SIMPLICIAL SETS INSIDE CUBICAL SETS

THOMAS STREICHER AND JONATHAN WEINBERGER

Abstract. As observed by various people recently the topos $\mathbf{sSet}$ of simplicial sets appears as essential subtopos of a topos $\mathbf{cSet}$ of cubical sets, namely presheaves over the category $\mathbf{FL}$ of finite lattices and monotone maps between them. The latter is a variant of the cubical model of type theory due to Cohen et al. for the purpose of providing a model for a variant of type theory which validates Voevodsky’s Univalence Axiom and has computational meaning.

One easily shows that sheaves, i.e. simplicial sets, are closed under most of the type theoretic operations of $\mathbf{cSet}$. Finally, we construct in $\mathbf{cSet}$ a fibrant univalent universe for those types that are sheaves.

1. Introduction

As observed in [HS94] intensional Martin–Löf type theory should have a natural interpretation in weak $\infty$-groupoids. In the first decade of this millenium it was observed that simplicial sets are a possible implementation of this idea and around 2006 Voevodsky proved that the respective universe validates the so-called Univalence Axiom (UA) which roughly speaking states that isomorphic types are propositionally equal, see [KL12] for a detailed proof.

But adding a constant inhabiting the type expressing UA gives rise to a type theory lacking computational meaning. To overcome this problem Coquand et al. [CCHM18] have developed a so-called Cubical Type Theory based on explicit box filling operations from which UA can be derived. Cubes are finite powers of an interval object $\mathbb{I}$ which itself is not a proper type. Although in [CCHM18] this type theory is interpreted in the topos of covariant presheaves over the category of finitely presented free de Morgan algebras it is obvious that an equally strong type theory can be interpreted in the presheaf category $\mathbf{cSet}$ over the site $\mathbb{F}$ which is the full subcategory of $\mathbf{Poset}$ on finite powers of the 2 element lattice $\mathbb{I}$. This site is o.p.-equivalent to the algebraic theory of distributive lattices as observed in [Spi16].

As observed independently in [KV17] the topos $\mathbf{sSet}$ appears as subtopos of $\mathbf{cSet}$ and actually as an essential subtopos, cf. [Sat18].

Starting from a universe of cubical types within $\mathbf{cSet}$ we will construct a universe for simplicial types within $\mathbf{cSet}$. As it turns out, this universe is itself fibrant and univalent.

2. Simplicial Sets Inside Cubical Sets

We write $\Delta$ for the full subcategory of $\mathbf{Poset}$ on finite ordinals greater 0 and we write $\square$ for the full subcategory of $\mathbf{Poset}$ on finite powers of $\mathbb{I}$, the 2 element

2010 Mathematics Subject Classification. 03B15, 03G30, 18F20, 18G55.

Key words and phrases. cubical sets, simplicial sets, universes, Univalence Axiom.

We are grateful to Christian Sattler for many discussions, pointing out mistakes as well as problems in previous versions, and finally for making [Sat18] available on the arXiv.
lattice. Presheaves over $\Delta$ are called *simplicial sets* and presheaves over $\Box$ are called *cubical sets*. We write $\mathbf{sSet}$ and $\mathbf{cSet}$ for the toposes of simplicial and cubical sets, respectively.

Kapulkin and Voevodsky have observed in [KV17] that one may obtain $\mathbf{sSet}$ as a subtopos of $\mathbf{cSet}$ in the following way. The nerve functor $N_v : \mathbf{Cat} \to \mathbf{sSet}$ is known to be full and faithful and so is its restriction $u : \Box \to \mathbf{sSet}$ to the full subcategory $\Box$ of $\mathbf{Cat}$. This functor $u$ induces an adjunction $u_! \dashv u^* : \mathbf{sSet} \to \mathbf{cSet}$ where $u^*(X) = \mathbf{sSet}(u(-), X)$ and $u_!$ is the left Kan extension of $u$ along $Y_u : \Box \to \mathbf{cSet}$. It follows from general topos theoretic results that $u_! \dashv u^*$ exhibits $\mathbf{sSet}$ as a subtopos of $\mathbf{cSet}$ induced by the Grothendieck topology $J$ consisting of those sieves in $\Box$ which are sent by $u$ to jointly epic families in $\mathbf{sSet}$.

