FINITE VON NEUMANN ALGEBRA FACTORS WITH PROPERTY Γ

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Dedicated to Professor R. V. Kadison on his 75th birthday

Abstract. Techniques introduced by G. Pisier in his proof that finite von Neumann factors with property Γ have length at most 5 are modified to prove that the length is 3. It is proved that if such a factor is a complemented subspace of a larger C*-algebra then there exists a projection of norm one from the larger algebra onto the smaller and a new proof of the fact that the continuous Hochschild cohomology group $H^2(M, M)$ vanishes is also included.

1. Introduction

The articles [CS1, CS2, Pi1, Pi2, Pi3] address among other things the question whether a von Neumann algebra, on a Hilbert space $H$ which is a complemented subspace of $B(H)$, is an injective von Neumann algebra, i.e. the image of a projection of norm 1 from $B(H)$.

The question was answered in the positive for all von Neumann algebras if the projection onto $M$ is completely bounded. By an averaging technique one can prove that for properly infinite von Neumann algebras the existence of a bounded projection onto $M$ implies the existence of a completely bounded one and hence show that $M$ is injective.

For finite continuous von Neumann algebras this sort of arguments are not usable in general so we only have a very scattered or sporadic list of answers to this complementation question for finite continuous factors. The list of algebras for which this problem has been settled is identical to the list of algebras to which we can answer the similarity question - which we will describe below. It seems that whenever a method has been developed to answer one of the questions for a particular algebra, then it more or less immediately yields a method to construct an answer to the other problem for this particular algebra. In this paper we do obtain a certain positive result for the similarity problem first and then uses this to get a result concerning algebras which are complemented subspaces.

We will now describe the so called similarity problem [Ka, Pi4]. In [Ka] Kadison asks whether a homomorphism $\varphi$ of a C*-algebra $A$ into the bounded operators on a Hilbert space $B(H)$ is similar to a *-homomorphism, meaning that there exists a bounded invertible operator on $H$ which intertwines the given homomorphism with a self-adjoint homomorphism. In [Pi4] Pisier lists a lot of known results and problems related to this question. Over the last couple of years Pisier has introduced several new ideas and obtained remarkable new insights with respect to several questions related to various forms of similarities. Especially the concepts called length and similarity degree [Pi5, Pi6, Pi7] will play a major role in this article. We will describe these terms to the extent to which we need them in the next section. For now we will just mention that Pisier has proved...
that Kadison’s question has a positive answer for a C*-algebra $A$ if and only if $A$ has finite length.

In the article [Ch2] we proved that Kadison’s question could be settled in the affirmative for finite continuous von Neumann factors with property $\Gamma$. In modern language we proved that the length was at most 44. In [Pi7] Pisier proves that the length is in the interval $[3, 5]$. Using some basic ideas from [Pi7] and combining them with the point of view found in [Ch1] we can actually see that the length is 3. This implies that the only computed values (for C*-algebras) actually are 2 and 3, and it is quite a mystery if $\infty$ or some other integers will occur as lengths for C*-algebras.

As a bonus we can use the methods to prove that a finite continuous factor with property $\Gamma$ which is a complemented subspace of a C*-algebra also is complemented via a completely positive and completely contractive projection of the C* algebra onto the von Neumann algebra.

If the theory of simultaneously ultra strongly continuous multilinear mappings were better understood it might be that the methods could also be used to prove that all the continuous Hochschild cohomology groups for a factor with property $\Gamma$ vanish. We do have some results which indicates this, but we have no way of controlling simultaneous ultra strong continuity except if the cochains are completely bounded. In the latter case we already know that that the completely bounded cohomology is in general trivial [CS2]. In the case of a 2-cocycle the continuity problems can be solved quite easily and we have included a new proof of the vanishing of $H^2_c(M, M)$ for the von Neumann factors with property $\Gamma$.

We refer to the books [EK, KR] for the theory of operator algebras, to [Pa] for results on completely bounded operators, to [Ch, Di, EK] for results on the property $\Gamma$ to [Pi4, Pi5, Pi6, Pi7] for results on the similarity question, similarity degree and on length and to [SS1] for results on Hochschild cohomology.

2. Length and the similarity problem for factors with property $\Gamma$

The concepts of similarity degree and length was introduced by Pisier [Pi5, Pi6] as a result of his deep investigations into many different versions of similarity problems. We will not go into the details here but use the results of these articles for our particular purpose. We start by recalling the definition of length and then describe its deep connections to the set of bounded homomorphisms of an operator algebra into $B(H)$.

