloor, \ldots, n - 1\}\) and a subset \(J \subset \{L, \ldots, n - 1\}\) such that

(a) the collection of output variables \((P_j)_{j=0,\ldots,L-1}\) does not depend on the input variable \(C'_m\), i.e., \(\mathcal{L}^{-\rightarrow}(m) \cap \{0, \ldots, L-1\} = \emptyset\), and

(b) the input variable \(C'_m\) influences at most \(8\ell\) output variables \(\{P_j\}_{j \in J}\), i.e., we have \(|J| \leq 8\ell = O(1)\) for \(J = \mathcal{L}^{-\rightarrow}(m)\).

Statement (b) implies that the function \(A_{J,m}\) defined by Algorithm \((1)\) can be computed by an \(\mathcal{NC}^0\)-circuit from \((P_0, \ldots, P_{n-1})\), see Lemma \((3.2)\). Claim (ii) follows from this.

It remains to prove claim (i). Note that \(A_{J,m}\) computes the function

\[
P\left((P_j)_{j \in J}, D\right)
\]

appearing in Lemma \((3.1)\), see Eq. \((33)\). Furthermore, the Pauli

\[
Q = Q\left((P_j)_{j \notin J}, C_1, \ldots, C_L\right)\tag{34}
\]

in Lemma \((3.1)\) does not depend on \(C'_m\) by \((3)\).

Consider an output variable \(P_j\) with \(j \notin J\). By definition, \(P_j\) is a function of \((C'_j)_{j \in \mathbb{Z}_n \setminus \{m\}}\).

Because the embedding \((21)\) fixes all but the first \(L\) Cliffords \((C'_0, \ldots, C'_{L-1}) = (C_1, \ldots, C_L)\), and the \(m\)-th Clifford \(C'_m = D\) to the identity, we conclude that \(P_j\) (for \(j \notin J\)) is a function of \((C_1, \ldots, C_L)\) only. Combining this with \((34)\), we conclude that

\[
Q = Q(C_1, \ldots, C_L)
\]

is a function of \((C_1, \ldots, C_L)\) only as required in the definition of \(\text{CliffP}(Q)\).

With Lemma \((3.1)\) as well as expressions \((22)\) and \((17)\) for the probabilities \(p_{\text{Telep}}(P_0, \ldots, P_{n-1} | C'_0, \ldots, C'_{n-1})\) and \(p_{\text{CliffP}(Q)}(P | C_1, \ldots, C_L, D)\), we conclude that \(\mathcal{B}_{L,m,J}^c\) indeed produces a valid solution for the problem \(\text{PSIM}[\text{CliffP}(Q)]\), as claimed. \(\square\)

In the next section, we will use a deterministic oracle for \(\text{PSIM}[\text{CliffP}(Q)]\) to learn stabilizers of states \(I_A \otimes (C_L \cdots C_1)\Phi\).
3.2 From PSIM $\text{CliffP}(Q)_L$ to learning stabilizers of $I_A \otimes (C_L \cdots C_1)\Phi$

By definition, the output $P$ of the circuit $\text{CliffP}(Q)_L$ (see Fig. 2) can be interpreted as outputting the result of measuring the state

$$(I_A \otimes (C_L \cdots C_1))\Phi$$  \hspace{1cm} (35)$$

in a basis of maximally entangled states obtained by rotating the Bell basis by $I_A \otimes Q^D$. Here the effect of the Pauli $Q = Q(C_1, \ldots, C_L)$ is simply to possibly flip the measurement result (i.e., it can equivalently be achieved by classical post-processing). By choosing different inputs $(C_1, \ldots, C_L, D)$ with $(C_1, \ldots, C_L)$ fixed and varying over $D \in C$, a deterministic oracle $\mathcal{O}$ for the problem PSIM $[\text{CliffP}(Q)_L]$ thus essentially allows to simulate the effect of measuring the state (35) in different bases of stabilizer states. A protocol using such measurement results to determine the state (35) (or learn some information about it) therefore allows to determine (or learn some information about) the product $C_L \cdots C_1$. As we will discuss in Section 3.3, such information can in turn be used to solve the so-called word problem for the Clifford group of a single qubit.

In this section, we give a protocol for learning the stabilizers of the state (35) using a deterministic oracle $\mathcal{O}$ for PSIM $[\text{CliffP}(Q)_L]$. More precisely, the corresponding algorithm learns two distinct stabilizers generators of the state (35) up to signs, see Theorem 3.4 below. In other words, it finds two distinct (non-identity) Pauli operators that have expectation 1 or $-1$ in this state. Allowing for this remaining sign ambiguity has the key advantage that the function $Q(C_1, \ldots, C_L)$ does not need to be evaluated in the protocol.

The protocol borrows heavily from the pioneering work [15]. Indeed, Theorem 3.4 below is obtained by specializing the results of [15] to the problem at hand.

3.2.1 On the need for possibilistic state tomography

We emphasize that the problem of determining stabilizers of the state (35) using an oracle $\mathcal{O}$ for PSIM $[\text{CliffP}(Q)_L]$ cannot directly be solved using standard techniques for (stabilizer) quantum state tomography. This is because of the possibilistic nature of the simulation provided by $\mathcal{O}$. Accordingly, we refer to the problem under consideration as possibilistic tomography. To clarify the special difficulties associated with possibilistic tomography, consider a setting where several copies of an unknown stabilizer state $\Psi \in \mathbb{C}^2 \otimes \mathbb{C}^2$ can be generated/are available. This is the setting of standard tomography. Here a set $\{S_1, S_2\}$ of generators of the stabilizer group (up to a phase) of $\Psi$ can efficiently be determined by a sequence of measurements with respect to different Pauli bases, see e.g., [3,18].

