Violation of the mean path length invariance property

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The invariance property of the mean path length is an astonishing law of Nature governing the motion of particles inside a disordered material. Whatever the strength of the disorder, the property states that the mean path length is exclusively determined by the ratio between the volume and the surface. Till now, the property has been reported as universal and valid in any kind of disordered medium and also beyond diffusion conditions. Nevertheless, we found out that the property fails in anomalous transport and in other kinds of random walk. By means of Monte Carlo simulations of light transport, we show that, in these cases, the invariance property loses its validity and the mean path length becomes dependent on the diffusive characteristics of the medium. The critical issue of such a violation lies in the breaking of isotropy and homogeneity of the radiance in the whole volume. These results are valid for all natural or artificial phenomena where random walkers, whatever their nature, are able to experiment anomalous transport.

Particles and light transport through random media covers many different phenomena, such as insects migration, neutrons scattering and photons propagation. Despite their completely different origin and nature, all these phenomena share an outstanding property: the mean path length followed by a random walker is fixed by geometry and size of the medium, whatever the strength of its internal disorder [1–3]. This important invariance property (IP) is expected to be a universal law of Nature and neither failure nor exception have been yet reported. Here we report for the first time, to the extent of our knowledge, a violation of the IP in some special kinds of scattering laws, such as anomalous transport (AT).

Under the conditions of isotropic incidence on the boundaries of a non-absorbing volume, the IP states that the mean path length of random walkers propagating inside, even in presence of chaotic trajectories, is a constant that only depends on the geometry of the medium. Such a property descends from a generalization of the mean chord length theorem, also used in the context of nuclear physics by Dirac [4, 5] and valid for the ballistic propagation. It is critically striking the fact that the property is condensed in a very simple formula that is able to express a general and universal law of Nature. Moreover, the IP represents an unique exact tool able to validate, with arbitrary precision, any Monte Carlo (MC) code used to simulate random walk processes.

Random walk [6] is a framework of great interest that encompasses many different fields, ranging from the study of random foraging in ethology [7] to the modeling of light propagation in turbid media in physics [8]. Despite the huge differences in the nature of such types of phenomena, it has been reported that the IP maintains its general validity.

Our evidence of violation of the IP has been carried out by means of the study on some kinds of scattering laws, with a particular attention to AT, where the step distribution of the propagation is characterized by an “heavy tail” [9, 10]. The study of “heavy-tailed” distributions is critical to understand complex phenomena belonging to different fields, such as animal foraging strategies [11–15], human transport [16], earthquakes [17], hydrology [18], light propagation in special optical materials (Lévy glasses) [19, 20], biological tissues [21] and random lasers [22–28].

In this contribution we focus our investigation to the optical case, having in mind that the presented results can be generalized to other fields. For this reason, we also consider no refractive index mismatch between the medium and the external environment.

Historically, the IP has been derived for random walks by Bardsley and Dubi [29]. Independently and in a more general case, it has also been reported by Blanco and Fournier [3], thinking about random walkers (for example ants moving around their nest) that enter isotropically and uniformly in a diffusing medium. Once inside it, the random walkers perform different path patterns, with total step lengths $L$, characterized by a probability density function (pdf) $\tilde{p}(L)$, with mean value $\langle L \rangle$, before leaving the medium. The authors of Ref. [3] considered the optical case of a medium with a surface illuminated by an isotropic and uniform radiance $I_0$ and with an exponential attenuation due to absorption with a coefficient $\mu_a$. Their procedure was based on the comparison of the absorbed power $P_a$ obtained with two different methods, in the limit $\mu_a \to 0$, i.e.:

$$\lim_{\mu_a \to 0} \frac{P_a}{\mu_a} = \frac{1}{\mu_a} \int _{4\pi} \mu_a I(\hat{r}, \hat{s})dVd\hat{s}, \quad (1a)$$

$$\lim_{\mu_a \to 0} \frac{P_a}{\mu_a} = \frac{\pi I_0 S}{\mu_a} \int _{0} ^{\infty} \left[ 1 - \exp (\mu_a L) \right] \tilde{p}(L)dL, \quad (1b)$$

where $I$ is the radiance, $\hat{r}$ the position and $\hat{s}$ the angular direction, while $V$ and $S$ are the volume and the external area of the medium, respectively. Can be worth to remind that the energy density $u(\vec{r})$ in the medium is
linked to $I$ by the speed of light in the medium $v$:

$$u(\vec{r}) = \frac{1}{v} \int \frac{I(\vec{r}, \vec{s})ds}{4\pi}.$$  

(2)

Under the hypothesis of uniform and isotropic radiance within the whole volume $V$, and thus a constant $u(\vec{r}) = u_0$, $P_n/\mu_a$ is reduced to $4\pi V I_0$ in Eq. (1a) and the mean path length can be written as:

$$\langle L \rangle_{IP} = \frac{4V}{S},$$  

(3)

that is an expression completely independent of the scattering characteristics of the medium and valid also in the ballistic case. Its validity has been theoretically reported for a more general case of random walk [30]. In optics the IP has been demonstrated also in presence of resonances [1], and then verified by an elegant experiment in conditions of classical transport (CT) regime [2].

In summary, at the basis of the IP there are two main hypotheses: 1) a Lambertian illumination at the external boundary of the medium, i.e. an uniform isotropic incident radiance; and 2) isotropic and uniform radiance inside the volume. In principle, in Eq. (1b) no hypothesis has been done for $p(L)$, that is determined by the structure of the distribution $p(\ell)$ of the individual step length $\ell$ between two consecutive scattering events. However, in our work, we have found that the form of $p(\ell)$ is crucial to preserve the isotropy of the radiance within the medium.

