Solving the Greenberger-Horne-Zeilinger paradox: an explicitly local and realistic model of hidden variables for the GHZ quantum state

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The Greenberger-Horne-Zeilinger (GHZ) version of the Einstein-Podolsky-Rosen (EPR) paradox is widely regarded as a conclusive logical argument that rules out the possibility of describing quantum phenomena within the framework of a local and realistic model of hidden variables in which the observers are free to choose their own experimental settings. In this paper we show, however, that the GHZ argument implicitly relies on an additional crucial assumption, which is not required by fundamental physical principles and had gone unnoticed. Namely, we note that the argument implicitly assumes the existence of an absolute angular frame of reference with respect to which the polarization properties of the hypothetical hidden configurations of the entangled particles as well as the orientation of the measurement apparatus that test the system can be defined. We further note that such an absolute frame of reference would not exist if the hidden configurations of the entangled particles spontaneously break the gauge rotational symmetry. Indeed, by skipping this unnecessary additional assumption we are able to build an explicitly local and realistic model of hidden variables for the GHZ state, which complies with the ‘free-will’ hypothesis and reproduces the quantum mechanical predictions, and thus completes the description of the system in the EPR sense.

1. The inability to accommodate the seemingly trivial notions of locality and physical realism within the current interpretation of the quantum mechanical wavefunction is at the core of a long lasting debate about the foundations of the quantum theory and the role played by measurements, whose origins go back to the formulation of the renowned Einstein-Podolsky-Rosen (EPR) paradox seventy years ago [1]. Solving these key issues would require to develop a description of quantum phenomena in terms of a local and realist model of hidden variables. Nonetheless, according to the current wisdom, such a description is not possible in so far as we insist on keeping the notion, also seemingly trivial, that the observers’ choice of their measurement settings is not constrained by the actual hidden configuration of the observed system (free-will). Indeed, several fundamental theorems state that generic models of hidden variables that share certain intuitive features cannot fully reproduce the predictions of quantum mechanics [2–10]. Moreover, carefully designed experimental tests have consistently confirmed the predictions of the quantum theory and, thus, ruled out these generic models of hidden variables [11, 12, 21].

The best known among these theorems is Bell’s theorem [2, 3, 5, 10], which proves that such generic models of hidden variables cannot reproduce the statistical correlations predicted by quantum mechanics for the outcomes of long sequences of strong polarization measurements performed along certain relative directions on two entangled particles, if the measurements are fully anti-correlated when performed along parallel directions.

The Greenberger-Horne-Zeilinger version of Bell’s theorem [6] is an even more conclusive proof of the limitations of these generic local and realistic models of hidden variables, since it proves that such models cannot reproduce even single outcomes of strong spin polarization measurements performed along certain relative directions on three or more entangled particles prepared in the so-called GHZ state,

$$|\text{GHZ}\rangle \equiv \frac{1}{\sqrt{2}} \left( |\uparrow \uparrow \text{ ... } \uparrow \rangle + |\downarrow \downarrow \text{ ... } \downarrow \rangle \right).$$

However, in a recent paper [22] we have shown that Bell’s theorem crucially relies on an implicit feature of the considered generic class of models of hidden variables, beyond the aforementioned features of locality, physical realism and ‘free-will’. This additional feature, unlike the other three explicit features, is not required by fundamental physical principles. Namely, we noted that Bell’s theorem implicitly assumes that there exists an absolute angular frame of reference with respect to which we can define the polarization properties of the hypothetical hidden configurations of the pair of entangled particles as well as the orientation of the measurement apparatus that test them. Indeed, we showed that this implicit assumption is not correct if these hidden configurations spontaneously break the gauge rotational symmetry along an otherwise arbitrary direction. Following this insight we actually built an explicitly local and realistic model of hidden variables that complies with the hypothesis of ‘free-will’ and fully reproduces the predictions of quantum mechanics for Bell’s states and, thus, completes the description of the quantum polarization singlet in the sense advocated by Einstein, Podolsky and Rosen in [1].

