Risk Aware Belief-dependent Constrained POMDP Planning

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Abstract

Risk awareness is fundamental to an online operating agent. However, it received less attention in the challenging continuous domain under partial observability. Existing constrained POMDP algorithms are typically designed for discrete state and observation spaces. In addition, current solvers for constrained formulations do not support general belief-dependent constraints. Crucially, in the POMDP setting, risk awareness in the context of a constraint was addressed in a limited way. This paper presents a novel formulation for risk-averse belief-dependent constrained POMDP. Our probabilistic constraint is general and belief-dependent, as is the reward function. The proposed universal framework applies to a continuous domain with nonparametric beliefs represented by particles or parametric beliefs. We show that our formulation better accounts for the risk than previous approaches.

1 Introduction

Decision making under uncertainty in partially observable domains is a key capability for reliable autonomous agents. Commonly, the basis of the state-of-the-art algorithms for decision making under uncertainty is the Partially Observable Markov Decision Process (POMDP). The robot does not have access to its state. Instead, it maintains the distribution, named the belief, over the state given all its current information, namely, its history of actions and the observations alongside the prior belief. The decision maker shall maintain and reason about the evolution of the belief within the planning phase. At the same time, the robot’s online goal is to find an optimal action for its current belief. Unfortunately, an exact solution of POMDP is unfeasible \cite{13}. A critical limitation of the classical POMDP formulation is the assumption that the belief-dependent reward is nothing more than the expectation of state-dependent reward with respect to belief \cite{11}. Another limiting assumption in many state-of-the-art algorithms is the discrete domain, e.g., the state and the observation \cite{16}, \cite{21}. In contrast, we tackle the continuous domain in terms of state and observation spaces.

Augmenting POMDP with general belief-dependent rewards is a long-standing problem. Unravelling it would allow information theoretic rewards, which are extremely important in numerous problems in Artificial Intelligence (AI) and Robotics, such as autonomous exploration, Belief Space Planning (BSP) \cite{9}, and active Simultaneous Localization and Mapping (SLAM) \cite{14}. The belief-dependent reward formulation is known as $\rho$-POMDP \cite{1}, \cite{7}. Earlier techniques focused on offline solvers and extended $\alpha$-vectors approach to piecewise linear and convex \cite{1}, \cite{6} or Lipschitz-continuous rewards \cite{7}. These approaches are also limited to discrete domains. More recent solvers, such as Sparse Sampling (SS) \cite{10} and Monte Carlo Tree Search (MCTS) \cite{17}, \cite{19} focus on online methods. These methods are suitable for continuous domains.

Continuous spaces and general belief-dependent rewards render many off-the-shelf POMDP solvers not applicable. One way to incorporate general belief-dependent reward is a reformulation of POMDP as Belief-MDP (BMDP) and using online solvers designed for MDP, e.g., SS or MCTS. Such an MCTS running on BMDP is called a Particle Filter Tree with Double Progressive Widening (PFT-DPW) \cite{17}. 

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The described limitation also applies to recently appeared chance constrained approaches, gaining attention. The motivation to add the chance constraints is to introduce the notion of risk into the problem. Initially, the planning community focused on collision avoidance, formulating it as a chance constraint. For example, [3] is limited to Gaussian beliefs, and the chance constraint is calculated with respect to the nominal trajectory provided by the controller. Such a formulation does not apply to general beliefs represented by particles, nor is it a truly risk-averse constraint. We extensively debate this claim in the paper.

More recent works examine a discrete domain and a belief-dependent constraint being the first moment of the state-dependent constraint [12]. Moreover, [12] provides only a local solution. Another line of work considers chance constraints [15]. The paper [15] introduces the algorithm RAO* which uses admissible heuristics for the action value function (Q-function) in the belief space. This aspect is problematic with general belief-dependent rewards.

Moreover, the RAO* algorithm never evaluates the chance constraint itself. Instead, the evaluation is done using an admissible execution risk heuristic, which can be loose; and therefore, the algorithm might declare no feasible solution exists, although this is not true. This aspect makes the RAO* algorithm overconservative. Although the chance constraint in [15] appears similar to ours, it is genuinely different. In striking contrast, our constraint formulation is general and belief-dependent. In particular, being distribution-aware, our probabilistic constraint formulation is truly risk-averse.

Typically algorithms designed for general beliefs represent the belief as a set of particles and use a particle filter [20] for nonparametric Bayesian updates. In this work, we assume the setting of nonparametric beliefs, although our formulations also support a parametric setting.

The contributions of this work are threefold. Firstly, we formulate a risk-averse belief-dependent constrained POMDP. Averaging the state-dependent reward/constraint to obtain the belief-dependent reward/constraint is a severe hindrance that we relax. We are unaware of prior works addressing POMDP with risk-averse or belief-dependent constraints (even with expectation). Our constraint is more general compared to previous approaches: The proposed constraint is probabilistic, whereas the state-of-the-art constrained formulations devise the constraint as an expectation with respect to observations. In particular, our probabilistic belief-dependent constraint supports risk-averse operators, such as Conditional Value at Risk (CVaR), and leads to a novel safety constraint formulation. Secondly, we rigorously derive theory and present an algorithm to evaluate our probabilistic constraint. In addition, we provably unveil the drawbacks of established chance constraints and propose a novel algorithm based on such constraints to serve as our baseline. Thirdly, we provide a way to guide the belief tree construction while planning. This permits to save time and focus on branches that satisfy the constraint. Our framework is universal for challenging continuous domains and can be applied in nonparametric and parametric settings.

The rest of this paper is organized as follows. We start from preliminaries [2]. We then define our novel framework [3] and give relevant examples of possible constraints [4]. Next, we adaptively evaluate the constraint while constructing the belief tree [5]. Further, we present an online algorithm [6] for our novel formulation and rigorously reveal the drawbacks of the conventional chance constraint [7]. Finally, we introduce the state-of-the-art online solver for chance-constrained POMDP in continuous setting [8]. Eventually, section [9] show simulations and results. The conclusions and final remarks are presented in section [10]. To allow fluid reading, we placed the proofs for all theorems and lemmas, and additional in-depth discussions in the appendix.
2 Preliminaries

The $\rho$-POMDP is a tuple $(X,A,Z,T,O,\rho,\gamma,b_0)$ where $X,A,Z$ denote state, action, and observation spaces with $x \in X$, $a \in A$, $z \in Z$ the momentary state, action, and observation, respectively. $T(x',a,x) = P_T(x'|x,a)$ is a stochastic transition model from the past state $x$ to the subsequent $x'$ through action $a$, $O(z,x) = P_Z(z|x)$ is the stochastic observation model, $\gamma \in (0,1]$ is the discount factor, $b_0$ is the belief over the initial state (prior), and $\rho$ is the belief-dependent reward operator. Let $h_k$ be a history, of actions and observations alongside the prior belief, obtained by the agent up to time instance $k$. The posterior belief $b_k$ is a shorthand for the probability density function of the state given all information up to current time index $b_k(x_k) \triangleq p(x_k|h_k)$. The policy is $a$, indexed by the time instances, mapping from belief to action to be executed $\pi_k : B \mapsto A$, where $B$ is the space of all the beliefs taken into account in the problem. The policy for $L$ consecutive steps ahead is denoted by $\pi_{k:k+L-1}$. Sometimes we will omit the time indices for clarity and write $\pi$. We hope the time indices will be evident from the context. When an information theoretic reward, for instance, information gain, is introduced to the problem, the reward can assume the following form $\rho(b,a,z',b') = (1-\lambda)r^a(b,a) + \lambda r^z(b,a,z',b')$, in this case it is a function of two subsequent beliefs, an action, and an observation. Note that in the setting of nonparametric beliefs, we shall resort to sampling approximations using $m_s$ samples of the belief. Such a reward is comprised of the expectation over the state and action dependent reward

$$r^x(b,a) = E_{x\sim b}[r^x(x,a)] \approx \frac{1}{m_s} \sum_{i=1}^{m_s} r^x(x^i,a),$$

and the information-theoretic reward $r^z(\cdot)$ weighted by $\lambda$, which in general can be dependent on consecutive beliefs and the elements relating them (e.g. information gain). The online decision making goal is to find an action to execute, maximizing the action value function

$$Q^\pi(b_k,a_k) = E_{z_{k+1}}[\rho(b_k,a_k,z_{k+1},b_{k+1}) + V^\pi(b_{k+1})|b_k,a_k],$$

where $\pi$ is the policy and the value function

$$V^\pi(b_k) = E_{z_{k+1},b_{k+1}}[\sum_{\ell=k}^{L-1} \rho(b_\ell,a_\ell,z_{\ell+1},b_{\ell+1})|b_k,\pi],$$

is expected cumulative reward under the particular policy $\pi$.

