Renormalization Mass Scale and Scheme Dependence in Inclusive Semileptonic $b \rightarrow u$ Decays

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July 25, 2018

PACS No.: 11.10Hi
Key Words: renormalization scheme, $b$ quark decays
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Abstract

The renormalization group (RG) equation is first used to sum all leading-logarithmic, next-to-leading logarithmic etc. contributions to the decay rate $\Gamma$ for the process $b \rightarrow u\ell^-\bar{\nu}_\ell$ when using minimal subtraction. Next, all logarithmic contributions to $\Gamma$ are summed, leaving $\Gamma$ in terms of the log-independent contributions and the renormalization group functions as well as a mass scale set by the pole mass $m_{\text{pole}}$ for the $b$ quark. The implicit and explicit dependence of $\Gamma$ on the renormalization induced mass scale $\mu$ (which is non-physical and arbitrary) cancels. The renormalization scheme (RS) dependence of $\Gamma$ is considered, and two particular renormalization schemes suggested. Both of these leave $\Gamma$ dependent on $m_{\text{pole}}$ and a set of RS invariant parameters $\tau_i$ and $\sigma_i$.

1 Introduction

Mass independent renormalization schemes $[1,2]$ are relatively easy to implement, but are often viewed as being “unphysical”. This is particularly true when the process being considered involves a heavy particle such as a $b$ quark. In addition to this problem, perturbative results obtained
using a mass independent renormalization scheme (RS) depend on a non-physical scale parameter $\mu$. Furthermore, results at a finite order of perturbation theory can be altered by making a finite renormalization of the quantities that characterize the theory (masses, couplings and field strengths).

In ref. [3] the renormalization group (RG) equation is used to sum logarithmic corrections to various processes. One can use the RG functions at $n$-loop order to sum the $N^{n-1}LL$ corrections (one loop to sum leading-log, two loop to sum next-to-leading-log, etc.). Alternatively, one can sum all logarithmic corrections in terms of the log-independent corrections and the RG functions. This latter approach was used in refs. [4,5] to examine $R_{e^+e^-}$, the cross section for $e^+e^- \rightarrow$ hadrons when using mass-independent renormalization. It was found that upon summing all logarithmic contributions to $R_{e^+e^-}$ the explicit dependence of $R_{e^+e^-}$ on $\mu$ cancelled with its implicit dependence on $\mu$, leaving $R_{e^+e^-}$ dependent only on the ratio $Q/\Lambda$ where $Q$ is the centre of mass energy and $\Lambda$ is a mass scale associated with the boundary value of the running coupling $a(\ln \mu/\Lambda)$. A set of RS invariant parameters $\tau_i$ was found, and $R_{e^+e^-}$ was seen to be expressible in terms of $a(\ln Q/\Lambda)$ and $\tau_i$.

In this paper, we use the RG equation to examine the perturbative expansion for the decay rate $\Gamma$ for the process $b \rightarrow u\ell^-\bar{\nu}_\ell$. Explicit diagrammatic computations of $\Gamma$ to two-loop order are in ref. [18]. The arbitrary renormalization scale parameter $\mu$ occurs explicitly in this result; there is also an implicit dependence on $\mu$ through dependence of $\Gamma$ on the parameters that characterize the theory (the renormalized couplings and masses). Various approaches have been used to minimize sensitivity of $\Gamma$ on $\mu$ and discussion of several of these appear in ref. [22]. In ref. [23] it is shown that even when the pole mass RS is used with a Padé estimate of the three-loop contribution to $\Gamma$, there still is a significant dependence on $\mu$.

We show in this paper that the RG equation can be used to treat dependence of $\Gamma$ on $\mu$ in various ways. This will be done using the mass independent RS [1,2] rather than the pole mass RS. This has the advantage of making the RG functions independent of the mass, and of facilitating renormalization beyond one-loop order. Its disadvantage is that the mass parameter that occurs in a perturbative calculation is not physical and also “runs”; that is, it is dependent on the renormalization parameter $\mu$. However, this shortcoming can be overcome as the pole mass and the running mass can be related [15]. We exploit this relationship to eliminate the running mass in favour of the pole mass once we have applied the RG equation to treat the $\mu$ dependence of $\Gamma$.

In section 2 we use the RG approach to carry out two distinct partial summations of the perturbative expansion of the decay rate $\Gamma$. First, we use the RG equation to sum the $LL, NLL, \ldots N^pLL$ contributions to $\Gamma$ in terms of the $1-$ to $(p+1)$-loop contributions to $\beta$ and $\gamma$, the RG functions associated with $a(\mu)$ and $m(\mu)$ respectively. This approach was used in ref. [3] to show that summation of $LL, NLL$ etc. contributions to $\Gamma$ serve to reduce (though not entirely eliminate) dependence of $\Gamma$ on $\mu$. Second, we then carry out a summation in which all of the logarithmic contributions to $\Gamma$ are summed so that $\Gamma$ is expressed in terms of its log independent part evaluated using an
auxiliary function $\eta$. In this log-summed form the explicit and implicit dependence on $\mu$ is shown to cancel, much as it did in the analysis of $R_{e^+e^-}$. The problem of dependence of the perturbative contributions of $\Gamma$ on $\mu$ is resolved.

Even within the mass independent renormalization schemes there is a degree of RS ambiguity. The renormalization scheme can be parameterized by the appropriate coefficients in the loop expansion of the RG functions associated with the coupling constant and the anomalous mass dimension [7,8]. In section 3 we use these parameters to investigate the RS dependence of both the running coupling and the running mass and we examine some of the implications of this dependence. In section 4 we show that the requirement that $\Gamma$ be RS independent (i.e. $\Gamma$ does not depend on these parameters) leads to a set of RS invariant parameters $\tau_i$ which can be computed perturbatively.

In section 5 we use RG summation to relate the pole mass for the $b$ quark $m_{pole}$ to the running mass $m(\mu)$ and we use this relationship to express $\Gamma$ in terms of a physical, RS invariant mass scale, $m_{pole}$. We also show that the RS independence of $m_{pole}$ leads to a second set of RS invariant parameters $\sigma_i$ which can also be computed perturbatively.

In section 6, two particular RS are considered. In one scheme, it proves possible to limit the perturbative expansion of $\Gamma$ to a single term; all higher loop contributions to $\Gamma$ in this scheme only serve to affect the behaviour of the running coupling and mass. In this scheme it is not necessary to consider the convergence of perturbative expansion for $\Gamma$ and so performing a Borel summation or considering renormalons as done in ref. [22] is not pertinent. In a second scheme, which is a natural generalization of the t’Hooft RS [16,17], the perturbative expansion for the RG functions $\beta(a)$ and $\gamma(a)$ terminate, making it possible to determine $a(\mu)$ and $m(\mu)$ in closed form, though now the perturbative expansion for $\Gamma$ is an infinite series.

