AN ANALYTIC KOSZUL COMPLEX
IN A BANACH SPACE

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ABSTRACT. We show that the holomorphic ideal sheaf of a linear section of
a pseudoconvex open subset Ω of, say, a Hilbert space $X = \ell_2$ is acyclic. We
also prove an analog of Hefer’s lemma, i.e., if $f : \Omega \times \Omega \to \mathbb{C}$ is holomorphic
and $f(x, x) = 0$ for $x \in \Omega$, then there is a holomorphic $g : \Omega \times \Omega \to X^*$ with
values in the dual space $X^*$ of $X$ such that $f(x, y) = g(x, y)(x - y)$.

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1. INTRODUCTION.

Variants of the Koszul complex play so important a role in commutative
algebra, and algebraic and analytic geometry that one can verily call it the
mother of all resolutions.

This paper establishes an exactness and acyclicity result for an analytic
Koszul complex in a Banach space that serves as preparation to show in [P2]
that the ideal sheaf of certain complex submanifolds of a Banach space be-
longs to a class of sheaves, to be called therein of type (S), which are studied
and proved acyclic in certain cases therein via a method of resolutions. In
effect, we do here the case of the ideal sheaf of a linear submanifold.

Let $X', X'', Z$ be complex Banach spaces, $X = X' \times X''$, $\Lambda_p$ the Banach
space of all continuous complex $p$-linear alternating maps $X'' \to Z$ for $p \geq 0$;
$\Lambda_0 = \Lambda_{-1} = Z$; and $\mathcal{O}^\Lambda_p \to X$ the sheaf of germs of holomorphic functions
$X \to \Lambda_p$. Let $E$ be the Euler vector field on $X''$ defined by $E(x'') = x''$,
$\mathcal{L}_E$ the Lie derivation, and $i_E$ the inner derivation determined by the vector
field $E$, i.e., $i_E$ is the contraction of $p$-forms with $E$: if $f$ is a local section
of $\mathcal{O}^\Lambda_p$, then let $i_Ef$ be the local section of $\mathcal{O}^{\Lambda_{p-1}}$ given for $p \geq 1$ by

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(i_E f)(x', x'')(\xi''_1, \xi''_2, \ldots, \xi''_{p-1}) = f(x', x'')(x'', \xi''_1, \ldots, \xi''_{p-1}), and for p = 0 by (i_E f)(x', x'') = f(x', 0). Let I be the subsheaf of \( O^2 \) of all sections that vanish on \( X' \). We consider the Koszul complex

\[ \ldots \to O^{A_p} \to O^{A_{p-1}} \to \ldots \to O^{A_1} \to I \to 0 \]

of analytic sheaves over \( X \), where each map is \( i_E \). Let \( K_p, p \geq 0 \), be the corresponding sequence of kernel sheaves: \( K_p(U) = \{ f \in O(U, \Lambda_p) : i_E f = 0 \text{ on } U \}, U \subset X \text{ open}; K_0 = I \).

Lempert [L1] introduced the notion of plurisubharmonic domination, and demonstrated its usefulness for proving vanishing theorems first in [L2]. Following him we say that plurisubharmonic domination holds in a complex Banach manifold \( \Omega \) if given any locally upper bounded function \( u : \Omega \to \mathbb{R} \) there is a continuous plurisubharmonic function \( \psi : \Omega \to \mathbb{R} \) such that \( u(x) < \psi(x) \) for all \( x \in \Omega \).

**Theorem 1.1.** (Lempert)

(a) [L1] If \( X \) is a Banach space with a countable unconditional basis, and \( \Omega \subset X \) is pseudoconvex open, then plurisubharmonic domination holds in \( \Omega \).

(b) [L2] If \( X \) is a Banach space with a Schauder basis, plurisubharmonic domination holds in an open subset \( \Omega \) of \( X \) (consequently, \( \Omega \) is pseudoconvex), and \( Z \) is any Banach space, then the sheaf cohomology groups \( H^q(\Omega, O^Z) \) vanish for all \( q \geq 1 \), where \( O^Z \to \Omega \) is the sheaf of germs of holomorphic functions \( \Omega \to Z \).

(Lempert proved in [L2] a theorem different from Theorem 1.1(b), but his method can be modified to give Theorem 1.1(b) as it stands above; see [P1, Thm. 1.3(a)].)

Theorem 1.2 below is this paper’s main theorem.

**Theorem 1.2.** With the above notation, let \( \Omega \subset X \) be pseudoconvex open. If \( X', X'' \) have Schauder bases, and plurisubharmonic domination holds in every pseudoconvex open subset of \( \Omega \), then

(a) the Koszul complex (1.1) is exact on the germ level and on the level of global sections over \( \Omega \), and

(b) the \( K_p \) are acyclic over \( \Omega \): \( H^q(\Omega, K_p) = 0 \) for all \( q \geq 1 \) and \( p \geq 0 \).

Note that Theorem 1.1 implies that Theorem 1.2 applies when \( X', X'' \) have countable unconditional bases; e.g., when \( X = \ell_2 \). In the proof of Theorem 1.2 we first look at the local exactness of (1.1), then prove vanishing by a Leray covering argument combined with Lempert’s method of exhaustion. See Theorem 5.1 for an extension of Theorem 1.2.

2. LOCAL EXACTNESS AND VANISHING.
In this section we show that the Koszul complex (1.1) is exact on the germ level and on the level of global sections over suitable pseudoconvex open neighborhoods $\Omega'$ of each point $x_0 \in \Omega$.

**Proposition 2.1.** Let $\Omega' \subset \Omega$ be pseudoconvex open. If (a) or (b) below holds, then (1.1) is exact on the level of global sections over $\Omega'$.

(a) $\Omega'$ is such that there is a $\lambda \in \mathcal{O}(X'')$ that is linear, or more generally homogeneous of degree $m \geq 1$, with $\lambda(x'') \neq 0$ for $(x', x'') \in \Omega'$.

(b) $\Omega'$ is such that the flow lines of the Euler vector field $E$ stay in $\Omega$ for time $-\infty \leq t \leq 0$, i.e., if $(x', x'') \in \Omega'$, then $(x', tx'') \in \Omega'$ for all $0 \leq t \leq 1$.

