Asymptotics of the eigenvalues of seven-diagonal Toeplitz matrices of a special form

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Abstract

We find uniform asymptotic formulas for all the eigenvalues of certain 7-diagonal symmetric Toeplitz matrices of large dimension. The entries of the matrices are real and we consider the case where the real-valued generating function such that its first five derivatives at the one endpoint of interval are equal to zero. This is not the simple-loop case considered earlier. We obtain nonlinear equations for the eigenvalues. It should be noted that our equations have a more complicated structure than the equations for the simple loop case.

Keywords Toeplitz matrices, eigenvectors, asymptotic expansions.

1 Introduction

Let $a(t)$ be a Lebesgue integrable function defined on the unit circle $\mathbb{T} = \{ t \in \mathbb{C} : |t| = 1 \}$. We denote by $T_n(a)$ the Toeplitz matrix $T_n(a) := (a_{j-k})_{j,k=1}^{n-1}$, where $n \in \mathbb{N}$ is a natural number, and $a_l$ denotes the $l$-th coefficient of the Fourier series of the function $a$. Note that the Toeplitz matrix can be viewed as an operator from a finite dimensional vector space. The function $a(t)$ is called the symbol of the Toeplitz matrix (Toeplitz operator) $T_n(a)$. This paper is devoted to finding asymptotic formulas for the eigenvalues of the Toeplitz matrix with the symbol $a(t) = (t - 2 + \frac{1}{t})^3$.

Toeplitz matrices, as well as closely related Toeplitz operators, have been intensively studied for various classes of symbols over the past, about a hundred years ([1], [2], [3], [4], [5]). The importance of this subject is largely due to the numerous applications of Toeplitz matrices in numerical methods of differential and integral equations, probability theory, statistical physics (see, for example, [6], [7], [8], [9]). As mentioned above, this work is devoted to finding asymptotic formulas for the eigenvalues of the Toeplitz matrix with the symbol $a(t) = (t - 2 + \frac{1}{t})^3$. Toeplitz matrices with this symbol are self-adjoint matrices. However, the study of non-self-adjoint Toeplitz
matrices can also be reduced to this case, the symbol of which is the cube of the linear Laurent polynomial and has a third-degree derivative at the end of the interval equal to zero. We note that all the asymptotic formulas for the eigenvalues obtained in this paper, in essence, admit an eigenvalue uniform with respect to the number, an estimate for the remainder term. It should be said that the symbol under consideration has specific properties: it is a real, symmetric function, and the first and second derivatives of the symbol vanish at the points \( t = \pm 1 \). The last condition, namely the vanishing of the second derivative, significantly complicates the problem of finding an asymptotic formula for the eigenvalues, since in this case the general research methods developed in the work [10] are inapplicable (see also works [11], [12], [13], [14], [15], which present general approaches to finding the asymptotics of the eigenvalues for various classes of Toeplitz matrices). In addition, the case we are considering is more complicated than that considered in the work [16].

The main idea of the study is that due to the results of Spitzer and Schmidt ([2]) the eigenvalues should be sought on the limiting spectrum, which is obtained from the condition of the coincidence of the moduli of the mean roots of the function \( b(t, \lambda) = a(t) - \lambda \). In our case, there are six such roots: \( \xi_i(\lambda), i = 1, \ldots, 6 \) and the limiting spectrum is determined by the condition \( |\xi_3(\lambda)| = |\xi_4(\lambda)| \). In the self-adjoint case, the limiting spectrum is an interval and the eigenvalues lie on this interval, which somewhat simplifies the problem. But in the case of the aforementioned symbol at the endpoints, the moduli of all the roots coincide, and this fact significantly complicates the problem of finding the asymptotic formula. Moreover, if you understand what the asymptotic formula looks like in the case of multiple roots of the function mentioned above \( b(t, \lambda) \), this will be an important step in finding such a formula for the eigenvalues of an arbitrary band Toeplitz matrix. Thus, the asymptotics of the eigenvalues of Toeplitz matrices with the symbol \( a(t) = (t - 2 + \frac{1}{2})^3 \) cannot be derived from the results known to us in this area, and the solution of such a problem is of interest in view of the emerging fundamental difficulties, the resolution of which will be an important step in the study of the general problem of finding the asymptotics of the spectrum of an arbitrary banded Toeplitz matrix.

2 Main results

In this section, we will present the main results of the article. We formulate a theorem describing an asymptotic formula for the eigenvalues of a Toeplitz matrix with the symbol mentioned above. The eigenvalues are calculated as the values of the function \( g(\varphi) = a(e^{i\varphi}) \), where \( a(t) = (t - 2 + \frac{1}{2})^3 \), for fixed values of the argument \( \varphi \). Furthermore, the function \( g(\varphi) = a(e^{i\varphi}) = -\left(2 \sin \frac{\varphi}{2}\right)^6 \) defined on \([0, 2\pi]\) has the following properties:

(i) The function \( g : [0, 2\pi] \to \mathbb{R} \), has range \([m, 0]\) with \( m < 0 \).

(ii) \( g(\pi) := m \), \( g^{(1)}(\pi) = 0 \), and \( g^{(2)}(\pi) > 0 \).

(iii) \( g(0) = g(2\pi) = 0 \), \( g^{(k)}(0) = g^{(k)}(2\pi) = 0 \) \((k = 1, \ldots, 5)\), and \( g^{(6)}(0) = g^{(6)}(2\pi) < 0 \).

Thus, the structure of the asymptotic formula for the eigenvalues is such that this formula is a refinement, on the one hand, of Szegő’s limit theorem, which describes the limit spectrum of Toeplitz matrices as the image of the unit circle \( \mathbb{T} \) under the action of the symbol, and on the other, as mentioned in the introduction, is a refinement of the results of Spitzer and Schmidt ([2]), which give the same answer in self-adjoint as Szegő’s limit theorem ([1], [5]).
Note that the problem is solved with respect to the variable $\phi$, from which the eigenvalues $\lambda$ are expressed by a simple substitution $\lambda = g(\phi)$. Let’s introduce some functions. All functions will be defined on the interval $\phi \in (0, \pi)$.

$$\beta(\phi) := \arccos \left(1 - (1 - \cos \phi) e^{\frac{2\pi i}{n}}\right)$$ (2.1)

Arcos is multivalued function, $\beta(\phi)$ is one of its regular branches. The existence of this branch when $\phi \in (0, \pi)$ will be shown in the section 4

\[ c(\phi) := \Re(\beta(\phi)), \quad b(\phi) := \Im(\beta(\phi)) \]
\[ B(\phi) := \Re(\sin(\beta) e^{-\frac{\pi i}{n}}), \quad C(\phi) := -\Im(\sin(\beta) e^{-\frac{\pi i}{n}}) \] (2.2)

**Theorem 2.1.** Let $\lambda = g(\phi)$. Then the equation $\det T_n(a - g(\phi)) = 0$ is equivalent to the following equations:

$$\tan \left(\frac{n+3}{2} \phi\right) = f(\phi),$$ (2.3)

and

$$\tan \left(\frac{n+3}{2} \phi\right) = \frac{1}{h(\phi)},$$ (2.4)

where

\[ f(\phi) = 2 B(\phi) \sin \left(\frac{(n+3)c(\phi)}{\sin(\phi)(\cos((n+3)c(\phi)) + \cosh((n+3)b(\phi)))}\right) \]
\[ h(\phi) = 2 B(\phi) \sin \left(\frac{(n+3)c(\phi)}{-\cos((n+3)c(\phi)) + \cosh((n+3)b(\phi)))}\right) \]

$\phi \in (0, \pi)$.

It is easy to see that equations 2.3 and 2.4 are equivalent the following set of equations correspondingly

$$\phi = \frac{2}{n+3} \left[\pi j + \arctan f(\phi)\right],$$ (2.5)

\[ j \in \left\{1, 2, \ldots, \left\lfloor\frac{n+1}{2}\right\rfloor\right\} \]

and

$$\phi = \frac{2}{n+3} \left[\pi j + \frac{\pi}{2} - \arctan h(\phi)\right],$$ (2.6)

\[ j \in \left\{1, 2, \ldots, \left\lfloor\frac{n}{2}\right\rfloor\right\} \]

Apply to a solution of this equations Fix Point Method. Put

$$\phi^{(0)}_{2j-1} = d_{2j-1}, \quad \phi^{(k+1)}_{2j-1} = \frac{2}{n+3} \left[\pi j + \arctan (f(\phi^{(k)}_{2j-1}))\right]$$

and

$$\phi^{(0)}_{2j} = d_{2j}, \quad \phi^{(k+1)}_{2j} = \frac{2}{n+3} \left[\pi j + \frac{\pi}{2} - \arctan (h(\phi^{(k)}_{2j}))\right]$$

where

$$d_m = \frac{\pi(m+1)}{n+3}, \quad m = 1, 2, \ldots, n$$
Theorem 2.2. If \( n \) is sufficiently large then

1) The equation (2.3) has exactly one root \( \varphi_{2j-1} \) on each of the intervals \( \left( \frac{\pi(2j-1)}{n+3}, \frac{\pi(2j+1)}{n+3} \right) \), where \( j \in \{1, \ldots, \left[ \frac{n+1}{2} \right] \} \). Moreover, we can write the following estimate:

\[
\left| \varphi_{2j-1}^{(k+1)} - \varphi_{2j-1}^{(k)} \right| \leq \frac{L}{(n+3)} (0.8)^k ,
\]

where \( k \) - iteration number, \( L \) does not depend on \( j \) and \( n \).

2) The equation (2.4) has exactly one root \( \varphi_{2j} \) on each of the intervals \( \left( \frac{2\pi j}{n+3}, \frac{2\pi (j+1)}{n+3} \right) \), where \( j \in \{1, \ldots, \left[ \frac{n}{2} \right] \} \). Moreover, we can write the following estimate:

\[
\left| \varphi_{2j}^{(k+1)} - \varphi_{2j}^{(k)} \right| \leq \frac{L}{(n+3)} (0.8)^k ,
\]

where \( k \) - iteration number, does not depend on \( j \) and \( n \).

Lemma 2.1. For a sufficiently large \( n \), the roots of the equations (2.3) and (2.4) are pairwise distinct.

This theorem shows that the equations (2.3) and (2.4) have at least \( \frac{n+1}{2} + \left[ \frac{n}{2} \right] = n \) roots, however, since the set of these roots coincides with the set of eigenvalues of the \( n \times n \) matrix, so there cannot be more roots. The problem was reduced to solving the equations

\[
\varphi = \frac{2}{n+3} \left[ \pi j + \arctan f(\varphi) \right]
\]

on intervals \( \left( \frac{\pi(2j-1)}{n+3}, \frac{\pi(2j+1)}{n+3} \right) \), where \( j \in \{1, \ldots, \left[ \frac{n+1}{2} \right] \} \), and

\[
\varphi = \frac{2}{n+3} \left[ \pi j + \frac{\pi}{2} - \arctan h(\varphi) \right]
\]

on intervals \( \left( \frac{2\pi j}{n+3}, \frac{2\pi (j+1)}{n+3} \right) \), where \( j \in \{1, \ldots, \left[ \frac{n}{2} \right] \} \)

Let \( q := \frac{n+3}{2} \). To solve the equation (2.9) we introduce the parameter \( d_{1,j} := d_{2j-1} = \frac{2\pi j}{n+3} \). Then \( \varphi \) can be represented as \( \varphi = d_{1,j} + \frac{u}{q} \), and equation (2.9) can be rewritten as:

\[
u = \arctan f \left( d_{1,j} + \frac{u}{q} \right)
\]

where \( u \in (-\frac{\pi}{2}, \frac{\pi}{2}) \). Similarly, to solve the equation 2.10, we introduce the parameter \( d_{2,j} := d_{2j} = \frac{2\pi j+1}{n+3} \), so if \( \varphi = d_{2,j} + \frac{w}{q} \), then equation (2.10) can be rewritten as:

\[
w = -\arctan h \left( d_{2,j} + \frac{w}{q} \right)
\]

where \( w \in (-\frac{\pi}{2}, \frac{\pi}{2}) \)

Theorem 2.3. Let \( a(t) = (t - 2 + \frac{1}{2})^3 \). Then, as \( n \to \infty \)
1. If $d_{1,j} : \frac{1}{2} e^{\pi(j-1)} > q^2$ then:

$$\varphi_{2j-1} = d_{1,j} + 2\frac{u_1^*}{n+3} + \frac{4u_2^*}{(n+3)^2} + O\left(\frac{1}{n^3}\right),$$

(2.13)

where $u_1^* = \arctan\left(2\frac{C(d_{1,j})}{\sin(d_{1,j})}\right)$ and $u_2^* = 2\frac{C'(d_{1,j})\sin(d_{1,j}) - C(d_{1,j})\cos(d_{1,j})}{\sin^2(d_{1,j}) + 4C^2(d_{1,j})}\arctan\left(2\frac{C(d_{1,j})}{\sin(d_{1,j})}\right)$

2. If $d_{2,j} : \frac{1}{2} e^{\pi(j-1)} > q^2$ then:

$$\varphi_{2j} = d_{2,j} + 2\frac{w_1}{n+3} + \frac{4w_2^*}{(n+3)^2} + O\left(\frac{1}{n^3}\right),$$

(2.14)

where $w_1^* = \arctan\left(2\frac{C(d_{2,j})}{\sin(d_{2,j})}\right)$ and $w_2^* = 2\frac{C'(d_{2,j})\sin(d_{2,j}) - C(d_{2,j})\cos(d_{2,j})}{\sin^2(d_{2,j}) + 4C^2(d_{2,j})}\arctan\left(2\frac{C(d_{2,j})}{\sin(d_{2,j})}\right)$

For brevity, we define:

$$Z_1^{(1)} = 2\frac{B_1^{(1)} \sin (P_a^{(1)} + C_1^{(1)} \sinh (P_b^{(1)}))}{(\cos (P_a^{(1)}) + \cosh (P_b^{(1}))}$$

(2.15)

and

$$Z_1^{(2)} = 2\frac{B_1^{(2)} \sin (P_a^{(2)} - C_1^{(2)} \sinh (P_b^{(2)}))}{(-\cos (P_a^{(2)}) + \cosh (P_b^{(2}))},$$

(2.16)

where $P_a^{(1)} = qd_{1,j} + u_1$, $P_b^{(1)} = \sqrt{3}(qd_{1,j} + u_1)$, $B_1^{(1)} = 1 + \frac{3}{16}d_{1,j}^2$, $C_1^{(1)} = \frac{\sqrt{3}}{16}d_{1,j}^2$

$P_a^{(2)} = qd_{2,j} + w_1$, $P_b^{(2)} = \sqrt{3}(qd_{2,j} + w_1)$, $B_1^{(2)} = 1 + \frac{3}{16}d_{2,j}^2$, $C_1^{(2)} = \frac{\sqrt{3}}{16}d_{2,j}^2$

Theorem 2.4. Let $a(t) = (t - 2 + \frac{1}{t})^3$. Then, as $n \to \infty$

1. If $d_{1,j} : \frac{1}{2} e^{\pi(j-1)} \leq q^2$ then:

$$\varphi_{2j-1} = d_{1,j} + 2\frac{u_1^*}{n+3} + \frac{4u_2^*}{(n+3)^2} + O\left(\frac{1}{n^3}\right),$$

(2.17)

where $u_1^*$ is the solution of equation $u_1 = \arctan (Z_1^{(1)})$ and $u_2^* = R^{(1)}(u_1^*)$ (see proof of the theorem)

2. If $d_{2,j} : \frac{1}{2} e^{\pi(j-1)} \leq q^2$ then:

$$\varphi_{2j} = d_{2,j} + 2\frac{w_1}{n+3} + \frac{4w_2^*}{(n+3)^2} + O\left(\frac{1}{n^3}\right),$$

(2.18)

where $w_1^*$ is the solution of equation $w_1 = -\arctan (Z_1^{(2)})$ and $w_2^* = R^{(2)}(w_1^*)$ (see proof of the theorem)

The following result gives us the asymptotic formulas for eigenvalues $\lambda$.