When the agent performs an action and receives an observation, it shall update its belief from $b$ to $b'$. Let us denote the update operator by $\psi$ such that $b' = \psi(b,a,z')$. In our context, it will be a particle filter since we focus on the setting of nonparametric beliefs. Moreover, we define a propagated belief $b'^\pi$ as the belief $b'$ after the robot performed an action $a$ and before it received and observation.

In this paper, we present a new risk-averse decision making problem. We augment the $\rho$-POMDP [1] objective with a novel, probabilistic general belief-dependent constraint. To our knowledge, all previous chance constrained formulations such as [3], [15] suffer from limiting assumptions. To be specific, they perform averaging over the state trajectories as we unveil in this work and therefore are not distribution-aware formulations. Moreover, a general belief-dependent constraint was not studied nor proposed. Nevertheless, such a constraint is of the highest importance. For instance, as we discuss in the sequel, such a formulation can be used to determine when to stop exploration, e.g. in an active SLAM context, which is an open problem currently [4]. [13].

Risk-averse planning has been actively investigated [5], [22], but risk aversion was not considered for the constraint to the best of our knowledge.
3 Risk Aware Belief-dependent Constrained $\rho$-POMDP

3.1 Problem Formulation

In this work we augment the classical formulation described above with a general probabilistic belief-dependent constraint. We introduce a new problem with the following objective

$$a^*_k \in \arg\max_{a_k \in A} \{Q^\pi(b_k, a_k)\} \text{ subject to}$$

$$P(c(b_{k+L}, \pi_{k+1:k+L-1}, a_k)) \geq 1 - \epsilon,$$  \hspace{1cm} (4)

where $c$ is a Bernoulli random variable. By $\pi^*$ we denote the belief tree policy defined by the planning algorithm.

The constraint (5) requires two parameters, $\epsilon$, and $\delta$. The former, $\epsilon$, is the probability margin within which we permit to the future, rendered by possible future observations generating the beliefs (see Fig. 1), violate the constraint, in other words, to be unprofitable or unsafe. The parameter $\delta$ is the margin for some particular sequence of the beliefs $b_{k+L}$. With a high probability of at least $1 - \epsilon$, we want the received sequence of future posterior beliefs to fulfill the constraint.

The constraint can be of two forms. The first form is cumulative

$$c(b_{k+L}) \triangleq 1\{(\sum_{i=k}^{k+L-1} \phi(b_{i+1}, b_i)) \geq \delta\},$$

and the second is multiplicative

$$c(b_{k+L}) \triangleq \prod_{i=k}^{k+L-1} 1\{\phi(b_{i+1}, b_i) \geq \delta\},$$

where $\phi$ denotes a general belief-dependent operator. Let us interpret the two forms, (6) and (7). The first form is formulated with respect to a cumulative value of the operator $\phi$ along a sequence of beliefs generated by a sequence of possible future observations. In this form we permit immediate value of the operator $\phi$ to deviate but the cumulative value shall fulfill the inequality (6). In contrast, (7) states that every value of $\phi$ in the sequence of the beliefs shall fulfill the inequality (7), meaning to be larger than or equal to $\delta$. Both formulations are novel, to the best of our knowledge. Furthermore, the form of (6) is motivated by the long standing question of stopping exploration. The form of (7) is motivated by safety, e.g., collision avoidance.

From now on, for clarity, in the constraint, we will use $\pi^*$ instead of $\pi_{k+1:k+L-1}$. When the problem (4) is augmented with the probabilistic constraint (5), ideally, every selection of the action following the policy shall take into account the constraint at the root of the belief tree.

4 Possible Constraints

This section focuses on possible operators $\phi$ as a constraint. One important example is a safety constraint, e.g., collision avoidance or energy consumption. We propose the following formulation,

$$P(1\{b_{k+L} \in B_{k+L}\} | b_k, \pi_{k+1:k+L-1}, a_k) \geq 1 - \epsilon,$$  \hspace{1cm} (8)

where $B_{k+L}$ is the space of safe belief sequences starting at time index $k$ and of length $L$. To relate to (5), in (8): $c(b_{k+L}) \triangleq 1\{b_{k+L} \in B_{k+L}\}$. Further, we show explicitly why this formulation is advantageous over previous formulations of safety constraint. The safeness of a sequence of beliefs $b_{k+1:k+L}$ can be defined in various ways. One possibility is

$$1\{b_{k+L} \in B_{k+L}\} \triangleq \prod_{i=k}^{k+L} 1\{p(1\{x \in \mathcal{X}_i^{safe}\} | b_i) \geq \delta\},$$  \hspace{1cm} (9)

where $\mathcal{X}_i^{safe}$ is the safe space, which generally can be time-dependent, e.g., due to moving obstacles in the context of collision avoidance.

Another possibility is to use the Conditional Value at Risk (CVaR) operator for collision avoidance as $-\phi$ (minus sign is needed merely to maintain $\geq$ in (7)). We define the deviation of the robot’s position from the safe region $\mathcal{Y}$ considering the obstacle $\ell$ as follows \text{dist}(x, \mathcal{Y}) = \min_{y \in \mathcal{Y}} \|x - y\|_2$. Note that $\int_{\ell=1}^{M} \mathcal{Y} = \mathcal{X}$.
Lemma 1

In this section, we delve into the evaluation of our novel formulation of the constraint. We start by presenting

where (13) defines the task of reaching the goal.

Further, let \( \phi \) assume and redefine the constraint of the first form as follows

The constraint formulated with probabilities of (9) as well as conventional [15] is unable to distinguish such a
tail. Meaning if the unsafe tail is extremely unsafe but with low probability, such a constraint will catch that.

Another example of a general belief-dependent constraint is Information Gain (IG), defined as follows

where \( H(\cdot) \) denotes differential entropy. Utilizing this constraint with the form [6] allows one to reason if
the cumulative information gain along a planning horizon is significant enough (above threshold \( \delta \)) with
the probability of at least \( 1 - \alpha \). Such a capability has a number of implications. For instance, in the context of
informative planning and active SLAM, instead of prompting the agent to maximize its information gain, we
informative planning and active SLAM, instead of prompting the agent to maximize its information gain, we
explain the meaning of such a constraint. Let \( \zeta = \text{dist}(x, Y^t) \). By definition \( \text{CVaR}_{\alpha}[\zeta] \triangleq \mathbb{E}[\zeta | \zeta \geq \text{VaR}_{\alpha}(\zeta)] \),
where the value-at-risk \( \text{VaR}_{\alpha}(\zeta) \) at confidence level \( \alpha \) is the \( 1 - \alpha \) quantile of \( \zeta \), namely,

The value \( \text{VaR}_{\alpha}[\text{dist}(x, Y^t)] \) is the minimal value such that with probability at least \( 1 - \alpha \) the deviation from
the safe space considering one obstacle is smaller than or equal to it. The \( \text{CVaR} \) is taking the average of the unsafe
tail. Meaning if the unsafe tail is extremely unsafe but with low probability, such a constraint will catch that.
The constraint formulated with probabilities of [8] as well as conventional [13] is unable to distinguish such a
behavior. The distribution over the unsafe part of the beliefs is unaccessible. We note that such a constraint
was suggested by [8], in the setting of randomly moving obstacles. However [8] assumes a fully observable state
and linear models, and not the general POMDP setting considered herein.

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the cumulative information gain along a planning horizon is significant enough (above threshold \( \delta \)) with
the probability of at least \( 1 - \epsilon \). Such a capability has a number of implications. For instance, in the context of
informative planning and active SLAM, instead of prompting the agent to maximize its information gain, we
can require that it does so only if it is able to decrease uncertainty in some tangible amount. This is a new
concept made possible by our general formulation, which therefore can be used to identify, e.g., when to stop
exploration. We aim to investigate this aspect in the future.

Let us discuss one more important constraint, the probability of reaching the goal (see, e.g., [3]). Throughout
the manuscript, for clarity, we assumed that the operator \( \phi \) is identical for all time indices. We now relax that
assumption and redefine the constraint of the first form as follows\(^3\)

Further, let \( \phi_{k+1}() \equiv 0 \quad \forall \ell \in k : k + L - 2 \) and

where [13] defines the task of reaching the goal.

5 Constraint Evaluation

In this section, we delve into the evaluation of our novel formulation of the constraint. We start by presenting
a helpful lemma.

