The classification of large spaces of matrices with bounded rank

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Abstract

Given an arbitrary (commutative) field $K$, let $V$ be a linear subspace of $M_n(K)$ consisting of matrices of rank lesser than or equal to some $r \in [1, n-1]$. A theorem of Atkinson and Lloyd states that, if $\dim V > n r - r + 1$ and $\# K > r$, then either all the matrices of $V$ vanish on some common $(n - r)$-dimensional subspace of $K^n$, or it is true of the matrices of the transposed space $V^T$. Following some arguments of our recent proof of the Flanders theorem for an arbitrary field, we show that this result holds for any field. We also show that the results of Atkinson and Lloyd on the case $\dim V = n r - r + 1$ are independent on the given field save for the special case $n = 3, r = 2$ and $K \simeq F_2$, where the same techniques help us classify all the exceptional cases up to equivalence. Similar theorems of Beasley for rectangular matrices are also successfully extended to an arbitrary field.

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1 Introduction

Here, we let $K$ be an arbitrary (commutative) field and $n$ a positive integer. We will use the French convention for the set of integers: we denote by $\mathbb{N}$ the set of positive integers.

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non-negative integers, and by $\mathbb{N}^*$ the set of positive ones. We denote by $M_{n,p}(\mathbb{K})$ the set of matrices with $n$ rows, $p$ columns and entries in $\mathbb{K}$, and by $M_n(\mathbb{K})$ the space of square matrices of order $n$. For $M \in M_{n,p}(\mathbb{K})$, we denote by $m_{i,j}$ its entry on the $i$-th row and $j$-th column.

We denote by $\mathfrak{sl}_n(\mathbb{K})$ the subspace of matrices with trace 0 in $M_n(\mathbb{K})$, and by $T^+_n(\mathbb{K})$ (resp. $T^-_n(\mathbb{K})$) the subspace of upper-triangular matrices (resp. of lower-triangular matrices). We equip $M_{n,p}(\mathbb{K})$ with the non-degenerate symmetric bilinear form $b : (A, B) \mapsto \text{tr}(A^T B)$. Given a subset $\mathcal{A}$ of $M_{n,p}(\mathbb{K})$, its orthogonal with respect to $b$ will always be written $\mathcal{A}^\perp$ unless specified otherwise.

We make the group $\text{GL}_n(\mathbb{K}) \times \text{GL}_p(\mathbb{K})$ act on the set of linear subspaces of $M_{n,p}(\mathbb{K})$ by

$$(P, Q).V := P V Q^{-1}.$$ Two linear subspaces of the same orbit will be called equivalent (this means that they represent, in a change of bases, the same set of linear transformations from a $p$-dimensional vector space to an $n$-dimensional vector space).

**Definition 1.** Given a subset $V$ of $M_{n,p}(\mathbb{K})$, we denote by $\text{rk} V$ the maximal rank for a matrix in $V$, and call it the rank of $V$.

Following Atkinson and Lloyd, we also introduce the notation:

**Notation 2.** Let $(s, t) \in [0, n] \times [0, p]$. Set then

$$\mathcal{R}(s, t) := \left\{ \begin{bmatrix} M & N \\ P & 0 \end{bmatrix} \mid M \in M_{s,t}(\mathbb{K}), N \in M_{s,p-t}(\mathbb{K}), P \in M_{n-s,t}(\mathbb{K}) \right\} \subset M_{n,p}(\mathbb{K})$$

(notice that we understate $n$ and $p$ in this notation; however, no confusion should arise when we use it).

In particular, $\mathcal{R}(r, 0)$ is the set of all matrices with all rows zero starting from the $(r + 1)$-th, and $\mathcal{R}(0, r)$ is the set of all matrices with all columns zero starting from the $(r + 1)$-th (hence they are rank $r$ linear subspaces of $M_{n,p}(\mathbb{K})$ when $r \leq \min(n, p)$). Notice also that $\mathcal{R}(s, t)$ has rank at most $s + t$.

Constructing various linear subspaces $V$ of $M_{n,p}(\mathbb{K})$ such that $\text{rk} V \leq r$ is straightforward: any linear subspace of $\mathcal{R}(r, 0)$ or $\mathcal{R}(0, r)$ will do, and also any equivalent subspace to any such subspace will do. Equivalently, we may choose an $r$-dimensional linear subspace $F$ of $\mathbb{K}^n$ and consider an arbitrary linear subspace of the space of matrices $M$ such that $\text{Im} M \subset F$, or choose a $(p - r)$-linear subspace $G$ of $\mathbb{K}^p$ and consider an arbitrary linear subspace of the
space of matrices $M$ such that $MG = \{ 0 \}$. When $r = 1$, it is widely known that every rank $r$ linear subspace of $M_{n,p}(\mathbb{K})$ falls into one of the above two types. This however fails for $r \geq 2$: notice indeed in this case that $R(1, r - 1)$ has rank $r$ although there is no common vector in the kernels of its matrices, and whilst the sum of the images of its matrices is $\mathbb{K}^n$.

Linear spaces of matrices with conditions on the rank have been extensively studied in the last sixty years. The foundational paper is the one of Dieudonné [3], in which he proved that a linear subspace of $M_n(\mathbb{K})$ consisting only of singular matrices must have a dimension lesser than or equal to $n^2 - n$ and that, when equality occurs, the subspace should be equivalent either to $R(n - 1, 0)$ or to $R(0, n - 1)$. This theorem was later generalized by Flanders [4] to subspaces of $M_{n,p}(\mathbb{K})$ with rank $\leq r$ in the case $\# \mathbb{K} > r$, proving in particular that such a subspace must have a dimension lesser or equal to $r \max(n, p)$ (later, Meshulam [6] generalized this to an arbitrary field, and very recently the author [7] gave a new proof that encompasses the affine subspaces).

In [1], Atkinson and Lloyd took the ideas of Flanders one step further. Still assuming $\# \mathbb{K} > r$, they proved that if a linear subspace $V$ of $M_n(\mathbb{K})$ satisfies $\rk V \leq r$ and $\dim V > nr - r + 1$, then it is equivalent to a subspace of $R(r, 0)$ or $R(0, r)$. With the same assumptions on the field, they also proved that if a linear subspace $V$ of $M_n(\mathbb{K})$ satisfies $\rk V \leq r$ and $\dim V = nr - r + 1$, then either it is equivalent to a subspace of $R(r, 0)$ or $R(0, r)$, or it is equivalent to $R(r - 1, 1)$ or $R(1, r - 1)$. This was later generalized by Beasley [2] to the case of rectangular matrices (see our Theorem 3 below), again with the assumption that $\# \mathbb{K} > r$. Most of the techniques used in those papers are based on the original ideas of Flanders, which ultimately rely on considerations of zeros of polynomials, and are therefore insufficient to handle the case of small finite fields. It was even thought at the time that the theorems of Atkinson and Lloyd could not be proven for an arbitrary field. The following beautiful counter-example was indeed given in a paper of Meshulam [6]: the linear subspace

$$\mathcal{J}_3(\mathbb{F}_2) := \left\{ \begin{bmatrix} a & 0 & 0 \\ c & b & 0 \\ d & e & a + b \end{bmatrix} \mid (a, b, c, d, e) \in \mathbb{F}_2^5 \right\}$$

of $M_3(\mathbb{F}_2)$ has rank 2, dimension 5 but it is an easy exercise[1] to prove that it is

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1 Notice that given some $x \in \mathbb{F}_2^3 \setminus \{ 0 \}$, the linear subspace $\mathcal{J}_3(\mathbb{F}_2)x$ can have any dimension between 1 and 3 (but never 0) depending on the choice of $x$. 

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neither equivalent to $\mathcal{R}(1, 1)$ nor to a linear subspace of $\mathcal{R}(2, 0)$ or $\mathcal{R}(0, 2)$. As we shall see, this counter-example is exceptional. We will prove the following theorem indeed:

**Theorem 1.** Let $K$ be an arbitrary field, and $n \in \mathbb{N}^*$. Let $r \in [1, n - 1]$. Let $V$ be a linear subspace of $M_n(K)$ such that $\text{rk} V \leq r$.

(a) If $\dim V > nr - r + 1$, then $V$ is equivalent to a linear subspace of $\mathcal{R}(r, 0)$ or $\mathcal{R}(0, r)$, i.e. either there exists an $r$-dimensional subspace $F$ of $K^n$ such that $\forall M \in V, \text{Im} M \subset F$, or there exists an $(n - r)$-dimensional subspace $G$ of $K^n$ such that $\forall M \in V, G \subset \text{Ker} M$.