**Proof.** Given any $f \in \mathcal{O}(\Omega', \Lambda_p)$ with $i_E f = 0$ on $\Omega'$ we need to produce a $g \in \mathcal{O}(\Omega', \Lambda_{p+1})$ with $i_E g = f$ on $\Omega'$. Let $d = d_{x''}$ be the usual operator of outer differentiation.

(a) Letting $g = \frac{d\lambda}{m\lambda} \wedge f$ will do. Here
\[
g(x', x'')(\xi_0'', \xi_1'', \ldots, \xi_p'') = \\
\sum_{i=0}^{p} (-1)^i \frac{d\lambda}{m\lambda}(x'')(\xi_i'')f(x', x'')(\xi_0'', \ldots, \xi_{i-1}, \xi_{i+1}, \ldots, \xi_p'').
\]
Then $i_E g = f$ since $(i_E g)(x', x'')(\xi_1'', \ldots, \xi_p'') = \frac{\lambda(x''p)}{\lambda(x'')} f(x', x'')(\xi_1'', \ldots, \xi_p') + 0 + \ldots + 0 = f(x', x'')(\xi_1'', \ldots, \xi_p'').$

(b) If $p = 0$, then $i_E df = \mathcal{L}_E f$ since both sides are just the derivative of the function $f$ with respect to $E$. For $p \geq 1$ recall the Cartan identity $d(i_E f) + i_E(dff) = \mathcal{L}_E f$, which is true also in any Banach space. As $i_E f = 0$, we have that $i_E df = \mathcal{L}_E f$. Our $g$ will be a suitable integral of $df$ that inverts the Lie derivation $\mathcal{L}_E$. Let $F_E^t$ be the flow of the Euler vector field $E$, i.e., $F_E^t(x'') = e^t x''$. Then $(F_E^t)^* (\mathcal{L}_E f) = \frac{df}{dt}((F_E^t)^* f)$, and $\mathcal{L}_E f = E = E$, where as usual $(F_E^t)^*$ is the pull back of forms, and $(F_E^t)_*$ is the push forward of tangent vectors by the diffeomorphism (biholomorphism) $F_E^t$. Recall another general identity: $(F_E^t)^* (i_E df) = i_{(F_E^t)_* E} ((F_E^t)^* df)$.

Define $g \in \mathcal{O}(\Omega', \Lambda_{p+1})$ by $g = \int_{t=-\infty}^{0} (F_E^t)^* (df) \, dt$. This integral converges and is holomorphic since on substituting $e^t x''$ in the form $df$ we gain at least one factor of $e^t$ in the integrand, and letting $s = e^t$ we can rewrite $g$ as a proper integral over $[0, 1]$ with respect to $s$. Now
\[
i_E g = \int_{t=-\infty}^{0} i_E ((F_E^t)^* df) \, dt = \int_{-\infty}^{0} i_{(F_E^t)_* E} ((F_E^t)^* df) \, dt
\]
\[
= \int_{-\infty}^{0} (F_E^t)^* (i_E df) \, dt = \int_{-\infty}^{0} (F_E^t)^* (\mathcal{L}_E f) \, dt
\]
\[
= \int_{-\infty}^{0} \frac{df}{dt}((F_E^t)^* f) \, dt = (F_E^0)^* f - (F_E^{-\infty})^* f = f,
\]
where $F_E^0$ and $(F_E^0)^*$ are the identity, and $(F_E^{-∞})^*f = 0$ because we have that $((F_E^{-∞})^*f)(x',x'') = f(x',0) = 0$ for $p = 0$ since $f \in I(Ω')$, and $((F_E^{-∞})^*f)(x',x'')(ξ''_1, \ldots, ξ''_p) = f(x',0)(0, \ldots, 0) = 0$ for $p ≥ 1$. The proof of Proposition 2.1 is complete.

Note that a Koszul complex can also be built over a suitable pseudoconvex open neighborhood of a split complex Banach submanifold $M$ of $Ω$ if a suitable global holomorphic vector field $E$ can be found at least on a neighborhood of $M$ in $Ω$ that can play the role of the Euler vector field. This is the case, e.g., when $M$ is a complete intersection in $Ω$.

**Proposition 2.2.** The sequence (1.1) is exact on the germ level over $Ω$.

**Proof.** It is enough to show that each point $x_0 = (x'_0, x''_0) \in Ω$ has arbitrary small neighborhoods $Ω'$ over which (1.1) is exact on the level of global sections over $Ω'$. We put a norm $∥x∥ = \max\{|x'|, |x''|\}$ on $x = (x',x'') \in X = X' × X''$ so that the open balls $B_X(x,ε)$ are direct products $B_X(x,ε) = B_{X'}(x',ε) × B_{X''}(x'',ε)$. We can now choose such neighborhoods $Ω'$ in the form of balls $Ω' = B_X(x_0,ε)$ as follows.

If $x''_0 ≠ 0$, then there is a linear functional $λ \in (X'')^*$ such that $∥λ∥ = 1$ and $λ(x''_0) = ∥x''_0∥$. Then $λ(x''_0) ≠ 0$ for $∥x'' - x''_0∥ < ε < ∥x''_0∥$, since $|λ(x'') - ∥x''_0∥| = |λ(x'') - λ(x''_0)| ≤ ε < ∥x''_0∥$. That is, $Ω' = B_X(x_0,ε)$ for $0 < ε < ∥x''_0∥$ satisfies Proposition 2.1(a) and lies in any neighborhood of $x_0$ in $Ω$ if $ε > 0$ is small enough.

If $x''_0 = 0$, then $Ω' = B_X(x_0,ε)$, being the product of a set in $X'$ by a convex set in $X''$ that contains $0 \in X''$, satisfies Proposition 2.1(b) and lies in any neighborhood of $x_0$ in $Ω$ if $ε > 0$ is small enough. The proof of Proposition 2.2 is complete.

Consider now an exact sequence

\[(2.1) \quad \ldots → O^{L_p} → O^{L_{p-1}} → \ldots O^{L_1} → K'_0 → 0\]

of analytic sheaves over $Ω$, where each differential is called $d$, and $L_p$ are arbitrary Banach spaces, and the short exact sequences

\[(2.2) \quad 0 → K'_p → O^{L_p} → K'_{p-1} → 0\]

of analytic sheaves, where $K'_p$ is the kernel of $d : O^{L_p} → O^{L_{p-1}}$ for $p ≥ 2$ and of $d : O^{L_1} → K'_0$ for $p = 1$. By Proposition 2.2 our Koszul complex (1.1) is as (2.1).

