EQUILIBRIUM TWO-DIMENSIONAL
DILATONIC SPACETIMES

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ABSTRACT

We study two-dimensional dilaton gravity coupled to massless scalar fields and look for time-independent solutions. In addition to the well-known black hole, we find another class of solutions that may be understood as the black hole in equilibrium with a radiation bath. We claim that there is a solution that is qualitatively unchanged after including Hawking radiation and back-reaction and is geodesically complete. We compute the thermodynamics of these spacetimes and their mass. We end with a brief discussion of the linear response about these solutions, its significance to stability and noise, and a speculation regarding the endpoint of Hawking evaporation in four dimensions.
I. INTRODUCTION

Two-dimensional gravity has emerged as a useful toy model for addressing some basic philosophic questions about semiclassical and quantum gravity. Dilation gravity is the low-energy effective theory of string theory and so solutions of this theory arise naturally as the $\sigma$-models backgrounds associated to conformal field theories. An example of these are the coset models\cite{1,2,3} among which was found two-dimensional $\sigma$-models, including that of the black hole,\cite{4,5,6} which is a solution of two-dimensional dilaton gravity with a Minkowski signature. Besides just the graviton and dilaton, a two-dimensional gravity also arises naturally from dimensional reductions of ordinary Einstein gravity in four and five dimensions (see Refs. \cite{7,8,9}). Realistic string theory should also have low-energy “matter” fields. Callan, et al.\cite{10} showed that including massless scalar fields to dilaton gravity in two dimensions yields a toy model rich with phenomena such as gravitational collapse and Hawking radiations. This model has subsequently been explored by many authors, who addressed a host of questions ranging from the structure of the singularity,\cite{11} of the Hawking radiation and its back reaction on the metric,\cite{12,13} information loss in black hole evaporation,\cite{14,15,16} and approaches to the full quantum theory.\cite{17–22} There are several very readable reviews on this subject; see for example Refs. \cite{8,23}.

Although two-dimensional dilaton gravity is somewhat different than ordinary Einstein–Hilbert gravity, there are many features of the model that are generic to other dimensions and other metric theories of gravitation. Certainly one difficulty in trying to answer some of the above questions is that they concern (inherently) non-equilibrium phenomena. Since the Green’s functions, that arise naturally when studying quantum fields in these backgrounds
are typically equilibrium Green’s functions they are often not suitable for addressing these non-equilibrium questions beyond low orders in perturbation theory.

There are, however, many interesting questions to be answered about the equilibrium properties of semiclassical gravity, such as the nature of the equilibrium state and the power spectrum of noise in the spacetime itself. These issues have been somewhat neglected, as they do not allow one to directly answer the interesting non-equilibrium questions posed above. Alternatively, these notions are computable and rigorously defined.

Here we study (static) solutions to two-dimensional dilatonic gravity coupled to matter and try to answer some of the equilibrium questions above. We will find that although the ordinary dilaton black hole is not an equilibrium solution (that is, semiclassically with realistic boundary conditions or once back-reaction is included) there does exist a solution which is non-singular, geodesically complete and static. This solution seems to persist even with the back-reaction included. It is interpreted as the black hole in equilibrium with a heat bath of radiation. We then compute its thermodynamic properties and describe the fluctuations about this solution. The idea of studying the two-dimensional dilatonic black hole in equilibrium with a radiation bath is not new (see for example the discussion in Refs. [24–26,16]). In some earlier works these solutions were discarded on the basis that they are not finite energy solutions. This is trivially true for black holes in a radiation bath in any open spacetime and so here we accept this as a simple fact and instead focus on what we feel are more pressing physics issues. At the end we remark on whether something like these solutions may be realized approximately. Principally, then, in this note we aim to clarify and extend those earlier works.

Let us consider a naive picture of a black hole of mass $M$ in thermal equilibrium in $1 + 1$ dimensions, in a box of length $L$. Let the radiation in the box be of massless bosons
of temperature $T$. Here we follow closely the line of reasoning presented in the initial pages of Ref. [27]. As suggested by earlier investigations of the two-dimensional black hole, the temperature of the black hole, $T_{BH}$ is fixed and independent of the mass. The total energy and entropy of the system is

$$U = M + aLT^2$$

$$S = \frac{M}{T_{BH}} + 2aLT$$

(1)

where $a$ depends on the number of species of bosons in the thermal bath. Extremizing the entropy at fixed energy yields $T = T_{BH}$ and the second derivative of the entropy with respect to temperature at this extremum is negative,

$$\frac{d^2S}{dT^2} = \frac{\partial^2 M}{2T^2} \bigg|_U = -\frac{2aL}{T_{BH}} < 0$$

(2)

suggesting that the system is stable. As we show later, when we write down a solution to the field equations that correspond to a black hole in a heat bath, this picture is too naive; in 1 + 1 dimensional dilaton gravity the presence of the bath very dramatically affects the geometry of the spacetime.

