Factorisation of $\mathcal{N} = 2$ theories on the squashed 3-sphere

Sara Pasquetti$^{1,2}$

$^1$School of Physics, Queen Mary University of London,
Mile End Road, London E1 4NS, UK

$^2$Pure Mathematics Section, Huxley Building,
Imperial College, Queen’s Gate London SW7 2AZ, UK

Abstract: Partition functions of $\mathcal{N} = 2$ theories on the squashed 3-sphere have been recently shown to localise to matrix integrals. By explicitly evaluating the matrix integral we show that abelian partition functions can be expressed as a sum of products of two blocks. We identify the first block with the partition function of the vortex theory, with equivariant parameter $\hbar = 2\pi i b^2$, defined on $\mathbb{R}^2 \times S_1$ corresponding to the $b \to 0$ degeneration of the ellipsoid. The second block gives the partition function of the vortex theory, with equivariant parameter $\hbar^L = 2\pi i /b^2$, on the dual $\mathbb{R}^2 \times S_1$ corresponding to the $1/b \to 0$ degeneration. The ellipsoid partition appears to provide the $\hbar \to \hbar^L$ modular invariant non-perturbative completion of the vortex theory.
1 Introduction

Partition functions of $\mathcal{N} = 2$ theories on the ellipsoid $S^3_b$ have been recently shown to reduce to matrix integrals in [1], by a generalisation of the localisation method developed in [2, 3]. The ellipsoid is a deformation of the three-sphere, preserving a $U(1) \times U(1)$ isometry, which can be parameterised as:

$$b^2|z_1| + \frac{1}{b^2}|z_2|^2 = 1, \quad z_1, z_2 \in \mathbb{C},$$

(1.1)

where $b$ is the squashing parameter.

In the $b \to 0$ limit the ellipsoid degenerates to $\mathbb{R}^2 \times S_1$ where the partition function counts finite-energy configurations on $\mathbb{R}^2$, known as vortices [4–6]. The vortex partition function can be evaluated via equivariant localisation, with equivariant parameter $\hbar = 2\pi ib^2$ [7]. More precisely on $\mathbb{R}^2 \times S_1$ we have the so-called K-theory vortex partition function, with equivariant parameter $q = e^{\beta \hbar}$ where $\beta$ is the $S_1$ radius. The vortex partition function $Z_V$ can be expressed as a perturbative series:

$$Z_V = e^{S_0/\hbar + S_1 + S_2\hbar + \cdots}.$$  

(1.2)

On a general basis we expect the perturbative series $S_0/\hbar + S_1 + S_2\hbar + \cdots$ to be asymptotic and as a consequence we expect non-perturbative corrections of the form $e^{-A/\hbar}$.

Clearly we can realise another completely equivalent construction by considering the opposite limit $1/b \to 0$. The ellipsoid degenerates to another copy of $\mathbb{R}^2 \times S_1$
where lives the dual vortex theory, that we call antivortex theory $\bar{Z}_V$, with equivariant parameter $h^L = 2\pi i/b^2$.

We expect that the ellipsoid partition function, in the $b \to 0$ limit, will have a perturbative part coming from the vortex sector and a non-perturbative part from the antivortex sector and conversely in the $1/b \to 0$ limit.

As an illustration of what we discussed above let us consider the double-sine function $s_b$ whose definition and relevant identities are collected in appendix A. The double-sine is the basic building block appearing in ellipsoid partition functions [1] and it is the partition function of a free chiral multiplet. The double-sine function has the following representation:

$$s_b(x) = e^{-i\pi x^2/2} \prod_{k=1}^\infty \left( 1 - q_1^{-(2k+1)} e^{-2\pi bx} \right) \prod_{k=1}^\infty \left( 1 - q_2^{-(2k+1)} e^{-2\pi x/b} \right)^{1/2}, \quad (1.3)$$

where $q_1 = e^{i\pi b^2}$ and $q_2 = e^{i\pi/b^2}$. This expression can be interpreted as the factorisation of the ellipsoid partition function into a vortex and an antivortex contribution. By observing the “free energy”:

$$\log s_b(x) = -i\pi x^2/2 + \left( \sum_k \frac{1}{k} \frac{e^{-2\pi x b}}{q_1^k - q_1^{-k}} \right) + \left( \sum_k \frac{1}{k} \frac{e^{-2\pi x/b}}{q_2^k - q_2^{-k}} \right), \quad (1.4)$$

we see that the small $\hbar$ expansion produces a perturbative part in $\hbar$, from the vortex sector, and a non-perturbative part in $e^{-1/\hbar}$ from the antivortex sector. A similar structure appears in the dual small $\hbar^L$ expansion.

On this respect the ellipsoid partition function appears as the non-perturbative completion of the vortex theory. It is interesting to observe that the inclusion of non-perturbative corrections restores the modular $b \to 1/b$ invariance which is indeed manifest in the ellipsoid.

In this note we will argue that the vortex-antivortex factorisation occurs also in interacting cases where the partition function is a matrix integral. By a direct evaluation of the integral we show that the partition function can be expressed as:

$$Z_{S^3} = \sum_{i=1}^N Z_{cl}^{(i)} Z_{1-loop}^{(i)} Z_V^{(i)} \times \bar{Z}_{1-loop}^{(i)} \bar{Z}_{\bar{V}}^{(i)}, \quad (1.5)$$

where, as we will discuss in the next section, the sum runs over vortex sectors.

