Skew Cyclic codes over $\mathbb{F}_q + u\mathbb{F}_q + v\mathbb{F}_q + uv\mathbb{F}_q$ *

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Abstract: In this paper, we study skew cyclic codes over the ring $R = \mathbb{F}_q + u\mathbb{F}_q + v\mathbb{F}_q + uv\mathbb{F}_q$, where $u^2 = u, v^2 = v, uv = vu, q = p^m$ and $p$ is an odd prime. We investigate the structural properties of skew cyclic codes over $R$ through a decomposition theorem. Furthermore, we give a formula for the number of skew cyclic codes of length $n$ over $R$.

Key words: linear codes; skew cyclic codes; Gray map; generator polynomial

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1 Introduction

Cyclic codes form an important subclass of linear block codes, studied from the fifties onward. Their clear algebraic structures as ideals of a quotient ring of a polynomial ring makes for an easy encoding. A landmark paper [11] has shown that some important binary nonlinear codes with excellent error-correcting capabilities can be identified as images of linear codes over $\mathbb{Z}_4$ under the Gray map.

Recently, in [3], D. Boucher et al. gave skew cyclic codes defined by using the skew polynomial ring with an automorphism $\theta$ over the finite field with $q$ elements. The definition generalizes the concept of cyclic codes over non-commutative polynomial rings. Soon afterwards, D. Boucher et al. studied skew constacyclic codes in [5]. Later, in [4], some important results on the duals of the skew cyclic codes over $\mathbb{F}_q[x; \theta]$ are given. In [12], I. Siap et al. presented the structure of skew cyclic codes of arbitrary length. Further, S. Jitman et al. in [10] defined skew constacyclic codes over the skew polynomial ring with coefficients from finite rings. In [1], T. Abualrub and P. Seneviratne studied skew cyclic codes over ring $\mathbb{F}_2 + v\mathbb{F}_2$ with $v^2 = v$. Moreover, J. Gao [6] and F. Gursoy et al. [8] presented skew cyclic

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codes over $\mathbb{F}_p + v\mathbb{F}_p$ and $\mathbb{F}_q + v\mathbb{F}_q$ with different automorphisms, respectively. In [7], J. Gao et al. also studied skew generalized quasi-cyclic codes over finite fields.

In this article, we mainly study skew cyclic codes over ring $R = \mathbb{F}_q + u\mathbb{F}_q + v\mathbb{F}_q + uv\mathbb{F}_q$, where $u^2 = u, v^2 = v, uv = vu$ and $q = p^m$.

In our work, the automorphism $\theta$ on the ring $R$ is defined to be

$$\theta(b_0 + b_1 u + b_2 v + b_3 uv) = b_0^p + b_2^p u + b_1^p v + b_3^p uv,$$

for all $b_0 + b_1 u + b_2 v + b_3 uv \in R$, where $b_i \in \mathbb{F}_q$, and $i = 0, 1, 2, 3$. In fact, for any $a_1 \eta_1 + a_2 \eta_2 + a_3 \eta_3 + a_4 \eta_4 \in R$, we have

$$\theta(a_1 \eta_1 + a_2 \eta_2 + a_3 \eta_3 + a_4 \eta_4) = \theta(a_1)\eta_1 + \theta(a_2)\eta_2 + \theta(a_3)\eta_3 + \theta(a_4)\eta_4.$$

Note that if $m$ is even, the order of the ring automorphism $|\langle \theta \rangle|$ is $m$, otherwise, $2m$.

The material is organized as follows. In Section 2, we show the basics of codes over ring $R$ that we need for further reference. Section 3 derives the structure of linear codes over $\mathbb{F}_q$ and gives the structural properties of skew cyclic codes over $R$ through a decomposition theorem. Section 5, we give an example to illustrate the discussed results.

2 Preliminary

Let $\mathbb{F}_q$ be a finite field with $q$ elements, where $q = p^m$, $p$ is an odd prime. Throughout, we let $R$ denote the commutative ring $\mathbb{F}_q + u\mathbb{F}_q + v\mathbb{F}_q + uv\mathbb{F}_q$, where $u^2 = u, v^2 = v$, and $uv = vu$. Let $\eta_1 = 1 - u - v + uv, \eta_2 = uv, \eta_3 = u - uv, \eta_4 = v - uv$. It is easy to verify that $\eta_i^2 = \eta_i, \eta_i \eta_j = 0$, and $\sum_{k=1}^{4} \eta_k = 1$, where $i, j = 1, 2, 3, 4$ and $i \neq j$. According to [2], we have $R = \eta_1 R \oplus \eta_2 R \oplus \eta_3 R \oplus \eta_4 R$. By calculating, we can easily obtain that $\eta_i R \cong \mathbb{F}_q$, $i = 1, 2, 3, 4$. Therefore, for any $r \in R$, $r$ can be expressed uniquely as $r = \sum_{i=1}^{4} \eta_i a_i$, where $a_i \in \mathbb{F}_q$ for $i = 1, 2, 3, 4$.

