Calculation of, and bounds for, the multipole moments of stationary spacetimes

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Abstract
In this paper, the multipole moments of stationary asymptotically flat spacetimes are considered. We show how the tensorial recursion of Geroch and Hansen can be replaced by a scalar recursion on $R^2$. We also give a bound on the multipole moments. This gives a proof of the ‘necessary part’ of a long-standing conjecture due to Geroch.

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1. Introduction

The relativistic multipole moments of asymptotically flat spacetimes have been defined by Geroch [7] for static spacetimes and then generalized to the stationary case by Hansen [8]. Together with Beig’s [1] generalized definition of centre of mass, this gives a coordinate-independent description of all asymptotically flat stationary spacetimes.

However, the tensorial recursion which defines the multipole moments (1) is computationally rather complicated as it stands. In the axisymmetric (static or stationary) case, on the other hand, it was shown [3, 4] that the recursion can be replaced by a scalar recursion on $R$ and that all the moments can be collected into one complex-valued function $y$ on $R$, where the moments are given by the derivatives of $y$ at 0.

In the general case, the multipole of order $2^n$ has $2^{n+1}$ degrees of freedom, as compared to one degree of freedom in the axisymmetric case. Therefore, apart from the technical problems, it is not obvious what form a generalization to the general case should take. In this paper, we show that also in the general stationary case, the recursion (1) can be simplified to a scalar recursion, this time on $R^2$. This is shown using normal coordinates, complex null geodesics, and exploiting the extra conformal freedom of the conformal compactification.

Using this simplification we can partially confirm an extension of a long-standing conjecture by Geroch [7]:

Given any set of multipole moments, subject to the appropriate convergence condition, there exists a static solution of Einstein’s equations having precisely those moments.
This conjecture has its natural extension to the stationary case.

In this paper, we will state the appropriate convergence condition in the general stationary case, i.e. we will prove that this condition is necessary for the existence of a stationary solution to Einstein’s equations.

2. Multipole moments of stationary spacetimes

In this section, we quote the definition of multipole moments given by Hansen in [8], which is an extension to stationary spacetimes of the definition by Geroch [7]. We thus consider a stationary spacetime \((M, g_{ab})\) with timelike Killing vector field \(\xi^a\). We let \(\lambda = -\xi^a \xi_a\) be the norm and define the twist \(\omega\) through \(\nabla_a \omega = \epsilon_{abcd} \xi^b \nabla^c \xi^d\). If \(V\) is the 3-manifold of trajectories, the metric \(g_{ab}\) (with signature \((-\, +\, +\, +\)) induces the positive definite metric

\[
h_{ab} = \lambda g_{ab} + \xi_a \xi_b
\]

on \(V\). It is required that \(V\) is asymptotically flat, i.e. there exist a 3-manifold \(\hat{V}\) and a conformal factor \(\Omega\) satisfying

(i) \(\hat{V} = V \cup \Lambda\), where \(\Lambda\) is a single point;
(ii) \(\hat{h}_{ab} = \Omega^2 h_{ab}\) is a smooth metric on \(\hat{V}\);
(iii) at \(\Lambda\), \(\Omega = 0\), \(\hat{D}_a \Omega = 0\), \(\hat{D}_a \hat{D}_b \Omega = 2 \hat{h}_{ab}\),

where \(\hat{D}_a\) is the derivative operator associated with \(\hat{h}_{ab}\). On \(M\) and/or \(V\), one defines the scalar potential

\[
\phi = \phi_M + i \phi_J, \quad \phi_M = \frac{\lambda^2 + \omega^2 - 1}{4\lambda}, \quad \phi_J = \frac{\omega}{2\lambda}.
\]

The multipole moments of \(M\) are then defined on \(\hat{V}\) as certain derivatives of the scalar potential \(\hat{\phi} = \phi / \sqrt{\Omega}\) at \(\Lambda\). More explicitly, following [8], let \(\hat{R}_{ab}\) denote the Ricci tensor of \(\hat{V}\) and let \(P = \phi\). Define the sequence \(P, P_{a_1}, P_{a_1a_2}, \ldots\) of tensors recursively:

\[
P_{a_1 \ldots a_n} = C \left[ D_{a_1} P_{a_2 \ldots a_n} - \frac{(n-1)(2n-3)}{2} \hat{R}_{a_1a_2} P_{a_3 \ldots a_n} \right],
\]

where \(C[\cdot]\) stands for taking the totally symmetric and trace-free part. The multipole moments of \(M\) are then defined as the tensors \(P_{a_1 \ldots a_n}\) at \(\Lambda\). The requirement that all \(P_{a_1 \ldots a_n}\) be totally symmetric and trace-free makes the actual calculations very cumbersome.

