Graph state representation of the toric code

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Given their potential for fault-tolerant operations, topological quantum states are currently the focus of intense activity. Of particular interest are topological quantum error correction codes, such as the surface and planar stabilizer codes that are equivalent to the celebrated toric code. While every stabilizer state maps to a graph state under local Clifford operations, the graphs associated with topological stabilizer codes remain unknown. We show that the toric code graph is composed of only two kinds of subgraphs: star graphs (which encode Greenberger-Horne-Zeilinger states) and half graphs. The topological order is identified with the existence of multiple star graphs, which reveals a connection between the repetition and toric codes. The graph structure readily yields a log-depth quantum circuit for state preparation, assuming geometrically non-local gates, which can be reduced to a constant depth including ancillae and measurements at the cost of increasing the circuit width. The results provide a new graph-theoretic framework for the investigation of topological order and the development of novel topological error correction codes.

I. INTRODUCTION

Since the idea of topological quantum computation was first introduced by Kitaev in the form of the celebrated toric code [1–3], interest in finding ways to generate topological states and implement topological operations has remained strong [4–6], due to the potential for the implementation of fault-tolerant quantum gates with extremely high error thresholds [7]. In condensed matter physics, topologically ordered states are usually framed as the degenerate ground states of a specially chosen gapped local Hamiltonian [8–15]. In the quantum information community, topological models are framed in terms of stabilizers, which underpin the framework of quantum error correction codes (QECC) [16]. For example, the toric code states are the four-fold degenerate eigenstates of a ‘Hamiltonian’ consisting of the the negative sum of \( N - 2 \) toric-code stabilizer generators, where \( N \) is the number of physical qubits and the degeneracy is connected to the non-zero genus of the torus. The code distance of toric code is said to be ‘macroscopic’ as it scales with the number of physical qubits. The macroscopic code distance is a characteristic of the toric code and is a suitable proxy for the existence of topological order [17]. The toric code is the most well-studied topological model, not only because of its apparent simplicity but also because all two-dimensional translationally invariant topological stabilizer codes (so-called surface or planar codes, depending on the boundary conditions) are equivalent to it [18].

Every stabilizer state is equivalent to a graph state under local Clifford (LC) operations [19, 20]. However, little is currently known about the structure of the graph states that are LC-equivalent to topological stabilizer code states. What is the signature of topological order in the graph connectivity? What new insights into topology and topological QECC might this mapping enable? And, can the graph structure point to a specific state preparation-and/or logical encoding procedure for topological QECC via a quantum circuit?

In this work, we make the first step of addressing these questions by mapping a toric code state to its LC-equivalent graph state, which is denoted as the toric graph state. The degenerate toric code has two fewer stabilizer generators than is the case for graph states. In order to effect the map, one may supplement the generators with two closed ‘string’ operators; these consist of contiguous strings of X and Z gates that encircle the torus, which commute with all toric-code stabilizer generators and with one other. In this way, one can obtain the graph state that is LC-equivalent to any of the ground states of the 2D toric code, depending on the orientation of the string operators; these states are denoted as toric graph states in this work. These kinds of string operators correspond to the logical X and Z gates in the toric code, and can therefore map the target toric graph state to any other after the state preparation. Furthermore, as the strings have length \( L \sim \sqrt{N} \), the toric code distance is macroscopic as it scales with the number of physical qubits. The macroscopic code distance is a characteristic of the toric code and is a suitable proxy for the existence of topological order.

We find that the toric graph can be decomposed into only two distinct subgraphs: star graphs, where one vertex is connected to all other vertices and which define Greenberger-Horne-Zeilinger (GHZ) states [21] (see for example Fig. 1(a)), and half graphs [22] (see for example Fig. 1(b)). Perhaps surprisingly, the macroscopic distance of the toric code is identified with the existence of multiple star subgraphs in the toric graph, which reveals a connection to the repetition code such as Shor’s nine-qubit code [23].

Despite the fact that the number of edges in the toric code graph increases as \( L^2 \), the binary quadratic function defining the graph adjacency matrix [24] is shown to be decomposable into a \( \log(N) \) number of operations.
Using this insight, we provide explicit quantum algorithms to generate toric graph states and to encode arbitrary quantum states into the toric QECC in log depth. Furthermore, because any graph state can be generated in constant depth by including ancillae that are subsequently projected out via measurements \[25\], our algorithm can be expressed in constant depth at the cost of increasing the circuit width from scaling as \(N^3\) to scaling as \(L^3 \sim N^{3/2}\). Given the result in Ref. \[15\], this work therefore provides an algorithm for the preparation of any 2D topological stabilizer code state in either log depth including only unitary gates or in constant depth allowing for measurements of ancillae. The mapping to graph states therefore provides an alternative method to prepare 2D topological stabilizer code states in either log depth including only unitary gates or in constant depth allowing for measurements of ancillae, complementing currently known schemes \[26\]–\[31\].

This manuscript is organized as follows. The technical background is reviewed in Sec. \[II\]. The mapping of the toric code stabilizer to the graph is covered in Sec. \[III\] and the resulting graph structure and its decomposition are discussed. Section \[IV\] covers the construction of the log-depth quantum circuit that generates a toric graph state and its generalization for the preparation of any 2D topological stabilizer code state. The results are discussed briefly in Sec. \[V\]. Various technical details and proofs are included in the Appendices.

II. BACKGROUND AND FORMALISM

A. Stabilizer states and graph states

Define the Pauli group on \(N\) qubits as \(\mathcal{P}_N := \{\pm 1, \pm i\} \times \{I, X, Y, Z\}^\otimes N\), where

\[
X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]

(1)

correspond to the Pauli matrices. The set \(\mathcal{S} := \{S \in \mathcal{P}_N | S |\psi\rangle = |\psi\rangle\}\) is said to stabilize a state \(|\psi\rangle \in \mathcal{H}_2^\otimes N\). The set of states simultaneously stabilized by \(m\) independent operators \(\{S_1, \ldots, S_m\}\) from \(\mathcal{P}_N\) then generate a state subspace of dimension \(2^{N-m}\). When \(m = N\), the subspace contains only one state called the stabilizer state, and the \(N\) independent operators are the generators of \(\mathcal{S}\).

Graph states are special stabilizer states where the stabilizer generators are related to simple graphs \[21\]. Given a graph \(G = (V, E)\), where \(|V| = N\), the corresponding graph state is

\[
|G\rangle = \prod_{(i,j) \in E} CZ(i,j)H^\otimes N |0^N\rangle,
\]

(2)

for which the stabilizer generators are

\[
S_i = X_i \prod_{(i,j) \in E} Z_j.
\]

(3)

A graph can be represented by its adjacency matrix \(A \in \mathbb{Z}_2^{N \times N}\), where \(A_{ij} = 1\) iff \((i, j)\) is an edge in \(E\). With adjacency matrix \(A\), the graph state can also be written in terms of its binary quadratic form \[24\]

\[
|G\rangle = \frac{1}{\sqrt{2^N}} \sum_{q \in \{0,1\}^N} (-1)^{f_G(q)} |q\rangle,
\]

(4)

in which \(f_G(q) = \sum_{i<j} A_{ij}q_iq_j \pmod{2}\) is a quadratic Boolean function. Unless stated otherwise, the addition of binary variables is performed \(\pmod{2}\). There is therefore a useful correspondence among simple graphs, graph states, and quadratic Boolean functions.

As discussed in Sec. \[III\] the toric code maps to a graph, called the toric graph in what follows, that can be decomposed into two types of subgraphs: star and half graphs. These are reviewed here, and examples are shown in Fig. \[1\]. The star graph on \(m\) vertices is the complete bipartite graph \(K_{m-1,1}\), as shown in Fig. \[1a\]. Because the star graph is LC-equivalent to the complete graph \(K_m\) \[21\], the label of the large-degree vertex is arbitrary. Without loss of generality, the non-zero elements of the adjacency matrix are then \(A_{mi} = A_{im} = 1\) for \(i = 1, \ldots, m - 1\), or alternatively

\[
A_{ij}^{(\text{star})} = \delta_{i,m}\theta_{j,m-1} + \delta_{j,m}\theta_{i,m-1}, i, j \in \{1, \ldots, m\},
\]

(5)

where \(\delta_{i,j}\) and \(\theta_{i,j}\) are the usual Kronecker and Heaviside theta functions, respectively:

\[
\delta_{i,j} = \begin{cases} 1 & j = i \\ 0 & \text{otherwise} \end{cases}
\]

\[
\theta_{i,j} = \begin{cases} 1 & i \leq j \\ 0 & \text{otherwise}. \end{cases}
\]

(6)

The star graph state can then be written as

\[
|G^{(\text{star})}\rangle = \frac{1}{\sqrt{2^{m-1}}} |0^m\rangle + |1^m\rangle \otimes |0\rangle
\]

\[
= \frac{1}{\sqrt{2^m}} \sum_{q \in \{0,1\}^m} (-1)^{f_{\text{star}}(q)} |q\rangle,
\]

(7)

where \(f_{\text{star}}(q) = (q_1 + \cdots + q_{m-1}) \cdot q_m = q_m P(q_{m-1})\); here \(P\) is the parity operator acting on the length \(m - 1\) bit string \(q_{m-1} \equiv q_1 \oplus \cdots \oplus q_{m-1}\). Evidently, \(|G^{(\text{star})}\rangle\) is locally equivalent to an \(m\)-qubit GHZ state

\[
|1^m\rangle \otimes H |G^{(\text{star})}\rangle = \frac{1}{\sqrt{2}} \left| + \right>^\otimes m + \left| - \right>^\otimes m.
\]

(8)
The corresponding graph state can be written as
\[ |G\rangle = \prod_v \sigma_v^x \prod_p \sigma_p^z, \quad \text{(11)} \]
where the first sum is over the nearest qubits surrounding a given vertex \( v \) of the lattice, corresponding to a 'star' operator, while the second is over the nearest qubits surrounding the center \( p \) of a square, corresponding to a 'plaquette' operator. These operators are depicted in Fig. 2.

The star and plaquette operators can share at most two edges, so that the \( A_v \) and \( B_p \) commute. However, they are not entirely independent because
\[ \prod_v A_v = I, \quad \prod_p B_p = I, \quad \text{(12)} \]
where \( I \) is the identity. There are \( N - 2 \) independent stabilizer generators and the ground subspace of the associated Hamiltonian
\[ H = -\sum_v A_v - \sum_p B_p \quad \text{(13)} \]
is four-fold degenerate. In order to specify one state in the degenerate subspace, we add two more stabilizer generators \( S_\alpha \) and \( S_\beta \). One choice for the \( S_\alpha (S_\beta) \) corresponds to a string of \( Z \) (\( X \)) gates applied to the qubits residing only on vertical (horizontal) edges of a given row of the lattice. The choice of row is unimportant because of the translational invariance of the system; also, the rows for \( S_\alpha \) and \( S_\beta \) can coincide because the operations are on a different set of qubits.

