Dependence on parameters of CW globalizations of families of Harish-Chandra modules and the meromorphic continuation of $C^\infty$ Eisenstein series

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Abstract

The first main result is that the Casselman-Wallach Globalization of a real analytic family of Harish-Chandra modules is continuous in the parameter and the family of globalizations is holomorphic if the family of Harish-Chandra modules is holomorphic. Our proof of this result uses results from the thesis of Vincent van der Noort in several critical ways. In his thesis the holomorphic result was proved in the case when the parameter space is a one dimensional complex manifold up to a branched covering. The second main result is a proof of the meromorphic continuation of $C^\infty$ Eisenstein series using Langlands’ results in the $K$ finite case as an application of the methods in the proof of the first part.

1 Introduction

The purpose of this article is to extend my work on smooth Fréchet globalizations of Harish-Chandra modules to include parameters. In [BK] it is asserted that their work carries that goal out. This may be so, but I could not find a proof of any case but the spherical principal series where the result is a simple consequence of the uniqueness of the smooth Fréchet globalization of moderate growth. Also, beyond the brilliant handling of the spherical
principal series in [BK] their proof of the Casselman-Wallach (CW) Theorem follows the main thread in [RRG] (almost verbatim) which does not take into account parameters. Also, non-linear groups do not appear in [BK] (e.g. the metaplectic group). In this paper a different tactic is taken to this problem. We approach it from the perspective of the excellent thesis of Vincent van der Noort who studies the question: Given an analytic family of Harish-Chandra modules, how does the corresponding family of CW completions depend on the parameter? The CW completion was first realized in terms of imbedding into parabolically induced representations. This paper considers another class of Harish-Chandra modules that were first studied in a special case in [HOW]. For lack of a name they were called J–modules. These Harish-Chandra modules are constructed using a free subalgebra of the center of the enveloping algebra generated by the split rank number of independent elements that was first studied in [HOW]. This algebra is denoted $D$ in this paper. In the category of Harish-Chandra modules with $D$ action by a fixed character the J–modules in the category are projective. Furthermore, every Harish-Chandra module has a resolution by J–modules. Much of the paper, involves analyzing the CW globalizations of families of J–modules using a key results of van der Noort, which also play an important role in other aspects of the paper. For the sake of completeness a complete proof of these results is included.

In van der Noort’s thesis the parametrization studied were holomorphic and results were proved about holomorphic dependence of CW globalizations. He essentially solved the problem in the case when the parameter space is one complex dimension modulo the possible necessity to go to a branched covering. In this paper I prove that if the dependence of the Harish-Chandra modules in the parameters is real analytic then the dependence of the CW completion is continuous (Corollary 41) and if it is holomorphic it is holomorphic (Corollary 44).

The final section of the paper gives a proof of the meromorphic continuation of $C^\infty$ Eisenstein series using the continuation of $K$–finite Eisenstein series in Langlands [L2], Chapter 7. This is done by reducing the problem to a general result on the holomorphic dependence of the extensions of what we call linear functionals of locally uniform moderate growth on holomorphic families of Harish-Chandra modules underlying holomorphic families of smooth Fréchet representations of moderate growth. For the reader whose only interest is in the continuation of $C^\infty$ Eisenstein series a more direct proof is sketched.
2 The subalgebra $D$ of $Z(\mathfrak{g})$

Let $G$ be a real reductive group of inner type. That is, if $\mathfrak{g} = \text{Lie}(G)$, $\mathfrak{g}_\mathbb{C} = \mathfrak{g} \otimes \mathbb{C}$ then $\text{Ad}(G)$ is contained in the identity component of $\text{Aut}(\mathfrak{g}_\mathbb{C})$. Let $K$ be a maximal compact subgroup of $G$ and let $\theta$ denote the corresponding Cartan involution of $G$ (and of $\mathfrak{g}$). On Set $\mathfrak{k} = \text{Lie}(K)$ and $\mathfrak{p} = \{X \in \mathfrak{g}|\theta X = -X\}$ let $p$ be the projection of $\mathfrak{g}_\mathbb{C}$ onto $\mathfrak{p}_\mathbb{C}$ corresponding to $\mathfrak{g}_\mathbb{C} = \mathfrak{k}_\mathbb{C} \oplus \mathfrak{p}_\mathbb{C}$. Fix a symmetric $\text{Ad}(G)$–invariant bilinear form, $B$, on $\mathfrak{g}$ such that $B|_{\mathfrak{k}}$ is negative definite and $B|_{\mathfrak{p}}$ is positive definite Extend $p$ to a homomorphism of $S(\mathfrak{g}_\mathbb{C})$ onto $S(\mathfrak{p}_\mathbb{C})$. Then $p$ is the projection corresponding to $S(\mathfrak{g}_\mathbb{C}) = S(\mathfrak{p}_\mathbb{C}) \oplus S(\mathfrak{g}_\mathbb{C})|_K$.

In [HOW] we found homogeneous elements $w_1, \ldots, w_l$ of $S(\mathfrak{g}_\mathbb{C})^G$ with $w_1 = \sum v_i^2$ with $\{v_1, \ldots, v_n\}$ orthonormal basis of $\mathfrak{g}_\mathbb{C}$ with respect $B$. Satisfying the following properties

1. $p(w_1), \ldots, p(w_l)$ are algebraically independent.

2. There exists a finite dimensional homogeneous subspace $E$ of $S(\mathfrak{p}_\mathbb{C})^K$ such that the map $\mathbb{C}[p(w_1), \ldots, p(w_l)] \otimes E \rightarrow S(\mathfrak{p}_\mathbb{C})^K$ given by multiplication is an isomorphism.

If $\mathfrak{g}_\mathbb{C}$ contains no simple ideals of type $E$ one can take $E = \mathbb{C}1$. If $\mathfrak{g}$ is split over $\mathbb{R}$ then $\mathbb{C}[w_1, \ldots, w_l] = S(\mathfrak{g}_\mathbb{C})^G$.

Let $\mathcal{H}$ denote the space of harmonic elements of $S(\mathfrak{p}_\mathbb{C})$, that is, the orthogonal complement to the ideal $S(\mathfrak{p}_\mathbb{C}) (S(\mathfrak{p}_\mathbb{C}) \mathfrak{p})^K$ in $S(\mathfrak{p}_\mathbb{C})$ relative to the Hermitian extension of inner product $B|_{\mathfrak{p}}$. Then the Kostant-Rallis theorem ([KR]) implies that the map

$$\mathcal{H} \otimes S(\mathfrak{p}_\mathbb{C})^K \rightarrow S(\mathfrak{p}_\mathbb{C})$$

given by multiplication is a linear bijection. This and 2. easily imply

**Lemma 1** The map

$$\mathcal{H} \otimes E \otimes \mathbb{C}[w_1, \ldots, w_l] \otimes S(\mathfrak{t}_\mathbb{C}) \rightarrow S(\mathfrak{g}_\mathbb{C})$$

given by multiplication is a linear bijection.

Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$ and let

$$W = \{s \in GL(\mathfrak{a})|s = \text{Ad}(k)|_\mathfrak{a}, k \in K\}.$$
Let \( H \in a \) be such that \( a = \{ X \in p \mid [H, X] = 0 \} \). If \( \lambda \in \mathbb{R} \) then set \( g^\lambda = \{ X \in g \mid [H, X] = \lambda X \} \). Set \( n = \bigoplus_{\lambda > 0} g_\lambda \) and \( \bar{n} = \theta n = \bigoplus_{\lambda > 0} g_{-\lambda} \). Then

\[
p = p(n) \oplus a
\]

and \( p(n) \) is the orthogonal complement to \( a \) in \( p \) relative to \( B \). Let \( q \) be the projection of \( p \) onto \( a \) corresponding to this decomposition. Then the Chevalley restriction theorem implies that

\[
q : S(p)^K \to S(a)^W
\]

is an isomorphism of algebras. Also, as above, if \( H \) is the orthogonal complement to \( (S(a)a)^W S(a) \) in \( S(a) \). Then the map

\[
S(a)^W \otimes H \to S(a)
\]

given by multiplication is a linear bijection. Putting these observations together the map

\[
S(n) \otimes S(a)^W \otimes H \otimes S(t) \to S(g)
\]

given by multiplication is a linear bijection. We also note that the map

\[
\mathbb{C}[w_1, \ldots, w_l] \otimes E \to S(a)^W
\]

given by

\[
w \otimes e \mapsto q(p(w))q(e)
\]

is a linear bijection. This in turn implies

**Lemma 2** The map

\[
S(n) \otimes \mathbb{C}[w_1, \ldots, w_l] \otimes E \otimes H \otimes S(t) \to S(g)
\]

given by multiplication is a linear bijection.

Let \( \text{symm} \) denote the symmetrization map from \( S(g_C) \) to \( U(g_C) \) then \( \text{symm} \) is a linear bijection and \( \text{symm} \circ \text{Ad}(g) = \text{Ad}(g) \circ \text{symm} \). Let \( Z(g_C) = U(g_C)^G \) denote the center of \( U(g_C) \). Set \( z_i = \text{symm}(w_i) \) and

\[
D = \mathbb{C}[z_1, \ldots, z_l].
\]

Note that if \( S_j(g_C) = \sum_{k \leq j} S^j(g_C) \) and if \( U^j(g_C) \subset U^{j+1}(g_C) \) is the standard filtration of \( U(g_C) \) then

\[
\text{symm}(S_j(g_C)) = U^j(g_C).
\]

The above and standard arguments ([HOW] Theorem 2.5 and Lemma 5.2) imply
Theorem 3 Let the notation be as above. Then
1. The map
\[ \mathcal{H} \otimes E \otimes D \otimes U(\mathfrak{k}_C) \to U(\mathfrak{g}_C) \]
given by
\[ h \otimes e \otimes D \otimes k \mapsto \text{symm}(h)\text{symm}(e)Dk \]
is a linear bijection.
2. The map
\[ U(n_C) \otimes E \otimes H \otimes D \otimes U(\mathfrak{k}_C) \to U(\mathfrak{g}_C) \]
given by
\[ n \otimes e \otimes h \otimes D \otimes k \mapsto n\text{symm}(e)\text{symm}(h)Dk \]
is a linear bijection.

3 A class of admissible finitely generated \((\mathfrak{g}, K)\)-modules

Retain the notation in the preceding section. Note that Theorem 3 implies that the subalgebra \(DU(\mathfrak{k}_C)\) of \(U(\mathfrak{g}_C)\) is isomorphic with the tensor product algebra \(D \otimes U(\mathfrak{k}_C)\) and that \(U(\mathfrak{g}_C)\) is free as a right \(DU(\mathfrak{k}_C)\) under multiplication. If \(R\) is a \(DU(\mathfrak{k}_C)\)-module then form
\[ J(R) = U(\mathfrak{g}_C) \otimes_{DU(\mathfrak{k}_C)} R. \]

Denote by \(H(\mathfrak{g}, K)\) the Harish–Chandra category of admissible finitely generated \((\mathfrak{g}, K)\)-modules. Let \(R\) be a finite dimensional continuous \(K\)-module that is also a \(D\)-module and the actions commute then \(K\) acts on \(J(R)\) as follows:
\[ k \cdot (g \otimes r) = Ad(k)g \otimes kr, k \in K, g \in U(\mathfrak{g}_C), r \in R. \]
Then as a \(K\)-module
\[ J(R) \cong \mathcal{H} \otimes E \otimes R \]
with \(K\) acting trivially on \(E\). Note that \(J(R) \in H(\mathfrak{g}, K)\) since the multiplicities of \(K\)-types in \(\mathcal{H}\) are finite and \(J(R)\) is clearly finitely generated as a \(U(\mathfrak{g}_C)\)-module. Let \(W(D, K)\) be the category of finite dimensional \((D, K)\)-modules with \(K\) acting continuously and the action of \(D\) and \(K\) commute.
Lemma 4 $R \rightarrow J(R)$ defines an exact faithful functor from the category $W(K, D)$ to $H(\mathfrak{g}, K)$.

Proof. This follows since $U(\mathfrak{g}_C)$ is free as a module for $D U(\mathfrak{t}_C)$ under right multiplication. □

As usual, denote the set of equivalence classes of irreducible, finite-dimensional, continuous representations of $K$ by $\hat{K}$. If $V \in H(\mathfrak{g}, K)$ set $V(\gamma)$ equal to the sum of all irreducible $K$–subrepresentations of $V$ in the class of $\gamma$. Then $V(\gamma)$ is invariant under the action of $Z(\mathfrak{g}_C)$ hence under the action of $D$.

If $V \in H(\mathfrak{g}, K)$ there is a finite subset $F \subset \hat{K}$ such that

$$U(\mathfrak{g}_C) \sum_{\gamma \in F} V(\gamma).$$

Set $R = \sum_{\gamma \in F} V(\gamma) \in W(D, K)$ and one has the canonical $(\mathfrak{g}, K)$–module surjection $J(R) \rightarrow V$ given by $g \otimes r \mapsto gr$. A submodule of an element of $H(\mathfrak{g}, K)$ is in $H(\mathfrak{g}, K)$ so

Proposition 5 If $V \in H(\mathfrak{g}, K)$ then there exists a sequence of elements $R_j \in W(\mathfrak{g}, K)$ and an exact sequence in $H(\mathfrak{g}, K)$

$$\cdots \rightarrow J(R_k) \rightarrow \cdots \rightarrow J(R_2) \rightarrow J(R_1) \rightarrow J(R_0) \rightarrow V \rightarrow 0.$$

Notice that this exact sequence us a free resolution of $V$ as a $U(n)$–module.

Let $\beta : D \rightarrow \mathbb{C}$ be an algebra homomorphism. Let $H(\mathfrak{g}, K)_\beta$ be the full subcategory of $H(\mathfrak{g}, K)$ consisting of modules $V$ such that if $z \in D$ then it acts by $\beta(z)I$. The next result is an aside that will not be used in the rest of this paper and is a simple consequence of the definition of projective object.

