Bulk flow scaling for turbulent channel and pipe flows

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Abstract – We report a theory deriving bulk flow scaling for canonical wall-bounded flows. The theory accounts for the symmetries of boundary geometry (flat plate channel vs. circular pipe) by a variational calculation for a large-scale energy length, which characterizes its bulk flow scaling by a simple exponent, i.e., $m = 4$ for the channel and $5$ for the pipe. The predicted mean velocity shows excellent agreement with several dozen sets of quality empirical data for a wide range of the Reynolds number ($Re$), with a universal bulk flow constant $\kappa \approx 0.45$. Predictions for dissipation and turbulent transport in the bulk flow are also given, awaiting data verification.

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An intriguing feature of practical turbulent flows is the presence of net spatial momentum and energy transports, which are constrained by boundaries [1,2]. Flows at the surface of a vessel or inside a pipeline, over the wings of an aircraft or close to the ground on windfarms, etc., develop abundant flow structures contributing to turbulent transports. However, even for canonical cases, i.e., flat-plate channels and cylinder pipes, the boundary effect has not been treated theoretically in predicting mean-flow scaling. A milestone for the mean-momentum scaling is von Karman’s velocity-defect law [3], i.e.,

$$U_c - U = u_g (y/R),$$

where $U$ is the streamwise mean velocity; $U_c$ is the mean centerline velocity; $u_g$ is the friction velocity (defined later) and $g$ is an unknown scaling function depending only on the wall distance $y$ (normalized with the half channel height or pipe radius $R$). Later, Millikan [4] developed a matching argument deriving the celebrated Prandtl-Karman log-law of (1). However, all theoretical accounts have been restricted to the scaling in an overlap region (typically for $y \leq 0.15$), without addressing the influence of geometries on more than 80% of the flow domain. Most efforts devoted for the whole domain description are empirical, such as Coles wake function [5], composite Padé approximation [6,7], with limited accuracy and unknown domain of applicability. A recent work by L’vov et al. [8] achieves a noteworthy description of channel and pipe flows, but its outer flow description invoking a fitting function derived from simulation data does not distinguish channel and pipe. Thus, a theoretical derivation for the complete expression of function $g$ is still lacking, which is particularly important to resolve recent vivid debates on the universality of the mean-velocity scaling in the canonical wall-bounded turbulent flows [9].

In this paper, we present a novel attempt which identifies a universal mechanism deriving the mean-velocity scaling for canonical wall-bounded flows, based on a symmetry consideration of wall constraints. Most importantly, the theory suggests a universal bulk flow constant for channel and pipe. This is accomplished by introducing a length function whose calculation based on a variational argument yields a geometry dependence (planar vs. circular) with an integer scaling exponent (4 for a flat channel and 5 for a cylindrical pipe). The analysis enables a prediction of (1) valid in the entire flow domain, and the predicted mean-velocity profiles are in excellent agreement with several dozen of recent, reliable experiments over a wide range of $Re$’s. The results also shed light on the debate between the log-law and power-law [10,11] in favor of the former, and have applications to other boundary effects (such as roughness, compressibility, pressure gradient — studied by us but not addressed here).

We start with the fully developed incompressible turbulent channel flow, between two parallel plates of height $2R$, driven by a constant pressure gradient $f_p = -\frac{1}{\rho} \frac{\partial p}{\partial x}$. 

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in the streamwise $x$-direction. The flow develops a mean-velocity profile $U$, depending on wall-normal $y$-direction only. The mean-momentum flux is described by the Reynolds-averaged Navier-Stokes (RANS) equation, i.e.,

$$\nu \partial_y U - \overline{u'v'} = \tau_p,$$  \hspace{1cm} (2)

where $S \equiv \partial_y U$ is mean shear; $W \equiv -\overline{u'v'}$ denotes Reynolds stress which is unknown; $\tau_p = \int_R^r \int_p d\tau' = u'_2 r / R$ is the total stress with $r = R - y$, the distance to the centerline, and $u_x \equiv \sqrt{\int p / R}$ is the friction velocity determined by the pressure force. Note that dimensionally, (2) has an alternative interpretation at the local position $r$: the pressure gradient force supplies the energy $\tau_p = \int_p / R$ which balances the viscous damp $\nu S$ and the turbulent shear fluctuation $W$. In this sense, for a cubic flow volume in a channel, namely $V_r = r \times R \times R$ ($R \times R$ indicates the $r$-surface area), the total turbulent shear stress is thus $M = \int_0^V W dV$ (we will return to this quantity later).

