Numerical algorithms for detecting embedded components

Robert Krone∗ Anton Leykin†
March 10, 2015

( with appendix by Jonathan D. Hauenstein‡)

Abstract

We produce algorithms to detect whether a complex affine variety presented numerically by the machinery of numerical algebraic geometry corresponds to an associated component of a polynomial ideal.

More precisely, we complete the task of numerical primary decomposition: methods known before produce a finite set of suspect components of a complex affine scheme including all associated components along with the so-called pseudocomponents, but stop short of distinguishing between the two types.

1 Introduction

An algorithmic approach to complex algebraic geometry known as numerical algebraic geometry (see [17, 16]) provides fast approximate methods to solve systems of polynomial equations. In case when the solution set is a finite set of points polynomial homotopy continuation techniques are able to find approximations to all solutions. In case when the solution set is positive-dimensional, it is a union of irreducible complex affine varieties and numerical irreducible decomposition [15] is performed to capture the information about the irreducible pieces with numerical data stored in the so-called witness sets. In ideal-theoretic terms, given a generating set of an ideal \( I \) in the polynomial ring \( R = \mathbb{C}[x] = \mathbb{C}[x_1, \ldots, x_N] \), the numerical irreducible decomposition gives a numerical description of the components corresponding to the prime ideals \( P_i \) in the decomposition of the radical \( \sqrt{I} = P_1 \cap \cdots \cap P_r \).

∗School of Mathematics, Georgia Tech, Atlanta GA, USA (rkrone3@math.gatech.edu). Partially supported by NSF grant DMS-1151297
†School of Mathematics, Georgia Tech, Atlanta GA, USA (leykin@math.gatech.edu). Partially supported by NSF grants DMS-0914802 and DMS-1151297
‡Department of Mathematics, North Carolina State University, Raleigh NC, USA (hauenstein@ncsu.edu). Partially supported by NSF DMS-1262428 and DARPA YFA.
The goal of numerical primary decomposition \cite{11} is to find a generic point on every component of the affine scheme \(\text{Spec}(R/I)\); in ideal-theoretic terms, find a generic point on the component \(\mathbb{V}(P)\) for every associated prime ideal \(P \in \text{Ass}(R/I)\). In general a primary decomposition will include embedded components not found in an irreducible decomposition, whose corresponding primes strictly contain other associated primes of \(I\).

The algorithm proposed in \cite{11} (outlined in §3) achieves the above goal but produces some extraneous points on the so-called pseudocomponents. Pseudo-components correspond to pieces of the singular locus and their appearance is unavoidable by the algorithm.

This work addresses one question that is important for understanding affine schemes via numerical techniques: Given a suspect component, how to determine whether it is a pseudocomponent or a true embedded component?

**Problem 1.1** (Main Problem). For

- an ideal \(I \subset R\) given by a finite generating set,
- a point \(y \in \mathbb{C}^n\), and
- generic points \(y_1, \ldots, y_r\) on a collection of components \(\mathbb{V}(P_1), \ldots, \mathbb{V}(P_r)\), \(P_i \in \text{Ass}(R/I)\), that contain \(y\),

decide whether there is a component \(\mathbb{V}(P), P \in \text{Ass}(R/I)\), that contains \(y\) and is distinct from the above.

We shall describe ideals of a polynomial ring \(R\) as well as the ideals of the localization \(R_y\) of \(R\) at a point \(y \in \mathbb{C}^N\) in terms of the Macaulay dual spaces.

The numerical approach we propose is radically different from the symbolic techniques such as Gröbner and standard bases, since it has approximate computation as its underpinnings: while the generators of a given ideal are assumed to be exact, we intend to compute a Macaulay dual space by a hybrid symbolic-numerical procedure. The algorithms rely conceptually on two oracles:

**O1** Given a polynomial system \(F\), return an approximation to a generic (in practice, random) point on each irreducible component of \(\mathbb{V}(F)\) with any prescribed error bound.

This task boils down to polynomial homotopy continuation techniques.

**O2** Compute an approximate kernel of an approximate matrix given a threshold for the singular values.

This task boils down to singular value decomposition techniques.

\(^2\)Here and throughout the paper we say a “generic point on component” to refer to a point in the complement of a proper Zariski closed subset of the component containing the “degeneracy locus” dictated by the context. One can trust numerical methods mentioned so far to produce random points on components that avoid a the degeneracy locus “with probability 1”.
The most important point to make about all algorithms in the paper is that they are numerically stable or, as we prefer to call them, consistent with respect to numerical error: the algorithms produce discrete output (a Boolean value, finite sets of integers, etc.) and there exists $\varepsilon > 0$ such that for all input with $|\text{error}| < \varepsilon$ the output is the same.

For example, suppose the task is to recover the rank of the Jacobian $\frac{\partial F}{\partial x}(y)$ at a generic point $y$ on a component of $V(F)$. Then the oracle $O1$ provides an approximation $y_\varepsilon$ to $y$ with $|y_\varepsilon - y| < \varepsilon$ and a part of $O2$ recovers the numerical rank by counting singular values of $\frac{\partial F}{\partial x}(y_\varepsilon)$ above the threshold $\delta > 0$. There exists $\delta$ and $\varepsilon$ such that the numerical rank is the same for any choice $y_\varepsilon$, in particular, for $y_\varepsilon = y$ and, therefore, coincides with the true rank.

The algorithms in this article are implemented in Macaulay2 \cite{3} with parts of code residing in the packages NumericalHilbert \cite{8} and NumericalAlgebraic-Geometry \cite{12, 10}. Instructions on steps necessary to reproduce results for the examples are posted at

\[ \text{www.math.gatech.edu/~rkrone3/embedded-component-test/} \]

The beginning of \S 2.1 mostly covers basic preliminaries: Macaulay dual spaces and their connection to local polynomial rings, (local) Hilbert function, regularity index, s- and g- corners. Also \S 2.1 reviews the operation of taking a colon ideal through the numerical lens and develops the local membership test. The numerical primary decomposition is revisited in \S 3. The main part of this work, \S 4, develops algorithms for embedded component testing. One important side result worth highlighting is Theorem 4.15. It concerns associated components of the generic hyperplane section of an affine scheme and makes the dimension reduction possible in our approach.

Acknowledgments. The main authors are grateful to the author of the Appendix for helpful discussions that started at the Institut Mittag-Leffler, which kindly hosted both Hauenstein and Leykin in the Spring of 2011.

We also would like to thank numerous people, in particular, Uli Walther and Karl Schwede, for discussions of matters in \S 4.2 as well as to Hailong Dao and Sasha Anan’in who produced a proof of Lemma 4.12 (via mathoverflow.net).

2 Preliminaries

For $\alpha \in (\mathbb{Z}_{\geq 0})^N$ and $x \in \mathbb{C}^N$, let

$$x^\alpha = x_1^{\alpha_1} \cdots x_N^{\alpha_N},$$

$$|\alpha| = \sum_{i=1}^{N} \alpha_i,$$

$$\alpha! = \alpha_1! \alpha_2! \cdots \alpha_N!,$$

$$\partial^\alpha = \frac{1}{\alpha!} \frac{\partial^{|\alpha|}}{\partial x^\alpha},$$

$$\partial^\alpha = \frac{1}{\alpha!} \frac{\partial^{|\alpha|}}{\partial x^\alpha},$$
and the map $\partial^\alpha[y] : R \to \mathbb{C}$ be defined by $\partial^\alpha[y](g) = (\partial^\alpha g)(y)$.

Instead of $\partial^\alpha[y]$ we sometimes write $\partial^x\alpha[y]$, for example, $\partial^1 - \partial^y + \partial^x^2yz$, and when the point $y$ is implied $\partial^\alpha[y]$ we write $\partial^\alpha$. For $y \in \mathbb{C}^N$, let

$$D_y = \text{span}_\mathbb{C}\{\partial^\alpha[y] \mid \alpha \in (\mathbb{Z}_{\geq 0})^N\}$$

be the vector space of differential functionals at $y$. This linear space is graded by the order, for a finite sum $q = \sum c_\alpha \partial^\alpha$,

$$\text{ord } q = \max_{c_\alpha \neq 0} |\alpha|.$$

The homogeneous part of order $i$ of $q \in D_y$ is referred to as $q_i$. This grading is the associated graded linear space of the filtration $D_y^*$:

$$D_y^0 \subset D_y^1 \subset D_y^2 \subset \ldots, \text{ where } D_y^i = \{q \in D_y \mid \text{ord } q \leq i\}.$$  

The Macaulay dual space, or simply dual space, is the $\mathbb{C}$-space of differential functionals that vanish at $y$ for an ideal $I \subset \mathbb{C}[x] = \mathbb{C}[x_1, \ldots, x_N]$ is

$$D_y[I] = \{q \in D_y \mid q(g) = 0 \text{ for all } g \in I\}. \quad (1)$$

The dual space $D_y[I]$ is a linear subspace of $D_y$, a basis of $D_y[I]$ is called a dual basis for $I$.