Lemma 6 Let $F$ be a finite dimensional $K$–module and let $D$ act on $F$ by $\beta(z)I$ yielding an object $R \in W(K, D)$. Then $J(R)$ is projective in $H(\mathfrak{g}, K)_\beta$.

4 The objects in $W(K, D)$

If $R \in W(K, D)$ then $R$ has an isotypic decomposition $R = \bigoplus_{\gamma \in \hat{K}} R(\gamma)$. Only a finite number of the $R(\gamma) \neq 0$. If $D \in D$ then $D R(\gamma) \subset R(\gamma)$ for all $\gamma \in \hat{K}$. If $\chi : D \rightarrow \mathbb{C}$ is an algebra homomorphism then we set
$R_{\chi} = \{ v \in R | (D - \chi(D))^k v = 0 \text{ for some } k > 0 \}$ Then setting $ch(D)$ equal to the set of all algebra homomorphisms of $D$ to $C$ we have the decomposition

$$R = \bigoplus_{\gamma \in \hat{K}, \chi \in ch(D)} R_{\chi}(\gamma).$$

Fix a $K$–module $(\tau_\gamma, F_\gamma) \in \gamma$. Then $R_{\chi}(\gamma)$ is isomorphic with

$$\text{Hom}_K(V_\gamma, R_{\chi}) \otimes F_\gamma$$

with $K$ acting on $F_\gamma$ and $D$ acting on $\text{Hom}_K(V_\gamma, R)$. If $R$ is an irreducible object in $W(K, D)$ then Schur’s lemma implies that $D$ acts by a single homomorphism to $C$ and $R$ is irreducible as a $K$–module. Set $V_\chi$, equal to the module with $D$ acting by $\chi$ and $K$ acting by an element of $\gamma$.

We next analyze the homomorphisms $\chi$. Let $\chi$ be such a homomorphism then $\chi(z_i) = \lambda_i \in C$. Thus one simple parametrization is by $(\lambda_1, \ldots, \lambda_l) \in C^l$. We use the notation $\beta_\lambda$ for the homomorphism such that $\beta_\lambda(z_i) = \lambda_i$. An alternate parametrization is through the Harish-Chandra homomorphism. Recall the exact sequence (c.f. [RRG], Theorem 3.6.6)

$$0 \rightarrow (U(g_C)p_C)^K \rightarrow U(g_C)^K \rightarrow U(a_C)^W \rightarrow 0.$$

It is standard that the linear map $\gamma \circ \text{symm} : S(p_C)^K \rightarrow U(a_C)^W$ is a linear bijection. This and the definition of $D$ imply that $U(a_C)^W$ is finitely generated as a $\gamma(D)$–module. This in turn implies that $U(a_C)$ is finitely generated as a $\gamma(D)$–module. Thus we have a morphism $\varphi : a_C^* \rightarrow C^l$ such that $\gamma(D)(\nu)$ is the homomorphism $z_i \mapsto \varphi_i(\nu)$. Hence $\gamma(D)(\nu) = \beta_\varphi(\nu)$ for $\nu \in a_C^*$. Set $\chi_{\nu|D} = \beta_\varphi(\nu)$.

**Definition 7** Let $X$ be a complex or real analytic manifold. An analytic family in $W(K, D)$ based on $X$ is a pair $(\mu, V)$ of a a finite dimensional continuous $K$–module, $V$, and a $\mu : X \times D \rightarrow \text{End}(V)$ such that $D \rightarrow \mu(x, D)$ is a representation of $D$ on $V$ and $x \mapsto \mu(x, D)$ is analytic for all $D \in D$.

## 5 Analytic families of $J$–modules

Throughout this section analytic will mean complex analytic in the context of a complex analytic manifold and real analytic in the contest of a real analytic manifold. Theorem 3 implies
Corollary 8 Let $R \in W(K,D)$ then
\[ J(R)/n^{k+1}J(R) \cong U(n)/n^{k+1}U(n) \otimes E \otimes H \otimes R_M \]
as an $(n,M)$-module with $n$ and $M$ acting trivially on $E \otimes H$ and $n$ acting trivially on $R$.

Let $(\mu, V)$ be an analytic family of objects in $W(K,D)$ based on $X$. Let $V_x, x \in X$ be the object in $W(K,D)$ with $K$ acting by its action on $V$ and $D$ action by $\mu_x = \mu(x, \cdot)$.

Theorem 9 Notation as above. Let $\sigma_{k,x}$ be the action of $a$ on
\[ U(n)/n^{k+1}U(n) \otimes E \otimes H \otimes V_x|_M \]
under the identification
\[ J(V_x)/n^{k+1}J(V_x) \cong U(n)/n^{k+1}U(n) \otimes E \otimes H \otimes V_x|_M. \]
If $u \in U(\mathfrak{g}_C)$ then the map $x \to \sigma_{k,x}(u)$ is an analytic map.

Proof. Theorem 3 implies that if $X_1, ..., X_m$ is a basis of $n$, $Y_1, ..., Y_n$ is a basis of $\mathfrak{t}$ and $h_1, ..., h_r$ is a basis for $\text{symm}(E)\text{symm}(H)$ then if $I, J, L$ are multi-indices of size $m, n, l$ respectively then
\[ X^J z^L h_i Y^L \]
is a basis of $U(\mathfrak{g}_C)$. Here, as usual, $X^J = X_1^{j_1} \cdots X_m^{j_m}$, ... This implies that if $u \in U(\mathfrak{g}_C)$ then
\[ uX^J z^L h_i Y^L = \sum a_{I, L, i, j, I, L, i, j} (u) X^J z^L h_i Y^L. \]
This implies that if we take a basis $v_1, ..., v_d$ of $V$ then the elements $X^J h_i \otimes v_j$ form a basis of $J(V_x)$. Thus if $u \in U(\mathfrak{g}_C)$ then
\[ uX^J h_i \otimes v_j = \sum \mu_x (z^L i) Y^L h_i \otimes \mu_x (z^L i) Y^L v_j = \sum a_{I, L, i, j, I, L, i, j} (u) X^J h_i \otimes \mu_x (z^L i) Y^L v_j = \]
The theorem follows from this formula. \hfill \blacksquare

If $X$ is a complex manifold or a real analytic manifold and $V$ is a vector space over $\mathbb{C}$ then a map $\phi : X \to V$ is said to be holomorphic, real analytic or continuous if for each $x \in X$ there exists a open neighborhood, $U$, of $X$ such that if $Z = \text{Span}_{\mathbb{C}} \{ \phi(x) | x \in U \}$ then $\dim Z < \infty$ and $\phi : U \to Z$ is holomorphic, real analytic or continuous respectively.
Definition 10 Let $X$ be a complex or real analytic manifold. Then an holomorphic, analytic or continuous family of admissible $(\mathfrak{g}, K)$–modules based on $X$ is a pair, $(\mu, V)$, of an admissible $(\mathfrak{k}, K)$–module, $V$, and

$$\mu : X \times U(\mathfrak{g}) \to \text{End}(V)$$

such that $x \mapsto \mu(x, y)v$ is holomorphic (resp. analytic, resp. continuous) for all $y \in U(\mathfrak{g})$, $v \in V$ and if we set $\mu_x(y) = \mu(x, y)$ for $y \in U(\mathfrak{g})$ then $(\mu_x, V)$ is an admissible finitely $(\mathfrak{g}, K)$–module. It will be called a family of objects in $H(\mathfrak{g}, K)$ if each $(\mu_x, V)$ is finitely generated.

Theorem 11 Let $X$ be an analytic or complex manifold. Let $(\lambda, R)$ be an family of objects in $W(K, D)$ based on $X$ and define $R_x \in W(K, D)$ to be the module with action $\lambda_x(\cdot, \cdot)$. Let $V = H \otimes E \otimes R$ (acting by the tensor product action with its action on $E$ trivial) and let $T_x : V \to J(R_x)$ be given by $T_x(h \otimes e \otimes r) = \alpha_x(\text{symm}(h) e) (1 \otimes r)$ with $\alpha_x$ the action of $U(U(\mathfrak{g}_C))$ on $J(R_x)$. If $\lambda(x, y) = T^{-1}_x \alpha_x(y) T_x$ then $(\lambda, V)$ is an analytic family of objects in $H(\mathfrak{g}, K)$ based on $X$.

Proof. We argue as in the proof of Theorem 9. Let $\{h_i\}$ be a basis of $H$ such that for each $i$ there exists $\gamma \in \hat{K}$ such that $h_i \in H(\gamma)$, let $e_j$ be a basis of $E$, let $r_m$ be a basis of $R$ and let $Y_1, ..., Y_n$ be a basis of $\mathfrak{k}$. Then if $y \in U(\mathfrak{g}_C)$

$$y \text{symm}(h_i) e_j z^L Y^J = \sum_{i,j_1,j_1,L_1} b_{i_1,j_1,j_1,L_1} \text{symm}(h_{i_1}) e_{j_1} z^{L_1} Y_{K_1}.$$ 

Thus

$$T^{-1}_x \alpha_x(y) T_x(h_i \otimes e_j \otimes r_k) = \sum b_{i,j_1,j_1,j_0} (y) h_{i_1} \otimes e_{j_1} \otimes \lambda_x(z^{L_1}) Y^{J_1} r_k.$$ 

The theorem follows.  

Next we define another type of analytic family. Let $A$ and $N$ be the connected subgroups of $G$ with $\text{Lie}(A) = \mathfrak{a}$ and $\text{Lie}(N) = \mathfrak{n}$. Let $M$ be the centralizer of $\mathfrak{a}$ in $K$. Set $Q = MAN$ then $Q$ is a minimal parabolic subgroup of $G$. 

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**Definition 12** An analytic family of finite dimensional $Q$–modules based on the manifold (real or complex analytic) $X$ is a pair $(\sigma, S)$ with $S$ a finite dimensional continuous $M$–module and a real analytic map $\sigma : X \times Q \to GL(S)$ such that $x \mapsto \sigma(x, q)$ is holomorphic and $\sigma(x, \cdot) = \sigma_x$ is a representation of $Q$.

Let $(\sigma, S)$ be a continuous representation of $Q$. Set $I^\infty(\sigma|_M)$ equal to the space of all smooth $f : K \to S$ satisfying $f(mk) = \sigma(m)f(k)$. Define and action $\pi_\sigma$ of $G$ on $I^\infty(\sigma|_M)$ as follows: if $f \in I^\infty(\sigma|_M)$ then extend $f$ to $G$ by $f_\sigma(qk) = \sigma(q)f(k)$, then, since $K \cap Q = M$ and $QK = G$, $f_\sigma$ is $C^\infty$ on $G$ set $\pi_\sigma(g)f(k) = f_\sigma(kg)$. Also set

$$\pi_\sigma(Y)f(k) = \frac{d}{dt}f_\sigma(k \exp ty)|_{t=0}$$

for $Y \in g$ and $k \in K, f \in I^\infty(\sigma|_M)$. Let $I(\sigma_M)$ be the space of all right $K$ finite elements of $I^\infty(\sigma|_M)$.

Put and $M$–invariant inner product, $\langle \cdot, \cdot \rangle$ on $S$. If $f, h \in I^\infty(\sigma|_M)$ then set

$$\langle f, h \rangle = \int_K \langle f(k), h(k) \rangle dk$$

with $dk$ normalized invariant measure on $K$. The following is standard.

**Proposition 13** Let $(\sigma, S)$ be an analytic family of finite dimensional representations of $Q$ based on the complex or real analytic manifold $X$. Set $\lambda(x, y) = \pi_\sigma(x)(y)$ for $x \in X, y \in U(g_C)$. If $\mu$ is the common value of $\sigma_x|_M$, then $(\lambda, I(\mu))$ is an analytic family of objects in $H(g, K)$ based on $X$.

**Proof.** It is standard that

$$x, g \mapsto \langle \pi_\sigma_x(g)f, h \rangle$$

is real analytic and holomorphic in $x$ for $f, g \in I(\mu)$. □

**Definition 14** If $(\lambda, V)$ and $(\mu, W)$ are analytic families of objects in $H(g, K)$ based on the manifold $X$ then a homomorphism of the analytic (resp. continuous) family $(\lambda, V)$ to $(\mu, W)$ is a map

$$T : X \to \text{Hom}_C(V, W)$$

such that

1. $x \mapsto T(x)v$ is an analytic map of $X$ to $W$ for all $v \in V$.
2. $T(x) \in \text{Hom}_{H(g,K)}(V_x, W_x)$ with $V_x = (\lambda_x, V), W_x = (\mu_x, W)$. 

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If \( R \in W(K, D) \) the space \( J(R)/n^{s+1}J(R) \) has a natural structure of an \( M \) module and an \( n+a \) module. Since \( \dim J(R)/n^{s+1}J(R) < \infty \) and \( AN \) is a simply connected Lie group \( J(R)/n^{s+1}J(R) \) has a natural structure of a finite dimensional continuous \( Q \)–module with action \( \sigma_{s,R} \). Let \( p_s \) denote the natural surjection
\[
p_s : J(R) \to J(R)/n^sJ(R).
\]
If \( k \in K, v \in J(R) \), define \( S_{s,R}(v)(k) = p_{s,R}(kv) \), then \( S_{s,R} \in \text{Hom}_{H(g,K)}(J(R), (\pi\sigma_{s,R}, I(\sigma_{s,R}|_M))) \). Combining the above results we have

**Theorem 15** Let \( (\mu, R) \) be an analytic (resp. continuous) family in \( W(K, D) \) based on the manifold \( X \). Let \( (\lambda, V) \) be the analytic family (as in Theorem 11) corresponding to \( x \to J((\mu_x, R)) \). Then recalling that \( V = \mathcal{H} \otimes E \otimes R \) define \( T_s(x)(h \otimes e \otimes r) = S_{s,R}(\text{symm}(h)e \otimes r) \). Then \( T_s \) defines a homomorphism of the analytic family \( (\lambda, V) \) to \( (\xi, I(\sigma_{s,R_x}|_M)) \) with \( \xi(x, y) = \pi\sigma_{s,R_x}(y) \) and \( \sigma_{s,R_x} \) is defined as in Theorem 9.

