Self-Adjointness of the Dirac Hamiltonian and Vacuum Quantum Numbers Induced by a Singular External Field

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Effects of fermion-vacuum polarization by a singular configuration of an external static vector field are considered in \( (2 + 1) \)-dimensional spacetime. Expressions for the induced vacuum charge and magnetic flux are obtained.

1 Introduction

Effects of singular external fields (zero-range potentials) are comprehensively studied in quantum mechanics (see [1] and references therein). In this study, we consider the effect of singular external fields on the fermion vacuum in quantum field theory. In contrast to the Schrodinger operator, the Dirac operator may be free of an explicit delta-function singularity; nonetheless, the problem of self-adjoint extension arises in both cases, albeit for different reasons (see, for example [2]). It is well known that a singular magnetic-mono pole external field leads to a \( \Theta \) vacuum violating CP symmetry [3, 4, 5, 6, 7]. It will be shown below (see also [8]) that a singular magnetic-string external field leads to a \( \Theta \) vacuum violating C symmetry.

Our analysis will be based on representing the second-quantized fermion-field operator in the form

\[
\Psi(x, t) = \sum_{E>0} e^{-iEt} \psi_E(x) a_E + \sum_{E<0} e^{-iEt} \psi_E(x) b_E^\dagger.
\]

(1)

Here, the symbol \( \sum_{E} \) denotes summation over the discrete spectrum of the energy \( E \) and integration (with some measure) over its continuum spectrum; \( a_E^\dagger \) and \( a_E \) \( (b_E^\dagger \) and \( b_E \) \) are the fermion (antifermion) creation and annihilation operators satisfying anticommutation relations; and \( \psi_E(x) \) is a solution to the equation

\[
H \psi_E(x) = E \psi_E(x),
\]

(2)

where

\[
H = -i\alpha \left[ \frac{\partial}{\partial x} - iV(x) \right] + \beta m
\]

(3)

is the Dirac Hamiltonian in a static external vector field \( V(x) \). If the condition

\[
\int_X [\tilde{\psi}^\dagger (H \psi) - (H^\dagger \tilde{\psi})^\dagger \psi] d\Omega = 0,
\]

(4)

where \( d\Omega \) is the volume element of a spatial region \( X \), is met, the Hamiltonian \( H \) is a Hermitian (symmetric) operator acting in the space of functions defined on \( X \). If, in addition, the spaces of the functions \( \psi \) and \( \tilde{\psi} \) coincide, the Hamiltonian \( H \) is a self-adjoint operator.
The integral on the left-hand side of (4) can be reduced to an integral over the surface \( \partial X \) bounding the spatial region \( X \). As a result, relation (4) takes the form

\[
- i \int_{\partial X} \tilde{\psi}^\dagger \alpha \psi \cdot d\sigma = 0,
\]

(5)

where \( d\sigma \) is an oriented element of the surface \( \partial X \), the normal to this element being directed outside the region \( X \).

A standard procedure leads to the expressions for the vacuum-charge density,

\[
\rho(x) = -\frac{1}{2} \sum_E \text{sgn}(E) \psi^\dagger_E(x) \psi(x),
\]

(6)

and for the vacuum current,

\[
j(x) = -\frac{1}{2} \sum_E \text{sgn}(E) \psi^\dagger_E(x) \alpha \psi(x),
\]

(7)

where

\[
\text{sgn}(u) = \begin{cases} 
1, & u > 0 \\
-1, & u < 0
\end{cases}.
\]

Performing integration in (6) over the spatial region \( X \), we obtain the total vacuum charge

\[
Q^{(l)} = \int_X \rho \, d\Omega.
\]

(8)

As to a global quantity associated with vacuum current (7), it follows from the Maxwell equation

\[
\frac{1}{e^2} \frac{\partial}{\partial \mathbf{x}} \times \mathbf{B}^{(l)}(x) = \mathbf{j}(x)
\]

(9)

(\( e \) is the coupling constant) that a magnetic field is induced in the vacuum. This magnetic field is characterized by the field strength

\[
\mathbf{B}^{(l)}(x) = e^2 \int_{x(\infty)}^x \mathbf{j}(x) \times dx \quad (\mathbf{j}(x(\infty)) = \mathbf{B}^{(l)}(x(\infty)) = 0)
\]

(10)

and by the total flux (in \( 2\pi \) units)

\[
\Phi^{(l)} = \frac{1}{2\pi} \int \mathbf{B}^{(l)} \cdot d\sigma,
\]

(11)

where \( d\sigma \) is an oriented element of the surface orthogonal to the lines of force of the magnetic field in the spatial region \( X \).

In this study, we consider a second-quantized fermion field in an external field generated by a source in the form of a singular magnetic string. If we direct the coordinate \( x^3 \) axis along the string, the strength of the string magnetic field is given by

\[
B^3(x) = 2\pi \Phi^{(0)} \delta(x).
\]

(12)

By \( x \), we will henceforth mean a two-dimensional vector in the plane orthogonal to the string axis \( [x = (x^1, x^2)] \); the parameter \( \Phi^{(0)} \) is the total magnetic flux (in \( 2\pi \) units) of a string. It is natural to choose the gauge

\[
V_3 = 0.
\]

(13)
The two-dimensional vector potential \( \mathbf{V} = (V_1, V_2) \) can then be defined as
\[
x \cdot \mathbf{V}(x) = 0, \quad x \times \mathbf{V}(x) = \Phi(0).
\] (14)

That the potential \( \mathbf{V}(x) \) is indeterminate at the point \( x = 0 \) of the plane is associated with the delta-function singularity in the strength \( B^3 \) (12) at this point. On the plane orthogonal to the string axis, the Dirac Hamiltonian has the form
\[
H = -i \alpha^r \partial_r - ir^{-1} \alpha^\varphi (\partial_\varphi - i \Phi(0)) + \beta m,
\] (15)
where
\[
\alpha^r = \alpha^1 \cos \varphi + \alpha^2 \sin \varphi, \quad \alpha^\varphi = -\alpha^1 \sin \varphi + \alpha^2 \cos \varphi,
\] (16)
and where we introduced the polar coordinates
\[
r = [(x^1)^2 + (x^2)^2]^\frac{1}{2}, \quad \varphi = \arctan\left(\frac{x^2}{x^1}\right).
\]

This article is organized as follows. In Section 2, we determine the complete system of solutions to the Dirac equation in the field of a singular magnetic string. In Section 3, we consider the vacuum charge induced on the plane orthogonal to the string axis. In Section 4, we analyze the vacuum magnetic flux through this plane. The results obtained in this study are discussed in Section 5. Some technical details concerning the derivation of basic relations are described in Appendices A and B.

2 Solving the Dirac equation in the field of a singular magnetic string

It is well known that, in \((2 + 1)\)-dimensional space-time, the Clifford algebra does not have a faithful irreducible representation; instead, it has two nonequivalent irreducible representations. Accordingly, the matrices \( \alpha^1, \alpha^2 \) and \( \beta \) admit the following realizations in terms of square rank-two matrices:
\[
\alpha^1 = -\sigma^2, \quad \beta = \sigma^3, \quad \alpha^2 = s\sigma^1, \quad s = \pm 1.
\] (17)
Here, \( \sigma^1, \sigma^2 \) and \( \sigma^3 \) are the Pauli matrices, and the two possible values of the parameter \( s \) correspond to the two nonequivalent representations.

