PHASELESS SIGNAL RECOVERY IN INFINITE DIMENSIONAL SPACES USING STRUCTURED MODULATIONS

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Abstract. This paper considers the recovery of continuous signals in infinite dimensional spaces from the magnitude of its frequency samples. It proposes a sampling scheme which involves a combination of oversampling and specific structured modulations. This allows for almost every signal with finite support to be reconstructed from its magnitude samples in the frequency domain up to a unimodular constant. Sufficient conditions on the signal and the sampling system are given such that signal recovery is possible, and it is shown that an average sampling rate of four times the Nyquist rate is enough to reconstruct almost every time limited signal.

1. Introduction

Signal reconstruction from magnitude measurements only is a necessary step in many different applications of engineering and science. It is well-known as the phase retrieval problem and originates from the fact that detectors can often only record the squared modulus of an electromagnetic wave but not its phase. One of its most important applications is during the image reconstruction stage in X-ray crystallography, widely used in material and biological science [21, 22]. There the phase of radiation, scattered from an object, carries information about the surface or the inner structure of this object. The phase retrieval problem also appears in numerous other areas of imaging science including electron microscopy, astronomical imaging, diffraction imaging, X-ray tomography, to mention only a few. Furthermore it plays an important role in other fields such as speech processing [24], radar [14], signal theory [12], quantum communication [10].

The underlying problem lies in the fact that in general magnitude and phase of a signal are independent. Therefore signal recovery from magnitude measurements is only possible if it is combined with additional information about the signal. If, for example, the signal is known to be causal or bandlimited, then the logarithm of the magnitude and its phase are related by the Hilbert transform which may be used to recover the phase from magnitude measurements [5, 25]. Also in the case where the z-transform satisfies certain sufficient conditions [12], the signal is completely determined by its magnitude.

When no or only little a priori knowledge is available on the original signal, one can still successfully perform reconstruction from magnitude measurements by

Date: 25. May 2013.

2000 Mathematics Subject Classification. Primary 30D10, 94A20; Secondary 42C15, 94A12.

Key words and phrases. Bernstein spaces, Interpolation, Phase retrieval, Sampling, Signal reconstruction.

This work was partly supported by the German Research Foundation (DFG) under Grant BO 1734/22-1.
taking several measurements of the same object under slightly different conditions. To this end various methods were proposed, such as a distorted-object approach by which the Fresnel diffraction pattern is measured at different distances [30], the usage of aperture-plane modulation [8, 33], or the recording of several fractional Fourier transforms [14]. Having obtained several measurements, signal recovery is mainly performed by iterative alternating projection algorithms [9]. Although these algorithms are usually easy to implement, they often come with problems in terms of convergence (see, e.g., [3, 20]), which strongly depends on specific signal constraints. Moreover, in connection with the described multiple measurement methods, there seems to be no systematic approach to design the different measurements such that iterative algorithms will definitely converge to the correct signal.

More recently, analytic investigations on the phase retrieval problem could derive sufficient conditions on the number of measurements such that a unique solution exists. In [2, 4] it was found that for an N-dimensional space slightly less than 4N suitably chosen amplitude measurements are sufficient. However, no corresponding reconstruction algorithm was given which achieves this bound. The closest result was presented in [1], where the authors proposed a method which guarantees signal recovery from amplitude measurements requiring N^2 measurements. This fundamentally limits its practicability to low dimensional spaces. Ideas of sparse signal representation and convex optimization were applied in [6, 19] to allow for lower computational complexity.

Note that all of the above approaches address finite dimensional signals. A natural follow-up question is whether similar results can be obtained for continuous signals in infinite dimensional spaces. In [28] it was shown that real valued bandlimited signals are completely determined simply by their magnitude samples taken at twice the Nyquist rate. It is not clear whether a similar result holds for complex valued signals, since results for finite dimensional spaces indicate that oversampling alone may not be sufficient. In [1, 2, 6], the particular choice of measurement vectors was the key to enabling signal recovery. The first attempt on complex valued time limited L^2-signals was made in [31] where recovery was guaranteed given specific amplitude measurements taken at four times the Nyquist rate. It provides a reconstruction algorithm which incorporates ideas from finite dimensional spaces in [1] and was inspired by the structured illuminations frequently used in optics [6, 8, 30, 33].

The present work extends the result of [31] to larger signal spaces, namely to finitely supported signals in L^p with 1 ≤ p ≤ ∞. As in [31], a bank of modulators is applied before the intensity measurement. We derive conditions on the sampling rate and the modulators such that generically (i.e. up to a meager set) every signal can be reconstructed, up to a unimodular constant, from samples taken at a rate which may be arbitrarily close to four times the Nyquist rate.

The paper is organized as follows. Basic notations and some preliminary results are presented in Section 2 whereas sampling and reconstruction in Bernstein spaces are recaptured in Sec. 3. Our sampling setup is described in Sec. 4. In particular two conditions on the sampling system are introduced which will enable signal recovery. Sec. 5 will prove that these two conditions are indeed sufficient for signal reconstruction from magnitude measurements, with the exception of a set of signals of first category. Sect. 6 shows that by a small change of the sampling system and
under mild restrictions on the signal space, even this limitation can be avoided. The paper closes with a short discussion in Sec. 7.

2. Notations and Preliminaries

2.1. Function spaces. Let $\mathcal{S} \subseteq \mathbb{R}$ be an arbitrary subset of the real axis $\mathbb{R}$. For any exponent $1 \leq p \leq \infty$ we write $\mathcal{L}^p(\mathcal{S})$ for the usual Lebesgue space on $\mathcal{S}$. The exponent $p'$ which satisfies $1/p + 1/p' = 1$ is the conjugate exponent of $p$ and by convention $p' = \infty$ is the conjugate exponent of $p = 1$. In particular, $\mathcal{L}^2(\mathcal{S})$ is the Hilbert space of square integrable functions on $\mathcal{S}$ where the bar denotes the complex conjugate. The set of continuous functions on $\mathbb{R}$ is denoted by $C^0(\mathbb{R})$ and its dual space, i.e. the set of all bounded linear functionals on $C^0(\mathbb{R})$, is equipped with the usual locally convex topology (see, e.g., [17, Chapter 20.1]) that there exists constants $C = C(p, \sigma)$ such that for all $x \in B^p_\sigma$ of all entire functions of exponential type $\leq \sigma$ whose restriction to the real axis belongs to $L^p(\mathbb{R})$. The norm in $B^p_\sigma$ is defined by the $L^p(\mathbb{R})$-norm on $\mathbb{R}$. In particular, $B^2_\sigma$ is the so called Paley-Wiener space. The theorem of Plancherel-Pólya implies (see, e.g. [17, Chapter 20.1]) that

$$\|x(\xi + i\eta)\| \leq C \|x\|_{L^2(\mathbb{R})} e^{\sigma|\eta|} \text{ for all } \xi, \eta \in \mathbb{R}.$$  

Consequently, $B^p_\sigma \subset B^\infty_\sigma$ for all $1 \leq p < \infty$ and convergence in $B^p_\sigma$ implies uniform convergence in each horizontal strip of the complex plane.

2.2. Schwartz space, distributions, and Fourier transform. The Schwartz space $\mathcal{S}$ of rapidly decreasing functions on the real axis $\mathbb{R}$, consists of smooth functions whose derivatives of all orders decay faster than any polynomial. It is assumed that $\mathcal{S}$ is equipped with the usual locally convex topology (see, e.g., [13,26]). Its dual space, i.e. the set of all bounded linear functionals on $\mathcal{S}$, will be denoted by $\mathcal{S}'$ and its elements are called tempered distributions. If $x \in \mathcal{S}'$ can be represented by a locally integrable function $\tilde{x} \in L^1_{\text{loc}}(\mathbb{R})$ such that

$$x(\phi) = \int_{\mathbb{R}} \tilde{x}(t) \phi(t) \, dt, \quad \phi \in \mathcal{S}$$

then $x$ is called regular. The support of a distribution $x \in \mathcal{S}'$ is the complement of the open set on which it vanishes, and $x$ is said to vanish on $V$ if $x(\phi) = 0$ for all $\phi \in \mathcal{S}$ with support in $V$.

