ON HANKEL-TYPE OPERATORS WITH DISCONTINUOUS
SYMBOLS IN HIGHER DIMENSIONS

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Abstract. We obtain an asymptotic formula for the counting function of the discrete
spectrum for Hankel-type pseudo-differential operators with discontinuous symbols.

1. Introduction

Consider on $L^2(a, b), 0 \leq a < b \leq \infty$ the integral operator of the form

\[(\Gamma_{a,b}(k)u)(x) = \int_a^b k(x + y)u(y)dy,\]

with some function $k = k(t), t > 0$. The operator $\Gamma_{0,\infty}(k)$ is called Hankel operator on
$L^2(0, \infty)$, see [9], p. 46. For $\Gamma_{a,b}(k)$ with $0 \leq a < b < \infty$ we use the term truncated
Hankel operator. The symbol $\mathcal{K} = \mathcal{K}(\xi), \xi \in \mathbb{R}$, of the operator $\Gamma_{0,\infty}(k)$ is defined (non-
uniquely) as a function such that its Fourier transform $\hat{\mathcal{K}}(t)$ coincides with $k(t)$ for all
$t > 0$. We are interested in the case, when the symbol $\mathcal{K}(\xi)$ is a bounded function
with jump discontinuities, which ensures that the operator $\Gamma_{0,\infty}(k)$ is not compact. The
leading example of such an operator is given by the Carleman kernel $k(t) = t^{-1}, t > 0$
(see [9], p. 54), for which one can choose

$$
\mathcal{K}(\xi) = \begin{cases} 
-\pi i, \xi \leq 0, \\
\pi i, \xi > 0.
\end{cases}
$$

The operator $\Gamma_{a,b}(k)$ with this symbol is bounded for all $a$ and $b, 0 \leq a < b \leq \infty$, and
$\|\Gamma_{a,\infty}(k)\| = \pi, a \geq 0$. If $a = 0$ and/or $b = \infty$, then $\Gamma_{a,b}(k)$ is not compact. Among other
results, H.S. Wilf investigated the asymptotics of the counting function of the discrete
spectrum of the truncated operator $\Gamma_{1,b}(k), k(t) = t^{-1}, as b \to \infty$ (see [14], Corollary 1).
He proved that the number of eigenvalues of $\Gamma_{1,b}(k)$ in the interval $(\lambda, \infty)$ for any $\lambda \in
(0, \pi)$ is asymptotically equal to $C(\lambda) \log b$ as $b \to \infty$, with some explicit constant $C(\lambda)$.
H. Widom in [12], Theorem 4.3 derived a similar asymptotic formula for the truncated
Hilbert matrix (i.e. matrix with the entries $(j + k + 1)^{-1}, j, k = 0, 1, 2, \ldots$), as well as
for some more general Hankel matrices. Later, a good deal of attention became focused
on the asymptotics of the determinants of truncated Hankel (and Toeplitz) matrices, see e.g. [1], [2], [3] and [6].

The Hankel operator $\Gamma_{0,\infty}(k)$ can be rewritten in the form

$$(\Gamma_{0,\infty}(k)u)(x) = (\tilde{G}(k)u)(-x), \ x > 0,$$

where

$$(\tilde{G}(k)u)(x) = (1 - \chi(0,\infty)(x)) \int_{-\infty}^{\infty} k(y - x)\chi(0,\infty)(y)u(y)dy.$$

Here $\chi(0,\infty)$ denotes the characteristic function of the half-axis $(0, \infty)$. As A. Pushnitski and D. Yafaev [10] indicated to the author, in the scattering theory context it is natural to consider alongside $\Gamma_{0,\infty}(k)$ the symmetrised Hankel operator

$$(\tilde{G}(k) + \tilde{G}^*(k)).$$

In this note we study a multi-dimensional analog of the truncated symmetrised Hankel operator with a discontinuous symbol. It is defined in the following way. Let $\text{Op}_l^\alpha(a)$ and $\text{Op}_r^\alpha(a)$ be the standard “left” and “right” pseudo-differential operators with the smooth symbol $a = a(x, \xi), x, \xi \in \mathbb{R}^d, d \geq 1$, i.e.

