An asymptotic solution to a passive biped walker model

Sergey A Yudaev\textsuperscript{1,*}, Dmitrii Rachinskii\textsuperscript{2} and Vladimir A Sobolev\textsuperscript{1}

\textsuperscript{1} Samara National Research University, Russia
\textsuperscript{2} Department of Mathematical Sciences, The University of Texas at Dallas, USA
\textsuperscript{*}Corresponding author: s.a.yudaev@gmail.com

Abstract. We consider a simple model of a passive dynamic biped robot walker with point feet and legs without knee. The model is a switched system, which includes an inverted double pendulum. Robot’s gait and its stability depend on parameters such as the slope of the ramp, the length of robot’s legs, and the mass distribution along the legs. We present an asymptotic solution of the model. The first correction to the zero order approximation is shown to agree with the numerical solution for a limited parameter range.

1. Introduction

Passive dynamic walkers have been widely developed since 1990 when they were introduced [1]. The problem of legged robot locomotion continues to generate interest of researchers attempting to improve the design of walkers.

Most of us do walking as a daily routine without any thought or care. Nevertheless, walking is a complex process depending on various factors. Human body, controlled by nervous system, involves skeletal muscles and limbs to reproduce an efficient and natural gait. From there, biped gait dynamics are examined by several disciplines. Passive walkers constitute a class of robots that use gravity to power their dynamics. Several experimental studies have shown that this kind of walking is possible with reasonable stability over a range of slopes without any actuation [2–4]. Further, passive dynamics designs were adopted for models with different kind of actuation strategies [5, 6]. However, relatively few analytic and asymptotic results are available for non-actuated walking. In this paper, we consider a simple biped model, which extends the model that has been comprehensively analyzed in [7]. We follow a similar approach, but extend the study by including additional point masses of the legs and allowing the positions of these masses to vary.

The paper organized as follows. Section 2 contains a brief description and assumptions of the model. Equations of motion and equations of transition are presented in Section 3. Section 4 presents asymptotic results, which are complemented by, and compared to, numerical simulations.

2. Model Description

Figure 1 presents a simple passive walker in two dimensions. Given appropriate initial conditions and placed at the top of the ramp with a small slope, this robot starts moving.

The model consists of two rigid legs. The sway leg and the stance leg make angles $\theta_{sw}$, $\theta_{st}$ with the upward direction, respectively. We consider kneeless legs of mass $m$ and length $l$ connected
through a hip joint of mass $M$. The point mass $m$ is placed at a distance $a$ from the hip joint and a distance $b = l - a$ from the foot (tip of the leg). The ramp slope is denoted by $\alpha$. All the angles are measured counterclockwise.

A few simplifying assumptions make the system better tractable. No friction is included at the leg joint. A perfectly rigid ground is assumed. Foot collision is assumed to be absolutely plastic. As a consequence of this, the foot impact and foot transition are instantaneous. At the time of the impact, $t_{imp}$, the distance between the foot of the swing leg and the walking surface becomes zero, hence the constraint $\theta_{sw}(t_{imp}) + \theta_{st}(t_{imp}) = -2\alpha \mod 2\pi$ is met. At this instant, the feet are renamed (switched), the stance leg becomes the swing leg and vice versa. The stance foot is assumed to be fixed on the ground until the impact occurs. The period of motion between the foot transitions is called the gait cycle.

3. Equations of Motion

The model is hybrid since continuous motion is interrupted by discrete events. More precisely, the continuous motion during the gait cycle is followed by a discrete legs’ switch. Equations of motion for this hybrid model were obtained in [8]. Introducing new parameters $\beta = m/M$, $\nu = b/l$, and $\mu = a/l = 1 - \nu$, one can describe the gait cycle as the second order system of the form

$$ M(\theta) \ddot{\theta} + N(\theta, \dot{\theta}) \dot{\theta} + G(\theta) = 0, \quad (1) $$

where the zero right hand side corresponds to passive dynamics. Here $\theta = (\theta_{sw}, \theta_{st})^T$ is the vector of angular coordinates; $M(\theta)$ represents the inertia matrix; the matrix $N(\theta, \dot{\theta})$ accounts for centrifugal and Coriolis forces; and, $G(\theta)$ contains the gravity terms:

