Forward kinematics for 6-UPS parallel robot using extra displacement sensor

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Abstract

Many assembly modes exist due to multi-solutions for forward kinematics of 6-UPS parallel robot. However, the existing theoretical research results have not been well applied to the real-time control, because of the complexity of the forward kinematics. This paper proposes an analytic algorithm to establish the forward kinematics model of a 6-UPS parallel robot with an extra displacement sensor mounted on the centers of upper and lower platform ((6+1)-UPS). Based on the unit quaternion method, the improved forward kinematics model of the proposed (6+1)-UPS system is theoretically derived to be four equations about four components of unit quaternion. The whole computation process is analytical. The method of measurement of the 7th link length and the utilization of unit quaternion reduces the complexity of computation process of forward kinematics, avoids the computer memory spillover and increases the computation efficiency to be suitable for the real-time control application. Then, the analytical solution of the position vector and orientation matrix of the (6+1)-UPS parallel robot are obtained. Finally, the correctness and effectiveness of the proposed approach are illustrated with a numerical example.

Keywords: Parallel robot, Forward kinematics, Extra displacement sensor, Unit quaternion, Orthogonal complement method, Analytical algorithm

1 Introduction

The research on the development of 6-UPS parallel robot has made great progress in recent years. The 6-UPS parallel robot has been widely applied to the VR, entertainment, medical, aerospace simulator, wave compensation simulator, radio telescope (FAST) and so on. With the development of science and technology, the requirements of high speed, high precision, high rigidity, high dynamic performance, low inertia and small structural size have been proposed for this kind of typical parallel robot. The key point to realize these high performance requirements is that the forward kinematics forms a global real-time feedback accurately and timely.

The existing methods for the solution of the forward kinematics of 6-UPS parallel robot can be basically divided into three categories: echelon-form, numerical iterative and extra-sensor approaches (Vertechy et al., 2008). In echelon-form approaches, 40th, 20th, 17th and 14th univariate polynomial algebraic equations, which obtain all solutions including unreasonable complex and extraneous roots, are obtained by dual quaternion method (Wampler, 1996) (Zhou et al., 2015), special geometric properties (Husty, 1996), Gröbner base algorithm (Cheng et al., 2010), Support Vector Regression (Morell et al., 2013), algebraic elimination method (Cheng et al., 2010) (Huang et al., 2010), and so on. Unfortunately, a variety of assembly modes exist due to the multi-solutions of forward kinematics for 6-UPS parallel robot. Moreover, the calculation of the kinematic model is complicated and time-consuming. Therefore, it is difficult to realize the real-time feedback control with the echelon-form schemes. In numerical iterative approaches, appropriate initial values of the actual position and orientation of 6-UPS parallel robot need to be chosen in advance to guarantee the robustness of the overall system. Yang et al. (2014) (2017) made use of the unit quaternion method to develop a fast numerical solution of forward kinematics. Furthermore, the numerical solution of redundant actuation has been analyzed, in which the singularity of the algorithm has been solved. However, further research is required to study the analytic solution. In summary, the first two
methods of theoretical analysis and numerical method are unable to realize the real-time feedback well. Therefore, a compromise approach by adding extra displacement sensor is used to solve the problem of real-time feedback of forward kinematics. This approach achieves the real-time feedback by avoiding the complicated calculation and, as a result, the approach is convenient for engineering application.

In 1991, Shi and Fenton (1991) firstly proposed a method to solve the forward kinematics of 3-RRP by adding extra sensors. However, this method leads to an explicit expression used in real-time. Moreover, it is difficult to be employed in the parallel mechanisms with different configurations. The unique position and orientation are obtained for general parallel mechanisms using 4 rotational sensors and for special mechanisms with 3 linear sensors (Merlet, 1993). This result of unique position and orientation was obtained also by Tancredi et al. (1995) by using 3 rotational sensors for a flat platform.

Bonev et al. (2001) presented a method by placing 3 linear extra sensors on the upper and lower platform of a general parallel robot 9-RRP. This method reduces the complexity of forward kinematics by getting distance information, even optimizes the placement location by using the condition number of sensibility of measurement errors sensors. In order to obtain an actual pose for the forward kinematics of the generalized Stewart-Gough parallel manipulators, two or three sensors must be used (Parenti-Castelli et al., 2000) (Chiu et al., 2001). Besides, Han et al. (1995) gets the orientation quickly by using 2 extra sensors measuring links length. Kim et al. (2016) proposed a new geometric approach to obtain a unique solution of forward kinematics of a 3-SPS/S manipulator with an extra sensor. However, the extra sensor constrains the moving platform motion.

In conclusion, echelon-form, numerical iteration and extra-sensor methods lead to obtaining the forward kinematics of parallel mechanism. Many scholars achieve real-time feedback control by adding more than one extra sensor. However, due to more than one sensor, the hardware cost increases greatly. The accuracy of forward kinematics is also influenced because of the presence of the testing error based on two or three extra sensors. Furthermore, the calculation process of forward kinematics is cumbersome. Increase of calculation time and influence of real-time feedback are caused for this reason. In this paper, one extra displacement sensor is installed on the connecting points on the upper and lower platform of 6-UPS parallel robot. Four equations involving four components of unit quaternion are obtained using unit quaternion method and orthogonal complement method. Finally, the analytic forward kinematics model is obtained. The model is adapted for closed-loop real-time applications of 6-UPS parallel robot. The correctness and effectiveness of the proposed method are verified by a specific numerical example.

