2-AUSLANDER ALGEBRAS ASSOCIATED WITH REDUCED WORDS IN COXETER GROUPS

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ABSTRACT. In this paper we investigate the endomorphism algebras of standard cluster tilting objects in the stably 2-Calabi-Yau categories Sub Λ w with elements w in Coxeter groups in [4]. They are examples of the 2-Auslander algebras introduced in [12]. Generalizing work in [9] we show that they are quasihereditary, even strongly quasihereditary in the sense of [17]. We also describe the cluster tilting object giving rise to the Ringel dual, and prove that there is a duality between Sub Λ w and the category F(Λ) of good modules over the quasihereditary algebra. When w = uv is a reduced word, we show that the 2-Calabi-Yau triangulated category Sub Λ v is equivalent to a specific subfactor category of Sub Λ w. This is applied to show that a standard cluster tilting object M in Sub Λ w and the cluster tilting object Λ w ⊕ Ω M lie in the same component in the cluster tilting graph.

INTRODUCTION

Let A be a finite dimensional algebra over an algebraically closed field k, where idAA ≤ 1 and id denotes injective dimension. Then the category Sub A of submodules of free A-modules of finite rank is an extension closed subcategory of the category mod A of finitely generated A-modules. Further, c = Sub A is a Frobenius category, that is, the projective and injective objects coincide and there are enough projective and injective objects.

In this paper we consider the important cases when c is stably 2-Calabi-Yau, that is, the stable category c is a 2-Calabi-Yau triangulated category. Let M be a cluster tilting object in c, that is, Ext1c(M, M) = 0 and if Ext1c(M, X) = 0, then X is a summand of a finite direct sum of copies of M. The endomorphism algebras Endc(M) belong to the class of algebras called 2-Auslander algebras (see [11][13]), and they are known to have global dimension at most 3 [11].

In this paper we deal with the finite dimensional factor algebras Λ w of preprojective algebras Λ associated with elements w in Coxeter groups [14][4]. Then idΛwΛ w ≤ 1, and we know that c w = Sub Λ w is stably 2-Calabi-Yau, and c w = Sub Λ w is triangulated 2-Calabi-Yau (see also [9] for the case of adaptable words). We consider mainly cluster tilting objects M which are associated with reduced expressions of w, which are called standard cluster tilting objects. In this case we show that the 2-Auslander algebras EndΛw(M) are quasihereditary, even strongly quasihereditary in the terminology of [17], and that the Ringel dual quasihereditary algebra of EndΛw(M) is EndΛw(Ω M), where Ω M denotes the direct sum of Λ w and the syzygy module Ω M of M.

Using mutation of cluster tilting objects in Sub Λ w or in Sub Λ w, there is an associated graph called the cluster tilting graph, where the vertices correspond to the isomorphism classes of basic cluster tilting objects. It is an important open problem whether this graph is connected. This is known to be the case for cluster categories of finite dimensional hereditary algebras (see [6]), and for cluster categories of coherent sheaves on weighted projective lines in the tubular case [3]. Here we show that if M is a standard cluster tilting object in Sub Λ w, then Ω M, which gives rise to the Ringel dual of
End$_{\Lambda_w}(M)$, lies in the same component as $M$. In order to prove this, we construct, for a reduced word $w = uv$, an embedding of $\mathcal{C}_v$ into $\mathcal{C}_w$, which induces an equivalence of triangulated categories from $\mathcal{C}_v$ to a subfactor triangulated category of $\mathcal{C}_w$, using a general construction investigated in [15]. This is also of interest in its own right.

The paper is organised as follows. In Section 1 we give some background material on Hom-finite 2-Calabi-Yau categories with cluster tilting objects, in particular we deal with those associated with elements in Coxeter groups. We also give basic definitions and facts about 2-Auslander algebras and about (strongly) quasihereditary algebras. In Section 2 we show that the endomorphism algebras of standard cluster tilting objects $M$ in $\text{Sub}\Lambda_w$ are strongly quasihereditary, with Ringel dual given by $\overline{\Omega}M$. We also show a strong relationship between the category $\text{Sub}\Lambda_w$ and the category $\mathcal{F}(\Delta)$ of $\text{End}_{\Lambda_w}(M)$-modules with good filtrations (see [15]). In Section 3 we discuss the embedding of $\mathcal{C}_v = \text{Sub}\Lambda_v$ into $\mathcal{C}_w = \text{Sub}\Lambda_w$, which induces our desired equivalence of triangulated categories. Then we apply this in Section 4 to show that $M$ and $\Omega M$ lie in the same component in the cluster tilting graph.

This work was inspired by the work on quasihereditary algebras for the case of adaptable words in [9], and was presented in Mexico City (December 2008), Bielefeld (June 2009) and Durham (July 2009). Most of the results in Sections 2 and 4 have later also been proved using different methods in [10]. A further generalization of our class of quasihereditary algebras in Section 2 has been announced in [19].

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1. Background

1.1. 2-CY categories. Let $A$ be a finite dimensional basic $k$-algebra. An extension closed subcategory $\mathcal{C}$ of $\text{mod}A$ is called Frobenius if the projective and injective objects coincide, and there are enough projective and enough injective objects. Then $\mathcal{C}$ is stably 2-CY if the stable category $\mathcal{C}^\perp$ is a 2-CY triangulated category. An object $M$ in $\mathcal{C}$ (or $\mathcal{C}^\perp$) is cluster tilting if $\text{Ext}^1_{\mathcal{C}}(M, X) = 0$ if and only if $X$ is in $\text{add}M$. Here $\text{add}M$ denotes the full additive subcategory of $\mathcal{C}$ (or $\mathcal{C}^\perp$) whose objects are finite direct sums of copies of $M$.

When $\mathcal{C}$ is Hom-finite triangulated 2-CY, there is a way of constructing subfactors of $\mathcal{C}$ which are again Hom-finite triangulated 2-CY [15] (see also [4]). Let $D$ be a rigid object in $\mathcal{C}$, that is $\text{Ext}^1_{\mathcal{C}}(D, D) = 0$, and consider $D^{\perp1} = \{X \in \mathcal{C}; \text{Ext}^1_{\mathcal{C}}(D, X) = 0\}$. Then the factor category $D^{\perp1}/\text{add}D$ is triangulated 2-CY, and there is a one-one correspondence between the cluster tilting objects in $\mathcal{C}$ containing $D$ as a summand, and the cluster tilting objects in $D^{\perp1}/\text{add}D$.

Let $M = M_1 \oplus \ldots \oplus M_n$ be a cluster tilting object in the stably 2-CY category $\mathcal{C}$, where the $M_i$ are indecomposable and nonisomorphic. Assume that $M_i$ is not projective for $1 \leq i \leq m$ and $M_i$ is projective for $m < i \leq n$. For each $i = 1, \ldots, m$ there is a unique indecomposable object $M_i^\perp \neq M_i$ such that $\mu_i(M) = (M/M_i) \oplus M_i^\perp$ is a cluster tilting object in $\mathcal{C}$. This gives rise to a graph, the cluster tilting graph, where the vertices correspond to cluster tilting objects up to isomorphism. For each $M$ there are $m$ vertices $\mu_1(M), \ldots, \mu_m(M)$ connected to $M$ by an edge.
1.2. 2-CY categories associated with words in Coxeter groups. An important class of (stably) 2-CY categories are those associated with reduced words in Coxeter groups \([4, 4]\). Let \(Q\) be a finite connected quiver with \(n\) vertices and no oriented cycles, and let \(W_Q\) be the associated Coxeter group, with generators \(s_1, \ldots, s_n\), and let \(\Lambda\) be the associated preprojective algebra. For each \(i = 1, \ldots, n\), let \(I_i := \Lambda(1-e_i)\Lambda\), where \(e_i\) is the idempotent element at the vertex \(i\) of \(Q\). Let \(w = s_{i_1} \cdots s_{i_l}\) be a reduced word in \(W_Q\). Then the ideal \(I_w := I_{i_1} \cdots I_{i_l}\) is independent of the reduced expression of \(w\), and if \(Q\) is non-Dynkin, \(I_w\) is a tilting \(\Lambda\)-module of projective dimension at most one. The algebra \(\Lambda_w := \Lambda/I_w\) is a finite dimensional algebra with \(\text{id}_{\Lambda_w} \Lambda_w \leq 1\), so that \(\text{Sub} \Lambda_w\) is a Frobenius category. Further \(\text{Sub} \Lambda_w\) is stably 2-CY, and the stable category \(\text{Sub} \Lambda_w\) is triangulated 2-CY.

