Structure of the largest idempotent-free sequences in finite semigroups

Guoqing Wang

Department of Mathematics, Tianjin Polytechnic University, Tianjin, 300387, P. R. China

Email: gqwang1979@aliyun.com

Abstract

Let $S$ be a finite semigroup, and let $E(S)$ be the set of all idempotents of $S$. Gillam, Hall and Williams in 1972 proved that every sequence of elements in $S$ of length at least $|S| - |E(S)| + 1$ contains a nonempty subsequence whose product is idempotent, which affirmed a question proposed by Erdős. They also gave a sequence of elements in a particular semigroup $S$ to show the value $|S| - |E(S)| + 1$ is best possible. Motivated by this work, in this paper we characterized the structure the extremal sequence $T$ provide that $T$ is a sequence of elements in any finite semigroup $S$ of length exactly $|S| - |E(S)|$ such that $T$ contains no nonempty subsequence whose product is idempotent.

Key Words: Idempotents; Idempotent-free sequence; Erdős-Burgess constant; Davenport constant

1 Introduction

Let $S$ be a finite semigroup. The operation on $S$ is denoted by $\ast$. An element $x \in S$ is called an idempotent if $x^2 = x \ast x = x$. P. Erdős had posed to D.A. Burgess (see [2]) a question with respect to the idempotents in any finite semigroup which is stated as follows:
‘In a finite semigroup $S$ of order $n$, any sequence $T$ of elements in $S$ of length $n$ must contains a nonempty subsequence whose product is idempotent.’

Burgess [2] in 1969 gave an answer to this question in case that $S$ is commutative and contains only one idempotent. Shortly after this, this question was completed affirmed in 1972 by Gillam, Hall and Williams, who proved the following improved result actually.

**Theorem A.** ([11]) Let $S$ be a finite semigroup of order $n$. Any sequence $T$ of elements in $S$ of length at least $n - |E(S)| + 1$ contains a nonempty subsequence whose product is idempotent.

In the same paper, they also gave a sequence of elements in a particular semigroup $S$ to show the value $|S| - |E(S)| + 1$ is best possible. So, a nature question arises:

**What can we say about the structure of the semigroup $S$ and the extremal sequence $T$, provide that $T$ is a sequence of elements in $S$ of length exactly $n - |E(S)|$ and contains no nonempty subsequence whose product is idempotent?**

In this paper, we answer the above inverse question by determining the structure of the semigroup $S$ and the extremal sequence $T$. For the notational convenience, we present the main theorem together with its proof in Section 3. Section 2 contains only some necessary notations and lemmas. In the final Section 4, further researches are proposed.

## 2 Some Preliminaries

- Throughout this paper, we shall always denote by $S$ a finite semigroup when there is no other specific denotation.

  We begin this section by introducing some notations (see [7]) used by the researchers worked in Zero-sum Theory to deal with such kind of additive problems on algebraic structures (mainly on commutative groups) in the past, which will be also of use to the additive problems on semigroups.

  Let $F(S)$ be the (multiplicatively written) free commutative monoid with basis $S$. Then
any \( T \in \mathcal{F}(S) \), say
\[
T = x_1 x_2 \cdots x_n = \prod_{x \in S} x^{(v_x(T))},
\]
is a sequence of elements in the semigroup \( S \), where \( v_x(T) \) denotes the multiplicity of \( x \) in the sequence \( T \). Note that the operation (joining two sequences) on \( \mathcal{F}(S) \) is represented by \( \cdot \), which are different to the operation \( * \) on \( S \), and therefore,
\[
x^n = x * x * \cdots * x
\]
is the product of \( n \) elements of \( S \) all equal to \( x \), and
\[
x^{(n)} = x \cdot \cdots \cdot x
\]
is the sequence of all terms equal to \( x \) with multiplicity \( n \). By
\[
\text{supp}(T) = \{ x \in S : v_x(T) > 0 \}
\]
we denote the set of all the elements of \( S \) with positive multiplicities in the sequence \( T \). Let \( T_1, T_2 \in \mathcal{F}(S) \) be two sequences on \( S \). We call \( T_2 \) a subsequence of \( T_1 \) if
\[
v_x(T_2) \leq v_x(T_1)
\]
for every element \( x \in S \), in particular, if \( T_2 \neq T_1 \), we call \( T_2 \) a proper subsequence of \( T_1 \), and write
\[
T_3 = T_1 \cdot T_2^{(-1)}
\]
to mean the unique subsequence of \( T_1 \) with \( T_2 \cdot T_3 = T_1 \).