For any $\phi \in \mathcal{S}$, the Fourier transform is the function $\hat{\phi} : \mathbb{R} \to \mathbb{C}$ defined by

$$\hat{\phi}(\omega) = (\mathcal{F}\phi)(\omega) = \int_{\mathbb{R}} \phi(t) e^{-i\omega t} \, dt, \quad \omega \in \mathbb{R}$$

and the inverse Fourier transform of any $\phi \in \mathcal{S}$ is

$$\phi(t) = (\mathcal{F}^{-1}\hat{\phi})(t) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{\phi}(\omega) e^{i\omega t} \, d\omega.$$ 

It is well known that $\mathcal{F} : \mathcal{S} \to \mathcal{S}$ is a continuous, injective mapping of $\mathcal{S}$ onto itself, and we have $x = \mathcal{F}^{-1}\mathcal{F} x$ for all $x \in \mathcal{S}$ (see [29]). Moreover, $\mathcal{F}$ can be extended to
any \( x \in \mathcal{S}' \) by defining its Fourier transform \( \hat{x} = \mathcal{F}x \) by \( \hat{x}(\phi) = x(\hat{\phi}) \) for all \( \phi \in \mathcal{S} \). Then \( \mathcal{F} : \mathcal{S}' \to \mathcal{S}' \) is a continuous, bijective mapping of \( \mathcal{S}' \) onto itself.

Of particular importance for us are the subspaces of regular distributions \( \mathcal{L}^p(\mathbb{R}) \subset \mathcal{S}' \). The Riesz-Thorin interpolation theorem implies that, if \( p' \) denotes the conjugate exponent of \( p \), then for \( 1 \leq p' \leq 2 \) the Fourier transform \( \mathcal{F} : \mathcal{L}^{p'}(\mathbb{R}) \to \mathcal{L}^p(\mathbb{R}) \) is bounded
\[
\| \mathcal{F}x \|_{\mathcal{L}^p} \leq \frac{1}{2\pi} \| x \|_{\mathcal{L}^{p'}}.
\]
However, apart from the case \( p' = 2 \), the range of \( \mathcal{F} : \mathcal{L}^{p'}(\mathbb{R}) \to \mathcal{L}^p(\mathbb{R}) \) is not the whole of \( \mathcal{L}^p(\mathbb{R}) \) and there exists functions \( \hat{x} \in \mathcal{L}^p(\mathbb{R}) \) whose inverse Fourier transform \( x = \mathcal{F}^{-1}\hat{x} \) is a tempered distribution \( x \in \mathcal{S}' \) which can not be identified with a function in \( \mathcal{L}^p(\mathbb{R}) \).

### 2.3. Fourier-Laplace transform and Paley-Wiener theorem

The classical Fourier transform in \( \mathbb{R} \) for \( \mathcal{S} \) can be extended to the whole complex plane by replacing the variable \( w = \mathbb{R} \) by \( z = \mathbb{C} \). Then \( \hat{x}(z) = (\mathcal{F}x)(z) \) defines an entire function. Similarly, we can define the so-called **Fourier-Laplace transform** on \( \mathcal{S}' \)
\[
\hat{x}(z) = x(e^{-z}) \quad \text{where} \quad e^{-z}(t) := e^{-izt} \in C^\infty(\mathbb{R})
\]
for every \( x \in \mathcal{S}' \) with compact support \([26, \text{Theorem 6.24.}]\). This expression is well defined because the space of test functions with compact support \( C^\infty_0(\mathbb{R}) \) is dense in \( C^\infty(\mathbb{R}) \). Since \( x \) itself also has compact support, \( \hat{x} \) is an entire function of \( z \). Note that the restriction of \( \hat{x} \) to \( \mathbb{R} \) is the Fourier transform of \( x \). The following theorem provides a relation between entire functions of exponential type and the Fourier-Laplace transform of tempered distributions with compact support. It is an extension of the classical Paley-Wiener theorem (see, e.g., \([13, 26]\)).

**Theorem 2.1** (Paley-Wiener). An entire function \( \hat{x}(z) \) is the Fourier-Laplace transform of a (tempered) distribution which has compact support in the interval \( I_\sigma = \{ t : |t| \leq \sigma \} \) if and only if there exists a constant \( C \) and an \( N \in \mathbb{N} \) such that
\[
|\hat{x}(\xi + i\eta)| \leq C (1 + |\xi + i\eta|)^N e^{\sigma|\eta|}, \quad \text{where} \quad z = \xi + i\eta.
\]
In particular, every entire function of exponential type \( \leq \sigma \) is the Fourier-Laplace transform of a (tempered) distribution with compact support contained in \( I_\sigma \).

### 3. Sampling and Reconstruction in Bernstein spaces

#### 3.1. Signal space

Let \( T > 0 \) be a real number. Throughout this paper \( \mathbb{T} := [-T/2, T/2] \) stands for the closed interval of length \( T \). Since our signals will be sampled in the frequency domain, we define our signal spaces in terms of its Fourier representation as follows: For any \( 1 \leq p \leq \infty \), our signal space will be the set
\[
\mathcal{Y}^p_{T/2} := \{ x \in \mathcal{S}' : \hat{x} = \mathcal{F}x \in \mathcal{B}^p_{T/2} \}
\]
of all tempered distributions whose Fourier-Laplace transform belongs to the Bernstein space \( \mathcal{B}^p_{T/2} \). In the following we call \( x \) the signal in the **time domain** and \( \hat{x} = \mathcal{F}x \) the signal in the **Fourier domain**. According to the Paley-Wiener Theorem \([21]\), every \( x \in \mathcal{Y}^p_{T/2} \) has a compact support contained in the interval \( \mathbb{T} \).

Note that for \( p > 2 \) our signal space \( \mathcal{Y}^p_{T/2} \) contains tempered distributions which are not regular (cf. Sec. \([22]\)). Conversely, since \( \mathcal{F} : \mathcal{L}^{p'}(\mathbb{R}) \subset \mathcal{S}' \to \mathcal{L}^p(\mathbb{R}) \) is injective for \( p' \in [1, 2) \), every \( x \in \mathcal{Y}^p_{T/2} \) is a function in \( \mathcal{L}^{p'}(\mathbb{T}) \) but \( \mathcal{Y}^p_{T/2} \) is not the
whole space $L^p(T)$ since there exists functions $x \in L^p(T)$ for which $Fx$ is not in $L^p(\mathbb{R})$ but only a tempered distribution.

3.2. Interpolating sequences. Let $\Lambda = \{\lambda_n\}_{n \in \mathbb{Z}}$ be a sequence of complex numbers and let $\hat{x} \in \mathcal{B}^p_{T/2}$ be arbitrary. Then the linear mapping $S_\Lambda : \hat{x} \mapsto \{\hat{x}(\lambda_n)\}_{n \in \mathbb{Z}}$ describes the sampling of $\hat{x} \in \mathcal{B}^p_{T/2}$ at the points $\Lambda$. We say that $\Lambda$ is complete interpolating for $\mathcal{B}^p_{T/2}$ if $S_\Lambda$ is an isomorphism between $\mathcal{B}^p_{T/2}$ and $\ell^p$.

**Definition 1.** A sequence $\Lambda = \{\lambda_n\}_{n \in \mathbb{Z}}$ of complex numbers is said to be complete interpolating for $\mathcal{B}^p_{T/2}$ if there exist two positive constants $A,B$ such that

$$A \|\hat{x}\|_{\mathcal{B}^p_{T/2}} \leq \|\{\hat{x}(\lambda_n)\}\|_{\ell^p} \leq B \|\hat{x}\|_{\mathcal{B}^p_{T/2}} \quad \text{for all } \hat{x} \in \mathcal{B}^p_{T/2}. $$

Equivalently, $\Lambda$ is complete interpolating for $\mathcal{B}^p_{T/2}$ if the above defined sampling operator $S_\Lambda : \mathcal{B}^p_{T/2} \to \ell^p$ is bounded and if the interpolation problem $\hat{x}(\lambda_n) = a_n$ for all $n \in \mathbb{Z}$ has a unique solution $\hat{x} \in \mathcal{B}^p_{T/2}$ for every sequence $\{a_n\}_{n \in \mathbb{Z}} \subseteq \ell^p$. We always assume that any interpolation sequence $\Lambda = \{\lambda_n\}_{n \in \mathbb{Z}}$ is ordered increasingly by the real parts of the $\lambda_n$.

