FUSION, CROSSING AND MONODROMY
IN CONFORMAL FIELD THEORY
BASED ON $SL(2)$ CURRENT ALGEBRA
WITH FRACTIONAL LEVEL

Jens Lyng Petersen\textsuperscript{1}, Jørgen Rasmussen\textsuperscript{2},
\textit{The Niels Bohr Institute, Blegdamsvej 17, DK-2100 Copenhagen Ø, Denmark}

Ming Yu\textsuperscript{3}
\textit{Institute of Theoretical Physics, Academia Sinica, P.O.Box 2735, Beijing 100080,}
\textit{Peoples Republic of China}

Abstract

Based on our earlier work on free field realizations of conformal blocks for conformal field theories with $SL(2)$ current algebra and with fractional level and spins, we discuss in some detail the fusion rules which arise. By a careful analysis of the 4-point functions, we find that both the fusion rules previously found in the literature are realized in our formulation. Since this is somewhat contrary to our expectations in our first work based on 3-point functions, we reanalyse the 3-point functions and come to the same conclusion. We compare our results on 4-point conformal blocks in particular with a different realization of these found by O. Andreev, and we argue for the equivalence. We describe in detail how integration contours have to be chosen to obtain convenient bases for conformal blocks, both in his and in our own formulation. We then carry out the rather lengthy calculation to obtain the crossing matrix between s- and t-channel blocks, and we use that to determine the monodromy invariant 4-point greens functions. We use the monodromy coefficients to obtain the operator algebra coefficients for theories based on admissible representations.

\textsuperscript{1}e-mail address: jenslyng@nbi.dk
\textsuperscript{2}e-mail address: jrasmussen@nbi.dk
\textsuperscript{3}e-mail address: yum@itp.ac.cn
1 Introduction

There are several reasons why conformal field theories based on affine $SL(2)_k$ are interesting to study, not only for $k$ positive integer and for unitary, integrable representations based on usual integer and and half integer spins, but also for fractional levels and for fractional spins, in particular for admissible representations \[1, 2\]. Thus for example it was shown in Refs. \[3, 4\] how 2-d quantum gravity coupled to minimal conformal matter could in principle be described by a topological $G/G$ model with the $G$’s being affine $SL(2)$’s with the same levels. This possibility is closely related to the realization that minimal conformal field theory is obtained via hamiltonian reduction of $SL(2)$ \[5, 6, 7\].

However, there a minimal theory labelled in the standard way by $(p, q)$ ($p$ and $q$ co-prime integers) is related to an $SL(2)$ theory with level $k$ given by

$$ t \equiv k + 2 = p/q $$

and where admissible representations with fractional spins of $SL(2)_k$ have to be used. This approach appears potentially very interesting since a proper understanding of it would seem to lead to obvious possibilities for generalizations based on different groups and super-groups.

Other completely different applications may be envisaged. For example very interesting string backgrounds describing black holes may be obtained from $SL(2)$ theories with fractional levels \[8\].

To set up in detail conformal field theory based on $SL(2)_k$ with fractional levels and spins, one must first understand how to write down conformal blocks. In Refs. \[9\] we have given a general description of how this can be done based on the free field Wakimoto realization \[10\]. A number of technical obstacles arising from the occurrence of fractional powers of free fields had to be overcome. Several other groups have also studied the conformal blocks from several different points of view with more or less complete results \[11, 12, 13, 14, 15, 16\]. In this paper we have compared in detail to the approach by Andreev \[13\] although so far only 3- and 4-point functions have been written down there, but the 4-point function turns out to be very convenient for calculation of the crossing matrix between s-and t-channel blocks \[17\].

However, once the conformal blocks are obtained the next step in the program is to determine the monodromy invariant greens functions; these are the ones for which physical applications can be made and they are necessary before for example an application to 2-d quantum gravity can be made. It is the principal goal of the present paper to obtain these monodromy invariant combinations, and from the ensuing monodromy coefficients to determine the operator algebra coefficients. As described by Dotsenko and Fateev \[17\] this problem is conveniently solved by means of the crossing matrix relating the conformal blocks in the s- and t-channels (in fact, just a particular row and column of that matrix). A central portion of this paper is to show how to generalize the treatment of Dotsenko and Fateev \[17\] from minimal models to the problem at hand.

The conformal blocks for 4-point functions are characterized conveniently in terms of couplings to intermediate states. These in turn are determined by the fusion rules of the theory. Fusion rules for $SL(2)_k$ theories based on admissible representations have been obtained in Refs. \[14, 15\], who agreed that two different fusion rules were operating.
only the first of which generalize in an obvious way the fusion rules found in integrable representations. In our first work on the subject \[9\] we presented a calculation of the 3-point function, which appeared to give rise only to the first of the two fusion rules. However, we shall show explicitly in this paper, that in fact the 4-point functions (also written down in \[9\]) clearly imply both fusion rules. In addition we shall show in section 2 that an analysis of the 3-point function based on the idea of over-screening will provide the same result.

The fusion rules provide a neat starting point for giving convenient bases for the conformal blocks in the s- and t-channels, namely by demanding that some of the monodromies become trivial. One must then understand how the general integral representations can reproduce these bases. Here we shall use either the ones provided by us \[9\] (to be referred to as PRY), or the one for the 4-point function obtained by Andreev \[13\]. In either case it was not previously specified how integration contours should be chosen in order to generate specific members of s- or t-channel bases. In the present paper we discuss that. In the case of our own integral representation, we show how to obtain conformal blocks in the s-channel corresponding to fusion rule I, and how to obtain conformal blocks in the t-channel corresponding to fusion rule II, using contours where the integration of the auxiliary variable, \(u\) (introduced in \[9\]) is carried out first. We also show how to obtain conformal blocks in the s-channel corresponding to fusion rule II, and how to obtain conformal blocks in the t-channel corresponding to fusion rule I, using contours where the \(u\)-integration is done last. In the integral representation of Andreev \[13\] there is no \(u\)-variable to worry about and the contours we find are more tractable. That his 4-point blocks are equivalent to ours is a priori rather clear since both he and we have checked that the blocks we write down satisfy the Knizhnik-Zamolodchikov equations \[18\]. Nevertheless we find it very instructive to attempt a direct analytic proof of how the equivalence may be obtained, in particular we shall see from the proof how our auxiliary integration may be gotten rid of in the process. The situation turns out to have a rather remarkable counterpart within the context of minimal models: in addition to the standard well known integral representation written down in Ref. \[17\], an alternative form also exists, as mentioned in Ref. \[13\]. We show in section 5 that our integral representation and the one of \[13\], are related in a closely analogous way.

Having written down the full s- and t-channel bases for conformal blocks and understood the corresponding integration contours, we go on to calculate the relevant parts of the crossing matrix in section 6. It turns out that only a moderate generalization will be needed compared to the similar calculation in minimal models \[17\]. In both cases the calculations are rather lengthy, however.

In section 7 we use our results to calculate the monodromy invariant greens functions for 4-point functions.

In section 8 we use the monodromy coefficients to obtain the operator algebra coefficients, in particular for fusion rule I.

In section 9 we show how to generalize the previous treatment in which it was assumed that both vertices in the 4-point blocks pertain to the same fusion rule (I or II). The idea of over-screening used in section 2 for the 3 point function may be employed to obtain additional 4-point blocks in which there are different fusion rules (I and II) operating at the two vertices of the block. These new 4-point blocks correspond to different sets of
external spins from the ones previously considered, and so they do not mix with these under crossing. Based on the calculations carried out in the previous parts of the paper it is fairly easy to read off what the new monodromy coefficients should be, and in particular we obtain in this way new expressions for the operator algebra coefficients in the case of fusion rule II. Now these are parametrized in a way quite different from what was obtained in section 7, but in a way which renders the comparison with the coefficients pertaining to fusion rule I much more natural. In this parametrization we find identical functional forms for the operator algebra coefficients and we explain some differences in the parameters.

Finally, section 10 contains some concluding remarks.

2 Notations

We shall be interested in N-point functions (in this paper mostly 3- and 4-point functions) of primary fields. These are taken to depend on a spin label, \( j \), a position variable \( z \) (we only need specify the chiral dependencies for now), and one more variable, \( x \) which represents an equivalent but more convenient way of keeping track of the \( SL(2) \) weight dependence on a weight, \( m \). More precisely, if the affine currents are denoted by \( J^a(z) \), \( a = 1, 2, 3 \) or \( a = +, -, 3 \), and if the primary field (chiral vertex operator) for short is denoted \( \phi_j(w, x) \), then the OPE’s take the form

\[
J^a(z)\phi_j(w, x) \sim \frac{1}{z - w}D_x^a\phi_j(w, x)
\]

\[
D^+_x = -x^2\partial_x + 2xj
\]

\[
D^3_x = -x\partial_x + j
\]

\[
D^-_x = \partial_x
\]

Correlators (conformal blocks) transform covariantly with respect to projective transformations of both \( z \) and \( x \) variables. Thus in a 4-point function

\[
\langle \phi_{j_1}(z_4, x_4)\phi_{j_3}(z_3, x_3)\phi_{j_2}(z_2, x_2)\phi_{j_1}(z_1, x_1) \rangle
\]

we consider as usual the limits

\[
z_4 \to \infty, \quad x_4 \to \infty \]

\[
z_3 \to 1, \quad x_3 \to 1 \]

\[
z_2 \to z, \quad x_2 \to x \]

\[
z_1 \to 0, \quad x_1 \to 0
\]

so that the 4-point conformal blocks will be (in general multi-valued) functions of \((z, x)\). We label s- and t-channel conformal blocks by tree graphs, the meaning of which is that in the limit \( z \to 0 \) followed by \( x \to 0 \) the s-channel block corresponding to Fig. 1 has the behaviour following from the OPE’s

\[
S(z, x) \sim z^{h-h_1-h_2}(-x)^{j_1+j_2-j}(\text{const.} + \mathcal{O}(z, -x))
\]
Figure 1: Graphs for s- and t-channel blocks

whereas for the t-channel block we have in the limit $z \to 1$ followed by $x \to 1$

$$T(z, x) \sim (1 - z)^{h - h_2 - h_3}(x - 1)^{j_2 + j_3 - j} (\text{const.} + \mathcal{O}(1 - z, x - 1))$$ \hfill (5)

Here the conformal weights are given by the standard expression

$$h_i = \frac{j_i(j_i + 1)}{t}$$ \hfill (6)

with $t = k + 2$, where $k$ is the level. Admissible representations exist for $t = p/q$ with $p$ and $q$ co-prime integers. Then the allowed values for the spins are given by $(r, s \text{ integers})$ \cite{1, 2}

$$2j^+_r + 1 = r - st \quad (r, s) \geq (1, 0)$$
$$2j^-_r + 1 = -r + st \quad (r, s) \geq (1, 1)$$ \hfill (7)

and we have the translation symmetry

$$j^\pm_{r+np,s+nq} = j^\pm_{r,s}$$ \hfill (8)

Any $j^-$ may be written in terms of $j^+$

$$j^-_{r,s} = -j^+_{r,s} - 1$$
$$= j^+_{p-r,q-s}$$ \hfill (9)

so we may choose to work with the latter. Then for a coupling between 3 spins, $j_1, j_2, j_3$, labelled accordingly by $r_i, s_i$, the fusion rules are \cite{14, 15}

**Fusion rule I**

$$1 + |r_1 - r_2| \leq r_3 \leq p - 1 - |r_1 + r_2 - p|$$
$$|s_1 - s_2| \leq s_3 \leq q - 1 - |s_1 + s_2 - q + 1|$$ \hfill (10)

**Fusion rule II**

$$1 + |p - r_1 - r_2| \leq r_3 \leq p - 1 - |r_1 - r_2|$$
$$1 + |q - s_1 - s_2| \leq s_3 \leq q - 2 - |s_1 - s_2|$$ \hfill (11)
In both cases, \( r_3 \) and \( s_3 \) jump in steps of 2. It is easily checked that both sets of fusion rules cannot be satisfied simultaneously. In the next subsection we will discuss how fusion rule II arises from our 3-point function \([9]\).

A very convenient way to think about the fusion rules in our case consists in the following. Consider the s-channel coupling of \( j_1, j_2 \) to a \( j \). When we parametrize \( j = j_1 \) for fusion rule I as \( j_1 + j_2 - j_1 = r - st \) we shall see that the integers \( r, s \) are related to the number of screenings of the first and second kinds \([7, 9]\) around the \( j_1 j_2 j_1 \) vertex. The singular behaviour of the s-channel block in the limit \( z \to 0, x \to 0 \) is then

\[
 z^{h-h_1-h_2}(-x)^{j_1+j_2-j_1} = z^{h-h_1-h_2}(-x)^{r-st} \tag{12}
\]

with \( h = j_1(j_1+1)/t \). For fusion rule II we may then parametrize the internal \( j \) as

\[
 j_{II} \equiv -j_1 - 1 \tag{13}
\]

Of course the conformal dimensions for \( j_1 \) and \( j_{II} \) are the same, but we find the singular behaviour of the s-channel block to be

\[
 z^{h-h_1-h_2}(-x)^{j_1+j_2-j_{II}} = z^{h-h_1-h_2}(-x)^{2j_1+2j_2-r+st+1} \tag{14}
\]

All these statements follow by analysing the fusion rules Eq. (10), Eq. (11). By analysing the s-channel 4-point blocks in the limit \( z \to 0, x \to 0 \) we indeed find both of these singular behaviours and hence verify that the blocks realize both fusion rules I and II. In the t-channel the discussion is analogous, with \( j_1 \leftrightarrow j_3, \ z \to 1-z \) and \( x \to 1-x \), so that we consider the limits \( z \to 1 \) followed by \( x \to 1 \).