In the rest of this section, we shall give some necessary preliminaries on semigroups. For more related terminologies and results, one is referred to [12].

The zero element of \( S \), denoted \( 0_S \) (if exists), is the unique element \( z \) of \( S \) such that \( z * x = z \) for every \( x \in S \). For any element \( x \in S \), the least integer \( r > 0 \) such that \( x^r = x^t \) for some positive integer \( t \neq r \) is the **index** of \( x \), denoted \( I(x) \). Then the least integer \( k > 0 \) such that \( x^{I(x)+k} = x^{I(x)} \) is the **period** of \( x \), denoted \( P(x) \). Let \( X \) be a subset of a semigroup \( S \). We say \( X \) generates \( S \) and the elements of \( X \) are generators of \( S \) provided that every element \( s \in S \) is
the product of one or more elements of \( X \), denoted \( S = \langle X \rangle \). A semigroup is cyclic when it is generated by a single element \( x \), denoted by \( \langle x \rangle \).

For any commutative semigroup \( S \), one fundamental congruence, denoted \( \mathcal{N}_S \), on \( S \) is given as follows. Let \( a, b \) be any two elements of \( S \). We write \( a \equiv_{\mathcal{N}_S} b \) to mean that \( a^m = b \ast c \) for some \( c \in S \) and some integer \( m > 0 \). If \( a \equiv_{\mathcal{N}_S} b \) and \( b \equiv_{\mathcal{N}_S} a \), we write \( a \mathcal{N}_S b \). We call the commutative semigroup \( S \) an archimedean semigroup provide that \( a \mathcal{N}_S b \) for any two elements \( a, b \) of \( S \).

**Lemma 2.1.** ([13], Chapter III, Theorem 1.2) Let \( S \) be a commutative semigroup. Then \( \mathcal{N}_S \) is the smallest semilattice congruence on \( S \). In particular, \( S/\mathcal{N}_S \) is a semilattice and every \( \mathcal{N}_S \)-class \( A(x) \) is archimedean, where \( x \) denotes an arbitrary element of \( S \) and \( A(x) \) denotes the archimedean component of \( x \) (the \( \mathcal{N}_S \)-class to which \( x \) belongs).

The semilattice \( Y(S) = S/\mathcal{N}_S \) is called the **universal semilattice** of \( S \). Furthermore, there exists a partition \( S = \bigcup_{y \in Y(S)} S_y \) into subsemigroups \( S_y \) (one for every \( y \in Y(S) \)) with respect to the universal semilattice \( Y(S) \), in particular, \( S_{y_1} \ast S_{y_2} \subseteq S_{y_1 \wedge y_2} \) for all \( y_1, y_2 \in Y(S) \), and each component \( S_y \) is archimedean. The following lemma to characterize the structure of any finite commutative archimedean semigroup will be useful for the proof later.

**Lemma 2.2.** ([13], Chapter I, Proposition 3.6, Proposition 3.7, Proposition 3.8, and Chapter III, Proposition 3.1) A finite commutative semigroup \( S \) is archimedean if and only if it is an ideal extension of a finite abelian group \( G \) by a finite commutative nilsemigroup \( N \). Moreover, the partial homomorphism \( \varphi^N_G : N \setminus \{0_N\} \to G \) to construct the ideal extension of the group \( G \) by the nilsemigroup is given by

\[
\varphi^N_G : a \mapsto a \ast e_G
\]

where \( a \) denotes an arbitrary element \( N \setminus \{0_N\} = S \setminus G \) and \( e_G \) denotes the identity element of the subgroup \( G \).

