PULLBACK DYNAMICS OF A NON-AUTONOMOUS MIXTURE PROBLEM IN ONE DIMENSIONAL SOLIDS WITH NONLINEAR DAMPING

MIRELSON M. FREITAS
Federal University of Pará, Raimundo Santana Street s/n, Salinópolis PA, 68721-000, Brazil

ALBERTO L. C. COSTA AND GERALDO M. ARAÚJO
Institute of Exact and Natural Sciences, Doctoral Program in Mathematics, Federal University of Pará, Augusto corrêa Street, Number 01, 66075-110, Belém PA, Brazil

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ABSTRACT. This paper is devoted to study the asymptotic behavior of a non-autonomous mixture problem in one dimensional solids with nonlinear damping. We prove the existence of minimal pullback attractors with respect to a universe of tempered sets defined by the sources terms. Moreover, we prove the upper-semicontinuity of pullback attractors with respect to non-autonomous perturbations.

1. Introduction. The theory of mixtures of solids has been widely investigated in the last decades, see for instance the references [3, 5, 6, 7] for a detailed presentation. In this paper, our interest is devoted to a special case of a theory of binary mixture of solids with nonlinear damping, sources terms and non-autonomous external forces. Qualitative properties of solutions to the problem defining this kind of material have been the scope of many investigations. In particular, several results concerning existence, uniqueness, continuous dependence and asymptotic stability can be found in the literature [1, 2, 14, 20].

Recently, [21] studied a one-dimensional binary mixture problem of solids with nonlinear damping and sources terms given by

\[
\begin{align*}
\rho_1 u_{tt} - a_{11}u_{xx} - a_{12}w_{xx} + g_1(u_t) + f_1(u, w) &= 0, & \text{ in } (0, L) \times (0, T), \\
\rho_2 w_{tt} - a_{12}u_{xx} - a_{22}w_{xx} + g_2(w_t) + f_2(u, w) &= 0, & \text{ in } (0, L) \times (0, T),
\end{align*}
\]

(1.1)

with corresponding boundary-initial condition

\[
\begin{align*}
u(0, t) &= u(L, t) = w(0, t) = w(L, t) = 0, & t &\geq 0, \\
u(0) &= u_0 \in H^1_0(0, L), & u_t(0) &= u_1 \in L^2(0, L), \\
w(0) &= w_0 \in H^1_0(0, L), & w_t(0) &= w_1 \in L^2(0, L).
\end{align*}
\]

(1.2)

The variables \( u \) and \( w \) are, respectively, the displacement of the material and the volume fraction, \( g_1(u_t), g_2(w_t) \) are nonlinear damping terms and \( f_1(u, w), f_2(u, w) \)}
are nonlinear sources terms. Finally, $a_{11}, a_{12}, a_{22}$ are constants satisfying the relation
\begin{equation}
0 < a_{11} > a_{11}a_{22} - a_{12}^2 > 0.
\end{equation}

By using the quasi-stability theory of Chueshov and Lasiecka [12], the authors proved that the dynamical system generated by the problem (1.1)-(1.2) has a compact global attractor with finite fractal dimension. They also proved that solutions on the global attractor are more regular. Moreover, the existence of a generalized exponential attractor was proved.

Our proposing is considering the mixture problem in one dimensional solids with nonlinear damping and non-autonomous forcing given by
\begin{equation}
\begin{cases}
\rho_1 u_{tt} - a_{11} u_{xx} - a_{12} w_{xx} + g_1(u_t) + f_1(u, w) = h_1, & \text{in } (0, L), \ t \geq \tau, \\
\rho_2 w_{tt} - a_{12} u_{xx} - a_{22} w_{xx} + g_2(w_t) + f_2(u, w) = h_2, & \text{in } (0, L), \ t \geq \tau,
\end{cases}
\end{equation}
with corresponding boundary-initial condition
\begin{align}
&u(0, t) = u(L, t) = w(0, t) = w(L, t) = 0, \ t > \tau, \\
&u(\tau) = u_0^* \in H^1_0(0, L), \ u_t(\tau) = u_1^* \in L^2(0, L), \\
w(\tau) = w_0^* \in H^1_0(0, L), \ w_t(\tau) = w_1^* \in L^2(0, L).
\end{align}

Here, we consider the functions $h_j = h_j(x, t), j = 1, 2$, as time-dependent perturbations, which make the system non-autonomous whose hypotheses will be given in the next section.

The main goal here is to prove the existence of minimal pullback $\mathcal{D}$-attractors for the evolution process generated by the problem (1.4)-(1.5) with respect to a universe of tempered sets defined by the growth of sources terms $f_1(u, w), f_2(u, w)$. We will also prove the upper-semicontinuity of pullback attractors as the non-autonomous perturbation tends to zero. In fact, we will prove that the family of pullback attractors associated to problem (1.4)-(1.5) with $h_j$ replaced by $\epsilon h_j$ converges to the corresponding compact global attractor associated with the autonomous limit problem (1.1)-(1.2) when $\epsilon \to 0$.

This paper is organized as follows: In Section 2, we establish the existence and uniqueness of weak and strong solutions. This is presented in the Theorem 2.8. In Section 3, we recall the key definitions and results which concern the non-autonomous dynamical systems and pullback attractors. In Section 4, we prove that the evolution process generated by the problem (1.4)-(1.5) has a pullback $\mathcal{D}$-absorbing family and is pullback $\mathcal{D}$-asymptotically compact and, consequently, the existence of minimal pullback attractors is established. Our main result is presented in the Theorem 4.6. In Section 5, we prove the upper-semicontinuity of global attractors as the non-autonomous perturbation tends to zero. More precisely, we prove that the family of pullback attractors associated to problem (1.4)-(1.5) with $h_j$ replaced by $\epsilon h_j$ converges to the corresponding global attractor associated with the limit problem (1.1)-(1.2) as $\epsilon \to 0$. This is presented in the Theorem 5.1.

2. Preliminaries and well-posedness. In this section, the existence and uniqueness of weak and strong solutions of the problem (1.4)-(1.5) will be studied.

2.1. Assumptions. In this part, we present some notations and assumptions. We will use the following notations
\[ \| u \|_p = \| u \|_{L^p(0, L)}, \quad p \geq 1, \quad (u, v)_2 = (u, v)_{L^2(0, L)}. \]
We recall the Poincaré’s inequality
\[ \lambda_0 \| \varphi \|_2^2 \leq \| \varphi_x \|_2^2, \quad \forall \varphi \in H^1_0(0, L), \]  
where \( \lambda_0 = \pi^2/L^2 \).

Our study is given on the phase space \( \mathcal{H} = (H^1_0(0, L))^2 \times (L^2(0, L))^2 \).

It is a Hilbert space with the inner product: If \( z = (u, w, u', w') \), \( \tilde{z} = (\tilde{u}, \tilde{w}, \tilde{u}', \tilde{w}') \) \( \in \mathcal{H} \), then we define
\[
(z, \tilde{z})_{\mathcal{H}} = \rho_1 \int_0^L u' \tilde{u}' \, dx + \rho_2 \int_0^L u' \tilde{w}' \, dx + \left( a_{22} - a_{12}^2/a_{11} \right) \int_0^L w_x \tilde{w}_x \, dx \\
+ \int_0^L \left( \frac{a_{12}}{\sqrt{a_{11}}} w_x + \sqrt{a_{11}} u_x \right) \left( \frac{a_{12}}{\sqrt{a_{11}}} \tilde{w}_x + \sqrt{a_{11}} \tilde{u}_x \right) \, dx.
\]
The associated norm is then given by
\[
\| z \|_{\mathcal{H}}^2 = \rho_1 \| u' \|_2^2 + \rho_2 \| w' \|_2^2 \\
+ \left( a_{22} - a_{12}^2/a_{11} \right) \| w_x \|_2^2 + \left\| \frac{a_{12}}{\sqrt{a_{11}}} w_x + \sqrt{a_{11}} u_x \right\|_2^2.
\]

**Lemma 2.1.** There exists a constant \( \kappa_0 > 0 \) such that
\[
\kappa_0 \left( \| u_x \|_2^2 + \| w_x \|_2^2 \right) \leq \left( a_{22} - a_{12}^2/a_{11} \right) \| w_x \|_2^2 + \left\| \frac{a_{12}}{\sqrt{a_{11}}} w_x + \sqrt{a_{11}} u_x \right\|_2^2.
\]  

**Proof.** Indeed, we observe that
\[
\| u_x \|_2^2 + \| w_x \|_2^2 = \| u_x \|_2^2 + \left\| \frac{a_{12}}{\sqrt{a_{11}}} w_x + \sqrt{a_{11}} u_x + \left( 1 - \frac{a_{12}}{\sqrt{a_{11}}} \right) w_x - \sqrt{a_{11}} u_x \right\|_2^2.
\]
It follows that
\[
\| u_x \|_2^2 + \| w_x \|_2^2 \leq \| u_x \|_2^2 + 2 \left\| \frac{a_{12}}{\sqrt{a_{11}}} w_x + \sqrt{a_{11}} u_x \right\|_2^2 + 2 \left( 1 - \frac{a_{12}}{\sqrt{a_{11}}} \right) \| w_x - \sqrt{a_{11}} u_x \|_2^2 \\
\leq (1 + 4a_{11}) \| u_x \|_2^2 + 2 \left\| \frac{a_{12}}{\sqrt{a_{11}}} w_x + \sqrt{a_{11}} u_x \right\|_2^2 \\
+ 4 \left( 1 - \frac{a_{12}}{\sqrt{a_{11}}} \right)^2 \| w_x \|_2^2 \\
= 2 \left\| \frac{a_{12}}{\sqrt{a_{11}}} w_x + \sqrt{a_{11}} u_x \right\|_2^2 + 4 \left( 1 - \frac{a_{12}}{\sqrt{a_{11}}} \right)^2 \| w_x \|_2^2 \\
+ \frac{1 + 4a_{11}}{a_{11}} \left\| \frac{a_{12}}{\sqrt{a_{11}}} w_x + \sqrt{a_{11}} u_x - \frac{a_{12}}{\sqrt{a_{11}}} w_x \right\|_2^2.
\]
that is,
\[
\|u_x\|^2 + \|w_x\|^2 \leq 2 \left( \frac{a_{12}}{\sqrt{a_{11}}} \|w_x + \sqrt{a_{11}} u_x\|^2 + 4 \left( 1 - a_{12}/\sqrt{a_{11}} \right)^2 \|w_x\|^2 \right) + \frac{1 + 4a_{11}}{a_{11}} \left( \frac{a_{12}}{\sqrt{a_{11}}} \|w_x + \sqrt{a_{11}} u_x\|^2 + \frac{a_{12}^2}{a_{11}} \|w_x\|^2 \right),
\]
and consequently
\[
\|u_x\|^2 + \|w_x\|^2 \leq 2 \left( \frac{a_{12}}{\sqrt{a_{11}}} \|w_x + \sqrt{a_{11}} u_x\|^2 + 4 \left( 1 - a_{12}/\sqrt{a_{11}} \right)^2 \|w_x\|^2 \right) + \frac{2(1 + 4a_{11})}{a_{11}} \left( \frac{a_{12}}{\sqrt{a_{11}}} \|w_x + \sqrt{a_{11}} u_x\|^2 + \frac{a_{12}^2}{a_{11}} \|w_x\|^2 \right). \tag{2.4}
\]
Grouping the terms in (2.4), we have
\[
\|u_x\|^2 + \|w_x\|^2 \leq \frac{2(1 + 5a_{11})}{a_{11}} \left( \frac{a_{12}}{\sqrt{a_{11}}} \|w_x + \sqrt{a_{11}} u_x\|^2 + \frac{4a_{11}^2(1 - a_{12}/\sqrt{a_{11}})^2 + 2a_{12}^2(1 + 4a_{11})}{a_{11}^2(a_{22} - a_{12}/a_{11})} \|w_x\|^2 \right).
\]
Choosing
\[
c := \max \left\{ \frac{2(1 + 5a_{11})}{a_{11}}, \frac{4a_{11}^2(1 - a_{12}/\sqrt{a_{11}})^2 + 2a_{12}^2(1 + 4a_{11})}{a_{11}^2(a_{22} - a_{12}/a_{11})} \right\}
\]
we conclude that
\[
\frac{1}{c} \left( \|u_x\|^2 + \|w_x\|^2 \right) \leq (a_{22} - a_{12}^2/a_{11}) \|w_x\|^2 + \left( \frac{a_{12}}{\sqrt{a_{11}}} \|w_x + \sqrt{a_{11}} u_x\|^2 \right).
\]
The proof is complete. \(\square\)