We will use the notation \( J(R) \) for the analytic family associated with \( x \to J((\mu_x, R)) \).

### 6 Some results of Vincent van der Noort

Throughout this section \( Z \) will denote a connected real or complex analytic manifold. We will use the terminology analytic to mean complex analytic or real analytic depending on the context.

We continue the notation of the previous sections. In particular \( G \) is a real reductive group of inner type.

We denote (as is usual) the standard filtration of \( U(g) \), by
\[
... \subset U^j(g) \subset U^{j+1}(g) \subset ...
\]
Let \( V \) be an admissible \((\text{Lie}(K), K)\) module. We note that if \( E \subset V \) is a finite dimensional \( K \)–invariant subspace of \( V \) then there exists a finite subset \( F_{j,E} \subset K \) such that
\[
U^j(g) \otimes E \cong \sum_{\gamma \in F_{j,E}} m_{\gamma,j}V_\gamma.
\]
If \( v \in V \) we denote by \( E_v \) the span of \( Kv \) in \( V \).
The purpose of this section is to prove a theorem of van der Noort which first appeared in his thesis [VdN]. Our argument follows his original line with a few simplifications. We include the details only because he is not expected to publish it. In his thesis he emphasized the holomorphic case.

Fix a maximal torus, $T$, of $M$ then $\text{Lie}(T) \oplus \mathfrak{a}$ is a Cartan subalgebra of $\mathfrak{g}$. Set $\mathfrak{h}$ equal to its complexification. We parametrize the homomorphisms of $Z(\mathfrak{g})$ to $\mathbb{C}$ by $\chi_{\Lambda}$ for $\Lambda \in \mathfrak{h}^*$ using the Harish–Chandra parametrization. Endow $\hat{M}$ with the discrete topology. Then we note that if $C$ is a compact subset of $\hat{M} \times a_C^*$ then there exist a finite number of elements $\xi_1, \ldots, \xi_r \in \hat{M}$ and compact subsets $D_j$, of $a_C^*$ such that

$$C = \bigcup_{j=1}^r \xi_j \times D_j.$$ 

If $\xi \in \hat{M}$ and $\nu \in a_C^*$ then set $\sigma_{\xi,\nu}(\text{man}) = \xi(m) a^{\nu + \rho} (\rho(H) = \frac{1}{2} \text{tr}(adH|_{\text{Lie}(N)}))$, $H \in \mathfrak{a}$, $a^\nu = \exp(\nu(H)) a = \exp(H)$, $\xi$ is taken to be a representative of the class $\xi$. $H^{\xi,\nu}$ is $I(\sigma_{\xi,\nu})$ which equals as a $K$–module $H^\xi = \text{Ind}_M^K(\xi)$. If $f \in H^\xi$ set $f_{\nu}(nak) = a^{\nu + \rho} f(k), n \in N, a \in A, k \in K$. $A_\rho(\nu)$ is the corresponding Kunze–Stein intertwining operator (c.f. [W1], 8.10.18. p.241).

**Proposition 16** Let $\xi \in \hat{M}$ and let $\Omega \subset a_C^*$ be open with compact closure. Then there exists $F \subset \hat{K}$ such that $\pi_{\xi,\nu}(U(\mathfrak{g})) \left( \sum_{\gamma \in F} H^\xi(\gamma) \right) = H^\xi$ for all $\nu \in \Omega$.

The proof of this result will use the following lemma.

**Lemma 17** If $\nu_0 \in a_C^*$ then there exists an open neighborhood of $\nu_0$, $U_{\nu_0}$, and a finite subset $F = F_{\nu_0}$ of $\hat{K}$ such that $\pi_{\xi,\nu}(U(\mathfrak{g})) \left( \sum_{\gamma \in F} H^\xi(\gamma) \right) = H^\xi$ for all $\nu \in U_{\nu_0}$.

**Proof.** If $\gamma \in \hat{K}$ fix $W_\gamma \in \gamma$. If $\text{Re}(\nu, \alpha) > 0$ for all $\alpha \in \Phi^+$ and if $\gamma \in \hat{K}$ and $A_\mathcal{T}(\nu) H^\xi(\gamma) \neq 0$ then $\pi_{\xi,\nu}(U(\mathfrak{g})) \left( H^\xi(\gamma) \right) = H^\xi$ (c.f. [RRG], Theorem 5.4.1 (1)). Fix such a $\gamma_\nu$ (which always exists since the operator $A_\mathcal{T}(\nu) \neq 0$), take $F_\nu = \{ \gamma_\nu \}$ and $U_\nu$ an open neighborhood of $\nu$ such that $A_\mathcal{T}(\mu) H^\xi(\gamma_\nu) \neq 0$ for $\mu \in U$. Let $\nu \in a_C^*$ be arbitrary. There exists a positive integer, $k$, such that $\text{Re}(\nu + k\rho, \alpha) > 0$ for all $\alpha \in \Phi^+$ and such that $k\rho$ is the highest weight of a finite dimensional spherical representation, $V^{k\rho}$, of $G$ relative to $\mathfrak{a}$. The lowest weight of $V^{k\rho}$ relative to $\mathfrak{a}$ is $-k\rho$ and $M$ acts trivially on that weight space thus $H_{\hat{K}}^{\xi,\nu + k\rho} \otimes V^{k\rho}$ has $H_{\hat{K}}^{\xi,\nu}$ as a quotient representation.
(see [W1], 8.5.14,15). Take $F_\nu$ to be the set of $K$–types that occur in both $W_{\gamma_\nu+k\rho} \otimes V^{k\rho}$ and $H^\xi$ and $U_\nu = U_{\nu+k\rho} - k\rho$. 

We now prove the proposition. By the lemma above for each $\nu \in \tilde{\Omega}$ there exists $F_\nu$ and $U_\nu$ as in the statement of the lemma. The $U_\nu$ form an open covering of $\tilde{\Omega}$ which is assumed to be compact. Thus there exist a finite number $\nu_1, \ldots, \nu_r \in \tilde{\Omega}$ such that 

$$\tilde{\Omega} \subset \bigcup_{i=1}^r U_{\nu_i}.$$ 

Take $F = \bigcup_{i=1}^r F_{\nu_i}$. This proves the proposition.

**Lemma 18** Let $\chi_{\xi,\nu}$ denote the infinitesimal character of $\pi_{\xi,\nu}$. If $C$ is a compact subset of $h^*_K$ then 

$$\{(\xi,\nu) \in \hat{\mathfrak{g}} \times \mathfrak{a}_C^* | \chi_{\xi,\nu} = \chi_{\Lambda}, \Lambda \in C\}$$

is compact.

**Proof.** Fix a system of positive roots for $(M^0, T)$ ($M^0$ the identity component of $M$). If $\lambda_\xi$ is the highest weight of $\xi$ relative to this system of positive roots and if $\rho_M$ is the half sum of these positive roots then $\chi_{\xi,\nu} = \chi_{\Lambda}$ with $\Lambda = \lambda_\xi + \rho_M + \nu$. This implies the lemma. 

**Lemma 19** Let $(\pi, V)$ be an analytic family of admissible $(\mathfrak{g}, K)$ modules based on $Z$. Assume that $z_0 \in Z$ is such that $(\pi_{z_0}, V)$ is finitely generated. If $T$ is an element of $Z(\mathfrak{g})$ there exist analytic functions $a_0, \ldots, a_{n-1}$ on $Z$ such that if $z \in Z$ and $\mu$ is an eigenvalue of $\pi_z(T)$ then $\mu$ is a root in $x$ of 

$$f(z, x) = x^n + \sum_{j=0}^{n-1} a_j(z)x^j.$$ 

**Proof.** Let $F$ be a finite number of elements of $\hat{K}$ such that $\pi_{z_0}(U(\mathfrak{g})) \sum_{\gamma \in F} V(\gamma) = V$. Let $L = \sum_{\gamma \in F} V(\gamma)$. Then we define the $a_j$ the by the formula 

$$f(z, x) = \det (xI - \pi_z(T)|_L) = x^n + \sum_{j=0}^{n-1} a_j(z)x^j.$$ 

The Cayley-Hamilton theorem implies that $h(z) = T^n + \sum_{j=0}^{n-1} a_j(z)T^j \in Z(\mathfrak{g})$ vanishes on $L$. Let $\gamma \in \hat{K}$ then there exist $x_1, \ldots, x_r \in U(\mathfrak{g})$ and $v_1, \ldots, v_r \in L$
such that \( \{ \pi_{z_0}(x_i)v_i \}_{i=1}^r \) is a basis of \( V(\gamma) \). Let \( P_\gamma \) be the projection onto the \( \gamma \)-isotypic component of \( V \). Thus
\[
(P_\gamma \pi_z(x_1)v_1) \wedge (P_\gamma \pi_z(x_2)v_2) \wedge \cdots \wedge (P_\gamma \pi_z(x_r)v_r) \in \wedge^r V(\gamma)
\]
(a one dimensional space) is non-zero for \( z = z_0 \). This implies that there exists an open neighborhood, \( U \), of \( z_0 \) in \( \Omega \) such that
\[
P_\gamma \pi_z(x_1)v_1, P_\gamma \pi_z(x_2)v_2, \ldots, P_\gamma \pi_z(x_r)v_r
\]
is a basis of \( V(\gamma) \) for \( z \in U \). That
\[
h(z)P_\gamma \pi_z(x_i)v_i = P_\gamma \pi_z(x_i)h(z)v_i = 0
\]
implies that \( h(z)V(\gamma) = 0 \) for \( z \in U \). The connectedness of \( Z \) implies that \( h(z)V(\gamma) = 0 \) for \( z \in Z \). Thus \( h(z) = 0 \) for all \( z \in Z \). This proves the Lemma. \( \blacksquare \)

If \( V \) is a \((g, K)\)-module then set \( ch(V) \) equal to the set of \( \Lambda \in \mathfrak{h}^* \) such that there exists \( v \in V \) with \( Tv = \chi_\Lambda(T)v \) for all \( T \in Z(g) \).

**Corollary 20** Keep the notation and assumptions of the previous lemma.

If \( \omega \subset Z \) is compact then there exists a compact subset \( C \) of \( \mathfrak{h}^* \) such that \( ch(\pi_z, V) \subset C \) for all \( z \in \omega \).

**Proof.** Let \( T_1, \ldots, T_m \) be a generating set for \( Z(g) \) and let \( f_j(z, x) \) be the function in the previous lemma corresponding to \( T_j \). Then
\[
f_j(z, x) = x^{n_j} + \sum_{i=0}^{n_j-1} a_{j,i}(z)x^i
\]
with \( a_{j,i} \) analytic in on \( Z \). If \( \chi_\Lambda \in ch(\pi_z, V) \) then
\[
|\chi_\Lambda(Z_j)| \leq \max_{0 \leq i < n_j} |a_{j,i}(z)| + 1
\]
(c.f. [RGG, 7.A.1.3]). If \( C \subset Z \) is compact then there exists a constant \( r < \infty \) such that \( |a_{j,i}(z)| \leq r \) for all \( i, j \) and \( z \in C \). This implies the corollary. \( \blacksquare \)

**Theorem 21** Let \( (\pi, V) \) be an analytic family of admissible \((g, K)\) modules based on \( Z \). Assume that there exists \( z_0 \in Z \) such that \( (\pi_{z_0}, V) \) is finitely generated. If \( \omega \) is a compact subset of \( Z \) then there exists \( S_\omega \subset K \) a finite subset such that if \( y \in \omega \) then
\[
\pi_y(U(g)) \left( \sum_{\gamma \in S_\omega} V(\gamma) \right) = V.
\]
Proof. Let $C$ as in the above corollary for $\omega$. Let

$$X = \{(\xi, \nu) \in \hat{M} \times a^*_C | \chi_{\xi,\nu} = \chi_\Lambda, \Lambda \in C\}.$$ 

$X$ is compact so there exist $\xi_1, \ldots, \xi_r \in \hat{M}$ and $D_1, \ldots, D_r$, compact subsets of $a^*_C$, such that $X = \bigcup_j \xi_j \times D_j$. Let $S_j \subset \hat{K}$ be the finite set corresponding to $\xi_j \times D_j$ in Proposition 16. Set $S_{\omega} = \bigcup S_j$. Let $L_1 \subset L_2 \subset \ldots \subset L_j \subset \ldots$ be an exhaustion of the $K$–types of $V$ with each $L_j$ finite.

