Operator product expansion in static-quark effective field theory: large perturbative correction

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Abstract. We calculate the coefficients of operators with dimensions $d \leq 7$ in the operator product expansion of correlators of $\bar{q}\Gamma Q$ currents, for the effective field theory of an infinite-mass quark, $Q$. Exact two-loop results are obtained, with an arbitrary gauge group and spacetime dimension, for the perturbative ($d = 0$) and quark-condensate ($d = 3$) contributions, confirming our previous result for the anomalous dimension of the current. Leading-order results are given for light-quark operators with $d = 5, 6, 7$ and gluon operators with $d = 4, 6$. The existence of a perturbative correction of order 100% precludes a reliable determination of $f_B$ from non-relativistic sum rules.

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1 Introduction

The effective field theory (EFT) of a static quark \cite{1,2} considerably simplifies the QCD analysis of hadrons containing a single heavy quark. In particular, mesons in EFT are analogous to the hydrogen atom in QED, rather than to positronium. To leading order in \(1/M\), EFT possesses a heavy-quark spin symmetry \cite{3}, enabling hadrons to be classified according to the angular momentum and parity, \(j^\pi\), of light fields. The \(Q\bar{q}\) ground state with \(j^\pi = \frac{1}{2}^-\) (S-wave antiquark) gives \(0^-\) and \(1^-\) mesons with identical properties \(\left(m_B = m_{B^*}, f_B = f_{B^*}\right)\). Excited states with \(j^\pi = \frac{1}{2}^+\) and \(\frac{3}{2}^+\) (\(P\)-wave antiquark) give degenerate pairs of mesons \(0^+, 1^+\) and \(1^+, 2^+\). The splitting in each pair (i.e. the hyperfine structure) is of order \(1/M\).

In QCD, however, even such a simplified problem requires nonperturbative methods, such as lattice simulation \cite{4}. An alternative method is provided by sum rules \cite{5} based on the operator product expansion (OPE).

In order to investigate meson properties in EFT, we consider correlators of bilinear currents, \(\bar{J} = \bar{q}\Gamma\bar{Q}\), where the tildes distinguish EFT quantities from those of conventional QCD and the static-quark field satisfies \(\bar{Q} = \gamma_0\bar{Q}\). The irrelevance of \(\gamma_0\) reduces the time-like component of the (pseudo)vector current to the (pseudo)scalar current; \(\sigma_{ij}\) to \(\gamma_i\); \(\epsilon_{ijk}\gamma_k\gamma_5\). There are thus four such currents in EFT, with \(\Gamma = \gamma_5, \gamma_i\) and \(\Gamma = 1, \gamma_5\). The quantum numbers of the first pair, with \(\Gamma\) anticommuting with \(\gamma_0\), are those of the ground-state \(j^\pi = \frac{1}{2}^-\) mesons; the quantum numbers of second pair, with \(\Gamma\) commuting with \(\gamma_0\), are those of the excited-state \(j^\pi = \frac{1}{2}^+\) mesons. Currents with quantum numbers of other mesons necessarily involve derivatives.

In the \(\overline{\text{MS}}\) scheme, the renormalized current \(\bar{J}(\mu)\) is obtained from the bare current \(\bar{J}_0 = \bar{q}_0\Gamma\bar{Q}_0\), in \(D = 4 - 2\varepsilon\) dimensions, by the multiplicative renormalization \(\bar{J}(\mu) = \bar{Z}_J \bar{J}_0(\mu)\bar{J}_0\), with \(\mu^2 = \mu^2 e^\gamma/4\pi\). The renormalization constant \(\bar{Z}_J\) was calculated at the one-loop level in \cite{6} and to two loops in \cite{7,8}. The EFT currents \(\bar{J}\) are related to the corresponding QCD currents \(J\) at the one-loop level by \cite{4}

\[
\bar{J}(\mu) = J(\mu) \left[ 1 + C_F \frac{\alpha_s(\mu)}{4\pi} \left( \frac{H^2 - 10}{4} \log \frac{M^2}{\mu^2} - \frac{3}{4} H^2 + H H' + \frac{1}{2} H + 4 \right) \right] + O \left( \frac{1}{M} \right) + O \left( \frac{1}{\Lambda^2} \right),
\]

where \(\gamma_\mu \Gamma \gamma_\mu = H \Gamma, H' = \partial H/\partial D\), and in this and all subsequent equations we adopt the convention that the upper sign corresponds to \(\Gamma\) anticommuting with \(\gamma_0\) and the lower sign to the commuting case. Each EFT current, \(\bar{q}\Gamma\bar{Q}\), is related to two QCD currents, \(\bar{q}\gamma_i\gamma_5\gamma_k\gamma_0\bar{Q}\) and \(\bar{q}\gamma_\mu\gamma_5\gamma_k\gamma_0\bar{Q}\), by formula \cite{4}, which we have verified by methods simpler than those in \cite{4}, using dimensional regularization of infrared singularities, as in \cite{5}.