One of the key achievements of [15] is to extend this stabilizer state tomography to the possibilistic setting, where the process of measuring a state $\Psi$ in a Pauli basis is replaced by an oracle $\mathcal{O}$ simulating the measurement. Remarkably, possibilistic simulation is sufficient, i.e., the simulated measurement results produced by $\mathcal{O}$ only need to lie in the support of the output distribution defined by Born’s rule for measurements. While this notion of simulation has been referred to as possibilistic in [25], the authors of [15] express this by saying that $\mathcal{O}$ can be adversarial.

3.2.2 Possibilistic state tomography from the magic square

The procedure of [15] reduces possibilistic stabilizer state tomography to possibilistic circuit simulation. Here we restate their result in a simplified and specialized form suitable for our purposes:

**Theorem 3.4 (Adapted from [15]).** Let $Q : C^L \to \mathcal{P}$ be arbitrary. Suppose $\mathcal{O}$ is a deterministic oracle (i.e., a function) solving PSIM $[\text{CliffP}(Q)_L]$. There is an algorithm $\mathcal{A}^\mathcal{O}$ with access to $\mathcal{O}$ with the following properties:
(i) \( A^O \) takes as input a sequence \((C_1, \ldots, C_L) \in C^L\).

(ii) The output of \( A^O \) is a pair \((S_1, S_2) \) of two distinct two-qubit Pauli operators of the form \( S_j = P_{A_j}^i \otimes P_{B_j}^i \) with \( P_{A_j}^i, P_{B_j}^i \in \mathcal{P} \setminus \{I\} \), for \( j = 1, 2 \).

(iii) Each operator \( S_j \) for \( j = 1, 2 \) has expectation value \( \{\pm 1\} \) in the state \((I_A \otimes C_L \cdots C_1) |\Phi\rangle\).

(iv) The algorithm \( A^O \) makes \( O(1) \) queries to \( O \).

(v) The algorithm \( A^O \) is realizable by a non-uniform \( \mathcal{AC}^0 \)-circuit.

In the remainder of this section, we outline the proof of Theorem 3.4. We first show that the oracle \( O \) can be used to learn a non-stabilizer \( M \) of the state \((I_A \otimes C_L \cdots C_1) |\Phi\rangle\). Here a non-stabilizer of a stabilizer state \( \Psi \) is a Pauli operator \( M \) such that \( \langle \Psi | M | \Phi \rangle = 0 \).

**Lemma 3.5.** Let \( O \) be a deterministic oracle for PSIM \([\text{CliffP}(Q)_L]\). Then there is an \( \mathcal{NC}^0 \)-algorithm \( N^O \) which takes as input an \( L \)-tuple \((C_1, \ldots, C_L) \in C^L\),

(i) makes \( 6 \) queries to \( O \), and

(ii) outputs a non-stabilizer \( M \) of the state \((I_A \otimes C_L \cdots C_1) |\Phi\rangle\).

The proof of Lemma 3.5 exploits the fact that measurements of the observables of the magic square illustrated in Fig. 9 cannot be explained by a non-contextual model.

**Proof.** Each row \( j \in [3] \) of the magic square in Fig. 9 consists of three commuting Pauli observables \((P_{j,1}, P_{j,2}, P_{j,3})\) which can be jointly measured, i.e., give rise to a measurement with outcomes \((m_{j,1}, m_{j,2}, m_{j,3}) \in \{\pm 1\}^3\) corresponding to the eigenvalues of the corresponding operators. Similarly, each column \( k \in [3] \) of the magic square defines a measurement with three (dependent) measurement outcomes \((m'_{1,k}, m'_{2,k}, m'_{3,k})\) associated with observables \((P_{1,k}, P_{2,k}, P_{3,k})\).

In total, the magic square thus defines six measurements, one for each row and column.

Suppose we have six copies of a two-qubit stabilizer state \( \Psi \). Using three copies to perform the measurements associated with each row gives a matrix \( m = (m_{j,k}) \in \text{Mat}_{3 \times 3}(\{\pm 1\}) \) of measurement outcomes. Using the remaining three copies to perform the measurements associated which each column similarly gives a matrix \( m' = (m'_{j,k}) \in \text{Mat}_{3 \times 3}(\{\pm 1\}) \). Because of the “magic square relations”

\[
\begin{align*}
P_{j,1}P_{j,2}P_{j,3} &= -I \quad \text{for every} \quad j \in [3] \\
P_{1,k}P_{2,k}P_{3,k} &= I \quad \text{for every} \quad k \in [3]
\end{align*}
\]

satisfied by the observables \( \{P_{j,k}\}_{j,k} \) in the magic square, we have

\[
\begin{align*}
m_{j,1}m_{j,2}m_{j,3} &= -1 \quad \text{for every} \quad j \in [3] \quad (36) \\
m'_{1,k}m'_{2,k}m'_{3,k} &= 1 \quad \text{for every} \quad k \in [3] \quad (37)
\end{align*}
\]

Because a magic square, i.e., a \( \{\pm 1\}\)-valued matrix satisfying both (36) and (37) does not exist, there must be a pair \((j, k) \in [3]^2\) such that \( m_{j,k} \neq m'_{j,k} \). The corresponding Pauli operator \( P_{j,k} \) thus leads to a measurement result +1 and a measurement result −1, depending on the context. This means that \( P_{j,k} \) must be a non-stabilizer of \( \Psi \). In summary, a non-stabilizer \( M \) of a two-qubit stabilizer state \( \Psi \) can be obtained with certainty by performing Pauli measurements on six copies of \( \Psi \).