(b) If $\dim V = nr - r + 1$ and $(n, r, \# K) \neq (3, 2, 2)$, then either $V$ is equivalent to a linear subspace of $\mathcal{R}(r, 0)$ or $\mathcal{R}(0, r)$, or $V$ is equivalent to $\mathcal{R}(1, r - 1)$ or $\mathcal{R}(r - 1, 1)$.

Notice that $\dim \mathcal{R}(1, r - 1) = \dim \mathcal{R}(r - 1, 1) = nr - r + 1$, hence the additional new cases when $\dim V = nr - r + 1$. The case where $n = 3$, $r = 2$ and $K \simeq \mathbb{F}_2$ is described in the following theorem:

**Theorem 2.** Let $V$ be a rank 2 linear subspace of $M_3(\mathbb{F}_2)$ with dimension 5. Then:

- either $V$ is equivalent to a linear subspace of $\mathcal{R}(0, 2)$ or $\mathcal{R}(2, 0)$;
- or $V$ is equivalent to $\mathcal{R}(1, 1)$;
- or $V$ is equivalent to the subspace $J_3(\mathbb{F}_2)$.

Using the same techniques, we shall also establish the following generalization, already proved by Beasley [2] in the case $\# K > r$.

**Theorem 3.** Let $K$ be an arbitrary field, and $n$ and $p$ denote two integers such that $n > p$. Let $r \in [1, p - 1]$. Let $V$ be a linear subspace of $M_{n,p}(K)$ such that $\text{rk} V \leq r$.

(a) If $\dim V > nr - r + 1 + p - n$, then $V$ is equivalent to a linear subspace of $\mathcal{R}(0, r)$.

(b) If $\dim V = nr - r + 1 + p - n$, then either $V$ is equivalent to a linear subspace of $\mathcal{R}(0, r)$, or it is equivalent to $\mathcal{R}(1, r - 1)$, or it is equivalent to $\mathcal{R}(r, 0)$ and then $n = p + 1$ or $r = 1$.

Using a transposition shows that the previous theorems encompass $M_{n,p}(K)$ for every pair $(n, p) \in (\mathbb{N}^*)^2$. 

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Structure of the paper

Our proof of Theorems 1 and 3 has two major steps. We will show that, in most cases, we can use a transposition and right and left-multiplication by non-singular matrices to reduce $V \subset M_{n,p}(\mathbb{K})$ to the form $\left\{ \begin{bmatrix} M & \varphi(M) \end{bmatrix} \mid M \in W \right\}$ where $W$ is a linear subspace of $M_{n,r}(\mathbb{K})$, and $\varphi$ a linear map. The second step is to prove, with the assumption $\dim W \geq nr - n + 2$, that, in this case, the matrices of $V$ vanish on some common linear subspace of dimension $p - r$: we will coin this as the Common Kernel Theorem. The first step will use some recent new ideas of the proof of the Flanders theorem found in [7]. The Common Kernel Theorem is completely independent from the first step and uses the affine version of the Flanders theorem (again, see [7]): since its proof involves no discussion of particular cases, we will actually start with it (Section 2), then work on the reduction to the situation of this theorem, both for square matrices and rectangular matrices (Sections 3 and 4). The case $\dim V = nr - r + p - n$ is a lot more involving than the case of sharp inequality (although it is based on the same core ideas) so we will devote the entire Section 4 to study it. In the final section, we will classify the 5-dimensional rank 2 subspaces of $M_3(\mathbb{F}_2)$ (there, we will use various results from the previous sections).

2 The Common Kernel Theorem

2.1 Statement of the theorem, and the structure of its proof

This section is devoted to the proof of the following theorem, which is a major tool for establishing Theorems 1 and 3 but is also quite interesting in itself.

Theorem 4 (Common Kernel Theorem). Let $(n, p, r) \in \mathbb{N}^3$ with $n > r$ and $p > r$. Let $W$ be a linear subspace of $M_{n,r}(\mathbb{K})$ such that $\dim W \geq nr - n + 2$. Let $\varphi : W \to M_{n,p-r}(\mathbb{K})$ be a linear map. Consider the linear subspace

$$V = \left\{ \begin{bmatrix} M & \varphi(M) \end{bmatrix} \mid M \in W \right\} \subset M_{n,p}(\mathbb{K})$$

and assume that $\text{rk} V \leq r$. Assume finally that $(n, r, \# \mathbb{K}) \neq (3, 2, 2)$ or $\dim W > nr - n + 2$. Then there exists a $(p - r)$-dimensional linear subspace $G$ of $\mathbb{K}^p$ such that $G \subset \text{Ker} N$ for every $N \in V$, i.e. $V$ is equivalent to a linear subspace of $\mathbb{R}(0,r)$. 

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Remarks 1. (a) Notice that $J_3(F_2)$ is a counter-example in the case $(n, r, \# K) = (3, 2, 2)$ and $\dim W = 5$.

(b) Notice also that the lower bound $n r - n + 2$ on $\dim W$ is tight if $r \geq 2$. Consider indeed the linear subspace $W := R(1, r-1)$, of dimension $n r - n + 1$ and define, for $M \in W$,

$$\varphi(M) := \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \in M_{n,p}(K).$$

A straightforward computation shows then that $\text{rk} V \leq r$ and $\bigcap_{M \in V} \text{Ker} M$ has dimension $p - r - 1$.

An important step of the proof will involve our recent result (cf. [7]):

**Theorem 5.** Given positive integers $n \geq p$, let $V$ be an affine subspace of $M_{n,p}(K)$ such that $\text{rk} V < p$. Then $\dim V \leq n(p - 1)$.

If in addition $\dim V = n(p - 1)$ and $(n, p, \# K) \neq (2, 2, 2)$, then $V$ is a linear subspace of $M_{n,p}(K)$.

We will actually use the following corollary:

**Corollary 6.** Let $n$ and $p$ be positive integers such that $n \geq p$. Let $V$ be a linear subspace of $M_{n,p}(K)$ such that $\dim V > np - n$, and assume $(n, p, \# K) \neq (2, 2, 2)$ or $\dim V > np - p + 1$. Then $V$ is spanned by its rank $p$ matrices.

**Proof.** Assume the contrary holds. Then there would be a linear hyperplane $H$ of $V$ containing all the rank $p$ matrices of $V$. Choosing $M_0 \in V \setminus H$, it would follow that $\text{rk}(M_0 + H) < p$, whereas $M_0 + H$ is an affine subspace of $M_{n,p}(K)$ with dimension greater than or equal to $n(p - 1)$. However, $M_0 + H$ is not a linear subspace of $M_{n,p}(K)$, which contradicts Theorem 5. \qed

**Remark 2.** Notice the exceptional case of

$$T_2^+(F_2) = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \mid (a, b, c) \in F_2^3 \right\}$$

which has dimension 3 but is not spanned by its non-singular matrices (there are only two of them!).

Theorem 4 will obviously follow should we prove Proposition 7 and Corollary 9 below:
Proposition 7. With the assumptions from Theorem 4, one has
\[ \forall M \in W, \text{rk} \left[ M \varphi(M) \right] = \text{rk} M. \]

Theorem 8 (Representation lemma). Let \((n, r, p) \in \mathbb{N}^3\). Let \(V\) be a linear subspace of \(M_{n,r}(\mathbb{K})\) such that \(\dim V \geq nr - n + 2\). Let \(\varphi : V \to M_{n,p}(\mathbb{K})\) be a linear map such that \(\text{Im} \varphi(M) \subseteq \text{Im} M\) for every \(M \in V\). Then there exists \(C \in M_{r,p}(\mathbb{K})\) such that \(\varphi(M) = MC\) for every \(M \in V\).

Corollary 9. Let \((n, p, r) \in \mathbb{N}^3\) such that \(p > r\). Let \(W\) be a linear subspace of \(M_{n,r}(\mathbb{K})\) such that \(\dim W \geq nr - n + 2\). Let \(\varphi : W \to M_{n,p-r}(\mathbb{K})\) be a linear map. Consider the linear subspace
\[ V = \left\{ \left[ M \varphi(M) \right] \mid M \in W \right\}, \]
and assume that \(\text{rk} \left[ M \varphi(M) \right] = \text{rk} M\) for every \(M \in W\). Then there exists a \((p - r)\)-dimensional subspace \(G\) of \(\mathbb{K}^p\) such that \(G \subseteq \ker N\) for every \(N \in V\).

2.2 Proof of Proposition 7

We make the assumptions of Theorem 4. Consider the linear subspace \(W'\) consisting of the matrices in \(W\) having the form \(M = \begin{bmatrix} P \\ 0 \end{bmatrix}\) for some \(P \in M_{n-1,r}(\mathbb{K})\).

