ELLiptic FerMAT numbErS AND Elliptic DIVISIBILITY
Sequence

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Abstract. For a pair \((E, P)\) of an elliptic curve \(E/\mathbb{Q}\) and a nontorsion point \(P \in E(\mathbb{Q})\), the sequence of elliptic Fermat numbers is defined by taking quotients of terms in the corresponding elliptic divisibility sequence \((D_n)_{n \in \mathbb{N}}\) with index powers of two, i.e. \(D_1, D_2/D_1, D_4/D_2, \) etc. Elliptic Fermat numbers share many properties with the classical Fermat numbers, \(F_k = 2^{2^k} + 1\). In the present paper, we show that for magnified elliptic Fermat sequences, only finitely many terms are prime. We also define generalized elliptic Fermat numbers by taking quotients of terms in elliptic divisibility sequences that correspond to powers of any integer \(m\), and show that many of the classical Fermat properties, including coprimality, order universality and compositeness, still hold.

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1. INTRODUCTION

Let \(E\) be an elliptic curve defined over \(\mathbb{Q}\) by a Weierstrass equation with integer coefficients

\[ E : y^2 + a_1y + a_3y = x^3 + a_2x^2 + a_4x + a_6. \]

We say \((E)\) is minimal if the discriminant \(|\Delta(E)|\) is minimal among all Weierstrass equations for \(E\). Moreover, we say a minimal Weierstrass equation is reduced if \(a_1, a_3 \in \{0, 1\}\) and \(a_2 \in \{-1, 0, 1\}\). It is worth noting that for all elliptic curves over \(\mathbb{Q}\), minimal models exist and a reduced minimal model is unique.

For a fixed nontorsion point \(P \in E(\mathbb{Q})\), we can define the elliptic divisibility sequence as follows:

**Definition 1.1.** The elliptic divisibility sequence (EDS) associated to the pair \((E, P)\) is the sequence \(D = (D_n)_{n \in \mathbb{N}} : \mathbb{N} \rightarrow \mathbb{N}\) defined by taking the positive square root of the denominator of successive iterations of \(P\) as a lowest fraction, i.e.,

\[ [n]P = \left( \frac{A_n}{D_n^2}, \frac{B_n}{D_n^2} \right), \]

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where \( \gcd(A_n, D_n) = \gcd(B_n, D_n) = 1 \). An EDS is minimal if the Weierstrass equation of \( E/\mathbb{Q} \) is minimal and reduced. Also, an EDS is normalized if \( D_1 = 1 \).

In this paper, we always assume \( D_1 = 1 \). Much of our work revolves around elliptic Fermat numbers, analogues of the classical Fermat numbers (\( F_n = 2^{2^n} + 1, \ n \geq 0 \)) defined by S. Binegar, R. Dominick, M. Kenney, J. Rouse, and A. Walsh in [1]. In the original version of the paper, they define elliptic Fermat numbers as follows:

**Definition 1.2.** Let \( D = (D_n)_{n \in \mathbb{N}} \) be an EDS. Define the sequence of elliptic Fermat numbers (EFN) \( \{F_k(E, P)\}_{k \geq 1} \) as follows:

\[
F_k(E, P) = \begin{cases} 
\frac{D_{2^k}}{D_{2^k-1}} & \text{if } k \geq 1 \\
\frac{D_1}{k} & \text{if } k = 0
\end{cases}
\]

Note that in the final version of the paper includes slightly different definition of elliptic Fermat numbers, on the other hand, it does not have much effect on our proof too much, which uses the original definition. In [1, Theorem 9], they list a certain set of conditions which force \( F_k(E, P) \) to be composite for all \( k \geq 1 \):

**Theorem 1.3 (BDKRW).** For an elliptic curve \( E : y^2 = x^3 + ax^2 + bx + c \), assume the following:

(i) \( E(\mathbb{Q}) = \langle P, T \rangle \), where \( P \) has infinite order and \( T \) is a rational point of order 2.

(ii) \( E \) has an egg.

(iii) \( T \) is on the egg.

(iv) \( T \) is the only integral point on the egg.

(v) \( P \) is not integral.

(vi) \( \gcd(b, m_0) = 1 \).

(vii) \( |\tau_k| = 2 \) for all \( k \).

(viii) \( 2 \nmid e_k \) for all \( k \).

(ix) The equations \( x^4 + ax^2y^2 + by^4 = \pm 1 \) has no integer solutions where \( y \not\in \{0, \pm 1\} \).

Then \( F_k(E, P) \) is composite for all \( k \geq 1 \).