A solution to the time-independent Dirac equation (2) with the Hamiltonian \( H \) in the form (15) is given by
\[
\psi_E(x) = \sum_{n \in \mathbb{Z}} < x|E, n > ,
\] (18)
where
\[
< x|E, n > = \begin{pmatrix} f_n(r, E)e^{in\varphi} \\ g_n(r, E)e^{i(n+s)\varphi} \end{pmatrix},
\] (19)
and \( \mathbb{Z} \) is the set of integers. The radial wave functions \( f_n \) and \( g_n \) satisfy the system of equations
\[
(-\partial_r + r^{-1}s\lambda)f_n = (E + m)g_n, \quad [\partial_r + r^{-1}(s\lambda + 1)]g_n = (E - m)f_n,
\] (20)
where \( \lambda = n - \Phi^{(0)} \). In the case of \( \Phi^{(0)} \neq n' \) (where \( n' \in \mathbb{Z} \)), two linearly independent solutions to (20) that correspond to the continuous spectrum, \(|E| > |m|\), \( \{ | } \) can be represented as
\[
\begin{pmatrix}
  f_n^{(\pm)}(r, E) \\
  g_n^{(\pm)}(r, E)
\end{pmatrix} = \begin{pmatrix}
  f^{(0)}(E)J_{\pm\lambda}(kr) \\
  \pm g^{(0)}(E)J_{\pm(s\lambda+1)}(kr)
\end{pmatrix},
\]
(21)
where \( k = \sqrt{E^2 - m^2} \), and \( J_\mu(z) \) is the Bessel function of order \( \mu \). It can be seen from (21) that a solution that is regular at the point \( r = 0 \) can be chosen for all modes with the exception of that which corresponds to \( n = n_0 \), where \( n_0 \) is determined from the condition
\[
-1 < s\lambda_0 < 0.
\]
(22)
For this mode, either the upper or the lower component of the spinor in (19) – depending on the choice of the plus or minus sign in (21) – proves to be irregular at \( r = 0 \), although it is square-integrable. It is also obvious that, for \( \Phi^{(0)} = n' \) (where \( n' \in \mathbb{Z} \)), a solution that is regular at the point \( r = 0 \) can be chosen for all modes. Let us introduce the quantity
\[
F = \frac{1}{2} + s \{ \Phi^{(0)} \} \frac{1}{2}.
\]
(23)
Here, \( \{ u \} \) stands for the fractional part of the quantity \( u \) – that is \( \{ u \} = u - [u] \) \((0 \leq \{ u \} < 1)\), where \([u]\) is the integral part of \( u \) (the closest integer to \( u \) from below or the integer equal to \( u \) if it is integral itself). Taking into account the relations
\[
n_0 = [\Phi^{(0)}] + \frac{1}{2} - \frac{1}{2} s, \quad s\lambda_0 = -F,
\]
(24)
and the condition of orthonormality for states of the continuous spectrum for \( \text{sgn}(E) = \text{sgn}(E') \),
\[
\int d^2x < E, n|x > < x|E', n' > = \frac{\delta(k - k')}{\sqrt{kk'}}\delta_{mm'},
\]
(25)
we obtain
\[
\begin{pmatrix}
  f_n \\
  g_n
\end{pmatrix} = \frac{1}{2\sqrt{\pi}} \begin{pmatrix}
  \sqrt{1 + mE^{-1}}J_{l-F}(kr) \\
  \text{sgn}(E)\sqrt{1 - mE^{-1}}J_{l+1-F}(kr)
\end{pmatrix}, \quad l = s\left(n - [\Phi^{(0)}] - \frac{1}{2}\right) + \frac{1}{2},
\]
(26)
for the regular modes with \( s\lambda > s\lambda_0(l \geq 1) \);
\[
\begin{pmatrix}
  f_n \\
  g_n
\end{pmatrix} = \frac{1}{2\sqrt{\pi}} \begin{pmatrix}
  \sqrt{1 + mE^{-1}}J_{l+F}(kr) \\
  -\text{sgn}(E)\sqrt{1 - mE^{-1}}J_{l-1+F}(kr)
\end{pmatrix}, \quad l' = -s\left(n - [\Phi^{(0)}] - \frac{1}{2}\right) - \frac{1}{2},
\]
(27)
for the regular modes with \( s\lambda < s\lambda_0(l' \geq 1) \); and
\[
\begin{pmatrix}
  f_{n_0} \\
  g_{n_0}
\end{pmatrix} = \frac{1}{2\sqrt{\pi}(1 + \sin 2\nu \cos F\pi)} \begin{pmatrix}
  \sqrt{1 + mE^{-1}}[\sin \nu J_{l-F}(kr) + \cos \nu J_F(kr)] \\
  \text{sgn}(E)\sqrt{1 - mE^{-1}}[\sin \nu J_{l-1-F}(kr) - \cos \nu J_{1-F}(kr)]
\end{pmatrix},
\]
(28)
\footnote{\( 1 \) It should be noted that, in \((2 + 1)\)-dimensional spacetime, as well as in any spacetime having an odd number of dimensions, the parameter \( m \) appearing in expression (3) for the Hamiltonian can take both positive and negative values.}
for the irregular mode ($s \lambda = s \lambda_0$). The parameter $\nu$ is determined by the requirement that Hamiltonian (15) be a self-adjoint operator. We are now going to consider this issue in some detail.

In the case being considered, the condition requiring that Hamiltonian be a Hermitian operator (5) takes the form

$$2\pi r \sum_{n \in \mathbb{Z}} (\tilde{f}_n g_n - \tilde{g}_n f_n) \bigg|_{r=\infty} = 0.$$  (29)

If the modes corresponding to $n \neq n_0$ are subjected to the regularity condition at $r = 0$, Hamiltonian (15) as defined on the space of these modes becomes a self-adjoint operator. The mode with $n = n_0$ cannot be subjected to the regularity condition, because we would then be obliged to discard solution (28), thereby spoiling the completeness of the set of solutions to the Dirac equation. Thus, there arises the problem of determining the boundary condition for the irregular mode at $r = 0$ – in other words, the problem of the self-adjoint extension of a Hermitian operator (precisely for this mode). This problem is solved with the aid of the Weyl-von Neumann theory of self-adjoint operators (see, for example, [1, 9]). Since the defect index of the operator defined on the space of regular functions is $(1, 1)$ in the case being considered, the self-adjoint extension represents a set of operators that is parametrized with the aid of one real continuous variable ($\Theta$), and the required boundary condition for the irregular mode has the form

$$\lim_{r \to 0} (|m|r)^F \cos \left( \frac{s}{2} + \frac{\pi}{4} \right) f_{n_0} = -\text{sgn}(m) \lim_{r \to 0} (|m|r)^{1-F} \sin \left( \frac{s}{2} + \frac{\pi}{4} \right) g_{n_0}.$$  (30)

Substituting the asymptotic form of solution (28) for $r \to 0$ into (30), we arrive at

$$\tan \nu = \text{sgn}(m E) \sqrt{E - m} \left( \frac{k}{|m|} \right)^{2F-1} A(F, \Theta),$$  (31)

where

$$A(F, \Theta) = 2^{1-2F} \frac{\Gamma(1-F)}{\Gamma(F)} \tan \left( \frac{s}{2} + \frac{\pi}{4} \right),$$  (32)

and $\Gamma(z)$ is the Euler gamma function. Formulas (31) and (32) establish the relation between the parameters $\nu$ and $\Theta$. The boundary condition (30) results in that the spectrum involves not only a continuum but also the bound state

$$\psi_{BS}(x) = \frac{\kappa}{\pi} \sqrt{\frac{\sin F \pi}{1 + (2F-1)m^{-1}E_{BS}}} \left( \frac{\sqrt{1 + m^{-1}E_{BS}K_F(kr)} e^{in_0\varphi}}{\text{sgn}(m) \sqrt{1 - m^{-1}E_{BS}K_{1-F}(kr)} e^{i(n_0+s)\varphi}} \right),$$  (33)

where $\kappa = \sqrt{m^2 - E_{BS}^2}$, $K_\mu(z)$ is the Macdonald function of order $\mu$, and the bound-state energy $E_{BS}$ ($|E_{BS}| < |m|$) is determined as a real-valued root to the algebraic equation

$$\sqrt{m + E_{BS}} \left( \frac{\kappa}{|m|} \right)^{1-2F} = -A(F, \Theta).$$  (34)

