RESIDUES OF CONNECTIONS AND THE CHEVALLEY-WEIL FORMULA FOR CURVES

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Abstract. Given a finite group of automorphisms of a compact Riemann surface, the Chevalley-Weil formula computes the character valued Euler characteristic of an equivariant line bundle. The goal of this article is to give a proof by computing using residues of a Gauss-Manin connection.

Let $G$ be a finite group of automorphisms of a compact Riemann surface $X$. Then $G$ will act on the space of holomorphic 1-forms, more generally on the space of sections of powers of the canonical bundle $K$, or even more generally on the cohomology of an equivariant holomorphic line bundle $L$. The virtual representation $H^0(X, L) - H^1(X, L)$ is determined by its character $\chi_G(L)$, and there are at least two ways in which one might try to compute it. First of all, one could try to find a formula for the character as a complex valued function on the group. The holomorphic Lefschetz fixed point theorem of Atiyah and Bott [AB] gives a formula for $\chi_G(L)$ in this form:

$$\chi_G(L)(g) = \sum_p \frac{\nu_p(g)}{1 - \tau_p(g)}, \quad g \neq 1$$

where the sum runs over fixed points of $g$, and $\tau_p$ and $\nu_p$ are the characters of the action of the isotropy groups $G_p$ on $T_p^*$ and $L_p$. The second, and arguably more useful type of formula, is an expression for $\chi_G(L)$ as a linear combination of characters of irreducible representations, the set of which we denote by $\hat{G}$. Chevalley and Weil [CW] gave such a formula when $L$ is a positive power of the canonical bundle. Formulas of this type for more general $L$ were considered by Ellingsrud and Lønsted [EL]. When $G$ acts freely on $X$, $\chi_G(L)$ is a multiple of the character of the regular representation $\chi_{reg}$. More specifically,

$$\chi_G(L) = \left( \frac{1}{|G|} \deg L + 1 - h \right) \chi_{reg}$$

where $h$ is the genus of $X/G$. If the action is not free, then

$$\chi_G(L) = \left( \frac{1}{|G|} \deg L + 1 - h \right) \chi_{reg} - \sum_{\xi \in \hat{G}} m_\xi(L) \xi$$

where the coefficients $m_\xi(L)$ are given by explicit formulas depending on $\xi$, $\tau_p$ and $\nu_p$. This can be made much more explicit when $G$ is cyclic, and we will see that the holomorphic Lefschetz fixed point theorem for finite order automorphisms of Riemann surfaces is a straightforward consequence of this.
Our main goal is to give a new proof of the (generalized) Chevalley-Weil formula. The argument is neither very efficient nor particularly general, but we hope that it is instructive. We start by proving a general residue theorem, which expresses the degree of a vector bundle on a compact Riemann surface as minus the sum of traces of residues of a singular holomorphic connection. Next, given an equivariant line bundle $L$, we decompose the direct image $\pi_* L = \bigoplus V_\xi$ into isotypic components, where $\pi : X \to X/G$ is the projection. The heart of the proof is the computation of the degrees of the bundles $V_\xi$ by using the residue theorem applied to certain Gauss-Manin connections. With these in hand, the formula follows easily from Riemann-Roch on $X/G$.

1. Characters

We start by recalling a few facts from representation theory needed below [S]. Let $G$ be a finite group of order $N$. Let $F(G)$ denote the ring of class functions on $G$. The vector space $F(G)$ has a basis given by the set $\hat{G}$ of characters of irreducible representations. In fact, $\hat{G}$ forms an orthonormal basis with respect to the inner product $\langle \chi, \eta \rangle = \frac{1}{N} \sum_{g \in G} \overline{\chi(g)} \eta(g)$. Thus any element $\eta \in F(G)$ can be expanded as a Fourier series $\eta = \sum_{\chi \in \hat{G}} \langle \eta, \chi \rangle \chi$.

We write $\chi_{reg}(g) = \sum_{\chi \in \hat{G}} \chi(1) \chi(g) = \begin{cases} N & \text{if } g = 1 \\ 0 & \text{otherwise} \end{cases}$ for the character of the regular representation $\mathbb{C}[G]$. If $\chi \in \hat{G}$, and $M$ is a $G$-module, then the $\chi$-isotypic component $M_\chi \subset M$ is given by $e_\chi M$, where $e_\chi = \frac{1}{N} \sum_{g \in G} \chi(g^{-1}) g \in \mathbb{C}[G]$ is a central idempotent.

