A CHARACTERIZATION OF SIMPLICIAL ORIENTED GEOMETRIES AS GROUPOIDS WITH ROOT SYSTEMS

MATTHEW DYER AND WEIJIA WANG

Abstract. This paper shows that simplicial oriented geometries can be characterized as groupoids with root systems having certain favorable properties, as conjectured by the first author. The proof first translates Handa’s characterization of oriented matroids, as acycloids which remain acycloids under iterated elementary contractions, into the language of groupoids with root systems, then establishes favorable lattice theoretic properties of a generalization of a construction which Brink and Howlett used in their study of normalizers of parabolic subgroups of Coxeter groups and uses Björner-Edelman-Ziegler’s lattice theoretic characterization of simplicial oriented geometries amongst oriented geometries.

1. Introduction

Groupoids with root systems have received attention recently in various mathematical contexts. When studying the normalizer of a parabolic subgroup of a Coxeter group, Brink and Howlett [7] considered certain groupoids which have presentations by generators and relations resembling those of a Coxeter group (see [5], [20] and [2] as standard references on Coxeter groups). There are also closely related notions of Weyl groupoid and Coxeter groupoid developed by Cuntz, Heckenberger and Yamane ([23], [25], [10], [12], [11], [24], [13]), which were initially studied for their connections with certain Hopf algebras.

Motivated in part by longstanding conjectures (surveyed in [17]) involving a conjectural ortholattice completion of weak order of an (infinite) Coxeter group, the first author defined in [15] and [16] the notions of protorootoid and rootoid by abstracting lattice-theoretic properties of weak order on Coxeter groups to a setting associated to groupoids with root systems. This provides in particular a unified framework in which Weyl and Coxeter groupoids, and the groupoids of Brink and Howlett, can be studied, along with other mathematical structures including oriented matroids. One of the advantages of this framework is that various classes of rootoids are stable under natural categorical constructions, including formation of categorical limits and functor rootoids. In particular, the construction of Brink and Howlett can be generalized and extended to the context of rootoids.

In this paper, we shall find it more convenient to work with the concrete notion of signed groupoid set rather than with the more abstract notion of protorootoids (the relation between these two notions is closely analogous to that between Boolean algebras of sets and Boolean algebras). A signed groupoid set is a triple \((G, \Phi, \Phi^+)\) where \(G\) is a groupoid and \(\Phi = \{\phi_a\}\) (called the root system) is a family of definitely involuted sets (i.e. sets with an involution map and a chosen “positive” part \(\phi_a^+\)) indexed by the objects of \(G\) such that the groupoid acts on \(\Phi\) with action maps \(a G_b \times \phi_b \rightarrow \phi_a\) where \(a G_b := \text{Hom}_G(b, a)\). Action by groupoid elements is required...
to commute with involution maps but is not required to preserve positive elements. A Coxeter group, viewed as a groupoid with one object, and with its standard root system is a typical example.

Oriented matroids are combinatorial structures which abstract basic convexity properties of sets of vectors in real vector spaces, or, from a dual point of view, they abstract real, finite, essential hyperplane arrangements. See [3] as a general reference. By a simplicial oriented geometry, we mean a reduced simplicial oriented matroid without parallel elements; these abstract some features of real, finite, central, simplicial hyperplane arrangements. The study of hyperplane arrangements and especially simplicial arrangements has been important in parts of algebra, algebraic geometry, combinatorics, representation theory and topology over the last several decades; amongst fundamental work in this area we mention [14] and [1].

Any reduced oriented matroid can be naturally given the structure of a connected and simply connected signed groupoid set. The objects of the groupoid are in bijection with the hemispaces of the oriented matroid, which correspond to chambers of a hyperplane arrangement, and the root system at a given object is precisely the original oriented matroid viewed as an involuted set (this corresponds to the set of unit normal vectors to the hyperplanes, with involution given by multiplication by $-1$). The positive roots at a given object are those elements in the hemispaces represented by that object, corresponding to roots with positive inner product with a vector in the corresponding open chamber. Each groupoid morphism acts trivially on the set of roots, but since the positive roots at its domain and codomain may differ, the morphism may change the signs of certain roots.

This work focuses on the problem of finding the properties that a signed groupoid set $(G, \Phi, \Phi^+)$ needs to have so that it comes from a simplicial oriented geometry (i.e. a reduced simplicial oriented matroid without parallel elements). We show that, as conjectured in [16], a simplicial oriented geometry can be characterized as a finite, connected, simply connected signed groupoid set which is real, principal and complete. Here, “finite”, “connected” and simply connected” have standard meanings. “Real” means every root has its sign changed by some morphism. “Principal” means that the groupoid is generated by “simple morphisms” which each make a single positive root negative, and that the length of a groupoid morphism with respect to the simple generators is equal to the cardinality of its inversion set (the set of positive roots made negative by the element’s inverse). Finally, “complete” means that the weak order at each object, given by inclusion of inversion sets of morphisms with that object as codomain, is a complete lattice. (These notions will be defined precisely in Section 2.18.) The fact that a simplicial oriented geometry gives rise to such a signed groupoid set is essentially a reformulation of facts known from [3]. The key ingredients used to prove the other direction of the correspondence are (1) Handa’s characterization of oriented matroids and (2) a generalized Brink-Howlett construction described in this paper. Handa’s characterization of oriented matroids uses the notion of hemispaces and contraction.

Starting with a weaker notion called acycloid, one can perform contraction operations on it. Handa’s theorem states that if after all sequences of elementary contractions, one still gets an acycloid, then the original acycloid is in fact an oriented matroid. On the other hand when an acycloid is viewed as a signed groupoid set, the non-trivial elementary contractions (that is, those at non-loops) closely correspond to taking a single, special connected component of the signed groupoid.
set obtained by applying the generalized Brink-Howlett construction. Our main theorem in Section 3 asserts that if the original signed groupoid set is finite, connected, simply connected, preprincipal and complete, then taking any component, not just a special one corresponding to an elementary contraction of the associated acycloid, produces another signed groupoid set having exactly the same properties. Since both the original and constructed signed groupoid set have the structure of an acycloid, Handa’s theorem ensures that the original signed groupoid set comes from an oriented matroid, which can be seen to be simplicial on lattice-theoretic grounds (lattice theoretic arguments are also needed in establishing that Handa’s result is applicable).

As a significant corollary, the simplicial oriented matroids (by which we mean those whose associated simple oriented matroid is a simplicial oriented geometry) are preserved by a more general construction than contraction, which we call hypercontraction, corresponding to taking an arbitrary component after applying the generalized Brink-Howlett construction. Hypercontraction is defined for arbitrary signed groupoid sets and is interesting for many other classes of them besides that of simplicial oriented matroids. We leave many natural questions about it open.

Several of the results we prove in Sections 2 and 3 of this paper are consequences of more general ones in the theory of rootoids. For simplicity and brevity, we prove in this paper only a little more than needed to establish the connection of these more general facts with oriented matroids, and don’t discuss in detail such more general results in relation to the ones here. We conclude the paper with some remarks and open problems related directly to the content of this paper. More complete discussion of further developments can be found in [15]–[16] and their planned sequels.

2. Preliminaries

2.1. Oriented Matroids. There are many equivalent axioms for oriented matroids (see [4]). We emphasize their characterization by closure operators as given in [20] and [8], using a formalization in terms of involuted sets instead of signed sets (see Remark 2.10). Full details on oriented matroids can’t be included here, and the reader unfamiliar with them may find it helpful to look at Example 1 below for motivation while reading our discussion.

2.2. By an involuted set, we mean a pair \((E, *)\) where \(E\) is a set and \(*\) is an involution of \(E\); that is, \(*: E \to E\) is a function, denoted by \(x \mapsto x^*\) for \(x \in E\), satisfying \(x^{**} = x\). We say that \(E\) is strictly involuted if the involution is fixed point free (that is, \(x^* \neq x\) for all \(x \in E\)).

Recall that a closure operator on a set \(E\) is a function \(c: \mathcal{P}(E) \to \mathcal{P}(E)\), (where \(\mathcal{P}(E)\) is the power set of \(E\)) such that (1) \(A \subseteq c(A)\) if \(A \subseteq E\), (2) \(c(A) \subseteq c(B)\) if \(A \subseteq B \subseteq E\) and (3) \(c(c(A)) = c(A)\) if \(A \subseteq E\). Subsets \(F\) of \(E\) which satisfy \(C(F) = F\) are said to be \(c\)-closed or just closed. One easily checks that the intersection of a family of closed sets is closed. We say \(c\) is reduced if \(c(\emptyset) = \emptyset\). Also, \(c\) is called finitary or of finite type if whenever \(A \subseteq E\) and \(x \in c(A)\) there exists a finite set \(B \subseteq A\) such that \(x \in c(B)\). Given a (finitary) closure operator \(c\) on \(E\), for any disjoint \(A, B \subseteq E\), one has a (finitary) closure operator \(c_{A,B}\) on \(B\) given by \(c_{A,B}(X) = c(A \cup X) \cap B\), for \(X \subseteq B\). We call \(c_{A,B}\) a contraction of \(c\) (by \(A\)) if \(B = E \setminus A\), a restriction of \(c\) (to \(B\)) if \(A = \emptyset\) and a minor of \(c\) in general.
2.3. An oriented matroid is a system $(E, *, cx)$ where $E$ is a set with a map $*: E \to E$ and a map $cx: P(E) \to P(E)$ such that

- $(M1)$ $(E, *)$ is a strictly involution set,
- $(M2)$ $cx$ is a closure operator on $E$,
- $(M3)$ $cx$ is finitary,
- $(M4)$ $cx(X)^* = cx(X^*)$ for all $X \subseteq E$,
- $(M5)$ if $X \subseteq E$ and $x \in cx(X \cup \{x^*\})$, then $x \in cx(X)$,
- $(M6)$ if $X \subseteq E$ and $x, y \in E$ with $x \in cx(X \cup \{y^*\})$ but $x \notin cx(X)$, then $y \in cx((X \setminus \{y\}) \cup \{x^*\})$.

We remark at once that if $A$ and $B$ are disjoint subsets of $E$ satisfying $A = A^*$ and $B = B^*$, then $(B, *, cx_{A,B})$ is an oriented matroid, where $*$ is the restriction of $*$ to an involution on $B$ and $cx_{A,B}$ is the minor of $cx$ from $A$ and $B$.

When the closure operator and the involution map are understood, we usually denote an oriented matroid by $E$ for simplicity. We recall some facts about an alternative description of oriented matroids using the concept of their circuits (see for instance [20] and [8]). A circuit of $(E, *, cx)$ is a minimal nonempty subset $X$ of $E$ such that $X^* \subseteq cx(X)$. A circuit is called improper if it is of the form $\{e, e^*\}$ for some $e \in E$. A circuit $C$ which is not improper is said to be proper, and satisfies $C \cap C^* = \emptyset$. The set of circuits (together with $*$ map) determines the oriented matroid by requiring

$$cx(X) = \{e \in E \mid \text{there exists } U \subseteq X \text{ such that } U \cup \{e^*\} \text{ is a circuit}\}.$$ 

Oriented matroids may be axiomatized in terms of their circuits and involution $*$.

2.4. We say a set $F \subseteq E$ is closed if $F$ is $cx$-closed. The elements of the closed set $L := cx(\emptyset) = L^*$ are called loops of $E$. We say $E$ is reduced if $cx$ is reduced (that is, $E$ has no loops, or $L = \emptyset$). We say $E$ is simple if it is reduced and all singleton subsets of $E$ are closed. An oriented matroid $(E, *, cx)$ has an associated reduced oriented matroid $(E_0, *, cx_0)$ where $E_0 := E \setminus L \subseteq E$, $e^* := e^*$ for $e \in E_0$ and $cx_0(X) := cx(X) \setminus L$ for $X \subseteq E_0$. There is also a simple oriented matroid $(E_1, *, cx_1)$ associated to $(E, *, cx)$, defined as follows. There is an equivalence relation $\sim$ on $E_0$ such that $e \sim f \iff cx_0(e) = cx_0(f)$, for $e, f \in E_0$. Let $E_0 := E_0/\sim$, $\hat{f} := [e]$ for $e \in E_0$, and $cx_1(X) = cx(\bigcup_{x \in X} x)/\sim$ for $X \subseteq E_1$, where $[e]$ denotes the $\sim$-equivalence class of $e \in E_0$. (In general, for any equivalence relation $\equiv$ on a set $X$, we write $Y/\equiv$ for the set of $\equiv$-equivalence classes which are contained in $Y$, for any subset $Y$ of $X$ which is a union of $\equiv$-equivalence classes.)

Using the relations between $E$ and $E_0$, we extend some terminology used in [S] from reduced oriented matroids to general oriented matroids. A closed set $F$ is called a sharp if $F \cap F^* = L$. A hemispace of $E$ is a closed set $H$ such that $E = H^* \cup H$ and $H \cap H^* = L$. Hemispaces are the same as maximal sharps (that is, inclusion maximal elements of the set of sharps) and any sharp, such as $L$, is contained in a hemispace; see [S] Theorem 7, Proposition 8. We also define a tope of $E$ to be a set of the form $H \setminus L$ for some hemispace $H$ of $E$. (Therefore if $L = E$.

\footnote{A largely equivalent theory may be developed in which “strictly” is dropped from (M1) (see [S]), but many statements and definitions then become more cumbersome (for example, in the definition of proper circuit, one then has to add a condition that $e \neq e^*$), while only a few become more natural (see footnote to Example 1). Similarly, the definitions of (pre)acycloids and signed groupoid sets given later may be modified to allow non-strictly involuted ground sets.}

\footnote{The conclusion $y \in cx((X \setminus \{y\}) \cup \{x^*\})$ in (M6) is routinely misstated in the literature as $y \in cx((X \cup \{x^*\}) \setminus \{y\})$; that version would imply that $cx(A) = E$ for all $A \subseteq E$.}
there exists a unique tope, i.e. \( \emptyset \). Using the circuit axioms for oriented matroids in \([3]\) (or in \([2]\)), it can be shown that the circuits are the inclusion-wise minimal elements in \( \mathcal{P}(E) \) which are contained in no tope (\( \mathbb{B} \) Theorem 1.1]), and the topes are the inclusion-wise maximal elements in \( \mathcal{P}(E) \) which contain no circuit.

**Lemma 2.5.** Let \( M = (E, \ast, cx) \) be a reduced oriented matroid. Suppose that \( C \) is a circuit of \( M \). Let \( x \in C \). Then \( (C \setminus \{x\}) \cup \{x^*\} \) contains no circuit.

**Proof.** Note that since \( M \) is reduced, \( C \setminus \{x\} \) is not empty. Suppose to the contrary, \( C' \) is a circuit contained in \((C \setminus \{x\}) \cup \{x^*\} \). By the circuit axioms of an oriented matroid (see \([3]\) Section 6 axiom (C1)), a circuit cannot be properly contained in another circuit. So \( C' \) cannot be contained in \( C \setminus \{x\} \). Therefore \( C' = D \cup \{x^*\} \) where \( D \subseteq C \setminus \{x\} \). But the circuit axioms of an oriented matroid ensures that there exists a circuit \( Z \) contained in \((C \setminus \{x\}) \cup D = C \setminus \{x\} \) (see \([3]\) Section 6 axiom (C3)). This is a contradiction. \( \square \)

We record the following explicit description of oriented matroid closure operators in terms of hemispaces.

**Theorem 2.6.** Let \( M = (E, \ast, cx) \) be an oriented matroid with the set \( \mathfrak{H} \) of hemispaces. (Note \( \mathfrak{H} = \{T \cup L \mid T \in \mathfrak{F} \} \) where \( \mathfrak{F} \) is the set of tops of \( M \) and \( L = cx(\emptyset) \) is the set of loops.) For \( H \in \mathfrak{H} \), let \( cx_H \) denote the restriction of \( cx \) to a closure operator on \( H \). That is, for \( X \subseteq H \), \( cx_H(X) = cx(X) \subseteq H \).

(a) For \( H \in \mathfrak{H} \) and \( X \subseteq H \), we have \( cx_H(X) = \bigcap_{K \supseteq X} K \).

(b) The closure operator \( cx \) is given by \( cx(X) = \bigcup_{H \in \mathfrak{H}} cx_H(X \cap H) \).

**Proof.** Let \( L := cx(\emptyset) \) be the set of loops. Note \( L \) is contained in each closed set, including each hemispace. By considering the associated reduced oriented matroid, one easily reduces to the case \( L = \emptyset \). So henceforth we assume \( M \) is reduced.

(a) Let \( X \subseteq H \in \mathfrak{H} \). Then \( cx_H(X) = cx(X) \subseteq \bigcap_{K \supseteq X} K \) since the right hand side is closed in \( E \) and contains \( X \). For the reverse inclusion, let \( X \subseteq H \) and \( x \in E \). Assume that \( X \) and \( x \) satisfy the property that for any hemispace \( K \), if \( K \supseteq X \) then \( x \) must also be in \( K \). Then we claim that there exists some \( V \subseteq X \) such that \( V \cup \{x^*\} \) is a circuit. Otherwise no subset of \( X \cup \{x^*\} \) is a circuit. So \( X \cup \{x^*\} \) must be contained in some hemispace \( K \) by the discussion of circuits and hemispaces above. But \( x \in K \) also, which contradicts that \( K \cap K^* = \emptyset \). Hence \( x \in cx(X) = cx_H(X) \), and (a) is proved.

(b) Let \( X \subseteq E \) be arbitrary. Then clearly \( \bigcup_{H \in \mathfrak{H}} cx_H(X \cap H) \subseteq cx(X) \). Now take \( x \in cx(X) \). Then there exists \( X' \subseteq X \) such that \( X' \cup \{x^*\} \) is a circuit. This implies that \( X' \cup \{x\} \) is contained in a hemispace \( H \) by Lemma 2.5 and the discussion of circuits and hemispaces above. So \( x \in cx(X') \subseteq cx(X \cap H) = cx_H(X \cap H) \), so we are done. \( \square \)

2.7. An anti-exchange (or convex) closure operator on a set \( U \) is a closure operator \( c \) on \( U \) such that for \( p, q \in U \) and \( X \subseteq U \), if \( q \in c(X \cup \{p\}) \) but \( q \notin c(X) \), then \( p \notin c(X \cup \{q\}) \). For any disjoint \( A, B \subseteq U \), the minor \( c_{A,B} \) is then an anti-exchange closure on \( B \). Let us say that a subset \( P \) of a poset \( Q \) is saturated in \( Q \) if every maximal chain of \( P \) (regarded as subposet of \( Q \)) is a maximal chain of \( Q \). If \( Q \) is the Boolean poset of all subsets of a set \( U \), then the maximal chains of \( Q \) are in natural bijection with total orders of \( U \), by a map sending each total order to its set of downsets. In that case, for any saturated subposet \( P \) of \( Q \), the map
$X \mapsto \bigcap_{Y \in P, Y \supseteq X} Y$ is an anti-exchange closure operator on $U$. On the other hand, it can be shown that if $c$ is a reduced, finitary anti-exchange closure on $U$, then the set $P$ of $c$-closed subsets of $U$ is saturated in the power set $Q$ of $U$ and the anti-exchange closure on $U$ from $P$ is just $c$.

Suppose that $M$ is a simple oriented matroid and that $H \in \mathfrak{H}$ is a hemispace. It can be shown that the poset $P = \{H \cap K \mid K \in \mathfrak{H}\}$ is saturated in the power set $Q$ of $H$. By Theorem 2.6, the associated anti-exchange closure operator on $H$ is $cx_H$ (which is also finitary). All these facts are well known for finite $E$ (see [19], [18] and [3]).

2.8. Recall (see for example [29]) that a (finitary, possibly infinite, unoriented) matroid is a pair $(F, c)$ where $F$ is a set and $c$ is a finitary closure operator on $F$ such that the following exchange condition holds: if $X \subseteq F$ and $x, y \in F$ satisfy $x \in c(X \cup \{y\}) \setminus c(X)$, then $y \in c(X \cup \{x\})$. For any disjoint sets $A, B \subseteq F$, the minor $(B, c_{A,B})$ is also a matroid. The rank of $(F, c)$ is the (well-defined) cardinality of any subset $B$ of $F$ which is inclusion minimal subject to $c(B) = F$ (such a set is called a basis of $F$).

For each oriented matroid $(E, *, cx)$ one can associate a matroid $(E, \overline{cx})$ to it where $\overline{cx}(X) = cx(X \cup X^*)$ for $X \subseteq E$.