In the same vein, we prove the non-primality of a sequence of elliptic Fermat numbers which is defined by magnified elliptic divisibility sequences. First, we recall the definition of magnified elliptic divisibility sequences.

**Definition 1.4.** Let \( E/\mathbb{Q} \) and \( E'/\mathbb{Q} \) be two elliptic curves. We say a rational point \( P \in E(\mathbb{Q}) \) is magnified if \( P = \phi(P') \) for some (nonzero) isogeny \( \phi : E' \to E \) over \( \mathbb{Q} \) and some \( P' \in E'(\mathbb{Q}) \). Moreover, an EDS \( D = (D_n)_{n \geq 1} \) is magnified if \( D \) is a minimal EDS associated to some magnified point on an elliptic curve over \( \mathbb{Q} \). We call a sequence of elliptic Fermat numbers \( \{F_k(E, P)\}_{k \geq 1} \) magnified if it is defined by using a magnified EDS.

In Section 3, we prove the following non-primality result of the sequence of magnified elliptic Fermat numbers.

**Theorem 1.5.** Let \( E/\mathbb{Q} \) be a minimal magnified elliptic curve with a fixed point \( P \in E(\mathbb{Q}) \) having a (nonzero) odd-degree isogeny \( \phi : E' \to E \) from a minimal elliptic curve \( E'/\mathbb{Q} \) satisfying \( \phi(P') = P \). Then the terms \( F_k(E, P) \) are composite for sufficiently large \( k \).

In section 4, we consider a generalization of elliptic Fermat numbers. Generalized classical Fermat numbers have the form \( a^{2^n} + b^{2^n} \) for some relatively prime integers \( a \) and \( b \). It is natural to consider a similar generalization of elliptic Fermat numbers:
Definition 1.6. Let $D = (D_n)_{n \in \mathbb{N}}$ be an EDS, and let $m \geq 1$ be an integer. We define the sequence of generalized elliptic Fermat numbers $\{F_k^{(m)}(E, P)\}$ as follows:

$$F_k^{(m)}(E, P) = \begin{cases} \frac{D_m}{D_{mk}} & \text{if } k \geq 1 \\ \frac{D_m}{D_1} & \text{if } k = 0. \end{cases}$$

Note that Definition 1.2 is the special case where $m = 2$. In [1] Theorem 3, Theorem 4], they prove the following theorems about elliptic Fermat numbers:

Theorem 1.7 (BDKRW). For all $k \neq \ell$, $\gcd(F_k(E, P), F_{\ell}(E, P)) \in \{1, 2\}$.

Theorem 1.8 (BDKRW). Let $\Delta(E)$ be the discriminant of $E$ and suppose that $N$ is a positive integer with $\gcd(N, 6\Delta(E)) = 1$. Then $P$ has order $2^k$ in $E(\mathbb{Z}/N\mathbb{Z})$ if and only if $N \mid F_0(E, P) \cdots F_k(E, P)$ and $N \nmid F_0(E, P) \cdots F_{k-1}(E, P)$.

For generalized elliptic Fermat numbers generated by odd $m$, we prove slightly weakened generalizations of these properties, namely:

Theorem 1.9 (Coprime). Let $F = (F_k^{(m)}(E, P))_{k \in \mathbb{N}}$ be the sequence of generalized elliptic Fermat numbers for a fixed elliptic curve $E$, a rational point $P \in E(\mathbb{Q})$ and an odd integer $m \geq 1$. Then for all distinct $k, \ell \geq 0$, $\gcd(F_k^{(m)}(E, P), F_\ell^{(m)}(E, P)) \mid m$.

Theorem 1.10 (Order Uniqueness). Let $F = (F_k^{(m)})_{k \in \mathbb{N}}$ be the sequence of generalized elliptic Fermat numbers for a fixed elliptic curve $E$, a rational point $P \in E(\mathbb{Q})$ and an odd integer $m \geq 1$. Then for all $N \in \mathbb{N}$ satisfying $\gcd(N, 6\Delta(E)) = 1$,

$P$ has order $m^k$ in $E(\mathbb{Z}/N\mathbb{Z}) \iff N \mid F_0^{(m)} \cdots F_k^{(m)}$ and $N \nmid F_0^{(m)} \cdots F_{k-1}^{(m)}$.

We also have an analogous non-primality result for generalized elliptic Fermat numbers.

Theorem 1.11. Let $E/\mathbb{Q}$ be a minimal magnified elliptic curve with a fixed point $P \in E(\mathbb{Q})$ having a (nonzero) degree $d$ isogeny $\phi: E' \to E$ from a minimal elliptic curve $E'/\mathbb{Q}$ satisfying $\phi(P') = P$. For $m$ relatively prime to $d$, $F_k^{(m)}(E, P)$ are composite for sufficiently large $k$.

