Synchronisation and MSW sharpening of neutrinos propagating in a flavour blind medium.

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We consider neutrino oscillations in a medium in which scattering processes are blind to the neutrino flavour. We present an analytical derivation of the synchronised behaviour obtained in the limit where the average scattering rate is much larger than the oscillation frequency. We also examine MSW transitions in these circumstances, and show that a sharpening of the transition can result.

I. INTRODUCTION

We examine a system consisting of an oscillating neutrino propagating through a dense medium. This falls into the very broad class of problems in which a quantum system cannot be considered closed or isolated, due to interaction of the system with an environment or heat bath. In particular we consider two-flavour neutrino oscillations in a medium such that the system-bath interactions do not distinguish the flavour of the neutrino.

An example is the density matrix of a single neutrino undergoing $\nu_\mu - \nu_\tau$ oscillations embedded in an electron-positron plasma. In some sense this example is similar to an early universe environment, though lacking some features of a realistic physical scenario, such as neutrino pair processes and the inclusion of other neutrino flavours. Another example in which flavour blind interactions are relevant is the case of $\nu_\mu - \nu_\tau$ oscillations of neutrinos coming from the core of a supernova. In this case, although the indices of refraction for $\nu_\mu$ and $\nu_\tau$ will be different due to loop graph effects, the collision rates for the neutrinos scattering from the medium of nucleons will be the same for the two flavours, to a good approximation. However, it is the coherence properties of the system that are most intriguing, and which are worth studying independently of the various circumstances in which they may arise.

An analogue to the neutrino system is that of a double well coupled to a thermal bath, in which the neutrino flavour oscillations are replaced by oscillations from the left to the right side of the well. Our work provides an analytic treatment of the distinctive behaviour that we noted previously in a numerical treatment of double well oscillations [1]. Since double well systems are of general interest in condensed matter physics [2,3], the type of behaviour we examine here may be expected to arise in various situations unrelated to neutrino physics.

As the medium cannot distinguish the flavour of the neutrino, the only effect of collisions with particles in the medium is to change the value of the neutrino’s energy (and to entangle the neutrino with the environment). This is in contrast with active-sterile oscillations in
which collisions may in some sense be thought of as measuring the flavour of the neutrino. It is the consideration of the neutrino energy variable, and the rapid energy changing collisional processes that is crucial, and when the average collision rate is much larger than average oscillation rate we find that a collective oscillation results [1]. That is, for a neutrino in a thermally distributed mixture of momentum states, the oscillations are synchronised despite being momentum dependent in the absence of collisions.

We stress that the novel synchronisation effect arises as a result of energy changing processes. In particular, the result is distinct from the synchronisation effect of ref. [4] which results from non-linear feedback, as we consider here only linear dynamics.

A. The Quantum Kinetic Equations

The evolution equations for the neutrino density matrix have been formulated in ref. [5] for active-active oscillations, and ref. [6] for active-sterile oscillations. These Quantum Kinetic Equations (QKEs) or Bloch equations, take into account both coherent oscillation behaviour and potentially decoherent effects of interaction of the neutrino system with an environment. The QKEs are essentially Boltzmann equations, generalised to allow for quantal coherence between particle species.

We choose to parametrise the neutrino density matrix as

\[ \rho(p) = \frac{1}{2} \left[ P_0(p) + P(p) \cdot \sigma \right] \]

with

\[ P(p) = P_x(p)\hat{x} + P_y(p)\hat{y} + P_z(p)\hat{z} \]

The QKEs for a single neutrino undergoing elastic scattering in a medium which does not distinguish flavour are [5]

\[ \dot{P}(k) = \int d^3 k' \left[ \Gamma(k', k)\bar{P}(k') - \Gamma(k, k')\bar{P}(k) \right] + V(k) \times P(k), \]

\[ \dot{P}_0(k) = \int d^3 k' \left[ \Gamma(k', k)P_0(k') - \Gamma(k, k')P_0(k) \right], \]

where \( \Gamma(k', k) \) determines the rate at which neutrinos are scattered from momentum state \( k' \) to state \( k \). Thermal equilibrium of the medium enforces

\[ \Gamma(k', k) = e^{(k'-k)/T} \Gamma(k, k'). \]

