A Replica Approach to Products of Random Matrices

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(January 31, 2018)

We analyse products of random $R \times R$ matrices by means of a variant of the replica trick which was recently introduced for one–dimensional disordered Ising models. The replicated transfer matrix can be block–diagonalized with help of irreducible representations of the permutation group. We show that the free energy (or the Lyapunov exponent) of the product corresponds to the replica symmetric representation, whereas non–trivial representations correspond to certain correlation functions.

PACS numbers : 02.10Sp, 05.20-y, 75.10 Nr

I. INTRODUCTION

The asymptotic properties of products of random matrices play an important role in many physical problems [1,2]. In models like disordered one–dimensional magnetic systems they describe the thermodynamic quantities such as free energy or correlations, for localization of electronic waves in random potentials they are related to the transport properties, see also [3]. Such products also appear in the context of chaotic dynamical systems characterizing the divergence of neighboring trajectories.

Although there are many known results on products of random matrices, some of them even mathematically rigorous, we want to present a general replica transfer matrix method. Replicas are known to be a very powerful but nevertheless somewhat mysterious tool in the statistical mechanics of disordered systems and related problems [4]. In the case of mean–field models replicas predict the concept of replica symmetry breaking which is related to a highly nontrivial ultrametric structure of states in the low temperature phase.

The existence of replica symmetry breaking in low dimensional systems is not yet clear, see e.g. the argumentation of [5]. In [6] a replica approach to one–dimensional disordered Ising models was presented. Although there does not exist any phase transition at nonzero temperature, a rich replica structure could be observed leading to a ‘natural’ criterion for replica symmetry breaking in this special system which is not related to Parisi’s replica symmetry breaking scheme for mean–field models. This criterion is based on the representation structure of the permutation group and could be deduced to a large extent with rigorous methods. In the present paper we generalize the approach of [6] to infinite products of random matrices of finite dimension. We mainly use the language of statistical mechanics, i.e. the random matrices are considered as transfer matrices of one–dimensional models with finite discrete degrees of freedom and random short–range interactions. We show that the representation theoretic approach to replica symmetry breaking is quite general and can be formulated without specifying a particular one–dimensional model.

The outline of the paper is the following. In Sec. II we introduce the replicated transfer matrix. For the analysis we need several tools from the representation theory of the symmetric group. These are presented in the third section. In Sec. IV the replica symmetric representation space is considered and the free energy is calculated. The connection between non–trivial representations of the symmetric group and connected correlation functions is analyzed in Sec. V. They will provide a natural criterion for replica symmetry breaking. In the last section we give a summary and outlook. Several appendices contain longer calculations or proofs.

II. THE REPLICATED TRANSFER MATRIX

We consider $N R \times R$ matrices $T_i, i = 1, ..., n$, drawn from a single probability distribution $P(T)$, where $R$ is any positive integer. In the case of an one–dimensional model with random Hamiltonian $H = \sum_i H_i(s_i, s_{i+1})$ ($s_i$ can take $R$ different values) and inverse temperature $\beta$ they are given by $T_i = (\exp\{H_i(s_i, s_{i+1})\})$. For a general distribution these matrices do not commute. Therefore we cannot find a common system of eigenvectors. In order to calculate
self-averaging quantities such as the free energy we introduce as usual the n-fold replicated and disorder averaged partition function,
\[
\ll Z^n \gg = \ll (\text{tr} \prod_{i=1}^{N} T_i)^n \gg = (\text{tr} \ll T^\otimes n \gg)^N,
\]
where \( \ll \cdot \gg \) denotes the average with respect to \( P(T) \) and \( \otimes \) the Kronecker product of matrices. With this relation we are able to replace the product of \( N \) random \( R \times R \) matrices by the \( N \)-th power of a single \( R^n \times R^n \) matrix which can be analyzed using standard transfer matrix techniques: we have to find expressions for the eigenvalues of \( T_n := \ll T^\otimes n \gg \) which enable an analytic continuation in \( n \). The free energy is then given by
\[
f = -\frac{1}{\beta} \ll \ln Z \gg = -\frac{1}{\beta} \lim_{n \to 0} \partial_n \ll Z^n \gg,
\]
which is dominated by the largest eigenvalue of \( T_n \) for \( n \to 0 \). Several correlation length can be described by smaller eigenvalues of the same matrix.

