Causal set d’Alembertians for various dimensions

Fay Dowker$^1$ and Lisa Glaser$^2$

$^1$ Blackett Lab, Imperial College, London SW7 2AZ, UK
$^2$ The Niels Bohr Institute, Copenhagen University, Blegdamsvej 17, DK-2100 Copenhagen Ø, Denmark

E-mail: glaser@nbi.dk

Received 12 May 2013, in final form 15 August 2013
Published 11 September 2013
Online at stacks.iop.org/CQG/30/195016

Abstract

We propose, for dimension $d$, a discrete Lorentz invariant operator on scalar fields that approximates the Minkowski spacetime scalar d’Alembertian. For each dimension, this gives rise to a scalar curvature estimator for causal sets, and thence to a proposal for a causal set action.

PACS numbers: 4.60.Nc, 02.40.−k, 11.30.Cp

(Some figures may appear in colour only in the online journal)

1. Discrete d’Alembertians

In a discrete spacetime either locality or Lorentz invariance must be sacrificed. Causal sets are Lorentz invariant discrete spacetimes and are therefore fundamentally nonlocal. In order to base an approach to quantum gravity on causal sets, one must confront this nonlocality and take up the challenge of recovering the local physics that describes the physical world so well. A major step in this direction was made when a discrete operator for scalar fields on causal sets was proposed that approximates the continuum d’Alembertian when the causal set is well-approximated by two-dimensional Minkowski spacetime $[1, 2]$. This was extended to four dimensions $[3]$ and we first review these two- and four-dimensional operators.

Let $(\mathcal{C}, \preceq)$ be a causal set and let $\phi : \mathcal{C} \rightarrow \mathbb{R}$ be a scalar field on $\mathcal{C}$. For $x, y \in \mathcal{C}$ and such that $y \preceq x$, we define the inclusive order interval $I(x, y) := \{ z \in \mathcal{C} : y \preceq z \preceq x \}$ and $n(x, y) := |I(x, y)| - 2$.\(^3\) The two- and four-dimensional causal set d’Alembertians are, respectively $[1, 3]$,

\[ B^{(2)} \phi(x) := \frac{1}{l^2} \left[ -2\phi(x) + 4 \left( \sum_{y \in L_1(x)} \phi(y) - 2 \sum_{y \in L_2(x)} \phi(y) + \sum_{y \in L_3(x)} \phi(y) \right) \right], \tag{1} \]

\(^3\) This is the convention for $n(x, y)$ used in $[1]$ and differs from that defined in $[3]$. 
where the sums run over the past 'layers' \( L_x := \{ y < x : n(x, y) = i - 1 \} \). \( l \) is a length, the analogue of the 'lattice spacing'.

Given a \( d \)-dimensional, causal Lorentzian manifold \( M \) with a scalar field \( \phi : M \to \mathbb{R} \) of compact support and a point \( x \in M \), the operator \( B^{(d)} \) gives rise to a random variable, \( B^{(d)}(x) \), in the following way. The Poisson process of sprinkling \([1]\) into \( M \) at density \( \rho = l^{-d} \) produces a random causal set \( (C, \preceq) \) to which the point \( x \in M \) is added. \( \phi \) restricts to a scalar field on the random causal set \( C \) and the value of the random variable \( B^{(d)}(x) \) is obtained by applying the operator \( B^{(d)} \) to \( \phi \) on \( C \) at the point \( x \). Its expectation value in the sprinkling process is denoted by

\[
\tilde{B}^{(d)}(x) := \mathbb{E}(B^{(d)}(x)). \tag{3}
\]

\( \tilde{B}^{(d)}(x) \) depends on the manifold \( M \) and \( \rho = l^{-d} \) the sprinkling density, but we suppress these labels for ease of notation. When the Lorentzian manifold is \( d \)-dimensional Minkowski spacetime, in the limit of infinite density \( \tilde{B}^{(d)}(x) \) tends to the continuum flat \( d \)-Laplacian of \( \phi \) for \( d = 2 \) \([1]\) and \( d = 4 \) \([3]\):

\[
\lim_{l \to 0} \tilde{B}^{(d)}(x) = \Box^{(d)}(x). \tag{4}
\]

Following the form of the operator in two and four dimensions, consider the following ansatz for dimension \( d \)

\[
B^{(d)}(x) = \frac{1}{l^2} \left( \alpha_d \phi(x) + \beta_d \sum_{i=1}^{n_d} C^{(d)}_i \sum_{y \in L_x} \phi(y) \right), \tag{5}
\]

where \( n_d \) is the number of layers summed over, and \( \alpha_d, \beta_d \) and \( C^{(d)}_i \) for \( i = 1, \ldots, n_d \) are constants to be determined. The first coefficient \( C^{(d)}_1 \) is fixed to be equal to 1.

