n-Coherence Relative to a Hereditary Torsion Theory

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Received 13 January 2020; Accepted 6 July 2020; Published 4 August 2020

1. Introduction

Throughout this paper, $R$ is an associative ring with identity and all modules considered are unitary. For any $R$-module $M$, $M^+ = \text{Hom}(M, (\mathbb{Q}/\mathbb{Z}))$ will be the character module of $M$.

Recall that a torsion theory $\tau = (\mathcal{T}, \mathcal{F})$ for the category of all right $R$-modules consists of two subclasses $\mathcal{T}$ and $\mathcal{F}$ such that

1. $\text{Hom}(T, F) = 0$ for all $T \in \mathcal{T}$ and $F \in \mathcal{F}$
2. If $\text{Hom}(T, F) = 0$ for all $F \in \mathcal{F}$, then $T \in \mathcal{T}$
3. If $\text{Hom}(T, F) = 0$ for all $T \in \mathcal{T}$, then $F \in \mathcal{F}$

In this case, $\mathcal{T}$ is called a torsion class and its objects are called $\tau$-torsion, $\mathcal{F}$ is called a torsion-free class, and its objects are called $\tau$-torsion free. From, Proposition 2.1, Chap VI in [9], a class $\mathcal{T}$ of right $R$-modules is a torsion class for some torsion theory if and only if $\mathcal{T}$ is closed under quotient modules, direct sums, and extensions. From Proposition 2.2, Chap VI in [9], a class $\mathcal{F}$ of right $R$-modules is a torsion-free class for some torsion theory if and only if $\mathcal{F}$ is closed under submodules, direct products, and extensions. A torsion theory $\tau = (\mathcal{T}, \mathcal{F})$ is called hereditary if $\mathcal{F}$ is closed under submodules.

We recall also that a right $R$-module $M$ is called FP-injective [2] or absolutely pure [3] if $\text{Ext}^1_R(A, M) = 0$ for every finitely presented right $R$-module $A$; a left $R$-module $M$ is flat if and only if $\text{Tor}_1^R(A, M) = 0$ for every finitely presented right $R$-module $A$; a ring $R$ is right coherent [4] if every finitely generated right ideal of $R$ is finitely presented, or equivalently, if every finitely generated submodule of a projective right $R$-module is finitely presented. FP-injective modules, flat modules, coherent rings, and their generalizations have been studied extensively by many authors. For example, in 1994, Costa introduced the concept of right $n$-coherent rings in [5]. Following [5], a ring $R$ is called right $n$-coherent if every $n$-presented right $R$-module is $(n+1)$-presented, where a right $R$-module $A$ is called $n$-presented in case there exists an exact sequence of right $R$-modules $F_n \rightarrow F_{n-1} \rightarrow \cdots \rightarrow F_1 \rightarrow F_0 \rightarrow A \rightarrow 0$, in which every $F_i$ is finitely generated free. It is easy to see that a ring $R$ is right coherent if and only if $R$ is right 1-coherent. In 1996, Chen and Ding introduced the concepts of $n$-FP-injective modules and $n$-flat modules in [6], using the two concepts and characterized right $n$-coherent rings. Following [6], a right $R$-module $M$ is called $n$-FP-injective in case $\text{Ext}^n_R(A, M) = 0$ for every $n$-presented right $R$-module $A$; a left $R$-module $M$ is called $n$-flat in case $\text{Tor}_n^R(A, M) = 0$ for every $n$-presented right $R$-module $A$.

Let $\tau = (\mathcal{T}, \mathcal{F})$ be a (hereditary) torsion theory for the category of all right $R$-modules. Then, according to [7], a right $R$-module $M$ is called right $\tau$-finitely generated (or $\tau$-FG for short) if there exists a finitely generated submodule $N$ such that $(M/N) \in \mathcal{T}$; a right $R$-module $A$ is called $\tau$-finitely generated (or $\tau$-FP for short) if there exists an exact sequence of right $R$-modules $0 \rightarrow K \rightarrow F \rightarrow A \rightarrow 0$ with $F$ finitely generated free and $K$ $\tau$-finitely generated; $R$ is called $\tau$-coherent if every finitely generated right ideal of $R$ is $\tau$-FP. In 1993, Nieves introduced the concept of $\tau$-n-presented (or $\tau$-$n$-FP for short) modules in [8]. Let $\tau = (\mathcal{T}, \mathcal{F})$ be a torsion theory for the category of all right $R$-modules; then, according to [8], a right
2. Characterizations of $\tau$-$\left(n+1\right)$-Presented Modules and Right $\tau$-$n$-Coherent Rings

