A time–dependent Cosmological ”Constant”

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Abstract

We extend the Wald cosmic no–hair theorem to a general class of scalar–tensor nonminimally coupled theories of gravity where ordinary matter is also present in the form of a perfect fluid. We give a set of conditions for obtaining a de Sitter expansion independently of any initial conditions, generalizing the treatment of such a problem as given by Wald. Finally we apply the results to some specific models.

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1 Introduction

In this paper we generalize the so called "cosmological no–hair theorem" [1], to the case of nonminimally coupled scalar–tensor theories of gravity. These ones were formulated years ago in order to better understand inertia and Mach’s Principle in the theoretical framework of General Relativity. Today, they are playing an important role in cosmology, due to the fact that they seem to solve a lot of shortcomings connected with inflationary models like those related to the phase transition from false vacuum to true vacuum in early universe and those related to the bubble spectrum able to produce seeds for the large scale structures (see [2],[3],[4] and references therein for a discussion of nonminimally coupled scalar-tensor theories).

The no–hair conjecture was introduced by Hoyle and Narlikar [5]: roughly speaking, they claimed that if there is a positive cosmological constant, all the expanding universes will approach the de Sitter behaviour. That is, if a cosmological constant is present, no matter which are the initial conditions, the universe will become homogeneous and isotropic. However, there is something vague in such a formulation. It is not clear the meaning of ”expanding universe” as well as ”approach the de Sitter behaviour”. Furthermore, there is no general proof (or disproof) of such a conjecture; on the contrary, there are counter–examples of initially expanding and then recollapsing universes which never become de Sitter [7].

In 1983, Wald gave a proof of a simplified version of the conjecture. He proved that:

All Bianchi cosmologies (except IX), in presence of a positive cosmological constant, asymptotically approach the de Sitter behaviour [8].

In all these discussions (in Wald’s paper too), the cosmological constant is a true constant and it is put by hand in the gravitational arena. It is relevant that in Wald’s proof are not used the Bianchi identities, then the proof is independent of the dynamical evolution of the material sources. Here we discuss how to introduce a time dependent ”cosmological constant” in the context of scalar–tensor theories of gravity in which a scalar field φ is nonminimally coupled to geometry being also present a standard perfect fluid matter (noninteracting with φ), (see also [9]) . So doing, we will introduce a more general set of conditions respect to those given by Wald, not based on various ”energy conditions” as in Wald, by which it is possible to get a de Sitter asymptotic behaviour for the scale factor of the universe; in other words, we introduce an ”effective cosmological constant” via the asymptotic de Sitter expansion. Furthermore, we will show that the de Sitter asymptotic behaviour is not depending on the (asymptotic) sign, as well as on the value, of the effective gravitational ”constant”, which is, in our units (\(8\pi G_N = \hbar = c = 1\)),

\[
G_{\text{eff}} = -\frac{1}{2F(\phi)},
\]

(1)

where \(F(\phi)\) describes the (nonminimal) gravitational coupling. In what follows, we take into account only Friedman–Robertson–Walker (FRW) flat cosmologies described by the scale factor \(a(t)\). The scalar field \(\phi\) is, of course, a function of time only.
The paper is organized as follows: we discuss first how to introduce a cosmological constant, then we generalize Wald’s theorem. Finally we give some concrete examples.

2 The Effective Cosmological Constant

The class of theories we are considering are described by the action

$$\mathcal{A} = \int d^4x \sqrt{-g} \left[ F(\phi) R + \frac{1}{2} g^{\mu\nu} \phi_{,\mu} \phi_{,\nu} - V(\phi) + \mathcal{L}_m \right],$$  \hspace{1cm} (2)

where $V(\phi)$ and $F(\phi)$ are generic functions of the field $\phi$ and $\mathcal{L}_m$ is the ordinary matter contribution to the total Lagrangian density.

