GENERATING MAPPING CLASS GROUPS OF NONORIENTABLE SURFACES WITH BOUNDARY

MICHAL STUKOW

Abstract. We obtain simple generating sets for various mapping class groups of a nonorientable surface with punctures and/or boundary. We also compute the abelianizations of these mapping class groups.

1. Introduction

Let $N_{g,s}^n$ be a smooth, nonorientable and compact surface of genus $g$ with $s$ boundary components and $n$ punctures. If $s$ and/or $n$ is zero then we omit it from the notation. If we do not want to emphasise the numbers $g, s, n$, we simply write $N$ for a surface $N_{g,s}^n$. Recall that $N_g$ is a connected sum of $g$ projective planes and $N_{g,s}^n$ is obtained from $N_g$ by removing $s$ open discs and specifying the set $\Sigma = \{z_1, \ldots, z_n\}$ of $n$ distinguished points in the interior of $N$.

Let $\text{Diff}(N)$ be the group of all diffeomorphisms $h: N \to N$ such that $h$ is the identity on each boundary component and $h(\Sigma) = \Sigma$. By $\mathcal{M}(N)$ we denote the quotient group of $\text{Diff}(N)$ by the subgroup consisting of maps isotopic to the identity, where we assume that isotopies fix $\Sigma$ and are the identity on each boundary component. $\mathcal{M}(N)$ is called the mapping class group of $N$. The mapping class group of an orientable surface is defined analogously, but we consider only orientation preserving maps.

By abuse of notation we will use the same letter for a map and its isotopy class and we will use the functional notation for the composition of diffeomorphisms.

For any $1 \leq k \leq n$, let $\mathcal{PM}^k(N)$ be the subgroup of $\mathcal{M}(N)$ consisting of elements which fix $\Sigma$ pointwise and preserve the local orientation around the punctures $\{z_1, \ldots, z_k\}$. For $k = 0$, we obtain the so-called pure mapping class group $\mathcal{PM}(N)$.

2000 Mathematics Subject Classification. Primary 57N05; Secondary 20F38, 57M99.

Key words and phrases. Mapping class groups, Nonorientable surfaces.

Supported by BW 5100-5-0205-6.
The main reason for introducing the groups $PM^k(N)$ with $k \geq 1$ is as follows. The usual way to study the mapping class group of a surface with boundary is via the homomorphism

$$i_* : PM(N_{g,s}) \to PM(N_{g+s}^{n+s}),$$

induced by the inclusion $i : N_{g,s}^n \to N_{g+s}^{n+s}$, where $N_{g+s}^{n+s}$ is the surface obtained by gluing a disk with a puncture to each boundary component of $N_{g,s}^n$. However, in the nonorientable case, the homomorphism $i_*$ is not an epimorphism. To be more precise, its image consists of maps which preserve the local orientation around the punctures coming from the boundary components of $N_{g,s}^n$. Hence the group $PM^k(N)$ occurs naturally as an image of $i_*$.

1.1. **Background.** One of the oldest problems concerning mapping class groups is to find a simple generating set or a generating set with some additional properties. In the case of an orientable surface there are many results in this direction – see for example [1, 3, 6, 13–15, 17–19, 21, 22, 27] and references there.

On the other hand, the nonorientable case has not been studied much. The first significant result is due to Lickorish [16], who proved that the mapping class group $M(N_g)$ is generated by Dehn twists and a so-called “crosscap slide” (or a “Y-homeomorphism”). Later Chillingworth [4] found a finite generating set for the group $M(N_g)$, which was extended by Korkmaz [12] to the case of groups $M(N_g^n)$ and $PM(N_g^n)$. It is also known that the group $M(N_g^n)$ is generated by involutions [25, 26].

Another natural question is to compute the (co)homology groups of mapping class groups. As above, there are many results concerning the orientable case – a good reference is a survey article [11]. In the nonorientable case, Korkmaz [10, 12] computed the first integral homology group of $M(N_g^n)$, and under additional assumption $g \geq 7$, of $PM(N_g^n)$.

1.2. **Main results.** The main goal of this paper is to extend some of the above results to the case of mapping class groups of nonorientable surfaces with boundary. More precisely, for every $g \geq 3$ we obtain finite generating sets for the groups $PM^k(N_{g,s}^n)$ and $M(N_{g,s}^n)$ – cf Theorems 5.2 and 5.3. Then using these generating sets we compute their first integral homology groups (abelianizations) – cf Theorems 6.21 and 6.22.

The reason for the assumption $g \geq 3$ is the exceptional (and non-trivial) nature of the cases $g = 1$ and $g = 2$. Most of our analysis make no sense in these cases, hence we leave them for future consideration.
Let us point out that although, using the results of [2], one can prove that the mapping class group of a closed nonorientable surface is an infinite index subgroup of the mapping class group of an orientable surface, we do not see any method to deduce our results from the orientable case.

It is worth mentioning that surfaces with boundary occur in a natural way when one considers the stabilisers in the mapping class group of sets of circles. Such a situation is very common when dealing with complexes of curves on a surface. In particular we believe that our work will be an important step toward finding a presentation for the mapping class group of a nonorientable surface [24].

2. Preliminaries

By a circle on $N$ we mean an oriented simple closed curve on $N \setminus \Sigma$, which is disjoint from the boundary of $N$. Usually we identify a circle with its image. Moreover, as in the case of diffeomorphisms, we will use the same letter for a circle and its isotopy class. According to whether a regular neighbourhood of a circle is an annulus or a Möbius strip, we call the circle two-sided or one-sided respectively. We say that a circle is generic if it bounds neither a disk with less than 2 punctures nor a Möbius strip disjoint from $\Sigma$.

Let $a$ be a two-sided circle. By definition, a regular neighbourhood of $a$ is an annulus, so if we fix one of its two possible orientations, we can define the right Dehn twist $t_a$ about $a$ in the usual way. We emphasise that since we are dealing with nonorientable surfaces, there is no canonical way to choose the orientation of $S_a$. Therefore by a twist about $a$ we always mean one of two possible twists about $a$ (the second one is then its inverse). By a boundary twist we mean a twist about a circle isotopic to a boundary component. It is known that if $a$ is not generic then the Dehn twist $t_a$ is trivial. In particular, a Dehn twist about the boundary of a Möbius strip is trivial – see Theorem 3.4 of [5].

Other important examples of diffeomorphisms of a nonorientable surface are the crosscap slide and the puncture slide. They are defined as a slide of a crosscap and of a puncture respectively, along a loop. The general convention is that one considers only crosscap slides along one-sided simple loops (in such a form they were introduced by Lickorish [16]), for precise definitions and properties see [12].

The following two propositions follow immediately from the above definitions.
Proposition 2.1. Let \( N_a \) be an oriented regular neighbourhood of a two–sided circle \( a \) in a surface \( N \), and let \( f: N \to N \) be any diffeomorphism. Then \( ft_a f^{-1} = t_{f(a)} \), where the orientation of a regular neighbourhood of \( f(a) \) is induced by the orientation of \( f(N_a) \).

Proposition 2.2. Let \( v \) be a slide of a puncture \( z \) along a simple closed loop \( \alpha \) on a surface \( N \), and let \( f: N \to N \) be any diffeomorphism. Then \( fv f^{-1} \) is a slide of the puncture \( f(z) \) along the loop \( f(\alpha) \).

The next proposition, which provides a relationship between puncture slides and twists is proved in Section 6.1 of [7].

Proposition 2.3. Let \( \alpha \) be a two–sided simple loop on a surface \( N \), based at the puncture \( z \). Define also \( a \) and \( b \) to be the boundary circles of a regular neighbourhood \( N_\alpha \) of \( \alpha \) such that the orientations of \( \alpha \) and of \( N_\alpha \) are as in Figure 1 (we indicate the orientation of \( N_\alpha \) by choosing the direction of a right twist about \( a \) ). Then \( t_a t_b^{-1} \) is the slide of \( z \) along \( \alpha \).