A more direct proof of a stronger result has been found by Sattler in [Sat18] and independently by the authors of this paper based on the well known fact that splitting idempotents in $\Box$ gives rise to the category of finite lattices and monotone maps between them. E.g. by restricting to subobjects of objects in $\Box$ one obtains an equivalent small full subcategory $\mathbf{FL}$ of $\mathbf{Poset}$. Thus $\mathbf{cSet}$ is equivalent to $\mathbf{FL} = \mathbf{Set}^{\mathbf{FL}^{op}}$ for which reason we write $\mathbf{cSet}$ for $\mathbf{FL}$.

The inclusion functor $i : \Delta \to \mathbf{FL}$ induces an essential geometric morphism $i^* \dashv i_* \dashv i^!$, which, moreover, is injective, i.e. $i_*$ and thus also $i^!$ is full and faithful. The inverse image part $i^*$ restricts presheaves over $\mathbf{FL}$ to presheaves over $\Delta$ (by precomposition with $i^{op}$). The direct image part $i_*$ is given by $i_*(X) = \mathbf{sSet}(N_v(-), X)$ since $N_v$ restricted to $\mathbf{FL}$ is given by $i^* \circ Y_{\mathbf{FL}}$. The cocontinuous functor $i^!$ is the left Kan extension of $Y_{\mathbf{FL}} \circ i$ along $Y_{\Delta}$. It sends $X \in \mathbf{sSet}$ to the colimit of $\Delta \downarrow X \xrightarrow{\eta_X} \Delta \xrightarrow{i^!} \mathbf{FL} \xrightarrow{Y_{\mathbf{FL}}} \mathbf{cSet}$.

The Grothendieck topology on $\mathbf{FL}$ corresponding to the injective geometric morphism $i^* \dashv i_* \dashv i^!$ can be described as follows: $S \subseteq Y_{\mathbf{FL}}(L)$ is a cover iff $i^* S = i^* Y_{\mathbf{FL}}(L)$, i.e. $S$ contains all chains in $L$, i.e. all monotone maps $c : [n] \to L$. Thus, a sieve $S \subseteq Y_{\mathbf{FL}}(\mathbb{I}^n)$ covers iff for every maximal chain $C \subseteq \mathbb{I}^n$ there is an idempotent $r \in S$ whose image is $C$. Obviously, such an $S$ contains all monotone maps to $\mathbb{I}^n$ whose image is contained in $C$. Thus the collection of all monotone maps to $\mathbb{I}^n$ whose image is contained in a (maximal) chain in $\mathbb{I}^n$ is the least covering sieve for $\mathbb{I}^n$.

## 3. Minimal Cisinski Model Structures on Simplicial and Cubical Sets

The representable object $\mathbb{I}$ in $\mathbf{cSet}$ and $\mathbf{sSet}$ induces a minimal Cisinski model structure on these presheaf toposes which is generated by open box inclusions as described in the following definition.

**Definition 3.1** (Minimal Cisinski Model Structure on $\mathbf{sSet}$ and $\mathbf{cSet}$). Let $\mathcal{E}$ be $\mathbf{cSet}$ or $\mathbf{sSet}$. The interval $\mathbb{I}$ is given by $Y_{\mathbf{FL}}(\mathbb{I})$ and $Y_{\Delta}(\mathbb{I})$, respectively.

An open box inclusion in $\mathcal{E}$ is a subobject of the form $(\{\varepsilon\} \times X) \cup (\mathbb{I} \times Y) \hookrightarrow I \times X$ where $Y \hookrightarrow X$ and $\varepsilon \in \{0, 1\}$.

Fibrations in $\mathcal{E}$ are those maps which are weakly right orthogonal to all open box inclusions. Cofibrations in $\mathcal{E}$ are precisely the monomorphisms. In fact, this Cisinski model structure on simplicial sets coincides with the well known Kan model structure since by a theorem in [GZ67], Sec. 2, open box inclusions and horn inclusions generate the same class of anodyne extensions. We will
show that it is obtained just by restricting the minimal Cisinski model structure on cubical sets defined above to simplicial sets.