**Remark 2.2.** Combining (2.1) and (2.3), we obtain a useful inequality
\[
\|u\|^2 + \|w\|^2 \leq \gamma \left( (a_{22} - a_{12}^2/a_{11}) \|w_x\|^2 + \left( \frac{a_{12}}{\sqrt{a_{11}}} \|w_x + \sqrt{a_{11}} u_x\|^2 \right), \tag{2.5}
\]
where \(\gamma = (\kappa_0 \lambda_0)^{-1}\).

Inspired by [17] we use the following assumption.

**Assumption 2.3** (Damping, sources and external forces).

(i) **Damping:** \(g_1, g_2 \in C^1(\mathbb{R})\) are increasing functions with \(g_1(0) = g_2(0) = 0\). Moreover, there exist \(M_1, M_2 > 0\) such that, for \(j = 1, 2\)
\[
M_1 \leq g_j'(s) \leq M_2, \quad \forall s \in \mathbb{R}. \tag{2.6}
\]
(ii) **Sources:** There exists a function \(F \in C^2(\mathbb{R}^2)\) such that
\[
\nabla F = (f_1, f_2). \tag{2.7}
\]
There exist $p \geq 1$ and $C > 0$ such that, for $j = 1, 2$
$$|\nabla f_j(u, w)| \leq C \left(|u|^{p-1} + |w|^{p-1} + 1 \right). \quad (2.8)$$

There exist $\beta, m_F > 0$ with
$$0 \leq \beta < \frac{1}{2\gamma}, \quad (2.9)$$
such that
$$F(u, w) \geq -\beta (|u|^2 + |w|^2) - m_F. \quad (2.10)$$

Moreover, we assume that
$$\nabla F(u, w) \cdot (u, w) - F(u, w) \geq -\beta (|u|^2 + |w|^2) - m_F. \quad (2.11)$$

(iii) **External forces:** For the external forces, we assume that
$$h_1, h_2 \in L^2_{\text{loc}}(\mathbb{R}; L^2(0, L))$$
and
$$\int_{-\infty}^{t} e^{-\sigma_0(t-s)} \left(\|h_1(s)\|^2_2 + \|h_2(s)\|^2_2\right) \, ds < \infty, \quad \forall t \in \mathbb{R}, \quad (2.12)$$
with $\sigma_0 \in (0, \sigma_1]$, where $\sigma_1 > 0$ is a constant dependent only on the parameters of model specified later in Lemma 4.1.

Remark 2.4. Observe that assumption (2.6) implies the monotonicity property, that is,
$$(g_j(u) - g_j(v))(u - v) \geq M_1 |u - v|^2, \quad \forall u, v \in \mathbb{R}, \quad j = 1, 2. \quad (2.13)$$

2.2. **Existence and uniqueness of solutions.** In order to describe the results, we introduce the definition of generalized solution to the problem (1.4)-(1.5).

**Definition 2.5.** A function $z = (u, w, u_t, w_t)$ is called a generalized (weak) solution to (1.4)-(1.5) if
$$z \in C([\tau, \infty); \mathcal{H}), \quad z(\tau) = (u_0^\tau, w_0^\tau, u_1^\tau, w_1^\tau) \in \mathcal{H},$$
and satisfies the following identity in the sense of distributions
$$\rho_1 \frac{d}{dt}(u_t, \varphi)_2 + \rho_2 \frac{d}{dt}(u_t, \psi)_2 + \left(a_{22} - \frac{a_{12}^2}{a_{11}}\right) (w_x, \psi_x)_2$$
$$+ \left(\frac{a_{12}}{\sqrt{a_{11}}} w_x + \frac{a_{12}}{\sqrt{a_{11}}} u_x, \frac{a_{12}}{\sqrt{a_{11}}} \psi_x + \frac{a_{12}}{\sqrt{a_{11}}} \varphi_x\right)_2$$
$$+ (g_1(u_t), \varphi)_2 + (g_2(w_t), \psi)_2 + (f_1(u, w), \varphi)_2 + (f_2(u, w), \psi)_2$$
$$= (h_1, \varphi)_2 + (h_2, \psi)_2, \quad \forall \varphi, \psi \in H^1_0(0, L). \quad (2.14)$$

If a weak solution satisfies further
$$z \in C((\tau, \infty); \mathcal{H}_1) \cap W^{1, \infty}_{\text{loc}}(\tau, \infty); \mathcal{H}),$$
where
$$\mathcal{H}_1 = (H^2(0, L) \cap H^1_0(0, L))^2 \times (H^1_0(0, L))^2,$$
then it is called strong.
We define, along a strong solution, the total energy by

$$E(t) = E(t) + \int_0^t F(u(t), w(t)) \, dx,$$

where $E(t)$ is the linear energy given by

$$E(t) = \frac{1}{2} \| (u, w, u_t, w_t) \|^2_H.$$

**Lemma 2.6.** Let $z = (u, w, u_t, w_t)$ be a strong solution of (1.4)-(1.5), this way, the total energy satisfies

$$\frac{d}{dt} E(t) = - \int_0^t \left( g_1(u_t) u_t + g_2(w_t) w_t \right) \, dx + \int_0^t \left( h_1 u_t + h_2 w_t \right) \, dx$$

$$\leq - \frac{M_1}{2} \left( \| u_t \|^2_H + \| w_t \|^2_H \right) + \frac{1}{2M_1} \left( \| h_1(t) \|^2_H + \| h_2(t) \|^2_H \right).$$

**Proof.** Multiplying the first equation in (1.4) by $u_t$ and integrating by parts over $[0, L]$, we obtain

$$\frac{d}{dt} \rho_1 \int_0^L |u_t|^2 \, dx + \frac{d}{dt} \frac{a_{11}}{2} \int_0^L |u_x|^2 \, dx + a_{12} \int_0^L w_x u_{xt} \, dx$$

$$+ \int_0^L g(u_t) u_t \, dx + \int_0^L f_1(u, w) u_t = \int_0^L h_1 u_t \, dx. \tag{2.17}$$

If in turn we multiply by $w_t$ the second equation in (1.4), we get

$$\frac{d}{dt} \rho_2 \int_0^L |w_t|^2 \, dx + \int_0^L u_x w_{xt} \, dx + \frac{d}{dt} \frac{a_{21}}{2} \int_0^L |w_x|^2 \, dx$$

$$+ \int_0^L g_2(w_t) w_t \, dx + \int_0^L f_2(u, w) w_t = \int_0^L h_2 w_t \, dx. \tag{2.18}$$

Add (2.17) and (2.18) and then use (2.7) to obtain

$$\frac{d}{dt} \frac{1}{2} \int_0^L \left( \rho_1 |u_t|^2 + \rho_2 |w_t|^2 + \rho_2 a_{12} w_x u_{xt} \right) \, dx + \int_0^L \nabla F(u, w) \cdot (u_t, w_t) \, dx$$

$$= - \int_0^L \left( g_1(u_t) u_t + g_2(w_t) w_t \right) \, dx + \int_0^L (h_1 u_t + h_2 w_t) \, dx.$$

It follows that

$$\frac{d}{dt} \frac{1}{2} \int_0^L \left( \rho_1 |u_t|^2 + \rho_2 |w_t|^2 + \left( a_{22} - \frac{a_{12}^2}{a_{11}} \right) |w_x|^2 + \frac{a_{12}}{\sqrt{a_{11}}} u_x \right)^2 \, dx$$

$$+ \frac{d}{dt} \int_0^L F(u, w) \, dx = - \int_0^L \left( g_1(u_t) u_t + g_2(w_t) w_t \right) \, dx + \int_0^L (h_1 u_t + h_2 w_t) \, dx.$$

This proves that

$$\frac{d}{dt} E(t) = - \int_0^L \left( g_1(u_t) u_t + g_2(w_t) w_t \right) \, dx + \int_0^L (h_1 u_t + h_2 w_t) \, dx.$$
By (2.6) and Young’s inequality we conclude that
\[
\frac{d}{dt} E(t) = - \int_0^L \left( g_1(u_t)u_t + g_2(w_t)w_t \right) dx + \int_0^L \left( h_1 u_t + h_2 w_t \right) dx \\
\leq - M_1 (\|u_t\|^2 + \|w_t\|^2) + \int_0^L \left( \rho_1^1 \|u_t\| + \rho_2^2 \|w_t\| \right) dx \\
\leq - \frac{M_1}{2} (\|u_t\|^2 + \|w_t\|^2) + \frac{1}{2M_1} (\|h_1(t)\|^2 + \|h_2(t)\|^2).
\]

The proof is complete. \(\square\)

2.3. **Semigroup formulation.** In this part, we show that the system (1.4)-(1.5) is well-posed using nonlinear semigroup theory and monotone operators (see e.g. [10, 4]).