In general, it is quite complicated to characterize complete interpolating sequences. However, for $\mathcal{B}^p_{T/2}$ with $1 < p < \infty$ a large class of interpolating sequences is known, namely the zero sets of so called sine-type functions (see, e.g., [17]):

**Definition 2.** An entire function of exponential type is said to be a sine-type function of type $\sigma$ if it has simple and separated zeros and if there exist positive constants $A,B,H$ such that

$$A e^{\sigma|\eta|} \leq |f(\xi + i\eta)| \leq B e^{\sigma|\eta|} \quad \text{for } |\eta| \geq H.$$ 

We also note that sequences which differ from an interpolation sequence derived from sine-type functions by their imaginary parts and which elements have separated real parts, are still zeros of sine-type functions [18]. Moreover, for the important Paley-Wiener space $\mathcal{B}^2_{T/2}$, Katsnelson’s theorem [16] states that if $\Lambda = \{\lambda_n\}_{n \in \mathbb{Z}}$ is the zero set of a sine-type function and $\delta_n := \inf_{k \notin n} |\lambda_n - \lambda_k|$ for all $n \in \mathbb{Z}$, then a set $\{\mu_n\}_{n \in \mathbb{Z}}$ which satisfies $|\mu_n - \lambda_n| < d \delta_n$ for all $n \in \mathbb{Z}$ and for some $0 < d < 1/4$ is again complete interpolating for $\mathcal{B}^2_{T/2}$.

Conversely, if the zero set $\Lambda = \{\lambda_n\}_{n \in \mathbb{Z}}$ of a sine-type function $S$ is known, then $S$ can be determined from $\Lambda$ by

$$S(z) = z^{\delta_\Lambda} \lim_{R \to \infty} \prod_{|\lambda_n| < R \atop \lambda_n \neq 0} \left(1 - \frac{z}{\lambda_n}\right)$$

with $\delta_\Lambda = 1$ if $0 \in \Lambda$ and $\delta_\Lambda = 0$ otherwise, and where the infinite product converges uniformly on each compact set in $\mathbb{C}$.

3.3. Signal reconstruction in the Fourier and time domain. If $\Lambda$ is the zero set of a sine-type function $S$ then the inverse mapping of the sampling operator $S_\Lambda : \hat{x} \mapsto \{\hat{x}(\lambda_n)\}_{n \in \mathbb{Z}}$ is explicitly known. More precisely, one has the following result [17 Lect. 22].
Lemma 3.1. Let \( \Lambda = \{\lambda_n\}_{n \in \mathbb{Z}} \) be the zero set of a sine-type function \( S \) of type \( T/2 \) and define
\[
\hat{\psi}_n(z) := \frac{S(z)}{S'((\lambda_n)(z - \lambda_n))}, \quad z \in \mathbb{C}.
\]
Then for each \( 1 \leq p < \infty \)
\[
\hat{x}(z) = \sum_{n \in \mathbb{N}} \hat{x}(\lambda_n) \hat{\psi}_n(z) \quad \text{for all } \hat{x} \in \mathcal{B}_{T/2}^p \tag{8}
\]
where the sum converges uniformly on each horizontal strip in the complex plane, and for \( 1 < p < \infty \) the sum \( \hat{x} \) also converges in the norm of \( \mathcal{B}_{T/2}^p \).

Remark 1. It follows in particular that the interpolation kernels \( \hat{\psi}_n \) themselves belong to \( \mathcal{B}_{T/2}^p \) for every \( 1 < p < \infty \). Consequently, \( \psi_n = F^{-1} \hat{\psi}_n \in \mathcal{L}^2(\mathbb{T}) \), and by the definition of \( \hat{\psi}_n \) it can be seen that \( \psi_n \in C_0(\mathbb{R}) \) for every \( n \in \mathbb{N} \).

Remark 2. Note that for \( p = 1 \) we have uniform convergence in the supremum norm because of \([2]\) though it does not converge in the \( \mathcal{B}_{T/2}^1 \) norm.

Lemma 3.2 does not hold for \( p = \infty \), i.e. the series \([9]\) does not converge uniformly on \( \mathbb{R} \) for every \( \hat{x} \in \mathcal{B}_{T/2}^\infty \). However, it is reasonable to consider regular distributions \( x \in \mathcal{Y}_{T/2}^\infty \). Then \( x \in \mathcal{L}^1(\mathbb{T}) \) such that \( \hat{x} \in C_0 \cap \mathcal{B}_{T/2}^\infty \). For such functions, it is known that \([9]\) converges if oversampling is applied. More precisely, the following result was proven in \([23]\).

Lemma 3.2. Let \( \Lambda = \{\lambda_n\}_{n \in \mathbb{Z}} \) be the zero set of a sine-type function \( S \) of type \( T'/2 \). If \( T < T' \) then for every \( \hat{x} \in C_0 \cap \mathcal{B}_{T/2}^\infty \) we have
\[
\lim_{N \to \infty} \max_{\omega \in \mathbb{R}} \left| \hat{x}(\omega) - \sum_{n=-N}^{N} \hat{x}(\lambda_n) \hat{\psi}_n(\omega) \right| = 0.
\]

Remark 3. In \([23]\) only the case of real zeros \( \lambda_n \) was treated. However, the corresponding proof can also be applied for complex zeros, using that all zeros of a sine-type function lie in a strip parallel to the real axis.

Lemma 3.3. Let \( \Lambda = \{\lambda_n\}_{n \in \mathbb{Z}} \) be the zero set of a sine-type function \( S \) of type \( T/2 \) and let \( \psi_n = F^{-1} \hat{\psi}_n \) be the inverse Fourier transforms of the functions defined in \([8]\). Then for each \( 1 \leq p < \infty \), we have
\[
x = \sum_{n \in \mathbb{N}} \hat{x}(\lambda_n) \psi_n \quad \text{for all } x \in \mathcal{Y}_{T/2}^p \tag{10}
\]
where the sum converges in the topology of \( \mathcal{S}' \).

Proof. Let \( \hat{x}_N := \sum_{n=-N}^{N} \hat{x}(\lambda_n) \hat{\psi}_n \), then \( \|\hat{x} - \hat{x}_N\|_{\mathcal{B}_{T/2}^p} \to 0 \) as \( N \to \infty \) for all \( \hat{x} \in \mathcal{B}_{T/2}^p \) by Lemma 3.1. Using \([4]\), it follows that
\[
\left| \int_{\mathbb{R}} [\hat{x}(\omega) - \hat{x}_n(\omega)] \phi(\omega) \, d\omega \right| \leq \|\hat{x} - \hat{x}_n\|_{\mathcal{B}_{T/2}^p} \int_{\mathbb{R}} |\phi(\omega)| \, d\omega \leq C_\phi \|\hat{x} - \hat{x}_n\|_{\mathcal{B}_{T/2}^p} \quad \text{for all } \phi \in \mathcal{S}
\]
with a constant $C_\phi$ which depends only on $\phi$, $p$, and $T$. Consequently, $\hat{x}_N \to \hat{x}$ in $S'$, and since $F$ is a continuous one-to-one mapping of $S'$ onto $S'$ (see, e.g., [29], §9) it follows that $x_N \to x$ in $S'$ for all $x \in \mathcal{Y}^p_{T/2}$.

**Remark 4.** Assume that $x \in \mathcal{Y}^p_{T/2}$ is a regular distribution, i.e. that $x \in L^p' (T)$. Then it follows from (11) that for $1 < p \leq 2$, the reconstruction series (10) converges even in the $L^p'$-norm to the desired signal $x \in \mathcal{Y}^p_{T/2}$. However, for $2 < p < \infty$, the series (10) need not converge in the norm of $L^p'$ even if $x \in L^p' (T)$.

**Remark 5.** Lemma 3.2 may also be formulated for all regular distributions in $\mathcal{Y}^\infty_{T/2}$. Then one has to suppose that $\Lambda$ is a the zero set of a sine-type function of type $T'/2 > T/2$ and one has to use Lemma 3.2 in the corresponding proof.

**Example 1.** The zeros of the sine-type function $S(z) = \sin\left(\frac{T'}{2}z\right)$ of type $T'/2$ are given by $\lambda_n = n \frac{T'}{2\pi}$, $n \in \mathbb{Z}$. The corresponding functions $\hat{\psi}_n(z)$ are then given by $\hat{\psi}_n(z) = \text{sinc}(\frac{T'}{2}[z - n \frac{2\pi}{T'}])$ where $\text{sinc}(x) := \sin(x)/x$. Then the reconstruction series (9) is known as Shannon sampling series, and it converges for all $\hat{x} \in \mathcal{B}^p_{T/2}$ with $1 < p < \infty$, provided that $T \leq T'$. The corresponding time-domain series (10) becomes $x(t) = \sum_{n \in \mathbb{Z}} \hat{x}(\lambda_n) e^{-in\frac{2\pi}{T'}t}$ for all $t \in T$.