---

2) For $s = 1$ and $m > 0$, this boundary condition was obtained in [10].
It is obvious that there is no bound state for

\[ 0 < A(F, \Theta) < \infty \]  

and that there arises a bound state for

\[ -\infty < A(F, \Theta) < 0. \]  

The bound-state energy is zero, \( E_{BS} = 0 \), at

\[ A(F, \Theta) = -1. \]  

We also have

\[
\text{sgn}(E_{BS}) = \text{sgn}(m), \quad -\infty < A(F, \Theta) < -1, \\
\text{sgn}(E_{BS}) = -\text{sgn}(m), \quad -1 < A(F, \Theta) < 0.
\]

Thus, we have constructed the complete system of solutions to the Dirac equation in the field of a singular magnetic string. It should be noted that for the case in which \( s = 1, m > 0 \) and \( E > 0 \), the results presented in this section were first obtained in [10].

### 3 Induced vacuum charge

By using the explicit form of solutions to the Dirac equation, we can find the vacuum-charge density averaged over all directions. We have

\[
\bar{\rho}(r) = \frac{1}{2\pi} \int_0^{2\pi} d\phi \rho(x),
\]

where \( \rho(x) \) is determined by relation (6).

For the contribution of the regular modes (26) and (27) to the averaged vacuum charge \( \bar{\rho}(r) \) (40), we obtain

\[
\bar{\rho}_{\text{REG}}(r) = -\frac{1}{8\pi} \int_0^\infty dk \frac{k}{\sqrt{k^2 + m^2}} \sum_{l} \{ (E + m)[J_{l,F}(kr) + J_{l,F}(kr)] + \\
+ (E - m)[J_{l+1,F}(kr) + J_{l-1,F}(kr)] \},
\]

where we combined summation over \( l \) and \( l' \). In general, the correct procedure should involve introducing the regularizing factor \( |E|^{-t}(t > 0) \) in (41) and going over to the limit \( t \to 0^+ \) upon performing summation and integration. However, the final result remains unchanged if, instead of introducing a regularizing factor, we perform first summation over the sign of the energy and then integration with respect to \( k \), the integral of the contribution of each modes in (26) and (27) with respect to \( k \) being convergent and integration with respect to \( k \) being commutative with summation over \( l \). In our subsequent calculations, summation performed first over the sign of the energy and then over \( l \) is followed by integration with respect to \( k \).

Summation over the sign of energy and over \( l \) in (41) yields

\[
\bar{\rho}_{\text{REG}}(r) = -\frac{m}{4\pi} \int_0^\infty dk \frac{k}{\sqrt{k^2 + m^2}} [J_{l,F}(kr) - J_{l,F}(kr)].
\]
With the aid of the relation
\[ \frac{1}{\sqrt{k^2 + m^2}} = \frac{2}{\pi} \int_0^\infty du \frac{1}{k^2 + m^2 + u^2}, \]
we can perform integration with respect to \( k \). Following the substitution \( u = \sqrt{q^2 - m^2} \), we eventually obtain
\[ \tilde{\rho}_{\text{REG}}(r) = -\frac{m}{2\pi^2} \int_{|m|}^{\infty} dq \frac{q}{\sqrt{q^2 - m^2}} [I_{1-F}(qr)K_{1-F}(qr) - I_F(qr)K_F(qr)], \tag{43} \]
where \( I_\mu(z) \) is the modified Bessel function of order \( \mu \). The expression coincident with (43) is obtained by deforming the contour of integration in the complex plane (see Appendix A).

An alternative representation of the contribution of the regular modes to the averaged vacuum-charge density has the form
\[ \tilde{\rho}_{\text{REG}}(r) = -\frac{m^{-1}}{(2\pi)^{\frac{3}{2}}} \int_0^\infty ds \exp(-s^2 - \frac{m^2r^2}{2s^2}) [I_{1-F}(s^2) - I_F(s^2)], \tag{44} \]
which can be obtained by using the relation
\[ \frac{1}{\sqrt{k^2 + m^2}} = \frac{2}{\sqrt{\pi}} \int_0^\infty du \exp[-u^2(k^2 + m^2)]. \]

Taking into account (33)-(39), we find that the contribution of the bound state is
\[ \tilde{\rho}_{\text{BS}}(r) = \text{sgn}(m)\text{sgn}(A + 1)[1 - \text{sgn}(A)] \frac{\sin F\pi}{(2\pi)^{\frac{3}{2}} m + E_{BS}(2F - 1)} \times \]
\[ \times [(m + E_{BS})K_0^2(kr) + (m - E_{BS})K_1^2(kr)], \tag{45} \]
where the quantity \( A \) is given by (32).

Taking into account (28) and (31), we obtain the contribution of the irregular mode in the form
\[ \tilde{\rho}_{\text{IRREG}}(r) = -\frac{1}{8\pi} \int_0^\infty dk \frac{k}{\sqrt{k^2 + m^2}} \{Ak^{2F}m|m|^{-2F}[L_+ + L_-]J^2_{-F}(kr) + \]
\[ + Ak^{-2(1-F)}m|m|^{-2F}[(m - \sqrt{k^2 + m^2})^2 L_+ + (m + \sqrt{k^2 + m^2})^2 L_-]J^2_{1-F}(kr) + \]
\[ + 2[(m + \sqrt{k^2 + m^2})L_+ + (m - \sqrt{k^2 + m^2})L_-]J_F(kr)J_F(kr) + \]
\[ + 2[(m - \sqrt{k^2 + m^2})L_+ + (m + \sqrt{k^2 + m^2})L_-]J_{1-F}(kr)J_{1-F}(kr) + \]
\[ + A^{-1}k^{-2F}m^{-1}|m|^{2F}[(m + \sqrt{k^2 + m^2})^2 L_+ + (m - \sqrt{k^2 + m^2})^2 L_-]J^2_F(kr) + \]
\[ + A^{-1}k^{2(1-F)}m^{-1}|m|^{2F}[L_+ + L_-]J^2_{1+F}(kr) \}, \tag{46} \]
where summation over the sign of energy has been performed, and
\[ L_{(\pm)} = [Ak^{-2(1-F)}m|m|^{-2F}(-m \pm \sqrt{k^2 + m^2}) + 2\cos F\pi + A^{-1}k^{-2F}m^{-1}|m|^{2F}(m \pm \sqrt{k^2 + m^2})]^{-1}. \tag{47} \]

In Appendix A, it is shown how expression (46) can be transformed by deforming the contour in the complex plane to arrive at the final form
\[ \tilde{\rho}_{\text{IRREG}}(r) = \frac{m}{2\pi^2} \int_{|m|}^{\infty} dq \frac{q}{\sqrt{q^2 - m^2}} [I_{1-F}(qr)K_{1-F}(qr) - I_F(qr)K_F(qr)] - \frac{\sin F\pi}{\pi^3 m} \int_{|m|}^{\infty} dq \times \]
\[
\frac{q^3}{\sqrt{q^2 - m^2}} \left[ 1 + A \left( \frac{q}{|m|} \right)^{-2(1-F)} \right] K_F^2(qr) - \left[ 1 + A^{-1} \left( \frac{q}{|m|} \right)^{-2F} \right] K_{1-F}^2(qr) - \\
- \text{sgn}(m) \text{sgn}(A + 1) \left[ 1 - \text{sgn}(A) \right] \left( \frac{q}{2(\pi)^2} \right) \frac{m^2}{m + E_{BS}(2F - 1)} \times \\
\times [(m + E_{BS}) K_F^2(kr) + (m - E_{BS}) K_{1-F}^2(kr)].
\]