In this paper we are only concerned with the perturbative contributions to the decay $b \to u \ell^- \bar{\nu}_\ell$. Other non-perturbative effects such as the Fermi motion of the $b$ quark must be taken before $|V_{ub}|$ can be inferred from experiment. These problems are considered in refs. [24,25]. An extended discussion of the properties of heavy quarks and leptons appears in ref. [26].

### 2 Renormalization Group Summation

A perturbative evaluation of the amplitude $\Gamma$ for the semi-leptonic decay process $b \to u \ell^- \bar{\nu}_\ell$ leads to the expression [18]

$$\Gamma = [m(\mu)]^5 \sum_{n=0}^{\infty} \sum_{k=0}^{n} T_{n,k} a^n(\mu) \ln^k \left( \frac{\mu}{m(\mu)} \right)$$

(1)

where $m(\mu)$ is the running mass for the $b$ quark and $a(\mu) (= \alpha_s(\mu)/\pi)$ is the strong coupling. (We assume five active quark flavours and have absorbed an overall factor of $\frac{G_F^2 |V_{ub}|^2}{192\pi^3}$ into the expansion coefficients $T_{n,k}$.) As $\Gamma$ is independent of the renormalization scale parameter $\mu$ we have the RG
where
\[ \beta(a) = \mu \frac{\partial a}{\partial \mu} = -ba^2(1 + ca + c_2a^2 + \ldots) \] (3)

and
\[ m\gamma(a) = \mu \frac{\partial m}{\partial \mu} = mfa(1 + g_1a + g_2a^2 + \ldots). \] (4)

We first organize the sums in eq. (1) using the functions
\[ S_n(\xi) = \sum_{k=0}^{\infty} T_{n+k,k} \xi^k \] (5)

\( (S_0: \text{leading-logs (LL)}, S_1: \text{next-to-leading-logs (NLL)}, \text{etc.}) \) so that
\[ \Gamma = m^5 \sum_{n=0}^{\infty} a^n S_n \left( a \ln \frac{\mu}{m} \right). \] (6)

Substitution of eqs. (3,4,6) into eq. (2) leads to a set of nested equations with the boundary conditions \( S_n(0) = T_{n,0} \equiv T_n; \) once \( S_0, S_1 \ldots S_{n-1} \) are known, it is possible to solve for \( S_n \) [3]. We find that
\[ (1 - b\xi)S'_0(\xi) + 5fS_0(\xi) = 0 \] (7)

and so
\[ S_0(\xi) = T_0(1 - b\xi)^{5f/b}. \] (8)

Similarly, we find that
\[ (1 - b\xi)S'_1 + (-b + 5f)S_1 + (-bc\xi - f)S'_0 + 5fg_1S_0 = 0. \] (9)

The solutions for \( S_0 \ldots S_3 \) appear in ref. [3].

Eq. (1) shows that \( \Gamma \) does not directly depend on a physical mass parameter (analogous to the centre of mass energy \( Q \) that occurs in refs. [4,5] where \( R_{e^+e^-} \) is discussed). Instead a “running mass” \( m(\mu) \) associated with the renormalized mass of the \( b \) quark appears. This makes discussions of how \( \Gamma \) depends on \( \mu \) more complicated than for \( R_{e^+e^-} \). For \( R_{e^+e^-} \) one can very simply sum all of the log-dependent contributions to \( R_{e^+e^-} \) to show immediately that all \( \mu \) dependence cancels. This is also possible for \( \Gamma \), but it proves to be more awkward to sum all of its log-dependent pieces.

We begin by defining
\[ A_n(a(\mu)) = \sum_{k=0}^{\infty} T_{n+k,n} a(\mu)^{n+k} \] (10)
so that eq. (1) becomes

\[ \Gamma = m^5(\mu) \sum_{n=0}^{\infty} A_n(a(\mu)) \ell^n \]  

(11)

where now \( \ell = \ln(\mu/m(\mu)) \). If eq. (11) is substituted into eq. (2) we find

\[ A_n(a(\mu)) = \frac{-1}{n} \left[ \hat{\beta}(a(\mu)) \frac{\partial}{\partial a} + 5\hat{\gamma}(a(\mu)) \right] A_{n-1}(a(\mu)) \]  

(12)

where

\[ \hat{\beta} = \beta/(1 - \gamma), \]  

(13a)

\[ \hat{\gamma} = \gamma/(1 - \gamma). \]  

(13b)

We now define

\[ E(\mu) = \exp \left[ \int_0^{a(\mu)} dx \frac{\gamma(x)}{\beta(x)} + \int_0^K dx \frac{fx}{bx^2(1 + cx)} \right] \]  

(14)

where \( K \) is some cut off. The second integral in eq. (14) is an infinite constant whose role is to ensure that the argument of the exponential is finite.

We also note that solutions to eqs. (3,4) can be written as

\[ \ln \left( \frac{\mu}{\Lambda} \right) = \int_0^{a(\mu)} dx \frac{\gamma(x)}{\beta(x)} + \int_0^K dx \frac{fx}{bx^2(1 + cx)} \]  

(15)

and

\[ m(\mu) = M E(\mu) \]  

(16)

where \( \Lambda \) and \( M \) are scale dependent quantities used to define boundary conditions on eqs. (3,4). We note that in eqs. (15,16) a change in \( K \) can be absorbed into changes in \( \Lambda \) and \( M \). In refs. [7,8], \( K \) is taken to be infinite. (Below we will often use \( a(\mu) \) to denote \( a \left( \ln \frac{\mu}{\Lambda} \right) \). Similarly, the \( \mu \) dependence of any dimensionless quantity, such as \( E \), will be written \( E(\mu) \) but will be understood to mean \( E(\ln \mu/\Lambda) \).

