Abstract. Let $\Phi^h(x)$ with $x = (t, y)$ denote the near-critical scaling limit of the planar Ising magnetization field. We take the limit of $\Phi^h$ as the spatial coordinate $y$ scales to infinity with $t$ fixed and prove that it is a stationary Gaussian process $X(t)$ whose covariance function is the Laplace transform of a mass spectral measure $\rho$ of the relativistic quantum field theory associated to the Euclidean field $\Phi^h$. Our analysis of the small distance/time behavior of the covariance functions of $\Phi^h$ and $X(t)$ shows that $\rho$ is finite but has infinite first moment.

1. Synopsis

In [3] (resp., [4]), it was shown that the critical Ising model (resp., near-critical model with external magnetic field $ha^{15/8}$) on the rescaled lattice $a\mathbb{Z}^2$ has a scaling limit $\Phi^0$ (resp., $\Phi^h$) as $a \downarrow 0$ — denoted then by $\Phi^\infty$ (resp., $\Phi^{\infty,h}$). $\Phi^h$ is a (generalized) random field on $\mathbb{R}^2$ — i.e., for suitable test functions $f$ on $\mathbb{R}^2$, there are random variables $\Phi^h(f)$, formally written as $\int_{\mathbb{R}^2} \Phi^h(x)f(x)dx$. Euclidean random fields such as $\Phi^h$ on the Euclidean “space-time” $\mathbb{R}^d := \{x = (x_0, y_1, \ldots, y_{d-1})\}$ (in our case $d = 2$) are related to quantum fields on relativistic space-time, $\{(t, y_1, \ldots, y_{d-1})\}$, essentially by replacing $x_0$ with a complex variable and analytically continuing from the purely real $x_0$ to a pure imaginary ($-it$) — see [23], Chapter 3 of [13] and [19] for background. One major reason for interest in $\Phi^h$ is that the associated quantum field is predicted [25, 26] to have remarkable properties including relations between the masses of particles within the quantum field theory and the Lie algebra $E_8$ — see [8, 2, 18].

In [5], exponential decay of truncated correlations in $\Phi^h$ was proved; this shows, roughly speaking, the existence of at least one particle with strictly positive mass and no smaller mass particles in the relativistic quantum field theory associated to $\Phi^h$. In this paper, we study a Gaussian process $X(t)$ which is the long spatial distance scaling limit of $\Phi^h$ and turns out to be related to a mass spectral measure $\rho(m)$ of the relativistic quantum field theory associated to $\Phi^h$. More precisely, the covariance function $K(t)$ of $X(t)$ is given by the Laplace transform of $\rho(m)$. For more about the appearance of the particle masses of [25, 26] in $K$ (and thus in the atoms of $\rho$), see (18) below.
2. Introduction

2.1. Background. In (classical) relativistic field theory, the existence of a conserved energy-momentum four-vector \( p = (E, \mathbf{p}) \) is guaranteed by time and translation invariance. In quantizing a classical theory to a relativistic quantum field theory, the energy-momentum four-vector becomes a self-adjoint operator \( P = (\mathcal{H}, \mathcal{P}) \) acting on quantum states in a Hilbert space. On physical grounds, it is natural to make certain assumptions on the spectrum of the operator \( P \). These include (see, e.g., [1]):

(i) The spectrum of \( P \) all lies within the forward light cone:
\[
E^2 - \mathbf{p}^2 \geq 0 \quad \text{and} \quad E \geq 0.
\] (1)

(ii) There exists a unique Lorentz-invariant ground state \( \Omega \) of lowest energy. This is the vacuum state, whose energy is chosen by convention to be zero: \( \mathcal{H}\Omega = 0 \). From this and (1), it follow that \( \mathcal{P}\Omega = 0 \) also. The Lorentz invariance of the vacuum ensures that \( \Omega \) appears as the vacuum state to all inertial observers.

(iii) There exist states \( \psi \) which are (generalized) eigenvectors of both \( \mathcal{H} \) and \( \mathcal{P} \) and whose eigenvalues satisfy the relation \( E_i^2 - \mathbf{p}_i^2 = m_i^2 \). Each \( \psi \) corresponds to a single-particle state describing a stable particle of mass \( m_i \).

(iv) For a field theory whose particle content is a single type of particle with mass \( m_1 \), the spectrum of the energy-momentum operator \( P \) has a discrete part containing the eigenvalue zero corresponding to \( \Omega \) as well as the one-particle hyperboloid \( E^2 - \mathbf{p}^2 = m_1^2 \) corresponding to \( \psi_1 \). The spectrum also contains a continuous part that lies above the hyperboloid \( E^2 - \mathbf{p}^2 = 4m_1^2 \).

Examples of quantum field theories with these properties have been constructed rigorously. The simplest of those is the free scalar field of mass \( \mu \), also known as free boson field or Klein-Gordon field because the classical field \( \varphi \) satisfies the Klein-Gordon equation
\[
\left( \frac{\partial^2}{\partial t^2} - \nabla^2 + \mu^2 \right) \varphi(t, \mathbf{y}) = 0.
\] (2)

This field theory describes a free particle of mass \( \mu \), and the spectrum of the energy-momentum operator has the form described in item (iv) above.

More complex, interacting quantum field theories can be constructed starting from Euclidean fields such as \( \Phi^h \). Generally speaking (see, e.g., [13]), a Euclidean (generalized) field \( \Phi \) is a random generalized function, that is, a random element of the space \( \mathcal{D}'(\mathbb{R}^d) \) of real distributions. Equivalently, a Euclidean field can be thought of as a probability measure \( \mathbb{P} \) on \( \mathcal{D}'(\mathbb{R}^d) \). \( \Phi \) acts on test functions \( f \in \mathcal{D}(\mathbb{R}^d) = C_0^\infty(\mathbb{R}^d) \), and for each \( f \), \( \Phi(f) \) is a real random variable in \( L^2(\mathbb{P}) \). Note that one can sometimes work with more general test functions and this will be the
case in this paper. As the name suggests, a Euclidean field is assumed to be invariant under Euclidean transformations.

Taking a suitable test function $f(y)$ on $\mathbb{R}^{d-1}$, $\Phi(f(y)\delta_0(x_0)) \in L^2(\mathbb{P}_0)$, where $\mathbb{P}_0$ is the restriction of $\mathbb{P}$ to the Euclidean “time” $x_0 = 0$ subspace. The space $\mathbb{H} := L^2(D'(\mathbb{R}^{d-1}), \mathbb{P}_0)$ is interpreted as a quantum mechanical Hilbert space and $\psi := \Phi(f(y)\delta_0(x_0))$ as a state vector. The Hilbert space $L^2(D'(\mathbb{R}^d), \mathbb{P})$ is the path space for the quantum operators in the Heisenberg representation.

