Benchmarking Problems for Robust Discrete Optimization

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Abstract

Robust discrete optimization is a highly active field of research where a plenitude of combinations between decision criteria, uncertainty sets and underlying nominal problems are considered. Usually, a robust problem becomes harder to solve than its nominal counterpart, even if it remains in the same complexity class. For this reason, specialized solution algorithms have been developed. To further drive the development of stronger solution algorithms and to facilitate the comparison between methods, a set of benchmark instances is necessary but so far missing. In this paper we propose a further step towards this goal by proposing several instance generation procedures for combinations of min-max, min-max regret, two-stage and recoverable robustness with interval, discrete or budgeted uncertainty sets. Besides sampling methods that go beyond the simple uniform sampling method that is the de-facto standard to produce instances, also optimization models to construct hard instances are considered. Using a selection problem for the nominal ground problem, we are able to generate instances that are several orders of magnitudes harder to solve than uniformly sampled instances when solving them with a general mixed-integer programming solver. All instances and generator codes are made available online.

Keywords: robust optimization; benchmarking; instance generator; combinatorial optimization

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1. Motivation

In real-world decision making, problem data and the consequences of our actions can rarely be predicted with full accuracy. Instead, we must make use of imprecise measurements, forecasts and estimates to be capable of acting in an uncertain environment. Several methods have been developed to treat uncertainty in decision making, such as stochastic programming [BL11] or fuzzy optimization [KY95]. This paper focuses on the robust optimization paradigm.

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for discrete optimization problems [KY13, BTEGN09, KZ16]. While this paradigm can be realized in many different variants, the basic unifying idea is to define an uncertainty set of possible parameter outcomes, and to protect against the worst case over this set.

Robust optimization problems for discrete optimization problems have been studied for decades under various angles. While many publications focus on the complexity side of these problems (e.g., determining which variants can be solved in polynomial time, and which become hard), an active research community has emerged that is concerned with solving such problem in the most efficient manner [GS16]. Frequently, such algorithms are developed for a very specific type of problem. The diversity of robust optimization variants in combination with the trove of classic or real-world combinatorial optimization problems with which they can be combined have created a partitioned rather than unified landscape of methods. This has become visible by a lack of standardized benchmarks, problem generators, or algorithm libraries. As explained in the algorithm engineering paradigm [San09, MHS01], having a strong foundation of shared instances with easy access is instrumental in the development of better algorithms. Currently, many papers need to reimplement generator methods. At the same time, these generators usually show little sophistication, but instead sample parameter values uniformly from an interval.

The main goal of this paper is to provide a starting point in this direction, by establishing a first collection of instances that shall be further extended in the future. To this end, there are several obstacles that need to be overcome. Firstly, the large variety of types of robust optimization methods in connection with the possibilities of underlying nominal combinatorial problems means that we cannot consider all combinations simultaneously. As a response, one could fix on type of robust optimization approach (e.g., one-stage min-max with discrete uncertainty) and vary the underlying nominal problem. What we present in this paper instead is to focus on one fixed nominal problem and to vary possible ways to create a robust counterpart. This underlying problem is the so-called selection problem (see, e.g., [DW13]), where the only constraint states that a subset of exactly \( p \) items have to be chosen from \( n \) available items. The advantage of this problem is that it provides as little structure on its own as possible, thus disentangling the difficulty that may come from a complicated structure of feasible solutions (e.g., the structure of a graph for a shortest path problem) with the difficulty that arises from the robust objective function. Furthermore, it allows us to focus on uncertainty in the objective function.

Secondly, a benchmark set should contain challenging problems that drive the development of new and improved solution methods. But to determine the computational challenge of an instance, we need to know which algorithm is used to solve it. This difficulty is avoided by using what we believe to be the most common universal solution approach, that is, by formulating a compact mixed-integer program (MIP) and solving this through a general-purpose MIP solver.

Our contributions are as follows. We consider min-max, min-max regret, two-stage and recoverable robust selection problems in combination with discrete, interval and budgeted uncertainty sets. For each combination we consider different instance generators. We always include the simple approach of choosing parameter values uniformly from an interval as the baseline heuristic. Additionally, we consider alternative ways to sample problem instances, and compare the resulting computation times. We show that this way it is possible to create instances that are several orders of magnitude harder to solve. Furthermore, we build upon the HIRO ("hard instances for robust optimization") approach from [GM20] to formulate the generation of hard instances as optimization problems. While such an approach is clearly
more time-intensive to use than a generator based on random sampling, we show that for most
problem variants, even harder instances can be found. All instances along with generator code
are provided in an easy-to-use format online, in an attempt to stimulate further research for
better solution methods.

The remainder of this paper is structured as follows. In Section 2 we formally introduce
all types of robust optimization problems and uncertainty sets that we use. Additionally, we
recall the high-level idea of the HIRO approach from GM20. We then present methods for
instance generation along with computational results for various problem variants: Min-max
problems are presented in Section 3, min-max regret problems in Section 4, two-stage problems
in Section 5 and recoverable problems in Section 6. We conclude this work by discussing the
structure and format of our benchmark library in Section 7 before summarizing our findings
and pointing out further research questions in Section 8. In Appendix A we give an overview
to all sampling methods that are introduced in this paper, while Appendix B shows detailed
generation times.

2. Problem Definitions

We write vectors in bold and use the notation [n] to denote sets \{1, \ldots, n\}. We focus on
combinatorial problems with uncertain objective function. That is, given a set of feasible
solutions \(X\subseteq \{0, 1\}^n\), the so-called nominal problem is to solve

\[
\min_{x \in X} c^t x
\]

for a known cost vector \(c \in \mathbb{R}_+^n\). In the selection problem that we use as a baseline nominal
problem, we have \(X = \{x \in \{0, 1\}^n : \sum_{i \in [n]} x_i = p\}\). It is easy to check that the optimal
solution of the selection problem is a set of \(p\) items with the smallest costs, which can be
found in \(O(n)\). Due to their simplicity, selection problems have been playing a key role in the
complexity analysis for robust combinatorial optimization, see, e.g., the results presented in
Bis11, DK12, DW13, KKZ15, KZ17, CGKZ18.

If costs are uncertain, the robust optimization approach is to define an uncertainty set
\(U \subseteq \mathbb{R}_+^n\) containing all possible cost vectors that we wish to protect against. A range of such
concepts have been defined; the most frequently used are listed here (for a more detailed
overview, see, e.g., KZ16).

In the min-max setting, we need to find a single solution \(x \in X\) that performs optimally
under the worst-case decision criterion, that is, we need to solve

\[
\min_{x \in X} \max_{c \in U} c^t x
\]

(Min-Max)

In the min-max regret criterion, we compare against the best possible outcome under each
scenario. Let \(\text{opt}(c) = \min_{x \in X} c^t x\) be the objective value if the cost scenario \(c\) is known. The
problem is then to solve

\[
\min_{x \in X} \max_{c \in U} (c^t x - \text{opt}(c))
\]

(Min-Max Regret)

In case that the decision maker can respond to a scenario after it has been revealed, other
approaches can be applied, which slightly differ in philosophy. For two-stage robust problems,
the decision maker only fixes part of the solution ahead of the uncertainty. After the scenario
has been revealed, he can complete the partial solution to a full solution. In recoverable robustness (which is essentially also a two-stage problem), the decision maker first fixes a complete solution, but is allowed to modify this solution after the scenario is known. More formally, a common definition is

$$\min_{x \in X'} \max_{c \in U} \min_{y \in X(x)} C'x + c'y$$

for two-stage robustness with \( X(x) = \{ y \in \{0,1\}^n : x + y \in X \} \) and \( X' = \{ x \in \{0,1\}^n : X(x) \neq \emptyset \} \), and

$$\min_{x \in X} \min_{y \in X : d(x,y) \leq \Delta} C'x + c'y$$

for recoverable robustness, where \( d : X \times X \rightarrow \mathbb{R} \) denotes some function measuring dissimilarity between solutions, and \( \Delta \) is the maximum recovery distance.

Many more problem variants have been considered as well, which are not covered in this paper; these include for example the ordered weighted averaging criterion \([YK12, CGKZ20]\) or two-stage regret \([JWZG13, GKZ20]\).

In all decision problems introduced so far, we require a set \( U \) of possible scenarios. The structure of this set is a deciding factor in the theoretical and practical hardness of instances. Amongst the most common sets for combinatorial problems are discrete uncertainty sets

\[ U_D = \{ c^1, \ldots, c^N \} \]

where \( N \) scenarios are simply given as an explicit list; interval sets

\[ U_I = \{ c \in \mathbb{R}_+^n : c_i \in [c_i, c_i + d_i] \} \]

where for each item \( i \in [n] \) a lower bound \( c_i \) and a deviation \( d_i \) are given; and budgeted uncertainty sets, which are essentially interval uncertainty sets where not all coefficients may deviate to the upper bound simultaneously, i.e.,

\[ U^p_B = \{ c \in \mathbb{R}_+^n : c_i = c_i + d_i \delta, \sum_{i \in [n]} \delta_i \leq \Gamma, \delta \in [0,1]^n \} \]

for some parameter \( \Gamma \in [0,n] \). This case is also known as continuous budgeted uncertainty, as opposed to discrete budgeted uncertainty, where each \( \delta_i \) is binary. Note that this does not impact min-max problems, but may make a difference in two-stage settings. Sometimes, a variant of this set is considered, where the budget \( \Gamma \) affects the sum of deviations, that is,

\[ U^p_{\text{Bvar}} = \{ c \in \mathbb{R}_+^n : c_i = \xi_i + d_i \delta_i, \sum_{i \in [n]} \delta_i \leq \Gamma, \delta_i \in [0,d_i] \forall i \in [n] \} \]

In the following, we may also write \( c_i = \xi_i + d_i \) to denote the upper bound on the costs of item \( i \) in interval or budgeted uncertainty sets.

Many more uncertainty sets have been considered, including ellipsoidal uncertainty sets \([BTN98, CG17]\), data-driven polyhedral sets based on statistical testing \([BGK18]\) or machine learning \([SHY17]\), or even combinations of multiple sets simultaneously \([BSS19, DGR20]\).

In Table 1 we present a non-exhaustive list of papers that have performed experimental studies within this framework for different types of robust optimization problem and uncertainty set. We note that in particular the combinations of min-max robust optimization with
Table 1: Experimental studies from the literature, combining different uncertainty sets with types of robust optimization problems.

|              | Interval Uncertainty | Discrete Uncertainty | Budgeted Uncertainty |
|--------------|----------------------|----------------------|----------------------|
| Min-max      | —                    | [GH19], [GM20]       | [HRS18], [MPS13]    |
|              |                      | [SLTW12], [TYK08]    | [BGKK19]             |
| Min-max regret| BMMG07, [MG04]       | [KMZ12], [FIMY15]    | [CG18] —            |
| Two-stage    | —                    | —                    | [CGKZ18], [ZZ13]    |
| Recoverable  | —                    | BKK11                | CGKZ18, [CG16],     |
|              |                      | —                    | BGKK19              |

discrete uncertainty and min-max regret robust optimization with interval uncertainty have been frequently considered.

Note that min-max robust optimization problems with interval uncertainty are equivalent to nominal problems, as

\[
\min_{\mathbf{x} \in \mathcal{X}} \max_{c \in \mathcal{U}} c^T \mathbf{x} = \min_{\mathbf{x} \in \mathcal{X}} \mathbf{c}^T \mathbf{x}.
\]

For this reason, min-max problems with interval uncertainty are excluded from the remainder of this study. Similarly, two-stage and recoverable robust problems with interval uncertainty are equivalent to problems with only a single scenario and are therefore not considered here, though they find some interest in the recent literature (see, e.g., [KZ17, FHLW20, BG21]). Furthermore, while some papers consider two-stage problems with discrete scenarios, they are most commonly not given by an explicit list, but described implicitly, see, for example, [HDJPR21, Sub21].

We conclude this section by briefly recalling the high-level ideas of the HIRO approach from [GM20] to generate uncertainty sets. The aim is to avoid optimal robust solutions that clearly outperform alternative solutions. To this end, a given uncertainty set is modified, such that the value of the optimal robust solution is increased. This creates a shorter left-hand tail in the distribution of robust objective values. Computational experience shows that such approaches can increase the computational difficulty of instances by orders of magnitude for min-max problems with discrete uncertainty sets. In the following sections, we extend this methodology to other types of robust optimization problem and other types of uncertainty sets.

