Pseudo-Hermiticity, and Removing Brownian Motion from Finance

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Abstract

In this article we apply the methods of quantum mechanics to the study of the financial markets. Specifically, we discuss the Pseudo-Hermiticity of the Hamiltonian operators associated to the typical partial differential equations of Mathematical Finance (such as the Black-Scholes equation) and how this relates to the non-arbitrage condition.

We propose that one can use a Schrödinger equation to derive the probabilistic behaviour of the financial market, and discuss the benefits of doing so. This presents an alternative approach to replace the use of standard diffusion processes (for example a Brownian motion or Wiener process).

We go on to study the method using the Bohmian approach to quantum mechanics. We consider how to interpret the equations for pseudo-Hermitian systems, and highlight the crucial role played by the quantum potential function.

1 Introduction:

The majority of mathematical models, used to capture the dynamics of the financial markets, are based on the underlying concept of Brownian motion. This is in many ways quite natural. If one assumes that the log returns (for example 1 day returns) of the price of a financial asset are independent, and (at least in the short term) identically distributed, then the central limit theorem implies that the distribution of returns will converge to a normal distribution. Furthermore, the variance will increase linearly with time. Thus the representation of the random variables, that drive the market price, as Brownian motions seems an obvious first step.

Also, by basing the random behaviour of the financial market on Brownian motion, one ends up with partial differential equations, such as the Black-Scholes equation, that can be mapped via changes in the coordinate space, to the standard heat equation (for example see [13] chapter 4). Crucially, this means that a wide array of analytic, and numerical, methods are available to find solutions.

However, the limitations of these simple approaches are well known. In this article we wish to present the means by which one can replace Brownian motion, and derive the equations of finance, such as the Black-Scholes equation, using the framework of quantum probability. The ultimate goal, is to show how this methodology can be applied, so that future research can be directed towards the modelling of more
The application of quantum probability to problems in finance has been studied in a number of sources. For example links between the Black-Scholes partial differential equation and a non-Hermitian Schrödinger equation have been investigated by Baaquie in [2]. Haven also discusses the implications of the modelling of derivative prices as state functions of a Schrödinger equation in [10] and [11]. Baaquie goes on to show how one can apply the path integral approach of quantum mechanics, to derivative pricing in [3] and [4].

In the articles discussed above, and many others on quantum finance, the links between the Black-Scholes equation of finance, and the Schrödinger equation of quantum mechanics, are applied to deriving new mathematical techniques, as well as to providing new ways of understanding the behaviour of the markets.

In this article, we develop these ideas by showing how one can derive equations of finance without resorting to standard building blocks such as the Martingale Representation Theorem. Rather than using quantum methods as a means to study the equations of finance, that have been derived using classical approaches, we seek to develop the models entirely within the quantum framework.

The intention is that this opens the door, in future research, to the application of exotic Hamiltonian functions that can be used to apply non-Brownian drivers to finance.

The approach is focused on representing the financial market as a quantum state. There is an alternative quantum approach, suggested by Segal & Segal in [27], that is based on focusing on quantum observables. Accardi & Boukas develop a quantum Black-Scholes by introducing randomness using the Hudson-Parthasarathy quantum stochastic calculus (see [1], [19]). Further analysis and development on operator focused approaches has been presented in [14]-[17].

In some respects the Accardi-Boukas approach is the dual of the approach considered in this article. In [15], the author shows how one can generate solutions to the Accardi-Boukas quantum Black-Scholes by building on the path integral methods introduced by Baaquie ([3], [4]). The principals outlined above, apply in this case as well. One can build a model directly using the Schrödinger equation, rather than by using Hudson-Parthasarathy stochastic calculus, and exploiting the purely algebraic similarities to Schrödinger equations.

With this in mind, in section 2, we start by explaining the classical derivation of equations, such as the Black-Scholes equation. One crucial ingredient being the assumption that the financial markets are free of arbitrage, and the consequent use of Martingale probability measures.

We go on to show how to replace the classical derivation, using quantum probability in section 4. We show how, by basing the underlying random variables on a Hermitian Schrödinger equation, the final equations end up with the pseudo-Hermiticity property linked to the Black-Scholes equation in [21].

Finally, in section 5, we show how one can view the methods shown in section 4 from a Bohmian perspective (see [18]), and discuss some of the insights this can give us. In particular we consider the meaning of the quantum potential that arises in the Bohmian approach, and how the form of this quantum potential impacts the properties of the diffusion.
2 The Martingale Representation Theorem, and the Classical Approach to Financial Modelling

In this section, we outline the fundamental building blocks to the classical approach to Mathematical Finance, based primarily on stochastic calculus. In the next section, we go on to show how one can replace the use of classical stochastic calculus entirely using quantum probability, and discuss the benefits of doing so. This is in contrast with some other approaches to quantum finance, where the form of the equations defining models (such as the Black-Scholes equation) are taken as given, having been derived using the classical approach, and quantum methods are then applied in the study of solutions.