2.9. In this subsection, we assume the oriented matroid is finite, i.e. $|E| < \infty$. An oriented geometry is a finite, simple oriented matroid. Let $(E, *, cx)$ be an oriented geometry and $H$ be a hemispace. There exists a unique minimal subset of $H$, called the set of extreme elements of $H$, and denoted by $\text{ex}(H)$, such that $cx(\text{ex}(H)) = H$ (see [3] Section 6); it follows from the fact $cx_H$ is anti-exchange). If $|\text{ex}(H)|$ is equal to the rank of the oriented matroid $(E, \overline{cx})$, then $H$ is called simplicial. If all hemispaces are simplicial then we call the oriented geometry a simplicial oriented geometry. By a simplicial oriented matroid, we mean an oriented matroid for which the associated simple oriented matroid is a simplicial oriented geometry.

**Example 1.** Let $V$ be a real vector space. For any $A \subseteq V$, define

$$\text{cone}(A) := \left\{ \sum_{i \in I} k_i v_i \mid v_i \in A, k_i \in \mathbb{R}_{\geq 0}, |I| < \infty \right\}$$

where by convention the empty sum has value $0 \in V$. Thus, $\text{cone}(A)$ is the pointed convex cone in $V$ spanned by $A$. Say that a subset $A$ of $V$ is positively independent if $\sum_{i \in I} k_i v_i = 0$ with $v_i \in A$, all $k_i \in \mathbb{R}_{\geq 0}$ and $|I| < \infty$ implies $k_i = 0$ for all $i \in I$.

For any subset $E = -E$ of $V \setminus \{0\}$, there is a reduced oriented matroid $M_E := (E, *, cx_E)$ with $cx_E(A) := \text{cone}(A) \cap E$ and $x^* = -x$ for $x \in E$. (Examples of non-reduced oriented matroids arise as minors of these.) The oriented matroid $M_E$ is simple if and only if for $\alpha \in E$ and $0 \neq c \in \mathbb{R}$, one has $c\alpha \in E$ if and only if $c = \pm1$.

For $X \subseteq E$, we have $\overline{cx}(X) = \text{span}(X) \cap E$, where span denotes linear span. The hemispaces of $M_E \setminus \{0\}$ are the sets of positive elements of vector space total orderings of $V$, and the hemispaces of $M_E$ are the intersections of such hemispaces with $E$. The circuits of $M_E$ are the minimal non-empty subsets of $E$ which are not positively independent. Suppose for example that the hemispace $H$ of $M_E$ is contained in some affine subspace $U$ (that is, a translate of a codimension one linear

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3 If we had not required $E$ to be strictly involuted, we could have allowed more generally $E = -E \subseteq V$ and all such oriented matroids would be minors of the one with $E = V$. 


subspace of $V$) such that $0 \not\in U$. Then the convex closure operator $\text{cx}_H$ from $M_E$ is given by $\text{cx}_H(X) = \text{conv}(X) \cap H$ for $X \subseteq H$, where $\text{conv}(X)$ denotes the convex hull of a subset of $U$.

Now suppose that $M_E$ is an oriented geometry. Let $V_0 = \text{span}(E)$. This is a finite-dimensional real vector space. Choose a positive definite inner product $(-\mid-): V_0 \times V_0 \to \mathbb{R}$ on $V_0$. Consider the set $A$ of linear hyperplanes of $V_0$ orthogonal to the elements of $E$. Then $A$ is a real, finite, essential, linear hyperplane arrangement in $V_0$ (essential means that $\bigcap_{H \in A} H = \{0\}$). The connected components $V \setminus \bigcap_{H \in A} H$ are called (open) chambers. Every chamber $C$ determines a hemispace $H$ of $E$, consisting of all $\alpha \in E$ which have positive inner product with some point (or equivalently, all points) of $C$, and every hemispace so arises. In fact, consider a hemispace $H$. It is positively independent and spans $V_0$, and $\text{cx}(H)$ is a set of representatives of the extreme rays of the polyhedral cone $K = \text{cone}(H)$ spanned by $H$. Then the dual cone $K^\vee := \{ \alpha \in V_0 \mid (\alpha \mid K) \subseteq \mathbb{R}_{\geq 0} \}$ also spans $V_0$ and its interior is a chamber yielding the hemispace $H$. Moreover, the facets (codimension one faces) of $K^\vee$ are in bijection with the elements of $\text{cx}(H)$, so $K^\vee$ is simplicial if and only if $|\text{cx}(H)| = \dim(V_0)$, which is the rank of $E$. Thus, the hyperplane arrangement is simplicial (that is, all its chambers are open simplicial cones) if and only if $E$ is a simplicial oriented geometry.

Oriented matroids and geometries arise in many other, quite different ways, and those arising as above will be said in this paper to be realizable. Not every oriented geometry is realizable; see [4] for more precise discussion of realizability and similar models (for instance, pseudosphere arrangements) which can be used to describe finite oriented matroids in general.

Remark 2.10. By definition, a signed subset (or signed vector) of a set $E'$ is a function $f: E' \to \{+ , 0, - \}$. Associated to $E'$, one has a set $E := E' \times \{\pm\}$ with a fixed point free involution $*$ defined by $(e, \pm)^* = (e, \mp)$ for $e \in E'$. Then the signed subsets of $E'$ correspond bijectively to the subsets $X$ of $E$ satisfying $X \cap X^* = \emptyset$, by a standard correspondence which attaches to a signed subset $f$ of $E'$ the subset $\{(e, f(e)) \in E \mid e \in E', f(e) \neq 0\}$. In much of the literature, oriented matroids, acycloids etc are defined by specifying certain collections of signed subsets of a set $E'$ (circuits, vectors, covectors, topes etc). In this paper, we always regard the sets of circuits, vectors, covectors, topes etc as subsets of $\{X \subseteq E \mid X \cap X^* = \emptyset\}$ for a strictly involuted set $E$, using the above standard correspondence. In particular, the proper circuits and topes of an oriented matroid $E$ in the framework of strictly involuted sets as considered above correspond to the circuits and topes, respectively, for oriented matroids in the framework of signed sets.

We state the following result more generally than needed in this paper, for application elsewhere.

Proposition 2.11. Let $M = (E, *, \text{cx})$ be an oriented matroid, and $F$ be a subset of $E$ with $F = F^*$. Let $M_F$ denote the restriction of $M$ to $F$. Let $L := \text{cx}(\emptyset)$ be the set of loops of $M$, so $L \cap F$ is the set of loops of $M_F$. Then the hemispaces of $M_F$ are the intersections of $F$ with the hemispaces of $M$. More precisely, the following conditions are equivalent for $A \subseteq F$:

(i) $A$ is a hemispace of $F$ (that is, $F = A \cup A^*$, $A \cap A^* = L \cap F$ and $\text{cx}(A) \cap F = A$).
(ii) $F = A \cup A^*$, $L \cap F \subseteq A$ and there is no circuit of $E$ contained in $A \setminus L$.
(iii) $A = H \cap F$ for some hemispace $H$ of $E$. 


(iv) $A \cup A^* = F$, $\text{cx}(A)$ is a sharp of $E$ and $L \cap F \subseteq A$.

Proof. By definition, $M_F := (F, \hat{f}, d)$ where $d$ is the restriction $d = c_{\emptyset, F}$ of $c$ (so $d(X) = c(X) \cap F$ for $X \subseteq F$), and $\hat{x} = x^*$ for $x \in F$). Thus, hemispaces of $M_F$ are as described in (i). We note for the proof that $(L \cap F)^* = L^* \cap F^* = L \cap F$. We show that (i) implies (ii). Assume that (i) holds. Suppose, contrary to (ii), that $\{a_1, \ldots, a_n\}$ is a circuit of $E$ which is contained in $A \setminus L$. Then $a_n^* \in \text{cx}(\{a_1, \ldots, a_{n-1}\}) \cap A^* \subseteq \text{cx}(A) \cap F = A$. Hence $a_n \in A \cap A^* = L \cap F$, a contradiction.

Now we show (ii) implies (iii). Assume (ii) holds. Since $A \setminus L$ contains no circuit of $E$, it is contained in some tope $T$, and hence $A$ is contained in some hemispace $H = T \cup L$ of $E$. Then $L \cap F \subseteq A \subseteq H \cap F$. We have

$$H \cap A^* = (H^* \cap A)^* = (H^* \cap H \cap A)^* \subseteq (L \cap F)^* = L \cap F \subseteq A$$

and hence

$$A \subseteq H \cap F = H \cap (A \cup A^*) = (H \cap A) \cup (H \cap A^*) \subseteq A,$$

so $A = H \cap F$ as required.

To show (iii) implies (iv), assume that $A = H \cap F$ where $H$ is a hemispace of $E$. Then $A^* = H^* \cap F^* \subseteq H^* \cap F$ so

$$A \cup A^* = (H \cap F) \cup (H^* \cap F) = (H \cup H^*) \cap F = E \cap F = F$$

and $A \supseteq A \cap A^* = (H \cap F) \cap (H^* \cap F) = L \cap F$. Suppose that $x \in (\text{cx}(A) \cap \text{cx}(A^*)) \setminus L$. Since $A^* = H^* \cap F$ and $\text{cx}(A)^* = \text{cx}(A^*)$, one has $x \in (H \cup H^*) \setminus L$. This contradicts the fact that $H$ is a sharp. Hence $\text{cx}(A)$ is a sharp of $E$.

Finally, assume that (iv) holds. We show (i) follows. Since $L \cap F = (L \cap F)^* \subseteq A$, we have

$$L \cap F \subseteq A \cap A^* \subseteq \text{cx}(A) \cap A^* \subseteq \text{cx}(A) \cap \text{cx}(A)^* \cap F = L \cap F.$$

Hence $A \cap A^* = \text{cx}(A) \cap A^* = L \cap F$ and so

$$\text{cx}(A) \cap F = \text{cx}(A) \cap (A \cup A^*) = (\text{cx}(A) \cap A) \cup (\text{cx}(A) \cap A^*) = A \cup (A \cap A^*) = A,$$

completing the proof.

We conclude our discussion of oriented matroids with the following technical fact, to be used in discussing realizations of signed groupoid sets in Section 4.

**Lemma 2.12.** Let $(E, +, \text{cx})$ and $(F, -, \text{cx})$ be oriented matroids and $f : E \to F$ be an injective map such that $f(\alpha^*) = -f(\alpha)$ for any $\alpha \in E$ and $f(\text{cx}(\emptyset)) = f(E) \cap c(\emptyset)$ (note this latter condition holds if $E$ and $F$ are both reduced). Suppose that for all $A \subseteq E$ such that $A \cup A^* = E$ and $A \cap A^* = \text{cx}(\emptyset)$, one has $c(f(A)) \cap f(E) = f(A)$ if and only if $\text{cx}(A) = A$ (that is, if and only if $A$ is a hemispace of $E$). Then for any $X \subseteq E$, we have $f(\text{cx}(X)) = c(f(X)) \cap f(E)$. In other words, $f$ induces an isomorphism of oriented matroids from $E$ to the restriction of $F$ to $f(E)$.

**Proof.** Note that for $A \subseteq E$, one has $A \cup A^* = E$ and $A \cap A^* = \text{cx}(\emptyset)$ if and only if $f(A) \cup -f(A) = f(E)$ and $f(A) \cap -f(A) = f(\text{cx}(\emptyset))$. Thus, the assumptions together with Proposition 2.11 imply that $f$ induces a bijection $A \mapsto f(A)$ between the hemispaces of $E$ and those of the restriction of $F$ to $f(E)$. Since the hemispaces of an oriented matroid determine its closure operator (by our discussion of the relation between hemispaces, topes, circuits and the closure operator, or more concretely from Theorem 2.6), the result follows.
2.13. **Acycloids.** In this paper we also use the notion of an acycloid (see [21], [22]) which is weaker than that of an oriented matroid.

We define a notion of a preacycloid. A **preacycloid** is a system \( A = (E, *, \mathcal{T}) \) where \( E \) is a finite set with a map \( * : E \to E \) and \( \mathcal{T} \) is a collection of subsets of \( E \) such that

(A1) \((E, *)\) is a strictly involuted set,
(A2) if \( H \in \mathcal{T} \), then \( H \cap H^* = \emptyset \) and \( H \cup H^* = E \setminus L \), where \( L := E \setminus \bigcup_{H' \in \mathcal{T}} H' \),
(A3) if \( H \in \mathcal{T} \), then \( H^* \in \mathcal{T} \).

We call the set \( L \) in (A2) the set of **loops** of \( A \). We say \((E, *, \mathcal{T})\) is loopless if \( L = \emptyset \). By an element \( e \) of \( A \), we mean an element \( e \) of the underlying set \( E \) of \( A \). We shall call an element of \( \mathcal{T} \) of a preacycloid a **tope**.

We say that two elements \( e, f \) of \( E \) are **parallel** if for all \( H \in \mathcal{T} \), one has \( e \in H \iff f \in H \). Parallelism is an equivalence relation, which we denote as \( \sim \) or \( \sim_A \), on \( E \). We denote the parallelism (that is, \( \sim \)-equivalence) class of \( e \in E \) by \([e]_A\). If \( e \) is a loop, then \([e] = L \supseteq \{e, e^*\}\) so \(|[e]| > 1\).

An **acycloid** is defined to be a preacycloid satisfying the following condition:

(A4) \( \mathcal{T} \neq \emptyset \) if and only if \( H_1, H_2 \in \mathcal{T} \) with \( H_1 \neq H_2 \), then there exists \( e \in H_1 \setminus H_2 \) such that \((H_1 \setminus [e]) \cup [e]^* \in \mathcal{T} \).

We say that a preacycloid (or acycloid) as above is **simple** if it satisfies the equivalent conditions (A5)–(A5)' below:

(A5) every parallelism class in \( E \) is a singleton set.
(A5)' if \( e, f \in E \) with \( e \neq f \), then there exists \( H \in \mathcal{T} \) such that \( e \in H \) and \( f \notin H \).

Note that any simple preacycloid is loopless, by the above remarks on the parallelism class of a loop.

Any preacycloid \( A = (E, *, \mathcal{T}) \) determines a simple preacycloid \( A^\circ := (E', \dagger, \mathcal{T}') \), which we call the simplification of \((E, *, \mathcal{T})\), where \( E' := (E \setminus L) / \sim, [e] = [e]^* \) for \( e \in E \setminus L \) and \( \mathcal{T}' = \{H/\sim | H \in \mathcal{T}\} \). It is easily seen that a preacycloid \((E, *, \mathcal{T})\) is an acycloid if and only if its simplification is a simple acycloid.

**Lemma 2.14.** Let \( A \) be an acycloid and \( H, H' \) be tops of \( A \). Let \( n \) be the number of parallelism classes contained in \( H \setminus H' \). Then there is a sequence \( H = H_0, \ldots, H_n = H' \) of tops of \( A \) and elements \( e_i \in H_{i-1} \setminus H_i \) such that \( H_i = (H_{i-1} \setminus [e_i]) \cup [e_i]^* \) for \( i = 1, \ldots, n \).

**Proof.** Note that \( H \setminus H' \) is in fact a finite union of parallelism classes (of non-loops) and that \( e_i \in H \setminus H' \) implies \([e_i] \subseteq H \setminus H' \). The result follows easily by induction on \( n \) using the acycloid axioms. Details are omitted. \( \square \)

**Lemma 2.15.** Let \( M = (E, *, \text{cx}) \) be a finite oriented matroid and \( \mathcal{T} \) denote the set of tops of \( E \). Then

(a) \((E, *, \mathcal{T})\) is an acycloid, called the **tope acycloid** (or preacycloid) of \( M \).
(b) \( M \) is an oriented geometry if and only if its tope acycloid is simple.

**Proof.** This is well known; for instance, it underlies the definition of (simple) acycloids (see [22, 4]). We sketch a proof for completeness.

We first show that \( A \) is a preacycloid; this is essentially trivial. Condition (A1) follows from (M1), and (A3) is easily checked using (M4). If \( H \in \mathcal{T} \), then \( H \cap H^* = \emptyset \) and \( H \cup H^* = E \setminus \text{cx}(\emptyset) = E \). Hence (A2) holds, with \( L = \text{cx}(\emptyset) \).

We next show that if \( M \) is an oriented geometry, then \( A \) is a simple acycloid.
To prove (A4), notice that \((H_1 \setminus H_2) \cap \text{ex}(H_1) \neq \emptyset\). To see this note that otherwise \(\text{ex}(H_1) \subseteq H_1 \cap H_2 \subseteq H_1\) and \(H_1 \cap H_2\) is a closed set, contradicting \(\text{cx}(\text{ex}(H_1)) = H_1\). Then take \(e \in (H_1 \setminus H_2) \cap \text{ex}(H_1)\). We claim that \(\text{cx}(H_1 \setminus \{e\}) = H_1 \setminus \{e\}\) because otherwise we will have another minimal subset whose closure is \(H_1\), contradicting the uniqueness of \(\text{ex}(H)\). Therefore \(H_1 \setminus \{e\}\) is closed. So by [8, Theorem 7], there exists a hemispace \(H_3\) such that \(H_3 \supseteq H_1 \setminus \{e\}\) and \(e \not\in H_3\). Then necessarily, \(H_3 = (H_1 \setminus \{e\}) \cup \{e^*\}\) as required.

We prove (A5). Take \(e, f \in E\) with \(e \neq f\) and \(e \neq f^*\). We claim that \(\text{cx}(\{e, f^*\})\) is a sharp. Note that \(e^* \not\in \text{cx}(\{e, f^*\})\). To see this, note that \(e^* \in \text{cx}(\{e, f^*\})\) will imply \(e^* \in \text{cx}(\{f^*\})\) by (M5). But singleton subsets are closed in a simplicial oriented geometry, so this is a contradiction. Similarly, \(f \not\in \text{cx}(\{e, f^*\})\). Take \(x \not\in \{e, f, e^*, f^*\}\). We prove that it cannot happen that \(\{x, x^*\} \subseteq \text{cx}(\{e, f^*\})\). For if so, by (M6), we will have \(f \in \text{cx}(\{x, x^*\})\) and \(f \in \text{cx}(\{e, x\})\). That will force \(f \in \text{cx}(\{x\})\) by [8, Proposition 2], contrary to \(\text{cx}(\{e\}) = \{e\}\). Therefore \(\text{cx}(\{e, f^*\})\) is a sharp. By [8, Theorem 7], this sharp is contained in a hemispace. Hence (A5) follows. Now \(A\) has no loops since \(M\) has no loops. Lemma 2.14 then easily implies that all parallelism classes in \(A\) are singletons, and \(A\) is simple as required.

Now suppose more generally that \(M\) is just an oriented matroid, and let \(M'\) be the corresponding oriented geometry. It is easy to see that the tope acycloid \(A'\) of \(M'\) is the simplification of the tope preacycloid \(A\) of \(M\), and \(A\) is therefore an acycloid. Finally, suppose that \(A\) is simple. Since \(A\) has no loops, \(M\) has no loops. Since \(A'\) has singleton parallelism classes, the parallelism classes in \(A\) must be precisely the sets \(\text{cx}(\{e\})\) for \(e \in E\) which are identified to singletons in forming \(M'\) from \(M\). It follows that if \(A\) is simple, these sets are singletons and so \(M\) is an oriented geometry.

Note that there exist acycoids which are not tope acycoids of oriented matroids; see [22].

2.16. Contraction of acycloids. We consider some constructions which preserve preacycloids. Let \(A = (E, \ast, \mathfrak{T})\) be a preacyloid. If \(\Gamma\) is any subset of \(E\), define

\[ \Tilde{\mathfrak{T}}_{\Gamma} := \{H \setminus \Gamma \mid H \in \mathfrak{T}, \Gamma \subseteq H\} \cup \{H \setminus \Gamma \cup \Gamma^* \in \mathfrak{T}\}. \]

Trivially, if \(\Gamma\) is not a union of parallelism classes of non-loops or if \(\Gamma \cap \Gamma^* \neq \emptyset\), then \(\Tilde{\mathfrak{T}}_{\Gamma} = \emptyset\). Define \(A/\Gamma\) to be the triple \((E, \ast, \Tilde{\mathfrak{T}}_{\Gamma})\). It is easily checked that in general, \(A/\Gamma\) is a preacycloid and \(A/\Gamma = A/\Gamma^*\). If \(L\) is the set of loops of \(A\), then \(\Gamma \cup \Gamma^* \cup L\) is contained in the set of loops of \(A/\Gamma\). We will use the construction \(A/\Gamma\) in this section only for parallelism classes \(\Gamma\), but use it more generally in Section 4.