$$(\text{Op}_l^\alpha a)u(x) = \left(\frac{\alpha}{2\pi}\right)^d \int \int e^{i\alpha(x-y)\xi} a(x, \xi)u(y)dyd\xi,$$

$$(\text{Op}_r^\alpha a)u(x) = \left(\frac{\alpha}{2\pi}\right)^d \int \int e^{i\alpha(x-y)\xi} a(y, \xi)u(y)dyd\xi,$$

for any function $u$ from the Schwartz class on $\mathbb{R}^d$. If the symbol $a$ depends only on $\xi$, then the above operators coincide with each other and we simply write $\text{Op}_\alpha(a)$. Here and below integrals without indication of the domain are assumed to be taken over the entire Euclidean space $\mathbb{R}^d$. The large constant $\alpha \geq 1$ can be thought of as a truncation parameter. The conditions imposed on the symbol $a$ in the main Theorem 1 below ensure that the above operators are trace class for all $\alpha \geq 1$.

In order to introduce the jump discontinuities, let $\Lambda, \Omega$ be two domains in $\mathbb{R}^d$, and let $\chi_\Lambda(x), \chi_\Omega(\xi)$ be their characteristic functions. We use the notation

$$P_{\Omega,\alpha} = \text{Op}_\alpha(\chi_\Omega).$$

Define the operator

$$T_\alpha(a) = T_\alpha(a; \Lambda, \Omega) = \chi_\Lambda P_{\Omega,\alpha} \text{Op}_l^\alpha(a)P_{\Omega,\alpha}\chi_\Lambda,$$

and its off-diagonal version

$$G_\alpha(a) = G_\alpha(a; \Lambda, \Omega) = (1 - \chi_\Lambda)P_{\Omega,\alpha} \text{Op}_l^\alpha(a)P_{\Omega,\alpha}\chi_\Lambda.$$

The central object for us is the following Hankel-type self-adjoint operator

$$H_\alpha(a) = H_\alpha(a; \Lambda, \Omega) = G_\alpha(a; \Lambda, \Omega) + G_\alpha^*(a; \Lambda, \Omega),$$
which is a natural multi-dimensional analogue of the truncated symmetrised operator (2). Note the following elementary property of $H_\alpha(a)$. Let $U$ be the unitary operator in $L^2(\mathbb{R}^d)$ defined by

$$U u = u \chi - u (1 - \chi), u \in L^2(\mathbb{R}^d),$$

so that $U^* = U$. Then $U^* H_\alpha(a) U = -H_\alpha(a)$. This implies, in particular, that the spectrum of $H_\alpha(a)$ is symmetric w.r.t. zero, i.e.

$$\dim \ker (H_\alpha(a) - \lambda) = \dim \ker (H_\alpha(a) + \lambda), \lambda \in \mathbb{R}. \tag{3}$$

Let $g$ be a function analytic in a disk of a sufficiently large radius, such that $g(0) = 0$. In 1982 H. Widom in [13] conjectured an asymptotic formula for the trace $\text{tr} g(T_\alpha)$, $\alpha \to \infty$, which was subsequently proved in [11]. In order to state this result define for any symbol $b = b(x, \xi)$, any domains $\Lambda, \Omega$ and any $C^1$-surfaces $S, P$, the coefficients

$$\mathfrak{M}_0(b) = \mathfrak{M}_0(b; \Lambda, \Omega) = \frac{1}{(2\pi)^d} \int_{\Lambda} \int_{\Omega} b(x, \xi) d\xi dx, \tag{4}$$

$$\mathfrak{M}_1(b) = \mathfrak{M}_1(b; S, P) = \frac{1}{(2\pi)^{d-1}} \int_{S} \int_{P} b(x, \xi) |n_S(x) \cdot n_P(\xi)| dS_\xi dS_x, \tag{5}$$

where $n_S(x)$ and $n_P(\xi)$ denote the exterior unit normals to $S$ and $P$ at the points $x$ and $\xi$ respectively. Define also

$$\mathfrak{A}(g; b) = \frac{1}{(2\pi)^2} \int_0^1 \frac{g(bt) - tg(b)}{t(1-t)} dt. \tag{6}$$

