ZERO CYCLES ON AFFINE VARIETIES

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Abstract. We show that the Chow group of 0-cycles $\text{CH}^d(A)$ of an affine algebra $A$ of dimension $d \geq 2$ over an algebraically closed field $k$ is torsion-free. This affirmatively answers a question of Murthy.

As a consequence, we show that the Euler class group of $A$ is isomorphic to its Chow group of 0-cycles. In particular, a projective $A$-module of rank $d$ splits off a free summand of rank one if and only if $c_d(P) = 0$ in $\text{CH}^d(A)$.

Along the way, we prove a pro-descent theorem for algebraic $K$-theory in positive characteristic and use this to prove the Bloch-Srinivas conjecture for the Chow group of 0-cycles on a quasi-projective scheme over any perfect field with isolated Cohen-Macaulay singularities. This implies that the Chow group of 0-cycles of the homogeneous coordinate ring of a smooth arithmetically Cohen-Macaulay projective variety vanishes in positive characteristic.

1. Introduction

It is well known that a smooth variety $X$ has associated to it, the theory of Chow groups of algebraic cycles on $X$. These Chow groups form a universal additive cohomology theory on the category of smooth varieties. However, no such theory is yet known for varieties with singularities. The only piece of this conjectural theory known today is the Chow group of 0-dimensional cycles, introduced by Levine-Weibel [33].

Since its introduction, the Levine-Weibel Chow group of 0-cycles on singular varieties has been extensively studied by various authors. As a result, we now know that this Chow group of 0-cycles satisfies many expected properties which were previously known for smooth varieties. However, one intriguing question of deep consequences, which is yet to be answered, is whether the Chow group of 0-cycles of an affine variety of dimension at least two over an algebraically closed field $k$ is torsion-free. One often refers to this as the affine Roitman torsion problem.

It is a consequence of the torsion theorems of Roitman [42] and Milne [36] that the above question has a positive answer if the underlying variety is smooth. If $k$ has characteristic zero, this question was shown to have a positive solution for singular affine varieties by Levine [31]. A result of Srinivas [48] and a joint work of the author with Srinivas [26] together imply that the above question has an affirmative answer in the positive characteristic as well, provided the underlying affine variety is normal. However, without the assumption of normality in positive characteristic, this question remains open in any dimension.

Almost all the known answers to the affine Roitman torsion problem were deduced from suitable Roitman torsion theorems for projective schemes. However, there is no Roitman torsion theorem available in positive characteristic in general. One reason for this is that there is a lack of understanding of the Albanese variety in positive characteristic. One such Albanese variety was constructed in [12]. But any further study of this Albanese has been a formidable task.

1.1. The main results. Our goal in this paper is to affirmatively settle the general case of the above torsion problem for all affine varieties over an algebraically closed field. As is already known, a positive solution to the torsion problem has outstanding applications. We establish several of these in this paper. In particular, we positively answer a question of Murthy [39]. Along the way, we prove a pro-descent theorem for the algebraic $K$-theory in positive

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characteristic and this allows us to prove the Bloch-Srinivas conjecture in positive characteristic. All these results were known in zero characteristic but completely unknown in positive characteristic.

We fix an algebraically closed field $k$ of any characteristic. The central focus of this paper is a proof of the following.

**Theorem 1.1.** For any reduced affine algebra $A$ over $k$ of dimension $d \geq 2$, the Chow group of 0-cycles $\text{CH}^d(A)$ is uniquely divisible.

We deduce the following results as consequences of the statement and the method of the proof of Theorem 1.1.

1.1.1. **Murthy’s question.** Recall that every smooth maximal ideal of a reduced affine $k$-algebra $A$ of dimension $d \geq 1$ has a class in $K_0(A)$ and $F^d K_0(A)$ is the subgroup of $K_0(A)$ generated by these classes. In [39, Question 2.12], Murthy posed the following open question.

**Question 1.2.** (Murthy) Let $A$ be a reduced affine algebra of dimension $d \geq 2$ over $k$. Is $F^d K_0(A)$ torsion-free?

As a consequence of Theorem 1.1, we prove the following.

**Theorem 1.3.** For any reduced affine algebra $A$ over $k$ of dimension $d \geq 2$, the group $F^d K_0(A)$ is torsion-free.

As Murthy already observed, Theorem 1.3 has important applications. One such application to projective modules is the following. Using this theorem and [39, Definition 3.5], one can show that (see Corollary 7.3) every projective $A$-module $P$ of rank $d$ has its top Chern class $c_d(P)$ in $\text{CH}^d(A)$. This Chern class was also defined by Levine [30] and it coincides with the known top Chern class of vector bundles in the Chow groups of smooth varieties. As a consequence of Theorem 1.1 and [39, Theorem 3.7], we obtain:

**Theorem 1.4.** Let $A$ be a reduced affine algebra over $k$ of dimension $d \geq 2$. Let $P$ be a projective $A$-module of rank $d$. Then $P$ splits off a free summand of rank one if and only if $c_d(P) = 0$ in $\text{CH}^d(A)$.

M. Schlichting [44] has very recently defined Euler classes of projective modules of top rank over an affine algebra in a cohomology group of some Milnor $K$-theory sheaf and has proven an analogue of Theorem 1.4 for his Euler classes. However, a good description of Schlichting’s cohomology group and its comparison with the Chow group of 0-cycles is a challenging problem.

1.1.2. **Euler class group.** If $k$ is not algebraically closed, Theorem 1.4 no longer holds even for smooth affine varieties, as the famous example of the tangent bundle on the real 2-sphere shows. To remedy this, Nori defined the notion of ‘Euler class group’ $E(A)$ of a smooth affine algebra $A$ over any field. Later, the notion of Euler class group was defined by Bhatwadekar-Sridharan [6] for any commutative Noetherian ring. This group admits the Euler class $e(P)$ of any projective module $P$ of rank $= \dim(A)$.

It was shown in [4] and [6] that for $A$ either smooth or containing $\mathbb{Q}$, the vanishing of $e(P)$ is a necessary and sufficient condition for $P$ to split off a free summand of rank one. Theorem 1.4 therefore suggests that the Euler class group of [6] should coincide with $\text{CH}^d(A)$ for any reduced affine algebra over $k$ if $k = \overline{k}$. As another application of Theorem 1.1, we show that this is indeed the case. This was proven in [4, Corollary 4.15] when $A$ is smooth.

**Theorem 1.5.** Let $A$ be a reduced affine algebra over $k$ of dimension $d \geq 2$. Then there is a canonical isomorphism $E(A) \cong \text{CH}^d(A)$.
As a consequence of this isomorphism and [27] Theorems 6.4.1, 6.4.2], we get:

**Corollary 1.6.** Let $A$ be a reduced affine algebra over $k$ of dimension $d \geq 2$. Assume that one of the following holds.

1. $k = \overline{\mathbb{F}}_p$.
2. $A = \bigoplus_{i \geq 0} A_i$ is a graded $k$-algebra with $A_0 = k = \overline{\mathbb{Q}}$.

Let $I \subset A$ be a local complete intersection ideal of height $d$. Then $I$ is a complete intersection in $A$. In particular, any finite intersection of distinct smooth maximal ideals of $A$ is a complete intersection in $A$.

A weaker version of Corollary [1.6] was shown in [27] Theorems 6.4.1, 6.4.2], namely, that every smooth maximal ideal of $A$ is a complete intersection.

1.1.3. **Bloch-Srinivas conjecture.** Let $X$ be a reduced quasi-projective variety of dimension $d$ with isolated singularities over a perfect field $k$. Let $F^d K_0(X)$ denote the subgroup of the Grothendieck group of vector bundles $K_0(X)$, generated by the classes of smooth codimension $d$ closed points on $X$. For any closed subscheme $Z$ of $X$, let $F^d K_0(X, Z)$ be the subgroup of the relative $K$-group $K_0(X, Z)$, generated by the classes of smooth closed points of $X \setminus Z$ (see [9]).

Assume there exists a resolution of singularities $\pi : \tilde{X} \to X$ with reduced exceptional divisor $E \subset \tilde{X}$ and let $r E$ denote the $r$th infinitesimal thickening of $E$. From the above definitions and [22] Lemma 3.1], one obtains the following commutative diagram for each $r > 1$ with surjective arrows.

\[
\begin{array}{ccc}
F^d K_0(X) & \xrightarrow{\pi^*} & F^d K_0(\tilde{X}, (r - 1)E) \\
\downarrow & & \downarrow \\
F^d K_0(\tilde{X}, rE) \to F^d K_0(\tilde{X}, rE)
\end{array}
\]

**Theorem 1.7.** Let $X$ be a quasi-projective variety of dimension $d \geq 2$ over a perfect field $k$ such that $X$ has only Cohen-Macaulay isolated singularities. Suppose there exists a resolution of singularities $\pi : \tilde{X} \to X$ with the reduced exceptional divisor $E \subset \tilde{X}$. Then for all $r \gg 1$, the map $F^d K_0(X) \to F^d K_0(\tilde{X}, rE)$ is an isomorphism.

The $d = 2$ case of this theorem was conjectured by Bloch and Srinivas [46] and proven in [26]. The $d \geq 3$ case in characteristic zero was proven in [24]. Morrow [38] has recently proven a stronger version of the theorem in characteristic zero. Theorem [1.7] is new when $d \geq 3$ and $\text{char}(k) > 0$.

If we consider only affine or projective schemes, we obtain the following sharper result.

**Corollary 1.8.** In Theorem [1.7] assume further that $k$ is algebraically closed of characteristic $p > 0$ and $X$ is either affine or projective. Then

$$CH^d(X) \cong F^d K_0(X) \cong F^d K_0(\tilde{X}, E).$$

Corollary [1.8] has the following consequence for the Chow group of 0-cycles on the affine cone of a projective embedding of a smooth variety.

**Corollary 1.9.** Let $k$ be an algebraically closed field of characteristic $p > 0$. Let $A$ be the homogeneous coordinate ring of a smooth projective variety $Z \to \mathbb{P}_k^d$ of dimension $d - 1 \geq 1$. Assume that $A$ is Cohen-Macaulay. Then $CH^d(A) = 0$. In particular, every projective $A$-module of rank $d$ splits off a free summand of rank one.
Note that if $A$ is normal, there is a cohomological criterion for $A$ to be Cohen-Macaulay (see [11, Ex. 18.16, p. 468]). This is given in terms of the vanishing of $H^i(Z, \mathcal{O}_Z(n))$ for $1 \leq i \leq d - 2$ and $n \geq 0$. In particular, $A$ is Cohen-Macaulay if it is normal and $Z \hookrightarrow \mathbb{P}_k^n$ is a complete intersection.

We remark that Corollary 1.9 is very specific to characteristic $p$ as there are counter-examples in characteristic zero (see [47, Corollary 2]). For a modified form of Corollary 1.9 in characteristic zero, see [24, Theorem 1.5].

1.1.4. A pro-descent theorem in positive characteristic. The proof of the above Bloch-Srinivas conjecture is deduced from the following pro-descent theorem for algebraic $K$-theory in positive characteristic. Given a closed immersion $Y \subset X$ of schemes, let $K(X,Y)$ denote the relative $K$-theory spectrum.

**Theorem 1.10.** Let $k$ be a perfect field and let $X$ be a quasi-projective Cohen-Macaulay $k$-scheme with only isolated singularities. Let $Y$ denote the singular locus of $X$ with reduced induced closed subscheme structure. Let $\pi : \tilde{X} \to X$ be a resolution of singularities of $X$ with reduced exceptional divisor $E$. Then the map of pro-abelian groups $\pi^* : \varprojlim_r K_n(X, rY) \to \varprojlim_r K_n(\tilde{X}, rE)$ is an isomorphism for each $n \in \mathbb{Z}$.

In characteristic zero, the above descent theorem was proven in [24, Theorem 1]. A stronger form of this theorem in characteristic zero was proven by Morrow [37]. The reader may recall that the algebraic $K$-theory does not satisfy descent for abstract blow-ups in general. The pro-descent theorems of above kind are significant in this context because they tell us that the algebraic $K$-theory does satisfy a descent if we work with pro-spectra.

1.2. Outline of the proofs. This paper is organized as follows. The proof of the torsion theorem is done by reducing the case of higher dimensions to certain kind of surfaces which are not necessarily affine. This reduction is done in §7. To prove the theorem for surfaces, we need to establish a divisibility property of the $SK_1$ of some normal surfaces. We do this in several steps. One crucial ingredient in this is a form of pro-descent theorem for the bi-relative $K$-theory for finite and some other abstract blow-ups in positive characteristic. This is the content of Sections 3 and 4.

As a byproduct of the descent theorem, we prove the Bloch-Srinivas conjecture in positive characteristic in §4. In Sections 5 and 6 we prove the divisibility properties. Apart from the pro-descent theorem, this also requires some vanishing results for mod-$p$ $K$-theory from [13]. A combination of these and a Mayer-Vietoris sequence yields the proof of the torsion theorem for surfaces. Most of the applications are obtained in §8.

1.3. Notations. Let $k$ be a field and let $\textbf{Sch}_k$ denote the category of separated schemes of finite type over $k$. We shall let $\textbf{Sm}_k$ denote the category of those schemes in $\textbf{Sch}_k$ which are smooth over $k$. For $X \in \textbf{Sch}_k$, the normalization of $X_{\text{red}}$ will be denoted by $X^N$. Given a closed immersion $Y \subset X$ in $\textbf{Sch}_k$ and a positive integer $r$, the scheme $rY \subset X$ will denote the closed subscheme of $X$ defined by the sheaf of ideals $I_Y^r$, where $I_Y$ is the sheaf of ideals in $\mathcal{O}_X$ defining $Y$. We shall call this the $r$th infinitesimal thickening of $Y$ inside $X$. We shall specify the nature of the field $k$ in each section.

2. Chow group of 0-cycles and some preliminary results

In this section, we recall the definition of the Chow group of 0-cycles on singular schemes from [33]. We prove some properties of this Chow group which will be used in this paper. We also prove some other preliminary results we need for our torsion theorem.
2.1. 0-cycles on singular schemes. Let $k$ be a field. Let $X$ be an equi-dimensional reduced quasi-projective scheme over $k$ of dimension $d \geq 1$. Let $X_{\text{sing}}$ and $X_{\text{reg}}$ denote the singular and regular loci of $X$, respectively. Let $Y \subset X$ be a closed subset not containing any component of $X$ such that $X_{\text{sing}} \subseteq Y$. Let $Z^{d}(X, Y)$ be the free abelian group on closed points of $X \setminus Y$. We shall often write $Z^{d}(X, X_{\text{sing}})$ as $Z^{d}(X)$. A (reduced) Cartier curve on $X$ relative to $Y$ is a purely 1-dimensional closed subscheme $C \hookrightarrow X$ that is reduced, has no component contained in $Y$ and is defined by a regular sequence in $X$ at each point of $C \cap Y$.

Let $C$ be a Cartier curve in $X$ relative to $Y$ and let $\{\eta_{1}, \ldots, \eta_{r}\}$ denote the set of its generic points. Let $O_{C, C \cap Y}$ denote the semilocal ring of $C$ at $(C \cap Y) \cup \{\eta_{1}, \ldots, \eta_{r}\}$. Let $k(C)$ denote the ring of total quotients of $C$. Notice that $O_{C, C \cap Y}$ and $k(C)$ coincide if $C \cap Y = \emptyset$.

