Positive mass theorems for asymptotically AdS spacetimes with arbitrary cosmological constant

Naqing Xie Xiao Zhang

Abstract

We formulate and prove the Lorentzian version of the positive mass theorems with arbitrary negative cosmological constant for asymptotically AdS spacetimes. This work is the continuation of the second author’s recent work on the positive mass theorem on asymptotically hyperbolic 3-manifolds.

1 Introduction

In general relativity, our spacetime is modelled by a 4-dimensional Lorentzian manifold \((N^{1,3}, \tilde{g})\) with the Lorentzian metric \(\tilde{g}\) of signature \((-1,1,1,1)\) which satisfies the Einstein field equations

\[
\tilde{Ric}(\tilde{g}) - \frac{\tilde{R}(\tilde{g})}{2} \tilde{g} + \Lambda \tilde{g} = T
\]

where \(\tilde{Ric}, \tilde{R}\) are the Ricci and scalar curvatures of \(\tilde{g}\) respectively, \(T\) is the energy-momentum tensor of matter, and \(\Lambda\) is the cosmological constant.

It is well known that the positive mass theorem plays an important role in general relativity. The definition of the total energy and the total linear momentum for asymptotically flat spaces was given by Arnowitt-Deser-Misner from the Hamiltonian point of view \([2]\). This ADM mass is in fact a geometric invariance \([3, 5]\). Physicists believe, with some justification, that the total mass for a nontrivial isolated gravitational system must be positive. This was the famous positive mass conjecture which was first proved by Schoen and Yau in a series of papers \([16, 17, 18]\) using minimal surface techniques and then by Witten \([21, 15, 3]\) using spinors.

It is natural to extend the positive mass theorem to asymptotically AdS spacetime where spatial infinities are asymptotically hyperbolic. Such a theorem was proved with a fixed negative cosmological constant for spacelike, asymptotically hyperbolic hypersurfaces with zero second fundamental form in \([20, 6, 7]\), and with nonzero second fundamental form in \([25, 13]\). In general, there are two versions of the positive mass theorem (cf. \([22]\)). One is the Riemannian setting to use the initial data set which is a 3-dimensional Riemannian manifolds equipped with another 2-tensor. The other is the Lorentzian setting to use a spacelike hypersurface in 4-dimensional Lorentzian manifolds. Although, technically, different spin structures are used in different settings, the two versions are essentially equivalent in asymptotically flat spacetimes. Interestingly, the situation changes in asymptotically AdS spacetimes and the Riemannian version of the positive mass theorem in \([25]\) is not equivalent to the Lorentzian version in \([13]\). For instance, for the maximal spacelike hypersurfaces in AdS spacetimes, the dominant energy condition in AdS spacetimes implies the energy condition in \([25]\), hence the theorem in \([25]\) holds. However, this theorem is not included in the nonnegativity of the energy-momentum matrix in \([13]\).
The present paper is essentially the continuation of the second author’s recent work in [25]. We will prove positive mass theorems for asymptotically AdS spacetimes with arbitrary negative cosmological constant. We first define $e_0$-Killing spinors and use it to obtain the corresponding Lorentzian version of the positive mass theorem in [25]. We then use imaginary-Killing spinors to prove another positive mass theorem analogous to the one in [13]. We would like to point out that we use a little different setting to study a spacelike hypersurface in asymptotically AdS spacetimes in the second case, instead of extending an initial data set to an asymptotically AdS spacetime in [13]. We note that it was used to study the quasi-local mass in [19] for the positive mass theorem with zero second fundamental form for asymptotically AdS spacetimes with arbitrary negative cosmological constant.

It is an interesting question whether the total angular momentum can be dominated by the total energy. In [8], Corvino and Schoen constructed regular solutions of vacuum Einstein constraint equations, which are Kerr at infinity. This initial data set indicates, in general, there is no relation between the total energy and the total angular momentum. However, certain extra energy conditions were found in asymptotically flat spacetimes in [22] that the total angular momentum is dominated by the total energy. But the analogue of this new energy condition does not imply the similar result in asymptotically AdS spacetimes.

This paper is organized as follows: In Section 2, we make a study of $e_0$- as well as imaginary-Killing spinors in AdS spacetime along the hyperbolic 3-space. Section 3 gives the definition of total energy-momenta for asymptotically AdS initial data sets. In Section 4, we derive a Weitzenb"ock formula for $e_0$-Killing hypersurface Dirac-Witten operator and state some known results on comparing two spin connections. Section 5 deals with the boundary value problem of the Dirac-Witten equation and a positive mass theorem is proved. In Section 6, by using the imaginary Killing spinors, we reach another positive mass theorem which corresponds to one of the energy-momentum inequalities from the definite positivity of Maerten’s operator.

## 2 The AdS Spacetime, the Hyperbolic Space, and the $e_0$- and imaginary Killing Spinors

The anti-de Sitter (AdS) spacetime $(N^{1,3}, \tilde{g}_{AdS})$ is a static spherically solution to the vacuum (i.e., $T = 0$) Einstein equation (1.1) with negative cosmological constant $\Lambda = -\frac{3}{l^2}$ which reads

$$
\tilde{g}_{AdS} = -(\frac{r^2}{l^2} + 1) dt^2 + (\frac{r^2}{l^2} + 1)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\psi^2).
$$

(2.1)

Set $\kappa = l^{-1} > 0$ and $\tilde{r} = \frac{\sinh(\kappa r)}{\kappa}$, then in terms of the polar coordinate system $(r, \theta, \psi)$ ($0 < r < \infty$, $0 \leq \theta < \pi$ and $0 \leq \psi < 2\pi$), the AdS metric can be rewritten as

$$
\tilde{g}_{AdS} = -\cosh^2(\kappa r) dt^2 + \tilde{g}
$$

(2.2)

where

$$
\tilde{g} = dr^2 + \frac{\sinh^2(\kappa r)}{\kappa^2} (d\theta^2 + \sin^2 \theta d\psi^2).
$$

(2.3)

The hyperbolic 3-space $H^3$ is the $t-$slice in $(N^{1,3}, \tilde{g}_{AdS})$ which is topologically $\mathbb{R}^3$ endowed with the metric $\tilde{g}$. Note that it is totally geodesic and has constant sectional curvature $-\kappa^2$. 
We use the convention that the Greek indices $\alpha, \beta, \gamma, \ldots$ run over the spacetime and the Latin ones $i, j, k, \ldots$ are the spatial indices. Denote the associated orthonormal frame $\{\overset{\circ}{e}_\alpha\}$ by

$$
\overset{\circ}{e}_0 = \frac{1}{\cosh(\kappa r)} \frac{\partial}{\partial t}, \quad \overset{\circ}{e}_1 = \frac{\partial}{\partial r}, \quad \overset{\circ}{e}_2 = \frac{\kappa}{\sinh(\kappa r)} \frac{\partial}{\partial \theta}, \quad \overset{\circ}{e}_3 = \frac{\kappa}{\sinh(\kappa r) \sin \theta} \frac{\partial}{\partial \psi}
$$

and its coframe $\{\overset{\circ}{e}^\alpha\}$ by

$$
\overset{\circ}{e}^0 = \cosh(\kappa r) dt, \quad \overset{\circ}{e}^1 = dr, \quad \overset{\circ}{e}^2 = \frac{\kappa}{\kappa} \sinh(\kappa r) d\theta, \quad \overset{\circ}{e}^3 = \frac{\kappa^2}{\kappa} \sinh(\kappa r) \sin \theta d\psi
$$

respectively.