### C. Local Clifford equivalence and the symplectic representation

In this work, we make heavy use of the symplectic representation of Pauli operators [16]. As this formalism is not widely employed, we briefly review the notation here, closely following the notation in Ref. [20]. Neglecting overall phases, a single Pauli matrix \( \sigma \) can be written as \( \sigma = Z^u X^v \), where \( u, v \in \{0,1\} \).

Alternatively, they can be represented as a binary tuple, \( I \rightarrow (00) \), \( X \rightarrow (01) \), \( Z \rightarrow (10) \) or equivalently the vectors
\[ I = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad X = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad Y = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \quad \text{(14)} \]

The generalization to \( N \) qubits \( \sigma_1 \otimes \cdots \otimes \sigma_N \) is then \( (u_1 \cdots u_N | v_1 \cdots v_N) \in \mathbb{Z}_2^N \), i.e., a 2\( N \)-dimensional binary vector. For example,
\[ X \otimes Z = (01|10) = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}. \quad \text{(15)} \]

The \( N \) stabilizer generators that uniquely define an \( N \)-qubit state can then expressed as full-rank \( 2N \times N \)-dimensional matrix \( S \); for example, the graph state corresponding to the two-vertex path graph \( P_2 \) is defined by...
the stabilizer generators \(S_1 = X \otimes Z\) and \(S_2 = Z \otimes X\), which are combined in the symplectic notation:

\[
S_{p_2} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{pmatrix},
\]

In general, the symplectic notation of the stabilizer generators for every graph state \(|G\rangle\) is

\[
S = \begin{pmatrix} A \\ I \end{pmatrix},
\]

where \(A\) is the adjacency matrix of graph \(G\). This form is referred to as the standard form of the stabilizer for graph states in this work. The central advantage of this formulation is that, because all of the stabilizer generators are mutually commuting, the matrix \(S\) is automatically self-orthogonal under the symplectic inner product \(S^T J S = 0\), where

\[
J = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}
\]

is the symplectic metric tensor (note that the sign convention is not the same as the symplectic algebra in classical Hamiltonian mechanics), and \(0\) represents the all-zero matrix.

Clifford operations, which maps the Pauli group to itself under conjugation, then correspond to \(2N \times 2N\) matrices \(Q\) that preserve the metric, i.e. \(Q^T J Q = J\) \([32]\). Local Clifford gates refer to those which are tensor products of local gates acting on single qubits. Two quantum states are local Clifford equivalent if they can be mapped to each other by local Clifford operations. Clifford operations that transform a stabilizer generator matrix \(S\) to another \(S'\) can always be written in the form \(S' = QSR\), where \(R\) is an \(N \times N\) invertible matrix corresponding to a basis change \([29]\). Restricting to local Clifford gates further implies that \(Q\) can be partitioned into four \(N \times N\) blocks, each of which is diagonal. For example, if we partition all qubits into two complementary sets \(R_1 \cup R_2 = \{1, \ldots, N\}\), then

\[
H_{R_2} = \bigotimes_{i \in R_1} I \bigotimes_{j \in R_2} H_j
\]

is a local Clifford operation and its symplectic representation is

\[
Q = \begin{pmatrix} I_{R_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{R_2} \\ 0 & 0 & I_{R_1} & 0 \\ 0 & I_{R_2} & 0 & 0 \end{pmatrix}.
\]

The transformation of the stabilizer is then effected by ordinary matrix-vector multiplication. These results imply a specific procedure to map any stabilizer generator matrix to standard (graph) form, which is used in this work to derive the (non-unique) graph that stabilizes the toric code.

III. TORIC GRAPH STATE

In this section, the toric code is first expressed in the symplectic notation. Then the toric graph state is obtained for one of the toric code states by specifying the two string operators. Finally, the toric graph is shown to be decomposable into star and half graphs, and prove that the subgraphs consisting of multiple star graphs contribute to the macroscopic distance of the toric QECC.

A. Symplectic representation of the toric code

In the toric code, qubits are located on the edges of a regular \(L \times L\) square lattice thus there are \(N = 2L^2\) qubits in total. It is convenient to distinguish the qubits situated on horizontal (\(x\)) and vertical (\(y\)) edges, whose locations on the grid are denoted by \((i,j,x)\) and \((i,j,y)\), respectively, where \(i,j \in \{1, L\}\). The star operator \(A_{ij}^x\) centered at coordinate \((i,j)\) then includes qubits with labels \{\((i−1,j,x),(i,j,x),(i,j−1,y),(i,j,y)\)\} \(mod\ L\); likewise, the plaquette operator \(B_{ij}^x\) with the coordinate \((i,j)\) located at the bottom left of a given plaquette includes qubits with labels \{\((i,j,x),(i,j+1,x),(i,j,y),(i+1,j,y)\)\} \(mod\ L\). Given that the star and plaquette terms apply \(X\) and \(Z\) gates, respectively, they can be expressed in the symplectic representation as

\[
A_{bin}^{ij} = \begin{pmatrix} 0 & v_{ij}^x \\ v_{ij}^x & 0 \end{pmatrix}; \quad B_{bin}^{ij} = \begin{pmatrix} p_{ij}^x & 0 \\ 0 & 0 \end{pmatrix},
\]

where \(A_{bin}^{ij}, B_{bin}^{ij}\) are \(4L^2\)-length vectors, and \(v_{ij}^x, p_{ij}^x \in \mathbb{Z}_2^{2L^2}\) are binary strings with elements defined by

\[
v_{ij}^{lmd} = \delta_{i,l}\delta_{j,m} + \delta_{i−1,l}\delta_{j,m}\delta_{d,x} + \delta_{i,l}\delta_{j−1,m}\delta_{d,y},
\]

\[
p_{ij}^{lmd} = \delta_{i,l}\delta_{j,m} + \delta_{i,l}\delta_{j+1,m}\delta_{d,x} + \delta_{i+1,l}\delta_{j,m}\delta_{d,y},
\]

where \(l, m \in \{1, \ldots, L\}\) and \(d \in \{x,y\}\). The toric code stabilizer, without the string operators, therefore consists of the antidiagonal block matrix

\[
S = \begin{pmatrix} 0 & Z_p \\ X_v & 0 \end{pmatrix},
\]

where the columns of the \(2L^2 \times 2L^2\) matrices \(X_v\) and \(Z_p\) correspond respectively to the \(v_{ij}^x\) and \(p_{ij}^x\), \(i,j \in \{1, L\}\).

With the expressions \([22]\), it is straightforward to prove the relations \([12]\) within the symplectic notation:

\[
\prod_v A_v = \prod_v \begin{pmatrix} 0 \\ X_v \end{pmatrix} = \left(\sum_{ij} v_{ij}^x\right).
\]

Note that evaluation of the terms in the sum above (and in what follows) is accomplished via bitwise exclusive or (XOR), with \(0 + 0 = 1 + 1 = 0\) and \(0 + 1 = 1 + 0 = 1\).
The resulting bitstring is then
\[
\sum_{ij} v^{ij} = \sum_{ij \ lmd} v^{ij}_{lmd} = \sum_{lmd} \left( \sum_{ij} \left( \delta_{1,l} \delta_{j,m} + \delta_{l-1,l} \delta_{j,m} \delta_{d,x} + \delta_{l,l} \delta_{j,1-m} \delta_{d,y} \right) \right) \\
= \sum_{lmd} \left( 1 + \sum_{d} \delta_{d,x} + \sum_{d} \delta_{d,y} \right) = 0, \tag{25}
\]
which yields
\[
\prod_{v} A_v = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = I, \tag{26}
\]
as expected. The second condition in Eq. (22) is found analogously. Note that in this work, the ⊕ notation represents a direct sum rather than an XOR operation.

### B. Star operators

To bring the stabilizer into the standard form as in Eq. (23), the $2L^2 \times 2L^2$ submatrix (0)$X_v$) must be transformed into an identity. Due to the constraints (22), the rank of $X_v$ is only $L^2 - 1$. The first step in the procedure is to form a full-rank $L^2 - 1$-dimensional matrix by taking linear combinations of the $X_v$ vectors to obtain a zero column vector. This column is replaced by $S\beta$, and finally column permutations yield an $L^2 \times L^2$ identity submatrix. The remaining $L^2 \times L^2$ identity submatrix will be obtained from the plaquette operators in Sec. IIIC.

The first linear combination is achieved by multiplying $X_v$ on the right by $R_v^x T^x$, where $T^x, R^x \in \mathbb{Z}_2^{L^2 \times L^2}$ are invertible matrices with elements defined by
\[
R^x_{kn,ij} = \theta_{i,k} \delta_{j,n}, \quad T^x_{kn,ij} = \begin{cases} \delta_{i,k} \theta_{j,n} & i = 1 \\ \delta_{i,k} \delta_{j,n} & \text{otherwise}. \end{cases} \tag{27}
\]

Consider first the action of $R^x$: it transforms the star operator represented by $v^{ij}$ to $v^{ij'}$. The column vectors of $X_v R^x$ are
\[
v^{ij'} = \sum_{kn} v^{kn} R^x_{kn,ij} = \sum_{kn} v^{kn} \theta_{i,k} \delta_{j,n} = \sum_{k} v^{kj} = \sum_{k=i}^{L} v^{kj} = \sum_{k=i}^{L} \left( \delta_{k,l} \delta_{j,m} + \delta_{k-1,l} \delta_{j,m} \delta_{d,x} + \delta_{k,l} \delta_{j,1-m} \delta_{d,y} \right) \\
= \sum_{k=i}^{L} \left( \delta_{d,x} \delta_{j,m} + \delta_{d,y} \delta_{k,l} \right) \\
= \sum_{k=i}^{L} \delta_{d,x} \delta_{j,m} (\delta_{k,l} + \delta_{k-1,l}) \\
+ \sum_{k=i}^{L} \delta_{d,y} (\delta_{j,m} + \delta_{j-1,m}). \tag{28}
\]
Let’s evaluate the first term above:
\[
\sum_{lmd} \delta_{d,x} \delta_{j,m} (\delta_{k,l} + \delta_{k-1,l}) \\
= \sum_{lmd} \delta_{d,x} \delta_{j,m} \left( \delta_{k,l} + \delta_{k-1,l} + \delta_{k+1,l} + \delta_{k,l} \\
+ \ldots + \delta_{L,l-1} + \delta_{L+1,l} + \delta_{L,l} \right) \\
= \sum_{lmd} \delta_{d,x} \delta_{j,m} (\delta_{k,l} + \delta_{L,l}). \tag{29}
\]
where only the endpoints in the sum are unpaired and therefore remain. One then obtains the matrix elements
\[
v^{ij'} = (\delta_{L,l} + \delta_{k-1,l}) \delta_{d,x} \delta_{j,m} \\
+ \theta_{i,l} (\delta_{j,m} + \delta_{j-1,m}) \delta_{d,y}. \tag{30}
\]