Since $C$ is a reduced curve, it is Cohen-Macaulay and hence the canonical map $k(C) \to \prod_{i=1}^{r} O_{C, \eta_{i}}$ is an isomorphism. In particular, the map $\theta_{C} : O_{C, C \cap Y}^{\times} \to \prod_{i=1}^{r} O_{C, \eta_{i}}^{\times}$ is injective.

Given $f \in O_{C, C \cap Y}^{\times}$, let $\{f_{i}\} = \theta_{C}(f)$ and let $(f_{i})_{\eta_{i}} := \text{div}(f_{i})$ denote the divisor of zeros and poles of $f_{i}$ on $\{\eta_{i}\}$ in the sense of [19]. We let $(f)_{C} := \sum_{i=1}^{r} (f_{i})_{\eta_{i}}$. As $f$ is an invertible regular function on $C$ along $Y$, we see that $(f)_{C} \in Z^{d}(X, Y)$.

Let $R^{d}(X, Y)$ denote the subgroup of $Z^{d}(X, Y)$ generated by $(f)_{C}$, where $C$ is a Cartier curve on $X$ relative to $Y$ and $f \in O_{C, C \cap Y}^{\times}$. The Chow group of 0-cycles on $X$ relative to $Y$ is the quotient

\[
\text{CH}^{d}(X, Y) = \frac{Z^{d}(X, Y)}{R^{d}(X, Y)}.
\]

The group $\text{CH}^{d}(X, X_{\text{sing}})$ is denoted in short by $\text{CH}^{d}(X)$, and is called the Chow group of 0-cycles on $X$. The Classical Chow moving lemma for smooth varieties has the following version for 0-cycles on singular schemes, proven in [12 Corollary 1.4].

**Lemma 2.1.** Let $Y \subset X$ be a closed subscheme such that $X_{\text{sing}} \subset Y$ and $X \setminus Y$ is dense in $X$. Then the canonical map $\text{CH}^{d}(X, Y) \to \text{CH}^{d}(X)$ is an isomorphism.

Bertini type theorems for singular schemes (see [31 § 1]) can be used to prove the following expression for the elements of $R^{d}(X, Y)$.

**Lemma 2.2.** (cf. [4 Lemma 2.1]) Let $k$ be an algebraically closed field and let $X$ be a reduced quasi-projective scheme over $k$ of dimension $d \geq 2$. Then any element $\alpha \in R^{d}(X, X_{\text{sing}})$ can be written as $\alpha = (f)_{C}$ for a single reduced (but possibly reducible) Cartier curve $C$ on $X$.

**Lemma 2.3.** Let $X \subset \mathbb{P}_{k}^{n}$ be a reduced quasi-projective scheme over $k$ of dimension $d$ and let $Y \subset X$ be a closed subscheme containing $X_{\text{sing}}$ and not containing any component of $X$. Let $\iota : X' \hookrightarrow X$ be the closed immersion of a hypersurface section of $X$ with respect to the embedding $X \subset \mathbb{P}_{k}^{n}$. Assume that $X'$ is reduced and $X' \setminus Y$ is regular. Then there is a push-forward map $\iota_{*} : \text{CH}^{d-1}(X', X' \cap Y) \to \text{CH}^{d}(X, Y)$.

**Proof.** It is clear from our assumption that there is an inclusion $\iota_{*} : Z^{d-1}(X', Y') \subset Z^{d}(X, Y)$, where $Y' = X' \cap Y$. Let $C \hookrightarrow X'$ be a reduced Cartier curve relative to $Y'$ and let $(f)_{C} \in R^{d-1}(X', Y')$. Since $X' \subset X$ is a complete intersection, it follows that $C \hookrightarrow X$ is also a Cartier curve in $X$ relative to $Y$ and clearly one has $(f)_{C} \in R^{d}(X, Y)$. This finishes the proof. \(\square\)

**Lemma 2.4.** Let $X \subset \mathbb{P}_{k}^{n}$ be a reduced quasi-projective scheme over $k$ of dimension $d$. Let $\iota : X' \hookrightarrow X$ be the closed immersion of a hypersurface section of $X$ with respect to the embedding $X \subset \mathbb{P}_{k}^{n}$. Assume that $X'$ is reduced and $X' \setminus X_{\text{sing}}$ is regular. Assume further
that there is a proper map \( \pi : X \to Z \) obtained by blowing up a finite collection of smooth closed points of a reduced quasi-projective scheme \( Z \) over \( k \). Then there is a push-forward map \((\pi \circ \iota)_* : \text{CH}^{d-1}(X', X' \cap X_{\text{sing}}) \to \text{CH}^d(Z)\).

Proof. It is known that there is a push-forward map \( \pi_* : \text{CH}^d(X) \to \text{CH}^d(Z) \) which is an isomorphism (see \[12\] Corollary 2.7). The lemma follows by combining this with \( \iota_* : \text{CH}^{d-1}(X', X' \cap Y) \to \text{CH}^d(X) \) from Lemma 2.6. \( \Box \)

Recall from \[33\] Proposition 2.1] that every regular closed point \( x \in X \) defines a class \( [k(x)] \in K_0(X) \) and this yields a cycle class map

\[
(2.2) \quad \text{cyc}_X : \text{CH}^d(X, Y) \to K_0(X).
\]

It follows from Lemma 2.5] that the image of this cycle class map does not depend on \( Y \). This image is classically denoted by \( F^dK_0(X) \).

2.2. Some preliminary results. For an equi-dimensional \( k \)-scheme \( X \), let \( \text{CH}_q^p(X) \) denote the (homological) Chow group of codimension \( q \) cycles on \( X \) in the sense of \[19\]. Recall from \[20\] that every normal projective variety \( X \) over \( k \) of dimension \( d \) has an Albanese variety \( \text{Alb}(X) \) which is an abelian variety and there is an Albanese map \( \text{alb}_X : \text{CH}^d(X, 0) \to \text{Alb}(X) \).

Lemma 2.5. Let \( X \) be a normal quasi-projective surface over \( k \) and let \( Y \subset X \) be a strict normal crossing divisor such that \( Y \cap X_{\text{sing}} = \emptyset \). Then there is a canonical isomorphism \( H^2_Y(X, K_2) \cong \text{CH}^q_p(Y) \).

Proof. Let \( U = X \setminus X_{\text{sing}} \) denote the smooth locus of \( X \). Then excision for the Zariski cohomology with support and our assumption imply that the map \( H^2_Y(X, K_2) \to H^2_Y(U, K_2) \) is an isomorphism. So we can assume that \( X \) is smooth.

We consider the Gersten resolution

\[
(2.3) \quad 0 \to K_{2, X} \xrightarrow{\iota_X} i_{k(X), *}(K_2(k(X))) \to \bigoplus_{z \in X(1)} i_{z,*}(K_1(k(z))) \to \bigoplus_{x \in X(2)} i_{x,*}(K_0(k(x))) \to 0.
\]

Setting \( \mathcal{F} = \text{Coker}(\epsilon_X) \), we get an exact sequence of Zariski sheaves

\[
0 \to \mathcal{F} \to \bigoplus_{z \in X(1)} i_{z,*}(K_1(k(z))) \to \bigoplus_{x \in X(2)} i_{x,*}(K_0(k(x))) \to 0.
\]

Since the above sequences are the flasque resolutions of \( K_{2, X} \) and \( \mathcal{F} \), respectively, it follows that the map \( H^0_Y(X, \mathcal{F}) \to H^2_Y(X, K_{2, X}) \) is an isomorphism and there is an exact sequence

\[
H^0_Y(X, \bigoplus_{z \in X(1)} i_{z,*}(K_1(k(z)))) \to H^0_Y(X, \bigoplus_{x \in X(2)} i_{x,*}(K_0(k(x)))) \to H^1_Y(X, \mathcal{F}) \to 0.
\]

Equivalently, we have an exact sequence

\[
(2.4) \quad \bigoplus_{Z} \text{K}(Z) \xrightarrow{\text{div}} \bigoplus_{x \in Y(1)} K_0(k(x)) \to H^1_Y(X, \mathcal{F}) \to 0,
\]

where \( Z \) runs through all 1-dimensional irreducible components of \( Y \). It is well known that the first arrow from the left takes a rational function on \( Y \) to its divisor. Hence, it is clear from the definition of \( \text{CH}^1_F(Y) \) that this exact sequence is equivalent to an isomorphism \( \text{CH}^1_F(Y) \cong H^1_Y(X, \mathcal{F}) \). Combining this with \( H^1_Y(X, \mathcal{F}) \cong H^2_Y(X, K_{2, X}) \), we get the desired isomorphism \( \text{CH}^1_F(Y) \cong H^2_Y(X, K_{2, X}) \). \( \Box \)

Lemma 2.6. Let \( Y \) be a reduced affine curve over an algebraically closed field \( k \). Then \( \text{SK}_1(Y) \) is uniquely divisible.


Proof. We can assume that $Y$ is connected. It follows from [2] Theorem 5.3] that $SK_1(Y) = V \oplus P$, where $V$ is uniquely divisible and $P$ is either zero (if char$(k) = 0$) or a $p$-group of bounded exponent (if char$(k) = p > 0$). We can thus assume that char$(k) = p > 0$.

It follows from [16, Theorem 3] that the cup product map Pic$(Y) \otimes k^\times \to SK_1(Y)$ is surjective. Since $k^\times$ is divisible, so is Pic$(Y) \otimes k^\times$. In particular, its quotient $SK_1 \simeq V \oplus P$ is also divisible. This in turn implies that $P$ is divisible. Since it is also of bounded exponent, it must be zero. This yields an isomorphism $SK_1 \simeq V$, which is uniquely divisible. □

Lemma 2.7. Let $f : A \to B$ be a surjective morphism of smooth commutative algebraic groups over $k$. Then $f$ is also surjective on the torsion subgroups.

Proof. If $A$ and $B$ are abelian varieties, then we can write Ker$(f)$ as an extension of a finite abelian group by an abelian variety. In particular, Ker$(f) \otimes \mathbb{Q}/\mathbb{Z} = 0$. But this easily implies that $f : A_{\text{tors}} \to B_{\text{tors}}$ is surjective.

In the general case, the structure theorem for smooth commutative algebraic groups says that there are unique linear subgroups $A_1 \subset A$, $B_1 \subset B$ and abelian variety quotients $A \to A_2$, $B \to B_2$ such that one has the diagram of exact sequence

$$0 \longrightarrow A_1 \longrightarrow A \longrightarrow A_2 \longrightarrow 0$$

Moreover, the corresponding maps on the linear and the abelian variety parts are also surjective. We have seen above that the map $A_2 \to B_2$ is surjective on the torsion subgroups.

One also knows (see [53, Theorem 2.3.3]) that in the above decomposition, the linear parts $A_1$ and $B_1$ are canonically products of their diagonalizable and unipotent parts. Thus we have $A_1 = D_1 \times U_1$, $B_1 = D'_1 \times U'_1$, where $D_1$ (resp. $D'_1$) and $U_1$ (resp. $U'_1$) are the unique diagonalizable and unipotent parts of $A_1$ (resp. $B_1$).

Since there are no nontrivial homomorphisms from a diagonalizable to a unipotent group and vice-versa (see [53, § 2.3.2]), we get $D_1 \to D'_1$ and $U_1 \to U'_1$. If char$(k) = 0$, then $U_1$ and $U'_1$ are both $k$-vector spaces by [53, § 3.1] and hence uniquely divisible. If char$(k) = p > 0$, then $U_1$ and $U'_1$ are both $p$-groups of bounded exponents by [53, § 3.2.1]. In particular, both groups are already torsion and hence

$$U_1 \otimes \mathbb{Q}/\mathbb{Z} = U'_1 \otimes \mathbb{Q}/\mathbb{Z} = 0.$$  

(2.5)

Next we note that the kernel of the map $D_1 \to D'_1$ is again a diagonalizable closed subgroup and by [53, Corollary 1.2.6], any diagonalizable group $D$ is a direct product of a torus and a finite abelian group of order prime to $p$ (if char$(k) = p > 0$). In particular,

$$D \otimes \mathbb{Q}/\mathbb{Z} = 0.$$  

(2.6)

(2.5) and (2.6) together imply that the map $A_1 \to B_1$ is surjective on the torsion subgroups. They also imply that the maps $A \to A_2$ and $B \to B_2$ are surjective on the torsion subgroups. A simple diagram chase above now shows that the middle vertical map is surjective on the torsion subgroups. □

Lemma 2.8. Let $X \in \text{Sch}_k$ and let $Y \subset X$ be a proper closed subscheme. Let $f : X' \to X$ be a finite morphism and let $Y' = Y \times_X X'$. Then the induced map $\tilde{f} : \text{Bl}_Y(X') \to \text{Bl}_Y(X)$ is also finite.

Proof. It is enough to prove the lemma when $X = \text{Spec}(A)$ is affine. Let $I \subset A$ denote the ideal of definition of $Y$. Let $X' = \text{Spec}(B)$ so that there is a finite ring homomorphism $A \to B$. Set $J = IB$. 


We can find a polynomial ring \( A[x_0, \ldots, x_n] \) and a surjective graded ring homomorphism \( A[x_0, \ldots, x_n] \to A[It] \), where \( A[It] \) is the Rees-algebra of \( I \) over \( A \). This yields a surjection \( B[x_0, \ldots, x_n] \to B[It] \) and hence a commutative diagram

\[
\begin{array}{ccc}
\text{Bl}_Y(X') & \longrightarrow & \mathbb{P}_X^n \\
\downarrow & & \downarrow \\
\text{Bl}_Y(X) & \longrightarrow & \mathbb{P}_X^n.
\end{array}
\]

It is clear that the horizontal arrows are closed immersions. Since the right vertical arrow is clearly finite, it follows that the left vertical arrow is also finite. \( \square \)

3. Bi-relative \( K \)-theory associated to blow-ups in positive characteristic

In this section, we use some results of [15] to study the bi-relative \( K \)-groups associated to certain abstract blow-ups in \( \text{Sch}_k \). The results of this section will be used to prove a pro-descent theorem for algebraic \( K \)-theory in positive characteristic. We shall also use these results to prove our torsion theorem for the Chow group of 0-cycles later in this paper.

3.1. Pro-descent for presheaf of spectra. Recall that a pro-object \( \{A_i, \alpha_i\}_{i \in \mathbb{N}^+} \) in a category \( C \) is a sequence \( \{A_i \overset{\alpha_i}{\leftarrow} A_2 \overset{\alpha_2}{\leftarrow} \cdots \} \) of objects in \( C \). A pro-object will be formally denoted by \( \text{"\( \lim \)"}_i A_i \). A morphism \( f : \text{"\( \lim \)"}_i A_i \to \text{"\( \lim \)"}_j B_j \) in the category \( \text{pro} C \) of pro-objects in \( C \) is an element of the set \( \text{\( \lim \)lim}_j \text{Hom}_C(A_i, B_j) \). We say that \( f : \text{\( \lim \)lim}_i A_i \to \text{\( \lim \)lim}_j B_j \) is strict if it takes each \( A_i \) to \( B_j \).