Let $\mathcal{S}$ be the (locally) spinor bundle of $(N^{1,3}, \tilde{g}_{AdS})$ and its restriction to $\mathbb{H}^3$ is globally defined since every orientable 3-manifold is spin. We say that a spin or $\Phi_0 \in \Gamma(\mathcal{S})$ is an $\overset{\circ}{e}_0$-Killing spinor (along $H^3$) if

$$
\nabla_{\overset{\circ}{e}_0} \Phi_0 + \frac{\kappa}{2} \overset{\circ}{e}_0 \cdot \Phi_0 = 0 \quad (2.4)
$$

for every tangent vector $X$ of $\mathbb{H}^3$.

Choose a standard symplectic basis as in [15] and [24], the spinors over AdS can be written as a 4-vector valued functions $\Phi = (\Phi^1, \Phi^2, \Phi^3, \Phi^4)^t \in \mathbb{C}^4$. We fix the following Clifford representation throughout the paper:

$$
\overset{\circ}{e}_0 \mapsto \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad \overset{\circ}{e}_1 \mapsto \begin{pmatrix} 1 & 1 \\ -1 & -1 \\ -1 & -1 \\ 1 & 1 \end{pmatrix},
$$

$$
\overset{\circ}{e}_2 \mapsto \begin{pmatrix} 1 & 1 \\ -1 & -1 \\ -1 & -1 \\ 1 & 1 \end{pmatrix}, \quad \overset{\circ}{e}_3 \mapsto \begin{pmatrix} -\sqrt{-1} & -\sqrt{-1} \\ \sqrt{-1} & \sqrt{-1} \\ \sqrt{-1} & \sqrt{-1} \\ -\sqrt{-1} & -\sqrt{-1} \end{pmatrix}.
$$

Using it, one has

**Lemma 2.1** The set of the solutions of the $\overset{\circ}{e}_0$-Killing equation (2.4) is 4-dimensional. Precisely,

$$
\Phi_0 = \left( \begin{array}{c} (\lambda_1 e^{\frac{\sqrt{-1}}{2} \theta} \sin \frac{\theta}{2} + \lambda_2 e^{-\frac{\sqrt{-1}}{2} \theta} \sin \frac{\theta}{2} \cos \frac{\theta}{2} e^{-\frac{\sqrt{-1}}{2} \theta}) \\ (\lambda_2 e^{-\frac{\sqrt{-1}}{2} \theta} \sin \frac{\theta}{2} - \lambda_1 e^{\frac{\sqrt{-1}}{2} \theta} \sin \frac{\theta}{2} \cos \frac{\theta}{2} e^{-\frac{\sqrt{-1}}{2} \theta}) \\ (\lambda_3 e^{\frac{\sqrt{-1}}{2} \theta} \sin \frac{\theta}{2} + \lambda_4 e^{-\frac{\sqrt{-1}}{2} \theta} \sin \frac{\theta}{2} \cos \frac{\theta}{2} e^{-\frac{\sqrt{-1}}{2} \theta}) \\ (\lambda_4 e^{-\frac{\sqrt{-1}}{2} \theta} \sin \frac{\theta}{2} - \lambda_3 e^{\frac{\sqrt{-1}}{2} \theta} \sin \frac{\theta}{2} \cos \frac{\theta}{2} e^{-\frac{\sqrt{-1}}{2} \theta}) \end{array} \right) \in \mathbb{C}^4 \quad (2.6)
$$

where $\lambda_1, \lambda_2, \lambda_3, \text{ and } \lambda_4$ are four arbitrary complex numbers.

Due to the fact that $\overset{\circ}{e}_0$ changes the chirality of spinors, the form of $\overset{\circ}{e}_0$-Killing spinors looks different from that of the imaginary Killing spinors [12, 10]. In fact,

**Lemma 2.2** The imaginary Killing spinors along $\mathbb{H}^3$ satisfying the imaginary Killing equations

$$
\nabla_X \Phi + \frac{\kappa \sqrt{-1}}{2} X \cdot \Phi = 0 \quad \text{for each tangent vector } X \text{ of } \mathbb{H}^3 \quad (2.7)
$$
are of the form

\[
\begin{pmatrix}
\Phi^1 \\
\Phi^2 \\
\Phi^3 \\
\Phi^4
\end{pmatrix} = \begin{pmatrix}
u^e\frac{\partial}{\partial u} - v^e\frac{\partial}{\partial v} \\
u^e\frac{\partial}{\partial u} + v^e\frac{\partial}{\partial v} \\
\sqrt{-1}\nu^e\frac{\partial}{\partial u} + \sqrt{-1}\nu^e\frac{\partial}{\partial v} \\
\sqrt{-1}\nu^e\frac{\partial}{\partial u} - \sqrt{-1}\nu^e\frac{\partial}{\partial v}
\end{pmatrix}
\]

(2.8)

where

\[
\begin{align*}
u^+ &= \lambda_1 e^{\frac{\sqrt{-1}}{2}\psi}\sin\frac{\theta}{2} + \lambda_2 e^{\frac{-\sqrt{-1}}{2}\psi}\cos\frac{\theta}{2}, \\
u^- &= \lambda_3 e^{\frac{\sqrt{-1}}{2}\psi}\sin\frac{\theta}{2} + \lambda_4 e^{\frac{-\sqrt{-1}}{2}\psi}\cos\frac{\theta}{2}, \\
v^+ &= -\lambda_1 e^{\frac{\sqrt{-1}}{2}\psi}\sin\frac{\theta}{2} + \lambda_4 e^{\frac{-\sqrt{-1}}{2}\psi}\cos\frac{\theta}{2}, \\
v^- &= -\lambda_3 e^{\frac{\sqrt{-1}}{2}\psi}\sin\frac{\theta}{2} + \lambda_2 e^{\frac{-\sqrt{-1}}{2}\psi}\cos\frac{\theta}{2}.
\end{align*}
\]

Here \(\lambda_1, \lambda_2, \lambda_3, \text{ and } \lambda_4\) are four arbitrary complex numbers.

Remark:
The imaginary Killing spinors in the full spacetime look like in the similar form but \(\lambda_\mu\) is time dependent [10], i.e.

\[
\lambda_1 = C_1 \cos\left(\frac{K}{2}t\right) + C_3 \sin\left(\frac{K}{2}t\right), \quad \lambda_3 = C_3 \cos\left(\frac{K}{2}t\right) - C_1 \sin\left(\frac{K}{2}t\right)
\]

\[
\lambda_2 = C_2 \cos\left(\frac{K}{2}t\right) + C_4 \sin\left(\frac{K}{2}t\right), \quad \lambda_4 = C_4 \cos\left(\frac{K}{2}t\right) - C_2 \sin\left(\frac{K}{2}t\right)
\]

where \(C_1, C_2, C_3, C_4\) are four arbitrary complex constants.