### 2.1 Duality

To help the reader, in this section we list key facts about ideal and dual space correspondence; see [9] for proofs and references.

Without a loss of generality, we may assume $y = 0 \in \mathbb{C}^N$. Consider the local ring $R_0 = R_m$ where $m = (x_1, \ldots, x_N)$. Let the space of dual functionals be defined as above replacing $R$ (polynomial) with $R_0$ (rational functions with denominators not vanishing at 0).

**Remark 2.1.** Ideals in $R$ with all primary components containing the origin are in one-to-one correspondence with ideals in the local ring $R_0$ given by extension ($I \subset R$ extends to $IR_0 \subset R_0$) and contraction ($I \subset R_0$ contracts to $I \cap R \subset R$, all of whose primary components contain the origin).

For ideal $I \subset R$, the dual space $D_0[I]$ is identical to the dual space of its extension in $R_0$, $D_0[IR_0]$. Note that, for $I \subset R$, $f \in IR_0 \subset R$ if and only if $q(f) = 0$ for all $q \in D_0[I]$.

As a corollary, for $J_1, J_2 \subset R_0$, we have $J_1 \subseteq J_2$ if and only if $D_0[J_1] \supseteq D_0[J_2]$. Hence, an ideal $J \subset R_0$ is uniquely determined by its dual space $D_0[J].$

$R$ naturally acts on the the dual space by differentiation.

$$x_i : D_y \to D_y$$

$$\partial^\alpha \mapsto \partial^{\alpha - e_i}, \quad (i = 1, \ldots, N),$$
where $\partial^\beta$ is taken to be 0 when any entry of $\beta$ is less than zero. For all $q \in D_0$ and $f \in R_0$, note that $(x_i \cdot q)(f) = q(x_i f)$, so the action of $x_i$ on a functional can be seen as pre-multiplication by $x_i$.

A subspace $L \subset D_0$ is the dual space of some ideal $I_L \subset R_0$ if and only if it is closed under differentiation: $x_i \cdot L \subset L$ for all $0 \leq i \leq N$.

The map

$$\text{Dual} : \{\text{ideals of } R_0\} \rightarrow \{\text{subspaces of } D_0 \text{ closed under differentiation}\}$$

defined by $\text{Dual}(J) = D_0[J]$ is a bijection and provides another way to characterize the dual space.

An alternative characterization of the dual space can be given via the following Proposition.

**Proposition 2.2.** For ideal $J = \langle f_1, \ldots, f_n \rangle \subset R_0$, let $L$ be the maximal subspace of $D_0$ that is closed under differentiation and satisfies $q(f_i) = 0$ for all $q \in L$ and each $0 \leq i \leq n$. Then $L = D_0[J]$.

From Proposition 2.2, it follows that for $I = \langle f_1, \ldots, f_n \rangle$, a dual element $q$ is in $D_0[I]$ if and only if $q(f_i) = 0$ for all $0 \leq i \leq n$ and $0 \leq j \leq N$. Note that this leads to a completion algorithm for computing $D^k_y[I]$ (see, e.g., [14]) assuming $y$ is in the vanishing set of $I$:

$$D^0_y[I] \leftarrow \text{span}_\mathbb{C}(\partial^0)$$

for $i = 1 \rightarrow k$ do

$$D^i_y[I] \leftarrow \{q \in D_y \mid x_j \cdot q \in D^{i-1}_y[I] \text{ for all } j = 1, \ldots, N \text{ and } q(f_i) = 0 \text{ for all } i = 1, \ldots, n\}$$

end for

For ideals $J_1, J_2 \subset R_0$, the following can be readily shown:

$$D_0[J_1 + J_2] = D_0[J_1] \cap D_0[J_2],$$
$$D_0[J_1 \cap J_2] = D_0[J_1] + D_0[J_2].$$

For the truncated dual spaces the second equality holds if $J_1$ and $J_2$ are homogeneous ideals. In general, we have only one inclusion:

$$D^k_0[J_1 \cap J_2] \supset D^k_0[J_1] + D^k_0[J_2].$$

Since $D^k_0[J_1 \cap J_2]$ is finite dimensional, it follows that

$$D^k_0[J_1 \cap J_2] \subset D^k_0[J_1] + D^k_0[J_2]$$

for some $l$.

### 2.2 Primal and dual monomial order

Let $\geq$ be a local monomial ordering (1 is the largest monomial), which we shall refer to as a **primal order**. For $g = \sum_{\alpha} a_\alpha x^\alpha$, a nonzero polynomial, the **initial**
term with respect to ≥ is the largest monomial with respect to ≥ that has a nonzero coefficient, namely
\[ \text{in}_{\geq}(g) = \max_{\geq}\{x^\alpha | a_\alpha \neq 0\}. \]

For an ideal \( I \), the initial terms of \( I \) with respect to ≥ is the set of initial terms with respect to ≥ of all the elements of \( I \), namely
\[ \text{in}_{\geq}(I) = \{\text{in}_{\geq}(f) | f \in I\}. \]

A monomial is called a standard monomial of \( I \) with respect to ≥ if it is not a member of \( \text{in}_{\geq}(I) \).

We shall order the monomial differential functionals via the dual order:
\[ \partial^\alpha \succeq \partial^\beta \iff x^\alpha \leq x^\beta, \]
the order opposite to ≥.

The initial term \( \text{in}_{\prec}(q) \) of \( q \) is the largest monomial differential functional that has a nonzero coefficient.

A dual basis that has distinct initial terms is called a reduced dual basis.

Using a (possibly infinite dimensional) Gaussian elimination procedure, it is easy to see that any dual basis can be transformed into a reduced dual basis.

**Theorem 2.3** (Theorem 3.3 of [9]). For an ideal \( I \subset R \) the monomial lattice \( N^N \) is a disjoint union of \( \text{in}_{\prec} D_0[I] \) and \( \text{in}_{\geq} I \).

### 2.3 Local Hilbert function, g- and s-corners

The Hilbert function of an ideal \( I \subset R_0 \) provides combinatorial information about \( I \) that can be computed numerically using truncated dual spaces.

For an ideal \( I \subset R_0 \) define the Hilbert function as
\[
H_I(k) = \dim_{\mathbb{C}}(gr(R_0/I)k) = \dim_{\mathbb{C}} \left( \frac{I + m^{k+1}}{I + m^k} \right) = \dim_{\mathbb{C}} \left( R_0/(I + m^{k+1}) \right) - \dim_{\mathbb{C}} \left( R_0/(I + m^k) \right).
\]

The Hilbert function is determined by the initial ideal with respect to the primal monomial order (that respects the degree): \( H_I(k) = H_{\text{in}_{\geq}(I \cap R_0)}(k) \), for all \( k \in \mathbb{N} \).

We can compute the Hilbert function using truncated dual spaces:
\[
H_I(k) = \dim_{\mathbb{C}} D_0^k[I] - \dim_{\mathbb{C}} D_0^{k-1}[I], \text{ for } k \geq 0,
\]
where \( \dim_{\mathbb{C}} D_0^{-1}[I] \) is taken to be 0.

