Mixed finite element formulations and energy-momentum time integrators for thermo-viscoelastic fiber-reinforced continua

Julian Dietzsch\textsuperscript{1,*}, Michael Groß\textsuperscript{1,**}, and Richard Friedrich Schuffenhauer\textsuperscript{1,***}

\textsuperscript{1} TU Chemnitz, Professor of applied mechanics and dynamics, Reichenhainer Straße 70, D-09126 Chemnitz, Germany.

Today, fiber-reinforced materials and their exact dynamic simulation play a significant role in the construction of lightweight structures. These materials are used in aircraft, automobiles and wind turbines, for instance. The low density and the high modulus of elasticity play a major role, but also the thermal properties should not be neglected. First of all, the thermal expansion of the matrix part and the ability to conduct the heat in a directional way with the fibers. For these materials, volumetric locking effects of an incompressible matrix material as well as locking effects due to stiff fibers can occur. On the one hand, there are combinations of well known mixed elements with an independent approximation of the volume dilatation and an independent approximation of the right Cauchy-Green tensor for the anisotropic part of the strain energy function to reduce these effects. On the other hand, we have developed mixed elements where fields for the fourth and fifth invariants are added, as well as a version with the corresponding tensor fields. For long-term simulations it is necessary to use higher order time integrators to perform an accurate dynamic simulation. Galerkin-based time integrators offer a good option for this application. To eliminate a huge energy error these have to be extended to an energy-momentum time integration scheme. It is logical to combine these methods and thus combine the advantages of these methods. We formulate the mixed elements using Hu-Washizu functionals and combine this with the mixed principle of virtual power. By adding a thermo-mechanical coupling part in the strain energy and introducing Fourier’s heat conduction we obtain a thermo-mechanical formulation for the different mixed elements and a higher order Galerkin-based time integrator. Dirichlet boundary conditions in the form of Lagrange multiplier methods as well as Neumann boundary conditions in the mechanical and thermal context are also provided. In addition, we extend the continuum so that we can model different fiber families and the directional heat conduction of the fibers. As numerical examples serve cook’s cantilever beam as well as a rotating heat pipe. We primarily analyze the spatial and time convergence, the conservation properties as well as the effect of the heat conduction of the fibers.

\[ \Pi^{\text{int}} = \int_{B_0} \frac{1}{2} \mathbf{S} : (\mathbf{C} - \tilde{\mathbf{C}}) d\mathbf{V} + \int_{B_0} \tilde{\mathbf{S}} : \dot{\mathbf{C}} d\mathbf{V} + \int_{B_0} \eta \left( \Theta - \tilde{\Theta} \right) d\mathbf{V} + \int_{B_0} \psi d\mathbf{V} + \int_{B_0} p \left( J(\tilde{\mathbf{C}}) - \tilde{J} \right) d\mathbf{V} + \int_{B_0} \tilde{p} \tilde{J} d\mathbf{V} \]

In order to obtain the discrete Euler-Lagrange equations, we use the mixed principle of virtual power \cite{5}, based on the total energy balance \( \mathcal{H} = \dot{T} + \Pi^{\text{ext}} + \Pi^{\text{int}} \), consisting of the time derivative of the internal power functional \( \Pi^{\text{int}} \), the kinetic power functional \( \dot{T} \) and the external power functional \( \Pi^{\text{ext}} \). The free energy function \( \Psi \) depends on the right Cauchy-Green tensor \( \mathbf{C} \) and the absolute temperature \( \Theta \), so that the applied internal energy functional \( \Pi^{\text{int}} \) is given by

\[ \Pi^{\text{int}} = \int_{B_0} \frac{1}{2} \mathbf{S} : (\mathbf{C} - \tilde{\mathbf{C}}) d\mathbf{V} + \int_{B_0} \tilde{\mathbf{S}} : \dot{\mathbf{C}} d\mathbf{V} + \int_{B_0} \eta \left( \Theta - \tilde{\Theta} \right) d\mathbf{V} + \int_{B_0} \psi d\mathbf{V} + \int_{B_0} p \left( J(\tilde{\mathbf{C}}) - \tilde{J} \right) d\mathbf{V} + \int_{B_0} \tilde{p} \tilde{J} d\mathbf{V} \]

We introduce an independent mixed field \( \tilde{\mathbf{C}} \) and the corresponding Lagrange multiplier \( \tilde{\mathbf{S}} \), the superimposed stress tensor \( \tilde{\mathbf{S}} \) to derive an energy–momentum scheme and the assumed temperature field \( \tilde{\Theta} \) with the corresponding entropy density field \( \eta \). The operators \( \text{cof}(\tilde{\mathbf{C}}) \) define the cofactor of \( \tilde{\mathbf{C}} \) and \( J(\tilde{\mathbf{C}}) \) the volume dilatation. The remaining terms include further independent fields. First, the independent volume dilatation \( \tilde{J} \) according to displacement-pressure formulation shown in \cite{1} (acronym DP). An independent field for the cofactor \( \tilde{\mathbf{H}} \) leads to the formulation introduced in \cite{2} (acronym CoFEM). By adding a field \( \tilde{\mathbf{C}}_A \) in the anisotropic strain energy function, we obtain the formulation shown in \cite{3} (CoSKA). At least we introduce independent fields for the cofactor \( \tilde{\mathbf{H}}_A \), and the volume dilatation \( \tilde{J}_A \) of the anisotropic strain energy function (CoCoA). For each independent field, the corresponding second term introduce the superimposed field to arrive at an energy–momentum scheme \cite{5}. The external power functional \( \Pi^{\text{ext}} \), including the Piola heat flux vector, the Dirichlet boundary conditions and the internal viscous dissipation, are shown in \cite{5, 6}. Note that, we introduce an independent field for the viscous internal variable. In both references, the complete definition of the kinetic power functional \( \dot{T} \) and the free energy function \( \Psi \) is also shown.

