CANTOR AND SIERPINSKI SURFACES AS SPECTRAL CURVES FOR SOLUTIONS TO THE KADOMTSEV–PETVIASHVILI EQUATION

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Abstract. In this note we will discuss singular surfaces that are homeomorphic to spheres except where they become doubly degenerate on singularities homeomorphic to a Cantor set or Sierpinski gasket that we call Cantor and Sierpinski surfaces. We will also discuss a family distributions that are holomorphic on the Riemann sphere except on the singular sets corresponding to the singular sets of the Cantor and Sierpinski surfaces. These distributions are limits of families of rational functions that can also be associated with holomorphic line bundles on surfaces with a finite number of doubly degenerate singular points. We conjecture that a subset of these solutions can be used to formulate a definition of a holomorphic line bundle on a Cantor or Sierpinski surface. One piece of evidence for this conjecture comes from the theory of primitive solutions to (1+1)D completely integrable systems. We then use the distributions to derive some new solutions to the (2+1)D Kadomtsev–Petviashvili equation, and its (1+1)D reduction — the Korteweg–de Vries equation. These equations also are related to solutions to the (1+1)D time independent Schrödinger equation, and allow the calculation of many potentials giving rise to Cantor like energy spectra.

1. Introduction

In this note we will discuss examples of distributions in the sense of Schwartz that we believe are promising for the study of the topology of the Sierpinski gasket. In this case our test functions/differentials will always be assumed to be on some domain \( \Omega \subset \mathbb{C} \) of interest. We will label the space of test functions (differentials) by \( \mathcal{D} \), we will the corresponding (scalar or one form valued) distributions by \( \mathcal{D}' \). The compactly supported distributions will be labeled by \( \mathcal{D}'_0 \). The effective definition of distribution for the purposes will therefore be elements of \( \mathcal{D}'_0 \). (We make this clear up front to avoid confusion due to lack of precision.) These distributions will allow us to produce holomorphic function that define line bundles on complex surfaces with complicated singular sets (such as a Cantor set or Sierpinski gasket) also allow the Kadomtsev–Petviashvili (KP) equation and the Korteweg–de Vries (KdV) equation.

This assumption is not always physically accurate, but compactly supported distributions can approximate most physically accurate solutions provided we are careful with a limiting procedure. For example, if we consider a periodic KdV solver using the IST, then there must be an infinite number of “gaps” in the “spectrum” of the initial condition that diverge to infinity. However, a finite gap potential can approximate any generic periodic potential \( u(x) \in L^\infty(\mathbb{R}) \) arbitrarily. Moreover, in [4, 3, 10, 7] we showed that any finite gap potential can be computed as (a possibly shifted [7]) primitive potentials, and all finite gap solutions to the KdV equation are primitive solutions. These primitive solutions to the KdV equation

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are a subset of the solutions that can be put in the framework of the construction in this paper.

The main mathematical method used in this paper is the nonlocal dbar problem. The nonlocal dbar problem was originally introduced by Ablowitz, Bar-Yaacov, and Fokas [1] to formulate the inverse scattering transform for the KP equation. This method was further generalized by Zakharov and Manakov [12] to other (2+1)D completely integrable systems. Recently, the nonlocal dbar problem has been used to study new classes of solutions to completely integrable systems called primitive solutions [3, 4, 5, 10, 9, 7, 8, 14, 13].

The distributions introduced in this paper are singular on a Cantor set or Sierpinski gasket, where the singularities can potentially correspond to the Riemann sphere with nonlocal identifications along two copies of a Cantor set or Sierpinski gasket. These singular surfaces will inherit a transition function to allow one to explicitly move through doubly degenerate singular points that are holomorphic off of the singularities. These spaces of distributions are formed using the nonlocal dbar problem, and a limit of rational functions on the plane as the number of poles diverges to infinity. In this limit, the poles coalesce into singularities along arcs in the plane, and the rational function limit to sectionally analytic functions away from the singular arcs.

1.1. Prerequisite Definitions.

Definition 1. Suppose $Q_n \subset \mathbb{C}$ is a sequence of complex point sets such that $Q_n \subset Q_{n+1}$ and

$$ Q = \lim_{n \to \infty} Q_n := \{x \in Q_n : n = 0, 1, 2, \ldots\} $$

such that

$$ \lim_{n \to \infty} \frac{1}{|Q_n|} \sum_{\lambda_n \in Q_n} \delta(\lambda - \lambda_n) d\lambda = d\mu_Q(\lambda) $$

where $\mu_Q(\lambda)$ is the uniform probability measure on $Q$.