To begin with, consider the low-energy effective action of string propagation in a two-dimensional manifold $(\mathcal{M}, g)$

$$I = I_G + I_M$$

$$I_G = \frac{1}{2\pi} \int_\mathcal{M} d^2x \sqrt{-g} e^{-2\phi} \left[ R + 4(\nabla \phi)^2 + 4\lambda^2 \right] + \frac{1}{\pi} \int_{\partial \mathcal{M}} e^{-2\phi} K d\epsilon$$

$$I_M = \frac{1}{2} \int_\mathcal{M} d^2x \sqrt{-g} \sum_{\Lambda=1}^{N} (\nabla f_i)^2$$

(3)

where $K$ is the trace of the second fundamental form on the boundary, (this term is necessary for the $I_G$ to depend on the fields $g, \phi$ and their first derivatives only) and where $f_i$ are massless scalars minimally coupled to gravity.
We look for equilibrium (i.e. globally static) solutions by requiring the metric possess a global time-like Killing vector. Any two-dimensional metric that has a global time-like Killing vector may be put in the form,

\[ ds^2 = -\Omega^2(r)dt^2 + dr^2 . \] (4)

The non-vanishing Christoffel symbols with this metric are \( \Gamma^1_{00} = \Omega \Omega' \) and \( \Gamma^0_{10} = \Omega'/\Omega \) where the prime denotes \( \partial/\partial r \). The scalar curvature is \( R = -2\Omega''/\Omega \) and the most general static and covariantly conserved tensor has the form

\[ T_{\mu\nu} = \begin{bmatrix} A - \frac{\Omega}{\Omega'} A' & \frac{b}{\Omega'} \\ \frac{b}{\Omega} & \frac{A}{\Omega'^2} \end{bmatrix} \] (5)

where \( A \) is an arbitrary function of \( r \) and \( b \) is a constant. The equations of motion that follow from Eqs. (1) are, in general,

\[ \frac{1}{\pi} e^{-2\phi} \left( \nabla_\mu \nabla_\nu \phi + g_{\mu\nu} \left( (\nabla \phi)^2 - \nabla^2 \phi - \lambda^2 \right) \right) + \frac{1}{2} T^{(f)}_{\mu\nu} = 0 \] (6a)

\[ \frac{1}{\pi} e^{-2\phi} \left( R + 4\nabla^2 \phi - 4(\nabla \phi)^2 + 4\lambda^2 \right) = 0 \] (6b)

\[ \frac{1}{\sqrt{-g}} \nabla_\mu g^{\mu\nu} \sqrt{-g} \nabla_\nu f_i = 0 \] (6c)

where

\[ T^{(f)}_{11} = \frac{1}{2} \sum_{i=1}^N \left[ (\partial_r f_i)^2 + \frac{1}{\Omega^2} (\partial_t f_i)^2 \right] \]

and \( T^{(f)}_{\mu\nu} \) is the stress tensor of the fields \( f_i \). Since \( D^\mu T^{(f)}_{\mu\nu} = 0 \), Eq. (6a) implies that

\[ R + 4\nabla^2 \phi - 4(\nabla \phi)^2 \]

is a constant, so Eq. (6b), while an independent equation, is consistent with Eq. (6a). Equation (6b) is thus essentially a statement of the gravitational Bianchi identity.
To look for static solutions to Eq. (6) with the metric of Eq. (4) we learn immediately that $T^{(f)}_{\mu\nu}$ may not be of the most general form of Eq. (5). Indeed, (6a) for $\mu = 1, \nu = 0$ implies that $b = 0$. Since the matter fields $f_i$ that we are minimally coupling to gravity are massless we expect that (at least classically) $T^{(f)}_{\mu\nu}$ is traceless. This forces $A = 2A_0$, a constant. $A_0$ may be thought of as the local energy density. It is easy to check that constant $A$ is consistent with the equations of motion for the $f_i$, Eq. (6c), but we are, for now, not interested in explicit classical solutions for the $f_i$. The remaining equations of motion in this metric ansatz read:

$$
\phi'' - (\phi')^2 + \lambda^2 + \frac{\pi A_0}{\Omega^2} e^{2\phi} = 0
$$

$$
\phi'' + \frac{\Omega'}{\Omega} \phi' - (\phi')^2 + \lambda^2 - \frac{\Omega''}{2\Omega} = 0
$$

$$
-\frac{\Omega'}{\Omega} \phi' + (\phi')^2 - \lambda^2 + \frac{\pi A_0}{\Omega^2} e^{2\phi} = 0
$$