We can apply this idea to construct an algorithm \( N^O \) as specified in the lemma using an oracle \( O \) for PSIM \([\text{CliffP}(Q)_L]\). Recall that on input \((C_1, \ldots, C_L, D) \in C^{L+1}\), the oracle \( O \)
produces a Pauli operator $P$ occurring with non-zero probability $p_{CliffP(Q)L}(P \mid C_1, \ldots, C_L, D)$ in the output distribution of CliffP$(Q)L$. We note that this output probability can be written as

$$p_{CliffP(Q)L}(P \mid C_1, \ldots, D) = |\langle \Phi_P(D) \mid (I_A \otimes Q(C_1 \cdots C_L)) \mid \Phi \rangle|^2,$$

where $Q = Q(C_1, \ldots, C_L)$ and where

$$|\Phi_P(D)\rangle := (I_A \otimes D^\dagger P) |\Phi\rangle \quad \text{for} \quad P \in \mathcal{P}$$

defines a rotated Bell basis $(I_A \otimes D^\dagger)\mathcal{B} := \{\Phi_P(D)\}_{P \in \mathcal{P}}$ (with $\mathcal{B} := \{\Phi_P(I)\}_{P \in \mathcal{P}}$ being the standard Bell basis), see expression (16). In other words, this output distribution is identical to that obtained when measuring the state

$$I_A \otimes (Q(C_1, \ldots, C_L)C_L \cdots C_1) |\Phi\rangle$$

(38)

using the von Neumann measurement with the orthonormal basis $(I \otimes D^\dagger)\mathcal{B}$.

We argue below that by suitably choosing $D$ and applying the von Neumann measurement with basis $(I \otimes D^\dagger)\mathcal{B}$, we can realize any row- and every column-measurement associated with the magic square. This means that query access to $\mathcal{O}$ with $(C_1, \ldots, C_L)$ fixed and $D \in \mathcal{C}$ arbitrary immediately translates to the (possibilistic) simulation of magic square-measurements in the state (38). In particular, six queries to $\mathcal{O}$ of the form $(C_1, \ldots, C_L, D)$ with $D \in \mathcal{C}$ reveal a non-stabilizer $M$ of the state (38). The claim (ii) follows from this using the following simple fact: If $M$ is a non-stabilizer of a stabilizer state $(I_A \otimes Q)\Psi$ and $Q \in \mathcal{P}$ is arbitrary, then $M$ is also a non-stabilizer of the state $\Psi$.

It remains to argue that the measurement of any row or column of operators in the magic square is equivalent to a von Neumann measurement in a rotated Bell basis $(I \otimes D^\dagger)\mathcal{B}$ for a suitably chosen $D \in \mathcal{C}$. To this end, observe that for any two single-qubit Cliffords $D_A, D_B \in \mathcal{C}$ and any Pauli $P' \in \mathcal{P}$, we have the identity

$$(D_A \otimes D_B)(I_A \otimes P') |\Phi\rangle = (I_A \otimes P')(D_A \otimes D_B) |\Phi\rangle \quad \text{where} \quad P'' := D_B P' D_B^\dagger$$

$$= (I_A \otimes P'')(I_A \otimes D_B D_A^T |\Phi\rangle$$

$$= (I_A \otimes D_B D_A^T P) |\Phi\rangle$$

$$= |\Phi_P(D)\rangle$$

where

$$D := (D_B D_A^T | \quad \text{(39)}$$

$$P := (D_B D_A^\dagger) P'' D_B D_A^T = (D_A^T)^\dagger P'' D_A^T \quad \text{(40)}$$

This is because the Bell state $|\Phi\rangle$ satisfies $(I \otimes L) |\Phi\rangle = (L^T \otimes I) |\Phi\rangle$ for any operator $L$ on $\mathbb{C}^2$. Eqs. (40) and (39) imply that – up to a relabeling of the measurement results as expressed by (40), the von Neumann measurement in the rotated Bell basis $(I \otimes D^\dagger)\mathcal{B}$ is equivalent to a von Neumann measurement in the basis $(D_A \otimes D_B)\mathcal{B}$ consisting of the rotated Bell states

$$(D_A \otimes D_B)(I_A \otimes P) |\Phi\rangle \quad \text{for} \quad P \in \mathcal{P}.$$
A von Neumann measurement in the rotated Bell basis \((D_A \otimes D_B)B\) – which, as shown above, is equivalent to a measurement in the basis \((I_A \otimes D^\dagger)B\) – reveals the basis state, that is, the pair of eigenvalues of \(X'_A \otimes X'_B\) and \(Z'_A \otimes Z'_B\), respectively.

We note that any pair \((P_{1,1} \otimes P_{2,2}, Q_{1,1} \otimes Q_{2,2})\) of Pauli observables with \(P_{1,1}, P_{2,2}, Q_{1,1}, Q_{2,2} \in \{X, Y, Z\}\) and \(P_{1,1} \neq Q_{1,1}\) for \(j = 1, 2\) can be written in the form \((41)\) for suitably chosen Clifford operators \(D_A, D_B\). It is straightforward to check that every row and column of the magic square in Fig. 9 contains a pair \((X'_A \otimes X'_B, Z'_A \otimes Z'_B)\) of two-qubit operators satisfying this condition (the third operator is equal to \(\pm(X'_A \otimes X'_B)(Z'_A \otimes Z'_B)\) and its eigenvalue can be obtained from the measurement results of the other two operators). Thus each row or column of the magic square can be measured by using a von Neumann measurement in a rotated Bell basis of the form \((I_A \otimes D^\dagger)B\), as claimed.

The fact that the algorithm \(N^O\) is realizable by an \(NC^0\)-circuit is obvious since it only manipulates a constant number of bits.