Following [22] but taking into account Remark 2.10 we define the elementary contraction \(A/e\) of the preacycloid \(A = (E, \ast, \mathfrak{T})\) at \(e \in E\) as follows. Let \(E' := E \setminus \{e, e^*\}\) and let \(\uparrow\) denote the restriction of \(\ast\) to an involution on \(E'\). If \(e\) is a loop of \(E\), define \(A/e := (E', \uparrow, \mathfrak{T}_{[e]}(\uparrow))\) where \([e]\) is the parallelism class of \(e\); this is a preacycloid too since \(A/e = (A/\{e\})/e\) where \(e\) is a loop of \(A/\{e\}\). (This elementary contraction corresponds to the contraction of an oriented matroid by the set \(\{e, e^*\}\), when the acycloid is the tope acycloid of an oriented matroid, though we shall not need this.)

Let \(A, B\) be preacycloids. Say that \(B\) is an elementary contraction of \(A\) if \(B = A/e\) for some element (loop or non-loop) \(e\) of \(A\). Call \(B\) an elementary quasi contraction of \(A\) if it is equal to \(A/\{e\}\) for some non-loop \(e\) of \(A\). We say that \(B\) is a contraction (respectively, quasi contraction) of \(A\) if there is \(n \in \mathbb{N}\) and a
sequence $A = A_0, A_1, \ldots, A_n = B$ of preacycloids such that for each $i = 1, \ldots, n$, $A_i$ is an elementary contraction (respectively, elementary quasicontraction) of $A_{i-1}$.

The next theorem states a key fact for the proof of the main results in this paper, namely Handa’s characterization of oriented matroids in terms of acycloids and their contractions, along with its trivial reformulation (replacing contractions by quasicontractions) which is better adapted for purposes here.

**Theorem 2.17.** Let $A = (E, *, \Sigma)$ be a preacycloid. Then the following are equivalent:

(i) there is some oriented matroid $M = (E, *, cx)$ whose tope (pre)acycloid is $\Sigma$.

(ii) every contraction of $A$ is an acycloid.

(iii) every quasicontraction of $A$ is an acycloid.

**Proof.** The equivalence of (i) and (ii) is from [22]. (The fact that (i) implies (ii) was previously known from descriptions of contractions of oriented matroids in terms of their topes, and that (ii) implies (i) was conjectured by Tomizawa.)

We sketch a proof of the equivalence of (ii) and (iii) using the following observations on “triviality of loops” (the proofs of which we omit).

Let $B$ be a preacycloid, $f$ be a loop of $B$, and $e$ be an element of $B$. Then

(a) $B$ is an acycloid if and only if $B/f$ is an acycloid.

(b) If $e \neq f, f^*$, then $e$ is a non-loop of $B$ if and only if it is a non-loop of $B/f$. In that case, the parallelism classes of $e$ in $B$ and $B/f$ are equal (that is, $[e]_B = [e]_{B/f}$).

(c) If $e \neq f, f^*$ and $e$ is not a loop of $B$, then $B/[e]/f = B/f/[e]$ where $[e] := [e]_B = [e]_{B/f}$.

(d) If $e \neq f, f^*$, then $B/e/f = B/f/e$.

Above and below, the omitted parentheses should be left justified; for example, $B/[e]/f := (B/[e])/f$. Using these facts, one sees by induction on $n$ that

$$A/e_1/\ldots/e_n = A/[e_{i_1}]/\ldots/[e_{i_p}]/e_1/\ldots/e_n$$

if the left hand side is defined (that is, if $e_j \neq e_{i_1}, e_{i_j}^*$ for $i < j$), where $i_1 < \ldots < i_p$ are the indices $i$ such that the elementary contraction $e_i$ on the left hand side is at a non-loop $e_i$ (of $A/e_1/\ldots/e_{i-1}$). Here, the parallelism class $[e_{i_j}]$ is taken in $A/[e_{i_1}]/\ldots/[e_{i_p}]/e_1/\ldots/e_n$, for $j = 1, \ldots, p$. The elementary contraction $e_j$ on the right hand side is at a loop (of $A/[e_{i_1}]/\ldots/[e_{i_p}]/e_1/\ldots/e_{j-1}$), for $j = 1, \ldots, n$.

The identity applies in particular if $A/[e_1]/\ldots/[e_n]$ is defined (that is, $e_j$ is not a loop in $A/[e_1]/\ldots/[e_j]$ for $j = 1, \ldots, n$), with $i_j = j$ for $j = 1, \ldots, n$. It follows that the set of preacycloids arising as a contraction of $A$ is the same as the set of preacycloids obtained by applying a (possibly empty) sequence of successive contractions at loops to some quasicontraction of $A$. Then (ii) $\iff$ (iii) follows from (a)-(d).

We leave the interested reader to check that, for a preacycloid $A$ as in the theorem, $A$ is not the tope (pre)acycloid of any oriented matroid (that is, the condition (2.17) fails) if and only if there a contraction (respectively, quasicontraction) of $A$ which has no topes.

2.18. **Groupoids with root systems.** In the remainder of this section we describe rudimentary properties of the notion of signed groupoid set as defined in [139]. We also discuss some additional conditions one may impose which give rise to classes of
structures which abstract, with varying degrees of generality, certain basic features of Coxeter groups and their root systems.

A groupoid is a small category in which every morphism has an inverse. A groupoid is called connected if it has at least one object and for any of its objects \(a, b\), there is at least one morphism from \(b\) to \(a\). A groupoid is called simply connected if for any of its objects \(a, b\), there is at most one morphism from \(b\) to \(a\). A groupoid is said to be finite if the set of its morphisms (and hence also the set of its objects) is finite. Let \(G\) be a groupoid and \(a, b\) be objects of \(G\). We denote by \(\alpha G\) the set of morphisms with codomain \(a\). Denote by \(a G_b\) the set of morphisms from \(b\) to \(a\). Denote by \(\text{Ob}(G)\) the set of objects of \(G\) and by \(\text{Mor}(G)\) the set of morphisms of \(G\). A subgroupoid \(H\) of \(G\) is a subcategory of \(G\) such that for any morphism in \(H\), its inverse in \(G\) is also contained in \(H\). A maximal connected subgroupoid of \(G\) is called a connected component of \(G\). For \(a \in \text{Ob}(G)\), the connected component containing \(a\) is denoted \(G[a]\). A subgroupoid \(H\) is said to be full if it contains all morphisms in \(G\) between any two objects of \(G\). We say a subgroupoid \(H\) is a union of components of \(G\) if every component of \(H\) is a component of \(G\). We denote by \(1_a\) the identity morphism at the object \(a\).

For an involuted set \((E, \ast)\), we often write \(-x := x^\ast\). A definitely involuted set \(E\) is a strictly involuted set \(E\) together with a chosen subset \(E^+ \subseteq E\) such that \(E = E^+ \cup (E^+)^*\) where we use \(\cup\) to denote a disjoint union. We denote \((E^+)^*\) by \(-E\).

2.19. A signed groupoid set is a triple \((G, \Phi, \Phi^+)\) where \(G\) is a groupoid and \(\Phi = (\Phi_a)\) is a family of definitely involuted sets indexed by the objects of \(G\) such that \(G\) acts on \(\Phi\), via maps \(\gamma G_b \times \gamma \Phi \to \gamma \Phi\) for \(a, b \in \text{Ob}(G)\), with the properties:

(i) \(1_b(x) = x\) where \(x \in \gamma \Phi\),
(ii) \(f(g(x)) = (fg)(x)\) where \(x \in \gamma \Phi\), \(g \in a G_b\) and \(f \in c G_a\),
(iii) \(f(-x) = -f(x)\) where \(x \in \gamma \Phi\) and \(f \in a G_b\).

We will call the elements in \(\gamma \Phi\) roots (at object \(a\)) and the elements in \(\gamma \Phi^+\) (respectively \(\gamma \Phi^-\)) positive roots (respectively negative roots) at object \(a\). The collection of definitely involuted sets \(\gamma \Phi\) is called the root system of \(G\). For convenience, sometimes we write \(\alpha > 0\) (respectively \(\alpha < 0\)) if \(\alpha\) is a positive root (respectively if \(\alpha\) is a negative root).

We shall not define a category of signed groupoid sets in this paper, but will occasionally use the obvious notion of an isomorphism of signed groupoid sets.

For a signed groupoid set \(R = (G, \Phi, \Phi^+)\) and a subgroupoid \(H\) of \(G\), let the restriction \(R_H\) of \(R\) to \(H\) be the signed groupoid set \(R_H := (H, \Psi, \Psi^+)\) defined as follows: for any \(a \in \text{Ob}(H)\), \(\alpha \Psi := a \Phi\) as definitely involuted set, and for any \(a, b \in \text{Ob}(H)\), the map \(a H_b \times a \Psi \to a \Psi\) is the restriction of the map \(a G_b \times a \Phi \to a \Phi\). If \(H\) is a component (respectively, a union of components) of \(G\), we say that \(R_H\) is a component (respectively, a union of components) of \(R\). For an object \(a\) of \(R\) (that is, an object \(a\) of the underlying groupoid \(G\) of \(R\)), we let \(R[a]\) denote the component of \(R\) whose underlying groupoid is \(G[a]\).

Example 2. (Coxeter groups and real reflection groups) A lot of the definitions and terminology we use for signed groupoid sets is motivated by standard notions in the study of Coxeter groups and reflection groups, but the setting is far more general and familiarity with Coxeter groups and root systems is not required in the proofs. We provide here some informal remarks for readers who are not familiar with these matters.
Consider a real vector space $V$ equipped with a symmetric bilinear (but not necessarily positive definite) form $(-|-)$: $V \times V \rightarrow \mathbb{R}$. A vector $\alpha \in V$ is said to be non-isotropic if $(\alpha|\alpha) \neq 0$. In that case, the corresponding reflection is defined to be the (unique) invertible linear map $s_\alpha: V \rightarrow V$ which fixes the hyperplane orthogonal to $\alpha$ pointwise and maps $\alpha$ to $-\alpha$. A real reflection group on $V$ is a group $W$ of invertible linear maps of $V$ which is generated by a set of reflections. By a root system for $W$ is generally meant a $W$-stable set $\Phi$ of $V \setminus \{0\}$ such that $W$ is generated by reflections in vectors in some subset of $\Phi$ (this subset is often but not always equal to $\Phi$, depending on the context in which $W$ and $\Phi$ arise).

Coxeter groups $W$ are a class of groups which are defined by existence of a certain simple type of presentation with a special set of generators $S$, the elements of which are called “simple reflections.” One then calls $(W,S)$ a Coxeter system. Coxeter groups always have faithful representations as reflection groups with root systems as above. The finite Coxeter groups arise by taking $(V,(-|-))$ to be an inner product space (i.e. $V$ is finite dimensional and the form is positive definite) and $W$ as a finite group generated by reflections on $V$. For the root system, one may take the set of all unit vectors $\alpha \in V$ such that $s_\alpha$ in $W$, but other choices are often more natural in applications (for instance, “crystallographic” root systems of finite Weyl groups arise naturally in the structure theory of semisimple complex Lie algebras).

Coxeter groups in general have a similar “standard geometric representation” defined from their presentation, as described in [2] or [26], and a standard root system $\Phi$ such that $s_\alpha \in W$ for all $\alpha \in \Phi$. Certain (crystallographic) Coxeter groups also arise naturally as Weyl groups of Kac-Moody Lie algebras ([27]); in this case, the root system is the disjoint union of a set of “real roots” (the reflections in which generate $W$) and a set of “imaginary roots” (roots in which may even be isotropic and so have no corresponding reflection).

The root system, as a subset of $V \setminus \{0\}$ stable under multiplication by $\pm 1$, may be regarded as a realizable oriented matroid. For Coxeter groups, its construction or properties gives rise to a standard (up to $W$-action) hemispace of $\Phi$, called the standard positive system $\Phi^+$, elements of which are called positive roots. Typically, $S$ may be characterized geometrically either as the reflections $s_\alpha$ in the roots $\alpha$ which span the extreme rays of cone($\Phi^+$), or combinatorially as the group elements which map only one positive root (and its other positive scalar multiples, if any) outside $\Phi^+$. For a reflection group $W$ which is not a Coxeter group, there is usually no canonical choice of positive system, but one may define $\Phi^+$ to be an arbitrarily chosen hemispace of $\Phi$; simple reflections in either sense above do not then typically generate $W$. For example, the orthogonal group of a real Euclidean space, with the root system being the unit sphere, is examined from this point of view in [28].

Let $\Phi$ be a root system of a Coxeter group (or more generally, real reflection group) $W$ as above. Regard the group $W$ as a groupoid $G$ with a single object $\bullet$, and the set of morphisms being the set of elements in $W$. Let $\Phi = \Phi, \Phi^+ = \Phi^+$. The action of morphisms on the roots is exactly the action of the corresponding elements in $W$ on the roots. Then $(G,\Phi,\Phi^+)$ is a signed groupoid set.

**Example 3.** (Preacycloids) Let $A = (E,*,\Sigma)$ be a preacycloid with $\mathcal{T} \neq \emptyset$ and $L$ as its set of loops. Choose a subset $L^+$ of $L$ such that $L = L^+ \cup (L^+)^*$. Consider a connected, simply connected groupoid $G$ whose objects are indexed by the toposes of $A$. For $H \in \Sigma$, denote the corresponding object by $\tilde{H}$. The set $\tilde{\Phi}$ of roots at
an object $\tilde{H}$ is $E$ and the set $\tilde{H}^+$ of positive roots at this object is $H \cup L^+$. Any morphism acts on $E$ as identity. Then $R = (G, \Phi, \Phi^+)$ is a signed groupoid set. In particular, associated to an oriented matroid, one can construct such a signed groupoid set (by taking $A$ above as the tope (pre)acycloid of the oriented matroid). We shall denote this signed groupoid set $R$ as $\text{SGS}(A)$ to denote its dependence on $A$. It is easy to see that the isomorphism type of $\text{SGS}(A)$ is independent of the choice of $L^+$.

Example 4. (Brink-Howlett Groupoid) Let $(W, S)$ be a Coxeter group with standard root system $\Phi$, $\Phi^+$ be the standard positive system and $\Delta \subseteq \Phi^+$ denote its simple system (the positive roots corresponding to the simple reflections). Construct a groupoid with objects being the subsets of $\Delta$ and a morphism from $I$ to $J$ being of the form $(I, w, J)$ where $I, J \subseteq \Delta$ and $w \in W$ with $w(I) = J$. At each object, let the root system and the set of positive roots be inherited from those of $W$; that is, $\rho \Phi = \Phi$ and $\gamma \Phi^+ = \Phi^+$. Equipped with such root systems at its objects, the groupoid becomes a signed groupoid set. This groupoid (though not its root system as defined here) is considered in [7] for the purpose of studying the normalizer of parabolic subgroups of $W$. We will later show that this groupoid set can be obtained by applying the generalized Brink-Howlett construction, described in Section 2.15 to the signed groupoid set in Example 2. (We remark that there are also more subtle choices of root system, which we do not discuss in this paper, for the Brink-Howlett groupoid.)

Example 5. (Weyl groupoid) Given a Cartan scheme $C$, one can associate to it a Weyl groupoid $G$. Suppose that $G$ has a root system of type $C$, in the sense of [24]. At each object, the root system is a (definitely involuted) subset of the free $\mathbb{Z}$–module of rank equal to the rank of $C$. The morphisms of the Weyl groupoid can be considered as automorphisms of $\mathbb{Z}^{\text{rank}(C)}$ and the action of them on the roots satisfies the required conditions of a signed groupoid set. For details of Weyl groupoids, see [23, 25, 10, 12, 11, 24] and [13].

2.20. We say that a signed groupoid set $(G, \Phi, \Phi^+)$ is finite (respectively connected, simply connected) if is $G$ is finite and $a \Phi$ is finite for all $a \in \text{Ob}(G)$ (respectively $G$ is connected, simply connected). For $g \in a \cdot G$, we define the inversion set

$$\Phi_g = a \cdot \Phi^+ \cap g(a \Phi^-) = \{ \alpha \in a \cdot \Phi^+ \mid g^{-1}(\alpha) \in a \cdot \Phi^- \}.$$ 

A positive (respectively negative) root $\alpha$ in $a \cdot \Phi^+$ (respectively $a \cdot \Phi^-$) is called imaginary if $\alpha \notin \Phi_g$ for any $g \in a \cdot G$ (respectively if $-\alpha \notin \Phi_g$ for any $g \in a \cdot G$). A root that is not imaginary is called real. If for all $a \Phi$, where $a \in \text{Ob}(G)$, the set of imaginary roots is empty then we call $(G, \Phi, \Phi^+)$ real.

Denote the set of positive (respectively, negative) imaginary roots of $a \Phi$ as $a \cdot \Phi^+_{\text{im}}$ (respectively, $a \cdot \Phi^-_{\text{im}} = -a \cdot \Phi^+_{\text{im}}$) and let $a \Phi^+_\text{re} = a \cdot \Phi^+ \setminus a \cdot \Phi^+_{\text{im}}$, $a \Phi^-_{\text{re}} = a \cdot \Phi^- \setminus a \cdot \Phi^-_{\text{im}}$ denote the sets of all positive and all negative real roots at $a$, respectively. Observe that for any morphism $g: b \to a$ in $G$, we have $g(a \Phi^+_{\text{im}}) = a \Phi^+_{\text{im}}$, $g(a \Phi^-_{\text{re}}) = a \Phi^-_{\text{re}}$ and $g(a \Phi^+_{\text{re}} \cap g(a \Phi^-_{\text{re}}))$. The following facts are from [14], where they are expressed in the language of protorootoids and checked by routine computations with 1–cycles.

Lemma 2.21. Let $R = (G, \Phi, \Phi^+)$ be a signed groupoid set.

(a) Let $f$ and $g$ be morphisms in $G$ such that the composite $fg$ is defined. Then we have $\Phi_{fg} = (\Phi_f \setminus -f \Phi_g) \cup (f \Phi_g \setminus -\Phi_f)$.
(b) For any morphism $h$ of $G$, we have $\Phi_{h^{-1}} = -h^{-1}\Phi_h$.

(c) Let $f, g$ be two composable morphisms in $G$. We have the equivalence: $\Phi_f \cap \Phi_g = \emptyset \iff \Phi_f \subseteq \Phi_{fg} \iff \Phi_{fg} = \Phi_f \cup f\Phi_g$.

(d) For morphisms $h, g \in \mathcal{G}$, we have $\Phi_h = \Phi_g$ if and only if $\Phi_{h^{-1}g} = \emptyset$.

Proof. (a) By definition

$$\Phi_{fg} = \{\alpha \mid \alpha \in \Phi_f, -f^{-1}(\alpha) \notin \Phi_g\} \cup \{\alpha \mid \alpha > 0, f^{-1}(\alpha) \in \Phi_g\}.$$  

Note that $\alpha < 0$ and $f^{-1}(\alpha) \in \Phi_g$ imply $-\alpha \in \Phi_f$. Therefore the equation holds.

(b) This follows from (a) on taking $f := h^{-1}$ and $g := h$.

(c) Suppose $\Phi_f \cap \Phi_g = \emptyset$. Note that it follows from (b) that $\Phi_{f^{-1}} = -f^{-1}\Phi_f$. Therefore $\Phi_f \cap -f\Phi_g = \emptyset$. By (a) $\Phi_f \subseteq \Phi_{fg}$. Suppose that $\Phi_f \subseteq \Phi_{fg}$. Again by (a) this implies that $\Phi_f \cap -f\Phi_g = \emptyset$ (and therefore $f\Phi_g \cap -\Phi_f = \emptyset$). Use (a) again we see that $\Phi_{fg} = \Phi_f \cup f\Phi_g$. Finally suppose that $\Phi_{fg} = \Phi_f \cup f\Phi_g$. By (a) this implies that $\Phi_f \cap -f\Phi_g = \emptyset$. Hence $-f^{-1}\Phi_f \cap f^{-1}f\Phi_g = \emptyset$. Therefore $\Phi_{f^{-1}} \cap \Phi_g = \emptyset$.

(d) This follows by taking $f := h^{-1}$ in (a) and using (b). \qed

2.22. Let $R = (G, \Phi, \Phi^+)$ be a signed groupoid set. If $\Phi_g = \emptyset$ implies that $g$ is the identity morphism then we say $(G, \Phi, \Phi^+)$ is faithful. By Lemma 2.21 this holds if and only if for any $a \in \text{Ob}(G)$ and any $g, h \in \mathcal{G}$ with $\Phi_g = \Phi_h$, one has $g = h$.

