Low-Energy Dynamics of String Solitons

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Abstract

The dynamics of a class of fivebrane string solitons is considered in the moduli space approximation. The metric on moduli space is found to be flat. This implies that at low energies the solitons do not interact, and their scattering is trivial. The range of validity of the approximation is also briefly discussed.
This note is concerned with the low energy dynamics of string theory solitons – more precisely the neutral wormhole fivebranes \[1,2,3\]. The method, which has now been applied to a number of soliton systems, is to approximate the field evolution as geodesic motion on the moduli space of static solutions, the metric being that induced by the field kinetic energy \[4\]. We find that the metric is flat, implying trivial scattering at low energies. This result was conjectured in \[5\], and is in agreement with more general considerations we shall mention later. The same problem has also been addressed in \[6\], where however a different result was obtained.

We first briefly review the soliton solutions of interest. The bosonic part of the low-energy effective action for Type-II string theory (or heterotic strings when the gauge field \(F\) is zero) in ten dimensions is given to lowest order in the Regge slope \(\alpha'\) by

\[
I = \frac{1}{2\kappa^2} \int d^{10}x \sqrt{g} e^{-2\phi} \left[ R + 4(\nabla \phi)^2 - \frac{1}{3} H^2 \right] + \text{Surface Term} ,
\]

where \(\kappa^2\) is proportional to the gravitational constant, \(\phi\) is the dilaton field, and \(H_{\mu\nu}\) is the field strength obtained from the antisymmetric tensor field \(B_{\mu\nu}\) \(\left( H_{\lambda\mu\nu} = \frac{1}{2} \partial_\lambda B_{\mu\nu} + \text{cyclic} \right)\). The corresponding equations of motion are

\[
R_{\mu\nu} + 2 \nabla_\mu \nabla_\nu \phi - H_{\mu\nu}^2 = 0 ,
\]

\[
\nabla^\lambda (e^{-2\phi} H_{\lambda\mu\nu}) = 0 ,
\]

\[
\nabla^2 \phi - 2(\nabla \phi)^2 + \frac{1}{3} H^2 = 0 .
\]

Restricting immediately to fields with five-dimensional translational symmetry, the system is essentially (4+1)-dimensional, and the remaining dimensions may be ignored. Field configurations are then characterized by the total flux \(Q\) (the “axion charge”), of the \(H\)-field through the 3-sphere at infinity. For a given \(Q\), the ADM mass \[7\] satisfies the Bogomol’nyi-type bound

\[
M_{ADM} \geq \frac{2\pi^2}{\kappa^2} Q .
\]

Saturating this bound gives the neutral fivebrane soliton solutions. Up to coordinate and gauge transformations, the general solution is

\[
g_{00} = -1, \quad g_{ij} = e^{2\phi} \delta_{ij} ,
\]

\[
e^{2\phi} = 1 + \sum_{n=1}^N \frac{Q^n}{|x - \bar{a}_n|^2} , \quad H_{ijk} = \omega_{ijkl} \nabla^l \phi ,
\]

\[1\] Without loss of generality we set the asymptotic dilaton field to zero. The ADM mass is then the same in both canonical and sigma-model variables.
all other components being zero. Here, $i$ runs from 1 to 4, $\vec{x}$ is the position vector in 4-dimensional space, and $\omega_{ijkl}$ is the volume form on this space. Physically, (4) describes a static configuration of $N$ solitons with positions $\vec{a}_n$ and charges $Q_n$. Each soliton is a semi-infinite wormhole, stabilized by the flux of the $H$-field through its throat (see Fig.1); one may regard the existence of the multi-soliton static solution as a consequence of the exact cancellation between the attractive forces due to gravity and the dilaton, and a repulsive force due to the $H$-field. The $Q_n$’s are quantized in units of $\alpha'$ \cite{8}, and so for given values, the moduli space of solutions, $\mathcal{M}_N$, is just the space of $\vec{a}_n$’s – in other words, $\mathbb{R}^{4N}$.

