Universality and Scaling
in Short-time Critical Dynamics

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Abstract

Numerically we simulate the short-time behaviour of the critical dynamics for the two dimensional Ising model and Potts model with an initial state of very high temperature and small magnetization. Critical initial increase of the magnetization is observed. The new dynamic critical exponent $\theta$ as well as the exponents $z$ and $2\beta/\nu$ are determined from the power law behaviour of the magnetization, auto-correlation and the second moment. Furthermore the calculation has been carried out with both Heat-bath and Metropolis algorithms. All the results are consistent and therefore universality and scaling are confirmed.
1 Introduction

For statistical systems at criticality in equilibrium or near equilibrium it is well known that the critical behaviour is characterized by universality and scaling. This is more or less due to the infinite spatial and time correlation lengths. It has long been challenging whether universality and scaling may also be present for the systems far from equilibrium or not. An interesting non-equilibrium system is the so-called ordering dynamic process: A statistical system initially at a high temperature state is suddenly quenched to the critical temperature or below where locates the symmetry broken phase, and then evolves according to a certain dynamics. Domain growth in this process has been studied since thirty years [1]. In the case of a system quenched to the neighbourhood of the critical temperature, recent investigations show that universality and scaling may appear in a quite early stage of the time evolution where the system is still far from equilibrium and the spatial correlation length is small. Essential here is the large time correlation length which induces a memory effect.

Let us consider that the Ising model initially in a random state with a small magnetization is suddenly quenched to the critical temperature and then evolves according to a dynamics of model A. Janssen, Schaub and Schmittmann [2] have argued by an \( \epsilon \)-expansion up to two–loop order that, besides the well known universal behaviour in the long-time regime, there exists another universal stage of the relaxation at early times, the so-called critical initial slip, which sets in right after the microscopic time scale \( t_{\text{mic}} \). The authors predicted that the magnetization undergoes a critical initial increase, and introduced a new dynamic critical exponent \( \theta \) to describe the power law increase of the magnetization. Previously \( \theta \) has been measured with Monte Carlo simulation in two dimensions somehow indirectly from the power law decay of the autocorrelation [3, 4], and recently in three dimensions directly from the power law increase of the magnetization [5]. The exponents \( \theta \) for two and three dimensions have also been measured from damage spreading [6]. They are in good agreement with the result from the \( \epsilon \)-expansion. Detailed scaling analysis reveals that the characteristic time scale for the critical initial slip is \( t_0 \sim m_0^{-z/x_0} \), where \( x_0 \) is the scaling dimension of \( m_0 \), related to \( \theta \) by \( x_0 = \theta z + \beta/\nu \). More interestingly, it was pointed out that the exponents \( \beta, \nu \) and \( z \) should be valued the same as those in the equilibrium or long-time stage of the relaxation. Therefore, based on the scaling relation in the initial stage of the time evolution, a new promising way for measuring the exponents \( z, \beta \) and \( \nu \) has been proposed [7, 8, 9]. In reference [8, 9] the investigation has also been extended to the critical relaxation starting from a completely ordered state. This indicates a possible broad application of the short-time dynamics since the universal behaviour of the short-time dynamics is found to be quite general [10, 11, 12, 13, 14, 15]. More and deeper understanding of this phenomenon becomes urgent.

Even though analytically the short-time dynamics has been studied in a variety of systems, most of the numerical simulations are limited to the Ising model.
In a previous paper [16], two of the authors have systematically extended the numerical investigation of the dynamic process discussed above to the critical three state Potts model in two dimension with Monte Carlo Heat-bath algorithm. They observed the critical initial increase of the magnetization and have directly measured the critical exponent $\theta$. The result shows that the direct measurement of $\theta$ is better and more reliable than that indirectly measured from the power law decay of the auto-correlation, especially when $\theta$ is smaller. With $\theta$ in hand, the exponent $z$ can be obtained from the power law decay of the auto-correlation, and then the exponent $\beta/\nu$ from the power law increase of the second moment of the magnetization. They are in good agreement with the known values. Such a procedure relates the direct and indirect measurements of $\theta$ to each other and provides a strong support for the scaling of the short-time dynamics for the Potts model. On the other hand, it also indicates an efficient way to estimate the critical exponent $z$ which normally is quite difficult to be measured due to the critical slowing down. For the Potts model, among the distributed values of $z$ from different numerical simulations [17, 18, 19, 20], the result confidently supports the relative small one [17, 18]. Compared with the method from the finite size scaling, its advantage is that the measurement from the power law behaviour is more direct and can be carried out in a single lattice.