### 4. Measurement Methodology

We consider signals in the signal space (5) and apply a measurement methodology which uses oversampling in connection with structured modulations of the desired signal. Suppose $x \in \mathcal{Y}^p_{T/2}$ is the signal of interest. Although the loss of phase information might be intrinsic to the measurement procedure, it is often possible to influence the desired signal before the actual measurement. In optical applications one may apply spatial light modulators for this purposes. An example of a corresponding measurement setup (see also, e.g., [6, 15, 33]) is sketched in Fig. 1. There the object of interest is illuminated by a coherent light source which produces a diffraction pattern $x(t)$ containing the information about the object. This diffraction pattern is modified by a spatial light modulator (a mask) with complex-valued spatial transmittance $p^{(m)}(t)$. In our setup we include $m = 1, 2, \ldots, M$ different masks with specific transmittance functions $p^{(m)}$. After spatial modulation, one

\[ \text{Figure 1. Schematic setup for structured modulation in optics using spatial light modulators (masks).} \]

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1 The variable $t$ stands here for the one or two dimensional spatial dimension.
obtains a diffraction pattern \( y^{(m)}(t) = x(t)p^{(m)}(t) \). This pattern is transformed into the frequency domain by the subsequent lens or diffraction imaging (see, e.g., [11]). There, the intensity is measured \( |\hat{y}^{(m)}(\omega)|^2 = |(\mathcal{F}y^{(m)})(\omega)|^2 \) and sampled at discrete points \( \omega_n = n\beta \). These measurements are repeated with different masks \( p^{(m)}, m = 1, \ldots, M \).

For the following explanations, we provide a more schematic illustration of the setup in Fig. 2. So the signal of interest \( x \) is multiplied with \( M \) known modulating functions \( p^{(m)} \). This way a collection of \( M \) representations (illuminations) \( y^{(m)} \) of \( x \) is obtained. Afterwards, the modulus of the Fourier spectra \( \hat{y}^{(m)} \) are measured and uniformly sampled with frequency spacing \( \beta \). Here we require that \( p^{(m)} \) has the following general form

\[
 p^{(m)}(t) = \sum_{k=1}^{K} \alpha_k^{(m)} e^{i\lambda_k t}
\]

(11)

where \( \lambda_k \) and \( \alpha_k^{(m)} \) are complex coefficients which are configured subsequently in a specific way in order to guarantee signal recovery. Then the samples in the \( m \)th branch are given by

\[
 c_n^{(m)} = |\hat{y}^{(m)}(n\beta)|^2 = \left| \sum_{k=1}^{K} \alpha_k^{(m)} \hat{x}(n\beta + \lambda_k) \right|^2
\]

(12)

with the length \( K \) vectors

\[
\alpha^{(m)} := \begin{pmatrix} \alpha_1^{(m)} \\ \vdots \\ \alpha_K^{(m)} \end{pmatrix} \quad \text{and} \quad \hat{x}_n := \begin{pmatrix} \hat{x}(n\beta + \lambda_1) \\ \vdots \\ \hat{x}(n\beta + \lambda_K) \end{pmatrix}.
\]

We will show that if the vectors \( \alpha^{(m)}, m = 1, \ldots, M \) and the interpolation points \( \lambda_{n,k} := n\beta + \lambda_k : n \in \mathbb{Z}, k = 1, \ldots, K \) are properly chosen, then it is possible to reconstruct \( x \) from all samples \( c = \{c_n^{(m)} : m = 1, \ldots, M ; n \in \mathbb{Z} \} \). The reconstruction procedure will consist of two steps. First for each \( n \in \mathbb{Z} \) one determines the
vector $\hat{x}_n$ from the $M$ measurements $\{c_{n,m}\}_{m=1}^M$. Secondly, one reconstructs the continuous $x$ from the entries of the vectors $\hat{x}_n$ using Lemma 5.3.

4.1. Choice of the coefficients $\alpha_k^{(m)}$. In order to determine the vector $\hat{x}_n \in \mathbb{C}^K$ from the $M$ intensity measurements $\{c_{n,m}\}_{m=1}^M$, we apply a result from [1]. It states that if the family of $\mathbb{C}^K$-vectors $A = \{\alpha^{(1)}, \ldots, \alpha^{(M)}\}$ constitutes a 2-uniform $M/K$-tight frame which contains $M = K^2$ vectors or if $A$ is a union of $K + 1$ mutually unbiased bases in $\mathbb{C}^K$, then every $\hat{x}_n \in \mathbb{C}^K$ can be reconstructed up to a constant phase from the magnitude of the inner products $\{c_{n,m}\}_{m=1}^M$. We will only discuss the first case here and therefore fix $M = K^2$. The adaption to the second case is obvious.

**Condition A.** A sampling system as in Fig. 2 is said to satisfy Condition A if the coefficients $\alpha_k^{(m)}$ in $\{c_{n,m}\}_{m=1}^M$ are such that $A = \{\alpha^{(1)}, \ldots, \alpha^{(M)}\}$ constitutes a 2-uniform $M/K$-tight frame.

Reconstruction will then be based on the following formula

$$Q_{\hat{x}_n} = \frac{K + 1}{K} \sum_{m=1}^M c_{n,m}^* Q_{\alpha^{(m)}} - \frac{1}{K} \sum_{m=1}^M c_{n,m}^* I$$

with rank-1 matrices $Q_k = xx^*$. For $K = 2$ a valid choice for $A$ reads $\{\alpha^{(1)} = (a, b), \alpha^{(2)} = (b, a), \alpha^{(3)} = (a, -b), \alpha^{(4)} = (-b, a)\}$ with $a = \sqrt{\frac{1}{2}(1 - \frac{1}{\sqrt{3}})}$ and $b = e^{i5\pi/4} \sqrt{\frac{1}{2}(1 + \frac{1}{\sqrt{3}})}$.

4.2. Choice of the interpolation points. Let $\{\lambda_k\}_{k=1}^K$ be ordered increasingly by their real parts. For each $n \in \mathbb{Z}$, the vector $\hat{x}_n$ contains the values of $\hat{x}$ at $K$ distinct interpolation points in the complex plane collected in the sequences

$$\lambda_n^a := \{\lambda_{n,k}^a\}_{k=1}^K \text{ with } \lambda_{n,k}^a = n\beta + \lambda_k, \quad n \in \mathbb{Z}.$$  

Therein, the parameter $a \in \mathbb{N}$ denotes the number of overlapping points of consecutive sets $\{\lambda_n^a\}$ (cf. also Fig. 3). More precisely, we require for every $n \in \mathbb{Z}$ that

$$\lambda_{n+1}^a = \lambda_{n+1,K}^a \text{ for all } i = 1, \ldots, a.$$  

In the following $\Lambda_{O,n}^a = \lambda_n^a \cap \lambda_{n+1}^a$ denotes the set of overlapping interpolation points between $\lambda_n^a$ and $\lambda_{n+1}^a$, and we define the overall interpolation sequences

$$\Lambda^a := \bigcup_{n \in \mathbb{Z}} \lambda_n^a.$$  

In general we allow for $a \geq 1$, but we will see that $a = 1$ is generally sufficient for reconstruction.

As explained in Sec. 3.2, $x \in \mathcal{Y}_{T/2}^p$ can be perfectly reconstructed by $\{\lambda\}$ if $\Lambda^a$ is complete interpolating for $\mathcal{B}_{T/2}^p$. Consequently, we require the following property of the interpolation points $\{\lambda_k\}$.

**Condition B.** A sampling system as in Fig. 2 is said to satisfy Condition B if the coefficients $\{\lambda_k\}_{k=1}^K$ in $\{\lambda_k\}_{k=1}^K$ are such that $\Lambda^a$ is complete interpolating for $\mathcal{B}_{T/2}^p$ and satisfies (15) for a certain $1 \leq a < K$ and $p \in (1, \infty)$.  

