EQUIVARIANT SYMPLECTIC GEOMETRY
OF COTANGENT BUNDLES, II

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ABSTRACT. We examine the structure of the cotangent bundle $T^*X$ of an algebraic variety $X$ acted on by a reductive group $G$ from the viewpoint of equivariant symplectic geometry. In particular, we construct an equivariant symplectic covering of $T^*X$ by the cotangent bundle of a certain variety of horospheres in $X$, and integrate the invariant collective motion on $T^*X$. These results are based on a “local structure theorem” describing the action of a certain parabolic in $G$ on an open subset of $X$, which is interesting by itself.

INTRODUCTION

An important class of symplectic manifolds with a Hamiltonian group action is formed by cotangent bundles of manifolds acted on by Lie groups. Cotangent bundles arise as phase spaces for many important Hamiltonian dynamical systems with symmetries. Therefore it is an important problem to study the equivariant symplectic geometry of cotangent bundles. In particular, one may address the problem of constructing a symplectic manifold which is locally equivariantly isomorphic to the cotangent bundle but has a simpler structure than the latter one. Morally, this should help in integrating Hamiltonian dynamical systems on cotangent bundles.

In this paper, we study the equivariant geometry of cotangent bundles in the framework of algebraic geometry. More precisely, let $X$ be a smooth algebraic variety over complex numbers (or, more generally, over any algebraically closed field of characteristic zero) acted on by a connected reductive group $G$. We examine the natural $G$-action on the cotangent bundle $T^*X$ from the viewpoint of transformation groups and symplectic geometry.

This problem has attracted the attention of several researchers. In particular, F. Knop studied the moment map $\Phi : T^*X \to g^*$ in [Kn90]. Using the moment map, he obtained deep results on the geometry of...
the action \( G : T^*X \) including: the existence and the description of the stabilizer in general position; the description of the collective functions (which are the integrals for any \( G \)-invariant Hamiltonian system on \( T^*X \)); the relation between symplectic invariants of \( T^*X \) (corank, defect) and important invariants of \( X \) (complexity, rank), which play a significant role, e.g., in studying equivariant embeddings of \( X \); etc. In [Kn94] Knop studied the invariant collective motion on \( T^*X \), i.e., the flow generated by the skew-gradients of invariant collective functions, under restriction that \( X \) is “non-degenerate”. (This class includes, e.g., all quasiaffine varieties.)

On the other hand, E. B. Vinberg [Vi01] constructed an equivariant symplectic rational Galois covering of \( T^*X \) by the cotangent bundle of the variety of generic horospheres provided that \( X \) is quasiaffine. (A horosphere in \( X \) is an orbit of a maximal unipotent subgroup of \( G \).) Since the variety of horospheres and its cotangent bundle have a relatively simple structure from the point of view of \( G \)-action, this result solves the problem posed above.

Our main objective here is to generalize Vinberg’s construction to arbitrary \( X \). The general scheme of reasoning is the same as in [Vi01], so that our paper may be regarded as a direct continuation of [Vi01].

On the other hand, the main technical tool in the paper is a refined version of the so-called “local structure theorem” (see Section 2), which in its turn yields simple proofs and generalizations of some results from [Kn90], [Kn94]. Roughly speaking, the main results of [Kn90] stem from a “quantization” argument, i.e., by passing from \( T^*X \) to differential operators on \( X \). We reprove them by pure “classical” arguments in the spirit of [Kn94] but dropping the “non-degeneracy” assumption. We are also able to deduce some results of Knop on invariant collective motion in full generality.

The structure of the paper is as follows. In Section 1 we describe a general construction from symplectic geometry used by Vinberg to construct his rational covering. Then we recall Vinberg’s result and explain why it does not generalize directly. In Section 2 we prove the refined local structure theorem. Using this theorem, we examine the geometry of \( T^*X \) in Section 3. In particular, we describe the image of the moment map and the stabilizer in general position. An equivariant symplectic rational Galois covering of \( T^*X \) by the cotangent bundle of a certain variety of “degenerate” horospheres is constructed in Section 4. In Section 5 we study the invariant collective motion on \( T^*X \) and show how it can be integrated after lifting via the above covering.

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1 The author is grateful to E. B. Vinberg for his kind permission to entitle this paper as Part II of [Vi01].
1. Polarization of cotangent bundles

1.1. In [Vi01 §4] Vinberg introduced a general construction from symplectic geometry which relates the cotangent bundles of two different varieties. The desired symplectic covering of a cotangent bundle is its particular application. So for convenience of the reader we recall this construction here.

Let \(X, Y,\) and \(Z \subseteq X \times Y\) be smooth irreducible algebraic varieties. Assume that the projections \(p : Z \rightarrow X,\) \(q : Z \rightarrow Y\) are smooth surjective maps. Then \(Z\) can be regarded as a family of smooth subvarieties \(Z_y = \{x \mid (x, y) \in Z\} \subseteq X\) of equal dimension parametrized by points \(y \in Y,\) or similarly, as a family of subvarieties \(Z_x \subseteq Y.\)

For any \(z = (x, y) \in X \times Y\) there are canonical isomorphisms

\[ T_z(X \times Y) \cong T_xX \oplus T_yY, \quad T_z^*(X \times Y) \cong T_{x}^*X \oplus T_{y}^*Y. \]
For any $\alpha \in T^*_z(X \times Y)$ we denote by $\alpha', \alpha''$ its projections to $T^*_xX$ and $T^*_yY$, respectively.

**Definition 1.** The *skew conormal bundle* of $Z$ is

$$SN^*Z = \{ \alpha \in T^*_z(X \times Y) \mid z \in Z, \; \alpha' - \alpha'' = 0 \text{ on } T_zZ \}.$$

**Remark 1.** The skew conormal bundle is obtained from the usual conormal bundle $N^*Z$ by an automorphism of $T^*(X \times Y) \simeq T^*X \times T^*Y$, namely, by multiplying the covectors over $Y$ conormal bundle in order to avoid superfluous signs in some formulæ. It is easy to see that $\hat{\rho}$ induces an isomorphism

$$SN^*Z \simeq \bigsqcup_{y \in Y} N^*Z_y.$$

Indeed, for any $z = (x, y) \in Z$ the linear map $\hat{\rho} : SN^*_zZ \to N^*_xZ_y$ is isomorphic, because $T_zZ \cap T_z(X \times \{ y \}) = T_z(Z_y \times \{ y \})$ and $T_zZ + T_z(X \times \{ y \}) = T_z(X \times Y)$ by smoothness and surjectivity of $q$. Similarly,

$$SN^*Z \simeq \bigsqcup_{x \in X} N^*_xZ_x.$$

Recall that the canonical symplectic structure on $T^*X$ arises from a certain 1-form $\ell'$, called the *action form*. Given $\alpha' \in T^*_xX$ and $\nu' \in T_\alpha T^*_xX$, we put $\ell'(\nu') = \langle \alpha', \xi' \rangle$, where $\xi' \in T_xX$ is the projection of $\nu'$. Then the symplectic form on $T^*X$ is defined as $\omega' = d\ell'$. Similarly, one defines the action form $\ell''$ and the symplectic form $\omega''$ on $T^*Y$.