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2. Non-primality Conjecture for Elliptic Fermat Numbers

Everest, Miller, and Stephens [2] proved the following conjecture for magnified elliptic divisibility sequences.

Conjecture 2.1 (Primality Conjecture). Let $D = (D_n)_{n \in \mathbb{N}}$ be an elliptic divisibility sequence. Then $D$ contains only finitely many prime terms.

Note that for the Fibonacci sequence, it is conjectured that primes occur infinitely many times. In this section, we will prove the following conjecture for magnified elliptic Fermat numbers.

Conjecture 2.2 (Primality Conjecture for elliptic Fermat numbers). Let $F = (F_n)_{n \in \mathbb{N}}$ be the sequence of elliptic Fermat numbers for a fixed elliptic curve $E$ and a rational point $P \in E(\mathbb{Q})$. Then $F$ contains only finitely many prime terms.
Corollary 2.3. For any odd prime \( p \nmid 6\Delta(E) \), \( P \) has order \( 2^k \) in \( E(\mathbb{F}_p) \) if and only if \( p | F_k(E, P) \).

Using the above corollary, we can prove the following result.

Lemma 2.4. Let \( E/\mathbb{Q} \) be a minimal magnified elliptic curve with a fixed point \( P \in E(\mathbb{Q}) \) having a (nonzero) odd-degree isogeny \( \phi : E' \to E \) from a minimal elliptic curve \( E'/\mathbb{Q} \) satisfying \( \phi(P') = P \). For sufficiently large \( k \), we have
\[
\gcd(F_k(E', P'), F_k(E, P)) \neq 1.
\]

Proof. Let \( p \) be a fixed prime which divides \( F_k(E', P') \). Let \( S \) be the set of primes for which \( E, E' \), and \( \phi \) cannot give an isogeny modulo \( p \). Note that \( S \) is a finite set. Without loss of generality, we can assume \( p \notin S \). From Corollary 2.3, it is sufficient to prove that the isogeny \( \phi \) reduction modulo \( p \) (which is again an isogeny) preserves the order \( 2^k \) of \( P' \)
\[
\phi : E'(\mathbb{F}_p) \to E(\mathbb{F}_p).
\]
We consider the dual isogeny \( \hat{\phi} \) of \( \phi \). We know
\[
\hat{\phi} \circ \phi = [d]
\]
where \([d]\) is the multiplication-by-\( d \) map on \( E' \). Then the map
\[
[d] : E'(\mathbb{F}_p) \phi \to E(\mathbb{F}_p) \xrightarrow{\hat{\phi}} E'(\mathbb{F}_p)
\]
preserves the order of \( 2^k \) of \( P' \) and so does \( \phi \). □

Proposition 2.5. Let \( E/\mathbb{Q} \) be a minimal magnified elliptic curve with a fixed point \( P \in E(\mathbb{Q}) \) having a (nonzero) odd-degree isogeny \( \phi : E' \to E \) from a minimal elliptic curve \( E'/\mathbb{Q} \) satisfying \( \phi(P') = P \). For sufficiently large \( k \), we actually have the following divisibility:
\[
F_k(E', P') | F_k(E, P).
\]

Proof. The proof of Lemma 2.4 essentially implies the result. □

Definition 2.6. Let \( E/\mathbb{Q} \) be an elliptic curve with a point \( P \in E(\mathbb{Q}) \), denote \( P \) as \( P = (A, B) \). We define the height of a point \( h(P) \) by using its \( x \)-coordinate:
\[
h(P) = \log(\max(|A|, D^2)).
\]
Moreover, we define the canonical height of a point \( \hat{h}(P) \) by
\[
\hat{h}(P) = \lim_{k \to \infty} \frac{h(2^k P)}{4^k}.
\]

Remark 2.7. Note that when we have \([l]P = \left(\frac{A_l}{D_l^2}, \frac{B_l}{D_l^2}\right) \) with \( \gcd(A_l, D_l) = 1 \), then
\[
\lim_{l \to \infty} \frac{\log(D_l^2)}{l^2} = \lim_{l \to \infty} \frac{|A_l|}{l^2} = \hat{h}(P).
\]
For instance, if we choose \( l = 3^k \) for some \( k \), we can also represent
\[
\hat{h}(P) = \lim_{k \to \infty} \frac{\log(D_{3^k}^2)}{9^k} = \lim_{k \to \infty} \frac{|A_{3^k}|}{9^k}.
\]
This observation will be useful in Section 3.2.