Lemma 1 (Representation of our general constraint).

\[
P \left( c(b_{k,k+L}) | b_k, \pi_{k+1:k+L-1}, a_k \right) = \mathbb{E}_{z_{k+1:k+L}} \left[ c(b_{k,k+L}) | b_k, \pi_{k+1:k+L-1}, a_k \right].
\]

\(^3\)We denote \( f \equiv g \) for two operators, if we have \( f(x) = g(x) \) \( \forall x \).
The reader can find the proof in Appendix A. From Lemma 1 we behold how to obtain the best sample approximation of the constraint, since the theoretical expectation (14) is out of the reach. In practice, we approximate expectation in (14) with a finite number of samples, such that the constraint becomes

\[
\frac{1}{m} \sum_{i=1}^{m} c(b^i_{k:k+L}) \geq 1 - \epsilon, \tag{15}
\]

where \( m \) is the number of the observation sequences \( z_{k+1:k+L} \) expanded from action \( a_k \) at the root of the belief tree. If the belief tree is given, we can traverse it from the bottom and calculate value of \( c(b^i_{k:k+L}) \) for \( i \in 1 \ldots m \) along the way such that when we reach the root, we have everything to evaluate (15). In general, since the parameter \( m \) has to be known, this applies to approaches that decouple belief tree construction from the solution, e.g. SS algorithm [10].

However, we would like to guide the belief tree construction such that if the action does not fulfill the constraint we will spend on it as less effort as possible. This brings us to the next section.

5.1 Adaptive Constraint Inquiry

In this section, we address a complete belief tree construction. We bound the expression of the constraint from each end using the already expanded part of the belief tree. Suppose the online algorithm at the root for each action expands upon termination \( m \) laces appropriate to the drawn observations \( \{z_{k+1:k+L}\}_{i=1}^{m} \). Each lace \( i \) corresponds to a particular realization of the return.

Suppose the algorithm already expanded \( n \leq m \) laces with some order. We denote expanded laces by a sub-sequence \( j = 1 \ldots n \), such that \( i_j \) is the index of the observation sequence, i.e, \( i_j \in 1 \ldots m \). The lower bound LB on the constraint expression is

\[
1 - \epsilon \leq \frac{1}{m} \sum_{j=1}^{n} c(b^j_{k:k+L}) \leq \frac{1}{m} \sum_{i=1}^{m} c(b^i_{k:k+L}). \tag{16}
\]

Whereas the upper bound UB reads

\[
\frac{1}{m} \sum_{i=1}^{m} c(b^i_{k:k+L}) \leq \frac{m-n}{m} + \frac{1}{m} \sum_{j=1}^{n} c(b^j_{k:k+L}) < 1 - \epsilon. \tag{17}
\]

By the question mark we denote the inequalities that shall be fulfilled online to check either the constraint is met (16) or violated (17). These bounds allow to evaluate the constraint adaptively before expanding the \( m \) laces of the belief sequences \( b_{k:k+L} \). Such a technique is applicable for both settings: open and closed-loop.

One example of an adaptive usage of (16) and (17) is to save time in open loop planning or alternatively spend more time on the action sequences which fulfill the constraint. Think about a static action sequence to be checked. After each expanded lace of constraint \( c(b^j_{k:k+L}) \) we are probing (16), if fulfilled, we know that the constraint is satisfied, and we can stop calculating the constraint. Else we are trying (17), if fulfilled, we know that the current action sequence violates the constraint. The third possibility is to add one more lace and check again. In such a way, we adaptively expand the lower possible number of constraint laces to be evaluated and validate or invalidate the action sequence depending on whether the probabilistic constraint is fulfilled or not.

Another example is the closed loop setting, where we deal with policies. We focus attentively on the closed loop setting in this paper.

![Figure 2: Visualization of the conventional chance constraint. The indicators over the teal and purple trajectories are averaged without any distinction (Section 7).](image-url)
5.2 Constraint Confidence $\epsilon$ and Adaptive Constraint Pruning

The constraint confidence $\epsilon$ controls the stiffness of the condition that the distribution of belief-dependent constraint shall fulfill. The maximal stiffness is reached when $\epsilon = 0$. In this case, (15) can be satisfied only if for every $i$ the value of $c(b_{k+L}) = 1$. This fact is a sufficient condition to prune an action at the root of the belief tree after a single lace resulting in $c(b_{k+L}) = 0$. Moreover, using the structure of (7) we know that

$$\phi(b_{k+1}, b) \geq \delta$$

shall be satisfied for every pair of subsequent belief nodes. Therefore, we can invalidate the lace from any belief in the lace violating (18). In other words, we adapt the number of constraint laces to minimal depending on the scenario. In the next section, we focus on this key observation and present a complete algorithm.

6 The Algorithm

Algorithm 1 Prob. Constrained BMDP Sparse Sampling

1: procedure PCSS(belief: $b$, depth: $d$)
2:     $c \leftarrow 1 \{ \phi(b) \geq \delta \}$
3:     $C(b) \leftarrow \{ \}$
4:     if $d = 0$ then
5:         $C(b) \leftarrow C(b) \cup \{ c \}$
6:         return (Null, $\rho(b)$)
7:     end if
8:     $(a^*, v^*) \leftarrow (\text{Null}, -\infty)$
9:     for $a \in A$ do
10:        $v \leftarrow 0.0$
11:        Calculate propagated belief $b'$
12:        $C(ba) \leftarrow \{ \}$
13:        status $\leftarrow$ true
14:        for $m = 1 : m_d$ do
15:            Sample $x^o \sim b'$
16:            Sample $z' \sim P(z'|x^o)$
17:            $b' \leftarrow \psi(h, a, z')$
18:            $r \leftarrow \rho(b, a, z', b')$
19:            $v' \leftarrow \text{PCSS}(b', d-1)$
20:            $v+ = (r + \gamma \cdot v')/m_d$
21:            $C(b') \leftarrow \{ \}$
22:            for each $c' \in C(b')$ do
23:                $C(b') \leftarrow C(b') \cup \{ c' \}$
24:            end for
25:            $C(ba) \leftarrow C(ba) \cup C(b')$
26:        end for
27:     end for
28:     if $0 \in C(ba)$ then
29:        status $\leftarrow$ false
30:     end if
31:     if status $\land v > v^*$ then
32:        $(a^*, v^*) \leftarrow (a, v)$
33:     end if
34:     end for
35: end procedure

In particular, inspired by SS [10] and adaptivity aspects in [2], we present an algorithm (Alg. 1) for the general form of constraint of (7). Notably, to our knowledge, it is the first algorithm in the continuous domain dealing with probabilistic constraints in a POMDP setting.
Alg. 1 is presented for $\epsilon = 0$. It uses the Bellman optimality criterion while traversing the tree from the bottom up symmetrically. In line 28, we check if the constraint is violated. In this case, before applying Bellman optimality, Alg. 1 discards branches of actions that violate the constraint at the root of the belief tree. This discarding happens in line 31 of Alg. 1 through the boolean variable “status”, which will be zero if the subtree emanating from current belief $b$ does not fulfill the constraint. Note that Alg. 1 is designed for the constraint of the form (7) since we perform logical AND in line 23. As described in the previous section, this is a mechanism to guide tree construction. As we traverse the belief tree, we discard branches that do not satisfy the constraint with respect to the root of the tree.

7 Safety Constraint

In this section, we further analyze our novel version of safety constraint (8) and discuss why it is more general, and its advantages over an existing state-of-the-art formulation, e.g., [3, 15]. Since the paper [3] presents a parametric method for Gaussian beliefs, it is relevant for us solely from the constraint formulation perspective. We also cannot compare to RAO$^+$ from [15] since it never evaluates the constraint itself, only the lower bound, which makes it over-conservative. We unveil this aspect in section 8.1.

We, however, shall compare two formulations of the constraint, ours (8) and the conventional. Let us rigorously show why the conventional chance constraint is insufficient to account for risk. Further, we will present our adaption of RAO$^+$, and this will be our baseline.

