Application and Implementation of Map Projection and Gauss Mapping

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Abstract. This paper introduces the map projection from the pole expansion and along the longitude, and gives the definition of Gauss mapping. Using Matlab programming, the expressions of each parameter in map projection and Gauss mapping are derived, and the process has been realized through animation. The experimental results show that mathematical logic can be used to transform the earth in 3D space into a 2D map projection, and the surface of the 3D space can be mapped to a single spherical surface by Gauss mapping. The wide application of mathematics in cartography shows that mathematics plays an important role in promoting the development of cartography.

1. Introduction

Map projection is a theory and method for transforming arbitrary points on the surface of the earth to a map plane by using certain mathematical laws. The non-flattenable surface can be projected onto a plane by establishing a mathematical conversion formula between the 3D space and the 2D projection, which makes the human understanding of the geography more intuitive and ensures the integrity of the spatial information. Undoubtedly, the projection process must be a certain deformation of the shape of the space. This paper will introduce the mathematical application in cartography, carry out rigorous mathematical formula calculation on the 3D earth, project the earth from the poles and along the longitude, and get the spherical surface of the Gauss mapping. The results show that mathematics can be well applied to cartography, especially map projection.

2. Map projection

The essence of map projection is to map the latitude and longitude lines on the earth to a plane or expandable surface according to a mathematical formula. Here, we use the mathematical analytic method, which is a kind of projection method that establishes the point-to-point function relationship between the spherical surface and the projection surface, and determines the intersection position of the latitude and longitude lines.

2.1 Map projection from the pole

After splitting the earth along the longitude from the pole, it expands into a "petal" shape around the pole, essentially centering on one of its vertices and expanding the 3D sphere into a 2D split shape. It turns out that, each longitude, a differential circle of the sphere's cross section, is projected onto the map to become a differential ellipse.
2.2 Map projection along the longitude
The expansion of the earth along the longitude is somewhat similar to that of the pole, except from the way of the earth mapped to two dimensions. Compared with the expansion from the pole, the expansion along the longitude can be visually seen as the earth rolling in a plane and spreading at the equator, each "time zone" will leave a map. Therefore, the expansion plane of the following figure is obtained.

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2.3 Gauss mapping
In differential geometry, a Gauss mapping is a mapping from a surface in Euclidean space $\mathbb{R}^3$ to a unit sphere $S^2$. It is also important to apply to map projection. Given the surface $X$ in Euclidean space $\mathbb{R}^3$, the Gauss mapping is a continuous map: $N: X \rightarrow S$, that makes $N(p)$ be the normal vector of the surface $X$ at the point $p$. The end point of $N(p)$ is a point on the unit sphere $S$ of the center of the sphere with $O$.

3. Application and programming implementation
3.1 Statement of problems
Earth is a complex system in 3D space. Through rigorous mathematical knowledge and simulation with maps, humans can more objectively understand the laws of geography, and the application of mathematics and geography has far-reaching significance.

3.2 Analysis and exploration

3.2.1 3D graphics of the earth
The surface of the earth is actually a rugged surface, so in the process of drawing, it is divided into small polygons.

1) Calculate the \( yoz \) circular radius and \( x \) axis coordinates corresponding to each polygon on the sphere
First, the earth is roughly regarded as a sphere. Given a sphere radius \( a \), the earth is divided into \( n=12 \) parts by the longitude, and the angle between the \( xoy \) planes is \( \frac{\pi}{4}, \frac{2\pi}{4}, \ldots, \frac{2\pi}{4} \), recorded as a vector \( \theta \). Each piece is divided into \( m=3l \) layers by the angle of \( [0, \pi] \) along the longitude, recorded as the angle \( \beta \), thus the earth has been divided into \( nm \) small polygons. Therefore, the \( x \) axis coordinates of each layer is \( \cos a \beta \), and the \( yoz \) circular radius of each layer is \( \sin a \beta \).

2) Calculate the vertex sequence of each polygon on the sphere
If you expand the earth from the pole, you can get \( n \) parts of the same shape starting from the pole, the angle between two adjacent parts should be \( \frac{2\pi}{n} \), thus, the radius of each small polygon is \( R_s = \frac{a \sin \beta}{\cos (\frac{2\pi}{2n})} \). Therefore, the length of each polygon of the \( yoz \) plane is \( dL = 2R_s \tan \left( \frac{2\pi}{2n} \right) \). Hence, the 3D vertex sequence of the polygon is \( x_0 = a \cos \beta, \quad y_0 = R_s \cos \theta, \quad z_0 = R_s \sin \theta \). Finally, you can draw the corresponding shape of the earth.

3.2.2 The expansion from the pole

1) Calculate the angle of each vertex of the polygon
In the procedure of such expansion, the sphere is actually extended step by step along the \( z \)-axis to the \( xoy \) plane. Therefore, in this process, we divide the sphere expansion into \( m \) layers according to the animation, then the transformation of the angle of each point is divided into \( m \) layers according to \( [0, \pi) \), and each angle is recorded as \( \omega \); each "petal" is divided into \( m \) layers by falling on the 2D plane. It is also divided into \( m \) layers by \( [0, \pi) \), and each angle is recorded as \( \gamma \).

