LOGISTIC ELLIPTIC EQUATION WITH A NONLINEAR BOUNDARY CONDITION ARISING FROM COASTAL FISHERY HARVESTING

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Abstract. Let $0 < q < 1 < p$. In this study, we investigate positive solutions of the logistic elliptic equation $-\Delta u = u(1 - u^{p-1})$ in a smooth bounded domain $\Omega$ of $\mathbb{R}^N$, $N \geq 1$, with the nonlinear boundary condition $\frac{\partial u}{\partial \nu} = -\lambda u^q$ on $\partial \Omega$. This nonlinear boundary condition arises from coastal fishery harvesting. When $p > 1$ is subcritical, we prove that in the case of $\lambda > 0$ sufficiently small but no positive solutions for $\lambda > 0$ large enough. In the case of $\lambda < 1$, there exists at least one positive solution for every $\lambda > 0$. Here, $\lambda_{\Omega} > 0$ is the smallest eigenvalue of $-\Delta$ under the Dirichlet boundary condition. An interpretation of our main results from an ecological viewpoint is presented.

1. Introduction and main results

Let $\Omega \subset \mathbb{R}^N$, $N \geq 1$, be a bounded domain with smooth boundary $\partial \Omega$. In this paper, we consider the following logistic elliptic equation with the nonlinear boundary condition arising from coastal fishery harvesting (1.5):

\[
\begin{aligned}
-\Delta u &= u(1 - u^{p-1}) \quad \text{in } \Omega, \\
u &\geq 0 \quad \text{in } \Omega, \\
\frac{\partial u}{\partial \nu} &= -\lambda u^q \quad \text{on } \partial \Omega.
\end{aligned}
\]

(1.1)

Here,

- $\Delta = \sum_{i=1}^N \frac{\partial^2}{\partial x_i^2}$ is the usual Laplacian in $\mathbb{R}^N$;
- $\lambda \geq 0$ is a parameter;
- $0 < q < 1 < p$, and $p < \frac{N+2}{N-2}$ if $N > 2$;
- $\nu$ is the unit outer normal to $\partial \Omega$.

A nonnegative function $u \in H^1(\Omega)$ is called a nonnegative (weak) solution of (1.1) if $u$ satisfies

\[
\int_{\Omega} (\nabla u \nabla \varphi - u \varphi + u^p \varphi) + \lambda \int_{\partial \Omega} u^q \varphi = 0, \quad \varphi \in H^1(\Omega)
\]

(1.2)

(we may regard $(\lambda, u)$ as a nonnegative solution of (1.1)). A nonnegative solution of (1.1) belongs to the Sobolev space $W^{1,r}(\Omega)$ for $r > N$ (and consequently to the H"older space $C^\theta(\Omega)$ for $\theta \in (0, 1)$) ([27 Theorem 2.2]), and moreover, belongs to $C^{2+\theta}(\Omega)$ for $\theta \in (0, 1)$ by Schauder’s interior estimate. A nonnegative solution $u$ of (1.1) is called positive if $u > 0$ in $\Omega$. The strong maximum principle (19) is applicable to show that a nontrivial nonnegative solution of (1.1)

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is positive. However, in terms of the sublinearity with \(0 < q < 1\), we do not know if a positive solution of (1.1) achieves the positivity on the entire boundary \(\partial \Omega\). In fact, Hopf’s boundary point lemma ([19]) is not necessarily applicable for a positive solution of (1.1) because the function \(t \mapsto -t^q\) \((t \geq 0)\) does not satisfy the slope condition, i.e., the slope of this function is not bounded below. Moreover, it does not necessarily hold that \(u \in C^2(\overline{\Omega})\) for a positive solution \(u\) of (1.1). As a matter of fact, we prove that \(u \not\in C^1(\Omega \cup \{x_0\})\) if \(u\) is a positive solution of (1.1) that takes a zero value at \(x_0 \in \partial \Omega\) ([30, Theorem 2]).

When \(p = 2\), the unknown function \(u > 0\) in \(\Omega\) ecologically represents the biomass of fish that inhabit a lake \(\Omega\), obeying the logistic law, where the nonlinear boundary condition \(\frac{\partial u}{\partial \nu} = -\lambda u^q\) on \(\partial \Omega\) with \(0 < q < 1\) means fishery harvesting with the harvesting effort \(\lambda\) on the lake coast \(\partial \Omega\) ([15, Subsection 2.1]).

The sublinear nonlinearity \(u^q\) with \(0 < q < 1\) appearing in (1.1) induces the absorption effect on \(\partial \Omega\). Sublinear boundary conditions were explored in [14, 33, 12, 24, 25, 16, 24, 22, 23, 26]. The case of sublinear incoming flux on \(\partial \Omega\), the mixed case of sublinear absorption and incoming flux on \(\partial \Omega\), and the sublinear term \(u^q\) multiplied by indefinite weight were studied in [14, 33, 12, 23], in [11], and in [5, 21, 22, 26], respectively. It should be emphasized that \(f(t) = t(1 - t^{p-1})\) for \(t \in \mathbb{R}\) is concave, whereas \(g(t) = -\lambda t^q\) is convex for \(t \geq 0\). The combined nonlinearity can achieve multiplicity of positive solutions of (1.1) in certain cases. The study of nonlinear problems with a concave-convex nature originates from Ambrosetti, Brezis, and Cerami [3] who consider the positive solutions of the problem \(-\Delta u = \lambda u^q + u^p\) in \(\Omega\) with \(0 < q < 1 < p\) under the Dirichlet boundary condition. Studies of concave-convex problems were presented in [8, 10, 9, 32, 12, 16, 24, 25, 17].

It is well known that the Neumann logistic problem, (1.1) with \(\lambda = 0\):

\[
\begin{aligned}
-\Delta u &= u(1 - u^{p-1}) \quad \text{in } \Omega, \\
u &\geq 0 \quad \text{in } \Omega, \\
\frac{\partial u}{\partial \nu} &= 0 \quad \text{on } \partial \Omega.
\end{aligned}
\]  

has exactly two nonnegative solutions, \(u \equiv 0\) and 1. For \(\lambda > 0\), it is easy to see that \(u \equiv c \geq 1\) is a supersolution of (1.1). By a comparison argument, a positive solution \(u\) of (1.1) satisfies that \(u < 1\) in \(\overline{\Omega}\) (Proposition 2.1). Let \(\lambda_\Omega > 0\) be the smallest eigenvalue of the Dirichlet eigenvalue problem

\[
\begin{aligned}
-\Delta \phi &= \lambda \phi \quad \text{in } \Omega, \\
\phi &= 0 \quad \text{on } \partial \Omega.
\end{aligned}
\]

We refer to \(\phi_\Omega \in C^{2+\theta}(\overline{\Omega})\) as a positive eigenfunction associated with \(\lambda_\Omega\), and it is well known that \(\lambda_\Omega\) is simple, \(\phi_\Omega > 0\) in \(\Omega\), and \(\frac{\partial \phi_\Omega}{\partial \nu} < 0\) on \(\partial \Omega\). The smallest eigenvalue \(\lambda_\Omega\) is characterized by the variational formula

\[
\lambda_\Omega = \inf \left\{ \int_\Omega (\nabla u)^2 : u \in H^1_0(\Omega), \int_\Omega u^2 = 1 \right\}.
\]

The sign of \(\lambda_\Omega - 1\) plays an essential role in determining the positive solution set \(\{(\lambda, u)\}\) of (1.1). In the case of \(\lambda_\Omega < 1\), \(\varepsilon \phi_\Omega\) is a subsolution of (1.1) for any \(\lambda > 0\), provided that \(\varepsilon > 0\) is sufficiently small; therefore, the sub and supersolution method ensures that problem (1.1) has at least one positive solution \(u\) for every \(\lambda > 0\) such that \(\varepsilon \phi_\Omega \leq u \leq 1\) in \(\overline{\Omega}\) (Proposition 2.5). However, the situation is different in the case of \(\lambda_\Omega > 1\). In this case, the variational
characterization (1.4) provides that \( \int_\Omega (|\nabla u|^2 - u^2) \geq 0 \) for \( u \in H^1_0(\Omega) \), where the equality holds only when \( u = 0 \). This enables us to prove that a positive solution \( u \) of (1.1) satisfies that \( \|u\|_{L^p(\Omega)} \geq C\lambda^{\frac{1}{p-1}} \) as \( \lambda \to \infty \). Combining this lower bound and the upper bound \( u < 1 \) in \( \Omega \) for the positive solution \( u \) of (1.1) shows that problem (1.1) has no positive solution for any \( \lambda > 0 \) large enough (Proposition 1.2).

Now, we present our main results in this paper. The first main result presents, unconditionally for every \( \lambda_\Omega > 0 \), the uniform upper bound for the positive solutions of (1.1) and their positivity on \( \partial \Omega \), and the existence and uniqueness of a smooth positive solution curve \( \{ (\lambda, u_{1,\lambda}) \} \) of (1.1) emanating from \( (\lambda, u) = (0, 1) \). The definitions of the asymptotic stability and the instability for positive solutions \( (\lambda, u) \) of (1.1) with the condition that \( u > 0 \) in \( \Omega \) are referred to in Figure 1 below. The existence part of Theorem 1.1 holds for all \( p > 1 \).

**Theorem 1.1.** Let \( u \) be a positive solution of (1.1) for \( \lambda > 0 \). Then, \( u < 1 \) in \( \Omega \) and \( u > 0 \) on \( \Gamma \) with some \( \Gamma \subset \partial \Omega \) satisfying that \( |\Gamma| > 0 \). Conversely, problem (1.1) has a smooth positive solution curve \( \{ (\lambda, u_{1,\lambda}) : 0 \leq \lambda < \Omega \} \) with some \( \Omega > 0 \) in \( \mathbb{R} \times C^{2+\theta}(\Omega) \), \( \theta \in (0, 1) \), such that \( u_{1,0} = 1 \), \( u_{1,\lambda} \) is asymptotically stable. Moreover, the positive solution set \( \{ (\lambda, u) \} \) of (1.1) forms the smooth positive solution curve in a neighborhood of \( (\lambda, u) = (0, 1) \).

If \( \lambda_\Omega < 1 \), then it is well known (4) that the logistic Dirichlet problem

\[
\begin{cases}
-\Delta u = u(1 - u^{p-1}) & \text{in } \Omega, \\
u \geq 0 & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]

has a unique positive solution \( u_\lambda \in C^{2+\theta}(\Omega) \), such that \( u_\lambda > 0 \) in \( \Omega \), and \( \frac{\partial u_\lambda}{\partial \nu} < 0 \) on \( \partial \Omega \). The second main result is in the case of \( \lambda_\Omega < 1 \), which provides a global existence result for the positive solutions of (1.1). The existence part of Theorem 1.2 holds for all \( p > 1 \).

**Theorem 1.2.** Assume that \( \lambda_\Omega < 1 \). Then, problem (1.1) has at least one positive solution \( u_\lambda \in C^{2+\theta}(\Omega) \), \( \theta \in (0, 1) \), for each \( \lambda > 0 \) such that \( u_\lambda > 0 \) in \( \Omega \). Furthermore, it holds that

(i) \( u_\lambda \) is unique for \( \lambda > 0 \) small (i.e., \( u_\lambda \) coincides with \( u_{1,\lambda} \) given by Theorem 1.1);

(ii) \( u_n \to u_\lambda \) in \( H^1(\Omega) \) for a positive solution \( u_n \) of (1.1) with \( \lambda = \lambda_n \to \infty \),

see Figure 1.

**Remark 1.3.**

(i) From Theorem 1.1, we observe that \( u_\lambda < 1 \) in \( \Omega \).

(ii) No bifurcation from the trivial line \( \{ (\lambda, 0) : \lambda \geq 0 \} \) for positive solutions of (1.1) occurs because of assertion (i) and Proposition 5.4.

The third main result is in the case of \( \lambda_\Omega > 1 \), which provides a local multiplicity result for the positive solutions of (1.1).

**Theorem 1.4.** Suppose that \( \lambda_\Omega > 1 \). Then, problem (1.1) has at least two positive solutions \( U_{1,\lambda}, U_{2,\lambda} \) for \( \lambda \in (0, \hat{\lambda}) \) with some \( \hat{\lambda} > 0 \) such that \( U_{1,\lambda} \to 1 \) in \( C^{2+\theta}(\Omega) \), \( \theta \in (0, 1) \), and \( U_{2,\lambda} \to 0 \) in \( H^1(\Omega) \) as \( \lambda \to 0^+ \) (implying that \( U_{1,\lambda} \) coincides with \( u_{1,\lambda} \) given by Theorem 1.1).

Additionally, the following two assertions hold:
Given \( U_{2,\lambda_n} \) with \( \lambda_n \to 0^+ \), we have that up to a subsequence, \( v_n = \lambda_n^{\frac{1}{1-q}} U_{2,\lambda_n} \to v_0 \) in \( H^1(\Omega) \). Here, \( v_0 \) is a positive solution of the problem

\[
\begin{align*}
-\Delta v &= v & \text{in } \Omega, \\
\frac{\partial v}{\partial \nu} &= -v^q & \text{on } \partial \Omega,
\end{align*}
\]

which admits that \( v_0 > 0 \) on \( \Gamma \) with some \( \Gamma \subset \partial \Omega \) satisfying that \( |\Gamma| > 0 \).

(ii) \( U_{2,\lambda} \) is unstable, provided that it is positive in \( \Omega \).

Meanwhile, there is no positive solution of (1.1) for \( \lambda > 0 \) large enough.

Furthermore, problem (1.1) possesses a bounded subcontinuum \( \{(\lambda, u)\} \) in \( [0, \infty) \times C(\Omega) \) of nonnegative solutions, joining \((0, 0)\) to \((0, 1)\), meaning that \((\lambda, U_{1,\lambda})\) and \((\lambda, U_{2,\lambda})\) could be linked with a bounded, closed connected subset of nonnegative solutions \((\lambda, u)\) of (1.1).

The fourth main result is the following:

**Theorem 1.5.** Suppose that \( \lambda_\Omega > 1 \). Then, problem (1.1) possesses a bounded subcontinuum (i.e., nonempty, closed, and connected subset) \( C_0 = \{(\lambda, u)\} \) of nonnegative solutions in \( [0, \infty) \times C(\Omega) \) such that:

(i) \((0, 0), (0, 1) \in C_0\);

(ii) \( C_0 \cap \{(\lambda, 0) \cup (0, u)\} = \{(0, 0), (0, 1)\} \), which implies that \( C_0 \setminus \{(0, 0)\} \) consists of positive solutions of (1.1);

(iii) \( \{(\lambda, u_{1,\lambda}) : 0 \leq \lambda < \lambda_\Omega\} \subset C_0 \), see Figure 2.

**Remark 1.6.**

(i) From Theorem 1.1, we observe that if \((\lambda, u) \in C_0 \setminus \{(0, 0)\}\) for \( \lambda > 0 \), then \( u < 1 \) in \( \Omega \).

(ii) No bifurcation from the trivial line \( \{(\lambda, 0) \setminus \lambda > 0\} \) for positive solutions of (1.1) occurs because of Proposition 5.4.

(iii) Actually, Theorem 1.5 provides that problem (1.1) has at least two positive solutions for \( \lambda > 0 \) small as stated in Theorem 1.4. It is worthwhile mentioning that Theorem 1.4 presents the asymptotic profile of the second positive solution \( U_{2,\lambda} \) as \( \lambda \to 0^+ \). Moreover, from the proof of Theorem 1.4, the parameter range of \( \lambda \) for which the multiplicity of positive solutions holds is estimated with \( p \) and \( q \) (3.11) and (3.13).

