A cohomological Seiberg–Witten invariant emerging from the adjunction inequality

Hokuto Konno

Abstract
We construct an invariant of closed spin\(^c\) 4-manifolds using families of Seiberg–Witten equations. This invariant is formulated as a cohomology class on a certain abstract simplicial complex consisting of embedded surfaces of a 4-manifold. We also give examples of 4-manifolds which admit positive scalar curvature metrics and for which this invariant does not vanish. This non-vanishing result of our invariant provides a new class of adjunction-type genus constraints on configurations of embedded surfaces in a 4-manifold whose Seiberg–Witten invariant vanishes.

MSC 2020
57R57 (primary)

Contents
1. INTRODUCTION .................................... 109
2. CONSTRUCTION OF THE INVARIANT I: FAMILY OF RIEMANNIAN METRICS .. 112
3. CONSTRUCTION OF THE INVARIANT II: PERTURBATIONS AND COUNTING ARGUMENTS .................................................. 132
4. NON-VANISHING RESULTS FOR THE COHOMOLOGICAL INVARIANT .......... 141
5. CONCLUDING REMARKS ............................... 164
APPENDIX A: SIMPLE PROPERTIES OF COMPLEX OF SURFACES .................. 165
ACKNOWLEDGEMENTS ................................ 166
REFERENCES ........................................ 166

© 2022 The Authors. The publishing rights in this article are licensed to the London Mathematical Society under an exclusive licence.
1 | INTRODUCTION

In this paper, we construct an invariant of closed spin\(^c\) 4-manifolds. This invariant is defined using families of Seiberg–Witten equations and formulated as a cohomology class on a certain abstract simplicial complex corresponding to a given spin\(^c\) 4-manifold. This simplicial complex encodes information about mutual intersections of embedded surfaces violating the adjunction inequalities with respect to the given spin\(^c\) structure. We also give examples of 4-manifolds which admit positive scalar curvature metrics and for which the invariant does not vanish. This non-vanishing result of our invariant provides a new class of adjunction-type genus constraints on configurations of embedded surfaces in a 4-manifold whose Seiberg–Witten invariant vanishes, for example, \(k\mathbb{C}\mathbb{P}^2\#l(-\mathbb{C}\mathbb{P}^2)\) for some \(k, l > 3\).

Starting with Witten [24], the Seiberg–Witten invariant of smooth spin\(^c\) 4-manifolds has occupied an important place in 4-dimensional geometry and topology. This invariant is defined, roughly speaking, by counting the points of the moduli space of the solutions to the Seiberg–Witten equations. For a given spin\(^c\) structure \(\mathfrak{s}\), the Seiberg–Witten invariant is defined to be trivial if the formal dimension \(d(\mathfrak{s})\), given by the formula \(d(\mathfrak{s}) = (c_1(\mathfrak{s})^2 - 2\chi(X) - 3\text{sign}(X))/4\), of the moduli space of the Seiberg–Witten equations with respect to \(\mathfrak{s}\) is negative. However, even in the case that \(d(\mathfrak{s}) < 0\), one can fruitfully consider a counting argument for the parameterized moduli space of a suitable family of Seiberg–Witten equations. (This type of ideas has been suggested in Donaldson [3, 4].) Indeed, the formal dimension of the moduli space of a family of Seiberg–Witten equations parameterized by a \(k\)-dimensional space is \(d(\mathfrak{s}) + k\), which is non-negative for \(k \geq -d(\mathfrak{s})\). Based on this idea, Ruberman [17–19] has defined several invariants of diffeomorphisms on 4-manifolds using moduli spaces parameterized by 1-dimensional spaces and Li–Liu [11] have defined an invariant of families of spin\(^c\) 4-manifolds. The main aim of this paper is to construct an invariant of spin\(^c\) 4-manifolds using families of Seiberg–Witten equations; we show that this invariant is non-trivial for some spin\(^c\) 4-manifold \((X, \mathfrak{s})\) with \(d(\mathfrak{s}) < 0\). Note that we construct an invariant of a single 4-manifold, while Li–Liu’s is of a family of 4-manifolds.

The invariant that we construct in this paper also has other roots in a classical problem in 4-dimensional topology: the genus bound problem. The Seiberg–Witten theory has been used to attack this problem. We first recall the celebrated paper due to Kronheimer–Mrowka [9] on the Thom conjecture. By arguments in the paper, for a given spin\(^c\) 4-manifold \((X, \mathfrak{s})\), one can show that a strong lower bound on genus called the adjunction inequality (with respect to \(\mathfrak{s}\)) holds for an embedded surface in \(X\) if the Seiberg–Witten invariant of \((X, \mathfrak{s})\) does not vanish and the surface has non-negative self-intersection number. However, for a surface embedded in a spin\(^c\) 4-manifold whose Seiberg–Witten invariant vanishes, the adjunction inequality does not hold in general. (For example, see Nouh [15] for the 4-manifold \(\mathbb{C}\mathbb{P}^2\#\mathbb{C}\mathbb{P}^2\).) Although one cannot expect any systematic result on the adjunction inequality for a single embedded surface in such a spin\(^c\) 4-manifold by this reason, one can do for configurations of embedded surfaces. The first result in this direction is due to Strle [21]. He has given certain constraints on configurations of embedded surfaces with positive self-intersection number. More precisely, Strle has considered not only a single surface but also several embedded surfaces, and he has shown that if they are mutually disjoint, then the adjunction inequality holds for at least one of them under some conditions. Inspired by this Strle’s work, the author [8] has given constraints on configurations of embedded surfaces with zero self-intersection number. We note that both of Strle’s and the author’s results are valid for spin\(^c\) 4-manifolds whose Seiberg–Witten invariants vanish. To obtain the constraints on configurations, the author used a higher dimensional family.
of Seiberg–Witten equations on a 4-manifold. This proof suggests that one might expect some systematic way to relate families of Seiberg–Witten equations and adjunction-type genus constraints on configurations.

In this paper we introduce an abstract simplicial complex associated with a spin\(^c\) 4-manifold \((X, \mathfrak{s})\) and define an invariant of \((X, \mathfrak{s})\) as a simplicial cohomology class on this simplicial complex. The simplicial complex encodes information about mutual intersections of all embedded surfaces violating the adjunction inequalities with respect to \(\mathfrak{s}\) and having zero self-intersection number. In order to define the simplicial complex, we start with the following ‘ambient’ simplicial complex \(\mathcal{K}\), which Mikio Furuta introduced to the author. Throughout this paper, we mean by a surface a smooth oriented closed connected 2-dimensional manifold.

**Definition 1.1.** Let \(X\) be a smooth oriented closed connected 4-manifold. We define an abstract simplicial complex \(\overline{\mathcal{K}} = \overline{\mathcal{K}}(X)\) as follows:

- The vertices of \(\overline{\mathcal{K}}\) are defined as smoothly embedded surfaces (that is, smooth oriented closed 2-dimensional submanifolds of \(X\)) with self-intersection number zero. We denote by \(V(\overline{\mathcal{K}})\) the set of vertices of \(\overline{\mathcal{K}}\).
- For \(n \geq 1\), a collection of \((n + 1)\) vertices \(\Sigma_0, \ldots, \Sigma_n \in V(\overline{\mathcal{K}})\) spans an \(n\)-simplex if and only if \(\Sigma_0, \ldots, \Sigma_n\) are mutually disjoint.

We call \(\overline{\mathcal{K}}\) the complex of surfaces of \(X\).

The complex of surfaces is a 4-dimensional analog of the complex of curves due to Harvey [6] in 2-dimensional topology. Note that, in the definition of the complex of surfaces above, we do not consider isotopy classes of embeddings of surfaces as in the definition of the complex of curves. (See Remark 3.7 on this point.)

We topologize \(\overline{\mathcal{K}}(X)\) as a simplicial complex, namely with the weak topology. In fact, \(\overline{\mathcal{K}}(X)\) is contractible for any \(X\) (Proposition A.2), and therefore one cannot use any homotopical information of \(\overline{\mathcal{K}}(X)\) to define a non-trivial invariant of \(X\). We can, however, find a suitable subcomplex of \(\overline{\mathcal{K}}\) which is not homotopically trivial in general and will be used as parameter space (up to homotopy) of families of Seiberg–Witten equations. The subcomplex is defined considering one of special phenomena in 4-dimensional topology, namely the adjunction inequality. This subcomplex is the main ingredient to study higher-dimensional families of Seiberg–Witten equations in this paper. We will denote by \(g(\Sigma)\) the genus of a surface \(\Sigma\). Set

\[
\chi^- (\Sigma) := \max \{-\chi(\Sigma), 0\},
\]

where \(\chi(\Sigma) = 2 - 2g(\Sigma)\) is the Euler characteristic of \(\Sigma\).

**Definition 1.2.** Let \(\mathfrak{s}\) be a spin\(^c\) structure on \(X\). We define \(\mathcal{K} = \mathcal{K}(X, \mathfrak{s})\) as the full subcomplex of \(\overline{\mathcal{K}}(X)\) spanned by

\[
V(\mathcal{K}) := \{\Sigma \in V(\overline{\mathcal{K}}) \mid \chi^- (\Sigma) < |c_1(\mathfrak{s}) \cdot [\Sigma]|\}.
\]

Namely a collection of vertices \(\Sigma_0, \ldots, \Sigma_n \in V(\mathcal{K})\) spans an \(n\)-simplex of \(\mathcal{K}\) if and only if it does in \(\overline{\mathcal{K}}\). We call \(\mathcal{K}\) the adjunction complex of surfaces of \((X, \mathfrak{s})\).
We note a similarity between the simplicial complex $\mathcal{K}$ above and the Kakimizu complex [7] in 3-dimensional topology and knot theory: in the definitions of both simplicial complexes one focuses one’s attention on surfaces of low genus.

We now introduce our invariant, which will be defined in Subsection 3.1, in the most basic setting. Let $(X, \mathfrak{s})$ be a smooth oriented closed spin$^c$ 4-manifold equipped with a homology orientation. Here a homology orientation means an orientation of the vector space $H^1(X; \mathbb{R}) \oplus H^+(X; \mathbb{R})$, where $H^+(X; \mathbb{R})$ is a maximal positive-definite subspace of $H^2(X; \mathbb{R})$ with respect to the intersection form of $X$. We denote by $b^+(X)$ the dimension of $H^+(X; \mathbb{R})$. Let $n$ be the integer given by $d(\mathfrak{s}) = -(n + 1)$. Assume that $n \geq 0$ and that $b^+(X) \geq n + 3$. Under these assumptions, we shall construct a cohomology class

$$SW(X, \mathfrak{s}) \in H^n(\mathcal{K}(X, \mathfrak{s}); \mathbb{Z})$$

using $(n + 1)$-dimensional families of Seiberg–Witten equations. We will show that this cohomology class $SW(X, \mathfrak{s})$ is an invariant of a spin$^c$ 4-manifold $(X, \mathfrak{s})$. We call $SW(X, \mathfrak{s})$ the cohomological Seiberg–Witten invariant associated with the adjunction complex of surfaces. (In fact, we can extend the definition of the cohomological Seiberg–Witten invariant to more general spin$^c$ structures. See Subsection 3.3 on this point.) Since $H^*(\mathcal{K}(X, \mathfrak{s}); \mathbb{Z})$, to which $SW(X, \mathfrak{s})$ belongs, is also a diffeomorphism invariant of $(X, \mathfrak{s})$, the calculation of $SW(X, \mathfrak{s})$ seems hard. Nevertheless, we can show that there are examples of $(X, \mathfrak{s})$ with $SW(X, \mathfrak{s}) \neq 0$. In particular, we obtain examples of $(X, \mathfrak{s})$ such that the homotopy type of $\mathcal{K}(X, \mathfrak{s})$ is non-trivial.

A remarkable point is that

$$SW(k\mathbb{C}P^2 \# l(-\mathbb{C}P^2), \mathfrak{s}) \neq 0$$

holds for some $k, l > 3$ and some spin$^c$ structure $\mathfrak{s}$ on $k\mathbb{C}P^2 \# l(-\mathbb{C}P^2)$. (See Corollary 4.10.) Note that the usual Seiberg–Witten invariant vanishes for any spin$^c$ structure on it and the Donaldson invariant of this manifold also vanishes. Furthermore, the refinement of the Seiberg–Witten invariant called the Bauer–Furuta invariant [2] of this 4-manifold also vanishes. Indeed, this 4-manifold admits a metric with positive scalar curvature, and the Bauer–Furuta invariant vanishes for 4-manifolds admitting a positive scalar curvature metric as the Seiberg–Witten invariant does. To the best of the author’s knowledge, $SW(X, \mathfrak{s})$ is the first invariant which is defined using the Seiberg–Witten theory and which is non-trivial for some 4-manifolds admitting positive scalar curvature metrics.

The non-vanishing of the cohomological invariant yields a concrete geometric result connected with the adjunction inequality. We will prove in fact not only that $SW(X, \mathfrak{s}) \neq 0$ but also that the evaluation $< SW(X, \mathfrak{s}), \cdot > : H_*(\mathcal{K}(X, \mathfrak{s}); \mathbb{Z}) \to \mathbb{Z}$ is non-zero for some $(X, \mathfrak{s})$. We therefore also obtain a non-trivial homology class in $H_*(\mathcal{K}(X, \mathfrak{s}); \mathbb{Z})$; we can concretely give such a class. We will see that each non-trivial homology class in $H_*(\mathcal{K}(X, \mathfrak{s}); \mathbb{Z})$ provides some adjunction-type genus constraints on configurations: the adjunction inequality for some embedded surface under a certain condition on geometric intersections with other embedded surfaces. Such constraints on configurations will be obtained for 4-manifolds whose Seiberg–Witten and Bauer–Furuta invariants vanish as explained above. This point is similar to Strle [21] and the author’s result [8]. On the other hand, we will also show that one non-trivial homology class in $H_*(\mathcal{K}(X, \mathfrak{s}); \mathbb{Z})$ provides constraints on infinitely many configurations of embedded surfaces. This is a phenomenon newly found through our use of the simplicial homology theory.
We show this non-vanishing result using the following two key ingredients: the first one is a combination of wall-crossing and gluing technique originally due to Ruberman [17–19], and the second is a description of higher dimensional wall-crossing phenomena in terms of embedded surfaces given in [8] by the author. Combining these two tools, we can obtain a solution to the Seiberg–Witten equations used to show the adjunction inequality on a 4-manifold whose Seiberg–Witten invariant vanishes.

We finish off this introduction with an outline of the contents of this paper. Section 2, 3 are devoted to constructing the invariant $\mathcal{SW}$. In Section 2, we construct certain families of Riemannian metrics on a given 4-manifold obtained by stretching neighborhoods of embedded surfaces. In Section 3, we consider the moduli spaces of the Seiberg–Witten equations parameterized by the families of Riemannian metrics constructed in Section 2, and define $\mathcal{SW}$ by counting the points of the parameterized moduli spaces. In Section 4, we prove that the invariants $\mathcal{SW}(X, \mathfrak{s})$ are non-trivial for some spin$^c$ 4-manifolds $(X, \mathfrak{s})$ as described above, and give a geometric application of this non-vanishing result. In particular, in Subsection 4.5, we give bounds on the complexity of configurations of surfaces using this non-vanishing, along an idea of Strle [21].

2 | CONSTRUCTION OF THE INVARIANT I: FAMILY OF RIEMANNIAN METRICS

In this section, we construct certain families of Riemannian metrics on a given 4-manifold obtained by stretching neighborhoods of embedded surfaces and study some properties of these families. These families will provide foundations of the construction of our cohomological invariant.

2.1 | Notation on simplicial complexes

We first summarize some notation and convention on simplicial complexes. Let $K$ be a simplicial complex.

• For $n \geq 0$, we will write $\Delta^n$ for the standard $n$-simplex that is defined by

$$\Delta^n = \text{Conv}\{e_0, \ldots, e_n\}.$$ 

Here $\{e_0, \ldots, e_n\} = \{(1, 0, \ldots, 0), \ldots, (0, \ldots, 0, 1)\}$ is the standard basis of $\mathbb{R}^{n+1}$, and $\text{Conv}\{A\}$ is the convex hull of a subset $A \subset \mathbb{R}^{n+1}$.

• We will denote by $V(K)$ the set of vertices of $K$.

• We will denote by $S(K)$ the set of simplices of $K$ and by $S_n(K)$ the set of $n$-simplices of $K$.

• If $v_0, \ldots, v_n \in V(K)$ span an $n$-simplex $\sigma \in S(K)$, we regard $\sigma$ as the set $\sigma = \{v_0, \ldots, v_n\}$. If $v$ is a vertex of $\sigma$, we will write $v \in \sigma$.

• For an $n$-simplex $\sigma = \{v_0, \ldots, v_n\}$, we will denote by $< \sigma > = < v_0, \ldots, v_n >$ an oriented simplex. The simplex with the opposite orientation is denoted by $- < \sigma >$.

• If $\tau$ is a face of $\sigma$, we will write $\tau < \sigma$.

• We will denote by $|K| = \bigcup_{\sigma \in S(K)} |\sigma|$ the geometric realization of $K$. 

For a simplex $\sigma \in S(K)$, we can define a simplicial complex $\mathcal{X}(\sigma)$ by

$$S(\mathcal{X}(\sigma)) = \{ \tau \mid \tau < \sigma \}.$$  

We will also simply write this simplicial complex $\sigma$ when no confusion can arise. The geometric realization of $\mathcal{X}(\sigma)$ can be identified with $|\sigma| \subset |K|$.  

- We will denote by $\text{Bd}(K)$ the barycentric subdivision of $K$. An $n$-simplex of $\text{Bd}(K)$ is given by a set $\{\sigma_0, ..., \sigma_n\}$, where $\sigma_0 \preceq \cdots \preceq \sigma_n$ is an increasing sequence in $S(K)$. We identify $|\text{Bd}(K)|$ with $|K|$ as usual. For a simplex $s = \{\sigma_0, ..., \sigma_n\} \in S(\text{Bd}(K))$ with $\sigma_0 \preceq \cdots \preceq \sigma_n$, set $\sigma_{\max}(s) := \sigma_n$ and $\sigma_{\min}(s) := \sigma_0$. If readers are not familiar with this combinatorial description of the barycentric subdivision, see [14] to connect this with a geometric description.

- We will denote by

$$C_n(K) = C_n(K; \mathbb{Z}), \quad C^*(K) = C^*(K; \mathbb{Z})$$

the simplicial chain complex and the cochain complex with integer coefficient, respectively. We will denote by

$$\partial : C_n(K) \to C_{n-1}(K), \quad \delta : C^*(K) \to C^{*+1}(K),$$

the boundary operator and the coboundary operator, respectively.

## 2.2 Outline of the construction of the cohomological invariant

Before starting technical details of the construction of the cohomological invariant, in this subsection we shall give a sketch of the whole construction and explain technical issues which will be contended with in subsequent subsections.

Let $(X, \mathfrak{s})$ be a smooth oriented closed spin$^c$ 4-manifold with a homology orientation. Let $n$ be an integer defined by $d(\mathfrak{s}) = -(n + 1)$, where $d(\mathfrak{s})$ denotes the formal dimension of the (unparameterized) moduli space of the Seiberg–Witten equations. Under the assumptions that $b^+(X) \geq n + 3$ and that $n \geq 0$, using families of Seiberg–Witten equations, we shall construct a cohomology class depending only on $(X, \mathfrak{s})$, which we call the cohomological Seiberg–Witten invariant:

$$\mathbb{S}\mathbb{W}(X, \mathfrak{s}) \in H^n(\mathcal{K}(X, \mathfrak{s}); \mathbb{Z}).$$  

Our construction of $\mathbb{S}\mathbb{W}(X, \mathfrak{s})$ consists of the following three steps:

Step 1 Construct a cochain on $\mathcal{K} = \mathcal{K}(X, \mathfrak{s})$ using a non-topological data, such as a metric.

Step 2 Show that the cochain is cocycle.

Step 3 Show that the cohomology class of the cocycle is independent of the choice made in the first step.

Let us explain these three steps in more detail. In this explanation, we assume that $n = 1$ for simplicity. The first step is to construct a cochain

$$SW(X, \mathfrak{s}, A) \in C^1(\mathcal{K}),$$
where $\mathcal{A}$ is a certain additional data: this cochain depends on not only $(X, \mathfrak{s})$ but also other various choices. The first choice used to construct $SW(X, \mathfrak{s}, \mathcal{A})$ is a Riemannian metric $g$ on $X$. To make a 1-cochain, we have to give an integer corresponding to an oriented 1-simplex $< \sigma >=< \Sigma_0, \Sigma_1 >$. Since $\Sigma_0$ and $\Sigma_1$ are disjoint, we can stretch some neighborhoods of $\Sigma_0$ and $\Sigma_1$ independently from the metric $g$. Thus we obtain a 2-parameter family of metrics. Strictly speaking, we have to modify $g$ to be cylindrical near the sphere bundles of the normal bundles of $\Sigma_0$ and $\Sigma_1$ to stretch their neighborhoods. This will cause the main technical difficulty, but, for the moment, we describe the whole construction as if we could forget the problem. Since $\Sigma_0$ and $\Sigma_1$ violate the adjunction inequalities, we can show that the moduli space of the Seiberg–Witten equations is empty for such a stretched metric by using a quantitative version of Kronheimer–Mrowka’s [9] argument.

We describe the 2-parameter family of metrics in Figure 1. Here $\Sigma_i$-direction means the stretching parameter for $\Sigma_i$. The coordinate $(R_0, R_1)$ means the metric obtained by stretching the neighborhood of $\Sigma_i$ by the length $R_i$ ($i = 0, 1$) from the initial metric $g$. For sufficiently large $R > 0$, any metric on the codimension 1 face $F$ of the ‘2-simplex’ in Figure 1 is stretched for at least one of the neighborhood of $\Sigma_0$ and one of $\Sigma_1$. Therefore the moduli space on the face $F$ is empty. Let us perturb the Seiberg–Witten equations on the ‘2-simplex’ except for the face $A$, and consider the moduli space of the perturbed Seiberg–Witten equations parameterized by the 2-simplex. Then, on any face of codimension $\geq 1$, the parameterized moduli space is empty since we have assumed that $d(\mathfrak{s}) = -2$. We can count the points of the parameterized moduli space on the 2-simplex and obtain an integer $SW(< \sigma >)$. (Here we have to use the given homology orientation and the orientation of the simplex $< \sigma >$.) Therefore, we obtain the 1-cochain

$$SW(X, \mathfrak{s}, \mathcal{A}) : C_1(\mathcal{K}) \to \mathbb{Z}$$

defined by

$$< \sigma > \mapsto SW(< \sigma >).$$

The second step is to show that the cochain $SW(X, \mathfrak{s}, \mathcal{A})$ is cocycle. To do this, let $< \sigma >=< \Sigma_0, \Sigma_1, \Sigma_2 >$ be a 2-simplex of $\mathcal{K}$. We will see that $SW(X, \mathfrak{s}, \mathcal{A})(\partial < \sigma >) = 0$. Write

$$< \tau_0 > = < \Sigma_1, \Sigma_2 >, \quad < \tau_1 > = < \Sigma_0, \Sigma_2 >, \quad < \sigma_2 > = < \Sigma_0, \Sigma_1 >,$$

(1)

then we have $\partial < \sigma > = \sum_{i=0}^{2}(-1)^i < \tau_i >$. We describe the 3-parameter family of metrics in Figure 2 obtained from the stretching neighborhoods of $\Sigma_0$, $\Sigma_1$, and $\Sigma_2$. Let $\mathbf{v}_0$, $\mathbf{v}_1$, and $\mathbf{v}_2$ denote the coordinates $(R, 0, 0)$, $(0, R, 0)$, and $(0, 0, R)$ in Figure 2, respectively. The integer
$SW(X, A)(\partial < \sigma >) = \sum_{i=0}^{2} (-1)^i SW(< \tau_i >)$ is the sum of the counting the points of the moduli space of the perturbed equations on codimension 1 faces of the '3-simplex' given by

\{g, v_0, v_1\}, \{g, v_1, v_2\}, \{g, v_2, v_0\}

in Figure 2. Here we denote the face spanned by \{v, v', v''\} briefly by \{v, v', v''\}, and we identify $g$ with the coordinate (0,0,0). Any two of these faces meet along a 1-simplex, and thus these faces form a 2-parameter family of metrics homeomorphic to $D^2$. This 2-parameter family can be continuously deformed to the face $\{v_0, v_1, v_2\}$ fixing all 1-simplices corresponding to stretched metrics:

\{v_0, v_1\}, \{v_1, v_2\}, \{v_2, v_0\}.

Since we have assumed that $b^+(X) \geq 3 + 1 = 4$, in this deformation of 2-parameter families along a 3-parameter family, we may assume that there is no reducible. Note that, on the face $\{v_0, v_1, v_2\}$, all metrics are stretched for at least one of $\Sigma_0, \Sigma_1, \Sigma_2$. Thus the parameterized moduli space on this face is empty. Therefore, we have $SW(X, A)(\partial < \sigma >) = 0$ using the argument by cobordisms.

The third step is to show that all the ambiguities arising from choices used to construct the cocycle $SW(X, A)$ are absorbed into coboundaries. From this, we can obtain the well-defined cohomology class $SW(X, A) = [SW(X, A, A_0)].$ To do this, let us take two choices $A_0$ and $A_1$ (for example, metrics $g_0$ and $g_1$) and make two cocycles $SW_0 := SW(X, A_0)$ and $SW_1 := SW(X, A_1)$ using these two choices, respectively. By connecting $g_0$ with $g_1$ by a path in the space of metrics, we obtain the family described in Figure 3. (Strictly speaking, to define the path between $R_0$ and $R_1$ in suitable sense, we need a quantitative version of an argument in Kronheimer–Mrowka [9].)
We define a 0-cochain $SW_{0,1} \in C^0(\mathcal{K})$ by

$$\langle \Sigma \rangle \mapsto SW(\langle \Sigma \rangle),$$

where $SW(\langle \Sigma \rangle) \in \mathbb{Z}$ is the counting on the square in Figure 3 under suitable orientation. Let us take a 1-simplex $\langle \sigma \rangle = \langle \Sigma_0, \Sigma_1 \rangle$. For this 1-simplex, we obtain the family described in Figure 4.

Let $\nu_0^i$ and $\nu_1^i$ denote $(R_i, 0)$ and $(0, R_i)$ in Figure 4, respectively. The integer $SW_0(\langle \sigma \rangle) - SW_1(\langle \sigma \rangle)$ is the counting on the two triangles $\{g_0, \nu_0^0, \nu_0^1\}$ and $\{g_0, \nu_1^0, \nu_1^1\}$ under suitable signs. We can see that the moduli space on the square $\{\nu_0^0, \nu_0^1, \nu_1^0, \nu_1^1\}$ is empty using a quantitative version of Kronheimer–Mrowka’s argument again. Hence we deduce that the total sum of the counting on two triangles $\{g_0, \nu_0^0, \nu_1^0\}$, $\{g_0, \nu_0^1, \nu_1^1\}$ and two squares $\{g_0, g_1, \nu_0^0, \nu_1^0\}$, $\{g_0, g_1, \nu_1^1, \nu_0^1\}$ is zero using the argument by cobordisms. Namely, we have $SW_0(\langle \sigma \rangle) - SW_1(\langle \sigma \rangle) = \pm \delta SW_{0,1}(\langle \sigma \rangle)$, since

$$SW(\langle \Sigma_1 \rangle) - SW(\langle \Sigma_0 \rangle) = SW_{0,1}(\langle \Sigma_1 \rangle - \langle \Sigma_0 \rangle) = SW_{0,1}(\partial(\langle \sigma \rangle)) = \delta SW_{0,1}(\langle \sigma \rangle).$$

(Here $\pm$ means the suitable sign.) Thus we obtain $SW_0 - SW_1 = \pm \delta SW_{0,1}$.

These three steps enable us to define the cohomology class $S\mathcal{W}(X, \mathfrak{s})$. However, there are, of course, a number of details to justify the above argument. The main technical issue is caused since the cardinality of the set of vertices of $\mathcal{K}$ is infinite. We here explain this issue for the motivation of the subsequent subsections. In the first step, we have to stretch some neighborhoods of $\Sigma_0$ and $\Sigma_1$ independently for a 1-simplex $\langle \sigma \rangle = \langle \Sigma_0, \Sigma_1 \rangle$. Therefore, we have to introduce a metric obtained by gluing the initial metric $g$ with some cylindrical metrics. Let $N(\sigma; \Sigma_i)$ denote a neighborhood of $\Sigma_i$ with $N(\sigma; \Sigma_0) \cap N(\sigma; \Sigma_1) = \emptyset$, and let $g(\sigma)$ denote a new metric made from $g$ and having a cylindrical metric near the boundary of $N(\sigma; \Sigma_i)$. Then we can do the argument in the first step under a small modification. However, to do the second step, we have to consider a 2-simplex $\langle \tau \rangle = \langle \Sigma_0, \Sigma_1, \Sigma_2 \rangle$ and stretch neighborhoods of $\Sigma_0, \Sigma_1,$ and $\Sigma_2$ independently. For 1-simplices $\sigma_j$ ($j = 0, 1, 2$) given in (1), we have already fixed the neighborhood $N(\sigma_j; \Sigma_i)$ for each $\Sigma_i \in \sigma_j$. However, for the independent stretching for these three surfaces, we need disjoint neighborhoods of the surfaces, and we may have to take a smaller neighborhood $N(\tau; \Sigma_i)$ in $N(\sigma_j; \Sigma_i)$ and define the metric $g(\tau)$ using these new neighborhoods. The stretching argument in the first step is done using $g(\sigma)$ as the initial metric, however, the argument in the second step should be done using $g(\tau)$. We have to mediate between these two arguments, and this (and its higher dimensional version) will be the main part of the rest part of Section 2 and Section 3.
Note that, if we only consider a finite subcomplex $K$ of $\mathcal{K}$, then we can choose a sufficiently small neighborhood $N(K; \Sigma)$ for each surface $\Sigma \in V(\mathcal{K})$ satisfying that $N(K; \Sigma) \cap N(K; \Sigma') = \emptyset$ if $\Sigma \cap \Sigma' = \emptyset$. However, for a vertex $\Sigma \in V(\mathcal{K})$, one can make a sequence $\Sigma', \Sigma'', \ldots \in V(\mathcal{K})$ closing to $\Sigma$ obtained as parallel copies (in the sense of the following Remark 2.1) of $\Sigma$. Therefore, we cannot take a neighborhood $N(K; \Sigma)$ depending only on each $\Sigma \in V(\mathcal{K})$ satisfying that $N(K; \Sigma) \cap N(K; \Sigma') = \emptyset$ if $\Sigma \cap \Sigma' = \emptyset$.