In [2, 10], it was shown that (when the mass is non-zero) there exist a conformal factor \(\Omega\) and a chart, such that all components of the metric \(\hat{h}_{ab}\) and the potential \(\hat{\phi}\) are analytic in terms of the coordinates, in a neighbourhood of the infinity point. Expressed in these coordinates, the exponential map becomes analytic. Therefore, we can use Riemannian normal coordinates and still have analyticity of the metric components and the potential. If the mass is zero, this analyticity condition will be assumed. Thus, henceforth we assume that \(\Omega\) is chosen such that the (rescaled) metric and potential are analytic in a neighbourhood of \(\Lambda\).

Remark. Although the potential and the metric are analytic in terms of the coordinates on the conformally compactified manifold of timelike Killing trajectories, the relevant physical fields on a spacelike hypersurface in the spacetime are in general not. However, Dain [5] showed that they are analytic in terms of a radial coordinate and the corresponding angles. Due to the specific choice of conformal factor, also this proof requires that the mass is non-zero.
3. Multipole moments through a scalar recursion on $\mathbb{R}^2$

Suppose that $(x^1, x^2, x^3) = (x, y, z)$ are normal coordinates (with respect to $\hat{h}_{ab}$) centred around $\Lambda$. This means that for any constants $a = a^1$, $b = a^2$, $c = a^3$, the curve $t \to (at, bt, ct)$ is a geodesic, i.e., in terms of coordinates that

\[ \dot{x}^i + \Gamma^i_{jk} \dot{x}^j \dot{x}^k = \Gamma^i_{jk} a^j a^k = 0, \]

where the Christoffel symbols are evaluated at $(at, bt, ct)$ for appropriate $t$. Due to analyticity, this relation holds for complex values of $a, b, c$.

\[ \gamma_{\phi} : t \to (t \cos \varphi, t \sin \varphi, u), \quad t \in [0, t_0], \quad \varphi \in [0, 2\pi), \]

for some suitable $t_0$. The tangent vector $\eta^a = \eta^a(t) = \cos \varphi \left( \frac{\partial}{\partial t} \right)^a + \sin \varphi \left( \frac{i \partial}{\partial \varphi} \right)^a$ is seen to be a complex null vector along $\gamma_{\phi}$. Namely, from $\eta^a \hat{D}_a \eta^b = 0$, we infer that $\eta^a \hat{D}_a(\eta^b \eta^b) = 0$. The (constant) value of $\eta^a \eta^b$ is then found to be 0 by evaluation at $t = 0$.

Next, consider the mapping $F : \mathbb{R}^3 \rightarrow \hat{V}_C : (\xi, \zeta) \rightarrow (\xi, \zeta, i\sqrt{\xi^2 + \zeta^2})$. We let $S$ denote the 2-surface $F(U) \subset \hat{V}_C$, where $U \subset \mathbb{R}^2$ is a suitable neighbourhood of $(\xi, \zeta) = (0, 0)$. $S$ is then a smooth surface, except at $\Lambda$ where it has a vertex point, closely resembling a null cone in a three-dimensional Lorentzian space. The curves $\gamma_{\phi}$ are given by $\gamma_{\phi}(t) = F(t \cos \varphi, t \sin \varphi)$, and in particular $\eta^a$ lies along $S$. This suggests that we use the polar coordinates $\rho, \phi$ around $\Lambda$ on $S$ defined via $\xi = \rho \cos \phi, \zeta = \rho \sin \phi$. We now follow the approach from [4], where a useful vector field $\eta^a$ on $\hat{V}$ was introduced. In [4], where the spacetime was axisymmetric, $\eta^a$ was explicitly expressed in terms of the metric cast in the Weyl–Papapetrou form [11] and was defined on the whole of $\hat{V}$ except on the symmetry axis. In this paper, the axisymmetry condition is dropped, which makes the construction of a corresponding $\eta^a$ more difficult. In addition, a general spacetime has $2n + 1$ degrees of freedom for the multipole moment of order $2n$, compared to one degree of freedom in the axisymmetric case [9]. Nevertheless, it will turn out to be sufficient to know the potential $\hat{\phi}$ on $S$ to determine all the moments. On $S$, $\eta^a$ has the following properties.