In co-ordinate space, the correlator \(i<T\bar{J}(x)\bar{J}^\dagger(0)>\) is proportional to \(\delta(x)\). Its Fourier transform

\[
\tilde{\Pi}(q_0) = i \int dx \ e^{i q x} <T\bar{J}(x)\bar{J}^\dagger(0)> \tag{2}
\]

therefore depends only on the energy \(q_0\) and is given by the dispersion relation:

\[
\tilde{\Pi} (\omega) = \int_0^\infty \frac{\tilde{\rho}(\omega') d\omega'}{\omega' - \omega} - \cdots,
\]
where the dots denote a quadratic subtraction polynomial. The contribution of the lowest meson, \( M \), is
\[
\tilde{\rho}_M(\omega) = \tilde{f}_M^2 \delta(\omega - \omega_M),
\]
where \( |<0|\tilde{f}(\mu)|M>| = \tilde{f}_M(\mu) \) is its coupling, and \( \omega_M \) its binding energy. Note that the EFT meson state \( |M> \) lacks the relativistic normalization factor \( \sqrt{2E} \) of QCD and that \( \omega \) and \( \omega_M \), like all energies in EFT, are measured relative to the pole mass \( M \) of the heavy quark.

The correlator (2) has the structure
\[
\Pi(\omega) = \text{Tr} \left[ \frac{1 + \gamma_0}{2} \Gamma \{ \gamma_0 A(\omega) + B(\omega) \} \right] = 2 \left\{ A(\omega) \mp B(\omega) \right\},
\]
(3)
where \( \Gamma = \gamma_0 \Gamma^\dagger \gamma_0 \) and \( A \) and \( B \) derive from operators with even and odd dimensions, respectively. Thus the correlators of \( \bar{q}\gamma_5 \tilde{Q} \) and \( \bar{q}\gamma_i \gamma_5 \tilde{Q} \) coincide, yielding the same ground-state sum rule. Similarly, the correlators of \( \bar{q}Q \) and \( \bar{q}\gamma_i \gamma_5 Q \) coincide, yielding an excited-state sum rule obtained by a change of sign of the odd-dimensional contributions.

Mesons with a single heavy quark have been considered in applications of nonrelativistic [10, 11], relativistic Borel-transform [12, 13, 14], and moment-method [15, 16, 17] QCD sum rules. (A comparison is made in [18].) EFT sum rules are similar to nonrelativistic [10] sum rules, but in the QCD case the perturbative corrections are plagued by powers of a large hybrid logarithm: \( \log(M/\omega) \). In EFT these are summed, \( \text{ab initio} \), by the introduction of a new anomalous dimension, leading to well defined radiative corrections, whose size has so far not received due attention. Here we remedy that state of affairs.

In Sections 2 and 3 we calculate the perturbative \((d = 0)\) and quark-condensate \((d = 3)\) contributions, up to two loops, using the method proposed in [7]. These contributions should dominate the sum rule. In the Section 4 we give higher-dimensional contributions, up to \( d = 7 \), to leading order. A Borel-transform EFT sum rule is derived in Section 5 and conclusions are drawn from a numerical investigation. Throughout, we omit \( 1/M \) corrections, though these may be systematically included. We believe that a proper account of the radiative corrections in the \( M \to \infty \) limit is required first, in order to determine whether the “QCD hydrogen atom” is amenable to sum-rule analysis. Our conclusion has importance for any more realistic analysis of the B meson.