(48)

Summing (43), (45), and (48), we find that the averaged vacuum-charge density is given by

\[
\bar{\rho}(r) = -\frac{\sin F\pi}{\pi^3 m} \int_{|m|}^{\infty} dq \frac{q^3}{\sqrt{q^2 - m^2}} \times \\
\left[ 1 + A \left( \frac{q}{|m|} \right)^{-2(1-F)} \right] K_F^2(qr) - \left[ 1 + A^{-1} \left( \frac{q}{|m|} \right)^{-2F} \right] K_{1-F}^2(qr) \\
\times A \left( \frac{q}{|m|} \right)^{2F} + 2 + A^{-1} \left( \frac{q}{|m|} \right)^{2(1-F)}. \tag{49}
\]

This expression tends to zero in proportion to \(|m|^{3/2} r^{-3} \exp(-2|m|r)\) for \(r \to \infty\) and diverges in proportion to \(|m|r^{-1}\) for \(r \to 0\). In the case of \(\Theta \not\equiv \frac{\pi}{2} \text{(mod } \pi\)), integration of (49) over the entire plane yields the expression for the vacuum charge induced by a singular magnetic string. The resulting expression has the form

\[
Q^{(I)} = -\frac{\text{sgn}(m)}{2\pi} \int_1^\infty dv \frac{F[1 + Av^{-1+F}] - (1 - F)[1 + A^{-1}v^{-F}]}{Av^F + 2 + A^{-1}v^{1-F}}. \tag{50}
\]

It should be emphasized once again that relations (49) and (50) hold only in the case of nonintegral values of the string flux \((0 < F < 1)\). For integral values of the string flux \((F = 0)\), the density and the flux vanish because all square-integrable modes are then regular for \(r \to 0\), and the latter case does not differ in the least from the string-free case \((\Phi^{(0)} = 0)\).

Expression (50) can be reduced to the form

\[
Q^{(I)} = -\frac{1}{2} \text{sgn}(m) \left[ F + \frac{2}{\pi} \arctan \left( \frac{1 + Av^F}{\sqrt{v - 1}} \right) \right]_{v = 0}^{v = \infty}, \tag{51}
\]

whence we eventually obtain

\[
Q^{(I)} = \begin{cases} 
-\frac{1}{2} \text{sgn}(m)[F - \text{sgn}(A + 1)], & 0 < F < \frac{1}{2} \\
-\frac{1}{2} \text{sgn}(m) \arctan[\tan(\frac{\Theta}{2})], & F = \frac{1}{2} \\
\frac{1}{2} \text{sgn}(m)[1 - F - \text{sgn}(A^{-1} + 1)], & \frac{1}{2} < F < 1
\end{cases} \tag{52}

\]

For the case of \(F = \frac{1}{2}, s = 1, \text{ and } m > 0\), the last relation was obtained in \([11]\). We also have

\[
\lim_{F \to 0} Q^{(I)} = \frac{1}{2} \text{sgn}(m), \tag{53}
\]

\[
\lim_{F \to 1} Q^{(I)} = -\frac{1}{2} \text{sgn}(m). \tag{54}
\]
In deriving the last two relations, we also considered that

\[
\begin{align*}
\lim_{F \to 0} A &= 0, \quad \Theta \neq s \frac{\pi}{2} (\text{mod } 2\pi), \\
\lim_{F \to 1} A^{-1} &= 0, \quad \Theta \neq -s \frac{\pi}{2} (\text{mod } 2\pi).
\end{align*}
\]

(55)

From relations (53) and (54), it can be seen that the vacuum charge as a function of the string flux undergoes discontinuities at integral values of its argument.

This is confirmed in the case of \( \Theta = \frac{\pi}{2} (\text{mod } \pi) \) as well. From (49), we can easily obtain

\[
\tilde{\rho}(r) = \left\{ \begin{array}{ll}
-\frac{s}{\pi} m \int_{|m|}^{\infty} dq \frac{q}{\sqrt{q^2 - m^2}} K_F(qr), & \Theta = s \frac{\pi}{2} (\text{mod } 2\pi) \\
\frac{s}{\pi} m \int_{|m|}^{\infty} dq \frac{q}{\sqrt{q^2 - m^2}} K_{1-F}(qr), & \Theta = -s \frac{\pi}{2} (\text{mod } 2\pi)
\end{array} \right.
\]

(56)

Integrating this relation over the entire plane, we arrive at

\[
Q^{(l)} = \left\{ \begin{array}{ll}
-\frac{s}{2} \text{sgn}(m) F, & \Theta = s \frac{\pi}{2} (\text{mod } 2\pi) \\
\frac{s}{2} \text{sgn}(m)(1 - F), & \Theta = -s \frac{\pi}{2} (\text{mod } 2\pi)
\end{array} \right.
\]

(57)

4 Induced vacuum magnetic flux

By using the explicit form of solutions to the Dirac equation, we can find the vacuum current averaged over all directions. We have

\[
\tilde{j}(r) = \frac{1}{2\pi} \int_{0}^{2\pi} d\phi \tilde{j}(\mathbf{x}),
\]

(58)

where \( \tilde{j}(\mathbf{x}) \) is given by (7). Having verified that the radial component of the vacuum current vanishes, \( \tilde{j}^r(r) = 0 \), we proceed to considering the angular component \( \tilde{j}^\phi(r) \).

For the contribution of the regular modes (26) and (27) to \( \tilde{j}^\phi(r) \), we obtain

\[
\tilde{j}^\phi_{\text{REG}}(r) = -\frac{s}{2\pi} \int_{0}^{\infty} dk \frac{k^2}{|E|} \sum_{l=1}^{\infty} [J_{l-F}(kr) J_{l+1-F}(kr) - J_{l+F}(kr) J_{l-1+F}(kr)].
\]

(59)

Performing summation over \( l \), we arrive at

\[
\tilde{j}^\phi_{\text{REG}}(r) = -\frac{sr}{4\pi} \int_{0}^{\infty} dk \frac{k^3}{\sqrt{k^2 + m^2}} \left[ J_{2-F}(kr) - J_{2-F}(kr) J_{-F}(kr) + J_{1+F}(kr) J_{1+F}(kr) - J_{1+F}(kr) \right].
\]

(60)

In the same way as that used to evaluate the contribution of the regular modes to the averaged vacuum-charge density, the integral in (60) can be reduced to an integral featuring modified Bessel functions in the corresponding integral. The result has the form

\[
\tilde{j}^\phi_{\text{REG}}(r) = -\frac{s}{2\pi^2} \int_{|m|}^{\infty} dq \frac{q^2}{\sqrt{q^2 - m^2}} \left[ I_{1-F}(qr) K_F(qr) - I_F(qr) K_{1-F}(qr) + \frac{2}{\pi} \sin F\pi (qr) K_{1-F}(qr) \right].
\]

(61)
With the aid of (33)-(39), we find that the contribution of the bound state is given by

\[ \tilde{j}^\phi_{bs}(r) = s \sgn(m) \sgn(A + 1)[1 - \sgn(A)] \frac{\sin F\pi}{2\pi^2} \frac{k^3}{m + E_{bs}(2F - 1)} K_F(kr)K_{1-F}(kr). \]  

(62)