We recall that a nonempty subclass $\mathcal{T}$ of right $R$-modules is called a weak torsion class [9] if $\mathcal{T}$ is closed under homomorphic images and extensions. Following [9], if a class $\mathcal{T}$ of right $R$-modules is a weak torsion class, then a right $R$-module $M$ is called $\mathcal{T}$-finitely generated (or $\mathcal{T}$-FG for short) if there exists a finitely generated submodule $N$ such that $(M/N) \in \mathcal{T}$; a right $R$-module $A$ is called $\mathcal{T}$-finitely presented (or $\mathcal{T}$-FP for short) if there exists an exact sequence of right $R$-modules $0 \longrightarrow K \longrightarrow F \longrightarrow A \longrightarrow 0$ with $F$ finitely generated free and $K\mathcal{T}$-finitely generated; a right $R$-module $A$ is called $(\mathcal{T},n)$-presented if there exists an exact sequence of right $R$-modules:

$$0 \longrightarrow K_{n-1} \longrightarrow F_{n-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0,$$

such that $F_0, \cdots, F_{n-1}$ are finitely generated free and $K_{n-1}$ is $\mathcal{T}$-finitely generated, where $n$ is a positive integer.

Theorem 1. Let $\tau$ be a torsion theory of mod-$R$, $n$ a non-negative integer, and $A$ an $n$-presented right $R$-module. Then, the following statements are equivalent for $A$:

1. $A$ is $\tau$-$\left(n+1\right)$-presented
2. The canonical map $\lim Ext^0_R(A,X) \longrightarrow Ext^0_R(A, X)$ is an isomorphism for each direct system $\{X_i\}_{i \in I}$ of $\tau$-torsion-free modules
3. $\text{Tor}_1^R(A,X^+) \cong Ext^0_R(A,X^+)$ for each $\tau$-torsion-free module $X$
4. $\text{Tor}_1^R(A,E^+) = 0$ for each $\tau$-torsion-free injective module $E$

Proof

(1)\Rightarrow(2) Use induction on $n$. If $n = 0$, then the result holds by Proposition 2.5(3) in [4]. Assume that the result holds when $n = k$. Then, when $n = k + 1$, suppose $A$ is a $\tau$-$(k+2)$-presented module. Let $0 \longrightarrow N \longrightarrow F \longrightarrow A \longrightarrow 0$ be an exact sequence of right $R$-modules, where $F$ is finitely generated free and $N$ is $\tau$-$(k+1)$-presented. Then, we have a commutative diagram:

$$\begin{array}{cccccc}
\lim Ext^0_R(F, X) & \longrightarrow & \lim Ext^0_R(N, X) & \longrightarrow & \lim Ext^0_R(A, X) & \longrightarrow 0 \\
\langle \phi_1 \rangle & & \langle \phi_2 \rangle & & \langle \phi_3 \rangle & \\
\lim Ext^0_R(F, limX_i) & \longrightarrow & Ext^0_R(N, limX_i) & \longrightarrow & Ext^0_R(A, limX_i) & \longrightarrow 0
\end{array}$$

with exact rows. Since $\phi_1$ is an isomorphism and hence epic and $\phi_2$ is an isomorphism by hypothesis, we have that $\phi_3$ is also an isomorphism by the Five Lemma.

(2)\Rightarrow(1) Use induction on $n$. If $n = 0$, then the result holds by Proposition 2.5 (3) in [4]. Assume that the result holds when $n = k$. Then, when $n = k + 1$, suppose $A$ is a $(k+1)$-presented right $R$-module. Let $0 \longrightarrow N \longrightarrow F \longrightarrow A \longrightarrow 0$ be an exact sequence of right $R$-modules, where $F$ is finitely generated free and $N$ is $k$-presented. Then, for any direct system $\{X_i\}_{i \in I}$ of $\tau$-torsion-free modules. If $k > 0$, then we have a commutative diagram:
with exact rows. From 25.4 (d) in [10], \( \phi_0 \) is an isomorphism and hence epic, and \( \phi_i \) is an isomorphism by condition. Note that \( \phi_1 \) is an isomorphism, so, by the Five Lemma, we have that \( \phi_2 \) is also an isomorphism. So, \( N \) is \( \tau \)-FP by Proposition 2.5 in [4], and it shows that \( A \) is \( \tau \)-2-FP.