Here we investigate the validity of Eq. (3) in a case of AT and in other kinds of random walk (constant steps length and Pareto distributions), where the LB law is not valid. As the most studied case of AT, we use the Generalized Lambert-Beer (GLB) law [31, 32], where random walkers perform path patterns characterized by individual step lengths $\ell$ drawn from a “heavy tailed” distribution $p(\ell)$ with diverging moments $\langle \ell^n \rangle$. It is also worth to note that the GLB is strictly linked to a fractional radiative transfer equation [32] and the fractional operators appearing in this equation are non-local. This means that the exterior domain cannot be decoupled from the interior by conventional boundary conditions. Thus, it is not trivial that an equation based on precise volume/surface ratios can be demonstrated by a model using non-local operators. A generalized radiative transfer equation which takes into account any suitable $p(\ell)$ has been previously derived by Larsen et al. [37]. By using this approach, photon migration in conditions of AT can be simulated by a classical MC method where the $p(\ell)$ derived by the LB law is replaced by a different kind of $p(\ell)$. Thus, in this case, the problem of the boundary can be circumvented with a MC solution of the generalized radiative transfer equation, describing the same physical system.

For the sake of clarity, we stress that the words “classical” and “anomalous” transport are used here with the following meaning: the former pertains to the extraction of scattering interactions by the LB law, while the latter by the GLB law [33]. In CT regime for a non absorbing medium ($\mu_a = 0$), a random walker subjected to scattering performs, between two independent interactions, steps of length $\ell$ distributed according to the LB law:

$$p(\ell) = \mu_a e^{-\mu_a \ell},$$  

(4)

where $\mu_a$ is scattering coefficient, that is the reciprocal of the scattering mean free path. In AT regime, the corresponding $p(\ell)$ is given by the GLB law [33, 34]:

$$p(\ell) = \gamma^\beta \ell^{\beta-1} \Gamma_{\beta, \beta} \left(-\gamma^\beta \ell^\beta\right),$$  

(5)

where $\gamma$ is the scale parameter representing the scattering strength and $\Gamma_{\beta, \beta}(x)$ is the two parameters Mittag-Leffler function [35]:

$$\Gamma_{\beta, \beta}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\beta (k+1))},$$  

(6)

where $\beta \in (0,1]$ and $\Gamma$ is the Gamma function. For $\beta = 1$ (CT), Eq. (5) becomes Eq. (4) and $\gamma$, in a non absorbing medium, has the meaning of scattering coefficient $\mu_a$. For $0 < \beta < 1$, the distribution has an “heavy-tail” with an asymptotic power-law trend $p(\ell) \sim \ell^{-\beta-1}$ [34, 36]. In the cases of AT, longest and shortest steps are more likely than in the diffusive case.

By the inversion of the cumulative distribution associated to Eq. (5), $\ell$ is obtained as [34, 38]:

$$\ell(u,v) = -\frac{\ln(u)}{\gamma} \left[ \frac{\sin[\beta \pi (1-v)]}{\sin[\beta \pi v]} \right]^{1/\beta},$$  

(7)

where $u,v \in (0,1)$ are two uniformly distributed random numbers. For $\beta = 1$ (CT), Eq. (7) is equivalent to the usual inversion formula used for Eq. (4). Examples and characteristics of $p(\ell)$ and paths generated by Eq. (5) are shown in Supplementary Materials.

To investigate the IP in non-absorbing media with AT, we have modified a previously developed MC code [39–41] adding the possibility to extract photons paths according to Eq. (7). We have firstly proceeded to validate the code in the regime of standard scattering, i.e. for $\beta = 1$ (LB law, infinite medium, point light source). Being the core of a MC program the generation of the trajectories, the MC results for the statistics of the positions, over a large scattering range, are checked to be in excellent agreement with exact analytical expressions [42]. Then, we have verified, in the case $\beta = 1$, that in finite geometries, i.e. in a sphere of radius $R = 5$ mm subjected to a uniform Lambertian illumination and without absorption, the results of the MC simulations for $\langle L \rangle$ are consistent, within the standard error, for $10^8$ simulated trajectories, with the value $\langle L \rangle_{IP} = \frac{4}{3}R$. This check guarantees that the employed MC code accounts correctly for the effect of boundaries on photon migration.
The AT regime has been studied by adding, to the rigorously validated MC code, the single implementation of Eq. (7). This fact increases the significance of this kind of investigation, since the outcome of the simulations is directly connected to the function used for $p(\ell)$. Different scattering properties are considered by varying $\gamma$, $\beta$, and $g$, where $g \in [0, 1]$ is the asymmetry factor of the Heneyy-Greenstein scattering function [43]. For each set of initial parameters, the final result is derived by a set of 100 independent simulations, each consisting in $10^6$ trajectories to evaluate the standard error on $\langle L \rangle$. Each photon injected is received at the boundaries. In Supplementary Material, graphical examples of trajectories are shown in case of CT and AT for a sphere of radius of 5 mm.

The main results are summarized in Fig. 1 ($g = 0$) and Fig. 2 ($g = 0.9$), that are referred to $10^6$ simulated trajectories. Other cases with lower statistics have been performed to evaluate the convergence of the results (see Supplementary Material). In both figures, for CT regime ($\beta = 1$), the obtained MC value for $\langle L \rangle$ agrees, within the standard error, with the expected value given by Eq. (3), i.e., $\langle L \rangle_{IP} = \frac{1}{2}R$. The standard error $\sigma_{\langle L \rangle}$ for $g = 0$ (Fig. 1) spans $(2 \div 4) \cdot 10^{-4}$ mm for $\gamma < 1$ mm$^{-1}$ and $(0.9 \div 7) \cdot 10^{-3}$ mm$^{-1}$ for $\gamma \geq 1$ mm$^{-1}$. For $g = 0.9$ (Fig. 2), $\sigma_{\langle L \rangle}$ spans $(2 \div 3) \cdot 10^{-4}$ mm for $\gamma < 1$ mm$^{-1}$ and $(0.9 \div 3) \cdot 10^{-3}$ mm$^{-1}$ for $\gamma \geq 1$ mm$^{-1}$.

In CT ($\beta = 1$), the MC simulations fully confirm the IP, giving an $\langle L \rangle$ independent of the scattering properties, here represented by $\gamma$ and $g$, and equal to $\langle L \rangle_{IP}$. This result confirms the precision and the accuracy of the MC code, posing a solid ground to the study of the AT regime, that only differs by the replacement of Eq. (4) with Eq. (7).