2 Two-loop perturbative contribution

In [7] we gave a method for calculating two-loop EFT diagrams, with one external momentum and zero light-quark mass, and implemented it as a REDUCE [19] package that evaluates these diagrams in terms of the basic structures \( \Gamma_{1,2} \), where
\[
\Gamma_n \equiv \left[ (-2\omega)^{-2n} \Gamma(-\varepsilon)/(4\pi)^{D/2} \right] \Gamma(1 + 2n\varepsilon).
\]
Using this package, we evaluate the bare perturbative correlator, up to two loops:
\[
\Pi_0^{\text{pt}}(\omega) = -\frac{4N_c \omega^2}{1 - 2\varepsilon} \left[ \Gamma_1 + C_T g_0^2 \frac{1 - \varepsilon}{\varepsilon} \left( \frac{1}{1 - 2\varepsilon} \Gamma_2^2 - \frac{1 - \varepsilon/2}{1 - 4\varepsilon} \Gamma_2 \right) \right] + O \left( g_0^4 \right).
\]
(4)

For the sum rule, we need the spectral density, which is easily found as the discontinuity of (4) divided by \( 2\pi i \). For the sake of reliability, we have also derived
MS mass. For the spectral results from [13, 20] by: transforming the heavy-quark sign of a quark mass in scalar and vector results. We may therefore obtain EFT each method, the renormalized spectral function \( \delta \)

\[
\overline{\rho}_0^{(1)}(\omega) = \frac{2 N_c (2 \omega)^{D-2}}{(4\pi)^{D/2}} \frac{\Gamma(1 - \varepsilon)}{(1 - 2\varepsilon)\Gamma(1 - 2\varepsilon)},
\]

\[
\delta_2 = \frac{C_F g_0^2 (-2\omega)^{-2\varepsilon} (1 - \varepsilon)\Gamma(1 + 2\varepsilon)\Gamma(1 - \varepsilon)}{\varepsilon^2(1 - 2\varepsilon)},
\]

\[
\delta_3 = \frac{2 C_F g_0^2 (2\omega)^{-2\varepsilon} (1 - \varepsilon)(1 - \varepsilon/2)\Gamma(1 - 2\varepsilon)\Gamma(1 - \varepsilon)}{\varepsilon^2(1 - 4\varepsilon)\Gamma(1 - 4\varepsilon)},
\]

where \( \overline{\rho}_0^{(1)} \) is the leading-order (one-loop) result, \( \delta_2 \) is the (complex) one-loop vertex correction [7] to the diagram with the two-particle cut, and \( \delta_3 \) is the contribution of three-particle cuts. Expressing \( g_0^2 \) in terms of \( \alpha_s(\mu) \) and using \( Z_J \) [4], we obtain, by each method, the renormalized spectral function

\[
\overline{\rho}^{pt}(\omega) = N_c \frac{\omega^2}{2\pi^2} \left[ 1 + 3 C_F \frac{\alpha_s(\mu)}{4\pi} \left( \frac{2\omega}{\mu} + \frac{4\pi^2}{9} + \frac{17}{3} \right) + O \left( \frac{\alpha_s^2(\mu)}{\mu^2} \right) \right],
\]

which we now compare with the results of more laborious QCD calculations.

The perturbative QCD (pseudo)scalar correlator, for any pair of quark masses, was calculated up to two loops by one of us in [20]. The (pseudo)vector correlator was found in [13], where the results of calculations of absorptive parts [21] were compared and corrected. (See also [16].) The pseudoscalar and pseudovector results were obtained in the scheme with \( \gamma_5 \) anticommuting with all \( \gamma_\mu \), by changing the sign of a quark mass in scalar and vector results. We may therefore obtain EFT results from [13, 20] by: transforming the heavy-quark \( \overline{\text{MS}} \) mass to the pole mass \( M \); using [3] in the anticommuting \( \gamma_5 \) scheme; taking the limit \( M \to \infty \), with \( \omega \) and \( m \) fixed, where \( q^2 = (M + \omega)^2 \) and \( m \) is the light-quark \( \overline{\text{MS}} \) mass. For the spectral function this gives

\[
\overline{\rho}^{pt}(\omega) = \frac{N_c \omega^2}{2\pi^2} \left[ 1 + \frac{C_F \alpha_s(\mu)}{4\pi} \left( \frac{6\omega^2 + 6\omega m - 18m^2}{\lambda^2} - 2 \log \frac{\omega^2}{\mu} + \frac{4\pi^2}{9} + \frac{17}{3} \right) + O \left( \frac{\alpha_s^2(\mu)}{\mu^2} \right) \right]
\]

\[
= \frac{N_c \omega^2}{2\pi^2} \left[ 1 + \frac{3 C_F \alpha_s(\mu)}{4\pi} \left( -2 \log \frac{\omega^2}{\mu} + \frac{4\pi^2}{9} + \frac{17}{3} \right) + O \left( \frac{\alpha_s^2(\mu)}{\mu^2} \right) \right]
\]