**Lemma 1.** Suppose $\hat{V}$ and $S$ are defined as above. Then there exists a regularly direction-dependent (at $\Lambda$) vector field $\eta^a$ on $S$ with the following properties.

(a) $\eta^a \hat{D}_a \eta^b$ is parallel to $\eta^b$.

(b) For all tensors $T_{a_1 \ldots a_n}$, $\eta^{a_1} \cdots \eta^{a_n} T_{a_1 \ldots a_n} = \eta^{a_1} \cdots \eta^{a_n} C[T_{a_1 \ldots a_n}]$.

(c) At $\Lambda$, $P_{a_1 \ldots a_n}$ (in $\hat{V}$) is determined by $\eta^{a_1} \cdots \eta^{a_n} P_{a_1 \ldots a_n}$ (on $S$).

**Proof.** (a) was demonstrated above, (b) follows as in [4], while (c) requires a different argument. A totally symmetric and trace-free tensor $P_{a_1 \ldots a_n}$ has $2n + 1$ degrees of freedom, and in Cartesian coordinates $(x, y, z)$ it can be expressed via the components $\quad P_{\overset{i}{x} \ldots \overset{j}{y} \ldots \overset{n-j-1}{z}}$

and $P_{\overset{j}{y} \ldots \overset{j}{x} \ldots \overset{n-j-1}{z}}$. Therefore, at $\Lambda$, we can write

\[ P_{a_1 \ldots a_n} = \sum_{j=0}^{n} a_j C[(dx)_{a_1} \cdots (dx)_{a_j} (dy)_{a_{j+1}} \cdots (dy)_{a_n}] + \sum_{j=0}^{n-1} b_j C[(dx)_{a_1} \cdots (dx)_{a_j} (dy)_{a_{j+1}} \cdots (dy)_{a_{n-1}} (dz)_{a_n}]. \quad (2) \]

In brief, any index occurrence of several $z$s can be removed via $P_{yz} + P_{zy} + P_{yy} = 0$. 

Contracting with $\eta^a \cdots \eta^a$ and using lemma 1(b) we find that

$$\eta^a \cdots \eta^a P_{a_1, \ldots, a_n} = \sum_{j=0}^{n} a_j \cos^j \varphi \sin^{n-j} \varphi + i \sum_{j=0}^{n-1} b_j \cos^j \varphi \sin^{n-1-j} \varphi.$$  

(3)

If the left-hand side is zero, the trigonometric polynomial to the right must be identically zero. This means that all the coefficients $a_n$ and $b_n$ are zero, and by (2) that $P_{a_1, \ldots, a_n}$ is zero. In particular, the $2n + 1$ components on the RHS of (2) are linearly independent. This proves (c).

Note that although the moments are encoded in the coefficients $a_n$ and $b_n$, this encoding is dependent on the choice of normal coordinates, i.e., the orientation of the coordinate axes in $T\Lambda V$. We can now replace the recursion (1) on $\hat{V}$ with a scalar recursion on $S$. Again, we follow [4] and define

$$f_n = \eta^a \eta^b \cdots \eta^a P_{a_{n-1}, \ldots, a_n}, \quad n = 0, 1, 2, \ldots$$

(4)

on $S$. In particular, $f_0 = P = \hat{\phi} = \phi/\sqrt{\Omega}$. The moments $P_{a_{n-1}, \ldots, a_n}(\Lambda)$ will now be encoded in the trigonometric polynomial given by the direction-dependent limit $\lim_{\rho \to 0} f_n(\rho, \phi)$, which takes the form in (3). See also lemma 3. Note that although $P_{a_{n-1}, \ldots, a_n}$ is analytic on $\hat{V}$, $f_n$ will not be analytic in terms of $\xi$, $\eta$ since $\eta^a$ is direction dependent at $\Lambda \in S$. In general, we have the following lemmas.