(1)\( \Rightarrow \) (3) In case, \( n = 0 \), then the result holds by Lemma 3.1 in [4]. In case, \( n = 1 \), then there is an exact sequence of right \( R \)-modules \( 0 \rightarrow K \rightarrow F \rightarrow A \rightarrow 0 \), where \( F \) is finitely generated free and \( K \) is \( \tau \)-FP. And then we have a commutative diagram:

\[
\begin{array}{cccccc}
0 & \rightarrow & \text{Tor}^R_1(A, X^+) & \rightarrow & K \otimes X^+ & \rightarrow & F \otimes X^+ \\
\downarrow f & & \downarrow g & & \downarrow h & & \\
0 & \rightarrow & \text{Ext}^R_1(A, X^+) & \rightarrow & \text{Hom}(K, X^+) & \rightarrow & \text{Hom}(F, X^+) \\
\end{array}
\]

with exact rows. By Lemma 3.1 [4], \( g \) and \( h \) are isomorphisms. So, by the Five Lemma, \( f \) is also an isomorphism. In case, \( n > 1 \), then we have an exact sequence of right \( R \)-modules \( 0 \rightarrow K_{n-2} \rightarrow F_{n-2} \rightarrow \cdots \rightarrow F_1 \rightarrow F_0 \rightarrow M \rightarrow 0 \), where each \( F_i \) is finitely generated free and \( K_{n-2} \) is \( \tau \)-2-FP, and hence we have \( \text{Tor}^R_1(K_{n-2}, X^+) \equiv \text{Tor}^R_1(K_{n-2}, X^+) \equiv \text{Ext}^R_0(A, X^+) \equiv \text{Ext}^R_0(A, X^+) \), as required.

(3)\( \Rightarrow \) (4) It is obvious.

(4)\( \Rightarrow \) (1) Since \( A \) is \( n \)-FP, there exists an exact sequence of right \( R \)-modules \( 0 \rightarrow K_{n-2} \rightarrow F_{n-2} \rightarrow \cdots \rightarrow F_1 \rightarrow F_0 \rightarrow M \rightarrow 0 \), where each \( F_i \) is finitely generated free and \( K_{n-2} \) is finitely presented. Thus, \( \text{Tor}^R_1(K_{n-2}, E^+) \equiv \text{Tor}^R_1(A, E^+) = 0 \) for any \( \tau \)-torsion-free injective module \( E \) by (4). It follows from Proposition 2 in [6] that \( K_{n-2} \) is \( \tau \)-2-FP, and therefore \( A \) is \( \tau \)-\((n + 1)\)-presented.

\[
\begin{array}{cccccc}
0 & \rightarrow & \text{limExt}^R_1(A, X_i) & \rightarrow & \text{limExt}^R_{i+1}(A, X_i) & \rightarrow & 0 \\
\downarrow \phi_2 & & \downarrow \phi_1 & & \downarrow \phi_0 & & \\
0 & \rightarrow & \text{Ext}^R_1(A, \text{lim}X_i) & \rightarrow & \text{Ext}^R_{i+1}(A, \text{lim}X_i) & \rightarrow & 0 \\
\end{array}
\]

Corollary 1. Let \( n \) be a nonnegative integer and \( A \) an \( n \)-presented right \( R \)-module. Then, the following statements are equivalent:

1. \( A \) is \((n + 1)\)-presented
2. The canonical map \( \lim \text{Ext}^R_1(A, X_i) \rightarrow \text{Ext}^R_1(A, \text{lim}X_i) \) is an isomorphism for each direct system \( \{X_i\}_{i \in I} \) of right \( R \)-modules
3. \( \text{Tor}^R_1(A, X^+) \equiv \text{Ext}^R_1(A, X)^+ \) for each right \( R \)-module \( X^+ \)
4. \( \text{Tor}^R_1(A, E^+) = 0 \) for each injective right \( R \)-module \( E^+ \)

Definition 1. Let \( \tau \) be a torsion theory of \( \text{mod-} R \). Then, the ring \( R \) is called right \( \tau \)-\( n \)-coherent, if every \( n \)-presented right \( R \)-module is \((\tau, n + 1)\)-presented.