The algorithm \(N^O\) of Lemma 3.5 only provides a single non-stabilizer. However, using a randomization procedure based on Kilian randomization and using \(N^O\) as a subroutine, it is possible to construct an algorithm which produces a uniformly random non-stabilizer of \((I_A \otimes C_L \cdots C_1)\Phi\). Let us call \(N^O\) by \(R\).

**Theorem 3.6** (Kilian-randomization-based algorithm). There is a randomized \(NC^0\)-algorithm \(N^R_{\text{uniform}}\) which uses polynomially many random bits. Given an input \((C_1, \ldots, C_L) \in C^L\) the algorithm \(N^R_{\text{uniform}}\)

(i) makes 1 query to \(R\), and

(ii) outputs a uniformly random non-stabilizer of \((I_A \otimes C_L \cdots C_1)\Phi\).

We defer the description of \(N^R_{\text{uniform}}\) and the proof of Theorem 3.6 to Appendix A.

An algorithm such as the one described in Theorem 3.6 can be run repeatedly to (eventually) learn the stabilizer group of \(\Psi\). For a sufficiently large but constant number of calls to such an algorithm, a pair \((S_1, S_2)\) of Pauli operators with the desired property can be computed with high probability. We can derandomize the learning algorithm in a non-uniform manner and obtain an \(AC^0/poly\) algorithm by using the result of Ajtai and Ben-Or [5] which builds on approximate counting [22].

Theorem 3.4 follows from these arguments, see Theorem 3.6 in Appendix A for details.

### 3.3 Quantum advantage against \(NC^0\) from \(PSIM[\text{Telep}]\)

In this section, we finish the argument for our result, establishing a quantum advantage for the \(PSIM[\text{Telep}]\) problem against (non-uniform) \(NC^0\).

In Section 3.1 we showed that a possibilistic \(NC^0/poly\) simulator for \(\text{Telep}\) circuits implies a possibilistic \(NC^0/poly\) simulator for \(\text{CliffP}(Q)\) for some fixed function \(Q\):

\[
(\text{PSIM}[\text{Telep}] \in NC^0/poly) \quad \Rightarrow \quad (\exists Q: \text{PSIM}[\text{CliffP}(Q)] \in NC^0/poly).
\]

(42)
We will use the possibilistic simulation for the CliffP\( (Q) \) problem together with possibilistic state tomography, Section 3.2 (specialized from [15]), to solve certain word problem for a fixed group that is as hard as PARITY. This will lead to the following implication:

\[
(\exists Q : \text{PSIM}[\text{CliffP}(Q)] \in \text{NC}^0/\text{poly}) \implies (\text{PARITY} \in \text{AC}^0/\text{poly}).
\]

Combining it with Eq. (42) we will obtain

\[
(\text{PSIM}[\text{Telep}] \in \text{NC}^0/\text{poly}) \implies (\text{PARITY} \in \text{AC}^0/\text{poly}).
\]

This contradicts \( \text{PARITY} \not\in \text{AC}^0/\text{poly} \) [16]. Hence, \( \text{PSIM}[\text{Telep}] \not\in \text{NC}^0/\text{poly} \).

We relate \( \text{PARITY} \) and word problem for a fixed group in Section 3.3.1 and use it to complete the proof of quantum advantage from \( \text{PSIM}[\text{Telep}] \) in Section 3.3.2.

### 3.3.1 PARITY as a word problem for the Clifford group

The word problem for groups is defined as follows. Let \( G \) be a group with identity element \( e \in G \). The word problem for the group \( G \) is to decide for an input sequence of group elements \((g_1, \ldots, g_L) \in G^L\) whether the product \( g_L \cdots g_1 \) equals the identity element \( e \). In our scenario we use the promise variant which promises that the product is either identity or some other chosen element \( g \). We denote this problem \( \text{WP}(G, g) \). A sequence of group elements \((g_1, \ldots, g_L)\) is a yes-instance if \( g_L \cdots g_1 = e \) and no-instance if \( g_L \cdots g_1 = g \).

Consider the word problem \( \text{WP}(C, H) \), where \( C \) is the group of one-qubit Clifford unitaries with the standard product operation and \( H \) the Hadamard unitary. We show that \( \text{PARITY} \) reduces to this problem. Let \( x \in \{0,1\}^L \) be a bit string. We will show how to use \( \text{WP}(C, H) \) with the input length \( L \) to compute \( \text{PARITY} \) of the input \( x \). Set

\[
g_i := H^{x_i} \quad \text{for all} \quad i \in [n].
\]

Since \( H^2 = I \), the product

\[
g_L \cdots g_1 = H^{\text{PARITY}(x)} = \begin{cases} I & \text{if } x \text{ has even parity, and} \\ H & \text{if } x \text{ has odd parity.} \end{cases}
\]

Hence we can use \( \text{WP}(C, H) \) to solve \( \text{PARITY} \).

We note that this discussion may appear overly detailed to simply compute the parity. However, one may use other (Clifford) groups and corresponding word problems to address further computational tasks, along the lines of [15], which we expand upon in Section 3.3.3. In particular, using for example two qubits and corresponding gate-teleportation circuits, one can construct circuits solving word problems associated with \( S_6 \), related to Barrington’s theorem [6]. However, because of the use of lightcone-type arguments, our line of reasoning cannot provide quantum advantage beyond \( \text{NC}^0 \).

### 3.3.2 PSIM [Telep] cannot be in NC^0

In the following lemma we use a deterministic oracle for the \( \text{PSIM}[\text{CliffP}(Q)] \) problem together with the possibilistic state tomography from Section 3.2 to solve \( \text{PARITY} \).