If \( C \) admits cofiltered limits, the limit of \( \text{\( \lim \)lim}_i A_i \) will be denoted by \( \lim\lim_i A_i \). If \( C \) is an abelian category, then so is \( \text{pro} C \). If \( f : \text{\( \lim \)lim}_i A_i \to \text{\( \lim \)lim}_j B_j \) is a strict morphism, then one checks easily that \( \text{Ker}(f) = \text{\( \lim \)lim}_i \text{Ker}(f_i) \) and \( \text{Coker}(f) = \text{\( \lim \)lim}_i \text{Coker}(f_i) \). In particular, a sequence of strict morphisms of pro-objects

\[
\text{\( \lim \)lim}_i A_i \to \text{\( \lim \)lim}_j B_j \to \text{\( \lim \)lim}_i C_i
\]

is exact in the abelian category \( \text{pro} C \) if it restricts to an exact sequence of objects in \( C \) for each \( i \in \mathbb{N}^+ \). We should warn that the exactness of (3.1) does not imply that the sequence remains exact if we replace \( \text{\( \lim \)lim}_i \) by \( \lim\lim_i \).

Consider a Cartesian square in \( \text{Sch}_k \):

\[
\begin{array}{ccc}
Y' & \longrightarrow & X' \\
\downarrow & & \downarrow \\
Y & \longrightarrow & X.
\end{array}
\]

Recall that (3.2) is called an abstract blow-up square if \( \iota \) is a closed immersion, \( f \) is proper and the induced map \( X' \setminus Y' \to X \setminus Y \) is an isomorphism. We say that (3.2) is a finite abstract blow-up square if it is an abstract blow-up square such that \( f \) is finite.

Given a presheaf of (possibly non-connective) \( S^1 \)-spectra \( L \) on \( \text{Sch}_k \), we let \( L(X, X') \) denote the homotopy fiber of the map of spectra \( g^* : L(X) \to L(X') \). We let \( L(X, X', Y, Y') \) denote the homotopy fiber of the map of spectra \( L(X, Y) \to L(X', Y') \). It is easy to check that \( L(X, X', Y, Y') \) is same as the homotopy fiber of the map of spectra \( L(X, X') \to L(Y, Y') \) in the
Lemma 3.1. Let \( L_n(X, X', Y, Y') \) for \( n \in \mathbb{Z} \). The groups \( K_n(X, Y) \) and \( K_n(X, X', Y, Y') \) will be called the relative and the bi-relative K-groups, respectively.

We shall say that a presheaf of spectra \( L \) on \( \text{Sch}_k \) satisfies pro-descent for the square if the induced square of pro-spectra

\[
\begin{array}{ccc}
L(X) & \longrightarrow & \text{lim}_{r} L(rY) \\
\downarrow & & \downarrow \\
L(X') & \longrightarrow & \text{lim}_{r} L(rY').
\end{array}
\]

is homotopy Cartesian. Equivalently, the map of pro-abelian groups \( \text{lim}_{r} \pi_n(L(X, rY)) \to \text{lim}_{r} \pi_n(L(X', rY')) \) is an isomorphism for all \( n \in \mathbb{Z} \).

3.2. Zariski descent for bi-relative K-theory and consequences. In this subsection, we shall assume that \( k \) is a field of characteristic \( p > 0 \). Let \( f : X' \to X \) be a finite morphism in \( \text{Sch}_k \). Let \( I_Y \) be a sheaf of ideals on \( X \) such that the map \( I_Y \to f_* \circ f^*(I_Y) \) is an isomorphism. Let \( \iota : Y \hookrightarrow X \) be the closed subscheme defined by \( I_Y \) and let \( Y' = Y \times_X X' \). In this case, we shall say that \( Y \) is a conducting subscheme and write \( K_n(X, X', Y, Y') \) in short as \( K_n(X, X', Y, Y) \). For any locally closed subscheme \( U \subset X \), we let \( Y_U = Y \times_X U \).

Lemma 3.1. Let \( U, V \subset X \) be two open subsets such that \( X = U \cup V \) and let \( W = U \cap V \). Then the induced square of spectra

\[
\begin{array}{ccc}
K(X, X', Y) & \longrightarrow & K(U, U', Y_U) \\
\downarrow & & \downarrow \\
K(V, V', Y_V) & \longrightarrow & K(W, W', Y_W)
\end{array}
\]

is homotopy Cartesian.

Proof. Let \( K^{X \setminus U}(X) \) denote the homotopy fiber of the restriction map \( K(X) \to K(U) \). Then it follows from [52, Theorem 8.1] that the map \( K^{X \setminus U}(X) \to K^{V \setminus W}(V) \) is a homotopy equivalence and the same holds if we replace \( X \) by \( Y \). The homotopy fiber sequence

\[
K^{X \setminus U}(X, Y) \to K^{X \setminus U}(X) \to K^{Y \setminus U}(Y)
\]

shows that the map \( K^{X \setminus U}(X, Y) \to K^{V \setminus W}(V, Y_V) \) is a homotopy equivalence. The same is also true if we replace \( X \) by \( X' \).

Notice now that \( K^{X \setminus U}(X, Y) \) is also the homotopy fiber of the map \( K(X, Y) \to K(U, Y_U) \). In particular, we have a commutative diagram of spectra

\[
\begin{array}{ccc}
K^{X \setminus U}(X, X', Y) & \longrightarrow & K(X, X', Y) \longrightarrow K(U, U', Y_U) \\
\downarrow & & \downarrow \\
K^{X \setminus U}(X, Y) & \longrightarrow & K(X, Y) \longrightarrow K(U, Y_U) \\
\downarrow & & \downarrow \\
K^{X \setminus U}(X', Y') & \longrightarrow & K(X', Y') \longrightarrow K(U', Y'_U),
\end{array}
\]
where all rows and columns are homotopy fiber sequences, and these fiber sequences uniquely define $K^{X\setminus U}(X, X', Y)$. The commutative diagram of homotopy fiber sequences

\begin{equation}
\begin{CD}
K^{X\setminus U}(X, X', Y) @>>> K^{X\setminus U}(X, Y) @>>> K^{X\setminus U'}(X', Y') \\
@. @. @. \\
K^{V\setminus W}(V, V', Y_V) @>>> K^{V\setminus W}(V, Y_V) @>>> K^{V\setminus W'}(V', Y_{V'})
\end{CD}
\end{equation}

now shows that the map $K^{X\setminus U}(X, X', Y) \to K^{V\setminus W}(V, V', Y_V)$ is a homotopy equivalence. But this is equivalent to saying that (3.7) is homotopy Cartesian. \qed

An argument identical to the proof of Lemma 3.1 also proves the following.

Lemma 3.2. Let $U, V \subset X$ be two open subsets such that $X = U \cup V$ and let $W = U \cap V$. Let $Y \subset X$ be a closed subscheme. Then the induced square of spectra

\begin{equation}
\begin{CD}
K(X, Y) @>>> K(U, Y_U) \\
@. @. @. \\
K(V, Y_V) @>>> K(W, Y_W)
\end{CD}
\end{equation}

is homotopy Cartesian.

Lemma 3.3. For every integer $n \in \mathbb{Z}$, the bi-relative $K$-group $K_n(X, X', Y)$ is a $p$-primary torsion group of bounded exponent.

Proof. If $X$ is affine, this follows from [15, Theorem C]. In general, we shall argue by induction on the minimal number of affine open subsets that cover $X$. Let us write $X = U_1 \cup \cdots \cup U_r$, where each $U_i$ is affine open in $X$. Set $U = U_1, V = U_2 \cup \cdots \cup U_r$ and $W = U \cap V = (U \cap U_2) \cup \cdots \cup (U \cap U_r)$.

It follows from Lemma 3.1 that for every $n \in \mathbb{Z}$, there is an exact sequence

$$K_{n+1}(W, W', Y_W) \to K_n(X, X', Y) \to K_n(U, U', Y_U) \oplus K_n(V, V', Y_V).$$

Since our schemes are all separated, each $U \cap U_i$ is affine for $2 \leq i \leq r$. Since $U$ is affine, $K_n(U, U', Y_U)$ is p-primary torsion of bounded exponent. It follows by induction on the minimal number of open subsets in an affine open cover that $K_{n+1}(W, W', Y_W)$ and $K_n(V, V', Y_V)$ are p-primary torsion of bounded exponents. We deduce easily that $K_n(X, X', Y)$ is also p-primary torsion of bounded exponent. \qed

Using [15, Theorem A] and Lemma 3.2 an argument identical to the proof of Lemma 3.3 proves the following result.

Lemma 3.4. Let $X \in \text{Sch}_k$ and let $Y \subset X$ be a closed subscheme whose sheaf of ideals in $\mathcal{O}_X$ is nilpotent. Then for every integer $n \in \mathbb{Z}$, the relative $K$-group $K_n(X, Y)$ is a $p$-primary torsion group of bounded exponent.

Lemma 3.5. Let $Z \subset Y \subset X$ be closed immersions in $\text{Sch}_k$ such that $Z_{\text{red}} = Y_{\text{red}}$. In the commutative square (3.2) of Lemma 3.3, let $Z' = Z \times_X X'$. Then for any $n \in \mathbb{Z}$, the group $K_n(X, X', Y, Y')$ is p-primary torsion of bounded exponent if and only if so is $K_n(X, X', Z, Z')$. 

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Proof. We have a commutative diagram of spectra

\[
\begin{array}{ccc}
K(X, X', Y, Y') & \rightarrow & K(X, X', Z, Z') \\
\downarrow & & \downarrow \\
K(X, Y) & \rightarrow & K(X, Z) \\
\downarrow & & \downarrow \\
K(X', Y') & \rightarrow & K(X', Z')
\end{array}
\]

where the rows and columns are homotopy fiber sequences. In particular, there is an exact sequence of bi-relative \(K\)-groups

\[
K_{n+1}(Y, Y', Z, Z') \rightarrow K_n(X, X', Y, Y') \rightarrow K_n(X, X', Z, Z') \rightarrow K_n(Y, Y', Z, Z').
\]

It suffices therefore to show that for every \(n \in \mathbb{Z}\), the bi-relative \(K\)-group \(K_n(Y, Y', Z, Z')\) is a \(p\)-primary torsion group of bounded exponent. But this is an immediate consequence of the vertical fiber sequence on the right end of (3.8) and Lemma 3.4.

\[\square\]

**Proposition 3.6.** Given a finite abstract blow-up square (3.2) and an integer \(n \in \mathbb{Z}\), the bi-relative \(K\)-group \(K_n(X, X', Y, Y')\) is a \(p\)-primary torsion group of bounded exponent.

**Proof.** By Lemma 3.5, it is enough to show that \(K_n(X, X', rY, rY')\) is a \(p\)-primary torsion group of bounded exponent for some \(r \gg 1\).

We can find a closed subscheme \(Z \subset X\) whose support is contained in \(Y\) such that \(\mathcal{I}_Z = f_* \circ f^*(\mathcal{I}_Z)\). Setting \(Z' = Z \times_X X'\), it follows that \(Z \subset rY\) for \(r \gg 1\). Hence we obtain a commutative diagram of spectra of the form (3.8) with \(Y\) and \(Y'\) replaced by \(rY\) and \(rY'\), respectively for \(r \gg 1\).

If \(Z_{red}\) is a proper closed subset of \(Y\), then the map \(rY' \rightarrow rY\) is also a finite abstract blow-up with \(Z \subset rY\) a conducting subscheme. Hence, Lemma 3.3 says that \(K_n(rY, rY', Z, Z')\) is \(p\)-primary torsion of bounded exponent for each \(n \in \mathbb{Z}\). Using the exact sequence

\[
K_{n+1}(rY, rY', Z, Z') \rightarrow K_n(X, X', rY, rY') \rightarrow K_n(X, X', Z, Z'),
\]

we see that \(K_n(X, X', rY, rY')\) will be \(p\)-primary torsion of bounded exponent if \(K_n(X, X', Z, Z')\) is so.

If \(Z\) has same support as \(Y\), then the inclusion \(Z \subset rY\) is a nilpotent thickening and again, \(K_n(X, X', rY, rY')\) will be \(p\)-primary torsion of bounded exponent if \(K_n(X, X', Z, Z')\) is so, by Lemma 3.5. We have thus reduced to showing that \(K_n(X, X', Z, Z')\) is \(p\)-primary torsion of bounded exponent. But this follows from Lemma 3.3.

\[\square\]

**Proposition 3.7.** Let \(X \in \text{Sch}_k\) be a Cohen-Macaulay scheme with only isolated singularities. Let \(Y\) denote the singular locus of \(X\) with reduced induced closed subscheme structure. Let \(Z \subset X\) be a closed subscheme such that \(Z_{red} = Y\). Let \(\pi : \tilde{X} \rightarrow X\) denote the blow-up of \(X\) along \(Z\) and let \(E \subset \tilde{X}\) denote the reduced exceptional divisor. Then for every \(n \in \mathbb{Z}\), the bi-relative \(K\)-group \(K_n(X, \tilde{X}, Y, E)\) is a \(p\)-primary torsion group of bounded exponent.

**Proof.** Recall that a Noetherian scheme is called Cohen-Macaulay if each of its local rings is a Cohen-Macaulay ring. Let \(I_X\) be the sheaf of ideals on \(X\) defining \(Z\). By the Northcott-Rees theory of reduction of ideals, there exists \(m \geq 1\) and a minimal reduction ideal \(J\) of \(I_X\), where one can take \(m = 1\) if \(k\) is infinite (see [55, Theorem 14.14]). Recall here that an ideal \(J \subset I_X\) is a reduction of \(I_X\) if the map \(\text{Proj}_X(\mathcal{R}(I_X)) \rightarrow \text{Proj}_X(\mathcal{R}(J))\) is a finite morphism, where \(\mathcal{R}(I_X)\) denotes the sheaf of Rees-algebras of \(I_X\) over \(\mathcal{O}_X\) (see [43, Theorem 1.5]). Since the blow-up of \(I_X\) is unchanged if we replace \(I_X\) by its powers, we can assume \(m = 1\) to obtain a reduction ideal.
Now, since $X$ is Cohen-Macaulay with only isolated singularities, we can further assume that the reduction ideal $\mathcal{J}$ is a local complete intersection ideal sheaf in $\mathcal{O}_X$ (see [24, Proposition 1.6]). Setting $X' = \text{Proj}_X(\mathcal{R}(\mathcal{J}))$, this gives rise to a commutative diagram

$$
\begin{array}{ccc}
\widetilde{X} & \xrightarrow{f} & X' \\
\pi & \searrow & \pi' \\
& X, & 
\end{array}
$$

where $f$ is finite and $\pi'$ is the blow-up along a regular closed immersion $W \subset X$ such that $Z \subset W$ with $W_{\text{red}} = Y$. By Lemma 3.5, it suffices to show that $K_n(X, \widetilde{X}, W, W')$ is a $p$-primary torsion group of bounded exponent, where $W = W \times_X \widetilde{X}$.

We set $W' = W' \times_X X'$ so that $\widetilde{W} = W' \times_{X'} \widetilde{X}$. It follows from [51, Théorème 2.1] that the map $K(X, W) \to K(X', W')$ is a homotopy equivalence. On the other hand, we have a commutative diagram

$$
\begin{array}{ccc}
K(X, X', W, W') & \to & K(X, W) \to K(X', W') \\
\downarrow & & \downarrow \\
K(X, \widetilde{X}, W, \widetilde{W}) & \to & K(X, W) \to K(\widetilde{X}, \widetilde{W})
\end{array}
$$

where the two rows are homotopy fiber sequences. In particular, we get a homotopy fiber sequence

$$K(X, X', W, W') \to K(X, \widetilde{X}, W, \widetilde{W}) \to K(X', \widetilde{X}, W', \widetilde{W}).$$

We have shown above that $K(X, X', W, W')$ is contractible and it follows from Proposition 3.6 that $K_n(X, \widetilde{X}, W, W')$ is $p$-primary torsion of bounded exponent. We conclude that the same holds for $K_n(X, \widetilde{X}, W, W)$ too. This finishes the proof. □

Let $X$ be singular scheme in $\text{Sch}_k$. Recall that a morphism of schemes $\pi : \widetilde{X} \to X$ is called a resolution of singularities if it is proper, its restriction to the regular locus of $X$ is an isomorphism and $\widetilde{X}$ is regular. It is well-known that a resolution of singularities always exists in characteristic zero. In characteristic $p > 0$, it exists if either $\dim(X) \leq 2$ or $\dim(X) = 3$ and $p > 3$.

