On the $\pi\mathfrak{F}$-norm and the $\mathfrak{H}$-$\mathfrak{F}$-norm of a finite group*

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Abstract

Let $\mathfrak{H}$ be a Fitting class and $\mathfrak{F}$ a formation. We call a subgroup $N_{\mathfrak{H},\mathfrak{F}}(G)$ of a finite group $G$ the $\mathfrak{H}$-$\mathfrak{F}$-norm of $G$ if $N_{\mathfrak{H},\mathfrak{F}}(G)$ is the intersection of the normalizers of the products of the $\mathfrak{F}$-residuals of all subgroups of $G$ and the $\mathfrak{H}$-radical of $G$. Let $\pi$ denote a set of primes and let $\mathfrak{G}_\pi$ denote the class of all finite $\pi$-groups. We call the subgroup $N_{\mathfrak{G}_\pi,\mathfrak{F}}(G)$ of $G$ the $\pi\mathfrak{F}$-norm of $G$. A normal subgroup $N$ of $G$ is called $\pi\mathfrak{F}$-hypercentral in $G$ if either $N = 1$ or $N > 1$ and every $G$-chief factor below $N$ of order divisible by at least one prime in $\pi$ is $\mathfrak{F}$-central in $G$. Let $Z_{\pi\mathfrak{F}}(G)$ denote the $\pi\mathfrak{F}$-hypercentre of $G$, that is, the product of all $\pi\mathfrak{F}$-hypercentral normal subgroups of $G$. In this paper, we study the properties of the $\mathfrak{H}$-$\mathfrak{F}$-norm, especially of the $\pi\mathfrak{F}$-norm of a finite group $G$. In particular, we investigate the relationship between the $\pi'\mathfrak{F}$-norm and the $\pi\mathfrak{F}$-hypercentre of $G$.

1 Introduction

All groups considered in this paper are finite, and all classes of groups $\mathfrak{X}$ mentioned are non-empty. $G$ always denotes a group, $p$ denotes a prime, $\pi$ denotes a set of primes, and $\mathbb{P}$ denotes the set of all primes. Also, let $\pi(G)$ denote the set of all prime divisors of the order of $G$, and let $\pi(\mathfrak{X}) = \bigcup\{\pi(G) : G \in \mathfrak{X}\}$ for a class of groups $\mathfrak{X}$.

Recall that a class of groups $\mathfrak{F}$ is called a formation if $\mathfrak{F}$ is closed under taking homomorphic images and subdirect products. A formation $\mathfrak{F}$ is said to be saturated if $G \in \mathfrak{F}$ whenever $G/\Phi(G) \in \mathfrak{F}$. The $\mathfrak{F}$-residual of $G$, denoted by $G^\mathfrak{F}$, is the smallest normal subgroup $N$ of $G$.

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$G$ with $G/N \in \mathfrak{F}$. The formation product $\mathfrak{F} \circ \mathfrak{F}$ of a class of groups $\mathfrak{F}$ and a formation $\mathfrak{F}$ is the class of all groups $G$ such that $G^{\mathfrak{F}} \in \mathfrak{F}$. A class of groups $\mathfrak{H}$ is called a Fitting class if $\mathfrak{H}$ is closed under taking normal subgroups and products of normal $\mathfrak{H}$-subgroups. The $\mathfrak{H}$-radical of $G$, denoted by $G_{\mathfrak{H}}$, is the maximal normal $\mathfrak{H}$-subgroup of $G$. The Fitting product $\mathfrak{H} \diamond \mathfrak{X}$ of a Fitting class $\mathfrak{H}$ and a class of groups $\mathfrak{X}$ is the class of all groups $G$ such that $G/G_{\mathfrak{H}} \in \mathfrak{X}$. A class of groups $\mathfrak{B}$ is called a Fitting formation if $\mathfrak{B}$ is both a formation and a Fitting class. Note that for a Fitting formation $\mathfrak{B}$, a formation $\mathfrak{F}$ and a Fitting class $\mathfrak{H}$, $\mathfrak{H} \circ (\mathfrak{B} \circ \mathfrak{F}) = (\mathfrak{H} \circ \mathfrak{B}) \circ \mathfrak{F}$ always holds, and we denote it by $\mathfrak{H} \circ \mathfrak{B} \circ \mathfrak{F}$.

The class of the groups of order 1 is denoted by 1, and the class of all finite groups is denoted by $\mathfrak{G}$. We use $\mathfrak{G}$ (resp. $\mathfrak{N}$, $\mathfrak{U}$, $\mathfrak{A}$) to denote the class of finite soluble (resp. nilpotent, supersolvable, abelian) groups and $\mathfrak{G}_\pi$ (resp. $\mathfrak{N}_\pi$, $\mathfrak{U}_\pi$) to denote the class of finite $\pi$-solvable (resp. $\pi$-nilpotent, $\pi$-supersolvable) groups. Also, the symbol $\mathfrak{G}_\pi$ denotes the class of all finite $\pi$-groups.

A formation function $f$ is a local function $f: \mathbb{P} \to \{\text{classes of groups}\}$ such that $f(p)$ is a formation for all $p \in \mathbb{P}$. Let $L F(f)$ denote the set of all groups $G$ whose chief factors $L/K$ are all $f$-central in $G$, that is, $G/C_G(L/K) \in f(p)$ for all $p \in \pi(L/K)$. The canonical local definition of a saturated formation $\mathfrak{F}$ is the uniquely determined formation function $F$ such that $\mathfrak{F} = L F(F)$, $F(p) \subseteq \mathfrak{F}$ and $\mathfrak{G}_\pi \circ F(p) = F(p)$ for all $p \in \mathbb{P}$ (for details, see [11] Chap. IV).

Following [11], for a class of groups $\mathfrak{X}$, we define closure operations as follows:

- $\mathfrak{N}(G) = \{N \in \mathfrak{X} : G \leq H \text{ for some } H \in \mathfrak{X}\}$
- $\mathfrak{S}_H \mathfrak{X} = \{G : G \text{ is subnormal in } H \text{ for some } H \in \mathfrak{X}\}$
- $\mathfrak{Q} \mathfrak{X} = \{G : \text{ there exist } H \in \mathfrak{X} \text{ and an epimorphism from } H \text{ onto } G\}$
- $\mathfrak{E} \mathfrak{X} = \{G : \text{ there exists a series of subgroups of } G : 1 = G_0 \leq G_1 \leq \cdots \leq G_n = G \text{ with each } G_i/G_{i-1} \in \mathfrak{X}\}$

Recall that the norm $\mathcal{N}(G)$ of $G$ is the intersection of the normalizers of all subgroups of $G$, and the Wielandt subgroup $\omega(G)$ of $G$ is the intersection of the normalizers of all subnormal subgroups of $G$. These concepts were introduced by R. Baer [11] and H. Wielandt [31] in 1934 and 1958, respectively. Much investigation has focused on using the concepts of the norm and the Wielandt subgroup to determine the structure of finite groups (see, for example, [2, 3, 6, 9, 12, 13, 18, 20, 22]).

Recently, Li and Shen [19] considered the intersection of the normalizers of the derived subgroups of all subgroups of $G$. Also, in [12] and [24], the authors considered the intersection of the normalizers of the nilpotent residuals of all subgroups of $G$. Furthermore, for a formation $\mathfrak{F}$, Su and Wang [29] investigated the intersection of the normalizers of the $\mathfrak{F}$-residuals of all subgroups of $G$ and the intersection of the normalizers of the products of the $\mathfrak{F}$-residuals of all subgroups of $G$ and $O_{\pi'}(G)$. As a continuation of the above ideas, we now introduce the notion of $\mathfrak{H} \mathfrak{F}$-norm as follows:

**Definition 1.1.** Let $\mathfrak{H}$ be a Fitting class and $\mathfrak{F}$ a formation. We call a subgroup $\mathcal{N}_{\mathfrak{H} \mathfrak{F}}(G)$ of $G$ the $\mathfrak{H} \mathfrak{F}$-norm of $G$ if $\mathcal{N}_{\mathfrak{H} \mathfrak{F}}(G)$ is the intersection of the normalizers of the products of
the \( \mathfrak{F} \)-residuals of all subgroups of \( G \) and the \( \mathfrak{S} \)-radical of \( G \), that is,

\[
N_{\mathfrak{S}, \mathfrak{F}}(G) = \bigcap_{H \leq G} N_G(H^{\mathfrak{F}}G_H).
\]

In particular, when \( \mathfrak{S} = 1 \), the subgroup \( N_{\mathfrak{F}}(G) \) of \( G \) is called the \( \mathfrak{F} \)-norm of \( G \), and we denote it by \( N_{\mathfrak{F}}(G) \), that is,

\[
N_{\mathfrak{F}}(G) = \bigcap_{H \leq G} N_G(H^{\mathfrak{F}}); \quad \text{when} \quad \mathfrak{S} = \mathfrak{G}_\pi, \text{the subgroup} \quad N_{\mathfrak{G}_\pi, \mathfrak{F}}(G) \text{ of } G \text{ is called the } \pi \mathfrak{F} \text{-norm of } G, \text{ and we denote it by } N_{\pi \mathfrak{F}}(G), \text{ that is,}
\]

\[
N_{\pi \mathfrak{F}}(G) = \bigcap_{H \leq G} N_G(H^{\mathfrak{F}}O_{\pi}(G)).
\]

**Definition 1.2.** Let \( N_{\mathfrak{G}_1, \mathfrak{F}}(G) = 1 \) and \( N_{\mathfrak{G}_i, \mathfrak{F}}(G) / N_{\mathfrak{G}_{i-1}, \mathfrak{F}}(G) = N_{\mathfrak{G}_i, \mathfrak{F}}(G / N_{\mathfrak{G}_{i-1}, \mathfrak{F}}(G)) \) for \( i = 1, 2, \ldots \). Then there exists a series of subgroups of \( G \):

\[
1 = N_{\mathfrak{G}_0, \mathfrak{F}}(G) \leq N_{\mathfrak{G}_1, \mathfrak{F}}(G) \leq N_{\mathfrak{G}_2, \mathfrak{F}}(G) \cdots \leq N_{\mathfrak{G}_n, \mathfrak{F}}(G) = N_{\mathfrak{G}_{n+1}, \mathfrak{F}}(G) = \cdots.
\]

Denote \( N_{\mathfrak{G}_\infty, \mathfrak{F}}(G) \) the terminal term of this ascending series. In particular, when \( \mathfrak{S} = 1 \), we denote \( N_{\mathfrak{G}_1, \mathfrak{F}}(G) \) by \( N_{\mathfrak{F}}(G) \); when \( \mathfrak{S} = \mathfrak{G}_\pi \), we denote \( N_{\mathfrak{G}_\infty, \mathfrak{F}}(G) \) by \( N_{\pi \mathfrak{F}}(G) \).

Let \( \mathfrak{F} \) be a formation. A \( G \)-chief factor \( L/K \) is said to be \( \mathfrak{F} \)-central in \( G \) if \( (L/K) \simeq (G/C_G(L/K)) \in \mathfrak{F} \). Following [17], a normal subgroup \( N \) of \( G \) is called \( \pi \mathfrak{F} \)-hypercentral in \( G \) if either \( N = 1 \) or \( N > 1 \) and every \( G \)-chief factor below \( N \) of order divisible by at least one prime in \( \pi \) is \( \mathfrak{F} \)-central in \( G \). Let \( Z_{\pi \mathfrak{F}}(G) \) denote the \( \pi \mathfrak{F} \)-hypercentre of \( G \), that is, the product of all \( \pi \mathfrak{F} \)-hypercentral normal subgroups of \( G \). The \( \mathfrak{F} \pi \mathfrak{F} \)-hypercentre of \( G \) is called the \( \mathfrak{F} \)-hypercentre of \( G \), and we denote it by \( Z_{\mathfrak{F}}(G) \).