To calculate this, some notations will be introduced. The original matrices \( T_i \) act on a \( R \)-dimensional vector space \( V \). As a basis we chose any orthonormalized set of \( R \) vectors and denote these by \( |s\rangle \), \( s = 1, ..., R \). Consequently \( T_n \) is a linear operator defined on the \( n \)-fold tensor product \( V^\otimes n \) of \( V \) with itself which has dimension \( R^n \). The orthonormalized basis-vectors of this space are chosen naturally as \( |s^1 \rangle \otimes |s^2 \rangle \otimes ... \otimes |s^n \rangle =: |s^1 s^2 ... s^n \rangle \) where \( s^a \in \{1, ..., R\} \) for all \( a = 1, ..., n \). The matrix elements of \( T_n \) are then given by
\[
\langle s^1 s^2 ... s^n | T_n | s'^1 s'^2 ... s'^n \rangle = \ll \prod_{a=1}^{n} (s'^a | T | s^a) \gg = \ll \prod_{a=1}^{n} T_{s^a, s'^a} \gg
\]
for any two basis vectors of \( V^\otimes n \).

The average over \( P(T) \) produces interactions between the replicas. Nevertheless the replicas are completely equivalent, a renumbering does not change the matrix \( T_n \). This leads to a symmetry of the transfer matrix under replica permutations, i.e. to replica symmetry of \( T_n \). The action of any permutations is given by the \( R^n \) dimensional representation \( D \) of the symmetric group \( S_n \):
\[
D(\pi) | s^1 s^2 ... s^n \rangle = | s^{\pi(1)} s^{\pi(2)} ... s^{\pi(n)} \rangle, \quad \forall \pi \in S_n,
\]
whose operator product with \( T_n \) commutes,
\[
D(\pi) T_n = T_n D(\pi), \quad \forall \pi \in S_n.
\]
A direct consequence of equation (3) is the closure of any eigenspace of \( T_n \) under permutations, these eigenspaces define a subrepresentations of \( D \) which in the most general case are irreducible. Further reducibilities would be a hint to a further hidden symmetry.

Consider an element \( Y \) of the group algebra \( s_n \) of \( S_n \), i.e. \( Y \) is a linear combination of permutations \( \pi \in S_n \). Due to (3) and the linearity of the action of the transfer matrix on \( V^\otimes n \) it also commutes with \( T_n \). The space \( U = Y V^\otimes n = \sum_{s^1, ..., s^n} R Y | s^1 s^2 ... s^n \rangle \) is therefore invariant under action of \( T_n \). If we are able to construct elements of \( s_n \) projecting \( V^\otimes n \) to a proper subspace we can thus achieve a block diagonalization of \( T_n \) by its restriction to \( U \) and to its orthogonal complement \( (1 - Y) V^\otimes n \).

\[\text{III. SOME REMARKS ON THE SYMMETRIC GROUP}\]

In this section we review some properties of the symmetric group and its irreducible representations. These are well-studied and numerous excellent presentations can be found, e.g. in [8]. Here we omit any proofs.

The symmetric group \( S_n \) contains the \( n! \) permutations of \( n \) distinguishable objects. Consider any representation \( D \) on a linear space \( V \). \( D \) is called to be irreducible iff there are no proper subspaces \( (\neq \{0\}) \) of \( V \) closed under \( D(S_n) \). A representation is called completely reducible iff it can be decomposed into a direct sum of irreducible
subrepresentations. This decomposition is unique up to isomorphisms. Our $D$ defined in the previous section is completely reducible.

The irreducible representations of $S_n$ are classified by the so–called standard Young tableaus. Each Young tableau is characterized by a partition of $n$, i.e. a set of integers $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_m > 0$, $m \leq n$, fulfilling $\sum \lambda_a = n$. One arranges $m$ rows of length $\lambda_1$, ..., $\lambda_m$ as shown in the figure and fills the boxes with the integers $1$, ..., $n$. The tableau is called standard iff the entries of the boxes are increasing within every row and within every column, see e.g. the figure.

At first we define the row symmetrizer $\text{SYM}_{[\lambda_1, ..., \lambda_m]} = \prod_{a=1}^{m} \text{SYM}_a$ with $\text{SYM}_a$ being the sum of all permutations within the $a$–th row. Then we still need the column antisymmetrizer $\text{ASYM}_{[\lambda_1, ..., \lambda_m]} = \prod_{b=1}^{\lambda_1} \text{ASYM}_b$ with $\text{ASYM}_b$ being the total antisymmetrizer of column $b$, i.e. the sum of $(-1)^\pi$ over all permutations of this column. $(-1)^\pi$ signifies whether $\pi$ is odd or even. The Young operator is then defined by

$$Y_{[\lambda_1, ..., \lambda_m]} = \text{ASYM}_{[\lambda_1, ..., \lambda_m]} \text{SYM}_{[\lambda_1, ..., \lambda_m]}$$

and is an element of the group algebra $\mathfrak{s}_n$.