Our key insight is that a conventional chance constraint does not fully depend on the observations since it averages state trajectories and is not explicitly formulated with respect to posterior beliefs. Let us recite the formulation presented in [15], [3], which reads as follows

$$P(1_{\{\tau \in L_{i=0}^{T-1} \chi_{i}^{safe}\}} | b_k, \pi) \geq \delta, \tag{19}$$

where $\tau = x_{k:k+L}$ is the trajectory of the states. This constraint means probability that the trajectory $\tau$ will be safe. Note that from the properties if the indicator variable

$$1_{\{\tau \in L_{i=0}^{T-1} \chi_{i}^{safe}\}} = 1\{\bigcap_{i=k}^{T-1} \{x_i \in \chi_i^{safe}\}\} = \bigwedge_{i=k}^{T-1} 1\{x_i \in \chi_i^{safe}\}.$$

Meaning, the safe trajectory is the trajectory comprised of safe states. Another property of the indicator variable is

$$P(1_{\{\tau \in L_{i=0}^{T-1} \chi_{i}^{safe}\}} | b_k, \pi) = \int_{\tau} \prod_{i=k}^{T-1} 1\{x_i \in \chi_i^{safe}\} \mathbb{P}(\tau | b_k, \pi_{k+1:k+L-1}, a_k) \, d\tau. \tag{20}$$

Crucially, as seen in (20), and in contrast to (8) and (9), such an approach is not distribution-aware since it sees the posterior solely through the lens of expectation. In such a formulation the observations are required merely to decide which action to take alongside the trajectory. As visualized in Fig. 2, we regard in that formulation trajectories coming from different posterior beliefs without a distinction from which belief they arrived. In contrast, we will see further our approach is truly distribution-aware since it accounts for the number of safe posteriors. Let us show the following lemma.

Lemma 2 (Probability of trajectory).

$$\mathbb{P}(\tau | b_k, \pi_{k+1:k+L-1}, a_k) = \mathbb{P}_T(x_{k+1} | x_k, a_k) b_k(x_k), \tag{21}$$

$$\int \prod_{i=k+1}^{T-1} \mathbb{P}(x_{i+1} | x_i, \pi(b_i(b_{i-1}, a_{i-1}, z_i))) \mathbb{P}_Z(z_i | x_i) \, dz_{k+1:k+L-1}. \tag{21}$$

We provide the proof in Appendix A.2. As we see, the observations are required solely for decision which action to take. In particular, when we deal with static action sequences $a_{k:k+L-1}$ the observations cancel out.

As mentioned, in this paper we focus on belief-dependent constraints and rewards. Moreover, the Belief MDP (BMDP) formulation will permit us to deal with belief-dependent rewards such as information gain and differential entropy. Therefore, our novel Probabilistic Constraint (PC) given by (9) and Alg. 1 which is based on BMDP, are formulated in terms of beliefs and not the trajectories.
In order to compare and benchmark our approach to the conventional Chance Constraint (CC) formulation, we now observe the latter from another angle and reformulate it in the context of posterior beliefs. To improve readability let us introduce another Bernoulli variable \( i \in \{ \in \mathcal{X}^{\text{safe}} \} \).

**Lemma 3** (Average over the safe posteriors).

\[
P \left( \bigwedge_{k=k-L}^{k+L} i_k | b_k, \pi \right) = P(i_k | b_k) \mathbb{E}_{z_{k+1}} \left[ P(i_{k+1} | b_{k+1}) \mathbb{E}_{z_{k+2}} \left[ P(i_{k+2} | b_{k+2}) \ldots a_{k+1}, b_{k+1} \right] a_k, b_{k}^{\text{safe}} \right],
\]

where \( \bar{b}_i = \psi(b_{i-1}^{\text{safe}}, a_{i-1}, z_i) \) which is different than \( b_i = \psi(b_{i-1}, a_{i-1}, z_i) \) used in [9], similarly \( \bar{z}_i \) is the observation given safe belief instead of original. We provide the proof in Appendix A.3. Here, \( \psi \) is a method for Bayesian belief update. Further details can be found in Appendix A.3 Note that since \( b_k \) is actual belief and we know that the agent is safe \( P(i_k | b_k) = 1 \); and therefore can be omitted. The statement (22) is very close to equation (12) of [15]. The difference is that we derived the recurrence relation until the horizon and we work with probabilities to be safe instead of probability to be unsafe (eq. 6 of [15]).

Let us elaborate on the notation of the safe belief. We define the safe belief as

\[
b_{\text{safe}}(x) = \begin{cases} 1 & \{ x \in \mathcal{X}^{\text{safe}} \}, \\ 0 & \{ x \notin \mathcal{X}^{\text{safe}} \}, \end{cases} \]

i.e., we nullify the unsafe portion of the belief and re-normalize.

We emphasize that Lemma 3 is just a reformulation of the conventional constraint [19], and now we proceed to analyzing the difference between that formulation and our constraint [5]. We rearrange our formulation [5] to arrive to an expression similar in its form to (22). To that end, we use (14) alongside with (8) and (9), and get

\[
\mathbb{E}_{z_{k+1}} \left[ \prod_{i=k}^{k+L} \mathbb{1} \{ P(i_k | b_k) \geq \delta \} | b_k, a_{k+L-1}, a_k \right] =
\]

\[
\mathbb{E}_{z_{k+1}} \left[ \prod_{i=k}^{k+L} \mathbb{1} \{ P(i_k | b_k) \geq \delta \} \mathbb{1} \{ P(i_{k+1} | b_{k+1}) \geq \delta \} \ldots \mathbb{1} \{ P(i_{k+L} | b_{k+L}) \geq \delta \} | a_k, b_k \right]. \tag{24}
\]

While appearing similar, the two formulations are genuinely different.

For clearly seeing the differences let us focus on the myopic setting \( (L = 1) \). Let us present our formulation (Probabilistic Constraint) versus the conventional formulation (Chance Constraint):

**PC:** \[
\mathbb{E}_{z_{k+1}} \left[ \mathbb{1} \{ P(i_{k+1} | b_k, a_k, z_{k+1}) \geq \delta \} | b_k, a_k \right] \geq 1 - \epsilon, \tag{25}
\]

**CC:** \[
\mathbb{E}_{z_{k+1}} \left[ P(i_{k+1} | b_k^{\text{safe}}, a_k, z_{k+1}) | b_k^{\text{safe}}, a_k \right] \geq \delta. \tag{26}
\]

From the above we immediately see that our approach (PC) is truly distribution aware as it counts the number of safe posteriors because of the indicator outside the inequality involving the probability value. In contrast, the conventional formulation (CC) merely averages the posterior probabilities and asks if on average they are larger than or equal to \( \delta \).

Let us give a specific example. Assume the safe belief \( b_k \). Assume that \( \delta = 0.7 \) and we have three equiprobable observations in a myopic setting such that \( P(1 \{ x_{k+1} \in \mathcal{X}^{\text{safe}} \} \psi(b_{k+1}^{\text{safe}}, a_k, z_{k+1}^j)) \) equals 0.1, 1.0. 1.0 for \( j = 1, 2, 3 \) respectively. On average we have exactly 0.7 such that (26) is fulfilled. However one belief is extremely unsafe. In contrast, as our formulation is distribution-aware, it is aware that only two out of the three observation sequences satisfy the constraint. For example, it will declare (the sampling-based approximation of) (25) is not satisfied if (e.g.) we select \( \epsilon = 0 \) and \( \delta = 0.7 \).

The reformulation of the conventional constraint allows to reveal its another undesired characteristic. With a growing horizon \( L \), the probability \( P(i_{k+L} | \mathbb{1} \{ x_k, x_{k+L} \in \mathcal{X}^{\text{safe}} \} | b_k, \pi) \) will decay. To see that explicitly let us start from the terminal time index \( k + L \). We observe that

\[
\mathbb{E}_{z_{k+L}} \left[ P(i_{k+L} | b_{k+L}) | a_{k+L-1}, b_{k+L-1} \right] \leq \max_{z_{k+L}} P(i_{k+L} | b_{k+L}). \tag{27}
\]
Algorithm 2 Chance Constrained BMDP Sparse Sampling