We write \(w\) when we mean the reduced expression of \(w\). For \(w = s_{i_1} \cdots s_{i_l}\), let

\[
M_j = P_{i_j}/(I_{i_1} \cdots I_{i_j})P_{i_j} \quad \text{and} \quad M_w = M_1 \oplus \cdots \oplus M_t,
\]

where \(\Lambda = P_1 \oplus \cdots \oplus P_n\), and \(P_i\) is the indecomposable projective \(\Lambda\)-module associated with the vertex \(i\). Then \(M_w\) is a cluster tilting object in \(\text{Sub} \Lambda_w\) and in \(\text{Sub} \Lambda_w\), which we call a standard cluster tilting object. There is clearly only a finite number of standard cluster tilting objects in \(\text{Sub} \Lambda_w\), and they are all known to lie in the same component of the cluster tilting graph \([4]\). For a vertex \(i\) in the quiver \(Q\), let \(i_1, \ldots, i_l\) be the ordered vertices of type \(i\) in \(w\). Then we have the epimorphisms \(M_{i_j} \to M_{i_{j+1}}\), and we denote the kernels by \(L_{i_j}\) and call them layers as in \([1]\).

1.3. 2-Auslander algebras. The Auslander algebras are finite dimensional algebras which by definition are the endomorphism algebras \(\text{End}_A(M)\) when \(M\) is an additive generator of \(\text{mod} \ A\) for a finite dimensional algebra \(A\) of finite representation type \([2]\). They are characterized as being algebras \(\Gamma\) of global dimension at most two and dominant dimension at least two, that is, in the minimal injective resolution \(0 \to \Gamma \to I_0 \to I_1 \to I_2 \to 0\) of \(\Gamma\), both \(I_0\) and \(I_1\) are projective.

In \([11, 13]\) the more general concept of \(n\)-Auslander algebras was introduced for \(n \geq 1\), where the \(1\)-Auslander algebras are the Auslander algebras. For \(n = 2\) we have the following. Let \(U\) be a cotilting module and \(^\perp U = \{X \in \text{mod} \ A; \text{Ext}_A^i(X, U) = 0 (i > 0)\}\). Then \(M\) is a cluster tilting object in \(^\perp U\) if \(\text{add} M = \{X \in ^\perp U; \text{Ext}_A^i(X, M) = 0\} = \{Y \in ^\perp U; \text{Ext}_A^i(M, Y) = 0\}\). When \(\text{id}_A U \leq 1\), the algebras \(\text{End}_A(M)\) are (1-relative) 2-Auslander algebras. They are the finite dimensional algebras \(\Gamma\) with \(\text{gl.dim} \Gamma \leq 3\), and if \(0 \to \Gamma \to I_0 \to I_1 \to I_2 \to I_3 \to 0\) is a minimal injective resolution of \(\Gamma\), then \(I_0, I_1\) and \(I_2\) have projective dimension at most 1. When \(\Gamma = \Lambda_w\) and \(U = \Lambda_w\) for an element \(w\) in a Coxeter group and \(M\) is a standard cluster tilting object in \(\text{Sub} \Lambda_w = ^\perp \Lambda_w\), the algebras \(\text{End}_A(M)\) are 2-Auslander algebras.

When \(M\) is a standard cluster tilting object in \(\text{Sub} \Lambda_w\), there is an explicit description of the 2-Auslander algebras \(\text{End}_A(M)\) in terms of quivers with relations \([4, 5]\).

1.4. Strongly quasihereditary algebras. Let \(A\) be a finite dimensional \(k\)-algebra. We say that an ideal \(I\) of \(A\) is heredity if \(I^2 = I\), \(I\) is a projective \(A\)-module and \(\text{End}_A(I)\) is a semisimple algebra. We say that \(A\) is quasihereditary if there exists a chain \(A \supset I_1 \supset \cdots \supset I_n = 0\) of ideals of \(A\) such that \(I_{i-1}/I_i\) is a hereditary ideal of \(A/I_i\) for any \(i\) (see \([8]\)). This is equivalent to the following condition: Let \(P_1, \ldots, P_n\) be nonisomorphic indecomposable projective \(A\)-modules. For each \(i = 1, \ldots, n\), denote by \(\Delta_i\) the largest factor of \(P_i\) with composition factors amongst the simple modules \(S_1, \ldots, S_j\), where \(S_j\) is associated to \(P_j\). Then \(A\) is quasihereditary (with respect to the ordering \(P_1, \ldots, P_n\)) if and only if each \(P_i\) has a filtration by the modules \(\Delta_1, \ldots, \Delta_n\), and \(A\) has finite global dimension (see \([18]\)). The algebra \(A\) is said to be (left) strongly quasihereditary (see \([17]\)) if each \(\Delta_i\) has projective dimension at most one.

The subcategory \(\mathcal{F}(\Delta)\) of \(\text{mod} \ A\), whose objects have a filtration using \(\Delta_1, \ldots, \Delta_n\), is contravariantly finite and resolving, that is extension closed, closed under kernels of epimorphisms and contains the projective. There is a cotilting module \(U\) associated with \(\mathcal{F}(\Delta)\), which is also a tilting module, given by the indecomposable Ext-injective modules in \(\mathcal{F}(\Delta)\). Then we have \(\mathcal{F}(\Delta) = ^\perp U\), and \(U\) is said to be
a characteristic tilting module. The algebra $\text{End}_A(U)$ is again quasihereditary, and is called the Ringel dual of $A$ (see [18]). We say that $A$ is $\Delta$-serial if the indecomposable projective $A$-modules have a unique $\Delta$-composition series.

2. CONSTRUCTION OF QUASIEREDITARY ALGEBRAS WITH ADDITIONAL PROPERTIES

Throughout this section, let $\Lambda$ be a preprojective algebra of a finite quiver $Q$ without oriented cycles, $w$ be an element in the Coxeter group, and $w$ be a reduced expression of $w$. We have a standard cluster tilting object $M = M_w$ in $\text{Sub} \Lambda_w$ and a 2-Auslander algebra $\Gamma := \text{End}_{\Lambda_w}(M)$. We show that $\Gamma$ is strongly quasihereditary and $\Delta$-serial, and that its Ringel dual is the 2-Auslander algebra $\text{End}_{\Lambda_w}(\tilde{\Omega}M)$ given by the cluster tilting object $\tilde{\Omega}M$ in $\text{Sub} \Lambda_w$. We give two different approaches to proving that $\Gamma$ is quasihereditary, where the second one depends heavily on [4], while the first one is more direct.

With the previous notation we have the following.

**Theorem 2.1.** $\Gamma = \text{End}_{\Lambda_w}(M)$ is a quasihereditary algebra.

**Proof.** We denote by $e$ the idempotent of $\Gamma$ corresponding to the simple direct summand $M_1 = P_{i_1}/l_{i_1}P_{i_1}$ of $M$. It suffices to show the following assertions.

(i) $\Gamma e \Gamma$ is a heredity ideal of $\Gamma$.

(ii) $\Gamma/\Gamma e \Gamma$ is isomorphic to $\Gamma' := \text{End}_{\Lambda_e}(M')$ for the cluster tilting object $M' = M_w$ in $\text{Sub} \Lambda_w$ associated to the reduced expression $w = s_{i_2} \cdots s_{i_t}$.

Then the theorem follows inductively. The statement (i) follows from the following general observation.

**Lemma 2.2.** Let $\Lambda$ be an algebra and $M$ be a finite length $\Lambda$-module with a simple direct summand $S$. Let $\Gamma = \text{End}_{\Lambda}(M)$ and $e$ the idempotent of $\Gamma$ corresponding to the direct summand $S$ of $M$. Then $I = \Gamma e \Gamma$ is a heredity ideal of $\Gamma$.

