Stability for line solitary waves of Zakharov–Kuznetsov equation

YOHEI YAMAZAKI*

Department of Mathematics, Kyoto University
Kitashirakawa-Oiwakecho, Sakyo, Kyoto 606-8502, Japan,
Osaka City University Advanced Mathematical Institute
3-3-138 Sugimoto, Sumiyoshi-ku Osaka 558-8585 Japan

Abstract

In this paper, we consider the stability for line solitary waves of the two dimensional Zakharov–Kuznetsov equation on $\mathbb{R} \times \mathbb{T}_L$ which is one of a high dimensional generalization of Korteweg–de Vries equation, where $\mathbb{T}_L$ is the torus with the period $2\pi L$. The orbital and asymptotic stability of the one soliton of Korteweg–de Vries equation on the energy space has been proved by Benjamin [2], Pego and Weinstein [35] and Martel and Merle [25]. We regard the one soliton of Korteweg–de Vries equation as a line solitary wave of Zakharov–Kuznetsov equation on $\mathbb{R} \times \mathbb{T}_L$. We prove the stability and the transverse instability of the line solitary waves of Zakharov–Kuznetsov equation by applying Evans’ function method and the argument of Rousset and Tzvetkov [38]. Moreover, we prove the asymptotic stability for the orbitally stable line solitary wave of Zakharov–Kuznetsov equation by using the argument Martel and Merle [25, 26, 27], a Liouville type theorem and a corrected virial type estimate.

1 Introduction

We consider the two dimensional Zakharov–Kuznetsov equation

$$u_t + \partial_x (\Delta u + u^2) = 0, \quad (t, x, y) \in \mathbb{R} \times \mathbb{R} \times \mathbb{T}_L,$$

(1.1)

where $\Delta = \partial_x^2 + \partial_y^2$, $u = u(t, x, y)$ is an unknown real-valued function, $\mathbb{T}_L = \mathbb{R}/2\pi L\mathbb{Z}$ and $L > 0$.

In [38], Zakharov and Kuznetsov derived the Zakharov–Kuznetsov equation to describe the propagation of ionic-acoustic waves in uniformly magnetized plasma. In [20], Lannes, Linares and Saut proved the rigorous derivation of the Zakharov–Kuznetsov equation from

* E-mail addresses: y-youhei@math.kyoto-u.ac.jp

AMS 1991 subject classifications. 35B32, 35B35, 35Q53.
the Euler–Poisson system for uniformly magnetized plasmas. The Cauchy problem of the Zakharov–Kuznetsov equation has been studied in the recent years. In [9], Faminskii proved the global well-posedness of the Zakharov–Kuznetsov in the energy space \( H^1(\mathbb{R}^2) \).

This result has been pushed down to \( H^s(\mathbb{R}^2) \) for \( s > \frac{3}{4} \) by Linares and Pastor [21]. This result was recently improved by Grünrock and Herr [16] and Molinet and Pilod [31] who proved local well-posedness in \( H^s(\mathbb{R}^2) \) for \( s > \frac{1}{2} \). In [31], Molinet and Pilod showed the global well-posedness of (1.1) in \( H^1(\mathbb{R} \times \mathbb{T}_L) \).

Moreover, the well-posedness of higher dimensional Zakharov–Kuznetsov equation and the generalized Zakharov–Kuznetsov have been studied by [14, 21, 22, 23, 36].

The equation (1.1) has the following conservation laws:

\[
M(u) = \int_{\mathbb{R} \times \mathbb{T}_L} |u|^2 dxdy, \tag{1.2}
\]

\[
E(u) = \int_{\mathbb{R} \times \mathbb{T}_L} \left( \frac{1}{2} |\nabla u|^2 - \frac{1}{3} u^3 \right) dxdy, \tag{1.3}
\]

where \( u \in H^1(\mathbb{R} \times \mathbb{T}_L) \).

In this paper, we show the orbital stability and the asymptotic stability of solitary waves of (1.1). By a solitary wave, we mean a non-trivial solution of (1.1) with form \( u(t, x, y) = Q(x - ct, y) \), where \( c > 0 \) and \( Q \in H^1(\mathbb{R} \times \mathbb{T}_L) \) is a solution of

\[
-\Delta Q + cQ - Q^2 = 0, \quad (x, y) \in \mathbb{R} \times \mathbb{T}_L. \tag{1.4}
\]

We can write the equation (1.4) as \( S'_c(Q) = 0 \), where

\[
S_c(u) = E(u) + cM(u)
\]

and \( S'_c \) is the Fréchet derivative of \( S_c \).

The orbital stability of solitary waves is defined as follows.

**Definition 1.1.** We say that a solitary wave \( Q(x - ct, y) \) is orbitally stable in \( H^1(\mathbb{R} \times \mathbb{T}_L) \) if for any \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that for all initial data \( u_0 \in H^1(\mathbb{R} \times \mathbb{T}_L) \) with \( \|u_0 - Q\|_{H^1} < \delta \), the solution \( u(t) \) of (1.1) with \( u(0) = u_0 \) exists globally in time and satisfies

\[
\sup_{t > 0} \inf_{(x_0, y_0) \in \mathbb{R} \times \mathbb{T}_L} \|u(t, \cdot, \cdot) - Q(\cdot - x_0, \cdot - y_0)\|_{H^1} < \varepsilon.
\]

Otherwise, we say the solitary wave \( Q(x - ct, y) \) is orbitally unstable in \( H^1(\mathbb{R} \times \mathbb{T}_L) \).

The orbital stability of positive solitary waves of the generalized Zakharov–Kuznetsov equation on \( \mathbb{R}^N \) was showed by de Bouard [8] under the assumption of well-posedness on the energy space. In [5], Côte, Muñoz, Pilod and Simpson have proved the asymptotic stability of positive solitary waves and multi-solitary waves of the Zakharov–Kuznetsov equation on \( \mathbb{R}^2 \) by adapting the argument of Martel and Merle [25, 26, 27] to a multidimensional model.
The solution $u$ to (1.1) is not depend on the variable of the transverse direction $T_L$ if and only if the solution $u$ is a solution to the Korteweg–de Vries equation

$$u_t + u_{xxx} + 2uu_x = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}. \tag{1.5}$$

The Korteweg–de Vries equation describes the propagation of ionic-acoustic waves in unmagnetized plasma. The equation (1.5) has the soliton solution $R_c(t, x) = Q_c(x - ct)$, where $Q_c$ is the positive symmetric solution to

$$-\partial_x^2 Q + cQ - Q^2 = 0, \quad Q \in H^1(\mathbb{R}). \tag{1.6}$$

Here, $Q_c$ has the explicit form

$$Q_c(x) = \frac{3c}{2} \cosh^{-2}\left(\frac{\sqrt{cx}}{2}\right).$$

The orbital stability of the soliton $R_c$ has been proved by Benjamin [2]. In [35], Pego and Weinstein have showed the asymptotic stability of the soliton $R_c$ on the exponentially weighted space by investigating a spectral property of linearized operator around $Q_c$. The argument of Pego and Weinstein [35] is useful to prove the asymptotic stability on the exponentially weighted space for nonintegrable equation. However, the assumption of the exponential decay of initial data yields that the solution does not have a small soliton other than the main soliton. To treat solutions with a small soliton other than the main soliton, Mizumachi [28] has improved this result, using polynomial weighted spaces. In [25, 26, 27], Martel and Merle have proved the asymptotic stability of the soliton for initial data on $H^1(\mathbb{R})$. To prove the asymptotic stability for initial data on $H^1(\mathbb{R})$, Martel and Merle have showed the Liouville type theorem for the Korteweg–de Vries equation. The main tool to show the Liouville type theorem is the virial type estimate for solutions with some decay in space.

Then, we regard the soliton solution $R_c$ of (1.5) as a line solitary wave of (1.1), namely we define the line solitary wave $\tilde{R}_c$ and the solution $\tilde{Q}_c$ of (1.4) by

$$\tilde{R}_c(t, x, y) = \tilde{Q}_c(x - ct, y) = R_c(t, x) = Q_c(x - ct), \quad (t, x, y) \in \mathbb{R} \times \mathbb{R} \times T_L.$$  

A natural question concerning $\tilde{R}_c$ is the stability of $\tilde{R}_c$ with respect to perturbations which are periodic in the transversal direction. The stability of the line solitary wave $\tilde{R}_c$ on Kadomtsev–Petviashvili equation have been studied by many papers. The stability of $\tilde{R}_c$ on KP-II was confirmed the heuristic analysis by Kadomtsev and Petviashvili [17]. In [42], Villarroel and Ablowitz have showed the stability of line solitons $\tilde{R}_c$ of KP-II against decaying perturbations by the inverse scattering method. In [30] Mizumachi and Tzvetkov have proved the orbital stability and the asymptotic stability of $\tilde{R}_c$ on KP-II in $L^2(\mathbb{R} \times \mathbb{T})$ by using the Bäcklund transformation. The asymptotic stability for line solitons $\tilde{R}_c$ of KP-II on $\mathbb{R}^2$ has been proved by Mizumachi [29]. On $\mathbb{R}^2$, because of finite speed propagations of local phase shifts along the crest of the modulating line soliton for the transverse direction, the line soliton $\tilde{R}_c$ is not orbitally stable in the usual sense. To prove the asymptotic stability, Mizumachi have showed that the local modulations of the amplitude and the phase shift of line solitons behave like a self-similar solution of
the Burgers equation. For KP-I equation, Rouset and Tzvetkov have proved the orbital stability and instability for line solitons $\tilde{R}_c$ of KP-I on $\mathbb{R} \times \mathbb{T}$ in \cite{38, 40} and on $\mathbb{R}^2$ in \cite{37}. For Zakharov–Kuznetsov equation, the instability for line solitons $\tilde{R}_c$ on $\mathbb{R}^2$ has been showed by Rouset and Tzvetkov in \cite{37}. On $\mathbb{T}_{L_1} \times \mathbb{T}_{L_2}$ with sufficiently large $L_2$, the linear instability of line periodic solitary waves of Zakharov–Kuznetsov equation have been showed by Johnson \cite{15} by using Evan’s function method.

The one of main results is the following:

**Theorem 1.2.** Let $c > 0$. Then, the following holds.

(i) If $0 < L \leq \frac{2}{\sqrt{3}c}$, then $\tilde{R}_c$ is orbitally stable.

(ii) If $L > \frac{2}{\sqrt{3}c}$, then $\tilde{R}_c$ is orbitally unstable.

In Theorem 1.2, the instability for line solitary waves follows a symmetry breaking bifurcation of line solitary waves in the following proposition.

**Proposition 1.3.** Let $c_0 > 0$ and $L = \frac{2}{\sqrt{3}c_0}$. Then, there exist $\delta_0 > 0$ and $\varphi_{c_0} \in C^2((-\delta_0, \delta_0)^2, H^2(\mathbb{R} \times \mathbb{T}_L))$ such that for $\bar{a} = (a_1, a_2) \in (-\delta_0, \delta_0)^2$ we have $\varphi_{c_0}(\bar{a}) > 0$, $\varphi_{c_0}(\bar{a})(x, y) = \varphi_{c_0}(\bar{a})(-x, y)$, and

$$-\Delta \varphi_{c_0}(\bar{a}) + \dot{c}(\bar{a})\varphi_{c_0}(\bar{a}) - (\varphi_{c_0}(\bar{a}))^2 = 0,$$

$$(\varphi_{c_0}(\bar{a}) = \tilde{Q}_{c_0} + a_1\tilde{Q}_{c_0}^2 \cos \frac{y}{L} + a_2\tilde{Q}_{c_0}^2 \sin \frac{y}{L} + O(|\bar{a}|^2) \quad \text{as} \ |\bar{a}| \to 0$$

and

$$\|\varphi_{c_0}(\bar{a})\|^2_{L^2(\mathbb{R} \times \mathbb{T}_L)} = \|\tilde{Q}_{c_0}\|^2_{L^2(\mathbb{R} \times \mathbb{T}_L)} + \frac{C_{2, c_0}}{2}|\bar{a}|^2 + o(|\bar{a}|^2) \quad \text{as} \ |\bar{a}| \to 0,$$

where $\dot{c}(\bar{a}) = c_0 + \frac{\dot{c}''(0)}{2}|\bar{a}|^2 + o(|\bar{a}|^2)$ as $|\bar{a}| \to 0$, $\dot{c}''(0) > 0$ and

$$C_{2, c_0} = \frac{3\dot{c}''(0)}{2c_0}\|\tilde{Q}_{c_0}\|^2_{L^2} - \frac{5}{2}\|\tilde{Q}_{c_0}^2 \cos \frac{y}{L}\|^2_{L^2} > 0$$

**Remark 1.4.** Proposition 1.3 follows Proposition 1 and the proof of Theorem 1.3 in \cite{46}. The positivity of the constant $\dot{c}''(0)$ follows the relation $L^4\dot{c}''(0) = \omega''(0)$ and the positivity of $\omega''(0)$ in the proof of Theorem 1.3 in \cite{46}, where $\omega''(0)$ have been defined in Proposition 1 in \cite{46}. The positivity of the constant $C_{2, c_0}$ have been proved from the inequality (2.25)

$$R(p) \leq \frac{4(p + 1)(p^6 + 18p^5 - 11p^4 - 130p^3 + 13p^2 + 16p - 3)}{(5 - p)(p + 3)^2(5p - 1)(3p + 1)(p - 1)}$$

$$+ \frac{32p^3(p + 1)^4(3p - 1)}{3(7p - 3)(5p - 1)(3p + 1)(p + 3)^3(p - 1)}$$

in \cite{46} and the relation

$$6LC_{2, c_0}\|\tilde{Q}_{c_0}\|^2_{L^2(\mathbb{R} \times \mathbb{T}_L)} = R(2) < 0,$$

where $R(p)$ is defined in the proof of Theorem 1.3 in \cite{46}.
We define a semi-norm $\|\cdot\|_{H^1(x>a)}$ on $H^1(\mathbb{R} \times \mathbb{T}_L)$ by

$$\|u\|_{H^1(x>a)}^2 = \int_{x>a} (|\nabla u(x,y)|^2 + |u(x,y)|^2) dx dy, \quad u \in H^1(\mathbb{R} \times \mathbb{T}_L).$$

The following theorem is a main theorem for the asymptotic stability.

**Theorem 1.5.** Let $c_0 > 0$.

(i) If $0 < L < \frac{2}{\sqrt{5c_0}}$, then the following holds. For any $\beta > 0$, there exists $\varepsilon_{L,\beta} > 0$ such that for $u_0 \in H^1(\mathbb{R} \times \mathbb{T}_L)$ with $\|u_0 - \tilde{Q}_{c_0}\|_{H^1} < \varepsilon_{L,\beta}$, there exist $\rho(t) \in C^1([0, \infty), \mathbb{R})$ and $c_+ > 0$ satisfying that

$$\|u(t, \cdot, \cdot) - \tilde{Q}_{c_+}(\cdot - \rho(t), \cdot)\|_{H^1(x>\beta t)} \to 0 \text{ as } t \to \infty,$$

and $|c_0 - c_+| \lesssim \|u_0 - \tilde{Q}_{c_0}\|_{H^1}$, where $u$ is the unique solution of (1.1) with $u(0) = u_0$.

(ii) If $L = \frac{2}{\sqrt{5c_0}}$, then the following holds. For any $\beta > 0$, there exists $\varepsilon_{\beta} > 0$ such that for $u_0 \in H^1(\mathbb{R} \times \mathbb{T}_L)$ with $\|u_0 - \tilde{Q}_{c_0}\|_{H^1} < \varepsilon_{\beta}$, there exist $\rho_1(t), \rho_2(t) \in C^1([0, \infty), \mathbb{R})$, $c_+ > 0$ and $\tilde{c}_+ \in \mathbb{R}^2$ satisfying that

$$\|u(t, \cdot, \cdot) - \Theta(\tilde{c}_+, c_+)(\cdot - \rho_1(t), \cdot - \rho_2(t))\|_{H^1(x>\beta t)} \to 0 \text{ as } t \to \infty,$$

$$\dot{\rho}_1(t) - \dot{c}_+ \to 0, \dot{\rho}_2(t) \to 0 \text{ as } t \to \infty,$$

and $|c_0 - c_+| + |\tilde{c}_+|^2 \lesssim \|u_0 - \tilde{Q}_{c_0}\|_{H^1}$, where

$$\tilde{c}_+ = \begin{cases} c_+, & \tilde{a}_+ = (0, 0), \\ \hat{c}(\tilde{a}_+), & c_+ = c_0, \end{cases} \quad (1.7)$$

and $u$ is the unique solution of (1.1) with $u(0) = u_0$.

**Remark 1.6.** Since a neighborhood of $\tilde{Q}_{c_0}$ in $H^1(\mathbb{R} \times \mathbb{T}_L)$ contains the branch corresponding to unstable line solitary waves in the case $L = \frac{2}{\sqrt{5c_0}}$, Theorem 1.5 shows that solutions away from unstable solitary waves approach one of solitary waves in the neighborhood $\tilde{Q}_{c_0}$ as $t \to \infty$ in the sense of the norm $H^1(x > \beta t)$.

**Remark 1.7.** In Theorem 1.5 the unique solution $u$ of (1.1) with $u(0) = u_0$ means that for $T > 0$ the function $u|_{[-T, T]}$ is a unique solution of (1.1) with $u(0) = u_0$ in $C([-T, T], H^1(\mathbb{R} \times \mathbb{T}_L)) \cap X_T^{1,\frac{1}{2}+}$ which is defined in [31].
Let us now explain the argument to prove Theorem 1.2. Since the solution \( \tilde{Q}_c \) of (1.4) is not a minimizer of the functional \( S_c(u) \) on \( \{ u \in H^1; M(u) = M(\tilde{Q}_c) \} \) for general \( c > 0 \), we cannot apply the variational argument to prove the orbital stability. Therefore, to prove the orbital stability of \( \tilde{Q}_{c_0} \) we use the argument in [12, 44] for \( 0 < L < \frac{2}{\sqrt{3}c_0} \). In the case \( L = \frac{2}{\sqrt{3}c_0} \), the linearized operator of (1.4) around \( \tilde{Q}_{c_0} \) has an extra eigenfunction corresponding to zero eigenvalue. Thus, we cannot show the orbital stability of \( \tilde{Q}_{c_0} \) by using the standard argument in [12, 13, 44]. Since any neighborhood of \( \tilde{Q}_{c_0} \) contains the two branches which are comprised of line solitary waves \( \tilde{Q}_c \) and solitary waves \( \varphi_{c_0}(\tilde{a}) \), we cannot apply the argument for the linearized operator of the evolution equation with an extra eigenfunction by Comech and Pelinovsky [6] and Maeda [24]. Because of the degeneracy of the third order term of Lyapunov functional, we cannot use the argument for the instability of a standing wave on a point of interaction of two branches of standing waves in Ohta [33]. To prove the stability of \( \tilde{Q}_{c_0} \), we apply the argument in [46, 47].

To show the nonlinear instability of \( \tilde{Q}_c \) from the existence of an unstable mode of the linearized operator around \( \tilde{Q}_c \), we apply the argument by Grenier [11] and Rousset and Tzvetkov [38]. Since the simple criterion in [37, 39] does not seem to be applicable to the linearized operator of (1.1) around \( \tilde{Q}_c \), it is difficult to get the existence of an unstable mode of the linearized operator by the implicit function theorem. For sufficiently large \( L \), Bridges [3] have showed the existence of an unstable mode by sophisticated arguments. To get the existence of an unstable mode of linearized operator for all \( L > \frac{2}{\sqrt{3}c_0} \), we use Evans’ function method in Pego and Weinstein [34] for gKdV equation.

Next we explain the main ideas and difficulties in the proof of Theorem 1.5. Since the equation (1.1) is not complete integrable, we cannot use the inverse scattering method to get the asymptotic behavior of solutions. To prove the asymptotic stability, we apply the argument by Martel and Merle [25, 26, 27] and Côte et al. [5]. This argument relies on a Liouville type theorem for decaying solutions around a solitary wave. From the orbital stability and the monotonicity property, solutions near by a solitary wave converge to an exponentially decaying function in \( H^1(x > a) \) up to subsequence of time. Due to the Liouville type theorem, this function has to be solitary waves. The main tool to prove Liouville type theorem is the virial type estimate. In the case \( 0 < L < \frac{2}{\sqrt{3}c_0} \), the linearized operator of (1.4) around \( \tilde{Q}_{c_0} \) is coercive on \( \{ u \in H^1; M(u) = M(\tilde{Q}_{c_0}) \} \) by modulating translation. Thus, applying the estimate of [27] we can show the virial type estimate. However, in the case \( L = \frac{2}{\sqrt{3}c_0} \), the linearized operator of (1.4) around \( \tilde{Q}_{c_0} \) is not coercive on the function space with the standard orthogonal condition. To get the coerciveness of linearized operator, we estimate the difference between the solution and \( \Theta \) instead of the difference between the solution and solitary waves, where \( \Theta \) is defined in Theorem 1.5. However, since \( \Theta \) is not solution of stationary equation, a term including \( S'_c(\Theta) \) appears in the virial type estimate. Therefore, we cannot get the coerciveness of the virial type estimate by the argument in [27]. To treat the term with \( S'_c(\Theta) \), we investigate the virial type estimate with a correction term \( S'_c(\hat{\Theta}) \), where \( \hat{c} \) is the suitable propagation speed of \( \Theta \). To get the coerciveness of the virial type estimate with a correction, we use the precise estimate for a quadratic form and interactions among main terms. Due to this virial type estimate with the correction, we get the Liouville type theorem around the bifurcation point \( \tilde{Q}_{c_0} \).
Our plan of the present paper is as follows. In Section 2, we show the well-posedness result on weighted space to prove the monotonicity property. The argument of this well-posedness result follows the argument by Kato in [18]. In Section 3, we prove the properties of the linearized operator of (1.1) and the estimate of semi-group corresponding to the linearized operator. To show the linear instability of linearized equation, we use the argument by Pego and Weinstein [34]. In Section 4, we prove (ii) of Theorem 1.2 by the argument of Rousset and Tzvetkov [38]. In Section 5, we show (i) of Theorem 1.2 by the argument of [12] and [46, 47]. In Section 6, we prove the coercive type estimate of a quadratic form and the Liouville property for orbitally stable solitary waves. To get the monotonicity property, we use the Kato type local smoothing effect in Section 2. In Section 7, we prove Theorem 1.5 by applying the Liouville property and the monotonicity property in Section 6.

2 Preliminaries

In this section, we show the regularity of solutions to (1.1) on weighted space. To prove of smoothness of solutions to (1.1) on weighted space, we apply the argument on KdV in [18]. For $u \in L^2(\mathbb{R} \times \mathbb{T}_L)$ we define $\hat{u}$ by the space-time Fourier transform of $u$.