More formally, given an uncertainty set \(\tilde{\mathcal{U}}\) (e.g., produced by a random sampling method), the HIRO approach is to solve

\[
\max_{\tilde{\mathcal{U}} \in N(\hat{\mathcal{U}})} \min_{\mathbf{x} \in \mathcal{X}} \max_{c \in \tilde{\mathcal{U}}} c^T \mathbf{x}
\]

where \(N(\hat{\mathcal{U}})\) denotes a neighborhood of set \(\hat{\mathcal{U}}\). In the original paper, the authors considered \(\tilde{\mathcal{U}} = \{\mathbf{c}^1, \ldots, \mathbf{c}^N\}\) and \(N(\hat{\mathcal{U}}) = \mathcal{U}(\mathbf{c}^1) \times \mathcal{U}(\mathbf{c}^2) \times \ldots \times \mathcal{U}(\mathbf{c}^N)\) with

\[
\mathcal{U}(\mathbf{c}^i) = \left\{ \mathbf{c} \in \mathbb{R}^n_+ : c_i \in [\max\{\tilde{c}_i^1 - b, 0\}, \min\{\tilde{c}_i^1 + b, C\}] \forall i \in [n], \sum_{i \in [n]} c_i \leq \sum_{i \in [n]} \tilde{c}_i^1 \right\}
\]
where \( b \geq 0 \) is a constant (the budget for change) and \( C \) is a desired global upper bound on costs, i.e., each scenario can be modified by changing costs by up to \( b \) units and not increasing the total sum of costs in this scenario. In this paper, we always use \( C = 100 \). Due to the added maximization layer, HIRO is at least as hard as the robust optimization problem that is being changed. In [GM20] the authors show that the HIRO problem is \( \Sigma_p^2 \)-complete. To solve it, an iterative solution procedure is possible, where we choose \( \mathcal{U} \in N(\tilde{\mathcal{U}}) \) in a master problem, and determine a robust solution \( \mathbf{x} \) in a subproblem.

3. Min-Max Problems

We now consider robust optimization problems with the min-max criterion. We discuss how to generate instances with discrete or with budgeted uncertainty and compare solution times when solving these problems with CPLEX. For all experiments we use an Intel(R) Xeon(R) Gold 6154 CPU @ 3.00GHz computer with 754 GB RAM. In addition, only one core has been used for each instance generation or evaluation. Recall that all sampling methods are summarized in Appendix A and generation times are presented in Appendix B.

3.1. Discrete Uncertainty

3.1.1. Problem Statement

We assume that the uncertainty set is modeled as a list of \( N \) scenarios with \( \mathcal{U}_D = \{c^1, \ldots, c^N\} \).

Using the epigraph reformulation, the corresponding mixed-integer program in case of the selection problem is as follows.

\[
\begin{align*}
\min & \quad t \\
\text{s.t.} & \quad t \geq \sum_{i \in [n]} c^j_i x_i \quad \forall j \in [N] \\
& \quad \sum_{i \in [n]} x_i = p \\
& \quad x_i \in \{0, 1\} \quad \forall i \in [n]
\end{align*}
\]

This problem is known to be weakly NP-hard even for \( N = 2 \), and strongly NP-hard when \( N \) is unbounded (see [Ave01, KZ09]).

3.1.2. Sampling

To generate uncertainty sets \( \mathcal{U}_D \), a basic (and often-used) approach is to choose values randomly and uniformly iid over some interval. We refer to this method as \( \text{MM-D-U} \). In this case, we choose each value \( c^j_i \) from the set \( \{1, \ldots, 100\} \). We consider two additional sampling methods. In the first method \( \text{MM-D-1} \), we choose \( c^j_i \) from the set \( \{1, \ldots, 10\} \cup \{91, \ldots, 100\} \), i.e., values are either small or large. In the second method \( \text{MM-D-2} \), we create a symmetry in the items. The first \( \lfloor n/2 \rfloor \) items have costs generated as with \( \text{MM-D-U} \), while the last \( \lceil n/2 \rceil \) items have costs \( c^j_i = 100 - c^j_{i-\lfloor n/2 \rfloor} \).
3.1.3. HIRO

We use the HIRO approach to increase the hardness of instances for robust discrete optimization problems first introduced in [GM20], where details of this method can be found. To solve the resulting max-min-max problem, an iterative solution method is used. For a current set of \( K \) candidate solutions \( x^k \), a scenario set is constructed by solving the following optimization problem:

\[
\begin{align*}
& \text{max } t \\
& \text{subject to } t \leq \sum_{j \in [N]} \sum_{i \in [n]} d_{ijk} x^k_i \\
& \quad \sum_{j \in [N]} \lambda^k_j = 1 \\
& \quad d_{ijk} \leq c^i_j \\
& \quad d_{ijk} \leq \tau^i_j \lambda^k_j \\
& \quad c^i \in \mathcal{U}(\tilde{c}^i) \\
& \quad \lambda^k_j \in \{0, 1\}
\end{align*}
\]

where sets \( \mathcal{U}(\tilde{c}^i) \) allow a deviation from the input scenario through a budget \( b \) as defined in Section 2. Here, \( \lambda \) variables represent the assignment of worst-case scenario to solutions. Having constructed an uncertainty set, we solve the corresponding robust problem to find the next candidate solution \( x \). This process is repeated until convergence or a time limit is reached. We refer to instances of this type as \( \text{MM-D-XH-B} \), where \( X \) refers to the sampling type of the input uncertainty set, and \( B \) to the budget.

3.1.4. Experimental Setup

We perform four experiments to test different parameter combinations. In Exp1, we consider the pairs \((20, 11), (25, 13), (30, 15), (35, 17), (40, 21)\) for \((n, p)\) and set \( N = n \) in all cases. We chose \( p \) to be odd as preliminary experiments showed higher computation times for \( \text{MM-D-2} \) in such cases. In Exp2, we fix \( n = N = 30 \) but change \( p \in \{5, 11, 15, 21, 25\} \). In Exp3, we fix \( n = 30 \) and \( p = 15 \) but change \( N \in \{5, 10, 15, 20, 25, 30, 35, 40\} \). Finally, Exp4 considers large-scale problems with \( N \in \{100, 500, 1000, 5000, 10000\} \) also using \( n = 30 \) and \( p = 15 \).

In each experiment, we generate 50 instances using \( \text{MM-D-U}, \text{MM-D-1} \) and \( \text{MM-D-2} \). Using each of these sets as input, we also apply \( \text{MM-D-XH-B} \) with budgets \( b \in \{1, 2, 5\} \) (recall that parameter \( b \) controls by how much coefficients of an input set may be changed), except for Exp4. For instance generation, we apply a 600-second time limit. Thus, the number of instances in the four experiments are \( 5 \times 50 \times 3 \times 4 = 3000 \), \( 5 \times 50 \times 3 \times 4 = 3000 \), \( 8 \times 50 \times 3 \times 4 = 4800 \), and \( 5 \times 50 \times 3 = 750 \), respectively.

All instances are solved using CPLEX-20-10 with a 600-second time limit.

3.1.5. Experimental Results

Figures 1–4 summarize the results of the four experiments. Each line connects average solution times for the parameter that is changed in the respective experiment.
Figure 1 demonstrates that while MM-D-U instances can be solved in less than one second, MM-D-2 instances can hit the time limit of 600 seconds when \( n = N = 40, p = 21 \). Also, it is observed that HIRO is effective for both MM-D-U and MM-D-1 with all budgets \( b \). As for MM-D-2, the HIRO approach can slightly increase the hardness of instances with \( b = 1 \), while it decreases the evaluation time of the instances with \( b = 2 \) and \( b = 5 \).

Figure 2 shows that instances generated using sampling methods for the cases \( p = 11 \) and \( p = 15 \) are harder to solve in comparison to the other values of \( p \). HIRO shows a similar performance as in Exp1.

Figure 3 implies that for the instances with lower number of scenarios HIRO also works for MM-D-2. However, when the number of scenarios grows, HIRO decreases the difficulty of the instances, as before. Problem hardness increases with \( N \), as can be expected.

Finally, Figure 4 illustrates that for large-scale problems (in terms of the number of scenarios), the gap between hardness of instances generated using MM-D-U, MM-D-1 and MM-D-2 decreases. In some cases (e.g., \( N = 1000 \)), performance of MM-D-1 even exceeds performance of MM-D-2. In addition, all instances generated with the three sampling methods hit the time limit for \( N = 5000 \) and \( N = 10000 \).

Figure 1: Min-max problems with discrete uncertainty for varying values of \((n, p)\), Exp1.

Figure 2: Min-max problems with discrete uncertainty for varying values of \( p \), Exp2.
Figure 3: Min-max problems with discrete uncertainty for varying values of $N$, Exp3.

Figure 4: Min-max problems with discrete uncertainty for varying values of $N$, Exp4.
3.2. Budgeted Uncertainty

3.2.1. Problem Statement

By dualizing the inner worst-case problem, a compact MIP of the robust min-max selection problem under $U_B$ can be formulated as follows:

$$\min_{x \in \mathcal{X}} \max_{c \in U_B^t} \mathbf{c}^T x = \min_{x \in \mathcal{X}} \sum_{i \in [n]} c_i x_i + \Gamma \pi + \sum_{i \in [n]} \delta_i$$

s.t. $\sum_{i \in [n]} x_i = p$

$\pi + \delta_i \geq d_i x_i \quad \forall i \in [n]$  
$\pi \geq 0$  
$\delta_i \geq 0 \quad \forall i \in [n]$  
$x_i \in \{0, 1\} \quad \forall i \in [n]$  

In [BS03] it is noted that candidate values of variable $\pi$ can be enumerated. As the resulting problems with fixed value of $\pi$ are equivalent to nominal problems, it is possible to solve this problem in polynomial time. More precisely, enumerating $\pi$ from the set $P = \{d_i : i \in [n]\} \cup \{0\} =: \{\pi^1, \ldots, \pi^K\}$ with $K = |P|$ and substituting $\delta_i$, we can rewrite the problem as follows.

$$\min_{x \in \mathcal{X}} \max_{c \in U_B^t} \mathbf{c}^T x = \min_{x \in \mathcal{X}} \min_{k \in [K]} \left( \sum_{i \in [n]} c_i x_i + \Gamma \pi^k + \sum_{i \in [n]} [d_i - \pi^k]_+ x_i \right)$$

3.2.2. Sampling

As a baseline sampling method, we use $\text{MM-B-U}$, where values $c_i$ and $d_i$ are chosen uniformly iid from $\{1, \ldots, 100\}$. Additionally, we consider an approach where each $c_i$ is chosen uniformly iid from $\{1, \ldots, 100\}$, but we set $d_i = 100 - c_i$, i.e., all items have the same upper bound $c_i + d_i = 100$. We refer to this approach as $\text{MM-B-1}$. Finally, we also choose $c_i$ uniformly iid from $\{1, \ldots, 10\}$ and $d_i$ uniformly from $\{100 - c_i - 1, \ldots, 100\}$. This means that items with smaller nominal costs $c_i$ are more likely to have higher deviations $d_i$. This approach is denoted as $\text{MM-B-2}$.

3.2.3. HIRO

We follow the HIRO idea to construct an optimization model to modify a given uncertainty set. For a fixed value $\Gamma$, note that the uncertainty is defined through the lower bounds $\underline{c}$ and deviations $\underline{d}$. Let $\bar{c}$ and $\bar{d}$ be given lower and upper bounds, respectively. A modification might affect just one of these vectors, or both. We consider all three possibilities. Whenever $\bar{c}$ or $\underline{d}$ are changed, we require that the sum of lower bounds (or deviations) must not increase, and that the lower bounds (or deviations) are changed by up to $b$ units in each item.

1. Modification of lower bounds.

As the set $P = \{d_i : i \in [n]\} \cup \{0\}$ is only affected by the choice of $\underline{d}$ we can dualize the inner nominal problems to obtain a compact linear programming formulation of HIRO:

$$\max_{\underline{d}} \min_{x \in \mathcal{X}} \max_{c \in U_B^t} \mathbf{c}^T x = \max t$$
2. Modification of deviations.

As $P$ depends on $d$, we enumerate $\pi$ through the index $k$ of the item where $\pi = d_k$. The resulting HIRO problem is then given as:

$$\max_{\mathbf{d}} \min_{\mathbf{x} \in X} \max_{\mathbf{c} \in \mathcal{U}(\tilde{\mathbf{c}})} t = \max t$$

s.t. $t \leq \Gamma d_k + p \cdot \alpha^k - \sum_{i \in [n]} \beta^k_i$ \hspace{1cm} $\forall k \in [K]$ 

$$\alpha^k - \beta^k_i \leq c_i + [d_i - \pi^k]_+$ \hspace{1cm} $\forall i \in [n], k \in [K]$ 

$c \in \mathcal{U}(\tilde{c})$ 

$$\alpha^k \geq 0 \hspace{2cm} \forall k \in [K]$$

$$\beta^k_i \geq 0 \hspace{2cm} \forall i \in [n], k \in [K]$$

To avoid the nonlinearity in $[d_i - d_k]_+$, we assume without loss of generality that all $\tilde{d}_i$ values are sorted in non-decreasing order. In the HIRO model, we require that this order must be preserved. That is, we can simplify

$$[d_i - d_k]_+ = \begin{cases} d_i - d_k & : k < i \\ 0 & : i \leq k \end{cases}$$

and add the constraint

$$d_1 \leq \cdots \leq d_n.$$ 

The modification of both lower bounds and deviations is possible through a model that combines both models above.

3.2.4. Experimental Setup

We use sampling methods MM-B-U, MM-B-1 and MM-B-2 to generate instances with $n = 40$, $p = 20$ and $\Gamma \in \{5, 10, 15, 20\}$. For all combinations of method and parameter we generate 50 instances. Then, we use generated instances as input to each of the HIRO formulations (solving with respect to $\mathbf{c}, \mathbf{d}$, or both) and solve them using CPLEX-20-10 with budgets $b \in \{1, 2, 5, 10, 20\}$ and a 600-second time limit. All problem instances are then solved using a 600-second time limit as well.
3.2.5. Experimental Results

In Figures 5-7, each line connects the average solution times for increasing values of $\Gamma$. It can be seen that while $\text{MM-B-U}$ instances are relatively easy to solve ($< 0.1$ seconds on average), finding the solution of instances generated by $\text{MM-B-1}$ and $\text{MM-B-2}$ are considerably more time consuming ($> 1000$ seconds on average). In particular, when $\Gamma$ increases the instances hit the given time limit.