In the case of AT ($\beta < 1$) and with a standard error of the same order of the case $\beta = 1$, the results show a clear violation of the IP as the scattering strength $\gamma$ increases, as well as the dependence of $\langle L \rangle$ on $g$. See Supplementary Material for the complete dataset. We have observed a similar behavior also for Mie scattering functions (results not shown). When the scattering strength $\gamma$ tends to 0, i.e. when a propagation approximately ballistic is established, $\langle L \rangle$ approaches the invariance value $\langle L \rangle_{IP}$ as expected by the mean chord length theorem. The same result emerges in the approach of the ballistic regime as $g \rightarrow 1$ (see Supplementary Material). The violation of the IP has been also found for other kinds of step distributions, such as the constant step length and the Pareto law (see Supplementary Material). Moreover, the violation of the IP has been also observed for other geometries such as the cylinder and the slab (results not shown in this paper).

The Fig. 3 shows the energy density $u(d)$, where $d$ is the depth from the surface to the center of the sphere of radius $R = 5$ mm. The results show that the form of $p(\ell)$ determines the behavior of $u$: for the CT-case...
FIG. 3. (color online) MC simulation of the energy density $u$ as a function of the depth $d$ inside the sphere, starting from the surface, presented for different scattering laws. $u$ is normalized to the superficial energy density in the case of ballistic propagation. The parameters are $g = 0$ and $\gamma = 1 \text{ mm}^{-1}$.

In Fig. 4 the angular distribution of the radiance outgoing from the medium is shown for different cases of $\beta$, normalized to the input isotropic radiance. While in the LB-case the output radiation preserves the isotropy of the input one, in the anomalous cases such a symmetry is broken. In the LB case, the isotropy of outgoing radiance $I$ (Fig. 4) and the constancy of $u$ (Fig. 2) imply that the radiance is constant in the whole medium. In the other cases, the non Lambertian angular distribution and the non homogeneous $u$ determine that the radiance is anisotropic and non-uniform in the volume. For $\beta < 1$, an anisotropic distribution of radiance is also found inside the volume (data not shown).

The results show that the LB is a sufficient condition for the validity of IP. It is worthwhile to note that any other used distribution leads to non-isotropic radiance, putting this behavior outside the basic requirements for obtaining Eq. (3) from Eqs. (1a) and (1b).

In conclusion, the main result of this work is the violation of the mean path length invariance property in anomalous transport regime, where the $p(\ell)$ is given by the generalized Lambert-Beer law, as well as in other tested distributions (constant step length and Pareto law). In these media the mean path length depends on the scattering characteristics of the medium, i.e., the scattering strength $\gamma$ and the scattering function. Such a violation appears to be determined by the breaking of isotropy of the radiance, as it is evident in the asymmetry between the outgoing and the input angular distribution of the radiation and in the behavior of the energy density inside the volume. Indeed, as shown in Fig. 3, if the radiation impinges isotropically onto the surface of the medium, the constancy of $u$ is a condition that can be only reached asymptotically inside the volume. Thus, when a scattering law different from LB is used, the IP can be satisfied only by considering a small portion of the actual volume, i.e., a sub-volume reached by radiation coming from far away.

In order to experimentally test such results, a possible mean could be to exploit “superdiffusive media”, such as Lévy glasses [19, 20]. Moreover, it should be of great interest to study how a coherent approach, that can include Anderson localization, overlaps with our results obtained in an incoherent approach.

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Violation of the mean path length invariance property
Supplementary Material

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I. INTRODUCTION

This report is aimed to provide detailed information of the Monte Carlo (MC) results plotted in Figs. 1 and 2 of the paper. The general purpose is to give direct access to the MC data obtained and also to show their convergence features. Furthermore, in this document additional results pertaining the violation with other scattering laws and the behavior in the limit condition of the ballistic regime as \( g \) approaching 1 are also shown.

In Sec. II the general information on MC simulations are reported, as well as the characteristics of the generalized Lambert-Beer (GLB) law. In Sec. III, the validation of the MC code is shown in the case of classical transport (CT) regime by providing the MC data with their standard error. Similarly, in Sec. IV, the MC data corresponding to the case of anomalous transport (AT) together with their standard error are shown. In Sec. V the behavior in the ballistic approaching is shown, whereas in Sec. VI ans in Sec. VII the results concerning the distribution with constant step length and the Pareto law are respectively reported.

II. CHARACTERISTICS OF THE GLB AND MC SIMULATIONS

In Fig. 1 three different \( p(\ell) \) for \( \gamma = 10 \text{ mm}^{-1} \) are reported: the LB case (solid black curve), GLB with \( \beta = 0.7 \) (dashed red curve) and GLB with \( \beta = 0.3 \) (dash-dot curve). In the cases of AT, longest and shortest steps are more likely than in the diffusive case. To visualize the differences, in Fig. (2) four different path patterns in a 2D open world are reported. The top-left case (\( \beta = 1 \)) is generated by the LB law (Eq. (3), whereas the other three cases are examples of AT. It can be noted the huge differences among the different scales due to the extremely long jumps that are more likely as \( \beta \) decreases in the cases of AT.

The MC simulations have been carried out by using an Henyey-Greenstein scattering phase function for two values of the asymmetry factor \( g = 0 \) and 0.9. The geometry of the sphere has been considered with a radius of 5 mm and with matched refractive index with the external medium. In Fig. 3, five examples of trajectories are shown for \( \gamma = 1 \text{ mm}^{-1} \) and \( g = 0 \) within a medium that consists in a sphere of radius 5 mm, and for two different values of \( \beta \): 1 (CT, above) and 0.7 (AT, below). The figure highlights that if \( \beta < 1 \), the probability to generate extremely long \( \ell \) (as well as extremely short \( \ell \), however not clearly visible) becomes increased.

A Lambertian uniform illumination is set at the external boundary of the sphere and the simulations were performed for the non-absorbing medium. No scatters are present outside the sphere, so photons are not able to came back to the medium. The results, for each set of initial parameters, were obtained by performing a set of 100 independent
FIG. 2. (color online) Four different random paths in a 2D infinite medium created with the LB probability density law (case $\beta = 1$, top-left plot) and with the GLB law (cases of AT: $\beta = 0.9, 0.7, 0.3$). In all the cases the scale parameter is the same ($\gamma = 10$ mm$^{-1}$). The huge differences in the scales highlight the likelihood of extremely long steps in the GLB cases.

