Topological full groups of minimal subshifts and quantifying local embeddings into finite groups

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Abstract

We investigate quantitative aspects of the LEF property for subgroups of the topological full group $\mathcal{F}$ of a two-sided minimal subshift over a finite alphabet, measured via the LEF growth function. We show that the LEF growth of $\mathcal{F}$ may be bounded from above and below in terms of the recurrence function and the complexity function of the subshift, respectively. As an application, we construct groups of previously unseen LEF growth types, and exhibit a continuum of finitely generated LEF groups which may be distinguished from one another by their LEF growth.

1 Introduction

Often in geometric group theory, one considers a growth function $F_\Gamma$, which describes some part of the asymptotic structure of a finitely generated group $\Gamma$. Examples include subgroup growth; word growth; conjugacy growth; Dehn function; Følner function; residual finiteness growth. Having introduced $F_\Gamma$, it is always natural to attempt the following: (i) estimate $F_\Gamma$ for some group $\Gamma$ of interest; (ii) relate the behaviour of $F_\Gamma$ to structural features of $\Gamma$ or its actions; (iii) explore the types of functions which can arise as $F_\Gamma$ for some $\Gamma$. In this paper we make contributions to all three of these themes for the LEF growth function, by examining some groups arising in Cantor dynamics.

1.1 Statement of results

A group $\Gamma$ is LEF (locally embeddable into finite groups) if every finite subset of $\Gamma$ admits an injective partial homomorphism (a local embedding) into a finite group. In other words, every finite subset of the multiplication table of $\Gamma$ occurs in the multiplication table of some finite group. If $\Gamma$ is finitely generated by $S$, the prototypical finite subsets are the balls $B_S(n)$ in the associated word metric. The LEF growth function $L^S_\Gamma$ sends $n \in \mathbb{N}$ to the minimal order of a finite group into which $B_S(n)$ locally embeds. The dependence of the function on $S$ is slight, so we suppress $S$ from our notation for the rest of the Introduction.

That the topological full group $\mathcal{F}$ of a Cantor minimal system $(X, \phi)$ is a LEF group was proved by Grigorchuk and Medynets [3]. We prove an effective version of
their result, in the case of a two-sided minimal subshift \((X, \sigma)\) over a finite alphabet. Let \(R_X : \mathbb{N} \to \mathbb{N}\) be the recurrence function of \((X, \sigma)\) (see Definition 2.12 below).

**Theorem 1.1.** Let \(\Gamma\) be a finitely generated subgroup of \(\|\sigma\|\). Then:

\[
\mathcal{L}_\Gamma(n) \leq (2R_X(n))!.
\]

In particular this inequality holds for \(\Gamma = \|\sigma\|\).

For nondecreasing unbounded functions \(F_1\) and \(F_2\) we write \(F_1 \preceq F_2\) if, up to constant rescaling of the argument, \(F_2\) bounds \(F_1\) from above (see Definition 2.4 below). Note that the Cantor minimal system \((X, \varphi)\) being isomorphic to a minimal subshift is a necessary and sufficient condition for \(\|\varphi\|\) to be finitely generated [14]. In the other direction, we have the following lower bound on the LEF growth of \(\|\sigma\|\). Let \(p_X : \mathbb{N} \to \mathbb{N}\) be the complexity function of the minimal subshift \((X, \sigma)\) (see Definition 2.10 below).

**Theorem 1.2.** There exists \(c > 0\) such that \(\exp \left( c p_X(n^{1/2}) \right) \leq \mathcal{L}_{\|\sigma\|}(n)\).

Our proof yields \(c = \log(60)/9 \approx 0.455\), but small modifications to the argument would enable us to make \(c\) arbitrarily large. The group \(\|\sigma\|\) always has exponential word growth, which immediately implies that \(\mathcal{L}_{\|\sigma\|}\) grows at least exponentially. One consequence of Theorem 1.1 is that when the subshift \(X\) is extremely “orderly”, then this exponential lower bound is close to best-possible.

**Example 1.3.** Let \(X\) be a linearly recurrent subshift. Then:

\[
\exp(n) \leq \mathcal{L}_{\|\sigma\|}(n) \leq n!.
\]

On the other hand, is \(X\) is highly non-deterministic, then Theorem 1.2 gives a novel lower bound for the LEF growth of \(\|\sigma\|\).

**Example 1.4.** Let \(X\) be a subshift of positive entropy. Then:

\[
\exp(\exp(n^{1/2})) \leq \mathcal{L}_{\|\sigma\|}(n).
\]

See Subsection 2.2 for definitions of linear recurrence and entropy. Although the upper and lower bounds proved in Theorems 1.1 and 1.2 are some distance apart, they are powerful enough to allow us to observe new phenomena in the kinds of functions which can arise as the LEF growth functions of groups.

**Theorem 1.5.** For any \(r \in [2, \infty)\), there exists a finitely generated LEF group \(\Gamma^{(r)}\) such that:

1. There exists \(C_r > 0\) such that \(\mathcal{L}_{\Gamma^{(r)}}(n) \leq \exp \left( C_r (\log n)^r \right)\);

2. For any \(2 \leq r' < r\), and for all \(C > 0\), \(\mathcal{L}_{\Gamma^{(r')}}(n) \preceq \exp \left( C (\log n)^{r'} \right)\).

Previously, the only functions known to arise as LEF growth functions of groups were polynomial, exponential, or comparable to \(\exp(n^C)\) or \(\exp(\exp(n))\) (see [5, Theorem 1.7]) and some inexplicit, very large functions (see [5, §5]). Theorem 1.5 also immediately implies that uncountably many inequivalent LEF growth functions occur among finitely generated LEF groups, answering a question posed in [5].
Theorem 1.6. There is an uncountable family $\mathcal{F}$ of pairwise nonisomorphic finitely generated LEF groups such that, for $\Gamma_1, \Gamma_2 \in \mathcal{F}$, if $\Gamma_1 \neq \Gamma_2$ then $\mathcal{L}_{\Gamma_1} \approx \mathcal{L}_{\Gamma_2}$.

We write $F_1 \approx F_2$ if $F_1 \preceq F_2$ and $F_2 \preceq F_1$ (see Definition 2.4 below). The groups constructed in Theorem 1.5 have the form $\langle \sigma \rangle$, for $(X, \sigma)$ a minimal subshift which we construct by adapting a construction of Jung, Lee and Park [10], as follows.

Theorem 1.7. For every $r \in [2, \infty)$, there are a minimal subshift $(X_r, \sigma)$, constants $C_r, c_r > 0$, and an increasing sequence $(n_1^{(r)})$ of integers such that:

(i) For all $n \geq 2$, $\mathcal{R}_{X_r}(n) \leq \exp \left( C_r (\log n)^r \right)$;

(ii) For all $i \in \mathbb{N}$, $p_{X_r}(n_i^{(r)}) \geq \exp \left( c_r (\log n_i^{(r)})^r \right)$.

The conclusions (i) and (ii) of Theorem 1.5 for the group $\Gamma = \langle \sigma \rangle$ then follow from Theorem 1.7(i) and (ii), by Theorems 1.1 and 1.2, respectively. Further examples of subshifts of “intermediate” complexity could be a rich source of examples of new exotic behaviours in the LEF growth of groups, and this should be investigated further.

1.2 Background and structure of the paper

The concept of a LEF group first appears in the work of Mal’cev, but was developed and popularised by Vershik and Gordon [17]. All residually finite groups are LEF, including all finitely generated nilpotent or linear groups, but LEF enjoys some closure properties that residual finiteness does not: for instance, the (regular restricted) wreath product of LEF groups is LEF. Among finitely presented groups, the classes of LEF and residually finite groups coincide, and this observation provides a useful tool for proving that certain groups are not finitely presentable (see [8] for a proof along these lines for derived subgroups of topological full groups). LEF groups have also been studied in connection with weaker approximation properties of groups, such as soficity and hyperlinearity, since they provide a source of examples beyond those arising from residual finiteness or amenability. For instance, Elek and Szabó [7] used the LEF property to construct the first examples of sofic groups which are not residually amenable.

The LEF growth function was introduced independently in [1] and [4] (in the latter under the name geometric full residual finiteness growth), and fits into the extensive literature on quantifying finite approximations of infinite groups which has developed over the last decade. This program started with the work of Bou-Rabee and collaborators on quantitative residual finiteness (see [3] and the references therein). Using results on quantitative residual finiteness, word growth, and finite presentability, the LEF growth function has been estimated for several natural classes of groups (see [5], §2.4).

Example 1.8. Let $\Gamma$ be a finitely generated group.

