Growing pseudo-eigenmodes and positive logarithmic norms in rotating shear flows

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Abstract. Rotating shear flows, when angular momentum increases and angular velocity decreases as functions of radiation coordinate, are hydrodynamically stable under linear perturbation. The Keplerian flow is an example of such a system, which appears in an astrophysical context. Although decaying eigenmodes exhibit large transient energy growth of perturbation which could govern nonlinearity in the system, the feedback of inherent instability to generate turbulence seems questionable. We show that such systems exhibiting growing pseudo-eigenmodes easily reach an upper bound of growth rate in terms of the logarithmic norm of the involved non-normal operators, thus exhibiting feedback of inherent instability. This supports the existence of turbulence of hydrodynamic origin in the Keplerian accretion disc in astrophysics. Hence, this answers the question of the mismatch between the linear theory and experimental/observed data and helps in resolving the outstanding question of the origin of turbulence therein.

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1. Introduction

Despite many efforts devoted to it, the origin of hydrodynamic turbulence is still poorly understood, because there is significant mismatch between the predictions of linear theory and the experimental data (see e.g. [1, 2]). The mismatch between theoretical results and observed data is also found in astrophysical contexts, where the accretion flow with Keplerian angular momentum profile is a common subject.

The hot ionized Keplerian disc flow, which is linearly stable otherwise, in the presence of magnetic coupling is shown to exhibit magneto-rotational instability (MRI) [3] based on a local analysis. Then the unstable modes were thought to lead to nonlinearity and then turbulence in the flow. However, it was also shown that the criterion drawn from the local dispersion relation is inadequate and may be misleading [4, 5]. Then the question arises as to how could a Keplerian accretion disc flow having a very low molecular viscosity generate turbulence, which could successively govern diffusive viscosity to support the transfer of mass inwards and angular momentum outwards, in the absence of any unstable mode of hydrodynamic origin?

On the other hand, the existence of large transient growth of perturbation in plane Couette flow with and without the Coriolis force was demonstrated [1, 2]. It was argued that the large transient growth plausibly could lead to subcritical transition to turbulence in laboratory experiments and in astrophysical flows, which in turn could lead to turbulent viscosity [6]. Note, however, that the shear flows with and without rotation are physically quite different. The three-dimensional (3D) growth factors and then the plausible turbulent viscosity in the presence of the Coriolis force were shown to be significant in Keplerian flow only if the elliptical vortices were present in the background flow [7, 8]. The Coriolis effect, which absorbs the pressure fluctuation, is the main culprit to kill any growth of energy in the background of linear shear. Moreover, very importantly, the feedback of inherent instability and then sustenance of turbulence based on large transient growth is questionable, in lieu of linear instabilities. Therefore, the problem of hydrodynamic transition to turbulence in rotating shear flow, particularly when angular momentum increases and angular velocity decreases as functions of radial coordinate, e.g. in the astrophysical accretion disc, remains unsolved. While Bech and Andersson [9] found turbulence persisting in numerical simulations of subcritical rotating flows for large enough Reynolds numbers, in general the problem of hydrodynamic transition to turbulence in rotating shear flows, particularly astrophysical ones, has not been paid enough attention in the literature so far.

Decades back, Craik [10] studied the impact of rotation (including the Keplerian one) and hence the Coriolis effects on flow with elliptical streamlines. Later on, Salhi [11] examined rotation and buoyancy effects on homogeneous shear flows and found similarities between rotation and stratification effects. In fact, much earlier, Bradshaw [12] discussed the formal algebraic analogy between the parameters describing streamline curvature and buoyancy in a turbulent flow. Very recently, the stability problem of unbounded rotating shear flows has been studied again in the presence of uniform vertical density stratification [13], which has important astrophysical applications, e.g. in geometrically thin radially stratified accretion discs. Note that indeed Johnson and Gammie [14] investigated non-axisymmetric radially stratified linearized accretion discs in a shearing sheet approximation neglecting vertical structure. Then they studied the time evolution of a plane wave perturbation co-moving with the shear flow. The shearing flow with rotation and stratification has many other applications, including that in geophysics, as described in detail by Salhi and Cambon [13]. Interestingly, the same/similar
physics is borrowed to understand rotating shear flows in different communities with the use of different terminologies. The engineering community compute the ‘rotational Richardson number’ introduced by Bradshaw [12] to understand the linear stability of a flow, which is proportional to the square of the epicyclic frequency used by astrophysicists and the Rayleigh number by fluid dynamicists in general for the same purpose. While the positive Richardson number and Rayleigh number correspond to linear stability in engineering and fluid physics, respectively, \( q < 2 \) (the real value of epicyclic frequency) is the equivalent condition in astrophysics with angular velocity \( \Omega \sim r^{-q} \) (will be discussed in detail in subsequent sections).