Assume for the rest of this subsection that $R$ is faithful, unless otherwise stated.

There is a partial order $\leq_a$ on $\mathcal{G}$, called the weak order of $G$ (at $a$), such that $g \leq_a h$ if and only if $\Phi_g \subseteq \Phi_h$. Note that $1_a$ is the minimum element of $(\mathcal{G}, \leq_a)$.

In what follows when we compare two morphisms at a given object, it is always understood that they are compared in the sense of weak order. If $g, h \in \mathcal{G}$, we may write $g \leq h$ instead of $g \leq_a h$ if confusion is unlikely.

A morphism $g$ of $G$ is said to be simple if $|\Phi_g| = 1$. By Lemma 2.21(b), the inverse of a simple morphism is simple. The set of simple morphisms with codomain $a$ is denoted $\mathcal{S}_a$. We call a morphism $g \in \mathcal{G}$ atomic if $g$ is an atom in the poset $\mathcal{G}$ (that is, if $h \in \mathcal{G}$ with $h \leq_a g$ implies $h = g$ or $h = 1_a$). Clearly a simple morphism is atomic.

We call $R$ interval finite if for any object $a$ and morphism $g \in \mathcal{G}$, the set $\{h \mid h \in \mathcal{G}, \Phi_h \subseteq \Phi_g\}$ is finite (that is, the closed interval $[1_a, g]$ in the weak order $\leq_a$ on $\mathcal{G}$ is finite). We say $(G, \Phi, \Phi^+)$ is inversion-set finite if for any $g \in \text{Mor}(\mathcal{G})$, $\Phi_g$ is finite. An inversion-set finite signed groupoid set is clearly interval finite.

Say that the groupoid $G$ is generated by a set $X \subseteq \text{Mor}(G)$ if every non-identity morphism in $G$ is expressible as a composite of morphisms in $X \cup X^{-1}$, where $X^{-1} := \{g^{-1} \mid g \in G\}$. For $g \in \text{Mor}(G)$, define the length $l_X(g) \in \mathbb{N}$ of $g$ (with respect to $X$) to be 0 if $g$ is an identity morphism, and otherwise to be the minimum number of factors occurring in products of elements of $X \cup X^{-1}$ with value $g$.

A (not necessarily faithful) signed groupoid set $R = (G, \Phi, \Phi^+)$ is called principal if the set $S := \bigcup_{a \in \text{Ob}(G)} \mathcal{S}_a$ of simple morphisms generates $G$ and for all $g \in \text{Mor}(G)$, $l(g) := l_S(g)$ satisfies $l(g) = |\Phi_g|$. A principal signed groupoid set is necessarily faithful, since if $g \in \text{Mor}(G)$ satisfies $\Phi_g = \emptyset$, then $l(g) = |\Phi_g| = 0$ and so $g$ is an identity morphism.

\footnote{Here and elsewhere in this paper, assumptions of faithfulness can sometimes be removed, though often at the expense of more cumbersome statements or notation.}
If $G$ is generated by its atomic morphisms, we say that $R$ is atomically generated. An interval finite, faithful signed groupoid set is called preprincipal if for any $g, s \in a G$, with $s$ being atomic, one has either $\Phi_g \supseteq \Phi_s$ or $\Phi_g \cap \Phi_s = \emptyset$.

We say that $R$ is antipodal if for each $a \in \text{Ob}(G)$, the weak order $(a G, \leq_a)$ has a maximum element (this notion is the only one from above that is not already considered in [15–16]). In general, a maximum element of $(a G, \leq_a)$ will be denoted $\omega_a: w_a \rightarrow a$ if it exists.

For completeness, the following lemma states and proves, using the terminology of signed groupoid sets, some additional facts formulated in terms of protoroitoids in [15].

**Lemma 2.23.** Let $R = (G, \Phi, \Phi^+)$ be a faithful signed groupoid set.

(a) Let $x \in a G_b$, $y \in a G_c$ and $w \in a G$. 
- If $x \leq_a xy$, then $y^{-1} \leq_a y^{-1} x^{-1}$.
- If $x <_a xy$ and $x <_a xw$, then $xy <_a xw$ if and only if $y <_b w$.

(b) The inverse of an atomic morphism is atomic.

(c) Suppose that $G$ is generated by its simple morphisms. Then $|\Phi_g| \leq l(g)$ for all $g \in \text{Mor}(G)$ and thus $R$ is inversion-set finite.

(d) Suppose that $R$ is interval finite. Then $R$ is atomically generated.

(e) Let $R$ be interval finite. Suppose that any atomic morphism of $R$ is simple. Then $R$ is principal.

(f) If $R$ is principal, then it is preprincipal and every atomic morphism of $R$ is simple.

**Proof.** (a) The two assertions follow directly from Lemma 2.21(c) and (b).

(b) It follows from the first assertion of (a).

(c) We prove this by induction (the argument does not require the assumption that $R$ is faithful). If $l(g) = 0$, then $g$ is the identity morphism at some object and thus $|\Phi_g| = 0$. Suppose $|\Phi_g| \leq l(g)$ for $g$ such that $l(g) < n$. Now assume that $l(g) = n$. Then $g = s_n s_{n-1} \cdots s_1$ where each $s_i$ is simple. By Lemma 2.21(a),

$$\Phi_g = (\Phi_{s_n} \setminus -s_n \Phi_{s_{n-1}} \cdots s_1) \cup (s_n \Phi_{s_{n-1}} \cdots s_1 \setminus -\Phi_{s_n}).$$

By definition one sees that $l(s_{n-1} \cdots s_1) = n - 1$. Therefore by induction, we have $|\Phi_{s_{n-1}} \cdots s_1| \leq n - 1$. Note $|\Phi_{s_n}| = 1$. Therefore $|\Phi_g| \leq n$.

(d) Take a morphism $g \in a G$. Denote the cardinality of the interval $[1_a, g]$ in the weak order at $a$ by $l'(g)$. We use induction on $l'(g)$ to show that if $g$ is not an identity morphism, then $g$ is a product of atomic morphisms. If $l'(g) = 1$ then $\Phi_g = \emptyset$ and $g = 1_a$ by faithfulness. Suppose that $g$ is a product of atomic morphisms if $1 < l'(g) < n$. Suppose $l'(g) = n$. Take an atom $r \in a G$ such that $\Phi_r \subseteq \Phi_g$. Lemma 2.21(c) implies that $\Phi_{r-1} \cap \Phi_{r-1}g = \emptyset$ and $\Phi_g = \Phi_r \cup r \Phi_{r-1}g$. Let $b$ denote the domain of $r$. We will show that $l'(r^{-1} g) < l'(g)$ and then the assertion follows from the induction. To that end, we show that there exists a bijection between the interval $[1_b, r^{-1} g]$ and the interval $[r, g]$ under the map $h \mapsto rh$.

Since $\Phi_b \subseteq \Phi_{r-1}g$ and $\Phi_{r-1} \cap \Phi_{r-1}g = \emptyset$, we have $\Phi_{rh} = \Phi_r \cup r \Phi_{r-1}g$ by Lemma 2.21(c). Hence $\Phi_r \subseteq \Phi_{rh} \subseteq \Phi_r \cup r \Phi_{r-1}g$. Therefore the map is well-defined. It follows from the invertibility of $r$ that the map is injective. Take $h' \in a G$ such that $r \leq_a h' \leq_a g$. Again by Lemma 2.21(c), $\Phi_{h'} = \Phi_r \cup r \Phi_{r-1}h'$. It follows that $\Phi_{r-1}h' \subseteq \Phi_{r-1}g$. Therefore one sees that the map is surjective. Hence $l'(r^{-1} g) = ||[r, g]| < ||[1_a, g]|| = l'(g)$. 

simple morphism. We have to show that for any morphism \( g \) \( | \Phi_g | = l(g) \).

We prove this by induction on \( | \Phi_g | \). If \( | \Phi_g | = 0 \), then \( \Phi_g = \emptyset \) and \( g \) is an identity morphism since \( R \) is finite, we can take an atomic morphism \( r \) is simple (and therefore \( r \) is also simple). We have \( \Phi_g = \Phi_r \cup r \Phi_{r^{-1} g} \) by Lemma 2.21(c) and therefore \( | \Phi_{r^{-1} g} | = | \Phi_g | - 1 \). By induction \( | \Phi_{r^{-1} g} | = l(r^{-1} g) = n - 1 \).

By the definition of \( l \), \( l(g) \leq n \). On the other hand, by (c), \( n \leq l(g) \). Therefore the assertion follows.

(f) Let \( R \) be a principal signed groupoid set. By (c), a principal signed groupoid set is inversion-set finite, and thus interval finite. We now show that every atomic morphism of \( (G, R, \Phi^+) \) is simple. Take \( r \) an atomic morphism. If it is not simple, suppose \( l(r) = k > 1 \) and \( r = s_k s_{k-1} \cdots s_1 \) with \( s_i \) simple. Since \( R \) is principal, \( | \Phi_r | = k \).

By Lemma 2.21(a),

\[
\Phi_r = (\Phi_{s_k} \setminus -s_k \Phi_{s_{k-1} \cdots s_1}) \cup (s_k \Phi_{s_{k-1} \cdots s_1} \setminus -\Phi_{s_k})
\]

By definition \( l(\Phi_{s_k} \setminus -s_k \Phi_{s_{k-1} \cdots s_1}) = k - 1 \). By (b) \( |\Phi_{s_{k-1} \cdots s_1}| \leq k - 1 \). Therefore this forces \( \Phi_r = \Phi_{s_k} \cup s_k \Phi_{s_{k-1} \cdots s_1} \). Therefore \( \Phi_{s_k} \subseteq \Phi_r \) by Lemma 2.21(c), a contradiction. Therefore for an atomic morphism \( r \) and a morphism \( g \) having the same codomain as \( r \), either \( \Phi_r \subseteq \Phi_g \) or \( \Phi_g \cap \Phi_r = \emptyset \), since \( \Phi_r \) is a singleton set.

Lemma 2.24. Let \( R = (G, \Phi, \Phi^+) \) be a faithful signed groupoid set, \( X \) be a set of generators of \( G \) such that \( \{ g^{-1} | g \in X \} = X \), \( a \in \text{Ob}(G) \) and \( \alpha \in \Phi^+ | \Phi_{re} \). Then there exist some \( y \in _G a \) and \( s \in X \cap _G a \) such that \( y(\alpha) \in \Phi_s \). In particular, this applies with \( R \) atomically generated (respectively, preprincipal, principal) and \( X \) equal to the set of atomic (respectively, atomic, simple) morphisms of \( R \).

Proof. Since \( \alpha \) is a real root, there is some morphism \( g \) such that \( \alpha = g(\alpha) \) are of opposite sign. Obviously, \( g \) is not an identity morphism, so it is a product \( g = s_n \cdots s_1 \) of morphisms \( s_i \) in the set \( X \). For some \( i = 1, \ldots, n \), \( s_i \cdots s_1 \) are of opposite sign. Hence there is \( x \in G \) and a morphism \( r \) in \( X \) such that \( x(\alpha) \) and \( rx(\alpha) \) are of opposite sign. If \( x(\alpha) > 0 \) and \( rx(\alpha) < 0 \), then \( x(\alpha) \in \Phi_{r^{-1}} \) where \( r^{-1} \) is in \( X \); in this case, we can choose \( y = x \) and \( s = r^{-1} \).

Otherwise, we have \( x(\alpha) = r^{-1}(rx(\alpha)) < 0 \) and \( rx(\alpha) > 0 \), so \( rx(\alpha) \in \Phi_{r^{-1}} \); in this case, we can take \( y = rx \) and \( s = r \).

2.25. Real compression. Let \( R = (G, \Phi, \Phi^+) \) be any signed groupoid set. Define a preorder (that is, a reflexive, transitive relation) \( \preceq_a \) on \( _G \Phi \) by the condition that for \( \alpha, \beta \in _G \Phi \), one has \( \alpha \preceq_a \beta \) if for all \( b \in \text{Ob}(G) \) and \( g \in _a \Phi_b \), \( g(\beta) \in _b \Phi^- \) implies \( g(\alpha) \in _b \Phi^- \). Following [6] (see also [2]), we call \( \preceq_a \) the dominance preorder at \( a \). Let \( \sim_a \) denote the corresponding equivalence relation on \( _G \Phi \), defined by

\[
\alpha \sim_a \beta \iff (\alpha \preceq_a \beta \text{ and } \beta \preceq_a \alpha), \text{ for } \alpha, \beta \in _G \Phi.
\]

We call the relation \( \sim_a \) parallelism on \( _G \Phi \). Thus, roots \( \alpha, \beta \in _G \Phi \) are said to be parallel if for all \( g \in _G \Phi, g^{-1}(\alpha) \) and \( g^{-1}(\beta) \) are of the same sign. Note that each of \( _G \Phi_{\sim_a} \) if non-empty, is a single parallelism class. If \( R \) contains no distinct parallel roots at any object, we call it compressed. Thus, \( R \) is compressed if and only if its dominance preorder at each object is a partial order (called dominance order).
We attach to $R$ another signed groupoid set $R^{rec} := (G, \Phi_{rec}, \Phi^+_{rec})$, called the real compression of $R$, as follows. For $a \in \text{Ob}(G)$, define the definitely involuted set $\Phi_{rec}^a$ by

$$\Phi_{rec}^a := a \Phi_{rec}/\sim_a, \quad -[\alpha]_a := [-\alpha]_a \quad \text{for} \quad \alpha \in a \Phi_{rec},$$

where $[\alpha]_a$ is the $\sim_a$-equivalence class of $\alpha$, and $a \Phi_{rec}^+ := a \Phi_{rec}^+ / \sim_a$. The maps $bG_a \times a \Phi_{rec}^a \to b\Phi_{rec}^a$ defining the action of $G$ are given by $g[\alpha]_a := [g\alpha]_b$. It is easy to check that this gives a well-defined signed groupoid set $R^{rec}$.

**Lemma 2.26.** Let $R$ be a faithful signed groupoids set. Assume that $R$ is preprincipal, with $A$ as its set of atomic morphisms. Then

(a) the parallelism classes of real roots in $a \Phi$ are the sets $x(\Phi_s)$ where $x \in G_b$ and $s \in G \cap A$.

(b) $R^{rec}$ is principal.

(c) if $g$ is in $aG$, then $l_A(g) = |\Phi_g/\sim_a|$ (the number of parallelism classes in $\Phi_g$).

**Proof.** (a) By definition of preprincipal signed groupoid sets, the sets $\Phi_s$ for $s \in A$ are parallelism classes, and hence so are the sets $x(\Phi_s)$. Every real root appears in such a parallelism class by Lemma 2.21 and (a) follows.

(b) It is easy to see that $g$ is an atomic morphism of $R$ if and only if $g$ is an atomic morphism of $R^{rec}$. Now we show that an atomic morphism $g$ is also simple in $R^{rec}$, i.e. $|\Phi_g| = 1$ in $R^{rec}$. Note that $R$ is atomically generated by Lemma 2.26 (d). Therefore $R^{rec}$ is also atomically generated. Atomic morphisms are simple in $R^{rec}$ by (a) applied to $R^{rec}$. The result follows from Lemma 2.26 (c).

(c) If $R$ is principal, this follows from the definition of principalness since the parallelism classes of the real roots are singletons by (a) and Lemma 2.26 (f). It follows for faithful, preprincipal signed groupoid sets since both the length of a groupoid element with respect to the set of atomic generators and the number of parallelism classes in an inversion set are invariant under real compression. □

2.27. In this subsection, $R = (G, \Phi, \Phi^+)$ denotes a faithful signed groupoid set. If for all $a \in \text{Ob}(G)$, the weak order $(\leq_a)_{\text{weak}}$ at $a$ is a complete lattice, we say that $R$ is complete.

For two morphism $g, h \in aG$, if $\Phi_g \cap \Phi_h = \emptyset$, we write $g \perp h$ and say they are orthogonal. By Lemma 2.21(c), orthogonality is expressible in terms of the family of weak orders of $R$ at the objects of $G$. We say $R$ is rootoidal if for any $a \in \text{Ob}(G)$, the weak order $(\leq_a)$ is a complete meet semilattice (that is, any of its non-empty subsets has a meet (greatest lower bound)) and the weak orders satisfy the following Join Orthogonality Property (JOP): if $h, g_i \in aG$, where $i \in I$, with $g_i \perp h$ for all $i$ and the join (least upper bound) $g = \bigvee_{i \in I} g_i$ exists in weak order at $a$, then $g \perp h$. (We remark that a subset of a complete meet semilattice has a join if and only if it is bounded above; its join is then the meet of the set of upper bounds of the subset.)

The condition that a signed groupoid set be rootoidal is crucial in extending many basic facts which hold for complete signed groupoid sets to non-complete ones. The main reason for mentioning it in this paper (where our main results concern complete signed groupoid sets anyway) is to make explicit the fact, which we shall use several times, that weak orders in complete signed groupoid sets have the JOP, as (e) of the following lemma shows.
Lemma 2.28. Let $R = (G, \Phi, \Phi^\circ)$ denote a faithful signed groupoid set.

(a) If $a \in \text{Ob}(G)$, an element $\omega_a \in \mathcal{G}$ is a maximum element of weak order at $a$ if and only if $\Phi_{w_a} = a_{\Phi}^\circ$.

In (b)–(c), we assume that $R$ is antipodal and for each $a \in \text{Ob}(G)$, let $\omega_a : w_a \rightarrow a$ denote the maximum element in weak order $(aG, \leq_a)$.

(b) For $a \in \text{Ob}(G)$, $\omega_{w_a} = \omega_a^{-1}$, $w_{w_a} = a$ and $\omega_a(\Phi_{w_a}) = a_{\Phi}^\circ$. For any morphism $g : a \rightarrow b$ in $G$, one has $\Phi_{gw_a} = b_{\Phi}^\circ \setminus \Phi_g$.

(c) Define a map $g \mapsto g^\perp : \text{Ob}(G) \rightarrow \text{Ob}(G)$ as follows: if $g : b \rightarrow a$, let $g^\perp := g\omega_b$. Then $g \mapsto g^\perp$ is an order reversing bijection of $(\text{Ob}(G), \leq_a)$ with itself, satisfying $(g^\perp)^\perp = g$, $g \land g^\perp = 1_a$ (where these meets and joins exist even if the weak order at $a$ is not a lattice). For $g, h \in G$, one has $g \perp h \iff g \leq h$.

(d) Assume that $R$ is complete. Then $R$ is rootoidal and for each $a \in \text{Ob}(G)$, the weak order at $a$ is a complete ortholattice with maximum element $\omega_a := \bigvee_{g \in aG} g$ and orthocomplement $g \mapsto g^\perp$ defined as in (c). In particular, $R$ is antipodal.

(e) $R$ is complete if and only if it rootoidal and antipodal.

(f) $R$ is complete (respectively, rootoidal, antipodal, preprincipal) if and only if $R^{\text{rec}}$ is complete (respectively, rootoidal, antipodal, preprincipal).

Proof. (a) This follows from definition of weak order since $a_{\Phi}^\circ = \bigcup_{g \in aG} g_{\Phi}$.

(b) By (a), we have $\omega_a^{-1}(a_{\Phi}^\circ) \subseteq w_a_{\Phi}^\circ$ and similarly with a replaced by $w_a$. This implies that $(\omega_a^{-1} \omega_a^{-1})(a_{\Phi}^\circ) \subseteq w_a_{\Phi}^\circ$. Also, $(\omega_a^{-1} \omega_a^{-1})(a_{\Phi}^\circ) \subseteq w_a_{\Phi}^\circ$. Hence $\Phi_{\omega_a \omega_a} = \emptyset$. Since $R$ is faithful, it follows that $\omega_a \omega_a$ is an identity morphism and so $\omega_a = \omega_a^{-1}$. In particular, $w_{w_a} = a$. Equality holds in the first inclusion in the proof of (b) since replacing $a$ by $w_a$ in it gives the reverse inclusion. To prove the equality $\Phi_{gw_a} = b_{\Phi}^\circ \setminus \Phi_g$, note both sides are contained in $b_{\Phi}^\circ$. If $a$ is in this set of roots, one has $(g\omega_b)^{-1} \alpha < 0 \iff g^{-1}(\alpha) > 0$, and the equality follows.