The most fundamental building block for models of the financial market is the Martingale. To define this concept, we start with a probability space: \((\Omega, \mathcal{F}, P)\). \(\Omega\) represents the space of possible outcomes which we are trying to model (real number line of asset prices for example), and \(\mathcal{F}\) represents a sigma algebra of subsets of \(\Omega\), each subset representing a measurable event. For example, if \(\Omega\) is mapped to \(\mathbb{R}\), then a measurable event \(U \in \mathcal{F} = [a, b]\) would represent the event that our price \(X\), is to be found in the range \(a < X < b\). Finally, \(P\) is a probability measure on \(\mathcal{F}\). In other words a function: \(P: \mathcal{F} \rightarrow [0, 1]\) such that:

- \(P(\Omega) = 1\)
- \(P(\emptyset) = 0\)
- For a collection of disjoint sets: \(\{A_i\}\), \(P(\bigcup_i A_i) = \sum_i P(A_i)\).

A stochastic process is given by a sequence of sigma algebras: \(\mathcal{F}_t\), and a sequence of real valued functions: \(X_t: \mathcal{F}_t \rightarrow \mathbb{R}^n\). We generally require that if \(s < t\) then \(\mathcal{F}_S \subset \mathcal{F}_t\).

Finally, a Martingale is a stochastic process: \(M_t\) such that:

- \(\mathbb{E}^P[|M_t|] < \infty\) for all \(t\).
- \(\mathbb{E}^P[M_t | M_s] = M_s\), for all \(s \leq t\).
- For any function \(f: \Omega \rightarrow \mathbb{R}^n\), we have \(\mathbb{E}^P[f] = \int_{\omega \in \Omega} f(\omega)dP(\omega)\).
- See [25] for more detail.

The first assumption for most models of the financial market is that the model should be free of arbitrage. The first fundamental theorem of mathematical finance (see for example [5] theorem 3.9) roughly states that a model is arbitrage free if and only if there exists an equivalent Martingale probability measure: \(Q\).

For an alternative way to think about this, let \(\Pi_t\) represent a trading strategy consisting of holding \(\Pi_t\) units of the asset \(X_t\), and time \(t\). Informally speaking, the trading strategy is considered self-financing in the event that the change in the value of the strategy is driven completely by changes in the asset price \(X_t\). In other words, after day 1, no further cash is injected into (or taken out from) the portfolio.

Our model is then arbitrageable, if there exists a self-financing strategy \(\Delta_t\) such that:
The first fundamental theorem tells us that there will be no such strategy in the event that we use a probability measure $Q$ such that the risky asset $X_t$, is a Martingale.

The link to the Black-Scholes equation, and to the other partial differential equations of finance, is provided by the Martingale representation theorem (see [25] theorem 4.3.4). Using this theorem, we represent the Martingale $X_t$ using an Ito integral:

$$X_t = X_0 + \int_0^t f(s, \omega) dB(s)$$  \hspace{1cm} (1)

Thus we see that:

- Whilst many financial models have complex dynamics, the fundamental processes at the heart of the random behaviour are largely assumed to be driven by Brownian motion. Behaviour such as ‘fat-tails’ and skew must be introduced exogenously.
- The Brownian motions are introduced largely based on mathematical tractability.
- Complex market dynamics are generally introduced either via the function $f(s, \omega)$ or by introducing new stochastic processes, also driven by Brownian motions, with correlated Wiener processes. We briefly touch on these, in section 3, below.
- The quantum approach outlined in this paper allows one to build the complex behaviours into the fundamental random processes themselves, by selection of different Hamiltonian operators in the Schrödinger equation. Essentially, the Schrödinger equation does the job the Martingale Representation Theorem does, in the classical approach.

### 3 Classical Approaches to Developing the Black-Scholes Model

The Black-Scholes model is based on the stochastic process below:

$$X_t = X_0 + \sigma \int_0^t X_s dB(s)$$  \hspace{1cm} (2)

where for simplicity $X_t$ represents the forward price (which is a Martingale) for the traded asset. In this section, we briefly discuss some of the most common ways in which to enrich this simple model using classical approaches.

Whilst each method has benefits and drawbacks, the compromise is generally modelling complex dynamics at the expense of computational efficiency in generating solutions. Furthermore, it is to be expected that no single approach can offer a perfect representation of the random behaviour of financial markets.

Therefore, by applying completely different theoretical building blocks, the quantum approach can offer new insights and new mathematical techniques. The path integral technique shown by Baaquie in [2]-[4], being a key example.
### 3.1 Local Volatility:

The first natural extension of this model is to allow the volatility $\sigma$ to itself depend on the current value for $X_t$ and the time $t$. This was first introduced by Dupire (see [7]), and crucially allows one to adapt the function $\sigma(X_t, t)$ so that the stochastic process simultaneously generates observed prices for vanilla options at different strikes & maturities. This is in contrast to the Black-Scholes model, whereby different values for the constant $\sigma$ are obtained by calibrating it to different strikes.

