A SOLUTION FOR DISCRETE COST SHARING PROBLEMS WITH NON RIVAL CONSUMPTION

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ABSTRACT. In this paper we show several results regarding to the classical cost sharing problem when each agent requires a set of services but they can share the benefits of one unit of each service, i.e. there is non rival consumption. Specifically, we show a characterized solution for this problem, mainly adapting the well-known axioms that characterize the Shapley value for TU-games into our context. Finally, we present some additional properties that the shown solution satisfy.

1. Introduction. In a general framework for cost sharing problems, there exists a finite set of \( n \) agents, \( N \), each one of them need to consume probably different quantities of \( m \) different goods or services, \( M \). Additionally, there exists a cost function that assigns a real number to any possible quantities of consumption of the goods. Depending on the kind of goods (divisible, indivisible, shared) or the kind of domain of the cost function we have several well-known models for cost sharing problems. In this paper we consider that each agent \( i \in N \) decides if he has to use each service \( k \in M \), where we are not allowing partial consumption of each service. That is called a discrete cost sharing problem. So, each agent \( i \in N \) has a set of consumption requirements \( M_i \subseteq M \). We assume that several agents can share the benefits of one unit of each service; therefore, we consider that there is no rival consumption. Then, if the agents decide to hire a service \( k \in M \), this service can be used for all the agents requiring it, but they need to agree about how much each agent have to pay for it. There are several papers studying different models for discrete cost sharing problems, for example [4], [3] and [6]. On the particular case of discrete cost sharing problems with non rival consumption, please refer to [1].

There are several situations where this kind of problems arise. For example, in a mall, the public garbage collection service comes only one day a week, causing dissatisfaction among some of the owners of the establishments. It is possible to hire an additional private garbage collection service. The company that provides the private service can make a trip to the mall once a day from Thursday to Sunday and

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the cost of this service does not depend on the number of serviced establishments, but only on which days the service is provided. Due to the nature of the different establishments, we can assume that each owner has a set of days when he prefers to have this additional private service (it could be empty, and this would mean that owner is satisfied with the public service); then we have a cost sharing problem with non rival consumption, and the owners must agree about how to share the cost of hiring the private service for the days they require.

For providing solutions to a discrete cost sharing problem, we use cooperative games techniques, specifically, the theory of transferable utility cooperative games. Lloyd S. Shapley, in 1953, characterized axiomatically a very well-known solution for this kind of cooperative games using the additivity, symmetry, nullity and efficiency properties. Since that day, there have been several works where the authors adapt those axioms into the context under their study: coalitional structures, multi-choice games, bankruptcy problems, etc, for finding a possible extension of Shapley’s result in their problems. We follow this line and then, the main result of the paper is that we show a solution for discrete cost sharing problems with non rival consumption adapting Shapley’s axioms, and it is not very surprising that it is related to the Shapley value of a certain cooperative game. In addition, we show that the solution also satisfies another well-know properties for solutions to cost sharing problems.

This paper is structured as follows: In the second section we present the definitions and notation that we are going to use in this work. In Section 3, we show a characterized solution for discrete cost sharing problems with non rival consumption based on the axioms that characterizes the Shapley value for transferable utility cooperative games. Finally, in Section 4, we show that the characterized solution satisfies several additional properties, providing strongness to the solution.

2. Notation and definitions. In this section, we provide the notation that we use along the paper and we present our model for discrete cost sharing problems with non rival consumption.