Eliminating the terms involving $A_0$ permits one to integrate once, finding

$$
\phi' = \frac{\Omega'}{\Omega} + \frac{c}{\Omega}
$$

where $c$ is a constant of integration. These equations of motion are invariant under together rescaling $\Omega$ and by shifting $\phi$ by a constant, and so there are only really three possibilities for $c$; either 0 or $\pm \lambda$. It is simple to show that the solutions with $c = \pm \lambda$ are: a) $\Omega = \tanh \lambda r$, $\phi = -\ln(\cosh \lambda r) + \phi_0$; the LDV $\Omega = 1$, $\phi = -\lambda r + \phi_0$; and b) $\Omega = \cosh \lambda r$, $\phi = \ln(\sinh \lambda r) + \phi_0$, respectively. Regions a) and b) correspond to those of the maximally extended black hole solutions\textsuperscript{10,28–31} and a), the LDV, and b) are solutions with $A_0 = 0$.

For $c = 0$, Eqs. (7) reduce to

$$
\frac{\Omega''}{2\Omega} - \left(\frac{\Omega'}{\Omega}\right)^2 + \lambda^2 = 0
$$
which has the general solution
\[
\frac{1}{\Omega} = B_+ e^{\sqrt{2} \lambda r} + B_- e^{-\sqrt{2} \lambda r}, \quad \phi = \ln \Omega + \phi_0
\]  
with \( B_\pm, \phi_0 \) being real numbers.

These solutions all have \( A_0 = \frac{\lambda^2}{\pi} e^{-2\phi_0} \neq 0 \). They are static spacetimes filled with radiation density, \( A_0 \).

Before including back-reaction or studying the thermodynamics of these spacetimes we briefly describe their geometry. These solutions, Eq. (10), all approach constant curvature \( R = -4\lambda^2 \) asymptotically, \( r \to \pm \infty \). There are three different geometries possible from Eq. (10), depending on the relative sign of \( B_+ \) and \( B_- \). Note that as described above, we may scale \( \Omega \) such that, without loss of generality, \( B_+ = 1 \). Three different geometries result from whether \( B_- \) is positive, zero, or negative. Essentially they are

\[
\Omega = \begin{cases} 
\frac{1}{2 \cosh \sqrt{2} \lambda r} & B_- > 0 \quad (I) \\
e^{-\sqrt{2} \lambda r} & B_- = 0 \quad (II) \\
\frac{1}{2 \sinh \sqrt{2} \lambda r} & B_- < 0 \quad (III) 
\end{cases}
\]

Solutions I and II are geodesically complete and III has a naked singularity at \( r = 0 \).

Solution II is another linear dilaton solution (LDV) \( \phi = -\sqrt{2} \lambda r + \phi_0 \). Solution II is a constant curvature metric and therefore has, in addition to the trivial time translation Killing vector, two other Killing vectors, one associated with translations in \( r \) and the other akin to a boost. This is in strong analogy to the LDV solution (flat space) with \( A_0 = 0 \). It is instructive to compare these solutions with the black hole solution. In null coordinates \( x^+ \) and \( x^- \) the black hole metric (with \( A_0 = 0 \)) is\(^6,10\)

\[
ds_{\text{BH}}^2 = -\frac{dx^+ dx^-}{M \lambda - \lambda^2 x^+ x^-}.
\]
In the coordinates 

\[ x^\pm = \sqrt{\frac{M}{4\lambda}} \left[ \frac{\sqrt{2}}{\lambda} \cosh \sqrt{2} \lambda r \pm t \right] \]

the metric of spacetime III is 

\[ ds^2_{\text{III}} = -\frac{dx^+dx^-}{M \left[ \frac{\lambda^2}{2} \left(x^+ + x^-\right)^2 \right]} . \quad (12) \]