Remark 2.1. For an embedded surface $\Sigma \subset X$ with self-intersection number zero, by pushing $\Sigma$ in the fiber direction of its normal bundle, we obtain an embedded surface whose homology class coincides with functions $\Sigma$ and which has no geometric intersections with $\Sigma$. We call such a surface a parallel copy of $\Sigma$.

2.3 Construction of a family of Riemannian metrics

In this subsection, we construct a family of Riemannian metrics on a 4-manifold by stretching neighborhoods of embedded surfaces. (We consider the simplicial complex $\mathcal{K}$ and construct a family of Riemannian metrics using $\mathcal{K}$ in this section in order to construct a cohomology class on $\mathcal{K}$ in Section 3. However, for any subcomplex of $\bar{\mathcal{K}}$, one can also consider a similar construction of a family of Riemannian metrics using it.)

Let $(X, \mathfrak{s})$ be a smooth oriented closed spin$^c$ 4-manifold. We denote by $\text{Met}(X)$ the space of Riemannian metrics. Henceforth in this subsection we fix a metric $g \in \text{Met}(X)$. We also fix functions $N(\cdot)$ and $a(\cdot)$ as follows. First, for each $\Sigma \in V(\mathcal{K})$, we fix a tubular neighborhood $N(\Sigma)$ of $\Sigma$ equipped with a diffeomorphism $f_{\Sigma} : N(\Sigma) \to D^2 \times \Sigma$ with $f_{\Sigma}|_{\Sigma} = t_\Sigma$, where $D^2$ is the unit disk in $\mathbb{C}$ and $t_\Sigma : \Sigma \to D^2 \times \Sigma$ is the inclusion. We will denote by $\bar{f}_{\Sigma} : N(\Sigma) \setminus \Sigma \to (0, 1] \times S^1 \times \Sigma$ the diffeomorphism defined as the composition of the restriction of $f_{\Sigma}$ and the diffeomorphism $D^2 \setminus \{0\} \times \Sigma \cong (0, 1] \times S^1 \times \Sigma$ obtained from the canonical identification $D^2 \setminus \{0\} \cong (0, 1] \times S^1$. We next take a function $a : S(\mathcal{K}) \to (0, 1]$ satisfying $a(\Sigma) = a(\{\Sigma\}) = 1$ for each $\Sigma \in V(\mathcal{K})$ and enjoying the following two conditions (a) and (b) for any $\sigma \in S(\mathcal{K})$:

**Condition 2.2.**

(a) For any (strictly small) face $\tau \preceq \sigma$, the inequalities

$$0 < a(\sigma) \leq a(\tau)$$

hold.

(b) The equality $U(\sigma, \Sigma) \cap U(\sigma, \Sigma') = \emptyset$ holds for vertices $\Sigma, \Sigma' \in \sigma$ with $\Sigma \neq \Sigma'$. Here $U(\sigma, \Sigma)$ is the subset of $N(\Sigma)$ defined as

$$U(\sigma, \Sigma) \cong (0, 6a(\sigma)] \times S^1 \times \Sigma$$

via the identification obtained from $\bar{f}_{\Sigma}$.

The existence of such a function $a(\cdot)$ easily follows from induction on the dimension of $\sigma \in S(\mathcal{K})$. 
We fix our convention on cut-off functions as follows. Let us consider the concrete monotonically increasing function $\rho_0 : [0, 1] \to [0, 1]$ defined as

$$
\rho_0(t) := \frac{\tilde{\rho}_0(t)}{\tilde{\rho}_0(t) + \tilde{\rho}_0(1-t)}, \quad \tilde{\rho}_0 : [0, 1] \to [0, 1], \quad \tilde{\rho}_0(t) = e^{-1/t} \text{ if } t > 0, \quad \tilde{\rho}_0(0) = 0.
$$

**Convention 2.3.** Let $a, b, a', b'$ be real numbers with $a < a' < b' < b$. In this paper, the cut-off function $\rho$ on $([a, b], [a', b'])$ is the following function: The function $\rho$ coincides with the constant function 1 on $[a', b']$, with the pull-back of $\rho_0(t)$ by the diffeomorphism

$$
[0, 1] \cong [a, a'] ; \quad t \mapsto (a' - a)t + a
$$
on $[a, a']$, and with the pull-back of $\rho_0(1-t)$ by the diffeomorphism

$$
[0, 1] \cong [b', b] ; \quad t \mapsto (b - b')t + b'
$$on $[b', b]$, respectively. We also define the cut-off function on $([a, b], [a', b'])$ as the pull-back of the cut-off function defined above on $([a, b], [a', b'])$ by the projection $[a, b] \times S^1 \times \Sigma \to [a, b]$.

We next define a metric $G(\cdot, \cdot)$ using the positive numbers $a(\cdot)$ and the fixed metric $g$. Let $\sigma \in S(K)$ be a simplex and $\lambda \in \mathbb{R}$ be a real number with $0 < \lambda \leq a(\sigma)$. We define the metric $G(\sigma, \lambda)$ by gluing the metric $g$ with the product metric on

$$
\bigcup_{\Sigma \in \sigma} [\lambda, 6\lambda] \times S^1 \times \Sigma,
$$
where we use the cut-off function on $([\lambda, 6\lambda], [2\lambda, 5\lambda]) \times S^1 \times \Sigma$ for each $\Sigma \in \sigma$ via the identifications obtained from $\tilde{f}_\Sigma$ in order to glue them. Here we equip $\Sigma$ with the metric $g_\Sigma$ of constant scalar curvature and of unit area, and equip $S^1$ with the metric $d\theta^2$ of unit length in (2). Note that $\{\tilde{f}_\Sigma\}_{\Sigma \in \sigma}$ gives an identification between mutually disjoint $(\dim \sigma + 1)$-subsets of $X$ and (2) since $\lambda \leq a(\sigma)$. Let us define

$$
\tilde{X}(\lambda, \Sigma) := [2\lambda, 5\lambda] \times S^1 \times \Sigma \subset N(\Sigma) \subset X
$$via the identification obtained from $\tilde{f}_\Sigma$.

We next define a 'stretched' metric. Let $\sigma, \tau \in S(K)$ be simplices with $\tau \subset \sigma$, $\lambda \in \mathbb{R}$ be a real number with $0 < \lambda \leq a(\sigma)$, and $\{r_\Sigma\}_{\Sigma \in \tau} \in [0, \infty)^\tau = \prod_{\Sigma \in \tau} [0, \infty)$ be a family of non-negative numbers. We define $G(\sigma, \lambda, \{r_\Sigma\}_{\Sigma \in \tau}) \in \text{Met}(X)$ as follows. For $\Sigma \in \tau$, the restricted metric $G(\sigma, \lambda)|_{\tilde{X}(\lambda, \Sigma)}$ can be expressed as

$$
G(\sigma, \lambda)|_{\tilde{X}(\lambda, \Sigma)} = dt^2 + d\theta^2 + g_\Sigma, \quad (3)
$$where $dt^2$ is the standard metric on the interval $[2\lambda, 5\lambda]$. Let

$$
\rho_\lambda : [2\lambda, 5\lambda] \to [0, 1]$$
be the cut-off function on $([2\lambda, 5\lambda], [3\lambda, 4\lambda])$. We modify the metric $G(\lambda)$ by replacing (3) with

$$(r_\Sigma \cdot \rho_\lambda(t) + 1)dt^2 + d\theta^2 + g_\Sigma$$

on $X(\lambda, \Sigma)$ for every $\Sigma \in \tau$; we write

$$G(\lambda, \Sigma, \{r_\Sigma\}_{\Sigma \in \tau})$$

for the modified metric. We set

$$X(\lambda, \Sigma) := [3\lambda, 4\lambda] \times S^1 \times \Sigma \subset N(\Sigma) \subset X$$

via the identification obtained from $f_\Sigma$. We define the restricted Riemannian manifold $C(\lambda, \Sigma, r_\Sigma)$ by

$$C(\lambda, \Sigma, r_\Sigma) := (X(\lambda, \Sigma), G(\sigma, \lambda, \{r_\Sigma\}_{\Sigma \in \tau}))$$

$$\subset (X, G(\sigma, \lambda, \{r_\Sigma\}_{\Sigma \in \tau})).$$

(Note that although $\sigma$ and $\tau$ appear in the right-hand side of (4), $C(\lambda, \Sigma, r_\Sigma)$ is independent of $\sigma$ and $\tau$ with $\Sigma \in \tau < \sigma$.) Then the Riemannian manifold $C(\lambda, \Sigma, r_\Sigma)$ is isometric to the cylinder with the standard metric

$$C_{std}(\lambda, \Sigma, r_\Sigma) := ([0, \lambda(r_\Sigma + 1)] \times S^1 \times \Sigma, dt^2 + d\theta^2 + g_\Sigma).$$

We also set

$$\overline{C}(\lambda, \Sigma, r_\Sigma) := (X(\lambda, \Sigma), G(\sigma, \lambda, \{r_\Sigma\}_{\Sigma \in \tau})).$$

Then there exists a positive number $\Lambda(\lambda, r_\Sigma) > \lambda(r_\Sigma + 1)$ such that $\overline{C}(\lambda, \Sigma, r_\Sigma)$ is isometric to the cylinder

$$\overline{C}_{std}(\lambda, \Sigma, r_\Sigma) := ([0, \Lambda(\lambda, r_\Sigma)] \times S^1 \times \Sigma, dt^2 + d\theta^2 + g_\Sigma).$$

The number $\Lambda(\lambda, r_\Sigma)$ depends only on $\lambda$ and $r_\Sigma$ since we fix convention on cut-off functions.

Let $n \geq 0$. Henceforth, we fix an $n$-simplex $\sigma \in S_0(\mathcal{K})$ in this subsection. We consider simplices of $\text{Bd}(\sigma)$, where we regard $\sigma$ as the simplicial complex $\mathcal{X}(\sigma)$ explained in Section 2.1. We shall construct a continuous map

$$\phi_\sigma : \bigcup_{s \in \mathcal{S}(\text{Bd}(\sigma))} |s| \times [0, \infty)^{\sigma_{\min}(s)} \to \text{Met}(X).$$

Here the domain of $\phi_\sigma$ is a subset of

$$\bigcup_{s \in \mathcal{S}(\text{Bd}(\sigma))} |s| \times [0, \infty)^{\sigma} = |\text{Bd}(\sigma)| \times [0, \infty)^{\sigma} = |\sigma| \times [0, \infty)^{\sigma}.$$
The family of Riemannian metrics (5) is what we aim to construct in this subsection. To construct the map (5), for each simplex \( s \in S(Bd(\sigma)) \), we will construct
\[
\phi_{\sigma,s} : |s| \times [0, \infty)^{\sigma_{\min}(s)} \to \text{Met}(X)
\]
so that the restriction of \( \phi_{\sigma,s} \) to dom(\( \phi_{\sigma,s} \)) \( \cap \) dom(\( \phi_{\sigma,s'} \)) coincides with that of \( \phi_{\sigma,s'} \) for another simplex \( s' \in S(Bd(\sigma)) \). Here we denote by dom(\( \phi_{\sigma,s} \)) the domain of \( \phi_{\sigma,s} \). Then we can obtain (5) by gluing functions \( \phi_{\sigma,s} \) together. Namely we define the map (5) by
\[
\phi_{\sigma} = \bigcup_{s \in S(Bd(\sigma))} \phi_{\sigma,s} : \bigcup_{s \in S(Bd(\sigma))} |s| \times [0, \infty)^{\sigma_{\min}(s)} \to \text{Met}(X).
\]

To construct the map (6), let us consider simplices of \( Bd(s) \), where we regard \( s \) as the simplicial complex \( \mathcal{K}(s) \) explained in Section 2.1. For each simplex \( \mathcal{S} \in S(Bd(s)) \), we will construct
\[
\phi_{\sigma,s,\mathcal{S}} : \mathcal{S} \times [0, \infty)^{\sigma_{\min}(s)} \to \text{Met}(X)
\]
so that the restriction of \( \phi_{\sigma,s,\mathcal{S}} \) to dom(\( \phi_{\sigma,s,\mathcal{S}} \)) \( \cap \) dom(\( \phi_{\sigma,s,\mathcal{S}'} \)) coincides with that of \( \phi_{\sigma,s,\mathcal{S}'} \) for another simplex \( \mathcal{S}' \in S(Bd(s)) \). We will define (6) by gluing functions \( \phi_{\sigma,s,\mathcal{S}} \) together.

The map (8) is constructed as follows. Let \( \mathcal{S} \) be a simplex of \( Bd(s) \). We can write \( \mathcal{S} = \{ s_0, \ldots, s_l \} \) with \( s_0 \leq \cdots \leq s_l < s \) for some \( l \geq 0 \). Recall that the geometric realization \( |\mathcal{S}| = \text{Conv}\{s_0, \ldots, s_l\} \) is a subset of the real vector space \( \mathbb{R}^{[s_0, \ldots, s_l]} \) generated by \( s_0, \ldots, s_l \). Let \( t = \{ t_i \}_{i=0}^l \) be a collection of non-negative numbers with \( \sum_{i=0}^l t_i = 1 \). We set
\[
\lambda(\mathcal{S}, t) := \sum_{j=0}^l t_j a(\sigma_{\max}(s_j)).
\]
Note that \( \lambda(\mathcal{S}, t) \leq a(\sigma_{\min}(s_i)) \) holds for any \( i \in \{0, \ldots, l\} \). Indeed, \( a(\sigma_{\max}(s_j)) \leq a(\sigma_{\min}(s_i)) \) holds for each \( j \), since we have
\[
\sigma_{\min}(s_l) < \cdots < \sigma_{\min}(s_0) < \sigma_{\max}(s_0) < \cdots < \sigma_{\max}(s_l).
\]
The metric \( G(\sigma_{\min}(s_i), \lambda(\mathcal{S}, t), \{ r_{\Sigma} \}_{\Sigma \in \sigma_{\min}(s)}) \) is therefore well-defined. Since we also have \( \sigma_{\min}(s) < \sigma_{\min}(s_i) \), we can define the metric
\[
G(\sigma_{\min}(s_i), \lambda(\mathcal{S}, t), \{ r_{\Sigma} \}_{\Sigma \in \sigma_{\min}(s)})
\]
for each \( \{ r_{\Sigma} \}_{\Sigma \in \sigma_{\min}(s)} \in [0, \infty)^{\sigma_{\min}(s)} \). We now define the value of \( \phi_{\sigma,s,\mathcal{S}} \) at the pair \( (p, \{ r_{\Sigma} \}) \) by
\[
\phi_{\sigma,s,\mathcal{S}}(p, \{ r_{\Sigma} \}) := \sum_{i=0}^l t_i G(\sigma_{\min}(s_i), \lambda(\mathcal{S}, t), \{ r_{\Sigma} \}_{\Sigma \in \sigma_{\min}(s)}),
\]
where
\[
p = \sum_{i=0}^l t_i s_i \in |\mathcal{S}|, \quad \{ r_{\Sigma} \} = \{ r_{\Sigma} \}_{\Sigma \in \sigma_{\min}(s)} \in [0, \infty)^{\sigma_{\min}(s)}, \quad \text{and} \quad t = \{ t_i \}_{i=0}^l.
\]
We note that the summation in the right-hand side of (10) makes sense since Met(X) is convex.
FIGURE 5  The domain of $\phi_\sigma$

It is straightforward to check that the restriction of $\phi_{\sigma,s}$ to $\text{dom}(\phi_{\sigma,s}, \gamma)$ $\cap$ $\text{dom}(\phi_{\sigma,s}, \gamma')$ coincides with that of $\phi_{\sigma,s', \gamma'}$ for another simplex $\gamma' \in S(Bd(s))$. We can therefore define the map $\phi_{\sigma,s}$ by gluing functions $\phi_{\sigma,s, \gamma}$ together. Namely we define the map (6) by

$$\phi_{\sigma,s} = \bigcup_{\gamma \in S(Bd(s))} \phi_{\sigma,s, \gamma} : \bigcup_{\gamma \in S(Bd(s))} |\gamma| \times \lbrack 0, \infty \rbrack^{\sigma_{\min}(s)} \to \text{Met}(X).$$

Here note that $\bigcup_{\gamma \in S(Bd(s))} |\gamma| = |Bd(s)| = |s|$. Similarly, one can check that the restriction of $\phi_{\sigma,s}$ to $\text{dom}(\phi_{\sigma,s}) \cap \text{dom}(\phi_{\sigma,s'})$ coincides with that of $\phi_{\sigma,s'}$ for another simplex $s' \in S(Bd(\sigma))$. We can therefore define the map $\phi_\sigma$ by (7) as explained.

Example 2.4. We here describe the domain of $\phi_\sigma$ in the case that $\sigma$ is a 1-simplex of $\mathcal{K}$. Write $\sigma = \{\Sigma_0, \Sigma_1\}$ and put $s_i = \{\Sigma_i\}, \sigma$ ($i = 0, 1$). The domain of $\phi_\sigma$ is described as the shadowed part in Figure 5. The shadowed part is a subspace of $|Bd(\sigma)| \times \lbrack 0, \infty \rbrack^{\sigma} = |\sigma| \times \lbrack 0, \infty \rbrack^{\sigma} \cong \Delta^1 \times \lbrack 0, \infty \rbrack^2$ and obtained as a union

$$\bigcup_{i=0}^1 \left( |s_i| \times \lbrack 0, \infty \rbrack^{\Sigma_i} \right) \cup \left( |\sigma| \times \lbrack 0, \infty \rbrack^{\sigma} \right) \cup \bigcup_{i=0}^1 \left( |\{\Sigma_i\}| \times \lbrack 0, \infty \rbrack^{\Sigma_i} \right)$$

$$\cong \bigcup_{i=0}^1 \left( \Delta^1 \times \lbrack 0, \infty \rbrack \right)_i \cup \lbrack 0, \infty \rbrack^2 \cup \bigcup_{i=0}^1 [0, \infty)_i,$$

where $(\Delta^1 \times \lbrack 0, \infty \rbrack)_i$ and $[0, \infty)_i$ are a copy of $\Delta^1 \times \lbrack 0, \infty \rbrack$ and of $\lbrack 0, \infty \rbrack$, respectively. Note that the last part $|\{\Sigma_i\}| \times \lbrack 0, \infty \rbrack^{\Sigma_i}$ is contained in the first part $|s_i| \times \lbrack 0, \infty \rbrack^{\Sigma_i}$. In Figure 5, the horizontal arrows $\rightarrow$ correspond to the ‘stretching parameter’ $r_{\Sigma_0}$ on $\Sigma_0$ and the vertical arrows $\uparrow$ correspond to $r_{\Sigma_1}$. The map $\phi_\sigma$ is obtained by stretching a neighborhood of $\Sigma_i$ on the part $|s_i| \times \lbrack 0, \infty \rbrack^{\Sigma_i}$, and by stretching neighborhoods of both of $\Sigma_0$ and $\Sigma_1$ on the part $|\sigma| \times \lbrack 0, \infty \rbrack^{\sigma}$. The decompositions of the domain of $\phi_\sigma$ obtained from the barycentric subdivision of $s_i$ are also described in Figure 5.

We now note the following lemma, which will be used to prove a vanishing property of the Seiberg–Witten moduli space for a stretched metric.
Lemma 2.5. Let \( \sigma \) be a simplex of \( \mathcal{K} \), \( s \) be a simplex of \( \text{Bd}(\sigma) \), and \( \mathcal{S} \) be a simplex of \( \text{Bd}(s) \). Let us write \( \mathcal{S} = \{ s_0, \ldots, s_l \} \) with \( s_0 \leq \cdots \leq s_l < s \), where \( l = \dim \mathcal{S} \). Let \( t = \{ t_i \}_{i=0}^l \) be a collection of non-negative numbers with \( \sum_{i=0}^l t_i = 1 \). Set \( p = \sum_{i=0}^l t_i s_i \in |\mathcal{S}| \). Then, for each \( \{ r_{\Sigma} \} = \{ r_{\Sigma} \}_{\Sigma \in \sigma_{\min}(s)} \in [0, \infty)^{\sigma_{\min}(s)} \), the subspaces of \( X \)

\[
\{ \tilde{X}(\lambda(\mathcal{S}, t), \Sigma) \mid \Sigma \in \sigma_{\min}(s) \}
\]

are mutually disjoint in \( X \) and we have

\[
\bigcup_{\Sigma \in \sigma_{\min}(s)} (\tilde{X}(\lambda(\mathcal{S}, t), \Sigma), \phi_{\Sigma}(p, \{ r_{\Sigma} \})) = \bigcup_{\Sigma \in \sigma_{\min}(s)} \overline{C_{\gamma}(\lambda(\mathcal{S}, t), \Sigma, r_{\Sigma})}.
\]

Proof. Since we have \( \sigma_{\min}(s) < \sigma_{\min}(s_l) \) and (9), we obtain \( \lambda(\mathcal{S}, t) \leq a(\sigma_{\min}(s)) \). The subspaces in (11) are therefore mutually disjoint. For each \( \Sigma \in \sigma_{\min}(s) \), we have

\[
\phi_{\Sigma}(p, \{ r_{\Sigma} \})|_{\tilde{X}(\lambda(\mathcal{S}, t), \Sigma)} = \sum_{i=0}^l t_i \cdot G(\sigma_{\min}(s_i), \lambda(\mathcal{S}, t), \{ r_{\Sigma} \})|_{\tilde{X}(\lambda(\mathcal{S}, t), \Sigma)}
\]

\[
= \sum_{i=0}^l t_i \cdot G(\sigma_{\min}(s), \lambda(\mathcal{S}, t), \{ r_{\Sigma} \})|_{\tilde{X}(\lambda(\mathcal{S}, t), \Sigma)}
\]

\[
= G(\sigma_{\min}(s), \lambda(\mathcal{S}, t), \{ r_{\Sigma} \})|_{\tilde{X}(\lambda(\mathcal{S}, t), \Sigma)}.
\]

This proves (12). \( \square \)

2.4 Structure of the parameter space

In this subsection, we investigate the structure of a subspace of the domain of the map (5).

Let \( n \geq 0 \) and \( \sigma \in S_n(\mathcal{K}) \) be an \( n \)-simplex. For a simplex \( s \in S(\text{Bd}(\sigma)) \) and a non-negative number \( R \geq 0 \), we set

\[
\text{ext} \tilde{\partial}(\sigma, s, R) := \left\{ \{ r_{\Sigma} \}_{\Sigma \in \sigma_{\min}(s)} \in [0, R]^{\sigma_{\min}(s)} \mid \exists \Sigma \in \sigma_{\min}(s), r_{\Sigma} = R \right\}
\]

\[
\subset [0, \infty)^{\sigma_{\min}(s)}.
\]

(The notation \( \text{ext} \tilde{\partial} \) stands for ‘the exterior half’ of the boundary of \( [0, R]^{\sigma_{\min}(s)} \).) We shall consider the subspace

\[
\text{ext} \tilde{\partial}(\sigma, s, R) := \bigcup_{s \in S(\text{Bd}(\sigma))} |s| \times \text{ext} \tilde{\partial}(\sigma, s, R)
\]

of the domain of the map (5). For example, in the case that \( \sigma \) is a 1-simplex, \( \text{ext} \tilde{\partial}(\sigma, s, R) \) is the union of dotted lines in Figure 5. We also set

\[
\text{ext} \partial(\sigma, R) := \left\{ \{ x_{\Sigma} \}_{\Sigma \in \sigma} \in [0, R]^\sigma \mid \exists \Sigma \in \sigma, x_{\Sigma} = R \right\}.
\]

The aim of this subsection is to prove the following proposition:
Proposition 2.6. There exists a homeomorphism

$$\psi_{\sigma,R} : \mathfrak{ext}\delta(\sigma, R) \to \mathfrak{ext}\delta(\sigma, R). \quad (13)$$

Before proving Proposition 2.6, we decompose the domain and the range of the map (13) into small pieces. Let $$\Sigma \in \sigma$$ be a vertex and $$\tau < \sigma$$ be a face with $$\Sigma \in \tau$$. We set

$$\mathfrak{ext}\delta(\sigma, \Sigma, \tau, R) : = \left\{ \{r_\Sigma'\}_{\Sigma' \in \tau} \in [0, R]^\tau \mid r_\Sigma = R \right\}. \quad (14)$$

For a simplex $$s \in S(Bd(\sigma))$$ satisfying $$\tau = \sigma_{\min}(s)$$, we have $$\mathfrak{ext}\delta(\sigma, \Sigma, \tau, R) \subset \mathfrak{ext}\delta(\sigma, s, R)$$. Moreover, we also have the decomposition

$$\mathfrak{ext}\delta(\sigma, R) = \bigcup_{\Sigma \in \sigma} \bigcup_{\Sigma' \in \tau \prec \sigma} \bigcup_{s \in S(Bd(\sigma)), \tau = \sigma_{\min}(s)} |s| \times \mathfrak{ext}\delta(\sigma, \Sigma, \tau, R) \quad (15)$$

Here the variables in the three unions in the right-hand side of (14) and (15) are $$\Sigma, \tau$$, and $$s$$, respectively. In (15), $$\dim s$$ and $$\dim \tau$$ denote the dimension of $$s$$ as a simplex of $$\text{Bd}(\sigma)$$ and that of $$\tau$$ as a simplex of $$\mathcal{K}$$, respectively. The space $$\mathfrak{ext}\delta(\sigma, R)$$ also decomposes as $$\mathfrak{ext}\delta(\sigma, R) = \bigcup_{\Sigma \in \sigma} \mathfrak{ext}\delta(\sigma, \Sigma, R)$$, where we set

$$\mathfrak{ext}\delta(\sigma, \Sigma, R) : = \left\{ \{x_\Sigma'\}_{\Sigma' \in \sigma} \in [0, R]^\sigma \mid x_\Sigma = R \right\} \subset \mathfrak{ext}\delta(\sigma, R).$$

For each $$\Sigma \in \sigma$$, we will construct a homeomorphism

$$\psi_{\sigma,\Sigma,R} : \bigcup_{\Sigma \in \tau \prec \sigma} \bigcup_{s \in S(Bd(\sigma)), \tau = \sigma_{\min}(s), \dim s + \dim \tau = n} |s| \times \mathfrak{ext}\delta(\sigma, \Sigma, \tau, R) \to \mathfrak{ext}\delta(\sigma, \Sigma, R) \quad (16)$$

and define $$\psi_{\sigma,R}$$ by gluing functions $$\psi_{\sigma,\Sigma,R}$$. Note that we have a homeomorphism

$$\tilde{\psi}_{\sigma,\Sigma,R} : \mathfrak{ext}\delta(\sigma, \Sigma, R) \to [0, R]^{\sigma \setminus \Sigma} \quad (17)$$

given by

$$\{x_{\Sigma'}\}_{\Sigma' \in \sigma} \mapsto \{x_{\Sigma'}\}_{\Sigma' \in \sigma \setminus \Sigma},$$

and we consider $$[0, R]^{\sigma \setminus \Sigma}$$ in order to construct (16). For a face $$\tau < \sigma$$ with $$\Sigma \in \tau$$, we define a collection of boundary points

$$\{b_{\Sigma'}(\tau)\} = \{b_{\Sigma'}(\tau)\}_{\Sigma' \in \sigma \setminus \Sigma} \in [0, R]^{\sigma \setminus \Sigma} \subset \partial([0, R]^{\sigma \setminus \Sigma})$$

by

$$\Sigma' \in \tau \setminus \{\Sigma\} \Rightarrow b_{\Sigma'}(\tau) = R, \text{ and } \Sigma' \not\in \tau \setminus \{\Sigma\} \Rightarrow b_{\Sigma'}(\tau) = 0.$$
We set
\[ Q(\sigma, \Sigma, \tau, R) := \left\{ \{x_{\Sigma'}\}_{\Sigma' \in \sigma \setminus \{\Sigma\}} \in [0, R]^{\sigma \setminus \{\Sigma\}} \mid |x_{\Sigma'} - b_{\Sigma'}(\tau)| \leq R/2 \right\}. \]

Then we obtain the decomposition
\[ [0, R]^{|\sigma \setminus \{\Sigma\}|} = \bigcup_{\Sigma \in \tau \prec \sigma} Q(\sigma, \Sigma, \tau, R). \]

Indeed, for any point \( \{x_{\Sigma'}\} \in [0, R]^{|\sigma \setminus \{\Sigma\}|} \), let us define \( \tau < \sigma \) by
\[ \tau := \left\{ \Sigma' \in \sigma \setminus \{\{\Sigma\} \mid |x_{\Sigma'} - R| \leq R/2 \right\} \cup \{\Sigma\}, \]
then we have \( \{x_{\Sigma'}\} \in Q(\sigma, \Sigma, \tau, R). \)

For \( \Sigma \in \sigma \) and \( \tau < \sigma \) with \( \Sigma \in \tau \), we will construct a homeomorphism
\[ \psi_{\sigma, \Sigma, \tau, R} : \bigcup_{s \in S(Bd(\sigma)), \tau = \sigma_{\min}(s), \dim s + \dim \tau = n} |s| \times \text{ext}(\sigma, \Sigma, \tau, R) \to Q(\sigma, \Sigma, \tau, R) \]
and define \( \psi_{\sigma, \Sigma, R} \) by gluing functions \( \psi_{\sigma, \Sigma, \tau, R} \). For a simplex \( s \in S(Bd(\sigma)) \) satisfying \( \tau = \sigma_{\min}(s) \) and \( \dim s + \dim \tau = n \), we define subsets \( \Delta(\sigma, \Sigma, \tau, s, R) \), \( \Box(\sigma, \Sigma, \tau, R) \), and \( Q(\sigma, \Sigma, \tau, s, R) \) of \( Q(\sigma, \Sigma, \tau, R) \) as follows. Note that we have \( \sigma = \sigma_{\max}(s) \) (that is, \( n = \dim \sigma_{\max}(s) \)) since
\[ n \geq \dim \sigma_{\max}(s) \geq \dim s + \dim \sigma_{\min}(s) = \dim s + \dim \tau = n. \]

Such a simplex \( s \) therefore corresponds to a sequence
\[ \tau = \sigma_0 \leq \cdots \leq \sigma_k = \sigma, \]
where \( k = n - \dim \tau \). We define
\[ \Delta(\sigma, \Sigma, \tau, s, R) := \text{Conv} \left\{ \left\{ \frac{1}{2} b_{\Sigma'}(\sigma_i) \right\}_{\Sigma' \in \sigma \setminus \tau} \mid 0 \leq i \leq k \right\} \]
\[ \subset [0, R/2]^{\sigma \setminus \tau}, \]
\[ \Box(\sigma, \Sigma, \tau, R) := Q(\sigma, \Sigma, \tau, R) \cap [0, R]^{\tau \setminus \{\Sigma\}} = [R/2, R]^{\tau \setminus \{\Sigma\}}, \]
and
\[ Q(\sigma, \Sigma, \tau, s, R) := \left\{ \{x_{\Sigma'}\}_{\Sigma' \in \sigma \setminus \{\Sigma\}} \in Q(\sigma, \Sigma, \tau, R) \mid \begin{cases} \{x_{\Sigma'}\}_{\Sigma' \in \sigma \setminus \{\Sigma\}} \in \Delta(\sigma, \Sigma, \tau, s, R), \\ \{x_{\Sigma'}\}_{\Sigma' \in \tau \setminus \{\Sigma\}} \in \Box(\sigma, \Sigma, \tau, R) \end{cases} \right\} \]
\[ \cong \Delta(\sigma, \Sigma, \tau, s, R) \times \Box(\sigma, \Sigma, \tau, R). \]
Here, in the case that \( \tau = \sigma \) and \( \tau = \{ \Sigma \} \), we identify \( Q(\sigma, \Sigma, \tau, s, R) \) with \( \blacksquare(\sigma, \Sigma, \tau, R) \) and \( \triangle(\sigma, \Sigma, \tau, s, R) \), respectively.