**Lemma 2.** Suppose that $f$ is an analytic function, on a ball of radius $r_0$ around $\Lambda$ on $\hat{V}$. Then the restriction of $f$ to $S$, $f_L$, can be decomposed as $f_L(\xi, \zeta) = f_1(\xi, \zeta) + i f_2(\xi, \zeta)$, where $f_1$ and $f_2$ are analytic in terms of $\xi$ and $\eta$ on the disc $\xi^2 + \zeta^2 < \frac{r_0^2}{2}$, and where $\rho = \sqrt{\xi^2 + \zeta^2}$. Furthermore, if $f$ is real valued then $f_1$ and $f_2$ are real valued.

**Proof.** We start by splitting $f = f(x, y, z)$ into its even, $f_e$, and odd, $f_o$, parts with respect to $z$. We can now rewrite $f_e(x, y, z) = \hat{f}_e(x, y, z^2)$ and $f_o(x, y, z) = z \hat{f}_o(x, y, z^2)$, where both $\hat{f}_e$ and $\hat{f}_o$ are analytic in their arguments (at least near $(0, 0, 0)$). The restriction of $\hat{f}_e$ to $S$ gives $f_1(\xi, \zeta) = \hat{f}_e(\xi, \zeta, -(\xi^2 + \zeta^2))$, while the restriction of $\hat{f}_o$ gives $f_2(\xi, \zeta) = \hat{f}_o(\xi, \zeta, -(\xi^2 + \zeta^2))$. Adding these, and also noting that $z \to i \rho$, gives the required decomposition. On $S$ we have $|z|^2 = \xi^2 + \zeta^2$, hence $\xi^2 + \zeta^2 < \frac{r_0^2}{2}$ implies $x^2 + y^2 + z^2 < r_0^2$. This gives the domain of analyticity. The reality follows from the construction.

**Remark.** In $f_L$, the subscript $L$ stands for the ‘leading term’.

Although tensor fields on $\hat{V}$ can be pulled back to $S \backslash \{\Lambda\}$, we will only need their contractions with the appropriate number of $\eta^a$ vectors. This contraction will introduce a direction dependence which shows up in the following lemma.

**Lemma 3.** Suppose $T_{a_{n-1}, b}$ is an analytic tensor field on a ball of radius $r_0$ around $\Lambda$ on $\hat{V}$. Then the scalar field $f_L = \eta^a \cdots \eta^b T_{a_{n-1}, b}$ on $S$ can be written as $f_L(\xi, \zeta) = \frac{1}{\rho^n}(f_1(\xi, \zeta) + i f_2(\xi, \zeta))$, where $f_1$ and $f_2$ are analytic (on the disc with radius $\frac{r_0}{\sqrt{2}}$ around the origin) in terms of $\xi$ and $\zeta$, and where $\rho = \sqrt{\xi^2 + \zeta^2}$. Furthermore, if $T_{a_{n-1}, b}$ is real valued then $f_1$ and $f_2$ are real valued.

**Proof.** Consider the scalar field $g = x^a \cdots x^b T_{a_{n-1}, b}$ on $\hat{V}$, where

$$x^a = x^a \left( \frac{\partial}{\partial x} \right)^a + y \left( \frac{\partial}{\partial y} \right)^a + z \left( \frac{\partial}{\partial z} \right)^a.$$
We remark that the boundedness of $\rho^{a}$ on $S$. Thus, $g_{l} = \rho^{a}f_{l}$. But $f_{l}$ is bounded near the origin, thus we can divide $g_{l}$ by $\rho^{a}$ and get the lemma. □

We remark that the boundedness of $f_{l}(\xi, \zeta) = (f_{1}(\xi, \zeta) + i\rho f_{2}(\xi, \zeta))$ when $\rho \to 0$ implies that both $f_{1}$ and $f_{2}$ have zeros of sufficient order at $(\xi, \zeta) = (0, 0)$. It also implies that $f_{l}$ will be direction dependent there.

We can now contract (1) with $\eta^{a}$ and get the following theorem.

