EQUIVARIANT BENJAMINI-SCHRAMM CONVERGENCE OF SIMPLICIAL COMPLEXES AND $\ell^2$-MULTIPLICITIES

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Abstract. We define a variant of Benjamini-Schramm convergence for finite simplicial complexes with the action of a fixed finite group $G$ which leads to the notion of random rooted simplicial $G$-complexes. For every random rooted simplicial $G$-complex we define a corresponding $\ell^2$-homology and the $\ell^2$-multiplicity of an irreducible representation of $G$ in the homology. The $\ell^2$-multiplicities generalize the $\ell^2$-Betti numbers and we show that they are continuous on the space of sofic random rooted simplicial $G$-complexes. In addition, we study induction of random rooted complexes and discuss the effect on $\ell^2$-multiplicities.

1. Introduction

Benjamini and Schramm [3] introduced the concept of random rooted graphs as probability measures on the space of connected rooted graphs. This allowed them to define convergence – today known as Benjamini-Schramm convergence – of sequences of finite graphs and to study the corresponding limit random rooted graphs. It was realized by Aldous and Lyons [2] that Benjamini-Schramm limits of finite graphs share a useful mass-transport property, called unimodularity, which, when added to the definition of random rooted graphs, allows to extend several results from quasi-transitive graphs to this setting. These two insights provide the basis of a large number subsequent developments.

The basic idea of Benjamini-Schramm convergence is not specific to graphs and was, in particular, generalized to simplicial complexes in the work of Elek [8], Bowen [5] and one of the authors [13]. The central result of [13] (extending similar results in [8, 5]) is that suitably defined $\ell^2$-Betti numbers of random rooted simplicial complexes are Benjamini-Schramm continuous (on the space of sofic random rooted complexes). For graphs a stronger result was proven in [1]. Here we introduce an equivariant Benjamini-Schramm convergence for simplicial complexes with the action of a fixed finite group; to our knowledge this is a new concept even for graphs. Then we define refinements of $\ell^2$-Betti numbers, namely $\ell^2$-multiplicities, and prove a corresponding continuity result.

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Statement of main results. We fix a finite group $G$ and consider the space $\mathcal{SC}^D(G)$ of isomorphism classes of rooted simplicial $G$-complexes of vertex degree at most $D$. Here a rooted simplicial $G$-complex $(K, o)$ consists of a simplicial complex with an action of $G$ and a distinguished $G$-orbit $o$ of vertices in $K$ which touches every connected component of $G$. A random rooted simplicial $G$-complex $\mu$ is a unimodular probability measure on $\mathcal{SC}^D(G)$; for details we refer to Section 2.

For instance, picking a root uniformly at random in a finite simplicial $G$-complex defines a random rooted $G$-complex. Limits of laws of finite simplicial $G$-complexes are called sofic; see Definition 2.4.

Let $K$ be a finite simplicial $G$-complex. The group $G$ acts on $K$ and thus acts on the homology groups $H_n(K, \mathbb{C})$, i.e., we have a finite dimensional representation of $G$ on $H_n(K, \mathbb{C})$. This representation decomposes as a direct sum of irreducible representations of $G$ and every irreducible representation $\rho \in \text{Irr}(G)$ occurs a finite number, say $m(\rho, H_n(K, \mathbb{C}))$, of times in this decomposition. The number $m(\rho, H_n(K, \mathbb{C}))$ is called the multiplicity of $\rho$ in $H_n(K, \mathbb{C})$. Drawing from this, we define $\ell^2$-multiplicities in Section 4; these are nonnegative real numbers $m_n^{(2)}(\rho, \mu)$ for every random rooted simplicial $G$-complex $\mu$ and every irreducible representation $\rho \in \text{Irr}(G)$. In fact, if $\mu_K$ is the law of a finite $G$-complex, then

$$m_n^{(2)}(\rho, \mu_K) = \frac{m(\rho, H_n(K, \mathbb{C}))}{|K^{(0)}|};$$

see Example 4.7. Our main result is the following approximation theorem.

**Theorem 1.1.** Let $(\mu_k)_{k \in \mathbb{N}}$ be a sequence of random rooted simplicial $G$-complexes. If every $\mu_k$ is sofic and the sequence converges weakly to $\mu_\infty$, then

$$\lim_{k \to \infty} m_n^{(2)}(\rho, \mu_k) = m_n^{(2)}(\rho, \mu_\infty)$$

for every $n \in \mathbb{N}_0$ and every irreducible representation $\rho$ of $G$.

Along the way (in Section 3) we investigate induced $G$-complexes. Given a subgroup $H \leq G$ and a simplicial $H$-complex $L$, it is natural to construct a simplicial $G$-complex $K = G \times_H L$ by inducing the action from $H$ to $G$. This can be promoted to an operation $\text{ind}_H^G$ which takes random rooted $H$-complexes to random rooted $G$-complexes. We provide a criterion to decide whether a sequence of finite $G$-complexes converges to an induced random rooted $G$-complex; see Proposition 3.3. This is relevant, since the $\ell^2$-multiplicities of the induced random rooted complex $\text{ind}_H^G(\mu)$ can be computed from the $\ell^2$-multiplicities of $\mu$ (provided $\mu$ is sofic). As a special case, for $H = \{1\}$, we have the following result.

**Theorem 1.2.** Let $\mu$ be a sofic random rooted simplicial complex and let $G$ be a finite group. For all $\rho \in \text{Irr}(G)$ and $n \in \mathbb{N}_0$ the following identity holds:

$$m_n^{(2)}(\rho, \text{ind}_1^G(\mu)) = \frac{\dim_{\mathbb{C}}(\rho) b_n^{(2)}(\mu)}{|G|};$$
In the topological setting $L^2$-multiplicities of CW-complexes have been defined and studied in [10]. Our results are mainly complementary to the results therein. However, for towers of finite sheeted covering spaces of simplicial complexes our results provide a different approach to [10, Theorem 1.2]. In this setting, we think that our investigation of induced complexes is useful. Combining Theorem 1.2 with Proposition 3.6 provides a new perspective on the centralizer condition which remained a bit obscure in [10, Proposition 5.5].

2. Random rooted simplicial $G$-complexes

We will use the language of simplicial complexes as introduced in many standard texts; for instance [14]. The \( n \)-skeleton of a simplicial complex \( K \) is denoted by \( K^{(n)} \). The number of 1-simplices containing a vertex \( x \in K^{(0)} \) is called the degree of \( x \). The supremum over the degrees of all vertices of \( K \) will be called the vertex degree of \( K \). To avoid some technical issues we will consider simplicial complexes whose vertex degree is bounded above by some large constant \( D > 0 \). For a simplicial complex \( K \), a non-empty set of vertices \( V \subseteq K^{(0)} \) and \( r > 0 \) we define \( B_r(K, V) \) to be the full subcomplex on the set of vertices of path distance at most \( r \) from a vertex in \( V \).

We fix a finite group \( G \). A rooted simplicial $G$-complex is a pair \((K, o)\) consisting of a simplicial complex \( K \) with \( G \)-action and a \( G \)-orbit \( o \) of vertices of \( K \) such that every connected component of \( K \) contains at least one vertex in \( o \). Two rooted simplicial $G$-complexes \((K, o), (L, o')\) are isomorphic if there is a simplicial isomorphism \( f : K \to L \) such that \( f(o) = o' \) and \( g \cdot f(x) = f(g \cdot x) \) for all \( x \in K^{(0)} \) and \( g \in G \); in this case we write \((K, o) \cong (L, o')\).

We denote by \( \mathcal{SC}^D_*(G) \) the space of isomorphism classes of rooted simplicial $G$-complexes with vertex degree bounded by \( D \). The following defines a metric on \( \mathcal{SC}^D_*(G) \):

\[
d([K, o], [L, o']) = \inf \left\{ \frac{1}{2r} \mid B_r(K, o) \cong B_r(L, o') \right\}
\]

for \([K, o], [L, o'] \in \mathcal{SC}^D_*(G)\). Note that subcomplexes of the form \( B_r(K, o) \) are stable under the \( G \)-action. With the induced topology \( \mathcal{SC}^D_*(G) \) is a compact totally disconnected space. For every finite rooted simplicial $G$-complex \( \alpha \) of radius at most \( r \), we obtain an open subset \( U(\alpha, r) \subseteq \mathcal{SC}^D_*(G) \) consisting of all isomorphism classes of rooted simplicial $G$-complexes \([K, o]\) such that \( B_r(K, o) \cong \alpha \). The sets \( U(\alpha, r) \) are open and compact and provide a basis for the topology.