We show that rotating flows exhibiting pseudo-eigenmodes, in the absence of any unstable eigenmode itself, quickly reach a growth rate of the order of the logarithmic norm under a very small perturbation. This plausibly triggers turbulence. Note that the logarithmic norm, as discussed later, is an upper bound of such growth rates. This establishes that the intrinsic perturbation in the system is enough to sustain the turbulence, which is a manifestation of the transient local instability. Thus, the turbulence of rotating shear flows and the viscosity of hydrodynamic origin in accretion discs are shown to be an inherent property of these systems.

In the next section, we describe hydrodynamic equations in shear flows, pseudo-eigenmodes, logarithmic norm and corresponding transient instability in the presence of the Coriolis effects. Subsequently, in section 3, we discuss solutions. Finally, in section 4, we summarize with the conclusions.

2. Equations describing shear flow in the presence of the Coriolis effect and the corresponding pseudo-eigenmode and logarithmic norm

Let us focus on a local analysis considering a small patch of rotating shear flow whose unperturbed velocity profile \( \vec{U} \) corresponds to the background shear and angular velocity \( \Omega \sim r^{-q} \), as described earlier [2], shown schematically in figure 1. Following earlier work [2], all the variables are written in dimensionless form. The figure describes our choice of local Cartesian coordinates: \( x \) is along the radial direction, \( y \) is along the azimuthal or streamwise direction and \( z \) is along the vertical direction. The unperturbed flow has a velocity in the \( y \)-direction and a gradient of velocity, i.e. shear, along the \( x \)-direction. The Coriolis force arising due to rotation of the patch is described by an angular velocity vector \( \Omega \) along the \( z \)-direction.

For convenience, we write down the corresponding equations for the evolution of perturbation in terms of the set of velocity and corresponding vorticity \( (u, \zeta) \) and hence the Orr–Sommerfeld and Squire equations in the presence of Coriolis acceleration are given by

\[
\left( \frac{\partial}{\partial t} + U \frac{\partial}{\partial y} \right) \nabla^2 u - \frac{\partial^2 U}{\partial x^2} \frac{\partial u}{\partial y} + \frac{2}{q} \frac{\partial \zeta}{\partial z} = \frac{1}{Re} \nabla^4 u, \tag{1}
\]

\[
\left( \frac{\partial}{\partial t} + U \frac{\partial}{\partial y} \right) \zeta - \frac{\partial U}{\partial x} \frac{\partial u}{\partial z} - \frac{2}{q} \frac{\partial u}{\partial z} = \frac{1}{Re} \nabla^2 \zeta, \tag{2}
\]

where \( u \) is the \( x \)-component of velocity and \( \zeta \) is the corresponding component of vorticity, \( q = 3/2 \) for a Keplerian flow and \( q = 2 \) for a constant angular momentum flow and \( Re \) is the Reynolds number. Note that in the presence of linear background shear, which is of present interest, \( \vec{U} = (0, -x, 0) \), and hence the term proportional to \( \frac{\partial^2 u}{\partial x^2} \) vanishes. For a detailed description, see Mukhopadhyay et al [2]. We further reduce the set of equations in the form...
Figure 1. Background unperturbed flow in the local comoving box. The size of arrows indicates the magnitude of the respective velocities.

of an eigenvalue equation
\[ \frac{\partial Q}{\partial t} = -i \mathcal{L} Q, \quad (3) \]
where
\[ Q = \left( \begin{array}{c} \hat{u} \\ \hat{\zeta} \end{array} \right), \]
\[ u(x, y, z, t) = \hat{u}(x, t) \exp[i(k_y y + k_z z)], \quad (4) \]
\[ \zeta(x, y, z, t) = \hat{\zeta}(x, t) \exp[i(k_y y + k_z z)], \]
and \( \mathcal{L} = \mathcal{L}(k_y, k_z, R_c) \) is a differential operator. Note that because of translation invariance of the unperturbed flow in the \( y \)- and \( z \)-directions, we decompose the perturbations in terms of Fourier modes.