From the result on well-posedness in $H^1(\mathbb{R} \times \mathbb{T}_L)$ by Molinet-Pilod [31], for initial data $u_0 \in H^1(\mathbb{R} \times \mathbb{T}_L)$ there exists the unique solution $u(t)$ of (1.1) such that $u(0) = u_0$ and for $T > 0$

$$u|_{[-T,T]} \in C([-T,T], H^1(\mathbb{R} \times \mathbb{T}_L)) \cap X_T^{1,\frac{1}{2}+}.$$  

Moreover, for any $T > 0$, there exists a neighborhood $\mathcal{U}$ of $u_0$ in $H^1(\mathbb{R} \times \mathbb{T}_L)$, such that the flow map data-solution

$$v_0 \in \mathcal{U} \mapsto v \in C([0,T), H^1(\mathbb{R} \times \mathbb{T}_L)) \cap X_T^{1,\frac{1}{2}+}$$

is smooth. Here, the function space $X^{1,\frac{1}{2}+}$ is defined in [31]. In this paper, we define $H^1$-solution by the solution in the function space $C([0, \infty), H^1(\mathbb{R} \times \mathbb{T}_L))$ satisfying the conservation laws $M(u(t)) = M(u(0))$ and $E(u(t)) = E(u(0))$.

Let $U_b(t) = \exp(-t(\partial_x - b)((\partial_x - b)^2 + \partial_y^2))$ for $b > 0$. Then, we have for $u \in L^2(\mathbb{R} \times \mathbb{T}_L)$ if $e^{bx}u \in L^2(\mathbb{R} \times \mathbb{T}_L)$ then

$$e^{-bx}U_b(t)e^{bx}u = U_0(t)u.$$  

The following lemma decay properties of the propagator $U_b$.

Lemma 2.1. Let $b > 0$, $s, s' \in \mathbb{R}$, $s < s'$ and $n \in \mathbb{Z}_+$. Then, there exists $C = C(n, s, b) > 0$ such that for $u \in H^s(\mathbb{R} \times \mathbb{T}_L)$, $0 \leq j \leq n$ and $t > 0$

\begin{align}
\|U_b(t)u\|_{H^{s'}} &\leq Ct^{-\frac{s-s'}{2}}e^{bt}\|u\|_{H^{s'}} \\
\|\partial_x^j \partial_y^{n-j}U_b(t)u\|_{L^2} &\leq Ct^{-\frac{n}{2}}e^{bt}\|u\|_{L^2} \\
\|\partial_t U_b(t)u\|_{L^2} &\leq Ct^{-\frac{n}{2}}e^{bt}\|u\|_{L^2}
\end{align}  

(2.1) \quad (2.2) \quad (2.3)
Proof. By the factorization we have
\[ U_b(t) = \exp(t b^3) \exp(-3t b^2 \partial_x) \exp(t b(3 \partial_x^2 + \partial_y^2)) \exp(-t \partial_x \Delta). \]
Since \( \exp(-3t b^2 \partial_x) \) and \( \exp(-t \partial_x \Delta) \) are unitary in \( H^s \) and \( \exp(t b(3 \partial_x^2 + \partial_y^2)) \) is the heat semigroup, we have the estimates (2.1)–(2.3).

Proposition 2.2. Let \( u \) be a \( H^1 \)-solution to (1.1) with \( e^{b u} \in L^2(\mathbb{R} \times \mathbb{T}_L) \) for some \( b > 0 \). Then we have \( e^{b u} \in C([0, \infty), L^2(\mathbb{R} \times \mathbb{T}_L)) \cap C^\infty((0, \infty), H^\infty(\mathbb{R} \times \mathbb{T}_L)) \), with
\[
\| e^{b u}(t) \|_{L^2} \leq e^{Kt} \| e^{b u}(0) \|_{L^2} \tag{2.4}
\]
where \( K \) denotes various constant depending only on \( b \) and \( \| u(0) \|_{L^2} \). Moreover, for any \( T > 0 \) and \( s \geq 0 \),
\[
\| e^{b u}(t) \|_{H^s} \leq K't^{-\frac{s}{2}}, \quad 0 < t \leq T, \tag{2.5}
\]
\[
\| e^{b (\partial_x)^n u}(t) \|_{H^s} \leq K't^{-\frac{snb}{2}}, \quad 0 < t \leq T, \quad n \in \mathbb{Z}_+	ag{2.6}
\]
where \( K' \) depend on \( s, n, T, b, \| e^{b u}(0) \|_{L^2}, M(u(0)) \) and \( E(u(0)) \).

Proof. Let
\[
q(x) = e^{b x} (1 + \varepsilon e^{2b x})^{-\frac{1}{4}}, \quad r(x) = e^{b x} (1 + \varepsilon e^{2b x})^{-1}, \quad p(x) = q(x)^2.
\]
Then, we have \( q, r, p \in L^\infty(\mathbb{R} \times \mathbb{T}_L) \) and
\[
\partial_x p = 2br^2, \quad |\partial_x^2 p| \leq 4b^2 r^2, \quad |\partial_x^4 p| \leq 12b^3 r^2, \quad |\partial_x r| \leq br.
\]
Therefore, we have
\[
\frac{d}{dt} (pu, u)_{L^2} \leq -2b(3\| \partial_x u \|_{L^2}^2 + \| r \partial_y u \|_{L^2}^2) + 12b^2 \| ru \|_{L^2}^2 + \frac{8b}{3}(r^2 u, u^2)_{L^2}. \tag{2.7}
\]
Then,
\[
(r^2 u, u^2)_{L^2} \leq \frac{1}{2} \| r \nabla u \|_{L^2}^2 + K_0 \| ru \|_{L^2}^2,
\]
where \( K_0 \) depend only \( b \) and \( \| u(0) \|_{L^2} \). From (2.7) and \( r < q \), we obtain
\[
\frac{d}{dt} \| qu \|_{L^2}^2 \leq -\frac{2b}{3} \| r \nabla u \|_{L^2}^2 + K \| ru \|_{L^2}^2 \leq -\frac{2b}{3} \| r \nabla u \|_{L^2}^2 + K \| qu \|_{L^2}^2.
\]
It follows that \( \| qu(t) \|_{L^2} \leq e^{K t} \| qu(0) \|_{L^2} \). Applying the monotone convergence theorem, we obtain that
\[
\| e^{b u}(t) \|_{L^2} \leq e^{Kt} \| e^{b u}(0) \|_{L^2}, \quad t \geq 0,
\]
where $K$ depend only $b$ and $\|u(0)\|_{L^2}$. Since $e^{bx}U_0(t) = U_0(t)e^{bx}$, by Lemma 2.1 we have for $t > 0$

$$\|e^{bx}u(t)\|_{L^2} \leq \|U_0(t)e^{bx}u(0)\|_{L^2} + \int_0^t \|U_0(t-\tau)e^{bx}\partial_x(u(\tau)^2)\|_{L^2}d\tau$$

$$\leq Ce^{bt}\|e^{bx}u(0)\|_{L^2} + \int_0^t C(M(u), E(u))(t-\tau)^{-3/4}\|e^{bx}u(\tau)\|_{L^2}d\tau.$$ 

Here, we use $\|u(t)\|_{H^1} \leq C(M(u), E(u))$. Therefore, $e^{bx}u(t) \in C([0, \infty), L^2(\mathbb{R} \times \mathbb{T}_L)).$ Next we show (2.5). By the Sobolev embedding and Hölder inequality, we have

$$\|e^{bx}\partial_x(u^2)\|_{H^{-\frac{1}{4}}} \leq b\|e^{bx}u^2\|_{H^{-\frac{1}{4}}} + \|e^{bx}u^2\|_{H^\frac{3}{4}}$$

$$\leq C(M(u), E(u))\|e^{bx}u\|_{H^1}. $$

Thus,

$$\|e^{bx}u(t)\|_{H^\frac{3}{4}} \leq Ct^{-\frac{s}{2}}e^{bT}\|e^{bx}u(0)\|_{L^2} + K'\int_0^t (t-\tau)^{-\frac{3}{4}}\|e^{bx}u(\tau)\|_{H^1}d\tau$$

$$\leq Ct^{-\frac{s}{2}}e^{bT}\|e^{bx}u(0)\|_{L^2} + K'\int_0^t (t-\tau)^{-\frac{3}{4}}\|e^{bx}u(\tau)\|_{H^\frac{3}{4}}d\tau,$$

where $K'$ depends only on $s, n, T, b, \|e^{bx}u(0)\|_{L^2}$, $M(u(0))$ and $E(u(0))$. Therefore, $e^{bx}u \in C((0, \infty), H^\frac{3}{4}(\mathbb{R} \times \mathbb{T}_L))$ and (2.5) holds for $s = \frac{5}{4}$. By the interpolation, we obtain (2.5) for $0 \leq s \leq \frac{5}{4}$. To prove $s > \frac{5}{4}$, we use the induction on $s$. Suppose (2.5) has been proved for $0 \leq s \leq s' - \frac{1}{2}$, where $s' \geq \frac{7}{4}$. We shall show (2.5) for $0 \leq s \leq s'$. By Duhamel formula, we have

$$t^\frac{s}{2}e^{bx}u(t) = \int_0^t U_b(t-\tau)\left(\frac{s}{2}t^\frac{s}{2} - 1\right)e^{bx}u(\tau) - \frac{s}{2}\tau^\frac{s}{2}e^{bx}\partial_x(u(\tau)^2)\right)d\tau.$$ 

Since $\|U_b(t-\tau)\|_{H^{s'}-\frac{3}{4} \to H^s} \leq C(t-\tau)^{-\frac{s}{4}},$

$$t^\frac{s}{2}\|e^{bx}u(t)\|_{H^s} \leq C \int_0^t (t-\tau)^{-\frac{s}{4}}\left(\frac{s}{2}t^\frac{s}{2} - 1\right)\|e^{bx}u(\tau)\|_{H^{s'}-\frac{3}{4}} + \tau^\frac{s}{2}\|e^{bx}\partial_x(u(\tau)^2)\|_{H^{s'}-\frac{3}{4}}\right)d\tau.$$ (2.8)

From the assumption of the induction we have

$$\tau^\frac{s}{2} - 1\|e^{bx}u(\tau)\|_{H^{s'}-\frac{3}{4}} \leq C' \tau^\frac{s}{2} - 1 - \frac{s'}{4} = K'\tau^\frac{s}{4}.$$ 

On the other hand, by Appendix A in [18] for $f, g \in H^s(\mathbb{R} \times \mathbb{T}_L)(s \geq \frac{5}{4})$

$$\|fg\|_{H^s} \lesssim \|f\|_{H^\frac{5}{4}}\|g\|_{H^\frac{5}{4}} \|f\|_{H^s} + \|g\|_{H^\frac{5}{4}}\|g\|_{H^\frac{5}{4}}\|f\|_{H^s}.$$ 

Thus, we have

$$\|e^{bx}\partial_x(u(\tau)^2)\|_{H^{s'}-\frac{3}{4}} \lesssim \|e^{bx}u(\tau)^2\|_{H^{s'}-\frac{3}{4}} \lesssim \|e^{bx}u(\tau)\|_{H^{\frac{5}{4}}}\|e^{bx}u(\tau)\|_{H^\frac{5}{4}}\|e^{bx}u(\tau)\|_{H^\frac{5}{4}}\|e^{bx}u(\tau)\|_{H^{s'}-\frac{3}{4}}.$$
From the assumption of the induction, we obtain

\[ \|e^{bx}\partial_x(u(\tau)^2)\|_{H^{\nu'-\frac{3}{4}}} \leq K'_t r^{-\frac{\nu'}{2}}, \]

where \( K'_t \) depends only on \( s, n, T, b, \|e^{bx}u(0)\|_{L^2}, M(u(0)) \) and \( E(u(0)) \). Since

\[ \|e^{bx}u(0)\|_{L^2} \leq (\|u(0)\|_{L^2}\|e^{bx}u(0)\|_{L^2})^{\frac{1}{2}}, \]

\( K'_t \) depends only on \( s, n, T, b, \|e^{bx}u(0)\|_{L^2}, M(u(0)) \) and \( E(u(0)) \). From (2.8) we obtain

\[ \|e^{bx}u(t)\|_{H^{\nu'}} \leq K'' t^{-\frac{\nu'}{2}} \]

where \( K'' \) depends only on \( s, n, T, b, \|e^{bx}u(0)\|_{L^2}, M(u(0)) \) and \( E(u(0)) \). This proves (2.5) for \( 0 \leq s \leq s' \), completing the induction.

Finally we prove (2.6) by induction on \( n \). For the case \( n = 0 \), it is known by (2.5). Assuming that it has been proved for all \( s \geq 0 \) up to a given \( n \), we prove it for \( n + 1 \). By the induction hypothesis,

\[ \|\partial_t^n \Delta (e^{bx}u)\|_{H^s} \lesssim \|\partial_t^n (e^{bx}u)\|_{H^{s+3}} \leq K'T^{-\frac{s+3+3n}{2}}. \] (2.9)

On the other hand,

\[ \|\partial_t^n e^{bx} \partial_x (u^2)\|_{H^s} \lesssim \|\partial_t^n e^{bx} u^2\|_{H^{s+1}} \lesssim \sum_{j=0}^n \|e^{bx} (\partial_t^j u)(\partial_t^{n-j} u)\|_{H^{s+1}}. \]

By Appendix A in [18],

\[ \|e^{bx} (\partial_t^j u)(\partial_t^{n-j} u)\|_{H^{s+1}} \lesssim \|e^{bx} \partial_t^j u\|_{H^{s+1}} \|e^{bx} \partial_t^{n-j} u\|_{H^4} \|e^{bx}\partial_t^{n-j} u\|_{H^{\frac{9}{4}}} \]

\[ + \|e^{bx}\partial_t^{n-j} u\|_{H^{s+1}} \|e^{bx}\partial_t^j u\|_{H^4} \|e^{bx}\partial_t^j u\|_{H^{\frac{9}{4}}}. \]

Therefore,

\[ \|\partial_t^n e^{bx} \partial_x (u^2)\|_{H^s} \leq K'T^{-\frac{s+3+3n}{2}}. \] (2.10)

From (2.9) and (2.10) we obtain (2.6) for \( n + 1 \), completing the induction. \( e^{bx}u \in C^\infty((0, \infty), H^\infty(\mathbb{R} \times \mathbb{T}_L)) \) follows the estimate (2.6).

\[ \square \]

### 3 Linearized operator

In this section, we show the properties of the linearized operator of (1.1) around \( \tilde{R}_c \). We define the linearized operator \( \mathbb{L}_c \) of (1.4) around \( \tilde{Q}_c \) by

\[ \mathbb{L}_c = S''(\tilde{Q}_c) = -\Delta + c - 2\tilde{Q}_c \]

10
and the linearized operator $\mathcal{L}_c$ of (1.6) around $Q_c$ by

\[ \mathcal{L}_c = -\partial_x^2 + c - 2Q_c. \]

Then, the linearized operator of (1.1) around $\tilde{R}_c$ is $\partial_x \mathbb{L}_c$. From Theorem 3.4 in [4], $\mathcal{L}_c$ has the only one negative eigenvalue

\[ -\lambda_c = -\frac{5c}{4} \]

and an eigenfunction $(Q_c)^\frac{3}{2}$ corresponding to $-\lambda_c$.

**Proposition 3.1.** Let $c > 0$.

(i) If $0 < L \leq \frac{2}{\sqrt{5c}}$, then $\partial_x \mathbb{L}_c$ has no eigenvalues with a positive real part.

(ii) If $0 < L < \frac{2}{\sqrt{5c}}$, then $\text{Ker}(\mathbb{L}_c) = \text{Span}\{\partial_x \tilde{Q}_c\}$.

(iii) If $L = \frac{2}{\sqrt{5c}}$, then $\text{Ker}(\mathbb{L}_c) = \text{Span}\{\partial_x \tilde{Q}_c, (\tilde{Q}_c)^\frac{3}{2} \cos \frac{y}{L}, (\tilde{Q}_c)^\frac{3}{2} \sin \frac{y}{L}\}$.

(iv) If $L > \frac{2}{\sqrt{5c}}$, then $\partial_x \mathbb{L}_c$ has a positive eigenvalue and the number of eigenvalue of $\partial_x \mathbb{L}_c$ with a positive real part is finite.

Here, Span\{u_1, \ldots, u_n\} is the vector space spanned by vectors $u_1, \ldots, u_n$.

**Proof.** By the Fourier expansion, we have for $u \in H^1(\mathbb{R} \times \mathbb{T}_L)$

\[ (\mathbb{L}_c u)(x, y) = \sum_{n=-\infty}^{\infty} \left( \mathcal{L}_c + \frac{n^2}{L^2} \right) u_n(x) e^{in\frac{y}{L}}, \]  

(3.1)

where

\[ u(x, y) = \sum_{n=-\infty}^{\infty} u_n(x) e^{in\frac{y}{L}}. \]

From the equation (3.1), we obtain that $\partial_x \mathbb{L}_c$ has an eigenvalue $\lambda$ if and only if there exists $n \in \mathbb{Z}$ such that $\partial_x(\mathcal{L}_c + n^2/L^2)$ has an eigenvalue $\lambda$. By Theorem 3.4 in [34], the essential spectrum of $\partial_x \mathbb{L}_c$ is the imaginary axis. Moreover, from Theorem 3.1 in [34], the number of eigenvalues of $\partial_x(\mathcal{L}_c + n^2/L^2)$ with a positive real part less than or equal to the number of negative eigenvalues of $\mathcal{L}_c + n^2/L^2$. In the case $L \leq \frac{2}{\sqrt{5c}}$, since $n^2/L^2 \geq \lambda_c$ for all $n \neq 0$, $\mathcal{L}_c + n^2/L^2$ has no negative eigenvalues and (i) is verified. The kernel of $\partial_x(\mathcal{L}_c + n^2/L^2)$ is trivial if and only if the kernel of $\mathbb{L}_c + n^2/L^2$ is trivial. Therefore, for $L > \frac{2}{\sqrt{5c}}$ the kernel of $\partial_x \mathbb{L}_c$ is spanned by $\partial_x \tilde{Q}_c$. In the case $L = \frac{2}{\sqrt{5c}}$, the kernel of $\partial_x \mathbb{L}_c$ is spanned by $\partial_x \tilde{Q}_c$, $(Q_c)^\frac{3}{2} \cos \frac{y}{L}$ and $(Q_c)^\frac{3}{2} \sin \frac{y}{L}$. Thus, (ii) and (iii) are verified.

To prove (iv), we apply Evans’ function method in [34]. We consider the following equation:

\[ \partial_x \left( \mathcal{L}_c + a \right) u - \lambda u = 0. \]  

(3.2)
The equation (3.2) is equivalent to the first order system
\[ \partial_x \bar{u} = A(a, \lambda, x) \bar{u} \] (3.3)
where
\[ \bar{u} = \begin{pmatrix} u \\ \partial_x u \\ \partial^2_x u \end{pmatrix}, \quad A(a, \lambda, x) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2\partial_x Q_c(x) - \lambda & c + a - 2Q_c(x) & 0 \end{pmatrix}. \]

First, we show that \( A(a, \lambda, x) \) satisfies the assumption \( \textbf{H1}, \textbf{H2}, \textbf{H3} \) and \( \textbf{H4} \) in Section 1 of [34]. Then, \( A(a, \lambda, x) \) is analytic in \( \lambda \) and \( a \) for each \( x \), so \( \textbf{H1} \) holds true. Let
\[ A_\infty(a, \lambda) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\lambda & c + a & 0 \end{pmatrix}. \]

Then, \( \lim_{|x| \to \infty} A(a, \lambda, x) = A_\infty(a, \lambda) \) and \( A(a, \lambda, x) \) satisfies \( \textbf{H2} \) and \( \textbf{H4} \). We define
\[ \mu_1(a, \lambda) := \inf \{ \Re \mu; \mu \text{ is an eigenvalue of } A_\infty(a, \lambda) \}, \]
\[ \mu_2(a, \lambda) := \inf \{ \Re \mu; \Re \mu > \Re \mu_1(a, \lambda), \mu \text{ is an eigenvalue of } A_\infty(a, \lambda) \}. \]

Let
\[ J = \{(a, \lambda) \in \mathbb{C}^2; A_\infty(a, \lambda) \text{ has some purely imaginary eigenvalue}\}. \]

We define \( J_+ \) be the connected component of \( \mathbb{C}^2 \setminus J \) which contains \( \{a \geq 0\} \times \{ \lambda > 0 \} \). Form the perturbation theory of matrices, the number of eigenvalues counting multiplicity of \( A_\infty(a, \lambda) \) having the negative real part is constant for \( (a, \lambda) \in J_+ \). Since \( A_\infty(0, \lambda) \) has the only one simple negative eigenvalue for \( \lambda > 0 \), the number of eigenvalues counting multiplicity of \( A_\infty(a, \lambda) \) having the negative real part is 1 for \( (a, \lambda) \in J_+ \). Therefore, for \( (a, \lambda) \in J_+ \)
\[ \mu_1(a, \lambda) < 0 < \mu_2(a, \lambda) \]
Moreover, for \( a > -c/2 \)
\[ \mu_1(a, 0) < 0 \leq \mu_2(a, 0). \]