3 Definition of the Total Energy-Momenta

In this section, we define the total energy-momenta for asymptotically AdS initial data sets. Suppose that \((N^{1,3}, \hat{g})\) is a Lorentzian manifold with the Lorentzian metric \(\hat{g}\) of signature \((-1, 1, 1, 1)\) satisfying the Einstein field equations. Usually, a triple \((M, g_{ij}, h_{ij})\) is served as a Cauchy surface on the initial problem of the Einstein equations. Here \(M\) is a 3-dimensional spacelike hypersurface with induced Riemannian metric \(g_{ij}\) and \(h_{ij}\) is the second fundamental form of \(M\) in \(N\).

We say that the initial data set \((M, g_{ij}, h_{ij})\) is asymptotically AdS if:

1. There is a compact set \(K \subset M\) such that \(M_\infty = M \setminus K\) is diffeomorphic to \(\mathbb{R}^3 \setminus \text{open ball}\);
2. Under this diffeomorphism, the metric \(g_{ij} = g(\tilde{e}_i, \tilde{e}_j)\) on the end \(M_\infty\) is of the form

\[
g_{ij} = \delta_{ij} + a_{ij}
\]

where \(a_{ij}\) satisfies

\[
a_{ij} = O(e^{-\tau Kr}), \quad \nabla_k a_{ij} = O(e^{-\tau Kr}), \quad \nabla_k \nabla_l a_{ij} = O(e^{-\tau Kr}); \quad (3.1)
\]

and the second fundamental form \(h_{ij} = h(\tilde{e}_i, \tilde{e}_j)\) satisfies

\[
h_{ij} = O(e^{-\tau Kr}), \quad \nabla_k h_{ij} = O(e^{-\tau Kr}) \quad (3.2)
\]
for $\tau > \frac{3}{2}$. Here $\tilde{\nabla}$ is the Levi-Civita connection with respect to the hyperbolic metric $\tilde{g}$;

(3) There exists a distance function $\rho_2$ such that

$$T_{00}e^{\rho_2} \in L^1(M), \quad T_{hi}e^{\rho_2} \in L^1(M).$$

(3.3)

Denote $n^0 = 1$, $n^i$ ($i = 1, 2, 3$) the restriction of the natural coordinate $x^i$ to the unit round sphere, i.e.

$$n^0 = 1, \quad n^1 = \sin \theta \cos \psi, \quad n^2 = \sin \theta \sin \psi, \quad n^3 = \cos \theta$$

and

$$e_i = \tilde{\nabla}^j g_{ij} - \tilde{\nabla}_i \text{tr}_g (g) - \kappa (a_{1i} - g_{1i} \text{tr}_g (a)),
\quad P_{ki} = h_{ki} - g_{ki} \text{tr}_g (h).$$

For such a spacetime, (under a fixed diffeomorphism) the total energy vector $E_{\{\nu\}}$ and for each $k$ the total linear momentum vector $P_{\{\nu\}}k$ are defined by

$$E_{\{\nu\}} = \frac{1}{16\pi} \lim_{r \to \infty} \int_{S_r} \varepsilon_1 \omega_{\nu},$$

$$P_{\{\nu\}}k = \frac{1}{8\pi} \lim_{r \to \infty} \int_{S_r} P_{k1} \omega_{\nu}$$

(3.4) (3.5)

where

$$\omega_{\nu} = n^\nu e^{e_0 e_2} \wedge e^3 \quad (\nu = 0, 1, 2, 3).$$

(3.6)

Remarks:

(1) For simplicity, we just assume that there is only one end. The extension of multi-ends case is straightforward.

(2) Our definition $E_{\{\nu\}}$ is the same as $p_\nu$ in [6]. The geometric invariance of the total energy was given in [7, 4] as well as in [20] for the case with a spherical conformal infinity.

(3) The definition of the total linear momentum vector can be found in [25]. In fact, the Lorentzian lengths of $P_{\{\nu\}1}$ is invariant but $P_{\{\nu\}A}$ ($A = 2, 3$) is not [25], Prof. 2.1]. Therefore, for $i = 1, 2, 3$,

$$(c_1 E_{\{0\}} + c_2 P_{\{0\}1})^2 - \sum_i (c_1 E_{\{i\}} + c_2 P_{\{i\}1})^2$$

(3.7)

gives a geometric invariant where $c_1$ and $c_2$ are real constants.

4 The Spin Connections, the Dirac-Witten Operators, and the Weitzenböck Formula

In this section, we establish a Weitzenböck formula for the Dirac-Witten operator associated with the $c_0$-Killing connection. We also state some known results (due to Min-Oo [14] and certain generalizations in [11, 11, 23]) on comparing two spin connections.

Recall that $(N^{1,3}, \bar{g})$ is a Lorentzian manifold with the Lorentzian metric $\bar{g}$ of signature $(-1, 1, 1, 1)$ satisfying the Einstein field equations. Let $(M, g, h)$ be a 3-dimensional spacelike hypersurface with induced Riemannian metric $g_{ij}$ and $h_{ij}$ is the second fundamental form of $M$ in $N$. Let $\mathcal{S}$ be the (locally) spinor bundle of $N$ and we still denote by $\mathcal{S}$ its restriction to $M$. 

5
Let $\nabla$ and $\overline{\nabla}$ be the Levi-Civita connections of $\tilde{g}$ and $g$ respectively. We also denote by the same symbols their lifts to the spinor bundle $\mathbb{S}$.

Fix a point $p \in M$ and an orthonormal basis $\{e_\alpha\}$ of $T_pN$ with $e_0$ normal and $\{e_i\}$ tangent to $M$. Extend $\{e_\alpha\}$ to a local orthonormal frame in a neighborhood of $p$ in $M$ such that $(\overline{\nabla}_i e_j)_p = 0$. Extend this to a local orthonormal frame $\{e_\alpha\}$ for $N$ with $(\nabla_0 e_j)_p = 0$. Let $\{e^\alpha\}$ be its dual frame. Then

$$\begin{align*}
(\overline{\nabla}_i e_j)_p &= h_{ij} e_0, \\
(\nabla_i e_0)_p &= h_{ij} e_j
\end{align*}$$

(4.1)

where $h_{ij} = \tilde{g}((\overline{\nabla}_i e_0, e_j)$ are the components of its second fundamental form at $p$. The two connections on the spinor bundle are related by

$$\nabla_i = \overline{\nabla}_i - \frac{1}{2} h_{ij} e_0 \cdot e_j \cdot .$$

(4.2)

We define the $e_0$-Killing connection by

$$\hat{\nabla}_X = \nabla_X + 2 \kappa e_0 \cdot X \cdot .$$

(4.3)

Then the associated hypersurface Dirac-Witten operators are

$$D = \sum_{k=1}^{3} e_k \cdot \nabla_k,$$

(4.4)

$$\hat{D} = \sum_{k=1}^{3} e_k \cdot \hat{\nabla}_k = D + \frac{3\kappa}{2} e_0.$$  

(4.5)

There are two choices of metrics on the spinor bundle \cite{15, 24}. Restricted $\mathbb{S}$ to $M$ inherits an Hermitian metric $(\phi, \psi)$ and a positive definite metric $< \phi, \psi >$. They are related by the equation

$$(\phi, \psi) = < e_0 \cdot \phi, \psi > .$$

With respect to $< \cdot, \cdot >$, $e_j$ is skew-Hermitian while $e_0$ is Hermitian \cite{15, 24}. We also note that $\nabla$ is compatible with $< \cdot, \cdot >$ but $\overline{\nabla}$ is not. Moreover,

$$\nabla_i (e_0 \cdot \phi) = e_0 \cdot \nabla_i \phi.$$  

(4.6)