Along with the above Corollary, let’s recall the following theorem from [1].
Theorem 2.8. Let $E/\mathbb{Q}$ be an elliptic curve with a fixed point $P \in E(\mathbb{Q})$. If $\hat{h}(P)$ denotes the canonical height of $P$, we get

$$\lim_{k \to \infty} \frac{\log(F_k(E, P))}{4^k} = \frac{3}{8} \hat{h}(P).$$

Proof. The result follows directly from Definition 1.2. See [1, Theorem 10]. \qed

Using Corollary 2.5 and Theorem 2.8, we can prove the Primality conjecture for magnified elliptic Fermat numbers.

Theorem 2.9. Let $E/\mathbb{Q}$ be a minimal magnified elliptic curve with a fixed point $P \in E(\mathbb{Q})$ having a (nonzero) odd-degree isogeny $\phi : E' \to E$ from a minimal elliptic curve $E'/\mathbb{Q}$ satisfying $\phi(P') = P$. Then the terms $F_k(E, P)$ are composite for sufficiently large $k$.

Proof. Using Siegel’s Theorem, we know $\hat{h}(P) = m \hat{h}(P')$, where $m$ is the degree of the given isogeny $\phi$. Therefore, for sufficiently large $k$, there is a prime divisor which is a proper divisor of $F_k(E, P)$ by Theorem 2.8. \qed

Example 2.10. We can see the divisibility of corresponding elliptic Fermat numbers using the following magnified elliptic divisibility sequence, which has a degree 3 isogeny $\phi$ that maps

$$E'_1 : y^2 = x^3 - 9x + 9, \quad \text{with } P' = [1, 1]$$

to

$$E_1 : y^2 = x^3 - 189x - 999, \quad \text{with } P = [-8, 1].$$

Then we get the following factorizations, and we can check the divisibility of corresponding elliptic Fermat numbers.

\[
\begin{array}{c|c}
F_1(E'_1, P') &= 1 \\
F_2(E'_1, P') &= 17 \\
F_3(E'_1, P') &= 53 \cdot 127 \\
F_4(E'_1, P') &= 89 \cdot 179 \cdot 307 \cdot 5813 \cdot 67211 \\
& \quad \times 838133 \\
& \quad \times 265666679 \\
& \quad \times 3205176128020873 \\
\end{array}
\]

\[
\begin{array}{c|c}
F_1(E_1, P) &= 2 \\
F_2(E_1, P) &= 2 \cdot 17 \cdot 19 \\
F_3(E_1, P) &= 2 \cdot 53 \cdot 127 \cdot 10799 \cdot 14867 \\
F_4(E_1, P) &= 2 \cdot 89 \cdot 179 \cdot 307 \cdot 757 \cdot 3205176128020873 \\
& \quad \times 838133 \\
\end{array}
\]

Similarly, for a degree 7 isogeny which maps

$$E'_2 : y^2 + xy = x^3 - x^2 + x + 1, \quad \text{with } Q' = [0, 1]$$

to

$$E_2 : y^2 + xy = x^3 - x^2 - 389x - 2859, \quad \text{with } Q = [26, 51],$$

we have

\[
\begin{align*}
F_1(E'_2, Q') &= 1 & F_1(E_2, Q) &= 1 \\
F_2(E'_2, Q') &= 3 & F_2(E_2, Q) &= 3 \cdot 701 \\
F_3(E'_2, Q') &= 11 & F_3(E_2, Q) &= 11 \cdot 233 \cdot 2887 \cdot 273001 \\
F_4(E'_2, Q') &= 1523 \cdot 15443 & F_4(E_2, Q) &= 103 \cdot 131 \cdot 311 \cdot 467 \cdot 1523 \cdot 11831 \cdot 15443 \cdot 12539851 \cdot 7015932452763098743789
\end{align*}
\]

3. Generalized elliptic Fermat numbers and their properties

3.1. Coprimality and order universality. The authors of [1], motivated by the coprimality of the classical Fermat numbers, show that any two distinct elliptic Fermat numbers are either relatively prime, or else their only common factor is 2. In this section, we prove an analogous result for generalized elliptic Fermat numbers generated by odd \( m \). We also generalize the order universality properties stated in [1, Theorem 4] and [1, Corollary 5].