**Proposition 3.1.** The inclusion $i_* : \text{sSet} \to \text{cSet}$ preserves and reflects fibrations.

*Proof.* From the preservation properties of the sheafification functor $i^*$ it follows immediately that open box inclusions in $\text{sSet}$ are precisely the sheafifications of open box inclusions in $\text{cSet}$.

Since both in $\text{cSet}$ and $\text{sSet}$ fibrations are those maps which are weakly right orthogonal to all open box inclusions a map $f$ in $\text{sSet}$ is a fibration iff $i_* f$ is a fibration in $\text{cSet}$.

□

Writing $F$ for the class of fibrations in $\text{cSet}$ the class of Kan fibrations in $\text{sSet}$ is given by $F \cap \text{sSet}$ (considering $\text{sSet}$ as full subcategory of $\text{cSet}$ via $i_*$).

Sattler has pointed out to us an elegant argument that $i_*$ preserves and reflects weak equivalences between fibrant objects.

**Proposition 3.2.** For fibrant objects $A, B \in \text{sSet}$ a map $f : A \to B$ is a weak equivalence in $\text{sSet}$ iff $i_* f$ is a weak equivalence in $\text{cSet}$.

*Proof.* In both $\text{sSet}$ and $\text{cSet}$ weak equivalences between fibrant objects are just homotopy equivalences and these are preserved by $i^*$ and $i_*$ since these functors preserve $\mathbb{I}$ and finite products. □

**Proposition 3.3.** The adjunction $i^* \dashv i_*$ is a Quillen adjunction. Moreover, $i^*$ preserves weak equivalences between arbitrary objects.

*Proof.* By Proposition 3.1 the functor $i_*$ preserves fibrations. As $i^*$ is in turn a right adjoint it preserves monomorphisms. Thus $i^* \dashv i_*$ is a Quillen adjunction.

As every cubical set is cofibrant, by Ken Brown’s Lemma $i^*$ preserves weak equivalences.

□

4. Universes in cubical sets

Given a Grothendieck universe $\mathcal{U}$ this induces a universe à la Yoneda $\pi : E \to U$ in $\mathcal{E}_{\mathcal{L}}$ which is generic for the class of $\mathcal{U}$-small maps in $\mathcal{E}_{\mathcal{L}}$ [Str05]. As described in [GS17, Sect. 9] there is a universe $\pi_c : E_c \to U_c$ generic for $\mathcal{U}$-small fibrations in $\text{cSet}$ such that $U_c$ is fibrant. Moreover, there is a morphism $u_c : U_c \to U$ which is full and faithful as a (split) cartesian functor (when considering $U_c$ and $U$ as (split) fibrations and not just as presheaves) with $\pi_c$ isomorphic to $u_c^* \pi$.

Note that as described in [GS17] $U_c(L)$ does not simply consist of $\mathcal{U}$-small fibrations over $\mathcal{E}_{\mathcal{L}}(L)$ but rather such fibrations together with a functorial choice of fillers which are forgotten by $u_c$.

---

1. *i.e.* all $\mathcal{U}$-small fibrations can be obtained as pullback of the generic one in a typically non-unique way.
We now give the construction of a fibrant univalent universe in cubical sets which is generic for small fibrations which are families of sheaves. Recall from [Str05] that a map \( a : A \to I \) is a family of sheaves ifff the naturality square

\[
\begin{array}{ccc}
A & \xrightarrow{\eta_A} & i_* i^* A \\
\downarrow a & & \downarrow i_* i^* a \\
I & \xrightarrow{\eta_I} & i_* i^* I
\end{array}
\]

is a pullback. Thus \( i_* i^* \pi_c \) is a generic family for small families of sheaves within \( \mathbf{cSet} \) but, alas, this universe is not univalent. In the rest of this section we construct a univalent universe from this making, however, use of the univalent universe \( \pi_s : E_s \to U_s \) in \( \mathbf{sSet} \) as constructed in [KL12].