**Definition 2.1.** A unital operator algebra $A$ has finite length at most $l \in \mathbb{N}$ if there exists a constant $C$ such that for any $k \in \mathbb{N}$ and any $x \in M_k(A)$ there exist $n \in \mathbb{N}$, scalar matrices $\alpha_0 \in M_{k,n}(\mathbb{C}), \alpha_1 \in M_n(\mathbb{C}), \ldots, \alpha_{l-1} \in M_n(\mathbb{C}), \alpha_l \in M_{n,k}(\mathbb{C})$ and diagonal matrices $D_1, \ldots, D_l$ in $M_n(A)$ such that 

\[
x = \alpha_0 D_1 \alpha_1 D_2 \ldots D_l \alpha_l \quad \text{and} \quad \left( \prod_0^l \| \alpha_i \| \right) \left( \prod_1^l \| D_i \| \right) \leq C \| x \| .
\]

The length $l(A)$ is defined to be the least possible $l$ for which these conditions are fulfilled.

Suppose $\varphi$ is a homomorphism of an operator algebra $A$ into $B(H)$, then it is quite easy to see that if $A$ has finite length $l$ with constant $C$ then for any $k \in \mathbb{N}$ the homomorphism $\varphi_k : M_k(A) \to M_k(B(H))$ has norm at most $C\|\varphi\|^l$, so $\varphi$ is completely bounded and $\|\varphi\|_{cb} \leq C\|\varphi\|^l$. In particular finite length implies that any bounded homomorphism is
completely bounded. The very surprising result result of [Pi3] is that the converse is also true, as formulated in the following theorem.

Theorem 2.2 (Pisier). Let $A$ be a unital operator algebra then any bounded unital homomorphism of $A$ into $B(H)$ is completely bounded if and only if there exist positive constants $C$ and $\alpha$ such that for any bounded unital homomorphism $\varphi : A \to B(K)$: 

$$
\|\varphi\|_{cb} \leq C\|\varphi\|^\alpha. 
$$

Moreover $A$ has this property if and only if $A$ has finite length and the length $l$ is the minimum over the possible values of $\alpha$.

The least $\alpha$ usable above is called the similarity degree and the theorem tells that the similarity degree and the length is the same integer.

In [Pi7] Pisier proves that von Neumann factors with property $\Gamma$ have length at most 5 by constructing a concrete factorization of the type above for certain elements of “rank” one or less in $M_k(A)$. Here we will prove that the length is at most 3 with help of the theorem mentioned above by showing that for any bounded homomorphism $\varphi$ of a finite continuous factor with property $\Gamma$ we have $\|\varphi\|_{cb} \leq \|\varphi\|^3$. We know already by [Ch2, Pi7] that any bounded homomorphism of such a factor into $B(H)$ is completely bounded and similar to a $^*$-homomorphism. A close examination of the invertible operator which intertwines these two homomorphisms - along the lines of the computations in [Pi7] gives the result.

Theorem 2.3. Let $M$ be a continuous finite von Neumann algebra factor with property $\Gamma$ on a Hilbert space $H$ then $M$ has length 3.

Proof. It follows from [Pi7, Remark 12] that the length is at least 3.

Let now $\varphi$ denote a bounded unital homomorphism of $M$ into $B(K)$ then by [Ch2] there exists a $^*$-representation $\pi$ of $M$ on $K$ and an invertible $x$ in $B(K)$ such that $\pi = Ad(x)\varphi$. Let $h = (x^*x)$ then it follows that $Ad(h^{\frac{1}{2}})\varphi$ also is a $^*$-representation and we can - and will - replace $x$ by $h^{\frac{1}{2}}$ and assume that $\pi = Ad(h^{\frac{1}{2}})\varphi$ and $h$ is a positive and invertible contraction.

The weak closure, or the bi-commutant $\pi(M)^{''}$ of the unital algebra $\pi(M)$ splits into a sum of an infinite von Neumann algebra - say $N$ - and a finite one - say $M_f$. If one of the summands is missing then the following computations will all be simplified, so we assume that both $N$ and $M_f$ are non trivial. The Hilbert space $K$ splits accordingly via a central projection - say $Q$ - in the weak closure of $\pi(M)$ such that $N$ acts on $(I - Q)K$ and $M_f$ acts on $QK$. Finally the representation $\pi$ splits via $Q$ into the sum of an infinite representation - say $\pi_{\infty}$ and a finite one - say $\pi_f$. The latter representation is automatically ultra weakly continuous since any ultra weakly continuous functional on $M_f$ factors through the trace on $M_f$. On the other hand this trace induces a trace on $M$ and since this algebra is a factor it has only got a single ( normalized ) trace. Hence the composition of the representation $\pi_f$ and an ultra weakly continuous functional on $M_f$ has a density with respect to the trace on $M$ and therefore is ultra weakly continuous on $M$. In particular this means that $\pi_f$ is a normal isomorphism of $M$ onto $M_f$.

