Higgs Decay into Two Photons, Revisited

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Abstract

The one-loop calculation of the amplitude for the Higgs decay $H \rightarrow \gamma \gamma$ due to virtual $W$’s in the unitary gauge is presented. As the Higgs does not directly couple to the massless photons, the one-loop amplitude is finite. The calculation is performed in a straightforward way, without encountering divergences. In particular, artifacts like dimensional regularization are avoided. This is achieved by judiciously routing the external momenta through the loop and by combining the integrands of the amplitudes before carrying out the integration over the loop momentum. The present result satisfies the decoupling theorem for infinite Higgs mass, and is thus different from the earlier results obtained in the $\xi = 1$ gauge using dimensional regularization. The difference between the results is traced to the use of dimensional regularization.

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1 Introduction

The Higgs decay into two photons, \( H \to \gamma\gamma \),

\[ (1.1) \]
is an important channel in the search for the Higgs particle \[1\] at the Large Hadron Collider. If the Higgs mass is near about 115 GeV/c\(^2\), as favored by the first possible evidence \[2\;3\] from the Large Electron-Positron collider (LEP) at CERN, then this decay is a good way to look for the Higgs, because the photons can be seen cleanly.

The decay (1.1) was studied theoretically many years ago \[4\;5\]. In particular, in the standard model of Glashow, Weinberg, and Salam \[6\], there are two major contributions, one from the top loop and one from the \( W \) loop. Interestingly, according to the previous calculations \[5\], these two contributions are qualitatively different: while the one from the top loop satisfies the decoupling theorem \[7\], that from the \( W \) does not. In this context, decoupling is the phenomenon of a particle to cease interaction with other particles when its mass grows arbitrarily large.

Although there is no solid argument for the decoupling theorem, it appears to be physically quite reasonable. With this in mind, it is the purpose of this paper to revisit the one-loop \( W \) contribution to the decay (1.1). To this end, we shall perform the calculation in a way that differs substantially from the earlier one, and we shall present the details in what follows.

The earliest calculation for the decay width is given by Ellis, Gaillard, and Nanopoulos \[4\]. That calculation can be characterized as follows:

(a) it is carried out in the \( R_\xi \) gauge with the choice \( \xi = 1 \), i.e., the \( R_1 \) gauge;

(b) dimensional regularization \[8\] is used; and

(c) the mass of the Higgs particle is taken to be much smaller than the \( W \) mass.

Concerning (a), in principle, all values of \( \xi \) are equivalent, but the algebra is vastly simpler when the value \( \xi = 1 \) is chosen. The (b) is to be discussed extensively in this paper. While the Higgs particle was believed to be much lighter than that of the \( W \) at the time this Ref. \[4\] was written, it is now known this is not so \[2\;3\].

The result obtained in Ref. \[4\] has been confirmed by later calculations \[5\] and extended to arbitrary values of the Higgs mass. Again, these are one-loop calculations in the \( R_1 \) gauge using dimensional regularization — see (a) and (b) above.

In contrast, the following different point of view is taken for the present study. Since the photon is massless, there is no coupling of the Higgs particle to the photon in the Lagrangian of the standard model. Since the standard model is certainly one-loop renormalizable \[9\], this absence of direct coupling implies that the one-loop contribution to the decay (1.1) through the \( W \) loop must necessarily be finite. It is thus emphasized that the quantity being studied is a finite one, and therefore it must be possible to carry out the calculation in a completely straightforward manner, without the introduction of, for example, regularization of any kind. Accordingly, the (a), (b), and (c) above are to be replaced, for the present study, by the following:
(a’) it is to be carried out in the simplest gauge;

(b’) throughout this study, the space-time dimension = 4; (1.2)

and

(c’) the mass of the Higgs particle is arbitrary.

The importance of the present study is due to the fact that the result of the calculation on the basis of (a’), (b’), and (c’) is different from the previous one of (a), (b), and (c). Furthermore, the present result does satisfy the decoupling theorem discussed above.

In Sec. 2, we present the formulae for the Feynman diagrams in the unitary gauge; these are the ones we want to calculate. In Sec. 3 these amplitudes are simplified and combined to give the present result that satisfies the decoupling theorem. Since our result is different from the previous, generally accepted one, we have chosen to present the calculation in detail, both for eventual verification and because of the unfamiliar nature of the unitary gauge. Finally, in Sec. 4 we trace back the reason why our results is different from the previous one, and find that the difference is due to (b) versus (b’). In other words, it is the use of dimensional regularization that is the cause of the violation of the decoupling theorem.

2 Formulation of the problem

In this section, the problem is to be formulated in accordance with the (a’), (b’), and (c’) above. As to the choice of the simplest gauge, a natural one is the unitary gauge — this is the gauge without any ghost and hence only the physical particles enter into the perturbative calculation. Nevertheless, this choice requires a careful discussion because it is the conventional wisdom that the unitary gauge is not suited for such calculations in quantum field theory.

In the unitary gauge, the $W$ propagator takes the form

$$P^{\alpha\beta}(p) = -i g^{\alpha\beta} - p^\alpha p^\beta / M^2 - i \epsilon, \quad (2.1)$$

where the quantity $M$ is the mass of the vector particle $W$ and $p$ its four-momentum. Thus, this $W$ propagator in the unitary gauge consists of two terms: a first one that behaves as $(p^2)^{-1}$ for large $p^2$, and a second one that behaves as $(p^2)^0$. The presence of this second term is the reason why the unitary gauge is rarely used, due to the difficulties, in general, of carrying out renormalization.

For the present problem, however, this difficulty does not enter, because there is no divergence as discussed above, and hence there is no need to renormalize. On the other hand, the absence of ghosts in the unitary gauge greatly simplifies the necessary calculation. There are only four relevant Feynman rules as given in Fig. 1 and these four Feynman rules lead to only three one $W$-loop diagrams for the decay $W$ loop. This is to be compared with fourteen Feynman diagrams in the $R_1$ gauge.
\[ \frac{i}{p^2 - M^2 + i\epsilon} \left[ -g^{\alpha\beta} + \frac{p^\alpha p^\beta}{M^2} \right] \]

\[ i\epsilon \left[ (p_2 - p_3)_\alpha g_{\beta\gamma} + (p_3 - p_1)_\beta g_{\gamma\alpha} + (p_1 - p_2)_\gamma g_{\alpha\beta} \right] \]

\[ -i\epsilon^2 \left[ 2g_{\alpha\beta} g_{\mu\nu} - g_{\alpha\mu} g_{\beta\nu} - g_{\alpha\nu} g_{\beta\mu} \right] \]

\[ ig M g_{\alpha\beta}; \]

Figure 1: The relevant Feynman rules in the unitary gauge for the decay \( H \to \gamma\gamma \) at the one \( W \)-loop level.