**Proof.** We denote by $\text{soc}_S(M)$ the sum of the simple submodules of $M$ which are isomorphic to $S$. Then the inclusion map $f : \text{soc}_S(M) \rightarrow M$ induces an isomorphism $\text{Hom}_\Lambda(M, \text{soc}_S(M)) \cong I$ of $\Gamma$-modules. In particular $I$ is a projective $\Gamma$-module. Dually $I$ is a projective $\Gamma^{\text{op}}$-module.

Since $\text{End}_I(\Gamma e)$ is Morita equivalent to $\Gamma e \cong \text{End}_\Lambda(S)$, which is a division algebra, we have that $I$ is a heredity ideal.

Now we show (ii). We have a functor $F : \text{mod} \Lambda \rightarrow \text{mod} \Lambda$ given by $F(X) := X/\text{soc}_{M_1}(X)$.

Clearly we have that $F(M)$ is $M'$ above. Thus we have an algebra homomorphism $\phi = F_{M,M} : \Gamma = \text{End}_{\Lambda}(M) \rightarrow \Gamma' = \text{End}_{\Lambda}(M')$.

We will show that $\phi$ induces an isomorphism $\Gamma/\Gamma e \Gamma \cong \Gamma'$. Clearly $f \in \Gamma$ satisfies $\phi(f) = 0$ if and only if $f$ factors through $M_1$ if and only if $f \in \Gamma e \Gamma$.

We only have to show that $\phi$ is surjective. Fix any $g \in \text{End}_{\Lambda}(M')$ and consider the exact sequence

$$0 \rightarrow \text{soc}_{M_1}(M) \rightarrow M \xrightarrow{p} M' \rightarrow 0.$$ 

By [1] the syzygy $\Omega_\Lambda M$ of the $\Lambda$-module $M$ satisfies $\text{Hom}_\Lambda(\Omega_\Lambda M, M_1) = 0$, so we have $\text{Ext}_\Lambda^1(M, M_1) = 0$. Hence the map $gp : M \rightarrow M'$ factors through $p$, and there exists $f \in \text{End}_{\Lambda}(M)$ such that $pf = gp$. Then $f$ satisfies $g = F(f) = \phi(f)$, and we have the assertion.

For the second proof we first give a sufficient condition for $\text{End}_e(M)$ to be strongly quasihereditary, for an object $M$ in some additive category $\mathcal{C}$.
Lemma 2.3. Let \( \mathcal{C} \) be a Hom-finite extension closed subcategory of an abelian \( k \)-category, and \( M \) an object in \( \mathcal{C} \) with \( \text{Ext}^1_{\mathcal{C}}(M, M) = 0 \). Let \( \Gamma = \text{End}_{\mathcal{C}}(M) \). Write \( M = M_1 \oplus \ldots \oplus M_n \), where the \( M_i \) are indecomposable and nonisomorphic. Assume that for each \( M_i \) the minimal left add(\( \bigoplus_{j \neq i} M_j \))-approximation \( f_i : M_i \to M'_i \) is surjective, and that \( \text{gl.dim} \Gamma < \infty \). Then we have the following:

(a) \( \Gamma \) is (left) strongly quasihereditary with respect to the ordering \( \text{Hom}_{\mathcal{C}}(M_n, M), \ldots, \text{Hom}_{\mathcal{C}}(M_1, M) \) of the nonisomorphic indecomposable projective \( \Gamma \)-modules, and the associated factor modules of the \( \text{Hom}_{\mathcal{C}}(M_i, M) \) are the \( \Delta_i = \text{Hom}_{\mathcal{C}}(\text{Ker} f_i, M) \), for \( i = 1, \ldots, n \).

(b) If \( M'_i \) is indecomposable or zero for each \( i = 1, \ldots, n \), then \( \Gamma \) is \( \Delta \)-serial.

Proof. (a) For each \( i = 1, \ldots, n \), consider the exact sequence \( 0 \to L_i \to M_i \xrightarrow{f_i} M'_i \to 0 \), where \( L_i = \text{Ker} f_i \). Since \( \text{Ext}^1_{\mathcal{C}}(M, M) = 0 \), we have an exact sequence \( 0 \to \text{Hom}_{\mathcal{C}}(M'_i, M) \to \text{Hom}_{\mathcal{C}}(M_i, M) \to \text{Hom}_{\mathcal{C}}(L_i, M) \to 0 \), and hence \( \text{pd}_{\Gamma} \text{Hom}_{\mathcal{C}}(L_i, M) \leq 1 \).

Denote by \( S_i \) the simple \( \Gamma \)-module which is the top of the indecomposable projective \( \Gamma \)-module \( \text{Hom}_{\mathcal{C}}(M_i, M) \). Assume that \( S_i \) is a composition factor of \( \text{Hom}_{\mathcal{C}}(L_i, M) \), for some \( r = 1, \ldots, n \). Then we have a nonzero map \( \text{Hom}_{\mathcal{C}}(M_r, M) \xrightarrow{g_i} \text{Hom}_{\mathcal{C}}(M_i, M) \) such that the composition \( \text{Hom}_{\mathcal{C}}(M_r, M) \to \text{Hom}_{\mathcal{C}}(M_i, M) \to \text{Hom}_{\mathcal{C}}(L_i, M) \) is nonzero. Hence the map \( \text{Hom}_{\mathcal{C}}(M_r, M) \to \text{Hom}_{\mathcal{C}}(M_i, M) \) does not factor through the map \( \text{Hom}_{\mathcal{C}}(M'_i, M) \xrightarrow{f_i} \text{Hom}_{\mathcal{C}}(M_i, M) \). Then \( g : M_i \to M_r \) does not factor through \( f_i : M_i \to M'_i \). Since \( f_i : M_i \to M'_i \) is a minimal left \( \text{add}(\bigoplus_{j \neq i} M_j) \)-approximation, it follows that \( M_i \) is not a summand of \( \bigoplus_{j \neq i} M_j \). Hence we have \( r \geq i \) for all simple composition factors \( S_r \) of \( \text{Hom}_{\mathcal{C}}(L_i, M) \).

We want to show that \( \text{Hom}_{\mathcal{C}}(L_i, M) \) is the largest factor of \( \text{Hom}_{\mathcal{C}}(M_i, M) \) with composition factors amongst \( S_r \) for \( r \geq i \). This follows from the exact sequence \( 0 \to \text{Hom}_{\mathcal{C}}(M'_i, M) \to \text{Hom}_{\mathcal{C}}(M_i, M) \to \text{Hom}_{\mathcal{C}}(L_i, M) \to 0 \), since the top of \( \text{Hom}_{\mathcal{C}}(M'_i, M) \) only has composition factors \( S_j \) with \( j < i \). Then we see that \( \Delta_i = \text{Hom}_{\mathcal{C}}(L_i, M) \), and hence \( \text{pd}_{\Gamma} \Delta_i \leq 1 \).

We show that each indecomposable projective \( \Gamma \)-module \( \text{Hom}_{\mathcal{C}}(M_i, M) \) has a \( \Delta \)-filtration by induction on the length. If the length of \( \text{Hom}_{\mathcal{C}}(M_i, M) \) is smallest possible, then \( M'_i = 0 \), so that \( \text{Hom}_{\mathcal{C}}(M_i, M) = \Delta_i \). The rest follows easily. This finishes the proof of (a).

(b) The top of the \( \Delta \)-filtration of the indecomposable projective \( \Gamma \)-module \( \text{Hom}_{\mathcal{C}}(M_i, M) \) has to be \( \Delta_i \), and the remaining part is \( \text{Hom}_{\mathcal{C}}(M'_i, M) \) which is zero or an indecomposable projective \( \Gamma \)-module. Thus the assertion follows by induction on the length of the indecomposable projectives.

We have the following direct consequence of Lemma 2.3.

Theorem 2.4. Let \( M \) be a standard cluster tilting object in \( \text{Sub} \Lambda_w \) associated with a reduced expression \( w = s_{i_1} \ldots s_{i_t} \) of an element in a Coxeter group. Then:

(a) \( \Gamma = \text{End}_{\text{Sub} \Lambda_w}(M) \) is (left) strongly quasihereditary.