We will use the notation $V_y$ for the $(g, \hat{K})$–module $(\pi_y, V)$. Let $y \in C$. Set $W_j = \pi_y(U(g))(\sum_{\gamma \in L_j} V(\gamma))$ then $W_j \subset W_{j+1}$ and $\cup W_j = V$. Each $W_j$ is finitely generated and admissible, hence of finite length. Therefore $V_y$ has a finite composition series

$$0 = V_y^0 \subset V_y^1 \subset \ldots \subset V_y^N$$

or a countably infinite composition series

$$0 = V_y^0 \subset V_y^1 \subset \ldots \subset V_y^n \subset V_y^{n+1} \subset \ldots$$

with $V_y^i/V_y^{i-1}$ irreducible. Thus by the dual form of the subrepresentation theorem there exists for each $i, \xi_i \in \hat{M}$ and $\nu_i \in a^*_C$ so that $V_y^i/V_y^i$ is a quotient of $(\pi_{\xi_i,\nu_i}, H^{\xi_i,\nu_i})$. Observe that $(\xi_i, \nu_i) \in X$. Thus $V_y^i/V_y^{i-1}(\gamma_i) \neq 0$ for some $\gamma_i \in S_{\omega}$. Let $M$ be a quotient module of $V_y$. Then $M = V_y/U$ with $U$ a submodule of $V_y$. There must be an $i$ such that $V_y^i/(V_y^{i-1} \cap U) \neq 0$. Let $i$ be minimal subject to this condition. Then $V_y^{i-1} \subset U$. Thus $V_y^i/V_y^{i-1}$ is a submodule of $M$. Hence $M(\gamma) \neq 0$ for some $\gamma \in S_{\omega}$. This implies that

$$\pi_y(U(g)) \left( \sum_{\gamma \in S_{\omega}} V(\gamma) \right) = V.$$ 

Indeed,

$$\left( V_y/\pi_y(U(g)) \left( \sum_{\gamma \in S_{\omega}} V(\gamma) \right) \right)(\gamma) = 0, \gamma \in S_{\omega}.$$ 

Corollary 22 (To the proof) Let $(\pi, V)$ be an analytic family of finitely generated admissible $(g, K)$ modules based on $Z$. Let $\omega$ be open in $Z$ with
compact closure. Let for each \( z \in \omega \), \( U_z \) be a \((g, K)\)–submodule of \( V_z \). Then there exists a finite subset \( F_{\omega} \subset \hat{K} \) such that

\[
\pi_z(U(g)) \left( \sum_{\gamma \in F_{\omega}} U_z(\gamma) \right) = U_z.
\]

**Proof.** In the proof of the theorem above all that was used was that the set of possible infinitesimal characters is compact. ■

7 Imbeddings of families of \( J \)–modules

Let \( X \) be a connected real or complex analytic manifold and let \((\mu, R)\) be an analytic family of objects in \( W(K, D) \) based on \( X \). The purpose of this section is to prove

**Theorem 23** Let the representation of \( Q \), \( \sigma_{k,x} \), on

\[
W_k = U(n)/n^{k+1} U(n) \otimes E \otimes H \otimes R|_M
\]

be as in Theorem 3 and let \( T_k(x) \) be the analytic family as in Theorem 15. If \( \omega \) is a compact subset of \( X \) then there exists \( k_{\omega} \) such that if \( x \in \omega \) then \( T_k(x) \) is injective.

This is a slight extension of a result in [HOW]. Given \( k \) then \((\sigma_{k,x}, W_k)\) as a composition series \( W_{k,x} = W_{k,x}^1 \supset W_{k,x}^2 \supset ... \supset W_{k,x}^r \supset W_{k,x}^{r+1} = \{0\} \) and each \( W_{k,x}^i/W_{k,x}^{i+1} \) is isomorphic with the representation \((\lambda_i, H_{\lambda})\) with \((\lambda_j, H_j)\) an irreducible representation of \( M \) and \( \nu_j \in a^*_C \) and \( \lambda_j,\nu(\text{man}) = a^{\nu_j+\rho} \lambda_j(m) \) with \( m \in M, a \in A \) and \( n \in N \). Also note that there is a natural \( Q \)–module exact sequence

\[
0 \to n^{k+2} U(n)/n^{k+1} U(n) \otimes E \otimes H \otimes R|_M \to W_{k+1,x} \to W_{k,x} \to 0.
\]

We may assume that the composition series is consistent with this exact sequence. This implies that the \( \nu_j \) that appear in \( W_k/W_{k+1} \) are of the form \( \mu + \alpha_1 + ... + \alpha_k \) with \( \alpha_i \) a restricted positive root (i.e. a weight of \( a \) on \( n \)).

Now consider the corresponding exact sequence of \((g, K)\)–modules.

\[
(*) 0 \to I(\eta_{k,x}) \to I(\sigma_{k+1,x}) \to I(\sigma_{k,x}) \to 0.
\]
The \((\mathfrak{g}, K)\)–modules \(I(\sigma_\nu)\) with \(\sigma\) an irreducible representation of \(M\) with Harish-Chandra parameter \(\Lambda_\sigma\) (for \(\text{Lie}(M)_\mathbb{C}\)) and \(\nu \in \mathfrak{a}_C^*\) have infinitesimal character with Harish-Chandra parameter \(\Lambda_\sigma + \nu\). We are finally ready to prove the theorem.

Let \(C_\omega\) be the compact set \(\bigcup_{x \in \omega} \text{ch}(J(R_x))\). Let \(C_\omega = \bigcup_{j=1}^k \Lambda_i + D_i\) with \(D_i\) compact in \(\mathfrak{a}_C^*\). Assume that the result is false for \(\omega\). Then for each \(j\) there exists \(k \geq j\) and \(x\) such that \(\ker T_k(x) \neq 0\) but \(\ker T_{k+1} = 0\). Label the Harish-Chandra parameters that appear in \(I(\sigma_o x)\), \(\Lambda_1 + \nu_1, \ldots, \Lambda_s + \nu_s\) with \(\Lambda_i \in \text{Lie}(T)^*\) and \(\nu_i \in \mathfrak{a}_C^*\) (recall that we have fixed a maximal torus of \(M\)).

The above observations imply that \(\text{ch}(J(R_x))\) contains an element of the form \(\Lambda + \nu_i + \beta_k\) with \(\beta_k\) a sum of \(k\) positive roots, \(\Lambda \in \text{Lie}(T)^*\) and \(1 \leq i_k \leq s\).

We now have our contradiction \(\nu_{i_k} + \beta_k \in \bigcup D_i\) which is compact. But the set of \(\nu_{i_k} + \beta_k\) is unbounded.

8 Families of Hilbert and Fréchet representations

**Definition 24** Let \(X\) be metric space. A continuous family of Hilbert representations based on \(X\) of \(G\) is a pair \((\pi, H)\) of a Hilbert space \(H\) and \(\pi : X \times G \rightarrow H\) strongly continuous such that if \(\pi_x(g) = \pi(x, g)\) then \((\pi_x, H)\) is a strongly continuous representation of \(G\). The family will be called admissible if \(\pi_x|_K\) is independent of \(x \in X\) and \(\dim H(\gamma) < \infty\) for each \(\gamma \in \hat{K}\).

**Lemma 25** Let \((\pi, H)\) be a continuous family of admissible Hilbert representations of \(G\) based on the connected real or complex analytic manifold \(X\) and denote by \(d\pi_x\) the action of \(\mathfrak{g}\) on \(H^K_\infty\) (the \(K\)–finite \(C^\infty\)–vectors). Then \((d\pi, H^K)\) is a continuous family of admissible \((\mathfrak{g}, K)\)–modules based on \(X\).

**Proof.** If \(\gamma \in \hat{K}\) then \(C^\infty_c(\gamma; G)\) denotes the space of all \(f \in C^\infty_c(G)\) such that

\[
\int_K \chi_\gamma(k) f(k^{-1} g) dk = f(g), g \in G
\]

with \(\chi_\gamma\) the character of \(\gamma\). Then

\[
H(\gamma) = \pi_x(C^\infty_c(\gamma; G)) H.
\]

We also note that if \(Y \in \mathfrak{g}, f \in C^\infty_c(\gamma; G)\) and \(v \in H\) then

\[
d\pi_x(Y) \pi_x(f) v = \pi_x(Yf) v
\]
with \( Yf \) the usual action of \( Y \in \mathfrak{g} \) on \( C^\infty(G) \) as a left invariant vector field. Thus, if \( v \in H_K \) and \( y \in U(\mathfrak{g}_C) \) then the map

\[
x \mapsto d\pi_x(y)v
\]
is continuous.

The following lemma is Lemma 1.1.3 in [RRG] taking into account dependence on parameters. The proof is essentially the same taking into account the dependence on parameters and using the local compactness of \( X \).

**Lemma 26** Let \( X \) be a locally compact metric space and let \( H \) be a Hilbert space. Assume that for each \( x \in X \), \( \pi_x : G \to GL(H) \) (bounded invertible operators such that

1) If \( \omega \subset X \) and \( \Omega \subset G \) are compact subsets of \( X \) and of \( G \) respectively then there exists a constant \( C_{\omega,\Omega} \) such that \( \|\pi_x(g)\| \leq C_{\omega,\Omega} \) for \( x \in \omega, g \in \Omega \).

2) The map \( x, g \to \langle \pi_x(g)v, w \rangle \) is continuous for all \( v, w \in H \).

Then \((\pi, H)\) is a continuous family of representations of \( G \) based on \( X \) and conversely if \((\pi, H)\) is a continuous family of Hilbert representations then 1) and 2) are satisfied.

An immediate corollary is

**Corollary 27** Let \((\pi, H)\) be an admissible, continuous family of Hilbert representations of \( G \) based on the locally compact metric space \( X \). Set for each \( x \in X \), \( \hat{\pi}_x(g) = \pi_x(g^{-1})^* \) then \((\hat{\pi}, H)\) is a continuous, admissible family of Hilbert representations of \( G \) based on \( X \).

Let \( \|g\| \) be a norm on \( G \), that is a continuous function from \( G \) to \( \mathbb{R}_{>0} \) (the positive real numbers) such that

1. \( \|k_1gk_2\| = \|g\|, k_1, k_2 \in K, g \in G \),
2. \( \|xy\| \leq \|x\| \|y\|, x, y \in G \),
3. The sets \( \|g\| \leq r < \infty \) are compact.
4. If \( X \in \mathfrak{p} \) then if \( t \geq 0 \) then \( \log \|\exp tX\| = t \log \|\exp X\| \).

If \((\sigma, V)\) is a finite dimensional representation of \( G \) with compact kernel and if \( \langle \ldots, \ldots \rangle \) is an inner product on \( V \) that is \( K \)-invariant then if \( \|\sigma(g)\| \) is the operator norm of \( \sigma(g) \) then \( \|g\| = \|\sigma(g)\| \) is a norm on \( G \). Taking the representation on \( V \oplus V \) given by

\[
\begin{bmatrix}
\sigma(g) \\
\sigma(g^{-1})^*
\end{bmatrix}
\]
then we may (and will) assume in addition
5. $\|g\| = \|g^{-1}\|$.

Note that 5. implies that $\|g\| \geq 1$.

Using the same proof as Lemma 2.A.2.1 in [RGG] (which we give for the sake of completeness) one can prove

**Lemma 28** If $(\pi, H)$ is a continuous family of Hilbert representations modeled on $X$ and if $\omega$ is a compact subset of $x$ then there exists constants $C_\omega, r_\omega$ such that

$$\|\pi_x(g)\| \leq C_\omega \|g\|^r_\omega.$$ 

**Proof.** Let $B_1 = \{g \in G | \|g\| \leq 1\}$. Then if $v \in H$ and $(x, g) \in \omega \times B_1$ then $\sup \|\pi_x(g)v\| < \infty$ by strong continuity. The principle of uniform boundedness (c.f. [RS],III.9,p.81) implies that there exists a constant, $R$, such that $\|\pi_x(g)\| \leq R$ for $(x, g) \in \omega \times B_1$. Let $r = \log R$. In particular if $k \in K$ then $\|\pi_x(kg)\| \leq \|\pi_x(k)\| \|\pi_x(g)\| \leq R \|\pi_x(g)\|$. Also,

$$\|\pi_x(g)\| = \|\pi_x(k^{-1})\pi_x(kg)\| \leq R \|\pi_x(kg)\|.$$ 

Thus for all $k \in K, g \in G$

$$R^{-1} \|\pi_x(g)\| \leq \|\pi_x(kg)\| \leq R \|\pi_x(g)\|.$$ 

Let $X \in p, X \neq 0$ and let $j$ be such that

$$j < \log \|\exp X\| \leq j + 1$$

then

$$\log \|\pi_x(\exp X)\| \leq \log \|\pi_x(\exp(\frac{X}{j + 1}))\|^{j+1} \leq r(j + 1) \leq r + r \log \|\exp X\|.$$ 

Thus

$$\|\pi_x(\exp X)\| \leq R \|\exp X\|^r, X \in p.$$ 

If $g \in G$ then $g = k \exp X$ with $k \in K$ and $X \in p$ so

$$\|\pi_x(g)\| = \|\pi_x(k \exp X)\| \leq R^2 \|\exp X\|^r = R^2 \|g\|^r.$$ 

Take $C_\omega = R^2$ and $r_\omega = r$. $lacksquare$
We define $\mathcal{S}(G)$ to be the space of all $f \in C^\infty(G)$ such that of $x \in U(\mathfrak{g}_C)$ (thought of as a left invariant differential operator) and $r > 0$ then

$$p_{r,x}(f) = \sup_{g \in G} |xf(g)| \|g\|^r < \infty.$$ 

$\mathcal{S}(G)$ is a Fréchet (using the semi-norms $p_{r,x}$) algebra (under convolution) of functions on $G$.

Lemma 2.A.2.4 in [RRG] implies that there exists $d > 0$ such that 

$$\int_G dg \|g\|^d < \infty.$$ 

This implies that $\mathcal{S}(G)$ acts on any Banach representation, $(\pi, V)$ of $G$ via

$$\pi(f) = \int_G f(g)\pi(g)dg.$$ 

Recall that a pair $(\pi, V)$ of a Fréchet space, $V$, and a representation of $G$, $\pi$, on $V$ is called a smooth Fréchet representation of moderate growth if the map $g \mapsto \pi(g)v$ is $C^\infty$ and if $p$ is a continuous seminorm on $V$ then there exists a continuous seminorm $q$ on $V$ and $r$ such that

$$p(\pi(g)v) \leq \|g\|^r q(v).$$ 

This implies that a smooth Fréchet representation of moderate growth is an $\mathcal{S}(G)$–module. A smooth Fréchet representation of moderate growth is defined to be admissible if the $(\mathfrak{g}, K)$–module $V_K$ is admissible. It is said to be of Harish-Chandra class if $V_K$ is admissible and finitely generated. Let $\mathcal{H}F(G)$ be the category of smooth Fréchet representations of moderate growth in the Harish-Chandra class.