All three HIRO methods are effective over both $\text{MM-B-U}$ and $\text{MM-B-1}$ instances. The figures represent better results when $b$ increases. The best case for $\text{MM-B-U}$ happens when modification is done for both $c$ and $d$ with $b = \Gamma = 20$. However, HIRO gives different results when applied to $\text{MM-B-2}$ instances. Although modifying $c$ always leads to the better results by increasing $b$, modification over $d$ or both $c$ and $d$ for these instances makes them generally easier to solve, with few exceptions.

Figure 5: Min-max problems with budgeted uncertainty for varying $\Gamma$. HIRO modifies $c$.

Figure 6: Min-max problems with budgeted uncertainty for varying $\Gamma$. HIRO modifies $d$. 
Figure 7: Min-max problems with budgeted uncertainty for varying $\Gamma$. HIRO modifies $c$ and $d$.

4. Min-Max Regret Problems

4.1. Interval Uncertainty

4.1.1. Problem Statement

To calculate the regret $\max_{c \in U}(c'x - \text{opt}(c))$, we can find a worst-case scenario (i.e., a maximizer of this problem) by setting

$$c_i^{w^c}(x) = \begin{cases} c_i + d_i & \text{if } x_i = 1 \\ c_i & \text{if } x_i = 0 \end{cases}$$

see, e.g. [ABV09]. Accordingly, the min-max regret problem can be reformulated:

$$\min_{x \in X} \max_{c \in U} (c'x - \text{opt}(c)) = \min_{x \in X} \max_{y \in X} \left( (c_i + d_i)x_i - \sum_{i \in [n]} (c_i + d_i)x_i) y_i \right)$$

If $X$ can be represented by a polyhedron with compact description, we can apply duality in the inner maximization problem to find a compact MIP formulation. In case of the selection problem, this leads to the following regret problem.

$$\min \sum_{i \in [n]} (c_i + d_i)x_i - p \pi + \sum_{i \in [n]} \rho_i$$

s.t. $\pi - \rho_i \leq d_i x_i + c_i$ \hspace{1cm} $\forall i \in [n]$

$$\sum_{i \in [n]} x_i = p$$

$$x_i \in \{0, 1\} \hspace{1cm} \forall i \in [n]$$

$$\pi \geq 0$$

$$\rho_i \geq 0 \hspace{1cm} \forall i \in [n]$$

It is possible to solve this problem in polynomial time, see [Ave01].
4.1.2. Sampling

We use a uniform sampling method (MMR-I-U) as a baseline, where $\zeta_i$ and $d_i$ are chosen iid from $\{1, \ldots, 100\}$. Two additional sampling methods are considered. In MMR-I-1, we choose $\zeta_i$ uniformly from $\{1, \ldots, 10\}$ and $d_i$ from $\{91, \ldots, 100\}$ with probability 0.5. Otherwise, we choose $\zeta_i \in \{91, \ldots, 100\}$ and $d_i \in \{1, \ldots, 10\}$, i.e., items either have low costs $\zeta_i$ but high deviations $d_i$ or vice versa. In MMR-I-2, we follow a similar approach, but sample items that either have both small $\zeta_i$ and $d_i$, or both large $\zeta_i$ and $d_i$.

4.1.3. HIRO

In the MIP formulation of the selection problem, note that there always exists an optimal solution where $\pi \in P = \{0\} \cup \{\zeta_i: i \in [n]\} \cup \{d_i: i \in [n]\}$. Let us write $P = \{\pi^1, \ldots, \pi^K\}$ with $K = |P|$. For fixed choice $\pi^k \in P$, the problem becomes

$$
\min \sum_{i \in [n]} (\zeta_i + d_i) x_i - p \pi^k + \sum_{i \in [n]} (|\pi^k - \zeta_i - d_i|_+ + [\pi^k - \zeta_i]_+) x_i + \sum_{i \in [n]} [\pi^k - \zeta_i]_+ \\
\text{s.t.} \sum_{i \in [n]} x_i \geq p \\
x_i \in \{0, 1\} \quad \forall i \in [n]
$$

The dual of this problem can be written as

$$
\max \quad p \alpha^k - \sum_{i \in [n]} \beta_i^k - p \pi^k + \sum_{i \in [n]} [\pi^k - \zeta_i]_+ \\
\text{s.t.} \quad \alpha^k - \beta_i^k \leq \zeta_i + d_i + [\pi^k - \zeta_i - d_i]_+ - [\pi^k - \zeta_i]_+ \quad \forall i \in [n] \\
\alpha^k \geq 0 \\
\beta_i^k \geq 0 \quad \forall i \in [n]
$$

To formulate the HIRO approach, we introduce binary variables to linearize the nonlinearities of this model. Let $z_i^k = 1$ if $\pi_i^k - \zeta_i \geq 0$ and $q_i^k = 1$ if $\pi_i^k - \zeta_i - d_i \geq 0$. The linearized model is then as follows.

$$
\max t \\
\text{s.t.} \quad t \leq \sum_{i \in [n]} (z_i^k \pi^k - q_i^k) - p \pi^k + \sum_{i \in [n]} \beta_i^k - \sum_{i \in [n]} \beta_i^k \\
\alpha^k - \beta_i^k \leq \zeta_i + d_i + \pi^k q_i^k - q_i^k - q_i^k - q_i^k + q_i^k \\
s_i^k \geq \pi^k - \zeta_i \quad \forall i \in [n], k \in [K] \\
z_i^k \geq \zeta_i - (\zeta_i + b)(1 - z_i^k) \quad \forall i \in [n], k \in [K] \\
q_i^k \geq \zeta_i - (\zeta_i + b)(1 - q_i^k) \quad \forall i \in [n], k \in [K] \\
q_i^k \geq \zeta_i - (d_i + b)(1 - q_i^k) \quad \forall i \in [n], k \in [K] \\
\xi \in \mathcal{U}(\xi) \\
d \in \mathcal{U}(\tilde{d}) \\
z_i^k, q_i^k \in \{0, 1\} \quad \forall i \in [n], k \in [K]
$$
\[\alpha^k \geq 0 \quad \forall k \in [K]\]
\[\sigma^k_i, \beta^k_i, z^k_i, q^k_i, \hat{q}^k_i \geq 0 \quad \forall i \in [n], k \in [K]\]

4.1.4. Experimental Setup

We generate instances with \( n = 100 \) and vary \( p \in \{10, 20, \ldots, 90\} \). For each sampling method, 50 instances are generated. These instances are also used as an input to the HIRO method with budgets \( b \in \{1, 2, 5\} \) and 600-second generation time limit. The total number of instances generated is therefore equal to \( 3 \times 9 \times 50 \times 4 = 5400 \). All instances are solved using CPLEX-20-10 with a time limit of 600 seconds.

4.1.5. Experimental Results

Figure 8 summarizes the results of this experiment. Each line connects the average solution times for increasing values of \( p \). For small values of \( p \), MMR-I-1 generates instances that are considerably harder to solve than those generated by MMR-I-U (time limit versus < 0.1 seconds), while for larger values of \( p \), method MMR-I-2 produces harder instances. The HIRO approach can successfully increase the hardness of instances. In particular, HIRO increases the solution time of MMR-I-U more than 100 times for \( p \in \{30, 40, \ldots, 80\} \) when \( b = 5 \). In addition, MMR-I-2H-5 generates instances that are hard for both small and large values of \( p \), thus resulting in instances that combine the best features of the other methods.

4.2. Discrete Uncertainty

4.2.1. Problem Statement

We consider using the min-max regret criterion for the selection problem with discrete uncertainty sets \( U_D = \{e^j, \ldots, e^N\} \). By calculating the optimal objective values \( \text{opt}(e^j) \) for each scenario \( j \in [N] \), the corresponding MIP formulation is as follows.

\[
\min t
\]
\[ t \geq \sum_{i \in [n]} c^j_i x_i - \text{opt}(c^j) \quad \forall j \in [N] \]
\[ \sum_{i \in [n]} x_i = p \]
\[ x_i \in \{0, 1\} \quad \forall i \in [n] \]

Similar to the min-max case, this problem is weakly NP-hard even for \( N = 2 \), and strongly NP-hard when \( N \) is unbounded (see [KZ16]).

### 4.2.2. Sampling

We consider the same sampling approaches as in Section 3.1.2. These are: uniform sampling in \( \{1, \ldots, 100\} \) (MMR-D-U), sampling in \( \{1, \ldots, 10\} \cup \{91, \ldots, 100\} \) (MMR-D-1) and sampling with item symmetry (MMR-D-2).

### 4.2.3. HIRO

We modify the model introduced in [GM20] for min-max problems with discrete uncertainty (see Section 3.1.3). To this end, we introduce a new variable \( y^j \) which represents an optimal solution under scenario \( j \in [N] \). For a set \( \{x^1, \ldots, x^K\} \) of candidate solutions, constructing an uncertainty set that maximizes the best robust objective value among \( x^k, k \in [K] \), is formulated as follows:

\[
\begin{align*}
\max & \ t \\
\text{s.t.} & \ t \leq \sum_{j \in [N]} \lambda^k_j (c^j x^k - c^j y^j) & \forall k \in [K] \\
& \sum_{j \in [N]} \lambda^k_j = 1 & \forall k \in [K] \\
& c^j \in \mathcal{U}(\tilde{c}^j) & \forall j \in [N] \\
& y^j \in \mathcal{X} & \forall j \in [N] \\
& \lambda^k_j \in \{0, 1\} & \forall j \in [N], k \in [K]
\end{align*}
\]

Note that the formulation is nonlinear due to the products between variables \( \lambda, c, \) and \( y \). We introduce new variables \( d_{ijk} = \lambda^k_j c^j_i \) and \( \alpha_{ijk} = \lambda^k_j c^j_i y^j_i \). The linearized MIP formulation we use is then as follows.

\[
\begin{align*}
\max & \ t \\
\text{s.t.} & \ t \leq \sum_{j \in [N]} \sum_{i \in [n]} (x^k_i d_{ijk} - \alpha_{ijk}) & \forall k \in [K] \\
& \sum_{j \in [N]} \lambda^k_j = 1 & \forall k \in [K] \\
& d_{ijk} \leq c^j_i & \forall i \in [n], j \in [N], k \in [K] \\
& d_{ijk} \leq c^j_i \lambda^k_j & \forall i \in [n], j \in [N], k \in [K] \\
& \alpha_{ijk} \geq c^j_i - c^j_i (2 - \lambda^k_j - y^j_i) & \forall i \in [n], j \in [N], k \in [K]
\end{align*}
\]
\[ c^j \in \mathcal{U} \bar{c}^j \quad \forall j \in [N] \]
\[ y^j \in \mathcal{X} \quad \forall j \in [N] \]
\[ \lambda_j^k \in \{0, 1\} \quad \forall j \in [N], k \in [K] \]
\[ d_{ijk}, \alpha_{ijk} \geq 0 \quad \forall i \in [n], j \in [N], k \in [K] \]

4.2.4. Experimental Setup

We follow a similar setup as in Section 3.1 and consider four experiments. In Exp1 all inputs change, which means that for parameters \((N, n, p)\), we consider combinations \((30, 30, 15), \quad (40, 40, 20)\) and \((40, 40, 21)\). In Exp2 we fix \(n = N = 30\) but change \(p \in \{10, 11, 15, 20, 21\}\). In Exp3 we fix \(n = 30\) and \(p = 15\) but change \(N \in \{20, 30, 40\}\). Finally, Exp4 has a similar setup as Exp3, but we consider large-scale problems with \(N \in \{100, 200, 500, 1000, 2000, 5000\}\).

For each parameter choice, we generate 50 instances using MMR-D-U, MMR-D-1 and MMR-D-2. All instances of Experiments 1, 2 and 3 are used as an input to the HIRO formulation with a budget \(b \in \{1, 2, 5\}\) and a 600-second generation time limit. Thus, Exp1 considers \(3 \times 50 \times 3 \times 4 = 1800\) instances, Exp2 considers \(5 \times 50 \times 3 \times 4 = 3000\) instances, and Exp3 considers \(3 \times 50 \times 3 \times 4 = 1800\) instances. For Exp4, only the sampling methods are used, which means that we consider \(6 \times 50 \times 3 = 900\) instances. All instances are solved using CPLEX-20-10 with a 600-second time limit.

4.2.5. Experimental Results

Figures 9, 10, 11 and 12 summarize the results of Exp1, Exp2, Exp3 and Exp4, respectively. It can be seen that HIRO can slightly increase solution times of some instances. Similarly, HIRO shows mixed results for the other experiments as well. In particular, instances generated by MMR-D-2 can increase solution times significantly over MMR-D-U. Interestingly, this does not apply for large-scale instances from Exp4, where MMR-D-1 tends to create instances that are harder to solve.

Figure 9: Min-max regret problems with discrete uncertainty for varying values of \((n, p, N)\), Exp1.
Figure 10: Min-max regret problems with discrete uncertainty for varying values of \( p \), Exp2.

Figure 11: Min-max regret problems with discrete uncertainty for varying values of \( N \), Exp3.