(c) By (b), we have $\Phi_{g^\perp} = \Phi_{\omega_a} \setminus \Phi_g$. Hence the map $g \mapsto g^\perp$ is order reversing. It is an involution since $\Phi(g^\perp)^\perp = \Phi_{\omega_a} \setminus \Phi_g = \emptyset$ and $R$ is faithful. One has $g \land g^\perp = \omega_a$ and $g \land g^\perp = 1_a$ since $\Phi_g \cup \Phi_{g^\perp} = \Phi_{\omega_a}$ and $\Phi_g \cap \Phi_{g^\perp} = \emptyset$. The final assertion of (c) just amounts to $\Phi_g \cap \Phi_{g^\perp} = \emptyset$ and holds since $\Phi_g \subseteq \Phi_{\omega_a}$.

(d) Assume $R$ is complete. Then the weak order at $a \in \text{Ob}(G)$ obviously has maximum element $\omega_a := \bigvee_{g \in aG} g$. The properties of the map $\perp$ in (c) show by definition (see [2]) that the weak order is a complete ortholattice with orthocomplement $\perp$. It remains to prove $R$ is rootoidal. Certainly each weak order, as a complete lattice, is a complete meet semilattice. Suppose $a \in \text{Ob}(G)$ and $g_i, h$ in $aG$ satisfy $g_i \perp h$ for all $i$. Then $g_i \leq_a h^\perp$ for all $i$. Hence the join $g = \bigvee_{i \in I} g_i$ satisfies $g \leq_a h^\perp$. That is, $g \perp h$, as required to verify the JOP.

(e) The “only if” direction follows from (d). Conversely, suppose $R$ is rootoidal and antipodal. Then for any object $a$ of $G$, the weak order at $a$ is a complete meet semilattice with a maximum element, which implies it is a complete lattice. Hence $R$ is complete by definition.

(f) The groupoid and its weak orders are preserved under real compression. Hence for any property, such as those in (f), which is expressible in terms of the groupoid and its weak orders, $R$ has that property if and only if $R^{\text{rec}}$ does. □
Coxeter group corresponding to conjugation by the longest element, can also be extended to antipodal signed groupoid sets.

**Example 6.** (1) Consider the signed groupoid set $R = (G, \Phi, \Phi^+)$ from the standard root system of a Coxeter group (Example 2). It is easily verified to be principal, real and compressed (these conditions reduce to well known elementary properties of Coxeter groups and their root systems). The weak order at the unique object is the usual weak (right) order of the Coxeter group, which is known to be a complete meet semilattice (see [2]). If $W$ is finite, it is well known that the longest element $w_0$ is a maximum element of weak order, which implies that weak order is a complete lattice. Lemma 2.28 implies that $R$ is complete, antipodal and rootoidal. (Similarly, it follows from results in [24] that the signed groupoid sets discussed in Example 5 are principal, real, compressed, complete, antipodal and rootoidal.) The JOP for infinite Coxeter groups $W$ is proved in [17], showing that $R$ above is principal, real, compressed and rootoidal (though not complete) in general. This example and its relation to conjectures in op. cit. (which, from the discussion in 2.7, are very closely related to various notions of convexity on root systems) provided the first author’s principal motivation for the study of rootoids, which underlies this paper.

(2) Suppose that $R = (G, \Phi, \Phi^+)$ is the signed groupoid set in Example 1 associated to the set of all (real or imaginary roots) of a Kac-Moody Lie algebra with Weyl group $W$. The real and imaginary roots of $R$ as defined here coincide with the real and imaginary roots in the usual sense of Kac-Moody Lie algebras, and parallelism classes of real roots are singletons. Then $R^{\text{rec}}$ identifies with the signed groupoid set associated similarly to the subsystem of all real roots, which is isomorphic (as signed groupoid set) to that in (1) (with possibly infinite $W$).

(3) Suppose that $R = (G, \Phi, \Phi^+)$ is the signed groupoid set associated in a similar way as in (1) to a non-reduced crystallographic root system of a finite Weyl group $W$ (see [5]). There are no imaginary roots and the parallelism class of a root is the set of all roots which are positive real scalar multiplies of it. Then $R^{\text{rec}}$ is isomorphic to the signed groupoid set attached as in (1) to a reduced root system of $W$.

Part (f) of the following proposition will play an important role in the proof of our main result. The crucial points, which characterize simplicial oriented geometries amongst oriented geometries by completeness properties, were stated in [15, Theorem 6.11], and are essentially just a translation of results from [3]. We include a proof, citing [3] for the key facts.

**Proposition 2.29.** Let $R = (G, \Phi, \Phi^+)$ be the signed groupoid set $SGS(A)$ associated to a preacycloid $A = (E, *, T)$, where $T \neq \emptyset$, in Example 3.

(a) $R$ is finite, faithful, connected, simply connected, antipodal and inversion set finite (hence interval finite).

(b) $R$ is real if and only if $A$ is loopless.

(c) $R$ is real and compressed if and only if $A$ is simple. More generally, the signed groupoid set attached to the simplification $A^\circ$ of $A$ canonically identifies with the real compression $R^{\text{rec}}$ of $R$.

(d) $R$ is preprincipal if and only if $A$ is an acycloid.

(e) $R$ is real and principal if and only if $A$ is a simple acycloid.
(f) Assume that $A$ is the tope (pre)acyloid of an oriented matroid $M$. Then $R$ is finite, faithful, connected, simply connected, preprincipal and antipodal. Further, $R$ is real and principal (equivalently, real and compressed) if and only if $M$ is an oriented geometry. Finally, $R$ is complete (or equivalently, rootoidal) if and only if $M$ is simplicial.

Proof. (a) Note that $R$ is connected and simply connected by definition.

Suppose that $H, K ∈ T$ and $u : K → H$ is a morphism in $G$. Then

$$Φ_u = (H ∪ L^+) ∩ - (K ∩ L^+) = H ∩ K^* = H \setminus K.$$ 

If $Φ_u = ∅$, then $H ⊆ K$. Since $H ∪ H^* = K ∪ K^* = E \setminus L$, this forces $H = K$ and $H = K$, proving $R$ is faithful. This also shows that the maximum element of weak order at $H$ is the morphism $m : H^* → H$, in $G$, since $Φ_m = H ⊇ Φ_u$ for all $u$ as above.

Since $R$ is finite, it is inversion set finite and therefore interval finite by subsection 2.22.

(b) From above, we see that the set of positive imaginary roots of $R$ at any object $H$ of $G$ is $L^+$, and that $L$ is therefore the set of all imaginary roots.

(c) By (b), we may assume for the proof of the first assertion of (c) that $A$ is loopless (that is, $L = ∅$) and $R$ is real. From the definitions and the above description of inversions sets, one sees that for any $H ∈ T$, $α, β ∈ \overline{H}Φ$ are parallel in $\overline{H}Φ$ if and only if they are parallel as elements of $E$ in the acycloid $A$. We sketch the (essentially trivial) proof of the second assertion. To form $R^{rec}$, one discards its imaginary roots and identifies parallelism classes of real roots to singletons. To form $A^*$, one discards loops of $A$ and identifies the parallelism classes of non-loops in $A$ to singletons. It is straightforward to check from the definitions and facts above that the construction of the associated signed groupoid set is compatible with these corresponding deletions and identifications.

(d) From the proof of (a), the morphism $u : K → H$ in $G$ has inversion set $Φ_u = H ∩ K^* = H \setminus K$. By definition, then, the morphism $u : K → H$ is atomic in $R$ if and only if $K \neq H$ and for each $J ∈ T$, $H ∩ J^* ⊆ H ∩ K^*$ implies $H ∩ J^* = ∅$ or $H ∩ J^* = H ∩ K^*$. Note

$$J = (J ∩ H) ∪ (J ∩ H^*) = (H \setminus (H ∩ J^*)) ∪ (H ∩ J^*)$$

and similarly for $K$. Therefore, $u$ is atomic if and only if $H ≠ K$ and the only $J ∈ T$ which satisfy $H ∩ J^* ⊆ H ∩ K^*$ are $J = H$ and $J = K$.

For any $J ∈ T$, $H ∩ J^*$ is a union of parallelism classes for $A$ which are contained in $E \setminus L$. So if $e ∈ H$ and $K = (H \setminus [e]) ∪ [e]^* ∈ T$, then $u$ is atomic, since $Φ_u = [e]$ is a single such parallelism class. The argument below will show that if $A$ is an acycloid or $R$ is preprincipal, every atomic morphism $u$ in $R$ so arises.

Assume first that $A$ is an acycloid. By (a), $R$ is interval finite. Suppose that $u : K → H$ in $G$ is atomic (so $H ≠ K$). Choose by (A4) some $e ∈ H \setminus K$ such that $J := (H \setminus [e]) ∪ [e]^*$. Then $H \setminus J = [e] ⊆ H \setminus K$, which implies that $K = (H \setminus [e]) ∪ [e]^*$ and $Φ_u = [e]$. Now let $u : K → H$ be any atomic morphism, and $v : J → H$ be any morphism. The inversion set $Φ_v = J ∩ H^*$ is a union of parallelism classes, while $Φ_u$ is a single parallelism class, so either $Φ_u ⊇ Φ_v$ or $Φ_u ∩ Φ_v = ∅$. This shows $R$ is preprincipal.

Conversely, suppose that $R$ is preprincipal. Suppose $u : K → H$ is atomic. By definition of preprincipalness, for any $v : J → H$ in $G$, either $Φ_v ∩ Φ_u = ∅$ or


\[ \Phi_u \subseteq \Phi_v \]. That is, for any \( J \in \mathcal{T} \), one has either \( (H \cap J^*) \cap (H \cap K^*) = \emptyset \) or \( H \cap K^* \subseteq H \cap J^* \) (that is, either \( J^* \cap (H \cap K^*) = \emptyset \) or \( J^* \supseteq H \cap K^* \)). This together with the definition of the parallelism of the roots implies that \( H \cap K^* \) is a single parallelism class: \( H \cap K^* = [e] \) for any \( e \in H \cap K^* \subseteq E \setminus L \), and so \( K = (H \setminus [e]) \cup [e]^* \).

Now we check axiom (A4) for acycloids. Let \( \tilde{H}, \tilde{J} \in \mathcal{T} \) with \( H \neq J \). Let \( v: \tilde{J} \to \tilde{H} \) in \( \tilde{G} \). Then there is some atom \( u: \tilde{K} \to \tilde{H} \) in \( \tilde{G} \) so \( \Phi_u \subseteq \Phi_v \). That is, \( H \setminus K \subseteq H \setminus J \). Choose any \( e \in H \setminus K \). We have \( K = (H \setminus [e]) \cup [e]^* \in \mathcal{T} \) where \( e \in H \setminus J \), as required. This completes the proof of (d).

For the proof of (e), one may assume by (c) that \( A \) is simple. The result may be proved using (d) and Lemma 2.23(e).

We prove (f). Its second sentence follows from (a), (d), (e) and Lemmas 2.15 and 2.20. The completeness of \( R \) is equivalent to \( R \) being rootoidal by Lemma 2.28(e). Finally, \( R \) is complete if and only if, in the terminology of [4], the poset of regions (or topes), oriented from any fixed base region, is a (complete) lattice; this holds if and only if the oriented geometry is simplicial by [3] Theorem 6.3 and 6.5.

3. Generalized Brink-Howlett construction of signed groupoid sets

In this section we will discuss the generalized Brink-Howlett construction for signed groupoid sets with certain properties. We will show that the construction preserves those favorable properties. This will play an important role in the proof and applications of our main theorem in the next section.

In this section, \( R = (G, \Phi, \Phi^+) \) is a faithful signed groupoid set, frequently satisfying additional stated conditions.

**Definition 3.1.** A square of \( R \) is a quadruple \((x, w, y, z)\) of morphisms of \( G \) such that \( xw = yz \) and \( x(\Phi_w) = \Phi_y \):

\[
\begin{array}{ccc}
  a & \xrightarrow{w} & b \\
  \downarrow & & \downarrow \\
  c & \xrightarrow{y} & d
\end{array}
\]

**Example 7.** Assume that \( R \) is antipodal, and let \( x: b \to d \) be a morphism. Then \((x, w, y, z) = (x, x^{-1} \omega_d, x^+, (x^+)^{-1} \omega_d)\) is a square. For one has \( x(x^{-1} \omega_d) = \omega_d = x^+((x^+)^{-1} \omega_d) \), and, by Lemma 2.28,

\[
x(\Phi_{x^{-1} \omega_d}) = x(\Phi^+_x \setminus \Phi_{x^{-1}}) = d\Phi^+_x \setminus \Phi_x = \Phi^+_x
\]

(\text{where the second equality follows by checking that} \( x(\Phi^+_x \setminus \Phi_{x^{-1}}) \subseteq d\Phi^+_x \setminus \Phi_x \) and \( b\Phi^+_x \setminus \Phi_{x^{-1}} \supseteq x^{-1}(d\Phi^+_x \setminus \Phi_x) \)).

**Remark 3.2.** Let \( H \) be a category (often a groupoid in applications). Suppose given two functors \( F_1, F_2: H \to G \) and a natural transformation \( \eta: F_1 \to F_2 \). We say that \( \eta \) is a square natural transformation if for each morphism \( f: p \to q \) in \( H \), the commutative diagram

\[
\begin{array}{ccc}
  F_1 p & \xrightarrow{F_1 f} & F_1 q \\
  \downarrow \eta_p & & \downarrow \eta_q \\
  F_2 p & \xrightarrow{F_2 f} & F_2 q
\end{array}
\]
from the definition of a natural transformation gives a square \((\eta_y, F_1 f, F_2 f, \eta_p)\). The properties of squares given below imply that there is a subcategory of the category of functors \(H \to G\) with all objects but only square natural transformations as morphisms. This observation underlies basic constructions in the theory of functor rootoids, which has been a principal motivation for the development of the theory of rootoids surveyed in [15–16] and for the approach in this paper.

**Lemma 3.3.** \((x, w, y, z)\) is a square of \(R\) if and only if \(xw = yz\), \(\Phi_{x^{-1}} \cap \Phi_w = \emptyset\), \(\Phi_x \cap \Phi_y = \emptyset\), \(\Phi_z \cap \Phi_{y^{-1}} = \emptyset\) and \(\Phi_{z^{-1}} \cap \Phi_{w^{-1}} = \emptyset\).

**Proof.** Assume that \((x, w, y, z)\) is a square of a signed groupoid set \(R = (G, \Phi, \Phi^+)\). Since \(x(\Phi_w) = \Phi_y \subseteq \delta \Phi^+\), \(\Phi_{x^{-1}} \cap \Phi_w = \emptyset\). Similarly since \(x^{-1}(\Phi_y) = \Phi_w \subseteq \delta \Phi^+\), \(\Phi_x \cap \Phi_y = \emptyset\).

Suppose \(\alpha \in \Phi_z \cap \Phi_{y^{-1}}\). We show that \(wz^{-1}(\alpha) \in \delta \Phi^+\). Otherwise \(wz^{-1}(\alpha) \in \delta \Phi^+\). \(w^{-1}wz^{-1}(\alpha) = z^{-1}(\alpha) \in \delta \Phi^+\) and \(xwz^{-1}(\alpha) = y(\alpha) \in \delta \Phi^+\). This contradicts the fact \(\Phi_w \cap \Phi_{x^{-1}} = \emptyset\). Now \(-y(\alpha) \in \Phi_y\) and \(-wz^{-1}(\alpha) \in \delta \Phi^+ \setminus \Phi_y\). But this contradicts \(x^{-1}\Phi_y = \Phi_w\) as \(x^{-1}(\pm y(\alpha)) = -wz^{-1}(\alpha)\). Hence \(\Phi_z \cap \Phi_{y^{-1}} = \emptyset\).

Similarly one can prove that \(\Phi_{z^{-1}} \cap \Phi_{w^{-1}} = \emptyset\).

Conversely assume that \(xw = yz\), \(\Phi_{x^{-1}} \cap \Phi_w = \emptyset\), \(\Phi_x \cap \Phi_y = \emptyset\), \(\Phi_z \cap \Phi_{y^{-1}} = \emptyset\) and \(\Phi_{z^{-1}} \cap \Phi_{w^{-1}} = \emptyset\). We need to show that \(x(\Phi_w) = \Phi_y\). Since \(xw = yz\), \(\Phi_{x^{-1}} \cap \Phi_w = \emptyset\) and \(\Phi_z \cap \Phi_{y^{-1}} = \emptyset\), by Lemma 2.21(c) we have \(\Phi_x \cup x\Phi_w = \Phi_y \cup y\Phi_z\). Since \(\Phi_x \cap \Phi_y = \emptyset\), \(\Phi_y \subseteq x\Phi_w\). Suppose there exists \(\beta \in x\Phi_w \cap y\Phi_z\). Then \(z^{-1}y^{-1}(\beta) = w^{-1}x^{-1}(\beta) < 0\). So \(z^{-1}y^{-1}(\beta) = w^{-1}x^{-1}(\beta) > 0\) and this root is contained in \(\Phi_{w^{-1}} \cap \Phi_{z^{-1}}\), a contradiction. Hence \(x\Phi_w \cap y\Phi_z = \emptyset\). So we conclude that \(\Phi_y = x\Phi_w\). 

**Lemma 3.4.** \(x, y, z\) and \(w\) be morphisms of \(G\).

(a) \((x, w, y, z)\) is a square if and only if \((y, z, x, w)\) is a square.

(b) \((x, w, y, z)\) is a square if and only if \((w, z^{-1}, x^{-1}, y)\) is a square.

**Proof.** This follows immediately from Lemma 3.3. 

The above lemma implies that the dihedral group of order 8 acts naturally on the squares of \(G\) (and less precisely, on the characterizations of a fixed square; for example, Lemma 3.4 (a) implies that \((x, w, y, z)\) is a square if and only if \(yz = xw\) and \(y(\Phi_z) = \Phi_x\).

**Lemma 3.5.** \((x, w, y, z)\) is a square of \(R\) if and only if \(xw = yz\), \(x \lor y = xw\) and \(x^{-1} \lor w = x^{-1} y\). (Here the joins are taken with respect to the weak order of the morphisms at the corresponding object and exist in this special situation even without any assumption that the weak orders are lattices.)

**Proof.** Suppose that \((x, w, y, z)\) is a square. By Definition 3.1 and Lemma 3.3 we have \(\Phi_{x^{-1}} \cap \Phi_w = \emptyset\). So one has \(\Phi_{xw} = \Phi_x \cup x\Phi_w = \Phi_x \cup \Phi_y\). Hence \(x \lor y = xw\). The equality \(x^{-1} \lor w = x^{-1} y\) can be proven similarly. Conversely assume \((x, w, y, z)\) has the properties: \(xw = yz\), \(x \lor y = xw\) and \(x^{-1} \lor w = x^{-1} y\). Note that \(\Phi_x \subseteq \Phi_{x \lor y}\). So \(\Phi_x \subseteq \Phi_{xw}\). By Lemma 2.21(c) \(\Phi_{x^{-1}} \cap \Phi_w = \emptyset\). Similarly from \(x^{-1} \lor w = x^{-1} y\) we have \(\Phi_x \cap \Phi_y = \emptyset\). By \(yz = y \lor x\) one obtains \(\Phi_{y^{-1}} \cap \Phi_x = \emptyset\). By \(wz^{-1} = w \lor x^{-1}\) one obtains \(\Phi_{w^{-1}} \cap \Phi_{z^{-1}} = \emptyset\). Then again by Definition 3.1 and Lemma 3.3 this implies that \(x\Phi_w = \Phi_y\).

The following lemma is straightforward from the definitions and Lemma 3.3.
Lemma 3.6. Let \( x, x', y, v, w, z, z' \) be morphisms of \( R \). Suppose \( xw = yz, x'w' = y'x \). If any two of \((x, w, y, z), (x', w', y', x), (x, w'w, y'y, z)\) are squares, then the third is also a square.

\[
\begin{array}{c}
c \xrightarrow{w} d \\
\downarrow z & \downarrow x \quad \downarrow x' \\
e \xrightarrow{y} f \quad \xrightarrow{y'} b
\end{array}
\]

Proof. Assume that \((x, w, y, z)\) and \((x', w', y', x)\) are squares. Then \( xw = yz \) and \( y(\Phi_z) = \Phi_x \), and \( x'w' = y'x \) and \( y'(\Phi_x) = \Phi_{x'} \). Hence \( x'w'w = y'xw = y'y'z \) and \( (y'y)(\Phi_z) = y'(\Phi_z) = \Phi_{x'} \). Therefore \((x', w', y'y, z)\) is a square. The other assertions can be proved similarly (or reduced to this one using symmetry properties of squares).