As above, for a scalar field \( \phi \) of compact support on a \( d \)-dimensional Lorentzian manifold \( M \), \( x \in M \) and a density \( \rho = l^{-d} \) the Poisson process of sprinkling into \( M \) gives rise to a random variable \( B^{(d)}(x) \), whose expected value we denote by \( \tilde{B}^{(d)}(x) \). The Poisson distribution implies that this expected value is

\[
\tilde{B}^{(d)}(x) = \alpha_d l^{-2} \phi(x) + \beta_d l^{-d+2} \int_{J^+(x)} d^d y \sqrt{-g(y)} \phi(y) \\
\times \sum_{i=1}^{n_d} C^{(d)}_i \frac{(V_d(y)l^{-d})^{i-1}}{(i-1)!} \exp(-V_d(y)l^{-d}). \tag{6}
\]

Here \( J^+(x) \) is the causal past of \( x \), and \( V_d(y) \) is the spacetime volume of the causal interval between \( x \) and \( y \) in \( M \), i.e. the intersection of the causal past of \( x \) with the causal future of \( y \). In the integral, \( l^{-d} \sqrt{-g(y)} d^d y \) is the probability that a point is sprinkled in the volume element at \( y \) and the factor \( \frac{(V_d(y)l^{-d})^{i-1}}{(i-1)!} \exp(-V_d(y)l^{-d}) \) is the probability that the element is in the \( i \)th layer i.e. that there are exactly \( i - 1 \) elements sprinkled in the interval between \( x \) and \( y \).

If we define integrals

\[
I_d(l) := \int_{J^+(x)} d^d y \sqrt{-g(y)} e^{-V_d(y)l^{-d}} \phi(y) \tag{7}
\]
it can be shown that
\[ \bar{B}^{(2)} \phi(x) = \alpha_2 l^{-2} \phi(x) + \beta_2 l^{-4} \mathcal{O}_2 l^2(l) \]  
(8)
\[ \bar{B}^{(4)} \phi(x) = \alpha_4 l^{-2} \phi(x) + \beta_4 l^{-6} \mathcal{O}_4 l^2(l) \]  
(9)
where
\[ \mathcal{O}_2 = \frac{1}{8} (H + 2)(H + 4) \quad \text{and} \quad \mathcal{O}_4 = \frac{1}{32} (H + 2)(H + 4)(H + 6) \]  
(10)
and
\[ H = -\frac{\partial}{\partial l} \]  
(11)
The differential operator \( \mathcal{O}_2 \) annihilates \( l^2 \) and \( l^4 \) and thus eliminates contributions to \( \bar{B} \phi(x) \) that would not tend to zero in the \( l \to 0 \) limit. Our strategy will be to choose the appropriate differential operators \( \mathcal{O}_d \) for general \( d \) which will fix the constants \( C_i \) via
\[ \mathcal{O}_d \exp(-l^{-d}V) = \sum_{i=1}^{n_d} C_i d^{(d)}(l^{-d}V)^{i-1} (i-1)! \exp(-l^{-d}V) \]  
(12)
and then to solve for \( \alpha_d \) and \( \beta_d \).

We propose the following form of \( \mathcal{O}_d \), for even dimensions \( d = 2n \), consistent with (10),
\[ \mathcal{O}_{2n} = \frac{(H + 2)(H + 4) \ldots (H + 2n + 2)}{2^{n+1} (n+1)!}, \]  
(13)
and for odd dimensions, \( d = 2n + 1 \), \( \mathcal{O}_{2n+1} = \mathcal{O}_{2n} \).