We now can re-express eq. (12) in the form

\[ B_n(a(\mu)) = \frac{-1}{n} \hat{\beta}(a(\mu)) \frac{\partial}{\partial a} B_{n-1}(a(\mu)) \]  

(17)

where

\[ B_n(a) = E^5(\mu) A_n(a). \]  

(18)

We now define an auxiliary quantity \( \eta \) so that

\[ \frac{\partial}{\partial \eta} = \frac{\hat{\beta}(\eta)}{\partial a} \]  

(19)
so that by eq. (13a)
\[ \eta(a(\mu)) = \int_0^{a(\mu)} dx \frac{1 - \gamma(x)}{\beta(x)} + \int_0^K dx \frac{1 - fx}{bx^2(1 + cx)}. \] (20)
Together eqs. (17,19) show that
\[ B_n(a(\mu)) = \frac{1}{n} \frac{\partial}{\partial \eta} B_{n-1}(a(\mu)) \]
or upon iteration,
\[ = \frac{(-1)^n}{n!} \frac{\partial^n}{\partial \eta^n} B_0(a(\mu)). \] (21)
Eqs. (11,18,21) together lead to
\[ \Gamma = m^5(\mu)E^{-5}(\mu) \sum_{n=0}^{\infty} \frac{(-\ell)^n}{n!} \left( \frac{\partial}{\partial \eta} \right)^n B_0(a(\mu)). \] (22)
Eq. (16) results in eq. (22) becoming
\[ \Gamma = M^5 B_0(a(\eta - \ell)). \] (23)
However, we now see by eqs. (14-16,20), that
\[ \eta - \ell = \ln \left( \frac{M}{\Lambda} \right) \] (24)
and using eqs. (18,23,24), that
\[ \Gamma = M^5 E^5 \left( \ln \frac{M}{\Lambda} \right) A_0 \left( a \left( \ln \frac{M}{\Lambda} \right) \right). \] (25)
By eq. (25) we see that \( \Gamma \) is now expressed in terms of its log-independent contribution \( A_0 \) and that all explicit and implicit dependence on \( \mu \) has cancelled. The presence of a factor of \( E^5 \left( \ln \frac{M}{\Lambda} \right) \) in eq. (25) means that this RG summed expression for \( \Gamma \) cannot be recovered by simply choosing a particular value for \( \mu \) in eq. (1); this is unlike \( R_{e^+e^-} \) where, by setting \( \mu = Q \) in the initial expression, the RG summed result is obtained [7,8]. The mass scale \( M \) will be shown to be a RS independent (though unphysical) quantity. However, in section five it will be shown that \( M \) can be expressed in terms of the pole mass of the \( b \) quark, \( m_{pole} \), a physical quantity that is RS invariant.

3 Renormalization Scheme Dependence of \( a \) and \( m \)
As has been noted above, even when using a mass independent RS such as \( \overline{\text{MS}} \), one can perform finite renormalizations of \( a(\mu) \) and \( m(\mu) \) [9], so that
\[ \overline{a} = a + x_2a^2 + x_3a^3 + \ldots \equiv F(a) \] (26a)
\[ \overline{m} = m(1 + y_1a + y_2a^2 + \ldots) \equiv mG(a). \] (26b)
If now,

\[ \mu \frac{d\bar{a}}{d\mu} = \bar{\beta}(\bar{a}) = -\bar{\beta} \bar{a}^2(1 + \bar{c} \bar{a} + \bar{c}_2 \bar{a}^2 + \ldots) \]  
(27a)

\[ \frac{\mu}{m} \frac{dm}{d\mu} = \bar{\gamma}(\bar{a}) = \bar{\gamma} \bar{a}(1 + \bar{g}_1 \bar{a} + \bar{g}_2 \bar{a}^2 + \ldots) \]  
(27b)

it follows from eqs. (3,4,26,27) that

\[ \bar{b} = b \]  
(28a)

\[ \bar{c} = c \]  
(28b)

\[ \bar{c}_2 = c_2 - 2c x^2 + x^3 - x_2^2 \]  
(28c)

\[ \bar{c}_3 = c_3 - 3c x^2 + 2(c_2 - 2c x_2) x_2 + 2x_4 - 2x_2 x_3 \]  
(28d)

etc. as well as

\[ \bar{f} = f \]  
(29a)

\[ \bar{g}_1 = g_1 - x_2 - (b/f) y_1 \]  
(29b)

\[ \bar{g}_2 = g_2 - x_3 - x_2 y_1 - \bar{g}_1(y_1 + 2x_2) + g_1 y_1 - (b/f)(2y_2 + cy_1) \]  
(29c)

etc.

In refs. [7,8] it has been suggested the RS ambiguities within mass independent renormalization could be parameterized by the mass scale \( \mu \) as well as the coefficients \( c_i (i \geq 2) \), \( g_i (i \geq 1) \) in eqs. (3,4). To see how \( a \) and \( m \) vary with changes in \( c_i \) and \( g_i \), we define

\[ \frac{\partial a}{\partial c_i} = B_i(a) \approx a^{i+1} \left( W_0^i + W_1^i a + \ldots \right) \]  
(30a)

\[ \frac{1}{m} \frac{\partial m}{\partial c_i} = \Gamma_i^c(a) \approx a^i \left( U_0^i + U_1^i a + \ldots \right) \]  
(30b)

\[ \frac{1}{m} \frac{\partial m}{\partial g_i} = \Gamma_i^g(a) \approx a^i \left( V_0^i + V_1^i a + \ldots \right) . \]  
(30c)

(It is evident that \( \frac{\partial a}{\partial g_i} = 0 \).) The consistency condition

\[ \left( \mu \frac{\partial}{\partial \mu} \frac{\partial}{\partial c_i} - \frac{\partial}{\partial c_i} \mu \frac{\partial}{\partial \mu} \right) a \equiv \left[ \mu \frac{\partial}{\partial \mu}, \frac{\partial}{\partial c_i} \right] a = 0 \]  
(31)

leads to [7]

\[ B_i(a) = -b \beta(a) \int_0^a dx \frac{x^{i+2}}{\beta^2(x)} \]  
(32a)

\[ \approx a^{i+1} \left[ \frac{1}{i-1} - c \left( \frac{i-2}{i(i-1)} \right) a + \frac{1}{i+1} \left( c^2 \frac{i-2}{i} - c_2 \frac{i-3}{i-1} \right) a^2 + \ldots \right] . \]  
(32b)
when the expansions of eq. (35) are substituted into eq. (36) we find that

$$\Gamma_i = \gamma(a) \frac{\beta(a)}{\beta(a)} B_i(a) + b \int_0^a dx \frac{x^{i+2} \gamma(x)}{\beta^2(x)}$$

(33a)

and

$$\approx \frac{f}{b} a^i \left[ -\frac{1}{i(i-1)} + 2 \left( \frac{c}{i(i+1)} - \frac{g_1}{(i+1)(i-1)} \right) a + \frac{1}{i+2} \left( \frac{2c_2}{i+1} - \frac{3c^2}{i+1} + \frac{4g_1c}{i} - \frac{3g_2}{i-1} \right) a^2 + \ldots \right]$$

(33b)

The commutator of any two operators \( [\mu \frac{\partial}{\partial \mu}, \frac{\partial}{\partial a}, \frac{\partial}{\partial c}, \frac{\partial}{\partial g}] \) acting on \( \beta(a), \gamma(a), B_k(a), \Gamma_k(a), \Gamma^g_k(a) \) gives zero (eg. \( [\frac{\partial}{\partial a}, \frac{\partial}{\partial g}] \Gamma_k(a) = 0 \)) which is a non-trivial consistency check on eqs. (32-34).