Let us write the problem (1.4)-(1.5) as a Cauchy problem
\[
\begin{cases}
\frac{dz(t)}{dt} + Az(t) = F(t, z(t)), \ t > \tau, \\
z(\tau) = z_\tau \in \mathcal{H},
\end{cases}
\tag{2.19}
\]
where
\[
z(t) = (u(t), w(t), u'(t), w'(t)) \in \mathcal{H}, \quad u' = u_t, \quad w' = w_t,
\]
with \(\mathcal{H}\) defined in (2.2), and
\[
z_\tau = (u_\tau^0, w_\tau^0, u_\tau^1, w_\tau^1) \in \mathcal{H},
\]
is the initial condition and \(A : D(A) \subset \mathcal{H} \rightarrow \mathcal{H}\) is the nonlinear operator defined by
\[
A z = \begin{pmatrix}
-u' \\
-w'
\end{pmatrix} - \begin{pmatrix}
a_{11} u_{xx} & a_{12} u_{xx} + \frac{1}{\rho_1^1} \rho_1 g_1(u') \\
a_{12} w_{xx} & a_{22} w_{xx} + \frac{1}{\rho_2^2} \rho_2 g_2(w')
\end{pmatrix}.
\]
The domain of \(A\) is given by
\[
D(A) = (H^2(0, L) \cap H_0^1(0, L))^2 \times (H_0^1(0, L))^2,
\]
and \(F : [\tau, \infty) \times \mathcal{H} \rightarrow \mathcal{H}\) is defined by
\[
F(t, z) = \begin{pmatrix}
0 \\
0 \\
-f_1(u, w) + h_1(t) \\
-f_2(u, w) + h_2(t)
\end{pmatrix}.
\tag{2.20}
\]

2.4. **Local and global solutions.** This part, is dedicated to prove the existence of local and global solutions for the problem (1.4)-(1.5). We start by proving an auxiliary result which will be used in the sequel.

**Lemma 2.7.** Suppose that \(z = (u, w, u_t, w_t)\) is a weak solution to (1.4)-(1.5). Then, there exist constants \(\beta_0, C_F > 0\) such that
\[
E(t) \geq \beta_0 \|(u, w, u_t, w_t)\|_{\mathcal{H}}^2 - C_F, \quad \forall t \geq \tau,
\tag{2.21}
\]
and
\[
E(t) \leq C_F \left( \|(u, w, u_t, w_t)\|_{\mathcal{H}}^{p+1} + 1 \right), \quad \forall t \geq \tau.
\tag{2.22}
\]
Proof. By (2.10) and (2.5) it follows that
\[
\int_0^LF(u,w)\,dx \geq -\beta(\|u\|_2^2 + \|w\|_2^2) - Lm_F - \beta \gamma \left( (a_{22} - a_{12}^2/a_{11})\|w_x\|_2^2 + \left\| \frac{a_{12}}{\sqrt{a_{11}}}w_x + \sqrt{a_{11}}u_x \right\|_2^2 \right) - Lm_F.
\]
We choose \(C_F = Lm_F\) to get
\[
\mathcal{E}(t) \geq \frac{1}{2}(\|u, w, u_t, w_t\|_H^2) - \beta \gamma \left( (a_{22} - a_{12}^2/a_{11})\|w_x\|_2^2 + \left\| \frac{a_{12}}{\sqrt{a_{11}}}w_x + \sqrt{a_{11}}u_x \right\|_2^2 \right) - C_F
\[
\geq \left( \frac{1}{2} - \beta \gamma \right) \|(u, w, u_t, w_t)\|_H^2 - C_F.
\]
Using (2.9), we obtain (2.21) with
\[
\beta_0 = \frac{1}{2} - \beta \gamma > 0.
\]

The assertion (2.22) follows from the assumption (2.8) and the fact that \(H_0^1(0, L) \hookrightarrow L^p(0, L)\). The proof is complete.

Theorem 2.8 (Well-posedness). Suppose that Assumption 2.3 holds. Then for any initial data \(z_\tau \in \mathcal{H}\), problem (1.4)-(1.5) has a unique weak solution \(z = (u, w, u_t, w_t)\) satisfying
\[
z \in C(\tau, \infty); \mathcal{H}, \quad z(\tau) = z_\tau.
\]
If \(z_\tau \in D(\mathcal{A})\), then the solution is strong. Moreover, the weak solutions depend continuously on the initial data \(z_\tau\) in the phase space \(\mathcal{H}\).

Proof. The proof of the present theorem is done by using the equivalent Cauchy problem (2.19). The existence of solution for the autonomous case \((\mathcal{F}(t, z) = \mathcal{F}(z))\) was proven in [21]. Here we will present the main ideas of the proof.

**Step 1: Local solutions.** Note that \(\mathcal{A}\) is maximal monotone, and that for each \(t \in [\tau, \infty)\) fixed, \(\mathcal{F}(t, \cdot) : \mathcal{H} \to \mathcal{H}\) is locally Lipschitz. Then by [10, Theorem 7.2] for all \(z_\tau \in D(\mathcal{A})\) there exists \(t_{\text{max}} \leq \infty\) and a unique strong solution \(z\) for (2.19) defined on the interval \([\tau, t_{\text{max}}]\). Moreover, if \(z_\tau \in \mathcal{H}\) then (2.19) has a unique weak solution \(z \in C([\tau, t_{\text{max}}]; \mathcal{H})\) and such solutions satisfy \(\limsup_{t \to t_{\text{max}}} \|z(t)\|_\mathcal{H} = \infty\), provided \(t_{\text{max}} < \infty\).

**Step 2: Global solutions.** By Lemma 2.6 we have
\[
\frac{d}{dt}\mathcal{E}(t) \leq -\frac{M_1}{2}(\|u_t\|_2^2 + \|w_t\|_2^2) + \frac{1}{2M_1}(\|h_1(t)\|_2^2 + \|h_2(t)\|_2^2).
\]
It follows that
\[
\mathcal{E}(t) \leq \mathcal{E}(\tau) + \frac{1}{2M_1} \int_\tau^t (\|h_1(s)\|_2^2 + \|h_2(s)\|_2^2)\,ds, \quad \forall t \geq \tau.
\]
Using (2.21) we obtain
\[
\|z(t)\|_\mathcal{H}^2 \leq C\mathcal{E}(t) + C, \quad \forall t \geq \tau,
\]
for some constant $C > 0$ independent of $t$. Hence, by (2.24) we conclude that
$$\|z(t)\|_\mathcal{H} < \infty, \quad \forall t \geq \tau,$$
which implies that $t_{\text{max}} = \infty$.

**Step 3: Continuous dependence.** Let $z^1 = (u^1, w^1, u^1_t, w^1_t)$ and $z^2 = (u^2, w^2, u^2_t, w^2_t)$ be the weak solutions of (1.4)-(1.5) with corresponding initial data $z^1_0, z^2_0 \in \mathcal{H}$. Let us denote
$$u = u^1 - u^2, \quad w = w^1 - w^2.$$
Then $(u, w, u_t, w_t)$ is the solution of
\begin{equation}
\begin{cases}
\rho_1 u_{tt} - a_{11} u_{xx} - a_{12} u_{xx} + g_1(u^1_t) - g_1(u^2_t) = f_1(u^2, w^2) - f_1(u^1, w^1), \\
\rho_2 u_{tt} - a_{12} u_{xx} - a_{22} u_{xx} + g_2(u^2_t) - g_2(u^1_t) = f_2(u^2, w^2) - f_2(u^1, w^1),
\end{cases}
\end{equation}
with Dirichlet boundary conditions and initial condition
$$(u(\tau), w(\tau), u_t(\tau), w_t(\tau)) = z^1_0 - z^2_0.$$ \hspace{1cm} (2.25)

Multiplying the first equation in (2.25) by $u_t$ and second by $w_t$ and then integrating over $[0, L]$ we obtain
\begin{align*}
\frac{1}{2} \frac{d}{dt} \|u, w, u_t, w_t\|_{\mathcal{H}}^2 &= -\int_0^L (g_1(u^1_t) - g_1(u^2_t)) u_t \, dx \\
&\quad - \int_0^L (g_2(w^1_t) - g_2(w^2_t)) w_t \, dx \\
&\quad + \int_0^L (f_1(u^2, w^2) - f_1(u^1, w^1)) u_t \, dx \\
&\quad + \int_0^L (f_2(u^2, w^2) - f_2(u^1, w^1)) w_t \, dx.
\end{align*}
\hspace{1cm} (2.26)

From (2.8), Hölder’s inequality and embedding $H^1_0(0, L) \hookrightarrow L^\infty(0, L)$ we deduce that
\begin{align*}
\int_0^L (f_1(u^2, w^2) - f_1(u^1, w^1)) u_t \, dx \\
&\leq C \left( \|z^1\|_{\mathcal{H}}^{p-1} + \|z^2\|_{\mathcal{H}}^{p-1} + 1 \right) \left( \|u\|_2 + \|w\|_2 \right) \|u_t\|_2 \, dx \\
&\leq C_0 \left( \|u\|_2^2 + \|w\|_2^2 \right) + M_1 \|u_t\|_2^2,
\end{align*}
\hspace{1cm} (2.27)

where $C_0 = C \left( \|z^1\|_{\mathcal{H}}^{2(p-1)} + \|z^2\|_{\mathcal{H}}^{2(p-1)} + 1 \right)$ and $C > 0$ is a constant independent of $z^1, z^2$.

In a similar way we obtain that
\begin{align*}
\int_0^L (f_2(u^2, w^2) - f_2(u^1, w^1)) w_t \, dx \\&\leq C_0 \left( \|u\|_2^2 + \|w\|_2^2 \right) + M_1 \|w_t\|_2^2.
\end{align*}
\hspace{1cm} (2.28)

It follows from (2.27), (2.28) and (2.5) that
\begin{align*}
\int_0^L (f_1(u^2, w^2) - f_2(u^1, w^1)) u_t \, dx + \int_0^L (f_2(u^2, w^2) - f_2(u^1, w^1)) w_t \, dx \\
&\leq 2C_0 \left( \|u\|_2^2 + \|w\|_2^2 \right) + M_1 \left( \|u_t\|_2^2 + \|w_t\|_2^2 \right)
\end{align*}
\hspace{1cm} (2.29)
By (2.13) we conclude that

\[
\int_0^L (g_1(u^1_t) - g_1(u^2_t)) u_t \, dx \geq M_1 \|u_t\|_2^2, \\
\int_0^L (g_2(w^1_t) - g_2(w^2_t)) w_t \, dx \geq M_1 \|w_t\|_2^2.
\]

Substituting the estimates (2.29) and (2.30) in (2.26), we get that

\[
\frac{d}{dt} \|(u(t), w(t), u_t(t), w_t(t))\|_{\mathcal{H}}^2 \leq 4\gamma C_0(t) \|(u(t), w(t), u_t(t), w_t(t))\|_{\mathcal{H}}^2.
\]

Applying Gronwall’s lemma to (2.31) we conclude that

\[
\|z^1(t) - z^2(t)\|_{\mathcal{H}}^2 \leq e^{4\gamma \int_\tau^T C_0(r) \, dr} \|z^1_\tau - z^2_\tau\|_{\mathcal{H}}^2, \quad t \in [\tau, T],
\]

for any \( T \geq \tau \). Since \( \int_\tau^T C_0(r) \, dr < \infty \) for \( t \in [\tau, T] \), the continuous dependence follows from (2.32). The proof of Theorem 2.8 is complete. \( \square \)

3. Abstract results on the theory of pullback attractors. To describe the next results, we need some notations, definitions, and results (see, for instance [19, 15, 8, 13, 9] and references therein) which will be used throughout the following sections. We begin with precise definitions of the notions of a evolution process in metric space \((X,d)\).