As usual, we have the following formulae.

$$d(< \phi, \nabla_i \psi > \ast e^i) = (e_i < \phi, \nabla_i \psi >) \ast 1$$

$$= (\nabla_i e_0) \cdot \phi, \nabla_i \psi > \ast 1 + < \nabla_i \phi, \nabla_i \psi > \ast 1 + < \phi, \nabla_i \nabla_i \psi > \ast 1$$

$$= (h_{ij} e_j \cdot \phi, \nabla_i \psi > \ast 1 + < \nabla_i \phi, \nabla_i \psi > \ast 1 + < \phi, \nabla_i \nabla_i \psi > \ast 1$$

$$= < h_{ij} e_0 \cdot e_j \cdot \phi, \nabla_i \psi > \ast 1 + < \nabla_i \phi, \nabla_i \psi > \ast 1 + < \phi, \nabla_i \nabla_i \psi > \ast 1$$

(4.7)

and

$$d(< e_i \cdot \phi, \psi > \ast e^i) = (D\phi, \psi) - < \phi, D\psi >) \ast 1$$

$$= (\hat{D}\phi, \psi) - < \phi, \hat{D}\psi >) \ast 1.$$  

(4.8)
Hence,
\[ \nabla^*_i = -\nabla_i - h_{ij}e_0 \cdot e_j, \]  
(4.10)

\[ \hat{\nabla}^*_i = \nabla^*_i + \frac{\kappa}{2} e_0 \cdot e_i \]  
(4.11)

\[ D^* = D \]  
(4.12)

\[ \hat{D}^* = \hat{D} = D + \frac{3\kappa}{2}e_0 \cdot e_0 = D^* + \frac{3\kappa}{2}e_0. \]  
(4.13)

with respect to \(<\cdot, \cdot>\).

The corresponding Weitzenböck formula for the Dirac-Witten operator \( \hat{D} \) is therefore:

**Lemma 4.1** One has
\[ \hat{D}^* \hat{D} = \hat{D} = \nabla^* \nabla + \hat{R} \]  
(4.14)

where
\[ \hat{R} = \frac{1}{4}(Scal\tilde{g} + 2\tilde{R}_{00} + 2\tilde{R}_{0i}e_0 \cdot e_i \cdot + 6\kappa^2 - 4tr(h)) \in \text{End}(S). \]

**Proof:** By straightforward computation, it follows that
\[
\hat{D}^* \hat{D} = \hat{D} = (D + \frac{3\kappa}{2}e_0) \circ (D + \frac{3\kappa}{2}e_0) =
\]
\[
D^2 + \frac{9\kappa^2}{4} + \frac{3\kappa}{2} e_i \cdot (\nabla_i e_0).
\]

On the other hand,
\[
\hat{D} = \hat{D} = (D + \frac{3\kappa}{2}e_0) \circ (D + \frac{3\kappa}{2}e_0) =
\]
\[
D^2 + \frac{9\kappa^2}{4} + \frac{3\kappa}{2} tr(h).
\]

The standard Weitzenböck formula \([21, 15]\) reads
\[ D^2 = \nabla^* \nabla + \frac{1}{4}(Scal\tilde{g} + 2\tilde{R}_{00} + 2\tilde{R}_{0i}e_0 \cdot e_i). \]

This proves the lemma. Q.E.D.

Since we work on a non-compact manifold we need the following integrated version of the Weitzenböck formula \([11, 14]\) involving a boundary term:
Lemma 4.2 We have
\[
\int_{\partial M} \langle \phi, \nabla_i \phi + e_i \cdot \hat{D} \phi \rangle \ast e^i = \int_M \left\{ |\nabla \phi|^2 - |\hat{D} \phi|^2 \right\} \ast 1 + \int_M \langle \phi, \hat{R} \phi \rangle \ast 1 \tag{4.15}
\]
for all \( \phi \in \Gamma(S) \).

The assumption we make in order to prove the positive mass theorem is a modified version of dominant energy condition:
\[
\text{Scal} \tilde{g} + 2R_{00} + 6\kappa^2 - 4\kappa \text{tr}(h) \geq \sqrt{\sum_i (2\tilde{R}_i)^2}. \tag{4.16}
\]
This ensures
\[
\langle \phi, \hat{R} \phi \rangle \geq 0, \ \forall \phi \in \Gamma(S). \tag{4.17}
\]
Remarks:
(1) Set \( \kappa = 1, p_{ij} = -h_{ij} + \delta_{ij} \). Let
\[
\mu = \frac{1}{2} (\text{Scal} \tilde{g} + (p_i^i)^2 - p_{ij}p^{ij}),
\]
\[
\overline{\omega}_j = \nabla_i p_{ji} - \nabla_j p_i^i.
\]
Then by the Gauss-Codazzi equations of \( M \) in \( N \), one can show that the energy condition (4.16) is equivalent to the following “dominant energy condition” in [25]
\[
\mu \geq \sqrt{\sum_i \overline{\omega}_i^2}.
\]
(2) If \( M \) is maximal, i.e. \( \text{tr}(h) = 0 \), then the energy condition (4.16) just reduces to the standard dominant energy condition
\[
T_{00} \geq \sqrt{\sum_i T_{0i}^2}. \tag{4.18}
\]
Einstein equation (1.1) gives
\[
\text{Scal} \tilde{g} = 2(\tilde{G}_{00} - \tilde{R}_{00})
\]
and
\[
\tilde{R}_{0i} e_0 \cdot e_i = e_0 \cdot (\tilde{G}_{0i} e_i).
\]
where \( \tilde{G}_{\mu\nu} = \tilde{R}_{\mu\nu} - \frac{\kappa^2}{2} g_{\mu\nu} \) is the Einstein tensor. Then
\[
\hat{R} = \frac{1}{2} \tilde{G}_{00} e_0 \cdot -\frac{3\kappa^2}{2} g_{00} e_0 \cdot -\frac{1}{2} \tilde{G}_{0i} e_i e_0 \cdot \\
= \frac{1}{2} (T_{00} e_0 \cdot -T_{0i} e_i) e_0 \cdot.
\]
In order to compute the boundary term which gives rise to the total energy and the total momentum, we define a new connection and see the difference of two connections on the spinor bundle. Most of the results here are due to the work in [14, 1, 23]. Recall that \( g = \tilde{g} + a \) with \( a = O(e^{-\tau\kappa r}) \), \( \nabla a = O(e^{-\tau\kappa r}) \), \( \tilde{\nabla} a = O(e^{-\tau\kappa r}) \). Orthonormalizing \( \overset{\circ}{e}_i \) with respect to \( \overset{\circ}{g} \) gives rise an orthonormal basis \( e_i \) with respect to \( g \), i.e.

\[
e_i = \overset{\circ}{e}_i - \frac{1}{2} a_{ik} \overset{\circ}{e}_k + o(e^{-\tau\kappa r}).
\]

This gives a gauge transformation

\[
\mathcal{A} : SO(\overset{\circ}{g}) \rightarrow SO(g)
\]

\( \overset{\circ}{e}_i \mapsto e_i \)

(and in addition \( e_0 \mapsto e_0 \)) which identifies the corresponding spin group and spinor bundles.