![Figure 1. Two–sided loop \( \alpha \) and its regular neighbourhood.](image)

Finally, let us recall the so–called lantern relation, which will be our main tool in studying properties of mapping class groups. The proof can be found in Section 4 of [8].

Proposition 2.4. Let \( S \) be a sphere with four holes embedded in a surface \( N \setminus \Sigma \) and let \( a_0, a_1, a_2, a_3 \) be the boundary circles of \( S \). Define also \( a_{1,2}, a_{2,3}, a_{1,3} \) as in Figure 2 and assume that the orientations of regular neighbourhoods of these seven circles are induced from the orientation of \( S \). Then
\[
t_{a_0} t_{a_1} t_{a_2} t_{a_3} = t_{a_{1,2}} t_{a_{2,3}} t_{a_{1,3}}.
\]
orientable surface and one or two projective planes (one for \( g \) odd and two for \( g \) even). Figures 3 and 4 show this model of \( N \) – in these figures the shaded disks represent crosscaps, hence their interiors are to be removed and then the antipodal points on each boundary component are to be identified.

Let \( \mathcal{C} \) be the set of circles indicated in Figure 3 for \( g = 2r + 1 \), and in Figure 4 for \( g = 2r + 2 \). Hence

\[
\mathcal{C} = \{a_1, \ldots, a_r, b_1, \ldots, b_r, c_1, \ldots, c_{r-1}, d_1, \ldots, d_r, e_1, \ldots, e_{n-1}\},
\]

**Figure 2.** Circles of the lantern relation.

**Figure 3.** Circles \( \mathcal{C} \) for \( g = 2r + 1 \).

**Figure 4.** Circles \( \mathcal{C} \) for \( g = 2r + 2 \).
for $g = 2r + 1$, and
\[ C = \{ a_1, \ldots, a_r, b_1, \ldots, b_{r+1}, c_1, \ldots, c_r, d_1, \ldots, d_r, e_1, \ldots, e_{n-1} \}, \]
for $g = 2r + 2$. The figures also indicate our chosen orientations of local neighbourhoods of circles in $C$, the orientation is such that the arrow points to the right if we approach the circle. Therefore by a twist about one of the circles in $C$ we will always mean the twist determined by this particular choice of orientation (recall that the general rule is that we consider right Dehn twists, i.e. if we approach the circle of twisting we turn to the right). In what follows we will often indicate the orientation of a regular neighbourhood of a circle by drawing the direction of a twist.

Let $v_j$ be a slide of a puncture $z_j$ along the loop $\alpha_j$ for $j = 1, \ldots, n$ as in Figure 5. If $g = 2r + 2$, let $w_1, \ldots, w_n$ be puncture slides along $\beta_1, \ldots, \beta_n$ – cf Figure 5.

Define also $y$ to be a crosscap slide such that $y^2$ is a twist about the circle $\xi$ indicated in Figure 6. To be more descriptive, let $N_1$ be the connected component of $N \setminus \xi$ diffeomorphic to a Klein bottle with one boundary component. Then $N_1$ is diffeomorphic to a disk with two crosscaps and we can define $y$ to be a slide of one of these crosscaps along the core of the second one. It turns out that in what follows, the ambiguity in the definition of $y$ is inessential.

**Theorem 3.1** (Theorem 4.13 of [12]). Let $g \geq 3$. Then the mapping class group $\mathcal{PM}(N^n_g)$ is generated by
- $\{ t_l, v_j, y \mid l \in C, 1 \leq j \leq n \}$ if $g$ is odd and
- $\{ t_l, v_j, w_j, y \mid l \in C, 1 \leq j \leq n \}$ if $g$ is even.
Now let us simplify the above generating set for $g$ even (we will replace all the $w_j$'s by a single twist).

Let $\lambda$ be the circle indicated in Figure 7.

**Theorem 3.2.** Let $g = 2r + 2 \geq 4$. Then the mapping class group $\mathcal{PM}(N^g_n)$ is generated by

$$\{ t_l, v_j, y, t_\lambda \mid l \in \mathcal{C}, 1 \leq j \leq n \}.$$ 

**Proof.** Let $G$ denote the group generated by the above elements. By Theorem 3.1, it is enough to prove that $w_j \in G$ for $j = 1, \ldots, n$. Our first claim is that if $\delta_j$ is a circle as in Figure 8(ii) then

$$ (t_{b+1}^{-1}(w_j)t_{b+1})v_j^{-1} = t_{\delta_j}^{-1}. $$

(3.1)

For notational convenience, for any simple loop $\alpha$ based at $z_j$, let $p(\alpha)$ be the slide of $z_j$ along $\alpha$. By Proposition 2.2, the left-hand side of (3.1) can be rewritten as follows (observe that we compose loops from left to right).

$$ (t_{b+1}^{-1}(w_j)t_{b+1})v_j^{-1} = p(t_{b+1}^{-1}(\beta_j))p(\alpha^{-1}) = p(\alpha^{-1}t_{b+1}^{-1}(\beta_j)) $$

It is not hard to check that Figure 8(i) shows the loop $t_{b+1}^{-1}(\beta_j)$. Hence $\alpha^{-1}t_{b+1}^{-1}(\beta_j)$ is the loop shown in Figure 8(ii). By Proposition 2.3, the slide along this loop is equal to $t_{\delta_j}^{-1}$ which completes the proof of (3.1).
Therefore, by equation (3.1), it is enough to prove that \( t_{\delta_i} \in G \). Before we do that we need two lemmas.

**Lemma 3.3.** Let \( e_{i,j} \) and \( \nu_{i,m} \) be the circles shown in Figure 9 for \( i = 1, \ldots, r \), \( j = 0, \ldots, n \) and \( m = 1, \ldots, n \). Then the twists \( t_{e_{i,j}} \) and \( t_{\nu_{i,m}} \) are in the group generated by \( \{ t_l | l \in C \} \).

**Proof.** Let \( H = \langle t_l | l \in C \rangle \). It is straightforward to check that
\[
e_{i,j} = t_{a_i} t_{c_i-1} t_{a_{i-1}} t_{b_i-1} t_{a_i-1} t_{b_{i-1}} t_{a_{i-1}} (e_{i-1,j}).
\]
Moreover, \( e_{0,0} = b_1 \), \( e_{0,n} = d_1 \) and \( e_{0,i} = e_i \) for \( i = 1, \ldots, n - 1 \). Therefore, by induction on \( i \), \( t_{e_{i,j}} \in H \) (cf Proposition 2.1).

The rest of the proof follows, by Proposition 2.1, from the relation
\[
\nu_{i,m} = t_{e_{i,m-1}} t_{a_i} (e_{i,m}).
\]

**Remark 3.4.** For further reference, observe that neither in the definition of the circles \( e_{i,j} \) and \( \nu_{i,m} \) nor in the proof of the above lemma, we used the assumption that \( g \) is even.

**Lemma 3.5.** Let \( \tau \) and \( \tau_j \) be the circles indicated in Figure 10 for \( j = 1, \ldots, n \). Then \( t_{\tau}, t_{\tau_j} \in G \).
Proof. By Proposition 2.1 and Lemma 3.3, the assertion follows from the relations
\[ \tau = t_{d_r} t_{a_r}^2 t_{b_r} t_{d_r}(\lambda), \]
\[ \tau_j = t_{e_{r,j-1}} t_{e_{r,j}}^{-1}(\tau). \]
□

Now let us come back to the proof of Theorem 3.2. As was observed, it is enough to prove that \( t_{\delta_j} \in G \). Observe that the seven circles indicated in Figure 10 form a configuration of the lantern relation (the four circles in Figure 10(i) bound a sphere with four holes). Therefore we have the relation
\[ t_{a_r} t_{\tau_j} = t_{\delta_j} t_{\tau} t_{\nu_{r,j}}. \]
By Lemmas 3.3 and 3.5, this implies that \( t_{\delta_j} \in G \).
\( \square \)

\[ \text{Figure 10. Circles of the lantern relation } t_{a_r} t_{\tau_j} = t_{\delta_j} t_{\tau} t_{\nu_{r,j}}. \]

4. Generators for the group \( \mathcal{PM}^k(N^g_n) \)

The following proposition can be found in any book on combinatorial group theory – see for example Chapter 9 of [9].