If $H \subset G$ is a subgroup, let $\text{Ind} : F(H) \to F(G)$ be induction. This takes the character of an $H$-module $M$ to the character of the induced $G$-module $\mathbb{C}[G] \otimes_{\mathbb{C}[H]} M$. In the opposite direction we have restriction. Frobenius reciprocity says that these operations are adjoint, i.e. if $\chi \in F(H), \eta \in F(G)$,

$$\langle \chi, \eta \rangle_H = \langle \text{Ind} \chi, \eta \rangle_G$$

2. Statement of the Chevalley-Weil formula

Let $X$ be a compact Riemann surface. Fix a finite subgroup $G \subseteq \text{Aut}(X)$ of order $N$, and let $\pi : X \to Y = X/G$ be the quotient. For each $p \in X$, let $G_p = \{ g \in G \mid gp = p \}$ be the isotropy group. It acts faithfully on the cotangent space $T^*_p = m_p/m^2_p$. In particular, it is cyclic of finite order, say $N_p$. Let $\tau_p$ be the character of the representation of $G_p$ on $T^*_p$. We can choose an isomorphism $G_p \cong \mathbb{Z}/N$ such that $\tau_p(1) = e^{2\pi i/N} = \zeta$, and a local coordinate $x$ at $p$ so that $\tau_p(1)$ acts by $x \mapsto \zeta x$. 

Let $\psi : L \to X$ be a line bundle in the geometric sense, i.e. $L$ is a complex manifold, with holomorphic projection $\psi$ satisfying the usual conditions. There is an action of $C^*$ on $L$ preserving the fibres. Let $Aut^{C^*}(L)$ denote the group of holomorphic automorphisms of the manifold $L$ which commute with the action of $C^*$. It follows easily that an element of $Aut^{C^*}(L)$ preserves the zero section. Thus we have a homomorphism

$$p : Aut^{C^*}(L) \to Aut(X)$$

which sends $\gamma$ to its restriction to the zero section. Let

$$Aut(X)_L = \{g \in Aut(X) \mid g^*L \cong L\}$$

denote the isotropy group of $L \in Pic(X)$. There is an exact sequence

$$1 \to \mathbb{C}^* \to Aut^{C^*}(L) \xrightarrow{p} Aut(X)_L \to 1$$

[B, lemma 3.4.1]. Let us suppose that $G \subseteq Aut(X)_L$, or equivalently that $L \in Pic(X)^G$ is invariant under $G$. A $G$-linearization of $L$ is a splitting of

$$1 \to \mathbb{C}^* \to Aut^{C^*}(L) \xrightarrow{p} Aut(X)_L \to 1,$$

or in other words a lifting of $G$ to a subgroup of $Aut^{C^*}(L)$. A $G$-equivariant line bundle on $X$ is a line bundle with a $G$-linearization. Equivariant bundles and invariant bundles are not quite the same thing. If $Pic_G(X)$ denotes the group of equivariant line bundles, then there is a forgetful homomorphism

$$Pic_G(X) \to Pic(X)^G$$

In general, it is neither injective nor surjective, but it will have finite kernel and cokernel. To see this, observe that by basic facts about group cohomology [Br, chap IV], the obstruction to splitting (1) lies in $H^2(G, \mathbb{C}^*)$. If the obstruction vanishes, then the set of splittings is parametrized by $H^1(G, \mathbb{C}^*) \cong Hom(G, \mathbb{C}^*)$. Both cohomology groups are finite.

Given a $G$-equivariant line bundle $L$, $G$ will act on $L$ by bundle automorphisms. Consequently, it will act on the space of holomorphic sections $\mathcal{L}(U) = \mathcal{O}_X(L)(U)$ for any $G$-invariant set $U$. Therefore $G$ will act on the Čech complex $\check{C}(U, \mathcal{L})$ for any covering by $G$-invariant open Stein sets. (Note that such a cover exists, because we may use the preimage of a Stein cover of the Riemann surface $X/G$.) Thus $G$ acts on $H^1(X, \mathcal{L})$. Let $\chi_G(L)$ denote the character of the virtual representation

$$H^0(X, \mathcal{L}) - H^1(X, \mathcal{L})$$

In other words, $\chi_G(L)$ is the character of $H^0(X, \mathcal{L})$ minus the character of $H^1(X, \mathcal{L})$.

Here are a few examples of equivariant line bundles.