**Lemma 2.7.** The space \( Q(\sigma, \Sigma, \tau, R) \) decomposes as

\[
Q(\sigma, \Sigma, \tau, R) = \bigcup_{s \in S(Bd(\sigma)), \\
\tau = \sigma_{\min}(s), \\
dim s + \dim \tau = n} Q(\sigma, \Sigma, \tau, s, R).
\]

**Proof.** Set \( k = n - \dim \tau \). It is easy to check the case when \( k \leq 1 \), therefore we assume \( k \geq 2 \). It suffices to show that the left-hand side of (19) is included in the right-hand side. Let \( \{ x_{\Sigma'} \}_{\Sigma' \in \sigma \setminus \{ \Sigma \} } \) be a point in \( Q(\sigma, \Sigma, \tau, R) \). We inductively take \( \sigma_0, \ldots, \sigma_k \) as follows. We first set \( \sigma_0 := \tau \). For \( i \) with \( 0 \leq i \leq k \), assume that \( \sigma_i \) is already given. Take \( \Sigma_i \in \sigma \setminus \sigma_i \) such that \( x_{\Sigma_i} \) attains the maximum

\[
\max_{\Sigma' \in \sigma \setminus \sigma_i} x_{\Sigma'}.
\]

Using this \( \Sigma_i \), we define \( \sigma_{i+1} := \sigma_i \cup \{ \Sigma_i \} \). Through this procedure, we eventually have \( \sigma_0, \ldots, \sigma_k \); we set \( s := \{ \sigma_0, \ldots, \sigma_k \} \). We note that

\[
\{ x_{\Sigma'} \}_{\Sigma' \in \sigma \setminus \tau} \in \triangle(\sigma, \Sigma, \tau, s, R)
\]

holds. Indeed, we have

\[
\triangle(\sigma, \Sigma, \tau, s, R) = \bigcap_{0 \leq i \leq k-2} \left\{ \{ x_{\Sigma'} \}_{\Sigma' \in \sigma \setminus \tau} \in [0, R/2]^{\tau} \mid x_{\Sigma_i} \geq x_{\Sigma_{i+1}} \right\}
\]

and \( x_{\Sigma_0} \geq \cdots \geq x_{\Sigma_{k-1}} \) by the definition of \( \Sigma_i \). Hence it follows that \( \{ x_{\Sigma'} \}_{\Sigma' \in \sigma \setminus \{ \Sigma \} } \in Q(\sigma, \Sigma, \tau, s, R) \). \( \square \)

Using the decomposition above, we now prove Proposition 2.6.

**Proof of Proposition 2.6.** Let \( \Sigma \) be a vertex of \( \sigma \), \( \tau \) be a face of \( \sigma \) with \( \Sigma \in \tau \), and \( s \) be a simplex of \( Bd(\sigma) \) satisfying \( \tau = \sigma_{\min}(s) \) and \( \dim s + \dim \tau = n \). We define a map

\[
\psi_{\sigma, \Sigma, \tau, s, R} : |s| \times \text{ext} \partial(\sigma, \Sigma, \tau, R) \to Q(\sigma, \Sigma, \tau, s, R)
\]

as follows. If we write \( s = \{ \sigma_0, \ldots, \sigma_k \} \), we have \( |s| = \text{Conv} \{ \sigma_0, \ldots, \sigma_k \} \). We define a map

\[
\psi_{\sigma, \Sigma, \tau, s, R} : |s| \to \triangle(\sigma, \Sigma, \tau, s, R)
\]

by

\[
\sum_{i=0}^{k} t_i \sigma_i \mapsto \sum_{i=0}^{k} t_i \cdot \left\{ \frac{1}{2} b_{\Sigma'}(\sigma_i) \right\}_{\Sigma' \in \sigma \setminus \tau}
\]

for \( t_i \geq 0 \) with \( \sum_{i=0}^{k} t_i = 1 \). We next define a map

\[
\psi_{\sigma, \Sigma, \tau, s, R} : \text{ext} \partial(\sigma, \Sigma, \tau, R) \to \blacksquare(\sigma, \Sigma, \tau, R)
\]
by

\[ \{ r_{\Sigma'} \}_{\Sigma' \in \tau \setminus \Sigma} \mapsto \left\{ \frac{1}{2} (r_{\Sigma'} + 1) \right\}_{\Sigma' \in \tau \setminus \Sigma} \]

using the identification \( \mathfrak{ext} \delta(\sigma, \Sigma, \tau, R) \cong [0, R]^{\tau \setminus \Sigma} \) given by

\[ \{ r_{\Sigma'} \}_{\Sigma' \in \sigma_{\text{min}}(s)} \mapsto \{ r_{\Sigma'} \}_{\Sigma' \in \tau \setminus \Sigma} \]

We define the map (20) as the product map of \( \psi^\bigtriangleup \sigma, \Sigma, \tau, s, R \) and \( \psi^\square \sigma, \Sigma, \tau, s, R \). The map \( \psi \sigma, \Sigma, \tau, s, R \) is obviously a homeomorphism. We denote by \( \psi^{-1} \sigma, \Sigma, \tau, s, R \) the inverse map.

It is straightforward to check that the restriction of \( \psi \sigma, \Sigma, \tau, s, R \) to \( \text{dom}(\psi \sigma, \Sigma, \tau, s, R) \cap \text{dom}(\psi \sigma, \Sigma, \tau, s', R) \) coincides with that of \( \psi \sigma, \Sigma, \tau, s', R \) for another simplex \( s' \). We can therefore define the map (18) by gluing functions \( \psi \sigma, \Sigma, \tau, s, R \) together. Namely we define the map (18) as

\[
\bigcup_{s \in S(Bd(\sigma)), \tau = \sigma_{\text{min}}(s), \dim s + \dim r = n} \psi \sigma, \Sigma, \tau, s, R : \bigcup_{s \in S(Bd(\sigma)), \tau = \sigma_{\text{min}}(s), \dim s + \dim r = n} |s| \times \mathfrak{ext} \delta(\sigma, \Sigma, \tau, R) \to Q(\sigma, \Sigma, \tau, R).
\]

In a similar vein, we obtain a map from \( \text{dom}(\psi \sigma, \Sigma, R) \) to \( \text{dom}(\psi \sigma, \Sigma, \tau, s, R) \cap \text{dom}(\psi \sigma, \Sigma, \tau, s', R) \) by gluing functions \( \psi \sigma, \Sigma, \tau, s, R \) together, and we define the map (16) by composing the inverse map of the homeomorphism (17) with the glued map. Similarly, we define the map (13) by gluing functions \( \psi \sigma, \Sigma, R \) together. We obtain the inverse map of \( \psi \sigma, R \) by gluing functions \( \psi^{-1} \sigma, \Sigma, \tau, s, R \) together and therefore \( \psi \sigma, R \) is a homeomorphism.

\[ \Box \]

**Example 2.8.** Let \( \sigma = \{ \Sigma_0, \Sigma_1 \} \) be a 1-simplex of \( \mathcal{K} \). Then the possibilities of triples \((\Sigma, \tau, s)\) such that \( \Sigma \in \tau < \sigma, \tau = \sigma_{\text{min}}(s), \) and \( \dim s + \dim r = n \) are as follows.

1. \( \Sigma = \Sigma_0, \tau = \{ \Sigma_0 \}, s = \{ \{ \Sigma_0 \}, \{ \Sigma_0, \Sigma_1 \} \}. \)
2. \( \Sigma = \Sigma_0, \tau = \{ \Sigma_0, \Sigma_1 \}, s = \{ \{ \Sigma_0 \}, \{ \Sigma_1 \} \}. \)
3. \( \Sigma = \Sigma_1, \tau = \{ \Sigma_1 \}, s = \{ \{ \Sigma_1 \}, \{ \Sigma_0 \}, \{ \Sigma_1, \Sigma_2 \} \}. \)
4. \( \Sigma = \Sigma_1, \tau = \{ \Sigma_0, \Sigma_1 \}, s = \{ \{ \Sigma_0, \Sigma_1 \}, \{ \Sigma_0, \Sigma_1, \Sigma_2 \} \}. \)

The decompositions discussed above of the domain and the range of \( \psi \sigma, R \) are described in Figure 6. (See also Example 2.4 and Figure 5.) The (upper and lower case) Roman numerals in Figure 6 correspond to Arabic numerals above. In Figure 6, the parts (I)–(IV) correspond to (i)–(iv) through \( \psi \sigma, R \), respectively.

**Example 2.9.** Let \( \sigma = \{ \Sigma_0, \Sigma_1, \Sigma_2 \} \) be a 2-simplex of \( \mathcal{K} \). For \( \Sigma = \Sigma_0 \), the possibilities of pairs \((\tau, s)\) such that \( \Sigma \in \tau < \sigma, \tau = \sigma_{\text{min}}(s) \) and \( \dim s + \dim r = n \) are as follows.

1. \( \tau = \{ \Sigma_0 \}, s = \{ \{ \Sigma_0 \}, \{ \Sigma_0, \Sigma_1 \}, \{ \Sigma_0, \Sigma_1, \Sigma_2 \} \}. \)
2. \( \tau = \{ \Sigma_0 \}, s = \{ \{ \Sigma_0 \}, \{ \Sigma_0, \Sigma_2 \}, \{ \Sigma_0, \Sigma_1, \Sigma_2 \} \}. \)
3. \( \tau = \{ \Sigma_0, \Sigma_1 \}, s = \{ \{ \Sigma_0, \Sigma_1 \}, \{ \Sigma_0, \Sigma_1, \Sigma_2 \} \}. \)
4. \( \tau = \{ \Sigma_0, \Sigma_2 \}, s = \{ \{ \Sigma_0, \Sigma_2 \}, \{ \Sigma_0, \Sigma_1, \Sigma_2 \} \}. \)
5. \( \tau = \{ \Sigma_0, \Sigma_1, \Sigma_2 \}, s = \{ \{ \Sigma_0, \Sigma_1 \}, \{ \Sigma_0, \Sigma_2 \} \}. \)
For each case that $\Sigma = \Sigma_1$ and $\Sigma = \Sigma_2$, we have similar five combinations of pairs $(\tau, s)$. The decompositions corresponding to all of these combinations of $(\tau, s)$ of the range of $\psi_{\sigma, R}$ are described in Figure 7. The lowercase Roman numerals in Figure 7 correspond to Arabic numerals above.

### 2.5 Vanishing of solutions to the Seiberg–Witten equations

A basic tool to construct the cohomological invariant is the vanishing property of the Seiberg–Witten moduli space for a metric obtained by stretching neighborhoods of embedded surfaces violating the adjunction inequalities. This is originally due to Kronheimer–Mrowka’s paper on the Thom conjecture [9]. While a cylinder with infinite length was used in [9], we here need some quantitative estimate for the length of stretched cylinders. Such an analytical argument has been given in the author’s paper [8]. In this subsection, we adjust it to the setting of this paper.

We use the following terminology and notation on the Seiberg–Witten equations. Fix a spin$^c$ structure $\mathfrak{s}$ on a smooth oriented closed connected 4-manifold $X$. Let $h$ be a Riemannian metric on
We denote by $\Omega^+ = \Omega^+_R = \Gamma(\Lambda^+_h)$ the space of self-dual 2-forms on $X$ with respect to $h$. Let $S^\pm \to X$ and $L \to X$ denote the spinor bundles and the determinant line bundle of $\mathfrak{s}$, respectively. A $U(1)$-connection $A$ on $L$ gives rise to the Dirac operator $D_A : \Gamma(S^+) \to \Gamma(S^-)$. For a $U(1)$-connection $A$, a positive spinor $\Phi \in \Gamma(S^+)$, and an imaginary self-dual 2-form $\mu \in i\Omega^+$, we call the equations

$$
\begin{align*}
\rho(F_A^+ - \mu) &= \sigma(\Phi, \Phi), \\
D_A \Phi &= 0
\end{align*}
$$

the (perturbed) Seiberg–Witten equations with respect to $(h, \mu)$. Here $\rho : \Lambda^+ \to \mathfrak{su}(S^+)$ is the map obtained from the Clifford multiplication, $F_A^+$ is the self-dual part of the curvature $F_A$ of $A$, and $\sigma(\cdot, \cdot)$ is the quadratic form given by $\sigma(\Phi, \Phi) = \Phi \otimes \Phi^* - |\Phi|^2 i d/2$. For $A$ and $\Phi$, we call the equations

$$
\begin{align*}
\rho(F_A^+) &= \sigma(\Phi, \Phi), \\
D_A \Phi &= 0
\end{align*}
$$

the unperturbed Seiberg–Witten equations with respect to $h$.

We here need the following analytical Lemmas 2.10, 2.11. Let $\| \cdot \|_{L^2(X, h)}$ denotes the $L^2$-norm on $X$ with respect to a given metric $h$. For a surface $\Sigma$ and $R > 0$, we equip $\Sigma$ with the metric of constant scalar curvature and of unit area, $S^1$ with the metric of unit length, and $[0, R] \times S^1 \times \Sigma$ with the standard product metric as in Subsection 2.3. The statement of Lemma 2.10 appears in the proof of Lemma 4 in Kronheimer–Mrowka [10].

**Lemma 2.10** (Kronheimer–Mrowka [10]). Let $h$ be a metric on $X$, $\Sigma$ be a surface embedded in $X$ with $[\Sigma]^2 = 0$, and $R$ be a positive number. Suppose that $(X, h)$ contains a Riemannian submanifold $(X', h)$ (with boundary) which is isometric to $[0, R] \times S^1 \times \Sigma$. Then, for any closed 2-form $\omega$ on $X$, the inequality

$$
||[\omega] \cdot [\Sigma]|^2 \leq \frac{||\omega||^2_{L^2(X', h)}}{R}
$$

holds.

**Proof.** By Fubini’s theorem, we have

$$
\begin{align*}
||\omega||^2_{L^2(X', h)} &= \int_{[0, R] \times S^1} |\omega|^2 d\mu_{[0, R] \times S^1 \times \Sigma} \\
&= \int_{[0, R] \times S^1} \left( \int_{\{(t, \theta)\} \times \Sigma} |\omega|^2 d\mu_{\{(t, \theta)\} \times \Sigma} \right) d\mu_{[0, R] \times S^1},
\end{align*}
$$

where $d\mu$ denotes the volume form. Note that $\Sigma$ and $\{(t, \theta)\} \times \Sigma$ have the same homology class in $X$. Therefore, by the Cauchy–Schwarz inequality, we obtain

$$
\text{Area}(\Sigma) \cdot \int_{\{(t, \theta)\} \times \Sigma} |\omega|^2 d\mu_{\{(t, \theta)\} \times \Sigma} \geq \left( \int_{\{(t, \theta)\} \times \Sigma} \omega \right)^2 = \left( \int_{\Sigma} \omega \right)^2.
$$
Therefore

\[
\|\omega\|^2_{L^2(X', h)} \geq \frac{1}{\text{Area}(\Sigma)} \int_{[0,R] \times S^1} \left\| \int_{\Sigma} \omega \right\|^2 d\mu_{[0,R] \times S^1} = \frac{\text{Area}([0,R] \times S^1)}{\text{Area}(\Sigma)} \left\| \int_{\Sigma} \omega \right\|^2
\]

holds. Since \text{Area}([0,R] \times S^1) = R and \text{Area}(\Sigma) = 1, this proves the lemma.

For a cohomology class \( c \in H^2(X; \mathbb{R}) \) and a metric \( h \) on \( X \), we denote by \( H_h(c) \) the harmonic representative of \( c \) with respect to \( h \). We denote by \( s(h) : X \to \mathbb{R} \) the scalar curvature of \( h \). Let us define \( \kappa(h) : X \to \mathbb{R} \) by

\[
\kappa(h) := \max\{-s(h), 0\}.
\]

The following Lemma 2.11 is essentially proven as Lemma 3 in Kronheimer–Mrowka [10], while in [10] the Seiberg–Witten equations on 3-manifolds were considered.

**Lemma 2.11 (Kronheimer–Mrowka [10]).** If there exists a solution to the Seiberg–Witten equations with respect to a metric \( h \), then the inequality

\[
\|H_h(c_1(\mathfrak{g}))\|^2_{L^2(X, h)} \leq \frac{\|\kappa(h)\|^2_{L^2(X, h)}}{(4\pi)^2} - \int_X c_1(\mathfrak{g})^2
\]

(22) holds.

**Proof.** Write \( \kappa = \kappa(h), s = s(h) \) and \( L^2 = L^2(X, h) \). Let \((A, \Phi)\) be a solution to the Seiberg–Witten equations with respect to \( h \) and \( \mathfrak{g} \). A standard argument combining the Seiberg–Witten equations with the Weitzenböck formula gives the equality

\[
\int_X |\nabla_A \Phi|^2 + \frac{1}{4} \int_X s|\Phi|^2 + \frac{1}{4} \int_X |\Phi|^4 = 0,
\]

and hence an inequality

\[
\int_X |\Phi|^4 \leq -\int_X s|\Phi|^2.
\]

This combined with the Cauchy–Schwarz inequality implies

\[
\int_X |\Phi|^4 \leq \int_X \kappa^2.
\]

(23)

Rewriting \( F_A^+ \) in terms of \( \Phi \) using the Seiberg–Witten equations, we have

\[
|F_A^+| \leq \frac{|\sigma(\Phi, \Phi)|}{2} \leq \frac{|\Phi|^2}{2\sqrt{2}}
\]

(24)
at each point. Now an $L^2$-a priori estimate for curvatures

$$\|F^+_A\|_{L^2} \leq \frac{\|\kappa\|_{L^2}}{2\sqrt{2}} \quad (25)$$

follows from (23) and (24).

Set $c = c_1(\mathcal{S})$. Take a representative $\omega \in \Omega^2(X, \mathbb{R})$ of $c$. We have

$$\|\omega\|_{L^2}^2 = \|\omega^+\|_{L^2}^2 + \|\omega^-\|_{L^2}^2$$

and

$$\int_X c^2 = \int_X \omega \wedge \omega = \|\omega^+\|_{L^2}^2 - \|\omega^-\|_{L^2}^2,$$

hence

$$\|\omega\|_{L^2}^2 = 2\|\omega^+\|_{L^2}^2 - \int_X c^2 \quad (26)$$

holds. In addition, we have

$$\|H_h(c)\|_{L^2} \leq \|\omega\|_{L^2} \quad (27)$$

since the $h$-harmonic form minimizes $L^2_h$-norm in the forms having the same cohomology class. Since we can take $\omega$ as $\omega = (-1/2\pi i)F_A$, from (25), (26) and (27), we obtain

$$\|H_h(c)\|_{L^2}^2 \leq \|\omega\|_{L^2}^2 = 2\|\omega^+\|_{L^2}^2 - \int_X c^2$$

$$= \frac{1}{2\pi^2} \|F^+_A\|_{L^2}^2 - \int_X c^2$$

$$\leq \frac{\|\kappa\|_{L^2}^2}{16\pi^2} - \int_X c^2.$$

This proves the lemma. \hfill \Box

Henceforth in this subsection, we fix a metric $g \in \text{Met}(X)$. We also fix $N(\cdot), a(\cdot)$ as in Subsection 2.3. For a simplex $\sigma$ in $\mathcal{K}$, we define

$$\lambda(\sigma) = \lambda(\sigma, a(\cdot)) := \min_{s \in S(Bd(\sigma)), \mathcal{S} \in S(Bd(s))} \left\{ \lambda(\mathcal{S}, t) \mid \begin{array}{c} t = \{t_{ij}\}_{j=0}^{\dim \mathcal{S}}, \\
\sum_{ij} t_{ij} = 1, \\
i_j \geq 0 \end{array} \right\}$$

and

$$C(\sigma) = C(\sigma, g, N(\cdot), a(\cdot)) := \max_{h \in \varphi(\sigma)} \frac{\|\kappa(h)\|_{L^2(X, h)}^2}{(4\pi)^2} - \int_X c_1(\mathcal{S})^2.$$
where we regard
\[ |\sigma| = \bigcup_{s \in S(Bd(\sigma))} |s| \times \{0\}^{\sigma_{\min}(s)}. \]

These real numbers \( \lambda(\sigma, a(\cdot)) \) and \( C(\sigma, g, N(\cdot), a(\cdot)) \) depend only on \( \sigma \) and \( a(\cdot) \), and on \( \sigma, g, N(\cdot), \) and \( a(\cdot) \), respectively, since we have fixed the spin\( ^c \) structure \( \mathfrak{s} \) in this subsection. Note that \( \lambda(\sigma) > 0 \) holds. We also note that \( |C(\sigma)| < \infty \) holds since \( \phi_\sigma(|\sigma|) \) is compact and the map \( h \mapsto \|\kappa(h)\|^2_{L^2(X,h)} \) is continuous. Set
\[ \bar{R}(\sigma) = \bar{R}(\sigma, g, N(\cdot), a(\cdot)) := \max \left\{ \frac{C(\sigma, g, N(\cdot), a(\cdot))}{\lambda(\sigma, a(\cdot))}, 0 \right\} \]
and
\[ R(\sigma) = R(\sigma, g, N(\cdot), a(\cdot)) := \max_{\tau < \sigma} \bar{R}(\tau, g, N(\cdot), a(\cdot)). \]

**Proposition 2.12.** Let \( \sigma \) be a simplex of \( K \). Then, for any \( R \geq \bar{R}(\sigma) \), there is no solution to the unperturbed Seiberg–Witten equations for any metric in \( \phi_\sigma(e^\xi \partial(\sigma, R)) \).

**Proof.** We argue by contradiction. Assume that for some non-negative number \( R \geq \bar{R}(\sigma) \) and some metric \( h \in \phi_\sigma(e^\xi \partial(\sigma, R)) \), there exists a solution to the unperturbed Seiberg–Witten equations for \( h \). For this \( h \), there exist
\[ s \in S(Bd(\sigma)), \ l > 0, \ \mathcal{J} = \{s_0, \ldots, s_l\} \in S(Bd(s)), \]
\[ \{r_{\Sigma}\}_{\Sigma \in \sigma_{\min}(s)} \in e^\xi \partial(\sigma, s, R), \ t = \{t_j\}_{j=0}^l \]
such that \( t_j \geq 0, \ \sum_{j=0}^l t_j = 1 \), and \( h = \phi_\sigma(p, \{r_{\Sigma}\}) \), where \( p = \sum_{j=0}^l t_j s_j \). From Lemma 2.5, we have
\[ \bigcup_{\Sigma \in \sigma_{\min}(s)} (X(\lambda(\mathcal{J}, t), \Sigma), \phi_\sigma(p, \{r_{\Sigma}\})) \cong \bigcup_{\Sigma \in \sigma_{\min}(s)} \mathbb{C}Y_{\text{std}}(\lambda(\mathcal{J}, t), \Sigma, r_{\Sigma}). \] (28)

Here \( \cong \) means an isometry. We set \( r'_{\Sigma} := \Lambda(\lambda(\mathcal{J}, t), r_{\Sigma}) \). Since \( \chi^{-}(\Sigma)^2 + 1 \leq |c_1(\mathfrak{s}) \cdot [\Sigma]|^2 \) holds for each \( \Sigma \in \sigma_{\min}(s) \), we have
\[ \sum_{\Sigma \in \sigma_{\min}(s)} r'_{\Sigma} \cdot \chi^{-}(\Sigma)^2 + \sum_{\Sigma \in \sigma_{\min}(s)} r'_{\Sigma} \leq \sum_{\Sigma \in \sigma_{\min}(s)} r'_{\Sigma} \cdot |c_1(\mathfrak{s}) \cdot [\Sigma]|^2 \] (29)
\[ \leq \sum_{\Sigma \in \sigma_{\min}(s)} \|\mathcal{H}_h(c_1(\mathfrak{s}))\|^2_{L^2(X(\lambda(\mathcal{J}, t), \Sigma), h)} \]
\[ \leq \|\mathcal{H}_h(c_1(\mathfrak{s}))\|^2_{L^2(X, h)} \]
\[ \leq \frac{\|\kappa(h)\|_{L^2(X,h)}^2}{(4\pi)^2} - \int_X c_1(\mathfrak{s})^2, \]
where the second and the last inequality follow from Lemma 2.10 and Lemma 2.11, respectively. Set $X' = X \setminus \bigcup_{\Sigma \in \sigma_{\min}(s)} X(\lambda(\mathcal{S}, t), \Sigma)$ and $\{0\} = \{0\}_{\sigma_{\min}(s)}$. We also have

$$
\|\kappa(h)\|^2_{L^2(X, h)} = \sum_{\Sigma \in \sigma_{\min}(s)} \|\kappa(h)\|^2_{L^2(X(\lambda(\mathcal{S}, t), \Sigma), h)} + \|\kappa(h)\|^2_{L^2(X', h)}
$$

$$
= \sum_{\Sigma \in \sigma_{\min}(s)} (4\pi)^2 r'_\Sigma \cdot \chi^- (\Sigma)^2 + \|\kappa(\phi_{\sigma}(p, \{0\}))\|^2_{L^2(X', \phi_{\sigma}(p, \{0\}))}
$$

$$
\leq \sum_{\Sigma \in \sigma_{\min}(s)} (4\pi)^2 r'_\Sigma \cdot \chi^- (\Sigma)^2 + \|\kappa(\phi_{\sigma}(p, \{0\}))\|^2_{L^2(X, \phi_{\sigma}(p, \{0\}))},
$$

From (29) and (30), we deduce

$$
\sum_{\Sigma \in \sigma_{\min}(s)} r'_\Sigma \leq \frac{\|\kappa(\phi_{\sigma}(p, \{0\}))\|^2_{L^2(X, \phi_{\sigma}(p, \{0\}))}}{(4\pi)^2} - \int_X c_1(\mathcal{S})^2 \leq C(\sigma).
$$

However, since we also have

$$
\sum_{\Sigma \in \sigma_{\min}(s)} r'_\Sigma \geq \max_{\Sigma \in \sigma_{\min}(s)} r'_\Sigma > \lambda(\mathcal{S}, t) \left( \max_{\Sigma \in \sigma_{\min}(s)} r_\Sigma + 1 \right)
$$

$$
\geq \lambda(\sigma)(R + 1) > \lambda(\sigma)R(\sigma),
$$

it follows that $R(\sigma) < C(\sigma)/\lambda(\sigma)$. This contradicts the definition of $R(\sigma)$. \[\square\]

3 | CONSTRUCTION OF THE INVARIANT II: PERTURBATIONS AND COUNTING ARGUMENTS

In this section, we complete the construction of the cohomological invariant. We consider the moduli spaces of the Seiberg–Witten equations parameterized by the families of Riemannian metrics given in Section 2; we construct a cohomology class by counting the points of the parameterized moduli spaces.