**Proposition 2.3.** Let $Ω' ⊂ Ω$ be pseudoconvex open.

(a) If (2.1) is exact on the level of global sections over $Ω'$, then we have $H^q(Ω', K'_0) = 0$ for all $q ≥ 1$ and $p ≥ q$. 

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(b) If \( H^1(\Omega', K'_p) = 0 \) for all \( p \geq 1 \), then (2.1) is exact on the level of global sections over \( \Omega' \).

(c) If \( H^q(\Omega', K'_p) = 0 \) for all \( q \geq 2 \) and \( p \geq 1 \), then \( H^q(\Omega', K'_p) = 0 \) for all \( q \geq 1 \) and \( p \geq 1 \).

Proof. Writing down the long exact sequence associated to (2.2) for \( p \geq 1 \) in cohomology we get that

\[
0 \rightarrow K'_p(\Omega') \rightarrow \mathcal{O}(\Omega', L_p) \rightarrow K'_{p-1}(\Omega') \rightarrow \\
\rightarrow H^1(\Omega', K'_p) \rightarrow 0 \rightarrow H^1(\Omega', K'_{p-1}) \rightarrow \\
\rightarrow H^2(\Omega', K'_p) \rightarrow 0 \rightarrow H^2(\Omega', K'_{p-1}) \rightarrow 
\]

(2.3)

where the zeros except the first are by Lempert’s Theorem 1.1(b).

(a) As the third map in (2.3) is an epimorphism by assumption, we see that \( H^1(\Omega', K'_p) = 0 \) for \( p \geq 1 \). Since \( H^q(\Omega', K'_{p-1}) \cong H^{q+1}(\Omega', K'_p) \) for \( q \geq 1 \) and \( p \geq 1 \), we find that \( H^q(\Omega', K'_p) = 0 \) for \( q \geq 1 \) and \( p \geq q \) as claimed.

(b) As \( H^1(\Omega', K_p) = 0 \) for \( p \geq 1 \) by assumption, we see from (2.3) that the third map is an epimorphism, thus (2.1) is exact on the level of global sections over \( \Omega' \) as claimed.

(c) This follows from the dimension shifting relation \( H^q(\Omega', K'_{p-1}) \cong H^{q+1}(\Omega', K'_p) \) for \( q \geq 1 \) and \( p \geq 1 \) in (a). The proof of Proposition 2.3 is complete.

**Proposition 2.4.** Let \( \Omega' \subset \Omega \) be as in the proof of Proposition 2.2. Then the kernel sheaves \( K_p \) of the Koszul complex (1.1) are acyclic over \( \Omega' \) for \( p \geq 0 \).

Proof. By the proof of Proposition 2.3 it is enough to prove that \( I = K_0 \) is acyclic over \( \Omega' \).

If \( \Omega' \cap X' = \emptyset \), then \( I = \mathcal{O}^Z \) over \( \Omega' \), and we are done by Theorem 1.1(b).

If \( \Omega' \cap X' \neq \emptyset \), then since \( \Omega' \) is a product set \( \Omega' = B_{x'}(x'_0, \varepsilon) \times B_{x''}(x''_0, \varepsilon) \), any \( h \in \mathcal{O}(\Omega' \cap X', Z) \) has an automatic holomorphic extension \( \tilde{h} \in \mathcal{O}(\Omega', Z) \) defined simply by \( \tilde{h}(x', x'') = h(x') \). Hence \( H^1(\Omega', I) = 0 \). Considering the short exact sequence \( 0 \rightarrow I \rightarrow \Omega' \mathcal{O}^Z \rightarrow \Omega' \cap X' \mathcal{O}^Z \rightarrow 0 \) of analytic sheaves over \( \Omega' \) we see that \( H^q(\Omega', I) = 0 \) for \( q \geq 2 \) since \( H^{\geq 1}(\Omega' \cap X', \mathcal{O}^Z) = 0 \) by Theorem 1.1(b). The proof of Proposition 2.4 is complete.

Note that if \( \Omega' \) has a finite covering by pseudoconvex open subsets that is a Leray covering for each \( K'_p \) for \( p \geq 0 \), then in Proposition 2.3(a) the range \( 'p \geq q' \) can be replaced by \( 'p > 0' \) as then the cohomology groups \( H^q(\Omega', K'_p) \) vanish for all high enough \( q \) and all \( p \geq 0 \).
Remark that a finite intersection $\Omega' = \bigcap \Omega_i'$ of balls $\Omega_i'$ as in the proof of Proposition 2.2 is again a set $\Omega'$ to which Proposition 2.1 applies. That is, we have a covering of $\Omega$ by balls that is a Leray covering of $\Omega$ for the sheaves $K_p$, $p \geq 0$, as in Proposition 2.4.

We end this section with a vanishing result in the midrange between local and global.

**Proposition 2.5.** Let $\Omega' \subset \Omega$ be pseudoconvex open. Suppose that there is a finite set $\mathcal{U}$ of pseudoconvex open subsets $U$ of $\Omega$ that is a Leray covering of $\bigcup \mathcal{U} \supset \Omega'$ for $K_p'$, $p \geq 1$, as in (2.2). Then $H^q(\mathcal{U}, K_p')(\mathcal{U}|\Omega') = 0$ for all $q \geq 1$ and $p \geq 0$.

**Proof.** The proof is by simple and standard homological algebra based on a double complex. Recall the complex (2.1) and note that its differential $d$ can be extended to (alternating) cochains componentwise, and that this extension, also called $d$, commutes with the Čech differential $\delta$. Given a cocycle $f \in Z^q(\mathcal{U}, K_p')$, $q \geq 1$, $p \geq 0$, of the finite covering $\mathcal{U}$, we need to find a cochain $g \in C^q(\mathcal{U}, K_p')$ with $\delta g = f|\Omega'$. To do that we determine cochains $\varphi_i \in C^{q+i-1}(\mathcal{U}, O^{L_{p+i}})$ and $\psi_i \in C^{q+i-2}(\mathcal{U}|\Omega', O^{L_{p+i}})$ for $i \geq 1$ such that $f = d\varphi_1, \delta \varphi_i = d\varphi_{i+1}$, and $\varphi_i|\Omega' = d\psi_{i+1} + \delta \psi_i$ for $i \geq 1$.