Let \( \tau = (\mathcal{T}, \mathcal{F}) \) be a torsion theory of \( \text{mod-} R \). Then, it is easy to see that \( R \) is right \( \tau \)-\( n \)-coherent if and only if every \( n \)-presented right \( R \)-module is \((\mathcal{T}, n + 1)\)-presented.

Example 1

1. Let \( \tau = (0, \text{mod-} R) \). Then, \( R \) is right \( \tau \)-\( n \)-coherent if and only if \( R \) is right \( n \)-coherent.
2. \( R \) is right \( \tau \)-coherent if and only if \( R \) is right \( \tau \)-1-coherent.
3. Let \( \tau = (\text{mod-} R, 0) \). Then, \( R \) is right \( \tau \)-\( n \)-coherent.

Proof. (1) and (3) are obvious. (2) follows from Theorem 3.3 (2) in [4].

Definition 2. Let \( \tau \) be a torsion theory of \( \text{mod-} R \) and \( n \) a positive integer. Then, a right \( R \)-module \( M \) is said to be \( \tau \)-\( n \)-FP-injective, if \( \text{Ext}^R_1(A, M) = 0 \) for each \( \tau \)-\((n + 1)\)-presented module \( M \); a right \( R \)-module \( M \) is said to be \( \tau \)-FP-injective if it is \( \tau \)-1-FP-injective.

Clearly, each \( n \)-FP-injective module is \( \tau \)-\( n \)-FP-injective. If \( \tau = (\text{mod-} R, 0) \), then it is easy to see that a right \( R \)-module \( M \) is \( \tau \)-\( n \)-FP-injective if and only if it is \( n \)-FP-injective. Now, we give our characterization of right \( \tau \)-coherent rings.
Theorem 2. Let $\tau$ be a hereditary torsion theory of mod-$R$ and $n$ a positive integer. Then, the following statements are equivalent for the ring $R$:

1. $R$ is right $\tau$-$n$-coherent
2. $\lim \text{Ext}^n_A(M, X_i) \cong \text{Ext}^n_A(M, \lim X_i)$ for any $n$-presented right $R$-module $M$ and a directed system $\{X_i\}_{i \in I}$ of $\tau$-torsion-free modules
3. $\text{Tor}^n_A(M, X^*) \cong \text{Tor}^n_A(M, X^*)$ for any $n$-presented right $R$-module $M$ and a directed system $\{X^*\}_{i \in I}$ of $\tau$-torsion-free injective modules
4. $\text{Tor}^n_A(M, X^*) = 0$ for any $n$-presented right $R$-module $M$ and each $\tau$-torsion-free injective module $E$
5. If $X$ is a $\tau$-$n$-FP-injective module, then $X$ is $n$-FP-injective
6. Any direct limit of $\tau$-torsion-free $n$-FP-injective modules is $n$-FP-injective
7. Any direct limit of $\tau$-torsion-free FP-injective modules is $n$-FP-injective
8. Any direct limit of $\tau$-torsion-free injective modules is $n$-FP-injective
9. A $\tau$-torsion-free module $X$ is $n$-FP-injective if and only if $X^*$ is $n$-flat
10. If $Y$ is a pure submodule of a $\tau$-torsion-free $n$-FP-injective module $X$, then $X/Y$ is $n$-FP-injective.

Proof

(1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) follows from Theorem 1.
(1) $\Rightarrow$ (5), (6) $\Rightarrow$ (7) $\Rightarrow$ (8), and (3) $\Rightarrow$ (9) are obvious.
(5) $\Rightarrow$ (6) Let $X = \lim X_i$, where each $\{X_i\}_{i \in I}$ is a $\tau$-torsion-free $n$-FP-injective module. Then, for any $\tau$-$(n + 1)$-FP module $A$, by Theorem 1, we have that $\text{Ext}^n_A(M, X) = \text{Ext}^n_A(M, \lim X_i) \cong \text{Ext}^n_A(M, X_i)$, so $X$ is $\tau$-$n$-FP-injective and thus it is $n$-FP-injective by (5).