![Fig. 1: Two-dimensional cross-section of a typical 3-soliton configuration. (The wormhole throats are infinitely long.)](image)

To provide the setting for the moduli space description of the dynamics we perform a split between space and time, extending a treatment given by Ruback for a similar analysis of Kaluza-Klein monopoles \cite{9}. We assume that the low energy dynamics involves only small perturbations away from $\mathcal{M}_N$. In the neighbourhood of a space-like surface $\Sigma$ we may choose synchronous coordinates, so that $g_{00} = -1$, $g_{0i} = 0$, and the metric takes the form

$$ds^2 = -dt^2 + g_{ij} dx^i dx^j.$$  \tag{5}

The $U(1)$ gauge-invariance associated with the antisymmetric tensor field allows in addition the choice $B_{0i} = 0$. Performing standard manipulations \cite{10}, the action may be written as

\footnote{In fact, under small perturbations, such as those caused by imparting small velocities to the solitons, event horizons will form at some distance down the wormhole throats. Our assumption is that these horizons are sufficiently far down that they cause no significant effects.}
the time integral of the Lagrangian $L = T - V$ where

$$V = -\frac{1}{2\kappa^2} \int_\Sigma d^4x \sqrt{g} \ e^{-2\phi} \left[ 4R + 4\nabla^i \phi \nabla_i \phi - \frac{1}{3} H^{ijk} H_{ijk} \right]$$

$$+ \frac{1}{\kappa^2} \int_{\partial \Sigma} d^3x \sqrt{h} \ e^{-2\phi} (K^S - K^S_0)$$

is to be regarded as the potential energy, and

$$T = \frac{1}{8\kappa^2} \int_\Sigma d^4x \sqrt{g} \ e^{-2\phi} \left[ g^{ik} g^{jl} (\dot{g}_{ij} + \dot{B}_{ij})(\dot{g}_{kl} + \dot{B}_{kl}) - (\dot{\gamma} - 4\dot{\phi})^2 \right]$$

the kinetic energy. In these expressions, $4R$ is the four-dimensional Ricci scalar evaluated on the spatial metric, $K^S$ is the extrinsic curvature scalar of the boundary at spatial infinity, $h$ is the induced metric on the boundary, and $K^S_0$ is the curvature scalar that the boundary would have were it embedded in flat space. Dots denote time derivatives, and $\dot{\gamma} = g^{ij} \dot{g}_{ij}$. The surface term in (6) is a remnant of the surface term in (1), which is required to compensate for the presence of second derivatives in the Ricci scalar; the surface terms involving time derivatives cancel against a total time derivative present in the expression relating the five-dimensional Ricci scalar to $4R$.

We must also take care to impose the equations of motion corresponding to $g_{00}$, $g_{0i}$ and $B_{0i}$. These are just the constraints associated with the diffeomorphism invariance and $U(1)$ gauge invariance of the theory. The Hamiltonian and momentum constraints associated with the former are

$$\frac{1}{4} g^{ik} g^{jl} (\dot{g}_{ij} + \dot{B}_{ij})(\dot{g}_{kl} + \dot{B}_{kl}) - \frac{1}{4} (\dot{\gamma} - 4\dot{\phi})^2$$

$$- 4R - 4\nabla^i \nabla_i \phi + 4\nabla^i \phi \nabla_i \phi + \frac{1}{3} H^{ijk} H_{ijk} = 0$$

and

$$\tilde{\nabla}^j [e^{-2\phi} (\dot{g}_{ij} + \dot{B}_{ij})] = e^{-2\phi} \tilde{\nabla}_i (\dot{\gamma} - 4\dot{\phi})$$

respectively, where $\tilde{\nabla}_i$ is a generalized covariant derivative in which the Christoffel connection $\Gamma^i_{jk}$ is replaced by $\tilde{\Gamma}^i_{jk} = \Gamma^i_{jk} - H^i_{jk}$; the (Gauss) constraint associated with the latter is