The Einstein equations can be written as

$$G_{\mu\nu} = \tilde{T}_{\mu\nu} = -\frac{1}{2F(\phi)} T^{(\text{tot})}_{\mu\nu},$$ \hspace{1cm} (3)

where $G_{\mu\nu}$ is the Einstein tensor and $T^{(\text{tot})}_{\mu\nu}$ is the total stress–energy tensor

$$T^{(\text{tot})}_{\mu\nu} = T^{(\phi)}_{\mu\nu} + T^{(m)}_{\mu\nu},$$ \hspace{1cm} (4)

where

$$T^{(\phi)}_{\mu\nu} = \phi_{,\mu} \phi_{,\nu} - \frac{1}{2} g_{\mu\nu} \phi^{,\alpha} \phi_{,\alpha} + g_{\mu\nu} V(\phi) + 2g_{\mu\nu} \Box F(\phi) - 2F(\phi)_{,\mu\nu},$$ \hspace{1cm} (5)

represents the scalar field source and $T^{(m)}_{\mu\nu}$ is the perfect fluid matter source. The Klein–Gordon (KG) equation is obtained from the action (2) by varying with respect to $\phi$:

$$\Box \phi - R F'(\phi) + V'(\phi) = 0,$$ \hspace{1cm} (6)

where the prime means the derivative with respect to $\phi$. Using the Einstein equations, from the contracted Bianchi identity [8], we obtain the relation:

$$[F(\phi) G^{(\phi)}_{\mu\nu} - T^{(\phi)}_{\mu\nu}]_{,\nu} = \frac{1}{2} \phi_{,\mu} [\Box \phi - R F'(\phi) + V'(\phi)] = T^{(m)}_{\mu\nu} ;$$ \hspace{1cm} (7)

then imposing the KG equation, we get $T^{(m)}_{\mu\nu} = 0$, which is the usual ”Bianchi identity” for standard matter.

The major point of our discussion is the following: it is possible to construct a time–dependent cosmological ”constant” coherently with the Einstein equations as well as to the (contracted) Bianchi identity? In other words, for constructing an effective (time–dependent) cosmological ”constant”, we cannot refer to the standard stress–energy tensor of the form $\Lambda g_{\mu\nu}$ since this implies the introduction of a truly constant $\Lambda$.

As we already said, we will restrict our considerations only to FRW (flat) universes. In other words, we will introduce an effective (time dependent) cosmological ”constant”...
only in a cosmological context. We will discuss this important point in the conclusions. In this case the metric is:

\[ ds^2 = dt^2 - a(t)^2(dx^2 + dy^2 + dz^2) , \]

where \( a = a(t) \) is the scale factor of the universe. From (3), using metric (8), we get the (cosmological) Einstein equations

\[ H^2 + \frac{\dot{F}}{F} H + \frac{\rho_\phi}{6F} + \frac{p_m}{6F} = 0 , \]

\[ \dot{H} = - \left( H^2 + \frac{V}{6F} \right) - H \frac{\dot{F}}{2F} + \frac{\dot{\phi}^2}{6F} - \frac{1}{2} \frac{\ddot{F}}{F} + \frac{3p_m + \rho_m}{12F} . \]

where \( H = \dot{a}/a \), \( \rho_\phi = \frac{1}{2} \dot{\phi}^2 + V(\phi) \), \( \rho_m, p_m \) are, respectively, the Hubble parameter, the energy density of scalar field, the energy density and the pressure of standard matter. We have not used the state equation of standard matter. Eq.(9) can be rewritten as:

\[ \mathcal{P}(H) \equiv (H - \Lambda_{\text{eff},1}) (H - \Lambda_{\text{eff},2}) = -\frac{\rho_m}{6F} , \]

where \( \mathcal{P}(H) \) is a second degree polynomial in \( H \), and

\[ \Lambda_{\text{eff},1,2} = -\frac{\dot{F}}{2F} \pm \sqrt{\left( \frac{\dot{F}}{2F} \right)^2 - \frac{\rho_\phi}{6F}} , \]

"1" is relative to choose the plus sign and "2" to the minus. Using the effective gravitational coupling (1) and its time (relative) variation, the quantities defined by Eq.(12) can be rewritten:

\[ \Lambda_{\text{eff},1,2} = \frac{\ddot{G}_{\text{eff}}}{2G_{\text{eff}}} \pm \sqrt{\left( \frac{\ddot{G}_{\text{eff}}}{2G_{\text{eff}}} \right)^2 + \frac{G_{\text{eff}} \rho_\phi}{3}} . \]

