On the approximations of slowly varying envelope and slowly varying profile in nonlinear optics

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Abstract. The comparative analysis of two specified in heading approximations is carried out. It is shown that both approximations mutually supplement each other, finding applications in nonlinear optics of pulse duration from nano- to femtoseconds.

1. Introduction
Since the 60s of the last century one began to create pulse lasers in laboratory conditions. Pulses of nanosecond duration were generated at that time. There were at once questions concerning the theoretical studies of interaction of such pulses with matter. In recent years duration of the generated pulses has reached femtoseconds. Such signals contain about one period of electromagnetic oscillations. Therefore their spectrum range is very wide. This spectrum can contain Fourier-components, which are resonant to various quantum transitions. As the result, it can lead to strong excitations of media. The simplest models of media lead to difficult nonlinear self-consistent equations. The two-level model well describes the interaction of medium with resonant or quasi-resonant optical pulses. In this case the carrier pulse frequency $\omega$ is close to the frequency $\omega_0$ of a quantum transition. To move ahead further, it is necessary to use various physically reasonable approximations. In the present article we will discuss two approximations.

From the first look at the heading of the present work, the difference between two designated approximations is not visible. There should not be confusion, as these approximations make absolutely different physical senses. The present work is devoted to the disclosure of the essence and determination of distinctions between these approximations.

2. Approximation of slowly varying envelope
Let the carrier frequency of an optical pulse belong to the visible or infrared spectral range ($\omega \sim 10^{15} \text{c}^{-1}$). Then the pulses whose duration $\tau_p$ lies in the range from nano- to picoseconds contain from one million to one thousand oscillations. If to designate the last number as $N$, then for such pulses it is possible to determine a small parameter

$$\mu_l = N^{-1} \sim (\omega \tau_p)^{-1} << 1.$$ (1)

The sense of condition (1) consists in the fact that here it is possible to define a concept of an envelope of a light pulse. This envelope is a function, which varies slowly at times of the order of the oscillation period and at distances about a wavelength. In this case the electric field of the pulse can be presented in the form

$$E(z,t) = \psi(z,t)e^{i(\omega t-kz)} + \psi^*(z,t)e^{-i(\omega t-kz)},$$ (2)
where \( z \) is the distance passed by the pulse, \( t \) is time, \( k \) is the wave number, corresponding to the carrier frequency \( \omega \), \( \psi \) is the complex pulse envelope, which satisfies the conditions

\[
|\psi_t| < \omega |\psi|, \quad |\psi_z| < k |\psi|.
\]

(3)

Let us illustrate the application of the slowly varying envelope approximation (SVEA) on the example of interaction of a light pulse with the two-level medium. The corresponding self-consistent set of equations looks like [1]

\[
E_{zz} - \left( \frac{n_m}{c} \right)^2 E_u = \left( \frac{4\pi}{c^2} \right) P_n = a U_n = -a \omega_0 V_t,
\]

(4)

\[
U_i = -\alpha_0 V, \quad V_i = \omega_0 U + 2 (dE/\hbar) W, \quad W_i = -2 (dE/\hbar) V .
\]

(5)

Here \( c \) is the speed of light, \( \hbar \) is the Planck constant, \( n_m \) is the refractive index of the matrix, \( a = 8\pi d n / c^2 \), \( n \) is the concentration of two-level atoms, \( d \) is the dipole moment of the considered quantum transition, the material variables \( U \) and \( W \) have the meaning of dimensionless induced atomic dipole moment and half the difference of population of the excited and basic quantum levels (population inversion), respectively, \( V \) is the growth rate of the change of the induced dipole moment.