The CW theorem

**Theorem 29** The functor $V \to V_F$ from $\mathcal{H}F(G)$ to $H(\mathfrak{g}, K)$ is an isomorphism of categories.

We will prove this as a consequence of the usual statement of the theorem is (see [RRG] Theorem 11.6.7 (2))

**Theorem 30** If $(\pi_i, V_i) \in \mathcal{H}F(G)$, for $i = 1, 2$ and if $T \in \text{Hom}_{H(\mathfrak{g}, K)}((V_1)_K, (V_2)_K)$ then $T$ extends to a continuous element of $\text{Hom}_{\mathcal{H}F(G)}(V_1, V_2)$ with closed image that is a topological summand.
If $V_1, V_2 \in \mathcal{HF}(G)$ have the property that $(V_1)_K = (V_2)_K = V$ then one has

$$V_i \subset \prod_{\gamma \in K} V(\gamma), i = 1, 2.$$ 

As the formal sums that converge relative to the continuous seminorms endowing the topology on $V_1$ and $V_2$ respectively. The identity map on $V$ induces an isomorphism of $V_1$ and $V_2$. But this is given by the identity map on $\prod_{\gamma \in K} V(\gamma)$. Hence $V_1 = V_2$. This implies the isomorphism of categories.

The inverse functor can be seen as follows. Let $V \in H(\mathfrak{g}, K)$ and let $(\pi, H)$ be a Hilbert representation of $G$ such that $(d\pi, H_K)$ is equivalent to $V$. Let $T \in \text{Hom}_{H(\mathfrak{g}, K)}(V, H_K)$ give the isomorphism. Let $\langle \ldots, \ldots \rangle$ the Hilbert space structure on $H$ and let $(v, w) = \langle Tv, Tw \rangle$. If $x \in U(\mathfrak{g})$ set $p_x(v) = \sqrt{(xv, xv)}$

$$\nabla = \{ \{v_\gamma\} \in \prod_{\gamma \in K} V(\gamma)| \sum_{\gamma \in K} p_x(v_\gamma)^2 < \infty \}.$$ 

Then $T$ extends to an isomorphism of $\bar{V}$ onto $\bar{H}^\infty$. Thus defining $\mu(g) = T^{-1}\pi(g)T$ on $\bar{V}$ we have $(\mu, \bar{V}) \in \mathcal{HF}(G)$ and $\bar{V}_K = V$. The uniqueness implies that $V \to \nabla$ defines the inverse functor.

Another corollary of the CW theorem is (see [HOW] Theorem 11.8.2)

**Theorem 31** If $(\pi, V) \in \mathcal{HF}(G)$ and if $v \in V$ then $\pi(S(G))v$ is closed in $V$ and a topological summand.

**Corollary 32** If $(\pi, H)$ is a Hilbert representation of $G$ such that $H^\infty_K \in H(\mathfrak{g}, K)$ and if $H_K$ is generated by the subspace $U$ then $\pi(S(G))U = H^\infty$.

**Definition 33** A continuous family of objects in $\mathcal{HF}(G)$ based on the metric space $X$ is a pair $(\pi, V)$ of a Fréchet space $V$ and a continuous map

$$\pi : X \times G \to \text{End}(V)$$

(here $\text{End}(V)$ is the algebra of continuous operators on $V$ with the strong topology) such that such that for each $x \in X$, if $\pi_x(g) = \pi(x, g)$ then $(\pi_x, V) \in \mathcal{HF}(G)$. We will say that the family has local uniform moderate growth if for each $\omega$ a compact subset of $X$ and each continuous seminorm on $V$, $p$, there exists a continuous seminorm $q_\omega$ on $V$ and $r_\omega$ such that if $v \in V$ then

$$p(\pi_x(g)v) \leq q_\omega(v) \|g\|^{r_\omega}.$$ 

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Definition 34 A holomorphic family of objects in $\mathcal{H}\mathcal{F}(G)$ based on the complex manifold $X$ is a continuous family $(\pi, V)$ such that the map $x \mapsto \pi_x(g)v$ is holomorphic from $X$ to $V$ for all $g \in G, v \in V$.

Lemma 35 If $(\pi, H)$ is a continuous family of Hilbert representations based on the metric space $X$ such that the representations $(d\pi_x, H^K_\infty) \in H(g, K)$ and the $K-C^\infty$ vectors are the $G-C^\infty$ vectors then $(\pi, H_\infty)$ is a continuous family of objects in $\mathcal{H}\mathcal{F}(G)$ based on the metric space $X$ that is of local uniform moderate growth.

Proof. We note that if $f \in S(G)$ and $v \in H$ then the map $x \mapsto \pi_x(f)v$ is continuous from $X$ into $H^\infty$. Also $\pi_x(h)\pi_x(f)v = \pi_x(L(h)f)v$ with $L(h)f(g) = f(h^{-1}g)$. The last assertion follows from Lemma 28. For want of a better place to put it we include the following simple Lemma in this section.

Lemma 36 Let $(\tau, V)$ be a finite dimensional continuous representation of $K$ and let $X$ be a locally compact metric space (resp. an analytic manifold). If $u \in X$ let $\langle \ldots, \ldots \rangle_u$ be an inner product on $V$ such that $\tau(k)$ acts unitarily with respect to $\langle \ldots, \ldots \rangle_u$ for $k \in K$ and such that the map $u \mapsto \langle v, w \rangle_u$ is continuous (resp. real analytic) for all $v, w \in V$. Then there exists, for each $u$ and an ordered orthonormal basis of $V, e_1(u), \ldots, e_n(u)$ such that the map $u \mapsto e_i(u)$ is continuous (resp. real analytic) and the matrix of $\tau(k)$ with respect to $e_1(u), \ldots, e_n(u)$ is independent of $u$. Furthermore, if $X$ is compact and contractible and $(\sigma, W)$ is a finite dimensional continuous representation of $K$ and $u \mapsto B(u) \in \text{Hom}_K(V, W)$ is continuous and surjective for $u \in X$ then $e_1(u), \ldots, e_r(u)$ with $r = \dim V - \dim W$ can be taken in $\ker B(u)$.

Proof. Fix an inner product, $(\ldots, \ldots)$, on $V$ such that $\tau$ is unitary. Then there exists a positive definite Hermitian operator (with respect to $(\ldots, \ldots)$), $A(u)$ such that $\langle v, w \rangle_u = (A(u)v, w), v, w \in V$. Then $A(u)$ is continuous (resp. real analytic) in $u$. Now,

$$\langle v, w \rangle_u = \langle \tau(k)v, \tau(k)w \rangle_u = (A(u)\tau(k)v, \tau(k)w) = (\tau(k)^{-1}A(u)\tau(k), v, w \in V, k \in K.$$ So

$$\tau(k)^{-1}A(u)\tau(k) = A(u), u \in X, k \in K.$$
Set $S(u) = A(u)\frac{1}{2}$ then $\langle v, w \rangle_u = (S(u)v, S(u)w)$. Thus if $T(u) = S(u)^{-\frac{1}{2}}$ then $\tau(k)T(u) = T(u)\tau(k), k \in K, u \mapsto T(u)$ is continuous (resp. real analytic) and

$$\langle T(u)v, T(u)w \rangle_u = \langle v, w \rangle, v, w \in V.$$  

Let $e_1, \ldots, e_n$ be an (ordered) orthonormal basis of $V$ with respect to $(\ldots, \ldots)$ then $e_1(u) = T(u)e_1, \ldots, e_n(u) = T(u)e_n$ is an orthonormal basis of $V$ with respect to $\langle \ldots, \ldots \rangle_u$. If $\tau(k)e_i = \sum j_ik_je_j$ then

$$\tau(k)e_i(u) = \tau(k)T(u)e_i = T(u)\tau(k)e_i = \sum k_je_j.$$  

To prove the second assertion note that $u \mapsto \ker B(u)$ is a $K$–vector bundle over $X$. Since $X$ is compact and contractible the bundle is a trivial $K$–vector bundle ($[\pi]$ Lemma 1.6.4). Thus there is a representation $(\mu, Z)$ of $K$ and $u \mapsto L(u) \in \text{Hom}_K(Z, V)$ continuous such that $L(u)Z = \ker B(u)$ and $L(u)$ is injective. Notice that $B(u) : \ker B(u) \to W$ is a $K$–module isomorphism. Now pull back the inner product $\langle \ldots, \ldots \rangle_u$ to $Z$ using $L(u)$ getting a $K$–invariant inner product, $(\ldots, \ldots)_u$, on $Z$ and push the inner product to $W$ getting a $K$–invariant inner product $(\ldots, \ldots)_u^1$ on $W$ Now apply the first part of the lemma to get an orthonormal basis $f_1(u), \ldots, f_r(u)$ of $Z$ with respect to $(\ldots, \ldots)_u$ and an orthonormal basis $f_{r+1}(u), \ldots, f_n(u)$ ($n = \dim V$) with respect to $(\ldots, \ldots)_u^1$ such that the matrices of the action of $K$ with respect to each of these bases is constant. Take $e_i(u) = L(u)f_i(u)$ for $i = 1, \ldots, r$ and $e_i(u) = (B(u)|_{\ker B(u)}\uparrow)^{-1} f_i(u)$ for $i = r + 1, \ldots, n.$ 

9 Continuous globalization of families of $J$–modules.

We maintain the notation of the previous sections. Let $Z$ be a connected analytic manifold and let $(\mu, L)$ be an analytic family of objects in $W(K, D)$ based on $Z$.

**Theorem 37** Let $U \subset Z$ be open with compact closure. There exists a continuous family $(\pi, H)$ of Hilbert representations of $G$ based on $U$ such that the continuous family of $(\mathfrak{g}, K)$–modules $(d\pi, H^\infty_K)$ is isomorphic with the analytic family $z \mapsto J(L_z)$ of objects in $H(\mathfrak{g}, K)$ based on on $U$ (thought of as a continuous family). Furthermore, the $K$–$C^\infty$ vectors of $(\pi_u, H)$ are the $G$–$C^\infty$ vectors for every $u \in U$. 

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Proof. Let $\gamma \in \hat{K}$ then Theorem 32 implies
\[
\dim J(L_z)(\gamma) = \dim E \dim \gamma \dim \text{Hom}_K(V_\gamma, \mathcal{H} \otimes L).
\]
for every $z \in Z$. In particular it is independent of $z$. Theorem 23 implies that there exists $k$ and for each $u \in U$ the map
\[
T_{k,u} : J(L_u) \to I(\sigma_{k,u})
\]
is injective. Note that the space of $K$–finite vectors in $I(\sigma_{k,u})$ is the $K$–finite induced representation $\text{Ind}_M^K(\sigma_{k,L_u})$ and hence independent of $u$. Let $(H_1, \langle \ldots, \ldots \rangle)$ be the Hilbert space completion of $\text{Ind}_M^K(\sigma_{k,L_u})$ corresponding to unitary induction from $M$ to $K$. This gives an analytic family of Hilbert representations of $G$, $\mu_z$. For each $\gamma \in \hat{K}$ the family of linear operators $T_{k,L_u}(J(L_u)(\gamma)) \in \text{Hom}_K(J(L_u)(\gamma), I(\sigma_{k,L_u})(\gamma))$ is analytic in $u \in U$ (see Theorem 15) and injective. On $(E \otimes \mathcal{H} \otimes L)(\gamma)$ put for each $u \in U$ the inner product $\langle \ldots, \ldots \rangle_u = T_{k,L_u}^*(\ldots, \ldots)$. Then $\pi(k)$ acts unitarily with respect to $\langle \ldots, \ldots \rangle_u$ for $u \in U$ and $u \mapsto (\ldots, \ldots)_u$ is real analytic. Let $e^1_1(u), \ldots, e^\gamma_{\gamma}(u)$ be as in Lemma 36. Then $u \mapsto e^\gamma_{\gamma}(u)$ is analytic. Put $f^\gamma_{\gamma}(u) = T_{k,L_u} e^\gamma_{\gamma}(u)$ and set
\[
P(u)_\gamma(v) = \sum \langle v, f^\gamma_{\gamma}(u) \rangle f^\gamma_{\gamma}(u)
\]
then $u \mapsto P(u)_\gamma$ is an analytic map of $U$ into the manifold of orthogonal projection operators of rank $m_\gamma$ on $H_1(\gamma)$. Set 37
\[
P(u) = \sum_{\gamma \in K} P(u)_\gamma.
\]
Then $P(u)H_1$ is the closure in the Hilbert space $H_1$ of $T_{k,L_u}(J(L_u))$. Observing that the $K$–finite vectors in $(\mu_u, H_1)$ are contained in the analytic vectors implies that $P(u)H_1$ is invariant under $\mu_u(G)$. Note $u \mapsto P(u)$ is continuous in the strong operator topology from $U$ to the bounded operators on $H_{1\text{definite}}$. This is proved by the following standard calculus style argument. Let $v \in H_1$ be a unit vector and $u_o \in U$. We can expand $v = \sum v_\gamma$ in $H_1$ with $v_\gamma \in H(\gamma)$ and let $\varepsilon > 0$ be given then there exists $F \subset \hat{K}$ a finite set such that $\left\| \sum_{\gamma \notin F} v_\gamma \right\| < \varepsilon$. Also $P(u)_F = \sum_{\gamma \in F} P(u)_\gamma$ is analytic in $u$ thus there exists a neighborhood $U_1$ of $z_o$ in $U$ such that $\left\| (P(u)_F - P(u_0)_F) v \right\| < \frac{\varepsilon}{2}$ for $u \in U_1$. Noting that $\left\| P(u) \right\| = 1$ we have
\[
\left\| (P(u) - P(u_o)) v \right\| \leq \left\| (P(u)_F - P(u_0)_F) v \right\| + \left\| (P(u) - P(u_o)) \sum_{\gamma \notin F} v_\gamma \right\| < \frac{\varepsilon}{2}.
\]
\[ \| (P(u)_F - P(u_0)_F ) v \| + \frac{\varepsilon}{2} < \varepsilon. \]

For each \( u \in U \) put the inner product \( \langle ..., ... \rangle_v = T_{u,L_0}^* \langle ..., ... \rangle_v \) on \( E \otimes H \otimes L \). Pull back the action of \( G \) on \( P(u)H_1 \) to the Hilbert space completion, \( H_u \) of \( E \otimes H \otimes L \) with respect to \( \langle ..., ... \rangle_v \) to get the representation \( \eta_u \) of \( G \) such that \( d\eta_u \) is equivalent with \( J(L_u) \). Note that \( \{ e_j(u) \} \) is an orthonormal basis of \( H_u \) for all \( u \in Z \). If \( v, u \in Z \) define \( T(v,u) : H_u \rightarrow P(v)H_1 \) by

\[ T(v,u) e_j(u) = e_j(v). \]

Then \( T(u,v) \) is a unitary \( K \)-isomorphism with inverse \( T(v,u) \). Fix \( u_o \in U \), set \( H = H_{u_o} \) and set \( \pi_u(g) = T(u,u_o) \eta_u(g) T(u_o,u) \). Then \( (\pi,H) \) is the desired continuous family. The last assertion follows from the fact that the \( K - C^\infty \) vectors of \( (\mu_u, H_1) \) are the \( G - C^\infty \) vectors.