If we go back to the representation $\tilde{D}(S_n)$, then the action of $Y_{[\lambda_1, ..., \lambda_m]}$ on any element $|v\rangle$ of $\tilde{V}$ maps this vector to an irreducible subrepresentation. A basis of the irreducible representation space can be constructed by applying all permutations to $Y_{[\lambda_1, ..., \lambda_m]} |v\rangle$ and selecting a maximal linearly independent subset. Every standard Young tableau gives a different irreducible representation, those corresponding to the same partition $[\lambda_1, ..., \lambda_m]$ but different entries are isomorphic. Depending on the structure of $\tilde{D}$, also the action of the same Young operator on different vectors from $\tilde{V}$ can give different irreducible subrepresentations of $\tilde{D}$. Every irreducible subrepresentation can be constructed in the prescribed way.

Another notion needed in the following is that of the associate representation. For any irreducible representation given by a standard Young tableau with partition $[\lambda_1, ..., \lambda_n]$ it is given by the transposed standard Young tableau, i.e. the rows become the columns and vice versa. The transposed partition is denoted by $[\lambda_1, ..., \lambda_m]$ with $\lambda_1 = m$ and $\tilde{m} = \lambda_1$. An example is given in the following figure.

IV. THE REPLICA SYMMETRIC EIGENSPACES

We now return to the problem of finding the eigenvalues of the replicated and disorder–averaged transfer matrix $T_n$. At the end of the second section we showed that the space $YV^\otimes n$ is invariant with respect to $T_n$ for every $Y \in \mathfrak{s}_n$. 3
In particular, this is the case for the Young operators which define minimal invariant sets obtainable without further knowledge of the exact form of $T_n$.

In this section we concentrate on a special irreducible subrepresentation described by the standard Young tableau with only one row. The Young operator $Y_n$ becomes the symmetrizer of the complete symmetric group, its image $Y_n V^\otimes n$ is therefore invariant under permutations. The corresponding irreducible subrepresentations of $D$ are thus one–dimensional, all permutations are represented trivially by the identity. Consequently the elements of $Y_n V^\otimes n$ are replica symmetric and therefore also the eigenvectors of $T_n$ constructed within this space.

As basis vectors for $Y_n V^\otimes n$ we introduce

$$|\rho_1, ..., \rho_{R-1}⟩ = \frac{1}{\rho_1! \cdots \rho_R!} Y_n |1⟩ \otimes \cdots \otimes |R⟩ \otimes \rho_R \sum_{s=1}^n |s⟩^{1 \otimes n}$$

(7)

where $\rho_s, s = 1, ..., R - 1$, and $\rho_R = n - \sum_{s=1}^{R-1} \rho_s$ have to be non–negative integers. The replica symmetric submatrix of $T_n$ can be calculated by

$$T_n[\rho_1, ..., \rho_{R-1}] = \frac{⟨\rho_1, ..., \rho_{R-1}⟩_{T_n[σ_1, ..., σ_{R-1}]}⟩}{⟨\rho_1, ..., \rho_{R-1}⟩_{[ρ_1, ..., \rho_{R-1}]}⟩ - 1 \cdots - \rho_{R-1}⟩_{[ρ_1, ..., \rho_{R-1}]}⟩ - 1}.$$  

(8)

The denominator results from the fact that the vectors (7) are orthogonal but not normalized. In order to send the replica number $n$ to zero we have to introduce generating functions into the eigenvalue equation

$$Λ_n Z(σ_1, ..., σ_{R-1}) = \sum_{\{'σ_1, ..., σ_{R-1}\}} T_n[\rho_1, ..., \rho_{R-1}] Z(ρ_1, ..., ρ_{R-1})$$

(9)

by writing

$$Φ[σ_1, ..., R−1] = \sum_{\{'σ_1, ..., σ_{R-1}\}} x^{R-1}_R Z(ρ_1, ..., ρ_{R-1}).$$

(10)

The eigenvalue equation [8] now reads

$$Λ_n [Φ[σ_1, ..., R−1]] = \left(\sum_{s=1}^R x_s T_{R,s} \right) n \cdot Φ \left[ \sum_{s=1}^R x_s T_{1,s}, ..., \sum_{s=1}^R x_s T_{R-1,s} \right]$$

