AdSS\textsubscript{5} Brane World Cosmology

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Abstract: The gravitational equations of the 5-dimensional analogue of the AdSS space-time, where all the matter fields are confined on the 3-brane are examined. The most general solutions are established in the generic case of a non-$Z_2$-symmetric bulk. Constraining these solutions we derive a number of remarkable metrics widely investigated in the literature. Finally, we make many important conclusions about the viability of the presented scenario and cosmology.

Keywords: Brane Worlds, Schwarzschild-anti-de Sitter space-time.
1. Introduction

Since the papers of Kaluza and Klein [1] it has been suggested a possibility that there exist extra dimensions beyond those of Minkowski space-time. In recent years, the ideas of extra dimensions has become much more compelling. According to [2, 3, 4] it has been suggested that additional dimensions could have a quite distinct nature from those of the Kaluza and Klein. In other words, ordinary matter would be confined to our 4-dimensional universe while gravity would live in extended space-time (bulk).

During the past 30 years, research in the theory of black holes has brought to light strong hints of a fundamental relationship between gravitation, thermodynamics, and the quantum theory. Indeed, the discovery of the thermodynamic behavior of black holes has given rise to most of our present physical insights into the nature of quantum phenomena occurring in strong gravitational fields. Recently several authors have found exact cosmological solutions of the brane worlds models described by 5-dimensional black-hole like geometries [5, 6, 7, 10, 11, 12, 13, 14, 15, 16]. Hence it is significant to examine the bulk modifications in the case of the non-$Z_2$-symmetry and the external finite temperature (see below).

In a typical brane world scenario [2] our 4-dimensional universe $q_{\mu\nu}$ is described by a 3-brane in 5-dimensional space-time $g_{ab}$. The extra dimension need not be small or compact; in [8] Randall and Sundrum (RS) shown that gravity can be localized on a single brane even though the fifth dimension is noncompact. The metric contains a warp factor which is an exponential function of the extra dimension

$$ds^2 = e^{-2k|y|}(-dt^2 + d\vec{x}^2) + dy^2.$$  \hspace{1cm} (1.1)

The bulk is described by the $AdS_5$ metric with $y = 0$ taken as the brane, so that $y < 0$ is identified with $y > 0$ illustrating thus the $Z_2$-symmetry.

The success of the $AdS/CFT$ correspondence [18, 19, 20, 21, 22, 23] has lead to the intense study of these space-times. The $AdS/CFT$ correspondence is a quite specific conjecture. For instance, this conjecture allows us to regard the entire $AdS_5$ geometry as a manifestation of the dynamics of a 4-dimensional conformal field theory. It has been realized that there are deep connections between the brane worlds and the $AdS/CFT$ correspondence [20, 23, 24]. In this case, one should require two copies of the conformal field theory, one for each of the $AdS_5$ patches. One important point is that for a stable vacuum state of the theory, the extrinsic curvature of the boundary surface (brane) should be proportional to the induced metric [18, 21, 25]. For this reason, the bulk is changed from $AdS_5$ to $AdSS_5$ given by the metric

$$ds^2 = e^{-2y/\Lambda}[-(1 - (\pi\Lambda T_0)^4 e^{4y/\Lambda}) dt^2 + d\vec{x}^2] + \frac{dy^2}{1 - (\pi\Lambda T_0)^4 e^{4y/\Lambda}},$$  \hspace{1cm} (1.2)

1Where the Latin indices $a, b = 0, 1, 2, 3, 4$; and the Greek indices $\mu, \nu = 0, 1, 2, 3$. 

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where Λ (< 0) is the bulk cosmological constant and $T_0$ is the Hawking temperature (see [13, 4, 5] for discussions and references). This is a constant parameter of the $AdSS_5$ solution. We stress that black-hole like geometries is an standard way to introduce the temperature [12, 13]. At the same time taking into account the brane worlds ideology one suitable candidate can be constructed by gluing together two patches of the $AdSS_5$ space-time along the brane world volume in a $Z_2$-symmetric way

$$
 ds^2 = e^{-2k|y|}[-\left(1 - \frac{U_T^4}{k^4} e^{4k|y|}\right)dt^2 + d\vec{x}^2] + \frac{dy^2}{1 - \frac{U_T^4}{k^4} e^{4k|y|}},
$$

(1.3)

where the horizon parameter $U_T$ is proportional to the external temperature $T_0$.

