REPDIGITS AS PRODUCTS OF CONSECUTIVE BALANCING OR
LUCAS-BALANCING NUMBERS

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Abstract. Repdigits are natural numbers formed by the repetition of a single digit. In this paper, we explore the presence of repdigits in the product of consecutive balancing or Lucas-balancing numbers.

Keywords: Balancing numbers, Lucas-balancing numbers, repdigits, divisibility sequence.

1. Introduction

The balancing sequence \{B_n : n \geq 0\} and the Lucas-balancing sequence \{C_n : n \geq 0\} are solutions of the binary recurrence \(x_{n+1} = 6x_n - x_{n-1}\) with initial terms \(B_0 = 0, B_1 = 1\) and \(C_0 = 1, C_1 = 3\) respectively. The balancing sequence is a variant of the sequence of natural numbers since natural numbers are solutions of the binary recurrence \(x_{n+1} = 2x_n - x_{n-1}\) with initial terms \(x_0 = 0, x_1 = 1\). The balancing numbers have certain properties identical with those of natural numbers [9]. It is important to note that the balancing sequence is a strong divisibility sequence, that is, \(B_m | B_n\) if and only if \(m | n\) [5].

In the year 2004, Liptai [2] searched for Fibonacci numbers in the balancing sequence and observed that 1 is the only number of this type. In a recent paper [6], the second author proved that there is no perfect square in the balancing sequence other than 1. Subsequently, Panda and Davala [8] verified that 6 is the only balancing number which is also a perfect number.

For a given integer \(g > 1\), a number of the form \(N = a\left(\frac{a^{m-1}}{g-1}\right)\) for some \(m \geq 1\) where \(a \in \{1, 2, \ldots, g-1\}\) is called a repdigit with respect to base \(g\) or \(g\)-repdigit. For \(g = 10\), \(N\) is simply called a repdigit and if, in addition, \(a = 1\), then \(N\) is called a repunit. Luca [3] identified the repdigits in Fibonacci and Lucas sequences. Subsequently, Faye and Luca [1] explored all repdigits in Pell and Pell-Lucas sequences. Marques and Togbé [4] searched for the repdigits which are product of consecutive Fibonacci numbers. In this paper, we search for repdigits in the balancing and Lucas-balancing sequences. In addition, we also explore repdigits which are product of consecutive balancing or Lucas-balancing numbers.

2. Main Results

In this section, we prove some theorems assuring the absence of certain class of repdigits in the balancing and Lucas-balancing sequences. As generalizations, we also show that the product of consecutive balancing or Lucas-balancing numbers is never a repdigit with more than one digit.

In the balancing sequence, the first two balancing numbers \(B_1 = 1\) and \(B_2 = 6\) are repdigits. We have checked the next 200 balancing numbers, but none is a repdigit. The following theorem excludes the presence of some specific types of repdigits in the balancing sequence.
Theorem 2.1. If $m, n$ and $a$ are natural numbers, $m \geq 2$, $a \neq 6$, and $1 \leq a \leq 9$, then the Diophantine equation

$$B_n = a\left(\frac{10^m - 1}{9}\right)$$

has no solution.

\textit{Proof.} To prove this theorem, we need all the least residues of the balancing sequence modulo 3, 4, 5, 7, 8, 9, 11, and 20 (see [7]). We list them in the following table.

| Row no. | $m$ | $B_n \mod m$ | Period |
|---------|-----|--------------|--------|
| 1       | 3   | 0, 1, 0, 2   | 4      |
| 2       | 4   | 0, 1, 2, 3   | 4      |
| 3       | 5   | 0, 1, 1, 0, 4, 4 | 6      |
| 4       | 7   | 0, 1, 6     | 3      |
| 5       | 8   | 0, 1, 6, 3, 4, 5, 2, 7 | 8      |
| 6       | 9   | 0, 1, 6, 8, 6, 1, 0, 8, 3, 1, 3, 8 | 12     |
| 7       | 10  | 0, 1, 6, 2, 6, 1, 0, 10, 5, 9, 5, 10 | 12     |
| 8       | 20  | 0, 1, 6, 15, 4, 9, 10, 11, 16, 5, 14, 19 | 12     |

\textit{Table 1.}

Since $m \geq 2$, it follows that $n \geq 3$. We claim that $m$ is odd. Observe that if $m$ is even, then

$$11 \mid \frac{10^m - 1}{9} \mid B_n$$

and from the seventh row of Table 1, it follows that $6 \mid n$ and consequently $B_6 \mid B_n$. Since $10 \mid B_6$, it follows that $10 \mid B_n = a \cdot \frac{10^m - 1}{9}$, which is a contradiction. Now, to complete the proof, we distinguish eight different cases corresponding to the values of $a$.

