A refinement of Sharkovskii’s theorem on orbit types characterized by two parameters

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Abstract

The so-called type problem or forcing problem is considered as a way to generalize Sharkovskii’s theorem. In this paper, by focusing on certain types of orbits, we obtain a solution of the type problem, which gives a refinement of Sharkovskii’s theorem on orbit types characterized by two parameters.

1 Introduction

For continuous interval maps, in 1964, the remarkable theorem of Sharkovskii [7] gave the complete answer of the following question: Given a periodic orbit of a specified period, find the other periods of periodic orbits that must exist (Theorem 1). One can classify types of orbits, up to symmetry, depending on the arithmetic ordering of the points in the real line; refer to [2]. For example, two period-4 orbits \( \{x_0 \to x_1 \to x_2 \to x_3 \to x_0\} \), one with \( x_0 < x_1 < x_2 < x_3 \) and the other one with \( x_0 < x_2 < x_1 < x_3 \), are considered to have different types. The so-called type problem or forcing problem is the following:

Given a period-\( n \) orbit of a specified type, find, for any positive integer \( m \), the types of period-\( m \) orbits that must exist.

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There are at least \((n - 1)!/2\) different types of period-\(n\) orbits. Due to the factorial growth of the number of different types, the complete solution of the type problem is not an easy task. The type problem is closely related to bifurcations of any one-parameter family of continuous interval maps in the sense that its solutions can show all possible routes of bifurcations of various types of periodic orbits. For more discussions on the type problem, refer to [1] and [4].

By considering the implications between existence of different orbit types characterized by one parameter, a well known solution of the type problem gives a refinement of Sharkovskii’s theorem (Theorem 2). In this paper, we extend the result by considering different orbit types characterized by two parameters (Theorem 3).

2 Definitions and Statements of Theorems

Let \(I\) be a nontrivial compact interval and let \(f\) be a continuous map from \(I\) into itself.

First of all, we give some basic definitions. The forward orbit of \(x_0\) for \(f\) is defined to be the sequence \(\{x_i : i \geq 0\}\), where \(x_i = f^i(x_0)\), i.e., the \(i\)-th iterate of \(f\) at \(x_0\). A sequence \(\{x_{-i} : i \geq 0\}\) is called a backward orbit of \(x_0\) for \(f\) if \(f(x_{-i}) = x_{-i+1}\) for all \(i \geq 1\). It is possible that a point has several backward orbits if \(f\) is not one-to-one. A point \(x_0\) is called a period-\(n\) point of \(f\) if its forward orbit \(\{x_i\}\) satisfies \(x_n = x_0\) and \(x_i \neq x_0\) for all \(0 < i < n\). A period-1 point is also called a fixed point.

We say that the property \(P(n)\) holds if \(f\) has a period-\(n\) point, and denote that \(P(n) \rightarrow P(m)\) if the property \(P(n)\) implies the property \(P(m)\).

\textit{Sharkovskii’s theorem} reads as follows.

\textbf{Theorem 1 ([7]).} Let \(f\) be a continuous map from \(I\) into itself. Then the following diagram holds:

\[
egin{align*}
P(3) & \rightarrow P(5) \rightarrow P(7) \rightarrow \cdots \\
\rightarrow P(2 \cdot 3) & \rightarrow P(2 \cdot 5) \rightarrow P(2 \cdot 7) \rightarrow \cdots \\
\rightarrow P(2^2 \cdot 3) & \rightarrow P(2^2 \cdot 5) \rightarrow P(2^2 \cdot 7) \rightarrow \cdots \\
\rightarrow \cdots & \rightarrow P(2^2) \rightarrow P(2) \rightarrow P(1).
\end{align*}
\]

In the following, some properties for \(f\) with specific types of orbits are defined.