As Proposition 2.3(b) shows that (2.1) is $d$-exact on the global level over the bodies of the simplices of $\mathcal{U}$ we can one after another find $\varphi_1, \varphi_2, \varphi_3, \ldots$ such that $f = d\varphi_1$, $\delta \varphi_1 = d\varphi_2$, $\delta \varphi_2 = d\varphi_3$, .... Since $\varphi_{i+1} = 0$ for $i$ large enough as $\varphi_{i+1} \in C^{q+i}(\mathcal{U}, O^{L_{p+i+1}})$ and this group is zero for $i$ large enough because $\mathcal{U}$ is finite, we see that $\delta \varphi_i = d\varphi_{i+1} = 0$, i.e., $\varphi_i|\Omega' \in C^{q+i-1}(\mathcal{U}|\Omega', O^{L_{p+i}})$ is a $\delta$-cocycle, and hence $\varphi_i|\Omega' = \delta \psi_i$, where $\psi_i \in C^{q+i}(\mathcal{U}|\Omega', O^{L_{p+i}})$. Let $\psi_j = 0$ for $j > i$, and determine $\psi_{i-1}, \psi_{i-2}, \ldots, \psi_1$ one after another.

As $\delta \varphi_{i-1}|\Omega' = d\varphi_i|\Omega' = d \delta \psi_i$ we find that $\delta(\varphi_{i-1}|\Omega' - d \psi_i) = 0$, i.e., $\varphi_{i-1}|\Omega' = d\psi_i + \delta \psi_{i-1}$, etc., $\varphi_1|\Omega' = d\psi_2 + \delta \psi_1$. Then as $f|\Omega' = d\varphi_1|\Omega' = \delta d \psi_1$ letting $g = d \psi_1$ will do. The proof of Proposition 2.5 is complete.

### 3. Exhaustion.

This section describes a way to exhaust a pseudoconvex open subset $\Omega$ of a Banach space $X$ that is convenient for proving vanishing results for sheaf cohomology over $\Omega$. We follow here [L3, §2].

We say that a function $\alpha$, call their set $\mathcal{A}'$, is an **admissible radius function** on $\Omega$ if $\alpha : \Omega \to (0, 1)$ is continuous and $\alpha(x) < \text{dist}(x, X \setminus \Omega)$ for $x \in \Omega$. We say that a function $\alpha$, call their set $\mathcal{A}$, is an **admissible Hartogs radius function** on $\Omega$ if $\alpha \in \mathcal{A}'$ and $-\log \alpha$ is plurisubharmonic on $\Omega$. Call $\mathcal{A}$ cofinal in $\mathcal{A}'$ if for each $\alpha \in \mathcal{A}'$ there is a $\beta \in \mathcal{A}$ with $\beta(x) < \alpha(x)$ for $x \in \Omega$. 

Proposition 3.1. Plurisubharmonic domination holds in $\Omega$ if and only if $A$ is cofinal in $A'$.

Proof. Write $\alpha = e^{-u} \in A'$ and $\beta = e^{-\psi} \in A$. As plurisubharmonic domination holds on $\Omega$ for $u$ continuous if and only if for $u$ locally upper bounded, the proof of Proposition 3.1 is complete.

It will be often useful to look at coverings by balls $B(x,\alpha(x))$, $x \in \Omega$, $\alpha \in A'$ and shrink their radii to obtain a finer covering by balls $B(x,\beta(x))$, $x \in \Omega$, $\beta \in A$.

Let $e'_n$, $n \geq 1$, be a Schauder basis in the Banach space $(X', \| \cdot \|')$, and similarly $e''_n$, $n \geq 1$, in $(X'', \| \cdot \|''%)$. One can change the norms $\| \cdot \|', \| \cdot \|''$ to equivalent norms so that

\[
\| \sum_{i=m}^n x'_i e'_i \|' \leq \| \sum_{i=M}^N x'_i e'_i \|'
\]

for $0 \leq M \leq m \leq n \leq N \leq \infty$, $x'_i \in \mathbb{C}$, and similarly for $X'', \| \cdot \|''$, $\{e''_i\}$. Let $X = X' \times X''$ with norm $\|x\| = \max\{\|x'\|', \|x''\|''\}$ on $x = (x', x'') \in X$, and Schauder basis $e_{2n-1} = e'_n$, $e_{2n} = e''_n$, for $n \geq 1$, if both $X', X''$ are infinite dimensional. If $k' = \dim(X') < \infty$, then let $e_n = e'_n$ for $n \leq k'$, and $e_n = e''_{n-k'}$ for $n > k'$. We assume that $X$ is infinite dimensional, since otherwise Theorem 1.2 reduces to a well-known classical theorem. Then $X', \| \cdot \|, \{e_i\}$ also satisfy the analog of (3.1). Introduce the projections $\pi_N : X \to X$, $\pi_N \sum_{i=1}^\infty x_i e_i = \sum_{i=1}^N x_i e_i$, $x_i \in \mathbb{C}$, $\pi_0 = 0$, $\pi_\infty = 1$, $\varphi_N = 1 - \pi_N$, and define for $\alpha \in A$ and $N \geq 0$ integer the sets

\[
D_N(\alpha) = \{ \xi \in \Omega \cap \pi_N \xi : (N+1)\alpha(\xi) > 1 \}
\]

\[
\Omega_N(\alpha) = \{ x \in \pi_N^{-1} D_N(\alpha) : \varphi_N(x) < \alpha(\pi_N x) \}.
\]

These $\Omega_N(\alpha)$ are pseudoconvex open in $\Omega$, and they will serve to exhaust $\Omega$ as $N = 0, 1, 2, \ldots$ varies.

Proposition 3.2. Let $\alpha \in A$, and suppose that plurisubharmonic domination holds in $\Omega$.

(a) There is an $\alpha' \in A$, $\alpha' < \alpha$, with $\Omega_n(\alpha') \subset \Omega_N(\alpha)$ for all $N \geq n$. So any $x_0 \in \Omega$ has a neighborhood contained in all but finitely many $\Omega_N(\alpha)$.