Taking into account (28) and (31), we obtain the contribution of the irregular mode in the form

\[ \tilde{j}^\phi_{irreg}(r) = \frac{s}{4\pi} \int_0^\infty dk \frac{k^2}{\sqrt{k^2 + m^2}} (Ak^{-2(1-F)}m|m|^{-2F}[(m - \sqrt{k^2 + m^2})L_+ + \right. \]

\[ + (m + \sqrt{k^2 + m^2})L_][J_F(kr)J_{1-F}(kr) + [L_+ + L_-][J_F(kr)J_{1+F}(kr) - J_F(kr)J_{1-F}(kr)] + \]

\[ \left. + A^{-1}k^{-2F}m^{-1}|m|^{2F}[(m + \sqrt{k^2 + m^2})L_+ + (m - \sqrt{k^2 + m^2})L_-]J_F(kr)J_{1+F}(kr)], \]  

(63)

where the quantities \( L_+ \) and \( L_- \) are given by (47). In Appendix B, it is shown how expression (63) can be transformed by deforming the contour of integration in the complex plane to arrive at

\[ \tilde{j}^\phi_{irreg}(r) = \frac{s}{2\pi^2} \int_{|m|}^\infty dq \frac{q^2}{\sqrt{q^2 - m^2}} [I_{1-F}(qr)K_F(qr) - I_F(qr)K_{1-F}(qr)] - \]

\[ - \frac{s \sin F\pi}{\pi^3} \int_{|m|}^\infty dq \frac{q^2}{\sqrt{q^2 - m^2}} A(q^2_F - A^{-1}(q^2_F)^{2(1-F)}) + \]

\[ - s \sgn(m) \sgn(A + 1)[1 - \sgn(A)] \frac{\sin F\pi}{2\pi^2} \frac{k^3}{m + E_{bs}(2F - 1)} K_F(kr)K_{1-F}(kr). \]  

(64)

Summing (61), (62), and (64), we find that the averaged vacuum current has the form

\[ \tilde{j}^\phi(r) = - \frac{s \sin F\pi}{\pi^3} \int_{|m|}^\infty dq \frac{q^2}{\sqrt{q^2 - m^2}} [qr[K^2_{1-F}(qr) - K^2_F(qr)] + \]

\[ + 2(F - 1) + \frac{A(q^2_F)^{2F} - A^{-1}(q^2_F)^{2(1-F)}}{A(q^2_F)^{2F} + 2 + A^{-1}(q^2_F)^{2(1-F)}} K_F(qr)K_{1-F}(qr)]. \]  

(65)

This expression tends to zero in proportion to \(|m|^{1/2}r^{-3/2}\exp(-2|m|r)\) for \( r \to \infty \) and diverges in proportion to \( r^{-2} \) for \( r \to 0 \).

Averaging relation (10), where any point that is infinitely remote from the string (that is, any point lying on the circle \( r = \infty \)) can be taken for \( x_{(\infty)} \), we find that the averaged field strength is given by

\[ \tilde{B}^3_{(l)}(r) = - \frac{s e^2 \sin F\pi}{\pi^3} \int_{r}^\infty dv' \int_{|m|}^\infty dq' \frac{q'^2}{\sqrt{q'^2 - m^2}} [qr'[K^2_{1-F}(qr') - K^2_F(qr')] + \]

\[ + 2(F - 1) + \frac{A(q^2_{(\infty)})^{2F} - A^{-1}(q^2_{(\infty)})^{2(1-F)}}{A(q^2_{(\infty)})^{2F} + 2 + A^{-1}(q^2_{(\infty)})^{2(1-F)}} K_F(qr')K_{1-F}(qr')]. \]  

(66)

For the total flux (11) of the vacuum magnetic field induced by a singular magnetic string, we obtain

\[ \Phi^{(l)} = - \frac{s e^2 F(1-F)}{2\pi |m|} \left[ \frac{1}{6}(F - 1/2) + \frac{1}{4\pi} \int_{1}^\infty dv \frac{Av - A^{-1}v^{1-F}}{v\sqrt{v-1}Av + 2 + A^{-1}v^{1-F}} \right]. \]  

(67)
The coupling constant $e$ has dimensions of $\sqrt{|m|}$. Expression (67) can also be recast into the form [compare with (50)]

$$
\Phi(I) = -\frac{e^2 F(1 - F)}{12\pi^2 |m|} \int_1^\infty \frac{dv}{v\sqrt{v - 1}} \frac{(1 + F)(1 + Av^F) - (2 - F)(1 + A^{-1}v^{1-F})}{Av^F + 2 + A^{-1}v^{1-F}}.
$$

(68)

In the case of half-integer values of the string flux, we have

$$
\Phi(I) = -\frac{e^2}{8\pi^2 |m|} \arctan[\tan(\frac{\Theta}{2})], \quad F = \frac{1}{2}.
$$

(69)

It should be emphasized that, in contrast to the vacuum charge, the vacuum magnetic flux is continuous at integral values of the string flux.

5 Discussion of results

It has been shown that, on the plane orthogonal to the singular magnetic string specified by (12), the fermion vacuum is characterized by the quantum number $s$ (52), (57), and (67), which depend on the parameter $\Theta$ of self-adjoint extension. In relation to the $\Theta$ vacuum for the case of a monopole $[3, 4, 5, 6, 7]$, the $\Theta$ vacuum for the case of a string possesses a richer structure [dependence on $\Phi(0)$, $s$ and $\text{sgn}(m)$]. In all probability, this is due to a nontrivial topology of the base space in the latter case: $\pi_1 = 0$ in the case of a space with a punctured point, and $\pi_1 = \mathbb{Z}$ in the case with a removed line (or in the case of a plane with a punctured point), where $\pi_1$ is the first homotopic group. It should be noted that the vacuum charge changes sign and the vacuum magnetic flux remains unchanged under either of the substitutions $s \rightarrow -s$ and $m \rightarrow -m$.

Let us compare expressions (52) and (57) obtained here for the vacuum charge with the expression for the vacuum charge induced by a regular configuration of an external magnetic field. In the latter case, one has $[12]$

$$
Q(I) = -\frac{1}{2} s \text{sgn}(m) \Phi, \quad \Phi = \frac{1}{2\pi} \int d^2x \tilde{B}^3(x),
$$

(70)

where $\tilde{B}^3(x)$ is a function that is continuous everywhere with the exception of integrable singularities at isolated points or on isolated lines. In contrast to (70), expressions (52) and (57) are periodic in the flux of an external magnetic field. This can be considered as a manifestation of the Aharonov-Bohm effect $[13]$ in quantum field theory (see $[14]$). Since it is sometimes stated that the vacuum charge is not periodic in $\Phi(0)$ (see, for example, $[15, 16]$), we will dwell on this point at greater length.

Under charge conjugation,

$$
C : V \rightarrow -V, \quad \psi \rightarrow \sigma_1 \psi^*,
$$

(71)

the charge operator and its vacuum expectation value, as well as the vacuum magnetic flux, must change sign, but this is not the case for expressions (52), (57), and (67), because the boundary condition (30) violates charge-conjugation symmetry. However, for a specific choice of the parameter $\Theta$, this symmetry can be conserved.

In particular, the choice of

$$
\Theta = s\frac{\pi}{2} (\text{mod } 2\pi), \quad s\Phi(0) > 0 \quad \left\{ \begin{array}{l}
\Theta = -s\frac{\pi}{2} (\text{mod } 2\pi), \quad s\Phi(0) < 0
\end{array} \right. \quad (\Phi(0) \neq n, \quad n \in \mathbb{Z}),
$$

(72)

\{
which corresponds to the boundary condition considered in \cite{17, 18}, leads to the expression (compare with the results presented in \cite{15, 16})

\[
Q^{(I)} = \left\{ \begin{array}{l}
-\frac{1}{2} s \text{sgn}(m) \{ \| \Phi^{(0)} \| \}, \quad \Phi^{(0)} > 0 \\
\frac{1}{2} s \text{sgn}(m) (1 - \| \Phi^{(0)} \|), \quad \Phi^{(0)} < 0
\end{array} \right\}, \quad \| \Phi^{(0)} \| \neq 0. \tag{73}
\]

This result changes sign under change conjugation, but it is not periodic in \( \Phi^{(0)} \).