(b) We have \( \Delta_i = \text{Hom}_{\text{Sub} \Lambda_w}(L_i, M) \).

(c) The indecomposable projective \( \Gamma \)-modules are \( \Delta \)-uniserial.

Proof. Since \( \Gamma \) is a 2-Auslander algebra, we have \( \text{gl.dim} \Gamma \leq 3 \).

Fix a vertex \( i \) in the quiver \( Q \). Let \( i_1, \ldots, i_t \) be the ordered vertices of type \( i \) in \( w \). Then we have a sequence of irreducible epimorphisms \( M_{i_t} \to \ldots \to M_{i_1} \) in add\( M \), which correspond to arrows going from right to left in the quiver of add\( M \) (see [4]). All other arrows in the quiver go from left to right. By Lemma 2.3, we only have to show that the epimorphisms \( M_{i_j} \to M_{i_{j-1}} \) for \( j \geq 2 \) and \( M_{i_1} \to 0 \) are minimal left \( \text{add}(\bigoplus_{\{j \neq i_j\}} M_{i_j}) \)-approximations of \( M_{i_j} \).
This is easy to see directly, or one can use an idea from [5 Th.6.6]: We consider a nonzero path $C : M_{ij} \to M_r$ with $r < l_j$. On the basis of the right turning points we define $\alpha(C)$, and show as in [5 Th.6.6] that we can replace $C$ by another path representing the same element, but with smaller $\alpha$-value. Then we assume that we have made a choice with $\alpha(C)$ minimal. Assume the start is not $M_{ij} \to M_{l_{j+1}}$. Then we have to go to the right from $M_{l_{j+1}}$, and hence have a right turning point. So we can reduce the $\alpha$-value and get a contradiction to the minimality, and we are done.

Our next aim is to show that $\text{Hom}_{\Lambda}(\Omega M, M)$ is the characteristic tilting module for the quasihereditary algebra $\Gamma$. Since the $\Delta_i$ have projective dimension at most one, and the characteristic tilting module is filtered by the $\Delta_i$s, we know that it must have projective dimension at most one. We recall the following information.

**Proposition 2.5.** Let $M$ in $\text{Sub} \Lambda_w$ be a standard cluster tilting object as before, and $\Gamma = \text{End}_{\Lambda_w}(M)$.

(a) Then $\Omega M$ is a cluster tilting object in $\text{Sub} \Lambda_w$, and $\text{End}_{\Lambda_w}(M) \cong \text{End}_{\Lambda_w}(\Omega M)$.

(b) $U = \text{Hom}_{\Lambda_w}(\Omega M, M)$ is a tilting $\Gamma$-module of projective dimension at most one such that $\text{End}_{\Gamma}(U) \cong \text{End}_{\Lambda_w}(\Omega M)$.

**Proof.** (a) This follows since $\text{id}_{\Lambda_w} \Lambda_w \leq 1$, and hence $\Omega : \text{Sub} \Lambda_w \to \text{Sub} \Lambda_w$ is an equivalence.

(b) This is [11 Th.5.3.2].

In order to show that $U = \text{Hom}_{\Lambda_w}(\Omega M, M)$ is the characteristic tilting module for $\Gamma$, it will be sufficient to prove that $U$ is in $\mathcal{F}(\Delta)$ and that $\mathcal{F}(\Delta)$ is in $\mathcal{U} U$. For the second statement it is sufficient to show that $\text{Hom}_{\Lambda_w}(\text{Sub} \Lambda_w, M)$ is contained in $\mathcal{U} U$, since we have already seen that the $\Delta_i$ are contained in $\text{Hom}_{\Lambda_w}(\text{Sub} \Lambda_w, M)$ and hence $\mathcal{F}(\Delta) \subset \text{Hom}_{\Lambda_w}(\text{Sub} \Lambda_w, M)$.

**Proposition 2.6.** For any $X$ in $\text{Sub} \Lambda_w$ we have the following.

(a) $\text{pd}_{\Gamma} \text{Hom}_{\Lambda_w}(X, M) \leq 1$.

(b) $\text{Ext}^i_{\Gamma}(\text{Hom}_{\Lambda_w}(X, M), U) = 0$ for all $i > 0$, so $\text{Hom}_{\Lambda_w}(X, M)$ is in $\mathcal{U} U$.

**Proof.** (a) Since $M$ is a cluster tilting object in $\text{Sub} \Lambda_w$, we have an exact sequence

$$0 \to X \to M_0 \xrightarrow{p} M_1 \to 0$$

in $\text{Sub} \Lambda_w$ with $M_0$ and $M_1$ in $\text{add} M$ (see [12716]). We apply $\text{Hom}_{\Lambda_w}(\cdot, M)$ to get the exact sequence

$$0 \to \text{Hom}_{\Lambda_w}(M_1, M) \to \text{Hom}_{\Lambda_w}(M_0, M) \to \text{Hom}_{\Lambda_w}(X, M) \to 0,$$

showing $\text{pd}_{\Gamma} \text{Hom}_{\Lambda_w}(X, M) \leq 1$.

(b) Apply $\text{Hom}_{\Gamma}(\cdot, \text{Hom}_{\Lambda_w}(\Omega M, M))$ to get the first exact sequence in the following diagram:

\[
\begin{array}{ccc}
\text{Hom}_{\Gamma}((M_0, M), (\Omega M, M)) & \longrightarrow & \text{Hom}_{\Gamma}((M_1, M), (\Omega M, M)) \\
\downarrow & & \downarrow \\
\text{Hom}_{\Lambda_w}(\Omega M, M_0) & \longrightarrow & \text{Hom}_{\Lambda_w}(\Omega M, M_1)
\end{array}
\]

The second exact sequence is obtained by applying $\text{Hom}_{\Lambda_w}(\Omega M, \cdot)$ to the exact sequence (2), and the two isomorphisms follow since $M_0$ and $M_1$ are in $\text{add} M$.

Fix any $f \in \text{Hom}_{\Lambda_w}(\Omega M, M_1)$. Since $\text{Hom}_{\Lambda_w}(\Omega M, M_1) = \text{Ext}^1_{\Lambda_w}(M, M_1) = 0$, we have that $f$ factors through a projective $\Lambda_w$-module $P$. Since $p$ in (2) is surjective, we have that $f$ factors through $p$.

\[
\begin{array}{ccc}
P & \longrightarrow & \Omega M \\
\downarrow & & \downarrow \\
M_0 & \longrightarrow & M_1
\end{array}
\]
Consequently, the above map \( \text{Hom}_{\Lambda_w}(\tilde{\Omega}M, M_0) \to \text{Hom}_{\Lambda_w}(\tilde{\Omega}M, M_1) \) is surjective. Thus \( \text{Ext}_i^1(\text{Hom}_{\Lambda_w}(X, M), U) = 0 \), and we are done by (a). 

\[ \square \]

We can now show the desired property for \( U \).

**Theorem 2.7.** With the previous notation we have the following:

(a) \( U = \text{Hom}_{\Lambda_w}(\tilde{\Omega}M, M) \) is in \( \mathcal{F}(\Delta) \).

(b) \( \mathcal{F}(\Delta) = \perp U \).

(c) \( U \) is the characteristic tilting module.