\textbf{Case I:} $a = 1$. Assume that $B_n$ is of the form $\frac{10^m - 1}{9}$ for some $m$. Since $m$ is odd, $B_n \equiv 1 \pmod{11}$ and also $B_n \equiv 11 \pmod{20}$. From the last row of Table 1, it follows that if $B_n \equiv 11 \pmod{20}$ then $n \equiv 7 \pmod{12}$. But, from the seventh row of Table 1, it follows that whenever $n \equiv 7 \pmod{12}$, $B_n \equiv 10 \pmod{11}$—a contradiction to $B_n \equiv 1 \pmod{11}$. Hence, no $B_n$ is of the form $\frac{10^m - 1}{9}$.

\textbf{Case II:} $a = 2$. If $B_n = 2 \cdot \frac{10^m - 1}{9}$, then $B_n \equiv 2 \pmod{5}$. But, in view of the third row of Table 1, it follows that for no value of $n$, $B_n \equiv 2 \pmod{5}$. Hence, $B_n$ cannot be of the form $2 \cdot \frac{10^m - 1}{9}$.

\textbf{Case III:} $a = 3$. If $B_n = 3 \cdot \frac{10^m - 1}{9}$, then $B_n \equiv 0 \pmod{3}$. But, in view of the first row of Table 1, $n \equiv 0, 2 \pmod{4}$. So, $B_2 \mid B_n$ and consequently $2 \mid \frac{10^m - 1}{9}$, which is a contradiction. Hence, $B_n$ cannot be of the form $3 \cdot \frac{10^m - 1}{9}$.

\textbf{Case IV:} $a = 4$. If $B_n = 4 \cdot \frac{10^m - 1}{9}$, then $B_n \equiv 0 \pmod{4}$ and in view of the second row of Table 1, it follows that $17 \mid B_n$. Since $17 \mid B_4$, it follows that $17 \mid (10^m - 1)$. But this is possible if $16 \mid m$, which is a contradiction since $m$ is odd. Hence, $B_n$ cannot be of the form $B_n = 4 \cdot \frac{10^m - 1}{9}$.
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Case V: \( a = 5 \). If \( B_n = 5 \cdot \frac{10^m - 1}{9} \), then \( B_n \equiv 0 \pmod{5} \) and in view of the third row of Table I this is possible only if \( 3 \mid n \). Hence, \( B_3 \mid B_n \) and since \( 7 \mid B_3 \), it follows that \( 7 \mid \frac{10^m - 1}{9} \), which implies that \( 6 \mid m \), a contradiction since \( m \) is odd. Hence, \( B_n \) cannot be of the form \( B_n = 5 \cdot \frac{10^m - 1}{9} \).

Case VI: \( a = 7 \). If \( B_n = 7 \cdot \frac{10^m - 1}{9} \), then \( B_n \equiv 0 \pmod{7} \) and in view of the fourth row of Table I this is possible only if \( 3 \mid n \). Hence, \( B_3 \mid B_n \) and since \( 5 \mid B_3 \), it follows that \( 5 \mid \frac{10^m - 1}{9} \), which is a contradiction. Hence, \( B_n \) cannot be of the form \( B_n = 7 \cdot \frac{10^m - 1}{9} \).

Case VII: \( a = 8 \). If \( B_n = 8 \cdot \frac{10^m - 1}{9} \), then \( B_n \equiv 0 \pmod{8} \) and in view of the fifth row of Table I this is possible only if \( 8 \mid n \). Hence, \( B_8 \mid B_n \) and since \( 17 \mid B_8 \), it follows that \( 17 \mid (10^m - 1) \). But this is possible if \( 16 \mid m \), which is a contradiction since \( m \) is odd. Hence, \( B_n \) cannot be of the form \( B_n = 8 \cdot \frac{10^m - 1}{9} \).

Case VIII: \( a = 9 \). If \( B_n = 9 \cdot \frac{10^m - 1}{9} \), then \( B_n \equiv 0 \pmod{9} \) and in view of the sixth row of Table I this is possible only if \( 6 \mid n \). Consequently, \( B_6 \mid B_n \) and since \( 11 \mid B_6 \), it follows that \( 11 \mid \frac{10^m - 1}{9} \). But this is possible only if \( m \) is even, which is a contradiction since \( m \) is odd. Hence, \( B_n \) cannot be of the form \( B_n = 9 \cdot \frac{10^m - 1}{9} \).

Thus, (2.1) has no solution if \( m \geq 2 \) and \( a \neq 6 \). This completes the proof. \( \square \)

We next study the presence of repdigits in the products of consecutive balancing numbers. The product \( B_1B_2 = 6 \) is a repdigit. So a natural question is: "Is there any other repdigit which is a consecutive product of balancing numbers?" In the following theorem, we answer this question in negative.

