Stability of Utility Maximization in Incomplete Markets

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joint work with Kasper Larsen, CMU

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Outline

1. Generalia
2. Well-posedness
3. Well-posedness: what we actually can do
4. Under the hood
5. Summary and Extensions
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Financial Markets

The model

- Pick your favorite viable (NA) financial model:
  - a semimartingale stock-price process 
    \[ (S_t)_{t \in [0, T]}, \text{ on } (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P}) \].
  - a bond (numeraire) is normalized to \( S_0^0 \equiv 1 \).

- Trading in this model is implemented by a choice of a portfolio \((H_t)_{t \in [0, T]} \in L(S) \) (+ admissibility requirements).

- The terminal wealth of the portfolio (strategy) \( H \) is
  \[ X_T^{x,H} = x + \int_0^T H_u \, dS_u, \]
  when the initial wealth is \( x \).
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Utility Maximization

Utility functions

- \( U : (0, \infty) \to \mathbb{R} : C^1 \), strictly concave, strictly increasing,
- \( \lim_{x \to 0} U'(x) = +\infty \), \( \lim_{x \to \infty} U'(x) = 0 \). (Inada conditions).
- \( \limsup_{x \to \infty} \frac{xU'(x)}{U(x)} < 1 \) (Reasonable asymptotic elasticity).

The utility-maximization problem

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    u(x) = \sup_{H \in \ldots} \mathbb{E}[U(x + \int_0^T H_u \, dS_u)].
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State of the art

Markowitz, Merton, Pliska, Cox, Huang, Karatzas, Lehoczky, Shreve, Xu, Kramkov, Schachermayer, . . .:

Under very mild regularity conditions, the optimal \( \hat{H} \) and \( \hat{X} \)
1) exist and 2) are unique.
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Requirements of Jacques Hadamard (1902) - still a template in applied mathematics

- existence
- uniqueness
- well-posedness (stability with respect to perturbations in problem data)

Another way of looking at utility maximization.

\[(x, U, (S_t)_{t \in [0, T]}) \mapsto \hat{H} \text{ (or } \hat{X})\]

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What are the continuity properties of the function \( S \mapsto \hat{X} \)?
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Marginal Utility-Based Prices

- When a contingent claim $B$ is replicable in the market, the **NA-principle uniquely defines the “price”** $p$ of $B$.
- When markets are incomplete, another ingredient is needed: the **risk-profile** of the investor, e.g.
- **MUBP (Davis’) price**: number $p \in \mathbb{R}$ such that

$$\sup_{H \in \ldots, q \in \mathbb{R}} \mathbb{E} \left[ U \left( X_{T}^{x,H} + q(B - p) \right) \right] \leq \sup_{H \in \ldots} \mathbb{E} \left[ U \left( X_{T}^{x,H} \right) \right]$$

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What can we say about the continuity of the MUBP, as a function of the model $(S_t)_{t \in [0,T]}$?

**Warning!** - Hugonnier, Kramkov and Schachermayer (2005) show that MUBP exists, but does not have to be unique; a whole interval $[\mathcal{P}, \mathcal{P}]$ of prices is possible.
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Market Price of Risk

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- $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P})$ - a filtered probability space (usual conditions)
- $(S_t)_{t \in [0, T]}$ - a continuous $(\mathcal{F}_t)_{t \in [0, T]}$-semimartingale.
- Schachermayer (1995): if NA holds, there exists
  - a continuous local martingale $M$, and
  - a predictable process $\lambda$ such that $\lambda^2 \cdot \langle M \rangle < \infty$ a.s., and
  \[
  S_t = M_t + \int_0^t \lambda_u d\langle M \rangle_u, \ t \in [0, T].
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- $(\lambda_t)_{t \in [0, T]}$ can be called the market price of risk - a sufficient statistic for pricing and utility maximization (not for hedging, though).
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3. Define \(S^\lambda_t = M_t + \lambda \cdot \langle M \rangle_t, \ t \in [0, T], (S^0)^\lambda \equiv 1\).

4. Solve the utility-maximization problem and for each \(\lambda \in \Lambda\) compute
   - \(u^\lambda(\cdot)\) - the value function
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The central problem

Questions reformulated

Find a natural pair of topologies on \( \Lambda \) and on the set \( \mathbb{L}^0 \) of finite random variables such that the mappings (correspondence)

1. \( \Lambda \ni \lambda \mapsto u^\lambda (x) \),
2. \( \Lambda \ni \lambda \mapsto \hat{X}^\lambda_t \), and
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- **$U$-dependence** Jouini and Napp (2004), Carasus and Rásonyi (2005)
- **market-dependence** Hubalek and Schachermayer (1998), El-Karoui, Jeanblanc-Picqué and Shreve (1998), Prigent (2003)
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Towards the main result

An example

\[ U(x) = \log(x). \]  
With \( Z(\lambda) = \mathcal{E}(-\lambda \cdot M)_T, \) \( ||\lambda||_2^2 = \mathbb{E}[\int_0^T \lambda_u^2 \, d\langle M\rangle_u], \)

\[ u^\lambda(x) = \log(x) + \frac{1}{2} ||\lambda||_2^2, \quad \hat{X}^\lambda_T = \frac{x}{Z(\lambda)}. \]

So, for convergence of \( u^\lambda, \) the topology on \( \Lambda \) should be aware of this.