(b) There are $\beta, \gamma \in A$, $\gamma < \beta < \alpha$, so that for all $N$ and $x \in \Omega_N(\gamma)$

\[
B(x, \gamma(x)) \subset \Omega_N(\beta) \cap \pi_N^{-1} B_X(\pi_N x, \beta(x)) \subset B_X(x, \alpha(x)),
\]

(c) and, additionally, for all $N$ there is a finite set of points $\xi_i = \pi_N \xi_i \in D_N(\gamma)$ such that for each $x \in \Omega_N(\gamma)$ there is a $\xi_i$ with

\[
B(x, \gamma(x)) \subset \Omega_N(\beta) \cap \pi_N^{-1} B_X(\xi_i, \beta(\xi_i)).
\]
(d) There is an \( \alpha' \in A, \alpha' < \alpha \), with \( \Omega_N(\alpha') \subset \Omega_N(\alpha) \cap \Omega_{N+1}(\alpha) \) for all \( N \geq 0 \).

**Proof.** See [L3, Prop. 2.1] and [L2, Prop. 4.3] for (a) and (b), and [L3, Prop. 2.3] for (d). We modify slightly Lempert’s definition of \( \beta, \gamma \) in his proof of (b) in [L3, Prop. 2.1] so as to work also for (c) here.

To complete part (b) choose the functions \( \beta \) and \( \gamma \) first in \( A' \) then in \( A \) applying plurisubharmonic domination in \( \Omega \) so that \( \beta < \alpha/8, \gamma < \beta/8, \alpha(x) < 2\alpha(y) \) for \( x, y \in B_X(z,2\beta(z)) \), and \( \beta(x) < 2\beta(y) \) for \( x, y \in B_X(z,2\gamma(x)) \). Then (b) is verified as in the proof of [L3, Prop. 2.1] arguing with the triangle inequality only.

(c) Let \( \varepsilon_0 = \frac{1}{2} \min_{D_N(\gamma)} \gamma \). As \( \gamma \) is strictly positive and continuous on the compact set \( D_N(\gamma) \) we see that \( \varepsilon_0 > 0 \). Choose a finite \( \varepsilon_0 \)-net \( \{\xi_i\} \) in the totally bounded set \( D_N(\gamma) \), i.e., for each \( \xi \in D_N(\gamma) \) there is a \( \xi_i \) with \( \|\xi - \xi_i\| < \varepsilon_0 \). We claim that this choice of points \( \{\xi_i\} \) will do. Indeed, let \( x \in \Omega_N(\gamma) \) be any point, and pick a \( \xi_i \) with \( \|\pi_N x - \xi_i\| < \varepsilon_0 \). We need to show that if \( \|y - x\| < \gamma(x) \), then \( y \in \Omega_N(\beta) \), and \( \|\pi_N y - \xi_i\| < \beta(\xi_i) \). The already proved part (b) implies that \( y \in \Omega_N(\beta) \), and we have

\[
\|\pi_N y - \xi_i\| \leq \|\pi_N (y-x)\| + \|\pi_N x - \xi_i\| \leq \|y-x\| + \varepsilon_0 < \gamma(x) + \gamma(\xi_i) < \frac{1}{8}\beta(x) + \frac{1}{8}\beta(\xi_i) < \frac{1}{4}\beta(\pi_N x) + \frac{1}{8}\beta(\xi_i) < \frac{1}{8}\beta(\xi_i),
\]

where we used the above properties of \( \beta \) and \( \gamma \) including the doubling inequality of \( \beta \). The proof of Proposition 3.2 is complete.

The meaning of Proposition 3.2(bc) is that certain refinement maps exist between certain open coverings.

**4. VANISHING.**

This section completes the proof of Theorem 1.2. Resume the notation and hypotheses of §1–3. For \( \alpha \in A, N \geq 0 \), put

\[
\mathcal{B}(\alpha) = \{B_X(x,\alpha(x)) : x \in \Omega\}, \quad \mathcal{B}_N(\alpha) = \{B_X(x,\alpha(x)) : x \in \Omega_N(\alpha)\}.
\]

We say that a complex (2.1) satisfies **condition (4.1)** (i.e., is tractable by our current methods) if there is an \( \alpha_0 \in A \) so that for any \( \alpha \in A, \alpha < \alpha_0 \), there are \( \beta, \gamma \in \mathcal{A} \) such that Proposition 3.2 holds for \( \alpha, \beta, \gamma \) and the coverings \( \mathcal{B}(\alpha), \mathcal{B}(\gamma), \{\pi_N^{-1}B_X(\pi_N x, \beta(x)) \cap B_X(x,\alpha(x)) : x \in \Omega_N(\gamma)\} \) are Leray coverings of their respective unions for all \( K'_p, p \geq 1 \).

The following more natural condition below on a complex (2.1) implies condition (4.1) above.
We say that a complex (2.1) satisfies condition (4.2) if there is an open covering $\mathcal{U}$ of $\Omega$ by pseudoconvex open subsets $U$ of $\Omega$ such that if $V$ is any pseudoconvex open subset of any member $U \in \mathcal{U}$, then (2.1) is exact over $V$ on the level of global sections.

The reason for using the artificial looking condition (4.1) is that it is easily verified a priori for our Koszul complex (1.1) while (4.2) not — our Theorem 1.2 is in fact equivalent to saying that (1.1) satisfies condition (4.2). It seems unknown whether acyclicity holds over $\Omega$ for all complexes (2.1) in general.

**Proposition 4.1.** The Koszul complex (1.1) satisfies condition (4.1).

**Proof.** We saw that there is an $\alpha_0 \in \mathcal{A}'$ such that if $\alpha \in \mathcal{A}$, $\alpha < \alpha_0$, then $\mathcal{B}(\alpha)$ is a Leray covering, and we know from Proposition 2.1 that (1.1) is acyclic on the level of global sections over any set (e.g., $\pi^{-1}_N B_X(\pi_N x, \beta(x)) \cap B_X(x, \alpha(x))$) that is the product of a bounded convex open set in $X'$ by a bounded convex open set in $X''$. The proof of Proposition 4.1 is complete.

**Proposition 4.2.** Let $X$ be a Banach space with a Schauder basis, $\Omega \subset X$ pseudoconvex open, (2.1) a complex, and suppose that plurisubharmonic domination holds in any pseudoconvex open subset of $\Omega$, and that (2.1) satisfies condition (4.1). Then for any $\alpha \in \mathcal{A}$ there is a $\gamma \in \mathcal{A}$ with $\gamma < \alpha$ and $H^q(\mathcal{B}_N(\alpha), K'_p)|\mathcal{B}_N(\gamma) = 0$ for all $N \geq 0$, $q \geq 1$, and $p \geq 0$.