**Definition 3.1.** A mapping \( U(t, \tau) : X \to X, t \geq \tau \in \mathbb{R} \), is said to be an evolution process if

1. \( U(\tau, \tau)x = x \) for all \( \tau \in \mathbb{R} \) and \( x \in X \),
2. \( U(t, \tau) = U(t, s)U(s, \tau) \), for all \( t \geq s \geq \tau \).

A process \( U \) is said to be closed if for any sequence \( x_n \to x \) in \( X \) and \( U(t, \tau)x_n \to y \) in \( X \), then \( U(t, \tau)x = y \). In addition, \( U \) is said to be continuous if the mapping \( U(t, \tau) : X \to X \) is continuous for each \( t \geq \tau \) fixed.

**Remark 3.2.** It is clear that every continuous process is closed.

Throughout this paper, we denote by \( \hat{D} \) a family of parameterised subsets in \( X \), that is,

\[
\hat{D} = \{D(t)\}_{t \in \mathbb{R}} \text{ with } D(t) \subset X, \ t \in \mathbb{R}.
\]

**Definition 3.3.** A universe in \( X \) is a class \( \mathcal{D} \) of elements \( \hat{D} = \{D(t)\}_{t \in \mathbb{R}} \) such that each section \( D(t) \) is a non-empty subset of \( X \) for all \( t \in \mathbb{R} \). We say that a universe \( \mathcal{D} \) is inclusion closed if given \( \hat{D} \in \mathcal{D} \) and \( \hat{C} \) such that \( C(t) \subset D(t) \) for all \( t \in \mathbb{R} \), then \( \hat{C} \in \mathcal{D} \).

**Definition 3.4.** A family \( \hat{B} \) of non-empty sets is pullback \( \mathcal{D} \)-absorbing for the process \( U \) if for any \( t \in \mathbb{R} \) and any \( \hat{D} \in \mathcal{D} \), there exists a \( \tau_0(\hat{B}, t) \leq t \) such that

\[
U(t, \tau)D(\tau) \subset B(t) \quad \text{for any } \tau \leq \tau_0(\hat{B}, t).
\]

Observe that in the above definition the set \( \hat{B} \) does not necessarily belong to the class \( \mathcal{D} \).
**Definition 3.5.** Given a family $\hat{D}$, an evolution process $U$ is said to be pullback $\hat{D}$-asymptotically compact if for any $t \in \mathbb{R}$ and any sequences $\tau_n \to -\infty$ and $x_n \in D(\tau_n)$ the set $\{U(t, \tau_n)x_n\}_{n \in \mathbb{N}}$ is precompact in $X$. If a process is pullback $\hat{D}$-asymptotically compact for any $\hat{D} \in D$, then we say it is pullback $D$-asymptotically compact.

Now, we recall a criterion, which is useful for verifying the pullback $D$-asymptotic compactness of evolutions process generated by non-autonomous hyperbolic equations (see [11, 12] for autonomous system and [23, 24, 16] for non-autonomous system).

**Definition 3.6.** Let $X$ be a metric space and $B$ be a bounded subset of $X$. A function $\Psi : X \times X \to \mathbb{R}$ is said to be contractive on $B$ if for any sequence $\{x_n\} \subset B$ there exists a subsequence $\{x_{n_k}\}$ such that

$$\lim_{k \to \infty} \lim_{l \to \infty} \Psi(x_{n_k}, x_{n_l}) = 0.$$ 

The proof of the following result can be found in [16, Theorem 3.2] and [18, Theorem 3.2].

**Theorem 3.7.** Let $U$ be a evolution process on a Banach space $X$. Assume that $U$ possesses a pullback $D$-absorbing family $\hat{B}_0$ and for any $t \in \mathbb{R}$ and $\epsilon > 0$ there exists $\tau_\epsilon \leq t$ and a contractive function $\Psi_\epsilon : B_0(\tau_\epsilon) \times B_0(\tau_\epsilon) \to \mathbb{R}$ such that

$$\|U(t, \tau_\epsilon)z_1 - U(t, \tau_\epsilon)z_2\| \leq \epsilon + \Psi_\epsilon(z_1, z_2), \quad \forall z_1, z_2 \in B_0(\tau_\epsilon).$$

Then the process is pullback $D$-asymptotically compact.

In the sequel we introduce the concept of pullback $D$-attractor.

**Definition 3.8.** A family $\hat{A}$ is said to be the (minimal) pullback $D$-attractor for the evolution process $U$ if the following conditions are hold

(i) $A(t)$ is compact for all $t \in \mathbb{R}$;
(ii) $\hat{A}$ invariant, i.e., $U(t, \tau)A(\tau) = A(t)$ for all $t \geq \tau$;
(iii) $\hat{A}$ pullback $D$-attracting, i.e.,

$$\lim_{\tau \to -\infty} \text{dist}(U(t, \tau)A(\tau), A(t)) = 0, \quad \text{for all } \hat{D} \in D \text{ and } t \in \mathbb{R}. $$

(iv) $\hat{A}$ is minimal in the sense that if $\hat{C}$ is a family of closed sets which is pullback $D$-attracting, then $A(t) \subset C(t)$ for all $t \in \mathbb{R}$.

The following result ensures the existence of a minimal pullback attractor (see [13, Theorem 3.11]).

**Theorem 3.9.** Let $U$ be a closed evolution process in a metric space $X$. Consider a universe $D$ in $X$ and suppose that $U$ admits a pullback $D$-absorbing family $\hat{B}_0$ and that $U$ is pullback $\hat{B}_0$-asymptotically compact. Then, the family $\hat{A}_D = \{A(t)\}_{t \in \mathbb{R}}$ defined by

$$A(t) = \bigcup_{\hat{D} \in D} \Lambda(\hat{D}, t),$$

where denotes the pullback omega-limit

$$\Lambda(\hat{D}, t) = \bigcap_{s \leq t} \overline{\bigcup_{\tau \leq s} U(t, \tau)D(\tau)},$$
is the minimal pullback $\mathcal{D}$-attractor for the process $U$. If $\hat{B}_0 \in \mathcal{D}$, then

$$A(t) = \Lambda(\hat{B}_0, t) \subset \overline{B}_0(t), \quad \forall t \in \mathbb{R}.$$  

Moreover, if $B_0(t)$ is closed for all $t \in \mathbb{R}$ and the universe $\mathcal{D}$ is inclusion closed, then the pullback attractor $\hat{A}_D \in \mathcal{D}$.

4. **Pullback $\mathcal{D}$-attractors.** This section is dedicated to prove the existence of pullback $\mathcal{D}$-attractors for the evolution process generated by the problem (1.4)-(1.5).

4.1. **Pullback $\mathcal{D}$-absorbing.** In this part, the existence of a pullback $\mathcal{D}$-absorbing family is proved. Theorem 2.8 indicates that the problem (1.4)-(1.5) defines a continuous evolution process $U(t, \tau) : \mathcal{H} \to \mathcal{H}$ given by

$$U(t, \tau)z_{\tau} = (u(t), w(t), u_t(t), w_t(t)), \quad t \geq \tau,$$

with $(u(t), w(t), u_t(t), w_t(t))$ being the weak solution of (1.4)-(1.5) with initial data $z_{\tau} \in \mathcal{H}$.

To define a suitable universe in $\mathcal{H}$ for our purposes, we first establish the following stability inequality.

**Lemma 4.1.** Suppose that Assumption 2.3 holds. Then, for any $z \in \mathcal{H}$ there exist constants $\sigma_1 > 0$ and $C_1, C_2, C_3 > 0$ such that

$$\|U(t, \tau)z\|^2_{\mathcal{H}} \leq C_1 \left( \|z\|^p_{\mathcal{H}} + 1 \right) e^{-\sigma_1 (t-\tau)} + C_2 \int_{-\infty}^{t} e^{-\sigma_1 (t-s)} \left( \|h_1(s)\|^2 + \|h_2(s)\|^2_2 \right) ds + C_3 C_F, \quad \forall t \geq \tau. \quad (4.1)$$

**Proof.** For each $\epsilon > 0$, we shall define the perturbed energy by

$$\mathcal{E}_\epsilon(t) = \mathcal{E}(t) + \epsilon N(t), \quad t \geq \tau,$$

where $\mathcal{E}(t)$ is defined in (2.15) and

$$N(t) = \rho_1 \int_{0}^{L} uu_t \, dx + \rho_2 \int_{0}^{L} ww_t \, dx.$$  

Firstly, let us prove that there exists $\epsilon_0 > 0$ such that

$$\frac{1}{2} \mathcal{E}(t) - \frac{CF}{2} \leq \mathcal{E}_\epsilon(t) \leq \frac{3}{2} \mathcal{E}(t) + \frac{CF}{2}, \quad \forall t \geq \tau, \quad 0 < \epsilon \leq \epsilon_0. \quad (4.2)$$

Indeed, by Young’s inequality, (2.5) and (2.21) we have that

$$\|N(t)\| \leq \rho_1 \|u\|_2 \|u_t\|_2 + \rho_2 \|w\|_2 \|w_t\|_2$$

$$\leq \frac{1}{2} \left( \rho_1^2 \|u_t\|^2_2 + \rho_2^2 \|w_t\|^2_2 \right) + \frac{1}{2} \left( \|u\|^2_2 + \|w\|^2_2 \right)$$

$$\leq \frac{1}{2} \left( \rho_1^2 \|u_t\|^2_2 + \rho_2^2 \|w_t\|^2_2 \right)$$

$$+ \frac{\gamma}{2} \left( a_{22} - \frac{a_{12}^2}{a_{11}} \right) \|u_{x}x\|^2_2 + \left\| \frac{a_{12}}{\sqrt{a_{11}}} w_{x} + \sqrt{a_{11}} u_{x} \right\|^2_2$$

$$\leq \frac{1}{2} \max \{ 1, \rho_1, \rho_2, \gamma \} \|(u, w, u_t, w_t)\|^2_{\mathcal{H}}$$

$$\leq \frac{1}{2\beta_0} \max \{ 1, \rho_1, \rho_2, \gamma \} (\mathcal{E}(t) + C_F).$$
Choosing
\[ \epsilon_0 = \beta_0 \min \left\{ 1, \frac{1}{\rho_1}, \frac{1}{\rho_2}, \frac{1}{\gamma} \right\}, \] (4.3)
then the inequality (4.2) holds.