Furthermore, \( \Lambda_{\text{eff},1,2} \) have to be real, then the restriction

\[ \left( \frac{\dot{F}}{2F} \right)^2 \geq \frac{\rho_\phi}{6F} , \]

has to be satisfied. From the definition we have given, we get:

\[ \Lambda_{\text{eff},1} + \Lambda_{\text{eff},2} = -\frac{\dot{F}}{F} , \quad \Lambda_{\text{eff},1} - \Lambda_{\text{eff},2} = 2 \sqrt{\left( \frac{\dot{F}}{2F} \right)^2 - \frac{\rho_\phi}{6F}} \geq 0 , \]

that is, in general,

\[ \Lambda_{\text{eff},1} \geq \Lambda_{\text{eff},2} . \]
In terms of $\mathcal{P}(H)$, Eq. (10) for $\dot{H}$ becomes

$$
\dot{H} + \mathcal{P}(H) = H \frac{\dot{F}}{2F} + \frac{\dot{\phi}^2}{4F} - \frac{1}{2} \frac{d}{dt} \left( \frac{\dot{F}}{F} \right) - \frac{1}{2} \left( \frac{\dot{F}}{F} \right)^2 + \frac{3p_m + \rho_m}{12F}.
$$

(17)

We make now the following two hypotheses on the asymptotic behavior of the $F(\phi)$, i.e. for $t \gg 0$, we suppose that:

$$
\frac{\dot{F}}{F} \rightarrow \Sigma_0,
$$

(18)

$$
\frac{\rho_0}{6F(\phi)} \rightarrow \Sigma_1;
$$

(19)

where $\Sigma_{0,1}$ are two constants depending, respectively, on the parameters of the coupling and of the coupling and of the potential. Under these two hypotheses we see that the two quantities $\Lambda_{\text{eff},1,2}$ asymptotically go to constants. Vice versa, if we assume that $\Lambda_{\text{eff},i} \rightarrow \Lambda_i$ (constants), we see that $\dot{F}/F$ and $\rho_0/6F$ become constants for large $t$. Then hypotheses (18) and (19) are necessary and sufficient conditions since the two $\Lambda$’s are asymptotically constants. It is important to stress that hypothesis (18) does not select a specific asymptotic behavior for $\dot{F}/F$ since a wide class of $G_{\text{eff}}$ is allowed.

We will also assume, in the following considerations, that asymptotically the sign of $F(\phi)$ is constant (this is our third, quite natural, assumption), and then we have to consider the two cases: $F(t \gg 0) \leq 0$ and $F(t \gg 0) \geq 0$. Since we are considering that, asymptotically, $\dot{F}/F$ is constant, each of the above cases has two subcases related to the sign of $\dot{F}$. Of course the case $F(t \gg 0) \leq 0$ is physically relevant: the other one (repulsive gravity) can be interesting if related to the possibility of recovering the de Sitter behaviour for $a(t)$. In this way it appears clear that recovering such an asymptotic behaviour for $a(t)$ in not even connected to recover the standard sign of gravity, as we will discuss below, in general and in connection to some concrete examples. Let us now consider the case $F(t \gg 0) \leq 0$ and $\dot{F}(t \gg 0) \leq 0$; from hypothesis (18) we have $\Sigma_0 \geq 0$. Furthermore the condition (14) is (asymptotically) satisfied. Eq. (11) gives

$$
\mathcal{P}(H) \geq 0,
$$

(20)

then we have $H \geq \Lambda_1$, $H \leq \Lambda_2$. For the two $\Lambda_i$, we obtain the asymptotic expressions:

$$
\Lambda_1 = -\frac{\Sigma_0}{2} + \sqrt{\left( \frac{\Sigma_0}{2} \right)^2 + |\Sigma_1|} \geq 0,
$$

(21)

$$
\Lambda_2 = -\frac{\Sigma_0}{2} - \sqrt{\left( \frac{\Sigma_0}{2} \right)^2 + |\Sigma_1|} \leq 0.
$$

(22)

Considering Eq. (10), we have

$$
\dot{H} = - \left( H^2 - \frac{V}{6|F|} \right) - H \frac{\dot{F}}{2F} - \frac{\dot{\phi}^2}{6|F|} - \frac{1}{2} \left( \frac{\dot{F}}{F} \right)^2 - \frac{3p_m + \rho_m}{12|F|}.
$$

(23)
If (this is our last hypothesis)
\[ H^2 \geq \frac{V}{6|F|}, \tag{24} \]
we obtain then
\[ \dot{H} \leq 0. \tag{25} \]
In other words, from the two disequalities on \( P(H) \) and on \( \dot{H} \) we find that \( H(t) \) has a horizontal asymptote, or, equivalently, \( H \) goes to a constant. Then the universe, for large \( t \), has a de Sitter behaviour, (i.e. \( a(t) \sim \exp(\alpha t) \), where \( \alpha \) is an unknown constant).

