Symmetry-assisted resonance transmission of identical particles

D. Sokolovski\textsuperscript{a,b}, J. Siewert\textsuperscript{a,b}, and L.M. Baskin\textsuperscript{c}

\textsuperscript{a} Departamento de Química-Física, Universidad del País Vasco, UPV/EHU, Leioa, Spain
\textsuperscript{b} IKERBASQUE, Basque Foundation for Science, E-48011 Bilbao, Spain
\textsuperscript{c} The Bonch-Bruevich State University of Telecommunications, 193232, Pr. Bolshevik 22-1, Saint-Petersburg, Russia

(Dated: February 19, 2015)

We report novel interference effects in wave packet scattering of identical particles incident on the same side of a resonant barrier, different from those observed in Hong-Ou-Mandel experiments. These include significant changes in the mean number of transmissions and full counting statistics, as well as “bunching” and “anti-bunching” effects in the all-particles transmission channel. With several resonances involved, pseudo-resonant driving of the two-level system in the barrier, may result in sharp enhancement of scattering probabilities for certain values of temporal delay between the particles.

PACS numbers: 37.10.Gh, 03.75.Kk, 05.30.Jp

Scattering of a single particle by a potential barrier is a fundamental topic routinely addressed in quantum mechanics textbooks. However, with many particles involved in a scattering event, the problem tends to become much more contrived and difficult. At first glance, one might attribute those difficulties to interaction between the particles. Surprisingly, the complexity of the problem increases dramatically even for non-interacting particles, if only they are indistinguishable. This is exemplified by the boson-sampling problem \cite{1}, which has received considerable attention in recent years.

One often expects indistinguishability of the particles to lead to intriguing effects in the statistics of scattering outcomes, even if the particles do not otherwise interact with each other (see, e.g., \cite{2,5}). One paradigmatic example of this behaviour is the Hong-Ou-Mandel (HOM) effect \cite{2}, where two indistinguishable photons, incident on a balanced beam splitter from opposite sides, always exit this at the same side. The HOM effect has numerous important applications, such as in quality testing of single-photon sources \cite{3}, entanglement detection \cite{5}, entanglement swapping \cite{6}, and quantum metrology \cite{7}.

This suggests generalization of the HOM setup in order to look for new effects viz. useful applications involving scattering of identical bosons, fermions, or fermionized cold atoms \cite{8}. In this vein, Tichy \textit{et al.} \cite{9} derived a zero-transmission law for photons in a multiport beam splitter (see also \cite{10}). This was developed further, and confirmed experimentally, e.g., by Spagnolo \textit{et al.} \cite{11}. In addition, substantial efforts continue to extend these results from indistinguishable photons to bosonic cold atoms are currently underway \cite{12} \cite{13}. Related work on many-body scattering in mesoscopic devices, and tunneling decay of interacting bosons from a trap can be found in Refs. \cite{14} \cite{15} and \cite{17} \cite{19}, respectively.

While the multiport setups of Refs. \cite{9} \cite{11} act as interferometers for identical particles in \textit{spatially distributed} modes, the case of particles \textit{distributed in time} has, to our knowledge, received much less attention until now. A recent example is the case of two initially uncorrelated bosons (or fermions) incident, with a time lag $T$, on the same side of a tunneling barrier \cite{20}. "Meeting" of such particles in a barrier with a transmission resonance, results in interference and may alter the transmission probabilities for the scattering outcomes.

In this article we develop a general formalism for studying the incidence of $N$ identical particles \textit{on the same side} of a resonance barrier, supporting one or more metastable states. By using it we will demonstrate that the symmetry of the initial state may considerably alter both the overall tunneling rate, and the full counting statistics of an $N$-particle system. For brevity we will refer as particles (fermions or bosons) to both cold atoms and photons, equally amenable to our analysis.

Consider, in one dimension, a source sending $N$ identical noninteracting particles of mass $\mu$ \cite{21} in wave packet states $\psi_n(x_n)$,

$$
\psi_n(x_n,t) = (2\pi)^{-1/2} \int A_n(p) \exp[ipx_n - iE(p)(t + t_n)] dp, 
$$

$$
E(p) = p^2/2\mu \quad (1)
$$
towards a finite-width potential barrier at times $0 = t_1 < t_2 < \ldots < t_N$.