**Lemma 3.7.** Let \( O \) be a deterministic oracle for the \( \text{PSIM}[\text{CliffP}(Q)] \) problem. There is a non-uniform \( \text{AC}^0 \) circuit that makes \( O(1) \) queries to \( O \) and solves the \( \text{PARITY} \) problem.

**Proof.** Let \( x \in \{0,1\}^L \) be an input instance of the \( \text{PARITY}_L \) problem. Set, as in Eq. (43), \((C_1, \ldots, C_L)\) where \( C_i := H^{x_i} \). Again \( C_L \cdots C_1 = H^{\text{PARITY}(x)} \).
We will use the learning procedure with the oracle for the PSIM $[\text{CliffP}(Q)_L]$ problem established in Theorem 3.4 to learn the stabilizer group (up to the sign factor) of the state 

$$(I_A \otimes C_L \cdots C_1) \Phi = I_A \otimes (H^\text{PARITY}(x)) \Phi.$$ 

This theorem tells us that there is an $\text{AC}^0/\text{poly}$ algorithm that makes $O(1)$ queries to the oracle $O$ and determines a pair $(S_1, S_2) \in (\mathcal{P} \otimes \mathcal{P})^2$ of two distinct (non-identity) Pauli operators. Such that, each $S_i$ has expectation value $\pm 1$ in the state $(I \otimes H^\text{PARITY}(x)) \Phi$. Let us list what are the possible outcomes for the two options, i.e., when $x$ has even and odd parity:

(i) If $x$ has even parity the algorithm learns the stabilizer group of the state $(I_A \otimes H^\text{PARITY}(x)) \Phi = \Phi$. It is straightforward to check that the valid outputs $(S_1, S_2)$ of the learning algorithm are two distinct elements of the set $S_{\text{even}} := \{X \otimes X, Z \otimes Z, Y \otimes Y\}$.

(ii) If $x$ has odd parity the algorithm learns the stabilizers of the state $(I \otimes H) \Phi$. The output $(S_1, S_2)$ are two distinct elements of the set $S_{\text{odd}} := \{X \otimes Z, Z \otimes X, Y \otimes Y\}$.

Since $|S_{\text{even}} \cap S_{\text{odd}}| = 1$, we can determine the parity of $x$ from the output $(S_1, S_2)$.

One of the most important results in circuit complexity is that $\text{PARITY}$ is not contained in the (non-uniform) complexity class $\text{AC}^0$. This result was first shown by Ajtai [4] and independently by Furst, Saxe, and Sipser [13]. Improvements were given by Yao [27], and Håstad [16] based on random restrictions (i.e., so-called “switching lemmas”). Razborov [19], and Smolensky [20] gave a proof based on polynomials.

**Theorem 3.8** ($\text{PARITY} \not\in \text{AC}^0/\text{poly}$ [21]). If $\text{PARITY}_n : \{0, 1\}^n \rightarrow \{0, 1\}$ is realisable by a (non-uniform) $\text{AC}^0$-circuit of depth $d$ then its size $S$ is

$$S \geq 2^{\Omega(n^{1/(d-1)})}.$$ 

We use this result to complete our argument and show that there cannot be a (non-uniform) $\text{NC}^0$ circuit for $\text{PSIM}$ $[\text{Telep}]$. Since the $\text{Telep}_n$ quantum circuit trivially solves $\text{PSIM}$ $[\text{Telep}_n]$ this completes the proof of the main theorem, i.e., Theorem 1.1.

**Theorem 3.9** ($\text{PSIM}$ $[\text{Telep}] \not\in \text{NC}^0/\text{poly}$). The possibilistic simulation of the $\text{Telep}$ circuit, i.e., the $\text{PSIM}$ $[\text{Telep}]$ problem, is not in (non-uniform) $\text{NC}^0$.

**Proof.** For the sake of contradiction, assume that there is a (non-uniform) circuit $C_n$ for the $\text{PSIM}$ $[\text{Telep}_n]$ problem for any $n$.

We will use $C_n$ in a non-black-box way. From Lemma 3.3 we have that there is an $L = \Theta(n)$ and a non-uniform $\text{NC}^0$ algorithm $A$ that solves $\text{PSIM}$ $[\text{CliffP}(Q)_L]$ for some fixed function $Q : C^L \rightarrow \mathcal{P}$. This algorithm can be constructed from the algorithm $B_{\epsilon, m, J}^I$ (Algorithm 2) by fixing the parameters $m, J$ that are specific for the $C_n$ in the non-uniform way, i.e., $A$ is in $\text{NC}^0/\text{poly}$.

Lemma 3.7 tells us that we can use $A$ to solve $\text{PARITY}$ in $\text{AC}^0/\text{poly}$ with oracle access to $A$. Since $A$ is in $\text{NC}^0/\text{poly}$, we can solve $\text{PARITY}$ by an $\text{AC}^0/\text{poly}$ algorithm. This gives a contradiction with $\text{PARITY} \not\in \text{AC}^0/\text{poly}$, i.e., with Theorem 3.8. Hence there is no (even non-uniform) $\text{NC}^0$-circuit for the $\text{PSIM}$ $[\text{Telep}]$ problem.

An alternative proof of Theorem 3.9 uses $\text{NC}^0/\text{poly}$ and an associated circuit depth lower-bound for the $\text{PARITY}$ problem in place of $\text{AC}^0/\text{poly}$. (The $\text{AC}^0/\text{poly}$ appeared due to the derandomization procedure in the possibilistic state tomography, see Theorem 3.4, and the related proofs.) We can argue that there exists an $\text{NC}^0$ circuit that solves at least $2/3$ instances of the tomography problem correctly, which is sufficient to prove Theorem 3.9. This is because an output of an $\text{NC}^0/\text{poly}$ circuit cannot depend on all input bits and therefore such a circuit cannot solve the parity problem for more than $1/2$ instances. This alternative approach does not change the result of Theorem 3.9 but arrives at it through a weaker reduction.
3.3.3 Other word problems for Clifford modulo Pauli groups

Here we argue that the class of problems we could solve if we had an \( \text{NC}^0 \) simulator for the PSIM \([\text{Telep}]\) problem is much broader than just computing \( \text{PARITY} \). To address more general problems, we follow the lines of [15].