A commutative nilsemigroup \( S \) is a commutative semigroup with a zero element \( 0_S \) in which every element \( x \) is nilpotent, i.e., \( x^n = 0_S \) for some \( n > 0 \). The following lemma (see [12], Chapter IV, p127) on finite commutative nilsemigroup will be useful for our arguments. For the readers’ convenience, we propose its one-line proof here.
Lemma 2.3. Let $N$ be a finite commutative nilsemigroup, and let $a, b$ be two elements of $N$. If $a * b = a$ then $a = 0_N$.

Proof. We see that $a = a * b = a * b^2 = \cdots = a * b^n = a * 0_N = 0_N$ for some $n \in \mathbb{N}$, done. □

The following lemma on finite cyclic semigroups will be crucial for the proof of the main result in this paper.

Lemma 2.4. ([13], Chapter I, Lemma 5.7, Proposition 5.8, Corollary 5.9) Let $S = \langle x \rangle$ be a finite cyclic semigroup generated by $x$. Then every element $s \in S$ can be written uniquely in the form $x^i$ with $i \in [1, I(x) + \mathcal{P}(x) - 1]$. In particular,

$$S = \{x, x^2, \ldots, x^{I(x)}, x^{I(x)+1}, \ldots, x^{I(x)+\mathcal{P}(x)-1}\}$$

with

$$x^i * x^j = \begin{cases} x^{i+j}, & \text{if } i + j \leq I(x) + \mathcal{P}(x) - 1; \\ x^k, & \text{if } i + j \geq I(x) + \mathcal{P}(x), \text{ where} \\ I(x) \leq k \leq I(x) + \mathcal{P}(x) - 1 \text{ and } k \equiv i + j \pmod{\mathcal{P}(x)}. \end{cases}$$

Moreover,

(i) there exists a unique idempotent, $x^\ell$, in the cyclic semigroup $\langle x \rangle$, where

$$\ell \in [I(x), I(x) + \mathcal{P}(x) - 1] \text{ and } \ell \equiv 0 \pmod{\mathcal{P}(x)};$$

(ii) $\{x^{I(x)}, x^{I(x)+1}, \ldots, x^{I(x)+\mathcal{P}(x)-1}\}$ is a cyclic subgroup of $S$ isomorphic to the additive group $\mathbb{Z}_{\mathcal{P}(x)}$ of integers modulo $\mathcal{P}(x)$.

It is worth remarking that both finite cyclic groups and finite cyclic nilsemigroups are finite cyclic semigroups. In fact, in case of $I(x) = 1$ or $\mathcal{P}(x) = 1$, the above finite cyclic semigroup $\langle x \rangle$ will be a finite cyclic group or a finite cyclic nilsemigroup, respectively.
3 The structure of the extremal sequence

To characterize the structure of the extremal sequence, the following notations will be useful.

Let $T = x_1 x_2 \cdots x_n \in \mathcal{F}(S)$ be a sequence. By $\sum(T)$ we denote the set of all the elements of $S$ that can be represented to be a product of some terms from $T$, i.e.,

$$\sum(T) = \{ s \in S : s = x_{\sigma(1)} * x_{\sigma(2)} * \cdots * x_{\sigma(t)}, \text{ and } 1 \leq t \leq n \}$$

where $\sigma$ takes every permutation of $\{1, 2, \ldots, n\}$. For any element $x$ of $S$, we define

$$\lambda_T(x) = | \sum(T \cdot x) \setminus \sum(T) |.
$$

We call a nonempty sequence $T \in \mathcal{F}(S)$ **idempotent-free** if $\sum(T) \cap E(S) = \emptyset$. Then we have the following property on any idempotent-free sequence.