Next consider the action of $T^x$:
\[
v^{ij''} = \sum_{kn} v^{kn} T^x \Theta_{kn,ij} \\
= \sum_{kn} v^{kn} \delta_{i,k} \times \begin{cases} \theta_{j,n} & i = 1 \\ \delta_{j,n} & \text{otherwise} \end{cases} \tag{31}
\]
which transform $v^{ij'}$ to $v^{ij''}$. When $i \neq 1$, one has $v^{ij''} = v^{ij'}$, but for $i = 1$ one obtains
\[
v^{1j''} = \sum_{n=j}^{L} v^{1n}. \tag{32}
\]
Using Eq. (30) and the fact that $i - 1 = L$ when $i = 1$ due to the periodic boundary conditions, the matrix elements become
\[
v^{1j''} = \sum_{n=j}^{L} \delta_{d,y} (\delta_{n,m} + \delta_{n-1,m}) \theta_{1,l} \\
= \delta_{d,y} (\delta_{L,m} + \delta_{L-1,m}). \tag{33}
\]
Again because of the periodic boundary conditions, $v^{1j''} = 0$, which shows explicitly that the rank of $X_v R^x T^x$ is reduced by one. In order to make $X_v$ a full-rank matrix, the zero column is replaced by the string operator $S\beta = (0_{\beta})$, where $S\beta = \sum_{lmd} \delta_{d,y} \delta_{L,m}$. One then obtains
\[
v^{1j''} = v^{1j+1''} + s\beta = \sum_{lmd} \delta_{d,y} \delta_{j,m}. \tag{34}
\]
All other column vectors remain unchanged: $v^{ij'''} = v^{ij''} = v^{ij'}$ when $i \neq 1$.

It remains to show that one can extract an $L^2 \times L^2$ identity submatrix from $X_v$, whose columns are the $v^{ij'''}$. Consider two complementary subsets of row indices $R_1$ and $R_2$, defined as
\[
R_1 := \{1, \ldots, L-1\}_x \times \{1, \ldots, L\}_x \\
\cup \{1\}_y \times \{1, \ldots, L\}_y; \\
R_2 := \{L\}_x \times \{1, \ldots, L\}_x \\
\cup \{2, \ldots, L\}_y \times \{1, \ldots, L\}_y, \tag{35}
\]
where \( \{ \cdots \}_d \times \{ \cdots \}_d \) represents the row and column indices of the original square lattice and \( d \) indicates a horizontal \( x \) or vertical \( y \) qubit. Clearly, both \( R_1 \) and \( R_2 \) contain \( L^2 \) rows. Combining the results of Eqs. (30) and (31), one obtains the elements

\[
\begin{align*}
\psi_{lmd}^{ij}'' &= \delta_{j,m} \times \begin{cases} 
\delta_{i,j} \delta_{d,y} & i = 1 \\
\delta_{i,j} \delta_{d,x} & i \geq 2;
\end{cases} \\
\psi_{lmd}^{ij}''' &= \begin{cases} 
\theta_{2,l} \delta_{j,m} \delta_{d,y} & i = 1 \\
\delta_{L,l} \delta_{j,m} \delta_{d,x} + \theta_{i,l} (\delta_{j,m} + \delta_{j-1,m}) \delta_{d,y} & i \geq 2.
\end{cases}
\end{align*}
\]

(36)

There is only one non-zero element in each bitstring \( v_{R_1}^{ij}''' \) and its location is unique. Therefore, \( X_v^{R_1}''' \) is an \( L^2 \times L^2 \) identity matrix after appropriate permutation of columns, which will be effected in Sec. IID.

### C. Plaquette operators

The plaquette operators are treated in much the same way as the star operators discussed in Sec. IID but form linear combinations of the \( Z_p \) column vectors to obtain a zero column vector, then replace this with the string operator \( S_\alpha \) to make \( Z_p \) a full-rank matrix, and finally use linear combinations again to extract an \( L^2 \times L^2 \) submatrix.

Define invertible operators \( T^z, R^z \in \mathbb{Z}_2^{L^2} \) with elements

\[
R_{kn,ij}^z = \delta_{k,i} \delta_{j,n}; \\
T_{kn,ij}^z = \delta_{i,k} \begin{cases} 
\theta_{n,j} & i = L \\
\delta_{n,j} & \text{otherwise}.
\end{cases}
\]

(37)

The column vector of \( Z_p R^z \) is

\[
p^{ij}''' = \sum_{kn} p^{kn} R_{kn,ij}^z = \sum_k p^{kj} \theta_{k,i} = \sum_k p^{kj} \\
= \sum_k \bigoplus_{lmd}(\delta_{k,l} \delta_{j,m} + \delta_{k,l} \delta_{j-1,m} \delta_{d,x} + \delta_{k+1,l} \delta_{j,m} \delta_{d,y}) \\
= \bigoplus_{lmd} \delta_{d,x}(\delta_{j,m} + \delta_{j+1,m}) \sum_{k=1}^i \delta_{k,l} \\
+ \bigoplus_{lmd} \delta_{d,y} \delta_{j,m} \sum_{k=1}^i (\delta_{k,l} + \delta_{k+1,l}) \\
= \bigoplus_{lmd} \theta_{l,i} (\delta_{j,m} + \delta_{j+1,m}) \delta_{d,x} + (\delta_{1,l} + \delta_{1+1,l}) \delta_{j,m} \delta_{d,y}.
\]

(39)

The column vector of \( Z_p R^z T^z \) is \( p^{ij}'''' = p^{ij}''' \) when \( i \neq L \). When \( i = L \), one can make use of the periodic boundary conditions to obtain

\[
p^{Lj}''' = \sum_{kn} p^{kn} T_{kn,Lj}^z = \sum_n p^{Ln} \theta_{n,j} = \sum_{n=1}^i p^{Ln} \\
= \bigoplus_{lmd} \sum_{n=1}^i (\delta_{n,m} + \delta_{n+1,m}) \theta_{i,L} \delta_{d,x} \\
= \bigoplus_{lmd} (\delta_{1,m} + \delta_{j+1,m}) \delta_{d,x}.
\]

(40)

Again, one obtains a zero vector, \( p^{LL}''' = 0 \). In order to yield a full-rank matrix for \( Z_p \), we include the string operator \( S_\alpha = (s_\alpha |0\rangle, \) where \( s_\alpha = \bigoplus_{lmd} \delta_{d,x} \delta_{m,1}. \) Then

\[
p^{Lj}'''' = p^{Lj-1}'''' + s_\alpha = \bigoplus_{lmd} \delta_{j,m} \delta_{d,x}.
\]

(41)

All other column vectors remain unchanged: \( p^{ij}''' = p^{ij}''' \) when \( i \neq L \).

The column vector of matrix \( Z_p''' \) is \( p^{ij}''' \), from which one can extract another \( L^2 \times L^2 \) identity submatrix. Again using the complementary subsets of row indices \( R_1 \) and \( R_2 \), Eq. (35), the elements of the plaquette bitstrings become

\[
p_{lmd}^{ij}''' = \begin{cases} 
\theta_{i,j} (\delta_{j,m} + \delta_{j+1,m}) \delta_{d,x} + \delta_{l,i} \delta_{j,m} \delta_{d,y} & i < L \\
\theta_{i,l} \delta_{j,m} \delta_{d,x} & i = L;
\end{cases} \\
p_{lmd}^{ij}'''' = \begin{cases} 
\delta_{j+1,m} \delta_{d,y} & i < L \\
\delta_{j,i} \delta_{d,x} & i = L.
\end{cases}
\]

(42)

As was the case for the star operators, there is only one non-zero element in each bitstring \( p_{R_2}^{ij}''' \) and its location is unique. Therefore, \( Z_{p_{R_2}}''' \) is another \( L^2 \times L^2 \) identity matrix after appropriate permutation of columns, which will be effected in Sec. IID. Combining these results with those in Sec. IID, the toric code stabilizer in symplectic form, Eq. (23), is now transformed to

\[
S \rightarrow \begin{pmatrix} 0 & Z_p''' \\
X_v''' & 0 \\
\end{pmatrix} = \begin{pmatrix} 0 & Z_p''' \\
X_v'''_R & 0 \\
\end{pmatrix}.
\]

(43)

### D. Transformation to standard form

To convert the stabilizer (43) to standard form, one first applies Hadamard operations to the \( R_2 \) qubits,

\[
\begin{pmatrix} 0 & Z_p''' \\
X_v''' & 0 \\
\end{pmatrix} \begin{pmatrix} 0 & Z_p'''_R \\
X_v'''_{R_2} & 0 \\
\end{pmatrix} = \begin{pmatrix} A & B \\
\end{pmatrix},
\]

(44)

where \( Q \) is defined in Eq. (20). It remains to convert \( B \) to a \( 2L^2 \times 2L^2 \) identity matrix, which is accomplished
by appropriate column permutations. The columns of $B$, expressed as $b^{ij}_{ld}$, are $v^{ij}_{lR_1} \oplus 0_{R_2}$, and $0_{R_1} \oplus p^{ij}_{lmdR_2}$, where according to Eqs. (36) and (42)

$$v^{ij}_{lmdR_1} = \delta_{l,m} \times \begin{cases} 
\delta_{l,i}\delta_{d,y} & i = 1; \\
\delta_{l-1,i}\delta_{d,x} & i \geq 2;
\end{cases}$$

$$v^{ij}_{lmdR_2} = \delta_{l,m} \times \begin{cases} 
\delta_{l+1,i}\delta_{d,y} & i < L; \\
\delta_{l,L}\delta_{d,x} & i = L,
\end{cases}$$

and these are to be converted to the matrix elements of the identity

$$v^{ij}_{lmd} = \delta_{l,i}\delta_{j,m}\delta_{d_1,d_2},$$

where $i, j, l, m \in \{1, \ldots, L\}$ and $d_1, d_2 \in \{x, y\}$. When $i \in \{2, \ldots, L\}$, the column vector of $B$ is

$$v^{ij}_{lmdR_1} \oplus 0_{R_2} = \bigoplus_{lmd \in R_1} \delta_{l-1,i}\delta_{j,m}\delta_{d,x} \oplus 0_{lmdR_2}$$

$$= \bigoplus_{lmd \in R_1} \delta_{l-1,i}\delta_{j,m},$$

where the only nonzero entry is at row $(i-1, j, x)$. To map to the form in Eq. (46), relabel this column vector:

$$v^{ijx}_{lR_1} = v^{i+1,jx}_{lR_1} \oplus 0_{R_2} = \bigoplus_{lmd \in R_1} \delta_{l,i}\delta_{j,m}\delta_{d,x},$$

where now $i \in \{1, \ldots, L-1\}$. Any column permutation on $B$ must also be performed on $A$. Recall from Eqs. (36) and (42) that the columns of $A$, expressed as $a^{ij}_{ld}$, are expressed as $p^{ij}_{lR_1} \oplus 0_{R_2}$ and $0_{R_1} \oplus v^{ij}_{lR_2}$, with elements

$$p^{ij}_{lmdR_1} = \begin{cases} 
\theta_{l,i}\delta_{j,m} + \delta_{j+1,m} & i < L; \\
\theta_{l,L}\delta_{j,m} & i = L;
\end{cases}$$

$$v^{ij}_{lmdR_2} = \begin{cases} 
\theta_{l+1,i}\delta_{j,m} & i = 1; \\
\theta_{l,L-1}\delta_{j,m} \delta_{d,x} + \theta_{l,i}\delta_{j,m} + \delta_{j-1,m} & i \geq 2.
\end{cases}$$