We will define $c$ by $c = \|\varphi\|^2$, then for any unitary $u$ in $M$ we have $0 \leq \varphi(u)\varphi(u)^* \leq c$. Since $\varphi(u) = h^{-\frac{1}{2}}\pi(u)h^{\frac{1}{2}}$ multiplication of this inequality from the left and from the right with $h^{\frac{1}{2}}$ yields, as in [Ch1 (7)] the following inequality

\begin{equation}
\forall u \text{ unitary in } M, \quad 0 \leq \pi(u)h\pi(u)^* \leq ch.
\end{equation}
The unitaries in $M$ form a group so we can extend the inequality above to the formally stronger inequality

$$(2.2) \quad \forall u, v \text{ unitaries in } M, \quad 0 \leq \pi(u)h\pi(u)^* \leq c\pi(v)h\pi(v)^*.$$ 

Since the Kaplansky Density Theorem makes it possible to approximate unitaries in the weak closure $\pi(M)''$ of $\pi(M)$ strongly with unitaries from $\pi(M)$ the validity of the inequality $\ref{eq:2.2}$ can be extended to

$$(2.3) \quad \forall u, v \text{ unitaries in } \pi(M)'', \quad 0 \leq uhv^* \leq cuhv^*.$$ 

It is hopefully clear that the inequality $\ref{eq:2.1}$ above also imply that the norm of $\varphi$ is at most $c^\frac{3}{2}$ and therefore in order to prove that $M$ has length $3$ it is by Theorem $\ref{eq:2.2}$ sufficient to prove that for any $n \in \mathbb{N}$ and any unitary $V = (v_{ij}) \in M_n(M)$ the inequality just below - named (Goal) - holds.

\begin{equation}
(\text{Goal}) \quad 0 \leq \pi_n(V)(I_{M_n(C)} \otimes h)\pi_n(V)^* \leq c^3(I_{M_n(C)} \otimes h).
\end{equation}

In the rest of the proof we will fix $n \in \mathbb{N}$ and the unitary $V = (v_{ij}) \in M_n(M)$. Further we will use the convention that for any operator $x \in B(K)$ we will let $\tilde{x} \in M_n(B(K))$ be given as $\tilde{x} = I_{M_n(C)} \otimes x$.

It is clearly enough to prove the inequality (Goal) for any vector state on $M_n(B(K))$ so we will also fix a unit vector $\xi = (\xi_1, \ldots, \xi_n) \in C^n \otimes K$, a positive real $\varepsilon$ and verify the inequality (Goal) in the state $\omega_\xi$ up to $\varepsilon$.

We will have to divide the computations according to the two representations $\pi_\infty$ and $\pi_f$. Here we are faced with the problem that $h$ does not commute with the central projection $Q$, so we will have to replace $h$ by one which does. In order to do so we first split $\xi$ as the sum $\xi_\infty + \xi_f$ by $\xi_\infty = (I - Q)\xi$ and $\xi_f = Q\xi$.

Since $M$ is assumed to have property $\Gamma$ we can by Dixmier’s result [11, Proposition 1.10] find a set $\{p_1, \ldots, p_n\}$ of pairwise orthogonal and equivalent projections in $M$ with sum $I$ such that all the norms $\|v_{ij}, p_i\|_2$ are small. Here the norm $\|\|_2$ is the one induced by the pre Hilbert space structure on $M$ coming from the unique trace state. It is a well known fact that for any (uniformly) bounded subset of $M$ the ultra strong topology is the same as the one coming from this norm. In particular this means that given the vector $\xi_f$ in $QK \oplus \cdots \oplus QK$ and the fact that the representation $\pi_f$ is normal we can find the set of projections $\{p_1, \ldots, p_n\} \subset M$ such that

$$(2.4) \quad \forall l \in \{1, \ldots, n\} : \quad \|\pi_f(I - p_l)(\pi_f)^n(V^*)\pi_f(p_l)\xi_f\| \leq \left(\frac{\varepsilon}{n^\frac{3}{2}}\right)^\frac{3}{2}.$$

Having the projections $\{p_1, \ldots, p_n\} \subset M$, we will replace $h$ by a positive contraction commuting with a finite dimensional subfactor of $M_f$ which contains $\{\pi_f(p_1), \ldots, \pi_f(p_n)\}$ in its main diagonal algebra. In order to do so we find a set of matrix units $(f_{ij})$ in $M_f$ such that $f_{ii} = \pi_f(p_i)$ and the set of matrix units generates a subfactor - say $F$ - of $M_f$ isomorphic to $M_n(C)$. In the infinite algebra $N$ we find a pair of unital and commuting subfactors $B$ and $L$ of $N$ such that, $B$ is isomorphic to $B(\ell^2(\mathbb{N}))$ and $N$ is isomorphic to the von Neumann algebra tensor product $B \bar{\otimes} L$. Finally we will let $G$ denote the von Neumann subalgebra of $\pi(M)''$ acting on $K$ given as the sum $G = B \oplus F$. This is a von Neumann algebra of type I and hence injective. Further since $B$ is infinite and $F$ is finite $Q$ is also a central projection in this algebra. We can then average over the unitary
translates $uhu^*$ of $h$ with unitaries from $G$ and we can find a positive contraction $k$ in the commutant $G'$ of $G$ such that

\begin{equation}
  k \in \overline{\text{conv} \{ uhu^* | u \text{ unitary in } G \}} \cap G'
\end{equation}

