PROJECTIVE DUALITY, UNEXPECTED HYPERSURFACES AND LOGARITHMIC DERIVATIONS OF HYPERPLANE ARRANGEMENTS

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Abstract. Several papers have been written studying unexpected hypersurfaces. We say a finite set of points $Z$ admits unexpected hypersurfaces if a general union of fat linear subspaces imposes less that the expected number of conditions on the ideal of $Z$. In this paper, we introduce the concept of a very unexpected hypersurface. This is a stronger condition which takes into account an explanation for some hypersurfaces previously considered unexpected.

We then develop a duality theory to relate the study of very unexpected hypersurfaces to the derivations of dual hyperplane arrangements. This allows us to generalize results in the plane of Cook, Harbourne, Migliore, Nagel [7] Faenzi, and Vallés [13] to higher dimensions. In particular, we give a criterion to determine if a set $Z$ admits very unexpected hypersurfaces in the case $Z$ is invariant under the action of an irreducible reflection group on the ambient projective space.

Our approach has new applications even in $\mathbb{P}^2$ where we are able to place strong conditions on sets of points $Z$ which admit certain types of unexpected curves. We close relating Terao’s Freeness Conjecture for line arrangements to a conjecture due to G. Dirac [9] on configurations of points in $\mathbb{R}^2$.

1. Introduction

Determining the dimension of a linear system is a fundamental problem in algebraic geometry, and many classic results are devoted to determining the dimensions of certain linear systems. Frequently, when working with families of linear systems, there is a naive dimension count, or an expected dimension, that should hold in the general case. The difficulty then lies in determining when this dimension count fails.

As an illustration, we consider the projective coordinate ring $R = \mathbb{C}[X_0, X_1, X_2, X_3]$ of $\mathbb{P}^3$. For a proper linear subspace $L \subset \mathbb{P}^3$ vanishing on $L$ imposes $\left( \dim \frac{L+d}{d} \right)$ conditions on forms of degree $d$. This gives a dimension count for the ideal $I(L)$ of $\dim[I(L)]_d = \dim[R_d] - \left( \dim \frac{L+d}{d} \right) = \left( \frac{d+3}{d} \right) - \left( \frac{d+2}{d} \right)$.

By extension, if $Z \subset \mathbb{P}^3$ is any proper subscheme we might expect that

$$\dim[I(Z) \cap I(L)]_d = \max \left\{ 0, \dim[I(Z)]_d - \left( \dim \frac{L+d}{d} \right) \right\}.$$ 

In fact the above equality holds for all sufficiently large $d$ as long as $L$ and $Z$ have empty intersection. Let $Z$ be a 0-dimensional scheme and let $L$ be a general linear subspace, to ensure that $L \cap Z = \emptyset$. We then get the expected dimension:

$$[I(Z) \cap I(L)^m]_d = \min \{ 0, \dim[I(Z)]_d - \dim[R/I(L)^m]_d \}. \tag{1}$$

Many papers have been written exploring when the above formula fails to give the actual dimension. If $Z$ is a set of double points with general support and $L$ is a general point with $m = 2$, the celebrated theorem of Alexander and Hirschowitz [2] gives a complete characterization of when equation 1 fails to give the correct count. Where $m = 1$ and $L \subset \mathbb{P}^3$ is a line, the paper [21] relates equation 1 to Lefschetz properties of general Artinian reductions of $R/I(Z)$.