The classical solution in conformal gauge has the general solution 

\[ ds^2 = -e^{2\rho}dx^+dx^- \quad (13) \]

with 

\[ \partial_+ \partial_- e^{-2\rho} = -\lambda^2 , \quad \partial_+ \partial_+ e^{-2\rho} = -t_+(x^+) , \quad \partial_- \partial_- e^{-2\rho} = -t_-(x^-) \quad (14) \]

where \( t_\pm(x^\pm) \) are the right and left moving parts of the \( T^{(f)}_{\mu\nu} \). Thus, solutions I and III may be interpreted as a two-dimensional dilatonic black hole in a radiations bath where the naive temperature of the black hole \( (T_{\text{BH}} \sim \frac{\lambda}{2\pi}) \) is the same as that of the radiation bath. This is quite different than known solutions to general relativity in four dimensions where classically there are static cosmological solutions with \( T_{\text{BH}} > T_{\text{cosmo}} \) (see Refs. [32,33]), where \( T_{\text{cosmo}} \) is a cosmological temperature associated with the cosmological horizon. In four dimensional Einstein gravity it seems technically formidable to find analytic solutions corresponding to a black hole in equilibrium with a heat bath. We will remark on the relation of the solutions of the two-dimensional model presented here and ordinary gravity in four dimensions later.

The interpretation of spacetimes I and III as the two-dimensional dilaton black hole in a radiation bath is, although correct, not quite the naive model of the black hole "in-a-box" discussed after the introduction. The presence of uniform, static energy density \( A_0 \) drastically
modifies the geometry of the spacetime, and there is no smooth limit as $A_0 \to 0$; the equations with and without the bath ($A_0 = 0$ and $A_0 \neq 0$, respectively) are very different. It is still useful for taxonomic purposes to regard the solutions I, II and III as related to a), the LDV and b) respectively, of the black hole spacetime but including the effects of a radiation bath.*

For example, the curvature scalar of I is positive near $r \approx 0$ while that for III is strictly more negative than $-4\lambda^2$. Also note that spacetimes I and III only have a single global Killing vector. That is, the $r$-translation isometry of spacetime II is broken spontaneously in spacetimes I and III, as expected of black hole backgrounds. Later, in computing asymptotic masses we will again see how natural it will be to think of the solutions I, II and III as corresponding to the patches a), the LDV and b), respectively, of the black hole spacetime.

It is easy to compute the Hawking radiation\textsuperscript{12} of, and back-reaction on these spacetimes. Following Refs. [8,10] this may be done in two steps. First, Hawking radiation arises from the conformal anomaly of the matter fields,\textsuperscript{34,35}

\[
\langle T_{\mu\nu}g^{\mu\nu} \rangle = -\frac{N}{12}R \tag{15}
\]

Now, following Ref. [10] and using metrics I, II and III, one may easily compute semiclassically what the Hawking flux at $r = \infty$. It is simple to show that this flux is negligibly small compared to the ambient matter ($A_0 \neq 0$) at infinity. Thus spacetimes I,II and III do not semiclassically decay, unlike the black hole without a radiation bath ($A_0 = 0$ and with trivial boundary conditions at $r = \infty$.) Furthermore, due to the fact that these metrics are functions of $(x^+ + x^-)$, we find that at infinity $x^+ \to \infty$ or $x^- \to \infty$ the components of

* Note although regions a) and b) (see discussion after Eq. (8)) of the black hole metric are related by duality,\textsuperscript{29–31,38} when matter is included, duality is no longer a symmetry; there is no duality symmetry relating solutions I and III.
the full renormalized stress tensor $T_{++}(x^+, x^-)$ and $T_{--}(x^+, x^-)$ are equal and in the limit proportional to $A_0$.

Of course it may be argued that there is nothing really “asymptotic” about $r \sim \infty$. It is simple to show that for the spacetimes I, II, III a time-like trajectory reaches $r = \infty$ in a finite proper time proportional to $1/\lambda$. Indeed it is easy to find coordinates in which one may study the maximal extensions of spacetimes I, II and III. Figure 1 contains the Penrose diagrams for these spacetimes. It is natural to regard spacetime II as the “vacuum” to which we should compare the solutions I and III, as was the LDV for the black hole without a radiation bath ($A_0 = 0$). In this sense, “asymptotic” will mean $r \sim \infty$ since there the solutions I and III approach solutions II.