Theorem 2.2. If \( m, n, k \) and \( a \) are natural numbers such that \( m > 1 \) and \( 1 \leq a \leq 9 \), then the Diophantine equation

\[
B_nB_{n+1} \cdots B_{n+k} = a\left(\frac{10^m - 1}{9}\right)
\]

has no solution.

Proof. Firstly, we show that (2.2) has no solution for \( k \geq 2 \). Assume to the contrary that (2.2) has a solution in positive integers \( n, m, a \) for \( k \geq 2 \). Then, \( 2 \mid (n + i) \) and \( 3 \mid (n + j) \) for some \( i, j \in \{0, 1, \ldots, k\} \). Since \( 2 \mid B_2 \) and \( 5 \mid B_3 \), it follows that \( 2 \mid B_{n+i} \) and \( 5 \mid B_{n+j} \). Hence, \( 10 \mid B_nB_{n+1} \cdots B_{n+k} = a\left(\frac{10^m - 1}{9}\right) \), which is a contradiction. Hence, (2.2) has no solution for \( k \geq 2 \).

We next show that (2.2) has no solution if \( k = 1 \). If \( k = 1 \), (2.2) reduces to

\[
B_nB_{n+1} = a\left(\frac{10^m - 1}{9}\right).
\]

One of \( n \) and \( n + 1 \) is even and consequently, either \( B_n \) or \( B_{n+1} \) is also even. Hence, \( a \in \{2, 4, 6, 8\} \). Since \( m > 1 \), \( B_nB_{n+1} \geq 11 \) and hence \( n \) must be greater than 1.

In the following table we list all the least residues of \( B_nB_{n+1} \) modulo 5 and 100, which will be useful in the proof.
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| $m$    | $B_nB_{n+1} \mod m$                          | Period |
|--------|---------------------------------------------|--------|
| 5      | 0, 1, 0                                     | 3      |
| 100    | 20, 0, 6, 10, 40, 56, 70, 30, 56, 80, 70, 6, 40, 60, 6, 50, 0, 56, 10, 90, 56, 20, 30, 6, 50, 20, 6, 90, 60, 56, 50, 56, 60, 90, 6, 20, 80, 6, 30, 20, 56, 90, 10, 56, 0, 50, 6, 60, 40, 6, 70, 80, 56, 30, 70, 56, 40, 10, 6, 0 | 60     |

Table 2.

If $a = 2$ or $a = 4$, then

$$B_nB_{n+1} = a \cdot \frac{10^m - 1}{9} \equiv a \pmod{5}.$$ 

If $a = 8$, then

$$B_nB_{n+1} = 8 \cdot \frac{10^m - 1}{9} \equiv 3 \pmod{5}.$$ 

Similarly, if $a = 6$, then

$$B_nB_{n+1} = 6 \cdot \frac{10^m - 1}{9} \equiv 66 \pmod{100}.$$ 

Since the least residues of the last three congruences do not appear in the appropriate row of Table 2, it follows that $B_nB_{n+1}$ is not a repdigit if $n > 1$. This completes the proof.

In Theorem 2.1, we proved the absence of certain type of repdigits in the sequence of balancing numbers. However, in case of Lucas-balancing numbers, $C_1 = 3$ and $C_3 = 99$ are two known repdigits. Thus, a natural question is: "Does this sequence contain any other larger repdigit?" In the following theorem, we answer this question in negative.

**Theorem 2.3.** If $m, n$ and $a$ are natural numbers and $1 \leq a \leq 9$, then the Diophantine equation

$$C_n = a \left( \frac{10^m - 1}{9} \right) \tag{2.3}$$

has the only solutions $(m, n, a) = (1, 1, 3), (2, 3, 9)$.

**Proof.** To prove this theorem, we need all the least residues of the Lucas-balancing sequence modulo 5, 7 and 8. We list them in the following table.

| Row no. | $m$ | $C_n \mod m$ | Period |
|---------|-----|--------------|--------|
| 1       | 5   | 1, 3, 2, 4, 2, 3 | 6      |
| 2       | 7   | 1, 3, 3      | 3      |
| 3       | 8   | 1, 3         | 2      |

Table 3.
Among the first three Lucas-balancing numbers $C_1 = 3$ and $C_3 = 99$ are repdigits and $(2.3)$ is satisfied for $(m, n, a) = (1, 1, 3), (2, 3, 9)$. Now, let $n \geq 4$ and hence $m \geq 3$. Since $C_n$ is always odd, $a \in \{1, 3, 5, 7, 9\}$. Since no zero appears in the first two rows of Table 3, it follows that $C_n$ is not divisible by 5 or 7 and hence the possible values of $a$ are limited to 1, 3, 9.