FIG. 3. (color online) Five examples of trajectories for $\gamma = 1$ mm$^{-1}$, $g = 0$ and for two different $\beta$: 1 (CT regime) above, 0.7 (a case of AT) below. The center of the sphere is located at (0, 0, 0) mm, whereas the point of incidence at (-5, 0, 0) mm.

simulations, each consisting in a number $N$ of trajectories, in order to evaluate the standard error $\sigma_{\langle L \rangle}$ on the mean path length $\langle L \rangle$. The routine for the random number generation is characterized by a periodicity far away to be
reached during all these simulations, as well as the maximum number of scattering events for each trajectory ($2 \cdot 10^6$). Indeed, each photon injected into the sphere is received at the boundaries and then no trajectories have been lost in the calculation of $\langle L \rangle$. The results obtained are summarized inside several tables where $\langle L \rangle$ and its standard error are reported. Three different kinds of statistics, i.e., total number of simulated trajectories equal to $N \cdot 100 = 10^6$, $N \cdot 100 = 10^7$ and $N \cdot 100 = 10^8$, are included in the next sections. These results provide an overview of the convergence features of the MC simulations and guarantee the correct convergence of the results plotted in Figs. 1 and 2 of the paper.

III. VALIDATION OF THE MONTE CARLO IN THE CLASSICAL TRANSPORT

| $\gamma$ (mm$^{-1}$) | $N = 10^4$ | $N = 10^5$ | $N = 10^6$ |
|----------------------|------------|------------|------------|
|                      | $\langle L \rangle$ (mm) | $\sigma_{\langle L \rangle}$ (mm) | $\langle L \rangle$ (mm) | $\sigma_{\langle L \rangle}$ (mm) | $\langle L \rangle$ (mm) | $\sigma_{\langle L \rangle}$ (mm) |
| 1 \cdot 10^{-1}      | 6.667      | 0.002      | 6.667      | 0.0007     | 6.6667     | 0.0003     |
| 1 \cdot 10^{-6}      | 6.667      | 0.002      | 6.674      | 0.0007     | 6.6666     | 0.0002     |
| 1 \cdot 10^{-5}      | 6.67      | 0.002      | 6.662      | 0.0007     | 6.6666     | 0.0002     |
| 1 \cdot 10^{-4}      | 6.667      | 0.002      | 6.672      | 0.0008     | 6.6666     | 0.0002     |
| 1 \cdot 10^{-3}      | 6.69      | 0.002      | 6.65      | 0.0007     | 6.6666     | 0.0002     |
| 1 \cdot 10^{-2}      | 6.69      | 0.002      | 6.666     | 0.0008     | 6.6666     | 0.0003     |
| 1 \cdot 10^{-1}      | 6.67      | 0.003      | 6.674      | 0.0011     | 6.665      | 0.0004     |
| 1 \cdot 10^{0}       | 6.65      | 0.009      | 6.665      | 0.003      | 6.672      | 0.0009     |
| 2 \cdot 10^{0}       | 6.657     | 0.011      | 6.668     | 0.003      | 6.669     | 0.0011     |
| 1 \cdot 10^{1}       | 6.61     | 0.03      | 6.649     | 0.009      | 6.68     | 0.003     |
| 1 \cdot 10^{2}       | 6.68     | 0.08      | 6.67     | 0.02      | 6.670     | 0.007     |

In classical transport regime, the value of the invariance property ($\langle L \rangle_{IP} = \frac{1}{2}R$, where $R = 5$ mm) is expected. Then, in such a case the results of the Monte Carlo can be validated for all $\gamma$ and $g$. Tables I and II show that, as $N$ increases, $\langle L \rangle$ converges to the expected value and becomes consistent with $\langle L \rangle_{IP}$ within a confidence interval defined by the standard error $\sigma_{\langle L \rangle}$, for all values of the scattering strength $\gamma$, in the case $N = 10^6$ (reported in Figs. 3 and 4 of the paper).

IV. VIOLATION OF THE INVARIANCE PROPERTY IN ANOMALOUS TRANSPORT

In the case of anomalous transport ($\beta < 1$), the $\langle L \rangle$ becomes clearly inconsistent with $\langle L \rangle_{IP} = \frac{1}{2}R$ as $\gamma$ increases, as reported in tables III to X. As $N$ increases, such an inconsistency is confirmed by the convergence of the values of $\langle L \rangle$ and by the confidence interval defined by $\sigma_{\langle L \rangle}$. In fact, the results of the tables show that $\langle L \rangle$, for increasing values of $\gamma$, progressively diverges from the expected value of the invariance property with differences heavily larger than the standard error affecting the MC data. The values reported in Figs. 1 and 2 of the paper correspond to the case of $N = 10^6$. $\langle L \rangle$ appears to be also dependent on $g$. 
A better understanding of the violation of the IP can be achieved studying the characteristics of the $p(\ell)$ reported in Eq. (5) and plotted in Fig. 1. In case of $\beta < 1$, extreme events, very short or very large values of $\ell$, become more likely to appear, compared to the condition $\beta = 1$. These effects progressively increase when $\beta$ decreases from 1 to zero, as shown in Fig. 1. For photon migration inside a finite medium, the extreme events with short values of $\ell$ are the most important, since the values of $\ell$ larger than the dimensions of the medium produce on the simulation, whatever their length, the same effect on $\langle L \rangle$. Hence, the behavior of $p(\ell)$ gives a qualitative explanation of the overall reduction of $\langle L \rangle$ observed in our results.