**Theorem 3.8.** Let $X$ be a quasi-projective Cohen-Macaulay $k$-scheme with only isolated singularities. Let $Y$ denote the singular locus of $X$ with reduced induced closed subscheme structure. Let $\pi : \widetilde{X} \to X$ be a resolution of singularities of $X$ with reduced exceptional divisor $E$. Then for every $n \in \mathbb{Z}$, the bi-relative $K$-group $K_n(X, \widetilde{X}, Y, E)$ is a $p$-primary torsion group of bounded exponent.

**Proof.** Since $X$ is quasi-projective over $k$, we can apply [32, Lemma 1.4] to find a map $X' \xrightarrow{\pi'} \widetilde{X} \xrightarrow{\pi} X$ such that $X'$ is the blow-up of $X$ along a closed subscheme $Z \subset X$ with $Z_{\text{red}} = Y$. Note that most of the results of loc. cit. are valid only in characteristic zero, but the above cited result is characteristic free.

On the other hand, the regularity of $\widetilde{X}$ implies that $\pi'$ is the blow-up of $\widetilde{X}$ along a closed subscheme $\widetilde{Z} \subset \widetilde{X}$ whose support is $E$ (see [21, Ex. II.7.11]). Since $E \subset \widetilde{X}$ is a divisor on a regular scheme, $\widetilde{Z}$ must be a Cartier divisor on $\widetilde{X}$. In particular, $\pi'$ must be an isomorphism. We conclude that $\pi$ is the blow-up of $X$ along a closed subscheme $Z \subset X$ with $Z_{\text{red}} = Y$. We now apply Proposition 3.7 to conclude the proof. □
4. Pro-descent for $K$-theory in positive characteristic

It is well known that algebraic $K$-theory of Quillen-Thomason-Trobaugh does not satisfy descent for an abstract blow-up. Even the descent for finite abstract blow-up fails. It was observed for the first time in [24] that the finite abstract blow-up for algebraic $K$-theory is true if we work with pro-spectra of $K$-theory in a suitable sense. Later, the descent for the pro-spectra of $K$-theory was proven by Morrow [37] for any abstract blow-up in the category of finite type schemes over a field of characteristic zero. In this section, we use the results of § 3.2 to prove some of these pro-descent theorems in positive characteristic. We shall later apply these results in suitable form to prove our torsion theorem.

**Theorem 4.1.** Let $k$ be any perfect field of characteristic $p \geq 0$. Then the Thomason-Trobaugh algebraic $K$-theory satisfies pro-descent for a finite abstract blow-up square in $\text{Sch}_k$.

**Proof.** If $\text{char}(k) = 0$, the theorem follows from [37] Theorem 01]. So we assume $p > 0$. Let (3.2) be a finite abstract blow-up square. We need to show that the map $\text{"lim}_r^\pi K_n(X, rY) \to \text{"lim}_r^\pi K_n(X, rY')$ is an isomorphism for all $n \in \mathbb{Z}$. Equivalently, we need to show that $\text{"lim}_r^\pi K_n(X, X', rY, rY') = 0$ for all $n \in \mathbb{Z}$.

We can find a closed subscheme $Z \subset X$ whose support is contained in $Y$ such that $I_Z \xrightarrow{\sim} f_* \circ f^*(I_Y)$. In particular, this property holds for $I_rZ$ for every $r \geq 1$. Using the exact sequence

$$\text{"lim}_r^\pi K_{n+1}(rY, rY', Z, Z') \to \text{"lim}_r^\pi K_n(X, X', rY, rY') \to \text{"lim}_r^\pi K_n(X, X', Z, Z')$$

from (3.10), it suffices to prove the theorem in the case when $Z$ has same support as $Y$. In this case, we have $Y \subseteq rZ \subseteq r'Y$ for all $r, r' \gg 1$. Hence, it follows that the map $\text{"lim}_r^\pi K_n(X, rZ) \to \text{"lim}_r^\pi K_n(X, rY)$ is an isomorphism, and a similar conclusion holds on $X'$.

We can thus assume that $I_rY \xrightarrow{\sim} f_* \circ f^*(I_rY)$ for all $r \geq 1$ and write $K_n(X, X', rY, rY')$ as $K_n(X, X', rY)$.

It follows from Lemma 3.3 and [13] Lemma 2.1] that the map of pro-abelian groups $K_n(X, X', rY) \to \text{"lim}_m^\mu K_n(X, X', rY; Z/p^m)$ is an isomorphism for each $r \geq 1$ and $n \in \mathbb{Z}$.

Here, the terms on the right-hand side of the map are the homotopy groups with coefficients. On the other hand, for every $m' \geq m$, $r' \geq r \geq 1$ and $n \in \mathbb{Z}$, there is a commutative diagram

$$K_n(X, X', rY) \to K_n(X, X', rY; Z/p^m)$$

and this shows that there is an isomorphism of pro-abelian groups

$$\text{"lim}_r^\pi K_n(X, X', rY) \xrightarrow{\sim} \text{"lim}_m^\mu \text{"lim}_r^\pi K_n(X, X', rY; Z/p^m).$$

Since there is a natural isomorphism of functors $\text{"lim}_r^\pi \text{"lim}_m^\mu \xrightarrow{\sim} \text{"lim}_m^\mu \text{"lim}_r^\pi$, we conclude that the map $\text{"lim}_r^\pi K_n(X, X', rY) \to \text{"lim}_m^\mu \text{"lim}_r^\pi K_n(X, X', rY; Z/p^m)$ of pro-abelian groups is an isomorphism. It is therefore sufficient to show that $\text{"lim}_r^\pi K_n(X, X', rY; Z/p^m) = 0$ for each $m \geq 1$ and $n \in \mathbb{Z}$. 
If $X$ is affine, it follows from [14, Theorem 1] that the cyclotomic trace map induces an isomorphism of pro-abelian groups

\[(4.1) \quad K_n(X, X', rY; \mathbb{Z}/p^m) \xrightarrow{\sim} \lim_{\substack{\rightarrow \cr \longleftarrow \cr m \rightarrow \infty}} \text{TC}_n^q(X, X', rY; p, \mathbb{Z}/p^m)\]

for each $m, r \geq 1$ and $n \in \mathbb{Z}$. Here, $\text{TC}_n^q(X, X', rY; p, \mathbb{Z}/p^m)$ denotes the bi-relative topological cyclic homology (see [14, § 2]).

We now apply [37, Theorem 3.5], which says that for each $m, q \geq 1$ and $n \in \mathbb{Z}$, the pro-abelian group $\lim_{\substack{\rightarrow \cr \longleftarrow \cr m \rightarrow \infty}} \text{TC}_n^q(X, X', rY; p, \mathbb{Z}/p^m)$ is zero (this uses the perfectness of $k$). In particular, we get

\[
\lim_{\substack{\rightarrow \cr \longleftarrow \cr m \rightarrow \infty}} K_n(X, X', rY; \mathbb{Z}/p^m) \xrightarrow{\sim} \lim_{\substack{\rightarrow \cr \longleftarrow \cr q \rightarrow \infty}} \lim_{\substack{\rightarrow \cr \longleftarrow \cr m \rightarrow \infty}} \text{TC}_n^q(X, X', rY; p, \mathbb{Z}/p^m) \\
\xrightarrow{\sim} \lim_{\substack{\rightarrow \cr \longleftarrow \cr q \rightarrow \infty}} \lim_{\substack{\rightarrow \cr \longleftarrow \cr m \rightarrow \infty}} \text{TC}_n^q(X, X', rY; p, \mathbb{Z}/p^m) \\
= 0.
\]

If $X$ is not affine, we write $X = U \cup V$ and repeat the proof of Lemma [3.3] to get an exact sequence of pro-abelian groups

\[
\lim_{\substack{\rightarrow \cr \longleftarrow \cr m \rightarrow \infty}} K_{n+1}(W, W', rY_W; \mathbb{Z}/p^m) \xrightarrow{\sim} \lim_{\substack{\rightarrow \cr \longleftarrow \cr m \rightarrow \infty}} K_n(X, X', rY; \mathbb{Z}/p^m) \\
\]
where the middle vertical arrow is an isomorphism for every \( r \geq 1 \) by \( \text{(4.2)} \). On the other hand, the long exact sequence of bi-relative \( K \)-theory and Theorem 4.1 tell us that the map \( A_r \rightarrow A_1 \) is zero for all \( r \gg 1 \). In particular, the map \( F^dK_0(X', rW') \rightarrow F^dK_0(\tilde{X}, r\tilde{W}) \) is an isomorphism for all \( r \gg 1 \).

In the commutative diagram

\[
\begin{array}{ccc}
F^dK_0(X', r'E') & \rightarrow & F^dK_0(X', r'W') \\
\downarrow & & \downarrow \\
F^dK_0(\tilde{X}, r'E) & \rightarrow & F^dK_0(\tilde{X}, r\tilde{W}),
\end{array}
\]

where \( r' \gg r \gg 1 \), all arrows are surjective. We have just shown that the right vertical arrow is an isomorphism. The top horizontal arrow is an isomorphism by \( \text{(4.2)} \). It follows that all arrows are isomorphisms. The theorem follows easily form this and \( \text{(4.2)} \). \( \square \)

Proof of Corollary 1.8: We first show that the cycle class map (see (2.2)) \( \text{cyc}_X : CH^d(X) \rightarrow F^dK_0(X) \) is an isomorphism. Since \( \dim(X) \geq 2 \) and it has isolated Cohen-Macaulay singularities, it follows that \( X \) is normal. It follows from [30, Corollary 5.4] that \( \ker(\text{cyc}_X) \) is a torsion group. In particular, it must be zero if \( X \) is affine by [26, Corollary 1.7]. If \( X \) is projective, then it follows from [28, Chap. II, Theorem 11] that the map \( \text{Alb}(X) \rightarrow \text{Alb}(\tilde{X}) \) is an isomorphism, where \( \text{Alb}(\tilde{X}) \) is the Albanese variety of a normal projective variety in the sense of Lang [28, Chap. 3, § 3]. In particular, it follows from [26 Theorem 1.6] that there is a commutative diagram

\[
\begin{array}{ccc}
CH^d(X) & \rightarrow & F^dK_0(X) \\
\downarrow & & \downarrow \\
CH^d(\tilde{X}) & \rightarrow & F^dK_0(\tilde{X})
\end{array}
\]

in which the left vertical arrow is an isomorphism on torsion. Since the bottom horizontal arrow is known to be an isomorphism (see [31 Theorem 3.2]), it follows that the top horizontal arrow must be injective on the torsion subgroup. We conclude that \( \ker(\text{cyc}_X) = 0 \).

We now have a commutative diagram with exact rows:

\[
\begin{array}{ccc}
0 & \rightarrow & F \rightarrow CH^d(X) \rightarrow F^dK_0(\tilde{X}, E) \rightarrow 0 \\
\downarrow & & \downarrow & & \downarrow \\
\pi^* & & \pi_* \\
CH^d(\tilde{X}) & \rightarrow & CH^d(\tilde{X}),
\end{array}
\]

where \( F = \ker(\text{cyc}_X) \). It follows that \( F \hookrightarrow \ker(\pi^*) \).

It follows from Theorem 1.7 and Lemma 3.3 that \( F \) is a \( p \)-primary torsion group. On the other hand, it follows from [26, Theorem 1.6, Corollary 1.7] that \( \ker(\pi^*) \) is torsion-free. We must therefore have \( F = 0 \). \( \square \)

Proof of Corollary 1.9: Set \( X = \text{Spec}(A) \) and let \( \pi : \tilde{X} \rightarrow X \) denote the blow-up of \( X \) at its vertex \( P \). It is easy to check that there is a projection map \( p : \tilde{X} \rightarrow Z \) which is an affine bundle of rank one. Moreover, the 0-section of this affine bundle \( t : Z \hookrightarrow \tilde{X} \) is also the exceptional divisor for \( \pi \). It particular, \( \pi \) is a resolution of singularities of \( X \) with reduced exceptional divisor \( Z \).

The homotopy invariance implies that the pull-back map \( t^* : K(\tilde{X}) \rightarrow K(Z) \) is a homotopy equivalence of spectra. Equivalently, \( K(\tilde{X}, Z) \) is contractible. In particular, \( F^dK_0(\tilde{X}, Z) = 0 \). It follows from Corollary 1.8 that \( CH^d(X) = 0 \). The last part of the corollary follows from the vanishing of \( CH^d(X) \) and [39, Theorem 3.7]. \( \square \)
5. Divisibility of $SK_1$ of a normal projective surface

Recall that for any noetherian scheme $X$, there is a natural map $K_1(X) \to H^0(X,\mathcal{K}_{1,X})$ and $SK_1(X)$ is defined to be the kernel of this map. Our goal in this section is to prove the divisibility property of the cohomology group $SK_1(X)$ for a normal projective surface $X$. In this section, we shall always assume that the base field $k$ is algebraically closed of characteristic $p > 0$ unless we specifically mention otherwise.

For any abelian group $A$ and a prime number $l$, we shall denote $A \otimes \mathbb{Z} \mathbb{Q}_l/\mathbb{Z}_l = \varinjlim A/l^m$ by $A/l^\infty$. The $l$-primary torsion subgroup of $A$ will be denoted by $\ell^\infty A$. For any $Z \in \text{Sch}_k$, we shall write $\varinjlim K_n(Z; \mathbb{Z}/l^m)$ by $K_n(Z; \mathbb{Z}/l^\infty)$. Here, $K(Z; \mathbb{Z}/m)$ is the $K$-theory spectrum with coefficients. For $n \in \mathbb{Z}$, let $K_{n,Z}$ denote the Zariski sheaf on $Z$ associated to the presheaf $U \mapsto K_n(U)$. The sheaves of relative and bi-relative $K$-theory $K_{n,(Z,Z')}$ and $K_{n,(Z,W,Z')}$ etc. are defined similarly.

Let $X$ be an irreducible normal projective surface over $k$ and let $Y \subset X$ denote its singular locus with the reduced induced subscheme structure. Using the resolution of singularities of surfaces, we can find a resolution of singularities $\pi : \tilde{X} \to X$ of such that the reduced exceptional divisor $E \subset \tilde{X}$ is a strict normal crossing divisor.