First we remark that since $Q$ is a central projection in $G$ we must have that $k$ commutes with $Q$ and secondly we see from the construction of $k$ and 2.3, that this inequality must hold for $k$ too.

\begin{equation}
  \forall u, v \text{ unitaries in } \pi(M)'', \quad 0 \leq uku^* \leq cvku^*.
\end{equation}

The next observations with respect to $k$ have to be performed according to the decomposition of the Hilbert space $K$ via $Q$ and $I - Q$, so we will split 2.6. Let us define $k_\infty = k(I - Q)$ and $k_f = kQ$ then we get from 2.6

\begin{equation}
  \forall u, v \text{ unitaries in } N, \quad 0 \leq uk_\infty u^* \leq cvk_\infty v^*.
\end{equation}

\begin{equation}
  \forall u, v \text{ unitaries in } M_f, \quad 0 \leq uk_f u^* \leq cvk_f v^*.
\end{equation}

With respect to 2.7 it is quite easy to see that this one extends to unitaries in any matrix algebras over $N$ with the same constant $c$ because $k_\infty$ is in the commutant of $B$ and $B$ is an infinite tensor factor of $N$. Hence we get immediately the following inequality with respect to $\xi_\infty$.

\begin{equation}
  0 \leq (\pi_\infty)_n(V)\tilde{k}_\infty(\pi_\infty)_n(V^*)\xi_\infty,\xi_\infty \leq (c\tilde{k}_\infty\xi_\infty,\xi_\infty).
\end{equation}

We will now show that we can obtain a similar inequality with respect to $\xi_f$. Here we will use an argument which is based on the one Pisier uses in the proof of [Pi7, Lemma 5] where it is proved that an operator in $M_n(M)$ which is supported on a projection of trace $1/n$ (with respect to the normalized trace) can be factored in the way described in Definition 2.1.

We then define partial isometries $w_l, \ 1 \leq l \leq n$ in $M_n(M_f)$. Let $(e_{ij})$ denote a set of matrix units for the $M_n(C)$ part of the product $M_n(M) = M_n(C) \otimes M_f$, then we can write the the partial isometries $w_l$ as sums of tensors as below and we get

\[ w_l = \sum_{i=1}^{n} e_{1i} \otimes f_{li}; \quad w_l w_l^* = \tilde{f}_{ll}; \quad w_l^* w_l = e_{11} \otimes I_{M_f}. \]

From the inequalities 2.4 we get since $p_j = f_{jj}$ and $f_{jj}$ commutes with $k_f$ and $k_f$ is a contraction that

\begin{equation}
  ((\pi_f)_n(V)\tilde{k}_f(\pi_f)_n(V^*)\xi_f,\xi_f)
  = \sum_{i,j,l=1}^{n} (\tilde{f}_{li}(\pi_f)_n(V)\tilde{f}_{lj}\tilde{k}_f(\pi_f)_n(V^*)\tilde{f}_{il}\xi_f,\xi_f)
  \leq \sum_{l=1}^{n} (\tilde{f}_{ll}(\pi_f)_n(V)\tilde{f}_{ll}\tilde{k}_f(\pi_f)_n(V^*)\tilde{f}_{ll}\xi_f,\xi_f) + \varepsilon.
\end{equation}