From the variational viewpoint, the functional (3.1) associated with (1.1) and introduced in Section 3 is coercive, and is bounded from below (Lemma 3.1). Therefore, problem (1.1) has a least energy solution for every \( \lambda > 0 \) with respect to the functional, which is nonnegative in \( \Omega \). In the case of \( \lambda_\Omega > 1 \), problem (1.1) has no positive solution for \( \lambda > 0 \) large (Theorem 1.4), which implies that the least energy solution is zero for such \( \lambda \). It should be noted that the nonnegative solutions of (1.1) are the steady state solutions of the nonlinear initial boundary
value problem

\begin{equation}
\begin{aligned}
\frac{\partial u}{\partial t} &= \Delta u + u(1 - u^{p-1}) \quad \text{in } (0, \infty) \times \Omega, \\
\frac{\partial u}{\partial \nu} &= -\lambda u^q \quad \text{on } (0, \infty) \times \partial \Omega, \\
u(0, x) &= u_0(x) \geq 0 \quad \text{for } x \in \Omega.
\end{aligned}
\end{equation}

(1.7)

In the case of $\lambda_\Omega > 1$, it would be expected that the global nonnegative solution of (1.7) vanishes as $t \to \infty$ for $\lambda > 0$ large enough, and that the positive steady state solution jumps down to zero at $\lambda = \lambda^* := \sup \{\lambda > 0 : (\lambda, u) \in \mathcal{C}_0 \}$ in terms of the existence of a $\supset$ shaped subcontinuum of positive solutions (Theorem 1.5). From the viewpoint of fishery harvesting, we could infer that by overfishing, the collapse of the stock for fish occurs as the harvesting effort $\lambda$ moves beyond $\lambda^*$ (Figure 2). However, the extinction wouldn’t occur in the case of $\lambda_\Omega < 1$ (Figure 1). The positive solution $u_\lambda$ by Theorem 1.2 is weakly stable in the sense of Amann [1], because it is constructed via sub and supersolutions (Proposition 2.5). Therefore, it would be expected that the global nonnegative solution of (1.7) converges to a positive steady state solution for every $\lambda > 0$. Nevertheless, Theorem 1.2 (ii) shows that the positive steady state solution remains positive in $\Omega$ but vanishes on $\partial \Omega$ as $\lambda \to \infty$. From the viewpoint of fishery harvesting, it can be inferred that the population of fish becomes extinct on the coast $\partial \Omega$ eventually. Mathematically, we know that the larger the size of the domain $\Omega$, the smaller $\lambda_\Omega$ is. Therefore, our ecological interpretation would be consistent with this mathematical fact.

![Figure 1. Possible positive solution set of (1.1) in the case of $\lambda_\Omega < 1$.](image)

![Figure 2. Possible positive solution set of (1.1) in the case of $\lambda_\Omega > 1$.](image)
In the one-dimensional case of the limiting problem \( (1.6) \)
\[
\begin{aligned}
  -v''(x) &= v(x), \quad x \in (a, b), \\
  -v'(a) &= -v(a)^q, \\
  v'(b) &= -v(b)^q,
\end{aligned}
\tag{1.8}
\]
with \( a < b \), a positive solution \( v \) of \( (1.8) \) is positive in \( [a, b] \). The equation \(-v'' = v\) admits the general solution \( v_c(x) = C_1 \sin(x + C_2) \in C^\infty(\mathbb{R}) \) with \( C_1, C_2 \in \mathbb{R} \). When \( C_1 \neq 0 \), \( v_c'(x_0) \neq 0 \) if \( v_c(x_0) = 0 \), and thus, the boundary condition implies that \( v_c(a)v_c(b) > 0 \) if \( v_c > 0 \) in \( (a, b) \).

Since we have Theorem 1.4 (i) with \( v_0 > 0 \) in \( [a, b] \) accordingly, Theorem 1.4 (i) and (ii) provide that \( U_{2, \lambda} > 0 \) in \( [a, b] \) is unstable for \( \lambda > 0 \) sufficiently small (using Lemma 3.3).

In this paper, we do not discuss if problem \( (1.1) \) admits a positive solution \( u \) satisfying that \( u = 0 \) somewhere on \( \partial \Omega \). It is an interesting open question whether such degenerate positive solutions exist or not.

To conclude the Introduction, we refer to the case \( \lambda < 0 \). For \( \lambda < 0 \), problem \( (1.1) \) possesses a concave-concave nature. Thus, the positive solution set \( \{(\lambda, u)\} \) is not complicated. Indeed, there exists a unique positive solution of \( (1.1) \) for \( \lambda < 0 \), and it is asymptotically stable. The proofs are carried out in the same spirit of that for \[12\] Theorem 1.2. The asymptotic profile of the unique positive solution as \( \lambda \to -\infty \) is still an open question.

**Notation:** \( \| \cdot \| \) denotes the usual norm of \( H^1(\Omega) \). \( u_n \rightharpoonup u_{\infty} \) means that \( u_n \) converges weakly to \( u_{\infty} \) in \( H^1(\Omega) \). \( u \neq 0 \) means that \( u \in H^1(\Omega) \setminus \{0\} \). \( H^1_0(\Omega) = \{ u \in H^1(\Omega) : u = 0 \text{ on } \partial \Omega \} \). \( \int_\Omega f \, dx \) for \( f \in L^1(\Omega) \) and \( \int_{\partial \Omega} g \, d\sigma \) for \( g \in L^1(\partial \Omega) \) are written simply as \( \int_\Omega f \) and \( \int_{\partial \Omega} g \), respectively. \( | \cdot | \) represents the Lebesgue measure in \( \Omega \) and the surface measure on \( \partial \Omega \) both.

The remainder of this paper is organized as follows.

In Section 2, we prove the local existence result for a positive solution of \( (1.1) \) emanating from \( (\lambda, u) = (0, 1) \), using the implicit function theorem. Additionally, we conduct stability analysis for the positive solution. Moreover, in the case when \( \lambda_0 < 1 \), we establish the global existence result for a positive solution of \( (1.1) \) for every \( \lambda > 0 \), applying the sub and supersolution method. We present a proof for Theorem 1.1 in this section.

In Section 3, we establish the local multiplicity result for positive solutions of \( (1.1) \), using the variational method on the Nehari manifold associated with \( (1.1) \).

In Section 4, using the variational technique, we evaluate the asymptotic profile for positive solutions of \( (1.1) \) as \( \lambda \to 0^+ \) and \( \lambda \to \infty \). We present proofs for Theorems 1.2 and 1.4 in this section.

In Section 5, with the aid of a topological argument by Whyburn \[31\], we prove Theorem 1.5 via the bifurcation technique. Our bifurcation approach to \( (1.1) \) from zero solutions is non standard, meaning that the well known local and global bifurcation results from simple eigenvalues by Crandall and Rabinowitz \[7\], Rabinowitz \[20\], or López-Gómez \[18\] are not directly applied because the function \( t^q \) (\( t \geq 0 \)) with \( 0 < q < 1 \) is not right differentiable at \( t = 0 \). We perform a regularization scheme for \( (1.1) \) to overcome this difficulty.
2. Existence of positive solutions via the implicit function theorem and sub and supersolutions

First, we establish the \textit{a priori} upper bound of the uniform norm on \( \Omega \) and positivity on \( \partial \Omega \) for the positive solutions of (1.1).

**Proposition 2.1.** Let \( u \) be a positive solution of (1.1). Then, the following two assertions hold:

(i) \( u < 1 \) in \( \Omega \);

(ii) \( u > 0 \) on some \( \Gamma \subset \partial \Omega \) satisfying that \( |\Gamma| > 0 \).

**Proof.** (i) Assume by contradiction that \( M := \max_{\Omega} u \geq 1 \) for a positive solution \( u \) of (1.1). Additionally assume that \( u(x_0) = M \) for \( x_0 \in \Omega \). Choose \( K > 0 \) such that \( Kt + f(t) \) is increasing for \( t \in [0, M] \), where \( f(t) = t(1 - t^{p-1}) \). Using the fact that \( -\Delta M = 0 \geq f(M) \), we deduce that

\[
(-\Delta + K)(M - u) \geq (KM + f(M)) - (Ku + f(u)) \geq 0 \quad \text{in } \Omega.
\]

Since \( u \in C^2(\Omega) \cap C(\overline{\Omega}) \), the strong maximum principle applies, and then, \( M - u \) is identically equal to zero, i.e., \( u \equiv M \), which is contradictory for the nonlinear boundary condition. Hence, we obtain that \( x_0 \in \partial \Omega \). However, \( u \in C^1 \) in a neighborhood of \( x_0 \) by the bootstrap argument based on the fact that \( u \in W^{1,r}(\Omega) \) with \( r > N \), thus, Hopf’s boundary point lemma applies at \( x_0 \). We then arrive at the contradiction that

\[
0 > -\lambda u^\theta(x_0) = \frac{\partial u}{\partial \nu}(x_0) > 0.
\]

(ii) Since \( u \) is a positive solution of (1.1), it follows from (1.2) with \( \varphi = 1 \) that

\[
\int_{\Omega} u(1 - u^{p-1}) = \lambda \int_{\partial \Omega} u^q.
\]

Assertion (i) shows that \( \int_{\partial \Omega} u^q > 0 \), as desired. \( \square \)

Set \( U = \{(\lambda, u) \in \mathbb{R} \times C^{2+\theta}([\Omega]) : u > 0 \text{ in } \overline{\Omega} \} \) with \( \theta \in (0, 1) \), and define the nonlinear mapping

\[
F : U \rightarrow C^\theta(\overline{\Omega}) \times C^{1+\theta}(\partial \Omega); \quad (\lambda, u) \mapsto \left(-\Delta u - u(1 - u^{p-1}), \frac{\partial u}{\partial \nu} + \lambda u^q\right). \quad (2.1)
\]

We then deduce that \( (\lambda, u) \) is a positive solution of (1.1) in \( U \) if and only if \( F(\lambda, u) = 0 \). Consider the following linearized eigenvalue problem of \( F(\lambda, u) = 0 \) at \( (\lambda, u) \):

\[
\begin{cases}
-\Delta \varphi = f'(u)\varphi + \gamma \varphi & \text{in } \Omega, \\
\frac{\partial \varphi}{\partial \nu} = -\lambda g'(u)\varphi + \gamma \varphi & \text{on } \partial \Omega,
\end{cases} \quad (2.2)
\]

where \( f(t) = t(1 - t^{p-1}) \) and \( g(t) = t^q \). We refer to \( \gamma_1 = \gamma_1(\lambda, u) \) and \( \varphi_1 = \varphi_1(\lambda, u) \) as the smallest eigenvalue and a corresponding positive eigenfunction associated with \( \gamma_1 \), respectively. It is well known that \( \gamma_1 \) is simple, and \( \varphi_1 > 0 \) in \( \Omega \). A positive solution \( (\lambda, u) \) of (1.1) satisfying that \( u > 0 \) in \( \overline{\Omega} \) is said to be asymptotically stable, unstable, and weakly stable if \( \gamma_1(\lambda, u) > 0 \), \( \gamma_1(\lambda, u) < 0 \), and \( \gamma_1(\lambda, u) \geq 0 \), respectively.

Second, the implicit function theorem proves the following existence and uniqueness result for the positive solutions of (1.1).
Proposition 2.2. Problem (1.1) has a smooth positive solution curve \( \{(\lambda, u_{1,\lambda}) : |\lambda| < \lambda\} \) with a small \( \lambda > 0 \) in \( \mathbb{R} \times C^{2+\theta}(\Omega) \) for \( \theta \in (0,1) \), which satisfies that \( u_{1,0} = 1 \), and \( 0 < u_{1,\lambda} < 1 \) in \( \overline{\Omega} \) for \( \lambda > 0 \). Moreover, there is a neighborhood of \( (\lambda, u) = (0,1) \) such that if \( u \) is a positive solution of (1.1) in the neighborhood, then \( u = u_{1,\lambda} \) for some \( \lambda > 0 \).

Proof. Choose an open neighborhood \( U \) of \( (\lambda, u) = (0,1) \) in \( \mathbb{R} \times C^{2+\theta}(\Omega) \), and consider (2.1) and (2.2). We then observe that \( \gamma_1(0,1) > 0 \). Therefore, the mapping \( (-\Delta - f'(1), \frac{\partial}{\partial \nu}) \) is a homeomorphism, and the implicit function theorem is applicable to deduce all the assertions except that of \( u_{1,\lambda} < 1 \) in \( \overline{\Omega} \). The assertion that \( u_{1,\lambda} < 1 \) in \( \overline{\Omega} \) follows from Proposition 2.1 (i).

Third, we prove the following stability result for \( u_{1,\lambda} \).

Proposition 2.3. If a positive solution \( u \) of (1.1) satisfies that \( u \geq \left(\frac{1-q}{p-q}\right)^{\frac{1}{q-1}} \) in \( \Omega \), then \( u \) is asymptotically stable. Particularly, \( u_{1,\lambda} \) given by Proposition 2.2 is asymptotically stable for \( \lambda > 0 \) close to 0.

Proof. Let \( u \) be a positive solution of (1.1) such that \( u > 0 \) in \( \Omega \). Proposition 2.1 (i) shows that \( f(u) > 0 \) in \( \Omega \). Consider the smallest eigenvalue \( \gamma_1(\lambda, u) \) of (2.2) with a corresponding positive eigenfunction \( \varphi_1(\lambda, u) \), and observe that

\[
\int_{\Omega} (-\Delta \varphi_1 f(u) + \Delta u f'(u) \varphi_1) = \gamma_1 \int_{\Omega} \varphi_1 f(u).
\]

By the divergence theorem, we deduce that

\[
\int_{\Omega} (-\Delta \varphi_1 f(u) + \Delta u f'(u) \varphi_1) \geq \int_{\Omega} \frac{d}{d\nu} \left(\frac{g(u)}{f(u)}\right) (f(u))^2 \varphi_1 - \gamma_1 \int_{\partial \Omega} \varphi_1 f(u).
\]

It follows that

\[
\gamma_1 = -\int_{\Omega} f''(u) |\nabla u|^2 \varphi_1 + \int_{\partial \Omega} \frac{d}{d\nu} \left(\frac{g(u)}{f(u)}\right) (f(u))^2 \varphi_1.
\]

By direct computations,

\[
f''(t) = -p(p-1)t^{p-2}, \quad \frac{d}{dt} \left(\frac{g(t)}{f(t)}\right) = \frac{(p-q) \left(t^{p-1} - \frac{1-q}{p-q}\right)}{t^{2-q}(1-t^{p-1})^2}.
\]

Hence, if \( u \geq \left(\frac{1-q}{p-q}\right)^{\frac{1}{q-1}} \) in \( \Omega \), then \( \gamma_1 > 0 \). The desired conclusion follows.

Remark 2.4. Propositions 2.2 and 2.3 hold for all \( \lambda_0 > 0 \) and \( p > 1 \).

We can then prove Theorem 1.1

Proof of Theorem 1.1 The first assertion follows from Proposition 2.1. The second assertion follows from Propositions 2.2 and 2.3.

To conclude this section, we prove the following global existence result for the positive solutions of (1.1) in the case when \( \lambda_0 < 1 \). The subsolution and the supersolution of (1.1) are defined as in [2].
Proposition 2.5. Assume that \( \lambda_\Omega < 1 \). Then, problem (1.1) has at least one positive solution \( u_\lambda \) for every \( \lambda > 0 \), which satisfies that \( u_\lambda > 0 \) in \( \Omega \).

**Proof.** It is clear that \( u \equiv 1 \) is a supersolution of (1.1). We construct a certain subsolution of (1.1) to ensure the existence of a positive solution of (1.1) that is positive on the entire boundary \( \partial \Omega \).

