FRAMING FUNCTIONS AND STRENGTHENED VERSION OF
DEHN’S LEMMA

TETSUYA ITO

Abstract. We give a lower estimate of the framing function of knots, and prove a strengthened version of Dehn’s lemma conjectured by Greene-Wiest.

Let \( K : S^1 \to S^3 \) be an oriented knot in \( S^3 \). A compressing disc of \( K \) is a continuous map \( D : D^2 \to M \) such that \( D|_{\partial D^2} = K \) and that \( D|_{\text{Int} D^2} \) is transverse to \( K \). The intersection points of \( D \) and \( K \) are called holes. In [GW], Greene and Wiest introduced the framing function \( n_K : \mathbb{Z} \to \mathbb{Z}_{\geq 0} \) of a knot \( K \), defined by

\[
n_K(k) = \min \{ \#D \cap \partial D \mid D \text{ is a compressing disc with } i(D, K) = k \}.
\]

Here \( i(D, K) \) denotes the algebraic intersection number of \( D \) and \( K \).

The aim of this paper is to show the following theorem conjectured in [GW].

**Theorem 1.** If \( K \) is not unknot in \( S^3 \), then \( n_K(0) \geq 4 \).

As is observed in [GW], this theorem can be understood as a strengthened form of Dehn’s lemma. Dehn’s lemma [P] says that if \( K \) admits a compressing disc \( D \) without holes, then \( K \) is unknot. Theorem 1 says that one can weaken the hypothesis as \( K \) admits a compressing disc with two holes of opposite sign.

First of all, we give a lower bound of \( n_K(0) \) in terms of the genus of \( K \). Although this estimate is a direct consequence of Gabai’s theorem on immersed Seifert genus, it gives rise to a new insight for the framing function: As is observed in [GW], the definition and several calculations of framing function suggest that the knot framing function \( n_K \) is related to four-dimensional invariants of knots, like the signature and the unknotting number. The following estimate demonstrates that \( n_K \) is also related to three-dimensional invariants of knots.

**Proposition 1.** Let \( g(K) \) be the genus of \( K \). Then \( n_K(0) \geq 2g(K) \).

**Proof.** Assume that \( n_K(0) = 2m \). Let \( D : D^2 \to S^3 \) be a compressing disc with \( m \) positive holes and \( m \) negative holes. As observed in [GW], we may assume that \( D \) is an immersion. For a pair of a positive hole \( p \) and a negative hole \( n \) of \( D \), we remove small neighborhoods of \( p \) and \( n \) and attach a thin tube contained in a neighborhood of \( K \), as shown in Figure 1. This gives rise to an immersion \( I : \Sigma = \Sigma_{m,1} \to S^3 \), where \( \Sigma_{m,1} \) denotes the closed oriented genus \( m \) surface minus disc.

\( I \) is an embedding near \( K = \partial \Sigma \) and \( I^{-1}(\partial \Sigma) = \partial \Sigma \), so \( I \) is an immersed Seifert surface of \( K \). Since the immersed Seifert genus is equal to the usual genus of \( K \) [Gal Corollary 6.22], \( g(K) \leq m \).

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Proof of Theorem. Assume that there exists a non-trivial knot $K$ with $n_K(0) = 2$. By Proposition if $g(K) > 1$, then $n_K(0) \geq 4$ so $g(K)$ must be one. Let $D$ be a compressing disc of $K$ with exactly one positive and one negative holes.

From the proof of Proposition by attaching a tube to $D$, we get an immersed Seifert surface $I: \Sigma_{1,1} \to S^3$ of genus one. Take a loop $\gamma \subset \Sigma_{1,1}$ so that it is homotopic to the co-core of the attached tube. Then the loop $I_\gamma : S^1 \to S^3 - K$ represents a meridian of $K$ so $[I(\gamma)] \neq 0 \in H_1(S^3 - K)$.

Let $F$ be a genus one, embedded Seifert surface of $K$ and let $M$ be the closed $3$-manifold obtained by $0$-framed surgery along $K$. By attaching discs to $F$, we get an incompressible torus $\hat{F}$ in $M$. Similarly by attaching discs to the boundary of $I(\Sigma_{1,1})$, we get an immersion of a torus $\hat{I} : T^2 \to M$. We will show that $[\hat{I}(\gamma)] = 0 \in H_1(M)$. Since the inclusion $S^3 - N(K) \hookrightarrow M$ induces an isomorphism on the first homology group, this implies that $[I(\gamma)] = 0 \in H_1(S^3 - K)$. This is a contradiction, which complete the proof of Theorem.

By homotopy, we put the immersed torus $\hat{I}(T^2)$ so that it is minimally transverse to $\hat{F}$. Here minimally means that the number of the connected components of the preimages of the intersection $\hat{I}(T^2) \cap \hat{F}$ is minimum.

First assume that $\hat{I}(T^2)$ does not intersect with $\hat{F}$. This implies that all loops on $\hat{I}(T^2)$ are null-homologous in $M$, so $[\hat{I}(\gamma)] = 0 \in H_1(M)$.

