Equilibrium states of interval maps for hyperbolic potentials

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Abstract
We study the thermodynamic formalism of sufficiently regular interval maps for Hölder continuous potentials. We show that for a hyperbolic potential there is a unique equilibrium state, and that this measure is exponentially mixing. Moreover, we show the absence of phase transitions: the pressure function is real analytic at such a potential.

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1. Introduction
In this paper we study the thermodynamic formalism of sufficiently regular interval maps for Hölder continuous potentials. The case of a piecewise monotone interval map $f : I \to I$, and a potential $\varphi : I \to \mathbb{R}$ of bounded variation satisfying

\[ \sup_{I} \varphi < P(f, \varphi), \]

where $P(f, \varphi)$ denotes the pressure, is very well understood. Most results apply under the following weaker condition:

For some integer $n \geq 1$, the function $S_n(\varphi) := \sum_{j=0}^{n-1} \varphi \circ f^j$ satisfies

\[ \sup_{I} \frac{1}{n} S_n(\varphi) < P(f, \varphi). \quad (1.1) \]

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In what follows, a potential $\varphi$ satisfying this condition is said to be hyperbolic for $f$. See for example [BK90, DKS90, HK82, Kel85, LSV98, Rue94] and references therein, as well as Baladi’s book [Bal00, section 3]. The classical result of Lasota and Yorke [LY73] corresponds to the special case where $f$ is piecewise $C^2$ and uniformly expanding, and $\varphi = -\log |Df|$.

For a complex rational map in one variable $f$, and a Hölder continuous potential $\varphi$ that is hyperbolic for $f$, a complete description of the thermodynamic formalism was given by Denker, Haydn, Przytycki, and Urbanski in [Hay99, DPU96, DU91a, Prz90], extending previous results of Freire et al [FLM83, Mañ83], and Ljubich [Lju83]. See also the alternative approach of Szostakiewicz et al [SUZ11] that also yields the absence of phase transitions.

In this paper we extend these results to the case of a sufficiently regular interval map and a Hölder continuous potential, with the purpose of applying them in the companion paper [LRL14]. We obtain our main results by constructing a conformal measure with the Patterson-Sullivan method, and then using a result of Keller from [Kel85]. This approach is more efficient than the inducing scheme approach of [BT08], as it does not rely on any bounded distortion hypothesis, and it applies to a larger class of maps, including maps with flat critical points.

We now proceed to describe our main results more precisely. The class of maps we consider is introduced in section 1.1, and in section 1.2 we recall the definition of the function spaces defined by Keller in [Kel85]. Our main results are stated in section 1.3.

### 1.1. Interval maps

Let $I$ be a compact interval in $\mathbb{R}$. A continuous map $f : I \to I$ is multimodal if it is not injective, and if there is a finite partition of $I$ into intervals on each of which $f$ is injective.

**Definition 1.1.** Let $f : I \to I$ be a multimodal map. The Julia set $J(f)$ of $f$ is the complement of the largest open subset of $I$ on which the family of iterates of $f$ is normal.

In contrast with the complex setting, the Julia set of a multimodal map might be empty, reduced to a single point, or might not be completely invariant. However, if the Julia set of such a map $f$ is not completely invariant, then it is possible to make an arbitrarily small smooth perturbation of $f$ outside a neighbourhood of $J(f)$, so that the Julia set of the perturbed map is completely invariant and coincides with $J(f)$. For background on the theory of Julia sets, see for example [dMvS93].

Given a differentiable map $f : I \to I$, a point of $I$ is critical for $f$ if the derivative of $f$ vanishes at it. We denote by $\text{Crit}(f)$ the set of critical points of $f$.

In what follows we denote by $\mathcal{M}$ the collection of all those differentiable multimodal maps $f$ such that:

- $Df$ is Hölder continuous;
- $\text{Crit}(f)$ is finite;
- $J(f)$ contains at least 2 points and is completely invariant.

### 1.2. Keller spaces

In this subsection, let $X$ be a compact subset of $\mathbb{R}$, and let $m$ be an atom-free Borel probability measure on $X$. We consider the equivalence relation on the space of complex valued functions

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2 In this setting, most of the results have been stated for a potential $\varphi$ satisfying the condition $\sup \varphi < P(f, \varphi)$ that is more restrictive than $\varphi$ being hyperbolic for $f$. General arguments show they also apply to hyperbolic potentials, see [IRRL12, section 3].

3 This last property can only happen if there is a turning point in the interior of the basin of a one-sided attracting neutral periodic point, that is eventually mapped to this neutral periodic point.
defined on \( X \), defined by agreement on a set of full measure with respect to \( m \). Denote by \( d \) the pseudo-distance on \( X \) defined by
\[
d(x, y) := m(\{ z \in X : x \leq z \leq y \text{ or } y \leq z \leq x \}).
\]
Note that for all \( x \) in \( X \) and \( \varepsilon > 0 \), the set
\[
B_d(x, \varepsilon) := \{ y \in X : d(x, y) < \varepsilon \}
\]
has strictly positive measure with respect to \( m \).

Given a function \( h : X \to \mathbb{C} \) and \( \varepsilon > 0 \), for each \( x \) in \( X \) put
\[
\text{osc}(h, \varepsilon, x) := \text{ess-sup} \{ |h(y) - h(y')| : y, y' \in B_d(x, \varepsilon) \}
\]
and
\[
\text{osc}_1(h, \varepsilon) := \int_X \text{osc}(h, \varepsilon, x) \, dm(x).
\]

Fix \( A > 0 \), and for each \( \alpha \) in \( (0,1] \) and each \( h : X \to \mathbb{C} \), put
\[
|h|_{\alpha,1} := \sup_{\varepsilon \in (0,A]} \frac{\text{osc}_1(h, \varepsilon)}{\varepsilon^\alpha}
\]
and
\[
\|h\|_{\alpha,1} := \|h\|_1 + |h|_{\alpha,1}.
\]

Note that \( |h|_{\alpha,1} \) and \( \|h\|_{\alpha,1} \) depend only on the equivalence class of \( h \). Let \( H^{\alpha,1}(m) \) be the space of equivalence classes of functions \( h : X \to \mathbb{C} \) such that \( \|h\|_{\alpha,1} < +\infty \). Note that \( |\cdot|_{\alpha,1} \) and \( \|\cdot\|_{\alpha,1} \) induce a semi-norm and a norm on \( H^{\alpha,1}(m) \), respectively; by abuse of notation we denote these functions also by \( |\cdot|_{\alpha,1} \) and \( \|\cdot\|_{\alpha,1} \). Keller shows in [Kel85] that \( H^{\alpha,1}(m) \) is a Banach space with respect to \( \|\cdot\|_{\alpha,1} \). Some properties of these spaces are gathered in section 4.1.

1.3. Statement of results

To state our main results, we recall a few concepts of thermodynamic formalism, see, for example [Kel98], or [PU10] for background. Let \( (X, \text{dist}) \) be a compact metric space, and let \( T : X \to X \) be a continuous map. Denote by \( M(X) \) the space of Borel probability measures on \( X \) endowed with the weak* topology, and by \( M(X, T) \) the subspace of \( M(X) \) of those measures that are invariant by \( T \). For each measure \( \nu \) in \( M(X, T) \), denote by \( h_\nu(T) \) the measure-theoretic entropy of \( \nu \). For a continuous function \( \phi : X \to \mathbb{R} \), denote by \( P(X, \phi) \) the topological pressure of \( T \) for the potential \( \phi \), defined by
\[
P(T, \phi) := \sup \left\{ h_\nu(T) + \int_X \phi \, d\nu : \nu \in M(X, T) \right\}.
\]

An equilibrium state of \( T \) for the potential \( \phi \) is a measure at which the supremum above is attained.

For a multimodal map \( f \) and a continuous function \( \varphi : J(f) \to \mathbb{R} \), we denote \( P(f|_{J(f)}, \varphi) \) just by \( P(f, \varphi) \). Moreover, we say \( \varphi \) is hyperbolic for \( f \) if (1.1) is satisfied with \( I \) replaced by \( J(f) \) and \( f \) replaced by \( f|_{J(f)} \).

**Definition 1.2.** Let \( f \) be a multimodal map. Given a Borel measurable function \( g : J(f) \to [0, +\infty) \), a Borel probability measure \( \mu \) on \( J(f) \) is \( g \)-conformal for \( f \), if for each Borel set \( A \) on which \( f \) is injective, we have
\[
\mu(f(A)) = \int_A g \, d\mu.
\]

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**Theorem A.** Let \( f \) be an interval map in \( \mathcal{A} \). Then for every Hölder continuous potential \( \varphi : J(f) \to \mathbb{R} \) that is hyperbolic for \( f \) there is an atom-free \( \exp(P(f, \varphi) - \varphi) \)-conformal measure for \( f \). If in addition \( f \) is topologically exact on \( J(f) \), then the support of this measure is equal to \( J(f) \).

Given a multimodal map \( f \), and a continuous potential \( \varphi : J(f) \to \mathbb{R} \), denote by \( \mathcal{L}_\varphi \) the transfer or Ruelle-Perron-Frobenius operator, acting on the space of bounded functions defined on \( J(f) \) and taking values in \( \mathbb{C} \), defined as follows

\[
\mathcal{L}_\varphi(\psi)(x) := \sum_{y \in f^{-1}(x)} \exp(\varphi(y)) \psi(y).
\]

The following is our main result. It is obtained by combining theorem A, a result of Keller in [Kel85], and known arguments.

**Theorem B.** Let \( f \) be an interval map in \( \mathcal{A} \) that is topologically exact on \( J(f) \). Let \( \alpha \) be in \((0, 1]\), and let \( \varphi : J(f) \to \mathbb{R} \) be a Hölder continuous potential of exponent \( \alpha \) that is hyperbolic for \( f \). Finally, let \( \mu \) be an atom-free \( \exp(P(f, \varphi) - \varphi) \)-conformal measure for \( f \) given by theorem A. Then there is \( A > 0 \) such that for the space \( H^{\alpha, 1}(\mu) \) defined in section 1.2 with \( X = J(f) \), the following properties hold.

- **Spectral gap:** The operator \( \mathcal{L}_\varphi \) maps \( H^{\alpha, 1}(\mu) \) to itself, and \( \mathcal{L}_\varphi |_{H^{\alpha, 1}(\mu)} \) is bounded. Moreover, the number \( \exp(P(f, \varphi)) \) is an eigenvalue of algebraic multiplicity 1 of \( \mathcal{L}_\varphi |_{H^{\alpha, 1}(\mu)} \), and there exists \( \rho \) in \((0, \exp(P(f, \varphi)))\) such that the spectrum of \( \mathcal{L}_\varphi |_{H^{\alpha, 1}(\mu)} \) is contained in \( B(0, \rho) \cup \{\exp(P(f, \varphi))\} \).

- **Equilibrium state:** There is a unique equilibrium state \( \nu \) of \( f \) for the potential \( \varphi \). Moreover, this measure is absolutely continuous with respect to \( \mu \), and the measure-theoretic entropy of \( \nu \) is strictly positive. Finally, there is a constant \( C > 0 \) such that for every integer \( n \geq 1 \), every bounded measurable function \( \phi : J(f) \to \mathbb{C} \), and every \( \psi \in H^{\alpha, 1}(\mu) \), we have

\[
C_n(\phi, \psi) := \left| \int_{J(f)} \phi \circ f^n \cdot \psi \, dv - \int_{J(f)} \phi \, dv \int_{J(f)} \psi \, dv \right| \leq C \|\phi\|_{\infty} \|\psi\|_{\alpha, 1} \rho^n.
\]

- **Real analyticity of pressure:** For each Hölder continuous function \( \chi : J(f) \to \mathbb{R} \), the function \( t \mapsto P(f, \varphi + t \chi) \) is real analytic on a neighbourhood of \( t = 0 \).