**Proof.** We consider some open coverings and refinement maps of them. Let $\alpha, \beta, \gamma, \{\xi_i\}$ be as in Proposition 3.2(bc). Consider the open coverings $\mathcal{B}_N(\alpha)$, $\mathcal{U}_N = \{U(x) = \pi^{-1}_N B_X(\pi_N x, \beta(x)) \cap B_X(x, \alpha(x)) : x \in \Omega_N(\gamma)\}$, $\mathcal{U}'_N = \{U(\xi_i) : i\}$, $\mathcal{B}_N(\gamma)$, and their refinement maps $\mathcal{U}_N \to \mathcal{B}_N(\alpha)$ given by $U(x) \mapsto B_X(x, \alpha(x))$, $\mathcal{U}'_N \to \mathcal{U}_N$ given by $U(\xi_i) \mapsto U(\xi_i)$, and $\mathcal{B}_N(\gamma) \to \mathcal{U}'_N|\Omega_N(\beta)$ given by $B_X(x, \gamma(x)) \mapsto \Omega_N(\beta) \cap U(\xi_i)$, where $\xi_i$ is the point assigned to $x \in \Omega_N(\gamma)$ in Proposition 3.2(c). Due to the inequalities (3.3) and (3.4) the above are indeed refinement maps and hence induce maps

\[
H^q(\mathcal{B}_N(\alpha), K'_p) \to H^q(\mathcal{U}_N, K'_p) \to H^q(\mathcal{U}'_N, K'_p) \to
\]

\[
H^q(\mathcal{U}'_N, K'_p)|\mathcal{U}'_N|\Omega_N(\beta) \to H^q(\mathcal{B}_N(\gamma), K'_p)
\]

in cohomology for $q \geq 1$, and $p \geq 0$. We see via condition (4.1) that Proposition 2.5 applies, and thus the fourth group in (4.3) is zero, and together with it so is the composite map $H^q(\mathcal{B}_N(\alpha), K'_p) \to H^q(\mathcal{B}_N(\gamma), K'_p)$ in (4.3). The proof of Proposition 4.2 is complete.

The upshot of Proposition 4.2 is that the refinement map $\mathcal{B}_N(\gamma) \to \mathcal{B}_N(\alpha)$ factors through a finite covering.

**Theorem 4.3.** Let $X, \Omega, (2.1)$ be as in Proposition 4.2.
(a) $H^q(\Omega, K'_p) = 0$ for all $q \geq 1$ and $p \geq 0$.
(b) The sequence (2.1) is exact over $\Omega$ on the level of global sections.

Proof. (a) Suppose first that $q \geq 2$. Let $f \in H^q(\Omega, K'_p)$ be a cohomology class. We saw earlier that due to plurisubharmonic domination in $\Omega$ there is an $\alpha$ with $10\alpha \in \mathcal{A}$ so that $f$ can be represented as a cocycle $f \in Z^q(\mathcal{B}(\alpha), K'_p)$. Choose $\gamma$ as in Proposition 4.2. Proposition 4.2 yields a $g_N \in C^{q-1}(\mathcal{B}(\gamma), K'_p)$ with $\delta g_N = f|\mathcal{B}(\gamma)$. We can extend the cochain $g_N$ to a cochain $g_{N+1} \in C^{q-1}(\mathcal{B}(\gamma), K'_p)$ simply by defining $g_{N+1}$ to be zero over simplices $\bigcap_{i=1}^q B_X(x_i, \gamma(x_i))$ if at least one vertex $x_i \notin \Omega(\gamma)$. Proposition 4.2 gives a $\gamma' \in \mathcal{A}$, $\gamma' < \gamma$, with $H^{q-1}(\mathcal{B}(\gamma'), K'_p)|\mathcal{B}(\gamma') = 0$ for all $N \geq 1$, and Proposition 3.2(d) a $\gamma'' \in \mathcal{A}$, $\gamma'' < \gamma'$ with $\Omega(\gamma') \subset \Omega(\gamma'') \cap \Omega(\gamma')$ for all $N \geq 1$. So similarly by extending a $(q-2)$-cochain there is an $h_N \in C^{q-2}(\mathcal{B}(\gamma''), K'_p)$ with $|g_{N-1}|(\mathcal{B}(\gamma'')) = \delta h_N|\mathcal{B}(\gamma'')$. Letting $g'_N = g_N|\mathcal{B}(\gamma'') - \sum_{n=1}^N \delta h_n \in C^{q-1}(\mathcal{B}(\gamma''), K'_p)$ Proposition 3.2(a) implies as $g'_N|\mathcal{B}(\gamma'') = g_{N-1}|\mathcal{B}(\gamma'')$ that $g'_N$ converges as $N \to \infty$ in a quasistationary manner to a $g \in C^{q-1}(\mathcal{B}(\gamma''), K'_p)$ with $\delta g = f|\mathcal{B}(\gamma'')$. Thus $f$ equals zero in $H^q(\Omega, K'_p)$ for $q \geq 2$ and $p \geq 0$. Part (a) follows then from Proposition 2.3(c) and (b) from (a) via Proposition 2.3(b). The proof of Theorem 4.3 is complete.

Proof of Theorem 1.2. Propositions 2.2 and 4.1 exhibit Theorem 1.2 as a special case of Theorem 4.3. The proof of Theorem 1.2 is complete.

5. AN ANALOG OF HEFER’S LEMMA.

In this section we extend Theorem 1.2 and draw some corollaries from it.

While it seems far from being currently proved, it is reasonable to hope that plurisubharmonic domination holds in every pseudoconvex open subset $\Omega$ of any Banach space $X$ that is a direct summand of a Banach space $Y$, which has a Schauder basis (i.e., if $X$ has the bounded approximation property, fondly called BAP). It is certainly a good question to ask. In this spirit we can relax the hypotheses of Theorem 1.2 as follows.

**Theorem 5.1.** Let $X', X'', Z$ be Banach spaces, $\Omega \subset X = X' \times X''$ pseudoconvex open, $I$ the sheaf of germs of holomorphic functions $\Omega \to Z$ that vanish on $X'$. Suppose that $X$ has a Schauder basis, and that plurisubharmonic domination holds in every pseudoconvex open subset of $\tilde{\Omega} = \Omega \times X' \times X''$. Then

(a) the Koszul complex (1.1) is exact on the germ level and on the level of global sections over $\Omega$, and
(b) the $K_p$ are acyclic over $\Omega$: $H^q(\Omega, K_p) = 0$ for all $q \geq 1$ and $p \geq 0$.