This expression is reminiscent of the partition function on the 4-sphere obtained by Pestun [2], with vortices playing here the role of 4d instantons. This analogy suggests

\footnote{For a similar discussion in the context of matrix models see [8].}
that it should be possible to derive the factorised form of the ellipsoid partition function
directly by an alternative localisation of the path integral.

The vortex partition function can be geometrically engineered [4, 9], in particular,
as shown in the next section, the vortex partition function is given by an open topolog-
ical string partition function $Z_V = Z_{\text{top}}$. From this perspective our expression (1.5) is
also evocative of the Ooguri-Strominger-Vafa relation between black hole entropy and
topological string partition function [10]:

$$Z_{\text{BH}} = |Z_{\text{top}}|^2.$$

This note is organised as follows. In the next section we study the abelian theory
with $N$ chirals and $N$ antichirals, beginning with the explicit evaluation of the matrix
integral that leads to the factorised form. We then interpret our expression in terms
of vortex theories and topological strings. In section 3 we repeat the analysis for the
abelian theory with $2N$ chirals. In section 4 we discuss more general theories and future
directions.

2 The non-chiral theory

Our first example is the $\mathcal{N} = 2$ $U(1)$ theory with $N$ flavours. We introduce vector
masses $m_i$ with $\sum_i m_i = 0$, axial masses $\mu_i$, $i = 1 \cdots N$ and a FI parameter $\xi$. The
ellipsoid partition function, as explained in [1], reduces to a matrix integral which reads:

$$Z_{S_b^3}^{U(1)}(N,N) = \int dx e^{2\pi i x \xi} \prod_{i=1}^N \frac{s_b(x + m_i + \mu_i/2 + iQ/2)}{s_b(x + m_i - \mu_i/2 - iQ/2)}, \quad (2.1)$$

where $Q = b + 1/b$. It is possible to directly evaluate the matrix integral (2.1) by closing
the integration contour in the upper half-plane and picking the contributions of the
simple poles of the double-sines in the numerators located at:

$$x = -m_i - \mu_i/2 + i m b + i n/b, \quad m, n \geq 0, \quad i = 1 \cdots N. \quad (2.2)$$

There are $N$ infinite towers of poles to take into account. Each tower contains poles
labelled by indices $m, n$ and the residue at each $(m, n)$ pole can be evaluated by means
of formulas A.4 and A.5. Putting all together we find:

$$Z_{S_b^3}^{U(1)}(N,N) = \sum_{i=1}^N e^{-2\pi i \xi (m_i + \mu_i/2)} \prod_{j \neq i}^N \frac{s_b(D_{ji} + iQ/2)}{s_b(C_{ji} - iQ/2)} Z^{(i)} \bar{Z}^{(i)}, \quad (2.3)$$

where:
\[ Z^{(i)} = \sum_{n=0}^{\infty} \prod_{k=1}^{n} \frac{(1 - q^{(k-1)} e^{-2\pi b C_{ji}})}{(1 - q^k)} \prod_{j \neq i} \frac{(1 - q^{(k-1)} e^{-2\pi b C_{ji}})}{(1 - q^k e^{-2\pi b D_{ji}})} z^n \equiv Z_V^{(i)}, \tag{2.4} \]

\[ \bar{Z}^{(i)} = \sum_{m=0}^{\infty} \prod_{l=1}^{m} \frac{(1 - \bar{q}^{(l-1)} e^{-2\pi / b C_{ji}})}{(1 - \bar{q}^l)} \prod_{j \neq i} \frac{(1 - \bar{q}^{(l-1)} e^{-2\pi / b C_{ji}})}{(1 - \bar{q}^l e^{-2\pi / b D_{ji}})} \bar{z}^m \equiv \bar{Z}_V^{(i)}, \tag{2.5} \]

with:
\[ D_{ji} = m_j + \mu_j / 2 - m_i - \mu_i / 2, \quad C_{ji} = m_j - \mu_j / 2 - m_i - \mu_i / 2. \tag{2.6} \]

We also introduced equivariant parameters:
\[ q = q_1^{-2} = e^{-\beta h}, \quad \bar{q} = q_2^{-2} = e^{-\beta h^L}, \tag{2.7} \]

where \( \beta \) is the \( S^1 \) radius in \( \mathbb{R}^2 \times S^1 \) and vortex counting parameters:
\[ \left( e^{-\pi b \sum_{j=1}^{N} (\mu_j + iQ)} e^{-2\pi \xi_b} \right) = e^{2\pi i \xi_{eff} / b} = z, \quad \left( e^{-\pi / b \sum_{j=1}^{N} (\mu_j + iQ)} e^{-2\pi \xi_b} \right) = e^{2\pi i \xi_{eff} / b} = \bar{z}, \tag{2.8} \]

with \( \xi_{\text{eff}} \) the effective FI parameter.

The factorisation of the matrix integral (2.3) at a merely technical level occurs because of the cancellation of the factors \((-1)^{nm}\) in formulas (A.4), (A.5) between double-sine functions in the numerator (chiral) and in the denominator (anti-chiral). For the same reason an integral involving an arbitrary number of chirals will factorise whenever the difference between the number of chirals and anti-chirals is even. This condition is however necessarily satisfied by physical partition functions. Indeed a one loop computation yields the following expression for the effective Chern-Simons coupling, obtained integrating out charge \( Q[\psi] \) fermions:
\[ k_{\text{eff}} = k + \sum_{\text{fermions}} Q[\psi]^2 / 2. \tag{2.9} \]

A necessary condition to ensure gauge invariance is to have \( k_{\text{eff}} \in \mathbb{Z} \) which is precisely the condition for the factorisation of the partition function.