We recall the definition of the Gray map over $R$ in [13]

$$\Phi : R = \mathbb{F}_q + u\mathbb{F}_q + v\mathbb{F}_q + uv\mathbb{F}_q \rightarrow \mathbb{F}_q^4$$

$$\eta_1 a + \eta_2 b + \eta_3 c + \eta_4 d \rightarrow (a, a + b, a + c, a + b + c + d).$$

Equivalently, if $r = a' + b'u + c'v + d'uv \in R$, then

$$\Phi(r) = (a', 2a' + b', c' + d', 2a' + b', 4a' + 2b' + 2c' + d').$$

This map can be naturally extended to the case over $R^m$.  

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For any element \( r = a + bu + cv + duv \in R \), we define the Lee weight of \( r \) as \( w_L(r) = w_H(a, a + b, a + c, a + b + c + d) \), where \( w_H \) denotes the ordinary Hamming weight for \( q \)-ary codes. The Lee distance of \( r \in R \) can be similarly defined.

From the definition of the Gray map \( \Phi \), we can easily check that \( \Phi \) is \( \mathbb{F}_q \)-linear and it is also a distance-reserving isometry from \((R^n, d_L)\) to \((F_q^{4n}, d_H)\), where \( d_L \) and \( d_H \) denote the Lee and Hamming distance in \( R^n \) and \( F_q^{4n} \), respectively.

### 3 Linear codes over \( R \)

In this section, we mainly show some familiar structural properties of \( R \). The proofs of the following theorems can be found in [13], so we omit them here.

If \( A_i \ (i = 1, 2, 3, 4) \) are codes over \( R \), we denote their direct sum by

\[
A_1 \oplus A_2 \oplus A_3 \oplus A_4 = \{ a_1 + a_2 + a_3 + a_4 | a_i \in A_i, i = 1, 2, 3, 4 \}.
\]

**Definition 3.1** Let \( C \) be a linear code of length \( n \) over \( R \), we define that

\[
C_1 = \{ a \in \mathbb{F}_q^n | \exists b, c, d \in \mathbb{F}_q^n | \eta_1 a + \eta_2 b + \eta_3 c + \eta_4 d \in C \},
\]

\[
C_2 = \{ b \in \mathbb{F}_q^n | \exists a, c, d \in \mathbb{F}_q^n | \eta_1 a + \eta_2 b + \eta_3 c + \eta_4 d \in C \},
\]

\[
C_3 = \{ c \in \mathbb{F}_q^n | \exists a, b, d \in \mathbb{F}_q^n | \eta_1 a + \eta_2 b + \eta_3 c + \eta_4 d \in C \},
\]

\[
C_4 = \{ d \in \mathbb{F}_q^n | \exists a, b, c \in \mathbb{F}_q^n | \eta_1 a + \eta_2 b + \eta_3 c + \eta_4 d \in C \}.
\]

It is clear that \( C_i \ (i = 1, 2, 3, 4) \) are linear codes over \( \mathbb{F}_q^n \). Furthermore, \( C = \eta_1 C_1 \oplus \eta_2 C_2 \oplus \eta_3 C_3 \oplus \eta_4 C_4 \), and \( |C| = |C_1| \cdot |C_2| \cdot |C_3| \cdot |C_4| \). Throughout the paper \( C_i \ (i = 1, 2, 3, 4) \) will be reserved symbols referring to these special subcodes.

According to Definition 3.1 and [13], we have the following theorem.

**Theorem 3.1** Let \( C = \eta_1 C_1 \oplus \eta_2 C_2 \oplus \eta_3 C_3 \oplus \eta_4 C_4 \) be a linear code of length \( n \) over \( R \). Then \( C^\perp = \eta_1 C_1^\perp \oplus \eta_2 C_2^\perp \oplus \eta_3 C_3^\perp \oplus \eta_4 C_4^\perp \).

According to the definition of the Gray map \( \Phi \), we can easily obtain the following theorem.

**Theorem 3.2** Let \( C \) be a linear code of length \( n \) over \( R \), \( |C| = q^k \) and \( d_L(C) = d \). Then \( \Phi(C) \) is a \( q \)-ary linear code with parameter \([4n, k, d]\).