For \( \varepsilon, \delta > 0 \), we set \( w = \varepsilon(\phi \Omega + \delta) \) such that \( w < 1 \). We then deduce that

\[
-\Delta w - w(1 - w^{p-1}) \leq \varepsilon \phi \Omega \{ \lambda_\Omega - 1 + \varepsilon^{p-1}(\phi \Omega + \delta)^{p-1} \} \quad \text{in } \Omega, \\
\frac{\partial w}{\partial \nu} + \lambda w^q \leq \varepsilon (\lambda_\Omega + \lambda \varepsilon^{q-1}\delta^q) \quad \text{on } \partial \Omega, 
\]

where \( \frac{\partial \phi \Omega}{\partial \nu} \leq -C \Omega \) on \( \partial \Omega \) for some \( C \Omega > 0 \). Choose \( \delta = \varepsilon^\tau \) with \( \tau > 1 - \frac{q}{q} \), and then, assertions (2.3) and (2.4) show that \( w \) is a subsolution of (1.1) if \( \varepsilon > 0 \) is small enough. Therefore, the sub and supersolution method [2, (2.1) Theorem] shows the existence of a positive solution \( u \) of (1.1) such that \( \varepsilon(\phi \Omega + \varepsilon^\tau) \leq u \leq 1 \) in \( \Omega \), as desired. \( \square \)

**Remark 2.6.**

(i) Proposition 2.5 holds for all \( p > 1 \).

(ii) We observe from Corollary 4.10 below that \( u_\lambda = u_{1,\lambda} \) for \( \lambda > 0 \) close to 0, where \( u_{1,\lambda} \) is the positive solution of (1.1) emanating from \( (\lambda, u) = (0,1) \), given by Proposition 2.2.

3. Multiplicity of positive solutions via Nehari manifold

In this section, we discuss the existence and multiplicity of positive solutions for (1.1) in the case of \( \lambda_\Omega > 1 \) by employing the variational method on the Nehari manifold associated with (1.1).

3.1. Nehari manifold and fibering map. We introduce the functional associated with (1.1):

\[
J_\lambda(u) := \frac{1}{2}E(u) + \frac{1}{p+1}A(u) + \frac{\lambda}{q+1}B(u), \quad u \in H^1(\Omega), 
\]

where

\[
E(u) := \int_\Omega (|\nabla u|^2 - u^2), \quad A(u) := \int_\Omega |u|^{p+1}, \quad \text{and } B(u) := \int_{\partial \Omega} |u|^{q+1}.
\]

In the sequel, we use the notations:

\[
E^\pm = \{ u \in H^1(\Omega) : E(u) \gtrless 0 \}, \quad A^+ = \{ u \in H^1(\Omega) : A(u) > 0 \}, \quad B^+ = \{ u \in H^1(\Omega) : B(u) > 0 \}.
\]

By a simple calculation, it is easy to verify the following lemma.

**Lemma 3.1.** Let \( \lambda > 0 \). Then, \( J_\lambda \) is coercive, and is bounded from below in \( H^1(\Omega) \). More precisely, we have \( C > 0 \) such that \( J_\lambda(u) \geq \frac{1}{2}\|u\|^2 - C \) for \( u \in H^1(\Omega) \). Here, \( C \) does not depend on \( \lambda \).

The following three lemmas are used frequently in our arguments.
Lemma 3.2. Assume that $\lambda_\Omega > 1$. Then, $E(u) \geq 0$ for $u \in H_0^1(\Omega)$, and moreover, $E(u)^{\frac{1}{2}}$ and $\|u\|_{H_0^1(\Omega)}$ are equivalent in $H_0^1(\Omega)$.

Proof. Since $\lambda_\Omega > 1$, it follows from (1.4) that $E(u) \geq 0$ for $u \in H_0^1(\Omega)$, and $u \equiv 0$ if $u \in H_0^1(\Omega)$ satisfies that $E(u) = 0$. We claim that there exists $C > 0$ such that $\|u\|^2_{H_0^1(\Omega)} \leq CE(u)$ for $u \in H_0^1(\Omega)$. We assume to the contrary that $u_n \in H_0^1(\Omega)$, $\|u_n\|_{H_0^1(\Omega)} = 1$, but $E(u_n) \to 0$. Then, up to a subsequence, $u_n \to u_0$ in $H_0^1(\Omega)$ and $u_n \to u_0$ in $L^2(\Omega)$. This implies that $0 \leq E(u_0) \leq \liminf_n E(u_n) \leq \limsup_n E(u_n) = 0$, thus, $u_0 = 0$. Consequently, $E(u_n) \to 0$, thus, $u_n \to 0$ in $H_0^1(\Omega)$. This is contradictory for $\|u_n\|_{H_0^1(\Omega)} = 1$.  

Lemma 3.3. Let $u_n$ be a positive solution of (1.1) for $\lambda = \lambda_n \geq 0$ such that $\lambda_n$ is bounded. Then, $u_n$ is bounded in $W^{1,r}(\Omega)$ for $r > N$ (implying that $u_n$ is bounded in $C^0(\overline{\Omega})$ for $\theta \in (0,1)$). Furthermore, up to a subsequence, $\lambda_n \to \lambda_0$, $u_n \to u_0$, and $u_n \to u_0$ in $C(\Omega)$.

Proof. We may infer that $\lambda_n \to \lambda_0$. Substituting $\varphi = u_n$ for (1.2), Proposition 2.1 (i) shows that

$$\int_{\Omega} |\nabla u_n|^2 = \int_{\Omega} \left( u_n^2 - A(u) - \lambda_n B(u_n) \right) \leq \int_{\Omega} u_n^2 \leq |\Omega|.$$  

It follows that $u_n$ is bounded in $H^1(\Omega)$, implying that up to a subsequence, $u_n \to u_0$, and $u_n \to u_0$ in $L^{p+1}(\Omega)$ and $L^2(\partial\Omega)$. As in the proof of [27, Theorem 2.2], we deduce that $u_n$ is bounded in $W^{1,r}(\Omega)$ for $r > N$. Sobolev’s embedding theorem ensures that this is the case in $C^\theta(\overline{\Omega})$ with $\theta = 1 - \frac{N}{r}$. The assertion that $u_n$ has a convergent subsequence in $C(\Omega)$ follows by the fact that $C^\theta(\overline{\Omega}) \subset C(\Omega)$ is compact. The desired conclusion follows.

Lemma 3.4. Let $\{u_n\} \subset H^1(\Omega)$ be such that $E(u_n) \leq 0$, $u_n \to u_0$, and $u_n \to u_0$ in $L^2(\Omega)$. Then, the following two assertions hold:

(i) If $\|u_n\| \geq C$ for some $C > 0$, then $u_0 \neq 0$.

(ii) Suppose that $\lambda_\Omega > 1$. If $u_0 \neq 0$, then $u_0 \not\in H_0^1(\Omega)$, i.e., $u_0 \in B^+$.  

Proof. (i) From the inequalities

$$E(u_0) \leq \liminf_n E(u_n) \leq \limsup_n E(u_n) \leq 0,$$

we infer that if $u_0 = 0$, then $\|u_n\| \to 0$, as desired.

(ii) If $u_0 \in H_0^1(\Omega)$, then $u_0 = 0$ from (3.2), using Lemma 3.2.

The Nehari manifold for (1.1) is then defined by

$$\mathcal{N}_\lambda := \{ u \in H^1(\Omega) \setminus \{0\} : E(u) + A(u) + \lambda B(u) = 0 \}.$$  

It should be noted that a positive solution of (1.1) belongs to $\mathcal{N}_\lambda$. Given $u \neq 0$, the fibering map for (1.1) is defined as

$$j_\lambda(t) := J_\lambda(tu) = \frac{t^2}{2} E(u) + \frac{t^{p+1}}{p+1} A(u) + \frac{t^{q+1}}{q+1} B(u), \quad t > 0.$$  

A direct computation gives us that

$$j_\lambda'(t) = tE(u) + t^p A(u) + \lambda t^q B(u).$$
where the derivative of a function is represented with a prime. Then, we observe that

\[ j''_u(t) = 0 \iff tu \in \mathcal{N}_\lambda \quad \text{(in particular } j'_u(1) = 0 \iff u \in \mathcal{N}_\lambda). \]

We next split \( \mathcal{N}_\lambda \) into three parts, using \( j_u \). By direct computation,

\[ j''_u(t) = E(u) + ptu^{-1}A(u) + \lambda qt^{-1}B(u), \]

and it follows that \( j''_u(1) = E(u) + pA(u) + \lambda qB(u) \). If \( j'_u(1) = 0 \), then we infer that

\[ j''_u(1) = \begin{cases} (1 - q)E(u) + (p - q)A(u), \\ -(p - 1)E(u) - \lambda(p - q)B(u). \end{cases} \]

We then define

\[ \mathcal{N}^\pm_\lambda := \{ u \in \mathcal{N}_\lambda : j''_u(1) \geq 0 \}, \]

that is,

\[ \mathcal{N}^\pm_\lambda = \left\{ u \in \mathcal{N}_\lambda : E(u) \geq -\frac{p - q}{1 - q}A(u) \right\} = \left\{ u \in \mathcal{N}_\lambda : E(u) \leq -\frac{\lambda(p - q)}{p - 1}B(u) \right\}. \]

The next lemma is a direct consequence from these definitions.

**Lemma 3.5.** \( \mathcal{N}_\lambda \subset A^+ \cap E^- \) and \( \mathcal{N}^-_\lambda \subset A^+ \cap B^+ \cap E^- \). Moreover, \( \mathcal{N}_\lambda \subset A^+ \cap B^+ \cap E^- \) if \( \lambda_{\Omega} > 1 \).

**Proof.** If \( u \in \mathcal{N}_\lambda \), then \( u \neq 0 \), i.e., \( u \in A^+ \). It follows that \( E(u) = -A(u) - \lambda B(u) \leq -A(u) < 0 \), thus, \( u \in E^- \). Additionally if \( u \in \mathcal{N}^-_\lambda \), then it follows that \( \lambda \frac{p - q}{p - 1}B(u) > E(u) > 0 \). Thus, \( u \in B^+ \). We assume that \( u \in \mathcal{N}_\lambda \) and \( B(u) = 0 \) under \( \lambda_{\Omega} > 1 \). Lemma [3.2] then shows that \( E(u) > 0 \) because \( u \neq 0 \) and \( u \in H^1_0(\Omega) \), which is contradictory for \( \mathcal{N}_\lambda \subset E^- \). Hence, we deduce that \( \mathcal{N}_\lambda \subset B^+ \). \( \Box \)

Using the change of variables

\[ \mu = \lambda^{\frac{p - 1}{q}} \quad \text{and} \quad v = \lambda^{-\frac{1}{q + 1}}u, \quad (3.3) \]

we also consider the functional

\[ I_\mu(v) := \frac{1}{2}E(v) + \frac{\mu}{p + 1}A(v) + \frac{1}{q + 1}B(v), \quad v \in H^1(\Omega), \]

associated with the problem

\[ \begin{cases} -\Delta v = v - \mu v^p & \text{in } \Omega, \\ \frac{\partial v}{\partial \nu} = -v^q & \text{on } \partial \Omega. \end{cases} \quad (3.4) \]

A nonnegative function \( v \in H^1(\Omega) \) is called a nonnegative (weak) solution of (1.1) if \( v \) satisfies

\[ \int_\Omega (\nabla v \nabla \varphi - v \varphi + \mu v^p \varphi) + \int_{\partial \Omega} v^q \varphi = 0, \quad \varphi \in H^1(\Omega). \quad (3.5) \]

It should be noted that (3.4) with \( \mu = 0 \) is (1.0).

The following result is the counterpart of Lemma [3.1] for \( J_\lambda \).

**Lemma 3.6.** Let \( \mu > 0 \). Then, \( I_\mu \) is coercive, and is bounded from below in \( H^1(\Omega) \). More precisely, we have \( C_\mu > 0 \) such that \( I_\mu(v) \geq \frac{1}{2}||v||^2 - C_\mu \) for \( u \in H^1(\Omega) \).
Similarly, we introduce the Nehari manifold associated with (3.4):
\[ \mathcal{M}_\mu := \{ v \in H^1(\Omega) \setminus \{0\} : E(v) + \mu A(v) + B(v) = 0 \}, \]
and the fibering map \( i_v(t) \) for \( v \neq 0 \):
\[ i_v(t) := I_\mu(tv) = \frac{t^2}{2} E(v) + \frac{\mu t^{p+1}}{p+1} A(v) + \frac{t^{q+1}}{q+1} B(v), \quad t > 0. \] (3.6)
By direct computation,
\[ i_v'(t) = tE(v) + \mu t^p A(v) + t^q B(v), \] (3.7)
and we deduce that \( i_v'(t) = 0 \) if and only if \( tv \in \mathcal{M}_\mu \). Particularly, \( i_v'(1) = 0 \iff v \in \mathcal{M}_\mu \).
Moreover, observing that
\[ i_v''(t) = E(v) + \mu tp^{p-1} A(v) + qt^{q-1} B(v), \]
we define similarly
\[ \mathcal{M}_\mu^\pm := \{ v \in \mathcal{M}_\mu : i_v''(1) \geq 0 \} = \{ v \in \mathcal{M}_\mu : E(v) \geq -\frac{\mu}{1-q} A(v) \}
= \{ v \in \mathcal{M}_\mu : E(v) \leq -\frac{\mu}{p-1} B(v) \}. \]
Clearly, if \( \mu = \lambda \frac{p-1}{p-q} \), then
\[ u \in \mathcal{N}_\lambda \iff v = \lambda^{-\frac{1}{p-q}} u \in \mathcal{M}_\mu, \quad (3.8) \]
\[ u \in \mathcal{N}_\lambda^\pm \iff v = \lambda^{-\frac{1}{p-q}} u \in \mathcal{M}_\mu^\pm. \quad (3.9) \]
We now look for a certain condition for \( j_u \) to possess two distinct critical points. Given \( u \in A^+ \cap B^+ \cap E^- \), we set \( j_u'(t) = t^q j_u(t) \) with
\[ j_u(t) := t^{1-q} E(u) + tp^{q} A(u) + \lambda B(u), \quad t > 0. \]
We note that \( \tilde{j}_u \) has the unique global minimum point
\[ t_0 = t_0(u) = \left( \frac{1-q}{p-q} \right)^{\frac{1}{p-q}} \left( \frac{-E(u)}{A(u)} \right)^{\frac{1}{p-q}} > 0, \]
and that \( \tilde{j}_u \) is decreasing and increasing for \( t < t_0 \) and \( t > t_0 \), respectively. Therefore, \( j_u \) has two distinct critical points if and only if \( \tilde{j}_u(t_0) < 0 \), and in this case, \( j_u \) possesses exactly two critical points. The desired condition is given by the following:
\[ \lambda < \frac{p-1}{p-q} \left( \frac{1-q}{p-q} \right)^{\frac{1-q}{p-q}} \left( \frac{-E(u)}{A(u)} \right)^{\frac{1-q}{p-q}} B(u)^{\frac{p-q}{p-q}}. \] (3.10)
On the basis of (3.10), we discuss a class of \( u \) for which \( j_u \) has two distinct critical points for \( \lambda > 0 \) small. We define
\[ \mathcal{F}_\delta := \{ u \in A^+ \cap B^+ : E(u) + A(u) \leq 0, \|u\| \geq \delta \}, \quad 0 < \delta \leq \frac{|\Omega|^{\frac{1}{2}}}{2}, \]
and introduce the value
\[ \lambda_\delta = \lambda_\delta(\delta) := \inf \left\{ \frac{p-1}{p-q} \left( \frac{1-q}{p-q} \right)^{\frac{1-q}{p-q}} \left( \frac{-E(u)}{A(u)} \right)^{\frac{1-q}{p-q}} B(u)^{\frac{p-q}{p-q}} : u \in \mathcal{F}_\delta \right\}. \] (3.11)
It should be noted that $u = \frac{1}{2} \in \mathcal{F}_\delta$, and we deduce the following lemma:

**Lemma 3.7.** $\mathcal{F}_\delta$ is bounded in $H^1(\Omega)$.

**Proof.** Assume by contradiction that $\|u_n\| \to \infty$ for $u_n \in \mathcal{F}_\delta$. Then, say $w_n = \frac{u_n}{\|u_n\|}$, and $\|w_n\| = 1$. Thus, $E(w_n)$ is bounded. Moreover, up to a subsequence, $w_n \to w_0$, and $w_n \to w_0$ in $L^{p+1}(\Omega)$ and $L^2(\partial\Omega)$. Lemma 3.4 (i) shows that $w_0 \neq 0$, i.e., $A(w_0) > 0$. The condition $E(u_n) \leq -A(u_n)$ implies that $E(w_n) \leq -A(w_0)\|w_n\| \to -\infty$, which is a contradiction. □

Using Lemma 3.7, we prove that $\lambda_\ast$ is positive.