Let $N = \{1, \ldots, n\}$ be a finite set of agents and $Q = \{1, \ldots, q\}$ a finite set of services. We consider that each agent $i \in N$ requires a subset of the set of services, and we denote it by $M_i \subseteq Q$. The requirements vector of all the agents is denoted by $M := (M_1, \ldots, M_n)$. Additionally, there exists a cost function $c$ assigning a real number to every subset of services, this is, $c : 2^Q \to \mathbb{R}$ with $c(\emptyset) = 0$. For $S \subseteq Q$, $c(S)$ denotes the cost of the subset of services $S$. The set of all possible cost functions for the set of services $Q$ is denoted by $G_Q := \{c : 2^Q \to \mathbb{R}, c(\emptyset) = 0\}$. It is straightforward that $G_Q$ is a vector space with the usual operations: for $c_1, c_2 \in G_Q$ and $\delta \in \mathbb{R}$, $(c_1 + c_2)(S) := c_1(S) + c_2(S)$ and $(\delta c_1)(S) := \delta c_1(S)$, for all $S \subseteq Q$.

We are interested in situations where every service in $Q$ is required by at least one agent. Thus, given a set of services $Q$, we denote the set of possible requirements vectors we are interested on by

$$E(Q) := \{M \in 2^Q \times \cdots \times 2^Q \mid \forall k \in Q, \exists i \in N \text{ with } k \in M_i\}.$$  

Formally, a discrete cost sharing problem (or simply a problem) is a pair $(c, M)$, with $c \in G_Q$ and $M \in E(Q)$. A solution is a mapping

$$\varphi : G_Q \times E(Q) \to \mathbb{R}^n$$

where the $i$-th element of the vector, $\varphi_i(c, M)$, indicates the amount that the agent $i \in N$ must pay for the set of services he requires, according to the problem $(c, M) \in G_Q \times E(Q)$. A very important assumption is that when a service is hired, any agent
requiring it can use it; that is, there is non rival consumption. This fact can be related also in the functional form of the cost function and in the definition of the problem.

**Example 1.** Retaking the motivation situation presented in the Introduction, assume a company providing the private garbage collection service on Wednesdays (W), Fridays (F) and Sundays (S) with the next cost function ($M = \{W, F, S\}$):

\[
\begin{align*}
    c(\emptyset) &= 0 \\
    c(\{W\}) &= 20 \\
    c(\{F\}) &= 30 \\
    c(\{S\}) &= 10 \\
    c(\{W, F\}) &= 40 \\
    c(\{W, S\}) &= 20 \\
    c(\{F, S\}) &= 35 \\
    c(\{W, F, S\}) &= 45.
\end{align*}
\]

Suppose there are three establishments in the mall, $N = \{1, 2, 3\}$, with requirements $M_1 = \{S\}$, $M_2 = \{W, S\}$ and $M_3 = \{W, F, S\}$. How must they share the cost of hiring the whole set of services? We are going to solve this situation after presenting the characterized solution.

In the next sections we use several results and definitions regarding to transferable utility cooperative games (TU-games). For further information about this topic, you could refer to the seminal paper [5] or a textbook as [2].

3. **A characterization.** Now, we present our main result: we characterize a solution for the discrete cost sharing problem with non rival consumption based on a characterization of the Shapley value for TU-games.

**Axiom 1.** (Additivity) Given two problems $(c_1, M), (c_2, M) \in G_Q \times E(Q)$, a solution $\varphi : G_Q \times E(Q) \to \mathbb{R}^n$ satisfies the additivity axiom if

$$\varphi(c_1 + c_2, M) = \varphi(c_1, M) + \varphi(c_2, M).$$

Suppose that the cost function can be divided into periods of time. Then it would be desirable that the payment made by an agent in the original problem is the same as if we apply the solution in each problem defined by the divided cost functions and then add the amounts that correspond to each period.

**Axiom 2.** (Efficiency) Given a problem $(c, M) \in G_Q \times E(Q)$, a solution $\varphi : G_Q \times E(Q) \to \mathbb{R}^n$ satisfies the efficiency axiom if

$$\sum_{i \in N} \varphi_i(c, M) = c(Q).$$

In an efficient solution, the cost of hiring the full set of services is distributed entirely among the agents.