We could have included in the action a level \( k \) Chern-Simons term, which, after localisation, contributes to the integrand as \( e^{-\pi i k x^2} \) [3]. When evaluated at the poles (2.2) the Chern-Simons term gives:
\[ e^{-\pi i k (-m_i - \mu_i / 2 + im b + i n / b)^2} = e^{-\pi i k (m_i + \mu_i / 2)^2} e^{-2\pi k (m_i + \mu_i / 2) (m b + n / b)} q_1^{k m_i^2} q_2^{k n / b^2} (-1)^{k n m}. \tag{2.10} \]
Clearly the inclusion of a Chern-Simons term with \( k \in \mathbb{Z} \) in our theory (2.1) will not spoil the factorisation property. It is also easy to realise that for theories where the difference between the number of chirals and anti-chirals is odd, a bare Chern-Simons term with \( k \in \mathbb{Z}/2 \) will ensure the factorisation.

We close this section by observing that \( Z^{(i)}_V \) is just a basic hypergeometric series:

\[
Z^{(i)}_V = N \Phi^{(i)}_{N-1}(a_1^{(i)}, \ldots, a_N^{(i)}; b_1^{(i)}, \ldots, b_{N-1}^{(i)}; z, q) = \sum_n \frac{\prod_{j=1}^N (a_j^{(i)}, q)_n}{(q, q)_n \prod_{j\neq i}^N (b_j^{(i)}, q)_n} z^n,
\]

where:

\[
a_j^{(i)} = e^{-2\pi b C_{ji}}, \quad b_j^{(i)} = q e^{-2\pi b D_{ji}},
\]

and the q-Pochhamers are defined as:

\[
(a, q)_n = \prod_{k=0}^{n-1} (1 - aq^k).
\]

In particular, by looking at the definition of coefficients \( C_{ji}, D_{ji} \) given in (2.6), it is easy to realise that the index \( i = 1, \ldots, N \) labels the \( N \) independent solutions of the basic-hypergeometric difference equation. Analogously \( \bar{Z}^{(i)}_V \) for \( i = 1, \ldots, N \) will be basic hypergeometric series defined as above with \( b \to 1/b \). In conclusion we find that the ellipsoid partition function will be annihilated by a \( q \)-difference and by a \( \bar{q} \)-difference operator. It would be interesting to explore the role of these operators in the dual Chern-Simons theory as discussed in [6].

### 2.1 Vortex-Antivortex factorisation

As we mentioned in the introduction, in the \( b \to 0 \) limit the ellipsoid degenerates to \( \mathbb{R}^2 \times S^1 \), therefore, in this limit, we expect the ellipsoid partition function to reduce to the vortex theory on \( \mathbb{R}^2 \times S^1 \) [4, 5, 11].

The vortex partition function can be computed via equivariant localisation and consists of a perturbative part and a vortex part [7]. Since we are considering abelian theories, we are interested in vortex configurations labelled by the trivial partition \( 1^n \). The (K-theory) vortex partition function for the abelian theory with \( N_f \) fundamentals and \( N_a \) anti-fundamentals reads:

\[
\sum_n z^n \frac{\prod_{i=1}^{N_f} \prod_{k=1}^n (1 - Q_i^f q^{k-1})}{\prod_{k=1}^n (1 - q^k) \prod_{i=1}^{N_a} \prod_{k=1}^n (1 - Q_i^a q^k)}, \tag{2.14}
\]

where \( Q_i^f = e^{-\beta m_i^f} \), \( Q_i^a = e^{-\beta m_i^a} \) and \( m_i^f, m_i^a \) are fundamental and anti-fundamental masses. In the \( \beta \to 0 \) limit we recover the (homological) vortex partition function on \( \mathbb{R}^2 \) computed in [7].
It is immediate to identify our expressions $Z_V^{(i)}$, for $i = 1, \cdots N$, as abelian vortex partition functions with $N$ chiral fundamental multiplets with masses $D_{ji}$ for $j \neq i$ (one is always massless) and $N$ chiral antifundamentals with masses $C_{ji}$.

Similarly $\bar{Z}_V^{(i)}$, for $i = 1, \cdots N$, can be identified with dual ($\hbar \rightarrow \hbar L$) vortex partition functions with $N$ fundamental and $N$ anti-fundamental chirals on the dual $\mathbb{R}^2 \times S^1$ corresponding to the $1/b \rightarrow 0$ degeneration of the ellipsoid.