3.1 | Definition of the invariant in a simple case

Let $X$ be a smooth oriented closed connected 4-manifold equipped with a homology orientation, $\mathcal{S}$ be a spin$^c$ structure on $X$, and $n$ be the integer given by $d(\mathcal{S}) = -(n + 1)$. We assume that $n \geq 0$ and that $b^+(X) \geq n + 3$. (These assumptions will be needed since we will use $(n + 1)$-parameter families in order to define the cohomological invariant.) We also assume that $V(\mathcal{K}(X, \mathcal{S})) \neq \emptyset$. In this subsection, we define the cohomological invariant

$$
\mathcal{S}\mathcal{W}(X, \mathcal{S}) \in H^n(\mathcal{K}(X, \mathcal{S}); \mathbb{Z})
$$
under this setting. Although we will consider more general spin\(^c\) structures in Subsection 3.3 using what is called \(\mu\)-maps, the case which we treat in this subsection is the most basic one; we can give a non-trivial example for this case in Section 4.

We fix a metric \(g \in \text{Met}(X)\) and data \(N(\cdot), a(\cdot)\) as explained in Subsection 2.3 throughout this subsection. We also fix certain kinds of perturbations \(\varphi\), explained below to define a cochain on \(\mathcal{K}\). While in Subsection 2.5 we have considered only the unperturbed Seiberg–Witten equations, we now need the perturbed ones. Fix an integer \(l \geq 2\). The space of perturbations \(\Pi = \Pi(X, \mathfrak{g})\) is given by

\[
\Pi(X, \mathfrak{g}) := \{ (h, \mu) \in \text{Met}(X) \times i\Omega^2 \mid \mu \in i\Omega^+_{h} \} = \bigcup_{h \in \text{Met}(X)} i\Omega^+_{h},
\]

where \(\Omega^+_{h}\) is the completion by the \(L^2_{l-1}\)-Sobolev norm defined by \(h\) of the space of self-dual 2-forms with respect to \(h\). (We here omit \('L^2_{l-1}(\cdot)' from our notation for simplicity.) The space \(\Pi \to \text{Met}(X)\) is a Hilbert bundle over \(\text{Met}(X)\). Let us identify \(\text{Met}(X)\) with the zero-section of this bundle, and denote by \(\Pi_h\) the fiber \(i\Omega^+_{h}\) of this bundle \(\Pi\) at \(h \in \text{Met}(X)\). Fix a smooth reference connection \(A_0\) of the determinant line bundle. We call the subset of perturbations \(\mathcal{W} = \mathcal{W}(X, \mathfrak{g}) \subset \Pi(X, \mathfrak{g})\) defined by

\[
\mathcal{W}(X, \mathfrak{g}) = \bigcup_{h \in \text{Met}(X)} \mathcal{W}(X, \mathfrak{g})_h, \quad \mathcal{W}(X, \mathfrak{g})_h := F^+_{A_0} + i \text{Im}d^+_{h} \subset i\Omega^+_{h}
\]

the wall. We define \(\hat{\Pi} = \bigsqcup_{h \in \text{Met}(X)} \hat{\Pi}_h\) by

\[
\hat{\Pi}_h := \Pi_h \setminus \mathcal{W}_h.
\]

The wall \(\mathcal{W}\) is of codimension-\(b^+(X)\) in \(\Pi\) since \(\Omega^+_{h}/\text{Im}(d^+_{h}) \cong H^+(X; \mathbb{R})\) for each \(h\). Recall that, for \((h, \mu) \in \Pi\), the perturbed Seiberg–Witten equations with respect to \((h, \mu)\) have a reducible solution if and only if \((h, \mu) \in \mathcal{W}\).

For \(k \in \{-1, 0, \ldots, n\}\), we now define maps

\[
\{ \varphi_\sigma : [0, R(\sigma)]^\sigma \to \hat{\Pi} \}_{\sigma \in S_b(\mathcal{K})}
\]

inductively on \(k\) as follows. Here we define \(S_{-1}(\mathcal{K})\) as \(\emptyset\); we regard \(\varphi_\emptyset\) as a map \(\varphi_\emptyset : \{\text{pt}\} \to \hat{\Pi}\). In the case when \(k = -1\), take a generic point \(*\) in \(\hat{\Pi}\) so that the moduli space for the perturbed Seiberg–Witten equations with respect to \(*\) is empty. (Recall that the formal dimension \(d(\mathfrak{g})\) is assumed to be negative.) Then we can define \(\varphi_\emptyset : \{\text{pt}\} \to \hat{\Pi}\) by \(\varphi_\emptyset(\text{pt}) = *\). Next we define \(\varphi_\sigma\) in the case that \(k = 0\). Let \(\{\Sigma\} \in S_0(\mathcal{K})\). We here simply write it \(\Sigma\). Note that there is no solution to the unperturbed Seiberg–Witten equations for the metric \(\phi_\Sigma(\rho) = R(\Sigma)\) from Proposition 2.12. Take a generic path from \(*\) to \(\phi_\Sigma(\rho) \times \{R(\Sigma)\}\) in \(\hat{\Pi}\) so that the parameterized moduli space with respect to this path is smooth. (Now we regard \(\text{Met}(X)\) as a subset of \(\Pi\).) Here, if \(d(\mathfrak{g}) < 1\), the smoothness of the parameterized moduli space means just that it is empty. This construction gives a map

\[
\varphi_\Sigma : [0, R(\Sigma)]^{[\Sigma]} = [0, R(\Sigma)] \to \hat{\Pi}.
\]
Note that \( \varphi_{\Sigma}^{|_0} = \varphi_{\emptyset} \) obviously holds. Next, for \( k \in \{1, \ldots, n\} \) assume that \( \varphi_\tau \) is already constructed for any \( l \leq k - 1 \) and any \( \tau \in S_l(\mathcal{K}) \) and that the restriction \( \varphi_\tau |_{\text{ext}\delta(\tau, R(\tau))} \) coincides with the composition

\[
\phi_\tau \circ \psi^{-1}_{\tau, R(\tau)} : \text{ext}\delta(\tau, R(\tau)) \cong \text{ext}\delta(\tau, R(\tau)) \rightarrow \hat{\Pi}.
\]

Here \( \psi_{\tau, R(\tau)} \) is the homeomorphism given in Proposition 2.6. We note that the image of this map \( \phi_\tau \circ \psi^{-1}_{\tau, R(\tau)} \) is a priori contained in \( \text{Met}(X) \), however, in fact in \( \hat{\Pi} \). Indeed, from Proposition 2.12, there is no solution to the unperturbed Seiberg–Witten equations for any metric in the image of the map \( \phi_\tau \circ \psi^{-1}_{\tau, R(\tau)} \). In particular, reducible solutions do not appear on the image, therefore it is contained in \( \hat{\Pi} \). (Here we regard \( \text{Met}(X) \) as a subset of \( \Pi \).)

We shall construct \( \varphi_\sigma \) for \( k \)-simplices under these assumptions. Let \( \sigma \in S_k(\mathcal{K}) \). Note that \( [0, R(\sigma)]^\sigma \) decomposes as

\[
[0, R(\sigma)]^\sigma = [0, R(\sigma)]^\sigma \cup (0, R(\sigma))^\sigma \cup \bigcup_{\tau < \sigma, \dim \tau = k - 1} [0, R(\sigma)]^\tau \cup (0, R(\sigma))^\sigma \cup \bigcup_{\tau < \sigma, \dim \tau = k - 1} \text{ext}\delta(\tau, R(\tau)) \cup (0, R(\sigma))^\sigma. \tag{34}
\]

Here we regard \( [0, R(\tau)]^\tau \) as a subset of \( [0, R(\sigma)]^\sigma \) by inserting some factors \( \{0\} \) in appropriate places. We take a map \( \varphi_\sigma \) satisfying following three conditions:

**Condition 3.1.**

1. On \( \text{ext}\delta(\sigma, R(\sigma)) \), the map \( \varphi_\sigma \) coincides with the composition

\[
\phi_\sigma \circ \psi^{-1}_{\sigma, R(\sigma)} : \text{ext}\delta(\sigma, R(\sigma)) \rightarrow \hat{\Pi}.
\]

2. For any face \( \tau < \sigma \) with \( \dim \tau = k - 1 \),
   a) on \( [0, R(\tau)]^\tau \), the map \( \varphi_\sigma \) coincides with \( \varphi_\tau : [0, R(\tau)]^\tau \rightarrow \hat{\Pi} \), and
   b) on \( \bigsqcup_{R(\tau) \leq R \leq R(\sigma)} \text{ext}\delta(\tau, R) \), the map \( \varphi_\sigma \) coincides with the composition

\[
\phi_\tau \circ \left( \bigsqcup_{R(\tau) \leq R \leq R(\sigma)} \psi^{-1}_{\tau, R} \right) : \bigsqcup_{R(\tau) \leq R \leq R(\sigma)} \text{ext}\delta(\tau, R) \rightarrow \hat{\Pi}.
\]

3. The restriction of \( \varphi_\sigma \) to \( (0, R(\sigma))^\sigma \) is generic in the sense that the parameterized moduli space with respect to \( \varphi_\sigma \) is smooth over \( (0, R(\sigma))^\sigma \).

By the construction of the homeomorphism given in Proposition 2.6, the map in the case of (2)b above is continuous. The maps in other cases are obviously continuous, and so is the whole map \( \varphi_\sigma \). Continuity of \( \varphi_\sigma \) and the compactness of \( [0, R(\sigma)]^\sigma \) imply that the parameterized moduli space, defined soon later, with respect to \( \varphi_\sigma \) is compact.
The procedure above can be continued to \( k = n + 1 \) since the codimension of \( \mathcal{W} \) is \( b^+ \) and we have assumed that \( b^+ > n + 2 \); however, we stop it at \( k = n \). (In the subsequent subsections, we will use cobordism arguments, and therefore consider \((n + 2)\)-parameter families. Our hypothesis \( b^+ \geq n + 3 \) will be needed in those arguments.) We eventually obtain perturbations (33) for each \( k \in \{-1, 0, \ldots, n\} \).

**Lemma 3.2.** Let \( \sigma \in S_k(\mathcal{K}) \) be a simplex of \( \mathcal{K} \) with \( k \in \{0, \ldots, n\} \). Then there is no solution to the Seiberg–Witten equations for any element in \( \varphi_\sigma(\mathcal{\partial}([0, R(\sigma)]^\mathcal{F})) \). There is also no solution to the Seiberg–Witten equations for \( \varphi_\emptyset(\text{pt}) \).

**Proof.** The proof in the case that \( k = -1 \) is obvious since \( \varphi_\emptyset(\text{pt}) = * \) and we choose \( * \) so that the moduli space with respect to \( * \) is empty. For \( k \in \{0, \ldots, n\} \), the proof is by induction on \( k \) as follows. First take \( \Sigma \in S_0(\mathcal{K}) \). Then we have \( \varphi_\Sigma(\mathcal{\partial}([0, R(\Sigma)]^\mathcal{K})) = \{*\} \cup \varphi_\Sigma(|\Sigma| \times \{ R(\Sigma) \}) \). There is no solution for \( * \) as above, and also for \( \varphi_\Sigma(|\Sigma| \times \{ R(\Sigma) \}) \) by Proposition 2.12. Next, for \( k \in \{1, \ldots, n\} \) assume that the statement holds for any \( l \leq k - 1 \). Take \( \tau \in S_k(\mathcal{K}) \). For elements in \( \mathcal{\partial}(\mathcal{\partial}\sigma, R(\sigma)) \) and \( \bigsqcup_{R(\tau) \leq R \leq R(\sigma)} \mathcal{\partial}(\tau, R) \) for some \( \tau \prec \sigma \) with \( \text{dim} \tau = k - 1 \), the non-existence of solutions follows from Proposition 2.12. For elements in \( \mathcal{\partial}([0, R(\tau)]^\mathcal{F}) \), the non-existence follows from the induction hypothesis. For elements in \( (0, R(\tau)]^\mathcal{F} \), the non-existence follows from that \( \text{dim} \tau < n \) and that the formal dimension for the unparametrized Seiberg–Witten equations is given by \( d(\mathcal{F}) = -(n + 1) \).

Let \((h, \mu) \in \Pi\). We denote by \( C(h) = C(h, \mathcal{F}) \) the \( L^2 \)-completion defined by \( h \) of the space of pairs consisting of a \( U(1) \)-connection of the determinant line bundle and a positive spinor for \( \mathcal{F} \). The gauge group \( \mathcal{G}(h) \) is defined as the \( L^2 \) completion using \( h \) of \( \text{Map}(X, U(1)) \). We define \( \mathcal{B}(h) = B(h, \mathcal{F}) \) by \( B(h) := C(h)/\mathcal{G}(h) \). We will denote by \( \mathcal{M}(h, \mu) = \mathcal{M}((h, \mu), \mathcal{F}) \subset \mathcal{B}(h) \) the moduli space for \((h, \mu) \) defined as the quotient of the space of solutions to the Seiberg–Witten equations with respect to \( (h, \mu) \) divided by \( \mathcal{G}(h) \) as usual. Let \( \varphi : T \to \Pi \) be a map from a topological space \( T \), which is called the parameter space. We define the parameterized moduli space parameterized by \( \varphi \) on \( T \) as

\[
\mathcal{M}(\varphi) = \mathcal{M}(\varphi, \mathcal{F}) := \bigsqcup_{t \in T} \mathcal{M}(\varphi(t), \mathcal{F})
\]

Now let us consider the parameterized moduli space given by \( \varphi \). Let \( \sigma \in S_k(\mathcal{K}) \) a simplex of \( \mathcal{K} \) with \( k \in \{0, \ldots, n\} \). The parameterized moduli space \( \mathcal{M}(\varphi_\sigma) \) is compact because of the compactness argument for the usual Seiberg–Witten theory and the compactness of the parameter space \([0, R(\sigma)]^\mathcal{F}\). Since we avoid the wall, \( \mathcal{M}(\varphi_\sigma) \) does not contain a reducible solution. We here note that

\[
\mathcal{M}(\varphi_\sigma, \mathcal{F}) = \bigsqcup_{t \in (0, R(\sigma)]^\mathcal{F}} \mathcal{M}(\varphi_\sigma(t), \mathcal{F})
\]

follows from Lemma 3.2. Recall that \( \varphi_\sigma \) is generic on \((0, R(\sigma)]^\mathcal{F}\) and that \( d(\mathcal{F}) = -(n + 1) \). We therefore have \( \mathcal{M}(\varphi_\sigma) = \emptyset \) if \( k \leq n - 1 \), and \( \mathcal{M}(\varphi_\sigma) \) is a 0-dimensional manifold if \( k = n \).

Let \( \sigma \) an oriented \( n \)-simplex of \( \mathcal{K} \). Then we have an orientation of \([0, R(\sigma)]^\mathcal{F}\). Putting the given homology orientation and this orientation together, we obtain the orientation of \( \mathcal{M}(\varphi_\sigma) \). (See, for example, Appendix A in Salamon’s book [20].) We follow the orientation convention in
this book.) We now eventually have an oriented compact 0-dimensional manifold $\mathcal{M}(\varphi_\sigma)$ from $<\sigma>$. We will denote by

$$SW(\varphi_\sigma) = SW(\varphi_\sigma, \mathfrak{s}) \in \mathbb{Z}$$

the algebraic count $#\mathcal{M}(\varphi_\sigma)$. Set $A = (g, N(\cdot), a(\cdot), \varphi_\sigma)$, which we call an additional data. We define an $n$-cochain

$$SW(X, \mathfrak{s}, A) \in C^n(\mathcal{K})$$

by

$$C_n(\mathcal{K}) \to \mathbb{Z}; \langle \sigma \rangle \mapsto SW(\varphi_\sigma).$$

We will prove the following proposition in Subsection 3.2:

**Proposition 3.3.** The cochain $SW(X, \mathfrak{s}, A)$ constructed above is a cocycle.

We now may write the definition of the cohomological invariant:

**Definition 3.4.** Let $X$ be a smooth oriented closed connected 4-manifold equipped with a homology orientation, $\mathfrak{s}$ be a spin$^c$ structure on $X$, and $n$ be the integer given by $d(\mathfrak{s}) = -(n + 1)$. Assume that $n \geq 0, b^+(X) \geq n + 3$, and $V(\mathcal{K}(X, \mathfrak{s})) \neq \emptyset$. We define a cohomology class $\mathbb{S}W(X, \mathfrak{s}) \in H^n(\mathcal{K}(X, \mathfrak{s}); \mathbb{Z})$ by

$$\mathbb{S}W(X, \mathfrak{s}) := [SW(X, \mathfrak{s}, A)],$$

where $A = (g, N(\cdot), a(\cdot), \varphi_\sigma)$ is an additional data.

In fact, $\mathbb{S}W(X, \mathfrak{s})$ is independent of the choice of $A$:

**Theorem 3.5.** The cohomology class $\mathbb{S}W(X, \mathfrak{s}) \in H^n(\mathcal{K}(X, \mathfrak{s}); \mathbb{Z})$ is an invariant of a spin$^c$ 4-manifold $(X, \mathfrak{s})$. Namely the cohomology class $[SW(X, \mathfrak{s}, A)]$ in Definition 3.4 is independent of the choice of $A$.

We will prove Theorem 3.5 in Subsection 3.2.

**Remark 3.6.** To define the number $SW(\varphi_\sigma) \in \mathbb{Z}$ used to define $SW(X, \mathfrak{s}, A)$, in fact we do not need to assume that $\varphi_\sigma$ is generic on $(0, R(\sigma))^\sigma$ for each $n$-simplex $\sigma$. To justify the counting argument in the case that $\varphi_\sigma$ is not generic, one can use Ruan’s virtual neighborhood technique [16]. More precisely, one needs a family version of the technique (for example, see [8]) in our case. An advantage of the use of the virtual neighborhood technique is that we can use a map $\varphi_\sigma$ which is continuous (not necessarily smooth) on $(0, R(\sigma))^\sigma$ to define $SW(X, \mathfrak{s}, A)$. This is based on the same mechanism in the following linear situation: to define the index of a family of Fredholm operators, we do not need to assume that the family smoothly depends on its parameter, but just need to assume it continuously depends on it.
3.2 Arguments by cobordisms

The purpose of this subsection is to prove Proposition 3.3 and Theorem 3.5. Both of them are shown using arguments by cobordisms. We follow all the settings and the notation of Subsection 3.1.

**Proof of Proposition 3.3.** As we noted in Subsection 3.1, perturbations $\varphi_\bullet$ can be constructed for $(n+1)$-simplices since $b^+ \geq n + 3$. Let us fix $\varphi_\bullet$ for $(n+1)$-simplices through the inductive procedure in Subsection 3.1.

Let us take an oriented $(n + 1)$-simplex $\langle \sigma \rangle = \langle \Sigma_0, \ldots, \Sigma_{n+1} \rangle$ of $\mathcal{K}$. Write $\langle \tau_1 \rangle = \langle \Sigma_0, \ldots, \hat{\Sigma}_i, \ldots, \Sigma_{n+1} \rangle$, then we have

$$SW(X, \mathcal{S}, \mathcal{A})(\partial \langle \sigma \rangle) = \sum_{i=0}^{n+1} (-1)^i SW(\varphi_{\tau_i}). \quad (35)$$

Recall the decomposition

$$\partial([0, R(\sigma)]^\sigma) = \text{ext}\partial(\sigma, R(\sigma)) \cup \bigcup_{i=0}^{n+1} [0, R(\tau_i)]^{\tau_i} \cup \bigcup_{R(\tau_i) \leq R \leq R(\sigma)} \text{ext}\partial(\tau_i, R).$$

By Proposition 2.12, the parameterized moduli space $\mathcal{M}(\varphi_{\sigma})$ vanishes on $\partial([0, R(\sigma)]^\sigma)$ except for the components $[0, R(\tau_i)]^{\tau_i}$ for $i = 0, \ldots, n + 1$. The moduli space also vanishes on $\partial([0, R(\tau_i)]^{\tau_i})$ by Lemma 3.2. Hence the count of the point of the moduli space on $\partial([0, R(\sigma)]^\sigma)$ coincides with that on $\bigcup_{i=0}^{n+1} (0, R(\sigma_{\tau_i}))^{\tau_i}$ for the perturbations $\varphi_{\tau_i}$. By taking the orientations into account, we therefore have

$$\sum_{i=0}^{n+1} (-1)^i SW(\varphi_{\tau_i}) = \#\mathcal{M}(\varphi_{\sigma} | \partial([0, R(\sigma)]^\sigma)). \quad (36)$$

Since the total moduli space $\mathcal{M}(\varphi_{\sigma})$ is an oriented compact 1-dimensional manifold, the right-hand side of (36) is zero. From this and (35), we have $SW(X, \mathcal{S}, \mathcal{A})(\partial < \sigma >) = 0$. This proves the proposition. \qed

**Proof of Theorem 3.5.** Take two additional data $\mathcal{A}_i = (g_i, \mathcal{N}_i(\cdot), a_i(\cdot), \varphi_{\star i})$ ($i = 0, 1$); let us write $SW(X, \mathcal{S}, \mathcal{A}_i)$ simply $SW_i$. We now connect these two data $\mathcal{A}_0$ and $\mathcal{A}_1$ by a 1-parameter family. Take a generic path

$$\{g_t\}_{t \in [0,1]} \quad (37)$$

from $g_0$ and $g_1$ in $\text{Met}(X) \cap \hat{\Pi}$. For each $\Sigma \in V(\mathcal{K})$, by the uniqueness of tubular neighborhoods, we may take a continuous family of tubular neighborhoods $N_t(\Sigma)$ of $\Sigma$. Then we have a family of diffeomorphisms $f_{\Sigma,t} : N_t(\Sigma) \to D^2 \times \Sigma$ with $f_{\Sigma,t}|_{\Sigma} = t_S$; we also obtain a family $\tilde{f}_{\Sigma,t} : N_t(\Sigma) \setminus \Sigma \to (0,1] \times S^1 \times \Sigma$ by the same way to define $\tilde{f}_\Sigma$ in Subsection 2.3. We may also take a family of positive numbers $a_t(\cdot)$ satisfying the following conditions: $a_t(\Sigma) = a_t((\Sigma)) = 1$ holds for each $\Sigma \in V(\mathcal{K})$ and each $t \in [0,1]$, and for each $\sigma \in S(\mathcal{K})$, the condition obtained by replacing $a(\cdot), U(\cdot), N(\cdot)$, and $f_{\Sigma}$ with $a_t(\cdot), U_t(\cdot), N_t(\cdot)$, and $\tilde{f}_{\Sigma,t}$ in Condition 2.2, respectively, holds. We next take a
path $\varphi_{*,t}$ as follows. By repeating the construction of $\varphi_*$ in Subsection 2.3 with the parameter $t$, we obtain $\varphi_{*,t}$. Set $R_t(\sigma) = R(\sigma, g_t, N_t(\cdot), a_t(\cdot))$. Note that $R_t(\sigma)$ continuously depends on $t$. Then we may prove the statement obtained by replacing $R(\sigma)$ and $\varphi_\sigma$ with $R_t(\sigma)$ and $\varphi_{\sigma,t}$ in Proposition 2.12, respectively, by the quite same way in Subsection 2.5. We now fix a map

$$\varphi_{*,t} = \bigsqcup_{t \in [0,1]} \varphi_{*,t} : \bigsqcup_{t \in [0,1]} [0, R_t(\sigma)]^2 \to \hat{\Gamma}$$

for each $\sigma \in S_k(K)$ with $k \in \{-1, 0, \ldots, n\}$ through the following inductive procedure. For $i = 0, 1$, denote by $\ast_i$ the image of $\varphi_{\cdot, t}$. Take a path $\ast_t$ in $\hat{\Gamma}$ from $\ast_0$ to $\ast_1$. For each $\{\Sigma\} \in S_0(K)$, recall that $\varphi_{\Sigma, t} : [0, R_i(\Sigma)] \to \hat{\Gamma}$ is given as a path from $\ast_i$ to $\varphi_{\Sigma, t}(\{\Sigma\} \times \{R_i(\Sigma)\})$. Let us take a map $\varphi_{\Sigma, \cdot} : \bigsqcup_{t \in [0,1]} [0, R_i(\Sigma)] \to \hat{\Gamma}$ satisfying the following three conditions.

1. On $\bigsqcup_{t \in [0,1]} \{0\} \cong [0,1]$, the map $\varphi_{\Sigma, \cdot}$ coincides with $g_t : [0,1] \to \text{Met}(X) \cap \hat{\Gamma}; t \mapsto g_t$

2. On $\bigsqcup_{t \in [0,1]} \{R_i(\Sigma)\} \cong [0,1]$, the map $\varphi_{\Sigma, \cdot}$ coincides with $\varphi_{\Sigma, t} : [0,1] \to \text{Met}(X) \cap \hat{\Gamma}; t \mapsto \varphi_{\Sigma, t}(\{\Sigma\} \times \{R_i(\Sigma)\})$.

3. For each $i = 0, 1$, on $[0, R_i(\Sigma)]$, the map $\varphi_{\Sigma, \cdot}$ coincides with $\varphi_{\Sigma, i}$.

Next we consider $k$-simplices with $k \in \{1, \ldots, n\}$. Assume that $\varphi_{\cdot, \cdot}$ is already constructed for any $l \leq k - 1$ and $\tau \in S_l(K)$, and that for each $t \in [0,1]$ the restriction $\varphi_{\cdot, t}|_{\text{ext}\delta(\tau, R_i(\tau))}$ coincides with the composition

$$\varphi_{\cdot, t} \circ \psi^{-1}_{\cdot, R_i(\tau)} : \text{ext}\delta(\tau, R_i(\tau)) \cong e \text{xt}\delta(\tau, R_i(\tau)) \to \hat{\Gamma}.$$

(We note that the homeomorphism $\psi_{\cdot, R}$ for $R \geq 0$ given in Proposition 2.6 is independent of any additional data.) Let us take $\sigma \in S_k(K)$. By replacing $R(\cdot)$ with $R_i(\cdot)$ in (34) and taking the disjoint union with respect to $t \in [0,1]$, we have a decomposition of $\bigsqcup_{t \in [0,1]} [0, R_i(\sigma)]^2$. We take $\varphi_{\sigma, \cdot}$ satisfying the two conditions: the first one is the condition obtained by replacing $R(\cdot)$, $\varphi_{\cdot, \cdot}$, and $\varphi_{\cdot, t}$ in (1) and (2) in Condition 3.1 with $R_i(\cdot), \varphi_{\cdot, t},$ and $\varphi_{\cdot, t},$ respectively, for any $t \in [0,1]$, and the second is that obtained by replacing $0, R(\sigma))^2$ and $\varphi_{\cdot, \cdot}$ in (3) in Condition 3.1 with $\bigsqcup_{t \in [0,1]} (0, R_i(\sigma))^2$ and $\varphi_{\cdot, \cdot}$, respectively. This procedure can be continued to $k = n$ since we have assumed that $b^+ > n + 2$.

Using the 1-parameter family connecting $A_0$ and $A_1$ obtained above, we define

$$SW_{0,1} \in C^{n-1}(K)$$

by

$$C^{n-1}(K) \to \mathbb{Z}; \langle \tau \rangle \mapsto (-1)^{n-1} \ SW(\varphi_{\cdot, \cdot}),$$

where $SW(\varphi_{\cdot, \cdot}) = \# M(\varphi_{\cdot, \cdot}).$ Here the orientation of $M(\varphi_{\cdot, \cdot})$ is defined by the product orientation of the parameter space via the identification $\bigsqcup_{t \in [0,1]} [0, R_i(\tau)]^2 \cong [0, R_0(\tau)]^2 \times [0, 1].$
Now let us take an oriented $n$-simplex $\sigma = \langle \Sigma_0, ..., \Sigma_n \rangle$ and write $\tau_j = \langle \Sigma_0, ..., \hat{\Sigma}_j, ..., \Sigma_n \rangle$. Then we have

$$(SW_1 - SW_0 - \delta SW_{0,1})(\sigma) = SW(\varphi_{\sigma,1}) - SW(\varphi_{\sigma,0}) - (-1)^{n-1}\sum_{j=0}^{n} (-1)^j SW(\varphi_{\tau_j,\cdot}) = \# M(\varphi_{\sigma,\cdot}|_{\mathbb{I} \subset [0,1][0,R(\sigma)]^2}) = 0,$$

where the last equality follows from the fact that $M(\varphi_{\cdot,\cdot})$ is an oriented compact 1-dimensional manifold. We therefore have $SW_1 - SW_0 = \delta SW_{0,1}$. This proves the theorem.