**Example 2.1.** For any homomorphism $h : G \to \mathbb{C}^*$, the trivial bundle $\mathbb{C}_X = \mathbb{C} \times X$ has a $G$-linearization, where $g(v, x) = (h(g)v, gx)$. We usually take $h = 1$. Then the action on sections $\mathcal{O}_X(U)$ is the usual action on functions $gf(x) = f(g^{-1}x)$

**Example 2.2.** The canonical bundle $K$ has a natural $G$-linearization such that the $G$-action on sections is the action $\omega \mapsto g^*\omega$ on 1-forms. There is an induced linearization on any power $K^n$.

**Example 2.3.** If $D = \sum n_p p$ is a divisor, then recall that we have a line bundle whose sheaf of sections is

$$\mathcal{O}_X(D)(U) = \{f \text{ meromorphic on } U \mid \forall p \in U, \text{ord}_p(f) \geq -n_p\}$$
If $D$ is $G$-invariant, $\mathcal{O}_X(D)$ has a $G$-linearization such that the action is $gf(x) = f(g^{-1}x)$. All $G$-equivariant line bundles arise this way; more precisely

$$\text{Pic}_G(X) \cong \frac{\text{Div}(X)^G}{\pi^* \text{Princ}(Y)}$$

[Bo] thm 2.3.

If $L$ is equivariant, then $G_p$ will act on the fibre $L_p$. Let $\nu_p$ denote the character of this representation.

**Theorem 2.4.** Let $X, G$ be as above, and let $L$ be an equivariant line bundle on $X$. Let $Y = X/G$ have genus $h$. Then

$$(2) \quad \chi_G(L) = \left( \frac{1}{N} \deg L + 1 - h \right) \chi_{\text{reg}} - \sum_{\xi \in \hat{G}} m_\xi(L) \chi_{\text{reg}}$$

where

$$m_\xi(L) = \sum_{p \in X} m_{\xi,p}(L)$$

**Remark 2.5.**

1. This is a special case of a more general result due to Ellingsrud and Lønsted [EL]. When $L = K^\otimes n$, the formula goes back to Chevalley and Weil [CW].
2. The sums $m_\xi(L)$ are clearly finite since $m_{\xi,p}(L) = 0$ unless $G_p \neq \{1\}$. We also note that $m_{\xi,p}(L)$ only depends on the orbit by the arguments of section 4. So we can rewrite

$$m_\xi(L) = \sum_{q \in Y} |G_p| m_{\xi,p}(L)$$

where we choose one $p \in X$ over each $q$.
3. The formula can be made more explicit in certain cases, see section 5.

**Corollary 2.6.** If $G$ acts freely, then

$$\chi_G(L) = \left( \frac{1}{N} \deg L + 1 - h \right) \chi_{\text{reg}}$$

In particular, $\chi_G(L)$ is a multiple of the regular representation.

**3. The Residue theorem**

Let $V$ be a holomorphic vector bundle on compact Riemann surface $Y$, and let $\mathcal{V} = \mathcal{O}_Y(V)$ be its sheaf of holomorphic sections. Let $\Omega^1_Y$ be the sheaf of holomorphic 1-forms. A holomorphic connection on $V$ (or $\mathcal{V}$) is a $\mathbb{C}$-linear map

$$\nabla : \mathcal{V} \to \Omega^1_Y \otimes \mathcal{V}$$

satisfying the Leibnitz rule. Note that the curvature necessarily vanishes, because it locally a matrix of holomorphic 2-forms. Therefore by Chern-Weil theory [GH chap 3, §3], the degree $\deg V := \int_Y c_1(V) = 0$. This provides an obstruction to the existence of such a connection. We can avoid it by allowing singularities. Given a finite set $S \subset Y$, let $\Omega^1_Y(\log S)$ (resp. $\Omega^1_Y(*)S)$ be the sheaf of
holomorphic forms on $Y - S$ (with poles of finite order, resp. order at most 1, on points of $S$). Given a vector bundle $V$ on $Y$, a $\mathbb{C}$-linear map 
\[ \nabla : V \to \Omega^1_Y(\infty S) \otimes V \]
satisfying the Leibniz rule will be referred to as a singular holomorphic connection with singularities along $S$. We call $\nabla$ meromorphic (resp. logarithmic) if it takes values in $\Omega^1_Y(S) \otimes V$ (resp. $\Omega^1_Y(\log S) \otimes V$). Any vector bundle $V$ carries a meromorphic connection, namely $\nabla = d$ where $V|_{X - S}\cong \mathcal{O}_X^n$ is a local trivialization of the underlying algebraic vector bundle (which exists by GAGA). When $V$ is a line bundle, we will give a more precise construction in lemma 4.3.