Then, the product of viscous and Reynolds stresses contributes to the growth of the turbulent kinetic energy $k = u'_2 u'_2 / 2$, which is described by the mean-kinetic-energy equation, i.e.,

$$SW + \Pi = \epsilon.$$ \hspace{1cm} (3)

Here $P = SW$ is the production; $\Pi$ represents the spatial energy transfer (including diffusion, convection and fluctuation transport); $\epsilon$ the viscous dissipation (for explicit expressions, see [12]).

In our recent work [13], a dimensional analysis among $\epsilon, S, W$ yields

$$\ell = W^{(3/4 + 1/2)} S^{(3/4 - 1)} \epsilon^{(-1/4)} \hspace{1cm} (4)$$

where $\ell$ is the characteristic length representing eddies responsible for the energy spatial transfer, and $n$ is an arbitrary real number, tentatively chosen to be integer. Note that as $n \rightarrow \infty$, the length becomes the classical mixing length of Prandtl: $\ell_\infty = \sqrt{W / S}$; while a unique $n = 4$ defines a physically meaningful length valid throughout the channel, i.e.,

$$\ell_\epsilon = W^{3/4} S^{1/2} \epsilon^{-1/4}.$$ \hspace{1cm} (5)

This length is similar to the crucial scaling function in the model of L’vov et al. [8] (restricted to the bulk zone only), but its interpretation follows a concept of order function developed by us [14]. Our main result of this paper is to give a physical derivation of $\ell_\epsilon$.

According to its definition (5), $\ell_\epsilon$ can also be expressed in terms of the eddy viscosity $\nu_t = W / S$, i.e.,

$$\ell_\epsilon = W^{3/4} / \epsilon^{1/4}.$$ \hspace{1cm} (6)

This expression reminds us of the Kolmogorov dissipation length $\eta = \nu^{3/4} / \epsilon^{1/4}$, where $\epsilon$ is the rate of energy cascade in the inertial range, while $\nu$ is the molecular viscosity, transferring the kinetic energy to heat. So, Kolmogorov’s $\eta$ can be interpreted as a balance length between the inertial energy cascade and molecular dissipation (i.e., $\epsilon = \nu^3 / \eta^4$). Similarly, the expression of the length $\ell_\epsilon = W^{3/4} / \epsilon^{1/4}$, suggests that the length $\ell_\epsilon$ is also a balance length between energy cascade rate $\epsilon$ and the energy transfer rate through the eddy viscosity $\nu_t$ (i.e., $\epsilon = \nu_t^3 / \ell_\epsilon^4$). But, the role of $\nu_t$ is different from $\nu$: in contrast to the molecular dissipation to heat, the eddy viscosity here acts as an energy source, since $SW = \nu_t S^2$ is the kinetic-energy production term. In other words, $\ell_\epsilon$ is the integral eddy size where the eddy viscosity $\nu_t$ produces the kinetic energy through the mean shear, and here we refer to $\ell_\epsilon$ as the length of energy production eddies (PE). Note that in analogy to Townsend [15], a possible materialization of PE is through the ensemble-averaged vortex packets [16] or clusters [17]. Moreover, recalling the attached eddy hypothesis by Perry [18], we assume similarly that PE distribute uniformly in the spanwise direction (since time average is applied), and the streamwise characteristic size is $\ell_\epsilon$—only depending on centerline distance $r$. In fig. 1, we depict the distribution of PE in a channel (and a pipe), where one can see a monotonic decrease of $\ell_\epsilon$ away from centerline, indicating the influence of wall constraint $\ell_\epsilon = 0$ approaching the wall. Note that one can also interpret the increment of $\ell_\epsilon$ approaching the centerline as the growth of PE, in analogy to the growth of the wall attached eddies as wall distance increases [18]. Under such a statistical point of view, we consider two important quantities associated with the PE, which are the total shear stress $M'$, and the total kinetic energy $E'$ associated with the growth of PE, as presented below.

Recall that the total shear stress for a given flow volume $V_r$ is $M$. Then, for PE sketched in fig. 1, they are uniformly distributed in the spanwise direction (similar assumption as in [18]) with a streamwise length size $\ell_\epsilon$ depending on the wall distance $y$. A volume integration of $W$ for these PE yields $M' = \int_0^{V_r} W \ell_\epsilon Rdy = \int_0^{V_r} W (\ell_\epsilon / R) dV$. Moreover, as we know that $u_y$, the friction velocity, is the global velocity scale of wall turbulence, so $u_x^2$ is a global
energy scale. Since $\nabla \ell_\varepsilon$ represents the growth rate of PE's length with increasing wall distance, then, $u_\varepsilon \nabla \ell_\varepsilon$ is an associated velocity increment due to the PE's growing length, and $|u_\varepsilon \nabla \ell_\varepsilon|^2$ the associated kinetic-energy increment. Integrated in the whole volume $V_r$, we obtain the total kinetic-energy increment associated with PE's growing length: $E' = \int_{V_r} |u_\varepsilon \nabla \ell_\varepsilon|^2 \, dv$.