By the perturbation theory of matrices, there exists a domain \( \bar{\Omega} \in \mathbb{C}^2 \) such that \( \{a \geq 0\} \times \{ \lambda \geq 0 \} \subset \bar{\Omega} \) for \( (a, \lambda) \in \bar{\Omega} \) and \( A_\infty(a, \lambda) \) has the unique eigenvalue with the smallest real part \( \mu_1(a, \lambda) \), which is simple and
\[ \mu_1(a, \lambda) < \mu_2(a, \lambda) \] (3.4)
which implies \( \textbf{H3} \). Therefore, \( A(a, \lambda, x) \) satisfies the assumption \( \textbf{H1}, \textbf{H2}, \textbf{H3} \) and \( \textbf{H4} \) in Section 1 of [34], so we can define Evans’ function \( D(a, \lambda) \) for \( (a, \lambda) \in \bar{\Omega} \) by Definition 1.8 in [34]. For \( (a_0, \lambda_0) \in \bar{\Omega} \) with \( \Re \lambda_0 > 0 \), from Proposition 1.9 in [34] the kernel of the operator \( \partial_x (\mathcal{L}_c + a_0) - \lambda_0 \) is non-trivial if and only if \( D(a_0, \lambda_0) = 0 \). Since \( A(a, \lambda, x) \) is analytic in \( a \) and \( \lambda \) for each fixed \( x \), Evans’ function \( D(a, \lambda) \) is also analytic in \( a \) and \( \lambda \) for \( (a, \lambda) \in \bar{\Omega} \).
Let 
\[ \mathcal{P}(\nu) = \nu^3 - (c + a)\nu + \lambda \]
denote the characteristic polynomial of \( A_\infty \) and 
\[ \tilde{\mathcal{P}}(\nu) = \nu^3 + \lambda, \quad \mathcal{Q}(\nu) = -(c + a)\nu. \]
Then, the roots \( \nu_0 \) of \( \tilde{\mathcal{P}} \) are the cube roots of \(-\lambda\), and for \( |\nu - \nu_0| = o(1) \) as \( |\lambda| \to \infty \) we have
\[ \mathcal{Q}(\nu) = -(c + a)\nu_0(1 + o(1)), \quad \left| \frac{\partial \tilde{\mathcal{P}}}{\partial \nu}(\nu) \right| = 3\nu_0^2(1 + o(1)), \quad \left| \frac{\partial^2 \tilde{\mathcal{P}}}{\partial \nu^2}(\nu) \right| = \frac{|c + a|}{2|\lambda|^{\frac{4}{3}}}. \]
We choose \( \rho(\lambda) = \rho_0 |c + a|/3|\lambda|^{\frac{1}{3}} \) for any \( \rho_0 > 1 \) Then, the assumption of Lemma 1.20 in [34] are satisfied and the roots of \( \tilde{\mathcal{P}}(\nu) = 0 \) are given by 
\[ \nu = (-\lambda)^{\frac{1}{3}} + O(|c + a| |\lambda|^{-\frac{1}{3}}) \quad (3.5) \]
as \( \lambda \to \infty \). From (3.5) for any labeling \( \nu_1(a, \lambda), \nu_2(a, \lambda), \nu_3(a, \lambda) \) of roots of \( A_\infty(a, \lambda) \) we have
\[ \left| \frac{\nu_k}{\partial \nu}(\nu_j) \right| = \frac{|\lambda|^\frac{1}{3}}{3|\lambda|^\frac{2}{3}}(1 + o(|c + a|)) = O((1 + |a|)|\lambda|^{-\frac{4}{3}}), \]
as \( |\lambda| \to \infty \) in \( \tilde{\Omega} \). To apply Corollary 1.19 in [34], we obtain that the hypotheses of Proposition 1.17 in [34] hold. By Corollary 1.18 in [34], it follows that \( D(a, \lambda) \to 1 \) as \( |\lambda| \to \infty \) in \( \tilde{\Omega} \) for each fixed \( a \). So for \( 0 \leq a \leq \lambda_c \),
\[ D(a, \lambda) \to 1 \text{ as } \lambda \to \infty. \quad (3.6) \]
Since
\[ \partial_x \mathcal{L}_c \partial_x Q_c = 0, \quad \mathcal{L}_c \partial_x Q_c = 0 \]
and
\[ \partial_x Q_c(x)e^{\sqrt{c}x} \to -6c^{\frac{3}{2}} \text{ as } x \to \infty, \quad Q_c(x)e^{-\sqrt{c}x} \to 6c \text{ as } x \to -\infty, \]
from (1.35) in [34] and \( D(0,0) = 0 \) we have
\[ \frac{\partial D}{\partial a}(0,0) = -\frac{1}{12c^2} \int_{-\infty}^{\infty} |\partial_x Q_c|^2 dx = 0. \quad (3.7) \]
From Theorem 3.4 in [4] we have that the kernel of \( \mathcal{L}_c + a \) on \( L^2(\mathbb{R}) \) is trivial for \( 0 < a < \lambda_c \). If there exists \( 0 < a_0 < \lambda_c \) satisfying \( D(a_0,0) = 0 \), then there exists a solution \( u_0 \) of \( \partial_x(\mathcal{L}_c + a_0)u = 0 \) such that for all \( \varepsilon > 0 \) there is \( C_\varepsilon > 0 \) satisfying that
\[ |u_0(x)| + |\partial_x u_0(x)| + |\partial_x^2 u_0(x)| \leq C_\varepsilon e^{-(\mu_1 - \varepsilon)x} \text{ as } x \to \infty \]
and
\[ |u_0(x)| \leq C_\varepsilon e^{-\varepsilon x} \text{ as } x \to -\infty. \]
Since \(((\mathcal{L}_c + a)u)(x) \to 0\) as \(x \to \infty\), \(u_0\) is a solution \((\mathcal{L}_c + a)u = 0\). By the property of solutions of ordinary differential equations, any solution of \((\mathcal{L}_c + a)u = 0\) decays or grows exponentially to \(-\infty\). Thus, there are no solutions of \((\mathcal{L}_c + a)u = 0\) which grows subexponentially tend to \(-\infty\) and decays exponentially tend to \(\infty\). This contradicts that \(u_0\) is a solution of \((\mathcal{L}_c + a)u = 0\). Thus, \(D(a, 0) \neq 0\) for \(0 < a < \lambda_c\). Since \(D(a, \lambda)\) is real and continuous for real numbers \(a\) and \(\lambda\) in \(J_c\), by (3.7) \(D(a, 0)\) is negative for \(0 < a < \lambda_c\). From (3.6), for a there exists \(\lambda(a) > 0\) such that \(D(a, \lambda(a)) = 0\). Therefore, \(\partial_x(\mathcal{L}_c + a)\) has a positive eigenvalue \(\lambda(a)\) for \(0 < a < \lambda(a)\). Thus, \(\partial_xL_c\) has a positive eigenvalue for \(L > \sqrt{\lambda_c}\).

To prove the estimate of the propagator \(e^{\partial_x(\mathcal{L}_c + a)t}\) we apply the following Gearhart–Greiner–Herbst–Prüss theorem, see [32].

**Theorem 3.2.** Let \(A\) be a generator of a strongly continuous semigroup on a complex Hilbert space \((\mathcal{H}, \|\cdot\|_\mathcal{H})\). Then for each \(t > 0\), the following spectral mapping theorem is valid

\[\sigma(e^A) \setminus \{0\} = \{e^\lambda; \text{ either } \mu_k := \lambda + 2\pi ik \in \sigma(A) \text{ for some } k \in \mathbb{Z}\]

or the sequence \(\{\|\mu_k - A\|^{-1}_{\mathcal{H} \rightarrow \mathcal{H}}\}_{k \in \mathbb{Z}}\) is unbounded

**Proposition 3.3.** Let \(a, s \geq 0\) and \(s \in \mathbb{Z}\). Then, for \(\varepsilon > 0\) there exists \(C = C(\varepsilon, s) > 0\) for \(u \in H^s(\mathbb{R})\) and \(t > 0\),

\[\|e^{\partial_x(\mathcal{L}_c + a)t}u\|_{H^s(\mathbb{R})} \leq Ce^{(\mu(a) + \varepsilon)t}\|u\|_{H^s(\mathbb{R})}\]  

(3.8)

where \(\mu(a)\) is the maximum of the real part of elements in \(\sigma(\partial_x(\mathcal{L}_c + a))\).

**Proof.** By the compact perturbation theory the essential spectrum of \(\partial_x(\mathcal{L}_c + a)\) is the essential spectrum of \(\partial_x^2\), so the essential spectrum of \(\partial_x(\mathcal{L}_c + a)\) is the imaginary axis. If we show the sequence \(\{\|\lambda + 2\pi ik - \partial_x(\mathcal{L}_c + a)\|^{-1}_{H^s \rightarrow H^s}\}_{k}\) is bounded for all \(\text{Re } \lambda > \mu(a)\), we can show the estimate (3.8) by applying Theorem 3.2 and Lemma 2 and 3 in [41] (see also the proof of Lemma 3.2 in [45]). If \(s \geq 1\), we have that for \(u \in H^s(\mathbb{R})\)

\[\|\lambda + 2\pi ik - \partial_x(\mathcal{L}_c + a)\|^{-1}_{H^s}u\|_{H^s} \leq \|\lambda + 2\pi ik - \partial_x(\mathcal{L}_c + a)\|^{-1}_{H^{s-1}} \|\lambda + 2\pi ik - \partial_x(\mathcal{L}_c + a)\|^{-1}u\|_{H^{s-1}}.

Here, we use the boundedness of \((-2(Q_c)_{xx} - 2(Q_c)_{x} \partial_x) (\lambda + 2\pi ik - \partial_x(\mathcal{L}_c + a))^{-1}\) on \(H^{s-1}(\mathbb{R})\). Therefore, the boundedness of the sequence \(\{\|\lambda + 2\pi ik - \partial_x(\mathcal{L}_c + a)\|^{-1}_{H^s \rightarrow H^s}\}_{k}\) follows the boundedness of the sequence \(\{\|\lambda + 2\pi ik - \partial_x(\mathcal{L}_c + a)\|^{-1}_{L^2 \rightarrow L^2}\}_{k}\). Thus, we prove the boundedness of the sequence \(\{\|\lambda + 2\pi ik - \partial_x(\mathcal{L}_c + a)\|^{-1}_{L^2 \rightarrow L^2}\}_{k}\). For \(\beta \in \mathbb{C}\) we have

\[(i\beta - (i\partial_x)(\mathcal{L}_c + a))^{-1} = (I + A_\beta B)^{-1}(i\beta - (i\partial_x)((i\partial_x)^2 + c + a))^{-1},

where

\[A_\beta = 2(i\partial_x)(i\beta - (i\partial_x)((i\partial_x)^2 + c + a))^{-1}\sqrt{Q_c}

\[B = \sqrt{Q_c}.

14
Since

\[(I + A_\beta B)^{-1} = I - A_\beta(I + BA_\beta)^{-1}B,\]

for \(\text{Re } \lambda > \mu(a)\) the sequence \(\{\|(\lambda + 2\pi ik - \partial_x(L_c + a))^{-1}\|_{L^2 \to L^2}\}_k\) is bounded if and only if \(\{\|(I + BA_{\lambda+2\pi ik})^{-1}\|_{L^2 \to L^2}\}_k\) is bounded. For \(u \in L^2(\mathbb{R})\) we have

\[
\|BA_{\lambda+2\pi ik}u\|_{L^2} = \|\sqrt{Q_c}2(i\partial_x)(i\lambda - 2\pi k - (i\partial_x)((i\partial_x)^2 + c + a))^{-1}(\sqrt{Q_c}u)\|_{L^2} \\
\leq \|\eta(i\lambda - 2\pi k + \eta(\eta^2 + c + a))^{-1}\|_{L^1}\|u\|_{L^2}.
\]

Let

\[p(\eta, k) = -\text{Im } \lambda - 2\pi k + \eta(\eta^2 + c + a).\]

From (3.31) in the proof of Proposition 3.1 for \(k \in \mathbb{Z}\) there exist roots \(\alpha_j(k)\) \((j = 1, 2, 3)\) of \(p(\eta, k) = 0\) satisfies

\[\alpha_j(k) = (2\pi k)^{\frac{1}{3}}\omega_3^j + O(|k|^{-\frac{1}{3}})\]

as \(|k| \to \infty\), where \(\omega_3\) is a primitive root of \(\eta^3 - 1 = 0\). Since \(|\text{Im } \alpha_j| = \sqrt{3}(2\pi k)^{\frac{1}{3}} + O(|k|^{-\frac{1}{3}})\) as \(|k| \to \infty\), we have

\[
\int_{-\infty}^{\infty} \frac{|\eta|}{|\text{Re } \lambda| + |\eta - \alpha_1(k)||\eta - \alpha_2(k)||\eta - \alpha_3(k)|} d\eta \\
\leq \frac{1}{\sqrt{2}} \int_{-\infty}^{\infty} \frac{|\eta|}{|\text{Re } \lambda| - |\eta - \alpha_1(k)||\eta - \alpha_2(k)||\eta - \alpha_3(k)|} d\eta \\
\leq \frac{|k|^{-\frac{1}{2}}}{|\text{Re } \lambda|} \sup_{-|k|^{-1} < \xi < |k|^{-1}} |\xi + \alpha_3(k)k^{-\frac{1}{3}}| \\
+ \frac{|k|^{-\frac{1}{2}}}{\sqrt{2}} \int_{(-1,-|k|^{-1})\cup(|k|^{-1},1)} |\xi|^{-1}d\xi \sup_{\xi \in \mathbb{R}} |\xi - (\alpha_1(k) - \alpha_3(k))k^{-\frac{1}{3}}||\xi - (\alpha_2(k) - \alpha_3(k))k^{-\frac{1}{3}}| \\
+ \frac{|k|^{-\frac{1}{2}}}{\sqrt{2}} \int_{(-\infty,-1)\cup(1,\infty)} |\xi - (\alpha_1(k) - \alpha_3(k))k^{-\frac{1}{3}}||\xi - (\alpha_2(k) - \alpha_3(k))k^{-\frac{1}{3}}|d\xi \\
\leq |k|^{-\frac{1}{4}} \log |k|.
\]

Hence, we obtain there exists \(C > 0\) such that

\[\|BA_{\lambda+2\pi ik}u\|_{L^2} \leq C|k|^{-\frac{1}{4}}(\log |k|)\|u\|_{L^2}.
\]

Since \(\|BA_{\lambda+2\pi ik}\|_{L^2 \to L^2} \to 0\) as \(|k| \to \infty\), \(\{\|(I + BA_{\lambda+2\pi ik})^{-1}\|_{L^2 \to L^2}\}_k\) is bounded. Thus, we obtain the conclusion.

\[
\square
\]

4 Orbital instability

In this section, we prove (ii) of Theorem 1.2 by applying the argument in [38]. We assume \(L > 2/\sqrt{5c}\). Let \(\mu_{\text{max}}\) be the largest eigenvalue of \(\partial_x(L_c)\). Then, there exists a positive integer \(k_0\) such that the largest eigenvalue of \(\partial_x(L_c + k_0^2/L^2)\) is \(\mu_{\text{max}}\). Let \(\chi\) be a eigenfunction of
\[ \partial_x(\mathcal{L}_c + \frac{k_0^2}{L^2}) \] corresponding to \( \mu_{\text{max}} \). Since \( \mu_{\text{max}} > 0 \), from the dichotomy for ordinary differential equations \( \chi \in H^s(\mathbb{R}) \) for \( s > 0 \). For \( \delta > 0 \) we define the solution \( u^\delta \) of (1.1) with initial data \( \delta \chi \cos \frac{k_0 y}{L} + Q_c \) and we set \( v^\delta(t, x, y) = u^\delta(t, x + ct, y) - Q_c(x) \). Then, we have \( v^\delta(0, x, y) = \delta \chi(x) \cos \frac{k_0 y}{L} \) and

\[
\partial_t v^\delta + \partial_x \mathbb{P}_c v^\delta + \partial_x (v^\delta)^2 = 0.
\]

We define \( V^s_K \) as the function space

\[
V^s_K = \left\{ u \in L^2(\mathbb{R} \times T_L); u(x, y) = \sum_{j=-K}^{K} u_j(x)e^{\frac{jky}{L}}, u_j \in H^s(\mathbb{R}) \right\},
\]

and we define a norm of \( V^s_K \) as

\[
\|u\|_{V^s_K} = \sup_{|j| \leq K} \|u_j\|_{H^s(\mathbb{R})}, \text{ for } u = \sum_{j=-K}^{K} u_j e^{\frac{jky}{L}} \in V^s_K.
\]

To show the smallness of the high frequency part of \( v^\delta \), we consider an approximate solution

\[
v^\delta_M = \sum_{l=1}^{M} \delta^l w_l, \quad w_l \in V^{s-l+1}_l
\]

where \( w_1 \) is the solution of

\[
\partial_t w + \partial_x \mathbb{P}_c w = 0, \quad w(0, x, y) = \chi(x) \cos \frac{k_0 y}{L},
\]

and \( w_1 \) is the solution of

\[
\partial_t w + \partial_x \mathbb{P}_c w + \partial_x \left( \sum_{l_1, l_2 \geq 1, l_1 + l_2 = l} w_{l_1} w_{l_2} \right) = 0, \quad w(0, x, y) = 0.
\]

Then, \( v^\delta_M \) satisfies

\[
\partial_t v^\delta_M + \partial_x \mathbb{P}_c v^\delta_M + \partial_x (v^\delta_M)^2 = F,
\]

where

\[
F = \delta^M \partial_x \left( \sum_{1 \leq l_1, l_2 \leq M, l_1 + l_2 > M} \delta^{l_1+l_2-M} w_{l_1} w_{l_2} \right).
\]

From Proposition 3.3, we have the following lemma.

**Lemma 4.1.** For \( K, s, \varepsilon > 0 \) there exists \( C_{K, s, \varepsilon} > 0 \) such that for \( u \in V^s_K \)

\[
\|e^{t\partial_x \mathbb{P}_c} u\|_{V^s_K} \leq C_{K, s, \varepsilon} e^{(\mu_{\text{max}} + \varepsilon)t} \|u\|_{V^s_K}.
\]
Let \( w^\delta = v^\delta - v^\delta_M \). Then, we have
\[
\partial_t w^\delta + \partial_x \mathbb{L}_c w^\delta + 2\partial_x (w^\delta v^\delta_M) + \partial_x (w^\delta w^\delta) + F = 0. 
\] (4.1)

Therefore,
\[
\frac{d}{dt} \left\| w^\delta \right\|_{L^2}^2 = \int_{\mathbb{R} \times T_L} \left( (w^\delta)^2 \partial_x \tilde{Q}_c - (w^\delta)^2 \partial_x v^\delta_M - F w^\delta \right) dx dy \\
\leq \left( 1 + \left\| \partial_x v^\delta_M \right\|_{L^\infty} + \left\| \partial_x \tilde{Q}_c \right\|_{L^\infty} \right) \left\| w^\delta \right\|_{L^2}^2 + \left\| F \right\|_{L^2}^2 
\] (4.2)

From Lemma 4.1, we have that for \( \varepsilon_0 > 0 \) there exists \( C_{M,s,\varepsilon_0} > 0 \) such that
\[
\left\| u_1(t) \right\|_{H^s} \leq C_{M,s,\varepsilon_0} e^{(M,max + \varepsilon_0)t}. 
\]

Therefore, there exists \( C_{M,\varepsilon_0} > 0 \) such that we have
\[
\left\| \partial_x v^\delta_M(t) \right\|_{L^\infty} \leq C_{M,\varepsilon_0} (\varepsilon e^{(M,\max + \varepsilon_0)t} + \delta^M e^{M,(\varepsilon,\max + \varepsilon_0)t}) 
\] (4.3)
\[
\left\| F \right\|_{L^2} \leq C_{M,\varepsilon_0} \delta^{M+1} e^{(M+1),(\varepsilon,\max + \varepsilon_0)t}. 
\] (4.4)

We set \( T_{\delta,\varepsilon} = (\log(\varepsilon) - \log(\delta))/2\mu_{\max} \). Since \( e^{(\max + \varepsilon_0)t} \leq \varepsilon/\delta \) for \( 0 < t \leq T_{\delta,\varepsilon} \), by (4.2) - (4.4), we have
\[
\frac{d}{dt} \left( e^{-1 + \left\| \partial_x \tilde{Q}_c \right\|_{L^\infty} + 2\varepsilon C_{M,\varepsilon_0}} \left\| w^\delta(t) \right\|_{L^2}^2 \right) \leq C_{M,\varepsilon_0}^2 \delta^{2(M+1)} e^{2(M+1),(\varepsilon,\max + \varepsilon_0)t} 
\]
for any \( 0 < \varepsilon < 1 \) and \( 0 < t \leq T_{\delta,\varepsilon} \). Thus,
\[
\frac{d}{dt} \left( e^{-1 + \left\| \partial_x \tilde{Q}_c \right\|_{L^\infty} + 2\varepsilon C_{M,\varepsilon_0}} \left\| w^\delta(t) \right\|_{L^2}^2 \right) \leq C_{M,\varepsilon_0}^2 \delta^{2(M+1)} e^{2(M+1),(\varepsilon,\max + \varepsilon_0)t} - (1 + \left\| \partial_x \tilde{Q}_c \right\|_{L^\infty} + 2\varepsilon C_{M,\varepsilon_0}) t.
\]

If we choose large \( M \) and small \( \varepsilon(M) \) satisfying
\[
2(M + 1)(\varepsilon,\max + \varepsilon_0) - (1 + \left\| \partial_x Q_x \right\|_{L^\infty} + 2\varepsilon C_{M,\varepsilon_0}) > 0,
\]
then we obtain
\[
\left\| w^\delta(t) \right\|_{L^2}^2 \leq C_{M,\varepsilon_0}^M \delta^{2(M+1)} e^{2(M+1),(\varepsilon,\max + \varepsilon_0)t} 
\]
for \( 0 < t \leq T_{\delta,\varepsilon} \). Hence, there exists \( C_{M,\varepsilon_0}^M > 0 \) such that
\[
\left\| w^\delta(T_{\delta,\varepsilon}) \right\|_{L^2} \leq C_{M,\varepsilon_0}^M \varepsilon^{M+1} 
\]
for small \( \varepsilon > 0 \). Let \( P_0 \) be a projection satisfying
\[
(P_0 u)(x, y) = \int_{T_L} u(x, z) dz, \quad \text{for } (x, y) \in \mathbb{R} \times T_L.
\]

From the definition of \( v^\delta_M \) and the estimate (4.1), we have
\[
\left\| (\text{Id} - P_0)v^\delta_M(t) \right\|_{L^2} \geq \sqrt{\pi} \left\| \chi \right\|_{L^2} \delta^{\varepsilon_0} e^{\mu_{\max} t} - C_{\varepsilon_0} (\delta^2 e^{2\mu_{\max} t} + \delta^M e^{M\mu_{\max} t})
\]

\[17\]
Lemma 5.1. Let $\inf \| u^\delta(T_{\delta, \varepsilon}, \cdot) - \tilde{Q}_c(\cdot + a, \cdot) \|_{L^2}$ \geq \| (Id - P_0)(u^\delta(T_{\delta, \varepsilon}) - R_c(T_{\delta, \varepsilon})) \|_{L^2}$
\[ = \| (Id - P_0)(v^\delta(T_{\delta, \varepsilon})) \|_{L^2} \]
\[ \geq \| (Id - P_0)v^\delta(T_{\delta, \varepsilon}) \|_{L^2} - \| w^\delta(T_{\delta, \varepsilon}) \|_{L^2} \]
\[ \geq \sqrt{\pi} \| \chi \|_{L^2} \varepsilon - C_M^{\varepsilon} \varepsilon^2. \]

Thus, if we choose
\[ \varepsilon_1 = \frac{\sqrt{\pi} \| \chi \|_{L^2}}{2C_M^{\varepsilon}}, \]
for any $\delta > 0$ there exists $T_{\delta, \varepsilon} > 0$ such that
\[ \inf \| u^\delta(T_{\delta, \varepsilon}, \cdot) - \tilde{Q}_c(\cdot + a) \|_{L^2} \geq \frac{\sqrt{\pi} \| \chi \|_{L^2} \varepsilon_1}{2}. \]

This completes the proof of (ii) in Theorem 1.2.

5 Orbital stability

In this section, we prove (i) of Theorem 1.2 by applying the arguments in [12] and [46]. We write the outline of the proof of (i) of Theorem 1.2.

Theorem 3.3 in [12] yields the following coercive type lemma for $L_{c_0}$.