If the Euclidean field $\Phi$ has the Markov property, as is typically the case for fields arising in statistical mechanics, then one has that

$$\text{Cov}(\Phi^h(f(y)\delta_0(x_0)), \Phi^h(f(y - x)\delta_0(x_0)))$$

$$= \text{Cov}(\Phi^h(f(y)\delta_0(x_0)), e^{-sH}\Phi^h(f(y - x)\delta_0(x_0)))$$

$$= (\psi, (e^{ix\mathcal{P}} e^{-sH} - \mathcal{P}_0)\psi)$$

(3)

$$= (\psi, (e^{ix\mathcal{P}} e^{itH} - \mathcal{P}_0)\psi),$$

(4)

where $t = is$, $(\cdot, \cdot)$ denotes the inner product in the Hilbert space $L^2(D'(\mathbb{R}^{d-1}), \mathbb{P}_0)$, $\mathcal{H}$ is the Hamiltonian/energy operator (the generator of the Markov semigroup of $\Phi$ for translations along Euclidean time), $\mathcal{P}$ is the momentum operator (the generator of spatial translations in Euclidean space), and $\mathcal{P}_0$ is the orthogonal projection onto the eigenspace of constant random variables (i.e., $\mathcal{P}_0\psi = \mathbb{E}_0\psi$, where $\mathbb{E}_0$ denotes expectation with respect to $\mathbb{P}_0$). Equations (3) and (4) follow from the Osterwalder-Schrader reconstruction theorem [23] and provide the connection between Euclidean fields and Relativistic Quantum Field Theory. Indeed, the Euclidean invariance of $\Phi$ implies that, after performing a so-called Wick rotation, i.e., after replacing $s$ by $-it$, the field becomes invariant under Lorentz transformations, hence a relativistic field.

2.2. Main results. Let $H(t, y)$ be the covariance function of $\Phi^h$. The existence of $H$ follows from Proposition 6.1.4 of [13]. Note that the analyticity of $\mathbb{E}(\exp(i\Phi^h(f)))$ (which is axiom OS0 on p. 91 of [13]) can be proved for $h = 0$ by using the GHS inequality [14] (as in Proposition 3.5 and Corollary 3.8 of [3]) or by arguments based on the Lee-Yang theorem [16]. A useful technical tool in the Lee-Yang setting is a result (Theorem 7 of [22]) that Gaussian tail bounds are preserved under convergence in distribution when the Lee-Yang property is valid. The GHS inequality can then be used to extend Gaussian tail bounds and hence analyticity from $h = 0$ to $h > 0$. We remark that $H$ actually depends on $h$; for most of this paper we fix $h > 0$ and only specify the dependence on $h$ when there might be confusion. Note that $H$ for different values of $h$ are related to each other by a scale transformation.
(see (35) or (36) below). Loosely speaking,
\[ H(t, y) = \text{Cov} \left( \Phi^h(t_0, y_0), \Phi^h(t_0 + t, y_0 + y) \right) \text{ for any } (t_0, y_0) \in \mathbb{R}^2. \]

(5)

To study the long spatial distance behavior, we define a family of stochastic processes \( \{X_L(s) : s \geq 0\} \) indexed by a parameter \( L > 0 \):
\[ X_L(s) := \frac{\Phi^h(1_{[-L,L]}(y)\delta_s(t)) - E\Phi^h(1_{[-L,L]}(y)\delta_s(t))}{\sqrt{2L}}, \]
(6)
where \( 1_{[-L,L]}(y)\delta_s(t) \) is the product of an interval indicator function in \( y \) and a delta function in \( t \), and \( E \) is the expectation with respect to the field \( \Phi^h \). Formally,
\[ \Phi^h(1_{[-L,L]}(y)\delta_s(t)) := \int_{\mathbb{R}^2} \Phi^h(t, y)1_{[-L,L]}(y)\delta_s(t) \ dtdy. \]
(7)

The integral in (7) is not a priori well-defined because of the delta function. We will show in Appendix A that it is well-defined by approximating the delta function with nice (say smooth) functions. The mean zero stationary Gaussian process \( \{X_s : s \in \mathbb{R}\} \) which will be the main focus of this paper is defined by the covariance function:
\[ \text{Cov}(X(s), X(t)) = K(t-s) := \int_{-\infty}^{\infty} H(t-s, y)dy \text{ for any } s, t \in \mathbb{R}. \]
(8)

Our first main result shows how \( X(s) \) arises naturally from \( \Phi^h \) and how it is related to the relativistic quantum field theory associated to \( \Phi^h \).

**Theorem 1.** For any \( n \in \mathbb{N} \) and distinct \( s_1, \ldots, s_n \in \mathbb{R} \), we have
\[ (X_L(s_1), \ldots, X_L(s_n)) \Rightarrow (X(s_1), \ldots, X(s_n)) \text{ as } L \to \infty, \]
where \( \Rightarrow \) denotes convergence in distribution. Moreover, there exists \( m_1 > 0 \) such that the covariance function \( K(s) \) of \( X(s) \) is given by
\[ K(s) = \int_{m_1}^{\infty} e^{-m|s|}d\rho(m), \]
(10)
where \( \rho(m) \) is a mass spectral measure of the relativistic quantum field theory obtained from \( \Phi^h \) via the Osterwalder-Schrader reconstruction theorem \[23\]. See (29) and (33) below for the precise definition of \( \rho \).

**Remark 1.** We believe, but have not yet proved, that as a process
\[ \{X_L(s) : 0 \leq s \leq 1\} \Rightarrow \{X(s) : 0 \leq s \leq 1\}, \]
where \( \Rightarrow \) denotes convergence in distribution in the space \( C[0,1] \) of continuous functions with the sup norm topology. However, if we define
\[ Y_L(t) := \int_0^t X_L(s)ds, \]
(12)
then one may apply a similar result as in Theorem 2 of \[21\] to show that \( Y_L(t) \) does converge in distribution to \( \int_0^t X(s)ds \).
Remark 2. The first part of Theorem 1 implies (by combining it with arguments like those in [20]) that as $\lambda \to \infty$

$$\sqrt{\lambda} \left[ \Phi^h(t, \lambda y) - B h^{1/15} \right] \Rightarrow \hat{\Phi}^h(t, y) ,$$

where $B \in (0, \infty)$ is a constant (see (11) and (14) below) and $\hat{\Phi}^h$ is a mean zero Gaussian field with covariance function

$$\text{Cov} \left( \hat{\Phi}^h(t_0, y_0), \hat{\Phi}^h(t_0 + t, y_0 + y) \right) = K(t) \delta_0(y) \quad \forall (t_0, y_0) \in \mathbb{R}^2 .$$

(14)

Remark 3. $K$ and $m_1$ in Theorem 1 depend on $h$, with $m_1 = Ch^{8/15}$ for some constant $C < \infty$ (see Corollary 1.6 of [5]).