$$ M(\theta) = \begin{bmatrix} \beta \nu^2 & -\beta \nu \cos \varphi \\ -\beta \nu \cos \varphi & 1 + \beta + \beta(1 - \nu)^2 \end{bmatrix}, $$

$$ N(\theta, \dot{\theta}) = \begin{bmatrix} 0 & \beta \nu \dot{\theta}_{st} \sin \varphi \\ -\beta \nu \dot{\theta}_{sw} \cos \varphi & 0 \end{bmatrix}, $$

$$ G(\theta) = \frac{g}{l} \begin{bmatrix} \beta \nu \sin \theta_{sw} \\ -(1 + 2\beta - \beta \nu) \sin \theta_{st} \end{bmatrix}, $$

where we denote $\varphi = \theta_{st} - \theta_{sw}$. 

![Figure 1. A simple biped robot walker on a slope.](image)
Since centrifugal and Coriolis forces affect the walker motion only slightly, we ignore the term \( N(\theta, \dot{\theta}) \dot{\theta} \). Then, equation (1) becomes
\[
M(\theta)\ddot{\theta} + G(\theta) = 0. \tag{2}
\]

To describe the switch of stance and swing legs, we use the algebraic transition equation
\[
\theta^+ = J\theta^- \quad \text{with} \quad J = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},
\]
which relates the pre-impact and post-impact coordinate values. Here, the “−” and “+” superscripts refer to the values of state variables before and after the impact, respectively. The conservation of the angular momentum imposes a relation between the pre- and post-impact angular velocities:
\[
\dot{\theta}^+ = [Q(\varphi)]^{-1}P(\varphi)\dot{\theta}^-,
\]
where
\[
P(\varphi) = \begin{bmatrix} (1 + 2\beta - 2\beta\nu)\cos \varphi & -\beta(1 - \nu)\nu \\ -\beta(1 - \nu)\nu & 0 \end{bmatrix},
\]
\[
Q(\varphi) = \begin{bmatrix} \beta\nu(1 - \cos \varphi) & \beta + \beta(1 - \nu)^2 + 1 - \beta\nu\cos \varphi \\ \beta\nu^2 & -\beta\nu\cos \varphi \end{bmatrix}.
\]

**Figure 2.** Periodic solution on the phase space diagram.

We reproduced some numerical results from [9] for the hybrid model. Figure 2 presents the phase diagram of one gait cycle for a periodic solution (of period \( \tau \)). Because the motion is periodic,
\[
\theta_{st}^+(\tau) = \theta_{sw}^+(0), \quad \theta_{sw}^+(\tau) = \theta_{st}^+(0), \quad \dot{\theta}_{st}^+(\tau) = \theta_{sw}^+(0), \quad \dot{\theta}_{sw}^+(\tau) = \theta_{st}^+(0).
\]
The switch corresponds to the instantaneous transition from \( \theta_{st}^- \) to \( \theta_{sw}^+ \) and from \( \theta_{sw}^- \) to \( \theta_{st}^+ \).
The time trace of a periodic solution for several gait cycles is presented in Figure 3. Vertical jumps correspond to the switching moments defined by the relationship \( \theta_{sw}(t_{imp}) + \theta_{st}(t_{imp}) = -2\alpha \pmod{2\pi} \). Parameters for both Figures 2 and 3 are \( \beta = 0.01, \nu = 0.2, \mu = 0.8, \) and \( \alpha = 0.0061 \) rad.

![Time trace of the periodic solution.](image)

**Figure 3.** Time trace of the periodic solution.

The dependence of the gait cycle on parameters has been studied in [10].