**Lemma 3.1.** Let $T$ be an idempotent-free sequence, and let $x$ be a term of $T$. Then

$$\lambda_T(x) - 1 \geq 1.$$

**Proof.** By Conclusion (i) of Lemma 2.4, we derive that $\langle x \rangle \not\subseteq \sum(T)$, and thus, $\langle x \rangle \not\subseteq \sum(T x^{(-1)})$. Let $k$ be the least positive integer such that $x^k \not\subseteq \sum(T x^{(-1)})$. Notice $k \leq \bar{I}(x) + \mathcal{P}(x) - 1$. If $k = 1$, i.e., $x \not\subseteq \sum(T x^{(-1)})$, then $x \in \sum(T) \setminus \sum(T x^{(-1)})$ which implies $\lambda_T(x) \geq 1$, done. Hence, we assume $k > 1$. Then there exists a subsequence, say $x_1 x_2 \cdots x_m$, of $T x^{(-1)}$ with $x_1 * x_2 * \cdots * x_m = x^{k-1}$. It follows that $x^k = x^{k-1} * x = x_1 * x_2 * \cdots * x_m \cdot x \in \sum(T)$, which implies $\lambda_T(x) \geq 1$. This completes the proof. \hfill $\square$

Now we are in a position to state the main result of this paper.

**Theorem 3.2.** Let $T \in \mathcal{F}(S)$ be a sequence with length $|T| = |S| - |E(S)|$ and $\sum(T) \cap E(S) = \emptyset$. Let $R = \langle \text{supp}(T) \rangle$. Then $R$ is commutative with $S \setminus R \subseteq E(S)$ and the universal semilattice $Y(R)$ is a chain such that $x_1 * x_2 = x_1$ for any elements $x_1, x_2 \in R$ with $x_1 \preceq_{R, x_2}$. Moreover,

(i) each archimedean component of $R$ is, either a finite cyclic semigroup $\langle x \rangle$ with $x \in \text{supp}(T)$ and $I(x) \equiv 1 \pmod{\mathcal{P}(x)}$, or an ideal extension of a nontrivial finite cyclic group $\langle x_2 \rangle$ by a
nontrivial finite cyclic nilsemigroup \( \langle x_1 \rangle \) with \( x_1, x_2 \in \text{supp}(T) \) and the partial homomorphism \( \varphi_{(x_2)}^{(x_1)} \) being trivial, i.e., \( \varphi_{(x_2)}^{(x_1)}(x_1) = e_{(x_2)} \) where \( e_{(x_2)} \) denotes the identity element of the subgroup \( \langle x_2 \rangle \).

(ii) \( v_x(T) = \overline{I}(x) + \mathcal{P}(x) - 2 \) for each element \( x \in \text{supp}(T) \).

Proof of Theorem 3.2. Let

\[
m = |T| = |S| - |E(S)|
\]

and

\[
T = \prod_{i=1}^{m} a_i,
\]

and let \( \tau \) denote an arbitrary permutation of \( \{1, 2, \ldots, m\} \). Take

\[
T_{\tau}^k = \prod_{i=1}^{k} a_{\tau(i)}
\]

for each \( k \in [1, m] \). Since \( \sum(T_{\tau}^k) \cap E(S) = \emptyset \) for all \( k \in [1, m] \), it follows from Lemma 3.1 that

\[
|T| = |S| - |E(S)|
\]

\[
\geq |\sum(T)| = |\sum(T_{\tau}^{m-1})| + \lambda_{T_{\tau}^{m-1}}(a_{\tau(m)})
\]

\[
\geq |\sum(T_{\tau}^{m-1})| + 1 = |\sum(T_{\tau}^{m-2})| + \lambda_{T_{\tau}^{m-2}}(a_{\tau(m-1)}) + 1
\]

\[
\geq |\sum(T_{\tau}^{m-2})| + 2
\]

\[
\vdash \geq |\sum(T_{\tau})| + m - 1 = m = |T|.
\]

It follows that

\[
|\sum(T_{\tau})| = k \quad (1)
\]

for each \( k \in [1, m] \), and that

\[
\sum(T) = S \setminus E(S). \quad (2)
\]

Then we have the following.