Note that $H$ is a function only of the radial variable $\sqrt{t^2 + y^2}$. Our second result is about the small distance behavior of $H$ and $K$. Before stating our result, we recall the covariance function of $\Phi^0$. From Remark 1.4 of [7], one has for all $(t, y) \in \mathbb{R}^2$ with $(t, y) \neq (0, 0)$,

$$H^0(t, y) = \text{Cov} \left( \Phi^0(0, 0), \Phi^0(t, y) \right) = C_1 (t^2 + y^2)^{-1/8} ,$$

(15)

where $C_1 \in (0, \infty)$ is independent of $t$ and $y$.

Theorem 2.

$$\lim_{\lambda \downarrow 0} \lambda^{1/4} H(0, \lambda y) = H^0(0, y) = C_1 |y|^{-1/4} , \quad y \in \mathbb{R} \setminus \{0\} .$$

(16)

Moreover,

$$\lim_{\epsilon \downarrow 0} \frac{K(0) - K(\epsilon)}{\epsilon^{3/4}} = 2 \int_0^\infty \left[ H^0(0, y) - H^0(1, y) \right] dy \in (0, \infty) .$$

(17)

Remark 4. The limit (17) implies that $K(s)$ is not differentiable at $0$. By a classic result of Fernique [12] (see also [17], (17) also implies that $X(t)$ has continuous sample paths. Loosely speaking, the sample path of $|X(t) - X(0)|$ behaves locally like $|t|^{3/8}$, which is rougher than a one-dimensional Brownian motion.

The next result summarizes what the behavior of $K$ described in Theorem 2 tells us about the mass spectral measure $\rho(m)$ of the relativistic quantum field theory associated with $\Phi^h$.

Corollary 1. The mass spectral measure $\rho(m)$ in (10) is a finite measure, but its first moment is infinite.

Proof. Theorem 2 implies that $K(0) < \infty$. This, together with (10), proves the first claim. The second claim follows from the observation that, if $\rho(m)$ had a finite first moment, $K(s)$ would be differentiable at $s = 0$, contradicting (17) in Theorem 2. □

$K$ should actually capture much more information about the particle masses of the quantum field theory associated with $\Phi^h$. Based on [25, 26, 8] (see, e.g., (4.60) of [8]) and (10), one expects that there should
be masses \( m_1, m_2, m_3 \in (0, \infty) \) and constants \( B_1, B_2, B_3 \in (0, \infty) \) such that, for large \( t \)

\[
K(t) = B_1 e^{-m_1|t|} + B_2 e^{-m_2|t|} + B_3 e^{-m_3|t|} + O\left(e^{-2m_1|t|}\right),
\]

(18)

where \( m_1 < m_2 < m_3 < 2m_1 \) and the \( m_1 \) here is the same as in (10). In Appendix B, we derive the first order behavior (i.e., \( B_1 e^{-m_1|t|} \)) based on a conjecture on the large distance behavior of \( H \), for which we also provide a heuristic argument.

3. Proof of the main results

A key property of \( H \) that we will use is that \( H(0, y) \) is continuous on \( \mathbb{R} \setminus \{0\} \). Indeed, because of the relation between Euclidean and quantum field theories, \( H \) is real analytic in the half-plane \( \{(t, y) : t > 0\} \). We will also use another important property: \( H(0, y) \) is nonincreasing on \( (0, \infty) \); this can be seen in a variety of ways — e.g., by spectral representation arguments like those used below in the second part of the proof of Theorem 1.

Proof of Theorem 1

By the classical convergence theorem (see, e.g., p.167 of [9]), (9) is equivalent to

\[
\lim_{L \to \infty} \mathbb{E} e^{i [z_1 X_L(s_1) + \cdots + z_n X_L(s_n)]} = \mathbb{E} e^{i [z_1 X(s_1) + \cdots + z_n X(s_n)]}
\]

(19)

for each \((z_1, \ldots, z_n) \in \mathbb{R}^n\).

We will first prove (19) for \((z_1, \ldots, z_n) \in (\mathbb{R}^+)^n\). Under the assumption that all \( z_i \)'s are nonnegative, we will show that the sequence

\[
\{Y_k := \sum_{j=1}^{\infty} z_j \Phi^h \left(1_{[k,k+1)}(y)\delta_{\delta_i}(t)\right) : k \in \mathbb{Z}\}
\]

(20)

satisfies all the conditions in Theorem 2 of [20]. We have

\[
\text{Cov}(Y_0, Y_k) = \sum_{j=1}^{\infty} \sum_{l=1}^{\infty} z_j z_l \text{Cov} \left( \Phi^h \left(1_{[0,1)}(y)\delta_{\delta_i}(t)\right), \Phi^h \left(1_{[k,k+1)}(y)\delta_{\delta_i}(t)\right)\right)
\]

\[
= \sum_{j=1}^{\infty} \sum_{l=1}^{\infty} z_j z_l \int_{0}^{1} \int_{k}^{k+1} H(s_l - s_j, y_2 - y_1) dy_2 dy_1
\]

\[
= \sum_{j=1}^{\infty} \sum_{l=1}^{\infty} z_j z_l \int_{k-1}^{k+1} H(s_l - s_j, u) [1 - |u - k|] du.
\]

(21)

Therefore, by the continuity of \( H \) on \( \mathbb{R} \setminus \{0\} \),

\[
\text{Var}(Y_k) = \text{Var}(Y_0) = \sum_{j=1}^{\infty} \sum_{l=1}^{\infty} z_j z_l \int_{-1}^{1} H(s_l - s_j, u) [1 - |u|] du < \infty.
\]

(22)

The translation invariance of \( \{Y_k : k \in \mathbb{Z}\} \) follows from the translation invariance of \( \Phi^h \). The FKG inequality for this sequence follows.
from the fact that \( \Phi^h \) is the scaling limit of the renormalized magnetization field and the later has the FKG inequality. Note that the FKG inequality only holds when \( z_1, \ldots, z_n \) are all non-negative and this is why we prove (19) first for non-negative \( z_j \)'s. The last condition we need to check is the finiteness of the “susceptibility”:

\[
A := \sum_{k \in \mathbb{Z}} \text{Cov}(Y_0, Y_k) = \sum_{j=1}^{n} \sum_{l=1}^{n} z_j z_l \int_{-\infty}^{\infty} H(s_l - s_j, u) du, \tag{23}
\]

where we have used (21) and some simplifications. Clearly, (23) is finite by the exponential decay of \( H \) (see [5]) and the continuity of \( H \) on \( \mathbb{R} \setminus \{0\} \). We can now apply Theorem 2 of [20] to show that for each \( (z_1, \ldots, z_n) \in (\mathbb{R}^+)^n \) and \( r \in \mathbb{R} \),

\[
\lim_{L \to \infty} \mathbb{E} e^{ir[z_1 X_L(s_1) + \ldots + z_n X_L(s_n)]} = e^{-Ar^2/2}
\]

where we have used (23) and (8) in the last equality.