If $a \in \{1, 9\}$, then

$$C_n = a \cdot \frac{10^m - 1}{9} \equiv 10^m - 1 \equiv 7 \pmod{8}.$$  

Similarly, if $a = 3$, then

$$C_n = 3 \cdot \frac{10^m - 1}{9} \equiv 5 \pmod{8}.$$  

Since, the least residues 5 and 7 do not appear in the last row of Table 3, it follows that $(2.3)$ has no solution for $n > 3$. This completes the proof.

In Theorem 2.2, we noticed that no product of consecutive balancing numbers is a repdigit with more than one digit, though the only product $B_1B_2 = 6$ is a single digit repdigit. The following theorem negates the possibility of any repdigit as product of consecutive Lucas-balancing numbers.

**Theorem 2.4.** If $m, n, k$ and $a$ are natural numbers and $1 \leq a \leq 9$, then the Diophantine equation

$$C_nC_{n+1} \cdots C_{n+k} = a \left( \frac{10^m - 1}{9} \right)$$

(2.4)

has no solution.

**Proof.** All the Lucas-balancing numbers are odd and in view of $(2.4)$, $a \in \{1, 3, 5, 7, 9\}$. It is easy to see that $(2.4)$ has no solution if $m = 1, 2$. In the following table we list all the nonnegative residues of Lucas-balancing numbers and their consecutive product modulo 5, 7 and 8 which will play an important role in proving this theorem.

| $m$ | $C_n \mod m$ | $C_nC_{n+1} \cdots C_{n+k} \mod m$ |
|-----|---------------|----------------------------------|
| 5   | 1, 3, 2, 4, 2, 3 | $\in \{1, 2, 3, 4\}$ |
| 7   | 1, 3, 3        | $\in \{1, 2, 3, 4, 5, 6\}$ |
| 8   | 1, 3           | $\in \{1, 3\}$                 |

**Table 4.**

For $m \geq 3$, $C_nC_{n+1} \cdots C_{n+k} = a \left( \frac{10^m - 1}{9} \right) \equiv 7a \pmod{8}$. But from the last row of Table 4, it follows that $7a \equiv 1, 3 \pmod{8}$ and hence $a = 5$ or $a = 7$. Now, reducing $(2.4)$ modulo $a$ we get $C_nC_{n+1} \cdots C_{n+k} \equiv 0 \pmod{a}$. Since, 0 does not appear as a residue of $C_nC_{n+1} \cdots C_{n+k}$ modulo 5 or 7, it follows that $(2.4)$ has no solution for $m \geq 3$. This completes the proof.
3. Conclusion

In the last section, we noticed that the Lucas-balancing sequence contains only two repdigits namely $C_1 = 3$ and $C_3 = 99$, while we could not explore all repdigits in the balancing sequence. In Theorem 2.1, we proved that $B_n$ is not a repdigit ($B_n \neq a \left( \frac{10^m - 1}{9} \right)$), with more than one digit, if $a \neq 6$. Thus, repdigits in the balancing sequence having all digits 6 is yet unexplored. In this connection, one can verify that if $n \not\equiv 14 \pmod{96}$ then $B_n$ is not a repdigit. Further, if $m \not\equiv 1 \pmod{6}$, then also $B_n$ is not a repdigit. We believe that, $B_1 = 1$ and $B_2 = 6$ are the only repdigits in the balancing sequence. It is still an open problem to prove the existence or nonexistence of repdigits that are 6 times of some repunit other than $B_2 = 6$.

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References

[1] B. Faye and F. Luca, Pell and Pell-Lucas numbers with only one distinct digit, Ann. Math. Inform. 45 (2015), 55–60.

[2] K. Liptai, Fibonacci balancing numbers, Fibonacci Quart., 42 (2004), 330–340.

[3] F. Luca, Fibonacci and Lucas numbers with only one distinct digit, Port. Math., 57 (2000).

[4] D. Marques and A. Togbé, On repdigits as product of consecutive Fibonacci numbers, Rend. Istit. Mat. Univ. Trieste, 44 (2012), 393–397.

[5] G.K. Panda, Some fascinating properties of balancing numbers, Congr. Numer., 194 (2009), 185–189.

[6] G.K. Panda, Arithmetic progression of squares and solvability of the Diophantine equation $8x^4 + 1 = y^2$, East-West J. Math., 14.2 (2012), 131–137.

[7] G.K. Panda and S.S. Rout, Periodicity of balancing numbers, Acta Math. Hungar., 143.2 (2014), 274–286.

[8] G.K. Panda and R.K. Davala, Perfect balancing numbers, Fibonacci Quart., 53.2 (2015), 261–264.

[9] P.K. Ray, Balancing and Cobalancing Numbers, Ph.D. Thesis, National Institute of Technology, Rourkela, (2009).

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