Remark 3.7. For a closed spin$^c$ 4-manifold $(X, \mathfrak{s})$, in the same vein of the definition of the complex of curves due to Harvey [6] in 2-dimensional topology, one can define an abstract simplicial complex $K^{\text{isol}} = K^{\text{isol}}(X, \mathfrak{s})$: vertices are isotopy classes of embeddings of surfaces and simplices are given as collections of such isotopy classes which can be represented by disjoint surfaces. A natural question one can ask is whether we can construct a cohomology class which is similar to $SW(X, \mathfrak{s})$ on this simplicial complex $K^{\text{isol}}$. However, to construct such a cohomology class on $K^{\text{isol}}$, the way used to construct $SW(X, \mathfrak{s})$ on $K$ cannot work similarly. An essential issue is that it is difficult to find an effective analogy of the proof of Theorem 3.5 for $K^{\text{isol}}$. Namely it seems hard to absorb new ambiguity arising from choices of representatives of isotopy classes into coboundaries. The author does not know the answer to the question above at this stage.

3.3 The case when the moduli space is of higher-dimension

In this subsection, using what is called $\mu$-maps, we extend the definition of the cohomological Seiberg–Witten invariant also to the case when higher dimensional moduli spaces appear. Let $(X, \mathfrak{s})$ be a smooth oriented closed connected spin$^c$ 4-manifold equipped with a homology orientation. We assume that $V(K(X, \mathfrak{s})) \neq \emptyset$. Let $n \geq 0$ be a natural number such that $b^+(X) \geq n+3$ holds and $d(\mathfrak{s}) + n + 1$ is a non-negative even number. For such a natural number, we define a cohomology class

$$SW^n(X, \mathfrak{s}) \in H^n(K(X, \mathfrak{s}); \mathbb{Z})$$

in this subsection. If there is no such a natural number $n$ for $\mathfrak{s}$, we define the cohomological Seiberg–Witten invariant of $(X, \mathfrak{s})$ as $0 \in H^*(K(X, \mathfrak{s}); \mathbb{Z})$. We denote by $m$ the non-negative number given by $2m = d(\mathfrak{s}) + n + 1$. The setting in Subsection 3.1 is the case that $d(\mathfrak{s}) + n + 1 = 0$ in this subsection; in this case, $SW^n(X, \mathfrak{s})$ coincides with $SW(X, \mathfrak{s})$.

For a metric $h \in \text{Met}(X)$, let $B^+(h) = B^+(h, \mathfrak{s})$ be the subspace of $\mathcal{B}(h)$ consisting of the image of irreducible configurations by the quotient map $\mathcal{C}(h) \to \mathcal{B}(h)$. We obtain, as usual, a complex line bundle $\mathcal{L}(h) = \mathcal{L}(h, \mathfrak{s}) \to B^+(h)$ equipped with a Hermitian metric using the based gauge group.

Let us fix an additional data $A = (g, N(\cdot), a(\cdot), \varphi_{\cdot})$ as in Subsection 3.1. For each $\sigma \in S_k(K)$ with $k \in \{-1, 0, ..., n\}$, let $\varphi_{\sigma} : [0, R(\sigma)]^2 \to \text{Met}(X)$ be the map given as the composition of $\varphi_{\sigma}$:
and the projection \( \tilde{\Pi} \to \text{Met}(X) \). We set

\[
B^r(\varphi_\sigma) := \bigcup_{t \in [0, R(\sigma)]^F} B^r(\varphi_\sigma'(t))
\]

and define a complex line bundle \( L(\varphi_\sigma) \to B^r(\varphi_\sigma) \) by

\[
L(\varphi_\sigma) := \bigcup_{t \in [0, R(\sigma)]^F} L(\varphi_\sigma'(t)).
\]

As in the construction of \( \varphi_* \), we can take sections

\[
\{ s_{\sigma, i} : B^r(\varphi_\sigma) \to L(\varphi_\sigma) \}_{\sigma \in S_k(\mathcal{K}), i \in \{1, \ldots, m\}}
\]

inductively on \( k \in \{-1, 0, \ldots, n\} \) satisfying the following conditions.

- The restriction of \( s_{\sigma, i} \) to \( B^r(\varphi_\sigma)|_{(0, R(\sigma))^\mathbb{F}} \) is smooth for each \( k \in \{-1, 0, \ldots, n\}, \sigma \in S_k(\mathcal{K}) \), and \( i \in \{1, \ldots, m\} \).

- The intersection

\[
\mathcal{M}(\varphi_\sigma, s_{\sigma, 1}, \ldots, s_{\sigma, m}) := \mathcal{M}(\varphi_\sigma) \cap s_{\sigma, 1}^{-1}(0) \cap \cdots \cap s_{\sigma, m}^{-1}(0)
\]

is a 0-dimensional compact oriented smooth manifold for each \( \sigma \in S_n(\mathcal{K}) \).

The last condition is achieved by taking all sections \( s_{\sigma, i} \) for \( k \in \{-1, 0, \ldots, n\}, \sigma \in S_k(\mathcal{K}) \), and \( i \in \{1, \ldots, m\} \) to be transverse to the zero-section. (Note that, for all \( \sigma \in S_n(\mathcal{K}) \), the ‘cut parametrized moduli space’ \( \mathcal{M}(\varphi_\sigma, s_{\sigma, 1}, \ldots, s_{\sigma, m}) \) is empty over \( \partial([0, R(\sigma)]^\mathbb{F}) \) by reason of formal dimension.) Set \( \mathcal{A}_n := (g, N(\cdot), a(\cdot), \varphi_*, \{s_{\cdot, i}\}_{i}) \), which we also call an additional data. We define an \( n \)-cochain

\[
SW^n(X, \mathfrak{s}, \mathcal{A}_n) \in C^n(\mathcal{K})
\]

by

\[
C_n(\mathcal{K}) \to \mathbb{Z} ; \ (\sigma) \mapsto \# \mathcal{M}(\varphi_\sigma, s_{\sigma, 1}, \ldots, s_{\sigma, m}).
\]

We then obtain the following proposition by the same way to prove Proposition 3.3:

**Proposition 3.8.** The cochain \( SW^n(X, \mathfrak{s}, \mathcal{A}_n) \) constructed above is a cocycle.

We now may write the definition of the generalized cohomological invariant:

**Definition 3.9.** Let \( X \) be a smooth oriented closed connected 4-manifold equipped with a homology orientation and \( \mathfrak{s} \) be a spin\(^c\) structure on \( X \). Assume that \( V(\mathcal{K}(X, \mathfrak{s})) \neq \emptyset \) and that there exists a natural number \( n \geq 0 \) such that \( b^+(X) \geq n + 3 \) holds and \( d(\mathfrak{s}) + n + 1 \) is a non-negative even number. We define a cohomology class \( SW^n(X, \mathfrak{s}) \in H^n(\mathcal{K}(X, \mathfrak{s}); \mathbb{Z}) \) by

\[
SW^n(X, \mathfrak{s}) := [SW^n(X, \mathfrak{s}, \mathcal{A}_n)],
\]

where \( \mathcal{A}_n = (g, N(\cdot), a(\cdot), \varphi_*, \{s_{\cdot, i}\}_{i}) \) is an additional data.
Note that $\mathbb{S}\mathbb{W}^n(X, \mathfrak{s}) = \mathbb{S}\mathbb{W}(X, \mathfrak{s})$ holds in the case that $d(\mathfrak{s}) + n + 1 = 0$ by definition. Since two sections $s_{\sigma, i}$ and $s'_{\sigma, i}$ can be connected by a path of sections, two additional data $A_n$ and $A'_n$ can be also connected by a path. So, as in Theorem 3.5, we have the following theorem by a cobordism argument:

**Theorem 3.10.** The cohomology class $\mathbb{S}\mathbb{W}^n(X, \mathfrak{s}) \in H^n(\mathbb{K}(X, \mathfrak{s}); \mathbb{Z})$ is an invariant of a spin$^c$ 4-manifold $(X, \mathfrak{s})$. Namely the cohomology class $[\mathbb{S}\mathbb{W}^n(X, \mathfrak{s}, A_n)]$ in Definition 3.9 is independent of the choice of $A_n$.

**Remark 3.11.** Although we will give a spin$^c$ 4-manifold $(X, \mathfrak{s})$ such that $\mathbb{S}\mathbb{W}(X, \mathfrak{s}) \neq 0$ in Section 4, it seems hard to give a non-trivial example for $\mathbb{S}\mathbb{W}^n(X, \mathfrak{s})$ with $d(\mathfrak{s}) + n + 1 \neq 0$ using the proof of the non-vanishing result for $\mathbb{S}\mathbb{W}(X, \mathfrak{s})$. Indeed, we will construct a spin$^c$ 4-manifold $(X, \mathfrak{s})$ with $\mathbb{S}\mathbb{W}(X, \mathfrak{s}) \neq 0$ as the connected sum of two spin$^c$ 4-manifolds $(M, \mathfrak{s}_0)$ and $(N, t)$, where the Seiberg–Witten invariant of $(M, \mathfrak{s}_0)$ does not vanish. If one tries to give an example of $(X, \mathfrak{s})$ with $\mathbb{S}\mathbb{W}(X, \mathfrak{s}) \neq 0$ and $d(\mathfrak{s}) + n + 1 \neq 0$ via this way, one has to find $(M, \mathfrak{s}_0)$ such that the Seiberg–Witten invariant of it does not vanish and $d(\mathfrak{s}_0) > 0$. The simple type conjecture due to Witten [24] tells us that it is difficult (or might be impossible). The author does not know whether there exists a non-trivial example for $\mathbb{S}\mathbb{W}^n(X, \mathfrak{s})$ with $d(\mathfrak{s}) + n + 1 \neq 0$ at this stage.

4 | NON-VANISHING RESULTS FOR THE COHOMOLOGICAL INVARIANT

In this section, we prove that the invariant $\mathbb{S}\mathbb{W}(X, \mathfrak{s})$ constructed in Subsection 3.1 is non-trivial for some spin$^c$ 4-manifolds. We also explain the non-vanishing result from a classical point of view: we will derive some constraints connected with the adjunction inequality on configurations of embedded surfaces from the non-vanishing result. An interesting point of our cohomological formulation is that one non-vanishing result provides certain constraints on infinitely many configurations. We use the following two key tools to prove the non-vanishing result: the first one is a combination of wall-crossing and gluing technique originally due to Ruberman [17–19], and the second is a description of higher dimensional wall-crossing phenomena in terms of embedded surfaces given in [8] by the author. These tools are explained in Subsection 4.1 and Subsection 4.2, respectively.

4.1 | Combination of wall-crossing and gluing arguments

Ruberman has discussed a combination of wall-crossing and gluing arguments in [17–19]. Ruberman considered wall-crossing over a 4-manifold with $b^+ = 1$, and Baraglia and the author generalized that argument for all $b^+ \geq 1$ in [1]. Here we shall use the result in [1].

Let $(N, t)$ be a smooth oriented closed spin$^c$ 4-manifold equipped with a homology orientation. Set $k := b^+(N)$ and assume that $k \geq 1$. Let $\varphi^N : D^k \to \Pi(N, t)$ be a generic map satisfying $\varphi^N(S^{k-1}) \subset \tilde{\Pi}(N, t)$ from the $k$-dimensional disk to the space of perturbations. Here ‘generic’ means that $\varphi^N$ is transverse to the wall $\mathcal{W}(N, t)$, defined in (32). Then we can define the ‘intersection number’ $\varphi^N \cdot \mathcal{W}(N, t) \in \mathbb{Z}$ of $\varphi^N$ and $\mathcal{W}(N, t)$. This intersection number can be interpreted as the mapping degree of the map $\varphi^N|_{S^{k-1}} : S^{k-1} \to \tilde{\Pi}(N, t) \simeq S^{k-1}$. Here the given orientation
of $H^+(N; \mathbb{R})$ is used to determine the sign of the mapping degree. The precise convention of the sign is given in Ruberman [19] (in the case that $b^+ = 1$) and we do not repeat it here. (In fact, for our non-vanishing result in Subsection 4.3, we do not have to determine the sign.)

Let $(M, \mathfrak{s}_0)$ be a smooth oriented closed spin$^c$ 4-manifold equipped with a homology orientation. Suppose that $b^+(M) \geq 2$ and $d(\mathfrak{s}_0) = 0$. For natural numbers $k \geq 1$ and $l \geq k$, we consider the 4-manifold $N$ defined by

$$N = k\mathbb{CP}^2 \# l(-\mathbb{CP}^2) = \#_{i=1}^k \mathbb{CP}^2_i \# (\#_{j=1}^l (-\mathbb{CP}^2)).$$

Here $\mathbb{CP}^2_i$ and $-\mathbb{CP}^2$ are copies of $\mathbb{CP}^2$ and $-\mathbb{CP}^2$, respectively, where $-\mathbb{CP}^2$ is the projective plane equipped with the opposite orientation. Let $H_i$ and $E$ be a generator of $H^2(\mathbb{CP}^2_i)$ and one of $H^2(-\mathbb{CP}^2)$, respectively. The basis $H_1, \ldots, H_k$ of $H^+(N; \mathbb{R})$ gives a homology orientation of $N$. Let $\mathfrak{t}$ be the spin$^c$ structure on $N$ determined by

$$c_1(\mathfrak{t}) = \sum_{i=1}^k H_i + \sum_{j=1}^l E. \quad (38)$$

We set

$$(X, \mathfrak{s}) := (M \# N, \mathfrak{s}_0 \# \mathfrak{t}).$$

Then we have $d(\mathfrak{s}) = -k$. Let $(g^M, \mu) \in \Pi(M, \mathfrak{s}_0)$ be a generic point and $\varphi^N : D^k \to \Pi(N, \mathfrak{t})$ be a generic map satisfying $\varphi^N(B^{k-1}) \subset \Pi(N, \mathfrak{t})$. Then we can consider the intersection number $\varphi^N \cdot \mathcal{W}(N, \mathfrak{t})$ as in the previous paragraph. Let $B^4_M \subset M$ and $B^4_N \subset N$ be small balls. Assume that the metric $g^M$ and the metric-component of each point in the image of $\varphi^N$ are cylindrical near the boundaries of $B^4_M$ and $B^4_N$, respectively, and the boundaries of $B^4_M$ and $B^4_N$ are mutually isometric. We also assume that $\mu$ and the self-dual 2-form-component of each point in the image of $\varphi^N$ are supported on the complement of $B^4_M$ and $B^4_N$, respectively. Then we may define $\varphi : D^k \to \Pi(X, \mathfrak{s})$ by $\varphi(x) := (g^M, \mu) \# \varphi^N(x)$, where $\#$ is the connected sum consisting of $M \setminus B^4_M, N \setminus B^4_N$, and a cylinder with the standard product metric and of sufficiently large length. Since $\varphi$ is also a generic map and we have $d(\mathfrak{s}) = -k$, the moduli space $\mathcal{M}(\varphi, \mathfrak{s})$ parameterized by $\varphi$ on $D^k$ is a 0-dimensional compact manifold.

The following proposition is discussed by Ruberman in the case that $b^+(N) = 1$ and $\varphi^N \cdot \mathcal{W}(N, \mathfrak{t}) = \pm 1$ in [19, Section 4]. A similar argument, spelled out by Baraglia and the author in [1], works for general $b^+(N) \geq 1$ and $\varphi^N \cdot \mathcal{W}(N, \mathfrak{t})$.

**Proposition 4.1** [19, 1]. Suppose that the length of the cylinder used to define the connected sum is sufficiently large. Then

$$\# \mathcal{M}(\varphi, \mathfrak{s}) = \pm (\varphi^N \cdot \mathcal{W}(N, \mathfrak{t})) \cdot \text{SW}(M, \mathfrak{s}_0) \quad (39)$$

holds in $\mathbb{Z}$. Here $\text{SW}(M, \mathfrak{s}_0)$ is the Seiberg–Witten invariant of $(M, \mathfrak{s}_0)$.

**Proof.** We may adapt Theorem 4.4 in [1] after a small modification. To explain it, first we summarize a part of the statement of the theorem. The setting needed to describe the part of the statement of the theorem is as follows. Let $(M, \mathfrak{s}_0)$ $(N, \mathfrak{t})$, and $(X, \mathfrak{s})$ be as above. Let $B$ be an
oriented closed manifold of dimension $k$. Let $M \to E_M \to B$ and $N \to E_N \to B$ smooth fiber bundles equipped with fiberwise spin$^c$ structures whose restriction on fibers are isomorphic to $\mathfrak{s}_0$ and $\mathfrak{t}$, respectively. Assume that $E_M$ and $E_N$ admit sections, and let us construct the fiberwise connected sum $X \to E_X \to B$ along these sections. This bundle $E_X$ is equipped with the fiberwise spin$^c$ structure obtained from that of $E_M, E_N$, and the restriction this fiberwise spin$^c$ structure on a fiber is isomorphic to $\mathfrak{s}$. Let us fix a homology orientation of $X$. Since we have $d(\mathfrak{s}) = -\dim B$ and $b^+(X) \geq \dim B + 2$, we can define the families Seiberg–Witten invariant $FSW(E_X, \mathfrak{s}) \in \mathbb{Z}$ by counting the parametrized moduli space for $E_X$ over $B$.

Theorem 4.4 in [1] gives away to compute $FSW(E_X, \mathfrak{s})$ as follows. Fix a fiberwise metric of $E_N$. Then we obtain a bundle of harmonic self-dual 2-forms, denoted by $H^+(N) \to H^+(E_N) \to B$. Since the space of fiberwise metrics is contractible, the isomorphism class of $H^+(E_N)$ is independent of the choice of fiberwise metric. Theorem 4.4 asserts that we have

$$FSW(E_X, \mathfrak{s}) = \text{SW}(M, \mathfrak{s}_0) \cdot \langle e(H^+(E_N)), [B] \rangle.$$  (40)

(Note that $e(H^+(E_N))$ is denoted by $e(H^+(N))$ in [1].) We shall explain how to modify the above gluing result to show (39).

First, note that, although the parameter space of the families is a closed manifold $B$ in Section 4 in [1], the proof of the gluing formula localizes only around the points in $B$ on which wall-crossing for $N$ happens. This localization, given as Corollary 7.9 in [1], can be understood easily as follows. Let $\{\eta_N(b)\}_{b \in B}$ be a families perturbation consisting of a fiberwise imaginary self-dual 2-form over $E_N$. (This corresponds to $\mathcal{F}^N$ we discussed, but for a while we fit our notation to that of [1] for explanation.) Since we assumed that $d(\mathfrak{t}) < 0$, if the families perturbation $\{\eta_N(b)\}_{b \in B}$ was taken to be generic, the moduli space on the fiber $(E_N)_b$ over $b$ with respect to $\eta_N(b)$ is empty unless the Seiberg–Witten equations on $(E_N)_b$ have a reducible solution for $\eta_N(b)$. The gluing argument for the moduli space on $M$, which is possibly non-empty, and the empty moduli space on $N$ yields just the empty moduli space on $X$. Therefore non-trivial gluing occurs only around the points functions $b$ in $B$ for which wall-crossing for $(E_N)_b$ happens for $\eta_N(b)$.

Because of this localization, we may use $B = D^k$ as the parameter space of families provided that wall-crossing does not happen over $\partial B$. Let us describe it more precisely. Assume that $E_M$ and $E_N$ are trivialized over $B = D^k$: $E_M = B \times M, E_N = B \times N$. Let $\eta_M = \{\eta_M(b)\}_{b \in B}$ and $\eta_N = \{\eta_N(b)\}_{b \in B}$ be generic families perturbations over $E_M$ and $E_N$, respectively. Assume that the Seiberg–Witten equations on $(N, \eta_N(b))$ do not have reducible solutions for all $b \in \partial B$. Then we obtain the following gluing formula:

$$FSW(E_X, \mathfrak{s}, \eta_M, \eta_N) = \langle e(H^+(E_N), \eta_N), [B, \partial B] \rangle \cdot \text{SW}(M, \mathfrak{s}_0).$$  (40)

Here let us explain notations appearing in (40). First, $FSW(E_X, \mathfrak{s}, \eta_M, \eta_N) \in \mathbb{Z}$ denotes the number obtained by counting the parameterized moduli space for $E_X$ with respect to the fiberwise metric on $E_N$ obtained from $\eta_M$ and $\eta_N$. (The precise construction of the fiberwise metric on $E_X$ is described in Section 7 in [1].) Second, $e(H^+(E_N), \eta_N) \in H^k(B, \partial B; \mathbb{Z})$ is the relative Euler class with respect to $\eta_N$ defined as follows. For each $b \in B$, denote by $g_N(b)$ the underlying metric of $\eta_N(b)$. Set $g_N = \{g_N(b)\}_{b \in B}$. For a metric $h$ on $N$, let $H^+(N, h)$ denote the space of imaginary self-dual harmonic 2-forms of $N$ with respect to the metric $h$. Then we can define a vector bundle $H^+(E_N, g_N) \to B$ with fiber $H^+(N, g_N(b))$. Regard $\eta_N$ as a section $\eta_N : B \to \Pi(N) \times B$ of the
trivial bundle $\Pi(N) \times B \to B$. Let
\[
p : \Pi(N) \to \bigsqcup_{h \in \text{Met}(N)} H^+(N, h)
\]
be the fiberwise $L^2$-projection. Let $s : B \to H^+(E_N, g_N)$ be the section defined by
\[
s(b) = p(F_{A_N}^+g_N(b) - \eta_N(b)),
\]
where $A_N$ is a fixed smooth reference connection on $N$. One may see that $s(b) \neq 0$ for all $b \in \partial B$ because of the assumption that the Seiberg–Witten equations on $(N, \eta_N(b))$ do not have reducible solutions for all $b \in \partial B$. Therefore, we may define the relative Euler class $e(H^+(E_N), \eta_N) \in H^k(B, \partial B; \mathbb{Z})$ with respect to $\eta_N$ as the pull-back of the Thom class of the bundle $H^+(E_N) \to B$ by $s$.

As in the proof of Proposition 7.2 of [1], we may take $\eta_N$ so that $\eta_N$ meets the wall $\mathcal{W}(N, t)$ transversely in a neighborhood of the zero points of the section $s : B \to H^+(E_N, g_N)$, and that $\eta_N$ does not meet the wall outside the neighborhood. More precisely, $\eta_N$ can be taken to meet the wall once around each zero point $b$ of $s$, and that the sign of the intersection between $s$ and the zero section of $H^+(N)$ at $b$ corresponds to that of between $\eta_N$ and the wall around $b$. Thus we have that $< e(H^+(E_N), \eta_N), [B, \partial B] > = \pm \eta_N \cdot \mathcal{W}(N, t)$. Coming back to our current notation of this paper, $\eta_N$ denotes $\varphi^N$. This and (40) complete the proof. □

4.2 Higher dimensional wall-crossing and embedded surfaces

In [8], the author has given a description of higher dimensional wall-crossing phenomena (that is, wall-crossing for general $b^+ \geq 1$) in terms of embedded surfaces. In this subsection, we recall and rewrite a part of it as a convenient form for the setting of this paper. We also introduce some concepts, which will be used to state the non-vanishing result in Subsection 4.3. For natural numbers $k \geq 1$ and $l \geq k$, set $N = k\mathbb{CP}^2#l(-\mathbb{CP}^2) = \#_{i=1}^k \mathbb{CP}^2 \#(\#_{j=1}^l (-\mathbb{CP}^2))$ and let $t$ be the spin$^c$ structure on $N$ given in Subsection 4.1.

Let $\Sigma^+_i, \Sigma^-_i \in V(\mathcal{K}(N, t)) (i = 1, \ldots, k)$ be vertices such that
\[
i \neq i' \Rightarrow \Sigma^+_i \cap \Sigma^+_{i'} = \Sigma^-_i \cap \Sigma^+_{i'} = \Sigma^-_i \cap \Sigma^-_{i'} = \emptyset
\]
holds. We write the collection of $2k$-surfaces $\Sigma^+_i, \Sigma^-_i \ (1 \leq i \leq k)$ as
\[
\Sigma = \{ \Sigma^+_i \}_{1 \leq i \leq k, \epsilon \in \{+,-\}}.
\]
Set
\[
\text{Sign}^k = \{+, -\}^{1, \ldots, k}.
\]
For each $\epsilon = \{ \epsilon_i \}_{i=1}^k \in \text{Sign}^k$, let $\sigma_\epsilon \in S(\mathcal{K}(N, t))$ be the $(k - 1)$-simplex defined by
\[
\sigma_\epsilon := \{ \Sigma^+_{\epsilon_i} \mid 1 \leq i \leq k \}.
\]
We may also define a subcomplex $K(\Sigma)$ of $\mathcal{K}(N, t)$ by giving the set of simplices as

$$S(K(\Sigma)) := \{ \{ \Sigma_i^\varepsilon \mid i \in J \} \mid \{ \varepsilon_i \} \in \{+, -\}^J, J \subset \{1, \ldots, k\} \}.$$ 

The set $\{ \sigma_\varepsilon \mid \varepsilon \in \text{Sign}^k \}$ can be regarded as the set of $(k - 1)$-simplices of $K(\Sigma)$. Note that the geometric realization $|K(\Sigma)|$ is homeomorphic to $S^{k-1}$.

**Remark 4.2.** Let $(M, \mathfrak{s}_0)$ be an oriented closed spin$^c$ 4-manifold. Let us consider the spin$^c$ 4-manifold $(X, \mathfrak{s}) := (M \# N, \mathfrak{s}_0 \# t)$. Here the connected sum $M \# N$ is defined by gluing a ‘punctured’ $M$ and $N$ along a puncture belonging to the complement of all functions $\Sigma_i^\varepsilon$ in $N$. Then each $\Sigma_i^\varepsilon$ can be regarded as an embedded surface in $X$ and it also violates the adjunction inequality with respect to $\mathfrak{s}$. Therefore, $K(\Sigma)$ can be regarded as a subcomplex of $\mathcal{K}(X, \mathfrak{s})$. We will fix an orientation of each $\sigma_\varepsilon$ as below. Then we can consider the fundamental class of $K(\Sigma)$, and thus obtain the homology class $[K(\Sigma)] \in H_{k-1}(\mathcal{K}(X, \mathfrak{s}); \mathbb{Z})$.

Fix data $g^N, N(\cdot)$, and $a(\cdot)$ on the 4-manifold $N$ as in Section 2. Then we can define $R^N(\sigma_\varepsilon) = R(\sigma_\varepsilon, g^N, N(\cdot), a(\cdot))$ as in Subsection 2.5. Set

$$R^N_{\Sigma} := \max_{\varepsilon \in \text{Sign}^k} R^N(\sigma_\varepsilon).$$

For a collection of positive numbers $\{R_{\tau}\}_{\tau \prec \sigma_\varepsilon, \varepsilon \in \text{Sign}^k}$, we may take maps

$$\{ \varphi^N_{\tau, R_{\tau}} : [0, R_{\tau}] \to \Pi(N, t) \}_{\tau \prec \sigma_\varepsilon, \varepsilon \in \text{Sign}^k}$$

inductively on the dimension of $\tau$ by applying the same procedure in Subsection 3.1. Here we should note that the image of $\varphi^N_{\sigma_\varepsilon, R_{\sigma_\varepsilon}}$ is not contained in $\hat{\Pi}(N, t)$ in general even if we take $\varphi^N_{\sigma_\varepsilon, R_{\sigma_\varepsilon}}$ to be generic; $\varphi^N_{\sigma_\varepsilon, R_{\sigma_\varepsilon}}$ generically intersects the wall at points. However, the partial image $\varphi^N_{\sigma_\varepsilon, R_{\sigma_\varepsilon}}(\text{ext} \hat{\partial}(\sigma_\varepsilon, R_{\sigma_\varepsilon}))$ satisfies that

$$\varphi^N_{\sigma_\varepsilon, R_{\sigma_\varepsilon}}(\text{ext} \hat{\partial}(\sigma_\varepsilon, R_{\sigma_\varepsilon})) \subset \text{Met}(N) \cap \hat{\Pi}(N, t)$$

if $R_{\sigma_\varepsilon} \geq R^N_{\Sigma}$ as in Subsection 3.1. This is based on the vanishing result, Proposition 2.12, and therefore we have a stronger property: there is no solution to the unperturbed Seiberg–Witten equations for any metric in $\varphi^N_{\sigma_\varepsilon, R_{\sigma_\varepsilon}}(\text{ext} \hat{\partial}(\sigma_\varepsilon, R_{\sigma_\varepsilon}))$. Furthermore, we also have

$$\varphi^N_{\sigma_\varepsilon, R_{\sigma_\varepsilon}}(\hat{\partial}([0, R_{\sigma_\varepsilon}]^\varepsilon)) \subset \hat{\Pi}(N, t)$$

for any positive number $R_{\sigma_\varepsilon}$ with $R_{\sigma_\varepsilon} \geq R^N_{\Sigma}$. Indeed, there is no solution to the Seiberg–Witten equations for any element in the left-hand side of (44). It follows from the same argument used to prove Lemma 3.2. We therefore obtain the relation (44) from consideration of reducible solutions.

For $R > 0$, we define a space $D^k_{\Sigma, R}$ as the quotient

$$D^k_{\Sigma, R} := \left( \bigsqcup_{\varepsilon \in \text{Sign}^k} [0, R]^\varepsilon \right) / \sim,$$
where \( \sim \) is the equivalence relation generated by the following identification: for \( \varepsilon, \varepsilon' \in \text{Sign}^k \) and for a common face \( \tau \) of \( \sigma_\varepsilon \) and \( \sigma_{\varepsilon'} \), we identify the subset \([0, R]^\tau \subset [0, R]^{\varepsilon} \) and the subset \([0, R]^\tau \subset [0, R]^{\varepsilon'} \) by the identity map \([0, R]^\tau \to [0, R]^\tau \).

**Definition 4.3.** We call a collection of orientations of functions \( \sigma_\varepsilon \) an orientation of \( \Sigma \) if the orientations of functions \([0, R]^{\varepsilon} \) defined by that of functions \( \sigma_\varepsilon \) induce an orientation of \( D^k_{\Sigma, R} \) for \( R > 0 \).