We now move to the study of the prefactor in eq (2.3). By using the the representation (A.7) it is easy to see that the prefactor is also factorised in terms of 1-loop contributions given in [7]:

$$
\frac{\prod_{j=1}^{N} s_b(D_{ji} + iQ/2)}{\prod_{j}^{N} s_b(C_{ji} - iQ/2)} \equiv e^{\pi i/2 \sum j ((D_{ji} + iQ/2)^2 - (C_{ji} - iQ/2)^2)} Z_{1\text{-loop}}^{(i)} \bar{Z}_{1\text{-loop}}^{(i)},
$$

(2.15)

with

$$
Z_{1\text{-loop}}^{(i)} = \prod_{k=1}^{\infty} \frac{\prod_{j=1}^{N} (1 - q^k e^{-2\pi b D_{ji}})}{\prod_{j}^{N} (1 - q^k - e^{-2\pi b C_{ji}})}, \quad \bar{Z}_{1\text{-loop}}^{(i)} = \prod_{k=1}^{\infty} \frac{\prod_{j=1}^{N} (1 - \bar{q}^k e^{2\pi b D_{ji}})}{\prod_{j}^{N} (1 - \bar{q}^k - e^{2\pi b C_{ji}})}.
$$

(2.16)

Finally the exponential terms in eq. (2.15) and in eq. (2.3) can be simplified and combined to give the $i$-th classical action:

$$
e^{-2\pi i (m_i + \mu_i/2) \bar{e} i \pi/2 ((D_{ji} + iQ/2)^2 - (C_{ji} - iQ/2)^2)} = e^{-\pi i \sum j \mu_j m_j} e^{-2\pi i \xi_{\text{eff}} (m_i + \mu_i/2)} \equiv e^{-\pi i \sum j \mu_j m_j} Z_{cl}^{(i)},
$$

(2.17)

where $\xi_{\text{eff}}$ was introduced in (2.8).

To summarise, up to a prefactor, we can write the ellipsoid partition function in the following form:

$$
Z_{S_b}^{U(1)}(N, N) = \sum_{i=1}^{N} Z_{cl}^{(i)} \times (Z_{1\text{-loop}}^{(i)} Z_V^{(i)}) \times (\bar{Z}_{1\text{-loop}}^{(i)} \bar{Z}_V^{(i)}),
$$

(2.18)

which makes visible the vortex-antivortex structure for finite $b$.

As we explained in the introduction, the semiclassical $\hbar$ expansion of the free energy will consist of a perturbative $\hbar$ part coming from the vortex sector in (2.18) and a non-perturbative part coming from the anti-vortex sector. Conversely, in the dual semiclassical $\hbar^L$ expansion, the anti-vortex sector will contribute perturbatively while the vortex sector non-perturbatively.

We can then regard the ellipsoid partition function as the modular $\hbar \leftrightarrow \hbar L$ invariant non-perturbative completion of the vortex theory.

Our final expression for the ellipsoid partition function (2.18) has the same structure as the partition function of $\mathcal{N} = 2$ theories on the four sphere $S^4$ obtained by Pestun.
\[ Z_{S^4} = \int d\mu_\alpha Z_{cl}(\alpha) Z_{1\text{-loop}}(\alpha)^2 |Z_{inst}(\alpha, \tau)|^2, \quad (2.19) \]

where \( d\mu_\alpha \) is the measure over the Cartan of the gauge group. In the ellipsoid case (2.18) the role of the instanton partition function \( Z_{inst}(\alpha, \tau) \) is played by the vortex partition function, with instanton and vortex counting parameters given respectively by the 4d gauge coupling and by the 3d FI.

However while the \( S^4 \) partition function (2.19) factorises into holomorphic and antiholomorphic blocks, the ellipsoid partition function factorises into two blocks related by \( h \to h^L \). Moreover while in the \( S^4 \) cases there is an integration over the Cartan of the gauge group, in the ellipsoid case (2.18) we have a discrete sum.

It should be possible to obtain our result (2.18) directly from the localisation of the path integral by choosing an appropriate \( \mathcal{Q} \)-exact term to add to the action.

### 2.2 Geometric engineering

As explained in [4] (see also [9]) the K-theory vortex partition function can be engineered in topological string theory. Here we observe that the relevant topological string geometry arises very naturally directly from the semiclassical analysis of the ellipsoid partition function. We start from the integral form of the partition function (2.1), shift the integration variable \( x \to x - m_i - \mu_i/2 + iQ/2 \) and use the \( b \to 0 \) limit of the double-sine function given in (A.10) to obtain:

\[
Z_{S^2}^{U(1)}(N, N) \sim \int dx \ e^{2\pi i x} e^{-\frac{\pi i}{2} \sum_j (x + D_{ji} + iQ)^2 - (x + C_{ji})^2} e^{\frac{1}{2\pi i} \sum_j \left[ \text{Li}_2(-e^{2\pi b(x + D_{ji} + iQ)}) - \text{Li}_2(-e^{2\pi b(x + C_{ji})}) \right]} \sim \int dx \ e^{W(x)}, \quad (2.20)
\]

where \( W(x) \) is the twisted superpotential. By extremising we find:

\[
0 = \partial_x W(x) = 2\pi \xi - \pi i \sum_j (D_{ji} - C_{ji} + iQ) - \frac{1}{ib} \sum_j \left( \log(1 + e^{2\pi b(x + D_{ji} + iQ)}) - \log(1 + e^{2\pi b(x + C_{ji})}) \right). \quad (2.21)
\]

If now we introduce \( \mathbb{C}^* \) coordinates \( X \equiv e^{2\pi b x} \) and \( Y \equiv e^{2\pi b \xi} \), we can rewrite the saddle point equation (2.21) as:

\[
Y = \prod_j \frac{1 + X e^{2\pi b(D_{ji} + iQ)}}{1 + X e^{2\pi b C_{ji}}}, \quad (2.22)
\]

which can be immediately identified with the the mirror curve \( \text{à la} \) Hori-Vafa [12] of the strip geometry depicted in Fig. 1.

\[ ^2 \text{We thank G. Bonelli and N. Drukker for discussion on this point.} \]
The leading contribution to the integral (2.20), the twisted superpotential $W(x)$, is the Abel-Jacobi map on the mirror curve, or in other words, it is the disk amplitude with boundary on the mirror of a toric brane [13, 14]. At this point we could use the Remodelling method [15] to reproduce all the subleading terms in the semiclassical expansion as in [16]. Equivalently we can use the topological vertex [17] to compute A model amplitudes directly reproducing the vortex counting partition function as in [4],[9]. We choose the second approach.