The Hilbert function \( H_I(k) \) is a polynomial for all \( k \geq m \) for a sufficiently large \( m \geq 0 \) (see, e.g., [4] Lemma 5.5.1). This polynomial is called the Hilbert polynomial \( HP_I(k) \). If the dimension of \( I \subset R_0 \) is \( d \), then \( HP_I(k) \) is a polynomial of degree \( d - 1 \).
The regularity index of the Hilbert function is

$$\rho_0(I) = \min\{ m : H_I(k) = HP_I(k) \text{ for all } k \geq m \}.$$ 

For a 0-dimensional ideal $I$, the multiplicity $\mu_0(I)$ is defined as $\dim_{\mathbb{C}}(R_0/I) = \dim D_0[I]$. For $I$ of dimension $d > 0$ with Hilbert polynomial $HP_I(k) = a_d k^{d-1} + O(k^{d-2})$ the multiplicity is defined as

$$\mu_0(I) = a_d (d-1)!.$$ 

The multiplicity of $I$ can be interpreted geometrically as follows. For $I \subset R_0$ with dimension $d$, let $L \subset R$ be a generic affine plane of codimension $d$. Then $J = (I \cap R) + L$ is a 0-dimensional ideal and the points of $\mathbb{V}(J)$ are smooth points of $\mathbb{V}(I \cap R)$. The multiplicity of $I$ is the same as that of $J$, which is the sum of the local multiplicities of the points in $\mathbb{V}(J)$. In particular, this means that the multiplicity of $I$ can be computed numerically: the points $\mathbb{V}(J)$ approximated by homotopy continuation and then the local multiplicities at these points obtained via dual spaces.

We refer to the minimal monomial generators of a monomial ideal $M$ as g-corners. We call a monomial $x^\alpha$ an $s$-corner of $M$ when $x_i x^\alpha \in M$ for all $i = 1, \ldots, n$. For a general ideal $I$, the g-corners and s-corners of $I$ will refer to the g-corners and s-corners of the monomial ideal $in_{\geq} I$, respectively.

![Figure 1: The "staircase" of monomial ideal $I = \langle x_1^3, x_1^2 x_2, x_2^3 \rangle$ in the lattice of monomials. The regularity index of the Hilbert function is $\rho_0(I) = 5$.](image)

**Remark 2.4.** For a 0-dimensional ideal $I$, The Hilbert regularity index

$$\rho_0(I) = \max\{ |\alpha| : x^\alpha \text{ is an s-corner of } in_{\geq} I \} + 1.$$ 

$^g$ and $^s$ stand for generators of $in_{\geq} I$ and monomials spanning the socle of the quotient $R_0/in_{\geq} I$, respectively.
Let $F \subset R$ be a finite set of generators of $I$ and let $F^h \subset R[h]$ denote the homogenization of $F$. Then it is possible to compute $\rho_0(I)$ using the relationship between the truncated dual spaces $D_0^0[I]$ and $D_0^0[(F^h)]$ together with a stop criterion for the latter that recovers all $g$-corners (and, therefore, all corners) of $\langle F^h \rangle$.

**Remark 2.5.** Here we outline the idea of the algorithm of [7] that computes the $g$-corners and, therefore, the $s$-corners and the regularity index.

Let $\varphi : R[h] \to R$ denote the dehomogenization map sending $h$ to 1. We equip $R[h]$ with the unique graded local order $\geq$ such that for monomials $a, b \in R[h]$ with the same total degree, $a \geq b$ if and only if $\varphi(a) \geq \varphi(b)$. By calculating a reduced dual basis of $D_0^0[(F^h)]$ for a given $k$, we find the monomials in the complement of $\text{in}_\geq D_0^0[(F^h)]$, which by Theorem 2.3 correspond to the monomials of $\text{in}_\geq (F^h)$ of degree $\leq k$. Examining these monomials, we deduce all $g$-corners of $(F^h)$ which have degree $\leq k$. The calculation is run for successively higher values of $k$ until all of the $g$-corners of $(F^h)$ are found.

If $C$ is a set of monomial generators of $\text{in}_\geq (F^h)$ then $\varphi(C)$ generates $\text{in}_\geq (F)$.

### 2.4 Quotient ideals and local ideal membership test

Recall that any polynomial $g \in R$ defines a differential operator on $D_0$ by $(g \cdot p)(f) = p(gf)$.

**Theorem 2.6** (Theorem 2.20 of [8]). $D_0[I : \langle g \rangle] = g \cdot D_0[I]$.

Let $\succ$ be a primal order on the monomials of the local ring $R_0$, and $\geq$ be the dual order for the dual monomials of $D_0$. For any $p \in D_0$, we must have $\text{deg}_{\succ}(x_1 \cdot p) \leq \text{deg}_{\succ}(p) - 1$, since differentiation reduces the degree of each monomial by 1, but may also annihilate the lead term. Therefore taking the derivative of the dual space truncated at degree $d + 1$ we have $x_1 \cdot D_0^{d+1}[I] \subset D_0^d[I : \langle x_1 \rangle]$. Equality may not hold since there may be some functionals $q \in D_0^d[I : \langle x_1 \rangle]$ with $q = x_1 \cdot p$ for some $p \in D_0[I]$ with lead term having degree higher than $d + 1$ and is annihilated by $x_1$. In general, finding $D_0^d[I : \langle x_1 \rangle]$ from the truncated dual space of $I$ may require calculating $D_0^c[I]$ up to a very high degree $c$.

Some of these issues can be side-stepped through homogenization. As in the algorithm described in Remark 2.5 for $f \in R$, let $f^h \in R[h]$ denote the homogenization of $f$. Let $\varphi : R[h] \to R$ be the dehomogenization map, which sends $h$ to 1.

**Proposition 2.7.** $\varphi(\langle (F^h) : \langle (g^h) \rangle \rangle) = \langle g \rangle$.

**Proof.** Suppose $j \in \langle (F^h) : \langle (g^h) \rangle \rangle$, so $jg^h \in \langle F^h \rangle$. Then by dehomogenizing, $\varphi(jg) \in \langle (F) \rangle$ so $\varphi(j) \in \langle (g) \rangle$.

Suppose $j \in \langle (F) \rangle$. Then $jg = \sum_{f \in F} a_f f$ for some $a_f \in R$. Homogenizing, $h^c \cdot j^h g^h = \sum_{f \in F} h^c \cdot a_f f^h$ for some non-negative integers $c$ and $c_f$. Therefore $h^c \cdot j^h \in \langle (F^h) : \langle (g^h) \rangle \rangle$ and $\varphi(h^c \cdot j^h) = j$. \hfill $\Box$
Since \( \langle F^h \rangle \) and \( g^h \) are both homogeneous,
\[
g^h \cdot (D^d_0(\langle F^h \rangle)) = D^{d-e}_0(\langle F^h \rangle : \langle g^h \rangle)
\]
where \( e \) is the degree of \( g^h \).

We will make use of this for an ideal membership test using the homogenized dual space. Let \( I \) be an ideal of the local ring \( R_0 \). If \( g \) is not in \( I \) then at some degree the Hilbert functions of \( I \) and \( I + \langle g \rangle \) will differ. We can compute the values of the Hilbert function for successive degrees using the dual space. If \( g \) is in \( I \) then \( I : \langle g \rangle = R_0 \). This can be checked by computing \( D^d_0(\langle F^h \rangle : \langle g^h \rangle) \) for some \( d \) and seeing that \( h^d \) is in its initial ideal. This implies that there is some \( f \in \langle F^h \rangle : \langle g^h \rangle \) with \( \varphi(\text{in}_> f) = 1 \). Running both tests simultaneously for successive degrees \( d \) guarantees termination.

**Algorithm 2.8.** \( B = \text{IdealMembership}(F, g) \)

**Require:** \( I = \langle F \rangle \), an ideal of \( R \); 
\( g \), a polynomial in \( R \).

**Ensure:** \( B = (g \in IR_0) \), a Boolean value.

\[
e \leftarrow \deg g^h; \\
d \leftarrow 0; \\
\text{loop} \\
\quad D_1 \leftarrow D^d_0[I]; \\
\quad D_2 \leftarrow D^d_0[I + \langle g \rangle]; \\
\quad \text{if } D_1 \neq D_2 \text{ then} \\
\quad\quad \text{return } \text{false}; \\
\quad \text{end if} \\
\quad C \leftarrow g^h \cdot D^{d+e}_0(\langle F^h \rangle); \\
\quad \text{if } h^d \in \text{in}_> C \text{ then} \\
\quad\quad \text{return } \text{true}; \\
\quad \text{end if} \\
\quad d \leftarrow d + 1; \\
\text{end loop}
\]

Algorithm 2.8 fills in the gap left by the local membership test proposed in Theorem 4.6 of [11], which missed the necessary assumption of homogeneity.

### 3 Numerical Primary Decomposition

There is a handful of methods for symbolic primary decomposition with implementations carried out for decomposition over \( \mathbb{Q} \). For a good overview see [1].