In this paper we develop these ideas to build an explicitly local and realistic model of hidden variables for the GHZ state of three entangled particles. The paper is organized as follows. In Section 2 we review
the argument put forward by Greenberger, Horne and Zeilinger against the impossibility to describe the quan-
tum mechanical predictions for the GHZ state within 
the framework of a local and realistic model of hidden 
variables. In Section 3 we introduce a simple explicit 
model of hidden variables that overcomes this argument. 
In Section 4 we extend this model and discuss it in 
detail. Our conclusions are summarized in section 5.

2. The Greenberger-Horne-Zeilinger spin polarization 
state of three entangled particles is described by the 
quantum wavefunction:

\[ |\Pi\rangle = \frac{1}{\sqrt{2}} \left( |\uparrow\rangle^{(A)} \otimes |\uparrow\rangle^{(B)} \otimes |\uparrow\rangle^{(C)} + \right. 
+ e^{i\Phi} |\downarrow\rangle^{(A)} \otimes |\downarrow\rangle^{(B)} \otimes |\downarrow\rangle^{(C)} \right), \tag{2} \]

where \{ |\uparrow\rangle, |\downarrow\rangle \} denotes a basis of single particle spin 
polarization eigenstates along its locally defined Z-axis. 
In this state all three outcomes in each single event of a 
long sequence of strong spin polarization measurements 
performed on each of the three particles along their 
corresponding Z-axes must be consistently equal, either 

\[ S_{Z}^{(A)}(n) = S_{Z}^{(B)}(n) = S_{Z}^{(C)}(n) = +1, \]

or

\[ S_{Z}^{(A)}(n) = S_{Z}^{(B)}(n) = S_{Z}^{(C)}(n) = -1, \]

for all \( n \in \{1, ..., N\} \), with each one of the two possibili-
ties happening with a probability of 1/2.

In fact, in the GHZ state (2) the expected average 
values of long sequences of strong spin polarization 
measurements performed along any arbitrary directions \( \Omega_{\alpha}^{(A)} \), 
\( \Omega_{\beta}^{(B)} \), \( \Omega_{\gamma}^{(C)} \) in the XY-planes orthogonal to the local Z-
axes are equal to zero:

\[ \langle S_{\Omega_{\alpha}}^{(A)}(n) \rangle_{n \in N} = \langle S_{\Omega_{\beta}}^{(B)}(n) \rangle_{n \in N} = \langle S_{\Omega_{\gamma}}^{(C)}(n) \rangle_{n \in N} = 0, \tag{3} \]

as well as their two-particles correlations:

\[ \langle S_{\Omega_{\alpha}}^{(A)}(n) \cdot S_{\Omega_{\beta}}^{(B)}(n) \rangle_{n \in N} = \langle S_{\Omega_{\alpha}}^{(B)}(n) \cdot S_{\Omega_{\beta}}^{(C)}(n) \rangle_{n \in N} = 
= \langle S_{\Omega_{\alpha}}^{(C)}(n) \cdot S_{\Omega_{\beta}}^{(A)}(n) \rangle_{n \in N} = 0. \tag{4} \]

Notwithstanding, the three-particles correlation is non-
zero, in general, and given by:

\[ \langle S_{\Omega_{\alpha}}^{(A)}(n) \cdot S_{\Omega_{\beta}}^{(B)}(n) \cdot S_{\Omega_{\gamma}}^{(C)}(n) \rangle_{n \in N} = 
= \cos \left( \Delta_{\Omega_{\alpha}^{(A)}} + \Delta_{\Omega_{\beta}^{(B)}} + \Delta_{\Omega_{\gamma}^{(C)}} + \Phi \right), \]

where \( \Delta_{\Omega_{\alpha}^{(A)}} \), \( \Delta_{\Omega_{\beta}^{(B)}} \) and \( \Delta_{\Omega_{\gamma}^{(C)}} \) describe the relative ori-
entations of each one of the measurement apparatus with 
respect to some implicit reference direction, which we de-
fine as X-axis.