(11)

where we introduced $x_R = 1$ for simplicity, for the calculations see appendix A. In this equation a sensible limit $n \to 0$ can be performed. The largest eigenvalue is $Λ = 1 - \beta nf + O(n^2)$ where $β$ is the inverse temperature and $f$ the free energy. Finally, we change from left to right eigenfunctions, introduce $x = (x_1, ..., x_{R-1}) ∈ R^{R-1}$ and

$$h_r(x) = \frac{\sum_{s=1}^R x_s T_{R,s}}{\sum_{s=1}^R x_s T_{R,s}}$$

(12)

and obtain an equation for a $(R - 1)$–dimensional invariant density

$$Φ(0)[x] = \int d^{R-1}y ≪ δ^{(R-1)}(x - h(y)) ≫ Φ(0)[x]$$

(13)

where we used the $(R - 1)$–dimensional Dirac distribution $δ^{(R-1)}(·)$. The density has to be normalized, $\int d^{R-1}x \ Φ(0)[x] = 1$. As in perturbation theory we calculate the $O(n)$–corrections of $Λ$ with the unperturbed eigenfunction,

$$f = -\frac{1}{β} \int d^{R-1}x \ Φ(0)[x] ≪ ln(\sum_{s=1}^R x_s T_{R,s}) ≫ .$$

(14)

In this paper we do not calculate this free energy for any special distribution. This task itself is very hard and has been solved only for a few distributions of quenched disorder, see e.g. [10] and references therein.

The same equations can be obtained also without using replicas. For the one–dimensional Ising model this was established by Derrida and Hilhorst in [10], their method using Riccati variables also generalizes to more complicated degrees of freedom than Ising spins. Another result reminiscent of ours was obtained by Lin [11], who showed the equivalence of an early replica approach by Kac with Dyson’s method for the phonon spectrum of a chain of random masses and springs.
V. TWO–POINT CORRELATIONS

To be sure that replica symmetry is not violated we have to consider the eigenvalues of the other representations. They have a very simple interpretation in terms of connected two–point correlation functions.

Consider the operator

\[ X(s) = x_s|s \]  

where \( x_s \) is any observable assigned to the basis vectors \( |s \rangle \), e.g. spin, location, or occupation number. It can be simply extended to the replicated vector space \( V^\otimes n \) by introducing the \( n \) operators \( X^{(n)}_a = 1^\otimes n-1 \otimes X \otimes 1^\otimes n-a, \ a = 1, \ldots, n \). They are commutative and measure the value of \( x \) at the \( a \)-th replica site. In addition we introduce the operators

\[ X^{(\lambda)} = \left\{ \begin{array}{ll} 1 & \text{if } \lambda = 1 \\ \prod_{1 \leq a < b \leq \lambda} (X^{(\lambda)}_a - X^{(\lambda)}_b) & \text{if } \lambda > 1 \end{array} \right. \]  

for any non–negative integer \( \lambda \). For every partition \( [\lambda_1, \ldots, \lambda_m] \) and its transpose \( [\tilde{\lambda}_1, \ldots, \tilde{\lambda}_m] \), they can be combined to the operator

\[ X_{[\lambda_1, \ldots, \lambda_m]} = X^{(\tilde{\lambda}_1)} \otimes \ldots \otimes X^{(\tilde{\lambda}_m)} \]  

acting on \( V^\otimes n \). Moreover, it maps any replica symmetric vector to a vector in a representation space belonging to the standard Young tableau with \( m \) rows of length \( \lambda_1, \ldots, \lambda_m \) where we fill one column after the other successively with integers 1, ..., \( n \). An example is given by the 2nd tableau in figure 1. A sketch of the proof will be shown in appendix B.

Using this we find that

\[ \text{tr}(T_{n}^{i}X_{[\lambda_1, \ldots, \lambda_m]}|T_{n}^{j-i}X_{[\lambda_1, \ldots, \lambda_m]}|T_{n}^{N-j}) \propto \Lambda^{[j-i]}_{[\lambda_1, \ldots, \lambda_m]} \]  

for large distances \( |j - i| \). \( \Lambda_{[\lambda_1, \ldots, \lambda_m]} \) is the largest eigenvalue if we consider only eigenfunctions in the subspace \( Y_{[\lambda_1, \ldots, \lambda_m]} \otimes V^\otimes n \).