Since $w_l$ commutes with $\tilde{k}_f$ we can use the equality $\tilde{f}_{ll} = w_l w_l^*$ to write $\tilde{f}_{ll}\tilde{k}_f = w_l\tilde{k}_f w_l^*$ and then transform 2.10 into a set of $n$ inequalities regarding operators on $K_f$ by identifying
K_f with the subspace of \( C^n \otimes K_f \) corresponding to the projection \( e_{11} \otimes I_{K_f} \). We will now look at each of the terms in the sum above. Hence we identify define vectors 
\[ \eta_l = w_l^* \xi_f \]
in \( K_f \) and operators \( x_l \) in \( M_f \) by 
\[ x_l = w_l^* (\pi_f)_n(V) w_l = \sum_{i,j=1}^{n} f_{ij} \pi_f(v_{ij}) f_{ij} \]
By construction, each operator \( x_l \) is a contraction and hence since \( M_f \) is a finite von Neumann algebra there exist unitary operators - say - \( y_l, z_l \) in \( M_f \) such that 
\[ x_l = \frac{1}{2} (y_l + z_l). \]
The inequalities \( 2.10 \) and \( 2.8 \) then yields
\[
(\pi_f)_n(V) \tilde{k}_f (\pi_f)_n(V^*) \xi_f, \xi_f) \leq \sum_{l=1}^{n} (x_l k_f x_l^* \eta_l, \eta_l) + \varepsilon
\]
\[
\leq \frac{1}{4} \sum_{l=1}^{n} ((y_l + z_l) k_f (y_l + z_l)^* \eta_l, \eta_l) + \frac{1}{4} \sum_{l=1}^{n} ((y_l - z_l) k_f (y_l - z_l)^* \eta_l, \eta_l) + \varepsilon
\]
\[
= \frac{1}{2} \sum_{l=1}^{n} (y_l k_f y_l^* \eta_l, \eta_l) + \frac{1}{2} \sum_{l=1}^{n} (z_l k_f z_l^* \eta_l, \eta_l) + \varepsilon
\]
\[
\leq c \sum_{l=1}^{n} (k_f \eta_l, \eta_l) + \varepsilon
\]
\[
= c (\tilde{k}_f \xi_f, \xi_f) + \varepsilon.
\]
The last equality is due to the fact that \( k_f \) commutes with all the \( f_{ij} \). Going back to the inequality \( 2.3 \) we get \( h \leq c k \) and \( k \leq c h \) so \( 2.8 \) and \( 2.11 \) show that
\[
(\pi_f)_n(V) \tilde{h} (\pi_f)_n(V^*) \leq c^3 \tilde{h}
\]
and consequently \( \varphi \) is completely bounded and satisfies \( \| \varphi \|_{cb} \leq c^2 = \| \varphi \|^3 \). Hence the similarity degree is 3 and the length is too.

3. Von Neumann factors with property \( \Gamma \) as complemented subspaces.

As mentioned in the introduction we here consider a continuous finite factor \( M \) with property \( \Gamma \) which is a complemented subspace of a C*-algebra \( A \) and we will prove that there exists a completely positive projection of norm one from \( A \) onto \( M \). The way we prove it, is by showing that there exists a completely bounded projection from \( A \) onto \( M \) and then refer to the articles \[ CS1, CS2, Pi2, Pi3 \] to get the result. The completely bounded projection is obtained as a point ultra weak limit of a bounded net of continuous linear mappings of \( A \) into \( M \).

**Theorem 3.1.** Let \( A \) be a unital C*-algebra on a Hilbert space \( H \) and let \( M \) be a continuous finite factor on \( H \) which is a subalgebra of \( A \) such that there exists a bounded projection of \( A \) onto \( M \). If \( M \) has property \( \Gamma \) then there exists a completely positive projection of norm one from \( A \) onto \( M \).

**Proof.** Let \( \pi \) denote a bounded projection from \( A \) onto \( M \). We will first prove that we may assume that \( M \) is generated - as a von Neumann algebra - by a countable set of operators. Suppose that the theorem has been proven for such von Neumann algebras.
Then for a general finite factor as $M$ and any von Neumann subalgebra say $N$ of $M$ there exists a conditional expectation of norm one $\rho_N$ of $M$ onto $N$ coming from the trace by 

$$\text{tr}(\rho_N(m)n) = \text{tr}(mn)$$

for $m \in M$ and $n \in N$, so in particular for each subfactor $N$ of $M$ with a countable set of generators we have the bounded projection $\rho_N\pi$ of $A$ onto $N$. Moreover it follows nearly immediately from the construction of $\rho_N$ that the property $\Gamma$ of $M$ is inherited by $N$. Then by assumption there exists a projection of norm one from $A$ onto $N$ and since the unit ball in $M$ is ultra weakly compact we can perform a limit over norm one projections from $A$ onto larger and larger subalgebras $N$ of $M$ and by compactness obtain a net of projections of norm one from $A$ onto subalgebras of $M$ which converges pointwise ultra weakly to a projection of norm one from $A$ onto $M$.

Let us now assume that $M$ is generated by a countable set of operators. Then there exists a conditional expectation of norm one — say $\rho$ — define a projection $\rho$ from $\Gamma$ onto larger and larger subalgebras of $M$ and since the unit ball in $M$ is ultra weakly compact we can perform a limit over norm one projections from $A$ onto $A$ being ultra strongly for $\rho_f$ - say $\pi$. Moreover it follows nearly immediately from the construction of $\rho_N$ that the property $\Gamma$ of $M$ is inherited by $N$. Then by assumption there exists a projection of norm one from $A$ onto $N$ and since the unit ball in $M$ is ultra weakly compact we can perform a limit over norm one projections from $A$ onto $A$ onto larger and larger subalgebras $N$ of $M$ and by compactness obtain a net of projections of norm one from $A$ onto $A$ onto $N$ which converges pointwise ultra weakly to a projection of norm one from $A$ onto $M$.