In particular, for \( \Phi = 0 \) the following four relationships 
follow:

\[ S_{X}^{(A)}(n) \cdot S_{X}^{(B)}(n) \cdot S_{X}^{(C)}(n) = +1, \quad n = 1, ..., N \]
\[ S_{X}^{(A)}(m) \cdot S_{Y}^{(B)}(m) \cdot S_{Y}^{(C)}(m) = -1, \quad m = 1, ..., M \]
\[ S_{Y}^{(A)}(k) \cdot S_{Y}^{(B)}(k) \cdot S_{Z}^{(C)}(k) = -1, \quad k = 1, ..., K \]
\[ S_{Y}^{(A)}(l) \cdot S_{Y}^{(B)}(l) \cdot S_{X}^{(C)}(l) = -1, \quad l = 1, ..., L, \tag{5} \]

for any four sequences of strong measurements per-
doned along directions \( (X, X, X) \), \( (X, Y, Y) \), \( (Y, Y, X) \) 
and \( (Y, Y, X) \).

These four relationships (5) lie at the core of the 
Greenberger-Horne-Zeilinger paradox [1]. On one hand, 
these relationships imply that we can gain certainty 
about the polarization properties of any of these three 
particles without in any sense disturbing them. Thus, 
according to the notion introduced by Einstein, Podol-
sky and Rosen [1], these polarization properties are el-
ements of reality whose values must be set at the time 
when the three entangled particles are produced. On the 
other hand, this notion seems to be inconsistent: by mul-
tiplying the last three equations in (5) and assuming that 
all polarizations components must take values either +1 
or −1, we would obtain that

\[ S_{X}^{(A)}(n) \cdot S_{X}^{(B)}(n) \cdot S_{X}^{(C)}(n) = -1, \quad n = 1, ..., N \tag{6} \]

which is in contradiction with the first one.

This argument is widely considered as the most 
clear-cut evidence against the possibility of giving the 
wavefunction (2) an statistical interpretation within 
the framework of a local and realistic model of hidden 
configurations, in which the observers are free to choose 
the setting of their measurements.

3. The above argument crucially relies on the implicit-
ity assumed existence of an absolute angular frame of 
reference, with respect to which the polarization prop-
erties of the hidden configurations of the triplet of en-
tangled particles, as well as the orientation of the three 
measurement apparatus that test them, can be defined. 
In such an absolute frame of reference all the polariza-
tion components of each possible hidden configuration 
must take a binary value, either +1 or −1, and relationships 
(5) immediately follow. However, as we already did no-
tice in [22], the existence of such an absolute angular 
frame of reference is not required by fundamental physi-
ical principles. Indeed, the polarization properties of 
the hidden configurations can only be properly defined with 
respect to the reference directions set by the orientation 
of the measurement apparatus. Furthermore, the ori-
tentation of the measurement apparatus can only be defined
with respect to each other, while their global orientation is actually an spurious gauge degree of freedom.

That is, the orientation of the three measurement apparatus, whose correlation is given by

\[
\langle S_{\Omega,\alpha}^{(A)}(n) \cdot S_{\Omega,\beta}^{(B)}(n) \cdot S_{\Omega,\gamma}^{(C)}(n) \rangle_{n \in \mathbb{N}} = \cos(\Delta + \Phi),
\]

is described by a single physical degree of freedom, namely, the angle \(\Delta\) measured with respect to a set of \(X\)-axes locally defined at the sites of each one of the three particles and for which the correlation is given by:

\[
\langle S_X^{(A)}(n) \cdot S_X^{(B)}(n) \cdot S_X^{(C)}(n) \rangle_{n \in \mathbb{N}} = \cos(\Phi),
\]

and, in particular for \(\Phi = 0\) by

\[
\langle S_X^{(A)}(n) \cdot S_X^{(B)}(n) \cdot S_X^{(C)}(n) \rangle_{n \in \mathbb{N}} = +1.
\]

Actually, condition (9) defines the notion of parallel directions at the sites of each one of the three entangled particles. All sets of axes for which this condition is fulfilled are gauge equivalent and, hence, physically indistinguishable through measurements performed on a triplet of entangled particles prepared in the GHZ state.