It is somewhat more interesting to study the back-reaction of the Hawking radiation in these solutions. For the black hole it was hoped that studying the back-reaction would further clarify gravitation collapse, subsequent evaporations, and the problem of information loss.

In the metric ansatz, Eq. (4), the most general, static, covariantly conserved stress tensor (see Eq. (5)) satisfying Eq. (15) is:

$$T_{\mu\nu} = \begin{bmatrix} 2A_0 - \alpha \left( \frac{(\Omega')^2}{2} - \Omega\Omega'' \right) & \frac{b}{\Omega} \\ \frac{b}{\Omega} & \frac{2A_0}{\Omega^2} - \frac{\alpha}{2} \left( \frac{\Omega'}{\Omega} \right)^2 \end{bmatrix}$$

(16)

where $A_0$ and $b$ are again constants and $\alpha = N/12$.

To understand how the Hawking radiation back-reacts on the metrics I, II and III, we can begin by putting Eq. (16) into Eq. (6a). This gives the equations of motion with back-reaction,

$$\phi'' - (\phi')^2 + \lambda^2 + \frac{\pi e^{2\phi}}{\Omega^2} \left( A_0 - \frac{\alpha}{2} \left( \frac{(\Omega')^2}{2} - \Omega\Omega'' \right) \right) = 0$$

(17a)

$$\phi'' + \frac{\Omega'}{\Omega} \phi' - (\phi')^2 + \lambda^2 - \frac{\Omega''}{2\Omega} = 0$$

(17b)

$$-\frac{\Omega'}{\Omega} \phi' + (\phi')^2 - \lambda^2 + \frac{\pi e^{2\phi}}{\Omega^2} \left( A_0 - \frac{\alpha}{4} \frac{(\Omega')^2}{4} \right) = 0$$

(17c)
Note that for metric II, the LDV solution, the terms in Eq. (17) due to the back-reaction are indeed negligibly small in the $r \to \infty$ region. This is also true for the other solutions (I,III). Thus our notion of “asymptotic,” as described above for these solutions is unchanged by including the back-reaction. Although for $A_0 = 0$ the LDV solution remains a solution even including back-reaction, for $A_0 \neq 0$ it is simple to show that there can be no exactly LDV solution (no solution with $\phi = -\beta r + \phi_0$ for some $\beta$).

The asymptotic solution to Eq. (17) are, $(r \to \infty)$

$$\phi(r) \to \sqrt{2} \lambda r + \phi_0 + e^{-2\sqrt{2} \lambda r} \left( \omega_0 + \frac{2\alpha}{A_0} \left( \frac{\sqrt{2}}{3} \lambda r - \frac{1}{6} \right) \right)$$

$$\Omega(r) \to e^{-\sqrt{2} \lambda r} \left( 1 + e^{-2\sqrt{2} \lambda r} \left( \omega_0 + \alpha \frac{2\sqrt{2}}{3} \lambda r \right) \right)$$

where $\omega_0 < 0$, $\omega_0 = 0$, $\omega_0 > 0$ correspond to asymptotic views of the solutions I, II and III, respectively.

Although no exact solution to Eq. (17) with $A_0 \neq 0$ (or $A_0 = 0$) is known, we can ascertain many properties of any solution. For example, letting $y = e^{-2\phi} + \pi \alpha$, it is straightforward to show that Eq. (17) imply

$$\left[ 4\lambda^2 + \frac{\Omega'}{\Omega} \left( \frac{\Omega''}{(\Omega')^2} \right) \right]' y = \pi \left( 4\alpha \lambda^2 - \frac{\alpha \Omega''}{\Omega} + 4A_0 \frac{\Omega'}{\Omega (\Omega')^2} \right)$$

$$(\Omega'y)' = -\frac{2\pi}{\Omega} \left( A_0 - \frac{\alpha (\Omega')^2}{4} \right) .$$

We now study these equations near a power law singularity, such as that found in spacetime III. For the metric to behave as

$$\Omega \sim r^\gamma , \quad \text{as } r \to 0 \quad (\gamma \neq 0, \text{ real})$$

(which could indicate either a horizon $\gamma > 0$ or a singularity $\gamma < 0$) using Eq. (19) we find that including back-reaction only allows $\gamma = 1$. That is, roughly speaking, including back-reaction
is not consistent with simple power-law singularities in the metric. Of course, Eq. (15) is a one-loop result and near curvature singularities we might expect that there may be other contributions to \( \langle T_{\mu \nu} g^{\mu \nu} \rangle \) that would dominate. By “singularity” we really mean curvatures approaching “1” in units of \( \lambda^2 / \alpha \). As a consequence of this we see that including back reaction will strongly modify the singularity at \( r = 0 \). This is also true for the singularity of the black hole in the absence of a bath \( (A_0 = 0) \).