The three diagrams in the unitary gauge are shown in Fig. 2. This reduction of the number of Feynman diagrams to be calculated is not free: the price to be paid is that, due to the second term in the \( W \) propagator (2.1), these three diagrams lead to integrals that are highly divergent. Because of this divergence, the integrands for these three diagrams must be added together before integrating with respect to the loop momentum. Since shifting the momentum variable is not allowed for such divergent integrals, the choices of the momentum variables for the three diagrams are interdependent.

Fortunately, such interdependence of momentum choices between different diagrams is
well known in quantum field theory [10, 11, 12], and has been studied for the present problem [13], leading to the choice of momenta already shown in Fig. 2. It is now straightforward to write down the amplitudes corresponding to these three diagrams.

The corresponding amplitudes are

$$
\mathcal{M}_1 = \frac{-ie^2 g M}{(2\pi)^4} \int d^4k \left[ g_\alpha^\beta - (k + \frac{k_1 + k_2}{2})_\alpha (k + \frac{k_1 + k_2}{2})^\beta / M^2 \right]
\times \left[ g^{\rho \sigma} (k + \frac{-k_1 + k_2}{2})^\rho (k + \frac{-k_1 + k_2}{2})^\sigma / M^2 \right]
\times \left[ g^{\alpha \gamma} (k - \frac{k_1 + k_2}{2})^\alpha (k - \frac{k_1 + k_2}{2})^\gamma / M^2 \right]
\times \left[ (k + \frac{3k_1 + k_2}{2})_\rho g_{\beta \mu} + (k + \frac{-3k_1 + k_2}{2})_\beta g_{\mu \rho} + (-2k - k_2)_\mu g_{\rho \beta} \right]
\times \left( \frac{(k - \frac{k_1 + 3k_2}{2})_\sigma g_{\gamma \nu} + (k + \frac{-k_1 + 3k_2}{2})_\gamma g_{\nu \sigma} + (-2k + k_1)_\nu g_{\sigma \gamma}}{\left[ (k + \frac{k_1 + k_2}{2})^2 - M^2 + i\epsilon \right] \left[ (k + \frac{k_1 + 3k_2}{2})^2 - M^2 + i\epsilon \right] \left[ (k - \frac{k_1 + k_2}{2})^2 - M^2 + i\epsilon \right]} \right),
$$

(2.2)

$$
\mathcal{M}_2 = \frac{ie^2 g M}{(2\pi)^4} \int d^4k \left[ g_\alpha^\beta - (k + \frac{k_1 + k_2}{2})_\alpha (k + \frac{k_1 + k_2}{2})^\beta / M^2 \right]
\times \left[ g^{\alpha \gamma} - (k - \frac{k_1 + k_2}{2})^\alpha (k - \frac{k_1 + k_2}{2})^\gamma / M^2 \right]
\times \left[ 2 g_{\mu \nu} g_{\beta \gamma} - g_{\mu \beta} g_{\nu \gamma} - g_{\mu \gamma} g_{\nu \beta} \right]
\times \left( \frac{\left[ (k + \frac{k_1 + k_2}{2})^2 - M^2 + i\epsilon \right] \left[ (k - \frac{k_1 + 3k_2}{2})^2 - M^2 + i\epsilon \right]}{\left[ (k + \frac{k_1 + k_2}{2})^2 - M^2 + i\epsilon \right]} \right),
$$

(2.3)
and

\[ M_3 = \frac{-ie^2 g M}{(2\pi)^4} \int d^4k \left[ g_\rho^\sigma - (k + \frac{k_1 + k_2}{2})^\rho (k + \frac{k_1 - k_2}{2})^\sigma / M^2 \right] \]

\[ \times \left[ g_\beta^\gamma - (k - \frac{k_1 + k_2}{2})^\beta (k - \frac{k_1 - k_2}{2})^\gamma / M^2 \right] \]

\[ \times \left[ (k + \frac{k_1 + 3k_2}{2})_\nu \gamma_\beta^\mu + (k + \frac{k_1 - 3k_2}{2})_\beta g_\gamma^\nu + (-2k - k_1)_\nu g_\gamma^\beta \right] \]

\[ \times \frac{(k - \frac{3k_1 + k_2}{2})_\sigma g_\gamma^\mu + (k + \frac{3k_1 - k_2}{2})_\gamma g_\mu^\sigma + (-2k + k_2)_\mu g_\sigma^\gamma}{\left[ (k + \frac{k_1 + k_2}{2})^2 - M^2 + ie \right] \left[ (k + \frac{k_1 - k_2}{2})^2 - M^2 + ie \right] \left[ (k - \frac{k_1 + k_2}{2})^2 - M^2 + ie \right]} \cdot \]

(2.4)

In these formulae, \( e \) is the electric charge and \( g \) is the SU(2) electroweak coupling constant.
One should note that, in the eqs. (2.2), (2.3), and (2.4), we have omitted the polarization vectors \((\epsilon^\mu)^*\) and \((\epsilon^\nu)^*\) for the outgoing photons. Also, since we are dealing with real photons, we have

\[
k_1^2 = k_2^2 = 0, \quad k_1\mu = k_2\nu = 0,
\]

where \(k_1\) and \(k_2\) are the four-momenta of the two photons, and \(k_1 + k_2\) is the four-momentum of the Higgs particle. Consequently,

\[
2 (k_1 \cdot k_2) = M_H^2,
\]

where \(M_H\) is the Higgs mass. Also note that the amplitude \(\mathcal{M}_2\) is symmetrical for the interchange of the two photons 1 and 2 \((k_1 \leftrightarrow k_2, \mu \leftrightarrow \nu)\).

For the evaluation of the amplitude, use is made of the Ward identities to simplify the algebra. For the \(VV\gamma\) vertex

\[
V_{\alpha\beta\gamma}(p_1, p_2, p_3) = (p_2 - p_3)_\alpha g_{\beta\gamma} + (p_3 - p_1)_\beta g_{\gamma\alpha} + (p_1 - p_2)_\gamma g_{\alpha\beta},
\]

with all four-momenta \(p_1, p_2, p_3\) \((p_1 + p_2 + p_3 = 0)\) incoming, this identity reads

\[
p^\alpha_1 V_{\alpha\beta\gamma}(p_1, p_2, p_3) = [p^2_3 g_{\beta\gamma} - p_3\beta p_3\gamma] - [p^2_2 g_{\beta\gamma} - p_2\beta p_2\gamma].
\]

For the special case that \(p_2\), e.g., is associated with one of the real outgoing photons, this identity reduces to

\[
p^\alpha_1 V_{\alpha\mu\gamma}(p_1, -k_1, p_3) = p^2_3 g_{\mu\gamma} - p_3\mu p_3\gamma,
\]

because of the relations (2.5). Of course, there is a similar relation for photon 2. In practice, we often use a slightly modified version of this equation in Sec. 3, i.e.,

\[
p^\alpha_1 V_{\alpha\mu\gamma}(p_1, -k_1, p_3) = [p^2 - M^2] g_{\mu\gamma} - p_3\mu p_3\gamma + M^2 g_{\mu\gamma},
\]

the reason being that the first term in (2.10) can be used to cancel a factor in the denominator of \(\mathcal{M}_1\) or \(\mathcal{M}_2\). In this way, cancelations with contributions from \(\mathcal{M}_2\) can be achieved.