\[
= \frac{N_c \omega m}{2\pi^2} \left[ 1 + \frac{3 C_F \alpha_s(\mu)}{4\pi} \left( -4 \log \frac{\omega^2}{\mu} + \frac{4\pi^2}{9} + 8 \right) + O \left( \frac{\alpha_s^2(\mu)}{\mu^2} \right) \right]
\]

\[
- \frac{N_c m^2}{4\pi^2} \left[ 1 + \frac{3 C_F \alpha_s(\mu)}{4\pi} \left( -6 \log \frac{\omega^2}{\mu} + 2 \right) + O \left( \frac{\alpha_s^2(\mu)}{\mu^2} \right) \right] + O \left( m^3 \right),
\]

where \( \text{Li}(x) \equiv - \int_0^x dy \log(1 - y)/y \) and \( \lambda \equiv (\omega^2 - m^2)^{1/2} \). The agreement of [3] with [3], as \( m \to 0 \), confirms the matching conditions [4] for (pseudo)scalar and (pseudo)vector currents. The expansion of the full correlator beyond order \( m^2 \) involves \( \log m \) terms, which must be absorbed into quark condensates [13].
3 Two-loop $<\bar{q}q>$ contribution

One- and two-loop diagrams for the $<\bar{q}q>$ contribution in any dimension $D$ and in an arbitrary covariant gauge can be systematically evaluated by the method developed in [7]. We have calculated the 11 contributing one-particle-irreducible diagrams using our REDUCE package. Combining these with the propagator and vertex results of [7], we find a gauge-invariant $<\bar{q}q>$ contribution to the bare correlator, of the form

$$\Pi_0^3(\omega) = \pm \frac{<\bar{q}q>}{2\omega} \left[ 1 + P_1 \Gamma_1 C_F g_0^2 + \left\{ (P_{A1} C_A + P_{F1} C_F) \Gamma_1 \right. \right.$$  \\
$$+ \left. (P_{A2} C_A + P_{F2} C_F + P_{L2} T_F N_L) \Gamma_2 \right\} C_F g_0^4 + O \left( g_0^6 \right) \right], \quad (8)$$

with the following coefficients, for any gauge group:

$$P_1 = \frac{(D-1)(D-4)}{D-3}$$
$$P_{A1} = \frac{(D^3 - 8D^2 + 19D - 10)(D - 5)}{2(D - 3)^2(D - 4)}$$
$$P_{F1} = -\frac{2(2D - 9)(D - 2)}{(D - 3)(D - 4)}$$
$$P_{A2} = \frac{2D^7 - 51D^6 + 552D^5 - 3245D^4 + 11064D^3 - 21626D^2 + 22080D - 8808}{4(2D - 7)(D - 3)^2(D - 4)(D - 6)}$$
$$P_{F2} = -\frac{D^6 - 20D^5 + 167D^4 - 728D^3 + 1704D^2 - 1968D + 816}{2(D - 3)^2(D - 4)(D - 6)}$$
$$P_{L2} = -\frac{4(D - 2)(D - 4)}{(2D - 7)(D - 3)(D - 6)}.$$

The infinities in (8) relate the anomalous dimensions of $J$ and $<\bar{q}q>$, confirming our result [7] for $\tilde{\gamma}_J$. Using $\tilde{T}_J$, $Z_{\bar{q}q}$, and $Z_\alpha$, we obtain the finite renormalized result

$$\Pi^3(\omega) = \pm \frac{<\bar{q}q>}{2\omega} \left\{ \frac{\alpha_s(\mu)}{\pi} + \frac{3}{2} C_F \left[ \left( -\frac{7}{4} + \frac{\zeta(2)}{2} \right) C_A + \left( \frac{1}{2} - 2\zeta(2) \right) C_F + T_F N_L \right] \log \frac{2\omega}{\mu} \right. \right.$$  \\
$$+ \left. \left( \frac{149}{48} - \frac{3\zeta(2)}{8} + \frac{\zeta(3)}{2} \right) C_A + \left( \frac{11}{8} + \frac{5\zeta(2)}{2} - 2\zeta(3) \right) C_F - \frac{4}{3} T_F N_L \right\} + O \left( \alpha_s^3(\mu) \right), \quad (9)$$