For any integer $r \geq 1$, the pull-back map $K_1(X,rY) \to K_1(U,rY_U)$ for any open subset $U \subset X$ gives a natural map $\alpha_X : K_1(X,rY) \to H^0(X,K_{1,(X,rY)})$. It is easy to check that the map $H^2(X,K_{2,(X,rY)}) \to H^2(X,K_{2,X})$ is an isomorphism. Using this isomorphism and the Bloch’s formula $H^2(X,K_{2,X}) = F^2K_0(X)$ of Levine [29], the Thomason-Trobaugh spectral sequence shows that the map $\alpha_X : K_1(X,rY) \to H^0(X,K_{1,(X,rY)})$ is surjective. Letting $SK_1(X,rY)$ denote its kernel, there is an exact sequence

(5.1) \[ 0 \to SK_1(X,rY) \to K_1(X,rY) \xrightarrow{\alpha_X} H^0(X,K_{1,(X,rY)}) \to 0. \]

There is an analogous exact sequence for the pair $(\tilde{X},rE)$ as well. It also follows from the Thomason-Trobaugh spectral sequence that there is an exact sequence

(5.2) \[ H^2(X,K_{3,X}) \to SK_1(X) \to H^1(X,K_{2,X}) \to 0. \]

Lemma 5.1. The pull-back map $\pi^*$ on $K$-theory induces a short exact sequence

\[ 0 \to \frac{SK_1(\tilde{X},rE)}{SK_1(X,rY)} \to \frac{K_1(\tilde{X},rE)}{K_1(X,rY)} \xrightarrow{\alpha_X} \frac{H^0(\tilde{X},K_{1,(\tilde{X},rE)})}{H^0(X,K_{1,(X,rY)})} \to 0. \]

Proof. Since the Thomason-Trobaugh spectral sequence is functorial with respect to pull-back maps on $K$-theory, there is a commutative diagram of short exact sequences

\[ \begin{array}{ccc}
0 & \to & SK_1(X,rY) \\
\downarrow & & \downarrow \\
0 & \to & SK_1(\tilde{X},rE)
\end{array} \]

\[ \begin{array}{ccc}
K_1(X,rY) & \xrightarrow{\alpha_X} & H^0(X,K_{1,(X,rY)}) \\
\downarrow & & \downarrow \\
K_1(\tilde{X},rE) & \xrightarrow{\alpha_{\tilde{X}}} & H^0(\tilde{X},K_{1,(\tilde{X},rE)})
\end{array} \]

where the vertical arrows are pull-back maps $\pi^*$. It is therefore sufficient to prove the stronger assertion that there is a commutative diagram

(5.3) \[ \begin{array}{ccc}
H^0(X,K_{1,(X,rY)}) & \xrightarrow{\beta_X} & K_1(X,rY) \\
\downarrow & & \downarrow \\
H^0(\tilde{X},K_{1,(\tilde{X},rE)}) & \xrightarrow{\beta_{\tilde{X}}} & K_1(\tilde{X},rE)
\end{array} \]
such that \( \alpha_X \circ \beta_X \) and \( \alpha_X \circ \beta_X \) are identity.

To prove this, we let \( f \in H^0(X, K_1(X, rY)) \). This is equivalent to a regular map \( f : X \to \mathbb{G}_m \)
over \( k \) such that \( f|_{rY} = 1 \) and hence a commutative diagram with exact rows

\[
\begin{array}{cccccc}
0 & \longrightarrow & K_1(\mathbb{G}_m, \{1\}) & \longrightarrow & K_1(\mathbb{G}_m) & \longrightarrow & K_1(\{1\}) & \longrightarrow & 0 \\
\downarrow{f^*} & & \downarrow{f^*} & & \downarrow{f^*} & & \downarrow{f^*} & & \\
0 & \longrightarrow & H^0(X, rY) & \longrightarrow & H^0(X) & \longrightarrow & H^0(rY) & \longrightarrow & 0
\end{array}
\]

(5.4)

Letting \( \mathbb{G}_m = \text{Spec} (k[t^{\pm 1}]) \), one knows that \( t \) defines a unique class \([t]\) \( K_1(\mathbb{G}_m, \{1\}) \). We set \( \beta_X (f) = f^*([t]) \in K_1(X, rY) \). One can check (as is well known) that \( \delta_X \circ f^*([t]) = f \) in \( H^0(X, K_1(X)) \). Since \( t \in K_1(\mathbb{G}_m, \{1\}) \), we see that \( \delta_X \circ f^*(t) \) lies in \( H^0(rY, K_1(X)) \). Hence, it
must lie in \( H^0(X, K_1(X, rY)) \). We conclude that \( \alpha_X \circ \beta_X (f) = f \).

Given any \( f, g \in H^0(X, K_1(X, rY)) \), we consider the maps

\[
\begin{array}{cccccc}
X \xrightarrow{h} \mathbb{G}_m \times \mathbb{G}_m \xrightarrow{\mu} \mathbb{G}_m \xrightarrow{\pi} \mathbb{G}_m
\end{array}
\]

(5.5)

where \( p_1, p_2 \) are the projections, \( h(x) = (f(x), g(x)) \) and \( \mu \) is the multiplication map. All
these maps take \( Y \to 1 \in \mathbb{G}_m \) or to \((1, 1) \in \mathbb{G}_m \times \mathbb{G}_m \).

Since \( \mu \) is induced by the map on functions \( \mu_* : k[t^{\pm 1}] \to k[x^{\pm 1}, y^{\pm 1}] \), given by \( \mu_*(t) = xy \), we get

\[
\beta_X (fg) = (fg)^*([t]) = (\mu \circ h)^*([t]) = h^*([x] + [y]) = (p_1 \circ h)^*([t]) + (p_2 \circ h)^*([t]) = f^*([t]) + g^*([t]) = \beta_X (f) + \beta_X (g).
\]

It follows that \( \beta_X \) is a group homomorphism. The same construction also defines \( \beta_X \).
Furthermore, it follows from the above construction that

\[
\beta_X^{-1} \circ \pi^*(f) = (f \circ \pi)^*([t]) = \pi^* \circ f^*([t]) = \pi^* \circ \beta_X (f)
\]

for every \( f \in H^0(X, K_1(X, rY)) \). In particular, (5.4) commutes and this finishes the proof. \( \square \)

Lemma 5.2. For every \( r \geq 1 \), the map

\[
SK_1(X, rY)/p^\infty \to SK_1(X)/p^\infty
\]

is an isomorphism.

Proof. We first claim that the map \( SK_1(X, rY) \to SK_1(X) \) is surjective. Using (5.1), it
suffices to show that the map \( H^i(X, K_{i+1}(X, rY)) \to H^i(X, K_{i+1, X}) \) is surjective for \( i = 1, 2 \).
We have an exact sequence of Zariski sheaves

\[
K_{n+1, rY} \to K_{n, (X, rY)} \to K_{n, X} \to K_{n, rY}.
\]

In particular, the kernel and the cokernel of the middle arrow are supported on \( Y \). The
desired surjectivity for the induced maps on the cohomology groups follows easily (e.g., see [11] Lemma 1.3)). This proves the claim.
The long exact sequence for the relative $K$-theory of the pair $(X, rY)$ and the inclusion $H^0(X, K_{1, (X, rY)}(1, X)) \hookrightarrow H^0(X, K_{1, X})$ now give us an exact sequence
\begin{equation}
(5.6) \quad K_2(rY) \to SK_1(X, rY) \to SK_1(X) \to 0.
\end{equation}

We can write $K_2(rY) = K_2(rY, Y) \oplus K_2(Y)$. It follows from Lemma 5.3 that $K_2(rY, Y)$ is a $p$-primary torsion group of bounded exponent. Since $\mathbb{Q}_p/\mathbb{Z}_p$ is divisible, we must have $K_2(rY, Y)/p^\infty = 0$. On the other hand, $K_2(Y)$ is divisible and hence $K_2(Y)/p^\infty \simeq \lim_{m \to \infty} K_2(Y)/p^m = 0$. All of this uses that $k$ is algebraically closed. It follows that the map $SK_1(X, rY)/p^\infty \to SK_1(X)/p^\infty$ is an isomorphism. \hfill \Box

Lemma 5.3. $SK_1(\tilde{X})/p^\infty = 0$.

Proof. Using the exact sequence
\begin{equation}
(5.7) \quad H^2(\tilde{X}, K_{3, \tilde{X}}) \to SK_1(\tilde{X}) \to H^1(\tilde{X}, K_{2, \tilde{X}}) \to 0,
\end{equation}
it suffices to show that $H^2(\tilde{X}, K_{3, \tilde{X}})/p^\infty = 0 = H^1(\tilde{X}, K_{2, \tilde{X}})/p^\infty$.

The Gersten resolution for $K_{3, \tilde{X}}$ tells us that $H^2(\tilde{X}, K_{3, \tilde{X}})$ is the quotient of the group
\[
\bigoplus_{x \in X^{(2)}} K_1(k(x)),
\]
which is clearly divisible (since $k = \overline{k}$). This yields $H^2(\tilde{X}, K_{3, \tilde{X}})/p^\infty = 0$.

The vanishing of $H^1(\tilde{X}, K_{2, \tilde{X}})/p^\infty$ follows from the exact sequence
\begin{equation}
(5.8) \quad 0 \to H^1(\tilde{X}, K_{2, \tilde{X}})/p^\infty \to \lim_{m \to \infty} H^1(\tilde{X}, K_{2, \tilde{X}}/p^m) \xrightarrow{\tau_{\tilde{X}}} p^\infty CH^2(\tilde{X}) \to 0
\end{equation}
and the Roitman torsion theorem for $\tilde{X}$, whose proof involves showing that $\tau_{\tilde{X}}$ is an isomorphism (see [36] proof of Theorem 0.1)]. \hfill \Box

Lemma 5.4. $SK_1(\tilde{X}, E)/p^\infty = 0$.

Proof. For $r \geq 1$, we have a commutative diagram
\[
\begin{array}{ccc}
SK_1(\tilde{X}, rE) & \to & SK_1(\tilde{X}) \to SK_1(rE) \\
\downarrow & & \downarrow \\
K_2(rE) & \to & K_1(\tilde{X}, rE) \to K_1(\tilde{X}) \to K_1(rE),
\end{array}
\]
where the vertical arrows are injective. Since $H^0(X, K_{1, (\tilde{X}, rE)}) \hookrightarrow H^0(\tilde{X}, K_{1, \tilde{X}})$, we get an exact sequence
\begin{equation}
(5.9) \quad K_2(rE) \to SK_1(\tilde{X}, rE) \to SK_1(\tilde{X}) \to SK_1(rE)
\end{equation}
For $r = 1$, we break this exact sequence into the smaller exact sequences
\[
K_2(E) \to SK_1(\tilde{X}, E) \to F_1 \to 0;
\]
\[
0 \to F_1 \to SK_1(\tilde{X}) \to F_2 \to 0;
\]
\[
0 \to F_2 \to SK_1(E) \to F_3 \to 0.
\]

It is shown in [26] Lemma 7.7] that the map $SK_1(\tilde{X}) \to SK_1(E)$ induces a surjective map $p^\infty SK_1(\tilde{X}) \to p^\infty SK_1(E)$. In particular, we get $p^\infty SK_1(\tilde{X}) \to p^\infty F_2$, which in turn shows that $F_1/p^\infty \hookrightarrow SK_1(\tilde{X})/p^\infty$. Lemma 5.3 implies that $F_1/p^\infty = 0$. To prove the lemma, we are now left with showing that $K_2(E)/p^\infty = 0$. Using the universal coefficient exact sequence
\begin{equation}
(5.10) \quad 0 \to K_n(E)/p^\infty \to K_n(E; \mathbb{Z}/p^\infty) \to p^\infty K_{n-1}(E) \to 0,
\end{equation}
it suffices to show that $K_2(E; \mathbb{Z}/p^\infty) = 0$. \hfill \Box
Letting $S \subset E$ be a finite set of closed points lying inside the smooth locus of $E$ and using excision plus devissage, one gets an exact sequence

$$K_2(S; \mathbb{Z}/p^m) \to K_2(E; \mathbb{Z}/p^m) \to K_2(E \setminus S; \mathbb{Z}/p^m).$$

It follows from (5.10) that the first term of this exact sequence is zero. We can thus assume that $E$ is affine.

We let $E = \text{Spec}(A)$, $B = \text{Spec}(A^N)$ and let $I \subset A$ denote the ideal of $E_{\text{sing}}$. Since $E$ is a strict normal crossing divisor on a smooth surface, $I$ is in fact a conducting ideal for the normalization map $A \to B$. We now have a commutative diagram of exact sequences

$$\begin{align*}
K_4(B/I; \mathbb{Z}/p^m) &\to K_3(B, I; \mathbb{Z}/p^m) \to K_3(B; \mathbb{Z}/p^m) \\
K_2(A, B, I; \mathbb{Z}/p^m) &\to K_2(A, I; \mathbb{Z}/p^m) \to K_2(A; \mathbb{Z}/p^m)
\end{align*}$$

(5.11)

$$\begin{align*}
K_3(A/I; \mathbb{Z}/p^m) &\to K_2(A, I; \mathbb{Z}/p^m) \to K_2(A; \mathbb{Z}/p^m) \\
K_3(B/I; \mathbb{Z}/p^m) &\to K_2(B, I; \mathbb{Z}/p^m) \to K_2(B; \mathbb{Z}/p^m) \to K_2(B/I; \mathbb{Z}/p^m).
\end{align*}$$

Since $B, A/I, B/I$ are all smooth of dimension at most one, it follows from [13, Theorem 8.4] that $K_2(A, B, I; \mathbb{Z}/p^m) \cong K_2(A, I; \mathbb{Z}/p^m) \cong K_2(A; \mathbb{Z}/p^m)$ for each $m \geq 1$. It therefore suffices to show that $K_2(A, B, I; \mathbb{Z}/p^\infty) = 0$.

To prove this, we use the short exact sequence

$$0 \to K_2(A, B, I)/p^\infty \to K_2(A, B, I; \mathbb{Z}/p^\infty) \to p^\infty K_1(A, B, I) \to 0.$$ 

It follows from Lemma 5.3 that $K_2(A, B, I)$ is a $p$-primary torsion group of bounded exponent. Since $\mathbb{Q}_p/\mathbb{Z}_p$ is divisible, we must have $K_2(A, B, I)/p^\infty = 0$. On the other hand, $K_1(A, B, I) \cong I/I^2 \otimes_{A/I} \Omega^1_{(B/I)/(A/I)}$ by [17, Theorem 0.2]. Since $B/I \cong k^r$ for some $r \geq 1$, we have $\Omega^1_{(B/I)/(A/I)} = 0$. We conclude that $K_2(A, B, I; \mathbb{Z}/p^\infty) = 0$. This finishes the proof.

\begin{lemma}
For every $r \geq 1$, we have $SK_1(\tilde{X}, rE)/p^\infty = 0$.
\end{lemma}

\begin{proof}
The case $r = 1$ follows from Lemma 5.4. So we assume $r \geq 2$. Using the long exact relative $K$-theory sequence and (5.11), we get a commutative diagram with exact rows

$$\begin{align*}
SK_1(\tilde{X}, rE) &\to SK_1(\tilde{X}, E) \to SK_1(rE, E) \\
K_2(rE, E) &\to K_1(\tilde{X}, rE) \to K_1(\tilde{X}, E) \to K_1(rE, E),
\end{align*}$$

where the vertical arrows are all injective.

Using Lemma 3.4, we get an exact sequence

$$0 \to A \to SK_1(\tilde{X}, rE) \to SK_1(\tilde{X}, E) \to B \to 0,$$

where $A$ and $B$ are $p$-primary torsion groups of bounded exponents. Setting $SK_1(\tilde{X}, rE) := SK_1(\tilde{X}, rE)/A$, the divisibility of $\mathbb{Q}_p/\mathbb{Z}_p$ implies that

(5.12) $SK_1(\tilde{X}, rE)/p^\infty \cong SK_1(\tilde{X}, rE)/p^\infty$. 