**Proof.** (a) For a fixed vertex \( k \) in the quiver \( Q \), we consider as before the ordered vertices \( i_{l_1}, \ldots, i_{l_t} \) of type \( k \). Then we have exact sequences \( 0 \to L_{i_j} \to M_{i_j} \to M_{i_{j-1}} \to 0 \) for \( j \geq 2 \), and we have \( L_{i_j} = M_{i_j} \). The \( M_{i_j} \) are factors of the indecomposable projective \( \Lambda_w \)-module \( P_k \). Hence we have the exact commutative diagram

\[
\begin{array}{ccc}
0 & \to & 0 \\
\downarrow & & \downarrow \\
\Omega M_{i_j} & \to & \Omega M_{i_{j-1}} \\
\downarrow & & \downarrow \\
P_k & \sim & P_k \\
\downarrow & & \downarrow \\
0 & \to & L_{i_j} \to M_{i_j} \to M_{i_{j-1}} \to 0
\end{array}
\]

which gives rise to the exact sequence

\[ 0 \to \Omega M_{i_j} \overset{i}{\to} \Omega M_{i_{j-1}} \to L_{i_j} \to 0. \quad (3) \]

Applying \( \text{Hom}_{\Lambda_w}(\ , M) \) to (3) we get an exact sequence

\[ 0 \to \text{Hom}_{\Lambda_w}(L_{i_j}, M) \to \text{Hom}_{\Lambda_w}(\Omega M_{i_{j-1}}, M) \to \text{Hom}_{\Lambda_w}(\Omega M_{i_j}, M). \]

Fix any \( g \in \text{Hom}_{\Lambda_w}(\Omega M_{i_j}, M) \). Since \( \text{Hom}_{\Lambda_w}(\Omega M_{i_j}, M) \cong \text{Ext}_1^{\Lambda_w}(M_{i_j}, M) = 0 \), we have that \( g \) factors through a projective \( \Lambda_w \)-module \( P \). Since \( P \) is injective in \( \text{Sub} \Lambda_w \) and \( L_{i_j} \in \text{Sub} \Lambda_w \), we have that \( g \) factors through \( i \) in (3).

\[
\begin{array}{ccc}
0 & \to & \Omega M_{i_j} \\
\downarrow & & \downarrow \\
\perp & & \perp \\
g \downarrow & & \downarrow \\
M & \to & L_{i_j} \to 0
\end{array}
\]

Consequently we have an exact sequence

\[ 0 \to \text{Hom}_{\Lambda_w}(L_{i_j}, M) \to \text{Hom}_{\Lambda_w}(\Omega M_{i_{j-1}}, M) \to \text{Hom}_{\Lambda_w}(\Omega M_{i_j}, M) \to 0. \]

We know that \( \text{Hom}_{\Lambda_w}(L_{i_j}, M) = \Delta_{i_j} \) by Theorem 2.4 (b). This implies that \( \text{Hom}_{\Lambda_w}(\Omega M_{i_{j-1}}, M) \) has a \( \Delta \)-filtration if \( \text{Hom}_{\Lambda_w}(\Omega M_{i_j}, M) \) has a \( \Delta \)-filtration. Using induction on the length, we have that \( U = \text{Hom}_{\Lambda_w}(\tilde{\Omega}M, M) \) has a \( \Delta \)-filtration. Hence \( U \) is in \( \mathcal{F}(\Delta) \).

(b)(c) Since all \( \Delta_j \) are in \( \text{Hom}_{\Lambda_w}(\text{Sub} \Lambda_w, W) \) by Theorem 2.4 (b), it follows from Proposition 2.6 (b) that \( \mathcal{F}(\Delta) \subset \text{Hom}_{\Lambda_w}(\text{Sub} \Lambda_w, M) \subset \perp U \). Since \( U \) is in \( \mathcal{F}(\Delta) \), then \( U \) is Ext-injective in \( \mathcal{F}(\Delta) \), and is hence a summand of the characteristic tilting module. Since \( U \) is already a tilting \( \Gamma \)-module, it must be the characteristic tilting module. \[ \square \]

We end this section by showing that there is induced a duality between \( \text{Sub} \Lambda_w \) and \( \mathcal{F}(\Delta) \), for any choice of standard cluster tilting object associated with \( w \).
Theorem 2.8. The functors \( \text{mod } \Lambda_w \xrightarrow{F=\text{Hom}_{\Lambda_w}(, M)} \text{mod } \Gamma \) induce dualities \( \text{Sub } \Lambda_w \xrightarrow{G=\text{Hom}(, M)} \), \( U = \mathcal{F}(\Delta) \).

Proof. By Theorem 2.7, we have \( \Delta = \mathcal{F}(\Delta) \), and hence \( F(\text{Sub } \Lambda_w) = \text{Hom}_{\Lambda_w}(\text{Sub } \Lambda_w, M) \subset \Delta = \mathcal{F}(\Delta) \) by Proposition 2.6. Let \( Y \in \text{mod } \Gamma \). Then we have a surjection \( \Gamma^n \rightarrow Y \) in \( \text{mod } \Gamma \). Applying \( G \) we get an injection \( G(Y) \rightarrow M^n \), showing that \( G(Y) \) is in \( \text{Sub } \Lambda_w \) since \( M \) is in \( \text{Sub } \Lambda_w \). Hence we have functors \( \text{Sub } \Lambda_w \xrightarrow{F=\text{Hom}(, M)} \mathcal{F}(\Delta) \).

We then show that \( GF \simeq \text{id on } \text{Sub } \Lambda_w \). For \( X \) in \( \text{Sub } \Lambda_w \) we have already mentioned that there is an exact sequence \( 0 \rightarrow X \rightarrow M_0 \rightarrow M_1 \rightarrow 0 \) with \( M_0 \) and \( M_1 \) in \( \text{add } M \), and hence an exact sequence \( 0 \rightarrow \text{Hom}_{\Lambda_w}(M_1, M) \rightarrow \text{Hom}_{\Lambda_w}(M_0, M) \rightarrow \text{Hom}_{\Lambda_w}(X, M) \rightarrow 0 \) in \( \text{mod } \Gamma \) since \( \text{Ext}^1_{\Lambda_w}(M_1, M) = 0 \).

Applying \( \text{Hom}_{\Lambda_w}(, M) \) to the last exact sequence we get an exact sequence
\[
0 \rightarrow \text{Hom}_\Gamma(\text{Hom}_{\Lambda_w}(X, M), M) \rightarrow \text{Hom}_\Gamma(\text{Hom}_{\Lambda_w}(M_0, M), M) \rightarrow \text{Hom}_\Gamma(\text{Hom}_{\Lambda_w}(M_1, M), M).
\]

When \( M' \) is a summand of \( M^n \), we have an isomorphism \( \text{Hom}_\Gamma(\text{Hom}_{\Lambda_w}(M', M), M) \simeq M' \), and we get the following commutative diagram
\[
\begin{array}{ccc}
0 & \rightarrow & \text{Hom}_\Gamma(\text{Hom}_{\Lambda_w}(X, M), M) \\
\downarrow & & \downarrow \\
0 & \rightarrow & X \\
\downarrow & & \downarrow \\
& & M_0 \\
\downarrow & & \downarrow \\
& & M_1
\end{array}
\]

It follows that \( GF(X) = \text{Hom}_\Gamma(\text{Hom}_{\Lambda_w}(X, M), M) \simeq X \) as desired.

Next we show that \( FG \simeq \text{id on } U = \mathcal{F}(\Delta) \). For \( \text{add } U \) this follows since \( F(\bigwedge M) = U \) and hence
\[
(FG)U = F((\bigwedge M)) = F(\bigwedge M) = U.
\]

Fix any \( Y \in \mathcal{F}(\Delta) \). Since \( U \) is a cotilting module, there exists an exact sequence \( 0 \rightarrow Y \rightarrow U_0 \rightarrow U_1 \).

Applying \( \text{Hom}_\Gamma(, M) \) we get an exact sequence \( \text{Hom}_\Gamma(U_1, M) \rightarrow \text{Hom}_\Gamma(U_0, M) \rightarrow \text{Hom}_\Gamma(Y, M) \rightarrow 0 \), using that \( M = \text{Hom}_{\Lambda_w}(\Lambda_w, M) \) is in \( \text{add } U = \text{add } \text{Hom}_{\Lambda_w}(\bigwedge M, M) \). Applying \( \text{Hom}_\Gamma(, M) \) we get the exact commutative diagram
\[
\begin{array}{ccc}
0 & \rightarrow & \text{Hom}_{\Lambda_w}(\text{Hom}_\Gamma(Y, M), M) \\
\downarrow & & \downarrow \\
0 & \rightarrow & Y \\
\downarrow & & \downarrow \\
& & U_0 \\
\downarrow & & \downarrow \\
& & U_1
\end{array}
\]

using that \( FG \simeq \text{id on } \text{add } U \). It follows that \( (FG)Y = \text{Hom}_{\Lambda_w}(\text{Hom}_\Gamma(Y, M), M) \simeq Y \), and we are done. \( \square \)

Since, as we have seen above, \( \text{Hom}_\Gamma(, M) : \mathcal{F}(\Delta) \rightarrow \text{Sub } \Lambda_w \) is an exact functor, we have the following direct consequence of Theorem 2.8.