### 3.2 Stochastic Volatility:

The local volatility modelling approach described above remains a common work-horse model for many practitioners in the financial markets. However, the model is designed as a simple means by which to match market prices, rather than a model to address the well-known deficiencies of Black-Scholes.

One key issue is that the single factor Ito process which reproduces all observed vanilla option prices, is unique. There are no degrees of freedom that can be used to ensure the underlying process has reasonable dynamics. For example, the evolution of the prices for short maturity options, or the implied correlation between the equity spot price, and the Black-Scholes implied volatility cannot be controlled, and are both generally unrealistic in local volatility models.

The very fact that from one business day to the next, the local volatilities implied by the traded prices for vanilla options change, leads practitioners to consider stochastic volatility models.

A general stochastic volatility model may be defined as follows (for example see [13] chapter 6):

$$X_t = X_0 + \int_0^t \nu_s \sigma(X_s, s) X_s dB_1(s)$$

$$\nu_t = \nu_0 + \int_0^t a(\nu_s, s) ds + \int_0^t b(\nu_s, s) dB_2(s)$$

$$dB_1(t) dB_2(t) = \rho dt$$

By defining suitable functions: $a(\nu_t, t)$ and $b(\nu_t, t)$, one is able to calibrate a stochastic process to observed option prices, whilst also retaining dynamics that make sense. However, this generally comes at the following cost:

- These solutions can replicate behaviours seen in the past. However, there is often a trade off between developing models based on financial principals, versus developing models specifically to fit historical observations.

- Generating solutions using available numerical methods (such as Monte-Carlo techniques) is often prohibitively computational intensive.

It should be noted that the quantum approach will also experience similar types of issues. However, by starting from a completely different set of ideas: that of the wave-function rather than that of the Ito diffusion process, one is likely to develop complex models with different strengths and weaknesses.
3.3 Other Classical Approaches:

In the sections above, we have introduced important classical approaches, applied by practitioners, in order to resolve some difficulties associated with the Black-Scholes model. There are a number of other extensions, some of which we mention below:

- When one measures the Hurst exponent \( H \) for the daily log returns of most traded equities, one does not generally find \( H = 1/2 \). This has lead to the application of fractional Brownian motion to problems in finance (see for example [26]). In these models incremental daily returns are no longer independent and the models are not arbitrage free. Incorporating fractional Brownian motion using a Schrödinger equation approach is an important potential future direction for research.

- The approaches discussed above, incorporate all the random behaviour to the traded equity price. In some cases it is important to incorporate some randomness to interest rates, and the discount factors applied to future cashflows. We discuss these ideas further in section 4.5.

- Some authors have investigated the introduction of random jumps in the stock price, using Lévy processes (see for example [22]). Investigating quantum representations for Levy processes is another possible future avenue for research.

4 The Schrödinger Equation, and the Quantum Approach to Financial Modelling.

To illustrate how the quantum approach can replace the Martingale representation theorem discussed above, and the use of Brownian motions & Wiener processes in Mathematical Finance, we first illustrate how this can be achieved using the example of the simple Black-Scholes equation.

Looking at the problem from another perspective, the key property that allows the use of the quantum approach to study the Black-Scholes equation is the Pseudo-Hermiticity of the Black-Scholes Hamiltonian. For further discussion on this, refer to [21].

4.1 Step 1: Choosing What to Model

The first step is to decide what we wish to model. For example, we may decide that we wish to model log returns for a listed equity. This seems reasonable, since relative movements in the listed share price seem more fundamental than the absolute amount. In this case, we write the stock price as:

\[ S = e^x \]  \hspace{1cm} (4)

where \( x \) represents the random variable we wish to model. In fact, for the majority of liquid listed equities, one can observed forward prices directly in the market, rather than constructing a forward curve using a discount curve, forecasting dividends etc. Therefore, we ignore dividends & interest rates, and assume that \( S \) represents the market price for the forward.

4.2 Step 2: Choose Dynamics

In this step, we will encode the dynamics for the random variable \( x \) into a Schrödinger equation. The form of the Schrödinger equation will control what kind of variable we get. For example, fat-tailed,
There are different ways of interpreting the use of the Schrödinger equation in this way. Equation 5 describes the evolution of a wave-function, and whilst real world measurements associated to this wave-function will have random outcomes, the wave-function evolution is deterministic. We discuss different financial interpretations in more detail in section 4.4.

For the time being, and for the sake of simplicity, we use a straightforward Hamiltonian function:

$$\frac{i}{\hbar} \frac{\partial \psi}{\partial t} = \hat{H} \psi \tag{5}$$

$$\hat{H} = -\frac{\sigma^2}{2} \frac{\partial^2}{\partial x^2} + C$$

$\psi(x, t)$ represents the wave function for our random variable $x$. We define the position operator: $X$ in the usual way:

$$\mathbb{E}^\psi[X] = \int_R x |\psi(x, t)|^2 dx$$

$$\mathbb{E}^\psi[X^2] = \int_R x^2 |\psi(x, t)|^2 dx$$

$C$ represents an operator that can be used to enforce the Martingale condition. For example, in this case we show below how we can use a constant potential to ensure $e^x$ is a Martingale. This will be addressed in section 4.3.