We now consider a commutative diagram of exact sequences

\[
\begin{array}{cccccc}
0 & \rightarrow & SK_1(X,rE) & \rightarrow & SK_1(\tilde{X},E) & \rightarrow B & \rightarrow 0 \\
& & \downarrow & & \downarrow & & \\
0 & \rightarrow & SK_1(\tilde{X},rE)[p^{-1}] & \rightarrow & SK_1(\tilde{X},E)[p^{-1}] & \rightarrow B[p^{-1}] & \rightarrow 0,
\end{array}
\]

where the vertical arrows are the localizations.

Since $B$ is $p$-primary torsion, the vertical arrow on the right is zero. Combining this with Lemma 5.4 and the isomorphism $\mathbb{Z}[p^{-1}]/\mathbb{Z} \cong \lim_{\rightarrow} \mathbb{Z}/p^m \cong \mathbb{Q}_p/\mathbb{Z}_p$, we get a surjection $B \twoheadrightarrow SK_1(X,rE) \otimes_{\mathbb{Z}} \mathbb{Z}[p^{-1}]/\mathbb{Z} \cong SK_1(X,rE) \otimes_{\mathbb{Z}} \mathbb{Q}_p/\mathbb{Z}_p$. It follows that $SK_1(X,rE)/\mathbb{Q}_p^\infty$ is a divisible group of bounded exponent, and hence it must be zero. We conclude from (5.12) that $SK_1(\tilde{X},rE)/\mathbb{Q}_p^\infty = 0$.

**Theorem 5.6.** Let $X$ be a normal projective surface over an algebraically closed field $k$. Then

\[ SK_1(X) \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z} = 0 = H^1(X,\mathcal{K}_{2,X}) \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z}. \]

**Proof.** It suffices to show that $SK_1(X) \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z} = 0$ as the remaining part follows from (5.2).

Since $\mathbb{Q}/\mathbb{Z} \cong \bigoplus_{l} \mathbb{Q}_l/\mathbb{Z}_l$, where $l$ runs over the set of prime numbers, it suffices to show that $SK_1(X)/l^\infty = 0$ for every prime $l$. For $l \neq \text{char}(k)$, this follows from [2, Theorem 7.9]. So we assume $\text{char}(k) = p > 0$ and show that $SK_1(X)/p^\infty = 0$. In view of Lemma 5.2, this is equivalent to showing that $SK_1(X,rY)/p^\infty = 0$ for every $r \geq 1$.

To prove this, we consider a commutative diagram with exact bottom row

\[
\begin{array}{cccccc}
SK_1(X,rY) & \rightarrow & SK_1(\tilde{X},rE) & \rightarrow & K_0(X,\tilde{X},rY,rE).
\end{array}
\]

Since $X$ is a normal surface, it is Cohen-Macaulay with isolated singularities. It follows from Lemma 5.1 and Theorem 3.8 that there is an exact sequence

\[ 0 \rightarrow A \rightarrow SK_1(X,rY) \rightarrow SK_1(\tilde{X},rE) \rightarrow B \rightarrow 0, \]

where $A$ and $B$ are $p$-primary torsion groups of bounded exponents. It follows from Lemma 5.5 that $SK_1(\tilde{X},rE)/p^\infty = 0$. We now repeat the proof of Lemma 5.5 in verbatim to conclude that $SK_1(X,rY)/p^\infty = 0$. \(\square\)

### 6. The torsion in the Chow group of surfaces

We prove Theorem 1.1 in this section for surfaces. In fact, we prove this torsion theorem for a certain class of non-affine surfaces as well. This more general result will be needed in the proof of the general case of Theorem 1.1 later in this paper. We fix an algebraically closed field $k$ of any characteristic. For an equi-dimensional $k$-scheme $X$, let $\text{CH}^d(X)$ denote the (homological) Chow group of codimension $q$ cycles on $X$ in the sense of [19]. Recall from [26] that every normal projective variety $X$ over $k$ of dimension $d$ has an Albanese variety $\text{Alb}(X)$ which is an abelian variety and there is an Albanese map $\text{alb}_X : \text{CH}^d(X)_0 \rightarrow \text{Alb}(X)$. 
6.1. **Divisibility of $H^1(X, \mathcal{K}_{2,X})$ for non-projective normal surfaces.** In order to prove our torsion theorem for surfaces, we need to extend Theorem 5.6 to certain non-projective normal surfaces. We do it in this subsection.

**Lemma 6.1.** Let $X$ be a reduced affine surface over $k$ and let $\pi : Y \to X$ be the blow-up of $X$ along a closed subscheme $Z \subset X$ whose support is finite. Let $W \subset X$ be a closed subscheme such that $Z \cap W = \emptyset$ and $\pi^{-1}(X \setminus W)$ is smooth. We can then find an open embedding $j' : Y^N \to Y$, where $Y$ is a projective normal surface over $k$ such that the following hold.

1. $\overline{Y}_{\text{sing}} = (Y^N)_{\text{sing}}$.
2. $Y \setminus Y^N$ is a strict normal crossing divisor on $Y$.
3. $Y \setminus Y^N$ is connected.

**Proof.** Let $E \subset Y$ denote the reduced exceptional divisor of the blow-up map $\pi$. Let $f : X^N \to X$ be the normalization map and let $Z' = Z \times_X X^N$. Consider the commutative square

$$
\begin{array}{ccc}
\Bl_{Z'}(X^N) & \to & Y \\
\pi^N & \downarrow & \downarrow \pi \\
X^N & \underset{f}{\to} & X.
\end{array}
$$

It follows from Lemma 2.5 that $g$ is finite and birational. The map $g^{-1}(Y \setminus E) \to Y \setminus E$ is clearly the normalization map. On the other hand, it is given that $Y$ is smooth in an open neighborhood of $E$. In particular, $g$ must be an isomorphism in that neighborhood. It follows that $\Bl_{Z'}(X^N) = Y^N$.

Since $X^N$ is an affine normal surface, we can find an open embedding $j : X^N \to \overline{X}$, where $\overline{X}$ is a normal projective surface over $k$ such that $\overline{X}_{\text{sing}} = X_{\text{sing}}$ and the complement $T = \overline{X} \setminus X^N$ is a strict normal crossing divisor on $\overline{X}$. Moreover, it follows from [IS Corollary 1] that $T$ is connected.

Since $Z'$ is a closed subscheme of $X^N$ with finite support, it is actually a closed subscheme of $\overline{X}$ and we get a Cartesian square

$$
\begin{array}{ccc}
\Bl_{Z'}(X^N) & \to & \Bl_{Z'}(\overline{X}) \\
\pi^N & \downarrow & \downarrow \pi \\
X^N & \underset{j}{\to} & \overline{X}.
\end{array}
$$

Since $\pi$ is an isomorphism away from $Z'$, it follows that $\Bl_{Z'}(\overline{X})$ is normal outside $\pi^{-1}(Z')$ and the singular locus of $\Bl_{Z'}(\overline{X})$ is same as that of $\Bl_{Z'}(X^N)$.

On the other hand, the normality of $Y^N = \Bl_{Z'}(X^N)$ implies that $\Bl_{Z'}(\overline{X})$ is already normal in an open neighborhood of $\pi^{-1}(Z')$. It follows that $\Bl_{Z'}(\overline{X})$ is normal. Furthermore, $\Bl_{Z'}(\overline{X}) \setminus \Bl_{Z'}(X^N) = \pi^{-1}(T)$ and the map $\pi^{-1}(T) \to T$ is clearly an isomorphism. Setting $\overline{Y} = \Bl_{Z'}(\overline{X})$, we see that $\overline{Y}$ is a normal projective surface with open embedding $j' : Y^N \to \overline{Y}$ and $\overline{Y} \setminus Y^N = T$, satisfies the desired properties. \qed

**Lemma 6.2.** Let $X$ be a reduced affine surface over $k$ and let $\pi : Y \to X$ be the blow-up of $X$ along a closed subscheme $Z \subset X$ whose support is finite. Let $Y^N \subseteq \overline{Y}$ be the inclusion obtained in Lemma 2.5 with complement $i : T \subseteq \overline{Y}$. Then the composite map $\psi : \text{CH}^1(T)_0 \to \text{CH}^2(\overline{Y})_0 \to \text{Alb}(\overline{Y})$ is surjective.
Proof. Keeping the notations of the proof of Lemma 6.1 let $\tilde{\pi} : \tilde{Y} \to Y$ be a resolution of singularities of $Y$ such that the reduced exceptional divisor $E$ is strict normal crossing. One knows that the map $\pi_* : \text{Alb}(\tilde{Y}) \to \text{Alb}(Y)$ is an isomorphism. Let $\phi : \tilde{Y} \to \text{Alb}(\tilde{Y})$ denote the universal morphism. Since $\tilde{Y}_{\text{sing}} \cap T = \emptyset$, we can assume $T$ to be a closed subscheme of $\tilde{Y}$.

Since $Y^N$ is smooth in an open neighborhood of $Z' = Z \times_X X^N$ by assumption, it follows that $F := \pi^{-1}(Z')$ is a closed subset of $\tilde{Y}$ which is disjoint from $E$ and $T$. Set $D = E + F + T$.

Since $T$ is connected, it follows from [10] Lemma 5.4 that $\psi(\text{CH}_F^x(T)_0)$ is an abelian subvariety of $\text{Alb}(Y)$. Let us denote this abelian subvariety by $B$.

Following the proof of [10] Theorem 5.3, let $A = \text{Alb}(Y)/B$ and let $\beta : \tilde{Y} \to A$ denote the composite map $\tilde{Y} \to \text{Alb}(\tilde{Y}) \to A$. It is then clear that $z := \beta(T)$ is a closed point of $A$. Since $A$ is divisible, we need to show that the image of every irreducible component of $\tilde{Y}$ in $A$ is a point. We can thus assume that $\tilde{Y}$ is irreducible. We then need to show that $\beta(\tilde{Y}) := \tilde{Z}$ is a point. Since $\beta$ is projective, $\beta(x) = z$ for every $x \in \tilde{Y} \setminus D$ will imply that $\dim(\tilde{Z}) = 0$, and we are done. Otherwise, we can assume that there is a closed point $x \in \tilde{Y} \setminus D$ such that $\beta(x) \neq z$. Let $Y' = \beta^{-1}(\beta(x))$.

Suppose first that $\dim(\tilde{Z}) = 1$. Since $\beta : \tilde{Y} \to \tilde{Z}$ is a projective and surjective morphism of relative dimension one, each of its fibers is at least one-dimensional. In particular, $\dim(Y') = 1$. Since $T \cap Y' = \emptyset$ and $Y' \not\subset E$, it follows that $\tilde{\pi}(Y')$ is a projective curve in $Y^N = Y \setminus T$. Since $Y' \not\subset F$, it also follows that $\tilde{\pi}(Y')$ is a projective curve in $Y^N$ which is not contained in $F$. Since $F$ is the exceptional divisor for the blow-up map $\pi^N : Y^N \to X^N$, it follows that $\pi^N(\tilde{\pi}(Y'))$ is a projective curve in $X^N$. But this is absurd as $X^N$ is affine.

Next assume that $\dim(\tilde{Z}) = 2$. Then it follows from [13] pg. 96 that the intersection matrix of $\beta^{-1}(z)$ is negative definite. In particular, we get $T^2 \leq 0$. On the other hand, we observe that $T$ is a closed subscheme of $\underline{X}$ (in the notations of the proof of Lemma 6.1) with affine complement $X^N$. Hence, it follows from [18] Theorem 1 that there is a monoidal transform $\eta : \underline{X}_1 \to \underline{X}$ with center in $T$ such that $\eta^{-1}(T)$ is the support of an ample divisor. This implies in particular that we must have $T^2 > 0$, which is again absurd. We have thus shown that $\dim(\tilde{Z}) = 0$ and this finishes the proof of the lemma. \hfill $\square$

Proposition 6.3. Let $X$ be a reduced affine surface over $k$ and let $\pi : Y \to X$ be the blow-up of $X$ along a closed subscheme $Z \subset X$ whose support is finite. Then

$$H^1(Y^N, \mathcal{K}_{2,Y^N}) \otimes_Z \mathbb{Q}/\mathbb{Z} = 0 = H^1(X^N, \mathcal{K}_{2,X^N}) \otimes_Z \mathbb{Q}/\mathbb{Z}.$$

Proof. Let $j : Y^N \hookrightarrow Y$ be an open embedding obtained in Lemma 6.1 with complement $Y \setminus Y^N = T$. We keep the notations of the proof of Lemma 6.1.

Using the Bloch’s formula for singular surfaces (see [20] Theorem 7), there is a commutative diagram

$$\begin{array}{c}
\text{CH}^2(Y) \xrightarrow{j_*} H^2(Y, \mathcal{K}_{2,Y}) \\
j^* \\
\text{CH}^2(Y^N) \xrightarrow{j^*} H^2(Y^N, \mathcal{K}_{2,Y^N})
\end{array}$$

where the two horizontal arrows are isomorphisms. Since the left vertical arrow is surjective by the choice of $Y$, it follows that the right vertical arrow is also surjective.

We now consider the long exact sequence of Zariski cohomology groups

$$H^1(Y, \mathcal{K}_{2,Y}) \xrightarrow{j_*} H^1(Y^N, \mathcal{K}_{2,Y^N}) \to H^2(Y, \mathcal{K}_{2,Y}) \xrightarrow{i_*} H^2(Y^N, \mathcal{K}_{2,Y^N}).$$
Let $i : T \to Y$ denote the inclusion map. Lemma 2.5, the isomorphisms of (0.3) and the Gersten resolution for $K_{2Y}$ (see [29] Theorem 7) for normal surfaces) together show that this exact sequence is equivalent to an exact sequence

$$H^1(Y, K_{2Y}) \xrightarrow{j^*} H^1(Y^N, K_{2Y^N}) \to CH^1_F(T) \xrightarrow{i_*} CH^2(Y) \xrightarrow{j_*} CH^2(Y^N) \to 0.$$  

Since $T$ is connected, we have a commutative diagram

$$
\begin{array}{ccc}
H^1(Y, K_{2Y}) & \xrightarrow{j^*} & H^1(Y^N, K_{2Y^N}) \\
\downarrow{deg} & & \downarrow{deg} \\
\mathbb{Z} & \xrightarrow{deg} & \mathbb{Z}.
\end{array}
$$

Let $CH^1_F(T)_0$ and $CH^2(Y)_0$ denote the kernels of the degree maps. Since $Y$ is smooth along $T$, it is easy to see that the map $CH^2(Y)_0 \to CH^2(Y^N)$ is surjective. Thus we get an exact sequence

$$H^1(Y, K_{2Y}) \xrightarrow{j^*} H^1(Y^N, K_{2Y^N}) \to CH^1_F(T)_0 \xrightarrow{i_*} CH^2(Y)_0 \xrightarrow{j_*} CH^2_0(Y^N) \to 0. \tag{6.4}$$

We next note that $T$ is a connected strict normal crossing divisor in $Y$ and hence the map $Pic^0(T^N) \to CH^1_F(T)_0$ is surjective. Combining this with Lemma 6.2, we see that the map $T^N \to Y$ induces a surjective morphism of abelian varieties $Pic^0(T^N) \to Alb(Y)$.

