REMARK ON MEROMORPHIC FUNCTIONS THAT SHARE
FIVE PAIRS

NORBERT STEINMETZ

Abstract. We determine all pairs \((f, g)\) of meromorphic functions that share
four pairs of values \((a_\nu, b_\nu)\), \(1 \leq \nu \leq 4\), and a fifth pair \((a_5, b_5)\) under some
mild additional condition.

1. Introduction

Meromorphic functions \(f\) and \(g\) are said to share the pair \((a, b)\) of complex numbers
(including \(\infty\)), if \(f - a\) and \(g - b\) (\(1/f\) and \(1/g\), if \(a = \infty\) and \(b = \infty\), respectively)
have the same zeros. Czubiak and Gundersen [1] proved that meromorphic func-
tions \(f\) and \(g\) that share six pairs \((a_\nu, b_\nu)\) are Möbius transformations of each
other, hence share all pairs \((a, L(a))\) for some Möbius transformation \(L\). On the
other hand, the functions

\[\hat{f}(z) = \frac{e^z + 1}{(e^z - 1)^2}\quad\text{and}\quad \hat{g}(z) = \frac{(e^z + 1)^2}{8(e^z - 1)}\]

share the values \(\infty, 0, 1,\) and \(-\frac{1}{2}\) with different multiplicities, and the pair \((-\frac{1}{2}, \frac{1}{4})\)
counting multiplicities. Thus

\[f(z) = \frac{1}{\hat{f}(z) + \frac{1}{2}}\quad\text{and}\quad g(z) = \frac{1}{\hat{g}(z) - \frac{1}{4}}\]

are not Möbius transformations of each other and share the pairs \((0, 0), (2, -4),\)
\((\frac{2}{3}, \frac{1}{3})\) and \((\frac{4}{3}, -\frac{2}{3})\) with different multiplicities, and the value \(\infty\) (the pair \((\infty, \infty))\)
counting multiplicities. Moreover, \(f\) and \(g\) have common counting function of poles
\(N(r, \infty) = T(r) + S(r)\), where \(T(r)\) and \(S(r)\) denote the common Nevanlinna char-
acteristic and remainder term of \(f\) and \(g\) (for notations and results of Nevanlinna
theory the reader is referred to Hayman’s monograph [5]), and \(f\) and \(g\) parametrise
the algebraic curve

\[4x^2 + 2xy + y^2 - 8x = 0.\]

Gundersen’s example \(\hat{f}, \hat{g}\) was the first to show that in Nevanlinna’s Four Value
Theorem [7] one cannot dispense with the condition ‘counting multiplicities’ for
each of the four values. This is possible for one value (Gundersen [2]) and also for
two of the values (Gundersen [3], Mues [6]), while the case of three such values
is still open. The state of art is outlined in [10]. Gundersen’s example also has
another characterisation due to Reinders [8, 9]: If \(f\) and \(g\) share four values \(a_\nu,\)

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and if \( f^{-1}(a) \subset g^{-1}(b) \) holds for some pair \((a,b)\) \((a,b \neq a_\nu)\), then either \(f\) and \(g\) are Möbius transformations of each other or else \(f = T \circ \hat{f} \circ h\) and \(g = T \circ \hat{g} \circ h\) holds for some Möbius transformation \(T\) and some non-constant entire function \(h\).

In [4], Gundersen considered functions \(f\) and \(g\) that share five pairs and are not Möbius transformations of each other. He proved several sharp inequalities for the corresponding Nevanlinna functions, including \(T(r,f) = T(r,g) + S(r)\) and
\[
\overline{N}(r; a_\nu, b_\nu) \geq \frac{1}{2}T(r) + S(r),
\]
where \(\overline{N}(r; a_\nu, b_\nu)\) denotes the counting functions of common \((a_\nu, b_\nu)\)-points of \((f,g)\), not counting multiplicities, and \(T(r)\) and \(S(r)\) denote the common Nevanlinna characteristic and remainder term, respectively.