**Remark 1.3.** It is well-known that the spectral gap property implies that the equilibrium state has strong stochastic properties. In theorem B we emphasize the exponential decay of correlations. See [Kel85, theorem 3.3] for the Central Limit Theorem and the Almost Sure Invariance Principle.

Given a compact subset \( X \) of \( \mathbb{R} \) and \( \alpha \) in \((0, 1]\), for each Hölder continuous function \( h : X \to \mathbb{C} \) of exponent \( \alpha \), put

\[
|h|_{\alpha} := \sup_{x, x' \in X, x \neq x'} \frac{|h(x) - h(x')|}{|x - x'|^\alpha} \quad \text{and} \quad \|h\|_{\alpha} := \|h\|_{\infty} + |h|_{\alpha}.
\]

Then for each \( A > 0 \) and each atom-free Borel probability measure \( m \) on \( X \), each Hölder continuous function \( h : X \to \mathbb{C} \) of exponent \( \alpha \) is in \( H^{\alpha, 1}(m) \) and we have

\[
\|h\|_{\alpha, 1} \leq 2^\alpha \max\{1, (\sup X - \inf X)^\alpha\}\|h\|_{\alpha}.
\]

see (4.1) and part 2 of proposition 4.1. Thus, the following corollary is a direct consequence of theorem B.
Corollary 1.4. Let $I$ be a compact interval of $\mathbb{R}$ and let $f : I \to I$ be an interval map in $\mathcal{A}$ that is topologically exact on $I$. Then for every Hölder continuous potential $\varphi : I \to \mathbb{R}$ that is hyperbolic for $f$, there is a unique equilibrium state $\nu$ of $f$ for the potential $\varphi$ and the measure-theoretic entropy of this measure is strictly positive. Moreover, if $\alpha$ in $(0, 1]$ is such that $\varphi$ is Hölder continuous of exponent $\alpha$, then there are constants $C > 0$ and $\rho$ in $(0, 1)$, such that for every integer $n \geq 1$, every bounded measurable function $\phi : J(f) \to \mathbb{C}$, and every Hölder continuous function $\psi : J(f) \to \mathbb{C}$ of exponent $\alpha$, we have

$$C_n(\phi, \psi) \leq C \|\phi\|_{\infty} \|\psi\|_\alpha \rho^n.$$  

Finally, for every Hölder continuous function $\chi : I \to \mathbb{R}$, the function $t \mapsto P(\varphi + t\chi)$ is real analytic on a neighbourhood of $t = 0$.

In [LRL14, theorem A] we show that for a map $f$ as in corollary 1.4, every Hölder continuous potential $\varphi$ is hyperbolic for $f$, provided that $f$ satisfies some additional regularity assumptions, that all the periodic points of $f$ are hyperbolic repelling, and that for every critical value $v$ of $f$ we have

$$\lim_{n \to +\infty} |Df^n(v)| = +\infty.$$  

Recall that a periodic point $p$ of $f$ of period $n$ is hyperbolic repelling, if $|Df^n(p)| > 1$. Thus, for such a map $f$ the conclusions of corollary 1.4 hold for all Hölder continuous functions $\varphi$, $\psi$, and $\chi$. See [LRL14, Main Theorem] for a more general formulation of this result.

For a Hölder continuous potential, corollary 1.4 improves [BT08, theorem 4] in various ways. The first is that in corollary 1.4 no bounded distortion hypothesis is assumed, in contrast with [BT08, theorem 4] where the existence of an induced map with bounded distortion is assumed. The second is that the hypothesis that the potential is hyperbolic in corollary 1.4 is weaker than the ‘bounded range’ condition assumed in [BT08, theorem 4]. See appendix A for the definition of the bounded range condition and for examples showing that this condition is more restrictive than hyperbolicity. The fact that our results hold for hyperbolic potentials, and not only for potentials satisfying the bounded range condition, is crucial to obtain the main results of the companion paper [LRL14]. Finally, corollary 1.4 applies to a larger class of interval maps, including maps having flat critical points.

1.4. Organization

The proof of theorem A occupies sections 2 and 3. In section 2 we show that for a hyperbolic potential the pressure function can be calculated using iterated preimages of any given point (corollary 2.2). This is then used in section 3 to apply the Patterson-Sullivan method to construct a conformal measure (proposition 3.2).

In section 4 we first recall some properties of the function spaces defined by Keller in section 4.1. In section 4.2 we recall the main results of [Kel85] (theorem 1) and deduce from it some known consequences needed for the proof of theorem B (corollary 4.4). The proof of theorem B is given in section 5.

2. Iterated preimages pressure

The main goal of this section is to prove the following proposition. For a multimodal map $f$ and a continuous potential $\varphi : J(f) \to \mathbb{R}$, put

$$\hat{\mathcal{L}}_{\varphi} := \exp(-P(f, \varphi)) \mathcal{L}_{\varphi}.$$  

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$$\hat{\mathcal{L}}_{\varphi} := \exp(-P(f, \varphi)) \mathcal{L}_{\varphi}.$$
Proposition 2.1. Let \( f \) be an interval map in \( \mathcal{A} \) that is topologically exact on its Julia set \( J(f) \), and denote by 1 the function defined on \( J(f) \) that is constant equal to 1. Then for every \( \varepsilon > 0 \), and every Hölder continuous potential \( \varphi : J(f) \to \mathbb{R} \) that is hyperbolic for \( f \), we have for every sufficiently large integer \( n \)

\[
\exp(-\varepsilon n) \leq \inf_{J(f)} \hat{L}_n^\varphi(1) \leq \sup_{J(f)} \hat{L}_n^\varphi(1) \leq \exp(\varepsilon n).
\]

The following corollary is a direct consequence of the proposition. This corollary is used in the next section.

Corollary 2.2. Let \( f \) be an interval map in \( \mathcal{A} \) that is topologically exact on \( J(f) \). Then for every Hölder continuous potential \( \varphi : J(f) \to \mathbb{R} \) that is hyperbolic for \( f \), and every point \( x_0 \) in \( J(f) \), we have

\[
P(f, \varphi) = \lim_{n \to +\infty} \frac{1}{n} \log \sum_{y \in f^{-n}(x_0)} \exp(S_n(\varphi)(y)).
\]

The remainder of this section is devoted to the proof of proposition 2.1, which depends on several lemmas.

Lemma 2.3. Let \( f \) be a Lipschitz multimodal map and let \( \varphi : J(f) \to \mathbb{R} \) be a Hölder continuous potential. Then for every integer \( N \geq 1 \), there is a constant \( C > 1 \) such that the function \( \tilde{\varphi} := \frac{1}{N} S_N(\varphi) \) satisfies the following properties:

1. The function \( \tilde{\varphi} \) is Hölder continuous of the same exponent as \( \varphi \), \( P(f, \tilde{\varphi}) = P(f, \varphi) \), and \( \varphi \) and \( \tilde{\varphi} \) share the same equilibrium states;
2. For every integer \( n \geq 1 \), we have

\[
\sup_{J(f)} |S_n(\varphi) - S_n(\tilde{\varphi})| \leq C.
\]

Proof. Let \( h : J(f) \to \mathbb{R} \) be defined by

\[
h := -\frac{1}{N} \sum_{j=0}^{N-1} (N-1-j) \varphi \circ f^j,
\]

and note that \( \varphi = \tilde{\varphi} + h \circ f \).

1. Since \( f \) is Lipschitz, \( \tilde{\varphi} \) is Hölder continuous of the same exponent as \( \varphi \). On the other hand, for every invariant measure \( \nu \) of \( f \), we have

\[
\int_{J(f)} \tilde{\varphi} \, d\nu = \int_{J(f)} (\varphi \circ f - h) \, d\nu = \int_{J(f)} \varphi \, d\nu,
\]

so \( P(f, \tilde{\varphi}) = P(f, \varphi) \), and \( \varphi \) and \( \tilde{\varphi} \) share the same equilibrium states.

2. Note that for every \( n \geq N \), we have

\[
S_n(\tilde{\varphi}) = S_n(\varphi \circ f - h) = S_n(\varphi) + h \circ f^n - h.
\]

This implies the desired inequality with \( C = (N-1)(\sup_{J(f)} \varphi - \inf_{J(f)} \varphi) \). □

Given an integer \( n \geq 1 \) and a point \( x \) in the domain of \( f \), a preimage \( y \) of \( x \) by \( f^n \) is critical if \( Df^n(y) = 0 \), and it is non-critical otherwise.
Lemma 2.4. Let $f$ be an interval map in $\mathcal{A}$ that is topologically exact on $J(f)$, and let $\tilde{\varphi} : J(f) \to \mathbb{R}$ be a Hölder continuous potential satisfying $\sup_{J(f)} \tilde{\varphi} < P(f, \tilde{\varphi})$. Then for every $\varepsilon > 0$ and every point $x_0$ of $J(f)$ having infinitely many non-critical preimages, there is $\delta > 0$ such that the following property holds: If for each integer $n \geq 1$ we denote by $\mathcal{D}_n$ the collection of diffeomorphic pull-backs of $B(x_0, \delta)$ by $f^n$, then

$$\liminf_{n \to +\infty} \frac{1}{n} \log \sum_{W \in \mathcal{D}_n} \inf_{W \cap J(f)} \exp(S_n(\tilde{\varphi})) \geq P(f, \tilde{\varphi}) - \varepsilon.$$ 

The proof of this lemma is based on Pesin’s theory, as adapted to interval maps in $\mathcal{A}$ by Dobbs in [Dob08, theorem 6], and on Katok’s theory, as adapted to complex rational maps by Przytycki and Urbański in [PU10, section 11.6]. Using [Dob08, theorem 6] instead of [PU10, corollary 11.2.4], the proofs in [PU10, section 11.6] apply without change to interval maps in $\mathcal{A}$.

Proof. In part 1 we show that there is $\delta_0 > 0$ and a forward invariant compact set $X$ on which $f$ is uniformly expanding, such that the desired assertion holds for every point $x_0$ in $X$ with $\delta = \delta_0$. In part 2 we deal with the general case using this special case.

1. Since by assumption $\sup_{J(f)} \tilde{\varphi} < P(f, \tilde{\varphi})$, there is $\varepsilon > 0$ so that $\varepsilon < P(f, \tilde{\varphi}) - \sup_{J(f)} \tilde{\varphi}$. Let $\nu$ be a measure in $\mathcal{M}(J(f), f)$ such that

$$h_\nu(f) + \int_{J(f)} \tilde{\varphi} \, d\nu \geq P(f, \tilde{\varphi}) - \varepsilon > \sup_{J(f)} \tilde{\varphi}.$$ 

Replacing $\nu$ by one of its ergodic components if necessary, assume $\nu$ is ergodic. We thus have

$$h_\nu(f) > \sup_{J(f)} \tilde{\varphi} - \int_{J(f)} \tilde{\varphi} \, d\nu \geq 0,$$

and then Ruelle’s inequality implies that the Lyapunov exponent of $\nu$ is strictly positive, see [Rue78]. By [PU10, theorem 11.6.1] there is a compact and forward invariant subset $X$ of $J(f)$ on which $f$ is topologically transitive, so that $f : X \to X$ is open and uniformly expanding, and so that

$$P(f|_X, \tilde{\varphi}|_X) \geq P(f, \tilde{\varphi}) - \varepsilon.$$ 

It follows that there is $\delta_0 > 0$ such that the desired property holds for every point $x_0$ in $X$ with $\delta = \delta_0$, see for example [PU10, proposition 4.4.3].