Claim A. If \( a, b \) are two distinct elements of \( \text{supp}(T) \), then \( a \cdot b = b \cdot a \in \{a, b\} \).

Proof of Claim A. By (1) and the arbitrariness of \( \tau \), we have that \( |\sum(a \cdot b)| = 2 \), which implies

\[
a \cdot b, \quad b \cdot a \in \{a, b\}.
\]
Suppose to the contrary without loss of generality that \(a * b \neq b * a\) with

\[
a * b = b
\]

and

\[
b * a = a.
\]

It follows that \(a * a = a * (b * a) = (a * b) * a = b * a = a\), and so \(a\) is idempotent, which is absurd. This proves Claim A.

By Claim A, then \(\mathcal{R} = \langle \text{supp}(T) \rangle\) is commutative. Moreover, we have the following.

**Claim B.**

\[
\mathcal{R} = \bigcup_{a \in \text{supp}(T)} \langle a \rangle.
\]

In particular, for any \(x \in \Sigma(T)\), there exists an element \(a \in \text{supp}(T)\) such that \(x = a^k\) for some integer \(k \in [1, v_a(T)]\).

**Proof of Claim B.** Take an arbitrary element \(x\) of \(\mathcal{R}\). There exists some distinct elements of \(\text{supp}(T)\), say \(x_1, x_2, \ldots, x_\ell\), such that

\[
x = x_1^{n_1} * x_2^{n_2} * \cdots * x_\ell^{n_\ell},
\]

where \(\ell > 0\) and \(n_1, n_2, \ldots, n_\ell > 0\). By applying Claim A, we conclude that \(x = x_t^{n_t}\) for some \(t \in [1, \ell]\). In particular, if \(x \in \Sigma(T)\), we can take all the integers \(n_1, n_2, \ldots, n_\ell\) with

\[
n_i \in [1, v_{x_i}(T)]
\]

for every \(i \in \{1, 2, \ldots, \ell\}\). This proves Claim B.

**Claim C.** For any \(a \in \text{supp}(T)\) and any integer \(k \in [1, I(a) + P(a) - 1]\) such that \(a^k \in \Sigma(T)\),

\[
v_a(T) \geq k.
\]

**Proof of Claim C.** By Claim B, we have that

\[
a^k = b'
\]
for some $b \in \text{supp}(T)$ with

$$t \in [1, v_b(T)].$$

Suppose

$$b \neq a.$$ 

It follows from Claim A that $a^k \ast b' = b' = a^k$, which implies that $a^k$ is idempotent, a contradiction. Hence, $b = a$ and $v_a(T) = v_b(T) \geq t \geq k$. This proves Claim C. \qed

Let $g$ and $h$ be two arbitrary elements of $R$ which belong to two distinct archimedean components of $R$. By Claim B, we have $g = a^k$ and $h = b'$ where $a, b$ are distinct elements of $\text{supp}(T)$ and $k, t > 0$. It follows from Claim A that

$$g \ast h = a^k \ast b' = a^k = g$$

or

$$g \ast h = a^k \ast b' = b' = h$$

which implies

$$g \preceq_{N_R} h$$

or

$$h \preceq_{N_R} g.$$ 

Since $N_R$ is a congruence on $R$, by the arbitrariness of $g$ and $h$, we conclude that the universal semilattice $Y(R) = R / N_R$ is a chain and $g \ast h = g$ for any elements $g, h \in R$ with $g \preceq_{N_R} h$.