Figure 12: Min-max regret problems with discrete uncertainty for varying values of \( N \), Exp4.
5. Two-Stage Problems

5.1. Discrete Uncertainty

5.1.1. Problem Statement

By introducing a set of variables $y^j_i$ for each scenario $j \in [N]$, the two-stage problem with discrete uncertainty is formulated as follows.

\[
\begin{align*}
\min & \quad \sum_{i \in [n]} C_i x_i + t \\
\text{s.t.} & \quad t \geq \sum_{i \in [n]} c^j_i y^j_i \quad \forall j \in [N] \\
& \quad \sum_{i \in [n]} x_i + y^j_i = p \quad \forall j \in [N] \\
& \quad x_i + y^j_i \leq 1 \quad \forall i \in [n], j \in [N] \\
& \quad x_i \in \{0, 1\} \quad \forall i \in [n] \\
& \quad y^j_i \in \{0, 1\} \quad \forall i \in [n], j \in [N]
\end{align*}
\]

5.1.2. Sampling

We consider the following sampling methods to determine first-stage costs $C$ and second-stage costs $c^j_i$ for all $j \in [N]$. As a baseline ($2ST-D-U$), all values are chosen uniformly from the set $\{1, \ldots, 100\}$. In $2ST-D-1$, we choose $C_i$ uniformly either from from $\{45, \ldots, 55\}$ with probability 0.5, or from set $\{25, \ldots, 75\}$ with probability 0.5. For each second-stage cost $c^j_i$, we choose uniformly from one of the sets $\{C_i - 5, \ldots, C_i + 5\}$, $\{1, \ldots, 10\}$ or $\{91, \ldots, 100\}$ with probabilities 0.5, 0.25, and 0.25, respectively. Finally, in $2ST-D-2$, $C_i$ is chosen uniformly from $\{1, \ldots, 100\}$ with probability 0.5, or is set to 50 with probability 0.5. If $C_i = 50$, then with probability equal to 0.5 we choose $c^j_i$ from $\{1, \ldots, 10\}$ and with probability equal to 0.5 we choose $c^j_i$ from $\{91, \ldots, 100\}$; otherwise, if $C_i \neq 50$, we choose $c^j_i$ from $\{C_i - 5, \ldots, C_i + 5\}$ and set negative values to be equal to zero.

5.1.3. HIRO

Given a set of candidate first-stage solutions $x^k_i$, $k \in [K]$, we would like to find first-stage costs $C$ and second-stage costs $c^j_i$, $j \in [N]$, such that the worst-case costs of the best candidate solution is maximized, i.e., to solve

\[
\begin{align*}
\max & \quad t \\
\text{s.t.} & \quad t \leq \sum_{i \in [n]} C_i x^k_i + \max_{j \in [N]} Q(x^k_i, c^j_i) \quad \forall k \in [K] \\
& \quad c^j_i \in \mathcal{U}(c^j_i) \quad \forall j \in [N] \\
& \quad C \in \mathcal{U}(C)
\end{align*}
\]
where $Q(x^k, c^j)$ denotes the second-stage costs of $x^k$ under scenario $c^j$. To calculate these for some scenario $c^j$, we solve

$$
\min \sum_{i \in [n]} c^j_i y^k_i \\
\text{s.t.} \sum_{i \in [n]} x^k_i + y^k_i = p \\
x^k_i + y^k_i \leq 1 \quad \forall i \in [n] \\
y^k_i \in \{0, 1\} \quad \forall i \in [n]
$$

By relaxing this problem, dualizing it and then using variables $\lambda$ to decide which scenario is assigned to which solution to maximize the objective value, we find the following formulation for HIRO.

$$
\max t \\
\text{s.t.} \quad t \leq \sum_{i \in [n]} C_i x^k_i + (p - \sum_{i \in [n]} x^k_i) \beta^k + \sum_{i \in [n]} (x^k_i - 1) \gamma^k_i \quad \forall k \in [K] \\
\sum_{j \in [N]} \lambda^k_j = 1 \quad \forall k \in [K] \\
\beta^k \leq \gamma^k_i + \sum_{j \in [N]} c^j_i \lambda^k_j \quad \forall i \in [n], k \in [K] \\
\lambda^k_j \in \{0, 1\} \quad \forall j \in [N], k \in [K] \\
\beta^k \geq 0 \quad \forall k \in [K] \\
\gamma^k_i \geq 0 \quad \forall i \in [n], k \in [K] \\
C \in \mathcal{U}(\hat{C}) \\
c' \in \mathcal{U}(\hat{c}') \quad \forall j \in [N]
$$

If we only optimize over first-stage costs $C$, we use nominal values $\hat{c}^j$ instead of treating them as variables. However, if we optimize over both $C$ and $c^j$, the nonlinearity in $c^j_i \lambda^k_j$ needs to be replaced using variables $d_{ijk}$ (see Section 3.1.3).

### 5.1.4. Experimental Setup

We use a similar setup as for the single-stage min-max problem from Section 3.1. We create four datasets to assess the effect of different parameters. In Exp1, we consider the two cases when $N = n = 50, p = 25$ and $N = n = 100, p = 50$. In Exp2, we fix $n = N = 50$ but change $p \in \{10, 20, 25, 30, 40\}$. In Exp3, we fix $n = 50$ and $p = 25$ but change $N \in \{10, 20, \ldots, 60\}$. Finally, in Exp4, we consider problems with a large-scale number of scenarios $N \in \{100, 200, 500, 1000, 2000, 5000\}$ with $n = 50$ and $p = 25$.

For each parameter configuration, we generate 50 instances using 2ST-D-U, 2ST-D-1 and 2ST-D-2. For Experiments 1-3, we also use HIRO with budget $b \in \{1, 2, 5\}$ and a 600-second generation time limit. We use HIRO for modifying only first-stage costs $C$, or both first- and second-stage costs $C$ and $c^j$. Thus, we consider $2 \times 50 \times 3 \times 7 = 2100$ in Exp1, $5 \times 50 \times 3 \times 7 = 5250$ instances in Exp2, $6 \times 50 \times 3 \times 7 = 6300$ instances in Exp3, and...
$6 \times 50 \times 3 = 900$ instances in Exp4, respectively. All instances are solved using CPLEX-20-10 with a 600-second time limit.

### 5.1.5. Experimental Results

In Figures 13 and 14 we show results on Exp1. HIRO modifying $C$ or both $C$ and $c^j$ for all $j \in [N]$ can increase the solution time of 2ST-D-U instances. In addition, in most cases it is possible to slightly increase the solution time of 2ST-D-1 instances, while this approach is less effective for most 2ST-D-2 instances, which result in highest solution times overall.

Figure 13: Two-stage problems with discrete uncertainty for varying values of $(n, p, N)$, Exp1. HIRO modifies $C$.

Figure 14: Two-stage problems with discrete uncertainty for varying values of $(n, p, N)$, Exp1. HIRO modifies $C$ and $c^j$.

The results shown in Figures 15 and 16 on Exp2 show that larger values of $p$ result in higher solution times. HIRO can slightly increase solution times on most instances, while particularly 2ST-D-1 gives good results for large values of $p$.

In Figures 17 and 18, we show results on Exp3 and finally, we show the results for large values of $N$ (Exp4) in Figure 19. We note that in this setting, 2ST-D-2 gives instances with the highest solution times, followed by 2ST-D-1.
Figure 15: Two-stage problems with discrete uncertainty for varying values of $p$, Exp2. HIRO modifies $C$.

Figure 16: Two-stage problems with discrete uncertainty for varying values of $p$, Exp2. HIRO modifies $C$ and $c^j$.

Figure 17: Two-stage problems with discrete uncertainty for varying values of $N$, Exp3. HIRO modifies $C$. 
Figure 18: Two-stage problems with discrete uncertainty for varying values of $N$, Exp3. HIRO modifies $C$ and $c^j$.

Figure 19: Two-stage problems with discrete uncertainty for varying values of $N$, Exp4.
5.2. Discrete Budgeted Uncertainty

5.2.1. Problem Statement

In this section we consider the two-stage selection problem under discrete budgeted uncertainty set. We use the linear formulation introduced in [CGKZ18] to assess the performance of different instance generators. Let

\[ S = \{0\} \cup \{c_i : i \in [n]\} \cup \{\bar{c}_i : i \in [n]\} =: \{\alpha^1, \ldots, \alpha^K\} \]

with \( K = |S| \), the model is as follows.

\[
\begin{align*}
\min \quad & t \\
\text{s.t.} \quad & t \geq \sum_{i \in [n]} C_i x_i + (p - \sum_{i \in [n]} x_i)\alpha^k - \sum_{i \in [n]} (1 - x_i)[\alpha^k - c_i]^+ + \Gamma \pi^k + \sum_{i \in [n]} \rho_i^k & \quad k \in [K] \\
& \sum_{i \in [n]} x_i \leq p \\
& \pi^k + \rho_i^k \geq (1 - x_i)([\alpha^k - \bar{c}_i]^+ - [\alpha^k - c_i - d_i]^+) & \quad i \in [n], k \in [K] \\
& x_i \in \{0, 1\} & \quad i \in [n] \\
& \pi^k \geq 0 & \quad k \in [K] \\
& \rho_i^k \geq 0 & \quad i \in [n], k \in [K]
\end{align*}
\]

5.2.2. Sampling

As a baseline sampling method, we use 2ST-DB-U, where all values \( C_i, c_i \) and \( d_i \) are chosen uniformly from \( \{1, \ldots, 100\} \). In 2ST-DB-1, we differ from this setting by sampling small lower costs \( c_i \) from \( \{1, \ldots, 10\} \) and high deviations \( d_i \) from \( \{100 - c_i, \ldots, 100\} \). Finally, in 2ST-DB-2, we differ from 2ST-DB-U by setting \( c_i = 100 - C_i \) and sampling \( d_i \) from \( \{c_i, \ldots, 100\} \).

5.2.3. Experimental Setup

In this experiment we consider the cases when \( n = 100 \) and \( p \in \{25, 50, 75\} \). We also use different \( \Gamma \) for each value of \( p \). If \( p = 25 \), we choose \( \Gamma \in \{5, 10, 15, 20\} \); if \( p = 50 \), we choose \( \Gamma \in \{10, 20, 30, 40\} \); finally, we choose \( \Gamma \in \{15, 30, 45, 60\} \) for \( p = 75 \). For all inputs we generate 50 instances using the sampling methods 2ST-DB-U, 2ST-DB-1 and 2ST-DB-2. As a consequence, \( 3 \times 4 \times 50 \times 3 = 1800 \) instances are generated. All instances are solved using CPLEX-20-10 with a 600-second time limit. Problem sizes were chosen so that results can be compared with those on recoverable robust optimization provided in Section 6.2.

5.2.4. Experimental Results

The results shown in Figure 20 show that computation times are very small and even 2ST-DB-1 and 2ST-DB-2 fail to generate harder instances compared to 2ST-DB-U in most cases. Although the parameter \( \Gamma \) have no considerable impact on the solution time when \( p = 25 \), it can be effective for 2ST-DB-U when \( p = 50 \); in particular, \( \Gamma = 20 \) increases the solution time to over one second. The best results of 2ST-DB-1 and 2ST-DB-2 could be observed for \( p = 75 \) and \( \Gamma = 30 \) when the solution time passes 5 and 10 seconds, respectively.
5.3. Continuous Budgeted Uncertainty

5.3.1. Problem Statement

In this section we consider the two-stage selection problem under continuous budgeted uncertainty $U_{Bvar}$. Using the formulation provided in [CGKZ18], the problem can be formulated as follows.

$$\min \sum_{i \in [n]} C_i x_i + \sum_{i \in [n]} c_i y_i + \Gamma \pi + \sum_{i \in [n]} d_i \rho_i$$

s.t. $\sum_{i \in [n]} (y_i + x_i) = p$

$x_i + y_i \leq 1$ \quad \forall i \in [n]

$\pi + \rho_i \geq y_i$ \quad \forall i \in [n]

$\pi \geq 0$

$\rho_i \geq 0$ \quad \forall i \in [n]

$x_i \in \{0, 1\}$ \quad \forall i \in [n]

$y_i \in [0, 1]$ \quad \forall i \in [n]

5.3.2. Sampling

We use the same instance generators as in the case of discrete budgeted uncertainty, described in Section 5.2.2. These are 2ST-CB-U (uniform sampling), 2ST-CB-1 (small lower bounds $c$ and large deviations $d$), and 2ST-CB-2 (lower bounds plus first-stage costs are constant).

5.3.3. Experimental Setup

We use a similar experiment as for discrete budgeted uncertainty, see Section 5.2.3. Thus, we consider selection problems with $n = 100$ and $p \in \{25, 50, 75\}$. Note that $\Gamma$ needs to be chosen differently, as it now denotes the budget of deviation instead of the number of items that can deviate. Hence, we choose $\Gamma \in \{400, 800, 1000, 1200\}$. For each parameter choice and
sampling method, 50 instances are generated. As a result, $3 \times 4 \times 50 \times 3 = 1800$ instances are generated in total. We use CPLEX-20-10 with a 600-second time limit to solve each instance.

### 5.3.4. Experimental Results

The results presented in Figure 21 show that all three methods generate instances which can be solved fast (less that 0.1 second), but 2ST-CB-1 stands out, as it can increase the solution time by almost a factor 10, compared to 2ST-CB-U when $p = 50$ and $\Gamma = 800, 1000, 1200$.