Because of \( \sum_a \lambda_a = n \) we have by definition \( X^{(\lambda)} \)

\[ X_{[\lambda_1, \ldots, \lambda_m]} = X^{(\lambda_1)}_{[\lambda_1, \lambda_2, \ldots, \lambda_m]} \otimes 1^{n-\sum_{a=2}^{m} \lambda_a} . \]  

Introducing this into \( (18) \) we can send \( n \to 0 \) and obtain

\[ \ll < X_{[\sum_{a=2}^{m} \lambda_a, \lambda_2, \ldots, \lambda_m]}(i) \cdot X_{[\sum_{a=2}^{m} \lambda_a, \lambda_2, \ldots, \lambda_m]}(j) > \gg \propto \lim_{n \to 0} \Lambda^{[j-i]}_{[n-\sum_{a=2}^{m} \lambda_a, \lambda_2, \ldots, \lambda_m]}, \]  

i.e. the two–point correlation function of \( X_{[\sum_{a=2}^{m} \lambda_a, \lambda_2, \ldots, \lambda_m]} \) decays exponentially with correlation length \( \xi = -1/\ln \Lambda \left[ \sum_{a=2}^{m} \lambda_a, \lambda_2, \ldots, \lambda_m \right] \). \( \ll \cdot \gg \) denotes the thermodynamic average in the disordered system with transfer matrices \( T_i, i = 1, \ldots, N \). In order to calculate this we still need 2 \( \sum_{a=2}^{m} \lambda_a \) real non–interacting replicas of the original quenched system.

Here we concentrate on Young tableaus having only two rows, i.e. to partitions \( [n - \lambda, \lambda] \). There the operator reads

\[ X_{[n - \lambda, \lambda]} = (X \otimes 1 - 1 \otimes X)^{\otimes \lambda} \otimes 1^{\otimes (n-2\lambda)} \]  

and consequently

\[ \ll < X_{[\lambda, \lambda]}(i) \cdot X_{[\lambda, \lambda]}(j) > \gg = \ll < x_{s_i} x_{s_j} > - < x_{s_i} > < x_{s_j} > > \Lambda \gg \propto \Lambda^{[j-i]}_{[-\lambda, \lambda]} \]  

describes the \( \lambda \)-th moment of the connected two–point correlation function with respect to the disorder distribution. The correlation length diverges whenever \( \lim_{n \to 0} \Lambda_{[n - \lambda, \lambda]} = 1 \). The criterion for replica symmetry breaking, i.e. the degeneracy of the largest replica symmetric eigenvalue with a non–symmetric one, thus coincides with the standard criterion for a phase transition. However, for one–dimensional systems with finite \( R \) we cannot expect any phase transition for non–vanishing temperature.

5
In App. C we will develop an equation for $\lim_{n\to 0} \Lambda_{(n-\lambda)}$. The calculations are quite similar to replica symmetric one, but due to the more complicated representation structure they are somewhat lengthy. Here we give only the final result, an eigenvalue equation for a function

$$\Phi^{[-\lambda,\lambda]} : \mathbb{R}^{R-1} \to \mathbb{R}^{((R-1)^{\lambda})}$$

(23)

given by its components $\Phi^{[-\lambda,\lambda]}_{s^1,\ldots,s^\lambda}(x)$:

$$\Lambda_{[-\lambda,\lambda]} \Phi^{[-\lambda,\lambda]}_{s^1,\ldots,s^\lambda}(x) = \int d^{R-1}y \sum_{r^1,\ldots,r^\lambda=1}^{R-1} \ll \delta^{(R-1)}(x-h(y)) \prod_{a=1}^{\lambda} \frac{\partial h_{sa}}{\partial y_a} \gg \Phi^{[-\lambda,\lambda]}_{r^1,\ldots,r^\lambda}(x)$$

(24)

For every eigenfunction $\Phi^{[-1,1]}_{s^1,\ldots,s^\lambda}(x)$ of $T^{[n-1,1]}_0$ for $n \to 0$ the function $\nabla \cdot \Phi^{[-1,1]}_{s^1,\ldots,s^\lambda}(x)$ is an eigenfunction of the replica symmetric transfer matrix given in (13) to the same eigenvalue. Only the largest replica symmetric eigenvalue ($=1$) cannot be reached in this way, because the integral of $\nabla \cdot \Phi^{[-1,1]}_{s^1,\ldots,s^\lambda}(x)$ over the definition space $\mathbb{R}^{R-1}$ vanishes. Therefore the largest eigenvalue of $T^{[-1,1]}_0$ equals the second largest of $T^{[0]}_0$ and so on. The first transfer matrix block which could produce a diverging correlation length outside the replica symmetric sector is the one corresponding to $[-2,2]$, i.e. to the second moment of the connected two–point correlation function. This is know to be just for spin glass transitions where the second moment of the connected two–point function describes the non–linear susceptibility, cf. [12].