Before proving any results, we present an example of a sequence of generalized elliptic Fermat numbers generated by \( m = 3 \):

**Example 3.1.** Let \( E : y^2 = x^3 + x^2 - 4x \), \( P = (-2, 2) \) and \( m = 3 \). The first four generalized elliptic Fermat numbers are listed below.

\[
\begin{align*}
P &= \left( \frac{-2}{1}, \frac{2}{1} \right) & F_0^{(3)}(E, P) &= 1 \\
3P &= \left( \frac{-2^3}{3}, \frac{26}{3} \right) & F_1^{(3)}(E, P) &= \frac{3}{1} = 3 \\
9P &= \left( \frac{-213293858}{10593^2}, \frac{2478721052834}{10593^3} \right) & F_2^{(3)}(E, P) &= \frac{10593}{3} = 3531 \\
27P &= \left( \frac{-2387\ldots4098}{4777\ldots2659^2}, \frac{7135\ldots8638}{4777\ldots2659} \right) & F_3^{(3)}(E, P) &= \frac{4777\ldots2659}{10593} = 4509 \ldots 2163 \ (33 \text{ digits})
\end{align*}
\]

We will now prove the following ”coprimality” theorem, which states that the gcd of any two generalized elliptic Fermat numbers generated by an odd integer \( m \) must divide \( m \):

**Theorem 3.2** (Coprimality). Let \( F = (F_k^{(m)}(E, P))_{k \in \mathbb{N}} \) be the sequence of generalized elliptic Fermat numbers for a fixed elliptic curve \( E \), a rational point \( P \in E(\mathbb{Q}) \) and an odd integer \( m \geq 1 \). Then for all distinct \( k, \ell \geq 0 \),

\[
\gcd(F_k^{(m)}(E, P), F_\ell^{(m)}(E, P)) \mid m.
\]

The heart of the proof relies on the following lemma from [3]:

**Lemma 3.3.** Let \( D = (D_n)_{n \geq 1} \) be a minimal EDS, let \( n \geq 1 \), and let \( p \) be a prime satisfying \( p \mid D_n \).

(a) For all \( m \geq 1 \) we have

\[
\operatorname{ord}_p(D_{mn}) \geq \operatorname{ord}_p(mD_n).
\]
(b) The inequality in (a) is strict,
\[ \text{ord}_p(D_{mn}) > \text{ord}_p(mD_n), \]
if and only if
\[ p = 2, 2 \mid m, \text{ord}_2(D_n) = 1 \text{ and } (E \text{ has ordinary or multiplicative reduction at } 2). \]

For our purposes, the conditions of (b) will never be met, since we are only working with odd \( m \). Thus, we always have equality, i.e., if a prime \( p \) satisfies \( p \mid D_n \), then
\[ \text{ord}_p(D_{mn}) = \text{ord}_p(mD_n). \]

We can apply Lemma 3.3 to generalized elliptic Fermat numbers in the following way:

**Proposition 3.4.** Let \( D = (D_n)_{n \geq 1} \) be a minimal EDS, and let \( m \geq 1 \) be an odd integer. If \( p \mid D_{m^{s-1}} \) for some \( s \geq 1 \), then
\[ \text{ord}_p(F_s^{(m)}) = \text{ord}_p(m). \]

**Proof.** Consider the case of Lemma 3.3 where \( n = m^{s-1} \). Then if \( p \mid D_{m^{s-1}} \) for some \( s \geq 1 \),
\[ \text{ord}_p(D_{m^s}) = \text{ord}_p(mD_{m^{s-1}}). \]

It immediately follows that
\[ \text{ord}_p(F_s^{(m)}) = \text{ord}_p\left(\frac{D_{m^s}}{D_{m^{s-1}}}\right) = \text{ord}_p(m). \]

We now use Proposition 3.4 to prove Theorem 1.9. For simplicity, we will let \( F_k^{(m)} \) denote \( F_k^{(m)}(E, P) \) whenever it appears in the rest of the paper.

**Proof.** Let \( p \) be prime, and suppose \( p \mid D_{m^{t-1}} \) for some \( s \geq 1 \). Let \( t \) be the smallest index for which \( p \mid D_{m^{t-1}} \), i.e., let \( t = \min\{s \geq 1 : p \mid D_{m^{s-1}}\} \). Since \( \{D_n\} \) is a divisibility sequence, it is given that \( p \mid D_{m^{t-1}} \) implies \( p \mid D_{m^{k-1}} \) for all \( k \geq t \). Without loss of generality, we assume \( k < \ell \) throughout the proof. Thus, for all distinct \( k, \ell \) with \( \ell > k \geq t \), we can use Proposition 3.4 in order to conclude that
\[ \text{ord}_p(F_k^{(m)}) = \text{ord}_p(F_\ell^{(m)}) = \text{ord}_p(m). \]