(i) If $\Gamma$ is virtually $\mathbb{Z}^d$, then $\mathcal{L}_{\Gamma}(n) \approx n^d$;

(ii) $\mathcal{L}_{\Gamma}$ is bounded above by a polynomial function iff $\Gamma$ is virtually nilpotent;

(iii) If $\Gamma \leq \text{GL}_d(\mathbb{Z})$ is finitely generated, not virtually nilpotent, then $\mathcal{L}_{\Gamma}(n) \approx \exp(n)$.
Groups of larger LEF growth can be explicitly constructed using wreath products.

**Theorem 1.9** ([5] Theorem 1.8). If $\Gamma$ is a finitely generated LEF group with word growth function $\gamma_\Gamma$, and $\Delta$ is a finite centreless group, then $\exp \{ \gamma_\Gamma(n) \} \leq L_{\Delta\Gamma}(n) \leq \exp (L_{\Gamma}(n))$.

In particular, using Example 1.8 Theorem 1.9 allows us to construct groups of LEF growth $\approx \exp(\exp(n))$ and $\approx \exp(n^d)$ (for any $d \in \mathbb{N}$). Beyond this, however, few types of LEF growth functions had hitherto been observed, and our Theorem 1.5 greatly extends the spectrum of known growth types.

The derived subgroup of the topological full group $[\sigma]$ of a minimal subshift $(X, \sigma)$ is a remarkable object in group theory. It is a finitely generated infinite simple group, which, as well as being LEF, is amenable [12] (and indeed was the first group discovered with this combination of properties). It is also a natural invariant from the point of view of topological dynamics. As shown in [2], for any Cantor minimal system $(X, \varphi)$, $[\varphi]'$ retains perfect information about the dynamics of $(X, \varphi)$.

**Theorem 1.10** (Bezuglyi-Medynets). Let $(X, \varphi)$ and $(Y, \psi)$ be Cantor minimal systems. Then $[\varphi]' \cong [\psi]'$ iff $(X, \varphi)$ and $(Y, \psi)$ are flip-conjugate.

It is therefore reasonable to expect that group-theoretic asymptotic invariants of $[\varphi]'$ should reflect asymptotic features of the dynamical system $(X, \varphi)$. Our Theorems 1.1 and 1.2 are in this spirit: knowing the LEF growth function of $[\sigma]'$ allows one to deduce some bounds on the recurrence or complexity functions of $X$.

Our proof of Theorem 1.1 is based on Elek’s streamlined proof of LEF for topological full groups [9]. Given $\Gamma \leq [\sigma]'$, finitely generated by $S$, and a two-sided sequence $x \in X$, the $\sigma$-orbit $O$ of $x$ is dense in $X$, so $\Gamma$ acts faithfully on $O$. Further, any short word in $S$ moves some cylinder set $C \subseteq X$, defined by a short string in $x$. Since only finitely many such cylinder sets $C$ arise, each of which intersects $O$, there exists $M \in \mathbb{N}$ such that no nonidentity element of $B_g(n)$ fixes $\{ \sigma^i x : 1 \leq i \leq M \}$ pointwise; moreover we can take $M \leq R_X(Cn)$ for some $C > 0$. Carefully choosing the exact value of $M$ to ensure consistency, we use this to construct a local embedding $B_g(n) \to \text{Sym}(M)$.

For our lower bound, we observe that $[\sigma]'$ contains many copies of the alternating group $\text{Alt}(5)$, acting on disjoint subsets of $X$ (hence generating their direct product). It follows that any finite group admitting a local embedding of a large ball in $[\sigma]'$ also contains a direct product of many copies of $\text{Alt}(5)$ as a subgroup, and so has large order. The supply of disjoint subsets on which to act, in this construction, is limited by the complexity function $p_x$, hence the appearance of $p_x$ in Theorem 1.2.

This paper is structured as follows: in Subsections 2.1, 2.2, and 2.3 we collect necessary background results about LEF growth of groups, symbolic dynamics, and topological full groups, respectively. In Section 5 we construct the local embeddings required to prove Theorem 1.1. In Section 4 we prove Theorem 1.2. In Section 5 we describe the construction of the minimal subshifts arising in Theorem 1.7 and deduce Theorems 1.5 and 1.6.
2 Preliminaries

2.1 LEF groups and Schreier graphs

**Definition 2.1.** For $\Gamma, \Delta$ groups and $F \subseteq \Gamma$, a partial homomorphism of $F$ into $\Delta$ is a function $\phi : F \to \Delta$ such that, for all $g, h \in F$, if $gh \in F$, then $\phi(gh) = \phi(g)\phi(h)$. A partial homomorphism $\phi$ is called a local embedding if it is injective. $\Gamma$ is locally embeddable into finite groups (LEF) if, for all finite $F \subseteq \Gamma$, there exists a finite group $Q$ and a local embedding of $F$ into $Q$.

Henceforth suppose that $\Gamma$ is LEF and generated by the finite set $S$. Let $B_S(n) \subseteq \Gamma$ denote those elements of length at most $n$, with respect to the word metric induced on $\Gamma$ by $S$.

**Definition 2.2.** The LEF growth of $\Gamma$ (with respect to $S$) is:
\[ L^S_1(n) = \min\{|Q| : \exists \phi : B_S(n) \to Q \text{ a local embedding}\} \]

and the LEF action growth is:
\[ L^A_1(n) = \min\{d : \exists \phi : B_S(n) \to \text{Sym}(d) \text{ a local embedding}\} \]

**Remark 2.3.** It is clear that $L^A_1(n) \leq L^S_1(n) \leq L^S_1(n)!$.

**Definition 2.4.** For $F_1, F_2 : \mathbb{N} \to \mathbb{N}$ nondecreasing functions, write $F_1 \preceq F_2$ if there exists $C > 0$ such that $F_1(n) \leq F_2(Cn)$ for all $n$. Write $F_1 \approx F_2$ if $F_1 \preceq F_2$ and $F_2 \preceq F_1$.

**Lemma 2.5.** Let $F = \mathcal{L}$ or $\mathcal{L} \Gamma$. Let $\Delta \leq \Gamma$ be finitely generated by $T$. Then there exists $C > 0$ such that for all $n$,
\[ F^T_\Delta(n) \preceq F^S_\Gamma(Cn) \]

In particular, for $T$ a second finite generating set for $\Gamma$, $F^S_\Gamma \approx F^T_\Gamma$.

**Proof.** This is proved for $F = \mathcal{L}$ as Corollary 2.7 in [5]; the proof for $F = \mathcal{L} \Gamma$ is identical. \(\square\)

The next Proposition is key to the proof of Theorem [2]. It uses an idea already exploited in [5] Theorem 3.4. to control the LEF growth of wreath products.

**Proposition 2.6.** Let $n \in \mathbb{N}$ and $m \geq 2$. Suppose $\Delta_1, \ldots, \Delta_m \subseteq \Gamma$ are finite centreless subgroups, generating their direct product, and that $\Delta_i \subseteq B_S(n)$. Suppose that $Q$ is a finite group and that $\phi : B_S(2n) \to Q$ is a local embedding. Then the $\phi(\Delta_i) \leq Q$ generate their direct product, and $|Q| \geq \prod_i |\Delta_i| \cdot |\Delta_i|$.

**Proof.** Since $\phi$ restricts to an injective homomorphism on each $\Delta_i$, $\phi(\Delta_i)$ is a subgroup of $Q$, isomorphic to $\Delta_i$. Certainly, for $i \neq j$ and $g_i \in \Delta_i, g_j \in \Delta_j$,
\[ \phi(g_i)\phi(g_j) = \phi(g_ig_j) = \phi(g_j)\phi(g_i). \]

Therefore, if the $\phi(\Delta_i)$ fail to generate their direct product, there exists $1 \leq i \leq m$ and $1 \neq g \in \Delta_i$ such that $\phi(g) \in P$, where $P = \langle \phi(\Delta_i) : i \neq j \rangle \leq Q$. $P$ centralizes $\phi(\Delta_i)$, since the $\phi(\Delta_j)$ do, so for $h \in \Delta_i, \phi(gh) = \phi(g)\phi(h) = \phi(h)\phi(g) = \phi(hg)$. By injectivity of $\phi$ restricted to $\Delta_i$, $g$ is central in $\Delta_i$, contradiction. \(\square\)
A based graph is a pair \((G, v)\), where \(G\) is a directed graph and \(v \in V(G)\). A morphism of based graphs \((G_1, v_1) \rightarrow (G_2, v_2)\) is a graph-morphism \(\phi : G_1 \rightarrow G_2\) with \(\phi(v_1) = v_2\). For \(C\) a set, an edge-colouring of the graph \(G\) in \(C\) is a function \(c : E(G) \rightarrow C\). A morphism \((G_1, v_1, c_1) \rightarrow (G_2, v_2, c_2)\) of based graphs with edge colourings in \(C\) is a morphism \(\phi : (G_1, v_1) \rightarrow (G_2, v_2)\) of based graphs such that, for all \(e \in E(G_1)\), \(c_1(e) = c_2(\phi(e))\).