Given a problem $(c, M) \in G_Q \times E(Q)$, an agent $i \in N$ is a null agent on $(c, M)$ if

$$c \left( \bigcup_{T \subseteq \{i\}} M_T \right) = c \left( \bigcup_{T \subseteq \{i\}} M_T \right), \quad \forall T \subseteq N \setminus \{i\}.$$  

Notice that if an agent does not need any service, then he is a null agent.

**Axiom 3.** (Null agent) A solution $\varphi : G_Q \times E(Q) \to \mathbb{R}^n$ satisfies the null agent axiom if

$$\varphi_i(c, M) = 0$$

for every null agent $i \in N$ on $(c, M)$.
According to this axiom, if the requirements of an agent $i$ does not affect the cost of the set of demands of every subset of agents $T \subseteq N \setminus \{i\}$, this agent must pay zero in the problem.

Two agents $i, j \in N$ have symmetric requirements in the problem $(c, M) \in G_Q \times E(Q)$, if

$$c \left( \bigcup_{t \in T \cup \{i\}} M_t \right) = c \left( \bigcup_{t \in T \cup \{j\}} M_t \right), \quad \forall T \subseteq N \setminus \{i, j\}.$$

**Axiom 4.** (Symmetric requirements) A solution $\varphi : G_Q \times E(Q) \to \mathbb{R}^n$ satisfies the symmetric requirements axiom if

$$\varphi_i(c, M) = \varphi_j(c, M)$$

for every pair of agents $i, j \in N$ with symmetric requirements on $(c, M)$.

Two agents have symmetric requirements if their sets of demands are indistinguishable from each other regarding to how much they contribute to the cost function. So, a solution which satisfies the previous axiom indicates that agents with this type of requirements have to pay the same amount for their services.

**Theorem 3.1.** There exists a unique solution $\varphi : G_Q \times E(Q) \to \mathbb{R}^n$ that satisfies the additivity, efficiency, null agent and symmetric requirements axioms, and it has the next formulation:

$$\varphi_i(c, M) = Sh_i(N, w_{i(c,M)}), \quad \forall i \in N$$

where $w_{i(c,M)} \in G_N$ is the characteristic function of a TU-game defined as follows:

$$w_{i(c,M)}(T) = c \left( \bigcup_{j \in T} M_j \right), \quad \forall T \subseteq N.$$

**Proof.** First, we prove the uniqueness of the solution. Defining $\delta_\emptyset = 0$ and $\delta_S = c(S) - \sum_{R \subseteq S} \delta_R$, any cost function $c \in G_Q$ can be written as follows:

$$c = \sum_{S \subseteq Q} \delta_S^* u_S^*$$

where each $u_S^* \in G_Q$, is defined as follows:

$$u_S^*(R) = \begin{cases} 
1, & \text{if } R \subseteq S \\
0, & \text{otherwise.}
\end{cases}$$

Additionally, we define the set

$$Q_M := \{S \subseteq Q \mid \emptyset \neq M_j \subseteq S, \text{ for some } j \in N\},$$

which groups the subsets of $Q$ containing the non-empty demand set of at least one agent. Let $\varphi : G_Q \times E(Q) \to \mathbb{R}^n$ be a solution that satisfies the axioms of the theorem. Because of the additivity axiom, we have

$$\varphi_i(c, M) = \sum_{S \subseteq Q} \varphi_i(\delta_S u_S^*, M).$$

Notice that all agents have symmetric requirements in $(\delta_S^* u_S^*, M)$ when $S \not\in Q_M$ and $u_S^*(Q) = 0$. Thus, because of the efficiency axiom and the symmetric requirements
axiom, \( \varphi_i(\delta_S^* u_S^*, M) = 0 \), for all \( i \in N \) if \( S \notin Q_M \). Thus,

\[
\varphi_i(c, M) = \sum_{S \in Q_M} \varphi_i(\delta_S^* u_S^*, M).
\]

If \( S \in Q_M \), there are three groups of agents in \((\delta_S^* u_S^*, M)\) who have symmetric requirements:

- \( i \in N \) such that \( \emptyset \neq M_i \subseteq S \)
- \( i \in N \) such that \( \emptyset \neq M_i \nsubseteq S \)
- \( i \in N \) such that \( M_i = \emptyset \).