2. Main result

The aim of this paper is to prove

**Theorem 1.** Suppose that meromorphic functions \(f\) and \(g\) share four pairs \((a_\nu, b_\nu)\), and a fifth pair \((a_5, b_5)\) counting multiplicities and such that
\[
m(r, 1/(f - a_5)) + m(r, 1/(g - b_5)) = S(r)
\]
holds. Then either \(f\) and \(g\) are Möbius transformations of each other or else \(f = T \circ \hat{f} \circ h\) and \(g = T \circ \hat{g} \circ h\) holds for suitably chosen Möbius transformations \(T\) and \(S\) and some non-constant entire function \(h\).

**Proof.** We note that (5) is automatically fulfilled if \(a_\nu = b_\nu\), \(1 \leq \nu \leq 4\). Three of the pairs \((a_\nu, b_\nu)\) may be prescribed. We will assume \((a_1, b_1) = (0, 0)\), \((a_2, b_2) = (2, -4)\), and, in particular, \(a_5 = b_5 = \infty\), to stay as close as possible with the modified example of Gundersen. Then \(f\) and \(g\) have the same poles counting multiplicities, and such that
\[
m(r, f) + m(r, g) = S(r)
\]
holds. We also assume that \(f\) and \(g\) are not Möbius transformations of each other.

Similar to the approach in [4] we consider
\[
P(x, y, \xi) = c_1x^2 + c_2xy + c_3y^3 + c_4x + c_5y.
\]
Then there are at least two linear independent vectors \(\xi = (c_1, \ldots, c_5) \in \mathbb{C}^5\) such that
\[
P(a_\nu, b_\nu, \xi) = 0 \quad (1 \leq \nu \leq 4)
\]
holds, that is, \(P(z) = P(f(z), g(z), \xi)\) vanishes whenever \(f(z) = a_\nu\) and \(g(z) = b_\nu\).

If \(P\) does not vanish identically, this yields
\[
\sum_{\nu=1}^{4} \overline{N}(r; a_\nu, b_\nu) \leq \overline{N}(r, 1/P) \leq T(r, P) + O(1) \leq 2T(r) + S(r);
\]
for the last inequality the additional hypothesis \((\nu)\) is used. On the other hand it follows from the Second Main Theorem that
\[
\sum_{\nu=1}^{4} \overline{N}(r; a_\nu, b_\nu) + \overline{N}(r, \infty) \geq 3T(r) + S(r),
\]
hence $T(r) \leq N(r, \infty) + S(r)$. Thus, still assuming $P \neq 0$, it follows that
\[
N(r, 1/P) = N(r, 1/P) + S(r) = 2T(r) + S(r)
\]
\[
m(r, 1/P) = S(r)
\]
\[
\sum_{\nu=1}^{4} N(r, a_{\nu}, b_{\nu}) = \sum_{\nu=1}^{4} N(r, 1/P) + S(r)
\]
\[
T(r) = N(r, \infty) + S(r).
\]

In particular, the quotient $\chi(z) = P(z; \bar{c})/P(z; c)$ satisfies $T(r, \chi) = S(r)$. In other words, $f$ and $g$ parametrise the algebraic curve
\[
F(x, y; z) = \chi_1 x^2 + \chi_2 yx + \chi_3 y^2 + \chi_4 x + \chi_5 y = 0 \quad (\chi_k = \chi c_k - \tilde{c}_k)
\]
over the field $\mathbb{C}(\chi)$. This is also true if $P(z; c)$ or $P(z; \bar{c})$ vanishes identically. It is obvious that $\chi_1 \chi_3 \neq 0$, since otherwise $g$ [resp. $f$] would be a Möbius transformation or a rational function of $f$ [resp. $g$] of degree two over the field $\mathbb{C}(\chi)$. In the first case it would follow that $g$ is an ordinary Möbius transformation of $f$, while in the second case we would obtain a contradiction: $T(r, g) = 2T(r, f) + S(r)$.