2. The hypothesis that $x_0$ has infinitely many non-critical preimages implies that there is a non-critical preimage $x_0'$ of $x_0$ such that all preimages of $x_0'$ are non-critical. Since $f$ is topologically exact on $J(f)$, there is a preimage of $x_0'$ in $B(X, \delta_0)$, and therefore there is an integer $n \geq 1$ and a non-critical preimage $x_0''$ of $x_0$ by $f^n$ in $B(X, \delta_0)$ all whose preimages are non-critical. It follows that there is $\delta > 0$ such that the pull-back of $B(x_0, \delta)$ by $f^n$ that contains $x_0''$ is contained in $B(X, \delta_0)$. Then the desired assertion follows from part 1. $\square$

Lemma 2.5. Let $f$ be an interval map in $\mathcal{A}$ that is topologically exact on $J(f)$, and let $\varphi : J(f) \to \mathbb{R}$ be a Hölder continuous potential that is hyperbolic for $f$. Then for every $\varepsilon > 0$ there is $N_0 > 0$ such that for every integer $n \geq N_0$, we have

$$\inf_{J(f)} \hat{\mathcal{L}}^n(\varphi)(1) \geq \exp(-\varepsilon n).$$
Proof. Let $C > 1$ be the constant given by lemma 2.3. Since $\varphi$ is hyperbolic for $f$, there is an integer $N \geq 1$ such that the function $\hat{\varphi} := \frac{1}{N} S_n(\varphi)$ satisfies $\sup_{j(f)} \hat{\varphi} < P(f, \varphi)$. By part 1 of lemma 2.3, the function $\hat{\varphi}$ is Hölder continuous and

$$P(f, \hat{\varphi}) = P(f, \varphi) > \sup_{j(f)} \hat{\varphi}.$$ 

In view of part 2 of lemma 2.3, to complete the proof of the lemma it suffices to prove that for every $\varepsilon > 0$ there is $N_0 > 0$ such that for every $n \geq N_0$ we have

$$\inf_{j(f)} \frac{L^0_{\hat{\varphi}}(1)}{\exp(-\varepsilon n)} \geq \exp(C) \exp(-\varepsilon n).$$

Let $x_0$ be a point of $J(f)$ that has infinitely many non-critical preimages. Let $\delta > 0$ and for each integer $n \geq 1$, let $\mathfrak{D}_n$ be as in lemma 2.4 with $\varepsilon$ replaced by $\varepsilon/2$. Then there is $n_0 \geq 1$ such that for every integer $n \geq n_0$ we have

$$\frac{1}{n} \log \sum_{w \in \mathfrak{D}_n} \inf_{w \cap J(f)} \exp(S_n(\hat{\varphi})) \geq P(f, \hat{\varphi}) - \varepsilon/2.$$ 

This implies for each $n \geq n_0$ and every $x^* \in B(x_0, \delta) \cap J(f)$, we have

$$L^0_{\hat{\varphi}}(1)(x^*) \geq \exp(n (P(f, \hat{\varphi}) - \varepsilon/2)). \quad (2.1)$$

On the other hand, since $f$ is topologically exact on $J(f)$, there is $n_1 \geq 1$ such that $f^{n_1}(B(x_0, \delta)) \supset J(f)$. Given $x \in J(f)$, let $x'$ in $B(x_0, \delta) \cap J(f)$ be such that $f^{n_1}(x') = x$. It follows that for every $n \geq n_1 + n_0$

$$L^0_{\hat{\varphi}}(1)(x) = \sum_{y \in f^{-n}(x)} \exp(S_n(\hat{\varphi})(y))$$

$$= \sum_{y' \in f^{-n_1}(x')} \sum_{y \in f^{-n-n_1}(y')} \exp(S_{n-n_1}(\hat{\varphi})(y) + S_{n_1}(\hat{\varphi})(y'))$$

$$\geq \exp(S_{n_1}(\hat{\varphi})(x')) L^{n-n_1}_{\hat{\varphi}}(1)(x')$$

$$\geq \exp(n_1 \inf_{j(f)} \hat{\varphi}) \exp((n-n_1) (P(f, \hat{\varphi}) - \varepsilon/2)).$$

Let $N_0 \geq 0$ be such that

$$\exp(n_1 \inf_{j(f)} \hat{\varphi}) \geq \exp(C) \exp(n_1 (P(f, \hat{\varphi}) - (\varepsilon N_0)/2).$$

Then for every $x \in J(f)$ and every integer $n \geq \max\{n_1 + n_0, N_0\}$, we have

$$L^0_{\hat{\varphi}}(1)(x) \geq \exp(C) \exp(n_1 (P(f, \hat{\varphi}) - (\varepsilon N_0)/2 + (n-n_1) (P(f, \hat{\varphi}) - \varepsilon/2))$$

$$\geq \exp(C) \exp(n (P(f, \hat{\varphi}) - \varepsilon) + (\varepsilon/2)(n + n_1 - N_0))$$

$$\geq \exp(C) \exp(n (P(f, \hat{\varphi}) - \varepsilon)).$$

This proves the desired inequality with $N_0 = \max\{n_1 + n_0, N_0\}$, and so the proof of the lemma is complete. \hfill $\square$

In order to complete the proof of proposition 2.1, we recall the definition of topological pressure using ‘$(n, \varepsilon)$-separated sets’. Let $(X, \text{dist})$ be a compact metric space and let $T : X \to X$ be a continuous map. For each integer $n \geq 1$ the function $\text{dist}_n : X \times X \to \mathbb{R}$ defined by

$$\text{dist}_n(x, y) := \max \{\text{dist}(T^i(x), T^i(y)) : i \in \{0, 1, \cdots, n-1\}\},$$

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is a metric on $X$. Note that \( \text{dist}_1 = \text{dist} \). For \( \varepsilon > 0 \), and an integer \( n \geq 1 \), a pair of points \( x \) and \( x' \) of \( X \) are \((n, \varepsilon)\)-close if \( \text{dist}_1(x, y) < \varepsilon \). Moreover, a subset \( F \) of \( X \) is \((n, \varepsilon)\)-separated, if it does not contain 2 distinct points that are \((n, \varepsilon)\)-close. For a continuous function \( \phi : X \to \mathbb{R} \), the pressure \( P(T, \phi) \), defined in (1.2), satisfies

\[
P(T, \phi) = \lim_{\varepsilon \to 0} \limsup_{n \to +\infty} \frac{1}{n} \log \sup_{f} \left( \sum_{y \in F} \exp(S_{\phi}(\phi(y))) \right),
\]

where the supremum is taken over all \((n, \varepsilon)\)-separated subsets \( F \) of \( X \), see for example [Kel98] or [PU10].

**Proof of proposition 2.1.** Put \( \text{Crit}'(f) := \text{Crit}(f) \cap J(f) \).

In view of lemma 2.5 it is enough to show the following: For every \( \varepsilon > 0 \) there is \( N_1 > 0 \) such that for every integer \( n \geq N_1 \), we have

\[\sup_{J(f)} \mathcal{H}_{\varepsilon}^{\#}(1) \leq \exp(n\varepsilon).\]

The proof of this fact can be adapted from [Prz90, lemma 4]. The details are as follows.

By (2.2) there is \( \varepsilon_0 > 0 \) such that for every \( \varepsilon_* \) in \((0, \varepsilon_0)\) there is an integer \( N(\varepsilon_*) \geq 1 \) such that for every \( n \geq N(\varepsilon_*) \) and every \((n, \varepsilon_*)\)-separated subset \( N \) of \( J(f) \), we have

\[
\sum_{y \in N} \exp(S_{\phi}(\phi(y))) \leq \exp\left(n(P(f, \phi) + \varepsilon/2)\right).
\]

Putting \( N := \#\text{Crit}(f) + 1 \), and let \( L \geq 2\#\text{Crit}(f) \) be large enough so that

\[N^{2\#\text{Crit}(f)^2/L} \leq \exp(\varepsilon/2).\]

Note that a point in the domain of \( f \) can have at most \( N \) preimages by \( f \). On the other hand, there is \( \varepsilon' \) in \((0, \varepsilon_*)\) such that for every \( c \) in \( \text{Crit}'(f) \), every \( x \) in \( B(c, 2\varepsilon') \), and every \( j \) in \( \{1, 2, \ldots, L\} \), we have either

\[
f^{j}(x) \notin B(\text{Crit}(f), 2\varepsilon') \quad \text{or} \quad f^{j}(c) \in \text{Crit}(f).
\]

Since no critical point of \( f \) in \( J(f) \) is periodic, for each critical point \( c \) of \( f \) in \( J(f) \) there are at most \#\text{Crit}'(f) \(-\) 1 integers \( j \geq 1 \) such that \( f^{j}(c) \) is in \( \text{Crit}(f) \). Reducing \( \varepsilon' \) if necessary, assume that for every \( x \) in \( J(f) \) satisfying \( \text{dist}(x, \text{Crit}'(f)) > 2\varepsilon' \) the map \( f \) is injective on \( B(x, \varepsilon') \).

Given an integer \( n \geq 1 \) and a point \( x \) of \( J(f) \), denote by \( P_{(n, \varepsilon')}(x) \) the number of points in \( f^{-n}(f^{n}(x)) \) that are \((n, \varepsilon')\)-close to \( x \). Note that \( P_{(n, \varepsilon')}(x) \geq 1 \) and put

\[P_{(n, \varepsilon')} := \sup_{x \in J(f)} P_{(n, \varepsilon')}(x).
\]

Let \( x_0 \) be a point of \( J(f) \). Then for every integer \( n \geq 1 \), the set \( f^{-n}(x_0) \) can be partitioned into \( P_{(n, \varepsilon')}(x_0) \) sets, each of which is \((n, \varepsilon')\)-separated. So there is a \((n, \varepsilon')\)-separated subset \( N \) of \( f^{-n}(x_0) \) such that

\[
\sum_{y \in N} \exp(S_{\phi}(\phi(y))) \geq \frac{1}{P_{(n, \varepsilon')} x_0} \sum_{y \in f^{-n}(x_0)} \exp(S_{\phi}(\phi(y)))
\]

Together with (2.3), for every \( n \geq N(\varepsilon') \) we have

\[
\sum_{y \in f^{-n}(x_0)} \exp(S_{\phi}(\phi(y))) \leq P_{(n, \varepsilon')} \exp(n(P(f, \phi) + \varepsilon/2)).
\]

Thus, to complete the proof of the proposition it suffices to prove there is an integer \( N' \geq 1 \) such that for every \( n \geq N' \) we have \( P_{(n, \varepsilon')} \leq \exp(n\varepsilon/2) \).
Fix a point $x_0$ in $J(f)$, and for each point $x$ of $J(f)$ put $P_{(a_0, e')}((x)) = 1$. Note that if for some $n \geq 2$ a point $y$ in $J(f)$ and a point $y'$ in $f^{-n}(f^n(x_0))$ are $(n, e')$-close, then $f(y)$ and $f(y')$ are $(n - 1, e')$-close. Therefore we have

$$P_{(a_0, e')}((x_0)) \leq N \cdot P_{(a_0 - 1, e')}(f((x_0))),$$

and when $f$ is injective on $B(x_0, e')$, we have $P_{(a_0, e')}((x_0)) \leq P_{(a_0 - 1, e')}(f((x_0)))$. In particular, when $\text{dist}(x_0, \text{Crit}(f)) \geq 2e'$ we have $P_{(a_0, e')}((x_0)) \leq P_{(a_0 - 1, e')}(f((x_0)))$. By induction and the definition of $e'$ we obtain

$$P_{(a_0, e')}((x_0)) \leq N \cdot P_{(a_0 - 1, e')}(f((x_0))).$$

This completes the proof of the proposition.

\[\square\]

3. Conformal measure

The main goal of this section is to prove theorem A, on the existence of a conformal measure. We use a general method of construction of conformal measures, usually known as the ‘Patterson-Sullivan method’. For rational maps, this method was introduced by Sullivan in [Sul83], see also [DU91b] and [PU10, section 12].