| $\gamma$ (mm$^{-1}$) | $N = 10^4$ | $N = 10^5$ | $N = 10^6$ |
|----------------------|------------|------------|------------|
|                      | $\langle L \rangle$ (mm) | $\sigma_{\langle L \rangle}$ (mm) | $\langle L \rangle$ (mm) | $\sigma_{\langle L \rangle}$ (mm) | $\langle L \rangle$ (mm) | $\sigma_{\langle L \rangle}$ (mm) |
| $1 \cdot 10^{-1}$    | 6.662      | 0.002      | 6.666      | 0.011      | 6.6666      | 0.0002      |
| $1 \cdot 10^{-6}$    | 6.668      | 0.002      | 6.666      | 0.011      | 6.6665      | 0.0002      |
| $1 \cdot 10^{-5}$    | 6.667      | 0.002      | 6.667      | 0.011      | 6.6669      | 0.0002      |
| $1 \cdot 10^{-4}$    | 6.667      | 0.002      | 6.666      | 0.011      | 6.6662      | 0.0002      |
| $1 \cdot 10^{-3}$    | 6.666      | 0.003      | 6.665      | 0.011      | 6.6644      | 0.0002      |
| $1 \cdot 10^{-2}$    | 6.644      | 0.003      | 6.649      | 0.011      | 6.6488      | 0.0002      |
| $1 \cdot 10^{-1}$    | 6.531      | 0.003      | 6.527      | 0.011      | 6.5281      | 0.0003      |
| $1 \cdot 10^0$       | 5.759      | 0.007      | 5.756      | 0.002      | 5.7541      | 0.0006      |
| $2 \cdot 10^0$       | 5.201      | 0.007      | 5.216      | 0.002      | 5.2137      | 0.0007      |
| $1 \cdot 10^1$       | 3.485      | 0.008      | 3.497      | 0.003      | 3.4954      | 0.0009      |
| $1 \cdot 10^2$       | 1.448      | 0.006      | 1.450      | 0.002      | 1.4502      | 0.0006      |

| $\gamma$ (mm$^{-1}$) | $N = 10^4$ | $N = 10^5$ | $N = 10^6$ |
|----------------------|------------|------------|------------|
|                      | $\langle L \rangle$ (mm) | $\sigma_{\langle L \rangle}$ (mm) | $\langle L \rangle$ (mm) | $\sigma_{\langle L \rangle}$ (mm) | $\langle L \rangle$ (mm) | $\sigma_{\langle L \rangle}$ (mm) |
| $1 \cdot 10^{-1}$    | 6.669      | 0.002      | 6.668      | 0.007      | 6.6667      | 0.0002      |
| $1 \cdot 10^{-6}$    | 6.667      | 0.002      | 6.667      | 0.008      | 6.6666      | 0.0002      |
| $1 \cdot 10^{-5}$    | 6.667      | 0.002      | 6.667      | 0.007      | 6.6661      | 0.0003      |
| $1 \cdot 10^{-4}$    | 6.662      | 0.002      | 6.661      | 0.008      | 6.6626      | 0.0002      |
| $1 \cdot 10^{-3}$    | 6.646      | 0.002      | 6.643      | 0.008      | 6.6445      | 0.0003      |
| $1 \cdot 10^{-2}$    | 6.558      | 0.003      | 6.560      | 0.009      | 6.5593      | 0.0002      |
| $1 \cdot 10^{-1}$    | 6.174      | 0.003      | 6.173      | 0.010      | 6.1740      | 0.0003      |
| $1 \cdot 10^0$       | 4.926      | 0.005      | 4.924      | 0.015      | 4.9251      | 0.0005      |
| $2 \cdot 10^0$       | 4.351      | 0.005      | 4.352      | 0.013      | 4.3497      | 0.0005      |
| $1 \cdot 10^1$       | 2.940      | 0.005      | 2.945      | 0.017      | 2.9463      | 0.0005      |
| $1 \cdot 10^2$       | 1.438      | 0.004      | 1.439      | 0.012      | 1.4390      | 0.0004      |

| $\gamma$ (mm$^{-1}$) | $N = 10^4$ | $N = 10^5$ | $N = 10^6$ |
|----------------------|------------|------------|------------|
|                      | $\langle L \rangle$ (mm) | $\sigma_{\langle L \rangle}$ (mm) | $\langle L \rangle$ (mm) | $\sigma_{\langle L \rangle}$ (mm) | $\langle L \rangle$ (mm) | $\sigma_{\langle L \rangle}$ (mm) |
| $1 \cdot 10^{-1}$    | 6.665      | 0.002      | 6.659      | 0.008      | 6.6654      | 0.0002      |
| $1 \cdot 10^{-6}$    | 6.667      | 0.002      | 6.664      | 0.008      | 6.6627      | 0.0002      |
| $1 \cdot 10^{-5}$    | 6.657      | 0.002      | 6.654      | 0.008      | 6.6548      | 0.0002      |
| $1 \cdot 10^{-4}$    | 6.629      | 0.002      | 6.630      | 0.008      | 6.6304      | 0.0002      |
| $1 \cdot 10^{-3}$    | 6.556      | 0.003      | 6.559      | 0.008      | 6.5527      | 0.0002      |
| $1 \cdot 10^{-2}$    | 6.326      | 0.003      | 6.324      | 0.009      | 6.3232      | 0.0003      |
| $1 \cdot 10^{-1}$    | 5.727      | 0.004      | 5.729      | 0.011      | 5.7287      | 0.0004      |
| $1 \cdot 10^0$       | 4.574      | 0.004      | 4.571      | 0.011      | 4.5721      | 0.0004      |
| $2 \cdot 10^0$       | 4.140      | 0.004      | 4.136      | 0.012      | 4.1377      | 0.0004      |
| $1 \cdot 10^1$       | 3.110      | 0.004      | 3.113      | 0.012      | 3.1124      | 0.0004      |
| $1 \cdot 10^2$       | 1.886      | 0.003      | 1.882      | 0.012      | 1.8888      | 0.0003      |
### TABLE VI. MC results for $\beta = 0.3$ and $g = 0$