We then find a linear subspace \(H\) of \(M_{n-1,r}(\mathbb{K})\) and a linear isomorphism \(i : H \cong W'\) such that \(i(N) = \begin{bmatrix} N \\ 0 \end{bmatrix}\) for every \(N \in H\). For \(N \in H\), we write
\[ \varphi(i(N)) = \begin{bmatrix} C(N) \\ \alpha(N) \end{bmatrix} \]
with \(C(N) \in M_{n-1,p-r}(\mathbb{K})\) and \(\alpha(N) \in M_{1,p-r}(\mathbb{K})\)
and notice that \(C\) and \(\alpha\) are linear maps. Given \(N \in H\) such that \(\text{rk} N = r\), the condition \(\text{rk} \left[ i(N) \varphi(i(N)) \right] \leq r\) means that \(\text{rk} \begin{bmatrix} N \\ C(N) \\ \alpha(N) \end{bmatrix} \leq r\), hence \(\alpha(N) = 0\). However, the rank theorem shows \(\dim H = \dim W' \geq \dim V - r > (n - 1)(r - 1)\). Corollary 6 then yields that \(H\) is spanned its rank \(r\) matrices, therefore \(\alpha = 0\) (notice, in the exceptional case \(n = 3, r = 2\) and \(\mathbb{K} \simeq \mathbb{F}_2\), that the assumptions of Theorem 4 imply that \(\dim \geq (n - 1)(r - 1) + 2\).

This shows that, for every \(M \in V\), if \(\text{Im} M \subseteq \mathbb{K}^{n-1} \times \{0\}\), then \(\text{Im} \varphi(M) \subseteq \mathbb{K}^{n-1} \times \{0\}\).
Notice finally that the assumptions remain essentially unchanged should \( V \) be replaced with \( PV \) for an arbitrary non-singular matrix \( P \in \text{GL}_n(\mathbb{K}) \). The previous situation then easily generalizes as follows: for every linear hyperplane \( \mathcal{H} \) of \( \mathbb{K}^n \), for every \( M \in V \), the condition \( \text{Im} \ M \subset \mathcal{H} \) implies \( \text{Im} \varphi(M) \subset \mathcal{H} \).

Let finally \( M \in V \). Writing \( \text{Im} M \) as the intersection of a family of linear hyperplanes of \( \mathbb{K}^n \), we then deduce from the above result that \( \text{Im} \varphi(M) \subset \text{Im} M \), hence \( \text{rk} [M \begin{bmatrix} 0 & \varphi(M) \end{bmatrix}] = \text{rk} M \).

### 2.3 Proof of Theorem \( \Xi \)

We proceed by induction on \( r \), with \( n \) and \( p \) fixed. The case \( r = 0 \) is trivial (the case \( r = 1 \) is also easy but the reader will carefully check that we actually start from \( r = 0 \)). Given some \( r \in \mathbb{N} \setminus \{0\} \), assume Proposition \( \Xi \) holds for \( (n, r-1, p) \). Let \( V \) and \( \varphi \) be as in Proposition \( \Xi \) for the triple \( (n, r, p) \). In \( V \), consider the linear subspace \( W \) of matrices of the form \( M = \begin{bmatrix} 0 & K(M) \end{bmatrix} \) with \( K(M) \in M_{n,r-1}(\mathbb{K}) \) (i.e. the matrices of \( V \) with zero as first column). Notice that the rank theorem shows \( \dim W \geq \dim V - n \geq n(r-1) - n + 2 \). Thus \( K(W) \) and the map \( M \mapsto \varphi(K^{-1}(M)) \) satisfy the assumptions of Theorem \( \Xi \), which yields a matrix \( C \in M_{r-1,p}(\mathbb{K}) \) such that \( \varphi(M) = K(M)C \) for every \( M \in W \).

Setting \( \tilde{C} := \begin{bmatrix} 0 \\ C \end{bmatrix} \in M_{r,p}(\mathbb{K}) \), and replacing \( \varphi \) with \( M \mapsto \varphi(M) - \tilde{C}M \), we obviously leave the assumptions unchanged, but we also have

\[
\forall M \in W, \varphi(M) = 0,
\]

and it suffices to prove the expected result in this new situation.

Set \( F := \{ C_1(M) \mid M \in V \} \subset \mathbb{K}^n \) where \( C_1(M) \) denotes the first column of \( M \).

The rank theorem then shows that

\[
\dim F^\perp + \dim K(W)^\perp = \dim M_{n,r}(\mathbb{K}) - \dim V \leq n - 2.
\]

We choose a basis \( (x_1, \ldots, x_s) \) of \( F^\perp \) and a basis \( (y_1, \ldots, y_t) \) of the subspace of \( K(W)^\perp \) spanned by its rank 1 matrices. Then \( \text{Im} A \subset \text{Im}(y_1) + \cdots + \text{Im}(y_t) \) for every rank 1 matrix \( A \) in \( K(W)^\perp \), and \( x \in \text{span}(x_1, \ldots, x_s) \) for every \( x \in F^\perp \).

Define finally

\[
G := \text{span}(x_1, \ldots, x_s) + \text{Im}(y_1) + \cdots + \text{Im}(y_t)
\]

and notice that \( \dim G \leq s + t \leq \dim F^\perp + \dim K(W)^\perp \leq n - 2 \) (and \( G \) does not depend on the choice of \( (y_1, \ldots, y_t) \)). We will now use the following lemma, to be proven later on:
Lemma 10. Let $E$ be an $n$-dimensional vector space, and $H$ be a linear subspace of $E$ such that $\text{codim}_E H \geq 2$. Then there is a basis $(e_1, \ldots, e_n)$ of $E$ such that each plane $\text{span}(e_1, e_2)$, $\text{span}(e_2, e_3)$, $\ldots$, $\text{span}(e_{n-1}, e_n)$ intersects $H$ trivially.

Replacing $V$ and $\varphi$ respectively with $PV$ and $M \mapsto \varphi(P^{-1}M)$ for some well-chosen non-singular $P \in \text{GL}_n(\mathbb{K})$, and denoting by $(e_1, \ldots, e_n)$ the canonical basis of $\mathbb{K}^n$, we may now assume that each plane $\text{span}(e_1, e_2)$, $\text{span}(e_2, e_3)$, $\ldots$, $\text{span}(e_{n-1}, e_n)$ intersects $G$ trivially. In this situation, we may now explicitly compute $\varphi$. Let $A \in V$ such that $a_{1,1} = 0$, and write

$$A = \begin{bmatrix} 0 & L_1 \\ \vdots & \vdots \end{bmatrix}$$

for some $L_1 \in M_{1,r-1}(\mathbb{K})$.

Since $e_1$ does not span the image of any $A \in K(W)^\perp$, we may find a matrix in $K(W)$ with $L_1$ as first row, i.e. some matrix in $V$ has the form

$$A' = \begin{bmatrix} 0 & L_1 \\ 0 & \vdots \end{bmatrix}.$$

Then $\varphi(A') = 0$, hence

$$\varphi(A) = \varphi(A - A') = \varphi \begin{bmatrix} 0 & 0 \\ \vdots & \vdots \end{bmatrix}.$$

The assumptions from Proposition 8 then show that the first row of $\varphi(A)$ must be 0. More generally, since none of the $e_i$’s belongs to $G$, we obtain that, for any $M \in V$ and any $i \in [1,n]$, the $i$-th row of $\varphi(M)$ is zero whenever $m_{i,1} = 0$ (notice that this only uses the fact that none of $e_1, \ldots, e_n$ belongs to $G$). By the factorization lemma for linear maps, this yields row matrices $L'_1, \ldots, L'_n$ in $M_{1,p}(\mathbb{K})$ such that

$$\forall M \in V, \varphi(M) = \begin{bmatrix} m_{1,1} & L'_1 \\ m_{2,1} & L'_2 \\ \vdots \\ m_{n,1} & L'_n \end{bmatrix}.$$

We claim that $L'_1 = L'_2 = \cdots = L'_n$. By performing the row operation $L_1 \leftarrow L_1 - L_2$, we transform the pair $(V, \varphi)$ into a new pair $(V', \varphi')$ which essentially has the same properties, save for the assumption on the relations between $e_1, \ldots, e_n$ and the subspace $G'$ (which we associate to $V'$ as we did with $G$ to $V$). However,
since \(e_1 + e_2 \notin G\), we now have \(e_1 \notin G'\) and the above arguments show that \(m_{1,1}L'_1 - m_{2,1}L'_2 = 0\) for every \(M \in V\) for which \(m_{1,1} = m_{2,1} = 0\). We finally notice that there actually exists some \(M \in W\) such that \(m_{1,1} - m_{2,1} = 1\). Indeed, if not, there would be a non-zero vector \(X \in F^\perp \cap \text{span}(e_1, e_2)\), yielding \(G \cap \text{span}(e_1, e_2) \neq \{0\}\). We deduce that \(L'_1 = L'_2\).