Corollary 2.9. The objects in \( \text{Sub } \Lambda_w \) are exactly the objects in \( \text{mod } \Lambda_w \) which have a filtration by the layers \( L_1, \ldots, L_t \).

Since the functor \( G : \text{Sub } \Lambda_w \rightarrow \mathcal{F}(\Delta) \) is not exact, it is not the case that every filtration of an object \( X \) in \( \text{Sub } \Lambda_w \) gives rise to a filtration of \( F(X) \). For example, while the indecomposable projective \( \Gamma \)-modules have a unique \( \Lambda \)-composition series, there is no analogous result for the \( M_j \). As we have seen, the \( M_i \) are filtered by \( L_1, \ldots, L_n \), but it can even happen that some \( L_j \) is filtered by two other \( L' \)'s, as the following example shows.
Example 2.10. Let $Q$ be the quiver and $w = s_1s_2s_3s_4s_3s_2s_1$, and $M$ the associated cluster tilting object

$$
1 \oplus 2 \oplus 1 \oplus 1 \oplus 1 \oplus 2 \oplus 1 \oplus 2 \oplus 1 \oplus 1 \oplus 2 \oplus 1 \oplus 1 \oplus 1 \oplus 1
$$

The $L_j$ are then the following:

$$
1 \ 2 \ 1 \ 1 \ 3 \ 2 \ 1 \ 2 \ 1 \ 3 \ 2 \ 1 \ 2 \ 1 \ 3 \ 2 \ 1 \ 1 \ 3
$$

Then $2 \ 1$ can be filtered by $1,2$, and $1 \ 3 \ 2 \ 1$ can be filtered by $1 \ 3 \ 2 \ 1$.

3. Relationship between 2-CY Frobenius categories associated with elements in Coxeter groups

In this section we first investigate the relationship between $\text{Sub} \Lambda_v$ and $\text{Sub} \Lambda_w$ when $w = uv$ is a reduced expression. Note that we have $\text{Sub} \Lambda_u \subset \text{Sub} \Lambda_w$ [4]. We show that there is a fully faithful functor $T_{u,v} : \text{Sub} \Lambda_v \to \text{Sub} \Lambda_w$, which preserves $\text{Ext}^1(\ ,

\text{mod} \Lambda_v \to \text{mod} \Lambda_w$.

Proof. (a) The exact sequence $0 \to I_v \to \Lambda \to \Lambda_v \to 0$ gives rise to an exact sequence $I_u \otimes \Lambda_v \to I_u \otimes \Lambda \to I_u \otimes \Lambda_v \to 0$. Then we have $I_u \otimes \Lambda_v = I_u/I_v \otimes I_v = I_u/I_w$.

(b) If $X$ is a $\Lambda_v$-module, then it is a factor module of a free $\Lambda_v$-module, so $I_u \otimes \Lambda X$ is a factor module of a direct sum of copies of $I_u/I_w$, which is a $\Lambda_w$-module. Thus $I_u \otimes \Lambda X$ is a $\Lambda_w$-module. □

Proposition 3.2. We have a fully faithful functor $T_{u,v} : \text{Sub} \Lambda_v \to \text{Sub} \Lambda_w$, which preserves $\text{Ext}^1(\ ,

Proof. Without loss of generality we can assume that $Q$ is not Dynkin.

(i) From the exact sequence of $\Lambda$-modules $0 \to I_v \to \Lambda \to \Lambda_v \to 0$ we get the exact sequence $0 = \text{Tor}_1^\Lambda(I_v, \Lambda) \to \text{Tor}_1^\Lambda(I_v, \Lambda_v) \to I_u \otimes \Lambda \to I_u \otimes \Lambda_v \approx I_u$. Since $I_u \otimes \Lambda I_v = I_u I_v = I_w \subset I_u$, using that the word $uv$ is reduced (see [5]), we see that $\text{Tor}_1^\Lambda(I_u, \Lambda_v) = 0$.

(ii) For $X$ in $\text{Sub} \Lambda_v$ we have an exact sequence $0 \to X \to \Lambda_v^n \to Y \to 0$, with $Y$ in $\text{Sub} \Lambda_v$ since $\Lambda_v$ is a cotilting $\Lambda_v$-module with $\text{id}_{\Lambda_v} \Lambda_v \leq 1$. Applying $I_u \otimes \Lambda \to$, we obtain an exact sequence

$$
\text{Tor}_2^\Lambda(I_u, Y) \to \text{Tor}_1^\Lambda(I_u, X) \to \text{Tor}_1^\Lambda(I_u, \Lambda_v^n) \to \text{Tor}_1^\Lambda(I_u, Y) \to I_u \otimes \Lambda X \to I_u \otimes \Lambda \Lambda_v^n.
$$

Since $\text{pd}_{\Lambda} I_u \leq 1$, and we conclude that $\text{Tor}_1^\Lambda(I_u, X) = 0$, and hence $\text{Tor}_1^\Lambda(I_u, \text{Sub} \Lambda_v) = 0$. It follows that $\text{Tor}_1^\Lambda(I_u, Y) = 0$, and $I_u \otimes \Lambda X$ is a submodule of $I_u \otimes \Lambda \Lambda_v^n$. Since $I_u \otimes \Lambda \Lambda_v = I_u/I_w$ is in $\text{Sub} \Lambda_w$, it follows that $I_u \otimes \Lambda X$ is in $\text{Sub} \Lambda_w$. Hence we have a functor $I_u \otimes \Lambda \to : \text{Sub} \Lambda_v \to \text{Sub} \Lambda_w$. 
(iii) We have shown that \( I_u \otimes \Lambda = I_u \otimes \tilde{\Lambda} \) – on \( \text{Sub} \Lambda \). We know that \( I_u \otimes \tilde{\Lambda} \) – is an autoequivalence of the derived category of \( \Lambda \). Since \( \text{Sub} \Lambda_v \) and \( \text{Sub} \Lambda_w \) are extension closed full subcategories of \( \text{mod} \Lambda \), we have
\[
\text{Ext}^i_{\Lambda_v}(\ , \ ) = \text{Ext}^i_{\Lambda}(\ , \ ) = \text{Ext}^i_{\Lambda_v}(I_u \otimes \Lambda , I_u \otimes \Lambda ) = \text{Ext}^i_{\Lambda_w}(I_u \otimes \Lambda , I_u \otimes \Lambda )
\]
for \( i = 0, 1 \). Thus we have the assertion. \( \square \)

By Proposition 2.5 we know that \( \tilde{\Omega}_{\Lambda_w}M_u \) is also a cluster tilting object in \( \text{Sub} \Lambda_w \). For a direct summand \( \tilde{\Omega}_{\Lambda_w}M_u \) of \( \tilde{\Omega}_{\Lambda_w}M_w \), we consider the subfactor category \( T_{u,v} := (\tilde{\Omega}_{\Lambda_w}M_u)^{1}/[\tilde{\Omega}_{\Lambda_w}M_u] \) of \( \text{Sub} \Lambda_w \), and we shall show that it is triangle equivalent to \( \text{Sub} \Lambda_w \). We start with the following.

**Lemma 3.3.** With the previous notation we have \( I_u \otimes \Lambda \text{ Sub} \Lambda_v = (\tilde{\Omega}_{\Lambda_w}M_u) \).