Let \( \mathfrak{X} \) be a class of groups. Recall that a subgroup \( U \) of \( G \) is called \( \mathfrak{X} \)-maximal in \( G \) if \( U \in \mathfrak{X} \) and \( G \) does not have a subgroup \( V \) such that \( U < V \) and \( V \in \mathfrak{X} \). Following [27], we use \( \text{Int}_{\mathfrak{X}}(G) \) to denote the intersection of all \( \mathfrak{X} \)-maximal subgroups of \( G \).

In [5] Remark 4, J. C. Beidleman and H. Heineken observed that \( N_{\mathfrak{G}_c, \mathfrak{F}}(G) \) coincides with \( \text{Int}_{\mathfrak{N}_c}(G) \) for every group \( G \), where \( \mathfrak{N}_c \) denotes the class of nilpotent groups of class at most \( c \). In [27], A. N. Skiba gave conditions under which the \( \mathfrak{F} \)-hypercentre \( Z_{\mathfrak{F}}(G) \) coincides with \( \text{Int}_{\mathfrak{F}}(G) \) for every group \( G \). Also, Guo and A. N. Skiba [10] gave conditions under which the \( \pi \mathfrak{F} \)-hypercentre \( Z_{\pi \mathfrak{F}}(G) \) coincides with \( \text{Int}_{\mathfrak{F}}(G) \) for every group \( G \).

Motivated by the above observations, the following questions naturally arise:

**Problem (I).** Under what conditions \( N_{\mathfrak{F}}(G) \) coincides with the \( \mathfrak{N} \circ \mathfrak{F} \)-hypercentre \( Z_{\mathfrak{N} \circ \mathfrak{F}}(G) \)? More generally, under what conditions \( N_{\pi \mathfrak{F}}(G) \) coincides with the \( \pi(\mathfrak{N} \circ \mathfrak{F}) \)-hypercentre \( Z_{\pi(\mathfrak{N} \circ \mathfrak{F})}(G) \)?

**Problem (II).** Under what conditions \( N_{\mathfrak{F}}(G) \) coincides with \( \text{Int}_{\mathfrak{N} \circ \mathfrak{F}}(G) \)? More generally, under what conditions \( N_{\pi \mathfrak{F}}(G) \) coincides with \( \text{Int}_{\pi(\mathfrak{N} \circ \mathfrak{F})}(G) \)?
For a class of groups $\mathfrak{X}$, a group $G$ is called $\mathfrak{S}$-critical for $\mathfrak{X}$ if $G \notin \mathfrak{X}$ but all proper subgroups of $G$ belong to $\mathfrak{X}$. Let $\text{Crit}_\mathfrak{S}(\mathfrak{X})$ denote the set of all groups $G$ which are $\mathfrak{S}$-critical for $\mathfrak{X}$. For convenience of statement, we give the following definition.

**Definition 1.3.** We say that a formation $\mathfrak{F}$ satisfies:

1. The $\pi$-boundary condition (I) if $\text{Crit}_\mathfrak{S}(\mathfrak{F}) \subseteq \mathfrak{N}_\pi \circ \mathfrak{F}$ (equivalently, $\text{Crit}_\mathfrak{S}(\mathfrak{F}) \subseteq \mathfrak{G}_\pi \circ \mathfrak{F}$, see Lemma 2.7 below).
2. The $\pi$-boundary condition (II) if for any $p \in \pi$, $\text{Crit}_\mathfrak{S}(\mathfrak{F}_p \circ \mathfrak{F}) \subseteq \mathfrak{G}_\pi \circ \mathfrak{F}$.
3. The $\pi$-boundary condition (III) if for any $p \in \pi$, $\text{Crit}_\mathfrak{S}(\mathfrak{F}_p \circ \mathfrak{F}) \subseteq \mathfrak{N}_\pi \circ \mathfrak{F}$.
4. The $\pi$-boundary condition (III) in $\mathfrak{G}$ if for any $p \in \pi$, $\text{Crit}_\mathfrak{S}(\mathfrak{F}_p \circ \mathfrak{F}) \cap \mathfrak{G} \subseteq \mathfrak{N}_\pi \circ \mathfrak{F}$.

Note that a formation $\mathfrak{F}$ satisfies the $\pi$-boundary condition (III) (resp. the $\pi$-boundary condition (I)) if and only if $\mathfrak{N}_\pi \circ \mathfrak{F}$ satisfies the $\pi$-boundary condition (resp. the $\pi$-boundary condition in $\mathfrak{G}$) in the sense of [17].

**Remark 1.4.** If a formation $\mathfrak{F}$ satisfies the $\pi$-boundary condition (II), then clearly, $\mathfrak{F}$ satisfies the $\pi$-boundary condition (I). However, the converse does not hold. For example, let $\pi = \mathbb{P}$ and $\mathfrak{F} = \mathfrak{N}_3$. By [17, Chap. IV, Satz 5.4], $\text{Crit}_\mathfrak{S}(\mathfrak{N}_3) \subseteq \mathfrak{N} \circ \mathfrak{N}_3$. Now let $G = A_5$, where $A_5$ is the alternating group of degree 5. Then $G \in \text{Crit}_\mathfrak{S}(\mathfrak{N}_3 \circ \mathfrak{N}_3)$, but $G \notin \mathfrak{G} \circ \mathfrak{N}_3$. Hence $\text{Crit}_\mathfrak{S}(\mathfrak{N}_3 \circ \mathfrak{N}_3) \not\subseteq \mathfrak{G} \circ \mathfrak{N}_3$.

**Remark 1.5.** If a formation $\mathfrak{F}$ satisfies the $\pi$-boundary condition (III), then $\mathfrak{F}$ satisfies the $\pi$-boundary condition (II). However, the converse does not hold. For example, let $\pi = \mathbb{P}$ and $\mathfrak{F} = \mathfrak{G}_3$. For any prime $p \neq 3$, $\text{Crit}_\mathfrak{S}(\mathfrak{G}_p \circ \mathfrak{G}_3) \subseteq \mathfrak{N}_3 \cup \text{Crit}_\mathfrak{S}(\mathfrak{N}_3)$. If there exists a group $H$ such that $H \in \text{Crit}_\mathfrak{S}(\mathfrak{G}_p \circ \mathfrak{G}_3) \setminus (\mathfrak{G} \circ \mathfrak{G}_3)$, then by [17, Chap. IV, Satz 5.4], we have that $H \in \mathfrak{N}_3$. Hence $H$ has the normal 3-complement $A$. If $A < H$, then $A \in \mathfrak{G}_p \circ \mathfrak{G}_3 \subseteq \mathfrak{G}$, and thereby $H \in \mathfrak{G}$, a contradiction. Therefore, $H = A \in \mathfrak{G}_p \cup \text{Crit}_\mathfrak{S}(\mathfrak{G}_p) \subseteq \mathfrak{G}$, also a contradiction. This shows that $\text{Crit}_\mathfrak{S}(\mathfrak{G}_p \circ \mathfrak{G}_3) \subseteq \mathfrak{G} \circ \mathfrak{G}_3$, and so $\mathfrak{G}_3$ satisfies the $\mathbb{P}$-boundary condition (II). Now let $G = S_3$, where $S_3$ is the symmetric group of degree 3. Then it is easy to see that $G \in \text{Crit}_\mathfrak{S}(\mathfrak{G}_2 \circ \mathfrak{G}_3)$, but $G \notin \mathfrak{N} \circ \mathfrak{G}_3$. Hence $\text{Crit}_\mathfrak{S}(\mathfrak{G}_2 \circ \mathfrak{G}_3) \not\subseteq \mathfrak{G} \circ \mathfrak{G}_3$.

Firstly, we give a characterization of $\mathfrak{H} \circ \mathfrak{N} \circ \mathfrak{F}$-groups by using their $\mathfrak{H}$-$\mathfrak{F}$-norms.

**Theorem A.** Let $\mathfrak{H}$ be a saturated Fitting formation such that $\mathfrak{G}_{\pi'} \subseteq \mathfrak{H} = \mathfrak{E}\mathfrak{H}$ and $\mathfrak{F}$ a formation such that $\mathfrak{F} \subseteq \mathfrak{S}\mathfrak{F}$. Suppose that one of the following holds:

(i) $G^{\mathfrak{H} \circ \mathfrak{N} \circ \mathfrak{F}} \in \mathfrak{G}_{\pi'}$.

(ii) $\mathfrak{F}$ satisfies the $\pi$-boundary condition (I).

Then the following statements are equivalent:

1. $G \in \mathfrak{H} \circ \mathfrak{N} \circ \mathfrak{F}$.
2. $G/N_{\mathfrak{H},\mathfrak{F}}(G) \in \mathfrak{H} \circ \mathfrak{N} \circ \mathfrak{F}$.
3. $G/N_{\mathfrak{H},\mathfrak{F}}(G) \in \mathfrak{H} \circ \mathfrak{N} \circ \mathfrak{F}$.
4. $N_{\mathfrak{H},\mathfrak{F}}(G/N) > 1$ for every proper normal subgroup $N$ of $G$.
5. $G = N_{\mathfrak{H},\mathfrak{F}}(G)$. 

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The main purpose of this paper is to give answers to Problem (I) and (II). In the universe of all groups, we prove:

**Theorem B.** Let \( \mathfrak{F} \) be a formation such that \( \mathfrak{F} = \mathfrak{SF} \). Then:

1. If \( \mathfrak{F} \) satisfies the \( \pi \)-boundary condition (II), then \( \mathcal{N}_{\pi \mathfrak{F}}(G) = Z_{\pi(\mathfrak{F})}(G) \) holds for every group \( G \).
2. If \( \mathcal{N}_{\pi \mathfrak{F}}(G) = Z_{\pi(\mathfrak{F})}(G) \) holds for every group \( G \), then \( \mathfrak{F} \) satisfies the \( \pi \)-boundary condition (I).
3. \( \mathcal{N}_{\pi \mathfrak{F}}(G) = Z_{\pi(\mathfrak{F})}(G) \) holds for every group \( G \) if and only if \( \mathcal{N}_{\pi \mathfrak{F}}(G) = Z_{\pi(\mathfrak{F})}(G) \) holds for every group \( G \in \bigcup_{\pi \mathfrak{F} \subseteq \mathfrak{F}} (\text{Crit}_{\mathfrak{F}}(\mathfrak{S}_p \circ \mathfrak{F}) \setminus (\mathfrak{S}_p \circ \mathfrak{F})) \).