Given two rooted simplicial $G$-complexes \((K, o)\) and \((L, o')\), such that \( o \) and \( o' \) are not isomorphic as \( G \)-sets, then \( d([K, o], [L, o']) = \infty \). Let \( \text{Orb}(G) \) denote the finite set of isomorphism classes of transitive $G$-sets. The space \( \mathcal{SC}^D_*(G) \) splits into
a disjoint union
\[ \mathcal{SC}_*^D(G) = \bigsqcup_{X \in \text{Orb}(G)} \mathcal{SC}_*^D(G, X), \]

where \( \mathcal{SC}_*^D(G, X) \) is the subspace of all \([K, o]\) where \( o \) belongs to \( X \).

In the same manner we define the space \( \mathcal{SC}_{**}^D(G) \) of isomorphism classes of doubly rooted simplicial \( G \)-complexes \((K, o, o')\) consisting of a simplicial \( G \)-complex \( K \) and two orbits \( o \) and \( o' \) of vertices which touch every connected component.

**Definition 2.1.** A random rooted simplicial \( G \)-complex is a unimodular probability measure \( \mu \) on \( \mathcal{SC}_*^D(G) \), where unimodular means
\[
\int_{\mathcal{SC}_*^D(G)} \sum_{x \in K(0)} f([K, o, Gx]) d\mu([K, o]) = \int_{\mathcal{SC}_*^D(G)} \sum_{x \in K(0)} f([K, Gx, o]) d\mu([K, o])
\]
for all Borel measurable functions \( f : \mathcal{SC}_{**}^D(G) \to \mathbb{R}_{\geq 0} \).

**Remark 2.2.** A weak limit of random rooted simplicial \( G \)-complexes is again a random rooted simplicial \( G \)-complex. This can be deduced by observing that a probability measure \( \mu \) on \( \mathcal{SC}_*^D(G) \) is unimodular if and only if for all \( n \in \mathbb{N} \) and all continuous functions \( f : \mathcal{SC}_{**}^D(G) \to \mathbb{R} \) the equality
\[
\int_{\mathcal{SC}_*^D(G)} \sum_{\substack{x \in K(0) \\text{ s.t. } d(x,o) \leq n}} f([K, o, Gx]) d\mu([K, o]) = \int_{\mathcal{SC}_*^D(G)} \sum_{\substack{x \in K(0) \\text{ s.t. } d(x,o) \leq n}} f([K, Gx, o]) d\mu([K, o])
\]
holds, where the sum now runs over the vertices of path distance at most \( n \) from the root orbit. This can be verified using the theorem of monotone convergence and the monotone class theorem.

**Example 2.3.** Let \( K \) be a finite simplicial \( G \)-complex. Then there is a unique random rooted simplicial \( G \)-complex which is fully supported on isomorphism classes of the form \([K, o]\); it is given by \( \mu_K^G := \sum_{x \in K(0)} \frac{\delta_{[K,Gx]}}{|K(0)|} \),

where \( \delta_{[K,Gx]} \) denotes the Dirac measure of the point \([K,Gx]\). That \( \mu_K^G \) is a probability measure is obvious, we only have to verify that it is unimodular:
\[
\int_{\mathcal{SC}_*^D(G)} \sum_{x \in L(0)} f([L, o, Gx]) d\mu_K^G([L, o]) = \sum_{y \in K(0)} \frac{1}{|K(0)|} \sum_{x \in K(0)} f([K, Gy, Gx])
\]
\[
= \sum_{x \in K(0)} \frac{1}{|K(0)|} \sum_{y \in K(0)} f([K, Gy, Gx])
\]
\[
= \int_{\mathcal{SC}_*^D(G)} \sum_{y \in L(0)} f([L, Gy, o]) d\mu_K^G([L, o]).
\]
**Definition 2.4.** A sequence $(K_n)_n$ of finite simplicial $G$-complexes (of vertex degree bounded by $D$) converges Benjamini-Schramm if the weak limit $\lim_{n \to \infty} \mu_{K_n}^G$ exist. A random rooted simplicial $G$-complex on $\text{SC}^D_s(G)$ which is the Benjamini-Schramm limit of a sequence of finite simplicial $G$-complexes is called sofic.

3. Induction of simplicial complexes

Let $G$ be a finite group and let $H \leq G$ be a subgroup. Given an $H$-set $Y$ one can construct the induced $G$-set $G \times_H Y$. The set

$$G \times_H Y = (G \times Y) / \sim$$

is obtained by forming the quotient of $G \times Y$ under the equivalence relation $(g, y) \sim (gh, h^{-1} \cdot y)$ for all $g \in G$, $y \in Y$ and $h \in H$. We write $[g, y]$ to denote the equivalence class of $(g, y)$. Clearly $g_2 \cdot [g_1, y] = [g_2 g_1, y]$ defines an action of $G$ on $G \times_H Y$.

A simplicial $H$-complex $K$ gives rise to a simplicial $G$-complex $G \times_H K$ by induction. The vertices of $G \times_H K$ are the elements of $G \times_H K^{(0)}$. For every simplex $\sigma \subseteq K^{(0)}$ of $K$ and every $g \in G$ we define a simplex $\sigma_g = \{[g, x] | x \in \sigma\}$ of $G \times_H K$. As a simplicial complex (without $G$-action) $G \times_H K$ is isomorphic to a disjoint union of $|G/H|$ copies of $K$. In particular, induction does not alter the vertex degree.

**Lemma 3.1.** The function $\text{ind}_H^G : \text{SC}^D_s(H) \to \text{SC}^D_s(G)$ which maps $[K, o]$ to $[G \times_H K, G[1, o]]$ is continuous. In particular, the push-forward of measures with $\text{ind}_H^G$ is weakly continuous.

**Proof.** We observe that the ball of radius $r$ in $G \times_H K$ around $G[1, o]$ is isomorphic to $G \times_H B_r(K, o)$. This implies that

$$d(\text{ind}_H^G([K, o]), \text{ind}_H^G([L, o'])) \leq d([K, o], [L, o'])$$

and proves the assertion. \qed

We use this to define induction of random rooted simplicial complexes.

**Lemma 3.2.** Let $\mu$ be a random rooted simplicial $H$-complex. The push-forward measure $\text{ind}_H^G(\mu)$ is a random rooted simplicial $G$-complex.

**Proof.** The push-forward preserves the total mass, so $\text{ind}_H^G(\mu)$ is a probability measure. It remains to verify unimodularity. Recall the general transformation rule [1, §6]

$$\int_{\text{SC}^D_s(G)} t(z) d\text{ind}_H^G(\mu)(z) = \int_{\text{SC}^D_s(H)} t(\text{ind}_H^G(\mu)) d\mu(w)$$
for all measurable nonnegative functions $t$ on $\mathcal{SC}^D_*(G)$. We obtain for all measurable $f$:

$$
\int_{\mathcal{SC}^D_*(G)} \sum_{x \in L(0)} f(L, o, Gx) d \text{ind}^G_H(\mu)([L, o])
$$

$$
= \int_{\mathcal{SC}^D_*(H)} \sum_{x \in G \times K(0)} f(G \times H, K, Go, Gx) d \mu([K, o])
$$

$$
= \int_{\mathcal{SC}^D_*(H)} |G/H| \sum_{y \in K(0)} f(G \times H, K, Go, G[y, Go]) d \mu([K, o])
$$

$$
= \int_{\mathcal{SC}^D_*(G)} \sum_{x \in L(0)} f(L, Gx, o) d \text{ind}^G_H(\mu)([L, o])
$$

where we use the transformation rule in steps (1) and (3), and the unimodularity of $\mu$ in step (2). □

The following criterion is useful to show that a sequence of finite simplicial $G$-complexes converges to an induced random rooted simplicial $G$-complex.