Then, as conjectured in [13] and proved in [11],

$$\text{tr} g(T_\alpha(a)) = \alpha^d \mathfrak{M}_0(g(a); \Lambda, \Omega) + \alpha^{d-1} \log \alpha \mathfrak{M}_1(\mathfrak{A}(g; a); \partial \Lambda, \partial \Omega) + o(\alpha^{d-1} \log \alpha), \tag{7}$$

The aim of this note is to establish a similar trace formula for the operator $H_\alpha(a)$, see Theorem 1.

We use the standard notation $\mathcal{S}_1$ for the trace class operators, and $\mathcal{S}_2$ for the Hilbert-Schmidt operators, see e.g. [4]. The underlying Hilbert space is assumed to be $L^2(\mathbb{R}^d)$. By $C, c$ (with or without indices) we denote various positive constants independent of $\alpha$, whose precise value is of no importance.

**Acknowledgment.** The author is grateful to G. Rozenblum for discussions, and to A. Böttcher and A. Pushnitski for stimulating remarks. This work was supported in part by EPSRC grant EP/F029721/1.

## 2. Main result

### 2.1. Definitions and main results.**

To state the result define for any function $g = g(t), t \in \mathbb{R}$, such that $|g(t)| \leq Ct$, the integral

$$\mathfrak{U}(g; b) = \frac{2}{\pi^2} \int_0^1 \frac{g(bt)}{t\sqrt{1-t^2}} dt. \tag{8}$$
Denote by 
\[ g_{ev}(t) = \frac{g(t) + g(-t)}{2} \]
the even part of \( g \). Let \( \mathfrak{W}_1 \) be as defined in (5). The next theorem contains the main result of the paper:

**Theorem 1.** Let \( \Lambda, \Omega \subset \mathbb{R}^d, d \geq 2 \) be bounded domains in \( \mathbb{R}^d \) such that \( \Lambda \) is \( C^1 \) and \( \Omega \) is \( C^3 \). Let \( a = a(x, \xi) \) be a symbol satisfying the condition
\[
\max_{0 \leq n \leq d+2} \sup_{0 \leq m \leq d+2} |\nabla_x^n \nabla_\xi^m a(x, \xi)| < \infty,
\]
and having a compact support in both variables. Let \( g = g(t), t \in \mathbb{R} \) be a function such that \( g_{ev}(t) t^{-2} \) is continuous on \( \mathbb{R} \). Then
\[
\text{tr} g(H_\alpha(a)) = \alpha^{d-1} \log \alpha \ \mathfrak{W}_1(\mu(\chi_I; |a|); \partial \Lambda, \partial \Omega) + o(\alpha^{d-1} \log \alpha),
\]
as \( \alpha \to \infty \).

Note that the coefficient on the right-hand side of (10) is well-defined for any function \( g \) satisfying the conditions of Theorem 1. Note also that in view of (3), we have
\[
\text{tr} g(H_\alpha) = \text{tr} g_{ev}(H_\alpha).
\]