To compare \( \nabla \) and \( \overset{\circ}{\nabla} \) in particular their lifts to the spinor bundles, one introduces a new connection \( \overset{\check{\nabla}}{\nabla} = \mathcal{A} \circ \overset{\circ}{\nabla} \circ \mathcal{A}^{-1} \). This new connection is compatible with the metric \( g \) but has a torsion

\[
\overset{\check{T}}{T}(X, Y) = \overset{\circ}{\nabla}_X Y - \overset{\circ}{\nabla}_Y X - [X, Y]
\]

\[
= -(\overset{\circ}{\nabla}_X A)\overset{\circ}{A}^{-1} Y + (\overset{\circ}{\nabla}_Y A)\overset{\circ}{A}^{-1} X.
\]

Then the difference of \( \nabla \) of \( \overset{\circ}{\nabla} \) is then expressible in terms of the torsion

\[
2g(\overset{\circ}{\nabla}_X Y - \overset{\circ}{\nabla}_X Y, Z) = g(\overset{\check{T}}{T}(X, Y), Z) - g(\overset{\check{T}}{T}(X, Z), Y) - g(\overset{\check{T}}{T}(Y, Z), X)
\]

(4.21)

for any tangent vectors \( X, Y, Z \in T M \).

Since both \( \nabla \) and \( \overset{\circ}{\nabla} \) are \( g \)-compatible, their induced connections on the spinor bundle \( S(M) \) differ by

\[
\overset{\circ}{\nabla}_j - \overset{\circ}{\nabla}_j = -\frac{1}{8} \sum_{k, l} (\omega_{kl}(e_j) - \overset{\check{\omega}}{\omega}_{kl}(e_j)) e_k \cdot e_l.
\]

(4.22)

where \( \omega_{kl}(e_j) = -g(\nabla_j e_k, e_l) \) and \( \overset{\check{\omega}}{\omega}_{kl}(e_j) = -g(\overset{\circ}{\nabla}_j e_k, e_l) \).

From (4.20) and (4.21) we have obtained the following asymptotic formula

\[
\nabla_j - \overset{\check{\nabla}}{\nabla}_j = \frac{1}{8} \sum_{k \neq l} (\overset{\circ}{\nabla}_k g_{jl} - \overset{\circ}{\nabla}_j g_{lk}) e_k \cdot e_l + o(e^{-\tau\kappa r})
\]

(4.23)

for the difference of the two connections acting on spinors. And further we have

**Lemma 4.3** (Prop. 3.2, [23]) Let \((M, g_{ij}, h_{ij})\) be a 3-dimensional asymptotically AdS initial data set. Then

\[
\sum_{j, j \neq i} Re \langle \phi, e_i \cdot e_j \cdot (\nabla_j - \overset{\check{\nabla}}{\nabla}_j) \phi \rangle = \frac{1}{4} (\overset{\circ}{\nabla} g_{ij} - \overset{\circ}{\nabla}^r g_{ij} + o(e^{-\tau\kappa r})) |\phi|^2
\]

(4.24)

for all \( \phi \in \Gamma(S) \).
We extend the $\epsilon_0$-Killing spinors $\Phi_0$ in (2.6) on the end to the inside smoothly. With respect to the metric $g$, these $\epsilon_0$-Killing spinors $\Phi_0$ can be written as $\Phi_0 = A\Phi_0$. Let $\hat{\nabla}_X = \nabla_X + \frac{\kappa}{2} \epsilon_0 \cdot X$. Then

$$\hat{\nabla}_j \Phi_0 = A(\nabla_j \Phi_0) + \frac{\kappa}{2} \epsilon_0 \cdot e_j \cdot (A\Phi_0) = \frac{\kappa}{4} a_{jk} \epsilon_0 \cdot (A \hat{e}_k) \cdot \Phi_0 + o(e^{-\tau r}) \Phi_0.$$

## 5 The Dirac-Witten Equation and Positive Mass Theorem I

As explained in the introduction, in this section, we will study an elliptic boundary value problem on $M$ with given boundary values as $r \to \infty$. We will solve for a spinor $\phi$ satisfying the first order elliptic Dirac-Witten equation $\bigwedge D\phi = 0$ on the manifold $M$ which is asymptotic to the $\epsilon_0$-Killing spinor $\Phi_0$ at infinity. Our positive mass theorem is then a consequence of the nice Weitzenböck formula (4.15).

Let $C_0^\infty(S)$ be the space of smooth sections of the spinor bundle $S$ with compact support. Define an inner product on $S$ by

$$(\phi, \psi)_1 = \int_M \left\{ <\nabla \phi, \nabla \psi > + \frac{3\kappa^2}{4} <\phi, \psi > \right\} * 1. \quad (5.1)$$

Let $H^1(S)$ be the closure of $C_0^\infty(S)$ with respect to this inner product. Then $H^1(S)$ with the above inner product is a Hilbert space. Now define a bounded bilinear form $B$ on $C_0^\infty(S)$ by

$$B(\phi, \psi) = \int_M <\hat{D}\phi, \hat{D}\psi > *1. \quad (5.2)$$

By the Weitzenböck formula (4.15), we have

$$B(\phi, \phi) = \int_M |\nabla \phi|^2 * 1 + \int_M <\phi, \hat{\mathcal{R}}\phi > *1. \quad (5.3)$$

Due to the energy condition (4.16), we can extend $B(\cdot, \cdot)$ to $H^1(S)$ as a coercive (not strictly coercive in general) bilinear form.

Take $\Phi_0$ as the $\epsilon_0$-Killing spinor in (2.6). The same as in [25], due to the asymptotic conditions (3.1) and (3.2), we know that $\hat{\nabla} \Phi_0 \in L^2(S)$ and hence $\hat{D} \Phi_0 \in L^2(S)$. Note that $\Phi_0$ itself is not in $L^2(S)$ since $|\Phi_0|^2 = O(e^{\kappa r})$.

**Lemma 5.1** Let $(M, g_{ij}, h_{ij})$ be a 3-dimensional asymptotically AdS initial data set which satisfies the energy condition (4.16). There exists a unique spinor $\Phi_1$ in $H^1(S)$ such that

$$\hat{D}(\Phi_1 + \Phi_0) = 0. \quad (5.4)$$

**Proof:** Here we follow the argument of Lemma 4.2 in [25]. Since $B(\cdot, \cdot)$ is coercive on $H^1(S)$, and $\hat{D} \Phi_0 \in L^2(S)$, $\hat{\nabla} \Phi_0 \in L^2(S)$, thanks to the Lax-Milgram theorem (cf. Theorem 7.21 [9]), there exists a spinor $\Phi_1 \in H^1(S)$ such that

$$\hat{D} \hat{D} \Phi_1 = -\hat{\nabla}^* \hat{D} \Phi_0,$$
weakly. Let $\phi = \Phi_1 + \Phi_0$ and $\psi = \phi^* D \phi$. The elliptic regularity tells us that $\psi \in H^1(\mathcal{S})$, and

$$\hat{\nabla}^* D \psi = 0$$

in the classical sense. Then (4.15) implies that $\hat{\nabla} \psi = 0$. We thus have $|\partial_i \log |\psi|^2| \leq (\kappa + |h|)$ on the complement of the zero set of $\psi$ on $M$. If there exists $x_0 \in M$ such that $|\psi(x_0)| \neq 0$, then integrating it along a path from $x_0 \in M$ gives

$$|\psi(x)|^2 \geq |\psi(x_0)|^2 e^{(\kappa + |h|)(|x_0| - |x|)}.$$

Obviously, $\psi$ is not in $L^2(\mathcal{S})$ which gives the contradiction. Hence $\psi = 0$, and the proof of this lemma is complete. Q.E.D.