Accordingly, the contributions of this paper are as follows. (a) Only one extra displacement sensor is required to guarantee obtaining the unique solution of position vector \( \mathbf{P} \) and orientation matrix \( \mathbf{R} \) respectively. (b) Unit quaternion is employed to represent the orientation matrix \( \mathbf{R} \) leading to obtaining four equations. Two are about the first and the fourth elements of unit quaternion and the other two are about the second and the third elements of unit quaternion respectively. According to these four equations, the orientation matrix is easier to be obtained. In addition, the calculation process of forward kinematics of 6-UPS parallel robot is improved greatly. (c) Based on (a) and (b), orthogonal complement method is used to ensure gain of the second one of two equations about the second and the third elements of unit quaternion.

What’s more, the proposed algorithm avoids the computer memory spillover and increases the computation efficiency to be suitable for the real-time control application.

This paper is organized as follows. In Section 2, the foundation of forward kinematics is presented and detailed. In Section 3 and 4, the solutions of unit quaternion components and position vector \( \mathbf{P} \) are done, respectively. Numerical validations and discussion are presented in Section 4. Finally, Section 5 summarizes this paper.

### 2 Foundation of forward kinematics

The structure diagram of (6+1)-UPS is shown in Fig.1. The (6+1)-UPS consists of six identical legs connecting the base to a common moving platform. Two passive joints connecting the base is universal joint \( \mathbf{U} \) and the moving platform is spherical hinge \( \mathbf{S} \), respectively. The active joint is prismatic joint \( \mathbf{P} \) fixed on each leg. The desired movement of moving platform is obtained by adding drive on each \( \mathbf{P} \). The position of six universal joints and six spherical hinges centers are restricted to a plane, respectively, that is, this (6+1)-UPS belongs to the plane type. The points \( \mathbf{P}_a \) and \( \mathbf{P}_b \) are selected as the connecting points of extra sensor on the base and moving platform plane, respectively. For convenient analysis, the absolute static coordinate system \( O_x y_z b \) is fixed on the base. At the same time, the relative dynamic coordinate system \( O_x y_z a \) is fixed on the moving platform. \( O_x b \) is the circle center of the two platforms circumcircle. \( z_a z_b \) axis perpendicular to the plane. Six pairs of vertices of the moving and static platform are circularly arranged symmetrically on a plane circle, as shown in Fig.2.
Fig. 1 The structure diagram of (6+1)-UPS. There are two platforms and six legs. A, B, are six identical legs connecting the base to the moving platform. The passive joints A, connecting the moving platform are spherical hinge S. The passive joints B, connecting the base are universal joint U. The active joints P, driving the legs are prismatic joint P fixed on each leg. The points A and B are selected as the connecting points of extra sensor on the base and moving platform plane, respectively. The relative dynamic coordinate system fixed on the moving platform. The absolute static coordinate system fixed on the base.

Fig. 2 The vertices arrangement schematic of (6+1)-UPS. A, are the vertices of the moving platform. B, are the vertices of the base. P, is the connecting points respectively. The circles in dashed line are the Circumscribed circle of the moving and base platforms respectively.

For Fig. 1, points A in moving coordinate system O x y z are expressed as \( a_i = [a_{ix}, a_{iy}, a_{iz}]^T \), and points B in static coordinate system O x y z are expressed as \( b_i = [b_{ix}, b_{iy}, b_{iz}]^T \). Because of the plane arrangement of the base and moving platform, the \( z \) component of \( a_i \), \( b_i \) is 0, that is, \( a_{iz} = b_{iz} = 0 \), as a result, \( a_i = [a_{ix}, a_{iy}, 0]^T \), \( b_i = [b_{ix}, b_{iy}, 0]^T \). \( L \) represents the length of A,B, \( e_i \) represents the unit vector of A,B. \( P = [P_x, P_y, P_z]^T \) is the position vector of the origin of coordinate system O x y z, \( R \) is the orientation matrix.

2.1 Representation of orientation matrix \( R \)

In order to solve the forward kinematics conveniently and deduce the process of the algorithm easily, the orientation matrix \( R \) is expressed as follows:

\[
R = \begin{bmatrix}
\epsilon_0 + \epsilon_1^2 - \epsilon_2^2 - \epsilon_3^2 & 2\epsilon_2\epsilon_3 - 2\epsilon_0\epsilon_1 & 2\epsilon_0\epsilon_2 + 2\epsilon_1\epsilon_3 \\
2\epsilon_2\epsilon_3 + 2\epsilon_0\epsilon_1 & \epsilon_0^2 - \epsilon_1^2 + \epsilon_2^2 - \epsilon_3^2 & -2\epsilon_0\epsilon_1 + 2\epsilon_2\epsilon_3 \\
-2\epsilon_0\epsilon_2 + 2\epsilon_1\epsilon_3 & 2\epsilon_0\epsilon_1 + 2\epsilon_2\epsilon_3 & \epsilon_0^2 - \epsilon_1^2 - \epsilon_2^2 + \epsilon_3^2
\end{bmatrix}
\]