We can now examine how \( a \) and \( m \) change under variations in \( \mu, c_i \) and \( g_i \). Beginning with the expansions [4,10,11,12]

$$a(\mu') = a(\mu) \left[ 1 + (\alpha_{i1} \ell)a(\mu) + (\alpha_{i2} \ell + 2)^2 a^2(\mu) + \ldots \right]$$

(35a)

$$m(\mu') = m(\mu) \left[ 1 + (\beta_{i1} \ell)a(\mu) + (\beta_{i2} \ell + 2)^2 a^2(\mu) + \ldots \right]$$

(35b)

where \( \ell = \ln \left( \frac{\mu}{\mu'} \right) \), we can obtain the coefficients \( \alpha_{ij}, \beta_{ij} \) either by directly integrating eqs. (3,4) or by using the fact that \( a(\mu') \) and \( m(\mu') \) are independent of \( \mu \) and so

$$\mu \frac{d}{d\mu} a(\mu') = \left( \mu \frac{\partial}{\partial \mu} + \beta(a) \frac{\partial}{\partial a} \right) a(\mu') = 0$$

(36a)

$$\mu \frac{d}{d\mu} m(\mu') = \left( \mu \frac{\partial}{\partial \mu} + \beta(a) \frac{\partial}{\partial a} + m \gamma(a) \frac{\partial}{\partial \mu} \right) m(\mu') = 0.$$ 

(36b)

When the expansions of eq. (35) are substituted into eq. (36) we find that

$$a(\mu') = a(\mu) \left[ 1 + (b\ell)a(\mu) + (b\ell + b^2 \ell^2) a^2(\mu) + \left( bc_2 \ell + \frac{5}{2} b^2 c_2 \ell^2 + b^3 \ell^3 \right) a^3(\mu) + \ldots \right]$$

(37a)
and

\[
m(\mu') = m(\mu) \left[ 1 - (f \ell)a(\mu) + \left( -f g_1 \ell + \frac{f}{2}(f - b)\ell^2 \right) a^2(\mu) \right.
\]
\[
+ \left. \left( -f g_2 \ell + \left( f^2 g_1 - f b g_1 - \frac{f b c}{2} \right) \ell^2 - \frac{f}{3}(f - b) \left( \frac{f}{2} - b \right) \ell^3 \right) a^3(\mu) + \ldots \right].
\]

Using eq. (37) to express \( a(\mu'') \) and \( m(\mu'') \) in terms of \( a(\mu') \) and \( m(\mu') \) and subsequently to express \( a(\mu) \) and \( m(\mu) \) in terms of \( a(\mu') \) and \( m(\mu') \) gives a relation between \( a(\mu'') \) and \( m(\mu'') \) and \( a(\mu) \) and \( m(\mu) \) that is consistent with eq. (37). This is a non-trivial check on eq. (37).

We now will show how \( m \) and \( a \) in two different mass independent RS are related. If we replace the parameters \((c, g)\) by \((c', g')\), then \( a' \equiv a(c') \) and \( m' = m(c', g') \) are related to \( a = a(c) \), \( m = m(c, g) \) by expansions of the form

\[
a' = a \left[ 1 + x_2 (c', c) a + x_3 (c', c) a^2 + \ldots \right]
\]

and

\[
m' = m \left[ 1 + y_1 (c', c; g', g) a + y_2 (c', c; g', g) a^2 + \ldots \right],
\]

where

\[
x_n(c, c) = 0,
\]
\[
y_n(c, c; g, g) = 0.
\]

Since \( a' \) is independent of \( c_i \), we have the equation

\[
\frac{d a'}{d c_i} = 0 = \left( \frac{\partial}{\partial c_i} + B_i(a) \frac{\partial}{\partial a} \right) a'
\]

with \( B_i(a) \) given by eq. (32) and the boundary condition of eq. (39a). It follows that [4,5]

\[
a' = a \left\{ 1 + (c'_2 - c_2) a^2 + \frac{1}{2} (c'_3 - c_3) a^3 + \left[ \frac{1}{3} (c'_4 - c_4) - \frac{c}{6} (c'_3 - c_3) \right.ight.
\]
\[
+ \left. \frac{1}{6} (c'_2 - c_2)^2 + \frac{3}{2} (c'_2 - c_2)^2 \right] a^4 + \ldots \left\} \right.
\]

Similarly, since \( \frac{d a'}{d c_i} = \frac{d a'}{d g_i} = 0 \) we find that

\[
m' = m \left[ 1 + \frac{f}{b} (g_1 - g'_1) a + \frac{f}{2b} \left( g_2 - g'_2 + c_2 - c'_2 - c (g_1 - g'_1) + \frac{f}{b} (g_1 - g'_1)^2 \right) a^2 + \ldots \right].
\]

By using eqs. (37,41) it is possible to evaluate \( a(\mu) \) and \( m(\mu) \) in any mass-independent RS at any value of \( \mu \) to any desired order in perturbation theory once we know these values at some
particular value of $\mu$ in some particular mass independent RS. Eq. (26) is compatible with eqs. (37,41) provided we use eqs. (28,29) and identify $x_2$ with $\ln(\mu/\mu')$.

It is interesting that the sort of summation used to obtain the sums $S_0$, $S_1$, etc. of eq. (5) can be used in conjunction with eq. (35). We begin by writing eq. (35) in the form

$$a' = \sum_{n=0}^{\infty} \rho_n(\alpha \ell) a^{n+1}$$

$$m' = \sum_{n=0}^{\infty} m \psi_n(\alpha \ell) a^n$$

where $a = a(\mu)$, $a' = a(\mu')$, $m = m(\mu)$, $m' = m(\mu')$ and

$$\rho_n(\xi) = \sum_{k=0}^{\infty} \alpha_{n+k,k} \xi^k, \quad \psi_n(\xi) = \sum_{k=0}^{\infty} \beta_{n+k,k} \xi^k$$

Eq. (36) leads to a set of nested equations since

$$\sum_{n=0}^{\infty} [\rho'_n(\xi) - b (1 + ca + c_2 a^2 + \ldots) (\xi \rho'_n(\xi) + (n + 1) \rho_n(\xi))] a^n = 0$$

$$\sum_{n=0}^{\infty} m [\psi'_n(\xi) - b (1 + ca + c_2 a^2 + \ldots) (\xi \psi'_n(\xi) + n \psi_n) + f (1 + g_1 a + g_2 a^2 + \ldots) \psi_n] a^n = 0$$