**Theorem 4.** Let $V$ and $S$ be defined as in sections 2 and 3. Let $\eta^{a}$ have the properties given by lemma 1 and let $f_{na}$ be defined by (4). Then the recursion (1) on $V$ takes the form

$$f_{n} = \eta^{a}D_{a}f_{n-1} - \frac{(n-1)(2n-3)}{2}\eta^{a}\eta^{b}\bar{R}_{ab}f_{n-2}$$

on $S$. The moments of order $2^a$ are captured in the direction-dependent limit $\lim_{\rho\to 0} f_{a}(\rho, \phi)$. □

**Proof.** That (1) takes the form (5) follows exactly as in [4] using that $\eta^{a}D_{a}\eta^{b} = 0$, although the recursion is defined only on $S$ rather than on $V$. The last statement is the content of lemma 1(c).

### 3.1. Simplified calculation of the moments

In this section, we will show that it is possible to obtain the recursion (5) without the term involving the Ricci tensor. This will be accomplished by using the conformal freedom at hand, i.e. change $\Omega$. The conformal freedom is $\Omega \to \tilde{\Omega} = \frac{\Omega}{\alpha}$, where $\alpha$ is analytic near $\Lambda$ with $\alpha(\Lambda) = 1^2$. $\tilde{D}_{a}(\frac{1}{\alpha})$ at $\Lambda$ gives a shift of the moments which corresponds to a ‘translation’ of the physical space, [7]. Hence, we can assume that $\tilde{D}_{a}(\frac{1}{\alpha}) = 0$ at $\Lambda$. It is to be noted that a change of $\Omega$ changes the (rescaled) potential $\phi/\sqrt{\tilde{\Omega}}$. It also changes the normal coordinates on $V$, and hence all conclusions must be made with some care. In order to derive the simplified recursion (14), we will specify an $\alpha$ through $\alpha_{l}$, the restriction of $\alpha$ to $S$. However, in order to deduce that there exists a real-valued function $\alpha$ which prescribed the values of $\alpha_{l}$, we need to say more on the representation of $\alpha_{l}$. This result and a useful estimate are the contents of lemma 5.

**Lemma 5.** Let $f_{l} = f_{1}(\xi, \zeta) + i\rho f_{2}(\xi, \zeta)$ where $f_{1}$ and $f_{2}$ are analytic on the ball $U = \{ |\xi|^2 + |\zeta|^2 < r^{2}_{0} \}$, and where $\rho = \sqrt{\xi^2 + \zeta^2}$. We can then write

$$f_{l}(\rho \cos \phi, \rho \sin \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} c_{l,m} e^{im\phi} \rho^l,$$

where

$$\sum_{l=0}^{\infty} \sum_{m=-l}^{l} |c_{l,m}| \rho^l < \infty, \quad \rho < r_{0}. \tag{7}$$

Furthermore, the converse is true; if $f_{l}$ is a function satisfying (6) and (7) then there are functions $f_{1}$ and $f_{2}$ analytic in $U$ such that $f_{l} = f_{1} + i\rho f_{2}$. The functions $f_{1}$ and $f_{2}$ are real valued if and only if the coefficients $c_{l,m}$ satisfy $c_{l,m} = (-1)^{l-m}c_{l,-m}$.

2 A more natural condition is the equivalent statement $\Omega \to \tilde{\Omega} = \Omega \alpha$. However, this formulation gives slightly neater calculations.
Although we will always take $\xi$ and $\zeta$ to be real, they are temporarily complexified in this proof.
We note that $a_0 = 1$ while $a_1$ can be chosen to be $0$ (translation). $\alpha_L$ will have the right regularity if we can show that each $a_n(\varphi)$ is a polynomial in $e^{i\varphi}$ and $e^{-i\varphi}$ of degree at most $n$. Equation (11) becomes

$$\sum_{n=0}^{\infty} n(n-1)a_n \rho^{n-2} = \sum_{n=0}^{\infty} \sum_{j=0}^{n} b_{n+2} a_j \rho^{n+j}.$$  

Equating powers of $\rho$ we get

$$-(n+2)(n+1)a_{n+2} = \sum_{m=0}^{n} b_{m+2} a_{n-m}, \quad n \geq 0. \quad (13)$$

The polynomials $a_0$ and $a_1$ have maximal degrees 0 and 1, respectively. The polynomials $b_j$ have maximal degree $n$. Straightforward induction shows that $a_n$ have maximal degree $n$. Induction and (13) also imply $a_n(e^{i\varphi}, e^{-i\varphi}) = (-1)^n a_n(-e^{i\varphi}, -e^{-i\varphi})$. Thus, due to lemma 5, there are real-valued analytic functions $\alpha_1$ and $\alpha_2$ such that $\alpha_L = \alpha_1 + i\alpha_2$. The function $\alpha = \alpha_1(x, y) + \alpha_2(x, y)$ is then a real-valued analytic extension of $\alpha_L$ to a ball in $\hat{V}$. This shows that there exists an $\alpha$ such that $\eta^a \eta^b \hat{R}_{ab} = 0$ on $S$.