Appropriate topologies I - we can do a little better:

- Fact: \( || \cdot ||_2 \) convergence implies convergence in probability of \( Z(\lambda). \)
  Thus, \( \hat{X}^\lambda_T \) converges in probability, if \( \lambda \) converges in \( || \cdot ||_2. \)
- A topology \( \tau \) on \( \Lambda \) is called *appropriate* if the mapping \( \lambda \mapsto Z(\lambda) \) is continuous in probability.
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A puzzling example

- Complete Itô-process market, 1-dim Brownian filtration,
  \[ dS_t^\lambda = S_t^\lambda (\lambda_t \, dt + dB_t) \]

- Choose market-price-of-risk \( \lambda^n \) such that \( Z(\lambda_n) = \frac{1}{\mathbb{E}[f_n]} f_n \),

\[ f^n(\omega) \triangleq \begin{cases} 
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and \( p_n = \mathbb{P}[f^n = n] = \frac{1}{2} n^{-5}, \ q_n = \mathbb{P}[f^n = n^{-1}] = \frac{1}{2} n^{-3} \).

- Choice of \( p_n \) and \( q_n \) implies that \( \mathbb{E}[(Z(\lambda_n) - 1)^2] \to 0 \) and even \( \|\lambda_n\|_2 \to 0 \): markets converge towards \( dS_t = S_t \, dB_t \) in a very strong way (more than appropriately).
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Example cont’d

- Pick a power-utility $U(x) = \frac{4}{3}x^{3/4}$, and an initial wealth $x = 1$.
- The optimal terminal wealth is given by

$$\hat{X}_{T}^{\lambda_{n}} = I_{3/4}(y_{n}Z(\lambda_{n})) = y_{n}^{-4}(Z(\lambda_{n}))^{-4}, \quad y_{n}^{-4} = \frac{2}{3} + o(n),$$

so $\hat{X}_{T}^{\lambda_{n}} \to \frac{2}{3}$ in probability.
- In the limiting market, the price process $S$ is a martingale, so the optimal policy is not to invest in it at all: $\hat{X}_{T} = 1$.
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**V-relative compactness**

- **Legendre-Fenchel dual** $V : (0, \infty) \to \mathbb{R}$ of the utility $U$ is given by

$$V(y) = \sup_{x > 0} (U(x) - xy).$$

- A subset $\Lambda'$ of $\Lambda$ is said to be \textit{V-relatively compact} if the family

$$\{ V^+(Z(\lambda)) : \lambda \in \Lambda' \}$$

is uniformly integrable.

- If $U$ is bounded from above, the whole $\Lambda$ is $V$-relatively compact.
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The main result

Theorem

Let $\Lambda' \subseteq \Lambda$ be $V$-relatively compact, and let $\tau$ be an appropriate topology on $\Lambda$. Then

- for any $\lambda \in \Lambda'$, the function $u^\lambda : (0, \infty) \rightarrow \mathbb{R}$ is finite-valued, and for each $x > 0$ there exists an a.s.-unique optimal terminal wealth $\hat{X}_{T}^{x, \lambda}$ for the utility maximization problem.

- the following mappings are jointly continuous

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\Lambda' \times (0, \infty) \ni (\lambda, x) \mapsto u^\lambda(x) \in \mathbb{R}, \text{ and } \\
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|-------------------------------|-----------------------------|
| 1    Generalia                |                             |
| 2    Well-posedness           |                             |
| 3    Well-posedness: what we actually can do |   |
| 4    Under the hood           |                             |
| 5    Summary and Extensions   |                             |
A quick glance under

**Duality**

- The dual problem is relaxed:
  \[ \nu^\lambda(y) = \inf_{Q \in D^\lambda} \mathbb{V}(yQ) = \inf_{Q \in D^\lambda} \mathbb{E}[\mathbb{V}(y \frac{dQ}{dP})], \]
  where \( D^\lambda \) is the weak-* closure of \( M^\lambda \) (martingale “measures”) in \((L^\infty)^*\): dual elements are finitely-additive measures.

- Compactness of \( D^\lambda \) makes things happen (lower semi-continuity becomes full continuity)

**Structure of \( M^\lambda \)**

A nice multiplicative structure

\[ \left\{ \frac{dQ}{dP} : Q \in M^\lambda \right\}^* = \{ Z(\lambda)H_T : \langle H, M \rangle = 0 \}. \]

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Still under the hood: two results

Theorem

The ($\mathbb{P}$-Radon-Nikodym derivative of the) regular part of the solution optimal $\hat{Q}^*$ is the last element of a local martingale (as opposed to a supermartingale). In fact, any maximal regular part is!

Theorem

Let $\hat{D}^\lambda(y)$ denote the set of all optimal solutions $\hat{Q} \in D^\lambda$ of the dual problem. Then the interval $[\mathcal{P}^\lambda(x, B), \overline{\mathcal{P}}^\lambda(x, B)]$ of the MUBP for a contingent claim $B \in L^\infty$ is exactly equal to

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Summary and Extensions

Messages

- Oversensitivity to initial data is bad - requires study.
- Care is needed when utility maximization is used for investment/pricing.
- Estimate your parameters with a quadratic loss function on $\lambda$.

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- How about jumps? Other market perturbations? Random endowment?
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