Strong Pseudo Transitivity and Intersection Graphs

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Abstract

A directed graph $G = (V, E)$ is strongly pseudo transitive if there is a partition $\{A, E - A\}$ of $E$ so that graphs $G_1 = (V, A)$ and $G_2 = (V, E - A)$ are transitive, and additionally, if $ab \in A$ and $bc \in E$ implies that $ac \in E$. A strongly pseudo transitive graph $G = (V, E)$ is strongly pseudo transitive of the first type, if $ab \in A$ and $bc \in E$ implies $ac \in A$. An undirected graph is co-strongly pseudo transitive (co-strongly pseudo transitive of the first type) if its complement has an orientation which is strongly pseudo transitive (co-strongly pseudo transitive of the first type). Our purpose is show that the results in computational geometry [2, 7] and intersection graph theory [8, 6] can be unified and extended, using the notion of strong pseudo transitivity. As a consequence the general algorithmic framework in [11] is applicable to solve the maximum independent set in $O(n^3)$ time in a variety of problems, thereby, avoiding case by case lengthily arguments for each problem.

We show that the intersection graphs of axis parallel rectangles intersecting a diagonal line from bottom, and half segments are co-strongly pseudo transitive. In addition, we show that the class of the interval filament graphs is co-strongly transitive of the first type, and hence the class of polygon circle graphs which is contained in the class of interval filament graphs (but contains the classes of chordal graphs, circular arc, circle, and outer planar graphs), and the class of incomparability graphs are strongly transitive of the first type. For class of chordal graphs we give two different proofs, using two different characterizations, verifying that they are co-strongly transitive of the first type. Furthermore, we note that the class of co-strongly pseudo transitive graphs of the first type is properly contained in the class co-pseudo transitive graphs, unless $P = NP$, and the class of tree filament graphs is properly contained in the class co-pseudo transitive graphs, unless $P = NP$. Computational consequences are presented.
1 Introduction

“The first genuine monograph on graph theory (König, 1936) had the following subtitle: Combinatorial Topology of Systems of Segments [5]. Although graph theory and topology stem from the same root, the connection between them has somewhat faded away in the past few decades. In the most prolific new areas of graph theory including Ramsey theory, extremal graph theory and random graphs, graphs are regarded as abstract binary relations rather than systems of segments. It is quite remarkable that traditional graph theory is often incapable of providing satisfactory answers for the most natural questions concerning the drawings of graphs” [10].

Recall that a directed graph $G = (V, E)$ is transitive if $ab \in E$ and $bc \in E$ implies $ac \in E$. One can view a transitive graph as an alternate way of defining a partial order [13]. By dropping the orientation on the edges of a transitive graph we obtain a comparability graph. The complement of a comparability graph is an incomparability graph. In [11] we introduced the concepts of pseudo transitivity and strong pseudo transitivity which are ways of generalizing the concept of transitivity in graphs. The motivation behind writing this paper arose from recent results in computational geometry [2, 7] and intersection graphs theory [6], [8] that can be unified and generalized using the concept of strong pseudo transitivity.

Let $S$ be a finite set and $\{S_1, S_2, ..., S_k\}$ be a collection of subsets of $S$. The intersection (overlap) graph of $S$ is a graph with the vertex set $S$, where $S_i \cap S_j \neq \emptyset$ (for $i \neq j$, $S_i \not\subseteq S_j$, $S_j \not\subseteq S_i$).

Many intersecting graphs are the intersection or the overlap graphs of combinatorial or geometric structures. For instance incomparability graphs are intersection graphs of $x$–monotone curves intersecting two lines parallel to the $y$ axis [4], overlap graphs are overlap graphs of intervals on a line, chordal graphs are the intersection graph of subtrees of a tree [9], polygon-circle graphs are the intersection graphs of convex polygons in a circle, Interval filament graphs are the intersection graphs of filaments over intervals [8] and, subtree filament are intersection graphs of filaments over trees [8], or overlap graph of subtrees of a tree [6].