Let \( C = \eta_1 C_1 \oplus \eta_2 C_2 \oplus \eta_3 C_3 \oplus \eta_4 C_4 \) be a linear code of length \( n \) over \( R \). Since \( C \) is a \( F_q \)-module, then we have the following lemma.
Lemma 3.1 If $G_i$ are generator matrices of $q$-ary linear codes $C_i$ ($i = 1, 2, 3, 4$), respectively, then the generator matrix of $C$ is

$$G = \begin{pmatrix} \eta_1 G_1 \\ \eta_2 G_2 \\ \eta_3 G_3 \\ \eta_4 G_4 \end{pmatrix}.$$ 

Moreover, if $G_1 = G_2 = G_3$, then $G = G_1$.

In light of the definition of Gray map $\Phi$, we can easily obtain the following proposition.

Proposition 3.1 If $C$ is a linear code of length $n$ over $R$ with generator matrix $G$, then we have

$$\Phi(G) = \begin{pmatrix} \Phi(\eta_1 G_1) \\ \Phi(\eta_2 G_2) \\ \Phi(\eta_3 G_3) \\ \Phi(\eta_4 G_4) \end{pmatrix} = \begin{pmatrix} G_1 & G_1 & G_1 \\ 0 & G_2 & 0 \\ 0 & 0 & G_3 \\ 0 & 0 & 0 \\ G_4 \end{pmatrix}.$$ 

4 Skew Cyclic codes over $\mathbb{F}_q + u\mathbb{F}_q + v\mathbb{F}_q + uv\mathbb{F}_q$

In this section, we assume $C_3$ and $C_4$ are equivalent. Before studying skew cyclic codes over $R$, we define a skew polynomial ring $R[X; \theta]$ and skew cyclic codes over $R$. Next, we determine the structural properties of skew cyclic codes over $R$ through a decomposition theorem.

Definition 4.1 We define the skew polynomial ring as $R[x; \theta] = \{a_0 + a_1 x + \cdots + a_n x^n | a_i \in R, i = 0, 1, \cdots, n\}$, where the coefficients are written on the left of the variable $x$. The multiplication is defined by the basic rule $(ax^i)(bx^j) = a\theta^i(b)x^{i+j}$, and the addition is defined to be the usual addition rule of polynomials.

It is easily checked that the ring $R[x; \theta]$ is not commutative unless $\theta$ is the identity automorphism on $R$.

Definition 4.2 A nonempty subset $C$ of $R^n$ is called a skew cyclic code of length $n$ if $C$ satisfies the following conditions: (1) $C$ is a submodule of $R^n$; (2) if $r = (r_0, r_1, \cdots, r_{n-1}) \in C$, then skew cyclic shift $\rho(r) = (\theta(r_{n-1}), \theta(r_0), \cdots, \theta(r_{n-2})) \in C$.

Theorem 4.1 Let $C = \eta_1 C_1 \oplus \eta_2 C_2 \oplus \eta_3 C_3 \oplus \eta_4 C_4$ be a linear code of length $n$ over $R$, where $C_i$ ($i = 1, 2, 3, 4$) are codes over $\mathbb{F}_q$ of length $n$. Then $C$ is a skew cyclic code with respect to the automorphism $\theta$ if and only if $C_i$ are skew cyclic codes over $\mathbb{F}_q$ with respect to the automorphism $\theta$.

Proof For any $r = (r_0, r_1, \cdots, r_{n-1}) \in C$, let $r_i = \eta_1 a_i + \eta_2 b_i + \eta_3 c_i + \eta_4 d_i$ for $0 \leq i \leq n-1$, where $a = (a_0, a_1, \cdots, a_{n-1}) \in C_1$, $b = (b_0, b_1, \cdots, b_{n-1}) \in C_2$, $c = (c_0, c_1, \cdots, c_{n-1}) \in C_3$,
and \(d = (d_0, d_1, \cdots, d_{n-1}) \in C_4\). If \(C_i\) are skew cyclic codes, then \(\rho(r) = \rho(\eta_1 a + \eta_2 b + \eta_3 c + \eta_4 d) = \eta_1 \rho(a) + \eta_2 \rho(b) + \eta_3 \rho(c) + \eta_4 \rho(d) \in C\). This implies that \(C\) is a skew cyclic code over \(R\).