We need to compute open topological $A$ model partition functions on the strip geometry with boundaries mapped into a single toric brane placed in one of the gauge legs.

The open strip partition function is given by [18]:

$$
\frac{K_{\alpha_1, \ldots, \alpha_N}^{\beta_1, \ldots, \beta_N}}{K_{\bullet, \bullet}} = \left( \prod_i s_{\alpha_i}(q^\rho) s_{\beta_i}(q^\rho) \right) \prod_k \prod_{i<j} (1 - q^k Q_{\alpha_i, \beta_j})^{C_k(\alpha_i, \beta_j)} \prod_{i<j} (1 - q^k Q_{\beta_i, \alpha_j})^{C_k(\beta_i, \alpha_j)},
$$

we are interested in the case where the $i$–th representations is equal to a column representation $\alpha_i = 1^n$ and all the other reps are trivial. By using that:

$$
C_k(1^n, \bullet) = 1 \quad \text{for} \quad k \in [0, n-1],
C_k(n, \bullet) = 1 \quad \text{for} \quad k \in [-n + 1, 0],
$$

and otherwise zero we find:

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{The strip geometry.}
\end{figure}
\[
A_n^{(i)} = \frac{K^{i \cdots n}}{K^{i \cdots n}} = \frac{1}{\prod_{k=1}^n (1 - q^{k-1})} \prod_{j<i} \prod_{k=1}^n (1 - Q_{\alpha_j \alpha_j} q^{-(k-1)})\]

(2.25)

where the Kähler parameter are defined by:

\[
Q_{\alpha_i \alpha_j} = \prod_{k=i}^{j-1} M_k F_k,
Q_{\alpha_i \beta_j} = Q_{\alpha_i \alpha_j} M_j,
Q_{\beta_i \alpha_j} = Q_{\alpha_i \alpha_j} M_i^{-1},
\]

(2.26)

with \(i < j\). We then construct the generating function:

\[
Z^{(i)}_{\text{top}} = \sum_n A_n^{(i)} z^n,
\]

(2.27)

which can be immediately identified with the vortex partition function \(Z_V^{(i)}\), given in (2.4), with the following dictionary:

\[
Q_{\alpha_i \alpha_j} = q e^{-2\pi b D_{ij}},
Q_{\alpha_i \alpha_j}^{-1} = q e^{-2\pi b D_{ji}},
Q_{\alpha_i \beta_j} = e^{-2\pi b C_{ij}},
Q_{\beta_i \alpha_j}^{-1} = e^{-2\pi b C_{ji}},
\]

(2.28)

together with a shift of the open modulus \(z \rightarrow M_{i-1}^{-1} z\). It is also easy to realise that the distinct vortex partition functions \(Z_V^{(i)}\) labelled by the index \(i = 1, \ldots, N\) correspond to open topological string amplitudes with the toric brane placed in the \(i\)-th gauge leg.

So far in our computations we have assumed canonical framing for the toric brane in the representation \(\alpha_i\). To allow \(k\) units of framing we need to multiply the strip amplitude (2.25) by the following term\(^3\):

\[
(-1)^{k\lambda} q^{k \kappa(\lambda)/2} = (-1)^{kn} q^{-k(n^2-n)/2}.
\]

(2.29)

By comparing the above expression with eq. (2.10) we realise that the inclusion of a level \(k\) Chern-Simons term in the ellipsoid partition function amounts to the addition of \(k\) units framing for the brane together with a redefinition of the open modulus\(^4\).

\[^3\kappa(\lambda) = \sum_{i} (\lambda_i - 2i + 1) = n - n^2\text{ for } \lambda = 1^n.\]

\[^4\text{In the context of geometric engineering of instanton partition functions the level of the 5-dimensional Chern-Simons term has also been related to framing [19].}\]
3 The chiral theory

In this section we study the $\mathcal{N} = 2 U(1)$ theory with $2N$ chiral fermions with charge $+1$. We introduce axial masses $\mu_i$ with $\sum_i^{2N} \mu_i = 0$ and FI parameter $\xi$. The partition function reads:

$$Z_{S^3_b}^{U(1)}(2N,0) = \int dx e^{2\pi i x \xi} \prod_{j=1}^{2N} s_b(x + \mu_j/2 + iQ/2).$$  \hspace{1cm} (3.1)

As in the previous section to evaluate the integral (3.1) we close the contour in the upper half-plane and pick the contributions of the poles located at:

$$x = -\mu_i/2 + imb + in/b, \quad m,n \geq 0.$$ \hspace{1cm} (3.2)