Theorem 2.5. Let $a(t) = (t - 2 + \frac{1}{t})^3$. Then, as $n \to \infty$

1. $\lambda_{2j-1}^{(n)} = g(d_{1,j}) + g'(d_{1,j})\frac{2u_1^*}{n+3} + \frac{4u_2^* g'(d_{1,j}) + 2(u_1^*)^2 g''(d_{1,j})}{(n+3)^2} + o\left(\frac{1}{n^2}\right)$,

where $u_1^*$ and $u_2^*$ is defined in the same way as in the theorem 2.3 if $d_{1,j} : \frac{1}{2} e^{\pi(j-1)} > q^2$, and in the same way as in the theorem 2.4 if $d_{1,j} : \frac{1}{2} e^{\pi(j-1)} \leq q^2$. 

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2.

\[ \lambda_{2j}^{(n)} = g(d_{2j}) + g'(d_{2j}) \frac{2w_1^*}{n + 3} + \frac{4w_2^* g'(d_{2j}) + 2(w_2^*)^2 g''(d_{2j})}{(n + 3)^2} + o \left( \frac{1}{n^2} \right), \]

where \( w_1^* \) and \( w_2^* \) is defined in the same way as in the theorem 2.3 if \( d_{2j} \) : \( \frac{1}{2} e^{\pi(j-1)} > q^2 \), and in the same way as in the theorem 2.4 if \( d_{2j} \) : \( \frac{1}{2} e^{\pi(j-1)} \leq q^2 \).

The following result gives us the asymptotic formulas for the extreme eigenvalues near zero.

**Theorem 2.6.** Let \( g(\varphi) = a(e^{i\varphi}) = -(2\sin \frac{\varphi}{2})^6 \).

i) If \( d_{1,j} \to 0 \) as \( n \to \infty \), then

\[ \lambda_{2j-1}^{(n)} = -\frac{(2\pi j + 2u_1^*)^6}{(n + 3)^6} - \frac{24u_2^*(2\pi j + 2v_1^*)^5}{(n + 3)^7} + \Delta_1(n, j), \]

where \( |\Delta_1(n, j)| \leq M_1 \left( \frac{d_{1,j}}{n^4} + d_{1,j}^{10} \right) \) and the constant \( M_1 \) does not depend in \( j \) and \( n \).

ii) If \( d_{2,j} \to 0 \) as \( n \to \infty \), then

\[ \lambda_{2j}^{(n)} = -\frac{(2j + 1)^6}{(n + 3)^6} - \frac{24w_2^*(2j + 1)^5}{(n + 3)^7} + \Delta_2(n, j), \]

where \( |\Delta_2(n, j)| \leq M_2 \left( \frac{d_{2,j}}{n^4} + d_{2,j}^{10} \right) \) and the constant \( M_2 \) does not depend in \( j \) and \( n \).

### 3 Auxiliary results

In this section, some auxiliary statements will be proved. Now we will introduce auxiliary functions.

\[ \alpha_2 = \alpha_2(\varphi) = 1 + (\cos \varphi - 1) e^{2\pi i \frac{\varphi}{3}} \]

\[ B_c = B_c(\varphi) = |\alpha_2| \]

\[ \psi_c = \psi_c(\varphi) = \arg(\alpha_2) \]

\[ B_s = B_s(\varphi) = |(1 - \alpha_2^*)^\frac{1}{2}| = \sqrt{(1 - \cos \varphi)^2 (7 - 4 \cos \varphi + \cos^2 \varphi)} \]

\[ \psi_s = \psi_s(\varphi) = \arg \left( (1 - \alpha_2^* )^\frac{1}{2} \right) = \frac{\pi}{2} + \frac{\arctan \sqrt{3(3 - \cos \varphi)}}{2} \]

Obviously, \( \arg \left( (1 - \alpha_2^* )^\frac{1}{2} \right) \) has two regular branches, we select one of them. All subsequent statements will be proved when \( \varphi \in (0, \pi) \).

1. (a) \( B_s(\varphi) \) is increasing function, and \( B_s(\varphi) \in (0, 2\sqrt{3}) \)

(b) \( \psi_s(\varphi) \) is decreasing function, and \( \psi_s(\varphi) \in \left( \frac{\pi}{3}, \frac{\pi}{3} \right) \)

(c) \( B_c(\varphi) \) is increasing function, and \( B_c(\varphi) \in (1, \sqrt{7}) \)

(d) \( \psi_c(\varphi) \) is decreasing function, and \( \psi_c(\varphi) \in (-\arctan \frac{\sqrt{3}}{2}, 0) \)
Proof. We differentiate the corresponding functions, and decompose them into multipliers. The values at the edges of the interval are found by simple substitution:

\[ B_s' = \frac{\sin(\varphi)(9 - 7 \cos(\varphi) + 2\cos^2(\varphi))}{2(1 - \cos(\varphi))(7 - 4 \cos(\varphi) + \cos^2(\varphi))^{\frac{3}{4}}} > 0. \]  

(3.5)

So, the function \( B_s(\varphi) \) is increasing.

\[ \psi_s' = -\frac{\sqrt{3}\sin(\varphi)}{2(7 - 4 \cos(\varphi) + \cos^2(\varphi))} < 0. \]  

(3.6)

So, the function \( \psi_s(\varphi) \) is decreasing.

\[ B_c' = \frac{\sin(\varphi)(3 - 2 \cos(\varphi))}{2\sqrt{3 - 3 \cos(\varphi) + 3\cos^2(\varphi)}} > 0. \]  

(3.7)

So, the function \( B_c(\varphi) \) is increasing.

\[ \psi_c' = -\frac{\sqrt{3}\sin(\varphi)}{2(3 - 3 \cos(\varphi) + \cos^2(\varphi))} < 0. \]  

(3.8)

So, the function \( \psi_c(\varphi) \) is decreasing.

2. (a) \( B_c \cos(\psi_c) - B_s \sin(\psi_s) > 0 \)

(b) \( B_c \cos(\psi_c) + B_s \sin(\psi_s) > 0 \)

Proof. We show that \( B_c > B_s \) and \( \cos(\psi_c) > \sin(\psi_s) \), from which the statement of this item will follow. Since \( B_c > 0 \) and \( B_s > 0 \), therefore \( B_c > B_s \) is equivalent to \( B_c^4 - B_s^4 > 0 \).

\[ B_c^4 - B_s^4 = 2 - \cos^2 \varphi > 0 \]

Now we show that \( \cos(\psi_c) > \sin(\psi_s) \). \( \cos(\psi_c) = \sin(\frac{\pi}{2} + \psi_c) \), at the same time, from the points (1b and 1d), it follows that \( (\frac{\pi}{2} + \psi_c) \) and \( \psi_s \), are in the first quadrant, so \( (\frac{\pi}{2} + \psi_c) > \sin(\psi_s) \) is equivalent to \( \frac{\pi}{2} + \psi_c - \psi_s > 0 \). It’s not hard to get that

\[ \psi_c' - \psi_s' = -\frac{\sqrt{3}\sin(\varphi)(4 - \cos(\varphi))}{2(7 - 4 \cos(\varphi) + \cos^2(\varphi))(3 - 3 \cos(\varphi) + \cos^2(\varphi))} < 0. \]

Which means \( \frac{\pi}{2} + \psi_c - \psi_s > \frac{\pi}{2} + \psi_c(\pi) - \psi_s(\pi) = \frac{\pi}{2} - \arctan \frac{\sqrt{3}}{2} > 0 \), so \( \cos(\psi_c) > \sin(\psi_s) \). This means that the statement (2a) is true. From the statements (1a - 1d), it follows that both terms in the expression \( B_c \cos(\psi_c) + B_s \sin(\psi_s) \) is positive, which means that the statement (2b) is also true.

3. (a) \( c(\varphi) \) is increasing function.

(b) \( b(\varphi) \) is increasing function.

(c) \( c'(\varphi) \) is decreasing function, and \( c'(0) = \frac{1}{2}, c'(\pi) = 0 \).

(d) \( b'(\varphi) \) is decreasing function, and \( b'(0) = \frac{\sqrt{3}}{2}, b'(\pi) = 0 \).
Proof.

\[(\cos \beta)' = -\beta' \sin \beta,\]

so

\[\beta' = -\frac{(\cos \beta)'}{\sin \beta},\]

from where it is not difficult to get that

\[\beta' = \frac{\sin \varphi}{\sin \beta} e^{\frac{2\pi i}{3}} \tag{3.9}\]

Then

\[c'(\varphi) = \Re(\beta') = \frac{\sin \varphi}{B_s} \cos \left( \frac{2\pi}{3} - \psi_s \right),\]

\[b'(\varphi) = \Re(\beta') = \frac{\sin \varphi}{B_s} \sin \left( \frac{2\pi}{3} - \psi_s \right).\]

Since \(\psi_s(\varphi) \in (\frac{\pi}{4}, \frac{\pi}{3})\), then \(c'(\varphi) > 0, b'(\varphi) > 0\) so the functions \(a(\varphi)\) and \(b(\varphi)\) are increasing. Let’s find \(\beta''\).

\[\beta'' = \left( \frac{\sin \varphi}{\sin \beta} e^{\frac{2\pi i}{3}} \right)' = \frac{\cos (\varphi) \sin (\beta) - \sin (\varphi) \cos (\beta) \beta' e^{\frac{2\pi i}{3}}}{\sin^3(\beta)}\]

Substituting the equality (3.9) and reducing to a common denominator, we get the following equality:

\[\beta'' = \frac{\cos (\varphi) \sin^2 (\beta) e^{\frac{2\pi i}{3}} - \sin^2 (\varphi) \cos (\beta) e^{-\frac{2\pi i}{3}}}{\sin^3 (\beta)}.\]

Given that \(\sin^2 t = 1 - \cos^2 t\) we get the equality:

\[\beta'' = \frac{\cos (\varphi)(1 - \cos^2 (\beta))e^{\frac{2\pi i}{3}} - (1 - \cos^2 (\varphi)) \cos (\beta) e^{-\frac{2\pi i}{3}}}{\sin^3 (\beta)}\]

Substituting the equality (4.2) we get

\[\beta'' = \frac{\cos (\varphi)(1 - (1 - \cos (\varphi)) e^{\frac{2\pi i}{3}})^2 e^{\frac{2\pi i}{3}} - (1 - \cos^2 (\varphi))(1 - \cos (\varphi)) e^{\frac{2\pi i}{3}} e^{-\frac{2\pi i}{3}}}{\sin^3 (\beta)}\]

Then expanding brackets and combining like terms we get that:

\[\beta'' = \frac{\sqrt{3}(1 - \cos (\varphi))^2}{\sin^3 (\beta)} e^{\frac{2\pi i}{3}} \tag{3.10}\]

Since \(c'' = \Re(\beta''), b'' = \Im(\beta'')\) and also that \(\psi_s \in (\frac{\pi}{4}, \frac{\pi}{3})\) we get

\[c'' = \frac{\sqrt{3}(1 - \cos (\varphi))^2}{B_s^3} \cos \left( \frac{\pi}{6} - 3\psi_s \right) < 0,\]

\[b'' = \frac{\sqrt{3}(1 - \cos (\varphi))^2}{B_s^3} \sin \left( \frac{\pi}{6} - 3\psi_s \right) < 0,\]
so the functions \( c'(\varphi) \) and \( b'(\varphi) \) are decreasing. Let’s find \( c'(0) = \Re(\beta'(0)) \).

\[
\beta'(0) = \lim_{\varphi \to 0} \frac{\sin(\varphi)}{B_s} e^{\left(\frac{2\pi i}{3}-\psi_s\right)}
\]

Expand \( \sin(\varphi) \) and \( B_s \) in a Taylor series up to the first term is not difficult to get that

\[
\beta'(0) = e^{\frac{2\pi i}{3}}.
\]

It follows that \( c'(0) = \frac{1}{2} \) and \( b'(0) = \sqrt{3} \). Well \( c'(\pi) \) and \( b'(\pi) \) can be found by a simple substitution.

4. (a) \( \frac{c(\varphi)}{\varphi} \) is decreasing function.
(b) \( \frac{b(\varphi)}{\varphi} \) is decreasing function.
(c) \( \frac{B_c}{B_s} \) is decreasing function.
(d) \( \frac{B_s}{\sin(\varphi)} \) is increasing function.

**Proof.** Find the derivative of the function \( \frac{c(\varphi)}{\varphi} \) and show that it is negative.

\[
\left( \frac{c(\varphi)}{\varphi} \right)' = \frac{c'(\varphi)\varphi - c(\varphi)}{\varphi^2}.
\]

In order to prove that this derivative is negative, it is sufficient to show that \( c'(\varphi)\varphi - c(\varphi) < 0 \).

\[
(c'(\varphi)\varphi - c(\varphi))' = c''(\varphi) < 0
\]

From which it follows that \( c'(\varphi)\varphi - c(\varphi) < c(0) = 0 \), this means that the statement (4a) is true. The statement (4b) is proved similarly.

Since \( \frac{B_c}{B_s} > 0 \), decreasing \( \frac{B_c}{B_s} \) is equivalent to decreasing \( \frac{B_c^4}{B_s} \). By taking the derivative of the function \( \frac{B_c^4}{B_s} \) and expand into factors we get that:

\[
\left( \frac{B_c^4}{B_s} \right)' = -\frac{2\sin(\varphi)(3 - 3\cos(\varphi) + \cos^2(\varphi))(2 - \cos(\varphi))(3 + \cos(\varphi))}{(1 - \cos(\varphi))^3(7 - 4\cos(\varphi) + \cos^2(\varphi))^2} < 0.
\]

This means that the function \( \frac{B_c}{B_s} \) decreases.

Since \( \frac{B_s}{\sin(\varphi)} > 0 \) increasing \( \frac{B_s}{\sin(\varphi)} \) is equivalent to decreasing \( \frac{B_s^4}{\sin^4(\varphi)} \). By taking the derivative of the function \( \frac{B_s^4}{\sin^4(\varphi)} \) and expand into factors we get that:

\[
\left( \frac{B_s^4}{\sin^4(\varphi)} \right)' = \frac{6(3 - \cos(\varphi))(1 - \cos(\varphi))^3}{\sin^5(\varphi)} > 0.
\]

This means that the function \( \frac{B_s}{\sin(\varphi)} \) increases.

5. (a) If \( \varphi \) is sufficiently small then \( c = \frac{1}{2}\varphi - \frac{3}{48}\varphi^3 + o(\varphi^3) \).
(b) If \( \varphi \) is sufficiently small then \( b = \frac{\sqrt{3}}{2}\varphi - \frac{\sqrt{3}}{48}\varphi^3 + o(\varphi^3) \).
(c) If $\varphi$ is sufficiently small then $B = \varphi - \frac{1}{48} \varphi^3 + o(\varphi^3)$.