**Proposition 3.3.** Let $(K_n)_n$ be a sequence of finite simplicial $G$-complexes with vertex degree bounded by $D$. Assume that the sequence of random rooted simplicial $H$-complexes $(\mu^H_{K_n})_n$ converges to a random rooted simplicial $H$-complex $\mu_\infty$ for some subgroup $H \leq G$. Then $(\mu^G_{K_n})_n$ converges to $\text{ind}^G_H(\mu_\infty)$ on $\mathcal{SC}^D_*(G)$ if and only if

$$
\lim_{n \to \infty} \frac{|E(K_n, g, C)|}{|K_n(0)|} = 0
$$

for all $C > 0$ and all $g \in G \setminus H$ where $E(K, g, C) = \{x \in K(0) \mid d(x, gx) \leq C\}$.

**Proof.** Define $E(K, C) = \bigcup_{g \in G \setminus H} E(K, g, C)$.

Assume that equation (3.1) holds for all $g \in G \setminus H$ and all $C > 0$. Let $r > 0$ be given. We want to verify that for all $x \in K_n(0) \setminus E(K_n, 2r + 1)$ the ball of radius $r$ around $Gx$ in $K_n$ is isomorphic to $G \times H B_r(K_n, Hx)$. This amounts to showing that

$$
B_r(K_n, Hx) \cap gB_r(K_n, Hx) = \emptyset
$$

for all $g \in G \setminus H$ and that there is no edge between these two sets. Suppose that there is an element in the intersection or an edge between $B_r(K_n, Hx)$ and $gB_r(K_n, Hx)$. In both cases we can find $h, h' \in H$ such that $d(hx, gh'x) \leq 2r + 1$. 

However this implies $x \in E(K_n, 2r + 1)$ since $d(x, h^{-1}gh'x) \leq 2r + 1$ and $h^{-1}gh' \notin H$.

Let $\alpha$ be a rooted simplicial $G$-complex of radius at most $r$. Let $\varepsilon > 0$ and take $n$ sufficiently large such that $|E(K_n, 2r + 1)| < |K_n^{(0)}|\varepsilon$. The inverse image $V = (\text{ind}^G_H)^{-1}(U(\alpha, r)) \subseteq \mathcal{SC}^G_*(H)$ is a finite (possibly empty) union of sets of the form $U(\alpha', r)$; thus it is open and compact. The weak convergence $\mu^H_{K_n} \xrightarrow{w} \mu_\infty$ shows that for all sufficiently large $n$ the inequality $|\mu_\infty(V) - \mu^H_{K_n}(V)| < \varepsilon$ holds. Moreover, by the observation above

$$|\mu^G_{K_n}(U(\alpha, r)) - \mu^H_{K_n}(V)| \leq \frac{|E(K_n, 2r + 1)|}{|K_n^{(0)}|} < \varepsilon.$$ 

As $\alpha$ was arbitrary, we deduce the convergence $\mu^G_{K_n} \xrightarrow{w} \text{ind}^G_H(\mu_\infty)$.

Conversely, suppose that the sequence $(\mu^G_{K_n})_n$ converges to $\text{ind}^G_H(\mu_\infty)$. Let $g \in G \setminus H, C > 0$ and $x \in E(K_n, g, C)$. The ball $B_C(K_n, Hx)$ contains a path from $x$ to $gx$ and hence it is not isomorphic to a simplicial complex induced from $H$. By assumption the limit $\lim_{n \to \infty} \mu^G_{K_n}$ is supported on induced complexes and thus (3.1) is satisfied. \hfill \Box

**Example 3.4 (Sierpinski’s triangle with rotation).** We describe a sequence $(T_n)_n$ of two-dimensional simplicial complexes which occur in the construction of the fractal Sierpinski triangle. It appeared to us that the example becomes clearer if we describe the geometric realizations of the $T_n$ as subsets of $\mathbb{R}^2$ instead of working with the abstract simplicial complexes. Let $e_1 = (1, 0) \in \mathbb{R}^2$ and let $e_2 = \frac{1}{2}(1, \sqrt{3}) \in \mathbb{R}^2$. The points 0, $e_1$ and $e_2$ are the vertices of an equilateral triangle $T_0$ with sides of length 1; we consider $T_0$ to be a 2-simplex. We define inductively

$$T_{n+1} = T_n \cup (T_n + 2^n e_1) \cup (T_n + 2^n e_2).$$

By induction it is easy to verify that $T_n$ is a simplicial complex with $3^n + 3$ vertices, $3^n + 1$ edges and $3^n$ 2-simplices. The vertex degree of $T_n$ is 4 for all $n \geq 1$. The three vertices of degree 2 will be called the corners of $T_n$. The distance between two corners of $T_n$ is $2^n$. 

![Diagram](image-url)
Claim: The sequence $(T_n)_n$ converges Benjamini-Schramm to a random rooted simplicial complex $\tau_S$.

Let $r > 0$ and let $\alpha$ be a finite rooted simplicial complex of radius at most $r$. Take $m$ so large that $2^{m-1} > r$. Then any $r$-ball in $T_m$ contains at most one of the three corners of $T_m$. Let $N(k, \alpha)$ denote the number of vertices $v$ in $T_{m+k}$ such that the ball of radius $r$ around $v$ is isomorphic to $\alpha$. We observe that

$$N(k + 1, \alpha) = 3N(k, \alpha) + c_{\alpha}$$

for all $k \geq 0$ for some constant $c_{\alpha} \in \mathbb{Z}$. Indeed, $r$-balls around vertices of distance at least $r$ from one of the corners in $T_{m+k}$ occur exactly 3-times in $T_{m+k+1}$. In the small set of vertices which lie close to a corner of $T_{m+k}$, we always see two copies of $T_m$ being glued at a corner. Which shows that the effect of this operation does not depend on $k$; compare Figure 3.4.

Now it follows from a short calculation that $(\frac{|N(k, \alpha)|}{|T_{m+k}|}_k)$ is a Cauchy sequence. Since $r$ and $\alpha$ were arbitrary, we conclude that the sequence $(T_n)_n$ converges in the sense of Benjamini-Schramm.

Now we introduce an action of the finite cyclic group $G = \langle \sigma \rangle$ of order 3. We let $\sigma$ act by rotation of $2\pi/3$ around the barycenter $c_n = 2^{n-1}(1, \sqrt{3}^{-1})$ of $T_n$. All vertices of $T_n$ have Euclidean distance at least $\frac{2^{n-2}}{\sqrt{3}}$ from the barycenter. Thus every vertex is moved by an Euclidean distance of at least $2^{n-2}$ under the non-trivial rotations $\sigma$ and $\sigma^2$, which in particular also holds for the path distance in $T_n$. Proposition 3.3 implies that the sequence $(T_n)_n$ of simplicial $G$-complexes converges to the induced random rooted simplicial $G$-complex $\text{ind}_G^1(\tau_S)$. Roughly speaking, the sequence converges to three copies of the Sierpinski triangle which are permuted cyclically by $G$.

![Figure 1. Small balls touch at most two copies of $T_m$](image-url)
3.1. **Towers of finite sheeted covering spaces.** In this section we discuss a prominent family of examples of Benjamini-Schramm convergent sequences: towers of finite sheeted covering spaces.

Let $\Gamma$ be a group and let $K$ be simplicial complex of vertex degree at most $D$. Assume that $\Gamma$ acts simplicially on $K$, this means that an element $\gamma \in \Gamma$ stabilizes a simplex of $K$ if and only if it stabilizes all of its vertices. Recall that this condition can always be achieved by passing to the barycentric subdivision of $K$. We assume further that the action is proper and cocompact, i.e., every vertex has a finite stabilizer and there are only finitely many orbits of vertices.

Let $G \leq \Gamma$ be a finite subgroup. For every normal subgroup $N \triangledown \Gamma$ the quotient simplicial complex $K/N$ carries a $G$-action. Suppose that $\Gamma$ is residually finite and let $(N_n)_{n \in \mathbb{N}}$ be a descending chain of finite index normal subgroups of $\Gamma$ with $\bigcap_{n \in \mathbb{N}} N_n = \{1\}$. It is well-known (cf. [13, Example 19]) that the sequence $(K/N_n)_{n}$ of finite simplicial complexes (without $G$-action) converges to the random rooted simplicial complex $\sum_{x \in F} |St_{\Gamma}(x)|^{-1} \delta_{[K,x]}$ where $F$ is a fundamental domain for the action of $\Gamma$ on $K^{(0)}$ and $w(\Gamma) = \sum_{x \in F} |St_{\Gamma}(x)|^{-1}$. This measure does not depend on the choice of the fundamental domain. The purpose of this section is to describe the limit taking the $G$-action into account.