Definition 2. The following definitions are either classical constructions of Cantor [2] and Sierpinski [11] or explicit:

(1) The $n$-th step of the Cantor iteration for the Cantor middle $\epsilon$ set $C_n \subset \mathbb{C}$.
(2) The point set $P_n \subset C_n$ of the endpoints of the intervals making up $C_n$.
(3) The $n$-th step $S_n \subset$ of the Sierpinski iteration consisting of for the Sierpinski gasket.
(4) The vertex sets $V_n \subset S_n$ of vertices in the Sierpinski gasket $S$.
(5) The Cantor set can be defined by

$$ C = \bigcap_{n=0}^{\infty} C_n. $$

(6) The Sierpinski gasket can be defined by

$$ S = \bigcap_{n=0}^{\infty} S_n. $$
Proposition 3. The following are true:

(1) The point sets $P_n$ and $C$ are related by
$$\lim_{n \to \infty} P_n = C$$
and satisfy the necessary properties from definition to be a choice of $Q_n$ and $Q$ respectively.

(2) The point sets $V_n$ and $S$ are related by
$$S = \lim_{n \to \infty} V_n$$
and satisfy the necessary properties from definition to be a choice of $Q_n$ and $Q$ respectively.

2. The Nonlocal $\bar{D}bar$ Problem and Topological Spaces of Distributions

Definition 4. We will use $Q_n$ to refer to either $P_n$ or $V_n$ and we will let
$$Q = \lim_{n \to \infty} Q_n.$$ We will consider some explicit space of compactly supported distributions $D : \mathbb{C} \to \mathbb{C}$ defined with respect to smooth compactly supported test functions as follows:

(1) From $Q_n$, $Q_n$ and a list of numbers $a_j$ we determine the following distributions
$$\chi_n(\lambda) = 1 + \frac{1}{\pi |Q_n|} \sum_{\lambda_j \in Q_n} \frac{a_j}{\lambda - \lambda_j},$$
$$\chi(\lambda) = \lim_{n \to \infty} \chi_n(\lambda).$$
(at this point we make no comment on the existence of the limiting distribution, but for many choices of $a_j$ the existence of $\chi$ is easy to justify)

(2) Let $\phi$ be some isometry of $\mathbb{C}$ such that $\phi : Q \to \mathbb{Q} \subset \mathbb{C}$ where $Q$ and $\mathbb{Q}$ are disjoint.

(3) If we enforce the nonlocal conditions
$$a_j = r(\lambda_j)\chi_n(\phi(\lambda_j))$$
for all $n$ then the rational functions $\chi_n$ are uniquely defined by the above formula.

The functions $\chi_n(\lambda)$ and $\chi(\lambda)$ are holomorphic function in a neighborhood of $\infty$ in
$$S^2 \equiv \mathbb{C} \cup \{\infty\}$$
with $\chi_n(\infty) = 1$.

The distributions $\chi_n(\lambda)$ can be identified with rational functions by definition, but the distributions $\chi(\lambda)$ often cannot be represented by rational functions. An equivalent construction can also be made using functions and measures instead of distributions and rational functions. In the previous definition, the closure of the rational functions in the topology of uniform convergence in compact sets that do not intersect the singular sets of the surface can intuitively be formulated formally using limits of rational functions, and can be made rigorous using Schwartz’s theory of distributions.
**Proposition 5.** Let \( Q_n \supset Q_{n-1} \) and \( R_n \supset R_{n-1} \) be two sequences of point sets such that
\[
\lim_{n \to \infty} Q_n = Q, \quad \lim_{n \to \infty} R_n = Q'
\]
but so that the point measures on \( Q_n \) and \( R_n \) limit to measures on \( Q \) and \( Q' \), and \( \phi(Q_n) \) and \( R_n \) are disjoint. Now consider the distributions of the form
\[
\tilde{\chi}_n(\lambda) = 1 + \frac{1}{\pi|Q_n|} \left( \sum_{\lambda_j \in Q_n} \frac{a_j}{\lambda - \lambda_j} \right) + \frac{1}{\pi|R_n|} \left( \sum_{\mu_k \in R_n} \frac{b_j}{\lambda - \mu_k} \right)
\]
where \( a_j \) and \( b_j \) are bounded sequences associated to the common points of \( Q_n \) (at this point we make no comment on the existence of the limiting distribution, but for many choices of \( a_j \) and \( b_j \) the existence of \( \tilde{\chi} \) is easy to justify). These distributions are well defined, and if we assume the following nonlocal conditions on the poles
\[
a_j = r_1(\lambda_j)\chi_n(\phi(\lambda_j)), \quad b_k = r_2(\mu_k)\chi_n(\phi(\mu_k))
\]
then the distribution \( \tilde{\chi}_n(\lambda) \) is uniquely defined.