**Remark 1.6.** The converse of statement (2) of Theorem B does not hold. For example, let \( \pi = \mathbb{P} \) and \( \mathfrak{F} = \mathfrak{U} \). By K. Doerk’s result \( \mathbb{11} \), \( \text{Crit}_{\mathfrak{U}}(\mathfrak{E}) \subseteq \mathfrak{N} \circ \mathfrak{U} \). This means that \( \mathfrak{U} \) satisfies the \( \mathbb{P} \)-boundary condition (I). Let \( A \) be the 2-Frattini module of \( A_5 \), where \( A_5 \) is the alternating group of degree 5. By \( \mathbb{14} \) Example 1], the dimension of \( A \) is 5. Then by \( \mathbb{11} \), Appendix \( \beta \), Proposition \( \beta.5 \], there exists a Frattini extension \( G \) such that \( \mathbb{G}/A \cong A_5 \) and \( A = Z(G) \). Now we show that \( \mathcal{N}_{\mathfrak{U}}(G) = \Phi(G) \). As \( \mathcal{N}_{\mathfrak{U}}(G) < G \), it will suffice to prove that for any subgroup \( H \) of \( G \), \( \Phi(G) \leq N_G(H^{\mathfrak{U}}) \). If \( H/H \cap \Phi(G) \subseteq \mathfrak{U} \), then \( H^{\mathfrak{U}} \leq \Phi(G) \), and so \( \Phi(G) \leq N_G(H^{\mathfrak{U}}) \). Hence, consider that \( H/H \cap \Phi(G) \notin \mathfrak{U} \). Since \( G/\Phi(G) \cong A_5 \), \( H\Phi(G)/\Phi(G) \cong A_4 \), where \( A_4 \) is the alternating group of degree 4. This implies that \( H\Phi(G) \) is a Hall 5'-subgroup of \( G \), and thereby \( H \) is a Hall 5'-subgroup of \( G \). Thus \( \Phi(G) \leq H \), and consequently \( \Phi(G) \leq N_G(H^{\mathfrak{U}}) \). Therefore, \( \mathcal{N}_{\mathfrak{U}}(G) = \Phi(G) \). If \( \mathcal{N}_{\mathfrak{U}}(G) = Z_{\mathfrak{U}_{2\mathcal{U}}}(G) \), then \( Z_{\mathfrak{U}_{2\mathcal{U}}}(G) = \Phi(G) \). Since \( G^{2\mathfrak{U}} = G \), by \( \mathbb{11} \) Chap. IV, Theorem 6.10], \( Z(G) = \Phi(G) \). It follows that \( G \) is quasisimple. By \( \mathbb{13} \), Table 4.1], the Schur multiplier of \( A_5 \) is a cyclic group of order 2, a contradiction. Hence \( \mathcal{N}_{\mathfrak{U}}(G) \neq Z_{\mathfrak{U}_{2\mathcal{U}}}(G) \). Besides, we currently do not know whether the converse of statement (1) of Theorem B is true or not.

**Theorem C.** Let \( \mathfrak{F} \) be a formation such that \( \mathfrak{F} = \mathfrak{SF} \). Then the following statements are equivalent:

1. \( \mathcal{N}_{\pi \mathfrak{F}}(G) = \text{Int}_{\pi \mathfrak{F}}(G) \) holds for every group \( G \).
2. \( Z_{\pi(\mathfrak{F})}(G) = \text{Int}_{\pi(\mathfrak{F})}(G) \) holds for every group \( G \).
3. \( \mathfrak{F} \) satisfies the \( \pi \)-boundary condition (III).

In the universe of all solvable groups, we prove:

**Theorem D.** Let \( \mathfrak{F} \) be a formation such that \( \mathfrak{F} = \mathfrak{SF} \). Then \( \mathcal{N}_{\pi \mathfrak{F}}(G) = Z_{\pi(\mathfrak{F})}(G) \) holds for every group \( G \in \mathfrak{S}_p \circ \mathfrak{F} \).

**Theorem E.** Let \( \mathfrak{F} \) be a formation such that \( \mathfrak{F} = \mathfrak{SF} \). Then the following statements are equivalent:

1. \( \mathcal{N}_{\pi \mathfrak{F}}(G) = \text{Int}_{\pi \mathfrak{F}}(G) \) holds for every \( G \in \mathfrak{S} \).
2. \( Z_{\pi(\mathfrak{F})}(G) = \text{Int}_{\pi(\mathfrak{F})}(G) \) holds for every \( G \in \mathfrak{S} \).
3. \( \mathfrak{F} \) satisfies the \( \pi \)-boundary condition (III) in \( \mathfrak{S} \).
2 Preliminaries

The following two lemmas are well known.

**Lemma 2.1.** Let \( \mathfrak{F} \) be a formation. Suppose that \( H \leq G \) and \( N \leq G \). Then:

1. \( G^\delta N/N = (G/N)^\delta \).

2. If \( \mathfrak{F} = S\mathfrak{F} \) (resp. \( \mathfrak{F} = S_n\mathfrak{F} \)), then \( H^\delta \leq G^\delta \cap H \) (resp. \( N^\delta \leq G^\delta \cap N \)).

**Lemma 2.2.** Let \( \mathfrak{F} \) be a Fitting class. Suppose that \( H \leq G \) and \( N \leq G \). Then:

1. \( G_\delta \cap N = N_\delta \).
2. If \( \mathfrak{F} = S\mathfrak{F} \), then \( G_\delta \cap H \leq H_\delta \).
3. If \( \mathfrak{F} = Q\mathfrak{F} \), then \( G_\delta N/N \leq (G/N)_\delta \).
4. If \( \mathfrak{F} = E\mathfrak{F} \) and \( N \leq G_\delta \), then \( (G/N)_\delta \leq G_\delta /N \).

**Lemma 2.3.** Let \( \mathfrak{F} \) be a Fitting class and \( \mathfrak{F} \) a formation. Suppose that \( H \leq G \) and \( N \leq G \). Then:

1. \( N_\delta,\mathfrak{F}(G) \cap N \leq N_{\delta,\mathfrak{F}}(N) \).
2. If \( \mathfrak{F} = S\mathfrak{F} \), then \( N_{\delta,\mathfrak{F}}(G) \cap H \leq N_{\delta,\mathfrak{F}}(H) \).
3. If \( \mathfrak{F} = Q\mathfrak{F} \), then \( N_{\delta,\mathfrak{F}}(G) N/N \leq N_{\delta,\mathfrak{F}}(G/N) \).
4. If \( \mathfrak{F} = S\mathfrak{F} \) and \( G \in \mathfrak{F} \circ \mathfrak{M} \circ \mathfrak{F} \), then either \( G = 1 \) or \( N_{\delta,\mathfrak{F}}(G) \geq 1 \).

**Proof.**

1. By definition and Lemma 2.2(1), \( N_{\delta,\mathfrak{F}}(G) N/N = (\cap_{H \leq G} N_{G/H}(H^\delta G_\delta)) \cap N \leq \cap_{H \leq G} N_{H^\delta G_\delta}(N) \).

The proof of statement (2) is similar to (1).

3. By definition, Lemma 2.1(1) and Lemma 2.2(3), \( N_{\delta,\mathfrak{F}}(G) N/N \geq (\cap_{H \leq G} N_{G/H}(H^\delta G_\delta)) N/N \leq \cap_{N \leq H \leq G} N_{G/H}(H^\delta G_\delta)/N \leq \cap_{N \leq H \leq G} N_{G/H}(H^\delta G_\delta) N/(G/N)_\delta = N_{\delta,\mathfrak{F}}(G/N) \).

4. We may suppose that \( G > 1 \) and \( G_\delta = 1 \). Since \( G \in \mathfrak{F} \circ \mathfrak{M} \circ \mathfrak{F} \), \( G = G / G_\delta \in \mathfrak{M} \circ \mathfrak{F} \). Then \( G^\delta \in \mathfrak{M} \), and so \( Z(G^\delta) > 1 \). As \( \mathfrak{F} = S\mathfrak{F} \), we have that \( H^\delta \leq G^\delta \) for every subgroup \( H \) of \( G \) by Lemma 2.1(2). It follows that \( N_{\delta,\mathfrak{F}}(G) \geq Z(G^\delta) > 1 \).

**Lemma 2.4.** Let \( f \) be a subgroup functor assigning to every group \( G \) a characteristic subgroup \( f(G) \) of \( G \). Define a subgroup functor \( f_i \) as follows: for every group \( G \), \( f_0(G) = 1 \); \( f_1(G)/f_i-1(G) = f(G/f_i-1(G)) \) for \( i = 1, 2, \ldots \).

1. If \( f(G) N/N \leq f(G/N) \) for every group \( G \) and every normal subgroup \( N \) of \( G \), then \( f_1(G) N/N \leq f_1(G/N) \) for every group \( G \) and every normal subgroup \( N \) of \( G \).

2. If \( f(G) N/N \leq f(G/N) \) and \( f(G) \cap N \leq f(N) \) for every group \( G \) and every normal subgroup \( N \) of \( G \), then \( f_1(G) \cap N \leq f_1(N) \) for every group \( G \) and every normal subgroup \( N \) of \( G \).

3. If \( f(G) N/N \leq f(G/N) \) for every group \( G \) and every normal subgroup \( N \) of \( G \), and \( f(G) \cap H \leq f(H) \) for every group \( G \) and every subgroup \( H \) of \( G \), then \( f_1(G) \cap H \leq f_1(H) \) for every group \( G \) and every subgroup \( H \) of \( G \).

**Proof.**

1. By induction, we may suppose that \( f_{i-1}(G) N/N \leq f_{i-1}(G/N) \). Let \( f_{i-1}(G/N) = A_{i-1}/N \) and \( f_i(G/N) = A_i/N \). Then \( f_{i-1}(G) \leq A_{i-1} \) and \( A_i/A_{i-1} = f(G/A_{i-1}) \). It follows
that \((f_1(G)A_{i-1}/f_{i-1}(G))/A_{i-1}/f_{i-1}(G)) \leq f((G/f_i(G))/(A_{i-1}/f_{i-1}(G))) = (A_i/f_i(G))/(A_{i-1}/f_{i-1}(G))\). Therefore, \(f_i(G) \leq A_i\), and so \(f_i(G)N/N \leq f_i(G/N)\).

(2) By induction, we may assume that \(f_{i-1}(G) \cap N \leq f_{i-1}(N)\). Let \(f_{i-1}(G) \cap N = C_{i-1}\) and \(f_{i}(G) \cap N = C_i\). Then \(f(N/C_{i-1})(f_{i-1}(N)/C_{i-1})/(f_{i-1}(N)/C_{i-1}) \leq f((N/C_{i-1})/(f_{i-1}(N)/C_{i-1})) = (f_i(N)/C_{i-1})/(f_{i-1}(N)/C_{i-1})\). This implies that \(f(N/C_{i-1}) \leq f_i(N)/C_{i-1}\). Clearly, \(C_i f_{i-1}(G)/f_{i-1}(G) = f(G/f_{i-1}(G)) \cap (f_{i-1}(G)N/f_{i-1}(G)) \leq f(f_{i-1}(G)N/f_{i-1}(G))\). It follows that \(C_i/C_{i-1} \leq f(N/C_{i-1}) \leq f_i(N)/C_{i-1}\). Therefore, \(C_i \leq f_i(N)\), and so \(f_i(G) \cap N \leq f_i(N)\).

The proof of statement (3) is similar to (2).

**Lemma 2.5.** Let \(\mathfrak{F}\) be a Fitting class and \(\mathfrak{G}\) a formation. Suppose that \(H \leq G\) and \(N \trianglelefteq G\). Then:

1. If \(\mathfrak{F} = \mathfrak{Q}\mathfrak{F}\), then \(\mathcal{N}_{\mathfrak{F}}(G/N) \leq \mathcal{N}_{\mathfrak{F}}(G/N)\).
2. If \(\mathfrak{F}\) is a Fitting formation such that \(\mathfrak{F} = \mathfrak{S}\mathfrak{F}\), then \(\mathcal{N}_{\mathfrak{F}}(G/N) \leq \mathcal{N}_{\mathfrak{F}}(G/N)\).
3. If \(\mathfrak{F} = \mathfrak{Q}\mathfrak{F}\) and \(N \leq \mathcal{N}_{\mathfrak{F}}(G)\), then \(\mathcal{N}_{\mathfrak{F}}(G/N) = \mathcal{N}_{\mathfrak{F}}(G/N)\).
4. If \(\mathfrak{F} = \mathfrak{Q}\mathfrak{F}\) and \(N \leq \mathcal{N}_{\mathfrak{F}}(G)\), then \(\mathcal{N}_{\mathfrak{F}}(G/N) = \mathcal{N}_{\mathfrak{F}}(G/N)\).
5. If \(\mathfrak{F} = \mathfrak{Q}\mathfrak{F}\), then \(\mathcal{N}_{\mathfrak{F}}(G/N) = \bigcap \{N \mid N \leq G, \mathcal{N}_{\mathfrak{F}}(G/N) = 1\}\).