### 2.1 Fusion rule II and the 3-point function

In \([9]\) we found using the Felder contours \([20]\) the following expression for the 3-point function (here corrected for minor misprints)

\[
 W_F^{r,s} = \frac{e^{i\pi r(r+1-2r_1)/t} e^{i\pi ts(s-1-2s_1)}}{r \prod_{j=1}^{r} \frac{(1 - e^{2\pi i(r-j)/t})(1 - e^{2\pi ij/t})}{1 - e^{2\pi itj}}} \cdot \frac{1}{s \prod_{j=1}^{s} \frac{(1 - e^{2\pi it(s_1+1-j))}(1 - e^{2\pi ij})}{1 - e^{2\pi it}}} \cdot \frac{\Gamma(2j_2+1)}{\Gamma(j_2+j_3-j_1+1)} \cdot \Gamma(2rs) \prod_{i=1}^{r} \frac{\Gamma(i/t)}{\Gamma(1/t)} \cdot \prod_{i=1}^{s} \frac{\Gamma(it-r)}{\Gamma(t)} \cdot \prod_{i=0}^{r-1} \frac{\Gamma(s_1 + 1 + (1 - r_1 + i)/t)\Gamma(s_2 + (1 - r_2 + i)/t)}{\Gamma(s_1 + s_2 + 1 - 2s + (r - r_1 - r_2 + i + 1)/t)} \cdot \prod_{i=0}^{s-1} \frac{\Gamma(r_1 - r + (i - s_1)t)\Gamma(r_2 - r + (1 - s_2 + i)t)}{\Gamma(r_1 - r + r_2 + (s - s_1 - s_2 + i)t)} \tag{15}
\]

It turns out that the Felder contours alone cannot produce a well-defined and non-vanishing 3-point function corresponding to fusion rule II. We need the combination that the \( r \) screening variables of the first kind are integrated along Dotsenko-Fateev contours, while the \( s \) screenings of the second kind are taken along Felder contours (or vice
versa). This leads to

\[
W_{r,s}^{\text{DF}} = \lambda_r(1/t)\chi_s^{(2)}(1; t)
\]

\[
\cdot \frac{\Gamma(2j_2 + 1)}{\Gamma(j_2 + j_3 - j_1 + 1)} \prod_{i=1}^{r} \frac{\Gamma(i/t)}{\Gamma(1/t)} \prod_{i=1}^{s} \frac{\Gamma(it - r)}{\Gamma(t)}
\]

\[
\cdot \prod_{i=1}^{r} \frac{\Gamma(s_1 + 1 + (-r_1 + i)/t)\Gamma(s_2 + (-r_2 + i)/t)}{\Gamma(s_1 + s_2 + 1 - 2s + (r - r_1 - r_2 + i)/t)}
\]

\[
\cdot \prod_{i=1}^{s} \frac{\Gamma(r_1 - r + (i - 1 - s_1)t)\Gamma(r_2 - r + (-s_2 + i)t)}{\Gamma(r_1 - r + r_2 + (s - s_1 - s_2 - 1 + i)t)}
\]  

(16)

Here we have introduced the functions similar to Ref. [17]

\[
\lambda_r(1/t) = \prod_{j=1}^{r} e^{-i\pi(j-1)/t} \frac{s(j/t)}{s(1/t)}
\]  

(17)

with

\[
s(x) \equiv \sin(\pi x)
\]  

(18)

and the functions

\[
\chi_s^{(2)}(s_1; t) = e^{i\pi ts(s_1 - 2s_1)} \prod_{j=1}^{s} \frac{(1 - e^{2\pi i(s_1 - 1)j})(1 - e^{2\pi ij})}{1 - e^{2\pi it}}
\]

\[
= (2i)^s e^{i\pi ts(s_1 - 1)} \prod_{j=1}^{s} \frac{s((j - s_1 - 1)t)s(jt)}{s(t)}
\]  

(19)

If one chooses the alternative combination where the \( r \) screenings of the first kind are integrated along Felder contours, while the \( s \) screenings of the second kind are taken along Dotsenko-Fateev contours, the pre-factor in (16) would be \( \chi_r^{(1)}(r_1; 1/t)\lambda_s(t) \) where

\[
\chi_r^{(1)}(r_1; 1/t) = e^{i\pi r(r_1 + 2r_1)/t} \prod_{j=1}^{r} \frac{(1 - e^{2\pi i(r_1 - j)/t})(1 - e^{2\pi ij})}{1 - e^{2\pi it}}
\]

\[
= (2i)^r e^{i\pi r(r_1 - 1)/t} \prod_{j=1}^{r} \frac{s((j - r_1)/t)s(jt)}{s(t)}
\]  

(20)

There is a considerable freedom in choosing the numbers of screenings subject to the charge conservation since \( p - qt = 0 \). We will denote the following choice

\[
2r = r_1 + r_2 - r_3 - 1 + p
\]

\[
2s = s_1 + s_2 - s_3 + q
\]  

(21)

as over-screening due to the addition of \( p, q \). The analysis of (16) in terms of fusion rules is standard and using (21) one finds (17), fusion rule II. In the process we encounter the cancellation \( \Gamma(0)/\Gamma(0) = 1 \). It should be mentioned that for fusion rule I the choice of contours only affects the normalization.
3 Conformal blocks for fusion rules I and II according to PRY

The new feature discussed here compared to our discussion in PRY, Ref. [9], is the precise specification of integration contours for the various variables in the integral representation. Using the results of PRY for the 4-point function, we want to show here that the integration contours we indicate will give rise first to a set of s-channel conformal blocks corresponding to the intermediate state \((j)\) in Fig. 1 being given by fusion rule I, and second to a set of conformal blocks in the t-channel corresponding to the intermediate \(j\) being given by fusion rule II.

We first describe the situation in the s-channel corresponding to fusion rule I. Here the \(u\)-integration is carried out first. We write for the conformal block (cf. [9])

\[
W_{(r,s)}^{(R,S)}(j_1, j_2, j_3, j_4; z, x) = z^{2j_1j_2/t} (1 - z)^{2j_2j_3/t} \prod_{i \in \mathcal{I}} \frac{dw_i}{2\pi i} \int_{\mathcal{C}_o} \prod_{j \in \mathcal{O}} \frac{dw_j}{2\pi i} \int_{\mathcal{C}_u} \frac{du}{2\pi i} w_i^{2k_{i}j_1/t} (w_i - z)^{2k_{i}j_2/t} (w_i - 1)^{2k_{i}j_3/t} \prod_{i < j} (w_i - w_j)^{2k_{i}k_{j}/t} \prod_{i \in \mathcal{A}} \left(-\frac{u}{w_i - 1} + \frac{x}{w_i - z}\right)^{-k_{i}} (1 - u)^{2j_2 + 2j_3 - R + St} u^{-2j_3 - 1} \tag{22}
\]

Here we are considering an integral representation of the 4-point conformal block with a total of \(R\) screening operators of the first kind (\([9]\)) and a total of \(S\) screening operators of the second kind. The \(w_i\)'s are the positions of the screening operators, and \(k_i = -1\) for screenings of the first kind and \(k_i = t\) for screenings of the second kind. When \(i \in \mathcal{O}\) the corresponding \(w_i\) is integrated along the contour of Fig. 2(a) (whether it is of the first or second kind) corresponding to a screening of the vertex, \(j_1j_2j\). Different \(i, i' \in \mathcal{O}\) are taken along slightly different contours in order to avoid the singularity coming from \((w_i - w_{i'})^{2k_{i}k_{i'}/t}\). Similarly, the \(w_j\)'s for \(j \in \mathcal{I}\) are integrated along the contour, Fig. 2(b), corresponding to a screening of the vertex, \(jj_3j_4\). \(\mathcal{A} = \mathcal{O} \cup \mathcal{I}\) is simply the combined index set. We denote the numbers of screenings of the first kind at the \(j_1j_2j\) and the \(jj_3j_4\) vertices respectively as \(r\) and \(R - r\). Similarly the corresponding numbers of screenings of the second kinds at the two vertices are denoted \(s\) and \(S - s\). In the product of factors \((w_i - w_j)^{2k_{i}k_{j}/t}\) an arbitrary ordering of the indices is implied.

For fixed \(w_i\)'s the integrand has singularities in the \(u\)-plane at \(u = 0, 1, \Delta_i\), where

\[
\Delta_i = \frac{w_i - 1}{w_i - z} \tag{23}
\]

The integration contour for \(u\) is to divide the singularities, \(\Delta_i\), so that the ones for \(i \in \mathcal{O}\) lie outside \(\mathcal{C}_u\) and the ones for \(i \in \mathcal{I}\) lie inside \(\mathcal{C}_u\). If \(\mathcal{C}_u\) should pass through \(u = 1\). For \(z\) and \(x\) sufficiently small, we may take \(\mathcal{C}_u\) to be the unit circle, Fig. 3. Remember that in order to identify the nature of the block and the value of the intermediate \(j\), we are going to investigate the limit \(z \to 0\) followed by \(x \to 0\). The different positions of the singularities, \(\Delta_i\) in \(u\) for \(i \in \mathcal{O}\) and \(i \in \mathcal{I}\) mean that they give rise to different
Figure 2: The integration contours \( C_{O} \) (a) and \( C_{I} \) (b) for an s-channel block corresponding to fusion rule I.

Figure 3: The integration contour \( C_{u} \) for an s-channel block corresponding to fusion rule I.
Figure 4: Integration contours for an s-channel block corresponding to fusion rule II. The corresponding u-integration is simply along the unit circle.

singularities in the corresponding $w_i$ planes after the u-integration has been performed. In fact, for $i \in \mathcal{O}$ there occurs a pinching of singularities when $\Delta_i$ collides with either 0 of 1 (the additional singularities in $u$). This happens for $w_i$ equal to 1, and for $w_i$ equal to
defining the singularities for $w_i$ is what we term “pure”, meaning that it is of the form

$$ (w_i - z)^\delta (1 + \mathcal{O}(w - z)) $$

One can easily check that this is enough to ensure that the corresponding block will satisfy the Knizhnik-Zamolodchikov equation, going over the proof presented in PRY [9].

In contrast, the singularity at $w_i = 0$ is “non-pure”: it is a mixture of different powers of $w_i$. Hence we cannot allow the contour to end in $w_i = 0$, it has to surround that point as indicated.

Turning to the singularities in $w_i$ for $i \in \mathcal{I}$, we see that pinching occurs only when $\Delta_i = 1$, so that there is no extra singularity produced at $w_i = 1$: it remains pure, and we may take the integration contour to start in $w_i = 1$ as indicated. If more convenient, one may take the contour to wrap around the real axis from 1 to $\infty$, which is a form closer to the one used by Dotsenko and Fateev [17].

Having established that the choice of contours indicated is allowed in the sense that the conformal block will satisfy the Knizhnik-Zamolodchikov equations, it is a relatively simple matter to find the leading singularity in the limit $z \to 0$ followed by $x \to 0$. In fact, we may scale all the $w_i$’s with $i \in \mathcal{O}$ as

$$ w_i \to zw_i $$
In the limit $z \to 0$ this is easily seen to result in a leading $z$ behaviour of the form

$$W^{(R,S)}_{(r,s)}(z, x) \sim z^{-h(j_1)-h(j_2)+h(j_1)}$$  \hspace{1cm} (24)

where

$$h(j) = j(j+1)/t$$

and where

$$j_I = j_1 + j_2 - r + st$$  \hspace{1cm} (25)

This is not enough to prove that indeed the intermediate state corresponds to a primary field with that value of $j$, since

$$h(j) = h(-j-1)$$  \hspace{1cm} (26)

In fact, according to our earlier discussion, the difference between fusion rules I and II is exactly that for fusion rule I we should obtain $j = j_I$ whereas for fusion rule II we should obtain $j = j_{II} = -j_I - 1$. In other words the $z$ behaviour is precisely unable to distinguish between the two fusion rules. To distinguish we must investigate the leading $x$ behaviour in the limit $x \to 0$ after we have taken $z \to 0$. However, it is an easy matter to do so and to find the behaviour

$$W^{(R,S)}_{(r,s)} \sim z^{-h(j_1)-h(j_2)+h(j_1)}(-x)^{r-st}$$  \hspace{1cm} (27)

This is the proof that the conformal block we have constructed corresponds to fusion rule I, since $r - st = j_1 + j_2 - j_I$.

We next describe how contours have to be chosen in order to produce a $t$-channel block corresponding to fusion rule II. This situation can occur only provided there is at least one screening operator of the second kind [9]. We use the same defining equation as in Eq. (23), but the sets of indices as well as $r$ and $s$ have different meanings. Again we have a total of $R$ and $S$ screenings of the first and second kinds. There are $r$ and $s$ screenings associated with the upper vertex, and the corresponding index set for the $w_i$’s is $\mathcal{O}$. There are $R - r$ and $S - s$ screening operators of the two kinds associated with the lower vertex and the corresponding index set for the $w_i$’s is $\mathcal{I}$. The integration contour, $C_u$, is indicated in Fig. 6. One checks that in the limit $z \to 1$ followed by $x \to 1$ the two sets of singularities, $\Delta_i$ for $i \in \mathcal{O}$ and $i \in \mathcal{I}$ respectively are well separated, so that the contour may be taken to separate them as indicated. The contours for the two sets of $w_i$ contours, $C_\mathcal{O}$ and $C_\mathcal{I}$ are shown in Fig. 7 (a) and (b) respectively. In all cases one checks as for the $s$-channel block that the nature of singularities is such that the contours may be chosen as indicated, and that the block thus defined will satisfy the Knizhnik-Zamolodchikov equations, following PRY [9]. Then we investigate the combined behaviour $z \to 1$ followed by $x \to 1$. To this end we perform the following scalings of the integration variables:

$$w_i \to \frac{w_i - 1}{z - 1}, \quad i \in \mathcal{O}$$

$$w_i \to \frac{w_i}{w_i - 1}, \quad i \in \mathcal{I}$$

$$u \to \frac{u - \Delta_{j_0}}{1 - \Delta_{j_0}}$$  \hspace{1cm} (28)
Figure 5: Integration contours for the screening charges in the case of a t-channel block corresponding to fusion rule I. The \( u \)-integration is along a closed contour starting in \( 1 \) and surrounding \( x \).

Figure 6: The integration contour \( \mathcal{C}_u \) for a t-channel block corresponding to fusion rule II.