The first purpose of this paper is to present a detailed description of the results for the critical Potts model reported briefly in the previous paper [16]. The results have been refined by taking the effect of $t_{mic}$ into account, whose importance will be clarified in the investigation of the universality described below. Furthermore the data have been extended to bigger lattice $L = 576$ for the simulation of $m_0 = 0.0$ and smaller initial magnetization $m_0 = 0.02$ for the measurement of $\theta$ in order to clarify some unclear points.

The second purpose of this paper is to investigate the universality for short-time dynamics. We perform all of the simulations both with Heat-bath and Metropolis algorithms. We numerically observe the microscopic time scale $t_{mic}$, after which the universality will be switched on. Depending on what kind of dynamics one takes, $t_{mic}$ is different. Such a knowledge becomes very important in the measurement of the universal critical exponents. In this paper we have also presented a systematic investigation of the two dimensional Ising model (two state Potts model). Even though there exist already several simulations for the Ising model, up to now no serious confirmation of the universality was presented in the sense of numerical simulation, especially for the direct measurement of $\theta$. In reference [21], the auto-correlation for the Ising model was calculated with both Heat-bath and Metropolis algorithms. However the results were rather rough. Our results complete the systematic simulation of the short-time dynamics for the Ising model with better statistics.

In section 2, we will briefly recapitulate the scaling behaviour for the short-time dynamics, which will serve as the theoretical base of the numerical simulation. In section 3, we will describe the result of a simulation for the Ising model in two dimensions, concentrating our attention to the universality. In section 4,
the results for the three state Potts model will be given. Conclusion and some remarks are given in section 5.

2 Scaling for the short-time dynamics

Let us consider a dynamic system of model A. Janssen, Schaub and Schmittmann have shown [2] that when a system initially in a state with very high temperature \( T \gg T_c \) is suddenly quenched to the critical temperature and then evolves according to a certain dynamics, besides the well-known universal behaviour in the long-time regime, there emerges another universal stage of the dynamic relaxation at the macroscopic short-time regime, which sets in right after a microscopic time scale \( t_{\text{mic}} \). For the \( O(N) \) vector model the renormalization of the Langevin dynamics with initial conditions has been investigated with \( \epsilon \)-expansion up to two loop. An interesting observation is that a new divergence is induced in the short-time dynamics which should be renormalized by the initial magnetization. Taking this point into account, a generalized dynamic scaling relation has been derived,

\[
M^{(k)}(t, \tau, m_0) = b^{-k\beta/\nu} M^{(k)}(b^{-z} t, b^{1/\nu} \tau, b^{x_0} m_0),
\]

where \( M^{(k)} \) is \( k \)-th moment of the magnetization, \( t \) is the dynamic relaxation time, \( \tau \) is the reduced temperature, the parameter \( b \) represents the spatial rescaling factor and in addition \( x_0 \) is the anomalous dimension of the initial magnetization \( m_0 \). It is shown that \( x_0 \) is a new independent exponent, i.e. it can not be expressed by other known critical exponents.

As an example, let us now consider the time evolution of the magnetization in the initial stage of the dynamic relaxation. From above scaling relation (1), taking \( \tau = 0 \) and \( b = t^{1/z} \),

\[
M(t, m_0) = t^{-\beta/\nu z} M(1, t^{x_0/z} m_0).
\]

In our notation \( M \equiv M^{(1)} \) and the argument \( \tau \) has been omitted. Assuming the initial magnetization \( m_0 \) is small, the time evolution of the magnetization may be expanded according to \( m_0 \)

\[
M(t, m_0) = m_0 F(t) + O(m_0^2).
\]

Here the condition \( M(t, m_0 = 0) \equiv 0 \) has been used. From equations (2) and (3) one can easily realize that the time evolution of the magnetization obeys a power law

\[
M(t) \sim m_0 t^\theta
\]

where the exponent \( \theta \) which we name ‘short-time dynamic exponent’, is related to \( x_0 \) by

\[
\theta = (x_0 - \beta/\nu)/z.
\]
Here we should stress that the power law behaviour is valid only in case that \( t^{x_0/z}m_0 \) is also small enough. Therefore the universal behaviour shown in (4) is expected in the initial stage of the time evolution. The time scale for it is \( t_0 \sim m_0^{-z/x_0} \). However, in the limit of \( m_0 = 0 \) the time scale \( t_0 \) goes to infinity. Hence the initial condition can leave its trace even in the long-time regime \([22, 13, 23]\). Interestingly, for the \( O(N) \) vector model (\( n = 1 \) corresponds to the Ising model) it is shown by \( \epsilon \)-expansion that \( x_0 > \beta/\nu \) and therefore \( \theta > 0 \), i.e. the magnetization really undergoes an initial increase. This has also been confirmed by the numerical simulation for the three dimensional Ising model directly and also by the study of the damage spreading [6].