![Figure 21: Two-stage problems with continuous budgeted uncertainty for varying $p$ (denoted in the top left corners) and $\Gamma$.](image)

### 6. Recoverable Problems

#### 6.1. Discrete Uncertainty

##### 6.1.1. Problem Statement

We consider recoverable robust selection problems under discrete uncertainty sets, which we formulate using the model presented in [CGKZ18]:

\[
\begin{align*}
\min & \quad \sum_{i \in [n]} C_i x_i + t \\
\text{s.t.} & \quad t \geq \sum_{i \in [n]} c_{ij} y_{ij}^j \quad \forall j \in [N] \\
& \quad \sum_{i \in [n]} x_i = p \\
& \quad \sum_{i \in [n]} y_{ij}^j = p \quad \forall j \in [N] \\
& \quad z_{ij} \leq x_i \quad \forall i \in [n], j \in [N] \\
& \quad z_{ij} \leq y_{ij}^j \quad \forall i \in [n], j \in [N] \\
& \quad \sum_{i \in [n]} z_{ij} \geq \Delta \quad \forall j \in [N]
\end{align*}
\]
Here, $\Delta$ is called the recovery factor and means that at least $\Delta$ items of the first-stage solution must remain in the second-stage solution. Variables $z^j_i$ measure if first-stage solution $x$ and second-stage solution $y^j$ both pack the same item $i \in [n]$.

6.1.2. Sampling

We use the same sampling methods as provided for two-stage problems in Section 5.1.2 which we denote here as RR-D-U, RR-D-1, and RR-D-2, respectively.

6.1.3. HIRO

To formulate the HIRO approach, we would like to optimize $C$ and $c^j$, $j \in [N]$, with respect to a set of candidate solution $x^k$, $k \in [K]$, as follows:

\[
\begin{align*}
\max \quad & t \\
\text{s.t.} \quad & t \leq \sum_{i \in [n]} C_i x^k_i + \max_{j \in [N]} Q(x^k, c^j) \quad \forall k \in [K] \\
& c^j \in U(c^j) \quad \forall j \in [N] \\
& C \in U(C)
\end{align*}
\]

where, as before, $Q(x^k, c^j)$ denotes the second-stage costs of $x^k$ under scenario $c^j$. To determine these costs, we need to solve the second-stage recovery problem, which is given as

\[
\begin{align*}
\min \quad & \sum_{i \in [n]} c^j_i y^k_i \\
\text{s.t.} \quad & \sum_{i \in [n]} y^k_i = p \\
& \sum_{i \in [n]} x^k_i y^k_i \geq p - \Delta \\
& x^k_i + y^k_i \leq 1 \quad \forall i \in [n] \\
& y^k_i \in \{0, 1\} \quad \forall i \in [n]
\end{align*}
\]

Note that there always exists an integer optimal solution for the LP-relaxation of this problem. Hence, using duality and variables $\lambda$ that decide which scenario is assigned to each candidate solution, the HIRO problem is formulated as follows.

\[
\begin{align*}
\max \quad & t \\
\text{s.t.} \quad & t \leq \sum_{i \in [n]} C_i x^k_i + p \beta^k + (p - \Delta) \eta^k - \sum_{i \in [n]} \gamma^k_i \quad \forall k \in [K] \\
& \sum_{j \in [N]} \lambda^k_j = 1 \quad \forall k \in [K]
\end{align*}
\]
\begin{align*}
\beta^k + \eta^k x_i^k & \leq \gamma_i^k + \sum_{j \in [N]} c_j^k \lambda_j^k & \forall i \in [n], k \in [K] \\
\lambda_j^k & \in \{0, 1\} & \forall j \in [N], k \in [K] \\
\beta^k, \eta^k & \geq 0 & \forall k \in [K] \\
\gamma_i^k & \geq 0 & \forall i \in [n], k \in [K] \\
c^j & \in \mathcal{U}(\tilde{c}^j) & \forall j \in [N] \\
C & \in \mathcal{U}(\tilde{C}) &
\end{align*}

If we only optimize over $C$, we can use nominal values $\tilde{c}^j$ for second-stage costs instead of treating them as variables. However, if we optimize over both $C$ and $c^j$, the nonlinearity in $c_j^k \lambda_j^k$ needs to be replaced with additional variables.

### 6.1.4. Experimental Setup

This setup is similar to the one introduced in for two-stage problems in Section 5.1.4. We consider four experiments to assess the effect of different parameters. In Exp1, we consider the cases when $N = n = 50$, $p = 25$, $\Delta \in \{13, 20\}$ and $N = n = 100$, $p = 50$, $\Delta \in \{25, 40\}$. In Exp2, we fix $n = N = 50$, but change $p$ and $\Delta$, using $p = 25$, $\Delta \in \{13, 20\}$, $p = 30$, $\Delta \in \{15, 25\}$, and $p = 40$, $\Delta \in \{20, 30\}$. In Exp3, we fix $n = 50$ and $p = 25$, but change $N \in \{40, 50, 60\}$ and $\Delta \in \{13, 20\}$. Finally, in Exp4, we use the parameters of Exp3, but with $N \in \{100, 200, 500, 1000, 2000\}$.

For each parameter choice and sampling method, we generate 50 instances. For Exp1, Exp2 and Exp3, we also apply the HIRO method to each instance, where we vary only the first-stage costs, or both first-stage and second-stage costs, using varying budgets $b \in \{1, 2, 5\}$ and a 600-second generation time limit. Thus, Exp1 contains $2 \times 2 \times 50 \times 3 \times 7 = 4200$ instances, Exp2 contains $5 \times 2 \times 50 \times 3 \times 7 = 10500$ instances, Exp3 contains $6 \times 2 \times 50 \times 3 \times 7 = 12600$ instances, and there are $6 \times 2 \times 50 \times 3 = 1800$ instances in Exp4. We solve all generated instances using CPLEX-20-10 with a 600-second time limit.

### 6.1.5. Experimental Results

Figures 22 and 23 show results for Exp1. While HIRO methods can increase solution times for most instances generated by RR-D-U or RR-D-1 when we only modify first-stage costs $C$, results are mixed in the other cases. Both RR-D-1 and RR-D-2 generate instances that are considerably harder to solve than those generated by RR-D-U, in particular for higher values of $\Delta$.

In Figures 24 and 25, we present results for Exp2. The HIRO approach can be effective to increase solution times, if only first-stage costs $C$ are modified. As in Exp1, a positive correlation between increasing $\Delta$ and solution time can be observed. While for $p = 40$, all solution times remain small, increasing $\Delta$ gives a large increase in solution time for $p = 25$ or $p = 30$. In particular RR-D-1 gives consistently good results.

In Figures 26 and 27, we summarize results on Exp3. Sampling methods without HIRO become slightly harder with increasing $N$, and considerably harder when using $\Delta = 20$ instead of $\Delta = 13$. Both RR-D-1 and RR-D-2 perform similarly and produce instances that are harder to solve than when using RR-D-U. For these instances, applying the HIRO method can have a large impact: for $\Delta = 13$ and $N = 50$, solution times increase by several orders of magnitude.
Figure 22: Recoverable robust problems with discrete uncertainty for varying values of $(N, n, p, \Delta)$, Exp1. HIRO modifies $C$.

Figure 23: Recoverable robust problems with discrete uncertainty for varying values of $(N, n, p, \Delta)$, Exp1. HIRO modifies $C$ and $c_j$.

Figure 24: Recoverable robust problems with discrete uncertainty for varying values of $(p, \Delta)$, Exp2. HIRO modifies $C$. 
This effect is not apparent for $N = 40$ or $N = 60$. For $\Delta = 20$ and $N = 50$, this effect is reversed, and solution times drop by orders of magnitude (while they can be slightly increased for other values of $N$ when only first-stage costs are modified).

Finally, we consider large values of $N$ for Exp4 in Figure 28. Especially for $\Delta = 20$, sampling methods RR-D-1 and RR-D-2 produce instances that are harder to solve than those generated by RR-D-U. For $\Delta = 13$, these differences are less pronounced. For large values of $N$, all solution times are at the time limit, which means that they cannot be further differentiated.

6.2. Discrete Budgeted Uncertainty

6.2.1. Problem Statement

We now discuss recoverable selection problems under discrete budgeted uncertainty. To model these problems, we apply the following formulation adapted from [CGKZ18]. Let

$$S = \{ (\alpha, \beta) : \exists i,j \in [n], \delta_i, \delta_j \in \{0,1\} \text{ s.t. } \alpha = c_i + \delta_i d_i, \beta = c_j - d_j \delta_j - \alpha \}$$

$$= \{ (\alpha^1, \beta^1), \ldots, (\alpha^K, \beta^K) \}$$

Figure 27: Recoverable robust problems with discrete uncertainty for varying values of \((N, \Delta)\), Exp3. HIRO modifies \(C\) and \(c'\).

Figure 28: Recoverable robust problems with discrete uncertainty for varying values of \((N, \Delta)\), Exp4.
with \( K = |\mathcal{S}| \). Then, a compact formulation for the robust problem is as follows.

\[
\min t \\
\text{s.t.} \quad t \geq \sum_{i \in [n]} C_i x_i + \Gamma x^k + \sum_{i \in [n]} p_i^k + p \alpha^k + (p - \Delta) \beta^k - \sum_{i \in [n]} \left[ \alpha^k + x_i \beta^k - c_i \right]_+ \\
\pi^k + \rho_i^k \geq \left[ \alpha^k + x_i \beta^k - c_i \right]_+ - \left[ \alpha^k + x_i \beta^k - c_i - d_i \right]_+ \\
\sum_{i \in [n]} x_i = p \\
x_i \in \{0, 1\} \\
\rho_i^k \geq 0 \\
\pi^k \geq 0 \\
\forall i \in [n], k \in [K]
\]

6.2.2. Sampling

We use the same sampling methods as for two-stage robust selection with discrete budgeted uncertainty, see Section 5.2.2. We refer to them as RR-DB-U for uniform sampling, RR-DB-1 for sampling with small costs \( c_i \) and high deviations \( d_i \), and RR-DB-2 for the setting where \( c_i = 100 - C_i \).

6.2.3. Experimental Setup

We consider problems where \( n = 100 \) and \( p \in \{25, 50, 75\} \). We also use different \( \Gamma \) for each value of \( p \). If \( p = 25 \), we choose \( \Gamma \in \{5, 10, 15, 20\} \); if \( p = 50 \), we choose \( \Gamma \in \{10, 20, 30, 40\} \); finally, we choose \( \Gamma \in \{15, 30, 45, 60\} \) for \( p = 75 \). Unlike the experiment presented in Section 5.2.3 we also need to consider parameter \( \Delta \) which depends on the value of \( p \), too. Here, we consider the similar value of \( \Gamma \) for \( \Delta \), so if \( p = 25 \), we choose \( \Delta \in \{5, 10, 15, 20\} \); if \( p = 50 \), we choose \( \Delta \in \{10, 20, 30, 40\} \); finally, we choose \( \Delta \in \{15, 30, 45, 60\} \) for \( p = 75 \). We generate 50 instances using the three sampling methods for each parameter choice. As a consequence, we generate \( 3 \times 4 \times 4 \times 50 \times 3 = 7200 \) instances for this experiment. All instances are solved using CPLEX-20-10 with a 600-second time limit.

6.2.4. Experimental Results

We present results in Figure 29 where values for each \( \Gamma \) are grouped together and within each group, the four values for different choices of \( \Delta \) are shown. Method RR-DB-2 only slightly outperforms RR-DB-U, while instances generated by RR-DB-1 can be amongst the easiest to solve. While for \( p = 25 \), all parameter choices of \( \Gamma \) and \( \Delta \) result in problems of similar difficulty, only \( \Gamma = 10 \) or \( \Gamma = 20 \) results in hard problems for \( p = 50 \), while only \( \Gamma = 15 \) gives hard problems for \( p = 75 \).
Figure 29: Recoverable robust problems with discrete budgeted uncertainty for varying $p$ (noted in top left corners) and $(\Gamma, \Delta)$.

6.3. Continuous Budgeted Uncertainty

6.3.1. Problem Statement

A compact formulation for recoverable robust selection problems with continuous budgeted uncertainty $U_{B\text{var}}^{\text{cont}}$ was presented in [CGKZ18] as follows.

$$
\begin{align*}
\min & \quad \sum_{i \in [n]} C_i x_i + \sum_{i \in [n]} c_i y_i + \pi + \sum_{i \in [n]} d_i \rho_i \\
\text{s.t.} & \quad \sum_{i \in [n]} y_i = p \\
& \quad \sum_{i \in [n]} x_i = p \\
& \quad \sum_{i \in [n]} z_i \geq p - \Delta \\
& \quad z_i \leq x_i \quad \forall i \in [n] \\
& \quad z_i \leq y_i \quad \forall i \in [n] \\
& \quad \pi + \rho_i \geq y_i \quad \forall i \in [n] \\
& \quad x_i \in \{0, 1\} \quad \forall i \in [n] \\
& \quad y_i \in [0, 1] \quad \forall i \in [n] \\
& \quad \pi \geq 0 \\
& \quad \rho_i \geq 0 \quad \forall i \in [n]
\end{align*}
$$

6.3.2. Sampling

We use the same sampling methods as provided for two-stage problems in Section 5.2.2, which we denote here as RR-$DB$-$U$, RR-$DB$-1, and RR-$DB$-2, respectively.
6.3.3. Experimental Setup

We use similar parameters as in Section 6.2.3 except for $\Gamma$. We set $n = 100$ and consider $p \in \{25, 50, 75\}$ and $\Gamma \in \{400, 800, 1000, 1200\}$. We consider $\Delta \in \{5, 10, 15, 20\}$ when $p = 25$, $\Delta \in \{10, 20, 30, 40\}$ when $p = 50$ and $\Delta \in \{15, 30, 45, 60\}$ when $p = 75$. For each parameter choice and sampling method, 50 instances are generated (a total of $3 \times 4 \times 4 \times 50 \times 3 = 7200$ instances) and solved using CPLEX-20-10 with a 600-second time limit.