The parameter \( \Theta \) can be chosen in such a way as to conserve both periodicity in \( \Phi^{(0)} \) and the discrete symmetry (71). This can be achieved, for example, by setting

\[
\begin{align*}
\Theta &= s \frac{\pi}{2} \pmod{2 \pi}, \quad -\frac{1}{2} < s(\| \Phi^{(0)} \| - \frac{1}{2}) < 0 \\
\Theta &= 0 \pmod{2 \pi}, \quad \| \Phi^{(0)} \| = \frac{1}{2} \\
\Theta &= -s \frac{\pi}{2} \pmod{2 \pi}, \quad 0 < s(\| \Phi^{(0)} \| - \frac{1}{2}) < \frac{1}{2}
\end{align*} \tag{74}
\]

which corresponds to the condition of minimal irregularity, i.e., to a radial wave function that diverges for \( r \to 0 \) no faster than \( r^{-p} \), where \( p \leq \frac{1}{2} \). It is with this boundary condition that the result reported in \cite{14} is recovered in the form

\[
Q^{(I)} = \frac{1}{2} s \text{sgn}(m) \left[ \frac{1}{2} \text{sgn}_0(\| \Phi^{(0)} \| - \frac{1}{2}) - \| \Phi^{(0)} \| + \frac{1}{2} \right], \tag{75}
\]

where

\[
\text{sgn}_0(u) = \begin{cases} 
\text{sgn}(u), & u \neq 0 \\
0, & u = 0
\end{cases}
\]

It should be noted that expression (75) is continuous for integral values of the string flux and displays discontinuities at half-integer values.

Another choice of \( \Theta \) that is also compatible both with periodicity in \( \Phi^{(0)} \) and with symmetry (71) is

\[
\Theta = 0 \pmod{2 \pi}, \quad 0 < \| \Phi^{(0)} \| < 1. \tag{76}
\]

We then arrive at the expression

\[
Q^{(I)} = -\frac{1}{2} s \text{sgn}(m) \left[ \frac{1}{2} \text{sgn}_0(\| \Phi^{(0)} \| - \frac{1}{2}) - \| \Phi^{(0)} \| + \frac{1}{2} \right], \tag{77}
\]

which is discontinuous both at integral and half-integer values of the string flux.

When the boundary condition (72) is used, expression (67) for the vacuum magnetic flux takes the form (see also \cite{15})

\[
\Phi^{(I)} = \left\{ \begin{array}{l}
-\frac{e^2}{12\pi|m|} \{ \| \Phi^{(0)} \| (1 - \{ \| \Phi^{(0)} \| \})^2, \quad \Phi^{(0)} > 0 \\
\frac{e^2}{12\pi|m|} (1 - \{ \| \Phi^{(0)} \| \})[1 - (1 - \{ \| \Phi^{(0)} \| \})^2], \quad \Phi^{(0)} < 0
\end{array} \right\}. \tag{78}
\]

We also have

\[
\Phi^{(I)} = -\frac{e^2}{12\pi|m|} \{ \| \Phi^{(0)} \| \left[ 1 - \{ \| \Phi^{(0)} \| \} \right] \left[ \frac{3}{2} \text{sgn}_0(\{ \| \Phi^{(0)} \| - \frac{1}{2} \}) - \| \Phi^{(0)} \| + \frac{1}{2} \right] \}, \tag{79}
\]

for the boundary condition (74) and

\[
\Phi^{(I)} = -\frac{se^2 F(1 - F)}{2\pi|m|} \left[ \frac{1}{6} (F - \frac{1}{2}) + \frac{1}{4\pi} \int_1^\infty \frac{dv}{v^\sqrt{v-1}C_Fv^F + 2 + C_F^{-1}v^{1-F}} \right] \tag{80}
\]

for the boundary condition (74).
for the boundary condition (76).

In the last expression, the quantity $F$ is given by (23), and

$$C_F = 2^{1-2F} \frac{\Gamma(1-F)}{\Gamma(F)}. \quad (81)$$

Expressions (79) and (80) are odd under charge conjugation and are periodic in the string flux.

In conclusion, we note that the general form of the boundary condition that conserves both $C$ symmetry and periodicity in the string flux is given by

$$\Theta = \Theta_C (\text{mod } 2\pi), \quad -\frac{1}{2} < s(\|\Phi^{(0)}\| - \frac{1}{2}) < 0$$

$$\Theta = 0 \ (\text{mod } 2\pi), \quad \|\Phi^{(0)}\| = \frac{1}{2}$$

$$\Theta = -\Theta_C (\text{mod } 2\pi), \quad 0 < s(\|\Phi^{(0)}\| - \frac{1}{2}) < \frac{1}{2}$$

where $-\pi < \Theta_C \leq \pi$.

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Appendix A

On the basis of relations [see, for example, [19]]

$$J_\mu(iz) = \exp\left(\frac{i}{2\mu\pi}I_\mu(z)\right), \quad -\pi < \arg z \leq \frac{\pi}{2},$$

$$I_\mu(-z) = \exp(i\mu\pi)I_\mu(z), \quad K_\mu(-z) = \exp(-i\mu\pi)K_\mu(z) - i\pi I_\mu(z), \quad -\pi < \arg z < 0,$$

we can easily obtain

$$J_\mu(ikr)J_\nu(ikr) = \frac{1}{2i\pi} \left\{ \exp\left[\frac{i}{2}((\mu - \nu)\pi)\right]I_\mu(-ikr)K_\nu(-ikr) - \exp\left[\frac{i}{2}((\nu - \mu)\pi)\right]I_\mu(ikr)K_\nu(ikr) + \exp\left[\frac{i}{2}(\nu - \mu)\pi\right]I_\nu(-ikr)K_\mu(-ikr) - \exp\left[\frac{i}{2}(\mu - \nu)\pi\right]I_\nu(ikr)K_\mu(ikr) \right\}. \quad (A.1)$$

With the aid of (A.1), expression (46) can be recast into the form

$$\tilde{\rho}_{\text{irreg}}(r) = \int_{C_{\rightarrow}} d\omega F(\omega). \quad (A.2)$$

Here, $\omega = k^2$ is the new variable of integration; the contour $C_{\rightarrow}$ circumvents the real positive semiaxis of the variable $\omega$, going along it at infinitely small distances from below and above;
and the integrand has the form

\[
\mathcal{F}(\omega) = \frac{i}{(4\pi)^2} \frac{1}{\varepsilon} \{ A \omega^F m|m|^{-2F} [L_+ + L_-] I_{-F}(r \sqrt{-\omega}) K_F(r \sqrt{-\omega}) + \\
+ A \omega^{-1+F} m|m|^{-2F} [(m - \varepsilon)^2 L_+ + (m + \varepsilon)^2 L_-] I_{-1-F}(r \sqrt{-\omega}) K_{1-F}(r \sqrt{-\omega}) + \\
+ \omega^F [(m + \varepsilon) L_+ + (m - \varepsilon) L_-] (-\omega)^{F} I_F(r \sqrt{-\omega}) K_{-F}(r \sqrt{-\omega}) + \\
+ \omega^{-F} [(m + \varepsilon) L_+ + (m - \varepsilon) L_-] (-\omega)^{F} I_{-1-F}(r \sqrt{-\omega}) K_{1-F}(r \sqrt{-\omega}) + \\
+ \omega^{1-F} [(m - \varepsilon) L_+ + (m + \varepsilon) L_-] (-\omega)^{-1+F} I_{-1-F}(r \sqrt{-\omega}) K_{1-F}(r \sqrt{-\omega}) + \\
+ \omega^{1-F} [(m - \varepsilon) L_+ + (m + \varepsilon) L_-] (-\omega)^{-1+F} I_{-1+F}(r \sqrt{-\omega}) K_{1-F}(r \sqrt{-\omega}) + \\
+ A^{-1}\omega^{-F} m^{-1} |m|^{2F} [L_+ + L_-] I_{1+F}(r \sqrt{-\omega}) K_{1-F}(r \sqrt{-\omega}) \},
\]