As the spatial correlation length in the beginning of the time evolution is small, for a finite system of dimension \( d \) with lattice size \( L \) the second moment \( M^{(2)}(t, L) \sim L^{-d} \). From the finite size scaling one can deduce

\[
M^{(2)}(t) \sim t^{(d-2\beta/\nu)/z}. \tag{6}
\]

Furthermore careful scaling analysis shows that auto-correlation also decays by a power law [10]

\[
A(t) \sim t^{-d/z + \theta}. \tag{7}
\]

The new short-time dynamic exponent \( \theta \) enters the auto-correlation. Actually the first numerical estimation of \( \theta \) is from the measurement of the exponent \( \theta - d/z \) [4, 5]. Taking the exponent \( z \) as an input, one obtains \( \theta \). However, usually \( z \) is not known so accurately. Since \( \theta \) is normally much smaller than \( z \) as well as \( -d/z + \theta \), a small relative error of \( z \) and \( -d/z + \theta \) may induce a big error for \( \theta \).

Our strategy is that we first measure the exponent \( \theta \) directly from the power law increase of the magnetization, then taking it as an input we estimate the exponent \( z \) from the auto-correlation, and with \( z \) in hand we finally obtain the static exponent \( 2\beta/\nu \) from the second moment. Such a procedure can provide strong confirmation for the scaling relation for the short-time dynamics.

Traditionally the exponent \( z \) is defined in the long-time regime of the dynamic process and normally measured from the exponential decay of the auto-correlation or the magnetization of the systems. This measurement is difficult due to the critical slowing down. However, if we can obtain \( \theta \) from the direct measurement of the initial increase of the magnetization (4), \( z \) obtained from the scaling behaviour of the auto-correlation in (6) can be quite rigorous. One may also expect that the measurement is to some extent free from the critical slowing down, since all of these quantities are measured in the short-time regime of the dynamic process.
3 The Ising model

The Hamiltonian for the Ising model is

\[ H = J \sum_{<ij>} S_i S_j , \quad S_i = \pm 1 , \]  

(8)

with \( <ij> \) representing nearest neighbours. In the equilibrium the Ising model is exactly solvable. The critical point locates at \( J_c = \log(1 + \sqrt{2})/2 \). In principle any type of the dynamics can be given to the system to study the non-equilibrium evolution processes. In this paper we concentrate our attention on the Monte Carlo Heat-bath and Metropolis algorithm, both of which belong to the dynamics of model A.

3.1 Magnetization

As discussed in section 2, we study the short-time behaviour of the dynamic process starting from an initial state with zero correlation length and small magnetization. Such initial configurations can easily be generated numerically. Starting from those initial configurations, the system is updated both with the Heat-bath and Metropolis algorithm, in order to confirm the universality. We measure the time evolution of the magnetization

\[ M(t) = \frac{1}{N} \langle \sum_i S_i(t) \rangle . \]  

(9)

where \( N \) is the number of the lattice sites and the average \( \langle \cdots \rangle \) is taken over the independent initial configurations and the random force. The total number of the independent initial configurations is 150 000 for \( m_0 = 0.08, 0.06 \) and 0.04 and 300 000 for \( m_0 = 0.02 \). Errors are estimated by dividing the data into five groups.

In Fig. 1a and b the time evolution of the magnetization in double log scale with \( m_0 = 0.02 \) for different lattice sizes for both the Heat-bath and Metropolis algorithm, respectively, is displayed. For the Heat-bath algorithm, one can clearly see the initial power law increase of the magnetization from a very early stage of the time evolution, i.e. the microscopic time scale \( t_{mic} \) is ignorable small. On the other hand, for the Metropolis algorithm, this is not the case in the very beginning of the time evolution. The power law increase of the magnetization becomes stable only after certain time steps, say \( t \sim 20 \) to 30. In other words, for the Metropolis algorithm \( t_{mic} \sim 20 \) to 30, which is bigger than that for Heat-bath algorithm. In order to see this more clearly, we plot in Fig. 2 the exponent \( \theta \) as a function of the time \( t \) for both Heat-bath and Metropolis algorithms for lattice size \( L = 128 \) and different initial magnetization \( m_0 \). Here, \( \theta \) at time \( t \) is measured from the slope of the curves by the least square fit in the time interval of \([t, t + 15]\).
As expected, the exponent $\theta$ for the Heat-bath algorithm is quite stable from the very beginning of the time evolution but not that for Metropolis. Taking into account the errors as well as the fluctuation in time direction, however, the exponents $\theta$ from both algorithms become the same after some time steps around $t \sim 30$. This is a real indication of the universality in the short-time dynamics. From the above procedure we may summarize a criterion to measure $\theta$ and also other exponents discussed later:

(i) We first scan the data by the exponent measured at each time $t$ by least square fitting in the time interval of $[t, t + 15]$. We call this ‘15-scan’ in the following. Of course, the number of data for the least square fit can differ from 15 which is used here.