1: procedure CCSS(belief: $b$, belief: $\bar{b}$ depth: $d$, scale: $sc$)
2: \hspace{1em} $\varphi \leftarrow \phi(b)$
3: \hspace{1em} if $d = 0$ then
4: \hspace{2em} return $(Null, \rho(b), \varphi)$
5: \hspace{1em} end if
6: \hspace{1em} Make $\bar{b}$ safe $\\triangleright$ as in equation (23)
7: \hspace{1em} $(a^*, v^*) \leftarrow (Null, -\infty)$
8: \hspace{1em} for $a \in \mathcal{A}$ do
9: \hspace{2em} $v \leftarrow 0.0$ $\triangleright$ Value function
10: \hspace{2em} Calculate propagated belief $b'$ from $b$
11: \hspace{2em} Calculate propagated belief $\bar{b}'$ from $\bar{b}$
12: \hspace{2em} $\Phi(ba) \leftarrow \{\}$
13: \hspace{2em} status $\leftarrow$ true
14: \hspace{2em} for $\in \{1 : m_d\}$ do
15: \hspace{3em} Sample $x^o \sim b'$
16: \hspace{3em} Sample $z^o \sim P(z|x^o)$
17: \hspace{3em} $b' \leftarrow \psi(b, a, z')$
18: \hspace{3em} $\bar{b}' \leftarrow \psi(b, a, z')$
19: \hspace{3em} $r \leftarrow \rho(b, a, z', b')$ $\triangleright$ Calculate reward
20: \hspace{3em} $(a', v', \varphi_{exp} \leftarrow CCSS(b', \bar{b}', d-1, sc)$
21: \hspace{3em} $v' = (r + \gamma \cdot v')/m_d$
22: \hspace{3em} $\Phi(ba) \leftarrow \Phi(ba) \cup \varphi_{exp}$
23: \hspace{2em} end for
24: \hspace{1em} $\varphi_{exp} \leftarrow \frac{1}{|\Phi(ba)|} \sum_{\varphi_{exp} \in \Phi(ba)} w_{ba} \cdot \varphi_{exp}$
25: \hspace{1em} if $\varphi_{exp} < \delta^d \cdot sc + \delta(1 - sc)$ then
26: \hspace{2em} status $\leftarrow$ false
27: \hspace{2em} end if
28: \hspace{1em} if status $\land u > v^*$ then
29: \hspace{2em} $(a^*, v^*) \leftarrow (a, v)$
30: \hspace{2em} end if
31: \hspace{1em} end for
32: \hspace{1em} return $(Null, \rho(b), \varphi \cdot \varphi_{exp})$
33: end procedure

Continuing until the present time we obtain the product of probabilities of multiplicands, making it harder and harder to fulfill the constraint (19) with growing horizon. In contrast, in our formulation (9) we require being larger or equal to $\delta$ solely from the multiplicands of (9), so it does not suffer from such a problem and scales to the growing horizon.

To summarize, the conventional collision avoidance constraint (22) has two key differences versus ours (24).

1. Our constraint formulation is distribution-aware, while the conventional is the first moment only. Moreover, in the open loop setting the latter is formulated without any notion of observations. In the closed loop setting access to the posterior is solely through average posterior and is oblivious to the posterior beliefs themselves.

2. When the horizon grows the conventional constraint becomes more and more conservative up to violation. In other words the formulation does not scale with horizon, unless $\delta$ is appropriately adjusted. Our formulation does not suffer from this limitation. For proper comparison with our formulation, we propose to do scaling to $\delta$. Our baseline is Alg. 2 which will become apparent shortly. It can be applied with scaling or without. To the best of our knowledge it is a novel formulation on its own.

Before we move to the next section let us mention that a specific variation of (15) in the context of safety
with \( \epsilon = 0 \) is

\[
\forall j = 1 \ldots m \left( \prod_{\ell=k}^{k+L} \mathbf{1} \{ P(x_\ell \in X^\text{safe}_\ell | b_j^\ell) \geq \delta \} \right) = 1, \tag{27}
\]

where \( m = m_d^L \), namely, \( m_d \) to the power of \( L \), and where \( m_d \) is the number of observations expanded from each node of the belief tree as in Algorithms 1 and 2.

8 Baseline Approach: Chance-constrained Continuous POMDP

As mentioned, there are two prominent online approaches for solving a continuous POMDP with belief-dependent rewards in a nonparametric domain: SS [10], and PFT-DPW [18]. In continuous domains, it is unclear how to apply heuristics guided forward search described by [15]. Instead of using the heuristics, we utilize the Bellman principle to resolve that issue. Moreover, as we observe from equation (22) the observations for the objective (4) and the chance constraint (22) have different distributions.

In [15], this is addressed by considering a discrete and finite observation space and exhaustively expanding all the observations. Such an approach is not possible in a continuous setting. To tackle this issue, we resort to importance sampling such that only a single set of observations is maintained. Let us emphasize that such a problem was not addressed so far and serves only as our baseline. Note that this issue does not exist in our probabilistic approach Alg. 1.

Our solver for the conventional Chance-constrained POMDP is formulated as Alg. 2. Similar to [15], the chance constraint in Alg. 2 is assured to be fulfilled from any belief node in the belief tree until the bottom. The boolean variable “sc” switches between the scaled version of the algorithm, which we call CCSSs, and unscaled.

Next, we analyze the RAO* algorithm and present our importance sampling approach for chance-constrained continuous POMDP, which will serve as our baseline.

8.1 Remark on Adaptivity

The paper [15] utilizes an upper bound on future execution risk. Let us restate their future execution risk definition \( \text{er}(b_{k+1}, \pi) \) and the constraint

\[
\text{er}(b_{k+1}, \pi) \triangleq 1 - P(1 \{ \tau \in x_{k+1} : X^\text{safe}_{k+1} \} | b_{k+1}, \pi) \leq \Delta \triangleq 1 - \delta. \tag{28}
\]

Their upper bound \( \hat{U} \) is based on an admissible heuristic and is not adaptive in striking contrast to the bounds (16) and (17) in our approach. Crucially, they never evaluate the actual constraint

\[
\text{er}(b_{k+1}, \pi) \leq \hat{U}(b_{k+1}, \pi) \leq \Delta. \tag{29}
\]

Such an approach results in an over-conservative pruning (Fig. 3). This pruning will happen when their upper bound \( \hat{U} \) is larger than \( \Delta \) (the policy pruned inadequately), and the execution risk is lower. In contrast, in our approach (the roles of upper and lower bounds are reversed), we will check the lower bound, and if it is not informative adapt the bounds as explained in section 5.1.

Further we present an importance sampling approach for chance-constrained continuous POMDP.
8.2 Importance Sampling Approach for Chance-constrained Continuous POMDP

As we have seen in the Lemma 3, the distributions of the observations of the chance constraint and the action value function are different.

Let us observe the myopic setting. Since we draw observations sequentially the extension to arbitrary horizon is straightforward. In the case of the objective function the desired probability density is \( P(z_{i+1} | b_i, a_i) \). Whereas for the constraint we are dealing with \( P(\bar{z}_{i+1} | b_i, 1 \{ x_i \in X_i^{\text{safe}} \}, a_i) \). To be consistent at each node of the belief tree, we shall ensure that two of the distributions are consistent. One way to do that is by importance sampling. Suppose we sampled \( m \) samples \( \{ z_{i+1}^j \}_{j=1}^m \sim \mathbb{P}(z_{i+1} | b_i, a_i) \). From now on, we can think about

\[
\hat{P}_{(m)}(z_{i+1} | b_i, a_i) = \frac{1}{m} \sum_{j=1}^m \delta(z_{i+1} - z_{i+1}^j),
\]

(30)

as density of the discrete probability (Fig. 4). In such a way we obtain the desired probability density through the following manipulation

\[
\hat{P}_{(m)}(\bar{z}_{i+1} | b_i, 1 \{ x_i \in X_i^{\text{safe}} \}, a_i) = \sum_{j=1}^m w_{i+1}^{z,j} \delta(\bar{z}_{i+1} - \bar{z}_{i+1}^j),
\]

(31)

where the weight \( j \) is given by

\[
w_{i+1}^{z,j} = \frac{1}{m} \frac{\mathbb{P}(\bar{z}_{i+1} = \bar{z}_{i+1}^j | b_i, 1 \{ x_i \in X_i^{\text{safe}} \}, a_i)}{\mathbb{P}(\bar{z}_{i+1} = \bar{z}_{i+1}^j | b_i, a_i)}.
\]

(32)

To calculate the likelihoods of the observations we shall do the following. Suppose that the belief is represented by samples.

\[
b_k(x_k) \approx \sum_{i=1}^N w_k^i \delta(x_k - x_k^i),
\]

(33)

Let us introduce another notation \( \delta(x_k - x_k^i) = \delta^{x_k^i}(x_k) \) so

\[
\mathbb{P}(x_{k+1} | b_k, a_k, 1 \{ x_k \in X_k^{\text{safe}} \}) = \int_{x_k \in X} 1 \{ x_k \in X_k^{\text{safe}} \} \mathbb{P}_T(x_{k+1} | x_k, a_k) b_k(x_k) dx_k \approx \frac{\int_{\xi_k \in X} 1 \{ \xi_k \in X_k^{\text{safe}} \} b_k(\xi_k) d\xi_k}{\int_{\xi_k \in X} 1 \{ \xi_k \in X_k^{\text{safe}} \} b_k(\xi_k) d\xi_k} \int_{x_k \in X} 1 \{ x_k \in X_k^{\text{safe}} \} \mathbb{P}_T(x_{k+1} | x_k, a_k) \left( \sum_{i=1}^N w_k^i \delta^{x_k^i}(x_k) \right) dx_k = \frac{\sum_{i=1}^N w_k^i 1 \{ x_k^i \in X_k^{\text{safe}} \} \mathbb{P}_T(x_{k+1} | x_k^i, a_k)}{\sum_{i=1}^N w_k^i 1 \{ x_k^i \in X_k^{\text{safe}} \} \delta^{x_k^i}(x_k)} = \frac{\sum_{i=1}^N w_k^i 1 \{ x_k^i \in X_k^{\text{safe}} \} \mathbb{P}_T(x_{k+1} | x_k^i, a_k)}{\sum_{i=1}^N w_k^i 1 \{ x_k^i \in X_k^{\text{safe}} \} \delta^{x_k^i}(x_k)}.
\]