Next, we will show that (19) actually holds for each \( (z_1, \ldots, z_n) \in \mathbb{R}^n \), which would complete the proof of the first part of the theorem. For fixed \( (z_1, \ldots, z_n) \in \mathbb{R}^n \), we define

\[
W^+_L := \sum_{j: z_j \geq 0} z_j X_L(s_j), W^-_L := \sum_{j: z_j < 0} |z_j| X_L(s_j). \tag{25}
\]

We just proved in (24) that for any \( a \geq 0 \) and \( b \geq 0 \), \( aW^+_L + bW^-_L \) converges (as \( L \to \infty \)) in distribution to a Gaussian random variable with mean 0 and variance

\[
\sum_{j=1}^{n} \sum_{l=1}^{n} (a_1 z_{j \geq 0} + b_1 z_{j < 0})(a_1 z_{l \geq 0} + b_1 z_{l < 0}) |z_j z_l| \text{Cov}(X(s_j), X(s_l)). \tag{26}
\]

In particular, this implies that \( \{(W^+_L, W^-_L) : L > 0\} \) is tight as \( L \to \infty \). Let \( (Z^+, Z^-) \) be a subsequential limit in distribution of \( \{(W^+_L, W^-_L) : L > 0\} \) as \( L \to \infty \). For \( a \geq 0 \) and \( b \geq 0 \), \( aZ^+ + bZ^- \) is a mean zero normal random variable with variance given by (26). By Theorem 3 of [15], we know that \( (Z^+, Z^-) \) is a bivariate normal vector whose distribution is determined by \( aZ^+ + bZ^- \) for \( a, b \geq 0 \). Therefore all subsequential limits agree and so

\[
(W^+_L, W^-_L) \Rightarrow (Z^+, Z^-) \text{ as } L \to \infty, \tag{27}
\]

where “\( \Rightarrow \)” denotes convergence in distribution. By the continuous mapping theorem,

\[
aW^+_L + bW^-_L \Rightarrow aZ^+ + bZ^- \text{ as } L \to \infty \text{ for each } a, b \in \mathbb{R}. \tag{28}
\]

Setting \( a = 1 \) and \( b = -1 \), we proved (19) for each \( (z_1, \ldots, z_n) \in \mathbb{R}^n \).
For the second part of the theorem, by the Källén-Lehmann spectral formula (valid for Euclidean fields satisfying the Osterwalder-Schrader axioms — see Theorem 6.2.4 of [13]), we have

\[
H(s, y) = \int_{0}^{\infty} \left( \int_{-\infty}^{\infty} \int_{0}^{\infty} e^{ipy} e^{-E|m|} \delta(m^2 + p^2 - E^2) dEdp \right) d\tilde{\rho}(m),
\]

where \( \tilde{\rho}(m) \) is a mass spectral measure of the relativistic quantum field theory obtained from \( \Phi^h \) via the Osterwalder-Schrader reconstruction theorem [23].

Note that for fixed \( p \) and \( m \),

\[
\delta(m^2 + p^2 - E^2) = \delta(\sqrt{m^2 + p^2} + E) + \delta(\sqrt{m^2 + p^2} - E).
\]

Combining (29), (30) and (8), we get that for any \( s \neq 0 \),

\[
K(s) = \int_{-\infty}^{\infty} H(s, y) dy = \int_{-\infty}^{\infty} \left[ \int_{0}^{\infty} \left( \int_{-\infty}^{\infty} \int_{0}^{\infty} e^{ipx} e^{-E|m|} \delta(\sqrt{m^2 + p^2 - E}) \frac{dEdp}{2\sqrt{m^2 + p^2}} \right) d\tilde{\rho}(m) \right] dy
\]

\[
= \pi \int_{0}^{\infty} \frac{e^{-|s|m}}{m} d\tilde{\rho}(m).
\]

Then the continuity of \( K(s) \) in \( s \), the monotone convergence theorem and the last displayed equation imply that

\[
K(s) = \pi \int_{0}^{\infty} \frac{e^{-|s|m}}{m} d\tilde{\rho}(m), \forall s \in \mathbb{R}.
\]

By Theorem 2 and Remark 3 in [5], the support of \( \tilde{\rho} \) is in \([m_1, \infty)\) for some \( m_1 > 0 \). Now we define a new measure \( \rho \) by the Radon-Nikodym derivative

\[
\frac{d\rho(m)}{d\tilde{\rho}(m)} = \frac{\pi}{m}.
\]

Then (11) follows from (32) and (33).

The main ingredient for the proof of the Theorem 2 is the scaling relation for \( \Phi^h \) which was proved in [5].

**Proof of Theorem 2.** By Theorem 5 of [5],

\[
\lambda^{1/8} \Phi^h(\lambda x) \overset{d}{=} \Phi^{15/8h}(x) \text{ for any } x \in \mathbb{R}^2, h > 0, \lambda > 0,
\]

where \( \overset{d}{=} \) means equal in distribution. Hence for any \( t, y \in \mathbb{R} \),

\[
\text{Cov} \left( \lambda^{1/8} \Phi^h(\lambda t), \lambda^{1/8} \Phi^h(\lambda y) \right) = \text{Cov} \left( \Phi^{15/8h}(t), \Phi^{15/8h}(y) \right).
\]
Here the Cov’s refer to the covariance function of $\Phi^h$ and $\Phi^{a_{15/8}h}$ respectively, which will be denoted by $H^h$ and $H^{a_{15/8}h}$ in the rest of the proof. Setting $t = 0$, we get

$$\lambda^{1/4}H^h(0, \lambda y) = H^{a_{15/8}h}(0, y).$$  \hspace{1cm} (36)

As $H$ is a function only of the radial variable, we define

$$H(\sqrt{t^2 + y^2}) = \hat{H}^h(\sqrt{t^2 + y^2}) := H^h(t, y), \ \forall (t, y) \in \mathbb{R}^2.$$  \hspace{1cm} (37)