4.3 Step 3: Changing Variables and Pseudo Hermicity

So far, we have defined a random variable using the Schrödinger equation, and obtained a random variable with the correct dynamics. However, we still do not have a partial differential equation with respect to the original market observable: $S = e^x$. We can construct a valid Schrödinger equation using the change of variables: $S = e^{\frac{x}{2}}$, although the resulting Hamiltonian is generally Pseudo-Hermitian, rather than Hermitian (see [2]).

4.3.1 Brief Introduction to Pseudo-Hermiticity:

The wave-function $\psi$ is defined as belonging to a Hilbert space. For example, we could define: $\psi \in L^2(\mathbb{R})$. For this choice, the Hilbert space inner product is given by:

$$\langle \phi | \psi \rangle = \int_{\mathbb{R}} \overline{\phi(y)} \psi(y) dy \tag{6}$$

In this case, the model defined by equation 5, will conserve probability in the event that:

$$\hat{H}^\dagger = \hat{H}$$
In fact, as detailed in [24], we can define a different inner product (and consequently a different Hilbert space), using a linear Hermitian automorphism: $\eta$. We define:

$$\langle \phi | \psi \rangle_\eta = \int_\mathbb{R} \overline{\phi(y)}(\eta \psi)(y)dy$$  \hspace{1cm} (7)

Now, our Hamiltonian: $\hat{H}$ will conserve probability, and have real valued spectrum, in the event that:

$$\hat{H}^\dagger = \eta \hat{H} \eta^{-1}$$

$$\hat{H}^\dagger \eta = \eta \hat{H}$$  \hspace{1cm} (8)

Equation 8, defines a Pseudo-Hermitian Hamiltonian.

4.3.2 Finding the New Hamiltonian:

In our case, when we transfer from one coordinate system to another, we must conserve the inner products. We have:

$$\frac{dS}{dx} = e^x$$

So, changing variables: $f(S) = \phi(x)$, and $g(S) = \psi(x)$, we get:

$$\langle \phi | \psi \rangle = \int_\mathbb{R} \overline{\phi(x)}\psi(x)dx = \int_\mathbb{R} \overline{f(S)}g(S)dS = \int_\mathbb{R} \overline{f(e^x)}g(e^x)e^{-x}dx$$

So, by writing: $(\eta \psi)(x) = e^{-x/2} \psi(x)$, we can translate from a Schrödinger equation defined with respect to $x$, to one with respect to $S$. First assume we can find a positive operator square root for $\eta$. Ie, a positive operator $\rho$, such that $\rho^2 = \eta$. Then, we write $\hat{K} = \rho^{-1}\hat{H}\rho$. We have (since $\hat{H}$, and $\hat{\rho}$ are assumed to be Hermitian):

$$\hat{K}^\dagger \rho^2 = \rho \hat{H} \rho^{-1} \rho^2$$

$$= \rho \hat{H} \rho$$

$$= \rho^2 \hat{K}$$

Therefore, if we can find such an operator: $\rho$, then the Hamiltonian we require is given by: $\rho^{-1}\hat{H}\rho$. In our case, we have: $\rho(x) = e^{-x/2}$, and so:

$$\hat{K} \psi = e^{x/2} \left( -\frac{\sigma^2}{2} \frac{\partial^2 (e^{-x/2} \psi)}{\partial x^2} + Ce^{-x/2} \psi \right)$$

$$= -\frac{\sigma^2}{2} \frac{\partial^2 \psi}{\partial x^2} + \frac{\sigma^2}{2} \frac{\partial \psi}{\partial x} + \left( C - \frac{\sigma^2}{8} \right) \psi$$  \hspace{1cm} (9)

We must now adjust the operator: $C$ to ensure the traded underlying: $S = e^x$ is a Martingale under the Hamiltonian: $\hat{K}$. 

8
4.3.3 The Martingale Requirement:

Note that if $\psi$ is a solution to the Schrödinger equation defined by $\hat{H}$ (equation 5), and if $\rho$ represents pointwise multiplication by an $x$ function (in this case $\rho(x) = e^{-x/2}$), then: $\rho^{-1}\psi$ is a solution to the Schrödinger equation defined by Hamiltonian: $\hat{K}$ (equation 5).