(with $\rho_n(0) = \psi_n(0) = \delta_{n,0}$)

which are satisfied to each order in $a$. This partially sums the series in eq. (37), much like eq. (6) can be used to partially sum the series in eq. (1). We find that by eq. (44a), for example,

$$\rho_0(\xi) = (1 - b \xi)^{-1}, \quad \rho_1 = \frac{-c \ln(1 - b \xi)}{(1 - b \xi)^2}.$$
quantity (i.e. is invariant under changes in $\mu$, $c_i$ and $g_i$) is now straightforwardly established using the definition of $E(a(\mu))$ in eq. (14). To this end we note that

$$\frac{\partial}{\partial c_i} \left( m \exp \left[ - \int_0^a dx \frac{\gamma(x)}{\beta(x)} \right] \right) = m \left( \Gamma_i^c(a) - B_i(a) \frac{\gamma(a)}{\beta(a)} \right) - \int_0^a \frac{bx_i^2 \gamma(x)}{\beta^2(x)} \exp \left[ - \int_0^a dx \frac{\gamma(x)}{\beta(x)} \right] = 0$$

(46a)

by eq. (33a) and

$$\frac{\partial}{\partial g_i} \left( m \exp \left[ - \int_0^a dx \frac{\gamma(x)}{\beta(x)} \right] \right) = m \left( \Gamma_i^g - \int_0^a dx \frac{f x_i^{i+1}}{\beta(x)} \right) \exp \left[ - \int_0^a dx \frac{\gamma(x)}{\beta(x)} \right] = 0$$

(46b)

by eq. (34a). Together eqs. (16,46) show that $\mathcal{M} = m(\mu)E^{-1}(a(\mu))$ is invariant under changes of $\mu$, $c_i$ and $g_i$.

Since $\mathcal{M}$ is RS independent, it follows from eq. (25) that $T_{n,k}$ must be explicitly RS dependent in order to cancel the implicit RS dependence occurring in $a$. In the next section we show that the RS dependence of $T_{n,k}$ can be determined.

### 4 Renormalization Scheme Dependence of $T_{n,k}$

In eq. (1) it is apparent that perturbative contribution to the semileptonic decay rate $\Gamma$ is independent of the RS used to define $a$ and $m$ but that the perturbation expansion coefficients $T_{n,k}$ are RS-dependent. In this section we show that this RS-dependence can be determined by solving sets of nested first order pde’s that occur naturally in the perturbative analysis.

It is evident from eqs. (6) and (10) that all of the coefficients $T_{n,k}(k > 0)$ can be expressed in terms of $T_n(\equiv T_{n,0})$ on account of the RG equation, eq. (2). In fact, by substitution of eq. (1) into eq. (2) we find that

$$m^5 \sum_{n=0}^{\infty} \sum_{k=0}^{n} T_{n,k} \left[ k a^n \ell^{k-1} (1 - f a (1 + g_1 a + \ldots)) \right] + 5f (1 + g_1 a + \ldots) a^{n+1} \ell^k - b (1 + ca + c_2 a^2 + \ldots) na^{n+1} \ell^k = 0.$$  

(47)

For example, at order $(a^1 \ell^0)$ this leads to

$$T_{1,1} = -5f T_0$$

(48a)

while at order $(a^2 \ell^0)$

$$T_{2,1} = (b - 5f) T_1 - 5f (g_1 + f) T_0.$$  

(48b)

We now turn to the equations

$$\frac{d\Gamma}{dc_i} = 0 = \left( \frac{\partial}{\partial c_i} + B_i(a) \frac{\partial}{\partial a} + m \Gamma_i^c \frac{\partial}{\partial m} \right) \left( m^5 \sum_{n=0}^{\infty} \sum_{k=0}^{n} T_{n,k} a^n \ln^k \left( \frac{\mu}{m} \right) \right)$$

(49a)
\[
\frac{d\Gamma}{dg_i} = 0 = \left( \frac{\partial}{\partial g_i} + \Gamma g \frac{\partial}{\partial m} \right) \left( m^5 \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} T_{n,k} u^n \ln^n \left( \frac{\mu}{m} \right) \right). \tag{49b}
\]

With the expansions of eq. (30), eq. (49) results in
\[
\frac{\partial T_0}{\partial c_i} = 0, \quad \frac{\partial T_1}{\partial c_i} = 0, \quad \frac{\partial T_2}{\partial c_i} + \delta_2 \left( 5T_0 U_0^i \right) = 0
\]
and
\[
\frac{\partial T_3}{\partial c_i} + \delta_3 \left( 5T_0 U_0^i \right) + \delta_2 \left( T_1 W_0^i + 5T_0 U_1^i + 5T_1 U_0^i - T_{1,1} U_0^i \right) = 0 \tag{50a-d}
\]
e.tc. as well as
\[
\frac{\partial T_0}{\partial g_i} = 0, \quad \frac{\partial T_1}{\partial g_i} + \delta_1 \left( 5T_0 V_0^i \right) = 0
\]
\[
\frac{\partial T_2}{\partial g_i} + \delta_2 \left( 5T_0 V_0^i \right) + \delta_1 \left( 5T_1 V_0^i + 5T_0 V_1^i - T_{1,1} V_0^i \right) = 0
\]
and
\[
\frac{\partial T_3}{\partial g_i} + \delta_3 \left( 5T_0 V_0^i \right) + \delta_2 \left( 5T_1 V_0^i + 5T_1 V_0^i - T_{1,1} V_0^i \right) + \delta_1 \left[ 5 \left( T_2 V_0^i + T_1 V_1^i + V_2 T_0 \right) - \left( T_{2,1} V_0^i + T_{1,1} V_1 \right) \right] = 0 \tag{51a-d}
\]
e.tc.

Reading off the appropriate coefficients \( W_n^i, U_n^i \) and \( V_n^i \) from eqs. (32b, 33b, 34b) and using eq. (48) results in a set of equations that can be used to solve for \( T_0, T_1, T_2, T_3 \) etc. Solving eqs. (50,51) leads to
\[
T_0 = \tau_0 \tag{52a-c}
\]
\[
T_1 = \tau_1 + \frac{5f}{b} \tau_0 g_1
\]
\[
T_2 = \tau_2 + \frac{5f}{b} \left[ \tau_1 g_1 + \frac{\tau_0}{2} \left( c_2 + g_2 + (2f - c) g_1 + \frac{5f}{b} g_1^2 \right) \right]
\]
e.tc.

where the \( \tau_n \) are constants of integration and consequently are RS invariants.