(d) If $\varphi$ is sufficiently small then $C = \frac{\sqrt{3}}{16} \varphi^3 + o(\varphi^3)$.

**Proof.** Let’s find $\sin \beta$ near the point $\varphi = 0$, with an accuracy of $o(\varphi)$. From the formulas (3.3) and (3.4) it follows:

$$B_s = \sqrt[4]{(1 - 1 + \varphi^2 \varphi^2 + o(\varphi^2))^2 (7 - 4 + 1 + o(\varphi))} = \varphi + o(\varphi)$$

$$\psi_s = \frac{\pi - \arctan \frac{\sqrt{3}(1 + \varphi^2)}{2 + o(\varphi)}}{2} = \frac{\pi - \arctan \sqrt{3} + o(\varphi)}{2} = \frac{\pi}{3} + o(\varphi)$$

From which it follows that near the point $\varphi = 0$

$$\sin \beta = \varphi e^{\frac{\pi i}{3}} + o(\varphi) \quad (3.11)$$

Then from the formulas (3.9) and (3.10) it is not difficult to get:

$$\beta'(0) = e^{\frac{\pi i}{3}} \quad (3.12)$$

$$\beta''(0) = 0 \quad (3.13)$$

Now we’ll find $\beta'''(0)$

$$\beta'''(0) = \lim_{\varphi \to 0} \frac{\sqrt{3}(2(1 - \cos(\varphi)) \sin(\varphi) \sin^3(\beta(\varphi)) - 3(1 - \cos(\varphi))^2 \sin^2(\beta(\varphi)) \cos(\beta(\varphi)) \beta'(\varphi))}{\sin^6(\beta(\varphi))} e^{\frac{\pi i}{3}}$$

$$= \lim_{\varphi \to 0} \frac{\sqrt{3}(\varphi^6 e^{\pi i} - \frac{3}{4} \varphi^6 e^{\pi i})}{\varphi^6} e^{\frac{\pi i}{3}} = \frac{\sqrt{3}}{4} e^{\frac{7 \pi i}{6}}$$

We get that:

$$\beta'''(0) = \frac{\sqrt{3}}{4} e^{\frac{7 \pi i}{6}} \quad (3.14)$$

Since $\beta(0) = 0$, and taking into account (3.12), (3.13), and (3.14) near the point $\varphi = 0$, we have:

$$\beta(\varphi) = \varphi \left( \frac{1}{2} + \frac{\sqrt{3} i}{2} \right) - \varphi^3 \left( \frac{3}{48} + \frac{\sqrt{3} i}{48} \right) + o(\varphi^3) \quad (3.15)$$

Hence the statements (5a) and (5b) is true.

Let’s introduce the function $D(\varphi) = \sin(\beta)e^{\frac{\pi i}{3}}$ then $B = \Re(D)$, $C = -\Im(D)$. Let’s find the value of the first three derivatives of the function $D(\varphi)$ at the point $\varphi = 0$.

$$D'(\varphi) = \cos(\beta)\beta' e^{\frac{\pi i}{3}}$$

Taking into account the formula (3.12)

$$D'(0) = 1 \quad (3.16)$$

$$D''(\varphi) = (\cos(\beta)\beta'' - \sin(\beta)\beta'') e^{\frac{\pi i}{3}}$$
Given that \( \beta(0) = 0 \), and the formulas (3.13), we get
\[
D''(0) = 0 \quad (3.17)
\]
\[
D'''(\phi) = (\cos(\beta)\beta''' - 3\sin(\beta)\beta'' - \cos(\beta)(\beta')^3)e^{i\pi/3}
\]

Using the formulas (3.12), (3.13), and (3.14), it is not difficult to get that:
\[
D'''(0) = \frac{1}{8} - \frac{3\sqrt{3}i}{8} \quad (3.18)
\]

Then, taking into account (3.16), (3.17) and (3.18) near the point \( \phi = 0 \), we have:
\[
D(\phi) = \phi + \phi^3 \left( \frac{1}{48} - \frac{\sqrt{3}i}{16} \right) + o(\phi^3) \quad (3.19)
\]

Hence the statements (5c) and (5d) is true.

\[\square\]

4 Chebyshev polynomial

To solve this problem, we need to solve the equation \( \det T_n(a - g(\phi)) = 0, \phi \in (0, \pi) \). To find the determinant we will use the results obtained in the paper [17]. Let's define Chebyshev polynomials \( \{Q_n\}, \{U_n\}, \{V_n\}, \{W_n\} \), which satisfy the same recurrent formula
\[
Q_{n+1}(x) = 2xQ_n(x) - Q_{n-1}(x), \quad n = 1, 2, \ldots
\]
and the different initial conditions are:

\[
Q_0(x) = U_0(x) = 1, \quad 2Q_1(x) = U_1(x) = 2x \\
W_0(x) = V_0(x) = 1, \quad W_1(x) = V_1(x) + 2 = 2x + 1
\]

It is easy to check that these polynomials satisfy the following conditions
\[
Q_n(\cos \theta) = \cos n\theta, \quad U_n(\cos \theta) = \frac{\sin (n + 1)\theta}{\sin \theta} \\
V_n(\cos \theta) = \frac{\cos (n + \frac{1}{2})\theta}{\cos \frac{\theta}{2}}, \quad U_n(\cos \theta) = \frac{\sin (n + \frac{1}{2})\theta}{\sin \frac{1}{2}\theta} \quad (4.1)
\]

In [17], for the generating polynomial \( a(t) = \sum_{k=-r}^{r} a_k t^k \), where \( a_r \neq 0, a_k = a_{-k} \), the following theorem was proved.

**Theorem 4.1** ([17] Theorem 1). Let \( \xi_j \) and \( \frac{1}{\xi_j} \) be the (distinct) zeros of the polynomial \( g_1(t) = t^r a(t) \). Then, for all \( p \geq 1 \) \( \det T_{2p} \) equals

\[
\frac{a_r^{2p}}{2^{2r-1}} \times \begin{vmatrix}
V_p(\alpha_1) & \cdots & V_p(\alpha_r) \\
V_p+1(\alpha_1) & \cdots & V_p+1(\alpha_r) \\
\vdots & \ddots & \vdots \\
V_{p+r-1}(\alpha_1) & \cdots & V_{p+r-1}(\alpha_r)
\end{vmatrix} \times \begin{vmatrix}
W_p(\alpha_1) & \cdots & W_p(\alpha_r) \\
W_p+1(\alpha_1) & \cdots & W_p+1(\alpha_r) \\
\vdots & \ddots & \vdots \\
W_{p+r-1}(\alpha_1) & \cdots & W_{p+r-1}(\alpha_r)
\end{vmatrix}
\]

\[
\prod_{1 \leq j \leq r} (\alpha_j - \alpha_i) \prod_{1 \leq j \leq r} (\alpha_j + \alpha_i)
\]
and \( \det T_{2p+1} \) equals

\[
\frac{(-1)^r a_r^{2p}}{2^{r(r-2)}} \times \begin{vmatrix}
U_p(\alpha_1) & \ldots & U_p(\alpha_r) \\
\vdots & \ddots & \vdots \\
U_{p+r-1}(\alpha_1) & \ldots & U_{p+r-1}(\alpha_r)
\end{vmatrix} \times \begin{vmatrix}
Q_{p+1}(\alpha_1) & \ldots & Q_{p+1}(\alpha_r) \\
\vdots & \ddots & \vdots \\
Q_{p+r-1}(\alpha_1) & \ldots & Q_{p+r-1}(\alpha_r)
\end{vmatrix}
\prod_{1\leq i\leq j\leq r} (\alpha_j - \alpha_i),
\]

where \( \alpha_k = \frac{1}{2}(\zeta_k + \frac{1}{\zeta_k}) \) (\( k = 1, \ldots, r \)) are the zeros of the polynomial \( h_1(x) = a_0 + 2 \sum_{k=1}^{r} a_k Q_k(x) \).

In our case \( g_1(t) = (t^2 - 2t + 1)^3 - \lambda t^3 \), taking into account that \( \lambda = g(\varphi) = (2 \cos \varphi - 2)^3 \) it is easy to get that:

\[
\begin{align*}
\alpha_1 &= \cos \varphi \\
\alpha_2 &= 1 + (\cos \varphi - 1)e^{\frac{2\pi i}{3}} \\
\alpha_3 &= 1 + (\cos \varphi - 1)e^{-\frac{2\pi i}{3}}
\end{align*}
\]

Next, we show that there are such regular functions \( \beta = \beta(\varphi) \) and \( \gamma = \gamma(\varphi) \), \( \varphi \in (-\pi, \pi] \) which will satisfy the equations:

\[
\begin{align*}
\cos \beta &= 1 + (\cos \varphi - 1)e^{\frac{2\pi i}{3}} = \alpha_2 \\
\cos \gamma &= 1 + (\cos \varphi - 1)e^{-\frac{2\pi i}{3}} = \alpha_3
\end{align*}
\]

(4.2)

To do this, it is enough to show that each of the multifunctions \( E_2 = -i \Log(\alpha_2 + i(1 - \alpha_2^3)\frac{1}{2}) \) and \( E_3 = -i \Log(\alpha_3 + i(1 - \alpha_3^3)\frac{1}{2}) \) has at least one regular branch for \( \varphi \in (-\pi, \pi] \). Given the notation (3.1)-(3.4), it is not difficult to make sure that

\[
\alpha_3 = B_c e^{-i\psi_c} = \overline{\alpha_2}
\]

And also

\[
(1 - \alpha_3^{\frac{1}{2}})^\frac{1}{2} = B_s e^{-i\psi_s}
\]

Note that in this case, we choose one of the two regular branches. With this in mind, it is sufficient to show that each of the functions \( \tilde{E}_2 = -i \Log(B_c e^{i\psi_c} + iB_s e^{i\psi_s}) \) and \( \tilde{E}_3 = -i \Log(B_c e^{-i\psi_c} + iB_s e^{-i\psi_s}) \), has at least one regular branch for \( \varphi \in (-\pi, \pi] \). Note also that \( B_c, \psi_c, B_s, \) and \( \psi_s \) are even functions.

**Lemma 4.1.** Let \( \varphi \in (-\pi, \pi] \) then the multifunctions

\[
\tilde{E}_2 = -i \Log(B_c e^{i\psi_c} + iB_s e^{i\psi_s})
\]

and

\[
\tilde{E}_3 = -i \Log(B_c e^{-i\psi_c} + iB_s e^{-i\psi_s})
\]

have regular branches \( \beta(\varphi) \) and \( \gamma(\varphi) \) respectively.

**Proof.** In order for the multifunction \( \tilde{E}_2 \) to have a regular branch, it is sufficient that the curve \( B_c e^{i\psi_c} + iB_s e^{i\psi_s} \) (which is smooth contour) lies inside a simply connected region that does not contain the point zero. In the item 2a of the section 3, it was shown that \( \Re(B_c e^{i\psi_c} + iB_s e^{i\psi_s}) > 0 \), with \( \varphi \in (0, \pi) \), since \( B_c, \psi_c, B_s, \) and \( \psi_s \) are even functions, and \( B_c(0) \cos(\psi_c(0)) = 1, B_s e^{i\psi_s} = 0 \) then \( \Re(\cos \beta + i\sin \beta) > 0 \), when \( \varphi \in (-\pi, \pi] \). This means that the smooth contour \( B_c e^{i\psi_c} +
iB_s e^{i\psi_s} lies in a simply connected region that does not contain the point zero, and therefore the multifunction \( \tilde{E}_2 \) have a regular branch, for \( \varphi \in (-\pi, \pi] \). The function \( \log \) has an infinite number of regular branches, but to choose one of them, it is enough to determine its value at one point, let’s put \( \beta(0) = 0 \). It is not difficult to make sure that this value meets the conditions (4.2). Similarly, the multifunction \( \tilde{E}_3 \) have a regular branch, when \( \varphi \in (-\pi, \pi] \) with the value \( \gamma(0) = 0. \)

So we proved that there are regular functions \( \beta(\alpha) \) and \( \gamma(\varphi) \) which satisfy the equalities (4.2), and

\[
\begin{align*}
\cos(\beta) &= B_c e^{i\psi_c} \\
\sin(\beta) &= B_s e^{i\psi_s} \\
\cos(\gamma) &= B_c e^{-i\psi_c} \\
\sin(\gamma) &= B_s e^{-i\psi_s}
\end{align*}
\]

5 Proof of main the results

Proof of theorem 2.1. If \( n = 2p \) then by theorem 4.1 we have

\[
det(T_{2p}(a - g(\varphi))) = \frac{1}{2^6} \times \frac{\begin{vmatrix}
V_p(\cos \varphi) & V_p(\cos \beta) & V_p(\cos \gamma) \\
V_{p+1}(\cos \varphi) & V_{p+1}(\cos \beta) & V_{p+1}(\cos \gamma) \\
V_{p+2}(\cos \varphi) & V_{p+2}(\cos \beta) & V_{p+2}(\cos \gamma)
\end{vmatrix}}{(\cos \gamma - \cos \beta)(\cos \gamma - \cos \varphi)(\cos \beta - \cos \varphi)}
\]

\[
\times \frac{\begin{vmatrix}
W_p(\cos \varphi) & W_p(\cos \beta) & W_p(\cos \gamma) \\
W_{p+1}(\cos \varphi) & W_{p+1}(\cos \beta) & W_{p+1}(\cos \gamma) \\
W_{p+2}(\cos \varphi) & W_{p+2}(\cos \beta) & W_{p+2}(\cos \gamma)
\end{vmatrix}}{(\cos \gamma - \cos \beta)(\cos \gamma - \cos \varphi)(\cos \beta - \cos \varphi)}
\]

\[\text{(5.1)}\]

It is easy to check that for \( \varphi \in (0, \pi) \) \( \cos \varphi, \cos \gamma, \cos \beta \) are pairwise distinct, which means that the equation \( \det(T_{2p}(a - g(\varphi))) = 0 \) is equivalent to the equation:

\[
\begin{vmatrix}
V_p(\cos \varphi) & V_p(\cos \beta) & V_p(\cos \gamma) \\
V_{p+1}(\cos \varphi) & V_{p+1}(\cos \beta) & V_{p+1}(\cos \gamma) \\
V_{p+2}(\cos \varphi) & V_{p+2}(\cos \beta) & V_{p+2}(\cos \gamma)
\end{vmatrix} \times \begin{vmatrix}
W_p(\cos \varphi) & W_p(\cos \beta) & W_p(\cos \gamma) \\
W_{p+1}(\cos \varphi) & W_{p+1}(\cos \beta) & W_{p+1}(\cos \gamma) \\
W_{p+2}(\cos \varphi) & W_{p+2}(\cos \beta) & W_{p+2}(\cos \gamma)
\end{vmatrix} = 0
\]

\[\text{(5.2)}\]
Taking into account the properties (4.1), the equation (5.2) will take the form:

\[
\begin{vmatrix}
\cos \left( \frac{p + \frac{1}{2}}{2} \varphi \right) & \cos \left( \frac{p + \frac{1}{2}}{2} \beta \right) & \cos \left( \frac{p + \frac{1}{2}}{2} \gamma \right) \\
\cos \frac{p + \frac{3}{4}}{2} & \cos \left( \frac{p + \frac{3}{4}}{2} \beta \right) & \cos \left( \frac{p + \frac{3}{4}}{2} \gamma \right) \\
\cos \frac{p + \frac{5}{4}}{2} & \cos \left( \frac{p + \frac{5}{4}}{2} \beta \right) & \cos \left( \frac{p + \frac{5}{4}}{2} \gamma \right)
\end{vmatrix}
\times
\begin{vmatrix}
\sin \left( \frac{p + \frac{1}{2}}{2} \varphi \right) & \sin \left( \frac{p + \frac{1}{2}}{2} \beta \right) & \sin \left( \frac{p + \frac{1}{2}}{2} \gamma \right) \\
\sin \frac{p + \frac{3}{4}}{2} & \sin \left( \frac{p + \frac{3}{4}}{2} \beta \right) & \sin \left( \frac{p + \frac{3}{4}}{2} \gamma \right) \\
\sin \frac{p + \frac{5}{4}}{2} & \sin \left( \frac{p + \frac{5}{4}}{2} \beta \right) & \sin \left( \frac{p + \frac{5}{4}}{2} \gamma \right)
\end{vmatrix} = 0
\]