Finally, from (2.10), it immediately follows that

\[
p^\alpha_1 p_3^\gamma V_{\alpha\mu\gamma}(p_1, -k_1, p_3) = 0,
\]

and, similarly,

\[
p_1^\alpha p_3^\gamma V_{\alpha\nu\gamma}(p_1, -k_2, p_3) = 0.
\]

### 3 The Evaluation of the Amplitude

Our procedure for the evaluation of the amplitude is straightforward, but somewhat lengthy. We examine successively the terms in \(M^{-n}\), \(n = 6, 4, 2, 0\) in \(\mathcal{M}_1\), \(\mathcal{M}_2\), and \(\mathcal{M}_3\) using the Ward identities listed in Sec. 2. Here, a term in \(M^{-n}\) means a term with an explicit overall factor of \(M^{-n}\), not counting the \(M\) in the factor \(\pmiegM\) in eqs. (2.2)-(2.4). We shall find that all the terms with negative powers of \(M\) give a vanishing contribution. The resulting amplitude is then seen to satisfy the decoupling theorem [7].
3.1 The terms in $M^{-6}$

In $M_1$ as given by (2.2), there is only one term proportional to $M^{-6}$. It is obtained by taking the longitudinal parts of all three $W$-propagators. It then follows that its contribution vanishes because of the Ward identities (2.11) or (2.12). A similar conclusion holds for $M_3$ given by (2.4). As $M_2$ from eq. (2.3) has no terms in $M^{-6}$, it follows that the entire amplitude has no such terms.

3.2 The terms in $M^{-4}$

The terms in $M_1$ proportional to $M^{-4}$ necessarily result from the combination of two longitudinal parts of propagators. Because of the Ward identities (2.11) and (2.12), only the two propagators adjacent to the Higgs vertex contribute. They give

$$M_{11} = \frac{-ie^2gM}{(2\pi)^4} \frac{1}{M^4} \int d^4k (k + \frac{k_1 + k_2}{2})_\alpha (k - \frac{k_1 + k_2}{2})^\beta (k - \frac{k_1 + k_2}{2})^\gamma \times g^{\rho\sigma} \left[ (k + \frac{3k_1 + k_2}{2})_\mu g_{\beta\mu} + (k - \frac{3k_1 + k_2}{2})_\beta g_{\mu\rho} + (-2k - k_2)_\mu g_{\rho\beta} \right] \times \left[ (k - \frac{k_1 + k_2}{2})_\sigma g_{\gamma\nu} + (k + \frac{k_1 + k_2}{2})_\gamma g_{\nu\sigma} + (-2k + k_1)_\nu g_{\sigma\gamma} \right] \times \frac{k^2 - (k_1 \cdot k_2)}{[(k + \frac{k_1 + k_2}{2})^2 - M^2 + i\epsilon] [(k - \frac{k_1 + k_2}{2})^2 - M^2 + i\epsilon]},$$

(3.1)

Using the Ward identity (2.10), this expression can be rewritten in three terms

$$M_{11} = M_{111} + M_{112} + M_{113}, \quad (3.2)$$

with

$$M_{111} = \frac{-ie^2gM}{(2\pi)^4} \frac{1}{M^4} \int d^4k \left[ k^2 - \frac{(k_1 \cdot k_2)}{2} \right] (k - \frac{k_1 + k_2}{2})^\gamma \times g^{\rho\sigma} g_{\rho\mu}$$

$$\times \left[ (k - \frac{k_1 + 3k_2}{2})_\sigma g_{\gamma\nu} + (k + \frac{-k_1 + 3k_2}{2})_\gamma g_{\nu\sigma} + (-2k + k_1)_\nu g_{\sigma\gamma} \right] \times \frac{k^2 - (k_1 \cdot k_2)}{[(k + \frac{k_1 + k_2}{2})^2 - M^2 + i\epsilon] [(k - \frac{k_1 + k_2}{2})^2 - M^2 + i\epsilon]},$$

(3.3)
The last contribution, $\mathcal{M}_{113}$, will be treated in subsection 3.3 together with the other terms in $M^{-2}$.

First, we apply the Ward identity (2.9) to $\mathcal{M}_{111}$:

$$\mathcal{M}_{111} = \mathcal{M}_{1111} + \mathcal{M}_{1112},$$

with

$$\mathcal{M}_{1111} = -\frac{ie^2 g M}{(2\pi)^4} \frac{1}{M^4} \int d^4 k \frac{[k^2 - (k_1 \cdot k_2) \cdot (k - \frac{k_1 + k_2}{2})]}{[(k + \frac{k_1 + k_2}{2})^2 - M^2 + i\epsilon][(k - \frac{k_1 + k_2}{2})^2 - M^2 + i\epsilon]},$$

and

$$\mathcal{M}_{1112} = \frac{ie^2 g M}{(2\pi)^4} \frac{1}{M^4} \int d^4 k \frac{[k^2 - (k_1 \cdot k_2) \cdot (k + \frac{k_1 + k_2}{2})]}{[(k + \frac{k_1 + k_2}{2})^2 - M^2 + i\epsilon][(k - \frac{k_1 + k_2}{2})^2 - M^2 + i\epsilon]}.$$
\[ \mathcal{M}_{1112} + \mathcal{M}_{3112} = \frac{ie^2 g M}{(2\pi)^4} \frac{2}{M^4} \int d^4 k \left[ k^2 - \frac{(k_1 \cdot k_2)}{2} \right] \left( k_\mu - k_{2\mu} k_{1\nu} \right) \frac{1}{\left( \left( k + \frac{k_1 + k_2}{2} \right)^2 - M^2 + i\epsilon \right) \left( \left( k - \frac{k_1 + k_2}{2} \right)^2 - M^2 + i\epsilon \right)} . \]  

(3.10)

However, the amplitude \( \mathcal{M}_2 \) also yields terms of order \( M^{-4} \). They are

\[ \mathcal{M}_2 = \frac{ie^2 g M}{(2\pi)^4} \frac{1}{M^4} \int d^4 k \left[ k^2 - \frac{(k_1 \cdot k_2)}{2} \right] \left( 2g_{\mu\nu} \left( k^2 - \frac{(k_1 \cdot k_2)}{2} \right) - 2k_\mu k_\nu + \frac{k_{2\mu} k_{1\nu}}{2} \right) \frac{1}{\left( \left( k + \frac{k_1 + k_2}{2} \right)^2 - M^2 + i\epsilon \right) \left( \left( k - \frac{k_1 + k_2}{2} \right)^2 - M^2 + i\epsilon \right)} . \]  