When the manifold is \( d \)-dimensional Minkowski spacetime, to evaluate the integrals we choose \( x \) as the origin of coordinates and use radial null coordinates \( v := \frac{1}{\sqrt{2}} (l + r) \) and \( u := \frac{1}{\sqrt{2}} (l - r) \), where \( r \) is the radius in \( d - 1 \)-dimensional spherical coordinates centred on \( x \). For \( d > 2 \), the integral \( I_d(l) \) is then
\[ I_d(l) = \int_{-\infty}^{0} du \int_{u}^{0} dv \int_{0}^{2\pi} d\phi \frac{1}{\sqrt{2}} (v - u) \phi(y) e^{-l^{-d}V_{d-2}(y)} \]  
(14)
where \( d\Omega_{d-2} \) is the integral over the \((d-2)\)-sphere.

1.1. Three dimensions

We will explicitly demonstrate the calculations in three-dimensional (3D) Minkowski spacetime, there the integral is:
\[ I_3(l) = \int_{-\infty}^{0} du \int_{u}^{0} dv \int_{0}^{2\pi} d\phi \frac{1}{\sqrt{2}} (v - u) \phi(y) e^{-l^{-3}V_3(y)}. \]  
(15)
The volume of the causal interval \( V_3 \) is \( \frac{\pi}{8} r^3 = \frac{\pi}{8} (uv)^{\frac{3}{2}} \). The operator in 3D is \( \mathcal{O} = \frac{1}{8} (H + 2)(H + 4) \). This gives,
\[ \bar{B}^{(3)} \phi(0) := \alpha_3 l^{-2} \phi(0) + \beta_3 l^{-5} \int_{-\infty}^{0} du \int_{u}^{0} dv \int_{0}^{2\pi} d\phi \frac{1}{\sqrt{2}} (v - u) \phi(y) \times \left( 1 - \frac{27}{8} l^{-3} (uv)^{\frac{3}{2}} + \frac{9}{8} l^{-6} (uv)^{3} \right) e^{-l^{-3}V_3(y)^{\frac{3}{2}}} \]  
(16)
where \( l' \) is defined by \( l^{-3}V_3(y) = l'^{-3} (uv)^{\frac{3}{2}} \). Note that \( l' \) and \( l \) are equal up to a factor of order one.

The leading contribution to \( \bar{B} \) as \( l \to 0 \) comes from a neighbourhood of \( x = 0 \), the origin of coordinates. This can be seen by analysing the integral (16). The integrand is exponentially
suppressed for \( uv \gg l^2 \) and away from a neighbourhood of the origin, the integration range where the integral could possibly contribute in the limit is close to the light cone and given by 
\[-L \leq u \leq -a \text{ and } \frac{Lu}{v} \leq v \leq 0 \]
where \( a > 0 \) is chosen to be much larger than \( l \) and \( L \) is a cutoff set by the size of the compact region of support of \( \phi \). In the limit as \( l \) gets small, \( a \) can be chosen small enough that \( \phi \) is approximately constant as a function of \( v \) over this range of integration. The integral over \( v \) at fixed \( u \) of the integrand in (16) can be done and gives a dependence on \( l \) which is a higher power than \( l^0 \) and so the contribution from close to the lightcone and away from the origin vanishes in the limit as \( l \) gets small.

In this neighbourhood, defined by \(-a \leq u, v \leq 0\), the field can be Taylor expanded in Cartesian coordinates \( \{y^\mu\} \)

\[
\phi(y) = \phi(0) + y^\mu \partial_\mu \phi(0) + \frac{y^\mu y^\nu}{2} \partial_\mu \partial_\nu \phi(0) + O(y^3).
\]  

Integrating over \( \varphi \) gives:

\[
\int_0^{2\pi} d\varphi(y) = 2\pi \phi(0) + \sqrt{2\pi (u + v)} \partial_\mu \phi(0) + \frac{\pi}{2} (u + v)^2 \partial_\mu \partial_\nu \phi(0)
\]

\[+ \frac{\pi}{4} (v - u)^2 (\partial_\mu \partial_\nu + \partial_\nu \partial_\mu) \phi(0) + O_3(u, v).
\]  

This can then be used in (16), after which the integral can be done to give:

\[
\bar{B}^{(3)} \phi(0) = \alpha_3 l^{-2} \phi(0) + \beta_3 \left( l^{-2} \phi(0) + \frac{3\sqrt{2}}{\pi} \left( \frac{5}{3} \right) (-\partial_\mu^2 + \partial_\nu^2 + \partial_\mu \partial_\nu) \phi(0) \right) + \cdots.
\]  