Lemma 5.1. Let $c_0 > 0$. There exists $k_0 > 0$ such that for $u \in H^1(\mathbb{R})$ with $(u, Q_{c_0})_{L^2(\mathbb{R})} = 0$, $(u, \partial_x Q_{c_0})_{L^2(\mathbb{R})} = 0$,
\[ \langle L_{c_0} u, u \rangle_{H^{-1}(\mathbb{R}), H^1(\mathbb{R})} \geq k_0 \| u \|_{H^1(\mathbb{R})}^2. \]

5.1 Non-critical case $L < \frac{2}{\sqrt{3c_0}}$

To show the orbital stability of $\tilde{R}_{c_0}$ for $L < \frac{2}{\sqrt{3c_0}}$, we apply the argument in [43] (see also [45, 47]). Let $L < \frac{2}{\sqrt{3c_0}}$. By the Fourier expansion (3.1) we have for $u \in H^1(\mathbb{R} \times \mathbb{T}_L)$
\[ \langle L_{c_0} u, u \rangle_{H^{-1}(\mathbb{R} \times \mathbb{T}_L), H^1(\mathbb{R} \times \mathbb{T}_L)} = \sum_{n = -\infty}^{\infty} \left\langle \left( L_{c_0} + \frac{n^2}{L^2} \right) u_n, u_n \right\rangle_{H^{-1}(\mathbb{R}), H^1(\mathbb{R})}, \]

where
\[ u(x, y) = \sum_{n = -\infty}^{\infty} u_n(x) e^{inx}. \]

Since $\lambda_{c_0} < L^2$, $L_{c_0} + n^2/L^2$ is positive for $|n| \geq 1$. From Lemma 5.1 there exists $K_0 > 0$ such that for $u \in H^1(\mathbb{R} \times \mathbb{T}_L)$ with $(u, \tilde{Q}_{c_0})_{L^2(\mathbb{R} \times \mathbb{T}_L)} = 0$, $(u, \partial_x \tilde{Q}_{c_0})_{L^2(\mathbb{R} \times \mathbb{T}_L)} = 0$, we have
\[ \langle L_{c_0} u, u \rangle_{H^{-1}(\mathbb{R} \times \mathbb{T}_L), H^1(\mathbb{R} \times \mathbb{T}_L)} \geq K_0 \| u \|_{H^1(\mathbb{R} \times \mathbb{T}_L)}^2. \]

Combining (5.1) and the proofs of Theorem 3.4 and Theorem 3.5 in [12], we obtain the orbital stability of $\tilde{R}_{c_0}$. 

18
5.2 Critical case $L = \frac{2}{\sqrt{5c_0}}$

The proof of the orbital stability of $\tilde{R}_{c_0}$ for $L = \frac{2}{\sqrt{5c_0}}$ is similar to the proof of (i) of Theorem 1.4 in [46] (see also the proof of (i) of Theorem 1.4 [47]). Let $L = \frac{2}{\sqrt{5c_0}}$. In this case, from (iii) of Proposition 3.1 the linearized operator $L_{c_0}$ has an extra eigenfunction corresponding to the zero eigenvalue. Therefore, we have to recover the degeneracy of the kernel of $L_{c_0}$ from nonlinearity of (1.1). We define the action $S_c(u)$ by $E(u) + cM(u)$.

**Lemma 5.2.** There exist a neighborhood $U$ of $(0,0)$ and a $C^2$ function $\gamma_c(\tilde{a}) : U \to \mathbb{R}$ such that $\gamma_c(0,0) = c$ and for $\tilde{a} \in U$ and $|c - c_0| < c_0/2$

$$M(\Theta(\tilde{a}, \gamma_c(\tilde{a}))) = M(\tilde{Q}_c),$$

$$\gamma_c(\tilde{a}) - c = -\frac{cC_{2,c_0}}{3\|\tilde{Q}_{c_0}\|_{L^2}^2} |\tilde{a}|^2 + o(|\tilde{a}|^2), \quad (5.2)$$

where $\Theta(\tilde{a}, c)(x, y) = cc_0^{-1}\varphi_{c_0}(\tilde{a})(\sqrt{cc_0^{-1}}x, y)$.

**Proof.** Let

$$\gamma_c(\tilde{a}) = c \left( \|\tilde{Q}_c\|_{L^2}^2 \|\varphi_{c_0}(\tilde{a})\|_{L^2}^{-2} \right)^\frac{1}{4}.$$ 

By the definition of $\Theta$ we have

$$M(\Theta(\tilde{a}, \gamma_c(\tilde{a}))) = M(\tilde{Q}_c).$$

Since $\|\tilde{Q}_c\|_{L^2}^2 \|\tilde{Q}_{c_0}\|_{L^2}^{-2} = c^\frac{3}{2}c_0^{-\frac{3}{2}}$, we have

$$\gamma_c(\tilde{a}) = c - c \frac{\|\varphi_{c_0}(\tilde{a})\|_{L^2}^{-\frac{5}{2}} - \|\tilde{Q}_{c_0}\|_{L^2}^{-\frac{1}{2}}}{\|\varphi_{c_0}(\tilde{a})\|_{L^2}^{-\frac{1}{2}}} = c - \frac{cC_{2,c_0}}{3\|\tilde{Q}_{c_0}\|_{L^2}^2} |\tilde{a}|^2 + o(|\tilde{a}|^2).$$

Next, we investigate the difference between $\Theta$ and $\tilde{Q}_c$ on the action $S_c$.

**Lemma 5.3.** For $\tilde{a} \in U$ and $|c - c_0| < c_0/2$,

$$S_c(\Theta(\tilde{a}, \gamma_c(\tilde{a}))) - S_c(\tilde{Q}_c) = \left( \frac{c}{c_0} \right)^5 5c_0C_{2,c_0} \left\| \tilde{Q}_{c_0}^\frac{3}{2} \cos \frac{\tilde{a}}{2} \right\|_{L^2}^2 |\tilde{a}|^4,$$

$$+ \left( 1 - \frac{c}{c_0} \right) \|\partial_y \Theta(\tilde{a}, \gamma_c(\tilde{a}))\|_{L^2}^2 + o(|\tilde{a}|^4) \quad (5.3)$$

as $|\tilde{a}| \to 0$. 

19
Proof. First, we consider the case $c = c_0$. From the expansion

$$\Theta(\bar{a}, \gamma_{c_0}(\bar{a})) = \varphi_{c_0}(\bar{a})(x, y) + (\gamma_{c_0}(\bar{a}) - c_0)\partial_c Q_{c_0} + O((|\bar{a}| + (\gamma_{c_0}(\bar{a}) - c_0))(\gamma_{c_0}(\bar{a}) - c_0)),$$

we have

$$S_{c_0}(\Theta(\bar{a}, \gamma_{c_0}(\bar{a})) - S_{c_0}(\tilde{Q}_{c_0})) = S_{c_0}(\varphi_{c_0}(\bar{a})) - S_{c_0}(\tilde{Q}_{c_0}) + (c_0 - \tilde{c}(\bar{a}))M(\tilde{Q}_{c_0})$$

$$+ \frac{1}{2}(\gamma_{c_0}(\bar{a}) - c_0)^2 (S''(\tilde{Q})_c \partial_c \tilde{Q}_{c_0}, \partial_c \tilde{Q}_{c_0})_{L^2} + o(|\bar{a}|^4),$$

where $\tilde{c}$ is defined in Proposition 1.3. Since $\frac{\partial \tilde{c}}{\partial a_1}(0, 0) = 0$ and $\frac{\partial^2 \tilde{c}}{\partial (a_1)^2}(0, 0) = c''(0) > 0$, there exist $\delta_1 > 0$ and the inverse function $a_1(c)$ of $\tilde{c}(a_1, 0)$ on from $[c_0, \tilde{c}(\delta_1, 0))$ to $[0, \delta_1)$. For $c_1, c_2$ with $c_1 \neq c_2$

$$\frac{S_{c_1}(\varphi_{c_0}(c_1)) - S_{c_2}(\varphi_{c_0}(c_2))}{c_1 - c_2}$$

$$= \frac{(S''_{c_2}(\varphi_{c_0}(c_2))(\varphi_{c_0}(c_1) - \varphi_{c_0}(c_2)), \varphi_{c_0}(c_1) - \varphi_{c_0}(c_2))_{L^2} + M(\varphi_{c_0}(c_1))}{2(c_1 - c_2)}$$

$$+ \frac{o((\varphi_{c_0}(c_1) - \varphi_{c_0}(c_2))^2)}{c_1 - c_2}$$

$$\to M(\varphi_{c_0}(c_2))$$

as $c_1 \to c_2$.

where $\varphi_{c_0}(c) = \varphi_{c_0}(a_1(c), 0)$. Since $S''_{c_0}(\tilde{Q}_{c_0})\partial_{a_1}\varphi_{c_0}(a_1, a_2)|_{(a_1, a_2) = (0, 0)} = \mathbb{L}_{c_0}(\tilde{Q}_{c_0}^\frac{3}{2} \cos \frac{y}{T}) = 0$, for $c > c_0$

$$\frac{S_{c}(\varphi_{c_0}(c)) - S_{c_0}(\tilde{Q}_{c_0})}{c - c_0}$$

$$= \frac{(S''_{c}(\tilde{Q}_{c_0}))(\varphi_{c_0}(c) - \tilde{Q}_{c_0}), \varphi_{c_0}(c) - \tilde{Q}_{c_0})_{L^2} + M(\varphi_{c_0}(c)) + \frac{o((\varphi_{c_0}(c) - \tilde{Q}_{c_0})^2)}{c''(0)a_1(c)^2 + o(a_1(c)^2)}$$

$$\to M(\tilde{Q}_{c_0})$$

as $c \downarrow c_0$.

Therefore, $S_c(\varphi_{c_0}(c))$ is $C^1$ and $\partial_c S_c(\varphi_{c_0}(c)) = M(\varphi_{c_0}(c))$. By the same way we obtain that $M(\varphi_{c_0}(c))$ is $C^1$ and

$$\lim_{c \to c_0} \frac{M(\varphi_{c_0}(c)) - M(\tilde{Q}_{c_0})}{c - c_0} = \frac{C_{2,c_0}}{2c''(0)}.$$

Thus, we have

$$S_{\tilde{c}(\bar{a}, 0)}(\varphi_{c_0}(|\bar{a}|, 0)) - S_{c_0}(\tilde{Q}_{c_0}) + (c_0 - \tilde{c}(|\bar{a}|, 0))M(\tilde{Q}_{c_0})$$

$$= \frac{C_{2,c_0}}{4c''(0)}(\tilde{c}(|\bar{a}|, 0) - c_0)^2 + o((\tilde{c}(|\bar{a}|, 0) - c_0)^2)$$

$$= \frac{C_{2,c_0}c''(0)}{16}|\bar{a}|^4 + o(|\bar{a}|^4).$$

(5.5)
From Lemma 5.2 and $S^\prime\prime_{c_0}(\tilde{Q}_{c_0})\partial_c\tilde{Q}_{c_0} = -\tilde{Q}_{c_0}$,

$$
(\gamma_{c_0}(\tilde{a}) - c_0)^2(\frac{S^\prime\prime_{c_0}(\tilde{Q}_{c_0})\partial_c\tilde{Q}_{c_0}, \partial_c\tilde{Q}_{c_0}}{L^2}) = -\frac{c_0C^2_{2,c_0}}{12}\|\tilde{a}\|^4 + o(\|\tilde{a}\|^4).
$$

(5.6)

Since

$$
S_{\varepsilon}(\varphi_{c_0}(\tilde{a})) - S_{c_0}(\tilde{Q}_{c_0}) + (c_0 - \varepsilon(\tilde{a}))M(\tilde{Q}_{c_0})
$$

$$
= S_{\varepsilon}(\varphi_{c_0}(\tilde{a})) - S_{c_0}(\tilde{Q}_{c_0}) + (c_0 - \varepsilon(\|\tilde{a}\|, 0))M(\tilde{Q}_{c_0}),
$$

from (5.5) and (5.6) we obtain (5.3) for $c = c_0$.

Next, we consider the general cases. Since $M(\Theta(\bar{a}, \gamma_c(\tilde{a}))) = M(\tilde{Q}_c)$, we have

$$
S_c(\Theta(\bar{a}, \gamma_c(\tilde{a}))) - S_c(\tilde{Q}_c)
$$

$$
= \left(\frac{c}{c_0}\right)^2 \left( S_{c_0}(\Theta(\bar{a}, \gamma_c(\tilde{a}))) - S_{c_0}(\tilde{Q}_{c_0}) \right) + \left(1 - \frac{c}{c_0}\right)\|\partial_x\Theta(\bar{a}, \gamma_c(\tilde{a}))\|_{L^2}^2.
$$

Therefore, we obtain (5.3) for $c > 0$.

We define a distance $\text{dist}_{c_0}$ and neighborhoods $N_{\varepsilon,c}$ and $N^0_{\varepsilon,c}$ of $\tilde{Q}_{c_0}$ by

$$
\text{dist}_{c_0}(u) = \inf_{x \in \mathbb{R}} \left\| u(\cdot, \cdot) - \tilde{Q}_c(\cdot - x, \cdot) \right\|_{H^1},
$$

$$
N_{\varepsilon,c} = \{ u \in H^1(\mathbb{R} \times \mathbb{T}_L); \text{dist}_{c_0}(u) < \varepsilon \},
$$

$$
N^0_{\varepsilon,c} = \{ u \in N_{\varepsilon,c}; M(u) = M(\tilde{Q}_c) \}.
$$

In the following lemma, to get an orthogonal condition we decompose functions in $N_{\varepsilon,c}$.

**Lemma 5.4.** Let $\varepsilon > 0$ sufficiently small. Then, there exist $K_1 > 0$, $C^2$ functions $\rho: N_{\varepsilon,c_0} \rightarrow \mathbb{R}$, $c: N_{\varepsilon,c_0} \rightarrow \mathbb{R}$, $\bar{a} = (a_1, a_2): N_{\varepsilon,c_0} \rightarrow U$ and $\eta: N_{\varepsilon,c_0} \rightarrow H^1(\mathbb{R} \times \mathbb{T}_L)$ such that for $u \in N_{\varepsilon,c_0}$

$$
u(\cdot + \rho(u), \cdot) = \Theta(\bar{a}(u), c(u))(\cdot, \cdot) + \eta(u)(\cdot, \cdot),
$$

$$
|c(u) - c_0| + |\bar{a}(u)| + \|\eta(u)\|_{H^1} \leq K_1 \text{dist}_{c_0}(u),
$$

(5.7)

and $(\eta(u), \Theta(\bar{a}(u), c(u)))_{L^2} = (\eta(u), \partial_x\Theta(\bar{a}(u), c(u)))_{L^2} = (\eta(u), \partial_{a_1}\Theta(\bar{a}(u), c(u)))_{L^2} = \eta(u), \partial_{a_2}\Theta(\bar{a}(u), c(u)))_{L^2} = 0.$

**Proof.** We define

$$
G(u, c, \rho, a_1, a_2) = \begin{pmatrix}
(u(\cdot + \rho(u), \cdot) - \Theta(\bar{a}(u), c(u)))_{L^2} \\
(u(\cdot + \rho(u), \cdot) - \Theta(\bar{a}(u), \partial_x\Theta(\bar{a}(u), c(u)))_{L^2} \\
(u(\cdot + \rho(u), \cdot) - \Theta(\bar{a}(u), \partial_{a_1}\Theta(\bar{a}(u), c(u)))_{L^2} \\
(u(\cdot + \rho(u), \cdot) - \Theta(\bar{a}(u), \partial_{a_2}\Theta(\bar{a}(u), c(u)))_{L^2}
\end{pmatrix}.
$$
Then, \( G(\tilde{Q}_{c_0}, c_0, 0, 0, 0) = 0 \). Since

\[
\frac{\partial G}{\partial(c, \rho, a_1, a_2)} \bigg|_{u=\tilde{Q}_{c_0}, \rho=a_1=a_2=0} = \begin{pmatrix}
-(\partial_c \tilde{Q}_{c_0}, \tilde{Q}_{c_0})_{L^2} & 0 & 0 \\
0 & \|\partial_c \tilde{Q}_{c_0}\|_{L^2}^2 & 0 \\
0 & 0 & -\|\tilde{Q}_{c_0} \cos \frac{\gamma}{L}\|_{L^2}^2 \\
0 & 0 & 0 & -\|\tilde{Q}_{c_0} \sin \frac{\gamma}{L}\|_{L^2}^2
\end{pmatrix}
\]

is regular, from the implicit function theorem for small \( \varepsilon > 0 \) there exists \( C^2 \) functions \( c, \rho, a_1, a_2 : N_{\varepsilon, c_0} \to \mathbb{R} \) such that for \( u \in N_{\varepsilon, c_0} \)

\[
G(u, c(u), \rho(u), a_1(u), a_2(u)) = 0.
\]

Therefore,

\[
\eta(u) = u(\cdot + \rho(u), \cdot) - \Theta(\tilde{a}(u), c(u))
\]

satisfies the orthogonal conditions, where \( \tilde{a}(u) = (a_1(u), a_2(u)) \). The inequality (5.7) follows the implicit function theorem and the definition of \( \eta \).

In the following lemma, we estimate \( \|\Theta(\tilde{a}(u), c(u)) - \Theta(\tilde{a}(u), \gamma(\tilde{a}(u)))\|_{H^1} \) on \( N^0_{\varepsilon, c_0} \).

**Lemma 5.5.** Let \( \varepsilon > 0 \) sufficiently small. There exists \( C > 0 \) such that for \( |l - c_0| < \varepsilon^{1/2} \) and \( u \in N^1_{\varepsilon, c_0} \),

\[
\|\Theta(\tilde{a}(u), \gamma_l(\tilde{a}(u))) - \Theta(\tilde{a}(u), c(u))\|_{H^1} \leq C \|\eta(u)\|_{L^2}^2,
\]

\[
|\gamma_l(\tilde{a}(u)) - c(u)| \lesssim M(\eta(u)). \tag{5.8}
\]

**Proof.** For \( u \in N^1_{\varepsilon, c_0} \),

\[
M(\Theta(\tilde{a}(u), \gamma_l(\tilde{a}(u)))) = M(\tilde{Q}_1) = M(\eta(u) + \Theta(\tilde{a}(u), c(u))) = M(\eta(u)) + M(\Theta(\tilde{a}(u), c(u))).
\]

For sufficiently small \( \varepsilon > 0 \), we have

\[
|c(u) - c_0| + |\gamma_l(\tilde{a}(u)) - c_0| < \frac{c_0}{2}.
\]

Therefore,

\[
M(\eta(u)) = M(\Theta(\tilde{a}(u), \gamma_l(\tilde{a}(u)))) = M(\Theta(\tilde{a}(u), c(u))) = (\gamma_l(\tilde{a}(u))^{3/2} - c(u)^{3/2}) M(\varphi_{c_0}(\tilde{a}(u))) \gtrsim \gamma_l(\tilde{a}(u)) - c(u) \geq 0.
\]

Since

\[
\Theta(\tilde{a}(u), \gamma_l(\tilde{a}(u))) - \Theta(\tilde{a}(u), c(u)) = (\gamma_l(\tilde{a}(u)) - c(u)) \partial_c \tilde{Q}_{c_0} + o(\gamma_l(\tilde{a}(u)) - c(u)),
\]

\[
\|\Theta(\tilde{a}(u), \gamma_l(\tilde{a}(u))) - \Theta(\tilde{a}(u), c(u))\|_{H^1} \lesssim \gamma_l(\tilde{a}(u)) - c(u) \lesssim M(\eta(u)).
\]

Next we show the coerciveness of \( S''_{c_0}(\tilde{Q}_{c_0}) \) on a subspace of \( H^1(\mathbb{R} \times \mathbb{T}_L) \).
Lemma 5.6. There exist $k_2 > 0$ and $\varepsilon_0 > 0$ such that for $a_1, a_2 \in (-\varepsilon_0, \varepsilon_0)$ and $c \in (c_0 - \varepsilon_0, c_0 + \varepsilon_0)$, if $w \in H^1(\mathbb{R} \times \mathbb{T}_L)$ satisfies
\[
(w, \Theta(\tilde{a}, c))_{L^2} = (w, \partial_x \Theta(\tilde{a}, c))_{L^2} = (w, \partial_a \Theta(\tilde{a}, c))_{L^2} = 0,
\]
then
\[
\langle S''_{c_0}(\Theta(\tilde{a}, c))w, w \rangle_{H^{-1}(\mathbb{R} \times \mathbb{T}_L), H^1(\mathbb{R} \times \mathbb{T}_L)} \geq k_2 \|w\|_{H^1}^2.
\]

Proof. By the definition of $S_{c_0}$, $S''_{c_0}(\tilde{Q}_{c_0}) = 0$. Since $\mathcal{L}_{c_0} + n^2 L^{-2}$ is positive for $|n| \geq 2$, from Lemma 5.1 we obtain that there exists $k_2 > 0$ such that for $u \in H^1(\mathbb{R} \times \mathbb{T}_L)$ satisfying
\[
(u, \tilde{Q}_{c_0})_{L^2} = (u, \partial_x \tilde{Q}_{c_0})_{L^2} = (u, \tilde{Q}_{c_0}^2 \cos \frac{y}{n})_{L^2} = (u, \tilde{Q}_{c_0}^2 \sin \frac{y}{n})_{L^2} = 0,
\]
\[
\langle S''_{c_0}(\tilde{Q}_{c_0})u, u \rangle_{H^{-1}(\mathbb{R} \times \mathbb{T}_L), H^1(\mathbb{R} \times \mathbb{T}_L)} \geq k_2' \|u\|_{H^1}^2.
\]
By a continuity argument we obtain the conclusion.

Next, we show (i) of Theorem 1.2

Proof of (i) of Theorem 1.2 Let $\varepsilon > 0$ sufficiently small. Applying Lemma 5.2, 5.6 we obtain that for $u \in N_{c_0}^{c_0}$
\[
S_{c_0}(u) - S_{c_0}(\tilde{Q}_{c_0})
= S_{c_0}(\Theta(\tilde{a}(u), c(u)) + \eta(u)) - S_{c_0}(\tilde{Q}_{c_0})
\]
\[
= S_{c_0}(\Theta(\tilde{a}(u), \gamma_{c_0}(\tilde{a}(u)))) - S_{c_0}(\tilde{Q}_{c_0})
\]
\[
+ \langle S'_{c_0}(\Theta(\tilde{a}(u), \gamma_{c_0}(\tilde{a}(u))))\eta(u) + \Theta(\tilde{a}(u), c(u)) - \Theta(\tilde{a}(u), \gamma_{c_0}(\tilde{a}(u))) \rangle_{H^{-1}, H^1}
\]
\[
+ \frac{1}{2} \langle S''_{c_0}(\Theta(\tilde{a}(u), \gamma_{c_0}(\tilde{a}(u))))\eta(u), \eta(u) \rangle_{H^{-1}, H^1} + o(\|\eta(u)\|_{H^1}^2)
\]
\[
\geq \frac{5c_0 C_{2,c_0} \|\tilde{Q}_{c_0}^2 \cos \frac{y}{n}\|_{L^2}^2 |\tilde{a}(u)|^4 + k_2 \|\eta(u)\|_{H^1}^2}{48 \|\tilde{Q}_{c_0}\|_{L^2}^2}
\]
\[
+ \langle S'_{c_0}(\Theta(\tilde{a}(u), \gamma_{c_0}(\tilde{a}(u))))\eta(u) \rangle_{H^{-1}, H^1} + o(\|\eta(u)\|_{H^1}^2 + |\tilde{a}(u)|^4).
\]
Since $S''_{c_0}(\tilde{Q}_{c_0})\partial_c \tilde{Q}_{c_0} = -\tilde{Q}_{c_0}$ and the expansion 5.4, from Lemma 5.2 we have
\[
\langle S'_{c_0}(\Theta(\tilde{a}(u), \gamma_{c_0}(\tilde{a}(u))))\eta(u) \rangle_{H^{-1}, H^1}
\]
\[
= \langle S'_{c_0}(\Theta(\tilde{a}(u), \gamma_{c_0}(\tilde{a}(u))))\eta(u) \rangle_{H^{-1}, H^1}
\]
\[
= \langle (S''_{c_0}(\tilde{a}(u)) - S''_{c_0}(\tilde{Q}_{c_0})) (\gamma_{c_0}(\tilde{a}(u)) - c_0) \partial_c \tilde{Q}_{c_0}, \eta(u) \rangle_{H^{-1}, H^1}
\]
\[
+ (\gamma_{c_0}(\tilde{a}(u)) - c_0) \langle \tilde{Q}_{c_0}, \eta(u) \rangle_{L^2} + o(|\tilde{a}(u)|^4 + \|\eta(u)\|_{H^1}^2)
\]
\[
= o(|\tilde{a}(u)|^4 + \|\eta(u)\|_{H^1}^2).
\]
Therefore, there exist $\varepsilon_*, k_* > 0$ such that for $u \in N^{c_0}_{\varepsilon_*, c_0}$
\[
S_{c_0}(u) - S_{c_0}(\tilde{Q}_{c_0}) \geq k_*(|\tilde{a}(u)|^4 + \|\eta(u)\|_{H^1}^2).
\]  (5.9)
Now we suppose there exist $\varepsilon_0 > 0$, a sequence $\{u_n\}_n$ of solutions to (1.1) and a sequence $\{t_n\}$ such that $t_n > 0$, $u_n(0) \to \tilde{Q}_{c_0}$ as $n \to \infty$ in $H^1$ and $\text{dist}_{c_0}(u_n(t_n)) > \varepsilon_0$. Let $v_n = M(\tilde{Q}_{c_0})^{-\frac{1}{2}}M(u_n)^{-\frac{1}{2}}u_n(t_n)$. Then we have $M(v_n) = M(\tilde{Q}_{c_0})$, $\lim_{n \to \infty} \|v_n - u_n(t_n)\|_{H^1} = 0$ and $\lim_{n \to \infty} S_{c_0}(v_n) = S_{c_0}(\tilde{Q}_{c_0})$. Thus, by (5.9) $\lim_{n \to \infty} \tilde{a}(v_n) = 0$ and $\eta(v_n) \to 0$ as $n \to \infty$ in $H^1$. Since $\lim_{n \to \infty} \gamma_{\tilde{Q}_{c_0}}(\tilde{a}(v_n)) = c_0$, we have $\lim_{n \to \infty} c(v_n) = c_0$. Hence, $\lim_{n \to \infty} \text{dist}_{c_0}(u_n(t_n)) = 0$. This is a contradiction. We complete the proof of (i) of Theorem 1.2.