Moreover, the polarization properties of the hidden configurations of the triplet of entangled particles can be properly defined only with respect to the local reference directions set by the three measurement apparatus. That is, the actual value \(S^{(A)}_{\Omega,\alpha}(\Omega,\omega)\) of the polarization component of, say, particle \(A\) along some direction \(\Omega\) may be, in general, a function of the reference direction \(\Omega,\alpha\) set by the measurement apparatus of observer \(A\) (and, of course, also of the coordinate \(\omega \in \mathbb{S}\) that labels the hidden configuration in which the system of three entangled particles occurs). This dependence does not conflict with the principles of locality and realism, which only demand that the value of the polarization components of particle \(A\) cannot depend on the orientations of the reference directions \(\Omega,\beta,\Omega,\gamma\) along which observers \(B\) and \(C\) choose to test their particles. Therefore, we must not restrict our models within the constraint that all polarization components of either one of the particles must take a binary value, either +1 or −1: only the polarization component of each one of the particles along the reference direction set by the orientation of the corresponding measurement apparatus must take a binary value. That is, on all possible hidden configurations of the triplet we must have:

\[
s^{(A)}_{\Omega,\alpha}(\Omega,\omega) = \pm 1, \quad s^{(B)}_{\Omega,\beta}(\Omega,\omega) = \pm 1, \quad s^{(C)}_{\Omega,\gamma}(\Omega,\omega) = \pm 1,
\]

but the polarization components along any other directions must not necessarily take either one of these two values. Indeed, the only experimental access that we can have to the spin polarization components along these other directions is through weak measurements, whose outcome can have absolute values larger and smaller than one and may even be complex [23].

Therefore, it is crucial to realize that in order to obtain a meaningful description of the system we must be careful to compare magnitudes defined with respect to the same reference directions. For example, we can state that with respect to a set of parallel reference directions \(X^{(A)}, X^{(B)}\) and \(X^{(C)}\) defined by condition (9), the polarization components of the particles along the orthogonal directions \(Y^{(A)}, Y^{(B)}, Y^{(C)}\) take values either +1 or −1, according to the relationship:

\[
\begin{align*}
\langle s^{(A)}_{X}(X,\omega) \rangle &= i \langle s^{(A)}_{X}(X,\omega) \rangle, \\
\langle s^{(B)}_{Y}(X,\omega) \rangle &= i \langle s^{(B)}_{Y}(X,\omega) \rangle, \\
\langle s^{(C)}_{Y}(X,\omega) \rangle &= i \langle s^{(C)}_{Y}(X,\omega) \rangle,
\end{align*}
\]

with \(s^{(A)}_{X}(X,\omega) = \pm 1, s^{(B)}_{X}(X,\omega) = \pm 1\) and \(s^{(C)}_{X}(X,\omega) = \pm 1\). Then, the four constraints (10) become trivially identical,

\[
\begin{align*}
\langle s^{(A)}_{X}(X,\omega) \rangle &\cdot \langle s^{(B)}_{X}(X,\omega) \rangle &\cdot \langle s^{(C)}_{X}(X,\omega) \rangle = +1, \\
\langle i s^{(A)}_{X}(X,\omega) \rangle &\cdot \langle i s^{(B)}_{X}(X,\omega) \rangle &\cdot \langle i s^{(C)}_{X}(X,\omega) \rangle = -1, \\
i \langle s^{(A)}_{X}(X,\omega) \rangle &\cdot \langle s^{(B)}_{X}(X,\omega) \rangle &\cdot \langle i s^{(C)}_{X}(X,\omega) \rangle = -1, \\
i \langle s^{(A)}_{X}(X,\omega) \rangle &\cdot \langle i s^{(B)}_{X}(X,\omega) \rangle &\cdot \langle s^{(C)}_{X}(X,\omega) \rangle = -1.
\end{align*}
\]