Therefore, for each prime \( p \) that divides a term in \( \{D_n\} \), and for all distinct \( k, \ell \geq t \), where \( t \) is the entry point of \( p \), we have shown that
\[ \text{ord}_p(\gcd(F_k^{(m)}, F_\ell^{(m)})) = \text{ord}_p(m). \]

For all distinct \( k, \ell \) with \( k < t - 1 \), \( p \) is not a factor of \( \gcd(F_k^{(m)}, F_\ell^{(m)}) \). To see this, note that \( t = \min\{s \geq 1 : p \mid D_{m^{s-1}}\} \) implies \( p \nmid D_{m^k} \). So \( p \nmid F_k^{(m)} \), and for all distinct \( k, \ell \) with \( k < t - 1 \), we have the desired
\[ \text{ord}_p(\gcd(F_k^{(m)}, F_\ell^{(m)})) = 0. \]

When \( k = t - 1 \) and \( \ell > k \), we have
\[ p \mid D_{m^{k-1}} \text{ and } p \mid D_{m^\ell} \]
and
\[ p \mid D_{m^{t-1}} \text{ and } p \mid D_{m^t}. \]

Therefore, we have \( \text{ord}_p(F_k^{(m)}) > 0 \) and \( \text{ord}_p(F_\ell^{(m)}) = \text{ord}_p(m) \) and we get
\[ \text{ord}_p(\gcd(F_k^{(m)}, F_\ell^{(m)})) = \text{ord}_p(m). \]
It follows from Equations (2), (3), and (4) that for any distinct $k, \ell$ and any prime $p$,

$$\text{ord}_p(\gcd(F_k^{(m)}, F_\ell^{(m)})) = \begin{cases} \text{ord}_p(m) & \text{if } p \mid D_{m^t} \text{ for some } t \leq k \\ 0 & \text{otherwise.} \end{cases}$$

This implies $$\gcd(F_k^{(m)}, F_\ell^{(m)}) \mid m.$$ \hfill \(\square\)

**Remark 3.5.** The proof of Theorem 1.9 actually implies a more specific result than the theorem states. For once a prime $p$ appears as a divisor of some $F_t^{(m)}$, then $p^{\text{ord}_p(m)} \mid F_k^{(m)}$ for all $k \geq t$. Thus, if for example $3 \mid F_t^{(15)}(E, P)$, then Theorem 1.9 only tells us that $\gcd(F_k^{(15)}, F_\ell^{(15)}) \in \{1, 3, 5, 15\}$ for all distinct $k, \ell \geq t$, but in actuality, we know that $\gcd(F_k^{(15)}, F_\ell^{(15)}) \in \{3, 15\}$ because $3 \nmid 1$ and $3 \nmid 5$. This is even stronger in the case where $m = p^a$ for some prime $p$, as stated in the proof of the corollary below.

**Corollary 3.6.** Let $F = (F_k^{(p^a)}(E, P))_{k \in \mathbb{N}}$ be the sequence of generalized elliptic Fermat numbers for a fixed elliptic curve $E$, a rational point $P \in E(\mathbb{Q})$ and an odd prime power $p^a$. Then for all distinct $k, \ell \geq 0$,

$$\gcd(F_k^{(p^a)}(E, P), F_\ell^{(p^a)}(E, P)) \in \{1, p^a\}.$$  

**Proof.** The proof follows from (5). \hfill \(\square\)

**Example 3.7.** We factor the generalized elliptic Fermat sequence from Example 3.1:

- $F_0^{(3)}(E, P) = 1$
- $F_1^{(3)}(E, P) = \frac{3}{1} = 3$
- $F_2^{(3)}(E, P) = \frac{10503}{3} = 3531 = 3 \times 11 \times 107$
- $F_3^{(3)}(E, P) = \frac{477772659}{10593} = 4509 \ldots 2163 = 3 \times 324076900879427 \times 46385324158085723$

We see that $\gcd(F_0^{(3)}, F_1^{(3)}) = 1$, while $\gcd(F_2^{(3)}, F_\ell^{(3)}) = 3$ for distinct $1 \leq k, \ell \leq 3$.

In addition to proving the quasi-coprimality of elliptic Fermat numbers, the authors of [1] include a result connecting divisibility with order, which they call order universality. This property holds in full force for generalized elliptic Fermat numbers. In fact, the proofs are nearly the same as the proofs for the case where $m = 2$.