The corrections represented by \( \cdots \) can be obtained by calculating integral (16) with the lower limit of integration in \( u \) set to \(-a\). The correction terms that are first and second order in derivatives of \( \phi \) are

\[
E(l, a) = \left( \frac{3\sqrt{2}}{\pi} \right)^{\frac{1}{4}} \frac{\sqrt{2}}{l \pi} \frac{\partial \phi(x)}{\partial^2} + \left( \frac{3\sqrt{2}}{\pi} \right)^{\frac{1}{4}} \frac{\partial^3 \phi(x)}{l^{-1} \pi a \Gamma(\frac{3}{2})} - \left( \frac{3\sqrt{2}}{\pi} \right)^{\frac{1}{4}} \frac{3}{2} \frac{\partial_\mu^2 + \partial_\nu^2 + \partial_\mu \partial_\nu} {\Gamma(\frac{3}{2})}.
\]  

There are also smaller corrections proportional to higher derivatives of \( \phi \) as well as terms that go like \( \exp(-\frac{\pi}{3\sqrt{2}} a l^{-3}) \), which are exponentially small for large \( al^{-1} \). From (19) we can read off, \(-\alpha_3 = \beta_3 = \left( \frac{3\sqrt{2}}{\pi} \right)^{\frac{1}{4}} \frac{\Gamma(\frac{3}{2})}{2}\), and then

\[
\lim_{l \to 0} \bar{B}^{(3)} \phi(0) = \square^{(3)} \phi(0).
\]  

### 1.2. Other dimensions

The calculation can be repeated for Minkowski spacetime in other dimensions. \( O_d \) is given by (13) for \( d \) even, and \( O_{2n+1} = O_{2n} \) so the number of layers is \( n_d = \frac{d}{2} + 2 \) for even dimensions and \( n_d = \frac{d}{2} + 2 \) for odd dimensions. The resulting \( C^{(d)}_d \) for various dimensions are given in table 1. The case of one dimension is included for completeness and is the usual discretization of the second derivative.

The coefficients \( a_d \) and \( \beta_d \) are given in table 2. The volume, \( V_{0,\alpha(u,v)} \) of the flat causal interval in \( d \) dimensions between \( x \) at the origin of coordinates and the point \( y \) with radial null coordinates \( (u, v) \) is also given in table 2.
For odd $\alpha$ and $\beta$, the volume is given by

$$V_{d}(u, v) = c_d(uv)^{d/2} = \frac{S_{d-2}}{d(d-1)}(uv)^{d/2}$$

(22)

where $S_{d-2}$ is the volume of the $(d-2)$-sphere.

For odd $d$, $\alpha_d = -\frac{c_d^{(2d)}}{Gamma\left(\frac{d+2}{2}\right)}$ while for even $\alpha_d = -\frac{2c_d^{(2d)}}{Gamma\left(\frac{d+2}{2}\right)}$. In four dimensions this simplifies to $\alpha_4 = -\frac{4}{\sqrt{6}}$.

In general

$$\frac{\alpha_d}{\beta_d} = \lim_{\epsilon \to 0} \int_{-\epsilon}^{\epsilon} du \int_{-\epsilon}^{\epsilon} dv \left[ \frac{u + v - \mu}{\sqrt{2}} \right]^{d-2} e^{-\epsilon \mu \omega du dv}$$

(23)

2. Nonlocality scale

For $d = 2$ and $d = 4$ the value of $\tilde{B}^{(d)}$ approaches its limiting value—the continuum flat d’Alembertian—when the discreteness scale is small enough that the field is slowly varying on this scale, however for each individual sprinkling the fluctuations around the mean value are enormous and the fluctuations grow with increasing sprinkling density. We expect the same behaviour in other dimensions. This makes numerical simulations difficult and in order to tame the fluctuations, Sorkin introduced a family of operators for $d = 2$ parameterized by a fixed physical ‘nonlocality scale’ which is larger than the discreteness scale. Each operator in the family gives approximately the same mean as $B^{(2)}$, so long as the field is slowly varying.
on the nonlocality scale but the fluctuations around the mean now diminish as the discreteness scale tends to zero [1]. This introduction of a nonlocality scale was extended to \( d = 4 \) [3]\(^4\) and we here give the family of nonlocal operators in \( d \) dimensions. Recall

\[
B^{(d)}_\epsilon \phi(x) = \frac{1}{l^2} \left( \alpha_d \phi(x) + \beta_d \sum_{y \prec x} C^{(d)}_i \sum_{y \in k} \phi(y) \right). \tag{24}
\]