The functions \( \tilde{\chi}_n(\lambda) \) and \( \tilde{\chi}(\lambda) \) are holomorphic function in a neighborhood of \( \infty \) in
\[
S^2 \equiv \mathbb{C} \cup \{ \infty \}
\]
with \( \chi_n(\infty) = 1 \).

The following lemma will be important, because it will allows us to define the limiting values of the rational functions \( \chi_n(\lambda) \) and \( \tilde{\chi}_n(\lambda) \) as \( n \to \infty \) in a manner that does not depend on the limiting procedure that produce \( r(s), r_1(s) \) and \( r_2(s) \). This lemma allows us to study functions on \( Q \) that solve singular integral equations instead of the nonlocal dbar problem. However, we first need to define some function spaces.

**Definition 6.** Let \( H_\mu(Q) \) be the set of Hölder continuous functions on \( Q \) with Hölder coefficient \( 0 < \mu < 1 \).

**Lemma 7.** The rational functions \( \chi_n \) and \( \tilde{\chi}_n \) solve the nonlocal dbar problems
\[
(1) \quad \frac{\partial \tilde{\chi}}{\partial \lambda}(\lambda) = R(\lambda)\tilde{\chi}(\phi(\lambda))
\]
where
\[
R(\lambda) = \frac{1}{|Q_n|} \sum_{\lambda_j \in Q_n} r(\lambda_j)\delta(\lambda - \lambda_j),
\]
or
\[
(3) \quad R(\lambda) = \frac{1}{|Q_n|} \sum_{\lambda_j \in Q_n} r_1(\lambda_j)\delta(\lambda - \lambda_j) + \frac{1}{|R_n|} \sum_{\mu_k \in R_n} r_2(\mu_k)\delta(\lambda - \mu_k)
\]
and \( \tilde{\chi}(\lambda) \to 1 \) as \( \lambda \to \infty \). Moreover, if we suppose that \( r, r_1 \in H_\mu(Q) \) and \( r_2 \in H_\mu(Q') \), then \( R(\lambda) \) limits to
\[
(4) \quad R(\lambda) = \int_Q r(s)\delta(\lambda - s)d\mu_Q(s),
\]
or

\[ R(\lambda) = \int_Q r_1(s)\delta(\lambda - s) + r_2(s)\delta(\lambda - \phi(s))d\mu_Q(s) \]  

respectively as \( n \to \infty \).

Because of the assumption that the uniform probability measure \( d\mu_Q \) exists as the limit of delta measure, the proof of this lemma is a simply application of the theory of distributions. One just needs to apply the distributions to test functions, and then take the limit. The proof is left to the reader.

The functions \( \chi_n(\lambda) \) and \( \tilde{\chi}_n \) determines a holomorphic line bundle on a singular rational curve [10]. This is because the simple poles with nonlocal residues allow a transition between the two points that are non-locally identified. As \( n \to \infty \), the poles coalesce into the singular sets \( Q \) and \( Q' \) that are identified via the nonlocal identification. For the case of finite and infinite gap solutions to the KdV equation, we can take \( Q \) and \( Q' \) to be intervals, and the limit gives the finite gap solutions to the KdV equation as primitive solutions to the KdV equation.

The nonlocal \( \overline{dbar} \) can instead be formulated as a system of two (local) singular integral equations. The justification is a simple implication of the constructions in [4, 3, 5, 10, 7, 8, 9]. The limiting procedure used in [4] that was the inspiration for further results in [3, 10, 7, 8] using the fact that a deterministic limiting procedure discussed in [4] can easily be justified. However, this limit is ineffective for some numerical and statistical calculations.

In [5], the authors discuss a random discrete soliton amplitude spectrum that leads to a Riemann–Stiljes integration that gives an alternative way of rigorously defining the functions \( r(s), r_1(s) \) and \( r_2(s) \) in the singular integral equations in the case of a primitive solution to the KdV equation. The universality of the Riemann–Stiljes integral for any choice of partition generating it, also guarantees the definiteness of the limiting distribution.

In that case considered in this paper, these methods lead to the following theorem.