**Definition 2.7.** Let \(\Gamma\) be a group, \(\Omega\) be a \(\Gamma\)-set, and \(S \subseteq \Gamma\). The associated Schreier graph \(\text{Schr}(\Gamma, \Omega, S)\) is the graph with vertex set \(\Omega\) and edge set \(\{ (\omega, s) : \omega \in \Omega, s \in S \} \). Then \(\text{Schr}(\Gamma, \Omega, S)\) is naturally an edge-coloured graph with colours in \(\{1, \ldots, |S|\}\), via \(c((\omega, s)) = c(s)\).

**Definition 2.8.** Let \(G_1, G_2\) be directed edge-coloured graphs (with colours in \(C\)) and let \(r \in \mathbb{N}\). We say that \(G_1\) is locally embedded in \(G_2\) at radius \(r\) if, for every \(v \in V(G_1)\), there exists \(w \in V(G_2)\) and an isomorphism of based coloured graphs \((B_v(r), v) \cong (B_w(r), w)\) (here \(B_v(r) \subseteq G_1\) is the induced subgraph on the closed ball of radius \(r\) around \(v\) in the path metric on \(G_1\), and likewise for \(B_w(r) \subseteq G_2\)). We say that \(G_1\) and \(G_2\) are locally colour-isomorphic at radius \(r\) if each is locally embedded in the other at radius \(r\), that is, \(G_1\) and \(G_2\) have the same set of isomorphism-types of balls of radius \(r\).

The next observation is our key tool for constructing the local embeddings needed in Theorem [1,1] it is proved as Lemma 4.2 in [5].

**Lemma 2.9.** For \(i = 1, 2\) let \(\Gamma_i\) be a group acting faithfully on a set \(\Omega_i\), and let \(S_i\) be an ordered generating \(d\)-tuple in \(\Gamma_i\). Suppose that the Schreier graphs \(\text{Schr}(\Gamma_i, \Omega_i, S_i)\) are locally colour-isomorphic at some radius at least \([3r/2]\). Then there is a local embedding \(B_{S_i}(r) \rightarrow \Gamma_2\) extending \((S_1)_j \mapsto (S_2)_j\).

### 2.2 Words

Throughout, \(A\) will be a finite discrete set with \(|A| \geq 2\), the alphabet. Let \(A^* = \cup_n A^n\) be the set of finite words over \(A\). An infinite word shall be an element of either \(A^\mathbb{Z}\) or \(A^\mathbb{N}\). We equip these latter sets with the product topology; note that both are thereby homeomorphic to the Cantor space.

**Definition 2.10.** Let \(w \in A^\mathbb{Z}\). For \(k \in \mathbb{Z}\) and \(n \in \mathbb{N}\), the \(k\)th \(n\)-factor of \(w\) is \(w_k w_{k+1} \cdots w_{k+n-1} \in A^n\). \(v \in A^n\) is an \(n\)-factor of \(w\) if it is the \(k\)th \(n\)-factor for some \(k\). \(n\)-factors of words in \(A^\mathbb{N}\) and \(A^*\) are defined similarly, with the requirement that \(k \in \mathbb{N}\) and, for \(w \in A^*\), that \(k \leq |w| - n + 1\).

In any case, the set of all \(n\)-factors of \(w\) is denoted \(F_n(w)\). For \(w\) an infinite word, the complexity function \(p_w : \mathbb{N} \rightarrow \mathbb{N}\) of \(w\) is given by \(p_w(n) = |F_n(w)|\).

It is immediate from the definitions that \(p_w(n+m) \leq p_w(n)p_w(m)\) and \(p_w(n) \leq |A|^n\) for all \(n, m \in \mathbb{N}\).
Definition 2.11. The entropy of \( w \in \mathbb{A}^\mathbb{Z} \) is:

\[
h(w) = \lim_{n \to \infty} \frac{\log p_w(n)}{n}
\]

(the limit is well-defined, by the preceding remarks).

**Remark 2.17.** Let \( u \in \mathbb{A}^n \) and \( 1 \leq i \leq n \).

(i) For any \( k, l \in \mathbb{N} \),

\[
\bigcup_{w \in \mathbb{A}^k} \langle uw \rangle_{k+i} = \langle u \rangle_i = \bigcup_{w \in \mathbb{A}^l} \langle uw \rangle_i
\]

In particular, for any \( m \in \mathbb{N} \), the set of all \( m \)-cylinders of \( X \) forms a clopen partition of \( X \), and for \( 1 \leq k \leq m \), any \( k \)-cylinder of \( X \) is the disjoint union of the \( m \)-cylinders of \( X \) which intersect it.

(ii) The cylinder sets form a basis for the topology on \( \mathbb{A}^\mathbb{Z} \). Hence, for any clopen subset \( Y \) of \( \mathbb{A}^\mathbb{Z} \), \( Y \) is the union of cylinder sets. By compactness, and by (i), there exists \( C = C(Y) > 0 \) such that, for all \( m \geq C \), \( Y \) is the disjoint union of \( m \)-cylinders of \( \mathbb{A}^\mathbb{Z} \).

The next Lemma shall be used in the proof of Theorem 1.7.

**Lemma 2.18.** Let \( (L_j) \) be an increasing sequence of positive integers; let \( x^{(j)} \in \mathbb{A}^{L_j} \). For each \( j \in \mathbb{N} \), let \( K_j \in \mathbb{N} \) with \( 2 \leq K_j \leq L_{j+1} - L_j \) and suppose that \( x^{(j)} \) is the \( K_j \)th \( L_j \)-factor of \( x^{(j+1)} \). Choose \( M_0 \in \mathbb{N} \) with \( 1 \leq M_0 \leq L_0 \), and define \( (M_j) \) recursively via \( M_{j+1} = M_j + K_j - 1 \). Then there is a unique point \( x \in \mathbb{A}^\mathbb{Z} \) lying in the intersection of the cylinder sets \( \langle x^{(j)} \rangle_{M_j} \). Moreover for all \( n \), \( F_n(x) = \bigcup_j F_n(x^{(j)}) \).
Proof. By construction the \( \langle x^{(j)} \rangle_{M_j} \) form a nested descending sequence of nonempty closed sets in the compact space \( A^\mathbb{Z} \), hence their intersection is nonempty. Let \( x \) lie in this intersection.

In \( x^{(j)} \), there are \( M_j - 1 = M_{j-1} + K_{j-1} - 2 \) letters strictly to the left of the \( M_j \)th letter, and \( L_j - M_j \geq (L_{j-1} - M_{j-1}) + 1 \) strictly to the right. Since both these quantities tend to \( \infty \), for every \( i \in \mathbb{Z} \), the \( i \)th letter of \( x \) is uniquely determined by \( x \in \langle x^{(j)} \rangle_{M_j} \), provided \( j \) is sufficiently large. Similarly, for any \( i \) and \( n \), the \( i \)th \( n \)-factor of \( x \) lies in \( x^{(j)} \) for \( j \) sufficiently large, so \( F_n(x) \subseteq \bigcup_j F_n(x^{(j)}) \). Conversely, \( x \in \langle x^{(j)} \rangle_{M_j} \) implies \( x^{(j)} \) (and hence every \( n \)-factor of \( x^{(j)} \)) is a factor of \( x \).

The utility of Lemma 2.18 shall be that it allows us to an infinite word whose asymptotic features (such as the behaviour of the complexity or recurrence functions) can be controlled in terms of the sets of factors of the finite words \( u^{(n)} \).

### 2.3 Topological full groups and minimal subshifts

Let \( C \) be the Cantor space. A Cantor dynamical system is a pair \((X, \varphi)\), where \( X \) is a space homeomorphic to \( C \), and \( \varphi \in \text{Homeo}(X) \) (we specify the space \( X \), rather than always taking \( X = C \), in case the homeomorphism \( \varphi \) is described in terms of a particular model \( X \) of Cantor space; in particular this will be the case when the system is a subshift). The system \((X, \varphi)\) is called minimal if every orbit in \( X \) under the action of \( \langle \varphi \rangle \) is dense in \( X \).

**Example 2.19.** Let \( A \) be a finite discrete space with \(|A| \geq 2\). Then \( A^\mathbb{Z} \cong C \). The shift over \( A \) is \( \sigma \in \text{Homeo}(A^\mathbb{Z}) \) given, for \( a = (a_i)_{i \in \mathbb{Z}} \), by \( \sigma(a) = a_{i+1} \). The Cantor dynamical system \((A^\mathbb{Z}, \sigma)\) is never minimal.

**Definition 2.20.** The topological full group \( \|\varphi\| \) of the system \((X, \varphi)\) is the set of all homeomorphisms \( g \) of \( X \) such that there exists a continuous function \( f_g : X \to \mathbb{Z} \) (called the orbit cocycle of \( g \)) such that for all \( x \in X \), \( g(x) = \varphi^f_g(x)(x) \) (here we assume \( \mathbb{Z} \) equipped with the discrete topology).