Under the conditions (18), (19), the constant asymptotic sign of \( F(\phi(t)) \) and under the condition (24), the universe, for large \( t \), expands as de Sitter, even if it is not fixed the parameter which specifies such an expansion, i.e. the effective cosmological constant. If we compare Wald’s conditions with ours, we have:

\[
\begin{align*}
\left( H - \frac{\Lambda}{3} \right) \left( H + \frac{\Lambda}{3} \right) & \geq 0 \\
\dot{H} & \leq \frac{\Lambda}{3} - H^2 \leq 0 \quad \Rightarrow \quad \dot{H} \leq 0.
\end{align*}
\]

Essentially the equations involving \( H \) are the same in both cases. The true difference concerns the equation for \( \dot{H} \); our condition (\( \dot{H} \leq 0 \)) is more general than \( \dot{H} \leq \left( \frac{\Lambda}{3} - H^2 \right) \leq 0 \). The hypothesis (24), when \( \phi \to \text{const} \) is nothing else but \( H^2 \geq \frac{\Lambda}{3} \) (in our unit \( G_{\text{eff}} \to G_N \) if \( F \to -\frac{1}{2} \)); that is we recover the standard case where \( V = \text{const} \) is interpreted as the cosmological constant. By some algebra, it is easy to show that such a hypothesis is equivalent to

\[
\frac{1}{12F^2} \dot{\phi}^2 \geq \left( \frac{F'}{F} \right)^2 = \left( \frac{G'_{\text{eff}}}{G_{\text{eff}}} \right)^2. \tag{26}
\]

That is the above hypothesis pose a constraint on the minimum value (given by the relative (quadratic) variation of \( G_{\text{eff}} \)) of the (effective) ratio of the kinetic energy and the potential energy of the scalar field. Having shown that \( a(t) \) behaves like de Sitter for large \( t \), we have to see if it is possible to fix \( \alpha \) in order to recover the effective cosmological constant. To this purpose, the Bianchi contracted identity for matter is needed (it is important to stress that we have not used any Bianchi contracted identity to find the asymptotic behaviour of \( a(t) \)). As usual, we get \( \rho_m = Da^{-3\gamma} \) (we have used the state equation \( p_m = (\gamma - 1)\rho_m \), with \( 1 \leq \gamma \leq 2 \); \( D \) is the integration constant giving the matter content of universe). Introducing this expression for the matter in Eq.(17), for large \( t \), we have

\[
(H - \Lambda_1)(H + |\Lambda_2|) = \frac{D}{|F_0|} e^{-(3\gamma c + \Sigma_0)t}, \tag{27}
\]
being $3\gamma\alpha + \Sigma_0 \geq 0$. Then we get

$$(H - \Lambda_1)(H + |\Lambda_2|) \to 0,$$

i.e. $H \to \Lambda_1$. The (effective) matter content, $\rho_m/6F(\phi)$, tells us how $H$ is "distant" from the de Sitter behaviour given by the cosmological constant $\Lambda_1$. In other words, we do not use the Bianchi identity for finding the type of expansion, we only use it to select (asymptotically) the specific value of what we call "cosmological constant". Of course we have that the universe undergoes a de Sitter asymptotic expansion independently of any initial data. Actually the effective cosmological constant that we have obtained via such a procedure will depend on the parameters connected to the effective gravitational coupling "constant" and on those connected to the potential $V(\phi)$. Essentially, we have introduced the (effective) cosmological constant in a "pragmatic" way, through the (asymptotic) de Sitter behaviour for $a(t)$. In a certain sense, the approach in [1] is reversed: there, $\Lambda$ (constant) is introduced a–priori and this leads, under certain hypotheses, to a de Sitter expansion. Here, the de Sitter expansion is recovered under completely different hypotheses, and this (together with the contracted Bianchi identity for matter) selects the effective cosmological constant. Moreover, we have obtained such a result without assuming to recover the standard gravity (i.e. we do not need that $G_{eff} \to G_N$). If we now consider also the KG equation, from the condition (19), we get, for large $t$,

$$\frac{\dot{\phi}^2}{F(\phi)} = C_1(\Sigma_0, \Sigma_1) = -2\Sigma_0 \left(\sqrt{\left(\frac{\Sigma_0}{2}\right)^2 + |\Sigma_1|} - \frac{3}{2}\Sigma_0\right),$$