The algebraic curve (8) has the rational parametrisation (set $x = ty$)
\[
x = \frac{p(z, t)}{s(z, t)} = -\frac{t(\chi_4 t + \chi_5)}{\chi_1 t^2 + \chi_2 t + \chi_3}, \quad y = \frac{q(z, t)}{s(z, t)} = -\frac{\chi_4 t + \chi_5}{\chi_1 t^2 + \chi_2 t + \chi_3}
\]
with $t = x/y$. In terms of $f$ and $g$ this yields
\[
f(z) = \frac{p(z, t(z))}{s(z, t(z))} = -\frac{t(z)(\chi_4 t(z) + \chi_5)}{\chi_1 t(z)^2 + \chi_2 t(z) + \chi_3}
\]
\[
g(z) = \frac{q(z, t(z))}{s(z, t(z))} = -\frac{\chi_4 t(z) + \chi_5}{\chi_1 t(z)^2 + \chi_2 t(z) + \chi_3}
\]
with $t(z) = \frac{f(z)}{g(z)}$.

Since by (4), $f$ and $g$ have 'many' zeros, there are three possibilities to be discussed: The zeros correspond to the

a) poles of $t$, in which case $\chi_4 \equiv 0$ and 'almost all' zeros of $f$ are simple, while the zeros of $g$ have order two. Moreover, $t$ has 'almost no' zeros $(N(r, 1/t) = S(r))$.

b) zeros of $t$, in which case $\chi_5 \equiv 0$ and 'almost all' zeros of $g$ are simple, while the zeros of $f$ have order two. Moreover, $t$ has 'almost no' poles $(N(r, t) = S(r))$.

c) zeros of $\chi_4(z)H(z) + \chi_5(z)$ with $\chi_4 \chi_5 \neq 0$. Then 'almost all' zeros of $f$ and $g$ are simple, and $t$ has 'almost no' zeros and poles $(N(r, 1/t) + N(r, t) = S(r))$.

Taking all pairs $(a_{\nu}, b_{\nu})$ (1 $\leq \nu \leq 4$), into account, the following holds: for every $\nu$ there exist $\phi_{\nu}, \psi_{\nu}, \alpha_{\nu}, \beta_{\nu}, \tilde{\beta}_{\nu} \in \mathbb{C}(\chi)$ such that $p(z, t) - a_{\nu}s(z, t) = \phi_{\nu}(t - \alpha_{\nu})(t - \beta_{\nu})$ and $q(z, t) - b_{\nu}s(z, t) = \psi_{\nu}(t - \alpha_{\nu})(t - \tilde{\beta}_{\nu})$, respectively; occasionally the factor $(t - \beta_{\nu})$ and $(t - \tilde{\beta}_{\nu})$ corresponding to $\beta_{\nu} \equiv \infty$ and $\tilde{\beta}_{\nu} \equiv \infty$, respectively, might be missing. The functions\(^{1}\) $\alpha_{\nu}$ are mutually distinct, and the same is true for $\beta_{\nu}$ and also $\tilde{\beta}_{\nu}$. It is also obvious that $\beta_{\nu} \neq \tilde{\beta}_{\nu}$, and that both functions are exceptional for $t$, except when one of them coincides with $\alpha_{\nu}$. Since $t$ has at most two exceptional functions, we obtain the following picture:

\(^{1}\)At first glance one would expect that $\alpha_{\nu}, \beta_{\nu}, \tilde{\beta}_{\nu}$ are algebraic over $\mathbb{C}(\chi)$. But this is not the case, since analytic continuation which permutes $\alpha_{\nu}$ and $\beta_{\nu}$ would also permute $\alpha_{\nu}$ and $\tilde{\beta}_{\nu}$, in contrast to $\beta_{\nu} \neq \tilde{\beta}_{\nu}$. 
For $\nu = 1$ and $\nu = 4$, say, we have $\beta_\nu \equiv \alpha_\nu$, that is, the pairs $(a_\nu, b_\nu)$, are attained by $(f, g)$ in a $(2 : 1)$ manner, while for $\nu = 2$ and $\nu = 3$ this happens the other way $(1 : 2)$. This means that, in addition to \((5)\), that also