The cross product term describes precession of the neutrino flavour, where \( V(k) \) is given by

\[ V(k) = \beta(k)\hat{x} + \lambda(k)\hat{z}, \]

and the oscillation parameters are defined as

\[ \beta(k) = \frac{\delta m^2}{2k} \sin 2\theta_0, \]

\[ \lambda(k) = -\frac{\delta m^2}{2k} \cos 2\theta_0 + V_{\text{eff}}, \]

with \( \theta_0 \) being the vacuum mixing angle, \( \delta m^2 \) the mass-squared difference, and \( V_{\text{eff}} \) the effective matter potential resulting from coherent forward scattering.
B. Qualitative features for different coupling strength.

By way of example, we consider the density matrix of a single \( \nu_\mu - \nu_\tau \) neutrino propagating in an electron-positron plasma. For simplicity, we shall ignore the interaction of the neutrino with other \( \nu_{\mu,\tau} \) or \( \bar{\nu}_{\mu,\tau} \) that may be produced in the plasma, and consider only the single neutrino and its interaction with electrons and positrons. That is, we consider in isolation a particular subset of the various processes that may affect the evolution of the neutrino density matrix.

For our considerations it should suffice to take the following form for the scattering rate,

\[
\Gamma(k, k') \simeq \frac{0.01}{4\pi} G^2_{kk'} e^{-k'/T},
\]

(7)

as this embodies the correct threshold behaviour, the reciprocity relation (4) and leads to the correct total transition rate. We have solved the QKEs (3) numerically, and the results are presented in the graphs below, for various temperatures. For small temperatures, where the average time between collisions is much longer than the oscillation period, we have just the superposition of (almost) undamped precession rates at different frequencies. For moderate collisions rates we rapidly lose coherence in the neutrino system. This occurs simply because the oscillation modes are momentum dependent. Collisional processes transfer quanta across the momentum states which quickly get out of phase, and hence result in a damping of the amplitude of the oscillations. The most fascinating effect, however, is obtained for larger values of \( \Gamma \) where the average time between collisions is much smaller than the oscillation period. In this limit we see a synchronisation of the oscillations of all the momentum states. Here, the precession proceeds at at a rate determined by the thermal average of \( V(k) \), and damping is eliminated as the rapid collisions force the momentum modes to stay in phase.

![Graph](image)

**FIG. 1.** The probability that the neutrino is a \( \nu_\mu \), where the initial condition is a \( \nu_\mu \) with a thermal distribution of energies. The solid, dotted, dashed and heavy solid curves correspond to \( T=5, 10, 20 \) and 50MeV respectively, and we have taken \( \delta m^2 = 0.001eV^2 \), \( \sin 2\theta_0 = 1 \). In each case, the time is scaled by the inverse of the thermally averaged oscillation frequency.
We define the entropy in the neutrino system as

\[ s = -\int d^3k \text{Tr} \left[ \rho(k) \ln \rho(k) \right], \]  

where the trace is over the neutrino flavour index. The entropy is plotted in fig.2 for various collision rates.

\[ \text{FIG. 2. Entropy plotted for the same parameters as fig.1, where } S = s(t) - s(t=0). \]

Let us compare this behaviour with the familiar Quantum Zeno behaviour in active-sterile neutrino systems. In both cases the rate of entropy growth increases as we increase the interaction rate between the neutrino system and the environment, and then decreases again if we make the interaction rate much larger than the oscillation rate. In the active-sterile case however, this occurs because the “measurement-like” interactions freeze the neutrino in its initial state. In contrast, in our case the flavour precession proceeds as though it were a single mode of an isolated system.