**Proof.** Statements (1)-(3) directly follow from Lemmas 2.3 and 2.4.

4. By definition and (3), we have that \(\mathcal{N}_{\mathfrak{F}}(G/N)/(\mathcal{N}_{\mathfrak{F}}(G/N)/N) \leq \mathcal{N}_{\mathfrak{F}}(G/N)/(\mathcal{N}_{\mathfrak{F}}(G/N)/N)\) = 1. Therefore, \(\mathcal{N}_{\mathfrak{F}}(G/N) = \mathcal{N}_{\mathfrak{F}}(G/N)/N\).

5. By definition, \(\mathcal{N}_{\mathfrak{F}}(G/N)/\mathcal{N}_{\mathfrak{F}}(G/N) = 1\). On the other hand, if \(N \trianglelefteq G\) such that \(\mathcal{N}_{\mathfrak{F}}(G/N) = 1\), then \(\mathcal{N}_{\mathfrak{F}}(G/N) = 1\). Hence by (3), \(\mathcal{N}_{\mathfrak{F}}(G) \leq N\). Therefore, we have that \(\mathcal{N}_{\mathfrak{F}}(G) = \bigcap \{N \mid N \leq G, \mathcal{N}_{\mathfrak{F}}(G/N) = 1\}\).

**Lemma 2.6.** Let \(\mathfrak{F}\) be a Fitting class and \(\mathfrak{G}\) a formation such that \(\mathfrak{G} \subseteq \mathfrak{S}\). Suppose that \(G_1\) and \(G_2\) are groups with \(|G_1|, |G_2| = 1\). Then \(\mathcal{N}_{\mathfrak{G}}(G_1 \times G_2) = \mathcal{N}_{\mathfrak{G}}(G_1) \times \mathcal{N}_{\mathfrak{G}}(G_2)\) and \(\mathcal{N}_{\mathfrak{G}}(G_1 \times G_2) = \mathcal{N}_{\mathfrak{G}}(G_1) \times \mathcal{N}_{\mathfrak{G}}(G_2)\).

**Proof.** We only need to prove that \(\mathcal{N}_{\mathfrak{G}}(G_1 \times G_2) = \mathcal{N}_{\mathfrak{G}}(G_1) \times \mathcal{N}_{\mathfrak{G}}(G_2)\). Let \(G = G_1 \times G_2\). Since \(|G_1|, |G_2| = 1\), for every subgroup \(H\) of \(G\), we have that \(H = (H \cap G_1) \times (H \cap G_2)\).

By [M] Chap. IV, Theorem 1.18, \(H^\delta = (H \cap G_1)^\delta \times (H \cap G_2)^\delta\). Then it is easy to see that \(G\delta = (G_1)\delta \times (G_2)\delta\), and so \(H^\delta G\delta = (H \cap G_1)^\delta (G_1)\delta \times (H \cap G_2)^\delta (G_2)\delta\). This implies that \(G\delta = \mathcal{N}_{G}(H^\delta G_\delta) = G_\delta((H \cap G_1)(G_1)\delta \times (H \cap G_2)(G_2)\delta)\). Hence \(\mathcal{N}_{\mathfrak{G}}(G) = \mathcal{N}_{G}(H^\delta G_\delta) = \mathcal{N}_{G}(H \cap G_1)^\delta (G_1)\delta \times \mathcal{N}_{G}(H \cap G_2)^\delta (G_2)\delta) = \mathcal{N}_{\mathfrak{G}}(G_1) \times \mathcal{N}_{\mathfrak{G}}(G_2)\).

**Lemma 2.7.** Let \(\mathfrak{G}\) be a formation. Then \(\mathfrak{G}\) satisfies the \(\pi\)-boundary condition (I) if and only if \(\text{Crit}(\mathfrak{G}) \subseteq \mathfrak{S}\).

**Proof.** The necessity is evident. So we only need to prove the sufficiency. Suppose that \(\text{Crit}(\mathfrak{G}) \subseteq \mathfrak{S}\). Let \(G \in \text{Crit}(\mathfrak{G})\). If \(G^\delta \not\leq \Phi(G)\), then there is nothing to prove. We may, therefore, assume that \(G^\delta \not\leq \Phi(G)\). Let \(G^\delta/L\) be a \(G\)-chief factor. Clearly, \(G^\delta/L \in \mathfrak{S}\).

If \(L \not\leq \Phi(G)\), then \(G\) has a maximal subgroup \(M\) such that \(G = LM\). Since \(M \in \mathfrak{G}\), \(G/L \cong M/L \cap M \in \mathfrak{G}\), and so \(G^\delta \leq L\), which is absurd. Hence \(L \leq \Phi(G)\). This implies
that $L = G^\delta \cap \Phi(G)$. Since $G^\delta \Phi(G)/\Phi(G) \cong G^\delta /G^\delta \cap \Phi(G) \in \mathfrak{N}_\pi$, we have that $G^\delta \in \mathfrak{N}_\pi$ by [4, Lemma 3.1]. This shows that $G \in \mathfrak{N}_\pi \circ \mathfrak{F}$, and thus $\text{Crit}_G(\mathfrak{F}) \subseteq \mathfrak{N}_\pi \circ \mathfrak{F}$.

**Lemma 2.8.** Let $\mathfrak{F}$ be a saturated formation and $\pi \subseteq \pi(\mathfrak{F})$. Suppose that $H \leq G$ and $N \leq G$. Then:

1. If $N \leq Z_{\pi(\mathfrak{F})}(G)$, then $Z_{\pi(\mathfrak{F})}(G/N) = Z_{\pi(\mathfrak{F})}(G)/N$.
2. If $\pi(\mathfrak{F}) \cap G \leq \pi(\mathfrak{F}) \cap N \leq \pi(\mathfrak{F})$.
3. If $\mathfrak{F} = \mathfrak{F}_0$ (resp. $\mathfrak{F} = \mathfrak{F}_n$), then $Z_{\pi(\mathfrak{F})}(G) \cap H \leq Z_{\pi(\mathfrak{F})}(H)$ (resp. $Z_{\pi(\mathfrak{F})}(G) \cap N \leq Z_{\pi(\mathfrak{F})}(N)$).
4. If $\mathfrak{G}_{\pi'} \circ \mathfrak{F} = \mathfrak{F}$ and $G/Z_{\pi(\mathfrak{F})}(G) \in \mathfrak{F}$, then $G \in \mathfrak{F}$.
5. If $\mathfrak{F} = \mathfrak{F}_0$ (resp. $\mathfrak{F} = \mathfrak{F}_n$), $\mathfrak{G}_{\pi'} \circ \mathfrak{F} = \mathfrak{F}$ and $H \in \mathfrak{F}$ (resp. $N \in \mathfrak{F}$), then $HZ_{\pi(\mathfrak{F})}(G) \in \mathfrak{F}$ (resp. $NZ_{\pi(\mathfrak{F})}(G) \in \mathfrak{F}$).
6. If $\pi(\mathfrak{F}) \cap G \leq \pi(\mathfrak{F}) \cap N \leq \pi(\mathfrak{F})$.
7. If $\mathfrak{F} = \mathfrak{F}_n$, then $Z_{\pi(\mathfrak{F})}(G) \in \mathfrak{G}_{\pi'} \circ \mathfrak{F}$.

**Proof.** Statement (1) is evident by definition.

Statements (2)-(5) were proved in [16, Lemma 2.2].

(6) Let $\mathfrak{F} = LF(F)$, where $F$ is the canonical local definition of $\mathfrak{F}$. Then by [11, Chap. IV, Theorem 3.13], $\mathfrak{G}_{\pi'} \circ \mathfrak{F} = LF(H)$, where $H(p) = F(p)$ for all $p \in \pi$ and $H(p) = \mathfrak{G}_{\pi'} \circ \mathfrak{F}$ for all $p \in \pi'$. Then by definition, it is easy to see that $Z_{\pi(\mathfrak{F})}(G) = Z_{\pi(\mathfrak{G}_{\pi'}, \mathfrak{F}_n)}(G)$.

Statement (7) follows from (5) and (6).

**Remark 2.9.** Note that there exist several minor mistakes in [16]. In [16, Lemmas 2.2(6) and 2.2(7)] and [16, Lemma 2.4(g)], “$\mathfrak{G}_{\pi'} \circ \mathfrak{F} = \mathfrak{F}$” should be corrected as “$\mathfrak{G}_{\pi'} \circ \mathfrak{F} = \mathfrak{F}'$”; and in [16, Lemma 2.2(5)], “$Z_{\pi(\mathfrak{F})}(H) \cap A$” should be corrected as “$Z_{\pi(\mathfrak{F})}(A) \cap H$”.

**Lemma 2.10.** [27, Lemma 2.5] Let $\mathfrak{F} = LF(F)$ be a saturated formation, where $F$ is the canonical local definition of $\mathfrak{F}$, and $E$ a normal $p$-subgroup of $G$. If $E \leq Z_{\mathfrak{F}}(G)$, then $G/C_G(E) \in F(p)$.

**Lemma 2.11.** Let $\mathfrak{F}$ be a formation and $\mathfrak{B} = \mathfrak{N}_\pi \circ \mathfrak{F}$. Then:

1. $\mathfrak{B} = LF(b)$ with $b(p) = \mathfrak{F}$ for all $p \in \pi$ and $b(p) = \mathfrak{N}_\pi \circ \mathfrak{F}$ for all $p \in \pi'$.
2. The canonical local definition $B$ of $\mathfrak{B}$ can be defined as follows: $B(p) = \mathfrak{G}_p \circ \mathfrak{F}$ for all $p \in \pi$ and $B(p) = \mathfrak{N}_\pi \circ \mathfrak{F}$ for all $p \in \pi'$.

**Proof.** Statement (1) directly follows from [25, Lemma 1], and Statement (2) follows from [11, Chap. IV, Lemma 3.13].

**Lemma 2.12.** Let $\mathfrak{F}$ be a formation. Then:

1. $Z_{\pi(\mathfrak{F})}(G) = 1$ if and only if $C_G(G^\mathfrak{F}) = 1$ and $O_{\pi'}(G) = 1$.
2. $Z_{\pi(\mathfrak{F})}(G) \cap G^\mathfrak{F} = Z_{\pi(\mathfrak{F})}(G^\mathfrak{F})$.
3. $Z_{\pi(\mathfrak{F})}(G)/Z_{\pi(\mathfrak{F})}(G^\mathfrak{F}) = Z_{\pi(\mathfrak{F})}(G/Z_{\pi(\mathfrak{F})}(G^\mathfrak{F}))$.

**Proof.** (1) Suppose that $C_G(G^\mathfrak{F}) = 1$ and $O_{\pi'}(G) = 1$. If $Z_{\pi(\mathfrak{F})}(G) > 1$, then let $N$ be a minimal normal subgroup of $G$ contained in $Z_{\pi(\mathfrak{F})}(G)$. Clearly, $N$ is not a $\pi'$-group. Then by Lemma 2.11(1), we have that $G/C_G(N) \in \mathfrak{F}$, and so $N \leq C_G(G^\mathfrak{F}) = 1$, a contradiction.
Thus \( Z_{\pi(p)}(G) = 1 \). Now assume that \( Z_{\pi(p)}(G) = 1 \). Then clearly, \( O_{\pi'}(G) = 1 \). Suppose that \( C_G(G^\pi) > 1 \), and let \( N \) be a minimal normal subgroup of \( G \) contained in \( C_G(G^\pi) \). Then \( G^\pi \leq C_G(N) \) and \( N \) is not a \( \pi'\)-group. Hence by Lemma 2.11(1) again, \( N \leq Z_{\pi(p)}(G) \), which is impossible. Therefore, \( C_G(G^\pi) = 1 \).