(5.3)

It is not difficult to check that \(\sin(\varphi) \neq 0\), \(\sin(\beta) \neq 0\) and \(\sin(\gamma) \neq 0\) if \(\varphi \in (0, \pi)\). Then, since \(n = 2p\), the set of solutions to the equation (5.3) coincides with the union of the sets of solutions to the equations:

\[
\begin{vmatrix}
\cos \left( \frac{n + 1}{2} \varphi \right) & \cos \left( \frac{n + 1}{2} \beta \right) & \cos \left( \frac{n + 1}{2} \gamma \right) \\
\cos \frac{n + 3}{2} & \cos \left( \frac{n + 3}{2} \beta \right) & \cos \left( \frac{n + 3}{2} \gamma \right) \\
\cos \frac{n + 5}{2} & \cos \left( \frac{n + 5}{2} \beta \right) & \cos \left( \frac{n + 5}{2} \gamma \right)
\end{vmatrix} = 0
\]

(5.4)

and

\[
\begin{vmatrix}
\sin \left( \frac{n + 1}{2} \varphi \right) & \sin \left( \frac{n + 1}{2} \beta \right) & \sin \left( \frac{n + 1}{2} \gamma \right) \\
\sin \frac{n + 3}{2} & \sin \left( \frac{n + 3}{2} \beta \right) & \sin \left( \frac{n + 3}{2} \gamma \right) \\
\sin \frac{n + 5}{2} & \sin \left( \frac{n + 5}{2} \beta \right) & \sin \left( \frac{n + 5}{2} \gamma \right)
\end{vmatrix} = 0
\]

(5.5)

If \(n = 2p + 1\), similar reasoning will lead to the same equations (5.4) and (5.5). Expanding the determinant in the formula (5.4) by the first column, we obtain the following equation:

\[
\cos \left( \frac{n + 1}{2} \varphi \right) S_1 - \cos \left( \frac{n + 3}{2} \varphi \right) S_2 + \cos \left( \frac{n + 5}{2} \varphi \right) S_3 = 0,
\]

(5.6)

where

\[
S_1 = \cos \left( \frac{n + 3}{2} \beta \right) \cos \left( \frac{n + 5}{2} \gamma \right) - \cos \left( \frac{n + 5}{2} \beta \right) \cos \left( \frac{n + 3}{2} \gamma \right),
\]

\[
S_2 = \cos \left( \frac{n + 1}{2} \beta \right) \cos \left( \frac{n + 5}{2} \gamma \right) - \cos \left( \frac{n + 5}{2} \beta \right) \cos \left( \frac{n + 1}{2} \gamma \right),
\]

\[
S_3 = \cos \left( \frac{n + 1}{2} \beta \right) \cos \left( \frac{n + 3}{2} \gamma \right) - \cos \left( \frac{n + 3}{2} \beta \right) \cos \left( \frac{n + 1}{2} \gamma \right).
\]

Consider the sum of the first and third terms, using the fact that:

\[
\cos \left( \frac{n + 1}{2} \varphi \right) = \cos \left( \frac{n + 3}{2} \varphi - \varphi \right) = \cos \left( \frac{n + 3}{2} \varphi \right) \cos(\varphi) + \sin \left( \frac{n + 3}{2} \varphi \right) \sin(\varphi)
\]
and
\[
\cos \left( \frac{n+5}{2} \varphi \right) = \cos \left( \frac{n+3}{2} \varphi + \varphi \right) = \cos \left( \frac{n+3}{2} \varphi \right) \cos (\varphi) - \sin \left( \frac{n+3}{2} \varphi \right) \sin (\varphi)
\]
then
\[
\cos \left( \frac{n+1}{2} \varphi \right) S_1 + \cos \left( \frac{n+5}{2} \varphi \right) S_3 = \cos \left( \frac{n+3}{2} \varphi \right) \cos (\varphi)(S_1 + S_3) + \sin \left( \frac{n+3}{2} \varphi \right) \sin (\varphi)(S_1 - S_3)
\]

(5.7)

Let’s find \( S_1 + S_3 \) by grouping the first term of \( S_1 \) with the second of \( S_3 \) and vice versa:
\[
S_1 + S_3 = \cos \left( \frac{n+3}{2} \beta \right) \left( \cos \left( \frac{n+5}{2} \gamma \right) - \cos \left( \frac{n+1}{2} \gamma \right) \right)
- \cos \left( \frac{n+3}{2} \gamma \right) \left( \cos \left( \frac{n+5}{2} \beta \right) - \cos \left( \frac{n+1}{2} \beta \right) \right)
\]

Using the cosine difference formula, it is not difficult to obtain
\[
S_1 + S_3 = 2 \cos \left( \frac{n+3}{2} \gamma \right) \sin \left( \frac{n+3}{2} \beta \right) \sin (\beta) - 2 \cos \left( \frac{n+3}{2} \beta \right) \sin \left( \frac{n+3}{2} \gamma \right) \sin (\gamma)
\]

(5.8)

From similar reasoning we get:
\[
S_1 - S_3 = 2 \cos \left( \frac{n+3}{2} \gamma \right) \cos \left( \frac{n+3}{2} \beta \right) \cos (\gamma) - 2 \cos \left( \frac{n+3}{2} \beta \right) \cos \left( \frac{n+3}{2} \gamma \right) \cos (\beta)
\]

If we take the total multiplier out of the brackets:
\[
S_1 - S_3 = 2 \cos \left( \frac{n+3}{2} \gamma \right) \cos \left( \frac{n+3}{2} \beta \right) (\cos (\gamma) - \cos (\beta))
\]

(5.9)

Consider \( S_2 \) by writing it as:
\[
S_2 = \cos \left( \frac{n+3}{2} \beta - \beta \right) \cos \left( \frac{n+3}{2} \gamma + \gamma \right) - \cos \left( \frac{n+3}{2} \beta + \beta \right) \cos \left( \frac{n+3}{2} \gamma - \gamma \right)
\]

Using the formula of the cosine of the sum, expanding the brackets and giving similar terms, we get the following:
\[
S_2 = 2 \cos \left( \frac{n+3}{2} \gamma \right) \sin \left( \frac{n+3}{2} \beta \right) \cos (\gamma) \sin (\beta) - 2 \cos \left( \frac{n+3}{2} \beta \right) \sin \left( \frac{n+3}{2} \gamma \right) \cos (\beta) \sin (\gamma)
\]

(5.10)

Substituting the equalities (5.7)-(5.10) into the equation (5.6), we get the following equation:
\[
2 \cos \left( \frac{n+3}{2} \varphi \right) \cos \left( \frac{n+3}{2} \gamma \right) \sin \left( \frac{n+3}{2} \beta \right) \sin (\beta)(\cos (\varphi) - \cos (\gamma))
-2 \cos \left( \frac{n+3}{2} \varphi \right) \cos \left( \frac{n+3}{2} \beta \right) \sin \left( \frac{n+3}{2} \gamma \right) \sin (\gamma)(\cos (\varphi) - \cos (\beta))
= 2 \sin \left( \frac{n+3}{2} \varphi \right) \sin (\varphi) \cos \left( \frac{n+3}{2} \gamma \right) \cos \left( \frac{n+3}{2} \beta \right)(\cos (\beta) - \cos (\gamma))
\]

(5.11)
It is not difficult to make sure that
\[ \cos(\beta) - \cos(\gamma) = (\cos(\varphi) - 1)\sqrt{3}i \]
\[ \cos(\varphi) - \cos(\beta) = (\cos(\varphi) - 1) \left(\frac{3}{2} - \frac{\sqrt{3}i}{2}\right) \]
\[ \cos(\varphi) - \cos(\gamma) = (\cos(\varphi) - 1) \left(\frac{3}{2} + \frac{\sqrt{3}i}{2}\right) \]

So:
\[ \cos(\varphi) - \cos(\beta) = -\frac{1}{2} - \frac{\sqrt{3}i}{2} = -e^{\pi i/3} \quad (5.12) \]
\[ \cos(\varphi) - \cos(\gamma) = 1 - \frac{\sqrt{3}i}{2} = -e^{2\pi i/3} \]

Dividing the right and left sides of the equation (5.11) by \( \cos(\gamma) - \cos(\beta) \), using the equalities (5.12), and also taking into account that \( \sin(\beta) = B_se^{i\psi_s} \) and \( \sin(\gamma) = B_s e^{-i\psi_s} \), we get the equation:
\[ 2\cos\left(\frac{n+3}{2}\varphi\right) \cos\left(\frac{n+3}{2}\beta\right) \sin\left(\frac{n+3}{2}\gamma\right) B_se^{-i\psi_s + \pi i/3} \]
\[ - 2\cos\left(\frac{n+3}{2}\varphi\right) \cos\left(\frac{n+3}{2}\gamma\right) \sin\left(\frac{n+3}{2}\beta\right) B_se^{i\psi_s + 2\pi i/3} \quad (5.13) \]
\[ = 2\sin\left(\frac{n+3}{2}\varphi\right) \sin(\varphi) \cos\left(\frac{n+3}{2}\gamma\right) \cos\left(\frac{n+3}{2}\beta\right) \]

Given that \( \beta = c + ib, \gamma = c - ib \), we get
\[ 2\cos\left(\frac{n+3}{2}\beta\right) \sin\left(\frac{n+3}{2}\gamma\right) = \sin\left(\frac{(n+3)c - i\sinh((n+3)b)}{2}\right) \]
\[ 2\cos\left(\frac{n+3}{2}\gamma\right) \sin\left(\frac{n+3}{2}\beta\right) = \sin\left(\frac{(n+3)c + i\sinh((n+3)b)}{2}\right) \quad (5.14) \]
\[ 2\cos\left(\frac{n+3}{2}\beta\right) \cos\left(\frac{n+3}{2}\gamma\right) = \cos\left(\frac{(n+3)c + \cosh((n+3)b)}{2}\right) \]

Denote by \( S_5 \) the left side of the equation (5.13) without the multiplier \( \cos\left(\frac{n+3}{2}\varphi\right) \), then taking into account (5.14) we get:
\[ S_5 = (\sin\left(\frac{(n+3)c - i\sinh((n+3)b)}{2}\right) B_s \left(\cos\left(-\psi_s + \frac{\pi}{3}\right) + i\sin\left(-\psi_s + \frac{\pi}{3}\right)\right) \]
\[ - (\sin\left(\frac{(n+3)c + i\sinh((n+3)b)}{2}\right) B_s \left(\cos\left(\psi_s + \frac{2\pi}{3}\right) + i\sin\left(\psi_s + \frac{2\pi}{3}\right)\right) \]

Which is equivalent to:
\[ S_5 = \sin\left(\frac{(n+3)c}{2}\right) B_s \left(\cos\left(-\psi_s + \frac{\pi}{3}\right) - \cos\left(\psi_s + \frac{2\pi}{3}\right)\right) \]
\[ - i\sinh\left(\frac{(n+3)b}{2}\right) B_s \left(\cos\left(-\psi_s + \frac{\pi}{3}\right) + \cos\left(\psi_s + \frac{2\pi}{3}\right)\right) \]
\[ + i\sin\left(\frac{(n+3)c}{2}\right) B_s \left(\sin\left(-\psi_s + \frac{\pi}{3}\right) - \sin\left(\psi_s + \frac{2\pi}{3}\right)\right) \]
\[ + \sinh\left(\frac{(n+3)b}{2}\right) B_s \left(\sin\left(-\psi_s + \frac{\pi}{3}\right) + \sin\left(\psi_s + \frac{2\pi}{3}\right)\right) \quad (5.15) \]
It is easy to make sure that:

\[
B_s \left( \cos \left( -\psi_s + \frac{\pi}{3} \right) - \cos \left( \psi_s + \frac{2\pi}{3} \right) \right) = 2B
\]
\[
B_s \left( \cos \left( -\psi_s + \frac{\pi}{3} \right) + \cos \left( \psi_s + \frac{2\pi}{3} \right) \right) = 0
\]
\[
B_s \left( \sin \left( -\psi_s + \frac{\pi}{3} \right) - \sin \left( \psi_s + \frac{2\pi}{3} \right) \right) = 0
\]
\[
B_s \left( \sin \left( -\psi_s + \frac{\pi}{3} \right) + \sin \left( \psi_s + \frac{2\pi}{3} \right) \right) = 2C.
\]

Then, taking into account (5.15), (5.16) and 5.14, the equation (5.13) will take the form

\[
\cos \left( n + \frac{3}{2} \phi \right) (2B \sin ((n + 3)c) + 2C \sinh ((n + 3)b)) = \sin \left( n + \frac{3}{2} \phi \right) \sin \left( 2 \varphi \right) (\cos ((n + 3)c) + \cosh ((n + 3)b))
\]

Since \( \sin (\varphi) \neq 0 \), \( b > 0 \) and hence \( \cos ((n + 3)c) + \cos ((n + 3)b) > 0 \) when \( \varphi \in (0, \pi) \), then the equation can be rewritten as

\[
\sin \left( n + \frac{3}{2} \varphi \right) = \cos \left( n + \frac{3}{2} \varphi \right) 2 \frac{B \sin ((n + 3)c) + C \sinh ((n + 3)b)}{\sin (\varphi)(\cos ((n + 3)c) + \cosh ((n + 3)b))}
\]

Since \( \sin \left( \frac{n+3}{2} \varphi \right) \) and \( \cos \left( \frac{n+3}{2} \varphi \right) \) cannot be equal to zero at the same time, then the equation (5.17) and therefore the equation (5.4) is equivalent to the equation (2.3). Doing similar calculations for the equation 5.5, you can get that it is equivalent to the equation 2.4.