**Proposition 4.1.** Let \( X \) be a generating set for a group \( G \) and let \( U \) be a left transversal for a subgroup \( H \) (i.e. \( U \) is a set of representatives of left cosets of \( H \)). Then \( H \) is generated by the set
\[ \{uxux^{-1} : u \in U, x \in X, ux \notin U \}, \]
where \( g = gH \cap U \) for \( g \in G \).
\( \square \)

Let \( f_1, \ldots, f_n \) be the circles indicated in Figure 11.

**Theorem 4.2.** Let \( g \geq 3 \) and \( 0 \leq k \leq n \). Then the mapping class group \( \mathcal{PM}^k(N^g_n) \) is generated by
- \( \{t_l, t_{f_1}, \ldots, t_{f_k}, v_{k+1}, \ldots, v_n, y \mid l \in C \} \) if \( g \) is odd and
- \( \{t_l, t_{f_1}, \ldots, t_{f_k}, v_{k+1}, \ldots, v_n, y, t_\lambda \mid l \in C \} \) if \( g \) is even.
Proof. The proof will be by induction on \( k \). For \( k = 0 \) the theorem follows from Theorems 3.1 and 3.2. Suppose that the theorem is true for \( k - 1 \), i.e. the group \( G = \mathcal{P}\mathcal{M}^{k-1}(N^n_g) \) is generated by the set

\[
\begin{align*}
&\bullet \{ t_l, t_{f_1}, \ldots, t_{f_{k-1}}, v_k, \ldots, v_n, y \mid l \in C \} \text{ if } g \text{ is odd and} \\
&\bullet \{ t_l, t_{f_1}, \ldots, t_{f_{k-1}}, v_k, \ldots, v_n, y, t_\lambda \mid l \in C \} \text{ if } g \text{ is even.}
\end{align*}
\]

If \( H = \mathcal{P}\mathcal{M}^k(N^n_g) \), then \( H \) is of index two in \( G \), hence as a transversal for \( H \) we can take \( U = \{1, v_k\} \). By Proposition 4.1, \( H \) is generated by

\[
\begin{align*}
&\bullet \{ t_l, v_k t_l v_k^{-1}, t_{f_1}, \ldots, t_{f_{k-1}}, v_k t_{f_1} v_k^{-1}, \ldots, v_k t_{f_{k-1}} v_k^{-1}, \\
&v_{k+1}, \ldots, v_n, v_k^{2}, v_k v_{k+1} v_k^{-1}, \ldots, v_k v_n v_k^{-1}, y, v_k y v_k^{-1} \mid l \in C \} \text{ if } g \text{ is odd and} \\
&\bullet \{ t_l, v_k t_{f_1} v_k^{-1}, t_{f_1}, \ldots, t_{f_{k-1}}, v_k t_{f_1} v_k^{-1}, \ldots, v_k t_{f_{k-1}} v_k^{-1}, \\
&v_{k+1}, \ldots, v_n, v_k^{2}, v_k v_{k+1} v_k^{-1}, \ldots, v_k v_n v_k^{-1}, y, v_k y v_k^{-1}, \\
&t_\lambda, v_k t_\lambda v_k^{-1} \mid l \in C \} \text{ if } g \text{ is even}.
\end{align*}
\]

Let \( K \leq G \) be the group generated by

\[
\begin{align*}
&\bullet \{ t_l, t_{f_1}, \ldots, t_{f_k}, v_{k+1}, \ldots, v_n, y \mid l \in C \} \text{ if } g \text{ is odd and} \\
&\bullet \{ t_l, t_{f_1}, \ldots, t_{f_k}, v_{k+1}, \ldots, v_n, y, t_\lambda \mid l \in C \} \text{ if } g \text{ is even.}
\end{align*}
\]

Since each generator of \( K \) preserves the local orientation around each of \( z_1, \ldots, z_k \), we have \( K \leq H \). To complete the proof it is enough to show that \( H \leq K \), i.e. that each generator of \( H \) is in \( K \). Since \( v_k \) commutes with \( t_{a_i}, t_{b_j}, t_{c_i}, t_{e_m}, t_\lambda \) for \( i = 1, \ldots, r, j = 1, \ldots, r - 1, m = 1, \ldots, k - 1 \), it is enough to prove that

\[
\begin{align*}
&\bullet v_k t_{d_1} v_k^{-1}, \ldots, v_k t_{d_r} v_k^{-1}, \\
&v_k t_{e_1} v_k^{-1}, \ldots, v_k t_{e_m} v_k^{-1}, \\
&v_k t_{f_1} v_k^{-1}, \ldots, v_k t_{f_{k-1}} v_k^{-1},
\end{align*}
\]
The rest of this section is devoted to the proof of the above statements.

4.1. Case of \(v_k t_{d_i} v_k^{-1}\).

Lemma 4.3. Let the circles \(\mu_{i,j}\) for \(i = 1, \ldots, r, j = 1, \ldots, k\), be as in Figure 12. Then \(t_{\mu_{i,j}} \in K\).

**Figure 12.** Circles \(\mu_{i,j}\).

Proof. One can check that

\[
\mu_{i,j} = t_{c_{i-1}} t_{a_i} t_{a_{i-1}} t_{c_{i-1}}(\mu_{i-1,j}) \quad \text{for } i = 2, \ldots, r, j = 1, \ldots, k.
\]

Moreover, \(\mu_{r,j} = f_j\). Therefore, by Proposition 2.1, the lemma follows by descending induction on \(i\).

By Lemma 3.3 and Remark 3.4, \(t_{\nu_{i,j}} \in K\), for \(i = 1, \ldots, r, j = 1, \ldots, n\). Therefore by Lemma 4.3, Proposition 2.1, and by the relation

\[
v_k(d_i) = t^{-1}_{\nu_{i,k}} t_{d_i}(v_k),
\]

we obtain \(v_k t_{d_i} v_k^{-1} \in K\), for \(i = 1, \ldots, r\).
4.2. **Case of** $v_k e_j v_k^{-1}$. As above, by the relation
\[ v_k(e_j) = t_{\mu,j}^{-1} t_{e_j}(\nu_{1,k}), \]
we obtain $v_k e_j v_k^{-1} \in K$, for $j = k, \ldots, n$.

4.3. **Case of** $v_k b_{r+1} v_k^{-1}$, $g$ - even. Let $\rho = t_{d_r}(c_r)$ and $\rho_k = t_{e_r,k}^{-1} t_{e_{r,k-1}}(\rho)$ (cf Figure 13). By Lemma 3.3 and Remark 3.4, $t_{\rho}, t_{\rho_k} \in K$. Now the seven circles indicated in Figure 13 form a configuration of the lantern relation (the four circles in Figure 13(i) bound a sphere with four holes). Moreover, one of the circles in Figure 13(ii) is $v_k(b_{r+1})$. Therefore we have the relation
\[ t_{a_r} t_{\rho_k} t_{b_{r+1}} = t_{\nu_{k,v_{r+k}}} t_{\rho} t_{\nu_{r+1,k}}. \]
This proves that $t_{\nu_{k,v_{r+k}}} = v_k b_{r+1} v_k^{-1} \in K$.

4.4. **Case of** $v_k^2$. First observe that if $\lambda$ is as in Figure 7, then $t_{\lambda} \in K$. For $g$ even this follows from the definition of $K$, and for $g$ odd we have the relation $\lambda = t_{b_r}^{-1}(a_r)$. Therefore if $\omega = t_{d_r}(\lambda)$ and $\omega_k = t_{e_{r,k-1}}^{-1} t_{e_{r,k}}(\omega)$ (cf Figure 14) then $t_{\omega}, t_{\omega_k} \in K$. Now the seven circles indicated in

![Diagram](image1.png)

**Figure 13.** Circles of the lantern relation $t_{a_r} t_{\rho_k} t_{b_{r+1}} = t_{\nu_{k,v_{r+k}}} t_{\rho} t_{\nu_{r+1,k}}$.