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In order to be able to use $3.2$ we have to make a modification of $\pi$ corresponding to each of the sets $p^k_n$, $1 \leq k \leq n$. Hence we choose for each $n \in \mathbb{N}$ a set of matrix units - say $f^k_n$ - for a subfactor - say $F^n$ - of $M$ such that $F^n$ is isomorphic to $M_n(\mathbb{C})$ and $\forall k \in \{1, \ldots, n\} : f^k_n = p^k_n$. Let $U^n$ denote the group of unitaries in $F^n$, then this is a compact group and it consequently has a Haar probability measure say $\mu$. we can now define a projection $\pi^n$ of $A$ onto $M$ which is modular with respect to elements from $F^n$ by

$$\forall a \in A \quad \pi^n(a) = \int_{u \in U^n} \int_{v \in U^n} u\pi(u^*av)v^*d\mu(u)d\mu(v).$$

We can not prove that these projections are completely bounded but we can construct a sequence $\rho^n$ of mappings from $A$ into $M$ which has the property that $\|\rho^n\|_n \leq \|\pi\|$ and $\forall m \in M \rho^n(m) \rightarrow m$ ultra strongly for $n \rightarrow \infty, n \in \mathbb{N}$. We define $\rho^n$ by

$$\forall a \in A \quad \rho^n(a) = \sum_{k=1}^n f^k_n \pi^n(a) f^k_n.$$
for $X = (x_{ij}) \in M_n(A)$

$$\rho^n_n(X) = \sum_{k=1}^n (I \otimes f_{kk}^n) \pi^n_n(X)(I \otimes f_{kk}^n)$$

Hence, since we are working with the operator norm;

$$\|\rho^n_n(X)\| = \max\{\|(I \otimes f_{kk}^n) \pi^n_n(X)(I \otimes f_{kk}^n)\| \mid 1 \leq k \leq n\}.$$  \hspace{1cm} (3.6)

Now for each $k$ we use the properties of $w_k^n$ and the $F^n$ modularity of $\pi^n$ to see that

$$\|(I \otimes f_{kk}^n) \pi^n_n(X)(I \otimes f_{kk}^n)\| = \|w_k^n \pi^n(X) w_k^n\| = \|\pi\left(\sum_{i,j=1}^n f_{ik} x_{ij} f_{kj}\right)\|.$$ \hspace{1cm} (3.7)

The last sum inside $\pi$ is obtained in $M$ via the identification mentioned above and the norm of the sum is dominated by $\|X\|$ since the sum is nothing but $w_k^n X w_k^n$. A combination of this and the results 3.6 and 3.7 give that

$$\|\rho^n_n\| \leq \|\pi\|$$ \hspace{1cm} (3.8)

The sequence of uniformly bounded mappings $\rho^n$ of $A$ into $M$ has a subnet which converges pointwise ultra weakly to a linear mapping say $\rho$ of $A$ into $M$. By 3.8 we get that $\|\rho\|_n \leq \|\pi\|$, so $\rho$ is completely bounded. Further we get from 3.2 and the fact that the $\pi^n$ all are projections onto $M$ that $\forall m \in M \rho(m) = m$. We have then proved that $\rho$ is a completely bounded projection from $A$ onto $M$. By [CS1, Pi2] it then follows that there exists a completely positive projection of norm one from $A$ onto $M$ and the proof is completed. \hfill \square

4. Continuous Hochschild cohomology of von Neumann factors with property $\Gamma$.

The preprint [CS3] which was never published contains the result that for a von Neumann factor $M$ with property $\Gamma$ the second continuous Hochschild cohomology group of $M$ with coefficients in $M$, $H^2_c(M, M)$ vanishes. The result was later published in the book [SS1] by Sinclair and Smith. We think that the methods used above should be applicable as an ingredient in a proof of a general vanishing theorem for the continuous Hochschild cohomology groups of factors with property $\Gamma$. We are not able to get that far but we can get a new and quite easy proof of the fact that $H^2_c(M, M) = 0$ by using the methods above and the nice result from [SS2] Proof of Theorem 5.1 which in a very short form says; that in order to show that a continuous $n$-cocycle is a coboundary it is sufficient $(n - 1)$ times to prove that it is cohomologous to one which is completely bounded in one variable only.

**Theorem 4.1.** Let $M$ be a continuous finite von Neumann factor with property $\Gamma$ then the continuous Hochschild cohomology group $H^2_c(M, M)$ vanishes.

**Proof.** As in the case above where the algebra is a complemented subspace of some larger C*-algebra we will like to show first that it is sufficient to prove the result for a von Neumann factor which is countably generated.