The action forms $\ell', \ell''$ lift to one and the same 1-form $\ell$ on $SN^*Z$. Indeed, given any $\alpha \in SN^*_zZ$, $z = (x, y)$, and any $\nu \in T_\alpha SN^*_zZ$, denote $\nu' = d\hat{\rho}(\nu) \in T_\alpha T^*_xX$, $\nu'' = d\hat{q}(\nu) \in T_\alpha T^*_yY$, and by $\xi, \xi', \xi''$ the projections of $\nu, \nu', \nu''$ to $T_zZ, T_zX, T_yY$, respectively. Then $\langle \alpha' - \alpha'', \xi \rangle = \langle \alpha', \xi' \rangle - \langle \alpha'', \xi'' \rangle = 0$ implies $\ell'(\nu') = \ell''(\nu'')$. 

There is a commutative diagram

$$
\begin{array}{ccc}
T^*X & \leftarrow & SN^*Z \\
\downarrow & & \downarrow \\
X & \leftarrow & Z \\
\end{array}
\begin{array}{ccc}
\downarrow & & \downarrow \\
& \leftarrow & q \\
\end{array}
\begin{array}{ccc}
& & T^*Y \\
\end{array}
$$

where $\hat{\rho}, \hat{q}$ take $\alpha$ to $\alpha', \alpha''$, respectively.

**Remark 2.** Consider the disjoint union $\bigsqcup_{y \in Y} N^*_yZ_y$ of the conormal bundles of the subvarieties $Z_y$ in $X$. It has a natural structure of a subbundle in the pullback of $T^*X$ to $Z$. By definition, this bundle consists of pairs $(\alpha', y)$ such that $\alpha' \in T^*_xX$, $Z_y \ni x$, and $Z_y$ is tangent to $\text{Ker} \alpha'$ at $x$. We call such pairs the *polarized covectors* over $X$, and the whole bundle is called the *polarized cotangent bundle* of $X$ with respect to the family of subvarieties $Z_y$.

It is easy to see that $\hat{\rho}$ induces an isomorphism

$$\hat{\rho} : SN^*Z \simeq \bigsqcup_{y \in Y} N^*_yZ_y.$$
It follows that \( \omega', \omega'' \) lift to one and the same closed (but maybe degenerate) 2-form \( \omega = d\ell \) on \( SN^*Z \).

**Lemma 1.** If \( \dim X = \dim Y \), then the following conditions are equivalent:

1. (ND) \( \omega \) is non-degenerate at points in general position;
2. (DX) \( \hat{p} : SN^*Z \to T^*X \) is dominant;
3. (DY) \( \hat{q} : SN^*Z \to T^*Y \) is dominant.

**Proof.** We have \( \dim SN^*Z = \dim T^*X = \dim T^*Y = 2 \dim X \). If \( \omega \) is generically non-degenerate, then \( \hat{p} \) has finite generic fibres, hence it is dominant by dimension count. Conversely, a dominant map between varieties of equal dimension is generically étale whence (ND) \( \iff \) (DX). Similarly, (ND) \( \iff \) (DY). \( \square \)

Under the conditions of the lemma, \( \omega \) defines a symplectic structure on an open subset of \( SN^*Z \), so that \( \hat{p}, \hat{q} \) are symplectic rational coverings.

1.2. Assume now that \( X \) is equipped with an action of a reductive connected group \( G \). The induced action \( G : T^*X \) is Hamiltonian, i.e., it preserves the symplectic structure and there exists a \( G \)-equivariant moment map \( \Phi : T^*X \to g^* \) with the following property: for any \( \xi \in g \), regarded as a linear function on \( g^* \), the skew gradient of \( \Phi^*\xi \) equals the velocity field of \( \xi \) on \( T^*X \). The moment map is defined by the formula

\[
\langle \Phi(\alpha), \xi \rangle = \langle \alpha, \xi x \rangle, \quad \forall x \in X, \alpha \in T_x^*X, \xi \in g.
\]

For instance, if \( X = G/H \), then

\[
T^*X \cong G *_{H} (g/\mathfrak{h})^* \cong G *_{H} \mathfrak{h}^\perp,
\]

where \( \mathfrak{h}^\perp \) is the annihilator of \( \mathfrak{h} \) in \( g^* \), and the moment map amounts to the coadjoint action: \( \Phi(g * \alpha) = g\alpha \).

A **horosphere** in \( X \) is an orbit of any maximal unipotent subgroup of \( G \). This terminology goes back to I. M. Gelfand and M. I. Graev [GG59], and it is justified by an observation that for \( X = S^n(\mathbb{C}) \) (the \( n \)-dimensional complex sphere), \( G = SO_{n+1}(\mathbb{C}) \), the (generic) horospheres are nothing else but the complexifications of usual horospheres in the Lobachevsky space \( L^n \).

The set of generic horospheres can be equipped with a structure of an algebraic \( G \)-variety of the same dimension as \( X \), see [2.1] for details. (Since the notion of a “generic horosphere” is not quite well defined, this variety is determined only up to a birational equivalence, but this suffices for our purposes.)

In the notation of [1,1], let \( Y \) be the variety of generic horospheres in \( X \), and \( Z = \{(x, H) \mid x \in H\} \subset X \times Y \) the incidence variety. (Perhaps, one has to shrink \( X, Y, Z \) a little bit in order to fulfil all necessary requirements on smoothness.) Applying the construction of [1,1] to this setting, Vinberg proved the following
Theorem 1 ([Vi01]). If $X$ is quasiaffine, then there exists a $G$-equivariant symplectic rational Galois covering $T^*Y \rightarrow T^*X$.

Actually, Knop proved implicitly in [Kn94] that under the assumptions of the theorem $\hat{p} : SN^*Z \rightarrow T^*X$ is a rational Galois covering, and Vinberg showed that $\hat{q} : SN^*Z \rightarrow T^*Y$ is birational. The desired covering is then defined as $\hat{p}\hat{q}^{-1}$.

The variety of horospheres $Y$ and its cotangent bundle $T^*Y$ have a simple structure as $G$-varieties compared with that of $X$ and $T^*X$ (see, e.g., [Kn90], 2.1). Thus Theorem 1 gives a good approximation to the structure of $T^*X$ as a symplectic $G$-variety.

However Theorem 1 does not generalize naively to arbitrary $X$. Indeed, in view of (D) and (1) a necessary condition is that $\bigcup_{H \in Y} N^*H$ be dense in $T^*X$, i.e., any covector in general position must vanish along the tangent space of a suitable horosphere. But there are simple counterexamples in the non-quasiaffine case:

Example 1. Let $X = G/P$ be a generalized flag variety (i.e., $P$ is a parabolic subgroup of $G$). Generic horospheres are just the open Schubert cells with respect to various choices of a maximal unipotent subgroup of $G$. For instance, $X = \mathbb{P}^n$, $G = GL_{n+1}$, and generic horospheres are complements to hyperplanes. Thus $\bigcup_{H \in Y} N^*H$ is the zero bundle over $X$.