Interpolating sequences \( \Lambda^a \) fulfilling Condition \( B \) are \( \beta \)-periodic. So one specific way to obtain such sequences is to choose \( \Lambda^a \) as the set of zeros of a \( \beta \)-periodic sine-type function of type \( T/2 \geq T/2 \). Based on such a zero set one may modify the imaginary part of the individual interpolation points or one can move the individual interpolation points slightly (according to Katsnelson’s theorem) without changing the complete interpolating property and such that \( \beta \)-periodicity is preserved. So apparently, it is possible to construct many non-uniform complex interpolation sequences \( \Lambda^a \) which satisfy Condition \( B \). One particularly simple construction is obtained by starting with the zeros of the sine-type function \( \sin(\frac{T}{2}z) \), which has equally spaced zeros on the real axis (cf. Example 1) and consequently \( \beta \) being an integer multiple of \( 4\pi/T \).

5. Phaseless Signal Recovery

We assume a sampling scheme as described in Section 4 (cf. Fig. 2) which satisfies Condition \( A \) and \( B \). For this setup, we show that for \( 1 < p < \infty \) generically every \( x \in \mathcal{Y}_p^{T/2} \) can be reconstructed from the samples in (12). The proof provides an explicit algorithm for signal recovery. In principle, it consist in a two step procedure. First, a finite block of \( K \) samples of the Fourier domain signal \( \hat{x} \) is reconstructed up to a constant phase factor from the \( m = 1, \ldots, M \) intensity measurements (12) taken at sampling instant \( n \), using a finite dimensional phase retrieval algorithm and utilizing Condition \( A \) of the sampling system. In the second step we exploit that by our construction of the interpolation points, consecutively blocks have an overlap. Therewith, it is possible to make the unimodular factors in each block consistent over all block.

**Theorem 5.1.** For \( 1 < p < \infty \) let \( x \in \mathcal{Y}_p^{T/2} \) be sampled according to the scheme in Sec. 4 which satisfies Condition \( A \) and \( B \), and let \( c = \{c^{(m)}_n : m = 1, \ldots, M; n \in \mathbb{Z}\} \) be the sampling sequence in (12). If the set \( \hat{x}(\Lambda_{a,n}^a) \) contains at least one non-zero element for each \( n \in \mathbb{Z} \), then \( x \) can be perfectly reconstructed from \( c \) up to a constant phase.

**Proof.** According to Condition \( B \) of the sampling system, \( \Lambda^a \) is complete interpolating for \( \mathcal{B}_p^{T/2} \). Therefore the signal \( x \) can be reconstructed from the vectors \( \{\hat{x}_n\}_{n \in \mathbb{Z}} \) using (10). It remains to show that \( \{\hat{x}_n\}_{n \in \mathbb{Z}} \) can be determined from \( c \).

Let \( n \in \mathbb{Z} \) be arbitrary. Since the sampling system satisfies Condition \( A \), we can use (13) to obtain the rank-1 matrix \( Q_n := \hat{x}_n \hat{x}_n^* \) from the measurements.
\{c_n^{(m)}\}_{m=1}^M. Then ˆx_n ∈ C^K is obtained by factorizing Q_n. However, such a factorization is only unique up to a constant phase factor. If the phase φ_{n,i} of one element [ˆx_n,i] is known, the vector ˆx_n can be completely determined from Q_n by
\begin{equation}
\hat{x}(nβ + λ_k) = \sqrt{|Q_n|} e^{i(φ_{n,i} - \text{arg}((Q_n,i)))}, \; ∀k \neq i.
\end{equation}

Assume that we start the recovery of the sequence \{x_n\}_{n∈Z} at a certain n_0 ∈ Z. In this initial step, we set the constant phase of ˆx_{n_0} arbitrarily to θ_0 ∈ [−π, π]. In the next step, we determine ˆx_{n_0+1}. After the factorization of Q_{n_0+1}, the vector ˆx_{n_0+1} is only determined up to a constant phase. However, since Λ^a_{O,n_0} is non-empty, and because ˆx(Λ^a_{O,n_0}) contains at least one non-zero element, we have phase knowledge of at least one entry of ˆx_{n_0+1}, say ˆx(λ^{a}_{n_0+1,i}), where λ^{a}_{n_0+1,i} is an overlapping interpolation point of Λ^{a}_{n_0} and Λ^a_{n_0+1}. Thus, we can completely determine ˆx_{n_0+1} and successively all n = n_0 ± 1, n_0 ± 2, . . . using (16) to obtain ˆx(Λ^a) e^{iθ_0}.

The arbitrary setting of the phase of the initial vector ˆx_{n_0} yields a constant phase shift θ_0 for all ˆx_n which persists after the reconstruction of the time signal by (10).

**Remark 6.** In the case p = 1 and p = ∞, for regular distributions x ∈ L^1(T) and correspondingly ˆx ∈ C_0 ∩ B^∞_2, or ˆx ∈ C_0 ∩ B^1_{1/2}, reconstruction is also possible according to Lemma 3.2. Then however, Condition B has to be reformulated such that the interpolation points Λ^a are the zero set of a sine-type function of type T'/2 > T/2.

Theorem 5.1 states that x ∈ Y^p_{T/2} can only be reconstructed if ˆx = Fx ∈ B^p_{T/2} has at most a − 1 zeros on the overlapping interpolation sets Λ^a_{O,n}. Thus, the set  G of all ˆx ∈ B^p_{T/2} which has a zeros in at least one of the overlapping sets Λ^a_{O,n} contains all those functions for which the reconstruction described in the proof of Theorem 5.1 will fail. We are going to show that  G is in a sense small, namely that  G is a set of first category.

**Lemma 5.2.** The set  G of all ˆx ∈ B^p_{T/2} for which the reconstruction procedure of Theorem 5.1 fails is of first category.

**Proof.** Let Λ^a = \{λ_n\}_{n∈Z} be a set of interpolation points as applied in the sampling scheme of Theorem 5.1 and set
\[ G_n := \{\hat{x} ∈ B^p_{T/2} : \hat{x}(λ_n) = 0\} \text{ for } n ∈ Z. \]

Let ˆy ∈ B^p_{T/2} be arbitrary but ˆy ∉ G_n, fix n ∈ Z and write λ_n = ξ_n + iη_n. Then for every arbitrary ˆx ∈ G_n, it follows from (2) that
\[ |\hat{y}(ξ + iη_n) - \hat{x}(ξ + iη_n)| ≤ C ||\hat{y} - \hat{x}||_{B^p_{T/2}}, \text{ for all } ξ ∈ R \]
where C is a constant which depends only on p, T, and η_n but not on ˆx and ˆy. For ξ = ξ_n it follows in particular that ||\hat{y} - \hat{x}||_{B^p_{T/2}} ≥ \frac{1}{C}|\hat{y}(λ_n)| > 0. This shows that there exists an open ball around every ˆy ∈ B^p_{T/2}, ˆy ∉ G_n which contains no element of G_n. Thus G_n is nowhere dense in B^p_{T/2}. Since  G ⊂ \bigcup_{n∈Z} G_n and because the right hand side is the countable union of nowhere dense sets,  G is of first category.

So the restriction on the signal space given in Theorem 5.1 is fairly mild. Intuitively, this follows also from the fact that the zeros of an entire function of exponential type can not be arbitrarily dense in C. For example, by defining
\[ z_n := \{ z \in \mathbb{C} : n\pi/T < |z| \leq (n + 1)\pi/T \} \]

implies that for every \( \hat{x} \in B_{T/2}^p \) there exist only finitely many sets \( Z_n \) which contain more than one zero of \( \hat{x} \). Consequently, by choosing the spacing of the interpolation points in the overlapping sets \( \Lambda_{O,n}^a \) less than \( \pi/T \), it is very unlikely that a randomly chosen function from \( B_{T/2}^p \) fails to satisfy the condition of Theorem 5.1 especially for \( a > 1 \).

6. Signal Reconstruction in Subspaces

However, in order avoid even such pathological cases, we may a priori minimally restrict the function space allowed in Theorem 5.1 to prevent the measured signal \( \hat{x} \) as in Fig. 2 from having zeros in \( \Lambda_{O}^{a} \). From the practical point of view, this can be achieved by adding a known test signal \( u \) to the desired signal \( x \) as in Fig. 2 from having zeros in \( \Lambda_{O}^{a} \).