In the following corollary, we estimates the size of the modulation parameters.

**Corollary 5.7.** Let $c_0 > 0$ and $L = \frac{2}{\sqrt{4c_0}}$. Then, there exist $\delta_0, C > 0$ such that for $0 < \delta < \delta_0$ and $u_0 \in H^1(\mathbb{R} \times \mathbb{T}_L)$ with $\|u_0 - \tilde{Q}_{c_0}\|_{H^1} < \delta$, the solution $u$ of (1.1) corresponding to the initial data $u_0$ satisfies

$$\|c(u(t)) - c_0\| + |\tilde{a}(u(t))|^2 \leq C\delta, \quad t \in \mathbb{R},$$

where $c(u)$ and $\tilde{a}(u)$ are defined in Lemma 5.3.

**Proof.** We choose $\varepsilon > 0$ which is sufficiently small. By (i) of Theorem 1.2, there exists $\delta_1 > 0$ such that for any solution $u$ with $\|u - \tilde{Q}_{c_0}\|_{H^1} = \delta < \delta_1$ satisfies $u(t) \in N_{\varepsilon, c_0}$ for $t \in \mathbb{R}$. We define $c_m > 0$ as

$$\|u_0\|_{L^2} = \|\tilde{Q}_{c_m}\|_{L^2}.$$

Applying Lemma 5.2 and 5.5 we obtain

$$S_{c_m}(u) - S_{c_m}(\tilde{Q}_{c_m}) = \frac{1}{2} \langle S_{c_0}(\Theta(\tilde{a}(u), \gamma_{c_m}(\tilde{a}(u))))\eta(u), \eta(u) \rangle_{H^{-1}, H^1} + \left(\frac{c_m}{c_0}\right) \frac{5c_0C_{2,c_0} \|\tilde{Q}_{c_0}\|_{L^2}^2}{48} |\tilde{a}(u)|^4 + \left(1 - \frac{c_m}{c_0}\right) \|\tilde{Q}_{c_0}\|_{L^2}^2 |\tilde{a}(u)|^4$$

as $\delta \to 0$. Since $|c_0 - c_m| \lesssim \delta$ and

$$\partial_y \Theta(\tilde{a}(u), \gamma_{c_m}(\tilde{a}(u)))(x,y) = -\frac{a_1(u)\gamma_{c_m}(\tilde{a}(u))}{c_0L} \tilde{Q}_{c_0}^{\frac{3}{2}} \left(\frac{\gamma_{c_m}(\tilde{a}(u))}{c_0}\right) \sin \frac{y}{L}$$

$$+ \frac{a_2(u)\gamma_{c_m}(\tilde{a}(u))}{c_0L} \tilde{Q}_{c_0}^{\frac{3}{2}} \left(\frac{\gamma_{c_m}(\tilde{a}(u))}{c_0}\right) \cos \frac{y}{L} + O(|\tilde{a}(u)|^2),$$

there exist $k_3, k_4 > 0$ such that $k_3$ and $k_4$ are not depend on $c_m$, and

$$S_{c_m}(u) - S_{c_m}(\tilde{Q}_{c_m}) \geq k_3 \|\eta(u)\|_{H^1}^2 + k_3 |\tilde{a}(u)|^2 (|\tilde{a}(u)|^2 - \delta k_4 + O(|\tilde{a}(u)|^4 + \|\eta(u)\|_{H^1}^2),$$

(5.10)
Using the conservation laws and (5.7), we obtain
\[
S_{cm}(u) - S_{cm}(\tilde{Q}_{cm}) = S_{cm}(u_0) - S_{cm}(\tilde{Q}_{cm}) \\
\lesssim \|\eta(u_0)\|_{H^1}^2 + |\tilde{a}(u_0)|^2 \lesssim \delta^2.
\]
From (5.10) and (5.11), we have that there exist \(\delta_*, k_5 > 0\) such that if \(0 < \delta < \delta_*\), then
\[
\|\eta(u)\|_{H^1}^2 + |\tilde{a}(u)|^2(|\tilde{a}(u)|^2 - \delta k_4) - k_5\delta^2 \leq 0.
\]
Therefore, there exists \(C(k_4, k_5) > 0\) such that
\[
|\tilde{a}(u)|^2 + \|\eta(u)\|_{H^1} \leq C(k_4, k_5)\delta.
\]
Applying (5.8), we have
\[
|c_0 - c(u)| \lesssim \|\eta(u)\|_{L^2}^2 + |\gamma_{cm}(\tilde{a}(u)) - c_m| + |c_0 - c_m| \lesssim \delta.
\]

\section{Liouville property}

In this section, we prove the Liouville property of (1.1). First, we show the following equation of the integration of \(Q_c\).

\begin{lemma}
Let \(p, c > 0\). Then, we have
\[
\int_{\mathbb{R}} Q_c^{p+1} dx = \frac{3pc}{2p + 1} \int_{\mathbb{R}} Q_c^p dx.
\]
\end{lemma}

\begin{proof}
Since
\[
-\partial_x^2 Q_c + cQ_c - Q_c^2 = 0,
\]
we have
\[
\int_{\mathbb{R}} Q_c^{p+1} dx = -\int_{\mathbb{R}} Q_c^{p-1} \partial_x^2 Q_c dx + c \int_{\mathbb{R}} Q_c^p dx \\
=(p - 1) \int_{\mathbb{R}} Q_c^{p-2} (\partial_x Q_c)^2 dx + c \int_{\mathbb{R}} Q_c^p dx.
\]
Multiplying (6.2) by \(\partial_x Q_c\) and integrating this, we obtain
\[
-(\partial_x Q_c)^2 + cQ_c^2 - \frac{2}{3}Q_c^3 = 0.
\]
Thus,
\[
\int_{\mathbb{R}} Q_c^{p+1} dx = (p - 1) \int_{\mathbb{R}} Q_c^{p-2} \left(cQ_c^2 - \frac{2}{3}Q_c^3\right) dx + c \int_{\mathbb{R}} Q_c^p dx
\]
\end{proof}
which implies (6.1).

Let
\[ \phi_c(x) = -\frac{\partial_x Q_c(x)}{Q_c(x)} = \sqrt{c} \tanh \frac{\sqrt{c} x}{2}. \]

Then, \( \phi_c(x) \rightarrow \pm \sqrt{c} \) as \( x \rightarrow \pm \infty \) and
\[ \partial_x \phi_c(x) = \frac{c}{2} \cosh^{-2} \frac{\sqrt{c} x}{2} = \frac{1}{3} Q_c. \]

We introduce the following coerciveness type lemma in [27].

**Lemma 6.2.** For \( u \in H^1(\mathbb{R}) \)
\[ -\int_{\mathbb{R}} \partial_x u \mathcal{L}_c(u \phi_c) dx = \frac{3}{2} \int_{\mathbb{R}} \left( \partial_x \left( \frac{u}{Q_c} \right) \right)^2 Q_c^2 \partial_x \phi_c dx \]
\[ \geq \frac{5c}{8} \left( \int_{\mathbb{R}} |u|^2 \partial_x \phi_c dx - \|Q_c\|_{L^3(\mathbb{R})}^{-3} \left( \int_{\mathbb{R}} u Q_c^2 dx \right)^2 \right) \]

**Proof.** Let \( v = \frac{u}{Q_c} \). Since
\[ \mathcal{L}_c(u \phi_c) = \mathcal{L}_c(v \partial_x Q_c) = -2 \partial_x v \partial_x^2 Q_c - \partial_x^2 v \partial_x Q_c, \]
we have
\[ -\int_{\mathbb{R}} \partial_x u \mathcal{L}_c(u \phi_c) dx = \int_{\mathbb{R}} \partial_x (Q_c v) \mathcal{L}_c(v \partial_x Q_c) dx = \frac{1}{2} \int_{\mathbb{R}} (\partial_x v)^2 Q_c^3 dx \]
Let \( w = v Q_c^3 \). Using
\[ \partial_x^2 Q_c = c Q_c - Q_c^2, \quad (\partial_x Q_c)^2 = c Q_c^2 - \frac{2}{3} Q_c^3, \]
we obtain that
\[ \frac{1}{2} \int_{\mathbb{R}} (\partial_x v)^2 Q_c^3 dx = \frac{1}{2} \int_{\mathbb{R}} w \left( -\partial_x^2 w + \frac{3}{2} \partial_x^2 Q_c Q_c^{-1} w + \frac{3}{4} (\partial_x Q_c)^2 Q^{-2} w \right) dx \]
\[ = \frac{1}{2} \int_{\mathbb{R}} w \left( \mathcal{L}_c + \frac{5c}{4} \right) w dx \]
From the properties of \( \mathcal{L}_c \) the operator \( \mathcal{L}_c + \frac{5c}{4} \) is non-negative and the kernel of \( \mathcal{L}_c + \frac{5c}{4} \) is spanned by \( Q_c^3 \). Moreover, the second eigenvalue of \( \mathcal{L}_c + \frac{5c}{4} \) is \( \frac{5c}{4} \). Therefore, we have
\[ \int_{\mathbb{R}} w \left( \mathcal{L}_c + \frac{5c}{4} \right) w dx \geq \frac{5c}{4} \left( \|w\|_{L^2}^2 - \|Q_c\|_{L^3(\mathbb{R})}^{-3} \left( \int_{\mathbb{R}} w Q_c^2 dx \right)^2 \right) \]
\[ = \frac{5c}{4} \left( \int_{\mathbb{R}} |u|^2 \partial_x \phi_c dx - \|Q_c\|_{L^3(\mathbb{R})}^{-3} \left( \int_{\mathbb{R}} u Q_c^2 dx \right)^2 \right) \]
6.1 Monotonicity properties

In this subsection, we show the monotonicity properties of (1.1). By Proposition 2.2, the equation (1.1) has the Kato type local smoothing effect. Therefore, the proof of the monotonicity properties is similar to one in [27, 5]. Thus, we omit the detail of proofs in this subsection, see Section 3 in [5].

We define \( \psi_R \in C^\infty(\mathbb{R}, \mathbb{R}) \) by

\[
\psi_R(x) = \frac{2}{\pi} \arctan(e^{x/R}), \quad x \in \mathbb{R}.
\]

Then, we have \( \lim_{x \to \infty} \psi_R(x) = 1, \lim_{x \to -\infty} \psi_R(x) = 0, \)

\[
\partial_x \psi_R(x) = \frac{1}{\pi R \cosh(x/R)} \quad \text{and} \quad |\partial_x^3 \psi_R(x)| \leq \frac{1}{R^2} \partial_x \psi_R(x).
\]

Let \( \varepsilon, \beta, c_0 > 0 \) and \( u \) be a solution to (1.1) satisfying that there exists \( \rho \in C(\mathbb{R}, \mathbb{R}) \) such that

\[
\left\| u(t, \cdot, \cdot) - \tilde{Q}_c(\cdot - \rho(t), \cdot) \right\|_{H^1} < \varepsilon_0, \quad t \in \mathbb{R}
\]

and

\[
|\dot{\rho}(t) - c_0| \leq c_0/2, \quad t \in \mathbb{R}.
\]

For \( x_0, t_0, t \in \mathbb{R} \) we define

\[
\tilde{x} = \tilde{x}(x_0, t_0, t) = x - \rho(t_0) + \frac{\beta(t_0 - t)}{2} - x_0,
\]

\[
\tilde{x}_- = \tilde{x}(-x_0, t, t_0),
\]

\[
I_{x_0, t_0}(u(t)) = \int_{\mathbb{R} \times \mathbb{T}_L} |u(t, x, y)|^2 \psi_R(\tilde{x}(t))dxdy,
\]

and

\[
I_{-x_0, t_0}(u(t)) = \int_{\mathbb{R} \times \mathbb{T}_L} |u(t, x, y)|^2 \psi_R(\tilde{x}_-(t))dxdy.
\]

In the following lemma, we show the property of the parameter \( \rho \) (see Lemma 3.2 in [5]).

Lemma 6.3. Assume that \( u \in C(\mathbb{R}, H^1(\mathbb{R} \times \mathbb{T}_L)) \) is a solution to (1.1) satisfying (6.5), (6.6) and that there exist \( \tilde{\rho} \in C(\mathbb{R}, \mathbb{R}) \) and \( C, \delta_0 > 0 \) such that

\[
\int_{\mathbb{T}_L} |u(t, x + \tilde{\rho}(t), y)|^2 dy \leq Ce^{-\delta_0|x|}, \quad (t, x) \in \mathbb{R}^2.
\]

If \( 0 < \varepsilon_0 < \frac{1}{2} ||\tilde{Q}_{c_0}||_{L^2(|x| \leq 1)} \), then \( u \) satisfies

\[
\int_{\mathbb{T}_L} |u(t, x + \rho(t), y)|^2 dy \lesssim e^{-\delta_0|x|}, \quad (t, x) \in \mathbb{R}^2,
\]

where

\[
\|u\|^2_{L^2(|x| \leq R)} = \int_{|x| \leq R} |u|^2 dxdy.
\]
The following two lemmas show the $L^2$-monotonicity property of (1.1).

**Lemma 6.4.** Let $0 < \beta < c_0/2$. Assume that $u \in C(\mathbb{R}, H^1(\mathbb{R} \times T_L))$ is a solution to (1.1) satisfying (6.5) and (6.6). Then, for $x_0 > 0, t_0 \in \mathbb{R}$, $R \geq 2/\sqrt{\beta}$ and $t \leq t_0$

$$I_{x_0,t_0}(u(t)) - I_{x_0,t_0}(u(t_0)) \lesssim e^{-x_0/R},$$

(6.9) if $\varepsilon > 0$ in (6.5) is chosen small enough. Moreover, if $u$ satisfies the decay assumption (6.8), then

$$\int_{\mathbb{R} \times T_L} |u(t_0, x, y)|^2 \psi_R(\bar{x}(t_0))dxdy$$

$$+ \int_{-\infty}^{t_0} \int_{\mathbb{R} \times T_L} (|\nabla u|^2 + |u|^2)(t, x, y)\partial_x \psi_R(\bar{x}(t))dxdydt \lesssim e^{-x_0/R}. (6.10)$$

**Lemma 6.5.** Let $0 < \beta < c_0/2$. Assume that $u \in C(\mathbb{R}, H^1(\mathbb{R} \times T_L))$ is a solution to (1.1) satisfying (6.5) and (6.6). Then, for $x_0 > 0, t_0 \in \mathbb{R}$, $R \geq 2/\sqrt{\beta}$ and $t \geq t_0$

$$I_{x_0,t_0}^-(u(t)) - I_{x_0,t_0}^-(u(t_0)) \lesssim e^{-x_0/R},$$

(6.11) if $\varepsilon > 0$ in (6.5) is chosen small enough.

The proof of Lemma 6.4 follows the proof of Lemma 3.3 in [5]. The proof of Lemma 6.5 is similar to the proof of Lemma 4.9 in [5].

We define a functional $J$ by

$$J_{x_0,t_0}(u(t)) = \int_{\mathbb{R} \times T_L} (|\nabla u|^2 - \frac{2}{3} u^3)(t, x, y)\psi_R(\bar{x})dxdy.$$

In the following lemma, we show the monotonicity property for $J$.

**Lemma 6.6.** Let $0 < \beta < c_0/2$. Assume that $u \in C(\mathbb{R}, H^1(\mathbb{R} \times T_L))$ is a solution to (1.1) satisfying (6.5) and (6.6). Then, for $x_0 > 0, t_0 \in \mathbb{R}$, $R \geq 2/\sqrt{\beta}$ and $t \leq t_0$

$$J_{x_0,t_0}(u(t_0)) - J_{x_0,t_0}(u(t)) \lesssim e^{-x_0/R}. (6.12)$$

Moreover, if $u$ satisfies the decay assumption (6.8), then

$$\int_{\mathbb{R} \times T_L} |\nabla u|^2(t_0)\psi_R(\bar{x}(t_0))dxdy$$

$$+ \int_{-\infty}^{t_0} \int_{\mathbb{R} \times T_L} (|\nabla^2 u(t)|^2 + |\nabla u(t)|^2 + u(t)^4)(\partial_x \psi_R)(\bar{x}(t))dxdydt \lesssim e^{-x_0/R}. (6.14)$$

The proof of Lemma 6.6 is similar to the proof of Lemma 3.4 in [5].

The following proposition shows the boundedness of higher Sobolev norm of solutions satisfying the decay assumption (6.8).
Proposition 6.7. Let $0 < \beta < c_0/2$ and $k \in \mathbb{Z}_+$. Assume that $u \in C(\mathbb{R}, H^1(\mathbb{R} \times T_L))$ is a solution to (1.1) satisfying (6.5), (6.6) and the decay assumption (6.8). If $\varepsilon > 0$ in (6.5) is sufficiently small, there exist $\delta, C = C(k) > 0$ such that

$$\sup_{t \in \mathbb{R}} \int_{\mathbb{R} \times T_L} (\partial^\alpha u(x, t, y))^2 dx dy \leq C,$$

for $\alpha \in (\mathbb{N}_0)^2$ satisfying $|\alpha| \leq k$.

The proof of this proposition is same as the proof of Corollary 3.9 in [5].

6.2 Critical case $L = \frac{2}{\sqrt{\delta c_0}}$

In this section, we show the Liouville property for $L = \frac{2}{\sqrt{\delta c_0}}$.

Lemma 6.8. There exist $\varepsilon_0, K_0 > 0$ such that for any $0 < \varepsilon < \varepsilon_0$ the following is true. For any solution $u \in C(\mathbb{R}, H^1(\mathbb{R} \times T_L))$ of (1.1) satisfying

$$\inf_{b \in \mathbb{R}} \|u(t, \cdot, \cdot) - Q_{c_0}(\cdot - b, \cdot)\|_{H^1} \leq \varepsilon$$

there exist $\tilde{a} = (a_1, a_2) \in C^1(\mathbb{R}, \mathbb{R}^2)$ and $\rho, c \in C^1(\mathbb{R}, \mathbb{R})$ uniquely such that

$$\eta(t, x, y) = u(t, x + \rho(t), y) - \Theta(\tilde{a}(t), c(t))$$

satisfies for all $t \in \mathbb{R}$

$$|c(t) - c_0| + |a_1(t)| + |a_2(t)| + \|\eta(t)\|_{H^1} \leq K_0 \varepsilon,$$  (6.17)

$$\int_{\mathbb{R} \times T_L} \eta(t) \partial_x \Theta(\tilde{a}(t), c(t)) dx dy = \int_{\mathbb{R} \times T_L} \eta(t) \Theta(\tilde{a}(t), c(t)) dx dy$$

$$= \int_{\mathbb{R} \times T_L} \eta(t) \Theta(\tilde{a}(t), c(t))^2 \frac{x^2}{L} dx dy = \int_{\mathbb{R} \times T_L} \eta(t) \Theta(\tilde{a}(t), c(t))^2 \frac{y^2}{L} dx dy = 0$$  (6.18)

and

$$|\hat{a}(t)| \leq K_0 \|\eta(t)\|_{L^2},$$  (6.19)

$$|\hat{c}(t)| \leq \varepsilon K_0 \|\eta(t)\|_{L^2}.$$  (6.20)

$$|\hat{\rho}(t) - \hat{c}(t)| \leq K_0 (\|\eta(t)\|_{L^2} + |c - c_0| |\tilde{a}|),$$  (6.21)

where $\hat{c}(t) = c_0^{-1}(c(t) \hat{c}(\tilde{a}(t)))$.