More generally, for every \(i \in [1, n-1]\), using the row operation \(L_i \leftarrow L_i - L_{i+1}\) and the fact that \(G \cap \text{span}(e_i, e_{i+1}) = \{0\}\) shows that \(L'_i = L'_{i+1}\).

Therefore \(\forall M \in V, \varphi(M) = MC\) for \(C := \begin{bmatrix} L'_1 & 0 \\ 0_{r-1,p} & \end{bmatrix}\).

It thus only remains to prove Lemma 10.

**Proof of Lemma 10.** We may assume \(E = \mathbb{K}^n\) and \(\text{codim}_E H = 2\). Since \(\text{GL}_n(\mathbb{K})\) acts transitively on the set of \((n-2)\)-dimensional subspaces of \(E\), we may also assume that \(H\) is the subspace defined by the following system of (independent) linear equations:

\[
\sum_{k=1}^{[n/2]} x_{2k} = 0 ; \quad \sum_{k=0}^{[(n-1)/2]} x_{2k+1} = 0
\]

where, for \(t \in \mathbb{R}\), we have denoted by \([t]\) the greatest integer \(k\) such that \(k \leq t\). It is then easily checked that the canonical basis \((e_1, \ldots, e_n)\) of \(\mathbb{K}^n\) satisfies the conditions of Lemma 10 for this particular \(H\).

**2.4 Proof of Corollary 9**

Obviously, \(\varphi : W \to M_{n,p-r}(\mathbb{K})\) satisfies the assumptions of Proposition 8 which yields a matrix \(C \in M_{r,p-r}(\mathbb{K})\) such that \(\varphi(M) = MC\) for every \(M \in W\).

Setting \(A := \begin{bmatrix} C & \end{bmatrix} \in M_{p,p-r}(\mathbb{K})\), we deduce that \(\forall N \in V, NA = 0\).

Therefore every matrix of \(V\) vanishes on \(G := \text{Im}(A)\), which has obviously dimension \(p-r\).

**3 Reduction to the Common Kernel Theorem (I)**

**3.1 One additional lemma**

The reduction to the assumptions of the Common Kernel Theorem will rely heavily on the following result, which is a consequence of the Dieudonné theorem for singular affine subspaces of matrices:
Proposition 11. Let $V$ be a linear subspace of $M_n(\mathbb{K})$ such that $\text{codim } V < n-1$. Then, for every $x \in \mathbb{K}^n \setminus \{0\}$, one has
\[
\text{span}\{A^{-1}x \mid A \in V \cap \text{GL}_n(\mathbb{K})\} = \mathbb{K}^n.
\]

Remark 3. The upper bound $n - 1$ is tight: consider the linear subspace
\[V := \left\{ \begin{bmatrix} \alpha & L \\ 0 & M \end{bmatrix} \mid (\alpha, L, M) \in \mathbb{K} \times M_{1,n-1}(\mathbb{K}) \times M_{n-1}(\mathbb{K}) \right\}
\]
with codimension $n - 1$. Letting $e_1$ denote the first vector of the canonical basis of $\mathbb{K}^n$, we find that $A^{-1}e_1 \in \text{span}(e_1)$ for any $A \in V \cap \text{GL}_n(\mathbb{K})$.

Proof. We denote by $(e_1, \ldots, e_n)$ the canonical basis of $\mathbb{K}^n$. Let $x \in \mathbb{K}^n \setminus \{0\}$. Here, we consider the symmetric bilinear form $b : (A, B) \mapsto \text{tr}(AB)$ on $M_n(\mathbb{K})$ and denote by $V^\perp$ the orthogonal of $V$ for $b$. In $V^\perp$, the rank 1 matrices span a linear subspace, and we choose a basis $(B_1, \ldots, B_p)$ of it consisting of rank 1 matrices. Set $F := \sum_{k=1}^p \text{Im } B_k$. Then $\dim F \leq p = \text{codim } V \leq n - 2$ and $\text{Im } B \subset F$ for every $B \in V^\perp$ such that $\text{rk } B = 1$. Notice then that there is a basis $(f_1, \ldots, f_n)$ of $\mathbb{K}^n$ in which none of the $f_i$'s belongs to $V$ (use Lemma 10 for example). Replacing $V$ with $PVQ$ for a well-chosen pair $(P, Q) \in \text{GL}_n(\mathbb{K})^2$, we reduce the situation to the one where:

(i) $x = e_1$;

(ii) for every $i \in [1, n]$, the subspace $V^\perp$ contains no matrix $B$ such that $\text{Im } B = \text{span}(e_i)$

By assumption (ii) for $i = 1$, we find that, for any $C \in M_{n,1}(\mathbb{K})$, there is a matrix in $V$ with $C$ as first column. In particular, $V$ contains a matrix of the form
\[A_0 = \begin{bmatrix} 1 & L_0 \\ 0 & M_0 \end{bmatrix}.
\]

Denote now by $G$ the linear subspace of $V$ consisting of its matrices with zero as first column. We write every $M \in G$ as
\[M = \begin{bmatrix} 0 & L(M) \\ 0 & K(M) \end{bmatrix}.
\]
By the rank theorem, we find that \( \dim K(G) > (n - 1)(n - 2) \), hence the Dieudonné theorem for affine subspaces (see [3] or [7]) shows that the affine subspace \( M_0 + K(G) \) contains a non-singular matrix. It follows that there is a non-singular matrix \( P \in \text{GL}_{n-1}(K) \) and a row matrix \( L_1 \in M_{1,n-1}(K) \) such that

\[
A_1 = \begin{bmatrix} 1 & L_1 \\ 0 & P \end{bmatrix} \in V,
\]

hence, \( A_1^{-1}x = e_1 \). Using the same method for every column, we find that, for every \( i \in [1, n] \), the subspace \( V \) contains a non-singular matrix \( A_i \) with \( \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \) as \( i \)-th column, hence \( A_i^{-1}x = e_i \). Therefore \( \text{span}\{A_i^{-1}x \mid A_i \in V \cap \text{GL}_{n}(K)\} = K^n \), as claimed.

### 3.2 The general starting point

From now on, we let \( n, p \) and \( r \) be three integers such that \( n \geq p > r \), and \( V \) be a linear subspace of \( M_{n,p}(K) \) such that \( \dim V \geq nr - r + 1 + p - n \) and \( \text{rk} V \leq r \). Hence \( \dim V > n(r - 1) \), so the Flanders theorem (see [6] or [7] for the case of an arbitrary field) forbids \( \text{rk} V \leq r - 1 \), hence \( \text{rk} V = r \). Replacing \( V \) by an equivalent subspace, we lose no generality assuming that \( V \) contains the matrix \( J_r := \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \).

Let \( M = \begin{bmatrix} P_1 & C_1 \\ L_1 & \alpha_1 \end{bmatrix} \in V \), with blocks \( P_1, L_1, C_1 \) and \( \alpha_1 \) respectively of size \( (r, r), (n - r, r), (r, p - r) \) and \( (n - r, p - r) \) (in the rest of the proof, the block decompositions will have the same configuration unless specified otherwise). We assume \( \text{rk} P_1 = r \).

Gaussian elimination shows that \( A \) is equivalent to \( \begin{bmatrix} P_1 & C_1 \\ 0 & \alpha_1 - L_1 P_1^{-1} C_1 \end{bmatrix} \), hence

\[
L_1 P_1^{-1} C_1 - \alpha_1 = 0. \tag{1}
\]