(1)

where remembering \( A \equiv \epsilon_0^2 - \epsilon_1^2, C \equiv 2\epsilon_1\epsilon_2, B \equiv \epsilon_2^2 - \epsilon_3^2, D \equiv 2\epsilon_0\epsilon_3 \) are further introduced. That is, the symbol A is
defined as \( \varepsilon_3^2 - \varepsilon_1^2 \). Similarity, \( C \) is defined as \( 2\varepsilon_0\varepsilon_4 \), \( B \) is defined as \( \varepsilon_1^2 - \varepsilon_2^2 \), \( D \) is defined as \( 2\varepsilon_1\varepsilon_2 \). Therefore, the following expressions:

\[
\sqrt{A^2 + C^2} = \left(\varepsilon_0^2 - \varepsilon_4^2\right) + (2\varepsilon_0\varepsilon_4)^2 = \left(\varepsilon_0^2 + \varepsilon_4^2\right) - 2(\varepsilon_0\varepsilon_4)^2 + 4(\varepsilon_0\varepsilon_4)^2 = \left(\varepsilon_0^2 + \varepsilon_2^2\right)^2 + 2\varepsilon_0^2\varepsilon_4^2 = \varepsilon_0^2 + \varepsilon_4^2
\]

and

\[
\sqrt{B^2 + D^2} = \left(\varepsilon_1^2 - \varepsilon_2^2\right) + (2\varepsilon_1\varepsilon_2)^2 = \left(\varepsilon_1^2 + \varepsilon_2^2\right) - 2(\varepsilon_1\varepsilon_2)^2 + 4(\varepsilon_1\varepsilon_2)^2 = \left(\varepsilon_1^2 + \varepsilon_2^2\right)^2 + 2\varepsilon_1^2\varepsilon_2^2 = \varepsilon_1^2 + \varepsilon_2^2
\]

are obtained. Namely,

\[
\sqrt{A^2 + C^2} = \varepsilon_0^2 + \varepsilon_4^2, \quad \sqrt{B^2 + D^2} = \varepsilon_1^2 + \varepsilon_2^2
\]

Combination with \( A \triangleq \varepsilon_0^2 - \varepsilon_4^2 \) and \( \sqrt{A^2 + C^2} = \varepsilon_0^2 + \varepsilon_4^2 \), we obtain the expressions as following

\[
\varepsilon_0^2 = \frac{1}{2}(\sqrt{A^2 + C^2} + A), \quad \varepsilon_4^2 = \frac{1}{2}(\sqrt{A^2 + C^2} - A)
\]

Similarly, combination with \( B \triangleq \varepsilon_1^2 - \varepsilon_2^2 \) and \( \sqrt{B^2 + D^2} = \varepsilon_1^2 + \varepsilon_2^2 \), we obtain the expressions as following

\[
\varepsilon_1^2 = \frac{1}{2}(\sqrt{B^2 + D^2} + B), \quad \varepsilon_2^2 = \frac{1}{2}(\sqrt{B^2 + D^2} - B)
\]

According to the provisions of the unit quaternion, \( \varepsilon_0 > 0 \) is taken to solve the so-called 2-1 Covering Problem. The sign of \( \varepsilon_0 \) depends on \( C \) knowing from \( C = 2\varepsilon_0\varepsilon_4 \), remembering as \( \text{Sign}[C] \). Similarly, if the sign of \( \varepsilon_1 \) has been determined, the sign of \( \varepsilon_2 \) depends on \( D \) knowing from \( D = 2\varepsilon_1\varepsilon_2 \), remembering as \( \text{Sign}[D] \). As a result, two groups of unit quaternion component are obtained, remembering as short:

\[
\varepsilon_0 = \frac{\sqrt{A^2 + C^2} + A}{2}, \quad \varepsilon_4 = \frac{\sqrt{A^2 + C^2} - A}{2}
\]

\[
\varepsilon_1 = \frac{\sqrt{B^2 + D^2} + B}{2}, \quad \varepsilon_2 = \frac{\text{Sign}[\varepsilon_1]\text{Sign}[D]\sqrt{B^2 + D^2 - B}}{2}
\]

2.2 Construction vector equations and separation primary & secondary variables

For Fig. 1, according to vector closed relationship, the seven extendable link vectors of six groups of closed vectors \( O_aO_1A_1, \ O_bB_1A_1 \) and one closed vector \( O_0P_aP_a, \ O_0P_bP_b \) are expressed as

\[
L_k e_k = P + R a_k - b_k \quad k = 1, 2, \ldots
\]

Generally, 28 parameters of the base and the moving platform vertices and points \( P_a, P_b \) are employed to describe the structure of the (6+1)-UPS. 24 parameters of six pairs of vertices depend on \( r_1, r_2, \theta_1, \theta_2 \). \( r_1, r_2 \) are circumradii of the base and the moving platform respectively. \( \theta_1, \theta_2 \) are center semi-angles corresponding to the short side of platform respectively. The seventh pair of point coordinates is determined by their own coordinates. Each point coordinates are shown in Table 1.

| Table 1 Vertex coordinates of moving and base platform. |
|------------------------|------------------------|------------------------|------------------------|
| \( a_{x1} \) | \( a_{y1} \) | \( b_{x1} \) | \( b_{y1} \) |
| 1 | \( r_1 \cos(-\theta_2) \) | \( r_2 \sin(-\theta_2) \) | \( r_1 \cos(-\theta_1) \) | \( r_2 \sin(-\theta_1) \) |
| 2 | \( r_1 \cos(\theta_2) \) | \( r_2 \sin(\theta_2) \) | \( r_1 \cos(\theta_1) \) | \( r_2 \sin(\theta_1) \) |
| 3 | \( r_1 \cos(2\pi/3-\theta_2) \) | \( r_2 \sin(2\pi/3-\theta_2) \) | \( r_1 \cos(2\pi/3-\theta_1) \) | \( r_2 \sin(2\pi/3-\theta_1) \) |
| 4 | \( r_1 \cos(2\pi/3+\theta_2) \) | \( r_2 \sin(2\pi/3+\theta_2) \) | \( r_1 \cos(2\pi/3+\theta_1) \) | \( r_2 \sin(2\pi/3+\theta_1) \) |
| 5 | \( r_1 \cos(4\pi/3-\theta_2) \) | \( r_2 \sin(4\pi/3-\theta_2) \) | \( r_1 \cos(4\pi/3-\theta_1) \) | \( r_2 \sin(4\pi/3-\theta_1) \) |
| 6 | \( r_1 \cos(4\pi/3+\theta_2) \) | \( r_2 \sin(4\pi/3+\theta_2) \) | \( r_1 \cos(4\pi/3+\theta_1) \) | \( r_2 \sin(4\pi/3+\theta_1) \) |
| 7 | \( a_{x2} \) | \( a_{y2} \) | \( b_{x2} \) | \( b_{y2} \) |