Let \( \xi \) be the nonlocality length scale such that \( \xi \gg l \), and define \( \epsilon := \left( \frac{\xi}{l} \right)^d \). Define, for each \( d \), a one parameter family of operators on scalar fields on a causal set

\[
B^{(d)}_\epsilon \phi(x) := \frac{\epsilon^2}{l^2} \left( \alpha_d \phi(x) + \beta_d \epsilon \sum_{y \in x} f_{\epsilon}(n(x, y), \epsilon) \phi(y) \right), \tag{25}
\]

where the sum is over elements \( y \) in the causal set to the past of \( x \) and

\[
f_{\epsilon}(n, \epsilon) := (1 - \epsilon)^n \sum_{i=1}^{n_d} C^{(d)}_i \left( \frac{\epsilon}{1 - \epsilon} \right)^{i-1}. \tag{26}
\]

As before, the sprinkling process at density \( \rho = l^{-d} \) into a \( d \)-dimensional spacetime with a marked point \( x \) and a scalar field turns this operator into a random variable whose mean, \( \bar{B}^{(d)}_\epsilon \phi(x) \) has the same form as equation (6) but with \( l \) replaced by \( \xi \):

\[
\bar{B}^{(d)}_\epsilon \phi(x) = \alpha_d \xi^{-2} \phi(x) + \beta_d \xi^{-(d+2)} \int_{J^{-}(x)} d^d y \sqrt{-g(y)} \phi(y)
\]

\[
\times \sum_{i=1}^{n_d} C^{(d)}_i \frac{(V_d(y)\xi^{-d})^{i-1}}{(i-1)!} \exp(-V_d(y)\xi^{-d}). \tag{27}
\]

To see that this is the case, first note that the first term in (25) is constant in the sprinkling process and so its mean is equal to it and this gives the first term in (27). To derive the integral we look at each term in (26). Consider the random variable

\[
C^{(d)}_i \epsilon \sum_{y \in x} (1 - \epsilon)^{n(x,y)} \left( \frac{\epsilon}{1 - \epsilon} \right)^{i-1} \phi(y).
\]

Its mean over sprinklings is

\[
C^{(d)}_i \epsilon \rho \int d^d y \sqrt{-g(y)} \phi(y) \sum_{n=1}^{\infty} (1 - \epsilon)^n \left( \frac{\epsilon}{1 - \epsilon} \right)^{i-1} (\rho V_d(y))^n \frac{e^{-\rho V_d(y)}}{n!},
\]

where \( \rho d^d y \sqrt{-g(y)} \phi(y) \) is the probability that an element is sprinkled in an element of volume at \( y \) and \( \frac{\rho V_d(y)}{n} e^{-\rho V_d(y)} \) is the probability that exactly \( n \) elements are sprinkled into a region of volume \( V \). Using \( \epsilon \rho = \xi^{-d} \), this becomes

\[
C^{(d)}_i \xi^{-d} \rho \int d^d y \sqrt{-g(y)} \phi(y) \epsilon^{i-1} (\rho V_d(y))^{i-1} \frac{e^{-\rho V_d(y)}}{(i-1)!}
\]

\[
= C^{(d)}_i \xi^{-d} \int d^d y \sqrt{-g(y)} \phi(y) \left( \xi^{-d} V_d(y) \right)^{i-1} \frac{e^{-\xi^{-d} V_d(y)}}{(i-1)!},
\]

hence the result.

\[\text{There is an error in [3] where, using the convention defined in that paper, } n(x, y) \text{ should be replaced by } n(x, y) - 2 \text{ in the formula for the nonlocal operator.}\]
2.1. Simulations

Numerical simulations of $\bar{B}_4^{(d)}(x)$ were done for small causal sets sprinkled into 3D Minkowski spacetime. The sprinkling region is a causal interval defined by $-L \leq u \leq 0$ with the cutoff parameter $L = 1$. The test functions used were $\phi(x) = 1, \phi(x) = x^2$ and $\phi(x) = t^2$ and they were measured on causal sets with a mean number of elements $\langle N \rangle = 1000$. The discreteness length $l$ is set by $l^3 = \frac{1}{1000 \sqrt[3]{2}}$ and the nonlocality length $\xi$ is parameterized by $\epsilon_1 \equiv l/\xi$ which was varied between values 0.1 and 0.9. The graph in figure 1 plots the mean over 100 000 sprinklings of $B_{\text{corr}} \phi$, where $B_{\text{corr}} \phi$ is defined as the discrete operator $B_{\phi} - E(\xi, L)$.