The main idea relies on the observation that the technique used in Section 3.2 to distinguish the state \( |\Phi^+\rangle \) from \( (I \otimes H)|\Phi^+\rangle \), where \( C \in \mathbb{C}/\{I\} \) is a non-Pauli Clifford gate. This is because by definition of the Clifford group, any non-Pauli Clifford takes a stabilizer state to another, different stabilizer state. This allows us to solve the word problem \( \text{WP}(C/\mathcal{P}, C) \) for any \( C \in \mathbb{C}/\mathcal{P} \setminus \{I\} \), i.e., the problem of deciding whether a series of single-qubit Clifford modulo Pauli gates multiplies to \( I \) or \( C \).

For example, the Clifford gate \( SH \) generates a cyclic group of order 3. Thus replacing the Hadamard gate \( H \) in our construction allows us to compute unbounded \( \text{MOD}_3 \) gates, i.e., to compute \( |x| \mod 3 \) of the Hamming weight of the input string \( x \). Note that \( \text{MOD}_3 \) cannot be computed by constant-depth circuits with unbounded \( \text{PARITY} \)-gates [19, 20] and vice versa, hence this problems are incomparable.

Combining the ability of computing \( \text{MOD}_3 \) with that of computing \( \text{PARITY} \), we conclude that access to an \( \text{NC}^0 \) simulator for the PSIM \([\text{Telep}]\) problem elevates \( \text{AC}^0 \) circuits to \( \text{AC}^0 \) [6], i.e., \( \text{AC}^0 \) circuit with \( \text{MOD}_6 \) gates (and similarly \( \text{NC}^0 \) to \( \text{NC}^0 \) [6]).

Alternatively, we can arrive at similar conclusions using Barrington’s result [10]. The latter states that the promise word problem for the solvable group \( S_3 \) characterizes width-3 permutation branching programs. The promise word problem for \( S_3 \) can be treated in our setting because \( S_3 \) is isomorphic to \( \mathbb{C}/\mathcal{P} \). The isomorphism is given by

\[
I \mapsto 1, \quad H \mapsto (13), \quad S \mapsto (12), \quad HS \mapsto (123), \quad SHS \mapsto (23), \quad (HS)^2 \mapsto (132). 
\]

Two-qubit Clifford gate teleportation circuits A similar construction can be obtained with two-qubit (instead of one-qubit) gate teleportation. Notice that this circuit can be implemented on a ring of pairs of qubits. Assuming an \( \text{NC}^0 \) possibilistic simulator for this problem and following the lightcone argument, we can use it to solve a possibilistic simulation of the intermediate problem depicted in Figure 10. We can ask what problems such a simulator allows us to solve.

Let us consider the word problem \( \text{WP}(C_2/\mathcal{P}_2, H \otimes H) \), i.e., deciding whether a series of two-qubit Clifford modulo Pauli gates multiplies to \( I \otimes I \) or \( H \otimes H \), as used also in Ref. [15]. If we have an \( \text{NC}^0 \) simulator for PSIM \([2\text{CliffP}(Q)]\) we can use it to do possibilistic tomography in a similar way to what we did in Section 3.2.2 on each of the two Bell states.

Notice that the two-qubit Clifford group modulo Pauli operators is isomorphic to \( S_6 \) (see Ref. [15]). Since \( S_6 \) is a non-solvable group, the associated word problem is complete for the complexity class \( \text{NC}^1 \), i.e., for the complexity class of bounded fan-in logarithmic-depth polynomial-size circuits, according to Barrington’s theorem [6]. A constant-depth classical simulator for the two-qubit gate-teleportation problem could therefore be used to solve any problem in \( \text{NC}^1 \).

Note, however, that none of these more general computational problems can be used to obtain a quantum advantage beyond \( \text{NC}^0 \) when only our techniques are applied. The lightcone argument we rely on constitutes a bottleneck for such a separation. The existence of a quantum advantage beyond \( \text{NC}^0 \) for 1D-local quantum circuits therefore remains an open problem.
Figure 10: The Pauli-corrected two-qubit Clifford circuit $2\text{CliffP}(Q)_L$ with correction function $Q : C_2^L \to P_2$. It takes an input $(C_1, \ldots, C_L, D) \in C_2^{L+1}$ and outputs two single-qubit Pauli operators $P_1, P_2 \in P$ obtained from the two Bell measurements.

A PSIM [CliffP(Q)] and possibilistic state tomography

In this appendix we finish the proof of Theorem 3.4 from Section 3.2.2. It follows by specializing the arguments given in [15].

For convenience, let us define the non-stabilizer problem. Given an input $(C_1, \ldots, C_L) \in C_L$ the non-stabilizer problem is to output $P_A \otimes P_B \in P^2$, such that $P_A \otimes P_B$ is a non-stabilizer of the stabilizer state $(I_A \otimes C_L \cdots C_1)\Phi$.

The remainder of the arguments of [15] shows that an oracle $O$ for the non-stabilizer problem can be used to produce a uniformly random non-stabilizer of $(I_A \otimes C_L \cdots C_1)\Phi$. We can then use this algorithm to produce generators of the stabilizer group of this state.