The first auxiliary lemma, similar to a result by Duffin, Schaeffer [7], shows that if the Fourier-Laplace transform of the additive test signal is a sine-type function of type \( T'/2 > T/2 \), then we can achieve that all zeros of the sum signal are located outside a strip parallel to the real axis.

**Lemma 6.1.** Let \( \hat{u} \) be sine-type function of type \( T'/2 \) and let \( T < T' \) and \( 1 \leq p \leq \infty \) be arbitrary. For any \( \hat{x} \in B_{T/2}^p \) define the function

\[
\hat{v}(z) := \hat{x}(z) + D \hat{u}(z).
\]

Then \( \hat{v} \in B_{T'}^\infty \) and to every \( \hat{u} \) and \( D > 0 \) there exists an \( H > 0 \) such that

\[
|\hat{v}(\xi + i\eta)| > 0 \quad \text{for all } |\eta| \geq H.
\]

**Proof.** Since every sine-type function is bounded on \( \mathbb{R} \) and because \( B_{T/2}^p \subset B_{T'}^p \subset B_{T/2}^\infty \), it is immediately clear that \( \hat{v} \in B_{T'}^\infty \), and for all \( z = \xi + i\eta \in \mathbb{C} \) we have

\[
|\hat{v}(z)| \geq |D \hat{u}(z)| - |\hat{x}(z)|.
\]

Since \( \hat{x} \in B_{T/2}^p \) it follows from (2) that there is a constant \( M = C \|\hat{x}\| \) such that

\[
|\hat{x}(z)| \leq M e^{\frac{C}{2}|\eta|}.
\]

Similarly, by the definition of a sine type function in (7) one has the lower bound

\[
|D \hat{u}(z)| \geq D A_u e^{-\frac{C}{2}|\eta|}, \quad \text{for all } |\eta| > H_u.
\]
with two constants $A_u$ and $H_u$ which depend on $u$. Using these two bounds in (18) and keeping in mind that $T' > T$, one obtains

$$|\hat{v}(z)| \geq D A_u e^{\frac{T'}{2} |\eta|} - M e^{\frac{T'}{2} |\eta|} > 0 , \text{ for all } |\eta| > H$$

and with $H = \max [H_u, \frac{2}{\gamma} \ln \left(\frac{M}{D A_u}\right)]$.

**Remark 7.** Conversely for a given $H > H_u$, we can choose the constant $D$ as

$$D > \frac{M}{A_u} e^{\left(\frac{T'}{2} - \frac{T}{T'}\right)H}$$

for (17) to hold.

**Example 2.** One special choice for the function $\hat{u}$, that has been dealt with in [7], is $\hat{u}(z) = \cos(\frac{T'}{2} z)$. For this function, the constants $A_u$ and $H_u$ can be derived by

$$|\cos(\frac{T'}{2} z)| = \frac{1}{2} \left| e^{\frac{T'}{2} \xi} e^{-\frac{T}{T'} |\eta|} - e^{-\frac{T'}{2} \xi} e^{\frac{T}{T'} |\eta|} \right|$$

$$\geq \frac{1}{2} \left( e^{\frac{T'}{2} |\eta|} - e^{-\frac{T'}{2} |\eta|} \right) \geq \frac{1}{2} \left( e^{\frac{T'}{2} |\eta|} - 1 \right)$$

$$\geq \frac{1}{2} \left( 1 - e^{-\frac{T'}{2} H_u} \right) e^{\frac{T'}{2} |\eta|} = A_u e^{\frac{T'}{2} |\eta|} \text{ for all } |\eta| \geq H_u$$

where $H_u > 0$ is arbitrary and $A_u = [1 - \exp(-\frac{T'}{2} H_u)]/2$. Note that the time domain signal corresponding to this sine-type function $\hat{u}$ is the non-regular tempered distribution $u = [\delta_{T'/2} + \delta_{-T'/2}] / 2$ which vanishes on $T$.

After adding a sine-type function $\hat{u}$ of type $T'/2$ to the desired signal $\hat{x} \in \mathcal{B}_{T'/2}^0$, we obtain a function in $\hat{v} \in \mathcal{B}_{T'/2}^\infty \setminus C_0$. For these signals, the reconstruction formulas in Lemma 5.1 and 5.2 are no longer valid. Therefore, we need the following extension of a result in [17, Lect. 21] which later leads to an interpolation formula for $\mathcal{B}_{T'/2}$ (Lemma 6.3).

**Lemma 6.2.** Let $S$ be a sine-type function of type $\tilde{T}/2$ and let $\{\lambda_n\}_{n \in \mathbb{Z}}$ be its zero set. For any sequence $c = \{c_n\}_{n \in \mathbb{Z}} \in \mathbb{C}^\infty$ there exists an entire function $f$ of exponential type $\tilde{T}/2$ which solves the interpolation problem

$$f(\lambda_n) = c_n , \quad n \in \mathbb{Z} .$$

Every such entire function admits the representation

$$f(z) = \sum_{n \in \mathbb{Z}} c_n \frac{S(z)}{S'(\lambda_n)} \left[ \frac{1}{z - \lambda_n} + \frac{1}{\lambda_n} \right] + C_s S(z)$$

with an arbitrary constant $C_s \in \mathbb{C}$ and where the sum converges uniformly and absolutely on compact subsets of $\mathbb{C}$.

**Remark 8.** For every $C_s \in \mathbb{C}$, the function $f$ in (20) has the property that

$$|f(\xi + i\eta)| e^{-\frac{T}{2} |\eta|} = o(|\xi + i\eta|) , \quad \text{as } |z| \to \infty .$$

The prime at the summation sign in (20) means that the second term in the braces is set to zero if $\lambda_0 = 0$, and we will omit this prime subsequently. Lemma 6.2 basically states that for $p = \infty$ the interpolation problem $f(\lambda_k) = c_k$ is uniquely solvable by an entire function $f$ of exponential type up to an additive sine-type...
term. However, it should be noted that the function defined in (20) may not be bounded on \( \mathbb{R} \) for some sequences \( c \in \ell^\infty \), i.e. (20) may not be in \( B_{l^2}^{1/2} \).

**Proof.** The proof is partly along the same lines as in [17, Lect. 21, Theorem 1]. Let \( \lambda_k = \xi_k + i\eta_k \forall k \). First, it is shown that the sum in (20) converges. To this end, one considers the partial sums

\[
\varphi_{m,n}(z) = \sum_{|k|=n+1}^{m} c_k \frac{1}{S'(\lambda_k)} \left[ \frac{1}{z - \lambda_k} + \frac{1}{\lambda_k} \right].
\]

Since \( S \) is a sine-type function, one knows [32] that \( \inf_{n \in \mathbb{Z}} |S'(\lambda_n)| = : C_0 > 0 \). Then by the triangular and Cauchy-Schwarz inequality

\[
|\varphi_{m,n}(z)| \leq \frac{\|c\|_\infty}{C_0} \sum_{|k|=n+1}^{m} \left| \frac{1}{z - \lambda_k} + \frac{1}{\lambda_k} \right| \leq \frac{\|c\|_\infty}{C_0} \sum_{|k|=n+1}^{m} \frac{1}{|\lambda_k| |z - \lambda_k|} \leq \frac{\|c\|_\infty}{C_0} \left( \sum_{|k|=n+1}^{m} \frac{1}{|\lambda_k|^2} \right)^{1/2} \left( \sum_{|k|=n+1}^{m} \frac{1}{|z - \lambda_k|^2} \right)^{1/2}.
\]

(22)

Since the zeros of a sine-type function are separated, there exists a \( d > 0 \) such that \( |\lambda_m - \lambda_n| > 2d \) for all \( m \neq n \). If \( \delta \in (0, d) \) is fixed and \( z \) lies outside the set \( \bigcup_{n \in \mathbb{Z}} \{ z \in \mathbb{Z} : |z - \lambda_n| \leq \delta \} \), then there exists one zero, say \( \lambda_0 \), which is closest to \( z \) and for which \( |z - \lambda_0| > \delta \). For all other zeros \( \lambda_k \) we have then

\[
|z - \lambda_k| \geq (\lambda_k - \lambda_0) - d \frac{\lambda_k - \lambda_0}{|\lambda_k - \lambda_0|} = |\lambda_k - \lambda_0| \left| 1 - \frac{d}{|\lambda_k - \lambda_0|} \right| \geq \frac{1}{2} |\lambda_k - \lambda_0|
\]

using for the last inequality that \( |\lambda_k - \lambda_0| \geq 2d \). Therewith, we get

\[
\sum_{|k|=n+1}^{m} \frac{1}{|z - \lambda_k|^2} \leq \sum_{k \in \mathbb{Z}} \frac{1}{|z - \lambda_k|^2} \leq \frac{1}{\delta^2} + \frac{1}{4} \sum_{k \in \mathbb{Z}, k \neq 0} \frac{1}{|\lambda_k - \lambda_0|^2} \leq \frac{1}{\delta^2} + \frac{1}{4} \sum_{k \in \mathbb{Z}, k \neq 0} \frac{1}{|\xi_k - \xi_0|^2}.
\]