**Theorem 8.** The limiting distributions can be written in the forms

\[ \chi(\lambda) = 1 + \frac{1}{\pi} \int_Q \frac{f(s)}{\lambda - s}d\mu_Q(s) \]  
\[ \tilde{\chi}(\lambda) = 1 + \frac{1}{\pi} \int_Q \left( \frac{f_1(s)}{\lambda - s} + \frac{f_2(s)}{\lambda - \phi(s)} \right)d\mu_Q(s). \]

Moreover, the function \( f(s) \) solves the integral equation

\[ r(t) = f(t) + \frac{r(t)}{\pi} \int_Q \frac{f(s)}{\phi(t) - s}d\mu_Q(s) \]

and the functions \( f_1(s) \) and \( f_2(s) \) solve the system of singular integral equations

\[ r_1(t) = f_1(t) + \frac{r_1(t)}{\pi} \left( \int_Q \frac{f_1(s)}{\phi(t) - s}d\mu_Q(s) + \int_Q \frac{f_2(s)}{\phi(t) - \phi(s)}d\mu_Q(s) \right) \]  
\[ r_2(t) = f_2(t) + \frac{r_2(t)}{\pi} \left( \int_Q \frac{f_1(s)}{t - s}d\mu_Q(s) + \int_Q \frac{f_2(s)}{t - \phi(s)}d\mu_Q(s) \right) \]
which determine \( f(s), f_1(s) \) and \( f_2(s) \) from \( r(s), r_1(s) \) and \( r_2(s) \). The principle value integrals are defined via the embedding of \( C \) or \( S \) into \( \mathbb{C} \).

This theorem is proven by plugging (6) and (7) into (1), using the functions \( R \) of the form (4) and (5) respectively. The nonlocal dbar problem leads to the following integral equation

\[
\tilde{\chi}(\lambda) = 1 + \frac{1}{\pi} \int_C \frac{R(\zeta)\chi(\phi(\zeta))}{\lambda - \zeta} d^2\zeta
\]

where \( d^2\zeta \) is the usual area form on \( C \equiv \mathbb{R}^2 \). This integral equation comes from combining the inversion formula for the dbar operator and the asymptotic condition \( \tilde{\chi}(\lambda) \to 1 \) as \( \lambda \to \infty \). The integral equation is determined by its behavior near the singular support, because \( \chi(\lambda) \) solves the Cauchy–Riemann equations off the singular support. This is because with no singular support, the Cauchy–Riemann equations would imply \( \tilde{\chi}(\lambda) = 1 \). Therefore, the singular support is acting as a source for the Cauchy–Riemann equations (in a manner analogous to the way electrostatic charge distributions generate solutions to the Laplace equation).

**Definition 9.** We will use \( \mathcal{D}_n \) to refer to the distribution \( \chi(\lambda) \) and \( \tilde{\chi}(\lambda) \) that are determined by the singular integral equations in theorem

It seems likely that \( \tilde{\chi}(\lambda) \) extends the idea of a holomorphic line bundle on a singular rational curve, due to the interpretations of \( \chi_n(\lambda) \) and \( \tilde{\chi}_n(\lambda) \) as giving holomorphic line bundles on singular surfaces [10]. The constructions in [4 5 10 9 7 8 14 13] give evidence to the conjecture that the idea of a holomorphic line bundles on surfaces can be extended to singular surface \( \Sigma_Q = \mathbb{S}^2/\langle \sim \rangle \) using certain choices of \( \tilde{\chi}(\lambda) \), or equivalently to certain choices of \( r_1(s) \geq 0 \) and \( r_2(s) \leq 0 \) — which is the topological surface formed by taking \( \mathbb{C} \) and identifying \( Q \) with \( Q' \) by \( \sim \) via restriction of the isometry \( \phi \) to \( Q \) — can be realized for certain choices of \( r_1(s) \) and \( r_2(s) \).

**Definition 10.** When \( Q = C \) we call \( \Sigma_Q \) a Cantor surface, and when \( Q = S \) we call \( \Sigma \) a Sierpinski surface. We will call \( \tilde{\chi}(\lambda) \) a primitive distribution due to the connection with primitive potentials.

An explicit link between a notion of holomorphic line bundles on these singular surfaces and the distributions discussed in the previous section would lead to an idea of a Picard group of the Cantor and Sierpinski surfaces.