Now, we claim that there exist constants \( C_4, C_5 > 0 \) independent of \( t \) such that
\[ \frac{d}{dt} N(t) \leq -\mathcal{E}(t) + C_4 \left( \| u_t \|_2^2 + \| w_t \|_2^2 \right) + C_5 \left( \| h_1(t) \|_2^2 + \| h_2(t) \|_2^2 \right) + C_F. \] (4.4)

From definition of \( N \) and (1.4) we have
\[
\frac{d}{dt} N(t) = \rho_1 \| u_t \|_2^2 + \rho_1 \int_0^L w u_t \, dx + \rho_2 \| w_t \|_2^2 + \rho_2 \int_0^L w w_t \, dx
\]
\[
= \int_0^L \left( a_{11} u_{xx} + a_{12} w_{xx} - g_1(u_t) - f_1(u, w) + h_1 \right) u \, dx
\]
\[
+ \int_0^L \left( a_{12} u_{xx} + a_{22} w_{xx} - g_2(w_t) - f_2(u, w) + h_2 \right) w \, dx
\] (4.5)
\[
= -\int_0^L \left( a_{11} u_x^2 + 2a_{12} w_x u_x + a_{22} w_x^2 \right) \, dx - \int_0^L \nabla F(u, w) \cdot (u, w) \, dx
\]
\[
- \int_0^L (g_1(u_t)u + g_2(w_t)w) \, dx + \int_0^L (h_1u + h_2w) \, dx.
\]
Integrating by parts over \([0, L]\) and using (2.7), we obtain
\[
\int_0^L \left( a_{11} u_{xx} + a_{12} w_{xx} - g_1(u_t) - f_1(u, w) + h_1 \right) u \, dx
\]
\[
+ \int_0^L \left( a_{12} u_{xx} + a_{22} w_{xx} - g_2(w_t) - f_2(u, w) + h_2 \right) w \, dx
\]
\[
= -\int_0^L \left( a_{22} - a_{12}^2/a_{11} \right) \| w_x \|_2^2 + \left\| \frac{a_{12}}{\sqrt{a_{11}}} w_x + \sqrt{a_{11}} u_x \right\|_2^2
\]
\[
\int_0^L \left( a_{11} u_x^2 + 2a_{12} w_x u_x + a_{22} w_x^2 \right) \, dx
\]
we have
\[
\int_0^L \left( a_{11} u_{xx} + a_{12} w_{xx} - g_1(u_t) - f_1(u, w) + h_1 \right) u \, dx
\]
\[
+ \int_0^L \left( a_{12} u_{xx} + a_{22} w_{xx} - g_2(w_t) - f_2(u, w) + h_2 \right) w \, dx
\]
\[
= -\left( a_{22} - a_{12}^2/a_{11} \right) \| w_x \|_2^2 + \left\| \frac{a_{12}}{\sqrt{a_{11}}} w_x + \sqrt{a_{11}} u_x \right\|_2^2
\] (4.6)
\[
- \int_0^L \nabla F(u, w) \cdot (u, w) \, dx - \int_0^L (g_1(u_t)u + g_2(w_t)w) \, dx
\]
\[
+ \int_0^L (h_1u + h_2w) \, dx.
\]
Substituting (4.6) in (4.5) and subtracting and adding $\mathcal{E}(t)$ yields
\[
\frac{d}{dt} N(t) = -\mathcal{E}(t) + \frac{3\rho_1}{2} \|u_t\|^2 + \frac{3\rho_2}{2} \|w_t\|^2 - \int_0^L (g_1(u_t)u + g_2(w_t)w) \, dx
\]
\[
- \frac{1}{2} \left( (a_{22} - a_{12}^2/a_{11}) \|w_x\|^2 + \left\| \frac{a_{12}}{\sqrt{a_{11}}} w_x + \sqrt{a_{11}} u_x \right\|^2 \right) + \int_0^L (F(u, w) - \nabla F(u, w) \cdot (u, w)) \, dx + \int_0^L (h_1u + h_2w) \, dx.
\] (4.7)

Thanks to (2.5) and (2.11) we deduce that
\[
\int_0^L (F(u, w) - \nabla F(u, w) \cdot (u, w)) \, dx
\]
\[
\leq \beta \gamma \left( (a_{22} - a_{12}^2/a_{11}) \|w_x\|^2 + \left\| \frac{a_{12}}{\sqrt{a_{11}}} w_x + \sqrt{a_{11}} u_x \right\|^2 \right) + CF. \tag{4.8}
\]

Using (2.6), (2.5) and Young’s inequality we have

\[
- \int_0^L (g_1(u_t)u + g_2(w_t)w) \, dx
\]
\[
\leq M_2 \int_0^L (|u_t||u| + |w_t||w|) \, dx
\]
\[
\leq M_2 \|u_t\|_2 \|u\|_2 + M_2 \|w_t\|_2 \|w\|_2
\]
\[
\leq \frac{\gamma M_2^2}{2\beta_0} (\|u_t\|^2 + \|w_t\|^2) + \frac{\beta_0}{2\gamma} (\|u\|^2 + \|w\|^2) \tag{4.9}
\]

\[
\leq \frac{\gamma M_2^2}{2\beta_0} (\|u_t\|^2 + \|w_t\|^2)
\]
\[
+ \frac{\beta_0}{2} \left( (a_{22} - a_{12}^2/a_{11}) \|w_x\|^2 + \left\| \frac{a_{12}}{\sqrt{a_{11}}} w_x + \sqrt{a_{11}} u_x \right\|^2 \right).
\]

Similarly

\[
\int_0^L (h_1u + h_2w) \, dx \leq \frac{\gamma}{2\beta_0} (\|h_1(t)\|^2 + \|h_2(t)\|^2)
\]
\[
+ \frac{\beta_0}{2} \left( (a_{22} - a_{12}^2/a_{11}) \|w_x\|^2 + \left\| \frac{a_{12}}{\sqrt{a_{11}}} w_x + \sqrt{a_{11}} u_x \right\|^2 \right). \tag{4.10}
\]

Using the estimates (4.8)-(4.10) in (4.7) and the fact that $\beta_0 + \beta \gamma = 1/2$ (cf. (2.23)), we deduce

\[
\frac{d}{dt} N(t) \leq -\mathcal{E}(t) + \left( \frac{3\rho_1}{2} + \frac{\gamma M_2^2}{2\beta_0} \right) \|u_t\|^2 + \left( \frac{3\rho_2}{2} + \frac{\gamma M_2^2}{2\beta_0} \right) \|w_t\|^2
\]
\[
+ \frac{\gamma}{2\beta_0} (\|h_1(t)\|^2 + \|h_2(t)\|^2) + CF.
\]

Then taking

\[
C_4 = \frac{1}{2} \max \left\{ \frac{3\rho_1 + \gamma M_2^2}{\beta_0}, \frac{3\rho_2 + \gamma M_2^2}{\beta_0} \right\} \quad \text{and} \quad C_5 = \frac{\gamma}{2\beta_0}, \tag{4.11}
\]
Lemma 4.3. Suppose that Assumption 2.3 holds. Then, the family \( \mathcal{B}_{a} \) is pullback absorbing.

From (4.1), (2.16) and by the choice of \( \epsilon \), we have

\[
\frac{d}{dt} \mathcal{E}(t) \leq -\epsilon \mathcal{E}(t) + C_{a}(\|h_{1}(t)\|^{2} + \|h_{2}(t)\|^{2}) + \epsilon C_{F}.
\]

Using the second inequality in (4.2) we conclude that

\[
\frac{d}{dt} \mathcal{E}(t) \leq -2\epsilon \mathcal{E}(t) + C_{a}(\|h_{1}(t)\|^{2} + \|h_{2}(t)\|^{2}) + \frac{4\epsilon}{3} C_{F}.
\]

Multiplying (4.13) by \( e^{\frac{4\epsilon}{3}t} \) and integrating from \( \tau \) to \( t \), we get

\[
\mathcal{E}(t) \leq e^{-\frac{4\epsilon}{3}(t-\tau)} \mathcal{E}(\tau) + C_{a} \int_{\tau}^{t} e^{-\frac{4\epsilon}{3}(t-s)} (\|h_{1}(s)\|^{2} + \|h_{2}(s)\|^{2}) \, ds + 2C_{F}.
\]

Choosing \( \sigma_{1} = \frac{4\epsilon}{3} \) and using again (4.2) we obtain

\[
\mathcal{E}(t) \leq 3e^{-\sigma_{1}(t-\tau)} \mathcal{E}(\tau) + 2C_{a} \int_{\tau}^{t} e^{-\sigma_{1}(t-s)} (\|h_{1}(s)\|^{2} + \|h_{2}(s)\|^{2}) \, ds + 6C_{F}.
\]

Combining the inequalities (2.21)-(2.22) with (4.14) it follows that for any \( z \in \mathcal{H} \),

\[
\|U(t, \tau)z\|_{\mathcal{H}}^{2} \leq \frac{3C_{F}}{\beta_{0}} \left( \|z\|_{\mathcal{H}}^{2} + 1 \right) e^{-\sigma_{1}(t-\tau)}
\]

\[
+ \frac{2C_{a}}{\beta_{0}} \int_{-\infty}^{t} e^{-\sigma_{1}(t-s)} (\|h_{1}(s)\|^{2} + \|h_{2}(s)\|^{2}) \, ds + \frac{7C_{F}}{\beta_{0}}, \quad \forall t \geq \tau,
\]

and thereby (4.1) holds with \( C_{1} = \frac{3C_{F}}{\beta_{0}}, \ C_{2} = \frac{2C_{a}}{\beta_{0}} \) and \( C_{3} = \frac{7}{\beta_{0}} \). The proof is complete.

Thanks to Lemma 4.1, we can define a suitable tempered universe \( \mathcal{D} \) in \( \mathcal{H} \).