(ii) Using the figure obtained from the 15-scan, we can estimate $t_{\text{mic}}$ from which the exponent becomes stable. If we perform a simulation with different algorithms, we can compare these results and see that the universality is switched on after the microscopic time $t_{\text{mic}}$.

(iii) Finally we perform the least square fit in the time interval of $[t_{\text{mic}}, T]$ to obtain the final values for the exponents. Here $T$ can normally be the maximum updation time where finite size effects and the finite $m_0$ effect have not shown up. But sometimes we may take a bit smaller $T$ in order to escape the big fluctuation due to the lack of statistics. This can be judged by an inspection of the result of the 15-scan.

Coming back to Fig.2, one may observe a slight tendency that $t_{\text{mic}}$ decreases as $m_0$ gets smaller. We should stress that the errors estimated here can not completely represent the fluctuations in the time direction due to the large time correlation length and also other systematic errors, e.g. those from the random numbers. In table 1, results for $\theta$ measured from a time interval $[30, 100]$ are listed. A detailed analysis of the data reveals that the finite size effect is quite small for a lattice size $L = 128$.

From $m_0 = 0.08$ down to $m_0 = 0.02$ the measured $\theta$ shows a smooth linear increase. By definition $\theta$ should be measured in the limit of $m_0 = 0$. Following the procedure in the previous paper [16] we have carried out a linear extrapolation to $m_0 = 0$ and listed the results also in the table 1. The value $\theta = 0.191(1)$ from the Heat-bath algorithm is well consistent with the value $\theta = 0.191(3)$ obtained from damage spreading [4] and those obtained from auto-correlation before [3, 4]. For Metropolis algorithm, our first direct measurement $\theta = 0.197(1)$ is very close to that for Heat-bath and gives strong support for universality. The slight difference of $\theta$ for Heat Bath and Metropolis algorithms may come from the remaining of the finite size effect, finite $m_0$ effects or other systematic errors. To check this point bigger simulation with high statistics for smaller $m_0$ and bigger lattice size or even for longer updation time may be needed.
### Table 1: The short-time dynamic exponent $\theta$ measured for lattice size $L = 128$

| $m_0$ | 0.08  | 0.06  | 0.04  | 0.02  | 0.00  |
|-------|-------|-------|-------|-------|-------|
| HeatB | 0.173(1) | 0.179(1) | 0.183(1) | 0.187(1) | 0.191(1) |
| MetroP | 0.173(1) | 0.182(1) | 0.187(1) | 0.192(1) | 0.197(1) |

with different initial magnetization for the Ising model.

#### 3.2 Auto-correlation

Now we set $m_0 = 0$. The auto-correlation is defined as

$$ A(t) = \frac{1}{N} \langle \sum_i S_i(0)S_i(t) \rangle, $$

where $A(t)$ is the auto-correlation function.

We have performed the simulation with both the Heat Bath and Metropolis algorithm for lattice sizes $L = 256$. The number of independent initial configurations for the average is 35 000.

In order to see the universality and the possible effect of $t_{\text{mic}}$ as well as the fluctuation in the time direction, we again perform the 15-scan and display in Fig. 3 the exponent $-d/z + \theta$ as a function of time $t$ for both Heat-bath and Metropolis algorithm.

After a microscopic time scale $t_{\text{mic}} \sim 30$, the results from both algorithms agree well and are presenting a quite stable power law behaviour. This again supports the universality. Within the errors both algorithms give almost the same results. However, the error for Metropolis is much bigger than that for the Heat-bath algorithm.

In table 2 the exponents $-d/z + \theta$ measured from the time interval $[30, 100]$ are listed. They are consistent with the previous measurement [21], but the errors are much smaller. The dynamical exponent $z$ obtained by taking $\theta$ measured in the previous subsection as input are also given in table 2. For the Heat-bath algorithm, the value $z = 2.155(3)$ is in good agreement with $z = 2.153(2)$ measured from the finite size scaling of the Binder cumulant [8]. The value $z = 2.137(11)$ for Metropolis algorithm is slightly smaller but roughly consistent within the errors. Actually in reference [8], depending on the observables and the dynamic processes used for estimating $z$, the values for $z$ from the finite size scaling are also varying within 1%. To get more accurate $z$ still requires high precision numerical measurement.
\[ \theta - d/z \] 
\[ z \] 
\[ (d - 2\beta/\nu)/z \] 
\[ 2\beta/\nu \] 

| Method    | \[ \theta - d/z \] | \[ z \] | \[ (d - 2\beta/\nu)/z \] | \[ 2\beta/\nu \] |
|-----------|---------------------|--------|--------------------------|----------------------|
| HeatB     | 0.737(1)            | 2.155(03) | 0.817(7)     | 0.240(15)             |
| MetroP    | 0.739(5)            | 2.137(11) | 0.819(5)     | 0.250(14)             |

Table 2: The exponents measured for lattice size \( L = 256 \) with initial magnetization \( m_0 = 0.0 \) for the Ising model.