However, it then also follows that $\tilde{\eta}^a \tilde{\eta}^b \tilde{R}_{ab} = 0$ on $\hat{S}$. From $\alpha$ we get $\hat{\Omega} = \Omega/\alpha$, and the corresponding new metric $\hat{h}_{ab} = \alpha^{-2} h_{ab}$. Note that in (5), where now $\eta^a \eta^b \tilde{R}_{ab} = 0$, the recursion is stated in terms of $\tilde{D}_a$, i.e., it is expressed in terms of $\tilde{h}_{ab}$ instead of $h_{ab}$. From $\hat{h}_{ab}$ we get new normal coordinates, i.e., $(x, y, z)$ in $T_\Lambda V_C$ are mapped into $\hat{V}_C$ using the exponential map belonging to $\hat{h}_{ab}$. We then construct $\tilde{\eta}^a$ and the mapping $F$ with respect to these coordinates. Now, null geodesics, of which $S$ consists, are conformally invariant, although they become non-affinely parametrized. This means that $\tilde{\eta}^a \propto \eta^a$ and that points in $S$ are mapped into points in $\hat{S}$. Thus, $\tilde{\eta}^a \tilde{\eta}^b \tilde{R}_{ab} \propto \eta^a \eta^b \hat{R}_{ab} = 0$ on $\hat{S}$ (or $S$).

Henceforth, we denote all entities defined via $\hat{h}_{ab}$ instead of $h_{ab}$ with a tilde. In particular, $\hat{D}_a$ will denote the derivative operator associated with $\hat{h}_{ab}$. Applying lemma 6 to theorem 4, we immediately get the following theorem.

**Theorem 7.** Let $\hat{V}$ and $\hat{S}$ be defined as in sections 2 and 3, where $\hat{S}$ is defined in terms of normal coordinates connected to $\hat{h}_{ab}$. Let $\tilde{\eta}^a$ have the properties given by lemma 1 with respect to $\hat{h}_{ab}$, and let $\tilde{f}_n$ be defined by (4) with $\tilde{\eta}^a$ replacing $\eta^a$. Then the recursion (1) on $\hat{V}$ takes the form

$$\tilde{f}_n = \tilde{\eta}^a \hat{D}_a \tilde{f}_{n-1} = (\tilde{\eta}^a \hat{D}_a)^n \tilde{f}_0 = \frac{\partial^n}{\partial \rho^n} \tilde{f}_0 \quad (14)$$

on $\hat{S}$. The moments of order $2^n$ are captured in the direction-dependent limit $\lim_{\rho \to 0} \tilde{f}_n(\rho, \varphi)$.

**Proof.** Since $\tilde{\eta}^a \hat{D}_a$ is $\frac{\partial}{\partial \rho}$, each $\tilde{f}_n$ is easily derived from $\tilde{f}_0$. Also, from lemma 1(c), we (again) know that the $2n + 1$ degrees of freedom of $P_{n_1, \ldots, n_k}$ at $\Lambda$ are encoded in $\tilde{f}_n$. \qed

**4. Bounds on the moments**

All multipole moments are encoded in $\tilde{f}_0$, and we note that the recursion (14) is identical to the recursion emanating from a scalar function in $\mathbb{R}^3$ (after inversion). It is clear that many different functions on $\mathbb{R}^3$ will produce the same moments, but if we also require that the function, $g$ say, is harmonic, $g$ is uniquely determined by the moments.