Figure 7: The integration contours for the \( w \)'s: \( \mathcal{C}_O \) (a) and \( \mathcal{C}_I \) (b) for a t-channel block corresponding to fusion rule II.
where $j_0$ is an arbitrary index in the set $I$, however with the restriction that $w_{j_0}$ is the position of a screening operator of the second kind. It is rather straightforward to check that this gives rise to the combined singular behaviour

$$W_{(r,s)}^{(R,S)}(z, x) \sim (1 - z)^{-h(j_2) - h(j_3) + h(j_{II})}(x - 1)^{2j_2 + 2j_3 - r + st + 1}$$

(29)

where

$$j_{II} = -j_I - 1$$
$$j_I = j_2 + j_3 - r + st$$

(30)

so that Eq. (29) exactly demonstrates that we have fusion rule II, since $j_2 + j_3 - j_{II} = 2j_2 + 2j_3 - r + st + 1$.

It follows that the conformal blocks defined on the basis of the free field realization elaborated in PRY [9], indeed do give rise to both the fusion rules previously found in the literature [14, 15].

We have also found realizations of conformal blocks corresponding to fusion rule II in the s-channel and of ones corresponding to fusion rule I in the t-channel. However, these are not given by quite as simple contours as above. It is the appearance of non-pure singularities which has prevented us from finding such simple contours. The new idea is to carry out first the integrations of the screening operators letting the contours depend on $u$. Then there are only pure singularities in the $w_i$ planes and there will be no problems caused by non-pure singularities. It turns out that it is possible to find contours like that leaving, upon integration, a simple $u$ integral. Let us first consider the conformal blocks in the s-channel corresponding to fusion rule II, where the contours are depicted in Fig. 4.

$$W_{(r,s)}^{(R,S)}(j_1, j_2, j_3, j_4, z, x)$$

$$= z^{2j_1j_2/2}(1 - z)^{2j_2j_3/2} \frac{du}{2\pi i} \int_{C_u} \frac{dw}{2\pi i} \int_{1}^{z} \prod_{i \in I} dw_i \int_{0}^{\infty} \prod_{i \in O} dw_i$$
$$\cdot w_i^{-k_i} \frac{2k_i}{t} (w_i - z)^{2k_i} \frac{1}{2k_i} \prod_{i, j \in A \, i < j} (w_i - w_j)^{2k_i}$$
$$\cdot \prod_{i \in A} \left( -\frac{u}{w_i - 1} + \frac{x}{w_i - z} \right)^{-k_i} (1 - u)^{2j_2 + 2j_3 - R + St} u^{-2j_3 - 1}$$

(31)

where

$$A = I \cup O \cup \{R + S\}$$
$$w = w_{R+S}$$
$$k_{R+S} = t$$

(32)

To see that the above formula produces the right singular behaviour in the limit $z \to 0$ followed by $x \to 0$, we may scale all the $w_j$’s with $j \in O$ as

$$w_j \to z w_j, \quad \text{for all } j \in O$$
and scale all the \( w_i \)'s with \( i \in \mathcal{I} \) as

\[
w_i \to 1/w_i, \quad \text{for all } i \in \mathcal{I}
\]

and also

\[
w \to \frac{uz - x}{u - x}w
\]

We can then show that

\[
W_{(r,s)}^{(R,S)}(z, x) \sim z^{-h(j_1) + h(j_2) + h(j_{II})} (-x)^{j_1 + j_2 - j_{II}}
\]

where

\[
j_{II} = -j_1 - j_2 + r - st - 1
\]

This is precisely the expected singular behaviour.

Finally we consider the conformal blocks in the t-channel corresponding to fusion rule I, see Fig. 5

\[
\mathcal{W}_{(r,s)}^{(R,S)}(j_1, j_2, j_3; z, x) = z^{2j_1j_2/t} (1 - z)^{2j_2j_3/t} \int_{C_u} \frac{du}{2\pi i} \int_{C_w} \frac{dw}{2\pi i} \int_{-\infty}^{1} \prod_{i \in \mathcal{I}} dw_i \prod_{j \in O} dw_j
\]

\[
\cdot \prod_{i \in A} \left( -\frac{u}{w_i - 1} + \frac{x}{w_i - z} \right)^{-k_i} (1 - u)^{2j_2 + 2j_3 - R - st} u^{-2j_3 - 1}
\]

where

\[
\begin{align*}
\mathcal{A} &= \mathcal{I} \cup \mathcal{O} \cup \{r + s\} \\
w &= w_{r+s} \\
k_{r+s} &= t
\end{align*}
\]

To see that the above formula produces the right singular behaviour in the limit \( z \to 1 \) followed by \( x \to 1 \), we may scale all the \( w_j \)'s with \( j \in \mathcal{O} \) as

\[
w_j \to 1 - (1 - z)w_j, \quad \text{for all } j \in \mathcal{O}
\]

and scale all \( w_i \)'s with \( i \in \mathcal{I} \) as

\[
w_i \to w_i/(w_i - 1), \quad \text{for all } i \in \mathcal{I}
\]

and also

\[
w \to 1 - \frac{(1 - z)}{u - x}uw
\]

We also scale \( u \) as

\[
u \to x + (1 - x)u
\]
It should be noticed that the final \( u \) contour starts at 1 and goes along the unit circle such that it surrounds 0 and the other points which are away from 0 by a distance of order \( (1 - z)/(1 - x) \). This means that we can not deform the \( u \) contour to the form \( \int_0^1 du \), or in terms of the original \( u \) variable, that cannot be deformed into \( \int_x^1 du \). Using these scalings, we show that in the presence of at least one screening charge of the second kind in the scaling region (the region close to 1 and \( z \)) the singular behaviour is

\[
W^{(R,S)}_{(r,s)}(z, x) \sim (1 - z)^{-h(j_2) - h(j_3) + h(j_I)}(x - 1)^{j_2 + j_3 - j_I} 
\]

where

\[
j_I = j_2 + j_3 - r + st
\]

What happens if there is no screening charge of the second kind in the scaling region? Then the above method does not apply, but in that case \( j_2 + j_3 - j_I \) is an integer, and

\[
W^{(R,S)}_{(r,0)}(z, x) \sim (1 - z)^{-h(j_2) - h(j_3) + h(j_I)}(x - 1)^{j_2 + j_3 - j_I}
\]

is a polynomial in \( x \). There will be no extra singularities present in \( w \)'s, such as at \( \delta = \frac{x - z}{x - 1} \), if we integrate over \( u \) first. Thus we could choose the following contours

\[
(w = w_{r+s} = w_r \in \mathcal{O})
\]

\[
\int_z^1 dw_j, \quad j \in \mathcal{O}
\]

\[
\int_0^\infty dw_i, \quad i \in \mathcal{I}
\]

\[
\int_0^1 du
\]

These contours are effectively closed in the sense that a total derivative integrated along them vanishes, such as is required for the Knizhnik-Zamolodchikov equations to be satisfied \( \square \). Notice, however, that these contours are not closed (in the same sense) when there is a screening charge of the second kind in the scaling region. However, it is difficult to determine explicitly the \((1 - x)\) behaviour for these contours, but since we know that our formula is both projective and \( SL(2) \) invariant, we could express the above formula in terms of \( x_3 = 0 \) and \( x_1 = 1 \), where the \((1 - x)\) behaviour is manifest.

It may seem surprising that one could not make use of the \( j_1 \leftrightarrow j_3 \) symmetry to obtain t-channel contours from s-channel ones and vice versa. The reason is that the simple form of the 4-point function we have given with only one auxiliary \( u \) integration, breaks this symmetry, since not all 4 primary fields are treated on the same footing. For a more symmetric treatment, more \( u \) integrations have to be introduced, which is also inconvenient, however.

In the next section we start our detailed comparison between the 4-point functions written down by us \( \square \) and by Andreev \( \square \). For those latter ones, it turns out that simpler contours may be devised.

### 4 Conformal blocks in Andreev’s representation

In this section we base our discussion on the integral realization of Andreev \( \square \). In the next section we discuss the equivalence between that realization and ours \( \square \), described in
the preceding section. In this section we show how to choose simple integration contours so that we produce both s- and t-channel blocks corresponding to both fusion rules I and II. It will turn out that the t-channel blocks are obtained in a very simple way from the s-channel blocks so we mostly concentrate on the latter. It is the specification of the integration contours which is our contribution here over Ref. [13]. The advantage of the realization of [13] is that contrary to the case with ours, there is no auxiliary integration in addition to the integration over positions of screening charges. The disadvantage is that the representation (so far) has no underlying free field realization and therefore only is known for 4-point blocks. For our purpose later on it is convenient to have different names for s- and t-channel blocks. We denote them by letters $S$ or $S$ and $T$ or $T$. The difference will be explained.

4.1 Fusion Rule I

We define the complex block in the s-channel for fusion rule I with $r$ screenings of the first kind and $s$ screenings of the second kind at the right vertex as follows:

$$S^{(R,S)}_{(r,s,0)}(z, x) = z^{2j_1j_2/t} (1 - z)^{2j_3j_4/t} \int_0^z \prod_{i \in I_1, k \in J_1} du_i dv_k \int_{j \in I_2, l \in J_2} \prod_{j < j', \in I_2} (u_j - u_{j'})^{2\rho'} \prod_{i \in I_1, j \in J_2} (u_j - u_i)^{2\rho'} \prod_{k \in I_1, l \in J_2} (z - v_k)^{2\rho} \prod_{k < k', \in J_1} (v_k - v_{k'})^{2\rho} \prod_{l \in I_2, \in J_2} (v_l - v_{r'})^{2\rho'} \prod_{i,k} (u_i - v_k)^{-2} \prod_{i,l} (u_i - v_l)^{-2} \prod_{j,k} (u_j - v_k)^{-2} \prod_{j,l} (u_j - v_l)^{-2} \prod_{i,j,k,l} (u_i - x)(u_j - x)(v_k - x)^{-\rho}(v_l - x)^{-\rho} \quad (41)$$

Here we have introduced the following index sets

$$I_1 = \{1, ..., r\}$$
$$I_2 = \{r + 1, ..., R\}$$
$$J_1 = \{1, ..., s\}$$
$$J_2 = \{s + 1, ..., S\} \quad (42)$$

where $R$ and $S$ are the total numbers of screenings of the first and second kinds respectively. Variables $u$ and $v$ belong to screenings of the first and second kind respectively, although this language is rather symbolic, since as yet there exists no known free field realization which directly gives this form. Also the integrals are taken along complex Dotsenko-Fateev contours shown in Fig. 8. Notice that expressions of the form $(u_i - u_{r'})^{2\rho'}$ have a phase defined by the fact that the first of the two integration variables have a
lower imaginary part than the last variable. Finally

\begin{align*}
a &= -2j_3 + t + R - St - 1 \\
b &= -2j_1 + t + R - St - 1 \\
c &= 2j_1 + 2j_2 + 2j_3 - R + St + 1 \\
\rho &= t, \quad \rho' = 1/t \\
a' &= -a/t, \quad b' = -b/t, \quad c' = -c/t
\end{align*}

The integrand of this expression is provided in a slightly different form in Ref. [13]. In fact there, the \( j \)'s are replaced by their parametrizations Eq. (7) giving rise to 4 independent forms for the integrand depending on whether the \( j^+_i \) or the \( j^-_i \) form is used. The above form holds in general. By analysing the small \( z \) and small \( x \) behaviour of this form it is easy to establish that this conformal block corresponds to the \( s \)-channel diagram Fig. 1 with the intermediate \( j \) given by fusion rule I. Indeed by scaling \( u_i \to zu_i, v_k \to zv_k, i \in I_1, k \in J_1 \) we find

\[ S_{(r,s,0)}^{(R,S)}(z, x) \sim z^{-h(j_1) - h(j_2) + h(j_1)}(-x)^{j_1 + j_2 - j_1} \]

with

\[ j_I = j_1 + j_2 - r + st \]

The contours in Fig. 8 are essentially equal to the contours in Ref. [17] for minimal models.

### 4.2 Fusion rule II

Fig. 8 shows the integration contours. The \( s \)-channel block for fusion rule II is given by

\[ S_{(r,s,1)}^{(R,S)}(z, x) \]

\[ = z^{2j_1 j_2/t}(1 - z)^{2j_1 j_3/t} \int_0^z du_i du_k \int_1^\infty du_j du_l \oint_{C_u} \frac{dv}{2\pi i} \]

\[ u_i^{a'}(1 - u_i)^{b'}(z - u_i)^{c'} \prod_{i < i'} (u_i - u_i')^{2\rho'} u_j^{a'}(u_j - 1)^{b'}(u_j - z)^{c'} \]

\[ \prod_{j < j'} (u_j - u_j')^{2\rho'} \prod_{i,j} (u_j - u_i)^{2\rho'} \]
of using complex contours close to the real axis (for real $z$) define additional ones in analogy to the case for minimal models \[17\]. Namely, instead both 0 and
ordered" integrations, with an ordering so that all terms in the integral expression for we find

\[
v^a(1 - v_k)^b(z - v_k)^c \prod_{k < k'} (v_k - v_{k'})^{2\rho} v^a_l (v_l - 1)^b (v_l - z)^c \prod_{l < l'} (v_l - v_{l'})^{2\rho}
\prod_{k,l} (v_l - v_k)^{2\rho} \prod_{i,k} (u_i - v_k)^{-2} \prod_{i,l} (u_i - v_l)^{-2} \prod_{j,k} (u_j - v_k)^{-2} \prod_{j,l} (u_j - v_l)^{-2}
(u_i - x)(u_j - x)(v_k - x)^{-\rho}(v_l - x)^{-\rho}
v^a(1 - v)^b(v - z)^c(v - v_k)^2(v - v_l)^2(v - v_i)^{-2}(v - u_j)^{-2}(v - x)^{-\rho}
\]

(46)

Here the variables, $u_i, u_j, v_k, v_l$ are taken along approximately real (for $z$ real) contours as for fusion rule I. The indices indicate: $i = 1, ..., r; j = r + 1, ..., R; k = 1, ..., s; l = s + 1, ..., S - 1$, whereas $v$ runs along the contour, $C_v$ which starts at $x$ and surrounds both 0 and $z$, cf. Fig. 9. In addition to the $s$-channel blocks we have defined above, we define additional ones in analogy to the case for minimal models \[17\]. Namely, instead of using complex contours close to the real axis (for real $z$), we may use real "time ordered" integrations, with an ordering so that all terms in the integral expression for $S_{(r,s,0)}^{(RS)}$ become real. This block is denoted $S_{(r,s,0)}^{(RS)}$. Using arguments similar to Ref. \[17\] we find

\[
S_{(r,s,0)}^{(RS)}(z, x) = \lambda_r(\rho')\lambda_{R-r}(\rho')\lambda_s(\rho)\lambda_{S-s}(\rho)S_{(r,s,0)}^{(RS)}(z, x)
\]

\[
S_{(r,s,0)}^{(RS)}(z, x) = s_{(r,s,0)}^{(RS)}(z, x)N_{(r,s,0)}^{(RS)}
\]