**Proposition 3.8.** $\lambda_\ast(\delta) > 0$ in any case of $\lambda_\Omega$.

**Proof.** Assume to the contrary that $u_n \in \mathcal{F}_\delta$ admits the condition that

$$\frac{E(u_n)}{A(u_n)\frac{1}{p-q}B(u_n)\frac{1}{p-q}} \not\to 0. \quad (3.12)$$

Since $u_n$ is bounded in $H^1(\Omega)$ from Lemma 3.7, we obtain a subsequence of $\{u_n\}$, still denoted by the same notation, such that $u_n \to u_0$, and $u_n \to u_0$ in $L^{p+1}(\Omega)$ and $L^2(\partial\Omega)$. Since $\|u_n\| \geq \delta$, Lemma 3.4 (i) shows that $u_0 \neq 0$, i.e., $A(u_0) > 0$, from which it follows that $\lim_n E(u_n) \leq -A(u_0) < 0$. Given $\varepsilon > 0$, we deduce from (3.12) that if $n$ is large enough, then

$$-\varepsilon A(u_n)^{\frac{1-q}{p-q}}B(u_n)^{\frac{p-1}{p-q}} \leq E(u_n) \leq 0,$$

so that

$$-\varepsilon A(u_0)^{\frac{1-q}{p-q}}B(u_0)^{\frac{p-1}{p-q}} \leq \lim_n E(u_n) \leq \lim_n E(u_n) \leq 0.$$

This implies that $E(u_n) \to 0$ because $\varepsilon > 0$ is arbitrary, which is a contradiction. □

The following result is derived as a corollary from Proposition 3.8.

**Corollary 3.9.** Let $0 < \delta \leq \frac{\|\partial\Omega\|^{1/q}}{1+q}$, and let $(\lambda, u) \in (0, \lambda_\ast(\delta)) \times \mathcal{F}_\delta$. Then, $j_u$ has exactly two critical points $t_1, t_2 > 0$, i.e., $j_u''(t_j) = 0$, $j = 1, 2$, such that $0 < t_1 < t_2$, and $j_u''(t_1) > 0 < j_u''(t_2)$. Consequently, $t_1u \in N_\gamma^-$ and $t_2u \in N_\gamma^+$.

We then establish a similar result for $i_v$ in (3.13). Let $v \in A^+ \cap B^+ \cap E^-$. From (3.13) and (3.10), we observe that $i_v$ has two distinct critical points if

$$\mu < \frac{1}{p-q} \left( \frac{1-q}{p} \right)^{\frac{1}{p-q}} \left( \frac{-E(v)}{A(v)\frac{1}{p-q}B(v)\frac{p-1}{p-q}} \right)^{\frac{1}{p-q}},$$

and characterize a class of $v$ for which $i_v$ possesses two distinct critical points for $\mu > 0$ small. We define

$$G_\delta := \{v \in A^+ \cap B^+: E(v) + B(v) \leq 0, \|v\| \leq \delta\}, \quad \delta \geq \frac{|\partial\Omega|^{1/q}}{|\Omega|^{2(1-q)}};$$

and introduce the value

$$\mu_\ast = \mu_\ast(\delta) := \inf \left\{ \frac{1-\delta}{p-q} \left( \frac{1-q}{p} \right)^{\frac{1}{p-q}} \left( \frac{-E(v)}{A(v)\frac{1}{p-q}B(v)\frac{p-1}{p-q}} \right)^{\frac{1}{p-q}} : v \in G_\delta \right\}. \quad (3.13)$$
It should be noted that \( v = \left( \frac{|\partial \Omega|}{|\Omega|} \right)^{\frac{1}{1-q}} \in \mathcal{G}_\delta \), and we obtain the following lemma:

**Lemma 3.10.** Assume that \( \lambda_\Omega > 1 \). Then, there exists \( C > 0 \) such that \( ||v|| \geq C \) for \( v \in \mathcal{G}_\delta \).

**Proof.** Assume by contradiction that \( ||v_n|| \to 0 \) for \( v_n \in \mathcal{G}_\delta \). Say \( w_n = \frac{v_n}{||v_n||} \), and \( ||w_n|| = 1 \). Then, \( E(w_n) \) is bounded, and moreover, up to a subsequence, \( w_n \to w_0 \) and \( w_n \to w_0 \) in \( L^{p+1}(\Omega) \) and \( L^2(\partial \Omega) \). Lemma 3.13 shows that \( w_0 \neq 0 \) and \( w_0 \notin H_0^1(\Omega) \), i.e., \( B(w_0) > 0 \). The condition \( E(v_n) \leq -B(v_n) \) implies that \( E(w_n) \leq -B(w_n)||v_n||^{-(1-q)} \to -\infty \), as desired. \( \square \)

The next proposition is the counterpart of Proposition 3.11 for \( \lambda_\delta(\delta) \).

**Proposition 3.11.** \( \mu_\delta(\delta) > 0 \) if \( \lambda_\Omega > 1 \).

**Proof.** The proof is similar as that of Proposition 3.8. Assume by contradiction that \( v_n \in \mathcal{G}_\delta \), and

\[
\frac{E(v_n)}{A(v_n)^{\frac{1}{p-2}}B(v_n)^{\frac{p-1}{p-q}}} \to 0. \tag{3.14}
\]

Since \( v_n \) is bounded in \( H^1(\Omega) \), it follows that up to a subsequence, \( v_n \to v_0 \) and \( v_n \to v_0 \) in \( L^{p+1}(\Omega) \) and \( L^2(\partial \Omega) \). Additionally, because \( ||v_n|| \geq C \) from Lemma 3.10. Lemma 3.4 ensures that \( v_0 \neq 0 \) and \( v_0 \notin H_0^1(\Omega) \), i.e., \( B(v_0) > 0 \), which implies that

\[
\lim_n E(v_n) \leq -B(v_0) < 0. \tag{3.15}
\]

Given \( \varepsilon > 0 \), we infer from 3.14 that if \( n \) is large enough, then

\[-\varepsilon A(v_n)^{\frac{1}{p-2}}B(v_n)^{\frac{p-1}{p-q}} \leq E(v_n) \leq 0,\]

thus,

\[-\varepsilon A(v_0)^{\frac{1}{p-2}}B(v_0)^{\frac{p-1}{p-q}} \leq \lim_n E(v_n) \leq \lim_n E(v_n) \leq 0,\]

which implies that \( E(v_n) \to 0 \) because \( \varepsilon > 0 \) is arbitrary. This is contradictory for 3.15. \( \square \)

The next result is then derived as a corollary from Proposition 3.11 which is the counterpart of Corollary 3.9 for \( j_n \).

**Corollary 3.12.** Let \( \delta \geq \frac{1}{2}||\partial \Omega||^{\frac{1}{1-q}}/||\Omega||^{\frac{1}{2(1-q)}} \), and let \((\mu, v) \in (0, \mu_\delta(\delta)) \times \mathcal{G}_\delta \). Then, \( i_v \) has exactly two critical points \( t_1, t_2 > 0 \), i.e., \( i_v'(t_j) = 0 \), \( j = 1, 2 \), such that \( 0 < t_1 < t_2 \), and \( i_v''(t_1) < 0 < i_v''(t_2) \). Consequently, \( t_1 v \in \mathcal{M}_\mu^- \) and \( t_2 v \in \mathcal{M}_\mu^+ \).

3.2. **Existence of a global minimizer on \( \mathcal{N}_\lambda^+ \).** First, we claim that \( \mathcal{N}_\lambda^+ \neq \emptyset \) for \( \lambda > 0 \) small.

**Lemma 3.13.** There exists \( \lambda_0 > 0 \) such that if \( \lambda \in (0, \lambda_0) \), then we have a unique constant \( c_+(\lambda) \in \mathcal{N}_\lambda^+ \) such that \( c_+(\lambda) < 1 \) and \( c_+(\lambda) \nearrow 1 \) as \( \lambda \to 0^+ \). Moreover, it holds that

\[
\sup_{\lambda \in (0, \lambda_0)} J_\lambda(c_+(\lambda)) \leq -\frac{p-1}{3(p+1)}|\Omega|. \tag{3.16}
\]

**Proof.** For a constant \( c > 0 \), it is easy to observe that \( c \in \mathcal{N}_\lambda^+ \) if and only if

\[
c^{1-q} - c^{p-q} = \lambda \frac{|\partial \Omega|}{|\Omega|}, \quad \text{and} \quad 1 < \frac{p-q}{1-q} c^{p-1}.\]

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Hence, the first assertion holds for \( \lambda > 0 \) sufficiently small. For (3.16), the following calculation is conducted:

\[
J_\lambda(c_+(\lambda)) = \left(\frac{1}{2} - \frac{1}{p+1}\right)E(c_+(\lambda)) + \lambda \left(\frac{1}{q+1} - \frac{1}{p+1}\right)B(c_+(\lambda))
\]
\[
= c_+(\lambda)^2 \left(-\frac{p-1}{2(p+1)}|\Omega| + \lambda \frac{p-q}{(p+1)(q+1)}|\partial\Omega|c_+(\lambda)^{q-1}\right)
\]
\[
\rightarrow -\frac{p-1}{2(p+1)}|\Omega| \quad \text{as} \ \lambda \rightarrow 0^+.
\]

The desired conclusion now follows.

For \( \lambda \in (0, \lambda_0) \), we deduce from Lemma 3.1 that

\[
\eta_\lambda^+ := \inf \{J_\lambda(u) : u \in \mathcal{N}_\lambda^+ \} > -\infty. \tag{3.17}
\]

In this subsection, we establish the following result.

**Proposition 3.14.** Assume that \( \lambda_\Omega > 1 \). Then, there exists \( \lambda_+ \in (0, \lambda_0) \) such that for \( \lambda \in (0, \lambda_+) \),

\[
\eta_\lambda^+ = J_\lambda(u_\lambda^+) = \min \{J_\lambda(u) : u \in \mathcal{N}_\lambda^+ \} < 0,
\]

and additionally, there exists \( C > 1 \) such that

\[
C^{-1} \leq \|u_\lambda^+\| \leq C \quad \text{as} \quad \lambda \rightarrow 0^+. \tag{3.18}
\]

From (3.17), let \( \{u_{\lambda,n}\} \subset \mathcal{N}_\lambda^+ \) be a minimizing sequence for \( J_\lambda \) on \( \mathcal{N}_\lambda^+ \) such that \( J_\lambda(u_{\lambda,n}) \downarrow \eta_\lambda^+ \). Lemma 3.1 then ensures that up to a subsequence,

\[ u_{\lambda,n} \rightharpoonup u_{\lambda,\infty}, \quad \text{and} \quad u_{\lambda,n} \rightarrow u_{\lambda,\infty} \quad \text{in} \quad L^{p+1}(\Omega) \quad \text{and} \quad L^2(\partial\Omega). \tag{3.19} \]

It follows from (3.16) that \( \|u_{\lambda,\infty}\| \) has an \textit{a priori} lower bound if \( \lambda > 0 \) is small enough:

**Lemma 3.15.** Let \( u_{\lambda,\infty} \) be as in (3.19). Then, there exist \( \delta_+ > 0 \) and \( \lambda_+ > 0 \) such that \( \|u_{\lambda,\infty}\| \geq \delta_+ \) for \( \lambda \in (0, \lambda_+) \), where \( \lambda_+, \delta_+ > 0 \) do not depend on the choice of \( u_{\lambda,\infty} \).

**Proof.** It follows from (3.19) that \( J_\lambda(u_{\lambda,\infty}) \leq \lim_{n} J_\lambda(u_{\lambda,n}) = \eta_\lambda^+ \). Therefore, assertions (3.16) and (3.17) provide that

\[
J_\lambda(u_{\lambda,\infty}) \leq \eta_\lambda^+ \leq J_\lambda(c_+(\lambda)) \leq -\frac{p-1}{3(p+1)}|\Omega| \quad \text{for} \ \lambda \in (0, \lambda_0). \tag{3.20}
\]

The desired conclusion follows.

We then prove Proposition 3.14.

**Proof of Proposition 3.14.** With \( \lambda_+, \delta_+ > 0 \) of Lemma 3.15 and \( \lambda_*(\delta_+) \) by (5.11), we fix

\[
0 < \lambda < \min(\lambda_+, \lambda_*(\delta_+)).
\]

Let \( u_{\lambda,n} \) and \( u_{\lambda,\infty} \) be as in (3.19). First, we verify that \( u_{\lambda,\infty} \in \mathcal{F}_{\delta_*} \), and apply Corollary 3.9

with \( \delta = \delta_* \). We may infer that \( \delta_* \leq \frac{\|\Omega\|^{\frac{1}{p}}}{\lambda^{\frac{1}{p}}} \). From Lemma 3.15, we note that \( \|u_{\lambda,\infty}\| \geq \delta_+, \) thus, \( u_{\lambda,\infty} \neq 0, \) i.e., \( u_{\lambda,\infty} \in A^+ \). Using \( u_{\lambda,n} \in \mathcal{N}_\lambda \), we deduce that

\[
E(u_{\lambda,\infty}) \leq \lim_{n} E(u_{\lambda,n}) = \lim_{n} (-A(u_{\lambda,n}) - \lambda B(u_{\lambda,n})) \leq -A(u_{\lambda,\infty}),
\]

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thus, $E(u_{\lambda, \infty}) + A(u_{\lambda, \infty}) \leq 0$. Moreover, we apply Lemma 3.4(ii) to obtain that $u_{\lambda, \infty} \notin H^1_\delta(\Omega)$, i.e., $u_{\lambda, \infty} \in B^+$, considering (3.19) and the condition that $\lambda_\Omega > 1$, as desired. Corollary 3.4 with $\delta = \delta_+$ now applies, and then, there exist $0 < t_1 < t_2$ such that $t_1 u_{\lambda, \infty} \in \mathcal{N}^-_\lambda$ and $t_2 u_{\lambda, \infty} \in \mathcal{N}^+$. Next, we prove that $u_{\lambda, n} \to u_{\lambda, \infty}$ in $H^1(\Omega)$. If not, using $u_{\lambda, \infty} \in A^+ \cap B^+ \cap E^-$, we then infer that $t_1 < t_2$ because we have a subsequence of $\{u_{\lambda, n}\}$, still denoted by the same notation, such that

$$j''_{u_{\lambda, \infty}}(1) = E(u_{\lambda, \infty}) + A(u_{\lambda, \infty}) + \lambda B(u_{\lambda, \infty}) < \lim_{n \to \infty} (E(u_{\lambda, n}) + A(u_{\lambda, n}) + \lambda B(u_{\lambda, n})) = 0.$$ 

Hence, we deduce that

$$J_\lambda(t_2 u_{\lambda, \infty}) = j_{u_{\lambda, \infty}}(t_2) < j_{u_{\lambda, \infty}}(1) = \lim_{n \to \infty} j_{u_{\lambda, n}}(1) = \lim_{n \to \infty} J_\lambda(u_{\lambda, n}) = \eta^+,$$

which is contradictory for $t_2 u_{\lambda, \infty} \in \mathcal{N}^+_\lambda$, as desired. Immediately, it follows that $J_\lambda(u_{\lambda, n}) \to J_\lambda(u_{\lambda, \infty}) = \eta^+.$

Finally, we verify that $t_2 = 1$. To this end, we only have to notice that if $0 = j''_{u_{\lambda, \infty}}(t) \leq j''_{u_{\lambda, \infty}}(t)$, then $t > 0$ is unique, and $j''_{u_{\lambda, \infty}}(t) > 0$. We infer that $0 = j_{u_{\lambda, n}}(1) < j''_{u_{\lambda, n}}(1)$. Passing to the limit provides that $0 = j''_{u_{\lambda, \infty}}(1) < j''_{u_{\lambda, \infty}}(1)$, thus, $j''_{u_{\lambda, \infty}}(1) > 0$, which means that $t_2 = 1$, i.e., $u_{\lambda, \infty} \in \mathcal{N}^+$. By (3.20), we observe that $\eta^+ = J_\lambda(u_{\lambda, \infty}) < 0$, which completes the proof with $u^+ = u_{\lambda, \infty}$. Indeed, (3.18) follows from the fact that $u_{\lambda, \infty} \in \mathcal{F}_{\delta_+}$ combined with Lemma 3.7.