Despite the simplicity of the model of the two-level medium, the set of equations (4) and (5) is very difficult. This set is nonlinear, and the coefficients in two last equations are quickly changing on times of the period of light oscillations. This set would be much simpler if the temporal scale of changes of dynamical variables increased up to the duration of light pulses. The SVEA allows to make this. Substituting (2) into (5) and neglecting the second derivatives of the complex envelope \( \psi \), according to inequalities (3), we obtain approximately

\[
E_u = \left( 2i \omega \psi_t - \omega^2 \psi \right) e^{i(\omega z - k z)} + c.c., \quad E_{zz} = \left( -2i k \psi_z - k^2 \psi \right) e^{i(\omega z - k z)} + c.c.
\]

(6)

Let us return to system of the material equations (5) now. Let us consider that \( S = U + i V \). Then from the first two equations (5) we have

\[
S_i = i \omega_0 S + 2i (dE/\hbar) W .
\]

(7)

In the absence of the light pulse field (\( E = 0 \)) this equation describes the movement of a free complex oscillator with eigen-frequency. In this case we have \( S = \exp(i \omega_0 t) \). Obviously, it corresponds to the fact that the dipole moment of the excited two-level atom oscillates on the frequency of its quantum transition. The last term in the right-hand part can be considered as an external force the frequency of which is equal to the carrier frequency \( \omega \) of the light pulse. It is clear, that steady state oscillations of the oscillator occur on this frequency. Therefore, we will write

\[
S(z,t) = R(z,t) e^{i(\omega z - k z)},
\]

(8)

where \( R(z,t) \) is the complex envelope of the atomic dipole moment.

Presenting the variable \( V \) in the form

\[
V = \left( S - S^* \right) / 2i = -i \left( R e^{i(\omega z - k z)} - R^* e^{-i(\omega z - k z)} \right) / 2
\]

(9)

and neglecting the derivative of the envelope, we will have

\[
V_i = \omega \left( R e^{i(\omega z - k z)} + R^* e^{-i(\omega z - k z)} \right) / 2 .
\]

Substituting this expression, and also (2) and (6) into (4), after equating expressions at \( e^{i(\omega z - k z)} \) and \( e^{-i(\omega z - k z)} \) in the left- and right-hand parts, we obtain

\[
\psi_z + \left( \frac{n_m}{c} \right) \psi_t = -i \beta R,
\]

(10)
where $\beta = 4\pi d^2 m n_{0b} / (\hbar c n_m)$.

By obtaining (10) we have equated to zero coefficients at the free terms $\psi$ and $\psi^*$. It has allowed to find the dispersion equation $k = \omega n_{0m} / c$.

Thus, the use of SVEA has allowed to reduce wave equation (4) from the second order to the first order with respect to the derivatives.

Substituting (8) and (2) into (7), we find

$$R_i = i(\omega_0 - \omega) R + i(\psi + \psi^* e^{-2i(\omega - \omega_0)}) W.$$  

Here it is possible to neglect the term oscillating on the frequency $2\omega$. Really, the characteristic temporal scale of change of $\psi$ corresponds to a pulse duration $\tau_p$ that is, owing to (3), much larger than the period of oscillations of the imaginary exponent. Then with a good accuracy we have

$$R_i = i \Delta R + i \psi W,$$  

where $\Delta = \omega_0 - \omega$ is the frequency detuning from resonance.

Substituting (2), (8) and (9) into third equation (5) and also neglecting the terms oscillating on the frequency $2\omega$, we obtain

$$W_i = i(\psi^* R - \psi R^*) / 2.$$  

Equations (10) – (12) are well-known and bear the name of the Maxwell — Bloch (MB) system. It was the first integrable system generating solutions in the form of optical solitons [2].

Let us consider a case when frequency detuning meets the quasi-resonance condition [3]

$$\mu_2 \equiv (\Delta \tau_p)^{-1} \ll 1.$$  

In this case it is possible to exclude the material variables from system (10) – (12). For this purpose we rewrite (11) in the form

$$R = -\psi W / \Delta - i R_i / \Delta.$$  

The ratio of the second term in the right-hand part of (14) to the left-hand part is $\sim \mu_2 << 1$. Therefore, it is possible to use the method of consecutive approximations with respect to the second term in the right-hand part of (14). In the zero-order approximation we have $R = -\psi W / \Delta$.