![Diagram](image2.png)

**Figure 14.** Circles of the lantern relation $t_{a_r} t_{\omega_k} = t_{h_k} t_{\omega} t_{\nu_{r,k}}$. Figure 14 form a configuration of the lantern relation. Hence we have
\[ t_{a_r} t_{\omega_k} = t_{h_k} t_{\omega} t_{\nu_{r,k}}. \]
This proves that $v_k^2 = t_{h_k}^{-1} \in K$.

4.5. **Case of** $v_k t_{e_r} v_k^{-1}$, $g$ **even**. By the relation

$$v_k^{-1}(c_r) = t_{e_r} t_{e_{r,k-1}} t_{e_{e_{r,k}}} t_{e_r}^{-1}(c_r),$$

we have $v_k^{-1} t_{e_r} v_k \in K$. Since we proved that $v_k^2 \in K$, this implies that $v_k t_{e_r} v_k^{-1} \in K$.

4.6. **Case of** $v_k t_{f_j} v_k^{-1}$. By the relation $v_k^{-1}(f_j) = t_{f_k}(f_j)$, we have $v_k^{-1} t_{f_j} v_k \in K$ for $j = 1, \ldots, k-1$. Since $v_k^2 \in K$, this implies that $v_k t_{f_j} v_k^{-1} \in K$.

4.7. **Case of** $v_k v_j v_k^{-1}$. Using Propositions 2.2 and 2.3, it is straightforward to check that

$$v_j^{-1}(v_k v_j v_k^{-1}) = t_{\varepsilon_{k,j}}^{-1},$$

where $\varepsilon_{k,j}$ is the circle indicated in Figure 15(ii) for $j = k+1, \ldots, n$. Therefore it is enough to prove that $t_{\varepsilon_{k,j}} \in K$. Let

**Figure 15.** Circles of the lantern relation $t_{a_r} t_{\chi_{k,j}} = t_{\nu_{r,j}} t_{\nu_{r,k}} t_{\varepsilon_{k,j}}$.

$$\chi_{k,j} = t_{e_{r,k}}^{-1} t_{e_{r,j}}^{-1} t_{e_{r,j-1}} t_{e_{r,k-1}}^{-1}(e_{r,k}).$$

Clearly $t_{\chi_{k,j}} \in K$. Now the seven circles indicated in Figure 15 form a configuration of the lantern relation. Hence we have

$$t_{a_r} t_{\chi_{k,j}} = t_{\nu_{r,j}} t_{\nu_{r,k}} t_{\varepsilon_{k,j}}.$$}

This proves that $t_{\varepsilon_{k,j}} \in K$.
4.8. **Case of** \( v_k y v_k^{-1} \). Let \( y' = t_{v_k} y v_k^{-1} y v_k t_{v_k}^{-1} \). Since \( v_k^2, t_{v_k} \in K \), it is enough to prove that \( y' \in K \). By the relation \( t_{v_k}(v_k^{-1}(\xi)) \simeq \xi \), we have \( y'(\xi) \simeq \xi \). Therefore we can assume that \( y'(\xi) = \xi \). Let \( N_1 \) be this of two connected components of \( N \setminus \xi \) which is a support for \( y' \) (i.e. \( y' \) acts as the identity on the second component – since \( y' \) is conjugate to \( y \) this component is well defined). It is known that the mapping class group \( \mathcal{M}(N_1) \) is generated by \( y \) and \( t_{a_r} \) (cf Theorem A.7 of [23]), hence \( y' \) as a composition of these elements is in \( K \).

\[ \square \]

5. **Generators for groups** \( \mathcal{PM}^k(N_{g,s}^n) \) and \( \mathcal{M}(N_{g,s}^n) \)

The following proposition is well known and easy to prove, hence we state it without proof.

**Proposition 5.1.** Suppose that we have a short exact sequence of groups and homomorphisms

\[
1 \longrightarrow H \xrightarrow{i} G \xrightarrow{p} K \longrightarrow 1
\]

and let \( X_H \) and \( X_K \) be generating sets for \( H \) and \( K \) respectively. Let \( \tilde{X}_K \) be any subset of \( G \) such that \( p(\tilde{X}_K) = X_K \). Then \( G \) is generated by \( i(X_H) \cup \tilde{X}_K \).

\[ \square \]

Let \( \tilde{N} = N_{g,s}^{n+s} \) be the surface obtained from \( N = N_{g,s}^n \) by gluing a disk with one puncture to each boundary component and let \( 1 \leq k \leq n \) be an integer. We choose the notation in such a way that the first \( s \) of \( n + s \) punctures of \( \tilde{N} \) correspond to the boundary components of \( N \). Identifying \( N \) with a subsurface of \( \tilde{N} \), we can consider the circles in \( C, f_1, \ldots, f_{s+k} \) and \( \lambda \) as circles on \( N \). Similarly we will consider the puncture slides \( v_{s+k+1}, \ldots, v_{s+n} \) and the crosscap slide \( y \) as elements of \( \mathcal{PM}^k(N) \).

By Theorem 2.2 of [20], we have an exact sequence

\[
1 \longrightarrow \mathbb{Z}^s \longrightarrow \mathcal{PM}^k(N) \xrightarrow{i_*} \mathcal{PM}^{s+k}(\tilde{N}) \longrightarrow 1
\]

where \( i_* \) is the homomorphism induced by the inclusion \( i: N \to \tilde{N} \). Moreover, the generators of \( \ker i_* \simeq \mathbb{Z}^s \) correspond to the boundary twists \( t_{u_1}, \ldots, t_{u_s} \) on \( N \). Let \( C' = C \cup \{f_1, \ldots, f_{s+k}\} \cup \{u_1, \ldots, u_s\} \). Theorem 4.2 together with Proposition 5.1 implies the following.

**Theorem 5.2.** Let \( g \geq 3 \). Then the mapping class group \( \mathcal{PM}^k(N_{g,s}^n) \) is generated by

- \( \{t_l, v_{s+k+1}, \ldots, v_{s+n}, y \mid l \in C'\} \) if \( g \) is odd and
- \( \{t_l, v_{s+k+1}, \ldots, v_{s+n}, y, t_{\lambda} \mid l \in C'\} \) if \( g \) is even.

\[ \square \]
Now let us turn to the group $M(N_{g,s}^n)$. Let $\sigma_{s+1}, \ldots, \sigma_{s+n-1}$ be elementary braids on $N$, such that $\sigma_j^2 = \epsilon_{j,j+1}$ (cf Figure 16), where $\epsilon_{j,j+1}$ is a circle defined by Figure 15(ii). Let $C'$ be defined as before, i.e. $C' = C \cup \{f_1, \ldots, f_s\} \cup \{u_1, \ldots, u_s\}$.

**Theorem 5.3.** Let $g \geq 3$ and $n \geq 2$. Then the mapping class group $\mathcal{M}(N_{g,s}^n)$ is generated by

- $\{t_l, v_{s+1}, \sigma_{s+1}, \ldots, \sigma_{s+n-1}, y \mid l \in C'\}$ if $g$ is odd and
- $\{t_l, v_{s+1}, \sigma_{s+1}, \ldots, \sigma_{s+n-1}, y, t_\lambda \mid l \in C'\}$ if $g$ is even.

**Proof.** From the short exact sequence

$$1 \longrightarrow PM(N_{g,s}^n) \overset{1}{\longrightarrow} \mathcal{M}(N_{g,s}^n) \overset{p}{\longrightarrow} S_n \longrightarrow 1$$

where $S_n$ is the symmetric group on $n$ letters, by Theorem 5.2, Proposition 5.1 and by the fact that $p(\sigma_{s+1}), \ldots, p(\sigma_{s+n-1})$ generate $S_n$, we conclude that $\mathcal{M}(N_{g,s}^n)$ is generated by

- $\{t_l, v_{s+1}, \ldots, v_{s+n}, \sigma_{s+1}, \ldots, \sigma_{s+n-1}, y \mid l \in C'\}$ if $g$ is odd and
- $\{t_l, v_{s+1}, \ldots, v_{s+n}, \sigma_{s+1}, \ldots, \sigma_{s+n-1}, y, t_\lambda \mid l \in C'\}$ if $g$ is even.