**Proof.** The indecomposable summands of \( \tilde{\Omega}_{\Lambda_w}M_u \) are the indecomposable summands of \( I_u/I_w \) for \( u = u_1u_2 \). We have \( I_{u_1} \otimes \Lambda \text{ Sub} \Lambda_v \approx I_{u_1}/I_w \). By Lemma 3.2 the functor \( I_{u_1} \otimes \Lambda : \text{Sub} \Lambda_v \to \text{Sub} \Lambda_w \) preserves \( \text{Ext}^i(\ , \ ) \), so we have isomorphisms
\[
\text{Ext}^i_{\Lambda_v}(I_{u_1}/I_w, I_u \otimes \Lambda \text{ Sub} \Lambda_v) \approx \text{Ext}^i_{\Lambda_v}(I_{u_1} \otimes \Lambda \text{ Sub} \Lambda_v, I_{u_1} \otimes \Lambda \text{ Sub} \Lambda_v) \approx \text{Ext}^i_{\Lambda_w}(I_{u_1} \otimes \Lambda \text{ Sub} \Lambda_v, I_{u_2} \otimes \Lambda \text{ Sub} \Lambda_v) = 0.
\]
Thus we have the assertion. \( \square \)

**Proposition 3.4.** With the above notation, we have the following:

(a) \( (I_u \otimes \Lambda M_v) \otimes \Omega_{\Lambda_w}M_u \) is a cluster tilting object in \( \text{Sub} \Lambda_w \).

(b) \( I_u \otimes \Lambda M_v \) is a cluster tilting object in \( T_{u,v} \).

**Proof.** (a) Let \( X := (I_u \otimes \Lambda M_v) \otimes \Omega_{\Lambda_w}M_u \). Since \( M_v \) is a cluster tilting object in \( \text{Sub} \Lambda_v \), we have \( \text{Ext}^i_{\Lambda_v}(M_v, M_v) = 0 \), and so \( \text{Ext}^i_{\Lambda_v}(I_u \otimes \Lambda M_v, I_u \otimes \Lambda M_u) = 0 \) by Lemma 3.2. Since \( I_u \otimes \Lambda M_v \in (\Omega_{\Lambda_w}M_u)^{1} \), by Lemma 3.3 we have \( \text{Ext}^i_{\Lambda_v}(\Omega_{\Lambda_w}M_u, I_u \otimes \Lambda M_v) = 0 \). Hence \( \text{Ext}^i_{\Lambda_v}(I_u \otimes \Lambda M_v, \Omega_{\Lambda_w}M_u) = 0 \), since \( \text{Sub} \Lambda_w \) is stably 2-CY. Further \( \text{Ext}^i_{\Lambda_v}(\Omega_{\Lambda_w}M_u, \Omega_{\Lambda_w}M_u) \approx \text{Ext}^i_{\Lambda_w}(M_u, M_u) = 0 \), using that \( M_u \) is a summand of \( M_v \). So \( X \) is a rigid object in \( \text{Sub} \Lambda_w \).

Let \( a \) be the number of \( i \in Q_0 \) appearing in \( w \). We know from [3] that a rigid object in \( \text{Sub} \Lambda_w \) is cluster tilting if and only if it has at least \( l(w) - a \) nonisomorphic indecomposable summands. We only have to show that the number of nonisomorphic nonprojective indecomposable summands of the \( \Lambda_w \)-module \( X \) is at least \( l(w) - a \). Consider the following two kinds of direct summands of \( X \), where \( u = u_1u_2 \) and \( v = v_1v_2 \) are arbitrary decompositions of words.

(i) \( \Omega_{\Lambda_w}(P_i/I_{u_1}P_i) \), where \( u_1 \) ends at \( i \) which is not the last \( i \) in \( w \).

(ii) \( I_{u_1} \otimes \Lambda (P_j/I_{v_1}P_j) \), where \( v_1 \) ends at \( j \) which is not the last \( j \) in \( w \).

We will show that these \( \Lambda_w \)-modules are nonprojective and pairwise nonisomorphic. Then the number of these modules is exactly \( l(w) - a \), so we have that the number of nonisomorphic nonprojective indecomposable summands of the \( \Lambda_w \)-module \( X \) is at least \( l(w) - a \). This completes the proof.

Consider the module in (i). Since \( \Omega_{\Lambda_w}(P_i/I_{u_1}P_i) \approx I_{u_1}P_i/I_wP_i \), this is nonprojective by the condition on \( i \). Moreover all modules in (i) are pairwise nonisomorphic since the functor \( \Omega_{\Lambda_w} \) is an autoequivalence of \( \text{Sub} \Lambda_w \).

Consider the module in (ii). Since \( I_{u_1} \otimes \Lambda (P_j/I_{v_1}P_j) \approx I_{u_1}P_j/I_{v_1}P_j \), this is nonprojective by the condition on \( j \). Moreover all modules in (ii) are pairwise nonisomorphic since the functor \( I_u \otimes \Lambda \) is fully faithful by Proposition 3.2.

It remains to show that the modules in (i) and (ii) are nonisomorphic. Otherwise we have
\[
I_{u_1} \otimes \Lambda (P_i/I_{u_2v_1}P_i) \approx \Omega_{\Lambda_w}(P_i/I_{u_1}P_i) \approx I_{u_1} \otimes \Lambda (P_j/I_{v_1}P_j) \approx I_{u_1} \otimes \Lambda (I_{u_2}P_j/I_{u_2v_1}P_j).
\]
Since the functor $I_u \otimes \Lambda$ is fully faithful by Proposition 3.2, we have $P_i/I_{u_2}P_j \cong I_{u_1}P_i/I_{u_2}P_j$. This means that $I_{u_2}P_j/I_{u_2}P_j$ is a projective $\Lambda_{u_2}$-module. This implies that $j$ of $u_2v_1$ is the last $j$ in $u_2v$, a contradiction to the condition in (ii).

(b) We have the assertion from (a) and [15].

We now prove the main result in this section.

**Theorem 3.5.** Let the notation be as before. Then the functor $I_u \otimes \Lambda : \text{Sub} \Lambda \rightarrow \text{Sub} \Lambda_w$ induces an equivalence of triangulated categories between $\text{Sub} \Lambda$, and the subfactor category $\mathcal{T}_{u,v}$ of $\text{Sub} \Lambda_w$.

**Proof.** We have seen that $M_v$ is a cluster tilting object in $\text{Sub} \Lambda_v$ and $I_u \otimes \Lambda M_v$ is a cluster tilting object in $\mathcal{T}_{u,v}$. To show our desired equivalence, it is by [17] 4.5 sufficient to show that there is an induced isomorphism $\tilde{\text{End}} \Lambda_v (M_v) \rightarrow \text{End}_{\Lambda_w} (I_u \otimes \Lambda I_v)$.

By Lemma 3.2 there is an isomorphism $\text{End}_{\Lambda_v} (M_v) \rightarrow \text{End}_{\Lambda_w} (I_u \otimes \Lambda M_v)$ which induces an isomorphism $[\Lambda_v](M_v) \rightarrow [I_u \otimes \Lambda \Lambda_v](I_u \otimes \Lambda M_v) = [I_u/I_w](I_u \otimes \Lambda M_v)$, where $[X](Y)$ is the ideal in $\text{End}(Y)$ whose elements are the maps factoring through objects in $\text{add} X$. It is sufficient to prove the equality $[I_u/I_w](I_u \otimes \Lambda M_v) = [\tilde{\Omega}_{\Lambda_w}M_u](I_u \otimes \Lambda M_v)$.

Note that the indecomposable summands of $\tilde{\Omega}_{\Lambda_w}M_u$ are the indecomposable summands of $I_{u_1}/I_w$, where $u = u_1u_2$. Thus the left side is contained in the right side. We now show the other inclusion. The indecomposable summands of $I_u \otimes \Lambda M_v$ are by Lemma 3.1 the indecomposable summands of $I_u/I_{uv_1}$, where $v = v_1v_2$. Assume that we have a commutative diagram

$$
\begin{array}{ccc}
I_u/I_{uv_1} & \xrightarrow{g} & I_u/I_w \\
\xrightarrow{f} & & \xrightarrow{h} \\
I_u/I_{uv_1} & \xrightarrow{g} & I_u/I_w
\end{array}
$$

It is sufficient to show that the image of $g$ lies in $I_u/I_w$, or equivalently, that the composition $I_u \rightarrow I_u/I_{uv_1} \xrightarrow{g} I_u/I_w$ has image in $I_u/I_w$.