A method for numerical primary decomposition (NPD) was introduced in [11] and is intended to compute an absolute primary decomposition, i.e., decomposition over \( \mathbb{C} \). Conceptually it relies on the numerical oracles mentioned in the Introduction and is very different from the symbolic techniques such as Gröbner bases and characteristic sets. There are several components of the NPD algorithm that are not detailed in [11]; here we fill in the gaps.
The following construction, inspired by the higher-order deflation \cite{13}, computes a superset of the primary components of an ideal. Consider an ideal \( I = (f_1, \ldots, f_N) \subset R = \mathbb{C}[x] \). Let \( q = \sum_{|\alpha| \leq d} a_{\alpha} \partial^{\alpha} \in \mathbb{C}[a][\partial] \) be a linear differential operator of order at most \( d \) with coefficients in the polynomial ring \( \mathbb{C}[a] \). Note there is a natural action of \( \mathbb{C}[a][\partial] \) on \( \mathbb{C}[a][x] \).

The ideal generated by \( f_1, \ldots, f_N \) and \( q(x^\alpha f_i) \) for all \( |\alpha| \leq d-1 \) and \( i = 1, \ldots, N \) is called the deflation ideal of \( I \) of order \( d \) and denoted by \( I^{(d)} \).

We also refer to the deflated variety of order \( d \), \( X^{(d)} = \mathbb{V}(I^{(d)}) \subset \mathbb{C}^{B(n,d)} \), where \( B(n, d) = n + \left( \begin{array}{c} n+d-1 \\ d \end{array} \right) \) is the number of variables in \( \mathbb{C}[x,a] \).

The deflation ideal \( I^{(d)} \) and, therefore, the deflated variety \( X^{(d)} \) does not depend on the choice of generators of the ideal \( I \) (see \cite{11} Proposition 2.7).

Denote by \( \pi_d : X^{(d)} \to X \) the restriction of the natural projection from \( \mathbb{C}^{B(n,d)} \) to \( \mathbb{C}^n \). Note that this map is a surjection onto \( X = X^{(0)} = \mathbb{V}(I) \).

**Remark 3.1.** For every point \( x \in \mathbb{C}^n \) the fiber of \( \pi_d \) is isomorphic to the truncated dual space of order \( d \), i.e.,

\[
\pi_d^{-1}(x) \simeq D_x^d(I).
\]

The following statement enables us to compute all (including embedded) components associated to \( I \).

**Theorem 3.2** (Theorem 3.8 of \cite{11}). Every component is visible at some order \( d \), i.e., for every prime \( P \in \text{Ass}(R/I) \), there exists \( d \) such that the preimage \( Y^{(d)} = \pi_d^{-1}(Y) \) of the variety \( Y = \mathbb{V}(P) \) is an irreducible (isolated) component of the variety \( X^{(d)} = \mathbb{V}(I^{(d)}) \).

The term “visible” reflects the tool that is used to “see” components: numerical irreducible decomposition (NID) algorithms such as in \cite{14}, which can detect isolated components numerically.

We call an isolated component \( Y^{(d)} \) of \( X^{(d)} \) a pseudocomponent if \( \pi_d(Y^{(d)}) \) is not a component of \( X \). We call pseudocomponents and embedded components of \( X \) collectively suspect components.

Here is an outline of Algorithm 5.3 of \cite{11} that computes a superset of all associated components.

**Algorithm 3.3.** \( \mathcal{N} = \text{NPD}(I) \)

**Require:** \( I \), ideal of \( R \).

**Ensure:** \( \mathcal{N} \), components associated to \( I \).

\[
\begin{align*}
\mathcal{N} & \leftarrow \emptyset \\
d & \leftarrow 0 \\
\text{repeat} & \\
C_1 & \leftarrow \text{isolated components of } I^{(d)} \text{ computed with an NID algorithm} \\
C_2 & \leftarrow \{Y \in C_1 \mid \pi_d(Y) \neq Z \text{ for all } Z \in \mathcal{N} \}
\end{align*}
\]
for all $Y \in C_2$ do
  if $Y$ is not a pseudocomponent then
    $\mathcal{N} \leftarrow \mathcal{N} \cup \{Y\}$
  end if
end for

$d = d + 1$;

until a stopping criterion holds for $d$

There are two parts of the algorithm that need clarification:

- a routine to determine whether a subvariety of $X$ is a pseudocomponent (the main topic of this article);
- a stopping criterion.

A stopping criterion can be provided by a bound on the regularity index of the (global) Hilbert function. However, this a priori bound doubly exponential in the number of variables is not practical.

We envision future improvement of the stopping criterion based on the ideas in this work.

**Remark 3.4.** Isosingular decomposition [6] can also be used as a source of suspect components, although it is not known whether the procedure one may derive from the isosingular decomposition recovers all embedded components.

Another way to produce suspect components is via iterated first-order deflation: consider the projections of the visible components of $X^{(1)}$, $(X^{(1)})^{(1)}$, etc.

## 4 Algorithms to detect embedded components

The problem of distinguishing embedded components from pseudocomponents can be condensed to the following.

**Problem 4.1.** Consider an ideal $I \subset R$ and a prime ideal $P \supset I$. Let $Q_1, \ldots, Q_r \supset I$ be the primary ideals in a primary decomposition of $I$ such that $\sqrt{Q_i} \subseteq P$.

Given generators of $I$ and generic points $y_0 \in V(P)$ and $y_i \in V(Q_i)$ ($i = 1, \ldots, r$), determine whether $P$ is an associated prime of $R/I$.

Equivalently, let $y_0 = 0 \in V(P)$ be a sufficiently generic point (we may assume the origin is a generic point without a loss of generality), determine whether

$$IR_0 = Q_1R_0 \cap \cdots \cap Q_rR_0.$$  \hfill (2)

We describe an algorithm for when the suspect component $P$ is zero-dimensional, and then finally extend it to the fully general case.
4.1 Suspect component of dimension 0

Suppose the suspect component is of dimension 0. Without the loss of generality we may assume that it is the origin and also that $I = IR_0 \cap \hat{R}$ because we may ignore components away from the origin. To simplify our notation, let $I = Q_0 \cap J$ where $J = Q_1 \cap \cdots \cap Q_r$ (as in Problem 4.1) and either

- $Q_0 = R$, i.e., $V_0$ is a pseudocomponent;
- $Q_0$ is a primary ideal with $\sqrt{Q_0} = \langle x_1, \ldots, x_n \rangle \in \text{Ass}(R/I)$ and $Q_0$ does not contain $J = Q_1 \cap \cdots \cap Q_r$, i.e., $V_0$ is a (true) component.

The goal is to distinguish the two cases above. Is $I = J$ or not?

For a generic linear form $\ell$ (so $\ell \notin \sqrt{I}$) we have

$I \subseteq (I : \langle \ell \rangle) \subseteq J$

with equality at the first inclusion if and only if there is no embedded component of $I$ at the origin. Our general strategy will be to compute information about $I : \langle \ell \rangle$ and $J$ and compare to $I$ in order to certify either that $I = I : \langle \ell \rangle$ in which case there is no embedded component, or that $I \neq J$ in which case there is.

A major stumbling block is that we cannot get our hands directly on $I : \langle \ell \rangle$ or $J$, or even on their truncated dual spaces. In the former case, as discussed in Section 2.4, we can compute $S_d := \ell \cdot D^{d+1}_0[I]$ which is a subspace of $D^d_0[I : \langle \ell \rangle]$. If for large enough $d$, $S_d$ contains all $s$-corners of $D_0[I]$, then we conclude that $D_0[I : \langle \ell \rangle] = D_0[I]$, certifying that the origin is not embedded, but we cannot use this test to certify the origin is embedded. On the other side, we compute subspaces $J_d := J \cap R_d$ of $J$, where $R_d$ denotes the space of polynomials with all terms of degree $\leq d$. If $J_d \notin I$ for some $d$ then this certifies that the origin is embedded. Similarly as $J_d$ is only a subset of $J$, we cannot use it to certify the origin is a pseudocomponent. Both procedures are simultaneously iterated over $d$ until one terminates.

This algorithm is below, with the procedure IdealTruncation to compute $J_d$ defined later as Algorithm 4.10. To find $\text{in}_\geq I$ (in particular, the $s$-corners of the staircase) we use the algorithm of [7]; see Remark 2.5.