Since $v_j = \sigma_{j-1}^{-1}v_{j-1}\sigma_{j-1}$, for $j = s+2, \ldots, s+n$, the generators $v_{s+2}, \ldots, v_{s+n}$ can be removed from the above generating sets. \qed

### 6. Homological results for mapping class groups

The main goal of this section is to compute the first homology groups of the mapping class groups $PM_k(N_{g,s}^n)$ and $\mathcal{M}(N_{g,k}^n)$ for $g \geq 3$. Before we do this, we need some technical preparations.

Let $N = N_{g,s}^n$ and $G = PM_k(N)$. Moreover, for $f \in G$ let $[f]$ denote the homology class of $f$ in $H_1(G)$ (we will use the additive notation in $H_1(G)$).

**6.1. Homology classes of twists.** It is well known that two right twists about nonseparating circles on an oriented surface are conjugate in the mapping class group of this surface. The description of conjugacy classes of twists about nonseparating circles on a nonorientable surface

![Figure 16. Elementary braid $\sigma_j$.](image-url)
is slightly more difficult – it can be shown [24] that there are at least $2^{k+s-1} + 1$ such classes in the group $\mathcal{PM}^k(N^n_{g,s})$. However, for our purposes it will suffice to consider a very simple case of this description, namely when the complement of a circle in $N$ is nonorientable – see Proposition 6.4.

**Lemma 6.1.** Let $c$ be a nonseparating two–sided circle on $N$ such that $N \setminus c$ is nonorientable. Then there exists $f \in G$ such that $f(c) = a_1$ or $f(c) = a_1^{-1}$, where $a_1$ is as in Figures 3 and 4.

**Proof.** Clearly $N \setminus c$ and $N \setminus a_1$ are diffeomorphic. It is a standard argument that we can choose this diffeomorphism in such a way that it extends to a diffeomorphism $f : N \to N$ such that $f \in G$ and $f(c) = a_1^\pm 1$. □

**Lemma 6.2.** Let $c$ be a nonseparating two–sided circle on a Klein bottle $N = N_{2,1}$ with one boundary component. Then there exists $f \in \mathcal{M}(N)$ such that $ft_c f^{-1} = t_c^{-1}$.

**Proof.** The assertion follows from the structure of $\mathcal{M}(N_{2,1})$ – cf Theorem A.7 of [23]. □

**Lemma 6.3.** Let $c$ be a nonseparating two–sided circle on $N$ such that $N \setminus c$ is nonorientable. Then $t_c$ and $t_c^{-1}$ are conjugate in $G$. In particular $2[t_c] = [t_c^2] = 0$.

**Proof.** By Lemma 6.1, $t_c$ is conjugate to $t_{a_1}$ or to $t_{a_1}^{-1}$. Hence the assertion follows form Lemma 6.2 applied to the Klein bottle cut off by a circle $\xi$ in Figure 6. □

**Proposition 6.4.** Let $c$ be a nonseparating two–sided circle on $N$ such that $N \setminus c$ is nonorientable. Then the twist $t_c$ is conjugate to $t_{a_1}$ in $G$. In particular $[t_c] = [t_{a_1}]$ in $H_1(G)$.

**Proof.** By Lemma 6.1, $t_c$ is conjugate to $t_{a_1}$ or to $t_{a_1}^{-1}$. Hence Lemma 6.3 implies that $t_c$ is conjugate to $t_{a_1}$. □

**Lemma 6.5.** Assume $g = 2r + 2 \geq 4$. Then $t_{b_{r+1}}$ and $t_{b_{r+1}}^{-1}$ are conjugate in $G$, where $b_{r+1}$ is as in Figure 4. In particular $2[t_{b_{r+1}}] = [t_{b_{r+1}}^2] = 0$.

**Proof.** To obtain the conclusion it is enough to apply Lemma 6.2 to the Klein bottle cut off by a circle $\xi'$ indicated in Figure 17. □

**Lemma 6.6.** Assume $g = 2r + 2 \geq 6$. Then $[t_{b_{r+1}}] = 0$.

**Proof.** Figure 18 shows that there is a lantern configuration with one twist $t_{b_{r+1}}$ and each of the remaining six circles is conjugate to $t_{a_1}$ (cf Proposition 6.4). Hence we have the relation (in $H_1(G)$)
Lemma 6.7. Assume $g \geq 7$. Then $|t_{a_1}| = 0$.

Proof. Figure 19 shows that there is a lantern configuration with all twists conjugate to $t_{a_1}$. Hence we have the relation

$$[t_{b_{r+1}} t_{a_1} t_{a_1} t_{a_1}] = [t_{a_1} t_{a_1} t_{a_1}].$$

\[\Box\]

6.2. Homology classes of crosscap slides.

Lemma 6.8. Let $y$ be a crosscap slide on a Klein bottle $N = N_{2,1}$ with one boundary component. Then there exists a diffeomorphism $f : N \to N$ such that $fyf^{-1} = y^{-1}$ and $f|_{\partial N} = -id$. 
Proof. The lemma can be easily deduced from the proof of Theorem 5.8 of [12]. It is also a direct consequence of the structure of $\mathcal{M}(N)$ – cf Theorem A.7 of [23].

Lemma 6.9. Let $y$ be a crosscap slide on $N$ such that $y^2$ is a twist $t_\xi$ about a circle which separates $N$ into two nonorientable components. Then $y$ and $y^{-1}$ are conjugate in $G$. In particular $2[y] = [y^2] = 0$.

Proof. Let $N_1$ and $N_2$ be components of $N \setminus \xi$ and assume that $N_1$ is the support for $y$. By Lemma 6.8, it is enough to prove that there exists a diffeomorphism of $N_2$ which preserves the local orientation around each of the punctures, acts as the $-id$ on $\partial N_2$ coming from $\xi$ and as the $id$ on each of the remaining boundary components. Since $N_2$ is nonorientable such diffeomorphism can be obtained by sliding the boundary component coming from $\xi$ along a one–sided loop. □

6.3. Homology classes of puncture slides.

Lemma 6.10. Assume that $g \geq 3$. Then $2[v_j] = [v_j^2] = [t_{h_j}^{-1}] = 0$, for $j = s + k + 1, \ldots, s + n$.

Proof. Let us recall the instance of the lantern relation from Section 4.4 (cf Figure 14). After replacing $k$ with $j$ we can write it as

$$t_a r t_\omega r t_\omega t_{v_{r,j}}.$$ 

By Proposition 6.4, $[t_a r] = [t_\omega r] = [t_\omega] = [t_{v_{r,j}}]$. Hence $[t_{h_j}] = 0$. □

Lemma 6.11. Assume that $n \geq 2$. Then all the puncture slides $v_{s+1}, \ldots, v_{s+n}$ are contained in the same conjugacy class in $\mathcal{M}(N)$. In particular all these elements are equal in $H_1(\mathcal{M}(N))$.

Proof. The assertion follows inductively by the relation

$$v_{j+1} = \sigma_j^{-1} v_j \sigma_j$$ for $j = s + 1, \ldots, s + n - 1$. □

6.4. Homology classes of boundary twists.

Lemma 6.12. Assume that $g \geq 5$. Then the boundary twists $t_{u_1}, \ldots, t_{u_s}$ are trivial in $H_1(G)$.