\textbf{Definition 1.} Let \(k\) and \(n\) be positive integers. We say that, for \(f\),

1. the property \(L^k(n)\) holds if \(f^k\) has a period-\(n\) point \(x_0\) with the forward orbit \(\{x_i\}\) of the type either

\[
x_0 < x_1 < x_2 < \cdots < x_{n-1}
\]

or all inequalities reversed;
2. the property \( S^k(2n+1) \) holds if \( f^k \) has a period-(\( 2n+1 \)) point \( x_0 \) with the forward orbit \( \{x_i\} \) of the type either

\[
x_{2n} < x_{2n-2} < \cdots < x_2 < x_0 < x_1 < x_3 < \cdots < x_{2n-3} < x_{2n-1}
\]

or all inequalities reversed; and

3. the property \( L^k(\infty) \) holds if \( f^k \) has a fixed point \( x_0 \) with a backward orbit \( \{x_{-i}\} \) of the type either

\[
x_0 < \cdots < x_{-i+1} < \cdots < x_{-2} < x_{-1}
\]

or all inequalities reversed.

In [3], it is proved that \( L^k(\infty) \rightarrow L^k(n) \) and \( L^k(n+1) \rightarrow L^k(n) \); see also [5] for independent work. In [4], it is shown that \( S^k(2n+1) \rightarrow S^k(2n+3) \). Combining these results, together with Sharkovskii’s theorem [7] (see also [8]), one easily has the following refinement of Sharkovskii’s theorem.

**Theorem 2** ([3], [4], [7], [8]). Let \( f \) be a continuous map from \( I \) into itself. Then the following diagram holds:

\[
\begin{align*}
L^1(\infty) & \rightarrow \cdots \rightarrow L^1(5) \rightarrow L^1(4) \rightarrow L^1(3) \\
\leftrightarrow P(3) & \leftrightarrow S(3) \rightarrow P(5) \rightarrow S(5) \rightarrow P(7) \rightarrow S(7) \rightarrow \cdots \\
\leftarrow L^2(\infty) & \rightarrow \cdots \rightarrow L^2(5) \rightarrow L^2(4) \rightarrow L^2(3) \\
\leftarrow P(2 \cdot 3) & \leftarrow S^2(3) \rightarrow P(2 \cdot 5) \rightarrow S^2(5) \rightarrow P(2 \cdot 7) \rightarrow S^2(7) \rightarrow \cdots \\
\leftarrow L^2^2(\infty) & \rightarrow \cdots \rightarrow L^2^2(5) \rightarrow L^2^2(4) \rightarrow L^2^2(3) \\
\leftarrow P(2 \cdot 2 \cdot 3) & \leftarrow S^2^2(3) \rightarrow P(2 \cdot 2 \cdot 5) \rightarrow S^2^2(5) \rightarrow P(2 \cdot 2 \cdot 7) \rightarrow S^2^2(7) \rightarrow \cdots \\
\leftarrow \cdots & \leftarrow \cdots \rightarrow P(2^2) \rightarrow P(2) \rightarrow P(1).
\end{align*}
\]

We consider more properties for \( f \) with certain types of orbits.

**Definition 2.** Let \( k \), \( m \) and \( n \) be positive integers. We say that, for \( f \),

1. the property \( L^k(m,n) \) holds if \( f^k \) has a period-(\( m+n \)) point \( x_0 \) with the forward orbit \( \{x_i\} \) of the type either

\[
x_{m+n} < x_{m+n-1} < \cdots < x_{n+1} < x_n < x_0 < x_1 < x_2 < \cdots < x_{n-1}
\]

or all inequalities reversed;

2. the property \( L^k(m,\infty) \) holds if \( f^k \) has a fixed point \( x_0 \) with a backward orbit \( \{x_{-i}\} \) of the type either

\[
x_{-1} < x_{-2} < \cdots < x_{-m} < x_0 < \cdots < x_{-i} < x_{-i+1} < \cdots < x_{-m-1}
\]

or all inequalities reversed; and
3. the property $L^k(\infty, \infty)$ holds if $f^k$ has a fixed point $x_0 = y_0$ with two backward orbits $\{x_i\}$ and $\{y_i\}$ of the type either

$$x_{-1} < \cdots < x_{-i+1} < x_{-i} < \cdots < x_0 = y_0 < \cdots < y_{-i} < y_{-i+1} < \cdots < y_{-1}$$

or all inequalities reversed.

