Boson sampling with realistic photon-number resolution

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Boson sampling is a model of non-universal quantum computations. Originally, it was proposed as a means of solving a typical computationally hard problem—sampling of events from probabilities given by permanents of matrices. In its basic optical configuration, the scheme consists of Fock-state sources, linear optical interferometer, and photon-number-resolving (PNR) detectors. Practical implementation of the latter is usually a sophisticated task, which can be solved only approximately in experiments. We generalize the analytical expression for photocounting distributions in boson-sampling schemes on the case with realistic PNR detectors. It is shown that for such a scenario properly postselected sampling probabilities are proportional to probabilities obtained for ideal PNR detectors. The corresponding correction coefficients contain specific information about detector characteristics. Our results are illustrated with examples of arrays of single-photon detectors and detectors affected by a finite dead time.

I. INTRODUCTION

In the most general case, predictions of the measurement outcomes in complex quantum systems require to solve computationally hard problems. This fact lies at the heart of the idea of quantum computations. Indeed, one can simulate a solution to such a problem via manipulations and measurements conducted with the corresponding quantum system. In this context, optical networks are promising candidates for implementations of both universal and non-universal quantum computations [1–5]. While the former requires strong technological effort, the latter is designed for solving only a special group of problems and can be implemented with current technologies.

Boson sampling [5] is a prominent example of non-universal quantum computations, which can be implemented with linear optical networks. In its standard configuration, cf. Fig. 1, N input modes of optical radiation are transformed by a linear interferometer as

$$\hat{a}_{\text{out}}^{(j)} = \sum_{i=1}^{N} U_{ij} \hat{a}_{\text{in}}^{(i)}.$$  (1)

Here $\hat{a}_{\text{in}}^{(i)}$ and $\hat{a}_{\text{out}}^{(j)}$ are field annihilation operators of input and output modes, respectively, and $U_{ij}$ is a unitary transformation matrix. Each input mode is prepared in the Fock state $|n_i\rangle$ such that the total number of input photons fulfills the condition

$$n = \sum_{i=1}^{N} n_i \leq N.$$  (2)

Each output mode is analyzed by a photon-number resolving (PNR) detector giving the random outcomes $m_i$.

Let us consider the lossless scenario. In this case, the photon-number distribution at the output reads [6, 7]

$$P_{m_1 \ldots m_N} = \frac{\text{Perm} U [1^{n_1} \ldots N^{n_N}] [1^{n_1} \ldots N^{n_N}]!}{m_1! \ldots m_N! n_1! \ldots n_N!}.$$  (3)

Here Perm $U [1^{n_1} \ldots N^{n_N}] [1^{n_1} \ldots N^{n_N}]!$ is the permanent of the matrix constructed from the elements of the original matrix $U_{ij}$ such that its row index $i$ and the column index $j$ appear $m_i$ and $n_j$ times, respectively. Calculation of the permanent is an example of #P-hard problem. Hence, photocounting distributions (3) cannot be effectively sampled with classical devices.

The core idea of boson sampling is to sample events from probability distributions expressed in terms of permanents of matrices by using the quantum setup described above. This means that such a technique may effectively solve a computationally hard problem. A similar idea works also for other nonclassical states at the interferometer inputs [8–12]. In general, it has been shown that effective classical simulations in optical networks can be possible with phase-space functions if special condi-
tions are fulfilled [13].

Practical implementations of boson-sampling schemes [14–20] require theoretical analysis of various practical issues. They can be subdivided into three groups by relation to sources, interferometer, and detectors. First, mode mismatch causes photons to be partially distinguishable, i.e. the interference between them is lost [21–26]. Next, unavoidable imperfections in the interferometer may affect the final results of sampling [27]. Finally, the issues related to the detector imperfections must be considered. It has been shown that acceptable losses are not a hard problem in the standard boson-sampling scheme [5, 21]. An elegant solution consists in postselection of the events with the same numbers of detected and injected photons. However, there is another problem—the impossibility of distinguishing between adjacent numbers of photons. By using on/off detectors, which can indicate only the presence of photons, one can still sample probabilities corresponding to \( m_i = 0, 1 \) for \( i = 1 \ldots N \) in Eq. (3). Other probabilities in such a scheme cannot be sampled.