The algorithm for uniformly random non-stabilizer relies on the following:

Lemma A.1. Let $\Psi$ be a stabilizer state and $M$ a non-stabilizer of $\Psi$. If $V$ is a Clifford unitary chosen uniformly at random among the set of Cliffords satisfying $V\Psi = \Psi$, then $VMV^\dagger$ is a uniformly chosen non-stabilizer of $\Psi$.

For a proof see [15, Proof of Theorem 24].

Consider the Algorithm 3 below. It builds on Kilian’s randomization [17] and satisfies the properties proven in Lemma A.2, see [15, Theorem 24].

Algorithm 3 Kilian-randomized function $M^O$. Here $O$ is an oracle outputting a non-stabilizer of $(I_A \otimes C_L \cdots C_1)\Phi$ on input $(C_1, \ldots, C_L) \in C_L$.

1: function $M^O(C_1, \ldots, C_L)$
2: $D_1, \ldots, D_L \leftarrow$ uniformly random Clifford unitaries
3: $C'_1 \leftarrow D_1 C_1$
4: for $j = 2, \ldots, L$ do
5: $C'_j \leftarrow D_j C_j D^{-1}_{j-1}$
6: $M \leftarrow O(C'_1, \ldots, C'_L)$
7: return $(I_A \otimes D_L)^\dagger M(I_A \otimes D_L)$

Lemma A.2. Let $O$ be a deterministic oracle for the non-stabilizer problem. The Algorithm 3 satisfies:
(i) On input \((C_1, \ldots, C_L)\), the algorithm \(\mathcal{M}^\phi\) makes one query to the oracle \(\mathcal{O}\) and returns a non-stabilizer of \((I_A \otimes \prod_{i=1}^L C_i)^\Phi\).

(ii) There is a random variable \(\widetilde{M}\) on \(\mathcal{P}\) such that for any sequence \(C_1, \ldots, C_L\) of Cliffords, the output \(\mathcal{M}^\phi(C_1, \ldots, C_L)\) satisfies

\[
\mathcal{M}^\phi(C_1, \ldots, C_L) = \left( I_A \otimes \prod_{i=1}^L C_i \right) \left( I_A \otimes \prod_{i=1}^L C_i \right)^\dagger \to \widetilde{M} \left( I_A \otimes \prod_{i=1}^L C_i \right)^\dagger
\]

with certainty. In particular, the dependence of the output of \(\mathcal{M}^\phi\) on the input \((C_1, \ldots, C_L)\) is only via the product \(\prod_{i=1}^L C_i\).

(iii) The support of the distribution \(P_{\widetilde{M}}\) is contained in the set of non-stabilizers of the state \(\Phi\).

\[\text{Proof.}\] Claim (i) follows immediately from the definition of \(\mathcal{M}^\phi\): The operator \(M\) obtained in Step (6) of the Algorithm 3 is a non-stabilizer of \(I_A \otimes (C'_L \cdots C'_1)^\Phi = (I_A \otimes D_L(\prod_{i=1}^L C_i))^\Phi\), hence \((I_A \otimes D_L)^\dagger M (I_A \otimes D_L)\) is a non-stabilizer of \((I_A \otimes \prod_{i=1}^L C_i)^\Phi\).

For the proof of (ii), consider the following procedure, given \((C_1, \ldots, C_L)\): Pick \(D_L\) uniformly at random from the Clifford group, then pick \((C'_1, \ldots, C'_L)\) uniformly from the set of sequences satisfying

\[C'_L \cdots C'_1 = D_L \prod_{i=1}^L C_i.\]

Then set

\[M = \mathcal{O}(C'_1, \ldots, C'_L)\]

and

\[\widetilde{M} = (I \otimes (C'_L \cdots C'_1))^\dagger M (I \otimes (C'_L \cdots C'_1)).\]

Observe first that the distribution \(P_{\widetilde{M}}\) is a function of \(\prod_{i=1}^L C_i\) only and does not depend on the choice of \((C_1, \ldots, C_L)\) otherwise. Indeed, the sequence \((C_1, \ldots, C_L)\) only determines \(\prod_{i=1}^L C_i\), implying that the product \(D_L \prod_{i=1}^L C_i\) in (44) has a distribution which is a function of the product \(\prod_{i=1}^L C_i\) only. Because the distribution of \((C'_1, \ldots, C'_L)\) is fixed by \(D_L \prod_{i=1}^L C_i\) by definition, it follows that the distribution of \(M\) is determined by the product \(\prod_{i=1}^L C_i\) (with no additional dependence on \((C_1, \ldots, C_L)\)). It follows similarly from (45) that the distribution of \(\widetilde{M}\) has this property.

By definition, we have

\[
\left( I_A \otimes \prod_{i=1}^L C_i \right) \left( I_A \otimes \prod_{i=1}^L C_i \right)^\dagger \to \widetilde{M} \left( I_A \otimes \prod_{i=1}^L C_i \right)^\dagger = \left( I_A \otimes \prod_{i=1}^L C_i \left( D_L \prod_{i=1}^L C_i \right)^\dagger \right) \left( I_A \otimes \prod_{i=1}^L C_i \right)^\dagger
\]

\[
= (I_A \otimes D_L)^\dagger M (I_A \otimes D_L)
\]

\[= (I_A \otimes D_L)^\dagger \mathcal{O}(C'_1, \ldots, C'_L) (I_A \otimes D_L)\]

(46)
where we used (44) in the second step. Observe that the marginal distributions $P_{C_1\cdots C_L D_L}$ are identical for both algorithms $\mathcal{M}$ and the procedure used in this proof: This is because steps [4] and [5] amount to Kilian randomization [17], producing a uniformly random sequence $(C'_1, \ldots, C'_L)$ of group elements with a fixed product as in (44). It follows that (46) is indeed identical to the output of $\mathcal{M}$ as claimed.