The Eq. (2) expresses the closed vector \( O_0 P_a P_a, \ O_0 P_b P_b \) when \( k = 7 \). The length of PaPb is obtained by substituting the coordinates of \( a_k = \{a_{x1}, a_{y1}, a_{y2}, a_{y3}\}^T, \ b_k = \{b_{x1}, b_{y1}, b_{y2}\}^T, \ P \) and \( R \), then taking modules. That is, \( P_a P_b \). The points \( P_a, P_b \) are chosen at the origin points \( O_a, O_b \) of the coordinate systems, namely, \( a_{x2} = 0, a_{y2} = 0, b_{x2} = 0, b_{y2} = 0 \).

\( W \) is the position vector of the origin point \( O_b \) in the coordinate system \( O_x, y_a, z_a \), denoting as \( W = \{P_a, P_b, P_b\}^T \),
then \( \mathbf{P} = \mathbf{R} \cdot \mathbf{W} \). Combing with the orthogonality of \( \mathbf{R} \), \( \mathbf{W} = \mathbf{R}^T \mathbf{P} \) is obtained. Substituting \( \mathbf{a}_1, \mathbf{b}_1, \mathbf{P}, \mathbf{R} \) and \( \mathbf{W} \) into Eq. (2) and dot product with their own of both sides, the scalar equations of seven links square are obtained as follows (here subscript \( k \) is omitted)

\[
\begin{align*}
L_1^2 - r_1^2 - r_2^2 &= -2b_2P_x - 2b_1P_y + 2a_P + 2a_P \\
-2a(a_1b_2 + a_2b_1) - 2b(a_2b_1 - a_1b_2) &
\end{align*}
\]

\[
+2C(a_1b_2 - a_2b_1) - 2D(a_1b_2 + a_2b_1) + P_p
\]

(3)

Where, \( P_p \) is the modulus length square of position vector \( \mathbf{P} \). \( P_p = P_x^2 + P_y^2 + P_z^2 \). \( P_p \) is the projection in the direction of \( \mathbf{P}_x \). \( \mathbf{P}_x = \mathbf{P} \cdot \mathbf{x} \), and so on. The expressions: \( \mathbf{P} = \mathbf{P} \cdot \mathbf{y} \), \( \mathbf{P} = \mathbf{P} \cdot \mathbf{a} \), \( \mathbf{P} = \mathbf{P} \cdot \mathbf{b} \) are obtained.

Where, nine unknown numbers, such as \( P_x, P_y, P_z, P_a, P_b, A, B, C, D \), are linked with each other by each parameter of the moving platform’s position vector and orientation matrix. Moreover, the coefficients of nine unknown numbers are determined by the platform structure parameters and links length.

Then the specific expressions (4) of \( \mathbf{n}_1 \) are exported by linear equations from Eq. (3) as follows

\[
\begin{align*}
P_x &= L_1^2 \\
P_y &= P_{a0} + k_1B \\
P_z &= P_{a0} - k_1D \\
P_a &= P_{b0} + k_1B \\
P_b &= P_{b0} - k_1D \\
C &= 0 \\
A &= A_0
\end{align*}
\]

where, \( L_1 \) is the 7th link \( P_{a0} \) length measured by displacement sensor. \( P_{a0}, P_{b0}, A_0, C_0 \) are the constants that depend on \( r_1, r_2, \theta_1, \theta_2 \), and links length \( L_i (i = 1 \sim 6) \). \( k_1, k_2 \) are the constants that depend on \( r_1, r_2, \theta_1, \theta_2 \).

The specific expressions of above each constant are as follows

\[
k_1 = r_1 \sin[\theta_1 + 2\theta_2] \cdot \csc[\theta_1 - \theta_2] \quad k_2 = r_1 \sin[2\theta_1 + \theta_2] \cdot \csc[\theta_1 - \theta_2]
\]

\[
P_{a0} = \frac{1}{12r_2} \cdot Csc[\theta_1 - \theta_2] \cdot [\sqrt{3}\sin[\theta_1] \cdot (L_1^2 + L_2^2 + L_3^2 + L_4^2 + L_5^2 + L_6^2 - L_7^2)]
\]

\[
P_{b0} = \frac{1}{12r_2} \cdot Csc[\theta_1 + \theta_2] \cdot [\sqrt{3}\sin[\theta_1] \cdot (L_1^2 + L_2^2 + L_3^2 + L_4^2 + L_5^2 + L_6^2 - L_7^2)]
\]

\[
A_0 = \frac{1}{12r_2^2} \cdot \sec[\theta_1 - \theta_2] \cdot [(L_1^2 + L_2^2 + L_3^2 + L_4^2 + L_5^2 + L_6^2 - 6r_2^2 - 6r_2^2)]
\]

\[
C_0 = \frac{1}{12r_2^2} \cdot \csc[\theta_1 + \theta_2] \cdot [(L_1^2 - L_2^2 + L_3^2 - L_4^2 + L_5^2 - L_6^2)]
\]

3 Solution of unit quaternion components

The forward kinematics problem is to solve the position vector \( \mathbf{P} \) and orientation matrix \( \mathbf{R} \) when the seven link lengths are given. The unit quaternion components, \( \varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3 \), that the orientation matrix \( \mathbf{R} \) depends on are solved firstly in this paper.

