ON THE LARGE GENUS ASYMPTOTICS OF
WEIL-PETERSSON VOLUMES

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Abstract. A relatively fast algorithm for evaluating Weil-Petersson volumes of moduli spaces of complex algebraic curves is proposed. On the basis of numerical data, a conjectural large genus asymptotics of the Weil-Petersson volumes is computed. Asymptotic formulas for the intersection numbers involving $\psi$-classes are conjectured as well. The precision of the formulas is high enough to believe that they are exact.

The aim of this note is to report on the recent progress in computing Weil-Petersson volumes of moduli spaces of complex algebraic curves (with or without marked points) that resulted from better programming, software and hardware as compared to [7]. The numerical evidence led us to a plausible guess about their large genus asymptotic behavior that may have further applications in algebraic geometry, combinatorics, dynamical systems and string theory.

1. Algorithms

Let $\overline{M}_{g,n}$ denote the moduli space of stable $n$-pointed genus $g$ complex algebraic curves. The universal curve $p: \overline{M}_{g,n+1} \rightarrow \overline{M}_{g,n}$ has $n$ canonical sections $x_1, \ldots, x_n$ given by the marked points. Put $\psi_i = c_1(x_i^*\omega) \in H^2(\overline{M}_{g,n}, \mathbb{Q})$, where $\omega$ is the relative dualizing sheaf on $\overline{M}_{g,n+1}$. The first Mumford class of $\overline{M}_{g,n}$ is the direct image class

$$\kappa_1 = p_*\psi^2_{n+1} = \int_{\text{fiber}} \psi^2_{n+1} \in H^2(\overline{M}_{g,n}, \mathbb{Q}).$$

The Weil-Petersson metric is Kähler on $\overline{M}_{g,n}$. Its symplectic form $\omega_{WP}$ extends to $\overline{M}_{g,n}$ as a closed current and represents the class $2\pi^2\kappa_1 \in H^2(\overline{M}_{g,n}, \mathbb{R})$ (see [6]). By definition, the (normalized) Weil-Petersson volume of $\overline{M}_{g,n}$ is just its standard symplectic volume with respect to the form $\frac{1}{2\pi^2}\omega_{WP}$:

$$V_{g,n} = \frac{1}{(3g-3+n)!} \int_{\overline{M}_{g,n}} \kappa_1^{3g-3+n}. \quad (1)$$

For all $g, n \geq 0$ with $2g + n \geq 3$ these are positive rational numbers.
Below we describe an algorithm for computing Weil-Petersson volumes (see [1] for details).

**Theorem 1.** Let
\[
\partial_0 = \frac{1}{t} \left( \frac{\partial}{\partial y} - x(y) \frac{\partial}{\partial t} \right), \quad \partial_1 = - \frac{\partial}{\partial t} + y \partial_0,
\]
where
\[
x(y) = -\sqrt{y} J'_0(2\sqrt{y}) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(k-1)!} \frac{y^k}{k!}
\]
(J \(_0\) denotes the Bessel function of the first kind).

Then
(i) the KdV equation
\[
\partial_1 u = \partial_0 \left( \frac{u^2}{2} + h^2 \frac{\partial^2 u}{12} \right)
\]
has a unique solution \(u(y, t) = y + h^{2g} \sum_{g=1}^{\infty} u_g(y, t)\) where each \(u_g(y, t)\) is a Laurent polynomial in \(t\) of the form
\[
u_g(y, t) = \sum_{k=2g+1}^{5g-1} u_{g,k}(y) t^{-k};
\]
(ii) for each \(g \geq 2\) the equation
\[
\partial_0^2 \phi_g(y, t) = u_g(y, t)
\]
has a unique solution of the form
\[
\phi_g(y, t) = \sum_{k=2g-1}^{5g-5} \phi_{g,k}(y) t^{-k};
\]
(iii) for any \(g, n \geq 0\) the Weil-Petersson volume of \(\mathcal{M}_{g,n}\) is given by the formula
\[
V_{g,n} = \partial_0^n \phi_g(y, t) \big|_{y=0, t=1}.
\]

The above theorem extends to the intersection numbers involving \(\psi\)-classes. Fix a set \(d = (d_1, \ldots, d_n)\) of non-negative integers and put \(|d| = d_1 + \cdots + d_n\). Consider
\[
V_{g,n;d} = \frac{1}{(3g-3+n-|d|)!} \int_{\mathcal{M}_{g,n}} \psi_1^{d_1} \cdots \psi_n^{d_n} \kappa_1^{3g-3+n-|d|}.
\]

**Theorem 2.** Let
\[
\partial_0 = x_1 \left( \frac{\partial}{\partial y} + x_1^2 \left( x_2 + x(y) \right) \frac{\partial}{\partial x_1} + \sum_{k=2}^{\infty} x_{k+1} \frac{\partial}{\partial x_k} \right),
\]
\[
\partial_1 = x_1^2 \frac{\partial}{\partial x_1} + y \partial_0,
\]
with \( x(y) = -\sqrt{y}J_0'(2\sqrt{y}) \) as above.