Lemma 3.7. Assume that \( R \) is complete. Suppose that \((x, w_i, y_i, z_i), i \in I\) is a family of squares. Then there exists a square \((x, w, y, z)\) with \( w = \bigvee_i w_i \) and \( y = \bigvee_i y_i \).

\[
\begin{array}{c}
a \xrightarrow{w} b \\
\downarrow z & \downarrow x \quad \downarrow x \\
c \xrightarrow{y} d \quad \xrightarrow{y_i} d
\end{array}
\]

Proof. Note that since \( R \) is complete, the join of a set of morphisms at a given object always exists. Let \( \bigvee_i w_i = w \) and \( \bigvee_i y_i = y \). We let \( z = y^{-1}xw \). Next we show that \( xw = \bigvee_i xw_i \). Note \( x^{-1} \perp w_i \) for all \( i \) by Lemma 3.3. So by JOP (since \( R \) is rootoidal) \( x^{-1} \perp w \). So \( x \leq xw = x \bigvee_i w_i \) by Lemma 2.21 (c).

So by Lemma 2.23 (a), \( xw_i \leq xw \) and hence \( \bigvee_i xw_i \leq xw \). On the other hand, \( xw_i \leq \bigvee_i xw_i \). Also we have \( x \leq xw_i \leq \bigvee_i xw_i \). So by Lemma 2.23 (a) again \( w_i \leq x^{-1} \bigvee_i xw_i \). So \( w = \bigvee_i w_i \leq x^{-1} \bigvee_i xw_i \). By Lemma 2.23 (a) again \( xw = x \bigvee_i x w_i \leq x \bigvee_i xw_i \). Therefore \( xw = \bigvee_i xw_i \).

Now by Lemma 3.3 we only need to show that \( x \vee y = xw \) and \( x^{-1} \vee w = x^{-1}y \). Note that \( xw = \bigvee_i xw_i = \bigvee_i (x \vee y_i) = x \vee \bigvee_i y_i = x \vee y \). The second identity can be proved similarly.

Lemma 3.8. Assume that \( R \) is finite and complete. Suppose \( x, y \in _G \) with \( x \perp y \). Then there exists a unique morphism \( y' \) such that \( y \leq y' \), \((x, w', y', z')\) is a square for some \( w', z' \) and if \((x, w'', y'', z'')\) is a square with \( y \leq y'' \), then \( y' \leq y'' \).

\[
\begin{array}{c}
a \xrightarrow{w'} b \\
\downarrow z' & \downarrow x \quad \downarrow x \\
c \xrightarrow{y'} d \quad \xrightarrow{y''} d
\end{array}
\]

Proof. Assuming the existence, the uniqueness of such a morphism is evident. Now we construct explicitly such a \( y' \). Let \( y_0 = y \). Then we define a sequence \( \{y_i\}, \{w_i\} \) as follows:

\[
x \vee y_i = xw_i, \quad x^{-1} \vee w_i = x^{-1}y_i+1.
\]

We verify that \( x \perp y_i, x^{-1} \perp w_i, y_0 \leq y_1 \leq y_2 \ldots \) and \( w_0 \leq w_1 \leq w_2 \leq \ldots \). Note that \( x \leq x \vee y_i = xw_i \) so \( x^{-1} \perp w_i \) by Lemma 2.21 (c). Similarly \( x^{-1} \leq x^{-1} \vee w_i = x^{-1}y_i+1 \). So \( x \perp y_{i+1} \). The inequalities follow from the fact \( \Phi_{yi} \leq x \Phi_{wi} \).
and $\Phi_{w_i} \subseteq x^{-1}\Phi_{y_{i+1}}$. By finiteness of $G$, for sufficiently large $i$, $y_i$, and $w_i$ stabilize. We take $y' := y_i$, $w' := w_i$ for large $i$ and $z' := y'^{-1}xw'$. Then by Lemma 3.5 $(x, w', y', z')$ is a square. Assume $(x, w'', y'', z'')$ is a square with $y_0 = y \leq y''$. Then $xw_0 = x \lor y_0 \leq x \lor y'' = xw''$. Since $x^{-1} \perp w_0, w'', \text{ we have } w_0 \leq w''$ by Lemma 2.23 (a). Now we show by induction that $w_n \leq w''$ and $y_n \leq y''$ for any $n$. Taking $n$ sufficiently large will then show that $w' \leq w''$ and $y' \leq y''$. Assume that $w_i \leq w''$ and $y_i \leq y''$. Note that

$$x^{-1}y_{i+1} = x^{-1} \lor w_i \leq x^{-1} \lor w'' = x^{-1}y''.$$  

Since $x \perp y_{i+1}, y''$, we have $y_{i+1} \leq y''$ by Lemma 2.23 (a). Similarly,

$$xw_{i+1} = x \lor y_{i+1} \leq x \lor y'' = xw''.$$  

Since $x^{-1} \perp w_{i+1}, w''$, we have $w_{i+1} \leq w''$ by Lemma 2.23 (a). Then we readily see $y' \leq y''$. \hfill $\square$

We will henceforward denote $y'$ in the above lemma as $\Box y_x$.

Remark 3.9. The above lemma is a key step in the proof of our main theorem and has important consequences (adjointness properties) we don’t go into here. The construction used in its proof is a simple instance of what was referred to in [10] as the “zig-zag construction”; there is also a closely related “loop construction” mentioned there (corresponding to the situation above when $b = d$ and one additionally requires $y = w$) which we do not use in this paper. Under completeness assumptions as imposed here, both can be regarded as special cases (corresponding to a groupoid generated by a single arrow, either not a loop or a loop respectively) of a non-standard construction of certain adjoint functors in the context of fibered categories, as the first author will discuss elsewhere. The proof of many basic facts about closure of the category of rootoids under constructions discussed in [10] can, under such completeness assumptions, be given using this general construction.

Corollary 3.10. Assume that $R$ is finite and complete. Suppose $x_1, x_2, \ldots, x_p, y \in \mathcal{G}$ with $x_i \perp y$ for each $1 \leq i \leq p$. Then there exists a unique morphism $y' \in \mathcal{G}$ such that $y \leq y'$ and for each $1 \leq i \leq p$ there exists a square $(x_i, w_i, y_i, z_i)$ for some $w_i, z_i$ and for any family of squares $(x_i, w'_i, y'', z'_i)$ with $y \leq y''$ one has $y' \leq y''$.

![Diagram](https://via.placeholder.com/150)

Proof. First extend $\{x_i\}$ to an infinite sequence by requiring $x_i = x_j$ if $i \equiv j \pmod{p}$. Let $y_0 = y$ and $y_i = \Box y_{i-1}, x_i$. Then by Lemma 3.5 we have $y_0 \leq y_1 \leq y_2 \ldots$. Since $R$ is a finite signed groupoid set, this ascending chain must stabilize. Let $y' := y_i = y_{i+1} = \ldots$ for sufficiently large $i$. Then let $w_i = x_i^{-1}(x_i \lor y')$ and $z_i = y'^{-1}x_iw_i$. By our construction $(x_i, w_i, y', z_i)$ is a square. Now we let $(x_i, w'_i, y'', z'_i)$ be another family of squares with $y \leq y''$. Then by Lemma 3.8 we see $y_i \leq y''$ for all $i$, so $y' \leq y''$. Uniqueness of $y'$ is clear. \hfill $\square$

We denote $y'$ in the above lemma by $\Box y(x_1, x_2, \ldots, x_p)$.

Remark 3.11. Note that it follows from the definition that for $y \perp x_i, 1 \leq i \leq p$, $y = \Box y(x_1, x_2, \ldots, x_p)$ if and only if there exists a square $(x_i, w_i, y, z_i)$ for each $1 \leq i \leq p$.  

...
Corollary 3.12. Assume that \( R \) is finite and complete. Suppose \( x_1, x_2, \ldots, x_p \) and \( y \) are elements of \( \mathcal{G} \) with \( x_i \downarrow y \) for all \( i \). Assume that \( y = \square_y(x_1, x_2, \ldots, x_p) \) and \( u \) is a morphism such that \( y \leq yu \) and \( x_i \downarrow yu \) for all \( i \). Then \( yu \leq \square_{yu}(x_1, x_2, \ldots, x_p) \).

\[
\begin{array}{ccc}
 a_i & \xrightarrow{u_i} & b_i \\
 \downarrow z_i & & \downarrow x_i \\
 e & \xrightarrow{u} & c & \xrightarrow{y} & d
\end{array}
\]

Proof. We only need to show the equality \( \square_{yu}(x_1, x_2, \ldots, x_p) = y\square_u(z_1, z_2, \ldots, z_p) \).

To see this, one notes that the inequalities \( \square_{yu}(x_1, x_2, \ldots, x_p) \leq y\square_u(z_1, z_2, \ldots, z_p) \) and \( y^{-1}\square_{yu}(x_1, x_2, \ldots, x_p) \geq \square_u(z_1, z_2, \ldots, z_p) \) follow readily from Corollary 3.11 and Lemma 3.6 (the composition of two squares is a square). Using Lemma 2.23 (a) one sees \( \square_{yu}(x_1, x_2, \ldots, x_p) \geq y\square_u(z_1, z_2, \ldots, z_p) \), and therefore we have equality in this as required. \( \square \)

3.13. Next we introduce the generalized Brink-Howlett construction, which may be defined for any faithful signed groupoid set, though its main properties of interest here require stronger assumptions. In 3.14, \( R = (G, \Phi, \Phi^+) \) denotes a faithful signed groupoid set. First, we define a groupoid \( G^{\square} \) as follows.

The objects of \( G^{\square} \) are pairs \((a, X)\) where \( a \in \text{Ob}(G) \) and \( X \subseteq aG \). Note that \( X \) must be finite if \( G \) is finite. If \( X = \{x\} \) is a singleton, we may write \((a, x)\) in place of \((a, X)\). Let \((b, Y)\) be another object of \( G^{\square} \). A morphism \( f : (a, X) \to (b, Y) \) in \( G^{\square} \) is by definition a morphism \( f : a \to b \) of \( G \) such that there exists a (necessarily unique, by faithfulness of \( R \)) bijection \( \sigma_f : X \to Y \) for which \((f, g, \sigma_f(g), (\sigma_f(g))^{-1}fg)\) is a square for all \( g \in X \). Diagrammatically,

\[
\begin{array}{ccc}
 d_g & \xrightarrow{g} & a \\
 \downarrow (\sigma_f(g))^{-1}fg & & \downarrow f \\
 d_{\sigma_f(g)} & \xrightarrow{\sigma_f(g)} & b
\end{array}
\]

is a square for all \( g \in X \), where \( d_g \) (respectively \( d_{\sigma_f(g)} \)) denotes the domain of a morphism \( g \) (respectively \( \sigma_f(g) \)). Equivalently, we require \( f(\Phi_g) = \Phi_{\sigma_f(g)} \) for all \( g \in X \). Composition of morphisms in \( G^{\square} \) is by composition of their underlying morphisms\(^{5}\) in \( G \). Lemmas 3.4 and 3.5 imply that that \( G^{\square} \) is a groupoid.

We associate the following root system \( \Psi \) to \( G^{\square} \). For \((a, X) \in \text{Ob}(G^{\square})\), define \((a,X)^{\Psi} := a\Phi \) and \((a,X)^{\Psi^+} := a\Phi^+ \). The action of the morphisms on the root systems is inherited naturally from that of \( G \). Then \( R^{\square} := (G^{\square}, \Psi, \Psi^+) \) is clearly a faithful signed groupoid set, which is finite if \( R \) is finite. The signed groupoid set \( R^{\square} \) is said to be obtained by applying the generalized Brink-Howlett construction to \( R \). For \( g \in (a,X)G^{\square}, \) write the corresponding inversion set as \( \Psi_g := \{\alpha \in (a,X)^{\Psi^+} \mid g^{-1}(\alpha) < 0\} \).

Note that the cardinality \(|X|\) is constant as \((a, X)\) ranges over the objects of any component of \( G^{\square} \). The full subgroupoid of \( G^{\square} \) on all objects \((a, \emptyset)\) with \( a \in \text{Ob}(G) \)

\(^{5}\) To accord with common conventions that each morphism determines its domain and codomain, one should strictly denote a morphism \( f : (a, X) \to (b, Y) \) as a tuple \((b, Y, f, a, X)\) and use composition \((c, Z, g, b, Y)(b, Y, f, a, X) = (c, Z, gf, a, X)\), or use some similar artifice, but we shall not adopt such cumbersome notation.
is therefore a union of components of $G^\Box$ and is canonically isomorphic to $G$. If $G$

is inversion set finite, then the multiset $(|\Phi_g| \mid g \in X)$ is constant as $(a, X)$ ranges

over the objects of any component of $G^\Box$.

**Example 8.** Suppose that $(W, S)$ is a Coxeter group with standard root system $\Phi$

(as in [26] or [2]) and simple roots $\Pi$. Let $(G, \Phi, \Phi^+)$ be the (principal, rootoidal)

signed groupoid set discussed in Example [2] and Example [3] (1). Recall that $G$ has

one object $(\bullet)$ and $\text{Mor}(G) = W$. The objects of $G^\Box$ are pairs $(\bullet, X)$ with $X \subseteq W$.

Let $H$ denote the full subgroupoid of $G^\Box$ with objects of the form $(\bullet, I)$ where $I \subseteq S$. Since $S = \{w \in W \mid |\Phi_w| = 1\}$, the preceding paragraph implies that $H$ is

a union of components of $G^\Box$.

For $I \subseteq S$, let $\Pi_I$ be the set of simple roots such that the corresponding reflection

is in $I$. A morphism $(\bullet, I) \rightarrow (\bullet, J)$ in $H$, where $I, J \subseteq S$, is then just an element $w$ of $W$ such that $w(\Pi_I) = \Pi_J$, with composition of morphisms induced by multiplication in $W$. Thus, $H$ is canonically isomorphic to the groupoid investigated by Brink and Howlett in [7] in their study of normalizers of parabolic subgroups of Coxeter groups (see Example [4]).

**Proposition 3.14.** Assume that $R$ is finite and complete. Then the signed groupoid

set $R^\Box$ is finite and complete.

**Proof.** There is a natural injective map of sets $(a, X)_G^\Box \rightarrow \underline{a}G$. This map induces an order-isomorphism of the weak order on $(a, X)_G^\Box$ with its image, viewed as a subposet of weak order on $\underline{a}G$, since $\Psi_x = \Phi_x$ for $x \in (a, X)_G$. Let $x, y \in (a, X)_G^\Box$. Let $z = x \lor y$ in $\underline{a}G$. Then by Lemma 3.4

$z \in (a, X)_G^\Box$ and hence is the join of $x, y$ in $(a, X)_G^\Box$. Therefore $(G^\Box, \Psi, \Psi^+)$ is complete, and the above injective, order-preserving map preserves joins of all subsets of its domain. To show $R^\Box$ is rootoidal we have to show that $(a, X)_G^\Box$ satisfies JOP. But this property is clearly inherited from $\underline{a}G$. □

**Proposition 3.15.** Assume that $R$ is finite, complete and preprincipal. Then $R^\Box$

is also finite, complete and preprincipal.

**Proof.** Let $B$ be the set of atomic morphisms of $R^\Box$ and $(a, X)_B$ be the set of

atomic morphisms at the object $(a, X)$. Let $A$ be the set of atomic morphisms of $R$

and $\underline{a}A$ be the atomic morphisms of $R$ at the object $a$. We first show that if $X = \{g_1, g_2, \ldots, g_p\}$, then

$$(a, X)_B = \{\Box_s(g_1, g_2, \ldots, g_p) \mid s \in \underline{a}A, s \perp g_i, 1 \leq i \leq p\}.$$ 

Let $r \in (a, X)_B \subseteq \underline{a}G$. Choose $s \in \underline{a}A$ such that $s \leq r$. By definition (of square)

$r \perp g_i$ for all $i$. Therefore $s \perp g_i$ for all $i$ as well. So $\Box_s(g_1, g_2, \ldots, g_p)$ can be defined and $s \leq \Box_s(g_1, g_2, \ldots, g_p)$. $\Box_s(g_1, g_2, \ldots, g_p)$ is a morphism in $(a, X)_G$. By Corollary 3.10

$\Box_s(g_1, g_2, \ldots, g_p) \leq r$. But $r$ is an atomic morphism. So this inequality is actually an equality. Therefore we proved that the left hand side (that is, $(a, X)_B$) is contained in the right hand side.

Conversely, take $s \in \underline{a}A$ such that $s \perp g_i$ for all $i$. Let $r = \Box_s(g_1, g_2, \ldots, g_p)$. Before we prove the reverse inclusion, we show that for any $y \in (a, X)_G^\Box$, we have either $\Psi_r \subseteq \Psi_y$ or $\Psi_r \cap \Psi_y = \emptyset$. If $s \leq y$, then by Corollary 3.10 again, $r \leq y$ (i.e. $\Psi_r \subseteq \Psi_y$.) Otherwise $\Psi_r \cap \Psi_y = \emptyset$. So $y^{-1}s \geq y^{-1}$ thanks to Lemma 2.21(c). Then
Corollary 3.12 together with Lemma 3.3 says that \( y^{-1}s \leq y^{-1}r \). So \( y^{-1} \leq y^{-1}r \). Hence \( \Psi_y \cap \Psi_r = 0 \) by Lemma 2.21(c) again.

Now we show the reverse inclusion (i.e. \( r \in (a, X)^G \)). We may take \( u \in (a, X)^G \) such that \( u \leq r \) (note \( r \neq 1_{(a, X)} \) since \( s \leq r \) in \( aG \)). But the above paragraph with \( y = u \) implies that \( u = r \). So we are done. Along the way we have also showed the atomic morphisms of \( R_\square \) have the properties of a preprincipal signed groupoid set.

**Lemma 3.16.** Let \( a \in \text{Ob}(G) \) and \( X \subseteq aG \). Then

(a) \( \frac{1}{a} \Phi^+_\text{im} \subseteq (a, X)^G \).
(b) \( \bigcup_{x \in X} \Phi_x \subseteq (a, X)^G \).
(c) If \( R \) is rootoidal and \( X_0 \subseteq X \) is such that the join \( x_0 := \bigvee X_0 \) exists in \( aG \) (for instance, \( R \) is complete and \( X_0 = X \)), then \( \Phi_{x_0} \subseteq (a, X)^G \).
(d) If \( X = \{ x \} \) is a singleton and \( R \) is antipodal, then \( (a, X)^G \) is antipodal signed groupoid set.

**Proof.** (a) We have

\[
(a, X)^G = \bigcup_{g \in (a, X)^G} \Psi_g \subseteq \bigcup_{g \in aG} \Phi_g = \frac{1}{a} \Phi^+_\text{re}
\]

and (a) follows by taking complements in \( \frac{1}{a} \Phi^+ \).

(b) If \( g : (b, Y) \to (a, X) \) is in \( (a, X)^G \), then there is a bijection \( \sigma : X \to Y \) such that for each \( x \in X \), \((g, \sigma(x), x, g_x)\) is a square for some morphism \( g_x \) of \( G \). In particular, if \( x \in X \), we have \( g \perp x \) in \( aG \) by Lemma 5.3. That is, \( \Psi_g \cap \Phi_x = \Phi_g \cap \Phi_x = \emptyset \) and so \( \Phi_x \cap (a, X)^G = \emptyset \). Therefore \( \Phi_x \subseteq (a, X)^G \) (see the proof of (a)), proving (b).

(c) This follows on noting in the proof of (b) that \( g \perp x \) for all \( x \in X_0 \) implies \( g \perp x_0 \), by the JOP, and arguing similarly as in the end of the proof of (b).

(d) By definition of morphisms in \( R_\square \), for any object \( (b, Y) \) in the same component as \((a, X)\), the set \( Y \) is a singleton. Hence it suffices to show that \( (a, X)^G \) has a maximum element in weak order. Write \( g := x^+ : b \to a \) in \( \text{Mor}(G) \) in Lemma 2.28. By Example 4.1 g provides a morphism \( g : (b, y) \to (a, x) \) in \( (a, x)^G \) with \( y = (x^+)^{-1} \omega_y \). One has \( \Psi_g = \frac{1}{a} \Phi^+ \setminus \Phi_x \). Hence \( g \) is necessarily the maximum element in weak order in \( (a, x)^G \) since the complement of \( \Psi_g \) in \( (a, x)^G \) is contained in \( (a, x)^G \) (see Lemma 2.28(a)).