The operators $a_n^\dagger$ and $a_n$, creating and annihilating an
incident particle in the state \( \psi_n \), are given by

\[
a_n^+ = (2\pi)^{-1/2} \int A(p) \exp[-iE(t + t_n)]a^+(p) dp, \quad (2)
\]

where \( a_n^+ \) and \( a_n \) obey the usual commutation relations,

\[
[a(p), a^+(p')]_\pm = \delta(p - p'),
\]

where the dagger denotes Hermitian conjugation, and the plane wave creation and annihilation operators \( a^+(p) \) and \( a(p) \) obey the usual commutation relations,

\[
[a(p), a^+(p')]_\pm = \delta(p - p'),
\]

with the upper and lower signs corresponding to bosons and fermions, respectively (cf. also Ref. [22]). Their (anti) commutators coincide with the overlaps between the wave packets [1],

\[
[a_m, a_n^+] = \int A_m^*(p)A_n(p) \exp[iE(p)(t_n - t_m)] dp \equiv I_{mn}
\]

where \( I_{nn} = 1 \) and \( I_{mn} = I_m^* \). The symmetry of the incident state has no effect on the initial probability density provided \( I_{mn} = \delta_{mn} \), e.g., for the delays between emissions, \( |t_m - t_n| \), large enough for the rapid oscillations of the exponential in (3) to destroy the integral for all \( m \neq n \). We will refer to such particles as initially uncorrelated. It is readily seen that spreading of freely moving wave packets doesn’t alter the commutation relations [3]. Thus, the symmetrized or anti-symmetrized wave function describing \( N \) incident particles is given by

\[
|\Psi_{in}(t)\rangle = K^{-1/2} \prod_{n=1}^{N} a_n^+(t)|0\rangle, \quad (4)
\]

where \( |0\rangle \) is the vacuum state, and \( K \) is the normalization constant. By Wick’s theorem, we have (the upper and lower signs are for bosons and fermions, respectively)

\[
K = \sum_{\sigma(N)} (\pm 1)^{p(\sigma(N))} \prod_{i=1}^{N} I_{\sigma_i} = S^\pm |I_{mn}| \quad (5)
\]

where \( \sigma(N) \) is a permutation of the indices (0, 1, ..., \( N \)) and \( p(\sigma) \) is its parity [23].

At large times, after all particles have left the barrier area, each wave packet ends up split into the transmitted \( (t) \) and reflected \( (r) \) parts. Thus, as \( t \to \infty \), the wave function has the form

\[
|\Psi_{out}(t)\rangle = K^{-1/2} \prod_{n=1}^{N} [t_n^+(t) + r_n^+(t)]|0\rangle, \quad (6)
\]

where the corresponding creation and annihilation operators are given by

\[
t_n^+ = \int T(p)A_n(p) \exp[-iE(t + t_n)]a^+(p) dp, \quad (7)
\]

\[
r_n^+ = \int R(p)A_n(p) \exp[-iE(t + t_n)]a^-(p) dp, \quad (7)
\]

\[
t_n = (t_n^+)\dagger, \quad r_n = (r_n^+)\dagger, \quad (7)
\]

and \( T(p) \) and \( R(p) \) are the barrier transmission and reflection amplitudes for a particle with a momentum \( p \). Since \( |T(p)|^2 + |R(p)|^2 = 1 \), as \( t \to \infty \) we also have

\[
T_{mn} = [t_m, t_n^+] = \int [T(p)]^2A_m^*(p)A_n(p) \exp[iE(p)(t_m - t_n)] dp,
\]

\[
R_{mn} = [r_m, r_n^+] = I_{mn} - T_{mn}, \quad (8)
\]

while all remaining commutators vanish. In Eqs. (8) \( T_{mn} = T^*_{mn} \) is a Hermitian matrix of the overlaps between the transmitted parts of the wave packets, and its diagonal elements \( T_{nn} \) coincide with the probabilities \( w_n \) for the \( n \)-th particle to be transmitted on its own, \( T_{nn} = \int [T(p)]^2|A_n(p)|^2 dp = w_n \).