**Definition 4.2.** Given a function \( R : \mathbb{R} \to \mathbb{R}^{+} \), we consider a family of closed balls in \( \mathcal{H} \)

\[
\mathcal{B}_{\mathcal{H}}(0, R(t)) = \{ z \in \mathcal{H} : \|z\|_{\mathcal{H}} \leq R(t) \}
\]

satisfying

\[
\lim_{\tau \to -\infty} R^{p+1}(\tau)e^{\sigma_{1}\tau} = 0,
\]

where \( \sigma_{1} > 0 \) is a constant given in Lemma 4.1. Then, we define our universe attraction as

\[
\mathcal{D} = \{ \mathcal{D} : D(t) \neq \emptyset \text{ and } D(t) \subset \mathcal{B}_{\mathcal{H}}(0, R_{D}(t)) \text{ with } R_{D}(t) \text{ satisfying (4.16)} \}.
\]

**Lemma 4.3.** Suppose that Assumption 2.3 holds. Then, the family \( \mathcal{B}_{0} = \{ B_{0}(t) \}_{t \in \mathbb{R}} \)
defined by \( B_{0} = \mathcal{B}_{\mathcal{H}}(0, R_{0}(t)) \), the closed ball in \( \mathcal{H} \) of center zero and radius \( R_{0}(t) \),
where

\[
R_{0}^{2}(t) = C_{2} \int_{-\infty}^{t} e^{-\sigma_{0}(t-s)} (\|h_{1}(s)\|^{2} + \|h_{2}(s)\|^{2}) \, ds + C_{3}C_{F} + 1,
\]

is pullback \( \mathcal{D} \)-absorbing.
Proof. Firstly we observe that (2.12) implies that (4.18) is well-defined. Now, let \( \hat{D} \in \mathcal{D} \) and \( \tau \in \mathbb{R} \). Since \( \sigma_0 \leq \sigma_1 \), we conclude
\[
e^{-\sigma_1(t-s)} \leq e^{-\sigma_0(t-s)}, \quad \forall t \geq s.
\]
Therefore, by (4.1) we have
\[
\|U(t, \tau)z_\tau\|_{\mathcal{H}}^2 \leq C_1 \left( R_{\hat{D}}^{p+1}(\tau) + 1 \right) e^{-\sigma_1(t-\tau)}
+ C_2 \int_{-\infty}^{t} e^{-\sigma_1(t-s)}\left(\|h_1(s)\|_2^2 + \|h_2(s)\|_2^2\right) ds + C_3 C_F \tag{4.19}
\leq C_1 e^{-\sigma_1 t} \left( R_{\hat{D}}^{p+1}(\tau) + 1 \right) e^{\sigma_1 \tau} + R_0^2(t) - 1,
\]
for all \( z_\tau \in D(\tau) \). Then, by the tempered condition of \( \hat{D} \) it follows that
\[
\left( R_{\hat{D}}^{p+1}(\tau) + 1 \right) e^{\sigma_1 \tau} \to 0, \quad \text{as } \tau \to -\infty.
\]
Then, there exists \( \tau_0(\hat{D}, t) < t \) such that
\[
\|U(t, \tau)z_\tau\|_{\mathcal{H}}^2 \leq R_0^2(t), \quad \forall \tau \leq \tau_0(\hat{D}, t), \quad z_\tau \in D(\tau),
\]
that is,
\[
U(t, \tau)D(\tau) \subset B_0(t), \quad \forall \tau \leq \tau_0(\hat{D}, t).
\]
This proves that \( \hat{B}_0 \) is a pullback \( \mathcal{D} \)-absorbing family. The proof is complete. \( \square \)

4.2. Pullback \( \mathcal{D} \)-asymptotic compactness. In this part, we prove that the evolution process generated by the problem (1.4)-(1.5) is pullback \( \mathcal{D} \)-asymptotically compact. We first prove a stabilization inequality.

**Lemma 4.4.** Suppose that Assumption 2.3 holds. Let \( \hat{B}_0 \) the pullback \( \mathcal{D} \)-absorbing family given in Lemma 4.3 and let \( U(t, \tau)z^1 = (w^1, w^2, u^1_{11}, u^1_{22}, u^2_{11}, u^2_{22}) \) be the weak solutions of (1.4)-(1.5) with initial condition \( z^1 \in B_0(\tau), \quad j = 1, 2 \). Then, there exists a constant \( \sigma_2 > \sigma_1 \), and a constant \( C_{\tau, t} > 0 \) depending on \( \tau \leq t \) such that
\[
\|U(t, \tau)z_1 - U(t, \tau)z_2\|_{\mathcal{H}}^2 \leq 3R_0^2(\tau)e^{-\sigma_2(t-\tau)}
+ C_{\tau, t} \int_{\tau}^{t} \left(\|u(s)\|_{p+1}^2 + \|w(s)\|_{p+1}^2\right) ds, \tag{4.20}
\]
where \( u = u^1 - u^2 \) and \( w = w^1 - w^2 \).

**Proof.** The differences \( u = u^1 - u^2 \) and \( w = w^1 - w^2 \) solves the problem
\[
\begin{cases}
\rho_1 u_{tt} - a_{11} u_{xx} - a_{12} w_{xx} = F_1(u, w) - G_1(u_t), \\
\rho_2 w_{tt} - a_{22} u_{xx} - a_{22} w_{xx} = F_2(u, w) - G_2(u_t),
\end{cases} \tag{4.21}
\]
where
\[
G_1(u_t) = g_1(u^1_t) - g_1(u^2_t), \quad G_2(w_t) = g_2(w^1_t) - g_2(w^2_t),
F_j(u, w) = f_j(u^2, w^2) - f_j(u^1, w^1), \quad j = 1, 2,
\]
with Dirichlet boundary conditions and initial condition
\[
(u(\tau), w(\tau), u_t(\tau), w_t(\tau)) = z^1 - z^2.
\]
Since it follows from (4.23) and (4.24) that
and integrating over \((0, L)\), we obtain

\[
\frac{d}{dt} E(t) = \int_0^L (F_1(u, w)u_t + F_2(u, w)w_t) \, dx
- \int_0^L (G_1(u_t)u_t + G_2(w_t)w_t) \, dx.
\]  

(4.22)

Using (2.8), Hölder’s inequality with exponents \(p_1 = \frac{2(p+1)}{p-1}\), \(p_2 = p + 1\) and \(p_3 = 2\) we obtain

\[
\int_0^L F_1(u, w)u_t \, dx = \int_0^L (f_1(u^2, w^2) - f_1(u^4, w^4)) \, u_t \, dx
\leq C \left( \|U(t, \tau)z^1\|_{\mathcal{H}}^{p-1} + \|U(t, \tau)z^2\|_{\mathcal{H}}^{p-1} + 1 \right) \left( \|u\|_{p+1} + \|w\|_{p+1} \right) \|u_t\|_2
\leq C \left( \|U(t, \tau)z^1\|_{\mathcal{H}}^{2(p-1)} + \|U(t, \tau)z^2\|_{\mathcal{H}}^{2(p-1)} + 1 \right) \left( \|u\|_{p+1}^2 + \|w\|_{p+1}^2 \right)
+ \frac{M_1}{2} \|u_t\|_2^2.
\]  

(4.23)

In a similar way we obtain

\[
\int_0^L F_2(u, w)w_t \, dx
\leq C \left( \|U(t, \tau)z^1\|_{\mathcal{H}}^{2(p-1)} + \|U(t, \tau)z^2\|_{\mathcal{H}}^{2(p-1)} + 1 \right) \left( \|u\|_{p+1}^2 + \|w\|_{p+1}^2 \right)
+ \frac{M_1}{2} \|w_t\|_2^2.
\]  

(4.24)

It follows from (4.23) and (4.24) that

\[
\int_0^L (F_1(u, w)u_t + F_2(u, w)w_t) \, dx
\leq C \left( \|U(t, \tau)z^1\|_{\mathcal{H}}^{2(p-1)} + \|U(t, \tau)z^2\|_{\mathcal{H}}^{2(p-1)} + 1 \right) \left( \|u\|_{p+1}^2 + \|w\|_{p+1}^2 \right)
+ \frac{M_1}{2} (\|u_t\|_2^2 + \|w_t\|_2^2).
\]  

(4.25)

Since \(z^1 \in B(\tau)\), by (4.19) we have

\[
\|U(t, \tau)z^1\|_{\mathcal{H}}^2 \leq C_1 \left( R_{0\tau}^{p+1}(\tau) + 1 \right) e^{-\sigma_1(t-\tau)} + R_0^2(t) - 1.
\]  

(4.26)

Then, by (4.25) and (4.26) there exists a constant \(\kappa_1(t, \tau) > 0\) such that

\[
\int_0^L (F_1(u, w)u_t + F_2(u, w)w_t) \, dx
\leq \kappa_1(t, \tau) (\|u\|_{p+1}^2 + \|w\|_{p+1}^2) + \frac{M_1}{2} (\|u_t\|_2^2 + \|w_t\|_2^2).
\]  

(4.27)

By (2.6) we know that

\[
\int_0^L G_1(u_t) \, dx \geq M_1 \|u_t\|_2^2, \quad \int_0^L G_2(w_t) \, dx \geq M_1 \|w_t\|_2^2.
\]  

(4.28)

Substituting the estimates (4.25) and (4.28) in (4.22), we obtain

\[
\frac{d}{dt} E(t) \leq -\frac{M_1}{2} \left( \|u_t\|_2^2 + \|w_t\|_2^2 \right) + \kappa_1(t, \tau) (\|u\|_{p+1}^2 + \|w\|_{p+1}^2).
\]  

(4.29)
For each $\delta > 0$, we consider the functional

$$E_\delta(t) = E(t) + \delta Y(t),$$

where

$$Y(t) = \rho_1 \int_0^L u_x \, dx + \rho_2 \int_0^L w_x \, dx.$$

Firstly, let us prove that there exists $\delta_0 > 0$ such that

$$\frac{1}{2} E(t) \leq E_\delta(t) \leq \frac{3}{2} E(t), \quad \forall t \geq \tau, \quad \forall \delta \leq \delta_0. \tag{4.30}$$

Indeed, by Young’s inequality and (2.5) we have that

$$|Y(t)| \leq \rho_1 \|u_x\| + \rho_2 \|w_x\| \leq \frac{1}{2} \left( \rho_1^2 \|u_x\|^2 + \rho_2^2 \|w_x\|^2 \right) + \frac{\gamma}{2} \left( a_{22} - \frac{a_{12}^2}{a_{11}} \right) \|u_t\|^2 + \frac{a_{12}}{\sqrt{a_{11}}} \|u_x\|^2 + \frac{\gamma}{2} \left( \|u_t\|^2 + \|w_t\|^2 \right) \tag{4.31}$$

$$\leq \max \left\{ 1, \rho_1, \rho_2, \gamma \right\} E(t).$$

Then, taking

$$\delta_0 = \frac{1}{2} \min \left\{ 1, \frac{1}{\rho_1}, \frac{1}{\rho_2}, \frac{1}{\gamma} \right\}, \tag{4.32}$$

we see that (4.30) holds.