Proof. From Lemma 5.4, there exist $C^1$ mappings $\rho(t) = \rho(u(t)), c(t) = c(u(t)), \tilde{a}(t) = \tilde{a}(u(t)), \eta(t) = \eta(u(t))$ satisfying (6.16)–(6.18). By the calculation we have

$$\eta_t = \partial_x (-\Delta \eta - 2 \Theta \eta - \eta^2 - \Delta \Theta - \Theta^2) + \hat{\rho} \partial_x (\eta + \Theta) - \hat{a} \cdot \partial_x \Theta - \hat{c} \partial_x \Theta$$

$$= \partial_x \left(L_{c_0} \eta - \eta^2 + \partial_x S_{c_0}'(\Theta) - 2 \partial_x ((\Theta - \hat{Q}_{c_0}) \eta) + (\hat{\rho} - c_0) \partial_x (\eta + \Theta) - \hat{a} \cdot \partial_x \Theta - \hat{c} \partial_x \Theta\right),$$  (6.22)
where \( S'_{c_0}(\Theta) = -\Delta \Theta + c_0 \Theta - \Theta^2 \) and \( \hat{a} \cdot \partial_a \Theta = \hat{a}_1 \partial_{a_1} \Theta + \hat{a}_2 \partial_{a_2} \Theta \). From (6.18) and \( \Theta(x, y) = \Theta(-x, y) \) we obtain that

\[
0 = \frac{d}{dt} \int_{\mathbb{R} \times T_L} \eta \Theta dx dy
\]

\[
= - \int_{\mathbb{R} \times T_L} \Theta \hat{a} \cdot \partial_a \Theta dx dy - \hat{c} \int_{\mathbb{R} \times T_L} \tilde{Q}_{c_0} \partial_c \tilde{Q}_{c_0} dx dy
\]

\[
+ O(\|\eta\|_{L^2}(\|\eta\|_{L^2} + |\tilde{a}| + |c - c_0| + |\hat{a}| + |\hat{c}|)).
\]

By the expansion

\[
\Theta(\tilde{a}, c) = \tilde{Q}_{c_0} + O(|\tilde{a}| + |c - c_0|), \quad \hat{a} \cdot \partial_a \Theta(\tilde{a}, c) = \hat{a}_1 \tilde{Q}_{c_0}^2 \cos \frac{y}{L} + \hat{a}_2 \tilde{Q}_{c_0}^2 \sin \frac{y}{L} + O((|\tilde{a}| + |c - c_0|)|\tilde{a}|),
\]

we have

\[
\int_{\mathbb{R} \times T_L} \Theta \hat{a} \cdot \partial_a \Theta dx dy = O(|\tilde{a}| + |c - c_0|)|\tilde{a}|).
\]

Since \( \int_{\mathbb{R} \times T_L} \tilde{Q}_{c_0} \partial_c \tilde{Q}_{c_0} dx dy \neq 0 \),

\[
|\hat{c}| = O(\|\eta\|_{L^2}(\|\eta\|_{L^2} + |\tilde{a}| + |c - c_0| + |\hat{a}| + (|\tilde{a}| + |c - c_0|)|\tilde{a}|)). \tag{6.23}
\]

From (6.18), (6.23) and \( \Theta(x, y) = \Theta(-x, y) \), we obtain that

\[
0 = \frac{d}{dt} \int_{\mathbb{R} \times T_L} \eta \Theta^2 \cos \frac{y}{L} dx dy
\]

\[
= - \hat{a}_1 \int_{\mathbb{R} \times T_L} \left( \Theta^2 \cos \frac{y}{L} \right) \partial_{a_1} \Theta dx dy + O(\|\eta\|_{L^2} + (|\tilde{a}| + |c - c_0|)|\tilde{a}|).
\]

Since

\[
\int_{\mathbb{R} \times T_L} \left( \Theta^2 \cos \frac{y}{L} \right) \partial_{a_1} \Theta dx dy = \int_{\mathbb{R} \times T_L} \left( \tilde{Q}_{c_0}^2 \cos \frac{y}{L} \right)^2 dx dy + O(|\tilde{a}| + |c - c_0|),
\]

we obtain

\[
|\hat{a}_1| = O(\|\eta\|_{L^2} + (|\tilde{a}| + |c - c_0|)|\tilde{a}_2|). \tag{6.24}
\]

By the same way, from (6.23) and (6.24) we get

\[
|\hat{a}_2| = O(\|\eta\|_{L^2}). \tag{6.25}
\]

The estimates (6.19) and (6.20) follow (6.23)–(6.24). By the similar computation to (6.22),

\[
\eta_t = \partial_x (\mathbb{L} \eta - \eta^2) + \partial_x S'_{c_0}(\Theta) - 2 \partial_x ((\Theta - \tilde{Q}_{c_0}) \eta) + (\hat{\rho} - \hat{c}) \partial_x (\eta + \Theta) - \hat{a} \cdot \partial_a \Theta - \hat{c} \partial_c \Theta.
\]

By the definition of \( \Theta \) and \( \hat{c} \) we have

\[
S'_{c_0}(\Theta) = \frac{c^2}{c_0} \frac{\partial^2}{\partial y^2} \varphi_{c_0} - \varphi_{c_0}^2 + \frac{c(c - c_0)}{c_0^2} \partial_y \varphi_{c_0}
\]

\[
= \frac{c - c_0}{c_0} \partial_y^2 \Theta. \tag{6.27}
\]
Since
\[ S'_c(\Theta) = \frac{c-c_0}{c_0} \frac{\partial^2 P}{\partial y^2} = O(|c_0 - c||\vec{a}|), \tag{6.28} \]
from (6.18), (6.23)–(6.25), we obtain that
\[
0 = \frac{d}{dt} \int_{\mathbb{R} \times T_L} \eta(x,\Theta) dx dy
= (\dot{\rho} - \dot{\hat{c}}) \int_{\mathbb{R} \times T_L} (\partial_x \tilde{Q}_c) dx dy + O(\|\eta\|_{L^2} + |c_0 - c||\vec{a}|). \tag{6.29}
\]
Thus, the estimate (6.21) holds.

Next we prove the following Liouville type theorem.

**Theorem 6.9.** Let $c_0 > 0$ and $L = \frac{2}{\sqrt{|c_0|}}$. There exists $\varepsilon_0 > 0$ satisfies the following. For any solution $u \in C(\mathbb{R}, H^1(\mathbb{R} \times T_L))$ to (1.1) satisfying (6.5) and (6.8), there exist $c_+ > 0, \vec{a}_+ = (a_{1,+}, a_{2,+})$ and $\rho_0$ such that
\[
u(t, x, y) = \Theta(\vec{a}_+, c_+) (x - \hat{\dot{c}}_+ t + \rho_0, y),
\]
where
\[
\hat{\dot{c}}_+ = \begin{cases} c_+, & a_+ = (0, 0), \\
\hat{\dot{c}}(\vec{a}_+), & c_+ = c_0. \end{cases}
\]

**Proof.** Let $u \in C(\mathbb{R}, H^1(\mathbb{R} \times T_L))$ be solution to (1.1) satisfying (6.5) and (6.8). From Lemma 6.8, $\rho$ in Lemma 6.8 satisfies (6.5) and (6.8). Let $\eta(t), c(t), \vec{a}(t), \hat{\dot{c}}(t)$ be in Lemma 6.8. We define
\[
v = \mathbb{L}_c \eta - \eta^2.
\]
Then, $v$ has the following almost orthogonal condition.
\[
\int_{\mathbb{R} \times T_L} v \partial_x \tilde{Q}_c dx dy = \int_{\mathbb{R} \times T_L} (\mathbb{L}_c \eta - \eta^2) \partial_x \tilde{Q}_c dx dy = O(\|\eta\|_{L^2}^2), \tag{6.29}
\]
\[
\int_{\mathbb{R} \times T_L} v \partial_t \tilde{Q}_c dx dy = - \int_{\mathbb{R} \times T_L} \eta \tilde{Q}_c dx dy + O(\|\eta\|_{L^2}^2) = O(\|\eta\|_{L^2}(|c - c_0| + |\vec{a}| + \|\eta\|_{L^2})), \tag{6.30}
\]
\[
\int_{\mathbb{R} \times T_L} v \tilde{Q}_c^2 \cos \frac{y}{L} dx dy = \int_{\mathbb{R} \times T_L} \eta \mathbb{L}_c \left( \tilde{Q}_c^2 \cos \frac{y}{L} \right) dx dy + O(\|\eta\|_{L^2}^2)
= O(\|\eta\|_{L^2}(|c - c_0| + |\vec{a}| + \|\eta\|_{L^2})), \tag{6.31}
\]

31
\[
\int_{\mathbb{R} \times T_L} v\tilde{Q}_c^2 \sin \frac{y}{L} dxdy = O(||\eta||_{L^2}(1 + |c - c_0| + |\tilde{a}| + ||\eta||_{L^2})). \tag{6.32}
\]

From the orthogonal conditions (6.18) and Lemma 5.6, we have
\[
(v, \eta)_{L^2(\mathbb{R} \times T_L)} = (\mathbb{L}_c \eta, \eta)_{L^2(\mathbb{R} \times T_L)} - ||\eta||_{L^3(\mathbb{R} \times T_L)}^3 \geq k_2 ||\eta||_{H^1}^2 + O(||\eta||_{H^1}^3).
\]

Therefore, if \( \varepsilon_0 > 0 \) is sufficiently small, then for \( t \in \mathbb{R} \)
\[
||\eta||_{H^1} \lesssim ||v||_{L^2}. \tag{6.33}
\]

By (6.26), we have
\[
v_t = \mathbb{L}_c \eta_t + (\partial_t \mathbb{L}_c) \eta - 2\eta \mathbb{L}_c v + \mathbb{L}_c \partial_x S'_c(\Theta) + R(\eta, \tilde{a}, c), \tag{6.34}
\]

where
\[
R(\eta, \tilde{a}, c) = -2\eta \partial_x(v + S'_c(\Theta)) + (\tilde{\rho} - \tilde{c}) \mathbb{L}_c \partial_x(\eta + \Theta) - 2(\tilde{\rho} - \tilde{c}) \eta \partial_x(\eta + \Theta)
\]
\[
+ (\mathbb{L}_c - 2\eta)(2\partial_x((\tilde{Q}_c - \Theta) \eta) - \tilde{a} \cdot \partial_\Theta - \tilde{c} \partial_c(\Theta) + (\partial_\mathbb{L}_c) \eta).
\]

Therefore,
\[
-\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R} \times T_L} (v + S'_c(\Theta))^2 \phi_c dxdy
\]
\[
= -\int_{\mathbb{R} \times T_L} (\mathbb{L}_c \partial_x v) \phi_c dxdy - \int_{\mathbb{R} \times T_L} (\mathbb{L}_c \partial_x S'_c(\Theta)) \phi_c dxdy - \int_{\mathbb{R} \times T_L} (\mathbb{L}_c \partial_x v) S'_c(\Theta) \phi_c dxdy
\]
\[
- \int_{\mathbb{R} \times T_L} (\mathbb{L}_c \partial_x S'_c(\Theta)) S'_c(\Theta) \phi_c dxdy - \int_{\mathbb{R} \times T_L} R(\eta, \tilde{a}, c)(v + S'_c(\Theta)) \phi_c dxdy
\]
\[
- \int_{\mathbb{R} \times T_L} (\partial_t S'_c(\Theta))(v + S'_c(\Theta)) \phi_c dxdy - \frac{1}{2} \int_{\mathbb{R} \times T_L} (v + S'_c(\Theta))^2 \partial_c \phi_c dxdy. \tag{6.35}
\]

We estimate each term in (6.35) separately.

(1) The estimate of \(-\int_{\mathbb{R} \times T_L} (\mathbb{L}_c \partial_x v) \phi_c dxdy\). From the Fourier expansion \( v(t, x, y) = v_0(t, x) + \sum_{n=1}^{\infty} (v_{n,1}(t, x) \cos \frac{n}{L} + v_{n,2}(t, x) \sin \frac{n}{L}) \) and Lemma 6.2 we have
\[
-\int_{\mathbb{R} \times T_L} (\mathbb{L}_c \partial_x v) \phi_c dxdy
\]
\[
= -2\pi L \int_{\mathbb{R} \times T_L} (\mathbb{L}_c \partial_x v_0) \phi_c dxdy - \pi L \sum_{n \in \mathbb{Z}, j = 1, 2} \left( \left( L_0 - \frac{n^2}{L^2} \right) \partial_x v_{n,j} \right) v_{n,j} \phi_c dxdy
\]
\[
\leq \frac{5\pi L\mathcal{C}}{4} \left( \int_{\mathbb{R}} |v_0|^2 Q_c dx - ||Q_c||_{L^3(\mathbb{R})}^3 \left( \int_{\mathbb{R}} v_0 Q_c^2 dx \right)^2 \right)
\]
\[
+ \pi L \left( \frac{5\mathcal{C}}{8} + \frac{1}{6L^2} \right) \int_{\mathbb{R}} (|v_{1,1}|^2 + |v_{1,2}|^2) Q_c dx - \frac{5\mathcal{C}}{8} ||Q_c||_{L^3(\mathbb{R})}^3 \left( \int_{\mathbb{R}} v_{1,1} Q_c^2 dx \right)^2
\]
\[
- \frac{5\mathcal{C}}{8} ||Q_c||_{L^3(\mathbb{R})}^3 \left( \int_{\mathbb{R}} v_{1,2} Q_c^2 dx \right)^2
\]
\[
+ \pi L \sum_{n=2}^{\infty} \frac{n^2}{6L^2} \int_{\mathbb{R}} (|v_{n,1}|^2 + |v_{n,2}|^2) Q_c dx. \tag{6.36}
\]
From H"older's inequality, (6.39) and (6.40), we obtain

\[ \left| \int_{\mathbb{R}} v_0 Q_{c_0}^2 dx \right|^2 \leq \int_{\mathbb{R}} v_0 \left( Q_{c_0}^2 - \| Q_{c_0} \|_{L^2(\mathbb{R})}^{-2} \int_{\mathbb{R}} Q_{c_0} \varphi_{c_0} dx' \varphi_{c_0} \right) dx \]
\[ + O((|\bar{a}| + |c - c_0| + \| \eta \|_{L^2}) \| \eta \|_{L^2} \| v \|_{L^2}) \]
\[ \leq \left\| v_0 Q_{c_0}^2 \right\|_{L^2(\mathbb{R})}^2 \left\| Q_{c_0} \right\|_{L^2(\mathbb{R})}^{-2} \int_{\mathbb{R}} Q_{c_0} \varphi_{c_0} dx' \varphi_{c_0} \left\| Q_{c_0} \varphi_{c_0} \right\|_{L^2(\mathbb{R})}^2 \]
\[ + O((|\bar{a}| + |c - c_0| + \| \eta \|_{L^2}) \| \eta \|_{L^2} \| v \|_{L^2}) \]

(6.37)

Since

\[ \left\| Q_{c_0}^{\frac{1}{2}} - \left\| Q_{c_0} \varphi_{c_0} \right\|_{L^2(\mathbb{R})}^{-2} \int_{\mathbb{R}} Q_{c_0} \varphi_{c_0} dx' \varphi_{c_0} \left\| Q_{c_0} \varphi_{c_0} \right\|_{L^2(\mathbb{R})}^2 \] < \| Q_{c_0} \|_{L^3(\mathbb{R})}^3 \]

from (6.37) we obtain

\[ \int_{\mathbb{R}} |v_0|^2 Q_{c_0} dx - \| Q_{c_0} \|_{L^3(\mathbb{R})}^{-3} \left( \int_{\mathbb{R}} v_0 Q_{c_0}^2 dx \right)^2 \]
\[ \geq \int_{\mathbb{R}} |v_0|^2 Q_{c_0} dx + O((|\bar{a}| + |c - c_0| + \| \eta \|_{L^2}) \| \eta \|_{L^2} \| v \|_{L^2}) \]

(6.38)

On the other hand, from the almost orthogonal conditions (6.31) and (6.32)

\[ \left| \int_{\mathbb{R}} v_{1,j} Q_{c_0}^2 dx \right|^2 = \int_{\mathbb{R}} v_{1,j} \left( Q_{c_0}^{\frac{1}{2}} - \left\| Q_{c_0} \right\|_{L^2(\mathbb{R})}^{-2} \int_{\mathbb{R}} Q_{c_0}^{\frac{1}{2}} dx' Q_{c_0} \right) \frac{Q_{c_0}^{\frac{1}{2}}}{Q_{c_0}^{\frac{1}{2}}} dx \]
\[ + O((|\bar{a}| + |c - c_0| + \| \eta \|_{L^2}) \| \eta \|_{L^2} \| v \|_{L^2}) \]
\[ \leq \left\| Q_{c_0}^{\frac{1}{2}} - \left\| Q_{c_0} \right\|_{L^2(\mathbb{R})}^{-2} \int_{\mathbb{R}} Q_{c_0}^{\frac{1}{2}} dx' Q_{c_0} \right\|_{L^2(\mathbb{R})}^2 \int_{\mathbb{R}} |v_{1,j}|^2 Q_{c_0} dx \]
\[ + O((|\bar{a}| + |c - c_0| + \| \eta \|_{L^2}) \| \eta \|_{L^2} \| v \|_{L^2}) \]

(6.39)

By Lemma 6.1

\[ \int_{\mathbb{R}} Q_{c_0}^{\frac{1}{2}} dx = \frac{5c_0}{4} \int_{\mathbb{R}} Q_{c_0}^{\frac{1}{2}} dx \int_{\mathbb{R}} Q_{c_0}^{\frac{1}{2}} dx = \frac{9c_0}{8} \int_{\mathbb{R}} Q_{c_0}^{\frac{1}{2}} dx. \]

(6.40)

From H"older's inequality, (6.39) and (6.40), we obtain

\[ \int_{\mathbb{R}} |v_{1,j}|^2 Q_{c_0} dx - \| Q_{c_0} \|_{L^3(\mathbb{R})}^{-3} \left( \int_{\mathbb{R}} v_{1,j} Q_{c_0}^2 dx \right)^2 \]
\[ \geq \left\| Q_{c_0} \right\|_{L^2(\mathbb{R})}^{-2} \left\| Q_{c_0} \right\|_{L^2(\mathbb{R})}^{-1} \left\| Q_{c_0} \right\|_{L^2(\mathbb{R})}^{-1} \left( \int_{\mathbb{R}} Q_{c_0}^{\frac{1}{2}} dx \right)^2 \int_{\mathbb{R}} |v_{1,j}|^2 Q_{c_0} dx \]
\[ + O((|\bar{a}| + |c - c_0| + \| \eta \|_{L^2}) \| \eta \|_{L^2} \| v \|_{L^2}) \]
\[ = \sqrt{\frac{9}{10}} \int_{\mathbb{R}} |v_{1,j}|^2 Q_{c_0} dx + O((|\bar{a}| + |c - c_0| + \| \eta \|_{L^2}) \| \eta \|_{L^2} \| v \|_{L^2}). \]

(6.41)
By (6.37), (6.38) and (6.41), we obtain that there exists $k_3 > 0$ such that

\[- \int_{\mathbb{R}^2} (\mathbb{L} \partial_x v) v \phi_c dxdy \geq k_3 \int_{\mathbb{R}} |v_0|^2 Q_{c_0} dx + \pi L \left( \frac{5c_0}{8} \sqrt{\frac{3}{10}} + \frac{1}{6L^2} \right) \int_{\mathbb{R}} (|v_{1,1}|^2 + |v_{1,2}|^2) Q_{c_0} dx \]

\[+ \pi L \sum_{n=2}^{\infty} \frac{n^2}{6L^2} \int_{\mathbb{R}} (|v_{n,1}|^2 + |v_{n,2}|^2) Q_{c_0} dx \]

\[+ O(|\bar{a}| + |c - c_0| + \|\eta\|_{L^2}) \|\eta\|_{L^2}^2). \]  

(II) The estimate of $- \int_{\mathbb{R}^2} (\mathbb{L} \partial_x S_c' (\Theta)) S_{c'} (\Theta) \phi_c dxdy$. Since

\[S_c' (\Theta) = \frac{c - c_0}{c_0} \left( a_1 Q_{c_0}^{\frac{3}{2}} \cos \frac{y}{L} + a_2 Q_{c_0}^{\frac{5}{2}} \sin \frac{y}{L} \right) + O(|c - c_0| + |\bar{a}|) |c - c_0| |\bar{a}|), \]  

we have

\[- \int_{\mathbb{R}^2} (\mathbb{L} \partial_x S_c' (\Theta)) S_{c'} (\Theta) \phi_c dxdy \]

\[= - \frac{\pi L (c - c_0)^2 |\bar{a}|^2}{c_0^2} \int_{\mathbb{R}} \left( (\mathcal{L}_{c_0} + \frac{1}{L^2}) \partial_x Q_{c_0}^{\frac{3}{2}} \right) Q_{c_0}^{\frac{3}{2}} \phi_{c_0} dx + O((|c - c_0| + |\bar{a}|) |c - c_0|^2 |\bar{a}|^2)). \]  

From (6.2) and (6.3)

\[\partial_x Q_{c_0}^{\frac{3}{2}} = \frac{3}{2} Q_{c_0}^{\frac{1}{2}} \partial_x Q_{c_0}, \]

(6.45)

\[\partial_x^2 Q_{c_0}^{\frac{3}{2}} = \frac{9c_0}{4} Q_{c_0}^{\frac{3}{2}} - 2Q_{c_0}^{\frac{5}{2}}, \]

(6.46)

\[\partial_x^3 Q_{c_0}^{\frac{3}{2}} = \frac{27c_0}{8} Q_{c_0}^{\frac{5}{2}} \partial_x Q_{c_0} - 5Q_{c_0}^{\frac{3}{2}} \partial_x Q_{c_0}, \]

(6.47)

\[\partial_x^3 Q_{c_0}^{\frac{5}{2}} = \frac{3}{2} \partial_x^3 (Q_{c_0}^{\frac{1}{2}} \partial_x Q_{c_0}) = \frac{81c_0^2}{16} Q_{c_0}^{\frac{5}{2}} - 17c_0 Q_{c_0}^{\frac{3}{2}} + 10Q_{c_0}^{\frac{7}{2}}. \]  

(6.48)

From (6.45)–(6.48) we have

\[\left( (\mathcal{L}_{c_0} + \frac{1}{L^2}) \partial_x Q_{c_0}^{\frac{3}{2}} \right) Q_{c_0}^{\frac{3}{2}} \phi_{c_0} = -2c_0 Q_{c_0}^{4} + \frac{4}{3} Q_{c_0}^{5}. \]

Applying Lemma 6.1 we obtain

\[- \int_{\mathbb{R}^2} (\mathbb{L} \partial_x S_c' (\Theta)) S_{c'} (\Theta) \phi_c dxdy \]

\[= \frac{\pi L (c - c_0)^2 |\bar{a}|^2}{6c_0^2} \int_{\mathbb{R}} Q_{c_0}^{5} dx + O((|c - c_0| + |\bar{a}|) |c - c_0|^2 |\bar{a}|^2)). \]  

(6.49)
Therefore, applying Lemma 6.1, we obtain that

\[
- \int_{\mathbb{R} \times T_L} (\mathbb{L}_c \partial_x S'_c(\Theta))v\phi_c dx dy - \int_{\mathbb{R} \times T_L} (\mathbb{L}_c \partial_x v)S'_c(\Theta)\phi_c dx dy
\]

By the similar computation we have

\[
- \pi L (c - c_0) \int_{\mathbb{R}} (\mathcal{L}_{c_0} + \frac{1}{L^2}) \partial_x Q_{c_0}^3 \left( a_1 v_{1,1} + a_2 v_{1,2} \right) \phi_{c_0} dx
\]

\[
+ \pi L (c - c_0) \int_{\mathbb{R}} (\partial_x (\mathcal{L}_{c_0} + \frac{1}{L^2}) (Q_{c_0}^3 \phi_{c_0}) (a_1 v_{1,1} + a_2 v_{1,2}) dx
\]

\[+ O((|c - c_0| + |\vec{a}|) |c - c_0||\vec{a}||v||_{L^2}). \]  

(6.50)

From (6.45)–(6.48) we have

\[
\left( (\mathcal{L}_{c_0} + \frac{1}{L^2}) \partial_x Q_{c_0}^3 \right) \phi_{c_0} = -2c_0 Q_{c_0}^2 + \frac{4}{3} Q_{c_0}^2.
\]

By the similar computation we have

\[
\partial_x (\mathcal{L}_{c_0} + \frac{1}{L^2}) (Q_{c_0}^3 \phi_{c_0}) = -\frac{10c_0}{3} Q_{c_0}^2 + \frac{8}{3} Q_{c_0}^2.
\]