II. ANALYTIC SOLUTION FOR LARGE COLLISION RATE

We now show analytically that a synchronised solution follows from the Eqs. (3) in the large \( \Gamma \) limit. We begin be discretizing the equations, by allowing the momentum to take the values \( k_1, k_2, ..., k_N \). Expressing the resulting \( 3N \) equations in matrix form we have

\[ \dot{\mathbf{P}} = \mathcal{M}\mathbf{P}, \]  

where \( \mathbf{P} \) is the vector

\[ \mathbf{P} = (P_{x1}, ..., P_{xN}, P_{y1}, ..., P_{yN}, P_{z1}, ..., P_{zN})^T. \]

The matrix \( \mathcal{M} \), has the following block form

\[ \mathcal{M} = \begin{pmatrix} \mathcal{G} & -\Lambda & 0 \\ \Lambda & \mathcal{G} & -\mathcal{B} \\ 0 & \mathcal{B} & \mathcal{G} \end{pmatrix}, \]  

where
where
\[
G_{ij} = \Gamma_{ji}k_j^2 - \delta_{ij} \sum_k \Gamma_{ik}k_k^2,
\]
\[
\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_N),
\]
\[
\mathcal{B} = \text{diag}(\beta_1, \ldots, \beta_N).
\]

(12)

To solve for \( P(t) \), we introduce a matrix \( \mathcal{U} \) which diagonalises \( \mathcal{M} \) such that
\[
P = \mathcal{U}Q,
\]
and
\[
\dot{Q} = \mathcal{M}_{\text{diag}}Q - \mathcal{U}^{-1}\dot{\mathcal{U}}Q,
\]
where
\[
\mathcal{M}_{\text{diag}} \equiv \mathcal{U}^{-1}\mathcal{M}\mathcal{U}.
\]

(15)

We shall consider time independent \( \lambda \) and \( \beta \), so that \( \dot{\mathcal{U}} = 0 \), and the solution to eq.(14) is
\[
P_i(t) = \sum_j \mathcal{U}_{ij}Q_j(t) = \sum_j \exp \left[ (\mathcal{M}_{\text{diag}})_{jj}t \right] \mathcal{U}_{ij}Q_j(0).
\]

(16)

The procedure may also be adapted for time dependent mixing where \( \lambda \) and \( \beta \) change sufficiently slowly that an adiabatic approximation is valid.

Since we wish to find the solution in the large \( \Gamma \) limit, we first drop the \( \lambda \) and \( \beta \) terms. Then we see that \( \mathcal{M} \) has three zero eigenvalues, with eigenvectors that are linear combinations of
\[
y_1 = \frac{1}{\sqrt{N}} (e^{-k_1/T}, 0, 0)^T, \quad y_2 = \frac{1}{\sqrt{N}} (0, e^{-k_2/T}, 0)^T \quad \text{and} \quad y_3 = \frac{1}{\sqrt{N}} (0, 0, e^{-k_N/T})^T,
\]
where \( e^{-k/T} \) denotes \( (e^{-k_1/T}, e^{-k_2/T}, \ldots, e^{-k_N/T}) \). The corresponding eigenvectors of \( \mathcal{M}^T \) are
\[
x_1 = \frac{1}{\sqrt{N^t}} (k_1^2, 0, 0)^T, \quad x_2 = \frac{1}{\sqrt{N^t}} (0, k_2^2, 0)^T \quad \text{and} \quad x_3 = \frac{1}{\sqrt{N^t}} (0, 0, k_N^2)^T,
\]
with \( \mathcal{U}\mathcal{U}^{-1} = 1 \) requiring \( \sqrt{NN^t} = \sum_k k^2 e^{-k/T} \). All the other eigenvalues are negative and proportional to \( \Gamma \). To observe this let \( \Omega \) be an eigenvalue of \( \mathcal{G}^T \) with eigenvector \( \mathcal{X} \). The eigenvalue may be expressed as
\[
\Omega = \sum_k \Gamma_j k_j^2 \left( \frac{X_k k_j^2}{X_j k_k^2} - 1 \right), \quad \forall j : X_j \neq 0.
\]

(19)