However, if we consider the “most degenerate” horospheres, which are just points in $X$, then everything becomes fine: conormal bundles are just cotangent spaces at points of $X$, $Y = X$, and the covering $T^*Y \rightarrow T^*X$ is the identity map.

This example suggests a remedy in the general case: to take for $Y$ a certain variety of non-generic horospheres. This idea is developed in Section 4 and leads to a generalization of Theorem 1.

2. Local structure theorem

2.1. Let $G$ be a connected reductive group acting on an irreducible algebraic variety $X$. In this section, we describe the action of a certain parabolic subgroup of $G$ on an open subset of $X$. Results of this kind are called “local structure theorems” and are ubiquitous in the study of reductive group actions. They arise from Brion, Luna, Vust [BLV86], and Grosshans [Gr87], cf. [Kn90], [Kn94], [Vi01].

Fix a Borel subgroup $B \subseteq G$ with the unipotent part $U \subseteq B$. Let $P \supseteq B$ be the largest subgroup of $G$ which stabilizes all $B$-orbits in general position in $X$. Consider a Levi decomposition $P = P_u \times L$, where $P_u \subseteq U$ is the unipotent radical of $P$ and $L$ a Levi subgroup.

Theorem 2 ([Kn90] 2.3, [Kn94] §2]). There is an open $P$-stable subset $X_0 \subseteq X$ and a closed $L$-stable subset $Z_0 \subseteq X_0$ such that the natural
\textit{P-equivariant map}

\[
P \ast_L Z_0 \longrightarrow X_0, \quad p \ast z \mapsto pz,
\]

is an isomorphism. Furthermore, the kernel \( L_0 \) of the action \( L : Z \) contains \([L, L]\), the torus \( A = L/L_0 \) acts on \( Z_0 \) freely, and \( Z_0 \simeq A \times C \) for a certain closed subvariety \( C \subseteq Z_0 \), so that

\[
X_0 \simeq P_u \times A \times C.
\]

From this theorem, one deduces that \( P \) is exactly the stabilizer in \( G \) of a generic \( B \)-orbit and \( P_0 = P_u \ltimes L_0 \) is the stabilizer of a generic \( U \)-orbit (these orbits are parametrized by points of the cross-sections \( C \) and \( Z_0 \), respectively) \cite{Kn93, §2}, \cite{Vi01, §3}, cf. §1. Moreover, an element \( g \in G \) translates a generic \( B \)- or \( U \)-orbit to another generic \( B \)- or \( U \)-orbit iff \( g \in P \).

Since all maximal unipotent subgroups of \( G \) are conjugate, each horosphere is a \( G \)-translate of a \( U \)-orbit. By the above, the set of generic horospheres, i.e., \( G \)-translates of \( U \)-orbits in \( X_0 \), is isomorphic to \( G \ast_P Z_0 \simeq G/P_0 \times C \) as a \( G \)-set and inherits the structure of an algebraic \( G \)-variety from the latter one. Note that its dimension equals \( \dim X \).

\textbf{2.2.} In order to formulate our version of the local structure theorem, we have to introduce more notation.

We fix a \( G \)-invariant inner product on \( \mathfrak{g} \) and identify \( \mathfrak{g}^* \) with \( \mathfrak{g} \) by means of this product whenever it is convenient for our purposes.

The torus \( A \) is not a subgroup of \( L \), but its Lie algebra \( \mathfrak{a} \) can be embedded in \( \mathfrak{l} \) as the orthocomplement to \( \mathfrak{l}_0 \). Then \( M = Z(\mathfrak{a}) \) is a Levi subgroup of \( G \) containing \( L \), and \( Q = BM \) is a parabolic subgroup containing \( P \), with a Levi decomposition \( Q = Q_u \ltimes M \). Also put \( M_0 = L_0[M, M] \) and \( Q_0 = Q_u \ltimes M_0 \); then \( A \simeq M/M_0 \simeq Q/Q_0 \).

By the superscript \( "-" \) we indicate opposite parabolic subgroups (i.e., the parabolics intersecting given parabolics in specified Levi subgroups) and related subgroups (e.g., their unipotent radicals). The correlation between the Lie algebras of the groups introduced above is represented at the picture (the blocks indicate direct summands of \( \mathfrak{g} \)):

\[
\begin{array}{|c|c|c|}
\hline
\mathfrak{l} & \mathfrak{a} & \mathfrak{q} \\
\hline
\mathfrak{m} \cap \mathfrak{p}_u & \mathfrak{l}_0 & \mathfrak{m} \cap \mathfrak{p}_u \\
\hline
\mathfrak{m} & \mathfrak{q}_u & \mathfrak{p}_u \\
\hline
\end{array}
\]

Under the identification \( \mathfrak{g} \simeq \mathfrak{g}^* \), the spaces \( \mathfrak{l}, \mathfrak{l}_0, \mathfrak{m}, \mathfrak{m}_0, \mathfrak{a} \) are self-dual, whereas \( \mathfrak{p}_u = \mathfrak{p}^\perp \simeq (\mathfrak{g}/\mathfrak{p})^* \simeq \mathfrak{p}_u^* \) and \( \mathfrak{q}_u = \mathfrak{q}^\perp \simeq (\mathfrak{g}/\mathfrak{q})^* \simeq \mathfrak{q}_u^* \).
Definition 2. The \textit{principal stratum} of $\mathfrak{a}$ is

$$a^{pr} = \{ \lambda \in \mathfrak{a} \mid Z(\lambda) = M, \ g\lambda \notin \mathfrak{a}, \ \forall g \in N(M) \setminus N(\mathfrak{a}) \}$$

This is an open subset in $\mathfrak{a}$ complementary to finitely many linear subspaces (the kernels of the nonzero weights of $\text{ad} \mathfrak{a}$ in $\mathfrak{g}$ and the proper subspaces of the form $\mathfrak{a} \cap g\mathfrak{a}$, $g \in N(M)$).

Here comes the refined version of the local structure theorem:

Theorem 3. There is an open $Q$-stable subset $X_1 \subseteq X$ and a closed $M$-stable subset $Z_1 \subseteq X_1$ such that the natural $Q$-equivariant map

$$Q \ast_M Z_1 \longrightarrow X_1$$

is an isomorphism. Furthermore,

$$Z_1 \simeq M/(M \cap P^-_0) \times C \simeq M_0/(M_0 \cap P^-_0) \times A \times C$$

as an $M$-variety and

$$X_1 \simeq Q_u \times (M/M \cap P^-_0) \times C. \quad (3)$$

(In the product decompositions, it is always assumed that the groups act trivially on $C$.)

In comparison with Theorem 2, this theorem displays the local action of a larger parabolic subgroup $Q$, so that the locally free action of the complementary part $M \cap P_u$ to $Q_u$ in $P_u$ is extended to the action of $M_0$ on a generalized flag variety $M_0/(M_0 \cap P^-)$.