Now, we derive $Y$ and then use (4.21) to obtain

$$\frac{d}{dt} Y(t) = -E(t) + \frac{3\rho_1}{2} \|u_t\|^2 + \frac{3\rho_2}{2} \|w_t\|^2 - \int_0^L (G_1(u_t)u + G_2(w_t)w) \, dx$$

$$- \frac{1}{2} \left( a_{22} - \frac{a_{12}^2}{a_{11}} \right) \|u_t\|^2 + \frac{a_{12}}{\sqrt{a_{11}}} \|u_x\|^2 + \frac{\gamma}{2} \left( \|u_t\|^2 + \|w_t\|^2 \right) \tag{4.33}$$

$$+ \int_0^L (F_1(u, w)u + F_2(u, w)w) \, dx.$$

From (2.6), (2.5) and Young’s inequality, we have

$$\int_0^L (G_1(u_t)u + G_2(w_t)w) \, dx$$

$$\leq M_2 \int_0^L (|u_t| |u| + |w_t| |w|) \, dx$$

$$\leq M_2 \|u_t\|_2 \|u\|_2 + M_2 \|w_t\|_2 \|w\|_2$$

$$\leq \frac{\gamma M_2^2}{2} (\|u_t\|^2 + \|w_t\|^2) + \frac{1}{2} (\|u_t\|^2 + \|w_t\|^2) \tag{4.34}$$

$$\leq \frac{\gamma M_2^2}{2} (\|u_t\|^2 + \|w_t\|^2)$$

$$+ \frac{1}{2} \left( a_{22} - \frac{a_{12}^2}{a_{11}} \right) \|u_t\|^2 + \frac{a_{12}}{\sqrt{a_{11}}} \|u_x\|^2.$$
Using (2.8), Hölder’s inequality with exponents \( p_1 = \frac{2(p+1)}{p-1}, \) \( p_2 = p+1 \) and \( p_3 = 2 \) we obtain that
\[
\int_0^L F_1(u, w)u \, dx = \int_0^L \left( f_1(u^2, w^2) - f_1(u^1, w^1) \right) u \, dx \\
\leq C \left( \|U(t, \tau)z^1\|_{H}^{p-1} + \|U(t, \tau)z^2\|_{H}^{p-1} + 1 \right) (\|u\|_{p+1} + \|w\|_{p+1}) \|u\|_2
\]
(4.35)
\[
\leq C \left( \|U(t, \tau)z^1\|_{H}^{p-1} + \|U(t, \tau)z^2\|_{H}^{p-1} + 1 \right) (\|u\|_{p+1}^2 + \|w\|_{p+1}^2).
\]

Similarly, we have
\[
\int_0^L F_2(u, w)w \, dx
\]
(4.36)
\[
\leq C \left( \|U(t, \tau)z^1\|_{H}^{p-1} + \|U(t, \tau)z^2\|_{H}^{p-1} + 1 \right) (\|u\|_{p+1}^2 + \|w\|_{p+1}^2).
\]

Combining (4.35), (4.36) and (4.26) we conclude that there exists a constant \( \kappa_2(t, \tau) \) such that
\[
\int_0^L (F_1(u, w)u + F_2(u, w)w) \, dx \leq \kappa_2(t, \tau)(\|u\|_{p+1}^2 + \|w\|_{p+1}^2).
\]
(4.37)

Substituting the estimates (4.34) and (4.37) in (4.33), we obtain
\[
\frac{d}{dt} Y(t) \leq -E(t) + \left( \frac{3\rho_1}{2} + \frac{\gamma M_2^3}{2} \right) \|u_t\|_2^2 + \left( \frac{3\rho_2}{2} + \frac{\gamma M_2^3}{2} \right) \|w_t\|_2^2
\]
\[
+ \kappa_2(t, \tau)(\|u\|_{p+1}^2 + \|w\|_{p+1}^2).
\]

Defining the constant \( C_7 > 0 \) by
\[
C_7 = \frac{1}{2} \max \left\{ 3\rho_1 + \gamma M_2^3, 3\rho_2 + \gamma M_2^3 \right\},
\]
(4.38)

we obtain
\[
\frac{d}{dt} Y(t) \leq -E(t) + C_7 (\|u_t\|_2^2 + \|w_t\|_2^2) + \kappa_2(t, \tau)(\|u\|_{p+1}^2 + \|w\|_{p+1}^2).
\]
(4.39)

Let \( \kappa(t, \tau) = \kappa_1(t, \tau) + \kappa_2(t, \tau) \), and we choose
\[
\delta = \min \left\{ \delta_0, \frac{M_1}{2C_7} \right\}.
\]
(4.40)

Using (4.29) and (4.39) we see that, since that \( \delta \leq 1 \),
\[
\frac{d}{dt} E_\delta(t) \leq -\delta E(t) + \kappa(t, \tau)(\|u\|_{p+1}^2 + \|w\|_{p+1}^2).
\]
(4.41)

Using the second inequality in (4.30) we conclude that
\[
\frac{d}{dt} E_\delta(t) \leq -\frac{2\delta}{3} E_\delta(t) + \kappa(t, \tau)(\|u\|_{p+1}^2 + \|w\|_{p+1}^2).
\]
(4.42)

Multiply (4.42) by \( e^{\frac{2\delta}{3} t} \) and then integrate over \([\tau, t]\) to obtain
\[
E_\delta(t) \leq e^{-\frac{2\delta}{3}(t-\tau)} E_\delta(\tau) + \sup_{s \in [\tau, t]} \kappa(t, s) \int_{\tau}^{t} e^{-\frac{2\delta}{3}(t-s)} (\|u(s)\|_{p+1}^2 + \|w(s)\|_{p+1}^2) \, ds.
\]

Choosing \( \sigma_2 = \frac{2\delta}{3} \), and using (4.30) again, we obtain
\[
E(t) \leq 3e^{-\sigma_2(t-\tau)} E(\tau) + 2 \sup_{s \in [\tau, t]} \kappa(t, s) \int_{\tau}^{t} (\|u(s)\|_{p+1}^2 + \|w(s)\|_{p+1}^2) \, ds.
\]
In addition, from definition of $E(t)$ we have $E(\tau) \leq \frac{1}{2}R_0^2(\tau)$, and thereby (4.20) follows with $C_{\tau,t} = 4\sup_{s \in \tau,t} \kappa(s,t)$. Now, we prove that $\sigma_2 > \sigma_1$. Indeed, since $\beta_0 < \frac{1}{2}$, by (4.3) and (4.32) we see that $\epsilon_0 < \delta_0$. Hence by (4.11), (4.12), (4.38) and (4.40) we conclude that $\epsilon < \delta$. This proves that $\sigma_2 > \sigma_1$. The proof is complete. □

Lemma 4.5. Suppose that Assumption 2.3 holds. Then the evolution process \( \{U(t, \tau)\}_{t \geq \tau} \) is pullback $\mathcal{D}$-asymptotically compact.

Proof. In order to prove this result we will apply Theorem 3.7. Given $\epsilon > 0$ and $t \in \mathbb{R}$, by (4.18) we know that

$$R_0^2(\tau) = \left( C_2 \int_{-\infty}^{\tau} e^{\sigma_{0}\epsilon} (\|h_1(s)\|^2 + \|h_2(s)\|^2) \, ds \right) e^{-\sigma_0 t} + C_3 + 1, \; t \geq \tau. \quad (4.43)$$

Since the integral involved in (4.43) does not increase as $\tau$ decreases, and $\sigma_0 < \sigma_2$, we get

$$\lim_{\tau \to -\infty} R_0^2(\tau) e^{-\sigma_2 (t-\tau)} = \lim_{\tau \to -\infty} e^{(\sigma_2 - \sigma_0) \tau} \left( C_2 \int_{-\infty}^{\tau} e^{\sigma_{0}\epsilon} (\|h_1(s)\|^2 + \|h_2(s)\|^2) \, ds \right) e^{-\sigma_2 t} = 0.$$

Then, there exists a $\tau_\epsilon = \tau_\epsilon(t, \epsilon) \leq t$ such that

$$3R_0^2(\tau_\epsilon) e^{-\sigma_2 (t-\tau_\epsilon)} < \epsilon^2.$$

Define $\Psi_\epsilon : B_0(\tau_\epsilon) \times B_0(\tau_\epsilon) \to \mathbb{R}$ by

$$\Psi_\epsilon(z^1, z^2)^2 = C_{\tau_\epsilon,t} \int_{\tau_\epsilon}^t \left( \|u^1(s) - u^2(s)\|^2_{p+1} + \|w^1(s) - w^2(s)\|^2_{p+1} \right) \, ds,$$

where $C_{\tau_\epsilon,t} > 0$ is given in (4.20). Then, by Lemma 4.4 we obtain that

$$\|U(t, \tau_\epsilon) z^1 - U(t, \tau_\epsilon) z^2\|_H \leq \epsilon + \Psi_\epsilon(z^1, z^2), \; \forall z^1, z^2 \in B_0(\tau_\epsilon).$$

From now on, we will show that $\Psi_\epsilon(z^1, z^2)$ is contractive on $B_0(\tau_\epsilon)$. Let $z^n \in B_0(\tau_\epsilon)$ and we denote $U(s, \tau_\epsilon) z^n = (u^n(s), w^n(s), u^n_t(s), w^n_t(s))$. Then from (4.1) we have

$$\|U(s, \tau_\epsilon) z^n\|_H \leq C_{\tau_\epsilon, t}, \; \forall s \in [\tau_\epsilon, t].$$

Therefore, there exists a constant $C_8 > 0$ such that

$$\|u^n, w^n\|_{L^2(\tau_\epsilon, t; (H^1_0(0, L))^2)} \leq C_8 \quad \text{and} \quad \|u^n_t, w^n_t\|_{L^2(\tau_\epsilon, t; (L^2(0, L))^2)} \leq C_8. \quad (4.44)$$

Thanks to Aubin-Simon compactness theorem [22] there exists a subsequence such that

$$(u^n, w^n) \to (u, w) \quad \text{strongly in} \; L^2(\tau_\epsilon, t; (L^{p+1}(0, L))^2). \quad (4.45)$$

Finally, by (4.45) we conclude that

$$\lim_{m \to \infty} \liminf_{n \to \infty} \Psi_\epsilon(z^m, z^n) = 0.$$

Then the pullback $\mathcal{D}$-asymptotic compactness follows from Theorem 3.7. The proof is complete. □
4.3. Existence of pullback $\mathcal{D}$-attractors. As a consequence of the previous results, we obtain the following theorem, which is the main result of this section.

**Theorem 4.6.** Suppose that Assumption 2.3 holds. Then the evolution process associated to the problem (1.4)-(1.5) possesses a minimal pullback $\mathcal{D}$-attractor, where $\mathcal{D}$ is defined in (4.17). If in addition, $\sigma_0 < \frac{2p}{p+1}$, then above pullback $\mathcal{D}$-attractor belongs to $\mathcal{D}$.

**Proof.** Lemmas 4.3 and 4.5 indicate that the evolution process generated by the problem (1.4)-(1.5) has a pullback $\mathcal{D}$-absorbing family and it is pullback $\mathcal{D}$-asymptotically compact. Then, Theorem 3.9 implies the existence of the minimal pullback $\mathcal{D}$-attractor.

Now, we will prove that the pullback attractor belongs to universe $\mathcal{D}$ under assumption $\sigma_0 < \frac{2p}{p+1}$. It is clear that $\mathcal{D}$ is inclusion closed. Then we must show that $\bar{B}_0 \in \mathcal{D}$, that is,

$$
\lim_{\tau \to -\infty} R_0^{p+1}(\tau)e^{\sigma_1 \tau} = 0. 
$$

(4.46)

By (4.18) we see that

$$
R_0^2(\tau)e^{\frac{2\sigma_1}{p+1}\tau} = \left( C_2 \int_{-\infty}^{\tau} e^{-\sigma_0(s)}(||h_1(s)||_2^2 + ||h_2(s)||_2^2) \, ds \right) e^{(\frac{2\sigma_1}{p+1} - \sigma_0)\tau} + (C_3 C_F + 1)e^{\frac{2\sigma_1}{p+1}\tau}. 
$$

(4.47)

Since the integral involved in (4.47) does not increase as $\tau$ decreases and $\sigma_0 < \frac{2p}{p+1}$, we conclude that

$$
\lim_{\tau \to -\infty} R_0^2(\tau)e^{\frac{2\sigma_1}{p+1}\tau} = 0.
$$

Hence, (4.46) holds and $\bar{B}_0 \in \mathcal{D}$. Then from Theorem 3.9 the pullback attractor belongs to $\mathcal{D}$. The proof is complete.