We require that $S = e^x$ is a Martingale under the measure defined by: $\eta$, and the Hamiltonian $\hat{K}$ given by 4.3.2. Therefore, the Martingale requirement in this case translates to:

$$E^\eta[e^x|x_0] = \langle \delta_x|e^{x/2}\psi\rangle_\eta = e^{x_0}$$

If we write the solution to 5 as:

$$\psi(x,t) = \frac{1}{\sqrt{2\pi\sigma^2 t}} e^{x} \exp \left( -\frac{x^2}{2\sigma^2 t} - Ct \right)$$

Writing (without loss of generality) $x_0 = 0$, and utilising 7 with $\eta\psi = \eta(x)\psi(x,t)$, we get:

$$E^\psi[e^x|x_0 = 0] = \int e^{y/2}\psi(y,t)e^{-y}dy = e^{x_0}e^{\sigma^2 t/8}e^{-Ct}$$

Therefore in this case we must set $C$ as a constant, with value $\sigma^2/8$. Finally, writing $\hat{K}$ in terms of the new variable $S = e^x$, we find:

$$i \frac{\partial \psi}{\partial t} = \hat{H}_{BS}\psi$$

$$\hat{H}_{BS} = -\frac{\sigma^2 S^2}{2} \frac{\partial^2}{\partial S^2}$$

Equation 11 transforms to the standard Black-Scholes equation, using the mapping $it \to \tau$. This transformation, from real time to complex time, is often referred to as a “Wick rotation”. A similar Wick rotation can be used to transform the standard (Gaussian) Schrödinger equation to the heat equation. In the same way that the Wick rotation transforms a Gaussian wave-function to a Gaussian heat diffusion, the Black-Scholes wave-function is transformed to a Black-Scholes diffusion process.

Under equation 11, randomness only enters the model at the point of making a measurement of the market price. After carrying the Wick rotation, randomness is introduced through diffusion.

4.4 Remarks on the Interpretation:

The objective in section 4 is to find a new method to build Martingales without relying on the representation using Ito integrals. This way, one is able to build complex market dynamics into the random processes themselves, and provides the opportunity for studying new models of the financial market.

Care though should be taken when applying equation 11 in a financial context. For example, the following are possible interpretations:

- $\psi$ could represent the value process for a self-financing trading strategy. Then one could use equation 11 for derivative pricing, assuming an initial condition defined by the final payout. One can ensure $\psi$ is real valued by carrying out the Wick rotation, although the financial interpretation of doing this is not clear.
B One could interpret derivative prices as observables on the market state function: $\psi$. This is the interpretation discussed in [17]. In this instance, derivative prices will be real valued, although strictly positive payouts, such as a call option payout, will not always have strictly positive value. In order to ensure a strictly positive payout has positive value, one must incorporate interaction between the closed market quantum system & noise from the environment.

C One can simply interpret the solution $\psi$ as giving the statistical properties of the market. Then integrate over a (Martingale) probability function to derive derivative prices.

For the time being, in this article we are applying the quantum framework to derive the statistical behaviour of the market, rather than assuming a fully quantum approach. This approach can fit within interpretations A or C, which we discuss in more detail below.

4.4.1 Interpretation A versus C:

In interpretation A, $\psi$ represents the value function for a financial derivative. For this reason, $|\psi(S, t)|^2 dS$ no longer represents a probability measure. However, if one considers the solutions relating to Arrow-Debreu securities, one can see that the interpretation of $\psi$ as a wave-function in the conventional sense is still necessary.

The Arrow-Debreu security is defined as the derivative that pays out $1, in the event that the traded underlying is exactly $S_T$ at maturity $T$. Risk neutral pricing theory requires that the derivative value $\psi$ is given (for $t < T$, and Martingale probability measure $Q$) by:

$$\psi = \mathbb{E}^Q[\delta_{S_T}] = \int_{\mathbb{R}} \delta(y - S_T) dQ(y) = p(S_T, T|S_t, t)$$

Alternatively, if our quantum state starts in the Dirac state: $\delta_{S_t}$, and $\psi$ represents the solution to the Schrödinger equation at time $T$, then we have:

$$p(S_T, T|S_t, t) = \langle \delta_{S_t} | \psi \rangle = \psi(S_t, t)$$

The other alternative, interpretation C, is that we represent the financial market using a quantum state: the ‘quantum market’. Equation 11 then represents the Schrödinger equation for the quantum market, in which case the interpretation of $\psi$ as a wave-function in the conventional sense is required, and the conventional initial conditions applied when solving a Black-Scholes partial differential equation (ie the final pay-out) have no meaning. In many ways, this is the cleaner interpretation in the sense that there is a single quantum state (the financial market) which one can measure in many different ways (different derivative instruments). Rather than considering each different derivatives as different quantum states. Either interpretation is acceptable mathematically:

- Interpretation A: Arrow-Debreu securities are quantum states, and we integrate over these to derive the value for conventional derivatives.

- Interpretation C: There is a single quantum state. Derivatives operate at the classical level, in the sense that by trading a derivative, one is making a measurement of the quantum market.

In either case, an important next step is to consider the impact of frequent measurement to the time development of the quantum state. In [23], the authors investigate the application of the continuous
quantum measurement method to derivative pricing. We defer further investigation of this to a future work. The main objective of the current work is to apply the methods of quantum mechanics in a purely statistical sense.

4.5 Remarks on the Operator \( C \), and Ideas for Further Development of the Method:

In the example studied in this article (the simple Black-Scholes case), the operator \( C \) is a constant potential and has the effect of adjusting the drift in order to ensure that the traded underlying is a Martingale. However, one can use non-constant \( C \) to enforce the Martingale condition in models incorporating more exotic effects. In this section we suggest some ideas for potential future development in this direction.