Now we state our first positive mass theorem.

**Theorem 5.1** Let $(M, g_{ij}, h_{ij})$ be a 3-dimensional asymptotically AdS initial data set which satisfies the energy condition (4.10). Then the following $4 \times 4$ Hermitian matrix

$$
\begin{pmatrix}
E_0^0 + P_0^0 & -E_1^0 - P_1^0 & E_2^0 + P_2^0 & E_3^0 + P_3^0 \\
E_1^0 + P_1^0 & E_0^0 & \sqrt{-1}(E_2^0 + P_2^0) & E_0^0 + P_0^0 \\
E_2^0 + P_2^0 & \sqrt{-1}(E_1^0 + P_1^0) & E_0^0 & E_0^0 + P_0^0 \\
E_3^0 + P_3^0 & \sqrt{-1}(E_1^0 + P_1^0) & E_0^0 + P_0^0 & E_0^0 + P_0^0 \\
\end{pmatrix}
$$

(5.5)

is positive definite. Moreover, if $E_0^0 + P_0^0 = 0$, then the following equations hold on $M$:

$$R_{ijkl} + \tilde{h}_{ik} \tilde{h}_{jl} - \tilde{h}_{il} \tilde{h}_{jk} = 0,$$

(5.6)

$$\nabla_i \tilde{h}_{jk} - \nabla_j \tilde{h}_{ik} = 0$$

(5.7)

where $R_{ijkl}$ is the Riemann curvature tensor of $(M, g)$ and $\tilde{h}_{ij} = -h_{ij} + \kappa \delta_{ij}$.

The positivity of the $2 \times 2$ principal minor in (5.5) also implies

**Corollary 5.1** In particular, we have

$$E_0^0 + P_0^0 \geq \sum_{i=1}^{3} (E_{i}^0 + P_{i}^0)^2.$$  

(5.8)

As mentioned in the previous section, when $M$ is maximal, the energy condition reduces to the standard dominant energy condition and can be expressed in terms of the energy-momentum tensor $T_{\mu \nu}$. One thus also has the result below.
Corollary 5.2 Let \((M, g_{ij}, h_{ij})\) be a 3-dimensional asymptotically AdS initial data set. Assume that \(M\) is maximal and satisfies the dominant energy condition (4.18), then the \(4 \times 4\) Hermitian matrix in (5.5) is positive definite.

Now we are going to prove Theorem 5.1.

Proof: Let \(\phi\) be the solution of the Dirac-Witten equation \(\wedge D\phi = 0\) as in Lemma 5.1. Submitting this \(\phi\) into the Weitzenböck formula (4.15), we obtain that the boundary term is nonnegative due to the energy condition (4.16).

Denote
\[
\alpha_i = \nabla^i g_{ij} - \nabla_i \text{tr}_g (g),
\]
\[
b_{ij} = -2h_{ij} + \kappa a_{ij},
\]
\[
\tau_{ki} = b_{ki} - g_{ki} \text{tr}_g (b),
\]
and
\[
\beta_\nu = \frac{1}{16\pi} \lim_{r \to \infty} \int_{S_r} (\alpha \tau_{11}) \omega_\nu
\]
where \(\omega_\nu\) is defined in (3.6).

Therefore we obtain
\[
\int_M |\nabla \phi|^2 * 1 + \int_M <\phi, \nabla \phi > * 1
\]
\[
= \lim_{r \to \infty} \text{Re} \int_{S_r} <\Phi_0, \sum_{i,j,i \neq j} e_i \cdot e_j \cdot \nabla_j \Phi_0 > * e^i
\]
\[
= \lim_{r \to \infty} \text{Re} \int_{S_r} <\Phi_0, \sum_{i,j,i \neq j} e_i \cdot e_j \cdot (\nabla_j - \tilde{\nabla}_j) \Phi_0 > * e^i
\]
\[
+ \lim_{r \to \infty} \text{Re} \int_{S_r} <\Phi_0, \sum_{i,j,i \neq j} \frac{1}{2} h_{ijk} e_i \cdot e_j \cdot e_k \cdot \Phi_0 > * e^i
\]
\[
= \frac{1}{4} \lim_{r \to \infty} \int_{S_r} (\nabla_i g_{ij} - \nabla_i \text{tr}_g (g)) |\Phi_0|^2 * e^1
\]
\[
+ \frac{1}{4} \lim_{r \to \infty} \int_{S_r} \kappa (a_{kj} - g_{kj} \text{tr}_g (a)) <\Phi_0, e_k \cdot \Phi_0 > * e^1
\]
\[
- \frac{1}{2} \lim_{r \to \infty} \int_{S_r} (h_{k1} - g_{k1} \text{tr}_g (h)) <\Phi_0, e_k \cdot \Phi_0 > * e^1.
\]

Using the Clifford representation (2.5), the boundary term is equal to (up to a constant)
\[
\beta_0 (|\lambda_1|^2 + |\lambda_2|^2 + |\lambda_3|^2 + |\lambda_4|^2) + \beta_1 (-\lambda_2 \lambda_1 + \lambda_1 \lambda_2 + \lambda_3 \lambda_4 + \lambda_4 \lambda_3)
\]
\[
+ \beta_2 (-\sqrt{-1} (\lambda_2 \lambda_1 - \lambda_1 \lambda_2) + \sqrt{-1} (\lambda_3 \lambda_4 - \lambda_4 \lambda_3)) + \beta_3 (|\lambda_1|^2 - |\lambda_2|^2 + |\lambda_4|^2 - |\lambda_3|^2).
\]
It can be rewritten as a quadratic form $(\bar{\lambda}_1, \bar{\lambda}_2, \bar{\lambda}_3, \bar{\lambda}_4)Q(\lambda_1, \lambda_2, \lambda_3, \lambda_4)^t$ where $Q$ is

$$
\begin{pmatrix}
\beta_0 + \beta_3 & \beta_0 - \beta_3 \\
-\beta_1 - \sqrt{-1}\beta_2 & \beta_1 - \sqrt{-1}\beta_2
\end{pmatrix}
- \begin{pmatrix}
\beta_0 - \beta_3 & \beta_1 + \sqrt{-1}\beta_2 \\
\beta_1 - \sqrt{-1}\beta_2 & \beta_0 + \beta_3
\end{pmatrix}.
$$

(5.10)

This completes the proof of the nonnegativity.