Let $a$ be an arbitrary element of $\text{supp}(T)$. By (2), we have that all the elements except for the unique idempotent of $\langle a \rangle$ must belong to $\sum(T)$. Combined with Lemma [2,4] and Claim C, we conclude that

$$v_a(T) = I(a) + P(a) - 2,$$

and that the unique idempotent in the cyclic semigroup $\langle a \rangle$ is $a^{I(a) + P(a) - 1}$ which implies

$$I(a) + P(a) - 1 \equiv 0 \pmod{P(a)},$$

equivalently,

$$I(a) \equiv 1 \pmod{P(a)}.$$ (4)
By (3), we have Conclusion (ii) proved. Now it remains to show Conclusion (i).

Let $A_y (y \in Y(R))$ be an arbitrary archimedean component of $R$. Since $x N_R x'$ for any element $x \in R$ and any integer $t > 0$, by Claim B, we conclude that $A_y$ is an union of several cyclic subsemigroups generated by the elements of $\text{supp}(T)$, i.e.,

$$A_y = \bigcup_{i=1}^{k_y} \langle x_i \rangle,$$

(5)

where $k_y \geq 1$ and $x_1, x_2, \ldots, x_{k_y}$ are distinct elements of $\text{supp}(T)$. By Lemma 2.2, we may assume that $A_y$ is an ideal extension of a group $G_y$ by a nilsemigroup $N_y$ (note that $G_y$ or $N_y$ may be trivial which shall be reduced to the case that $A_y$ is a nilsemigroup or a group). Now we show that

$$|G_y \cap \text{supp}(T)| \leq 1 \quad (6)$$

and

$$|(A_y \setminus G_y) \cap \text{supp}(T)| \leq 1. \quad (7)$$

Suppose $a, b$ are two distinct elements of $A_y \cap \text{supp}(T)$. Recalling Claim A, we see $a \ast b \in \{a, b\}$. If $a, b \in G_y$, then $a$ or $b$ is the identity element of the group $G_y$, a contradiction. If $a, b \in A_y \setminus G_y = N_y \setminus \{0_{N_y}\}$, by Lemma 2.3 we derive a contradiction. This proves (6) and (7).

By (6) and (7), we have that

$$k_y \in \{1, 2\}$$

in (5).

Consider the case of $k_y = 1$, i.e., $A_y = \langle x \rangle$ for some $x \in \text{supp}(T)$. By (3), we have Conclusion (i) proved.

Consider the case of $k_y = 2$, i.e., $A_y = \langle x_1 \rangle \cup \langle x_2 \rangle$ where $x_1$ and $x_2$ are distinct elements of $\text{supp}(T)$. By (6) and (7), we may assume without loss of generality that $x_2 \in G_y$ and $x_1 \in A_y \setminus G_y = N_y \setminus \{0_{N_y}\}$. Combined with Claim A, we see $x_1 \ast x_2 = x_2$. Then we conclude that the partial homomorphism $\varphi_{\langle x_1 \rangle}^{\langle x_2 \rangle}$ is trivial, and $G_y = \langle x_2 \rangle$ and $N_y = \langle x_1 \rangle$, and so Conclusion (i) holds.

This completes the proof of Theorem 3.2 □
Remark 3.3. It is easy to notice that the sequence given in Theorem 3.2 is also an extremal sequence, i.e., the condition given in Theorem 3.2 is in fact sufficient too.

In the final part of this section, we give an equivalent form of Theorem 3.2 as follows. The equivalence of both theorems can be easily verified, which is left to the readers.