VI. SUMMARY AND OUTLOOK

In this paper we developed a general replica transfer matrix method capable of handling products of random finite–dimensional matrices. We obtained expressions for the free energy (or Lyapunov exponent) from the replica symmetric vectors $\Phi^{[-\lambda,\lambda]}_{s^1,\ldots,s^\lambda}(x)$ where the replica symmetric transfer matrix is given by

$$\Lambda_{[-\lambda,\lambda]} \Phi^{[-\lambda,\lambda]}_{s^1,\ldots,s^\lambda}(x) = \int d^{R-1}y \sum_{r^1,\ldots,r^\lambda=1}^{R-1} \ll \delta^{(R-1)}(x-h(y)) \prod_{a=1}^{\lambda} \frac{\partial h_{sa}}{\partial y_a} \gg \Phi^{[-\lambda,\lambda]}_{r^1,\ldots,r^\lambda}(x)$$

(24)

APPENDIX A: LAPLACE TRANSFORM OF THE REPLICA SYMMETRIC EIGENVALUE EQUATION

In this appendix we calculate the Laplace transform of the replica symmetric eigenvalue equation (13). We start with

$$\Lambda_{[\sigma]} Z(\sigma_1,\ldots,\sigma_{R-1}) = \sum_{\{\rho_1,\ldots,\rho_{R-1}\}} T^{[n]}_{[\sigma]}(\rho_1,\ldots,\rho_{R-1}|\sigma_1,\ldots,\sigma_{R-1}) Z(\rho_1,\ldots,\rho_{R-1})$$

(A1)

where the replica symmetric transfer matrix is given by

$$T^{[n]}_{[\sigma]}(\rho_1,\ldots,\rho_{R-1}|\sigma_1,\ldots,\sigma_{R-1}) = \frac{\langle \rho_1,\ldots,\rho_{R-1}|T^{[n]}|\sigma_1,\ldots,\sigma_{R-1} \rangle}{\langle \rho_1,\ldots,\rho_{R-1}|\rho_1,\ldots,\rho_{R-1} \rangle}.$$  

(A2)

using the replica symmetric vectors

$$|\rho_1,\ldots,\rho_{R-1}\rangle = \sum_{\{s^1,\ldots,s^n\} \sum_a \delta_{a,s} = \rho_a \ \forall s=1,\ldots,R-1 \} |s^1\ldots s^n \rangle,$$

(A3)

see Sec. IV. If we introduce the Laplace transformation (10) on the left side of (A1) we obtain (introducing $x_R = 1, \rho_R = n - \rho_1 - \ldots - \rho_{R-1}$)
\[
\Lambda_{[n]} \Phi[x_1, \ldots, x_{R-1}] = \sum_{\rho_1, \ldots, \rho_{R-1}} \sum_{\sigma_1, \ldots, \sigma_{R-1}} x^\sigma_1 \cdots x^\sigma_{R-1} T^n_{\sigma_1, \ldots, \sigma_{R-1}} (\rho_1, \ldots, \rho_{R-1}, \sigma_1, \ldots, \sigma_{R-1}) = \sum_{\rho_1, \ldots, \rho_{R-1}} \sum_{\{a_s\}} x_1^{a_{\rho_1}} \cdots x_{R-1}^{a_{\rho_{R-1}}} \langle 1 \rangle_{\rho_1} \cdots \langle R \rangle_{\rho_{R-1}} T_n^{s_1 \cdots s_n} \]

\[
\text{APPENDIX B: PROOF TO SEC. V}
\]

In this appendix we show that the operators \(X_{[\lambda_1, \ldots, \lambda_m]}\) defined in (17) map any replica symmetric vector to a representation space for an irreducible representation with a \(\Lambda\) Young tableau described by \([\lambda_1, \ldots, \lambda_m]\). This can be done by proving the equation

\[
Y_{[\lambda_1, \ldots, \lambda_m]} X_{[\lambda_1, \ldots, \lambda_m]} Y_{[n]} = c_{[\lambda_1, \ldots, \lambda_m]} X_{[\lambda_1, \ldots, \lambda_m]} Y_{[n]} \tag{B1}
\]

where \(c_{[\lambda_1, \ldots, \lambda_m]}\) is a real number given by \(Y_{[\lambda_1, \ldots, \lambda_m]}^2 = c_{[\lambda_1, \ldots, \lambda_m]} Y_{[\lambda_1, \ldots, \lambda_m]}^2\). Here we concentrate on the case \([n - \lambda, \lambda]\), i.e. to Young tableaus with only two rows. These are the most important cases for our needs, and the proof can be generalized directly to more complicated tableaus as well using analogous procedures.