6.3.4. Experimental Results

We present results in Figure 30, where as before values for each $\Gamma$ are grouped together and within each group, the four values for different choices of $\Delta$ are shown. Methods $\text{RR-DB-1}$ and $\text{RR-DB-2}$ generate instances that are harder to solve in almost all cases compared to the $\text{RR-DB-U}$ method. In all cases, solution times tend to increase with $\Gamma$. For $\Delta$, behavior is very different, with an increase with $\Delta$ in some cases, and a decrease in others. Furthermore, instances tend to become easier for increasing $p$.

![Figure 30: Recoverable robust problems with discrete budgeted uncertainty for varying $p$ (noted in top left corners) and $(\Gamma, \Delta)$.](image)

7. Instance Library and Formats

An outcome of this paper is to create a benchmark library that collects all instances presented in this paper, along with their generating code, in order to drive further development of improved solution algorithms. The library is available at a dedicated website https://robust-optimization.com. The instances included so far will form a basic stock of problems to be extended further.

For ease of usability, we decided to use a format that is as simple as possible. All instances are given as comma separated values (csv). The first line of each instance represents the main parameters depending on its uncertainty set and criterion. The instances generated for the min-max and min-max regret criteria under discrete uncertainty set have the same structure. Their first line consists of $n$, $p$ and $N$, and the following $N$ lines represents each scenario. Instances generated for the two-stage and recoverable criteria under discrete uncertainty set have also similar structure. The difference is that the second line represent the first stage.
scenario and the the following $N$ line show each scenario. In addition, for the recoverable criterion the first line contains the forth parameter $\Delta$.

The interval uncertainty set is only used for the min-max regret criterion. The instances generated for the related section has only three lines, the first one provides $n$ and $p$, the second line shows the lower bound of the interval assigned to each item and the third line represents the deviation of the cited intervals. The same structure also is used for the min-max criterion under budgeted uncertainty set. The only difference is that the first line contains the third parameter $\Gamma$.

The two-stage and recoverable criteria under both budgeted uncertainty set have similar structure to the budgeted uncertainty for the min-max criterion. The difference is that they have four lines, the second line represents the first stage scenario and the third and fourth line shows the lower bound and deviation of each interval, respectively. Moreover, for the recoverable criterion, the first line has a forth parameter $\Delta$.

8. Conclusions

Having benchmark sets of instances available is of central importance for the development of improved solution algorithms. For robust discrete optimization, it seems that no such benchmark currently exists, despite an active research community developing specialized solution methods. In this paper, we considered a wide range of robust decision criteria (min-max, min-max regret, two-stage, recoverable robust) in combination with different uncertainty sets (interval, discrete, continuous and discrete budgeted) to generate instances for selection problems that are harder to solve than the baseline method of sampling parameters uniformly. Besides alternative sampling methods, we proposed optimization models that modify uncertainty sets to increase robust objective values with the aim to find instances that are even harder to solve. For all problem combinations we were able to generate instances that are harder to solve than those generated by the baseline method, sometimes by several orders of magnitude.

This work can be considered as a starting point for an ongoing community effort to standardize benchmarking of robust optimization algorithms. While the sampling methods proposed here can be applied to any other combinatorial optimization problem on principle, it is likely that more problem-tailored sampling methods will be more effective (e.g., profits that are correlated to weights for knapsack problems; or graphs that need to be generated for network problems). On a dedicated webpage (robust-optimization.com), we started collecting instances, including those generated in this paper, and cordially invite interested researchers to participate.

In terms of further research, it would be interesting to develop solution algorithms that make use of properties of hard instances to alleviate this hardness. For example, some difficult min-max regret instances with interval uncertainty we generated have either small $c_i$ and $d_i$, or both large $c_i$ and $d_i$. Along the paradigm of algorithm engineering, methods that can use such structure have the potential to perform better than the results produced by the general mixed-integer programming solver we used.
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A. Overview of Sampling Generators

A.1. Min-Max Problems with Discrete Uncertainty

- \textit{MM-D-U}: For all \(i \in [n]\) and \(j \in [N]\), we choose \(c_{ij} \in \{1, \ldots, 100\}\) iid uniformly.
- \textit{MM-D-1}: For all \(i \in [n]\) and \(j \in [N]\), with probability \(\frac{1}{2}\) we choose \(c_{ij} \in \{1, \ldots, 10\}\), and with probability \(\frac{1}{2}\) we choose \(c_{ij} \in \{91, \ldots, 100\}\) iid uniformly.
- \textit{MM-D-2}: For each \(i \in [n]\) and \(j \in [N]\), if \(i \leq \lfloor n/2 \rfloor\), we choose \(c_{ij} \in \{1, \ldots, 100\}\) iid uniformly; otherwise, we set \(c_{ij} = 100 - c_{ij} - \lfloor n/2 \rfloor\).

A.2. Min-Max Problems with Budgeted Uncertainty

- \textit{MM-B-U}: For all \(i \in [n]\), we choose both \(c_i, d_i\) uniformly as an integer number from \(\{1, \ldots, 100\}\).
- \textit{MM-B-1}: For all \(i \in [n]\), we choose \(c_i\) uniformly as an integer number from \(\{1, \ldots, 100\}\) and set \(d_i = 100 - c_i\).
- \textit{MM-B-2}: For all \(i \in [n]\), we choose \(c_i\) uniformly as an integer number from \(\{1, \ldots, 10\}\) and then choose \(d_i \in \{100 - c_i - 1, \ldots, 100\}\).

A.3. Min-Max Regret Problems with Interval Uncertainty

- \textit{MMR-I-U}: For all \(i \in [n]\), we choose both \(c_i, d_i\) uniformly as an integer number from \(\{1, \ldots, 100\}\).
- \textit{MMR-I-1}: For all \(i \in [n]\), with probability \(\frac{1}{2}\), we choose \(c_i\) from \(\{1, \ldots, 10\}\) and \(d_i\) from \(\{91, \ldots, 100\}\) uniformly. Otherwise, we choose \(c_i\) from \(\{91, \ldots, 100\}\) and \(d_i\) from \(\{1, \ldots, 10\}\).
- \textit{MMR-I-2}: For all \(i \in [n]\), with probability \(\frac{1}{2}\), we choose \(c_i\) and \(d_i\) from \(\{1, \ldots, 10\}\) uniformly. Otherwise, we choose \(c_i\) and \(d_i\) from \(\{91, \ldots, 100\}\) uniformly.
A.4. Min-Max Regret Problems with Discrete Uncertainty

- **MMR-D-U**: For all \( i \in [n] \) and \( j \in [N] \), we choose \( c_{ij}^d \in \{1, \ldots, 100\} \) iid uniformly (\( \equiv \text{MM-D-U} \)).

- **MMR-D-1**: For all \( i \in [n] \) and \( j \in [N] \), with probability \( \frac{1}{2} \) we choose \( c_{ij}^d \in \{1, \ldots, 10\} \), and with probability \( \frac{1}{2} \) we choose \( c_{ij} \in \{91, \ldots, 100\} \) iid uniformly (\( \equiv \text{MM-D-1} \)).

- **MMR-D-2**: For each \( i \in [n] \) and \( j \in [N] \), if \( i \leq \lfloor \frac{n}{2} \rfloor \), we choose \( c_{ij}^d \in \{1, \ldots, 100\} \) iid uniformly; otherwise, we set \( c_{ij}^d = 100 - c_{i-[n/2]} \) (\( \equiv \text{MM-D-2} \)).

A.5. Two-Stage Problems with Discrete Uncertainty

- **2ST-D-U**: For all \( i \in [n] \), we choose \( C_i, c_i^d \) iid uniformly from \( \{1, \ldots, 100\} \).

- **2ST-D-1**: For all \( i \in [n] \), with probability equal to 0.5, we choose \( C_i \) from \( \{45, \ldots, 55\} \) and with probability equal to 0.5, from \( \{25, \ldots, 75\} \). For all \( i \in [n] \) and \( j \in [N] \), with probability equal to 0.5 we choose \( c_{ij}^d \) from \( \{C_i - 5, \ldots, C_i + 5\} \), with probability equal to 0.25 we choose \( c_{ij}^d \) from \( \{1, \ldots, 10\} \) and with probability equal to 0.25, \( c_{ij}^d \) from \( \{91, \ldots, 100\} \).

- **2ST-D-2**: For all \( i \in [n] \), with probability equal to 0.5 we choose \( C_i \) from \( \{1, \ldots, 100\} \) and with probability equal to 0.5 we set \( C_i = 50 \). Then, for all \( i \in [n] \) and \( j \in [N] \), if \( C_i = 50 \) with probability equal to 0.5 we choose \( c_{ij}^d \) from \( \{1, \ldots, 10\} \) and with probability equal to 0.5 we choose \( c_{ij}^d \) from \( \{91, \ldots, 100\} \); else, we choose \( c_{ij}^d \) from \( \{C_i - 5, \ldots, C_i + 5\} \) and set negative values to be equal to zero.

A.6. Two-Stage Problems with Discrete Budgeted Uncertainty

- **2ST-DB-U**: For all \( i \in [n] \), we choose \( C_i, c_i, d_i \) iid uniformly from \( \{1, \ldots, 100\} \).

- **2ST-DB-1**: For all \( i \in [n] \), we choose \( C_i \) iid uniformly from \( \{1, \ldots, 100\} \), \( c_i \) from \( \{1, \ldots, 10\} \) and \( d_i \) from \( \{100 - c_i, \ldots, 100\} \).

- **2ST-DB-2**: For all \( i \in [n] \), we choose \( C_i \) iid uniformly from \( \{1, \ldots, 100\} \), set \( c_i = 100 - C_i \) and choose \( d_i \) iid uniformly from \( \{c_i, \ldots, 100\} \).

A.7. Two-Stage Problems with Continuous Budgeted Uncertainty

- **2ST-CB-U**: For all \( i \in [n] \), we choose \( C_i, c_i, d_i \) iid uniformly from \( \{1, \ldots, 100\} \) (\( \equiv \text{2ST-DB-U} \)).

- **2ST-CB-1**: For all \( i \in [n] \), we choose \( C_i \) iid uniformly from \( \{1, \ldots, 100\} \), \( c_i \) from \( \{1, \ldots, 10\} \) and \( d_i \) from \( \{100 - c_i, \ldots, 100\} \) (\( \equiv \text{2ST-DB-1} \)).

- **2ST-CB-2**: For all \( i \in [n] \), we choose \( C_i \) iid uniformly from \( \{1, \ldots, 100\} \), set \( c_i = 100 - C_i \) and choose \( d_i \) iid uniformly from \( \{c_i, \ldots, 100\} \) (\( \equiv \text{2ST-DB-2} \)).
A.8. Recoverable Robust Problems with Discrete Uncertainty

- RR-D-U: For all \( i \in [n] \), we choose \( C_i, c_i, d_i \) iid uniformly from \{1, \ldots, 100\} (=2ST-D-U).

- RR-D-1: For all \( i \in [n] \), with probability equal to 0.5, we choose \( C_i \) from \{45, \ldots, 55\} and with probability equal to 0.5, from \{25, \ldots, 75\}. For all \( i \in [n] \) and \( j \in [N] \), with probability equal to 0.5 we choose \( c_i^j \) from \{\( C_i - 5, \ldots, C_i + 5 \}\}, with probability equal to 0.25 we choose \( c_i^j \) from \{1, \ldots, 10\} and with probability equal to 0.25, \( c_i^j \) from \{91, \ldots, 100\} (=2ST-D-1).

- RR-D-2: For all \( i \in [n] \), with probability equal to 0.5 we choose \( C_i \) from \{1, \ldots, 100\} and with probability equal to 0.5 we set \( C_i = 50 \). Then, for all \( i \in [n] \) and \( j \in [N] \), if \( C_i = 50 \) with probability equal to 0.5 we choose \( c_i^j \) from \{1, \ldots, 10\} and with probability equal to 0.5 we choose \( c_i^j \) from \{91, \ldots, 100\}; else, we choose \( c_i^j \) from \{\( C_i - 5, \ldots, C_i + 5 \}\} and set negative values to be equal to zero (=2ST-D-2).

A.9. Recoverable Robust Problems with Discrete Budgeted Uncertainty

- RR-DB-U: For all \( i \in [n] \), we choose \( C_i, c_i, d_i \) iid uniformly from \{1, \ldots, 100\} (=2ST-DB-U).

- RR-DB-1: For all \( i \in [n] \), we choose \( C_i \) iid uniformly from \{1, \ldots, 100\}, \( c_i \) from \{1, \ldots, 10\} and \( d_i \) from \{100 - c_i, \ldots, 100\} (=2ST-DB-1).