**Theorem 3.4.** Let $T \in F(S)$ be a sequence with length $|T| = |S| - |E(S)|$ and $\sum(T) \cap E(S) = \emptyset$. Let $R = \langle \text{supp}(T) \rangle$. Then $R$ is commutative such that $S \setminus R \subseteq E(S)$ and

\[ R = \bigcup_{i=1}^{k} \langle x_i \rangle \]

such that supp$(T) = \{x_1, x_2, \ldots, x_k\}$ with $x_i \ast x_j = x_j$ and $(x_i) \cap (x_j) = \emptyset$ for all $1 \leq i < j \leq k$, where $(x)^{\circ}$ denotes the subset of all non-idempotent elements in the finite cyclic semigroup $(x)^{\circ}$. Moreover, $I(x_i) \equiv 1 \pmod{\mathcal{P}(x_i)}$ and $v_{x_i}(T) = I(x_i) + \mathcal{P}(x_i) - 2$ for every $i \in \{1, 2, \ldots, k\}$.

## 4 Concluding remarks

The value $|S| - |E(S)| + 1$ to ensure that a sequence $T$ is not idempotent-free is best possible, which has been shown in [11], in the sense that $S$ is a general finite semigroup. However, this value may be no longer best possible for a particular kind of finite semigroups. Precisely, we give the following.

**Definition 4.1.** Let $S$ be a finite semigroup. Define $I(S)$ to be the least positive integer $m$ such that every sequence $T$ of elements in $S$ with length at least $m$ is not idempotent-free, i.e., $T$ contains a nonempty subsequence whose product is idempotent. The constant $I(S)$ is called the **Erdős-Burgess constant** of the finite semigroup $S$.

From the result given in [11] and Theorem 3.2 of this paper, we see that

\[ I(S) \leq |S| - |E(S)| + 1 \]

and the equality holds if and only if the structure of $S$ is given as in Theorem 3.2. Therefore, the following problem would be very interesting.
Problem 1. Determine the values of the Erdős-Burgess constant $I(S)$ for finite semigroups $S$.

Coincidentally, the Erdős-Burgess constant seems to be closely related to a classical combinatorial constant, the **Davenport constant**, in Zero-sum Theory which was originated from the work of Davenport [3], Erdős [5] together with Ginzburg and Ziv. Davenport constant is the most important constant in Zero-sum Theory which has been extensively investigated for abelian groups since the 1960s (see [4, 6–8, 10, 14–17]), and recently was also studied for finite commutative semigroups (see [1, 18], and P. 110 in [9]). For the readers’ convenience, we state the definition of Davenport constant for finite commutative semigroups below.

**Definition 4.2.** ([18]) Let $S$ be a finite commutative semigroup. Define $D(S)$ to be the least positive integer $\ell$ such that every sequence $T$ of elements in $S$ of length $|T| \geq \ell$ contains a proper subsequence $T'$ ($T' \neq T$) whose product is equal to the product of all terms in $T$.

It is easy to see that for the case that $S$ is a finite abelian group, both constants really mean the same thing, i.e., $I(S) = D(S)$. While, for the case that the finite commutative semigroup $S$ is not a group, both $I(S) < D(S)$ and $I(S) > D(S)$ could happen which can be noticed from the following example. The verifications of it will be left to the readers.

**Example.** Let $S = \langle x_1 \rangle \cup \langle x_2 \rangle$ where $\langle x_1 \rangle$ is a finite cyclic group and $\langle x_2 \rangle$ is a finite cyclic nilsemigroup with $x_1 \ast x_2 = x_2$ and $|\langle x_1 \rangle| = n_1$ and $|\langle x_2 \rangle| = n_2$. Then we obtain that $I(S) = (n_1 - 1) + (n_2 - 1) + 1$ and $D(S) = \max(n_1, n_2 + 1)$. By taking proper $n_1, n_2$, we have that both $I(S) < D(S)$ and $I(S) > D(S)$ could happen.

Therefore, we close this paper by proposing the following problem.

**Problem 2.** Let $S$ be a finite commutative semigroup. Does there exist any relation between the Erdős-Burgess constant $I(S)$ and the Davenport constant $D(S)$?

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