(3.11)

Combining the results from (3.9), (3.10), and (3.11) then gives

\[ \mathcal{M}_{1111} + \mathcal{M}_{3111} + \mathcal{M}_{1112} + \mathcal{M}_{3112} + \mathcal{M}_{21} = 0 . \]  

(3.12)

Next, we again apply the Ward identity (2.9), this time to the expression \( \mathcal{M}_{112} \) as given by (3.4) yielding two terms, i.e.,

\[ \mathcal{M}_{112} = C_{1121} + C_{1122} , \]  

(3.13)

with

\[ C_{1121} = \frac{ie^2 g M}{(2\pi)^4} \frac{1}{M^4} \int d^4 k \left[ k^2 - \frac{(k_1 \cdot k_2)}{2} \right] \left( k + \frac{k_2}{2} \right)_\mu \left( k - \frac{k_1}{2} \right)_\nu \]

\[ \times \frac{1}{\left( \left( k + \frac{k_1 + k_2}{2} \right)^2 - M^2 + i\epsilon \right) \left( \left( k - \frac{k_1 + k_2}{2} \right)^2 - M^2 + i\epsilon \right) \left( \left( k + \frac{k_1 + k_2}{2} \right)^2 - M^2 + i\epsilon \right) \left( \left( k - \frac{k_1 + k_2}{2} \right)^2 - M^2 + i\epsilon \right)} . \]  

(3.14)

and

\[ C_{1122} = -\frac{ie^2 g M}{(2\pi)^4} \frac{1}{M^4} \int d^4 k \left[ k^2 - \frac{(k_1 \cdot k_2)}{2} \right] \left( k - \frac{k_1 + k_2}{2} \right)_\gamma \left( k + \frac{k_2}{2} \right)_\mu \left( k - \frac{k_1}{2} \right)_\nu \]

\[ \times \frac{1}{\left( \left( k + \frac{k_1 + k_2}{2} \right)^2 - M^2 + i\epsilon \right) \left( \left( k - \frac{k_1 + k_2}{2} \right)^2 - M^2 + i\epsilon \right) \left( \left( k + \frac{k_1 + k_2}{2} \right)^2 - M^2 + i\epsilon \right) \left( \left( k - \frac{k_1 + k_2}{2} \right)^2 - M^2 + i\epsilon \right)} . \]  

(3.15)

It is readily seen that the these two terms (3.14) and (3.15) cancel:

\[ C_{1121} + C_{1122} = 0 . \]  

(3.16)

Hence, by virtue of eq. (3.13),

\[ \mathcal{M}_{1112} = 0 . \]  

(3.17)

We have thus shown that all the terms of order \( M^{-4} \) cancel. In the process, we generated a term of order \( M^{-2} \) [see (3.5)], which will have to be combined with the terms to be treated in the next subsection 3.3.
3.3 The terms in $M^{-2}$

Because there are three $W$ propagators in the first Feynman diagram, we can distinguish three contributions of order $M^{-2}$ from the first amplitude $\mathcal{M}_1$. They are

\[
\mathcal{M}_{12} = \frac{ie^2 g M}{(2\pi)^4} \frac{1}{M^2} \int d^4k \, g^\rho (k - \frac{k_1 + k_2}{2})^\beta (k - \frac{k_1 + k_2}{2})^\gamma \\
\times \left[ \left( k + \frac{3k_1 + k_2}{2} \right)_\rho g_{\beta\mu} + \left( k + \frac{-3k_1 + k_2}{2} \right)_\beta g_{\mu\rho} + (-2k - k_2)_\mu g_{\rho\beta} \right] \\
\times \left( k - \frac{k_1 + 3k_2}{2} \right)_\sigma g_{\gamma\nu} + \left( k + \frac{-k_1 + 3k_2}{2} \right)_\gamma g_{\nu\sigma} + (-2k + k_1)_\nu g_{\sigma\gamma} \\
\left[ (k + \frac{k_1 + k_2}{2})^2 - M^2 + i\epsilon \right] \left[ (k + \frac{-k_1 + k_2}{2})^2 - M^2 + i\epsilon \right] \left[ (k - \frac{k_1 + k_2}{2})^2 - M^2 + i\epsilon \right],
\]

(3.18)

\[
\mathcal{M}_{13} = \frac{ie^2 g M}{(2\pi)^4} \frac{1}{M^2} \int d^4k \, g^\beta (k + \frac{-k_1 + k_2}{2})^\rho (k + \frac{-k_1 + k_2}{2})^\sigma \\
\times \left[ \left( k + \frac{3k_1 + k_2}{2} \right)_\rho g_{\beta\mu} + \left( k + \frac{-3k_1 + k_2}{2} \right)_\beta g_{\mu\rho} + (-2k - k_2)_\mu g_{\rho\beta} \right] \\
\times \left( k - \frac{k_1 + 3k_2}{2} \right)_\sigma g_{\gamma\nu} + \left( k + \frac{-k_1 + 3k_2}{2} \right)_\gamma g_{\nu\sigma} + (-2k + k_1)_\nu g_{\sigma\gamma} \\
\left[ (k + \frac{k_1 + k_2}{2})^2 - M^2 + i\epsilon \right] \left[ (k + \frac{-k_1 + k_2}{2})^2 - M^2 + i\epsilon \right] \left[ (k - \frac{k_1 + k_2}{2})^2 - M^2 + i\epsilon \right],
\]

(3.19)

and

\[
\mathcal{M}_{14} = \frac{ie^2 g M}{(2\pi)^4} \frac{1}{M^2} \int d^4k \, g^\rho (k + \frac{k_1 + k_2}{2})^\gamma (k + \frac{k_1 + k_2}{2})^\beta \\
\times \left[ \left( k + \frac{3k_1 + k_2}{2} \right)_\rho g_{\beta\mu} + \left( k + \frac{-3k_1 + k_2}{2} \right)_\beta g_{\mu\rho} + (-2k - k_2)_\mu g_{\rho\beta} \right] \\
\times \left( k - \frac{k_1 + 3k_2}{2} \right)_\sigma g_{\gamma\nu} + \left( k + \frac{-k_1 + 3k_2}{2} \right)_\gamma g_{\nu\sigma} + (-2k + k_1)_\nu g_{\sigma\gamma} \\
\left[ (k + \frac{k_1 + k_2}{2})^2 - M^2 + i\epsilon \right] \left[ (k + \frac{-k_1 + k_2}{2})^2 - M^2 + i\epsilon \right] \left[ (k - \frac{k_1 + k_2}{2})^2 - M^2 + i\epsilon \right],
\]

(3.20)

Applying the Ward identity (2.10) to $\mathcal{M}_{12}$ yields three terms, i.e.,

\[
\mathcal{M}_{12} = \mathcal{M}_{121} + \mathcal{M}_{122} + \mathcal{M}_{123},
\]