2) Calculate the radius of the expansion, the radius and the coordinates of the middle line
In the shape of the sphere, there is a special line, that is, the middle line of each two intersecting longitude, whose \( x \) axis coordinate is \( a \cos (\pi - \omega + \gamma) \). The radius of the expanded 3D graphic is \( ryz = a \cdot (\pi - \omega) + a \sin (\pi - \omega + \gamma) \). Therefore, falling on the polygon, the radius of the polygon of the middle line is \( R_s = \frac{ryz}{\cos \left( \frac{\pi}{n} \right)} \), recorded as \( R_s \). Finally, the sequence of vertices in the middle line is: \( x = a \cos (\pi - \omega + \gamma), \quad y = R_s \cos \theta, \quad z = R_s \sin \theta \).

3) Calculate the coordinates of intersecting longitudes on both sides of the middle line
Since the two curves have the same \( x \) axis coordinates as the middle line, hence, \( X_s = X_s = x = a \cos (\pi - \omega + \gamma) \). About the \( y \) axis coordinate and the \( z \) axis coordinate, the product of
the side length of the polygon and the cosine of the angle should be added. By calculating,
\[ Y_a = x + \frac{dL}{2} \cos\left(\frac{\pi}{2} + \theta\right), \quad Y_b = y + \frac{dL}{2} \cos\left(-\frac{\pi}{2} + \theta\right), \quad Z_a = z + \frac{dL}{2} \sin\left(\frac{\pi}{2} + \theta\right), \quad Z_b = z + \frac{dL}{2} \sin\left(-\frac{\pi}{2} + \theta\right). \]

3.2.3 The expansion along the longitude
In the process of expanding along the longitude, we envision the earth as a wheel passing, leaving a trajectory in the 2D plane, mapping each "time zone" to a 2D plane, which is repeating the first part trajectory. Therefore, the point \(y\) axis coordinates and \(z\) axis coordinates of each part are the same. For \(y\) axis coordinates, the distance per movement is the maximum length of the polygon, \(\max dL\). Therefore, the sequence of curve coordinates is: \(x(i) = x(1)\), \(y(i) = y(1) + (i - 1) \max dL\), \(z(i) = z(1)\).

3.2.4 Gauss mapping
(1) A spherical equation in 3D space
\[ X = R, \sin \theta \cos \varphi, \quad Y = R, \sin \theta \sin \varphi, \quad Z = R, \cos \theta. \]
(2) Calculate the radius and the coordinates of the spherical cap
Assuming that the height of the spherical cap is \(h = 0.1 * R\), the \(z\) axis coordinate is \(z h = R_1 - h\). Dividing it into \(t = 51\) equal parts on the \(xoy\) plane, the angle of each is also divided into \(t\) equal parts, recorded as \(t s\). Thus, the radius and the coordinates of the spherical cap are: \(r h = \sqrt{R_1^2 - h^2}\), \(x h = r h * \cos(ts)\), \(y h = r h * \sin(ts)\), \(z h = R_1 - h\).
(3) Calculate the end point of the normal vector
Normal vector in all directions: \(n x = x h * \left(1 \over R_1\right), \quad n y = y h * \left(1 \over R_1\right), \quad n z = z h * \left(1 \over R_1\right). \)

3.3 Implementation and results
Through Matlab programming, there is no difference in the process of expanding the earth from the pole and along the longitude. The specific difference is in the core code of the implementation. The specific flow chart is as follows:

- Sphere and polygon curve initialization
- Calculate the polygon radius, side length, and vertex sequence of the sphere
- Draw a sphere
- Calculate the radius of the expansion process and the 3D coordinates of the middle line
- Calculate the 3D coordinates of the curves on both sides of the middle line of the expansion process
- Draw an expanded view

Fig. 4. The flow chart of map projection.

Through the above process, the process of expanding the earth from the pole and along the longitude can be obtained, and the results are as follows:
Fig. 5. The results of expanding from the pole

Fig. 6. The results of expanding along the longitude

Through Matlab programming, a part of the earth can be Gauss mapped to the unit sphere, the specific flow chart is as follows:

- Sphere and spherical equation initialization
- Draw a sphere
- Calculate the 3D coordinates and radius of the sphere cap
- Calculate the end of the normal vector
- Draw a truncated circle by Gauss mapping

Fig. 7. The flow chart of Gauss mapping

Through the above process, the spherical surface can be obtained by Gauss mapping, and the result is as follows:
4. Summary and conclusion

In this paper, by studying the map projection, two projection processes from the pole and along the longitude are realized. The expressions of various parameters in the process of changing the earth to the map are derived, and Matlab programming is used to visually display the process and results of map projection. Through analysis and exploration, the essence of expanding the earth from the pole is to divide the 3D sphere into \( n \times m \) polygons. In order to transform them into 2D, each part has been expanded along the \( z \) axis, along the shape of the sphere. The essence of the expanding along the longitude is that after cutting off the 3D sphere along the longitude, and expanding the trace on the plane, it is actually the process of repeating the first part projection. The whole process has witnessed the inseparability of mathematics and map projection. It is more concise to express the concept of cartography in mathematical form; at the same time, cartography provides a broad stage for the application of mathematics.

Acknowledgments

Thank the teachers for giving me patient guidance and practical advice during my writing of this thesis. At the same time, this paper is supported by the Beijing Education Union.

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