The standard reference for logarithmic connections is Deligne [D], although we will not really need much beyond the definition of residue, which we now explain. Given $s \in S$, and a basis $\{v_i\}$ of the stalk $V_s$ as an $\mathcal{O}_s$-module, a singular holomorphic connection is determined by a matrix of 1-forms $A = (a_{ij})$ given by 
\[ \nabla v_i = \sum_j a_{ij} \otimes v_j \]
The forms $a_{ij}$ are holomorphic in a punctured disk about $s$. Therefore given a local parameter $y$ at $s$, we can form a Laurent expansion
\[ A = \sum_{n=-\infty}^{\infty} A_n y^n dy \]
where the coefficients are constant matrices. In the logarithmic case, $A_{-1}$ gives a well defined endomorphism of the fibre $\text{End}(V_s)$ [D p 78]. For a general singular holomorphic connection, we claim something weaker:

**Lemma 3.1.** Given a singular holomorphic connection, the trace $\text{tr} A_{-1} \in \mathbb{C}$ is well defined, i.e. it is independent of the choice of basis of $V_s$ and local parameter.

**Proof.** Suppose that $v'_i = \sum f_{ij} v_j$ is a new basis. If $F = (f_{ij})$, then the new connection matrix is given by $A' = F A F^{-1} + dF \cdot F^{-1}$. Therefore $\text{tr} A' = \text{tr} A + \text{tr}(dF \cdot F^{-1})$, and so $\text{tr} A'_{-1} = \text{tr} A_{-1}$. Cauchy’s integral formula shows that $\text{tr} A_{-1}$ is independent of the local parameter. \qed

We denote $\text{tr} A_{-1}$ by $\text{tr} \text{Res}_s(\nabla)$.

**Theorem 3.2.** If $(V, \nabla)$ is vector bundle equipped with a singular holomorphic connection as above, then
\[ \deg V = -\sum_{s \in S} \text{tr} \text{Res}_s(\nabla) \]

**Remark 3.3.**

1. The theorem generalizes the well known fact that the sum of residues of meromorphic 1-form $\omega$ is zero [GH p 222]. This follows by applying the theorem when $V$ is trivial with connection $d + \omega$.
2. When the connection is logarithmic, this is a special case of a theorem of Esnault and Verdier [EV, appendix B].

**Proof.** Choose $\epsilon > 0$, so that the closed disks $D_s(\epsilon)$ of radius $\epsilon$ (with respect to a metric) around $s$ are disjoint and contained in coordinate neighbourhoods. By a standard partition of unity argument, we can construct a $C^\infty$ connection $\nabla'$ on $V$ which agrees with $\nabla$ on $\bigcup_s D_s(\epsilon/2)$. The curvature $\Theta$ of $\nabla'$ must vanish on
Let $B_s = (\beta_{ij})$ denote the connection matrix for $\nabla$ with respect to a trivialization of $V$ on $D_s(\epsilon)$. Then

$$\Theta = dB_s + B_s \wedge B_s$$

by the usual formula for curvature (see [GH, p 401] although the sign there differs from ours). Therefore

$$\text{tr } \Theta = \text{tr}(dB_s) + \sum_{ij} \beta_{ij} \wedge \beta_{ij} = d \text{tr } B_s$$

Since $B_s$ is the connection matrix of $\nabla$ on the annulus $D_s(\epsilon) - D_s(\epsilon/2)$,

$$\text{tr } B_s = (y^{-1} \text{tr Res}_s \nabla + f(y))dy$$

on the annulus, where $f$ is holomorphic on $D_s(\epsilon) - \{s\}$ with zero residue at $s$. Therefore by Stokes’ and Cauchy’s theorems

$$\frac{i}{2\pi} \int_{D_s(\epsilon)} \text{tr}(\Theta) = -\frac{1}{2\pi i} \int_{\partial D_s(\epsilon)} \text{tr}(B_s) = -\text{tr Res}_s(\nabla)$$

□

**Corollary 3.4.**

$$\sum_{s \in S} \text{tr Res}_s(\nabla) \in \mathbb{Z}$$

4. Proof of Chevalley-Weil

Let $\pi : X \to Y$ be a nonconstant holomorphic map between compact Riemann surfaces. Suppose that $L$ is a holomorphic line bundle on $X$ with a logarithmic connection

$$\nabla : L \to \Omega^1_X(\log S) \otimes L$$

After increasing $S$ if necessary, we can assume that it contains all the ramification points of $\pi$. Then $T = \pi(S)$ will contain the branch points. Let $\mathcal{L} = \mathcal{O}_X(L)$, and $\mathcal{V} = \pi_* \mathcal{L}$. The direct image

$$\mathcal{V} = \pi_* \mathcal{L} \xrightarrow{\pi_* \nabla} \pi_* (\Omega^1_X(\log S) \otimes \mathcal{L}) \cong \Omega^1_Y(\log T) \otimes \mathcal{V}$$

defines a logarithmic connection $\nabla$, which we refer to as the Gauss-Manin connection.