A directed graph $G = (V, E)$ is pseudo transitive if there is a partition $\{A, E - A\}$ of $E$ so that graphs $G_1 = (V, A)$ is transitive, and additionally, if $ab \in A$ and $bc \in E$ implies that $ac \in E$. A directed graph $G = (V, E)$ is strongly pseudo transitive if there is a partition $\{A, E - A\}$ of $E$ so that graphs $G_1 = (V, A)$ and $G_2 = (V, E - A)$ are transitive, and additionally, if $ab \in A$ and $bc \in E$ implies that $ac \in E$. Let $G = (V, E)$ be pseudo transitive (pseudo transitive) with the underlying partition $\{A, E - A\}$, then $G$ is pseudo transitive of the first type (pseudo transitive of the first type) if $ab \in A$ and $bc \in E$ implies that $ac \in A$. An undirected graph $G$ is co-pseudo transitive (co-pseudo transitive of the first type) if the complement of $G$ has
an orientation that is pseudo transitive (pseudo transitive of the first type one. $G$ is co-strongly pseudo transitive (co-strongly transitive of the first type), if the complement of $G$ has an orientation which is strongly pseudo transitive (co-strongly transitive of the first type). Co-pseudo transitive graphs contain the intersection graph of many geometric structures. For instance, the following result was shown in [11], with a slightly altered language.

**Theorem 1.1** Let $P$ be a finite collection of bounded closed subsets of $R^k$, then the intersection graph of $P$ is co-pseudo transitive of the first type.

A half segment is a straight line segment that has one end point on the $x$-axis, another end point in the upper half plane, and makes an acute angle with $x-$axis. Motivation behind introducing these segments arose from the work of Pach and Torocsik [14] on geometric graphs. Biro and Trotter [1] studied properties of partial orders arising from half segments. Computing the maximum independent set in the intersection graph of a set of rectangles is a fundamental problem arising in map labeling [15]. Since the general version of this problem is known to be NP-hard, some researchers have focused to solve the special version of the problem, including the instances where all rectangles are intersected by a diagonal line from below. See the work of Lubiw [7], and Correa, Feuilloley, Perez-Lantero, and Soto [2], which provide an $O(n^3)$ and $O(n^2)$ time algorithm, respectively.

In this paper we explore the connections between the graph classes mentioned above and the class of co-strongly pseudo transitive graphs. Specifically we show that the intersection graphs of half line segments and axis parallel rectangles intersecting a diagonal line from bottom are co-strongly pseudo transitive. Moreover, we show that the class of the interval filament graphs is co-strongly transitive of the first type, and hence the class of polygon circle graphs which is contained in the class of interval filament graphs (but contains the classes of chordal, circular arc, circle, and outer planar graphs), and the class of incomparability graphs are strongly transitive of the first type. For the class of chordal graphs which is contained in the class of polygon circle graphs, we provide two direct direct proofs, showing that they are co-strongly transitive of the first type. Additionally, we present some results concerning the Containment of different classes. A contribution of our work is to connect and unify the problems in computational geometry [2, 7], intersection graph theory [8, 6] and combinatorics [1] using the notion of strong pseudo transitivity, thereby, showing they are all amenable to the algorithmic framework in [11] for solving the maximum independent set in $O(n^3)$ time, thereby, avoiding case by case or lengthy arguments for each scenario.
2 Structural Results

**Theorem 2.1** Let $R$ be a set of axis parallel rectangles in the plane all of them are intersected by a diagonal line $l$ with the property that if two elements of $R$ intersect, then they also intersect below $l$. Let $G = (V, E)$ be the intersection graph of $S$, then $G$ is co-strongly pseudo transitive.

**Proof.** For any $C \in R$, let $x_C$ and $y_C$ denote the smallest $x$ coordinate, and smallest $y$ coordinate of four corners of $C$, respectively. Now let $W, Z \in R$ with $WZ \notin E$, so that $x_W \leq x_Z$ (the case $x_W > x_Z$ is symmetric). If $y_W \leq y_Z$, then, orient $WZ$ from $W$ to $Z$, and place it in $A$. Otherwise if $y_W > y_Z$, then, still orient $WZ$ from $W$ to $Z$ but place it in $B$. It can be verified that $A \cap B = \emptyset$, and that any non edge of $G$ has a orientation in $\hat{E} = A \cup B$. Furthermore, it can be shown that this case, the directed acyclic graph $H = (V, \hat{E})$ is strongly pseudo-transitive, with the partition $\{A, \hat{E} - A\}$. □.