3.3. Existence of a global minimizer on $\mathcal{N}^-_\lambda$. First, we prove that $\mathcal{M}^-_\mu \neq \emptyset$ for $\mu > 0$ small, which shows that $\mathcal{N}^-_\lambda \neq \emptyset$ for $\lambda > 0$ small.

**Lemma 3.16.** There exists $\mu_0 > 0$ such that if $\mu \in (0, \mu_0)$, then we have a unique constant $c_-(\mu) \in \mathcal{M}^-_\mu$ such that $c_-(\mu) > \left(\frac{\partial \Omega}{|\Omega|}\right)^{1-q}$, and $c_-(\mu) \searrow \left(\frac{\partial \Omega}{|\Omega|}\right)^{1-q}$ as $\mu \to 0^+$. Moreover, it holds that

$$\sup_{\mu \in (0, \mu_0)} I_\mu(c_-(\mu)) < \left(1 - \frac{q}{1+q}\right) \frac{|\partial \Omega|^{1-q}}{|\Omega|^{1+q}}.$$  

(3.21)

**Proof.** Consider a positive constant $c \in \mathcal{M}^-_\mu$, i.e., $E(c) + \mu A(c) + B(c) = 0$ and $E(c) > -\frac{p-q}{p-1} B(c)$. Observing that

$$c \in \mathcal{M}^-_\mu \iff \begin{cases} \mu = c^{-(p-1)} - c^{-(p-q)} |\partial \Omega|^{1-q} \\ c < \left(\frac{p-q}{p-1}\right)^{1-q} \left(\frac{|\partial \Omega|}{|\Omega|}\right)^{1-q} \end{cases},$$

the first assertion holds for $\mu > 0$ sufficiently small. For (3.21), we conduct the calculation

$$I_\mu(c_-(\mu)) = \left(1 - \frac{q}{1+q}\right) E(c_-(\mu)) + \mu \left(1 - \frac{1}{q+1}\right) A(c_-(\mu))$$

$$= c_-(\mu)^2 |\Omega| \left\{ \frac{1-q}{2(q+1)} - \frac{p-q}{(p+1)(q+1)} c_-(\mu)^{p-1} \right\}$$

$$\leq c_-(\mu)^2 |\Omega| \frac{1-q}{2(q+1)} \to \frac{|\partial \Omega|^{1-q}}{|\Omega|^{1+q}} \frac{1-q}{2(q+1)} \text{ as } \mu \to 0^+. $$

The desired conclusion now follows. \[\square\]
In this subsection, we establish the following result:

**Proposition 3.17.** Assume that \( \lambda_\Omega > 1 \). Then, there exists \( \lambda^-_\Omega > 0 \) such that for every \( \lambda \in (0, \lambda^-_\Omega) \),

\[
J_\lambda(u^-_\lambda) = \min \left\{ J_\lambda(u) : u \in \mathcal{N}^-_\lambda \right\} > 0,
\]

and additionally, there exists \( C > 1 \) such that

\[
C^{-1} \lambda^\frac{1}{1-q} \leq \|u^-_\lambda\| \leq C \lambda^\frac{1}{1-q} \quad \text{as} \quad \lambda \to 0^+.
\] (3.22)

For this purpose, we discuss the existence of a minimizer for the functional \( I_\mu \) on \( \mathcal{M}^-_\mu \). First, we verify that \( I_\mu \) is nonnegative on \( \mathcal{M}^-_\mu \).

**Lemma 3.18.** Let \( \mu > 0 \). If \( v \in \mathcal{M}^-_\mu \), then \( I_\mu(v) \geq \frac{(p-q)(1-q)}{2(p+1)(q+1)} B(v) \).

**Proof.** Using the fact that \( \mu A(v) = -E(u) - B(u) \) and \( E(v) > -\frac{p-1}{p+1} B(v) \) for \( v \in \mathcal{M}^-_\mu \), we deduce the assertion by direct calculations. \( \square \)

Let \( \mu \in (0, \mu_0) \). From Lemma 3.18, we define

\[
\xi^-_\mu := \inf \left\{ I_\mu(v) : v \in \mathcal{M}^-_\mu \right\} \geq 0. \quad (3.23)
\]

Let \( \{v_{\mu,n}\} \subset \mathcal{M}^-_\mu \) be a minimizing sequence for \( I_\mu \) on \( \mathcal{M}^-_\mu \) such that \( I_\mu(v_{\mu,n}) \searrow \xi^-_\mu \). Then, it follows from Lemma 3.10 that up to a subsequence,

\[
v_{\mu,n} \to v_{\mu,\infty}, \quad \text{and} \quad v_{\mu,n} \to v_{\mu,\infty} \quad \text{in} \quad L^{p+1}(\Omega) \quad \text{and} \quad L^2(\partial\Omega). \quad (3.24)
\]

We then prove the following result:

**Proposition 3.19.** Assume that \( \lambda_\Omega > 1 \). Then, there exists \( \lambda^-_\Omega > 0 \) such that if \( \mu \in (0, \lambda^-_\Omega) \), then

\[
\xi^-_\mu = I_\mu(v^-_\mu) = \min \left\{ I_\mu(v) : v \in \mathcal{M}^-_\mu \right\} > 0,
\]

and additionally, there exists \( C > 1 \) such that

\[
C^{-1} \leq \|v^-_\mu\| \leq C \quad \text{as} \quad \mu \to 0^+. \quad (3.25)
\]

As a matter of fact, in view of (3.8) and (3.9), Proposition 3.17 is a direct consequence of Proposition 3.19. Indeed, \( u^-_\lambda = \lambda^\frac{1}{1-q} v^-_\mu \) with \( \mu = \lambda^\frac{p-1}{p} \).

First, we prove the following lemma:

**Lemma 3.20.** Assume that \( \lambda_\Omega > 1 \). Let \( \mu \in (0, \mu_0) \), and let \( \xi^-_\mu \) be as in (3.23). Then, \( \xi^-_\mu > 0 \).

**Proof.** Let \( v_{\mu,n}, v_{\mu,\infty} \) be as in (3.24). Lemma 3.18 shows that

\[
\frac{(p-q)(1-q)}{2(p+1)(q+1)} B(v_{\mu,\infty}) = \lim_n \frac{(p-q)(1-q)}{2(p+1)(q+1)} B(v_{\mu,n}) \leq \lim_n I_\mu(v_{\mu,n}) = \xi^-_\mu.
\]

Therefore, the desired assertion holds if \( v_{\mu,\infty} \notin H^1_0(\Omega) \). Since \( v_{\mu,n} \in \mathcal{M}_\mu \), we infer that

\[
E(v_{\mu,\infty}) \leq \lim_n E(v_{\mu,n}) \leq \lim_n E(v_{\mu,n}) \leq 0.
\]

If \( v_{\mu,\infty} = 0 \), then \( v_{\mu,n} \to 0 \) in \( H^1(\Omega) \). Say \( w_n = \frac{u_{\mu,n}}{\|u_{\mu,n}\|} \), and \( \|w_n\| = 1 \). Then, up to a subsequence, \( w_n \to \hat{w}_\infty \) and \( w_n \to \hat{w}_\infty \) in \( L^{p+1}(\Omega) \) and \( L^2(\partial\Omega) \). Lemma 3.4 shows that \( \hat{w}_\infty \neq 0 \) and \( \hat{w}_\infty \notin H^1_0(\Omega) \), i.e., \( B(\hat{w}_\infty) > 0 \). From the condition that \( E(v_{\mu,n}) < -B(v_{\mu,n}) \), we deduce that \( E(w_n) < -B(w_n)\|v_{\mu,n}\|^{q-1} \to -\infty \), which is a contradiction because \( E(w_n) \) is bounded.
We thus obtain that \( v_{\mu,\infty} \neq 0 \). Lemma 3.4 (ii) is used again to obtain that \( v_{\mu,\infty} \notin H^1_0(\Omega) \). The proof is complete.

Next, we prove that \( v_{\mu,\infty} \) has an a priori upper bound if \( \mu > 0 \) is small, which is the counterpart of Lemma 3.15 for \( u_{\lambda,\infty} \).

**Lemma 3.21.** Assume that \( \lambda_\Omega > 1 \). Let \( v_{\mu,\infty} \) be as in (3.24). Then, there exist \( \delta_- > 0 \) and \( \mu_- \in (0, \mu_0) \) such that \( \| v_{\mu,\infty} \| \leq \delta_- \) for \( \mu \in (0, \mu_-) \), where \( \mu_- \), \( \delta_- > 0 \) do not depend on the choice of \( v_{\mu,\infty} \).

**Proof.** We argue by contradiction. Assume that \( \| v_{\mu_n,\infty} \| \to \infty \) for \( \mu_n \to 0^+ \). Let \( n \geq 1 \) be fixed. Following (3.24), we choose a sequence \( \{ v_{\mu_n, k} \}_k \subset \mathcal{M}_{\mu_n}^- \) such that \( v_{\mu_n, k} \to v_{\mu_n,\infty} \) in \( L^{p+1}(\Omega) \) and \( L^2(\partial\Omega) \). Then, it follows from (3.21) and (3.23) that

\[
I_{\mu_n}(v_{\mu_n,\infty}) = \frac{1 - q}{1 + q} \| \partial\Omega \|^{2 - q} |\Omega|^{q + \frac{1}{q}}.
\]

for some \( \varepsilon_0 > 0 \). This implies that up to a subsequence of \( \{ v_{\mu_n, k} \}_k \),

\[
I_{\mu_n}(v_{\mu_n, k}) < \frac{1 - q}{1 + q} \| \partial\Omega \|^{2 - q} |\Omega|^{q + \frac{1}{q}}.
\]

Using Lemma 3.18 it follows that

\[
\frac{(p - q)(1 - q)}{2(p + 1)(q + 1)} B(v_{\mu_n, k}) < \frac{1 - q}{1 + q} \| \partial\Omega \|^{2 - q} |\Omega|^{q + \frac{1}{q}}.
\]

Passing to the limit \( k \to \infty \) shows that

\[
\frac{(p - q)(1 - q)}{2(p + 1)(q + 1)} B(v_{\mu,\infty}) \leq \frac{1 - q}{1 + q} \| \partial\Omega \|^{2 - q} |\Omega|^{q + \frac{1}{q}}.
\]  

(3.26)

Say \( w_n = \frac{v_{\mu_n,\infty}}{\| v_{\mu_n,\infty} \|} \), and \( \| w_n \| = 1 \). We then have a subsequence of \( \{ w_n \} \), still denoted by the same notation, such that \( w_n \to w_\infty \), and \( w_n \to w_\infty \) in \( L^{p+1}(\Omega) \) and \( L^2(\partial\Omega) \). Lemma 3.4 ensures that \( w_\infty \neq 0 \) and \( w_\infty \notin H^1_0(\Omega) \), i.e., \( B(w_\infty) > 0 \). Observing from (3.26) that

\[
\frac{(p - q)(1 - q)}{2(p + 1)(q + 1)} B(w_n) \leq \frac{1 - q}{1 + q} \| \partial\Omega \|^{2 - q} |\Omega|^{q + \frac{1}{q}} \| v_{\mu,\infty} \|^{-(q + 1)} \to 0,
\]

we deduce that \( B(w_\infty) = 0 \), as desired. \( \square \)

We are then ready to prove Proposition 3.19.

**Proof of Proposition 3.19.** With \( \mu_- \), \( \delta_- \) of Lemma 3.21 and \( \mu_\ast(\delta_-) \) by (3.13), we fix

\[
0 < \mu < \min (\mu_-, \mu_\ast(\delta_-))
\]

Let \( v_{\mu, n}, v_{\mu, \infty} \) be as in (3.21), and we verify that \( v_{\mu,\infty} \in \mathcal{G}_{\delta_-} \) for applying Corollary 3.12 with \( \delta = \delta_- \). We may infer that \( \delta_- \geq |\partial\Omega|^{1 - q}/|\Omega|^{q + \frac{1}{q}} \), and Lemma 3.21 shows that \( \| v_{\mu,\infty} \| \leq \delta_- \). We prove that \( v_{\mu,\infty} \in A^+ \cap B^+ \) and \( E(v_{\mu,\infty}) + B(v_{\mu,\infty}) \leq 0 \). Since \( v_{\mu, n} \in E_- \), it follows that 

\[
E(v_{\mu,\infty}) \leq \lim n E(v_{\mu, n}) \leq \lim n E(v_{\mu, n}) \leq 0. \text{ If } v_{\mu,\infty} = 0, \text{ then } \| v_{\mu, n} \| \to 0. \text{ Say } w_n = \frac{v_{\mu, n}}{\| v_{\mu, n} \|} \text{, and } \| w_n \| = 1. \text{ We then obtain a subsequence of } \{ w_n \}, \text{ still denoted by the same notation, such}.
\]

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that \( w_n \to w_\infty \), and \( w_n \to w_\infty \) in \( L^{p+1} (\Omega) \) and \( L^2 (\partial \Omega) \). Immediately, Lemma 3.4 shows that \( w_\infty \neq 0 \) and \( w_\infty \not\in H^1_0 (\Omega) \). From \( v_{\mu,n} \in M_\mu \), we observe that

\[
I_\mu (v_{\mu,n}) = \left( \frac{1}{2} - \frac{1}{p+1} \right) E (v_{\mu,n}) + \mu \left( \frac{1}{p+1} - \frac{1}{q+1} \right) A (v_{\mu,n}),
\]

and Lemma 3.18 deduces that

\[
\frac{(p-q)(1-q)}{2(p+1)(q+1)} B (v_{\mu,n}) \leq \left( \frac{1}{2} - \frac{1}{q+1} \right) E (v_{\mu,n}) + \left( \frac{1}{p+1} - \frac{1}{q+1} \right) A (v_{\mu,n}).
\]

It follows that

\[
\frac{(p-q)(1-q)}{2(p+1)(q+1)} B (w_n) \leq \left( \frac{1}{2} - \frac{1}{q+1} \right) E (w_n) ||v_{\mu,n}||^{1-q} + \left( \frac{1}{p+1} - \frac{1}{q+1} \right) A (w_n) ||v_{\mu,n}||^{p-q}.
\]

Passing to the limit provides that \( B (w_\infty) = 0 \), which is contradictory for \( w_\infty \not\in H^1_0 (\Omega) \). Consequently, \( v_{\mu,\infty} \neq 0 \), i.e., \( v_{\mu,\infty} \in A^+ \). Moreover, Lemma 3.4 (ii) is used again to deduce that \( v_{\mu,\infty} \not\in H^1_0 (\Omega) \), i.e., \( v_{\mu,\infty} \in B^+ \). Finally, observing that

\[
E (v_{\mu,\infty}) + \mu A (v_{\mu,\infty}) + B (v_{\mu,\infty}) \leq \lim \frac{E (v_{\mu,n}) + \mu A (v_{\mu,n}) + B (v_{\mu,n})}{n} = 0,
\]

we obtain that \( E (v_{\mu,\infty}) + B (v_{\mu,\infty}) \leq 0 \), as desired. Corollary 3.12 now applies, and then, there exist \( 0 < t_1 < t_2 \) such that \( t_1 v_{\mu,\infty} \in M_\mu^+ \) and \( t_2 v_{\mu,\infty} \in M_\mu^- \).