Note that an orientation of \( \Sigma \) induces an orientation of \( |K(\Sigma)| \cong S^{k-1} \). Let \( \{ R_\varepsilon \}_{\varepsilon \in \text{Sign}^k} \) be a collection of positive numbers and \( R \) be a positive number such that \( R = R_{\sigma_\varepsilon} \) holds for any \( \varepsilon \in \text{Sign}^k \). Then we can take maps (42). By the construction of \( \varphi_* \), the collection of maps \( \{ \varphi^N_{\sigma_\varepsilon, R} \}_{\varepsilon \in \text{Sign}^k} \) gives rise to a map

\[
\varphi^N_{\Sigma, R} : D^k_{\Sigma, R} \to \Pi(N, t).
\]

The space \( D^k_{\Sigma, R} \) is homeomorphic to a \( k \)-dimensional disk. The boundary \( S^{k-1}_{\Sigma, R} := \partial D^k_{\Sigma, R} \) can be expressed as

\[
S^{k-1}_{\Sigma, R} = \left( \bigsqcup_{\varepsilon \in \text{Sign}^k} \text{ext}(\sigma_\varepsilon, R) \right) / \sim.
\]

We note that \( \varphi^N_{\Sigma, R}(S^{k-1}_{\Sigma, R}) \subset \text{Met}(N) \cap \overset{\circ}{\Pi}(N, t) \) holds by (43) and moreover there is no solution to the unperturbed Seiberg–Witten equations for any metric in \( \varphi^N_{\Sigma, R}(S^{k-1}_{\Sigma, R}) \). We can therefore define the intersection number \( \varphi^N_{\Sigma, R} \cdot \mathcal{W}(N, t) \) for this \( \varphi^N_{\Sigma, R} \).

**Remark 4.4.** Let \( \pi : \bigsqcup_{\varepsilon \in \text{Sign}^k} [0, R]^{\varepsilon} \to D^k_{\Sigma, R} \) be the quotient map. We note that the subspace

\[
\left\{ p \in \bigsqcup_{\varepsilon \in \text{Sign}^k} [0, R]^{\varepsilon} \mid \# \pi^{-1}(\pi(p)) > 1 \right\}
\]

of \( \bigsqcup_{\varepsilon \in \text{Sign}^k} [0, R]^{\varepsilon} \) is contained in

\[
\bigsqcup_{\varepsilon \in \text{Sign}^k} \partial([0, R]^{\varepsilon}).
\]

We also note that there is no solution to the Seiberg–Witten equations for any element in the image of (47) by functions \( \varphi^N_{\sigma_\varepsilon, R} \) if \( R \geq R^N_{\Sigma} \) as we mentioned to obtain (44).

We here introduce a term in order to describe wall-crossing phenomena in terms of embedded surfaces. In general, for an oriented closed 4-manifold \( W \), we often identify \( H^2(W; \mathbb{Z}) \) with \( H_2(W; \mathbb{Z}) \) by the Poincaré duality, and we denote the evaluation \( < a, b > \) for \( a \in H^2(W; \mathbb{Z}) \), \( b \in H_2(W; \mathbb{Z}) \) by \( a \cdot b \).
Definition 4.5. For vertices \( \Sigma^+_i, \Sigma^-_i \in V(\mathcal{K}(N, t)) \) \( (i = 1, \ldots, k) \) satisfying (41), we call the collection \( \Sigma = \{ \Sigma^\epsilon_{i} \}_{1 \leq i \leq k, \epsilon \in \{+, -\}} \) a wall-crossing collection of surfaces in \( (N, t) \) if the following two conditions are satisfied.

1. Let \( i, i' \in \{ 1, \ldots, k \} \). If \( i \neq i' \), then \( H_{i'} \cdot [\Sigma^\pm_i] = 0 \) holds.
2. For each \( i \in \{ 1, \ldots, k \} \), the two integers
   
   \[ (c_1(t) \cdot [\Sigma^+]_i) \cdot (H_i \cdot [\Sigma^+_i]) \]

   and

   \[ (c_1(t) \cdot [\Sigma^-_i]) \cdot (H_i \cdot [\Sigma^-_i]) \]

   are non-zero and have different signs.

Here \( H_i \) is the one in Subsection 4.1.

Concrete examples of wall-crossing collections of surfaces will be given in Subsection 4.4. The description of wall-crossing phenomena in term of embedded surfaces is given as the following proposition. An essential part of its proof has been given also in the proof of [8, Lemma 3.3].

**Proposition 4.6** ([8]). Let \( \Sigma = \{ \Sigma^\epsilon_{i} \}_{1 \leq i \leq k, \epsilon \in \{+, -\}} \) be a wall-crossing collection of surfaces. Assume that \( \Sigma^+_i \) and \( \Sigma^-_i \) intersect transversally for each \( i \). Let us take a collection of positive numbers \( \{ R_{\tau} \}_{\tau \prec \sigma, \epsilon \in \text{Sign}^k} \) and a positive number \( R \). Suppose that \( R = R_{\sigma \epsilon} \) holds for any \( \epsilon \in \text{Sign}^k \) and that \( R \geq R_\Sigma^N \). If we take \( \{ R_{\tau} \}_{\tau \prec \sigma, \epsilon \in \text{Sign}^k} \) so that \( R \) is large enough, then \( \varphi^N_{\Sigma, R} : D^k \Sigma, R \to \Pi(N, t) \) satisfies that \( \varphi^N_{\Sigma, R} \cdot W(N, t) = \pm 1 \).

**Proof.** Assuming that \( \Sigma^+_i \) and \( \Sigma^-_i \) intersect transversally for each \( i \), we have the following alternative simple description of the family \( \varphi^N_{\Sigma, R} \) by taking suitable additional data \( g^N, N(\cdot), \) and \( a(\cdot) \) in Section 2. This construction is originally due to Frøyshov [5]. The construction is based on \( k \)-parameter families of metrics obtained from simultaneous stretches of neighborhoods of disjoint \( k \)-surfaces. Roughly, the desired whole family is given by patching all of the possible stretching families, corresponding to all combinations of disjoint \( k \)-surfaces belonging to \( \Sigma \). Note that such a simultaneous stretching cannot be considered for surfaces intersecting each other, say \( \Sigma^+_i \) and \( \Sigma^-_i \).

To spell out this construction of Riemannian metrics precisely, take a Riemannian metric \( g^N \in \text{Met}(N) \) such that a neighborhood of the sphere bundle of the normal bundle \( \nu(\Sigma^+) \) is isometric to \( [0, 1] \times S^1 \times \Sigma^+ \). Here \( [0, 1] \times S^1 \times \Sigma^\epsilon \) is equipped with the standard product metric as in Subsection 2.3, 2.5. For \( i \neq j \), we may assume that the cylindrical parts \([0, 1] \times \Sigma^+ \) and \([0, 1] \times S^1 \times \Sigma^\epsilon \) are disjoint for any \( \epsilon \in \{+, -\} \). This is achieved by taking suitable \( N(\cdot) \) and \( a(\cdot) \). We therefore obtain a family of Riemannian metrics

\[ \bar{\varphi}^N_{\Sigma} : \mathbb{R}^k \to \text{Met}(N) ; (r_1, \ldots, r_k) \mapsto G(r_1, \ldots, r_k), \]

where \( G(r_1, \ldots, r_k) \) is defined by

- replacing \([0, 1] \times S^1 \times \Sigma^+ \) with \([0, r_i] \times S^1 \times \Sigma^+ \) from \( g^N \) if \( r_i \geq 0 \), and
- replacing \([0, 1] \times S^1 \times \Sigma^- \) with \([0, -r_i] \times S^1 \times \Sigma^- \) from \( g^N \) if \( r_i \leq 0 \)
for each $i$. For $R > R_\Sigma^N$, let $D^k(R)$ be the cuboid in $\mathbb{R}^k$ centered at the origin and with the length of each edge $2R$. Then the restriction
\[
\phi^N_\Sigma := \phi^N_\Sigma \big|_{D^k(R)} : D^k(R) \to \text{Met}(N)
\]
coincides with the metric-component of $\varphi^N_{\Sigma,R}$ under the natural identification $D^k_{\Sigma,R} \cong D^k(R)$.

Denote by $V^+$ the subspace of $H^2(N; \mathbb{R})$ spanned by $H_1, \ldots, H_k$. For a Riemannian metric $g$ on $N$, let $\mathcal{H}^{+g}(N)$ be the space of self-dual harmonic 2-forms on $N$. Define a linear isomorphism
\[
\varphi_g : \mathcal{H}^{+g}(N) \to V^+
\]
as the composition of the following maps $i_g$ and $p_{V^+}$. First, the map $i_g$ is an injective map
\[
i_g : \mathcal{H}^{+g}(N) \to H^2(N; \mathbb{R})
\]
defined as the restriction of the Hodge-isomorphism
\[
H^2_g(N) \to H^2(N; \mathbb{R})
\]
from the space of harmonic 2-forms to the second cohomology. Note that the image of $i_g$ varies with $g$: each metric gives a particular maximal-dimensional positive-definite subspace of $H^2(N; \mathbb{R})$. Second, $p_{V^+}$ is the projection
\[
p_{V^+} : H^2(N; \mathbb{R}) = V^+ \oplus (V^+)^\perp \to V^+.
\]
Here $(V^+)^\perp$ is the orthogonal complement with respect to the intersection form. Set $c = c_1(t)$ and define
\[
\Psi : \text{Met}(N) \to \mathbb{R}^k
\]
by
\[
\Psi(g) := (c \cdot \varphi^{-1}_g(H_i))_{i=0}^k.
\]
Here we regard $\varphi^{-1}_g(H_i)$ as an element of $H^2(X; \mathbb{R})$ via $i_g$ and take the cup product with the cohomology class $c$. For $R > 0$, define
\[
F = F_R : D^k(R) \to \mathbb{R}^k
\]
by
\[
F := \Psi \circ \phi^N_\Sigma.
\]
We claim that, if $0 \not\in F(\partial D^k(R))$, the mapping degree of $F : \partial D^k(R) \to \mathbb{R}^k \setminus \{0\}$ coincides with that of $\phi^N_\Sigma \big|_{\partial D^k(R)} : \partial D^k(R) \to \text{Met}(N) \cap \bar{\Pi}$. (Note that the image of the restriction map $\varphi^N_{\Sigma,R} \big|_{\partial D^k(R)}$ is contained in $\text{Met}(N) \cap \bar{\Pi}$ as we mentioned, and so is the image of $\phi^N_\Sigma \big|_{\partial D^k(R)}$.) First, we see that the intersection points of the image of $\phi^N_\Sigma$ with the wall $\mathcal{W}(N,t)$ correspond to the zero set of $F$. Recall that, for a metric $g$ on $N$, the condition that there exists a reducible solution to the unperturbed Seiberg–Witten equation for $g$ and $t$ is equivalent to the condition that $c$ is orthogonal to $\mathcal{H}^{+g}(N)$ with respect to the intersection form. Since $\varphi_g$ is an isomorphism, the latter condition is equivalent to $\Psi(g) = 0$. This shows that $\left(\phi^N_\Sigma^{-1}(\mathcal{W}(N,t))\right) = F^{-1}(0)$. Next, to complete the proof of the coincidence of the degrees of $F \big|_{\partial D^k(R)}$ and $\phi^N_\Sigma \big|_{\partial D^k(R)}$, we see that, generically, the signed
count of each intersection point of the image of $\phi_N^\Sigma$ with the wall $W(N, t)$ coincides with the signed count of the corresponding intersection point of $F$ with 0. Considering each intersection point separately, it suffices to prove the correspondence in the case that the family of metrics $\phi_N^\Sigma$ makes $F : \partial D^k(R) \to \mathbb{R}^k \setminus \{0\}$ a degree one map. Let $p_g : \Omega^+_g(N) \to \mathcal{H}^+(N)$ be the $L^2$-projection for each metric $g$. Since we have $c \cdot \varphi_g^{-1}(H_i) = p_g\left(\frac{1}{2\pi \sqrt{-1}} F^+_g \wedge \varphi_g^{-1}(H_i)\right)$ and supposed that $F|_{\partial D^k(R)}$ is a degree one map, the family $\{p_g(F^+_g)\}_{g \in \phi_N^\Sigma(D^k(R))}$ generically intersects with 0 once. In other words, the family of metrics $\phi_N^\Sigma(D^k(R))$ generically intersects with the wall $W(N, t)$ once.

The mapping degree of $\phi_N^\Sigma|_{\partial D^k(R)}$, which is that of $\varphi_N^\Sigma, R|_{S^{k-1}} : S^{k-1} \to \tilde{\Pi}(N, t) \simeq S^{k-1}$, is the intersection number $\varphi_N^\Sigma, R \cdot W(N, t)$. By the last paragraph, we have that $\varphi_N^\Sigma, R \cdot W(N, t)$ is the mapping degree of $F$, and our remaining task is to show that the mapping degree of $F$ is ±1. Set $\alpha_i^\pm := [\Sigma_i^\pm]$ for $i = 1, \ldots, k$, and define

$$\text{Hyp}_{\Sigma_i^\pm} := \left\{ (a_1, \ldots, a_k) \in \mathbb{R}^k \mid a_i = \frac{c \cdot \alpha_i^\pm}{H_i \cdot \alpha_i^\pm} \right\}.$$  

We shall show that, for all $i$, the image $F(F(\Sigma_i^\pm, R))$ is contained in an $\epsilon$-neighborhood of Hyp$_{\Sigma_i^\pm}$ for sufficiently small $\epsilon > 0$. Then the conditions on $(c \cdot \alpha_i^\pm) \cdot (H_i \cdot \alpha_i^\pm)$ in Definition 4.5 imply that $F$ has mapping degree ±1.

Let $i \in \{1, \ldots, k\}$. For a metric $g$ on $N$, define

$$\omega_i^\pm := \varphi^{-1}_g(p_{V^+}(\alpha_i^\pm)) \in \mathcal{H}^+(N).$$

Since we have that $\omega_i^\pm = \varphi^{-1}_g((H_i \cdot \alpha_i^\pm)H_i) = (H_i \cdot \alpha_i^\pm)\varphi_g^{-1}(H_i)$, we obtain that

$$\frac{c \cdot [\omega_i^\pm]}{H_i \cdot \alpha_i^\pm} = c \cdot \varphi_g^{-1}(H_i). \quad (48)$$

Let $r \in F(\Sigma_i^\pm, R)$ and set $g = \phi_N^\Sigma(r)$. By [8, Lemma 3.2], we have that

$$|\alpha_i^\pm - [\omega_i^\pm]| \leq \frac{C}{R^{1/4}},$$

where $C$ is a constant depends only on the metric outside neighborhoods of $\Sigma_i^\pm$. Therefore, by (48), $c \cdot \varphi_g^{-1}(H_i)$ is close to $(c \cdot \alpha_i^\pm)/(H_i \cdot \alpha_i^\pm)$ for sufficiently large $R$. Namely, the $i$th component of $F(r)$ is close to $(c \cdot \alpha_i^\pm)/(H_i \cdot \alpha_i^\pm)$, and this implies that $F(F(\Sigma_i^\pm, R))$ is contained in an $\epsilon$-neighborhood of Hyp$_{\Sigma_i^\pm}$ for sufficiently small $\epsilon$. This completes the proof of the proposition.

### 4.3 Non-vanishing theorem

In this subsection, we prove a non-vanishing theorem for the cohomological invariant $\mathbb{S}^W(X, \mathfrak{s})$. We also derive constraints connected with the adjunction inequality on configurations of embedded surfaces from the non-vanishing theorem.
Let \((M, \mathfrak{s}_0)\) be a smooth oriented closed spin\(^c\) 4-manifold with \(b^+(M) \geq 2\) and \(d(\mathfrak{s}_0) = 0\), and \(k, l\) be natural numbers with \(l \geq k > 0\). We set \(N = k\mathbb{C}{\mathbb{P}}^2 \# l(-\mathbb{C}{\mathbb{P}}^2)\) and \((X, \mathfrak{s}) = (M \# N, \mathfrak{s}_0 \# \mathfrak{t})\), where \(\mathfrak{t}\) is the spin\(^c\) structure defined by (38) on \(N\). Since we have \(b^+(X) \geq k + 2\) and \(d(\mathfrak{s}) = -k\), the invariant \(\mathbb{S}\mathbb{W}(X, \mathfrak{s})\) is defined and belongs to \(H_{k-1}(\mathcal{K}(X, \mathfrak{s}); \mathbb{Z})\). Recall that, for a wall-crossing collection of surfaces \(\Sigma = \{\Sigma_i\}_{1 \leq i \leq k, \varepsilon \in \{+, -\}}\) equipped with an orientation in the sense of Definition 4.3 in \((N, \mathfrak{t})\), we can consider the element \([K(\Sigma)] \in H_{k-1}(\mathcal{K}(X, \mathfrak{s}); \mathbb{Z})\) as in remark 4.2.

We now have the following non-vanishing theorem:

**Theorem 4.7.** Let \(\Sigma = \{\Sigma_i\}_{1 \leq i \leq k, \varepsilon \in \{+, -\}}\) be a wall-crossing collection of surfaces in \((N, \mathfrak{t})\). Assume that \(\Sigma^+_i\) and \(\Sigma^-_i\) intersect transversally for each \(i\). Let us equip \(\Sigma\) with an orientation in the sense of Definition 4.3. Then

\[
\langle \mathbb{S}\mathbb{W}(X, \mathfrak{s}), [K(\Sigma)] \rangle = \pm \text{SW}(M, \mathfrak{s}_0)
\]

holds. In particular, if \(\text{SW}(M, \mathfrak{s}_0) \neq 0\), then we obtain

\[
\mathbb{S}\mathbb{W}(X, \mathfrak{s}) \neq 0 \text{ in } H_{k-1}(\mathcal{K}(X, \mathfrak{s}); \mathbb{Z})
\]

and

\[
[K(\Sigma)] \neq 0 \text{ in } H_{k-1}(\mathcal{K}(X, \mathfrak{s}); \mathbb{Z}).
\]

**Remark 4.8.** There could be a generalization of Theorem 4.7 which includes the case that \(b^+(M) = 1\). However, the proof of Theorem 4.7 depends on the gluing argument in [1], and the case that \(b^+ = 1\) is not discussed in [1]. Therefore we do not deal with such a generalization in this paper.

**Proof of Theorem 4.7.** To prove the theorem, we combine the setting of Subsection 4.2 with that of Subsection 4.1 as follows. Let \(B^i_M \subset M\) and \(B^k_N \subset N\) be small balls such that \(B^i_N\) is contained in the complement of some neighborhood of the surfaces belonging to \(\Sigma\), namely functions \(\Sigma_i^\varepsilon\) for \(i \in \{1, \ldots, k\}\) and \(\varepsilon \in \{+, -\}\). Take a metric \(g^N \in \text{Met}(N)\) which is cylindrical near the boundary of \(B^k_N\). We also fix functions \(N(\cdot)\) and \(a(\cdot)\) for the surfaces belonging to \(\Sigma\). Then we can define the map \(\phi^\varepsilon_N\) for each \(\tau \prec \sigma_\varepsilon\) and \(\varepsilon \in \text{Sign}_k\) using \(g^M\), \(N(\cdot)\), and \(a(\cdot)\) for the surfaces belonging to \(\tau\). Recall that surfaces belonging to \(\Sigma\) can be regarded as vertices of \(\mathcal{K}(X, \mathfrak{s})\). Let us consider the metric \(g = g^M \# g^N \in \text{Met}(X)\) defined by connected sum. Then we obtain the map \(\phi_\tau\) defined as in (5) for each \(\tau \prec \sigma_\varepsilon\) and \(\varepsilon\) using \(g, N(\cdot),\) and \(a(\cdot)\) for the surfaces belonging to \(\sigma_\varepsilon\). We now see that \(R^X(\tau) := R(\tau, g, N(\cdot), a(\cdot))\) is independent of the choice of the length of the cylinder used to define the connected sum. First note that \(\lambda(\tau) = \lambda(\tau, a(\cdot))\) is only determined by \(a(\cdot)\) for faces of \(\tau\). To see that \(C(\tau) = C(\tau, g, N(\cdot), a(\cdot))\) is independent of the choice of the length, take any metric \(h \in \phi_\tau(\tau)\). Since \(h\) can be written as the connected sum of \(g^M\) and a metric changed from \(g^N\) only near the surfaces belonging to \(\tau\), the restriction of \(h\) to the cylinder in the connected sum coincides with the standard cylindrical metric, and hence is a metric with positive scalar curvature. From this, we deduce that the restriction of \(\kappa(h)\) to the cylinder is identically zero. Therefore \(\|\kappa(h)\|_{L^2(X, h)}\) is invariant under any change of the length of the cylinder. We thus have that \(R^X(\tau)\) is independent of the choice of the length of the cylinder, and we write it without length parameter of the cylinder.

Let us consider the collection of positive numbers \(\{R^X(\tau)\}_{\tau \prec \sigma_\varepsilon, \varepsilon \in \text{Sign}_k}\). Then the maps

\[
\{\phi^N_{\tau, R^X(\tau)} : [0, R^X(\tau)]^\tau \to \Pi(N, \mathfrak{s})\}_{\tau \prec \sigma_\varepsilon, \varepsilon \in \text{Sign}_k}
\]
are obtained as in Subsection 4.2. We extend $N(\cdot)$ and $a(\cdot)$ to any vertices of $K(X, s)$. Let $(g^M, \mu) \in \Pi(M, s_0)$ be a generic point. If we need, we may arrange that the metric $g^M$ and the metric-component of each point in the image of $\varphi^N_{\tau, R^X(\tau)}$ for each $\tau < \sigma_e$ and $\varepsilon \in \text{Sign}^k$ are cylindrical near the boundaries of $B^4_M$ and of $B^4_N$, respectively. Then the map $\phi_{\tau}$ for each $\tau < \sigma_e$ and each $\varepsilon$ can be written as

$$\phi_{\tau}(x) = g^M \# \phi_{\tau}^N(x)$$

(49)

for each $x \in \text{dom}(\phi_{\tau}) = \text{dom}(\phi_{\tau}^N)$. We may also assume that $\mu$ and the self-dual 2-form-component of each point in the image of $\varphi^N_{\tau, R^X(\tau)}$ for each $\tau$ are supported on the complement of $B^4_M$ and of $B^4_N$, respectively; as in Subsection 4.1, we may define

$$\varphi_{\tau} : [0, R^X(\tau)]^\tau \to \Pi(X, s)$$

by

$$\varphi_{\tau}(x) := (g^M, \mu) \# \varphi^N_{\tau, R^X(\tau)}(x)$$

(50)

using long-stretched connected sum. We also take $\varphi_{\tau}$ for any vertices of $K(X, s)$ as an extension of $\varphi_{\tau}$ for functions $\tau$. Then we can use the tuple $A = (g, N(\cdot), a(\cdot), \varphi_{\tau})$ as an additional data to construct a cochain on $K(X, s)$ through the procedure in Subsection 3.1.

For each $\varepsilon \in \text{Sign}^k$, we set $R^N(\sigma_e) := R(\sigma_e, g^N, N(\cdot), a(\cdot))$ and

$$R_\Sigma := \max_{\varepsilon \in \text{Sign}^k} \{R^X(\sigma_e), R^N(\sigma_e)\}.$$  

The map $\varphi_{\sigma_e, R^X} : [0, R^X(\sigma_e)]^{\sigma_e} \to \hat{\Pi}(X, s)$ coincides with the composition

$$\phi_{\sigma_e} \circ \bigg( \bigcup_{R^X(\sigma_e) \leq R \leq R_\Sigma} \psi^{-1}_{\sigma_e, R} \bigg) : \bigcup_{R^X(\sigma_e) \leq R \leq R_\Sigma} \text{ext}\delta(\sigma_e, R) \to \hat{\Pi}(X, s)$$

(51)

on $\text{ext}\delta(\sigma_e, R^X(\sigma_e))$. By gluing $\varphi_{\sigma_e}$ and (51) together, we obtain a map from $[0, R_\Sigma]^{\sigma_e}$ to $\hat{\Pi}(X, s)$. We define $\varphi_{\sigma_e, R_\Sigma}$ as this glued map. Note that, by Proposition 2.12, there is no solution to the unperturbed Seiberg–Witten equations for any metric in the image of $\text{ext}\delta(\sigma_e, R)$ by $\phi_{\sigma_e} \circ \psi^{-1}_{\sigma_e, R}$ if $R \geq R^X(\sigma_e)$. We therefore have

$$\# M(\varphi_{\sigma_e}, s) = \# M(\varphi_{\sigma_e, R_\Sigma}, s).$$

(52)

We also note that, by the same argument to prove Lemma 3.2, there is no solution to the Seiberg–Witten equations for any element in $\varphi_{\sigma_e, R_\Sigma}(\partial([0, R_\Sigma]^{\sigma_e}))$. 


Let us define $\varphi^N_{\sigma_\varepsilon, R_\Sigma} : [0, R_\Sigma]^{\sigma_\varepsilon} \to \Pi(N, t)$ using the same way to define $\varphi_{\sigma_\varepsilon, R_\Sigma}$ above. By the definition of $\varphi_{\sigma_\varepsilon}$ and (49), we have

$$\varphi^N_{\sigma_\varepsilon, R_\Sigma}(x) = (g^M, \mu) \# \varphi^N_{\sigma_\varepsilon, R_\Sigma}(x)$$

for any $x \in [0, R_\Sigma]^{\sigma_\varepsilon}$. By applying the construction of $\varphi^N_{\Sigma, R}$ in Subsection 4.2 to $R = R_\Sigma$, we have the map $\varphi^N_{\Sigma, R} : D^k_{R_\Sigma} \to \Pi(N, t)$ from functions $\varphi^N_{\sigma_\varepsilon, R_\Sigma}$. Note that $\varphi^N_{\Sigma, R_\Sigma}$ satisfies that $\varphi^N_{\Sigma, R_\Sigma}(S^k_{R_\Sigma}) \subset \hat{\Pi}(N, t)$. In a similar vein, functions $\varphi_{\sigma_\varepsilon, R_\Sigma}$ induce the map $\varphi_{\Sigma, R_\Sigma} : D^k_{R_\Sigma} \to \hat{\Pi}(X, \mathcal{S})$. This map $\varphi_{\Sigma, R_\Sigma}$ is also written as the connected sum of $(g^M, \mu)$ and $\varphi^N_{\Sigma, R_\Sigma}$ by (53). Let $\pi : \bigsqcup_{\varepsilon \in \text{Sign}_k} [0, R_\Sigma]^{\sigma_\varepsilon} \to D^k_{R_\Sigma}$ be the quotient map. If we need, we replace $\varphi_{\Sigma, R_\Sigma}$ with an approximation of it to be smooth on the complement of $\pi(\bigsqcup_{\varepsilon \in \text{Sign}_k} \partial([0, R_\Sigma]^{\sigma_\varepsilon}))$. From the first note in Remark 4.4 and the argument above, it follows that there is no solution on the space defined by substituting $R_\Sigma$ for $R$ in (46). We therefore have

$$\#M(\varphi_{\Sigma, R_\Sigma}, \mathcal{S}) = \sum_{\varepsilon \in \text{Sign}_k} \#M(\varphi_{\sigma_\varepsilon, R_\Sigma}, \mathcal{S}).$$

From the definition of $\mathcal{W}(X, \mathcal{S})$, (52), and (54), we obtain

$$\langle \mathcal{W}(X, \mathcal{S}), [K(\Sigma)] \rangle = S\mathcal{W}(X, \mathcal{S}, \mathcal{A})([K(\Sigma)]) = \sum_{\varepsilon \in \text{Sign}_k} \#M(\varphi_{\sigma_\varepsilon}, \mathcal{S}) = \sum_{\varepsilon \in \text{Sign}_k} \#M(\varphi_{\sigma_\varepsilon, R_\Sigma}, \mathcal{S}) = \#M(\varphi_{\Sigma, R_\Sigma}, \mathcal{S}) = \pm(\varphi^N_{\Sigma, R_\Sigma} \cdot \mathcal{W}(N, t)) \cdot \text{SW}(M, \mathcal{S}_0) = \pm \text{SW}(M, \mathcal{S}_0),$$

where the fifth and the last equalities are deduced from Proposition 4.1 and Proposition 4.6, respectively. (The sign might change in the last equality.) This proves the theorem.

Remark 4.9. Note that the usual Seiberg–Witten invariant on $X = M \# N$ in Theorem 4.7 vanishes for any spin$^c$ structure on it since we have assumed that $b^+(M) \geq 2$ and $b^+(N) \geq 1$. On the other hand, the invariant $\mathcal{W}(X, \mathcal{S})$ does not vanish if $l \geq 4k$ holds and if we take $(M, \mathcal{S}_0)$ so that $\text{SW}(M, \mathcal{S}_0) \neq 0$. Indeed, we may give a wall-crossing collection of surfaces in $(N, t)$ with $l \geq 4k$. (See Example 4.22.)