Let us now hark back to the RG summed expression for \( \Gamma \) appearing in eq. (25). If an explicit perturbative calculation of \( T_n, c_i, g_i \) has taken place to \( N^{th} \) order in perturbation theory by evaluation of Feynman diagrams using some mass-independent RS such as \( \overline{MS} \) [2,13], then \( \tau_0, \tau_1 \ldots \tau_N \) can be computed for this process by use of eq. (52).

### 5 The Pole Mass and the Running Mass

In eq. (25) we have an expression for \( \Gamma \) that depends on a mass scale \( M \) which is essentially a boundary value for the equation for the running mass \( m(\mu) \) of eq. (4). We will now relate this
mass scale $\mathcal{M}$ to a physical mass, the pole mass $m_{\text{pole}}$ of the $b$ quark. This pole mass is a RS independent, gauge invariant and infrared finite quantity [14]. Since quarks are always in a bound state, one cannot directly measure this pole mass; it is a quantity that is realized in perturbation theory. In a number of papers the self energy of the quark is discussed in detail and from this the relationship between $m_{\text{pole}}$ and $\mathcal{M}$ can be derived [15].

If one uses a mass independent RS, the renormalized quark propagator has the form

$$S^{-1}(p^\mu, m(\mu)) = A\left(p^2, m(\mu)\right)\not{p} - m(\mu)B\left(p^2, m(\mu)\right); \quad (53)$$

the pole mass is defined by the transcendental equation

$$\lim_{\not{p} \to m_{\text{pole}}} S^{-1}(p^\mu, m(\mu)) = 0. \quad (54)$$

It results in an expansion

$$m_{\text{pole}} = m(\mu) \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \kappa_{n,k} a^n L^k \quad (55)$$

where $\kappa_{0,0} = 1$ and $L = \ln \left(\frac{\mu}{m_{\text{pole}}}\right)$. The approach used to derive eq. (25) can now be applied to eq. (55).

Upon defining (as in eq. (10))

$$F_n(a(\mu)) = \sum_{k=0}^{\infty} \kappa_{n+k,n} a(\mu)^{n+k} \quad (56)$$

we can use the RG equation

$$\mu \frac{dm_{\text{pole}}}{d\mu} = 0 = \left(\mu \frac{\partial}{\partial \mu} + \beta(a) \frac{\partial}{\partial a} + m \gamma(a) \frac{\partial}{\partial m}\right) \left[m \sum_{n=0}^{\infty} F_n(a) \ln^n \left(\frac{\mu}{m_{\text{pole}}}\right)\right] \quad (57)$$

to show that

$$F_{n+1}(a) = -\frac{1}{n+1} \left(\beta(a) \frac{\partial}{\partial a} + \gamma(a)\right) F_n(a). \quad (58)$$

With $E$ defined in eq. (14) and

$$\phi_n = E F_n \quad (59)$$

then by eq. (57)

$$\phi_{n+1}(a) = -\frac{1}{n+1} \beta(a) \frac{d}{da} \phi_n(a) \quad (60)$$

or

$$\phi_{n+1}\left(a\left(\ln \frac{\mu}{A}\right)\right) = -\frac{1}{n+1} \frac{d}{d\left(\ln \frac{\mu}{A}\right)} \phi_n\left(a\left(\ln \frac{\mu}{A}\right)\right). \quad (61)$$

We thus find that

$$m_{\text{pole}} = m(\mu) E^{-1}(a(\mu)) \sum_{n=0}^{\infty} \frac{(-L)^n}{n!} \frac{d^n}{d \ln \left(\frac{\mu}{A}\right)^n} \phi_0\left(a\left(\ln \frac{\mu}{A}\right)\right) \quad (62)$$
which by eq. (16) becomes

$$\mathcal{M} \phi_0 \left(a \left(\ln \frac{\mu}{\Lambda} - L\right)\right),$$  \hspace{1cm} (63)

or by eq. (59)

$$m_{\text{pole}} = \mathcal{M} E \left(a \left(\ln \frac{m_{\text{pole}}}{\Lambda}\right)\right) F_0 \left(a \left(\ln \frac{m_{\text{pole}}}{\Lambda}\right)\right)$$  \hspace{1cm} (64)

since \(L = \ln \frac{\mu}{m_{\text{pole}}}\). It is this expression for \(m_{\text{pole}}\) that makes it feasible to eliminate \(\mathcal{M}\) in eq. (25) so that \(\Gamma\) is expressed in terms of the physical quantity \(m_{\text{pole}}\).

$$\Gamma = \left[ \frac{m_{\text{pole}} E \left(a \left(\ln \frac{\mu}{\Lambda}\right)\right)}{E \left(a \left(\ln \frac{m_{\text{pole}}}{\Lambda}\right)\right) F_0 \left(a \left(\ln \frac{m_{\text{pole}}}{\Lambda}\right)\right)} \right] \delta_0 \left(a \left(\ln \frac{\mu}{\Lambda}\right)\right)$$  \hspace{1cm} (65)

with \(\mathcal{M}\) given by eq. (64).

It is now possible to examine the RS dependency of the expansion coefficients \(\kappa_n \equiv \kappa_{n,0}\) of the function \(F_0\) defined in eq. (56). We find that since \(m_{\text{pole}}\) in eq. (64) is RS invariant, then

$$\mu \frac{dm_{\text{pole}}}{d\mu} = 0 = \frac{dm_{\text{pole}}}{dc_i} = \frac{dm_{\text{pole}}}{dg_i}. $$  \hspace{1cm} (66a-c)

Using eqs. (3,4), eq. (66a) becomes

$$\sum_{n=0}^{\infty} \sum_{k=0}^{n} m \left(\gamma(a)a^nL^k + n\beta(a)a^{n-1}L^k + ka^nL^{k-1}\right) \kappa_{n,k} = 0 $$  \hspace{1cm} (67)

which leads to relations such as

$$\kappa_{1,1} = -f \kappa_0 $$  \hspace{1cm} (68a)

$$\kappa_{2,1} = -f g_1 \kappa_0 + (b - f) \kappa_1 $$  \hspace{1cm} (68b)

$$\kappa_{2,2} = -\frac{f}{2}(b - f) \kappa_0. $$  \hspace{1cm} (68c)

Next, by eqs. (30a,b), eq. (66b) leads to

$$\frac{\partial \kappa_0}{\partial c_i} = 0, \quad \frac{\partial \kappa_1}{\partial c_i} = 0, \quad \frac{\partial \kappa_2}{\partial c_i} - \frac{f}{2b} \kappa_0 \delta^i_2 = 0. $$  \hspace{1cm} (69a-c)