**Algorithm 4.2.** $B = \text{IsOriginEmbedded}(I)$

**Require:** $I = \langle F \rangle$, an ideal of $R$  

**Ensure:** $B = \text{"origin is an embedded component of $I$", a boolean value.}$

1: compute $\text{in}_\geq I$
2: $d \leftarrow 0$
3: $\ell \leftarrow$ a generic linear form
4: **loop**
5: $J_d \leftarrow \text{IdealTruncation}(F, d)$
6: **if** $\text{in}_\geq J_d \notin \text{in}_\geq I$ **then**
7: **return** $\text{true}$
8: **end if**
9: $S_d \leftarrow \ell \cdot D^{d+1}_0[I]$
Proof of correctness and termination. If the condition in Line 10 holds then there is some \( f \in J_d \subset J \) such that \( f \notin I \). Hence \( J \neq I \) which implies the origin is an embedded component. Because \( J = \bigcup_d J_d \), if \( J \neq I \) then there is large enough \( d \) for which \( J_d \) will provide such a certificate.

Suppose \( I \neq J \) and let \( M_I \) denote the set of standard monomials of \( I \). Because \( I \) and \( I : \langle \ell \rangle \) differ only by a component at the origin, \( (I : \langle \ell \rangle)/I \) has finite \( \mathbb{C} \) dimension, and so \( M_I \setminus M_{I : \langle \ell \rangle} \) is also finite. \( M_{I : \langle \ell \rangle} \) is closed under division, so \( M_I \setminus M_{I : \langle \ell \rangle} \) contains a monomial which is maximal in \( M_I \), which is an s-corner of \( I \). Therefore if the condition in Line 10 holds then \( I = I : \langle \ell \rangle \). Because \( D_0[I : \langle \ell \rangle] = \bigcup_d S_d \), if \( I = I : \langle \ell \rangle \) then there is large enough \( d \) for which \( S_d \) will provide such a certificate.

One way to think about the algorithm is as follows. The staircases of \( J \) and \( I : \langle \ell \rangle \) sit “below” the staircase of \( I \). Since \( J_d \) is a subset of \( J \), it provides an upper bound on the staircase of \( J \), which can bound it away from \( I \), proving that \( J \neq I \). On the other hand, since \( S_d \) is a subset of \( D_0[I : \langle \ell \rangle] \), it provides a lower bound on the staircase of \( I : \langle \ell \rangle \). If it includes the s-corners of \( I \), then the staircases must agree. See Figure 2.

Figure 2: Both \( I \) and \( \langle J_d \rangle \) are contained in \( J \). In general, no other containments hold. For \( d \gg 0 \), \( \langle J_d \rangle = J \).

The set \( \text{in}_I \setminus \text{in}_J \) of monomials is finite.
4.1.1 Ideal truncation algorithm

To complete Algorithm 4.2 it remains to produce an algorithm for ideal truncations.

**Problem 4.3** (Local Interpolation). Let \( d > 0 \) and \( J = Q_1 \cap \cdots \cap Q_r \) with each \( Q_i \) a primary ideal such that each \( V_i = \mathcal{V}(Q_i) \) contains the origin (equivalently \( J = JR_0 \cap R \)). Compute \( J_d = J \cap R_d \).

We assume access to oracle \( O_J \) which can sample random generic points \( x \) on any \( V_i \), and for any such \( x \) and any \( e \geq 0 \) can compute \( D_x^e[J] \).

**Remark 4.4.** We can use the tools of NPD to sample points on the suspect components of \( I = J \cap Q_0 \), which in particular means generic points on \( \mathcal{V}(Q_i) \) can be produced. We can also compute truncated dual spaces \( D_x^e[I] \) using the generators of \( I \). The local properties of \( J \) and \( I \) agree away from the origin and the origin is not a primary component of \( J \). Therefore simply by excluding the origin from consideration, we have access to the tools promised by \( O_J \) and our oracle assumption is justified.

To solve Problem 4.3 we will use a form of interpolation. We will sample generic points \( x \) on the components of \( J \), and compute dual spaces \( D_x^e[J] \), which provide certain linear constraints on the evaluation and derivatives of polynomials \( f \in JR_x \). Finally we require a check to know when we have enough constraints to exactly define \( J_d \).

For discussion of interpolation see Appendix A In a special case when all \( Q_i, i \neq 0 \) are isolated, \( J_d \) can be obtained using a simpler interpolation procedure in §A.1. This technique, however cannot be extended to the general case.

For the general case, we first consider the double truncations of \( J \):

\[
J_d = \{ f \in R_d \mid \text{for all } i, D_x^e[Q_i]f = 0 \text{ for any generic point } x \in V_i \}. \tag{3}
\]

The following is a probabilistic algorithm to compute \( J_d \) whenever we have a procedure to compute \( D_x^e[J] \) for any sufficiently generic point \( x \in Q_i \) and any \( e \). In our case we have access to such a procedure because for any point \( x \) away from the origin \( D_x^e[J] = D_x^e[I] \). Note \( D_x^e[I] \) can be computed by the usual methods since the generators of \( I \) are known.

**Algorithm 4.5.** \( J_d = \text{DoubleTruncation}(O_J, d, e) \)

**Require:** \( O_J \) an oracle as in Problem 4.3; \( d, e \in \mathbb{N} \).

**Ensure:** \( J_d \) is as defined in (3)

\[
K \leftarrow R_d
\]

repeat

\[
\text{oldK} \leftarrow K
\]

with \( O_J \) choose generic points \( x_i \in V_i \) for \( i = 1, \ldots, r \).

\[
K \leftarrow J_d \cap (D_x^1[Q_i])^\perp \cap \cdots \cap (D_x^r[Q_i])^\perp
\]

until \( \text{oldK} = K \)
Proof of correctness and termination. Note that at every step $K \supseteq J_d$. Suppose at some step that $K \neq J_d$. There is $f \in K$ such that for some $V_i$ and any generic point $x \in V_i$, $f$ is not orthogonal to $D_x^e[J]$ by the definition of $J_d$. The point $x_i$ chosen on $V_i$ is chosen generically, so the new value of $K$ is strictly contained in $oldK$. Therefore when $K$ stabilizes, it must be equal to $J_d$. Since $K$ is finite dimensional at every step, termination is guaranteed.

Proposition 4.6. For any $d$, the chain $J_d^0 \supseteq J_d^1 \supseteq J_d^2 \supseteq \cdots$ stabilizes to $J_d$. That is, $J_d^e = J_d$ for all $e$ sufficiently large.

Proof. For any point $x$ recall from Remark 2.1 that polynomial $f$ has $p(f) = 0$ for all $p \in D_x[I]$ if and only if $f \in IR_x \cap Q_1$. Choosing a point $x_i$ from each $V_i$, the set $\bigcup_i J_d^e$ is the set of polynomials $f \in R_d$ orthogonal to each dual space $D_x_i[I]$. Because every $V_i$ contains at least one of the points $x_1, \ldots, x_r$, $\bigcap_i (D_x_i[I])^\perp = Q_1 \cap \cdots \cap Q_r = J$. Therefore $\bigcup_e J_d^e = J_d$. Since $J_d$ has finite $\mathbb{C}$-dimension, there must be some $e$ at which stabilization occurs.

This fact suggests an algorithm for computing $J_d$ from the double truncations, in particular for each value of $e \geq 0$ compute $J_d^e$ until some $J_d^e \subseteq J$. A naive stopping criterion for this procedure might be when $J_d^e = J_d^{e+1}$ for some $e$, but this will not work as the following example illustrates.

Example 4.7. Let $I = \langle x^k + y, y^k \rangle \subset R = \mathbb{C}[x, y, z]$, a positive-dimensional primary ideal. The reader may check that

\[
\begin{align*}
I_1^1 &= y \\
I_2^1 &= y \\
&\vdots \\
I_r^1 &= I_1 = 0
\end{align*}
\]

This example shows that equality of two subsequent $J_d^e$ and $J_d^{e+1}$ is not a valid stopping criterion. Also, note that $I_d^e \not\subset I$ for $e < k$.

Instead we require an method to check if $J_d^e \subseteq J$. First note that for any finite dimensional $\mathbb{C}$-vector subspace $V$ and any subspace $W$, a generic vector $v \in V$ is in $W$ if and only if $V \subseteq W$. Therefore it is sufficient for our purposes to check if a randomly chosen polynomial $g \in J_d^e$ is contained in $J$. Such a membership test was described in Algorithm 2.8 when generators for the ideal were known, but in this case we do not know generators of $J$, only for $I$, so the algorithm must be modified.
Proposition 4.8. Let $I = Q_0 \cap Q_1 \cap \cdots \cap Q_r$ be an irredundant primary decomposition with $V(Q_i) \ni 0$ for all $i$ and $\dim Q_0 = 0$. Let $J = Q_1 \cap \cdots \cap Q_r$.