Further, motivating by (1.2), (1.3) and other possible $AdSS_5$ like space-times we assume the ansatz

$$
 ds^2 = e^{2\sigma(y)}[-\zeta(y)dt^2 + d\vec{x}^2] + \zeta(y)^{-1}dy^2.
$$

(1.4)

In fact, this anzatz gives a possibility to establish the most general solution of such family of space-times. This solution can be useful in order to clarify the symmetries and the main features of the $AdSS_5$ bulk and therefore to derive the induced Friedmann-Robertson-Walker (FRW) cosmology on the brane (see [19] for the radiation-dominated example). For the same reason, it seems necessary to extend this scenario by adding matter to the brane.

## 2. Model Building

Following methods of [8, 10] we suppose the location of the brane in the form: $t = t(\tau)$, $y = y(\tau)$ parametrized by the local time $\tau$ on the brane; then the induced metric on the brane can be expressed as follows

$$
 ds^2_{ind} = -d\tau^2 + a(\tau)^2d\vec{x}^2 \equiv q_{\mu\nu}dx^\mu dx^\nu.
$$

(2.1)

Obviously, the embedding of a brane is specified by the extrinsic curvature $K_{ab} = -(\delta_a^c - n_a n^c)\nabla_c n_b$ on both sides, where $\nabla_*$ is the covariant derivative compatible with (1.4). Suppose the extrinsic curvature being associated with the outward unit normal; then the proportionality condition between the extrinsic curvature and the induced metric takes the form

$$
 K_{\mu\nu} = -\frac{1}{\Lambda} q_{\mu\nu}
$$

(2.2)

There is a set of solutions to (2.2) that relates the functions $t$ and $y$ in terms of $\sigma(y)$, $\zeta(y)$. In other words, we have only one function $y(\tau)$ that locates the brane. All the other surfaces have the same induced metric since they are given by translations in
Next step to proceed is to parametrize a particular surface by \((t_*, \bar{x})\), where \(t_*\) is a solution of (2.2) for \(t\) in terms of \(y\). Then the induced metric becomes
\[
ds_{\text{ind}}^2 = -e^{2\sigma(y)}(y)dt_*^2 + e^{2\sigma(y)}d\bar{x}^2,
\]
which can be expressed in the standard FRW form (2.1) defining the \(y = y(\tau)\) function appropriately. Notice that the authors of [8, 10] have introduced the relation between \(t(\tau)\) and \(y(\tau)\) in the form of a handy normalization condition. However in our case this relation is required by (2.2) in order to retain the vacuum state of the theory on the boundary [19, 21, 25].

For simplicity, we consider the confinement of the matter on a surface given by \(\tau = \tau_*\) such that \(y = y(\tau_*) = 0\), but keeping in mind that all the other surfaces can be obtained by translations in \(\tau\). Then the 5-dimensional Einstein equations are
\[
(5)R_{ab} = \frac{1}{2}g_{ab}(5)R = \tilde{k}^2T_{ab}; \quad T_{ab} = -\Lambda g_{ab} + \sqrt{|\zeta(y)|}S_{ab}\delta(y),
\]
where \(\tilde{k}\) is the 5-dimensional gravitational coupling constant such that \(\tilde{k}^2 = \frac{8\pi}{M_5^3}\) with \((5)M_p^3\) being the fundamental 5-dimensional Planck mass and
\[
S_{ab} = (-\lambda q_{\mu\nu} + \tau_{\mu\nu})\delta_a^\mu\delta_b^\nu.
\]
Notice that \(\lambda (> 0)\) and \(\tau_{\mu\nu}\) are the intrinsic brane tension and the energy-momentum tensor of the brane that we require to be expressed in the perfect fluid form \(\tau_{\mu\nu} = \text{diag}\{-\tilde{\rho}, \tilde{p}_1, \tilde{p}_2, \tilde{p}_3\}\). The equations (2.4) can be derived by taking the variation of the action
\[
S = \int d^5x\sqrt{|g|}\left(\frac{(5)M_p^3}{16\pi}(5)R - \Lambda\right) + \int d^4x\sqrt{|\tilde{g}|}\left(\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - V(\varphi)\right) - \lambda).
\]
The energy density and the pressure given by \(\tau_{\mu\nu}\) are expressed as follows
\[
\bar{\rho} = -q_{00}\left(q^{00}\frac{1}{2}(\partial_0\varphi)^2 - V(\varphi)\right) \equiv q_{00}\rho, \quad \bar{p}_{\mu} = -q_{\mu\nu}\left(q^{00}\frac{1}{2}(\partial_0\varphi)^2 + V(\varphi)\right) \equiv q_{\mu\nu}\rho.
\]
Suppose that \(p = \omega\rho\); then it is straightforward to show that the equations (2.4) yield three relations
\[
\zeta(y)\sigma'(y)^2 + \frac{1}{4}\zeta'(y)\sigma'(y) = k^2,
\]
\[
2\zeta'(y)\sigma'(y) + \frac{1}{2}\zeta''(y) = \tilde{k}^2\sqrt{|\zeta(y)|}(1 + \omega)\rho \delta(y),
\]
\[
-3\sigma''(y)\sqrt{|\zeta(y)|} = \tilde{k}^2(\lambda + \rho)\delta(y),
\]
where \(k^2 \equiv -\tilde{k}^2\Lambda/6\) and the prime denotes differentiation with respect to \(y\). We stress that these relations are similar to the result for the first time obtained in [4] for \(\rho = 0\).
It is possible to derive the most general solution of \((2.7)\) in a closed form
\[
\sigma(y) = \alpha |y| + \beta y + \chi; \quad (2.8)
\]
\[
\zeta(y) = \gamma^2 + \left( \frac{k^2 \Lambda}{6(\alpha \epsilon(y) + \beta)^2} + \gamma^2 \right) (e^{-4(\alpha |y| + \beta y)} - 1) \quad (2.9)
\]
with the Israel junction conditions \([3]\) at the brane
\[
\frac{\beta^2}{\alpha^2} = \frac{12 \Lambda}{k^2[2(\lambda + \rho)^2 - 3(\lambda + \rho)(1 + \omega)\rho]} + 1; \quad \gamma^2 \alpha^2 = \frac{k^4(\lambda + \rho)^2}{36}, \quad (2.10)
\]
where \(\lim_{y \to \pm 0} \zeta(y) \equiv \gamma^2; \lim_{y \to \pm 0} \sigma(y) \equiv \chi\) and \(\epsilon(y) \equiv |y|' = \begin{cases} 1, & y > 0 \\ -1, & y < 0 \end{cases}\).