So suppose that the result has been established in this case and let $\Phi$ denote a continuous 2-cocycle on $M$. Then for any subalgebra $N$ of $M$ we have - as demonstrated in the proof of [3.1] - a completely positive projection $\pi_N$ of norm one from $M$ onto $N$. This
projection is also an \( N \)-bi-module mapping, so a simple algebraic manipulation shows that the composed bilinear map given by \( \pi_N \Phi : N \times N \to N \) is a continuous 2 cocycle on \( N \). In the proof below we will show that when \( N \) is countably generated then this cocycle is the coboundary of a continuous linear mapping \( \varphi_N : N \to N \) which satisfies \( \| \varphi_N \| \leq 651 \| \Phi \| \). If one defines \( \psi_N = \varphi_N \pi_N \) one gets a bounded net of mappings of \( N \) into \( M \) indexed by the set of countably generated subalgebras of \( M \). We may then find a subnet which converges pointwise ultra weakly to a continuous 1-cocycle \( \psi \) on \( M \) and in turn get that \( \Phi \) is the coboundary of \( \psi \).

Let us now suppose that \( M \) is countably generated and that \( M \) acts standardly on \( H \) with \( \xi \) a cyclic and separating unit trace vector for \( M \). The involution induced by \( \xi \) is denoted \( J \).

The start of the proof follows the proof of [SS1, Theorem 6.4.2] where the stage is set. The pre dual of \( M \) is now separable and by Popa’s result [Po, Corollary 4.1] there exists an injective subfactor \( R \) in \( M \) such that \( R' \cap M = CI \).

By [SS1, Theorem 3.1.1] we may assume that \( \Phi \) is multimodular with respect to \( R \), separately ultra weakly continuous and vanishes whenever any of the arguments is in \( R \). Further by [SS1, Corollary 5.2.4] there is an ultra weakly continuous \( R \)-bimodular mapping \( \varphi : M \to JR'J \) such that the coboundary \( \partial \varphi \) equals \( \Phi \). Further by [SS1, Lemma 3.4.2] we can choose \( \varphi \) such that

\[
\| \varphi \| \leq 65 \| \Phi \| \quad \text{and trivially} \quad \| \Phi \| \leq 3 \| \varphi \|
\]

Since we are only dealing with 2-cocycles some direct estimates can be made to show that much less than a factor of 65 will do as well.

By [SS1, Lemma 5.4.7, (2)] we get a norm estimate for the action of \( \varphi_k : M_k(C) \otimes M \to M_k(C) \otimes JR'J \), but in order to understand the inequality below we must say that we use the term \( \| \|_2 \) on any finite factor to mean the 2-norm with respect to the trace state - or the normalized trace on the algebra. Then we can quote [SS1] as:

\[
\| \varphi_k(x) \|^2 \leq 4 \| \varphi \|^2 (\| x \|^2 + k \| x \|_2^2).
\]

We are now in the position to use the \( \Gamma \)-property in a similar way as above. Let \( \{ m_i \mid i \in \mathbb{N} \} \) be a sequence in the unit ball of \( M \) which is dense with respect to the \( \| \|_2 \) topology. For each \( n \in \mathbb{N} \) we choose using [Dl] a set of pairwise orthogonal and equivalent projections \( \{ p_1^n, \ldots, p_n^n \} \) in \( M \) with sum \( I \) such that

\[
\| m_j, p^n_i \|_2 \leq n^{-3} \quad \text{and} \quad \| \Phi(m_j, m_k, p^n_i) \|_2 \leq n^{-3}.
\]

For each \( n \in \mathbb{N} \) we will modify \( \Phi \) by a coboundary say \( \Psi^n \) which is related to the set of projections \( \{ p_1^n, \ldots, p_n^n \} \) in such a way that we can prove that \( \Phi \) is cohomologous to a 2-cocycle which is completely bounded in the first variable. By [SS2] this is sufficient in order to see that \( \Phi \) is a coboundary too.

\[
\forall n \in \mathbb{N} \forall m \in M \quad \text{let} \quad \psi^n(m) = \sum_{i=1}^{n} p_i^n \Phi(p_i^n, m)p_i^n.
\]

This \( \psi^n \) is clearly an ultra weakly continuous linear map of \( M \) into \( M \) such that \( \| \psi^n \| \leq \| \Phi \| \), consequently for the coboundary - say \( \Psi^n = \partial \psi^n \) - we have \( \| \Psi^n \| \leq 3 \| \Phi \| \).
In order to clarify the following computations we introduce some bilinear operators from $M \times M$ to $M$ by

\begin{equation}
\Omega^n = \Phi - \Psi^n
\end{equation}

\begin{align*}
\Delta^n_1(x, y) &= \sum_{i=1}^{n} p_i^n \Phi(p_i^n, x)[y, p_i^n] \\
\Delta^n_2(x, y) &= \Phi(x, y) - \sum_{i=1}^{n} p_i^n \Phi(x, y)p_i^n \\
\Delta^n &= \Delta^n_1 + \Delta^n_2 \\
\Theta^n(x, y) &= \sum_{i=1}^{n} (p_i^n \Phi(p_i^n, x)p_i^n - xp_i^n \Phi(p_i^n, y)p_i^n).
\end{align*}