**Theorem 3.8.** Let $F = (F_k^{(m)})_{k \in \mathbb{N}}$ be the sequence of generalized elliptic Fermat numbers for a fixed elliptic curve $E$, a rational point $P \in E(\mathbb{Q})$ and an odd integer $m \geq 1$. Then for all $N \in \mathbb{N}$ satisfying $\gcd(N, 6\Delta(E)) = 1$,

$$P \text{ has order } m^k \text{ in } E(\mathbb{Z}/N\mathbb{Z}) \iff N \mid F_0^{(m)} \cdots F_k^{(m)} \text{ and } N \nmid F_0^{(m)} \cdots F_{k-1}^{(m)}.$$  

**Proof.** The proof is identical to that of Theorem 4 in [1]. Namely, we define a homomorphism $\phi : E(\mathbb{Q}) \to E(\mathbb{Z}/N\mathbb{Z})$ that maps $P \to P \mod (n)$, then use the fact that $\phi(p^kP) = p^k\phi(P)$ to demonstrate that $P$ has order $m^k$ in $E(\mathbb{Z}/N\mathbb{Z})$ exactly when $N \mid F_0^{(m)} \cdots F_k^{(m)}$ and $N \nmid F_0^{(m)} \cdots F_{k-1}^{(m)}$. \hfill \(\square\)
Corollary 3.9. Let $F = (F_k^{(m)})_{k \in \mathbb{N}}$ be the sequence of generalized elliptic Fermat numbers for a fixed elliptic curve $E$, a rational point $P \in E(\mathbb{Q})$ and an odd integer $m \geq 1$. Let $p$ be an odd prime. Then

$$P \text{ has order } m^k \text{ in } E(\mathbb{F}_p) \iff p \mid F_k^{(m)}.$$  

Proof. The proof is similar to that of Corollary 5 in [1]. However, whereas the proof in [1] relies on the fact that $\gcd(F_k^{(2)}, F_\ell^{(2)}) \in \{1, 2\}$, here we require that $p \nmid m$ in order to make use of Theorem 1.9 which tells us that $\gcd(F_k^{(m)}, F_\ell^{(m)}) \mid m$.

The adapted proof proceeds as follows:

If $p \mid F_0^{(m)}(E, P) \cdots F_k^{(m)}(E, P)$ and $p \nmid F_0^{(m)}(E, P) \cdots F_k^{(m)}(E, P)$, then $p \mid F_k^{(m)}(E, P)$. Conversely, if $p \mid F_k^{(m)}(E, P)$, then $p \mid F_0^{(m)}(E, P) \cdots F_k^{(m)}(E, P)$. Theorem 1.9 gives us $p \mid F_i^{(m)}(E, P)$ for all $i \neq k$, implying $p \mid F_0^{(m)}(E, P) \cdots F_{k-1}^{(m)}(E, P)$. Thus we have shown that $p \mid F_k^{(m)}(E, P)$ if and only if $p \mid F_0^{(m)}(E, P) \cdots F_n^{(m)}(E, P)$ and $p \mid F_0^{(m)}(E, P) \cdots F_{k-1}^{(m)}(E, P)$, and the desired result follows from Theorem 3.8. □

Example 3.10. Using the curve $E$, point $P$ and integer $m$ from Example 3.7, observe that the order of $P \in E(\mathbb{F}_{593}) = 3^2$, and indeed, $593 \mid F_2^{(3)} = 3 \ast 593 = 1779$.

3.2. Primality conjecture for generalized elliptic Fermat numbers. We can also extend previous discussions about the Primality of elliptic Fermat numbers to generalized elliptic Fermat numbers. First, we state an analogous conjecture for generalized elliptic Fermat numbers.

Conjecture 3.11 (Primality Conjecture for generalized elliptic Fermat numbers). Let $F = (F_k^{(m)}(E, P))_{k \in \mathbb{N}}$ be the sequence of generalized elliptic Fermat numbers for a fixed elliptic curve $E$ and a rational point $P \in E(\mathbb{Q})$. Then $F$ contains only finitely many prime terms.

Using Corollary 3.9, we can prove the following result.

Lemma 3.12. Let $E/\mathbb{Q}$ be a minimal magnified elliptic curve with a fixed point $P \in E(\mathbb{Q})$ having a (nonzero) degree $d$ isogeny $\phi : E' \to E$ from a minimal elliptic curve $E'/\mathbb{Q}$ satisfying $\phi(P') = P$. For sufficiently large $k$ and $m$ with $\gcd(m, d) = 1$, we have

$$\gcd(F_k^{(m)}(E', P'), F_k^{(m)}(E, P)) \neq 1.$$  

Proof. Let $p$ be a fixed prime which divides $F_k^{(m)}(E', P')$. Let $S$ be the set of primes for which $E$, $E'$, and $\phi$ cannot give an isogeny modulo $p$. Note that $S$ is a finite set. Without loss of generality, we can assume $p \notin S$. From Corollary 2.3 it is sufficient to prove that the isogeny $\phi$ reduction modulo $p$ (which is again an isogeny) preserves the order $m^k$ of $P'$