By setting $y = 1$ and taking a limit in (36), we get

$$\lim_{\lambda \downarrow 0} \lambda^{1/4} \hat{H}^h(\lambda) = \lim_{\lambda \downarrow 0} \hat{H}^{a_{15/8}h}(1),$$  \hspace{1cm} (38)

provided that the limit on the RHS exists. This is indeed the case since by the GHS inequality [14], $\hat{H}^h$ is decreasing in $h$. Let us denote this limit by $\tilde{C}_1$. The GHS inequality also implies $\tilde{C}_1 \leq C_1$ where $C_1$ is defined in [15]. Next, we will show $\tilde{C}_1 \geq C_1$. We first define $f_\epsilon(t, y) = \epsilon^{-2} 1_{[-\epsilon/2, \epsilon/2]}(t) 1_{[-\epsilon/2, \epsilon/2]}(y)$. Then using the analyticity of $\hat{H}^h(r)$ on $\mathbb{R} \setminus \{0\}$, it is not hard to show that for any $h \geq 0$ and $s \in \mathbb{R} \setminus \{0\}$,

$$\lim_{\epsilon \downarrow 0} \text{Cov}(\Phi^h(f_\epsilon(t, y)), \Phi^h(f_\epsilon(t - s, y))) = \hat{H}^h(s).$$  \hspace{1cm} (39)

Note that

$$\text{Cov}(\Phi^h(f_\epsilon(t, y)), \Phi^h(f_\epsilon(t - s, y)))$$

$$= \mathbb{E}^h(\Phi^h(f_\epsilon(t, y)) \Phi^h(f_\epsilon(t - s, y))) - \left[\mathbb{E}^h(\Phi^h(f_\epsilon(t, y)))\right]^2$$

$$\geq \mathbb{E}^0(\Phi^0(f_\epsilon(y)\delta(t)) \Phi^0(f_\epsilon(y)\delta(s))) - \left[\mathbb{E}^h(\Phi^h(f_\epsilon(t, y)))\right]^2,$$  \hspace{1cm} (40)

where we have applied the FKG inequality in the last inequality. We next apply Theorem 4 of [6] to prove that $\mathbb{E}^h(\Phi^h(f_\epsilon(t, y)))$ is independent of $\epsilon$. One can rephrase Theorem 4 of [6] as follows:

$$\lim_{a \downarrow 0} \frac{\langle \sigma_0 \rangle_{a,h}}{a^{1/8}} = Bh^{1/15},$$  \hspace{1cm} (41)

where $\langle \cdot \rangle_{a,h}$ is the expectation for the critical Ising model on $a\mathbb{Z}^2$ with external field $a_{15/8}h$ and $B \in (0, \infty)$. The argument right after (16) of [6] implies that

$$\mathbb{E}^h(\Phi^h(f_\epsilon(t, y))) = \lim_{a \downarrow 0} \epsilon^{-2} a^{15/8} \left\langle \sum_{x \in Q_\epsilon^a} \sigma_x \right\rangle_{a,h},$$  \hspace{1cm} (42)

where $Q_\epsilon^a := a\mathbb{Z}^2 \cap [-\epsilon/2, \epsilon/2]^2$. Let $\mathcal{N}(a, \epsilon)$ be the cardinality of $Q_\epsilon^a$. Then it is clear that, for each $\epsilon > 0$,

$$\lim_{a \downarrow 0} \frac{\mathcal{N}(a, \epsilon)}{(\epsilon/a)^2} = 1.$$  \hspace{1cm} (43)
By (42) and translation invariance, we have

\[ E^h (\Phi^h (f_\epsilon(t, y))) = \lim_{a\downarrow 0} \epsilon^{-2} a^{15/8} N(a, \epsilon)(\sigma_0)_{a,h} \]

\[ = \lim_{a\downarrow 0} \frac{N(a, \epsilon)(\sigma_0)_{a,h}}{(\epsilon/a)^2 a^{1/8}} \]

\[ = Bh^{1/15}, \tag{44} \]

where we have used (41) and (43) in the last equality.

Taking \( \epsilon \downarrow 0 \) in (40) and using (39) and (44), we get

\[ \hat{H}^h(s) \geq \hat{H}^0(s) - (Bh^{1/15})^2. \tag{45} \]

Multiplying each side of the last displayed inequality by \( s^{1/4} \) and setting \( s \downarrow 0 \), we obtain \( \hat{C}_1 \geq C_1 \). Therefore,

\[ \lim_{\lambda \downarrow 0} \lambda^{1/4} \hat{H}^h(\lambda) = C_1. \tag{46} \]

This completes the proof of (16) with \( \lambda \) replaced by \( \lambda |y| \) for \( y \neq 0 \).

By the change of variable \( y = u\epsilon \) and (36), we have

\[ \int_0^\infty \left[ \hat{H}^h(y) - \hat{H}^h(\sqrt{y^2 + \epsilon^2}) \right] dy \]

\[ = \int_0^\infty \left[ \hat{H}^h(u\epsilon) - \hat{H}^h(\epsilon\sqrt{u^2 + 1}) \right] du \]

\[ = \int_0^\infty \left[ \hat{H}^{15/8h}(u) - \hat{H}^{15/8h}(\sqrt{u^2 + 1}) \right] du. \tag{47} \]

Note that

\[ \int_0^\infty \left[ \hat{H}^{15/8h}(u) - \hat{H}^{15/8h}(u + 1) \right] du = \int_0^1 \hat{H}^{15/8h}(u) du. \tag{48} \]

By the GHS inequality, (36) and (16), \( \hat{H}^{15/8h}(u) \uparrow \hat{H}^0(u) \) as \( \epsilon \downarrow 0 \) for each \( u \neq 0 \). So the monotone convergence theorem implies

\[ \lim_{\epsilon \downarrow 0} \int_0^1 \hat{H}^{15/8h}(u) du = \int_0^1 \hat{H}^0(y) dy < \infty, \tag{49} \]

where the last inequality follows from (16). To summarize, we just proved

\[ \lim_{\epsilon \downarrow 0} \int_0^\infty \left[ \hat{H}^{15/8h}(u) - \hat{H}^{15/8h}(u + 1) \right] du \]

\[ = \int_0^\infty \left[ \hat{H}^0(u) - \hat{H}^0(u + 1) \right] du = \int_0^1 \hat{H}^0(u) du < \infty. \tag{50} \]

By the monotonicity of \( \hat{H} \), we have for each \( u > 0 \),

\[ \hat{H}^{15/8h}(u) - \hat{H}^{15/8h}(\sqrt{u^2 + 1}) \leq \hat{H}^{15/8h}(u) - \hat{H}^{15/8h}(u + 1). \tag{51} \]
Hence, by the generalized dominated convergence theorem (see Exercise 20 in Section 2.3 of [10]), we have

\[
\lim_{\varepsilon \downarrow 0} \int_0^\infty \left[ \hat{H}^{15/8}(u) - \hat{H}^{15/8}(\sqrt{u^2 + 1}) \right] du \\
= \int_0^\infty \left[ \hat{H}^0(u) - \hat{H}^0(\sqrt{1 + u^2}) \right] du \leq \int_0^1 \hat{H}^0(u) du < \infty. \tag{52}
\]

This, (53), (37) and (47) complete the proof of (17). \qed

**Appended A. \( \Phi^h \) paired with a 1-d delta function**

In [4], it was shown that \( \Phi^h(f) \) is a well-defined random variable for any \( f \) in the Sobolev space \( \mathcal{H}^{-3}(\mathbb{R}^2) \). This was later generalized in [11] to any \( f \) in the Besov space \( \mathcal{B}_{p,\sigma}^{1/2,\text{loc}}(\mathbb{R}^2) \) where \( \sigma > 0 \) and \( p, q \in [1, \infty] \). But the test function \( 1_{[-L,L]}(y)\delta_\varepsilon(t) \) is in neither of those two spaces. The next lemma justifies the pairing of \( \Phi^h \) with such a test function.