The solution of equation (3), describing the evolution of perturbation, can be written in terms of an eigenfunction expansion
\[ Q(x, t) = \sum_{j=1}^{\infty} C_j \exp(-i\sigma_j t) \tilde{Q}_j(x), \quad (5) \]
\[ \tilde{Q}_j(x) = \left( \begin{array}{c} \tilde{u}_j(x) \\ \tilde{\zeta}_j(x) \end{array} \right), \]
where the complex eigenvalue of the problem \( \sigma_j = \sigma_{Rj} + i\sigma_{Ij} \). Therefore, for the \( j \)th mode, equation (3) reduces to
\[ \mathcal{L} \tilde{Q}_j = \sigma_j \tilde{Q}_j. \quad (6) \]
To obtain the set of eigenvalues and eigenvectors, we convert the differential operator \( L \) into an \( N \times N \) matrix in a finite-difference representation and then compute the eigenvalues and eigenvectors of the matrix. The physical parameters involved in the problem (mainly \( R_c \) and the components of perturbation wave vector \( k_y, k_z \) described below) determine the required order \( N \) of the matrix for adequate accuracy.

It is well known (see e.g. [2]) that all the eigenmodes are stable, i.e. \( \sigma_j < 0 \) for all \( j \), when \( q < 2 \). However, the corresponding pseudo-eigenmodes may be unstable. Following previous work discussing non-rotating shear flows [1], let us try to understand it. If the linearized fluid system is driven by a signal of the form \( S(x, t) = e^{-i\sigma t}s(x) \), then equation (3) becomes

\[
\frac{dQ}{dt} = -iLQ + e^{-i\sigma t}s.
\]

It can be verified that the response will be of the form \( Q(x, t) = e^{-i\sigma t}\phi(x) \), where \( \phi = (\sigma I - L)^{-1}s \) and \( I \) is the identity matrix. The operator \( (\sigma I - L)^{-1} \) transforms the input \( s \) to the linearized flow at frequency \( \sigma \) into the corresponding output \( \phi \). The maximum amplification rate that occurs in the process can be written as

\[
\|L^{-1}\| = \text{maximum}(\frac{\|\phi\|}{\|s\|}),
\]

where \( \| \cdot \| \) denotes a norm of the respective quantities, in this work 2-norm [2].

Now an eigenvalue of \( L \) is \( \sigma \) if

\[
L\phi = \sigma I\phi,
\]

where \( \phi \) is the corresponding eigenvector. On the other hand, using equations (8) and (9) \( \sigma \) can be identified as an eigenvalue of \( L \) if the perturbation with frequency \( \sigma \) brings an unbounded amplification so that

\[
A(k_y, k_z, R) \equiv \|L^{-1}\| = \infty.
\]

Generalizing the above definition of an eigenvalue, we can define \( \sigma \) as an \( \epsilon \)-pseudo-eigenvalue [16] of \( L \) for any \( \epsilon \geq 0 \) if

\[
A(k_y, k_z, R) \equiv \|L^{-1}\| = \epsilon^{-1},
\]

when \( \phi \) is the corresponding \( \epsilon \)-pseudo-eigenvector/mode such that

\[
\frac{\|L - \sigma I\|}{\|\phi\|} = \|\epsilon\phi\|.
\]

Naturally as \( \epsilon \to 0 \), \( \epsilon \) and \( \sigma \) tend to be the pure eigenmode and eigenvalue, respectively. The set of \( \epsilon \)-pseudo-eigenvalues of an operator is called the \( \epsilon \)-pseudo-eigenspectrum.