4.5.1 Stochastic Discount Factors: One Factor Model

We are representing our traded underlying as \( S = e^x \). In this case, \( S \) represents the forward contract for a particular maturity. If \( S_0 \) represents the spot price for an equity, \( S_T \) represents the forward price for maturity: \( T \), and \( r \) represents a funding rate, then non-arbitrage principals (ignoring dividends) require:

\[
S_T = S_0 e^{rT}
\]

If holding the stock pays a dividend: \( D \) at time \( t_d < T \), then again this cash flow should be subtracted from the price we can achieve at time \( T \), in order for the forward price to be arbitrage free, so that:

\[
S_T = S_0 e^{rT} - De^{r(T-t_d)}
\]

In general, especially for the major equity indices (FTSE100, Euro Stoxx50, S&P500) as well as the most liquid individual stocks, there are market makers for forward contracts and dividend forecasts are backed out from these. Furthermore, the volatility of the differentials between forward prices at different maturities is of a lower order of magnitude compared to the volatility of the spot price. For this reason, most models of the equity markets, used by practitioners, only incorporate the randomness of \( S_0 \). Interest rates, and funding costs, are treated as deterministic variables. Dividend cash flows are either treated as fixed, or treated as a fixed percentage of the spot price: \( S_0 \).

There are various classical approaches to modelling the randomness of a term structure of discount factors. For example one can model stochastic interest rates using the Heath Jarrow Morton framework (see [12]). However, classical models have a number of drawbacks. In general one must compromise between the fact that a continuous discount curve is an infinite dimensional object, and the tractability of methods for calibration and generating solutions.

In fact, one can use the operator \( C \) to incorporate some of the randomness of these factors into the model. Whilst the difficulties mentioned with classical models will also impact quantum models, these do offer different angles from which to approach the issue. For example, we assume:

- The dividends & interest rates can be incorporated into a single discount factor. We write this:
  \[
  S = DFe^x.
  \]
This discount factor is driven by the same combination of factors (both specific to the equity in question and general) that drive the spot price. The discount factor is therefore represented using a function: \( DF(x) \). So that we now write our forward price: \( DF(x)e^x \).

The majority of the time the dividends and interest are stable. In other words we have \( DF(x) \approx 1 \) for small \( x \).

Without loss of generality we can incorporate the initial spot price (\( S_0 \) say) into \( DF(x) \) so that the starting value of \( x \) is 0.

By modelling interest rates & dividends using the same underlying variable (\( x \)) as the main market price, the intention is to keep the model simple from the perspective of calibration & generating solutions, whilst simultaneously incorporating factors such as cuts to dividends after a market crash.

Now we require \( DF(x)e^x \) to be a Martingale. Consider, for example, that we wish to take account of the possibility of a cut in interest rates or a cut in the dividend yield under extreme market conditions. This would lead to respectively an increase or a decrease in the relevant discount factor. Then we could apply:

\[
DF(x) = \frac{1}{1 + \varepsilon x^2}
\]

In this instance, positive \( \varepsilon \) would lead to a reduction in the discount factor under extreme market conditions, corresponding to a cut in the dividend yield expectations. Alternatively, a negative value for \( \varepsilon \) would lead to an increase in the discount factor under extreme market conditions. This would correspond to a cut in interest rates. Figure 1 shows a chart of the discount factor against the driver: \( x \).

Now in order to ensure: \( DF(x)e^x \) is a Martingale we require a non-constant potential \( C(x) \). Although we can no longer necessarily write out a closed form solution (such as equation 10) we can still apply the Feynman-Kac formula (see for example [9] Theorem 20.3) and use Path integral methods to find solutions.

### 4.5.2 Stochastic Discount Factors: Multi-Factor Models

One could relax the above assumption that the random behaviour is driven by a single random variable, by simply extending our Hilbert space from \( L^2(\mathbb{R}) \) to \( L^2(\mathbb{R}^n) \), where \( n \) represented the number of random variables. For example, we could model a forward price: \( S_T = DF(y)e^x \), whereby our Hilbert space could be: \( L^2(\mathbb{R}^2) \). Now, the financial market is represented by wave function: \( \psi(x,t) \), where \( x = (x, y) \), and \( x \) was the random variable driving the equity markets, and \( y \) the random variable driving the interest rates.

Now, we must use the operator \( C \) to ensure \( DF(y)e^x \) is a Martingale, and therefore \( C \) must again be a non-constant potential: \( C(x, y) \). Whilst modelling in a higher number of dimensions may increase the difficulty in finding solutions, the same principals apply. One can still use Path integral methods, based on the Feynman-Kac result.
5 Hamilton Jacobi Theory and the Bohmian Approach:

5.1 The Bohmian Approach to Step 2:

In this section, we briefly review the strategy presented in section 4, from the perspective of Hamilton-Jacobi theory and the Bohmian approach to quantum mechanics. For an overview of this approach, see [18]. For an example of an application to quantum finance, see [20].