(3.21)
with

\[ \mathcal{M}_{121} = \frac{i e^2 g M}{(2\pi)^4} \frac{1}{M^2} \int d^4k \, g_\nu^\rho (k - \frac{k_1 + k_2}{2})^\beta \]

\[ \times \frac{(k + \frac{3k_1 + k_2}{2})_\rho g_{\beta\mu} + (k + \frac{-3k_1 + k_2}{2})_\beta g_{\mu\rho} + (-2k - k_2)_\mu g_{\beta\rho}}{\left[\left(k + \frac{k_1 + k_2}{2}\right)^2 - M^2 + i\epsilon\right]\left[\left(k - \frac{k_1 + k_2}{2}\right)^2 - M^2 + i\epsilon\right]}, \quad (3.22) \]

\[ \mathcal{M}_{122} = \frac{-i e^2 g M}{(2\pi)^4} \frac{1}{M^2} \int d^4k (k - \frac{k_1 + k_2}{2})^\beta (k - \frac{-k_1 + k_2}{2})^\nu (k + \frac{-k_1 + k_2}{2})_\nu \]

\[ \times \frac{(k + \frac{3k_1 + k_2}{2})_\rho g_{\beta\mu} + (k + \frac{-3k_1 + k_2}{2})_\beta g_{\mu\rho} + (-2k - k_2)_\mu g_{\beta\rho}}{\left[\left(k + \frac{k_1 + k_2}{2}\right)^2 - M^2 + i\epsilon\right]\left[\left(k - \frac{k_1 + k_2}{2}\right)^2 - M^2 + i\epsilon\right]}, \quad (3.23) \]

and

\[ \mathcal{M}_{123} = \frac{i e^2 g M}{(2\pi)^4} \int d^4k (k - \frac{k_1 + k_2}{2})^\beta \]

\[ \times \frac{(k + \frac{3k_1 + k_2}{2})_\nu g_{\beta\mu} + (k + \frac{-3k_1 + k_2}{2})_\beta g_{\mu\nu} + (-2k - k_2)_\mu g_{\beta\nu}}{\left[\left(k + \frac{k_1 + k_2}{2}\right)^2 - M^2 + i\epsilon\right]\left[\left(k - \frac{k_1 + k_2}{2}\right)^2 - M^2 + i\epsilon\right]}, \quad (3.24) \]

Adding to \( \mathcal{M}_{121} \) the analogous \( 1 \leftrightarrow 2 \) term from the amplitude \( \mathcal{M}_3 \) yields

\[ \mathcal{M}_{121} + \mathcal{M}_{321} = \frac{i e^2 g M}{(2\pi)^4} \frac{1}{M^2} \int d^4k (k - \frac{k_1 + k_2}{2})^\beta \]

\[ \times \frac{(-k + \frac{k_1}{2})_\nu g_{\beta\mu} + (2k - k_1 - k_2)_\beta g_{\mu\nu} + (-k + \frac{k_2}{2})_\mu g_{\beta\nu}}{\left[\left(k + \frac{k_1 + k_2}{2}\right)^2 - M^2 + i\epsilon\right]\left[\left(k - \frac{k_1 + k_2}{2}\right)^2 - M^2 + i\epsilon\right]}, \quad (3.25) \]

However, also the amplitude \( \mathcal{M}_2 \) has terms of order \( M^{-2} \):

\[ \mathcal{M}_{22} = \frac{-i e^2 g M}{(2\pi)^4} \frac{1}{M^2} \int d^4k (k + \frac{k_1 + k_2}{2})^\gamma (k + \frac{k_1 + k_2}{2})^\beta \]

\[ \times \frac{2 g_{\mu\nu} g_{\beta\gamma} - g_{\mu\beta} g_{\nu\gamma} - g_{\mu\gamma} g_{\nu\beta}}{\left[\left(k + \frac{k_1 + k_2}{2}\right)^2 - M^2 + i\epsilon\right]\left[\left(k - \frac{k_1 + k_2}{2}\right)^2 - M^2 + i\epsilon\right]}, \quad (3.26) \]

and

\[ \mathcal{M}_{23} = \frac{-i e^2 g M}{(2\pi)^4} \frac{1}{M^2} \int d^4k (k - \frac{k_1 + k_2}{2})^\beta (k - \frac{k_1 + k_2}{2})^\gamma \]

\[ \times \frac{2 g_{\mu\nu} g_{\beta\gamma} - g_{\mu\beta} g_{\nu\gamma} - g_{\mu\gamma} g_{\nu\beta}}{\left[\left(k + \frac{k_1 + k_2}{2}\right)^2 - M^2 + i\epsilon\right]\left[\left(k - \frac{k_1 + k_2}{2}\right)^2 - M^2 + i\epsilon\right]}, \quad (3.27) \]
From (3.25) and (3.27), it is readily seen that

\[ M_{12} + M_{32} + M_{23} = 0. \]  \tag{3.28}

What we did for \( M_{12} \) and \( M_{23} \) can be repeated, \textit{mutatis mutandis}, for \( M_{14} \) and \( M_{22} \).

The Ward identity (2.10) on \( M_{14} \) yields three terms

\[ M_{14} = M_{141} + M_{142} + M_{143}, \]  \tag{3.29}

with

\[ M_{141} = \frac{i e^2 g M}{(2\pi)^4} \frac{1}{M^2} \int d^4 k \, g^\sigma (k + \frac{k_1 + k_2}{2})^\gamma \]
\[ \times \left( (k - \frac{k_1 + 3k_2}{2})_\sigma g_{\gamma\nu} + (k + \frac{-k_1 + 3k_2}{2})_\gamma g_{\nu\sigma} + (2k_1)_\nu g_{\gamma\sigma} \right) \left[ (k - \frac{k_1 + k_2}{2})^2 - M^2 + i\epsilon \right] \left[ (k + \frac{k_1 + k_2}{2})^2 - M^2 + i\epsilon \right], \]  \tag{3.30}

\[ M_{142} = \frac{-i e^2 g M}{(2\pi)^4} \frac{1}{M^2} \int d^4 k \, (k + \frac{k_1 + k_2}{2})^\gamma (k + \frac{-k_1 + k_2}{2})^\sigma \left( (k - \frac{k_1 + k_2}{2})_\mu + \frac{k_1 + k_2}{2} \sigma \right) \]
\[ \times \left( (k - \frac{k_1 + 3k_2}{2})_\sigma g_{\gamma\nu} + (k + \frac{-k_1 + 3k_2}{2})_\gamma g_{\nu\sigma} + (2k_1)_\nu g_{\gamma\sigma} \right) \left[ (k + \frac{k_1 + k_2}{2})^2 - M^2 + i\epsilon \right] \left[ (k + \frac{k_1 + k_2}{2})^2 - M^2 + i\epsilon \right], \]  \tag{3.31}

and

\[ M_{143} = \frac{i e^2 g M}{(2\pi)^4} \int d^4 k \, g^\sigma \left( k + \frac{k_1 + k_2}{2} \right)^\gamma \]
\[ \times \left( (k - \frac{k_1 + 3k_2}{2})_\sigma g_{\gamma\nu} + (k + \frac{-k_1 + 3k_2}{2})_\gamma g_{\nu\sigma} + (2k_1)_\nu g_{\gamma\sigma} \right) \left[ (k + \frac{k_1 + k_2}{2})^2 - M^2 + i\epsilon \right] \left[ (k + \frac{k_1 + k_2}{2})^2 - M^2 + i\epsilon \right]. \]  \tag{3.32}