In the case of two rows we have

\[
Y_{[n - \lambda, \lambda]} = (1 - (1, 2))(1 - (3, 4)) \cdots (1 - (2\lambda - 1, 2\lambda)) \cdot \text{SYM}_{[n - \lambda, \lambda]} \tag{B2}
\]

where \((a, b)\) denotes the transposition permuting \(a\) and \(b\), and

\[
X_{[n - \lambda, \lambda]} = (X \otimes 1 - 1 \otimes X)^{\otimes \lambda} \otimes 1^{\otimes (n - 2\lambda)} \tag{B3}
\]

(i) As a first step we note that

\[
\forall \pi \in \mathcal{S}_n : \quad \pi X_{a_1}^{(n)} \cdots X_{a_l}^{(n)} Y_{[n]} = X_{\pi(a_1)}^{(n)} \cdots X_{\pi(a_l)}^{(n)} \pi Y_{[n]} = X_{\pi(a_1)}^{(n)} \cdots X_{\pi(a_l)}^{(n)} Y_{[n]} \tag{B4}
\]

It follows that \(\text{SYM}_{[n - \lambda, \lambda]} X_{[n - \lambda, \lambda]} Y_{[n]}\) is a sum of certain \(X_{a_1}^{(n)} \cdots X_{a_l}^{(n)} Y_{[n]}\) with integer prefactors depending on \(a_1 < \cdots < a_l\).

(ii) The action of \(\text{ASYM}_{[n - \lambda, \lambda]} = (1 - (1, 2))(1 - (3, 4)) \cdots (1 - (2\lambda - 1, 2\lambda))\) on these gives

\[
\text{ASYM}_{[n - \lambda, \lambda]} X_{a_1}^{(n)} \cdots X_{a_l}^{(n)} Y_{[n]} = \begin{cases} 
\pm X_{[n - \lambda, \lambda]} Y_{[n]} & \text{if} \ a_\rho \in \{2\rho - 1, 2\rho\} \forall \rho = 1, \ldots, \lambda \\
0 & \text{else}
\end{cases} \tag{B5}
\]

If there were the factors \(X_{2\rho - 1}^{(n)}\) and \(X_{2\rho}^{(n)}\) for any \(\rho \leq \lambda\), the action of \((1 - (2\rho - 1, 2\rho))\) would annihilate the term. The same happens, if there is any \(\rho < \lambda\) for which neither \(X_{2\rho - 1}^{(n)}\) nor \(X_{2\rho}^{(n)}\) appear in the product. The sign in (B5) can be obtained by counting the even indices \(a_\rho\) in \(X_{a_1}^{(n)} \cdots X_{a_l}^{(n)}\).

Altogether we find that the action of the Young operator produces only a constant of proportionality, and the proof is complete.
APPENDIX C: CALCULATION OF NON–SYMMETRIC EIGENVALUE EQUATIONS

In this appendix we present the calculation of the eigenvalue equations for non–trivial irreducible representations at the example of \([n-1,1]\). This case is surely the simplest non–trivial one, but the ideas of the calculation are the same also for higher representations.

We consider the standard Young tableau for the partition \([n-1,1]\) having entries \(1,3,4,...,n\) in the first row and 2 in the second. The corresponding Young operator

\[
Y_{[n-1,1]} = (1 - (1,2)) \cdot \text{SYM}(1,3,4,...,n)
\]  

maps an arbitrary basis vector \(|s_1...s_n\rangle\) up to a normalization constant to

\[
|s;\sigma_1, ..., \sigma_{R-1}\rangle := \left\{ \begin{array}{l}
\sum_{s \neq s_2} (|ss_2\rangle - |s^2\rangle) \otimes |\sigma_1, ..., \sigma_{\min(s,s_2)} - 1, ..., \sigma_{\max(s,s_2)} - 1, ..., \sigma_{R-1}\rangle
\end{array}\right.
\]

where the last term in the product is a symmetrized vector in the \((n-2)\)-fold replicated vector space, cf. equation (C1), and \(\sigma_s = \sum_{s=1}^{n} \delta_{s,s'}\). Due to \(Y_{[n-1,1]}T_n = T_nY_{[n-1,1]}\) these vectors form an invariant set with respect to \(T_n\). For given \(\sigma_1, ..., \sigma_{R-1}\) there are \(R-1\) linearly independent vectors of this type, so without loss of generality we can choose \(s^2 = 1, ..., R-1\).