Since \( \{\lambda_n\}_{n \in \mathbb{Z}} \) are zeros of a sine-type function, we can define \( H := \sup_{k \in \mathbb{Z}} |\eta_k| \). Now let \( m := \frac{3H}{2} \) denote the maximal number of zeros in the vertical strip \( \{ z : |Re z - \Re \lambda_0| < d \} \). Because the sequence \( \{\xi_n\}_{n \in \mathbb{Z}} \) is assumed to be ordered increasingly, one can easily see that

\[
|\xi_{m(k-1)+j} - \xi_0| \geq \sqrt{3} dk
\]

for \( j = 1 \ldots m \) and \( k \in \mathbb{N}_+ \), noting the geometry of the densest circle packing. One can now rewrite the last sum and obtain

\[
\sum_{k=1}^{\infty} \frac{1}{|\xi_k - \xi_0|^2} = \sum_{k=1}^{\infty} \sum_{j=1}^{m} \frac{1}{|\xi_{m(k-1)+j} - \xi_0|^2} \leq \sum_{k=1}^{\infty} \frac{m}{k^2 3d^2} = c \sum_{k=1}^{\infty} \frac{1}{k^2}
\]

for some constant \( c \).
with \( c = \frac{n}{32T} \). The same holds for the sum over \( k \in \mathbb{N}_- \). Hence we have that

\[
\sum_{|k|=n+1}^{m} \frac{1}{|z - \lambda_k|^2} \leq \frac{1}{g^2} + \frac{c}{2} \sum_{k=1}^{\infty} \frac{1}{k^2} =: M_5 < \infty
\]

with a constant \( M_5 \) which does not depend on \( z \). Moreover, since the sequence \( \{\lambda_n\}_{n \in \mathbb{Z}} \) is the zero set of a function of exponential type, the series \( \sum_{n \in \mathbb{Z}} |\lambda_n|^{-2} \) is convergent \(^{[32]}\). Consequently, \(^{[22]}\) and \(^{[23]}\) imply that to every \( \epsilon > 0 \) there exists an \( N(\epsilon) > 0 \) such that

\[
|\varphi_{m,n}(z)| \leq \epsilon |z| \quad \text{for all } m, n > N(\epsilon).
\]

It follows that \( \varphi_{m,0}(z) \) converges to a function \( \varphi(z) \) uniformly on each compact subset of \( \mathbb{C} \) as \( m \to \infty \). We set \( g(z) := S(z) \varphi(z) \). Since \( S \) is a sine-type function of type \( T/2 \), it follows from \(^{[21]}\) that \( g \) satisfies \(^{[21]}\), and it is obvious that \( g \) solves the interpolation problem \(^{[19]}\).

Now, let \( f \) be an entire function which satisfies \(^{[19]}\) and \(^{[21]}\). Then also the difference \( f(z) - g(z) \) satisfies \(^{[19]}\) and \(^{[21]}\). Since \( S \) is of exponential type \( T/2 \), it follows that the function \( h(z) = [f(z) - g(z)]/S(z) \) satisfies \( |h(z)| = o(|z|) \) as \( |z| \to \infty \). By Liouville's theorem \( h(z) \) is a constant, which proves \(^{[20]}\). \( \square \)

In Lemma \(^{[6.2]}\) the sequence \( \{c_n\}_{n \in \mathbb{Z}} \in \ell^\infty \) was arbitrary. In our application however, it arises from sampling an entire function \( v \in B_{T/2}^{\infty} \) so that the existence and boundedness of the solution in \(^{[20]}\) is naturally given. The next lemma shows that once oversampling is applied, the additional term \( C_s \) \( S(z) \) in the interpolation formula \(^{[20]}\) vanishes which allows for a simple and unique reconstruction of every \( v \in B_{T/2}^{\infty} \) from its samples.

**Lemma 6.3.** Let \( S \) be a sine-type function of type \( T/2 \) and let \( \Lambda = \{\lambda_n\}_{n \in \mathbb{Z}} \) be its zero set. If \( T > T' \) then

\[
v(z) = \sum_{n \in \mathbb{Z}} v(\lambda_n) \frac{S(z)}{S'(\lambda_n)} \left[ \frac{1}{z - \lambda_n} + \frac{1}{\lambda_n} \right] \quad \text{for all } v \in B_{T/2}^{\infty}
\]

where the sum converges absolutely and uniformly on compact subsets of \( \mathbb{C} \). Moreover, there exists a constant \( C_u \) such that

\[
\lim_{N \to \infty} \max_{t \in \mathbb{R}} \left| v(t) - \sum_{n = -N}^{N} v(\lambda_n) \frac{S(t)}{S'(\lambda_n)} \left[ \frac{1}{t - \lambda_n} + \frac{1}{\lambda_n} \right] \right| \leq C_u \|v\|_{B_{T/2}^{\infty}}
\]

for all \( v \in B_{T/2}^{\infty} \).

**Remark 9.** Relation \(^{[20]}\) states that for every \( v \in B_{T/2}^{\infty} \) the approximation error is uniformly bounded on the whole real axis \( \mathbb{R} \). So in particular, it follows from \(^{[20]}\) and \(^{[20]}\) that the right hand side of \(^{[25]}\) converges to \( v \) in the topology of \( S' \).

**Proof.** Since \( v \in B_{T/2}^{\infty} \) and \( \Lambda \) is the zero set of a sine-type function, the sequence \( \{c_n = v(\lambda_n)\}_{n \in \mathbb{Z}} \) is in \( \ell^\infty \). Because \( v \in B_{T/2}^{\infty} \subset B_{T/2}^{\infty} \), we know from Lemma \(^{[6.2]}\) that every entire function \( \tilde{v} \) of exponential type \( T/2 \) which satisfies \( \tilde{v}(\lambda_n) = c_n \) for all \( n \in \mathbb{Z} \) has the form \( \tilde{v}(z) = f(z) + C_s S(z) \) where \( f(z) \) stands for the sum on the right hand side of \(^{[25]}\). Consequently, also \( v \) has to have this form, i.e. \( v(z) = f(z) + C_s S(z) \) and we have to prove that \( C_s = 0 \). To this end, it is sufficient
to show that $f$ is of exponential type $T'/2$. Then, also $v - f$ is of exponential type $T'/2$ such that in the equation

$$v(z) - f(z) = C_s S(z)$$

the modulus of the left hand side can be upper bounded by

$$|v(z) - f(z)| \leq B_1 e^{\frac{T'}{2}|z|}$$

for all sufficiently large $|z|$ and with a certain constant $B_1 > 0$. Now assume $C_s > 0$. Because $S$ is of sin-e-type $\tilde{T}/2$, the modulus of the right hand side can be lower bounded by $|C_s S(z)| \geq B_2 \exp(\frac{\tilde{T}}{2}|z|)$ for all sufficiently large $|z|$ and with a constant $B_2 > 0$. But since $\tilde{T} > T'$ this yields a contradiction to the upper bound in (27). It follows that $C_s = 0$, i.e. that (29) holds.

It remains to show that $f$ is of exponential type $T'/2$. Without loss of generality, we assume $v(0) = 0$ (otherwise, one applies the following reasoning to $v(z) - v(0)$) and consider the function $g(z) := v(z)/z$. Since $v$ is of exponential type $T'/2$, to every $\epsilon > 0$ there exists a constant $C(\epsilon)$ such that

$$|g(z)| = \frac{1}{|z|} |v(z)| \leq \frac{1}{|z|} C(\epsilon) e^{\frac{T'}{2}+\epsilon|z|} \leq C(\epsilon) e^{\frac{T'}{2}+\epsilon|z|} \quad \text{for all } |z| \geq 1.$$ 

This shows that $g$ is of exponential type $T'/2$. Moreover the restriction of $g$ to $\mathbb{R}$ is square integrable. Thus $g \in B^2_{T'/2} \subset B^2_{\tilde{T}/2}$. Therefore, Lemma 3.1 can be applied to $g$, which gives

$$g(z) = \frac{v(z)}{z} = \sum_{n \in \mathbb{Z}} \frac{v(\lambda_n)}{\lambda_n} \frac{S(z)}{S'(\lambda_n)} \frac{1}{z - \lambda_n}.$$ 

By multiplying the whole equation by $z$, the right hand side becomes equal to $f(z)$ and shows that $f$ is of exponential type $T'/2$.