For \( N \) identical particles, there are \( N + 1 \) outcomes, with \( n = 0, 1, ...N \) particles crossing in the barrier who’s probabilities, \( W(n, N) \) we will study next. It is convenient to construct a generating function \( G(\alpha) \),

\[
G^\pm(\alpha) = \lim_{t\to\infty} \langle \Psi_{out}(t)|\Psi(t, \alpha)\rangle, \quad (9)
\]

where \( |\Psi(t, \alpha)\rangle = K^{-1/2} \prod_{n=1}^{N} [\alpha t_n^+(t) + \alpha r_n^+(t)]|0\rangle \), and \( K \) is defined by Eq. (5). By Wick’s theorem, we have

\[
G^\pm(\alpha) = S^\pm [I_{mn}^\pm]/S^\pm [I_{mn}], \quad (10)
\]

where the matrix \( \Delta \) is given by

\[
\Delta_{mn}^\pm = I_{mn} + (\alpha - 1)T_{mn}, \quad n, m = 1, 2, ..., N. \quad (11)
\]

For the mean number of transmissions, \( \bar{n}_T = \sum_{n=0}^{N} nW(n, N) \), we have

\[
\bar{n}_T = \partial_{\alpha}G^\pm(\alpha)|_{\alpha = 1} = \sum_{j=1}^{N} S^\pm [I_{mn}^j]/S^\pm [I_{mn}], \quad (12)
\]

where \( I_{mn}^j \) is the matrix obtained from \( I_{mn} \) by replacing the elements of the \( j \)-th row, \( I_{j1}, ..., I_{jN} \) with \( T_{j1}, ..., T_{jN} \). The full counting statistics of the \( N \)-particle process are evaluated by noting that

\[
W^\pm(n, N) = n!^{-1}\partial_{\alpha}^nG^\pm|_{\alpha = 0} = \sum_{j_1 < j_2 < ... < j_n} S^\pm [I_{mn}^{(j_1,j_2,...,j_n)}]/S^\pm [I_{mn}], \quad (13)
\]

where \( I_{mn}^{(j_1,j_2,...,j_n)} \) is the matrix obtained from \( R_{mn} \) by replacing the elements of the rows \( j_1, j_2, ..., j_n \), with the corresponding rows of the matrix \( T_{mn} \). In the simplest case of just two particles, \( N = 2 \), Eqs. (12) and (13) yield

\[
W^\pm(2, 2) = |w_1w_2 \mp |T_{12}|^2]/(1 \pm |I_{12}|^2), \quad (14)
\]

\[
\bar{n}_T = (w_1 + w_2 \mp 2Re(T_{12}I_{12}))/|1 \pm |I_{12}|^2|, \quad (14)
\]

which coincides with the results of [20] if the particles have the same momentum distribution, \( A_1(p) = A_2(p) \).
Equations (12) and (13) allow us to make several general observations. We note that in the limit of long delays between emission, \(|t_m - t_n| \to \infty\), all operators in Eqs. (2) and (7) commute, both \(I_{mn}\) and \(T_{mn}\) are diagonal, we recover the statistics for independent distinguishable particles (DP), \(W(n, N) = \sum_{\sigma_1(\sigma_2)} w_{i_1} \cdots w_{i_n}(1 - w_{i_n+1}) \cdots (1 - w_{i_n+1})\) and \(\pi_T = \sum_{i=1}^{N} w_i\).

The mean number of transmissions may be affected by the symmetry of the initial state, \(\pi_T \neq \sum_{i=1}^{N} w_i\), only if the particles are correlated initially, \(I_{mn} \neq \delta_{mn}\). For \(t_m - t_n \to 0\), \(T_{mn} \to \text{const.}\), this effect disappears for bosons, but not for fermions [25].

For initially uncorrelated particles, \(I_{mn} = \delta_{mn}\), the symmetry changes the probabilities \(W(n, N)\), but not \(\pi\), provided \(T_{mn} \neq w_n\delta_{mn}\). In this case, “bunching” and “anti-bunching” types of behaviour can be observed for bosons and fermions in the probability for all \(N\) particles to be transmitted, \(W(N, N)\). Since the matrix \(T_{mn}\) is positive definite, the Hadamard inequality for determinants [26], and its analog for permanents [27] ensure that \(W \pm (n, N) = \prod_{i=1}^{N} w_i\). Thus, \(N\) bosons (fermions) are more (less) likely to be transmitted together than DPs in the same one-particle states. Note that this argument cannot be extended to the probabilities \(W(n < N, N)\), or for initially correlated initial states, \(I_{mn} \neq \delta_{mn}\).