From Eq. (44), the column vector in $A$ with the same column index as $v^{ij}_{lR_1} \oplus 0_{R_2}$ is $0_{R_1} \oplus v^{ij}_{lR_2}$. Then

$$a^{ijx}_{l} = 0_{R_1} \oplus v^{i+1,jy}_{lR_2}$$

$$= \bigoplus_{lmd} \delta_{l,i}\delta_{j,m}\delta_{d,x} + \theta_{l+1,i}\left(\delta_{j,m} + \delta_{j-1,m}\right)\delta_{d,y},$$

again with $i \in \{1, \ldots, L-1\}$. Likewise, when $i = L$:

$$b^{ijx}_{l} = 0_{R_1} \oplus b^{i,jy}_{lR_2} = \bigoplus_{lmd} \delta_{l,L}\delta_{j,m}\delta_{d,x},$$

$$a^{ijy}_{l} = p^{ij}_{lR_1} \oplus 0_{R_2} = \bigoplus_{lmd} \theta_{l,L-1}\delta_{j,m}\delta_{d,x};$$

when $i \in \{2, \ldots, L\}$:

$$b^{ijy}_{l} = 0_{R_1} \oplus b^{i,jy}_{lR_2} = \bigoplus_{lmd} \delta_{l,i}\delta_{j,m}\delta_{d,y},$$

$$a^{ijy}_{l} = p^{i-1,jy}_{lR_1} \oplus 0_{R_2} = \bigoplus_{lmd} \theta_{l,i-1}\left(\delta_{j,m} + \delta_{j+1,m}\right)\delta_{d,x} + \delta_{l,i}\delta_{j,m}\delta_{d,y};$$

and when $i = 1$:

$$b^{ijy}_{l} = v^{ij}_{lR_1} \oplus 0_{R_2} = \bigoplus_{lmd} \delta_{l,1}\delta_{j,m}\delta_{d,y},$$

$$a^{ijy}_{l} = 0_{R_1} \oplus v^{i+1,jy}_{lR_2} = \bigoplus_{lmd} \theta_{2,i}\delta_{j,m}\delta_{d,y}. $$

That $B$ is an identity matrix after column permutations is clear from Eqs. (48), (51), (53), and (55).

Finally, the adjacency matrix $A$ for the toric code graph is obtained by combining Eqs. (50), (52), (54), and (56):

$$a^{ij}_{lmd_2} = \delta_{d_1,x}\delta_{d_2,z}\delta_{m,j}\left(\delta_{l,1}\delta_{l,-1} + \delta_{l,L}\delta_{l,-1}\right) + \delta_{d_1,y}\delta_{d_2,y}\delta_{m,j}\left(\delta_{l,1}\delta_{l,2} + \delta_{l,1}\delta_{l,2}\right) + \delta_{d_1,y}\delta_{d_2,y}\delta_{m,j}\delta_{m-1,j}\delta_{l,-1}\delta_{l,2} + \delta_{d_1,x}\delta_{d_2,y}\delta_{m,j}\left(\delta_{l,-1}\delta_{l,1}\delta_{l,2}\right) \theta_{l+1,1}\delta_{l,-1}\delta_{l,1}. $$

Note that the matrix elements in the expression above are symmetric under $i \leftrightarrow l, j \leftrightarrow m$ and $d_1 \leftrightarrow d_2$; the apparent lack of symmetry in the last two lines is resolved by noting that

$$\theta_{l+1,1}\delta_{l,-1}\delta_{l,1} = \theta_{l,1}\delta_{l,2}\delta_{l,-1} = \theta_{l,-1}\delta_{l,2}. $$

Eq. (57) is the first of two key results of the present work. The graph represented by the adjacency matrix will be referred to as the toric graph, illustrated in Fig. 3 and the corresponding graph state is called the toric graph state.

The result (57) was checked in two ways. First, the graph state for small systems was generated explicitly and compared with the toric code state on the same number of qubits. Second, the reduced density matrices and entanglement entropies for various bipartitions of the two systems were compared and found to agree in all cases.
E. Decomposition of the toric graph

The elements of the graph adjacency matrix, Eq. (57), and the associated graph shown in Fig. 3 appear too complicated to gain any insights about why this particular structure corresponds to a topological quantum state. However, it turns out this graph can be decomposed into three subgraphs, all of which have a rather simple structure. In particular, the adjacency matrix given can be decomposed into three terms:

$$A = A_{\text{mstar}} + A_{\text{mhalf1}} + A_{\text{mhalf2}},$$

(59)

where the entries of those three matrices are

$$A_{i_1j_1d_1,i_2j_2d_2}^{\text{mstar}} = \delta_{d_1,0}\delta_{d_2,0}\delta_{j_1,j_2}(\delta_{i_1,1}\theta_{2,i_2} + \delta_{i_1,-1}\theta_{2,i_2}) + \delta_{d_1,1}\delta_{d_2,2}\delta_{j_1,j_2}(\delta_{i_1,1}\theta_{2,i_2} + \delta_{i_1,-1}\theta_{2,i_2});$$

$$A_{i_1j_1d_1,i_2j_2d_2}^{\text{mhalf1}} = \delta_{d_1,1}\delta_{d_2,2}\delta_{j_1,j_2}(\delta_{i_1,1}\theta_{2,i_2} + \theta_{2,i_2}) + \delta_{d_1,1}\delta_{d_2,2}\delta_{j_1,j_2}(\delta_{i_1,1}\theta_{2,i_2} + \theta_{2,i_2});$$

$$A_{i_1j_1d_1,i_2j_2d_2}^{\text{mhalf2}} = \delta_{d_1,1}\delta_{d_2,2}\delta_{j_1,j_2}(\delta_{i_1,1}\theta_{2,i_2} + \theta_{2,i_2}) + \delta_{d_1,1}\delta_{d_2,2}\delta_{j_1,j_2}(\delta_{i_1,1}\theta_{2,i_2} + \theta_{2,i_2}).$$

(60)

These three matrices corresponding to three subgraphs $G_{\text{mstar}}, G_{\text{mhalf1}},$ and $G_{\text{mhalf2}}$, respectively.

First consider the subgraph $G_{\text{mstar}}$ and the subset of vertices

$$Q^1_{jd} := \{1, \ldots, L\}_d \times \{j\}_d,$$

(61)

following the notation of Eq. (55). Because of the restriction $\delta_{j_1,j_2}$ in the definition of $A_{\text{mstar}}$ above, there is no edge in $G_{\text{mstar}}$ connecting qubits in different subsets $Q^1_{jd}$. There are $2L$ disconnected components in total, and the adjacency matrices of components $Q^1_{jx}$ and $Q^1_{jy}$ have elements

$$(A_{jx}^{\text{mstar}})_{i_1,i_2} = A_{i_1,jx,i_2,jx} = \delta_{i_1,1}\theta_{2,i_2} + \delta_{i_1,-1}\theta_{2,i_2};$$

$$(A_{jy}^{\text{mstar}})_{i_1,i_2} = A_{i_1,jy,i_2,jy} = \delta_{i_1,1}\theta_{2,i_2} + \delta_{i_1,-1}\theta_{2,i_2}$$

(62)

respectively. From Eq. (55), the induced subgraphs on $Q^1_{jx}$ and $Q^1_{jy}$ are $L$-vertex star graphs with $(L,j,x)$ and $(1,j,y)$ being the central vertices, respectively, as shown in Fig. 4(a). Thus, ‘mstar’ is an abbreviation for ‘multiple star graphs.’

The graph $G_{\text{mhalf1}}$ is similarly made up of $L$ disconnected components, indexed by $j$. The adjacency matrix elements are

$$(A_{j}^{\text{mhalf1}})_{i_1,i_2d_1,i_2d_2} = A_{i_1j,jx,i_2,jy} = \delta_{d_1,0}\delta_{d_2,0}\delta_{j_1,j_2}(\delta_{i_1,1}\theta_{2,i_2} + \delta_{i_1,-1}\theta_{2,i_2}) + \delta_{d_1,1}\delta_{d_2,2}\delta_{j_1,j_2}(\delta_{i_1,1}\theta_{2,i_2} + \delta_{i_1,-1}\theta_{2,i_2}).$$

(63)

When $i_1 = 1$, $\theta_{2,i_1}$ is always zero, so vertex $(1,j,y)$ is isolated; likewise $\theta_{i_1,1}\theta_{2,i_2}$ is always zero when $i_1 = L$ and $(L,j,x)$ is also isolated. Based on Eqs. (63) and (59), $G_{\text{mhalf1}}$ corresponds to multiple copies of a $2(L-1)$-vertex half graph, as shown in Fig. 4(a).

$G_{\text{mhalf2}}$ is almost the same as $G_{\text{mhalf1}}$: composed of $L$ disconnected components and each component is a $2(L-1)$-vertex half graph. The only difference is that the vertices in each of the components are different:

$$Q^2_{jd} := \{1, \ldots, L\}_d \times (\{j\}_y \cup \{j+1\}_x).$$

(64)

The adjacency matrix of induced subgraph on each component $Q^2_{j}$ is the same as in Eq. (63), and the graph is shown in Fig. 4(c).

F. Observations on the toric graph structure

As discussed in Sec. II A, the stabilizer state represented by the star graph is LC-equivalent to the GHZ state, so the graph state represented by $G_{\text{mstar}}$ is LC-equivalent to multiple copies of the GHZ state. In fact,
such a multiple-copy GHZ state is already ‘topologically ordered,’ in the sense that it is a code state in a quantum error correction code with macroscopic distance \( d \sim \sqrt{N} \).