In order to prove (29), we assume without loss of generality that $0 \notin A$ and we write $(T_N v)(t)$ for the finite sum in (29), and $(A_N v)(t)$ for the finite Shannon series used in Lemma 3.1, i.e.

$$(A_N v)(t) = \sum_{n=-N}^{N} v(\lambda_n) \frac{S(t)}{S'(\lambda_n)} \frac{1}{t - \lambda_n}, \quad t \in \mathbb{R}.$$ 

Both approximation series are obvious related by

$$(T_N v)(t) = (A_N v)(t) - \frac{(A_N v)(0)}{S(0)} S(t).$$ 

For $v \in B^2_{T'/2}$, the triangle inequality yields

$$|v(t) - (T_N v)(t)| \leq |v(t) - (A_N v)(t)| + \left| \frac{(A_N v)(0)}{S(0)} \right| |S(t)|.$$

Now it is known [23, Theorem 5] that the first term on the right hand side is uniformly bounded by $C_1 \|v\|_\infty$ for all $t \in \mathbb{R}$ and all $N \in \mathbb{Z}$ with a certain constant $C_1$ independent of $t$ and $N$. The same result also implies that the second term is uniformly bounded by a constant of the form $C_2 \|v\|_\infty$ which is why (29) holds and the proof is complete. \qed
Now we are ready to state a corollary of Theorem 5.1. By adding an appropriate test signal $Du(t)$ prior to our sampling scheme (cf. Fig. 4) we are able to ensure the “non-zero requirement” of Theorem 5.1 and therefore every signal in our signal space $Y^p_{T/2}$ can be reconstructed from its magnitude measurement.

Corollary 6.4. Consider a sampling scheme according to Sec. 4 which satisfies Condition A and B and with the additional pre-processing as shown in Fig. 4. Then for every $1 \leq p \leq \infty$ there exists a function $Du(t)$ and an interpolation sequence $\Lambda^a$ with overlap $a \geq 1$ such that every

$$x \in \{x \in Y^p_{T/2} : \|x\| \leq 1\}$$

(28) can be perfectly reconstructed (up to a constant phase) from the measurements (12).

Remark 10. The additional assumption $\|x\| \leq 1$ only requires that an upper bound on the signal norm is known. Practically, this is necessary to calibrate the measurement system by an appropriate amplitude $D$ of the additive test signal $u$.

Proof. Let $x$ as in (28). Using (2), it follows that there is a constant $M$ such that $|\hat{x}(\xi)| \leq M$ for all $\xi \in \mathbb{R}$. Furthermore, without loss of generality we assume that $p = \infty$ since $B^p_{T/2} \subset B^\infty_{T/2} \forall p \in [1, \infty)$. Fix $T' > T$ and choose an arbitrary sine-type function $\hat{u}$ of type $T'/2$. Then it follows from Lemma 6.1 that there exist constants $H, D$ such that the function

$$\hat{v}(z) = \hat{u}(z) + D \hat{u}(z)$$

(29)

has no zeros for all $z = \xi + i\eta \in \mathbb{C}$ with $|\eta| > H$. Choose $\Lambda^a = \{\lambda_k = \xi_k + i\eta_k\} \subset \mathbb{Z}$ as the zero set of a sine-type function of type $T'/2 > T'/2$. By (18) we can shift the imaginary parts of the interpolation points $\lambda_k$ such that $|\eta_k| > H$ for all $k$ while $\Lambda^a$ remains the zero set of a sine-type function. Denote the corresponding sine-type function by $S$.

Now the signal (29) is modulated and sampled exactly as described in Sec. 4. Then our intensity measurements are given, similar as in (12), by $c_n^{(\alpha)} = |\langle \hat{v}, \alpha^{(m)} \rangle|^2$. Following the same steps as in the proof of Theorem 5.1, we obtain the values of $\hat{v}$ at the sampling set $\Lambda^a$ up to a constant phase $\theta_0$. Since by our construction overlapping interpolation points do not coincide with zeros of $\hat{v}$ the phase information can be propagated and we are able to recover $\hat{v}(\Lambda^a) e^{i\theta_0}$ from the intensity measurements for every signal $\hat{v}$ of the form (29).

Since $\hat{v} \in B^\infty_{T'/2} \subset B^\infty_{T/2}$ and $\sup_{k \in \mathbb{Z}} |\eta_k| < H < \infty$, it follows that the sequence $\{\hat{v}(\Lambda^a) e^{i\theta_0}\}$ is in $\ell^\infty$ such that Lemma 6.3 can be applied to interpolate $\hat{v}(z) e^{i\theta_0}$ from the sampling sequence $d_n := \hat{v}(\lambda_n) e^{i\theta_0}$, $n \in \mathbb{Z}$, using (25):

$$\hat{v}(z) e^{i\theta_0} = \sum_{n \in \mathbb{Z}} d_n \frac{S(z)}{S'(\lambda_n)} \left[ \frac{1}{z - \lambda_n} + \frac{1}{\lambda_n} \right].$$

But since $\theta_0$ is unknown, it is not possible to determine $\hat{x}(z)$ directly from $\hat{v}(z) e^{i\theta_0}$ using (29). Instead, one determines

$$\hat{x}(z) := \hat{v}(z) e^{i\theta_0} - D \hat{u}(z) = \hat{x}(z) e^{i\theta_0} - D \hat{u}(z) (1 - e^{i\theta_0}).$$

If we now choose $\hat{u}(z) = \cos(\frac{\theta_0}{2} z)$ and apply the inverse Fourier-Laplace transform to $\hat{x}$ one obtains $x(t) e^{i\theta_0}$ for $t \in \mathbb{T}$ since the inverse Fourier transform of the cosine function vanishes on $\mathbb{T}$ (cf. Example 2). □
7. Discussion and Outlook

To determine the sampling system in Fig. 2 one has to fix $K, M, a$ and $\beta$. The number $K \geq 2$ can be chosen arbitrarily. Then $M = K^2$ is fixed, and $1 \leq a \leq K - 1$. The sampling period $\beta$ has to be chosen such that the sampling system satisfies Condition B. As discussed before, one possible choice may start with the zeros of the function $\sin(\frac{\tilde{T}}{2}z)$ with $\tilde{T} \geq T$. Then $\delta := \lambda_k - \lambda_{k-1} = 2\pi/\tilde{T}$ such that $\beta = (K - a)\delta$. Therewith, the total sampling rate becomes

$$R(a, K, \tilde{T}) = \frac{M}{\beta} = \frac{K^2}{(K - a)\delta} = \frac{K^2}{K - a} \frac{\tilde{T}}{2\pi} = \frac{K^2}{K - a} \frac{\tilde{T}}{T} R_{Ny}$$

where $R_{Ny} := T/(2\pi)$ is the Nyquist rate. It is apparent that $R(a, K, \tilde{T})$ grows asymptotically proportional with $K$, increases with the overlap $a$, and is bounded below by

$$\inf_{1 \leq a < K, K \geq 1, \tilde{T} > T} R(a, K, \tilde{T}) = \inf_{T > T} R(1, 2, \tilde{T}) = 4R_{Ny}.$$ 

Since $\tilde{T}/T$ can be made arbitrarily close to 1 using Theorem 5.1 and Corollary 6.4 we can sample at a rate which is almost as small as $4R_{Ny}$ while still ensuring perfect reconstruction. This corresponds to the findings in [2] for finite dimensional spaces, where it was shown that basically any $x \in \mathbb{C}^N$ can be reconstructed from $M \geq 4N - 2$ magnitude samples.

We note that the above framework can be applied exactly the same way for bandlimited signals. To this end, one only has to exchange the time and frequency domain. Then the modulators in Fig. 2 have to be replaced by linear filters and the sampling of the magnitudes has to be done in the time domain.

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