(47)

where $s_{(r,s,0)}^{(RS)}(z, x)$ is normalized in such a way that the behaviour as $z \to 0, x \to 0$ is

\[
s_{(r,s,0)}^{(RS)}(z, x) = z^{-h(j_1) - h(j_2) + h(j_3)}(-x)^{j_1 + j_2 - j_3}(1 + O(z, x))
\]

(48)

The $\lambda$-functions were defined in Eq. \[17\]. Similarly for fusion rule II we write

\[
S_{(r,s,1)}^{(RS)}(z, x) = \lambda_r(\rho')\lambda_s(\rho)\lambda_{R-r}(\rho')\lambda_{S-s-1}(\rho)S_{(r,s,1)}^{(RS)}(z, x)
\]

\[
S_{(r,s,1)}^{(RS)}(z, x) = s_{(r,s,1)}^{(RS)}(z, x)N_{(r,s,1)}^{(RS)}
\]

\[
s_{(r,s,1)}^{(RS)}(z, x) = z^{-h(j_1) - h(j_2) + h(j_3)}(-x)^{j_1 + j_2 - j_3}(1 + O(z, x))
\]

(49)

where

\[
j_{II} = -j_I - 1
\]

(50)
The leading behaviour (for \( z \to 0 \) followed by \( x \to 0 \)) in this case of fusion rule II is determined by the scalings

\[
\begin{align*}
u_i & \to zu_i \\
v_k & \to zv_k \\
v & \to (-x)v
\end{align*}
\]

The normalization constants, \( N^{(R,S)}_{(r,s,0)} \) and \( N^{(R,S)}_{(r,s,1)} \), are found in terms of the famous Dotsenko-Fateev integral (last paper Ref. [17], appendix A, here with a minor misprint corrected):

\[
J_{nm}(a, b; \rho) = \rho^{2nm} \prod_{i=1}^{n} \frac{\Gamma(i\rho')}{\Gamma(\rho')} \prod_{i=1}^{m} \frac{\Gamma(i\rho - n)}{\Gamma(\rho)} \times \prod_{i=0}^{n-1} \frac{\Gamma(1 + a' + i\rho')\Gamma(1 + b' + i\rho')}{\Gamma(2 - 2m + a' + b' + (n - 1 + i)\rho')} \times \prod_{i=0}^{m-1} \frac{\Gamma(1 - n + a + i\rho)\Gamma(1 - n + b + i\rho)}{\Gamma(2 - n + a + b + (m - 1 + i)\rho)}
\]

We shall need these normalizations in the calculation of crossing matrices. After some calculations we obtain

\[
N^{(R,S)}_{(r,s,0)} = (-)^{R-r+S-s} J_{r,s}(a, c; \rho) J_{R-r,s-r}(a + c - 2(r - \rho s) - \rho, b; \rho) \prod_{i=0}^{R-r-1} \frac{s(a' + c' - 2(s - \rho' r) + 1 + i\rho')}{s(a' + b' + c' - 2(s - \rho' r) + 1 + \rho'(R - r - 1 + i))} \prod_{i=0}^{S-s-1} \frac{s(a + c - 2(r - \rho s) - \rho + i\rho)}{s(a + b + c - 2(r - \rho s) - \rho + \rho(S - s - 1 + i))}
\]

and

\[
N^{(R,S)}_{(r,s,1)} = N^{(R,S)}_{(r,s+1,0)} \frac{\Gamma(\rho)\Gamma(1 - \rho)\Gamma(2 - 2r + a + c + 2s\rho)}{\Gamma((-a - c - 2ps + 2r)\Gamma(1 - r + a + s\rho)\Gamma(1 - r + c + s\rho)} \frac{1}{\Gamma((s + 1)\rho - r)\Gamma(2 - r + a + c + (s - 1)\rho)}
\]

For the integral realization considered here [13] it is trivial to obtain the t-channel forms once the s-channel forms above are given. In fact we have in an obvious notation (\( \varepsilon = 0, 1 \) for fusion rules I and II)

\[
J_{(r,s,\varepsilon)}(z; x; j_1, j_2, j_3, j_4) = S^{(R,S)}_{(r,s,\varepsilon)}(1 - z, 1 - x; j_3, j_2, j_1, j_4)
\]

We notice the following. When in the integral realization, we also transform all integration variables as \( u \to 1 - u, v \to 1 - v \), the integrand for the t-channel block is identical to the one for the s-channel block, up to phases. In particular, whenever we have \((u - x)\) or \((v - x)^{-\rho}\) in the s-channel, we would have \((x - u)\) and \((x - v)^{-\rho}\) in the t-channel. Also, after transformation of the variables, the integration contours in the t-channel are between \( z \) and \( 1 \) and between \( 0 \) and \(-\infty\), and the complex contour for \( v \) in the case of fusion rule II surrounds \( z \) and \( 1 \). The above factors, \((u - x)\) etc. are real provided \( x < 0 \) in the s-channel, or \( x > 1 \) in the t-channel. These two possibilities map to each other under \( x \to 1 - x \).
5 Analysis of the equivalence between the integral realizations of PRY and of Andreev

The equality between our form of the conformal blocks and the one provided by Andreev turns out to be very closely related to an identity in itself remarkable for minimal models, which was mentioned in Ref. 13. The proof of this identity is simpler but very similar to what we shall need. Hence we start by discussing the identity for minimal models.

5.1 An identity for minimal models

Theorem

\[
\int_0^1 \prod_{k=1}^N dv_k (1 - v_k)^{b'} (1 - zv_k)^{c'} \prod_{k<k'} (v_k - v_{k'})^{2\rho'} \cdot \prod_{i=1}^M dw_i (1 - w_i)^{b} (1 - zw_i)^{c} \prod_{i<i'} (w_i - w_{i'})^{2\rho} \prod_{k,i} (v_k - w_i)^{-2} = K_{NM} \int_0^1 \prod_{k=1}^N dv_k (1 - v_k)^{b'-\delta'} (1 - zv_k)^{c'-\delta'} \prod_{k<k'} (v_k - v_{k'})^{2\rho'} \cdot \prod_{i=1}^M dw_i (1 - w_i)^{b+\delta} (1 - zw_i)^{c-\delta} \prod_{i<i'} (w_i - w_{i'})^{2\rho} \prod_{k,i} (v_k - w_i)^{-2} \tag{56}
\]

where

\[
a' = -\rho' a \quad b' = -\rho' b \quad c' = -\rho' c \quad \delta' = -\rho' \delta \quad \rho' = 1/\rho
\]

\[
\delta = a + c + 1 - N + (M - 1)\rho
\]

\[
\delta' = a' + c' + 1 - M + (N - 1)\rho'
\]

(57)

and

\[
K_{NM} = \prod_{i=0}^{N-1} \frac{\Gamma(a' + 1 + i\rho')\Gamma(b' + 1 + i\rho')}{\Gamma(-c' + M + (-N + 1 + i)\rho')\Gamma(a' + b' + c' + 2 - M + (N - 1 + i)\rho')} \cdot \prod_{i=0}^{M-1} \frac{\Gamma(a + 1 - N + i\rho)\Gamma(b + 1 - N + i\rho)}{\Gamma(-c + (-M + 1 + i)\rho)\Gamma(a + b + c + 2 - 2N + (M - 1 + i)\rho)} \tag{58}
\]

The left hand side of Eq. (56) has the structure of the standard integral realization for minimal models 17, leaving out some irrelevant pre-factors. There are \(N, M\) screening charges of the two kinds and they are at positions \(v_k\) and \(w_i\) (in Ref. 17 and in the previous section they were denoted by \(u\) and \(v\)), except they have been scaled by \(z\). Also of course here the meaning of the letters \(a, b, c\) are different from their meaning in the previous section. The integrations are taken to be (“time”-) ordered, and we only consider one kind of conformal block, where all the integrations are between 0 and \(z\).

The proof we present of this identity is by brute force and takes several lengthy calculations, although not so bad as the case we shall be mostly interested in: the identity between our \(SL(2)\) block and that of Andreev.
The idea is simply to consider both sides of Eq. (56) as functions of \( z \). Both versions of the conformal blocks have singularities only when \( z \to 0, 1, \infty \). The singularity at \( z \to 0 \) has been explicitly removed by the scaling, and it is rather trivial that both sides had the same power of \( z \). The limit \( z \to 0 \) is used for normalization. We isolate the other singularities and show that they have the same structure and the same strengths.

The limit \( z \to 0 \) is simple. In that limit both sides of Eq. (56) are holomorphic in \( z \) (since suitable pre-factors have been removed) and we may simply put \( z = 0 \). Then both sides may be done in terms of the Dotsenko-Fateev integral ([17], appendix A). This gives immediately the normalization, \( K_{NM} \).

The limits \( z \to 1, \infty \) are much more complicated. Here there will be several different power singularities of the form \((1 - z)^A \) and \( z^B \). We must isolate those and compute their strengths and demonstrate that we get the same results for both sides of Eq. (56). The basic analytic tool was described in Ref. [17].

\( z \to 1 \)

We split the integration region from 0 to 1 using a small positive \( \epsilon \) as follows (the integrations are time-ordered throughout)

\[
\int_0^1 = \int_0^{1-\epsilon} + \int_{1-\epsilon}^1
\]

\[
\int_0^1 \prod_{k=1}^N dv_k \prod_{i=1}^M dw_i = \sum_{n,m} \int_0^{1-\epsilon} \prod_{k=N-n+1}^N dv_k \prod_{i=M-m+1}^M dw_i \int_{1-\epsilon}^1 \prod_{l=1}^{N-n} dv_l \prod_{j=1}^{M-m} dw_j \tag{59}
\]

It is not difficult to check that a particular \( n, m \) term will give rise to a particular power of \((1 - z)\). One performs the following scalings of the \( \int_{1-\epsilon}^1 \) integration variables, \( w \sim v_l, w_j, \)

\[
w \rightarrow 1 - (1 - z) 1 - \frac{w}{w}
\]

\[
dw \rightarrow \frac{1 - z}{w^2} \, dw
\]

\[
1 - w \rightarrow (1 - z) 1 - \frac{w}{w}
\]

\[
1 - zw \rightarrow (1 - z) \frac{(1 - z)w + z}{w} \sim \frac{1 - z}{w}
\]

\[
\int_{1-\epsilon}^1 \sim \int_0^1 \tag{60}
\]

One rather easily finds that the power of \((1 - z)\) occurring on both sides of Eq. (56) is

\[
(1 - z)^{(N-n)(b+c+1)+(N-n)(N-n-1)p'+(M-m)(b+c+1)+(M-m)(M-m-1)p-2(N-n)(M-m)}
\]

The coefficient of this singularity may also be evaluated on both sides in terms of Dotsenko-Fateev integrals. It is not, however, immediately obvious that these coefficients are equal. Both sides involves many products of terms involving ratios of \( \Gamma \) functions. One employs over and over again the simple identity

\[
\frac{\Gamma(X)}{\Gamma(X - L)} = \prod_{j=0}^{L-1} (X - 1 - j) \tag{61}
\]
Thus it turns out for example that there are factors on both sides involving $\Gamma$ functions with argument involving $a'$ only (no $b', c'$). On the left hand side we have

$$\prod_{i=0}^{n-1} \Gamma(a' + 1 + i\rho')$$

whereas on the right hand side there are similar factors of the form

$$\prod_{i=0}^{N-n-1} \Gamma(a' + 1 + i\rho')$$

$$\prod_{i=0}^{N-M-1} \Gamma(a' + 1 - M + (n+i)\rho')$$

Using the identity Eq. (61) we find that the ratio between these two is

$$\prod_{i=0}^{N-n-1} \prod_{j=0}^{M-1} \frac{1}{(a' - j + (n+i)\rho')}$$

By going over the several other different factors on both sides and working out the ratios, one finally shows that the product of all ratios equals 1.

This completes the proof that the singularities are identical in the limit $z \to 1$.

\[ z \to \infty \]

The strategy is entirely analogous. We make the split (time-ordered integrations)

$$\int_0^1 = \int_0^\epsilon + \int_\epsilon^1$$

$$\int_0^1 \prod_{k=1}^{N} dv_k \prod_{i=1}^{M} dw_i = \sum_{n,m} \int_0^\epsilon \prod_{k=N-n+1}^{N} dv_k \prod_{i=M-m+1}^{M} dw_i \int_\epsilon^1 \prod_{l=1}^{N-n} dv_l \prod_{j=1}^{M-m} dw_j$$

(62)

The $\int_0^\epsilon$ integration variables are scaled according to

$$w \to \frac{1 - w}{-zw}$$

$$\int_0^\epsilon dw \to \int_1^{1/(1-z)} \frac{dw}{zw^2} \sim (-z)^{-1} \int_0^1 \frac{dw}{w^2}$$

(63)

However, this time we must identify the left hand side with $n, m$ with the right hand side with $N - n, M - m$. It is then simple to verify that the power of $z$ on both sides are

$$(-z)^{-na' + (N-n)c' - n(n-1)\rho' - ma + (M-m)c - m(m-1)\rho + 2nm}$$

To check that the coefficients also agree, as before one carries out explicitly the integrations in terms of Dotsenko-Fateev integrals resulting in many products of ratios of $\Gamma$ functions. Finally one laboriously checks that the ratios multiply up to 1.

5.2 Integral identity in $SL(2)$ current theory

As previously indicated there is no absolute need for proving the equivalence between the PRY 4-point function, [9], and the one by Andreev, [13], since both satisfy the Knizhnik-Zamolodchikov equations. Nevertheless, it is of some interest to understand better how
two such seemingly very different expressions can agree, and it is rather nice to be aware of the clarification provided by the relation to the minimal model case treated in the previous subsection. In this subsection we go over several of the steps needed for a direct analytic proof. In fact, we investigate the singularity structure of the two expressions in the double limits, $z, x \to 0, z, x \to 1, z, x \to \infty$ and in the single limit, $z \to x$. We restrict ourselves to just one of the s-channel conformal blocks.