(2) Firstly, we prove that \( Z_{\pi(p)}(G^\pi) \leq Z_{\pi(p)}(G) \). If \( Z_{\pi(p)}(G) > 1 \), then by induction, \( Z_{\pi(p)}((G/Z_{\pi(p)}(G))(G^\pi)) \leq Z_{\pi(p)}(G/Z_{\pi(p)}(G)) = 1 \). By Lemmas 2.1(1) and 2.8(2), \( Z_{\pi(p)}(G^\pi) \leq Z_{\pi(p)}(G) \). We may, therefore, assume that \( Z_{\pi(p)}(G) = 1 \). Then by (1), \( C_G(G^\pi) = 1 \) and \( O_{\pi'}(G) = 1 \). It follows that \( Z(G^\pi) = 1 \) and \( O_{\pi'}(G^\pi) = 1 \). By (1) again, \( Z_{\pi(p)}(G^\pi) = 1 \). Consequently, \( Z_{\pi(p)}(G^\pi) \leq Z_{\pi(p)}(G) \).

Suppose that \( Z_{\pi(p)}(G^\pi) > 1 \). Then by induction and Lemma 2.1(1), \( Z_{\pi(p)}(G/Z_{\pi(p)}(G^\pi)) \cap (G^\pi/Z_{\pi(p)}(G^\pi)) = Z_{\pi(p)}(G^\pi/Z_{\pi(p)}(G^\pi)) = 1 \). Hence by Lemma 2.8(1), \( Z_{\pi(p)}(G) \cap G^\pi = Z_{\pi(p)}(G^\pi) \). We may, therefore, assume that \( Z_{\pi(p)}(G^\pi) = 1 \). Then by (1), \( Z(G^\pi) = 1 \) and \( O_{\pi'}(G^\pi) = 1 \). If \( Z_{\pi(p)}(G) \cap G^\pi > 1 \), then let \( N \) be a minimal normal subgroup of \( G \) contained in \( Z_{\pi(p)}(G) \cap G^\pi \). Since \( O_{\pi'}(G^\pi) = 1 \), \( N \) is not a \( \pi'\)-group. It follows from Lemma 2.11(1) that \( G/C_G(N) \in \mathfrak{F} \), and so \( N \leq Z(G^\pi) \), a contradiction. Therefore, \( Z_{\pi(p)}(G) \cap G^\pi = 1 \).

(3) If \( Z_{\pi(p)}(G^\pi) > 1 \), then by induction, \( Z_{\pi(p)}(G/Z_{\pi(p)}(G^\pi)) = Z_{\pi\pi}(G/Z_{\pi(p)}(G^\pi)) \). Hence by (2) and Lemma 2.8(1), \( Z_{\pi(p)}(G)/Z_{\pi(p)}(G^\pi) = Z_{\pi\pi}(G/Z_{\pi(p)}(G^\pi)) \). We may, therefore, assume that \( Z_{\pi(p)}(G^\pi) = 1 \). Then by (2), \( Z_{\pi(p)}(G) \cap G^\pi = 1 \), and so \( Z_{\pi(p)}(G) \leq C_G(G^\pi) \). By [11] Chap. IV, Theorem 6.13, \( C_G(G^\pi) = Z_{\pi\pi}(G) \leq Z_{\pi\pi}(G) \leq Z_{\pi(p)}(G) \). This implies that \( Z_{\pi(p)}(G) = Z_{\pi\pi}(G) \).

Lemma 2.13. [27] Lemma 2.10] Let \( \mathfrak{F} = LF(F) \) be a saturated formation with \( p \in \pi(\mathfrak{F}) \), where \( F \) is the canonical local definition of \( \mathfrak{F} \). Suppose that \( G \) is a group of minimal order in the set of all groups \( G \in \text{Crit}_S(F(p)) \) and \( G \notin \mathfrak{F} \). Then \( G^\pi \) is the unique minimal normal subgroup of \( G \) and \( O_{\pi}(G) = \Phi(G) = 1 \).

3 Proofs of Main Results

Lemma 3.1. Let \( \mathfrak{H} \) be a saturated Fitting formation such that \( \mathfrak{H}^\pi \subseteq \mathfrak{H} = E\mathfrak{H} \) and \( \mathfrak{F} \) a formation. Then \( N_{\pi\pi}(G) \in \mathfrak{H} \cap \mathfrak{F} \) if one of the following holds:

(i) \( \mathfrak{F} = S_{\pi} \mathfrak{F} \) and \( G^{\delta_{\pi\pi\pi}} \in \mathfrak{S}_{\pi} \).

(ii) \( \mathfrak{F} \) satisfies the \( \pi \)-boundary condition (I).

Proof. Assume that the result is false and let \( G \) be a counterexample of minimal order. Note that if the condition (i) holds, then since \( \mathfrak{H} \cap \mathfrak{F} \cap \mathfrak{F} = S_{\pi}(\mathfrak{H} \cap \mathfrak{F} \cap \mathfrak{F}) \), we have that \( (N_{\pi\pi\pi}(G))^{\delta_{\pi\pi\pi\pi}} \leq G^{\delta_{\pi\pi\pi\pi}} \in \mathfrak{S}_{\pi} \) by Lemma 2.1(2). Hence the condition (i) holds for \( N_{\pi\pi\pi}(G) \) when the condition (i) holds for \( G \). If \( N_{\pi\pi\pi}(G) < G \), then by the choice of \( G \) and Lemma 2.5(1), \( N_{\pi\pi\pi}(G) = N_{\pi\pi\pi}(N_{\pi\pi\pi}(G)) \in \mathfrak{H} \cap \mathfrak{F} \cap \mathfrak{F} \), a contradiction. We may, therefore, assume that \( N_{\pi\pi\pi}(G) = G \). Let \( N \) be any minimal normal subgroup of \( G \). Then by Lemma 2.1(1), the condition (i) holds for \( G/N \) when the condition (i) holds for \( G \). Hence by the choice of \( G \) and Lemma 2.5(3), \( G/N = N_{\pi\pi\pi}(G/N) \in \mathfrak{H} \cap \mathfrak{F} \cap \mathfrak{F} \). Clearly, \( \mathfrak{H} \cap \mathfrak{F} \cap \mathfrak{F} \) is a saturated
formation by \[11\] Chap. IV, Theorem 4.8]. This implies that $N$ is the unique minimal normal subgroup of $G$.

If $G_\delta > 1$, then $N \leq G_\delta$. By Lemmas 2.2(3) and 2.2(4), $(G/N)_\delta = G_\delta/N$. Since $G/N \in \mathfrak{H} \circ \mathfrak{N} \circ \mathfrak{F}$, $G/G_\delta \in \mathfrak{N} \circ \mathfrak{F}$, and thus $G \in \mathfrak{H} \circ \mathfrak{N} \circ \mathfrak{F}$, a contradiction. Therefore, $G_\delta = 1$, and so $O_{\pi'}(G) = 1$. If $N \notin \Phi(G)$, then $G \in \mathfrak{H} \circ \mathfrak{N} \circ \mathfrak{F}$, which is impossible. Hence $N \notin \Phi(G)$. It follows that $G$ has a maximal subgroup $M$ such that $N \nsubseteq M$. Since $\mathcal{N}_{\delta, \delta}(G) > 1$ and $N$ is the unique minimal normal subgroup of $G$, we have that $N \leq \mathcal{N}_{\delta, \delta}(G)$. Then by the definition of $\mathcal{N}_{\delta, \delta}(G)$, $N \leq N_G(M^\delta)$. This induces that $M^\delta \leq G$. Hence $M^\delta = 1$, and so $M \in \mathfrak{F}$. It follows that $G/N \cong M/N \cap M \in \mathfrak{F}$, and thereby $G^\delta \leq N$. Since $1 < G^{G_\delta \circ \rho_\pi \delta} \leq G^\delta \leq N$, $N = G^{G_\delta \circ \rho_\pi \delta} = C^\delta$.

We claim that $N \in \mathfrak{N}$. If the condition (i) holds, then $N \in \mathfrak{S}_\pi$. As $O_{\pi'}(G) = 1$, $N \in \mathfrak{N}$. Now assume that the condition (ii) holds. Then since $G \notin \mathfrak{F}$, we may take a subgroup $K$ of $G$ such that $K \in \text{Crit}_\mathfrak{F}(\mathfrak{F}) \subseteq \mathfrak{N}_\pi \circ \mathfrak{F}$. If $N \notin \mathfrak{N}$, then $C_G(N) = 1$. Since $N \leq \mathcal{N}_{\delta, \delta}(G)$ and $G_\delta = 1$, we have that $N \leq N_G(K^\delta)$, and so $N \cap K^\delta \leq N$. As $K^\delta \in \mathfrak{N}_\pi$ and $O_{\pi'}(G) = 1$, $K^\delta \in \mathfrak{N}$. By \[11\] Chap. A, Proposition 4.13(b)], $N \cap K^\delta = 1$. It follows that $K^\delta \leq C_G(N) = 1$, and thus $K \in \mathfrak{F}$, a contradiction. Hence $N \in \mathfrak{N}$. Therefore, our claim holds. This induces that $G \in \mathfrak{N} \circ \mathfrak{F} \subseteq \mathfrak{F} \circ \mathfrak{N} \circ \mathfrak{F}$. The final contradiction completes the proof.

**Proof of Theorem A.** It is obvious that (1) implies (2) and (2) implies (3). Suppose that (3) holds, that is, $G/\mathcal{N}_{\delta, \delta}^\infty(G) \in \mathfrak{H} \circ \mathfrak{N} \circ \mathfrak{F}$. If $\mathcal{N}_{\delta, \delta}(G/N) = 1$ for some proper normal subgroup $N$ of $G$, then by Lemma 2.5(5), $\mathcal{N}_{\delta, \delta}^\infty(G) \leq N$, and so $G/N \in \mathfrak{H} \circ \mathfrak{N} \circ \mathfrak{F}$. Hence by Lemma 2.3(4), either $G = N$ or $\mathcal{N}_{\delta, \delta}(G/N) > 1$, a contradiction. This induces that (3) implies (4). Now assume that (4) holds. Then since $\mathcal{N}_{\delta, \delta}(G/\mathcal{N}_{\delta, \delta}^\infty(G)) = 1$, we have that $G = \mathcal{N}_{\delta, \delta}^\infty(G)$. Hence (4) implies (5). Finally, by Lemma 3.1, we get that (5) implies (1). This finishes the proof of the theorem.

Since $\mathfrak{N}_\pi = \mathfrak{S}_\pi \circ \mathfrak{N} = \mathfrak{S}_\pi \circ \mathfrak{N}$, the next corollary directly follows from Theorem A, which is also a generalization of \[29\] Theorem A] and \[29\] Theorem B].

**Corollary 3.2.** Let $\mathfrak{F}$ be a formation such that $\mathfrak{F} = \mathfrak{S}_\pi \circ \mathfrak{F}$. Suppose that one of the following holds:

(i) $G \in \mathfrak{S}_\pi \circ \mathfrak{F}$.

(ii) $\mathfrak{F}$ satisfies the $\pi$-boundary condition (I).

Then the following statements are equivalent:

(1) $G \in \mathfrak{N}_\pi \circ \mathfrak{F}$.

(2) $G/\mathcal{N}_{\pi, \delta}^\infty(G) \in \mathfrak{N}_\pi \circ \mathfrak{F}$.

(3) $G/\mathcal{N}_{\pi, \delta}^\infty(G) \in \mathfrak{N}_\pi \circ \mathfrak{F}$.

(4) $\mathcal{N}_{\pi, \delta}^\infty(G/N) > 1$ for every proper normal subgroup $N$ of $G$.

(5) $G = \mathcal{N}_{\pi, \delta}^\infty(G)$.