**Remark 2.** Denote by \( n_\pm(\lambda; \alpha) \) with \( \lambda > 0 \) the number of eigenvalues of the operator \( \pm H_\alpha(a) \) which are greater than \( \lambda \). In other words,
\[
n_\pm(\lambda; \alpha) = \text{tr} \chi_I(\pm H_\alpha(a)), \quad I = (\lambda, \|H_\alpha(a)\| + 1).
\]
Since the interval \( I \) does not contain the point 0, this quantity is finite. Due to (11), \( n_+(\lambda; a) = n_-(\lambda; a) \). In order to find the leading term of the asymptotics of \( n_\pm(\lambda; \alpha) \), \( \alpha \to \infty \), approximate the characteristic function \( \chi_I \) from above and from below by smooth functions \( g \). Then it follows from Theorem 1 that
\[
n_\pm(\lambda; \alpha) = \frac{1}{2} \alpha^{d-1} \log \alpha \ \mathfrak{W}_1(\mu(\chi_I; |a|)) + o(\alpha^{d-1} \log \alpha).
\]
A straightforward calculation shows that
\[
\mu(\chi_I; |a(x, \xi)|) = \begin{cases} 
\frac{2}{\pi^2} \cosh^{-1} \frac{|a(x, \xi)|}{2\lambda}, & |a(x, \xi)| > 2\lambda; \\
0, & |a(x, \xi)| \leq 2\lambda.
\end{cases}
\]
The formula (12) can be viewed as a multi-dimensional analogue of the asymptotics derived for the Carleman kernel in [14], Corollary 1.

Theorem 1 will be derived from the following theorem, which is simply formula (10) for even polynomial functions \( g \):
Theorem 3. Let the domains $\Lambda, \Omega \subset \mathbb{R}^d$, $d \geq 2$ and the symbol $a = a(x, \xi)$ be as in Theorem 1. Then for $g_p(t) = t^p$, $p = 1, 2, \ldots$,

\begin{equation}
\text{tr} \, g_{2p}(H_\alpha(a)) = \alpha^{d-1} \log \alpha \, \mathcal{W}_1(\mathcal{U}(g_{2p}; |a|); \partial \Lambda, \partial \Omega) + o(\alpha^{d-1} \log \alpha),
\end{equation}

as $\alpha \to \infty$.

Once this theorem is proved, the asymptotics is extended to functions $g$ satisfying the conditions of Theorem 1 with the help of an elementary trace estimate for $g(H_\alpha)$.

3. Auxiliary estimates

The proof relies on various trace class estimates, some of which were obtained in [11]. In these estimates we always assume that the symbols $a$ and $b$ satisfy the condition (9) and that the domains $\Lambda, \Omega$ are as in Theorem 1.

We begin with well known estimates for operators with smooth symbols:

Proposition 4. Let $d \geq 1$ and $\alpha \geq c$. Suppose that the symbols $a, b$ satisfy (9). Then

$$\| \text{Op}_\alpha^l(a) \| + \| \text{Op}_\alpha^r(a) \| \leq C.$$

If, in addition, $a$ and $b$ are compactly supported in both variables, then

$$\| \text{Op}_\alpha^l(a) - \text{Op}_\alpha^r(a) \|_{\mathcal{E}_1} \leq C\alpha^{d-1},$$

$$\| \text{Op}_\alpha^l(a) \text{Op}_\alpha^l(b) - \text{Op}_\alpha^l(ab) \|_{\mathcal{E}_1} \leq C\alpha^{d-1}.$$

The above boundedness is a classical fact, and it can be found, e.g. in [5], Theorem $B'_{11}$, where it was established under smoothness assumptions weaker than (9). For the trace class estimates see e.g. [11], Lemma 3.12, Corollary 3.13.

The following estimates are for operators with discontinuous symbols.

Proposition 5. Suppose that the symbol $a$ satisfies (9) and has a compact support in both variables. Assume that $\alpha \geq c$. Let $\text{Op}_\alpha(a)$ denote any of the operators $\text{Op}_\alpha^l(a)$ or $\text{Op}_\alpha^r(a)$. Then

$$\| [\text{Op}_\alpha(a), P_{\Lambda, \Omega}] \|_{\mathcal{E}_1} + \| [\text{Op}_\alpha(a), \chi_L] \|_{\mathcal{E}_1} \leq C\alpha^{d-1}.$$

See [11], Lemmas 4.3 and 4.5.