Therefore, we can classify the amplification in three regimes. Firstly, if \( R_c \) is equal to or greater than a certain critical value \( R_{ec} \) \( (R_c \geq R_{ec}) \), and \( L \) has at least one eigenvalue in the upper half-plane of the Argand diagram, then \( \epsilon = 0 \) and \( A = \infty \); the flow in this regime is linearly unstable (i.e. the system has at least one exponential growing mode). Secondly, if \( R_c \) is smaller than a certain value, \( R_{eg} \) \( (R_c < R_{eg}) \), and all the eigenvalues lie in the lower half-plane of the Argand diagram, then \( \epsilon = 1 \) and \( A = 1 \); this implies no amplification (i.e. there is no exponential growing mode). Finally, if \( R_{eg} < R_c < R_{ec} \) and \( L \) has at least one pseudo-eigenvalue in the upper half-plane (although all the eigenvalues still lie in the lower half-plane), then \( 1 > \epsilon > 0 \) and \( 1 < A < \infty \); this is the regime of linearly stable flow with the existence of transient growth (i.e. the system has at least one growing pseudo-eigenmode).
2.1. Logarithmic norm and minimum perturbation

Now a very important related quantity is the logarithmic norm that is the upper bound of the possible transient growth rate. The logarithmic norm, $\mu_p$, of a square matrix operator $A$ using a matrix $p$-norm is defined [15] as

$$\mu_p := \lim_{h \to 0, h > 0, h \in \mathbb{R}} \frac{\|I + hA\|_p - 1}{h}$$  \hspace{1cm} (13)$$

for a natural number $p = 1, 2, \ldots, \infty$, where $h$ is the time interval and $\mathbb{R}$ is the set of real numbers. For $p = 2$, it is readily shown that

$$\mu_2(A) = \lambda_{\text{max}} \left( \frac{A + A^\dagger}{2} \right) \geq \max \{ \text{Re}[\lambda(A)] \},$$  \hspace{1cm} (14)$$

where $\lambda$ denotes an eigenvalue of the symmetric operand matrix and Re indicates the real part. The growth of the solution $Q(x, t)$ of equation (7) over a non-zero time interval $h$ can be upper bounded [15] as follows:

$$\|Q(x, t + h)\|_p \leq \|I + (-iL)h\|_p \|Q(x, t)\|_p + h|e^{-i\sigma h}|\|s\|_p + O(h^2)$$  \hspace{1cm} (15)$$

and the limiting maximum growth rate then can be written as

$$\|Q(x, t + h)\|_p \leq \lim_{h \to 0, h > 0} \left( \frac{\|I + (-iL)h\|_p - 1}{h} \|Q(x, t)\|_p + |e^{-i\sigma h}|\|s\|_p + O(h) \right)$$

$$\leq \mu_p(-iL)\|Q(x, t)\|_p + |e^{-i\sigma h}|\|s\|_p$$  \hspace{1cm} (16)$$

from which one can see that the logarithmic norm of $-iL$ determines the upper bound of the transient growth of the system in equation (7). Hence, for a suitably small perturbation if we obtain a pseudo-eigenvalue that is close to the logarithmic norm and is in the upper half-plane of the Argand diagram, then the system will exhibit sustainable transient instability. The non-negative real parts in the pseudo-eigenspectrum of $-iL$, i.e. the values in the upper half plane of the Argand diagram for the pseudo-eigenspectrum of $L$, drive the transient growth. The pseudo-eigenvalue of $-iL$ can be defined, equivalently to equation (11), as [16]

$$\lambda_\epsilon(-iL) := \lambda(-iL + E), \quad \|E\| = \epsilon > 0, \epsilon \in \mathbb{R}.$$  \hspace{1cm} (17)$$

Using the triangular inequality property of the logarithmic norm, we can write

$$\max \{ \text{Re}[\lambda_\epsilon(-iL)] \} \leq \mu_2(-iL + E) \leq \mu_2(-iL) + \epsilon.$$  \hspace{1cm} (18)$$

The above inequality leads to the computation of a minimum $\epsilon$ (namely $\epsilon_{\text{min}}$) for which the equality holds. If $\epsilon_{\text{min}}$ is small (compared to the matrix 2-norm of the operator, i.e. $\|\cdot\|_2$) such that $\max\{\text{Re}[\lambda_\epsilon(-iL)]\} > 0$, i.e. there are pseudo-eigenvalues in the upper half-plane of the Argand diagram, then we can conclude that the rotational shear flow system with the Coriolis effect exhibits transient growth sustained by small intrinsic perturbations.