We proceed to describe a preliminary fact needed in the construction. Given a sequence $(a_n)_{n=1}^{+\infty}$ of real numbers such that

$$c := \limsup_{n \to +\infty} \frac{a_n}{n} < +\infty,$$

the number $c$ is called the transition parameter of $(a_n)_{n=1}^{+\infty}$. It is uniquely determined by the property that the series

$$\sum_{n=1}^{+\infty} \exp(a_n - ns)$$

converges for $s > c$ and diverges for $s < c$. For $s = c$ the sum may converge or diverge.

For a proof of the following simple fact, see, for example, [PU10, lemma 12.1.2].

**Lemma 3.1.** Let $(a_n)_{n=1}^{+\infty}$ be a sequence of real numbers having transition parameter $c$. Then there is a sequence $(b_n)_{n=1}^{+\infty}$ of positive real numbers such that

$$\sum_{n=1}^{+\infty} b_n \exp(a_n - ns) \begin{cases} < +\infty & \text{if } s > c \\ = +\infty & \text{if } s \leq c, \end{cases}$$

and $\lim_{n \to +\infty} b_n / b_{n+1} = 1$.

In view of corollary 2.2, theorem A is a direct consequence of the following proposition. A turning point of a multimodal map $f : I \to I$ is a point in $I$ at which $f$ is not locally injective.

**Proposition 3.2.** Let $f : I \to I$ be a multimodal map, let $X$ be a compact subset of $I$ that contains at least 2 points and satisfies $f^{-1}(X) \subset X$, and let $\varphi : X \to \mathbb{R}$ be a continuous function. Assume that $f$ has no periodic turning point in $X$, and that there is a point $x_0$ of $X$ and an integer $N \geq 1$ such that the number

$$P_{x_0} := \limsup_{n \to +\infty} \frac{1}{n} \log \sum_{y \in f^n(x_0)} \exp(S_n(\varphi)(y))$$

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satisfies $P_{x_0} > \sup_X \frac{1}{N} S_N(\psi)$. Then there is an atom-free $\exp(P_{x_0} - \psi)$-conformal measure for $f$. If in addition $f$ is topologically exact on $X$, then the support of this conformal measure is equal to $X$.

**Proof.** In part 1 below we construct a measure that is conformal outside $\text{Tur}(f) \cup \partial I$. In parts 2 and 3 we prove that this measure has no atoms, and in part 4 we conclude the proof of the proposition.

Denote by $\text{Tur}(f)$ the set of turning points of $f$.

1. **Construction of the measure $\mu$.** For each integer $n \geq 1$ put

   $$a_n := \log \sum_{y \in f^{-n}(x_0)} \exp(S_n(\psi)(y)).$$

   The hypotheses imply that $P_{x_0}$ is the transition parameter of $(a_n)_{n=1}^{+\infty}$. Let $(b_n)_{n=1}^{+\infty}$ be given by lemma 3.1, and for each real number $s > P_{x_0}$ define

   $$M_s := \sum_{n=1}^{+\infty} b_n \exp(a_n - ns),$$

   and the probability measure

   $$m_s := \frac{1}{M_s} \sum_{n=1}^{+\infty} \sum_{y \in f^{-n}(x_0)} b_n \exp(S_n(\psi)(y) - ns) \delta_y.$$  \hspace{1cm} (3.1)

   Let $\mu$ be a weak\(^\ast\) accumulation point, as $s \to P_{x_0}^+$, of the measures $\{m_s : s > P_{x_0}\}$ defined by (3.1). Since $X$ is compact and $f^{-1}(X) \subset X$, the support of $\mu$ is contained in $X$.

   Let $A$ be a Borel subset of $I$ on which $f$ is injective. Using (3.1) and

   $$f^{-1}(f^{-n}(x_0)) = f^{-(n+1)}(x_0),$$

   for each integer $n \geq 1$ we have

   $$m_s(f(A)) = \frac{1}{M_s} \sum_{n=1}^{+\infty} \sum_{y \in f^{-n}(x_0) \cap f(A)} b_n \exp(S_n(\psi)(f(y)) - ns)$$

   $$= \frac{1}{M_s} \sum_{n=1}^{+\infty} \sum_{y \in A \cap f^{-n}(x_0)} b_n \exp(S_n(\psi)(f(y)) - ns).$$

   It follows that

   $$\Delta_A(s) := \left| m_s(f(A)) - \int_A \exp(P_{x_0} - \psi) \, dm_s \right|$$

   $$= \left| \frac{1}{M_s} \sum_{n=1}^{+\infty} \sum_{y \in A \cap f^{-n}(x_0)} b_n \exp(S_n(\psi)(f(y)) - ns) \right|$$

   $$- \left| \sum_{n=1}^{+\infty} \sum_{y \in A \cap f^{-n}(x_0)} b_n \exp(S_n(\psi)(y) - ns) \exp(P_{x_0} - \psi(y)) \right|$$

   $$= \left| \frac{1}{M_s} \sum_{n=1}^{+\infty} \left( b_n - b_{n+1} \exp(P_{x_0} - s) \sum_{y \in A \cap f^{-n+1}(x_0)} \exp(S_n(\psi)(f(y)) - ns) \right) \right|$$

   $$- b_1 \sum_{y \in A \cap f^{-1}(x_0)} \exp(P_{x_0} - s) \right|$$
\begin{align*}
\leq \frac{1}{M_s} \sum_{n=1}^{+\infty} \left( 1 - \frac{b_{n+1}}{b_n} \exp(P_{x_0} - s) \sum_{y' \in f^{-1}(x_0), y \in A \cap f^{-1}(y')} \exp \left( S_n(\varphi)(y') - ns \right) \right) \\
+ \frac{1}{M_s} b_1 \exp(P_{x_0} - s) \left( \#(A \cap f^{-1}(x_0)) \right).
\end{align*}

Thus, if we put
\[
K := \sup_{x \in X} \#(f^{-1}(x))
\]
and
\[
\Delta(s) := \frac{K}{M_s} \sum_{n=1}^{+\infty} \left| 1 - \frac{b_{n+1}}{b_n} \exp(P_{x_0} - s) \sum_{y' \in f^{-1}(x_0)} \exp \left( S_n(\varphi)(y') - ns \right) \right|
\]
\[
+ \frac{K}{M_s} b_1 \exp(P_{x_0} - s),
\]
then for every Borel subset \(A\) of \(I\) on which \(f\) is injective we have
\[
\Delta_A(s) \leq \Delta(s).
\]

On the other hand, by lemma 3.1 and our hypotheses, we know that
\[
\lim_{n \to +\infty} \frac{b_n}{b_{n+1}} = 1 \quad \text{and} \quad \lim_{s \to P_{x_0}} M_s = +\infty,
\]
so we obtain
\[
\lim_{s \to P_{x_0}} \Delta_A(s) = \lim_{s \to P_{x_0}} \Delta(s) = 0.
\]

Thus, if in addition \(\mu(\partial A) = \mu(\partial f(A)) = 0\) and the closure of \(A\) is contained in \(I \setminus (\text{Tur}(f) \cup \partial I)\), then we have
\[
\mu(f(A)) = \int_A \exp(P_{x_0} - \varphi) \, d\mu.
\]

By [PU10, lemma 12.1.3] or [DU91b, lemma 3.3], it follows that the equality above holds for every Borel subset \(A\) of \(I \setminus (\text{Tur}(f) \cup \partial I)\) on which \(f\) is injective.

2. For every point \(c\) in \(\text{Tur}(f) \setminus \partial I\) we have
\[
2\mu(\{f(c)\}) \geq \mu(\{c\}) \exp(P_{x_0} - \varphi(c)),
\]
and for every \(x\) in \(\partial I\) we have
\[
\mu(\{f(x)\}) \geq \mu(\{x\}) \exp(P_{x_0} - \varphi(x)).
\]

First we prove inequality (3.4). Let \(c\) be in \(\text{Tur}(f) \setminus \partial I\), and let \((C_n)_{n=1}^{+\infty}\) be a sequence of compact neighbourhoods of \(c\) in \(I\) that decreases \(c\), and so that for each \(n\) the map \(f\) is injective on each connected component of \(C_n \setminus \{c\}\) and we have \(\mu(\partial C_n) = 0\). Put
\[
C_n^- := C_n \cap (-\infty, c] \quad \text{and} \quad C_n^+ := C_n \cap [c, +\infty).
\]

From (3.2) we obtain
\[
\lim_{s \to P_{x_0}} \left| m_s(f(C_n^+)) - \int_{C_n^+} \exp(P_{x_0} - \varphi) \, dm_s \right| = 0,
\]
and
\[
\lim_{s \to P_{x_0}} \left| m_s(f(C_n^-)) - \int_{C_n^-} \exp(P_{x_0} - \varphi) \, dm_s \right| = 0.
\]
By the construction of \( m_\ast \), we have \( m_\ast(|c|) \to 0 \) as \( s \to P_{s_0}^\ast \). On the other hand, note that \( f(C_n^-) \) and \( f(C_n^+) \) are compact. Thus, if we let \( (s_j)_{j=1}^\infty \) be a sequence in \( (P_{s_0}, +\infty) \) such that \( s_j \to P_{s_0} \) and \( m_j \to \mu \) as \( j \to +\infty \), then

\[
2\mu(f(C_n^-)) \geq \mu(f(C_n^+)) + \mu(f(C_n^-)) \\
\geq \limsup_{j \to +\infty} m_j(f(C_n^+)) + \limsup_{j \to +\infty} m_j(f(C_n^-)) \\
= \limsup_{j \to +\infty} \int_{C_n^+} \exp(P_{s_0} - \varphi) \, dm_j + \limsup_{j \to +\infty} \int_{C_n^-} \exp(P_{s_0} - \varphi) \, dm_j \\
\geq \liminf_{j \to +\infty} \left( \int_{C_n^+} \exp(P_{s_0} - \varphi) \, dm_j + \int_{C_n^-} \exp(P_{s_0} - \varphi) \, dm_j \right) \\
\geq \int_{C_n} \exp(P_{s_0} - \varphi) \, d\mu \\
\geq \mu(|c|) \exp(P_{s_0} - \varphi(c)).
\]

Letting \( n \to +\infty \), we obtain (3.4).

To prove (3.5), let \( x \in \partial I \), and let \( (B_n)_{n=1}^\infty \) be a sequence of compact neighbourhoods of \( x \) in \( I \) that decreases to \( x \), and so that for each \( n \) the map \( f \) is injective on \( B_n \) and we have \( \mu(\partial B_n) = 0 \). The argument above gives us that for every integer \( n \geq 1 \) we have

\[
\mu(f(B_n)) \geq \mu(|x|) \exp(P_{s_0} - \varphi(x)).
\]

Letting \( n \to +\infty \), we obtain (3.5).