Because of the symmetric requirement and null agent axioms, there exist \( \lambda_S, \mu_S \in \mathbb{R} \) for all \( S \in Q_M \) such that:

\[
\varphi_i(\delta_S^* u_S^*, M) = \begin{cases} 
\lambda_S, & \text{if } \emptyset \neq M_i \subseteq S \\
0, & \text{if } M_i = \emptyset \\
\mu_S, & \text{otherwise.}
\end{cases}
\]

For every \( T, R \subseteq Q \), we define the cost functions:

\[
\tilde{c}_T(R) = \begin{cases} 
1, & \text{if } T \cap R \neq \emptyset \\
0, & \text{otherwise.}
\end{cases}
\]

Notice that for each \( S \in Q_M \), \( \delta_S^* u_S^* = \delta_S^*(u_{Q}^* - \tilde{c}_Q(S)) \). All agents have symmetric requirements in \((\delta_S^* u_Q^*, M)\) (except for the agents that have an empty demand set, they are null agents). Because \( \varphi \) satisfies the axioms of the theorem,

\[
\varphi_i(\delta_S^* u_S^*, M) = \varphi_i(\delta_S^* u_Q^*, M) - \varphi_i(\delta_S^* \tilde{c}_Q \setminus S, M) = \frac{\delta_S^*}{n - d} - \varphi_i(\delta_S^* \tilde{c}_Q \setminus S, M)
\]

where \( d \) denotes the number of agents that don’t have any requirements.

Now, if \( M_i \subseteq S \), \( i \) is a null agent in \((\delta_S^* \tilde{c}_T, M)\). Then, \( \varphi_i(\delta_S^* \tilde{c}_T, M) = 0 \). Thus, for all \( S \in Q_M \) with \( M_i \subseteq S \),

\[
\lambda_S = \frac{\delta_S^*}{n - d}.
\]

For each \( S \in Q_M \) and each \( P \subseteq N \) we define

\[
P^S(M) = \{ j \in P \mid M_j \subseteq S \}
\]

the set of agents in \( P \) whose requirements are completely contained in \( S \). Finally, because of the efficiency and null agent axioms, \( (|N^S(M)| - d)\lambda_S + (n - |N^S(M)|)\mu_S = 0 \), for all \( S \in Q_M \) and

\[
\varphi_i(\delta_S^* u_S^*, M) = \begin{cases} 
\frac{\delta_S^*}{n - d}, & \text{if } \emptyset \neq M_i \subseteq S \\
0, & \text{if } M_i = \emptyset \\
-\frac{\delta_S^*}{(n - d)(n - |N^S(M)|)}, & \text{otherwise.}
\end{cases}
\]

This expression determines uniquely the solution. To prove that (1) satisfies the axioms given in the theorem, we just have to remember the axioms that characterize the Shapley value. Efficiency and additivity axioms are inherited directly from the efficiency and linearity of Shapley value; a null agent in \((c, M)\) is a null player in the TU-game \((N, w_{(c, M)})\) so he pays zero, and if \( i, j \in N \) have symmetrical requirements in \((c, M)\), then they have equal marginal contributions in the TU-game \((N, w_{(c, M)})\), so \( Sh_i(N, w_{(c, M)}) = Sh_j(N, w_{(c, M)}) \).
(Continuation of Example 1). For the situation given in Example 1, we can construct the associated TU-game \((N, w(c,M)):\)

\[
\begin{align*}
    w_{c,M}(\emptyset) &= 0 & w_{c,M}(\{1\}) &= 10 & w_{c,M}(\{2\}) &= 20 \\
    w_{c,M}(\{3\}) &= 45 & w_{c,M}(\{1, 2\}) &= 20 & w_{c,M}(\{1, 3\}) &= 45 \\
    w_{c,M}(\{2, 3\}) &= 45 & w_{c,M}(\{1, 2, 3\}) &= 45.
\end{align*}
\]