To be concrete, we give the following non-vanishing of $\mathcal{W}(X, \mathcal{S})$:

Corollary 4.10. Let $k$ and $l$ be positive integers with $l \geq 4k$. Then there is a spin$^c$ structure $\mathcal{S}$ on $X = (k + 3)\mathbb{CP}^2 \# (l + 19)(-\mathbb{CP}^2)$. 

such that
\[ \mathcal{SW}(X, \mathcal{S}) \neq 0 \text{ in } H^{k-1}(\mathcal{K}(X, \mathcal{S}); \mathbb{Z}). \]

**Proof.** In Theorem 4.7, let us take \( (K3, \mathcal{S}_{\text{can}}) \) as \( (M, \mathcal{S}_0) \), where \( K3 \) is a K3 surface and \( \mathcal{S}_{\text{can}} \) is the canonical spin\(^c\) structure of \( K3 \). Then \( \mathcal{SW}(M, \mathcal{S}_0) \neq 0 \) holds and \( M \# N \) is diffeomorphic to \( X \) in the statement of this corollary [12, 13]. Let \( \mathcal{S} \) be the spin\(^c\) structure on \( X \) corresponding to \( \mathcal{S}_0 \# t \) via \( X \cong M \# N \). A wall-crossing collection of surfaces in \( (N, t) \) will be given in Example 4.22, and therefore we have \( \mathcal{SW}(X, \mathcal{S}) \neq 0 \). \( \square \)

**Remark 4.11.** The 4-manifold \( X \) in Corollary 4.10 admits a metric with positive scalar curvature, though \( \mathcal{SW}(X, \mathcal{S}) \) does not vanish. Note that, in contrast, both of the Seiberg–Witten invariant and its refinement called the Bauer–Furuta invariant [2] vanish on any 4-manifold admitting a positive scalar curvature metric.

**Remark 4.12.** More generally, if we take sufficiently large numbers as \( k \) and \( l \), and a simply connected 4-manifold as \( M \) in Theorem 4.7, by Wall’s result [23], \( X \) in Theorem 4.7 is diffeomorphic to \( k'\mathbb{C}\mathbb{P}^2 \# l'(-\mathbb{C}\mathbb{P}^2) \) for some \( k' \) and \( l' \).

We here interpret Theorem 4.7 from a classical point of view: Theorem 4.7 yields certain adjunction-type genus constraints on configurations of embedded surfaces. The most general formulation of the constraints can be described using the following concept, which we call **bounding collections of surfaces**. Henceforth we follow the notation \( (M, \mathcal{S}_0) \), \( (N, t) \), and \( (X, \mathcal{S}) \) given at the beginning of this subsection. Recall that if we give an orientation of \( \Sigma \) in the sense of Definition 4.3, then we can consider the fundamental chain/class of \( K(\Sigma) \). We also simply write the fundamental chain \( K(\Sigma) \).

**Definition 4.13.** Let \( \Sigma = \{ \Sigma^c \}_{1 \leq i \leq k, c \in \{+, -\}} \) be a wall-crossing collection of surfaces in \( (N, t) \). We call a finite subset \( S \subset V(\mathcal{K}(X)) \) a **bounding collection of surfaces in** \( X \) for \( \Sigma \) if there exist a finite set \( \Lambda \), an integer \( c_\lambda \), and a \( k \)-simplex \( \sigma_\lambda \) of \( \mathcal{K}(X) \) for each \( \lambda \in \Lambda \) such that:

1. \( \Sigma \subset \bigcup_{\lambda \in \Lambda} \sigma_\lambda \);
2. \( S = \bigcup_{\lambda \in \Lambda} \sigma_\lambda \setminus \Sigma \); and
3. \( \partial(\sum_{\lambda \in \Lambda} c_\lambda \sigma_\lambda) = K(\Sigma) \) holds in \( C_{k-1}(\mathcal{K}(X)) \) for some orientations of \( \sigma_\lambda \)'s and some orientation of \( \Sigma \) in the sense of Definition 4.3.

Here in (3) above \( K(\Sigma) \) denotes the fundamental chain and is regarded as an element in \( C_{k-1}(\mathcal{K}(X)) \) via the natural inclusion \( C_*(\mathcal{K}(X, \mathcal{S})) \hookrightarrow C_*(\mathcal{K}(X)). \)

**Example 4.14.** For any wall-crossing collection of surfaces \( \Sigma = \{ \Sigma_i^c \}_{1 \leq i \leq k, c \in \{+, -\}} \) in \( (N, t) \), there exists a bounding collection of surfaces in \( X \). (In particular, if we give a suitable orientation of \( \Sigma \),

\[ [K(\Sigma)] = 0 \text{ in } H_{k-1}(\mathcal{K}(X); \mathbb{Z}) \]

holds. Compare this equality in \( H_{k-1}(\mathcal{K}(X); \mathbb{Z}) \) with the non-vanishing result in \( H_{k-1}(\mathcal{K}(X, \mathcal{S}); \mathbb{Z}) \) in Theorem 4.7.) Indeed, let us take \( S \in V(\mathcal{K}(X)) \) with

\[ S \cap \Sigma_i^c = \emptyset \quad (\forall \epsilon \in \{+, -\}, \forall i \in \{1, \ldots, k\}). \]  

(55)
Then $\mathcal{S} = \{S\}$ is a bounding collection of surfaces. We will simply write the condition (55)

$$S \cap \Sigma = \emptyset.$$ 

To give $S$ with $S \cap \Sigma = \emptyset$ corresponds to considering a ‘cone’ of $K(\Sigma)$. Since $|K(\Sigma)| \cong S^{k-1}$, the cone of $K(\Sigma)$ is a $k$-dimensional disk and therefore $K(\Sigma)$ is homologically trivial. For example, in the case that $k = 2$, the bounding by $S$ of $\Sigma$ is described in the left picture in Figure 8.

**Example 4.15.** In fact, there are infinitely many distinct bounding collections of surfaces in $X$ for any wall-crossing collection of surfaces $\Sigma = \{\Sigma^\pm \}_{1 \leq i \leq k, \varepsilon \in \{+,-\}}$ in $(N, t)$. Indeed, there are infinitely many triangulations of a disk, which is regarded as the cone of $K(\Sigma)$. If we fix such a triangulation, roughly speaking, by assigning elements of $V(\mathcal{K}(X))$ to vertices of the triangulation, we have a bounding collection of surfaces $\mathcal{S}$ for $\Sigma$. For example, in the case that $k = 2$, let us consider the triangulation of the 2-dimensional disk described in the right picture in Figure 8. The corresponding bounding collection of surfaces is as follows: let us consider $S, S' \in V(\mathcal{K}(X))$ such that

$$S \cap \Sigma^\pm = S \cap \Sigma^+ = \emptyset, \quad S' \cap \Sigma^\pm = S' \cap \Sigma^- = \emptyset,$$

and $S \cap S' = \emptyset$. (56)

Then $\mathcal{S} = \{S, S'\}$ is a bounding collection of surfaces for $\Sigma$.

We now have the following adjunction-type genus constraints for any bounding collection of surfaces:

**Corollary 4.16.** Suppose that $SW(M, s_0) \neq 0$. Let $\Sigma = \{\Sigma^\pm \}_{1 \leq i \leq k, \varepsilon \in \{+,-\}}$ be a wall-crossing collection of surfaces in $(N, t)$. Assume that $\Sigma^+$ and $\Sigma^-$ intersect transversally for each $i$. Then, for any bounding collection of surfaces $\mathcal{S}$ in $X$ for $\Sigma$, there exists a surface $S \in \mathcal{S}$ such that the inequality

$$\chi^-(S) \geq |c_1(\mathcal{S}) \cdot [S]|$$

(57)

holds.

In particular, for $S \in V(\mathcal{K}(X))$ with $S \cap \Sigma = \emptyset$, the inequality (57) holds.

**Proof.** Assume that all elements in $\mathcal{S}$ were to violate the adjunction inequalities. Then we would have

$$\partial(\sum_{\lambda \in \Lambda} c_1(\sigma_\lambda)) = K(\Sigma)$$

(58)
for some $\Lambda, c_\Lambda, \sigma_\Lambda$, and some orientations of $\sigma_\lambda$ and of $K(\Sigma)$, where $\sigma_\Lambda$ is a simplex not only of $\mathcal{K}(X)$ but also of $K(X, \mathfrak{s})$. Hence equation (58) would imply that $[K(\Sigma)] = 0$ in $H_{k-1}(K(X, \mathfrak{s}); \mathbb{Z})$. This contradicts Theorem 4.7.

Remark 4.17. As we mentioned in Remark 4.9, the usual Seiberg–Witten invariant vanishes on $X$ for any spin$^c$ structure on it. Corollary 4.16 therefore gives the adjunction inequality for a spin$^c$ 4-manifold whose Seiberg–Witten invariant vanishes under some condition on geometric mutual intersections of embedded surfaces. This point is similar to Strle [21] and the author’s result [8].

On the other hand, from Example 4.15, Corollary 4.16 asserts that one non-trivial element in $H_*(\mathcal{K}(X, \mathfrak{s}); \mathbb{Z})$ provides constraints on infinitely many configurations of embedded surfaces. This is a phenomenon newly found through our use of the simplicial homology theory.

Remark 4.18. In the statement of Corollary 4.16, if an element of $\mathcal{S}$ is contained in $M$ before taking the connected sum with $N$, the adjunction inequality for the element follows from an argument of Kronheimer–Mrowka [9] since we have assumed that $\text{SW}(M, \mathfrak{s}_0) \neq 0$ in Corollary 4.16.

We can generalize the constraint on configurations given in Corollary 4.16 to surfaces with positive self-intersection number. To state it, let us introduce a variant of $\mathcal{K}$:

Definition 4.19. For a smooth oriented closed connected 4-manifold $X$, we define an abstract simplicial complex $\mathcal{K}_{\geq 0} = \mathcal{K}_{\geq 0}(X)$ as follows.

- The vertices of $\mathcal{K}_{\geq 0}$ are defined as smoothly embedded surfaces with non-negative self-intersection number.
- For $n \geq 1$, a collection of $(n + 1)$ vertices $\Sigma_0, \ldots, \Sigma_n \in V(\mathcal{K}_{\geq 0})$ spans an $n$-simplex if and only if $\Sigma_0, \ldots, \Sigma_n$ are mutually disjoint.

We now extend the concept of bounding collections of surfaces to finite subsets of $V(\mathcal{K}_{\geq 0})$ by the quite same way in Definition 4.13.

Definition 4.20. Let $\Sigma$ be a wall-crossing collection of surfaces in $(N, t)$. We call a finite subset $S \subset V(\mathcal{K}_{\geq 0}(X))$ a bounding collection of surfaces in $X$ for $\Sigma$ if the condition obtained by replacing $\mathcal{K}$ with $\mathcal{K}_{\geq 0}$ in the defining condition in Definition 4.13 is satisfied.

Using this extension of the concept of bounding collections of surfaces, we can generalize corollary 4.16 as follows:

Corollary 4.21. Suppose that $\text{SW}(M, \mathfrak{s}_0) \neq 0$. Let $\Sigma = \{\Sigma_i^\epsilon\}_{1 \leq i \leq k, \epsilon \in \{+, -\}}$ be a wall-crossing collection of surfaces in $(N, t)$. Assume that $\Sigma_i^+$ and $\Sigma_i^-$ intersect transversally for each $i$. Then, for any bounding collection of surfaces $S \subset V(\mathcal{K}_{\geq 0}(X))$ in $X$ for $\Sigma$, there exists a surface $S \in S$ such that the inequality

$$
\chi^-(S) \geq |c_1(\mathfrak{s}) \cdot [S]| + [S]^2
$$

holds.

In particular, for $S \in V(\mathcal{K}_{\geq 0}(X))$ with $S \cap \Sigma = \emptyset$, the inequality (59) holds.
Proof. Let us write $S = \{S_1, \ldots, S_m\}$ ($m > 0$) and set $l_i := [S_i]^2 \geq 0$ for each $i \in \{1, \ldots, m\}$. We also set $l_0 := 0$ and $l'_i := \sum_{a=0}^{i-1} l_a$ for each $i \in \{1, \ldots, m+1\}$. Consider the blowup

$$M' := M \# l_1(-\mathbb{CP}^2) \# \cdots \# l_m(-\mathbb{CP}^2).$$

Let $\mathcal{S}'_0$ be the blowup spin$^c$ structure on $M'$, whose first Chern class is given as

$$c_1(\mathcal{S}'_0) = c_1(\mathcal{S}_0) + \sum_{j=1}^{l'_{m+1}} E,$$

where $E$ is a generator of $H^2(-\mathbb{CP}^2) \cong H_2(-\mathbb{CP}^2)$. We set $(X', \mathcal{S}') := (M' \# N, \mathcal{S}'_0 \# \mathfrak{t})$. Let $S'_i$ ($1 \leq i \leq m$) be an embedded surface in $X'$ which is obtained from the connected sum of $S_i$ and exceptional curves in $l_i(-\mathbb{CP}^2)$ and whose homology class is given as

$$[S'_i] = [S_i] - \sum_{j=l'_{i+1}}^{l'_{i+1}} E$$

if $c_1(\mathcal{S}) \cdot [S_i] \geq 0$, and

$$[S'_i] = [S_i] + \sum_{j=l'_{i+1}}^{l'_{i+1}} E$$

if $c_1(\mathcal{S}) \cdot [S_i] < 0$. Set $S' := \{S'_0, \ldots, S'_m\}$. Then $S' \subset V(\mathcal{K}(X'))$ holds and $S'$ is a bounding collection of surfaces in $X'$ for $\Sigma$. Since the Seiberg–Witten invariant of $(M, \mathcal{S}_0)$ coincides with that of the blowup $(M', \mathcal{S}'_0)$ by the blowup formula, we can apply Corollary 4.16 for the new bounding collection of surfaces $S'$. Then we have the adjunction inequality

$$\chi^{-}(S'_i) \geq |c_1(\mathcal{S}') \cdot [S'_i]|$$

for some $i$. If $c_1(\mathcal{S}) \cdot [S_i] \geq 0$, we have

$$\chi^{-}(S_i) = \chi^{-}(S'_i) \geq |c_1(\mathcal{S}') \cdot [S'_i]| = |c_1(\mathcal{S}) \cdot [S_i] + [S_i]^2| = |c_1(\mathcal{S}) \cdot [S_i]| + [S_i]^2,$$

and if $c_1(\mathcal{S}) \cdot [S_i] < 0$, we have

$$\chi^{-}(S_i) = \chi^{-}(S'_i) \geq |c_1(\mathcal{S}') \cdot [S'_i]| = |c_1(\mathcal{S}) \cdot [S_i] - [S_i]^2| = |c_1(\mathcal{S}) \cdot [S_i]| + [S_i]^2.$$

Hence the inequality (59) holds for some $S \in S$. □

4.4 Example of wall-crossing collections of surfaces

We give some examples of wall-crossing collections of surfaces to complete the proof of Corollary 4.10 and to understand Corollary 4.21 concretely.
Example 4.22. Let $k, l_i, d_{+,i}$, and $d_{-,i}$ $(1 \leq i \leq k)$ be positive integers satisfying

$$l_i \geq d_{+,i}^2 \geq 4$$

for each $\epsilon \in \{+,-\}$ and $i \in \{1, \ldots, k\}$. We set $l := \sum_{i=1}^{k} l_i$, $l_0 := 0$, and $l_i' := \sum_{a=0}^{i-1} l_a$ for each $i \in \{1, \ldots, k\}$. Let us consider $N = kCP^2 \# l(-CP^2) = \#_{i=1}^{k} CP^2 \# \left( \#_{j=1}^{l} (-CP^2) \right)$ and let $H_i$, $E$, and $t$ be the one in Subsection 4.1. Henceforth we identify $H_2(CP^2_i)$ with $H_2(CP^2_i)$ by the Poincaré duality and similarly $H_2(-CP^2)$ with $H_2(-CP^2)$. Let $\Sigma^+_i, \Sigma^-_i$ $(1 \leq i \leq k)$ be embedded surfaces in $N$ satisfying (41) and the following two conditions for each $i \in \{1, \ldots, k\}$:

$$[\Sigma^\pm_i] = d_{\pm,i} H_i \pm \sum_{j=l_i'+1}^{l_i'+d_{\pm,i}^2} E,$$  

$$g(\Sigma^\pm_i) = \frac{(d_{\pm,i} - 1)(d_{\pm,i} - 2)}{2}.$$  

Here double-signs $\pm$ correspond in the both sides of (61) and (62). (One can make such surfaces, for example, using connected sum in $N$ of a smooth algebraic curve of degree $d_{\pm,i}$ in $CP^2_i$ $(1 \leq i \leq k)$ and the exceptional curves in functions $-CP^2$. In this section, if we say an algebraic curve it means a smooth algebraic curve.) It is easy to check that $\Sigma^\epsilon_i \in V(K(N, t))$ holds for each $\epsilon \in \{+, -, \}$, $i \in \{1, \ldots, k\}$ and that the collection $\Sigma = \{\Sigma^\epsilon_i \}_{1 \leq i \leq k, \epsilon \in \{+, -\}}$ is a wall-crossing collection of surfaces in $(N, t)$.

From Corollary 4.21, we therefore have the following constraint on configurations. Let $(M, \mathcal{S}_0)$ be a smooth oriented closed spin$^c$ 4-manifold with $b^+(M) \geq 2$, $d(\mathcal{S}_0) = 0$, and $\text{SW}(M, \mathcal{S}_0) \neq 0$. Set $(X, \mathcal{S}) := (M \# N, \mathcal{S}_0 \# t)$. Let $\mathcal{S} \subset V(K_{\geq 0}(X))$ be a bounding collection of surfaces in $X$ for $\Sigma$. (As we mentioned in Example 4.15, there are infinitely many bounding collections of surfaces for $\Sigma$.) Then the inequality

$$\chi^-(S) \geq |c_1(\mathcal{S}) \cdot [S]| + [S]^2$$

holds for some surface $S \in \mathcal{S}$.

Similarly one can easily make other examples of wall-crossing collections of surfaces. For example, the signs of functions $E$ in (61) have some freedom for many $d_{\pm,i}$. We also give other examples as follows.

Example 4.23. Let $k, l_i, d_{+,i}$, and $d_{-,i}$ $(1 \leq i \leq k)$ be positive integers satisfying $l_i \geq d_{\epsilon,i}^2 - 3$ and $d_{\epsilon,i} \geq 3$ for each $\epsilon \in \{+,-\}$ and $i \in \{1, \ldots, k\}$. Let $l_0, l'_i$ and $(N, t)$ be the one in Example 4.22. Let $\Sigma^+_i, \Sigma^-_i$ $(1 \leq i \leq k)$ be embedded surfaces in $N$ satisfying (41), (62), and

$$[\Sigma^\pm_i] = d_{\pm,i} H_i + 2E_{l_i'+1} \pm \sum_{j=l_i'+2}^{l_i'+d_{\pm,i}^2-3} E.$$
for each $i \in \{1, \ldots, k\}$. Since $2E_{l'+1}$ can be represented by a sphere in $\mathbb{CP}^2$, one can make such surfaces $\Sigma^+_i$ and $\Sigma^-_i$ as the connected sum of an algebraic curve of degree $d_{\pm,i}$ in $\mathbb{CP}^2$, the sphere in $\mathbb{CP}^2$, and the exceptional curves in $-\mathbb{CP}^2$ with $l' + 2 \leq j \leq l' + d^2_{\pm,i} - 3$. It is easy to check that $\Sigma^\epsilon_i \in V(K(N, t))$ holds for each $\epsilon, i$ and that the collection $\Sigma = \{\Sigma^\epsilon_i\}_{1 \leq i \leq k, \epsilon \in \{+,-\}}$ is a wall-crossing collection of surfaces in $(N, t)$.

**Example 4.24.** Let $k, l, d_{+i},$ and $d_{-i}$ ($1 \leq i \leq k$) be positive integers satisfying $d_{-i} \in \{2, 3\}$ for each $i \in \{1, \ldots, k\}$ and (60) for each $\epsilon \in \{+, -\}$ and $i$. Let $l, l_0, l'$ and $(N, t)$ be the one in Example 4.22. For each $i \in \{1, \ldots, k\}$, let $\Sigma^+_i$ be an embedded surface in $N$ satisfying the $+\text{-part}$ of the conditions (61) and (62), and $\Sigma^-_i$ be an embedded surface in $N$ satisfying

$$[\Sigma^-_i] = 2d_{-i}H_i - 2 \sum_{j=l'+1}^{l'_i+d^2_{-i}} E_j, \quad \text{and}$$

$$g(\Sigma^-_i) = \frac{(2d_{-i} - 1)(2d_{-i} - 2)}{2}.$$

We also assume that $\Sigma^+_i$ and $\Sigma^-_i$ satisfy (41). One can make such surfaces $\Sigma^+_i$ and $\Sigma^-_i$ as the connected sum of an algebraic curve in $\mathbb{CP}^2$ and spheres in functions $-\mathbb{CP}^2$. It is easy to check that $\Sigma^\epsilon_i \in V(K(N, t))$ holds for each $\epsilon, i$ and that the collection $\Sigma = \{\Sigma^\epsilon_i\}_{1 \leq i \leq k, \epsilon \in \{+,-\}}$ is a wall-crossing collection of surfaces in $(N, t)$.

**Remark 4.25.** We note that the consequences of Corollary 4.21 in Examples 4.22, 4.23, 4.24 are generalizations of the following result obtained from the blowup formula. For simplicity we here focus our attention on the case of Example 4.22. Consider the wall-crossing collection of surfaces $\Sigma$ in Example 4.22 obtained from the connected sum of algebraic curves and the exceptional curves. Suppose that there exists $\epsilon_i \in \{+, -\}$ such that $d_{\epsilon_i,i} = 2$ for each $i \in \{1, \ldots, k\}$. Let $S$ be a bounding collection of surfaces. Assume that $S \cap \Sigma^0_i = \emptyset$ ($1 \leq i \leq k$) holds for some surface $S \in S$. For example, this assumption is satisfied in the both case that $S$ consists of a single surface and that $S$ consists of two surfaces as in Example 4.15 and $k = 2$. Then the adjunction inequality for $S$ follows from a surgery argument and the blowup formula for the usual Seiberg–Witten invariants. (This note on the blowup formula is due to Kouichi Yasui.) The surgery argument, however, does not work in the case that $d_{\epsilon,i} > 2$ holds for any $i \in \{1, \ldots, k\}$, and therefore the constraint obtained from Corollary 4.21 on elements in $S$ cannot be shown using the blowup formula in this case.

More precisely, the surgery argument above is used to construct a negative definite closed 4-manifold from $N$. In the case that $d_{\epsilon,i} > 2$ holds for any $i \in \{1, \ldots, k\}$, a closed 4-manifold cannot be obtained from usual surgeries since they can be used only for spheres. One cannot therefore use the usual blowup formula in this case. Nevertheless, we can use the combination of wall-crossing and gluing arguments due to Ruberman [17–19]; information about the Seiberg–Witten equations on $M$ survives on the connected sum $X = M \# N$. This suggests a possibility that one can extend the notion of 'negative definite closed 4-manifolds' from the gauge theoretic point of view: in our case, the complement of neighborhoods of suitable embedded surfaces in $N$ behaves as if it were a negative definite closed 4-manifold.
4.5 Application to bounds on the complexity of configurations of surfaces

In this subsection, we give a concrete application of Corollary 4.21 to give bounds on the complexity of configurations of surfaces. The basic idea of this subsection is due to Strle [21], in particular, [21, Section 15].

Let \((M, \mathfrak{s}_0)\) be a smooth oriented closed spin\(^c\) 4-manifold with \(b^+(M) \geq 2\) and \(d(\mathfrak{s}_0) = 0\), and \(k, l\) be natural numbers with \(l \geq k > 0\). We set \(N = k\mathbb{CP}^2 \# l(\mathbb{CP}^2)\) and \((X, \mathfrak{s}) = (M \# N, \mathfrak{s}_0 \# \mathfrak{t})\), where \(\mathfrak{t}\) is the spin\(^c\) structure defined by (38) on \(N\).

First, we give a bound on the complexity in the case that a bounding collection of surfaces consists of a single surface. Recall that, for embedded surfaces \(S, \Sigma\) in \(X\), if \([S] \cdot [\Sigma] = 0\) and \(S\) intersects transversally with \(\Sigma\), geometric intersection points between \(S\) and \(\Sigma\) appears as pairs of \(\pm 1\)-intersection points. Denote by \(I(S, \Sigma)\) the number of the pairs of \(\pm 1\)-intersection points between \(S\) and \(\Sigma\). For several embedded surfaces \(S_1, \ldots, S_n\), we say that \(S_1, \ldots, S_n\) are in general position if:

- for each \(i, j \in \{1, \ldots, n\}\) with \(i \neq j\), either \(S_i\) intersects with \(S_j\) transversally or \(S_i \cap S_j = \emptyset\) holds; and
- for each distinct three numbers \(i, j, k \in \{1, \ldots, n\}\), \(S_i \cap S_j \cap S_k = \emptyset\) holds.

**Theorem 4.26.** Suppose that \(\text{SW}(M, \mathfrak{s}_0) \neq 0\). Let \(\Sigma = \{\Sigma^\varepsilon_i\}_{1 \leq i \leq k, \varepsilon \in \{+, -\}}\) be a wall-crossing collection of surfaces in \((N, \mathfrak{t})\). Let \(S \in \mathcal{V}((K_{\geq 0}(X)))\) be a surface satisfying that \([S] \cdot [\Sigma^\varepsilon_i] = 0\) and that \(S\) and \(\Sigma^\varepsilon_i\) \((\varepsilon \in \{+, -\}, i \in \{1, \ldots, k\})\) are in general position. Then we have

\[
\max\{g(S) + I(S; \Sigma) - 1, 0\} \geq \frac{|c_1(\mathfrak{s}) \cdot [S]| + [S]^2}{2},
\]

where

\[
I(S; \Sigma) := \sum_{1 \leq i \leq k, \varepsilon \in \{+, -\}} I(S, \Sigma^\varepsilon_i).
\]

**Proof.** First, we recall a procedure replacing a pair of \(\pm 1\)-intersection points with increasing the genus by 1. For a pair \(p = \{p_+, p_-\}\) of \(\pm 1\)-intersection points between \(S\) and \(\Sigma^\varepsilon_i\) for some \((i, \varepsilon)\), we may construct a 3-dimensional 1-handle \(h^1_p\) as follows. Take a path \(\gamma_p : [0, 1] \to \Sigma^\varepsilon_i\) on \(\Sigma^\varepsilon_i\) from \(p_+\) to \(p_-\) so that \(\gamma_p([0, 1])\) does not intersect with \(\Sigma^\varepsilon_i'\) for all \((i', \varepsilon')\) with \((i', \varepsilon') \neq (i, \varepsilon)\). This condition can be achieved since the definition of general position implies that \(S \cap \Sigma^\varepsilon_i \cap \Sigma^\varepsilon_i' = \emptyset\) if \((i, \varepsilon) \neq (i', \varepsilon')\). Fattening \(\gamma_p\) to an extra 2-dimension in the normal direction in \(X\), we obtain a 3-dimensional 1-handle \(h^1_p \cong D^1 \times D^2\). More precisely, \(h^1_p\) is defined to be (the total space of) the restriction of the normal disk bundle \(D(\nu(\Sigma^\varepsilon_i))\) of \(\Sigma^\varepsilon_i\) in \(X\) to the arc \(\gamma_p\). Eliminating neighborhoods \(D^2_+\), \(D^2_-\) of \(p_+, p_-\) in \(S\) and attaching the part of \(\partial h^1_p\) which corresponds to \(D^1 \times \partial D^2\), we obtain a surface \(S_p\) satisfying \(g(S_p) = g(S) + 1\). In other words, \(S_p\) is the sum of \(S \setminus (D^2_+ \cup D^2_-)\) with the normal sphere bundle \(S(\nu(\Sigma^\varepsilon_i))\) restricted on the arc \(\gamma_p\). Then \(S_p\) does not intersect with \(\Sigma^\varepsilon_i\) anymore. Taking \(h^1_p\) thin enough, we may assume that \([S_p] = [S]\) and

\[
h^1_p \cap \Sigma^\varepsilon_i' = \emptyset
\] (63)

for all \((i', \varepsilon')\) with \((i', \varepsilon') \neq (i, \varepsilon)\).
Let \( \hat{S} \) be the surface obtained by doing the above procedure to all pairs of \( \pm 1 \)-intersection points between \( S \) and all \( \Sigma_i^\epsilon \). Here, taking the 1-handles in the construction thin enough, we may assume that all of these 1-handles do not intersect with each other. Thus we may assume that \( \hat{S} \) is an embedded surface, not immersed. Note that \( \hat{S} \) satisfies \([\hat{S}] = [S]\) and \(g(\hat{S}) = g(S) + I(S; \Sigma)\) by construction. By condition (63), \( \hat{S} \) does not intersect with any \( \Sigma_i^\epsilon \), namely \( \hat{S} \cap \Sigma = \emptyset \). Because of the general position assumption, \( \Sigma_i^+ \) and \( \Sigma_i^- \) intersect transversally for each \( i \). Therefore, from Corollary 4.21, we obtain that

\[
2 \max\{g(S) + I(S; \Sigma) - 1, 0\} = \chi(\hat{S}) \geq |c_1(\hat{\theta}) \cdot [\hat{S}]] + [\hat{S}]^2 = |c_1(\hat{\theta}) \cdot [S]| + [S]^2.
\]

This proves the theorem. \( \square \)

Example 4.27. Let \( M = K3 \) be the underlying smooth 4-manifold of a \( K3 \) surface. Let \( N = k\mathbb{CP}^2 # l(-\mathbb{CP}^2) = \#_{i=1}^k \mathbb{CP}^2 \#_{j=1}^l (-\mathbb{CP}^2) \), where \( k, l \) are constrained as in Example 4.22. Let \( t \) be the spin\(^c\) structure on \( N \) determined by (38). Let \( \Sigma = \{ \Sigma_i^\epsilon \}_{1 \leq i \leq k, \epsilon \in \{+, -\}} \) be the wall-crossing collection of surfaces in \((N, t)\) given in Example 4.22. Set \( X = M \# N \). Recall that the intersection form of \( M \) can be written as \( 2(-E_8) \oplus 3(011^0) \) with respect to a suitable basis of \( H_2(M; \mathbb{Z}) \). Let

\[
\xi = \left( 0_{2(-E_8)}, (p_i, q_i)_{i=1}^3, 0_N \right)
\]

be a homology class in \( H_2(X; \mathbb{Z}) = H_2(M; \mathbb{Z}) \oplus H_2(N; \mathbb{Z}) \) which is trivial except over the part of the basis corresponding to \( 3(011^0) \). Assume that \( p_i, q_i > 0 \). Then we have \( \xi^2 = 2 \sum p_i q_i > 0 \). Let \( S \) be a smooth representative of \( \xi \) such that \( S \) and \( \Sigma_i^\epsilon \) (\( \epsilon \in \{+,-\}, i \in \{1, \ldots, k\} \)) are in general position. Then we have

\[
g(S) + I(S; \Sigma) \geq \sum_{i=1}^3 p_i q_i + 1. \tag{64}
\]

To deduce (64), let us choose the canonical spin\(^c\) structure on \( M \) as \( \hat{\theta}_0 \). Then we have \( \text{SW}(M, \hat{\theta}_0) \neq 0 \). It follows that \( c_1(\hat{\theta}_0) \cdot [S] = 0 \) since \( c_1(\hat{\theta}_0) = 0 \). Thus we obtain (64) from Theorem 4.26.