We conclude from Lemma 2.7 that the map $Pic^0(T^N)_{\text{tors}} \to Alb(Y)_{\text{tors}}$ is surjective. The factorization $Pic^0(T^N)_{\text{tors}} \to (CH^1_F(T)_0)_{\text{tors}} \to Alb(Y)_{\text{tors}}$ implies that the composite map $(CH^1_F(T)_0)_{\text{tors}} \to (CH^2(Y)_{\text{tors}} \to Alb(Y)_{\text{tors}}$ is surjective. On the other hand, the Roitman torsion theorem for projective normal surfaces (see [26] Theorem 1.6) says that the map $CH^2(Y)_{\text{tors}} \to Alb(Y)$ is an isomorphism on the torsion subgroups. We thus conclude that the map

$$i_* : (CH^1_F(T)_0)_{\text{tors}} \to (CH^2(Y)_0)_{\text{tors}}$$

is surjective.

We set $H = \ker(i_*)$ so that (6.4) yields a commutative diagram of short exact sequences

$$
\begin{array}{ccc}
H^1(Y, K_{2Y}) & \xrightarrow{j^*} & H^1(Y^N, K_{2Y^N}) \\
\downarrow{deg} & & \downarrow{deg} \\
H^1(Y, K_{2Y})_{\mathbb{Q}} & \xrightarrow{j^*} & H^1(Y^N, K_{2Y^N})_{\mathbb{Q}} \\
\downarrow{deg} & & \downarrow{deg} \\
H_{\mathbb{Q}} & \to & H_{\mathbb{Q}}.
\end{array}
$$

where the vertical arrows are all localization maps. Since $Pic^0(T^N)$ is divisible and it surjects onto $CH^1_F(T)_0$, it follows that $CH^1_F(T)_0$ is divisible. In particular, the map $CH^1_F(T)_0 \to (CH^1_F(T)_0)_{\mathbb{Q}}$ is surjective. Combining this with (6.4) and (6.5), we conclude that the right vertical arrow in (6.6) is surjective. The left vertical arrow in (6.6) is surjective by Theorem 5.6. In particular, the middle vertical arrow is also surjective. Equivalently, $H^1(Y^N, K_{2Y^N})_{\mathbb{Q}} \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z} = 0$. To prove that $H^1(X^N, K_{2X^N})_{\mathbb{Q}} \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z} = 0$, we repeat the above proof in verbatim, assuming that the map $\pi$ is identity. This finishes the proof.

**Lemma 6.4.** Let $X$ be a reduced affine surface over $k$. Let $\pi : Y \to X$ be the blow-up of $X$ along a closed subscheme $Z \subset X$ whose support is finite. Then $CH^2(Y^N)$ is torsion-free.

**Proof.** Continuing the notations of the proof of Proposition 6.3, we have shown in (6.4) that there is an exact sequence

$$CH^1(T)_0 \to CH^2(Y)_0 \to CH^2(Y^N) \to 0.$$
We have also shown above that $\text{CH}^1(T)_0$ is divisible. It follows that the map $(\text{CH}^2(Y)_0)_{\text{tors}} \to \text{CH}^2(Y^N)_{\text{tors}}$ is surjective. On the other hand, it is shown in (6.5) that the map $(\text{CH}^1(T)_0)_{\text{tors}} \to (\text{CH}^2(Y)_0)_{\text{tors}}$ is surjective. We conclude that $\text{CH}^2(Y^N)_{\text{tors}} = 0$.

We now prove Theorem 1.1 for affine surfaces and their blow-ups.

**Theorem 6.5.** Let $X$ be a reduced affine surface over an algebraically closed field $k$. Let $\pi : Y \to X$ be the blow-up of $X$ along a closed subscheme $Z \subset X$ whose support is finite. Assume that $Y_{\text{sing}}$ is affine. Then $\text{CH}^2(X)$ and $\text{CH}^2(Y)$ are uniquely divisible.

**Proof.** Let $g : Y^N \to Y$ denote the normalization map. By [2, Theorem 3.3] (see also [23, Proposition 2.3] for a refined version), there exists a conducting subscheme $W \subset Y$ with $W_{\text{red}} = Y_{\text{sing}}$ such that setting $\overline{W} = W \times_X Y^N$, there is a Mayer-Vietoris exact sequence

$$
\text{SK}_1(Y^N) \to \frac{SK_1(\overline{W})}{SK_1(W)} \to \text{CH}^2(Y) \xrightarrow{g^*} \text{CH}^2(Y^N) \to 0,
$$

where we have identified $SK_0(Y) = F^2K_0(Y)$ with $\text{CH}^2(Y)$ (and similarly for $Y^N$) by [29, Theorem 7].

Since $W_{\text{red}} = Y_{\text{sing}}$ is affine, it follows that $W$ is affine (see [21, Ex. III.3.1]). Since the map $\overline{W} \to W$ is finite, it follows $\overline{W}$ is also affine. It is classically known (e.g., see [3, Chap. IX, Propositions 1.3, 3.9]) that $SK_1(R) = SK_1(R_{\text{red}})$ for any commutative ring $R$. We can therefore assume that $W$ and $\overline{W}$ are reduced in (6.7).

It follows easily from the Thomason-Trobaugh spectral sequence $E_2^{p,q} = H^p(Y, \mathcal{K}_qY) \Rightarrow K_{q-p}(Y)$ that there is a commutative diagram

$$
\begin{array}{ccc}
SK_1(Y^N) & \to & H^1(Y, \mathcal{K}_{2,Y^N}) \\
\downarrow & & \downarrow \\
\frac{SK_1(\overline{W})}{SK_1(W)} & \to & H^1(\overline{W}, \mathcal{K}_{2,\overline{W}}) \\
\end{array}
$$

where the top horizontal arrow is surjective and the bottom horizontal arrow is an isomorphism. We thus get an exact sequence

$$
H^1(Y^N, \mathcal{K}_{2,Y^N}) \to \frac{SK_1(\overline{W})}{SK_1(W)} \to \text{CH}^2(Y) \xrightarrow{g^*} \text{CH}^2(Y^N) \to 0.
$$

Since $W$ as well as $\overline{W}$ are reduced affine curves, Lemma 2.6 tells us that there is an exact sequence

$$
SK_1(W) \to SK_1(\overline{W}) \to \frac{SK_1(\overline{W})}{SK_1(W)} \to 0,
$$

where the first two terms from the left are uniquely divisible. It follows that the last term is also uniquely divisible.

We now consider a commutative diagram with exact rows

$$
\begin{array}{ccc}
H^1(Y^N, \mathcal{K}_{2,Y^N}) & \longrightarrow & \frac{SK_1(\overline{W})}{SK_1(W)} \\
\downarrow & & \downarrow \\
H^1(Y^N, \mathcal{K}_{2,Y^N})_\mathbb{Q} & \longrightarrow & \frac{(SK_1(\overline{W})}{SK_1(W)}_\mathbb{Q} \\
\end{array}
\text{Ker}(g^*) \longrightarrow 0 \text{ Ker}(g^*_\mathbb{Q}) \longrightarrow 0.
$$

We have just shown that the middle vertical arrow is an isomorphism. It follows from Proposition 6.3 that the left vertical arrow is surjective. We conclude that $\text{Ker}(g^*)$ is uniquely divisible.
Since $\text{CH}^2(Y^N)$ is generated by the images of $\text{Pic}(C) \to \text{CH}^2(Y^N)$, where $C$ is a smooth curve, it follows easily that $\text{CH}^2(Y^N)$ is divisible. It follows from Lemma 6.4 that $\text{CH}^2(Y^N)$ is torsion-free. In particular, it is uniquely divisible. We conclude from (6.9) that $\text{CH}^2(Y)$ is uniquely divisible. Exactly the same argument, using Proposition 6.3 and [26, Theorem 1.6], also shows that $\text{CH}^2(X)$ is uniquely divisible.

7. TORSION THEOREM IN HIGHER DIMENSION

In this section, we prove our torsion theorem for affine varieties by reducing it to the case of surfaces using some blow-up techniques. We shall use the following straightforward commutative algebra result.

**Lemma 7.1.** Let $R$ be commutative Noetherian ring such that $R_p$ is reduced for every associated prime ideal $p$ of $R$. Then $R$ must be reduced.

**Proof.** Let $a \neq 0$ be an element of $R$ such that $a^n = 0$ for some $n \geq 1$. Let $\{p_1, \ldots, p_r\}$ be the set of associated primes of $R$ and set $S = R \setminus (\cup_i p_i)$. It follows from our assumption that $R_S$ is reduced. In particular, $a$ dies in $R_S$. Equivalently, there exists $b \in S$ such that $ab = 0$. But this means $a = 0$ because $b$ is not a zero-divisor in $R$. □

**Theorem 7.2.** Let $X$ be a reduced affine scheme over $k$ of dimension $d \geq 2$ over an algebraically closed field $k$. Then $\text{CH}^d(X)$ is uniquely divisible.

**Proof.** In view of Theorem 6.5 we can assume $d \geq 3$. To show that $\text{CH}^d(X)$ is divisible, we let $\alpha \in Z^d(X, X_{\text{sing}})$ be a 0-cycle. We fix a locally closed embedding $X \hookrightarrow \mathbb{P}^N_k$. It follows from [1, Theorem 7] and [31, Lemma 1.3] that there exists a complete intersection reduced affine surface $Y \subset X$ containing the support of $\alpha$ such that $Y$ is smooth away from $X_{\text{sing}}$.

Since the inclusion $i : Y \hookrightarrow X$ is a complete intersection and $Y$ is smooth away from $X_{\text{sing}}$, it follows from Lemmas 2.1 and 2.4 that there exists a push-forward map $i_* : \text{CH}^2(Y) \to \text{CH}^d(X)$ which contains $\alpha$ in its image. Since $\text{CH}^2(X)$ is divisible by Theorem 6.5, we conclude that $\text{CH}^d(X)$ is divisible. The rest of the proof will be devoted to showing that $\text{CH}^d(X)$ is torsion-free.

Let $\alpha \in Z^d(X, X_{\text{sing}})$ be such that $n\alpha = 0$ in $\text{CH}^d(X)$ for some integer $n \geq 1$. Equivalently, $n\alpha \in R^d(X, X_{\text{sing}})$. We can thus use Lemma 2.2 to find a (reduced) Cartier curve $C$ on $X$ and a function $f \in \mathcal{O}_{C,S}^\times$ such that $n\alpha = (f)_C$, where $S = C \cap X_{\text{sing}}$. Since part of $n\alpha$ supported on any connected component of $C$ is also of the form $n\alpha' = (f')_{C'}$ for some Cartier curve $C'$ and some $f'$, we can assume that $C$ is connected. Let $\{C_1, \ldots, C_r\}$ denote the set of (irreducible) components of $C$.

We set $U = X \setminus X_{\text{sing}}$ so that $U$ is smooth. In particular, it is a dense open subset of the disjoint union of the smooth loci of the components of $X$. Let $\pi : X' \to X$ be a successive blow-up at smooth points such that the following hold.

1. The strict transform $D_i$ of each $C_i$ is smooth along $\pi^{-1}(U)$.
2. $D_i \cap D_j \cap \pi^{-1}(U) = \emptyset$ for $i \neq j$.
3. Each $D_i$ intersects the exceptional divisor $E$ (which is reduced) transversely at smooth points.

It is clear that there exists a finite set of blown-up closed points $T \subset U$ such that $\pi : \pi^{-1}(X \setminus T) \to X \setminus T$ is an isomorphism. In particular, $\pi : X'_{\text{sing}} \to X_{\text{sing}}$ is an isomorphism. This in turn implies that $X'_{\text{sing}}$ is an affine scheme of dimension at most $d - 1$. Set $U' = X' \setminus X'_{\text{sing}} = \pi^{-1}(U)$. Let $D$ denote the strict transform of $C$ with components $\{D_1, \ldots, D_r\}$.

Since $\pi$ is an isomorphism over an open neighborhood of $X_{\text{sing}}$, it follows that $\pi^{-1}(S) \simeq S$ and the map $D \to C$ is an isomorphism along $X_{\text{sing}}$. In particular, we have $f \in \mathcal{O}^\times_{D,S}$ and
\( \pi_*(f)_D = (f)_C = n\alpha. \) Since \( \text{Supp}(\alpha) \subset C, \) we can find \( \alpha' \in \mathcal{Z}_0(X', X'_{\text{sing}}) \) supported on \( D \) such that \( \pi_*(\alpha') = \alpha. \) This implies that \( \pi_*(n\alpha' - (f)_D) = 0. \) Setting \( \beta = n\alpha' - (f)_D, \) it follows that \( \beta \) must be a 0-cycle on the projective scheme \( E \) such that \( \deg(\beta) = 0. \) Since \( E \) is a disjoint union of successive blow-ups of \( \mathbb{P}^{d-1}_k \) at closed points, it follows that there are finitely many smooth projective rational curves \( g_j \in \mathcal{Z}_j \) such that \( \beta = \sum_{j=1}^s (g_j)_L_j. \) Using the argument of \([3, \text{Lemma 5.2}]\), we can further choose \( L_j's \) so that \( D' := D \cup (\bigcup_j L_j) \) is a connected reduced curve with following properties.

1. Each component of \( D' \) is smooth along \( U'. \)
2. \( D' \cap U' \) has only ordinary double point singularities in the sense that exactly two components of \( D' \cap U' \) meet at any of its singular points with distinct tangent directions.

In particular, the embedding dimension of \( D' \) at each of its singular points lying over \( U \) is two. Furthermore, \( D' \cap X'_{\text{sing}} = (D' \setminus (\bigcup_j L_j)) \cap X'_{\text{sing}} = D \cap X'_{\text{sing}}. \) This implies that \( D' \) is a Cartier curve on \( X'. \)

We now fix a locally closed embedding \( X' \hookrightarrow \mathbb{P}^n_k. \) Since \( X' \) is reduced of dimension \( d \geq 3, \) and since \( D' \subset X' \) is a local complete intersection along \( X'_{\text{sing}}, \) it follows from \([30, \text{Lemma 1.2}]\) (see also the proof of \([31, \text{Lemma 1.4}]\)) that for all \( m \gg 1, \) there is an open dense subset \( \mathcal{U}_m(D', m) \) of the scheme \( \mathcal{H}_m(D', m) \) of hypersurfaces of \( \mathbb{P}^n_k \) of degree \( m \) containing \( D' \) such that the following hold.

a) For general distinct \( H_1, \cdots, H_{d-2} \in \mathcal{U}_m(D', m), \) the scheme-theoretic intersection \( L = H_1 \cap \cdots \cap H_{d-2} \) has the property that \( L \cap X' \) is reduced away from \( D'. \)

b) \( D' \subset (L \cap X') \) is a local complete intersection along \( X'_{\text{sing}}. \)

Setting \( W' := D' \cap U' \) and following \([1, \text{§ 5}]\), let \( W'(\Omega_{D}, e) \) denote the locally closed subset of points in \( W' \) where the embedding dimension of \( W' \) is \( e. \) It follows from the above description of \( D' \) that \( \max \{ \dim(W'(\Omega_{D}, e)) + e \} \leq 2. \) We conclude from \([1, \text{Theorem 7}]\) that for all \( m \gg 1, \) there is an open dense subset \( \mathcal{W}_m(W', m) \) of the scheme \( \mathcal{H}_m(W', m) \) of hypersurfaces of \( \mathbb{P}^n_k \) of degree \( m \) containing \( W' \) such that for general distinct \( H_1, \cdots, H_{d-2} \in \mathcal{W}_m(W', m), \) the scheme-theoretic intersection \( L = H_1 \cap \cdots \cap H_{d-2} \) has the following properties.

a') \( L \cap X' \) is a complete intersection in \( X' \) of dimension two.

b') \( L \cap U' \) is smooth.