**Lemma 5.1.**

1. If \( \varphi \in \left( \frac{\pi}{n+3}, \pi \right) \) then

\[
\frac{2}{n + 3} \frac{f'(\varphi)}{1 + f^2(\varphi)} < 0.8
\]

2. If \( \varphi \in \left( \frac{2\pi}{n+3}, \pi \right) \) then

\[
\frac{2}{n + 3} \frac{h'(\varphi)}{1 + h^2(\varphi)} < 0.8
\]

**Proof.**

\[
\frac{2}{n + 3} f'(\varphi) = \frac{4}{n + 3} \left( I_1 + I_2 + I_3 + I_4 \right),
\]

where

\[
I_1 = \frac{B' \sin ((n + 3)c) + C' \sinh ((n + 3)b)}{\sin(\varphi)(\cos ((n + 3)c) + \cosh ((n + 3)b))}
\]
\[
I_2 = \frac{(n + 3)(Bc' \cos ((n + 3)c) +Cb' \cosh ((n + 3)b))}{\sin(\varphi)(\cos ((n + 3)c) + \cosh ((n + 3)b))}
\]
\[
I_3 = \frac{(n + 3)(B \sin ((n + 3)c) + C \sinh ((n + 3)b))(c' \sin ((n + 3)c) - b' \sinh ((n + 3)b))}{\sin(\varphi)(\cos ((n + 3)c) + \cosh ((n + 3)b))^2}
\]
\[
I_4 = \frac{(B \sin ((n + 3)c) + C \sinh ((n + 3)b)) \cos(\varphi)}{\sin^2(\varphi)(\cos ((n + 3)c) + \cosh ((n + 3)b))}.
\]
Consider the sum \( \frac{4(I_2 + I_3)}{n+3} \), reduce to a common denominator, expand the brackets and give similar summands.

\[
\begin{align*}
4(I_2 + I_3) &= 4(Bc' \cos ((n + 3)c) + Cb' \cosh ((n + 3)b)) (\cos ((n + 3)c) + \cosh ((n + 3)b)) \\
&\quad + \frac{4\sin(\varphi)(\cos ((n + 3)c) + \cosh ((n + 3)b))}{\sin(\varphi)(\cos ((n + 3)c) + \cosh ((n + 3)b))^2} \\
&\quad + \frac{4Ba' \cos^2 ((n + 3)c) + \cos ((n + 3)c) \cosh ((n + 3)b)(Bc' + Cb')) + Cb' \cosh^2 ((n + 3)b)}{\sin(\varphi)(\cos ((n + 3)c) + \cosh ((n + 3)b))^2} \\
&\quad + \frac{4Ba' \sin^2 ((n + 3)c) + \sin ((n + 3)c) \sinh ((n + 3)b)(Cc' - Bb')) - Cb' \sinh^2 ((n + 3)b)}{\sin(\varphi)(\cos ((n + 3)c) + \cosh ((n + 3)b))^2} \\
&= 4(Bc' + Cb') + \cos ((n + 3)c) \cosh ((n + 3)b)(Bc' + Cb')) + \sin ((n + 3)c) \cosh ((n + 3)b)(Cc' - Bb')) \\
&\quad \sin(\varphi)(\cos ((n + 3)c) + \cosh ((n + 3)b))^2
\end{align*}
\]

We will find \( Ba' + Cb' \) and \( Ca' - Bb' \). Taking into account the formula (3.9) and that \( c' = \Re(\beta') \), \( b' = \Im(\beta') \)

\[
Bc' + Cb' = \sin(\varphi) \left[ \cos \left( \frac{2\pi}{3} - \psi_s \right) \cos \left( \psi_s - \frac{\pi}{3} \right) - \sin \left( \frac{2\pi}{3} - \psi_s \right) \sin \left( \psi_s - \frac{\pi}{3} \right) \right] = \sin(\varphi) \cos \left( \frac{\pi}{3} \right) = \frac{1}{2} \sin(\varphi). \quad (5.21)
\]

Similarly

\[
Cc' - Bb' = -\frac{\sqrt{3}}{2} \sin(\varphi).
\]

Then:

\[
\begin{align*}
4(I_2 + I_3) &= \frac{4\left(\frac{1}{2} + \frac{1}{2} \cos ((n + 3)c) \cosh ((n + 3)b) - \frac{\sqrt{3}}{2} \sin ((n + 3)c) \sinh ((n + 3)b) \right)}{(\cos ((n + 3)c) + \cosh ((n + 3)b))^2} \\
&= \frac{4\left(1 + \frac{1}{2} \cos ((n + 3)c) \cosh ((n + 3)b) - \frac{\sqrt{3}}{2} \sin ((n + 3)c) \sinh ((n + 3)b) \right)}{\cos ((n + 3)c) + \cosh ((n + 3)b))^2}
\end{align*}
\]

It is not difficult to check that

\[
B'(\varphi) = \Re(\cot(\beta)e^{\frac{\pi}{3}}) \sin(\varphi)
\]
\[
C'(\varphi) = -3(\cot(\beta)e^{\frac{\pi}{3}}) \sin(\varphi).
\]

Then:

\[
I_1 = \frac{\left| \cot(\beta) \left[ \cos \left( \frac{\pi}{3} + \psi_c - \psi_s \right) \sin((n + 3)c) - \sin \left( \frac{\pi}{3} + \psi_c - \psi_s \right) \sinh((n + 3)b) \right] \right|}{\left( \cos((n + 3)c) + \cosh((n + 3)b) \right)},
\]
\[
I_4 = \frac{\left| \sin(\beta) \left[ \cos(\psi_s - \frac{\pi}{3}) \sin((n + 3)c) - \sin(\psi_s - \frac{\pi}{3}) \sinh((n + 3)b) \right] \right|}{\sin^2(\varphi) \left( \cos((n + 3)c) + \cosh((n + 3)b) \right)}.
\]

Next, consider three cases \( \varphi \in \left( \pi \frac{n+3}{n+3}, 2\pi \frac{n+3}{n+3} \right) \), \( \varphi \in \left[ \frac{2\pi}{n+3}, \frac{\pi}{2} \right) \) and \( \varphi \in \left( \frac{\pi}{2}, \pi \right) \).

Case 1. \( \varphi \in \left( \frac{\pi}{n+3}, 2\pi \frac{n+3}{n+3} \right) \)

First, let’s evaluate the value from below \( \cosh((n + 3)b) \). In (3b) of the section 3, it was shown
that \( b(\varphi) \) is an increasing function, so \( \cosh ((n+3)b) < \cosh ((n+3)b(\frac{\pi}{n+3})) \). From (5b) of the section 3 it follows that

\[
(n + 3)b \left( \frac{\pi}{n + 3} \right) = (n + 3)\frac{\sqrt{3}}{2} \frac{\pi}{n + 3} + o \left( \frac{1}{n + 3} \right) = \frac{\sqrt{3}\pi}{2} + o \left( \frac{1}{n + 3} \right)
\]

Then if \( n \) is sufficiently large:

\[
(n + 3)b \left( \frac{\pi}{n + 3} \right) > 2.72 \tag{5.25}
\]

Where do we get the lower bound for \( \cosh ((n+3)b) \)

\[
\cosh ((n+3)b) > \cosh (2.72) > 7.62 \tag{5.26}
\]

Consider \( \frac{4(I_2 + I_3)}{n+3} \), add and subtract in the numerator the summand \( \frac{\sqrt{3}}{2} \sin ((n+3)c) \cosh ((n+3)b) \), then we get the following

\[
\frac{4(I_2 + I_3)}{n + 3} = \frac{4\left( \frac{1}{2} \cos ((n+3)c) \cosh ((n+3)b) - \frac{\sqrt{3}}{2} \sin ((n+3)c) \cosh ((n+3)b) \right)}{(\cos ((n+3)c) + \cosh ((n+3)b))^2} + \frac{4\left( \frac{1}{2} + \frac{\sqrt{3}}{2} \sin ((n+3)c) \cosh ((n+3)b) - \sinh ((n+3)b) \right)}{(\cos ((n+3)c) + \cosh ((n+3)b))^2} \tag{5.27}
\]

\[
= 4\cos ((n+3)b) \cos ((n+3)c + \frac{\pi}{3}) + \frac{4\left( \frac{1}{2} + \frac{\sqrt{3}}{2} \sin ((n+3)c) \cosh ((n+3)b) - \sinh ((n+3)b) \right)}{(\cos ((n+3)c) + \cosh ((n+3)b))^2}
\]

Let’s introduce the notation

\[
L_1 = \frac{4\cos ((n+3)b) \cos ((n+3)c + \frac{\pi}{3})}{(\cos ((n+3)c) + \cosh ((n+3)b))^2},
\]

\[
L_2 = \frac{4\left( \frac{1}{2} + \frac{\sqrt{3}}{2} \sin ((n+3)c) \cosh ((n+3)b) - \sinh ((n+3)b) \right)}{(\cos ((n+3)c) + \cosh ((n+3)b))^2}
\]

From (5a) of the section 3 it follows that when \( \varphi \in (\frac{\pi}{n+3}, \frac{2\pi}{n+3}) \) \( c(n+3) + \frac{\pi}{3} \in (\frac{5\pi}{6} + o(1), \frac{8\pi}{6} + o(1)) \), this means that for a sufficiently large \( n \cos ((n+3)c + \frac{\pi}{3}) < 0 \), so \( L_1 < 0. \)

Let’s estimate \(|L_1|\)

\[
|L_1| \leq \frac{4\cos ((n+3)b)}{(\cos ((n+3)c) + \cosh ((n+3)b))^2} \leq \frac{4\cos ((n+3)b)}{(-1 + \cosh ((n+3)b))^2}
\]

Taking into account (5.26):

\[
|L_1| < \frac{4 \cdot 7.6}{(-1 + 7.6)^2} < 0.7 \tag{5.28}
\]

Let’s estimate \( L_2 \). From (4a) and (5a) it follows that \( c < \frac{\pi}{2} \), and since we consider the case when \( \varphi \in (\frac{\pi}{n+3}, \frac{2\pi}{n+3}) \), then \( c(n+3) < \pi \), which means \( \sin ((n+3)c) > 0 \), so both terms in \( L_2 \) greater than zero. It is not difficult to understand that with the growth of \( b \), \( L_2 \) also grows, then taking into account (5.25) and (5.26)

\[
L_2 < \frac{4\left( \frac{1}{2} + \frac{\sqrt{3}}{2} e^{-2.72} \right)}{(-1 + 7.6)^2} < 0.06 \tag{5.29}
\]
Then taking into account (5.28) and (5.29):

\[-0.7 < \frac{4(I_2 + I_3)}{n + 3} < 0.06 \quad (5.30)\]

Let’s estimate \(\frac{4I_1}{n+3}\). From (1a) and (1c) and the formulas (3.5) and (3.7) of the section 3, it is not difficult to get that in the neighborhood of the point \(\varphi = 0\) \(B_c = 1 + o(\varphi), B_s = \varphi + o(\varphi)\), then

\[|\cot \beta| = \frac{B_c}{B_s} = \frac{1}{\varphi} + o(1) \quad (5.31)\]

In (4c) of the section 3, it was shown that \(|\cot \beta|\) is a decreasing function, so for a sufficiently large \(n\) we have the following formula:

\[\frac{4|\cot \beta|}{n + 3} < \frac{4}{n + 3} \left(\frac{n + 3}{\pi} + o(1)\right) < 1.28 \quad (5.32)\]

From (1b) and (1d) of the section 3, and the formulas (3.6) and (3.8), it follows that near the point \(\varphi = 0\) \(\sin \left(\frac{\pi}{3} + \psi_c - \psi_s\right) = o(1)\). Since \((n + 3)b < (n + 3)\sqrt{3\varphi} < \sqrt{3}\pi\), then \(\sin((n + 3)b)\) is bounded. From which it follows that for a sufficiently large \(n\):

\[|\sin \left(\frac{\pi}{3} + \psi_c - \psi_s\right)\sinh((n + 3)b)| < 0.01 \quad (5.33)\]

Then substituting inequalities (5.26), (5.32) and (5.33) into the formula (5.23) we get that

\[\frac{4I_1}{n + 3} < \frac{1.28(1 + 0.01)}{-1 + 7.6} < 0.2\]

It was shown above that in the case under consideration \(\sin((n + 3)c) > 0\), and hence

\[\cos \left(\frac{\pi}{3} + \psi_c - \psi_s\right)\sin((n + 3)c) > 0.\]

Taking into account (5.33) we get

\[-0.01 < \frac{4I_1}{n + 3}\]

As a result

\[-0.01 < \frac{4I_1}{n + 3} < 0.2 \quad (5.34)\]

Let’s estimate \(\frac{4I_4}{n+3}\). Using the similar reasoning as in the estimation of \(\frac{4I_1}{n+3}\), it is easy to get that, near the point \(\varphi = 0\):

\[\frac{|\sin (\beta)|}{\sin (\varphi)} = 1 + o(1) < 1.01\]

\[\frac{4}{n + 3} \frac{1}{\sin (\varphi)} < \frac{4}{\pi} < 1.28\]

And also that

\[\sin (\psi_s - \frac{\pi}{3}) = o(1)\]

Then

\[\frac{4I_4}{n + 3} < \frac{1.01 \ast 1.28(1 + 0.01)}{-1 + 7.6} < 0.2\]
And also

\[-0.01 < \frac{4I_4}{n+3}\]

As a result

\[-0.01 < \frac{4I_4}{n+3} < 0.2 \quad (5.35)\]

From the formula (5.20) and the inequalities (5.30), (5.34) and (5.35) we get that when \( \varphi \in (\frac{\pi}{n+3}, \frac{2\pi}{n+3}) \)

\[\left| \frac{2}{n+3} f' \right| < 0.8\]

Which means

\[\left| \frac{2}{n+3} f' f^2 + 1 \right| < 0.8\]

Case 2. \( \varphi \in [\frac{2\pi}{n+3}, \frac{\pi}{2}] \). Using the similar reasoning as in the first case, it is not difficult to show that for a sufficiently large \( n \)

\[
\cosh ((n + 3)b) < 115. \quad (5.36)
\]

Let’s estimate \( \frac{4(I_2 + I_3)}{n+3} \).

\[
\left| \frac{4(I_2 + I_3)}{n+3} \right| = \frac{4\left( \frac{1}{2} + \frac{1}{2} \cos ((n + 3)c) \cosh ((n + 3)b) - \frac{\sqrt{3}}{2} \sin ((n + 3)c) \sinh ((n + 3)b) \right)}{(\cos ((n + 3)c) + \cosh ((n + 3)b))^2}
\]

\[
\leq \frac{4\left( \frac{1}{2} + \frac{1}{2} \cos ((n + 3)c) \cosh ((n + 3)b) \right) + \frac{\sqrt{3}}{2} \sin ((n + 3)c) \sinh ((n + 3)b))}{(\cos ((n + 3)c) + \cosh ((n + 3)b))^2}
\]

\[
< \frac{4\left( \frac{1}{2} + \frac{1}{2} \cos ((n + 3)c) \cosh ((n + 3)b) \right) \left( \frac{1}{2} \cos ((n + 3)c) + \frac{\sqrt{3}}{2} \sin ((n + 3)c) \cosh ((n + 3)b) \right)}{(\cos ((n + 3)c) + \cosh ((n + 3)b))^2}
\]

\[
\leq \frac{4\left( \frac{1}{2} + \cosh ((n + 3)b) \right) \left( \frac{1}{2} \cos ((n + 3)c) + \frac{\sqrt{3}}{2} \sin ((n + 3)c) \cosh ((n + 3)b) \right)}{(\cos ((n + 3)c) + \cosh ((n + 3)b))^2} < \frac{4\left( \frac{1}{2} + 115 \right)}{(-1 + 115)^2} < 0.04
\]

So

\[
\left| \frac{4(I_2 + I_3)}{n+3} \right| < 0.04. \quad (5.37)
\]

Let’s estimate \( \frac{4I_1}{n+3} \). Similarly, as in the first case, it is not difficult to get that for a sufficiently large \( n \)

\[
\frac{4|\cot \beta|}{n+3} < 0.64
\]