There exist several experimental techniques, which enable to distinguish, at least approximately, between adjacent numbers of photons. The first method consists in separating the light beam in spatial [28–31] or temporal [32–34] modes and detecting each of them with an on/off detector. The number of obtained clicks or triggered detectors can be approximately associated with the number of photons. The theory of such a detection has been presented in Ref. [35]. The other technique assumes counting the number of photocurrent pulses appearing inside a measurement time window. Each pulse is interpreted as an evidence of a detected photon. However, this correspondence is an approximation since detectors cannot count any photons during their dead-time intervals after each pulse. Classical photocounting theory for this detection scheme has been earlier presented in Refs. [36–42].

In this paper we analyze the effect of realistic photon-number resolution on the photocounting statistics of the standard boson-sampling scheme. We demonstrate that a proper postselection can still be useful for sampling events from probabilities expressed via permanents appearing in Eq. (3). Unlike in the case of detection losses, these probabilities contain coefficients depending on specific characteristics of detectors. Our results are illustrated with arrays of on/off detectors and with single-photon detectors affected by dead time.

The rest of the paper is organized as follows. In Sec. II we present the general consideration of detectors with realistic photon-number resolution and discuss the corresponding photocounting statistics at the output of the boson-sampling scheme. In Sec. III our results are applied to arrays of on/off detectors. The effect of detector dead time is considered in Sec. IV. Summary and concluding remarks are given in Sec. V.

II. REALISTIC PHOTON-NUMBER RESOLUTION

In this Section, we address general issues related to boson-sampling schemes with realistic photon-number resolution. Let us consider photon-number distribution at the output ports of the interferometer assuming the usage of ideal PNR detectors [6, 7],

\[
P_{m_1 \ldots m_N} = \text{Tr} (\hat{\rho} |m_1\rangle \langle m_1| \otimes \cdots \otimes |m_N\rangle \langle m_N|).
\]

(4)

Here \( \hat{\rho} \) is the density operator of light modes at the interferometer outputs, and \( |m_i\rangle \langle m_i| \) are projectors on the Fock states. An important property of this distribution is that it has non-zero values only if the total number of detected photons,

\[
n = \sum_{i=1}^{N} m_i,
\]

(5)

is exactly the same as the number of injected photons, cf. Eq. (2).

As the next step, we look into realistic photon-number resolution. In the most general case, the photocounting distribution is given by

\[
\rho_{n_1 \ldots n_N} = \text{Tr} \left( \hat{\rho} \hat{\Pi}_{n_1} \otimes \cdots \otimes \hat{\Pi}_{n_N} \right).
\]

(6)

Here \( \hat{\Pi}_n \) is the positive operator-valued measure (POVM) [43] describing the photocounting procedure. This POVM can always be expanded by projectors on the Fock states [44] as

\[
\hat{\Pi}_n = \sum_{m=0}^{+\infty} P_{n|m} |m\rangle \langle m|,
\]

(7)

where the coefficients

\[
P_{n|m} = |\langle m|\hat{\Pi}_n|m\rangle|
\]

(8)

can be interpreted as the probability to get \( n \) counts, e.g. clicks or pulses, given \( m \) injected photons. In the following Sections we will obtain these coefficients for specific photocounting techniques. In practice, they can also be reconstructed from the measurement data given certified Fock-state sources. Substitution of Eq. (7) into Eq. (6) yields

\[
\rho_{n_1 \ldots n_N} = \sum_{i=1}^{N} P_{n_1|m_1} \cdots P_{n_N|m_N} P_{m_1 \ldots m_N},
\]

(9)

where sum is taken over the indices \( m_i \) obeying the condition (5).