Substituting this expression into the mentioned small term, we will find as a first approximation

$$R = -(\psi / \Delta) W + i \Delta^{-2} (\psi W).$$  

Continuing the approximations, we will come to the Crisp expansion [4]

$$R = -(\psi / \Delta) W + i \Delta^{-2} (\psi W) + \Delta^{-3} (\psi W).$$  

Let us take into consideration that the interaction of the pulse with the medium under quasi-resonance conditions is weak. Therefore, in the second and third terms of the right-hand part of (15) we will neglect the change of the difference of population densities, believing $W = W_\infty$. Then we have

$$R = -(\psi / \Delta) W + i W_\infty \Delta^{-2} \psi_t + W_\infty \Delta^{-3} \psi_{tt}.$$  

As the difference of the population densities of atomic quantum levels under quasi-resonance conditions changes slightly, we will substitute (15a) into (12), restricting ourselves to the first two terms of this expression. Then $W_i = -W_\infty (\psi_\psi^* + \psi^* \psi_t^*) / 2 \Delta^2$. After integration of this equation we obtain

$$W = W_\infty \left(1 - \left|\psi_t\right|^2 / 2 \Delta^2\right).$$  

After substitution of (15a) and (16) into (10) we arrive at the Nonlinear Schrödinger (NLS) equation
\[ i\Phi_z = -\frac{k_z}{2}\Phi_{\tau\tau} - b|\Phi|^2\Phi, \quad (17) \]

where \( \Phi = \psi \exp(-iBw_c z/\Delta) \), \( \tau = t - z/v_z \), \( b = -\beta w_c / 2\Delta^3 \), \( k_z = \partial(v_z^{-1})/\partial\omega \) is the coefficient of group velocity dispersion, the linear group velocity \( v_g \) is determined by the expression \( 1/v_g = n_m/c - \beta w_c / \Delta^2 \).

Generally, NLS describes propagation of optical solitons in the isotropic non-resonant dielectrics having cubic (Kerr) nonlinearity \([5, 6]\). The model of the two-level medium accepted here is only an illustration of this fact.

3. Approximation of slowly varying profile

The pulse with the duration near 1 femtosecond may contain one or several periods of light oscillations. According to the terminology which has developed so far such signals are called the "few-cycle pulses" (FCP) \([7]\). In this case the parameter \( \mu_t \) stops being small and its value becomes unit order. It is clear, that in such conditions it is impossible to tell about pulse envelope. Here it is necessary to look for other methods and approaches. In 1973 in work \([8]\) an alternative approach to SVEA has been offered. Authors of the mentioned work used the approach of the medium of small concentration of two-level atoms. This approximation corresponds to the inequality

\[ \mu_3 = 8\pi d^2 n/\hbar\omega_0 << 1. \quad (18) \]

The right-hand part of (4) is proportional to small parameter \( \mu_t \), and therefore it can be considered small. In such a situation it is possible to apply the slowly varying profile approximation (SVPA) the essence of which is revealed below. If to put the right-hand part of (4) equal to zero, then we have the well-known solution consisting of superposition of two waves. These waves propagate respectively along and against a \( z \)-axis with a velocity \( c/n_m \). In approximation (18), the part of the pulse field \( E(z,t) \) scattered back, against a \( z \)-axis, is negligible. Therefore, it is possible to consider that the pulse propagates only along the \( z \)-axis. That is, when the right-hand part equals zero, we have the solution \( E(z,t) = E(t - n_m z/c) \). This assumption allows to lower the order of wave equation (4). To account for the right-hand part of (4), we will determine the "local" time and "slow" coordinate: \( \tau = t - n_m z/c, \zeta = \mu_3 z \). We assume that \( E(\tau,\zeta) \). Then

\[ E_z = \tau \partial_z E + \zeta \partial_z E = -(n_m/c)E_t + \mu_3 E_\zeta, \quad E_t = \tau \partial_t E + \zeta \partial_t E = E_z. \]

Neglecting the small term, proportional to \( \mu_3^2 \), we will write down

\[ E_{zz} \approx (n_m/c^3)E_{\tau\tau} - 2\mu_3 (n_m/c)E_{\zeta\zeta}, \quad E_\tau = E_{\tau\tau}. \]

As a result, equation (4) after integration with respect to \( g \) and return to the initial independent variables will take the form

\[ E_z + (n_m/c)E_t = gV. \quad (19) \]

Here \( g = 4\pi d n_0 c / n_m \).