| $\gamma$ (mm$^{-1}$) | $N = 10^4$ | $N = 10^5$ | $N = 10^6$ |
|-----------------------|-------------|-------------|-------------|
|                       | $(L)$ (mm)  | $\sigma_L$ (mm) | $(L)$ (mm)  | $\sigma_L$ (mm) | $(L)$ (mm)  | $\sigma_L$ (mm) |
| $1 \cdot 10^{-7}$     | 6.632       | 0.002       | 6.6343     | 0.0008       | 6.6343     | 0.0002       |
| $1 \cdot 10^{-6}$     | 6.603       | 0.003       | 6.6026     | 0.0008       | 6.6018     | 0.0002       |
| $1 \cdot 10^{-5}$     | 6.541       | 0.003       | 6.5387     | 0.0007       | 6.5397     | 0.0003       |
| $1 \cdot 10^{-4}$     | 6.419       | 0.003       | 6.4194     | 0.0008       | 6.4192     | 0.0003       |
| $1 \cdot 10^{-3}$     | 6.198       | 0.003       | 6.1966     | 0.0008       | 6.1969     | 0.0003       |
| $1 \cdot 10^{-2}$     | 5.815       | 0.003       | 5.8124     | 0.0010       | 5.8132     | 0.0003       |
| $1 \cdot 10^{-1}$     | 5.218       | 0.004       | 5.2165     | 0.0010       | 5.2152     | 0.0003       |
| $1 \cdot 10^0$        | 4.421       | 0.003       | 4.4145     | 0.0009       | 4.4152     | 0.0003       |
| $2 \cdot 10^0$        | 4.154       | 0.004       | 4.1500     | 0.0013       | 4.1492     | 0.0004       |
| $1 \cdot 10^1$        | 3.516       | 0.004       | 3.5174     | 0.0011       | 3.5179     | 0.0004       |
| $1 \cdot 10^2$        | 2.659       | 0.004       | 2.6635     | 0.0012       | 2.6633     | 0.0004       |

### TABLE VII. MC results for $\beta = 0.9$ and $g = 0$

| $\gamma$ (mm$^{-1}$) | $N = 10^4$ | $N = 10^5$ | $N = 10^6$ |
|-----------------------|-------------|-------------|-------------|
|                       | $(L)$ (mm)  | $\sigma_L$ (mm) | $(L)$ (mm)  | $\sigma_L$ (mm) | $(L)$ (mm)  | $\sigma_L$ (mm) |
| $1 \cdot 10^{-7}$     | 6.666       | 0.002       | 6.6665     | 0.0006       | 6.6668     | 0.0002       |
| $1 \cdot 10^{-6}$     | 6.670       | 0.002       | 6.6675     | 0.0008       | 6.6668     | 0.0002       |
| $1 \cdot 10^{-5}$     | 6.666       | 0.003       | 6.6674     | 0.0007       | 6.6666     | 0.0002       |
| $1 \cdot 10^{-4}$     | 6.665       | 0.002       | 6.6674     | 0.0007       | 6.6666     | 0.0002       |
| $1 \cdot 10^{-3}$     | 6.670       | 0.002       | 6.6664     | 0.0008       | 6.6662     | 0.0002       |
| $1 \cdot 10^{-2}$     | 6.667       | 0.002       | 6.6643     | 0.0008       | 6.6649     | 0.0003       |
| $1 \cdot 10^{-1}$     | 6.651       | 0.002       | 6.6526     | 0.0008       | 6.6521     | 0.0003       |
| $1 \cdot 10^0$        | 6.555       | 0.003       | 6.5531     | 0.0009       | 6.5535     | 0.0003       |
| $2 \cdot 10^0$        | 5.896       | 0.005       | 5.896      | 0.002       | 5.8957     | 0.0006       |
| $1 \cdot 10^1$        | 5.413       | 0.006       | 5.418      | 0.002       | 5.4163     | 0.0007       |
| $1 \cdot 10^2$        | 3.772       | 0.008       | 3.773      | 0.002       | 3.7707     | 0.0009       |

### TABLE VIII. MC results for $\beta = 0.7$ and $g = 0$

| $\gamma$ (mm$^{-1}$) | $N = 10^4$ | $N = 10^5$ | $N = 10^6$ |
|-----------------------|-------------|-------------|-------------|
|                       | $(L)$ (mm)  | $\sigma_L$ (mm) | $(L)$ (mm)  | $\sigma_L$ (mm) | $(L)$ (mm)  | $\sigma_L$ (mm) |
| $1 \cdot 10^{-7}$     | 6.667       | 0.002       | 6.6676     | 0.0007       | 6.6661     | 0.0002       |
| $1 \cdot 10^{-6}$     | 6.668       | 0.002       | 6.6670     | 0.0008       | 6.6669     | 0.0002       |
| $1 \cdot 10^{-5}$     | 6.669       | 0.002       | 6.6659     | 0.0008       | 6.6665     | 0.0002       |
| $1 \cdot 10^{-4}$     | 6.667       | 0.002       | 6.6647     | 0.0008       | 6.6660     | 0.0002       |
| $1 \cdot 10^{-3}$     | 6.664       | 0.002       | 6.6650     | 0.0008       | 6.6647     | 0.0002       |
| $1 \cdot 10^{-2}$     | 6.662       | 0.002       | 6.6549     | 0.0008       | 6.6554     | 0.0002       |
| $1 \cdot 10^{-1}$     | 6.609       | 0.002       | 6.6082     | 0.0007       | 6.6093     | 0.0002       |
| $1 \cdot 10^0$        | 6.396       | 0.003       | 6.3942     | 0.0011       | 6.3932     | 0.0003       |
| $2 \cdot 10^0$        | 5.554       | 0.004       | 5.5613     | 0.0013       | 5.5612     | 0.0005       |
| $1 \cdot 10^1$        | 5.101       | 0.004       | 5.1026     | 0.0013       | 5.1029     | 0.0004       |
| $1 \cdot 10^2$        | 3.758       | 0.005       | 3.7624     | 0.0017       | 3.7615     | 0.0005       |
### TABLE IX. MC results for $\beta = 0.5$ and $g = 0.9$