Equivalently, \( g \in \text{Homeo}(X) \) lies in \( \|\varphi\| \) if \( g \) is a finite clopen partition \( C_1, \ldots, C_d \) of \( X \) and integers \( a_1, \ldots, a_d \) such that for \( i \leq i \leq d \), \( g_{C_i} = \varphi^{a_i} \) (taking \( \{a_1, \ldots, a_d\} = \text{im}(f_g), C_i = f_g^{-1}(a_i) \)). It is straightforward to check that \( \|\varphi\| \) is a subgroup of \( \text{Homeo}(X) \).

**Remark 2.21.** If \((X, \varphi)\) is a minimal system, then the orbit cocycle \( f_g \) is uniquely determined by \( g \in \|\varphi\| \), since \( \varphi \) has no periodic points.

The next result gives the key source of minimal systems for our purposes.

**Proposition 2.22.** Suppose \( x \in A^\mathbb{Z} \) is uniformly recurrent non-periodic. Then \( \overline{\varphi(x)} \cong C \), and the system \((\overline{\varphi(x)}, \sigma)\) is minimal.

**Proof.** For minimality, see [11] Theorem 1.0.1. To prove \( \overline{\varphi(x)} \cong C \), note that the only property not automatically inherited by subspaces of \( C \) is being perfect; then, if \( Y = \overline{\varphi(x)} \), a non-periodic \( x \) must have \( |Y| = \infty \), which implies that \( Y \) has an accumulation point in \( C \); but \( Y \) contains its set of accumulation points \( \text{Acc}(Y) \) in \( C \) (by compactness), and \( \text{Acc}(Y) \) is closed and \( \sigma \)-invariant, so by minimality \( Y = \text{Acc}(Y) \). \( \square \)
Remark 2.24. Suppose \( s \in A^\mathbb{Z} \) is uniformly recurrent non-periodic. For any \( y \in \mathcal{O}(s) \) and \( n \in \mathbb{N} \), \( F_n(s) = F_n(y) \). In particular, \( p_s \) and \( R_s \) depend only on \( X \), and we may henceforth write \( p_x \) or \( R_x \) instead.

Let \( \langle \varphi \rangle' \) denote the derived subgroup of \( \langle \varphi \rangle \). The reason for our focus on minimal subshifts, among all minimal systems, is made clear by the next result.

Theorem 2.25 ([14] Theorem 5.4). For \( (X, \varphi) \) a Cantor minimal system, \( \langle \varphi \rangle' \) is a finitely generated group iff \( (X, \varphi) \) is isomorphic to a minimal subshift.

Theorem 2.26 ([8] Proposition 2.4). Let \( (X, \sigma) \) be a minimal subshift, and let \( S \) be a finite generating set for \( \langle \sigma \rangle' \). Then \( |B_S(n)| \geq \exp(n) \).

Corollary 2.27. We have \( \mathcal{L}_{\langle \varphi \rangle'}(n) \geq \exp(n) \).

Proof. This is immediate from the preceding Theorem: if \( \phi : B_S(n) \to Q \) is a local embedding, then \( \phi \) is injective, so \( |Q| \geq |B_S(n)| \).

Thus our Theorem 1.2 is only new in the case \( p_\mathcal{Y}(n) \notin n^2 \).

3 Construction of local embeddings

In this Section we prove Theorem 1.1. Let \( (X, \sigma) \) be a minimal subshift over the alphabet \( A \).

Proposition 3.1. Let \( S \subseteq \langle \sigma \rangle \) be finite. Then there exists an integer \( C_1 = C_1(S) \geq 1 \) such that for all \( r \in \mathbb{N} \) and all \( g \in B_S(r) \):

(i) \( \max \{|f_g(x)| : x \in X \} \leq C_1 r \);

(ii) For all \( m \geq C_1 r \), \( f_g \) is constant on every \( m \)-cylinder of \( X \).

Proof. For (i), for \( g \in \langle \sigma \rangle \) write \( \lambda(g) = \max \{|f_g(x)| : x \in X \} \). Since \( S \) is finite, we may choose \( C_1 \geq \max \{|\lambda(s) : s \in S\} \). For \( g, h \in \langle \sigma \rangle \), \( \lambda(gh) \leq \lambda(g) + \lambda(h) \), so by induction \( \lambda(g) \leq C_1 r \) for all \( g \in B_S(r) \).

For (ii), for every \( s \in S \cup S^{-1} \) there is a finite clopen partition \( C_s \) of \( X \), such that \( f_s \) is constant on the parts of \( C_s \). By Remark 2.17(ii), we may choose \( C_1 \) sufficiently large that for all \( s \in S \cup S^{-1} \) and \( m \geq C_1 \), \( f_{gh} \) is constant on every \( m \)-cylinder of \( X \). Let \( r \geq 2 \) and suppose by induction that the claim holds for smaller \( r \). Let \( m \geq C_1 r \), let \( U \) be an \( m \)-cylinder of \( X \), and let \( g \in B_S(r) \). Then there exist \( h \in B_S(r - 1) \) and \( s \in S \cup S^{-1} \cup \{e\} \) such that \( g = sh \). By inductive hypothesis, \( f_h \) is constant on \( U \), say with value \( i \). Observe that, for any \( k, l \in \mathbb{N} \), if \( x, y \in X \) lie in the same \((k + l)\)-cylinder of \( X \), then \( \sigma^k x, \sigma^l y \) lie in the same \( k \)-cylinder of \( X \). Thus, there is a unique \((m - |l|)\)-cylinder containing \( h(U) \). By (i), \( i \in [-C_1(r - 1), C_1(r - 1)] \), so \( m - |l| \geq C_1 \); this implies that \( f_{gh} \) is constant on \( h(U) \), hence \( f_g \) is constant on \( U \).
Remark 3.2. By Proposition\[3.1\](i), for \( k \in \mathbb{N} \) and \( m \geq C_1r \), if \( x, y \in X \) lie in the same \((k + m)\)-cylinder of \( X \), then \( g(x), g(y) \) lie in the same \( k \)-cylinder of \( X \) for any \( g \in B_S(r) \).

Lemma 3.3. Let \( C_1 \) be as in Proposition\[3.1\] For all \( r \in \mathbb{N} \), there exists \( M = M(r) \in \mathbb{N} \) satisfying:

(i) \( \{ \sigma^i x : 1 \leq i \leq M \} \) intersects every \( C_1(r + 1) \)-cylinder of \( X \);

(ii) \( x \) and \( \sigma^{M} x \) lie in the same \( C_1(2r + 1) \)-cylinder of \( X \);

(iii) \( 10C_1r \leq M \leq 2R_X(10C_1r) \).

Proof. Let \( u \in A^{2C_1(r + 1) + 1} \), \( \langle u \rangle \subset C_1(r + 1) \cap X \) is a \( C_1(r + 1) \)-cylinder of \( X \) iff \( u \in F_{2C_1(r + 1) + 1}(x) \). Therefore (i) holds for any \( M \geq R_X(2C_1(r + 1) + 1) \). For the same reason, a value of \( M \) satisfying (ii) occurs at least once in every interval of length \( R_X(2C_1(2r + 1) + 1) \). Therefore we can choose \( M \leq R_X(2C_1(2r + 1) + 10C_1r) \) satisfying (i), (ii) and the first inequality of (iii). Since \( 10C_1r \leq R_X(5C_1r) \) by Theorem\[2.13\] and since \( R_X \) is nondecreasing, the second inequality of (iii) also holds.\( \square \)

Corollary 3.4. Let \( M \) be as in Lemma\[3.3\] For any \( |i| \leq C_1r \), \( \sigma^{i} x \) and \( \sigma^{M+i} x \) lie in the same \( C_1(r + 1) \)-cylinder of \( X \).

Proof. This is immediate from Lemma\[3.3\](ii), by Remark\[3.2\]\( \square \)

Fix \( x \in X \) and let \( \mathcal{O} = \mathcal{O}(x) \) be the orbit of \( x \) under \( \sigma \). Then \( [\sigma] \) acts on \( \mathcal{O} \). Since \( X \) has no periodic points, this action induces a well-defined homomorphism \( \phi : [\sigma] \to \text{Sym}(\mathbb{Z}) \) by:

\[
g(\sigma^n x) = \sigma^{\phi([\sigma]) n}(x).
\]

(4)

In other words, \( \phi(g)[n] = n + f_g(\sigma^n x) \). Since \( \mathcal{O} \) is dense in \( X \), \( \phi \) is injective.