It is seen that these solutions do not exhibit, in the general case, the \(Z_2\)-symmetry and hence a number of restrictions should be imposed. To be precise, suppose that \(\alpha = -k, \beta = 0, \chi = 0, \text{ and } \rho \neq 0\); then \((2.8), (2.9)\) and \((2.10)\) yield
\[
\sigma(y) = -k |y|, \quad \zeta(y) = 1, \quad \lambda = \left( \frac{3(5)M_p^3}{4\pi} \right) k. \quad (2.11)
\]
This straightforwardly leads us to the warp factor and the intrinsic brane tension exactly as in the original RS scenario \([2, 3]\). It follows that in the \(Z_2\)-symmetric case with no matter on the brane, we can achieve only an original RS like scenario. If one relax the assumption of \(Z_2\)-symmetry \((\alpha = -k, \beta \neq 0, \chi = 0, \rho = 0)\); then it can readily be checked that \((2.8), (2.9)\) and \((2.10)\) yield
\[
\zeta(y) = \frac{k^2}{(\beta - k\epsilon(y))^2} - \left( \frac{k^2}{(\beta - k\epsilon(y))^2} + \frac{k^2}{\beta^2 - k^2} \right) e^{4(\beta |y| - \beta y)}, \quad (2.12)
\]
\[
\sigma(y) = -k |y| + \beta y, \quad \text{where} \quad \beta^2 = k^2 \left( 1 - \frac{9(5)M_p^6 k^2}{16\pi^2 \lambda^2} \right). \quad (2.13)
\]
Notice that one similar case was considered in the paper \([\text{I}]\). However the authors has focused on the equations like \((2.7)\) being solved in the form of infinite series
\[
\zeta(y) = 1 + \chi \zeta_1(y) + \chi^2 \zeta_2(y) + \chi^3 \zeta_3(y) + \chi^4 \zeta_4(y) + \mathcal{O}(\chi^5),
\]
\[
\sigma(y) = k |y| + \chi \sigma_1(y) + \chi^2 \sigma_2(y) + \chi^3 \sigma_3(y) + \chi^4 \sigma_4(y) + \mathcal{O}(\chi^5)
\]
with \(\zeta_i(y), \sigma_i(y)\) illustrating both the \(Z_2\)-symmetric and \(Z_2\)-antisymmetric behavior alternately. Apparently, summing these infinite series one should obtain expressions of the type \((2.12), (2.13)\). The inclusion of the matter gives us a possibility to consider the case \(\alpha = -k, \beta = 0, \chi = 0, \text{ and } \rho \neq 0\). The reader will have no difficulty in showing that the solution \((2.8), (2.9)\) can be reduced to the metric \((1.3)\), where
\[
\sigma(y) = -k |y|; \quad \zeta(y) = 1 - \frac{U_T^4}{k^4} e^{4k|y|}. \quad (2.14)
\]
with two additional conditions in the form
\[ \lambda = \frac{1}{4} \left( \rho (3 \omega - 1) \pm \sqrt{9 \rho^2 (1 + \omega)^2 - 12 \Lambda (5) M_p^3 / \pi} \right), \tag{2.15} \]
\[ (1 + \omega) = \frac{1}{\rho} \left( \frac{U_T^4 (5) M_p^3}{2 \pi k (k^4 + U_T^4)^{1/2}} \right) \equiv \frac{1}{\rho} \theta > 0. \tag{2.16} \]