3. \( \mu \) has no atoms. Assume that there is \( x_\ast \) in \( I \) such that \( \mu(|x_\ast|) \neq 0 \). Since by the construction \( \mu \) is supported on \( X \), the point \( x_\ast \) is in \( X \). Since \( f \) has no periodic turning point in \( X \), there is an integer \( n_1 \geq 1 \) such that the point \( y_\ast := f^{n_1}(x_\ast) \) satisfies for every integer \( n \geq 0 \) that \( f^n(y_\ast) \notin \text{Tur}(f) \). Applying (3.3), (3.4), or (3.5) repeatedly, we conclude that

\[
\mu(|y_\ast|) = \mu(|f^{n_1}(x_\ast)|) \geq \frac{1}{2^{n_1}} \exp(n_1 P_{s_0} - S_{n_1}(\varphi)(x_\ast)) \mu(|x_\ast|) > 0. \tag{3.6}
\]

and that for every integer \( n \geq 1 \)

\[
\mu(|f^n(y_\ast)|) \geq \exp(n P_{s_0} - S_n(\varphi)(y_\ast)) \mu(|y_\ast|). \tag{3.7}
\]

On the other hand, by hypothesis there is an integer \( N \geq 1 \) such that

\[
N P_{s_0} \geq \sup_X S_N(\varphi). \tag{3.8}
\]

Therefore, for each integer \( k \geq 1 \) we have by (3.7) with \( n = kN \),

\[
\mu(|f^k(y_\ast)|) \geq \exp(k N P_{s_0} - S_{kN}(\varphi)(y_\ast)) \mu(|y_\ast|) \\
\geq \exp(k \left( N P_{s_0} - \sup_X S_N(\varphi) \right)) \mu(|y_\ast|).
\]

Together with (3.6) and (3.8), this implies

\[
\lim_{k \to +\infty} \mu(|f^k(y_\ast)|) = +\infty,
\]

a contradiction. Thus \( \mu \) is atom-free.
4. **µ is \(\exp(P_{s_0} - \varphi)\)-conformal.** By part 3 and (3.3), for each Borel subset \(A\) of \(I\) on which \(f\) is injective, we have
\[
\mu(f(A)) = \mu(f(A \setminus (\text{Tur}(f) \cup \partial I)))
= \int_{A \setminus (\text{Tur}(f) \cup \partial I)} \exp(P_{s_0} - \varphi) \, d\mu
= \int_A \exp(P_{s_0} - \varphi) \, d\mu.
\]
Hence, \(\mu\) is \(\exp(P_{s_0} - \varphi)\)-conformal.

To prove the last statement of the proposition, suppose \(f\) is topologically exact on \(X\). Together with (3.3), with \(\exp(P_{s_0} - \varphi) > 0\), and the fact that \(\mu\) is atom-free, this implies that the support of \(\mu\) is equal to \(X\). This completes the proof of the proposition. \(\square\)

### 4. Keller spaces and quasi-compactness

We start this section by recalling in section 4.1 some properties of the function spaces defined by Keller in [Kel85]. In section 4.2 we first recall some of the results of [Kel85] that we state as theorem 1, and then we deduce from them some known consequences (corollary 4.4) that we use in the next section to prove theorem B.

Throughout this section, fix a compact subset \(X\) of \(\mathbb{R}\) endowed with the distance induced by the norm distance on \(\mathbb{R}\), and fix an atom-free Borel probability measure \(m\) on \(X\). We consider the equivalence relation on the space of complex valued functions defined on \(X\), defined by agreement on a set of full measures with respect to \(m\).

#### 4.1. Properties of Keller spaces

In the following proposition we compare the function spaces introduced by Keller in [Kel85], see section 1.2, with several other function spaces.

Given \(p \geq 1\) and a function \(h : X \to \mathbb{C}\), put
\[
\text{Var}_p(h) := \sup \left\{ \left( \sum_{i=1}^{k} |h(x_i) - h(x_{i-1})|^p \right)^{\frac{1}{p}} : k \geq 1, x_0, \ldots, x_k \in X, x_0 < \cdots < x_k \right\}
\]
and
\[
\|h\|_{BV_p} := \text{Var}_p(h) + \|h\|_{\infty}.
\]
The function \(h\) is of bounded \(p\)-variation, if \(\|h\|_{BV_p} < +\infty\). Let \(BV_p\) be the space of all bounded \(p\)-variation functions defined on \(X\). Then \(\cdot \|_{BV_p}\) is a norm on \(BV_p\), for which \(BV_p\) is a Banach space.

Given \(\alpha\) in \((0, 1]\), denote by \(H^\alpha\) the space of Hölder continuous functions of exponent \(\alpha\) defined on \(X\) and taking values in \(\mathbb{C}\). Then \(\| \cdot \|_{\alpha}\), defined in section 1.3, is a norm on \(H^\alpha\) and \((H^\alpha, \| \cdot \|_{\alpha})\) is a Banach space. Note that the definitions immediately imply that for each \(h\) in \(H^\alpha\), we have
\[
\text{Var}_{1/\alpha}(h) \leq |\sup X - \inf X|^\alpha |h|_{\alpha},
\]
and therefore
\[
\|h\|_{BV_{1/\alpha}} \leq \max \{ 1, |\sup X - \inf X|^\alpha \} \|h\|_{\alpha}.
\]
Proposition 4.1. Fix $A > 0$, let $X$ be a compact subset of $\mathbb{R}$, and let $m$ be an atom-free Borel probability measure on $X$. Then for each $\alpha$ in $(0, 1]$, the space $H^{\alpha, 1}(m)$ defined in section 1.2 satisfies the following properties:

1. $(H^{\alpha, 1}(m), \| \cdot \|_{\alpha, 1})$ is a Banach space;
2. For each function $h : X \to \mathbb{C}$ in $BV_{1/\alpha}$, we have $\| h \|_{\alpha, 1} \leq 2^\alpha \| h \|_{BV_{1/\alpha}}$;
3. Each function in $H^{\alpha, 1}(m)$ is essentially bounded. In fact, there is a constant $C_\alpha > 0$ such that each element $h$ of $H^{\alpha, 1}(m)$ satisfies $\| h \|_\infty \leq C_\alpha \| h \|_{\alpha, 1}$.

Note that by combining (4.1) with parts 2 and 3 of the proposition above, we obtain $H^\alpha \subset BV_{1/\alpha} \subset H^{\alpha, 1}(m) \subset L^\infty(m)$. We conclude that each of the spaces $BV_{1/\alpha}$ and $H^{\alpha, 1}(m)$ is dense in $L^1(m)$.

Part 1 of proposition 4.1 is part b of [Kel85, theorem 1.13]. Since this property is important for what follows, we provide a proof.

The rest of this subsection is devoted to the proof of proposition 4.1.

Lemma 4.2. For all $p \geq 1$, $h : X \to \mathbb{C}$, and $\epsilon > 0$, we have

$$\int_X osc(h, \epsilon, x)^p \, dm(x) \leq 2\epsilon \text{Var}_p(h)^p.$$ 

\textbf{Proof.} Put $a := \inf X$ and for each $t$ in $[0, m(X)]$, put $x(t) := \sup \{ y \in X : d(y, a) = t \}$.

Suppose first $\epsilon \geq m(X)/2$. Using that for every $x$ in $X$ we have $osc(h, \epsilon, x)^p \leq \text{Var}_p(h)^p$, we obtain

$$\int_X osc(h, \epsilon, x)^p \, dm(x) \leq m(X) \text{Var}_p(h)^p \leq 2\epsilon \text{Var}_p(h)^p.$$ 

It remains to consider the case $\epsilon < m(X)/2$. For every $\xi$ in $[0, 2\epsilon]$, put $n_\epsilon(\xi) := \max\{\text{non-negative integer } n : \xi + 2n\epsilon \leq m(X)\}$.

Since the balls $(B_d(x(\xi + 2k\epsilon), \epsilon))_{k=0}^{n_\epsilon(\xi)}$ are pairwise disjoint, we have

$$\sum_{k=0}^{n_\epsilon(\xi)} osc(h, \epsilon, x(\xi + 2k\epsilon))^p \leq \text{Var}_p(h)^p.$$ 

It follows that

$$\int_X osc(h, \epsilon, x)^p \, dm(x) = \int_0^{2\epsilon} \sum_{k=0}^{n_\epsilon(\xi)} osc(h, \epsilon, x(\xi + 2k\epsilon))^p \, d\xi \leq 2\epsilon \text{Var}_p(h)^p,$$

and so we obtain the lemma. $\square$

Lemma 4.3 ([Kel85], lemma 1.12). Let $\alpha$ be in $(0, 1]$, and let $(h_n)_{n=1}^{\infty}$ be a sequence in $H^{\alpha, 1}(m)$. If there is $h$ in $L^1(m)$ such that $(h_n)_{n=1}^{\infty}$ converges to $h$ in $L^1(m)$, then

$$|h|_{\alpha, 1} \leq \liminf_{N \to \infty} |h_n|_{\alpha, 1}.$$
Proof of proposition 4.1. To prove part 1 it suffices to check that $H^{s,1}(m)$ is complete with respect to $\| \cdot \|_{a,1}$. Let $(h_n)_{n=1}^\infty$ be a Cauchy sequence in $H^{s,1}(m)$. Then $(h_n)_{n=1}^\infty$ is also a Cauchy sequence in $L^1(m)$, and therefore there is $h$ in $L^1(m)$ such that $\| h_n - h \|_1 \to 0$ as $n \to +\infty$. By lemma 4.3, we have

$$|h|_{a,1} \leq \liminf_{n \to +\infty} |h_n|_{a,1} < +\infty.$$ 

It follows that $h$ is in $H^{s,1}(m)$. To complete the proof of part 1, it is enough to prove that for every $\delta > 0$ there is $N > 0$ such that for each integer $n \geq N$ we have

$$|h_n - h|_{a,1} \leq \delta.$$ 

In fact, since $(h_n)_{n=1}^\infty$ is a Cauchy sequence in $H^{s,1}(m)$, there is $N > 0$ such that for each pair of integers $k, n \geq N$, we have $\| h_k - h_n \|_{a,1} \leq \delta$. Fix $n \geq N$, and note that $h - h_k + (h_n - h)$ converges to $h_n - h$ in $L^1(m)$ as $k \to +\infty$. It follows from lemma 4.3 again that

$$|h_n - h|_{a,1} \leq \liminf_{k \to +\infty} |h - h_k + (h_n - h)|_{a,1} = \liminf_{k \to +\infty} |h_n - h_k|_{a,1} \leq \delta.$$ 

The proof of part 1 is complete.

Let us prove part 2. By Hölder’s integral inequality and lemma 4.2 with $p = 1/\alpha$, for every $\epsilon$ in $[0, A)$ we have

$$\text{osc}_1(h, \epsilon) \leq \left( \int_X \text{osc}(h, \epsilon, x)^{1/\alpha} \, dm(x) \right)^{\alpha} \leq 2^\alpha \epsilon^\alpha \text{Var}_1(h).$$

It follows that $|h|_{a,1} \leq 2^\alpha \text{Var}_1(h)$. On the other hand, since $\| h \|_1 \leq \| h \|_{\infty}$, we have

$$\| h \|_{a,1} = \| h \|_1 + |h|_{a,1} \leq \| h \|_{\infty} + 2^\alpha \text{Var}_1(h) \leq 2^\alpha \| h \|_{BV_1}.$$ 

It remains to prove part 3. Let $h$ be in $H^{s,1}(m)$ and fix $\epsilon > 0$. Then there are subsets $X_1$ and $X_2$ of $X$ of positive measure for $m$, such that

$$\sup_{X_1} |h| \leq \| h \|_1 \quad \text{and} \quad \inf_X |h| \geq \text{ess-sup}_X |h| - \epsilon.$$ 

It follows that

$$\text{ess-sup}_X |h| - \epsilon \leq \inf_{X_2} |h| \leq \sup_{X_1} |h| + \text{ess-sup}_{(x_1,x_2) \in X_1 \times X_2} |h(x_1) - h(x_2)| \quad (4.2)$$

$$\leq \| h \|_1 + \text{ess-sup}_{(x_1,x_2) \in X_1 \times X_2} |h(x_1) - h(x_2)|.$$ 

Let $N$ be the least integer such that $N \geq m(X)/(2A)$, put $\ell := m(X)/(4N)$, and note that $A \geq 2\ell$. Given $\xi$ in $[0, 2\ell]$, let $n(\xi)$ be the largest integer $n \geq 0$ such that $\xi + 2n\ell \leq m(X)$. Note that if for each $t \geq 0$ we put $x(t) := \sup \{ y \in X : d(y, \text{inf} \, X) \leq t \}$, then $\bigcup_{n=0}^{m(X)} B_d(x(\xi + 2k\ell), A)$ has full measure in $X$ with respect to $m$. We thus have

$$\text{ess-sup}_{(x_1,x_2) \in X_1 \times X_2} |h(x_1) - h(x_2)| \leq \sum_{k=0}^{n(\xi)} \text{osc}(h, A, x(\xi + 2k\ell)).$$ 

Since this holds for every $\xi$ in $[0, 2\ell]$, we have

$$\text{ess-sup}_{(x_1,x_2) \in X_1 \times X_2} |h(x_1) - h(x_2)| \leq \frac{1}{2\ell} \int_0^{2\ell} \sum_{k=0}^{n(\xi)} \text{osc}(h, A, x(\xi + 2k\ell)) \, d\xi$$

$$= \frac{1}{2\ell} \int_X \text{osc}(h, A, x) \, dm(x)$$

$$\leq \frac{1}{2\ell} |h|_{a,1} A^\alpha.$$ 

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Together with (4.2) this implies,
\[
\text{ess-sup}_{X} |h| - \varepsilon \leq \|h\|_1 + \frac{A^\alpha}{2\ell} |h|_{\alpha,1} \leq \max \left\{ 1, \frac{A^\alpha}{2\ell} \right\} \|h\|_{\alpha,1}.
\]
Since this holds for every \(\varepsilon > 0\), this proves part 3 with \(C_* = \max\{1, \frac{A^\alpha}{\ell}\}\), and completes the proof of the proposition.