On the other hand, if \(C\) is a skew cyclic code over \(R\), we have \(\rho(r) = (\theta(r_{n-1}), \theta(r_0), \cdots, \theta(r_{n-2})) = \eta_1 \rho(a) + \eta_2 \rho(b) + \eta_3 \rho(c) + \eta_4 \rho(d) \in C\), which implies \(\rho(a) \in C_1, \rho(b) \in C_2, \rho(c) \in C_3, \rho(d) \in C_4\). Thus \(C_i\) are skew cyclic codes over \(\mathbb{F}_q\).

According to [4, Corollary 18], we know that the dual code of every skew cyclic code over \(\mathbb{F}_q\) is also skew cyclic. By using this connection and Theorem 4.1, we get the following corollary.

**Corollary 4.1** If \(C\) is a skew cyclic code over \(R\), then the dual code \(C^\perp\) is also skew cyclic.

The following theorem determines the generator polynomials of a skew cyclic code of length \(n\) over \(R\).

**Theorem 4.2** Let \(C = \eta_1 C_1 \oplus \eta_2 C_2 \oplus \eta_3 C_3 \oplus \eta_4 C_4\) be a skew cyclic code of length \(n\) over \(R\) and suppose that \(g_i(x)\) are generator polynomials of \(C_i\) (\(i = 1, 2, 3, 4\)) respectively. Then \(C = \langle \eta_1 g_1(x), \eta_2 g_2(x), \eta_3 g_3(x), \eta_4 g_4(x) \rangle\) and \(|C| = q^{4n-\sum_{i=1}^4 \deg(g_i(x))}\).

**Proof** Since \(C_i = \langle g_i(x) \rangle\), for \(i = 1, 2, 3, 4\), and \(C = \eta_1 C_1 \oplus \eta_2 C_2 \oplus \eta_3 C_3 \oplus \eta_4 C_4\), then

\[
C = \left\{ c(x) = \sum_{i=1}^4 \eta_i r_i(x) g_i(x) | r_i(x) \in \mathbb{F}_q[x; \theta] \right\}.
\]

Hence \(C \subseteq \langle \eta_1 g_1(x), \eta_2 g_2(x), \eta_3 g_3(x), \eta_4 g_4(x) \rangle\). Conversely, for any \(\sum_{i=1}^4 \eta_i k_i(x) g_i(x) \in \langle \eta_1 g_1(x), \eta_2 g_2(x), \eta_3 g_3(x), \eta_4 g_4(x) \rangle\), where \(k_i(x) \in R[x; \theta]/(x^n - 1)\), then there exist \(r_i \in \mathbb{F}_q[x; \theta]\) such that \(\eta_i k_i(x) = \eta_i r_i(x)\), \(i = 1, 2, 3, 4\). Thus \(\langle \eta_1 g_1(x), \eta_2 g_2(x), \eta_3 g_3(x), \eta_4 g_4(x) \rangle \subseteq C\), which implies \(C = \langle \eta_1 g_1(x), \eta_2 g_2(x), \eta_3 g_3(x), \eta_4 g_4(x) \rangle\). Since \(|C| = |C_1| \cdot |C_2| \cdot |C_3| \cdot |C_4|\), we obtain that \(|C| = q^{4n-\sum_{i=1}^4 \deg(g_i(x))}\).

**Theorem 4.3** Let \(C_i (i = 1, 2, 3, 4)\) be skew cyclic codes over \(\mathbb{F}_q\) and \(g_i(x)\) be the monic generator polynomials of these codes respectively, then there is a unique polynomial \(g(x) \in R[x; \theta]\) such that \(C = \langle g(x) \rangle\) and \(g(x)\) is a right divisor of \(x^n - 1\), where \(g(x) = \sum_{i=1}^4 \eta_i g_i(x)\).

**Proof** By Theorem 4.2, we know \(C = \langle \eta_1 g_1(x), \eta_2 g_2(x), \eta_3 g_3(x), \eta_4 g_4(x) \rangle\). We take \(g(x) = \eta_1 g_1(x) + \eta_2 g_2(x) + \eta_3 g_3(x) + \eta_4 g_4(x)\), obviously, we have \(\langle g(x) \rangle \subseteq C\). On the other hand, one can check that \(\eta_i g_i(x) = \eta_i g(x) (i = 1, 2, 3, 4)\), which implies \(C \subseteq \langle g(x) \rangle\). Hence \(C = \langle g(x) \rangle\). Since \(g_i(x)\) are monic right divisors of \(x^n - 1 \in \mathbb{F}_q[x; \theta]\), then there exist
\[ r_i(x) \in \mathbb{F}_q[x; \theta] \text{ such that } x^n - 1 = r_i(x)g_i(x). \] Thus
\[
[\eta_1r_1(x) + \eta_2r_2(x) + \eta_3r_3(x) + \eta_4r_4(x)]g(x) = \sum_{i=1}^{4} \eta_ir_i(x) \cdot \sum_{i=1}^{4} \eta_ig_i(x)
\]
\[
= \sum_{i=1}^{4} \eta_ir_i(x)g_i(x)
\]
\[
= \sum_{i=1}^{4} \eta_i(x^n - 1)
\]
\[
= x^n - 1.
\]
This implies \( g(x) \) is a right divisor of \( x^n - 1 \).