The situation is drastically simplified if we assume that all counts of the considered detectors are solely related to absorbed signal photons, i.e. no dark counts, afterpulses, etc. take place. This yields \( P_{n|m} = 0 \) for \( m < n \), which
means that the number of photocounts does not exceed the number of photons. For the measurement outcomes obeying condition (5), Eq. (9) is reduced to

$$\rho_{n_1 \ldots n_N} = C_{n_1 \ldots n_N} P_{n_1 \ldots n_N}, \quad (10)$$

where

$$C_{n_1 \ldots n_N} = \prod_{i=1}^{N} P_{n_i | n_i} \quad (11)$$

are the correction coefficients between photocounting statistics of ideal and realistic detectors. Therefore, Eq. (10) directly links sampling probabilities of realistic and ideal PNR detectors. The latter are expressed in terms of permanents appearing in Eq. (3). The total efficiency of the corresponding postselection, i.e. the probability that the number of counts is equal to the number of injected photons, is given by sum of all $\rho_{n_1 \ldots n_N}$ satisfying condition (5). In the general case, its calculation involves values of permanents and thus cannot be evaluated with classical devices.

An elementary example is given by losses in the PNR detectors. The corresponding POVM reads

$$\Pi_n = \hat{F}_n(\eta) = \frac{\eta \hat{n}^n}{n!} \exp(-\eta \hat{n}); \quad (12)$$

where $\eta \in [0, 1]$ is the detection efficiency, $\hat{n}$ is the photon-number operator, and $\ldots$ : means normal ordering. In this case, $P_{n_i | n_i} = \eta^n$, see Refs. [45–47]. Hence, each correction coefficient is given by

$$C_{n_1 \ldots n_N} = \eta^n, \quad (13)$$

which coincides with the total postselection efficiency. A distinctive feature of this example is that the correction coefficients are the same for all events with $n_i$ obeying Eq. (5). Therefore, the photon-number statistics (3) can be obtained as the statistics of properly postselected events. This property does not hold for general realistic PNR detectors.

III. ARRAYS OF ON/OFF DETECTORS

As it has been already discussed in Introduction, arrays of on/off detectors [28–31] represent an efficient experimental tool enabling an approximate resolution between numbers of photons. The corresponding measurement layout, see Fig. 2, is based on splitting of a light beam on $K$ spatial modes and detecting each of them by an on/off detector. For such a scenario the measurement outcome is given by the number of triggered detectors. Equivalently, one can use a temporal separation of light modes by loops of optical fibers as it is discussed in Refs. [32–34]. In this Section, we consider measurement outcomes of boson-sampling schemes assuming that each output mode of the interferometer is analyzed by an array of on/off detectors or by an equal scheme with time-mode separation.

The POVM for this detection scheme [35] is given by

$$\Pi_n = \left( \frac{K}{n} \right) : (1 - e^{-\eta \hat{n}/K})^n e^{-\eta \hat{n}(K-n)/K}; \quad (14)$$

Here $K$ is the number of on/off detectors in the array, $n = 0 \ldots K$ is the number of triggered detectors, and $\eta$ is the detection efficiency. In this case, the probability to get $n_i$ clicks given $n_i$ photons, cf. Eq. (8), is directly obtained [35, 44] as

$$P_{n_i | n_i} = \left( \frac{\eta}{K} \right)^{n_i} \frac{K!}{(K-n_i)!}. \quad (15)$$

Substituting it into Eq. (11) we arrive at the corresponding correction coefficients,

$$C_{n_1 \ldots n_N} = \left( \frac{\eta}{K} \right)^{n_i} \prod_{i=1}^{N} \frac{K!}{(K-n_i)!}. \quad (16)$$

This expression includes the multiplier related to the detector losses given by Eq. (13). In addition, it explicitly depends on the set of numbers $n_i$, which is not the case for ordinary losses.

The correction coefficients (16) characterize the detector impact on the reconstructed photocounting statistics at the interferometer output. It establishes a link between data obtained with arrays of on/off detectors and photon-number statistics related to ideal PNR detectors. Its direct application in Eqs. (10) and (3) demonstrates how the probabilities of postselected events are related to permanents of matrices.