Now recall that the pressure is related to the cosmological energy density via \( \omega \). This implies that \( p = -\rho + \theta \). In fact, one can suppose that \( \theta \ll \rho \); then \( \rho \simeq -p \) and the matter sector has a suitable form for the slow-roll regime [see (2.6)].

Since we are interested in cosmological solutions, let us remember that we considered the confinement of the matter on a surface given by \( \tau = \tau_* \) such that \( y = y(\tau_*) = 0 \). All the other surfaces have the same induced metric (2.1) related by translations in \( \tau \). The solutions (2.8), (2.9) and the junction conditions (2.10) are relevant in order to understand symmetries and the main features of the \( AdS_S^5 \) bulk and therefore to derive the induced cosmology on the brane. Typical brane worlds scenarios assume the \( Z_2 \) symmetry motivated by the M-Theory origin. Nevertheless many recent papers examine models that are not derived from M-Theory. For instance, there have been suggested multi-brane models and scenarios, where the bulk is not manifestly \( Z_2 \)-symmetric \([8, 10, 11, 16]\) or one basically different action from (2.6) is proposed \([17]\). The conventional results that underline the non-\( Z_2 \)-symmetry reside in the effective Friedmann equations. Notice that one general brane embedding formalism have been constructed \([15]\) and the extra terms beyond the standard \( Z_2 \)-symmetric case, which characterize the non-\( Z_2 \)-symmetric embedding have been established. In our case, we do not derive such field equations. However the proposed brane model can be compared to the scenario \([8, 11]\), where one similar space-time is taken. Then the extra term must behaves like a positive Lambda-term for radiation or like a negative curvature term for dust (see also \([11]\)). Further, I avoid presenting details since the calculations are not original to me. We stress that constraints on extra terms from nucleosynthesis must be taken into account as well \([11, 17]\). It can be expected that effects of the \( Z_2 \)-symmetry breaking terms are decreasing with the evolution of universe and hence at late times the standard brane worlds cosmology should be recovered \([11, 16]\).

On the other hand, the bulk metrics (1.2) and (1.3) contain the terms that are proportional to the Hawking temperature. This can be understood as an external temperature that, in general, can be related to the boundary conformal field temperature via the \( AdS/CFT \) correspondence \([19]\). In our case, the expression (2.16) can suggests one possible candidate of such relation. However the finite temperature universe and the temperature induced on the brane are the topics in the course of development \([12, 13, 14]\) and hence we leave the details for a forthcoming paper. Usually, the authors evaluate the energy of the quantum bulk matter fields on a \( AdS_5(AdS_5) \) background at nonzero temperature. The brane seems these thermal
quantum effects via the effective potential that supplements the effective energy-momentum tensor. Finally, the existence of the thermal Casimir brane effect is also a subject of growing interest \cite{12, 21} and should be taken into account.

3. Conclusions

In this paper we have examined a brane world cosmological scenario based on the $AdS_5$ space-time. Taking account of the $AdS/CFT$ correspondence and the black-hole analogy we noticed a number of brane worlds candidates \cite{1, 2, 3, 4} (see also \cite{8, 9, 10, 12}), which can be formulated in the $AdS_5$ bulk and further, we introduced the anzatz \cite{1, 2}. In the basic section we have obtained the most general solutions for this anzatz and the action \cite{2, 6}. These solutions, in the general case, do not exhibit the $Z_2$-symmetry and a number of restrictions should be imposed. We have shown that the solutions \cite{2, 8}, \cite{2, 9} can lead us to the solutions of the original RS scenario \cite{2, 3, 4} or to the setup of the scenario proposed in \cite{7}.

The $AdS_5$ brane cosmology is an widely investigated topic in the literature. However in our case, the $AdS/CFT$ generic condition \cite{2, 2} imposes one physically relevant relation between the location functions $t(\tau)$ and $y(\tau)$. Hence it can be important to revise $AdS_5$ based cosmologies taking account of the solutions \cite{2, 8}, \cite{2, 9} and the relation \cite{2, 2}. Anyway, the arguments in this paper extend the results in the literature on the black-hole like geometries and the induced brane cosmologies.

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