A system likely to show these effects is a resonance barrier, where, due to the long delay in traversing it, even the particles well separated initially have a chance to “meet” during transmission. Transmission coefficient of such a barrier can be written as a sum of narrow Breit-Wigner peaks,

\[
[T(p)]^2 = \sum_{l=1}^{N} \frac{\Gamma_l^2}{(p^2/2\mu - \epsilon_l^2)^2 + \Gamma_l^2},
\]

and even for initially uncorrelated particles, the shape of \([T(p)]^2\) may be narrow enough to ensure that \(T_{mn}\) is not diagonal even if \(I_{mn} = \delta_{mn}\).

Figure 3 shows the mean number of transmissions for \(N\) particles emitted after equal intervals, \(|t_m - t_n| = T\), in identical Gaussian states of a coordinate width \(\sigma\) and a mean momentum \(p_0\), \(A(p) = A_n(p) = (\sigma^2/2\pi)^{1/4}\exp((p - p_0)^2/\sigma^2/4)\) (cf. Fig. 2). The scatterer supports two resonant metastable states with the energies \(\epsilon_{1,2}\) and widths \(\Gamma_{1,2}\) of which one, or both can be accessed by the incident particle, as shown in the insets in Figs. 3 a and b. With only one level involved, the mean number of transmissions \(\pi_T\) for bosons raises to a maximum value for some correlated initial state (cf. Fig.2), and then returns to the DP limit for initially uncorrelated particles (see Fig.3a). For fermions, the Pauli principle mostly reduces \(\pi_T\) to levels below the DP level, which for the maximally correlated states, obtained as \(T \to 0\) [24], is considerably reduced. With two metastable states involved, interference between resonances reverses the effect: for \(0.1 < p_0^2/2\mu T < 0.25\), \(\pi_T\) is suppressed for bosons, and enhanced for fermions (see Fig.3b).

The scattering probabilities \(W(n, N)\), plotted in Fig.4 for \(N = 4\), show that the increase or decrease in \(\pi_T\) results from a similar increase or decrease in the probability of the one-particle transmission channel, \(W(1, N)\).

With two resonances accessible to bosons, \(W(n, M)\) exhibit maxima, whenever the time between the emissions coincides with a multiple of the difference of resonant energies, \(T \approx T_k = 2\pi k/(E_2 - E_1)\), \(k = 1, 2, \ldots\). The peaks are most pronounced for the \((N,N)\) channel (cf. Fig. 4a) and, as shown in Fig.5a, become sharper as \(N\) increases. This is another consequence of the symmetrization of the initial state which, with each particle distributed between the wave packets in Fig.1, appears to produce quasi-periodic excitation of the metastable two-level system supported by the barrier. With the number of particles increasing, the excitation looks more periodic, and the “resonance” condition \(T \approx T_k\) needs to be satisfied with even greater accuracy.

For fermions, probing to resonance states, the peaks at
but for fermions. The parameters are as in Fig. 3b. Incident particles may be considered uncorrelated for $E_0T \gtrsim 0.4$

FIG. 5. (Color online) a)Probabilities for all bosons to be transmitted for $N = 4$ (two resonances); b) same as a), but for fermions. The parameters are as in Fig. 3b.

$$T = T_k$$

are replaced by dips, which appear, for example in the probability $W(2, 4)$ shown in Fig. 4b. In contrast to the bosonic case, these dips are never seen in the $(N, N)$ channel, where $W(N, N)$ undergoes sinusoidal oscillations, no matter how large is the numbers of particles $N$ (see Fig. 5b).

Experimental observation of effects of the Pauli principle on resonance tunneling would be possible for cold atoms in the Tonks-Girardeau regime, injected into quasi-one-dimensional trap with laser-induced barriers [22]. An optical realization of the bosonic experiment would consist in sending identically polarized photons toward a Fabry-Perot interferometer, or injecting them in a waveguide with narrowing, imitating a one-dimensional barrier. If required, a correlated initial state can be produced by scattering several uncorrelated particles off a long-lived resonance, and selecting the outcome in one of the $n$-particle transmission channels.

In summary, resonance scattering of "trains" of identical particles, similar to the one shown in Fig. 1, offers a plethora of interference phenomena, very different from those observed in the HOM experiments. For a correlated initial state, indistinguishability of the particles affects the mean number of transmission events. For initially uncorrelated particles, "piling up" in the barrier leads to significant changes in full counting statistics. Observation of both effects is within the capability of modern experimental techniques.