Hence we assume that $\hat{I}(T^2)$ has non-empty intersection with $\hat{F}$. We take a base point of $M$ so that it lies on $\hat{F}$. Take a simple closed curve $\alpha$ on $T^2$ so that the image of the loop $\hat{I}_\alpha$ lies on the intersection $\hat{F} \cap \hat{I}(T^2)$. Since we have assumed that $\hat{I}(T^2)$ is minimally transverse to $\hat{F}$, $\alpha$ is an essential simple closed curve on $T^2$. We denote the loop $\hat{I}_\alpha$ by $A$.

The loop $A$ is not null-homotopic in $M$, because otherwise by compressing $\hat{I}(T^2)$ along $\alpha$, we get an immersed sphere. Since $[\hat{I}(T^2)] = [\hat{F}] \neq 1 \in H_2(M)$, this sphere is not null-homotopic. This contradicts the fact $\pi_2(M) = 0$ [Ga2 Corollary 8.3].

Take a loop $B$ on $\hat{F}$ so that $\{A, B\}$ generates a free abelian group of rank two in $\pi_1(M)$. For a group $G$ and its element $x \in G$, let $Z_G(x) = \{y \in G \mid yx = xy\}$ be the centralizer of $x$ in $G$. As an element of $\pi_1(M)$, $\hat{I}_\gamma$ commutes with $A = \hat{I}_\alpha$. Thus both $B$ and $\hat{I}_\gamma$ belong to $Z_{\pi_1(M)}(A)$. There are three possibilities for the structure of the centralizer $Z_{\pi_1(M)}(A)$ [IJJS GT]:

(i) $Z_{\pi_1(M)}(A) \cong \mathbb{Z}$. 

Figure 1. Compressing disc $D$ and immersion $I$
(ii) There exist $h \in \pi_1(M)$ and an incompressible torus $T$ in $M$ such that $Z_{\pi_1(M)}(A) = h\pi_1(T)h^{-1}$.

(iii) There exist $h \in \pi_1(M)$ and a Seifert fibered component $S$ in the geometric decomposition of $M$ such that $Z_{\pi_1(M)}(A) = hZ_{\pi_1(S)}(h^{-1}Ah)h^{-1}$.

First assume that the case (i) occurs, and let $g$ be the generator of $Z_{\pi_1(M)}(A)$. Then $A = g^n$ and $\hat{I}|_\gamma = g^m$ for some $n, m \neq 0$. Since $[A] = n[g] = 0 \in H_1(M) \cong \mathbb{Z}$, $[g] = 0 \in H_1(M)$. Therefore $[\hat{I}(\gamma)] = 0 \in H_1(M)$.

Next assume that the case (ii) occurs. Since $A$ and $B$ are loops on the essential torus $\hat{F}$, this shows that the loop $\hat{I}|_\gamma$ is homotopic to a loop contained in the incompressible torus $\hat{F}$. This again implies that $[\hat{I}(\gamma)] = 0 \in H_1(M)$.

Finally assume that the case (iii) occurs. Then $\hat{F}$ is an incompressible torus in the Seifert fibered component $S$. If $\hat{F}$ is a non-separating torus in $S$, that is, $S - \hat{F}$ is connected, then $S$ is a Seifert fibered 3-manifold which is a torus bundle over the circle. This happens only if $K$ is a trefoil, since 0-framed surgery on a knot in $S^3$ yields a fibered 3-manifold if and only if the knot is fibered [Ga2 Corollary 8.19]. However, $n_{\text{trefoil}}(0) = 4$ [GW], this cannot happen. Thus, $\hat{F}$ is a separating incompressible torus of $S$. Now we put $\hat{I}(\gamma)$ so that it is contained in $\hat{F}$ and transverse to $\hat{F}$. $\hat{F}$ is separating, so the algebraic intersection number of $\hat{I}(\gamma)$ and $\hat{F}$ is zero. This implies that $[\hat{I}(\gamma)] = 0 \in H_1(M)$.

We close the paper by posting the following conjecture which generalizes Greene-Wiest’s one.

**Conjecture 1.** $n_K(0) \geq 4g(K)$.

Our results shows that this is the case if $g(K) = 1$. This is also true for the case $K$ is a torus knot [GW].

**References**

[Fr] S. Friedl, *Centralizers in 3-manifold groups*, arXiv:1205.2386

[Ga] D. Gabai, *Foliations and the topology of 3-manifolds*, J. Differential Geom. 18 (1983) 445–503.

[Ga2] D. Gabai, *Foliations and the topology of 3-manifolds, III*, J. Differential Geom. 18 (1987) 479–536.

[GW] M. Greene and B. Wiest, *A natural framing of knots*, Geom. Topol. 2 (1998) 31–64.

[JS] W. Jaco and P. Shalen, *Seifert fibered spaces in 3-manifolds*, Mem. Amer. Math. Soc. 21 (1979), no. 220.

[J] K. Johannson, *Homotopy equivalences of 3-manifolds with boundaries*, Lecture Notes in Mathematics, 761, Springer, Berlin, 1979

[P] D. Papakyriakopoulos, *On Dehn’s lemma and the asphericity of knots*, Ann. of Math. 66 (1957), 1–26.

**Research Institute for Mathematical Sciences, Kyoto University Kyoto, 606-8502, Japan**

*E-mail address: tetitoh@kurims.kyoto-u.ac.jp*

*URL: http://kurims.kyoto-u.ac.jp/~tetitoh/*