We say that an element $\gamma \in \Gamma$ is FC if it has a finite conjugacy class, i.e., $|\Gamma : C_{\Gamma}(\gamma)| < \infty$, where $C_{\Gamma}(\gamma)$ is the centralizer of $\gamma$. Consider the subgroup $H = \{g \in G \mid g$ is FC in $\Gamma\} \leq G$ of FC-elements which lie in $G$. Let $\Gamma_0 \leq_{f.i.} \Gamma$ be a finite index subgroup which satisfies

$$\Gamma_0 \subseteq \bigcap_{h \in H} C_{\Gamma}(h).$$

**Lemma 3.5.** Let $\mathcal{F}_0 \subseteq K^{(0)}$ be a fundamental domain for the action of $\Gamma_0$ on $K^{(0)}$. The measure

$$\mu^K_{\mathcal{F}} = \frac{1}{w(\Gamma_0)} \sum_{x \in \mathcal{F}_0} |St_{\Gamma_0}(x)|^{-1} \delta_{[K,Hx]}$$

on $\mathcal{S}\mathcal{C}_c^D(H)$ is unimodular and does not depend on the choices of $\Gamma_0$ and $\mathcal{F}_0$.

**Proof.** Observe that any element $\gamma \in \Gamma_0$ commutes with all $h \in H$ and thus defines an isomorphism between $(K, Hx)$ and $(K, H\gamma x)$ as simplicial $H$-complexes. This shows that the measure is independent of the fundamental domain.

In order to verify that $\mu^K_{\mathcal{F}}$ does not depend on $\Gamma_0$, it is sufficient to show that we can replace $\Gamma_0$ by some finite index normal subgroup $\Gamma_1 \leq_{f.i.} \Gamma_0$. Let $\mathcal{F}_1$ be a
fundamental domain for $\Gamma_1$. We obtain

$$
\sum_{x \in \mathcal{F}_1} |\text{St}_{\Gamma_1}(x)|^{-1}\delta_{[K,Hx]} = \sum_{y \in \mathcal{F}_0} |\text{St}_{\Gamma_0}(y)|^{-1} \sum_{\gamma \in \Gamma_0} |\text{St}_{\Gamma_1}(\gamma y)|^{-1}\delta_{[K,H\gamma y]}
$$

$$
= \sum_{y \in \mathcal{F}_0} |\text{St}_{\Gamma_0}(y)|^{-1}\delta_{[K,Hy]} \sum_{\gamma \in \Gamma_0} |\text{St}_{\Gamma_1}(y)|^{-1} = |\Gamma_0 : \Gamma_1| \sum_{y \in \mathcal{F}_0} |\text{St}_{\Gamma_0}(y)|^{-1}\delta_{[K,Hy]}
$$

and, from a similar calculation, also $w(\Gamma_1) = |\Gamma_0 : \Gamma_1|w(\Gamma_0)$.

It remains to show that $\mu_K^H$ is unimodular. Let $f : \mathcal{SC}_{\ast\ast}(H) \to \mathbb{R}_{\geq 0}$ be a measurable function. Unimodularity follows from a short calculation.

$$
\int_{\mathcal{SC}_{\ast\ast}^D(H)} \sum_{y \in L(0)} f(L,o,Hy) d\mu_K^H([L,o])
$$

$$
= \frac{1}{w(\Gamma_0)} \sum_{x \in \mathcal{F}_0} |\text{St}_{\Gamma_0}(x)|^{-1} \sum_{y \in K(0)} f(K,Hx,Hy)
$$

$$
= \frac{1}{w(\Gamma_0)} \sum_{x \in \mathcal{F}_0} |\text{St}_{\Gamma_0}(x)|^{-1} \sum_{y \in \mathcal{F}_0} \sum_{\gamma \in \Gamma_0} |\text{St}_{\Gamma_0}(y)|^{-1} f(K,Hx,H\gamma y)
$$

$$
= \frac{1}{w(\Gamma_0)} \sum_{x,y \in \mathcal{F}_0} |\text{St}_{\Gamma_0}(x)|^{-1} |\text{St}_{\Gamma_0}(y)|^{-1} \sum_{\gamma \in \Gamma_0} f(K,H\gamma^{-1}x,Hy)
$$

$$
= \cdots = \int_{\mathcal{SC}_{\ast\ast}^D(H)} \sum_{x \in L(0)} f(L,Hx,o) d\mu_K^H([L,o])
$$

\[\square\]

**Proposition 3.6.** Let $G \leq \Gamma$ be a finite subgroup and let $H \leq G$ be the subgroup of $\text{FC}$-elements for $\Gamma$. Let $(N_n)_n$ be a descending chain of finite index normal subgroups in $\Gamma$ with $\cap_{n \in \mathbb{N}} N_n = \{1\}$. The sequence of simplicial $G$-complexes $(K/N_n)_n$ converges to the random rooted simplicial complex $\mu_K^G := \text{ind}_H^G(\mu_K^H)$.

**Proof.** The proof consists of two steps. First we show that $(K/N_n)_n$ converges as a sequence of $H$-complexes to $\mu_K^H$ (Claims 1 and 2) and in the second step (Claim 3) we apply Proposition 3.3.

**Claim 1:** Let $r > 0$ be fixed. For all sufficiently large $n$ and all $x \in K(0)$, the $r$-ball $B_r(K,Hx)$ in $K$ and the $r$-ball $B_r(K/N_n,HN_nx)$ in $K/N_n$ are isomorphic as $H$-complexes.

Let $\Gamma_0 \leq \Gamma$ be as above and let $\mathcal{F}_0$ be a fundamental domain for $\Gamma_0$ acting on $K(0)$. The action is proper, the sets $\mathcal{F}_0$ and $H$ are finite and the vertex degree of $K$ is bounded, hence the set $S$ of elements $\gamma \in \Gamma$ such that

$$
B_{r+1}(K,Hx_0) \cap \gamma B_{r+1}(K,Hx_0) \neq \emptyset
$$

for some $x_0 \in \mathcal{F}_0$ is finite.
Take $n \in \mathbb{N}$ so large that $S \cap N_n = \{1\}$. Then for all $x_0 \in K^{(0)}$ and $\gamma \in N_n$ property (3.2) implies that $\gamma = 1$. Indeed, find $\gamma_0 \in \Gamma_0$ with $\gamma_0 x_0 \in F_0$ then multiplication with $\gamma_0$ yields $B_{r+1}(K, H\gamma_0 x_0) \cap \gamma_0 \gamma \gamma_0^{-1} B_{r+1}(K, H\gamma_0 x_0) \neq \emptyset$. We deduce $\gamma_0 \gamma \gamma_0^{-1} \in S \cap N_n = \{1\}$. In particular, the quotient map takes the vertices of the $r$-ball $B_r(K, hx)$ injectively to the $r$-ball $B_r(K/N_n, HN_n x)$. We have to verify that every simplex in $B_r(K/N_n, HN_n x)$ lifts to a unique simplex in $B_r(K, Hx)$. Let $\sigma$ be a simplex in $B_r(K/N_n, HN_n x)$ and let $\tilde{\sigma}$ be a lift in $K$ such that at least one vertex lies in $B_r(K, Hx)$. As a consequence $\tilde{\sigma}$ is a simplex in $B_{r+1}(K, Hx)$. Let $y$ be any vertex of $\tilde{\sigma}$. There is an element $k \in N_n$ such that $d(ky, Hx) \leq r$. This means that $B_{r+1}(K, Hx) \cap kB_{r+1}(K, Hx) \neq \emptyset$ and shows that $k = 1$. In particular, the simplex $\tilde{\sigma}$ lives in $B_r(K, Hx)$.

Claim 2: The sequence $\mu_{K/N_n}^H(U(\alpha, r))$ converges to $\mu_K^H$.