We can then examine $\Omega^n$. The fact that $\Phi$ is a 2-cocycle is used from the second to the third line just below.

\begin{equation}
\Omega^n(x, y) = \Phi(x, y) + \sum_{i=1}^{n} (-xp_i^n \Phi(p_i^n, y)p_i^n + p_i^n \Phi(p_i^n, x)p_i^n - p_i^n \Phi(p_i^n, x)p_i^n y)
\end{equation}

\begin{align*}
&= \Phi(x, y) + \Delta^n_1(x, y) + \sum_{i=1}^{n} (-xp_i^n \Phi(p_i^n, y)p_i^n + p_i^n \Phi(p_i^n, x)p_i^n - p_i^n \Phi(p_i^n, x)yp_i^n) \\
&= \Phi(x, y) + \Delta^n_1(x, y) + \sum_{i=1}^{n} (-xp_i^n \Phi(p_i^n, y)p_i^n - p_i^n \Phi(x, y)p_i^n + p_i^n \Phi(p_i^n, x)p_i^n) \\
&= \Delta^n(x, y) + \Theta^n(x, y).
\end{align*}

We know that $\Phi = \partial \varphi$ so we can see that $\Theta^n$ is expressed by

\begin{equation}
\Theta^n(x, y) = \sum_{i=1}^{n} (p_i^n x \varphi(y)p_i^n - p_i^n \varphi(p_i^n x)p_i^n + p_i^n \varphi(p_i^n x)yp_i^n - xp_i^n \Phi(p_i^n, y)p_i^n)
\end{equation}

\begin{align*}
&= \sum_{i=1}^{n} (p_i^n x \varphi(y)p_i^n - xp_i^n \Phi(p_i^n, y)p_i^n) + \sum_{i=1}^{n} p_i^n (\varphi(p_i^n x) - \varphi(p_i^n x)y)p_i^n.
\end{align*}

This decomposition shows that for a fixed $y$ in $M$ and a fixed natural number $k$ we get using (1.2) and (1.7) that for $X \in M_k(\mathbb{C}) \otimes M$, $\tilde{y} = I_{M_k(\mathbb{C})} \otimes y$ and any $n \geq k$ $n \in \mathbb{N}$

\begin{equation}
\|\Theta^n_k(X, \tilde{y})\| \leq 4\|\varphi\|\|X\||\|y\| + \max\{\|\tilde{p}_i^n \varphi_k(p_i^n X \tilde{y})\tilde{p}_i^n\| + \|\tilde{p}_i^n \varphi_k(p_i^n X)\tilde{y}\tilde{p}_i^n\| | 1 \leq i \leq n\}
\end{equation}

\begin{align*}
&\leq 4\|\varphi\|\|X\||\|y\| + 2(2\sqrt{2})\|\varphi\|\|X\||\|y\|
&\leq 10\|\varphi\|\|X\||\|y\|.
\end{align*}

It is clear from the construction of $\Delta^n$ in (1.3) that this sequence of bilinear operators on $M$ is uniformly bounded and converges pointwise ultra strongly towards 0. A combination of this with the estimates from (1.8) shows that if we take a subnet of the sequence $(\psi^n)_{n \in \mathbb{N}}$ which converges pointwise ultra weakly to a continuous 1-cochain - say $\psi$ on $M$ then the
2-cocycle - say $\Omega$ on $M$ given by $\Omega = \Phi - \partial \psi$ is completely bounded in the left variable, and by the methods from SS2, Proof of Theorem 5.1 we can show that a 2-cocycle which is completely bounded in the left variable is the coboundary of a continuous 1-cochain - say $\omega$ - on $M$ which is in the pointwise ultra weakly closed convex hull of the set of continuous 1-cochains of the form below:

\[(4.9) \quad m \to \sum_{j \in J} x^*_j \Omega(x_j, m) \quad \text{where} \quad x_j \in M \quad \text{and} \quad \sum_{j \in J} x^*_j x_j = I_M.\]

We do have $\Phi = \partial (\psi + \omega)$ and by a combination of 4.1, 4.4, 4.8 and 4.9 we get that

\[(4.10) \quad \|\psi + \omega\| \leq \|\Phi\| + \|\omega\| \leq \|\Phi\| + 650\|\Phi\| = 651\|\Phi\|\]

We have now proved that if $M$ is countably generated any continuous 2-cocycle $\Phi$ is inner and we have moreover obtained a universal bound on the cochains implementing $\Phi$ so we may conclude that the theorem is proved for a general continuous von Neumann factor with property $\Gamma$. 

\[\square\]

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