**Lemma A.** For any \( \varepsilon > 0 \), let \( g_\varepsilon \) be the probability density function of \( N(0, \varepsilon) \) (i.e., Gaussian with mean 0 and variance \( \varepsilon \)). For any \( f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \), we have that

\[
\{ \Phi^h(f(y)g_\varepsilon(t)) ; \varepsilon > 0 \}
\]

is a Cauchy sequence in \( L^2 \).

**Remark A.** Lemma A holds for more general probability density functions than \( g_\varepsilon \); we choose a Gaussian distribution to simplify the proof.

**Proof.** For \( f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \), let

\[
h(u) := \int_{\mathbb{R}} f(y) f(u + y) dy. \tag{54}
\]

Our assumption on \( f \) implies \( |h(u)| \leq \|f\|_2 \) for each \( u \in \mathbb{R} \). Therefore,

\[
\int_{\mathbb{R}} |h(u)| H(0, u) du < \infty, \tag{55}
\]

where the last inequality follows from the exponential decay of \( H(0, u) \) for large \( |u| \) and \( H(0, |u|) = O(|u|^{-1/4}) \) as \( |u| \downarrow 0 \) (see Theorem 2).

In the next calculation we use the fact that the random variables \( \Phi^h(f(y)g_\varepsilon(t)) \) and \( \Phi^h(f(y)g_\varepsilon(t)) \) have the same expectation, which can be proved with an argument analogous to that used in the proof of Theorem 2 to show that \( \mathbb{E}^h(\Phi^h(f, t, y)) \) is independent of \( \varepsilon \):

\[
\mathbb{E} \left[ \Phi^h(f(y)g_\varepsilon(t)) - \Phi^h(f(y)g_\varepsilon(t)) \right]^2 = \mathbb{E} \left[ \Phi^h(f(y)g_\varepsilon(t) - f(y)g_\varepsilon(t)) \right]^2
\]

\[
= \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} [f(y_1)g_\varepsilon(t_1) - f(y_1)g_\varepsilon(t_1)] [f(y_2)g_\varepsilon(t_2) - f(y_2)g_\varepsilon(t_2)]
\]

\[
\times H(t_2 - t_1, y_2 - y_1) dt_2 dy_2 dt_1 dy_1. \tag{56}
\]
By the change of variables \( u = y_2 - y_1, v = t_2 - t_1, w = y_1, x = t_1, \) this equals
\[
\int_{\mathbb{R}^2} h(u) H(v, u) \left[ G_{\epsilon, \tilde{\epsilon}}(v) + G_{\epsilon, \tilde{\epsilon}}(v) - G_{\epsilon, \tilde{\epsilon}}(v) - G_{\epsilon, \tilde{\epsilon}}(v) \right] dvdu,
\] (57)
where \( h(u) \) is as in (54) and \( G_{\epsilon, \tilde{\epsilon}} := g_{\epsilon} * g_{\tilde{\epsilon}} \) is the convolution. (We have used the fact that \( g_{\epsilon} \) is even). Since \( g_{\epsilon} \) is the density of \( N(0, \epsilon) \), we have
\[
G_{\epsilon, \tilde{\epsilon}} = g_{\epsilon + \tilde{\epsilon}}.
\] (58)
Another change of variables and using the explicit formula for \( g_{\epsilon} \) give that (57) equals
\[
\int_{\mathbb{R}^2} h(u) \sqrt{2\pi e^{-v^2/2}} \left[ H(\sqrt{2\epsilon v}, u) + H(\sqrt{2\tilde{\epsilon} v}, u) - 2H(\sqrt{\epsilon + \tilde{\epsilon} v}, u) \right] dvdu.
\] (59)
Note that by the monotonicity of \( H, \)
\[
\left| H(\sqrt{2\epsilon v}, u) + H(\sqrt{2\tilde{\epsilon} v}, u) - 2H(\sqrt{\epsilon + \tilde{\epsilon} v}, u) \right| \leq 4H(0, u).
\] (60)
For any \( \eta > 0, \) by (55), one may choose \( \xi > 0 \) and \( N > \xi \) such that
\[
4 \int_{\{u:|u|<\xi \text{ or } |u|>N\}} \int_{\mathbb{R}} \frac{|h(u)|}{\sqrt{2\pi}} e^{-v^2/2} H(0, u) dvdu + 4 \int_{\mathbb{R}} \int_{\{v:|v|>N\}} \frac{|h(u)|}{\sqrt{2\pi}} e^{-v^2/2} H(0, u) dvdu < \eta/2.
\] (61)
Since
\[
H(\sqrt{2\epsilon v}, u) + H(\sqrt{2\tilde{\epsilon} v}, u) - 2H(\sqrt{\epsilon + \tilde{\epsilon} v}, u)
\]
\[
= \left| H(0, \sqrt{u^2 + 2\epsilon v^2}) + H(0, \sqrt{u^2 + 2\tilde{\epsilon} v^2}) - 2H(0, \sqrt{u^2 + (\epsilon + \tilde{\epsilon}) v^2}) \right|
\]
is uniformly continuous on \( \{(u, v): \xi \leq |u| \leq N, |v| \leq N\}, \) we have for all small enough \( \epsilon \) and \( \tilde{\epsilon}, \)
\[
\int_{\{u:|u|\leq N\}} \int_{\{v:|v|\leq N\}} \frac{|h(u)|}{\sqrt{2\pi}} e^{-v^2/2} H(\sqrt{2\epsilon v}, u) + H(\sqrt{2\tilde{\epsilon} v}, u) - 2H(\sqrt{\epsilon + \tilde{\epsilon} v}, u) dvdu < \eta/2.
\] (62)
This combined with (59)-(61) completes the proof of the lemma. \( \square \)