**Theorem**

$$
\int_0^1 du \prod_{k=1}^N dv_k v_k^{a'} (1 - v_k)^{b'} (1 - z v_k)^{c'} \left( 1 - \frac{1 - v_k}{1 - z v_k} \frac{z}{x} \right) \prod_{k < k'} (v_k - v_{k'})^{2\rho'} \\
\cdot \prod_{i=1}^M dw_i w_i^b (1 - w_i)^b (1 - z w_i)^c \left( 1 - \frac{1 - w_i}{1 - z w_i} \frac{z}{x} \right) \prod_{i < i'} (w_i - w_{i'})^{2\rho} \\
\cdot \prod_{k,i}^N (v_k - w_i)^{-2} u^{-c-1} (1 - u)^{b+c-N+(M-1)\rho}
$$

$$= K_{NM}^x \int_0^1 du \prod_{k=1}^N dv_k v_k^{a'-b'} (1 - v_k)^{b'+\delta'} (1 - z v_k)^{c'-\delta'} \left( 1 - \frac{z}{x} v_k \right) \prod_{k < k'} (v_k - v_{k'})^{2\rho'} \\
\cdot \prod_{i=1}^M dw_i w_i^b (1 - w_i)^b (1 - z w_i)^c \left( 1 - \frac{z}{x} w_i \right) \prod_{i < i'} (w_i - w_{i'})^{2\rho} \\
\cdot \prod_{k,i}^N (v_k - w_i)^{-2}$$

(64)

where

$$K_{NM}^x = \frac{\Gamma(-c)\Gamma(b + c + 1 - N + (M - 1)\rho)}{\Gamma(b + 1 - N + (M - 1)\rho)} K_{NM}$$

(65)

Here, up to irrelevant common pre-factors, the left hand side is our form of the conformal block Eq. (22) for $r = R = N$ and $s = S = M$ in the s-channel. We denote this by $S^{PRY}$. Similarly up to the same pre-factors and the new normalization constant, $K_{NM}^x$, the right hand side is essentially Eq. (II). We denote it by $S^A$. Notice in particular that now we put

$$a = 2j_1, \ b = 2j_2 + \rho, \ c = 2j_3, \ \rho = t$$

(66)

$\delta, \delta'$ are given by the same expressions as for the minimal models:

$$\delta = a + c + 1 - N + (M - 1)\rho$$

$$\delta' = a' + c' + 1 - M + (N - 1)\rho'$$

(67)

Then $a - \delta$ is what was called $a$ in previous sections, $b + \delta$ was previously called $c$ and $c - \delta$ was previously called $b$. In subsequent sections we shall revert to this notation, but in this section we stick to the present notation in order to emphasize the similarity with minimal models. Also notice, that because all integrations are between 0 and 1 (after scaling by $z$), it is possible to deform the $u$-integration in Fig. 3 to being along the real axis from 0 to 1.

In this case we demonstrate that both the left hand side and the right hand side of the claimed identity have the same singularities in the limits $z, x \to 1, z, x \to \infty$ and
\[ z \to x. \] The proof turns out to be rather more laborious than for the minimal models. This is due to the \( x \)-dependence and the \( u \)-integration in the case of \( S^{PRY} \). However, the general strategy is entirely analogous, so we only indicate some of the steps on the way.

The limit \( z, x \to 0 \) is simple to deal with and it gives rise to the normalization constant, \( K_{NM}^x \) differing from the one in the minimal models, \( K_{NM} \) because of the \( u \)-integration.

\[ z, x \to 1 \]

We first deal with \( S^A \). Exactly as in the case of minimal models, we split the ordered integration ranges for the \( v \)'s in \( n v_k \)'s and \( m w_j \)'s in \((1, 1 - \epsilon)\) and \( N-n \) \( v_j \)'s and \( M-m \) \( w_j \)'s in \((1 - \epsilon, 1)\). Omitting integration signs and products for brevity we find

\[
S^{A}_{nm} \sim (1 - z)^{(N-n)(b'+c'+1)+(N-n)(N-n-1)\rho' + (M-m)(b+c+1)+(M-m)(M-m-1)\rho - 2(N-n)(M-m)}
\]

\[
\cdot (x - 1)^{N-n-(M-m)\rho} K_{NM}^x
\]

\[
\cdot v_k^{\delta - \delta'} (1 - v_k)^{b'+c'+(N-n)2\rho' - 2(M-m)} (x - v_k)(v_k - v_{k'})^{2\rho'}
\]

\[
\cdot w_i^{a - \delta}(1 - w_i)^{b+c+(M-m)2\rho - 2(N-n)} (x - w_i)^{-\rho}(w_i - w_i')^{2\rho}(v_k - w_i)^{-2}
\]

\[
\cdot v_i^{b' - c' - 2-(N-n)2\rho' + 2(M-m)} (1 - v_i)^{b + \delta'} (v_i - v_i')^{2\rho'}
\]

\[
\cdot w_j^{b - c - 2-(M-m)2\rho + 2(N-n)} (1 - w_j)^{b + \delta} (w_j - w_j')^{2\rho}(v_i - w_j)^{-2}
\]

where we have performed the same scalings as for minimal models. The above has to be summed over \( n \) and \( m \), but for a fixed value we pick up the pure \((1 - z)\) and \((x - 1)\) singularity indicated. The \( l, j \) part of the integration gives immediately rise to a standard Dotsenko-Fateev integral. For the \( k, i \) part we perform the further split and scalings

\[
\int_0^1 = \int_0^{1-\epsilon} + \int_{1-\epsilon}^1
\]

\[
v \to 1 - (1 - 1/x) \frac{1-v}{v}
\]

\[
\int_{1-\epsilon}^1 dv \to \int_0^1 (1 - 1/x) \frac{dv}{v^2}
\]

\[
\int_0^1 \prod_{k=1}^n dv_k \prod_{i=1}^m dw_i = \sum_{n_0,m_0} \int_0^{1-\epsilon} \prod_{k_0=n-n_0+1}^n dv_{k_0} \prod_{i_0=m-m_0+1}^m dw_{i_0}
\]

\[
\int_{1-\epsilon}^1 \prod_{k=1}^{n-n_0} dv_k \prod_{i=1}^{m-m_0} dw_i
\]

In the limit \( x \to 1 \) we extract the \((1 - x)\) power and find the coefficient again to be given by the product of two Dotsenko-Fateev integrals. Analysing the \( \Gamma \) functions of these we see that we can only get a non vanishing result if \( m_0 = m \) or if \( n_0 = m - 1 \). We denote these cases by \( S^{A_{I}}_{nm} \) and \( S^{A_{II}}_{nm} \). They will turn out to be related to fusion rules I and II. Combining everything we find the following singularities in the limit, \( z, x \to 1 \),

\[
S^{A_{I}}_{nm} = (1 - z)^{(N-n)(b'+c'+1)+(N-n)(N-n-1)\rho' + (M-m)(b+c+1)+(M-m)(M-m-1)\rho - 2(N-n)(M-m)}
\]

\[
\cdot (x - 1)^{N-n-(M-m)\rho} N(S^{A_{I}}_{nm})
\]

\[
S^{A_{II}}_{nm} = (1 - z)^{(N-n)(b'+c'+1)+(N-n)(N-n-1)\rho' + (M-m)(b+c+1)+(M-m)(M-m-1)\rho - 2(N-n)(M-m)}
\]

\[
\cdot (x - 1)^{b+c+1-N+n+(M-m-1)\rho} N(S^{A_{II}}_{nm})
\]
One checks that the singularities exactly correspond to fusion rules I and II. The normalizations, $N(S_{nm}^{IA})$ and $N(S_{nm}^{AI})$ are found explicitly in terms of products of Dotsenko-Fateev integrals to be lengthy expressions involving many products of ratios of $\Gamma$ functions.

We now turn to a similar analysis of $S_{nm}^{PRY}$ in the same limit $z \to 1$ followed by $x \to 1$. We replace $u \to 1 - u$ and perform the same split and the same scalings of the $v$ and $w$ variables as in the case of minimal models. Omitting again integration signs and products we find

$$S_{nm}^{PRY} \sim (1 - z)^{(N-n)(b'c'+1) + (N-n)(N-n-1)\rho + (M-m)(b+c+1) + (M-m)(M-m-1)\rho - 2(N-n)(M-m)} \cdot v_k^{\rho'}(1 - v_k)^{b'+c'}(N-n)2\rho - 2(M-m) \cdot \left( v_k - v_{k'} \right)^{2\rho'}$$

with similar expressions for $w_i$.

Here the $k, i$ integrations are independent of $x$ and $u$ and are readily evaluated in terms of Dotsenko-Fateev integrals, so we concentrate on the $l, j$ part. We perform the split and the scalings

$$\int_0^1 \prod_{l=1}^{N-n} dv_l \prod_{j=1}^{M-m} dw_j = \sum_{n_0, m_0} \int_0^\epsilon \prod_{l=0}^{N-n} dv_{l_0} \prod_{j_0=0}^{M-m} dw_{j_0}$$

$$\int_1^\epsilon \prod_{l=1}^{N-n-n_0} dv_l \prod_{j=1}^{M-m-m_0} dw_j$$

and similarly for $\int_0^1 du$. To be able to distinguish we write

$$\int_0^1 du \to \int_0^1 du$$

$$\int_0^\epsilon du \to \int_0^1 dy \int_0^1 \frac{dy}{y^2} \sim (x - 1) \int_0^1 \frac{dw}{w^2}$$

and denote them the $u$- and $y$- cases respectively. In the $u$-case the arising Dotsenko-Fateev integrals turn out to vanish unless $n_0 = m_0 = 0$, and we find in the $u$-case

$$S_{nm}^{PRY,u} \sim (1 - z)^{(N-n)(b'c'+1) + (N-n)(N-n-1)\rho + (M-m)(b+c+1) + (M-m)(M-m-1)\rho - 2(N-n)(M-m)} \cdot \prod_{k=1}^{n} \prod_{i=1}^{m} v_k^{\rho'}(1 - v_k)^{b'+c'}(N-n)2\rho - 2(M-m) \cdot \left( v_k - v_{k'} \right)^{2\rho'}$$

24
\[ w^b_j (1 - w_j)^{b+c+(M-m)2\rho - 2(N-n)} (w_i - w_{i'})^{2\rho} (v_k - w_i)^{-2} \]
\[ \prod_{i=1}^{N-n} M-M \prod_{j=1}^{M-m} v_i^{b'-c'-2-(N-n-1)2\rho' + 2(M-m)} (1 - v_i)^{b'/(v_i - v_{i'})^{2\rho'}} \]
\[ w_j^{b-c-2-(M-m-1)2\rho + 2(N-n)} (1 - w_j)^{b(w_j - w_{j'})^{2\rho}} (v_i - w_j)^{-2} \]
\[ (1 - (1 - v_i)(1 - u))(1 - (1 - w_j)(1 - u))^{-\rho} \]
\[ u^{b+c-N+n+(M-m-1)\rho} (1 - u)^{-c-1} \]

Now we want to establish
\[ S_{NM}^{PRY,u} = S_{NM}^A \]  

and
\[ S_{nm}^{PRY,u} = 0 \quad (n, m) \neq (N, M) \]

A straightforward analysis shows these to be the satisfied.

In the \( y \) case we introduce a similar further splitting resulting in objects \( S_{nm;n_0m_0}^{PRY} \). It turns out to be possible to demonstrate that
\[ S_{nm;00}^{PRY} = S_{nm}^A \]
\[ S_{nm;01}^{PRY} = S_{nm}^A \]

(In principle we should check that higher values of \( n_0, m_0 \) give zero. We anticipate no interesting problems here). The analysis contains no new ideas over the situation encountered for minimal models, but again the calculations involved are somewhat lengthy.

\( z, z/x, x \to \infty \)

Again we first analyse the \( S^A \) case. We introduce the same splitting of integrations and the same variable transformations as for minimal models, and find
\[ S_{nm}^A \sim K_{NM}^{\gamma} z^{-n(a' + \delta' - 1) + (N-n)(c' - \delta' + 1) - n(n-1)\rho'} \]
\[ (-z)^{m(n-a + \delta - 1) + (M-m)(c - \delta - \rho) - m(m-1)\rho + 2nm} \]
\[ x^{-n(N-n) + (M-m)\rho} \]
\[ v_k^{a' c' + 2\delta' - 2 + 2m - (n-1)2\rho} (1 - v_k)^{a' \delta'} (v_k - v_{k'})^{2\rho'} \]
\[ w_i^{a - c + 2\delta - 2 + 2m - (m-1)2\rho} (1 - w_i)^{a - \delta} (w_i - w_{i'})^{2\rho'} (v_k - w_i)^{-2} \]
\[ (1 + \frac{1 - v_k}{x v_k}) (1 + \frac{1 - w_i}{x w_i})^{-\rho} \]
\[ v_l^{a' c' + 2\delta' + 1 - 2m + n 2\rho} (1 - v_l)^{b' + \delta'} (v_l - v_{l'})^{2\rho'} \]
\[ w_j^{a + c + 2\delta - 2 + (2m-1)\rho} (1 - w_j)^{b + \delta} (w_j - w_{j'})^{2\rho'} (v_l - w_j)^{-2} \]

(79) again omitting integration signs and products, which are just as for the case of minimal models. The \( l, j \) integration is seen to result in Dotsenko-Fateev integrals. In the \( k, i \) integrals we perform a split of integrals form 0 to \( \epsilon \) and from \( \epsilon \) to 1. In the \( \int_0^\epsilon \) we transform variables like
\[ u \to \frac{1 - v}{x v} \]
\[ 1 + \frac{1 - v}{xv} \rightarrow \frac{1}{1 - v} - \frac{1}{x} \sim \frac{1}{1 - v} \]

\[ \int_0^\epsilon dv \rightarrow x^{-1} \int_0^1 \frac{dv}{v^2} \]

\[ \int_0^1 \prod_{k=1}^n dv_k \prod_{i=1}^m dw_i = \sum_{n_0, m_0} \int_0^\epsilon \prod_{k_0=n-n_0+1}^n dv_{k_0} \prod_{i_0=m-m_0+1}^m dw_{i_0} \]

\[ \int_\epsilon^1 \prod_{k=1}^{n-n_0} dv_k \prod_{i=1}^{m-m_0} dw_i \]  

(80)

An analysis of the coefficients of the singularities reveals that this is non vanishing only if

\[ (n_0, m_0) = (0, 0), (0, 1) \]

These two cases we term again (we use the same notation as before, even though now we consider a different limit), \( S_{nm}^{AI} \) and \( S_{nm}^{AII} \) for what turns out to be fusion rules I and II. We find

\[
S_{nm}^{AI} = (-z)^{(N-n)a' + nc' - (N-n)(N-n-1)p' - (M-m)a +mc - (M-m)(M-m)\rho + (M-m)(2N-2n-1)} \cdot x^{-(N-n)+(M-m)\rho} N(S_{nm}^{AI})
\]

\[
S_{nm}^{AII} = (-z)^{(N-n)a' + nc' - (N-n)(N-n-1)p' - (M-m)a +mc - (M-m)(M-m)\rho} \cdot (-z)^{(M-m)(2N-2n-1)} \cdot x^{-a-c-1+N-n+(-M+m)\rho} N(S_{nm}^{AII})
\]

(81)

where the normalizations (different of course to the ones in the previous limit \( z, x \rightarrow 1 \)), \( N(S_{nm}^{AI}) \) and \( N(S_{nm}^{AII}) \) are given (in terms of Dotsenko-Fateev integrals) by lengthy products of ratios of \( \Gamma \) functions. The singularities shown indicate that indeed we are dealing with fusion rules I and II.