4.2. Quasi-compactness and spectral gap

In this subsection, let \(I\) be a compact interval of \(\mathbb{R}\), \(N \geq 2\) an integer, and \(\mathcal{P} := \{I_1', \ldots, I_N\}\) a partition of \(I\) into intervals. Let \(T : I \to I\) be a transformation on \(I\) that is continuous and monotone on each \(I_i'\) in \(\mathcal{P}\). Furthermore, let \(X\) be a compact subset of \(I\) such that \(T^{-1}(X) = X\), for each \(i\) in \(\{1, \ldots, N\}\) put \(I_i := I_i' \cap X\), and put \(\mathcal{P} := \{I_1, \ldots, I_N\}\). Fix \(p \geq 1\), let \(g : X \to [0, +\infty)\) be a function of bounded \(p\)-variation, and let \(L_g\) be the operator acting on the space
\[
\mathcal{E}_b(X) := \{h : X \to \mathbb{C} \text{ measurable and bounded in absolute value}\},
\]
defined by
\[
L_g(h)(x) := \sum_{y \in T^{-1}(x)} h(y)g(y) = \sum_{i \in \{1, \ldots, N\}, x \in T(I_i)} (h \cdot g) \circ T^{-1}(x).
\] (4.3)
Assume in addition that there is an atom-free Borel probability measure \(m\) on \(X\) such that the following properties hold:

**H1.** For each \(I_i\) in \(\mathcal{P}\), the map \(T^{-1}_{|I_i}\) is non-singular with respect to \(m\), so that for every subset \(E\) of \(I_i\) of measure zero, the set \((T^{-1}_{|I_i})^{-1}(E) = T(E)\) is also of measure zero;

**H2.** On a set of full measure with respect to \(m\), we have
\[
g^{-1} = \sum_{i=1}^{N} \frac{d(T^{-1}_{|I_i})}{dm} \cdot m;
\]

**H3.** For each \(h\) in \(\mathcal{E}_b(X)\) we have \(\int_X L_g(h) \, dm = \int_X h \, dm\), and \(L_g\) extends to a positive linear map from \(L^1(m)\) to itself satisfying \(\|L_g(h)\|_1 \leq \|h\|_1\).

In the following theorem, we gather several results from [Kel85].

**Theorem 1 ([Kel85], theorems 3.2 and 3.3).** Let \(T\), \(g\), \(L_g\), and \(m\) be as above, and assume that there is an integer \(n \geq 1\) such that the function
\[
g_n(x) := g(x) \cdot \ldots \cdot g(T^{n-1}(x))
\]
satisfies \(\sup_{x} g_n < 1\). Then there is an integer \(k \geq 1\) and constants \(A > 0\), \(\beta\) in \((0, 1)\), and \(C > 0\), such that for every function \(h\) in \(H^{1/p,1}(m)\), we have
\[
\|L_g^k(h)\|_{1/p,1} \leq \beta \|h\|_{1/p,1} + C \|h\|_1.
\] (4.4)
Moreover, the following properties hold:

1. The set \(\mathcal{E}\) of eigenvalues of \(L_g^1|_{L^1(m)}\) of modulus 1 is finite. Moreover, for each \(\lambda\) in \(\mathcal{E}\), the space
\[
\mathcal{E}(\lambda) := \{h \in L^1(m) : L_g(h) = \lambda h\}
\]
is contained in \(H^{1/p,1}(m)\) and it is of finite dimension;
2. If for each \( \lambda \) in \( \mathcal{E} \) we denote by \( \mathcal{P}(\lambda) \) the projection in \( \mathbb{L}^1(m) \) to \( E(\lambda) \), then the operator
\[
\mathcal{D} := \mathcal{L}_g - \sum_{\lambda \in \mathcal{E}} \mathcal{P}(\lambda)
\]
satisfies \( \sup \{ \| \mathcal{D}^n \|_{1} : n \geq 0 \text{ integer } \} < +\infty \). Moreover, \( \mathcal{D} \) maps \( \mathbb{H}^{1/p, 1}(m) \) to itself, and there is \( \rho \) in \((0, 1)\) and a constant \( M > 0 \) such that for every integer \( n \geq 0 \) we have \( \| \mathcal{D}^n \|_{a, 1} \leq M \rho^n \). Finally, for each \( \lambda \) in \( \mathcal{E} \) the operators \( \mathcal{D}\mathcal{P}(\lambda) \) and \( \mathcal{P}(\lambda)\mathcal{D} \) are both identically zero, and for each \( \lambda' \) different from \( \lambda \), the operators \( \mathcal{P}(\lambda)\mathcal{P}(\lambda') \) and \( \mathcal{P}(\lambda')\mathcal{P}(\lambda) \) are also identically zero.

3. The set \( \mathcal{E} \) contains 1, and if we put \( h := \mathcal{P}(1)(1) \), then \( v := hm \) is a probability measure that is invariant by \( T \) and that is an equilibrium state of \( T_{x} \) for the potential \( g \).

The following corollary follows from the previous theorem using known arguments. We include its proof for completeness.

**Corollary 4.4.** Under the assumptions of theorem 1, and assuming in addition that \( T \) is topologically exact on \( X \), we have the following properties:

1. The number 1 is an eigenvalue of \( \mathcal{L}_0 \) of algebraic multiplicity 1. Moreover, there is \( \rho \) in \((0, 1)\) such that the spectrum of \( \mathcal{L}_0|_{\mathbb{H}^{1/p, 1}(m)} \) is contained in \( B(0, \rho) \cup \{ 1 \} \).

2. There is a constant \( C > 0 \) such that for every bounded measurable function \( \psi : X \to \mathbb{C} \), and every function \( \psi \) in \( \mathbb{H}^{1/p, 1}(m) \), the measure \( v \) given by part 3 of theorem 1 satisfies for every integer \( n \geq 1 \)
\[
C_n(\phi, \psi) \leq C \| \phi \|_\infty \| \psi \|_{1/p, 1} \rho^n.
\]

3. Given \( \psi \) in \( \mathbb{H}^{1/p, 1}(m) \), for each \( \tau \) in \( \mathbb{C} \) the operator \( \mathcal{L}_\tau \) defined by
\[
\mathcal{L}_\tau(h) := \mathcal{L}_g(\exp(\tau \psi) \cdot h)
\]
maps \( \mathbb{H}^{1/p, 1}(m) \) to itself and the restriction \( \mathcal{L}_\tau|_{\mathbb{H}^{1/p, 1}(m)} \) is analytic in the sense of Kato on \( \mathbb{C} \), and the spectral radius of \( \mathcal{L}_\tau|_{\mathbb{H}^{1/p, 1}(m)} \) depends on a real analytic way on \( \tau \) on a neighbourhood of \( \tau = 0 \).

The proof of this corollary is after the following lemma.

**Lemma 4.5.** Let \( \alpha, m, \) and \( \mathbb{H}^{0, 1}(m) \) be as above, and let \( C_\alpha \) be given by proposition 4.1. Then for every \( h \) and \( g \) in \( \mathbb{H}^{0, 1}(m) \), we have
\[
\| h \cdot g \|_{a, 1} \leq 2 C_\alpha \| h \|_{a, 1} \cdot \| g \|_{a, 1}.
\]

**Proof.** Using that each of the functions \( h \) and \( g \) is represented by a bounded function, we have
\[
\text{osc}(h \cdot g, \varepsilon, x) \leq \| h \|_\infty \text{osc}(g, \varepsilon, x) + \| g \|_\infty \text{osc}(h, \varepsilon, x).
\]
We thus have
\[
\| h \cdot g \|_{a, 1} \leq \| h \|_\infty \cdot \| g \|_{a, 1} + \| g \|_\infty \cdot \| h \|_{a, 1},
\]
and using part 3 of proposition 4.1 twice, we have
\[
\| h \cdot g \|_{a, 1} \leq \| h \|_\infty \cdot \| g \|_{a, 1} + \| h \|_\infty \cdot \| g \|_{a, 1} + \| g \|_\infty \cdot \| h \|_{a, 1} \\
\leq \| h \|_\infty \cdot \| g \|_{a, 1} + C_\alpha \| h \|_{a, 1} \| g \|_{a, 1} \\
\leq 2 C_\alpha \| h \|_{a, 1} \cdot \| g \|_{a, 1}.
\]

**Proof of corollary 4.4.** By part 3 of theorem 1 the function \( h := \mathcal{P}(1)(1) \) satisfies \( \int_X h \, dm = 1 \) and the measure \( v := hm \) is a probability. On the other hand, each of the spaces
\[
E_0 := \left\{ \psi \in \mathbb{H}^{1/p, 1}(m) : \int_X \psi \, dm = 0 \right\} \quad \text{and} \quad E_1 := \{ \alpha h : \alpha \in \mathbb{C} \}
\]
is invariant by $\mathcal{L}_g$. Moreover, since each function $\psi$ in $H^{1/p, 1}(m)$ can be decomposed as

$$
\psi = \left( \int_X \psi \, dm \right) h + \left( \psi - \left( \int_X \psi \, dm \right) h \right),
$$

we have $H^{1/p, 1}(m) = E_1 \oplus E_0$.

1. Let $\lambda$ be an eigenvalue of $\mathcal{L}_g\vert_{W^{1/p, 1}(m)}$ satisfying $|\lambda| = 1$, and let $\phi$ be a nonzero element of $H^{1/p, 1}(m)$ such that $\mathcal{L}_g(\phi) = \lambda \phi$. By hypothesis H3, for each integer $n \geq 1$ we have

$$
\int_X |\phi| \, dm = \int_X |\lambda^n \phi| \, dm = \int_X |\mathcal{L}_g^n(\phi)| \, dm \leq \int_X |\mathcal{L}_g^n(\phi)| \, dm = \int_X |\phi| \, dm.
$$

and therefore

$$
\int_X \mathcal{L}_g^n(\phi) - |\mathcal{L}_g^n(\phi)| \, dm = 0.
$$

Since $\mathcal{L}_g^n(\phi) - |\mathcal{L}_g^n(\phi)| \geq 0$, it follows that we have

$$
\mathcal{L}_g^n(\phi) = |\mathcal{L}_g^n(\phi)| = |\phi| \tag{4.5}
$$

on a set of full measure with respect to $m$.