In other words, the argument put forward by Greenberger, Horne and Zeilinger as a proof of the impossibility to reproduce the predictions of quantum mechanics for the GHZ state within the framework of a local and realistic statistical model of hidden variables can be overcome by realizing that there does not necessarily exist an absolute frame of reference with respect to which the polarization properties of the entangled particles can be defined and, in consequence, allowing their actual values depend on the reference direction with respect to which they are described.

4. In this section we build and discuss in detail an explicitly local and realistic statistical model of hidden variables for the GHZ state of three entangled particles. The model complies with the ‘free-will’ assumption and reproduces the quantum mechanical predictions for the average values and correlations of long sequences of strong spin polarization measurements performed along any three arbitrary directions.

Our statistical model consists of a continuous set \(\mathbb{S}\) of possible hidden configurations labelled by a coordinate \(\omega \in \mathbb{S}\), with a well-defined probability density \(G(\omega)\) for each one of these configurations to occur and a locally defined binary response function \(S^{(A,B,C)}_{\Omega,\alpha,\beta,\gamma}(\omega)\) that specifies the outcome that each one of these hidden configurations
would produce in each one of the three measurement apparatus as a function of their orientations.

In particular, the phase space of our model consists of infinitely many possible hidden configurations uniformly distributed over a unit torus \( S_1 \times C_1 \). An observer that strongly measures the polarization of, say, particle A along an arbitrary direction \( \Omega_\alpha^{(A)} \) orthogonal to its locally defined Z-axis fixes a reference frame of angular coordinates \((\omega_A, \eta) \in [-\pi, \pi] \times [-\pi, \pi]\) over this torus.

We assume that the probability density distribution of each one of the hidden configurations to occur is given by

\[
G(\omega_A, \eta) = \frac{1}{2\pi} \cdot g(\omega_A), \quad \text{with} \quad g(x) = \frac{1}{4} |\sin(x)|
\]

and the outcome of a strong measurement of the polarization component of particle A along the chosen reference direction is given by:

\[
S^{(A)}(\omega_A, \eta) = s^{(A)}(\Omega_\alpha, \omega_A, \eta),
\]

with:

\[
s^{(A)}(\Omega_\alpha, \omega_A, \eta) = \left\{ \begin{array}{ll}
S(\omega_A), & \text{if } \eta \in (0, +\pi], \\
-S(\omega_A), & \text{if } \eta \in (-\pi, 0].
\end{array} \right.
\]

and

\[
S(\omega) = \left\{ \begin{array}{ll}
+1, & \text{if } \omega \in (0, +\pi], \\
-1, & \text{if } \omega \in (-\pi, 0].
\end{array} \right.
\]

The observer of particle B measures its polarization along some other arbitrary direction \( \Omega_\beta^{(B)} \) orthogonal to its local Z-axis, which sets a different reference direction with its own frame of angular coordinates \((\omega_B, \eta) \in [-\pi, +\pi] \times [-\pi, \pi]\) over the torus \( S_1 \times C_1 \). By symmetry considerations we demand that the outcome of this measurement is described by the same response function:

\[
S^{(B)}(\omega_B, \eta) = s^{(B)}(\Omega_\beta, \omega_B, \eta) = \left\{ \begin{array}{ll}
S(\omega_B), & \text{if } \eta \in (0, +\pi], \\
-S(\omega_B), & \text{if } \eta \in (-\pi, 0].
\end{array} \right.
\]