Then

(i) the KdV equation

\[ \partial_1 v = \partial_0 \left( \frac{v^2}{2} + h^2 \frac{\partial^2 v}{12} \right) \]

has a unique solution

\[ v(y, x_1, x_2, \ldots) = y + h^2 \sum_{g=1}^{\infty} v_g(y, x_1, x_2, \ldots), \]

where each \( v_g(y, t) \) is a polynomial in \( x_1 \) of the form

\[ v_g(y, x_1, x_2, \ldots) = \sum_{k=2g+1}^{5g-1} v_{g,k}(y, x_2, x_3, \ldots) x_1^k, \]

(ii) for each \( g \geq 2 \) the equation

\[ \partial_0^2 \psi_g(y, x_1, x_2, \ldots) = v_g(y, x_1, x_2, \ldots) \]

has a unique solution of the form

\[ \psi_g(y, x_1, x_2, \ldots) = \sum_{k=2g-1}^{5g-5} \psi_{g,k}(y, x_2, x_3, \ldots) x_1^k, \]

(iii) the intersection number \( V_{g,n;0} \) is given by the formula

\[ V_{g,n;0} = \partial_{x_1}^{l_1} \partial_{x_2}^{l_2} \ldots \partial_{x_n}^{l_n} \psi_g(y, x_1, x_2, \ldots) \bigg|_{y=0, x_1=1, x_2=x_3=\cdots=0}, \]

where \( l_k \) is the number of \( d_i \)'s equal to \( k \).

The proof follows the same lines as that of Theorem 1 and utilizes an observation of M. Kazarian on how to explicitly express mixed intersection numbers of \( \psi \)- and \( \kappa \)-classes in terms of intersection numbers of \( \psi \)-classes alone [3]. The details will appear elsewhere. Note that for \( d_1 = \cdots = d_n = 0 \) it reduces to Theorem 1 with the obvious change of variable \( x_1 = 1/t \).

The main advantage of our algorithm is its speed, and in this respect it is superior to the algorithms of C. Faber [1] and M. Kazarian [3], though it loses to both of them in generality.

2. Asymptotics

It may be instructive to begin with the large \( n \) asymptotics of Weil-Petersson volumes. The following exact asymptotic formula was proven in [4] for any fixed \( g \):

\[ V_{g,n} = n! C^n n^{(5g-7)/2} \left( a_g + O\left(1/n\right)\right), \quad n \to \infty, \quad (3) \]

where \( C = -z_0 J_0'(z_0) \) and \( z_0 \) is the first positive zero of the Bessel function \( J_0(z) \). The coefficients \( a_g \) can also be explicitly computed [4]
(in fact, one can even get the complete asymptotic expansion of $V_{g,n}$ as $n \to \infty$).

The problem seems more challenging when $n$ is fixed and $g \to \infty$. We implemented the algorithm of Theorem 1 in a Maple program and computed all numbers $V_{g,n}$ for $g \leq 50$ and $1 \leq n \leq 4$. These data led us to

**Conjecture 1.** For any fixed $n \geq 0$

$$V_{g,n} = (2g)! \left( \frac{2}{\pi^2} \right)^g g^{n-7/2} \frac{22n-6}{\sqrt{\pi}} \left( 1 + \frac{c_n}{g} + O \left( \frac{1}{g^2} \right) \right), \quad g \to \infty.$$  

This formula agrees with the earlier results of [2, 5]. Approximate values of the constants $c_n$ with $n \leq 4$ are: $c_0 \approx 1.8$, $c_1 \approx 0.75$, $c_2 \approx 0.1$, $c_3 \approx -0.15$, $c_4 \approx -0.001$.

Our Maple implementation of the algorithm of Theorem 2 evaluates the intersection numbers $V_{g,n,d}$ given by (2). In particular, we computed all $V_{g,n,d}$ with $g \leq 40$ and $l_k \leq 2$, $k = 1, 2, 3)$, and from that we get

**Conjecture 2.** For any fixed $n > 0$ and a fixed set $d = (d_1, \ldots, d_n)$ of non-negative integers

$$\lim_{g \to \infty} V_{g,n,d} \div V_{g,n} = \prod_{k \geq 1} \left( \frac{\pi^{2k}}{2^k (2k + 1)!!} \right)^{l_k},$$

where $l_k$ denotes the number of $d_i$’s that are equal to $k$.

Both these conjectures hold numerically with a high precision, so there is a good reason to believe that they are actually true. However, at the moment these asymptotic formulas lack theoretical justification. The next section contains some data and heuristics.