In QCD, with $N_L = 4$, the ratio of next-next-to-leading to next-to-leading terms is large: $(-3.00 \log(2\omega/\mu) + 7.14)\alpha_s(\mu)/\pi$. This vanishes for $\mu = 0.185\omega$, which is too low a renormalization scale to use in EFT sum rules. However, the two-loop correction is not large by itself: about 15% for $\mu \sim 2\omega \sim 1$ GeV. We note that two-loop corrections to coefficient functions of quark condensates in light-quark QCD sum rules were calculated in [22], where the corresponding ratios were found to be even larger.
4 Higher-dimensional condensates

We have calculated the contributions of all quark and gluon condensates with \( d \leq 7 \), to leading order. They are most easily found \cite{23} in the fixed-point gauge, \( x_\mu A_\mu(x) = 0 \), in which the quark and gluon fields have gauge-covariant Taylor expansions and external gluons do not interact with the static quark. Using a systematic method \cite{24} to reduce vacuum expectation values to a minimal basis, we find the quark-condensate contributions

\[
\tilde{\Pi}^q(\omega) = \pm \frac{\langle \bar{q} q \rangle}{2\omega} + \frac{\langle \bar{q} g_{\mu\nu} i\sigma_{\mu\nu} q \rangle}{16\omega^3} - \frac{\langle \bar{q} J_\mu q \rangle}{96\omega^4} + \left[ 6\langle \bar{q} G_{\mu\nu} G_{\mu\nu} q \rangle - 3\langle \bar{q} G_{\mu\nu} \tilde{G}_{\mu\nu} i\gamma_5 q \rangle - 6\langle \bar{q} G_{\mu\nu} G_{\lambda\omega} \sigma_{\mu\nu} q \rangle \right] + O \left( \frac{1}{\omega^6} \right),
\]

(10)

where \( G_{\mu\nu} = gG_{\mu\nu}^{a} t^{a} \), \( \tilde{G}_{\mu\nu} \) is its dual, \( J_\mu = gJ_\mu^{a} t^{a} \), and \( J_\mu^{a} = D_\nu G_{\mu}^{a\nu} = g \sum_q q_{\mu\nu} t^{a} q \). The first three terms agree with \cite{10} and may also be confirmed by taking the \( M \rightarrow \infty \) limit of the relativistic results in \cite{12, 13}.

In the fixed-point gauge, no \( G^2 \) term from the light quark-propagator \( S(0,x) \) survives vacuum averaging. Hence there is no one-loop \( d = 4 \) gluon-condensate contribution in the non-relativistic limit. We obtain the \( G^3 \) contribution from the \( M \rightarrow \infty \) limit of the results of \cite{13} and neglect the \( J^2 \) term, which is commensurate with the unknown radiative correction to the \( d = 6 \) quark-condensate contribution in \cite{10}. The gluon-condensate contribution is thus:

\[
\tilde{\Pi}^g(\omega) = -\frac{g^2 f_{abc} G_{\lambda\mu}^{a} C_{\mu\nu}^{b} C_{\nu\lambda}^{c}}{4608\pi^2\omega^4} + O \left( \frac{1}{\omega^6} \right).
\]

(11)

5 Sum rule and conclusions

To obtain a sum rule, we specialize to the SU(3) gauge group of QCD, with \( C_A = N_c = 3 \), \( C_F = 4/3 \), \( T_F = 1/2 \), and \( N_L = 3 \) or 4 light flavours. From the two-loop anomalous dimension \cite{10} of \( \bar{J} \) and the two-loop \( \beta \)-function, we have

\[
\tilde{f}_M(\mu) = \hat{f}_M(\mu) s^{-2/3}(\mu) \left( 1 - K \frac{\alpha_s(\mu)}{\pi} + O \left( \frac{\alpha_s^2(\mu)}{\pi^2} \right) \right),
\]

(12)

\[
K = \frac{5}{12} - \frac{285 - 7\pi^2}{27b} + \frac{107}{2b^2}, \quad b = 11 - \frac{2}{3} N_L,
\]

where \( \hat{f}_M \) is a renormalization-group invariant. The radiative correction here is small: \( K = 0.189 \) or \( 0.227 \) for \( N_L = 3 \) or 4. The conventional QCD decay constants \( f_M \) are related by \cite{10} to \( \hat{f}_M \), as follows:

\[
f_M = \sqrt{\frac{2}{M}} \tilde{f}_M(M) \left( 1 - C \frac{\alpha_s(M)}{\pi} + O \left( \frac{\alpha_s^2(M)}{\pi^2} \right) \right) + O \left( \frac{1}{M^2} \right),
\]