**Corollary 4.2** Every left submodule of \( R[x; \theta]/(x^n - 1) \) is principally generated.

Let \( g(x) = g_0 + g_1x + \cdots + g_tx^t \) and \( h(x) = h_0 + h_1x + \cdots + h_{n-t}x^{n-t} \) be polynomials in \( \mathbb{F}_q[x; \theta] \) such that \( x^n - 1 = h(x)g(x) \) and \( C \) be the skew cyclic code generated by \( g(x) \) in \( \mathbb{F}_q[x; \theta]/(x^n - 1) \), according to Corollary 18 in [4], then the dual code of \( C \) is a skew cyclic code generated by \( \bar{\eta}(x) = h_{n-t} + \theta(h_{n-t-1})x + \cdots + \theta^{n-t}(h_0)x^{n-t} \). Therefore, we have the following corollary.

**Corollary 4.3** Let \( C_i \) be skew cyclic codes over \( \mathbb{F}_q \) and \( g_i(x) \) be their generator polynomial such that \( x^n - 1 = h_i(x)g_i(x) \) in \( \mathbb{F}_q[x; \theta] \). If \( C \) is a skew cyclic code over \( R \), then \( C^\perp = \langle \sum_{i=1}^{4} \eta_i\bar{h}_i(x) \rangle \) and \( |C^\perp| = q^{\sum_{i=1}^{t} \deg(g_i(x))} \).

In light of previous introduction, we know that the order of \( \theta \) is even. Therefore, we always assume that \( n \) be odd in the rest of the paper.

**Theorem 4.4** [6] Let \( n \) be odd and \( C \) be a skew cyclic code of length \( n \), then \( C \) is equivalent to a cyclic code of length \( n \) over \( R \).

By Theorem 4.4, we can determine the number of distinct skew cyclic codes of odd length \( n \) over \( R \).

**Corollary 4.4** Let \( n \) be odd and \( x^n - 1 = \prod_{i=1}^{t} p_i^{s_i}(x) \), where \( p_i(x) \in \mathbb{F}_q[x; \theta_i] \) is irreducible, then the number of distinct skew cyclic codes of length \( n \) over \( R \) is equal to the number of ideals in \( R[x]/(x^n - 1) \), i.e. \( \prod_{i=1}^{n} (s_i + 1)^4 \).

## 5 Application Examples

In this section, we will exhibit an example of skew cyclic codes and their Gray images over \( GF(9) \). Before giving an example, we first give the definition of Plotkin Sum.

Let \( C \oplus D \) denote the Plotkin sum of two linear codes \( C \) and \( D \), also called \( (u|u + v) \) construction, where \( u \in C, v \in D \). For more information on the Plotkin sum, one can see a
good survey [9].

In the following, we assume $G_i$ are generator matrices of 9-ary linear codes $C_i$ for $i = 1, 2, 3, 4$, respectively. Let $C = \eta_1 C_1 \oplus \eta_2 C_2 \oplus \eta_3 C_3 \oplus \eta_4 C_4$ be a linear code of length $n$ over $R$, then its Gray image $\Phi(C)$ is none other than

$$(C_1 \oplus_P C_2) \oplus_P (C_3 \oplus_P C_4).$$

We construct skew cyclic codes over $GF(9)$ with some conditions. If $C_1$ is a $[20, 1, 20]$ code, $C_2$ is a $[20, 9, 4]$ code, $C_3$ is a $[20, 10, 2]$ code and $C_4$ is a $[20, 10, 2]$ code, then the Gray image of $C$ has parameters $[80, 30, 4]$ over $GF(9)$.

6 Conclusion

This paper is devoted to studying skew cyclic codes over $R = \mathbb{F}_q + u\mathbb{F}_q + v\mathbb{F}_q + uv\mathbb{F}_q$, where $u^2 = u, v^2 = v, uv = vu, q = p^m$ and $p$ is an odd prime. First, we introduce the structure of linear codes over $R$ and show the structural properties of skew cyclic codes over $R$. Next, we give the enumeration of distinct skew cyclic codes over $R$ when $n$ is odd.

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