Since \( \mathbb{CP}^2 \# 2(-\mathbb{CP}^2) \cong S^2 \times S^2 \# (-\mathbb{CP}^2) \), by [22, Theorem 3] by Wall, there exists a representative \( S \) of \( \xi \) with \( g(S) = 0 \) provided that \( \xi \) is primitive. Note that, because of the adjunction inequality for \( M = K3 \), we cannot take such \( S \) inside the direct summand \( M \) in \( X \). The inequality (64) implies that \( S \) with \( g(S) = 0 \) must have intersection points with functions \( \Sigma_i^\epsilon \) whose number is estimated by

\[
I(S; \Sigma) \geq \sum_{i=1}^3 p_i q_i + 1.
\]

Next, we shall give an example of a bounding collection of surfaces consisting of two surfaces. To make the following discussion simple, let us consider \( N = 2\mathbb{CP}^2 \# K(-\mathbb{CP}^2) \) and the configuration given in the right picture in Figure 8. For a wall-crossing collection of surfaces \( \Sigma \), a surface \( \Sigma \in \Sigma \), and another embedded surface \( S \), if \( [S] \cdot [\Sigma'] = 0 \) for all \( \Sigma' \in \Sigma \setminus \{\Sigma\} \), let us define

\[
I(S; \Sigma \setminus \{\Sigma\}) := \sum_{\Sigma' \in \Sigma \setminus \{\Sigma\}} I(S, \Sigma').
\]
Theorem 4.28. Suppose that $\text{SW}(M, g_0) \neq 0$. Let $\Sigma = \{\Sigma^\pm_1\}_{1 \leq i \leq 2, \epsilon \in \{+, -\}}$ be a wall-crossing collection of surfaces in $(N, t) = (2\mathbb{C}\mathbb{P}^2 \# l(-\mathbb{C}\mathbb{P}^2), t)$. Let $S, S' \in V(K_{g_0}(X))$ be surfaces satisfying that

$$[S] \cdot [\Sigma^\pm_2] = [S] \cdot [\Sigma^+_2] = 0, [S'] \cdot [\Sigma^\pm_2] = [S'] \cdot [\Sigma^-_2] = 0,$$

and that $S, S', \Sigma^+_2, \Sigma^-_2$ are in general position. Assume that we have

$$\chi^-(S) + 2I(S; \Sigma \setminus \{\Sigma^+_1\}) < |c_1(\mathcal{S}) \cdot [S]| + [S]^2,$$  \hspace{1cm} (65)

$$\chi^-(S') + 2I(S; \Sigma \setminus \{\Sigma^+_1\}) < |c_1(\mathcal{S}) \cdot [S']| + [S']^2.$$  \hspace{1cm} (66)

Then the inequality

$$\min \left\{ \max\{g(S) - 1, 0\} + g(S'), \max\{g(S') - 1, 0\} + g(S) \right\} + I(S, S') \hspace{1cm} (67)$$

$$\geq \frac{|c_1(\mathcal{S}) \cdot [S]| + [S]^2 + |c_1(\mathcal{S}) \cdot [S']| + [S']^2}{2}$$

holds.

Proof. Let $\tilde{S}$ be a surface obtained from $S$ so that all pairs of $\pm 1$-intersection points with $\Sigma^+_1, \Sigma^\pm_2$ are replaced with 1-handles via the procedure in the first paragraph in the proof of Theorem 4.26. Similarly, let $\tilde{S}'$ be a surface obtained from $S'$ so that all intersection points with $\Sigma^-_1, \Sigma^\pm_2$ are resolved. Then we have

$$g(\tilde{S}) = g(S) + I(S; \Sigma \setminus \{\Sigma^-_1\})$$

and

$$g(\tilde{S}') = g(S') + I(S'; \Sigma \setminus \{\Sigma^+_1\}).$$

Therefore, by the assumptions (65) and (66), both of $\tilde{S}$ and $\tilde{S}'$ violate the adjunction inequalities, namely

$$\chi^-(\tilde{S}) < |c_1(\mathcal{S}) \cdot [S]| + [S]^2,$$

$$\chi^-(\tilde{S}') < |c_1(\mathcal{S}) \cdot [S']| + [S']^2.$$

Now we argue precisely as in the proof of [21, Theorem 15.1] First, note that $\tilde{S}$ and $\tilde{S}'$ have intersection points. This is deduced as follows. If $\tilde{S}$ and $\tilde{S}'$ are disjoint, since we have made these surfaces so that there do not exist intersection points with functions $\Sigma^\epsilon_i$, the configuration of $\tilde{S}, \tilde{S}'$, and functions $\Sigma^\epsilon_i$ are as in the right picture in Figure 8. Then, as noted in Example 4.15, $\{\tilde{S}, \tilde{S}'\}$ is a bounding collection of surfaces for $\Sigma$. Since both of $\tilde{S}$ and $\tilde{S}'$ violate the adjunction inequalities, by Corollary 4.21, $\tilde{S}$ and $\tilde{S}'$ must intersect.

Let us attach a 1-handle $h^1_1$ to $S$ to eliminate one pair of $\pm 1$-intersection points between $S$ and $S'$. If the resulting surface $\tilde{S} \cup h^1_1$ still violates the adjunction inequality, $\tilde{S} \cup h^1_1$ and $\tilde{S}'$ have intersection points by Corollary 4.21 again, and therefore we may repeat this procedure and obtain a
new surface $\hat{S} \cup h_1^1 \cup h_2^1$. Continuing this procedure until when the resulting surface satisfies the adjunction inequality, we deduce that there exists uniquely $k \in \{1, \ldots, I(S, S')\}$ such that
\[
\chi^-(\hat{S} \cup h_1^1 \cup \cdots \cup h_k^1) \geq |c_1(\xi) \cdot [S]| + [S]^2, \tag{68}
\]
\[
\chi^-(\hat{S} \cup h_1^1 \cup \cdots \cup h_{k-1}^1) < |c_1(\xi) \cdot [S]| + [S]^2. \tag{69}
\]
Define $\tilde{S} := \hat{S} \cup h_1^1 \cup \cdots \cup h_{k-1}^1$, and define $\tilde{S}'$ as the surface obtained by attaching all the remaining handles to eliminate all the remaining intersection points: $\tilde{S}' = \hat{S}' \cup h_k^1 \cup \cdots \cup h_{k-1}^1$. Note that $\tilde{S}$ satisfies
\[
\chi^-(\tilde{S}) \geq |c_1(\xi) \cdot [S]| + [S]^2 - 2 \tag{70}
\]
because of (68). By (69) and Corollary 4.21, $\tilde{S}'$ satisfies the adjunction inequality:
\[
\chi^-(\tilde{S}') \geq |c_1(\xi) \cdot [S']| + [S']^2.
\]
This inequality and the inequality (70) imply
\[
\chi^-(\tilde{S}) + \chi^-(\tilde{S}') \geq |c_1(\xi) \cdot [S]| + [S]^2 + |c_1(\xi) \cdot [S']| + [S']^2 - 2. \tag{71}
\]
Let us calculate the left-hand side of this inequality. First, since $g(S') - 1 + I(S, S') - (k - 1) \geq 0$, we have that
\[
\frac{\chi^-(\tilde{S}) + \chi^-(\tilde{S}')}{2} = \max\{g(S) - 1 + k - 1, 0\} + g(S') - 1 + I(S, S') - (k - 1)
\]
\[
= \max\{g(S) - 1, -(k - 1)\} + g(S') - 1 + I(S, S')
\]
\[
\leq \max\{g(S) - 1, 0\} + g(S') - 1 + I(S, S'). \tag{72}
\]
It follows from the inequalities (71) and (72) that
\[
\max\{g(S) - 1, 0\} + g(S') + I(S, S') \geq \frac{|c_1(\xi) \cdot [S]| + [S]^2 + |c_1(\xi) \cdot [S']| + [S']^2}{2}. \tag{73}
\]
Repeating the argument in the last paragraph under swapping the roles of $S$ and $S'$, we obtain
\[
\max\{g(S') - 1, 0\} + g(S) + I(S, S') \geq \frac{|c_1(\xi) \cdot [S]| + [S]^2 + |c_1(\xi) \cdot [S']| + [S']^2}{2}. \tag{74}
\]
Then the desired inequality (67) is obtained combining (73) and (74).

\textbf{Example 4.29.} As well as Example 4.27, let $M = K3$ be the underlying smooth 4-manifold of a $K3$ surface. Let $N_1$ be a copy of $\C P^2 \# l_1(-\C P^2)$, where $l_1$ is constrained as in Example 4.22. Let $t_1$ be the spin$^c$ structure on $N_1$ determined by (38) for $k = 1$. Let $\Sigma_t = \{\Sigma^t_+, \Sigma^-\}$ be the wall-crossing
collection of surfaces in $(N_1, t_1)$ given in Example 4.22. As well as Example 4.27, let

$$\xi = \left(0_{2(0,0,0,0,0)}, 0_{N_1}, (p_1, q_1, 0, 0, 0, 0)_{N_1}\right),$$

$$\xi' = \left(0_{2(0,0,0,0,0)}, 0_{N_1}, (0, 0, p_2, q_2, 0, 0)_{N_1}\right)$$

be homology classes in $H_2(M \# N_1; \mathbb{Z})$ which are trivial except over the part of the basis corresponding to the first and second direct summands of $3(0,0,0,0,0)_{N_1}$, respectively. Assume that $p_1, q_1 > 0$ and that $\xi$ and $\xi'$ are primitive. Applying Theorem 3 in [22] by Wall to $M \# N_1 \cong M \# (S^2 \times S^2) \# (-\mathbb{C}\mathbb{P}^2)$, we may take representatives $S$ and $S'$ in $M \# N_1$ of $\xi$ and $\xi'$ so that $g(S) = g(S') = 0$. Assume that $S, S', \Sigma^+_1, \Sigma^-_1$ are in general position. We shall deduce from Theorem 4.28 that at least one of the following three inequalities holds:

$$\begin{align*}
I(S, \Sigma^+_1) &\geq 2p_1q_1, \\
I(S', \Sigma^-_1) &\geq 2p_2q_2, \\
I(S, S') &\geq p_1q_1 + p_2q_2.
\end{align*} \tag{75}$$

Before starting to deduce this constraint, it is interesting to compare it with the result obtained in Example 4.27. Example 4.27 ensures that both of the following two inequalities hold:

$$\begin{align*}
I(S; \{\Sigma^+_1, \Sigma^-_1\}) &\geq p_1q_1 + 1, \\
I(S'; \{\Sigma^+_1, \Sigma^-_1\}) &\geq p_2q_2 + 1.
\end{align*} \tag{76}$$

Obviously the condition that both of the inequalities in (76) holds does not imply the condition that at least one of the inequalities in (75) holds, and vice versa. These two constraints are obtained from Theorem 4.26 and Theorem 4.28, respectively, and the main difference between these two theorems are choices of bounding collections of surfaces: Theorem 4.26 involves a bounding collection consisting of a single surface, and Theorem 4.28 involves a bounding collection consisting of two surfaces. Therefore, these two different constraints exhibit the freedom of choice of bounding collections provides several different geometric constraints.

Now let us deduce that at least one of the inequalities in (75) holds. To see this, it is enough to show that we have the last inequality in (75) provided that both of

$$\begin{align*}
I(S, \Sigma^+_1) &< 2p_1q_1, \\
I(S', \Sigma^-_1) &< 2p_2q_2.
\end{align*} \tag{77}$$

hold. Let $N_2$ be a copy of $\mathbb{C}\mathbb{P}^2 \# l_2(-\mathbb{C}\mathbb{P}^2)$, where $l_2$ is constrained as in Example 4.22. Set $N = N_1 \# N_2$ and $X = M \# N$. Let $t$ be the spin$^c$ structure on $N$ determined by (38) for $k = 2$, and $\mathfrak{s}_0$ be the canonical spin$^c$ structure on $M$, that is, the spin$^c$ structure coming from the unique spin structure on $M$. Adding new surfaces $\{\Sigma^+_2, \Sigma^-_2\}$ in $N_2$ to $\Sigma_1$, we obtain a wall-crossing collection of surfaces $\Sigma = \{\Sigma^+_1, \Sigma^-_1, \Sigma^+_2, \Sigma^-_2\}$ in $(N, t)$ as in Example 4.22. Since $S, S'$ are inside $M \# N_1$, they do not intersect with $\Sigma^+_2$. Therefore, we have

$$I(S, \Sigma^+_1) = I(S; \Sigma \setminus \{\Sigma^-_1\}), \quad I(S', \Sigma^-_1) = I(S'; \Sigma \setminus \{\Sigma^+_1\}).$$
This and (77) imply that $S$ and $S'$ satisfy the all assumptions of Theorem 4.28. Therefore, it follows from Theorem 4.28 that the last inequality in (75) holds.

5 | CONCLUDING REMARKS

Finally, we remark some further potential developments of the invariant $\mathbb{S}\mathbb{W}(X, \$).

Remark 5.1. For a closed spin$^c$ 4-manifold $(X, \$)$, one can define a subcomplex $\mathcal{K}_{\geq 0}(X, \$)$ of $\mathcal{K}_{\geq 0}(X)$ by considering all embedded surfaces violating the adjunction inequalities with respect to $\$ and having non-negative self-intersection number. The author expects that we can construct an invariant which is formulated as a cohomology class on $\mathcal{K}_{\geq 0}(X, \$)$ in a similar vein of the construction of $\mathbb{S}\mathbb{W}(X, \$)$. There are two differences between $\mathcal{K}_{\geq 0}(X, \$)$ and $\mathcal{K}(X, \$)$ from the point of view of the construction of a cohomological invariant. The first one is on the stretching construction of families of Riemannian metrics in Subsection 2.3, and the second is on the quantitative estimate given as Proposition 2.12 for the length of stretched cylinders to assure the non-existence of solutions to the Seiberg–Witten equations. For the first difference, the stretching argument can be replaced with the stretching of metrics for the fiber-direction of the normal bundles of embedded surfaces in general. However, for the second difference, the author does not know any quantitative estimate as in Proposition 2.12 for surfaces of positive self-intersection number at this stage.

Remark 5.2. Bauer–Furuta [2] have given a refinement, called the Bauer–Furuta invariant, of the usual Seiberg–Witten invariant. This is defined as the stable cohomotopy class of a finite-dimensional approximation of the Seiberg–Witten equations. The author expects that we may define such a refinement of our invariant $\mathbb{S}\mathbb{W}(X, \$)$. If we can define such an invariant, we can expect a generalization of Theorem 4.7: we might replace $(M, \$, \$_0)$ in Theorem 4.7 with a spin$^c$ 4-manifold whose Bauer–Furuta invariant does not vanish. Since there are spin$^c$ 4-manifolds such that their Bauer–Furuta invariants do not vanish but their Seiberg–Witten invariants do vanish, this expected non-vanishing theorem will provide new constraints on configurations of embedded surfaces as in Corollaries 4.16, 4.21.

Remark 5.3. We have considered only closed 4-manifolds in this paper. However, in general, one can expect that a gauge theoretical invariant of closed 4-manifolds is generalized to an invariant of 4-manifolds with boundary using the Floer homology theory. The author conjectures that our invariant $\mathbb{S}\mathbb{W}(X, \$)$ can be also generalized for 4-manifolds with boundary. The first thing we have to do is to define the complex of surfaces for 4-manifolds with boundary; two candidates for it might be considered. The first one is the simplicial complex obtained by considering closed surfaces embedded into the interior of the given 4-manifold, and the second is that obtained by considering surfaces with boundaries embedded into the given 4-manifold such that the boundary of each surface is in that of the 4-manifold. The author expects that $\mathbb{S}\mathbb{W}(X, \$)$ can be generalized to a cohomology class at least on the first candidate. On the other hand, he also expects that, if $\mathbb{S}\mathbb{W}(X, \$)$ can be generalized to the second candidate, it might relate to some variant of the Kakimizu complex [7] in 3-dimensional topology and knot theory.
**APPENDIX A: SIMPLE PROPERTIES OF COMPLEX OF SURFACES**

In this appendix, we prove that the complex of surfaces $\mathcal{K}(X)$ is contractible for any 4-manifold $X$.

**Lemma A.1.** Let $K = \hat{K}$ or $K$. Let $z = \sum_{k=1}^{m} a_k <\sigma_k > \in C_n(K)$ be an $n$-cycle, where $n \geq 1$, $m \geq 1$, $a_k \in \mathbb{Z}$ and $\sigma_k \in S_n(K)$. If there exists $\Sigma \in V(K)$ such that $\Sigma \notin \sigma_k$ holds for any $k$, then

$$[z] = 0 \text{ in } H_n(K; \mathbb{Z})$$

holds.

**Proof.** Let us define an oriented $(n + 1)$-simplex $<\tau_k>$ of $K$ by $<\tau_k> := <\Sigma, \sigma_k>$. More precisely, when we write $<\sigma_k> = <\Sigma_{0,k}, \ldots, \Sigma_{n,k}>$, we define

$$<\tau_k> := <\Sigma, \Sigma_{0,k}, \ldots, \Sigma_{n,k}>.$$

Then we obtain

$$\partial \left( \sum_{k=1}^{m} a_k <\tau_k> \right) = z + \sum_{k=1}^{m} a_k \sum_{l=0}^{n} (-1)^{l+1} <\Sigma, \Sigma_{0,k}, \ldots, \hat{\Sigma}_{l,k}, \ldots, \Sigma_{n,k}>.$$

We note an isomorphism between free modules

$$\mathbb{Z}\{ <\Sigma_{0,k}, \ldots, \hat{\Sigma}_{l,k}, \ldots, \Sigma_{n,k}> | 1 \leq k \leq m, 0 \leq l \leq n \} \rightarrow \mathbb{Z}\{ <\Sigma, \Sigma_{0,k}, \ldots, \hat{\Sigma}_{l,k}, \ldots, \Sigma_{n,k}> | 1 \leq k \leq m, 0 \leq l \leq n \}$$

defined by

$$<\Sigma_{0,k}, \ldots, \hat{\Sigma}_{l,k}, \ldots, \Sigma_{n,k}> \mapsto <\Sigma, \Sigma_{0,k}, \ldots, \hat{\Sigma}_{l,k}, \ldots, \Sigma_{n,k}>.$$

The equality $\partial z = 0$ in the first module hence implies that $\partial (\sum_{k=1}^{m} a_k <\tau_k>) = z$. □

**Proposition A.2.** The simplicial complex $\hat{K} = \hat{K}(X)$ is contractible for any $X$.

**Proof.** We show that $\pi_n(\hat{K}) = 0$ for any $n \geq 0$. For any continuous map $f : S^n \rightarrow |\hat{K}|$, there exist finite simplices $\sigma_1, \ldots, \sigma_m$ such that the image of $f$ is contained in the union of the geometric realizations of these simplices. By the argument in the proof of Lemma A.1, the union $\bigcup_{k=1}^{m} |\sigma_k|$ is contained in its cone $C(\bigcup_{k=1}^{m} |\sigma_k|) \subset |\hat{K}|$, and the map $f$ can be deformed into the constant map to the vertex point of the cone in $\hat{K}$. Hence we have $\pi_n(\hat{K}) = 0$, and $\hat{K}$ is contractible by the Whitehead theorem. □

**Remark A.3.** For an embedded surface $\Sigma \subset X$ with self-intersection number zero, by pushing $\Sigma$ into the fiber direction of its normal bundle, we obtain an embedded surface whose homology class coincides with functions $\Sigma$ and which has no geometric intersections with $\Sigma$. We call such a surface a **parallel copy** of $\Sigma$. The complex $K$ is a very huge space, however, homology group absorbs the hugeness in the following sense. For $n \geq 0$, let us take a cycle $z \in C_n(K)$. Let $z' \in C_n(K)$ be a
cycle obtained by replacing a surface which is a part of \( z \) with a parallel copy of the surface. Then \([z] = [z']\) in \( H_n(\mathcal{K}; \mathbb{Z})\) holds.

To see this, we write \( z = \sum_{k=1}^{m} a_k \langle \sigma_k \rangle \), where \( m \geq 1 \), \( a_k \in \mathbb{Z} \) and \( \sigma_k \in S_n(\mathcal{K}) \). We also write \( \langle \sigma_k \rangle = \langle \Sigma_{0,k}, ..., \Sigma_{n,k} \rangle \). Let \( \Sigma_{0,1} \) be the surface which is replaced with its parallel copy in the definition of \( z' \), and \( \Sigma'_{0,1} \) be the parallel copy of \( \Sigma_{0,1} \). We define an oriented simplex \( \langle \sigma'_k \rangle \) as follows. If \( \Sigma_{0,1} \in \sigma_k \) holds, we define \( \langle \sigma'_k \rangle \) as the simplex obtained by replacing \( \Sigma_{0,1} \) with \( \Sigma'_{0,1} \) from \( \langle \sigma_k \rangle \). If \( \Sigma_{0,1} \notin \sigma_k \) holds, we define \( \langle \sigma'_k \rangle = \langle \sigma_k \rangle \). Then \( z' = \sum_{k=1}^{m} a_k \langle \sigma'_k \rangle \) holds. By changing the order of indices, we may assume that

\[
1 \leq k \leq m' \Rightarrow \langle \sigma'_k \rangle \neq \langle \sigma_k \rangle, \text{ and}
\]
\[
m' + 1 \leq k \leq m \Rightarrow \langle \sigma'_k \rangle = \langle \sigma_k \rangle
\]
hold for some \( m' \). In addition, by changing the sign of \( a_k \), we may assume that \( \Sigma_{0,k} = \Sigma_{0,1} \) for each \( k \in \{1, ..., m'\} \). We set

\[
\langle \tau_k \rangle := \langle \Sigma'_{0,1}, \Sigma_{0,k}, \Sigma_{1,k}, ..., \Sigma_{n,k} \rangle = \langle \Sigma_{0,1}', \Sigma_{0,1}, \Sigma_{1,k}, ..., \Sigma_{n,k} \rangle
\]
for each \( k \in \{1, ..., m'\} \). Then one can check that \( \partial(\sum_{k=1}^{m'} a_k < \tau_k >) = z - z' \).

**ACKNOWLEDGEMENTS**

The author would like to express his deep gratitude to Mikio Furuta for helpful suggestions, in particular the definition of the complex of surfaces, and for continued encouragement during this work. The author would also like to express his appreciation to Nobuhiro Nakamura for telling him Ruberman’s work on invariants of diffeomorphisms using 1-parameter family. The author also wishes to thank Kouichi Yasui for comment on Example 4.22. A note in Remark 4.25 on the blowup formula is due to him. The author also thanks Genki Sato for a discussion on the simplicial complex theory. Finally, the author would like to express his deep gratitude to an anonymous referee for many valuable comments and suggestions. The author was supported by JSPS KAKENHI Grant Numbers 16J05569, 17H06461, 19K23412, 21K13785, and the Program for Leading Graduate Schools, MEXT, Japan.

**JOURNAL INFORMATION**

The *Journal of Topology* is wholly owned and managed by the London Mathematical Society, a not-for-profit Charity registered with the UK Charity Commission. All surplus income from its publishing programme is used to support mathematicians and mathematics research in the form of research grants, conference grants, prizes, initiatives for early career researchers and the promotion of mathematics.

**REFERENCES**

1. D. Baraglia and H. Konno, *A gluing formula for families Seiberg-Witten invariants*, Geom. Topol. 24 (2020), no. 3, 1381–1456.
2. S. Bauer and M. Furuta, *A stable cohomotopy refinement of Seiberg-Witten invariants. I*, Invent. Math. 155 (2004), no. 1, 1–19. MR2025298.
3. S. K. Donaldson, *Yang-Mills invariants of four-manifolds*, Geometry of low-dimensional manifolds, London Math. Soc. Lecture Note Ser., 150, Cambridge Univ. Press, Cambridge, 1990, pp. 5–40. MR1171888.
4. S. K. Donaldson, *The Seiberg-Witten equations and 4-manifold topology*, Bull. Amer. Math. Soc. (N.S.) 33 (1996), no. 1, 45–70. MR1339810.

5. K. A. Frøyshov, *An inequality for the h-invariant in instanton Floer theory*, Topology 43 (2004), no. 2, 407–432. MR2052970 (2005c:57043).

6. W. J. Harvey, *Boundary structure of the modular group*, Riemann surfaces and related topics: Proceedings of the 1978 Stony Brook Conference (State Univ. New York, Stony Brook, N.Y., 1978), Ann. of Math. Stud., 97, Princeton Univ. Press, Princeton, N.J., 1981, pp. 245–251. MR624817.

7. O. Kakimizu, *Finding disjoint incompressible spanning surfaces for a link*, Hiroshima Math. J. 22 (1992), no. 2, 225–236. MR1177053.

8. H. Konno, *Bounds on genus and configurations of embedded surfaces in 4-manifolds*, J. Topol. 9 (2016), no. 4, 1130–1152.

9. P. B. Kronheimer and T. S. Mrowka, *The genus of embedded surfaces in the projective plane*, Math. Res. Lett. 1 (1994), no. 6, 797–808. MR1306022 (96a:57073).

10. P. B. Kronheimer and T. S. Mrowka, *Scalar curvature and the Thurston norm*, Math. Res. Lett. 4 (1997), no. 6, 931–937. MR1492131 (98m:57039).

11. T.-J. Li and A.-K. Liu, *Family Seiberg-Witten invariants and wall crossing formulas*, Comm. Anal. Geom. 9 (2001), no. 4, 777–823. MR1868921 (2002k:57074).

12. R. Mandelbaum, *Decomposing analytic surfaces*, Geometric topology (Proc. Georgia Topology Conf., Athens, Ga., 1977), Academic Press, New York-London, 1979, pp. 147–217. MR537731.

13. B. Moishezon, *Complex surfaces and connected sums of complex projective planes*, Lecture Notes in Math., 603, Springer-Verlag, Berlin-New York, 1977. (With an appendix by R. Livne.) MR0491730.

14. J. R. Munkres, *Elements of algebraic topology*, Addison-Wesley Publishing Company, Menlo Park, CA, 1984. MR755006.

15. M. A. Nouh, *The minimal genus problem in $\mathbb{C}P^2$#$\mathbb{C}P^2$*, Algebr. Geom. Topol. 14 (2014), no. 2, 671–686. MR3159966.

16. Y. Ruan, *Virtual neighborhoods and the monopole equations*, Topics in symplectic 4-manifolds (Irvine, CA, 1996), First Int. Press Lect. Ser., I, Int. Press, Cambridge, MA, 1998, pp. 101–116. MR1635698 (2000e:57054).

17. D. Ruberman, *An obstruction to smooth isotopy in dimension 4*, Math. Res. Lett. 5 (1998), no. 6, 743–758. MR1671187 (2000c:57061).

18. D. Ruberman, *A polynomial invariant of diffeomorphisms of 4-manifolds*, Proceedings of the Kirbyfest (Berkeley, CA, 1998), Geom. Topol. Monogr., vol. 2, Geom. Topol. Publ., Coventry, 1999, pp. 473–488. MR1734421 (2001b:57073).

19. D. Ruberman, *Positive scalar curvature, diffeomorphisms and the Seiberg-Witten invariants*, Geom. Topol. 5 (2001), 895–924. MR1874146 (2002k:57076).

20. D. Salamon, *Spin geometry and Seiberg–Witten invariants*. available at https://people.math.ethz.ch/~salamon/ PREPRINTS/witsei.pdf.

21. S. Stie, *Bounds on genus and geometric intersections from cylindrical end moduli spaces*, J. Differential Geom. 65 (2003), no. 3, 469–511. MR2064429 (2005c:57042).

22. C. T. C. Wall, *Diffeomorphisms of 4-manifolds*, J. London Math. Soc. 39 (1964), 131–140. MR163323.

23. C. T. C. Wall, *On simply-connected 4-manifolds*, J. London Math. Soc. 39 (1964), 141–149. MR0163324.

24. E. Witten, *Monopoles and four-manifolds*, Math. Res. Lett. 1 (1994), no. 6, 769–796. MR1306021.