In part 1.1 below we prove that $\phi$ is nonzero on a set of full measure with respect to $m$, and in part 1.2 we show that there is $\theta_0$ in $\mathbb{R}$ such that $\phi = \exp(i\theta_0)\vert\phi\vert$ in $H^{1/p, 1}(m)$. Using these facts, we complete the proof of part 1 of the corollary in part 1.3.

1.1. Since $\phi$ is nonzero, there is $\kappa_0 > 0$ such that $\{x \in X : |\phi| \geq \kappa_0\}$ has positive measure with respect to $m$. Let $Y$ be the set of density points of this set, so $m(Y) > 0$, and put

$$
\varepsilon_0 := \min \left\{ \left( \frac{m(Y)\kappa_0}{2\|\phi\|_{L^1}} \right)^{\frac{1}{2}}, \lambda \right\}.
$$

Note that for each $y$ in $Y$, the number

$$
\kappa(y) := \text{ess} - \inf\{|\phi(x)| : x \in B_d(y, \varepsilon_0)\}
$$

satisfies

$$
\text{osc}(|\phi|, \varepsilon_0, y) \geq \kappa_0 - \kappa(y),
$$

so

$$
\int_Y \kappa_0 - \kappa(y) \, dm(y) \leq \text{osc}(|\phi|, \varepsilon_0) \leq \varepsilon_0^2 \|\phi\|_{L^1} \leq m(Y)\kappa_0/2.
$$

This implies that there is $y_0$ in $Y$ such that

$$
\text{ess} - \inf\{|\phi(x)| : x \in B_d(y_0, \varepsilon_0)\} = \kappa(y_0) \geq \kappa_0/2. \tag{4.6}
$$

Since by hypothesis $f$ is topologically exact on $X$, there is an integer $n \geq 1$ such that $T^n(B_d(y_0, \varepsilon_0)) = X$. Combined with hypothesis H2 and (4.5), the estimate (4.6) implies that $|\phi|$ is nonzero on a set of full measure with respect to $m$.

1.2. By part 1.1 there is $\nu > 0$ such that

$$
W := \{x \in X : |\phi(x)| \geq \nu\}
$$

satisfies $m(W) \geq 1/2$. Let $W'$ be the set of density points of $W$, so $m(W') \geq 1/2$.

For each $x$ in $X$ such that $\phi(x) \neq 0$, let $\theta(x)$ in $\mathbb{R}/2\pi\mathbb{Z}$ be such that $\phi(x) = \exp(i\theta(x))|\phi(x)|$. Suppose by contradiction that the function $\theta : X \to \mathbb{R}$ so defined is not constant on a set of full measure with respect to $m$. Then there are disjoint closed intervals $\Theta$
and Θ′ such that the sets θ⁻¹(Θ) and θ⁻¹(Θ′) are disjoint, and such that each of these sets has positive measure with respect to m. By property (4.5), for every integer n ≥ 1 we have

\[ T^{-n} \left( T^n (\theta^{-1}(\Theta)) \right) \setminus \{ |φ| = 0 \} \subset \theta^{-1}(\Theta). \]

Combined with part 1, hypothesis H2, and with our hypothesis that T is topologically exact on X, this implies that for every y and every ε > 0 the set \( \theta^{-1}(\Theta) \) intersects \( B_\delta(y, \varepsilon) \) on a set of positive measure. The same property holds replacing \( \Theta \) by \( \Theta′ \). Thus, if we denote by \( \delta \) the distance between \( \Theta \) and \( \Theta′ \) in \( \mathbb{R}/2\pi \mathbb{Z} \), then for every \( y \) in \( W′ \) and every \( \varepsilon \) in \( (0, A] \), we have

\[ \text{osc}(\phi, \varepsilon, y) \geq 2\nu \sin(\delta/2). \]

Therefore

\[ \|φ\|_a,1 \geq \frac{\text{osc}_1(φ, \varepsilon)}{ε^a} \geq \frac{m(W′)(2\nu \sin(\delta/2))}{ε^a} \geq \frac{\nu \sin(\delta/2)}{ε^a}. \]

Since this holds for an arbitrary \( ε \) in \( (0, A] \), we obtain a contradiction. This contradiction shows that the function \( θ \) is constant on a set of full measure with respect to \( m \).

1.3. By part 1.2 there is \( θ_0 \) in \( \mathbb{R} \) such that \( φ = \exp(iθ_0)|φ| \) in \( H^{a,1}(m) \). It follows that \( \mathcal{L}_g(|φ|) = λ|φ| \) is non-negative, and therefore that \( λ = 1 \). Since by part 3 of theorem 1 the number 1 is an eigenvalue of \( \mathcal{L}_g |W^{a,1}(m)| \), this proves that the number 1 is the only eigenvalue of \( \mathcal{L}_g \) of modulus 1.

The existence of \( ρ \) in \( (0, 1) \) such that the spectrum of \( \mathcal{L}_g |W^{a,1}(m)| \) is contained in \( B(0, ρ) \cup \{ 1 \} \) follows from part 2 of theorem 1.

It remains to prove that the algebraic multiplicity of 1 as an eigenvalue of \( \mathcal{L}_g |W^{a,1}(m)| \) is 1. Denote by Id the identity operator of \( H^{1/p,1}(m) \), and let \( φ \) be in the kernel of \( (\mathcal{L}_g |W^{a,1}(m)| - \text{Id})^2 \). Then \( φ := \mathcal{L}_g(φ) - φ \) satisfies \( \mathcal{L}_g(φ) = φ \). Suppose \( φ \) is nonzero. Then we can apply parts 1.1 and 1.2 with \( φ \) replaced by \( φ̂ \), to conclude that there is \( θ_0 \) in \( \mathbb{R} \) such that \( φ = \exp(iθ_0)|φ| \) in \( H^{a,1}(m) \). Using hypothesis H3 we obtain

\[ 0 < \int_X |φ̂|^2 dm \]

\[ = \exp(-iθ_0) \int_X \mathcal{L}_g(φ) - φ dm \]

\[ = \exp(-iθ_0) \left( \int_X \mathcal{L}_g(φ) dm - \int_X φ dm \right) \]

\[ = 0. \]

This contradiction proves that \( \mathcal{L}_g(φ) - φ = 0 \) is zero, and completes the proof of part 1.

2. Let \( C_α \) be the constant given by part 3 of proposition 4.1 and let \( ρ \) and \( M \) be the constants given by part 2 of theorem 1. Putting \( \psi = ψ - \int_X ψ dv \), we have

\[ C_α(φ, ψ) = \left| \int_X φ \circ f^n \cdot \psi dm \right| \]

\[ = \left| \int_X (φ \circ f^n) \cdot \psi \cdot h dm \right| \]

\[ = \left| \int_X \mathcal{L}_g^n \left( (φ \circ f^n) \cdot \psi \cdot h \right) dm \right| \]

\[ ≤ \|φ\|_∞ \cdot \left\| \mathcal{L}_g^n (ψ \cdot h) \right\|_{1/p,1}. \]
Noting that \( \hat{\psi} \cdot h \) is in \( E_0 \) and using part 2 of theorem 1, we conclude that

\[
C_n(\phi, \psi) \leq M\|\phi\|_{\infty}\|\hat{\psi} \cdot h\|_{1/p,1}^{\rho^p}.
\] (4.7)

On the other hand, by lemma 4.5 we have

\[
\|\hat{\psi} \cdot h\|_{1/p,1} \leq 2C_*\|\hat{\psi}\|_{1/p,1} \cdot \|h\|_{1/p,1}
\leq (2C_*\|h\|_{1/p,1}) (\|\psi\|_{1/p,1} + \|\psi\|_{1/p,1} \cdot \|h\|_{\infty})
\leq (2C_*\|h\|_{1/p,1} (1 + \|h\|_{\infty}) \|\psi\|_{1/p,1}.
\]

Together with (4.7) this implies the desired inequality with \( C = 2MC_*\|h\|_{1/p,1} (1 + \|h\|_{\infty}) \).

3. Let \( C_* \) be the constant given by part 3 of proposition 4.1. Observe that for each \( \tau \) in \( C \), we have

\[
|\exp(\tau \psi)|_{1/p,1} \leq \exp(|\tau| \cdot \|\psi\|_{\infty})|\tau| \cdot |\psi|_{1/p,1},
\]

so the function \( \exp(\tau \psi) \) is in \( H^{1/p,1}(m) \). Thus, by lemma 4.5 for every \( \chi \) in \( H^{1/p,1}(m) \) we have

\[
\|\mathcal{L}_\tau(\chi)\|_{1/p,1} \leq (2C_*\|\mathcal{L}_\tau\|_{1/p,1} \cdot \|\exp(\tau \psi)\|_{1/p,1}) \|\chi\|_{1/p,1}.
\]

This proves that \( \mathcal{L}_\tau \) maps \( H^{1/p,1}(m) \) to itself and that \( \mathcal{L}_\tau|_{H^{1/p,1}(m)} \) is bounded.

To prove that \( \tau \mapsto \mathcal{L}_\tau|_{H^{1/p,1}(m)} \) is analytic in the sense of Kato, for each \( \epsilon \) in \( \mathbb{C} \) let \( \eta_{\epsilon} : \mathbb{C} \to \mathbb{C} \) be defined by \( \eta_{\epsilon}(z) := \frac{\exp(\epsilon z) - 1}{\epsilon} - z \) and put \( \psi_{\epsilon} := \eta_{\epsilon} \circ \psi \). Noting that \( D\eta_{\epsilon}(z) = \exp(\epsilon z) - 1 \), we have

\[
|\psi_{\epsilon}| \leq (\exp(|\epsilon| \cdot \|\psi\|_{\infty}) - 1) |\psi|
\]
on \( X \), and

\[
|\psi_{\epsilon}|_{1/p,1} \leq (\exp(|\epsilon| \cdot \|\psi\|_{\infty}) - 1) |\psi|_{1/p,1}.
\]

It follows that

\[
|\psi_{\epsilon}|_{1/p,1} \leq (\exp(|\epsilon| \cdot \|\psi\|_{\infty}) - 1) |\psi|_{1/p,1}.
\] (4.8)

On the other hand, if for each \( \tau \) in \( C \) we define the operator \( \mathcal{D}_{\tau} \) by \( \mathcal{D}_{\tau}(\chi) := \mathcal{L}_\tau(\psi \cdot \chi) \), then for every \( \epsilon \) in \( \mathbb{C} \) and every \( \chi \) in \( H^{1/p,1}(m) \) we have

\[
\mathcal{L}_{\tau+\epsilon}(\chi) - \mathcal{L}_\tau(\chi) = \mathcal{D}_\epsilon(\chi).
\]

Combined with (4.8), we have by lemma 4.5

\[
\left\|\mathcal{L}_{\tau+\epsilon}(\chi) - \mathcal{L}_\tau(\chi) \right\|_{1/p,1} \leq 2C_*\left\|\mathcal{L}_\tau\right\|_{1/p,1} \|\psi_{\epsilon} \|_{1/p,1} \|\chi\|_{1/p,1}
\]

\[
\leq (2C_*\|\mathcal{L}_\tau\|_{1/p,1} \|\psi_{\epsilon}\|_{1/p,1})
\cdot (\exp(|\epsilon| \cdot \|\psi\|_{\infty}) - 1) \|\chi\|_{1/p,1}.
\]

This implies that the operator norm \( \|\mathcal{L}_{\tau+\epsilon} - \mathcal{L}_\tau\|_{1/p,1} \) converges to 0 as \( \epsilon \) converges to 0, and completes the proof that \( \tau \mapsto \mathcal{L}_\tau|_{H^{1/p,1}(m)} \) is analytic in the sense of Kato.