5. Upper-semicontinuity. In this section, we assume that the external forces $h_1, h_2 \in L^2_{loc}(\mathbb{R}; L^2(0, L))$ and

$$
\int_{-\infty}^{t} e^{-\sigma_0(t-s)} \left( ||h_1(s)||_2^2 + ||h_2(s)||_2^2 \right) \, ds < \infty, \quad \forall t \in (-\infty, 0].
$$

(5.1)

Consider the problem (1.4)-(1.5) with $h_1, h_2$ replaced by $c h_1, c h_2$

$$
\begin{cases}
\rho_1 u_{tt} - a_{11} u_{xx} - a_{12} w_{xx} + g_1(u) + f_1(u, w) = c h_1, & \text{in } (0, L), \ t \geq \tau, \\
\rho_2 w_{tt} - a_{21} u_{xx} - a_{22} w_{xx} + g_2(w) + f_2(u, w) = c h_2, & \text{in } (0, L), \ t \geq \tau,
\end{cases}
$$

(5.2)

with corresponding boundary-initial condition

$$
\begin{cases}
u(0, t) = u(L, t) = w(0, t) = w(L, t) = 0, \ t > \tau, \\
u(\tau) = u_0 \in H_0^1(0, L), \ u_t(\tau) = u_1^0 \in L^2(0, L), \\
w(\tau) = w_0^\tau \in H_0^1(0, L), \ w_t(\tau) = w_1^\tau \in L^2(0, L).
\end{cases}
$$

(5.3)

Our goal is to investigate the system (5.2) when $\epsilon \to 0$. Throughout this section, we assume $\epsilon \in (0, 1]$. To indicate the dependence on $\epsilon$, we will write the process generated by the problem (5.2)-(5.3) as $\{U_\epsilon(t, \tau)\}_{t \geq \tau}$. For each $\epsilon \in (0, 1]$ fixed, Theorem 4.6 implies the existence of a minimal $\mathcal{D}$-pullback attractor $\mathcal{A}_\epsilon = \{A_\epsilon(t)\}_{t \in \mathbb{R}}$.

When $\epsilon = 0$ the non-autonomous problem (5.2)-(5.3) reduces to a autonomous one. The existence of global attractor $\mathcal{A}_0$ for the autonomous problem is proved in

...
We are in position to formulate and prove the result on the upper semicontinuity of pullback attractors.

**Theorem 5.1.** The pullback $\mathcal{D}$-attractors $\hat{A}_\epsilon$ is upper semicontinuous at $\epsilon = 0$, that is,

$$\lim_{\epsilon \to 0} \text{dist}(A_\epsilon(t), A_0) = 0, \quad \forall t \in \mathbb{R}.$$ 

**Proof.** In view of [9, Proposition 1.20], we only need to prove that

(i) There exist $\delta > 0$ and $t_0 \in \mathbb{R}$ such that

$$\bigcup_{\epsilon \in (0, \delta)} \bigcup_{s \leq t_0} A_\epsilon(s)$$

is bounded in $\mathcal{H}$.

(ii) For any $t \in \mathbb{R}$, any $T \geq 0$ and any bounded set $D \subset \mathcal{H}$,

$$\sup_{\tau \in [t-T, t]} \| U_\epsilon(t, \tau) z - U_0(t, \tau) z \|_{\mathcal{H}} \underset{\epsilon \to 0}{\longrightarrow} 0. \quad (5.4)$$

To prove (i), observe that the radius of the absorbing ball given by (4.18) can be estimated from a fixed final time. Indeed, let $t_0 \in \mathbb{R}$ be fixed. Then by (4.18) and (5.1) we have

$$R^2_\epsilon(t) \leq C_2 \sup_{\tau \leq t_0} \left( \int_{-\infty}^{\tau} e^{-\sigma_0(t-s)} \left( \| h_1(s) \|_2^2 + \| h_2(s) \|_2^2 \right) ds \right) + C_3 C_F + 1$$

$$= R^2(t_0) < \infty, \quad \forall t \in (-\infty, t_0], \quad \epsilon \in (0, 1].$$

Hence, by the invariance of $\hat{A}_\epsilon$ we conclude that

$$A_\epsilon(t) \subset \overline{B}_{\mathcal{H}}(0, R(t_0)), \quad \forall t \in (-\infty, t_0], \quad \forall \epsilon \in (0, 1].$$

Consequently

$$\bigcup_{\epsilon \in (0,1)} \bigcup_{s \leq t_0} A_\epsilon(s) \subset \overline{B}_{\mathcal{H}}(0, R(t_0)),$$

and thereby (i) holds.

In order to prove (ii), given $z \in D$ and $t \geq \tau$, we denote by

$$U_\epsilon(t, \tau) z = (u^\epsilon(t), w^\epsilon(t), u^\epsilon_t(t), w^\epsilon_t(t)),$$

and

$$u = u^\epsilon - u^0, \quad w = w^\epsilon - w^0.$$

Then, $(u, w, u_t, w_t)$ solves the equation

$$\begin{cases}
\rho_1 u_{tt} - a_{11} u_{xx} - a_{12} w_{xx} + g_1(u^\epsilon) - g_1(u^0) = f_1(u^0, w^0) - f_1(u^\epsilon, w^\epsilon) + \epsilon h_1, \\
\rho_2 w_{tt} - a_{12} u_{xx} - a_{22} w_{xx} + g_2(w^\epsilon) - g_2(w^0) = f_2(u^0, w^0) - f_2(u^\epsilon, w^\epsilon) + \epsilon h_2,
\end{cases} \quad (5.5)$$

with Dirichlet boundary conditions and initial condition

$$(u(\tau), w(\tau), u_t(\tau), w_t(\tau)) = 0.$$
Multiplying the first equation in (5.5) by $u_t$ and second by $w_t$ and then integrating over $[0, L]$ we obtain

$$\frac{1}{2} \frac{d}{dt} \|(u, w, u_t, w_t)\|_{H^t}^2 = - \int_0^L (g_1(u'_t) - g_1(u^0_t))u_t \, dx$$

$$- \int_0^L (g_2(w'_t) - g_2(w^0_t))w_t \, dx$$

$$- \int_0^L (f_1(u', u^c) - f_1(u^0, w^0))u_t \, dx$$

$$- \int_0^L (f_2(u', u^c) - f_2(u^0, w^0))w_t \, dx \quad (5.6)$$

By (2.6) we see that

$$\int_0^L (g_1(u'_t) - g_1(u^0_t))u_t \, dx \geq M_1 \|u_t\|_{L^2}^2,$$

$$\int_0^L (g_2(w'_t) - g_2(w^0_t))w_t \, dx \geq M_1 \|w_t\|_{L^2}^2. \quad (5.7)$$

From (2.8), Hölder’s inequality and embedding $H^1_0(0, L) \hookrightarrow L^\infty(0, L)$ we deduce that

$$\int_0^L (f_1(u', u^c) - f_1(u^0, w^0))u_t \, dx \leq C \left(\|u'\|_{L^\infty} + \|u^0\|_{L^p} + \|u^c\|_{L^p} + \|w^0\|_{L^p} + 1\right) (\|u\|_2 + \|w\|_2) \|u_t\|_2 \, dx \quad (5.8)$$

$$\leq C(D, T, t)(\|u\|_{L^2}^2 + \|w\|_{L^2}^2) + \frac{M_1}{2} \|u_t\|_{L^2}^2.$$ 

Similarly,

$$\int_0^L (f_2(u', u^c) - f_2(u^0, w^0))w_t \, dx \leq C(D, T, t)(\|u\|_{L^2}^2 + \|w\|_{L^2}^2) + \frac{M_1}{2} \|w_t\|_{L^2}^2.$$ 

Therefore,

$$\int_0^L (f_1(u', u^c) - f_1(u^0, w^0))u_t \, dx + \int_0^L (f_2(u', u^c) - f_2(u^0, w^0))w_t \, dx \quad (5.9)$$

Also, we observe that

$$\varepsilon \int_0^L h_1 u_t \, dx + \varepsilon \int_0^L h_2 w_t \, dx \leq \varepsilon^2 \left(\|h_1(t)\|_{L^2}^2 + \|h_2(t)\|_{L^2}^2\right) + \frac{M_1}{2} \left(\|u_t\|_{L^2}^2 + \|w_t\|_{L^2}^2\right). \quad (5.10)$$

Substituting the estimates (5.7), (5.9) and (5.10) in (5.6) we obtain

$$\frac{d}{dt} \|(u, w, u_t, w_t)\|_{H^t}^2 \leq 4C(D, T, t)(\|u\|_{L^2}^2 + \|w\|_{L^2}^2)$$

$$+ \frac{\varepsilon^2}{M_1} \left(\|h_1(t)\|_{L^2}^2 + \|h_2(t)\|_{L^2}^2\right). \quad (5.11)$$
Combining (2.5) and (5.11) yields
\[
\frac{d}{dt} \| (u, w, u_t, w_t) \|_H^2 \leq 4C(D, T, t) \| (u, w, u_t, w_t) \|_H^2 + \frac{\epsilon^2}{M_1} (\| h_1(t) \|_2^2 + \| h_2(t) \|_2^2).
\] (5.12)

Applying Gronwall’s lemma to (5.12) we conclude that
\[
\| (u(t), w(t), u_t(t), w_t(t)) \|_H^2 \leq \frac{\epsilon^2}{M_1} \int_t^{\tau} e^{4C(D, T, t)(t-s)} (\| h_1(s) \|_2^2 + \| h_2(s) \|_2^2) \, ds,
\]
and thus
\[
\| U_\epsilon(t, \tau) z - U_0(t, \tau) z \|_H^2 \leq \frac{\epsilon^2}{M_1} \int_{t-T}^{t} e^{4C(D, T, t)(t-s)} (\| h_1(s) \|_2^2 + \| h_2(s) \|_2^2) \, ds,
\]
for any \( \tau \in [t-T, t] \) and \( z \in D \). Since \( h_1, h_2 \in L^2_{\text{loc}}(\mathbb{R}; L^2(0, L)) \) it follows that
\[
\sup_{\tau \in [t-T, t], z \in D} \| U_\epsilon(t, \tau) z - U_0(t, \tau) z \|_H \xrightarrow{\epsilon \to 0} 0,
\]
and thereby (ii) holds. The proof is complete. \( \square \)

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E-mail address: mirelson@ufpa.br
E-mail address: leandrocorreiacosta@hotmail.com
E-mail address: gera@ufpa.br