We give a local description. Choose $q \in Y$, and let $\{p_1, \ldots, p_m\} = \pi^{-1}(q)$. Let $n_i$ denote the ramification index of $p_i$. Choose local coordinates about $p_i$ and $q$ so that $y = x^{n_i}$. Let us also choose a local generator $\lambda$ for $L$ at $p_i$. Let $\mathcal{W}_i$ be the free $\mathcal{O}_{p_i}$-module with basis $\lambda, x\lambda, \ldots, x^{n_i-1}\lambda$. Then

$$\mathcal{V}_q = \bigoplus_{i=1}^m \mathcal{W}_i$$
The Gauss-Manin connection respects this decomposition. We will compute it on \( W_i \). Let us suppose that \( \nabla \) has residue \( r_i \) at \( p_i \). We can write

\[
\tilde{\nabla}(x^j\lambda) = \frac{dx}{x} \otimes r_i x^j \lambda + \frac{dx}{x} \otimes jx^j \lambda + \ldots
\]

where the omitted terms are holomorphic.

**Lemma 4.1.**

\[
\text{tr Res}_q \tilde{\nabla} = \sum_{i=1}^{m} \text{Res}_{p_i} \nabla + \frac{1}{2} \sum_{i=1}^{m} (n_i - 1)
\]

**Proof.** Using (3), we obtain

\[
\text{tr Res}_q \tilde{\nabla} = \sum_{i=1}^{m} \sum_{j=0}^{n_i-1} \frac{r_i + j}{n_i} = \sum_{i=1}^{m} \left( \frac{r_i + n_i - 1}{2} \right)
\]

□

**Corollary 4.2.** Let \( R \) be the ramification divisor of \( \pi \) then

\[
\deg V = \deg L - \frac{1}{2} \deg R
\]

**Proof.** This follows from the lemma and theorem [3.2] □

Let us now assume that \( G \subseteq \text{Aut}(X) \) is a subgroup of order \( N \), and that \( Y = X/G \). The ramification index of \( p \in X \) is exactly \( N_p = |G_p| \). In particular, \( p \) is ramified if and only if it is a fixed point for some \( g \neq 1 \). The fibre \( \pi^{-1}(\pi(p)) \) is precisely the orbit \( G \cdot p \).

**Lemma 4.3.** If \( L \) is a \( G \)-equivariant line bundle on \( X \), then there exists a \( G \)-invariant logarithmic connection

\[
\nabla : L \to \Omega^1_X(\log S) \otimes L
\]

for some invariant \( S \subset X \).

**Proof.** First suppose that \( \deg L < 0 \), then for \( m \gg 0 \), \( L^{\otimes -m} \) will have a nonzero section \( \sigma' \). Then \( \sigma' = \sigma' \otimes \ldots \otimes \sigma' \) will give an invariant section of \( L^{\otimes -n} \), where \( n = mN \). The dual gives an invariant injective map \( \sigma : \mathcal{O}_X(L^{\otimes n}) \to \mathcal{O}_X \). Then \( \nabla(\lambda) = d\sigma(\lambda^{\otimes n})/\sigma(\lambda^{\otimes n}) \otimes \lambda \) will have the desired properties. We can do the general case, by writing \( L = L_1 \otimes L_2^{-1} \), with \( \deg L_i < 0 \), and using the tensor product and dual connections. □

The direct image \( V = \pi_*L \) is locally free with \( G \)-action. We can decompose

\[
V = \bigoplus_{\xi \in G} V_\xi, \quad V_\xi = e_\xi V
\]

We have

\[
e_\xi \chi^1(X, L) = \chi^1(Y, V_\xi)
\]

Therefore by Riemann-Roch

\[
\langle \chi_G(L), \xi \rangle = \chi(Y, V_\xi) = \deg V_\xi + \xi(1)(1 - h)
\]

\[
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\]
where \( \chi(Y, V_\xi) \) is the Euler characteristic of \( V_\xi \). The proof of theorem 2.4 will follow from the computation of the degree \( \deg V_\xi \), which we now explain.