3. Optimal perturbations for growing pseudo-eigenmodes and logarithmic norms

We solve the eigenvalue equation (6) with no-slip boundary conditions

$$u = \frac{\partial u}{\partial x} = \zeta = 0, \text{ at } x = \pm 1.$$  \hspace{1cm} (19)$$
Table 1. Parameters for optimal/maximal perturbation with $R_e = 2000$.

| $q$  | $k_y$ | $k_z$ | $G_{\text{max}}$ | $\max(\frac{\text{Im}(\sigma)}{\epsilon})^2$ | $\max[\text{Im}(\sigma)]$ |
|------|------|------|-----------------|-----------------|-----------------|
| 1.5  | 1.2  | 0    | 21.67           | 2.81            | 0.165           |
| 1.7  | 1.05 | 0.6  | 23              | 2.95            | 0.159           |
| 1.99 | 0.14 | 1.64 | 174.43          | 38.11           | $4.285 \times 10^{-2}$ |
| 2    | 0    | 1.66 | 4661            | 2199.96         | $3.56 \times 10^{-3}$ |

Figure 2. Pseudo-eigenspectra of the flow under optimum perturbation when $R_e = 2000$ for $q = 2$ (upper left), $q = 1.99$ (upper right), $q = 1.7$ (lower left) and $q = 1.5$ (lower right). Various curves in each figure sequentially from outside to inside, respectively, indicate the results with $\epsilon = 10^{-1}$, $10^{-1.5}$, $10^{-2}$ and $10^{-2.5}$.

In Table 1, we enlist the optimum sets of parameters governing the perturbation that induce maximum growth for various values of $q$. We compute the maximum transient growth $G_{\text{max}}$ induced by the perturbation and the pseudo-eigenvalues determining the lower bound of the transient growth of energy: $\max((\text{Im}(\sigma))/\epsilon)^2 \leq G_{\text{max}}$, where Im indicates the imaginary part. Interestingly, while for $q = 2$ the optimum perturbation giving rise to the maximum growth is purely vertical, for $q = 1.5$ it is purely 2D. For $1.5 < q < 2$ it is 3D with $k_y$ and $k_z$ both non-zero. Figure 2 shows the optimum pseudo-eigenspectra for various values of $q$. Naturally, for
Table 2. Parameters for maximal pure vertical perturbation with $R_e = 2000$.

| $q$  | $k_z$ | $G_{max}$ | $\text{max}(\text{Im}(\sigma))/\epsilon^2$ | $\text{max}[\text{Im}(\sigma)]$ |
|------|-------|-----------|---------------------------------|---------------------------------|
| 1.5  | 2.5   | 3.84      | 1.54                            | 0.945                           |
| 1.7  | 2.4   | 6.31      | 2.15                            | 0.516                           |
| 1.99 | 1.9   | 153.4     | 42.74                           | $5.419 \times 10^{-2}$         |
| 2    | 1.66  | 4661      | 2199.96                         | $3.56 \times 10^{-3}$         |

$q = 2$ the spectrum is marginally unstable and the pseudo-eigenspectra for all values of $\epsilon$ extend to the upper half-plane in the Argand diagram. However, as $q$ decreases, particularly from 1.7 to 1.5, the pseudo-eigenvalues for particular values of $\epsilon$, e.g. $10^{-2}$ and $10^{-2.5}$, decrease, which in turn decrease the rate of maximum growth as shown in table 1.