In the sequel of this section, we restrict our attention to $\pi \mathfrak{F}$-norms.
Lemma 3.3. Let \( \mathfrak{F} \) be a formation such that \( \mathfrak{F} = S\mathfrak{F} \). Then \( Z_{\pi(\mathfrak{G})}(G) \leq N_{\pi(\mathfrak{G})}^\infty(G) \).

Proof. If \( N_{\pi(\mathfrak{G})}(G) > 1 \), then by induction, \( Z_{\pi(\mathfrak{G})}(G/N_{\pi(\mathfrak{G})}^\infty(G)) \leq N_{\pi(\mathfrak{G})}^\infty(G/N_{\pi(\mathfrak{G})}^\infty(G)) = 1 \). By Lemma 2.8(2), \( N_{\pi(\mathfrak{G})}(G)N_{\pi(\mathfrak{G})}^\infty(G)/N_{\pi(\mathfrak{G})}^\infty(G) = 1 \), and so \( Z_{\pi(\mathfrak{G})}(G) \leq N_{\pi(\mathfrak{G})}^\infty(G) \). Hence we may assume that \( N_{\pi(\mathfrak{G})}(G) = 1 \). Since \( \mathfrak{F} = S\mathfrak{F} \), \( H^\mathfrak{F} \leq G^\mathfrak{F} \) for every subgroup \( H \) of \( G \) by Lemma 2.1(2), and thereby \( C_G(G^\mathfrak{F})O_{\pi}(G) \leq N_G(H^\mathfrak{F}O_{\pi}(G)) \). This implies that \( C_G(G^\mathfrak{F}) = 1 \) and \( O_{\pi}(G) = 1 \). Then by Lemma 2.12(1), \( 1 = Z_{\pi(\mathfrak{G})}(G) \leq N_{\pi(\mathfrak{G})}^\infty(G) \).

Proofs of Theorem B(1) and Theorem D. We need to prove that if either \( G \in \mathfrak{S}_\pi \circ \mathfrak{F} \) or \( \mathfrak{F} \) satisfies the \( \pi \)-boundary condition (II), then \( N_{\pi(\mathfrak{G})}^\infty(G) = Z_{\pi(\mathfrak{G})}(G) \). Suppose that the result is false and let \( L \) be a counterexample of minimal order. By Lemma 3.3, \( Z_{\pi(\mathfrak{G})}(L) \leq N_{\pi(\mathfrak{G})}^\infty(L) \). We may, therefore, assume that \( N_{\pi(\mathfrak{G})}(L) > 1 \). If \( N_{\pi(\mathfrak{G})}(L) > 1 \), then by the choice of \( L \) and Lemma 2.8(1), \( N_{\pi(\mathfrak{G})}^\infty(L/Z_{\pi(\mathfrak{G})}(L)) = Z_{\pi(\mathfrak{G})}(L/Z_{\pi(\mathfrak{G})}(L)) = 1 \). Hence by Lemma 2.5(4), \( N_{\pi(\mathfrak{G})}^\infty(L) = Z_{\pi(\mathfrak{G})}(L) \), a contradiction. Therefore, \( Z_{\pi(\mathfrak{G})}(L) = 1 \), and thereby \( O_{\pi}(L) = 1 \).

Now let \( N \) be any minimal normal subgroup of \( L \) contained in \( N_{\pi(\mathfrak{G})}(L) \). If \( L \) has a minimal normal subgroup \( R \) which is different from \( N \), then by the choice of \( L \), \( N_{\pi(\mathfrak{G})}^\infty(L/R) = Z_{\pi(\mathfrak{G})}(L/R) \). It follows from Lemma 2.5(3) that \( NR/R \leq N_{\pi(\mathfrak{G})}^\infty(L/R) = Z_{\pi(\mathfrak{G})}(L/R) \). By \( L \)-isomorphism \( N \cong NR/R \), we have that \( N \leq Z_{\pi(\mathfrak{G})}(L) \), which is absurd. Thus \( N \) is the unique minimal normal subgroup of \( L \). If \( N \not\leq \phi(L) \), then \( L \) has a maximal subgroup \( M \) such that \( N \not\leq M \). Since \( N \leq N_{\pi(\mathfrak{G})}(L) \), \( N \leq N_{\pi(\mathfrak{G})}(L) \in \mathfrak{F} \). Note that \( \mathfrak{F} \) satisfies the \( \pi \)-boundary condition (I) if \( \mathfrak{F} \) satisfies the \( \pi \)-boundary condition (II). By Corollary 3.2, we have that \( L \in \mathfrak{N}_\pi \circ \mathfrak{F} \). This induces that \( L = Z_{\pi(\mathfrak{G})}(L) = Z_{\pi(\mathfrak{G})}(L) \) by Lemma 2.8(6), a contradiction. Hence \( N \not\leq \phi(L) \), and so \( N \) is an elementary abelian \( p \)-group with \( p \in \pi \). Let \( M \) be any maximal subgroup of \( L \). By the choice of \( L \), \( N_{\pi(\mathfrak{G})}^\infty(M) = Z_{\pi(\mathfrak{G})}(M) \). Then by Lemma 2.5(2), \( N \leq N_{\pi(\mathfrak{G})}^\infty(M) \cap M \leq N_{\pi(\mathfrak{G})}^\infty(M) = Z_{\pi(\mathfrak{G})}(M) \). Thus \( N \leq Z_{\pi(\mathfrak{G})}(M) \). By Lemmas 2.10 and 2.11(2), \( M/C_M(N) \in \mathfrak{S}_p \circ \mathfrak{F} \). If \( C_L(N) \not\leq M \), then \( L = C_L(N) \), and so \( L/C_L(N) \cong M/C_M(N) \in \mathfrak{S}_p \circ \mathfrak{F} \). This shows that \( N \leq Z_{\pi(\mathfrak{G})}(L) \) by Lemma 2.11(2), which is impossible. Hence \( C_L(N) \leq M \), and thereby \( C_L(N) \leq \phi(L) \). Since \( N \) is the unique minimal normal subgroup of \( L \), \( \phi(L) \) is a \( p \)-subgroup of \( L \). Therefore, \( C_L(N) \) is also a \( p \)-subgroup of \( L \). This implies that \( M \in \mathfrak{S}_p \circ \mathfrak{F} \). If \( L \in \mathfrak{S}_p \circ \mathfrak{F} \), then \( L \in \mathfrak{N}_\pi \circ \mathfrak{F} \), a contradiction as above. Hence \( L \in \text{Crit}_\mathfrak{F}(\mathfrak{S}_p \circ \mathfrak{F}) \). Then, in both cases, \( L \in \mathfrak{S}_\pi \circ \mathfrak{F} \).

Let \( F_p(L) \) be the \( p \)-Fitting subgroup of \( L \), that is, the \( \mathfrak{N}_p \)-radical of \( L \). As \( N \) is the unique minimal normal subgroup of \( L \), we have that \( O_{\pi}(L) = 1 \), and so \( F_p(L) = O_p(L) \). By [11 Ch. A, Theorem 13.8(4)], \( F_p(L) \leq C_L(N) \leq \phi(L) \). This induces that \( F_p(L) = \phi(L) \). Since \( L \in \mathfrak{S}_\pi \circ \mathfrak{F} \), \( L^\mathfrak{F} \in \mathfrak{S}_\pi \). If \( L^\mathfrak{F} \not\leq \phi(L) \), then \( L \in \mathfrak{S}_p \circ \mathfrak{F} \), which is absurd. Thus \( L^\mathfrak{F} \not\leq \phi(L) \). Let \( A/\phi(L) \) be an \( L \)-chief factor contained in \( L^\mathfrak{F}\phi(L)/\phi(L) \). Then \( A/\phi(L) \in \mathfrak{N}_\pi \), and so \( A/\phi(L) \in \mathfrak{N}_p \). Hence by [14 Lemma 3.1], we have that \( A \in \mathfrak{N}_p \). This implies that \( A \leq F_p(L) = \phi(L) \), a contradiction. The proof is thus completed.
Proofs of Theorems B(2) and B(3). (2) Suppose that \( N_{\pi \delta}^\infty(G) = Z_{\pi(\delta_0 \delta)}(G) \) holds for every group \( G \). Obviously, for any group \( G \in \text{Crit}_{G} \), \( G = N_{\pi \delta}^\infty(G) = N_{\pi \delta}^\infty(G) \). It follows that \( G = Z_{\pi(\delta_0 \delta)}(G) \), and so \( G \in \mathfrak{N}_\pi \circ \mathfrak{F} \) by Lemma 2.8(7). Hence \( \mathfrak{F} \) satisfies the \( \pi \)-boundary condition (I).

(3) The necessity is obvious. So we only need to prove the sufficiency. For any group \( G \in \text{Crit}_{G} \) and any \( p \in \pi \), either \( G \in \mathfrak{G}_\pi \circ \mathfrak{F} \) or \( G \in \text{Crit}_{G}(\mathfrak{G}_\pi \circ \mathfrak{F}) \). In the former case, \( G \in \mathfrak{N}_\pi \circ \mathfrak{F} \) by Lemma 2.7. In the latter case, a same discussion as in the proof of (2) shows that \( G \in \mathfrak{N}_\pi \circ \mathfrak{F} \). Therefore, \( \mathfrak{F} \) satisfies the \( \pi \)-boundary condition (I). The rest of the proof is similar to the proofs of Theorem B(1) and Theorem D.

Corollary 3.4. Let \( \mathfrak{F} \) be a formation such that \( \mathfrak{F} = \mathfrak{S}_\mathfrak{F} \). Suppose that one of the following holds:

(i) \( G \in \mathfrak{G}_\pi \circ \mathfrak{F} \).

(ii) \( \mathfrak{F} \) satisfies the \( \pi \)-boundary condition (II).

Then \( N_{\pi \delta}^\infty(G)/Z_{\pi(\delta)}(G) = N_{\pi \delta}^\infty(G)/Z_{\pi(\delta_0 \delta)}(G) \). In particular, if \( \mathfrak{F} \) is saturated, then \( N_{\pi \delta}^\infty(G)/Z_{\pi(\delta)}(G) = N_{\pi \delta}^\infty(G)/Z_{\pi(\delta_0 \delta)}(G) \).

Proof. Obviously, \( Z_{\pi(\delta_0 \delta)}(G) \) is 1. By induction, Lemma 2.5(4) and Lemma 2.8(1), we may assume that \( Z_{\pi(\delta)}(G) = 1 \). Then by Lemma 2.12(2), \( Z_{\pi(\delta_0 \delta)}(G) \cap G = 1 \), and thus \( Z_{\pi(\delta_0 \delta)}(G) \leq C_G(G) \). Since \( \mathfrak{F} = \mathfrak{S}_\mathfrak{F} \), \( H \leq G \) for every subgroup \( H \) of \( G \), and so \( C_G(G) \leq N_{\pi \delta}^\infty(G) \). Hence \( Z_{\pi(\delta_0 \delta)}(G) \leq N_{\pi \delta}^\infty(G) \).

Lemma 3.5. Let \( \mathfrak{F} \) be a formation such that \( \mathfrak{F} = \mathfrak{S}_\mathfrak{F} \). Then \( N_{\pi \delta}^\infty(G) \leq \text{Int}_{\pi \delta}(G) \) if one of the following holds:

(i) \( G \in \mathfrak{G}_\pi \circ \mathfrak{F} \).

(ii) \( \mathfrak{F} \) satisfies the \( \pi \)-boundary condition (I).

Proof. Let \( H \) be any subgroup of \( G \) such that \( H \in \mathfrak{N}_\pi \circ \mathfrak{F} \). Then we only need to prove that \( HN_{\pi \delta}^\infty(G) \in \mathfrak{N}_\pi \circ \mathfrak{F} \). By Lemma 2.5(2), we have that \( N_{\pi \delta}^\infty(G) \leq N_{\pi \delta}^\infty(HN_{\pi \delta}^\infty(G)) \). It follows that \( HN_{\pi \delta}^\infty(G)/N_{\pi \delta}^\infty(HN_{\pi \delta}^\infty(G)) \cong (HN_{\pi \delta}^\infty(G)/N_{\pi \delta}^\infty(G))/N_{\pi \delta}^\infty(G)(HN_{\pi \delta}^\infty(G))/N_{\pi \delta}^\infty(G) \in \mathfrak{N}_\pi \circ \mathfrak{F} \).