The dependence of the correction coefficients (16) on the number $K$ of on/off detectors in the array is shown in Fig. 3. Here we assume $\eta = 1$ without loss of generality. This coefficient is equal to unity if $n_i = 0, 1$ for all $i = 1 \ldots N$, which corresponds to the boson-sampling scheme with on/off detectors. In all other cases, the correction coefficient has non-trivial dependence on the set of $n_i$. The general rule is that these coefficients rapidly vanish while the maximal value of $n_i$ in the set increases.
When growing the number $K$ of on/off detectors in the array, the correction coefficients approach unity. However, in the most cases its contribution must be taken into account.

![Correction coefficients](image)

**FIG. 3.** The diagram shows the value of the correction coefficients (16) for a ten-port interferometer depending on the number $K$ of on/off detectors in the array. Without loss of generality, we assume $\eta = 1$.

### IV. DETECTION WITH FINITE DEAD TIME

In this Section we consider boson-sampling scheme with a method based on counting photons detected during a measurement time window $\tau_m$. This time characterizes a nonmonochromatic mode of light, the quantum state of which is analyzed. The light incident on a single-photon detector induces photocurrent pulses. Each pulse can be related to a single detected photon. The measurement outcome of such a scheme is the number of pulses appearing during the measurement time window $\tau_m$.

The main problem of the considered detection type is that the photons cannot be detected during a dead-time interval $\tau_d$ that follows each pulse. This results in non-ideal photon-number resolution. Another problem appears when the dead-time interval related to the last pulse overlaps with the next measurement time window. This leads to nonlinear memory effects in photocounting statistics. In order to avoid them and make time windows independent, detector input should be darkened after each measurement time window, see Fig. 4.

In the context of boson sampling, we address two scenarios related to this type of detection. First, we consider the case when the input light is a monochromatic continuous wave. It is split into parts, each of them related to a measurement time window. Next, we consider a more realistic scenario in which Fock-state sources irradiate nonmonochromatic modes of light. As an example we consider light modes in the form of exponential decay.

![Measurement windows](image)

**FIG. 4.** The idea of measurements with independent time windows is depicted. A light mode (solid line) arrives at the beginning of the time-window $\tau_m$. Detected photons induce photocurrent pulses (dashed line). After each pulse, detector cannot register photons during the dead time $\tau_d$ (shaded area). Each time window is followed by a time interval (hatched area) of darkened detector input exceeding $\tau_d$.

#### A. POVM for the detection with dead time

In this Subsection, we derive the POVM for detection with independent time windows. In the most general scenario, the light mode inside the measurement time interval $[0, \tau_m]$ is nonmonochromatic; see, e.g., Refs. [48, 49] for analysis of nonmonochromatic modes. The field intensity is a function of time, $I(t)$, normalized in the measurement time window by the detection efficiency $\eta$,

$$\int_0^{\tau_m} I(t) dt = \eta.$$  \hspace{1cm} (17)

The POVM should explicitly include this function.

Instead of using the POVM in the operator form, $\hat{\Pi}_n$, we will use its phase-space Q symbol defined as the average with coherent states $|\alpha\rangle$, cf. Refs. [50, 51],

$$\Pi_n(\alpha) = \langle \alpha | \hat{\Pi} | \alpha \rangle.$$  \hspace{1cm} (18)

The original operator form can be reconstructed from the Q symbol using the rule

$$\langle \alpha | : f (\hat{a}, \hat{a}^\dagger) : | \alpha \rangle = f (\alpha, \alpha^*),$$  \hspace{1cm} (19)

where $\hat{a}$ and $\hat{a}^\dagger$ are field annihilation and creation operators, respectively. The Q symbols of the POVM can be interpreted as photocounting distributions for the coherent state $|\alpha\rangle$.