And also taking into account (1b) and (1d) of the section 3, we get that

\[
\left| \sin \left( \frac{\pi}{3} + \psi_c - \psi_s \right) \right| < \sin \left( \frac{\pi}{4} \right) = \frac{1}{\sqrt{2}}
\]
Then
\[
\frac{4I_1}{n+3} = \frac{4}{n+3} \left| \cot (\beta) \right| \left| \left( \cos \left( \frac{\pi}{3} + \psi_s - \psi_c \right) \sin ((n+3)c) - \sin \left( \frac{\pi}{3} + \psi_c - \psi_s \right) \sinh ((n+3)b) \right) \right|
\]
\[
\leq \frac{4}{n+3} \left| \cot (\beta) \right| \left( \left| \cos \left( \frac{\pi}{3} + \psi_s - \psi_c \right) \sin ((n+3)c) \right| + \left| \sin \left( \frac{\pi}{3} + \psi_c - \psi_s \right) \cosh ((n+3)b) \right| \right)
\]
\[
< 0.64 \left( 1 + \frac{115}{\sqrt{2}} \right) \frac{1}{(-1 + 115)} < 0.79
\]

So
\[
\left| \frac{4I_1}{n+3} \right| < 0.79 \quad (5.38)
\]

Let’s estimate \( \frac{4I_2}{n+3} \). From 4d of the section 3 it follows that:
\[
\frac{B_s}{\sin (\varphi)} \leq \frac{B_s(\pi/2)}{\sin (\pi/2)} < 1.63
\]

And also taking into account 1b of the section 3
\[
\left| \sin \left( \frac{\pi}{3} - \psi_s \right) \right| \leq \sin \left( \frac{\pi}{12} \right) < 0.26
\]

It is also obvious that
\[
\frac{4}{(n+3)} \sin (\varphi) \leq \frac{4}{(n+3)} \frac{(n+3)}{2\pi} < 0.64
\]

then
\[
\frac{4I_4}{n+3} = \frac{4}{n+3} \left| \cos (\varphi) \right| \left| \left( \cos (\psi_s - \frac{\pi}{3}) \sin ((n+3)c) - \sin (\psi_s - \frac{\pi}{3}) \sinh ((n+3)b) \right) \right|
\]
\[
\leq \frac{4}{n+3} \left| \sin (\varphi) \right| \left( \left| \cos (\psi_s - \frac{\pi}{3}) \sin ((n+3)c) \right| + \left| \sin (\psi_s - \frac{\pi}{3}) \cosh ((n+3)b) \right| \right)
\]
\[
< 1.63 * 0.64 \frac{1 + 0.26 * 115}{-1 + 115} < 0.29
\]

So
\[
\left| \frac{4I_4}{n+3} \right| < 0.29 \quad (5.39)
\]

Then, taking into account (5.37), (5.38), and (5.39), we get that for \( \varphi \in \left( \frac{2\pi}{n+3}, \frac{\pi}{2} \right) \)
\[
\frac{2f'}{n+3} = \frac{4}{n+3} \left| I_1 + I_2 + I_3 + I_4 \right| \leq \frac{4}{n+3} \left( |I_1| + |I_2 + I_3| + |I_4| \right) < 0.8
\]

Which means
\[
\left| \frac{4}{n+3} \frac{f'}{f^2 + 1} \right| < 0.8
\]

Case 3. \( \varphi \in (\frac{\pi}{2}, \pi) \). In this case, the estimates (5.37) and (5.38) remain true, which means that the inequalities are met:
\[
\left| \frac{4}{(n+3)} \frac{I_1}{1 + f^2} \right| \leq \left| \frac{4I_1}{(n+3)} \right| < 0.47 \quad (5.40)
\]
Let's estimate \( \frac{4I_4}{(n+3)f^2} \).

\[
\left| \frac{4I_4}{(n+3)f^2} \right| = \left| \frac{4\cos(\varphi)(\cos((n+3)c) + \cosh((n+3)b))}{(n+3)(B\sin((n+3)c) + C\sinh((n+3)b))} \right|
\]

Since \( B = B_s \cos(\frac{\pi}{3} - \psi_s) \), \( C = -B_s \sin(\frac{\pi}{3} - \psi_s) \) and \( B_s \) is increases, then

\[
B < B_s(\pi) < 4,
\]

and also

\[
C > B_s(\frac{\pi}{2}) \sin(\frac{\pi}{3} - \psi_s(\frac{\pi}{2})) > 0.15
\]

From 3b and 4b of the section 3 it follows that

\[
b > b(\frac{\pi}{2}) > \frac{\pi}{4}
\]

Then

\[
\left| \frac{4I_4}{(n+3)f^2} \right| < \frac{4}{(n+3)} \left| \frac{1 + \cosh(\frac{(n+3)\pi}{4})}{(-4 + 0.15 \sinh(\frac{(n+3)\pi}{4}))} \right| < 0.26
\]

So

\[
\left| \frac{4I_4}{(n+3)(f^2 + 1)} \right| < \left| \frac{4I_4}{(n+3)f^2} \right| < 0.26
\]

Then, taking into account (5.40), (5.41) and (5.43), we get

\[
\left| \frac{2f'}{(n+3)(1 + f^2)} \right| = \frac{4}{n+3} \left| \frac{I_1 + I_2 + I_3 + I_4}{(1 + f^2)} \right| \leq \frac{4|I_1|}{(n+3)(1 + f^2)} + \frac{4|I_2 + I_3|}{(n+3)(1 + f^2)} + \frac{4|I_4|}{(n+3)(1 + f^2)}
\]

\[
< 0.04 + 0.47 + 0.26 = 0.77 < 0.8
\]

The inequality (5.19) is proved in the same way as in the cases 2 and 3 above.

Let's introduce two new functions:

\[
F(\varphi, n) := \frac{n+3}{2} \varphi - \arctan(f(\varphi))
\]

\[
G(\varphi, n) := \frac{n+3}{2} \varphi - \frac{\pi}{2} + \arctan(h(\varphi))
\]

**Proof of the theorem 2.2.**

\[
F'(\varphi, n) = \frac{n+3}{2} - \frac{f'(\varphi)}{1 + f^2(\varphi)}. 
\]

From the lemma 5.1 it follows that the recurrent formula (2.7) converges. If \( j = 1, 2, \ldots, \left[\frac{n+1}{2}\right] \) then \( F(\frac{\pi}{n+3}, n) = \frac{\pi}{2} - \arctan(f(\frac{\pi}{n+3})) < \pi j \), \( F(\pi, n) = \frac{(n+3)\pi}{2} - \arctan(f(\pi)) > \pi j. \) Which

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means that the equation $F(\varphi, n) = \pi j$, has exactly one root on the interval $(\frac{\pi}{n+3}, \pi)$, for each $j = 1, 2, \ldots, \lfloor \frac{n+1}{2} \rfloor$. From which follows the statement of the first part of the theorem.

$$G'(\varphi, n) = \frac{n+3}{2} - \frac{h'(\varphi)}{1 + h^2(\varphi)}.$$ 

From the lemma 5.1 it follows that the recurrent formula (2.8) converges. And also that $G'(\varphi, n) > 0$ for $\varphi \in (\frac{2\pi}{n+3}, \pi)$. Then if $j = 1, 2, \ldots, \lfloor \frac{n}{2} \rfloor$ then $G'(\frac{2\pi}{n+3}, n) = \frac{\pi}{2} + \arctan (h(\frac{\pi}{n+3})) < \pi j$, $G(\pi, n) = \frac{n+3}{2} + \arctan (h(\pi)) > \pi j$. It means that the equation $G(\varphi, n) = \pi j$, has exactly one root on the interval $(\frac{\pi}{n+3}, \pi)$, for each $j = 1, 2, \ldots, \lfloor \frac{n}{2} \rfloor$. From which follows the statement of the second part of the theorem.

**Proof of the lemma 2.1.** We show that all the roots of the equation are different. To do this it is sufficient to show that the following inequalities hold for all roots $\varphi_{2j-1} < \varphi_{2j}$ and $\varphi_{2j} < \varphi_{2j+1}$. We prove the first inequality. Suppose this is incorrect, then

$$\frac{2}{n+3} [\pi j + \arctan (f(\varphi_{2j-1}))] \geq \frac{2}{n+3} \left[ \pi j + \frac{\pi}{2} - \arctan (h(\varphi_{2j})) \right]$$

which is the same as

$$\arctan (f(\varphi_{2j-1})) \geq \frac{\pi}{2} - \arctan (h(\varphi_{2j})) \tag{5.44}$$

Since $\arctan x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ then to perform the inequality [5.44], it is necessary that

$$h(\varphi_{2j}) \geq 0$$

which is equivalent to

$$B(\varphi_{2j}) \sin ((n+3)a(\varphi_{2j})) \geq C(\varphi_{2j}) \sinh ((n+3)b(\varphi_{2j}))$$

From 4b it follows, that $\frac{b(\varphi)}{\varphi} \geq \frac{b(\pi)}{\pi} > \frac{1}{2}$, which means $b(\varphi_{2j}) > \frac{\varphi_{2j}}{2}$, then we get the inequality

$$B(\varphi_{2j}) > C(\varphi_{2j}) \sinh \left( \frac{n+3}{2} \varphi_{2j} \right)$$

which is equivalent to

$$\frac{B(\varphi_{2j})}{C(\varphi_{2j})} > \sinh \left( \frac{n+3}{2} \varphi_{2j} \right)$$

Since

$$\frac{B(\varphi_{2j})}{C(\varphi_{2j})} = -\frac{\Re (\sin (\beta) e^{-\frac{\pi i}{3}})}{3 (\sin (\beta) e^{-\frac{\pi i}{3}})} = -\frac{B_s \Re (e^{-\frac{\pi i}{3} + i \psi_s})}{B_s 3 (e^{-\frac{\pi i}{3} + i \psi_s})} = \cot \left( \frac{\pi}{3} - \psi_s \right)$$

we get the inequality

$$\cot \left( \frac{\pi}{3} - \psi_s \right) > \sinh \left( \frac{n+3}{2} \varphi_{2j} \right) \tag{5.45}$$

From 1b it follows that the left side of the equation (5.45) decreases, and the right side increases. Also note that the left part does not depend on $n$, and the right part grows with increasing $n$, respectively, with a sufficiently large $n \varphi_{2j}$ and therefore $\varphi_{2j-1}$ (since $\varphi_{2j-1} - \varphi_{2j} < \frac{\pi}{n+3}$) can be made arbitrarily small. Let

$$\varphi_{2j} < \varphi_{2j-1} < \frac{\pi}{12}. \tag{5.46}$$
In fact, this will be true already at \( n > 7 \). Taking into account (5.46), find the upper estimate for \(|f(\varphi_{2j-1})|\) and \(|h(\varphi_{2j})|\):

\[
\frac{B(\varphi)}{\sin(\varphi)} = \frac{B_s \cos \left( \frac{\pi}{3} - \psi_s \right)}{\sin(\varphi)}
\]

Then according to 4d we have

\[
\frac{B(\varphi_{2j})}{\sin(\varphi_{2j})} = \frac{B_s(\varphi_{2j}) \cos \left( \frac{\pi}{3} - \psi_s \right)}{\sin(\varphi_{2j})} \leq \frac{B_s(\varphi_{2j})}{\sin(\varphi_{2j})} < \frac{B_s(\varpi_{12})}{\sin(\varpi_{12})} < 1.1 \tag{5.47}
\]

Then according to 4d and 1b we have

\[
\frac{C(\varphi)}{\sin(\varphi)} = \frac{B_s \cos \left( \frac{\pi}{3} - \psi_s \right)}{\sin(\varphi)}
\]

Then

\[
|h(\varphi_{2j})| < 2 \left| \frac{B(\varphi_{2j})}{\sin(\varphi_{2j})} \right| \left| \frac{C(\varphi_{2j})}{\sin(\varphi_{2j})} \right| < \frac{2B_s(\varphi_{2j})}{\sin(\varphi_{2j})} < 0.01 \tag{5.48}
\]

Then

\[
|\varphi_{2j}| < 2 \left| \frac{B(\varphi_{2j})}{\sin(\varphi_{2j})} \right| \left| \frac{C(\varphi_{2j})}{\sin(\varphi_{2j})} \right| < \frac{2B_s(\varphi_{2j})}{\sin(\varphi_{2j})} \left| \frac{C(\varphi_{2j})}{\sin(\varphi_{2j})} \right| < 0.01 \tag{5.49}
\]

From similar reasoning, we get that

\[
|f(\varphi_{2j-1})| < \frac{2.2}{6.6} + \frac{0.02 \times 7.6}{6.6} < 0.5 < \frac{1}{\sqrt{3}} \tag{5.50}
\]

Which contradicts the inequality (5.44), which means \( \varphi_{2j-1} < \varphi_{2j} \). Similarly, it is shown that \( \varphi_{2j} < \varphi_{2j+1} \).

Proof of the theorem 2.3. Denote by \( j_m \) the smallest \( j \) for which the inequality \( \frac{1}{2} e^{\pi(j-1)} > q^2 \) is satisfied. Let \( d_{1,j} = \frac{\varpi}{q} \). Since \( 2b > \varphi \), as shown in the 4b of section 3, then

\[
\cosh(2qb) > \frac{1}{2} e^{d_{1,j} - \frac{\varpi}{q}} > \frac{1}{2} e^{\pi(j-1)} > q^2.
\]

In this case

\[
\frac{\sin(2qc)}{\cosh(2qb)} = O \left( \frac{1}{q^2} \right),
\]

\[
\frac{\cos(2qc)}{\cosh(2qb)} = O \left( \frac{1}{q^2} \right),
\]

and

\[
\tanh((2qb)) = 1 + O \left( \frac{1}{q^2} \right).
\]
Then the equation (2.11) can be rewritten as:

\[
    u = \arctan \left( 2 \frac{C(d_{1,j} + \frac{u_1}{q}) + O(\frac{1}{q^2})}{\sin (d_{1,j} + \frac{u_1}{q}) (1 + O(\frac{1}{q^2}))} \right)
\]

(5.51)

Let’s consider two cases. Case 1. \( d_{1,j} < (1 - \varepsilon)\pi \), where \( \varepsilon \) small positive number. Then, assuming \( u = u_1 + \frac{u_2}{q} \), the equation (5.51) takes the form:

\[
    u_1 + \frac{u_2}{q} = \arctan \left( 2 \frac{C(d_{1,j} + \frac{u_1}{q}) + O(\frac{1}{q^2})}{\sin (d_{1,j} + \frac{u_1}{q}) (1 + O(\frac{1}{q^2}))} \right)
\]

arctan \( \left( 2 \frac{C(d_{1,j} + \frac{u_1}{q}) + O(\frac{1}{q^2})}{\sin (d_{1,j} + \frac{u_1}{q}) + O(\frac{1}{q^2})} \right) \)

\[
= \arctan \left( 2 \frac{C'(d_{1,j}) \sin (d_{1,j}) - C(d_{1,j}) \cos (d_{1,j}) u_1}{\sin^2 (d_{1,j})} + O \left( \frac{1}{q^2} \right) \right)
\]

\[
= \arctan \left( 2 \frac{C'(d_{1,j}) \sin (d_{1,j}) - C(d_{1,j}) \cos (d_{1,j}) u_1}{\sin^2 (d_{1,j}) + 4C^2(d_{1,j})} \right) + O \left( \frac{1}{q^2} \right)
\]

In this case, assuming

\[
    u_1^* = \arctan \left( 2 \frac{C(d_{1,j})}{\sin (d_{1,j})} \right),
\]

and

\[
    u_2^* = 2 \frac{C'(d_{1,j}) \sin (d_{1,j}) - C(d_{1,j}) \cos (d_{1,j})}{\sin^2 (d_{1,j}) + 4C^2(d_{1,j})} \arctan \left( 2 \frac{C(d_{1,j})}{\sin (d_{1,j})} \right),
\]

we get that \( |u_1 - u_1^*| = O(\frac{1}{q^2}) \) and \( |u_2 - u_2^*| = O(\frac{1}{q^2}) \).