(8) $\Rightarrow$ (1). Let $A$ be an $n$-presented right $R$-module with a finite $n$-presentation $F_n \longrightarrow F_{n-1} \longrightarrow F_{n-2} \longrightarrow \cdots \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow d_0 A \longrightarrow 0$. Write $K_{n-1} = \text{Ker}(d_{n-1})$ and $K_{n-2} = \text{Ker}(d_{n-2})$. Then, $K_{n-1}$ is finitely generated, and we get an exact sequence of right $R$-modules $0 \longrightarrow K_{n-1} \longrightarrow F_{n-1} \longrightarrow K_{n-2} \longrightarrow 0$. Let $\{E_i\}_{i \in I}$ be any direct system of $\tau$-torsion-free injective right $R$-modules (with $I$ directed). Then, $E_i^* = n$-FP-injective by (8), so $\text{Ext}^n_A(M, E_i^*) = 0$ and hence $\text{Ext}^n_A(K_{n-1}, E_i^*) = 0$. Thus, we have a commutative diagram:

\begin{equation}
\begin{array}{ccc}
\lim \text{Hom}(K_{n-2}, E_i) & \longrightarrow & \lim \text{Hom}(F_{n-1}, E_i) \\
\downarrow f & & \downarrow g \\
\text{Hom}(K_{n-2}, \lim E_i) & \longrightarrow & \text{Hom}(F_{n-1}, \lim E_i)
\end{array}
\end{equation}

with exact rows. Since $f$ and $g$ are isomorphisms by 25.4(d) in [10], $h$ is an isomorphism by the Five Lemma. Now, let $\{X_i\}_{i \in I}$ be any direct system of $\tau$-torsion-free modules (with $I$ directed). Then, we have a commutative diagram with exact rows:

\begin{equation}
\begin{array}{cccc}
0 & \longrightarrow & \text{Hom}(K_{n-1}, X) & \stackrel{\phi_1}{\longrightarrow} & \text{Hom}(K_{n-1}, E(X_i)) & \longrightarrow & \text{Hom}(K_{n-1}, E(X_i)/X_i) \\
\downarrow 0 & & \downarrow 0 & & \downarrow \phi_2 & & \downarrow \phi_3 \\
0 & \longrightarrow & \text{Hom}(K_{n-1}, \lim X_i) & \longrightarrow & \text{Hom}(K_{n-1}, \lim E(X_i)) & \longrightarrow & \text{Hom}(K_{n-1}, \lim E(X_i)/X_i)
\end{array}
\end{equation}

where $E(X_i)$ is the injective hull of $X_i$. Since $K_{n-1}$ is finitely generated by 24.9 in [10], the maps $\phi_1$, $\phi_2$, and $\phi_3$ are monic. Since $\tau$ is a hereditary torsion theory and $X_i$ is $\tau$-torsion-free, by Proposition 3.2, Chap VI in [9], $E(X_i)$ is $\tau$-torsion-free. And so, by the above proof, $\phi_2$ is an isomorphism. Hence, $\phi_3$ is also an isomorphism by the Five Lemma again, and then $K_{n-1}$ is $\tau$-finitely presented by Proposition 2.5 (3) in [4], and thus $A$ is $\tau$-$n$-coherent. Therefore, $R$ is right $\tau$-$n$-coherent.

(9) $\Rightarrow$ (10) Since $Y$ is a pure submodule, the pure exact sequence $0 \longrightarrow Y \longrightarrow X \longrightarrow (X/Y) \longrightarrow 0$ induces a split exact sequence $0 \longrightarrow (X/Y)^+ \longrightarrow X^+ \longrightarrow Y^+ \longrightarrow 0$. Since $X$ is $\tau$-torsion-free and $n$-FP-injective, by (9), $X^+$ is $n$-flat, so $(X/Y)^+$ is also $n$-flat, and thus $(X/Y)$ is $n$-FP-injective by Corollary 2.8 in [2].