We then treat the \( S_{PRY} \) case. Again we first perform the same splittings and variable transformations as for \( S^A \) with the same meaning of \( v_k, v_l, w_i, w_j \). The \( i, k \) part is again simple, whereas the \( l, j \) part is treated with a split of the ordered integrations as

\[
\int_0^1 \prod_{l=1}^{N-n} dv_l \prod_{j=1}^{M-m} dw_j = \sum_{n_0, m_0} \int_0^\epsilon \prod_{l_0=N-n-n_0+1}^{N-n} dv_{l_0} \prod_{j_0=M-m-m_0+1}^{M-m} dw_{j_0} \]

\[ \int_\epsilon^1 \prod_{l=1}^{N-n-n_0} dv_l \prod_{j=1}^{M-m-m_0} dw_j \]  

(82)

followed by the scalings

\[
v_{l_0} \rightarrow \frac{1 - v_{l_0}}{xv_{l_0}}
\]

\[
w_{j_0} \rightarrow \frac{1 - w_{j_0}}{xw_{j_0}}
\]

(83)
We then seek to demonstrate that

\[
S_{nm;00}^{PRY} = S_{N-n,M-m}^{AF}
\]

\[
S_{nm;01}^{PRY} = S_{N-n,M-m}^{AF}
\]

\[
S_{nm;numo}^{PRY} = 0, \quad (n_0, m_0) \neq (0, 0), (0, 1)
\]

(84)

The proof here is lengthy again, but with no new ideas introduced.

\[z \rightarrow x\]

This case is the most complicated. We omit nearly all the details, most of which are similar to what have been described above, and we concentrate on the strategy. More details will be presented elsewhere [21]. First one may check that the nature of the singularity is a linear combination of just two different powers of \((z-x)\) namely either \((z-x)^0\) or \((z-x)^{c+1-\rho}\). Second one must investigate whether the coefficients of these two powers are the same for the two sides of Eq. (84). That coefficient is a function of \(x\) in the limit \(z \rightarrow x\), so we must investigate whether the coefficient functions defined by the two sides of Eq. (84) are equal. As above the technique is to investigate the singularity structure in the singular limits \(x = 0, 1, \infty\). It turns out that the sought equality depends on the following identities:

\[
\int_0^1 dw dy w^{-a-2+\rho}(1-w)^a(1-(1-w)(1-(1-x)y))^{-\rho} y^{-b-c+2}\cdot y^{-M+2+(-M+2)\rho} (1-y)^{b+c-N+(M-1)\rho} = (1-x)^{-a-1} \cdot \Gamma(a+1)\Gamma(b+c+1-N+(M-1)\rho)\Gamma(-a-b-c-2+N+(-M+2)\rho) \Gamma(\rho) \tag{85}
\]

and

\[
\int_0^1 dw \int_0^w dy w^{-a}(1-w)^{a+c-2}(w-y)^{-c} y^{b+c-2}(1-y)^{-b} = 0 \tag{86}
\]

which are not too difficult to prove. Next define the Dotsenko-Fateev integrand:

\[
DF(N, M; a, b, \rho; \{v_k\}, \{w_i\}) \equiv \prod_{k-1}^N v_k^{a'}(1-v_k)^b \prod_{k<k'}^M (v_k - v_{k'})^{2\rho} \prod_{i=1}^M w_i^{a}(1-w_i)^b \prod_{i<i'}^N (w_i - w_{i'})^{2\rho} \prod_{k,i}^{N,M} (v_k - w_i)^{-2} \tag{87}
\]

Then we find that the equality of the two sides of Eq. (84) depends on the following three identities (generalized Dotsenko-Fateev integrals):

\[(I) \quad \int_0^1 du \prod_{k=1}^N dv_k \prod_{i=1}^M dw_i DF(N, M; a, b, \rho; \{v_k\}, \{w_i\}) \]

\[
\cdot (1-(1-v_k)u)(1-(1-w_i)u)^{-\rho} u^{-c-1}(1-u)^{b+c-N+(M-1)\rho} = \frac{\Gamma(-c)\Gamma(b+c+1-N+(M-1)\rho)\Gamma(a+b+c+2-N+(M-2)\rho)}{\Gamma(b+1-\rho)\Gamma(a+b+c+2N+(2M-2)\rho)}
\]
As explained in Ref. [17], it is enough to calculate one column and one row of this matrix

\[ \begin{align*}
\text{(II)} & \quad \int_0^1 du \prod_{k=1}^n \int_{v_k}^{v_k} dv_k \int_{w_i}^{w_i} dw_i \frac{\Gamma(b + c + 1 - N + (M - 1)\rho) \Gamma(a + b + c - 2 - N + (M - 2)\rho)}{\Gamma(\rho)} \\
& \quad \cdot \int_0^1 \prod_{k=1}^n dv_k \prod_{i=1}^m dw_i \quad \text{DF}(n, m; a - c + 2n - 2 + (2m + 2)\rho, \rho; \{v_k\}, \{w_i\}) \\
& \quad \cdot \frac{1}{\Gamma(a + b + c - 2 - 2N + (2M - 2)\rho)} \cdot \frac{1}{\Gamma(\rho)}
\end{align*} \]

All the final integrals are of course Dotsenko-Fateev integrals. In the second identity there is a phase depending on the precise choice of the integration contour for \( u \). These last three identities we have not proven directly. (We have checked for low values of \( N, M \).) One might take the attitude that the undoubted identity of our realization and that of Andreev, i.e. the unquestionable correctness of Eq. [64], implies these somewhat remarkable integral identities.

### 6 Calculation of the crossing matrix

The crossing matrix, \( \alpha_{(r,s,e),(r',s',e')}^{(R,S)} \), is defined by the equation

\[ S_{(r,s,e),(r',s',e')}^{(R,S)}(z, x) = \sum_{r''=0}^{R} \sum_{s''=0}^{S} \alpha_{(r,s,e),(r',s',0)}^{(R,S)} T_{(r',s',0)}^{(R,S)}(z, x) \]

\[ + \sum_{r''=0}^{R} \sum_{s''=0}^{S-1} \alpha_{(r,s,e),(r',s',1)}^{(R,S)} T_{(r',s',1)}^{(R,S)}(z, x) \]

As explained in Ref. [17], it is enough to calculate one column and one row of this matrix in order to determine monodromy invariant green functions. We find that a moderate
Following the idea of Ref. \[17\] we define the following object (suppressing several variables)

\[
J(r_1, s_1, r_2, s_2, r_3, s_3) = z^{2ij + j3/t}(1 - z)^{2j3/t}
\]

\[
\int_0^z \prod_{i=1}^{r_1} du_i \prod_{k=1}^{s_1} dv_k \int_1^{z + \infty} \prod_{m=r_1+1}^{r_1+s_1} du_m \prod_{n=s_1+1}^{s_1+s_2} dv_n \int_1^{r_1+r_2+\infty} \prod_{j=r_1+r_2+1}^{r_1+r_2+r_3} du_j \prod_{l=s_1+s_2+1}^{s_1+s_2+s_3} dv_l
\]

\[
\prod_{i} u_i^\alpha (1 - u_i)^\beta (z - u_i)^\gamma \prod_{i<j'} (u_i - u_{i'})^{2\rho} (u_i - x)
\]

\[
\prod_{m} u_m^\alpha (1 - u_m)^\beta (u_m - z)^\gamma \prod_{m<m'} (u_m - u_{m'})^{2\rho} (x - u_m)
\]

\[
\prod_{j} u_j^\alpha (u_j - 1)^\beta (u_j - z)^\gamma \prod_{j<j'} (u_j - u_{j'})^{2\rho} (u_j - x)
\]

\[
\prod_{m,i} (u_m - u_i)^{2\rho} \prod_{j,i} (u_j - u_i)^{2\rho} \prod_{j,m} (u_j - u_m)^{2\rho}
\]

\[
\prod_{k} v_k^\alpha (1 - v_k)^\beta (z - v_k)^\gamma \prod_{k<k'} (v_k - v_{k'})^{2\rho} (v_k - x)^\rho
\]

\[
\prod_{n} v_n^\alpha (1 - v_n)^\beta (v_n - z)^\gamma \prod_{n<n'} (v_n - v_{n'})^{2\rho} (x - v_n)^\rho
\]

\[
\prod_{l} v_l^\alpha (v_l - 1)^\beta (v_l - z)^\gamma \prod_{l<l'} (v_l - v_{l'})^{2\rho} (v_l - x)^\rho
\]

\[
\prod_{n,k} (v_n - v_k)^{2\rho} \prod_{l,k} (v_l - v_k)^{2\rho} \prod_{l,n} (v_l - v_n)^{2\rho}
\]

\[
\prod_{\alpha,\beta} (v_\alpha - u_\beta)^{2\rho}
\]

In other words, there are \(r_1\) and \(s_1\) \(u\) and \(v\) integrations between \(0\) and \(z\), \(r_2\) and \(s_2\) \(u\) and \(v\) integrations between \(z\) and \(1\) and \(r_3\) and \(s_3\) \(u\) and \(v\) integrations between \(1\) and \(\infty\). Also the variables, \(u_i, v_k, u_j, v_l\) are taken along contours similar to the ones in Fig. 8, whereas the variables, \(u_m, v_n\) are taken along similar ones lying between \(z\) and \(1\). We notice that

\[
J(r, s, 0, 0, R - r, S - s) = J^\text{(R,S)}_{(r,s,0)}
\]

\[
J(0, 0, R, S, 0, 0) = J^\text{(R,S)}_{(R,S,0)}
\]

Therefore we may start from \(J(r, s, 0, 0, R - r, S - s)\) and gradually move integration contours by contour deformations on to the interval \((z, 1)\). In the process we pick up contributions from integrals between \(-\infty\) and \(0\), but these may be neglected in the calculation of the column, \(\alpha^\text{(R,S)}_{(r,s,e),(R,S,0)}\). The calculational procedure \[17\] consists in deforming upper
and lower $u$ and $v$ contours in appropriate ways, and forming suitable linear combinations of the result. As explained in Ref. [17], one may then derive identities for the functions, $J(r_1, s_1, r_2, s_2, r_3, s_3)$, by carefully keeping track of the phases arising between the result of the deformations, and the definitions of the $J$’s. The useful identities turn out to be after some calculations:

$$J(r_1, s_1, r_2, s_2, r_3, s_3) = e^{i\pi \rho'(r_2 - r_1 + 1)} \frac{s(b' + \rho'(r_2 + r_3))}{s(b' + c' + \rho'(r_1 - 1 + 2r_2 + r_3))} J(r_1 - 1, s_1, r_2 + 1, s_2, r_3, s_3) + ...$$

$$= -e^{i\pi \rho(s_2 - s_1 + 2)} \frac{s(b + \rho(s_2 + s_3))}{s(b + c + \rho(s_1 - 1 + 2s_2 + s_3))} J(r_1, s_1 - 1, r_2, s_2 + 1, r_3, s_3) + ...$$

$$J(0, 0, r_2, s_2, r_3, s_3) = e^{i\pi \rho(r_2 - r_3 + 1)} \frac{s(c' + \rho' r_2)}{s(b' + c' + \rho'(2r_2 + r_3 - 1))} J(0, 0, r_2 + 1, s_2, r_3 - 1, s_3) + ...$$

$$= -e^{i\pi \rho(s_2 - s_3 + 2)} \frac{s(c + \rho s_2)}{s(b + c + \rho(2s_2 + s_3 - 1))} J(0, 0, r_2 + 1, r_3, s_3 - 1) + ...$$

(92)

Here the dots stand for terms that cannot contribute to the crossing matrix element. After several further but in principle straightforward calculations we obtain

$$\alpha_{(r,s,0),(R,S)}^{(R,S)} = (-)^s e^{i\pi S\rho} \alpha_{r,R}^{(R)}(a', b', c'; \rho') \alpha_{s,S}^{(S)}(a, b, c; \rho)$$

$$\alpha_{s,S}^{(S)}(a, b, c; \rho) = \frac{\prod_{j=1}^S s(j\rho)}{\prod_{k=1}^S s(k\rho) \prod_{m=1}^S s(m\rho)} \prod_{j=0}^{S-1} \frac{s(b + \rho(S - s + j))}{s(b + c + \rho(S + j - 1))} \prod_{l=0}^{S-1} \frac{s(c + \rho(s + l))}{s(b + c + \rho(s + S + l - 1))}$$

(93)

and where $\alpha_{r,R}^{(R)}(a', b', c'; \rho')$ is given by a completely similar expression. The phase is the result of multiplying many phases together. This completes the calculation of the matrix elements of the relevant column as far as fusion rule I is concerned. The result has a form identical to what is found for minimal models [17].