If the equality holds, then there exists at least one non-vanishing spinor such that $\nabla \phi = 0$. If $E_{(0)} + P_{(0)} = 0$, then there is $\{\phi_\alpha\}$ which forms a basis of the spinor bundle everywhere on $M$ such that $\nabla \phi_\alpha = 0$. So in a local frame $\{e_\alpha\}$ we have

$$
\nabla_i \phi_\alpha = \frac{1}{2} h_{ik} e_0 \cdot e_k \cdot \phi_\alpha - \frac{\kappa}{2} e_0 \cdot e_i \cdot \phi_\alpha.
$$

Then

$$
\nabla_j \nabla_i \phi_\alpha = \frac{1}{2} (\nabla_j h_{ik}) e_0 \cdot e_k \cdot \phi_\alpha + \frac{1}{2} h_{ik} e_0 \cdot e_k \cdot \left(\frac{1}{2} h_{jl} e_0 \cdot e_l - \frac{\kappa}{2} e_0 \cdot e_j\right) \cdot \phi_\alpha
$$

$$
- \frac{\kappa}{2} e_0 \cdot e_i \cdot \left(\frac{1}{2} h_{jl} e_0 \cdot e_l - \frac{\kappa}{2} e_0 \cdot e_j\right) \cdot \phi_\alpha
$$

$$
= \frac{1}{2} (\nabla_j h_{ik}) e_0 \cdot e_k \cdot \phi_\alpha - \frac{1}{4} h_{ik} \tilde{h}_{jl} e_0 \cdot e_k \cdot \phi_\alpha.
$$

It is therefore,

$$
-\frac{1}{4} R_{ijkl} e_k \cdot e_l \cdot \phi_\alpha = (\nabla_i \nabla_j - \nabla_j \nabla_i) \phi_\alpha - \nabla_{[e_i, e_j]} \phi_\alpha
$$

$$
= -\frac{1}{2} (\nabla_j \tilde{h}_{ik} - \nabla_i \tilde{h}_{ik}) e_0 \cdot e_k \cdot \phi_\alpha + \frac{1}{4} (\tilde{h}_{ik} \tilde{h}_{jl} - \tilde{h}_{il} \tilde{h}_{jk}) e_0 \cdot e_k \cdot \phi_\alpha
$$

for a basis $\{\phi_\alpha\}$. This implies

$$
\sum_{k<l} (R_{ijkl} + \tilde{h}_{ik} \tilde{h}_{jl} - \tilde{h}_{il} \tilde{h}_{jk}) e_k \cdot e_l = \sum_k (\nabla_j \tilde{h}_{ik} - \nabla_i \tilde{h}_{ik}) e_0 \cdot e_k
$$

as an endomorphism of $\mathbb{S}$. Set

$$
\tilde{R}_{ijkl} = R_{ijkl} + \tilde{h}_{ik} \tilde{h}_{jl} - \tilde{h}_{il} \tilde{h}_{jk}
$$

and

$$
\tilde{h}_{ijk} = \nabla_i \tilde{h}_{jk} - \nabla_j \tilde{h}_{ik}.
$$

In terms of Clifford representation, we obtain

$$
\begin{pmatrix}
\sqrt{-1} \tilde{R}_{i23} & \tilde{R}_{ij23} & 0 & 0 \\
-\tilde{R}_{ij12} + \sqrt{-1} \tilde{R}_{ij13} & -\sqrt{-1} \tilde{R}_{ij23} & \sqrt{-1} \tilde{R}_{ij23} & \tilde{R}_{ij12} + \sqrt{-1} \tilde{R}_{ij13} \\
0 & 0 & \tilde{R}_{ij12} + \sqrt{-1} \tilde{R}_{ij13} & -\sqrt{-1} \tilde{R}_{ij23}
\end{pmatrix}
$$
This gives

\[ R_{ijkl} + \tilde{h}_{ik} \tilde{h}_{jl} - \tilde{h}_{il} \tilde{h}_{jk} = 0, \]

\[ \nabla_i \tilde{h}_{jk} - \nabla_j \tilde{h}_{ik} = 0, \]

and the theorem is proved. Q.E.D.

## 6 The Imaginary Killing Spinors and Positive Mass Theorem II

As mentioned in the introductory section, Maerten obtained the positivity of a sequilinear form under the relative energy condition. Classical linear algebra tells us that each principal minor of this form must be nonnegative which give rise to a set of energy-momentum inequalities (cf. Appendix [13]). Among them, the most interesting one might be the second order principal minor which gives the positivity of the Lorentzian length of the mass vector, i.e. \( m_0^2 - \vert m \vert^2 \geq 0 \). This special inequality is recovered in our formulism here by using the imaginary Killing spinor.

Define the modified imaginary Killing connection as

\[ \nabla_i = \nabla_i - \frac{1}{2} \tilde{h}_{ij} e_0 \cdot e_j + \frac{\kappa \sqrt{-1}}{2} e_i \cdot. \]

Here \( \nabla \) is the Levi-Civita connection with respect to the induced Riemannian metric on the spacelike hypersurface. The associated hypersurface Dirac-Witten operator is

\[ \hat{D} = \sum_{k=1}^{3} e_k \cdot \nabla_k = D - \frac{3 \kappa \sqrt{-1}}{2}. \]

The corresponding Weitzenböck formula is then

\[ \int_{\partial M} < \phi, \nabla_i \phi + e_i \cdot \hat{D} \phi > * e^i = \int_M (\vert \nabla \phi \vert^2 - \Vert \hat{D} \phi \Vert^2 + < \phi, \hat{R} \phi >) * 1. \]

Here

\[ \hat{R} = \frac{1}{4} (\text{Scal}^\flat + 2 \tilde{R}_{00} + 2 \tilde{R}_{0i} e_0 \cdot e_i + 6 \kappa^2). \]

(See also [13] for \( \kappa = 1 \).) By the Einstein equation (1.1), \( \hat{R} = \frac{1}{2} (T_{00} e_0 \cdot - T_{0i} e_i e_0 \cdot) \) whose positivity is ensured by the standard dominant energy condition (4.18).

Take \( \Phi_0 \) as an imaginary Killing spinor along the hyperbolic space (2.8) and extend it smoothly the whole manifold. Consider the elliptic boundary problem \( \hat{D} \phi = 0 \) with \( \phi \) asymptotic to \( \Phi_0 \) at infinity. Using the Clifford representation (2.5), the boundary term thus can be written
as a quadratic form \((\bar{\lambda}_1, \bar{\lambda}_2, \bar{\lambda}_3, \bar{\lambda}_4)Q(\lambda_1, \lambda_2, \lambda_3, \lambda_4)^t\). Here \(Q\) is a Hermitian \(4 \times 4\) matrix

\[
\begin{pmatrix}
E_{(0)} + E_{(3)} & -\sqrt{-1}E_{(1)} & -P_{(0)2} + P_{(3)2} & P_{(1)2} - P_{(1)3} \\
-\sqrt{-1}E_{(2)} & E_{(0)} - E_{(3)} & -P_{(1)2} + P_{(2)3} & -E_{(1)} \\
+\sqrt{-1}E_{(2)} & E_{(0)} - E_{(3)} & -P_{(1)2} + P_{(2)3} & -E_{(1)} \\
-P_{(0)2} + P_{(3)2} & +\sqrt{-1}(P_{(0)3} + P_{(3)3}) & +\sqrt{-1}(P_{(1)3} + P_{(2)2}) & E_{(0)} + E_{(3)} -E_{(1)} \\
+\sqrt{-1}(P_{(0)3} + P_{(3)3}) & +\sqrt{-1}(P_{(1)3} + P_{(2)2}) & E_{(0)} + E_{(3)} -E_{(1)} \\
-P_{(0)2} + P_{(3)2} & +\sqrt{-1}(P_{(0)3} + P_{(3)3}) & +\sqrt{-1}(P_{(1)3} + P_{(2)2}) & E_{(0)} + E_{(3)} -E_{(1)} \\
-P_{(0)2} + P_{(3)2} & +\sqrt{-1}(P_{(0)3} + P_{(3)3}) & +\sqrt{-1}(P_{(1)3} + P_{(2)2}) & E_{(0)} + E_{(3)} -E_{(1)} \\
-P_{(0)2} + P_{(3)2} & +\sqrt{-1}(P_{(0)3} + P_{(3)3}) & +\sqrt{-1}(P_{(1)3} + P_{(2)2}) & E_{(0)} + E_{(3)} -E_{(1)} \\
\end{pmatrix}
\]

Therefore, we have reached our second positive mass theorem.