The two sets of angular coordinates \( \omega_A \) and \( \omega_B \) are related by the relationship:

\[
\omega_B = \omega_A, \quad \text{if } \eta \in (0, \pi],
\]

\[
\omega_B = \pi + \omega_A, \quad \text{if } \eta \in (-\pi, 0],
\]

where

\[
\omega_A = L(\omega_A; \Delta),
\]

and

- If \( \Delta \in [0, \pi) \),

\[
L(\omega; \Delta) = \left\{ \begin{array}{ll}
q(\omega) \cdot \cos(\Delta) - \cos(\omega), & \text{if } -\pi \leq \omega < \Delta - \pi, \\
q(\omega) \cdot \cos(\Delta) + \cos(\omega) - 1, & \text{if } \Delta - \pi \leq \omega < 0,
\end{array} \right.
\]

- If \( \Delta \in [-\pi, 0) \),

\[
L(\omega; \Delta) = \left\{ \begin{array}{ll}
q(\omega) \cdot \cos(\Delta) + \cos(\omega) + 1, & \text{if } -\pi \leq \omega < \Delta, \\
q(\omega) \cdot \cos(\Delta) - \cos(\omega) - 1, & \text{if } \Delta \leq \omega < 0,
\end{array} \right.
\]

with

\[
q(\omega) = \text{sign}(\omega - \Delta) \text{mod}(\pi),
\]

and the function \( y = \cos(x) \) is defined in its main branch, such that \( y \in [0, \pi] \) while \( x \in [-1, +1] \). The parameter \( \Delta \) in this transformation law denotes now the relative orientation of the measurement setting \( \Delta + \Phi \), as defined in (17).

It is straightforward to check that the probability density remains invariant under the coordinates transformation (19,20), since

\[
|d\omega^\prime g(\omega^\prime)| = \frac{1}{4} |d\omega^\prime \sin(\omega^\prime)| = \frac{1}{4} |d\cos(\omega^\prime)| = \frac{1}{4} |d\cos(\omega)| = \frac{1}{4} |\sin(\omega)| = |d\omega g(\omega)|,
\]

and

\[
g(\pi + \omega) = \frac{1}{4} |\sin(\omega + \pi)| = \frac{1}{4} |\sin(\omega)| = g(\omega).
\]

In fact, these equalities state in precise terms that the probability of each hidden configuration to occur does not depend on the orientation of the reference direction chosen by the observers to describe their particles or, in other words, that our model complies with the requirements of 'free-will'.

We can now define a partition of the torus into four disjoint regions, \( S_1 \times C_1 = \mathcal{I}_{++} \cup \mathcal{I}_{+-} \cup \mathcal{I}_{-+} \cup \mathcal{I}_{--} \), as follows:

\[
\mathcal{I}_{++} = \{ \omega_A : \omega_A \in (\Delta, +\pi) \} = \{ \omega_B : \omega_B \in (0, -\Delta + \pi) \}, \text{ if } \eta > 0
\]

\[
\mathcal{I}_{-+} = \{ \omega_B : \omega_B \in (0, -\Delta) \}, \text{ if } \eta > 0
\]

\[
\mathcal{I}_{-+} = \{ \omega_B : \omega_B \in (-\pi, -\Delta) \}, \text{ if } \eta \leq 0
\]
\[ \mathcal{I}_{+} = \{ \omega_A : \omega_A \in (0, \Delta] \} = \begin{cases} \{ \omega_B : \omega_B \in (-\Delta, 0) \}, & \text{if } \eta > 0 \\ \{ \omega_B : \omega_B \in (-\Delta, \pi + \Delta) \}, & \text{if } \eta \leq 0 \end{cases} \]

\[ \mathcal{I}_{-} = \{ \omega_A : \omega_A \in (\Delta - \pi / 2, 0] \} = \begin{cases} \{ \omega_B : \omega_B \in (-\pi / 2, -\Delta) \}, & \text{if } \eta > 0 \\ \{ \omega_B : \omega_B \in (0, -\Delta + \pi) \}, & \text{if } \eta \leq 0 \end{cases} \]

where we have assumed without any loss of generality that \( 0 \leq \Delta \leq \pi \).