Proof of Theorem C. By Lemma 2.11(2), the canonical local definition \( F \) of \( \mathfrak{N}_\pi \circ \mathfrak{F} \) can be defined as follows: \( F(p) = \mathfrak{G}_p \circ \mathfrak{F} \) for all \( p \in \pi \); \( F(p) = \mathfrak{N}_\pi \circ \mathfrak{F} \) for all \( p \in \pi' \). Note that \( Z_{\pi(\delta_0 \delta)}(G) = Z_{\pi(\delta)}(G) \) by Lemma 2.8(6). Then by [13, Theorem A], (2) is equivalent to (3).

Next we show that (1) is equivalent to (3). Suppose that (3) holds, that is, \( \mathfrak{F} \) satisfies the \( \pi \)-boundary condition (III). Then clearly, \( \mathfrak{F} \) satisfies the \( \pi \)-boundary condition (I). Therefore, for every group \( G \), we have that \( N_{\pi \delta}^\infty(G) \leq \text{Int}_{\pi \delta}(G) \) by Lemma 3.5. Since (2) is equivalent to (3), \( \text{Int}_{\pi \delta}(G) = Z_{\pi(\delta_0 \delta)}(G) \) by Lemma 3.3. Consequently,
\( \mathcal{N}_\pi^\infty(G) = \text{Int}_{\mathfrak{N}_\pi}(G) \) holds for every group \( G \), and so (3) implies (1).

Now suppose that \( \mathcal{N}_\pi^\infty(G) = \text{Int}_{\mathfrak{N}_\pi}(G) \) holds for every group \( G \), and there exists a prime \( p \in \pi \) such that \( \text{Crit}_\pi(\mathfrak{G}_p \circ \mathfrak{F}) \not\subseteq \mathfrak{N}_\pi \circ \mathfrak{F} \). Let \( L \) be a group of minimal order in the set of all groups \( G \in \text{Crit}_\pi(\mathfrak{G}_p \circ \mathfrak{F}) \setminus (\mathfrak{N}_\pi \circ \mathfrak{F}) \). Then by Lemma 2.13, \( L^\mathfrak{N}_\pi \circ \mathfrak{F} \) is the unique minimal normal subgroup of \( L \) and \( O_p(L) = \Phi(L) = 1 \). Hence by [11] Chap. B, Theorem 10.3], there exists a simple \( \mathbb{F}_pL \)-module \( P \) which is faithful for \( L \). Let \( V = P \rtimes L \). For any subgroup \( H \) of \( V \) such that \( H \in \mathfrak{N}_\pi \circ \mathfrak{F} \), if \( PH = V \), then \( P \cap H \leq V \). This implies that \( P \cap H = 1 \) for \( P \) is a simple \( \mathbb{F}_pL \)-module, and so \( H \not\leq \mathfrak{N}_\pi \circ \mathfrak{F} \), a contradiction. Hence \( PH < V \). Then clearly, \( PH \cap L < L \), and thus \( PH/P = P(PH \cap L)/P \cong PH \cap L \in \mathfrak{G}_p \circ \mathfrak{F} \). It follows that \( PH \in \mathfrak{G}_p \circ \mathfrak{F} \subseteq \mathfrak{N}_\pi \circ \mathfrak{F} \). Therefore, \( P \leq \text{Int}_{\mathfrak{N}_\pi}(V) = \mathcal{N}_\pi^\infty(V) \). If \( P \not\leq \mathcal{N}_\pi(\mathfrak{F}) \), then \( P \leq \mathcal{N}_\pi(\mathfrak{F}) \). This induces that \( \mathcal{N}_\pi(\mathfrak{F}) = 1 \). Hence by [11] Chap. B, Theorem 10.3, there exists an \( \mathbb{F}_pL \)-module, and so \( \mathcal{N}_\pi(\mathfrak{F}) \subseteq \mathfrak{N}_\pi \circ \mathfrak{F} \), a contradiction. Hence \( \mathcal{N}_\pi(\mathfrak{F}) = 1 \). Therefore, \( L \in \mathfrak{F} \), a contradiction. This shows that (1) implies (3). Consequently, (1) is equivalent to (3). The theorem is thus proved.

**Proof of Theorem E.** We can prove the theorem similarly as in the proof of Theorem C by using [16] Theorem 4.22.

Now we give some conditions under which the formations satisfy the \( \mathbb{P} \)-boundary condition (I) (resp. the \( \mathbb{P} \)-boundary condition (II), the \( \mathbb{P} \)-boundary condition (III) in \( \mathfrak{G} \)). Recall that if \( \sigma \) denotes a linear ordering on \( \mathbb{P} \), then a group \( G \) is called a Sylow tower group of complexion (or type) \( \sigma \) if there exists a series of normal subgroups of \( G \): \( 1 = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_n = G \) such that \( G_i/G_{i-1} \) is a Sylow \( p_i \)-subgroup of \( G_j/G_{i-1} \) for \( 1 \leq i \leq n \), where \( p_1 < p_2 < \cdots < p_n \) is the ordering induced by \( \sigma \) on the distinct prime divisors of \( |G| \). Let \( \mathfrak{S}_\sigma \) denote the class of all Sylow tower groups of complexion \( \sigma \). By [11] Chap. IV, Example 3.4(g)], \( \mathfrak{S}_\pi \) is a saturated formation. Also, a formation \( \mathfrak{F} \) is said to be a \( \mathcal{S} \)-formation (or have the Shemetkov property) if \( \text{Crit}_\mathcal{S}(\mathfrak{F}) \subseteq \text{Crit}_\sigma(\mathfrak{N}) \cup \{ \text{cyclic groups of prime order} \} \). Clearly, \( \mathfrak{N}_\pi \) is a \( \mathcal{S} \)-formation. For details and more examples, see [15] Section 3.5. Moreover, a group \( G \) is said to be \( \pi \)-closed if \( G \) has a normal Hall \( \pi \)-subgroup. Let \( \mathfrak{C}_\pi \) denote the formation of all \( \pi \)-closed groups.

**Proposition 3.6.** A formation \( \mathfrak{F} \) satisfies the \( \mathbb{P} \)-boundary condition (I) if one of the following holds:

1. \( \mathfrak{F} \subseteq \mathfrak{S}_\sigma \).
2. \( \mathfrak{F} \) is a \( \mathcal{S} \)-formation.
3. \( \mathfrak{F} \subseteq \mathfrak{C}_2 \).
4. \( \mathfrak{F} \subseteq \mathfrak{N}_2 \).

**Proof.** (1) By [21], Theorem 8], \( \text{Crit}_\mathcal{S}(\mathfrak{S}_\sigma) \subseteq \mathfrak{G} \), and so \( \text{Crit}_\mathcal{S}(\mathfrak{F}) \subseteq \mathfrak{S}_\sigma \cup \text{Crit}_\mathcal{S}(\mathfrak{S}_\sigma) \subseteq \mathfrak{G} \).

Statement (2) is clear by definition.

(3) Note that \( \mathfrak{C}_2 \) is a \( \mathcal{S} \)-formation by [28], Remark]. Then by Feit-Thompson Theorem,
Proposition 3.7. A formation \( \mathfrak{F} \) satisfies the \( \mathbb{P} \)-boundary condition (II) if one of the following holds:

(1) \( \mathfrak{F} \subseteq \mathfrak{N} \).

(2) \( \mathfrak{F} \subseteq \mathfrak{S}_2 \) (equivalently, \( 2 \notin \pi(\mathfrak{F}) \)).

Proof. (1) By [17, Chap. IV, Satz 5.4], for any \( p \in \mathbb{P} \), \( \text{Crit}_s(\mathfrak{S}_p \circ \mathfrak{N}) = \text{Crit}_s(\mathfrak{N}_p) \subseteq \text{crit}_s(\mathfrak{N}) \subseteq \mathfrak{S} \). Hence \( \text{Crit}_s(\mathfrak{S}_p \circ \mathfrak{N}) \subseteq \mathfrak{N}_p \cup \text{Crit}_s(\mathfrak{N}_p) \subseteq \mathfrak{S} \).

(2) Note that by [28, Remark], \( \text{Crit}_s(\mathfrak{S}_2 \circ \mathfrak{S}_2) = \text{Crit}_s(\mathfrak{S}_2) \subseteq \mathfrak{S} \), and for any odd prime \( p \), \( \text{Crit}_s(\mathfrak{S}_p \circ \mathfrak{S}_2) = \text{Crit}_s(\mathfrak{S}_2) = \{ \text{cyclic group of order } 2 \} \subseteq \mathfrak{S} \). Hence for any \( p \in \mathbb{P} \), \( \text{Crit}_s(\mathfrak{S}_p \circ \mathfrak{F}) \subseteq (\mathfrak{S}_p \circ \mathfrak{S}_2) \cup \text{Crit}_s(\mathfrak{S}_p \circ \mathfrak{S}_2) \subseteq \mathfrak{S} \).

Recall that a group \( G \) is called \( p \)-decomposable if there exists a subgroup \( H \) of \( G \) such that \( G = P \times H \) for some Sylow \( p \)-subgroup \( P \) of \( G \). Also, we use \( \mathfrak{N}_r \) to denote the class of all groups \( G \) with \( l(G) \leq r \), where \( l(G) \) is the Fitting length of \( G \).

Proposition 3.8. (1) Let \( \mathfrak{F} \) be a formation with \( \pi(\mathfrak{F}) = \mathbb{P} \) such that \( \mathfrak{F} \subseteq \mathfrak{N} \). Then \( \mathfrak{F} \) satisfies the \( \mathbb{P} \)-boundary condition (III).

(2) Let \( \mathfrak{L} \) be the formation of all \( p \)-decomposable groups. Then \( \mathfrak{N} \circ \mathfrak{L} \) satisfies the \( \mathbb{P} \)-boundary condition (III) in \( \mathfrak{S} \).

(3) Let \( \mathfrak{F} \) be a formation with \( \pi(\mathfrak{F}) = \mathbb{P} \) such that \( \mathfrak{F} \subseteq \mathfrak{N} \). Then \( \mathfrak{N} \circ \mathfrak{F} \) satisfies the \( \mathbb{P} \)-boundary condition (III) in \( \mathfrak{S} \).

Proof. Statement (1) was proved in [16, Proposition 4.9(ii)], and statement (2) follows from [26, Lemma 5.2] and [16, Proposition 4.9(i)].

(3) By [11, Chap. IV, Theorem 1.16], we have that \( \mathfrak{F} = \mathfrak{N} \mathfrak{F} \). It follows from (1) and [16, Proposition 4.9(i)] that \( \mathfrak{N} \circ \mathfrak{F} \) satisfies the \( \mathbb{P} \)-boundary condition (III) in \( \mathfrak{S} \).

4 Applications

In this section, we investigate the structure of groups \( G \) whose minimal subgroups are contained in \( \mathcal{N}_{\pi,\mathfrak{S}}(G) \). Let \( \Psi_p(G) = \langle x | x \in G, o(x) = p \rangle \) if \( p \) is odd, and \( \Psi_2(G) = \langle x | x \in G, o(x) = 2 \rangle \) if the Sylow 2-groups of \( G \) are quaternion-free, otherwise \( \Psi_2(G) = \langle x | x \in G, o(x) = 2 \text{ or } 4 \rangle \).