Thanks to repeated applications of eq. (A.4) it is possible to express the partition function in the following factorised form:

$$Z_{S^3_b}^{U(1)}(2N,0) = \sum_{i=1}^{2N} e^{-\pi i \xi \mu_i} \prod_{j \neq i}^{2N} s_b(E_{ji} + iQ/2) Z^{(i)} \bar{Z}^{(i)},$$ \hspace{1cm} (3.3)

where we introduced $E_{ji} = \mu_j/2 - \mu_i/2$ and defined:

$$Z^{(i)} = \sum_{n=0}^{\infty} \frac{(q^n(1)2(-1)^n)^N}{\prod_{k=1}^{n}(1-q^k)} \frac{e^{-2\pi bn(\xi - N\mu_i/2)}}{\prod_{j \neq i}^{2N} \prod_{k=1}^{n}(1-q^k e^{-2\pi b E_{ji}})};$$ \hspace{1cm} (3.4)

and

$$\bar{Z}^{(i)} = \sum_{m=0}^{\infty} \frac{(q^m(1)2(-1)^m)^N}{\prod_{k=1}^{m}(1-q^m)} \frac{e^{-2\pi bm(\xi - N\mu_i/2)}}{\prod_{j \neq i}^{2N} \prod_{k=1}^{m}(1-q^m e^{-2\pi b E_{ji}})}.$$ \hspace{1cm} (3.5)

We can include a Chern-Simons term in the partition function (3.1) which, when evaluated at the poles (3.2) will contribute as:

$$e^{-\pi i k(\mu_i/2 + imb + in/b)^2}.$$ \hspace{1cm} (3.6)

In particular a Chern-Simons term with level $k = N$ simplifies our expressions since it cancels the factors $(q^n(1)2(-1)^n)^N$ in (3.4) and $(q^m(1)2(-1)^m)^N$ in (3.5) and shifts the FI parameter. So in this case we obtain:

$$Z^{(i)}_{k=N} = \sum_{n=0}^{\infty} \frac{1}{\prod_{k=1}^{n}(1-q^k)} \frac{z^n}{\prod_{j \neq i}^{2N} \prod_{k=1}^{n}(1-q^k e^{-2\pi b E_{ji}})} \equiv Z^{(i)}_V,$$ \hspace{1cm} (3.7)

and

$$Z^{(i)}_{k=N} = \sum_{m=0}^{\infty} \frac{1}{\prod_{k=1}^{m}(1-q^m)} \frac{z^m}{\prod_{j \neq i}^{2N} \prod_{k=1}^{m}(1-q^m e^{-2\pi b E_{ji}})} \equiv Z^{(i)}_V,$$ \hspace{1cm} (3.8)
with
\[ z_i = e^{-2\pi b (\xi + iNQ/2)} = e^{2\pi ib\xi_{\text{eff}}}, \quad \bar{z}_i = e^{-2\pi b (\xi + iNQ/2)} = e^{2\pi ib\xi_{\text{eff}}} \] (3.9)

As expected, \( Z_V^{(i)} \), \( \bar{Z}_V^{(i)} \) are identified with K-theory abelian vortex partition functions with \( 2N \) chiral.

We now use eq. (A.7) to rewrite the prefactor in eq (3.3) in terms of one-loop factors:
\[ \prod_{j \neq i}^{2N} s_b(E_{ji} + iQ/2) = e^{\pi i/2 \sum_{j,i}(E_{ji} + iQ/2)^2} Z_{1-\text{loop}}^{(i)} \tilde{Z}_{1-\text{loop}}^{(i)}, \] (3.10)

with
\[ Z_{1-\text{loop}}^{(i)} = \prod_{k=1}^{\infty} \left( 1 - q^k e^{-2\pi b E_{ji}} \right), \quad \tilde{Z}_{1-\text{loop}}^{(i)} = \prod_{k=1}^{\infty} \left( 1 - \bar{q}^k e^{-2\pi b E_{ji}} \right). \] (3.11)

Finally we introduce the \( i \)-th classical action:
\[ Z_{cl}^{(i)} = e^{-\pi \xi_{\text{eff}} \mu_i}. \] (3.12)

Putting all together the ellipsoid partition function of the \( U(1) \) theory with \( 2N \) chiral and level \( k = N \) Chern-Simons can be expressed (up to a prefactor) as:
\[ Z_{S^3_b}^{U(1), k=N}(2N, 0) = \sum_{i=1}^{N} Z_{cl}^{(i)} \times \left( Z_{1-\text{loop}}^{(i)} \tilde{Z}_{1-\text{loop}}^{(i)} \right) \times \left( Z_V^{(i)} \bar{Z}_V^{(i)} \right). \] (3.13)

We will now show how to geometrically engineer this theory. The relevant geometry is now the “half SU(\( N \)) geometry” that is the resolved \( A_{N-1} \times \mathbb{C} \) fibration depicted in Fig. (2). Actually, as explained in [20], for fixed \( N \), there are \( N+1 \) possible inequivalent geometries labelled by the integer \( m = 0, \ldots N \). Here we restrict to the \( m = 0 \) case, however other choices of \( m \) correspond simply to the inclusion of extra framing factors.