Let $r > 0$ and let $\alpha$ be a rooted simplicial $H$-complex of radius at most $r$. Let $n \in \mathbb{N}$ sufficiently large such that $N_n$ acts freely on $K$ and so that Claim 1 applies. In addition, we may take $\Gamma_0 \leq N_n$: the action of $\Gamma_0$ is also free. Now every point in $K/N_n$ is covered by exactly $|N_n : \Gamma_0|$ points in $F_0$ and we deduce

$$
\mu_{K/N_n}^H(U(\alpha, r)) = \frac{|\{x \in F_0 \mid B_r(K, Hx) \overset{H}{\cong} \alpha \}|}{|F_0|} = \frac{|\{x \in K^{(0)} / N_n \mid B_r(K, HN_n x) \overset{H}{\cong} \alpha \}|}{|K^{(0)} / N_n|} = \mu_{K/N_n}^H(U(\alpha, r))
$$

Claim 3: (3.1) holds for all $C > 0$ and all $g \in G \setminus H$.

Let $i \in \mathbb{N}$ be chosen so that $N_i$ acts freely on $K$ and let $F$ be a fundamental domain for the action of $N_i$ on $K^{(0)}$. Let $Z \subseteq \Gamma$ be the finite set of elements $\gamma \in \Gamma$ such that $d(\gamma x, x) \leq C$ for some $x \in F$. For $n \geq i$ the vertices of $K/N_n$ correspond bijectively to $N_i/N_n \times F$.

Take $x \in K^{(0)}$ and write $\bar{x} = N_n x \in K^{(0)}/N_n$. Suppose that $\bar{x} \in E(K/N_n, g, C)$; i.e., there is $\gamma_n \in N_n$ with $d(gx, \gamma_n x) \leq C$. There is a unique $x_0 \in F$ and an element $\gamma_i \in N_i$ satisfying $x = \gamma_i x_0$. This shows that $d(\gamma_i^{-1} \gamma_n^{-1} g \gamma_i x_0, x_0) \leq C$ and so $\gamma_i^{-1} g \gamma_i \in Z N_n$. How many elements has the finite set $e_n(g, Z) = \{k \in N_i/N_n \mid k^{-1} g k \in Z N_n / N_n\}$? Clearly, its cardinality is bounded above by $|Z| \cdot |C_{N_i/N_n}(g N_n)|$. The element $g \in G$ has an infinite conjugacy class in $\Gamma$ and thus

$$
\lim_{n \to \infty} \frac{|C_{\Gamma/N_n}(g N_n)|}{|\Gamma : N_n|} = 0;
$$

see the proof of [10] Lemma 4.12. We deduce that

$$
\lim_{n \to \infty} \frac{|E(K/N_n, g, C)|}{|K^{(0)}/N_n|} \leq \lim_{n \to \infty} \frac{|e_n(g, Z)|}{|K^{(0)}/N_n|} \leq \lim_{n \to \infty} \frac{|Z| \cdot |C_{\Gamma/N_n}(g N_n)|}{|N_i : N_n| |K^{(0)}/N_n|} = 0.
$$

This proves the last claim and Proposition 3.3 completes the proof. \qed
4. Homology and $\ell^2$-multiplicities of random rooted complexes

4.1. The homology of a random rooted complex. In order to define $\ell^2$-multiplicities we introduce the $\ell^2$-homology of random rooted simplicial $G$-complexes. To this end, we will construct a chain complex for each probability measure on $SC^D_*(G)$. We begin with a technical ingredient which allows us to pick a representative for each isomorphism class $[K,o] \in SC^D_*(G)$ of rooted simplicial $G$-complexes in a measurable way.

Let

$$
\mathbb{N}_G = \bigsqcup_{X \in \text{Orb}(G)} \mathbb{N}_0 \times X
$$

and let $\Delta^D(\mathbb{N}_G)$ be the simplicial $G$-complex consisting of all finite nonempty subsets of $\mathbb{N}_G$ with at most $D + 1$ elements. The action of $G$ is defined via the second coordinate. Every subcomplex $S$ of $\Delta^D(\mathbb{N}_G)$ can be encoded by an element $f_S \in \{0,1\}^{\Delta^D(\mathbb{N}_G)}$ such that $f_S(\sigma) = 1$ exactly if the simplex $\sigma$ is contained in the subcomplex $S$. We endow $\{0,1\}^{\Delta^D(\mathbb{N}_G)}$ with the product topology, i.e., the topology generated by all cylinder sets. Let $\text{Sub}(\Delta^D(\mathbb{N}_G)) \subseteq \{0,1\}^{\Delta^D(\mathbb{N}_G)}$ be the subset which consists of elements encoding $G$-invariant subcomplexes which contain a unique orbit of the form $\{0\} \times X$; this is a closed subspace.

**Lemma 4.1.** There is a continuous map $\Psi : SC^D_*(G) \to \text{Sub}(\Delta^D(\mathbb{N}_G))$ such that $(\Psi([K,o]),\{0\} \times X)$ is a representative of $[K,o]$ for all $[K,o] \in SC^D_*(G)$.

**Proof.** We only sketch the proof; a detailed treatment of the nonequivariant case can be found in [13, Lemma 1].

We enumerate $\mathbb{N}_G$, the set of vertices of $\Delta^D(\mathbb{N}_G)$, in the following way. First we enumerate the (isomorphism classes of) $G$-sets $X_1, \ldots, X_k \in \text{Orb}(G)$ and for every $i$ we order the elements of $X_i = \{x_{i,1}, \ldots, x_{i,m_i}\}$. Finally, we enumerate $\mathbb{N}_G$ diagonally:

$$(0,x_{1,1}), \ldots, (0,x_{1,m_1}), (0,x_{2,1}), \ldots, (0,x_{k,m_k}), (1,x_{1,1}), \ldots$$

Once the set of vertices is ordered, a diagonal enumeration provides an order on the set of all simplices of $\Delta^D(\mathbb{N}_G)$. The lexicographic order on $\{0,1\}^{\Delta^D(\mathbb{N}_G)}$, given by $a < b$ if there is a simplex $\sigma_0$ such that $a(\sigma) = b(\sigma)$ for all $\sigma < \sigma_0$ and $a(\sigma_0) = 1$ but $b(\sigma_0) = 0$, defines an order on the subcomplexes of $\Delta^D(\mathbb{N}_G)$. We define $\Psi$ to map an isomorphism class $[K,o] \in SC^D_*(G)$ to the $\prec$-minimal subcomplex $\Lambda$ of $\Delta^D(\mathbb{N}_G)$ such that $(\Lambda, \{0\} \times X_i) \in [K,o]$, where $X_i$ is the $G$-set in $\text{Orb}(G)$ with $X_i \cong o$, and such that the elements of $\{0\} \times X_i$ are the only vertices of $\Lambda$ with first coordinate 0. For a finite simplicial $G$-complex the existence of the minimal subcomplex of $\Delta^D(\mathbb{N}_G)$ follows from the well-ordering principle. In the situation of an infinite simplicial $G$-complex $[K,o]$ it is a direct consequence of the fact that

$$B_r(\Psi([B_{r+1}(K,o)])) = \Psi([B_r(K,o)]).$$
The preimage of a cylinder set under $\Psi$ is a countable union of open sets $U(\alpha, r)$ in $\text{SC}_D^\ast(G)$ and therefore open, hence $\Psi$ is continuous. $\square$

For any simplicial complex $L$, we write $C_n^{(2)}(L)$ to denote the complex Hilbert space of square-summable oriented $n$-chains of $L$. The map $\Psi$ from the preceding lemma gives rise to a field of Hilbert spaces $[K, o] \rightarrow C_n^{(2)}(\Psi([K, o]))$ on $\text{SC}_D^\ast(G)$ for each $n \in \mathbb{N}$. In addition, every oriented $n$-simplex $s$ of $\Delta^D_{\text{rooted}}(G)$ yields a characteristic vector field $\xi_s$ defined as

$$
\xi_s([K, o]) = \begin{cases} 
s & \text{if } s \text{ belongs to } \Psi([K, o]) \\
0 & \text{otherwise}.
\end{cases}
$$