Write every \( M \in V \) as \( M = \begin{bmatrix} K(M) & C(M) \\ L(M) & \alpha(M) \end{bmatrix} \), and set \( W := \text{Ker} \ K \) i.e. \( W \) is the linear subspace of \( V \) consisting of its matrices of the form \( \begin{bmatrix} 0_r & ? \\ ? & ? \end{bmatrix} \). For every
\( M \in W \), the matrix \( A + M \) belongs to \( V \) and has \( P_1 \) as left-upper block, hence (1) shows:
\[
(L(M) + L_1) P_1^{-1} (C(M) + C_1) - \alpha_1 - \alpha(M) = 0.
\]
Subtracting (1), we deduce:
\[
\forall M \in W, \ L(M) P_1^{-1} C(M) = \alpha(M) - L(M) P_1^{-1} C_1 - L_1 P_1^{-1} C(M). \tag{2}
\]
Notice that the left-hand side of the (2) is a quadratic function \( q \) of \( M \) on \( V \), whilst the right-hand side is a linear one. By computing the polar function of \( q \) as defined by \( b_q(M, N) := q(M + N) - q(M) - q(N) \), we deduce that:
\[
\forall (M, N) \in W^2, \ L(M) P_1^{-1} C(N) + L(N) P_1^{-1} C(M) = 0. \tag{3}
\]
Consider now the linear subspace \( H \) consisting of matrices of \( W \) which have the form \[
\begin{bmatrix}
0_r & ? \\
0_{n-r,r} & ?
\end{bmatrix}
\]
Notice that identity (2) shows that the linear map \( M \mapsto C(M) \) is one-to-one on \( H \), hence \( \dim H = \dim C(H) \). By the rank theorem, we deduce that:
\[
\dim V = \dim K(V) + \dim L(W) + \dim C(H).
\]
Identity (2) then shows
\[
\forall M \in W, \forall N \in H, \forall P \in K(V) \cap GL_r(\mathbb{K}), \ L(M) P_1^{-1} C(N) = 0. \tag{4}
\]
Notice in particular that \( I_r \in K(V) \cap GL_r(\mathbb{K}) \) hence:
\[
\forall M \in W, \forall N \in H, \ L(M) C(N) = 0.
\]
Notice finally that if we have \( H = \{0\} \) (which will not always be the case), then the factorization lemma for linear maps (cf. [5] proposition I p.45) shows that \( V \) has the form given in the Common Kernel Theorem, hence \( V \) is equivalent to a subspace of \( \mathcal{R}(0, r) \) unless \((n, p, r, \# \mathbb{K}) = (3, 3, 2, 2)\).

We will finally set
\[
G := \sum_{B \in C(H)} \text{Im} \ B \quad \text{and} \quad q := \dim G.
\]
In the rest of the section, we will focus on the case \( \dim V > n r - r + 1 + p - n \), and wait until Section 4 to tackle the case of equality (in that prospect, the following two paragraphs will serve as a necessary warm-up).
3.3 The case $\dim V > nr - r + 1$ for square matrices

Here, we assume $n = p$ and $\dim V > nr - r + 1$. On the one hand $\dim C(H) \leq q (n - r)$ since $\text{Im} B \subset G$ for every $B \in C(H)$; on the other hand, every matrix of $L(W)$ vanishes on $G$ hence $\dim L(W) \leq (r - q)(n - r)$. We deduce that $\dim L(W) + \dim C(H) \leq r(n - r)$, hence

$$\dim K(W) > r^2 - r + 1.$$ 

Assume then $C(H) \neq \{0\}$ and choose $B \in C(H) \setminus \{0\}$. Applying proposition [11] we see that

$$\sum_{A \in K(W) \cap GL_r(\mathbb{K})} \text{Im} A^{-1}B = \mathbb{K}^r,$$

hence identity (4) shows $\forall M \in W, L(M) = 0$, in which case, replacing $V$ with $V^T$ helps us see that we lose no generality assuming $C(H) = \{0\}$, hence $H = \{0\}$. In that case, the Common Kernel Theorem readily gives the desired conclusion.

3.4 The case $\dim V > nr - r + 1 + p - n$ for non-square matrices

Here, we assume $n > p$ and $\dim V > nr - r + 1 + p - n$. Notice then that $\dim V > (n - 1)r \geq pr$, hence $L(W) \neq \{0\}$. If $\dim K(V) > r^2 - r + 1$, then the same argument as in the previous paragraph shows that $H = \{0\}$. Assume now $\dim K(V) \leq r^2 - r + 1$. By the same line of reasoning as above, we see that $\dim C(H) \leq q(p - r)$ and $\dim L(W) \leq (r - q)(n - r)$, which yields:

$$\dim V \leq \dim K(V) + \dim L(W) + \dim C(H) \leq nr - (n - p)q - r + 1.$$ 

This shows $q = 0$ since $n - p > 0$, hence $H = \{0\}$ and we may conclude again using the Common Kernel Theorem.

4 Reduction to the common kernel theorem (II)

In this section, we keep all the assumptions from Section 3.2 but also assume $\dim V = nr - r + 1 + p - n$. Our goal is to prove the following facts:

- if $n > p$, then either $V$ is equivalent to a subspace satisfying the assumptions of the Common Kernel Theorem, or it is equivalent to $\mathcal{R}(1, r - 1)$, or it is equivalent to $\mathcal{R}(r, 0)$ in which case $r = 1$ or $p = n - 1$;
- if $n = p$, then either $V$ or $V^T$ is equivalent to a subspace satisfying the assumptions of the Common Kernel Theorem, or $V$ is equivalent to $\mathcal{R}(1, r - 1)$ or $\mathcal{R}(r - 1, 1)$, or $n = 3$, $r = 2$, $\mathbb{K} \simeq \mathbb{F}_2$ and $V$ is equivalent to $\mathcal{J}_3(\mathbb{K})$. 

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Notice that we do not discard the case \((n, r, \# \mathbb{K}) = (3, 2, 2)\) at this point. This will be useful in the prospect of Section 5.

Before tackling this, we will need two more lemmas which will be derived from the decidedly ubiquitous Corollary 6. Afterwards, we will finish the proof of Theorems 1 and 3.

4.1 Two more lemmas

Lemma 12. Let \(Y\) be a linear subspace of \(M_r(\mathbb{K})\) such that \(\dim Y = r^2 - r + 1\). Assume there is a non-trivial linear subspace \(F\) of \(\mathbb{K}^r\) which is stabilized by all the non-singular matrices of \(Y\).
Then either \(F\) has dimension \(r - 1\) and \(Y\) is similar to
\[
H_r := \left\{ \begin{bmatrix} N & C \\ 0 & \alpha \end{bmatrix} \bigg| (N, C, \alpha) \in M_{r-1}(\mathbb{K}) \times M_{r-1,1}(\mathbb{K}) \times \mathbb{K} \right\},
\]
or \(F\) has dimension 1 and \(Y\) is similar to
\[
K_r := \left\{ \begin{bmatrix} \alpha & L \\ 0 & N \end{bmatrix} \bigg| (N, L, \alpha) \in M_{r-1}(\mathbb{K}) \times M_{1,r-1}(\mathbb{K}) \times \mathbb{K} \right\}.
\]

Proof. Assume first that \(F\) is actually stabilized by all the matrices of \(Y\). Set \(s := \dim F\). Then choosing a basis with the first \(s\) vectors in \(F\) shows that \(Y\) is similar to a linear subspace of
\[
\left\{ \begin{bmatrix} A & B \\ 0 & C \end{bmatrix} \bigg| (A, B, C) \in M_s(\mathbb{K}) \times M_{s,r-s}(\mathbb{K}) \times M_{r-s}(\mathbb{K}) \right\}.
\]
If \(1 < s < r - 1\), then we would find \(\dim Y \leq r^2 - s(r - s) < r^2 - r + 1\). It follows that either \(s = 1\), in which case the equality of dimensions shows that \(Y\) is similar to \(K_r\), or \(s = r - 1\) and then we see that \(Y\) is similar to \(H_r\). It now remains to prove the first claim.

- If \(Y\) is spanned by its non-singular matrices, then the statement is trivial. By Corollary 6 this covers all cases save the one where \(r = 2\) and \(\mathbb{K} \simeq \mathbb{F}_2\).
- Assume now that \(r = 2\) and \(\mathbb{K} \simeq \mathbb{F}_2\), and choose \(B \in Y^\perp\).
Assuming that \(B\) is non-singular, we show that \(Y\) is spanned by its non-singular matrices: indeed, since the problem is invariant by replacing \(Y\) with an equivalent subspace, we lose no generality assuming \(B = I_2\); in
that case, we have $Y = \mathfrak{sl}_2(\mathbb{K})$, which is generated by the three matrices
\[
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix},
\begin{bmatrix}
1 & 1 \\
0 & 1
\end{bmatrix}
\text{ and }
\begin{bmatrix}
1 & 0 \\
1 & 1
\end{bmatrix}.
\]
Assume finally that $B$ has rank 1. Then, up to similarity, only two cases remain: if $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, then we see that $Y$ contains $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$, which has no eigenvalue in $\mathbb{F}_2$ and therefore cannot stabilize a non-trivial linear subspace of $\mathbb{F}_2^2$; if $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ then $Y = T_2^+(\mathbb{F}_2) = K_2$, QED.

\[\square\]

**Lemma 13.** Let $Y$ be a rank $r$ linear subspace of $M_{n,p}(\mathbb{K})$ such that $\dim V = n r - r + 1 + p - n$, with $n \geq p > r$. Assume that $K(Y) = K_r$ and, for every $C_1 \in M_{n-r,r-1}(\mathbb{K})$ and $L_1 \in M_{1,p-r}(\mathbb{K})$, that the subspace $Y$ contains the matrix
\[
\begin{bmatrix}
0 & 0 & L_1 \\
0 & 0 & 0 \\
0 & C_1 & 0
\end{bmatrix}.
\]