That the spectral radius of \( \mathcal{L}_\tau|_{H^{1/p,1}(m)} \) depends on a real analytic way on \( \tau \) on a neighbourhood of \( \tau = 0 \) follows from part 1 and from the fact that \( \tau \mapsto \mathcal{L}_\tau|_{H^{1/p,1}(m)} \) is analytic in the sense of Kato, see for example [RS78, theorem XII.8]. This completes the proof of the corollary. \( \Box \)
5. Proof of theorem B

The purpose of this section is to prove theorem B. Throughout this section, fix an interval map \( f : I \to I \) in \( \mathcal{A} \) that is topologically exact on its Julia set \( J(f) \). Then there is \( N \geq 2 \), and a partition \( \mathcal{P} := \{ I^i_1, \cdots, I^i_N \} \) of \( I \) into intervals, such that \( f \) is continuous and strictly monotone on each \( I^i_1 \) in \( \mathcal{P} \). For each \( i \) in \( \{1, \cdots, N\} \), put \( \hat{I}_i := I^i_1 \cap J(f) \).

Let \( \alpha \) be in \( (0, 1] \), let \( \psi : J(f) \to \mathbb{R} \) be a Hölder continuous potential of exponent \( \alpha \) that is hyperbolic for \( f \), and let \( N \geq 1 \) be an integer such that the function \( \bar{\psi} := \frac{1}{N} \sum_{i=1}^{N} \psi(\chi) \) satisfies \( \sup_{J(f)} \bar{\psi} < P(f, \psi) \). By part 1 of lemma 2.3, the function \( \bar{\psi} \) is Hölder continuous of exponent \( \alpha \), the potentials \( \psi \) and \( \bar{\psi} \) share the same equilibrium states, and \( P(f, \bar{\psi}) = P(f, \psi) > \sup_{J(f)} \bar{\psi} \). On the other hand, let \( \hat{\chi} : J(f) \to \mathbb{R} \) be a Hölder continuous function, and note that \( \hat{\chi} := \frac{1}{N} \sum_{i=1}^{N} \chi \) is Hölder continuous of the same exponent as \( \chi \) and that for every \( t \) in \( \mathbb{R} \) we have \( P(f, \bar{\psi} + t \hat{\chi}) = P(f, \psi + t \chi) \). Thus, replacing \( \psi \) by \( \bar{\psi} \) and \( \chi \) by \( \hat{\chi} \) if necessary, throughout the rest of this section we can assume \( \sup_{J(f)} \psi < P(f, \psi) \).

Recall that the operator \( \mathcal{L}_\psi \) is defined by

\[
\mathcal{L}_\psi(\psi)(x) := \sum_{y \in f^{-1}(x)} \exp(\psi(y)) \psi(y),
\]

see section 1.3, and that \( \hat{\mathcal{L}}_\psi := \exp(-P(f, \psi)) \mathcal{L}_\psi \). Note that if we put \( g := \exp(\psi - P(f, \psi)) \), then \( \hat{\mathcal{L}}_\psi \) coincides with the operator \( \mathcal{L}_\psi \) defined in (4.3) with \( T \) replaced by \( f \), and \( X \) replaced by \( J(f) \).

The following lemma is well-known. We include its short proof for completeness.

**Lemma 5.1.** Let \( \mu \) be an atom-free \( \exp(P(f, \psi) - \psi) \)-conformal measure for \( f \). Then for every function \( \psi \) in \( L^1(\mu) \), we have

\[
\int_{J(f)} \hat{\mathcal{L}}_\psi(\psi) \, d\mu = \int_{J(f)} \psi \, d\mu.
\]

**Proof.** Using that \( \mu \) is \( \exp(P(f, \psi) - \psi) \)-conformal, for every \( i \) in \( \{1, 2, \cdots, N\} \) and every measurable subset \( S \) of \( \hat{I}_i \), we have

\[
\mu(f^{-1}(S)) = \mu(S) = \int_S \exp(\psi - P(f, \psi)) \, d\mu = \int_S \exp(\psi - P(f, \psi)) \, d\mu_{f^{-1}}(S).
\]

Hence, if we define \( v_i := (f^{-1})_{\mu_{f^{-1}}}(S) \), then \( \frac{dv_i}{d\mu} = \exp(P(f, \psi) - \psi) \) on a subset of \( \hat{I}_i \) of full measure with respect to \( \mu \). It follows that for every \( \psi \) in \( L^1(\mu) \), we have

\[
\int_{f^{-1}(S)} \psi \circ f^{-1} \, d\mu_{f^{-1}}(S) = \int_{S} \psi \, dv = \int_{S} \psi \exp(\psi - P(f, \psi)) \, d\mu.
\]

Replacing \( \psi \) by \( \psi \exp(-P(f, \psi) + \psi) \) above, we obtain

\[
\exp(-P(f, \psi)) \int_{f^{-1}(S)} (\psi \exp(\psi)) \circ f^{-1} \, d\mu = \int_{S} \psi \, d\mu.
\]

(5.1)

It follows that

\[
\int_{J(f)} \hat{\mathcal{L}}_\psi(\psi) \, d\mu = \exp(-P(f, \psi)) \sum_{i=1}^{N} \int_{f^{-1}(S)} (\psi \exp(\psi)) \circ f^{-1} \, d\mu = \sum_{i=1}^{N} \int_{S} \psi \, d\mu = \int_{J(f)} \psi \, d\mu.
\]

The proof of the lemma is complete. \( \square \)
To prove theorem B, let μ be an atom-free exp(P(f, ϕ)−ϕ)-conformal measure for f given by theorem A. By (4.1) the function ϕ is of bounded (1/α)-variation. Since ϕ is continuous and hence bounded, it follows that C := supJ(f) exp(ϕ) < +∞. So for all x₁ and x₂ in [0, 1], we have

| exp(ϕ)(x₁) − exp(ϕ)(x₂) | ≤ C|ϕ(x₁) − ϕ(x₂)|.

This implies supJ(f) exp(ϕ), and so g, is of bounded (1/α)-variation. On the other hand, our hypothesis supJ(f) ϕ < P(f, ϕ) implies supJ(f) g < 1. Using that μ is a (g⁻¹)-conformal measure for f, properties H1 and H2 in section 4.2 hold with T, X, and m replaced by f, J(f), and μ, respectively. Moreover, lemma 5.1 implies property H3 of section 4.2 also holds. Therefore, for each equilibrium state

and note that the operator exp(ϕ) is real analytic on a neighbourhood of J(f) and from [Dob13, theorem 6].

It remains to show that the function t ↦ P(f, ϕ+tχ) is real analytic on a neighbourhood of t = 0. For each τ in C, let Lₜ be the operator defined in part 3 of corollary 4.4 with T replaced by f and X by J(f). On the other hand, for each t in R put

ϕᵣ := ϕ + tχ and gᵣ := exp(ϕᵣ − P(f, ϕᵣ)),

and note that the operator exp(−P(f, ϕᵣ))Lₜ coincides with the operator Lₜ defined in (4.3) with g replaced by gᵣ, T replaced by f, and X replaced by J(f). Let a in (0, 1) be sufficiently small so that both ϕ and χ are Hölder continuous of exponent α, let p₁ and p₂ in (supJ(f), ϕ, P(f, ϕ)) be such that p₁ < p₂, and let ϵ₀ > 0 be small enough so that

ε₀ supₗ|χ| < min{p₂ − p₁, P(f, ϕ) − p₂}.

Note that by our choice of ϵ₀, for every t in (−ϵ₀, ϵ₀) we have on J(f) that

ϕᵣ > ϕ − (P(f, ϕ) − p₂)1J(f),

so

P(f, ϕᵣ) ≥ P(f, ϕ − (P(f, ϕ) − p₂)1J(f)) = P(f, ϕ) − (P(f, ϕ) − p₂)

= p₂ > supϕ + sup_{J(f)} tχ ≥ supϕᵣ. (5.2)

Thus supJ(f) gᵣ < 1 and by theorem A there is an atom-free exp(P(f, ϕᵣ)−ϕᵣ)-conformal measure μᵣ for f. Since the function gᵣ is of bounded (1/α)-variation, it follows that properties H1 and H2 in section 4.2 hold with p = 1/α, and with g, T, X, and m replaced by gᵣ, f, J(f), and μᵣ, respectively. Moreover, lemma 5.1 implies that property H3 of section 4.2 also holds. We can thus apply part 1 of corollary 4.4 to conclude that exp(P(f, ϕᵣ)) is equal to the spectral radius of Lᵣ. Moreover, by part 3 of the same corollary, the function t ↦ exp(P(f, ϕᵣ)) is real analytic on (−ϵ₀, ϵ₀). The proof of theorem B is thus complete.
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Appendix A. Hyperbolic potentials and the bounded range condition

Let $X$ be a compact metric space and $T : X \to X$ be a continuous map. Denote by $h_{\text{top}}(T)$ the topological entropy of $T$. Recall that 2 continuous functions $\varphi : X \to \mathbb{R}$ and $\tilde{\varphi} : X \to \mathbb{R}$ are co-homologous, if there is a continuous function $\chi : X \to \mathbb{R}$ such that

$$\tilde{\varphi} = \varphi + \chi - \chi \circ T.$$ 

It is easy to see that every continuous potential $\varphi : X \to \mathbb{R}$ satisfying the bounded range condition:

$$\sup_X \varphi - \inf_X \varphi < h_{\text{top}}(T),$$

(A.1)

also satisfies

$$\sup_X \varphi < P(T, \varphi),$$

(A.2)

and it is therefore hyperbolic for $T$.

The purpose of this section is to show that, under a fairly general condition on $T$, there is a potential $\varphi$ satisfying (A.2) that is not co-homologous to any potential $\tilde{\varphi}$ satisfying (A.1) with $\varphi$ replaced by $\tilde{\varphi}$. When $f$ is a map in $\mathcal{A}$ that is topologically exact on $J(f)$, this general condition is easily seen to be satisfied when $T = f|_{J(f)}$.

**Lemma A.1.** Let $X$ be a compact metric space, let $T : X \to X$ be a continuous map, and let $h > 0$ be given. Suppose there are disjoint compact subsets $X'$ and $X''$ of $X$ that are forward invariant by $T$, and such that $h_{\text{top}}(T|_{X'}) > 0$. Let $\varphi : X \to (-\infty, 0]$ be a continuous function that is constant equal to $0$ on $X'$, and such that $\sup_{X''} \varphi < -h$. Then

$$\sup_X \varphi < P(T, \varphi),$$

and for every continuous function $\tilde{\varphi} : X \to \mathbb{R}$ that is co-homologous to $\varphi$, we have

$$\sup_X \tilde{\varphi} - \inf_X \tilde{\varphi} > h.$$ 

(A.3)

**Proof.** By the variational principle there is a probability measure $\nu'$ on $X$ that is supported on $X'$, that is invariant by $T$, and such that $h_{\nu'}(T) = h_{\nu'}(T|_{X'}) > 0$. Then we have

$$P(T, \varphi) \geq h_{\nu'}(T) + \int_X \varphi \, d\nu' = h_{\nu'}(T) > 0 = \sup_X \varphi.$$ 

On the other hand, if $\tilde{\varphi} : X \to \mathbb{R}$ is a continuous function that is co-homologous to $\varphi$, then we have

$$\sup_X \tilde{\varphi} \geq \int_X \tilde{\varphi} \, d\nu' = \int_X \varphi \, d\nu' = 0.$$
Moreover, if \( \nu'' \) is a probability measure on \( X'' \) that is invariant by \( T \), then
\[
\inf_{\tilde{\phi}} \int_{X''} \tilde{\phi} d\nu'' = \int_{X''} \phi d\nu'' = \int_{X''} \phi d\nu'' \leq \sup_{X''} \phi < -h.
\]
Together with the inequality \( \sup_{X'} \tilde{\phi} \geq 0 \) shown above, this implies (A.3). \( \square \)

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