Thus, provided that we can connect a function which is harmonic in a neighbourhood of $0 \in \mathbb{R}^3$ to each $\tilde{f}_0$, we have the following theorem.
Theorem 8. Suppose that \((M, g_{ab})\) is a stationary asymptotically flat spacetime, admitting an analytic (rescaled) potential and an analytic chart\(^4\) on the conformally compactified manifold of timelike Killing trajectories, around the infinity point \(\Lambda\). Then there exists a (flat) harmonic function \(g\) in a neighbourhood of \(0 \in T_\Lambda \mathcal{V} \cong \mathbb{R}^3\), such that all multipole moments of \(M\) are given by

\[ P_{m_1, \ldots, m_n}(\Lambda) = (\nabla_{a_1} \cdots \nabla_{a_n} g)(0), \]

(15)

where \(\nabla_a\) on the LHS of (15) is the flat derivative operator in \(\mathbb{R}^3\). This puts a bound on the multipole moments, since the Taylor expansion \(\sum_{|\alpha| \geq 0} \frac{\partial^\alpha g}{\partial x^\alpha}(0)\) of \(g\) converges in a neighbourhood of the origin in \(\mathbb{R}^3\).

Proof. By lemma 5 we have

\[ \tilde{f}_L(\rho \cos \varphi, \rho \sin \varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} c_{l,m} e^{i m \varphi} \rho^l, \quad \rho < r_0, \]

for some \(r_0 > 0\), and we know that \(\tilde{f}_L\) fully determines the moments of \(M\). We will now define the function \(g\), which will be shown to be harmonic in a neighbourhood of the origin. Finally, we will argue that \(g\) has the correct derivatives at \(0\), i.e. that the equality (15) is valid (for all \(n \geq 0\)). First, we define the coefficients

\[ a_{l,m} = c_{l,m} i^{-m-l} 2^{-l-1} \pi \sqrt{(l+m)!(l-m)!} \Gamma(l+\frac{1}{2}) \sqrt{2l+1} \]

and the function

\[ g(r, \theta, \varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} a_{l,m} Y^m_l(\theta, \varphi) r^l. \]

Due to the construction, \(g\) is harmonic at those (interior) points for which the sum converges. We now establish convergence. From the identity

\[ \sum_{m=-l}^{l} |Y^m_l(\theta, \varphi)|^2 = \frac{2l+1}{4\pi}, \]

we get

\[ \left| \sum_{m=-l}^{l} a_{l,m} Y^m_l \right| \leq \left( \sum_{m=-l}^{l} |a_{l,m}|^2 \right)^{\frac{1}{2}} \left( \sum_{m=-l}^{l} |Y^m_l|^2 \right)^{\frac{1}{2}} = \left( \sum_{m=-l}^{l} \frac{2l+1}{4\pi} |a_{l,m}|^2 \right)^{\frac{1}{2}} = \sqrt{\pi} \left( \frac{2l+1}{2\Gamma(l+\frac{1}{2})} \right)^{\frac{1}{2}} \left( \sum_{m=-l}^{l} |c_{l,m} \sqrt{(l+m)!(l-m)!}|^2 \right)^{\frac{1}{2}}. \]

Furthermore, the inequality \((l+m)!(l-m)! \leq (2l)!\), when \(-l \leq m \leq l\) follows from the convexity of \(\ln(\Gamma(x))\). Therefore,

\[ \left| \sum_{m=-l}^{l} a_{l,m} Y^m_l \right| \leq \sqrt{\pi} \left( \frac{2l+1}{2\Gamma(l+\frac{1}{2})} \right)^{\frac{1}{2}} \left( \sum_{m=-l}^{l} |c_{l,m}|^2 \right)^{\frac{1}{2}}. \]

\(^4\) As discussed before, the analyticity has been proved for the case with non-zero mass.
Next, the inequality \( \frac{\pi(2l)!}{4^l(l+1)} = \frac{(2l+1)!}{(2l+1)!} = (2l + 1) \frac{(2l)!}{(2l+1)!} \leq 2l + 1 \) gives
\[
\left| \sum_{m=-l}^{l} a_{l,m} Y_{lm}^{m} \right| \leq \sqrt{2l + 1} \left( \sum_{m=-l}^{l} |c_{l,m}|^2 \right)^{1/2} \leq \sqrt{2l + 1} \sum_{m=-l}^{l} |c_{l,m}|.
\]
But for all \( \epsilon > 0 \) we have \( \sqrt{2l + 1}(1 + \epsilon)^{-l} \to 0 \) as \( l \to \infty \). So the factor \( \sqrt{2l + 1} \) will not affect the radius of convergence.