(29)

that is $\dot{\phi}^2/F(\phi)$ goes to a constant. Being $F(\phi(t \gg 0)) \leq 0$, such a constant has to be negative: this request implies the following relation between $\Sigma_0$ and $\Sigma_1$, which has to be satisfied for the sign compatibility:

$$|\Sigma_1| \geq 2\Sigma_0^2.$$  

(30)

By Eq.(30) and condition (19), we get also

$$\frac{V}{6F} = C_2(\Sigma_0, \Sigma_1) = \left(\sqrt{\left(\frac{\Sigma_0}{2}\right)^2 + |\Sigma_1|} \right) \left(\frac{\Sigma_0}{6} - \sqrt{\left(\frac{\Sigma_0}{2}\right)^2 + |\Sigma_1|}\right).$$

(31)

That is the potential has to be (asymptotically) nonnegative. From the above relations we see that in the case $\Sigma_0 = 0$, we get that only $\frac{\dot{V}}{6F}$ is different from zero, giving rise to the expression $\frac{V}{6F}(t \gg 0) = -\Sigma_1^2$ which identifies, in this case, the cosmological (asymptotically) constant. Finally from Eqs.(29),(31), we find:

$$\frac{\dot{\phi}^2}{V} = C_3(\Sigma_0, \Sigma_1) = \frac{2\Sigma_0 \sqrt{\left(\frac{\Sigma_0}{2}\right)^2 + |\Sigma_1| - 3\Sigma_0^2}}{\Sigma_0 \sqrt{\left(\frac{\Sigma_0}{2}\right)^2 + |\Sigma_1| - 6\left(\frac{\Sigma_0}{2}\right)^2 - 6|\Sigma_1|}}.$$  

(32)

6
We will show the relevance of these relations discussing some concrete examples at the end of the paper. Let us consider now the other possibility connected to the case $F(\phi(t \gg 0)) \leq 0$, that is $\dot{F}(\phi(t \gg 0)) \geq 0$. In this case, $\Sigma_0 \leq 0$ while everything else is the same as in the case discussed above. In particular, the signs of the asymptotic values of $\Lambda_{1,2}$ are the same. Referring to our previous analysis, it is easy to show that now everything goes as in the Wald case (as it is clear looking at (17), that is it is possible to get the same two inequalities which are found in his proof). It is interesting that the compatibility of all the hypotheses that we have made with the KG equation gives rise again to Eq.(29), but being $\Sigma_0 \leq 0$, we get $\dot{\phi}^2/F(\phi) \geq 0$. Then the compatibility between (19) and the KG equation implies, for large $t$, that the scalar field has to go to a constant. In our units, $F \rightarrow -1/2$, and $\Lambda \rightarrow \sqrt{V(t \gg 0)}/3$.

Finally, let us consider the case of asymptotically repulsive gravity, that is

$$F(\phi(t \gg 0)) \geq 0.$$  \hfill (33)

Also here we have two subcases, $\dot{F}(\phi(t \gg 0)) \leq 0$ and $\dot{F}(\phi(t \gg 0)) \geq 0$. As we have already stressed, even if this situation seems unphysical, it gives a better understanding of the non-necessary correlation between the (asymptotic) de Sitter behaviour (i.e. between the no-hair) theorem and the recovering of standard gravity. Of course, the condition on the reality of $\Lambda_i$ now has to be carefully considered. The most interesting case is $\dot{\Lambda} \leq 0$.

Here, we have two (asymptotic) positive cosmological constants, that is

$$\Lambda_{\text{eff}1,2} \rightarrow \Lambda_{1,2} \geq 0, \quad \text{with} \quad \Lambda_1 \geq \Lambda_2.$$  \hfill (34)

Being $-\rho_m/6F \leq 0$, we have $\Lambda_1 \leq H \leq \Lambda_2$. Then, it is crucial to know the sign of $\dot{H}$: if $\dot{H} \geq 0$ the effective $\Lambda$ is given by the $\max(\Lambda_1, \Lambda_2)$; viceversa, if $\dot{H} \leq 0$, $\Lambda$ is given by the minimum between them. We will discuss an example of this last situation.