Then:

(i) either $Y$ is equivalent to $\mathcal{R}(1, r - 1)$;

(ii) or $n = p = 3$, $r = 2$, $\mathbb{K} \simeq \mathbb{F}_2$ and $Y$ is equivalent to
\[
\mathcal{J}_3(\mathbb{K}) = \left\{ \begin{bmatrix} a & 0 & 0 \\ c & b & 0 \\ d & e & a+b \end{bmatrix} \mid (a, b, c, d, e) \in \mathbb{K}^5 \right\}.
\]

**Proof.** Since $\dim Y = n r - r + 1 + p - n$, the rank theorem shows that every matrix of $\ker K$ must have the form
\[
\begin{bmatrix}
0 & 0 & L_1 \\
0 & 0 & 0 \\
0 & C_1 & 0
\end{bmatrix}
\] for some $(L_1, C_1) \in M_{1,p-r}(\mathbb{K}) \times M_{n-r,r-1}(\mathbb{K})$. We keep the notations from Section 3.2. By the factorization lemma for linear maps, we deduce that there are linear maps $\beta : K_r \to M_{n-r,1}(\mathbb{K})$, $\gamma : K_r \to M_{r-1,p-r}(\mathbb{K})$ and $\delta : K_r \to M_{n-r,p-r}(\mathbb{K})$ such that for every $M \in Y$, one has
Let now $P \in K_r \cap \text{GL}_r(\mathbb{K})$, and choose $M \in Y$ of the form $M = \begin{bmatrix} P & C_0 \\ L_0 & \alpha_1 \end{bmatrix}$.

Then formula (2) applied to $M$ and the matrix $\begin{bmatrix} 0 & 0 & L_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, for an arbitrary $L_2 \in M_{1,p-r}(\mathbb{K})$, shows $\delta(P) = L_0 P^{-1} \begin{bmatrix} L_2 \\ 0 \end{bmatrix}$: taking $L_2 = 0$ shows $\delta(P) = 0$, then a non-zero $L_2$ shows $\beta(P) = 0$ since $P \in K_r$. Symmetrically, we obtain that $\gamma(P) = 0$. We deduce that $\beta, \gamma$ and $\delta$ vanish on span($K_r \cap \text{GL}_r(\mathbb{K})$), hence on $K_r$ if $(r, \# K) \neq (2, 2)$ (we may use again Corollary 6, although a more elementary proof can be given); in the case $\beta, \gamma$ and $\delta$ are all zero, we then have an inclusion

$$Y \subset \left\{ \begin{bmatrix} a & L & L' \\ 0 & N & 0 \end{bmatrix} \mid (a, L, L', N) \in \mathbb{K} \times M_{1,r-1}(\mathbb{K}) \times M_{1,p-r}(\mathbb{K}) \times M_{n-1,r-1}(\mathbb{K}) \right\},$$

with equal dimensions on both sides, hence this is an equality, and this last subspace is obviously equivalent to $\mathcal{R}(1, r-1)$.

Assume now $r = 2$ and $\mathbb{K} = \mathbb{F}_2$. First, we prove that $\beta$ and $\gamma$ are zero. Notice already that $\beta, \gamma$ and $\delta$ vanish on the matrices $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, and the linear subspace of $K_2$ they span is $\{ M \in K_2 : \text{tr}(M) = 0 \}$. By the factorization lemma, we find three matrices $C_0 \in M_{n-1,r-1}(\mathbb{F}_2), L_0 \in M_{1,p-r}(\mathbb{F}_2)$ and $N_0 \in M_{n-r,p-r}(\mathbb{K})$ such that

$$\forall P \in K_2, \quad \beta(P) = (\text{tr } P) \cdot C_0, \quad \gamma(P) = (\text{tr } P) \cdot L_0 \quad \text{and} \quad \delta(P) = (\text{tr } P) \cdot N_0.$$

Applying this to the matrices $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ shows that $Y$ contains, for
every \((L, C) \in M_{1, p-r}(\mathbb{K}) \times M_{n-r, 1}(\mathbb{K})\), the matrices
\[
\begin{pmatrix}
1 & 0 & L \\
0 & 0 & L_0 \\
C_0 & C & N_0
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
0 & 0 & L \\
0 & 1 & L_0 \\
C_0 & C & N_0
\end{pmatrix}.
\]
Assume \(L_0 \neq 0\). Then taking \(L = 0\) and an arbitrary \(C \neq 0\) in the first matrix yields a matrix in \(Y\) with rank \(\geq 3\), which is forbidden. Hence \(L_0 = 0\). By the same line of reasoning with the second type of matrices, we find \(C_0 = 0\). If \(N_0 = 0\), then \(\beta, \gamma\) and \(\delta\) are all zero and we are done.
Assume now that \(N_0 \neq 0\). Taking \(L = 0\) and \(C = 0\) in the first type of matrices shows that \(\text{rk} N_0 \leq 1\), hence \(\text{rk} N_0 = 1\). If \(n - r > 1\), we may choose \(L = 0\) and \(C \in \mathbb{K}^{n-r} \setminus \text{Im} N_0\) in the first type of matrices, which yields a contradiction.
Hence \(n = r + 1 = 3\), and a similar line of reasoning shows that \(p = r + 1 = 3\). It follows that \(N_0 = 1\) and
\[
Y = \left\{ \begin{pmatrix}
a & c & d \\
0 & b & 0 \\
0 & e & a+b
\end{pmatrix} \mid (a, b, c, d, e) \in \mathbb{F}_2^5 \right\}.
\]
Finally, by a series of obvious row and column operations, we see that \(Y\) is equivalent to the subspace
\[
\left\{ \begin{pmatrix}
a & d & c \\
0 & a+b & c \\
0 & 0 & b
\end{pmatrix} \mid (a, b, c, d, e) \in \mathbb{F}_2^5 \right\}
\]
and then to
\[
\left\{ \begin{pmatrix}
b & 0 & 0 \\
e & a+b & 0 \\
c & d & a
\end{pmatrix} \mid (a, b, c, d, e) \in \mathbb{F}_2^5 \right\} = \mathcal{J}_3(\mathbb{F}_2),
\]
which finishes the proof.

### 4.2 The case \(\dim V = nr - r + 1\) for square matrices

Here, we assume \(n = p\), hence \(\dim V = nr - r + 1\). Once again, the assumption \(\dim K(V) > r^2 - r + 1\) would show, using Proposition 11, that \(L(W) = \{0\}\) or \(C(H) = \{0\}\), and in either case, we may use the Common Kernel Theorem...
to conclude immediately (save for the exceptional case \((n, r, \# \mathbb{K}) = (3, 2, 2)\)).

We will thus assume \(L(W) \neq \{0\} \), \(C(H) \neq \{0\} \) and \(\dim K(V) \leq r^2 - r + 1\).

Recall the notations \(G := \sum_{B \in C(H)} \text{Im} B \) and \(q = \dim G\). Using an appropriate conjugation, we may actually assume \(G = \mathbb{K}^q \times \{0\}\) (still assuming that \(J_r \in V\)).

Again, \(\dim C(H) \leq q (n - r)\) and \(\dim L(W) \leq (n - q) r\), and the rank theorem now shows that

\[
nr - r + 1 = \dim V \leq \dim K(V) + \dim L(W) + \dim C(H)
\]

\[
\leq r^2 - r + 1 + (n - q) r + q (n - r) = nr - r + 1.
\]

It follows that \(\dim K(V) = r^2 - r + 1\), \(\dim L(W) = (n - q) r\) and \(\dim C(H) = q (n - r)\), which in turns proves:

- that \(C(H)\) is the set of all matrices of \(M_{r,n-r}(\mathbb{K})\) whose image is included in \(\mathbb{K}^q \times \{0\}\);
- that \(L(W)\) is the set of all matrices of \(M_{n-r,n}(\mathbb{K})\) vanishing on \(\mathbb{K}^q \times \{0\}\).

We then deduce from identity (3) that \(P^{-1}\) stabilizes \(G\) for every \(P \in K(V) \cap \text{GL}_r(\mathbb{K})\), hence every non-singular element of \(K(V)\) stabilizes \(G\). Using Lemma 2 we find that \(q = 1\) or \(q = r - 1\). Assume first that \(q = 1\). Then we lose no generality assuming also that \(K(V) = K_r\) (notice that \(\mathbb{K} \times \{0\}\) is the only non-trivial subspace stabilized by the non-singular matrices of \(K_r\), so the assumption \(G = \mathbb{K}^q \times \{0\}\) remains unchanged).