Case 2. \( (1 - \varepsilon)\pi \leq d_{1,j} \leq \pi \). Since \( C(d_{1,j} + \frac{u_1}{q}) > 0 \) and \( \sin (d_{1,j} + \frac{u_1}{q}) > 0 \), then the equation (5.51) can be rewritten as:

\[
    u_1 + \frac{u_2}{q} = \arccot \left( 2 \frac{\sin (d_{1,j} + \frac{u_1}{q}) + O(\frac{1}{q^2})}{C(d_{1,j} + \frac{u_1}{q}) + O(\frac{1}{q^2})} \right)
\]

then similarly to the first case:

\[
\]

\[
= \arccot \left( 2 \frac{\sin (d_{1,j}) + \frac{C'(d_{1,j}) \cos (d_{1,j}) - C(d_{1,j}) \sin (d_{1,j}) u_1}{\sin^2 (d_{1,j}) + 4C^2(d_{1,j})}}{C(d_{1,j}) + O \left( \frac{1}{q^2} \right)} \right)
\]

\[
= \arccot \left( \frac{\sin (d_{1,j})}{C(d_{1,j})} - \frac{1}{2} \frac{C^2(d_{1,j})}{4C^2(d_{1,j}) + \sin^2 (d_{1,j})} \right) + O \left( \frac{1}{q^2} \right).
\]

As a result we get the same result as in the first case.

The second part of the theorem is proved in a similar way.
Lemma 5.2. 1. If $d_{1,j} : \frac{1}{2}e^{\pi(j-1)} \leq q^2$ then, for a sufficiently large $n$, the equation

$$u_1 = \arctan \left( Z_1^{(1)} \right)$$

has a unique solution on the interval $\left( -\frac{\pi}{2}, \frac{\pi}{2} \right)$.  

2. If $d_{2,j} : \frac{1}{2}e^{\pi(j-1)} \leq q^2$ then, for a sufficiently large $n$, the equation

$$w_1 = -\arctan \left( Z_1^{(2)} \right)$$

has a unique solution on the interval $\left( -\frac{\pi}{2}, \frac{\pi}{2} \right)$. 

Proof. Considering that $(P_a^{(1)})' = 1$, $(P_b^{(1)})' = \sqrt{3}$

$$(Z^{(1)})' = 2 \frac{((1 + \frac{3}{8}d_{1,j}^2) \cos (P_a^{(1)}) + \frac{3}{16}d_{1,j}^2 \cosh (P_b^{(1)}))(\cos (P_a^{(1)}) + \cosh (P_b^{(1))))}{(\cos (P_a^{(1)}) + \cosh (P_b^{(1))))^2}$$

$$+ 2 \frac{((1 + \frac{3}{8}d_{1,j}^2) \sin (P_a^{(1)}) + \frac{3}{16}d_{1,j}^2 \sinh (P_b^{(1)}))(-\sin (P_a^{(1)}) + \sqrt{3} \sin (P_b^{(1)}))}{(\cos (P_a^{(1)}) + \cosh (P_b^{(1))))^2}$$

Expanding the brackets and giving similar terms we get:

$$(Z^{(1)})' = 2 \frac{(1 + \frac{3}{8}d_{1,j}^2) + (1 + \frac{3}{8}d_{1,j}^2) \cos (P_a^{(1)}) \cosh (P_b^{(1)}) - (1 + \frac{3}{8}d_{1,j}^2) \sin (P_a^{(1)}) \sin (P_b^{(1)})}{(\cos (P_a^{(1)}) + \cosh (P_b^{(1))))^2}$$

From the condition of the lemma, it follows that for a sufficiently large $n$ $d_{1,j} = \frac{\pi j}{q} < 0.1$, then

$$\frac{\sqrt{3}}{8}d_{1,j}^2 < \frac{3}{8}d_{1,j}^2 < 0.01,$$

this means that the following inequality is true:

$$(Z^{(1)})' < 2 \frac{1.01 + 1.01 \cosh (P_b^{(1)}) + 1.01 \sinh (P_b^{(1)})}{(\cos (P_a^{(1)}) + \cosh (P_b^{(1))))^2} < \frac{2.02(1 + 2 \cosh (P_b^{(1)}))}{(-1 + \cosh (P_b^{(1)})^2} \quad (5.52)$$

Let’s estimate $\cosh (P_b^{(1)})$.

$$P_b^{(1)} = \sqrt{3}(\pi j - u_1) \leq \sqrt{3}(\pi - \frac{\pi}{2}) = \frac{\sqrt{3}\pi}{2}$$

Then

$$\cosh (P_b^{(1)}) < \cosh \left( \frac{\sqrt{3}\pi}{2} \right) < 7.6$$

Obviously, the right-hand side of the inequality (5.52) decreases with the growth of $\cosh (P_b^{(1)})$, then

$$(Z^{(1)})' < \frac{2.02(1 + 15.2)}{6.6^2} < 0.76$$

From this, first, it follows that the recurrent formula $u_1^{(k+1)} = \arctan \left( Z_1^{(1)}(u_1^{(k+1)}) \right)$ converges. Second, that the function $H(u_1) = u_1 - \arctan \left( Z_1^{(1)} \right)$ increases. Since $H\left( -\frac{\pi}{2} \right) < 0$, $H\left( \frac{\pi}{2} \right) > 0$ then the root exists, and given that $H(u_1)$ is increasing function then this root is the only one. The second part of the lemma is proved similarly. □
Proof of the theorem 2.4. Let \(d_{1,j} = \frac{\pi j}{q}, q = \frac{n+3}{2}, u = u_1 + \frac{u_2}{q}\). We have the equation:

\[
u = \arctan f
\]

where

\[
f = 2\frac{B(\varphi) \sin (2qc) + C(\varphi) \sinh (2qb)}{\sin(\varphi)(\cos (2qc) + \cosh (2qb))}
\]

and the following inequality is also true

\[
\frac{1}{2}e^{\pi(j-1)} \leq q^2
\]

From 5c of the section 3 it is not difficult to get that

\[
\frac{B(\varphi)}{\sin(\varphi)} = 1 + \frac{3}{16} \varphi^2 + O(\varphi^4)
\]

From the inequality (5.54), it follows that \(O(\varphi^4) = O\left(\frac{1}{q^2}\right)\). \(\varphi = d_{1,j} + \frac{u_1}{q} + O\left(\frac{1}{q^2}\right)\), so

\[
\frac{B(\varphi)}{\sin(\varphi)} = 1 + \frac{3d_{1,j}^2}{16} + \frac{3d_{1,j}u_1}{8} + O\left(\frac{1}{q^2}\right) = B_1^{(1)} + B_2^{(1)} + O\left(\frac{1}{q^2}\right),
\]

where

\[
B_1^{(1)} = 1 + \frac{3d_{1,j}^2}{16}, \quad B_2^{(1)} = \frac{3d_{1,j}u_1}{8}
\]

Similarly

\[
\frac{C(\varphi)}{\sin(\varphi)} = \frac{\sqrt{3}d_{1,j}^2}{16} + \frac{\sqrt{3}d_{1,j}u_1}{8} + O\left(\frac{1}{q^2}\right) = C_1^{(1)} + C_2^{(1)} + O\left(\frac{1}{q^2}\right),
\]

where

\[
C_1^{(1)} = \frac{\sqrt{3}d_{1,j}^2}{16}, \quad C_2^{(1)} = \frac{\sqrt{3}d_{1,j}u_1}{8}
\]

In 5a of the section 3, it was shown that

\[
c(\varphi) = \frac{1}{2} \varphi - \frac{1}{16} \varphi^3 + O(\varphi^5)
\]

\[
\varphi^3 = \left(d_{1,j} + \frac{u_1}{q} + \frac{u_2}{q^2} + O\left(\frac{1}{q^3}\right)\right)^3 = d_{1,j}^3 + 3d_{1,j}^2 \frac{u_1}{q} + 3d_{1,j} \frac{u_1^2}{q^2} + 3d_{1,j}^2 \frac{u_2}{q^4} + 3d_{1,j} \frac{u_2^2}{q^6} + 6d_{1,j} \frac{u_1 u_2}{q^3} + \frac{u_1^3}{q^2} + \frac{u_2^3}{q^6} + \frac{u_1^3}{q^3} + \frac{u_2^3}{q^2} + \frac{u_1^2}{q^4} + \frac{u_2^2}{q^4} + \frac{u_1}{q^5} + \frac{u_2}{q^5} + O\left(\frac{1}{q^7}\right)
\]

From the inequality (5.54), it follows that for a sufficiently large \(n\), on the right side of the equality (5.58), all the terms after the third one are \(O\left(\frac{1}{q^4}\right)\). Then

\[
2qa = d_{1,j}q + u_1 + (u_2 - \frac{1}{8}(d_{1,j}^2 q^2 + 3d_{1,j}^2 u_1 q + 3d_{1,j}u_1^2)) + \frac{1}{q} + O\left(\frac{1}{q^2}\right) = P_1^{(1)} + (u_2 + Q_2^{(1)}) + O\left(\frac{1}{q^2}\right),
\]
where
\[ P_a^{(1)} = d_{1,j}q + u_1, \quad Q_a^{(1)} = -\frac{1}{8}(d_{1,j}^2q^2 + 3d_{1,j}^2u_1q + 3d_{1,j}u_1^2). \]

In 5b of the section 3, it was shown that
\[ b(\varphi) = \frac{\sqrt{3}}{2}\varphi - \frac{\sqrt{3}}{48}\varphi^3 + O(\varphi^5) \]

Then from similar reasoning:
\[
2qb = P_b^{(1)} + (\sqrt{3}u_2 + Q_b^{(1)})\frac{1}{q} + O\left(\frac{1}{q^2}\right),
\]

where
\[ P_b^{(1)} = \sqrt{3}(d_{1,j}q + u_1), \quad Q_b^{(1)} = -\frac{\sqrt{3}}{24}(d_{1,j}^2q^2 + 3d_{1,j}^2u_1q + 3d_{1,j}u_1^2). \]

The numerator and denominator of the function \( f \) are divided by \( \cos P_b \) and expand each of the terms in a Taylor series to \( O\left(\frac{1}{q^2}\right) \)

\[
\frac{\sin(P_a^{(1)} + (Q_a^{(1)} + u_2)\frac{1}{q} + O(\frac{1}{q^2}))}{\cosh P_b^{(1)}} = \frac{\sin P_a^{(1)}}{\cosh P_b^{(1)}} + \frac{\cos P_a^{(1)} P_a^{(1)} + u_2}{\cosh P_b^{(1)} q} + O\left(\frac{1}{q^2}\right)
\]

\[
\frac{\sin(P_a^{(1)} + (Q_a^{(1)} + u_2)\frac{1}{q} + O(\frac{1}{q^2}))}{\cosh P_b^{(1)} B_1^{(1)}} = \frac{\sin P_a^{(1)}}{\cosh P_b^{(1)}} + \left(\frac{\sin P_a^{(1)}}{\cosh P_b^{(1)}} B_2^{(1)} + \frac{B_1^{(1)} Q_a^{(1)} \cos P_a^{(1)}}{\cosh P_b^{(1)}} + \frac{B_1^{(1)} u_2 \cos P_a^{(1)}}{\cosh P_b^{(1)}}\right)\frac{1}{q} + O\left(\frac{1}{q^2}\right)
\]

\[
\frac{\sinh(P_b^{(1)} + (Q_b^{(1)} + \sqrt{3}u_2)\frac{1}{q} + o(\frac{1}{q}))}{\cosh P_b^{(1)}} = \frac{\sinh P_b^{(1)}}{\cosh P_b^{(1)}} + \frac{Q_b^{(1)} + \sqrt{3}u_2}{q} + O\left(\frac{1}{q^2}\right)
\]

\[
\frac{\sinh(P_b^{(1)} + (Q_b^{(1)} + \sqrt{3}u_2)\frac{1}{q} + O(\frac{1}{q^2}))}{\cosh P_b^{(1)} C_1^{(1)}} = \frac{\sinh P_b^{(1)}}{\cosh P_b^{(1)}} + \left(\frac{\sinh P_b^{(1)}}{\cosh P_b^{(1)}} C_2^{(1)} + \frac{C_1^{(1)} Q_b^{(1)} + C_1^{(1)} \sqrt{3}u_2}{q}\right)\frac{1}{q} + O\left(\frac{1}{q^2}\right)
\]

Then the numerator of the function \( f \) will take the form:
\[
X_1^{(1)} + (X_2^{(1)} + X_3^{(1)} u_2)\frac{1}{q} + O\left(\frac{1}{q^2}\right)
\]
where

\[ X_1^{(1)} = 2 \left( \frac{\sin P_a^{(1)}}{\cosh P_b^{(1)}} B_1^{(1)} + \frac{\sinh P_b^{(1)}}{\cosh P_b^{(1)}} C_1^{(1)} \right) \]

\[ X_2^{(1)} = 2 \left( \frac{\sin P_a^{(1)}}{\cosh P_b^{(1)}} B_2^{(1)} + B_1^{(1)} \frac{\cos P_a^{(1)}}{\cosh P_b^{(1)}} + \frac{\sin P_b^{(1)}}{\cosh P_b^{(1)}} C_2^{(1)} + C_1^{(1)} Q_b^{(1)} \right) \]

\[ X_3^{(1)} = 2 \left( B_1^{(1)} \frac{\cos P_a^{(1)}}{\cosh P_b^{(1)}} + C_1^{(1)} \sqrt{3} \right) \]

Consider the denominator of the function \( f \).

\[ 1 + \frac{\cos(P_a^{(1)} + (Q_a^{(1)} + u_2)\frac{1}{q}) + O(\frac{1}{q^2})}{\cosh P_b^{(1)}} = 1 + \frac{\cos P_a^{(1)}}{\cosh P_b^{(1)}} - \frac{\sin P_a^{(1)}}{\cosh P_b^{(1)}} \frac{Q_a^{(1)} + u_2}{q} + O \left( \frac{1}{q^2} \right) = Y_1^{(1)} + (Y_2^{(1)} + Y_3^{(1)} u_2) \frac{1}{q} + O \left( \frac{1}{q^2} \right), \]

where

\[ Y_1^{(1)} = 1 + \frac{\cos P_a^{(1)}}{\cosh P_b^{(1)}} \]

\[ Y_2^{(1)} = -\frac{Q_a^{(1)} \sin P_a^{(1)}}{\cosh P_b^{(1)}} + \frac{Q_b^{(1)} \sinh P_b^{(1)}}{\cosh P_b^{(1)}} \]

\[ Y_3^{(1)} = -\frac{\sin P_a^{(1)}}{\cosh P_b^{(1)}} + \sqrt{3} \sinh P_b^{(1)} \]