Calculating the Shapley value of the previous TU-game, we get

\[
\text{Sh}(w_{c,M}) = (10/3, 25/3, 100/3) \equiv \varphi(c, M).
\]

Notice that, according to this solution, it is profitable the cooperation among owners for diminishing the cost for individual requirements.

4. Additional properties. Now, we are going to study additional properties of a solution for a discrete cost sharing problem \((c, M) \in G_Q \times E(Q).\) We say a cost function \(c\) is non-decreasing if \(S \subseteq R \subseteq Q\) implies \(c(S) \leq c(R).\)

Property 1 (Monotonicity). A solution \(\varphi : G_Q \times E(Q) \to \mathbb{R}^n\) is monotone if for every \(i, j \in N\) such that \(M_i \subseteq M_j\) we have \(\varphi_i(c, M) \leq \varphi_j(c, M)\) for all cost non-decreasing function \(c.\)

This property indicates that if an agent \(j\) requires at least the same set of services as an agent \(i,\) then the amount the agent \(j\) pays must be greater than or equal to what agent \(i\) pays if the cost function is non-decreasing.

A cost function \(c\) is concave if

\[
c(S) + c(R) - c(S \cap R) \geq c(S \cup R),
\]

for all \(S, R \subseteq Q.\) In a concave function, it is more affordable hiring the total set of services.

Property 2 (Stability). A solution \(\varphi : G_Q \times E(Q) \to \mathbb{R}^n\) is stable if \(\sum_{i \in T} \varphi_i(c, M) \leq c \left( \bigcup_{i \in T} M_i \right)\) for all \(T \subseteq N\) when the cost function \(c\) is concave.

If the cost function is concave and the solution is stable, any coalition of agents pays at most the amount they would pay if they hired separately the set of services that satisfies their demands; in other words, no coalition is punished by the cooperation.

We say that an agent \(i \in N\) is dummy in a requirements vector \(M \in E(Q)\) if \(M_i = \emptyset.\) So, it could be reasonable to think that a dummy agent does not have to pay anything because he requires nothing.

Property 3 (Dummy agent). A solution \(\varphi : G_Q \times E(Q) \to \mathbb{R}^n\) satisfies the dummy agent property if

\[
\varphi_i(c, M) = 0
\]

for every dummy agent \(i \in N\) in \(M.\)

Given a problem \((c, M),\) let \(D_M = \{ i \in N | M_i = \emptyset \}\) be the set of dummy agents in \(E(Q).\) We define the problem \((c, M_{-D})\) such that \(M_{-D}\) is the requirements vector of \(n - d\) elements were we dismiss the dummy agent demand sets (notice that \(M_{-D} \in E(Q)\) therefore, the problem is well defined).

Property 4 (Dummy reduction). A solution \(\varphi : G_Q \times E(Q) \to \mathbb{R}^n\) satisfies dummy reduction if

\[
\varphi_i(c, M) = \varphi_i(c, M_{-D}), \quad \forall i \in N \setminus D_M.
\]
This property indicates that if the dummy agents are dismissed from the problem, the payoff of the rest of the agents does not change.

Two agents \( i, j \in N \) are equivalents in \( M \in E(Q) \) if \( M_i = M_j \).

**Property 5** (Equivalent agents). A solution \( \varphi : G_Q \times E(Q) \to \mathbb{R}^n \) satisfies the equivalent agents axioms if

\[
\varphi_i(c, M) = \varphi_j(c, M)
\]

for each pair of equivalents agents \( i, j \in N \) in \( M \), with \((c, M) \in G_Q \times E(Q)\).