Let us choose an invariant connection 
\[
\nabla : L \to \Omega^1_X(\log S) \otimes L
\]
as in the above lemma. Then the Gauss-Manin connection \( \tilde{\nabla} \) is \( G \)-invariant, so we can decompose \( \tilde{\nabla} \) as a sum \( \bigoplus \tilde{\nabla}_\xi \) with respect to (4). Choose \( p = p_1 \in T \), and let \( q = \pi(p) \). Let \( n = N_p, m = N/n, \) and \( \{p_1, \ldots, p_m\} = G \cdot p_1 = \pi^{-1}(q) \). The modules with connection \( (W_i, \tilde{\nabla}|_{W_i}) \) are now isomorphic, so it suffices to look at \( W = W_1 \).

Let \( W = \C \otimes_{\O_T} W, \) and \( V = \C \otimes_{\O_T} V_q \) denote the fibres. The first vector space \( W \) is a \( \C[G_p] \)-module. The basis vectors \( x^i \lambda \) span \( G_p \)-submodules \( \C\nu_{\tau^i} \). Therefore
\[
W = \bigoplus_{i=0}^{n-1} \C\nu_{\tau^i}
\]
We note that \( \{\nu, \nu_{\tau}, \ldots, \nu_{\tau^{n-1}}\} = \hat{G}_p \). Since \( G_p \) is abelian, the \( G_p \)-module
\[
W \cong \C[G_p]
\]
The vector space \( V \) is the \( \C[G] \)-module induced from the \( G_p \)-module \( W \). Consequently \( V \cong \C[G] \). It will useful to weight the summands, by considering the class function
\[
\rho_q = \sum_{i=0}^{n-1} \left( \frac{r + i}{n} \right) \Ind(\nu_{\tau^i})
\]
We can see from (4) and the above remarks that
\[
\tr \Res_q \tilde{\nabla}_\xi = \langle \rho_q, \xi \rangle_G
\]

**Proof of theorem 2.4**. It suffices to prove that
\[
\langle \chi_G(L), \xi \rangle = \left( \frac{1}{N} \deg L + 1 - h \right) \xi(1) - m_\xi(L)
\]
for every \( \xi \in \hat{G} \). By comparing with (4), we see that if suffices to prove that
\[
\deg V_\xi = \frac{1}{N}(\deg L)\xi(1) - m_\xi(L)
\]

By theorem 3.2,
\[
\deg V_\xi = - \sum_{q \in T} \tr \Res_q \tilde{\nabla}_\xi = - \sum_{q \in T} \langle \rho_q, \xi \rangle_G
\]
Let us write \( \rho_q = \rho_q' + \rho_q'' \), where
\[
\rho_q' = \sum_{i=0}^{n-1} \left( \frac{r}{n} \right) \Ind(\nu_{\tau^i})
\]
\[
\rho_q'' = \sum_{i=0}^{n-1} \left( \frac{i}{n} \right) \Ind(\nu_{\tau^i})
\]
As noted above,
\[
\rho_q' = \left( \frac{r}{n} \right) \Ind W = \left( \frac{r}{n} \right) \chi_{reg}
\]
Therefore
\[ \langle -\rho'_q, \xi \rangle = \left( \frac{r}{n} \right) \xi(1) = rm \frac{\xi(1)}{N} \]
The quantities \( r \) and \( m \) depend on \( q \), even though the notation does not reflect this.

Summing over \( q \), and using theorem 3.2 yields
\[ \sum_q rm = -\deg L \]
Therefore
\[ \sum_q \langle -\rho'_q, \xi \rangle = \frac{1}{N}(\deg L)\xi(1) \]
From Frobenius reciprocity, we obtain
\[ \sum_q \langle \rho''_q, \xi \rangle = m \xi(L) \]
This proves (8).

\section*{5. Cyclic and dihedral groups}

We make the Chevalley-Weil formula more explicit in a couple of cases. With the previous notation, we calculate the local coefficients \( m_{\xi,p}(L) \) when \( G \) is cyclic or dihedral.

Let \( G = (g|g^n = 1) \). Then \( \hat{G} \) is also a cyclic group of order \( N \) generated by \( \xi(g^j) = \exp(2\pi ij/N) \).