Proof. Figure 20 shows that for each $1 \leq j \leq s$, there exists a lantern configuration with one twist $t_{u_j}$ and all other twists conjugate to $t_{a_1}$. Hence we have

$$[t_{u_j} t_{a_1} t_{a_1} t_{a_1}] = [t_{a_1} t_{a_1} t_{a_1}].$$ □
Lemma 6.13. Let $\kappa$ be a circle on $N$ as in Figure 21 for $g = 3$ and as in Figure 22 for $g = 4$. Then $[t_\kappa] = 0$.

Proof. Suppose first that $g = 3$ and let the circles $\kappa_1, \kappa_2, \kappa_3$ be as in Figure 21. For each $i = 1, 2, 3$ we have $t_{\kappa_i} = y_i^2$, where $y_i$ is a crosscap slide satisfying the assumptions of Lemma 6.9. In fact, we can define $y_i$ to be a crosscap slide on the component of $N \setminus \kappa_i$ which is diffeomorphic to a Klein bottle with one boundary component. By Lemma 6.9, $[t_{\kappa_1}] = [t_{\kappa_2}] = [t_{\kappa_3}] = 0$. Moreover, Figure 21 shows that there is a lantern relation

$$[t_\kappa] = [t_{\kappa_1}t_{\kappa_2}t_{\kappa_3}].$$

The case $g = 4$ is very similar. Observe that by the reasoning for $g = 3$, the homology classes of all the circles but $\kappa$ in Figure 22 are trivial. Hence the homology class of $\kappa$ is also trivial. □
By a region of a surface $N$ we will mean any closed connected sub-surface $\Delta$ of $N$ of genus 0 which satisfies the following two conditions.

1. $\Delta$ has one distinguished boundary component which is disjoint from the set of punctures and from the boundary of $N$. We will denote this distinguished boundary component by $\partial \Delta$.
2. Every boundary component of $\Delta$ different from $\partial \Delta$ is a boundary component of $N$.

In other words $\Delta$ is a disk with punctures and/or boundary components of $N$ imbedded in $N$.

Lemma 6.14. Let $\Delta$ be a region of a surface $N = N_{g,s}^n$ for $g \geq 3$, such that

- $\{ j \mid j \in \{1, \ldots, s\} \text{ and } u_j \subset \Delta \} = \Theta$,
- $\{ j \mid j \in \{s + 1, \ldots, s + n\} \text{ and } z_j \in \Delta \} = \Omega$,
- if $\Theta \neq \emptyset$ then the orientation of $\Delta$ agrees with the orientation of a neighbourhood of $u_\theta$ for $\theta \in \Theta$ (in other words $t_{u_\theta}$ is a right twist on $\Delta$).
- the orientation of a neighbourhood of $\partial \Delta$ agrees with the orientation of $\Delta$.

Then

$$[t_{\partial \Delta}] = \left[ \prod_{\theta \in \Theta} t_\theta \right] = \sum_{\theta \in \Theta} [t_\theta].$$

Proof. The proof is by induction on $|\Omega \cup \Theta|$. For $|\Omega \cup \Theta| = 1$ there is nothing to prove, and to make the inductive step, assume that $\Delta'$ is obtained from $\Delta$ by adding one puncture/boundary component. Figure 23 shows that we have the lantern relation

$$[t_{\partial \Delta} t_u t_{a_1} t_{a_1}] = [t_{\partial \Delta'} t_{a_1} t_{a_1}],$$

where $u$ is a circle around the added puncture/boundary component.

![Figure 23. Lantern relation $[t_{\partial \Delta} t_u t_{a_1} t_{a_1}] = [t_{\partial \Delta'} t_{a_1} t_{a_1}]$.](image)

Therefore $[t_{\partial \Delta'}] = [t_{\partial \Delta}] + [t_u]$ and $t_u$ is trivial if we added a puncture.
Lemma 6.15. Let \( g = 3 \) or \( g = 4 \), then \( [t_{u_1}] + [t_{u_2}] + \cdots + [t_{u_s}] = 0 \).

Proof. By Lemma 6.14, the homology class of \( t_{u_1}t_{u_2}\cdots t_{u_s} \) is equal to the homology class of a twist about a circle which is the boundary of a region containing all the punctures and boundary components, i.e. along the circle \( \kappa \) in Figure 21 for \( g = 3 \) and Figure 22 for \( g = 4 \) respectively. By Lemma 6.13, \([t_\kappa] = 0\). \(\square\)

Proposition 6.16. Let \( c \) be a two–sided circle on \( N \) such that \( c \) separates \( N \) into two nonorientable surfaces. Then \( t_c \) and \( t_c^{-1} \) are conjugate in \( G \). In particular \( 2[t_c] = [t_c^2] = 0 \).

Proof. Let \( N_1 \) and \( N_2 \) be components of \( N \setminus c \). For each \( i = 1, 2 \) there exists a diffeomorphism \( f_i \) of \( N_i \) which preserves the local orientation around each of the punctures, acts as the \(-id\) on \( \partial N_i \) coming from \( c \) and as the \( id \) on each of the remaining boundary components. In fact, since \( N_i \) is nonorientable such a homeomorphism can be obtained by sliding the boundary component coming from \( c \) along a one–sided loop. Now if \( f \) is a diffeomorphism of \( N \) obtained by gluing \( f_1 \) and \( f_2 \), then \( ft_c f^{-1} = t_c^{-1} \). \(\square\)

Lemma 6.17. Let \( g \geq 3 \), then \( 2[t_{u_j}] = [t_{u_j}^2] = 0 \).

Proof. Figure 24 shows that there is a lantern relation

\[
[t_{u_j}t_{a_1}t_{a_1}] = [t_{\eta_j}t_{a_1}t_{a_1}].
\]

Hence \( [t_{u_j}] = [t_{\eta_j}] \) and by Proposition 6.16, \( [t_{\eta_j}^2] = 0 \). \(\square\)

Remark 6.18. Observe that since the identity \( i \): \( N_{g,s}^n \to N_{g,s}^n \) induces a homomorphism \( i_\ast: \mathcal{PM}^k(N_{g,s}^n) \to \mathcal{M}(N_{g,s}^n) \), Lemmas 6.3, 6.5, 6.6, 6.7, 6.9, 6.10, 6.12, 6.14, 6.15, 6.17 and Proposition 6.4 remain true if we replace the group \( \mathcal{PM}^k(N_{g,s}^n) \) with \( \mathcal{M}(N_{g,s}^n) \).
6.5. Homology classes of elementary braids.

**Lemma 6.19.** Let $\sigma_{s+1}, \ldots, \sigma_{s+n-1}$ be elementary braids on $N = N_{g,s}^n$ as in Theorem 5.3. Then $\sigma_{s+1}, \ldots, \sigma_{s+n-1}$ are in a single conjugacy class in $\mathcal{M}(N)$. In particular all these elements are equal in $H_1(\mathcal{M}(N))$.

**Proof.** The assertion follows inductively from the braid relation

$$\sigma_{j+1} = (\sigma_j \sigma_{j+1}) (\sigma_j \sigma_{j+1})^{-1} \quad \text{for} \quad j = s+1, \ldots, s+n-2.$$

□

**Lemma 6.20.** Let $g \geq 3$. Then $[\sigma_{s+1}^2] = 0$ in $H_1(\mathcal{M}(N))$.

**Proof.** Since $\sigma_{s+1}^2$ is a twist about the boundary of a region containing the punctures $z_{s+1}$ and $z_{s+2}$, the assertion follows from Lemma 6.14 (cf Remark 6.18).