We observe that, since $\Psi$ is continuous, the function $[K, o] \rightarrow \langle \delta_s([K, o]), \delta_{s'}([K, o]) \rangle$ is continuous for all oriented simplices $s$ and $s'$; the $\xi_s$ form a fundamental sequence for a measurable field of Hilbert spaces; see [6, Prop. 4]. A vector field $\sigma : [K, o] \rightarrow C_n^{(2)}(\Psi([K, o]))$ is called measurable, if

$$
[K, o] \rightarrow \langle \sigma([K, o]), \xi_s([K, o]) \rangle
$$

is measurable for every oriented $n$-simplex $s$ of $\Delta^D_{\text{rooted}}(G)$. Let $\mu$ be a random rooted simplicial $G$-complex. The measurable vector fields $\sigma$ with the property

$$
\|\sigma\|^2 := \int_{\text{SC}_D^\ast(G)} \|\sigma([K, o])\|^2 d\mu < \infty
$$

form a pre-Hilbert space using the inner product

$$
\langle \sigma, \sigma' \rangle = \int_{\text{SC}_D^\ast(G)} \langle \sigma([K, o]), \sigma'([K, o]) \rangle d\mu.
$$

By factoring out the subspace of vector fields which vanish almost everywhere, we obtain a Hilbert space: the associated direct integral;

$$
C_n^{(2)}(\text{SC}_D^\ast(G), \mu) := \int_{\text{SC}_D^\ast(G)} C_n^{(2)}(\Psi([K, o])) d\mu,
$$

for details see [6, p. 168]. The differentials $\partial_n, [K, o]$ and their adjoints $d_{[K, o]}^n$ of the fibres $C_n^{(2)}(\Psi([K, o]))$ define bounded operators (compare to [13])

$$
\partial_n : C_n^{(2)}(\text{SC}_D^\ast(G), \mu) \rightarrow C_{n-1}^{(2)}(\text{SC}_D^\ast(G), \mu),
$$
$$
d_n : C_{n-1}^{(2)}(\text{SC}_D^\ast(G), \mu) \rightarrow C_n^{(2)}(\text{SC}_D^\ast(G), \mu),
$$

which commute with the induced unitary $G$-action on $C_n^{(2)}(\text{SC}_D^\ast(G), \mu)$, since they commute fibrewise and $G$ preserves fibres. Therefore, we have for each random rooted simplicial $G$-complex a chain complex $C_n^{(2)}(\text{SC}_D^\ast(G), \mu)$ and a Laplace operator $\Delta_n = \partial_{n+1} \circ d_{n+1} + d_n \circ \partial_n$ which also commutes with the $G$-action.
Definition 4.2. We define the $n$-th simplicial $ℓ^2$-homology of a random rooted simplicial complex $μ$ as the Hilbert space

$$H_n^{(2)}(SC^D(G), μ) := \ker Δ_n$$
equipped with the natural unitary action of $G$.

We would like to have a notion of dimension for a subspace of $C_n^{(2)}(SC^D(G), μ)$, to this end we introduce a von Neumann algebra with a trace. A bounded linear operator $T$ on $C_n^{(2)}(SC^D(G), μ)$ is decomposable, if there is an essentially bounded measurable field of operators $[K, o] \mapsto T_{[K, o]}$ such that $T = \int_Ω T_{[K, o]} dμ([K, o])$; see [6, p. 183]. The bounded decomposable operators $T$ on $C_n^{(2)}(SC^D(G), μ)$ such that for almost all $[K, o]$ and for all isomorphisms $\varphi: Ψ([K, o]) → Ψ([K, o'])$ of simplicial $G$-complexes the identity

$$⟨T_{[K, o]}σ([K, o]), σ([K, o])⟩ = ⟨T_{[K, o]}′σ([K, o]), σ([K, o])⟩$$

holds, form a von Neumann algebra $A_n(μ)$. In fact, to see that $A_n(μ)$ is closed in the strong operator topology one can use [6] Prop. 4, p. 183. Of course, the operators defined by elements of $G$ are contained in $A_n(μ)$, since we only consider isomorphisms which commute with the $G$-action. Moreover, $Δ_n \in A_n(μ)$, because $∂_s$ and $d_s$ commute with the chain map $φ_2: C_n^{(2)}(Ψ([K, o])) → C_n^{(2)}(Ψ([K, o']))$ induced by an isomorphism. For $T \in A_n(μ)$ we define

$$\text{tr}(T) = \sum_{X \in \text{Orb}(G)} \sum_{x ∈ X} \sum_{s ∈ Δ^D(NG)(n)} \frac{⟨Tξ_s, ξ_s⟩}{|X|(n + 1)},$$

where $Δ^D(NG)(n)$ denotes the set of $n$-simplices of $Δ^D(NG)$. Note that the formula does not depend on the chosen orientation of $s$. As in the nonequivariant case one can verify that $\text{tr}(ST) = \text{tr}(TS)$; see [13] after Definition 5. We obtain a normal, faithful and finite trace on $A_n(μ)$; compare to [13] Prop. 3.

Definition 4.3. Let $K$ be a field of $G$-invariant subspaces of $C_n^{(2)}(SC^D(G), μ)$ such that $φ_2 K([K, o]) = K([K, o'])$ for every isomorphism $φ: Ψ([K, o]) → Ψ([K, o'])$. Then the projection $\text{pr}_K: [K, o] → \text{pr}_K([K, o])$ is an element of $A_n(μ)$ and we define the von Neumann dimension of $K$ as

$$\dim_{vN}(K) = \text{tr}(\text{pr}_K).$$

Example 4.4. Let $L$ be a finite simplicial $G$-complex and $μ_L^G$ the associated random rooted simplicial $G$-complex from Example 2.3. Let $K$ be a field of $G$-invariant subspaces of $C_n^{(2)}(SC^D(G), μ_L^G)$ as in Definition 4.3. Given an orbit $o \subseteq L^{(0)}$ and an isomorphism $η: L → Ψ([L, o])$ we define $K(L) = η^{-1}(K([L, o])) \subseteq C_n^{(2)}(L)$; this
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The subspace does not depend on \( o \) and \( \eta \). We compute the dimension:

\[
\dim_v \mathcal{N}(\mathcal{K}) = \text{tr}(pr_K) = \sum_{X \in \text{Orb}(G)} \sum_{s \in \Delta^D(N_G)} \int_{\mathcal{S}C^D(G)} \langle pr_K \xi_s, \xi_s \rangle |X|(n+1) d\mu^G_L
\]

\[
= \frac{1}{|L^{(0)}|} \sum_{y \in L^{(0)}} \sum_{x \in G_y} \sum_{s \in L(n)} \langle pr_{KL(s)}, s \rangle |Gy|(n+1)
\]

\[
= \frac{1}{|L^{(0)}|} \sum_{s \in L(n)} \sum_{x \in s} \langle pr_{KL(s)}, s \rangle |Gy|(n+1) = \dim_C \mathcal{K}(L)
\]

4.2. Approximation of ℓ²-multiplicities. In order to fix our notation we recall some facts from the representation theory of finite groups. Let \( G \) be a finite group and let \((\rho, V) \in \text{Irr}(G)\) be an irreducible representation of \( G \) on the complex vector space \( V \). Usually we will not mention the underlying vector space and simply speak of the representation \( \rho \). Every irreducible representation is finite dimensional and is uniquely determined by its character \( \chi_\rho : G \to \mathbb{C} \) which maps \( g \in G \) to the trace of \( \rho(g) \). In particular, \( \chi_\rho(1) \) is the dimension of the underlying space \( V \).

Let \((\sigma, W)\) be any finite dimensional complex representation of \( G \), then \( W \) can be decomposed into isotypic components

\[
W = \bigoplus_{\rho \in \text{Irr}(G)} W_\rho
\]

where each \( W_\rho \) is (noncanonically) isomorphic to a direct sum of copies of \( \rho \), i.e., \( W_\rho \cong \rho^{m(\rho, \sigma)} \) where the number \( m(\rho, \sigma) \) of copies is called the multiplicity of \( \rho \) in \( \sigma \).

The irreducible representations correspond bijectively to the central idempotents in the group ring \( \mathbb{C}[G] \). The central idempotent corresponding to \( \rho \) is

\[
P_\rho = \frac{\chi_\rho(1)}{|G|} \sum_{g \in G} \chi_\rho(g) g \in \mathbb{C}[G];
\]

see [9 (2.12)]. The element \( P_\rho \) defines the orthogonal projection onto the \( \rho \)-isotypic component in every unitary representation of \( G \).