Symmetrizing \( M_{141} \) in the two photons by adding the analogous part of amplitude \( M_3 \), we obtain

\[ M_{141} + M_{341} = \frac{i e^2 g M}{(2\pi)^4} \frac{1}{M^2} \int d^4 k \, g^\sigma \left( k + \frac{k_1 + k_2}{2} \right)^\gamma \]
\[ \times \left( -k_1 + \frac{k_2}{2} \right)_\mu g_{\gamma\nu} + (2k_1 + k_2)_\gamma g_{\mu\nu} + (2k_1)_\nu g_{\mu\gamma} \right) \left[ (k - \frac{k_1 + k_2}{2})^2 - M^2 + i\epsilon \right] \left[ (k - \frac{k_1 + k_2}{2})^2 - M^2 + i\epsilon \right]. \]  \tag{3.33}
which is readily seen as the opposite of $\mathcal{M}_{22}$ in eq. (3.26). Hence,

$$\mathcal{M}_{141} + \mathcal{M}_{341} + \mathcal{M}_{22} = 0.$$  (3.34)

To continue the treatment of the $M^{-2}$ terms, we collect the remaining terms from $\mathcal{M}_{113}$, $\mathcal{M}_{13}$, $\mathcal{M}_{122}$, and $\mathcal{M}_{142}$ as given by eqs. (3.35), (3.19), (3.23), and (3.31) respectively:

$$\mathcal{M}_{113} + \mathcal{M}_{13} + \mathcal{M}_{122} + \mathcal{M}_{142} = -\frac{ie^2gM}{(2\pi)^4} \frac{1}{M^2}$$

$$\times \int d^4k \frac{A}{[(k + \frac{k_1 + k_2}{2})^2 - M^2 + i\epsilon] [(k - \frac{-k_1 + k_2}{2})^2 - M^2 + i\epsilon] [(k - \frac{k_1 + k_2}{2})^2 - M^2 + i\epsilon]},$$

with

$$A = -(k + \frac{k_1 + k_2}{2})^2 (k - \frac{k_1 + k_2}{2})^2 g_{\mu\nu} - (k^2 - \frac{(k_1 \cdot k_2)}{2}) (k + \frac{k_2}{2}) (k - \frac{k_1}{2})_{\mu} (k - \frac{k_1}{2})_{\nu}$$

$$+ (k + \frac{k_1 + k_2}{2})^2 (k - \frac{k_2}{2})_{\mu} (k - \frac{k_1}{2})_{\nu} + (k - \frac{k_1 + k_2}{2})^2 (k + \frac{k_2}{2})_{\mu} (k + \frac{k_1}{2})_{\nu}$$

$$+ (k^2 - \frac{(k_1 \cdot k_2)}{2}) [(k^2 - (k \cdot k_1) + (k \cdot k_2) - \frac{(k_1 \cdot k_2)}{2}) g_{\mu\nu} + (k + \frac{k_2}{2})_{\mu} (k - \frac{k_1}{2})_{\nu}]$$

$$+ (k + \frac{k_2}{2})_{\mu} [(k - \frac{k_1 + k_2}{2})^2 (k + \frac{k_1}{2})_{\nu} - (k + \frac{k_1 + k_2}{2}) \cdot k - \frac{k_1 + k_2}{2}) (k - \frac{k_1}{2})_{\nu}]$$

$$+ (k - \frac{k_1}{2})_{\nu} [(k + \frac{k_1 + k_2}{2})^2 (k - \frac{k_2}{2})_{\mu} - (k + \frac{k_1 + k_2}{2}) \cdot k - \frac{k_1 + k_2}{2}) (k + \frac{k_2}{2})_{\mu}].$$

Some elementary algebra shows that

$$A = 4 (k_1 \cdot k_2) k_{\mu} k_{\nu} + 2 k^2 k_{2\mu} k_{1\nu} - 2 (k_{\mu} k_{1\nu} + k_{2\mu} k_{\nu})(k \cdot k_1 + k_2)$$

$$+ g_{\mu\nu} [-2k^2 (k_1 \cdot k_2) + (k \cdot k_1 + k_2)^2]$$

$$+[k^2 - \frac{(k_1 \cdot k_2)}{2}][-g_{\mu\nu} (k \cdot k_1 - k_2) + 2(k_{\mu} k_{1\nu} - k_{2\mu} k_{\nu})].$$  (3.37)

We want to rewrite the last line in this expression for $A$ as follows:

$$[k^2 - \frac{(k_1 \cdot k_2)}{2}] [-g_{\mu\nu} (k \cdot k_1 - k_2) + 2(k_{\mu} k_{1\nu} - k_{2\mu} k_{\nu})]$$

$$= [k^2 - (k \cdot k_1 - k_2) - \frac{(k_1 \cdot k_2)}{2} - M^2 + (k \cdot k_1 - k_2) + M^2]$$

$$\times [-g_{\mu\nu} (k \cdot k_1 - k_2) + 2(k_{\mu} k_{1\nu} - k_{2\mu} k_{\nu})],$$  (3.38)
the reason being that the first four terms in the first bracket cancel the middle denominator in (3.34), which makes the denominator an even function of \( k \). As the second bracket is odd in \( k \), we can drop that term entirely. In the process, we have introduced a term in the numerator proportional to \( M^2 \), which will have to be treated in the next subsection 3.4, together with the remaining terms. Thus,

\[
M_{113} + M_{13} + M_{122} + M_{142} = M_{1131} + M_{1132}.
\] (3.39)

In eq. (3.39), we have

\[
M_{1131} = -ie^2 gM \left(\frac{1}{(2\pi)^4} \frac{1}{M^2}\right) \times \int d^4k \frac{A'}{\left[\left(k + \frac{k_1 + k_2}{2}\right)^2 - M^2 + i\epsilon\right]\left[\left(k + \frac{-k_1 + k_2}{2}\right)^2 - M^2 + i\epsilon\right]\left[\left(k - \frac{k_1 + k_2}{2}\right)^2 - M^2 + i\epsilon\right]},
\] (3.40)

with

\[
A' = 4(k_1 \cdot k_2) k_\mu k_\nu + 2k^2 k_\mu k_{1\nu} - 4k_\mu k_{1\nu} (k \cdot k_2) - 4k_2 \mu k_\nu (k \cdot k_1)
+ g_{\mu\nu} [-2k^2 (k_1 \cdot k_2) + 4(k \cdot k_1) (k \cdot k_2)],
\] (3.41)

and

\[
M_{1132} = -ie^2 gM \left(\frac{1}{(2\pi)^4}\right) \times \int d^4k \frac{-g_{\mu\nu} (k \cdot k_1 - k_2) + 2(k_\mu k_{1\nu} - k_2 \mu k_\nu)}{\left[\left(k + \frac{k_1 + k_2}{2}\right)^2 - M^2 + i\epsilon\right]\left[\left(k + \frac{-k_1 + k_2}{2}\right)^2 - M^2 + i\epsilon\right]\left[\left(k - \frac{k_1 + k_2}{2}\right)^2 - M^2 + i\epsilon\right]},
\] (3.42)