**Theorem 4.1.** A univalent small fibration generic for small fibrations which are also families of sheaves can be obtained by pulling back \( i_* i^* \pi_c \) along the homotopy equalizer of \( i_* e \circ i_* p \) and \( \text{id}_{i_* U_s} \) where \( e \) and \( p \) are maps such that both squares in

\[
\begin{array}{ccc}
E_s & \xrightarrow{i^* E_c} & E_s \\
\downarrow \pi_s & & \downarrow \pi_s \\
U_s & \xrightarrow{i^* \pi_c} & U_s
\end{array}
\]

are pullbacks.

**Proof.** Since by [Sat18] the functor \( i^* \) preserves small fibrations \( i^* \pi_c \) is generic for small fibrations in \( \mathbf{sSet} \). Notice that \( p e \sim \text{id}_{U_s} \) (i.e. \( p e \) and \( \text{id}_{U_s} \) are homotopy equivalent) since the universe \( \pi_s \) is univalent. Since \( i_* \) preserves fibrations, pullbacks and \( \sim \) (homotopy equivalence) for maps between fibrant objects we have

\[
\begin{array}{ccc}
i_* E_s & \xrightarrow{i_* i^* E_c} & i_* E_s \\
i_* \pi_s & \downarrow & i_* \pi_s \\
i_* U_s & \xrightarrow{i_* e} & i_* U_s
\end{array}
\]

with \( i_* p \circ i_* e = i_* (e \circ p) \sim i_* (\text{id}_{U_s}) = \text{id}_{i_* U_s} \). Thus \( i_* i^* \pi_c \) is generic for small fibrations which are families of sheaves and \( i_* \pi_s \) is a univalent such universe since by Prop. 3.2 the functor \( i_* \) preserves and reflects weak equivalences between fibrant objects and families.

Thus, pulling back \( i_* i^* \pi_c \) along the homotopy equalizer of \( i_* e \circ i_* p \) and \( \text{id}_{i_* U_s} \) gives rise to a univalent universe \( U_{cs} \) in \( \mathbf{cSet} \) generic for small fibrations which are families of sheaves. Since the construction of homotopy equalizers can be expressed in the internal language and this does not lead out of types the object \( U_{cs} \) is fibrant as well. \( \square \)
5. Does the minimal Cisinski model structure on \( \text{cSet} \) coincide with the test model structure?

Finally, we discuss the question whether the minimal Cisinski model structure on \( \text{cSet} \) coincides with the test model structure on it.

The inclusion functor \( i : \Delta \to \text{FL} \) is aspherical in the sense of [Mal05], i.e. \( N_{\text{v}}(\text{Elts}(i)) \) is a weak equivalence in \( \text{sSet} \). Thus by [Mal05, Th. 1.2.9] the functor \( i^* : \text{cSet} \to \text{sSet} \) preserves and reflects weak equivalences of the respective test model structures. Since \( i^* \) also preserves monos the adjunction \( i^* \dashv i_* \) is a Quillen equivalence between \( \text{cSet} \) and \( \text{sSet} \) endowed with the respective test model structures.

Let \( \varepsilon_X : i_i i^* X \to X \) be the counit of \( i_! \dashv i^* \) and \( \eta_X : X \to i_* i^* X \) be the unit of \( i^* \dashv i_* \). These are weak equivalences w.r.t. the test model structure on \( \text{cSet} \) since both are sent to isos and thus weak equivalences by applying \( i^* \).

We know that both \( i_! \dashv i^* \) and \( i^* \dashv i_* \) are Quillen pairs when \( \text{cSet} \) is endowed with the minimal Cisinski model structure. Thus, if the minimal Cisinski model structure on \( \text{cSet} \) coincides with the test model structure then all \( \eta_X : X \to i_* i^* X \) and \( \varepsilon_X : i_i i^* X \to X \) are weak equivalences w.r.t. the minimal Cisinski model structure on \( \text{cSet} \). But if all \( \varepsilon_X \) are weak equivalences w.r.t. the minimal Cisinski model structure then it coincides with the test model structure which can be seen as follows. Suppose \( m : Y \to X \) is an anodyne cofibration w.r.t. the test model structure then \( i_i i^* m \) is an anodyne cofibration in \( \text{sSet} \) from which it follows that \( i_i i^* m \) is an anodyne cofibration w.r.t. the minimal Cisinski model structure on \( \text{cSet} \). But since