For quasiaffine $X$ one verifies that $M = L$ and $Q = P$ \cite{Kn94, 3.1}, \cite{Vi01, §1}, so that Theorem 3 specializes to Theorem 2. More generally, this property characterizes \textit{non-degenerate} varieties \cite{Kn94, §3}. A smooth $G$-variety $X$ is non-degenerate iff the action $G : T^*X$ is \textit{symplectically stable}, i.e., generic $G$-orbits in $\text{Im} \Phi$ are closed in $\mathfrak{g}^*$, see Remark 3 below.

On the contrary, if $X$ is a generalized flag variety, then $M = Q = G$ and $X_1 = X$. This is the opposite extremity in Theorem 3.

Proof of Theorem 3. In the notation of Theorem 2 consider the cotangent bundle over $X_0$ and the conormal bundle $\mathcal{U}$ to the foliation of $U$-orbits in $X_0$. We have

$$T^*X_0 \simeq P_u \times p^-_u \times A \times \mathfrak{a} \times T^*C,$$

$$\mathcal{U} \simeq P_u \times \{0\} \times A \times \mathfrak{a} \times T^*C.$$

Claim 1. $\Phi(\mathcal{U}) \subseteq \mathfrak{a} + \mathfrak{q}_u$.

Indeed, as $\Phi$ is equivariant and $\mathfrak{a} + \mathfrak{q}_u$ is $P$-stable, it suffices to consider $\Phi(\mathcal{U}|_C)$. Take any $\alpha \in \mathcal{U}_x$, $x \in C$, and let $\lambda$ be its projection to $\mathfrak{a}$. We have

$$(\Phi(\alpha), \xi) = \langle \alpha, \xi x \rangle = \begin{cases} 0, & \xi \in p_0; \\ (\lambda, \xi), & \xi \in \mathfrak{a}. \end{cases}$$
Hence $\Phi(\alpha) \in \lambda + p_u$. But $\alpha$ is fixed by $L_0$ whence

(4) \[ \Phi(\alpha) \in \lambda + p_u^{L_0} \subseteq \lambda + q_u, \]

because $p_u^{L_0} \cap m = p_u^L = 0$.

Claim 2. $m_0 x \subseteq p_u x$, $\forall x \in X_0$.

Otherwise there exists $\alpha \in U_x$ that does not vanish on $m_0 x$. But then $\Phi(\alpha) \not\in m_0$, a contradiction with Claim I.

It follows from Claim II that $P_0 x = Q_0 x$, $\forall x \in X_0$. Recall that the orbits $P_u x = P_0 x = U x$ of points $x \in X_0$ are parametrized by $Z_0$. Hence the open set $X_1 = QX_0$ carries a foliation of $Q_0$-orbits parametrized by $Z_0$ and $\mathcal{U}$ extends to the conormal bundle of this foliation (denoted by the same letter). The quotient space $X_1/Q_0$ is $Q$-isomorphic to $Z_0 \simeq A \times C$ and $\mathcal{U}$ is the pullback of $T^*Z_0 \simeq A \times a \times T^*C$. (Here $Q$ acts on $Z_0$ through $A \simeq Q/Q_0$.)

We can lift $X_1$ into $\mathcal{U}$ as the pullback of a $Q$-invariant section $A \times \{\lambda\} \times C$ of $T^*Z_0$ ($\lambda \in a$). Formula (4) yields a commutative diagram of $Q$-equivariant maps

\[ T^*X \xrightarrow{\Phi} \mathfrak{g} \]

\[ X_1 \xrightarrow{\varphi} \lambda + q_u. \]

Now suppose $\lambda \in \mathfrak{a}^{pr}$; then $[q, \lambda] = [q_u, \lambda] = q_u$. As $Q\lambda = Q_u \lambda$ is closed in $\mathfrak{g}$ [Kr85, Satz III.1.1-4], we have $Q \lambda = \lambda + q_u \simeq Q/M$. It follows that $X_1 \simeq Q^* Z_1$ is a homogeneous fibering over $Q\lambda$ with fibre map $\varphi$ and fibre $Z_1 = \varphi^{-1}(\lambda)$.

Claim 3. $M x \simeq M/M \cap P_0^-$, $\forall x \in Z_1$.

Indeed, the action $M \cap P_0 : Z_1 \cap X_0$ is free and $(M \cap P_0)x = Q_0 x \cap Z_1 = M_0 x$, $\forall x \in Z_1 \cap X_0$. As $M_0 \cap P$ normalizes $M \cap P$, we obtain $(M \cap P_0)x = (M_0 \cap P)x$ and, without loss of generality, $(M_0 \cap P)_x = L_0$. Hence $(M_0)_x = M_0 \cap P^-$, the unique subgroup of $M_0$ containing $L_0$ and transversal to $M \cap P$. Since $A \simeq M/M_0$ acts on $Z_1/M_0 \simeq X_1/Q_0$ freely, we have $M x = (M_0)_x = M \cap P^-_0$.

Finally, replacing $C$ by the set of points in $Z_1$ with stabilizer $M \cap P_0^-$ (which is a cross-section for $Q$-orbits in $X_1$ by Claim II) yields $Z_1 \simeq (M/M \cap P^-_0) \times C$. \hfill \square

3. Geometry of cotangent bundle

Making use of the local structure theorem, we investigate here the equivariant geometry of the cotangent bundle $T^*X$ of a smooth $G$-variety $X$.

Consider the conormal bundle to the foliation of generic $Q_u$-orbits:

$\mathcal{N} = \{ \alpha \in T^*_x X \mid x \in X_1, \langle \alpha, q_u x \rangle = 0 \}$
By (3) and (2) we have

\[ N \cong Q_u \times T^*Z_1 \cong Q_u \times M_{\mu}^{\mathfrak p_u^-}(a + m \cap p_u^-) \times T^*C. \]

Lemma 2. \( \Phi(N) = a + M + q_u \), where \( M = M(m \cap p_u^-) = M(m \cap p_u) \) is the closure of a Richardson nilpotent orbit in \( m \).

Proof. Take any \( \alpha \in N_x, x \in C \). Let \( \mu \) be the projection of \( \alpha \) to \( N^*_x(Q_u C) \cong T^*_xMx \cong a + m \cap p_u^- \), with the Jordan decomposition \( \mu = \mu_s + \mu_n \) (\( \mu_s \in a, \mu_n \in m \cap p_u^- \)). We have

\[
(\Phi(\alpha), \xi) = (\alpha, \xi x) = \begin{cases} 0, & \xi \in q_u; \\ (\mu, \xi), & \xi \in m. \end{cases}
\]

Hence \( \Phi(\alpha) \in \mu + q_u \). It follows that

\[
\Phi(N) = Q\Phi(N|_C) \subseteq Q(a + m \cap p_u^- + q_u) = a + M + q_u.
\]

On the other hand, if \( \mu_s \in a^{pr} \), then \( Q\mu = \mu_s + M\mu_n + q_u \). Indeed, \( Z(\mu) \subseteq Z(\mu_s) = M \) whence \( [q_u, \mu] = q_u \) and \( Q_u \mu = \mu + q_u \) by [Kr85, Satz III.1.1.4]. Thus \( \Phi(N) \supseteq a^{pr} + M + q_u \) whence the assertion on \( \Phi(N) \).