Lemma 4.1. Suppose that \( \Psi_p(G^{\mathfrak{N}_p}) \leq Z_{\mathfrak{N}_p}(G) \). Then \( G \in \mathfrak{N}_p \).

Proof. By Lemma 2.8(3), for any subgroup \( H \) of \( G \), \( \Psi_p(H^{\mathfrak{N}_p}) \leq H \cap Z_{\mathfrak{N}_p}(G) \leq Z_{\mathfrak{N}_p}(H) \). Then by induction, \( H \in \mathfrak{N}_p \). We may, therefore, assume that \( G \in \text{Crit}_s(\mathfrak{N}_p) \). It follows from [17, Chap. IV, Satz 5.4] that \( G^{\mathfrak{N}_p} \) is a Sylow \( p \)-subgroup of \( G \). By [23, Theorem 1.1], \( G^{\mathfrak{N}_p} \) is a \( G \)-chief factor, and the exponent of \( G^{\mathfrak{N}_p} \) is \( p \) or 4 (when \( p = 2 \) and \( G^{\mathfrak{N}_p} \) is non-abelian). If \( p = 2 \) and \( G^{\mathfrak{N}_2} \) is non-abelian and quaternion-free, then by [30, Theorem...
3.1], $G^{n_2}$ has a characteristic subgroup $L$ of index 2. This induces that $L = \Phi(G^{n_2})$, and so $G^{n_2}$ is cyclic, which is contrary to our assumption. Hence $p$ is odd or $p = 2$ and $G^{n_2}$ is either abelian or not quaternion-free. This implies that $G^{\eta_p} = \Psi_p(G^{\eta_p}) \leq Z_{\eta_p}(G)$, and thereby $G \in \mathcal{N}_p$. The lemma is thus proved.

**Lemma 4.2.** Let $\mathfrak{F}$ be a saturated formation such that $\mathfrak{F} = \mathcal{S}\mathfrak{F}$ and $\pi \subseteq \pi(\mathfrak{F})$. Suppose that $\Psi_p(G^{\mathfrak{F}}) \leq Z_{\pi^\mathfrak{F}}(G)$ for every $p \in \pi$. Then $G \in \mathcal{S}\mathfrak{F} \circ \mathfrak{F}$.

**Proof.** Assume that the result is false and let $G$ be a counterexample of minimal order. If $O_\pi'(G) > 1$, then for every $p \in \pi$, $\Psi_p(G^{\mathfrak{F}}O_\pi'(G)) \leq Z_{\pi^\mathfrak{F}}(G)$ by Lemma 2.8(1). Hence by the choice of $G$, $G/O_\pi'(G) \in \mathcal{S}\mathfrak{F} \circ \mathfrak{F}$, and thereby $G \in \mathcal{S}\mathfrak{F} \circ \mathfrak{F}$, which is impossible. Therefore, $O_\pi'(G) = 1$. Let $M$ be any maximal subgroup of $G$. Since $M^{\mathfrak{F}} \leq G^{\mathfrak{F}}$ by Lemma 2.1(2), for every $p \in \pi$, $\Psi_p(M^{\mathfrak{F}}) \leq M \cap Z_{\pi^\mathfrak{F}}(G) \leq Z_{\pi^\mathfrak{F}}(M)$ by Lemma 2.8(3). Then by the choice of $G$, $M \in \mathcal{S}\mathfrak{F} \circ \mathfrak{F}$. We may, therefore, assume that $G \in \text{Crit}_\mathfrak{F}((\mathcal{S}\mathfrak{F} \circ \mathfrak{F}) \mathfrak{F})$.

If $Z_{\pi^\mathfrak{F}}(G) \leq \Phi(G)$, then $G$ has a maximal subgroup $M$ such that $G = Z_{\pi^\mathfrak{F}}(G)M$. It follows that $G/Z_{\pi^\mathfrak{F}}(G) \in \mathcal{S}\mathfrak{F} \circ \mathfrak{F}$. By Lemmas 2.8(4) and 2.8(6), $G \in \mathcal{S}\mathfrak{F} \circ \mathfrak{F}$, a contradiction. Hence $Z_{\pi^\mathfrak{F}}(G) \leq \Phi(G)$ is nilpotent. Since $O_\pi'(G) = 1$, $Z_{\pi^\mathfrak{F}}(G)$ is a $\pi$-group. Then it is easy to see that $Z_{\pi^\mathfrak{F}}(G) = Z_{\pi}(G)$. By [23], Chap. IV, Theorem 6.10, for every $p \in \pi$, $\Psi_p(G^{\mathfrak{F}}) \leq Z_{\pi}(G) \cap G^{\mathfrak{F}} \leq Z(G^{\mathfrak{F}})$. It follows from Lemma 4.1 that $G^{\mathfrak{F}} \in \mathcal{N}_\pi$. As $O_\pi'(G) = 1$, we have that $G^{\mathfrak{F}} \in \mathcal{N}_\pi \cap \mathcal{S}\mathfrak{F}$. Since $G \in \text{Crit}_\mathfrak{F}((\mathcal{S}\mathfrak{F} \circ \mathfrak{F}) \mathfrak{F})$ and $\mathfrak{F} = \mathcal{S}\mathfrak{F}$, $G \in \text{Crit}_\mathfrak{F}(\mathfrak{F})$. Then a similar discussion as in the proof of Lemma 4.1 shows that $G^{\mathfrak{F}}$ is a $p$-group with $p \in \pi$ such that the exponent of $G^{\mathfrak{F}}$ is $p$ or 4 (when $p = 2$ and $G^{\mathfrak{F}}$ is not quaternion-free) by using [23], Theorem 1.1. This implies that $G^{\mathfrak{F}} = \Psi_p(G^{\mathfrak{F}}) \leq Z_{\mathfrak{F}}(G)$, and so $G \in \mathfrak{F}$. The final contradiction ends the proof.

**Theorem 4.3.** Let $\mathfrak{F}$ be a formation such that $\mathfrak{F} = \mathcal{S}\mathfrak{F}$. Suppose that $\Psi_p(G^{\eta_\mathfrak{F}}) \leq \mathcal{N}_{\pi^\mathfrak{F}}(G)$ for every $p \in \pi$ and one of the following holds:

(i) $G \in \mathcal{S}\mathfrak{F} \circ \mathfrak{F}$.

(ii) $\mathfrak{F}$ satisfies the $\pi$-boundary condition (II).

(iii) $2 \in \pi$ and $\mathfrak{F}$ satisfies the $\{2\}$-boundary condition (II).

(iv) $\{2, q\} \subseteq \pi$, where $q$ is an odd prime, and $\mathfrak{F}$ satisfies the $\{2, q\}'$-boundary condition (II).

Then $G \in \mathcal{N}_\pi \circ \mathfrak{F}$.

**Proof.** If either the condition (i) or the condition (ii) holds, then by Theorem B(1), Theorem D and Lemma 2.8(6), $\Psi_p(G^{\eta_\mathfrak{F}}) \leq \mathcal{N}_{\pi^\mathfrak{F}}(G) = Z_{\pi^\mathfrak{F}}(G) = Z_{\pi^\mathfrak{F}}(G)$ for every $p \in \pi$. Hence in both cases, the theorem follows from Lemma 4.2.

Now suppose that the condition (iii) holds. Then it is easy to see that $\mathcal{N}_{\pi^\mathfrak{F}}(G) \leq \mathcal{N}_{2^\mathfrak{F}}(G)$ by definition, and so $\mathcal{N}_{\pi^\mathfrak{F}}(G) \leq \mathcal{N}_{2^\mathfrak{F}}(G)$. It follows that $\Psi_2(G^{\eta_\mathfrak{F}}) \leq \Psi_2(G^{\eta_\mathfrak{F}}) \leq \mathcal{N}_{\pi^\mathfrak{F}}(G) \leq \mathcal{N}_{2^\mathfrak{F}}(G)$. By applying the condition (iii), $G \in \mathcal{N}_2 \circ \mathfrak{F} \subseteq \mathcal{S} \circ \mathfrak{F}$, and thereby the condition (i) holds. Therefore, $G \in \mathcal{N}_\pi \circ \mathfrak{F}$. 

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Finally, we assume that the condition (iv) holds. Then for every $p \in \{2, q\}'$, $\Psi_p(G^{\pi_{(2,q)'}\circ \mathfrak{F}}) \leq \Psi_p(G^{\mathfrak{F}_{2,q}}} \leq N^\infty_{\mathfrak{F}_2}(G) \leq N^\infty_{\mathfrak{F}_{(2,q)}}(G)$. By applying the condition (ii), $G \in \mathfrak{N}_{(2,q)'} \circ \mathfrak{F} \subseteq \mathfrak{S} \circ \mathfrak{F}$ by Burnside’s $p^aq^b$-theorem, and so the condition (i) holds. Hence $G \in \mathfrak{N}_\pi \circ \mathfrak{F}$.

The next two corollaries can be regarded as generalizations of [24, Theorem 5.2] and [24, Theorem 5.3], respectively.

**Corollary 4.4.** Let $\mathfrak{F}$ be a formation such that $\mathfrak{F} = \mathfrak{S}\mathfrak{F}$ and $\mathfrak{F} \subseteq \mathfrak{U}$. Suppose that all cyclic subgroups of $G$ of odd prime order are contained in $N^\infty_{\mathfrak{F}}(G)$. Then:

1. $G \in \mathfrak{S}$.
2. The $p$-length of $G$ is at most 2 for every odd prime $p$, and if $\mathfrak{F} \subseteq \mathfrak{U}$, then the $p$-length of $G$ is at most 1 for every odd prime $p$.
3. The Fitting length of $G$ is bounded by 4, and if $\mathfrak{F} \subseteq \mathfrak{U}$, then the Fitting length of $G$ is bounded by 3.

**Proof.** By [17] Chap. IV, Satz 5.4], $\mathfrak{N}_2$ satisfies the $2'$-boundary condition (II), and so $\mathfrak{F}$ also satisfies the $2'$-boundary condition (II) for $\mathfrak{F} \subseteq \mathfrak{U} \subseteq \mathfrak{N}_2$. Since $\Psi_p(G^{\mathfrak{F}_{2,q}}} \leq N^\infty_{\mathfrak{F}}(G) \leq N^\infty_{\mathfrak{F}_{(2,q)}}(G)$ for every odd prime $p$, by Theorem 4.3, $G \in \mathfrak{N}_{(2,q)} \circ \mathfrak{F} \subseteq \mathfrak{S}$. Hence for every odd prime $p$, $G^p \leq G^q \in \mathfrak{N}_{2q} \subseteq \mathfrak{N}_p$, and so the $p$-length of $G$ is at most 2 for every odd prime $p$.

It is clear that $G^{2q^2} \leq G^0 \in \mathfrak{N}_{2q^2} \subseteq \mathfrak{N}_2$. This implies that the Fitting length of $G$ is bounded by 4. Now consider that $\mathfrak{F} \subseteq \mathfrak{U}$. The discussion is similar as above.

**Corollary 4.5.** Let $\mathfrak{F}$ be a formation such that $\mathfrak{F} = \mathfrak{S}\mathfrak{F}$ and $\mathfrak{F} \subseteq \mathfrak{U}$. Suppose that all cyclic subgroups of $G$ of order prime or 4 are contained in $N^\infty_{\mathfrak{F}}(G)$. Then:

1. $G \in \mathfrak{S}$.
2. The $p$-length of $G$ is at most 2 for every $p \in \mathbb{P}$, and if $\mathfrak{F} \subseteq \mathfrak{U}$, then the $p$-length of $G$ is at most 1 for every $p \in \mathbb{P}$.
3. The Fitting length of $G$ is bounded by 3, and if $\mathfrak{F} \subseteq \mathfrak{U}$, then the Fitting length of $G$ is bounded by 2.

**Proof.** The corollary can be proved similarly as in the proof of Corollary 4.4.

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