The open partition function is given by [20]:
\[ \frac{\mathcal{K}^{\alpha_1 \cdots \alpha_{2N}}}{\mathcal{K}^{\bullet \cdots \bullet}} = \left( \prod_{l}^{2N} s_{\alpha_l}(q^l) \right) \frac{1}{\prod_k \prod_{i<j}(1 - q^k Q_{j-1}^{-1}) C_k(\alpha_i, \alpha_j^*)}. \] (3.14)

We are interested in configurations involving a single toric brane placed in one of the \( 2N \) gauge legs, which correspond to take \( \alpha_i = 1^n \) and \( \alpha_j = 0 \) for \( j \neq i \) in (3.14). Using the definition of the \( C_k \) coefficients given in eq. (2.24) we obtain:
\[ A^{(i)}_n \equiv \frac{\mathcal{K}^{\bullet \cdots 1^n \cdots \bullet}}{\mathcal{K}^{\bullet \cdots \bullet}} = \frac{1}{\prod_{k=1}^{n} (1 - q^k)} \prod_{k=1}^{n} \prod_{l \neq i} (1 - q^{k(i-1)} Q_{l-1})^{-1} \prod_{j \neq i} (1 - q^{(k-1)} Q_{j-1})^{-1}, \] (3.15)
We now introduce a framing factor \( \left( q^n(n-1)/2(-1)^n \right)^{(i-1)} \) for the toric brane placed in the \( i \)-th leg and construct the generating function:

\[
Z_{top}^{(i)} = \sum_n A_n^{(i)} z^n,
\]  

(3.16)

where we also included in the open modulus the factor \( (Q_1 Q_2^2 Q_3^3 \cdots Q_{i-1}^{i-1}) \). It is now easy to see that the topological string partition function (3.16) matches the vortex partition function (3.7) with the following identifications:

\[
Q_1 \cdots Q_{j-1} = q e^{-2\pi i b E_{ji}}, \quad (Q_j \cdots Q_{i-1})^{-1} = q e^{-2\pi i b E_{ij}}.
\]

(3.17)

4 Discussion

We conclude with few comments about the generality of the factorisation.

As we explained in section 2 partition functions of abelian theories with arbitrary number of chiralss and a Chern-Simons term can be expressed in a factorised form. Indeed we observed that the condition ensuring factorisation, being equivalent to the cancelation of the \( \mathbb{Z}_2 \) anomaly, is always satisfied.

Non abelian theories, such as the \( U(N) \) theory with \( N_f \) flavours can also be argued to factorise. The strategy is to proceed as in [21] and ‘abelianise’ the integral by means of repeated applications of the Cauchy formula so to reduce the non-abelian partition function to a sum of abelian ones. It should be possible to interpret the final expression as a counting of more general vortex configurations labelled by Young tableaux.
However rather than proving factorisation in a case by case analysis, it would be more illuminating to derive it directly from an alternative localisation of the path integral.

In this note we did not discuss possible applications of our results in the context of the recently proposed correspondence [4–6, 22, 23] relating 3d $\mathcal{N} = 2$ gauge theories to analytically continued Chern-Simons theory. This topic will be addressed elsewhere [24].

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A Double-sine identities

The double-sine function is defined as:

$$s_b(x) = \prod_{m,n \geq 0} \frac{mb + n/b + Q/2 - ix}{mb + n/b + Q/2 + ix}, \quad (A.1)$$

and satisfies the following identities:

$$s_b(x)s_b(-x) = 1, \quad s_b(ib/2 - x)s_b(ib/2 + x) = \frac{1}{2\cosh \pi bx}, \quad (A.2)$$

or equivalently:

$$s_b(iQ/2 + x) = \frac{s_b(iQ/2 + x - ib)}{2i \sinh \pi bx}. \quad (A.3)$$

From the above identities we deduce the following expressions useful to evaluate resides:

$$\frac{s_b(x + iQ/2 + imb + in/b)}{s_b(x + iQ/2)} = \frac{(-1)^{mn}}{\prod_{k=1}^n 2i \sinh \pi b(x + ikb) \prod_{l=1}^m 2i \sinh \pi/b(x + il/b)} =$$

$$= \frac{(-1)^{nm}(-i)^{n+m} q_1^{-n(n+1)/2} q_2^{-m(m+1)/2} e^{-\pi b x n} e^{-\pi b x m}}{\prod_{k=1}^n (1 - q_1^{-2k} e^{-2\pi bx}) \prod_{l=1}^m (1 - q_2^{-2l} e^{-2\pi x/b})}, \quad (A.4)$$
\[
\frac{s_b(x - iQ/2 + \text{im}b + \text{in}b)}{s_b(x - iQ/2)} = \frac{(-1)^{nm}}{\prod_{k=1}^{n} 2i \sinh \pi b(x - iQ + ik b) \prod_{l=1}^{m} 2i \sinh \pi / b(x - iQ + il / b)} = \\
\frac{(-1)^{nm}(-i)^{n+m}q_1^{-n(n+1)/2} q_2^{-m(m+1)/2} e^{-\pi b(x - iQ)n} e^{-\pi b(x - iQ)m}}{\prod_{k=1}^{n} (1 - q_1^{-2(k-1)} e^{-2\pi b x}) \prod_{l=1}^{m} (1 - q_2^{-2(l-1)} e^{-2\pi b x})},
\]

where we defined:

\[
q_1 = e^{ibz^2}, \quad q_2 = e^{i\pi / b^2}.
\]