For $n = 0$ the POVM element is given by $\hat{\Pi}_0 = \hat{\mathcal{F}}_0(\eta)$, cf. Eq. (12). The Q symbols of the POVM for $n \geq 1$ can be obtained via integration of the probability density $\pi_n(t|\alpha)$ to register $n$ pulses in the time moments $t = (t_1, \ldots, t_n)$ given the coherent state $|\alpha\rangle$,

$$\Pi_n(\alpha) = \int_{\mathcal{T}_n} d^n t \pi_n(t|\alpha).$$  \hspace{1cm} (20)

where $t \in \mathcal{T}_n$ obeys the condition $0 \leq t_1 \leq t_2 \ldots \leq t_n \leq \tau_m$. In order to derive the probability density $\pi_n(t|\alpha)$ we first concentrate on the example of $\pi_1(t|\alpha)$. For this
purpose we consider the infinitesimal probability to get
a pulse at the time \( t_1 \) during the time-interval \( dt_1 \),
\[
|\alpha|^2 I(t_1) \, dt_1 \exp \left( -|\alpha|^2 I(t_1) \, dt_1 \right) \rightarrow |\alpha|^2 I(t_1) \, dt_1. \tag{21}
\]
It should be multiplied with the probabilities to get no
pulses before and after the time moment \( t_1 \),
\[
\exp \left( -|\alpha|^2 \int_{0}^{t_1} dt \, I(t) \right) \tag{22}
\]
and
\[
\exp \left( -|\alpha|^2 \int_{t_1}^{t_1 + \tau_d} dt \, I(t) \theta(t - t_1 - \tau_d) \right), \tag{23}
\]
respectively. Here \( \theta(t - t_1 - \tau_d) \) is the Heaviside step
function, which describes the impossibility of appearing
pulses during the dead-time interval \( \tau_d \) after the time
moment \( t_1 \). Introducing the function
\[
\Xi_1(t_1) = \int_{0}^{t_1} dt \, I(t) + \int_{t_1}^{t_1 + \tau_d} dt \, I(t) \theta(t - t_1 - \tau_d), \tag{24}
\]
we derive the expression
\[
\pi_1(t_3|\alpha) = |\alpha|^2 I(t_1) \exp \left( -|\alpha|^2 \Xi_1(t_1) \right) \tag{25}
\]
for the probability distribution to get a pulse at the time
moment \( t_1 \) given the coherent state \(|\alpha\rangle\).

The probability density \( \pi_n(\mathbf{t}|\alpha) \) to register \( n \geq 2 \)
pulses in the time moments \( \mathbf{t} = (t_1, \ldots, t_n) \) given the
coherent state \(|\alpha\rangle\) is constructed in a similar way. It is
composed of the following constituents:

\begin{itemize}
  \item The probability density of registering a pulse in the
time moment \( t_1 \) given by \(|\alpha|^2 I(t_1)\);
  \item The probability density of registering pulses at the
time moments \( t_i \) for \( i = 2 \ldots n \) given by \(|\alpha|^2 I(t_i)\);
  \item The zero-probability for the \( i \)th pulses to appear af-
after the \((i-1)\)th pulses during the dead time \( \tau_d \)
described by the Heaviside step function \( \theta(t_i - t_{i-1} - \tau_d) \);
  \item The probability density of registering no pho-
tons from the time moment \( t = 0 \) and up to
the first pulse in the time moment \( t_1 \) given by
\[
\exp \left( -|\alpha|^2 \int_{0}^{t_1} dt \, I(t) \right); \tag{26}
\]
  \item The probability of registering no photons in the
time domain between the \( i \)th and \((i+1)\)th pulses
given by \( \exp \left( -|\alpha|^2 \int_{t_i}^{t_{i+1}} dt \, I(t) \theta(t - t_i - \tau_d) \right) \),
where the dead time is taken into account;
  \item The probability of registering no photons in the
time domains between the \( n \)th pulse and the time moment
\( t = \tau_m \) given by
\[
\exp \left( -|\alpha|^2 \int_{t_n}^{\tau_m} dt \, I(t) \theta(t - t_n - \tau_d) \right), \tag{27}
\]
where the dead time is taken into account.
\end{itemize}