Concerning fusion rule II it turns out that a simple trick allows to obtain the result rather easily. In fact, a suitable contour deformation of the complex contour $C_v$ allows one to obtain an equation of the form

$$S_{(r,s,1)}^{(R,S)} = -\frac{1}{\pi} \left\{ e^{i\pi \rho s} s(c + \rho s) S_{(r,s+1,0)}^{(R,S)} + s(a + c + 2\rho s) \int_x^0 \right\}$$

(94)

The integral from $x$ to 0 (we imagine $x < 0$ in the s-channel) cannot have a contribution with a $(1 - z)$ singularity appropriate for $T_{(r,s,0)}^{(R,S)}$, and so we do not specify the integrand, and we drop the integral in the calculation. Now it is an easy matter to obtain the missing matrix elements from the ones we have already given. One finds

$$\alpha_{(r,s,1),(R,S,0)}^{(R,S)} = -\frac{1}{\pi} \frac{s(c + \rho s) s((s + 1)\rho)}{s(\rho)} \alpha_{(r,s+1,0),(R,S,0)}^{(R,S)}$$

(95)

where

$$s = 0, ..., S - 1$$
6.2 The row of the transformation matrix, \( \alpha^{(R,S)}_{(R,S,0),(r,s,\epsilon)} \)

The procedure is to consider the s-channel block, \( S^{(R,S)}_{(r,s,\epsilon)} \) and then isolate the t-channel singularities in \( (1 - z) \) and \( (x - 1) \). The strengths of these singularities will tell us which t-channel block is obtained. In this way we determine modified crossing matrix elements

\[
S^{(R,S)}_{(r,s,\epsilon)}(z, x) = \sum_{r'=0}^{R} \sum_{s'=0}^{S} \alpha^{(R,S)'}_{(r,s,\epsilon),(r',s',0)} I^{(R,S)}_{(r',s',0)}(z, x) + \sum_{r'=0}^{R} \sum_{s'=0}^{S-1} \alpha^{(R,S)'}_{(r,s,\epsilon),(r',s',1)} I^{(R,S)}_{(r',s',1)}(z, x) \tag{96}
\]

These matrix elements are related to the ones we have previously considered by the normalization constants of the last section. Denoting the corresponding normalizations in the t-channel by \( \tilde{N}^{(R,S)}_{(r,s,0)}(a, b, c; \rho) \), we have (cf. also Ref. [17])

\[
\alpha^{(R,S)}_{(R,S,0),(r,s,0)} = \alpha^{(R,S)'}_{(R,S,0),(r,s,0)} / \tilde{N}^{(R,S)}_{(r,s,0)}(a, b, c; \rho) = \alpha^{(R,S)'}_{(R,S,0),(r,s,0)} / N^{(R,S)}_{(r,s,0)}(b, a, c; \rho) \tag{97}
\]

We consider the real \( T \)-ordered form of the integral representation:

\[
e^{i\pi(R - S\rho)} S^{(R,S)}_{(R,S,0)} = z^{2j_1 j_2 / t} (1 - z)^{2j_3 j_4 / t} z^{P T} \int_0^1 \prod_{i=1}^R du_i \prod_{K=1}^S dv_K u_i^{\alpha'}(1 - z u_i)^{\beta'}(1 - u_i)^{\gamma'}(x - z u_i) \prod_{i < i'} (u_i - u_{i'})^{2\rho'} v_K^\rho (1 - z v_K)^\beta (1 - v_K)^\gamma (x - z v_K)^{-\rho} \prod_{K<K'} (v_K - v_{K'})^{2\rho} \prod_{I,K} (u_I - v_K)^{-2} \tag{98}
\]

The phase on the left hand side takes into account that s- and t-channel blocks are defined with different phase conventions as far as the factors \( (x - u) \) and \( (x - v)^{-\rho} \) are concerned. The pre-factor \( z^{P T} \) is obtained from scaling the integration variables with \( z \). When \( z \to 1 \) it is regular and we shall ignore it in the following.

Next, we use analytic tricks similar to Ref. [17] and similar to what was used in section 5, in order to isolate the singularities in \( (1 - z) \). Here we shall need to similarly isolate the singularities in \( (x - 1) \). We consider the integration region, where the first \( r \) \( u_i \)'s are integrated (ordered) from \( 1 - \epsilon \) to \( 1 \), (\( \epsilon \) small > 0) the first \( s \) \( v_k \)'s similarly from \( (1 - \epsilon) \) to \( 1 \), and the remaining variables from \( 0 \) to \( 1 - \epsilon \). The first \( u_i \)'s and \( v_k \)'s are indexed by \( i \) and \( k \) and the remaining ones by \( j, l \). The first \( u_i \)'s are transformed as \( u_i \to 1 - u_i \), followed by \( u_i \to (1 - z) u_i \), and likewise \( v_k \to 1 - v_k \) followed by \( v_k \to (1 - z) v_k \). Inserting that in Eq. (98) we find the singular behaviour as \( z \to 1 \):

\[
(1 - z)^{-h(j_2) - h(j_3) + h(j_1)}
\]

where

\[
j_I = j_2 + j_3 - r + st \tag{99}
\]
Furthermore one isolates the \( x \to 1 \) behaviour

\[
(x - 1)^{r-st} = (x - 1)^{j_2+j_3-j_1}
\]

Therefore we may calculate the coefficient of \( t_{(r,s,0)}^{(R,S)} \) in the expansion of \( S_{(R,S,0)}^{(R,S)} \). After some work one finds the result

\[
\alpha_{(R,S), (r,s,0)}^{(R,S)} = e^{i\pi(-R+S\rho)} J_{r,s}(-b - c + 2(r - 1 - (s - 1)\rho), c; \rho)
\]

\[
J_{R-r,S-s}(b + c - \rho - 2(r - \rho s), a; \rho)
\]

(100)

Some further calculations give

\[
\alpha_{(R,S), (r,s,0)}^{(R,S)} = (-S) e^{i\pi s\rho} \alpha_{(R,s)}^{(R)}(a', b', c', d'; \rho') \alpha_{S,s}^{(S)}(a, b, c, d; \rho)
\]

\[
\alpha_{S,s}^{(S)}(a, b, c, d; \rho) = \prod_{i=0}^{s-1} \frac{s(b + i\rho)}{s(b + c + (s - 1 + i)\rho)} \prod_{i=0}^{S-s-1} \frac{s(a + b + c + d + 2(S - 1)\rho - i\rho)}{s(b + c + d + 2(S - 1)\rho - (S - s - 1 + i)\rho)}
\]

(101)

Here for convenience of writing we have defined

\[
d \equiv -\rho \\
d' \equiv -d/\rho = 1
\]

(102)

The presence of the \( d \) dependence is the only difference from the corresponding expression in minimal models [17]. It originates directly from the factors \((u - x)\) and \((v - x)^{-\rho}\) in the integral realization. Such factors are not present in the case of minimal models.

In order to isolate the singularity which corresponds to the t-channel blocks for fusion rule II, we supplement the above specification of the integration region by the requirement that the variable \( v_{s+1} \) should be integrated between 1 - \( \epsilon \) and 1, and then transformed as \( v_{s+1} \to 1 - v_{s+1} \) followed by \( v_{s+1} \to (x - 1)v_{s+1} \). After some calculations we find

\[
\alpha_{(R,S), (r,s,1)}^{(R,S)} = e^{i\pi(-R+S\rho)} J_{r,s}(-b - c + 2(r - s\rho) + 2(\rho - 1), c; \rho)
\]

\[
J_{R-r,S-s-1}(a, b + c + \rho - 2(r - \rho s); \rho)
\]

\[
\frac{\Gamma(\rho - 1 - b - c - 2\rho s + 2r)\Gamma(b + c + 2\rho s - 2r + 1)}{\Gamma(\rho)}
\]

(103)

and using the normalizations and various \( \Gamma \) function identities

\[
\alpha_{(R,S), (r,s,0)}^{(R,S)} = \frac{\pi \alpha_{(R,S), (r,s,0)}^{(R,S)} s(\rho)}{s(b + \rho s)s(b + c + \rho(s - 1))}
\]

(104)

7 Monodromy invariant 4-point greens functions

Following the discussion in [17], monodromy invariant 4-point greens functions,

\[
G_{j_1,j_2,j_3,j_4}(z, \bar{z}, x, \bar{x})
\]

32
can be obtained by writing

\[ G_{j_1,j_2,j_3,j_4}(z,x) = \sum_{r,s,t} |S^{(R,S)}_{(r,s,\epsilon)}(j_1,j_2,j_3,j_4; z,x)|^2 X^{(R,S)}_{(r,s,\epsilon)} \]  

(105)

This form ensures single valuedness in the limits \( z \to 0 \) and \( x \to 0 \). Single valuedness in the limits \( z \to 1 \) and \( x \to 1 \) is ensured provided the \( X \)'s are chosen to satisfy \([17]\)

\[ X^{(R,S)}_{(r,s,\epsilon)} \propto \alpha^{(R,S)}_{(r,s,\epsilon),(r,s,\epsilon)}(b, a, c, d; \rho) \]

(106)

Using rescaling tricks similar to [17] we obtain

\[ X^{(R,S)}_{(r,s,0)} = X^{(R)}_{r}(a', b', c', d'; \rho') X^{(S)}_{s}(a, b, c, d; \rho) \]

\[ X^{(S)}_{s}(a, b, c, d; \rho) = \prod_{i=1}^{s} s(i \rho) \prod_{i=1}^{s} s(i \rho) \prod_{i=0}^{s-1} s(a + i \rho) s(c + i \rho) \]

\[ \prod_{i=0}^{s-1} s(b + i \rho) s(1 - a - b - c - d - 2(S - 1) \rho + i \rho) \]

(107)

This expression is very similar to the result for minimal models except for the presence of the \( d = -\rho \) and \( d^\prime = 1 \). Finally

\[ X^{(R,S)}_{(r,s,1)} = -\pi^2 \frac{s^2(\rho)}{s(c + \rho s)(s + 1) \rho s(a + \rho s) s(a + c + \rho(s - 1))} X^{(R,S)}_{(r,s,0)} \]

(108)

Consistency requires that the green functions thus defined automatically are single valued also around \( z \to x \). We have checked that indeed for one screening charge (of the second kind) this is the case. We expect it to be true generally.

It is convenient for studies of the operator algebra to also introduce the following expansion

\[ G_{j_1,j_2,j_3,j_4}(z,x) = \sum_{r,s,\epsilon} |S^{(R,S)}_{(r,s,\epsilon)}(j_1,j_2,j_3,j_4; z,x)|^2 \tilde{f}^{(R,S)}_{(r,s,\epsilon)} \]

(109)

where \( s^{(R,S)}_{(r,s,\epsilon)} \) are defined in Eq. [17] and Eq. [19]. The coefficients, \( f^{(R,S)}_{(r,s,\epsilon)} \), differ from the coefficients, \( X^{(R,S)}_{(r,s,\epsilon)} \) by a factor \( N^{(R,S)}_{(r,s,\epsilon)} \). It is fairly straightforward to collect all the results and obtain the expression for \( \tilde{f}^{(R,S)}_{(r,s,\epsilon)} \). On the way we use the formula \( [17] \) (A.35) alternative to Eq. [52]:

\[ J_{nm}(a, b; \rho) = \rho^{2nm} \prod_{i,j=1}^{n,m} \frac{1}{(-i + j \rho)} \prod_{i=1}^{n} \Gamma(i \rho') \prod_{i=1}^{m} \Gamma(i \rho) \]

\[ \prod_{i,j=0}^{n-1,m-1} \frac{1}{(a + j \rho - i)(b + j \rho - i)(a + b + (m - 1 + j) \rho - (n - 1 + i))} \]

\[ \prod_{i=0}^{n-1} \frac{\Gamma(1 + a' + i \rho') \Gamma(1 + b' + i \rho')}{\Gamma(2 - 2m + a' + b' + (n - 1 + i) \rho')} \]

\[ \prod_{i=0}^{m-1} \frac{\Gamma(1 + a + i \rho) \Gamma(1 + b + i \rho)}{\Gamma(2 - 2n + a + b + (m - 1 + i) \rho)} \]

(110)
We then obtain
\[
f^{(R,S)}_{(r,s,0)} = \Lambda^{(R,S)}_{r,s}(\rho)
\]
\[
\begin{align*}
&= \prod_{i,j=0}^{r-1, s-1} \frac{1}{(a + j\rho - i)^2(c + j\rho - i)^2(a + c + \rho(s - 1 + j) - (r - 1 + i))^2} \\
&\cdot \prod_{i,j=0}^{R-r-1, S-s-1} \frac{1}{(b - i + j\rho)^2(e - i + j\rho)^2(e + b - (R - r - 1 + i) + (S - s - 1 + j)\rho)^2} \\
&\cdot \prod_{i=0}^{r-1} \frac{G(1 + a' + i\rho')G(1 + c' + i\rho')}{G(2 - 2s + a' + c' + (r - 1 + i)\rho') G(2 - 2r + a + c + (s - 1 + i)\rho)} \\
&\cdot \prod_{i=0}^{R-r-1} \frac{G(1 + b' + i\rho)G(1 + e' + i\rho)}{G(2 + e' + b' - 2(S - s) + (R - r - 1 + i)\rho')} \\
&\cdot \prod_{i=0}^{S-s-1} \frac{G(1 + b + i\rho)G(1 + e + i\rho)}{G(2 + e + b - 2(R - r) + (S - s - 1 + i)\rho)}
\end{align*}
\]
where we have defined
\[
G(x) = \frac{\Gamma(x)}{\Gamma(1-x)} = \frac{1}{G(1-x)}
\]
and where
\[
\begin{align*}
\Lambda^{(R,S)}_{r,s}(\rho) &= \rho^{4rs+4(R-r)(S-s)} \sum_{i=1}^{S-s} G(i\rho) \prod_{i=1}^{S-s} G(i\rho) \prod_{i=1}^{r} G(i\rho') \prod_{i=1}^{R-r} G(i\rho') \\
&\cdot \prod_{i,j=1}^{r,s} \frac{1}{(i - j\rho)^2} \prod_{i,j=1}^{R-r,S-s-1} \frac{1}{(i - j\rho)^2}
\end{align*}
\]
and where we have defined
\[
\begin{align*}
e &\equiv -a - b - c - d - 2\rho(S - 1) + 2(R - 1) \\
e' &\equiv -e/\rho
\end{align*}
\]
These expressions are like the ones for minimal models except for the appearance of the terms \(d, d'\) in the definition of \(e, e'\). Finally
\[
\begin{align*}
f^{(R,S)}_{(r,s,1)} &= f^{(R,S)}_{(r,s+1,0)} \\
&= \frac{G(2 + a + c + 2s\rho - 2r)G(1 + a + c + 2s\rho - 2r)G(1 - (s + 1)\rho + r)}{G(1 - r + a + s\rho)G(1 - r + c + s\rho)G(2 - r + a + c + (s - 1)\rho)}
\end{align*}
\]
\section{Operator algebra coefficients}
As explained in \cite{17} the monodromy coefficients in the normalization Eq. \eqref{111} and Eq. \eqref{114} determine the operator algebra coefficients of the theory, \(C^{\lambda}_{\lambda_1,\lambda_2}\), defined by \((\lambda \equiv 2j + 1)\)
\[
\phi_{jz}(z, \overline{z}, x, \overline{x})\phi_{j1}(0, 0; 0, 0) = \sum_{j} \frac{(x^{\overline{x}})^{j_1 + j_2 - j}}{(z^{\overline{z}})^{\rho(j_1) + \rho(j_2) - \rho(j)}} C^{\lambda}_{\lambda_1,\lambda_2} \phi_{j}(0, 0; 0, 0)
\]
where the contribution from the conformal family is to be understood. Indeed

\[ f^{(R,S)}_{(r,s;\epsilon)}(j_1; j_2, j_3; j_4) = C_{\lambda_1\lambda_2}^{\lambda_\epsilon} C_{\lambda_3\lambda_4}^{\lambda_\epsilon} \]  