□

6.6. Main theorems.

**Theorem 6.21.** Let $N = N_{g,s}^n$. Then

$$H_1(\mathcal{PM}^k(N), \mathbb{Z}) \cong \begin{cases} \mathbb{Z}_{2}^{2+n-k} & \text{if } g = 3 \text{ and } s = 0, \\ \mathbb{Z}_{2}^{1+n-k+s} & \text{if } g = 3 \text{ and } s \geq 1, \\ \mathbb{Z}_{2}^{3+n-k} & \text{if } g = 4 \text{ and } s = 0, \\ \mathbb{Z}_{2}^{2+n-k+s} & \text{if } g = 4 \text{ and } s \geq 1, \\ \mathbb{Z}_{2}^{2+n-k} & \text{if } g = 5, 6, \\ \mathbb{Z}_{2}^{1+n-k} & \text{if } g \geq 7. \end{cases}$$

**Proof.** By Theorem 5.2 and Proposition 6.4, $H_1(\mathcal{PM}^k(N))$ is generated by

- $[t_{u_1}], [y], [v_{s+k+1}], \ldots, [v_{s+n}], [t_{u_1}], \ldots, [t_{u_s}]$ if $g$ is odd and
- $[t_{u_1}], [t_{b_{r+1}}], [y], [v_{s+k+1}], \ldots, [v_{s+n}], [t_{u_1}], \ldots, [t_{u_s}]$ if $g$ is even.

By Lemmas 6.3, 6.5, 6.9, 6.10 and 6.17, each of these generators has order at most 2. Moreover,

- $[t_{u_1}] = 0$ if $g \geq 7$ (Lemma 6.7),
- $[t_{u_{r+1}}] = 0$ if $g \geq 6$ (Lemma 6.6),
- $[t_{u_1}] = \ldots = [t_{u_s}] = 0$ if $g \geq 5$ (Lemma 6.12),
- $[t_{u_s}] = -([t_{u_1}] + \cdots + [t_{u_{s-1}}])$ if $g = 3, 4$ (Lemma 6.15).

Therefore $H_1(\mathcal{PM}^k(N))$ is generated by

- $[t_{u_1}], [y], [v_{s+k+1}], \ldots, [v_{s+n}], [t_{u_1}], \ldots, [t_{u_{s-1}}]$ if $g = 3$,
- $[t_{u_1}], [t_{b_{r+1}}], [y], [v_{s+k+1}], \ldots, [v_{s+n}], [t_{u_1}], \ldots, [t_{u_{s-1}}]$ if $g = 4$,
- $[t_{u_1}], [y], [v_{s+k+1}], \ldots, [v_{s+n}]$ if $g = 5, 6$,
- $[y], [v_{s+k+1}], \ldots, [v_{s+n}]$ if $g \geq 7$. 


To finish the proof it is enough to show that these elements are independent over \( \mathbb{Z}_2 \).

Suppose that
\[
[t^\alpha_{a_1} t_{b_{r+1}}^\beta y^\gamma v_{s+k+1}^{\nu_{s+k+1}} \ldots v_{s+n}^{\nu_{s+n}} u_1^{\mu_1} \ldots u_{s-1}^{\mu_{s-1}}] = 0,
\]
for some \( \alpha, \beta, \gamma, \nu, \mu \in \mathbb{Z}_2 \) such that \( \alpha = 0 \) for \( g \geq 7 \), \( \beta = 0 \) for \( g \neq 4 \) and \( \mu_1 = \ldots = \mu_{s-1} = 0 \) for \( g \geq 5 \). Our first goal is to show that
\( \alpha = \beta = \gamma = 0 \).

Let \( \tilde{N} \) be the closed surface obtained from \( N \) by forgetting the punctures and gluing a disk to each boundary component of \( N \). The inclusion \( N \hookrightarrow \tilde{N} \) induces the homomorphism
\[
\Phi : \mathcal{M}(N) \rightarrow \mathcal{M}(\tilde{N}).
\]

Clearly \( \Phi(v_i) = \Phi(t_{u_i}) = 1 \), hence equation (6.1) yields
\[
\alpha[t_{a_1}] + \beta[t_{b_{r+1}}] + \gamma[y] = [t^\alpha_{a_1} t_{b_{r+1}}^\beta y^\gamma] = 0 \quad \text{in } H_1(\mathcal{M}(\tilde{N})).
\]

By Theorem 1.1 of [10],
\[
H_1(\mathcal{M}(\tilde{N})) \cong \begin{cases} 
\langle t_{a_1}, y \rangle & \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \text{if } g = 3, 5, 6, \\
\langle t_{a_1}, t_{b_{r+1}}, y \rangle & \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \text{if } g = 4, \\
\langle y \rangle & \cong \mathbb{Z}_2 & \text{if } g \geq 7.
\end{cases}
\]

Hence \( \alpha = \beta = \gamma = 0 \) and equation (6.1) takes form
\[
[v_{s+k+1}^{\nu_{s+k+1}} \ldots v_{s+n}^{\nu_{s+n}} u_1^{\mu_1} \ldots u_{s-1}^{\mu_{s-1}}] = 0.
\]

In order to show that \( v_{s+k+1} = \ldots = v_{s+n} = 0 \), define
\[
\Psi_j : \mathcal{M}(N) \rightarrow \mathbb{Z}_2,
\]
for \( j = s + k + 1, \ldots, s + n \), as follows: \( \Psi_j(f) = 1 \) if \( f \) reverses the local orientation around the puncture \( z_j \), and \( \Psi_j(f) = 0 \) otherwise. Since \( \Psi_j(v_i) = 0 \) for \( i \neq j \) and \( \Psi_j(t_{u_i}) = 0 \), equation (6.2) implies that
\[
\nu_j = \nu_j[\Psi_j(v_j)] = [\Psi_j(v_j^\nu_j)] = 0 \quad \text{in } H_1(\mathbb{Z}_2) = \mathbb{Z}_2.
\]

Therefore equation (6.2) becomes
\[
[t^\mu_1_{u_1} \ldots t_{u_{s-1}}^{\mu_{s-1}}] = 0.
\]

This completes the proof for \( g \geq 5 \), hence assume that \( g = 3 \) or \( g = 4 \). We can also assume that \( s \geq 2 \).

Let \( \tilde{N}_j \), for \( j = 1, \ldots, s - 1 \), be the surface obtained from \( N \) by forgetting the punctures, gluing a cylinder to the circles \( u_j \) and \( u_{s} \) and finally, gluing a disk to each of the remaining boundary components.
Then $\tilde{N}_j$ is a nonorientable surface of genus either 5 or 6, with neither punctures nor boundary components. Let

$$\Upsilon_j : \mathcal{M}(N) \to \mathcal{M}(\tilde{N}_j)$$

be the homomorphism induced by inclusion. Since $\Upsilon_j(t_{u_l}) = 0$ for $l \neq j$ and $l \neq s$, equation (6.3) gives us

$$\mu_j[t_{u_j}] = [t_{v_j}] = 0 \quad \text{in } H_1(\mathcal{M}(\tilde{N}_j)).$$

Since $u_j$ is a nonseparating circle on $\tilde{N}_j$ and $\tilde{N}_j \setminus u_j$ is nonorientable, by Theorem 1.1 of [10], $[t_{u_j}] \neq 0$ in $H_1(\mathcal{M}(\tilde{N}_j))$, hence $\mu_j = 0$. □

**Theorem 6.22.** Let $N = N_{g,s}^n$ with $n \geq 2$. Then

$$H_1(\mathcal{M}(N), \mathbb{Z}) \cong \begin{cases} 
\mathbb{Z}_2^4 & \text{if } g = 3 \text{ and } s = 0, \\
\mathbb{Z}_2^{s+3} & \text{if } g = 3 \text{ and } s \geq 1, \\
\mathbb{Z}_2^5 & \text{if } g = 4 \text{ and } s = 0, \\
\mathbb{Z}_2^{s+4} & \text{if } g = 4 \text{ and } s \geq 1, \\
\mathbb{Z}_2^4 & \text{if } g = 5, 6, \\
\mathbb{Z}_2^3 & \text{if } g \geq 7.
\end{cases}$$

**Proof.** The proof follows similar lines to the proof of Theorem 6.21. By Theorem 5.3, Propositions 6.4 and 6.19, Lemmas 6.6, 6.7, 6.11, 6.12 and 6.15 (cf Remark 6.18), $H_1(\mathcal{M}(N))$ is generated by

- $[t_{a_1}], [v_{s+1}], [y], [\sigma_{s+1}], [t_{u_1}], \ldots, [t_{u_{s-1}}]$ if $g = 3$,
- $[t_{a_1}], [v_{s+1}], [t_{b_{s-1}}], [y], [\sigma_{s+1}], [t_{u_1}], \ldots, [t_{u_{s-1}}]$ if $g = 4$,
- $[t_{a_1}], [v_{s+1}], [y], [\sigma_{s+1}]$ if $g = 5, 6$,
- $[v_{s+1}], [y], [\sigma_{s+1}]$ if $g \geq 7$.