(34)

(35)

(36)

(37)

We got that

\[
\mathbb{P}(\bar{z}_{k+1} = \bar{z}_{k+1}^j | b_k, 1 \{ x_k \in X_k^{\text{safe}} \}, a_k) \approx \frac{\sum_{i=1}^N w_k^i 1 \{ x_k^i \in X_k^{\text{safe}} \} \mathbb{P}(\bar{z}_{k+1} = \bar{z}_{k+1}^j | x_k^i)}{\sum_{i=1}^N w_k^i 1 \{ x_k^i \in X_k^{\text{safe}} \}}.
\]

(38)

In case of the denominator we arrive to same expression, only without the indicator.

\[
\mathbb{P}(z_{k+1} = z_{k+1}^j | b_k, a_k) \approx \frac{\sum_{i=1}^N w_k^i \mathbb{P}(z_{k+1} = z_{k+1}^j | x_k^i)}{\sum_{i=1}^N w_k^i}.
\]

(39)

In reality, however, it is possible that after we discard all the samples of the belief which are not safe we are left with a very small set of samples or an empty set. To alleviate this issue we resample the safe particles to a constant number of samples \( N \).
The hyperparameters are $m_d = 10$, $L = 2$, $m_z = 70$, $\delta = 0.7$:

(a) The PCSS algorithm - no collision.
(b) Scaled CCSS with the same seed collided with the obstacle.
(c) CCSS without scaling.

Table 1: 50 Simulations of 21 planning sessions of PCSS versus CCSS.

| Parameters | num collisions | mean cum. reward (V) | mean cum. reward (V) no coll |
|------------|----------------|----------------------|-----------------------------|
| $Lm_a m_d$ |                | PCSS, CCSS, CSS | PCSS, CCSSs, CSS | PCSS, CCSSs, CSS |
| 1/100      | 5/50           | 10/50               | -116.5 ± 8.4, -111.0 ± 9.3 | -111.0 ± 9.3, -116.3 ± 8.3 |
| 1/100      | 5/50           | 5/50                | -118.5 ± 13.21, -110.3 ± 10.0 | -110.3 ± 10.0, -118.5 ± 13.5, -111.0 ± 9.1 |
| 1/100      | 6/50           | 5/50                | -122.2 ± 14.5, -107.6 ± 12.3 | -107.6 ± 12.3, -122.3 ± 14.6, -107.1 ± 12.3 |
| 2/100      | 5/50           | 24/50               | -110.63 ± 12.8, -103.1 ± 10.0 | -106.5 ± 8.9, -111.0 ± 12.9, -105.9 ± 8.4, -106.8 ± 8.0 |
| 2/100      | 5/50           | 25/50               | -111.3 ± 11.0, -103.8 ± 10.4 | -105.9 ± 10.4, -119.9 ± 10.9, -105.8 ± 11.3, -106.0 ± 10.0 |
| 3/100      | 5/50           | 34/50               | -109.3 ± 10.9, -107.1 ± 10.7 | -106.6 ± 10.9, -109.1 ± 11.5, -112.8 ± 11.0, -108.1 ± 11.0 |

9 Results

We present results obtained with Alg. [1] (PCSS) that uses our formulation [24] and the second proposed algorithm (CSSS) (Alg. [2]) based on the conventional constraint formulation [22]. We adopt the well known problem of navigation to the goal with collision avoidance. Our reward is $\rho(b, a, z') = \frac{1}{m_o} \sum_{i=1}^{m_o} r^f(x_i, a)$, where $m_o$ is the number of the obstacle particles and $r^f(x, a) = -\| x - x^g \|^2$. By $x^g$ we denote the location of the goal. Note that such $\rho(b, a, z', b')$ accounts for belief uncertainty as we show in Appendix [C]. Our obstacles have a circular shape with a center at $x^g$ and radius $r^g$. We approximate the probability of not having the collision by $\mathbb{P}(\{ x_k \in \mathbb{X}_{afe} \} | b_k) = 1 - \frac{1}{m_o} \sum_{i=1}^{m_o} 1 \{ \| x_k - x^g \|^2 \leq r^g \}$. Our action space is the space of motion primitives of unit vectors $A = \{ \rightarrow, \uparrow, \downarrow, \leftarrow \}$. Motion and observation models, and the initial belief are $P_T(|x, a) = \mathcal{N}(x + a, \Sigma_T)$, $P_E(|x; (x^{b,i})_{i=1}) = \mathcal{N}(x, \Sigma_O)$, $b_0 = \mathcal{N}(x_0, \Sigma_0)$ respectively. The covariance matrices are diagonal $\Sigma_T = I \cdot \sigma^2 e$ and

$$\Sigma_O(x) = \begin{cases} \sigma^2_o I_{d_i} & \text{if } d_i \geq r_{\min} \\ \sigma^2_o I_{d_i} & \text{else} \end{cases}$$

(40)

where $d_i = \| x - x^{b,i} \|_2$, $x^{b,i}$ is the 2D location of the beacon $i$. We set the parameters to be $r_{\min} = 1$, $\sigma^2_e = 0.1$ and $\sigma^2_o = 0.01$. The obstacles are of unit diameter. We apply the planner described by Alg. [1] and compare it against our second algorithm CCSS. Our simulation is in an MPC framework, i.e. re-planning after each step. In Fig. [5] we present an example where our algorithm succeeds to maneuver between the obstacles and reach the goal while scaled and unscaled CCSS fails. In the three algorithms in all runs, we use the same seed. Moreover, we conduct the ablation study shown in Table [1]. We observe the superiority of our approach in all configurations. The reader can find the code used for all simulations here [https://github.com/andreyzhitnikov/constrainedPOMDP].

10 Conclusions

We formulated a continuous POMDP with novel probabilistic risk-averse constraints. Our constraints and rewards are general and belief-dependent. Furthermore, we extended the conventional chance-constrained POMDP to the level of Belief MDP. Moreover, we showed that chance constraint is oblivious to posterior
beliefs. We presented PCSS and CCSS algorithms to tackle the two formulations in the challenging continuous domains and possibly a nonparametric setting. Our simulations corroborate that probabilistic constraints are paramount in terms of collision avoidance to the conventional formulation of chance constraints.

Appendix A  Proofs

A.1 Proof of Lemma 1 (representation of our general constraint).

\[ P(c(b_{k,k+L})|b_k, \pi, a_k) = \int_\mathbb{X}_{k+1:k+L} P(c(b_{k,k+L})|b_k, \pi, a_k, b_{k+1:k+L}) \cdot \]  
\[ \mathbb{P}(b_{k+1:k+L}|b_k, \pi, a_k, z_{k+1:k+L})p(z_{k+1:k+L}|b_k, \pi, a_k)dz_{k+1:k+L} = \]  
\[ \mathbb{E}_{z_{k+1:k+L}} [c(b_{k,k+L})|b_k, \pi, a_k] \]  
\[ (41) \]

We used the fact that \( p(b_{k+1:k+L}|b_k, \pi, a_k, z_{k+1:k+L}) \) is Dirac’s delta function. ■

A.2 Proof of Lemma 2 (probability of the trajectory)

\[ \mathbb{P}(x_{k:k+L}|b_k, \pi_{k+1:k+L-1}, a_k) = \int_{\mathbb{X}_{k+1:k+L-1}} \mathbb{P}(x_{k:k+L}, z_{k+1:k+L-1}|b_k, \pi)dz_{k+1:k+L-1} = \]  
\[ \int_{\mathbb{X}_{k+1:k+L-1}} \mathbb{P}(x_{k+L}|x_{k:k+L-1}, a_{k+L-1})\mathbb{P}(z_{k+L-1}|x_{k:k+L-1}, z_{k+1:k+L-2}, b_k, \pi)dz_{k+1:k+L-1} = \]  
\[ \int_{\mathbb{X}_{k+1:k+L-1}} \mathbb{P}_T(x_{k+L}|x_{k+L-1}, a_{k+L-1})\mathbb{P}_Z(z_{k+L-1}|x_{k+L-1})dz_{k+1:k+L-1} = \]  
\[ \int_{\mathbb{X}_{k+1:k+L-1}} \mathbb{P}_T(x_{k+L}|x_{k+L-1}, a_{k+L-1})\mathbb{P}_Z(z_{k+L-1}|x_{k+L-1})dz_{k+1:k+L-1} = \]  
\[ (42) \]