Define now \(H'\) as the linear subspace of \(W\) consisting of its matrices which have the form \(\begin{bmatrix} 0 & * \\ 0 & 0 \end{bmatrix}\). If \(H' = \{0\}\), then the Common Kernel Theorem applied to \(V^T\) shows again that \(V\) is equivalent to a subspace of \(\mathcal{R}(r, 0)\).

Assume \(H' \neq \{0\}\). Then applying the previous line of reasoning to \(L(H')\) and \(C(W)\) shows that there is a non-trivial linear subspace \(G'\) of \(\mathbb{K}^p\) which is stabilized by every non-singular \(P \in K(V)\), and such that \(L(H') = \{M \in M_{n-r,n}(\mathbb{K}) : G' \subset \text{Ker} M\}\) and \(C(W) = \{M \in M_{r,n-r}(\mathbb{K}) : \text{Im} M \subset G'\}\). Then \(G' = G = \mathbb{K} \times \{0\}\) (again, because \(\mathbb{K} \times \{0\}\) is the only non-trivial subspace stabilized by the non-singular matrices of \(K_r\)). Thus \(L(W) = L(H')\), \(C(W) = C(H)\) and we deduce that \(W = H + H'\). Let \(L \in M_{n-r,n}(\mathbb{K})\) and \(C \in M_{1,n-r}(\mathbb{K})\).

Then \(V\) contains a matrix \(M = \begin{bmatrix} 0 & 0 & C \\ 0 & 0 & 0 \\ 0 & L & N \end{bmatrix}\) for some \(N \in M_{n-r}(\mathbb{K})\). Recall
now that $V$ contains $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ so applying (2) to $J_r$ and $M$ shows $N = 0$. It then follows that $V$ satisfies the assumptions of Lemma 13 which essentially finishes our proof.

Assume finally that $q = r - 1$. Then the same line of reasoning as before shows that we may assume $W = H + H'$, with $L(H') = \{ M \in M_{n-r,r}(\mathbb{K}) : \mathbb{K}^{r-1} \times \{0\} \subseteq \ker M \}$ and $C(H) = \{ M \in M_{r,n-r}(\mathbb{K}) : \text{Im} M \subset \mathbb{K}^{r-1} \times \{0\} \}$. Using the easy fact that $H'^T$ is similar to $K_r$, we then see that, for $V' := V'^T$, we now have $q' = 1$, which reduces us to the previous situation. Notice finally, in the exceptional case $n = 3$, $r = 2$ and $\mathbb{K} \simeq \mathbb{F}_2$, that $q = r - 1$ implies $q = 1$, therefore $V$ is equivalent to $\mathcal{R}(1,1)$ or to $\mathcal{J}_3(\mathbb{K})$.

This completes the proof of Theorem 1.

4.3 The case $\dim V = nr - r + 1 + p - n$ for non-square matrices

Here, we assume $n > p$. If $q = 0$, then $H = \{0\}$ and we are reduced to the situation of the Common Kernel Theorem. We now assume that $q \geq 1$. Again, we denote by $H'$ the linear subspace of $V$ consisting of its matrices which have the form $\begin{bmatrix} 0_r & 0 \\ ? & ? \end{bmatrix}$.

- Assume $L(H') = \{0\}$. Then $\dim V \leq pr$ and $V^T$ is equivalent to a subspace of $M_{p,n}(\mathbb{K})$ satisfying the assumptions of the Common Kernel Theorem (with $n$ and $p$ reversed). Notice also that $nr + (p-n) - r + 1 - r p = (r-1)(n-p-1)$, hence $r = 1$ or $n = p+1$, and $\dim V = pr$. However, in the exceptional case $n = 4$, $p = 3$, and $r = 2$, we then have $\dim V > rp - p + 2$, so the Common Kernel Theorem may still be applied, and we recover that $V$ is equivalent to $\mathcal{R}(r,0)$.

From now on, we assume that $L(H') \neq \{0\}$.

- If $\dim K(V) > r^2 - r + 1$, we may then use Proposition 11 again to see that $L(W) = \{0\}$, hence $L(H') = \{0\}$ which contradicts our assumptions.

- Assume now that $\dim K(V) \leq r^2 - r + 1$. Recall the inequalities:

$$\dim C(H) \leq q(n-r), \quad \dim L(W) \leq (r-q)(n-r),$$

hence the rank theorem shows

$$\dim V \leq r^2 - r + 1 + \dim C(H) + \dim L(W) \leq n r - (n - p) q - r + 1,$$
which in turn yields \( q = 1 \). We may then assume \( G = \mathbb{K} \times \{ 0 \} \). From there, we may use the exact same line of reasoning as in paragraph 4.2 and show successively:

→ that \( K(V) \) satisfies the assumptions of Lemma 12 and deduce from there that \( K(V) = K_r \) since \( G = \mathbb{K} \times \{ 0 \} \);
→ that \( C(H) = \{ A \in M_{r,p} : \text{Im} A \subset \mathbb{K} \times \{ 0 \} \} \);
→ that \( L(H') = \{ A \in M_{n-r,r} : \mathbb{K} \times \{ 0 \} \subset \text{Ker} A \} \);
→ and finally that \( V \) satisfies all the assumptions of Lemma 13.

Hence either \( r = 1 \) and \( V \) is equivalent to \( \mathcal{R}(r,0) \), or \( V \) is equivalent to \( \mathcal{R}(1,r-1) \) (notice that \( n > p \) which discards the exceptional case of Lemma 13).

This completes the proof of Theorem 3.

5 The case of \( M_3(\mathbb{F}_2) \)

Here, we will classify all the 5-dimensional rank 2 subspaces of \( M_3(\mathbb{F}_2) \), as stated in Theorem 2. Let \( V \) be such a subspace. Notice that the proofs from Section 4 show that either \( V \) satisfies the assumptions in the Common Kernel Theorem, or \( V^T \) satisfies them, or \( V \) is equivalent to \( J_3(\mathbb{F}_2) \) (and then it also satisfies them). Notice also that \( J_3(\mathbb{F}_2) \) and \( J_3(\mathbb{F}_2)^T \) are actually equivalent, and even similar: this is easily seen by remarking that \( J_3(\mathbb{F}_2)^T = sl_3(\mathbb{F}_2) \cap T_J^3(\mathbb{F}_2) \) and by performing the row and column operations \( L_1 \leftrightarrow L_3 \) and \( C_1 \leftrightarrow C_3 \). It will thus suffice to prove the following proposition, which describes some exceptional cases in the Common Kernel Theorem:

**Proposition 14.** Let \( W \) be a linear subspace of \( M_{3,2}(\mathbb{F}_2) \) such that \( \dim W = 5 \). Let \( \varphi : W \to M_{3,1}(\mathbb{F}_2) \) be a linear map. Consider the linear subspace

\[
V = \left\{ \begin{bmatrix} M & \varphi(M) \end{bmatrix} \mid M \in W \right\} \subset M_3(\mathbb{F}_2)
\]

and assume that \( \operatorname{rk} V \leq 2 \). Then either \( V \) is equivalent to a subspace of \( \mathcal{R}(0,2) \) or it is equivalent to \( J_3(\mathbb{F}_2) \).

Notice that \( W^\perp \) contains only one non-zero matrix \( B \). Using a series of row and column operations, we then lose no generality assuming that \( B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \)
or \( B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \). We will deal with these two cases separately (in the second one, we will show that \( V \) is equivalent to a subspace of \( \mathcal{R}(0,2) \)).

5.1 The case \( \text{rk } B = 1 \)

We assume \( B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \), hence \( W = \left\{ \begin{bmatrix} a & 0 \\ c & b \\ d & e \end{bmatrix} \mid (a, b, c, d, e) \in \mathbb{F}_2^5 \right\} \).

Since \( M_2(\mathbb{F}_2) \) is spanned by \( \text{GL}_2(\mathbb{F}_2) \), the line of reasoning from the proof of Proposition 7 shows that the first entry of \( \varphi(M) \) only depends on the entry of \( M \) at the \((1,1)\) spot, hence either \( \forall M \in W, \varphi(M) = \begin{bmatrix} 0 \\ ? \\ ? \end{bmatrix} \) or \( \forall M \in W, \varphi(M) = \begin{bmatrix} m_{1,1} \\ ? \\ ? \end{bmatrix} \).

In the latter case, the column operation \( C_3 \leftarrow C_3 + C_1 \) shows we lose no generality assuming that every matrix of \( V \) decomposes as \( \begin{bmatrix} a & 0 & 0 \\ c & b & ? \\ d & e & ? \end{bmatrix} \). Notice now that \( V \) has a unique matrix of the form \( A_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & x \\ 0 & 0 & y \end{bmatrix} \). By performing an elementary row operation on the second and third rows, we may assume \( x = 0 \).