**APPENDIX B. LARGE DISTANCE/TIME BEHAVIOR OF \( H \) AND \( K \)**

In this appendix, we first derive the large \( t \) behavior of \( \hat{H}(t) \) (recall the definition of \( \hat{H} \) in (37)), based on the assumption that the mass spectrum has an upper gap \( (m_1, m_1 + \epsilon) \) for some \( \epsilon > 0 \). Then, based on this behavior, we prove the (first order) large distance behavior for \( K \).
Using the Källén-Lehmann spectral representation (see (29) and (30)), we have for $t > 0$ that

$$
\hat{H}(t) = \int_0^\infty \int_{-\infty}^\infty \frac{e^{-\sqrt{m^2+p^2} t}}{2\sqrt{m^2 + p^2}} dp \tilde{\rho}(m).
$$

(63)

Recall that the support of $\tilde{\rho}$ is in $[m_1, \infty)$. If we assume that the mass spectrum has an upper gap $(m_1, m_1 + \epsilon)$ for some $\epsilon > 0$, then we get

$$
\hat{H}(t) = \tilde{\rho} (\{m_1\}) \int_{-\infty}^\infty \frac{e^{-\sqrt{m_1^2+p^2} t}}{2\sqrt{m_1^2 + p^2}} dp + \int_{m_1+\epsilon}^\infty \int_{-\infty}^\infty \frac{e^{-\sqrt{m^2+p^2} t}}{2\sqrt{m^2 + p^2}} dp \tilde{\rho}(m).
$$

(64)

By the change of variable $y = \sqrt{1 + \frac{p^2}{m^2}}$, we have

$$
\int_{-\infty}^\infty \frac{e^{-\sqrt{m^2+p^2} t}}{2\sqrt{m^2 + p^2}} dp = \int_1^\infty \frac{e^{-mty}}{\sqrt{y^2-1}} dy = \int_1^\infty \frac{e^{-mt}}{2(y-1)} dy + \int_1^\infty \left[ \frac{1}{\sqrt{y^2-1}} - \frac{1}{\sqrt{2(y-1)}} \right] e^{-mt} dy.
$$

(65)

Using the inequality

$$
\sqrt{y + 1} - \sqrt{2} \leq \frac{y - 1}{2\sqrt{2}} \quad \text{for all } y \geq 1,
$$

we have

$$
\left| \int_1^\infty \left[ \frac{1}{\sqrt{y^2-1}} - \frac{1}{\sqrt{2(y-1)}} \right] e^{-mt} dy \right| \leq \frac{1}{8} \int_1^\infty \sqrt{y-1} e^{-mt} dy = \frac{1}{8} t^{-3/2} e^{-mt} \int_0^\infty y^{1/2} e^{-my} dy.
$$

(66)

On the other hand, the change of variable $u = \sqrt{t(y-1)}$ gives

$$
\int_1^\infty \frac{e^{-mt}}{\sqrt{2(y-1)}} = (2t)^{-1/2} e^{-mt} \int_0^\infty 2e^{-mu^2} du = \sqrt{\pi/(2m)} t^{-1/2} e^{-mt}.
$$

(68)

Combining (64)-(68), we get that there exists a constant $C_3 \in (0, \infty)$ such that

$$
\lim_{t \to \infty} \frac{\hat{H}(t)}{t^{-1/2} e^{-mt}} = C_3 := \tilde{\rho}(\{m_1\}) \sqrt{\pi/(2m_1)}.
$$

(69)

Based on this, we will show the following proposition.

**Proposition A.** Suppose (69) holds. Then we have

$$
\lim_{t \to \infty} \frac{K(t)}{e^{-m_1 t}} = C_3 \sqrt{2\pi/m_1} = \pi \tilde{\rho}(\{m_1\}) / m_1.
$$

(70)

We need the following ancillary lemma.
Lemma B. Let $m > 0$. For any $\alpha, \beta$ satisfying $0 < \beta < 1/2 < \alpha < 3/4$, we have

$$\lim_{t \to \infty} \frac{\int_0^\infty (u^2 + t^2)^{-1/4} e^{-m\sqrt{u^2 + t^2}} du}{e^{-mt}} = \lim_{t \to \infty} \frac{\int_0^\infty e^{-m\sqrt{u^2 + t^2}} du}{t^{1/2} e^{-mt}} = \sqrt{\frac{\pi}{2m}}.$$  

(71)

Proof. We first note that

$$\int_0^\infty (u^2 + t^2)^{-1/4} e^{-m\sqrt{u^2 + t^2}} du \leq \int_0^t t^{-1/2} e^{-m\sqrt{u^2 + t^2}} du$$

where we have used the bound (which can be proved by the Cauchy-Schwarz inequality)

$$\sqrt{u^2 + t^2} \geq t^{-1/2}u + \sqrt{1-t^{-1}t}$$

for all $u, t > 0$.

(73)

in the second inequality, and

$$1 - e^{-y} \leq y$$

for all $y \geq 0$, $\sqrt{t^2 - t} \geq t - 1$ for all $t \geq 1$.

(74)

in the last inequality. Using the same inequalities, one can show that

$$\int_0^\infty (u^2 + t^2)^{-1/4} e^{-m\sqrt{u^2 + t^2}} du \leq \int_0^\infty t^{-1/2} e^{-m\sqrt{u^2 + t^2}} du$$

which is

(75)

$$\leq \frac{e^m}{m} e^{-mt^{\alpha-1/2}} e^{-mt}.$$

Combining (72) and (75), we get

$$\lim_{t \to \infty} \frac{\int_0^\infty (u^2 + t^2)^{-1/4} e^{-m\sqrt{u^2 + t^2}} du}{e^{-mt}} = \lim_{t \to \infty} \frac{\int_t^{\infty} (u^2 + t^2)^{-1/4} e^{-m\sqrt{u^2 + t^2}} du}{e^{-mt}}$$

(76)

provided the limit on the LHS exists. It is easy to see that

$$(t^{2\alpha} + t^2)^{-1/4} \int_t^{\infty} e^{-m\sqrt{u^2 + t^2}} du \leq \int_t^{\infty} (u^2 + t^2)^{-1/4} e^{-m\sqrt{u^2 + t^2}} du$$

where

(77)

This and (76) prove the first equality in the lemma provided the limit exists. By the Taylor expansion, one can show that there exist constants $C_4, C_5 \in (0, \infty)$ such that for any $u \in [t^\beta, t^\alpha]$

$$1 + \frac{u^2}{2t^2} - C_4 \left(\frac{u}{t}\right)^4 \leq \sqrt{1 + (u/t)^2} \leq 1 + \frac{u^2}{2t^2} - C_5 \left(\frac{u}{t}\right)^4.$$  