### 3. Numerics

It is known that the order of magnitude of $V_{g,n}$ is $(2g)!$ for large $g$ (see [2, 3]), so the problem was to find the asymptotics up to the factors of smaller order. A question of M. Mirzakhani about the behavior of the ratio $V_{g-1,n+2}/V_{g,n}$ as $g \to \infty$ served us as a starting point. Computations show that this ratio decreases with $g$ for $n = 0, 1$, and increases for any fixed $n \geq 2$. At the same time for any fixed $g$ this ratio decreases when $n$ grows. Below is the table of decimal approximations (rounded up to 10 digits) for $n = 1, 2$ and $41 \leq g \leq 50$:

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The behavior of $V_{g-1,n+2}/V_{g,n}$ suggests that there is a limit as $g \to \infty$ independent of $n$. D. Zagier numerically identified this limit with $2\pi^2 = 19.7392088\ldots$ (private communication).

The next step is to analyze the behavior of the ratio $2gV_{g,n-1}/V_{g,n}$:

| $g$  | $2gV_{g,1}/V_{g,2}$ | $2gV_{g,2}/V_{g,3}$ | $2gV_{g,3}/V_{g,4}$ |
|------|---------------------|---------------------|---------------------|
| 40   | 0.5082406948        | 0.5031382404        | 0.4981366818        |
| 41   | 0.5080365079        | 0.5030613837        | 0.4981822417        |
| 42   | 0.5078421948        | 0.5029882014        | 0.4982256270        |
| 43   | 0.5076570564        | 0.5029184361        | 0.4982669896        |
| 44   | 0.5074804578        | 0.5028518541        | 0.4983064678        |
| 45   | 0.5073118214        | 0.5027882423        | 0.4983441875        |
| 46   | 0.5071506208        | 0.5027274063        | 0.4983802636        |
| 47   | 0.5069963746        | 0.5026691682        | 0.4984148013        |
| 48   | 0.5068486423        | 0.5026133653        | 0.4984789069        |
| 49   | 0.5067070199        | 0.5025598478        | 0.4984796388        |

It is not hard to see that expected limit in each of the columns is $1/2$ as $g \to \infty$.

These two observations combined together give the factor $2^{g+2n}/\pi^{2g}$ in the Weil-Petersson volume asymptotics. Similar to (3) it is natural to assume that the ratio $2gV_{g,n}/(2g)!\pi^{2g}$ behaves like $a_ng^{b_n}$, and it works. Moreover, it appears that $a_n = 2^{-6}\pi^{-1/2}$ is independent of $n$ and $b_n = n - 7/2$. In the table below $C_{g,n} = (2g)!\left(\frac{2}{\pi}\right)^g g^{n-7/2}2^{2n-6}/\sqrt{\pi}$:
We see that the ratio $V_{g,n}/C_{g,n}$ apparently tends to 1 as $g \to \infty$ for any $n = 1, 2, 3, 4$ (a standard extrapolation gives 1 up to at least 6 decimal digits). For other values of $n$ the situation is the same. It is worth mentioning that the case $n = 0$ is computationally harder because of an additional non-trivial integration [7], so currently we are able to compute $V_{g,0}$ only up to $g = 30$. However, Conjecture 1 is rather precise even in this case:

| $g$ | $V_{g,1}/C_{g,1}$ | $V_{g,2}/C_{g,2}$ | $V_{g,3}/C_{g,3}$ | $V_{g,4}/C_{g,4}$ |
|-----|------------------|------------------|------------------|------------------|
| 40  | 1.019018429      | 1.002495904      | 0.9962430037     | 0.9999695265     |
| 41  | 1.018547428      | 1.002435270      | 0.9963349432     | 0.9999703519     |
| 42  | 1.018099193      | 1.002377513      | 0.9964224911     | 0.9999711349     |
| 43  | 1.017672110      | 1.002322431      | 0.9965059531     | 0.9999718768     |
| 44  | 1.017264718      | 1.002269843      | 0.9965856093     | 0.9999725812     |
| 45  | 1.016875685      | 1.002219584      | 0.9966617148     | 0.9999732510     |
| 46  | 1.016503797      | 1.002171501      | 0.9967345009     | 0.9999738890     |
| 47  | 1.016147948      | 1.002125457      | 0.9968041809     | 0.9999744975     |
| 48  | 1.015807118      | 1.002081325      | 0.9968709494     | 0.9999750781     |
| 49  | 1.015480379      | 1.002038988      | 0.9969349845     | 0.9999756329     |

Clearly, this sequence converges and its evaluated limit is 1 as well.

Considerations that led us to Conjecture 2 are very similar to the ones described in this section.

**Acknowledgments.** Part of this work was done at the Max-Planck-Institut für Mathematik (Bonn) in winter 2007-08, whose support is gratefully acknowledged. Special thanks are to M. Kazarian, M. Mirzakhani and D. Zagier for numerous helpful discussions.

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