For the simulations it is important that we subtract the corrections, defined in equation (20), before comparing to the continuum prediction, since our causal sets are quite small and the corrections are non-negligible.

The data are consistent with the expected result. The best results are achieved with $\epsilon_1$ between 0.4 and 0.5. For epsilon close to one, the nonlocality length is approximately the discreteness length and the fluctuations in the results are large, as expected. For small $\epsilon$ the nonlocality length becomes of order the cutoff length and the deviations from the continuum d’Alembertian have two origins. One is that the sprinkled region is not large enough to accommodate the three ‘layers’ that give the necessary cancellations and the second is that we have ignored the ‘exponentially small’ corrections which are becoming non-negligible for these parameter values. For the causal sets used in the simulations the linear size is only about ten times the discreteness length and the good approximation to the continuum values is evidence that causal sets can efficiently encode geometrical information.

3. Scalar curvature

Having identified the coefficients in the operator $B_{\phi}$ we now investigate the mean, $\bar{B}_{\phi}$, of the random variable when the manifold, $\mathcal{M}$, is a $d$-dimensional curved Lorentzian manifold:

$$
\bar{B}_{\phi}(x) = a_d l^{-2} \phi(x) + \beta_d l^{-(d+2)} \mathcal{O}_d \int_{J^-(x)} d^d y \sqrt{-g(y)} \phi(y) \exp(-V_d l^{-d}),
$$

where $V_d := V_d(y)$, the volume of the causal interval between $x$ and $y$. 

Figure 1. $B_{\text{corr}} \phi$ plotted against $\epsilon_1^\frac{1}{3}$. The error bars show the standard error.
We assume that the spacetime region of compact support of the field is small compared to the radius of curvature of the spacetime. Then all curvature corrections to the flat space integral will be small and we can assume that the mean tends to a local limit as \( l \to 0 \) as it does in the flat space case. Then by dimensional arguments the limit will be a linear combination of \( \Box^{(d)} \phi(x) \) and \( R(x)\phi(x) \) where \( \Box^{(d)} = g^{\mu\nu} \nabla_\mu \nabla_\nu \) is the d’Alembertian and \( R \) is the Ricci scalar curvature of \( \mathcal{M} \):

\[
\lim_{l \to 0} \tilde{B}^{(d)}(x) = \Box^{(d)} \phi(x) + a_d R(x)\phi(x),
\]

where \( a_d \) is a dimensionless constant. For \( d = 2 \) and \( d = 4 \), the value of the coefficient \( a \) is \(-\frac{1}{2}\) [3]. We seek to determine \( a_d \) for dimension \( d \).

To obtain the local result (30), the integral (29) is performed over a normal neighbourhood of \( x, \mathcal{N} \), with Riemann normal coordinates, \( \{y^\mu\} \), centred at \( x \) as the origin of coordinates. \( \sqrt{-g} \) and \( V_d \) can be expanded to first order in the curvature:

\[
\sqrt{-g} = 1 - \frac{1}{6} R_{\mu\nu} y^\mu y^\nu + O(R^2) = 1 + \delta \sqrt{-g} + O(R^2),
\]

\[
V_d = V_{0,d} \left( 1 - \frac{d}{24(d+1)(d+2)} R_{\mu\nu} y^\mu y^\nu + \frac{d}{24(d+1)} R_{\mu\nu} y^\mu y^\nu + O(R^2) \right)
\]

\[
= V_{0,d} + \delta V_d + O(R^2),
\]

where \( V_{0,d} \) is the volume of the causal interval in \( d \)-dimensional flat space between the origin and a point with Cartesian coordinates \( \{y^\mu\} \) and all curvature components are evaluated at the origin. The formula (32) is from [4] where the curvature components are evaluated at the centre of the causal interval between the origin and \( y \): to this order in \( R \), the formulae are the same.