Consider two \( m\)-qubit GHZ states

\[
|\phi^+_m\rangle = \frac{|0^m\rangle + |1^m\rangle}{\sqrt{2}}, \quad |\phi^-_m\rangle = \frac{|0^m\rangle - |1^m\rangle}{\sqrt{2}}.
\]

and their \( m\)-copy states on \( m^2 \) qubits

\[
|\varphi^+\rangle = (|\phi^+_m\rangle)^\otimes m, \quad |\varphi^-\rangle = (|\phi^-_m\rangle)^\otimes m.
\]

Then, \( \text{span}_C \{ |\varphi^+\rangle, |\varphi^-\rangle \} \) is a quantum error correction code with distance \( d = m \). The proof is given in Appendix [A]. The reader might recognize that when \( m = 3 \), the code \( \text{span}_C \{ |\varphi^+\rangle, |\varphi^-\rangle \} \) is nothing but Shor’s celebrated nine-qubit (repetition) code [23,32]. The decomposition of the toric graph thus reveals an intriguing and apparently novel connection between the toric code and the repetition code.

The close connection between multiple GHZ states and the toric code is perhaps surprising. On the one hand, the GHZ states represent the long-range entanglement exhibited by topological states, spanning the length of the system. On the other hand, GHZ states are the most fragile many-qubit entangled states; a single measurement of any of the constituent qubits deletes all of the edges within the star graph. Because of the translational invariance, however, the resulting toric graph maintains the same connectivity on the remaining qubits. Thus, the toric graph is effectively invariant under single-qubit measurements, demonstrating the robustness of the underlying topology.

Moreover, the multi-copy GHZ state is also a simple example of the distance balancing technique proposed in Ref. [34]. Consider the code \( \text{span}_C \{ |\varphi^+\rangle, |\varphi^-\rangle \} \), in which \( d_X = m \) and \( d_Z = 1 \), where \( d_X \) and \( d_Z \) are the distances with respect to \( X \) and \( Z \), respectively. The distance balance method takes \( d_X / d_Z \) copies of such a code and outputs a new code with distance \( d_X = d_Z = \sqrt{d_X} \), at the cost of increasing the number of physical qubits. By adding another layer of structure corresponding to the half graphs, the degree of many vertices is not still bounded, yet the weight of the original stabilizer generators in the original (toric) code remains constant. The star graphs and half graphs thus play different key roles: the multi-star graphs contribute the large code distance while the half graphs ensure local stabilizer generators.

There is also a close resemblance between the code \( \text{span}_C \{ |\varphi^+\rangle, |\varphi^-\rangle \} \) and the recently proposed repetition cat code [35] in the continuous variable setting, where two approximately orthogonal coherent states are used as the qubit registers \( |0\rangle_c = |+\alpha\rangle \) and \( |1\rangle_c = |-\alpha\rangle \). Similar to \( \text{span}_C \{ |\varphi^+_0\rangle, |\varphi^-_0\rangle \} \), the bit flip error is exponentially suppressed \( (d_X = m) \) while the phase error is likely to occur \( (d_Z = 1) \). One therefore defines the repetition cat qubit state as \( |\pm\rangle_{L} = |\pm\rangle_{c^\otimes r} \) in order to correct the phase error, which corresponds to \( \text{span}_C \{ |\varphi^+\rangle, |\varphi^-\rangle \} \) if \( r = m \).

### IV. TORIC GRAPH STATE GENERATION

This section focuses on how to generate toric code states, and encode arbitrary quantum states in the toric QECC, within the quantum circuit model. We show that this can be accomplished in log depth in the absence of ancillae, and in constant depth including ancillae. The key step is to construct log-depth quantum circuits that generate the two toric code subgraphs: the star and half graphs.

The goal is to prepare the quantum state

\[
|f\rangle = \frac{1}{\sqrt{2^N}} \sum_{q \in \{0,1\}^N} (-1)^{f(q)} |q\rangle
\]

where the Boolean function \( f : \{0,1\}^N \rightarrow \{0,1\} \) is associated with the binary quadratic form for a graph, Eq. (4). Given an operator \( U_f \), which implements

\[
U_f |q\rangle = (-1)^{f(q)} |q\rangle, \quad \forall q \in \{0,1\}^N,
\]

one has \( |f\rangle = U_f H^\otimes N |0^N\rangle \). Moreover, if \( f \) can be decomposed into the sum (module 2) of other Boolean functions, i.e. \( f(q) = f_1(q) + \cdots + f_k(q) \), then \( |f\rangle \) can be generated by applying the commuting operators in sequence:

\[
|G\rangle = U_f^{(k)} \cdots U_f^{(1)} H^\otimes N |0^N\rangle.
\]

The target is the toric graph state

\[
|G_{\text{toric}}\rangle = \frac{1}{\sqrt{2L^2}} \sum_{q \in \{0,1\}^2L^2} (-1)^{f_{\text{toric}}(q)} |q\rangle,
\]

where \( f_{\text{toric}}(q) = \sum_{n_1 < n_2} q_{n_1} q_{n_2} (A_{\text{toric}})_{n_1 n_2} \) is the quadratic Boolean form related to toric graph and

\[
(A_{\text{toric}})_{n_1 n_2} = (A_{\text{toric}})_{i j d_1 d_2} \equiv \{ i j d_1 d_2 \}
\]

are the matrix elements of the graph adjacency matrix, Eq. (57). The variables \( n_1, n_2 \in \{1, \ldots, 2L^2\} \) are mapped to the qubit lattice coordinates \( i j d \) via

\[
(i, j, d) \leftrightarrow i + (j - 1) L + \delta_{d,y} L^2.
\]

Based on the decomposition of the toric graph and the relation between quadratic Boolean forms and the graph adjacency matrices, \( f_{\text{toric}} \) can be decomposed into

\[
f_{\text{toric}} = f_{\text{star}} + f_{\text{half1}} + f_{\text{half2}} = \sum_{j d} f_{\text{star}} + \sum_{d} f_{\text{half1}} + \sum_{j} f_{\text{half2}},
\]

where the adjacency matrices for \( f_{\text{star}}, f_{\text{half1}}, \) and \( f_{\text{half2}} \) are defined in Eqs. (62) and (63) respectively. Thus, \( |G_{\text{toric}}\rangle \) can be generated by the following circuits

\[
|G_{\text{toric}}\rangle = U_{\text{half}} U_{\text{star}} H^\otimes 2L^2 |0^\otimes 2L^2\rangle.
\]
where
\[ U_{\text{half1}}^{m} = \prod_{j} U_{f,j}^{\text{half1}} \prod_{j} U_{f,j}^{\text{half2}}, \] (75)
\[ U_{\text{star}}^{m} = \prod_{j \delta} U_{f,j \delta}^{\text{star}}. \] (76)

Different \( U_{f,j \delta}^{\text{star}} \) operators have the same circuit depth, as they all compute the same quadratic Boolean function associated with an \( L \)-vertex star graph; moreover, each acts on different subsets of qubits, so all can be performed in parallel. The situation is similar for \( U_{f,j}^{\text{half1}} \) and \( U_{f,j}^{\text{half2}} \). To summarize: the depth of the toric code quantum circuit in Eq. \( \text{(74)} \) corresponds to the sum of the circuit depths for \( U_{f,1}^{\text{star}} \), \( U_{f,1}^{\text{half1}} \) and \( U_{f,1}^{\text{half2}} \).

**A. Generation of the star graph state**

The \( f^{\text{star}} \) corresponds to the quadratic form for each \( L \)-vertex star graph, which is generically expressed as \( f(q) = (q_1 + \cdots + q_{L-1} + q_{c+1} + \cdots + q_L) q_c \) with \( c \) labeling the high-degree central vertex, the value of which is unimportant. The term in parentheses corresponds to the parity \( P(q_1, \ldots, q_n) \) of an \( n \)-length string \( q_1 \oplus \cdots \oplus q_n \). The log-depth quantum circuit is inspired by the classical parity algorithm in a parallel setting. First, divide all \( n \) elements into \( n/2 \) disjoint pairs and calculate the parity of each subset in parallel; then continue subdividing until only one pair remains. It requires \( \log_2 n \) iterations to obtain the parity of the bitstring (this of course ignores the \( n \) bits of classical communication required). In the quantum setting, the parity doesn’t need to be calculated; only the two-qubit gates need to be implemented that generate the appropriate contribution to the Boolean function. The operations at each iteration commute, and therefore they can be truly implemented in parallel.

Without loss the generality, suppose the central vertex is the last one, and the unitary operation implementing Eq. \( \text{(68)} \) with \( f(q) = (q_1 + \cdots + q_{L-1} - 1) q_L \) is
\[ U_{f}^{\text{star}} = P^{-1} CZ(QL-1,QL)P, \] (77)
in which \( P \) is a quantum operation that generates the parity linear form:
\[ P \ket{q} = \ket{q'}, \] (78)
where \( q'_{L-1} = q_1 + \cdots + q_{L-1} \). The operator \( P \) can be implemented using a series of \( CX = CNOT \) gates, which have the action
\[ CX(1, 2) \ket{q_1, q_2} = \ket{q_1, q_1 + q_2}. \] (79)
One first divides the first \( L - 1 \) qubits into \( (L - 1)/2 \) disjoint pairs and calculates the parity of each pair in parallel:
\[ \prod_{i=1}^{(L-1)/2} CX(2i - 1, 2i) \ket{q} = \ket{q'}, \] (80)
where \( \ket{q'} \) denotes the state in quantum register after first iteration and \( \hat{q}_{2i} = q_{2i-1} + q_{2i} \). If \( L - 1 \) is odd, then the \( L - 1 \)-th qubit does not need to be explicitly paired. One then divides the \( \lceil L/2 \rceil \) quantum registers with the parity result to \( (L - 1)/4 \) pairs, and repeats the procedure until all of the clauses have been paired. After \( \log(L - 1) \) iterations, one obtains the result in the \( L - 1 \)-th register as \( \hat{q}_{L-1} = q_1 + \cdots + q_{L-1} \). Next, the CZ gate implements the required phase:
\[ CZ(L-1, L) \ket{q'} = (-1)^{\hat{q}_{L-1} \cdot (q'_L)} \ket{q'} = (-1)^{(q_1 + \cdots + q_{L-1})q_L} \ket{q'}. \] (81)
As \( U_{f}^{\text{star}} \) should yield \((-1)^{f(q)} \ket{q} \) as the output state, one must implement the inverse of \( P \) to change \( \ket{q'} \) to \( \ket{q} \):
\[ P^{-1}(-1)^{f(q)} \ket{q'} = (-1)^{f(q)} \ket{q}. \] (82)

The nine-qubit example for \( U_{f}^{\text{star}} \) is shown in Fig. 5. The construction of \( P \) for arbitrary number of qubits, Algorithm \( \text{A} \) and the proof of its log depth are given in Appendix \( \text{B} \). Thus, the depth of \( U_{f}^{\text{star}} \) is \( O(\log L) \). Note that a log-depth circuit for the realization of GHZ states has been obtained recently by other means \( \text{[36]} \).