(13)

where \( C = 4/3 \) for vector mesons and \( C = 2/3 \) for pseudoscalar mesons (if a fully anticommuting \( \gamma_5 \) is used). In analogy with \cite{12}, we also have

\[
\langle \bar{q} q \rangle = \langle \bar{q} q \rangle s^{-4/3}(\mu) \left( 1 - K' \frac{\alpha_s(\mu)}{\pi} + O \left( \frac{\alpha_s^2(\mu)}{\pi^2} \right) \right),
\]
\[ K' = \frac{5}{6} - \frac{34}{3b} + \frac{107}{b^2}, \]

where \( \langle \bar{q}q \rangle \) is a (negative) renormalization-group invariant.

We adopt the standard model of the continuum spectral density, setting it equal to the perturbative one, starting at an effective threshold \( \omega_c \). We use the vacuum factorization approximation [3] for \( d = 6 \) and \( d = 7 \) quark condensates. For the latter, this is unreliable but serves as a rough guide, to ensure that the contribution is kept small. The \( d = 5 \) quark condensate is denoted by \( m_0^2 \langle \bar{q}q \rangle \). We omit the two-loop term in (9); to use it consistently we would need the unknown three-loop term \( \gamma_J \). Applying a Borel transform [5, 10] to the correlator, we obtain

\[
\tilde{f}_M^2 e^{-\omega_M/E} \approx \frac{3E^3}{\pi^2} \left[ 1 - \left( 1 + \frac{\omega_c}{E} + \frac{\omega_c^2}{2E^2} \right) e^{-\omega_c/E} \right]
\]

\[ \times a^{4/b}(\mu) \left[ 1 + \frac{a_s(\mu)}{\pi} \left( -2 \log \frac{2\omega_c}{\mu} + 2L \left( \frac{\omega_c}{E} \right) + 2K + \frac{4\pi^2}{9} + \frac{17}{3} \right) \right] \]

\[ + \frac{\langle \bar{q}q \rangle}{2} \left[ 1 + \frac{a_s(\mu)}{\pi} \left( 2 - \frac{264 - 14\pi^2}{27b} \right) \right] - \frac{m_0^2}{16E^2} + \frac{\pi\langle \alpha_s G^a_{\mu\nu} G^b_{\mu\nu} \rangle}{288E^4}, \tag{14} \]

\[
L(x) = \log x + \gamma + E_1(x) + \frac{1}{2}(3 + x)e^{-x} - \frac{3}{2}
\]

\[
1 - \left( 1 + x + \frac{1}{2}x^2 \right) e^{-x}
\]

\[
= \begin{cases} 
\frac{1}{3} + \frac{1}{15}x + \frac{43}{1000}x^2 + O(x^3), & x \ll 1, \\
\log x + \gamma - \frac{3}{2} + O(e^{-x}x^2 \log x), & x \gg 1,
\end{cases}
\]

where the upper (lower) sign is for ground (excited) state mesons, \( \gamma \) is Euler’s constant, and \( E_1(x) = \int_x^\infty dy \; e^{-y}/y \).

The left-hand side of sum rule (14) is a renormalization-group invariant; so is the right-hand side, to the given order in \( a_s(\mu) \). Numerical investigation, over a suitable range of the Borel variable \( E \), reveals that all terms may be kept under control, save one. The glaring exception is the perturbative correction, which invariably exceeds \( -2 \log(2\omega_c/\mu) + 11.1\alpha_s(\mu)/\pi \). For \( \mu \sim \omega_c \sim 1 \) GeV, this is of order 100%, an order of magnitude greater than in the QCD sum rule for \( \rho \) and, in our opinion, far too large for one to have any confidence in the neglect of unknown three-loop terms. An alternative way to express this problem is to say that fastest apparent convergence, with a vanishing next-to-leading correction, occurs at a scale \( \mu < 0.008\omega_c \sim 10 \) MeV, an order of magnitude smaller than \( \Lambda_{\overline{MS}} \). The radiative correction to the quark-condensate contribution, on the other hand, is acceptably small: around 15%.

We conclude that non-relativistic sum rules for mesons with a heavy quark are intrinsically unreliable. Whilst EFT solves the hybrid log problem, it also leaves one with a unique perturbative correction whose numerical size precludes any reliable determination of \( \tilde{f}_M \) and hence any estimate of \( f_B \).

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