3 Examples

Now we present some realizations of the above discussion. First of all, we have that the field Lagrangian (density), giving rise to the action (2), becomes in the FRW (flat) case:

$$\mathcal{L} = \mathcal{L}_\phi + D a^{3(1-\gamma)},$$  \hfill (35)

where

$$\mathcal{L}_\phi = 6a\dot{a}^2F(\phi) + 6a\dot{a}^2F(\phi)\dot{\phi} + a^3 \left( \frac{1}{2} \dot{\phi}^2 - V(\phi) \right).$$  \hfill (36)

We restrict our analysis to a dust-dominated universe ($\gamma = 1$), that is to the case $\mathcal{L} = \mathcal{L}_\phi + D$, since we are interested in asymptotic regimes.

1. The simplest example is given by $\phi = \text{const}$, $F(\phi) = -1/2$ and $V(\phi) = \Lambda$, that is the standard de Sitter case. In this case we have $\Sigma_0 = 0$ and $\Sigma_1 = -\Lambda/3$. 7
2. Let us consider an *a–priori* generic nonminimal coupling $F(\phi)$ and the potential $V(\phi) = \Lambda$. Using the Nöther Symmetry Approach [4,8], we get $F(\phi) = \frac{1}{12} \phi^2 + F'_0 \phi + F_0$, where $F'_0$ and $F_0$ are two generic parameters. We have already discussed such a case in [4,8]. From the relation between the asymptotic behaviour of the potential and the coupling (relation that we have found using the compatibility between the hypotheses we have done and the KG eq.) we see that the coupling has to go asymptotically to a constant. Anyway the general solution, for $\gamma = 1$, is

$$a(t) = \left[ c_1 e^{\lambda t} + c_2 e^{-\lambda t} \right]^{1/2},$$

and

$$\phi(t) = \frac{J_0}{\sqrt{c_1 e^{\lambda t} + c_2 e^{-\lambda t}}} \, \mathcal{K} + \frac{c_3}{\sqrt{c_1 e^{\lambda t} + c_2 e^{-\lambda t}}} - 6F'_0,$$

where $c_1$, $c_2$ and $c_3$ are the three integration constants and $\lambda = \sqrt{-2\Lambda/3H}$. $J_0$ is a constant of motion, $H = F_0 - 3F'_0^2 \leq 0$ is the $\phi$–part of Hessian determinant of ${\cal L}$, which depends only on the parameters connected to the function describing the coupling, and

$$\mathcal{K} = \int \frac{dt}{\sqrt{c_1 e^{\lambda t} + c_2 e^{-\lambda t}}},$$

is an elliptical integral of first kind. In this case, the effective cosmological constant is asymptotically given by

$$-\frac{\Lambda}{6F(\phi(t \gg 0))} = \frac{V(\phi(t \gg 0))}{3|H|}, \quad |\Sigma_1| = \frac{\Lambda}{6|H|}. \quad (40)$$

since $\Sigma_0 \rightarrow 0 \ (\phi \rightarrow -6F'_0)$. Then, for $t \rightarrow \infty$, we have $a(t) \sim e^{\frac{\lambda}{2} t}$. In this case, the conditions (15) and (19) hold, and the standard Einstein gravity ($G_{\text{eff}} \rightarrow G_N$) is restored. Of course $\lambda = 2 \sqrt{\frac{V(\phi(t \gg 0))}{3|H|}}$.