Finally, the claim (iii) is an immediate consequence of (i) and (ii).

\begin{algorithm}

\SetKwFunction{M}{M}

{Randomization function $\mathcal{M}'$} \\
1: function $\mathcal{M}'(C_1, \ldots, C_L)$ \\
2: \hspace{1em} $V \leftarrow$ a random Clifford satisfying $V\Phi = \Phi$. \\
3: \hspace{1em} $C'_1 \leftarrow C_1 V$ \\
4: \hspace{1em} for $j = 2, \ldots, L$ do \\
5: \hspace{2em} $C'_j \leftarrow C_j$ \\
6: \hspace{1em} $M \leftarrow \mathcal{M}^O(C'_1, \ldots, C'_L)$ \\
7: return $M$

\end{algorithm}

Let $\mathcal{M}$ be as specified in Algorithm 3 with an oracle access to the oracle $O$ that solves the non-stabilizer problem. Consider the algorithm $\mathcal{M}'$ specified in Algorithm 4. Then we have the following (see [15, Theorem 24]):

\textbf{Lemma A.3.} On input $(C_1, \ldots, C_L)$, the algorithm $\mathcal{M}'$ returns a uniformly distributed non-stabilizer of $\left(I_A \otimes \prod_{i=1}^L C_i\right)\Phi$.

\textbf{Proof.} Let $\tilde{M}$ denote the Pauli-valued random variable from Lemma A.2. Let $V$ be a uniformly distributed one-qubit Clifford unitary subject to the constraint $V\Phi = \Phi$. Then we have

$$P_{M'(C_1,\ldots,C_L)} = P_{M^O(C_1V,C_2,\ldots,C_L)} = P_{\prod_{i=1}^L C_i V \tilde{M} V^\dagger \prod_{i=1}^L C_i}$$

by definition of the algorithm $\mathcal{M}'$ and Lemma A.2 (ii). Since $\tilde{M}$ is supported on the set of non-stabilizers of the state $\Phi$ (see Lemma A.2 (iii)), it follows from Lemma A.1 that $V\tilde{M}V^\dagger$ is a random non-stabilizer of $\Phi$ with uniform distribution. The claim follows from this.

Finally, we combine a deterministic oracle for the non-stabilizer problem and the randomization procedure that we specified in Algorithms 3 and 4 to an algorithm learning the stabilizer group of the state $(I_A \otimes C_L \cdots C_1)\Phi$ (up to a sign factor).

\textbf{Theorem A.4.} Let $O$ be a deterministic oracle for the non-stabilizer problem. There is an $AC^0/poly$ algorithm with an oracle access to $O$ that, on input $(C_1, \ldots, C_L)$

(i) makes $O(1)$ oracle calls to $O$,

(ii) outputs two distinct non-identity Pauli operators $S_1, S_2$ such that

$$\langle \Phi | (I_A \otimes (C_L \cdots C_1))^\dagger S_j (I_A \otimes (C_L \cdots C_1)) | \Phi \rangle \in \{\pm 1\} \quad \text{for} \quad j = 1, 2$$

\textbf{Proof.} Let $\Psi$ be a stabilizer state. A Pauli operator $M$ has expectation value $\langle \Psi | M | \Psi \rangle \in \{\pm 1\}$ if and only if $M$ commutes with each stabilizer of $\Psi$. If this is not the case, then $M$ is a non-stabilizer. It follows that a two-qubit stabilizer state has exactly 12 non-stabilizers of the form $P_A \otimes P_B$ with $(P_A, P_B) \in P^2$. The remaining 4 operators constitute (up to $\pm$-signs) the stabilizer group of $\Psi$. In particular, given the 12 non-stabilizers of $\Psi$, it is straightforward
to output two distinct non-identity Pauli operators $S_1 = P_A \otimes P_B$, $S_2 = P'_A \otimes P'_B$ such that $\langle \Psi | S_j | \Psi \rangle \in \{ \pm 1 \}$ for $j = 1, 2$.

We apply this to the state $\Psi = (I_A \otimes C_L \cdots C_1) \Phi$. Let $\mathcal{M}'$ be the algorithm defined in Algorithm 4, using the function $\mathcal{M}^\mathcal{O}$ given in Algorithm 3. Notice that this algorithm returns a uniformly random non-stabilizer of $\Psi$ given the input $C_L \cdots C_1$. Then, there is an algorithm $\mathcal{M}''$ that uses $O(1)$ queries to $\mathcal{M}'$ and determines, with probability at least $2/3$, all 12 non-stabilizers of $\Psi$ of the form $P_A \otimes P_B$ with $(P_A, P_B) \in P^2$ or it reports failure. We can exploit the fact that we are allowing for non-uniformity to derandomize this algorithm using the result of Ajtai and Ben-Or [5] (see Theorem A.5). Thus we end up with an $\text{AC}^0/\text{poly}$ algorithm with the desired properties that makes $O(1)$ oracle queries to the oracle $\mathcal{O}$. The claim follows.

We used the following derandomization result in the proof of Theorem A.4.

**Theorem A.5** (Ajtai and Ben-Or [5, Theorem 4.]). Let $\mathcal{L}$ be a decision problem. Let $\mathcal{A}$ be an algorithm realizable by a probabilistic (non-uniform) $\text{AC}^0$ circuit that uses polynomially many random bits. Assume there is a fixed constant $r$ such that $\mathcal{A}$ accepts any $x \in \mathcal{L}$ with probability at least $1/2 + (\log n)^{-r}$, where $n$ is the length of the input; and falsely accepts any $x \notin \mathcal{L}$ with probability at most $1/2$. Then there is non-uniform $\text{AC}^0$ circuit that decides $\mathcal{L}$.

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