In each one of these four segments, the two measurements are fully correlated or anti-correlated:

- If \( \eta \in (0, +\pi] \),

\[
\begin{align*}
S_{\Omega_\alpha}^{(A)}(\omega_A, \eta) \cdot S_{\Omega_\beta}^{(B)}(\omega_B, \eta) & \big|_{\mathcal{I}_{++} \cup \mathcal{I}_{--}} = +1, \\
S_{\Omega_\alpha}^{(A)}(\omega_A, \eta) \cdot S_{\Omega_\beta}^{(B)}(\omega_B, \eta) & \big|_{\mathcal{I}_{+-} \cup \mathcal{I}_{-+}} = -1,
\end{align*}
\]

- If \( \eta \in (-\pi, 0] \),

\[
\begin{align*}
S_{\Omega_\alpha}^{(A)}(\omega_A, \eta) \cdot S_{\Omega_\beta}^{(B)}(\omega_B, \eta) & \big|_{\mathcal{I}_{++} \cup \mathcal{I}_{--}} = -1, \\
S_{\Omega_\alpha}^{(A)}(\omega_A, \eta) \cdot S_{\Omega_\beta}^{(B)}(\omega_B, \eta) & \big|_{\mathcal{I}_{+-} \cup \mathcal{I}_{-+}} = +1,
\end{align*}
\]

It is straightforward to notice that

\[
\mu \left( \mathcal{I}_{++} \cup \mathcal{I}_{--} \right) - \mu \left( \mathcal{I}_{+-} \cup \mathcal{I}_{-+} \right) = \cos(\Delta),
\]

where \( \mu(\cdot) \) denotes the normalized measure over the torus according to the probability density distribution \( \mu \). Hence,

- Over the subpopulation of states with \( \eta > 0 \),

\[
\langle S_{\Omega_\alpha}^{(A)} \cdot S_{\Omega_\beta}^{(B)} \rangle = \cos(\Delta),
\]

- Over the subpopulation of states with \( \eta \leq 0 \),

\[
\langle S_{\Omega_\alpha}^{(A)} \cdot S_{\Omega_\beta}^{(B)} \rangle = -\cos(\Delta).
\]

Therefore, over the whole population the two measurements are completely uncorrelated,

\[
\langle S_{\Omega_\alpha}^{(A)} \cdot S_{\Omega_\beta}^{(B)} \rangle = 0,
\]

since each one of the two populations \( \eta > 0 \) and \( \eta \leq 0 \) happens with probability \( 1/2 \).

Furthermore, if we now define the outcome of the measurement apparatus of observer C as:

\[
S_{\Omega_{\gamma}}^{(C)}(\omega, \eta) = \begin{cases} +1, & \text{if } \eta > 0, \\
-1, & \text{if } \eta \leq 0,
\end{cases}
\]

we obtain that the three-particles correlation is given by:

\[
\langle S_{\Omega_\alpha}^{(A)} \cdot S_{\Omega_\beta}^{(B)} \cdot S_{\Omega_{\gamma}}^{(C)} \rangle = \cos(\Delta)
\]

which reproduces the quantum mechanical predictions for the GHZ state.

Indeed, by an appropriate choice of the two subpopulations over which the outcome of the measurement apparatus of observer C is defined either positive or negative we can arbitrarily modify the angle \( \Phi \) in \( \text{(18)} \) and, hence, the actual value for the three-particles correlation. For the sake of concreteness we shall show now how we can choose two subpopulations such that over one of them \( S_{\Omega_{\gamma}}^{(C)} = +1 \) and over the other \( S_{\Omega_{\gamma}}^{(C)} = -1 \) and

\[
\langle S_{\Omega_\alpha}^{(A)} \cdot S_{\Omega_\beta}^{(B)} \cdot S_{\Omega_{\gamma}}^{(C)} \rangle = +1.
\]

We shall identify the two new subpopulations by new coordinates \( \eta^* \leq 0 \) and \( \eta^* > 0 \), where

\[
\eta^* = \begin{cases} -\eta, & \text{if } (\omega, \eta) \in \mathcal{I}_{+-} \cup \mathcal{I}_{-+}, \\
\eta, & \text{if } (\omega, \eta) \in \mathcal{I}_{++} \cup \mathcal{I}_{--}.
\end{cases}
\]