| $\gamma$ (mm$^{-1}$) | $N = 10^4$ | $N = 10^5$ | $N = 10^6$ |
|----------------------|-------------|-------------|-------------|
|                      | $\langle L \rangle$ (mm) | $\sigma_{\langle L \rangle}$ (mm) | $\langle L \rangle$ (mm) | $\sigma_{\langle L \rangle}$ (mm) | $\langle L \rangle$ (mm) | $\sigma_{\langle L \rangle}$ (mm) |
| $1 \cdot 10^{-7}$    | 6.668       | 0.002       | 6.6664      | 0.0007       | 6.6663      | 0.0002       |
| $1 \cdot 10^{-6}$    | 6.667       | 0.002       | 6.6653      | 0.0007       | 6.6666      | 0.0002       |
| $1 \cdot 10^{-5}$    | 6.668       | 0.002       | 6.6645      | 0.0008       | 6.6652      | 0.0002       |
| $1 \cdot 10^{-4}$    | 6.667       | 0.002       | 6.6630      | 0.0008       | 6.6624      | 0.0002       |
| $1 \cdot 10^{-3}$    | 6.654       | 0.002       | 6.6536      | 0.0007       | 6.6542      | 0.0002       |
| $1 \cdot 10^{-2}$    | 6.631       | 0.002       | 6.6260      | 0.0008       | 6.6267      | 0.0002       |
| $1 \cdot 10^{-1}$    | 6.546       | 0.002       | 6.5440      | 0.0008       | 6.5427      | 0.0003       |
| $1 \cdot 10^0$       | 6.305       | 0.003       | 6.2973      | 0.0010       | 6.2979      | 0.0003       |
| $2 \cdot 10^0$       | 5.672       | 0.003       | 5.6735      | 0.0009       | 5.6740      | 0.0004       |
| $1 \cdot 10^1$       | 5.379       | 0.004       | 5.3789      | 0.0012       | 5.3778      | 0.0004       |
| $1 \cdot 10^2$       | 4.502       | 0.004       | 4.4987      | 0.0013       | 4.5003      | 0.0004       |

### TABLE X. MC results for $\beta = 0.3$ and $g = 0.9$

| $\gamma$ (mm$^{-1}$) | $N = 10^4$ | $N = 10^5$ | $N = 10^6$ |
|----------------------|-------------|-------------|-------------|
|                      | $\langle L \rangle$ (mm) | $\sigma_{\langle L \rangle}$ (mm) | $\langle L \rangle$ (mm) | $\sigma_{\langle L \rangle}$ (mm) | $\langle L \rangle$ (mm) | $\sigma_{\langle L \rangle}$ (mm) |
| $1 \cdot 10^{-7}$    | 6.663       | 0.003       | 6.6637      | 0.0007       | 6.6635      | 0.0002       |
| $1 \cdot 10^{-6}$    | 6.660       | 0.002       | 6.6611      | 0.0007       | 6.6590      | 0.0002       |
| $1 \cdot 10^{-5}$    | 6.652       | 0.002       | 6.6524      | 0.0007       | 6.6518      | 0.0002       |
| $1 \cdot 10^{-4}$    | 6.637       | 0.002       | 6.6375      | 0.0007       | 6.6373      | 0.0002       |
| $1 \cdot 10^{-3}$    | 6.610       | 0.003       | 6.6090      | 0.0008       | 6.6088      | 0.0002       |
| $1 \cdot 10^{-2}$    | 6.553       | 0.002       | 6.5522      | 0.0008       | 6.5523      | 0.0003       |
| $1 \cdot 10^{-1}$    | 6.445       | 0.003       | 6.4452      | 0.0009       | 6.4450      | 0.0003       |
| $1 \cdot 10^0$       | 6.246       | 0.003       | 6.2453      | 0.0009       | 6.2450      | 0.0003       |
| $2 \cdot 10^0$       | 5.899       | 0.003       | 5.8985      | 0.0009       | 5.8977      | 0.0003       |
| $1 \cdot 10^1$       | 5.756       | 0.003       | 5.7542      | 0.0009       | 5.7555      | 0.0003       |
| $1 \cdot 10^2$       | 5.344       | 0.003       | 5.3491      | 0.0009       | 5.3496      | 0.0003       |
V. BALLISTIC APPROXIMATION

In this section we show how the ballistic regime is approached by putting $g \to 1$ (totally forward scattering), for $\beta = 0.9$ and for three different $\gamma$s ($0.01, 0.1$ and $1$ mm$^{-1}$). For each value of $g$, $\beta$ and $\gamma$, the number of independent simulations, each with $N = 10^6$ simulated trajectories, is 100.

The results are shown in table XI. In Fig. 4, $\langle L \rangle$ is shown as a function of $g$ for different $\gamma$s. As $g$ approaches the limit value of 1, $\langle L \rangle$ converges to $\langle L \rangle_{IP}$. Such a result is in agreement with the mean chord theorem, that states that the invariance property of the mean path length holds for the pure ballistic case.

Moreover, as a further check for the MC code, the ballistic case can be also found by setting the scattering probability to 0. In this case, with 100 independent simulation with $10^6$ trajectories, the value of $\langle L \rangle_{IP}$ is $(6.66666 \pm 0.00002)$ mm, in agreement with $\langle L \rangle_{IP}$.

| $g$  | $\gamma = 0.01$ (mm$^{-1}$) | $\gamma = 0.1$ (mm$^{-1}$) | $\gamma = 1$ (mm$^{-1}$) |
|------|-----------------------------|-----------------------------|-----------------------------|
|      | $\langle L \rangle$ (mm)    | $\sigma_{\langle L \rangle}$ (mm) | $\langle L \rangle$ (mm)    | $\sigma_{\langle L \rangle}$ (mm) | $\langle L \rangle$ (mm)    | $\sigma_{\langle L \rangle}$ (mm) |
| 0.00 | 6.6488                      | 0.0002                      | 6.6521                      | 0.0004                      | 6.6541                      | 0.0006                      |
| 0.10 | 6.6507                      | 0.0002                      | 6.5414                      | 0.0003                      | 6.8290                      | 0.0007                      |
| 0.20 | 6.6524                      | 0.0002                      | 6.5546                      | 0.0003                      | 5.9047                      | 0.0006                      |
| 0.30 | 6.6545                      | 0.0002                      | 6.5682                      | 0.0003                      | 5.9831                      | 0.0006                      |
| 0.40 | 6.6556                      | 0.0002                      | 6.5820                      | 0.0003                      | 6.0670                      | 0.0006                      |
| 0.50 | 6.6580                      | 0.0002                      | 6.5951                      | 0.0003                      | 6.1544                      | 0.0005                      |
| 0.60 | 6.6598                      | 0.0003                      | 6.6097                      | 0.0003                      | 6.2458                      | 0.0005                      |
| 0.70 | 6.6616                      | 0.0002                      | 6.6236                      | 0.0003                      | 6.3427                      | 0.0004                      |
| 0.80 | 6.6629                      | 0.0002                      | 6.6374                      | 0.0002                      | 6.4451                      | 0.0003                      |
| 0.90 | 6.6649                      | 0.0003                      | 6.6521                      | 0.0003                      | 6.5535                      | 0.0003                      |
| 0.95 | 6.6661                      | 0.0003                      | 6.6596                      | 0.0003                      | 6.6093                      | 0.0003                      |
| 0.98 | 6.6662                      | 0.0003                      | 6.6639                      | 0.0002                      | 6.6437                      | 0.0002                      |
| 0.99 | 6.6667                      | 0.0002                      | 6.6652                      | 0.0002                      | 6.6548                      | 0.0002                      |