4. Asymptotic Solution

We use perturbation method in order to study walking cycles for \( \alpha \ll 1 \) and \( \beta \ll 1 \). System (2) written componentwise has the form

\[
\nu \ddot{\theta}_{sw} - \cos \varphi \ddot{\theta}_{st} + \frac{g}{l} \sin \theta_{sw} = 0, \\
-\beta \nu \cos \varphi \ddot{\theta}_{sw} + (1 + \beta + \beta(1 - \nu^2)) \ddot{\theta}_{st} - \frac{g}{l} (1 + 2\beta - \beta \nu) \sin \theta_{st} = 0.
\]

Rearranging these equations, one obtains

\[
q(\varphi) \ddot{\theta}_{st} - \beta \cos \varphi \sin \theta_{sw} - p \sin \theta_{st} = 0, \quad (3)
\]

\[
\nu q(\varphi) \ddot{\theta}_{sw} - p \cos \varphi \sin \theta_{st} - (p(\nu - 1) + \nu + \beta \nu) \sin \theta_{sw} = 0, \quad (4)
\]

where \( p = \beta \nu - 2\beta - 1 \) and

\[
q(\varphi) = (\beta \cos \varphi^2 - 2\beta + 2\beta \nu - \beta \nu^2 - 1) \frac{l}{g}.
\]
Following [7], we introduce the scaling
\[ \alpha = \varepsilon^3, \quad \theta_{st} = \varepsilon \theta_{st}, \quad \theta_{sw} = \varepsilon \theta_{sw}, \]
where \( \theta_{sw} \equiv \theta_{sw}(t), \theta_{st} \equiv \theta_{st}(t) \). Substituting these formulas into equations (3), (4) and expanding the equations in a power series with respect the small parameter \( \varepsilon \) results in two governing equations with no zero order terms. After dividing both equations by \( \varepsilon \), the expansion contains only even powers of \( \varepsilon \). Therefore, the parameter \( \delta = \varepsilon^2 \) is introduced to obtain
\[
k \ddot{\theta}_{st} + p \dot{\theta}_{st} + \beta \theta_{sw} + \left( s(\theta_{st}, \theta_{sw}) \ddot{\theta}_{st} - \frac{p}{6} \dot{\theta}_{st}^3 - \frac{\beta}{2} \theta_{sw} \theta_{st}^2 + \beta \theta_{sw}^2 \theta_{st} - \frac{2 \beta}{3} \theta_{sw}^3 \right) \delta = 0,
\]
\[
k \nu \ddot{\theta}_{sw} + p \dot{\theta}_{sw} + j \theta_{sw} + \left( s(\theta_{st}, \theta_{sw}) \nu \ddot{\theta}_{sw} - p \left( \frac{2}{3} \dot{\theta}_{st}^3 - \theta_{sw} \theta_{st}^2 + \frac{1}{2} \theta_{sw}^2 \theta_{st} \right) - \frac{j}{6} \theta_{sw}^3 \right) \delta = 0,
\]
where \( j = \beta \nu^2 - 2 \beta \nu + 2 \beta + 1 \), \( k = (\beta \nu^2 - 2 \beta \nu + \beta + 1)l/g \), and
\[ s(\theta_{st}, \theta_{sw}) = \frac{l}{g} (\theta_{sw} - \theta_{st})^2. \]

Next, we use the regular expansion with respect to two independent small parameters \( \delta \) and \( \beta \) up to the linear terms:
\[
\theta_{sw} = \theta_{sw0} + \delta \theta_{sw1} + \beta \theta_{sw2}, \quad \theta_{st} = \theta_{st0} + \delta \theta_{st1} + \beta \theta_{st2}.
\]
The regular expansion results in the following equations for the zero order approximations:
\[
\ddot{\theta}_{st0} - \frac{g}{l} \theta_{st0} = 0, \quad \tag{6}
\ddot{\theta}_{sw0} - \frac{g}{\nu l} (\theta_{st0} - \theta_{sw0}) = 0; \quad \tag{7}
\]
equations for the first corrections with respect to the parameter \( \delta \):
\[
\ddot{\theta}_{st1} - \frac{g}{l} \theta_{st1} = -\frac{g}{6l} \theta_{sw0}^3, \quad \tag{8}
\ddot{\theta}_{sw1} + \frac{g}{\nu l} \theta_{sw1} = \frac{g}{\nu l} \left( \theta_{st1} - \frac{2}{3} \theta_{st0}^3 + \theta_{sw0}^2 \theta_{st0} - \frac{1}{2} \theta_{st0} \theta_{sw0}^2 + \frac{1}{6} \theta_{sw0}^3 \right); \quad \tag{9}
\]
and, equations for the first corrections with respect to the parameter \( \beta \):
\[
\ddot{\theta}_{st2} - \frac{g}{l} \theta_{st2} = (1 + \nu - \nu^2) \frac{g}{l} \theta_{st0} - \frac{g}{l} \theta_{sw0}, \quad \tag{10}
\ddot{\theta}_{sw2} + \frac{g}{\nu l} \theta_{sw2} = (1 - \nu + 1/\nu) \frac{g}{l} \theta_{st0(t)} + \frac{g}{\nu l} (\theta_{st2} - \theta_{sw0}); \quad \tag{11}
\]