(10) $\Rightarrow$ (6) Let $\{X_i\}_{i \in I}$ be a direct system of $\tau$-torsion-free $n$-FP-injective modules. Then by Proposition 1 in [7], we have a map-pure, and hence pure exact sequence $0 \longrightarrow K \longrightarrow \oplus_{i \in I} X_i \longrightarrow \lim X_i \longrightarrow 0$. Observing that $\oplus_{i \in I} X_i$ is $\tau$-torsion-free and $n$-FP-injective, by (10), we have that $\lim X_i$ is $n$-FP-injective. $\square$
We call a right $R$-module $X$ weakly $n$-FP-injective if $\text{Ext}^i_R(A, X) = 0$ for any $(n+1)$-presented right $R$-module $A$. Let $\tau = (0, \text{mod} - R)$. Then, we have the following results.

**Corollary 2.** Let $n$ be a positive integer. Then, the following statements are equivalent for a ring $R$:

1. $R$ is right $n$-coherent
2. $\lim \text{Ext}^i_R(A, X_i) \equiv \text{Ext}^i_R(A, \lim X_i)$ for any $n$-presented right $R$-module $A$ and direct system $\{X_i\}_{i \in I}$ of right $R$-modules
3. $\text{Tor}^R_n(A, X^+) \equiv \text{Ext}^i_R(A, X^+)$ for any $n$-presented right $R$-module $A$ and each injective right $R$-module $X$
4. $\text{Tor}^R_n(A, E^+) = 0$ for any $n$-presented right $R$-module $A$ and each injective right $R$-module $E$
5. If $X$ is a weakly $n$-FP-injective module, then $X$ is $n$-FP-injective
6. Any direct limit of $n$-FP-injective modules is $n$-FP-injective
7. Any direct limit of FP-injective modules is $n$-FP-injective
8. A right $R$-module $X$ is $n$-FP-injective if and only if $X^+$ is $n$-flat
9. If $Y$ is a pure submodule of an $n$-FP-injective right $R$-module $X$, then $X/Y$ is $n$-FP-injective

We note that the equivalences of (1), (2), (6), and (9) in Corollary 2 appeared in Theorem 3.1 in [2].

**Corollary 3.** Let $\tau$ be a hereditary torsion theory of $\text{mod}$-$R$ and $n$ a positive integer. If $R$ is right $\tau$-$n$-coherent, then a $\tau$-torsion-free module $X$ is $n$-FP-injective if and only if $X^+$ is $n$-FP-injective.

**Proof.** $\Rightarrow$ Let $X$ be a $\tau$-torsion-free $n$-FP-injective module. Since $R$ is right $\tau$-$n$-coherent, by Theorem 2 (9), $X^+$ is $n$-flat, and so $X^+$ is $n$-FP-injective by Proposition 2.3 in [2].

$\Leftarrow$ Since $X^+$ is $n$-FP-injective and $X$ is a pure submodule of $X^+$, so, by Proposition 2.6 in [2], $X$ is $n$-FP-injective.

Let $n = 1$; then, by Theorem 2, we can obtained several results on right $\tau$-coherent rings.

**Corollary 4.** Let $\tau$ be a hereditary torsion theory of $\text{mod}$-$R$. Then, the following statements are equivalent for the ring $R$:

1. $R$ is right $\tau$-coherent
2. $\lim \text{Ext}^i_R(A, X_i) \equiv \text{Ext}^i_R(A, \lim X_i)$ for any finitely presented right $R$-module $A$ and direct system $\{X_i\}_{i \in I}$ of $\tau$-torsion-free modules
3. $\text{Tor}^R_n(A, X^+) \equiv \text{Ext}^i_R(A, X^+)$ for any finitely presented right $R$-module $A$ and each $\tau$-torsion-free module $X$
4. $\text{Tor}^R_n(A, E^+) = 0$ for any finitely presented right $R$-module $A$ and each $\tau$-torsion-free module $E$
5. If $X$ is a $\tau$-1-FP-injective module, then $X$ is $F$-injective
6. Any direct limit of $\tau$-torsion-free FP-injective modules is $\tau$-FP-injective
7. Any direct limit of $\tau$-torsion-free injective modules is $\tau$-FP-injective
8. A $\tau$-torsion-free module $X$ is FP-injective if and only if $X^+$ is flat
9. If $Y$ is a pure submodule of a $\tau$-torsion-free FP-injective module $X$, then $X/Y$ is FP-injective

**Data Availability**

No data were used to support this study.

**Conflicts of Interest**

The authors declare that they have no conflicts of interest.

**Acknowledgments**

This research was supported by the Natural Science Foundation of Zhejiang Province, China (LY18A010018).

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