The technique in the proof of the Theorem above involving the bases \( \{ e_j(u) \} \) will be used several times in the next section.

10 Continuous globalization of families of objects in \( H(\mathfrak{g}, K) \)

**Theorem 38** Let \( (\pi,V) \) be an analytic family of objects in \( H(\mathfrak{g}, K) \) based on the analytic manifold \( X \). Let \( x_o \in X \) then there exists, \( U \), an open neighborhood of \( x_o \) in \( X \) and a continuous family of Hilbert representations \( (\mu_U, H_U) \) such that the family \( (d\mu_U, (H_U)^\infty) \) is isomorphic with \( (\pi|_U, V) \) (as a continuous family). Furthermore, the \( K - C^\infty \) vectors of \( \mu_{U,x} \) are the \( G - C^\infty \) vectors.

**Proof.** Let \( U_1 \) be an open neighborhood of \( x_o \) in \( X \) with compact closure. Then Theorem [21] implies that there exists \( F^0_U \subset \hat{K} \) a finite subset such that \( \pi_x(U(\mathfrak{g}_C)) \sum_{\gamma \in F^0_U} V(\gamma) = V \). Let \( R^0 = \sum_{\gamma \in F_U} V(\gamma) \). \( R^0 \) is invariant under the action \( \pi_x(D) \) for all \( x \in X \). This implies that \( ((\pi|_U)|_D, R^0) \) defines an analytic family of objects in \( W(K,D) \) based on \( U_1 \). Let \( J(R^0) \) be the corresponding \( J \)-family. Then we have the surjective analytic homomorphism of families

\[ T_0 \]
\[ J(R^0) \rightarrow V|_U \rightarrow 0 \]

with \( T_0(x) \) mapping \( J(R^0_x) \) onto \( V \) for all \( x \in U_1 \). Let \( \mu^0_x \) be the action of \( U(\mathfrak{g}_C) \) on the space \( J(R^0) \) (which we regard to be the fixed \( K \)-representation \( \mathcal{H} \otimes E \otimes R^0 \)) the Corollary [22] implies that there exists a finite subset \( F^1_U \subset \hat{K} \)
such that

\[ \ker T_0(x) = \mu_x(U(\mathfrak{g}_C)) \sum_{\gamma \in F_0^i} \ker T_0(x)|_{J(R^0)(\gamma)}. \]

If \( \gamma \in \hat{K} \) then

\[ \dim \ker T_0(x)|_{J(R^0)(\gamma)} = \dim J^0(R)(\gamma) - \dim V(\gamma) \]

for \( x \in U_1 \). Let \( (\sigma, (H^0, (\ldots, \ldots))) \) be the continuous family of Hilbert representations based on \( U_1 \) corresponding to \( J(R^0) \) as in Theorem 37. Let \( U \) be an open neighborhood of \( x_0 \) contained in \( U \) such that \( \overline{U} \) is contractible. Let \( \gamma \in \hat{K} \) if \( x \in \overline{U} \) let \( e^\gamma_1(x), \ldots, e^\gamma_n(x) \) and orthonormal basis of \( \langle H \otimes E \otimes R^0 \rangle(\gamma) \) with respect to the pull back of \((\ldots, \ldots)\) to \( \langle H \otimes E \otimes R^0 \rangle(\gamma) \) such that \( x \mapsto e^\gamma_i(x) \) is continuous on \( U \) and \( e^\gamma_1(x), \ldots, e^\gamma_n(x) \) is an orthonormal basis of \( \ker T_0(x)|_{(H \otimes E \otimes R^0)(\gamma)} \) and the matrix of \( k \in K \) with respect to the \( e^\gamma_i(x) \) is independent of \( x \) (see the second part of Lemma 36) for \( x \in U \). Define

\[ f^\gamma_i(x) = T_0(x)(e^\gamma_{n+i}(x)), i = 1, \ldots, \dim V(\gamma). \]

Let if \( x \in U \) let \( \langle \ldots, \ldots \rangle_x \) be the inner product on \( V \) such that \( \{ f^\gamma_i(x) \}_{\gamma,i} \) is an orthonormal basis of \( V \) with respect to \( \langle \ldots, \ldots \rangle_x \). Let \( H_x \) be the Hilbert space completion of \( V \) with respect to \( \langle \ldots, \ldots \rangle_x \). Let \( H^0_x \) be the closure of \( \ker T_0(x) \) in \( H \). then since \( \ker T_x(x) \) is a \((g,K)\) invariant subspace of the analytic vectors \( H_x \) is \( \sigma(G) \) invariant (c.f. [RRG] Proposition 1.6.6). The argument using the \( e^\gamma_i \) in the proof of Theorem 37 one proves that \( (\sigma_x|_{H^0_x}, H^0_x) \) defines a continuous family of Hilbert representations based on \( U \). Also the space of \( K \)–finite vectors of \( H^0_x/H^0_x \) is isomorphic with \( (\pi_x, V) \). Let \( \mu_x \) be the quotient representation on \( H^0_x/H^0_x \). Note that the quotient map on \( J(R^0)/\ker T_0(x) \) corresponding to \( T_0(x), S(x), \) extends to a unitary map of \( H^0_x/H^0_x \) onto \( H_x \) by the definition of \( \langle \ldots, \ldots \rangle_x \). Set \( \eta_x(g) = S(x)\mu_x(g)S(x)^{-1} \) for \( x \in U \). Finally, if \( x, y \in U \) then define \( L(x,y) : H_y \rightarrow H_x \) by

\[ L(x,y)f^\gamma_i(y) = f^\gamma_i(x) \]

for all \( i, \gamma \). Then \( x, y \rightarrow L(x,y) \) is unitary, strongly continuous and \( L(x,y)^{-1} = L(y,x) \). Set \( H = H_{x_0} \) and \( \nu_x(g) = L(x,x_0)\eta_x(g)L(x_0,x) \) then \( (\nu, H) \) is the desired continuous family of Hilbert representations based on \( U \). ■

We include the following corollary however as noted at the end of the section it is not necessary to prove the main results that follow.
**Corollary 39** Let $X$ be a connected analytic manifold and let $(\pi, V)$ be an analytic family of objects in $H(\mathfrak{g}, K)$ based on $X$ such that $\dim V = \infty$ then there exists a continuous family of Hilbert representations $(\lambda, H)$ such that the family $(d\lambda, (H)^{\infty}_K)$ is isomorphic with $(\pi, V)$ (as a continuous family). Furthermore, the $K - C^\infty$ vectors of $\mu_{U,x}$ are the $G - C^\infty$ vectors.

**Proof.** The previous theorem implies that there exists an open covering $\{U_\alpha\}$ of $X$ such that for each $\alpha$ there exists a continuous family of Hilbert representations $(\mu_{U_\alpha}, H_{U_\alpha})$ such that $(d\mu_{U_\alpha}, (H_{U_\alpha})^{\infty}_K)$ is continuously isomorphic with $(\pi|_{U}, V)$. For each $\alpha, \beta$ the definition of the Hilbert spaces $H_{U_\alpha}$ implies that one has $g_{U_\beta, U_\alpha}(x) : H_{U_\alpha} \to H_{U_\beta}$ a unitary isomorphism depending strongly continuously on $x \in U_\alpha \cap U_\beta$. This defines a Hilbert vector bundle over $X$.

Kuiper’s Theorem implies that all Hilbert bundles with infinite dimensional fibers are trivial ([BB], p.67). Thus there exists a fixed Hilbert space, $H$, and for each $\alpha$ and each $x \in U_\alpha$, $h_\alpha(x) : H_{U_\alpha} \to H$ a unitary isomorphism that depends strongly continuously on $x$ such that $g_{U_\beta, U_\alpha}(x) = h_\beta(x)^{-1} h_\alpha(x)$.

Define $\lambda_x(g) = h_U(x) \mu_x(g) h_U(x)^{-1}$.

This and Lemma 35 imply our main results

**Theorem 40** There exists a continuous family $(\lambda, Z)$ of objects in $\mathcal{H} \mathcal{F}(G)$ based on $X$ of local uniform moderate growth that globalizes the family $(\pi, V)$.

This can be interpreted in the following way:

**Corollary 41** Let $T$ be the inverse functor to the $K$–finite functor $\mathcal{H} \mathcal{F}(G) \to H(\mathfrak{g}, K)$ and let $(\pi, V)$ is an analytic family of objects in $H(\mathfrak{g}, K)$ based on the connected analytic manifold $X$ such that $\dim V = \infty$. If $T((\pi_x, V)) = (\lambda_x, \bar{V}_x)$ then

1. For all $x, y \in X, \bar{V}_x = \bar{V}_y$ as subspaces of $\prod_{\gamma \in K} V(\gamma)$. Set $\bar{V}$ equal to the common value.

2. The map $x, g, v \mapsto \lambda_x(g)v$ is continuous from $X \times G \times \bar{V}$ to $\bar{V}$, linear in $v$ and $C^\infty$ in $g$.

With this interpretation it is clear that this result follows from the local version of the Hilbert globalization (that is the first theorem in this section).
11 The dual functor.

We now consider a dual functor. Let \((\pi, V) \in H\mathcal{F}(G)\) let \(\lambda \in V'\) (the continuous dual). If \(v \in V_K\) then the following assertions are true

1. There exists \(f_{\lambda,v}\) a real analytic function on \(G\) such that \(f_{\lambda,v}(1) = \lambda(v)\).

2. If \(R_g\) denotes the right regular action of \(g \in G\) on \(C(G)\) then \(R_kf_{\lambda,v} = f_{\lambda,kv}\) for \(k \in K\). If \(x \in U(\mathfrak{g})\) is thought of as a left invariant differential operator then \(xf_{\lambda,v} = f_{\lambda,xv}\).

3. There exists \(d\) depending only on \(\lambda\) and \(C_v > 0\) such that \(|f_{\lambda,v}(g)| \leq C_v \|g\|^d\)

We note that conditions 1. and 2 uniquely specify \(f_{\lambda,v} = \lambda(\pi(g)v)\) which satisfies 3.

If \(Z\) is an object of \(H(\mathfrak{g}, K)\) then denote by \(Z^*_\text{mod}\) set of \(\lambda\) in the algebraic dual, \(Z^*\), of \(Z\) such that if \(v \in Z\) then there exists a real analytic function \(f_{\lambda,v}\) satisfying 1., 2., and 3.

A variant of the CW theorem proved in \([RRG]\) Theorem 11.6.6, Corollary 11.6.3 is

**Theorem 42** Let \(Z \in H(\mathfrak{g}, K)\) then if \(V \in H\mathcal{F}(G)\) and if \(V_K = Z\) then \(V'_Z = Z^*_\text{mod}\).

The purpose of this section is to prove a version of this theorem depending on parameters. Let \(X\) be a complex manifold and let \((\pi, V)\) be a holomorphic family of objects in \(H\mathcal{F}(G)\) based on the connected complex manifold \(X\) (see Definition 34). Then \((d\pi, V_K)\) is an holomorphic family of objects in \(H(\mathfrak{g}, K)\).

If \((\mu, Z)\) is a holomorphic family of objects in \(H(\mathfrak{g}, K)\) based on the complex manifold \(X\) then a correspondence \(x \mapsto \lambda_x \in Z^*\) will be called Holomorphic if \(x \mapsto \lambda_x(v)\) is holomorphic for all \(v \in Z\). A holomorphic correspondence \(x \mapsto \lambda_x\) with \(\lambda_x \in (Z_x)^*_\text{mod}\) is said to be of local uniform moderate growth if for each compact subset \(\omega \subset X\) there exists \(d_\omega\) such that if \(x \in \omega\) and \(v \in Z\) then

\[|f_{\lambda_x,v}(g)| \leq C_v \|g\|^{d_\omega}\]

for \(v \in Z, x \in X, g \in G\).

The purpose of this section is to prove

**Theorem 43** Let \(X\) be a complex manifold and let \((\pi, V)\) be a holomorphic family of objects in \(H\mathcal{F}(G)\) based on the connected complex manifold \(X\) and
if \( x \mapsto \lambda_x \) is a holomorphic correspondence with \( \lambda_x \in (V_x|K)^*_\text{mod} \) of local uniform moderate growth then the extension of \( \lambda_x \) to an element of \( V' \) (also denoted \( \lambda_x \)) is weakly holomorphic in \( x \) (i.e. \( x \mapsto \lambda_x(v) \) is holomorphic for \( v \in V \)).