The dual of the inclusion \( G_p \subseteq G \) gives a surjective homomorphism \( \hat{G} \to \hat{G}_p \).
Since the character \( \tau_p \) generates \( \hat{G}_p \), we can assume without loss of generality that \( \xi|_{G_p} = \tau_p \).
The character \( \nu_p \) is given by \( \xi^m|_{G_p} \) for nonnegative integer \( m < N_p \).

Given an integer \( x \), let \( \overline{x} \in \{0, 1, \ldots, N_p - 1\} \) be its residue modulo \( N_p \).

\textbf{Proposition 5.1.} With the notation above
\[ m_{\xi^k,p}(L) = \frac{k - m}{N} \]

\textbf{Proof.} The orthonormality of \( \hat{G} \) implies that
\[ \langle \xi^k|_{G_p}, \nu_p \cdot \tau_p^i \rangle_{G_p} = \begin{cases} 1 & \text{if } k \equiv i + m \mod N_p \\ 0 & \text{otherwise} \end{cases} \]
Substituting into
\[ m_{\xi^k,p}(L) = \frac{\xi^k(1)}{N} \sum_{i=0}^{N_p-1} i \langle \xi^k|_{G_p}, \nu_p \cdot \tau_p^i \rangle_{G_p} \]
proves the proposition.

Next suppose that \( G = D_n \) is the dihedral group of order \( N = 2n \). This group has a presentation
\[ D_n = \langle r, s \mid r^n = 1, s^2 = 1, srs = r^{-1} \rangle \]
Let \( R \subseteq D_n \) be the subgroup generated by \( r \). The characters in \( \hat{G} \) are easy to write down \cite{[S], §5.3}. When \( n \) is even,
\[ \hat{G} = \{ \chi_h \mid 0 < h < n/2 \} \cup \{ \psi_i \mid i = 1, 2, 3, 4 \} \]
where the characters are given by the table

|    | $r^k$ | $sr^k$ |
|----|-------|--------|
| $\chi_h$ | $2 \cos \frac{\pi n k}{n}$ | 0 |
| $\psi_1$ | 1 | 1 |
| $\psi_2$ | 1 | $-1$ |
| $\psi_3$ | $(-1)^k$ | $(-1)^k$ |
| $\psi_4$ | $(-1)^k$ | $(-1)^k+1$ |

In more conceptual terms, $\chi_h$ is induced from $\xi_h$, where $\xi$ is a generator of $\hat{R}$. We also have $\chi_h|_R = \xi^h + \xi^{n-h}$. When $n$ is odd $\psi_3$ and $\psi_4$ are omitted, otherwise $\hat{G}$ is the same.

A nontrivial cyclic subgroup of $D_n$ is easily seen to be either a subgroup $R$, or the subgroup $\{1, sr^k\}$ for some $k$. For simplicity, we just treat the case where $G_p \subseteq R$.

**Proposition 5.2.** Suppose that $G_p \subseteq R$. Lift $\tau_p$ to a generator $\xi \in \hat{R}$, and $\nu_p$ to $\xi^m$. Then

$$m_{\chi_{h,p}}(L) = 2 \left( \frac{n-m+n-h-m}{N} \right)$$

and

$$m_{\psi_{i,p}}(L) = \frac{-m}{N}, \quad i = 1, 2$$

When $n$ is even

$$m_{\psi_{i,p}}(L) = \frac{n/2-m}{N}, \quad i = 3, 4$$

**Proof.** From the properties above

$$\langle \chi_h|_{G_p}, \nu_p \cdot \tau^i_{p}\rangle_{G_p} = \begin{cases} 1 & \text{if } h \equiv i + m \mod N_p \text{ or } n - h \equiv i + m \mod N_p \\ 0 & \text{otherwise} \end{cases}$$

Substituting into the formula for $m_{\chi_{h,p}}(L)$ yields the first equation. The proofs of the remaining equations are similar. \(\square\)

6. THE LEFSCHETZ FORMULA

In this section, we want to explain the relation to the holomorphic Lefschetz fixed point formula of Atiyah and Bott [AB thm 4.12]. We refer to their paper for the general statement, as well as for some historical information about it. First let us briefly digress from Riemann surfaces to explain that the conclusion of corollary [20] is true in much greater generality.

**Proposition 6.1.**

1. If a finite group $G$ acts freely on a finite simplicial complex $X$, then the character $\chi_G(X)$ of the virtual representation $\sum (-1)^i H^i(X, \mathbb{C})$ is an integer multiple of a regular representation.