* Corresponding author: e-mail julian.dietzsch@mb.tu-chemnitz.de, phone +49 371 531 31929, fax +49 371 531 831929
** e-mail michael.gross@mb.tu-chemnitz.de
*** e-mail tmd@mb.tu-chemnitz.de

© 2021 The Authors. Proceedings in Applied Mathematics & Mechanics published by Wiley-VCH GmbH
2 Numerical Examples

Fig. 1: Convergence rates of the tip, deformed configurations \( \mathcal{B}_t \) with v. Mises equivalent stress \( \sigma_{VM} \) and the absolute temperature \( \theta \), respectively, of the Cook’s cantilever beam. Approximation with \( n_{el} = 32 \) spatial finite elements and linear time elements for time step size \( h_n = 0.002 \). The digits of the element name represent the polynomial degrees of the mixed variables in the order \( C, H, J, C_A, H_A, J_A \).

First, we consider the example of Cook’s cantilever beam prescribed in [6]. Here, we use a first fiber for a mechanical reinforcement, and a second non-stiff fiber with a very high thermal conductivity. In Fig. 1 on the left, we recognize that the CoCoA elements with a low polynomial degree for the anisotropic fields have the highest convergence rate at the upper tip, followed by the CoFEM elements and the standard elements. In the other plots, we see the multiaxial bending caused by the fibers as well as the typical stress distribution. Regarding the temperature distribution, we recognize that due to the viscous dissipation and the thermomechanical coupling, the temperature increases along the fibers.

Our second example is a rotating heatpipe [4], in which we use a carbon fiber reinforced epoxy resin (see [7]) with thermal and viscous material parameters. The heatpipe rotation with rotational speed \( \Omega = 10 \) is realized via a transient mechanical Dirichlet boundary on the left. A transient thermal Dirichlet boundary is applied to the left side with a frequency of \( f = 10 \) and an amplitude \( \Theta = 100 \). Additionally, an inner pressure \( \hat{p} = 0.5 \cdot 10^8 \) acts in the pipe. Two families of circumferential fibers are arranged in counter directions with angles of 45 degrees. Here, we apply the CoSKA2101 element. In Fig. 2, we show that the errors of the balance laws are below the Newton-Raphson tolerance. In Fig. 3, we see the v. Mises equivalent stress \( \sigma_{VM} \) and the absolute temperature \( \theta \), respectively. As expected, the stress is increased due to the inner pressure, especially where the wall thickness is less as under the ribs of the right side. The temperature is increased on the left due to the boundary condition.

Fig. 2: Error of the balance laws pertaining to the rotating heatpipe with the CoSKA2101 element and quadratic time finite elements with the time step size \( h_n = 10^{-4} \). The Newton-Raphson tolerance is denoted with the constant parameter TOL.

Fig. 3: V. Mises equivalent stress \( \sigma_{VM} \) and the absolute temperature \( \theta \) of the heatpipe for \( n_{el} = 8184 \) CoSKA2101 spatial finite elements.

3 Conclusion

The time integration of an anisotropic thermo-viscoelastic material formulation with an energy-momentum scheme, does not affect the excellent convergence rates of mixed finite elements. This saves computational time, which is especially advantageous with a local iteration of internal variables and the higher computational effort of an energy momentum scheme.

Acknowledgements The authors thank the ‘Deutsche Forschungsgesellschaft (DFG)’ for financial support under the grant GR 3297/4-1. Open access funding enabled and organized by Projekt DEAL.

References

[1] J. Simo, R. Taylor, and K. Pister, Comput. Methods Appl. Mech. Engrg. 1, 177-208 (1985).
[2] J. Schröder, P. Wriggers, and D. Balzani, Comput. Methods Appl. Mech. Engrg. 49, 3583-3600 (2011).
[3] J. Schröder, V. Viebahn, P. Wriggers, and D. Balzani, Comput. Methods Appl. Mech. Engrg. 310, 475-494 (2016).
[4] M. Groß, and J. Dietzsch, Comput. Methods Appl. Mech. Engrg. 320, 509-542 (2017).
[5] M. Groß, J. Dietzsch, and M. Bartelt, Comput. Methods Appl. Mech. Engrg. 336, 353-418 (2018).
[6] J. Dietzsch, M. Groß, and L. Flessing, Proceedings of 8th GACM Colloquium on Computational Mechanics, Kassel, Germany (2019), pp. 129-132.
[7] H. Dal, O. Gueltekin, F. Aksu Denli, and G. Holzapfel, PAMM 17, 91-94 (2017).