1 An adiabatic approximation was applied to the case of partially coherent oscillations in Ref. [7].

2 Note that since \( \mathcal{M} \) is neither hermitian nor symmetric, \( \mathcal{U} \) will neither be unitary nor orthogonal.
Since we may choose \( j \) such that \( |X_j/k_j^2| \geq |X_k/k_k^2|, \forall k \), it follows that \( \text{Re}\Omega \leq 0 \). Thus the eigenvectors corresponding to the non-zero eigenvalues will make a negligible contribution to the solution in the large \( \Gamma \) limit (for \( t > 1/\Gamma \)). Then the solution is simply
\[
P(t) \simeq e^{-k_i/T} \times \text{const} \equiv \text{thermal solution}.
\]

Treating \( \lambda \) and \( \beta \) as perturbation parameters, we are led to consider the \( 3 \times 3 \) matrix
\[V_{ij} = x_i^T V y_j,\]
where \( V \) is the matrix obtained by setting \( G = 0 \) in \( M \). Using standard degenerate perturbation theory, to first order the three eigenvalues become
\[
\pm iw, \quad 0
\]
where \( \omega \), the synchronised oscillation frequency, is\(^3\)
\[
\omega = \sqrt{\langle \lambda \rangle^2 + \langle \beta \rangle^2}
\]
and \( \langle \lambda \rangle \) and \( \langle \beta \rangle \) are the thermally averaged values
\[
\langle \lambda \rangle \equiv \sum_k k^2 e^{-k/T} \lambda_k / \sum_{k'} k'^2 e^{-k'/T}, \quad \langle \beta \rangle \equiv \sum_k k^2 e^{-k/T} \beta_k / \sum_{k'} k'^2 e^{-k'/T}.
\]

Note that in the absence of a matter potential, the synchronised precession frequency is just
\[
\omega = \langle \delta m^2 / 2k \rangle.
\]

For time independent mixing the solution becomes,
\[
P_x(k, t) = ne^{-k/T} c_{2\Theta} s_{2\Theta} [A' - A \cos(\omega t - \alpha)]
\]
\[
P_y(k, t) = -ne^{-k/T} A s_{2\Theta} \sin(\omega t - \alpha)
\]
\[
P_z(k, t) = ne^{-k/T} [A' c_{2\Theta}^2 + A s_{2\Theta}^2 \cos(\omega t - \alpha)],
\]
where
\[
c_{2\Theta} = \langle \lambda \rangle / w,
\]
\[
s_{2\Theta} = \langle \beta \rangle / w,
\]
\[
n = \frac{1}{\sum_{k'} k'^2 e^{-k'/T}} \sum_{k'} k'^2 P_z(k', t = 0).
\]

That is, we have undamped oscillations with frequency \( \omega \) at an effective mixing angle of \( \Theta \).

If the initial condition is such that the neutrino is in the pure flavour state described by \((1 + \sigma_3)/2\) then \( \alpha = 0 \) and \( A' = A = 1 \). Allowing for general initial conditions, such as an initial distribution of phases, we have
\[
\tan \alpha = \frac{\sum_k k^2 P_y(k, 0)}{\sum_{k'} k'^2 [s_{2\Theta} P_z(k', 0) - c_{2\Theta} P_x(k', 0)]}
\]
\[
A' = 1 + \frac{s_{2\Theta} \sum_k k^2 P_x(k, 0)}{c_{2\Theta} \sum_{k'} k'^2 P_z(k', 0)}
\]
\[
A = \left( \frac{\sum_k k^2 P_y(k, 0)}{s_{2\Theta} \sum_{k'} k'^2 P_z(k', 0)} \right)^2 + \left( 1 - \frac{c_{2\Theta} \sum_k k^2 P_x(k, 0)}{s_{2\Theta} \sum_{k'} k'^2 P_z(k', 0)} \right)^2 \right]^{1/2}
\]

\(^3\)If one assumes the existence of a synchronised solution, the oscillation frequency must be as given in eq.(22), from the self-consistency of eq.(3).
III. MSW SHARPENING