Expanding (29) in powers of curvature we get

\[
\tilde{B}^{(d)}(x) = \tilde{B}_0^{(d)}(x) + \delta \tilde{B}^{(d)}(x) + O(R^2).
\]

The zeroth order term is exactly the Minkowski space integral and will give \( \Box^{(d)} \phi(x) \) in the limit because the scalar d’Alembertian at \( x \) in Riemann normal coordinates is \( \eta^{\mu\nu} \partial_\mu \partial_\nu \).

To determine the coefficient \( a_d \) it is sufficient to consider \( \phi(x) = 1 \) and in that case the term first order in curvature is

\[
\delta \tilde{B}^{(d)} := \beta l^{-(d+2)} \mathcal{O}_d \int_{\mathcal{N}} d^d y \left( \delta \sqrt{-g} - l^{-d} \delta V \right) e^{-l^{-d} V_d}
\]

\[
= \beta l^{-(d+2)} \mathcal{O}_d \int_{\mathcal{N}} d^d y \left( -\frac{1}{6} R_{\mu\nu} y^\mu y^\nu - l^{-d} V_0 \right) - \frac{d}{24(d+1)(d+2)} R \eta^{\mu\nu} y^\mu y^\nu + \frac{d}{24 d+2} R \eta^{\mu\nu} y^\mu y^\nu
\]

\[
+ \frac{d}{24(d+1)} R \eta^{\mu\nu} y^\mu y^\nu \right) e^{-l^{-d} V_0}.
\]

Letting \( t := y^0 \) and \( r^2 := \sum (y^i)^2 \) and transforming to spherical polar coordinates, we can integrate over the angular coordinates. \( V_{0,d} \) depends only on \( t \) and \( r \) and is rotationally symmetric so the only terms in \( R_{\mu\nu} y^\mu y^\nu \) that remain are the diagonals \( R_{00} \mu y^\mu y^\nu \) and \( R_{00} \mu y^\mu y^\nu \). Since we are operating in Riemann normal coordinates we can use \( \Sigma R_{00} = (R_{00} + R) \) to get rid of \( R_{00} \). The result is

\[
\beta l^{-(d+2)} \mathcal{O}_d \int_{\mathcal{N}} dt \, dr \, r^{d-2} \left[ \left( R_{00}^2 + \frac{1}{d-1} (R_{00} + R)^2 \right) - \frac{1}{6} - \frac{1}{24(d+1)} \frac{\partial}{\partial t} \right] e^{-l^{-d} V_0}
\]

\[
+ R(t^2 - r^2) \frac{1}{24(d+1)(d+2)} \frac{\partial}{\partial t} e^{-l^{-d} V_0}.
\]
We can then split this into an integral multiplying \( R_{00} \) and a different integral multiplying \( R \), which can be solved independent of each other. In lightcone coordinates, the integral multiplying \( R_{00} \) is

\[
I_{00}^{(d)} := \mathcal{O}_d \left( -\frac{1}{6} - \frac{1}{24(d+1)} \frac{l}{\partial l} \int_{-a}^{0} du \int_{u}^{0} dv \left( \frac{v-u}{\sqrt{2}} \right)^{d-2} \right.
\]

\[
\times \left( \frac{(v+u)^2}{2} + \frac{(v-u)^2}{2(d-1)} \right) e^{-l^2 v_{0d}}, \tag{37}
\]

and the coefficient multiplying \( R \) is

\[
I_{R}^{(d)} := \beta l^{-(d+2)} S_{d-2} \mathcal{O}_d \left[ \int_{-a}^{0} du \int_{u}^{0} dv \left( \frac{v-u}{\sqrt{2}} \right)^{d-2} \right.
\]

\[
\times \left( \frac{(v-u)^2}{2(d-1)} \left( -\frac{1}{6} - \frac{1}{24(d+1)} \frac{l}{\partial l} \right) e^{-l^2 v_{0d}} \right.
\]

\[
+ 2uv \frac{1}{24(d+1)(d+2)} \frac{l}{\partial l} e^{-l^2 v_{0d}} \right]. \tag{38}
\]

The volume \( V_{0d} \) is given as a function of \( u \) and \( v \) in table 2 for dimensions \( d = 2 \rightarrow 7 \) and direct calculation of the integrals in these cases gives that \( \lim_{l \to 0} I_{00}^{(d)} = 0 \) and \( \lim_{l \to 0} I_{R}^{(d)} = -\frac{1}{2} \) for each of these dimensions. So, in these dimensions, if there is a local limit, then

\[
\lim_{l \to 0} \bar{R}^{(d)} \phi(x) = \Box^{(d)} \phi(x) - \frac{1}{2} R(x) \phi(x) . \tag{39}
\]

We make the obvious conjecture that the result (39) holds in all dimensions \( d \).