### 3.3 Second moment

For \( m_0 = 0 \), the second moment is defined by

\[
M^{(2)}(t) = \frac{1}{N^2} \left( \sum_i S_i(t) \right)^2. \tag{11}
\]

For the Heat-bath algorithm the exponent \( (d - 2\beta/\nu)/z \) is quite stable after \( t_{mic} \sim 20 \) to 30. However, for the Metropolis algorithm, \( t_{mic} \) seems to be somewhat bigger, \( t_{mic} \sim 60 \). In order to obtain more reliable results, we have extended the number of time steps for the Metropolis algorithm up to 150. In table 2 the measured values for \( (d - 2\beta/\nu)/z \) together with the exponent \( 2\beta/\nu \) deduced by taking the value of \( z \) from the previous subsection as input are given. All the results are consistent and confirm universality.

We should point out that the determination of the exponent \( 2\beta/\nu \) is not very accurate here since the exponent \( 2\beta/\nu \) is much smaller than \( d \) and \( z \), and therefore small relative errors in \( z \) and \( (d - 2\beta/\nu)/z \) will induce big errors for \( 2\beta/\nu \).

In this section we have investigated the universality in short-time dynamics for the Ising model. We have numerically observed the existence of the microscopic time scale \( t_{mic} \), after which the universality will be switched on. Depending on the algorithm, Metropolis or Heat-bath, \( t_{mic} \) is different. Such a knowledge is very important in the measurement of the critical exponents in short-time dynamics. In the next section we perform a detailed numerical simulation for the Potts model, taking this knowledge into account carefully.

### 4 The Potts model

The Hamiltonian for the \( q \) state Potts model is given by

\[
H = J \sum_{<ij>} \delta_{\sigma_i,\sigma_j}, \quad \sigma_i = 1, \ldots, q \tag{12}
\]

where \( <ij> \) represents nearest neighbours. It is known that the critical points locate at \( J_c = \log(1 + \sqrt{q}) \). The Ising model is the two state \((q = 2)\) Potts
model. In this section, we investigate the three state \((q = 3)\) Potts model in two dimensions.

### 4.1 Magnetization

We measure the time evolution of the magnetization defined as

\[
M(t) = \frac{3}{2N} \left\langle \sum_i \left( \delta_{\sigma_i(t),1} - \frac{1}{3} \right) \right\rangle.
\]

The total number of the independent initial configurations used for taking the average is 80 000 for \(m_0 = 0.06\) and 0.08 and 600 000 for \(m_0 = 0.02\) and 0.04. Similar to the case of the Ising model, errors are estimated by dividing the data into two or four groups.

In Fig. 4 the time evolution of the magnetization in double log scale with initial value \(m_0 = 0.04\) for different lattices and for both the Heat-bath and Metropolis algorithm is displayed. Somewhat different from the case of the Ising model, the outlooks of the curves from the Heat-bath and Metropolis algorithms appear very different. For the curves from Heat-bath algorithm the power law behaviour starts right at the very beginning of the time evolution. This means the microscopic time scale \(t_{mic}\) is small and ignorable. However for the Metropolis algorithm they first decrease and then increase after some time steps. Later analyses show that the power law behaviour becomes stable only after around 20 updation time steps, i.e. \(t_{mic} \sim 20\).

In the previous paper [16], the simulation was carried out using only the Heat-bath algorithm. Due to the fact that the error at the beginning of the time evolution is the smallest, we simply measured the exponent from the first 15 time steps. However, the result of the simulation for the Metropolis algorithm indicates that we should carefully analyse the data, with special attention to the effect of \(t_{mic}\). We show in Fig. 5 the exponent \(\theta\) vs. \(t\), obtained by the 15-scan for both the Heat-bath and Metropolis algorithm with initial magnetization \(m_0 = 0.04\). It is clear that \(\theta\) from the Heat-bath algorithm is quite stable from the very beginning of the time evolution but for the Metropolis this is apparently not the case. However, as was the case of the Ising model, the exponent \(\theta\) from both the Heat-bath and Metropolis algorithm coincide after some time steps around \(t_{mic} \sim 20\), showing again universality in short-time dynamics.