3.1 Solution of \( \varepsilon_0 \) and \( \varepsilon_3 \)

By the relationship \( A = \varepsilon_0^2 - \varepsilon_3^2, C = 2\varepsilon_0\varepsilon_3 \) between \( A, C \) and unit quaternion, two equations are obtained as follows

\[
\varepsilon_0^2 - \varepsilon_3^2 = A_0
\]

(5)

\[
2\varepsilon_0\varepsilon_3 = C_0
\]

(6)

where \( \varepsilon_0 \) is positive, that is, \( \varepsilon_0 > 0 \). The solutions of \( \varepsilon_0 \) and \( \varepsilon_3 \) are solved by substituting structure parameters
and length into Eq. (5)(6).

Therefore, there are unique solutions for \( e_0 \) and \( e_3 \).

### 3.2 Solution of \( e_1 \) and \( e_2 \)

Furthermore, two equations are constructed to solve \( e_1 \) and \( e_2 \). According to the normalization condition of unit quaternion, \( e_0^2 + e_1^2 + e_2^2 + e_3^2 = 1 \), the first quadratic equation about \( e_1 \) and \( e_2 \) is obtained

\[
e_1^2 + e_2^2 = 1 - (e_0^2 + e_3^2) = 1 - \sqrt{A_n^2 + C_n^2} \triangleq C_k^2
\]

(7)

The second equation about \( e_1 \) and \( e_2 \) is constructed in the following.

Substituting \( P_i = P \alpha \), \( P_j = P \beta \), \( B = e_1^2 - e_2^2 \), \( D = 2e_1e_2 \) into from the second to fifth equations of Eq. (4), the following expressions are obtained

\[
\begin{align*}
P_x &= P_{00} + k_1 \left( e_1^2 - e_2^2 \right) \\
P_y &= P_{00} - 2k_1e_1e_2 \\
P_\mathbf{a} &= P_{00} + k_1 \left( e_1^2 - e_2^2 \right) \\
P_\mathbf{b} &= P_{00} - 2k_1e_1e_2
\end{align*}
\]

(8)

In Eq. (8), \( x = \{1,0,0\}^T \), \( y = \{0,1,0\}^T \) are the base vector of the static coordinate system \( O_x y_z \), \( \mathbf{a} = R \{1,0,0\}^T \), \( \mathbf{b} = R \{0,1,0\}^T \) are the base vector of the moving coordinate system \( O_x y_z \).

The second employed equation about \( e_1 \) and \( e_2 \) is obtained by eliminating the three components of the position vector \( P \) from the Eq. (8). The orthogonal complement matrix method is used to help the derivation process. The coefficient matrix \( T_i \) of \( P \) is formed in Eq. (8)

\[
T_i = \begin{bmatrix}
1 & 0 & e_1^2 + e_2^2 - e_1^2 - e_2^2 & 2e_1e_2 - 2e_0e_3 \\
0 & 1 & 2e_0e_3 + 2e_1e_2 & e_0^2 - e_1^2 + e_2^2 - e_3^2 \\
0 & 0 & 2e_0e_3 - 2e_1e_2 & 2e_0e_1 + 2e_2e_3
\end{bmatrix}
\]

(9)

The orthogonal complement matrix \( T_i \) is obtained by solving \( T_i \)

\[
T_i = \begin{bmatrix}
e_0e_1 - e_0e_1 & -e_0e_2 - e_1e_3 & e_0e_1 + e_2e_3 & e_0e_2 - e_1e_3
\end{bmatrix}
\]

(10)

Multiply the right hand side of Eq. (8) by the orthogonal complement matrix \( T_i \), the expression is given by

\[
\begin{align*}
P_{00} + k_1 \left( e_1^2 - e_2^2 \right) & P_{00} - 2k_1e_1e_2 & P_{00} + k_2 \left( e_1^2 - e_2^2 \right) & P_{00} - 2k_2e_1e_3 \end{align*}^T = 0
\]

(11)

After substituting (10) into (11), a quartic equation for unit quaternion component is obtained as following expression:

\[
\begin{align*}
P_{00} + k_1 \left( e_1^2 - e_2^2 \right) \left( e_0e_1 - e_0e_1 \right) & + P_{00} - 2k_1e_1e_2 \left( e_0e_2 - e_1e_3 \right) & + \left( P_{00} + k_2 \left( e_1^2 - e_2^2 \right) \right) \left( e_0e_3 + e_2e_3 \right) \end{align*}
\]

(12)

Substituting \( e_0 \) and \( e_3 \) obtained from Eq. (5)(6) into Eq. (12), the second equation, which is a cubic equation about \( e_1 \) and \( e_2 \), is obtained

\[
f_i e_1 + f_1 e_2 + f_2 e_1 e_2 + f_3 e_1^2 e_2^2 + f_1 e_1 + f_0 e_2 = 0
\]