A useful factorised expression for the double sine function:

\[
s_b(z) = e^{-i\pi z^2/2} \prod_{k=1}^{\infty} (1 + e^{2\pi b z} e^{2\pi i b^2(k-1/2)}) =
\]

\[
= \prod_{k=1}^{\infty} (1 + q_1^{2k} q_1 e^{2\pi b z}) \prod_{k=1}^{\infty} (1 + q_2^{2k} q_2 e^{2\pi i b^2}) =
\]

\[
= \prod_{k=0}^{\infty} (1 + q_1^{-(2k+1)} e^{2\pi b z}) \prod_{k=0}^{\infty} (1 + q_2^{-(2k+1)} e^{2\pi i b^2})
\]

and for its log:

\[
\log s_b(z) = \frac{-i\pi z^2}{2} + \left( \sum_k \frac{(-1)^k}{k} \frac{e^{2\pi z b k}}{q_1^k - q_1^{-k}} \right) + \left( \sum_k \frac{(-1)^k}{k} \frac{e^{2\pi i b^2 k}}{q_2^k - q_2^{-k}} \right),
\]

can be derived by using the following analytic continuation:

\[
\prod_{n=0}^{\infty} (1 + q^{-(2n+1)} w) = \exp \left( - \sum_{k=1}^{\infty} \frac{(-1)^k w^k}{k (q^k - q^{-k})} \right) = \frac{1}{\prod_{n=0}^{\infty} (1 + q^{2n+1} w)}.
\]

We also need the \( b \to 0 \) limit:

\[
s_b(z) \to e^{-\pi i z^2/2} e^{i\pi (2 - Q^2)} / 24 \exp \left( \frac{1}{2\pi i b^2} \text{Li}_2(-e^{2\pi b z}) \right).
\]

**References**

[1] N. Hama, K. Hosomichi and S. Lee, “SUSY Gauge Theories on Squashed Three-Spheres,” JHEP **1105** (2011) 014 [arXiv:1102.4716 [hep-th]].

[2] V. Pestun, “Localization of gauge theory on a four-sphere and supersymmetric Wilson loops,” arXiv:0712.2824.
[3] A. Kapustin, B. Willett and I. Yaakov, “Exact Results for Wilson Loops in Superconformal Chern-Simons Theories with Matter,” JHEP 1003 (2010) 089 [arXiv:0909.4559 [hep-th]].

[4] T. Dimofte, S. Gukov, L. Hollands, “Vortex Counting and Lagrangian 3-manifolds,” [arXiv:1005.0977 [hep-th]].

[5] T. Dimofte, S. Gukov, “Chern-Simons Theory and S-duality,” [arXiv:1106.4550 [hep-th]].

[6] T. Dimofte, D. Gaiotto, S. Gukov, “Gauge Theories Labelled by Three-Manifolds,” [arXiv:1108.4389 [hep-th]].

[7] S. Shadchin, “On F-term contribution to effective action,” JHEP 0708 (2007) 052 [arXiv:hep-th/0611278].

[8] B. Eynard and M. Marino, “A holomorphic and background independent partition function for matrix models and topological strings,” J. Geom. Phys. 61 (2011) 1181 [arXiv:0810.4273 [hep-th]].

[9] G. Bonelli, A. Tanzini, J. Zhao, “Vertices, Vortices and Interacting Surface Operators,” [arXiv:1102.0184 [hep-th]].

[10] H. Ooguri, A. Strominger and C. Vafa, “Black hole attractors and the topological string,” Phys. Rev. D 70 (2004) 106007 [arXiv:hep-th/0405146].

[11] V. P. Spiridonov and G. S. Vartanov, “Elliptic hypergeometry of supersymmetric dualities II. Orthogonal groups, knots, and vortices,” arXiv:1107.5788 [hep-th].

[12] K. Hori and C. Vafa, “Mirror symmetry,” arXiv:hep-th/0002222.

[13] M. Aganagic and C. Vafa, “Mirror symmetry, D-branes and counting holomorphic discs,” hep-th/0012041.

[14] M. Aganagic, A. Klemm and C. Vafa, “Disk instantons, mirror symmetry and the duality web,” Z. Naturforsch. A 57, 1 (2002) hep-th/0105045.

[15] V. Bouchard, A. Klemm, M. Marino and S. Pasquetti, “Remodeling the B-model,” Commun. Math. Phys. 287 (2009) 117 [arXiv:0709.1453 [hep-th]].

[16] C. Kozcaz, S. Pasquetti and N. Wyllard, “A & B model approaches to surface operators and Toda theories,” JHEP 1008 (2010) 042 [arXiv:1004.2025 [hep-th]].

[17] M. Aganagic, A. Klemm, M. Mariño and C. Vafa, “The topological vertex,” Commun. Math. Phys. 254 (2005) 425, hep-th/0305132.

[18] A. Iqbal and A. K. Kashani-Poor, “The vertex on a strip,” Adv. Theor. Math. Phys. 10 (2006) 317 [arXiv:hep-th/0410174].

[19] Y. Tachikawa, “Five-dimensional Chern-Simons terms and Nekrasov’s instanton counting,” JHEP 0402 (2004) 050 [arXiv:hep-th/0401184].
[20] A. Iqbal and A.-K. Kashani-Poor, “SU(N) geometries and topological string amplitudes,” Adv. Theor. Math. Phys. 10 (2006) 1–32, hep-th/0306032;

[21] S. Benvenuti and S. Pasquetti, “3D-partition functions on the sphere: exact evaluation and mirror symmetry,” arXiv:1105.2551 [hep-th].

[22] Y. Terashima and M. Yamazaki, “SL(2,R) Chern-Simons, Liouville, and Gauge Theory on Duality Walls,” JHEP 1108 (2011) 135 [arXiv:1103.5748 [hep-th]].

[23] S. Cecotti, C. Cordova and C. Vafa, “Braids, Walls, and Mirrors,” arXiv:1110.2115 [hep-th].

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