Now, a variational argument is postulated here: for a given $M'$, $E'$ should be minimum. This hypothesis follows from an alternative interpretation of (2) that in the bulk flow (where mean shear is small and negligible), the Reynolds shear stress ($W$) is always balanced by the total stress, so that a fixed amount of $W$ corresponds to a certain work of the pressure driving force. Our argument goes to examine how the PE respond to the pressure driving in a fully developed turbulent state. It is thus natural to assume that a fully developed turbulent state corresponds to such a “most active” state in which turbulent cascade process is so fully developed that the $E'$ of PE reaches minimum. In other words, fully developed cascade is the essential mechanism to drive the minimum of $E'$. This is indeed the cornerstone assumption of this paper, which is validated in fig. 2 using direct numerical simulation (DNS) data. All above analysis can be equally applied to turbulent pipe flow. The difference from channel is that, the flow volume corresponding to the cylindrical boundary (see fig. 1) is $V = R \pi r^2$, and $M' = \int_0^r W \ell_\varepsilon 2 \pi r \, dr = \int_{V_0} W(\ell_\varepsilon / R) \, dv$, $V_R = \pi R^3$. In this case, $\alpha = a_0^2 / R^2$, and a similar calculation of (7) for pipe flow yields

$$\ell_\varepsilon^{pipe} / R \approx \kappa (1 - r^5) / 5. \quad (9)$$

Here the coefficient $\kappa / 5$ is determined by requiring the same near wall asymptotic scaling as (8), i.e., $\ell_\varepsilon^{pipe} / R \approx \ell_\varepsilon^{CH} / R \approx \kappa (y / R)$ when $r' \to 1$. This is reasonable because the geometry effect vanishes close to the wall, hence channel and pipe should share the same $\kappa$ (validated later). Also note that for turbulent boundary layer (TBL), though $W$ may different from channel to TBL, a leading order expansion $W \propto r / R$ ($r$ is the distance to the boundary layer thickness $R$) in TBL would also lead to (8) based on (7). In other words, (8) in channel should also apply in TBL when the latter flow becomes nearly parallel (at large $Re$'s); this result is in consistent with the same flat plate wall condition in the two flows.

Interestingly, a joint solution of (2) and (3) can be obtained using $\ell_\varepsilon$ and $\Theta = \varepsilon / (SW)$. From (5) one has $S = \sqrt{W / (\ell_\varepsilon \Theta^{1/4})}$. Integrating $S$ with $r$ and using $W \approx \tau_p$, one obtains the velocity-defect law (1), i.e.

$$U_\varepsilon - U = \int_0^r S \, dr = \int_0^r \frac{\sqrt{\tau_p}}{\ell_\varepsilon \Theta^{1/4}} \, dr. \quad (10)$$

Here, $\Theta$ can be derived analytically [13] as follows. When $r' \to 1$, $\Theta \approx 1$, corresponding to the well-known quasi-balence regime (QBR). On the other hand, near the centerline ($r' \to 0$), $\Theta = \varepsilon / (SW) \propto 1 / r'^2$, since $\varepsilon \to O(1)$ while $S \propto r'$ (and also $W \propto r'$) due to central mirror symmetry. Let $\Theta \approx c / r'^2$ as $r' \to 0$, where $c$ is a dimensionless coefficient. To match $\Theta \to 1$ as $r' \to 1$, a composite expression is $\Theta = 1 + c / r'^2 - c$, which can be rewritten as

$$\Theta(r') = [1 + (r_0 / r')^2] / (1 + r_0^2), \quad (11)$$
where \( r_c = \sqrt{C/(1-c)} \) indicates the thickness of the core layer. Thus, through a simple matching argument we derived an expression for \( \Theta \) valid from QBR to the central core.

Note that (10) also rewrites as

\[
\kappa U d / u_r = G(r') \approx \int_0^{r'} f(\tilde{r}) d\tilde{r}, \tag{12}
\]

where \( U d = U_c - U, f = m[(1 + r^2)/(r^2 + r^2)]^{1/4} / (1 - \tilde{r} m) \) \((m = 4 \) for channel and 5 for pipe\). The parameter \( r_c \) indicates the thickness of the core layer in channel and pipe (zero in TBL due to the absence of opposite wall), which has a slight Re-dependence at moderate Re's. It is obtained by fitting \( G \) with the velocity-defect data, which yields \( r_c \approx 0.27 \) for channels (DNS) and \( r_c \approx 0.5 \) for Princeton pipes (EXP). The predicted velocity defect is shown in fig. 3, where the universal bulk flow constant \( \kappa \approx 0.45 \) is remarked by the linear slope agreeing well with 25 sets of mean-velocity profiles. Note that the result for smooth pipe is also applied to rough pipe (with the same \( \kappa \) and \( r_c \)), consistent with Townsend's similarity hypothesis [15].