**Definition 4.5.** Let \( \mu \) be a random rooted simplicial \( G \)-complex and \( \rho \) an irreducible representation of \( G \). The \( \ell² \)-multiplicity of \( \rho \) in the homology of \( \mu \) is

\[
m^{(2)}(\rho, \mu) = \frac{1}{\chi_\rho(1)} \dim_v \mathcal{N}(H^2_n(\mathcal{S}C^D(G), \mu)_\rho)
\]

where \( H^2_n(\mathcal{S}C^D(G), \mu)_\rho \) denotes the \( \rho \)-isotypic component in the homology of \( \mu \).

In addition, we define the \( n \)-th \( \rho \)-Laplacian to be

\[
(Id - P_\rho) + \Delta_n =: \Delta_{n,\rho},
\]
where $\Delta_n$ is the Laplacian of $C_n^{(2)}(SC^D_\ast(G), \mu)$.

**Remark 4.6.** If $G$ is the trivial group $\{1\}$ and $\rho$ is the unique irreducible representation of $G$, i.e., the trivial $1$-dimensional representation, then $m_n^{(2)}(\rho, \mu) = b_n^{(2)}(\mu)$ is simply the $n$-th $\ell^2$-Betti number of a random rooted simplicial complex defined in [13, Def. 6].

**Example 4.7.** Let $K$ be a finite simplicial $G$-complex and $\mu_K^G$ the associated random rooted simplicial $G$-complex. Then it follows from Example 4.4 that $m_n^{(2)}(\rho, \mu_K^G)$ is the ordinary multiplicity of the representation $\rho$ in $H_n(K, \mathbb{C})$ divided by the number of vertices, i.e.,

$$m_n^{(2)}(\rho, \mu_K^G) = \frac{m(\rho, H_n(K, \mathbb{C}))}{|K^{(0)}|}.$$

**Lemma 4.8.** The operator $\Delta_{n,\rho}$ is positive self-adjoint and the operator norm $\|\Delta_{n,\rho}\|$ is bounded above by a constant $b(n, D)$ which depends only on $n$ and $D$. Moreover, the kernel of $\Delta_{n,\rho}$ is $H_n^{(2)}(SC^D_\ast(G), \mu_\rho)$.

**Proof.** It is easy to see that the operator $\Delta_{n,\rho}$ is positive self-adjoint using that $\Delta_n$ and $\text{Id} - P_\rho$ have these properties and commute. It is well-known (see [13, Proposition 2]) that the operator norm of the Laplacian $\Delta_n$ is bounded, since the bound on the vertex degree yields a bound for the number of $(n+1)$-simplices which contain a given $n$-simplex. Now, the operator $\text{Id} - P_\rho$ is a projection and we find $\|\Delta_{n,\rho}\| \leq \|\Delta_n\| + 1$.

Observe that a vector $x$ lies in $\ker(\Delta_{n,\rho})$ if and only if $\langle \Delta_{n,\rho}x, \Delta_{n,\rho}x \rangle = 0$. We note further that

$$\langle \Delta_{n,\rho}x, \Delta_{n,\rho}x \rangle = \|(\text{Id} - P_\rho)x\|^2 + \|\Delta_n x\|^2 + 2\langle \Delta_n (\text{Id} - P_\rho)x, x \rangle$$

since $P_\rho$ and $\Delta_n$ are self-adjoint and commute. All three summands are nonnegative, since $\Delta_n (\text{Id} - P_\rho)$ is a positive operator. We conclude that

$$\ker(\Delta_{n,\rho}) = \ker \Delta_n \cap \ker (\text{Id} - P_\rho) = \ker \Delta_n \cap \text{im}(P_\rho) = H_n^{(2)}(SC^D_\ast(G), \mu_\rho).$$

\[\square\]

Let $E_{\Delta_{n,\rho}}$ be the unique projection valued measure obtained from the spectral calculus for the bounded and self-adjoint operator $\Delta_{n,\rho}$. Then $E_{\Delta_{n,\rho}}$ has the property that for all bounded Borel functions $f$ on $\mathbb{R}$

$$f(\Delta_{n,\rho}) = \int_{\mathbb{R}} f(\lambda) dE_{\Delta_{n,\rho}}(\lambda).$$

Further, we define the **spectral measure** of $\Delta_{n,\rho}$ as

$$\nu_{n,\rho}(B) := \text{tr}(E_{\nu_{n,\rho}}(B)).$$

for every Borel set $B$. The spectral measure satisfies $\text{tr}(f(\Delta_{n,\rho})) = \int_{\mathbb{R}} f(\lambda) d\nu_{n,\rho}$ for all bounded Borel functions $f$ on $\mathbb{R}$.
Lemma 4.9. Let \((\mu^k)_{k \in \mathbb{N}}\) be a sequence of random rooted simplicial \(G\)-complexes which converges weakly to \(\mu^\infty\) and let \(\nu^k_{n,\rho}\) be the associated spectral measures of the \(n\)-th \(\rho\)-Laplacians \(\Delta^r_{n,\rho}\). Then \((\nu^k_{n,\rho})_k\) converges weakly to \(\nu^\infty_{n,\rho}\).

Proof. For the sake of simplicity we denote \(\nu^k_{n,\rho}\) and \(\nu^\infty_{n,\rho}\) by \(\nu^k\) and \(\nu^\infty\) respectively. Since, by Lemma 4.8, \(\text{spec}(\Delta^r_{n,\rho}) \subseteq [0, R]\) for some \(R > 0\), Weierstraß approximation implies that it is enough to check the identity
\[
\lim_{k \to \infty} \int_0^R f(\lambda) d\nu^k = \int_0^R f(\lambda) d\nu^\infty
\]
for all polynomials \(f \in \mathbb{R}[x]\). By linearity we can further assume that \(f = x^r\).

\[
\int_0^R f(\lambda) d\nu^k = \text{tr}(\Delta^r_{n,\rho}) = \sum_{X \in \text{Orb}(G)} \sum_{x \in X} \sum_{s \in \Delta^r_{\text{orb}(G)}(n)} \int_{\mathcal{S}^D_\rho(G)} \frac{\langle \Delta^r_{n,\rho} \xi_s, \xi_s \rangle}{|X|(n + 1)} d\mu^k
\]

Let us consider \(\Delta^r_{n,\rho}\) and observe that
\[
\Delta^r_{n,\rho} = ((\text{Id} - P_\rho) + \Delta_n)^r = \sum_{j=0}^{r} \binom{r}{j} (\text{Id} - P_\rho)^{r-j} \Delta_n^j
\]
\[
= \Delta_n^r + \sum_{j=0}^{r-1} \binom{r}{j} \Delta_n^j (\text{Id} - P_\rho).
\]

Let \(s \in \Delta(N_G)(n)\) with \((0, x) \in s\) and \(x \in X\) and suppose that \(s \in \Psi([K, o])\) for a rooted isomorphism class \([K, o]\). Then \((\text{Id} - P_\rho)(s)\) is supported in the 1-ball of the orbit \(\{0\} \times X\), since \(P_\rho\) is a linear combination of elements \(g \in G\) which only act on the second coordinate. Further, \(\Delta_n(s)\) is a linear combination of simplices in the 2-ball around \((0, x)\). Therefore, \(\langle \Delta^r_{n,\rho} \xi_s, \xi_s \rangle\) only depends on the \(2r + 1\)-neighbourhood of the orbit \(\{0\} \times X\). By the weak convergence of the sequence \((\mu^k)_k\) we obtain that
\[
\lim_{k \to \infty} \int_{U(\alpha, 2r + 1)} \langle \Delta^r_{n,\rho} \xi_s, \xi_s \rangle d\mu^k = \int_{U(\alpha, 2r + 1)} \langle \Delta^r_{n,\rho} \xi_s, \xi_s \rangle d\mu^\infty,
\]
for all finite rooted simplicial \(G\)-complexes \(\alpha\). Now the claim follows from the fact that \(\mathcal{S}^D_\rho(G)\) is a finite union of open sets of the form \(U(\alpha, 2r + 1)\).

Now we can prove Theorem 1.1. We follow the well-known strategy, going back to Lück [11] and Schick [12], of bounding the Fuglede-Kadison determinant. Here we have to use nonrational algebraic coefficients and the corresponding method is inspired from [7, Section 3]; compare also [10, Lemma 3.14].