Let \( \Gamma \leq [\sigma] \) be finitely generated, and let \( S = (s_1, \ldots, s_d) \) be an ordered generating \( d \)-tuple of elements of \( \Gamma \). Then \( \Gamma \) acts faithfully on \( \mathbb{Z} \) via \( \phi \). Let \( G = \text{Schr}(\Gamma, \mathbb{Z}, S) \) be the associated Schreier graph of the action of \( \Gamma \) on \( \mathbb{Z} \) with respect to \( S \). Let \( C_1 = C_1((s_1, \ldots, s_d)) \geq 1 \) be as in Proposition\[3.1\]

Lemma 3.5. For \( n \in \mathbb{Z} \) and \( r \in \mathbb{N} \), the isomorphism type of \( B_G(n, r) \) depends only on the \( C_1(r + 1) \)-cylinder of \( X \) in which \( \sigma^n x \) lies.

Proof. By Proposition\[3.1\](i), \( B_G(n, r) \) \( \cong \) \([n - C_1r, n + C_1r] \cap \mathbb{Z} \). For \( |i| \leq C_1r \), \( n + i \in B_G(n, r) \) iff, for some \( g \in B_S(r) \), \( f_g(\sigma^n x) = i \), and by Proposition\[3.1\](ii), this condition depends only on the \( C_1r \)-cylinder of \( X \) containing \( \sigma^n x \).

For \( n + i, n + j \in B_G(n, r) \), and \( 1 \leq c \leq d, (n + i, n + j) \) is a \( c \)-coloured edge in \( G \) iff \( f_{g}(\sigma^{n+c} x) = j - i \). This depends only on the \( C_1 \)-cylinder of \( X \) containing \( \sigma^{n+c} x \), which in turn depends only on the \( C_1(r + 1) \)-cylinder of \( X \) containing \( \sigma^n x \).\( \square \)

Let \( r \geq 1 \) and let \( M = M(r) \) be as in Lemma\[3.3\] For \( 1 \leq c \leq d \) define \( \overline{\overline{\tau}}_c : \mathbb{Z}/M \mathbb{Z} \to \mathbb{Z}/M \mathbb{Z} \) by:

\[
\overline{\overline{\tau}}_c(n + M \mathbb{Z}) = \phi(s_c)[n] + M \mathbb{Z} \text{ for } 1 \leq n \leq M.
\]

Proposition 3.6. The function \( \overline{\overline{\tau}}_c \) lies in \( \text{Sym}(\mathbb{Z}/M \mathbb{Z}) \).
Proof. We check that \( \overline{\sigma_c} \) is injective, and therefore is indeed a well-defined permutation. If \( 1 \leq m < n \leq M \) with \( \phi(s_c)[n] \equiv \phi(s_c)[m] \mod M \), then:

\[
 n - m \equiv f_{s_c}(\sigma^m x) - f_{s_c}(\sigma^n x) \mod M
\]

but by Proposition 3.1(i),

\[
|f_{s_c}(\sigma^m x) - f_{s_c}(\sigma^n x)| \leq 2C_1
\]

while \( 1 \leq n - m - 1 \leq M - 1 \), so either \( 1 \leq n - m \leq 2C_1 \) or \( M - 2C_1 \leq n - m \leq M - 1 \). In the former case,

\[
\phi(s_c)[n] \equiv \phi(s_c)[m] \mod M \text{ but } |\phi(s_c)[n] - \phi(s_c)[m]| \leq 4C_1,
\]

and since \( M \geq 10C_1 \) by Lemma 3.3(iii), \( \phi(s_c)[n] = \phi(s_c)[m] \), contradicting the injectivity of \( \phi(s_c) \). The latter case is similar; we have:

\[
1 \leq m \leq 2C_1 \text{ and } 1 \leq m + M - n \leq 2C_1.
\]

By Corollary 3.4 \( \sigma^{m+M} x \) and \( \sigma^m x \) lie in the same \( C_1(r+1) \)-cylinder of \( X \), so by Proposition 3.1 \( f_{s_c}(\sigma^{m+M} x) = f_{s_c}(\sigma^m x) \). We may therefore argue as in the former case, with \( m + M \) replacing \( m \).

\[ \square \]

Theorem 3.7 is now immediate, by Remark 2.3 from the next result.

**Theorem 3.7.** We have \( \mathcal{L}(r/3) \leq 2R_X(10C_1r) \).

**Proof.** Let \( M \) be as in Lemma 3.3. We claim that there is a local embedding \( B_g([2r/3]) \to \text{Sym}(Z/MZ) \) sending \( s_c \to \overline{\sigma_c} \) for \( 1 \leq c \leq d \). By the upper bound on \( M \) from Lemma 3.3(iii), the result follows. Let \( \overline{\mathcal{S}} = \{\overline{s}_1, \ldots, \overline{s}_d\} \); let \( \overline{\mathcal{T}} = \text{Schr}(\overline{\mathcal{S}}, Z/MZ) \), and let \( \overline{G} = \text{Schr}(\overline{\mathcal{T}}, Z/MZ, \overline{\mathcal{S}}) \). By Lemma 2.3 it suffices to show that \( G \) and \( \overline{G} \) are locally colour-isomorphic at radius \( r \).

By Lemma 3.3 and Lemma 3.3(i), for every \( m \in Z \), \( B_g(m, r) \subseteq G \) is isomorphic (as a based, edge-coloured graph) to \( B_g(n, r) \), for some \( 1 \leq n \leq M \). Let \( \pi_M : Z \to Z/MZ \) be reduction modulo \( M \). It suffices to check that for every \( 1 \leq n \leq M \) the restriction of \( \pi_M \) to \( V(B_g(n, r)) \subseteq Z \) induces an isomorphism of based edge-coloured graphs \( B_g(n, r) \to B_g(\pi_M(n, r)) \).

Consider \( I_n = [n - C_1r, n + C_1r] \cap Z \). Since \( 2C_1r < M \) (by Lemma 3.3), the restriction of \( \pi_M \) to \( I_n \) is injective. By Proposition 3.4(i), \( V(B_g(n, r)) \subseteq I_n \), so the restriction of \( \pi_M \) to \( V(B_g(n, r)) \) is a bijection onto its image. Then it suffices to show that \( \pi_M(\phi(s_c)[k]) = \overline{\sigma_c}(\pi_M(k)) \) for all \( k \in I_n \) and all \( c \).

Since \( -C_1r \leq k \leq M + C_1r \), there exists \( \epsilon \in \{0, \pm 1\} \) such that \( k' = k + \epsilon M \) is the representative of \( k + MZ \) in \([1, M]\). By Corollary 3.4 and Proposition 3.4(ii) (or by \( k = k' \)) we have \( f_{s_c}(\sigma^k x) = f_{s_c}(\sigma^{k'} x) \), which means \( \phi(s_c)[k] = \phi(s_c)[k'] + \epsilon M \). Then,

\[
\pi_M(\phi(s_c)[k]) = \pi_M(\phi(s_c)[k'] + \epsilon M) = \pi_M(\phi(s_c)[k']) = \overline{\sigma_c}(k' + MZ) = \overline{\sigma_c}(\pi_M(k))
\]

as claimed.
4 Obstructions to small local embeddings

Let \((X, \sigma)\) be a minimal subshift over \(A\), and fix \(x \in A^\mathbb{Z}\) with \(X = \overline{\{x\}}\). Let \(S \subseteq \|\sigma\|'\) be a finite generating set (such exists by Theorem 2.25), and \(|\cdot|_S\) be the word length function induced on \([\|\sigma\|']\) by \(S\). Recall that \(p_x : \mathbb{N} \to \mathbb{N}\) is the complexity function of \(x\) and \(R_x = R_x : \mathbb{N} \to \mathbb{N}\) is the recurrence function of \(x\).

Lemma 4.1. Let \(\text{Cyl}_X(m)\) be the set of \(m\)-cylinders of \(X\). Then \(|\text{Cyl}_X(m)| = p_x(2m + 1)\).

Proof. \(\mathcal{O}(x) = \{\sigma^i(x) : i \in \mathbb{Z}\}\) is dense in \(X\), so for \(u_i \in A, U = \langle u_m, \ldots, u_{-1}, u_0, u_1, \ldots, u_m \rangle\) intersects \(X\) if and only if \(u_j = x_{i+j}\) for \(-m \leq j \leq m\). That is, \(U\) intersects \(X\) if \(u_m \cdots u_0\) is a factor of \(x\).

Lemma 4.2. Suppose that \(m \geq (R_x(4) - 1)/2\), and let \(U \in \text{Cyl}_X(m)\). Then the sets \(\sigma^i(U), \sigma^j(U)\), for \(-2 \leq i \leq j \leq 2\), are pairwise disjoint.

Proof. Suppose to the contrary that \(y \in \sigma^i(U) \cap \sigma^j(U)\), for some \(-2 \leq i < j \leq 2\). Writing \(k = j - i\), we have \(y_l = y_{l+k}\) for \(-m \leq l \leq m - k\). Letting \(w = y_{-m}y_{-m+1} \cdots y_m \in F_{2m+1}(x)\), we have \(|F_k(w)| \leq k\). On the other hand, \(2m + 1 \geq R_x(k)\), so by the definition of \(R_x\), \(|F_k(w)| = |F_k(x)| = p_x(k) \geq k + 1\) (the last inequality holding by Theorem 2.13), a contradiction.