$$\nabla^j [e^{-2\phi} \dot{B}_{ij}] = 0.$$
space of fields \( f = (g_{ij}, \phi, B_{ij}) \). In this space one must of course regard as equivalent the points along each orbit generated by diffeomorphisms and \( U(1) \) transformations. The momentum and Gauss constraints, (9) (10), restrict tangent vectors, \( \dot{f} \), to be orthogonal to these orbits with respect to the metric defined by \( T \). However, this still leaves some gauge freedom in \( \dot{f} \). In particular we can add a vector corresponding to an infinitesimal diffeomorphism generated by a vector field (on \( \Sigma \)) of the form \( \dot{\xi}_i = \nabla_i \lambda \). By an appropriate choice of \( \lambda \) we may set
\[
\dot{\gamma} - 4\dot{\phi} = 0 .
\]
(11)
This condition fixes the gauge completely, and renders the kinetic energy \( T \) positive definite. In summary then, we have a gauge invariant positive definite metric, and potential \( V \). Furthermore, if the Hamiltonian constraint (8) is satisfied, \( T + V \) is given by a boundary term which can be shown to be equivalent to the ADM mass \( M_{ADM} \).

The idea of the moduli space description is that given initial data corresponding to giving the solitons some small velocities (i.e. a slow motion tangent to \( \mathcal{M}_N \)), \( V \) will force the motion to remain close to \( \mathcal{M}_N \), and the evolution will be well approximated by geodesic motion on \( \mathcal{M}_N \) with respect to the metric induced by \( T \). This requires of course that \( \mathcal{M}_N \) is at a (local) minimum of \( V \). To see that this is indeed the case, consider a small perturbation away from a point \( f_0 \) of the moduli space:
\[
f = f_0 + \epsilon v + O(\epsilon^2) \quad \dot{f} = 0 ,
\]
(12)
where the tangent vector \( v \) satisfies (9), (10) and (11). To \( O(\epsilon) \), the Hamiltonian constraint (8) is simply the divergence of the momentum constraint (9) evaluated on \( v \) and is thus automatically satisfied. The potential \( V \) is then just the ADM mass, and the Bogomol’nyi bound (3) gives the result.

To calculate the metric on \( \mathcal{M}_N \) we require an expression for a general tangent to \( \mathcal{M}_N \) satisfying the constraints as well as (11). Such a vector is given by
\[
\dot{g}_{ij} = 2g_{ij}\dot{\phi} + \nabla_i\dot{\xi}_j + \nabla_j\dot{\xi}_i ,
\]
(13)
\[
\dot{\phi} = \dot{\phi} + \dot{\xi}^k \nabla_k \phi ,
\]
(14)

\(^3\)The boundary term consists of the boundary part of (3), together with a piece which involves the normal derivative of the dilaton. The former is equivalent to the canonical ADM formula (12); the latter is precisely the term which generalizes this for the sigma-model metric (13).
and
\[ B_{ij} = \omega_{ij}^k \nabla_k \dot{\xi}_m + 2 \dot{\xi}_k H_{ijk}, \quad (15) \]
together with (4), where
\[ \dot{\varphi} = \sum_{n=1}^N \frac{\partial \phi}{\partial a^i_n} \dot{a}^i_n, \quad (16) \]
and
\[ \dot{\xi}_i = \sum_{n=1}^N \frac{Q_n \dot{a}^i_n}{|\vec{x} - \vec{a}_n(t)|^2}. \quad (17) \]
The first term in each case represents the variation of the fields (4) by an infinitesimal change in the moduli; the other terms represent the effect of an infinitesimal diffeomorphism generated by \( \dot{\xi}^k \) together with an infinitesimal \( U(1) \) rotation. This is just the solution obtained in [6] by giving each soliton an independent boost, but in a different gauge. (An appealing feature of this gauge is that the fractional variations of the fields are small at every point in space, in particular near the soliton cores. This makes regularization unnecessary.) It is straightforward to check the required properties. The condition (11) is equivalent to the identity
\[ \dot{\varphi} = -\frac{1}{2} \nabla_i \dot{\xi}^i + \dot{\xi}^i \nabla_i \phi. \quad (18) \]
Furthermore, since we are on \( \mathcal{M}_N \), the Hamiltonian constraint is automatically satisfied to \( O(\dot{a}_n) \), while the Gauss constraint is automatically satisfied by (15) for any \( \dot{\xi}_i \). The momentum constraint (9) is a little more tricky, but can be reduced to the form
\[ \frac{1}{2} \nabla^j (\nabla_j \dot{\xi}_i - \nabla_i \dot{\xi}_j) = e^{-2\phi} \nabla_i (e^{2\phi} \dot{\varphi}). \quad (19) \]
Expressed in terms of ordinary derivatives, this is equivalent to
\[ e^{-2\phi} \left[ \partial^2 \dot{\xi}_i - \partial_i (\partial_j \dot{\xi}_j + 2e^{2\phi} \dot{\varphi}) \right] = 0 \quad (20) \]
where \( \partial^2 \) is the flat space Laplacian, and it is straightforward to check that \( e^{-2\phi} \partial^2 \dot{\xi}_i \) and \( \partial_j \dot{\xi}_j + 2e^{2\phi} \dot{\varphi} \) vanish separately.