The equality \( M(m \cap p_u^-) = M(m \cap p_u) \) stems from a well-known property of induced nilpotent orbits [CM93, 7.1.3]. \( \square \)

Corollary 1. \( \overline{G\mathcal N} = T^*X \).

Proof. Since \( \mathcal N \) is \( Q \)-stable, \( \overline{G\mathcal N} = \overline{Q_u\mathcal N} \). Observe that \( \text{codim } \mathcal N = \dim Q_u \). Hence it suffices to prove that \( Q_u^- \alpha \approx Q_u^- \) is transversal to \( \mathcal N \) for some \( \alpha \in \mathcal N \).

Choose \( \alpha \in \mathcal N \) such that \( \mu = \Phi(\alpha) \in a^{pr} + M \). Then \( [q_u^- , \mu] = q_u^- \) is transversal to \( \Phi(\mathcal N) \). The assertion follows. \( \square \)

Corollary 2 ([Kn90, 5.4]). \( \overline{\text{Im } \Phi} = G(a + M + q_u) = \overline{G(a + M) = G(a + p_u)} = G(a + p_u^+) \).

Proof. It remains only to note that \( G(a + M + q_u) \) is closed, because \( a + M + q_u \) is stable under a parabolic \( Q \subseteq G \); similarly for \( G(a + p_u^+) \). \( \square \)

Remark 3. In particular, generic \( G \)-orbits in \( \text{Im } \Phi \) are represented by \( \mu \in a + M \). The orbit \( G\mu \) is closed in \( g \) iff \( \mu = \mu_s \in a \). Thus the action \( G : T^*X \) is symplectically stable iff \( M = 0 \) iff \( M = L \).

Corollary 3 ([Kn90, §8]). The stabilizers of points in general position for the action \( G : T^*X \) are conjugate to the stabilizer of a point from the open orbit for the action \( M \cap P_0^- : m \cap p_u^- \).

Proof. Take \( \alpha \in T^*X \) in general position. Without loss of generality we may assume \( \alpha \in \mathcal N, \Phi(\alpha) = \mu \in a^{pr} + M \). Then \( G_\alpha \subseteq G_\mu \subseteq G_\mu = M \subseteq Q \). However the stabilizer in general position for \( Q : \mathcal N \) coincides with that for \( M \cap P_0^- : m \cap p_u^- \). \( \square \)
The image of the moment map contains the principal open stratum \((\text{Im } \Phi)^{pr} = G(a^{pr} + M)\). Let \(T^*X\) denote its preimage in \(T^*X\), called the principal stratum of the cotangent bundle.

**Proposition 1.** The principal stratum of \(T^*X\) has the structure

\[
T^{pr}X \simeq G \ast_{N_X} \Sigma,
\]

where \(\Sigma\) is the unique component of \(\Phi^{-1}(a^{pr} + M)\) intersecting \(N\) and \(N_X\) is the stabilizer of \(\Sigma\) in \(N(a)\).

**Proof.** It is easy to deduce from Definition 2 that

\[
(\text{Im } \Phi)^{pr} \simeq G \ast_{N(a)} (a^{pr} + N(a)M)
\]

and \(N(a)M\) is a union of Richardson orbit closures in \(M\) permuted transitively by the Weyl group \(W(a) = N(a)/M\) of \(a\). Hence

\[
T^{pr}X \simeq G \ast_{N(a)} \tilde{\Sigma},
\]

where \(\tilde{\Sigma} = \Phi^{-1}(a^{pr} + N(a)M)\). The fibre \(\tilde{\Sigma}\) is smooth and its components are permuted transitively by \(W(a)\), because \(T^{pr}X\) is irreducible. Thus \(5\) holds for any component \(\Sigma\) of \(\tilde{\Sigma}\).

By Corollary 1, \(N\) intersects \(T^{pr}X\), and Lemma 2 implies that \(N^{pr} = N \cap T^{pr}X \subseteq Q_u \tilde{\Sigma} \simeq Q_u \times \tilde{\Sigma}\). Since \(N\) is irreducible, it intersects a unique component \(\Sigma\) of \(\tilde{\Sigma}\), and \(\Phi(\Sigma) = a^{pr} + M\) by Lemma 2 again. \(\Box\)

**Remark 4.** The variety \(\Sigma\) is a cross-section of \(T^*X\) in the terminology of Guillemin–Sternberg [GS84] and Knop [Kn97 5.4].

**Remark 5.** The subgroup \(W_X = N_X/M \subseteq W(a)\), i.e., the stabilizer of \(\Sigma\) in \(W(a)\), is nothing else but the little Weyl group of \(X\) defined in [Kn90]. To see this, consider a morphism \(\Psi : T^{pr}X \to Ga^{pr}\) taking \(\alpha\) to the semisimple part of \(\Phi(\alpha)\). It splits as

\[
T^{pr}X \xrightarrow{\ast_{N_X}} G \ast_{N_X} a^{pr} \xrightarrow{\ast_{N(a)}} G \ast_{N(a)} a^{pr} \simeq Ga^{pr}.
\]

The right arrow is a finite morphism, and the left one has irreducible generic fibres.

More precisely, restrict \(\Psi\) to an open subset \(\Sigma \cap T^*X_1 = \Sigma \cap N\) of \(\Sigma\). Since \(N^{pr} \simeq Q_u \times (\Sigma \cap N) \simeq Q_u \times T^{pr}Z_1\), we have \(M\)-isomorphisms

\[
\Sigma \cap N \simeq T^{pr}Z_1 \simeq T^*(M_0/M_0 \cap P^\perp) \times A \ast a^{pr} \times T^*C
\]

and \(\Psi\) is just the projection to \(a^{pr}\). So the fibres are isomorphic to \(A \times T^*(M_0C)\).

On the other hand, \(a\) embeds into \(N\) as the conormal space \(N^*_x(Q_0 C)\), \(\forall x \in C\). Factoring \(6\) by \(G\) we obtain a commutative diagram

\[
\begin{array}{c}
T^{pr}X \xrightarrow{} G \ast_{N_X} a^{pr} \xrightarrow{} G \ast_{N(a)} a^{pr} \\
\uparrow \quad \quad \quad \downarrow \\
a^{pr} \xrightarrow{} a^{pr}/W_X \xrightarrow{} a^{pr}/W(a),
\end{array}
\]
so that the composite morphism $T^{pr}X \to a^{pr}/W_X$ has irreducible generic fibres. Thus $W_X$ satisfies the definition of [Kn90, §6].

4. Horospheres and symplectic covering

4.1. Now we define a family of (possibly non-generic) horospheres which is used below to polarize $T^*X$ and generalize Theorem II.

Consider the set of $Q_u$-orbits in $X_1$ (parametrized by $Z_1$) and the set $Y$ of their $G$-translates. One may say that $Y$ is the set of generic orbits of unipotent subgroups conjugate to $Q_u$. On the other hand, $Y$ consists of horospheres: every $Q_u$-orbit in $X_1$ is an $M$-translate of $Q_u x$, $x \in Z_1$, such that $M_x \supset M \cap U$ whence $Q_u x = U x$.