3.17. In 3.17 3.18 assume that \( R = (G, \Phi, \Phi^+) \) is a faithful, connected, simply connected signed groupoid set. Since for any objects \( a, b \) in \( G \), there is a unique morphism \( a \to b \) in \( G \), and since it is invertible, we may use the map \( \frac{1}{a} \Phi \to \frac{1}{b} \Phi \) given by action of this morphism to canonically identify \( \frac{1}{a} \Phi \) and \( \frac{1}{b} \Phi \) for all \( a, b \in \text{Ob}(G) \). We thereby identify all \( \frac{1}{a} \Phi \) with a single set \( \Phi \), and identify the function \( \frac{1}{a} \Phi \to \frac{1}{b} \Phi \) induced by action of the groupoid morphism \( a \to b \) with the identity map on \( \Phi \). Note however that \( \frac{1}{a} \Phi^+ \subseteq \Phi \) still depends on \( a \) under this identification.

Using the identification \( \Phi = \frac{1}{a} \Phi \) for any \( a \in \text{Ob}(G) \), we may unambiguously transfer the relations of dominance order \( \preceq_a \) and parallelism \( \sim_a \), and the subsets of real and imaginary roots, from \( \frac{1}{a} \Phi \) to \( \Phi \). We denote them as \( \preceq, \sim, \Phi_{\text{re}} \) and \( \Phi_{\text{im}} \) respectively.
The real compression \( R_{rec} = (G, \Phi_{rec}, \Phi_{rec}^+) \) is connected and simply connected, so one may similarly identify all its systems \( a, \Phi_{rec} \) with a single definitely involuted set \( \Phi_{rec} \) on which all groupoid morphisms act by the identity map. Concretely, \( \Phi_{rec} := \Phi_{rec}/\sim = \{[a] \mid a \in \Phi_{rec} \} \) where \([a]\) denotes the parallelism class of \( a \) in \( \Phi \). Also, \( \Phi_{rec} \) is a finally involuted set with \(-[a] = [\bar{a}]\) for \( a \in \Phi_{rec} \). Finally, we have \( a\Phi_{rec}^+ := \{[a] \mid a \in a\Phi_{rec} \cap a\Phi^+\} \) for all \( a \in \text{Ob}(G) \).

The following result follows immediately from Proposition 3.14, Proposition 3.15 and Lemma 2.26.

**Corollary 3.18.** Let \( R = (G, \Phi, \Phi^+) \) be a finite, connected, simply connected, preprincipal and complete signed groupoid set, and \( S \) denote any connected component of \( R^2 \). Then \( S \) is also a finite, connected, simply connected, preprincipal and complete signed groupoid set. Further, the real compression has the properties of \( S \) listed above, but is also real and principal.

**Example 9.** We now discuss the generalized Brink Howlett construction when \( R \) is (faithful), connected and simply connected. We consider a component \( R^2[(a, X)] \) where \( a \in \text{Ob}(G) \) and \( X \subseteq aG \). Write \( R^2[(a, X)] = (H, \Psi, \Psi^+) \). For any morphism \( g: c \to a \) in \( aG \) (for instance, \( g \in X \)), we have \( \Phi_g = a\Phi^+ \cap -\Phi^+ \) or equivalently, \( c\Phi^+ = (a\Phi^+ \setminus \Phi_g) \cup -\Phi_g \). In these formulae, terms of the form \( u\Phi^+ \) can be equivalently replaced throughout by \( u\Phi_{rec}^+ \).

Given \( g: c \to a \) as above, for any morphism \( h: d \to b \) in \( G \), there is a unique commutative diagram

\[
\begin{array}{ccc}
c & \xrightarrow{g} & a \\
\downarrow k & & \downarrow f \\
d & \xrightarrow{h} & b
\end{array}
\]

since \( G \) is connected and simply connected. Since \( f \) acts trivially on \( \Phi \), this diagram is a square of \( R \) if and only if \( \Phi_h = \Phi_g \), or equivalently, if and only if \( \Phi_g \subseteq b\Phi^+ \) and \( d\Phi^+ = (b\Phi^+ \setminus \Phi_g) \cup -\Phi_g \). In particular, for fixed \( g \) and \( b \), a square of \( R \) as above is unique if it exists: if \( f \) would have to be the unique morphism \( a \to b \) and \( h \) would have to be the morphism, if such exists, in \( aG \) with \( \Phi_h = \Phi_g \), which would be unique by faithfulness of \( R \). Still for fixed \( g \) and \( b \), such a square exists if and only if \( \Phi_g \subseteq b\Phi^+ \) and \( (b\Phi^+ \setminus \Phi_g) \cup -\Phi_g = d\Phi^+ \) for some (necessarily unique) \( d \in \text{Ob}(G) \), in which case \( h \) is the unique morphism \( d \to b \) in \( \text{Ob}(G) \).

It follows that for \( b \in \text{Ob}(G) \), there is a morphism \( f: (a, X) \to (b, Y) \) in \( H \), for some \( Y \subseteq bG \), if and only if for each \( g \in X \), there is some (necessarily unique) object \( u_g \) of \( G \) such that \( \Phi_g \subseteq b\Phi^+ \) and \( (b\Phi^+ \setminus \Phi_g) \cup -\Phi_g = u_g\Phi^+ \). Then \( Y \) is uniquely determined as

\[
Y = \{h \in bG \mid h: u_g \to b \text{ for some } g \in X\}
\]

or by

\[
Y = \{h \in bG \mid \Phi_h = \Phi_g \text{ for some } g \in X\}
\]

and \( f: (a, X) \to (b, Y) \) is the unique morphism \( f: a \to b \) in \( G \).

In particular, there is a faithful functor \( H \to G \) which maps \( (b, Y) \to b \) on objects of \( H \) and sends a morphism \( f: (b, Y) \to (c, Z) \in H \) to \( f: b \to c \) in \( G \). One may use this to canonically embed \( H \) as a subgroupoid of \( G \).
3.19. We shall use the following notation and terminology. For any faithful signed groupoid set $R = (G, \Phi, \Phi^+)$ and $X \subseteq G$, write $R//((a, X) := R^{[a, X]}$ and call it the hypercontraction of $R$ at $(a, X)$. (If $X \neq \emptyset$, we may use the convention that morphisms determine their codomains and domains to write this more compactly as $R//X$. If $X = \{g\}$, we may also write $R//g$ for $R//X$.) We commend to the reader, as an instructive exercise, checking the fact (which is not used in this paper) that a hypercontraction of a hypercontraction of $R$ is (canonically isomorphic to) a hypercontraction of $R$.

We call a signed groupoid set of the form $R//\left(\left(\left(\ldots \right)\right)\right)$, where $a_0$ is an object of $R$ and $s_i$ is an atomic morphism of $R//\left(\left(\left(\ldots \right)\right)\right)$ for $i = 1, \ldots, n$, a quasicontraction of $R$. If $n = 0$, this quasicontraction is just the component $R//\left(\left(\left(\right)\right)\right)$ of $R$ and if $n > 0$, it is equal to $R//s_1//\left(\left(\ldots \right)\right)$. In particular, quasicontractions are always connected. We call $R//s$, where $s$ is an atomic morphism of $R$, an elementary quasicontraction of $R$. Thus, the quasicontractions of $R$ form the smallest set $Q$ of signed groupoid sets with the properties that any component of $R$ is in $Q$ and $Q$ is closed under the formation of elementary quasicontractions.

Remark 3.20. The above development shows that hypercontraction (and hence quasicontraction) preserves the following subclasses of faithful, connected signed groupoid sets: (1) the finite complete ones (2) the finite, complete, and preprincipal ones, and (3) the simply connected ones. It also preserves the classes of (4) finite ones (5) complete ones (6) rootoidal ones and (7) rootoidal and preprincipal ones. Preservation of class (4) is trivial, and that of classes (5)–(7) can be proved by similar but slightly more technical arguments to those in this paper, or deduced as an application of the theory of functor rootoids sketched in [15]–[16] (for which proofs will be given elsewhere, though the main ideas for the original proofs all appear either in op. cit., in this paper or in [17]).

4. Main Result and the Proof

In this section we state and prove our main theorem that a finite, connected, simply connected, real, principal, complete signed groupoid set has the structure of a simplicial oriented geometry. Throughout this section, we adopt the conventions of [1]–[7] regarding any connected, simply connected signed groupoid set: the sets of roots at the various objects are all canonically identified so all groupoid elements act by identity maps.

**Lemma 4.1.** Let $R = (G, \Phi, \Phi^+)$ be a faithful, finite, connected, simply connected signed groupoid set. Let $A := (\Phi, *, T)$ where $T = \{a \Phi^+|a \in \text{Ob}(G)\}$ and $* : \Phi \to \Phi$ is the map $x \mapsto -x$.

(a) $A$ is a preacycloid if and only if $R$ is antipodal. This holds in particular if $R$ is complete.

(b) Assume that $A$ is a preacycloid. Then $R$ is isomorphic to the signed groupoid set $SGS(A)$ attached to $A$ in Example 2.7 (taking $L^+ := a \Phi^+_m$). Hence properties of $A$ and $R$ are related as in Proposition 2.29.

**Proof.** (a) We consider the conditions (A1)–(A3) for $A$ to be a preacycloid. Condition (A1) is trivial. Condition (A2) holds with $L := \Phi_m$. If the weak order of

\footnote{The reader may show as an exercise that “finite” is redundant in this statement, but we retain it for emphasis.}
the signed groupoid set \( R \) at \( a \) has a maximum element \( \omega_a : w_a \to a \), then Lemma 2.28(b) implies that for each \( a \in \text{Ob}(G) \), one has \( (a \Phi^{+}_{re})^* = w_{a} \Phi^{+}_{re} \), so (A3) holds. On the other hand, if (A3) holds, then for any \( a \in \text{Ob}(G) \), there is some \( w_{a} \in \text{Ob}(G) \) such that \( w_{a} \Phi^{+}_{re} = (a \Phi^{+}_{re})^* \). Letting \( \omega_{a} \) be the unique morphism \( w_{a} \to a \) in \( G \), we then have \( \Phi^{+}_{w_{a}} = a \Phi^{+}_{re} \) and so \( \omega_{a} \) is the maximum element of the weak order at \( a \) by Lemma 2.28(a). The final assertion of (a) follows from Lemma 2.28(d).

(b) Note that \( G \) has at least one object since it is connected, and so \( T \neq \emptyset \). Thus the signed groupoid set \( R' \) attached to \( A \) in Example 3 is defined. Its underlying groupoid is a connected, simply connected groupoid with objects \( \tilde{H} \) for \( H \in T \), acting trivially on the strictly involuted set \( \Phi \), with \( \Phi \). The final assertion of (a) follows from Lemma 2.28(d).

4.2. Let \( \mathcal{R} \) be the class of all faithful, finite, connected, simply connected, antipodal signed groupoid sets. Let \( A \) denote the class of all acycloids with at least one (possibly empty) tope. We note that the elementary quasicontraction of an acycloid in \( A \) may have no topes, so \( A \) is not closed under quasicontractions.

For any \( R = (G, \Phi, \Phi^{+}) \) in \( \mathcal{R} \), denote the precycloid \((\Phi, \ast, \{a \Phi^{+}_{re} \mid a \in \text{Ob}(G)\})\) in \( A \) constructed from \( R \) in Lemma 4.1 as \( \mathcal{P}A(R) \). For any \( A \) in \( A \), let \( \mathcal{SGS}(A) \) be the finite, faithful, connected, simply connected, signed groupoid set attached to \( A \) in Example 3.

**Proposition 4.3.** The maps \( \mathcal{SGS} : A \to \mathcal{R} \) and \( \mathcal{P}A : \mathcal{R} \to A \) induce inverse bijections between the set of isomorphism classes of precycloids in \( A \) and the set of isomorphism classes of signed groupoid sets in \( \mathcal{R} \).

**Proof.** By Lemma 4.1 (b), it suffices to show that if \( A \) is a precycloid in \( A \) and \( \mathcal{P}A(R) = A \), then \( \mathcal{SGS}(A) \) is an acycloid. Write \( A = (E, \ast, T) \). Let \( L = L^{+} \cup (L^{+})^* \) be the set of loops of \( A \). Write \( R = (G, \Phi, \Phi^{+}) \). By definition, \( \Phi = E \) as involuted set, so \( \mathcal{P}A(R) = (E, \ast, T') \) where \( T' = \{a \Phi^{+}_{re} \mid a \in \text{Ob}(G)\} \). By definition, \( \text{Ob}(G) = \{H \mid H \in T\} \) and \( H^{+} = H \cup L^{+} \). Since \( H \cup H^{*} = E \setminus L \) and \( H^{*} \in T \) for all \( H \in T \neq \emptyset \), it follows that \( L^{+} = \Phi^{+}_{im} \). Therefore \( \Phi^{+}_{re} = \Phi^{+}_{im} \setminus \Phi^{+}_{im} = H \). Hence \( T' = T \) as required.

**Lemma 4.4.** Let \( R = (G, \Phi, \Phi^{+}) \) be in \( \mathcal{R} \).

(a) Let \( g \in \text{Mor}(G) \). Then \( R/\!/g \) is in \( \mathcal{R} \) and \( \mathcal{P}A(R/\!/g) = \mathcal{P}A(R)/\!/\Phi_{g} \) as precycloids, where \( \mathcal{P}A(R)/\!/\Phi_{g} \) is as defined in subsection 2.1. In particular, \( \mathcal{R} \) is closed under quasicontractions.

(b) Assume that \( R \) is preprincipal (or equivalently, \( \mathcal{P}A(R) \) is an acycloid). Then the elementary quasicontractions of \( \mathcal{P}A(R) \) are precisely the precycloids \( \mathcal{P}A(R/\!/s) \) where \( s \) is an atomic morphism of \( R \).

**Proof.** (a) By Lemma 3.10(d), \( R/\!/g \) is in \( \mathcal{R} \), and denoting it as \( (H, \Psi, \Psi^{+}) \), we have \( \Psi = \Phi \) as involuted set and \( \Psi^{+}_{im} = \Phi^{+}_{im} \cup \Phi_{g} \). We have \( \mathcal{P}A(R) = (\Phi, \ast, T) \) where \( T = \{a \Phi^{+}_{re} \mid a \in \text{Ob}(G)\} \). Similarly, \( \mathcal{P}A(R/\!/g) = (\Phi, \ast, T') \) where \( T' = \{a \Phi^{+}_{re} \mid (a, h) \in \text{Ob}(H)\} \). Let \( \Gamma := \Phi_{g} \). By Example 2 (a, h) \in \text{Ob}(H) \) if and only if \( a \in \text{Ob}(G) \) is such that \( \Gamma \subseteq a \Phi^{+}_{re} \) and \( (a \Phi^{+}_{re} \setminus \Gamma) \cup -\Gamma = d \Phi^{+}_{re} \) for some \( d \in \text{Ob}(G) \);
in that case, \( \Phi_g = \Phi_h \) and

\[
(a, h) \Psi^+_{\text{re}} = (a, h) \Psi^+ = a \Phi^+ \setminus (a \Phi^+ \cup \Phi_g) = a \Phi^+ \setminus \Gamma.
\]

So, in terms of \( \mathcal{F} \), we have

\[
\mathcal{F}' = \{ H \setminus \Gamma \mid H \in \mathcal{F}, \Gamma \subseteq H \text{ and } (H \setminus \Gamma) \cup \Gamma^* \in \mathcal{F}\} = \mathcal{F}_\Gamma.
\]

where \( \mathcal{F}_\Gamma \) is as in the definition of \( \mathcal{P}(\mathcal{R})/\Gamma \) in subsection 2.10. Taking \( g \) to be an atomic morphism of \( \mathcal{R} \) proves that \( \mathcal{R} \) is closed under elementary quasicontractions, and hence it is closed under quasicontractions, proving (a).

(b) The equivalence of the two assumptions is from Proposition 2.29(d). The conclusion follows from (a) and the definition of elementary quasicontraction, since the parallelism classes in \( \Phi_{\text{re}} \) are the inversion sets of atomic morphisms, by Lemma 2.26(a) and the fact groupoid morphisms act trivially on \( \Phi \).

We say that \( R \) in \( \mathcal{R} \) and \( A \) in \( \mathcal{A} \) (or their respective isomorphism classes) correspond if \( \mathcal{R} \cong SGS(A) \) (or equivalently, \( A \cong \mathcal{P}(\mathcal{R}) \)). Part (c) of the following proposition may be viewed as a reformulation of Handa’s characterization of oriented matroids.

**Proposition 4.5.** Suppose that \( R \) in \( \mathcal{R} \) and \( A \) in \( \mathcal{A} \) correspond.

(a) \( A \) is simple if and only if \( R \) is real and compressed. More generally, the simplification of \( A \) corresponds to the real compression of \( R \).

(b) \( A \) is an acycloid if and only if \( R \) is preprincipal. In that case, the isomorphism classes of elementary quasicontractions of \( R \) correspond bijectively to the isomorphism classes of elementary quasicontractions of \( A \).

(c) \( A \) is the tope (pre)acycloid of an oriented matroid if and only if every quasicontraction of \( R \) is preprincipal.

**Proof.** (a) This follows from Lemma 2.29(c).

(b) This follows from Proposition 2.29(d) and Lemma 4.4(b). Note however that the corresponding statement with “elementary” omitted does not follow, since \( R \) may have a contraction which is not preprincipal, and to which Lemma 4.4(b) need not apply.

(c) By Theorem 2.17 \( A \) is a preacycloid attached to an oriented matroid if and only if every quasicontraction of \( A \) is an acycloid. By (b), this holds if and only if every quasicontraction of \( R \) is preprincipal.

Let us define an arbitrary signed groupoid set \( R \) to be hereditarily preprincipal if every quasicontraction of \( R \) is preprincipal. In particular, since every component of \( R \) is a quasicontraction of \( R \), by our conventions, it follows that a hereditarily preprincipal signed groupoid set is preprincipal.

**Corollary 4.6.** Let \( R \) be a signed groupoid set. Then \( R \) corresponds to the tope acycloid of some finite oriented matroid (which is then uniquely determined up to isomorphism) if and only if \( R \) is faithful, finite, connected, simply connected, hereditarily preprincipal and antipodal.

The following is the main result of this paper.

**Theorem 4.7.** (a) Let \( A \) be the tope (pre)acycloid in \( \mathcal{A} \) of a simplicial oriented matroid \( M \). Then \( R := SGS(A) \) is a faithful, finite, connected, simply connected, preprincipal and complete signed groupoid set.
(b) Let $R$ be a faithful, finite, connected, simply connected, preprincipal and complete signed groupoid set. Then $R$ is in $\mathcal{R}$ and $A := PA(R)$ is the tope (pre)acycloid of a (uniquely determined) simplicial oriented matroid $M$.

(c) In either (a) or (b), the following are equivalent:

1. $R$ is real and principal
2. $R$ is real and compressed
3. $A$ is a simple acycloid
4. $M$ is a simplicial oriented geometry.

(d) In either (a) or (b), every hypercontraction of $R$ is a faithful, finite, connected, simply connected, preprincipal and complete signed groupoid set, and so also corresponds to a (unique up to isomorphism) simplicial oriented matroid.

Proof. (a) This follows from Lemma 2.21(f).

(b) By Corollary 3.18 and Lemma 2.28, all hypercontractions of $R$ are finite, connected, simply connected, preprincipal, complete (and therefore antipodal) signed groupoid sets. In particular, all quasicontractions of $R$ are in $\mathcal{R}$ and are preprincipal, so $A$ is the tope acycloid of an oriented matroid $M$ by Proposition 4.5(c). Since $R$ is complete, Proposition 2.29(f) implies that $M$ is simplicial.

(c) This follows from Proposition 2.29 and Lemma 2.15.

(d) As observed in the proof of (b), any hypercontraction of $R$ has all the properties assumed of $R$, so this follows by applying (b) to the hypercontractions. □

Example 10. We give a reformulation of Theorem 4.7 (d) directly in terms of simplicial oriented matroids, leaving the reader to check details. Let $M = (E, \ast, \mathcal{S})$ be a simplicial oriented matroid and $A = (E, \ast, T)$ be its tope acycloid (that is, $T$ is the set of topes of $M$). Fix a tope $H \in \mathcal{S}$ and a set $X \subseteq T$ of topes. Define $U := \{H \cap K^* \mid K \in X\}$. Let $T' := \{F \in \mathcal{S} \mid U \subseteq \{F \cap K^* \mid K \in \mathcal{S}\}\}$ and $L := \bigcap_{F \in T'} F$ (note $L \supseteq \bigcup_{Y \in U} Y$). Finally, let $\mathcal{S} := \{F \setminus L \mid F \in T'\}$. Then $A//(H, X) := (E, \ast, \mathcal{S})$ is the tope acycloid of a uniquely determined simplicial oriented matroid, which one might denote by $M//(H, X)$.