3. In the case $F(\phi) = k_0 \phi^2, \ V(\phi) = \lambda \phi^2, \ \gamma = 1$, where $k_0 < 0$ and $\lambda > 0$ are free parameters, the de Sitter regime is recovered even if solutions do not converge toward standard gravity. The coupling $F(\phi)$ is always negative, whereas $V(\phi)$ is always positive and $\dot{F}(\phi(t \gg 0)) < 0$. Infact the general solutions are [4,8]

$$a(t) = \left[ c_1 e^{\lambda_0 t} + c_2 e^{-\lambda_0 t} \right] \times \exp \left\{ -\frac{2}{3} \left[ c_3 \arctan \left( \frac{\sqrt{c_1}}{\sqrt{c_2}} e^{\lambda_0 t} + c_4 \ln(c_1 e^{\lambda_0 t} + c_2 e^{-\lambda_0 t}) \right) \right] \right\}, \quad (41)$$

which is clearly de Sitter for $t \gg 0$, and

$$\phi(t) = \frac{\exp \left[ c_3 \arctan \left( \frac{\sqrt{c_2}}{\sqrt{c_1}} e^{\lambda_0 t} + c_4 \ln(c_1 e^{\lambda_0 t} + c_2 e^{-\lambda_0 t}) \right) \right]}{c_1 e^{\lambda_0 t} + c_2 e^{-\lambda_0 t}}. \quad (42)$$
where
\[
\Lambda_0 = \sqrt{\frac{2\lambda\xi_2}{\zeta_1(\xi_1 - \xi_2)}}, \quad c_3 = \frac{F_0\sqrt{c_1c_2}}{c_2\Lambda_0}, \quad c_4 = \frac{\xi_1}{\xi_2}, \quad \xi_1 = 1 - 12k_0, \quad \xi_2 = 1 - \frac{32}{3}k_0.
\]

The constants \(c_1, c_2, c_3\) are the initial data and \(F_0\) is a constant of motion related to the existence of the Nöther symmetry \([4],[8]\). We get asymptotically
\[
\Sigma_0 = \sqrt{-\frac{32\lambda k_0}{(1 - 12k_0)(3 - 32k_0)}}, \quad \Sigma_1 = \frac{\lambda (128k_0^2 - 24k_0 + 1)}{2k_0(3 - 32k_0)(1 - 12k_0)}.
\]

In this case, it is always \(\Sigma_0 > 0\). Finally
\[
\Lambda_{\text{eff},1} = -\frac{\dot{F}}{2F} + \sqrt{\left(\frac{\dot{F}}{2F}\right)^2 - \frac{\rho_0}{6F}} \rightarrow \Lambda = \sqrt{\frac{\lambda(1 - 8k_0)^2}{2k_0(12k_0 - 1)(3 - 32k_0)}} > 0,
\]
which is exactly the constant that appears in the asymptotic behaviour of the scale factor \(a(t) \sim \exp(\Lambda t)\), i.e. the effective cosmological constant. It is relevant to stress that, we have
\[
F(\phi(t \gg 0)) \rightarrow k_0 \exp \left[2\Lambda \left(\frac{4|k_0|}{3 + 32|k_0|}t\right)\right] < 0,
\]
and \(F(\phi(t))\) diverges. We do not recover asymptotically the standard \(G_N\). Actually we have (plus infinity) asymptotic gravitational freedom \([11]\); nevertheless we have a de Sitter behaviour at infinity for \(a(t)\). Furthermore, the condition \([19]\) is always satisfied.

4. Another interesting case is \(F(\phi) = k_0\phi^2,\ V(\phi) = \lambda\phi^2,\ \gamma = 1,\) with \(k_0 > 0\) (precisely \(1/12 < k_0 < 3/32\)). The solutions are essentially the same as in the case \(k_0 < 0\), except we have to change "\(\arctan\)" with "\(\text{arctanh}\)". The asymptotic behaviours of \(a(t)\) and \(\phi(t)\) are:
\[
a(t) \sim e^{\Lambda_0(1 - 8k_0)/(3 - 32k_0)t}, \quad \phi(t) \sim e^{\Lambda_0(1k_0)/(32k_0 - 3)t},
\]
Now we have \(32k_0 - 3 < 0\) and then \(\phi(t)\) is a decreasing function of time, which implies \(\dot{F}(\phi(t \gg t)) \leq 0\) and \(\Sigma_0 \leq 0\). We see that \(\dot{H} \leq 0\) and then the effective cosmological constant is given by the min \((\Lambda_1, \Lambda_2)\). By some algebra, it is possible to verify that the true cosmological constant is \(\Lambda_2\) (which is always less than \(\Lambda_1\)). This example is useful to stress that the de Sitter asymptotic behaviour, connected with the presence of a cosmological constant, is independent of the sign of gravitational coupling.