In table 3 the values of \(\theta\) measured from the time interval \([20, 100]\) are listed. In principle for the Heat-bath algorithm the measurement may be carried out from the beginning. However for the reason of comparison we like to treat both algorithms the same. If we compare the results for the Heat-bath algorithm obtained in this paper with those measured from the first fifteen time steps in the previous paper [14], they are quite near.

In order to see the finite size effect, we have plotted in Fig. 6 the results of lattices \(L = 72\) and \(L = 144\) with the initial magnetization \(m_0 = 0.02\) for the
Heat-bath algorithm. Within the errors they are overlapping. Therefore we are satisfied with the lattice size $L = 72$ for the measurement of $\theta$. In table 3 the averaged values of $\theta$ for both algorithms still show some slight difference even though for smaller $m_0$ it looks as if they can be covered by the errors. Similar to the case of the Ising model this may be the remnant of the finite size or finite $m_0$ effects, or other systematic errors.

From $m_0 = 0.08$ down to $m_0 = 0.02$ the measured $\theta$ shows also a smooth linear decrease. Therefore we carry out a linear extrapolation for $\theta$ to the initial magnetization $m_0 = 0$. Since here we have measured $\theta$ in a different time regime, the result given in the previous paper [16] for $\theta$ has slightly been modified. Due to the extra data for $m_0 = 0.02$ the result extrapolated to $m_0 = 0$ is more reliable.

| $m_0$  | 0.08 | 0.06 | 0.04 | 0.02 | 0.00 |
|--------|------|------|------|------|------|
| HeatB  | 0.110(1) | 0.100(1) | 0.092(2) | 0.084(3) | 0.075(3) |
| MetroP | 0.100(1) | 0.092(1) | 0.084(1) | 0.077(2) | 0.070(2) |

Table 3: The short-time dynamic exponent $\theta$ measured for lattice size $L = 72$ with different initial magnetization for the Potts model.

4.2 Auto-correlation

The auto-correlation for the Potts model is given by

$$A(t) = \frac{1}{N} \langle \sum_i \left( \delta_{\sigma_i(0), \sigma_i(t)} - \frac{1}{3} \right) \rangle.$$

In Fig. 7 the auto-correlation for different lattice sizes for both algorithms is displayed in double log scale. Although in case of the magnetization already for a lattice size $L = 72$ a nice power law increase is already observed, for the auto-correlation the lattice size $L = 72$ is not big enough to present the power law behaviour. The convergence to a power law decay only starts around a lattice size $L = 144$. This can especially be seen for the Heat-bath algorithm. Compared with the Heat-bath algorithm the results from the Metropolis algorithm fluctuate a bit depending on the size of the lattice. As in the previous section, in order to show the universality and the possible effect of $t_{mic}$ as well as the fluctuation in the time direction, we present the exponent $-d/z + \theta$ obtained by the 15-scan for $L = 288$ as a function of time $t$ for both the Heat-bath and Metropolis algorithm in Fig. 8.

For the lattices bigger than $L = 144$, after a microscopic time scale $t_{mic} \sim 5$ the curves for both algorithms coincide and are presenting quite stable power law behaviour even though the fluctuations after time $t \sim 50$ become very big.
Such a small $t_{\text{mic}}$ here is consistent with the scenario in the last section that the smaller initial magnetization the shorter $t_{\text{mic}}$ is. Within the errors both algorithms give almost the same results. However we should point out that the error for Metropolis is much bigger than that for the Heat-bath algorithm. All these show that for the study of short-time dynamics Heat-bath algorithm is more efficient. In table 4 the exponent $-d/z+\theta$ measured from the time interval $[5, 50]$ is listed. To avoid too big fluctuations we have not made the measurement up to $t = 100$. In the previous paper [16] from lattice size $L = 144$ and $L = 288$, a linear extrapolation to infinite lattice size was carried out. However, the result for lattice size $L = 576$ does not go in this direction. Actually the difference among the results for lattice $L = 144, 288, 576$ is already very small as it was also pointed out in the previous paper. Therefore in this paper the result for infinite lattice is given as a simple average of the three lattices. The situation for Metropolis is less satisfactory. Anyway we also give the same average over the three lattices. From these values as well as those for $\theta$ in the previous subsection we can obtain the exponent $z$.