Simultaneously, we rename the angular coordinates of the hidden states as follows:

\[
\omega^*_A = \begin{cases} -\omega_A, & \text{if } (\omega, \eta) \in \mathcal{I}_{+-} \cup \mathcal{I}_{-+}, \\
-\omega_A + \pi, & \text{if } (\omega, \eta) \in \mathcal{I}_{++} \cup \mathcal{I}_{--}.
\end{cases}
\]

\[
\omega^*_B = \begin{cases} \omega_B + \pi, & \text{if } (\omega, \eta) \in \mathcal{I}_{++} \cup \mathcal{I}_{--}, \\
\omega_B, & \text{if } (\omega, \eta) \in \mathcal{I}_{+-} \cup \mathcal{I}_{-+}.
\end{cases}
\]

such that the definition of the outcomes of measurement apparatus A and B remains invariant for all hidden configurations:

\[
S(\omega^*, \eta^*) = \text{sign}(\omega^*) \cdot \text{sign}(\eta^*) = \text{sign}(\omega) \cdot \text{sign}(\eta) = S(\omega, \eta).
\]
Furthermore, it is straightforward to check that neither the probability to happen of each one of these configurations gets modified, since the probability density remains invariant,

\[ g(\omega) = g(-\omega) = g(\omega + \pi), \]  

as well as the actual actual value of the defined outcomes of strong measurement performed by observers A and B.

It is now straightforward to check that the outcomes of the strong polarization measurements of particles A and B along directions \( \Omega_\alpha \) and \( \Omega_\beta \) are:

- fully correlated \( \langle S^{(A)}_{\Omega_\alpha} \cdot S^{(B)}_{\Omega_\beta} \rangle = +1 \), over the subpopulation of states with \( \eta^* > 0 \).
- fully anticorrelated \( \langle S^{(A)}_{\Omega_\alpha} \cdot S^{(B)}_{\Omega_\beta} \rangle = -1 \), over the subpopulation of states with \( \eta^* \leq 0 \).

Therefore, by identifying these two new subpopulations as corresponding to positive or negative outcomes for the measurement of the polarization of particle C along the direction \( \Omega_\gamma \):

\[
S^{(C)}_{\Omega_\gamma}(\omega^*, \eta^*) = +1, \quad \text{if} \quad \eta^* > 0, \tag{34}
\]
\[
S^{(C)}_{\Omega_\gamma}(\omega^*, \eta^*) = -1, \quad \text{if} \quad \eta^* \leq 0, \tag{35}
\]

we obtain that the three-particles correlation is now given by eq. \( (31) \) as we had advanced.

6. We have shown in this paper that the argument behind the renowned GHZ paradox crucially relies on an implicit assumption that is not required by fundamental physical principles. Thus the argument can be overcome if we give up this unnecessary assumption. Namely, the argument put forward by Greenberger, Horne and Zeilinger thirty years ago implicitly assumes that there exists an absolute angular frame of reference with respect to which we can define the polarization properties of the hypothetical hidden configurations of the entangled particles, as well as the orientation of the measurement apparatus that test them. We remarked in this paper that such an absolute frame of reference does not necessarily exist. Indeed, only the relative orientation of the set of measurement apparatus with respect to a reference setting is an actual physical degree of freedom, while the individual orientations of each one of the apparatus are spurious gauge degrees of freedom. Moreover, the polarization properties of each one of the particles can only be properly defined with respect to the orientation of the corresponding measurement apparatus. With these observations in mind we built an explicitly local and realistic model of hidden variables that complies with the ‘free-will’ assumption and reproduces the predictions of quantum mechanics for the GHZ state of three entangled particles. This model resembles closely the model of hidden variables for the Bell’s polarization states of two entangled particles that we recently described in [22].

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