FIG. 4. (color online) $\langle L \rangle$ as a function of $g$ in the case of anomalous transport with $\beta = 0.9$ for different $\gamma$s.
VI. VIOLATION IN THE CASE OF RANDOM WALK WITH CONSTANT STEP LENGTHS

In this section, the results for the case of random walk of constant steps are reported for \( g = 0 \) and \( g = 0.9 \) within the sphere of radius 5 mm and with Lambertian illumination at the surface. In this case, the step size \( l \) is the reciprocal of \( \gamma \) (\( l = 1/\gamma \)). This is a case of distribution with finite mean and variance.

As reported in Fig. 5 and in table XII, the invariance property becomes violated as \( \gamma \) increases. For each value of \( g \) and \( \gamma \), the number of independent simulations, each with \( N = 10^6 \) simulated trajectories, is 100.

**TABLE XII.** MC results for \( g = 0.9 \) and \( g = 0 \) for a random walk with constant steps of length \( l = 1/\gamma \)

| \( \gamma \) (mm\(^{-1}\)) | \( g = 0.9 \) | \( g = 0 \) |
|---------------------------|--------------|--------------|
| \( \langle L \rangle \) (mm) | \( \sigma(L) \) (mm) | \( \langle L \rangle \) (mm) | \( \sigma(L) \) (mm) |
| \( 1 \cdot 10^{-5} \) | 6.6666 | 0.0002 | 6.6665 | 0.0002 |
| \( 1 \cdot 10^{-4} \) | 6.6665 | 0.0002 | 6.6671 | 0.0002 |
| \( 1 \cdot 10^{-3} \) | 6.6663 | 0.0002 | 6.6666 | 0.0002 |
| \( 1 \cdot 10^{-2} \) | 6.6666 | 0.0002 | 6.6661 | 0.0002 |
| \( 1 \cdot 10^{-1} \) | 6.8860 | 0.0003 | 6.6665 | 0.0002 |
| \( 1 \cdot 10^{0} \) | 6.8891 | 0.0009 | 9.4906 | 0.0011 |
| \( 1 \cdot 10^{1} \) | 6.885 | 0.002 | 9.636 | 0.004 |

**FIG. 5.** (color online) \( \langle L \rangle \) as a function of \( \gamma \) in the case of random walk of steps of length \( l = 1/\gamma \).

VII. VIOLATION IN THE CASE OF RANDOM WALK WITH PARETO LAW OF STEPS LENGTHS

In this section the results for the case of random walk with \( g = 0 \) and with steps length drawn from a Pareto distribution:

\[
p(\ell) = \frac{\alpha \ell_m^\alpha}{\ell^{\alpha+1}}
\]

where \( \alpha \) is a parameter \( \in (1, 2) \) and \( \ell_m \) chosen to fix the mean of the distributions equal to the chosen \( \gamma \)s in a way independent of \( \alpha \). These are cases of distribution with finite mean and diverging variance. The medium is a sphere.
of radius 5 mm and with Lambertian illumination at the surface.

As reported in Fig. 6 and in table XIII, the invariance property becomes violated as $\gamma$ increases. For each value of $g$ and $\gamma$, the number of independent simulations, each with $N = 10^6$ simulated trajectories, is 100.

### TABLE XIII. MC results for $\beta = 0.9$ for different $\gamma$ and $g$

| $\alpha$ | $\gamma = 10^{-4}$ (mm$^{-1}$) | $\gamma = 10^{-3}$ (mm$^{-1}$) | $\gamma = 10^{-2}$ (mm$^{-1}$) | $\gamma = 10^{-1}$ (mm$^{-1}$) | $\gamma = 1$ (mm$^{-1}$) |
|----------|---------------------------------|---------------------------------|---------------------------------|---------------------------------|--------------------------|
| $\langle L \rangle$ (mm) | $\sigma_{\langle L \rangle}$ (mm) | $\sigma_{\langle L \rangle}$ (mm) | $\sigma_{\langle L \rangle}$ (mm) | $\sigma_{\langle L \rangle}$ (mm) | $\sigma_{\langle L \rangle}$ (mm) |
| 1.3 | 6.6670 | 0.0002 | 6.7156 | 0.0002 | 6.7996 | 0.0003 | 8.3798 | 0.0009 | 8.592 | 0.003 |
| 1.5 | 6.6669 | 0.0002 | 6.6691 | 0.0002 | 6.7416 | 0.0003 | 8.4031 | 0.0003 | 8.748 | 0.003 |
| 1.7 | 6.6666 | 0.0002 | 6.6676 | 0.0002 | 6.7145 | 0.0002 | 8.4009 | 0.0008 | 8.873 | 0.002 |
| 1.9 | 6.6667 | 0.0002 | 6.6670 | 0.0002 | 6.6969 | 0.0002 | 8.3854 | 0.0006 | 8.967 | 0.002 |

FIG. 6. (color online) $\langle L \rangle$ as a function of $\gamma$ in the case of random walk with Pareto law step length distribution ($p(\ell) \sim \ell^{-(\alpha - 1)}$).