If two agents are equivalent, they demand exactly the same services; so, a solution that satisfies the above axiom allocates the same amount to pay to both agents.

Given \( S \subseteq Q \), we define the \( S \)-reduced problem \((c|_{Q \setminus S}, M|_{Q \setminus S}) \in G_{Q \setminus S} \times E(Q \setminus S)\) as follows:

\[
(c|_{Q \setminus S})(R) = c(R), \quad \forall R \subseteq Q \setminus S \\
(M|_{Q \setminus S})_i = M_i \cap (Q \setminus S), \quad \forall i \in N.
\]

In a \( S \)-reduced problem, the set of services in \( S \) are eliminated from the cost function and from the requirements vector.

Given \((c, M) \in G_Q \times E(Q)\), the set of services \( S \subseteq Q \) is free in \( c \) if:

\[
c(T) = 0, \quad \forall T \subseteq S
\]

\[
c(T \cup R) = c(R), \quad \forall T \subseteq S \quad \forall R \subseteq Q \setminus S.
\]

**Property 6** (Reduction). A solution \( \varphi : G_Q \times E(Q) \to \mathbb{R}^n \) satisfies the reduction axiom if

\[
\varphi(c, M) = \varphi(c|_{Q \setminus S}, M|_{Q \setminus S})
\]

for all \( S \subseteq Q \) free in \( c \), with \((c, M) \in G_Q \times E(Q)\).

This axiom states that a solution should not be affected if the original problem is reduced, discarding any set of free services.

**Proposition 1.** The solution \( \varphi : G_Q \times E(Q) \to \mathbb{R}^n \) given by (1) satisfies (a) monotonicity, (b) stability, (c) dummy agent, (d) dummy reduction, (e) equivalent agents and (f) reduction properties.

**Proof.** (a) Denoting by 

\[
\gamma_t = \frac{(n - t)!}{n!}
\]

we have

\[
\varphi_i(c, M) = \sum_{T \supseteq N} \gamma_t [w_{(c, M)}(T) - w_{(c, M)}(T \setminus \{i\})]
\]

\[
= \sum_{T \supseteq N} \gamma_t \left[ c \left( \bigcup_{t \in T} M_t \right) - c \left( \bigcup_{t \in T \setminus \{i\}} M_t \right) \right].
\]

Notice that if \( j \in T, \bigcup_{t \in T} M_t = \bigcup_{t \in T \setminus \{i\}} M_t \), since \( M_i \subseteq M_j \); and if \( j \notin T \), then

\[
c \left( \bigcup_{t \in T} M_t \right) \leq c \left( \bigcup_{t \in T \cup \{j\}} M_t \right) = c \left( \bigcup_{t \in (T \setminus \{i\}) \cup \{j\}} M_t \right).
\]

Thus

\[
\varphi_i(c, M) = \sum_{T \supseteq N} \gamma_t \left[ c \left( \bigcup_{t \in T} M_t \right) - c \left( \bigcup_{t \in T \setminus \{i\}} M_t \right) \right]
\]
\[ \leq \sum_{T \subseteq N \setminus \{j\}} \gamma_t \left[ c \left( \bigcup_{t \in T \cup \{j\}} M_t \right) - c \left( \bigcup_{t \in T \setminus \{i\}} M_t \right) \right] \]

\[ = \sum_{T \subseteq N \setminus \{j\}} \gamma_t \left[ c \left( \bigcup_{t \in (T \setminus \{i\}) \cup \{j\}} M_t \right) - c \left( \bigcup_{t \in T \setminus \{i\}} M_t \right) \right] \]

\[ = \sum_{T \subseteq N \setminus \{j\}} \gamma_{t+1} \left[ c \left( \bigcup_{t \in T \cup \{j\}} M_t \right) - c \left( \bigcup_{t \in T} M_t \right) \right]. \]