One can now realize what was mentioned in section 2, i.e., from the measurement of the auto-correlation a quite rigorous value for $z$ can be obtained in case that $\theta$ is known. Compared to the values of $z$ distributed between $z = 2.2$ and $z = 2.7$ from different numerical measurement [17, 19, 20], our result supports the relative small $z$ [17, 18]. The results for both algorithms coincide very well.

| $L$      | 144  | 288  | 576  | $\infty$ | $z$        |
|----------|------|------|------|-----------|------------|
| HeatB    | 0.839(1) | 0.834(1) | 0.835(1) | 0.836(2) | 2.196(08)  |
| MetroP   | 0.849(3) | 0.831(2) | 0.843(6) | 0.841(5) | 2.198(13)  |

Table 4: The exponent $\theta - d/z$ measured for different lattice sizes with initial magnetization $m_0 = 0.0$ for the Potts model. The last column gives the values for $z$.

| $L$      | 144  | 288  | $\infty$ | $2\beta/\nu$ |
|----------|------|------|-----------|--------------|
| HeatB    | 0.789(2) | 0.787(2) | 0.788(1) | 0.269(07)    |
| MetroP   | 0.787(7) | 0.789(9) | 0.788(6) | 0.269(16)    |

Table 5: The exponent $(d - 2\beta/\nu)/z$ measured for different lattice sizes with initial magnetization $m_0 = 0.0$ for the Potts model. The last column gives the values for the exponent $2\beta/\nu$. 

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4.3 The second moment

The second moment for the Potts model is defined by

\[ M^{(2)}(t) = \frac{9}{4N^2q} \sum_q \left\langle \left( \sum_i (\delta \sigma_i(t), 1 - \frac{1}{3}) \right)^2 \right\rangle. \tag{15} \]

Here we have taken the average of different components \( q \) of the variable \( \sigma_i(t) \).

(In the simulation of the magnetization \( M(t) \) starting from a non-zero initial magnetization, in principle we may also measure the exponent \( \theta \) from the power law behaviour of other components. However we did not do since the quality of the power law behaviour for different components is not exactly the same.)

In Fig. 9 the time evolution of the second moment for different lattices for both algorithms is plotted. We again plot the exponent \( (d - 2\beta/\nu)/z \) for each time step \( t \) in Fig. 10. It is clear that \( t_{mic} \) for Heat-bath is shorter than that for Metropolis. Results for both algorithms coincide after around \( t_{mic} \sim 30 \) which is also bigger than \( t_{mic} \) for \( \theta \) and \( -d/z + \theta \). In Table 5 the measured values for \( (d - 2\beta/\nu)/z \) are given. For the Heat-bath algorithm one may start the measurement from \( t \sim 10 \). However for the reason of comparison we treat both algorithms the same and perform measurements within the time interval \([35, 100]\). The result for the lattice \( L = 576 \) is a bit fluctuating and is not given in Table 5. Since the finite size effect is already smaller than the statistical fluctuation, the value of \( (d - 2\beta/\nu)/z \) for infinite lattice is given as a simple average over \( L = 144, 288 \).

5 Conclusions

We have simulated the universal short-time behaviour of critical dynamics for the Ising model and the Potts model with an initial state of very high temperature and small magnetization. The critical initial increase of the magnetization is observed. From the power law behaviour of the magnetization \( M(t) \), the second moment \( M^{(2)}(t) \) and the auto-correlation \( A(t) \), we obtain the related critical exponents \( \theta, z \) and \( \beta/\nu \) and confidently confirm the scaling properties in the short-time dynamics. The direct measurement of \( \theta \) shows the advantage of the method. Especially it allows a rigorous determination of the exponent \( z \), which is normally quite difficult to be measured from the long-time regime of the dynamic process. Furthermore all the simulations are carried out with both the Heat-bath and Metropolis algorithm. The results are consistent and the universality in the short-time dynamics is confirmed.

Here we should mention that not all the models would have a positive \( \theta \). For example, for the Potts model with \( q = 4 \) the exponent \( \theta \) is likely negative or very close to zero. In this case the measurement of \( \theta \) will become more difficult. On the other hand, how to determine the exponent \( \nu \) as well as the critical point
from the power law behaviour of the observables in the short-time dynamics is also very interesting.

Finally we would like to point out that the investigation of the short-time dynamics for statistical systems may be extended to the dynamic field theory, e.g. the stochastically quantized field theory where a fictitious dynamic process is introduced and the conventional field theory is approached in the equilibrium \[24, 25\]. Detailed investigations have been performed, especially for gauge theory and complex systems \[26, 27\]. However, up to now all these studies are only concentrated to the long-time behaviour of the dynamic process and its equilibrium. It will be very interesting to know whether the properties of the conventional field theory may be already obtained from the short-time behaviour of the dynamic system or not. Such a knowledge will be important for the numerical simulation of the lattice gauge theory.