(13)

\( f_i \) (i = 1 ~ 6) in Eq. (13) are the constants that depends on the structure parameters of (6+1)-UPS and length. Six groups of \( e_1 \) and \( e_2 \) results are calculated by combining two equations of (13) and (7). However, not all solutions meet the requirements because of the extraneous roots caused by elimination process and other reasons in mathematics. Therefore, a method of eliminating elements to solve \( e_1 \) and \( e_2 \) is developed in this paper. Furthermore, the Eq. (13) is further dealt with to obtain the simplified results are as follows:

\[
\begin{align*}
f_1 e_1^2 + f_2 e_1 + f_3 e_2 \end{align*} e_1 + \left( f_2 e_2^2 + f_3 e_1 + f_0 \right) e_2 = 0
\]

(14)

Remembering:

\[
f_1 e_1^2 + f_3 e_1 + f_0 = F
\]

(15)

\[
f_2 e_2^2 + f_3 e_2 + f_0 = G
\]

(16)

Eq. (14) is recorded as

\[
F e_1 + G e_2 = 0
\]

(17)

Therefore, there exists
The following expressions are obtained from Eq. (7)

\[ \varepsilon_2 = C_0^2 - C_1^2 \]  
\[ \varepsilon_3 = C_2^2 - C_3^2 \]  
\[ \varepsilon_1 = -\frac{G}{F} \varepsilon_2 \]  

Substituting (19) into (15) and (16), the following expressions are gotten

\[ F = f_1 (C_0^2 - C_1^2) + f_4 \varepsilon_2^2 + f_2, \]  
\[ G = f_2 \varepsilon_3^2 + f_1 (C_2^2 - C_3^2) + f_6. \]  

Substituting (18) into (7), the expression is written as follows

\[ \left(-\frac{G}{F} \varepsilon_2 \right)^2 + \varepsilon_3^2 = C_4^2 \]  

Substituting (21) and (22) into (23), the equation about \( \varepsilon_2 \) is obtained

\[ h_1 \varepsilon_0^6 + h_2 \varepsilon_4^2 + h_3 \varepsilon_2^2 + h_4 = 0 \]  

By the similar solution process of Eq. (24), the Eq. (25) about \( \varepsilon_1 \) is obtained as follows.

\[ g_0 \varepsilon_1^6 + g_1 \varepsilon_2^2 + g_3 \varepsilon_2^4 + g_4 = 0 \]  

The Eq. (24) is an even polynomial equation about \( \varepsilon_2 \). The coefficient \( h_i \) are constants depend on structural parameters, links length, \( A_0 \) and \( C_n \). According to a large number of calculations, only one positive real solution about \( \varepsilon_2 \) is obtained, defined by \( \varepsilon_2 = N_2 \). Then, \( \varepsilon_2 = \pm N_2 \). \( \varepsilon_2 \) owns two solutions opposite to each other. Similarly, Eq. (25) is also an even polynomial equation about \( \varepsilon_1 \). The coefficient \( g_i \) are constants depend on structural parameters, links length, \( A_0 \) and \( C_n \). Only one positive real solution about \( \varepsilon_1 \) is obtained, defined by \( \varepsilon_1 = N_1 \). \( \varepsilon_1 \) owns two solutions opposite to each other. Accordingly, the \( B \) of the orientation matrix \( R \) gets the unique solution, \( B = \varepsilon_1^2 - \varepsilon_2^2 = N_1^2 - N_2^2 \).

Therefore, \( \varepsilon_1 \) possesses two solutions opposite to each other, and \( \varepsilon_2 \) also possesses two solutions opposite to each other. Now, a linear proportional relationship between \( \varepsilon_1 \) and \( \varepsilon_2 \) is obtained by substituting \( \varepsilon_1^2 = N_1^2 \), \( \varepsilon_2^2 = N_2^2 \) into the cubic polynomial of unit quaternion components of Eq. (14) as follows

\[ \left( f_1 N_1^2 + f_4 N_2^2 + f_2 \right) \varepsilon_1 + \left( f_2 N_1^2 + f_3 N_2^2 + f_6 \right) \varepsilon_2 = 0 \]  

where remembering the coefficient of \( \varepsilon_1 \) and \( \varepsilon_2 \) as \( \varepsilon_{10} \) and \( -\varepsilon_{20} \), which are constants, then, Eq. (26) is denoted as

\[ \varepsilon_{10} \varepsilon_1 - \varepsilon_{20} \varepsilon_2 = 0 \]  

From Eq. (27), \( \varepsilon_1 = \frac{\varepsilon_{20}}{\varepsilon_{10}} \varepsilon_2 \) is given. Therefore, the \( D \) of the orientation matrix \( R \) also gets the unique solution,

\[ D = 2 \varepsilon_1 \varepsilon_2 = 2 \frac{\varepsilon_{20}}{\varepsilon_{10}} \varepsilon_2. \]  

At the same time, the sign of \( \varepsilon_1 \) depends on \( \varepsilon_2 \) and \( \frac{\varepsilon_{20}}{\varepsilon_{10}} \).