We also find by eq. (30c) that eq. (66c) leads to

$$\frac{\partial \kappa_0}{\partial g_i} = 0, \quad \frac{\partial \kappa_1}{\partial g_i} - \frac{f}{b} \kappa_0 \delta^i_2 = 0, \quad \frac{\partial \kappa_2}{\partial g_i} + \left( -\frac{f}{b} \kappa_1 + \frac{f c}{2b} \kappa_0 \right) \delta^i_1 - \frac{f}{2b} \kappa_0 \delta^i_2 = 0. $$  \hspace{1cm} (70a-c)

Eq. (68) has been used in deriving eqs. (69,70). From eqs. (69,70) we find that

$$\kappa_0 = \sigma_0 $$  \hspace{1cm} (71a)

$$\kappa_1 = \sigma_1 + \frac{f}{b} \sigma_0 g_1 $$  \hspace{1cm} (71b)

$$\kappa_2 = \sigma_2 + \sigma_1 g_1 + \frac{f}{2b} \sigma_0 \left(c_2 + g_1^2 + g_2\right). $$  \hspace{1cm} (71c)

In eq. (71), the quantities \(\sigma_i\) are constants of integration and thus are RS invariants. Similar expressions can be found for all \(\kappa_n (n \geq 0)\). We can find \(\sigma_i\) by using Feynman diagrams to compute \(\kappa_n, g_i, c_i\) in some prescribed RS such as \(\overline{\text{MS}}\) and then solving eq. (71).
6 Two Renormalization Schemes

We have found the RS dependency of both \( T_n \) and \( \kappa_n(n = 0, 1, 2) \) in eqs. (52,71). One could, for example, choose a RS in which \( T_n = 0 \) \( (n \geq 1) \); by eq. (52) this leads to

\[
g_1 = -\frac{b\tau_1}{5f\tau_0}
\]

\[
c_2 + g_2 = \frac{2b}{5f\tau_0}\left[-\tau_2 + \frac{\tau_1^2}{2\tau_0} + \frac{\tau_1}{2}(2f - c)\right].
\]

This in turn fixes \( \kappa_n \) in eq. (71). This is a scheme which we will call \( RS_1 \). (In principle one could also choose a scheme in which \( \kappa_n = 0 \) \( (n \geq 1) \), which fixes \( T_n \) in eq. (52).) A second alternative, \( RS_2 \), is to choose a RS, akin to that of 't Hooft [16], in which \( c_i = 0(i \geq 2) \) and \( g_i = 0(i \geq 1) \). We then have by eqs. (52,71)

\[
T_n = \tau_n, \quad \kappa_n = \sigma_n.
\]

In the scheme \( RS_2 \), by eq. (15)

\[
\ln\left(\frac{m_{\text{pole}}}{\Lambda}\right) = \int_K^{a_2\left(\ln\frac{m_{\text{pole}}}{\Lambda}\right)} dx\frac{1}{-bx^2(1+cx)}
\]

or

\[
-b\ln\frac{m_{\text{pole}}}{\Lambda} = \left(\frac{1}{a_2\left(\ln\frac{m_{\text{pole}}}{\Lambda}\right)} + \frac{1}{K}\right) + c\left(\frac{1 + ca_2\left(\ln\frac{m_{\text{pole}}}{\Lambda}\right)}{a_2\left(\ln\frac{m_{\text{pole}}}{\Lambda}\right)}\frac{K}{1 + cK}\right)
\]

and by eq. (14)

\[
E\left(a_2\left(\ln\frac{m_{\text{pole}}}{\Lambda}\right)\right) = \exp\int_K^{a_2\left(\ln\frac{m_{\text{pole}}}{\Lambda}\right)} dx\frac{fx}{-bx^2(1+cx)} = \left(\frac{a_2}{1 + ca_2}\frac{1 + Kc}{K}\right)^{-f/b}.
\]

We can let \( K \to \infty \) in eqs. (74,75). Eq. (74b) shows that \( a_2 \) can be written in terms of the Lambert \( W \) function \( (W \exp(W) = x) \) [17].

We know from ref. [18] that when using the \( \overline{MS} \) RS the quantities, \( T_{n,0} \equiv T_n \), appearing in eq. (1) are to two-loop order

\[
T_0 = 1, T_1 = C_F\left(\frac{65}{8} - 3\zeta_2\right) \approx 4.25360
\]

\[
T_2 = C_AC_F\left(\frac{19057}{648} - 19\zeta_2\ln(2) + \frac{55}{54}\zeta_2 - \frac{259}{36}\zeta_3 + \frac{101}{16}\zeta_4\right)
\]

\[
+ C_F^2\left(\frac{281113}{10368} + 38\zeta_2\ln(2) - \frac{9455}{216}\zeta_2 - \frac{22}{9}\zeta_3 + \frac{67}{8}\zeta_4\right)
\]

\[
+ C_FT_Fn_f\left(-\frac{1037}{144} - \frac{13}{36}\zeta_2 + \frac{8}{3}\zeta_3\right)
\]
\[ + C_F T_F \left( \frac{5615}{192} + 3\zeta_2 - 24\zeta_3 \right) \]
\[ \approx 26.7846 \quad (76c) \]

up to a factor of \( G_F^2 |V_{ub}|^2 / 192\pi^3 \).

Again using the \( \overline{\text{MS}} \) RS, it is known that \([15]\) the quantities \( \kappa_{n,0} (n = 0, 1, 2) \) appearing in eq. (55) are, to two loop order

\[ \kappa_0 = 1, \quad \kappa_1 = C_F = \frac{4}{3} \]  
\[ \kappa_2 = C_A C_F \left( \frac{1111}{384} + \frac{3}{2} \zeta_2 \ln(2) - \frac{1}{2} \zeta_2 - \frac{3}{8} \zeta_3 \right) \]
\[ + C_F^2 \left( \frac{121}{128} - 3\zeta_2 \ln(2) + \frac{15}{8} \zeta_2 + \frac{3}{4} \zeta_3 \right) \]
\[ + C_F T_F n_f \left( -\frac{71}{96} - \frac{1}{2} \zeta_2 \right) + C_F T_F \left( -\frac{3}{4} + \frac{3}{2} \zeta_2 \right) \]
\[ \approx 9.27793. \quad (77c) \]

We also know that to two loop order in eqs. (3,4) \([19]\)

\[ b = \frac{11 C_A - 4n_f T_F}{6} = \frac{33 - 2n_f}{6} \]
\[ bc = \frac{17 C_A^2 - n_f T_F (10 C_A + 6 C_F)}{12} \]
\[ f = \frac{-3 C_F}{2} = -2 \]
\[ fg_1 = \frac{-1}{8} \left( \frac{3}{2} C_F^2 + \frac{97}{6} C_F C_A - \frac{10}{3} C_F T_F n_f \right) \]
\[ = \frac{-1}{8} \left( \frac{202}{3} - \frac{20}{9} n_f \right) \quad (78d) \]

when using \( \overline{\text{MS}} \).

