THE BAIRE CATEGORY OF THE HYPERSPACE
OF NONTRIVIAL CONVERGENT SEQUENCES

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Abstract. Assume that $X$ is a regular space. We study topological properties of the family $S_c(X)$ of nontrivial convergent sequences in $X$ equipped with the Vietoris topology. In particular, we show that if $X$ has no isolated points, then $S_c(X)$ is a space of the first category which answers the question posed by S. Garcia-Ferreira and Y.F. Ortiz-Castillo.

1. Introduction

Let $X$ be a topological Hausdorff space. The Vietoris topology on the family $K(X)$ of all compact subsets of $X$ is generated by a base consisting of sets

$$\langle V_1, \ldots, V_n \rangle := \left\{ K \in K(X) : K \subset \bigcup_{i=1}^{n} V_i \text{ and } K \cap V_i \neq \emptyset \text{ for } 1 \leq i \leq n \right\},$$

where $n$ runs over $\mathbb{N}$ and $V_1, \ldots, V_n$ are nonempty open sets in $X$.

Our notation is consistent with that used in [1]. We say that $S \subset X$ is a nontrivial convergent sequence in $X$ if $S = \{x_n\}_{n \in \mathbb{N}} \cup \{\lim S\}$ for some injective sequence $(x_n)_{n \in \mathbb{N}}$ in $X$ which is convergent to some $\lim S \in X \setminus \{x_n\}_{n \in \mathbb{N}}$. In other words, $S$ is a set of terms of an injective convergent sequence with its limit. Obviously, $S$ is compact for any space $X$, and $S_c(X)$ is empty for a discrete space $X$. Hence each closed subset $F$ of $S$ is compact. Consequently, the spaces $K(X)$, $CL(X)$, and the family of all closed subsets of $X$ with the topology generated by an analogous base, given by (1), introduce the same topology in their subspace $S_c(X)$.

In general, separation axioms of the spaces $CL(X)$ and $K(X)$ depend on $X$. More precisely, $CL(X)$ is normal if and only if $X$ is compact (see [3]), and $K(X)$ is metrizable if and only if $X$ is metrizable ([2]).

The main aim of this paper is to show that $S_c(X)$ is of first category in itself under the assumption that $X$ is regular and crowded (i.e. has no isolated points). This result gives a positive answer to Question 3.4 in [1]. The authors of [1] asked whether $S_c(X)$ is of the first category in itself if $X$ is a metric crowded space. From now on, sets of the first category will be called also meager. In our considerations we will use the Banach Category Theorem (see [3]) which states that in any topological space the union of a family of open meager sets is meager, too. Thanks to this fact, it suffices to construct a meager open neighbourhood of any $S \in S_c(X)$. Such a construction is possible in the case of regular crowded space $X$. It happened that no precise assumptions on a space $X$ were formulated in some theorems and proofs in [1]. Hence we have decided to repeat or modify the respective arguments from [1], with an explicit evidence of properties of $X$, which would be helpful to understand all details.
2. MAIN RESULT

We will follow some ideas taken from the paper \[1\] while considering some specific subsets of \( S_c(X) \). For given \( k, m \in \mathbb{N}, 1 \leq i \leq m \) and pairwise disjoint nonempty closed sets \( C_1, \ldots, C_k \), we denote

\[
N^i_k(m, \{C_j : j \leq k\}) := \left\{ S \in S_c(X) : S \subseteq \bigcup_{j=1}^{k} C_j \text{ and } 1 \leq \text{card}(S \cap C_j) \leq m \text{ for all } l \leq k, l \neq i \right\}
\]

and \( N_k(m, \{C_j : j \leq k\}) := \bigcup_{i=1}^{k} N^i_k(m, \{C_j : j \leq k\}) \).

It can be immediately seen that, if \( S \in N^i_k(m, \{C_j : j \leq k\}) \), then \( \text{card}(S \cap C_i) = \omega \).

Here we present a fact which can be derived directly from \[1\] Lemma 3.1.

**Lemma 2.1.** Let \( X \) be a crowded topological space. Then the set \( N^i_k(m, \{C_j : j \leq k\}) \) is nowhere dense, whenever \( k, m \in \mathbb{N}, 1 \leq i \leq m \) and \( C_1, \ldots, C_k \) are pairwise disjoint, nonempty and closed sets.

Note that the assertion of the above fact can be lost if the set \( N_k(m, \{C_j : j \leq k\}) \) is considered as a subset of a closed subspace of \( X \), which is not crowded.

**Example 1.** Consider sets \( X := [0, 1], Y := [0, 1] \cup \{2\}, Z := [0, 1] \cup [2, 3] \) with the Euclidean topology. Note that \( X \) is a closed subspace of \( Y \), \( Y \) is a closed subspace of \( Z \) and both spaces \( X \) and \( Z \) are crowded but \( Y \) has one isolated point. Take \( k := 2, m := 1, C_1 := [0, 1], C_2 := \{2\}, i := 2 \).