The procedure of reduction of equation (4) to look (19) is carried out in detail here in order that it is clear that this reduction has no slightest hint on that how many oscillations of the light field can be contained in the pulse. Then there can be any quantity. This is the advantage of SVPA in comparison with SVEA. On the other hand, it follows from the procedure of reduction of wave equation (4), which is carried out above, that the velocity of an optical pulse slightly differs from linear velocity. Therefore, under condition of SVPA it is impossible to describe considerable reduction of velocity of the pulse propagation.
The procedure which is carried out above allows to understand the sense of the discussed approximation. The dependence of the form \( E = E(\tau, \zeta) \) reflects the fact of slow deformation of the profile of electric field in the fame of reference moving with a linear velocity \( c / n_m \).

Let us notice that the term "the slowly varying profile approximation" has passed to optics from works on acoustics [9].

System (19), (5) is called the reduced Maxwell – Bloch (RMB) system. As was shown in [8], this system is integrable, possessing soliton solutions.

Further in various works (see, e.g. [10 – 14]) the refusal from the approximation of the medium of small atomic concentration has been made. The approximations of weak nonlinearity and (or) dispersion were used. Also the refusal from the model of the two-level medium has been made [15]. Various phenomenological models of media were offered [16].

Let us shortly pay attention to the medium with square nonlinearity and weak dispersion. The polarizing response of such medium can be presented as \( P = P_{\text{lin}} + P_{\text{non}} \), where linear and nonlinear parts of the response have respectively the appearance \( P_{\text{lin}} = \int_0^\infty \chi(t')E(t-t')dt' \), \( P_{\text{non}} = \chi^{(2)} E^2 \), \( \chi(\tau) \) and \( \chi^{(2)} \) are linear and nonlinear susceptibilities. Making expansion \( E(t-t') = E(t) + t'^2 E_g / 2 - \ldots \) and substituting the obtained expression for \( P \) into the right-hand part of equation (4) at \( n_m = 1 \), after using SVPA we will come to the well-known Korteweg – de Vries equation:

\[
E_\varepsilon + qE_\varepsilon E_\tau - \alpha E_{\tau\tau\tau} = 0 .
\]

Here \( \tau = t - n_0 z / c \), \( n_0 = \sqrt{1 + 4\pi \chi_0} \) is the low-frequency refractive index, \( \chi_0 = \int_0^\infty \chi(\tau)d\tau \), \( \chi_\omega = \int_0^\infty \chi(\tau)e^{-i\omega\tau}d\tau \) are the low-frequency and high-frequency susceptibilities, respectively, \( q = 4\pi \chi^{(2)} / (cn_0) \), \( \alpha = - (cn_0)^{-1} \int_0^\infty \tau^2 \chi(\tau)d\tau = \left( \frac{\partial^2 \chi_\omega}{\partial \omega^2} \right)_{\omega=0} \) are the nonlinear and dispersion coefficients, respectively.

Various solutions of equation (20) are in detail discussed in monographs (see, e.g. [2]). Here it is more important to notice that on the basis of SVEA and SVPA it is possible to obtain the systems describing well-known effects in nonlinear optics. As an example we will consider the effect of optical rectification [17]. For this purpose we will present electric field in the form

\[
E(z,t) = \Sigma + \psi(z,t)e^{i(\omega - k_z z)} + \psi(z,t)e^{-i(\omega - k_z z)},
\]

(21)

where \( \Sigma \) is the low-frequency pulse component, generated by means of optical rectification. Usually the spectrum of this component lies in the terahertz area. Therefore, we will call it a terahertz component. Let us neglect the own nonlinearity and dispersion of terahertz components. Then,\n
\[
P_{\text{non}} = \left( 2\chi^{(2)} \Sigma \psi^{(\omega - k_z)} + c.c. \right) + 2\chi^{(2)} \left| \psi \right|^2, \]

\[
P_{\text{lin}} = \chi_\omega \psi^{(\omega - k_z)} - i(\partial \chi_\omega / \partial \omega)\psi - 0.5 \left( \frac{\partial^2 \chi_\omega}{\partial \omega^2} \right) \psi + \ldots + c.c.
\]

From here, from (21) and (4) after using SVPA we will obtain [15]

\[
i \psi_{\tau\tau} + \left( k_z / 2 \right) \psi_{\tau\tau} = \beta \Sigma \psi ,
\]

(22)

\[
\Sigma_{\tau} = - \beta \left( \left| \psi \right|^2 \right)_{\tau},
\]