Similarly one may show that back-reaction also strongly modifies solutions II in the strong coupling region \( r \to -\infty \). However, for metric I the corrections due to including the back-reaction appear to be mild almost everywhere. It can also be shown that Eqs. (19) are consistent with \( \Omega' \) vanishing linearly. Since this is the distinctive feature of the “throat” region \( (r \approx 0) \) of the metric in I, this indicates that there may be an exact semiclassical solution (i.e. with back-reaction included) that is qualitatively metric I.

We now associate “asymptotic” masses to the spacetimes discussed above. This will support the notion that spacetimes I, II, and III should be thought of as a), the LDV and b), respectively but with spacetime filled with a radiation bath.

Before continuing we note that, as described earlier, we have made a special choice in defining what we mean by “asymptotic.” With this definition we are not \textit{a priori} guaranteed that all “definitions” of mass will lead to the same expressions. Below we discuss a thermodynamic definition of “asymptotic” mass. However, first, we would like to point out that one may compute an “asymptotic” mass by simply comparing the geodesics near \( r = \infty \) of, say, spacetime I to those of the “vacuum” spacetime II. Recall the “vacuum” spacetime II is anti-deSitter space and, as usual, we consider geodesics in the covering space shown in Fig. 1. One may compute the “transit time” to cross a wedge: that is, the proper time for an
observer to enter and leave the region marked II. As expected, it is the same for all time-like trajectories and is $\sim 1/\lambda$. Now, when one solves the geodesic equation for geodesics near $r \sim \infty$ in spacetime I, one finds that their transit time is a little bit slower than that of spacetime II. This difference is attributable to an extra overall attractive mass $M \sim A_0/\lambda$ in spacetime I as compared with spacetime II. The natural position to associate to this mass would be near $r = 0$ in spacetime I, since that is where the metric deviates appreciable from the ”vacuum” metric of spacetime II. Again, one may even more simply compare the instantaneous accelerations of a body near $r \to \infty$ but initially at rest in both spacetimes I and II. One again finds that the extra acceleration towards $r = 0$ in space I may be attributable to a mass $M \sim A_0/\lambda$. These masses are to be thought of as contravariant quantities with respect to the metric II. It should also be possible to derive an ADM-like mass formula directly from the equations of motion, but we do not pursue that here.

We now compute the equilibrium thermodynamic functions for the spacetimes I, II and III. This will give a deeper understanding of these solutions and will be convenient for discussing issues of stability and fluctuation below. In what follows we proceed essentially from Refs. [28,36].

To understand thermodynamic properties of the spacetime semiclassically, it is most convenient to compute the free energy. This is done by simply evaluating the action Eq. (3) on a Euclidean continuation of the solutions in question. In our metric ansatz Eq. (4), this yields

$$I = \frac{1}{\pi} \int_{\partial \mathcal{M}} e^{-2\phi} \left( \frac{\Omega'}{\Omega} - 2\phi' \right) d\Sigma + B_{\text{mat}}$$

(21)

where we have computed the action in some box, the boundary $\partial \mathcal{M}$ of which is located at some fixed parameter distance $r_w$. $B_{\text{mat}}$ is a bulk term due entirely to the $I_M$ of the radiation bath.
Call $T_w$ the temperature that is seen by an inertial observer at the wall. Tolman’s relation (which is a consequence of thermal equilibrium) indicates,

$$T_w = \frac{\tilde{T}}{\Omega(r_w)}$$

(22)

where $\tilde{T}$ is some constant temperature. Later it will be identified with the temperature of the ambient radiation. This relation implies that in the Euclidean continuation of these spacetimes, the $t$-direction of the manifold at $r = r_w$ is periodic, with extent $\Omega/\tilde{T}$. Thus we find that, for fixed $A_0$, $B_{\text{mat}}$ is the same linear function in $r_w$ for any solution. We are fundamentally interested in distinguishing the solutions I, II and III from one another and so one can show $B_{\text{mat}}$, the bulk term, may be ignored; differences between the solutions will be manifest in differences in their $I_G$ only.