The following result was mentioned in [11] without an explicit proof. Next, we specifically state and prove it.

**Theorem 2.2** Let $R$ be a set of half segments in the plane, and let $G = (V, E), V = R$ be the intersection graph of $R$, then $G$ is co-strongly pseudo transitive.

**Proof.** For any $r \in R$, let $x_1(r)$ and $x_2(r)$ denote the smallest $x$ coordinate, and the largest $x$ coordinate of any points in $r$. Now let $r, s \in R$ so that $rs \notin E$ so that $x_1(r) < x_1(s)$. If $x_2(s) \geq x_2(r)$, then orient $rs$ from $r$ to $s$ and place it in $A$. Otherwise, if $x_2(s) < x_2(r)$ (note the assumption $rs \notin E$), then still orient $rs$ from $r$ to $s$, but place the directed edge $rs$ in $B$. The remaining of the proof copies previous the theorem. □.

**Theorem 2.3** Let $G = (V, E)$ be an interval filament graph, then $G$ is co-strongly pseudo transitive of the first type.

**Proof.** Consider a representation of $G$, where $I$ is a set of intervals on the real line, and for each $i \in I, C_i$ is a collection of curves on the half plane above $i$ that connects the end points of $i$. Then $V = \cup_{i \in I} C_i$ and furthermore $xy \in E$, if $x, y \in C_i$ for some $i \in I$, and $x$ and $y$ intersect. Now let $x, y \in V, xy \notin E$. If $x \in C_i, y \in C_j, i < j$, then orient $xy$ from $x$ to $y$ and place $xy$ in $A$. Otherwise $x, y \in C_i$ for some $i \in I$. Since $x$ and $y$ do not intersect and connect the endpoint for interval $i$, the area under one of them (say $x$) contains the area under the other (say $y$). In this case orient $xy$ from $x$ to $y$ and place it in $B$. Note that $A \cap B = \emptyset$, every $xy \notin E$ has an orientation in $\hat{E} = A \cap B$. Furthermore the directed acyclic graph $H = (V, \hat{E})$, is strongly pseudo transitive, and for any $xy \in A$ and $yz \in \hat{E}$, we have $xy \in A$. □.
Conjecture 2.1 The class of interval filament graphs is properly contained in the class co-strongly pseudo transitive of the first type.

Since chordal graphs play an important role in graph theory, we give two different direct proofs, based on different representations, showing that they are co-strongly pseudo transitive. The first proof uses the characterization that every chordal graph is the intersection graph of subtrees of a tree, where, the second assumes a perfect elimination ordering is given.

Let \( L = v_1, v_2, \ldots, v_n \) be a perfect elimination ordering (PEO) of a chordal graph \( G = (V, E) \). A canonical depth first search tree of \( G \) (with respect to \( L \)) is a depth first search spanning tree rooted at \( v_1 \) constructed applying the following simple rule for visiting vertices: Assume vertex \( v_i \) is currently visited, then, select the next vertex to visit to be the smallest indexed vertex \( v_j \), \( j > i \) among all unvisited vertices adjacent to \( v_i \) in \( G \).

Theorem 2.4 Every chordal graph \( G = (V, E) \) is co-strongly pseudo transitive of the first type.

**First Proof.** Let \( T \) be a tree, let \( V = \{T_1, T_2, \ldots, T_k\} \) be a set of subtrees of \( T \). Assign a root \( r \) to \( T \), embed \( T \) in the plane, and assign root \( r_i \) to each \( T_i \in V \), which is the closest vertex of \( T_i \) to \( r \). Let \( G = (V, E) \) be the intersection graph of these subtrees, let \( \bar{G} = (V, \bar{E}) \) be the complement of \( G \). To prove the claim we will show there is a suitable orientation on \( \bar{E} \). For any \( e = T_i T_j \in \bar{E} \) so that \( r_i \) is not an ancestor of \( r_j \) in \( T \), and \( r_j \) to the left of \( r_i \) in the planar embedding of \( T \), orient \( e \) from \( T_i \) to \( T_j \) and place the resulting oriented edge in \( A \). For any \( e = T_i T_j \in \bar{E} \) so that \( r_i \) is an ancestor of \( r_j \) in \( T \) orient \( e \) from \( T_i \) to \( T_j \) and place the resulting oriented edge in \( B \).