The following is a straightforward consequence of Theorems 4.7 and 2.6.

Corollary 4.8. Let $R = (G, \Phi, \Phi^+)$ be a finite, faithful, connected, simply connected, preprincipal and rootoidal signed groupoid set. Let $M = (\Phi, \ast, cx)$ be the finite oriented matroid with tope acycloid $A = PA(R)$. Then for $X \subseteq \Phi$, one has

$$cx(X) = \Phi_{\text{im}} \cup \left( \bigcup_{a \in \text{Ob}(G)} \bigcap_{b \in \text{Ob}(G)} \Phi_{\text{re}}^+ \right)_{\lambda_{\phi_{\text{re}}} = X \cap \lambda_{\phi_{\text{re}}}}.$$

For the remainder of this subsection, let $R = (G, \Phi, \Phi^+)$ be a finite, connected, simply connected, real, compressed, hereditarily preprincipal and antipodal signed groupoid set. By Proposition 1.5, the corresponding acycloid $PA(R)$ is the tope acycloid of an oriented geometry $M = (\Phi, \ast, cx)$. We shall give a condition for realizability of $M$ (or more generally, its embeddability in another oriented matroid) involving an analogue of a standard condition on the relation of simple roots and positive roots of root systems in real vector spaces. First we check that two possible notions of simple roots for $R$ agree, though this is not strictly necessary.
Lemma 4.9. Let $S$ be the set of simple morphisms of $G$. Then for any $a \in \text{Ob}(G)$, one has $\bigcup_{s \in S \cap G} \Phi_s = \text{ex}(\Phi^+)$. We denote this set by $a \Pi$ and call it the set of simple roots at $a$.

Proof. We shall use the properties of extreme elements mentioned in 2.9. Suppose $s: b \to a$ is simple. Write $\Phi_s = \{\alpha\}$ where $\alpha \in \Phi$. Then $\Phi^+ \cap -b\Phi^+ = \{\alpha\}$. This implies that $a\Phi^+ \setminus \{\alpha\} = a\Phi^+ \cap -b\Phi^+$ is cx-closed. So $\alpha \in \text{ex}(\Phi^+)$, or else we would have

$$a\Phi^+ = \text{cx}(\text{ex}(\Phi^+)) \subseteq \text{cx}(\Phi^+ \setminus \{\alpha\}) = a\Phi^+ \setminus \{\alpha\}$$

which is a contradiction. This proves $\bigcup_{s \in S \cap G} \Phi_s \subseteq \text{ex}(\Phi^+)$. For the reverse inclusion, let $\alpha \in \text{ex}(\Phi^+)$. Then $\text{cx}(\Phi^+ \setminus \{\alpha\}) = \Phi^+$, and there would be a minimal subset $\Gamma \subseteq a\Phi^+ \setminus \{\alpha\}$ with $\text{cx}(\Gamma) = \Phi^+$. But then $\Gamma$ would be a minimal subset of $\Phi^+$ satisfying $\text{cx}(\Gamma) = \Phi^+$, so $\Gamma = \text{ex}(\Phi^+)$, contrary to $\alpha \in \text{ex}(\Phi^+) \setminus \Gamma$. Since $a\Phi^+ \setminus \{\alpha\}$ is a closed subset of a hemispace, it is an intersection of hemispaces by Theorem 2.10. This implies that $\text{ex}(\Phi^+ \setminus \{\alpha\}) \cup \{\alpha\}$ is a hemispace, say equal to $\Phi^+$. The unique morphism $s: b \to a$ has $\Phi_s = \{\alpha\}$. Thus, $\alpha \in \Phi_s$ where $s \in S \cap aG$, as required. \hfill \Box

For the proof of the theorem below, we use the following definition.

Definition 4.10. We say that a subset $A$ of $\Phi$ is a half set if $A \cup A^* = \Phi$. For half sets $A$ and $B$ of $\Phi$, we define $d(A, B) = \frac{|A + B|}{2} = |A| + |B| - |A \cap B|$ where $+$ denotes the symmetric difference operation (that is, $A + B := (A \cup B) \setminus (A \cap B)$).

Definition 4.11. Let $R = (G, \Phi, \Phi^+)$ as above and let $M' = (E, -, c)$ be any (possibly infinite) oriented matroid. We define an embedding of $R$ in $M'$ to be an injective map $f: \Phi \to E$ such that $f(\alpha^*) = -f(\alpha)$ for any $\alpha \in \Phi$ and one has $c(f(a\Pi)) \cap f(\Phi) = f(a\Phi^+)$ for any $a \in \text{Ob}(G)$.

If there is real vector space $V$ such that $M'$ is the standard oriented matroid $M' = (E, -, c)$ (where $E = V \setminus \{0\}$, $-x$ is the additive inverse of $x \in E$ and $c(X) = \text{cone}(X) \cap E$ for all $X \subseteq E$), we also call an embedding $f$ of $R$ in $M'$ a realization of $R$ in $V$.

Theorem 4.12. Let $R$ be as above, $M' = (E, -, c)$ be any (possibly infinite) oriented matroid and $f: \Phi \to E$ be an embedding of $R$ in $M'$ in the above sense. Then $f$ induces an isomorphism of oriented matroids from $M$ (the oriented matroid associated to $R$) to the restriction of $M'$ to $f(\Phi)$.

Proof. We use Lemma 2.12 for $M$ and $M'$. We need only verify its hypotheses. The definition of embedding guarantees that for any hemispace $a\Phi^+$ of $M$, one has $c(f(a\Phi^+)) \cap f(\Phi) = f(a\Phi^+)$. If the hypotheses in Lemma 2.12 fails, there is therefore some half set $A$ of $\Phi$ such that $c(f(A)) \cap f(\Phi) = f(A)$ but $A \neq a\Phi^+$ for any $a \in \text{Ob}(G)$. Let $\Phi^+$ be such that $d(\Phi^+, A)$ is minimal. We claim that $a\Pi \subseteq A$. Suppose that there exists $\beta \in A \setminus a\Pi$. Then $\{\beta\}$ is simple. Write $\Phi = \{\alpha\}$ where $\alpha \in \Phi$. Then $\Phi^+ \setminus \{\beta\} \cup \{-\beta\} = a\Phi^+$ for some $c \in \text{Ob}(G)$. Then $d(\Phi^+, A) = d(\Phi^+, A) - 1$. This is a contradiction. So we have established the claim. But $f(A) = c(f(A)) \cap f(\Phi) \supseteq c(f(\Pi)) \cap f(\Phi) = f(\Phi^+)$. So this forces $f(A) = f(\Phi^+)$ and thus $A = A\Phi^+$, which is a contradiction. \hfill \Box

4.13. Suppose $R$ has a realization $f$ in the above sense. Then the associated oriented geometry $M$ is isomorphic to the oriented geometry attached to the set of vectors $f(\Phi)$ in $V$. Let $V_0 := \text{span}(f(\Phi))$ in $V$. As discussed in Example 1.
the linear hyperplanes in $V_0$ orthogonal to the elements of $f(\Phi)$ give a real, finite, essential, linear hyperplane arrangement in $V_0$, associated to $M$. In particular, the chambers of this arrangement correspond bijectively to the tope s of $M$ and thus also to the objects of $G$.

The above all applies in particular when $R$ is a finite, connected simply connected, principal and complete signed groupoid set. In that case $M$ is a simplicial oriented geometry and the above hyperplane arrangement is simplicial.

5. Final comments and open questions

5.1. In parts of this paper, we have worked only with connected and simply connected signed groupoid sets. For many purposes (though of course not, for example, in parameterizing the components of signed groupoids sets), this does not involve a significant loss of generality: one can work with the (closely related) universal covers of the components. In particular, our main results apply to universal covers of signed groupoid sets attached to finite Coxeter groups and Weyl and Coxeter groupoids. It is already well known that these covers correspond to special realizable simplicial geometries and to simplicial hyperplane arrangements, which have been studied quite deeply in the finite Coxeter group case.

5.2. One consequence of our main theorem is that it permits a purely algebraic and combinatorial construction and study of the underlying oriented matroid of the standard root system of a finite Coxeter group, and associated structures, without involving the standard root system in a real vector space, as follows.

Given a Coxeter system $(W, S)$, let $T = \{ wsw^{-1} | w \in W, s \in S \}$ be the set of abstract reflections. Regard $\Phi := T \times \{ \pm \}$ as strictly involuted set with involution $-(t, \pm) = (t, \mp)$ and $W$-action determined by $s(s, \pm) = (s, \mp)$ and $s(t, \pm) = (sts, \pm)$ if $t \neq s$, for $s \in S$ and $t \in T$. We call $\Phi$ the standard abstract root system of $(W, S)$; see [5, Ch IV, §1, no. 4] or [2, 1.3]. Define

$$\Phi^+ := T \times \{ + \} \quad \text{and} \quad \Xi := \{ w(\Phi^+) | w \in W \}.$$ 

If $W$ is finite (as we assume for now) then $A = (\Phi, -, \Xi)$ is the tope acycloid of a simplicial oriented geometry on which $W$ acts as a group of automorphisms.

This is easy to see using the natural identification of $\Phi$ as $W$-set with the standard root system of $(W, S)$ in a real vector space, but it is not so straightforward to verify otherwise (for example, from Handa’s (or other) characterizations of oriented matroids in terms of their tope s, or in terms of axioms for the oriented circuits or closure operator as may be defined in terms of $A$; see [4]). Using our theorem, the result follows directly, though we won’t give details; the most delicate fact required is that the weak order of a (finite) Coxeter group is a meet semilattice (which may be proved without recourse to realized root systems, for instance as in [2]).

5.3. Let $R$ denote the (finite, principal, complete) signed groupoid set attached above to the finite Coxeter system $(W, S)$. Recall we defined hypercontractions of $R$ as components of the generalized Brink Howlett construction applied to $R$. We remark that in general, many hypercontractions will be “trivial” or “small”, and

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A better understanding of such matters is relevant to study of root systems of infinite Coxeter groups (for which there is no canonical $W$-stable oriented matroid structure on the abstract root system, often many non-isomorphic such structures arising from various realized root systems, and conjecturally many more non-realizable such structures, though none are known so far.)
many sets of them will all be canonically isomorphic for general reasons related to
certain Galois connections. Nonetheless, there is still considerable richness in the
class of hypercontractions (for example, they include the components of the original
groupoid studied by Brink and Howlett).

Our main theorem implies the hypercontractions all have the same properties
as listed for $R$, so they (more precisely, the real compressions of their universal
covers) are isomorphic to signed groupoid sets associated to simplicial oriented
geometries. Though we haven’t given details in this paper, there are more general
constructions in categories of rootoids (functor rootoids and categorical limits, for
instance) which may be interpreted as constructions preserving the relevant class
of signed groupoid sets, and so produce simplicial oriented geometries from families
of simplicial oriented geometries, by the main theorem of this paper.

An important point is that while $R$ is known to be associated to a realizable
simplicial oriented geometry, it is not known whether simplicial oriented geometries
associated to its hypercontractions are realizable. Realizability is currently
known to hold only in very special classes of examples, by techniques which do not
extend in an obvious way to the general situation. Similar remarks apply to signed
groupoid sets $R$ associated to Coxeter groupoids and indeed to those associated
to realizable simplicial oriented geometries in general. Approaches to the study
of Coxeter groupoids and Weyl groupoids in the literature require the existence of
a suitable realized root system in order to develop their basic properties, and an
abstract construction of the associated signed groupoid set and its corresponding
simplicial oriented geometry, as discussed above for finite Coxeter groups, is not
currently available.

5.4. With these general comments in hand, we list below a few of many ques-
tions and problems we leave open in this work. It is quite possible that simple
counterexamples, constructions or arguments could settle some of them.

(1) Is the simplicial oriented geometry associated to (the real compression of
the) hypercontraction of the signed groupoid set attached to a realizable
simplicial oriented geometry itself realizable? If so, is the analogous state-
ment true for other constructions from [15]–[16]. If not, are there conditions
under which realizability is preserved by such constructions, either in gen-
eral or for natural subclasses such as signed groupoid sets from Coxeter
groups or Weyl or Coxeter groupoids. When realizability is preserved, one
has additional questions of whether rationality properties are preserved; if
one starts with a crystallographic, in a suitable sense, realized root system
for the original signed groupoid set, is there a natural crystallographic root
system for the signed groupoid set constructed from it.

(2) Develop a general theory of signed groupoids sets enriched by compatible
(possibly infinite) oriented matroid structures on their root systems at the
various objects. Use these in particular as a framework for an extended
theory of (possibly infinite) Weyl and Coxeter groupoids. Study these in
particular for (infinite) Coxeter groups with a view to constructing non-
realizable examples.

(3) Under what conditions are (real compressions of) hypercontractions
of signed groupoid sets from finite Coxeter groups (and from Weyl or Cox-
eter groupoids) themselves signed groupoid sets from finite Weyl or Coxeter
groupoids (in a suitably generalized sense as in (2)). Similarly for infinite Coxeter groups and groupoids.

(4) A faithful, connected, simply connected, antipodal signed groupoid set can be attached to any (possibly infinite) oriented matroid, in a very similar way as for the case of finite oriented matroids (see 2.13). Develop a natural characterization for the class of these signed groupoid sets (the finite ones are characterized by 4.6). Does the subclass of those, all of whose weak orders are complete lattices, form a reasonable generalization to infinite ground sets of the class of simplicial oriented matroids? See [28] for an example.

(5) Are signed groupoid sets attached to natural subclasses of finite oriented matroids stable under (natural subclasses of) hypercontractions? This paper proves that this holds for the class of simplicial oriented matroids and all hypercontractions, but it is open for the class of all finite oriented matroids. Similarly for other constructions and for signed groupoid sets from infinite oriented matroids.

(6) Study hereditarily preprincipal signed groupoid sets as a generalization of the class of signed groupoid sets attached to finite oriented matroids (which forms a subclass of hereditarily preprincipal ones by 4.6).

(7) The combinatorics of squares is enormously rich (see [10] for some indications of this). However, not much is known about the detailed combinatorics of squares in special cases. It is a natural problem to classify or describe more concretely the squares (or more generally, hypercubes: hypercubical diagrams all of the two dimensional faces of which are squares) in various special settings; for example, in signed groupoid sets from symmetric groups, classical Weyl groups, finite Coxeter groups and Coxeter groupoids, finite simplicial oriented matroids, finite oriented matroids, general Coxeter groups, general oriented matroids etc.

(8) We finish with a more concrete question related to (3) and (7). Suppose that $R = (G, \Phi, \Phi^+)$ is the (not simply connected) signed groupoid set attached to a Coxeter group (or perhaps a Coxeter groupoid) $W$. Consider some hypercontraction $R/(a, X) = (H, \Psi, \Psi^+)$ and a self-composable simple morphism $s: a \rightarrow a$ of $H$. Is $s$ necessarily an involution (that is, does it satisfy $s^2 = 1_a$)? This would be necessary for $R/(a, X)$ to come from a Coxeter groupoid (with the atomic morphisms of $H$ as its simple morphisms).

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References

[1] David Bessis. Finite complex reflection arrangements are $K(\pi,1)$. Ann. of Math. (2), 181(3):809–904, 2015.
[2] Anders Björner and Francesco Brenti. Combinatorics of Coxeter groups, volume 231 of Graduate Texts in Mathematics. Springer, New York, 2005.
[3] Anders Björner, Paul H. Edelman, and Günter M. Ziegler. Hyperplane arrangements with a lattice of regions. Discrete Comput. Geom., 5(3):263–288, 1990.
[4] Anders Björner, Michel Las Vergnas, Bernd Sturmfels, Neil White, and Günter M. Ziegler. 
Oriented matroids, volume 46 of Encyclopedia of Mathematics and its Applications. 
Cambridge University Press, Cambridge, second edition, 1999.
[5] Nicolas Bourbaki. Lie groups and Lie algebras. Chapters 4–6. Elements of Mathematics 
(Berlin). Springer-Verlag, Berlin, 2002. Translated from the 1968 French original by Andrew 
Pressley.
[6] Brigitte Brink and Robert B. Howlett. A finiteness property and an automatic structure for 
Coxeter groups. Math. Ann., 296(1):179–190, 1993.
[7] Brigitte Brink and Robert B. Howlett. Normalizers of parabolic subgroups in Coxeter groups. 
Invent. Math., 136(2):323–351, 1999.
[8] J. Richard Büchi and William E. Fenton. Large convex sets in oriented matroids. J. Combin. 
Theory Ser. B, 45(3):293–304, 1988.
[9] Raul Cordovil. A combinatorial perspective on the non-radon partitions. J. Combin. Theory, 
Ser A, 38:38–47, 1985.
[10] M. Cuntz and I. Heckenberger. Weyl groupoids with at most three objects. J. Pure Appl. 
Algebra, 213(6):1112–1128, 2009.
[11] M. Cuntz and I. Heckenberger. Finite Weyl groupoids of rank three. Trans. Amer. Math. 
Soc., 364(3):1369–1393, 2012.
[12] Michael Cuntz and István Heckenberger. Weyl groupoids of rank two and continued fractions. 
Algebra Number Theory, 3(3):317–340, 2009.
[13] Michael Cuntz and István Heckenberger. Finite Weyl groupoids. J. Reine Angew. Math., 
702:77–108, 2015.
[14] Pierre Deligne. Les immeubles des groupes de tresses généralisés. Invent. Math., 17:273–302, 
1972.
[15] M. J. Dyer. Groupoids, root systems and weak order I. [arXiv:1110.3217 [math.GR], 2011.
[16] M. J. Dyer. Groupoids, root systems and weak order II. [arXiv:1110.3657 [math.GR], 2011.
[17] Matthew Dyer. On the Weak Order of Coxeter Groups. Canad. J. Math., 71(2):299–336, 
2019.
[18] Paul H. Edelman. The lattice of convex sets of an oriented matroid. J. Combin. Theory Ser. 
B, 33(3):239–244, 1982.
[19] Paul H. Edelman and Robert E. Jamison. The theory of convex geometries. Geom. Dedicata, 
19(3):247–270, 1985.
[20] Jon Folkman and Jim Lawrence. Oriented matroids. J. Combin. Theory Ser. B, 25(2):199– 
236, 1978.
[21] Komei Fukuda and Keiichi Handa. Antipodal graphs and oriented matroids. Discrete Math., 
111(1-3):245–256, 1993. Graph theory and combinatorics (Marseille-Luminy, 1990).
[22] Keiichi Handa. A characterization of oriented matroids in terms of topes. European J. Combin., 
11(1):41–45, 1990.
[23] I. Heckenberger. The Weyl groupoid of a Nichols algebra of diagonal type. Invent. Math., 
164(1):175–188, 2006.
[24] István Heckenberger and Volkmar Welker. Geometric combinatorics of Weyl groupoids. J. 
Algebraic Combin., 34(1):115–139, 2011.
[25] István Heckenberger and Hiroyuki Yamane. A generalization of Coxeter groups, root systems, 
and Matsumoto’s theorem. Math. Z., 259(2):255–276, 2008.
[26] James E. Humphreys. Reflection groups and Coxeter groups, volume 29 of Cambridge Studies in 
Advanced Mathematics. Cambridge University Press, Cambridge, 1990.
[27] Victor G. Kac. Infinite-dimensional Lie algebras. Cambridge University Press, Cambridge, 
1990.
[28] Annette Pilkington. On the weak order of orthogonal groups. Comm. Algebra, 42(5):1965– 
1993, 2014.
[29] D. J. A. Welsh. Matroid theory. Academic Press [Harcourt Brace Jovanovich, Publishers], 
London-New York, 1976. L. M. S. Monographs, No. 8.
Department of Mathematics, 255 Hurley Building, University of Notre Dame, Notre Dame, Indiana 46556, U.S.A.
E-mail address: dyer@nd.edu

School of Mathematics (Zhuhai), Sun Yat-sen University, Zhuhai, Guangdong, 519082, China
E-mail address: wangweij5@mail.sysu.edu.cn