We acknowledge support of the Basque Government (Grant No. IT-472-10), and the Ministry of Science and Innovation of Spain (Grant No. FIS2009-12773-C02-01). LB acknowledges the Russian Fund of Fundamental Investigations (Grant 12-01-00247) and the Saint Petersburg State University (Grant 11.38.666.2013).

[1] S. Aaronson and A. Arkhipov, Theory of Computing 9, 143 (2013).
[2] C.K. Hong, Z.Y. Ou, and L. Mandel, Phys. Rev. Lett. 59, 2044 (1987).
[3] D. Sokolovski, Phys. Rev. Lett. 110, 115302 (2013).
[4] P.W. Sun and C.W. Wong, Phys. Rev. A 79, 013824 (2009).
[5] Y.H. Shih and C.O. Alley, Phys. Rev. Lett. 61, 2921 (1988).
[6] M. Halder, A. Beveratos, N. Gisin, V. Scarani, C. Simon, and H. Zbinden, Nat. Phys. 3, 692 (2007).
[7] P. Walther, J.-W. Pan, M. Aspelmeyer, R. Ursin, S. Gasparoni, and A. Zeilinger, Nature 429, 6988 (2004).
[8] M. Girardeau, J. Math. Phys. 1, 516 (1960).
[9] M.C. Tichy, M. Tiersch, F. de Melo, F. Mintert, and A. Buchleitner, Phys. Rev. Lett. 104, 220405 (2010).
[10] K. Mayer, M.C. Tichy, F. Mintert, T. Konrad, and A. Buchleitner, Phys. Rev. A 83, 062307 (2011).
[11] N. Spagnolo, C. Vitelli, L. Sansoni, E. Maiorino, P. Mataloni et al., Phys. Rev. Lett. 111, 130503 (2013).
[12] M. Bonneau, J. Rualdel, R. Lopes, J.-C. Jaskula, A. Aspect, D. Boiron, and C.I. Westbrook, Phys. Rev. A 87, 061603(R) (2013).
[13] R. Lopes, A. Imanaliev, A. Aspect, M. Cheneau, D. Boiron, and C.I. Westbrook, e-print arXiv:1501.03065 (2015).
[14] M.A. Khan and M.N. Leuenberger, Phys. Rev. B 90, 075439 (2014).
[15] D. Marian, E. Colomés, and X. Oriols, e-print arXiv:1408.1990 (2014).
[16] J.-D. Urbina, J. Kuipers, Q. Hummel, and K. Richter, e-print arXiv:1409.1558 (2014).
[17] G. Zürn, F. Serwane, T. Lompe, A.N. Wenz, M.G. Ries, J.E. Bohn, and S. Jochim, Phys. Rev. Lett. 108, 075303 (2012).
[18] M. Rontani, Phys. Rev. Lett. 108, 115302 (2012).
[19] S. Hunn, K. Zimmermann, M. Hiller, and A. Buchleitner, Phys. Rev. A 87, 043626 (2013).
[20] D. Sokolovski and L.M. Baskin, Phys. Rev. A, 90, 024101 (2014).
[21] For a photon in a waveguide $\mu$ is the effective mass acquired as a result of transversal confinement.
[22] R. Loudon, Phys. Rev. A 58, 4904 (1998).
[23] Thus, $S^{+}[I_{mn}]$ is the permanent (per) of the matrix $I_{mn}$, and $S^{-}[I_{mn}]$ is its determinant (det).
[24] One must also check that $T_{mn} \neq \text{const.} \times I_{mn}$.
[25] Note that for fermions, $|\Psi_{in}\rangle$ in Eq. (4), always normalised to unity tends to a finite limit as $t_{m} - t_{n} \rightarrow 0$. For example, for two atoms prepared in Gaussian states of a width $\sigma$, the one-particle density $\rho(x)$ tends to $\exp(-2x^{2}/\sigma^{2})[1 + 4x^{2}/\sigma^{2}]$ as $t_{2} \rightarrow t_{1}$. We study the scattering of this maximally correlated fermionic state, $I_{mn} = 1$, whenever the limit $t_{m} - t_{n} \rightarrow 0$ is taken in the rest of the paper.
[26] R.A. Horn and C.R. Johnson, Matrix Analysis (Cambridge University Press, New York, 1986).
[27] M. Marcus, Proc. Am. Math. Soc. 15, 967 (1964).
[28] T. P. Mayrath, F. Schreck, J. L. Hanssen, C. S. Chuu, and M. G. Raizen, Phys. Rev. A 71, 041604 (2005).