5. The last case we discuss is \(F = -1/2,\ V(\phi) = V_0(Ae^{2\lambda\phi} + Be^{2\lambda\phi})^2,\) where \(A, B\) and \(V_0\) are constants, and \(\lambda\) is given in terms of \(G_N\) (see \([10]\) for details). We are in a
situation similar to that discussed in our second example. Using the asymptotic relation between the coupling and the potential, also in this case the potential has to go (asymptotically) to a constant. Anyway, we have used our Noether symmetry approach for solving exactly the model. Asymptotically, using the behaviour of the exact solutions we find

\[ a(t) \sim e^{\Lambda t}, \quad \phi \sim \text{const}, \]

where \( \Lambda \) at the exponent of \( a(t) \) is:

\[ \Lambda = \sqrt{\frac{4|AB|}{3}}. \]

If we compute the effective (positive) \( \Lambda \) from the definition (12), we find

\[ \sqrt{-V(\phi) / 6F} = \sqrt{\frac{4|AB|}{3}}; \]

i.e. the same quantity as given by the asymptotic behaviour of the scale factor. Of course the standard matter has no role in this asymptotic regime.

We conclude the discussion of these examples stressing, again, that it appears clear that the (asymptotic) cosmological constant, as introduced in our approach, is a function of the parameters appearing into the two functions \( F(\phi), V(\phi) \).

4 Conclusions

We have discussed the cosmic no–hair theorem in the framework of nonminimally coupled scalar–tensor theories. We have introduced a time dependent cosmological ”constant” not using the ”geometrical side” of such theories (i.e. \( \Lambda g_{\mu \nu}, \) as usual) but the ”scalar side”. That is the effective cosmological ”constant” has been reconstructed by \( \dot{G}_{\text{eff}} / G_{\text{eff}} \) and by \( \rho_\phi / 6F(\phi) \). Actually \( \Lambda_{\text{eff}} \) has been introduced only in the case of homogeneous–isotropic flat cosmologies but it is not difficult to extend the above considerations to any Bianchi model (except Bianchi IX). The way we have followed to reconstruct the no–hair theorem is opposite of that usually adopted: instead of introducing by hand a cosmological constant and then searching for the conditions for an asymptotic de Sitter behaviour, we find the conditions to get such an asymptotic behaviour, and then we define an effective cosmological ”constant” (actually function of time), which becomes a (true) constant for \( t \gg 0 \). Of course, the time behaviour of \( \Lambda_{\text{eff}} \) can be of any type with respect to the asymptotic constant value [12]. Under the hypotheses we used, the de Sitter asymptotic regime is obtained and this is not necessarily connected with recovering the standard Einstein gravity (which is restored, in our units, for the value \( F(\phi)_\infty = -1/2 \) of the coupling). In other words, the cosmic no–hair theorem holds even if we are not in the Einstein regime (it is not even necessary that the right (attractive gravity) sign of the coupling is recovered). Furthermore, the role of the Bianchi contracted identity for the
standard) matter is to fix (only) the specific value of Λ, not the kind of the (de Sitter) asymptotic behaviour of a(t). It is interesting to stress that, by this mechanism, the "amount of Λ" is strictly related to the matter content of the universe. This is worthwhile in connection to the Ω problem since it seems that cold dark matter models, with non trivial amount of cosmological constant, have to be taken into serious consideration for large scale structure formation. In conclusion, we want to make two final remarks. The first concerns an important question which we have only mentioned. The way we have followed to introduce the (effective) cosmological "constant" seems to confine its meaning only to the cosmological arena. In the standard way used to define such a quantity, the problem does not exists since it is a true constant of the theory and then it is defined independently of any cosmological scenario. We believe that this question can be solved stressing that cosmology has to be taken into account in any other specific physical situation in relativity. Then the effective time–dependent cosmological constant we have introduced gets a role of the same kind of the standard Λ. From this point of view, the question we are discussing can be answered still using the (standard) way to define the cosmological constant, i.e.,(the cosmological) $T_{00}$. This is what we actually have done and what we believe to be the ingredient to use for understanding the role of (effective) cosmological "constant" also in different contexts than cosmology. Finally, in our construction of Λ, there is a contribution given by the (relative) time variation of the effective gravitational coupling: this implies that it would be possible to compute it, for example, via the density contrast parameter. This will be our next step in this kind of research.

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