All that now remains is to substitute (13) – (15) into the expression (7) for the kinetic energy. Using (18), the kinetic energy integrand can be written
\[
\nabla^i \left[ e^{-2\phi} \dot{\xi}^j \nabla_j \dot{\xi}_i \right] \\
-\dot{\xi}_i e^{-2\phi} \left[ \nabla^2 \dot{\xi}_i - \nabla_j \nabla_i \dot{\xi}^j - 2 \nabla_i \dot{\varphi} - 4 \dot{\varphi} \nabla_i \phi \right] \\
-\dot{\xi}_i \dot{\xi}_j e^{-2\phi} \left[ R_{ij} + 2 \nabla_i \nabla_j \phi - H_{ij}^2 \right]. \quad (21)
\]
The second term vanishes by (19). The third term is just the graviton equation of motion for the static solution (2) and is also zero. The final answer is thus given by a surface integral

\[ T = \frac{1}{4\kappa^2} \int_{\partial \Sigma} d^3x \sqrt{h} e^{-2\phi} \nabla_n \dot{\xi}^2, \]  

(22)

where \( \nabla_n \) is the normal derivative on the surface. Examination of the integrand reveals that the surface at space-like infinity contributes nothing, but that there is a finite contribution from the asymptotic \( S^3 \) associated with each wormhole throat. A simple calculation shows that the result is

\[ T = \sum_{n=1}^{N} \frac{1}{2} M_n |\vec{a}_n|^2 \]  

(23)

where \( M_n = \frac{2\pi^2}{\kappa^2} Q_n \) is the ADM mass of a single soliton of charge \( Q_n \). This means that the total kinetic energy is just the sum of the individual energies with no interaction terms. The metric on moduli space is therefore flat, and the low energy scattering trivial.

It remains to consider the validity of the approximation. For general scattering processes one expects the main corrections to be due to radiation in both the gravitational and matter fields, with scalar radiation dominating. In very close collisions however, the approximation will break down more seriously. Consider two solitons in collision. Since their total energy exceeds the Bogomol’nyi bound by the kinetic energy of the relative motion, there will be an event horizon for the combined system. For initial data with some non-zero impact parameter there will be a critical velocity below which the treatment we have given is valid, but for velocities which exceed this value, the solitons will fall within the horizon, and will therefore coalesce.

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4 The result is in agreement with more general considerations [14]. The Type-II soliton we have been considering possesses \( N = 4 \) worldsheet supersymmetry [3]. It can be argued that this implies that the moduli space is hyperKähler and so Ricci flat. The moduli space for two solitons (factoring out the center of mass motion) is topologically \( \mathbb{R}^4 \), and for this case at least Ricci flat is equivalent to flat.
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