Combining all these factors we arrive at the probability
density
\[
\pi_n(\mathbf{t}|\alpha) = |\alpha|^2 \mathcal{I}_n(\mathbf{t}) \exp \left[ -|\alpha|^2 \Xi_n(\mathbf{t}) \right], \tag{28}
\]
where
\[
\mathcal{I}_n(\mathbf{t}) = I(t_1) \prod_{i=2}^{n} I(t_i) \theta(t_i - t_{i-1} - \tau_d), \tag{29}
\]
and
\[
\Xi_n(\mathbf{t}) = \int_{0}^{t_1} dt \, I(t) + \sum_{i=1}^{n-1} \int_{t_i}^{t_{i+1}} dt \, I(t) \theta(t - t_i - \tau_d)
+ \int_{t_n}^{\tau_m} dt \, I(t) \theta(t - t_n - \tau_d). \tag{30}
\]
Equation (26) can also be applied for \( n = 1 \), where \( \Xi_1(t_1) \)
is given by Eqs. (24) and \( \mathcal{I}(t_1) = I(t_1) \).

The \( Q \) symbol of the POVM is obtained via integration
as it is given by Eq. (20). Applying the rule (19), we
obtain the POVM
\[
\hat{\Pi}_n = \hat{n} \int_{\mathcal{S}_{\tau_m}} d^n \mathbf{t} \, \mathcal{I}_n(\mathbf{t}) \exp \left[ -\hat{n} \Xi_n(\mathbf{t}) \right]: \tag{31}
\]
for \( n = 1 \ldots K + 1 \). Here \( K = [\tau_m/\tau_d] \) is the number of
whole dead time intervals fitting inside the measurement
time window.

\section*{B. Monochromatic light}

Let us consider the case of a nonmonochromatic light
mode. In this case, the intensity is given by
\[
I(t) = \frac{\eta}{\tau_m}. \tag{32}
\]
Substituting this form of the intensity in Eq. (29) and
performing integration, we arrive at the POVM
\[
\hat{\Pi}_0 = \hat{F}_0(\eta), \tag{33}
\]
\[
\hat{\Pi}_n = \sum_{k=0}^{n} \hat{F}_k(\eta \eta_n) - \sum_{k=0}^{n-1} \hat{F}_k(\eta \eta_{n-1}) \tag{34}
\]
for \( n = 1 \ldots K \),
\[
\hat{\Pi}_{K+1} = 1 - \sum_{k=0}^{K} \hat{F}_k(\eta \eta_K). \tag{35}
\]
Here $\hat{F}_k(\eta)$ is the POVM of the ideal PNR detectors with losses, cf. Eq. (12), and

$$\eta_n = \frac{\tau_m - n\tau_d}{\tau_m}$$  \hspace{1cm} (34)

is the adjusting efficiency. These equations can be considered as a straightforward generalization of the corresponding results in the classical photodetection theory [36–42].

In the considered scenario, the probability to get $n_i$ clicks given $n_i$ photons is obtained by substituting Eqs. (31), (32), and (33) into Eq. (8),

$$P_{n_i|n_i} = (\eta \eta_{n_i-1})^{n_i}.$$  \hspace{1cm} (35)

Thus, the corresponding correction coefficients are given by

$$C_{n_1...n_N} = \eta^n \prod_{i=1}^{N} \eta_{n_i-1}.$$  \hspace{1cm} (36)

Similar to the case of array detectors, this expression includes the multiplier related to the detection efficiency, cf. Eq. (13). At the same time, the correction coefficients are characterized by a non-trivial dependence on the set of numbers $n_i$.

The dependence of the correction coefficients on the dead time is shown in Fig. 5, where we also assume $\eta = 1$. It can be seen that the correction coefficient is equal to unity in two cases: when the dead time tends to zero and when each detector registers not more than one pulse, i.e. $n_i = 0, 1$ for $i = 1 \ldots N$. The correction coefficients vanish when the dead time and the maximal number $n_i$ increase. Moreover, the larger the maximal number $n_i$, the faster the coefficient $C_{n_1...n_N}$ as a function of the dead time tends to zero. When the dead time $\tau_d$ approaches the measurement time $\tau_m$, all coefficients except for those with $n_i = 0, 1$ tend to zero. For small values of the correction coefficients, the sampling can be problematic since such events are rare.