(23)

where \( k_z = \partial \left( 1 / n_\varepsilon \right) / \partial \omega = \left( 4\pi \right) \left( cn_\omega \right) \left[ \partial \chi_\omega / \partial \omega + 0.5 \omega^2 \partial^2 \chi_\omega / \partial \omega^2 - \left( \omega / 4\pi \right) \left( \partial n_\omega / \partial \omega \right)^2 \right], \)

\( n_\omega = \sqrt{1 + 4\pi \chi_\omega / c^2} \) is the refractive index corresponding to the frequency \( \omega \). Here we have put that the
group velocity \( v_g = c \left( n_\omega + \alpha c n_\omega \frac{\partial}{\partial \omega} \right)^{-1} \) of optical components is equal to the phase velocity
\( v_p = c / n_0 \) of the terahertz component. In this case generation of terahertz components happens most intensively, \( \beta_1 = 4\pi \chi(2) \omega/(cn_0), \beta_2 = 4\pi \chi(2)/(cn_0) \).

System (22), (23) is known as the Yajima - Oikawa system [18]. The soliton solution of this system describes the process of an optical way of generation of terahertz radiation [19], finding now various applications [20].

Let us highlight that in system (22), (23) \( \psi \) is the envelope of the pulse electric field. At the same time, \( \Sigma \) is the electric field of a terahertz component. Thus, by derivation of system (22), (23) both approaches – SVEA and SVPA were used.

4. Conclusion
The analysis which is carried out above allows to draw a conclusion that both approaches (SVEA and SVPA) play an important role in nonlinear optics of light pulses. On the one hand, the approach of SVPA allows to describe propagation in the media of both quasi-monochromatic pulses, and FCP. At the same time by means of SVEA it is possible to consider only dynamics of quasi-monochromatic pulses. On the other hand, by means of the approach of SVEA it is possible to describe propagation of pulses in the media with the strong dispersion (linear and nonlinear) leading to considerable reduction of group velocity in comparison with linear velocity of light. The approach of SVPA is adapted only for cases when dispersion is small, and reduction of group velocity is insignificant.

Thus, it is possible to state that both approaches (SVEA and SVPA) are absolutely different in the physical sense. At the same time, both approaches mutually supplement each other at the studies of problems of the modern nonlinear optics.

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References
[1] Allen L and Eberly J H 1978 Optical resonance and two-level atoms (New York: John Wiley & Sons)
[2] Lamb G L 1980 Elements of soliton theory (New York: John Wiley & Sons)
[3] Basharov A M and Maimistov A I 2000 Optika i Spektroskopiya 88 428
[4] Crisp M D 1973 Phys. Rev. A 8 2128
[5] Agrawal G P 1989 Nonlinear fiber optics (New York: Academic Press)
[6] Kivshar Yu S and Agrawal G P 2003 Optical solitons (New York: Academic Press)
[7] Brabec T and Krausz F 2000 Rev. Mod. Phys. 72 545
[8] Eilbeck J C, Gibbon J D, Caudrey P J and Bullough R K 1973 J. Phys. A 6 1337
[9] Akhmanov S A 1986 Uspekhi Fizicheskikh Nauk 49 361
[10] Belenov E M, Nazarkin A V and Ushchapovskii V A 1991 Sov. Phys. JETP 73 422
[11] Sazonov S V and Trifonov E V 1994 J. Phys. B 27 L7
[12] Leblond H, Sazonov S V, Mel’nikov I V, Mihalache D and Sanchez F 2006 Phys. Rev. A 74 063815
[13] Maimistov A I 2000 Quantum Electron 30 287
[14] Sazonov S V 2016 Opt. Commun. 380 480
[15] Sazonov S V and Sobolevskii 2003 JETP 96 1019
[16] Kozlov S A and Sazonov S V 1997 JETP 84 221
[17] Klyshko D N 1986 Physical Fundamentals of Quantum Electronics (in Russian, Moscow: Mir)
[18] Yajima N and Oikawa M 1976 Prog. Theor. Phys. 56 1719
[19] Sazonov S V 2012 JETP Lett. 96 263
[20] Hiroi H and Tanaka T 2016 J. Phys. Soc. Japan 85 082001