The proof follows the method of the proof of Proposition 11.6.2 in \cite{RRG} to prove a continuous version of the theorem. The holomorphic version will be derived from the continuous version. Let \( x_o \in X \) and let \( U \subset X \) be an open neighborhood of \( x_o \) with compact closure in \( X \) such that there exists \( (\mu, H) \) a continuous family of Hilbert representations based on such that \( (d\pi|U, V_K) \) is isomorphic with \( (d\mu, H^{\infty}_K) \) (Theorem \[38\]). Then there exists \( d = d_U \) and \( C = C_U \) such that \( \|f\mu_x(g)\| \leq C \|g\|^d \) for \( g \in G \) and \( x \in U \). Also, since \( \lambda \) has uniform moderate growth there exists \( m \) and for each \( v \in V_K \) there exists \( A_v \) such that if \( g \in G \) then

\[
|f_{\lambda_x,v}(g)| \leq A_v \|g\|^m 
\]

for \( x \in U \). Set \( s = \max\{d, m\} \). Let \( v_1, \ldots, v_n \) be an orthonormal basis of a \( K \)- and \( Z(g) \)-invariant subspace of \( W \) in \( H_K \) such that \( V_K = d\pi_x(U(g))W \) for \( x \in U \) (Theorem \[21\]). Also choose \( d_o \) such that

\[
L = \int_G \|g\|^{-d_o} \, dg < \infty. 
\]

If \( v, w \in H \) then define a new inner product

\[
\langle v, w \rangle_{1,x} = \sum_{i=1}^n \int_G \langle \mu_x(g)v_i, w \rangle \langle v, \mu_x(g)v_i \rangle \|g\|^{-2s-d_o} \, dg. 
\]

This integral converges uniformly in \( x \in U \) since

\[
|\langle \mu_x(g)v_i, w \rangle \langle v, \mu_x(g)v_i \rangle| \leq \|\mu_x(g)\|^{2s} \|v\| \|w\| 
\]

which also implies

\[
\|v\|_{1,x}^2 \leq L \|v\|^2. 
\]

Set \( H_{1,x} \) equal to the Hilbert space completion of \( H \) with respect to \( \langle \ldots, \ldots \rangle_{1,x} \) for \( x \in U \). Noting that relative to \( \langle \ldots, \ldots \rangle_{1,x} \) the action of \( K \) is unitary, so the action of \( K \) on \( H \) extends to \( H_{1,x} \). Let \( \hat{\mu}_x(g) = \mu_x(g^{-1})^* \) with respect to \( \langle \ldots, \ldots \rangle \). Note that \( (\hat{\mu}, H) \) is a continuous family of Hilbert representations based on \( U \) (Corollary \[27\]). Also note that

\[
\|\hat{\mu}_x(h)v\|_{1,x}^2 = \sum_{i=1}^n \int_G \langle \mu_x(g)v_i, \hat{\mu}_x(h)v_i \rangle^2 \|g\|^{-2s-d_o} \, dg = 
\]
\[
\sum_{i=1}^{n} \int_{G} |\langle \mu_{x}(h^{-1}g)v_{i}, v \rangle|^{2} \|g\|^{-2s-\delta_{0}} \, dg = \sum_{i=1}^{n} \int_{G} |\langle \mu_{x}(g)v_{i}, v \rangle|^{2} \|hg\|^{-2s-\delta_{0}} \, dg.
\]

If \( h, g \in G \)

\[
\|g\| = \|h^{-1}hg\| \leq \|h^{-1}\| \|hg\| = \|h\| \|hg\|
\]

so

\[
\|hg\| \geq \|h\|^{-1} \|g\|.
\]

Hence \( \|hg\|^{-2s-\delta_{0}} \leq \|h\|^{2s+\delta_{0}} \|g\|^{-2s-\delta_{0}} \). Thus

\[
\|\mu_{x}(h)v\|_{1,x}^{2} \leq \|h\|^{2s+\delta_{0}} \|v\|_{1,x}^{2}.
\]

Using this, it is easily seen that for each \( x \in U \), \( \mu_{x} \) extends to a strongly continuous representation of \( G \) on \( H_{1,x} \).

I. The \( K - C^{\infty} \) vectors of \( (\mu_{x}, H_{1,x}) \) are the same as the \( G - C^{\infty} \) vectors.

To prove this assertion note that if \( C \) is the Casimir operator of \( G \) corresponding to the invariant form \( B \) (see the beginning of section 2). Also set \( C_{K} \) equal to the Casimir operator of \( K \) corresponding to \( B_{t} \). Set \( \Delta = C - 2C_{K} \). Then elliptic regularity implies that the \( C^{\infty} \) vectors of \( G \) in \( H_{1,x} \) are the completion of \( V_{K} = H_{K} = (H_{1})_{K} \) with respect to the seminorms \( p_{k,x}(v) = \|d\mu_{x}(\Delta^{k})v\|_{1,x} \).

One has

\[
\Delta^{k} = \sum_{j=0}^{k} (-2)^{j} \binom{k}{j} C^{k-j}C_{K}^{j}.
\]

If \( v \in V \) then

\[
\|d\mu_{x}(C)v\|_{1,x}^{2} = \sum_{i=1}^{n} \int_{G} |\langle v_{i}, \mu_{x}(g)d\mu_{x}(C)v \rangle|^{2} \|g\|^{-2s-\delta_{0}} \, dg =
\]

\[
\sum_{i=1}^{n} \int_{G} |\langle d\mu_{x}(C)v_{i}, \mu_{x}(g)v \rangle|^{2} \|g\|^{-2s-\delta_{0}} \, dg.
\]

Now \( d\mu_{x}(C)v_{i} = \sum a_{ij}(x)v_{j} \) hence \( \langle d\mu_{x}(C)v_{i}, \mu_{x}(g)v \rangle = \sum a_{ij}(x) \langle v_{j}, \mu_{x}(g)v \rangle \). Hence setting \( A = \max_{i,j, x \in U} \{|a_{ij}(x)|\} \)

\[
|\langle d\mu_{x}(C)v_{i}, \mu_{x}(g)v \rangle| = |\sum_{j} a_{ij} \langle v_{j}, \mu_{x}(g)v \rangle| \leq A \sum_{j} |\langle v_{j}, \mu_{x}(g)v \rangle|
\]

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so
\[
\sum_i |\langle d\mu_x(C)v_i, \hat{\mu}_x(g)v \rangle|^2 \leq nA^2 \left( \sum_j |\langle v_j, \hat{\mu}_x(g)v \rangle| \right)^2 \\
= n^2A^2 \sum_i |\langle v_i, \hat{\mu}_x(g)v \rangle|^2.
\]

Thus
\[
\|d\mu_x(C)v\|_{1,x} \leq nA \|v\|_{1,x}.
\]

Set \(B = nA\). Then
\[
\left\|d\hat{\mu}_x(D^k) v\right\|_{1,x} = \left\|\sum_{j=0}^k (-2)^j \binom{k}{j} d\hat{\mu}_x(C^{k-j}) C^j_k v\right\|_{1,x} \\
\leq \sum_{j=0}^k (2)^j \binom{k}{j} B^{k-j} \left\|C^j_k v\right\|_{1,x} \leq \sum_{j=0}^k (2)^j \binom{k}{j} B^{k-j} \|(I + C_K)^j v\|_{1,x}
\]
The \(K-C^\infty\) vectors are the completion of \(H_K\) using the seminorms \(q_{k,x}(v) = \|(I + C_K)^k v\|_{1,x}\). This proves I.

Set \(\mu_{1,x}(g)\) equal to the adjoint of \(\hat{\mu}_x(g^{-1})\) with respect to \(\langle ..., ..., \rangle_{1,x}\). Then the space \(K\)-finite vectors of \(\mu_{1,x}\) is \((H_{1,x})_K = H_K = V_K\) and the corresponding \((g, K)\)-module is the conjugate dual to \(V_K\) and \(d\mu_x\) which is the same as the action of \(d\mu_x\). The CW theorem implies that the space of \(C^\infty\)-vectors of \(\mu_{1,x}\) is \(V\). In particular, if \(u \in H_{1,x}\) then the functional \(v \mapsto \langle v, u \rangle_{1,x}\) is a continuous functional on \(V\).

II. Let \(\lambda_x\) be as in the statement of the theorem. Then there exists \(w_x \in H_{1,x}\) such that for \(v \in V_K\), \(\lambda_x(v) = \langle v, w_x \rangle_{1,x}\) and if \(v \in V\) then the map \(x \mapsto \langle v, w_x \rangle_{1,x}\) is continuous on \(U\).

Note that if \(\mu \in V_K^*\) then for each \(\gamma \in \hat{K}\) there exists \(w_\gamma \in V(\gamma)\) such that \(\mu(v) = \langle v, w_\gamma \rangle\) for \(v \in V(\gamma)\). Let \(E_\gamma\) denote the projection of \(V_K\) to \(V(\gamma)\) corresponding to the direct sum decomposition \(V_K = \bigoplus_{\gamma \in \hat{K}} V(\gamma)\). Set \(\tau_\gamma(\mu) = w_\gamma\). \(\mu_\gamma = \mu \circ E_\gamma\). If \(v \in V_K\) and if \(\gamma \in \hat{K}\) and if \(\chi_\gamma\) is the character of \(\gamma\) and \(d(\gamma)\) is the dimension of \(\gamma\) and if
\[
f_{\lambda_x,v,\gamma}(g) = d(\gamma) \int_K \chi(k)f_{\lambda_x,v}(k^{-1}g)dk
\]

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then 1. and 2. above imply

\[ f_{\lambda_x,v,\gamma}(1) = d(\gamma) \int_K \chi(k) f_{\lambda_x,v}(k^{-1}) dk = \lambda_x(E_\gamma v). \]

Also 3. implies that

\[ u f_{\lambda_x,v,\gamma} = f_{\lambda_x,d\sigma_x(u),v,\gamma}, R(k) f_{\lambda_x,v,\gamma} = f_{\lambda_x,\sigma(k) v,\gamma}, \quad u \in U(g), \quad k \in K. \]

Thus

\[ f_{\lambda_x,v,\gamma} = f(\lambda_x)_v. \]

Also

\[ f(\lambda_x)_v(g) = \langle \mu_x(g)v, \tau_\gamma(\lambda_x) \rangle. \]

We will now show that the series \( \sum \| \tau_\gamma(\lambda_x) \|_{1,x}^2 \) converges uniformly in \( x \in U \). Indeed, the Schur orthogonality relations and the \( K \) bi-invariance of the norm on \( G \) imply that if \( v \in V_K \) then

\[ \int_G |f_{\lambda_x,v}(g)|^2 \| g \|^{-2s_U-d_o} dg = \sum_{\gamma \in K} \int_G |f(\lambda_x)_v(g)|^2 \| g \|^{-2s_U-d_o} dg. \]

Hence

\[ \infty > \sum_{\gamma \in K} \int_G |f(\lambda_x)_v(g)|^2 \| g \|^{-2s_U-d_o} dg = \sum_{\gamma \in K} \int_G |\mu_x(g)v, \tau_\gamma(\lambda_x)\|^2 \| g \|^{-2s_U-d_o} dg = \]

\[ \sum_{\gamma \in K} \| \tau_\gamma(\lambda_x) \|_{1,x}^2. \]

Since \( V \subset H_{1,x} \) for all \( x \in U \) the if \( v \in V \) then series

\[ \sum_{\gamma \in K} \langle v, \tau_\gamma(\lambda_x) \rangle_{1,x} \]

converges absolution an uniformly in \( x \in U \) defining the continuous family of extensions of \( \lambda_x \) to \( V \) for \( x \in U \). This proves II.

III. The extension of \( \lambda_x \) to \( V \) for \( x \in X \) depends weakly continuously on \( x \) (that is, if \( v \in V \) then \( x \mapsto \lambda_x(v) \) is continuous).

This follows from II. since continuity is a local property.
We now complete the proof of the theorem. Let $x_0 \in X$ and let $(\Psi, U)$ be a holomorphic chart for $X$ with $x_0 \in U$ and such that $\Psi(x_0) = 0$ and $\Psi(U)$ contains the closure of the polydisk $D = \{(z_1, ..., z_m)||z_i| < 1, i = 1, ..., m\}$. It is enough to prove the holomorphy assertion on $\Psi^{-1}(D)$. If $x \in \Psi^{-1}(D)$ write $\Psi(x) = (x_1, ..., x_m)$. Define $\xi_x$ for $x \in \Psi^{-1}(D)$ by

$$\xi_x(v) = \frac{1}{(2\pi i)^m} \int_{S^1 \times S^1 \times \cdots \times S^1} \frac{\lambda_z(v)dz_1 \cdots dz_m}{\prod_{j=1}^m z_j - x_j}.$$  

This integral defines a holomorphic function of $x$ on $D$ for each $v \in V$. If $v \in V_K$ then $\xi_x(v) = \lambda_x(v)$. Thus $\xi_x = \lambda_x$ on $V$.

The proof of this result implies our main theorem in the holomorphic case.

**Corollary 44** (To the proof) Let $(\pi, V)$ be a Holomorphic family of objects in $H(g, K)$ based on the complex manifold $X$ then the continuous family $(\lambda, V)$ in Theorem 40 and Corollary 41 is a holomorphic family of objects in $H_\mathcal{F}(G)$ based on $X$.