We observe the recurrence relation. Overall

\[ \mathbb{P}(\tau|b_k, \pi_{k+1:k+L-1}, a_k) = \mathbb{P}_T(x_{k+1}|x_k, a_k)b_k(x_k) \]  
\[ \int_{\mathbb{X}_{k+1:k+L-1}} \prod_{i=1}^{k+L-1} \mathbb{P}_T(x_{i+1}|x_i, \pi(b_i(b_{i-1}, a_{i-1}, z_i)))\mathbb{P}_Z(z_i|x_i)dz_{k+1:k+L-1} \]  
\[ (43) \]

\[ (a) \]

A.3 Proof of Lemma 3 (average over the safe posteriors)

\[ P\left(\bigcap_{i=k}^{k+L} 1 \{ x_i \in \mathcal{X}_i^{\text{safe}} \} | b_k, \pi\right) = P\left(1 \{ x_k \in \mathcal{X}_k^{\text{safe}} \} | b_k\right) \int_{\mathbb{X}_{k+1:k+L-1}} \prod_{i=1}^{k+L-1} \mathbb{P}_T(x_{i+1}|x_i, \pi(b_i(b_{i-1}, a_{i-1}, z_i)))\mathbb{P}_Z(z_i|x_i)dz_{k+1:k+L-1} \]  
\[ (44) \]

\[ (b) \]
Let us focus on the expression we marked by (b)

$$P \left( \bigwedge_{i=k+1}^{k+L} 1 \{ x_i \in X_i^{safe} \} | 1 \{ x_k \in X_k^{safe} \} \right) b_k, \pi =$$

$$\int_{b_{k+1}} P \left( \bigwedge_{i=k+1}^{k+L} 1 \{ x_i \in X_i^{safe} \} | b_{k+1}, 1 \{ x_k \in X_k^{safe} \}, b_k, \pi \right) P \left( b_{k+1} | 1 \{ x_k \in X_k^{safe} \} \right) db_{k+1} = (49)$$

$$\int_{b_{k+1}} P \left( b_{k+1} | 1 \{ x_k \in X_k^{safe} \}, b_k, \pi \right) P \left( \bigwedge_{i=k+1}^{k+L} 1 \{ x_i \in X_i^{safe} \} | b_{k+1}, \pi \right) db_{k+1} = (50)$$

Merging the two expressions we obtain

$$P \left( \bigwedge_{i=k}^{k+L} 1 \{ x_i \in X_i^{safe} \} | b_k, \pi \right) = P \left( 1 \{ x_k \in X_k^{safe} \} | b_k \right) \cdot$$

$$\int_{b_{k+1}} P \left( b_{k+1} | 1 \{ x_k \in X_k^{safe} \}, b_k, \pi \right) P \left( \bigwedge_{i=k+1}^{k+L} 1 \{ x_i \in X_i^{safe} \} | b_{k+1}, \pi \right) db_{k+1} = (51)$$

We observe that expression (a) is very similar to (c), namely

$$P \left( \bigwedge_{i=k}^{k+L} 1 \{ x_i \in X_i^{safe} \} | b_{k+1}, \pi \right) = P \left( 1 \{ x_k \in X_k^{safe} \} | b_{k+1} \right) \cdot$$

$$\int_{b_{k+2}} P \left( b_{k+2} | 1 \{ x_k \in X_k^{safe} \}, b_{k+1}, \pi \right) P \left( \bigwedge_{i=k+2}^{k+L} 1 \{ x_i \in X_i^{safe} \} | b_{k+2}, \pi \right) db_{k+2} = (52)$$

Merging the two we got

$$P \left( \bigwedge_{i=k}^{k+L} 1 \{ x_i \in X_i^{safe} \} | b_k, \pi \right) = P(1 \{ x_k \in X_k^{safe} \} | b_k).$$

$$\int_{b_{k+1}} P \left( b_{k+1} | 1 \{ x_k \in X_k^{safe} \}, b_k, \pi \right) P(1 \{ x_k \in X_k^{safe} \} | b_{k+1}).$$

$$\int_{b_{k+2}} P \left( b_{k+2} | 1 \{ x_k \in X_k^{safe} \}, b_{k+1}, \pi \right) P \left( \bigwedge_{i=k+2}^{k+L} 1 \{ x_i \in X_i^{safe} \} | b_{k+2}, \pi \right) db_{k+2} db_{k+1} = (53)$$

We behold the recurrence relation. Now we show that marginalization can be done with respect to the observations. Let us assume that $i$ is
the last index \((i = k + L)\)

\[
\int P(1 \{ x_i \in \mathcal{X}^{-1}_{i-1} \} \mid b_i) P(b_i \mid 1 \{ x_{i-1} \in \mathcal{X}^{\text{safe}}_{i-1} \} , b_{i-1} , \pi) db_i = \tag{54}
\]

\[
\int \int_{z_i \in z} P(1 \{ x_i \in \mathcal{X}^{\text{safe}}_{i} \} \mid b_i) \delta(b_i - \psi(b_{i-1}, 1 \{ x_{i-1} \in \mathcal{X}^{\text{safe}}_{i-1} \} , a_{i-1} , z_i)).
\]

\[
p(z_i | a_{i-1}, b_{i-1}, 1 \{ x_{i-1} \in \mathcal{X}^{\text{safe}}_{i-1} \}) dz_i db_i = \tag{55}
\]

\[
\int \int_{z_i \in z} P(1 \{ x_i \in \mathcal{X}^{\text{safe}}_{i} \} \mid b_i) \delta(b_i - \psi(b_{i-1}, 1 \{ x_{i-1} \in \mathcal{X}^{\text{safe}}_{i-1} \} , a_{i-1} , z_i)).
\]

\[
p(z_i | a_{i-1}, b_{i-1}, 1 \{ x_{i-1} \in \mathcal{X}^{\text{safe}}_{i-1} \}) dz_i = \tag{56}
\]

\[
p(z_i | a_{i-1}, b_{i-1}, 1 \{ x_{i-1} \in \mathcal{X}^{\text{safe}}_{i-1} \}) dz_i = \tag{57}
\]

\[
\mathbb{E}_{z_i} \left[ P(1 \{ x_i \in \mathcal{X}^{\text{safe}}_{i} \} \mid \psi(b_{i-1}, 1 \{ x_{i-1} \in \mathcal{X}^{\text{safe}}_{i-1} \} , a_{i-1} , z_i)) \right] \tag{58}
\]

We plug this result into expression for \(i - 1\) and do the same trick to \(b_{i-1}\) □

**Appendix B**  Calculating the Posterior Conditioned on the Safe Prior

\[
P(b'|b, a, 1 \{ x \in \mathcal{X}^{\text{safe}} \}) = \int_{z \in z} \mathbb{P}(b'|b, a, z, 1 \{ x \in \mathcal{X}^{\text{safe}} \}) \mathbb{P}(z|a, b, 1 \{ x \in \mathcal{X}^{\text{safe}} \}) dz = \tag{59}
\]

\[
\int_{z \in z} \delta(b' - \psi(b^{\text{safe}}, a, z)) \mathbb{P}(z|a, b, 1 \{ x \in \mathcal{X}^{\text{safe}} \}) dz
\]

We first calculate the propagated belief conditioned on the safe prior.

\[
\mathbb{P}(x'|b, a, 1 \{ x \in \mathcal{X}^{\text{safe}} \}) = \frac{\int_{x \in X} 1 \{ x \in \mathcal{X}^{\text{safe}} \} \mathbb{P}_T(x'|x, a)b(x)dx}{\int_{x \in X} 1 \{ x \in \mathcal{X}^{\text{safe}} \} b(x)dx} \tag{60}
\]

\(b\) and event safe, meaning that belief supposed to be zero at non safe places. Finally,

\[
\mathbb{P}(z|a, b, 1 \{ x \in \mathcal{X}^{\text{safe}} \}) = \int_{x \in X'} \mathbb{P}_Z(z|x') p(x'|b, a, 1 \{ x \in \mathcal{X}^{\text{safe}} \}) dx'. \tag{61}
\]

We can also look at the above from slightly different angle. We define \(b^{\text{safe}}\)

\[
b^{\text{safe}}(x) = \frac{1 \{ x \in \mathcal{X}^{\text{safe}} \} b(x)}{\int_{x \in X} 1 \{ \xi \in \mathcal{X}^{\text{safe}} \} b(\xi)} \tag{62}
\]

Now the \(\psi\) is conventional belief update operator receiving as input \(\psi(b^{\text{safe}}, a, z')\).