(78)
Therefore,
$$e^{mC_4 t^{4-\beta}} \int_{\ell^\beta}^{t^\beta} e^{-\frac{m^2 u}{2t}} du \leq \int_{\ell^\beta}^{t^\beta} \exp \left( mt \left[ 1 - \sqrt{1 + (u/t)^2} \right] \right) du \leq e^{mC_4 t^{4-\beta}} \int_{\ell^\beta}^{t^\beta} e^{-\frac{m^2 u}{2t}} du. \quad (79)$$

By our assumption on $\alpha$ and $\beta$, this implies
$$\lim_{t \to \infty} \int_{\ell^\beta}^{t^\beta} e^{-m\sqrt{u^2+t^2}} du = \lim_{t \to \infty} \int_{\ell^\beta}^{t^\beta} e^{-\frac{m^2 u}{2t}} du = \frac{\pi}{2m}, \quad (80)$$
if the limit on the LHS exists. Since
$$\int_{0}^{r} e^{-\frac{m^2 u}{2t}} du \leq \int_{0}^{r} \exp \left( -m(\ell^\alpha)^{1/\alpha} u^{2-1/\alpha} \right) du < \infty,$$
we have
$$\lim_{t \to \infty} \int_{\ell^\alpha}^{t^\alpha} e^{-\frac{m^2 u}{2t}} du = \lim_{t \to \infty} \int_{0}^{\infty} e^{-\frac{m^2 u}{2t}} du = \sqrt{\frac{\pi}{2m}}, \quad (82)$$
where we have used the fact that $(1/\sqrt{2\pi t/m})e^{-mu^2/(2t)}$ is the density of $N(0, t/m)$.

**Proof of Proposition A.** The limit (69) implies that, for any fixed $\epsilon > 0$,
$$(C_3 - \epsilon)r^{-1/2}e^{-mr} \leq \hat{H}(r) \leq (C_3 + \epsilon)r^{-1/2}e^{-mr}$$
for all large $r$. \quad (83)
Lemma B and the definition of $K$ (see (8)) imply that
$$(C_3 - \epsilon)\sqrt{\frac{2\pi}{m}} \leq \lim_{t \to \infty} \frac{K(t)}{e^{-mt}} \leq (C_3 + \epsilon)\sqrt{\frac{2\pi}{m}}, \quad (84)$$
This completes the proof of the proposition by sending $\epsilon \downarrow 0$. \qed

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**References**

[1] J.D. Bjorken and S.D. Drell (1965). *Relativistic Quantum Fields*. McGraw-Hill.

[2] D. Borthwick and S. Garibaldi (2011). Did a 1-dimensional magnet detect a 248-dimensional Lie algebra? *Notices Amer. Math. Soc.* 58 1055-1066.

[3] F. Camia, C. Garban and C.M. Newman (2015). Planar Ising magnetization field I. Uniqueness of the critical scaling limits. *Ann. Probab.* 43 528-571.

[4] F. Camia, C. Garban and C.M. Newman (2016). Planar Ising magnetization field II. Properties of the critical and near-critical scaling limits. *Ann. Inst. H. Poincaré Probab. Statist.* 52 146-161.
[5] F. Camia, J. Jiang and C.M. Newman (2017). Exponential decay for the near-critical scaling limit of the planar Ising model. To appear in Commun. Pure Appl. Math. Available as arXiv:1707.02668v3.

[6] F. Camia, J. Jiang and C.M. Newman (2019). FK-Ising coupling applied to near-critical planar models. To appear in Stoch. Proc. Appl. Available as arXiv:1709.00582v2.

[7] D. Chelkak, C. Hongler and K. Izyurov (2015). Conformal invariance of spin correlations in the planar Ising model. Ann. Math. 181 1087-1138.

[8] G. Delfino (2004). Integrable field theory and critical phenomena: the Ising model in a magnetic field. J. Phys. A: Math. Gen. 37 R45-R78.

[9] R. Durrett (2005). Probability: Theory and Examples. 3nd ed., Duxbury.

[10] G.B. Folland (1999). Real Analysis, Modern Techniques and Their Applications. 2nd ed., John Wiley and Sons, New York.

[11] M. Furlan and J.-C. Mourrat (2017). A tightness criterion for random fields, with application to the Ising model. Electron. J. Probab. 22 1-29.

[12] X. Fernique (1964). Continuité des processus Gaussiens. C. R. Acad. Sci. Paris 258 6058-6060.

[13] J. Glimm and A. Jaffe (1987). Quantum Physics: A Functional Integral Point of View. 2nd ed., Springer-Verlag.

[14] R.B. Griffiths, C.A. Hurst and S. Sherman (1970). Concavity of magnetization of an Ising ferromagnet in a positive external field. J. Math. Phys. 11 790-795.

[15] G.G. Hamedani and M.N. Tata (1975). On the determination of the bivariate normal distribution from distributions of linear combinations of the variables Amer. Math. Monthly 82 913-915.

[16] T.D. Lee and C.N. Yang (1952). Statistical theory of equations of state and phase transition. II. lattice gas and Ising model. Phys. Rev. 87 410-419.

[17] M.B. Marcus and L.A. Shepp (1970). Continuity of Gaussian processes. Trans. Amer. Math. Soc. 151 377-391.

[18] B. McCoy and J.M. Maillard (2012). The importance of the Ising model. Prog. Theor. Phys. 127 791-817.

[19] I. Montray and G. Münster (1997). Quantum Fields on a Lattice. Cambridge University Press, Cambridge.

[20] C.M. Newman (1980). Normal fluctuations and the FKG inequalities. Commun. Math. Phys. 74 119-128.

[21] C.M. Newman and A.L. Wright (1981). An invariance principle for certain dependent sequences. Ann. Probab. 9 671-675.

[22] C.M. Newman and W. Wu (2019). Lee-Yang property and Gaussian multiplicative chaos. Commun. Math. Phys. 369 153-170.

[23] K. Osterwalder and R. Schrader (1973). Axioms for Euclidean Green’s functions. Commun. Math. Phys. 31 83-112.

[24] T.T. Wu (1966). Theory of Toeplitz determinants and the spin correlations of two-dimensional Ising model. I. Phys. Rev. 149 380-401.

[25] A.B. Zamolodchikov (1989). Integrals of motion and S-matrix of the (scaled) $T = T_c$ Ising model with magnetic field. Int. J. Mod. Phys. 04 4235-4248.

[26] A.B. Zamolodchikov (1989). Integrable Field Theory from Conformal Field Theory. Adv. Stud. Pure Math. 19. Integrable Systems in Quantum Field Theory and Statistical Mechanics, M. Jimbo, T. Miwa and A. Tsuchiya, eds. (Tokyo: Mathematical Society of Japan, 641 - 674.)