**Proposition 2.** The set $Y$ carries a structure of an algebraic $G$-variety such that $Y \cong G *_Q Z_1 \cong G/Q_u(M \cap P^-_0) \times C$ and $\dim Y = \dim X$. (Here $Q$ acts on $Z_1$ through $M = Q/Q_u$.)

**Proof.** There is a natural map $G *_Q Z_1 \to Y$, $g * x \mapsto g Q_u x$. It suffices to verify that it is a bijection. The assertion on dimensions is obvious since $\dim G *_Q Z_1 = \dim G/Q + \dim Z_1 = \dim Q_u + \dim Z_1 = \dim X$. The bijectivity stems from the following

**Lemma 3.** $g Q_u x = Q_u x'$ ($x, x' \in Z_1$) iff $g \in Q$.

**Proof of the lemma.** As $Z_1 = MC$ and $M$ normalizes $Q_u$, we may assume without loss of generality that $x, x' \in C$.

First suppose that $x = x'$. Let $S$ be the stabilizer of $Q_u x$ in $G$. Then $S = Q_u \cdot S_x \supseteq Q_u(M \cap P^-_0)$ contains a maximal unipotent subgroup of $G$. The structure of such groups is well known [Kn90, §2]: we have $S = S_u \cdot \tilde{S}_x$ where $S_u$ is the unipotent radical of a parabolic subgroup $N(S)$ and $\tilde{S}$ is intermediate between the Levi subgroup of $N(S)$ and its semisimple part. Furthermore, $S_u \subseteq Q_u(M \cap P^-_0)$ and $\tilde{S} \supseteq L_0$.

We have $\tilde{S} = \tilde{Q}_u \cdot \tilde{S}_x$, where $\tilde{K}$ denotes the projection of $K \subseteq S$ to $\tilde{S}$. Hence $\tilde{S}/\tilde{S}_x$ is an affine $\tilde{S}$-variety with the transitive action of a unipotent subgroup of $\tilde{S}$. This is possible only if it is a point, i.e., $\tilde{S} = \tilde{S}_x$ whence $S_x$ contains a conjugate of $\tilde{S}$. Therefore $\tilde{S} \subseteq S_y$ for some $y \in Q_u x$.

The subgroup $\tilde{S} \cap M$ is reductive, contains $L_0$, and is contained in $S \cap M = M \cap P^-$. Hence $\tilde{S} \cap M = L_0$. If $\tilde{S} \neq L_0$, then $\tilde{S}$ intersects $Q_u$, a contradiction with $Q_u \cap S_y = \emptyset$. It follows that $\tilde{S} = L_0$, $N(S) = Q_u(M \cap P^-)$, and $S = Q_u(M \cap P^-) \subseteq Q$.

Finally, for arbitrary $x, x' \in C$ we have $g \in N(S) \subseteq Q$. \hfill $\Box$

4.2. Let $Y$ denote the variety of degenerate horospheres introduced in 4.1. Consider the polarized cotangent bundle with respect to $Y$:

$$\hat{T}^*X = \bigsqcup_{H \in Y} N^*H \cong G *_Q N.$$
(The latter isomorphism stems from Proposition 2.)

**Remark 6.** The polarized cotangent bundle can be interpreted in a different manner. Consider its principal open stratum

\[ \tilde{T}^{pr} X = \tilde{p}^{-1}(T^{pr} X) \simeq G_{\ast \mathcal{Q}} \mathcal{N}^{pr} \simeq G_{\ast \mathcal{M}} (\Sigma \cap \mathcal{N}). \]

By (5) the fibre product

\[ T^{pr} X \times a^{pr} / W_X \simeq G_{\ast \mathcal{M}} \Sigma \]

is birationally $G$-isomorphic to $\tilde{T}^{\ast} X$. Thus we generalize the definition of the polarized cotangent bundle in [Kn94 §3].

Now we prove our second main result generalizing Theorem 4.

**Theorem 4.** There exists a $G$-equivariant symplectic rational Galois covering $T^{\ast} Y \rightarrow T^{\ast} X$ with the Galois group $W_X$.

**Proof.** We argue as in [6,2]. It follows from (5) and (8) that $\tilde{p} : \tilde{T}^{\ast} X \rightarrow T^{\ast} X$ is a rational Galois covering with the Galois group $W_X$. It remains to prove that $\hat{q} : \tilde{T}^{\ast} X \rightarrow T^{\ast} Y$ is birational.

The morphism $\hat{q}$ maps $N^{\ast} H$ to $T^{\ast} Y$, $\forall H \in Y$. As $\hat{q}$ is equivariant, we may assume $H = Q_u x$, $x \in C$. The action $Q_u : N^{\ast} H$ is free and any orbit intersects $N^{\ast} H \simeq T^{\ast} x Z_1$ in exactly one point, i.e.,

\[ N^{\ast} H \simeq Q_u \times T^{\ast} x Z_1 \simeq Q_u \times (a + m \cap p^-) + T^{\ast} x C. \]

On the other hand,

\[ T^{\ast} H Y \simeq (g/(q_u + m \cap p^-))^* \oplus T^{\ast} x C \simeq a + m \cap p^- + q_u + T^{\ast} x C \]

by Proposition 4. It follows that the $Q_u$-action is free on an open subset $T^{\ast} H Y \simeq a^{pr} + m \cap p^- + q_u + T^{\ast} x C$ of $T^{\ast} H Y$, and the orbits are just parallel planes with the direction subspace $q_u$.

We already know that $\tilde{p}$ is a rational covering, hence $\hat{q}$ is a rational covering by Lemma 4. By definition, $\hat{q} : N^{\ast} H \rightarrow N^{\ast} H Z_x$ is a linear isomorphism (where $Z_x \subset Y$ is the set of horospheres containing $x$). Hence $N^{\ast} H Z_x$ intersects generic $Q_u$-orbits in exactly one point, i.e., it is transversal to $q_u$ and projects onto $a + m \cap p^- + T^{\ast} x C \simeq T^{\ast} x Z_1$ isomorphically.

More specifically, the composed map

\[ T^{\ast} x Z_1 \rightarrow N^{\ast} H \rightarrow N^{\ast} H Z_x \rightarrow T^{\ast} x Z_1 \]

is identity. Indeed, the projection of $\alpha' \in N^{\ast}_x X$ or $\alpha'' \in N^{\ast}_H Z_y$ to $T^{\ast} x Z_1$ means the restriction of $\alpha'$ to $T^{\ast} x Z_1 \subset T^{\ast} x X$, or of $\alpha''$ to $T^{\ast} x Z_1 \hookrightarrow T^{\ast} H Y$ (where $Z_1$ is regarded as a subvariety of $Y$), respectively. However, if we restrict the construction of [11,1] to $Z_1 \times Z_1 \subset X \times Y$, then the incidence variety $Z \subset X \times Y$ transforms to $\text{diag} Z_1$, so that $\alpha \in S N^{\ast}_{x,x}(\text{diag} Z_1)$, $\alpha' = \hat{p}(\alpha)$, $\alpha'' = \hat{q}(\alpha)$ correspond to one and the same covector in $T^{\ast} x Z_1$. 
Finally, we conclude that
\[
N^\text{pr} H \simeq Q_u \times (\mathfrak{a}^\text{pr} + \mathfrak{m} \cap \mathfrak{p}_0^-) + T_x^* C \xrightarrow{\hat{q}} T_H^\text{pr} Y
\]
is an isomorphism, which completes the proof. \qed

5. Invariant collective motion

5.1. The pullbacks of functions on $\mathfrak{g}^*$ along $\Phi$ are called collective functions on $T^* X$. Their skew gradients generate the tangent spaces to $G$-orbits at every point. Hence collective functions are in involution with $G$-invariant functions on $T^* X$, i.e., they serve as simultaneous integrals for all $G$-invariant Hamiltonian dynamical systems on $T^* X$.