- RR-DB-2: For all \( i \in [n] \), we choose \( C_i \) iid uniformly from \{1, \ldots, 100\}, set \( c_i = 100 - C_i \) and choose \( d_i \) iid uniformly from \{\( C_i \), \ldots, 100\} (=2ST-DB-2).

A.10. Recoverable Robust Problems with Continuous Budgeted Uncertainty

- RR-CB-U: For all \( i \in [n] \), we choose \( C_i, c_i, d_i \) iid uniformly from \{1, \ldots, 100\} (=2ST-CB-U).

- RR-CB-1: For all \( i \in [n] \), we choose \( C_i \) iid uniformly from \{1, \ldots, 100\}, \( c_i \) from \{1, \ldots, 10\} and \( d_i \) from \{100 - c_i, \ldots, 100\} (=2ST-CB-1).

- RR-CB-2: For all \( i \in [n] \), we choose \( C_i \) iid uniformly from \{1, \ldots, 100\}, set \( c_i = 100 - C_i \) and choose \( d_i \) iid uniformly from \{\( C_i \), \ldots, 100\} (=2ST-CB-2).

B. Generation Times

In addition to solution times of instances, the generation time of these instances is of importance, too. Preferably, there should be a balance between both solution and generation time of instances, as spending a lot of time on generating instances that might only slightly increase the solution time in comparison to the corresponding sampling method is not acceptable.

In this sense, it should be noticed that some HIRO methods hit the 600-second time limit. In the following tables we illustrate the generation time of our HIRO methods for all cited criteria in this paper. It also can be easily observed that the generation time of all sampling methods can be ignored as they are considerably fast.
| $n$ | $p$ | $N$ | $b$ | $UH$ | $1H$ | $2H$ |
|-----|-----|-----|-----|------|------|------|
| 20  | 10  | 20  | 1   | 1.21 | 37.71| 586.87|
|     |     |     | 2   | 27.17| 346.83| 599.17|
|     |     |     | 5   | 596.35| 508.87| 599.22|
| 20  | 11  | 20  | 1   | 1.15 | 16.93| 195.01|
|     |     |     | 2   | 27.61| 242.24| 472.15|
|     |     |     | 5   | 591.55| 409.46| 599.85|
| 25  | 13  | 25  | 1   | 10.19| 125.55| 553.69|
|     |     |     | 2   | 293.57| 390.39| 599.84|
|     |     |     | 5   | 599.91| 479.44| 599.78|
| 30  | 15  | 30  | 1   | 62.94| 329.47| 573.91|
|     |     |     | 2   | 570.01| 488.36| 599.87|
|     |     |     | 5   | 599.92| 534.38| 599.89|
| 35  | 17  | 35  | 1   | 288.29| 342.12| 599.29|
|     |     |     | 2   | 599.91| 442.22| 599.52|
|     |     |     | 5   | 599.99| 556.39| 599.71|
| 40  | 20  | 40  | 1   | 553.75| 469.73| 599.92|
|     |     |     | 2   | 599.98| 515.58| 599.86|
|     |     |     | 5   | 600.01| 589.19| 599.44|
| 40  | 21  | 40  | 1   | 530.51| 441.24| 596.07|
|     |     |     | 2   | 599.99| 534.46| 597.15|
|     |     |     | 5   | 600.01| 598.27| 596.71|

Table 2: Min-Max problem under discrete uncertainty set - Exp1
| $p$ | $b$ | UH | 1H | 2H |
|-----|-----|----|----|----|
| 5   | 1   | 0.81 | 7.47 | 189.19 |
|     | 2   | 5.75 | 106.33 | 457.56 |
|     | 5   | 436.99 | 234.42 | 599.99 |
| 10  | 1   | 8.19 | 202.81 | 599.94 |
|     | 2   | 212.83 | 445.85 | 599.97 |
|     | 5   | 600.03 | 469.48 | 599.92 |
| 11  | 1   | 21.89 | 216.37 | 560.04 |
|     | 2   | 385.08 | 383.39 | 599.96 |
|     | 5   | 600.01 | 436.44 | 599.92 |
| 15  | 1   | 62.94 | 329.47 | 573.91 |
|     | 2   | 570.01 | 488.36 | 599.87 |
|     | 5   | 599.92 | 534.38 | 599.89 |
| 20  | 1   | 12.33 | 119.56 | 599.92 |
|     | 2   | 275.58 | 372.14 | 599.92 |
|     | 5   | 599.98 | 555.09 | 599.83 |
| 21  | 1   | 8.36  | 57.71  | 533.68 |
|     | 2   | 161.51 | 317.91 | 588.92 |
|     | 5   | 573.73 | 510.98 | 599.83 |
| 25  | 1   | 0.99  | 6.29   | 297.04 |
|     | 2   | 8.44  | 74.96  | 529.49 |
|     | 5   | 240.71 | 434.21 | 599.89 |

Table 3: Min-Max problem under discrete uncertainty set - Exp2 ($n = N = 30$)
|   |   |   |   |
|---|---|---|---|
| 5 | 1  | 0.41 | 7.86 | 16.26 |
| 5 | 2  | 34.03 | 210.78 | 146.79 |
| 5 | 5  | 551.48 | 430.74 | 572.03 |
| 10 | 1  | 2.43 | 89.53 | 148.61 |
| 5 | 2  | 149.27 | 339.52 | 450.97 |
| 5 | 5  | 599.88 | 505.95 | 599.77 |
| 15 | 1  | 7.08 | 181.06 | 259.81 |
| 5 | 2  | 335.47 | 398.85 | 548.47 |
| 5 | 5  | 599.91 | 461.67 | 599.67 |
| 20 | 1  | 17.81 | 170.39 | 438.02 |
| 5 | 2  | 439.65 | 405.91 | 587.37 |
| 5 | 5  | 599.88 | 449.26 | 599.69 |
| 25 | 1  | 47.27 | 202.91 | 528.54 |
| 5 | 2  | 557.81 | 419.55 | 597.05 |
| 5 | 5  | 599.91 | 512.57 | 599.79 |
| 30 | 1  | 62.94 | 329.47 | 573.91 |
| 5 | 2  | 570.01 | 488.36 | 599.87 |
| 5 | 5  | 599.92 | 534.38 | 599.89 |
| 35 | 1  | 142.93 | 294.53 | 593.41 |
| 5 | 2  | 584.25 | 489.83 | 599.89 |
| 5 | 5  | 599.96 | 543.21 | 599.91 |
| 40 | 1  | 149.74 | 329.88 | 599.87 |
| 5 | 2  | 596.37 | 493.11 | 599.89 |
| 5 | 5  | 599.96 | 507.77 | 599.91 |

Table 4: Min-Max problem under discrete uncertainty set - Exp3 (n = 30, p = 15)
| $\Gamma$ | $b$ | HIRO-$c$ |   | HIRO-$d$ |   | HIRO-$c-d$ |   |
|--------|-----|---------|-----|---------|-----|-----------|-----|
|        |     | UH      | 1H  | 2H      | UH  | 1H        | 2H  |
| 5      | 1   | 0.017   | 0.012| 0.017   | 0.013| 0.008     | 0.006| 0.014 | 0.009| 0.008|
| 2      |     | 0.015   | 0.012| 0.021   | 0.014| 0.009     | 0.007| 0.015 | 0.009| 0.008|
| 5      | 10  | 0.016   | 0.012| 0.054   | 0.015| 0.009     | 0.008| 0.019 | 0.009| 0.009|
| 20     | 1   | 0.019   | 0.014| 0.046   | 0.019| 0.009     | 0.008| 0.031 | 0.010| 0.012|
| 10     | 2   | 0.027   | 0.021| 0.045   | 0.024| 0.009     | 0.023| 0.051 | 0.039| 0.046|
| 10     | 1   | 0.016   | 0.011| 0.019   | 0.013| 0.009     | 0.006| 0.015 | 0.010| 0.008|
| 2      |     | 0.016   | 0.012| 0.027   | 0.014| 0.009     | 0.006| 0.016 | 0.010| 0.008|
| 5      | 10  | 0.016   | 0.013| 0.051   | 0.016| 0.011     | 0.007| 0.021 | 0.014| 0.008|
| 10     | 2   | 0.021   | 0.017| 0.038   | 0.023| 0.012     | 0.007| 0.041 | 0.042| 0.009|
| 20     | 1   | 0.028   | 0.047| 0.039   | 0.035| 0.017     | 0.011| 0.076 | 0.057| 0.009|
| 15     | 2   | 0.013   | 0.014| 0.021   | 0.012| 0.011     | 0.006| 0.013 | 0.012| 0.007|
| 5      | 10  | 0.013   | 0.013| 0.027   | 0.013| 0.011     | 0.006| 0.012 | 0.014| 0.008|
| 10     | 2   | 0.015   | 0.018| 0.045   | 0.013| 0.014     | 0.007| 0.016 | 0.056| 0.008|
| 20     | 1   | 0.015   | 0.041| 0.041   | 0.018| 0.035     | 0.008| 0.026 | 0.081| 0.008|
| 15     | 2   | 0.021   | 0.066| 0.038   | 0.028| 0.052     | 0.011| 0.066 | 0.101| 0.009|
| 20     | 1   | 0.012   | 0.016| 0.013   | 0.011| 0.009     | 0.006| 0.015 | 0.027| 0.009|
| 2      |     | 0.012   | 0.015| 0.017   | 0.009| 0.011     | 0.006| 0.015 | 0.031| 0.011|
| 5      | 10  | 0.012   | 0.017| 0.019   | 0.011| 0.011     | 0.007| 0.017 | 0.028| 0.012|
| 10     | 2   | 0.014   | 0.019| 0.018   | 0.011| 0.015     | 0.008| 0.024 | 0.052| 0.011|
| 20     | 1   | 0.015   | 0.024| 0.018   | 0.016| 0.018     | 0.011| 0.059 | 0.112| 0.011|

Table 5: Min-Max problem under budgeted uncertainty set
| n  | p | b | UH  | 1H  | 2H  |
|----|---|---|-----|-----|-----|
| 10 | 1 | 10.27 | 3.56 | 7.32 |
|    | 2 | 13.34 | 9.23 | 18.91 |
|    | 5 | 32.11 | 342.77 | 592.56 |
| 20 | 1 | 9.82 | 2.33 | 7.08 |
|    | 2 | 12.27 | 18.59 | 18.01 |
|    | 5 | 32.55 | 599.88 | 592.41 |
| 30 | 1 | 16.33 | 1.76 | 7.85 |
|    | 2 | 19.36 | 3.97 | 18.43 |
|    | 5 | 42.98 | 174.09 | 578.04 |
| 40 | 1 | 17.68 | 1.94 | 6.28 |
|    | 2 | 21.51 | 4.29 | 21.54 |
|    | 5 | 44.54 | 8.22 | 340.26 |
| 50 | 1 | 17.94 | 3.29 | 5.54 |
|    | 2 | 24.54 | 8.23 | 16.93 |
|    | 5 | 61.32 | 47.44 | 115.22 |
| 60 | 1 | 31.28 | 7.79 | 6.86 |
|    | 2 | 48.32 | 57.67 | 25.38 |
|    | 5 | 96.75 | 82.52 | 452.97 |
| 70 | 1 | 40.91 | 7.55 | 7.54 |
|    | 2 | 59.51 | 34.48 | 18.66 |
|    | 5 | 161.15 | 74.91 | 599.88 |
| 80 | 1 | 44.51 | 9.07 | 7.14 |
|    | 2 | 67.74 | 24.51 | 21.46 |
|    | 5 | 180.21 | 31.91 | 593.85 |
| 90 | 1 | 39.34 | 9.14 | 9.92 |
|    | 2 | 59.58 | 19.11 | 41.36 |
|    | 5 | 171.41 | 144.04 | 452.59 |

Table 6: Min-Max Regret problem under interval uncertainty set (n = 100)

| n  | p | N  | b | UH  | 1H  | 2H  |
|----|---|----|---|-----|-----|-----|
| 30 | 15| 30 | 1 | 573.87 | 519.23 | 572.47 |
|    |   |    | 2 | 599.89 | 512.64 | 599.95 |
|    |   |    | 5 | 599.88 | 490.18 | 599.92 |
| 40 | 20| 40 | 1 | 599.86 | 558.33 | 599.89 |
|    |   |    | 2 | 599.88 | 585.66 | 599.91 |
|    |   |    | 5 | 599.87 | 556.48 | 599.89 |
| 40 | 21| 40 | 1 | 599.91 | 599.94 | 599.90 |
|    |   |    | 2 | 599.91 | 599.95 | 599.93 |
|    |   |    | 5 | 599.93 | 599.93 | 599.93 |