We proceed to show that the expression \( M_{1131} \) vanishes. To this end, note that the integral in (3.40) is only logarithmically divergent. It follows that a shift in the integration variable is allowed and that it does not produce any surface term. Combining the three factors in the denominator with the Feynman variables \( \alpha_1, \alpha_2, \) and \( \alpha_3 \), we obtain the denominator

\[
D = k^2 + \alpha_1 (k \cdot k_1 + k_2) + \alpha_3 (k \cdot -k_1 + k_2) - \alpha_2 (k \cdot k_1 + k_2)
+ (\alpha_1 - \alpha_3 + \alpha_2) \frac{(k_1 \cdot k_2)}{2} - M^2 + i\epsilon
= k^2 - (1 - 2\alpha_1) (k \cdot k_1) + (1 - 2\alpha_2) (k \cdot k_2) + (1 - 2\alpha_3) \frac{(k_1 \cdot k_2)}{2} - M^2 + i\epsilon,
\] (3.43)
because of the $\delta$-function $\delta(1 - \alpha_1 - \alpha_2 - \alpha_3)$ in the Feynman combination of denominators. With the shift
\[ k = \ell + \frac{1}{2}(1 - 2\alpha_1)k_1 - \frac{1}{2}(1 - 2\alpha_2)k_2, \] the denominator becomes
\[ D = \ell^2 - M^2 + 2\alpha_1\alpha_2(k_1 \cdot k_2) + i\epsilon. \] (3.45)

It is a simple matter to perform the shift (3.44) in the numerator $A'$ in (3.41). Dropping the terms odd in $\ell$ and using the relations (2.5), one finds that effectively $A' = 0$, meaning that our amplitude does not contain terms in $M^{-2}$. This result is closely related to the fact that our amplitude obeys the decoupling theorem.

### 3.4 The result

The last set of terms to be treated are those without negative powers of $M$. First, we list the term that derives from the amplitude $M_{15}$ as given by (2.2):

\[ M_{15} = \frac{-ie^2gM}{(2\pi)^4} \int d^4k \, g^{\rho\sigma} g^{\beta\gamma} \times [(k + \frac{3k_1 + k_2}{2})_\rho g_{\beta\mu} + (k + \frac{-3k_1 + k_2}{2})_\beta g_{\mu\rho} + (-2k - k_2)_\mu g_{\rho\beta}] \]

\[ \times \frac{(k - \frac{k_1 + 3k_2}{2})_\sigma g_{\gamma\nu} + (k + \frac{-k_1 + 3k_2}{2})_\gamma g_{\nu\sigma} + (-2k + k_1)_\nu g_{\sigma\gamma}}{[(k + \frac{k_1 + k_2}{2})^2 - M^2 + i\epsilon][(k - \frac{k_1 + k_2}{2})^2 - M^2 + i\epsilon]}, \]

\[ = \frac{-ie^2gM}{(2\pi)^4} \int d^4k \left\{ g_{\mu\nu} \left[ 2k^2 - (k \cdot k_1) + (k \cdot k_2) - 5(k_1 \cdot k_2) \right] \right. \]

\[ \left. + 2k_\mu k_\nu + \frac{9}{2}k_2\mu k_1\nu + 8(k + \frac{k_2}{2})_\mu (k - \frac{k_1}{2})_\nu \right\} \]

\[ [(k + \frac{k_1 + k_2}{2})^2 - M^2 + i\epsilon][(k - \frac{k_1 + k_2}{2})^2 - M^2 + i\epsilon][(k - k_1 + k_2)^2 - M^2 + i\epsilon], \] (3.46)

and from $M_{24}$ as given by (2.3)

\[ M_{24} = \frac{ie^2gM}{(2\pi)^4} \int d^4k \frac{g^{\beta\gamma}[2g_{\mu\nu}g_{\beta\gamma} - g_{\mu\beta}g_{\nu\gamma} - g_{\mu\gamma}g_{\nu\beta}]}{[(k + \frac{k_1 + k_2}{2})^2 - M^2 + i\epsilon][(k - \frac{k_1 + k_2}{2})^2 - M^2 + i\epsilon]}, \]

\[ = \frac{ie^2gM}{(2\pi)^4} \int d^4k \frac{6g_{\mu\nu}}{[(k + \frac{k_1 + k_2}{2})^2 - M^2 + i\epsilon][(k - \frac{k_1 + k_2}{2})^2 - M^2 + i\epsilon]} . \] (3.47)
The remaining contributions are $M_{123}$, $M_{143}$, and $M_{1132}$ as given by eqs. (3.24), (3.32), and (3.42) respectively. Adding these contributions gives

$$M_{15} + \frac{1}{2}M_{24} + M_{123} + M_{143} + M_{1132} = \frac{-ie^2gM}{(2\pi)^4} \int d^4k \left\{ g_{\mu\nu} \left[ -3k^2 + 3(k \cdot k_1) - 3(k \cdot k_2) - \frac{9}{2}(k_1 \cdot k_2) + 3M^2 \right] + 12k_\mu k_\nu + 3k_2\mu k_1\nu - 6k_\mu k_1\nu + 6k_2\mu k_\nu \right\} \left[ (k + \frac{k_1 + k_2}{2})^2 - M^2 + i\epsilon \right] \left[ (k + \frac{-k_1 + k_2}{2})^2 - M^2 + i\epsilon \right] \left[ (k - \frac{k_1 + k_2}{2})^2 - M^2 + i\epsilon \right],$$

(3.48)

We perform the same shift of integration variable as in eq. (3.41), we drop the odd terms in $\ell$, and we symmetrize $\ell \ell_\mu \ell_\nu \rightarrow \frac{1}{4} \ell^2 g_{\mu\nu}$, leading to

$$M_{15} + \frac{1}{2}M_{24} + M_{123} + M_{143} + M_{1132} = \frac{-2ie^2gM}{(2\pi)^4} \int d^4\ell \int_0^1 d\alpha_1 \int_0^{1-\alpha_1} d\alpha_2 \frac{g_{\mu\nu} \left[ (k_1 \cdot k_2) (-6 + 6 \alpha_1 \alpha_2) + 3M^2 \right] + 3 \left( 2 - 4 \alpha_1 \alpha_2 \right) k_2\mu k_1\nu}{\left[ \ell^2 - M^2 + 2\alpha_1 \alpha_2 (k_1 \cdot k_2) + i\epsilon \right]^3}.$$