Since \( D' \) is the closure of \( W' \) in \( X', \) we must have \( \mathcal{H}_m(W', m) = \mathcal{H}_m(D', m) \) and in particular, \( D' \subset L \cap X'. \) Combining (a) - (b) and (a') - (b') together, we conclude that there is a closed immersion \( Y' \subset X' \) with the following properties.

1. \( \dim(Y') = 2. \)
2. \( Y' \subset X' \) is a complete intersection.
3. \( Y' \) is reduced away from \( S. \)
4. \( D' \subset Y' \) is a local complete intersection along \( X'_{\text{sing}}. \)
5. \( Y' \cap U' \) is smooth.

If \( Y' \subset X' \) is as above, then (4) says that the local rings of \( Y' \) at \( S = D' \cap X'_{\text{sing}} \) contain regular elements. In particular, \( S \) contains no embedded prime of \( D'. \) We conclude from (3) and Lemma \([7, \text{Lemma 1}]\) that a surface \( Y' \) as above must be reduced. Let \( \iota : Y' \hookrightarrow X' \) denote the inclusion map.

Setting \( Z' = Y' \cap X'_{\text{sing}} = (\pi \circ \iota)^{-1}(X'_{\text{sing}}), \) it follows from the above constructions that \( Y'_{\text{sing}} \subset Z' \) and \( \dim(Z') \leq 1. \) Furthermore, \( \alpha' \) is an element of \( \mathcal{Z}^2(Y', Z') \) such that \( n\alpha' = (f)_D + \sum_j (g_j)_L_j, \) where \( D \) and \( L_j's \) are Cartier curves in \( Y'. \) In particular, \( n\alpha' = 0 \) in \( \text{CH}^2(Y'). \)
If we let $X_1 = \pi(Y')$, then it follows that $X_1 \subset X$ is a reduced closed subscheme and hence affine. Moreover, $Y'$ is the strict transform of $X_1$ under the blow-up map $\pi$ which is an isomorphism over $X_1 \setminus T$. In particular, $Y' \to X_1$ is the blow-up of $X_1$ along a closed subscheme supported on $T$. Since $T \cap X_{\text{sing}} = \emptyset$ and since $Y'$ is smooth away from $\pi^{-1}(X_{\text{sing}})$ by (5), it follows that the map $Y'_{\text{sing}} \to \pi(Y'_{\text{sing}})$ is an isomorphism. We conclude that $Y' \to X_1$ is the blow-up of an affine reduced surface $X_1$ along a closed subscheme with finite support such that $Y'_{\text{sing}}$ is affine.

It follows from Theorem 6.5 that $\alpha' = 0$ in $\text{CH}^2(Y')$. Hence $\alpha' = 0$ in $\text{CH}^2(Y',Z')$ by Lemma 2.1. Finally, it follows from Lemma 2.4 that $\alpha = \pi_* \circ \iota_* (\alpha') = 0$. This finishes the proof of the theorem. □

As the first application of Theorem 7.2, we can now answer Question 1.2 of Murthy as follows.

**Corollary 7.3.** Let $A$ be a reduced affine algebra of dimension $d \geq 2$ over an algebraically closed field $k$. Then the cycle class map $\text{cyc}_A : \text{CH}^d(A) \to F^dK_0(A)$ is an isomorphism. In particular, $F^dK_0(A)$ is torsion-free.

**Proof.** The map $\text{cyc}_A$ is surjective by definition. It follows from [30, Corollary 5.4] (see also [31, Corollary 2.7]) that there is a top Chern class map $c_{d,A} : F^dK_0(A) \to \text{CH}^d(A)$ such that $c_{d,A} \circ \text{cyc}_A$ is multiplication by $(d-1)!$. The corollary now follows from Theorem 7.2. □

## 8. Euler class group and 0-cycles

Let $k$ be an algebraically closed field of any characteristic. The goal of this section is to apply Theorem 7.2 to relate the Euler class groups of rings with their Chow groups and extend known results about the Euler class group of smooth affine algebras to arbitrary affine algebras over $k$.

### 8.1. Euler and weak Euler class groups.

Let $A$ be a reduced affine algebra over $k$ of dimension $d \geq 2$. We recall the following definition of the Euler class groups of $A$ from [5, § 2].

**Definition 8.1.** Let $F(A)$ be the free abelian group on the set of pairs $(J, \omega_J)$, where:

1. $J$ is an ideal of $A$ of height $d$.
2. $J$ is $m$-primary for some maximal ideal $m$ of $A$.
3. $\omega_J : (A/J)^d \to J/J^2$ is a surjective map of $A/J$-modules.

Let $[(J, \omega_J)]$ denote the class of $(J, \omega_J)$ in $F(A)$. Given an ideal $I$ of height $d$, let $I = \cap J_i$ be an irredundant primary decomposition of $I$, where each $J_i$ is $m_i$-primary for some maximal ideal $m_i$ of $A$. If $\omega_I : (A/I)^d \to I/I^2$ is a surjection, then the Chinese remainder theorem shows that $\omega_I$ uniquely defines a surjection $\omega_{J_i} : (A/J_i)^d \to J_i/J_i^2$. In particular, each $[(J_i, \omega_{J_i})] \in F(A)$. We write $[(I, \omega_I)] = \sum [(J_i, \omega_{J_i})] \in F(A)$.

Let $K(A)$ be the subgroup of $F(A)$ generated by the elements of the type $[(I, \omega_I)]$, where $I \subset A$ is an ideal of height $d$ with an $A$-linear surjection $\bar{\omega}_I : A^d \to I$ such that the diagram

\[
\begin{array}{ccc}
A^d & \xrightarrow{\bar{\omega}_I} & I \\
\downarrow & & \downarrow \\
(A/I)^d & \xrightarrow{\omega_I} & I/I^2
\end{array}
\]

commutes. The quotient $F(A)/K(A)$ is called the Euler class group of $A$ and is denoted by $E(A)$. 
Definition 8.2. Let $G(A)$ be the free abelian group on the set of ideals $J \subset A$ such that:

1. $J$ is of height $d$.
2. $J$ is $m$-primary for some maximal ideal $m$ of $A$.
3. There is a surjection of $A/J$-modules $\omega_I : (A/J)^d \twoheadrightarrow J/J^2$.

Let $[J]$ denote the class of $J$ in $G(A)$. Given an ideal $I$ of height $d$, let $I = \cap_i J_i$ be an irredundent primary decomposition of $I$, where each $J_i$ is $m_i$-primary for some maximal ideal $m_i$ of $A$. If $I/I^2$ is generated by $d$ elements, the Chinese remainder theorem shows that each $[J_i] \in G(A)$. We write $[I] = \sum_i [J_i] \in G(A)$.

Let $H(A)$ be the subgroup of $G(A)$ generated by the elements of the type $[I]$, where $I \subset A$ is an ideal of height $d$ with an $A$-linear surjection $\omega_I : A^d \rightarrow I$. The quotient $G(A)/H(A)$ is called the weak Euler class group of $A$ and is denoted by $E_0(A)$.

There is an evident surjection $E(A) \rightarrow E_0(A)$. But the following stronger result holds.

Lemma 8.3. Let $A$ be a reduced affine algebra of dimension $d \geq 2$ over an algebraically closed field $k$. Then the map $E(A) \rightarrow E_0(A)$ is an isomorphism.

Proof. This is a well known consequence of various results of [4], [5] and [6]. We only give a sketch. It follows from [5, Lemma 3.3] (see also the proof of [6, Corollary 7.9]) that the kernel of the map $E(A) \rightarrow E_0(A)$ is generated by the elements of the type $(I, \omega_I)$, where $I = (a_1, \cdots, a_d)$.

Hence, we need to show that if $I$ is an ideal of height $d$ generated by $d$ elements and if $\omega_I : (A/I)^d \rightarrow I/I^2$ is a given surjection, then $[(I, \omega_I)] = 0$ in $E(A)$. But this is proven in [10, Proposition 5.1]. This result is stated in loc. cit. for fields of characteristic zero, but the proof works without any change for any algebraically closed field. This is because the only two results needed in the proof are Suslin’s cancellation theorem [49] and [45, Lemma 2.4], both of which hold for reduced affine algebras when the base field is algebraically closed of any characteristic.

8.2. Euler class group and Chow group. To connect Euler class group with the Chow group of 0-cycles, we define a variant of weak Euler class group as follows. When $A$ is a smooth, this was defined by Bhatwadekar-Sridharan [5, Definition 2.2] and was shown to be isomorphic to $\text{CH}^d(A)$.

Let $A$ be a reduced affine algebra of dimension $d \geq 2$ over an algebraically closed field $k$. Let $G_s(A)$ denote the free abelian group on the set of smooth maximal ideals of $A$. Given an ideal $I$ of $A$ of height $d$ of the form $I = \cap_i m_i$, where each $m_i$ is a smooth maximal ideal with $m_i \neq m_j$ for $i \neq j$, we set $[J] = \sum_i [m_i] \in G_s(A)$. Let $H_s(A)$ denote the subgroup of $G_s(A)$ generated by elements $[J]$ as above such that $J$ is a complete intersection in $A$. We set $E_s(A) = G_s(A)/H_s(A)$.

There is an evident group homomorphism $\phi_A : E_s(A) \rightarrow E_0(A)$.

Lemma 8.4. The map $\phi_A : E_s(A) \rightarrow E_0(A)$ is an isomorphism.

Proof. We prove the surjectivity of $\phi_A$ using the Bertini theorems of Murthy [39, Theorem 2.3] and Swan [50, Theorem 1.3]. Let $I$ be an ideal of $A$ of height $d$ such that $I/I^2$ is generated by $d$ elements. It follows from this Bertini theorem that there exists an ideal $J \subset A$ (called a residual ideal of $I$) such that the following hold (see [39, Remarks 2.8, 3.2]).

1. $IJ$ is generated by $d$ elements.
2. $I + J = A$.
3. $J$ is a product of distinct smooth maximal ideal of $A$ of height $d$.

It follows from (1) and (2) that $[I] + [J] = [IJ] = 0$ in $E_0(A)$. In particular, $[I] = -[J]$ and (3) shows that $[J] \in E_s(A)$. This shows that $\phi_A$ is surjective.
To show that $\phi_A$ is injective, let $\alpha \in E_s(A)$ be such that $\phi_A(\alpha) = 0$. By repeating the above argument using the Bertini theorem again, we can write $\alpha = [J]$, where $J$ is a product of distinct smooth maximal ideals of height $d$ in $A$. It follows from [6] Theorem 4.2 and Lemma 8.3 that $J$ is generated by $d$ elements. On the other hand, $J$ is a product of distinct smooth maximal ideals of height $d$ in $A$, and hence it is already a local complete intersection in $A$. We conclude that $J$ is a complete intersection in $A$ and hence $[J] = 0$ in $E_s(A)$. □

As an application of Theorem 7.2 we can now prove the following result. When the ring $A$ is smooth, this was earlier proven by Bhatwadekar-Sridharan [4 Corollary 4.15].

**Theorem 8.5.** Let $A$ be a reduced affine algebra of dimension $d \geq 2$ over an algebraically closed field $k$. Then the assignment $m \mapsto [A/m] \in \text{CH}^d(A)$ induces a canonical isomorphism

$$\theta_A : E(A) \xrightarrow{\cong} \text{CH}^d(A).$$

**Proof.** In view of Lemmas 8.3 and 8.4 it suffices to show that the assignment $m \mapsto [A/m] \in \text{CH}^d(A)$ induces a canonical isomorphism $E_s(A) \xrightarrow{\cong} \text{CH}^d(A)$. To do this, we can assume that $X = \text{Spec}(A)$ is connected.

The assignment $m \mapsto [A/m] \in \text{CH}^d(A)$ clearly defines a group homomorphism $G_s(A) \to \text{CH}^d(A)$. It follows from [33] Lemma 2.2 that this assignment kills the classes of complete intersection ideals. In particular, it defines a group homomorphism $\theta_A : E_s(A) \to \text{CH}^d(A)$.

The map $\theta_A$ is surjective by the definition of $\text{CH}^d(A)$. To prove its injectivity, let $\alpha \in E_s(A)$ be such that $\phi_A(\alpha) = 0$. Using Murthy-Swan Bertini theorem as before, we can write $\alpha = [J]$, where $J$ is a product of distinct smooth maximal ideals of height $d$ in $A$. In particular, $J$ is a local complete intersection in $A$.

We have the maps $E_s(A) \xrightarrow{\theta_A} \text{CH}^d(A) \xrightarrow{\text{cyc}_A} F^dK_0(A)$, where the composite map takes $[J]$ to the class of $A/J$ in $F^dK_0(A)$ (see [39] Corollary 2.7). Then $\theta_A([J]) = 0$ implies that the class $[A/J]$ is zero in $F^dK_0(A)$. It follows from Corollary 7.3 that $F^dK_0(A)$ is torsion-free. We can thus apply [39] Corollary 3.4 to conclude that $J$ is a complete intersection in $A$. Equivalently, $[J] = 0$ in $E_s(A)$. This shows that $\theta_A$ is an isomorphism. □

**Proof of Corollary 1.6** It follows from Theorem 8.5 and 27 Theorems 6.4.1, 6.4.2 that in both cases, one has $E_0(A) = 0$. Suppose now that $I \subset A$ is a local complete intersection ideal of height $d$. Then $I/I^2$ is generated by $d$ elements as an $A/I$-module. In particular, $I$ defines a class in $E_0(A)$, which is zero. It follows from [6] Theorem 4.2 and Lemma 8.3 that $I$ is generated by $d$ elements and hence is a complete intersection. The last part of the corollary is obvious because a product of smooth maximal ideals is a local complete intersection. □

8.3. Applications to projective modules. Let $A$ be a reduced affine algebra of dimension $d \geq 2$ over an algebraically closed field $k$. We can assume that $X = \text{Spec}(A)$ is connected. Let $P$ be a projective $A$-module of rank $d$. One can then find a surjective $A$-linear map $\lambda : P \to J$, where $J \subset A$ is ideal such that $[J] \in E_0(A)$. It is known that the class $[J]$ depends only on $P$ (e.g., see [6] § 4). Hence, we get a unique class $e(P) \in E_0(A)$, called the Euler class of $P$.

On the other hand, the top Chern class of $P$ in $K_0(A)$ is defined as the class $c_d(P) = \sum_{i=0}^{d} (-1)^i[A^i(P)]$. It follows from [39] Remark 3.6 that $\text{cyc}_A \circ \theta_A(e(P)) = c_d(P) \in F^dK_0(A)$. As a consequence of Corollary 7.3, we can think of $e(P) = c_d(P)$ as the top Chern class of $P$ in $\text{CH}^d(A)$. This agrees with the classical Chern class when $A$ is smooth.

As a combination of Corollary 7.3 and 39 Theorem 3.7, we obtain the following.
**Theorem 8.6.** Let $A$ be a reduced affine algebra of dimension $d \geq 2$ over an algebraically closed field $k$. Let $P$ be a projective $A$-module of rank $d$. Then $P$ splits off a free summand of rank one (i.e., $P$ has a unimodular element) if and only if $c_d(P) = 0$ in $\text{CH}^d(A)$.

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