We then verify that

\[
v_{\mu,n} \to v_{\mu,\infty} \quad \text{in} \quad H^1 (\Omega), \quad (3.27)
\]

It should be noted that the inclusion \( M^- \subset A^+ \cap B^+ \cap E^\infty \) holds similarly as in Lemma 3.5. If \( (3.27) \) does not hold, because \( v_{\mu,n} \in M^- \), we then infer that up to a subsequence,

\[
I_\mu (t_1 v_{\mu,n}) = i_{v_{\mu,n}} (t_1) < \lim \inf i_{v_{\mu,n}} (t_1) \leq \lim \sup i_{v_{\mu,n}} (1) = \lim I_\mu (v_{\mu,n}) = \xi^-_\mu, \quad (3.28)
\]

provided that \( t_1 \leq 1 \). In fact, \( (3.28) \) holds even if \( t_1 > 1 \). From (3.17), we set \( \tilde{i}_{v_{\mu,n}} (t) = t^q \tilde{i}_n (t) \) with

\[
\tilde{i}_n (t) = t^{1-q} E (v_{\mu,n}) + \mu t^{p-q} A (v_{\mu,n}) + B (v_{\mu,n}).
\]

If \( \tilde{i}_n (t_1) < 0 \) for \( n \) large enough, then \( \tilde{i}_{v_{\mu,n}} (t_1) < \tilde{i}_{v_{\mu,n}} (1) \), thus, \( (3.28) \) proceeds well. To this end, we use the condition that \( v_{\mu,n} \in M^- \) to deduce that

\[
\tilde{i}_n (t_1) = \mu (t_{1, n}^{p-q} - t_{1, n}^{1-q}) A (v_{\mu,n}) + (1 - t_{1, n}^{1-q}) B (v_{\mu,n})
\]

\[
\to \mu (t_{1, n}^{p-q} - t_{1, n}^{1-q}) A (v_{\mu,\infty}) + (1 - t_{1, n}^{1-q}) B (v_{\mu,\infty}) =: \tilde{i}_\infty (t_1).
\]

If \( \tilde{i}_\infty (t_1) < 0 \), then this is the case that we desire. We use the condition that \( t_1 v_{\mu,\infty} \in M^- \), i.e., \( \mu A (v_{\mu,\infty}) < \frac{1-q}{p-1} t_{1, n}^{1-q} B (v_{\mu,\infty}) \) to deduce that

\[
\tilde{i}_\infty (t_1) < h (t_1) B (v_{\mu,\infty}), \quad \text{where} \quad h (t) := \frac{p-q}{p-1} \frac{1-q}{t^{p-q} - t^{1-q}}, \quad t > 1.
\]

Observing that \( h (1) = 0 \) and

\[
h' (t) = (1-q) (t^{-p} - t^{-q}) < 0, \quad t > 1,
\]

we find that \( \tilde{i}_\infty (t_1) < 0 \), as desired, and thus, \( (3.28) \) proceeds for any case of \( t_1 > 0 \). However, this is contradictory for \( t_1 v_{\mu,\infty} \in M^- \). Claim \( (3.27) \) is thus verified. Immediately, we deduce from Lemma 3.20 that \( I_\mu (v_{\mu,n}) \to I_\mu (v_{\mu,\infty}) = \xi^-_\mu > 0 \).
The assertion that $t_1 = 1$ can be verified similarly as in the proof of Proposition 3.14. Thus, $v_{\mu, \infty} \in M^{-}_\mu$, which completes the proof with $v^- = v_{\mu, \infty}$, since (3.20) follows from the fact that $v_{\mu, \infty} \in G_{\delta_-}$ combined with Lemma 3.10.

4. Asymptotic profiles of positive solutions

4.1. Asymptotic profiles of positive solutions as $\lambda \to \infty$. Proposition 2.5 proves that when $\lambda_\Omega < 1$, problem (1.1) has a positive solution $u > 0$ in $\Omega$ for every $\lambda > 0$. In this subsection, we evaluate the asymptotic profile of the positive solution as $\lambda \to \infty$.

**Proposition 4.1.** Assume that $\lambda_\Omega < 1$. If $u_n$ is a positive solution of (1.1) for $\lambda = \lambda_n \to \infty$, then $u_n \to u_D$ in $H^1(\Omega)$, where $u_D$ is the unique positive solution of the Dirichlet logistic problem (1.5).

**Proof.** First, we claim that $u_n$ is bounded in $H^1(\Omega)$. We infer from Proposition 2.1 that $u_n < 1$ in $\Omega$. From the definition of $u_n$ (1.2) with $(\lambda, u) = (\lambda_n, u_n)$ and $\varphi = u_n$, we deduce that

$$\int_\Omega |\nabla u_n|^2 = \int_\Omega (u_n^2 - w_n^{p+1}) - \lambda_n \int_{\partial \Omega} u_n^{q+1} \leq \int_\Omega u_n^2 \leq |\Omega|,$$

as desired. Immediately, up to a subsequence, $u_n \to u_0$, and $u_n \to u_0$ in $L^{p+1}(\Omega)$ and $L^2(\partial \Omega)$. We then deduce that

$$\int_{\partial \Omega} u_n^{q+1} = \frac{1}{\lambda_n} \left( - \int_\Omega |\nabla u_n|^2 + \int_\Omega (u_n^2 - w_n^{p+1}) \right) \leq \frac{1}{\lambda_n} \int_\Omega u_n^2 \to 0,$$

which implies that $\int_{\partial \Omega} u_n^{q+1} = 0$, thus, $u_0 \in H^1_0(\Omega)$. From (1.2) with $(\lambda, u) = (\lambda_n, u_n)$, it follows that

$$\int_\Omega (\nabla u_n \nabla \varphi - u_n \varphi + u_n^p \varphi) = 0, \quad \varphi \in H^1_0(\Omega).$$

Passing to the limit provides that $u_0$ is a nonnegative solution of (1.5).

Next, we claim that $u_0 \neq 0$. Since $E(u_n) \leq 0$, we infer that

$$E(u_0) \leq \lim_n E(u_n) \leq \lim_n E(u_n) \leq 0.$$

If $u_0 = 0$, then it follows that $\|u_n\| \to 0$. Say $w_n = \frac{u_n}{\|u_n\|}$, and $\|w_n\| = 1$. Then, up to a subsequence, $w_n \to w_0 \geq 0$, and $w_n \to w_0$ in $L^{p+1}(\Omega)$ and $L^2(\partial \Omega)$. Lemma 3.24 (i) shows that $w_0 \neq 0$. However, we observe from (4.11) that

$$\int_{\partial \Omega} w_n^{q+1} \leq \frac{1}{\lambda_n} \int_\Omega w_n^2 \|u_n\|^{1-q} \to 0.$$

This implies that $w_0 \in H^1_0(\Omega)$. By a similar argument as above, we deduce that $w_0$ is a nonnegative solution of the problem

$$\begin{cases}
-\Delta w = w & \text{in } \Omega, \\
w = 0 & \text{on } \partial \Omega.
\end{cases}$$

Since $w_0 \neq 0$ from Lemma 3.4 (i), we deduce that $\lambda_\Omega = 1$, which is contradictory for the assumption. The claim follows, thus, $u_0$ is the unique positive solution $u_D$ of (1.5), ensured by the strong maximum principle.
Finally, we prove that \( u_n \to u_0 \) in \( H^1(\Omega) \). It suffices to verify that \( E(u_n) \to E(u_0) \). Observing that \( E(u_0) + A(u_0) = 0 \) and \( E(u_n) \leq -A(u_n) \), we deduce that

\[
E(u_0) \leq \liminf_n E(u_n) \leq \limsup_n E(u_n) \leq -\limsup_n A(u_n) = -A(u_0) = E(u_0),
\]

as desired. \( \square \)

However, if \( \lambda_\Omega > 1 \), then problem (1.1) has no positive solutions for \( \lambda > 0 \) large enough.

**Proposition 4.2.** Assuming that \( \lambda_\Omega > 1 \),

\[
\sup \{ \lambda > 0 : (1.1) \text{ has at least one positive solution} \} < \infty.
\]

**Proof.** Let \( u \) be a positive solution of (1.1) for \( \lambda > 0 \). By the change of variables (3.3), \( v = \lambda^{\frac{1}{1-p}} u \) is a positive solution of (3.4) with \( \mu = \lambda^{\frac{p-1}{q-p}} \). It thus suffices to show that \( \mu \) possesses an upper bound. Substituting \( \varphi = 1 \) for (3.5), Hölder’s inequality is used to deduce that

\[
\mu \int_\Omega v^p = \int \Omega v - \int_{\partial \Omega} v^q \leq \int \Omega v \leq |\Omega|^\frac{p-1}{p} \left( \int_\Omega v^p \right)^{\frac{1}{p}}.
\]

It thus follows that

\[
\mu \|v\|_{L^p(\Omega)}^{p-1} \leq |\Omega|^{\frac{p-1}{p}}. \tag{4.2}
\]

We then claim that

\[
\|v\|_{L^p(\Omega)} > C, \quad \mu \to \infty \tag{4.3}
\]

for some \( C > 0 \). We assume by contradiction that \( \|v_n\|_{L^p(\Omega)} \to 0 \) for a positive solution \( v_n \) of (3.4) with \( \mu = \mu_n \to \infty \). Then, up to a subsequence, we deduce that \( v_n \to 0 \) a.e. in \( \Omega \). In view of (3.3), we may infer that \( v_n < 1 \) in \( \overline{\Omega} \) because \( v_n = \mu_n^{\frac{1}{p-1}} u_n \) and \( u_n < 1 \). Therefore, the Lebesgue dominated convergence theorem ensures that \( \|v_n\|_{L^2(\Omega)} \to 0 \). It follows that

\[
\int_\Omega |\nabla v_n|^2 = \int_\Omega v_n^2 - \mu_n \int_{\partial \Omega} v_n^{p+1} - \int_{\partial \Omega} v_n^{q+1} \leq \int_\Omega v_n^2 \to 0,
\]

that is, \( \|v_n\| \to 0 \). Say \( w_n = \frac{v_n}{\|v_n\|} \), and \( \|w_n\| = 1 \). Then, up to a subsequence, \( w_n \to w_0 \geq 0 \) and \( w_n \to w_0 \) in \( L^{p+1}(\Omega) \) and \( L^2(\partial \Omega) \). Lemma 3.4 shows that \( w_0 \neq 0 \) and \( w_0 \notin H^1_0(\Omega) \), i.e., \( \int_{\partial \Omega} w_0^{q+1} > 0 \). However, we find that

\[
\int_{\partial \Omega} w_n^{q+1} = -E(v_n) - \mu_n \int_\Omega v_n^{p+1} \leq -E(v_n);
\]

thus,

\[
\int_{\partial \Omega} w_n^{q+1} \leq -E(w_n)\|v_n\|^{1-q} \to 0.
\]

This implies that \( \int_{\partial \Omega} w_0^{q+1} = 0 \), which is a contradiction. Claim (4.3) is thus verified. The desired conclusion now follows from (4.2) and (4.3). \( \square \)
4.2. Asymptotic profiles of positive solutions as $\lambda \to 0^+$. The next proposition asserts that if $u_n$ is a positive solution of (1.1) for $\lambda = \lambda_n \to 0^+$, then $(\lambda_n, u_n)$ converges to $(0,0)$, except that $u_n$ is the positive solution given by Proposition 2.2.

**Proposition 4.3.** Let $u_n$ be a positive solution of (1.1) for $\lambda = \lambda_n \to 0^+$ such that $u_n \neq u_{1,\lambda_n}$, where $u_{1,\lambda_n}$ is the positive solution emanating from $(\lambda, u) = (0,1)$, ensured by Proposition 2.2. Then, $u_n \to 0$ in $C(\Omega)$ (and consequently in $H^1(\Omega)$).

**Proof.** Using Proposition 2.1 and Lemma 3.3, the proof is conducted on the basis of the fact that the Neumann logistic problem, i.e., (1.1) with $\lambda = 0$ has exactly two nonnegative solutions $u \equiv 0, 1$.

By employing the change of variables [3.3] for (1.1), we prepare some results concerning the positive solutions of (3.4) for $\mu = 0$ and $\mu > 0$ small, which play a crucial role in characterizing the asymptotic profile of a positive solution $u_n$ of (1.1) for $\lambda = \lambda_n > 0$ satisfying that $(\lambda, u_n) \to (0,0)$ in $\mathbb{R} \times H^1(\Omega)$.

We prove the following three lemmas:

**Lemma 4.4.** There exists $C > 0$ such that if $u_n$ is a positive solution of (1.1) for $\lambda = \lambda_n > 0$ such that $(\lambda_n, u_n) \to (0,0)$ in $\mathbb{R} \times H^1(\Omega)$, then $\|v_n\| \leq C$ for $v_n = \lambda_n^{-\frac{1}{q-1}} u_n$.

**Proof.** Let $u_n$ be a positive solution of (1.1) for $\lambda = \lambda_n \to 0^+$ such that $\|u_n\| \to 0$. Assume by contradiction that $\|v_n\| \to \infty$ for $v_n = \lambda_n^{-\frac{1}{q-1}} u_n$. Set $w_n := \frac{v_n}{\|v_n\|}$, and $\|w_n\| = 1$. Then, up to a subsequence, $w_n \to w_0 \geq 0$, and $w_n \to w_0$ in $L^{p+1}(\Omega)$ and $L^2(\partial \Omega)$. By Lemma 3.3 (i), we obtain $w_0 \neq 0$.

Substituting $\varphi = 1$ for (3.5) with $(\mu, v) = (\mu_n, v_n) = (\frac{\mu_n^{p-1}}{p-1}, \lambda_n^{-\frac{1}{q-1}} u_n)$, it follows that

$$\int_{\Omega} v_n = \int_{\Omega} u_n^{p-1} v_n + \int_{\partial \Omega} v_n^q.$$

thus,

$$\int_{\Omega} w_n = \int_{\Omega} u_n^{p-1} w_n + \int_{\partial \Omega} w_n^q \|v_n\|^{-(1-q)}.$$

Since $\|u_n\| \to 0$ and $\|w_n\| = 1$, we infer that $\int_{\Omega} u_n^{p-1} w_n \to 0$, thus, $\int_{\Omega} w_n \to 0$. Consequently, $\int_{\Omega} w_0 = 0$, and $w_0 = 0$ as desired.

**Lemma 4.5.** Assume that $\lambda_\Omega \neq 1$. Then, for $\mu_0 > 0$, there exist $C > 0$ such that $\|v\| \geq C$ for a positive solution $v$ of (3.4) for $\mu \in [0, \mu_0]$.