Start by inserting the wave function:

\[ \psi = Re^{iS} \]  

Into the Schrödinger equation with Hermitian Hamiltonian:

\[ \hat{H} = -\frac{\sigma^2}{2} \hat{P}^2 \]

one obtains 2 partial differential equations (see for example [18] chapter 3):

\[ \frac{\partial S}{\partial t} + \frac{\sigma^2 (\nabla S)^2}{2} - \frac{\sigma^2}{2} \nabla^2 R = 0 \]  \hspace{1cm} (13)

\[ \frac{\partial^2 R^2}{\partial t^2} + \nabla (\sigma^2 R^2 \nabla S) = 0 \]  \hspace{1cm} (14)

Now, equation 13 can be interpreted as a Hamilton-Jacobi equation with “quantum potential”: \( Q = -\frac{\sigma^2}{2} \nabla^2 R \). A classical solution \( S \) to this equation represents a particle with momentum:

\[ P = \nabla S \]
From a quantum perspective $|\psi|^2 = R^2$ represents the probability density function for the particle. Therefore, again from a classical perspective, if we interpret $R^2$ as the density ($\rho$) of a cloud of particles, then the particle flux will be defined by:

$$\vec{q} = \frac{\sigma^2}{2} \nabla S$$

Finally therefore, we can interpret equation 14 as the continuity equation for the particle:

$$\frac{\partial \rho}{\partial t} + \nabla \vec{q} = 0 \quad (15)$$

### 5.2 A Pseudo-Hermitian Case: The Black-Scholes Hamiltonian

We now investigate how this works for the Black-Scholes Hamiltonian. For now, we work using the pseudo-Hermitian Hamiltonian given by equation 9. We proceed by inserting the wave function 12 into the Schrödinger equation defined by the Black-Scholes Hamiltonian.

$$\frac{\partial \psi}{\partial t} = \frac{\partial R}{\partial t} e^{iS} + iR \frac{\partial S}{\partial t} e^{iS}$$

$$\frac{\partial^2 \psi}{\partial x^2} = \frac{\partial^2 R}{\partial x^2} e^{iS} + 2i \frac{\partial R}{\partial x} \frac{\partial S}{\partial x} e^{iS} - R \left( \frac{\partial S}{\partial x} \right)^2 e^{iS}$$

So, inserting $\psi = Re^{iS}$ into the Schrödinger equation:

$$i \frac{\partial \psi}{\partial t} = -\frac{\sigma^2}{2} \frac{\partial^2 \psi}{\partial x^2} + \frac{\sigma^2}{2} \frac{\partial \psi}{\partial x}$$

We get:

$$i \frac{\partial R}{\partial t} e^{iS} - R \frac{\partial S}{\partial t} e^{iS} = \frac{\sigma^2}{2} e^{iS} \left( \frac{\partial R}{\partial x} + R \left( \frac{\partial S}{\partial x} \right)^2 - \frac{\partial^2 R}{\partial x^2} \right) + \frac{i\sigma^2}{2} e^{iS} \left( R \frac{\partial S}{\partial x} - 2 \frac{\partial R}{\partial x} \frac{\partial S}{\partial x} \right)$$

We now divide by $e^{iS}$, and collect together real & imaginary terms:

$$\frac{\partial R}{\partial t} + \frac{\sigma^2}{2} \left( \frac{\partial S}{\partial x} \right)^2 + \frac{\sigma^2}{2} \frac{\partial R}{\partial x} - \frac{\sigma^2}{2} \frac{\partial^2 R}{\partial x^2} = 0$$

$$i \frac{\partial R}{\partial t} + i\sigma^2 \frac{\partial R}{\partial x} \frac{\partial S}{\partial x} - i \frac{\sigma^2}{2} \frac{\partial S}{\partial x} = 0$$

Finally, we divide the real equation by $R$, multiply the imaginary terms by $R$, and replace $\partial/\partial x$ with the more general $\nabla$ to get:

$$\frac{\partial S}{\partial t} + \frac{\sigma^2 (\nabla S)^2}{2} + \frac{\sigma^2}{2} \nabla R - \frac{\sigma^2}{2} \nabla^2 R = 0 \quad (16)$$

$$\frac{\partial R^2}{\partial t} + \nabla \left( \frac{\sigma^2 R^2 \nabla S}{2} \right) - \sigma^2 R^2 \nabla S = 0 \quad (17)$$

$$\frac{\partial S}{\partial t} + \frac{\sigma^2 (\nabla S)^2}{2} + \frac{\sigma^2}{2} \nabla R - \frac{\sigma^2}{2} \nabla^2 R = 0 \quad (16)$$
Now, equation 16 can still be interpreted as a Hamilton-Jacobi equation, with new quantum potential:
\[ Q = \frac{\sigma^2}{2} \nabla^2 R - \frac{\sigma^2}{2} \frac{\nabla^2 R}{R} \]. However, the interpretation of equation 17 as a continuity equation no longer works in the same way. This is a consequence of the fact that the pseudo-Hermitian Hamiltonian given by equation 9, does not conserve probability in the standard Hilbert space inner product.