C. Nonmonochromatic modes of light

Let us consider a scenario with identical nonmonochromatic field modes at the interferometer inputs. For example, it can be chosen in the form of exponential decay,

$$I(t) = \frac{\eta \gamma}{\tau_m [1 - \exp (-\gamma)]} \exp \left( -\gamma \frac{t}{\tau_m} \right),$$  \hspace{1cm} (37)

where $\gamma > 0$ is the decay rate. Such a mode can represent a pulse escaping from a leaky cavity, see e.g. [52–54]. For $\gamma \to 0$, one gets $I(t) = 1/\tau_m$ that corresponds to the monochromatic field. Here we consider a simple scenario for which all nonmonochromatic modes at the interferometer inputs are identical and, consequently, can be considered as interfered. In a different case, when the modes are non-identical, the photons are partially distinguishable and interference is lost; see Refs. [21–26].

Substitution of Eq. (29) into Eq. (8) for $n = m = n_i$ yields the probability to get $n_i$ pulses given $n_i$ photons,

$$P_{n_i|n_i} = n_i! \int_{T_{ni}} d^m t^n [\mathcal{I}_{n_i}(t)].$$  \hspace{1cm} (38)

Applying the explicit form (37) of $I(t)$, we arrive at the analytical form of this probability,

$$P_{n_i|n_i} = \left[ \frac{\sinh (\gamma \eta_{n_i-1}/2)}{\sinh (\gamma/2)} \right]^{n_i},$$  \hspace{1cm} (39)

where $\eta_n$ is given by Eq. (34). For $\gamma \to 0$ this expression tends to Eq. (35).

The correction coefficients (11) for the scenario with the nonmonochromatic modes in the form of exponential decay are given by

$$C_{n_1..n_N} = \eta^n \prod_{i=1}^{N} \left[ \frac{\sinh (\gamma \eta_{n_i-1}/2)}{\sinh (\gamma/2)} \right]^{n_i}.$$  \hspace{1cm} (40)

Their dependence on the dead time is shown in Fig. 6. In general, this dependence resembles properties of the same function in the scenario of monochromatic light, cf. Fig. 5. However, in the considered case, it vanishes much faster when the dead time increases.

V. SUMMARY AND CONCLUSIONS

To summarize, we have theoretically considered the standard boson-sampling schemes assuming realistic photon-number resolution. Our analysis is based on the derived linear equations linking photocounting distributions for such detectors to the proper photon-number
distributions. The latter is expressed in terms of permanents, the values of which cannot be effectively obtained with classical procedures. The equation coefficients are given in terms of detector characteristics. They can be calculated theoretically via the Fock-state representation of the POVM or, alternatively, they can be reconstructed in experiments.

If all photocounts in the considered detection schemes are solely related to the signal radiation—i.e. there are no dark counts, afterpulses, etc.—the analysis is significantly simplified. In this case, we can restrict ourselves by the events with equal numbers of photocounts and injected photons. The corresponding probabilities are the probabilities for ideal PNR detectors scaled by the correction coefficients. Therefore, the proper postselection of such events enables us to sample the probabilities proportional to the ones appearing in the boson-sampling problem with ideal PNR detectors.

Our theoretical consideration is illustrated with examples of two widely-used measurement schemes. First, we have dealt with arrays of on/off detectors. Next, we have considered photocounting with the dead time of detectors. For both measurement techniques, the correction coefficients take very small values for the cases with a large number of counts at least in one output port of the interferometer. This poses a restriction on the considered method since the corresponding sampling events will be rare.

Our results can be further generalized, which enables us to take into consideration other practical issues. For example, partial distinguishability of photons can be easily incorporated into our consideration. We believe that our results will be useful for experimental implementations of boson sampling schemes.

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