$$\phi : E'(\mathbb{F}_p) \to E(\mathbb{F}_p).$$  

We consider the dual isogeny $\hat{\phi}$ of $\phi$. We know

$$\hat{\phi} \circ \phi = [d]$$  

where $[d]$ is the multiplication-by-$d$ map on $E'$. Then the map

$$[d] : E'(\mathbb{F}_p) \xrightarrow{\phi} E(\mathbb{F}_p) \xrightarrow{\hat{\phi}} E'(\mathbb{F}_p)$$  

preserves the order of $m^k$ of $P'$ and so does $\hat{\phi}$. □
Proposition 3.13. Let $E/\mathbb{Q}$ be a minimal magnified elliptic curve with a fixed point $P \in E(\mathbb{Q})$ having a (nonzero) degree $d$ isogeny $\phi : E' \to E$ from a minimal elliptic curve $E'/\mathbb{Q}$ satisfying $\phi(P') = P$. For sufficiently large $k$, we actually have the following divisibility: For $m$ relatively prime to $d$, \[ F_k^{(m)}(E', P') \mid F_k^{(m)}(E, P). \]

Example 3.14. We can see the divisibility of corresponding generalized elliptic Fermat numbers using the following magnified elliptic divisibility sequence, which has a degree 2 isogeny $\phi$ that maps 
\[ E' : y^2 = x^3 + x^2 - 4x, \quad \text{with } P' = [-2, 2] \]
to 
\[ E : y^2 = x^3 + x^2 + 16x + 16, \quad \text{with } P = [0, 4], \]
then we get following list of $F_k^{(3)}(E', P')$ and $F_k^{(3)}(E, P)$.
\[
\begin{align*}
F_1^{(3)}(E', P') &= 3 & F_1^{(3)}(E, P) &= 3 \\
F_2^{(3)}(E', P') &= 3 \cdot 11 \cdot 107 & F_2^{(3)}(E, P) &= 3 \cdot 11 \cdot 23 \cdot 107 \cdot 449 \\
F_3^{(3)}(E', P') &= 3 \cdot 3240769000879427 \cdot 46385324158085723 & F_3^{(3)}(E, P) &= 3 \cdot 114078700999 \cdot 3240769000879427 \cdot 46385324158085723 \cdot 927508107491526089159 \\
& \vdots & & \vdots
\end{align*}
\]

From Remark 2.7, we can also describe the growth of generalized elliptic Fermat numbers using the canonical height of $P$.

Theorem 3.15. Let $E/\mathbb{Q}$ be an elliptic curve with a fixed point $P \in E(\mathbb{Q})$. Denote by $\hat{h}(P)$ the canonical height of $P$. For any $m$, we get
\[
\lim_{k \to \infty} \frac{\log(F_k^{(m)}(E, P))}{m^{2k}} = \left( \frac{1}{2} - \frac{1}{2m^2} \right) \cdot \hat{h}(P).
\]

Proof.
\[
\begin{align*}
\lim_{k \to \infty} \frac{\log(F_k^{(m)}(E, P))}{m^{2k}} &= \lim_{k \to \infty} \frac{\log(D_{m^k})}{m^{2k}} \\
&= \lim_{k \to \infty} \frac{\log(D_{m^k})}{m^{2k}} \cdot \lim_{k \to \infty} \frac{1}{2m^2} \cdot \frac{\log(D_{m^k - 1})}{m^{2(k-1)}} \\
&= \left( \frac{1}{2} - \frac{1}{2m^2} \right) \cdot \hat{h}(P).
\end{align*}
\]

Note that when $m = 2$, the result coincides with Theorem 2.8.

Theorem 3.16. Let $E/\mathbb{Q}$ be a minimal magnified elliptic curve with a fixed point $P \in E(\mathbb{Q})$ having a (nonzero) degree $d$ isogeny $\phi : E' \to E$ from a minimal elliptic curve $E'/\mathbb{Q}$ satisfying $\phi(P') = P$. For $m$ relatively prime to $d$, $F_k^{(m)}(E, P)$ are composite for sufficiently large $k$. 
Proof. Using Siegel’s Theorem, we know
\[ \hat{h}(P) = d\hat{h}(P'), \]
where \( d \) is the degree of the given isogeny \( \phi \). Therefore, for sufficiently large \( k \), there is a prime divisor which is a proper divisor of \( F_k^{(m)}(E, P) \) by Theorem 3.15. \( \square \)

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