Finally, because \( c \) is non-decreasing, \( w(c,M)(T \cup \{j\}) - w(c,M)(T) \geq 0 \) for all \( T \subseteq N \setminus \{j\} \) with \( T \ni i \); hence

\[ \varphi_i(c,M) \leq \sum_{T \subseteq N \setminus \{i\}} \gamma_{t+1} \left[ c \left( \bigcup_{t \in T \cup \{j\}} M_t \right) - c \left( \bigcup_{t \in T} M_t \right) \right] \]

\[ + \sum_{T \subseteq N \setminus \{j\}} \gamma_{t+1} \left[ c \left( \bigcup_{t \in T \cup \{j\}} M_t \right) - c \left( \bigcup_{t \in T} M_t \right) \right] \]

\[ = \sum_{T \subseteq N} \frac{t!(n-t-1)!}{n!} \left[ w(c,M)(T \cup \{j\}) - w(c,M)(T) \right] \]

\[ = Sh_j(N,w(c,M)) = \varphi_j(c,M). \]

(b) Notice that if \( c \) is concave, the TU-game \((N,w(c,M))\) is concave. If \( P \subseteq T \subseteq N \), let us denote

\[ S = \bigcup_{j \in P \cup \{i\}} M_j \quad \text{and} \quad R = \bigcup_{j \in T} M_j. \]

Notice that

\[ S \cap R = \left( \bigcup_{j \in P \cup \{i\}} M_j \right) \cap \left( \bigcup_{j \in T} M_j \right) = \bigcup_{j \in P} M_j \]

and

\[ S \cup R = \left( \bigcup_{j \in P \cup \{i\}} M_j \right) \cup \left( \bigcup_{j \in T} M_j \right) = \bigcup_{j \in T \cup \{i\}} M_j. \]

Thus

\[ c \left( \bigcup_{j \in P \cup \{i\}} M_j \right) \cup c \left( \bigcup_{j \in T} M_j \right) - c \left( \bigcup_{j \in P} M_j \right) \geq c \left( \bigcup_{j \in T \cup \{i\}} M_j \right), \]

o equivalently,

\[ c \left( \bigcup_{j \in P \cup \{i\}} M_j \right) - c \left( \bigcup_{j \in P} M_j \right) \geq c \left( \bigcup_{j \in T \cup \{i\}} M_j \right) - c \left( \bigcup_{j \in T} M_j \right) \]

\[ \Rightarrow w(c,M)(P \cup \{i\}) - w(c,M)(P) \geq w(c,M)(T \cup \{i\}) - w(c,M)(T). \]
This proves that \((N, w_{(c,M)})\) is concave, then the Shapley value is in the core of the TU-game \((N, w_{(c,M)})\); this is, for all \(T \subseteq N\), \(\sum_{i \in T} Sh_i(N, w_{(c,M)}) \leq w_{(c,M)}(T) = c(\bigcup_{i \in T} M_i)\). (c) It is easy to verify that a dummy agent is a null agent; then, since the solution satisfies the null agent axiom, \(\varphi_i(c, M) = 0\). (d) Since every dummy agent in \(M\) is a null agent in \((N, w_{(c,M)})\), the Shapley value is not affected if these agents are dismissed from the TU-game. (e) If two agents are equivalent, they have symmetric requirements and the result follows. (f) Let \(S \subseteq Q\) be a free set of service in \(c\); then, \(c(M_i) = c(M_i \cap (Q \setminus S))\) for all \(i \in N\). Thus,
\[
c \left( \bigcup_{i \in T} M_i \right) = c \left( \bigcup_{i \in T} [M_i \cap (Q \setminus S)] \right) = c \left( \bigcup_{i \in T} M_i |_{Q \setminus S} \right), \quad \forall T \subseteq N
\]
so we have \(\varphi(c, M) = \varphi(c |_{Q \setminus S}, M |_{Q \setminus S})\). \(\square\)

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