Then $g \in J$ if and only if $I : \langle g \rangle$ is a zero-dimensional ideal.

Proof. If $g \notin J$, then $g \notin Q_i$ for some $i > 0$, so $I : g \subset P_i$ where $P_i$ is the prime associated to $Q_i$. Since $P_i$ has positive dimension, so does $I : \langle g \rangle$. Conversely if $I : \langle g \rangle$ is positive-dimensional, it is contained in some positive-dimensional prime $P$. Then $I$ has a primary component $Q_i$ with $Q_i \subset P$ and $g \notin Q_i$. Since $Q_i \subset P$, it has positive dimension so $g \notin J$.

To check that this condition holds we use the dual space of $\langle F^h \rangle : \langle g^h \rangle$, where $I = \langle F \rangle$, to find $g$-corners of $I : \langle g \rangle$, just as in Algorithm 2.8. $I : \langle g \rangle$ is zero-dimensional if and only if for every variable $x_i$ there is a $g$-corner of $I : \langle g \rangle$ of the form $x_i^{a_i}$.

We do not know a method to show when $I : \langle x \rangle$ is not zero-dimensional. As a result, our algorithm to determine if $g \in J$ will stop at some cutoff degree $c$, return true if it can certify that $g \in J$, and return false if the cutoff value is reached.

Algorithm 4.9. $B = \text{IsWitnessPolynomial}(F, g, c)$

Require: $I = \langle F \rangle$, an ideal of $R$;

$g$, a polynomial in $R$;

c, a degree cutoff.

Ensure: $B = \text{false}$ if $g \notin J$ and true if $g \in J$ and $c$ sufficiently large.

(Here $J$ and $I$ differ by a component at the origin as in Proposition 4.8.)

\begin{verbatim}
\text{e} \leftarrow \text{deg } g^h \\
\text{d} \leftarrow 0 \\
G \leftarrow \{\} (the g\text{-corners of } I : \langle g \rangle) \\
\text{repeat} \\
\qquad C \leftarrow \text{new } g\text{-corners of } I : \langle g \rangle \text{ computed from } g^h \cdot D_0^{d+e}[\langle F^h \rangle] \\
\qquad \text{append } C \text{ to } G \\
\qquad \text{if } x_i^{a_i} \in G \text{ for all } i = 1, \ldots, n \text{ and any } a_i \text{ then} \\
\qquad \qquad \text{return true} \\
\text{end if} \\
\qquad d \leftarrow d + 1 \\
\text{until } d > c \\
\text{return false}
\end{verbatim}

Equipped with this algorithm for checking if a polynomial $g$ is in $J$, and the double truncation algorithm above, we can now compute $J_d$ as follows.

Algorithm 4.10. $J_d = \text{IdealTruncation}(F, d)$

Require: $I = \langle F \rangle$, an ideal of $R$;

d $\in \mathbb{N}$.

\begin{verbatim}
\text{e} \leftarrow 0 \\
\text{loop} \\
\text{end loop}
\end{verbatim}
\[ J_d^e \leftarrow \text{DoubleTruncation}(O_d, d, e) \]
\[ g \leftarrow \text{random polynomial chosen from } J_d^e \]
\[ \text{if } \text{IsWitnessPolynomial}(F, g, e) \text{ then} \]
\[ \quad \text{return } J_d = J_d^e \]
\[ \text{end if} \]
\[ e \leftarrow e + 1 \]
\[ \text{end loop} \]

Proof of correctness and termination. If \( \text{IsWitnessPolynomial}(F, g, e) \) returns true then \( g \) must be in \( J_d \). By Proposition 4.6, \( J_d^e \supseteq J_d \), so randomly chosen \( g \) from \( J_d^e \) has \( g \in J_d \) if and only if \( J_d^e = J_d \) almost surely. This proves correctness.

To prove termination, first note that there is \( e_0 \) such that \( J_d^e = J_d \) for all \( e \geq e_0 \) by Proposition 4.6. It remains to show that \( \text{IsWitnessPolynomial}(F, g, e) \) will return true for some \( e \geq e_0 \).

For any \( g \in J_d \), let \( c(g) \) denote the minimum cutoff value \( c \) such that \( \text{IsWitnessPolynomial}(F, g, c) \) returns true. Let \( \{b_1, \ldots, b_s\} \) be a \( \mathbb{C} \)-basis for \( J_d \), so we can express \( g \in J_d \) as \( g = \sum_{i=1}^s a_i b_i \). For any given value of \( c \), the set of polynomials
\[ W_c = \{ g \in J_d \mid c(g) = c \} \]
can be described by a finite set of algebraic conditions on \( a_1, \ldots, a_s \), so \( W_c \) is a constructible set. In particular, there is some \( c_0 \) such that \( W_{c_0} \) is Zariski open, so \( \text{IsWitnessPolynomial}(F, g, c_0) \) will return true for generic \( g \in J_d \). For \( e \geq \max\{e_0, c_0\} \), a generic polynomial \( g \) sampled from \( J_d^e \) will be in \( J_d \), and \( \text{IsWitnessPolynomial}(F, g, e) \) will certify this fact.

This completes Algorithm 4.2 for determining if the origin is a zero-dimensional embedded component of ideal \( I \).

4.1.2 An example computation

Example 4.11. Consider the cyclic 4-roots problem:

\[
I = (x_1 + x_2 + x_3 + x_4, x_1x_2 + x_2x_3 + x_3x_4 + x_4x_1, \\
x_1x_2x_3 + x_2x_3x_4 + x_3x_4x_1 + x_4x_1x_2, x_1x_2x_3x_4 - 1).
\]

Computing \texttt{numericalIrreducibleDecomposition} of the first-order deflated variety \( X^{(1)} = \mathbb{V}(I^{(1)}) \) we obtain witness sets representing isolated components of \( X^{(1)} \) that project to

- two irreducible curves, isolated components that are visible and can be discovered by \texttt{numericalIrreducibleDecomposition} of \( X = \mathbb{V}(I) \), and

- eight points, approximations to \( \{(a, b, -a, -b) \mid a \in \{\pm 1, \pm i\}, b = \pm a\} \) which are suspect components.

For an approximation of the point \((i, -i, -i, i)\), \texttt{isPointEmbedded} produces a witness polynomial,
witness poly: \( (d', d) = (1, 4) \)
\[
(0.586169+0.361093\cdot i)\cdot x_1 + (0.776351+0.36685\cdot i)\cdot x_2 +
(0.586169+0.361093\cdot i)\cdot x_3 + (0.776351+0.36685\cdot i)\cdot x_4
\]
showing that this point is an embedded component. Same conclusion holds for all suspect points.

The associated primes (computed over \( \mathbb{Q} \) with a symbolic Macaulay2 routine) are
\[
\text{Ass}(R/I) = \{(x_2 + x_4, x_1 + x_3, x_3 x_4 + 1),
(x_2 + x_4, x_1 + x_3, x_3 x_4 - 1),
(x_4 - 1, x_3 + 1, x_2 + 1, x_1 - 1),
(x_4 - 1, x_3 - 1, x_2 + 1, x_1 + 1),
(x_4 + 1, x_3 + 1, x_2 - 1, x_1 - 1),
(x_4 + 1, x_3 - 1, x_2 - 1, x_1 + 1),
(x_3 + x_4, x_2 + x_4, x_1 - x_4, x_4^2 + 1),
(x_3 - x_4, x_2 + x_4, x_1 + x_4, x_4^2 + 1)\}
\]
confirming the numerical results.

4.2 Suspect component of positive dimension

Let \( P_0 \) be the vanishing (prime) ideal of suspect component \( V_0 \); let \( d_0 = \dim V_0 > 0 \).

We would like to deduce and rely on a Bertini-type theorem (Theorem 4.15) that, roughly, says that given an ideal \( I \subset R \) with \( \min_{P \in \text{Ass}(R/I)} \dim P \geq d_0 \) we have a correspondence between \( \text{Ass}(R/I) \) and \( \text{Ass}(R/(I + L)) \) where \( L \) is a generic affine plane of codimension \( d_0 \). This correspondence is one-to-one for components of dimension \( d_0 + 1 \); there could be multiple 0-dimensional components in \( \text{Ass}(R/(I + L)) \) “witnessing” components of dimension \( d_0 \) in \( \text{Ass}(R/I) \).