By Lemma 2.1, the set \( N^2_2(1, \{C_1, C_2\}) \) is nowhere dense in the space \( S_c(Z) \). Nevertheless, this set is not nowhere dense in \( S_c(Y) \). To check it, set \( V_1 := [0, 1], V_2 := \{2\} \), and consider the open set \( \langle V_1, V_2 \rangle \). Obviously, each \( S \in \langle V_1, V_2 \rangle \) has exactly one point in \( S \cap V_2 \). Thus \( S \in N^2_2(1, \{C_1, C_2\}) \) and consequently, \( \langle V_1, V_2 \rangle \subset N^2_2(1, \{C_1, C_2\}) \). Moreover, all sets \( N^i_k(m, \{C_j : j \leq k\}) \) as in Lemma 2.1 are nowhere dense in \( S_c(X) \).

The next result (see \[1\] Theorem 3.2) is an application of the previous lemma.

**Lemma 2.2.** Suppose that \( U_1, U_2 \) are nonempty, closed and disjoint subsets of a crowded space \( X \). Then \( \langle \text{Int}(U_1), \text{Int}(U_2) \rangle \) is meager in \( S_c(X) \).

**Proof.** Thanks to Lemma 2.1 it suffices to observe that \( \langle \text{Int}(U_1), \text{Int}(U_2) \rangle \subset \bigcup_{m \in \mathbb{N}} N_2(m, \{U_1, U_2\}) \). But this follows from the disjointness of closed sets \( U_1, U_2 \). \( \square \)

Now, we will construct the respective neighbourhoods of nontrivial convergent sequences.

**Lemma 2.3.** Suppose \( X \) is a Hausdorff space and \( S = \{x_n\}_{n \in \mathbb{N}} \cup \{\lim S\} \in S_c(X) \). There are neighbourhoods \( V_n, n \in \mathbb{N} \) of \( x_n \)'s and neighbourhood \( V_S \) of \( \lim S \) such that

\[
V_1 \cap V_n = \emptyset \text{ for all } n \geq 2 \quad \text{and} \quad V_1 \cap V_S = \emptyset.
\]

**Proof.** Use the Hausdorff axiom to find disjoint neighbourhoods \( V'_1 \) of \( x_1 \), and \( V_S \) of \( \lim S \). Since \( (x_n)_{n \in \mathbb{N}} \) is convergent to \( \lim S \), the set \( M := \{m \geq 2 : x_m \notin V_S\} \) is finite. Again, for each \( m \in M \) use the Hausdorff axiom to find disjoint neighbourhoods \( V'_{1,m} \) of \( x_1 \), and \( V_m \) of \( x_m \). The intersection \( V_1 = V'_1 \cap \bigcap_{m \in M} V_{1,m} \) satisfies

\[
V_1 \cap V_m = \emptyset \text{ for each } m \in M.
\]

Then it suffices to define \( V_k := V_S \) for all \( k \notin M \cup \{1\} \). \( \square \)
Proposition 2.4. Suppose $X$ is a regular space and $S = \{x_n\}_{n \in \mathbb{N}} \cup \{\lim S\} \in S_c(X)$. Then there are nonempty, closed and disjoint sets $U_1, U_2$ with $S \in \langle \text{Int}(U_1), \text{Int}(U_2) \rangle$.

Proof. Let $V_S, V_n, n \in \mathbb{N}$, be the respective neighbourhoods of $\lim S, x_n, n \in \mathbb{N}$ considered in Lemma 2.3. Since $V_1 \cap (V_S \cup \bigcup_{n \geq 2} V_n) = \emptyset$, we have $V_1 \cap \text{cl}(V_S \cup \bigcup_{n \geq 2} V_n) = \emptyset$. Then we use the regularity of $X$ to find a neighbourhood $W_1$ of $x_1$ such that $\text{cl}(W_1) \subset V_1$. Put $U_1 := \text{cl}(W_1), U_2 := \text{cl}(V_S \cup \bigcup_{n \geq 2} V_n)$. Obviously, these sets are nonempty, closed and disjoint. We need to show that $S \in \langle \text{Int}(U_1), \text{Int}(U_2) \rangle$.

Indeed, $x_1 \in W_1 \subset \text{Int}(U_1)$ and $\{\lim S\} \cup \bigcup_{n \geq 2} \{x_n\} \subset V_S \cup \bigcup_{n \geq 2} V_n \subset \text{Int}(U_2)$.

□

Theorem 2.5. Suppose that $X$ is a regular crowded space. Then the space $S_c(X)$ is of the first category in itself.

Proof. Take $S \in S_c(X)$ and a neighbourhood of $S$ of the form $\langle \text{Int}(U_1), \text{Int}(U_2) \rangle$ considered in Proposition 2.4. Then by Lemma 2.2, this neighbourhood is meager. Therefore, by the Banach Category Theorem, $S_c(X)$ is of the first category as a union of meager neighbourhoods of its points.

□

References

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