**Appendix C**  Proof that Mean Distance to Goal Accounts for Uncertainty

In this section, we discuss in depth why the mean distance to goal intrinsically accounts for belief uncertainty. We show a geometrical visualization in Fig. 6. Since the distance is no negative, the less spread of belief implies lower distances and vise versa. Further, let us show that algebraically.
The goal
belief particles

(a)

(b)

(c)

(d)

Figure 6: Geometrical visualization of the natural belief uncertainty measure imprinted in the mean distance to the goal. (a) Less spread results in lowering all the distances, thereby the mean. (b) The reciprocal situation. (c) Another situation, here, to decrease the mean distance to the goal, one has to reduce the spread and the distance between the expected value of the belief and the goal. (d) The spread is decreased, but the distance between the expected value of the belief and the goal is large (Appendix C).

Theorem 1. Let $y$ be an arbitrary distributed random vector with $\mu_y$ and $\Sigma_y$ being the expected value and covariance matrix of $y$, respectively; and let $\Lambda$ be arbitrary matrix. The following relation is correct

$$
\mathbb{E} [y^T \Lambda y] = \text{tr}(\Lambda \Sigma_y) + \mu_y^T \Lambda \mu_y,
$$

where by $\text{tr}$ we denote the trace operator.

Proof. Since the quadratic form is a scalar quantity, $y^T \Lambda y = \text{tr}(y^T \Lambda y)$. Next, by the cyclic property of the trace operator,

$$
\mathbb{E} [\text{tr}(y^T \Lambda y)] = \mathbb{E} [\text{tr}(\Lambda y y^T)].
$$

(64)

Since the trace operator is a linear combination of the components of the matrix, it therefore follows from the linearity of the expectation operator that

$$
\mathbb{E} [\text{tr}(\Lambda y y^T)] = \text{tr}(\Lambda \mathbb{E}(y y^T)).
$$

(65)

A standard property of variances then tells us that this is

$$
\text{tr}(\Lambda (\Sigma_x + \mu_x \mu_x^T)).
$$

(66)

Applying the cyclic property of the trace operator again, we get

$$
\text{tr}(\Lambda \Sigma_y) + \text{tr}(\Lambda \mu_y \mu_y^T) = \text{tr}(\Lambda \Sigma_y) + \text{tr}(\mu_y^T \Lambda \mu_y) = \text{tr}(\Lambda \Sigma_y) + \mu_y^T \Lambda \mu_y.
$$

(67)

Now, we set $y = x - x^g$, where $x \sim \mathcal{b}$ with the mean $\mu_x$ and covariance matrix $\Sigma_x$; $x^g$ is the deterministic goal location. Recall that covariance matrix is invariant to the deterministic translational shifts of a random vector, so $\Sigma_{x-x^g} = \Sigma_x$. Moreover, by setting $\Lambda = I$ we obtain

$$
\mathbb{E} [y^T \Lambda y] = \mathbb{E} [(x - x^g)^T I (x - x^g)] = \mathbb{E} [\|x - x^g\|^2_2] = \text{tr}(\Sigma_x) + \|\mu_x - x^g\|^2_2.
$$

(68)

We arrived at the desired result. As we observe in Fig. 6 the trace of the covariance matrix controls the spread of the belief in the first summand; the second summand is the distance between the expected value of the belief.
References

[1] Mauricio Araya, Olivier Buffet, Vincent Thomas, and François Charpillet. A pomdp extension with belief-dependent rewards. In Advances in Neural Information Processing Systems (NIPS), pages 64–72, 2010. 1

[2] M. Barenboim and V. Indelman. Adaptive information belief space planning. In the 31st International Joint Conference on Artificial Intelligence and the 25th European Conference on Artificial Intelligence (IJCAI-ECAI), July 2022. 7

[3] A. Bry and N. Roy. Rapidly-exploring random belief trees for motion planning under uncertainty. In IEEE Intl. Conf. on Robotics and Automation (ICRA), pages 723–730, 2011. 2, 3, 5, 8

[4] Cesar Cadena, Luca Carlone, Henry Carrillo, Yasir Latif, Davide Scaramuzza, Jose Neira, Ian D Reid, and John J Leonard. Simultaneous localization and mapping: Present, future, and the robust-perception age. IEEE Trans. Robotics, 32(6):1309 – 1332, 2016. 3

[5] Yinlam Chow, Aviv Tamar, Shie Mannor, and Marco Pavone. Risk-sensitive and robust decision-making: a cvar optimization approach. Advances in Neural Information Processing Systems, 28:1522–1530, 2015. 3

[6] Louis Dressel and Mykel J. Kochenderfer. Efficient decision-theoretic target localization. In Laura Barbuiescu, Jeremy Frank, Mausam, and Stephen F. Smith, editors, Proceedings of the Twenty-Seventh International Conference on Automated Planning and Scheduling, ICAPS 2017, Pittsburgh, Pennsylvania, USA, June 18-23, 2017, pages 70–78. AAAI Press, 2017. 1

[7] Mathieu Fehr, Olivier Buffet, Vincent Thomas, and Jilles Dibangoye. rho-pomdps have lipschitz-continuous epsilon-optimal value functions. In S. Bengio, H. Wallach, H. Larochelle, K. Grauman, N. Cesa-Bianchi, and R. Garnett, editors, Advances in Neural Information Processing Systems 31, pages 6933–6943. Curran Associates, Inc., 2018. 3

[8] Astghik Hakobyan and Insoon Yang. Wasserstein distributionally robust motion planning and control with safety constraints using conditional value-at-risk. In 2020 IEEE International Conference on Robotics and Automation (ICRA), pages 490–496. IEEE, 2020. 5

[9] V. Indelman, L. Carlone, and F. Dellaert. Planning in the continuous domain: a generalized belief space approach for autonomous navigation in unknown environments. Intl. J. of Robotics Research, 34(7):849–882, 2015. 1

[10] Michael Kearns, Yishay Mansour, and Andrew Y Ng. A sparse sampling algorithm for near-optimal planning in large markov decision processes. Machine learning, 49(2):193–208, 2002. 1, 6, 7, 11

[11] M. Kochenderfer, T. Wheeler, and K. Wray. Algorithms for Decision Making. MIT Press, 2022. 1

[12] Jongmin Lee, Geon-Hyeong Kim, Pascal Poupart, and Kee-Eung Kim. Monte-carlo tree search for constrained pomdps. Advances in Neural Information Processing Systems, 31, 2018. 2

[13] C. Papadimitriou and J. Tsitsiklis. The complexity of Markov decision processes. Mathematics of operations research, 12(3):441–450, 1987. 1

[14] Julio A Placed and José A Castellanos. Enough is enough: Towards autonomous uncertainty-driven stopping criteria. arXiv preprint arXiv:2204.10631, 2022. 1, 5

[15] Pedro Santana, Sylvie Thiebaux, and Brian Williams. Rao*: An algorithm for chance-constrained pomdps. In Proceedings of the AAAI Conference on Artificial Intelligence, volume 30, 2016. 2, 3, 5, 8, 9, 11

[16] David Silver and Joel Veness. Monte-carlo planning in large pomdps. In Advances in Neural Information Processing Systems (NIPS), pages 2164–2172, 2010. 1

[17] Zachary Sunberg and Mykel Kochenderfer. Online algorithms for pomdps with continuous state, action, and observation spaces. In Proceedings of the International Conference on Automated Planning and Scheduling, volume 28, 2018. 1
[18] Zachary Sunberg and Mykel J. Kochenderfer. POMCPow: an online algorithm for pomdps with continuous state, action, and observation spaces. CoRR, abs/1709.06196, 2017.

[19] Ori Sztyglic, Andrey Zhitnikov, and Vadim Indelman. Simplified belief-dependent reward mcts planning with guaranteed tree consistency. arXiv preprint arXiv:2105.14239, 2021.

[20] S. Thrun, W. Burgard, and D. Fox. Probabilistic Robotics. The MIT press, Cambridge, MA, 2005.

[21] Nan Ye, Adhiraj Somani, David Hsu, and Wee Sun Lee. Despot: Online pomdp planning with regularization. JAIR, 58:231–266, 2017.

[22] A. Zhitnikov and V. Indelman. Simplified risk aware decision making with belief dependent rewards in partially observable domains. Artificial Intelligence, Special Issue on “Risk-Aware Autonomous Systems: Theory and Practice”, 2022.