It is important to note that our current results support the the asymptotic log-law instead of power-law. Note that at large Re's, there would be an asymptotic interval where \( \Theta \approx 1, \tau_p \approx u_r^2 \) and \( \ell_c \approx \kappa y \). In this region, \( \ell_c \approx \ell_\infty \), so the present calculation is identical to Prandtl-Karman's log-law [1,12], \( \ell_\infty \approx \kappa K y \). This implies that the bulk flow constant \( \kappa \) derived in the present study is exactly the Karman constant \( \kappa K \). Furthermore, rewriting (12) as \( U / u_r = U_c / u_r - G / \kappa \), and the leading order contribution in \( G \) is \( \int_0^{y/R} f(\tilde{r}) d\tilde{r} \propto \ln(y/R) \), we thus have \( U / u_r - U_c / u_r \propto \kappa^{-1} \ln(y/R) \), indicating the logarithmic scaling. Thus our results support the asymptotic log-law. In fig. 4, using the empirical \( U_c \) (and \( \kappa, r_c \)), we plot the mean velocity by (12), displaying impressive agreement with data (smooth and rough pipes only; channel data also agree well, but not shown). The relative errors are bounded within 1% (except for first several points near wall) — the same level as the data uncertainty. Note that (12) asymptotes to \( \ln(y_u/\nu) / \kappa \) in the log layer near the wall, and the comparison with data (e.g., \( Re = 25167 \) in fig. 4) shows that it describes the data well as close to the wall as \( y' = y_u/\nu \approx 50 \) (the beginning of the log layer). Note that our bulk flow scaling is not consistent with the near-wall linear scaling \( (U' \approx y') \) in the viscous sublayer; the latter needs to be described by an additional matching with a wall function; this topic is beyond the scope of this paper, addressed elsewhere [25].

Finally, the dissipation and transport in the mean-kinetic-energy equation (3) are predicted:

\[
\varepsilon = SW \Theta \approx \frac{u_r^3}{R} \frac{m(1 + r^2)^{3/4}}{\kappa(1 - r^m)(1 + r^2)^{3/4}}, \tag{13}
\]

\[
\Pi = SW (\Theta - 1) \approx \frac{u_r^3}{R} \frac{m r_c^2 (1 - r^2)(1 + r^4)}{\kappa(1 - r^m)(1 + r^2)^{3/4}}.
\]

Particularly, the centerline dissipation (equaling centerline transport) is

\[
\varepsilon_0 = \Pi_0 \approx \frac{u_r^3}{R} \frac{m r_c^{3/2}}{\kappa(1 + r^2)^{3/4}}. \tag{14}
\]

With the value \( \kappa \approx 0.45 \) and \( r_c \approx 0.5 \), we have \( \varepsilon_0 \approx 3.3 u_r^3 / R \) for pipe. For current channel data, using \( \kappa \approx 0.45 \) and \( r_c \approx 0.27 \) yields \( \varepsilon_0 \approx 1.2 u_r^3 / R \). However, since \( r_c \) has a moderate Re-effect (indicating a growth of central core layer), the predicted centerline dissipation may increase with increasing Re. Assuming the same \( r_c \) for high Re channels and pipes, one would have \( \varepsilon_0^{\text{Pipe}} / \varepsilon_0^{\text{CH}} = 5/4 \). These await verifications when \( \varepsilon_0 \) data are available.
In summary, we have developed an analytical theory for joint closure solutions of the mean momentum and kinetic-energy equations for turbulent channel and pipe flows in the bulk region. The variational assumption leads to an analytical formula of the eddy length function, where a universal bulk flow constant $\kappa \approx 0.45$ is identified to be valid for the entire flow domain (much beyond the overlap region). Note that (8) and (9) indicate a breaking of dilation invariance for $\ell_\varepsilon$ because of the presence of a characteristic length $\ell_0 = \kappa R/m$ at the centreline. However, the dilation invariance is preserved in its gradient, i.e., $d\ell_\varepsilon/dr' \propto r'^{m-1}$. Such a symmetry perspective is further explored in connection of the dilation symmetry in the direction normal to the wall for turbulence model equations widely used in engineering applications (i.e., $k - \omega$ equation), which will be reported elsewhere.

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