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References

[1] A. J. Bray Advances in Physics 43 (1994) 357.
[2] H. K. Janssen, B. Schaub and B. Schmittmann Z. Phys. B 73 (1989) 539.
[3] D. A. Huse Phys. Rev. B 40 (1989) 304.
[4] K. Humayun and A. J. Bray J. Phys. A 24 (1991) 1915.
[5] Z.B. Li, U. Ritschel and B. Zheng J. Phys. A: Math. Gen. 27 (1994) L837.
[6] P. Grassberger Physica A 214 (1995) 547.
[7] Z.B. Li, L. Schülke and B. Zheng Phys. Rev. Lett. 74 (1995) 3396.
[8] Z.B. Li, L. Schülke and B. Zheng, Finite Size Scaling and Critical Exponents in Critical Relaxation, Phys. Rev. E, to be published.
[9] L. Schülke and B. Zheng, Determination of the Critical Point and Exponents from Short-time Dynamics, Univ. of Siegen, 1995, preprint SI-95-10.
[10] H. K. Janssen, in Topics in Modern Statistical Physics, edited by G. Györgyi, I. Kondor, L. Sasvári and T. Tél (World Scientific, Singapore, 1992).
[11] K. Oerding and H. K. Janssen J. Phys. A: Math. Gen. 26 (1993) 3369,5295.
[12] K. Oerding and H. K. Janssen J. Phys. A: Math. Gen. 27 (1994) 715.
[13] U. Ritschel and P. Czerner Phys. Rev. Lett. 75 (1995) 3882.
[14] S. J. R. Lee and B. Kim Phys. Rev. E 51 (1995) R4.
[15] A. D. Rutenberg and A. J. Bray Phys. Rev. E 51 (1995) R1641.
[16] L. Schülke and B. Zheng Phys. Lett. A 204 (1995) 295.
[17] M. Aydin and M. C. Yalabik J. Phys. A, Math. Gen. 18 (1985) 1741.
[18] S. Tang and D. P. Landau Phys. Rev. B 36 (1987) 567.
[19] K. Binder Z. Phys. B 43 (1981) 119.
[20] M. Aydin and M. C. Yalabik J. Phys. A, Math. Gen. 21 (1988) 769.
[21] N. Menyhárd J. Phys. A: Math. Gen. 27 (1994) 663.
[22] H. W. Diehl and U. Ritschel J. Stat. Phys. 73 (1993) 1.
[23] U. Ritschel and H. W. Diehl Phys. Rev. E 51 (1995) 5392.
[24] G. Parisi and Y. Wu Sci. Sin. 24 (1981) 483.
[25] M. Namiki, Stochastic Quantization (Springer–Verlag, Berlin, Heidelberg, 1992).
[26] K. Okano, L. Schülke and B. Zheng, in STOCHASTIC QUANTIZATION, edited by M. Namiki and K. Okano (Progress of Theoretical Physics, Supplement No. 111, Kyoto, 1993), pp. 313–347.
[27] K. Fujimura, K. Okano, L. Schülke, K. Yamagishi, B. Zheng Nucl. Phys. B 424 (1994) 675.
Figure 1.a: Time evolution of the magnetization in double log scale for the Ising model with $m_0=0.02$ for the Heat-bath algorithm.
Figure 1.b: Time evolution of the magnetization in double log scale for the Ising model with $m_0=0.02$ for the Metropolis algorithm.
Figure 2.a: $\theta$ vs. $t$ for the Ising model with $m_0=0.02$. 
Figure 2.b: $\theta$ vs. $t$ for the Ising model with $m_0=0.04$. 

$\times$ Metropolis

$\square$ Heat-bath, $m_0 = 0.04$, $L=128$
Figure 2.c: $\theta$ vs. $t$ for the Ising model with $m_0=0.06$. 
Figure 2.d: $\theta$ vs. $t$ for the Ising model with $m_0 = 0.08$.

Legend:
- $\times$ Metropolis
- $\blacksquare$ Heat-bath, $m_0 = 0.08$, $L=128$
Figure 3: $\theta - d/z$ vs. $t$ for the Ising model.
Figure 4: The time evolution of the magnetization in double log scale for the Potts model with $m_0=0.04$.
Figure 5: $\theta$ vs. $t$ for the Potts model with $m_0=0.04$. 
Figure 6: $\theta$ vs. $t$ for the Potts model with $m_0=0.02$ and $L = 72, 144$ for the Heat-bath algorithm.
Figure 7: The auto-correlation as a function of $t$ in double log scale for the Potts model.
Figure 8: $\theta - d/z$ vs. $t$ for the Potts model.
Figure 9: The time evolution of the second moment in double log scale for the Potts model.
Figure 10: \((d - 2\beta/\nu)/z\) vs. \(t\) for the Potts model.