Proof. We consider a Koszul complex (5.1) as in (1.1) over a bigger
Banach space. Let \( \tilde{x} = (x', x'', y', y'') \in \tilde{X} = X' \times X'' \times X' \times X'' = \tilde{X'} \times \tilde{X}'' \ni (\tilde{x}') = (x', y'), \tilde{x}'' = (x'', y') \), \( \tilde{\Lambda}_p \) the Banach space of all continuous complex \( p \)-linear alternating maps \( \tilde{X}'' \to Z \) for \( p \geq 0 \), \( \tilde{\Lambda}_0 = \Lambda_{-1} = Z \), \( \mathcal{O}^{\tilde{\Lambda}_p} \to \tilde{X} \) the sheaf of germs of holomorphic functions \( \tilde{X} \to \tilde{\Lambda}_p, \tilde{E} \) the Euler vector field on \( \tilde{X}'' \) defined by \( \tilde{E}(x'') = \tilde{x}'' = (x'', y') \), \( \tilde{E} \) the inner derivation determined by \( \tilde{E} \), i.e., \( \tilde{i}_{\tilde{E}} \) is the contraction of \( p \)-forms with \( \tilde{E} \): if \( f \) is a local section of \( \mathcal{O}^{\tilde{\Lambda}_p} \), then let \( \tilde{i}_{\tilde{E}} f \) be the local section of \( \mathcal{O}^{\tilde{\Lambda}_{p-1}} \) given for \( p \geq 1 \) by \( (\tilde{i}_{\tilde{E}} f)(x', x'', y', y'')((\xi''_1, \eta''_1), \ldots , (\xi''_{p-1}, \eta''_{p-1})) = f(x', x'', y'')(x''', y'')(\xi''_1, \eta''_1), \ldots , (\xi''_{p-1}, \eta''_{p-1})) \), and for \( p \geq 0 \) by the formula \( (\tilde{i}_{\tilde{E}} f)(x', x'', y', y'') = f(x', 0, 0, y'') \). Let \( \tilde{I} \) be the subsheaf of \( \mathcal{O}^{\tilde{\Lambda}} \to \tilde{X} \) of sections that vanish on \( \tilde{X}' \). We consider the Koszul complex

\[
(5.1) \quad \ldots \to \mathcal{O}^\tilde{\Lambda}_p \to \mathcal{O}^\tilde{\Lambda}_{p-1} \to \ldots \to \mathcal{O}^\tilde{\Lambda}_1 \to \tilde{I} \to 0
\]

of analytic sheaves over \( \tilde{X} \), where each map is \( \tilde{i}_{\tilde{E}} \). Let \( \tilde{K}_p, p \geq 0 \), be the corresponding sequence of kernel sheaves: \( \tilde{K}_p(U) = \{ f \in \mathcal{O}(U, \tilde{\Lambda}_p) : \tilde{i}_{\tilde{E}} f = 0 \text{ on } U \} \), \( U \subset \tilde{X} \) open; \( \tilde{K}_0 = \tilde{I} \). Let \( \tilde{\Omega} = \{(x', x'', y', y'') \in \tilde{X} : (x', x'') \in \Omega \} = \Omega \times X' \times X'' \).

Theorem 1.2 applies to (5.1) over \( \tilde{\Omega} \) since \( \tilde{X}' \cong X' = X' \times X'' \) have a Schauder basis by assumption. Now Theorem 5.1 follows easily by considering the extension of forms \( f \in \mathcal{O}(U, \Lambda_p) \) to forms \( \tilde{f} \in \mathcal{O}(U, \tilde{\Lambda}_p) \) where \( U \subset \Omega \) open, \( \tilde{U} = U \times X' \times X'' \), \( p \geq 0 \), and the restriction \( f \) of forms \( \tilde{f} \) to \( \Omega \), defined for \( f \mapsto \tilde{f} \) by \( \tilde{f}(x', x'', y')((\xi''_1, \eta''_1), \ldots , (\xi''_{p}, \eta''_{p})) = f(x', x'', (\xi''_1, \ldots , \xi''_{p}), (\eta''_1, \ldots , \eta''_{p})) \), and for \( \tilde{f} \mapsto f \) by \( f(x', x'')(\xi''_1, \ldots , \xi''_{p}) = \tilde{f}(x', x'', 0, 0)((\xi''_1, 0), \ldots , (\xi''_{p}, 0)) \). As these simple maps intertwine \( \tilde{i}_{\tilde{E}} \) and \( i_{\tilde{E}} \), the proof of Theorem 5.1 is complete.

Similarly one could replace in Theorem 5.1 the assumption that ‘X have a Schauder basis’ by ‘X be a direct summand of a Banach space Y with a Schauder basis.’

The next item is an infinite dimensional analog of the classical Hefer lemma (without bounds).

**Theorem 5.2.** Let \( X, Z \) be Banach spaces, \( \Omega \subset X \) pseudoconvex open, and \( f \in \mathcal{O}(\Omega \times \Omega, Z) \). If \( X \) has a Schauder basis and plurisubharmonic domination holds in every pseudoconvex open subset of \( \Omega \times \Omega \), and \( f(x, x) = 0 \) for \( x \in \Omega \), then there is a \( g \in \mathcal{O}(\Omega \times \Omega, \text{Hom}(X, Z)) \) such that \( f(x, y) = g(x, y)(x - y) \) for \( x, y \in \Omega \).

**Proof.** Writing \( x' = \frac{1}{2}(x + y), x'' = \frac{1}{2}(x - y), x' + x'' = x, x' - x'' = y \) a direct decomposition of \( (x, y) \in X \times \tilde{X} \cong X' \times X'' \ni (x', x'') \) is obtained with \( X' \cong X'' \cong X \). As \( f = 0 \) for \( x'' = 0 \) an application of Theorem 1.2(a) completes the proof of Theorem 5.2.
The above form of Hefer’s lemma can be applied to give an algebraic
definition of the Fréchet differential $df \in \mathcal{O}(\Omega, X^*)$ of a function $f \in \mathcal{O}(\Omega)$
defined on a pseudoconvex open subset $\Omega$ of a Banach space $X$, which is just
like the classical case of polynomials in finitely many variables.