We apply $\text{Hom}_{\Lambda}(I_{u_i}, \cdot)$ to the exact sequence $0 \rightarrow I_w \rightarrow \Lambda \rightarrow \Lambda_w \rightarrow 0$ to get the exact sequence $\text{Hom}_{\Lambda}(I_{u_i}, \Lambda) \rightarrow \text{Hom}_{\Lambda}(I_{u_i}, \Lambda_w) \rightarrow \text{Ext}^1_{\Lambda}(I_{u_i}, I_w)$. The last term is 0 by [3] III.1.13]. Hence any map $I_u \rightarrow \Lambda_w$ factors through $\Lambda$. From [4] III.1.14 we know that any map $I_u \rightarrow \Lambda$ is given by the right multiplication with an element in $\Lambda$. Thus any map $I_u \rightarrow \Lambda_w$ has image in $I_u/I_w$. This finishes the proof of the theorem. 

4. **Application to components**

In this section let $w$ be an element in a Coxeter group and $w$ be a reduced expression of $w$. Then we have cluster tilting objects $M_w$ and $\tilde{\Omega}_{\Lambda_w}M_w$ in $\text{Sub} \Lambda_w$ by Proposition 2.5. Our main result here is the following.

**Theorem 4.1.** There is a sequence of mutations of cluster tilting objects from $M_w$ to $\tilde{\Omega}_{\Lambda_w}M_w$ in $\text{Sub} \Lambda_w$ (respectively, from $M_w$ to $\Omega_{\Lambda_w}M_w$ in $\text{Sub} \Lambda_w$).

We use induction on $l(w)$. If $l(w) = 1$, there is nothing to prove. So assume $l(w) > 1$, and write $w = s_jv$, where $w$ is a reduced expression and $s_j$ is one of the (distinguished) generators for the Coxeter group. Assume that the claim has been proved for reduced expressions of length less than $l(w)$. We show that there is a sequence of mutations between $\Omega_{\Lambda_w}M_w$ and $I_{i_j} \otimes \Lambda M_v$, and between $I_{i_j} \otimes \Lambda M_v$ and $M_w$ in $\text{Sub} \Lambda_w$. 

Lemma 4.2. Let \( w = s_{i_1} \ldots s_{i_t} = s_j w \) be a reduced expression.

(a) \( I_i \otimes_{\Lambda} M_v \) is a cluster tilting object in \( \text{Sub}_{\Lambda, w} \).

(b) There is a sequence of mutations of cluster tilting objects from \( I_i \otimes_{\Lambda} M_v \) to \( \Omega_{\Lambda, w} M_w \) in \( \text{Sub}_{\Lambda, w} \).

**Proof.** We have \( \Omega_{\Lambda, w}(P_{i_1}/I_{i_1} P_{i_1}) \approx I_i P_{i_1}/I_w P_{i_1} \approx I_i \otimes_{\Lambda} P_{i_1}/I_v P_{i_1} \) and

\[
\Omega_{\Lambda, w}(P_{i_1}/I_{i_1} \ldots I_{i_l} P_{i_l}) \approx I_i \ldots I_{i_l} P_{i_l}/I_w P_{i_l} \approx I_i \otimes_{\Lambda} (I_{i_1} \ldots I_{i_l} P_{i_l}/I_v P_{i_l}) \approx I_i \otimes_{\Lambda} \Omega_{\Lambda, w}(P_{i_1}/I_{i_1} \ldots I_{i_l} P_{i_l})
\]

for \( r = 2, \ldots, t \). Thus we have \( \Omega_{\Lambda, w} M_w \approx I_i \otimes_{\Lambda} \Omega_{\Lambda, w} M_v \) in \( \text{Sub}_{\Lambda, w} \).

By the induction assumption there is a sequence of mutations from \( M_v \) to \( \Omega_{\Lambda, w} M_v \) in \( \text{Sub}_{\Lambda, w} \). Then by Theorem 3.5 there is a sequence of mutations from \( I_i \otimes_{\Lambda} M_v \) to \( I_i \otimes_{\Lambda} \Omega_{\Lambda, w} M_v \) in \( \text{Sub}_{\Lambda, w} \). We have an induced sequence of mutations from \( I_i \otimes_{\Lambda} M_v \) to \( I_i \otimes_{\Lambda} \Omega_{\Lambda, w} M_v \) in \( \text{Sub}_{\Lambda, w} \) since \( \Omega_{\Lambda, w} M_{i_l} = I_i P_{i_1}/I_w P_{i_1} \approx I_i \otimes_{\Lambda} (P_{i_1}/I_v P_{i_1}) \) is a common direct summand of \( I_i \otimes_{\Lambda} M_v \) and \( I_i \otimes_{\Lambda} \Omega_{\Lambda, w} M_v \).  

In addition we have the following key step.

**Lemma 4.3.** There is a sequence of mutations of cluster tilting objects from \( M_w \) to \( I_i \otimes_{\Lambda} M_v \) in \( \text{Sub}_{\Lambda, w} \).

**Proof.** Let \( 1 = l_1 < l_2 < \ldots < l_k \) be all integers with \( i := i_{l_1} = i_{l_2} = \ldots = i_{l_k} \). We shall show that \( \mu_{l_{k-1}} \ldots \mu_{l_1}(M_w) \approx I_i \otimes_{\Lambda} M_v \), where \( \mu_i \) denotes the mutation at the vertex \( i \).

The summand of \( I_i \otimes_{\Lambda} M_v \) corresponding to some \( l \) which is not one of \( l_1, \ldots, l_k \) is

\[
I_i \otimes_{\Lambda} (P_{i_1}/I_{i_2} \ldots I_{i_{l-1}} P_{i_l}) \approx I_i P_{i_1}/I_{i_2} \ldots I_{i_{l-1}} P_{i_l} = P_{i_1}/I_{i_2} \ldots I_{i_{l-1}} P_{i_l}
\]

which is also a summand of \( \mu_{l_{k-1}} \ldots \mu_{l_1}(M_w) \). In the rest we shall show that the summand of \( \mu_{l_{k-1}} \ldots \mu_{l_1}(M_w) \) corresponding to \( l_u \) for \( u = 1, \ldots, k - 1 \) is

\[
I_i \otimes_{\Lambda} (P_{i_1}/I_{i_2} \ldots I_{i_{l_{u-1}}} P_{i_{l_u}}) \approx I_i P_{i_1}/I_{i_2} \ldots I_{i_{l_{u-1}}} P_{i_{l_u}}
\]

Consider the chain \( P_1 \supset I_{i_1} P_1 \supset I_{i_2} \ldots \supset I_{i_{l_{u-1}}} P_1 \supset \ldots \supset I_{i_{l_u}} P_1 = I_w P_1 \) of submodules of \( P_1 \). Here we have \( I_{i_1} \ldots I_{i_{l_u}} P_1 = I_w P_1 \) since after \( i_{l_u} \) there are no vertices of type \( i \). Then we know that

\[
P_{i_1}/I_{i_2} P_1 \leftarrow P_{i_1}/I_{i_1} \ldots I_{i_{l_u}} P_1 \leftarrow P_{i_1}/I_{i_1} \ldots I_{i_{l_{u-1}}} P_1 \leftarrow \ldots \leftarrow P_{i_1}/I_w P_1
\]

is part of the quiver of \( \text{add} M_w \), or equivalently, the quiver of \( \text{End}_{\Lambda, w}(M_w) \).

We show that after applying \( \mu_{l_{u-1}} \ldots \mu_{l_1} \) for \( u = 2, \ldots, k - 1 \), there are exactly two arrows ending at a vertex \( l_u \) which are \( l_{u-1} \rightarrow l_u \) and \( l_{u+1} \rightarrow l_u \). The arrows starting or ending at \( l_1, \ldots, l_k \) in the quiver of \( \text{add} M_w \) associated to an arrow \( a \) between \( i \) and \( j \) are indicated in the following picture.

(Other neighbours are omitted in this picture since the mutation behaviour is the same even if there are multiple arrows.) The assertion is easily seen from performing the sequence of mutations as follows:
We then have the following direct consequence. Putting the lemmas together we get the following main result Theorem 4.1 of this section.

Corollary 4.4. $M_w$ and $\Omega_{\Lambda_w} M_w$ lies in the same component of the cluster tilting graph of $\text{Sub}\Lambda_w$.

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