**B. Generation of the half graph state**

The Boolean quadratic forms \( f^{\text{half1}} \) and \( f^{\text{half2}} \) are associated with the \( 2(L-1) \)-vertex half graph, so the current task is to construct a quantum circuit that computes \( f_h \). For a \( 2n \)-vertex half graph, \( f_h(q, p) = \sum_{i,j} q_i p_j \), where \( i, j \in \{1, \ldots, n\} \). Moreover, \( f_h(q, p) \) can be decomposed in the following way:
\[ f_h(q, p) = \sum_{k=0}^{\lceil \log n \rceil} f_h^{(k)}(q, p), \] (83)
assuming that $n$ is a power of two. Otherwise, one need only replace the sum upper bound by $n/2^k \rightarrow \lceil n/2^k \rceil$, $2^k(i + 1/2) \rightarrow \min\{2^k(i + 1/2), n\}$, and $2^k(i + 1) \rightarrow \min\{2^k(i + 1), n\}$. For simplicity of analysis one can assume that $n$ is a power of two, but the results hold for arbitrary integer values. A few decomposed Boolean functions are listed as follows:

$$f_h^{(0)}(q, p) = \sum_{i=0}^{n} q_{i+1} p_{i+1};$$

$$f_h^{(1)}(q, p) = \sum_{i=0}^{n/2} q_{2i+1} p_{2i+2};$$

$$f_h^{(2)}(q, p) = \sum_{i=0}^{n/4-1} (q_{4i+1} + q_{4i+2})(p_{4i+3} + p_{4i+4});$$

$$f_h^{(\log_2 n)}(q, p) = \left( \sum_{j=1}^{n/2} q_{j} \right) \left( \sum_{j=n/2+1}^{n} p_{j} \right).$$ (85)

The decomposition for $f_h(q, p)$ when $n = 2^3$ is shown in Fig 6 from which one may obtain an intuition of why the decomposition Eq. (83) holds in general. In Fig 6, the columns correspond to variable $q_i$, and the rows correspond to variable $p_i$. Each element represents a term $q_i p_j$ appearing in $f_h(q, p)$. From the condition $i \leq j$ in the sum of $f_h(q, p)$, one obtains the representation as a triangle. The decomposition Eq. (83) corresponds to separating the triangle into a square and two triangles of half size iteratively. For example, the $2^3 \times 2^3$ triangle in Fig 6 is decomposed into the left bottom $2^2 \times 2^2$ square (corresponding to $f_h^{(3)}$) and two $2^2 \times 2^2$ triangles above and on the right of it. Decomposing these triangles in turn yields $2 \times 2$ square (corresponding to the $f_h^{(2)}$), with the $f_h^{(1)}$ and $f_h^{(0)}$ terms remaining.

Each square corresponds to a term in the Boolean function of the form $\left( \sum_{j} q_{j} \right) \left( \sum_{j} p_{j} \right)$, where $\left( \sum_{j} q_{j} \right)$ again corresponds to the parity operation which can be implemented in log depth. The sum in Eq. (84) consists of $n/2^k$ squares of side length $2^{k-1}$. As different squares associated to the same $k$ share no common variables, their parity operations can be implemented in parallel. Combining Eqs. (69) and (83), the half graph operators can therefore be implemented with a $O(\log^2 n)$-depth quantum circuit.

This circuit depth can be further reduced by a more careful construction. Note that one of the sums in the penultimate term of Eq. (83)

$$f_h^{(\log_2 n-1)}(q, p) = \left( \sum_{j=1}^{n/4} q_{j} \right) \left( \sum_{j=n/4+1}^{n} p_{j} \right) + \left( \sum_{j=n/2+1}^{3n/4} q_{j} \right) \left( \sum_{j=3n/4+1}^{n} p_{j} \right),$$ (86)

already includes half of the terms required by the final term $f_h^{(\log_2 n)}(q, p)$. Thus, if one were to also include the calculation of $\sum_{j=1+1/n/4} q_{j}$ and $\sum_{j=n/2+1} p_{j}$ at level $\log_2 n - 1$ (which shares no variables with other terms at this level), the level-$\log_2 n$ calculation would require only $O(1)$ operations (one multiplication). Applying this idea recursively, we can compute $f_h$ using a $O(\log n)$-depth quantum circuit, so the depth of $f_{\text{half1}}$ and $f_{\text{half2}}$ are both $O(\log L)$.

Let’s consider each term more carefully. The $f_h^{(0)}$ term is the sum of multiplications, all of which share no common variables, so that $U_{f_h^{(0)}}$ can be implemented as a depth-one quantum circuit:

$$U_{f_h^{(0)}}(q, p) = \prod_{i=1}^{n} CZ(q_i, p_i)|q, p\rangle = (-1)^{f_h^{(0)}(q, p)}|q, p\rangle.$$ (87)

Similarly,

$$U_{f_h^{(1)}}(q, p) = \prod_{i=0}^{n/2-1} CZ(q_{2i+1}, p_{2i+2})|q, p\rangle = (-1)^{f_h^{(1)}(q, p)}|q, p\rangle,$$ (88)

is also obtained with a depth-one quantum circuit. To construct $U_{f_h^{(2)}}$, one requires CX gates because the sum term involves the parity of two bits:

$$|q^{(2)}, p^{(2)}\rangle = \prod_{i=2}^{n/4-1} CX(q_{4i+1}, q_{4i+2}) CX(q_{4i+3}, q_{4i+4}) \times CX(p_{4i+1}, p_{4i+2}) CX(p_{4i+3}, p_{4i+4})|p, q\rangle.$$ (89)
C. Encoding an arbitrary unknown state

For fault-tolerant quantum computation within the toric QECC, it suffices to prepare only one of the degenerate states of the code, followed by fault-tolerant logical operations to transform the initial state to the desired state. However, the procedure is necessarily different if one is provided with an unknown quantum and asked to encode this within the toric QECC. In Ref. [28], it was shown that such an encoding can be effected in log-depth by modifying surface code stabilizer elements. In this section, we show that encoding arbitrary states into the toric code can be performed in log depth using the graph state insights above.

Let \( Z_a := \otimes_{i=1}^L Z_{(L, i, x)} \) denote the string of \( Z \) operators acting on the central vertices of star graphs on \( x \) edges of the lattice, and \( Z_\beta := \otimes_{i=1}^L Z_{(1, i, y)} \) defined analogously but for \( y \) edges. The four logical states after Hadamard conjugation are \( |G_{\text{toric}}\rangle, |G_\alpha\rangle, |G_\beta\rangle, |G_{\alpha Z \beta}\rangle \), and \( |G_{\alpha Z \beta}\rangle \) is defined in Eq. (74). Given an unknown two-qubit state

\[
|\psi\rangle = c_1 |00\rangle + c_2 |01\rangle + c_3 |10\rangle + c_4 |11\rangle,
\]

the aim is to prepare the encoded logical state

\[
|\psi_{\text{logical}}\rangle = (c_1 + c_2 Z_\beta + c_3 Z_\alpha + c_4 Z_\alpha Z_\beta) |G_{\text{toric}}\rangle.
\]

Start with

\[
H^\otimes L |0\rangle^\otimes (L-1) |1\rangle = \frac{1}{\sqrt{2 L}} \sum_{q \in \{0,1\}^L} (-1)^q L |q\rangle.
\]

After applying \( U^{\text{star}} \), Eq. (77), one obtains

\[
\frac{1}{\sqrt{2 L}} \sum_{q \in \{0,1\}^L} (-1)^{q_1 + \cdots + q_{L-1}} |q\rangle
\]

\[
= \frac{1}{\sqrt{2 L}} \sum_{q \in \{0,1\}^{L-1}} ((q, 0) - (-1)^{q_1 + \cdots + q_{L-1}} |q, 1\rangle),
\]

so that

\[
(H^\otimes L - I) U^{\text{star}} H^\otimes L |0\rangle^\otimes (L-1) |1\rangle = \frac{|0\rangle^\otimes L - |1\rangle^\otimes L}{\sqrt{2}}.
\]

Likewise,

\[
(H^\otimes L - I) U^{\text{star}} H^\otimes L |0\rangle^\otimes (L-1) |0\rangle = \frac{|0\rangle^\otimes L + |1\rangle^\otimes L}{\sqrt{2}}.
\]

Denote the operator \( (H^\otimes L - I) U^{\text{star}} (H^\otimes L - I) \) as \( U^{\text{GHZ}} \). Combining Eq. (96) and Eq. (97), one readily obtains

\[
U^{\text{GHZ}} |0\rangle^\otimes L - |0\rangle^\otimes L
\]

\[
U^{\text{GHZ}} |0\rangle^\otimes L - |1\rangle^\otimes L.
\]
Introducing 2(L − 1) ancillary qubits, all initialized in state |0⟩, the state (92) can be written as

\[
|\psi\rangle = c_1 |0⟩^\otimes(L-1) |0⟩ |0⟩^\otimes(L-1) |0⟩ + c_2 |0⟩^\otimes(L-1) |0⟩ |0⟩^\otimes(L-1) |1⟩ + c_3 |0⟩^\otimes(L-1) |1⟩ |0⟩^\otimes(L-1) |0⟩ + c_4 |0⟩^\otimes(L-1) |1⟩ |0⟩^\otimes(L-1) |1⟩ .
\]

(99)

After applying \(U^{\text{GHZ}} \otimes^2\) on state |\psi⟩, one obtains

\[
U^{\text{GHZ}} \otimes U^{\text{GHZ}} |\psi⟩ = c_1 |0⟩^\otimes(L) |0⟩^\otimes(L) + c_2 |0⟩^\otimes(L) |1⟩^\otimes(L) + c_3 |1⟩^\otimes(L) |0⟩^\otimes(L) + c_4 |1⟩^\otimes(L) |1⟩^\otimes(L) .
\]

(100)