In summary, there are only two groups of real roots for \( \varepsilon_1 \) and \( \varepsilon_2 \) which are opposite to each other. Thus, the employed unit quaternion \( \varepsilon_0, \varepsilon_1, \varepsilon_2 \) and \( \varepsilon_3 \) of orientation matrix \( R \) have been solved completely. According to the conclusion of section 3.1 and 3.2, there are two groups of solution for \( \varepsilon_0, \varepsilon_1, \varepsilon_2 \) and \( \varepsilon_3 \), that is, suppose that \( \{ \varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3 \} \) is the solution, then \( \{ \varepsilon_0, -\varepsilon_1, -\varepsilon_2, \varepsilon_3 \} \) is also the solution. And then, two orientation matrices, \( R_1 \) and \( R_2 \), are obtained by substituting the gained \( \varepsilon_0, \varepsilon_1, \varepsilon_2 \) and \( \varepsilon_3 \) into Eq. (1).

4 Solution of position vector P

At the basis of two groups of orientation matrix \( R \) of (6+1)-UPS parallel robot aforementioned, the position vector \( P \) is easily obtained by substituting the orientation parameters, \( \varepsilon_0, \varepsilon_1, \varepsilon_2 \) and \( \varepsilon_3 \), into the first three equations of Eq. (4). For the sake of the completeness of the paper, the specific results for the forward kinematics of (6+1)-UPS parallel robot are given here as follows

\[ P_x = P_{x0} + 2k_1 \varepsilon_0 \varepsilon_2 = P_{x10} + 2k_1 \frac{\varepsilon_{20}}{\varepsilon_{10}} N_2^2 \]  
\[ P_y = P_{y0} + k_2 (\varepsilon_0^2 - \varepsilon_2^2) = P_{y10} + k_1 (N_1^2 - N_2^2) \]
\[ P_z = \pm \sqrt{P_x^2 - P_y^2} = \pm \sqrt{L_z^2 - \left( P_{a_0} + 2 k_2 \frac{\varepsilon_2}{\varepsilon_{10}} N_i^2 \right)^2 - (P_{a_0} + k_1 \left( N_i^2 - N_i^2 \right))^2} \]  

(30)

Expect for the structure itself, the Eq. (30) is only related to \( N_1^2 \) and \( N_2^2 \), which are unique respectively. Therefore, two \( P_z \) values opposite to each other are obtained.

Thus, the two orientation matrices, \( R_1 \) and \( R_2 \), and two \( P_z \) are obtained under independent circumstances with each other. There is no one-to-one correspondence between the orientation matrix and \( P_z \). Therefore, in order to make the two orientation matrices, \( R_1 \) and \( R_2 \), and two groups of position vector, \( P_1, P_2, P_3 \), are one-to-one correspondence, the \( P_z \) is obtained through the following three different ways according to different situations.

1) When the third element of the first column in the orientation \( R \) is not zero, the third equation in Eq. (8) is used to solve \( P_z \).

   Remembering \( \alpha = R, x = R, \{1,0,0\} \), the vector component form is equivalent to be expressed as follows

\[ P_z \alpha_i + P_x \alpha_x + P_y \alpha_y = P_{a_0} + k_2 \left( \varepsilon_1^2 - \varepsilon_2^2 \right) \]  

(31)

2) When the third element of the first column in the orientation \( R \) equals to zero, but the third one of the second column is not zero, the fourth equation in Eq. (8) is used to solve \( P_z \).

   Remembering \( \beta = R, y = R, \{0,1,0\} \), the vector component form is equivalent to be expressed as follows

\[ P_z \beta_i + P_x \beta_y + P_y \beta_y = P_{a_0} - 2 k_2 \varepsilon_1 \varepsilon_2 \]  

(32)

3) When both the third element of the first column and the third one of the second column in the orientation \( R \) equal to zero, \( P_z \) is solved and taken positive by Eq. (31). At the same time, the orientation matrix \( R \) is the identity matrix, i.e.

\[ R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \]  

(33)

Thus, \( P_{z_1} \) is obtained by substituting \( R_1, P_1, P_2 \) the structure parameters, links length and the first group of \( \varepsilon_1 \) and \( \varepsilon_3 \) into Eq. (31) or Eq. (32) or Eq. (33). Similarly, \( P_{z_2} \) is also obtained by substituting the second group of \( \varepsilon_1 \) and \( \varepsilon_2 \), \( R_2 \) and the structure itself parameters into Eq. (31) or Eq. (32) or Eq. (33). Therefore, \( P_{z_1} \) and \( P_{z_2} \) are opposite to each other.

The moving platform is located on the top of the static platform, thus, the positive number between \( P_{z_1} \) and \( P_{z_2} \) is taken. A group of orientation matrix \( R \) corresponding to the positive \( P_z \) is taken too. The analytic forward kinematics of (6+1)-UPS has been fully completed.

In conclusion, the unique solution of \( P \) and \( R \) is obtained respectively.

5 Numerical example and analysis

A specific numerical example is used to verify the correctness and effectiveness of the previous method. Specific thinking is as following: 1) to calculate the seven links length by the \( P \) and \( R \) given in advance, 2) to solve the forward kinematics by substituting the seven links length, structure parameters into the proposed method, namely: (5), (6), (7) and (13), 3) to verify the correctness of the proposed method whether the \( P \) and \( R \) obtained by forward kinematics are in accordance with the ones given by inverse kinematic conditions. And the solution process is represented in Fig.3.
Obtaining the orientation matrix $R$ and the position vector $P$ respectively, then the orientation matrix $\alpha, \beta$ and $\epsilon$ as following: 

Given $x^2, y^2, \alpha^2, \beta^2$, 

Solving the orthogonal complement matrix $T$: of $[x, y, \alpha, \beta]$ in Eq. (8).

Dot-multiplying equal right of Eq. (8) by $T$ to eliminate $P$.

Obtaining a bivariate cubic equation about $\epsilon_1, \epsilon_2$.