Equations (6)-(11) can be easily solved consecutively.

The functions \( \theta_{st}(t) = \varepsilon \theta_{st}(t) \), \( \theta_{sw}(t) = \varepsilon \theta_{sw}(t) \) obtained from formulas (5)-(11) can be used to approximate the motion during the gait cycle between the moments when the legs are switched. Figure 4 compares this approximation with a numerical solution of the equations of motion (2) for \( \alpha^2/3 = \varepsilon^2 = 0.0086 \), \( \beta = 0.01 \).
Figure 4. A numerical solution of equation (2) (dotted line) and the asymptotic approximation (solid lines).

Conclusions
We considered a hybrid model of a biped passive walker with two independent small parameters, the ramp slope, $\alpha$, and the ratio of the mass of the leg to the mass at the hip joint, $\beta$. Using the regular asymptotic expansion, we obtained an approximating system and showed that its stable periodic solution is in a good agreement with the numerical solution of the model for small $\alpha$ and $\beta$. The asymptotic solution including the first correction to the zero order approximation improves the agreement between the solutions. It should be noted that the model of the biped walker naturally reproduces a stable gait on small slopes (small $\alpha$) for a relatively wide range of possible ratios between leg and body masses, $\beta$. The asymptotic approximation that we obtained by the perturbation method works well when both $\alpha$ and $\beta$ are small.

Acknowledgements
SY and VS were supported in part by the Russian Foundation for Basic Research through grant 16-41-630524 and by the Ministry of Education and Science of the Russian Federation under the Competitiveness Enhancement Program of Samara University (2013-2020). DR acknowledges the support of NSF through grant DMS-1413223.

References
[1] McGeer T 1990 Passive Dynamic Walking. International Journal of Robotics Research, 9(2), 62-82
[2] Spong M W and Bhatia G 2003 Further results on control of the compass gait biped. International Conference on Intelligent Robots and Systems, 1933-1938
[3] Goswami A, Espiau B and Keramane A 1996 Limit cycles and their stability in a passive bipedal gait. IEEE Conference on Robotics and Automation, 246-251
[4] McGeer T 1993 Dynamics and control of bipedal locomotion. Journal of Theoretical Biology, 166(3), 277-314
[5] Arthur D. Kuo 2002 Energetics of actively Powered locomotion using the simplest walking model. Journal of Biomechanical Engineering 124, 113-120
[6] Collins S, Ruina A, Tedrake R and Wisse M 2005 Efficient bipedal robots based on passive dynamic walkers
Science 307, 1082-1085

[7] Garcia M, Chatterjee A, Ruina A and Coleman M.J 1998 The Simplest Walking Model: Stability, Complexity,
and Scaling. ASME Journal of Biomechanical Engineering, 120, 281-288.

[8] Goswami A, Espiau B and Keramane A 1997 Limit cycles in a passive compass gait biped and passivity-
mimicking control laws. Journal of Autonomous Robots, 4(3),

[9] Shah N and Yeolekar M 2015 Influence of Slope Angle on the Walking of Passive Dynamic Biped Robot.
Applied Mathematics, 6(3): 456-465

[10] Goswami A, Thuliot B, Espiau B 1996 Compass-Like Biped Robot Part I: Stability and Bifurcation of Passive
Gaits. INRIA RR-2996,