**Proof.** Assume by contradiction that $\mu_n \geq 0$ and $(\mu_n, v_n) \to (\mu_\infty, 0)$ in $\mathbb{R} \times H^1(\Omega)$ for some $\mu_\infty \geq 0$. Say $w_n = \frac{v_n}{\|v_n\|}$, and $\|w_n\| = 1$. Then, up to a subsequence, $w_n \to w_\infty \geq 0$, and $w_n \to w_\infty$ in $L^{p+1}(\Omega)$ and $L^2(\partial \Omega)$. By Lemma 3.4 (i), we obtain $w_\infty \neq 0$. Substituting $\varphi = 1$ for (3.5) with $(\mu, v) = (\mu_n, v_n)$, it follows that

$$\int_{\partial \Omega} v_n^q = \int_{\Omega} v_n - \mu_n \int_{\Omega} u_n^{p-1} \leq \int_{\Omega} v_n,$$

thus,

$$\int_{\partial \Omega} w_n^q \leq \int_{\Omega} w_n \|v_n\|^{1-q} \to 0.$$
This implies that \( w_0 \in H_0^1(\Omega) \). Back to (3.3) with \((\mu, v) = (\mu_n, v_n)\), we obtain that 
\[
\int_\Omega (\nabla v_n \nabla \varphi - v_n \varphi + \mu_n v_n^p \varphi) = 0, \quad \varphi \in H_0^1(\Omega),
\]
and 
\[
\int_\Omega (\nabla w_n \nabla \varphi - w_n \varphi + \mu_n w_n^p \|v_n\|^{p-1} \varphi) = 0.
\]
Passing to the limit yields 
\[
\int_\Omega (\nabla w_\infty \nabla \varphi - w_\infty \varphi) = 0.
\]
Since \( w_\infty \geq 0 \) and \( w_\infty \neq 0 \), this implies that \( \lambda_\infty = 1 \), as desired. \( \square \)

**Lemma 4.6.** Assume that \( \lambda_\Omega < 1 \). Then, problem (3.4) has no positive solution for \( \mu = 0 \).

**Proof.** By the assumption \( \lambda_\Omega < 1 \), we obtain the unique positive solution \( u_D \) of (1.5). If \( v \) is a positive solution of (3.4) for \( \mu = 0 \), then, substituting \( \varphi = u_D \) for (3.5), we deduce that 
\[
0 = \int_\Omega (\nabla v \nabla u_D - vu_D) + \int_{\partial \Omega} v^q u_D = \int_\Omega (\nabla v \nabla u_D - vu_D).
\]
Using the fact that \( v \in W^{1,r}(\Omega) \) with \( r > N \), we deduce by the divergence theorem that 
\[
\int_\Omega (u_D - u_D^p) v = \int_\Omega (-\Delta u_D) v = \int_\Omega \nabla u_D \nabla v - \int_{\partial \Omega} \frac{\partial u_D}{\partial \nu} v.
\]
Combining these two assertions leads us to the contradiction 
\[
0 < \int_\Omega u_D^p v = \int_{\partial \Omega} \frac{\partial u_D}{\partial \nu} v < 0.
\]
\( \square \)

In the case of \( \lambda_\Omega > 1 \), we then establish the following asymptotic profile of a positive solution \( u_n \) of (1.1) for \( \lambda = \lambda_n > 0 \) such that \((\lambda_n, u_n) \to (0, 0)\) in \( \mathbb{R} \times H^1(\Omega) \).

**Proposition 4.7.** Assume that \( \lambda_\Omega > 1 \). If \( u_n \) is a positive solution of (1.1) for \( \lambda = \lambda_n > 0 \) such that \((\lambda_n, u_n) \to (0, 0)\) in \( \mathbb{R} \times H^1(\Omega) \), then up to a subsequence, 
\[
v_n = \lambda_n^{-\frac{1}{p-1}} u_n \longrightarrow v_0 \quad \text{in} \quad H^1(\Omega).
\]
Here, \( v_0 \) is a positive solution of (1.6), which satisfies that \( v_0 > 0 \) on \( \Gamma \subset \partial \Omega \) with the condition that \( |\Gamma| > 0 \).

**Proof.** Similarly as in the proof of Proposition 2.1 (ii), we deduce that \( v > 0 \) on \( \Gamma \subset \partial \Omega \) with \( |\Gamma| > 0 \) for a positive solution \( v \) of (1.3).

From Lemma 4.4, \( v_n \) is bounded in \( H^1(\Omega) \). Immediately, up to a subsequence, \( v_n \rightharpoonup v_0 \geq 0 \), and \( v_n \to v_0 \) in \( L^{p+1}(\Omega) \) and \( L^2(\partial \Omega) \). From (3.3), it follows that 
\[
\int_\Omega \left( \nabla v_n \nabla \varphi - v_n \varphi + \lambda_n \frac{v_n^p}{v_n^p} \varphi \right) + \int_{\partial \Omega} v_n^q \varphi = 0, \quad \varphi \in H^1(\Omega),
\]
and passing to the limit deduces 
\[
\int_\Omega (\nabla v_0 \nabla \varphi - v_0 \varphi) + \int_{\partial \Omega} v_0^q \varphi = 0.
\]
This means that \( v_0 \) is a nonnegative solution of (1.6) (i.e., (3.4) with \( \mu = 0 \)). Moreover, combining Lemma 4.5 and Lemma 3.4 (i) provides that \( v_0 \neq 0 \), thus, \( v_0 \) is a positive solution of (1.6).

Finally, we prove that \( v_n \to v_0 \) in \( H^1(\Omega) \). It suffices to show that \( E(v_n) \to E(v_0) \). Observing that \( E(v_n) \leq -B(v_n) \), we deduce that \( E(v_0) \leq \liminf E(v_n) \leq \limsup E(v_n) \leq -B(v_0) = E(v_0) \), as desired. \( \square \)

As a byproduct of Proposition 4.7, we obtain the following instability result for the positive solutions of (1.1) as \( \lambda \to 0^+ \).

**Proposition 4.8.** Assume that \( \lambda_\Omega > 1 \). Let \( u_n \) be a positive solution of (1.1) for \( \lambda = \lambda_n > 0 \) such that \( (\lambda_n, u_n) \to (0, 0) \) in \( \mathbb{R} \times H^1(\Omega) \). If \( u_n > 0 \) in \( \overline{\Omega} \), then \( u_n \) is unstable for \( n \) large enough.

**Proof.** Let \( u_n \) be a positive solution of (1.1) for \( \lambda = \lambda_n > 0 \) such that \( (\lambda_n, u_n) \to (0, 0) \) in \( \mathbb{R} \times H^1(\Omega) \), and \( u_n > 0 \) in \( \overline{\Omega} \). Considering the linearized eigenvalue problem (2.2) at \( (\lambda, u) = (\lambda_n, u_n) \), we prove that the smallest eigenvalue \( \gamma_{1,n} \) is negative for \( n \) large enough.

Recall that \( f(t) = t - t^p \) and \( g(t) = t^q \) for \( t > 0 \). Using a positive eigenfunction \( \varphi_{1,n} > 0 \) in \( \overline{\Omega} \) associated with \( \gamma_{1,n} \), we deduce that

\[
- \left( \frac{u_n}{\varphi_{1,n}} \right) \sum_{j=1}^{N} \frac{\partial}{\partial x_j} \left( \varphi_{1,n}^2 \frac{\partial}{\partial x_j} \left( \frac{u_n}{\varphi_{1,n}} \right) \right) = u_n \left( f(u_n) - f'(u_n)u_n \right) - \gamma_{1,n} u_n^2.
\]

Both sides are integrated over \( \Omega \), and we use the divergence theorem to deduce that

\[
\gamma_{1,n} \left( \int_{\Omega} u_n^2 + \int_{\partial \Omega} u_n^2 \right) = - \int_{\Omega} \varphi_{1,n} \left| \nabla \left( \frac{u_n}{\varphi_{1,n}} \right) \right|^2 - \int_{\Omega} u_n^2 F'(u_n) + \lambda_n \int_{\partial \Omega} u_n^2 G'(u_n)
\]

\[
\leq - \int_{\Omega} u_n^2 F'(u_n) + \lambda_n \int_{\partial \Omega} u_n^2 G'(u_n) =: I_n,
\]

where \( F(t) = \frac{f(t)}{t} \) and \( G(t) = \frac{g(t)}{t} \). Once we verify that up to a subsequence, \( I_n < 0 \), the proof is completed.

Since \( F'(t) = -(p-1)t^{p-2} \) and \( G'(t) = (q-1)t^{q-2} \), it follows that

\[
I_n = (p-1) \int_{\Omega} u_n^{p+1} + (1-q) \lambda_n \int_{\partial \Omega} u_n^{q+1}
\]

\[
= (p-1) \int_{\Omega} \lambda_n^{\frac{p+1}{q+1}} v_n^{p+1} + (1-q) \lambda_n^{\frac{2}{q+1}} \int_{\partial \Omega} v_n^{q+1}
\]

\[
= \lambda_n^{\frac{2}{q+1}} \left\{ (p-1) \lambda_n^{\frac{p-1}{q+1}} \int_{\Omega} v_n^{p+1} + (1-q) \int_{\partial \Omega} v_n^{q+1} \right\} =: \lambda_n^{\frac{2}{q+1}} \hat{I}_n,
\]

where \( \lambda_n > 0 \) and \( \lambda_n \to 0 \) as \( n \to \infty \). Proposition 4.7 enables us to deduce that up to a subsequence, \( \hat{I}_n \to - (1-q) \int_{\partial \Omega} v_n^{q+1} < 0 \), as desired. \( \square \)

In the case of \( \lambda_\Omega < 1 \), there is no positive solution \( u_n \) of (1.1) for \( \lambda = \lambda_n > 0 \) such that \( (\lambda_n, u_n) \to (0, 0) \) in \( \mathbb{R} \times H^1(\Omega) \).

**Proposition 4.9.** Assume that \( \lambda_\Omega < 1 \). Then, there exists \( C_0 > 0 \) and \( \lambda_0 > 0 \) such that \( \|u\| \geq C_0 \) for a positive solution \( u \) of (1.1) with \( \lambda \in (0, \lambda_0) \).
Proof. Assume by contradiction that \((\lambda_n, u_n) \to (0,0)\) in \(\mathbb{R} \times H^1(\Omega)\) for a positive solution \(u_n\) of \((1.1)\) for \(\lambda = \lambda_n > 0\). Using \(\mu_n = \frac{\lambda_n}{\lambda_n - q}\) and \(v_n = \frac{1}{\lambda_n - q} u_n\) from \((3.3)\), \((\mu, v) = (\mu_n, v_n)\) is a positive solution of \((3.4)\), and Lemma \(4.4\) shows that \(v_n\) is bounded in \(H^1(\Omega)\). It follows that up to a subsequence, \(v_n \to v_0 \geq 0\), and \(v_n \to v_0\) in \(L^{p+1}(\Omega)\) and \(L^2(\partial \Omega)\). Combining Lemma \(3.3\) (i) and Lemma \(1.5\) deduces that \(v_0 \neq 0\).

Observing that \((\mu, v) = (\mu_n, v_n)\) admits \((3.5)\), we deduce by passing to the limit that
\[
\int_{\Omega} (\nabla v_0 \nabla \varphi - v_0 \varphi) + \int_{\partial \Omega} v_0^2 \varphi = 0, \quad \varphi \in H^1(\Omega),
\]
which implies that \(v_0\) is a nonnegative solution of \((3.4)\) with \(\mu = 0\) (i.e., \((1.6)\)). Lemma \(1.6\) shows that \(v_0 = 0\), as desired. \(\square\)

As a consequence of Proposition \(4.9\) we have the uniqueness of a positive solution of \((1.1)\) for \(\lambda > 0\) small.

**Corollary 4.10.** Assume that \(\lambda_\Omega < 1\). Then, a positive solution of \((1.1)\) is unique for \(\lambda > 0\) small, which is given by \(u_{1,\lambda}\) of Proposition \(2.2\).

**Proof.** Let \(u_n\) be a positive solution of \((1.1)\) for \(\lambda = \lambda_n \to 0^+\). Lemma \(3.3\) then shows that up to a subsequence, \(u_n \to u_0 \geq 0\), and \(u_n \to u_0\) in \(C(\overline{\Omega})\). From the condition that
\[
\int_{\Omega} (\nabla u_n \nabla \varphi - u_n \varphi + u_n^p \varphi) + \lambda_n \int_{\partial \Omega} u_n^q \varphi = 0, \quad \varphi \in H^1(\Omega),
\]
we deduce by passing to the limit that
\[
\int_{\Omega} (\nabla u_0 \nabla \varphi - u_0 \varphi + u_0^p \varphi) = 0.
\]
This implies that \(u_0\) is a nonnegative solution of \((1.1)\) with \(\lambda = 0\), thus, \(u_0 \equiv 0\) or \(1\).

Proposition \(4.9\) shows that \(\|u_n\| \geq C_0\), and Lemma \(3.3\) (i) yields that \(u_0 \neq 0\). Thus, \(u_n \to 1\) in \(C(\overline{\Omega})\). Using a bootstrap argument and the compactness result, it follows that up to a subsequence, \(u_n \to 1\) in \(C^{2+\beta}(\overline{\Omega})\) for some \(\beta \in (0,1)\). Therefore, Proposition \(2.2\) shows that \(u_n = u_{1,\lambda_n}\) for \(n\) large, thus, the desired conclusion follows. \(\square\)

We then prove Theorems \(1.2\) and \(1.4\).

**Proof of Theorem 1.2.** The existence result for a positive solution of \((1.1)\) is due to Proposition \(2.5\).Assertions (i) and (ii) follow from Corollary \(4.10\) and Proposition \(4.1\) respectively. \(\square\)

**Proof of Theorem 1.4.** The multiplicity result for the positive solutions follows from Propositions \(3.13\) and \(3.17\). We choose the minimizers \(u_\lambda^\pm\) as \(u_\lambda^\pm \geq 0\) and \(u_\lambda^\pm \neq 0\). Therefore, by \(6\) Theorem 2.3], we deduce that \(u_\lambda^\pm\) is usual critical points for \(J_\lambda\), and \(u_\lambda^\pm\) is nontrivial, nonnegative weak solutions of \((1.1)\). This implies that \(u_\lambda^\pm\) are positive solutions of \((1.1)\), as stated in the Introduction. The behavior of \(U_{1,\lambda}(= u_\lambda^+)\) as \(\lambda \to 0^+\) is verified in a similar manner as in the proof of Corollary \(1.10\) (by \(3.18\)), we see that \(u_\lambda^+\) admits the assertion of Proposition \(4.9\).

The assertion that \(U_{2,\lambda}(= u_\lambda^-)\) converges to 0 in \(H^1(\Omega)\) as \(\lambda \to 0^+\) is verified by \((3.22)\). The nonexistence result for positive solutions comes from Proposition \(4.2\). Finally, assertions (i) and (ii) follow from Propositions \(4.7\) and \(4.8\) respectively. \(\square\)
5. EXISTENCE OF BOUNDED, CLOSED, AND CONNECTED SUBSETS OF POSITIVE SOLUTIONS

This section is devoted to the proof of Theorem 1.5. First of all, we consider the existence of a principal eigenvalue of the Steklov eigenvalue problem
\[
\begin{cases}
-\Delta \psi = \psi & \text{in } \Omega, \\
\frac{\partial \psi}{\partial \nu} = \sigma \psi & \text{on } \partial \Omega,
\end{cases}
\]
(5.1)
where \(\sigma \in \mathbb{R}\) is an eigenvalue parameter. We recall that a principal eigenvalue of (5.1) is referred to as an eigenvalue with constant sign eigenfunctions. A nonnegative eigenfunction of (5.1) is positive in \(\Omega\) by the strong maximum principle and boundary point lemma.

For (5.1), we present the following result that comes from [13, Lemma 9].

**Lemma 5.1.** If (5.1) has a principal eigenvalue \(\sigma_1\), then \(\sigma_1 < 0\) and \(\lambda_\Omega > 1\). Conversely, if \(\lambda_\Omega > 1\), then (5.1) possesses a unique principal eigenvalue \(\sigma_1 < 0\), which is characterized by the variational formula
\[
\sigma_1 = \inf \left\{ E(\psi) : \psi \in H^1(\Omega), \int_{\partial \Omega} \psi^2 = 1 \right\}.
\]
Additionally, the infimum is attained, and \(\sigma_1\) is simple.