(A.3)

where \( \varepsilon = \sqrt{\omega + m^2} \). By continuously deforming the contour of integration in the complex \( \omega \) plane as is shown in the Figure, we arrive at the relation

\[
\int_{C_C} d\omega \mathcal{F}(\omega) = \int_{C_C} d\omega \mathcal{F}(\omega) + \int_{C_C} d\omega \mathcal{F}(\omega) + \int_{C_C} d\omega \mathcal{F}(\omega). \quad (A.4)
\]

The integrals along the semicircles \( C_C \) and \( C_C \) of infinite radii vanish, whereas the integral along the contour circumventing the cut for \( \omega < -m^2 \) can be represented as

\[
\int_{C_C} d\omega \mathcal{F}(\omega) = -\frac{1}{(4\pi)^2} \int_{m^2}^{\infty} \frac{du}{\sqrt{u - m^2}} \left( A \omega^F m|m|^{-2F} \left\{ e^{iF\pi} [R^{(+)}_{1-F} + R^{(-)}_{1-F}] + \\
+ e^{-iF\pi} [R^{(-)}_{1-F} + R^{(+)}_{1-F}] \right\} I_{-F}(r \sqrt{u}) K_F(r \sqrt{u}) - A \omega^{-1+F} m|m|^{-2F} \left\{ e^{iF\pi} [m - i \sqrt{u - m^2}]^{2} R^{(+)}_{1-F} + \\
+ (m + i \sqrt{u - m^2})^{2} R^{(-)}_{1-F} \right\} e^{-iF\pi} [m - i \sqrt{u - m^2}] R^{(+)}_{1-F} + \\
+ (m + i \sqrt{u - m^2})^{2} R^{(-)}_{1-F}] \right\} I_{-F}(r \sqrt{u}) K_F(r \sqrt{u}) + \\
+ \left\{ e^{-iF\pi} [m + i \sqrt{u - m^2}] R^{(+)}_{1-F} + (m + i \sqrt{u - m^2})^{2} R^{(-)}_{1-F} \right\} \right\} I_{-F}(r \sqrt{u}) K_F(r \sqrt{u}) - \\
+ \left\{ e^{-iF\pi} [m + i \sqrt{u - m^2}] R^{(+)}_{1-F} + (m + i \sqrt{u - m^2})^{2} R^{(-)}_{1-F} \right\} \right\} I_{-F}(r \sqrt{u}) K_F(r \sqrt{u}) + \\
+ A^{-1} \omega^{-F} m^{-1} |m|^{2F} \left\{ e^{-iF\pi} [m \mp i \sqrt{u - m^2}]^{2} R^{(+)}_{1-F} + (m - i \sqrt{u - m^2})^{2} R^{(+)}_{1-F} + \\
+ (m - i \sqrt{u - m^2})^{2} R^{(-)}_{1-F} \right\} e^{-iF\pi} [m \mp i \sqrt{u - m^2}] R^{(-)}_{1-F} \right\} I_{-F}(r \sqrt{u}) K_{1-F}(r \sqrt{u}) + \\
+ A^{-1} \omega^{-F} m^{-1} |m|^{2F} \left\{ e^{-iF\pi} [R^{(+)}_{1-F} + R^{(-)}_{1-F}] + e^{iF\pi} [R^{(-)}_{1-F} + R^{(+)}_{1-F}] \right\} I_{1+F}(r \sqrt{u}) K_{1-F}(r \sqrt{u}) \right) \) (A.5)

where

\[
R^{(+)}_{1-F} = \frac{[A \omega^{-1+F} m|m|^{-2F} e^{iF\pi} (m \mp i \sqrt{u - m^2}) + 2 \cos F\pi + \\
+ A^{-1} \omega^{-F} m^{-1} |m|^{2F} e^{-iF\pi} (m \mp i \sqrt{u - m^2})]^{-1},
\]
\[ R_{(+)}^{(-)} = [A u^{-1 + F} m | m|^{-2 F} e^{-i F \pi} (m \mp i \sqrt{u - m^2}) + 2 \cos F \pi + A^{-1} u^{-F} m^{-1} | m|^{2F} e^{i F \pi} (m \pm i \sqrt{u - m^2})]^{-1}. \] (A.6)

Expression (A.5) can be reduced to the form:

\[
\int_{C_{\infty}} d\omega \mathcal{F}(\omega) = -\frac{1}{8\pi^2} \int_{m_2}^{\infty} \frac{du}{\sqrt{u - m^2}} \left( \frac{1}{\pi} \sin F \pi \{ A u^{-F} m | m|^{-2F} e^{i F \pi} (R_{(+)}^{(+)}) + R_{(-)}^{(+)}) + e^{-i F \pi} (R_{(+)}^{(-)}) + m + i \sqrt{u - m^2}) (e^{-i F \pi} R_{(+)}^{(+)} + e^{i F \pi} R_{(-)}^{(-)}) + (m - i \sqrt{u - m^2}) (e^{-i F \pi} R_{(-)}^{(+)} + e^{i F \pi} R_{(-)}^{(-)}) \right) + m_2 [I_1(r \sqrt{u}) K_F(r \sqrt{u}) - I_{1-F}(r \sqrt{u}) K_{1-F}(r \sqrt{u})] - \sin F \pi \{(m - i \sqrt{u - m^2}) (e^{-i F \pi} R_{(+)}^{(+)}) + e^{-i F \pi} R_{(+)}^{(-)} + (m + i \sqrt{u - m^2}) (e^{i F \pi} R_{(-)}^{(+)}) + e^{-i F \pi} R_{(-)}^{(-)} + A^{-1} u^{-1} m^{-1} | m|^{2F} e^{-i F \pi} (R_{(+)}^{(+)}) + e^{i F \pi} (R_{(+)}^{(-)} + R_{(-)}^{(-)})] K^2_{1-F}(r \sqrt{u}). \] (A.7)

As the result of further simplifications, we will arrive at the terms in relation (48) that are represented as integrals with respect to the variable \( q = \sqrt{u} \).

It remained to consider the integral along the contour circumventing the pole of the function \( \mathcal{F}(\omega) \). We have

\[
\int_{C_{\infty}} d\omega \mathcal{F}(\omega) = 2 \pi i \text{ Res}_{\omega = \kappa^2} \mathcal{F}(\omega), \quad \text{where } \kappa^2 = m^2 - E_{BS}^2, \quad \text{and the quantity } E_{BS} \text{ is given by (34). Choosing the branch for fractional exponents according to the prescription}
\]

\[ (-k^2)^\mu = k^{2\mu} \exp(i \mu \pi), \quad 0 < \mu < 1, \] (A.9)

we arrive at

\[
\int_{C_{\infty}} d\omega \mathcal{F}(\omega) = -\frac{1}{8\pi |E_{BS}|} [Ak^{-2F} m | m|^{-2F} e^{i F \pi} I_{-F}(kr) K_F(kr) - A^{-2F} m | m|^{-2F} e^{i F \pi} \left( m \mp |E_{BS}| \right)^2 I_{1-F}(kr) K_{1-F}(kr) + e^{i F \pi} (m \mp |E_{BS}|) I_F(kr) K_F(kr) + e^{-i F \pi} (m \mp |E_{BS}|) I_{-F}(kr) K_{-F}(kr) - A^{-1} k^{-2F} m^{-1} | m|^{2F} e^{-i F \pi} (m \mp |E_{BS}|) I_{1+F}(kr) K_{1-F}(kr) + e^{i F \pi} (m \mp |E_{BS}|) I_F(kr) K_F(kr) - e^{-i F \pi} (m \mp |E_{BS}|) I_{-F}(kr) K_{-F}(kr) \right) \text{ Res}_{\omega = -\kappa^2} L_{(\pm)} = \]

\[
\frac{i \sin^2 F \pi}{2 \pi^2 |E_{BS}|} [(m \mp |E_{BS}|) K^2_F(kr) + (m \mp |E_{BS}|) K^2_{1-F}(kr)] \text{ Res}_{\omega = -\kappa^2} L_{(\pm)}, \quad E_{BS} \gtrless 0. \] (A.10)