The next proposition is a crucial ingredient in our proof: Theorem 3 is derived from the following local version of the asymptotics (7).

Proposition 6. Suppose that the symbols $a$ and $b$ satisfy (9), and that $b$ has a compact support in both variables. Then for $g_p(t) = t^p$, $p = 1, 2, \ldots$,

$$(14) \quad \text{tr} \, \big( \text{Op}_\alpha^l(b)g_p(T_\alpha(a)) \big) = \alpha^{d-1} \mathcal{W}_0(bg_p(a); \Lambda, \Omega) + \alpha^{d-1} \log \alpha \, \mathcal{W}_1(bA(g_p; a); \partial \Lambda, \partial \Omega) + o(\alpha^{d-1} \log \alpha),$$

as $\alpha \to \infty$. 


Proposition 7. Suppose that the symbol \( a \) satisfies (9) and has a compact support in both variables. Assume that \( \alpha \geq 2 \). Then
\[
\| G_\alpha(a) \|_{S^2}^2 \leq C \alpha^{d-1} \log \alpha.
\]

See [11], Lemma 6.1, and also papers [7, 8] by D. Gioev.

4. Proof of Theorem 1

4.1. Polynomial functions, reduction to the operator \( T_\alpha \). We begin with studying the modulus of the operator \( G_\alpha(1) \), i.e. the operator \( | G_\alpha(1) | = \sqrt{G_\alpha^*(1) G_\alpha(1)} \).

Lemma 8. Suppose that the symbol \( b \) satisfies (9) and has a compact support in both variables. Then
\[
\text{tr } \text{Op}(b) g_p(G_\alpha^*(1) G_\alpha(1)) = \frac{1}{2} \alpha^{d-1} \log \alpha \ \mathcal{M}_1 \left( b \mathcal{M}(g_{2p}; 1); \partial \Lambda, \partial \Omega \right) + o(\alpha^{d-1} \log \alpha),
\]
as \( \alpha \to \infty \).

Proof. Denote \( G = G_\alpha(1) \) and \( T = T_\alpha(1) \). By inspection, \( G^* G = T - T^2 \), so \( g_p(G^* G) = h(T) \), with the polynomial \( h(t) = g_p(t - t^2) \). Thus one can use Proposition 6. Note that \( h(1) = 0 \), so that the first asymptotic coefficient in (14) vanishes, see (4). Let us find the second asymptotic coefficient, calculating \( \mathcal{A}(h; 1) \). Using (6) and (8) we get
\[
\mathcal{A}(h; 1) = \frac{2}{(2\pi)^2} \int_0^1 \frac{g_p(t - t^2)}{t(t - t)} dt = \frac{1}{\pi^2} \int_0^1 \frac{g_p(t^2)}{s \sqrt{1 - s^2}} ds = \frac{1}{2} \mathcal{M}(g_{2p}; 1).
\]
Thus, by Proposition 6,
\[
\text{tr } \left( \text{Op}_\alpha^l(b) h(T) \right) = \frac{1}{2} \alpha^{d-1} \log \alpha \ \mathcal{M}_1 \left( b \mathcal{M}(g_{2p}; 1); \partial \Lambda, \partial \Omega \right) + o(\alpha^{d-1} \log \alpha),
\]
which leads to the proclaimed formula. \( \square \)

Now we can prove the asymptotics for the operator \( H_\alpha(a) \):