Following Ref. [28] it is useful to consider the free energy $F$ as a function of $T_w$ and $D$, the total dilaton charge within the box. $D$ may be conveniently defined as the charge associated with the current

$$j^\mu = \epsilon^{\mu\nu} \partial_\nu e^{-2\phi}$$

(23)

where $\epsilon^{\mu\nu}$ is the antisymmetric covariant tensor ($\epsilon_{01} = \sqrt{-g}$) so $\nabla_\mu j^\mu = 0$. The charge within the box is thus

$$D = \int_{r}^{r_w} j_\alpha d\Sigma^\alpha = e^{-2\phi_w}$$

(24)

where $\phi_w = \phi(r_w)$ is the value of the dilation field at the wall.

It is now easy to compute the free energy $F = -T_w I$ from Eq. (21). For the spacetime II,

$$F^{\text{II}} = -\frac{\sqrt{2} \lambda}{\pi} D$$

(25)
and so, the entropy in the gravitational field is

\[ S^{\text{II}} = - \frac{\partial F^{\text{II}}}{\partial T_w} \bigg|_{D} = 0 \, . \]

As in the case without a radiation bath \((A_0 = 0)\), the pure LDV solution has zero entropy but finite energy \(U^{\text{II}} = F^{\text{II}} + T_w S^{\text{II}} = F^{\text{II}} \neq 0\). This may seem curious for a vacuum solution, but is expected since this is energy tied up in the dilaton field.

We now compute the free energy of spacetime I. With

\[ D = 4e^{-2\phi_0} \cosh^2 \left( \sqrt{2} \lambda r_w \right) \, , \quad T_w = T \cosh \left( \sqrt{2} \lambda r_w \right) \]

we find the free energy of spacetime I to be

\[ F^{\text{I}} = - \frac{\sqrt{2} \lambda D T}{\pi T_w} \sqrt{\left( \frac{T_w}{T} \right)^2 - 1} \]  

(27)

where \(T\) is a constant that we will relate to the temperature of the radiation bath (see Eq. (22)). Again using \(S = - \frac{\partial F}{\partial T_w} \bigg|_{D}\) we find the entropy to be

\[ S^{\text{I}} = \frac{\sqrt{2} \lambda D T}{\pi T_w^2 \sqrt{\left( \frac{T_w}{T} \right)^2 - 1}} \, . \]  

(28)

Note that the entropy is positive and that its “asymptotic” value is zero, which is consistent with the fact that the metric of I has no horizon. The energy \(U^{\text{I}} = F^{\text{I}} + T_w S^{\text{I}}\) is

\[ U^{\text{I}} = - \frac{\sqrt{2} \lambda D \left( \frac{T_w}{T} \right)^2 - 2}{\pi \left( \frac{T_w}{T} \right) \sqrt{\left( \frac{T_w}{T} \right)^2 - 1}} \, . \]  

(29)

Thus, an “asymptotic” observer would ascribe a mass to this spacetime,

\[ M^{\text{I}} = \lim_{r_w \to \infty} (U^{\text{I}} - U^{\text{II}}) = \frac{6\sqrt{2} \lambda e^{-2\phi_0}}{\pi} = \frac{6\sqrt{2} A_0}{\lambda} \]  

(30)
where we have made use of the fact that $e^{-2\phi_0} = \frac{A_0}{\lambda^2}$. Thus, as in the black hole case ($A_0 = 0$) we again find that the mass of the spacetime is related to the constant term of the dilaton. This reinforces the interpretation of metric I as that of the two-dimensional dilaton black hole in equilibrium with a radiation bath. It also concurs with the comparison of the transit times described in the beginning of this section.

The fact that this analysis has yielded an asymptotic mass for a spacetime that asymptotically has zero entropy seems to be at odds with the expectations that $S = \frac{M}{T}$. This is simply due to the fact that the quantities $F, M, T_w$ are really to be thought of as contravariant (i.e. possessing an upper index) quantities while $S$ and $T$ are really globally defined (scalar quantities). Roughly speaking, if the “asymptotic” region is not simply flat space, general covariance suggests one should expect a relation rather like $S = \frac{M}{T_w}$. Thus $S$ may vanish although $M$ is non-zero.

A final aside: Were we to repeat this computation of the mass for spacetime III we would find $M_{III} = -2\sqrt{2} A_0/2\lambda$. Negative mass solutions are not necessarily excluded since the dilaton contributes to the equations of motion in such a way that the dominant energy condition is violated and so the mass of spacetime is not bounded below. For spacetime III, it is related to the fact that this spacetime has a naked singularity. This identification of the mass is also consistent with the taxonomy described earlier, that we really should think of I, II and III as region a), LDV, b) of the extended black hole solution but in a bath of radiation.