Then the function \( f \) will take the form

\[ f = \frac{X_1^{(1)} + (X_2^{(1)} + X_3^{(1)} u_2)\frac{1}{q} + O(\frac{1}{q^2})}{Y_1^{(1)} + (Y_2^{(1)} + Y_3^{(1)} u_2)\frac{1}{q} + O(\frac{1}{q^2})} = Z_1^{(1)} + (Z_2^{(1)} + Z_3^{(1)} u_2) \frac{1}{q} + O \left( \frac{1}{q^2} \right) \]

where

\[ Z_1^{(1)} = \frac{X_1^{(1)}}{Y_1^{(1)}} \]

\[ Z_2^{(1)} = \frac{X_2^{(1)} Y_1^{(1)} - X_1^{(1)} Y_2^{(1)}}{(Y_1^{(1)})^2} \]

\[ Z_3^{(1)} = \frac{X_3^{(1)} Y_1^{(1)} - X_1^{(1)} Y_3^{(1)}}{(Y_1^{(1)})^2} \]

As a result, we have the equation:

\[ u_1 + \frac{u_2}{q} = \arctan \left( Z_1^{(1)} + (Z_2^{(1)} + Z_3^{(1)} u_2) \frac{1}{q} + O \left( \frac{1}{q^2} \right) \right) = \arctan Z_1^{(1)} + \frac{Z_2^{(1)} + Z_3^{(1)} u_2}{1 + (Z_1^{(1)})^2} \frac{1}{q} + O \left( \frac{1}{q^2} \right) \]

expand \( \arctan \) in a Taylor series:

\[ u_1 + \frac{u_2}{q} = \arctan Z_1^{(1)} + \frac{Z_2^{(1)} + Z_3^{(1)} u_2}{1 + (Z_1^{(1)})^2} \frac{1}{q} + O \left( \frac{1}{q^2} \right) \]

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then \( u_1 \) is defined from the equation

\[
u_1 = \arctan Z_1^{(1)}
\]

\[
u_2 = \frac{Z_2^{(1)}}{1 + (Z_1^{(1)})^2 - Z_3^{(1)}} = R^{(1)}(w_1)
\]

Consider the second equation. Let \( d_{2,j} = \frac{n^2 + \frac{n}{2}}{q} \), \( q = \frac{n+3}{2} \), \( w = w_1 + \frac{w_2}{q} \). We have the equation:

\[
w = \arctan (-h) \tag{5.60}
\]

where

\[
-h = 2 - B(\varphi) \sin (2qc) + C(\varphi) \sinh (2qb) \\
\sin(\varphi)(-\cos (2qc) + \cosh (2qb))
\]

Performing similar operations we get the following

\[
P_a^{(2)} = d_{2,j} q + w_1, \quad Q_a^{(2)} = -\frac{1}{8}(d_{2,j}^2 q^2 + 3d_{2,j}^2 w_1 q + 3d_{2,j} w_1^2).
\]

\[
P_b^{(2)} = \sqrt{3}(d_{2,j} q + w_1), \quad Q_b^{(1)} = -\frac{\sqrt{3}}{24}(d_{2,j}^2 q^2 + 3d_{2,j}^2 w_1 q + 3d_{2,j} w_1^2).
\]

\[
B_1^{(2)} = 1 + \frac{3d_{2,j}^2}{16}
\]

\[
B_2^{(2)} = \frac{3d_{2,j} w_1}{8}
\]

\[
C_1^{(2)} = 1 + \frac{\sqrt{3}d_{2,j}^2}{16}
\]

\[
C_2^{(2)} = \frac{\sqrt{3}d_{2,j} w_1}{8}
\]

\[
X_1^{(2)} = 2 \left( -\frac{\sin P_a^{(2)}}{\cosh P_b^{(2)}} B_1^{(2)} + \frac{\sinh P_b^{(2)}}{\cosh P_b^{(2)}} C_1^{(2)} \right)
\]

\[
X_2^{(2)} = 2 \left( -\frac{\sin P_a^{(2)}}{\cosh P_b^{(2)}} B_2^{(2)} - B_1^{(2)} \frac{Q_a^{(2)} \cos P_a^{(2)}}{\cosh P_b^{(2)}} + \frac{\sinh P_b^{(2)}}{\cosh P_b^{(2)}} C_2^{(2)} + C_1^{(2)} Q_b^{(2)} \right)
\]

\[
X_3^{(2)} = 2 \left( -B_1^{(2)} \frac{\cos P_a^{(2)}}{\cosh P_b^{(2)}} + C_1^{(2)} \sqrt{3} \right)
\]

\[
Y_1 = 1 - \frac{\cos P_a^{(2)}}{\cosh P_b^{(2)}}
\]

\[
Y_2^{(2)} = \frac{Q_a^{(2)} \sin P_a^{(2)}}{\cosh P_b^{(2)}} + \frac{Q_b^{(2)} \sinh P_b^{(2)}}{\cosh P_b^{(2)}}
\]

\[
Y_3^{(2)} = \frac{\sin P_a^{(2)}}{\cosh P_b^{(2)}} + \frac{\sqrt{3} \sinh P_b^{(2)}}{\cosh P_b^{(2)}}
\]
then
\[-h = \arctan \left( \frac{X_1^{(2)} + (X_2^{(2)} + X_3^{(2)} w_2) \frac{1}{q} + o(\frac{1}{q})}{Y_1^{(2)} + (Y_2^{(2)} + Y_3^{(2)} w_2) \frac{1}{q} + o(\frac{1}{q})} \right) = Z_1^{(2)} + (Z_2^{(2)} + Z_3^{(2)} u_2) \frac{1}{q} + O \left( \frac{1}{q^2} \right) \]

and we get similar expressions for the second equation:
\[w_1 = \arctan Z_1^{(2)},\]
\[w_2 = \frac{Z_2^{(2)}}{1 + (Z_1^{(2)})^2 - Z_3^{(2)}} = R^{(2)}(w_1).\]

\[\boxdot\]

**Proof of theorem 2.5.**

\[\lambda_{2j-1}^{(n)} = g(\varphi_{2j-1}^{(n)}) = g \left( d_{1,j} + \frac{2u_1^*}{n + 3} + \frac{4u_2^*}{(n + 3)^2} + O \left( \frac{1}{n^3} \right) \right)\]
\[= g(d_{1,j}) + g'(d_{1,j}) \left( \frac{2u_1^*}{n + 3} + \frac{4u_2^*}{(n + 3)^2} + O \left( \frac{1}{n^3} \right) \right)\]
\[+ \frac{1}{2} g''(d_{1,j}) \left( \frac{2u_1^*}{n + 3} + \frac{4u_2^*}{(n + 3)^2} + O \left( \frac{1}{n^3} \right) \right)^2 + O \left( \frac{1}{n^3} \right)\]

Expanding the brackets and leaving the terms of order no more than \(O \left( \frac{1}{n^3} \right)\), we get the statement of the first part of the theorem. The second part is proved similarly. \(\boxdot\)

**Proof of theorem 2.6.** We know that \(\lambda_j^{(n)} = g(\varphi_j^{(n)})\) for all \(j\) and \(n\).

\(i\) Given that \(q = \frac{n+3}{2}\) and \(\varphi_{2j-1} = d_{1,j} + \frac{u_1^*}{q} + \frac{u_2^*}{q^2} + O \left( \frac{1}{q^3} \right)\) we get

\[\lambda_{2j-1}^{(n)} = m \sin^6 \left( \frac{d_{1,j}}{2} + \frac{u_1^*}{2q} + \frac{u_2^*}{2q^2} + O \left( \frac{1}{q^3} \right) \right) \quad (q \to \infty).\]

Let \(C_j := 1 + \frac{u_1^*}{q_j}\). Since \(\sin x = x - \frac{1}{6} x^3 + O(x^5), \ x \to 0\), a simple calculation shows that

\[\lambda_{2j-1}^{(n)} = \frac{m C_j^6 d_{1,j}^6}{2^6} \left[ 1 + \frac{u_1^*}{d_{1,j} C_j q^2} + O \left( \frac{1}{d_{1,j} q^3} \right) \right]
\[= - \frac{d_{1,j} C_j^2}{24} \left( 1 + \frac{u_1^*}{d_{1,j} C_j q^2} + O \left( \frac{1}{d_{1,j} q^3} \right) \right)^3 + O \left( d_{1,j}^4 \right) \right] \]
\[= - C_j^6 d_{1,j}^6 \left[ 1 + \frac{u_2^*}{d_{1,j} C_j q^2} + O \left( \frac{1}{d_{1,j} q^3} \right) + O \left( d_{1,j}^4 \right) \right] \]
\[= - C_j^6 d_{1,j}^6 \left[ 1 + \frac{6u_2^*}{d_{1,j} C_j q^2} + O \left( \frac{1}{d_{1,j} q^3} \right) + O \left( d_{1,j}^4 \right) \right] \]
\[= - C_j^6 d_{1,j}^6 \left( \frac{6u_2^* C_j^5 d_{1,j}}{q^2} + O \left( \frac{d_{1,j}}{q^3} \right) + O \left( d_{1,j}^10 \right) \right) \quad (q \to \infty).\]

Finally, taking into account that \(C_j = 1 + \frac{u_1^*}{q_j}\) and \(q = \frac{n+3}{2}\), we obtain

\[\lambda_{2j-1}^{(n)} = \left( \frac{2\pi j + 2u_1^*}{n + 3} \right)^6 - \frac{24u_2^* (2\pi j + 2u_1^*)^5}{(n + 3)^7} + \Delta_1(n, j) \quad (n \to \infty).\]
ii) In a similar fashion, given that \( \lambda_{2j}^{(n)} = g(\varphi_{2j}^{(n)}) \) it is possible to deduce that

\[
\lambda_{2j}^{(n)} = -\frac{(2j + 1)\pi + 2w_1^*}{(n + 3)^6} - \frac{24w_2^*((2j + 1)\pi + 2w_1^*)^5}{(n + 3)^7} + \Delta_2(n, j), \quad (n \to \infty).
\]

Now, we prove the equivalence between the asymptotic formula presented in [18] with our result in Theorem 2.6.

In [18] the author considered the class of functions \( g \) satisfying:

(a) \( g \) is real, continuous, and periodic with period \( 2\pi \); \( \min g = g(0) = m^* \) and \( \varphi = 0 \) is the only value of \( \varphi \) \((\mod 2\pi)\) for which this minimum is attained.

(b) If \( g \) satisfies (a), then it has continuous derivatives of order \( 2k \) \((k \in \mathbb{N})\) in some neighborhood of \( \varphi = 0 \) and \( g^{(2k)}(0) = \sigma^2 > 0 \) is the first non-vanishing derivative of \( g \) at \( \varphi = 0 \).

**Theorem 5.1** ([18] Theorem 4). Let \( g \) be a function which satisfies Conditions (a) and (b). Let \( \lambda_{1,n} \) be the minimal eigenvalue of \( T_n(a) \). Then

\[
\lambda_{1,n} = m^* + O(n^{-2k}),
\]

where \( O \) cannot be replaced by \( o \).

Let \( g_1(\varphi) = -g(\varphi) = (2\sin \frac{\varphi}{2})^6 \). Notice that \( g_1 \) satisfies Conditions (a) and (b) with \( m^* = 0 \) and \( g^{(6)}(0) = 720 > 0 \). Therefore, from Theorem 5.1 we get

\[
\lambda_{1,n} = m^* + O \left( \frac{1}{n^6} \right) \quad (n \to \infty). \quad (5.61)
\]

On the other hand, if \( j = 1 \) in Theorem 2.6, then we easily obtain the following asymptotic expansion

\[
\lambda_1^{(n)} = m^* + O \left( \frac{1}{n^6} \right) \quad (n \to \infty),
\]

which coincides with Eq. (5.61).

### 6 Numerical experiments

The experiments in this section will be carried out using the Maple mathematical package. The first graph 6.1 shows the dependence of the relative error of the eigenvalue on the iteration number in the formulas of the theorem 2.2. The size of the matrix is 200 \( \times \) 200, the error is calculated for the first, average and last eigenvalues (ordered by modulus). Here \( k \) is the number of iterations and \( m \) is the number of the eigenvalue. The same results are shown in the table 1

Table 2 shows the maximum relative errors when using the formula from Theorem 2.3. The maximum was considered for all eigenvalues starting from the seventh. In Table 3 and Table 3 \( n \) is matrix size.

Table 3 shows the maximum relative deviations when using the formulas from Theorem 2.4. To find \( u^*_1 \) and \( w^1_1 \) a recursive formula was used. The number of iterations was taken equal to four. The maximum was found for the first six eigenvalues.
Figure 6.1: Dependence of the error on the number of iterations

Table 1: Dependence of the error on the number of iterations

| k  | 1            | 2            | 3            | 4            | 5            |
|----|--------------|--------------|--------------|--------------|--------------|
| δΛ/λ | 1.33 \cdot 10^{-1} | 1.9 \cdot 10^{-5} | 2.55 \cdot 10^{-6} | 3.31 \cdot 10^{-7} | 4.21 \cdot 10^{-8} |
| m=100 | 8.28 \cdot 10^{-5} | 4.7 \cdot 10^{-7} | 2.66 \cdot 10^{-9} | 1.51 \cdot 10^{-11} | 8.55 \cdot 10^{-14} |
| m=200 | 2.61 \cdot 10^{-6} | 1.88 \cdot 10^{-8} | 1.36 \cdot 10^{-10} | 9.28 \cdot 10^{-13} | 7.1 \cdot 10^{-15} |
| k  | 6            | 7            | 8            | 9            | 10           |
| δΛ/λ | 1.11 \cdot 10^{-14} | 1.92 \cdot 10^{-16} | 3.33 \cdot 10^{-18} | 5.78 \cdot 10^{-20} | 1 \cdot 10^{-21} |
| m=100 | 4.85 \cdot 10^{-16} | 2.75 \cdot 10^{-18} | 1.56 \cdot 10^{-20} | 8.83 \cdot 10^{-23} | 5.01 \cdot 10^{-25} |
| m=200 | 5.13 \cdot 10^{-17} | 3.71 \cdot 10^{-19} | 2.68 \cdot 10^{-21} | 1.94 \cdot 10^{-23} | 1.4 \cdot 10^{-25} |
### Table 2: Maximum relative error when using the formula from Theorem 2.3

| n   | 32     | 64     | 128    | 256    | 512    |
|-----|--------|--------|--------|--------|--------|
| \( \Delta \) \( \lambda \) | \( 1.33 \cdot 10^{-4} \) | \( 1.9 \cdot 10^{-5} \) | \( 2.55 \cdot 10^{-6} \) | \( 3.31 \cdot 10^{-7} \) | \( 4.21 \cdot 10^{-8} \) |

### Table 3: Maximum relative error when using the formula from Theorem 2.4

| n   | 32     | 64     | 128    | 256    | 512    |
|-----|--------|--------|--------|--------|--------|
| \( \Delta \) \( \lambda \) | \( 4.17 \cdot 10^{-4} \) | \( 2.94 \cdot 10^{-5} \) | \( 1.97 \cdot 10^{-6} \) | \( 1.28 \cdot 10^{-7} \) | \( 8.19 \cdot 10^{-9} \) |

### 7 Acknowledgment

This work is funded by RSCF-21-11-00283

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