**Definition 11.** We can form the holomorphic one forms

\[
\tilde{\omega}_n = \tilde{\chi}_n(\lambda)d\lambda, \quad \tilde{\omega} = \tilde{\chi}(\lambda)d\lambda.
\]

We can define the space \( \mathcal{A} \) of holomorphic differentials on

\[
\mathbb{C} \setminus Q
\]

of the form

\[
\omega = \chi(\lambda)d\lambda,
\]

and the space \( \tilde{\mathcal{A}} \) of holomorphic differentials on

\[
\mathbb{C} \setminus (Q \cup Q')
\]
of the form
\[ \tilde{\omega} = \tilde{\chi}(\lambda) d\lambda. \]

It seems likely that for some choices of functions \( r_1(s) \) and \( r_2(s) \) these one forms can be interpreted as holomorphic one forms on \( \Sigma_Q \). This interpretation would be important to singularity theory in complex geometry because it is an example of a complicated singular set for which the surface \( \Sigma_Q \) can still be given a basis of one forms. These one forms could also potentially be used to define an idea of a holomorphic line bundle on \( \Sigma_Q \).

The conjectures discussed at the end of this section also have evidence in connections between solutions to the nonlocal dbar problem as solutions to the KP equation.

3. THE (2+1)D KADOMTSEV–PETVIASHVILI, THE (1+1)D KORTEWEG–DE VRIES, AND THE (1+1)D SCHRÖDINGER EQUATION.

Define the phase function
\[ \psi(s, x, y, t) = sx + \alpha s^2 y - 4s^3 t. \]

To produce a solution to the KP equation
\[ (u_t - 6uu_x + u_{xxx})_x + \alpha^2 u_{yy} = 0. \]
we simply need to assume
\[ r_1(s) = e^{\Phi(s, x, y, t) - \psi(s, x, y, t)} \tilde{r}_1(s), \quad r_2(s) = e^{\Phi(s, x, y, t) - \psi(s, x, y, t)} \tilde{r}_2(s) \]
which is a simple implication of [8]. It is not clear if this solution is nonsingular and real. However, if we were to assume
\[ \tilde{r}_1(s) \geq 0, \quad \tilde{r}_2(s) \leq 0 \]
are supported on some positive intervals and \( \Phi(\lambda) = -\lambda \), then these solutions reduce to primitive solutions to the KdV equation, which are real, smooth and bounded [3, 7, 10, 4].

For the choices
\[ r(s) \geq 0, \quad r_1(s) \geq 0, \quad r_2(s) \leq 0 \]
supported on
\[ Q = C \subset \mathbb{R}^+, \]
and \( \Phi(\lambda) = -\lambda \), this construction gives a solution to the KdV equation. This reasoning is discussed in detail in [12, 4, 7, 8]. For most other choice of embedding of \( C \) into \( \mathbb{C} \), the construction will produce solutions to the KP equation. For \( Q = S \), the construction will always produce a solution to the KP equation rather than the KdV equation since \( S \) can not be restricted to a one dimensional set.

Consider a solution in the case of \( C \) and a solution \( u(x, t) \) to the KdV equation. Let us consider the \( t \) dependent family of one dimensional Schrödinger operators
\[ \hat{H}(t) = -\partial_x^2 + u(x, t). \]
The (energy) spectrum \( \sigma(\hat{H}) = \sigma(\hat{H}(t)) \) is constant in \( t \) and
\[ \sigma(\hat{H}) = \{ E = \lambda^2 : \lambda \in \mathbb{R}, \ -i\lambda \in C, \ or \ i\lambda \in C \}. \]
In other words, we can produce a potential with the above spectrum, and an explicit basis of (generalized) eigenfunctions of $\hat{H}$. This is precisely the information we need to explicitly construct the spectral projection operators.

We can use intuition from finite and infinite gap theory and the trace formula to make some conjectures on the behavior of the solutions to the KP equation, the KdV equation and the inverse spectral theory of one dimensional Schrödinger operators:

First, it seems likely that such a solution would either be quasi-periodic, or asymptotic to a quasi-periodic solution given the results in [6, 5]. It also seems likely that by associating gaps to the intervals $C_n$, computing finite gap solutions, and then taking an infinite gap limit, would lead to an equivalent constructions of some solutions/potentials in the isospectral set of $u(x, 0)$. Due to the fact that that gaps in the spectrum occurs at essentially all length scales smaller than the smallest interval $I$ such that $C \subset I$, it is likely that the resulting solution has interesting multi-soliton interactions at all length scales. These solutions to the KdV equation could therefore potentially exhibit complicated universal soliton gas dynamics.

Second, while the connection to spectral gaps is currently not clear for the Sierpinski surface, it still may be true that the Sierpinski surface discussed in this paper leads to solutions to the KP equation that have interesting multi-soliton interaction at all length scales. Further study into the Sierpinski surface and the nonlocal dbar problem could potentially lead to complicated soliton gasses when the support of the soliton spectrum is homeomorphic to $C$. Even if soliton gasses of this type turn out to not be physically relevant, they may still be interesting as a completely solvable model of a soliton gas.

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