Invariant collective functions, i.e., pullbacks of invariant functions on $\mathfrak{g}^*$, have the property that their skew gradients are both tangent and skew orthogonal to $G$-orbits, and even generate the kernel of the symplectic form on $\mathfrak{g}_\alpha$ for $\alpha \in T^* X$ in general position [GS84]. Since invariant collective functions are in involution, their skew gradients generate an Abelian flow of $G$-equivariant symplectomorphisms of $T^* X$ preserving $G$-orbits, which is called the invariant collective motion.

Restricted to any orbit $G \alpha \subset T^* X$, the invariant collective motion gives rise to a connected Abelian subgroup of $G$-automorphisms $A_\alpha \subseteq N(G_\alpha)/G_\alpha$. It is known [GS84] that $A_\alpha \simeq (G_{\Phi(\alpha)}/G_\alpha)^\circ$ for $\alpha$ in general position, cf. Remark 8 below.

A tempting problem is to integrate the invariant collective motion, i.e., to find an algebraic group of $G$-equivariant symplectomorphisms of $T^* X$ whose restriction to every orbit $G \alpha$ coincides with $A_\alpha$. However the problem has generally no solution in this formulation, as the following example shows.

**Example 2.** Let $X = G$ and $G$ act on $X$ by left translations. Then $T^* X \simeq G \times \mathfrak{g}^*$ is a trivial bundle with the $G$-action by left translations of the first factor, and the moment map $\Phi : G \times \mathfrak{g}^* \rightarrow \mathfrak{g}^*$ is just the coadjoint action map.

Invariant collective functions on $T^* X$ are of the form $F(g, \mu) = f(\mu)$, where $f$ is a $G$-invariant function on $\mathfrak{g}^*$. The differential $dF$ vanishes along $G \times \{\mu\}$ and equals $df$ on $\{e\} \times \mathfrak{g}^*$. It follows that the skew gradient of $F$ at $(e, \mu)$ is $(d_\mu f, 0)$. (Note that $d_\mu f \in \mathfrak{g}^{**} \simeq \mathfrak{g}$.) If $\mu$ is a regular point (i.e., $\dim G_\mu = \max$), then the $d_\mu f$ span the annihilator $\mathfrak{g}_\mu$ of $\mathfrak{g}_\mu$, whence $a_{(g,\mu)} \simeq \mathfrak{g}_\mu$ and $A_{(g,\mu)} \simeq (G_\mu)^\circ$.

On the principal stratum $T^\text{pr} X \simeq G \times \mathfrak{g}^\text{pr}$ (where $\mathfrak{g}^\text{pr}$ is the set of all regular semisimple elements in $\mathfrak{g} \simeq \mathfrak{g}^*$) all groups $A_{(g,\mu)}$ are maximal tori of $G$. Hence the hypothetical automorphism group integrating the invariant collective motion must be a torus. However for any regular $\mu \notin \mathfrak{g}^\text{pr}$ the group $A_{(g,\mu)}$ contains unipotent elements. Thus one might hope to integrate the invariant collective motion only on a proper open subset $T^\text{pr} X$. 
But even there it is not possible, because the family of tori \( A_{(g,\mu)} = G_\mu \) cannot be trivialized globally on \( G \times g^\pr \). Indeed, the family of \( G_\mu, \mu \in g^\pr \simeq G^*_{N(a)}a^\pr \), has the non-trivial monodromy group \( W = W(a) \), the Weyl group of \( g \). (Here \( A \) is a maximal torus of \( G \).) Only if we unfold \( T^\pr X \) by taking a Galois covering of \( g^\pr \) by \( G^* A a^\pr \simeq G/A \times a^\pr \) with the Galois group \( W = N(a)/A \), then the family of \( A_{(g,\mu)} \) lifts to a trivial family of tori, so that we can integrate the invariant collective motion on the covering space.

This example suggests the following reformulation of the problem: to integrate the invariant collective motion on a suitable étale covering of an open subset in \( T^* X \). This problem was solved by Knop \[Kn94\] for non-degenerate \( X \). Here we consider the general case.

Recall from (5) and (8) that the polarization map \( \hat{\rho} : \hat{T}^\pr X \to T^* X \) is a \( G \)-equivariant étale Galois covering of an open subset \( G^*_{N} \Sigma \simeq T^\pr X \). The symplectic structure on \( T^* X \) lifts to \( \hat{T}^\pr X \), so that \( G : \hat{T}^\pr X \) is a Hamiltonian action with the moment map \( \hat{\Phi} = \Phi \circ \hat{\rho} \). Hence the invariant collective motion on \( T^* X \) lifts to the invariant collective motion on \( \hat{T}^\pr X \).

We have a commutative diagram:

\[
\begin{array}{ccc}
\hat{T}^\pr X & \xrightarrow{\text{open}} & G^*_{M} \Sigma \\
& \downarrow & \downarrow \Phi \\
& G^*_{M} (a^\pr + \mathfrak{M}) & \xrightarrow{\text{étale}} & G(a^\pr + \mathfrak{M}) = (\text{Im } \Phi)^{\pr} \\
& \downarrow & \downarrow \\
& a^\pr & \xrightarrow{\text{étale}} & a^\pr/W(a).
\end{array}
\]

Note that the collective invariant functions on \( T^\pr X \) are exactly those pulled back from \( a^\pr/W(a) \). Consider the composed map

\[
\Pi : \hat{T}^\pr X \to a^\pr, \quad \Pi(g \ast \alpha) = \Phi(\alpha).s.
\]

**Theorem 5.** There is a Hamiltonian \( G \)-equivariant \( A \)-action on \( \hat{T}^\pr X \) with the moment map \( \Pi \) which integrates the invariant collective motion.

**Remark 7.** Instead of \( \hat{T}^\pr X \) we may consider the fibre product (9), which covers the whole \( T^\pr X \) and contains \( \hat{T}^\pr X \) as an open subset.

**Proof.** The torus \( Z(M) \) acts on \( \Sigma \) with kernel \( Z(M) \cap L_0 \). Indeed,

\[
\Sigma \cap \mathcal{N} \simeq \mathcal{N}_{Z_1} \simeq T^\pr (M/M \cap P_0) \times T^* C
\]

by (7), and the kernel of \( Z(M) : M/M \cap P_0 \) is exactly \( Z(M) \cap L_0 \). This yields an action of \( A = Z(M)/(Z(M) \cap L_0) \) on \( \Sigma \) which is free on \( \Sigma \cap \mathcal{N} \). The \( A \)-action commutes with \( M \) and immediately extends to
the whole $\hat{T}^{pr}X \simeq G \ast_M (\Sigma \cap N)$ by $G$-equivariance: $a(g \ast \alpha) = g \ast z\alpha$, where $z \in Z(M)$ represents $a \in A$.