The aim is to prepare a 2L^2-qubit state, so one must introduce another 2L(L−1) ancillary qubits all initialized in the state |0⟩. The state Eq. (100) then becomes

\[
|\psi'⟩ = c_1 \left( |0⟩^\otimes(L-1) |0⟩ \right)^{\otimes L} \left( |0⟩^\otimes(L-1) |0⟩ \right)^{\otimes L} + c_2 \left( |0⟩^\otimes(L-1) |0⟩ \right)^{\otimes L} \left( |0⟩^\otimes(L-1) |1⟩ \right)^{\otimes L} + c_3 \left( |0⟩^\otimes(L-1) |1⟩ \right)^{\otimes L} \left( |0⟩^\otimes(L-1) |0⟩ \right)^{\otimes L} + c_4 \left( |0⟩^\otimes(L-1) |1⟩ \right)^{\otimes L} \left( |0⟩^\otimes(L-1) |1⟩ \right)^{\otimes L} .
\]

(101)

From Eqs. (96–97), it is easy to see that

\[
U^{\text{star}} H^\otimes L |0⟩^\otimes(L-1) |0⟩ = \left( |+⟩^\otimes(L-1) |0⟩ + |−⟩^\otimes(L-1) |1⟩ \right) / \sqrt{2} ;
\]

\[
U^{\text{star}} H^\otimes L |0⟩^\otimes(L-1) |1⟩ = \left( |+⟩^\otimes(L-1) |0⟩ − |−⟩^\otimes(L-1) |1⟩ \right) / \sqrt{2} .
\]

(103)

State Eq. (102) is the L-qubit star graph state. Then we have

\[
( (U^{\text{star}} H^\otimes L )^\otimes 2L |\psi'⟩ = c_1 |φ^+_1⟩ |φ^+_1⟩ + c_2 |φ^+_1⟩ |φ^+_1⟩ + c_3 |φ^+_1⟩ |φ^+_1⟩ + c_4 |φ^+_1⟩ |φ^+_1⟩ ,
\]

(104)

where the states defined by

\[
|φ^+_1⟩ = \left( |+⟩^\otimes(L-1) |0⟩ + |−⟩^\otimes(L-1) |1⟩ \right) / \sqrt{2} ,
\]

\[
|φ^+_1⟩ = \left( |+⟩^\otimes(L-1) |0⟩ − |−⟩^\otimes(L-1) |1⟩ \right) / \sqrt{2} .
\]

(105)

(106)

satisfy

\[
Z_\alpha |φ^+_1⟩ \otimes |φ^+_1⟩ = |φ^-_1⟩ \otimes |φ^+_1⟩ ,
\]

\[
Z_\beta |φ^+_1⟩ \otimes |φ^+_1⟩ = |φ^+_1⟩ \otimes |φ^-_1⟩ ,
\]

\[
Z_\alpha Z_\beta |φ^+_1⟩ \otimes |φ^+_1⟩ = |φ^-_1⟩ \otimes |φ^-_1⟩ ;
\]

(107)

here \(Z_\alpha\) and \(Z_\beta\) act on the last qubits of the first and second \(L\) copies of star graph states, respectively. Last, one must apply the multiple half-graph operator \(U^{\text{mhalf}}\)
given in Eq. (75), which commutes with the \(Z_\alpha\) and \(Z_\beta\) because these operators only change the phase, as discussed in Sec. IV B. The whole quantum circuit is depicted in Fig. 8. From the analysis in Sec. IV one has \(U^{\text{mhalf}} |φ^+_1⟩ |φ^+_1⟩ = |G_\text{toric}⟩\). The encoded logical state is therefore

\[
|\psi_{\text{logical}}⟩ = U^{\text{mhalf}} (c_1 |φ^+_1⟩ |φ^+_1⟩ + c_2 |φ^+_1⟩ |φ^+_1⟩ + c_3 |φ^+_1⟩ + c_4 |φ^+_1⟩ ) .
\]

(108)

The quantum circuit for the full encoding procedure is depicted in Fig. 8.

The procedure discussed above encodes an arbitrary two-qubit state in the code space spanned by the vectors

\[
\text{span}_C \{ |G_\text{toric}⟩ , Z_\alpha |G_\text{toric}⟩ , Z_\beta |G_\text{toric}⟩ , Z_\alpha Z_\beta |G_\text{toric}⟩ \},
\]

(109)

which is locally equivalent to toric code. Without implementing the portion of the circuit corresponding to \(U^{\text{mhalf}}\) in the last step, the resulting state is instead given by Eq. (104), which is the logical state of (92) in the code space spanned by the vectors

\[
\text{span}_C \{ |φ^+_1⟩ , |φ^+_1⟩ , |φ^+_1⟩ , |φ^+_1⟩ , |φ^-_1⟩ , |φ^-_1⟩ , |φ^-_1⟩ , |φ^-_1⟩ \} .
\]

This is a repetition code, as discussed in Sec. III F. Thus, the repetition encoding procedure is a subroutine of full encoding circuit for the toric code.

D. State preparation via measurements

Any graph can be expanded to a graph containing vertices with at most degree three, by introducing ancillae [25] which are then measured in the X basis. Given that all the measurements commute and can therefore be performed simultaneously, graph state preparation can
be performed in (constant) depth three. For example, the star graphs on $L$ qubits with degree $L-1$ that are induced subgraphs of the toric graph can be represented by a totally asymmetric tree graph with $2L-4$ vertices with maximum degree three. Given that there are $2L$ star graphs comprising the toric graph, the contribution of the star-graph ancillae to the circuit width scales as $L^2 \sim N$.

Next consider the half graphs on $2(L-1)$ vertices that constitute the remaining subgraphs. The first vertex in the first of the two bipartite vertex subsets has an edge with all $L-1$ vertices in the second vertex subset and thus has the connectivity of a degree-$L-1$ star graph; following the procedure described for the star graphs, one can add $L-4$ ancillae to this central vertex to ensure that all resulting vertices have at most degree three. The second vertex in the first subset shares $L-2$ neighbors in the second subset, requiring the addition of $L-5$ ancillae, etc. The total number of required ancillae therefore scales as $L^2$; and, given that the number of half graphs scales as $L$, the total number of half-graph ancillae scales as $L^3$. Thus, preparing the toric graph of size $N$ in constant depth as a maximum degree-three graph requires a circuit width $O(L^3) \sim O(N^{3/2})$.

It is worthwhile to point out that the $O(L^3)$-qubit graph state with bounded degree-three vertices is topologically trivial, while the state that remains after the $X$ measurements is the topologically ordered toric graph state. Thus, one can obtain a topologically ordered state by projective measurement on a topologically trivial state in a higher-dimensional Hilbert space.

There are other methods to prepare one of the logical states in the toric code state by measurement. For example, one can perform projective measurements for all star operators and $S_z$ on the state $|0\rangle^\otimes N$, then correct the signs according to the measurement result. There is also a measurement-based method to generate toric code states [37, 38], but the depth is not analyzed explicitly.

V. DISCUSSION

In this work, we map a toric code state to its LC-equivalent graph state, which is found to consist solely of star and half graphs. Given that the star graphs encode GHZ states, the graph construction reveals a novel connection between the toric code and the nine-qubit (i.e. by a non-identity) by $O$. Here, $P^d_N \subset P_N$ contains all the operators whose weight is less than $d$. Based on the quantum error correction condition [33, 59], the subspace $\text{span}_c \{|\psi_1\rangle, |\psi_2\rangle\} \subset H^d_{2N}$ is a quantum error correction code with distance $d = m$.

Proof. The weight of the operator $O \in P_N$ is the number of qubits which are acted on non-trivially (i.e. by a non-identity) by $O$. Here, $P^d_N \subset P_N$ contains all the operators whose weight is less than $d$. Based on the quantum error correction condition [33, 59], the subspace $\text{span}_c \{|\psi_1\rangle, |\psi_2\rangle\} \subset H^d_{2N}$ is a quantum error correction code with distance $d$ if and only if the following conditions always hold:

$$\langle \psi_1 | O | \psi_1 \rangle = \langle \psi_2 | O | \psi_2 \rangle, \quad \langle \psi_1 | O | \psi_2 \rangle = 0,$$  

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Appendix A: Multiple copies of the GHZ state

**Remark.** Given two $m$-qubit GHZ states

$$|\phi^+_m\rangle = \frac{|0^m\rangle + |1^m\rangle}{\sqrt{2}}, \quad |\phi^-_m\rangle = \frac{|0^m\rangle - |1^m\rangle}{\sqrt{2}}.$$  

and their $m$-copy states on $m^2$ qubits

$$|\varphi^+\rangle = |\phi^+_m\rangle \otimes m, \quad |\varphi^-\rangle = |\phi^-_m\rangle \otimes m.$$  

Then, $\text{span}_c \{|\varphi^+\rangle, |\varphi^-\rangle\}$ is a quantum error correction code with distance $d = m$. 

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Algorithm 1 Generate star graph state

Require:
\textbf{unsigned integer \texttt{NUM}}

Ensure:
\textbf{binary[\texttt{NUM}] \texttt{qsTate}}

\begin{enumerate}
\item \textbf{function} \texttt{GENERATE STAR GRAPH STATE(\texttt{NUM})}
\item \textbf{unsigned integer \texttt{dePth}}
\item \textbf{unsigned integer \texttt{qtARG}}
\item \texttt{qsTate} \leftarrow \texttt{0}^{	exttt{NUM}}
\item \texttt{qsTate} \leftarrow \texttt{H[\texttt{NUM}]} \ast \texttt{qsTate}
\item \texttt{dePth} \leftarrow \left\lceil \log(\texttt{NUM} - 1) \right\rceil
\item \texttt{for} \texttt{d} \texttt{from 1} \texttt{to \texttt{dePth}} \texttt{do}
\item \texttt{for} \texttt{i} \texttt{from} \texttt{2}^{d-1} \texttt{to \texttt{NUM}-1} \texttt{do}
\item \texttt{if} \texttt{i} = \texttt{2}^{d-1} \mod \texttt{2}^d \texttt{then}
\item \texttt{qtArg} \leftarrow \min(\texttt{i} + \texttt{2}^{d-1}, \texttt{NUM} - 1)
\item \texttt{qsTate} \leftarrow \texttt{CX}(\texttt{i}, \texttt{qtArg}) \ast \texttt{qsTate}
\item \texttt{end if}
\item \texttt{end for}
\item \texttt{end for}
\item \texttt{qsTate} \leftarrow \texttt{CZ(\texttt{NUM} - 1, \texttt{NUM})} \ast \texttt{qsTate}
\item \texttt{for} \texttt{d} \texttt{from \texttt{dePth} to 1} \texttt{do}
\item \texttt{for} \texttt{i} \texttt{from} \texttt{2}^{d-1} \texttt{to \texttt{NUM}} \texttt{do}
\item \texttt{if} \texttt{i} - \texttt{2}^{d-1} = 0 \mod \texttt{2}^d \texttt{then}
\item \texttt{qtArg} \leftarrow \min(\texttt{i} + \texttt{2}^{d-1}, \texttt{NUM} - 1)
\item \texttt{qsTate} \leftarrow \texttt{CX}(\texttt{i}, \texttt{qtArg}) \ast \texttt{qsTate}
\item \texttt{end if}
\item \texttt{end for}
\item \texttt{end for}
\item \texttt{end function}
\end{enumerate}

for every operator \(O \in \mathcal{P}_N^d\).