It can be verified that \( A \cap B = \emptyset \), every \( e \in \bar{E} \) has an orientation in \( \bar{E} = A \cap B \), and that directed acyclic graph \( H = (V, \bar{E}) \), is strongly pseudo transitive, thus verifying the claim. Moreover, in this case \( xy \in A \) and \( yz \in \bar{E} \) implies \( xy \in A \).

**Second Proof.** Let \( L = \{v_1, v_2, \ldots, v_n\} \) be a PEO of \( G \). Let \( T \) be a canonical depth first search spanning tree of \( G \) rooted at \( v_1 \). Let \( \bar{e} = xy \notin E \) with \( df s(x) < df s(y) \). If \( x \) and \( y \) are on two different branches of \( T \) then orient \( \bar{e} \) from \( x \) to \( y \) and place \( xy \in A \). Otherwise, \( x = v_i \) and \( y = v_j \) are on the same branches of \( T \). Observe in this case that we must have \( i < j \). Since \( T \) is a canonical depth first tree, and, orient \( \bar{e} \) from \( x \) to \( y \) and place \( xy \in B \).

**Claim.** Let \( v_i, v_j, v_k, i < j < k \) be three vertices on the same branch of \( T \). If \( v_i v_j \notin E \), and \( v_j v_k \notin E \), then \( v_i v_k \notin E \).

It can verified (using the claim and properties of \( T \)) that \( A \cap B = \emptyset \), every \( e \in \bar{E} \) has an orientation in \( \bar{E} = A \cap B \), and that directed acyclic graph
\( H = (V, \hat{E}) \) is strongly pseudo transitive. Additionally, for any \( xy \in A \) and \( yz \in \hat{E} \), we have \( xy \in A \). □

We finish this section by establishing some containment properties.

**Theorem 2.5**

(i) The class of co-pseudo (co-strongly pseudo) transitive graphs of the first type is contained in the class of co-pseudo (co-strongly pseudo) transitive graphs.

(ii) The class of co-strongly pseudo transitive graphs of the first type is properly contained in the class co-pseudo transitive graphs of the first type, unless \( P = NP \).

(iii) The class of tree filament graphs is contained in the class co-pseudo transitive graphs.

(iv) The class of tree filament graphs is properly contained in the class co-pseudo transitive graphs, unless \( P = NP \).

**Proof.** Clearly (i) holds. For (ii), first note that by Theorem 1.1 the intersection graph \( G \) of a set of rectangles is co-pseudo transitive of the first time. Next note that computing the maximum independent set of \( G \) is NP-hard, but can be done in \( O(n^3) \) time for any co-strongly pseudo transitive graph. We omit proof of (iii). (iv) follows that graphs of boxicity two are co-pseudo transitive and computing their maximum independent set is known to be NP-hard, but computing maximum independent set in tree filament graphs can be done in polynomial time. □.

### 3 Algorithmic Consequences

The following result was shown in [11]

**Theorem 3.1** Let \( H = (V, F) \) be strongly pseudo-transitive. The maximum weighted chain can be computed in \( O(\sum_{x \in V} \deg^2(x) + n^2) \).

Using our notations the above theorem implies.

**Theorem 3.2** Let \( G = (V, E) \) be co-strongly pseudo-transitive. The maximum weighted independent set can be computed in \( O(n^3) \).

The above theorem implies the following general result.

**Theorem 3.3** Let \( G = (V, E) \) be one of the following graphs (i) incomparability (ii) overlap, (iii) chordal, (iii) polygon circle (iv) interval filament (v), or, (vi) intersection graph half segments, (vii) intersection graph axis parallel rectangles intersected by a diagonal line. Then, the maximum weighted independent set can be computed in \( O(n^3) \).
Note that our general framework extends the work of Lubiw [7] who showed the weighted maximum independent set of rectangles all which have their right most corner on a line can be computed in $O(n^3)$ time, but gives weaker result than a more recent work of Correa, Feuilloley, Perez-Lantero, Soto [2] that had $O(n^2)$ time complexity.

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