**Theorem 6.1** Let \((M, g_{ij}, h_{lj})\) be a 3-dimensional asymptotically AdS initial data set which satisfies the standard dominant energy condition \((4.18)\). Then \(Q\) is nonnegative. Moreover, if \(Q = 0\), then we have the following equations on \(M\):

\[
R_{ijkl} = (-\kappa^2)(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + h_{il}h_{jk} - h_{ik}h_{jl},
\]

\[
\nabla_i h_{jk} - \nabla_j h_{ik} = 0
\]

where \(R_{ijkl}\) is the Riemann curvature tensor of \((M, g)\). These are the Gauss and Codazzi equations of the isometric embedding in the AdS spacetime.

**Remarks:**

1. The dominant energy condition in physics also implies that \(T^{00} \geq |T^{\alpha\beta}|\). If \(Q = 0\), then \(T^{\alpha\beta} = 0\). This together with \((6.2)\), \((6.3)\) imply that \(N\) is AdS along \(M\), i.e. \(N\) has constant sectional curvature \(-\kappa^2\) along \(M\).

2. The energy-momentum matrix in \((6.1)\) is different from the one obtained in \([13]\) for \((1+3)\)-dimensional spacetimes. It should be related to the representation of spin group. However, the method presented in our paper is consistent.

The positivity of the \(2 \times 2\) principal minor in \((6.1)\) also implies the positivity of hyperbolic mass:

**Corollary 6.1** In particular, we have

\[
E_{(0)} \geq \sqrt{\sum_{i=1}^{3} E_{(i)}^2}.
\]

Clearly, the rigidity conclusion follows from the fact that when \(Q = 0\), there exists \(\{\phi_\alpha\}\) which forms a basis of the spinor bundle everywhere on \(M\) such that \(\nabla \phi_\alpha = 0\). Maerten \([13]\) also obtained this via the construction of the Killing initial data in \([4]\). In addition, he discusses the isometric embedding in a stationary pp-wave spacetime when the energy-momentum matrix is degenerate.
Acknowledgements

N. Xie is partially supported by Doctoral Foundation of Ministry of Education of China under grant 20030246001, Eurasia-Pacific Uninet Technologiestipendien China & Mongolei 2005/2006 (Doktorat) and Fudan Postgraduate Students Innovation Project. He would also thank Profs. C.H. Gu and H.S. Hu for their consistent encouragements. X. Zhang is partially supported by National Natural Science Foundation of China under grant 10421001, NKBPRC(2006CB805905) and Innovation Project of Chinese Academy of Sciences.

References

[1] Andersson, L., Dahl, M.: Scalar curvature rigity for asymptotically locally hyperbolic manifolds. Ann. Glob. Anal. Geom. 16, 1-27(1998)

[2] Arnowitt, S., Deser, S., Misner, C.: Coordinate invariance and energy expressions in general relativity. Phys. Rev. 122, 997-1006(1961)

[3] Bartnik, R.: The mass of an asymptotically flat manifold. Comm. Pure. Appl. Math. 36, 661-693(1986)

[4] Beig, R., Chruściel, P.: Killing initial data, Class. Quantum Gravity 14, A83-A92(1997)

[5] Chruściel, P.: Boundary conditions at spatial infinity from a Hamiltonian point of view, in Topological properties and global structure of space-time (Erice 1985), eds. P. Bergmann/V. de Sabbata, NATO, Adv. Sci. Inst. Ser. B, Plenum Press NY, 49-59(1986)

[6] Chruściel, P., Herzlich, M.: The mass of asymptotically hyperbolic Riemannian manifolds. Pacific J. Math. 212, 231-264(2003)

[7] Chruściel, P., Nagy, G.: The mass of spacelike hypersurfaces in asymptotically anti-de Sitter space-times, Adv. Theor. Math. Phys. 19, 697-754(2001)

[8] Corvino, J., Schoen, R.: On the asymptotics for the vacuum Einstein constraint equations, J. Diff. Geom. 73, 185-217(2006)

[9] Folland, G.: Introduction to partial differential equations. Princeton, NJ: Princeton University Press, 1995

[10] Henneaux, M., Teitelboim, C.: Asymptotically anti-de Sitter spaces. Commun. Math. Phys. 98, 391-424(1985)

[11] Herzlich, M.: Scalar curvature and rigidity of odd-dimensional complex hyperbolic spaces. Math. Ann. 312, 641-657(1998)

[12] Leitner, F.: Imaginary Killing spinors in Lorentzian geometry. J. Math. Phys. 44, 4795-4806(2003)

[13] Maerten, D.: Positive energy-momentum theorem for AdS-asymptotically hyperbolic manifolds, Ann. Henri Poincaré 7, 975-1011(2006)
[14] Min-Oo, M.: Scalar curvature rigidity of asymptotically hyperbolic spin manifolds. Math. Ann. 285, 527-539(1989)

[15] Parker, T., Taubes, C.: On Witten’s proof of the positive energy theorem. Commun. Math. Phys. 84, 223-238(1982)

[16] Schoen, R., Yau, S.T.: On the proof of the positive mass conjecture in general relativity. Commun. Math. Phys. 65, 45-76(1979)

[17] Schoen, R., Yau, S.T.: The energy and the linear momentum of spacetimes in general relativity. Commun. Math. Phys. 79, 47-51(1981)

[18] Schoen, R., Yau, S.T.: Proof of the positive mass theorem. II. Commun. Math. Phys. 79, 231-260(1981)

[19] Wang, M.T., Yau, S.T.: A generalization of Liu-Yau’s quasi-local mass, math.DG/0602321

[20] Wang, X.: Mass for asymptotically hyperbolic manifolds. J. Diff. Geom. 57, 273-299(2001)

[21] Witten, E.: A new proof of the positive energy theorem. Commun. Math. Phys. 80, 381-402(1981)

[22] Zhang, X.: Angular momentum and positive mass theorem, Commun. Math. Phys. 206, 137-155(1999)

[23] Zhang, X.: Strongly asymptotically hyperbolic spin manifolds. Math. Res. Lett. 7, 719-728(2000)

[24] Zhang, X.: Positive mass theorem for modified energy conditions, in Morse Theory, Minimax Theory and their Applications to Nonlinear Differential Equations, ed. H. Brezis, et al., New Stud. Adv. Math. 1, MA: Int. Press, Somerville, 275(2003)

[25] Zhang, X.: A definition of total energy-momenta and the positive mass theorem on asymptotically hyperbolic 3-manifolds I. Commun. Math. Phys. 249, 529-548(2004)

N. Xie
Institute of Mathematics
School of Mathematical Sciences
Fudan University
Shanghai 200433, PR China
nqxie@fudan.edu.cn

X. Zhang
Institute of Mathematics
Academy of Mathematics and System Sciences
Chinese Academy of Sciences
Beijing 100080, PR China
xzhang@amss.ac.cn