\[
\begin{array}{ccc}
i_i i^* Y & \xrightarrow{\varepsilon_Y} & Y \\
i_i i^* m & \downarrow & \downarrow m \\
i_i i^* X & \xrightarrow{\varepsilon_X} & X
\end{array}
\]

commutes it follows by the 2-out-of-3 property for weak equivalences that \( m \) is a weak equivalence and thus an anodyne cofibration w.r.t. the minimal Cisinski model structure on \( \text{cSet} \).

Thus, summarizing the above considerations we conclude that the minimal Cisinski and the test model structure on \( \text{cSet} \) coincide if and only if all \( \varepsilon_X : i_i i^* X \to X \) are weak equivalences in the minimal Cisinski model structure on \( \text{cSet} \). Alas, we do not know whether this is the case in general.

6. Conclusion

Using the fact that \( \text{sSet} \) is an essential subtopos of \( \text{cSet} \) we have constructed a fibrant univalent universe inside \( \text{cSet} \) which is generic for small families of sheaves, i.e. simplicial sets.

However, this construction makes use of the univalent universe inside \( \text{sSet} \). Formally speaking this universe can be constructed in the internal language of \( \text{cSet} \) but only at the price of importing the inconstructive universe \( \pi_s \) from \( \text{sSet} \) via \( i_* \).

Nevertheless, this may still model an extension of the cubical type theory of [CCHM18] providing a univalent universe for small simplicial sets the precise formulation of which we leave for future work.
References

[CCHM18] C. Cohen, T. Coquand, S. Huber, and A. M"ortberg, Cubical Type Theory: A constructive interpretation of the Univalence Axiom, 21st International Conference on Types for Proofs and Programs (TYPES 2015) (Dagstuhl, Germany) (Tarmo Uustalu, ed.), Leibniz International Proceedings in Informatics (LIPIcs), vol. 69, Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik, 2018, pp. 5:1–5:34.

[Cis19] D.-C. Cisinski, Higher categories and homotopical algebra, Cambridge University Press, 2019.

[GS17] N. Gambino and C. Sattler, The Frobenius condition, right properness, and uniform fibrations, Journal of Pure and Applied Algebra 221 (2017), no. 12, 3027 – 3068.

[GZ67] P. Gabriel and G. Zisman, Calculus of fractions and homotopy theory, Springer, 1967.

[HS94] M. Hofmann and Th. Streicher, The groupoid model refutes uniqueness of identity proofs, Proceedings Ninth Annual IEEE Symposium on Logic in Computer Science, July 1994, pp. 208–212.

[KL12] K. Kapulkin and P. Lumsdaine, The simplicial model of univalent foundations (after Voevodsky), Journal of the European Mathematical Society (2012), forthcoming, https://arxiv.org/abs/11211.2851

[KV17] K. Kapulkin and V. Voevodsky, Cubical approach to straightening. Submitted, https://www.math.uwo.ca/faculty/kapulkin/papers/cubical-approach-to-straightening.pdf 2017.

[Mal05] G. Maltsiniotis, La th\’eorie de l’homotopie de Grothendieck, Soci\’et\’e Math\’ematique de France, 2005.

[Sat18] C. Sattler, Idempotent completion of cubes in posets, https://arxiv.org/abs/1805.04126v2 2018.

[Spi16] B. Spitters, Cubical sets and the topological topos, https://arxiv.org/abs/111610.0527v0 2016.

[Str05] Th. Streicher, Universes in Toposes, From Sets and Types to Topology and Analysis, Towards Practicable Foundations for Constructive Mathematics 48 (2005), 78–90.

E-mail address: streicher@mathematik.tu-darmstadt.de, weinberger@mathematik.tu-darmstadt.de

Department of Mathematics, TU Darmstadt, Schloßgartenstrasse 7, 64289 Darmstadt, Hesse, Germany.