As in two and four dimensions [3], these results about the d’Alembertian operator in other dimensions give rise to proposals for scalar curvature estimators and actions for causal sets. For a causal set \( C \) and \( x \in C \) we define

\[
\mathcal{R}^{(d)}(x) = -\frac{2}{l^2} \left( \alpha_d^2 + \beta_d \sum_{i=1}^{n_i} C_i^{(d)} N_i(x) \right) , \tag{40}
\]

where \( N_i(x) \) is the cardinality of the \( i \)th layer \( L_i(x) \) to the past of \( x \). When evaluated on causal sets sprinkled into a \( d \)-dimensional Lorentzian manifold, this becomes a random variable whose mean will under appropriate conditions be close to the continuum scalar curvature \( R(x) \). Summing \( \mathcal{R}^{(d)}(x) \) over all elements of the causal set, gives the following proposal for an action for the causal set,

\[
\frac{1}{R} S^{(d)}(C) = \xi_d \left[ N + \frac{\beta_d}{\alpha_d} \sum_{i=1}^{n_i} C_i^{(d)} N_i \right] , \tag{41}
\]

where \( \xi_d = -\alpha_d \left( \frac{l}{\bar{l}} \right)^{d-2} \) and \( l^{d-2} = 8\pi G l \). The case \( d = 2 \) has been studied for flat regions of Minkowski spacetime, a cylinder spacetime and the topology changing trousers [5].

The nonlocal discrete d’Alembertian, (25) gives rise, similarly, to the nonlocal action,

\[
\frac{1}{R} S^{(d)}(C) = \xi_d \left[ \epsilon \sum x \in C \sum_{i=1}^{n_i} f_i (n(x, y), \epsilon) \right] . \tag{42}
\]

For the random variable we expect fluctuations around the mean to be dampened by the nonlocal averaging.
4. Summary

This paper generalizes the work of [1–3] from two and four dimensions to \(d\) dimensions. We gave a general procedure to obtain a scalar d’Alembertian operator for causal sets in any dimension. We have shown numerical evidence that the operator approximates the flat space scalar d’Alembertian in three dimensions. In curved spacetime, when the mean of the operator on a scalar field \(\phi\) has a local limit as the discreteness length tends to zero, the limit will be \(\Box^{(d)} - \frac{1}{2} R\), where \(R\) is the scalar curvature, for \(d = 2, 3, \ldots, 7\). It would be good to prove the conjecture for all \(d\). We used the discrete d’Alembertian to propose causal set actions in all dimensions. The operators and actions have nonlocal versions which give a way to damp the fluctuations about the mean. It would be interesting to explore these actions using Monte Carlo simulations of the path integral [6].

Acknowledgments

We thank D Benincasa, S Johnston and B Schmitzer, for helpful discussions. Special thanks go to N Hustler who did the simulations. FD thanks the Perimeter Institute for Theoretical Physics for support. Research at Perimeter Institute is supported by the Government of Canada through Industry Canada and by the Province of Ontario through the Ministry of Economic Development and Innovation.

References

[1] Sorkin R D 2006 Does locality fail at intermediate length-scales? Approaches to Quantum Gravity: Towards a New Understanding of Space and Time ed D Oriti (Cambridge: Cambridge University Press) pp 26–43
[2] Henson J 2006 The causal set approach to quantum gravity Approaches to Quantum Gravity: Towards a New Understanding of Space and Time ed D Oriti (Cambridge: Cambridge University Press)
[3] Benincasa D M T and Dowker F 2010 The scalar curvature of a causal set Phys. Rev. Lett. 104 181301
[4] Gibbons G W and Solodukhin S N 2007 The geometry of small causal diamonds Phys. Lett. B 649 317–24
[5] Benincasa D M T, Dowker F and Schmitzer B 2011 The random discrete action for 2-dimensional spacetime Class. Quantum Grav. 28 105018
[6] Surya S 2012 Evidence for a phase transition in 2D causal set quantum gravity Class. Quantum Grav. 29 132001