In evaluating the various quantities in eqs. (76-78) the following standard results have been used: \( \zeta_n \) is the Riemann zeta function \((\zeta_2 = \frac{\pi^2}{6}, \zeta_3 = 1.2020569 \ldots, \zeta_4 = \frac{\pi^4}{90})\); the invariants \( T_F = \frac{1}{2}, \)
\( N_F = C_A = N, \) \( C_F = \frac{N^2 - 1}{2N} \) of the \( SU(N) \) strong gauge group, where we have specialized to \( N = 3 \).

The effective number of light quarks \( n_f \) we take to be five for \( b \) decay. For the \( b \) quark we have \([3]\)

\[ m_{\text{pole}} = 4.7659 \text{ GeV}. \]

We now are in a position to use these two-loop \( \overline{\text{MS}} \) results in eqs. (76-78) to compute \( \Gamma \) using the RG summed result of eq. (25,65). When using \( \overline{\text{MS}} \), the running coupling at the mass of the Z Boson is

\[ \alpha_{\overline{\text{MS}}} \left( \ln \frac{M_Z^2}{\Lambda^2} \right) = 0.1185/\pi. \]  
\[ (80) \]
It is then possible to find $a_{\overline{\text{MS}}} \left( \ln \frac{m_{\text{pole}}}{\Lambda} \right)$ to second order either by direct integration of eq. (3) with $b$ and $c$ given by eqs. (78a,b) or by using the first two terms in eq. (42a) with $\ell = \ln \left( \frac{M}{m_{\text{pole}}} \right)$ and $\rho_0$ and $\rho_1$ given by eq. (45). This gives

$$a_{\overline{\text{MS}}} \left( \ln \frac{m_{\text{pole}}}{\Lambda} \right) = 0.0426569.$$  
(81)

From eqs. (14, 77, 78), the quantity $M$ appearing in eq. (64) is fixed in the $\overline{\text{MS}}$ scheme (recall $M$ is RS independent by eq. (46)). Only terms to second order are included in $E$ and $F_0$ in this calculation. This gives

$$M = 12.8135881 \ \text{GeV}.$$  
(82)

Keeping the first three terms in eq. (42a) and using $\ell = \ln \left( \frac{M}{M_M} \right)$ and eq. (79) we obtain

$$a_{\overline{\text{MS}}} \left( \ln \frac{M}{\Lambda} \right) = 0.0404511.$$  
(83)

We consequently have all of the ingredients needed to compute $\Gamma$ in the $\overline{\text{MS}}$ scheme to second order in perturbation theory. From eq. (25) we have

$$\Gamma_{\overline{\text{MS}}}^{(2)} = 1887.607292 \ \text{GeV}^5.$$  
(84)

Fig. 1 illustrates the behaviour of $\Gamma$ with respect to $\mu$. We have plotted the perturbative result of eq. (1) to two-loop order, the two loop RG summed result of eq. (6) (which includes $S_0$ and $S_1$), as well as a three-loop estimate obtained via Padé approximation techniques [21]. It is indeed striking that all expressions which depend on the renormalization scale parameter $\mu$ seem to converge near our scale independent result (i.e. $\mu$ independent result) provided in eq. (84).

It is of interest to compute $\Gamma$ in the ’t Hooft RS in which $c_i = 0 (i \geq 2)$, $g_i = 0 (i \geq 1)$ and $T_n$ and $S_n$ are given by eq. (52,71). From these equations and the computations given in eqs. (76-78), we find that

$$\tau_0 = 1, \quad \tau_1 = -14.18843$$  
(85a-b)

and

$$\sigma_0 = 1, \quad \sigma_1 = -2.355073.$$  
(86a-b)

We then find that in this “generalized ’t Hooft scheme”, $a_{tH} \left( \ln \frac{m_{\text{pole}}}{\Lambda} \right)$ can be determined using (41a). It is apparent that since we are working only to second order (i.e. order $a^2$),

$$a_{tH} = a_{\overline{\text{MS}}}.$$  
(90)

Eq. (25) can now be used to compute $\Gamma$ in this ’t Hooft RS; we find that to second order in perturbation theory it coincides with the perturbative result using $\overline{\text{MS}}$ RS

$$\Gamma_{tH}^{(2)} = \Gamma_{\overline{\text{MS}}}^{(2)} = 1887.607292 \ \text{GeV}^5.$$  
(93)
7 Discussion

We have applied RG summation to the perturbative contribution to the decay rate $\Gamma$ for the semileptonic decay of the $b$ quark. In ref. [3] it has been shown how RG summation of $LL \ldots N^3LL$ etc. contributions to this decay rate considerably diminishes its dependence on the renormalization scale parameter $\mu$, an unphysical parameter whose value considerably affects purely perturbative results. In section two of this paper we have demonstrated that RG summation can be used to sum all logarithmic correction to $\Gamma$ and that, when this is done, the implicit and explicit dependence of $\Gamma$ on $\mu$ appearing in eq. (1) cancels, as can be seen in eq. (25).

In section three we have considered the RS dependency within the context of mass independent renormalization of the running coupling and the running mass $m$. With the RS being characterized by the expansion coefficients of the RG functions $\beta$ and $\gamma$, it is possible to see how $a$ and $m$ vary with a change in RS. Using these results, in section four we have considered the RS dependence of the expansion coefficients of $\Gamma$ and find a set of RS invariant quantities $\tau_i$ in eq. (51). Similar considerations are used in section five to discuss the relationship of the pole to the running mass, leading to eq. (63) and a set of RS invariant parameters $\sigma_i$ appearing in eq. (70). In section six, two distinct RS’s have been considered. In the first of these, $RS_1$, the perturbation series for the decay rate $\Gamma$ is seen to truncate, which allows one to avoid the question of convergence of the expansion of the perturbative series for $\Gamma$ in powers of $a$. In the second (the ’t Hooft RS), $RS_2$, the series expansions for the RG functions $\beta$ and $\gamma$ both truncate. Using second order perturbation theory, we have computed the decay rate $\Gamma$ in the framework of our approach to RG summation using both the $\overline{MS}$ and ’t Hooft RS’s with the interesting outcome that the two predictions are identical in value.

The approach to RG summation further developed in this paper is expected to prove helpful in the determination of other physical quantities not fully known, for example the CKM parameter $|V_{ub}|$ [20-26].

Acknowledgements
R. Macleod provided some useful suggestions.

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Figure 1: The $\mu$ dependence of $\Gamma$ from perturbative and RG summation expressions in the $\overline{MS}$ scheme as compared to the scale independent result.