Taking into account the relation

\[
\text{Res}_{\omega = -\kappa^2} L_{(\pm)} = \frac{1}{i \sin F \pi |E_{BS}|} \frac{|E_{BS}| k^2}{|E_{BS}|/(2F - 1) \pm m}, \quad E_{BS} \gtrless 0, \] (A.11)

we obtain

\[
\int_{C_{\infty}} d\omega \mathcal{F}(\omega) = \text{sgn}(E_{BS}) \frac{\sin F \pi}{2 \pi^2} \frac{k^2}{m + E_{BS}(2F - 1)} \left[ (m + E_{BS}) K^2_F(kr) + (m - E_{BS}) K^2_{1-F}(kr) \right]. \] (A.12)
Naturally, the same expression is obtained if the branch for fractional exponents is chosen alternatively as
\[ (-\kappa^2)^\mu = \kappa^{2\mu} \exp(-i\mu\pi), \quad 0 < \mu < 1. \] (A.13)

With the aid of relations (35)-(39), expression (A.12) is reduced to the form coincident with that of the last term in (48).

It should be also noted that the above method can be used to reduce expression (42) to the form (43), in which case the integrand naturally does not have poles on the segment \(-m^2 < k^2 < 0\).

**Appendix B**

The contribution of the irregular mode to the averaged vacuum current can be found by a method that is similar to that used to calculate the contribution of this mode to the averaged density of the vacuum charge. In the case being considered, the function \(\mathcal{F}(\omega)\) has the form

\[
\mathcal{F}(\omega) = \frac{s}{i(4\pi)^2} \left( A\omega F m|m|^{-2F} \left[ (m - \varepsilon)L_{(+)\varepsilon} + (m + \varepsilon)L_{(-)\varepsilon} \right] (-\omega)^{-\frac{1}{2}} \times 
\times [I_{1-F} (r \sqrt{-\omega}) K_F (r \sqrt{-\omega}) - I_{-F} (r \sqrt{-\omega}) K_{1-F} (r \sqrt{-\omega})] - [L_{(+)\varepsilon} + L_{(-)\varepsilon}] \times \right)
\times \left\{ \omega^{1-F} (-\omega)^{-\frac{1}{2}+F} \left[ I_{1-F} (r \sqrt{-\omega}) K_F (r \sqrt{-\omega}) - I_{-F} (r \sqrt{-\omega}) K_{1-F} (r \sqrt{-\omega}) \right] + 
+ \omega^{F} (-\omega)^{\frac{1}{2}-F} \left[ I_F (r \sqrt{-\omega}) K_{1-F} (r \sqrt{-\omega}) - I_{1+F} (r \sqrt{-\omega}) K_F (r \sqrt{-\omega}) \right] \right\} + 
+ A^{-1}\omega^{1-F} m^{-1}|m|^{2F} \left[ (m + \varepsilon)L_{(+)\varepsilon} + (m - \varepsilon)L_{(-)\varepsilon} \right] (-\omega)^{-\frac{1}{2}+F} \left[ I_F (r \sqrt{-\omega}) K_{1-F} (r \sqrt{-\omega}) - 
- I_{1+F} (r \sqrt{-\omega}) K_F (r \sqrt{-\omega}) \right]. \] (A.14)

The integral along the contour circumventing the cut for \(\omega < -m^2\) (see Figure) can be reduced to the form

\[
\int_{C_{\omega}} d\omega \mathcal{F}(\omega) = \frac{s}{(4\pi)^2} \int_{m^2}^{\infty} du \left( 1 - \frac{m^2}{u} \right)^{-\frac{1}{2}} \left( A u^{-1+F} m|m|^{-2F} \left\{ e^{iF\pi} \left[ (m - i\sqrt{u - m^2}) R_{(+)\varepsilon}^{(+)\varepsilon} + 
+ (m + i\sqrt{u - m^2}) R_{(-)\varepsilon}^{(-)\varepsilon} \right] \times 
\times [I_{1-F} (r \sqrt{u}) K_F (r \sqrt{u}) - I_{-F} (r \sqrt{u}) K_{1-F} (r \sqrt{u})] + \left\{ e^{-iF\pi} R_{(+)\varepsilon}^{(+)\varepsilon} + R_{(-)\varepsilon}^{(-)\varepsilon} \right\} \right\} \right) \times 
\times \left\{ e^{-iF\pi} \left[ R_{(+)\varepsilon}^{(+)\varepsilon} + R_{(-)\varepsilon}^{(-)\varepsilon} \right] \right\} \left[ I_F (r \sqrt{u}) K_{1-F} (r \sqrt{u}) - I_{1+F} (r \sqrt{u}) K_F (r \sqrt{u}) \right] - 
- A^{-1} u^{-F} m^{-1}|m|^{2F} \left\{ e^{-iF\pi} \left[ (m + i\sqrt{u - m^2}) R_{(+)\varepsilon}^{(+)\varepsilon} + (m - i\sqrt{u - m^2}) R_{(-)\varepsilon}^{(-)\varepsilon} \right] \right\} \right\} \left[ I_F (r \sqrt{u}) K_{1-F} (r \sqrt{u}) - 
- I_{1+F} (r \sqrt{u}) K_F (r \sqrt{u}) \right], \] (A.15)

where the quantity \(R_{(+)\varepsilon}^{(+)}\) is given by (A.6). Simplifying the last expression, we arrive at the terms in relation (64) that are represented as integrals.
For the integral along the contour circumventing the pole of the function \( F(\omega) \) (see Figure), we have

\[
\int_{C\bigcirc} d\omega \ F(\omega) = \frac{s \kappa}{8\pi |E_{BS}|} \left\{ \left[ A \kappa^{-2(1-F)} m|m|^{-2F} e^{iF\pi} (m \mp |E_{BS}|) + e^{-iF\pi} \right] [I_{1-F}(\kappa r) K_F(\kappa r) - I_{-F}(\kappa r) K_{1-F}(\kappa r)] - [e^{iF\pi} + A^{-1} \kappa^{-2F} m^{-1}|m|^{2F} e^{-iF\pi} (m \pm |E_{BS}|)] [I_{F}(\kappa r) K_{1-F}(\kappa r) - I_{-1+F}(\kappa r) K_F(\kappa r)] \right\} \text{Res}_{\omega=-\kappa^2} \ L(\pm), \quad E_{BS} \gtrless 0,
\]

(A.16)

where we choose the branch \((-\kappa^2)^\mu\) according to (A.9). Taking into account (A.11), we obtain

\[
\int_{C\bigcirc} d\omega \ F(\omega) = s \text{sgn}(E_{BS}) \sin \frac{F\pi}{\pi^2} \kappa^3 \frac{m + E_{BS}(2F-1)}{K_F(\kappa r)K_{1-F}(\kappa r)}. \quad \text{(A.17)}
\]

With the aid of (35)-(39), expression (B.4) is reduced to the same form as that of the last term in (64).

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