We now consider the case where there is an MSW resonance, but the collision rates are still flavour blind. For example, imagine that our plasma has a nucleon component, or a lepton excess. The effective potential arises at 1 loop level, and is given by

\[
V = \frac{3G_F^2 m_e^2}{2\pi^2} \left[ (N_p + N_n) \ln \left( \frac{m_W^2}{m_e^2} \right) - N_p - \frac{2}{3} N_n \right].
\]  

(27)

We need only a very small nucleon component for this potential to be significant. For example, assuming a mass squared difference of \(\delta m^2 \sim 10^{-3}\text{eV}^2\), a nucleon density of \(N_n \sim 10^{-3} N_e\) would give rise to an MSW resonance at a temperature of roughly 10 MeV. If the nucleon density were smaller, the resonance would occur at higher temperature.

To illustrate the type of effect that could arise, we assume, by way of example, an exponentially decreasing effective matter potential. We take for our initial condition a neutrino in a definite flavour state with a thermal distribution of momentum values, and plot below the probability that the neutrino remains in that initial flavour state. In the absence of neutrino-environment interaction, each momentum state goes through the resonance at a different time resulting in a broad transition region, while for moderate interaction rate the oscillations get dephased and the neutrino ends up in an equal mixture of the two flavour states. For large interaction rate, we see the effect of synchronised behaviour - all momentum states go through the resonance at the same time, resulting in a much sharper transition.

![Fig. 3. “MSW sharpening”. The probability that the neutrino remains a \(\nu_\mu\), given an initial thermal distribution of \(\nu_\mu\). The dashed, dotted and solid curves are for \(T=1, 10\) and 50 MeV respectively, with \(\delta m^2 = 0.001\text{eV}^2\) and \(\sin^2 \theta_0 = 0.2\). The time is in units of \(1/|\omega(V = 0)|\).]

The sharpened MSW resonance when occurs \(\langle \lambda \rangle = 0\), where the matter potential has the value

\[
V_{\text{eff}}(\text{res}) = \frac{\sum k^2 e^{-k/T} c_{2\theta_0} \delta m^2 / (2k)}{\sum k^2 e^{-k/T}}.
\]

(28)
IV. CONCLUSION

When rapid scattering processes are flavour blind, a neutrino in a mixture of momentum states undergoes synchronised flavour oscillations. This collective, dissipationless behaviour has been demonstrated numerically and derived analytically. Under these circumstances MSW transitions become sharper as all momentum states go through the resonance together.

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[1] N. F. Bell, R. F. Sawyer and R. R. Volkas, quant-ph/0008133.
[2] A. J. Leggett et al, Rev. Mod. Phys. 59, 1 (1987), and references contained therein.
[3] J. R. Friedman, V. Patel, W. Chen, S. K. Tolpygo and J. E. Lukens, Nature, vol 406, 6 July 2000, p43.
[4] A. Kostelecky and S. Samuel, Phys. Rev. D52, 621 (1995); J. Pantaleone, Phys. Rev. D58, 073002 (1998).
[5] B. H. J. McKellar and M. J. Thomson, Phys. Rev. D49, 2710 (1994). See also G. Raffelt, G. Sigl and L. Stodolsky, Phys. Rev. Lett. 70, 2363 (1993).
[6] R. A. Harris and L. Stodolsky, Phys. Lett. 116B, 464 (1982); A. Dolgov, Sov. J. Nucl. Phys. 33, 700 (1981); K. Enqvist, K. Kainulainen and J. Maalampi, Nucl. Phys. B 349, 754 (1991); See P. N. Loreti and A. B. Balantekin, Phys. Rev. D 50 4762 (1994) for the derivation of a similar equation for the case of noisy matter density.
[7] N. F. Bell, R. R. Volkas and Y. Y. Y. Wong, Phys. Rev. D 59 113001 (1999); R. R. Volkas and Y. Y. Y. Wong, Phys. Rev. D 62 093025 (2000); K. S. M. Lee, R. R. Volkas and Y. Y. Y. Wong, Phys. Rev. D 62 093026 (2000).
[8] F. J. Botella, C. -S. Lim and W. J. Marciano. Phys. Rev D35, 896 (1987).