Set now \( V' := \{ M \in V : m_{1,1} = 0 \} \). For every \( M \in V' \), we write

\[
M = \begin{bmatrix} 0 \\ ? \\ \eta(M) \end{bmatrix}
\]

for some \( \eta(M) \in M_2(\mathbb{F}_2) \).

Since \( \text{rk}(A_0 + M) \leq 2 \) for every \( M \in V' \), the affine subspace \( \begin{bmatrix} 0 & 0 \\ 0 & y \end{bmatrix} + \eta(V') \) has a rank less than 1, and since obviously \( \dim \eta(V') \geq 2 \), we actually have \( \dim \eta(V') = 2 \). It follows from the rank theorem that \( \eta(V') \) contain no non-zero matrix of the form \( \begin{bmatrix} 0 & ? \\ 0 & ? \end{bmatrix} \) (since any matrix of \( M_{2,1}(\mathbb{F}_2) \) is the first column of a matrix of \( \eta(V') \)). We deduce that \( \varphi(M) \) does not depend on the entries \( m_{2,1} \) and \( m_{3,1} \). We finally distinguish between two cases:

- Assume \( y = 0 \). Then \( \eta(V') \) is a rank 2 linear subspace of dimension 2 of \( V' \), and it is obviously not equivalent to \( \mathcal{R}(1,0) \), so it is equivalent to
We actually lose no generality assuming that $\eta(V') = \mathcal{R}(0, 1)$ (if $\eta(V') \neq \mathcal{R}(0, 1)$, then the operation $C_3 \leftarrow C_2 + C_3$ on the elements of $V$ brings us back to this case). In this case, we thus have $V \subset \mathcal{R}(0, 2)$.

- Assume now that $y = 1$. Then we find two (unique) matrices in $\eta(V')$ with the form $\begin{bmatrix} 1 & \alpha \\ 0 & \beta \end{bmatrix}$ and $\begin{bmatrix} 0 & \alpha' \\ 1 & \beta' \end{bmatrix}$. Adding $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ to them, we obtain rank 1 matrices, hence $\beta = 1$ and $\alpha' = 0$. Adding their sum to $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ then yields $\alpha + \beta' = 0$. If $\alpha = \beta' = 0$, then we readily obtain $V = \mathcal{J}_3(\mathbb{F}_2)$.

Assume finally that $\alpha = \beta' = 1$. Then we have

$$V = \left\{ \begin{bmatrix} a & 0 & 0 \\ c & b & b \\ d & e & a + b + e \end{bmatrix} \mid (a, b, c, d, e) \in \mathbb{F}_2^5 \right\},$$

and the operation $C_3 \leftarrow C_3 - C_2$ shows that $V$ is equivalent to $\mathcal{J}_3(\mathbb{F}_2)$.

This completes the case $\text{rk } B = 1$.

### 5.2 The case $\text{rk } B = 2$

We assume $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$, hence

$$W = \left\{ \begin{bmatrix} N \\ L \end{bmatrix} \mid N \in \mathfrak{sl}_2(\mathbb{F}_2), \ L \in M_{1,2}(\mathbb{F}_2) \right\}.$$ 

Noticing that $\mathfrak{sl}_2(\mathbb{F}_2)$ is spanned by its non-singular matrices, we may use the arguments from the proof of Proposition 7 to see that the third entry of $\varphi(M)$ depends only on the last row of $M$. We may then write $\varphi \begin{bmatrix} N \\ L \end{bmatrix} = \begin{bmatrix} \gamma(N) \\ \delta(L) \end{bmatrix}$ for some linear form $\delta : M_{1,2}(\mathbb{F}_2) \rightarrow \mathbb{F}_2$. Using the operation $C_3 \leftarrow C_3 + a.C_1 + b.C_2$ for some well-chosen $(a, b) \in \mathbb{F}_2^2$, we may then assume that every matrix of $V$ has a zero entry at the $(3, 3)$ spot. It follows that:

- for every $N \in \mathfrak{sl}_2(\mathbb{F}_2)$, there is a unique $\gamma(N) \in M_{2,1}(\mathbb{F}_2)$ such that $V$ contains the matrix $\begin{bmatrix} N & \gamma(N) \\ 0 & 0 \end{bmatrix}$;
• for every $L \in M_{1,2}(\mathbb{F}_2)$, there is a unique $\alpha(L) \in M_{2,1}(\mathbb{F}_2)$ such that $V$ contains the matrix $\begin{bmatrix} 0 & \alpha(L) \\ L & 0 \end{bmatrix}$.

Let $N \in \mathfrak{sl}_2(\mathbb{F}_2)$ and $L \in M_{1,2}(\mathbb{F}_2)$. Computing the determinant of $\begin{bmatrix} N & \gamma(N) + \alpha(L) \\ L & 0 \end{bmatrix}$ then shows

$$L \tilde{N}^T (\gamma(N) + \alpha(L)) = 0,$$

where $\tilde{N}$ denotes the matrix of cofactors of $N$. Since $N \in \mathfrak{sl}_2(\mathbb{F}_2)$, we actually have $\tilde{N}^T = N$, hence the identity:

$$\forall (N, L) \in \mathfrak{sl}_2(\mathbb{F}_2) \times M_{1,2}(\mathbb{F}_2), \quad LN\gamma(N) = LN\alpha(L). \quad (5)$$

With $L$ fixed, notice that the left hand-side of (5) is a quadratic for $M$ of $N$, and the right hand-side a linear form. We deduce that

$$\forall (M, N, L) \in \mathfrak{sl}_2(\mathbb{F}_2)^2 \times M_{1,2}(\mathbb{F}_2), \quad L (M\gamma(N) + N\gamma(M)) = 0,$$

$$\forall (M, N) \in \mathfrak{sl}_2(\mathbb{F}_2)^2, \quad M\gamma(N) + N\gamma(M) = 0. \quad (6)$$

Notice that $\text{Ker} \gamma \neq \{0\}$ by the rank theorem.

• Assume that $\text{Ker} \gamma$ contains only singular matrices. Then $\dim \text{Ker} \gamma = 1$ since the three rank 1 matrices of $\mathfrak{sl}_2(\mathbb{F}_2)$ are linearly independent. It follows that $\gamma$ is onto on $M_{2,1}(\mathbb{F}_2)$. However, choosing $M_0 \in \text{Ker} \gamma \setminus \{0\}$, we find that $\forall N \in \mathfrak{sl}_2(\mathbb{F}_2), \quad M_0 \gamma(N) = 0$, hence $M_0 = 0$, a contradiction.

• We deduce that $\text{Ker} \gamma$ contains a non-singular matrix $M_0$. Then (6) entails $M_0 \gamma(N) = 0$ for every $N \in \mathfrak{sl}_2(\mathbb{F}_2)$, hence $\gamma = 0$.

• It follows that, given an arbitrary $L \in M_{1,2}(\mathbb{F}_2) \setminus \{0\}$, one has $LN\alpha(L) = 0$ for every $N \in \mathfrak{sl}_2(\mathbb{F}_2)$. However $\text{span}(L\mathfrak{sl}_2(\mathbb{F}_2)) = M_{2,1}(\mathbb{F}_2)$ (otherwise $\mathfrak{sl}_2(\mathbb{F}_2)$ would be similar to a subspace of $T_2^-(\mathbb{F}_2)$, hence to $T_2^+(\mathbb{F}_2)$), and we deduce that $\alpha(L) = 0$.

We conclude that $\alpha = 0$ and $\gamma = 0$, which yields $V \subset R(0, 2)$. This finishes the proof of Proposition [14] and Theorem [2] then follows as explained earlier.
5.3 A final remark

Using the previous results, the reader will easily prove the following generalization of Proposition 14, which fully describes the exceptional case in the Common Kernel Theorem:

**Proposition 15.** Let $W$ be a linear subspace of $\text{M}_{3,2}(\mathbb{F}_2)$ such that $\dim W = 5$. Let $p \geq 3$, and $\varphi : W \to \text{M}_{3,p-2}(\mathbb{F}_2)$ be a linear map. Consider the linear subspace

$$V = \left\{ \begin{bmatrix} M & \varphi(M) \end{bmatrix} \mid M \in W \right\} \subset \text{M}_{n,p}(\mathbb{F}_2)$$

and assume that $\text{rk} V \leq 2$. Then either $V$ is equivalent to a subspace of $\mathcal{R}(0, 2)$ or it is equivalent to the vector space:

$$\left\{ \begin{bmatrix} M & 0 \end{bmatrix} \mid M \in \mathcal{J}_3(\mathbb{F}_2) \right\} \subset \text{M}_{n,p}(\mathbb{F}_2).$$

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