Lemma 4.3. Let \(m \geq \max\{(R_x(4) - 1)/2, 5\}\). There exists a set \(\text{DCyl}_X(m) \subseteq \text{Cyl}_X(m)\) such that the sets \(\sigma^i(U), \sigma^j(U)\), for \(U \in \text{DCyl}_X(m)\) and \(-2 \leq i \leq j \leq 2\), are pairwise-disjoint and:

\[|\text{DCyl}_X(m)| \geq p_x(2m - 7)/9.\] (5)

Proof. We construct \(\text{DCyl}_X(m)\) via an iterative process, as follows. At step 0, let \(U_0\) be any \(m\)-cylinder. Then \(\sigma^i(U_0), \sigma^j(U_0)\), for \(-2 \leq i \leq 2\), are pairwise disjoint, by Lemma 4.2. Let \(A_0\) be the set of \((m - 4)\)-cylinders containing one of the \(\sigma^i(U_0), \sigma^j(U_0)\), for \(-4 \leq j \leq 4\), so that \(|A_0| \leq 9\), and let \(B_0 = \{U_0\}\).

At each subsequent step \(k + 1\), we start with \(B_k\) a family of \(m\)-cylinders and \(A_k\) a family of \((m - 4)\)-cylinders such that, for any \(U \in B_k\) and \(-4 \leq j \leq 4\), there exists \(V \in A_k\) such that \(\sigma^j(U) \subseteq V\). If \(A_k\) is a cover of \(X\), then set \(\text{DCyl}_X(m) = B_k\) and stop. Otherwise, choose \(y_k \in X \setminus \bigcup_{V \in A_k} V\) and let \(U_{k+1}\) be the \(m\)-cylinder of \(X\) containing \(y_k\). Then \(\sigma^i(U_{k+1}) \ (\leq -2 \leq i \leq 2)\) are pairwise disjoint (by Lemma 4.2). If there exists \(0 \leq l \leq k + 1\), let \(l \leq i, j \leq 2\) such that \(\sigma^i(U_{k+1}) \cap \sigma^j(U_l) \neq \emptyset\), then, letting \(V \in A_k\) be such that \(\sigma^j(U_l) \subseteq V\), \(U_{k+1} \subseteq V\) (by Remark 2.17), contradicting the choice of \(y_k\).
Proof. Write 

We may therefore let \( B_{k+1} = B_k \cup \{ U_{k+1} \} \) and produce \( A_{k+1} \) by adding to \( A_k \) all \((m - 4)\)-cylinders containing one of the \( \sigma^j(U_{k+1}) \) \((-4 \leq j \leq 4)\). At every stage \( |A_{k+1}| \leq |A_k| + 9, |B_{k+1}| \geq |B_k| + 1 \) and by Lemma 4.3 the process terminates only when \( |A_k| = p_\alpha(2m - 7) \).

\[\square\]

**Notation 4.4.** Let \( U \subseteq X \) be a nonempty clopen set, such that \( \sigma^{-1}(U), U \) and \( \sigma(U) \) are pairwise disjoint. We denote by \( f_U \) the element of \( [[\sigma]]' \) given by:

\[
f_U(y) = \begin{cases} 
\sigma(y) & y \in U \cup \sigma^{-1}(U), \\
\sigma^{-2}(y) & y \in \sigma(U), \\
y & \text{otherwise}.
\end{cases}
\]

**Lemma 4.5.** All \( f_U \) lie in \( [[\sigma]]' \).

Proof. Define \( h_U \in [[\sigma]] \) by:

\[
h_U(y) = \begin{cases} 
\sigma(y) & y \in \sigma^{-1}(U), \\
\sigma^{-1}(y) & y \in U, \\
y & \text{otherwise}.
\end{cases}
\]

Then \( f_U = f_U^{-1} h_U^{-1} f_U h_U \).

\[\square\]

The following identities appear as \([14] \) Lemma 5.3; they are proved by direct calculation, some of which is explained in \([11] \) Lemma 3.0.11.

**Lemma 4.6.** Let \( U, V \subseteq X \) be nonempty clopen subsets.

(i) If \( \sigma^i(V) \) are pairwise disjoint for \(-2 \leq i \leq 2\), and \( U \subseteq V \), then:

\[\tau_V f_U \tau_V^{-1} = f_{\sigma(U)}, \quad \tau_V^{-1} f_U \tau_V = f_{\sigma^{-1}(U)}\]

where \( \tau_V = f_{\sigma^{-1}(V)} f_{\sigma(V)} \);

(ii) If \( \sigma^{-1}(U), U, \sigma(U) \cup \sigma^{-1}(V), V, \sigma(V) \) are pairwise disjoint, then:

\[f_{\sigma(U) \cup \sigma^{-1}(V)} = f_V f_{U}^{-1} f_{V}^{-1} f_{U}\]

**Proposition 4.7.** There exists \( C_2 = C_2(X, S) > 0 \) such that for all \( m \geq (R_\alpha(4) - 1)/2 \), if \( W \subseteq X \) is an \( m \)-cylinder of \( X \), then \( f_{\sigma^{-1}(W)}, f_W, f_{\sigma(W)} \in B_S(C_2 m^2) \).

Proof. Write \( C_0 = [(R_\alpha(4) - 1)/2] + 1 \), and define \( m_n = C_0(2^n + 1) \), so that \( m_{n+1} = 2m_n - C_0 \). We inductively construct a nondecreasing sequence \( (l_n) \) of positive integers, such that, if \( W \subseteq X \) is an \( m \)-cylinder of \( X \), with \( C_0 \leq m \leq m_n \), then \( f_{\sigma^{-1}(W)}, f_W, f_{\sigma(W)} \in B_S(l_n) \). We then analyze the growth of \( l_n \). Since there are finitely many \( m \)-cylinders \( W \) of \( X \) with \( C_0 \leq m \leq m_0 = 2C_0 \), and for each, \( f_W \in [[\sigma]]' \) by Lemma 4.5, there is a
constant $l_0$ such that $f_{\sigma^{-1}(W)}, f_W, f_{\sigma(W)} \in B_S(l_0)$ for all such $W$. For $n \geq 1$, suppose $m_{n-1} < m \leq m_n$ and let:

$$W = \langle w_{-m}, \ldots, w_{-1}, w_0, w_1, \ldots, w_m \rangle \cap X$$

be an $m$-cylinder of $X$. Set $C = C_0$ or $C_0 - 1$, such that $m + C$ is even, and let:

$$U = \langle w_{-m}, \ldots, w_{-2}, w_{-1}, w_0, \ldots, w_C \rangle$$

$$V = \langle w_{-C}, \ldots, w_0, w_1, w_2, \ldots, w_m \rangle$$

so that $\sigma(U) \cap \sigma^{-1}(V) = W$, and:

$$\sigma(U) \cup \sigma^{-1}(V) \subseteq \langle w_{-C}, \ldots, w_{-1}, w_0, w_1, \ldots, w_C \rangle,$$

so by Lemma 4.2, $U$ and $V$ satisfy the conditions of Lemma 4.6 (ii), and:

$$|f_W|_S \leq 2(|f_U|_S + |f_V|_S). \tag{6}$$

Now, $\sigma^{-(C+m-2)/2}(U)$ and $\sigma^{(C+m-2)/2}(V)$ are $(m + C)/2$-cylinders of $X$, and $(m + C)/2 \leq m_{n-1}$, so by induction,

$$|f_{\sigma^{-(C+m-2)/2}}(U)|_S, |f_{\sigma^{(C+m-2)/2}}(V)|_S \leq l_{n-1}. \tag{7}$$

Let $U'_i$ be the $C_0$-cylinder of $X$ containing $\sigma^{-i}(U)$, for $1 \leq i \leq (C + m - 2)/2$. Then by Lemma 4.6 (i), $f_U$ can be obtained from $f_{\sigma^{-(C+m-2)/2}(U)}$ by conjugating by $\tau_{U'_i} \cdot \tau_{U'_i} \cdots \tau_{U'_1}$ in sequence. By our base case, $|\tau_{U'_i}|_S \leq 2l_0$ for all $i$, so

$$|f_U|_S \leq |f_{\sigma^{-(C+m-2)/2}}(U)|_S + 2l_0(C + m - 2)$$

and arguing similarly for $V$,

$$|f_V|_S \leq |f_{\sigma^{(C+m-2)/2}}(V)|_S + 2l_0(C + m - 2).$$

Combining with (6), (7), and our bound on $m_n$, we have:

$$|f_W|_S \leq 4l_{n-1} + C'2^n + C'' \tag{8}$$

for some constants $C', C'' > 0$. Finally, applying Lemma 4.6 (i) a final time, we conjugate $f_W$ by $\tau_{W'}$ or $\tau_{W'}^{-1}$ where $W'$ is the $C_0$-cylinder of $X$ containing $W$, and $|f_{\sigma(W)}|_S, |f_{\sigma^{-1}(W)}|_S$ also satisfy a bound as in (8) (for some larger $C''$). We may therefore take $l_n = 4l_{n-1} + C'2^n + C''$, so that $f_{\sigma^{-1}(W)}, f_W, f_{\sigma(W)} \in B_S(l_n)$. Solving the recurrence for $l_n$ we obtain $l_n \leq C'2^4 \leq (4C'/C_0^2)m_{n-1} \leq (4C'/C_0^2)m^2$ for some $C'_2 > 0$. \hfill \square