Table 7: Min-Max Regret problem under discrete uncertainty set - Exp1
Table 8: Min-Max Regret problem under discrete uncertainty set - Exp2 \((n = N = 30)\)

| \(p\) | \(b\) | \(N = 30\) | \(b\) | \(UH\) | \(1H\) | \(2H\) |
|---|---|---|---|---|---|---|
| 10 | 1 | 557.76 | 275.91 | 567.07 |
|   | 2 | 533.49 | 462.03 | 575.28 |
|   | 5 | 573.87 | 519.23 | 572.47 |
| 11 | 1 | 576.31 | 483.14 | 563.74 |
|   | 2 | 571.21 | 564.83 | 593.08 |
|   | 5 | 599.84 | 456.07 | 599.85 |
| 15 | 1 | 599.89 | 574.49 | 599.85 |
|   | 2 | 599.86 | 558.33 | 599.89 |
|   | 5 | 599.92 | 564.22 | 599.86 |
| 20 | 1 | 599.91 | 579.45 | 599.88 |
|   | 2 | 599.94 | 585.79 | 599.91 |
|   | 5 | 599.92 | 599.98 | 599.93 |
| 21 | 1 | 599.91 | 599.94 | 599.91 |
|   | 2 | 599.93 | 595.42 | 599.93 |
|   | 5 | 599.93 | 599.95 | 599.91 |

Table 9: Min-Max Regret problem under discrete uncertainty set - Exp3 \((n = 30, p = 15)\)

| \(N\) | \(b\) | \(UH\) | \(1H\) | \(2H\) |
|---|---|---|---|---|
| 20 | 1 | 530.04 | 402.71 | 524.44 |
|   | 2 | 573.87 | 519.23 | 572.47 |
|   | 5 | 599.92 | 540.41 | 599.46 |
| 30 | 1 | 599.91 | 472.94 | 596.06 |
|   | 2 | 599.86 | 558.33 | 599.89 |
|   | 5 | 599.91 | 575.77 | 599.91 |
| 40 | 1 | 599.92 | 564.02 | 599.92 |
|   | 2 | 599.91 | 599.94 | 599.91 |
|   | 5 | 599.92 | 599.94 | 599.92 |

Table 10: Two-stage problem under discrete uncertainty set - Exp1

| \(n\) | \(p\) | \(N\) | \(b\) | \(UH\) | \(1H\) | \(2H\) | \(UH\) | \(1H\) | \(2H\) |
|---|---|---|---|---|---|---|---|---|---|
| 50 | 25 | 50 | 1 | 1.63 | 8.37 | 182.54 | 160.74 | 546.89 | 114.88 |
|   |   |   | 2 | 2.91 | 17.15 | 241.46 | 339.47 | 599.99 | 157.22 |
|   |   |   | 5 | 4.64 | 441.08 | 388.92 | 559.85 | 599.94 | 599.99 |
| 100 | 50 | 100 | 1 | 18.05 | 131.22 | 404.11 | 599.97 | 599.97 | 71.41 |
|   |   |   | 2 | 31.21 | 202.34 | 475.57 | 600.01 | 599.97 | 118.48 |
|   |   |   | 5 | 55.03 | 546.54 | 552.05 | 600.03 | 599.93 | 599.99 |
|   |   | HIRO-C | HIRO-C-e^j |
|---|---|--------|-----------|
|   |   | UH | 1H | 2H | UH | 1H | 2H |
| 10 | 1 | 0.65 | 0.85 | 96.25 | 30.08 | 69.27 | 5.36 |
| 2  | 0.96 | 1.53 | 108.35 | 97.83 | 212.83 | 32.34 |
| 5  | 2.21 | 5.12 | 108.56 | 351.17 | 526.31 | 352.21 |
| 20 | 1  | 1.09 | 5.12 | 152.27 | 74.85 | 497.68 | 67.16 |
| 2  | 1.54 | 9.23 | 175.87 | 222.41 | 599.99 | 135.35 |
| 5  | 15.32 | 198.44 | 241.21 | 547.27 | 599.99 | 574.21 |
| 25 | 1  | 1.63 | 8.37 | 182.54 | 160.74 | 546.89 | 114.88 |
| 2  | 2.91 | 17.15 | 241.46 | 339.47 | 599.99 | 157.22 |
| 5  | 4.64 | 441.08 | 388.92 | 559.85 | 599.99 | 599.99 |
| 30 | 1  | 2.15 | 19.52 | 241.31 | 226.81 | 587.84 | 124.45 |
| 2  | 14.98 | 62.57 | 427.35 | 454.03 | 599.99 | 159.71 |
| 5  | 10.56 | 517.46 | 599.99 | 579.28 | 599.99 | 599.99 |
| 40 | 1  | 5.01 | 422.42 | 464.64 | 356.72 | 599.99 | 119.85 |
| 2  | 7.96 | 588.31 | 525.73 | 490.83 | 599.99 | 145.79 |
| 5  | 37.23 | 600.01 | 593.88 | 570.59 | 599.99 | 585.47 |

Table 11: Two-stage problem under discrete uncertainty set - Exp2 (n = N = 50)

|   |   | HIRO-C | HIRO-C-e^j |
|---|---|--------|-----------|
|   |   | UH | 1H | 2H | UH | 1H | 2H |
| 10 | 1 | 0.18 | 1.13 | 111.96 | 7.86 | 152.35 | 67.71 |
| 2  | 0.27 | 3.21 | 167.35 | 22.12 | 485.25 | 168.04 |
| 5  | 0.68 | 283.64 | 293.59 | 243.51 | 587.99 | 568.67 |
| 20 | 1  | 0.41 | 2.43 | 132.94 | 44.54 | 438.42 | 114.21 |
| 2  | 0.63 | 6.08 | 192.45 | 151.48 | 577.16 | 168.15 |
| 5  | 1.66 | 312.48 | 323.89 | 459.03 | 599.99 | 599.99 |
| 30 | 1  | 0.68 | 3.68 | 167.11 | 98.25 | 484.89 | 111.01 |
| 2  | 1.08 | 19.56 | 188.54 | 240.41 | 599.41 | 168.47 |
| 5  | 2.21 | 364.68 | 378.91 | 499.55 | 599.99 | 599.99 |
| 40 | 1  | 0.75 | 5.51 | 174.31 | 124.59 | 532.34 | 120.91 |
| 2  | 1.23 | 9.99 | 226.56 | 274.29 | 599.96 | 156.78 |
| 5  | 2.49 | 382.72 | 406.91 | 531.51 | 599.96 | 599.99 |
| 50 | 1  | 1.63 | 8.37 | 182.54 | 160.74 | 546.89 | 114.88 |
| 2  | 2.91 | 17.15 | 241.46 | 339.47 | 599.99 | 157.22 |
| 5  | 4.64 | 441.08 | 388.92 | 559.85 | 599.94 | 599.99 |
| 60 | 1  | 1.46 | 10.99 | 221.81 | 175.53 | 588.94 | 122.13 |
| 2  | 2.25 | 22.88 | 287.59 | 373.31 | 599.97 | 157.84 |
| 5  | 4.48 | 412.46 | 409.51 | 552.21 | 599.97 | 599.99 |

Table 12: Two-stage problem under discrete uncertainty set - Exp3 (n = 50, p = 25)
| $n$ | $p$ | $N$ | $\Delta$ | $b$ | HIRO-$C$ | HIRO-$C$-c$^j$ |
|-----|-----|-----|---------|----|---------|---------|
|     |     |     |         |    | UH      | 1H      | 2H      | UH      | 1H      | 2H      |
| 50  | 25  | 50  | 13      | 1  | 2.04    | 47.91   | 464.73  | 142.56  | 529.29  | 120.15  |
|     |     |     |         | 2  | 8.81    | 216.72  | 517.62  | 341.47  | 599.98  | 156.14  |
|     |     |     |         | 5  | 57.94   | 599.15  | 581.39  | 563.49  | 599.97  | 599.98  |
| 50  | 25  | 50  | 20      | 1  | 1.97    | 42.73   | 439.31  | 92.38   | 512.24  | 120.09  |
|     |     |     |         | 2  | 6.51    | 216.47  | 501.02  | 256.51  | 599.97  | 156.08  |
|     |     |     |         | 5  | 65.99   | 593.15  | 574.47  | 559.54  | 599.96  | 599.98  |
| 100 | 50  | 100 | 25      | 1  | 29.11   | 498.21  | 599.99  | 539.75  | 599.95  | 13.12   |
|     |     |     |         | 2  | 109.04  | 599.91  | 599.98  | 599.99  | 599.97  | 61.04   |
|     |     |     |         | 5  | 541.29  | 599.83  | 599.98  | 599.99  | 599.96  | 599.95  |
| 100 | 50  | 100 | 40      | 1  | 30.51   | 516.23  | 599.98  | 497.07  | 599.94  | 12.69   |
|     |     |     |         | 2  | 122.73  | 599.93  | 599.98  | 591.01  | 599.98  | 60.64   |
|     |     |     |         | 5  | 525.51  | 599.81  | 599.98  | 599.99  | 599.98  | 599.95  |

Table 13: Recoverable problem under discrete uncertainty set - Exp1

| $p$ | $\Delta$ | $b$ | HIRO-$C$ | HIRO-$C$-c$^j$ |
|-----|-------|----|---------|---------|
|     |       |    | UH      | 1H      | 2H      | UH      | 1H      | 2H      |
| 25  | 13    | 1  | 2.04    | 47.91   | 464.73  | 142.56  | 529.29  | 120.15  |
|     |       | 2  | 8.81    | 216.72  | 517.62  | 341.47  | 599.98  | 156.14  |
|     |       | 5  | 57.94   | 599.15  | 581.39  | 563.49  | 599.97  | 599.98  |
| 20  | 1     | 1  | 1.97    | 42.73   | 439.31  | 92.38   | 512.24  | 120.09  |
|     |       | 2  | 6.51    | 216.47  | 501.02  | 256.51  | 599.97  | 156.08  |
|     |       | 5  | 65.99   | 593.15  | 574.47  | 559.54  | 599.96  | 599.98  |
| 30  | 15    | 1  | 1.46    | 15.44   | 351.81  | 37.23   | 254.71  | 120.13  |
|     |       | 2  | 2.75    | 271.54  | 441.22  | 141.73  | 570.93  | 156.12  |
|     |       | 5  | 45.39   | 596.62  | 548.04  | 328.28  | 599.96  | 599.98  |
| 25  | 1     | 1  | 1.22    | 13.07   | 339.32  | 27.18   | 209.19  | 120.08  |
|     |       | 2  | 2.24    | 285.11  | 425.85  | 100.48  | 587.53  | 156.08  |
|     |       | 5  | 42.16   | 588.54  | 531.46  | 319.18  | 599.96  | 599.98  |
| 40  | 20    | 1  | 0.85    | 1.24    | 158.35  | 20.26   | 46.921  | 36.35   |
|     |       | 2  | 1.12    | 3.21    | 182.96  | 30.67   | 246.08  | 36.95   |
|     |       | 5  | 2.38    | 362.68  | 209.19  | 163.83  | 546.48  | 261.71  |
| 30  | 1     | 1  | 0.76    | 1.12    | 156.33  | 15.71   | 41.19  | 36.33   |
|     |       | 2  | 0.96    | 2.75    | 184.68  | 29.24   | 219.76  | 36.91   |
|     |       | 5  | 2.11    | 366.69  | 207.55  | 150.08  | 548.66  | 261.34  |

Table 14: Recoverable problem under discrete uncertainty set - Exp2 ($n = N = 50$)
| $N$ | $\Delta$ | $b$ | HIRO-$C$ | HIRO-$C^{+}$ |
|-----|----------|-----|---------|-----------|
|     |          |     | UH   | 1H | 2H | UH   | 1H | 2H |
| 40  | 13       | 1   | 2.06 | 38.92 | 413.02 | 87.71 | 505.09 | 120.11 |
|     |          | 2   | 8.01 | 168.72 | 491.43 | 309.91 | 599.96 | 156.11 |
|     |          | 5   | 45.21 | 577.48 | 577.88 | 556.12 | 599.97 | 599.98 |
| 20  | 1        | 1   | 1.96 | 38.88 | 413.02 | 66.57 | 495.11 | 120.11 |
|     |          | 2   | 6.58 | 196.81 | 491.43 | 230.15 | 599.96 | 156.11 |
|     |          | 5   | 49.69 | 569.71 | 577.88 | 545.59 | 599.97 | 599.98 |
| 50  | 13       | 1   | 2.04 | 47.91 | 464.73 | 142.56 | 529.29 | 120.15 |
|     |          | 2   | 8.81 | 216.72 | 517.62 | 341.47 | 599.98 | 156.14 |
|     |          | 5   | 57.94 | 599.15 | 581.39 | 563.49 | 599.97 | 599.98 |
| 20  | 1        | 1   | 1.97 | 42.73 | 439.31 | 92.38 | 512.24 | 120.09 |
|     |          | 2   | 6.51 | 216.47 | 501.02 | 256.51 | 599.97 | 156.08 |
|     |          | 5   | 65.99 | 593.15 | 574.47 | 559.54 | 599.96 | 599.98 |
| 60  | 13       | 1   | 2.81 | 48.62 | 496.17 | 185.21 | 555.02 | 120.18 |
|     |          | 2   | 9.91 | 231.19 | 531.54 | 375.75 | 599.96 | 156.16 |
|     |          | 5   | 82.23 | 599.98 | 585.02 | 569.62 | 599.96 | 599.98 |
| 20  | 1        | 1   | 2.71 | 47.78 | 466.45 | 133.82 | 530.02 | 120.12 |
|     |          | 2   | 9.89 | 236.18 | 513.78 | 299.86 | 599.96 | 156.11 |
|     |          | 5   | 70.25 | 599.32 | 579.41 | 568.27 | 599.96 | 599.98 |

Table 15: Recoverable problem under discrete uncertainty set - Exp3 ($n = 50, p = 25$)