(116)

with \( \lambda_\epsilon = \lambda_I, \lambda_{II} \) for \( \epsilon = 0, 1 \). However, this requires that the monodromy coefficients are properly normalized. The normalization adopted so far follows the prescription of Dotsenko and Fateev in the case of minimal models, but turns out to be inadequate here. Indeed it is completely essential that the above factorization takes place in such a way that the operator algebra coefficients only depend on the variables indicated and not on anything else. In particular, \( C_{\lambda_1\lambda_2}^{\lambda_\epsilon} \) is allowed to depend on \( r, s \) which are given in terms of the spins (the \( \lambda \)'s) indicated, but it is not allowed to depend on \( R, S \) for example. Likewise \( C_{\lambda_3\lambda_4}^{\lambda_\epsilon} \) is allowed to depend on \( R - r, S - s \) but again, not on \( R, S \). However, it is allowed (as utilized above) to renormalize the coefficients by arbitrary functions of \( R, S, \lambda_1, \lambda_2, \lambda_3, (j_4 = j_1 + j_2 + j_3 - R + St) \). It turns out to be possible to devise such a normalization with the above criterion satisfied. This we have done below.

We have to use (cf. Eq. (43))

\[
\begin{align*}
  a &= -\lambda_3 + R - St + t \\
  b &= -\lambda_1 + R - St + t \\
  e &= -\lambda_2 + R - St + t \\
  c &= \lambda_4 + R - St \\
  e + b &= -\lambda_I + 2(R - r) - 1 - 2t(S - s - 1) = +\lambda_{II} + 2(R - r) - 1 - 2t(S - s - 1) \\
  a + c &= \lambda_I + 2r - 2st - 1 + t = -\lambda_{II} + 2r - 2st - 1 + t
\end{align*}
\]

(117)

We then find

\[
C_{\lambda_1\lambda_2}^{\lambda}(r, s; I) = t^{-2rs} \prod_{i=1}^{r} G(i/t) \prod_{i=1}^{s} G(it - r) \cdot \prod_{i=0}^{r-1} \frac{G(1 - s + (1 - \lambda_1 + i)/t) G(1 - s + (1 - \lambda_2 + i)/t)}{G(1 + s - (1 + \lambda + i)/t)} \cdot \prod_{i=0}^{s-1} \frac{G(\lambda_1 + it) G(\lambda_2 + it)}{G(1 + \lambda - (1 + i)t)}
\]

\[
C_{\lambda_3\lambda_4}^{\lambda}(R - r, S - s; I) = t^{-2(R-r)(S-s)} \prod_{i=1}^{R-r} G(i/t) \prod_{i=1}^{S-s} G(it - (R - r)) \cdot \prod_{i=0}^{R-r-1} \frac{G(1 - (S - s) + (1 - \lambda_3 + i)/t)}{G(S - s - (1 - \lambda + i)/t)} \cdot \prod_{i=0}^{R-r-1} G((-S - s) + (1 + \lambda_4 + i)/t) \cdot \prod_{i=0}^{S-s-1} \frac{G(\lambda_3 + it) G(-\lambda_4 + (1 + i)t)}{G(1 - \lambda - it)}
\]

\[
= C_{\lambda_3, -\lambda_4 + t}^{\lambda}(R - r, S - s; I) = C_{\lambda_3\lambda_4}^{\lambda}(R - r, S - s; I)
\]

(118)
Here, for clarity we have indicated the dependencies on $r, s$ or $R - r, S - s$ and on the fusion rule (here I). In fact this is somewhat superfluous, as we shall see in particular in the next section. The point is, that from three spins, it is always clear by which of the two fusion rules they couple. And for each case there is a unique possible value of $(r, s)$ (or $(R - r, S - s)$). The last identity in (118) confirms the fact that treating the left vertex $(j_3j_4)$ in terms of conjugate fields (indicated by bars) is equivalent to simply considering the coupling $(j_3j_4)$. Hence the factorization of the 4-point function into 3-point functions is made manifest. According to the discussion in the next section of cases including fusion rule II this fact remains true. This indicates that a factorization of N-point functions for any combination of fusion rules is similarly manifest.

Let us consider the case of fusion rule I and parametrize the intermediate spin, $j$, as

$$2j + 1 = \lambda = \rho - \sigma t \quad \text{(119)}$$

Then we have

\begin{align*}
(r, s) & = \left( \frac{1}{2}(r_1 + r_2 - \rho - 1), \frac{1}{2}(s_1 + s_2 - \sigma) \right) \\
(R - r, S - s) & = \left( \frac{1}{2}(\rho + r_3 - r_4 - 1), \frac{1}{2}(\sigma + s_3 - s_4) \right) \quad \text{(120)}
\end{align*}

where the label (I,I) indicates that we have fusion rule I at both vertices of the 4-point block, $(j_1j_2)$ and $(j_3j_4)$. These $(r, s)$ and $(R - r, S - s)$ are integers precisely in that case. Modified expressions for $(r, s)$ or $(R - r, S - s)$ have to be used for fusion rule II (see next section).

The expression for the monodromy coefficients for the case where fusion rule II is operating at both vertices may similarly be obtained from section 7. In the new normalization adopted here it has the correctly factorized form. However, in the operator algebra coefficients constructed from it the parameters $r, s, R - r, S - s$ have very different significance, since it is based on

$$\lambda_{II} = -\lambda_I, \quad j_1 + j_2 - j = r - st$$

where $j = j_I$. Hence in the next section for fusion rule II we shall base our discussion on a quite different treatment, but one with a parametrization similar to the one for fusion rule I.

9 Blocks with mixed fusion rules and the operator algebra coefficients

The 4-point blocks considered so far are ones where we have either fusion rule I operating at both vertices, or fusion rule II operating at both vertices. We now describe how to obtain 4-point blocks for the case where we have either fusion rule I for $(j_1j_2j)$ and fusion rule II for $(jj_3j_4)$, we denote that case by (II,I), or fusion rule II for $(j_1j_2j)$ and fusion rule I for $(jj_3j_4)$, we denote that case by (I,II). We emphasize that for a collection of
spins considered so far, so that fusion rule I (or fusion rule II) is possible at both vertices, neither (I,II) nor (II,I) will be possible. Hence there will be no mixing in the crossing matrix calculations.

Our technique is based on the discussion of fusion rules I and II for the 3-point function in section 2. Namely we modify Eq. (120) as follows:

\[(I,II) (r, s) = \left( \frac{1}{2}(r_1 + r_2 - \rho - 1 + p), \frac{1}{2}(s_1 + s_2 - \sigma + q) \right) \]
\[(R - r, S - s) = \left( \frac{1}{2}(\rho + r_3 - r_4 - 1), \frac{1}{2}(\sigma + s_3 - s_4) \right) \quad (121)\]

\[(II,I) (r, s) = \left( \frac{1}{2}(r_1 + r_2 - \rho - 1), \frac{1}{2}(s_1 + s_2 - \sigma) \right) \]
\[(R - r, S - s) = \left( \frac{1}{2}(\rho + r_3 - r_4 - 1 + p), \frac{1}{2}(\sigma + s_3 - s_4 + q) \right) \quad (122)\]

Notice that these numbers of screenings (numbers of integrations in Andreev’s case) are integers precisely when the fusion rules indicated are operating.

Now, in all cases, we have for the intermediate spin, \(j\), \((\lambda = 2j + 1 = \rho - \sigma t)\):

\[j_1 + j_2 - j = r - st, \quad j + j_3 - j_4 = (R - r) - (S - s)t\]

independent of whether we have fusion rule I or II. This means that our parametrization of the internal spin is quite different for fusion rule II from what was used throughout the paper so far, but much more symmetric. It would in fact be somewhat natural to consider now an alternative representation of the case (II,II), namely

\[(II,II) (r, s) = \left( \frac{1}{2}(r_1 + r_2 - \rho - 1 + p), \frac{1}{2}(s_1 + s_2 - \sigma + q) \right) \]
\[(R - r, S - s) = \left( \frac{1}{2}(\rho + r_3 - r_4 - 1 + p), \frac{1}{2}(\sigma + s_3 - s_4 + q) \right) \quad (123)\]

This would give us an alternative form of that block from the one considered so far, but one with more screenings (more integrations in the case of Andreev’s representation). Our previous treatment is the most economic as far as the numbers of screenings are concerned, but does not lead to a convenient expression for the operator algebra coefficients. However, we do not consider the case (II,II) further, since a discussion of that is analogous to the following discussion of the cases (I,II) and (II,I).

It is quite clear from the discussion in section 2 that the “over-screened” expressions for \((r, s)\) and \((R - r, S - s)\) produce couplings of the spins according to the fusion rules indicated. From the preceding discussion it is then also almost trivial that the blocks built (in Andreev’s representation) with these numbers of screenings, using the contours in Fig. 8, will have the correct singular properties. However, closer inspection shows that the block built in that way is not well defined due to a \(\Gamma(0)\) singularity. But there is a
very simple remedy, like the one employed for the 3-point function in section 2. Namely, suppose we need to build the block corresponding to the case (I, II). Then we modify slightly the contours in Fig. 8 so that we choose Felder contours \cite{20} for the variables, \( v_1, \ldots v_s \), more precisely, we may take these variables to run along circle like contours passing through the point \( z \), and surrounding the contours for \( u_1, \ldots, u_r \), i.e. surrounding the point 0, in such a way that the \( v_1 \)-contour lies inside the \( v_2 \)-contour, etc. Similarly for the case of (II, I), we modify the contours for \( v_{s+1}, \ldots v_S \) into circle like contours surrounding 0 (actually surrounding \( \infty \)) and passing through the point 1, in such a way that the contour for \( v_{s+1} \) lies inside the contour for \( v_{s+2} \) etc. With these contours, we have checked in great detail that the blocks are well defined and non-vanishing. Actually in these mixed cases, we find cancellations between Gamma functions of negative integer arguments between numerator and denominator \cite{21}.

The construction of crossing matrices and monodromy coefficients is made essentially trivial by the following observations. We have previously seen that the integral for the Dotsenko-Fateev contours, Fig. 8, is related to a corresponding integral for (time-) ordered integrations by the factor

\[
\lambda_r(1/t)\lambda_s(t)\lambda_{R-r}(1/t)\lambda_{S-s}(t)
\]

Similarly for the present case of some contours being of Felder type, we get instead a factor

\[
\lambda_r(1/t)\chi_s^{(2)}(S - s_3 - 1; t)\lambda_{R-r}(1/t)\lambda_{S-s}(t)
\]

for (I, II) and a factor

\[
\lambda_r(1/t)\lambda_s(t)\lambda_{R-r}(1/t)\chi_{S-s}^{(2)}(S - s_2 - 1; t)
\]

for (II, I). These rather trivial new normalizations allow us to follow completely the treatment for fusion rule I (i.e., the case (I, I)) described above and insert appropriate \( \chi/\lambda \) factors as normalizations. It is rather easy to see, that for the new monodromy coefficients, \( f^{(R,S)}_{(r,s,(I,I))} \) and \( f^{(R,S)}_{(r,s,(II,I))} \) (in a self explanatory notation), the only \( \chi/\lambda \) factors multiplying \( f^{(R,S)}_{(r,s,(I,I))} \) which survive are ones which do not depend on \( (r, s) \) or \( (R-r, S-s) \), but only on \( (R, S) \) and hence may be absorbed into renormalizations.

The crucial conclusion of all these observations is the expected one, that the operator algebra coefficients in all cases look the same namely as given by Eq. (118). However, in each case we should investigate whether the indicated spins couple via fusion rule I or II, and accordingly use the relevant expression for \( (r, s) \).

10 Conclusions

In this paper we have investigated in detail the 4-point blocks for conformal field theories based on \( SL(2) \) current algebra with fractional levels and based on admissible representations. In particular we have devised integration contours appropriate for suitable conformal blocks, both using our own representation based on free fields \cite{3} and the one by Andreev applicable only to 4-point functions \cite{13}. We have found both fusion rules already known in the literature and we have investigated the relation between the two
representations of 4-point functions. We have then performed the lengthy calculations to obtain crossing matrices and monodromy coefficients. Based on the latter we have isolated the operator algebra coefficients of the theory for both fusion rules. They may in some sense be considered the principal result of our investigation. They are given by Eq. (118) for both fusion rules, even though the numbers of screening integrations are given by different expressions for fusion rule I and II. In his work [13], Andreev also gives operator algebra coefficients. They are obtained without many details and appear to differ from ours. He has avoided the entire discussion of monodromy invariant combinations by using integrations over the complex plane, rather than the careful and cumbersome treatment based on complex contours which we have used, following [17].

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