Then, we discuss bifurcation from \(\{ (\lambda, 0) : \lambda > 0 \}\) for the positive solutions of (1.1). However, this is a non standard bifurcation problem in the sense that Crandall and Rabinowitz’ local bifurcation theory from simple eigenvalues [7] is not directly applicable, because the function \(t^q (t \geq 0)\) with \(0 < q < 1\) is not right differentiable at \(t = 0\). To overcome this difficulty, we consider a regularization for (1.1) as follows.
\[
\begin{cases}
-\Delta u = u(1 - |u|^{p-1}) & \text{in } \Omega, \\
\frac{\partial u}{\partial \nu} = -\lambda \left\{ \frac{(q-1)(u + \varepsilon)^{q-2}u + (u + \varepsilon)^{q-1}}{u + \varepsilon} \right\} & \text{on } \partial \Omega,
\end{cases}
\]
(5.2)
with a regularization parameter \(\varepsilon \in (0, 1)\). Formally, we regard (1.1) as (5.2) with \(\varepsilon = 0\) for nonnegative solutions. This formal observation will be justified by a topological argument by Whyburn [31].

First, we evaluate bifurcation from \(\{ (\lambda, 0) : \lambda > 0 \}\) for the positive solutions of (5.2) with a fixed \(\varepsilon \in (0, 1)\) and second, how the bifurcating positive solution set \(\{ (\lambda, u) \}\) behaves as \(\varepsilon \to 0^+\). We remark that a positive solution of (5.2) is positive in \(\Omega\) by the strong maximum principle and boundary point lemma because (5.2) is a regular problem. For the bifurcation analysis for the positive solutions of (5.2), we consider the linearized eigenvalue problem
\[
\begin{cases}
-\Delta \varphi = (1 - p|u|^{p-1})\varphi + \gamma \varphi & \text{in } \Omega, \\
\frac{\partial \varphi}{\partial \nu} = -\lambda \left\{ (q-1)(u + \varepsilon)^{q-2}u + (u + \varepsilon)^{q-1} \right\} \varphi + \gamma \varphi & \text{on } \partial \Omega.
\end{cases}
\]
(5.3)
Substitute \(u = 0\) for (5.3), and consider the case \(\gamma = 0\):
\[
\begin{cases}
-\Delta \varphi = \varphi & \text{in } \Omega, \\
\frac{\partial \varphi}{\partial \nu} = -\lambda \varepsilon^{q-1} \varphi & \text{on } \partial \Omega.
\end{cases}
\]
(5.4)
If \(\lambda_\Omega > 1\), then Lemma 5.1 ensures that problem (5.4) has a unique principal eigenvalue \(\lambda_\varepsilon > 0\), which is simple and satisfies that
\[
\lambda_\varepsilon = \lambda_1 \varepsilon^{1-q} \quad \text{(implying that } \lambda_\varepsilon \to 0 \text{ as } \varepsilon \to 0^+\).
\]
(5.5)
Before stating our bifurcation result for (5.2), we establish several \textit{a priori} bounds for the positive solutions of (5.2).

**Proposition 5.2.** The following assertions hold:

(i) If $u$ is a positive solution of (5.2) for $\lambda > 0$, then $u < 1$ in $\Omega$.

(ii) Assume that $\lambda_0 > 1$. Then, we have the following:

(a) There exists $\Lambda_0 > 0$ such that if problem (5.2) has a positive solution for $\lambda > 0$, then $\lambda \leq \Lambda_0$, where $\Lambda_0$ does not depend on $\varepsilon \in (0,1)$.

(b) For each $\delta > 0$ that is small, there exists $C = C_{\varepsilon, \delta} > 0$ such that $\|u\|_{C(\overline{\Omega})} \geq C$ for a positive solution of (5.2) with $\lambda \in [0, \lambda_0 - \delta] \cup [\lambda_0 + \delta, \Lambda_0]$.

(iii) For $0 < \delta_1 < \delta_2 < 1$, there exists $\Lambda_1 > 0$ such that if problem (5.2) has a positive solution $u$ of (5.2) such that $\delta_1 < \|u\|_{C(\overline{\Omega})} < \delta_2$, then $\lambda \geq \Lambda_1$. Here, $\Lambda_1$ does not depend on $\varepsilon \in (0,1)$.

**Proof.** (i) The proof is carried out in the same spirit of that for Proposition 4.1 (i). Indeed, it suffices to notice that a constant $u = c \geq 1$ is a supersolution of (5.2).

(ii-a) We assume by contradiction that problem (5.2) has a positive solution $u_n$ for $\lambda = \lambda_n \to \infty$ and $\varepsilon = \varepsilon_n \in (0,1)$. Then, it follows that

$$
\int_{\partial \Omega} \frac{u_n^2}{(u_n + \varepsilon_n)^{1-q}} = \frac{1}{\lambda_n} \left\{ \int_{\Omega} \left( |\nabla u_n|^2 + u_n^2 \right) - \int_{\partial \Omega} u_n^{p+1} \right\} \leq \frac{1}{\lambda_n} \int_{\Omega} u_n^2.
$$

Say $w_n = \frac{u_n}{\|u_n\|}$ and then, up to a subsequence, $w_n \to w_\infty \geq 0$, $w_n \to w_\infty$ in $L^{p+1}(\Omega)$ and $L^2(\partial \Omega)$ for some $w_\infty \in H^1(\Omega)$. Since $u_n < 1$ and $\varepsilon_n < 1$, we infer from (5.6) that

$$
\frac{1}{2^{1-q}} \int_{\partial \Omega} w_n^2 \leq \frac{1}{\lambda_n} \int_{\Omega} w_n^2 \to 0.
$$

This implies that $\int_{\partial \Omega} w_\infty^2 = 0$, i.e., $w_\infty \in H^1_0(\Omega)$. The rest of the proof is in the same manner as that for Proposition 4.2.

(ii-b) This is a direct consequence of [1, Proposition 18.1].

(iii) Assume by contradiction that for some $0 < \delta_1 < \delta_2 < 1$, $u_n$ is a positive solution of (5.2) with $\varepsilon_n \in (0,1)$ for $\lambda = \lambda_n \to 0^+$ such that $\delta_1 < \|u_n\|_{C(\overline{\Omega})} < \delta_2$. It follows that $u_n$ is bounded in $H^1(\Omega)$. Immediately, up to a subsequence, $u_n \to u_0$, $u_n \to u_0$ in $L^{p+1}(\Omega)$ and $L^2(\partial \Omega)$, $u_n \to u_0$ a.e., and $\varepsilon_n \to \varepsilon_0 \in [0,1]$. Additionally, $u_n \to u_0 \geq 0$ in $C(\overline{\Omega})$ as in the proof of Lemma 3.1. We deduce that

$$
\frac{u_n}{u_n + \varepsilon_n} \leq 1 \quad \text{in} \quad \Omega,
$$

and so we can prove that $u_n$ is bounded in $W^{1, r}(\Omega)$ for $r > N$, following the bootstrap argument developed in the proof of [27, Theorem 2.2]. Consequently, $\delta_1 \leq \|u_0\|_{C(\overline{\Omega})} \leq \delta_2$ and particularly, $u_0 \neq 0, 1$.

Since $u_n$ is a positive solution of (5.2) with $(\lambda, \varepsilon) = (\lambda_n, \varepsilon_n)$, we observe

$$
\int_{\Omega} (\nabla u_n \nabla \varphi - u_n \varphi + u_n^{p-1} u_n \varphi) + \lambda_n \int_{\partial \Omega} (u_n + \varepsilon_n)^{q-1} u_n \varphi = 0, \quad \varphi \in H^1(\Omega).
$$

Passing to the limit shows that

$$
\int_{\Omega} (\nabla u_0 \nabla \varphi - u_0 \varphi + u_0^{p-1} u_0 \varphi) = 0,
$$

where
because 
\[(u_n + \varepsilon_n)^{q-1} u_n = \left( \frac{u_n}{u_n + \varepsilon_n} \right)^{1-q} u_n^q \leq \delta_n^q \quad \text{in } \Omega.\]

This means that \( u_0 \) is a nonnegative solution of the Neumann logistic problem (1.3). Therefore, \( u_0 \equiv 0 \) or 1, as desired. \( \square \)

Using Proposition 5.2, we prove the following existence result for bifurcating positive solutions of (5.2) at \((\lambda, u) = (\lambda_\varepsilon, 0)\). Taking into account that problem (5.2) is regular near \((\lambda_\varepsilon, 0)\), the proof of Proposition 5.3 is carried out in the same spirit of that for [29, Proposition 2.2] (see also [28, Theorem 1.1]), which is thus omitted here.

**Proposition 5.3.** Suppose that \( \lambda_\Omega > 1 \). Let \( \varepsilon \in (0, 1) \) be fixed. Then, problem (5.2) possesses a bounded component (i.e., maximal, closed, and connected subset) \( C_\varepsilon = \{(\lambda, u)\} \) of nonnegative solutions in \([0, \infty) \times C(\Omega)\) for some \( \theta \in (0, 1) \) that bifurcates from \( \{(\lambda, 0)\} \) at \((\lambda_\varepsilon, 0)\). Moreover, it holds that

(i) \((0, 1), (\lambda_\varepsilon, 0) \in C_\varepsilon\);

(ii) \( C_\varepsilon \) does not meet any \((0, u)\) or \((\lambda, 0)\) except for \((0, 1), (\lambda_\varepsilon, 0)\);

(iii) \( C_\varepsilon \setminus \{(\lambda_\varepsilon, 0)\} \) is contained in the positive solution set of (5.2), see Figure 3.

![Figure 3. Component \( C_\varepsilon \).](image_url)

The following nonexistence result is derived from Lemma 4.5 by considering (3.3), which plays an important role in determining the limiting behavior of \( C_\varepsilon \) as \( \varepsilon \to 0^+ \).

**Proposition 5.4.** Assume \( \lambda_\Omega \neq 1 \). Let \( \lambda_0 > 1 \). Then, there exists \( C > 0 \) such that \( \|u\| \geq C \) for a positive solution \( u \) of (1.1) with \( \lambda \in [1/\lambda_0, \lambda_0] \), meaning that there is no bifurcation point on \( \{(\lambda, 0) : \lambda > 0\} \) for positive solutions of (1.1).

We then end the proof of Theorem 1.5.

**End of proof of Theorem 1.5.** Let \( X \) be a metric space, and \( \mathcal{E}_n \subset X \). Set
\[
\lim_{n \to \infty} \mathcal{E}_n := \{x \in X : \lim_{n \to \infty} \text{dist} (x, \mathcal{E}_n) = 0\},
\]
\[
\lim_{n \to \infty} \mathcal{E}_n := \{x \in X : \lim_{n \to \infty} \text{dist} (x, \mathcal{E}_n) = 0\}.
\]
We then obtain the following (31 (9.12) Theorem):

**Theorem 5.5.** Assume that \( \{E_n\} \) is a sequence of connected sets which satisfies that

(i) \( \bigcup_{n \geq 1} E_n \) is precompact;

(ii) \( \lim_{n \to \infty} E_n \neq \emptyset. \)

Then, \( \lim_{n \to \infty} E_n \) is nonempty, closed and connected.

We then use Theorem 5.5 to obtain a closed and connected limit set of \( C_\varepsilon \) as \( \varepsilon \to 0^+ \), as Theorem 12 demands. We introduce the metric space \( X := \mathbb{R} \times C(\Omega) \) with the metric function given by

\[
d((\lambda, u), (\mu, v)) := |\lambda - \mu| + \|u - v\|_{C(\Omega)} \quad \text{for} \quad (\lambda, u), (\mu, v) \in \mathbb{R} \times C(\Omega).
\]

Let \( \varepsilon_n \in (0, 1) \) be such that \( \varepsilon_n \to 0^+ \), and \( C_n := C_{\varepsilon_n} \) the bounded component ensured by Proposition 5.3. From Proposition 5.2 (i) and (ii-a), we then deduce that

\[
\bigcup_{n \geq 1} C_n \subset \{(\lambda, u) \in \mathbb{R} \times C(\Omega) : 0 \leq \lambda \leq \Lambda_0, \ u \leq 1 \ \text{in} \ \Omega\}. \tag{5.8}
\]

Additionally, from (5.5), we also deduce that

\[
(0, 0), (0, 1) \in \lim_{n \to \infty} C_n. \tag{5.9}
\]

We claim that

\[
\bigcup_{n \geq 1} C_n \quad \text{is precompact.} \tag{5.10}
\]

For \( \{ (\lambda_k, u_k) \}_{k=1}^\infty \subset \bigcup_{n \geq 1} C_n \), we deduce that \( (\lambda_k, u_k) \in C_{n_k} \) for some \( n_k \). Note that

\[
\int_\Omega (\nabla u_k \nabla \varphi - u_k \varphi + u_k^p \varphi) + \lambda_k \int_{\partial \Omega} \left( \frac{u_k}{u_k + \varepsilon_{n_k}} \right)^{1-q} u_k^q \varphi = 0, \quad \varphi \in H^1(\Omega). \tag{5.11}
\]

From (5.5), we may infer that \( \{\lambda_k\} \) is a convergent sequence. Moreover, since \( u_k \) is bounded in \( C(\Omega) \), it is bounded in \( H^1(\Omega) \). Indeed, (5.11) with \( \varphi = u_k \) implies that

\[
\int_\Omega |\nabla u_k|^2 = \int_\Omega (u_k^q - u_k^{q+1}) - \lambda_k \int_{\partial \Omega} \left( \frac{u_k}{u_k + \varepsilon_{n_k}} \right)^{1-q} u_k^{q+1} \leq C.
\]

Hence, \( u_k \) has a convergent subsequence in \( C(\Omega) \), which is deduced in the same argument as in proof of Proposition 5.2 (iii). Claim (5.10) is thus verified.

Assertions (5.9) and (5.10) then enable us to apply Theorem 5.5 deducing that \( C_0 := \lim_{n \to \infty} C_n \) is nonempty, closed, and connected in \( [0, \infty) \times C(\Omega) \) such that \( (0, 0), (0, 1) \in C_0 \). From (5.8), \( C_0 \) is bounded in \( [0, \infty) \times C(\Omega) \). We claim that \( C_0 \) consists of nonnegative solutions of (1.1). Given \( (\lambda, u) \in C_0 \), there exists \( (\lambda_k, u_k) \in C_{n_k} \) such that \( \varepsilon_{n_k} \to 0^+ \) and \( (\lambda_k, u_k) \to (\lambda, u) \) in \( \mathbb{R} \times C(\Omega) \), deducing that \( u \geq 0 \). It follows that \( (\lambda_k, u_k) \) satisfies (5.11), and \( u_k \) is bounded in \( C(\Omega) \) and \( H^1(\Omega) \). By the same argument above, we infer that up to a subsequence, \( u_k \to u \). We observe that

\[
\left( \frac{u_k}{u_k + \varepsilon_{n_k}} \right)^{1-q} u_k^q \leq u_k^q \to 0 \quad \text{for} \ x \in \partial \Omega \ \text{with} \ u(x) = 0,
\]

for \( x \in \partial \Omega \) with \( u(x) = 0 \).
Thus
\[
\left( \frac{u_k}{u_k + \varepsilon n_k} \right)^{1-q} u_k^q \longrightarrow u^q \quad \text{for} \ x \in \partial \Omega \ \text{with} \ u(x) > 0,
\]

thus
\[
\left( \frac{u_k}{u_k + \varepsilon n_k} \right)^{1-q} u_k^q \longrightarrow u^q \quad \text{on} \ \partial \Omega.
\]

From (5.7), \( \left( \frac{u_k}{u_k + \varepsilon n_k} \right)^{1-q} u_k^q \) is bounded in \( C(\overline{\Omega}) \), and passing to the limit yields
\[
\int_{\Omega} (\nabla u \nabla \varphi - u \varphi + u^p \varphi) + \lambda \int_{\partial \Omega} u^q \varphi = 0, \quad \varphi \in H^1(\Omega),
\]
by the Lebesgue dominated convergence theorem, as desired.

Finally, we verify that \( C_0 \) is the desired subcontinuum. Assertion (i) follows from (5.9). Assertion (ii) follows from Proposition 5.2 (iii) and Proposition 5.4. Assertion (iii) follows from the combination of assertion (i) with Proposition 2.2. The proof of Theorem 1.5 is now complete.

\[\square\]

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