Now we prove that $\Pi$ is the moment map. Obviously $\Pi$ is $A$-invariant. For any $\xi \in \mathfrak{a}$ consider two functions $\Pi^*\xi$ and $\hat{\Phi}^*\xi$ on $\hat{T}^{pr}X$. (We think of $\xi$ as of a linear function on $\mathfrak{a}^* \simeq \mathfrak{a}$, or on $\mathfrak{g}^* \simeq \mathfrak{g}$, respectively.)

\textbf{Claim.} $d_{\alpha} \Pi^* \xi = d_{\alpha} \hat{\Phi}^* \xi$, $\forall \xi \in \mathfrak{a}$, $\alpha \in \Sigma$

Indeed, $T_{\alpha} \hat{T}^{pr}X = \mathfrak{g}\alpha + T_{\alpha} \Sigma$. The differentials coincide on $T_{\alpha} \Sigma$, because the functions coincide on $\Sigma$:

$$\Pi^* \xi(\alpha) = (\xi, \Phi(\alpha)_a) = (\xi, \Phi(\alpha)) = \hat{\Phi}^* \xi(\alpha), \quad \forall \alpha \in \Sigma.$$

On the other hand, $d(\Pi^* \xi)$ vanishes on $\mathfrak{g}\alpha$ for $\Pi^* \xi$ is $G$-invariant, and $d \hat{\Phi}^* \xi(\mathfrak{g}\alpha) = (\xi, d\hat{\Phi}(\mathfrak{g}\alpha)) = (\xi, [\mathfrak{g}, \Phi(\alpha)]) = (\xi, [\mathfrak{m}_0, \Phi(\alpha)]_n + \mathfrak{q}_n + \mathfrak{q}_n^{-}) = 0$.

It follows from the claim that the skew gradient of $\Pi^* \xi$ at any $\alpha \in \Sigma$ coincides with that of $\hat{\Phi}^* \xi$, i.e., with $\xi\alpha$. By $G$-invariance, we conclude that the skew gradient of $\Pi^* \xi$ on $\hat{T}^{pr}X$ is the velocity field of $\xi$ with respect to the above $A$-action. Thus the $A$-action is symplectic and $\Pi$ is the moment map.

Finally, since the horizontal arrows in (10) are étale maps, the skew gradients of $\Pi^* \xi$ ($\xi \in \mathfrak{a}$) span the same subspace in $T_{\alpha} \hat{T}^{pr}X$ as the skew gradients of invariant collective functions. It follows that the $A$-action integrates the invariant collective motion.

Now we have three symplectic actions on $\hat{T}^{pr}X$: of $G$, of $A$ (integration of the invariant collective motion), and of $W_X$ (the Galois group). They patch together in the following picture.

\textbf{Corollary.} There is a Hamiltonian action $G \times (W_X \ltimes A): \hat{T}^{pr}X$ with the moment map $\hat{\Phi}, \Pi : \hat{T}^{pr}X \rightarrow \mathfrak{g}^* \oplus \mathfrak{a}^*$.

\textbf{Proof.} The group $N_X$ acts on $\Sigma$ and on $Z(M)$ by conjugation. Hence $N_X$ preserves the kernel $Z(M) \cap L_0$ of $Z(M) : \Sigma$, i.e., $W_X$ acts on $A = Z(M)/(Z(M) \cap L_0)$. The actions of $W_X$ and $A$ on $\hat{T}^{pr}X$ patch together into the $(W_X \ltimes A)$-action, as the following calculation shows: let $n \in N_X$, $z \in Z(M)$ represent $w \in W_X$, $a \in A$, respectively; then

$$w \cdot a \cdot (g \ast \alpha) = w(g \ast z\alpha) = gn^{-1} \ast nz\alpha =$$

$$gn^{-1} \ast (nz^{n^{-1}})n\alpha = (wa) \cdot (gn^{-1} \ast n\alpha) = (wa) \cdot w(g \ast \alpha).$$

It remains to note that $\Pi$ is $W_X$-equivariant.

5.2. The orbits of the invariant collective motion in $T^{pr}X$ (or $\hat{T}^{pr}X$) can be defined intrinsically as $Z(M(\alpha)) \cdot \alpha$, where $M(\alpha) = Z(\mu_\alpha)$ for $\mu = \Phi(\alpha)$ (or $\mu = \hat{\Phi}(\alpha)$). Note that the $A$-action on $\hat{T}^{pr}X$ is free, hence $Z(M(\alpha)) \alpha \simeq A$, $\forall \alpha \in \hat{T}^{pr}X$. 

The projections of these orbits to $X$ are called flats. Namely, a flat through $x \in X$ is $F_\alpha = Z(M(\alpha))x$, where $\alpha$ is any (polarized) covector over $x$ with $\Phi(\alpha) \in (\text{Im } \Phi)^{pr}$ (or $\widehat{\Phi}(\alpha) \in (\text{Im } \Phi)^{pr}$, respectively). Clearly, $F_\alpha$ does not depend on the polarization of $\alpha$.

It is easy to see that $F_\alpha \simeq A$, $\forall \alpha \in \widehat{T}^{pr} X$. Indeed, without loss of generality we may assume $\alpha \in \Sigma \cap N$, $x \in X_1$. Then $M(\alpha) = M$, and $Z(M)_x \subseteq L_0$ by Theorem 3. Hence $Z(M)_x = Z(M) = Z(M) \cap L_0$, which implies the claim.

Remark 8. If $\alpha \in \widehat{T}^{pr} X$ is in general position, namely, the $G$-orbit of $\mu = \widehat{\Phi}(\alpha)$ has maximal dimension in $\text{Im } \Phi$, then the orbit of the invariant collective motion coincides with $(G\mu)^\circ$, $F_\alpha = (G\mu)^\circ x$, cf. [GS84], [Kn94 §4]. Indeed, assuming $\alpha \in \Sigma \cap N$, $\mu \in a^{pr} + m \cap p_u^-$, we have $G_\mu = M_\mu n$ and $M_\mu n$ is open in $\mathfrak{M}$. Then $(M_\mu n)^\circ \subseteq M \cap P^-$ and $(M \cap P^-)_\mu n = Z(M) \cdot (M \cap P^-)_\mu n = Z(M) \cdot G_\alpha$.

Now let $X \hookrightarrow \overline{X}$ be a $G$-equivariant open embedding. The closures of flats in $X$ are (possibly non-normal) toric varieties. Rigidity of tori implies that the closures of generic flats are isomorphic. It is an interesting problem to describe the closure of a generic flat. This has important applications in the equivariant embedding theory, see [Kn94]. The problem was solved by Knop for non-degenerate $X$ [Kn94], and it would be desirable to extend his solution to arbitrary $X$.

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