Lemma 4.12. Let \( I \) be an ideal and \( f \) be an element of \( R \). Then for a generic (affine) linear function \( h \in R \)
\[
(I + H) : F = (I : F) + H, \text{ where } F = \langle f \rangle, H = \langle h \rangle.
\]

Proof. (The proof follows closely the argument at mathoverflow.net/questions/143076 given by Hailong Dao.)

If \( I + F = R \) then \( I : F = I \) and \( (I + H) : F = I + H \); therefore, assume \( I + F \neq R \). The set of associated primes \( A = \text{Ass}(R/(I + F)) \) is finite, hence, a generic \( h \) would be a non-zerodivisor on \( R/(I + F) \). To see that it is enough to notice that the set of zerodivisors is exactly \( \bigcup_{P \in A} P \) and that \( n + 1 \) generic linear functions generate \( R \).

Consider the exact sequence
\[
0 \to R/(I : F) \to R/I \to R/(I + F) \to 0
\]
with first map being the multiplication by \( f \). Tensoring with \( R/H \) we get another exact sequence,
\[
0 \to R/(I : F + H) \to R/(I + H) \to R/(I + F + H) \to 0,
\]
coming from a long exact sequence for \( \text{Tor}^R(\cdot, R/H) \) and the fact that \( \text{Tor}_1^R(R/(I + F), R/H) = 0 \) as \( H \) is a non-zerodivisor on \( R/(I + H) \).

On the other hand, the first exact sequence with \( I \) replaced by \( I + H \) says that the leftmost term in the second sequence should be isomorphic to \( R/((I + H) : F) \), which proves the Lemma.

***Lemma 4.13.*** In the notation of the previous proposition, if \( I \) defines a scheme with no embedded components, then so does \( I + H \) for a generic \( H \).

*Proof.* See [2, Example 3.4.2(6)]: the condition of “having no embedded components” satisfies the Generic Principle [2, Theorem 3.3.10].

**Lemma 4.14.** Let \( I = Q_1 \cap \ldots \cap Q_r \) be a primary decomposition. Then for a generic hyperplane \( H \) the natural injection \( R/I \hookrightarrow \bigoplus_i (R/Q_i) \) induces an injection \( R/(I + H) \hookrightarrow \bigoplus_i (R/(Q_i + H)) \).

In particular, \( \text{Ass}(R/(I + H)) \subset \{ P + H \mid P \in \text{Ass}(R/I) \} \).

*Proof.* Consider the short exact sequence
\[
0 \to R/I \to \bigoplus_i (R/Q_i) \to C \to 0.
\]
As in the proof of Lemma 4.12 we see that \( \text{Tor}_1(C, R/H) = 0 \) for a generic hyperplane \( H \). Indeed, this follows from a generic \( H \) being a non-zerodivisor due to the finiteness of \( \text{Ass} C \).

**Theorem 4.15.** Let \( I \) be an ideal of \( R = \mathbb{C}[x_1, \ldots, x_n] \) and let \( L \) be the vanishing ideal for a generic affine \((n - k)\)-plane. Then
\[
\text{Ass}(R/I + L) = \{ P + L \mid P \in \text{Ass}(R/I), \dim(P) > k \} \cup \bigcup_{P \in \text{Ass}(R/I), \dim(P) = k} \text{Ass}(R/(P + L)).
\]

*Proof.* Lemma 4.13 says, in particular, that for a primary ideal \( Q \) the ideal \( Q + L \) has no embedded components; therefore, \( Q + L \) is either primary or 0-dimensional (in case \( \dim(Q) = \text{codim}(L) \)).

Now, on one hand, Lemma 4.14 says that \( I + L \) has no extraneous associated primes: all components have to come from \( Q + L \) where \( Q \) is an ideal in a primary decomposition of \( I \). On the other hand, Lemma 4.12 implies that every \( P \in \text{Ass}(R/I) \) is witnessed by \( \text{Ass}(R/(P + L)) \), since one can arrange an \( f \in R \) so that \( \text{Ass}(R/(I : f)) = \{ P \} \).
Finally, \( \text{Ass}(R/(P + L)) \) contains one element \( P + L \) when \( \dim(P) > k \), is empty when \( \dim(P) < k \), and is a finite set of maximal ideals when \( \dim(P) = k \).

Using this theorem we can reduce the case of a component of positive dimension to the embedded component test in the 0-dimensional case, i.e., the algorithms in previous subsections of this section. Indeed, for a suspect component \( V \) of dimension \( k \) one can intersect the scheme with a random affine plane \( \mathbb{V}(L) \) of codimension \( k \) and ask whether a point of \( V \cap \mathbb{V}(L) \) is an embedded component of that intersection.

**Example 4.16.** The radical ideal

\[
I = \langle x, z \rangle \cap \langle x^2 - y^2, y + z \rangle \cap \langle x^2 - z^2, x + 2y \rangle \cap \langle (x - 1)y \rangle
\]

describes a union of 5 lines and 2 planes.

A Macaulay2 script that takes a set of generators of \( I \) proceeds to construct the first deflation ideal \( I^{(1)} \) discovering 13 isolated components of \( \mathbb{V}(I^{(1)}) \) that project to suspect components in \( \mathbb{C}^3 \). Its summary reads

\begin{itemize}
  \item total: 13 suspect components
  \item true components: \{0, 3, 6, 9, 10, 11, 12\}
\end{itemize}

displaying the correct list of 7 true components and correctly discarding all pseudocomponents.

This example is built primarily to test various scenarios for pseudocomponents: there is a positive-dimensional pseudocomponent – the intersection of two isolated planes – and several 0-dimensional pseudocomponents. For the former, Theorem 4.15 is utilized to reduce to the 0-dimensional case. One of the latter – the origin – has a non-empty set of s-corners, which engages non-trivially one of the termination modes of Algorithm 4.2. Here is the corresponding excerpt:

\[
\begin{align*}
&\text{s-corners: } \{y z\} \\
&\text{LM(dual of colon ideal): } \{x, x y, x^2 y, y, x z, x^2 y z, y z, \ldots\}
\end{align*}
\]

\( \mathbb{V}(z, y, x) \), contained in 6 other components, is a PSEUDO-component.

The output can be interpreted to say that \( \partial^2_y \partial_z \) belongs to \( \ell \cdot D^4_0[I] \), for a generic linear form \( \ell \), hence the conclusion.

A Appendix: Local interpolation

(by Jonathan D. Hauenstein)

Here we describe an approach to the local interpolation problem stated in Problem 4.3. It is again assumed that \( I = Q_1 \cap \cdots \cap Q_r \) with \( I = IR_0 \cap R \).

Below, in \( \S A.1 \) we outline how to solve the problem in a special case and discuss an extension of this natural approach to the general case in \( \S A.2 \).
A.1 Isolated components

A simpler version of this Local Interpolation problem is when each \( Q_i \) is isolated. That is, each \( V_i = \mathcal{V}(Q_i) \) is an irreducible component of \( \mathcal{V}(I) \). In this case, we can simply interpolate a zero-scheme to compute \( I_d \). The zero-scheme is defined by the union of sufficiently many sufficiently generic points on each \( Q_i \) each with the scheme structure defined by the local dual space at the point for the ideal \( I + L \) where \( L \) defines a sufficiently generic linear space whose codimension is equal to the local dimension at the point. The number of points needed is algorithmically based on the stabilization of the rank of a matrix constructed in \([5]\). The justification for this follows from looking at primary ideals.

Let \( J \subset \mathbb{C}[x_1, \ldots, x_n] \) be a primary ideal with \( V = \mathcal{V}(J) \) such that \( \dim V > 0 \). Suppose that \( S \) is the scheme associated with \( J \) so that \( J = I(S) \) and \( J \subset \sqrt{J} = \mathcal{V}(V) \). In particular, if \( f \in \mathbb{C}[x_1, \ldots, x_n] \) is constructed using the setup described above, we know that \( f \in \sqrt{J} \) and \( \mu(J + \langle f \rangle + L) = \mu(J + L) \) for all generic linear space \( L \) of codimension equal to \( \dim V \), where \( \mu \) is the global multiplicity of a 0-dimensional scheme which is equal to the sum of local multiplicities \( \mu_0 \) at the points of the scheme.