Recall that for $g \in \text{Homeo}(X)$, the support of $g$ is:

$$\text{supp}(g) = \{ x \in X : g(x) \neq x \}.$$

**Proposition 4.8.** Let $m$ and $\text{DCyl}_X(m)$ be as in Lemma 4.3. There exists $C_3 = C_3(X, S) > 0$ such that, for all $U \in \text{DCyl}_X(m)$, there is a subgroup $\Delta_U \subseteq [\sigma]'$ satisfying: 

\[ \text{Proposition 4.8.} \]
\( (i) \Delta_U \cong \text{Alt}(5); \)
\( (ii) \Delta_U \subseteq B_S(C_3m^2); \)
\( (iii) \text{For all } g \in \Delta_U, \)
\[ \text{supp}(g) \subseteq \bigcup_{i=-2}^2 \sigma^i(U). \]  

**Proof.** For \( U \in \text{DCyl}_X(m) \), let \( \Delta_U = \langle f_{\sigma^{-1}(U)}, f_U, f_{\sigma(U)} \rangle \). Then (iii) holds, since it holds for \( g \in \{ f_{\sigma^{-1}(U)}, f_U, f_{\sigma(U)} \} \). Now \( \Delta_U \) acts, faithfully, on \( \{ \sigma^i(U) : |i| \leq 2 \} \). Identifying this set in the obvious way with \( \{-2, -1, 0, 1, 2\}, f_{\sigma^{-1}(U)}, f_U, f_{\sigma(U)} \) act, respectively, as the 3-cycles \( (-2 - 1 0), (-1 0 1) \) and \( (0 1 2) \). As such, \( \Delta_U \) acts as the alternating group on \( \{-2, -1, 0, 1, 2\} \), and we have (i).

By Proposition \[4,7\] \( f_{\sigma^{-1}(U)}, f_U, f_{\sigma(U)} \in B_S(C_3m^2) \), and since \( |\Delta_U| = 60 \), \( \Delta_U \subseteq B_S(60C_3m^2) \), whence (ii).

**Proof of Theorem 1.2.** Let \( Q \) be a finite group and \( \pi : B_S(r) \to Q \) be a local embedding. For any \( m \leq (r/2C_3)^{1/2} \), we have \( \Delta_U \leq B_S(r/2) \), where \( C_3 > 0 \) and \( \Delta_U \) are as in Proposition \[4,8\]. We apply Proposition \[2,6\] to the family \( \{ \Delta_U : U \in \text{DCyl}(m) \} \) for \( m \geq c' r^{1/2} \) (\( c' > 0 \) sufficiently small), and \( r \) larger than a constant such that \( m \geq (R_3(4) - 1)/2 \). Since the \( \Delta_U \) are disjointly supported (by Lemma \[4,3\] and \[9\]), they do indeed generate their direct product. From Proposition \[2,6\] we conclude:

\[ |Q| \geq \prod_{U \in \text{DCyl}_X(m)} |\Delta_U| \]
\[ \geq \exp(c\rho_X(2m - 7)) \quad \text{(by } \[5\]\text{)} \]
\[ \geq \exp(c\rho_X(2c'r^{1/2} - 7)) \]

for \( c = \log(60)/9 \). \( \square \)

**Remark 4.9.** We can improve the constant \( c \) by modifying the construction so as to take \( \Delta_U \cong \text{Alt}(2d + 1) \) for large \( d \), instead of \( \text{Alt}(5) \). To do this we would need to take \( \text{DCyl}_X(m) \) to be a family of \( m \)-cylinders \( U \) such that all sets \( \sigma^i(U) \) are pairwise disjoint for \( -d \leq i \leq d \), so that our construction in Lemma \[4,3\] would lead to a bound \( |\text{DCyl}_X(m)| \geq p_X(2m - C)/(2d - 1) \). We could nevertheless take \( c = \log(d!)/2d - 1 \), which grows in \( d \).

**Remark 4.10.** The same argument also gives a lower bound on the LEF action growth of \( \|\sigma\|^{\prime} \), which is a little stronger than that obtained by applying Remark \[2,3\] to the conclusion of Theorem \[1,2\]. Suppose \( \pi : B_S(r) \to \text{Sym}(d) \) is a local embedding. Then by Propositions \[2,6\] and \[4,8\], \( \text{im}(\pi) \) contains a subgroup isomorphic to the direct product of \( P = p_X(2c'r^{1/2} - 7) \) copies of \( \text{Alt}(5) \), which in turn contains the direct product of \( P \) copies of \( C_5 \). By \[9\], Theorem 2, the minimal degree of a faithful permutation representation of the latter is \( SP \). Hence \( \mathcal{L}A_4^{\|\sigma\|^{\prime}}(n) \geq p_X(n^{1/2}) \).
5 Systems of intermediate growth

In this Section we prove Theorem 1.7 and deduce Theorems 1.5 and 1.6. The following example is modelled on the construction in [10, §3]. In this Section, “large” means larger than a certain absolute constant (which we do not compute), so as to make true some needed inequalities.

Choose a real number \( r \geq 2 \). First, fix a large \( x \) divisible by 3. We work over the alphabet \( A = \{a, b\} \). Fix two words \( u(0), u'(0) \in A^* \) of length \( x/3 \). Among all words \( u(0)u'(0) \) for \( |v| = x/3 \), take a subset \( C_0 \) with \( |C_0| = x \) and such that no factor of any element of \( C_0 \) is equal to \( u(0), u'(0) \) except for the prefixes and suffixes themselves. This is easily achieved: for instance taking \( u(0) = a^{x/3}, u'(0) = b^{x/3} \), we can form \( C_0 \) by choosing \( v \) from among those words starting in \( b \) and ending in \( a \) (of which there are \( 2(\ell/3)^{-2} \geq x \) for \( x \) large).

Assuming that we have already defined \( C_j \), we are going to define \( C_{j+1} \): for all \( i \), we set \( N_i = |C_i| \) and \( l_i \) the length of any element of \( C_i \), so that \( N_0 = l_0 = x \). We prove the following for all \( j \geq 0 \):

(i) \( N_j l_j \) are large and increasing in \( j \); 3\( |N_j|, 3|l_j| \): all words in \( C_j \) have the same prefix of length \( l_j/3 \), and the same suffix of length \( l_j/3 \);

(ii) \( N_j + 2 \leq \exp(2(\log N_j)' N_j + 1 \geq N_j^2 \), and for \( j \) even \( N_j + 1 \geq \exp(\frac{1}{2}(\log N_j)') \);

(iii) \( N_j \leq l_j+1 \leq N_j^2 \).

We prove (i) and (iii) by induction with base case \( j = 0 \), for which all claims are true (for \( l_1 \) see below), and (ii) directly.

Fix an ordering of \( C_j \), arbitrarily: the elements of \( C_j \) are words \( u(i)^{(j)} \), where the index \( i \) follows the ordering. Define the word:

\[
u(i+1)^{(j)} = u_1^{(j)} u_2^{(j)} u_3^{(j)} \ldots u_{N_j}^{(j)} u_{N_j+1}^{(j)} \ldots u_{N_j}^{(j)} u_{N_j+2}^{(j)} \ldots u_{N_j}^{(j)} u_{N_j+1}^{(j)} u_{N_j+2}^{(j)} \ldots u_{N_j}^{(j)} \}
\]

where \( u(i+1) \) and \( u'(i+1) \) collect the first and last third of the \( u(i)^{(j)} \), respectively: this is possible since \( 3|N_j| \) by induction, and it implies \( 3|l_j+1| = |u(i+1)| \) for \( l_j N_j \) is large. Take a collection \( P_j \subset \text{Sym}(N_j/3) \): for \( j \) even, choose \( |P_j| = 3 \exp((\log N_j)' / 3) \), otherwise choose \( |P_j| = N_j^2 \) (note that this is possible for \( x \) large). Then, define \( C_{j+1} \) to be the set of all words \( \{u(i^{(j)} + u'(j^{(j+1)}) : \pi \in P_j \} \)