The transfer matrix block to be calculated is

\[
T_{[n-1,1]}^{(n-1,1)}(s;\sigma_1, ..., \sigma_{R-1}|r;\rho_1, ..., \rho_{R-1}) = \left\{ \begin{array}{l}
\langle s;\sigma_1, ..., \sigma_{R-1}|T_n|r;\rho_1, ..., \rho_{R-1}\rangle
\end{array}\right.
\]

\[
= \sum_{t \neq r} (T_{R,t}T_{s,t} - T_{R,t}T_{s,t})
\]

\[
\times (T^{\otimes(n-2)}|n-2\rangle (\sigma_1, ..., \sigma_{s} - 1, ..., \sigma_{R-1} - 1, ..., \rho_1, ..., \rho_{R-1}) .
\]

The matrix \((T^{\otimes(n-2)}|n-2\rangle\) is nothing but the replica symmetric matrix \(T_{[n-2]}^{[n-2]}\) without the average over the quenched disorder. For the eigenvalue equation

\[
\Lambda_{[n-1,1]} \cdot C(r;\rho_1, ..., \rho_{R-1}) = \sum_{s,\sigma_1, ..., \sigma_{R-1}} T_{[n-1,1]}^{[n-1,1]}(s;\sigma_1, ..., \sigma_{R-1}|r;\rho_1, ..., \rho_{R-1}) C(s;\sigma_1, ..., \sigma_{R-1})
\]

we introduce again a Laplace transform by

\[
\Phi_s[x_1, ..., x_{R-1}] = \sum_{\sigma_1, ..., \sigma_{R-1}} x_1^{\sigma_1} \cdot ... \cdot x_{s-1}^{\sigma_{s-1}} \cdot ... \cdot x_{R-1}^{\sigma_{R-1}} C(s;\sigma_1, ..., \sigma_{R-1}) .
\]

Due to \((x_R := 1)\)

\[
\sum_{\rho_1, ..., \rho_{R-1}} x_1^{\rho_1} \cdot ... \cdot x_{s-1}^{\rho_{s-1}} \cdot ... \cdot x_{R-1}^{\rho_{R-1}} T_{[n-1,1]}^{[n-1,1]}(s;\sigma_1, ..., \sigma_{R-1}|r;\rho_1, ..., \rho_{R-1})
\]

\[
= \sum_{t \neq r} (T_{R,t}T_{s,t} - T_{R,t}T_{s,t}) x_t \left( \sum_{\rho_1, ..., \rho_{R-1}} x_1^{\rho_1} \cdot ... \cdot x_{s-1}^{\rho_{s-1}} \cdot ... \cdot x_{R-1}^{\rho_{R-1}} (T^{\otimes(n-2)}|n-2\rangle (\sigma_1, ..., \sigma_{s} - 1, ..., \sigma_{R-1} - 1, ..., \rho_1, ..., \rho_{R-1}) \right)
\]

\[
= \sum_{t \neq r} (T_{R,t}T_{s,t} - T_{R,t}T_{s,t}) x_t \left( \sum_{p=1}^{R} x_p T_{R,p} \right)^{n-2} \left( \prod_{q=1}^{R} \left( \sum_{p=1}^{R} x_p T_{R,p} \right)^{\sigma_q-\delta_{s,q}} \right)
\]

\[
= \left( \sum_{p=1}^{R} x_p T_{R,p} \right)^n \cdot \partial h_s(x) \cdot \prod_{q=1}^{R-1} h_q(x)^{\sigma_q-\delta_{s,q}}
\]

where the 2nd last step is the same as in App. A for the replica symmetric case, and where we use the function \(h(x)\) defined in (I2). The eigenvalue equation becomes

\[
\Lambda_{[n-1,1]} \cdot \Phi_r[x_1, ..., x_{R-1}] = \left( \sum_{p=1}^{R} x_p T_{R,p} \right) \sum_{s=1}^{R-1} \partial h_s \Phi_r[h_1(x), ..., h_{R-1}(x)] .
\]
In the limit $n \to 0$ this results in equation (24) for $\lambda = 1$. The calculations for larger $\lambda$ are analogous.

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