Hence,
\[
\sum_{l=0}^{\infty} \sum_{m=-l}^{l} a_{l,m} Y_{lm}^{m} r^l \text{ converges if } r < r_0.
\]

This shows that \( g \) is well defined in a neighbourhood of \( \vec{0} \) in \( T_{\vec{\nabla}} \), and we must now show that we have equality in (15). We will do this by forming \( g_L \) and then compare with \( \tilde{f}_L \). Note, however, that \( f_L \) is defined on \( \bar{S} \subset V_C \), while \( g_L \) will be defined on the corresponding surface \( \bar{S} = S_{ab}(\Lambda) \) in \( T_{\bar{\nabla}} \). By \( \bar{S} \) we denote the surface defined by \( F \) as previously, but where \( F \) now maps \((\xi, \zeta)\) into \( T_{\bar{\nabla}} V_C \). (By \( \bar{S} \) \( f_L \) and \( g_L \) really can be compared only at \( \Lambda \) (i.e. \( \vec{0} \in T_{\bar{\nabla}} \)), where also the equality (15) is evaluated. On the other hand, the radial derivatives of both entities are well defined and comparable at \( \Lambda \). In other words, if both \( \bar{f}_L \) and \( g_L \) are equal when expressed in terms of the coordinates \( \vec{\xi}, \vec{\zeta} \), they will produce the same derivatives/moments. To summarize, in the case of \( g \), the function \( F \) and the vector \( \nabla \cdot \vec{v} \) are simply interpreted in \( T_{\bar{\nabla}} V_C \) rather than in \( V_C \).

To proceed, we recall that
\[
Y_{lm}^{m}(\theta, \varphi) = i^{m+|m|} \sqrt{\frac{(2l+1)(l-|m|)!}{4\pi(l+|m|)!}} P_{l}^{m}(\cos \theta) e^{i\rho \sin \varphi}
\]
for \(-l \leq m \leq l\) and that
\[
P_{l}^{m}(\cos \theta) = (-1)^{m} 2^{-l} \sin^{l} \theta \sum_{k=0}^{\lfloor l/2 \rfloor} \frac{(-1)^{k} (2l-2k)!}{k!(l-k)!(l-2k-m)!} \cos^{l-2k} \theta,
\]
where \( 0 \leq \theta \leq \pi \) and \( m \geq 0 \). Therefore,
\[
g(r, \theta, \varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \sum_{k=0}^{\lfloor m/2 \rfloor} c_{l,m} \sqrt{\pi} (l-|m|)! (2l-2k)! e^{i\rho \sin \varphi} \rho^{|m|-l-2k} r^{2k} \frac{1}{i^{l-|m|} (-1)^k 4^k k!(l-k)!(l-|m|-2k)! \Gamma \left( l + \frac{1}{2} \right)},
\]
where \( z = r \cos \theta, \rho = r \sin \theta \). When we take the restriction of \( g \) to \( \bar{S} \), i.e. form \( g_L \), only the terms with \( k = 0 \) survive since \( r_L = 0 \). Thus,
\[
g_L(\rho \cos \theta, \rho \sin \theta) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} c_{l,m} \sqrt{\pi} (2l)! e^{i\rho \sin \varphi} \rho^l = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} c_{l,m} e^{i\varphi} \rho^l = \tilde{f}_L,
\]
which means that we have equality in (15).

\[\square\]

5. Discussion

In this paper, we have studied the multipole moments of stationary asymptotically flat spacetimes. By using normal coordinates, and by exploiting the conformal freedom, we could show that the tensorial recursion (1) could be replaced by the scalar recursion (14). This
recursion is a direction-dependent recursion on $\mathbb{R}^2$, where the moments are encoded in the direction-dependent limits at $A$.

Using this setup, we could also show that the multipole moments cannot grow too fast. In essence, the rescaled potential behaves (locally) in the manner of a harmonic function on $\mathbb{R}^3$. The bounds on the moments given in theorem 8 give the necessary part in a conjecture due to Geroch [7], and it is of course tempting to conjecture that this condition on the moments also will be sufficient (as long as the monopole is real valued).

Whether this can be proved using the techniques presented here is still an open question.

We also remark that similar questions concerning the convergence of asymptotic expansions in the static case are currently being studied by Friedrich, using a different technique [6].

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