Note that the properties $L^k(m, n)$ and $L^k(n, m)$ are equivalent.

Now we state the main result of this paper.

**Theorem 3.** Let $f$ be a continuous map from $I$ into itself. Then for all integers $k \geq 1$ and $i \geq 0$, the following two diagrams hold:

![Diagram for Theorem 3](image_url)

Remark 1. The above theorem, together with Theorem 2, gives a refinement of Sharkovskii’s theorem on orbit types characterized by two parameters.

### 3 Proof of Theorem 3

First, we recall a basic lemma, the proof of which can be found in [6].
Lemma 1. Let $f$ be a continuous map from $I$ into itself. Then the following statements are true.

(a) If $J$ is a closed subinterval of $I$ with $f(J) \supset J$, then there exists a fixed point of $f$ in $J$.

(b) Let $J_i, 0 \leq i \leq n - 1$, be closed subintervals of $I$. If $f(J_i) \supset J_{i+1}$ for all $0 \leq i \leq n - 2$ and $f(J_{n-1}) \supset J_0$, then there exists a periodic point $y$ of $f$ in $J_0$ such that $f^i(y) \in J_i$ for all $1 \leq i \leq n - 1$ and $f^n(y) = y$.

The statements $S^2(2m + 1) \rightarrow P(2^i(2m + 3)) \rightarrow S^2(2m + 3)$ for all integers $i \geq 0$ and $m \geq 1$ come trivially from Theorem 2.

To prove the theorem, we only need to show that, for all integers $m \geq 2$, $n \geq 2$, $k \geq 1$ and $i \geq 0$,

(i) $L^k(m, n) \rightarrow L^k(m, n + 1)$ and $L^k(m, n) \rightarrow L^k(m + 1, n)$;

(ii) $L^k(m, n) \rightarrow L^k(m, \infty) \rightarrow L^k(m + 1, \infty) \rightarrow L^k(\infty, \infty) \rightarrow L^k(\infty)$;

(iii) $S^2(3) \rightarrow L^i(2, 2)$;

(iv) $S^2(2n + 1) \rightarrow L^i(n, n + 1)$.

It suffices to prove items (i) and (ii) with $k = 1$ and items (iii) and (iv) with $i = 0$ because the other cases with $k > 1$ and $i > 0$ follow immediately by considering $f^k$ and $f^2$, respectively.

For item (i) with $k = 1$, we may assume that $f$ has a period-$(m + n)$ point $x_0$ with the forward orbit $\{x_i\}$ of the type $x_m, x_{m+1}, \ldots, x_{m+n}$ such that $x_0 < x_i < x_{i+1} < \cdots < x_{m+n-1} < x_n < x_1 < x_2 < \cdots < x_{n-1}$. Since $f([x_n, x_0]) \supset [x_n, x_0]$, Lemma 1(a) implies that $f$ has a fixed point $z$ in $[x_n, x_0]$. In fact, $z$ lies in $(x_n, x_0)$ since $x_0$ and $z$ have different periods. Since $f([z, x_0]) \ni x_0$, there is a point $w \in (z, x_0)$ such that $f(w) = x_0$. Now let $J_0 = [z, w]$, $J_i = [w, x_0]$, $J_i = [x_{i-2}, x_{i-1}]$ for $2 \leq i \leq n$, $J_{n+1} = [x_{n}, z]$, and $J_i = [x_{i-1}, x_{i-2}]$ for $n + 2 \leq i \leq m + n$. Then $f(J_i) \supset J_{i+1}$ for $0 \leq i \leq m + n - 1$ and $f(J_{m+n}) \supset J_0$. By Lemma 1(b), $f$ has a periodic point $y$ in $J_0$ such that $f^i(y) \in J_i$ for all $1 \leq i \leq m + n$ and $f^{m+n+1}(y) = y$. It is clear that $y$ is neither $z$ nor $w$. Since $f^i(y) \in J_i$ for all $0 \leq i \leq m + n$, $y$ is a period-$(m + n + 1)$ point and its forward orbit has the type so that the property $L^1(m, n + 1)$ holds. This completes the proof of the first statement of item (i).