It is straightforward to go beyond the static ansatz Eq. (4) and study the full linear response of the equations of motion. Linearizing Eq. (6) reveals that the static solutions are stable only to time-dependent perturbations that vanish spatially at least as $\Omega^4$ in the asymptotic region. However, such perturbations seem not to contribute to the asymptotic
mass, and so do not correspond to throwing mass into the “hole” (neck at $r = 0$). Time-dependent perturbations larger than this represent spacetimes that are asymptotically not constant curvature. Indeed there are two possible scenarios with regard to “throwing mass” into the “hole” in these spacetimes. Either the perturbation causes a shift in the dilaton, thereby increasing the mass of the spacetime or it causes collapse to ensue in which some of the radiation energy of the bath falls, with the perturbation, into the hole, perhaps forming a singularity. A cursory analysis indicates that the latter possibility occurs, but more work remains to be one. For a related discussion in four dimensions, see Refs. [27,39].

More fundamentally, the study of equilibrium spacetimes including effects like Hawking radiation should include a discussion of the equilibrium power spectrum of noise in the spacetime. The most natural way to do this is by treating the thermal bath quantum mechanically and understanding Hawking radiation as dissipation of (gravitational) energy from the geometry of the space to the bath. For a stable, equilibrium spacetime it should be possible to understand the coupling between the hole and the bath that is represented by the Hawking radiation in terms of linear transport coefficients. At this time it seems that to compute linear transport coefficients one would need a clearer picture (for example, a realization of Hawking radiation as arising from a Hamiltonian in the bath’s phase-space co-ordinates) of Hawking’s effect than the cursory one presented here. Nonetheless, the power spectrum of noise could be expressed in terms of these transport coefficients. Work is underway to ascertain how complete this view of equilibrium is for the spacetimes presented here and their cousins in four dimensions.
CONCLUSION AND SPECULATIONS

We have found static solutions of two-dimensional dilaton gravity coupled to matter that may be interpreted as a two-dimensional black hole at equilibrium in a bath of radiation. We have computed the thermodynamic potentials and identified the mass attributable to the black hole. We have also included the Hawking radiation’s back-reaction on the metric and dilaton in these solutions.

Of course, this toy model of gravitations is somewhat different than higher-dimensional Einstein–Hilbert gravity. However, we feel some ideas discussed here may be fruitfully explored in Einstein–Hilbert gravity in four dimensions.

We conclude with one intriguing speculation. There are many issues surrounding the final stages of black hole evaporation. Information loss and related questions aside, it is valid to ask simply how would one describe the final stage of the evaporation. Note that our spacetime I has two interesting properties: it has no horizon and the “mean density” ($\sim M\lambda$) of the “hole” is equal to the energy density of the bath $A_0$. This is essentially due to the fact that in two dimensions Newton’s constant is dimensionless. Our intuition would suggest that, for four-dimensional black holes, immersing the black hole in a radiation bath at the same temperature as that of the hole would make neither the horizon evaporate nor tame the singularity. This results from the fact that for macroscopic black holes in four dimensions, the radiant energy density at the hole’s temperature is much smaller than the “mean density” of the hole. As evaporation proceeds, and somewhat before one gets to temperatures of the Planck scale, the equilibrium radiation energy density outside the hole becomes comparable to the “mean density” of the hole itself. Furthermore, near the endpoint of the evaporation, since the area of the horizon is so minute and the radiation density is so high, any single
(interacting) quanta would likely see many radiation lengths of other quanta between it and spatial infinity. Thus locally, near the evaporating hole, something akin to a quasi-equilibrium state (with respect to time scales of $\mathcal{O}(1/M_p)$) may be achieved. In such a state in analogy to the solution I presented here, we conjecture the horizon may “evaporate” and the singularity melt away. It would be interesting to investigate the validity of this “picture” of the final stage of black hole evaporation in a more realistic model.

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**NOTE ADDED**

After completion of this project, the author received two papers, Refs. [40,41] which describe cosmological solutions that are somewhat related to the static spacetimes discussed here.
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Figure 1  The physical maximally extended spacetime diagrams of I, II and III  (There is no physical reason to extend spacetime beyond a strong coupling region.)