It is straightforward to verify that \(\langle \phi^+_m | O | \phi^+_m \rangle = \langle \phi^-_m | O | \phi^-_m \rangle\) is satisfied \(\forall O \in \mathcal{P}_m^m\), while the second condition fails to hold because \(\langle \phi^+_m | Z_i | \phi^-_m \rangle = 1\), where \(Z_i\) is the Pauli \(Z\) operator acting on qubit \(i\). Consider multiple copies of the GHZ state instead. One can again verify that \(\langle \varphi^+ | O | \varphi^+ \rangle = \langle \varphi^- | O | \varphi^- \rangle\) still holds \(\forall O \in \mathcal{P}_m^m\). On the other hand, one obtains

\[
\langle \varphi^+ | O | \varphi^- \rangle = \langle \varphi^+ | \bigotimes_{i=1}^{m} O_i | \varphi^- \rangle = \prod_{i=1}^{m} \langle \phi^+_i | O_i | \phi^-_i \rangle, \quad (A5)
\]

where \(O_i \in \mathcal{P}_m\) acts on the \(i\)-th copy. If \(\langle \varphi^+ | O_i | \varphi^- \rangle \neq 0\), then \(\langle \phi^+_i | O_i | \phi^-_i \rangle \neq 0, \forall i \in \{1, \ldots, m\}\), so \(O\) acts on at least \(m\) qubits non-trivially and \(O \notin \mathcal{P}_m^m\). Therefore, condition Eq. (A3) and Eq. (A4) hold \(\forall O \in \mathcal{P}_m^m\) for \(|\varphi^+\rangle\) and \(|\varphi^-\rangle\).

Appendix B: Quantum circuit generating star graph states

In this section, we prove that Algorithm 1 generates a star graph state in log depth. Line 1 to line 5 describe the initialization process and the state in quantum register after line 5 is \(H^\otimes n |0^n\). Next, the circuit described from line 6 to line 14 is the gate \(P\) in Eq. (77) that implements \(\text{Parity}(q_1, \ldots, q_{n-1})\). Line 15 adds a global phase \((-1)^{\text{Parity}(q_1, \ldots, q_{n-1})} q_n\) and the remaining operation is exactly inverse of \(P\). It only remains to prove that line 6 to line 14 indeed implements \(\text{Parity}(q_1, \ldots, q_{n-1})\).

Consider the action from line 6 to line 14 on input \(|q\rangle\), a computational basis state and \(q \in \{0, 1\}^n\). \(d\) denotes the times of iteration and \(|q^d\rangle\) is the state in quantum register after the \(d\)-th iteration. During the \(d\)-th iteration, the gate \(\text{CX}(2^{d-1} + c \cdot 2^d, (c+1)2^d)\), where \(c \in \mathbb{N}\), is executed. One notices that all CX gates within the same iteration act on no common qubits so they can all be performed in parallel. The depth of the circuit from line 6 to line 14 is then \(\log(n-1)\).

After \(d\)-th iteration, the state in the quantum register should satisfy \(q_{c2^d} = \sum_j \epsilon_{c-1}^{2d+1} q_j\), where \(c \in \{1, \ldots, 2^{\lceil \log(n-1) \rceil - d}\}\). We prove this claim using induction. First, it holds trivially when \(d = 0\). Next, suppose it holds when \(d = k\). Then in the \((k+1)\)-th loop, gate \(\text{CX}(c2^{k+1} - 2^k, c2^{k+1})\) is executed, where
Algorithm 2 Generate half graph state

Require:

\[\text{unsignedinteger} \ N\text{UM}\]
\[\text{\triangleright} \text{size of the half graph}\]

Ensure:

\[\text{binary}[2\text{NUM}] \ q\text{STATEx}\]
\[\text{\triangleright} \text{2NUM-qubit half graph state}\]

\[\begin{align*}
&\text{function GENERATE HALF(\text{NUM})} \\
&\text{unsignedinteger} \ d\text{EPTH} \\
&\text{unsignedinteger} \ c\text{ONXQBIT} \\
&\text{unsignedinteger} \ c\text{OXYQBIT} \\
&\text{unsignedinteger} \ t\text{ARXQBIT} \\
&\text{unsignedinteger} \ t\text{ARYQBIT} \\
&\text{qsTatEx} \leftarrow 0^\text{NUM} \\
&\text{qsTatEx} \leftarrow \text{HAD(\text{NUM})} * \text{qsTatEx} \\
&\text{qsTatEy} \leftarrow 0^\text{NUM} \\
&\text{qsTatEy} \leftarrow \text{HAD(\text{NUM})} * \text{qsTatEy} \\
&\text{for} \ i \text{ from 1 to } \text{NUM} \text{ do} \\
&\quad \text{COXQBIT} \leftarrow i \\
&\quad \text{TARQBIT} \leftarrow i \\
&\quad \text{qsTatEx} \leftarrow \text{CZ(cOXQBIT, TARQBIT)} * (\text{qsTatEx, qsTatEy}) \\
&\text{end for} \\
&\text{dEPTH} \leftarrow \lceil \log(\text{NUM}) \rceil \\
&\text{for} \ d \text{ from 1 to } \text{dEPTH} \text{ do} \\
&\quad \text{for} \ i \text{ from 1 to } \text{NUM} \text{ do} \\
&\quad\quad \text{if} \ i = 2^{d-1} \mod 2^d \text{ then} \\
&\quad\quad\quad \text{COXYQBIT} \leftarrow i \\
&\quad\quad\quad \text{TARXQBIT} \leftarrow \min\{i + 2^{d-1}, \text{NUM}\} \\
&\quad\quad\quad \text{TARYQBIT} \leftarrow \min\{i + 2^{d-1}, \text{NUM}\} \\
&\quad\quad\quad \text{qsTatEx} \leftarrow \text{CZ(cOXQBIT, TARQBIT)} * (\text{qsTatEx, qsTatEy}) \\
&\quad\quad\quad \text{qsTatEy} \leftarrow \text{CZ(cOXQBIT, TARQBIT)} * (\text{qsTatEx, qsTatEy}) \\
&\quad\quad \text{end if} \\
&\quad \text{end for} \\
&\text{end for} \\
&\text{for} \ d \text{ from } \text{dEPTH} \text{ to } 1 \text{ do} \\
&\quad \text{for} \ i \text{ from } 2^d-1 \text{ to } \text{NUM} \text{ do} \\
&\quad\quad \text{if} \ i \equiv 2^{d-1} \mod 2^d \text{ then} \\
&\quad\quad\quad \text{COXYQBIT} \leftarrow i \\
&\quad\quad\quad \text{TARXQBIT} \leftarrow \min\{i + 2^{d-1}, \text{NUM}\} \\
&\quad\quad\quad \text{TARYQBIT} \leftarrow \min\{i + 2^{d-1}, \text{NUM}\} \\
&\quad\quad\quad \text{qsTatEx} \leftarrow \text{CZ(cOXQBIT, TARQBIT)} * (\text{qsTatEx, qsTatEy}) \\
&\quad\quad\quad \text{qsTatEy} \leftarrow \text{CZ(cOXQBIT, TARQBIT)} * (\text{qsTatEx, qsTatEy}) \\
&\quad\quad \text{end if} \\
&\quad \text{end for} \\
&\text{end for} \\
&\text{end function}
\end{align*}\]

\[c \in \{1, \ldots, 2^{\log(n-1)}-k-1\}, \text{ so} \]
\[q_{c2^k+1} = q_{c2^k+1}^k + q_{c2^k+1-2^k}^k = q_{2c2^k}^k + q_{(2c-1)2^k}^k \]
\[= \sum_{j=(2c-1)2^k+1}^{2c2^k} q_j + \sum_{j=(2c-2)2^k+1}^{(2c-1)2^k} q_j \]
\[= \sum_{j=(c-1)2^k+1}^{c2^k+1} q_j \]  \hspace{1cm} \text{(B1)}

If \(\log_2(n-1) \in \mathbb{N}\), then after \(\log_2(n-1)\)-th iteration, \(q_{n-1}^{\log_2(n-1)} = \text{Parity}(q_1, \ldots, q_{n-1})\). When \(\log_2(n-1) \notin \mathbb{N}\), then in the \(\log_2(n-1)\)-th iteration, the target qubit is replace as \(n-1\)-th qubit when it exceed \(n-1\) in line 10 so \(q_{n-1}^{\log_2(n-1)} = \text{Parity}(q_1, \ldots, q_{n-1})\) still holds. Therefore, we conclude Algorithm 1 indeed generates a star graph state in log depth.

Appendix C: Quantum circuit generating half states

The Unitary described from line 11 to line 15 adds the phase \((-1)^{f(k,q,p)}\). Next, consider what unitary lines 16.
computes when the input is in the computational basis \(|q,p\rangle\). \(|q,p\rangle\) denotes the state in the register after the \(d\)-th iteration. The operation within the quantum register for \(|q,p\rangle\) is the same as in Algorithm 1, so after \(k-1\)-th iteration

\[
q_{c2^k-1}^{k-1} = \sum_{j=(c-1)2^{k-1}+1}^{c2^{k-1}} q_j, \quad (C1)
\]

\[
p_{c2^k-1}^{k-1} = \sum_{j=(c-1)2^{k-1}+1}^{c2^{k-1}} p_j, \quad (C2)
\]

holds.

In addition, in the \(k\)-th iteration, gate \(\text{CZ}(q_{c2^k-2^{k-1}}, p_{c2^k})\) is executed with the quantum register in state \(|q^k-1, p^{k-1}\rangle\), adding a phase \((-1)^{f_k(q,p)}\), where

\[
f_k^f(q,p) = \sum_{c=0}^{n/2^k} \left( \sum_{j=(c+1/2)2^k+1}^{c2^k} q_j \right) \cdot \left( \sum_{j=(c+1/2)2^k+1}^{c2^k} p_j \right)
\]

\[
= f_k^{(q,p)}.
\]

When \(n\) is not the power of two, the sum index upper bound is replaced in line 22 and line 23. Therefore, the quantum circuit in the \(k\)-th iteration adds a global phase \((-1)^{f_k(q,p)}\) and all \(\text{CZ} (\text{CX})\) gates within the same iteration can be performed in parallel. Lines 30-41 uncompute the garbage and leave only the phase. Because of the decomposition Eq. (53), this algorithm prepares a half graph state in log depth.
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