where for \( \pi \in P_j \), \( u^{(j+1)} \) is obtained by permuting the factors \( u(i)^{(j)} \) of \( u^{(j+1)} \) with \( i \in (N_j/3, 2N_j/3) \) according to \( \pi \); that is:

\[
u(i^{(j+1)}) = u_1^{(j)} u_2^{(j)} u_3^{(j)} \ldots u_{N_j/3}^{(j)} u_{N_j/3+1}^{(j)} \ldots u_{N_j/3}^{(j)} u_{N_j/3+1}^{(j)} \ldots u_{N_j/3}^{(j)} \}
\]
By definition we have $3|N_{j+1}$ and $N_{j+1}$ large; $I_{j+1} > I_j$, $N_{j+1} > N_j$, and for all $j \geq 0$,

$$N_{j+2} \leq \begin{cases} \exp \left( 2(\log N_j) \right) & (2j) \\ \exp \left( (\log N_j^2) \right) & (2j+1) \end{cases} \leq \exp \left( 2' \log N_j \right);$$

$$N_{j+1} \geq \min \left\{ N_j^2, \exp \left( (\log N_j) \right) - 3 \right\} = N_j^2;$$

$$2|j \Rightarrow N_{j+1} \geq \exp \left( (\log N_j) \right) - 3 \geq \exp \left( (\log N_j) / 2 \right);$$

$$I_1 = x^2 = N_0^2 > N_0;$$

$$j \geq 1 \Rightarrow N_j < I_{j+1} = I_j N_j \leq N_{j-1}^2 N_j \leq N_j^2.$$

so (i)-(iii) do indeed hold. Two key features of this construction are that, for all $j$:

(a) All words in $C_j$ have the same prefix $w^{(j)}$ and suffix $w^{(j)}$ of length $\frac{1}{3} I_j$;

(b) Every word in $C_{j+1}$ is the product of all the words from $C_j$ (in some order).

We now construct a uniformly recurrent non-periodic word $x = x(r)$, such that the subshift $X_r = \mathcal{O}(x)$ satisfies the conditions of Theorem 1.7. Consider the words $x^{(j)} = u^{(j)}_{N_j/3}$. Then $x^{(j)}$ is the $K_j = (I_j(N_j/3 - 1) + 1)$th $I_j$-factor of $x^{(j+1)}$, for all $j$, and $2 \leq K_j \leq I_{j+1} - I_j$. For any $1 \leq M_0 \leq I_0$, apply Lemma 2.18 to the sequence $x^{(j)}$ (with $L_j = I_j$) to obtain $x$.

**Proposition 5.1.** The complexity function $p_x$ of $x$ satisfies $p_x(I_j) \geq \exp \left( (\log I_j)^{r} / 2^{r+1} \right)$, for all $j \in \mathbb{N}$.

**Proof.** Since $u^{(j+1)}_{N_j/3} \in C_{j+1}$ is a factor of $x$, we have (b) above that all elements of $C_j$ are distinct factors of $x$ of length $I_j$, so $p_x(I_j) \geq N_j$ for all $j \in \mathbb{N}$. For $j$ odd, it follows that:

$$p_x(I_j) \geq N_j \geq \exp \left( (\log N_j) / 2 \right) \geq \exp \left( (\log I_j) / 2^{r+1} \right)$$

by (ii) and (iii) above. \qed

**Proposition 5.2.** The recurrence function $R_x$ of $x$ satisfies:

$$R_x(n) \leq C \exp \left( 4^{r+1}(\log n) \right)$$

for some $C > 0$ and all $n \in \mathbb{N}$.

**Proof.** Fix any factor $w$ of $x$ of length $n$, and suppose that $j$ is such that $n < 2I_j/3$. $w$ is a factor of some $x^{(k)} = u^{(k)}_{N_k/3}$, for $k > j$, and applying (b) above repeatedly, $x^{(k)}$ is expressible as a product of words $u^{(j)}_i$. Therefore, there are indices $i_1$ and $i_2$ such that $w$ is a factor of some $u^{(j)}_{i_1} u^{(j)}_{i_2}$; moreover it intersects the middle third of at most one of $u^{(j)}_{i_1}$ or $u^{(j)}_{i_2}$, so by (a) above, it sits entirely inside either $u^{(j)}_{i_1} w^{(j)}$ or $w^{(j)} u^{(j)}_{i_2}$.
Now let \( v \) be any factor \( x \) of length \( 3l_{j+1} \). As before, \( u \) is a factor of some \( x^{(k')} = u^{(k')}_{N_{x^{(k')}}/3} \), for \( k' > j+1 \), and applying (b) above repeatedly, \( x^{(k')} \) is expressible as a product of words \( u^{(j+1)}_{i_1} u^{(j+1)}_{i_2} \). Therefore, there are indices \( i'_1 \) and \( i'_2 \) such that \( u^{(j+1)}_{i_1} u^{(j+1)}_{i_2} \) is a factor of \( v \). By (b) above, and the fact that every \( u^{(j)} \) has \( u^{(j)} \) as a prefix and \( u^{(j)}_u \) as a suffix, it follows that \( u^{(j)}_{i_1} u^{(j)}_{i_2} \) are factors of \( u^{(j+1)}_{i_1} u^{(j+1)}_{i_2} \), hence \( u \) is a factor of \( v \).

That is, \( R_x(n) \leq 3l_{j+1} \).

Partitioning the integers, we have \( R_x(n) \leq 3l_{j+1} \) for any \( n \in [2l_{j-1}/3, 2l_{j}/3) \). For \( j \geq 2 \), we have:

\[
\log R_x(n) - \log 3 \leq \log l_{j+1} \leq 2 \log N_j \leq 2^{r+1}(\log N_{j-1})^r
\]

\[
< 2^{r+1}(\log l_{j-1})^r \leq 2^{r+1}(\log (3n/2))^r \leq 4^{r+1}(\log n)^r
\]

by (ii) and (iii), and our bounds on \( n \), as required.

**Proof of Theorem 1.7.** Let \( x = x(r) \) be as above. Set \( X_r = \overline{O(r)} \). By Remark 2.24, items (i) and (ii) follow from Propositions 5.2 and 5.1, respectively, with \( n^{(r)}_1 = l_{2r+1} \).

**Proof of Theorem 1.5.** Recall that \( N! \leq \exp(N \log N) \) for all \( N \). Applying Theorem 1.1 to \( \Gamma^{(r)} = [\sigma]' \), where \( (X_r, \sigma) \) is as in Theorem 1.7 we have:

\[
L_{\Gamma^{(r)}}^S(n) \leq \exp \left( 2R_X(Cn) \log (2R_X(Cn)) \right)
\]

for all \( n \) and for some \( C > 0 \), where \( X = X_r \), so that \( R_X(n) \leq \exp (C_r(\log n)^r) \). Hence for \( n \geq 2 \):

\[
\log (2R_X(Cn)) \leq C_r(\log n + \log C)^r + \log 2 \leq C'_r(\log n)^r
\]

for a possibly larger constant \( C'_r \), so that:

\[
\log (2R_X(Cn)) + \log (2R_X(Cn)) \leq C''_r(\log n)^r + r \log \log n + \log C'_r
\]

\[
\leq C''_r(\log n)^r
\]

again, for \( C''_r \) a possibly larger constant. Thus, by (10), \( L_{\Gamma^{(r)}}^S(n) \leq \exp(\exp(C''_r(\log n)^r)) \), and we have (i). For (ii), suppose for a contradiction that \( 2 \leq r' < r \) and that \( C, C' > 0 \) are such that, for all \( n \) sufficiently large,

\[
L_{\Gamma^{(r)}}^S(n) \leq \exp \left( C(\log n)^r \right)
\]

By Theorem 1.2 and Theorem 1.7(ii), we have:

\[
L_{\Gamma^{(r)}}^S(n) \geq \exp \left( c \exp \left( \frac{1}{2} \log n^{(r)} + \log c \right)^r \right)
\]

for some \( c > 0 \) and all \( i \in \mathbb{N} \). Hence by (11).
\[ C(\log n_i^{(r)} + \log C') \geq c_r \left( \frac{1}{2} \log n_i^{(r)} + \log c \right)^r + \log c \]

for all \( i \) sufficiently large, a contradiction. \( \Box \)

**Proof of Theorem 1.6.** Let \( F = \{ \Gamma^{(r)} : r \geq 2 \} \), with \( \Gamma^{(r)} \) as in Theorem 1.5. Let \( 2 \leq r' < r \), and suppose for a contradiction that \( \mathcal{L}_{\Gamma^{(r)}} \leq \mathcal{L}_{\Gamma^{(r')}} \). By Theorem 1.5(i),

\[ \mathcal{L}_{\Gamma^{(r')}}(n) \leq \exp \left( \exp \left( C_{\phi'}(\log n)^{r'} \right) \right), \]

contradicting Theorem 1.5(ii). \( \Box \)

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