Proof of Theorem 3. For any pair of operators \( B_1, B_2 \) we write \( B_1 \sim B_2 \) if
\[
\| B_1 - B_2 \|_{\mathfrak{S}_1} \leq C \alpha^{d-1}, \ \alpha \geq 1.
\]
Denote \( H = H_\alpha(a), G = G_\alpha(a) \), and note that \( g_{2p}(H) = g_p(H^2) \). Since \( G^2 = 0, (G^*)^2 = 0 \), it follows that
\[
H^2 = GG^* + G^* G.
\]
This sum is in fact an orthogonal sum of two operators acting on the mutually orthogonal subspaces \( L^2(\mathcal{C} \Lambda) \) and \( L^2(\Lambda) \), where \( \mathcal{C} \Lambda \) denotes the complement of \( \Lambda \). Moreover, the non-zero spectra of \( G^* G \) and \( GG^* \) are the same. Thus
\[
\text{tr } g_{2p}(H) = \text{tr } g_p(H^2) = 2 \text{tr } g_p(G^* G).
\]
Using Propositions 5 and 4, we conclude that

\[ G \sim \text{Op}_\alpha^l(a) G_\alpha(1) \sim G_\alpha(1) \text{Op}_\alpha^l(a), \]

\[ G^* \sim \text{Op}_\alpha^l(\overline{a}) G_\alpha^*(1) \sim G_\alpha^*(1) \text{Op}_\alpha^l(\overline{a}), \]

\[ G^* G \sim \text{Op}_\alpha^l(|a|^2) G_\alpha^*(1) G_\alpha(1). \]

Thus referring again to Proposition 4, we obtain that

\[ g_p(G^* G) \sim \text{Op}_\alpha^l(|a|^{2p}) g_p(G_\alpha^*(1) G_\alpha(1)). \]

Now it follows from Lemma 8 and formula (15) that

\[ \text{tr} g_{2p}(H) = 2 \text{tr} \left( \text{Op}_\alpha^l(|a|^{2p}) g_p(G_\alpha^*(1) G_\alpha(1)) \right) + O(a^{d-1}) \]

\[ = \alpha^{d-1} \log \alpha \mathcal{W}_1 \left( |a|^{2p} \mathcal{U}(g_{2p}; 1); \partial \Lambda, \partial \Omega \right) + o(\alpha^{d-1} \log \alpha). \]

Since \(|a|^{2p} \mathcal{U}(g_{2p}; 1) = \mathcal{U}(g_{2p}; |a|)\), this implies the asymptotics (13). The proof of Theorem 3 is now complete. \(\square\)

4.2. Proof of Theorem 1. By virtue of (11) we may assume that \(g = g_{ev}\). Since \(h(t) = t^{-2} g_{ev}(t)\) is continuous, for any \(\epsilon > 0\) there exists an even polynomial \(p = p(t)\) and a continuous function \(r = r(t)\) such that

\[ g_{ev}(t) = p(t) + t^2 r(t), \quad \max_{t \in [-\|H_\alpha\|, \|H_\alpha\|]} |r(t)| \leq \epsilon. \]

In view of (13),

\[ \text{tr} p(H_\alpha(a)) = \alpha^{d-1} \log \alpha \mathcal{W}_1 \left( \mathcal{U}(p; a) \right) + o(\alpha^{d-1} \log \alpha). \]

Moreover, by the definition (8),

\[ |\mathcal{W}_1(\mathcal{U}(p; a)) - \mathcal{W}_1(\mathcal{U}(g_{ev}; a))| \leq C \epsilon. \]

Let us estimate the error term:

\[ \| g_{ev}(H_\alpha(a)) - p(H_\alpha(a)) \|_{\mathcal{S}_1} \leq \| r(H_\alpha(a)) \| \| H_\alpha^2(a) \|_{\mathcal{S}_1} \]

\[ \leq 2\epsilon \| G_\alpha(a) \|_{\mathcal{S}_2}^2 \leq C \epsilon \alpha^{d-1} \log \alpha, \]

where we have used Proposition 7. Collecting the above estimates together, we obtain

\[ \limsup_{\alpha \to \infty} \frac{1}{\alpha^{d-1} \log \alpha} \left| \text{tr} g(H_\alpha(a)) - \alpha^{d-1} \log \alpha \mathcal{W}_1(\mathcal{U}(g_{ev}; a)) \right| \leq C \epsilon. \]

Since \(\epsilon > 0\) is arbitrary, the asymptotics (10) follows.
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