Therefore, applying Lemma 6.1 we obtain that

\[
\left| - \int_{\mathbb{R} \times T_L} (\mathbb{L}_c \partial_x S'_c(\Theta))v\phi_c dx dy - \int_{\mathbb{R} \times T_L} (\mathbb{L}_c \partial_x v)S'_c(\Theta)\phi_c dx dy \right|
\]

\[
= \pi L (c - c_0) \int_{\mathbb{R}} \left( -\frac{4c_0}{3} Q_{c_0}^2 + \frac{4}{3} Q_{c_0}^2 \right) (a_1 v_{1,1} + a_2 v_{1,2}) dx
\]

\[+ O((|c - c_0| + |\vec{a}|) |c - c_0||\vec{a}||v||_{L^2}) \]

\[\leq \frac{3c_0 \pi L}{4} \int_{\mathbb{R}} (|v_{1,1}|^2 + |v_{1,2}|^2) Q_{c_0} dx + \frac{20|c - c_0|^2 |\vec{a}|^2 \pi L}{297c_0^2} \int_{\mathbb{R} \times T_L} Q_{c_0}^5 dx
\]

\[+ O((|c - c_0| + |\vec{a}|) |c - c_0||\vec{a}||v||_{L^2}). \]  

(6.51)

(IV) The estimate of \(- \int_{\mathbb{R} \times T_L} R(\eta, \vec{a}, c)(v + S'_c(\Theta))\phi_c dx dy. Since

\[
(\partial_\lambda \mathbb{L}_c) \eta = \dot{\eta} - 2\dot{c}(\partial_\lambda \mathbb{Q}_c) \eta
\]

and

\[
\mathbb{L}_c \partial_x \eta = 2(\partial_x \mathbb{Q}_c) \eta + v_x + \partial_x \eta^2,
\]

we have

\[
R(\eta, \vec{a}, c)
\]

\[
= -2\eta \partial_x (v + S'_c(\Theta)) + (\dot{c} - \dot{c})(2(\partial_x \mathbb{Q}_c) \eta + v_x + \partial_x \eta^2) + (\dot{c} - \dot{c}) \mathbb{L}_c \partial_x \Theta - 2(\dot{c} - \dot{c}) \eta \partial_x (\eta + \Theta)
\]

\[-2(\partial_x \mathbb{Q}_c) (\dot{\eta} + \dot{c})(\partial_x \mathbb{Q}_c) \eta + 6(\partial_x^2 \mathbb{Q}_c - \Theta) \eta x + 2(\partial_x (\mathbb{Q}_c - \Theta)) v + 2(\partial_x (\mathbb{Q}_c - \Theta))^2)
\]

\[-4(\partial_x (\mathbb{Q}_c - \Theta)) \eta_{xx} + 4(\partial_x (\mathbb{Q}_c - \Theta))(\partial_x \mathbb{Q}_c) \eta + 2(\mathbb{Q}_c - \Theta) v_x + 2(\mathbb{Q}_c - \Theta) \partial_x \eta^2
\]

\[-4 \eta\partial_x ((\mathbb{Q}_c - \Theta) \eta) - (\mathbb{L}_c - 2\eta)(\hat{\alpha} \partial_x \Theta + \dot{c} \partial_x \Theta) + \dot{\eta} - 2\dot{c}(\partial_\lambda \mathbb{Q}_c) \eta. \]  

(6.52)
From Lemma 6.8, integration by parts, $\mathbb{L}_c \partial_x \Theta = O(|c - c_0| + |\bar{a}|)$ and $\mathbb{L}_c \partial_\bar{a} \Theta = O(|c - c_0| + |\bar{a}|)$, we obtain

$$- \int_{\mathbb{R} \times \mathbb{T}_L} R(\eta, \bar{a}, c)(v + S_c'(\Theta))\phi_c dx dy = O(|c - c_0| + |\bar{a}| + \|\eta\|_{H^1}(|c - c_0|^2 |\bar{a}|^2 + \|v\|_{H^1}^2)).$$

(6.53)

(V) The estimate of $- \int_{\mathbb{R} \times \mathbb{T}_L} (\partial_t S_c'(\Theta))(v + S_c'(\Theta))\phi_c dx dy - \frac{1}{2} \int_{\mathbb{R} \times \mathbb{T}_L} (v + S_c'(\Theta))^2 \partial_t \phi_c dx dy$

Since

$$\partial_t S_c'(\Theta) = \frac{c}{c_0} \partial_y \Theta + \frac{c - c_0}{c_0} \bar{a} \cdot \partial_\bar{a} \partial_y \Theta + \frac{c - c_0}{c_0} \partial_\bar{a} \partial_y \Theta = O(|c - c_0| + |\bar{a}|\|\eta\|_{L^2}),$$

from Lemma 6.8 we have

$$- \int_{\mathbb{R} \times \mathbb{T}_L} (\partial_t S_c'(\Theta))(v + S_c'(\Theta))\phi_c dx dy - \frac{1}{2} \int_{\mathbb{R} \times \mathbb{T}_L} (v + S_c'(\Theta))^2 \partial_t \phi_c dx dy = O(|c - c_0| + |\bar{a}|(|c - c_0|^2 |\bar{a}|^2 + \|v\|_{L^2}^2)).$$

(6.54)

Therefore, from (I)–(V) we deduce gathering (6.42)–(6.54) that there exists $k_4 > 0$ such that

$$- \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R} \times \mathbb{T}_L} (v + S_c'(\Theta))^2 \phi_c dx dy \geq k_4 \left( \int_{\mathbb{R} \times \mathbb{T}_L} v^2 \tilde{Q}_{c_0} dx dy + |c - c_0|^2 |\bar{a}|^2 \right) + O\left(|c - c_0| + |\bar{a}| + \|\eta\|_{H^1}(|c - c_0|^2 |\bar{a}|^2 + \|v\|_{H^1}^2) \right).$$

(6.55)

On the other hand, by (6.54), we have

$$- \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R} \times \mathbb{T}_L} v^2 x dx dy$$

$$= \frac{1}{2} \int_{\mathbb{R} \times \mathbb{T}_L} (3|\partial_x v|^2 + |\partial_y v|^2 + \partial_t v^2) dx dy - \int_{\mathbb{R} \times \mathbb{T}_L} v^2 \partial_x (x \tilde{Q}_c) dx dy - \int_{\mathbb{R} \times \mathbb{T}_L} \left( \mathbb{L}_c \partial_\bar{a} S_c'(\Theta) \right) v x dx dy - \int_{\mathbb{R} \times \mathbb{T}_L} R(\eta, \bar{a}, c) v x dx dy.$$  

(6.56)

From Proposition 6.7,

$$\left| \int_{\mathbb{R} \times \mathbb{T}_L} \eta v x dx dy \right| \leq \left\| x^2 \eta \right\|_{L^1} \left\| \eta \right\|_{H^1} \left\| v \right\|_{H^1}^2 = O\left(\left\| \eta \right\|_{H^1}^{\frac{1}{2}} \left\| v \right\|_{H^1}^2 \right).$$

(6.57)

By the similar calculation to (6.57), we have

$$\left| \int_{\mathbb{R} \times \mathbb{T}_L} R(\eta, \bar{a}, c) v x dx dy \right| = O\left(\left| c - c_0 \right| + |\bar{a}| + \left\| \eta \right\|_{H^1} \left\| v \right\|_{H^1}^2 \right).$$

(6.58)
By the Hölder inequality and Proposition 6.7, we have
\[ \int_{\mathbb{R} \times T_L} v^2 \partial_x (x \bar{Q}_c) dx dy \leq \left( 1 + \gamma \| x^2 (\partial_x \bar{Q}_c)^2 \bar{Q}_c^{-1} \|_{L^\infty} \right) \int_{\mathbb{R} \times T_L} v^2 \bar{Q}_c dx dy + \frac{\gamma}{8} \| v \|_{L^2}^2. \tag{6.59} \]

By the Hölder inequality, we obtain there exists \( C > 0 \) such that
\[ \left\| \int_{\mathbb{R} \times T_L} (\mathcal{L}_c \partial_x S'_c(\Theta)) v x dx dy \right\| \leq \frac{\gamma}{8} \| v \|_{L^2}^2 + C |c - c_0| |\bar{a}|. \tag{6.60} \]

We deduce gathering (6.56)–(6.60) that
\[ - \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R} \times T_L} v^2 x dx dy \geq \frac{1}{4} \int_{\mathbb{R} \times T_L} (|\nabla v|^2 + \hat{c} v^2) dx dy + \left( 1 + \gamma \| x^2 (\partial_x \bar{Q}_c)^2 \bar{Q}_c^{-1} \|_{L^\infty} \right) \int_{\mathbb{R} \times T_L} v^2 \bar{Q}_c dx dy \]
\[ - C |c - c_0| |\bar{a}| + O((|c - c_0| + |\bar{a}| + \| \eta \|_{H^1}^2) \| v \|_{H^1}^2). \tag{6.61} \]

From (6.55) and (6.61), we obtain
\[ - \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R} \times T_L} (v + S'_c(\Theta))^2 \phi_c + \varepsilon_+ x v^2 \right) dx dy \]
\[ \geq \frac{\varepsilon_+}{4} \int_{\mathbb{R} \times T_L} (|\nabla v|^2 + c_0 v^2) dx dy + \frac{k_4}{2} |c - c_0|^2 |\bar{a}|^2 \]
\[ + O((|c - c_0| + |\bar{a}| + \| \eta \|_{H^1}^2)(\| v \|_{H^1}^2 + |c - c_0|^2 |\bar{a}|^2)), \tag{6.62} \]

where
\[ \varepsilon_+ = \frac{k_4}{2} \left( 1 + C + c_0 \left\| x^2 (\partial_x \bar{Q}_c)^2 \bar{Q}_c^{-1} \right\|_{L^\infty} \right)^{-1} > 0. \]

Integrating (6.62) between \( t_1 \) and \( t_2 \), we have for sufficiently small \( \varepsilon_0 > 0 \)
\[ \int_{\mathbb{R} \times T_L} \left( (v(t_1) + S'_c(t_1)) \Theta(\bar{a}(t_1), c(t_1)))^2 \phi_c(t_1) - (v(t_2) + S'_c(t_2)) \Theta(\bar{a}(t_2), c(t_2)))^2 \phi_c(t_2) \]
\[ + x v(t_1)^2 - x v(t_2)^2 \right) dx dy \]
\[ \geq \int_{t_1}^{t_2} \left( \frac{\varepsilon_+}{4} \int_{\mathbb{R} \times T_L} (|\nabla v|^2 + c_0 v(t)^2) dx dy + k_4 |c(t) - c_0|^2 |\bar{a}(t)|^2 \right) dt. \tag{6.63} \]

From Proposition 6.7
\[ \int_{-\infty}^{\infty} \left( \frac{\varepsilon_+}{4} \int_{\mathbb{R} \times T_L} (|\nabla v|^2 + c_0 v(t)^2) dx dy + k_4 |c(t) - c_0|^2 |\bar{a}(t)|^2 \right) dt \]
\[ \lesssim \sup_{t \in \mathbb{R}} \int_{\mathbb{R} \times T_L} \left( (v + S'_c(\Theta))^2 \phi_c + \varepsilon_+ x v^2 \right) dx dy < \infty. \]

Therefore, there exist sequences \( \{t_{1,n}\}_n \) and \( \{t_{2,n}\}_n \) such that
\[ \lim_{n \to \infty} t_{1,n} = -\infty, \quad \lim_{n \to \infty} t_{2,n} = \infty \]
and
\[
\lim_{n \to \infty} \frac{\varepsilon_+}{4} \int_{\mathbb{R} \times T_L} (|\nabla v(t_{1,n})|^2 + c_0 v(t_{1,n})^2) dxdy + k_4 |c(t_{1,n}) - c_0|^2 |\tilde{a}(t_{1,n})|^2 = 0.
\]

Combining (6.63) and (6.64), we obtain that
\[
\int_{-\infty}^{\infty} \frac{\varepsilon_+}{4} \int_{\mathbb{R} \times T_L} (|\nabla v(t)|^2 + c_0 v(t)^2) dxdy + k_4 |c(t) - c_0|^2 |\tilde{a}(t)|^2 dt = 0
\]
which implies \( v \equiv 0 \) and \( |c - c_0||\tilde{a}| \equiv 0 \). By (6.33) and \( v \equiv 0 \), we have \( \eta \equiv 0 \). Therefore, we obtain the conclusion.

6.3 Non-critical case \( L < \frac{2}{\sqrt{3c_0}} \)

In this subsection, we show the Liouville property for \( L < \frac{2}{\sqrt{3c_0}} \). Since the proof of the Liouville property for \( L < \frac{2}{\sqrt{3c_0}} \) is similar to the proof of the Liouville property for \( L = \frac{2}{\sqrt{3c_0}} \), we omit the detail of the proof.

**Lemma 6.10.** Let \( c_0 > 0 \). There exist \( \varepsilon_0, K > 0 \) such that for any \( 0 < \varepsilon < \varepsilon_0 \) the following is true. For any solution \( u \in C(\mathbb{R}, H^1(\mathbb{R} \times T_L)) \) of (1.1) satisfying
\[
\inf_{b \in \mathbb{R}} \| u(t, \cdot, \cdot) - Q_{c_0}(\cdot - b, \cdot) \|_{H^1} \leq \varepsilon
\]
there exist \( \rho_1, c \in C^1(\mathbb{R}, \mathbb{R}) \) uniquely such that
\[
\eta(t, x, y) = u(t, x + \rho(t), y) - Q_{c(t)}(x)
\]
satisfies for all \( t \in \mathbb{R} \)
\[
|c(t) - c_0| + \|\eta(t)\|_{H^1} \leq K_0 \varepsilon,
\]
\[
\int_{\mathbb{R} \times T_L} \eta(t) \partial_x Q_{c(t)} dxdy = \int_{\mathbb{R} \times T_L} \eta(t) Q_{c(t)} dxdy = 0
\]
and
\[
|\dot{c}(t)|^\frac{1}{2} + |\dot{\rho}(t) - c(t)| \leq K_0 \|\eta(t)\|_{L^2}.
\]

The following is Liouville property in the non-critical case.

**Theorem 6.11.** Let \( c_0 > 0 \) and \( L < \frac{2}{\sqrt{3c_0}} \). There exists \( \varepsilon_0 > 0 \) satisfies the following. For any solution \( u \in C(\mathbb{R}, H^1(\mathbb{R} \times T_L)) \) to (1.1) satisfying (6.3) and (6.8), there exist \( c_+ > 0 \) and \( \rho_0 \in \mathbb{R} \) such that
\[
u(t, x, y) = Q_{c_+}(x - c_+ t + \rho_0, y).
\]
Remark 6.12. The proof of Theorem 6.11 is easier than the proof of Theorem 6.9. In the case $L < \frac{2}{\sqrt{c_0}}$, $L_{c_0}$ has the following coercive type estimate. There exists $k_5 > 0$ such that for $\eta \in H^1(\mathbb{R} \times T_L)$ with $(\eta, \partial_x \tilde{Q}_{c_0})_{L^2} = (\eta, \tilde{Q}_{c_0})_{L^2} = 0$,

$$(L_{c_0} \eta, \eta)_{H^{-1}, H^1} \geq k_5 \| \eta \|^2_{H^1}.$$ 

Therefore, from Lemma 6.2 we can show a coercive type estimate for the virial identity

$$- \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R} \times T_L} v^2 (\phi_c + \varepsilon_+ x) dx dy \geq \frac{\varepsilon_+}{4} \int_{\mathbb{R} \times T_L} (|\nabla v|^2 + c_0 v^2) dx dy + o(\|v\|^2_{H^1})$$

as $\varepsilon_0 \to 0$, for sufficiently small $\varepsilon_+ > 0$.

7 Asymptotic stability

In this section, we prove Theorem 1.5 by applying the monotonicity property and the Liouville property in Section 6. We follow the argument Martel and Merle [25, 26, 27] for the generalized KdV equation and Côte et al. [5] for the Zakharov–Kuznestov equation on $\mathbb{R}^2$.

7.1 Critical case $L = \frac{2}{\sqrt{c_0}}$

In this subsection, we consider the critical case $L = \frac{2}{\sqrt{c_0}}$. The following proposition shows the compactness of the orbit of solutions in $H^1(x > -A)$.

**Proposition 7.1.** Let $c_0 > 0$ and $L = \frac{2}{\sqrt{c_0}}$. There exists $0 < \varepsilon_* < \varepsilon_0$ such that if $0 < \varepsilon \leq \varepsilon_*$ and $u \in C(\mathbb{R}, H^1(\mathbb{R} \times T_L))$ is a solution to (1.1) satisfying $\sup_{t \in \mathbb{R}} dist_{c_0}(u(t)) < \varepsilon$ then the following holds true. For any sequence $\{t_n\}_n$ with $\lim_{n \to \infty} t_n = \infty$, there exists a subsequence $\{t_{n_k}\}_k$ and $\tilde{u}_0 \in H^1(\mathbb{R} \times T_L)$ such that

$$\|u(t_{n_k}, \cdot) + \rho(t_{n_k}, \cdot) - \tilde{u}_0\|_{H^1(x > -A)} \to 0 \text{ as } k \to \infty$$

for any $A > 0$, where $\rho$ is the function associated to $u$ given by Lemma 6.8. Moreover, the solution $\tilde{u}$ of (1.1) with $\tilde{u}(0) = \tilde{u}_0$ satisfies

$$\left\| \tilde{u}(t, \cdot + \tilde{\rho}(t), \cdot) - \tilde{Q}_{c_0} \right\|_{H^1} \lesssim \varepsilon_0, \quad t \in \mathbb{R}$$

(7.1)

and

$$\int_{T_L} |\tilde{u}(t, x + \tilde{\rho}(t), y)|^2 dy \lesssim e^{-\delta_1|x|}, \quad (t, x) \in \mathbb{R}^2$$

(7.2)

for some $\delta_1 > 0$, where $\tilde{\rho}$ is the function associated to $\tilde{u}$ given by Lemma 6.8 and $\tilde{\rho}(0) = 0$.

Since the proof of Proposition 7.1 is similar to the proof of Proposition 4.1 in [5], we omit the proof.

Next, we show (ii) of Theorem 1.5
Proof of (ii) of Theorem 1.5. Let \( \beta > 0 \) and \( u \) be a solution to (1.1) with \( \text{dist}_{c_0}(u(0)) = \varepsilon \). By Theorem 1.2 if \( \varepsilon \) is sufficiently small, then \( \text{dist}_{c_0}(u(t)) < \varepsilon \). Let \( \rho, c \) and \( \tilde{a} \) be functions associated to \( u \) given by Lemma 6.8. From Proposition 7.1 for any sequence \( \{t_n\}_n \) with \( \lim_{n \to \infty} t_n = \infty \), there exist a subsequence \( \{t_{n_k}\}_k \), \( \tilde{c}_0 > 0 \), \( \tilde{a}_0 \in \mathbb{R}^2 \) and \( \tilde{u}_0 \in H^1(\mathbb{R} \times \mathbb{T}_L) \) such that for \( A > 0 \)

\[
    u(t_{n_k}, \cdot + \rho(t_{n_k}), \cdot) \to \tilde{u}_0 \text{ in } H^1(x > -A), \quad c(t_{n_k}) \to \tilde{c}_0 \quad \text{and} \quad \tilde{a}(t_{n_k}) \to \tilde{a}_0.
\]

Moreover, the solution \( \tilde{u} \) of (1.1) with \( \tilde{u}(0) = \tilde{u}_0 \) satisfies (7.1) and (7.2). Let \( \tilde{\rho}, \tilde{c} \) and \( \tilde{a} \) be functions associated to \( \tilde{u} \) given by Lemma 6.8. By uniqueness of the decomposition in Lemma 6.8 we have \( \tilde{\rho}(0) = 0, \tilde{c}(0) = \tilde{c}_0 \) and \( \tilde{a}(0) = \tilde{a}_0 \). Applying Theorem 6.9 we obtain that there exist \( \rho_0 \in \mathbb{R}, c_1 \geq 0 \) and \( \tilde{a}_1 \in \mathbb{R}^2 \) such that \( |c_1 - c_0||\tilde{a}_1| = 0 \) and

\[
    \tilde{u}(t, x, y) = \Theta(\tilde{a}_1, c_1)(x - \tilde{c}_1 t - \rho_0, y),
\]

where

\[
    \tilde{c}_1 = \begin{cases} 
        c_1, & \tilde{a}_1 = (0, 0), \\
        \tilde{c}(\tilde{a}_1), & c_1 = c_0.
    \end{cases}
\]

By uniqueness of the decomposition in Lemma 6.8 \( \rho_0 = 0, c_1 = \tilde{c}_0, \tilde{a}_1 = \tilde{a}_0 \) and \( |\tilde{c}_0 - c_0||\tilde{a}_0| = 0 \). Since for any sequence \( \{t_n\}_n \) with \( \lim_{n \to \infty} t_n = \infty \) there exists a subsequence \( \{t_{n_k}\}_k \) such that

\[
    u(t_{n_k}, \cdot + \rho(t_{n_k}), \cdot) - \Theta(\tilde{a}(t_{n_k}), c(t_{n_k})) \to 0 \text{ in } H^1(x > -A) \quad \text{and} \quad |c(t_{n_k}) - c_0||\tilde{a}(t_{n_k})| \to 0,
\]

we obtain

\[
    u(t, \cdot + \rho(t), \cdot) - \Theta(\tilde{a}(t), c(t)) \to 0 \text{ in } H^1(x > -A) \quad (7.3)
\]

and

\[
    |c(t) - c_0||\tilde{a}(t)| \to 0. \quad (7.4)
\]

Moreover, (7.3) implies that for \( R > 0 \) and \( x_0 \in \mathbb{R} \)

\[
    \lim_{t \to \infty} \int_{\mathbb{R} \times \mathbb{T}_L} (|\nabla \eta|^2 + |\eta|^2)(t, x - \rho(t), y) \psi_R(x - \rho(t) + x_0) dx dy = 0, \quad (7.5)
\]

where \( \eta(t, x, y) = u(t, x + \rho(t), y) - \Theta(\tilde{a}(t), c(t)) \). By (7.3), for any \( \alpha > 0 \) and \( R > 2/\sqrt{3} \) there exist \( x_1 \in \mathbb{R} \) and \( T_1 > 0 \) such that for \( x_0 > x_1 \) and \( t > T_1 \)

\[
    \int_{\mathbb{R} \times \mathbb{T}_L} |u(t, x, y)|^2 \psi_R(x - \rho(t) + x_0) dx dy - \int_{\mathbb{R} \times \mathbb{T}_L} |\Theta(\tilde{a}(t), c(t))|^2 dx dy < \alpha, \quad (7.6)
\]

where \( \psi_R \) is defined by (6.4). From Lemma 6.5 there exists \( x_2 \in \mathbb{R} \) such that for \( x_0 \geq x_2 \) and \( t \geq t' \)

\[
    \int_{\mathbb{R} \times \mathbb{T}_L} |u(t, x, y)|^2 \psi_R(x - \rho(t) + x_0) dx dy
    \quad - \quad \int_{\mathbb{R} \times \mathbb{T}_L} |u(t', x, y)|^2 \psi_R(x - \rho(t') + x_0) dx dy \leq \alpha. \quad (7.7)
\]
By (7.6) and (7.14) we have that for $t \geq t' > T_0$
\[
\int_{\mathbb{R} \times T_L} |\Theta(\bar{a}(t), c(t))|^2 \, dx \, dy \leq \int_{\mathbb{R} \times T_L} |\Theta(\bar{a}(t'), c(t'))|^2 \, dx \, dy + 3\alpha.
\]