(3.50)

Performing the integration over $d^4\ell$, and adding the contribution from $M_3$ (and the other half of $M_2$), which merely gives a factor of 2 because the result (3.50) is already $1 \leftrightarrow 2$ symmetric, we obtain the total amplitude $\mathcal{M}$

$$\mathcal{M} = M_1 + M_2 + M_3 = \frac{-e^2gM}{8\pi^2} \times \int_0^1 d\alpha_1 \int_0^{1-\alpha_1} d\alpha_2 \frac{g_{\mu\nu} \left[ (k_1 \cdot k_2) (-6 + 6 \alpha_1 \alpha_2) + 3M^2 \right] + 3 \left( 2 - 4 \alpha_1 \alpha_2 \right) k_2\mu k_1\nu}{M^2 - 2\alpha_1 \alpha_2 (k_1 \cdot k_2) - i\epsilon}.$$

(3.51)

Following Dyson’s prescription [14], we perform a subtraction of the amplitude for $k_1 = k_2 = 0$, to obtain the finite and gauge invariant result

$$\mathcal{M} = \frac{-e^2gM}{8\pi^2} \int_0^1 d\alpha_1 \int_0^{1-\alpha_1} d\alpha_2 \frac{6(1 - 2\alpha_1 \alpha_2) \left[ k_2\mu k_1\nu - g_{\mu\nu} (k_1 \cdot k_2) \right]}{M^2 - \alpha_1 \alpha_2 M_H^2 - i\epsilon}.$$

(3.52)

The integral over the Feynman parameters can be expressed in terms of elementary functions. Using

$$\tau = \frac{M_H^2}{4M^2},$$

(3.53)
we have
\[ \mathcal{M} = -\frac{3e^2g}{8\pi^2M} \left[ k_{2\mu} k_{1\nu} - g_{\mu\nu} (k_1 \cdot k_2) \right] \left[ \tau^{-1} + (2\tau^{-1} - \tau^{-2}) f(\tau) \right], \] 
(3.54)
with
\[ f(\tau) = \begin{cases} 
\text{arcsin}^2(\sqrt{\tau}) & \text{for } \tau \leq 1, \\
-\frac{1}{4} \left[ \ln \frac{1 + \sqrt{1 - \tau^{-1}}}{1 - \sqrt{1 - \tau^{-1}}} - i\pi \right]^2 & \text{for } \tau > 1.
\end{cases} \]
(3.55)
Clearly, for large Higgs masses (\( \tau \to \infty \)), we have from eq. (3.54) that \( \mathcal{M} \to 0 \), i.e., we have decoupling of the \( W \) contribution \([7]\).

4 Discussions and conclusions

Several points concerning our result eq. (3.54) deserve further discussion.

(a) As announced in the Introduction, our result (3.54) differs from the previous one \([5]\), which reads
\[ \mathcal{M}(\xi = 1) = -\frac{e^2g}{8\pi^2M} \left[ k_{2\mu} k_{1\nu} - g_{\mu\nu} (k_1 \cdot k_2) \right] \left[ 2 + 3\tau^{-1} + 3(2\tau^{-1} - \tau^{-2}) f(\tau) \right]. \] 
(4.1)
The first term in the second bracket of (4.1) is not present in our result (3.54). It is precisely the one which violates the decoupling theorem because it does not vanish for \( \tau \to \infty \).
The observation that two honest calculations for the same process in quantum field theory can lead to two different answers is both disturbing and intriguing. It is disturbing because the question naturally arises: which is the right answer from the physics point of view? One rather compelling argument in favor of our answer is the fact that our amplitude does satisfy the decoupling theorem for large Higgs masses.

In our opinion, the present calculation is reliable because it is straightforward: it is a calculation that is convergent and, therefore, it does not appeal to artifacts such as regularization.

(b) In the pioneering paper of Ellis, Gaillard, and Nanopoulos \([4]\) on the decay process (1.1) through one \( W \) loop, the assumption (c) of Sec. 1 was used: at that time, thirty-five years ago, it was believed that the Higgs particle had a small mass. In this limit of small Higgs mass, the present result given by eq. (3.54) is smaller by a factor of 5/7 compared with that of Ref. \([4]\). In the opposite limit of a large Higgs mass compared with the mass of the \( W \), the present result, which satisfies the decoupling theorem \([7]\), is in absolute value much smaller than the previous one \([4, 5]\).

(c) For a Higgs mass \( M_H = 115 \text{ GeV}/c^2 \), the quantity \( \tau = 0.511 \) [see eq. (3.53)]. A comparison of the two expressions (3.54) and (4.1) then shows that the amplitude \( \mathcal{M} \) is 24.9% smaller in absolute value than the \( \mathcal{M}(\xi = 1) \) amplitude. If the top loop is also taken into account \([16]\), the decay width for \( H \to \gamma\gamma \) is reduced by 54.2%.
The origin of the extra term 2 in (4.1) can be traced to the use of dimensional regularization. The previous calculations [5] were indeed performed in the $R_1$ gauge with an implementation of dimensional regularization.

This regularization scheme requires that the algebra be performed in $n$ dimensions. Hence, the symmetrization (3.49) is to be replaced by

$$\ell_\mu \ell_\nu \to \frac{1}{n} \ell^2 g_{\mu\nu} \quad (4.2)$$

and one must take

$$g^{\mu\mu} = n. \quad (4.3)$$

It is readily verified that the additional $n$-dependence from eq. (4.3) does not change the result (4.1). Therefore, the difference between the two results (3.54) and (4.1) stems from the behavior of the following integral:

$$I_{\mu\nu}(n) = \int d^n \ell \ell^2 g_{\mu\nu} - 4 \ell_\mu \ell_\nu \left[ \ell^2 - M^2 + i\varepsilon \right]^{3/2}. \quad (4.4)$$

By symmetric integration (3.49), this integral is

$$I_{\mu\nu}(4) = 0 ; \quad (4.5)$$

on the other hand, a direct evaluation with (4.2) yields

$$I_{\mu\nu}(n) \simeq -\frac{i\pi^2}{2} g_{\mu\nu}, \quad (4.6)$$

when $n$ is close to but less than 4. Thus, the integral $I_{\mu\nu}(n)$ is discontinuous at $n = 4$. Is such a behavior “pathological”? In view of the simple nature of the integral (4.4) and the fact that this integral is not defined for $n > 4$, the answer must be “no”. This raises the question how often dimensional regularization can lead to wrong answers. For the present case, the previous result is suspicious because of its failure to satisfy the decoupling theorem; in other cases, there may be no such guidance to suggest the necessity of repeating the calculation with the space-time dimension kept at four.

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