\[(9) \quad F_y(a_\nu, b_\nu; z) \equiv 0 \quad (\nu = 1, 4) \quad \text{and} \quad F_x(a_\nu, b_\nu; z) \equiv 0 \quad (\nu = 2, 3)\]

holds. To stay close with the modified example of Gundersen we assume $\chi_3 \equiv 1$ (this is possible since $\chi_3 \not\equiv 0$ is already known). From \((9)\), that is

\[\chi_5 = 4\chi_1 - 4\chi_2 + \chi_4 = 2\chi_1 a_3 + \chi_3 b_3 + \chi_4 = \chi_2 a_4 + 2b_4 = 0,\]

one can compute the coefficients $\chi_k$ in terms of $a_3$, $b_3$, $a_4$, $b_4$, namely

\[(10) \quad \chi_1 = \frac{b_4(b_3 + 4)}{a_4(a_3 - 2)}, \quad \chi_2 = -\frac{2b_4}{a_4}, \quad \chi_3 = 1, \quad \chi_4 = \frac{2b_4(2b_3 + 4a_3)}{a_4(2 - a_3)}.\]

In particular, the functions $\chi_k$ are constant, and $f$ and $g$ are rational functions (now over $\mathbb{C}$) of the meromorphic function $t = f/g$. Having determined the coefficients \((10)\) we now use \((8)\) to express $b_3$ and $b_4$ in terms of $a_3$ and $a_4$. The solutions to $F(a_\nu, b_\nu; z) = 0$ for $\nu = 2, 4$ are given by \((4)\)

- $b_4 = 2a_4 - 8$ and $b_3 = \frac{2(8 - 4a_4 + a_4a_3)}{a_4 - 4}$.

Since, however, $F(a_3, b_3; z) = \frac{32(a_3 - 2)(a_3 - 2)(a_3 - a_4 + 2)}{(a_4 - 4)^2}$ also has to vanish, we just have to discuss the sub-case $a_3 = a_4 - 2$, since $a_3 = 2$ and also $a_4 = 2$ would contradict $a_2 = 2$. Thus $a_3 = a_4 - 2$, $b_3 = 2a_4 - 4$ and $b_4 = 2a_4 - 8$, and $(a_1, a_2, a_3, a_4)$ and $(b_1, b_2, b_3, b_4)$ have the same cross-ratio $\frac{a_2 - a_3}{a_2 - a_4} : \frac{a_1 - a_4}{a_2 - a_4} = \frac{b_2 - b_3}{b_2 - b_4} : \frac{b_1 - b_4}{b_2 - b_4} = \frac{(a_4 - 2)^2}{a_4(a_4 - 3)}$. In other words, there exists some M"obius transformation $L$ such that $f$ and $L \circ g$ share four values $a_1, a_2, a_3, a_4$ and the pair $(\mathcal{X}, L(\mathcal{X}))$. By Reinders' characterisation this implies $f = T \circ \hat{f} \circ h$ and $g = S \circ \hat{g} \circ h$, where $S$ and $T$ are suitably chosen M"obius transformations, and $h$ is some non-constant entire function. $\square$

**Final remark.** It remains open whether or not—and how—the hypothesis \((8)\) may be relaxed. Is it sufficient to assume that the pair $(a_5, b_5)$ is shared 'counting multiplicities' by $f$ and $g$? Is it even true that functions sharing five pairs are either M"obius transformations of each other or else have the form $f = T \circ \hat{f} \circ h$ and $g = S \circ \hat{g} \circ h$?

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\(^2\)We note that *maple* was not able to determine all solutions to the system \((7)\), but note also that the coefficients $\chi_k$ are functions of the $a_\nu$, $b_\nu$; to get an impression: one has to solve

$$(b_3 + 4)b_4 + (8 - 4a_3)a_4 = a_3^2b_4(3a_3 + 4) + b_3^2(2 - a_3) = a_4(b_3 + 4) + a_3(8 - b_4) + 2b_4 + 4a_3 = 0$$

for $b_3, b_4$. 
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Institut für Mathematik · TU Dortmund · D-44221 Dortmund · Germany

E-mail address: stein@math.tu-dortmund.de