By the definition, the properties $L^1(m, n)$ and $L^1(n, m)$ are equivalent. Thus the second statement follows from the first one.

For item (ii) with $k = 1$, under the same assumption as above, we have shown the existence of a fixed point $z$ in $(x_n, x_0)$. First, we find a backward orbit $\{z_i\}$ of $z_0 = z$ with the type so that the property $L^1(m, \infty)$ holds. Since $f^i((x_m, x_{m+n-1})) \ni z_0$ for all $1 \leq i \leq m - 1$, we can, by induction on $i$, find $z_i$ with $f^i(z_i) = z_0$ in $(x_{m+n-i}, x_{m+n-i-1})$ for all $1 \leq i \leq m - 1$. Since $z_{m-1} \in (x_{n+1}, x_n) \subset (x_{n+1}, z_0) \subset f((x_n, z_0))$, we can find $z_{-m}$ in $(x_n, z_0)$. Again, by induction on $i$, we can find $z_{-i}$ in $(x_{m+n-i-1}, x_{m+n-i})$ for $m + 1 \leq i \leq m + n - 1$, since $f^{-i}(x_{m+n-i-1}, x_{m+n-i}) \ni z_{-m}$. Since
We have proved that the property $L^1(m, \infty)$ holds. Next, we find another backward orbit \( \{z'_m\} \) of \( z'_m = z \) with type so that the property $L^1(m + 1, \infty)$ holds. Let \( z'_{i+1} = z_i \) for \( 1 \leq i \leq m \). Since \( f((z'_{m}, z'_0)) \ni z'_{-m} \), we can find \( z'_{m-1} \) in \( (z'_{m}, z'_0) \). By induction on \( i \), we can find \( z'_{-i} \) in \( (z'_{-i-1}, z_{-i}) \) for all \( i \geq m + 1 \), since \( f((z_{-i-1}, z_{-i})) \ni z'_{-i} \). This shows the truth of the property $L^1(m + 1, \infty)$. By using the same argument, one can show that the property $L^1(\infty, \infty)$ holds. By the definition, we have $L^1(\infty, \infty) \rightarrow L^1(\infty)$. The proof of item (ii) is complete.

For item (iii) with \( i = 0 \), we may assume that \( f \) has a period-3 point \( x_0 \) with the forward orbit \( \{x_i\} \) of the type \( x_2 < x_0 < x_1 \). Since \( f((x_0, x_1)) \ni x_0 \), there is a point \( w \in (x_0, x_1) \) such that \( f(w) = x_0 \). Let \( g = f^2 \). Then \( g(x_2) = x_1 \), \( g(x_0) = x_2 \), \( g(w) = f(x_0) = x_1 \), and \( g(x_1) = x_0 \). By using Lemma 1(a), there are fixed points \( a, b \) and \( c \) for \( g \) such that \( x_2 < a < x_0 < b < w < c < x_1 \). Let \( J_0 = [x_0, b] \), \( J_1 = [x_2, a] \), \( J_2 = [w, c] \), and \( J_3 = [c, x_1] \). Then \( g(J_i) \ni J_{i+1} \) for \( 0 \leq i \leq 2 \) and \( g(J_3) \ni J_0 \). Lemma 1(b) implies that \( g \) has a period-4 point in \( J_0 \) so that the property $L^2(2, 2)$ holds. The proof of item (iii) is complete.

For item (iv) with \( i = 0 \), we may assume that \( f \) has a period-(2n + 1) point \( x_0 \) with the forward orbit \( \{x_i\} \) of the type \( x_{2n} < x_{2n-2} < \cdots < x_2 < x_0 < x_1 < x_3 < \cdots < x_{2n-3} < x_{2n-1} \). It is clear that the property $L^2(n + 1, n)$ holds, which is equivalent to the truth of the property $L^2(n, n + 1)$, and so the proof of item (iv) is complete.

We have finished the proof of Theorem 3.

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