As the maximum growth for $q = 2$ is due to the pure vertical perturbation, we show in figure 3 the pseudo-eigenspectra due to the pure vertical perturbation, giving rise to the maximum possible growth (maximal pure vertical perturbation), for $q < 2$ as well. Corresponding parameters are given in table 2. Interestingly, the spectra and thus growth for
Figure 4. Comparison of the pseudo-eigenspectra of flows under maximal pure vertical perturbation between $R_e = 1000$ (upper left) and $R_e = 2000$ (upper right) when $q = 1.5$, and $R_e = 1000$ (lower-left) and $R_e = 2000$ (lower-right) when $q = 1.99$. Various curves in each figure sequentially from outside to inside, respectively, indicate the results with $\epsilon = 10^{-1}$, $10^{-1.5}$, $10^{-2}$ and $10^{-2.5}$.

a particular $q$ appear quite different between that for the optimum and vertical perturbation (particularly for smaller values of $q$). This is due to the Coriolis effect generating epicyclic oscillations appearing in the latter case, which is responsible for hindering growing pseudo-eigenmodes. Note that the energy growth due to pure vertical perturbation is bounded by the dynamics of the system and is independent of $R_e$. In figure 4, we compare pseudo-eigenspectra of the flow with $q = 1.5$ and 1.99 for two different values of $R_e$ and find them very similar, reflecting their independence of $R_e$ under pure vertical perturbation.

Table 3 shows the minimum perturbation ($\epsilon_{\text{min}}$) for which the logarithmic norm rate of growth is approached. It may be noted that $\epsilon_{\text{min}} \ll \| -i\mathcal{L} \|_2$. Also, at $q = 2$ the flow is most susceptible to transient growth since the required perturbation is the least. A decrease in $q$ reduces the susceptibility to transient growth as the magnitude of the required perturbation increases. However, $\epsilon_{\text{min}}$ is still quite small even for $q = 1.5$, ensuring local instability and sustained turbulence.
Table 3. Minimum perturbation for the logarithmic norm growth rate at $R_e = 2000$.

| $q$  | $k_y$ | $k_z$ | $\mu_2$ | $\epsilon_{\text{min}}$ | $\|\mathbf{-iL}\|_2$ |
|------|-------|-------|---------|--------------------------|---------------------|
| 1.5  | 1.2   | 0     | 1.1830  | $4.05 \times 10^{-2}$   | 5.1063              |
| 1.7  | 1.05  | 0.6   | 1.0430  | $1.05 \times 10^{-2}$   | 5.0079              |
| 1.99 | 0.14  | 1.64  | 0.3851  | $1.98 \times 10^{-3}$   | 5.0089              |
| 2    | 0     | 1.66  | 0.3632  | $1.86 \times 10^{-5}$   | 5.0087              |

4. Summary and conclusions

While the existence of unstable pseudo-eigenmodes favors instability and then plausible turbulence under finite amplitude perturbation, the question remained whether the local instability is sustained or not. The present analysis argues that the rotating shear flows with the Coriolis force not only exhibit unstable pseudo-eigenmodes under suitable small perturbations, but also the growth rate in the pseudo-eigenmodes approaches a strictly positive logarithmic norm and thus ensures feedback of instability as an intrinsic property of the flows. Therefore, a very small amplitude of perturbation is able to trigger instability and then plausible turbulence in the system. In lieu of linear instabilities (of hydrodynamic and hydromagnetic origin) unstable pseudo-eigenmodes repopulating the growing disturbance ensure sustained turbulence. This argues, for the first time, in favor of sustained hydrodynamic instability and then turbulence in a rotating Couette-like flow, when angular momentum increases and angular velocity decreases with radius. This is practically the flow of many astrophysical systems, e.g. accretion disc around compact objects, some active galactic nuclei and star-forming systems. This is possible to conceive because of the non-self adjoint nature of the underlying operator, giving rise to non-normal eigenmodes and hence unstable pseudo-eigenspectrum and positive logarithmic norms. Note that molecular viscosity is too small to explain the observed accretion efficiencies in the systems mentioned above. Therefore, any viscosity is expected to arise due to turbulence in the flow. Hence, in the absence of any linear instability in the Keplerian accretion flows, the existence of small perturbations triggering local transient growth rates of the order of logarithmic norm, which is strictly positive for the associated spatial operator matrix, resolves the question of instability and the corresponding feedback process that replenishes the growth to an ultimate turbulence state. This is a new alternative picture of turbulence emergence in astrophysical flows explaining the transport of mass inward removing the angular momentum outward, particularly when the transport due to the nonlinearly developed MRI is expected to be very small [4] in experimental setups with small magnetic Prandtl numbers.

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