Proof of Theorem 1.1. Since all \(\mu_k\) are sofic, it is sufficient to prove the theorem under the assumption that the sequence \(\mu_k = \mu^G_k\) is actually a sequence of finite
simplicial $G$-complexes. Let $\nu^k_{n,\rho}$ (resp. $\nu^\infty_{n,\rho}$) be the spectral measure of the $n$-th $\rho$-Laplacian of $K_k$ (resp. of $\mu^\infty$).

By Lemma 4.8 we have to show that $\nu^k_{n,\rho}(\{0\}) = \chi_\rho(1)m_n^{(2)}(\rho, \mu_{K_k})$ converges to $\nu^\infty_{n,\rho}(\{0\})$. In view of Lemma 4.9 it remains to show that the Fuglede-Kadison determinant

$$\det(\nu^k_{n,\rho}) = \exp \int_{(0)} \log(\lambda) d\nu^k_{n,\rho}$$

of $\nu^k_{n,\rho}$ is uniformly bounded away from zero; see [10, Lemma 2.20].

Let $E \subseteq \mathbb{C}$ be a finite Galois extension of $\mathbb{Q}$ which is a splitting field for the finite group $G$; see [9, (9.10)] for the existence. In particular, all irreducible characters of $G$ take values only in the ring of integers $\mathcal{O}_E$ of $E$; see [9, (3.6)].

Fix $k \in \mathbb{N}$. We pick a basis of $C^{(2)}_n(K_k)$ by choosing an orientation for every $n$-simplex of $K = K_k$. The transformation matrix $A_\rho$ of the $\rho$-Laplacian $\Delta_{n,\rho}$ on $K_{n,\rho}$ with respect to this basis has entries in $\mathcal{O}_E$. The spectral measure $\nu^k_{n,\rho}$ agrees with the spectral measure of $A_\rho$ normalized by the number of vertices $|K^{(0)}|$; cf. Example 4.4. In particular, the power $\det(\nu^k_{n,\rho})^{\lambda} \Delta_{n,\rho}$ of the Fuglede-Kadison determinant is just the product over all non-zero eigenvalues of $A_\rho$, i.e., the lowest non-zero coefficient $c_\rho$ of the characteristic polynomial of $A_\rho$. We note that $c_\rho \in |G|^{-|K(n)|} \mathcal{O}_E$ and since, by Lemma 4.8, the operator norm of $\Delta_{n,\rho}$ is bounded above, there is an upper bound $|c_\rho| \leq b|K(n)|$ where $b$ depends only on $n$ and $D$.

Consider the action of the Galois group $\text{Gal}(E/\mathbb{Q})$ on the irreducible representations of $G$; cf. [9, p.152]. If $\tau \in \text{Gal}(E/\mathbb{Q})$ be a Galois automorphism of $E$, then $\tau(A_\rho) = A_{\tau(\rho)}$ and $\tau(c_\rho) = c_{\tau(\rho)}$. In particular, we obtain

$$|G|^{|E: \mathbb{Q}|K(n)|} \prod_{\tau \in \text{Gal}(E/\mathbb{Q})} \tau(c_\rho) \in \mathbb{Z} \setminus \{0\}$$

and therefore $|c_\rho| \geq |G|^{-|E: \mathbb{Q}|K(n)|}b^{-|(E: \mathbb{Q})|K(n)|}$. Since the vertex degree is bounded above by $D$, there is a constant $t > 0$ only depending on $n$ and $D$ such that $|K(n)| \leq t|K^{(0)}|$. We conclude that

$$\det(\nu^k_{n,\rho}) = |c_\rho|^{1/|K^{(0)}|} \geq |G|^{-|E: \mathbb{Q}|t}b^{-|(E: \mathbb{Q})|t}$$

and this lower bound does not depend on $K_k$. \hfill \Box

Let $G$ be a finite group and $H$ be a subgroup. Let $(\sigma, V)$ be a finite dimensional complex representation of $H$. Recall that the multiplicity of an irreducible representation $\rho \in \text{Irr}(G)$ in $\mathbb{C}[G] \otimes_{\mathbb{C}[H]} V$ is determined by the Forbenius reciprocity formula

$$m(\rho, \mathbb{C}[G] \otimes_{\mathbb{C}[H]} V) = \sum_{\theta \in \text{Irr}(H)} m(\theta, \rho|_H) \cdot m(\theta, V);$$
see [9 (5.2)]. The following theorem provides a reciprocity formula for induced sofic random rooted $G$-complexes. In particular, with $H = \{1\}$ we obtain Theorem 1.2 stated in the introduction.

**Theorem 4.10.** Let $\mu$ be a random rooted simplicial $H$-complex. If $\mu$ is sofic, then $\text{ind}_H^G(\mu)$ is sofic and, moreover,

$$m_n^{(2)}(\rho, \text{ind}_H^G(\mu)) = \frac{|H|}{|G|} \sum_{\theta \in \text{Irr}(H)} m(\theta, \rho|_H) m_n^{(2)}(\theta, \mu)$$

for every $\rho \in \text{Irr}(G)$ and all $n \in \mathbb{N}_0$.

**Proof.** We write $\nu = \text{ind}_H^G(\mu)$. Since $\mu$ is sofic, we can find a sequence $K_k$ of finite simplicial $H$-complexes such that the associated random rooted simplicial complexes $\mu_k$ converge weakly to $\mu$. Continuity of induction (Lemma 3.1) implies that the sequence $\text{ind}_H^G(\mu_k)$ converges to $\nu$. This shows that $\nu$ is sofic, since $\text{ind}_H^G(\mu_k)$ is the random rooted simplicial $G$-complex defined by $W_k = G \times_H K_k$.

Finally, using that $W_k$ is a disjoint union of $|G/H|$ copies of $K_k$ which are permuted by the action it follows that

$$H_n(W_k, \mathbb{C}) \cong \mathbb{C}[G] \otimes_{\mathbb{C}[H]} H_n(K_k, \mathbb{C}).$$

In particular, Frobenius reciprocity implies that

$$m(\rho, H_n(W_k, \mathbb{C})) = \sum_{\theta \in \text{Irr}(H)} m(\theta, \rho|_H) m(\theta, H_n(K_k, \mathbb{C})).$$

Based on the relation of multiplicities and $\ell^2$-multiplicities from Example 4.7 an application of Theorem 1.1 completes the proof. \qed

**Remark 4.11.** It appears to us that formula (4.1) should hold without the assumption of soficity. However, due to some technical problems and based on the fact that we do not know a single example of a non-sofic random rooted simplicial complex, we decided to restrict to the sofic case.

**Example 4.12.** (The $\ell^2$-multiplicities of Sierpinski’s triangle) We return to the setting of Example 3.4 and we compute the $\ell^2$-multiplicities of Sierpinski’s triangle with the rotation action of the cyclic group $G = \langle \sigma \rangle$ of order 3. Recall that, with this action, the Sierpinski triangle is an induced random rooted simplicial complex $\text{ind}_1^G(\tau_S)$ where $\tau_S$ is the limit of a sequence of finite 2-dimensional simplicial complexes $T_n$. Therefore, by Theorem 1.2 it is sufficient to compute the $\ell^2$-Betti numbers of $\tau_S$.

In order to use the Approximation Theorem 1.1 for the action of the trivial group (compare also [13]), we compute the normalized Betti numbers of every $T_n$. Note that $T_n$ is homotopy equivalent to a 1-dimensional complex, thus it is sufficient to calculate the normalized Euler characteristic of $T_n$. Using the formulas given in
Example 3.4 we find
\[
\frac{\chi(T_n)}{|T_n^{(0)}|} = 1 - \frac{2 \cdot 3^{n+1}}{3^{n+1} + 3} + \frac{2 \cdot 3^n}{3^{n+1} + 3} \xrightarrow{n \to \infty} -\frac{1}{3}.
\]
Every \( T_n \) is connected, so \( b_0(T_n) = 1 \) for all \( n \) and the normalized zeroth Betti numbers tend to zero; i.e., \( b_0^{(2)}(\tau_S) = 0 \). We deduce that \( b_1^{(2)}(\tau_S) = \frac{1}{3} \). Finally, we apply Theorem 1.2 to deduce that \( m_1^{(2)}(\rho, \text{ind}_1^G(\tau_S)) = \frac{1}{9} \) for every irreducible representation \( \rho \in \text{Irr}(G) \). All \( \ell^2 \)-multiplicities of Sierpinski’s triangle vanish outside of degree 1.

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