2. If $G$ acts freely on a compact complex manifold $X$, and $V$ is an equivariant vector bundle, then the character $\chi_G(V)$ of the virtual representation $\sum (-1)^i H^i(X, V)$ is an integer multiple of a regular representation.
Proof. By the Lefschetz fixed point theorem [H, p 179] \( \chi_G(X)(g) = 0 \) when \( g \neq 1 \). This implies that \( \chi_G(X) \) is an integer multiple of the regular representation by \([S, ex 2.7]\). The proof of the second statement is identical except that we use the holomorphic Lefschetz fixed point theorem \([AB, thm 4.12]\).

\[ \square \]

Corollary 6.2. In case (1) (resp. 2), if \( \chi(X) \) (resp. \( \chi(V) \)) denotes the topological (resp. holomorphic) Euler characteristic of \( X \) (resp. \( V \)), then

\[ \chi_G(X) = \frac{\chi(X)}{|G|} \chi_{reg} \]

\[ \chi_G(V) = \frac{\chi(V)}{|G|} \chi_{reg} \]

The coefficients on the right hand side of the two equations are integers.

Proof. We know that \( \chi_G(X) = c\chi_{reg} \) (resp. \( \chi_G(V) = c\chi_{reg} \)) for some \( c \in \mathbb{Z} \). Evaluation at \( g = 1 \) gives the value of \( c \). \( \square \)

We will deduce the holomorphic Lefschetz formula for finite order automorphisms of a Riemann surface from the Chevalley-Weil theorem by a fairly elementary reduction.

Theorem 6.3. Let \( g \) be a nontrivial finite order automorphism of a compact Riemann surface \( X \). Let \( L \) be a line bundle, which is equivariant for the group generated by \( g \), and let \( X_\sigma = \{ p \mid g(p) = p \} \), and let \( \tau_p, \nu_p \) be as in section 2. Then

\[ \sum_{i=0}^{1} (-1)^i \text{tr} [g : H^i(X, L) \to H^i(X, L)] = \sum_{p \in X_\sigma} \frac{\nu_p(g)}{1 - \tau_p(g)} \]

Proof. Let \( G \subseteq \text{Aut}(X) \) be the group generated by \( g \), and let \( N \) denote its order. Since \( g \neq 1 \), we have \( \chi_{reg}(g) = 0 \). Therefore theorem [2.4] reduces to

\[ \chi_G(L)(g) = \sum_{p \in X} M_p \]

where

\[ M_p = -\sum_{\xi \in \hat{G}} m_{\xi, p}(L) \xi(g) \]

Of course \( M_p = 0 \) unless \( N_p \neq 1 \). Let us assume this. Let \( n = N/N_p \) denote the index of \( G_p \) in \( G \). Employing the statement and notation of proposition [2.3] we find

\[ M_p = -\frac{1}{N} \sum_{k=m}^{N-1+m} \frac{1}{k-m} \zeta^k \]

\[ = -\frac{1}{N} \sum_{i=0}^{N_p-1} \sum_{\ell=0}^{n-1} i \zeta^{i+m+\ell N_p} \]

where the last expression results from the substitutions \( i = k-m \) and \( \ell = (k-m-i)/N_p \). Therefore

\[ (10) \]

\[ M_p = -\frac{\zeta^m}{N} \left( \sum_{i=0}^{N_p-1} i \zeta^i \right) \left( \sum_{\ell=0}^{n-1} (\zeta^{N_p})^\ell \right) \]
We now separate the argument into two cases depending on $n$. First suppose that $n \neq 1$. Then we claim that $M_p = 0$, which would be predicted by the theorem because $X^g = \{p \mid g \in G_p\} = \{p \mid N = N_p\}$. To see this, observe that $\zeta^{N_p}$ is a primitive $n$th root of unity, therefore
\[
\sum_{\ell=0}^{n-1} (\zeta^{N_p})^\ell = 0
\]
Substituting this into (10) shows that $M_p = 0$ as claimed.

Let us now assume that $n = 1$, which means that $N_p = N \neq 1$ so that $p \in X^g$. One checks the identity
\[
\sum_{i=0}^{N-1} i \zeta^i = \frac{N}{\zeta - 1}
\]
by cross multiplying and observing
\[
\sum_{i=0}^{N-1} i \zeta^i (\zeta - 1) = N \zeta^N - \sum_{i=1}^{N} \zeta^i = N
\]
Therefore by (10)
\[
M_p = -\frac{\zeta^m}{N}\left(\sum_{i=0}^{N-1} i \zeta^i\right)
= \frac{\zeta^m}{1 - \zeta}
= \frac{\nu_p(g)}{1 - \tau_p(g)}
\]

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