However, we know from [21] and [24], that the Hamiltonian conserves probabilities under the inner product 7, with \( \eta = e^{-x} \). Therefore, the probability density is now given by: \( \rho = e^{-x} R^2 \). Inserting this into the classical continuity equation 15 we get:
\[
\frac{e^{-x} \partial R^2}{\partial t} + \nabla (e^{-x} \sigma^2 R^2 \nabla S) = 0
\] (18)
After multiplying through by \( e^x \) we get back to equation 17.

5.3 Interpretation of the Quantum Potential:

To start with, consider the Hamilton-Jacobi equation for a particle, obeying classical mechanics with zero potential (see [18]). The partial differential equation is given by:
\[
\frac{\partial S}{\partial t} + \frac{\sigma^2 (\nabla S)^2}{2} = 0
\] (19)
The function \( S \), which is itself determined by the relevant initial conditions, determines the motion of the free particle through the relationship:
\[ P = -\nabla S \]
In the absence of any potential, the particle simply moves at constant velocity. If we introduce a large number of particles, each with different initial conditions, then each particle will move with a different velocity, depending on the initial condition. In the absence of an external potential, these velocities will be constant.

Therefore, consider the situation of \( N \) such particles moving freely along the real number line, having started at \( x = 0 \). The position of the particle: \( i \), with velocity: \( v_i \), after time: \( t \), will be given by: \( X_i = v_i t \). Therefore the variance of the particles after time: \( t \) is given by:
\[
\mathbb{E}[X^2] = \frac{1}{N} \sum_{i=1}^{N} (v_i t)^2
\]
\[
\mathbb{E}[X] = \frac{1}{N} \sum_{i=1}^{N} v_i t
\]
\[
\mathbb{E}[X^2] - \mathbb{E}[X]^2 = \frac{t^2}{N} \left( \sum_{i=1}^{N} (v_i)^2 - \left( \sum_{i=1}^{N} v_i \right)^2 \right)
\] (20)
We note the following:
- The variance increases with \( t^2 \). Therefore, this cannot represent a diffusion with independent increments.
In fact, each time step for one of the particles is 100% correlated, with the previous time-step. This is natural, since each particle is moving freely under constant velocity.

If we wish the $N$ particles to represent particles moving under a random diffusion, such as Brownian motion, then we must introduce a potential function. Thus, if we interpret the wave-function as defining the probability distribution for a classical diffusion, then the quantum potential $Q$ defines the potential function that will ensure the classical particles have the correct statistical properties.

Put another way, consider $N$ particles moving under quantum potential: $Q$. These particles, whilst obeying the laws of classical mechanics, and depending on the initial conditions, would have a probability distribution consistent with the Schrödinger wave-function. In some senses one could interpret the randomness as arising simply from the variation in the initial conditions. Once the particles were set in motion, the system is deterministic.

In fact, the setting of the initial conditions, can be defined by the moments one wishes the probability density function to have. This is the equivalent to specifying the mean and variance of the diffusion process:

$$\lim_{N \to \infty} \sum_{i=1}^{N} v_i^2 \to \sigma^2$$

$$\lim_{N \to \infty} \sum_{i=1}^{N} v_i \to \mu$$

### 5.3.1 Brownian Example:

The fundamental solution to equation 5 with $C = 0$, is given by:

$$\psi(x, t) = K_t(x) * \tilde{\psi}_0(p)$$

$$K_t(x) = \frac{1}{\sqrt{2\pi\sigma^2 t}} \exp\left(\frac{ix^2}{2\sigma^2 t}\right)$$

(21)

Where $f * g$ represents the convolution: $\int f(x - y)g(y)dy$, and $\tilde{h}(p)$ the Fourier transform of $h(x)$. We note, that under the Wick rotation, $\tau = it$ we get back to a heat kernel:

$$K_{\tau}(x) = \frac{1}{\sqrt{2\pi\sigma^2 \tau}} \exp\left(-\frac{x^2}{2\sigma^2 \tau}\right)$$

Following the analysis in [18] chapter 3, we set:

$$R = (\bar{\psi}\psi)^{1/2}$$

$$= \frac{1}{2\pi\sigma^2 \tau} \exp\left(\frac{x^2}{2\sigma^2 \tau}\right)$$

16
Finally, we can calculate the quantum potential:

\[ Q = -\frac{\sigma^2 \nabla^2 R}{2 R} \]

\[ = \frac{\sigma^2}{2} \left( \left( \frac{x}{\sigma^2 t} \right)^2 - \frac{1}{\sigma^2 t} \right) \]  

(22)

So finally, the Quantum potential in this case acts as a quadratic potential, that becomes shallower over time.

5.3.2 Choosing the Quantum Potential:

Therefore, it is clear from the analysis above, that one potential future avenue of research is to design diffusion processes, to be applied to finance, by starting with the base Hamilton-Jacobi equation for a free particle. The simplest case being equation 19, before choosing the quantum potential: \( Q \).

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