Moreover, by Lemmas 6.3, 6.5, 6.9, 6.10, 6.17 and 6.20 (cf Remark 6.18), each of these generators has order at most 2.

As in the proof of Theorem 6.21 suppose that

$$[\nu_{a_1}^{\alpha} \nu_{b_{s-1}}^{\beta} \nu_{y_{s+1}}^{\gamma} \nu_{\sigma_{s+1}}^{\varepsilon} \nu_{t_{u_1}}^{\mu_1} \cdots \nu_{t_{u_{s-1}}}^{\mu_{s-1}}] = 0,$$

for some $\alpha, \beta, \gamma, \varepsilon, \nu, \mu_1 \in \mathbb{Z}_2$. As in the proof of Theorem 6.21 we conclude that $\alpha = \beta = \gamma = \mu_1 = \ldots = \mu_{s-1} = 0$ and equation (6.4) becomes

$$[\nu_{v_{s+1}}^{\nu_{\sigma_{s+1}}^{\varepsilon}}] = 0.$$

Let

$$\Theta : \mathcal{M}(N) \to \mathbb{Z}_2$$

be the homomorphism defined as follows: $\Theta(f)$ is the sign of a permutation

$$(z_{s+1}, \ldots, z_{s+n}) \mapsto (f(z_{s+1}), \ldots, f(z_{s+n}))$$
(we use the additive notation for $\mathbb{Z}_2$, hence 0 and 1 means even and odd permutation respectively). Clearly $\Theta(v_{s+1}) = 0$ and $\Theta(\sigma_{s+1}) = 1$, hence by equation (6.5),

$$\varepsilon = \varepsilon[\Theta(\sigma_{s+1})] = [\Theta(\sigma_{s+1}^c)] = 0 \quad \text{in} \quad H_1(\mathbb{Z}_2) = \mathbb{Z}_2.$$ 

It remains to show that $\nu = 0$. Let $\Delta_{s+1}, \ldots, \Delta_{s+n}$ be the collection of disjoint oriented disks on $N$ such that $\Delta_j \cap \{z_{s+1}, \ldots, z_{s+n}\} = z_j$. Up to isotopy, every $f \in \mathcal{M}(N)$ restricts to $\bigcup_{j=s+1}^{s+n} \Delta_j$. For each $f \in \mathcal{M}(N)$ define

$$\Psi_j : \mathcal{M}(N) \to \mathbb{Z}_2$$

as follows: $\Psi_j(f) = 0$ if $f|_{\Delta_j} : \Delta_j \to f(\Delta_j)$ is orientation preserving and $\Psi_j(f) = 1$ otherwise (observe that $\Psi_j$ is just a map – it is not a homomorphism). Let $\Psi = \Psi_{s+1} + \cdots + \Psi_{s+n}$. It is not difficult to check that

$$\Psi : \mathcal{M}(N) \to \mathbb{Z}_2$$

is a homomorphism. Since $\Psi(v_{s+1}) = 1$, $[v_{s+1}] \neq 0$ in $H_1(\mathcal{M}(N))$, hence $\nu = 0$. \hfill $\square$

REFERENCES

[1] J. S. Birman. Mapping class groups and their relationship to braid group. *Comm. Pure Appl. Math.*, 22:213–238, 1969.

[2] J. S. Birman and D. R. J. Chillingworth. On the homeotopy group of a non–orientable surface. *Math. Proc. Cambridge Philos. Soc.*, 71:437–448, 1972.

[3] T. E. Brendle and B. Farb. Every mapping class group is generated by 6 involutions. *J. Algebra*, 278(1):187–198, 2004.

[4] D. R. J. Chillingworth. A finite set of generators for the homeotopy group of a non–orientable surface. *Math. Proc. Cambridge Philos. Soc.*, 65:409–430, 1969.

[5] D. B. A. Epstein. Curves on 2–manifolds and isotopies. *Acta Math.*, 115:83–107, 1966.

[6] S. P. Humphries. Generators for the mapping class group. In *Topology of low-dimensional manifolds*, volume 722 of *Lecture Notes in Math.*, pages 44–47. Springer–Verlag, 1979. Proc. Second Sussex Conf., Chelwood Gate, 1977.

[7] N. V. Ivanov. *Automorphisms of Teichmüller modular groups*, pages 199–270. Number 1346 in Lecture Notes in Math. Springer–Verlag, 1988.

[8] D. L. Johnson. Homeomorphisms of a surface which act trivially on homology. *Proc. Amer. Math. Soc.*, 75(1):119–125, 1979.

[9] D. L. Johnson. *Presentations of Groups*. Number 15 in London Math. Soc. Stud. Texts. Cambridge Univ. Press, 1990.

[10] M. Korkmaz. First homology group of mapping class groups of nonorientable surfaces. *Math. Proc. Cambridge Philos. Soc.*, 123(3):487–499, 1998.

[11] M. Korkmaz. Low–dimensional homology groups of mapping class groups: a survey. *Turkish J. Math.*, 26(1):101–114, 2002.

[12] M. Korkmaz. Mapping class groups of nonorientable surfaces. *Geom. Dedicata*, 89:109–133, 2002.
[13] M. Korkmaz. Generating the surface mapping class group by two elements. *Trans. Amer. Math. Soc.*, 357(8):3299–3310, 2005.

[14] C. Labruèře and Luis Paris. Presentations for the puctured mapping class groups in terms of Artin groups. *Algebr. Geom. Topol.*, 1:73–114, 2001.

[15] W. B. R. Lickorish. A representation of orientable combinatorial 3-manifolds. *Ann. of Math.*, 76:531–540, 1962.

[16] W. B. R. Lickorish. Homeomorphisms of non–orientable two–manifolds. *Math. Proc. Cambridge. Philos. Soc.*, 59:307–317, 1963.

[17] W. B. R. Lickorish. A finite set of generators for the homeotopy group of a 2-manifold. *Math. Proc. Cambridge. Philos. Soc.*, 60:769–778, 1964.

[18] C. Maclachlan. Modulus space is simply-connected. *Proc. Amer. Math. Soc.*, 29(1):85–86, 1971.

[19] J. McCarthy and A. Papadopoulos. Involutions in surface mapping class groups. *Enseign. Math.*, 33(3–4):275–290, 1987.

[20] M. Stukow. Commensurability of geometric subgroups of mapping class groups. arXiv:0707.3430v1.

[21] M. Stukow. The extended mapping class group is generated by 3 symmetries. *C. R. Math. Acad. Sci. Paris*, 338(5):403–406, 2004.

[22] M. Stukow. Small torsion generating sets for hyperelliptic mapping class groups. *Topology Appl.*, 145(1–3):83–90, 2004.

[23] M. Stukow. Dehn twists on nonorientable surfaces. *Fund. Math.*, 189:117–147, 2006.

[24] B. Szepietowski. A presentation for the mapping class group of a non-orientable surface from the action on the complex of curves. arXiv:0707.2776v1.

[25] B. Szepietowski. Mapping class group of a non–orientable surface and moduli space of Klein surfaces. *C. R. Math. Acad. Sci. Paris*, 335(12):1053–1056, 2002.

[26] B. Szepietowski. Involutions in mapping class groups of nonorientable surfaces. *Collect. Math.*, 55(3):253–260, 2004.

[27] B. Wajnryb. Mapping class group of a surface is generated by two elements. *Topology*, 35(2):377–383, 1996.

Institute of Mathematics, University of Gdańsk, Wita Stwosza 57, 80-952 Gdańsk, Poland

E-mail address: trojkat@math.univ.gda.pl