Since for any $\alpha > 0$
\[
\limsup_{t \to \infty} \int_{\mathbb{R} \times T_L} |\Theta(\bar{a}(t), c(t))|^2 \, dx \, dy \leq \liminf_{t' \to \infty} \int_{\mathbb{R} \times T_L} |\Theta(\bar{a}(t'), c(t'))|^2 \, dx \, dy + 3\alpha,
\]

\[
\int_{\mathbb{R} \times T_L} |\Theta(\bar{a}(t), c(t))|^2 \, dx \, dy
\]
has the limit as $t \to \infty$. By the definition of $\Theta$, we have
\[
\lim_{t \to \infty} \int_{\mathbb{R} \times T_L} |\Theta(\bar{a}(t), c(t))|^2 \, dx \, dy = \lim_{t \to \infty} \left( \frac{c(t)}{c_0} \right)^2 \|\varphi_{c_0}(\bar{a}(t))\|^2_{L^2}.
\]

Since $\|\varphi_{c_0}(\bar{a}(t))\|^2_{L^2}$ is strictly increasing with respect to $|\bar{a}|$, from (7.3) and (7.8) the $\omega$-limit set of $(|\bar{a}(t)|, c(t))$ is at most two points. By the continuity of $\bar{a}(t)$ and $c(t)$, the $\omega$-limit set of $(|\bar{a}(t)|, c(t))$ is the one point set which implies there exist $a_+ \geq 0$ and $c_+ > 0$ such that
\[
\lim_{t \to \infty} |\bar{a}(t)| = a_+, \quad \lim_{t \to \infty} c(t) = c_+.
\]

Therefore, by Corollary 5.7 we have
\[
|c_+ - c_0| + |a_+|^2 \lesssim \left\| u(0) - \bar{Q}_{c_0} \right\|_{H^1}.
\]

Next, we improve the convergence of (7.17). By Lemma 6.4 for all $t_1 \leq t_2$, $x_0 > 0$ and $R > 2/\sqrt{\beta}$
\[
\int_{\mathbb{R} \times T_L} |u(t_2, x, y)|^2 \psi_R(\bar{x}(t_1, t_2)) \, dx \, dy - \int_{\mathbb{R} \times T_L} |u(t_1, x, y)|^2 \psi_R(\bar{x}(t_1, t_1)) \, dx \, dy \leq C e^{-x_0/R},
\]

(7.10)

where $\bar{x}(t, \tau) = x - \rho(t) - \frac{\beta}{2}(\tau - t) + x_0$. By (6.18) if $R > \frac{1}{4c_0}$ and $\varepsilon_0$ is sufficiently small, then we have
\[
\left| \int_{\mathbb{R} \times T_L} \eta(t, x, y) \Theta(\bar{a}(t), c(t)) \psi_R(x + x_0) \, dx \, dy \right|
\]
\[
= \left| \int_{\mathbb{R} \times T_L} \eta(t, x, y) \Theta(\bar{a}(t), c(t))(1 - \psi_R(x + x_0)) \, dx \, dy \right|
\]
\[
= \|\eta(t)\|_{L^2} \|\Theta(\bar{a}(t), c(t))(1 - \psi_R(x + x_0))\|_{L^2} \lesssim e^{-x_0/R}.
\]

(7.11)

Since
\[
(\eta(t, x - \rho(t), y))^2 = (u(t, x, y))^2 - 2\eta(t, x - \rho(t), y)\Theta(\bar{a}(t), c(t))(x - \rho(t), y)
\]
\[
- (\Theta(\bar{a}(t), c(t))(x - \rho(t), y))^2,
\]

(7.12)

Since $\eta(t, x - \rho(t), y)$ is sufficiently small compared to $u(t, x, y)$, we have
\[
\int_{\mathbb{R} \times T_L} |\Theta(\bar{a}(t), c(t))|^2 \, dx \, dy \leq C e^{-x_0/R}.
\]

(7.13)
from (7.11) and (7.11) we have that there exists $C_0 > 0$ such that
\[
\int_{\mathbb{R} \times \mathbb{T}_L} (\eta(t_2, x - \rho(t_2), y))^2 \psi_R(\bar{x}(t_1, t_2))dxdy
- \int_{\mathbb{R} \times \mathbb{T}_L} (\eta(t_1, x - \rho(t_1), y))^2 \psi_R(\bar{x}(t_1, t_1))dxdy
\leq C_0(e^{-x_0/R} + |c(t_1) - c(t_2)| + ||\bar{a}(t_1)||^2 - ||\bar{a}(t_2)||^2).
\]

For $t > 0$ large enough, we define $0 < t' < t$ such that $\rho(t') + \frac{\beta}{2}(t - t') - x_0 = \beta t$. Then, we have $t' \to \infty$ as $t \to \infty$.
\[
\int_{\mathbb{R} \times \mathbb{T}_L} (\eta(t, x - \rho(t), y))^2 \psi_R(x - \beta t)dxdy
\leq \int_{\mathbb{R} \times \mathbb{T}_L} (\eta(t', x - \rho(t'), y))^2 \psi_R(x - \rho(t') + x_0)dxdy
+ C_0(e^{-x_0/R} + |c(t_1) - c(t_2)| + ||\bar{a}(t_1)||^2 - ||\bar{a}(t_2)||^2).
\]

From (7.3) and (7.9), we obtain for any $x_0 > 0$
\[
\lim_{t \to \infty} \sup \int_{\mathbb{R} \times \mathbb{T}_L} (\eta(t, x - \rho(t), y))^2 \psi_R(x - \beta t)dxdy \leq C_0e^{-x_0/R}.
\]

Therefore,
\[
\lim_{t \to \infty} \int_{\mathbb{R} \times \mathbb{T}_L} (\eta(t, x - \rho(t), y))^2 \psi_R(x - \beta t)dxdy = 0.
\] (7.12)

From Lemma 6.6 we have for all $t_1 \leq t_2$, $x_0 > 0$ and $R > 2/\sqrt{\beta}$
\[
J_{x_0,t_1}(u(t_2)) - J_{x_0,t_1}(u(t_1)) \leq Ce^{-x_0/R}.
\] (7.13)

Moreover, we have
\[
\left| \int_{\mathbb{R} \times \mathbb{T}_L} (u(t_2, x, y))^3 \psi_R(\bar{x}(t_1, t_2))dxdy - \int_{\mathbb{R} \times \mathbb{T}_L} (u(t_1, x, y))^3 \psi_R(\bar{x}(t_1, t_1))dxdy \right|
\leq \left( \int_{\mathbb{R} \times \mathbb{T}_L} (\eta(t_1, x - \rho(t_1), y))^2 \psi_R(\bar{x}(t_1, t_2))dxdy \right)^{1/2}
+ \left( \int_{\mathbb{R} \times \mathbb{T}_L} (\eta(t_2, x - \rho(t_2), y))^2 \psi_R(\bar{x}(t_1, t_2))dxdy \right)^{1/2}
+ (e^{-x_0/R} + |c(t_1) - c(t_2)| + ||\bar{a}(t_1)|| - ||\bar{a}(t_2)||).
\] (7.14)
By (7.13) and (7.14)
\[\int_{\mathbb{R} \times T_L} |\nabla \eta(t_2, x - \rho(t_2), y)|^2 \psi_R(\tilde{x}(t_1, t_2)) \, dx \, dy \]
\[- \int_{\mathbb{R} \times T_L} |\nabla \eta(t_1, x - \rho(t_1), y)|^2 \psi_R(\tilde{x}(t_1, t_1)) \, dx \, dy \]
\[\lesssim \left( \int_{\mathbb{R} \times T_L} (\eta(t_1, x - \rho(t), y))^2 \psi_R(\tilde{x}(t_1, t_2)) \, dx \, dy \right)^{\frac{1}{2}} \]
\[+ \left( \int_{\mathbb{R} \times T_L} (\eta(t_2, x - \rho(t_2), y))^2 \psi_R(\tilde{x}(t_1, t_2)) \, dx \, dy \right)^{\frac{1}{2}} \]
\[+ (e^{-x_0/R} + |c(t_1) - c(t_2)| + ||\tilde{a}(t_1)| - |\tilde{a}(t_2)||) \quad (7.15)\]

From (7.15) with \(t_1 = t'\) and \(t_2 = t\) we obtain that there exists \(C > 0\) such that
\[\int_{\mathbb{R} \times T_L} |\nabla \eta(t, x - \rho(t), y)|^2 \psi_R(x - \beta t) \, dx \, dy \]
\[\leq \int_{\mathbb{R} \times T_L} |\nabla \eta(t', x - \rho(t'), y)|^2 \psi_R(x - \rho(t') + x_0) \, dx \, dy \]
\[+ C \left( \int_{\mathbb{R} \times T_L} (\eta(t, x - \rho(t), y))^2 \psi_R(x - \beta t) \, dx \, dy \right)^{\frac{1}{2}} \]
\[+ C \left( \int_{\mathbb{R} \times T_L} (\eta(t', x - \rho(t'), y))^2 \psi_R(x - \rho(t') + x_0) \, dx \, dy \right)^{\frac{1}{2}} \]
\[+ C(e^{-x_0/R} + |c(t') - c(t)| + ||\tilde{a}(t')| - |\tilde{a}(t)||) \quad \text{for} \quad x_0 > 0 \]

Therefore, it follows form (7.5), (7.9) and (7.12) that for \(x_0 > 0\)
\[\limsup_{t \to \infty} \int_{\mathbb{R} \times T_L} |\nabla \eta(t, x - \rho(t), y)|^2 \psi_R(x - \beta t) \, dx \, dy \leq C e^{-x_0/R} \]
which implies
\[\lim_{t \to \infty} \int_{\mathbb{R} \times T_L} |\nabla \eta(t, x - \rho(t), y)|^2 \psi_R(x - \beta t) \, dx \, dy = 0. \quad (7.16)\]

Then, we define \(\rho_2(t)\) by
\[\rho_2(t) = \begin{cases} \Phi^{-1} \left( \frac{\tilde{a}(t)}{|\tilde{a}(t)|} \right), & \text{if } |\tilde{a}(t)| \neq 0 \text{ and } a_+ \neq 0, \\ 0, & \text{if otherwise,} \end{cases} \]
where \(\Phi(\theta) = (\cos \theta, -\sin \theta)\) for \(\theta \in \mathbb{R}/2\pi \mathbb{Z}\). Using
\[\Theta(\tilde{a}(t), c(t))(x, y) = \Theta((|\tilde{a}(t)|, 0), c(t))(x, y - \rho_2(t)),\]
from (7.12) and (7.16) we obtain
\[u(t, \cdot + \rho(t), y + \rho_2(t)) - \Theta((a_+, 0), c_+) \to 0 \text{ in } H^1(x > \beta t). \quad (7.17)\]
From (6.19)–(6.21), (7.3) and (7.9), we have

$$\lim_{t \to \infty} \dot{c}(t) = \lim_{t \to \infty} |\dot{a}(t)| = \lim_{t \to \infty} |\dot{\rho}(t) - \hat{c}_+| = 0,$$

where $\hat{c}_+$ is defined by (1.7). If $a_+ = 0$, then $\dot{\rho}_2(t) = 0$ for $t > 0$. On the other hand, if $a_+ \neq 0$, then $|\dot{\rho}_2(t)| \lesssim |\dot{a}(t)| \to 0$ as $t \to \infty$.

7.2 Non-Critical case $L < \frac{2}{\sqrt{5}\sqrt{c_0}}$

In this subsection, we show (i) of Theorem 1.5. The proof of (i) of Theorem 1.5 is similar to the proof of (ii) of Theorem 1.5. We omit the detail of the proof.

Proposition 7.2. Let $c_0 > 0$ and $L < \frac{2}{\sqrt{5}\sqrt{c_0}}$. There exists $0 < \varepsilon_* < \varepsilon_0$ such that if $0 < \varepsilon \leq \varepsilon_*$ and $u \in C(\mathbb{R}, H^1(\mathbb{R} \times T_L))$ is a solution to (1.1) satisfying $\sup_{t \in \mathbb{R}} \text{dist}_{c_0}(u(t)) < \varepsilon$ then the following holds true. For any sequence $\{t_n\}_n$ with $\lim_{n \to \infty} t_n = \infty$, there exists a subsequence $\{t_{n_k}\}_k$ and $\tilde{u}_0 \in H^1(\mathbb{R} \times T_L)$ such that

$$u(t_{n_k}, \cdot + \rho(t_{n_k}), \cdot) \to \tilde{u}_0 \text{ in } H^1(x > -A) \text{ as } k \to \infty$$

for any $A > 0$, where $\rho$ is the function associated to $u$ given by Lemma 6.10. Moreover, the solution $\tilde{u}$ of (1.1) with $\tilde{u}(0) = \tilde{u}_0$ satisfies

$$\left\| \tilde{u}(t, \cdot + \tilde{\rho}(t), \cdot) - \tilde{Q}_{c_0} \right\|_{H^1} \lesssim \varepsilon_0, \quad t \in \mathbb{R}$$

(7.18)

and

$$\int_{T_L} |\tilde{u}(t, x + \tilde{\rho}(t), y)|^2 dy \lesssim e^{-\delta_1|x|}, \quad (t, x) \in \mathbb{R}^2$$

(7.19)

for some $\delta_1 > 0$, where $\tilde{\rho}$ is the function associated to $\tilde{u}$ given by Lemma 6.10 and $\tilde{\rho}(0) = 0$.

By applying Theorem 6.11 and Proposition 7.2 and the similar proof to the proof of (ii) of Theorem 1.5, we obtain (i) of Theorem.

Acknowledgments

The author would like to express his great appreciation to Professor Yoshio Tsutsumi for a lot to helpful advices and encouragements. The author would like to thank Professor Tetsu Mizumachi for his helpful advices.

References

[1] J. C. Alexander, R. L. Pego and R. L. Sachs, On the transverse instability of solitary waves in the Kadomtsev-Petviashvili equation, Phys. Lett. A (1997), no. 3-4, 187–192.
[2] T. B. Benjamin, *The stability of solitary waves*, Proc. Roy. Soc. (London) Ser. A *328* (1972), 153–183.

[3] T.J. Bridges, *Universal geometric conditions for the transverse instability of solitary waves*, Phys. Rev. Lett. *84* no. 12 (2000) 2614–2617.

[4] S.-M. Chang, S. Gustafson, K. Nakanishi and T.-P. Tsai, *Spectra of linearized operators for NLS solitary waves*, SIAM J. Math. Anal. *39* (2007/08), no. 4, 1070–1111.

[5] R. Côte, C. Muñoz, D. Pilod and G. Simpson, *Asymptotic Stability of High-dimensional Zakharov-Kuznetsov Solitons*, Arch. Ration. Mech. Anal. *220* (2016), no. 2, 639–710.

[6] A. Comech and D. E. Pelinovsky, *Purely nonlinear instability of standing waves with minimal energy*, Comm. Pure Appl. Math. *56* (2003), no. 11, 1565–1607.

[7] B. Deconinck, D. E. Pelinovsky and J. D. Carter, *Transverse instabilities of deep-water solitary waves*, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. *462* (2006), no.2071, 2671–2694.

[8] A. de Bouard, *Stability and instability of some nonlinear dispersive solitary waves in higher dimension*, Proc. Roy. Soc. Edinburgh Sect. A *126* (1996), no. 1, 89–112.

[9] A. V. Faminskii, *The Cauchy problem for the Zakharov-Kuznetsov equation*, translation in Differential Equations A *31* (1995), no. 6, 1002–1012

[10] V. Georgiev and M. Ohta, *Nonlinear instability of linearly unstable standing waves for nonlinear Schrödinger equations*, Journal of the Mathematical Society of Japan *64* (2012), no. 2, 533–548.

[11] E. Grenier, *On the nonlinear instability of Euler and Prandtl equations*, Comm. Pure Appl. Math. *53* (2000) 1067–1091.

[12] M. Grillakis, J. Shatah and W. Strauss, *Stability theory of solitary waves in the presence of symmetry. I*, J. Funct. Anal. A *74* (1987), no. 1, 160–197.

[13] M. Grillakis, J. Shatah and W. Strauss, *Stability theory of solitary waves in the presence of symmetry II*, J. Funct. Anal. *94* (1990), no. 2, 308–348.

[14] A. Grünrock, *A remark on the modified Zakharov-Kuznetsov equation in three space dimensions*, Math. Res. Lett. *21* (2014), no. 1, 127–131.

[15] K. Johnson, *The transverse instability of periodic waves in Zakharov-Kuznetsov type equations*, Stud. Appl. Math. *124* (2010), no. 4, 323–345.

[16] A. Grünrock and S. Herr, *The Fourier restriction norm method for the Zakharov-Kuznetsov equation*, Discrete Contin. Dyn. Syst. *34* (2014), no. 5, 2061–2068.

[17] B. B. Kadomtsev and V. I. Petviashvili, *On the stability of solitary waves in weakly dispersive media*, Sov. Phys. Dokl. *15* (1970), 539–541.
[18] T. Kato, *On the Cauchy problem for the (generalized) Korteweg-de Vries equation*. Studies in applied mathematics, Adv. Math. Suppl. Stud., 8 93–128, Academic Press, New York, 1983.

[19] E. Kirr, P. G. Kevrekidis and D. E. Pelinovsky, *Symmetry-breaking bifurcation in the nonlinear Schrödinger equation with symmetric potentials*, Comm. Math. Phys. 308 (2011), no. 3, 795–844.

[20] D. Lannes, F. Linares and J.-C. Saut, *The Cauchy problem for the Euler-Poisson system and derivation of the Zakharov-Kuznetsov equation*, Studies in phase space analysis with applications to PDEs, 181–213, Progr. Nonlinear Differential Equations Appl., 84, Birkhauser/Springer, New York, 2013.

[21] F. Linares and A. Pastor, *Well-posedness for the two-dimensional modified Zakharov-Kuznetsov equation*, SIAM J. Math. Anal. 41 (2009), no. 4, 1323–1339.

[22] F. Linares and A. Pastor, *Local and global well-posedness for the 2D generalized Zakharov-Kuznetsov equation*, J. Funct. Anal. 206 (2001), no. 3, 219–254.

[23] F. Linares and J.-C. Saut, *The Cauchy problem for the 3D Zakharov-Kuznetsov equation*, Discrete Contin. Dyn. Syst. 24 (2009), no. 2, 547–565.

[24] M. Maeda, *Stability of bound states of Hamiltonian PDEs in the degenerate cases*, J. Funct. Anal. 263 (2012), no. 2, 511–528.

[25] M. Martel and F. Merle, *Asymptotic stability of solitons for subcritical generalized KdV equations*, Arch. Ration. Mech. Anal. 157 (2001), no. 3, 219–254.

[26] M. Martel and F. Merle, *Asymptotic stability of solitons of the subcritical gKdV equations revisited*, Nonlinearity 18 (2005), no. 1, 55–80.

[27] M. Martel and F. Merle, *Asymptotic stability of solitons of the gKdV equations with general nonlinearity*, Math. Ann. 341 (2008), no. 2, 391–427.

[28] T. Mizumachi, *Large time asymptotics of solutions around solitary waves to the generalized Korteweg-de Vries equations*, SIAM J. Math. Anal. 32 (2001), no. 5, 1050–1080.

[29] T. Mizumachi, *Stability of line solitons for the KP-II equation in $\mathbb{R}^2$*, Mem. Amer. Math. Soc. 238 (2015), no. 1125, vii+95 pp.

[30] T. Mizumachi and N. Tzvetkov, *Stability of the line soliton of the KP-II equation under periodic transverse perturbations*, Math. Ann. 352 (2012), no. 3, 659–690.

[31] L. Molinet and D. Pilod, *Bilinear Strichartz estimates for the Zakharov-Kuznetsov equation and applications*, Ann. Inst. H. Poincare Anal. Non Lineaire 32 (2015), no. 2, 347–371.

[32] R. Nagel (ed.), *One Parameters Semigroups of Positive Operators*, Lecture Notes in Math., 1184, Springer-Verlag, Berlin, 1984.
[33] M. Ohta, Instability of bound states for abstract nonlinear Schrödinger equations, J. Funct. Anal. 261 (2011), no. 1, 90-110.

[34] R. Pego and M. I. Weinstein, Eigenvalues, and instabilities of solitary waves, Philos. Trans. Roy. Soc. London Ser. A 340 (1992), no. 1656, 47–94.

[35] R. Pego and M. I. Weinstein, Asymptotic stability of solitary waves, Comm. Math. Phys. 164 (1994), no. 2, 305–349.

[36] F. Ribaud and S. Vento, Well-posedness results for the three-dimensional Zakharov-Kuznetsov equation, SIAM J. Math. Anal. 44 (2012), no. 4, 2289–2304.

[37] F. Rousset and N. Tzvetkov, Transverse nonlinear instability of solitary waves for some Hamiltonian PDE’s, J. Math. Pures. Appl. 90 (2008) 550–590.

[38] F. Rousset and N. Tzvetkov, Transverse nonlinear instability for two-dimensional dispersive models, Ann. I. Poincaré-AN 26 (2009) 477–496.

[39] F. Rousset and N. Tzvetkov, A simple criterion of transverse linear instability for solitary waves, Math. Res. Lett., 17 (2010), no. 1, 157–169.

[40] F. Rousset and N. Tzvetkov, Stability and instability of the KdV solitary wave under the KP-I flow, Comm. Math. Phys., 313 (2012), no. 1, 155–173.

[41] J. Shatah and W. Strauss, Spectral condition for instability, Contemp. Math., 255 (2000), 189–198.

[42] J. Villarroel and M. J. Ablowitz, On the initial value problem for the KPII equation with data that do not decay along a line, Nonlinearity, 17 (2004), no. 5, 1843–1866.

[43] M. I. Weinstein, Modulational stability of ground states of nonlinear Schrödinger equations, SIAM J. Math. Anal., 16 (1985), 472–491.

[44] M. I. Weinstein, Lyapunov stability of ground states of nonlinear dispersive evolution equations, Comm. Pure Appl. Math., 39 (1986), no. 1, 51–67.

[45] Y. Yamazaki, Transverse instability for a system of nonlinear Schrödinger equations, Discrete Contin. Dyn. Syst. Ser. B 19 (2014), no.2, 565–588.

[46] Y. Yamazaki, Stability of line standing waves near the bifurcation point for nonlinear Schrödinger equations, Kodai Math. J. 38 (2015), no. 1, 65–96.

[47] Y. Yamazaki, Transverse instability for nonlinear Schrödinger equation with a linear potential, Adv. in Differential Equations 21 (2016), no. 5-6, 429–462.

[48] V. E. Zakharov and E. A. Kuznetsov, On three dimensional solitons, Sov. Phys.-JETP, 39 (1974), no.2, 285–286.
Yohei Yamazaki
Department of Mathematics
Kyoto University
Kyoto 606-8502
Japan
E-mail address: y-youhei@math.kyoto-u.ac.jp