If \( f \in \sqrt{J} \setminus J \), then \( J \subset J + \langle f \rangle \subset \sqrt{J} \). Let \( T \) be the scheme defined by \( J + \langle f \rangle \) which is not equal to \( S \). Since \( \mathcal{V}(J) = \mathcal{V}(J + \langle f \rangle) \) is irreducible and \( J = I(S) \) is primary, the introduction of \( f \) reduces the multiplicity, i.e., the multiplicity of \( T \) is less than the multiplicity of \( S \). Since the construction above precludes this possibility, we know \( f \in J \).

A.2 General case

In the general case, we also interpolate a zero-scheme to compute \( I_d \) as follows. For each \( i = 1, \ldots, r \), we need to consider the index set \( \mathcal{A}_i = \{ j \mid \mathcal{V}(Q_i) \subset \mathcal{V}(Q_j) \} \). If \( \mathcal{A}_i = \{ i \} \), then \( Q_i \) is isolated so we may use \([A.J]\) to compute \((Q_i)_e \) for any \( e \geq 0 \).

Now, suppose that \( i \in \{ 1, \ldots, r \} \) such that, for any \( e \geq 0 \), we can compute

\[
(J_i)_e = \bigcap_{j \in \mathcal{A}_i \setminus \{ i \}} Q_j. 
\]

Fix a generic point \( x \in \mathcal{V}(Q_i) \) and an ideal \( L \) which defines a generic linear space passing through \( x \) of codimension equal to \( \dim \mathcal{V}(Q_i) \). Thus, we know that \( D_x[I + L] \) is spanned by the union of \( D_x[J_i + L] \) and a finite dimensional space. Therefore, there exists \( k \) such that \( D_x[I + L] \) is spanned by the union of \( D_x[J_i + L] \) and \( D_x^k[I + L] \). In some sense, \( D_x^k[I + L] \) is a finite dimensional subspace that describes all of the additional local scheme structure at \( x \). By taking sufficiently many generic points, we are thus able to compute \((J_i \cap Q_i)_d \) for any \( d \geq 0 \), provided we can find \( e \) such that \( J_i = ((J_i)_e)_d \).
The proposed method clearly relies upon constructing \( k \) and \( e \) sufficiently large to extract the additional local information provided by \( Q_i \). We can produce theoretical bounds on \( k \) and \( e \), which are, however, not useful for a practical computation.

In summary, let us remark on a way to approach Problem 4.3 in general case.

**Remark A.1.** One can compute \( \mathbb{I}_d \) by looking at the polynomials orthogonal to the functionals in \( D_{\mathbb{I}}^{k_i} [I + L_i] \), \( i = 1, \ldots, r \), where

- \( \text{codim}(L_i) = \text{dim}(Q_i) \),
- \( x_i \) is a generic point on \( V(Q_i) \), and
- \( k_i \) is a large enough number; determining \( k_i \) (in a practical manner) is an open problem.

Once the numbers \( k_i \) are known, taking sufficiently many pairs \((x_i, L_i)\) for each \( i \), one can use an iterative approach similar to the numerical local interpolation in special cases.

**Example A.2.** For \( I = \langle x(y^2 - x^3), y(y^2 - x^3) \rangle \), we have \( I = Q_1 \cap Q_2 \) where \( Q_1 = \langle y^2 - x^3 \rangle \) and \( Q_2 = \langle x^4, y \rangle \). Since \( V(Q_1) = V(I) \) and \( Q_1 \) is radical, the local dual space when intersecting with a generic line is simply the evaluation map. Thus, we simply interpolate points on \( V(Q_1) \) to compute

\[
(Q_1)_e = \left\{ \begin{array}{ll}
\langle 0 \rangle & e \leq 2 \\
\langle y^2 - x^3 \rangle \cdot x^j \cdot y^k & \begin{array}{l}
ej \geq 0, e \geq j + k - 3 \\
e \geq 3.
\end{array}
\end{array} \right.
\]

Since \( Q_2 \) is zero dimensional, we just need consider \( D_0^{k} [I] \) and \( D_0^{k} [(Q_1)_e] \) for various \( k \) and \( e \).

| \( k \) | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| \( \text{dim} \mathbb{C} D_0^{k} [I] \) | 1 | 3 | 6 | 8 | 10 |

| \( \text{dim} \mathbb{C} D_0^{k} (Q_1)_1 \) | 1 | 3 | 6 | 10 | 15 |
| \( \text{dim} \mathbb{C} D_0^{k} (Q_1)_2 \) | 1 | 3 | 6 | 10 | 15 |
| \( \text{dim} \mathbb{C} D_0^{k} (Q_1)_3 \) | 1 | 3 | 5 | 7 | 9 |

Hence, we see that \( D_0^{k} [I] \) is enough to obtain the proper scheme structure. In particular, this removes \( 1, x, y, x^2, xy, y^2 \) so that the interpolation finds that \( x(y^2 - x^3) \) and \( y(y^2 - x^3) \) are the lowest degree polynomials in \( I \).

**References**

[1] Wolfram Decker, Gert-Martin Greuel, and Gerhard Pfister. Primary decomposition: algorithms and comparisons. In Algorithmic algebra and number theory (Heidelberg, 1997), pages 187–220. Springer, Berlin, 1999.
[2] H. Flenner, L. O’Carroll, and W. Vogel. *Joins and intersections*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 1999.

[3] Daniel R. Grayson and Michael E. Stillman. Macaulay2, a software system for research in algebraic geometry. Available at [www.math.uiuc.edu/Macaulay2/](http://www.math.uiuc.edu/Macaulay2/).

[4] Gert-Martin Greuel and Gerhard Pfister. A *Singular introduction to commutative algebra*. Springer, Berlin, extended edition, 2008. With contributions by Olaf Bachmann, Christoph Lossen and Hans Schönemann, With 1 CD-ROM (Windows, Macintosh and UNIX).

[5] Zachary A Griffin, Jonathan D Hauenstein, Chris Peterson, and Andrew J Sommese. Numerical computation of the Hilbert function of a zero-scheme. *Springer Proceedings in Mathematics & Statistics*, 2014. To appear.

[6] Jonathan D. Hauenstein and Charles W. Wampler. Isosingular sets and deflation. *Found. Comput. Math.*, 13(3):371–403, 2013.

[7] Robert Krone. Numerical algorithms for dual bases of positive-dimensional ideals. *Journal of Algebra and Its Applications*, 12(06):1350018, 2013.

[8] Robert Krone. NumericalHilbert package for Macaulay2. 2014. Preprint available at [http://arxiv.org/abs/1405.5293](http://arxiv.org/abs/1405.5293).

[9] Robert Krone and Anton Leykin. Eliminating dual spaces. 2015. Preprint available at [http://arxiv.org/abs/1503.02038](http://arxiv.org/abs/1503.02038).

[10] Anton Leykin. NumericalAlgebraicGeometry package for Macaulay2. Available at [people.math.gatech.edu/~aleykin3/NAG4M2/](http://people.math.gatech.edu/~aleykin3/NAG4M2/).

[11] Anton Leykin. Numerical primary decomposition. In *International Symposium on Symbolic and Algebraic Computation*, pages 165–172. ACM, 2008.

[12] Anton Leykin. Numerical algebraic geometry. *Journal of Software for Algebra and Geometry*, 3:5–10, 2011.

[13] Anton Leykin, Jan Verschelde, and Ailing Zhao. Higher-order deflation for polynomial systems with isolated singular solutions. In *Algorithms in algebraic geometry*, volume 146 of *IMA Vol. Math. Appl.*, pages 79–97. Springer, New York, 2008.

[14] B. Mourrain. Isolated points, duality and residues. *J. Pure Appl. Algebra*, 117/118:469–493, 1997. Algorithms for algebra (Eindhoven, 1996).

[15] A.J. Sommese, J. Verschelde, and C.W. Wampler. Numerical decomposition of the solution sets of polynomial systems into irreducible components. *SIAM J. Numer. Anal.*, 38(6):2022–2046, 2001.
[16] A.J. Sommese, J. Verschelde, and C.W. Wampler. Introduction to numerical algebraic geometry. In A. Dickenstein and I. Emiris, editors, *Solving polynomial equations*, pages 301–338. Springer-Verlag, 2005.

[17] Andrew J. Sommese and Charles W. Wampler, II. *The numerical solution of systems of polynomials*. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2005.