**Theorem 5.3.** Let $X, Z$ be Banach spaces, $\Omega \subset X$ pseudoconvex open, and $f \in \mathcal{O}(\Omega, Z)$. If $X$ has a Schauder basis and plurisubharmonic domination
holds in every pseudoconvex open subset of $\Omega \times \Omega$, then the function $f(x) - f(y)$ can be written as $f(x) - f(y) = g(x,y)(x - y)$ for $x, y \in \Omega$, where $g \in \mathcal{O}(\Omega \times \Omega, \text{Hom}(X, Z))$, and $df(x)\xi = g(x, x)\xi$ for $x \in \Omega$, $\xi \in X$.

**Proof.** Theorem 5.2 provides such a $g$. Taking an $x$ partial derivative of
the identity in Theorem 5.2 in the $\xi$ direction and letting $y = x$ complete
the proof of Theorem 5.3.

Theorem 5.2 can also be formulated for arbitrary Banach vector bundles
instead of just trivial ones.

**Theorem 5.4.** Let $X$ be a Banach space, $\Omega \subset X$ pseudoconvex open, $E \rightarrow
\Omega \times \Omega$ a holomorphic Banach vector bundle, and $f \in \mathcal{O}(\Omega \times \Omega, E)$ a holomor-
phic section. If $X$ has a Schauder basis and plurisubharmonic domination
holds in every pseudoconvex open subset of $\Omega \times \Omega$, and $f(x, x) = 0$ for all $x \in
\Omega$, then there is a $g \in \mathcal{O}(\Omega \times \Omega, \text{Hom}(X, E))$ such that $f(x, y) = g(x, y)(x - y)$
for $x, y \in \Omega$.

**Proof.** Theorem 1.3(b) in [P1] gives a Banach space $Z_1$, an embedding
$I \in \mathcal{O}(\Omega \times \Omega, \text{Hom}(E, Z_1))$, and a projection $P \in \mathcal{O}(\Omega \times \Omega, \text{End}(Z_1))$ that
$I(x, y)(E_{x,y}) = P(x, y)(Z_1)$ for all $x, y \in \Omega$. Look at the function $f' \in \mathcal{O}(\Omega \times
\Omega, Z_1)$ defined by $f'(x, y) = I(x, y)f(x, y)$, which vanishes for $x = y$ in $\Omega$.

Theorem 5.2 gives a $g' \in \mathcal{O}(\Omega \times \Omega, \text{Hom}(X, Z_1))$ with $f'(x, y) = g'(x, x)(x - y)$.

Define $g \in \mathcal{O}(\Omega \times \Omega, \text{Hom}(X, E))$ by $g(x, y) = I(x, y)^{-1}P(x, y)g'(x, y)$. As $g(x, y)(x - y) = I(x, y)^{-1}P(x, y)I(x, y)f(x, y) = I(x, y)^{-1}I(x, y)f(x, y) = f(x, y)$, since $P(x, y)I(x, y) = 0$ the identity, the proof of Theorem 5.4 is com-
plete.

Theorem 5.1, just like Theorem 5.2, has a version with Banach vector bundles.

Let $X', X''$ be Banach spaces, $X = X' \times X''$, $\Omega \subset X$ pseudoconvex open,
$F \rightarrow \Omega$ a holomorphic Banach vector bundle, $\Lambda_p(F) \rightarrow \Omega$ the Banach vector
bundle of all the Banach spaces $\Lambda_p(F_x), x \in \Omega$, of all continuous complex $p$-
linear alternating maps from $X''$ to the fiber $F_x$ of $F$ over the point $x$, $p \geq 0$; $\Lambda_0(F_x) = F_x$, and $\mathcal{O}^{\Lambda_p(F)} \rightarrow \Omega$ the sheaf of germs of holomorphic sections
$\Lambda_p(F) \rightarrow \Omega$. Let $E$ be the Euler vector field on $X''$ defined by $E(x'') = x''$, $i_E$ the inner derivation determined by $E$, i.e., $i_E$ is the contraction of $p$-
forms in $\Lambda_p(F)$ with $E$: if $f$ is a local section of $\mathcal{O}^{\Lambda_p(F)}$, then let $i_E f$ be the
local section of $\mathcal{O}^{\Lambda_p-1}(F)$ given for $p \geq 1$ by $(i_E f)(x', x'')(\xi_1'', \ldots, \xi_{p-1}'') = f(x', x'')(x'', \xi_1'', \ldots, \xi_{p-1}'')$. Let $I$ be the subsheaf of $\mathcal{O}^F \to \Omega$ of all sections that vanish on $\Omega \cap (X' \times \{0\})$. We consider the Koszul complex

$$
\cdots \to \mathcal{O}^{\Lambda_p}(F) \to \mathcal{O}^{\Lambda_p-1}(F) \to \cdots \to \mathcal{O}^{\Lambda_1}(F) \to I \to 0
$$

of analytic sheaves over $\Omega$, where each map except the rightmost is $i_E$. Let $K_p$, $p \geq 0$, be the corresponding sequence of kernel sheaves: $K_p(U) = \{f \in \mathcal{O}(U, \Lambda_p(F)) : i_E f = 0 \text{ on } U\}$, $U \subset \Omega$ open, $p \geq 1$; $K_0 = I$.

**Theorem 5.5.** With the above notation, suppose that $X$ has a Schauder basis, and that plurisubharmonic domination holds in every pseudoconvex open subset of $\Omega \times X$. Then

(a) the Koszul complex (5.2) is exact on the germ level and on the level of global sections over $\Omega$, and

(b) the $K_p$ are acyclic over $\Omega$: $H^q(\Omega, K_p) = 0$ for all $q \geq 1$ and $p \geq 0$.

**Proof.** Just like in the proof of Theorem 5.4, rely on Theorem 1.3(b) in [P1] to exhibit the Banach vector bundle $F$ as a direct summand of a trivial Banach vector bundle $\Omega \times Z_1$, and thus the complex (5.2) as a direct summand of a complex (1.1). An application of Theorem 5.1 completes the proof of Theorem 5.5.

We conclude by remarking that the Koszul complex (1.1) of the ideal sheaf of the origin in a Banach space $X$ is also useful in connection with the projectivization $P(X)$ of $X$, and that Hefer’s lemma as above helps us understand the universal derivation of the algebra $\mathcal{O}(\Omega)$ for $\Omega \subset X$ pseudoconvex open; see [NW] for the latter if $X$ is finite dimensional.

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