Combination Eq. (7).

Two $P_i$ are obtained respectively, and they are opposite to each other. According to the actual platform structure, $P_i$ takes the positive number.

Obtaining two groups of orientation $R$ respectively.

(a) Inverse kinematics: the vertices coordinates, $a_i, b_i (i=1 \sim 6)$, of the moving platform and base are determined by structural parameters of 6-UPS parallel robot, $\theta_i = 0.046988\pi, \theta_2 = 0.28618\pi, r_1 = 0.849864, r_2 = 0.849864$. The initial values of position vector and unit quaternion are assumed to be $P = [-0.08, -0.08, 1.12203]^T, \epsilon_0 = 0.99291, \epsilon_1 = -0.0655277, \epsilon_2 = -0.055966, \epsilon_3 = -0.0818821$ respectively, then the orientation matrix $R$ is

$$R = \begin{bmatrix} 0.980326 & 0.169938 & -0.100407 \\ -0.155268 & 0.978003 & 0.139291 \\ 0.121869 & -0.120961 & 0.985148 \end{bmatrix}$$

Accordingly, the links length obtained by inverse kinematic are $L_1 = 1.53086, L_2 = 1.19957, L_3 = 1.26712, L_4 = 1.1733, L_5 = 1.25707, L_6 = 1.32348, L_7 = 1.12772$.

(b) Forward kinematics: according to the proposed method, $A = 0.979165, C = -0.162603$ are obtained by substituting the structure parameters of 6-UPS parallel robot and links length obtained by inverse kinematics into Eq. (4). Then the unique solution $\epsilon_0 = 0.99291$ and $\epsilon_1 = -0.0818821$ are obtained after $A$ and $C$ are substituted into (5) and (6) and $\epsilon_0$ is taken positive.

Combination of $\epsilon_0, \epsilon_1$ and Eq. (7), (9)\sim(14), two equations about $\epsilon_1, \epsilon_2$ are obtained. The two equations lead to two groups of real solutions of $\{\epsilon_i, \epsilon_2\}$ as following: $\{\epsilon_i = -0.0655277, \epsilon_2 = -0.055966\}, \{\epsilon_i = 0.0655277, \epsilon_2 = 0.055966\}$, which are kept real roots after rejecting the imaginary roots. Therefore, two groups of solutions about unit quaternion are obtained as follows:

$$\mathbf{e}_1 = \{\epsilon_0 = 0.99291, \epsilon_1 = -0.0655277, \epsilon_2 = -0.055966, \epsilon_3 = -0.0818821\}$$
$$\mathbf{e}_2 = \{\epsilon_0 = 0.99291, \epsilon_1 = 0.0655277, \epsilon_2 = 0.055966, \epsilon_3 = -0.0818821\}$$

As a result, two orientation matrices corresponding to the two groups of unit quaternion are obtained

$$R_1 = \begin{bmatrix} 0.980326 & 0.169938 & -0.100404 \\ -0.155268 & 0.978003 & 0.139291 \\ 0.121869 & -0.120961 & 0.985148 \end{bmatrix}$$
$$R_2 = \begin{bmatrix} 0.980326 & 0.169938 & -0.100404 \\ -0.155268 & 0.978003 & 0.139291 \\ 0.121869 & -0.120961 & 0.985148 \end{bmatrix}$$

$P_i, P_j$ are obtained by substituting structure parameters of 6-UPS parallel robot and links length into Eq. (4). Furthermore, the $P_i = 1.2203, P_j = -1.12203$ are obtained by substituting structure parameters of 6-UPS parallel robot, links length, $R_i$ and $R_j$ into Eq. (31), that is, $P_i = [-0.8, -0.8, 1.12203]^T, P_j = [-0.8, -0.8, -1.12203]^T$.

The moving platform is located on the top of the static platform, therefore the positive number is $P_2$, namely, $P_2$ takes $P_i = 1.2203$. Now, the orientation matrix is $R_i$ and the position vector is $P_i$, which is a given inverse solution condition. It shows that the presented method is correct in this paper, as shown in Fig. 4.

Fig.3 Solution process of forward kinematics.
6 Conclusions and future work

This paper presents an analytic approach for forward kinematics of 6-UPS parallel robot with only one extra displacement sensor. First, the forward kinematics model of (6+1)-UPS is established by using a unit quaternion and a series of equations are derived to be four equations about four components of unit quaternion. Furthermore, the forward kinematics model is made come down to the simultaneous solution of two equations about two components of unit quaternion, $\varepsilon_1$ and $\varepsilon_2$. The new forward kinematics model makes the calculation process is completely analytical. What’s more, the whole computation process is made to be very simple even on a regular computer. It is impossible to appear the problem of computer memory spillover or other problems. Finally, the analytic results of position vector $\mathbf{P}$ and orientation matrix $\mathbf{R}$ of (6+1)-UPS are easily obtained.

The proposed method reduces the complexity of computation process of forward kinematics, avoids the computer memory spillover and increases the computation efficiency by using unit quaternion and adding only one extra sensor, gaining additional 7th leg information available to guarantee a unique solution. The unique solution for the 6-UPS parallel robot is computed quickly, solves the problem of multiple solutions for forward kinematics of 6-UPS parallel robot and meets the final real-time closed-loop feedback control. In the end, the correctness and effectiveness of the proposed method are verified by a numerical example. However, the influence of testing error of extra sensor on calculation accuracy for forward kinematics will be researched in further study.

Declaration of conflicting interests

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