**Proof.** The first part of the proof of the previous theorem uses only the continuity of the family of objects in $H_\mathcal{F}(G)$. Thus if $\xi \in V$ and if $v \in V$ then $\xi(\lambda_x(g)v)$ is continuous in $x$ and smooth in $g$. $\xi \circ \lambda_x(g)|v \in V^*_{mod}$. and $x \mapsto \mu(\lambda_x(g)v)$ is holomorphic for $v \in V$. The first part of the proof shows that $x \mapsto \mu(\lambda_x(g)v)$ is continuous for $v \in V$. If $x_o \in X$ and the notation is as in the last part of the proof set for $x \in \Psi^{-1}(D)$ and $g \in G$

$$\eta_{x,g}(v) = \frac{1}{(2\pi i)^m} \int_{S^1 \times S^1 \times \cdots \times S^1} \frac{\mu(\lambda_z(g)v)dz_1 \cdots dz_m}{\prod_{j=1}^m z_j - x_j}.$$  

Then $x \mapsto \eta_{x,g}(v)$ is holomorphic in $x \in \Psi^{-1}(D)$, $C^\infty$ in $g$ and continuous in $v \in V$. But $\eta_{x,g}(v) = \mu(\lambda_x(g)v)$ for $v \in V$. This implies $\eta_{x,g}(v) = \mu(\lambda_x(g)v)$ for $v \in V$. We conclude that $x \mapsto \lambda_x(g)v$ is weakly holomorphic with values in $V$. This implies that it is strongly holomorphic (Grothendieck [G], Theorem 8 and its corollary).

## 12 Application to $C^\infty$ Eisenstein series

This section will involve terminology that would take us too far afield to explain completely. Also, only those that would be bored with the explanations
would be interested in the results. For details in what is omitted we suggest Langlands [L1]. Let $G$ be a real reductive group of inner type. Let $\Gamma$ be a discrete subgroup of $G$ such that $\Gamma \backslash G$ has finite volume. The results will be true for the class of $G$ and $\Gamma$ described in Chapter 1 of Langlands [L2]. However, we will only consider the subclass of $G = G_\mathbb{R}$ the real points of an algebraic group, $G$, defined over $\mathbb{Q}$ satisfying one more condition which we will describe later in this paragraph, and $\Gamma$ a subgroup that is of finite index in the points of a $\mathbb{Z}$–form of $G_\mathbb{Q}$ (the $\mathbb{Q}$–points), i.e. an arithmetic subgroup. A cuspidal parabolic subgroup of $G$ is the normalizer $P$ of a parabolic subgroup $P$ of $G$ defined over $\mathbb{Q}$. Then $P$ has a $\mathbb{Q}$–Langlands decomposition $P = MAN$ with $N$ the unipotent radical of $P$ and $M$ the intersection of the kernels of $\chi^2$ with $\chi : M_P \rightarrow \mathbb{R}^\times$ a character defined over $\mathbb{Q}$ and $M_P$ is a Levi-factor of $P$ that is defined over $\mathbb{Q}$. The other condition is that the “$A$” in the Langlands decomposition of $G$ is trivial. Then $\Gamma \cap P \subset MN$ and identifying $M$ with $MN/N$ then $\Gamma_M = (\Gamma \cap P)/(\Gamma \cap N)$ is an arithmetic subgroup of $M$.

Throughout this section $P$ will be a fixed Let $V$ be space of $C^\infty$ vectors of a closed, $M$–invariant, irreducible subspace of $L^2(\Gamma_M \backslash M)$. Let $\sigma$ denote the right regular action of $M$ on $V$. Let $K$ be a maximal compact subgroup of $G$ such that $M \cap K$ is maximal compact in $M$. We consider the smooth representation $(\pi_\nu, I^\infty_V)$ where $\nu \in \mathfrak{a}_C^\ast$, $\mathfrak{a} = \text{Lie}(A)$ and $I^\infty_V$ is the space of all $C^\infty$ functions from $K$ to $V$ such that $f(mk) = \sigma(m)f(k)$ for $m \in K \cap M$ and $k \in K$. If $f \in I^\infty_V$ define $f_\nu(nmak) = a^{\nu + \rho}(m)f(k)$ for $n \in N, m \in M, a \in A, k \in K$ and $\rho(h) = \frac{1}{2}\text{tr}(ad(h)|_{\text{Lie}(N)})$ for $h \in \mathfrak{a}$. Then since the ambiguity in the expression of an element $g \in G$ as $g = namk, n \in N, a \in A, m \in M, k \in K$ is in $M \cap K$. $f_\nu$ is a $C^\infty$ map of $G$ to $V$. We define $\pi_\nu(g) = f_\nu(kg)$. Endow $I^\infty_V$ with the $C^\infty$ topology so $I^\infty_V$ is a Fréchet space. Note that if we set $\pi(\nu, g) = \pi_\nu(g)$ then $(\pi, I^\infty_K)$ is a holomorphic family of objects in $\mathcal{H}(\mathcal{F}(G))$ based on $\mathfrak{a}_C^\ast$.

If $f \in I^\infty_V$ set $f_\nu(nmak) = a^{\nu + \rho}(m)f(k)(m)$ for $n \in N, m \in M, a \in A, k \in K$. Then $f_\nu \in C^\infty((\Gamma \cap P) \backslash G)$. Consider

$$E(P, f, \nu)(g) = \sum_{\gamma \in (\Gamma \cap P) \backslash \Gamma} f_\nu(\gamma g).$$

This series converges uniformly in $\nu$ in the set of all $\nu \in \mathfrak{a}_C^\ast$ with $\nu(\hat{\alpha}) > \rho(\hat{\alpha})$ for all roots $\alpha$ of $A$ acting on $\text{Lie}(N)$. Langlands has shown that if $f$ is in $(I^\infty_V)_K$ then this series has a meromorphic continuation to all $\nu \in \mathfrak{a}_C^\ast$. In this
section a proof will be given that the meromorphic continuation is true for all $f \in \hat{I}_V^\infty$.

Note that where the defining series converges $E(P, f, \nu)$ is an automorphic function. Here, a smooth function, $\varphi$, on $\Gamma \backslash G$ is called an automorphic function if

1. $\varphi$ is $Z(\mathfrak{g})$–finite and
2. There exists $d$ such that if $u \in U(\mathfrak{g})$ looked upon as a left invariant vector field that $|u \varphi(g)| \leq C_u \|g\|^d$ for all $g \in G$.

Usually the condition that $\varphi$ is right $K$–finite is also included in the definition.

**Lemma 45** If $g \in G$, $f \in \hat{I}_V^\infty$ then in the range of convergence

$$E(P, \pi_\nu(g)f, \nu) = \pi_{\Gamma}(g)E(P, f, \nu).$$

Here $\pi_{\Gamma}$ is the right regular representation of $G$ on $\Gamma \backslash G$.

**Proof.** This follows from

$$f_\nu(g) = f_\nu(e)$$

with $e$ the identity element of $G$ hence of $M$. □

If $f \in (\hat{I}_V^\infty)_K$ and $\nu_o \in \mathfrak{a}_C^*$ then there exists an open neighborhood of $\nu_o$, $U$ in $\mathfrak{a}_C^*$ and $\alpha$ a non-zero holomorphic function on $U$ such that $\nu \mapsto \alpha(\nu)E(P, f, \nu)(g)$ is holomorphic. $\mathcal{S}(f, \nu_o)$ be the set of pairs $(U, \alpha)$ with $U$ in $\mathfrak{a}_C^*$ and $\alpha$ a non-zero holomorphic function on $U$ such that

$$\nu \mapsto \alpha(\nu)E(P, f, \nu)(g)$$

is holomorphic on $U$. If $W$ is an open subset of $\mathfrak{a}_C^*$ with compact closure then there exists a finite subset $F_W \subset \hat{K}$ such that

$$d\pi_\nu(U(\mathfrak{g}_C))\left(\sum_{\gamma \in F_W} I_\nu^\infty(\gamma)\right) = (I_V^\infty)_K$$

for $\nu \in W$ (Theorem [21]). Let $f_1, \ldots, f_m$ be a basis of $\sum_{\gamma \in F_W} I_\nu^\infty(\gamma)$ then if $\nu_o \in W$ and $(U_i, \alpha_i) \in \mathcal{S}(f_i, \nu_o)$ then if $\beta = \alpha_1 \cdots \alpha_m$, $Z = U_1 \cap \cdots \cap U_m \cap W$ then $\nu \mapsto \beta(\nu)E(P, f, \nu)(g)$ is holomorphic in $\nu \in Z$ for all $g \in G$ and $f \in (I_V^\infty)_K$. 35
Proposition 46 If $Z$ is open in $\mathfrak{a}^\ast_c$ with compact closure such that if $\beta$ is holomorphic on $Z$ such that $\nu \mapsto \beta(\nu)E(P, f, \nu)(g)$ is holomorphic on $Z$ for all $g \in G$ and $f \in (I^\infty_V)_K$ then there exists $d_Z$ such that

$$|\beta(\nu)E(P, f, \nu)(g)| \leq C_f \|g\|^{d_Z}, f \in (I^\infty_V)_K, g \in G.$$ 

Proof. Lemma 5.1 in [L2] implies that if the constant terms of $\beta(\nu)E(P, f, \nu)$ relative to $Q$–rank one parabolic subgroups $P_i$ containing $P$ have exponents $a_{\mu_i,j}(\nu)$ and if $a_{\text{Re} \mu_i,j}(\nu) \leq C \|a\|^{d_{ij}}$ then

$$|\beta(\nu)E(P, f, \nu)(g)| \leq C_1 \|g\|^\text{max}_{i,j} d_{ij}+1$$

for $g \in G$ (1 added to the exponent is to dominate the logarithmic term in Langlands’ inequality). On the other hand, the main observation in [W2] implies that the $\mu_{ij}$ are restrictions of exponents of the $(g, K)$–module $(\pi_\nu, (I^\infty_V)_K)$. This implies that $\|\mu_{ij}(\nu)\|$ is bounded by the maximum of the norms of the Harish-Chandra parameters of $(\pi_\nu, (I^\infty_V)_K)$ (here the norms are with respect to the Hermitian extension of the inner product $-B(X, \theta Y)$ on $g$). Thus since the closure of $Z$ is compact there exists $s$ such that $a_{\text{Re} \mu_{ij}(\nu)} \leq C \|a\|^s$ for $\nu \in Z$. Take $d_Z = s + 1$. \[\Box\]

We are now ready to prove

Theorem 47 If $f \in I^\infty_V$ then $E(P, f, \nu)(g)$ has a meromorphic continuation to $\mathfrak{a}^\ast_c$.

Proof. Let $\nu_o \in \mathfrak{a}^\ast_c$ and let $Z$ be an open neighborhood of $\nu_o$ with compact closure such that there exists $\beta$ and $d_Z$ as above. If $f \in (I^\infty_V)_K$ define $\lambda_\nu(f) = \beta(\nu)E(P, f, \nu)(e)$. Then $\lambda_\nu \in (I^\infty_V)_K^\ast$, and if we set $f_{\lambda_\nu,f}(g) = \beta(\nu)E(P, f, \nu)(g)$ then the above lemma shows that $\nu \mapsto \lambda_\nu$ is of uniform moderate growth on $Z$ hence satisfies the hypotheses of Theorem 43 which implies that the extension of $\lambda_\nu$ to $I^\infty_V$ is weakly holomorphic in $\nu$. Since $\nu_o$ is arbitrary in $\mathfrak{a}^\ast_c$ this completes the proof. \[\Box\]

If we put the $L^2$ inner product $\langle \cdot, \cdot \rangle$ on and the inner product

$$\langle f_1, f_2 \rangle = \int_K (f_1(k), f_2(k)) dk.$$ 

on $I^\infty_V$ and complete to a Hilbert space $H_V$ then the operators $\pi_\nu(g)$ extend to $H_V$ and the family $(\pi, H_V)$ defines an analytic family of Hilbert representations based on $\mathfrak{a}^\ast_c$. Now one can apply the argument in the proof of
Theorem 43 directly to give a proof of the meromorphic continuation of $C^\infty$ Eisenstein series.

References

[A] M. Atiyah, K–Theory, W.A.Benjamin Inc. New York, 1969.

[BN] J. Barros-Neto, Spaces of vector valued real analytic functions. Trans. Amer. Math. Soc. 112 1964 381–391.

[BB] D. Booss and D.D. Bleeker, Topology and Analysis, The Atiyah-Singer Theorem and Guage-Thoeretic Physics, Universitext, Springer-Verlag, New York, 1985.

[BK] J. Bernstein and B. Krötz, Smooth Fréchet globalizations of Harish-Chandra modules. Israel J. Math. 199 (2014), no. 1, 45–111.

[BW] A. Borel and N. Wallach, Continuous cohomology, discrete subgroups and representations of reductive groups, Second Edition, Mathematical Surveys and Monographs, Volume 67, AMS, Providence, RI, 2000.

[G] A. Grothendieck, Sur certains espaces de fonctions holomorphes. II. J. Reine Angew. Math. 192, (1953). 77–95.

[HOW] Jing-Song Huang, Toshio Oshima and Nolan Wallach, Dimensions of spaces of generalized spherical functions. Amer. J. Math. 118 (1996), no. 3, 637–652.

[KR] Bertram Kostant and Stephen Rallis, Orbits and Lie group representations associated to symmetric spaces, Amer. Jour. Math. 93 (1971), 753-809.

[L1] R. P. Langlands, Eisenstein Series, Proceedings of Symposia in Pure Mathematics, Volume IX, American Mathematical Society, 1966, 235-252.

[L2] Robert P. Langlands, On the functional equations satisfied by Eisenstein series, Springer–Verlag Lecture Notes in Math., Springer–Verlag, Berlin–Heidelberg, New York, 1976.
[RS] M. Reed and B. Simon, Functional Analysis, Method of Mathematical Physics I, Academic Press, New York, 1972.

[VdN] Vincent van der Noort, Analytic parameter dependence of Harish-Chandra modules for real reductive groups, A family affair, Thesis University of Utrecht, 2009.

[W1] Nolan R. Wallach, Harmonic analysis on homogeneous spaces, Second Edition, Dover Publications, Mineola, 2018.

[W2] N. R. Wallach, On the constant term of a square integrable automorphic form. Operator algebras and group representations, Vol. II (Nep- tun, 1980), 227–237, Monogr. Stud. Math., 18, Pitman, Boston, MA, 1984.

[RRG] Nolan R. Wallach, Real reductive groups, I, II, Academic Press, San Diego, 1988, 1992.