More recently, a number of papers have been released (see for instance [17],[10],[4] and [16]), which studied failure of expected dimension under the name unexpected hypersurfaces or unexpected curves. These papers all study the above linear systems, where $Z$ is a reduced sets of points in $\mathbb{P}^n$, and $L$ is (possibly multiple) general linear subspace with an associated multiplicity. Many of these papers took inspiration from the paper [7]. The authors of [7] built off of earlier work in [8] and introduced the concept...
of unexpected curves in \( \mathbb{P}^2 \). Namely, they said that \( Z \) admits unexpected curves in degree \( d \) if for a general point \( X \),

\[
\dim [I(Z) \cap I(X)^{d-1}]_d > \max \left\{ 0, \dim [I(Z)]_d - \dim[R/I(X)^{d-1}]_d \right\}.
\]

The authors of [7] were able to give a full characterization of the degrees in which a set of points admits unexpected curves. Surprisingly, this characterization does not directly depend on the dimensions of either \( [I(Z)]_d \) or \( [I(Z) \cap I(X)^{d-1}]_d \). Namely, this information can be replaced with combinatorial information about \( Z \), and data coming from the (reduced) module of derivations, \( D_0(\mathcal{A}_Z) \), of the line arrangement, \( \mathcal{A}_Z \), dual to \( Z \). See definition 3.18 for the definition of splitting type.

**Theorem 5.1 ([7]).** For a finite set of points \( Z \subseteq \mathbb{P}^2 \), let \( \mathcal{A}_Z \) denote the dual line arrangement, and let \( (a_1, a_2) \) denote the splitting type of the bundle defined by \( D_0(\mathcal{A}_Z) \). Then exactly one of the following statements holds:

(i) There is some line \( L \subseteq \mathbb{P}^2 \) with \( |L \cap Z| > a_1 + 1 \), in which case \( |L \cap Z| = a_2 + 1 \) and \( Z \) never admits unexpected curves.

(ii) \( Z \) admits unexpected curves in degree \( d \) for precisely those \( d \) with \( a_1 < d < a_2 \).

This result allowed researchers to discover many new examples of unexpected curves by taking advantage of decades of prior research on line arrangements.

Given the observed connection between certain line arrangements and unexpected curves, it is natural to wonder if a similar connection exists in higher dimensions. In this paper we show that this is true at least to a certain extent. More specifically, if \( Z \subseteq \mathbb{P}^n \) is a finite set of points and \( L \) is a general codimension 2 linear subspace, we establish a general duality connecting the module of derivations \( D_0(\mathcal{A}_Z) \) of the dual hyperplane arrangement to the intersection of ideals \( [I(Z) \cap I(L)]_{d+1} \). In particular this allows us to recover \( \dim [I(Z) \cap I(L)]_d \) from knowledge of the splitting type of \( D_0(\mathcal{A}_Z) \).

In order to generalize 5.1, we introduce a modified definition of unexpected hypersurface which we call very unexpected hypersurfaces. Given a generic linear subspace \( L \subseteq \mathbb{P}^n \) admits very unexpected \( L \)-hypersurfaces if the intersection \( [I(Z) \cap I(L)]_d \) is larger than expected, as long as this failure is not “easily explained” (see definition 5.7). Our definition of very unexpected hypersurfaces is more technical than that of unexpected hypersurfaces. However the two definitions agree in \( \mathbb{P}^2 \), in that a set of points \( Z \subseteq \mathbb{P}^2 \) admits very unexpected curves if and only if it admits unexpected curves.

This new definition has a few advantages compared with the definition for unexpected hypersurfaces. The first is that with the standard definition of unexpected hypersurfaces a generalization of theorem 5.1 to higher dimensions is impossible. The second is that, as we mentioned, in certain cases the “unexpectedness” can be relatively easily explained. For instance, if all of the points in \( Z \) lie on a proper subspace \( H \), “unexpectedness” may simply be a consequence of the fact that \( [I(H) \cap I(L)]_d \subseteq [I(Z) \cap I(L)]_d \) (for further discussion, see example 5.4). It is then somewhat surprising that by merely accounting for cases where “unexpectedness” is well explained, we are able to recover a generalization of theorem 5.1. More specifically, if \( L \) is a generic codimension 2 subspace, the degrees in which \( Z \) admits very unexpected \( L \)-hypersurfaces can again be characterized by the combinatorial data of \( Z \) in conjunction with the splitting type of the Derivation Bundle of \( \mathcal{A}_Z \). We define, for every integer \( d \geq 0 \), a number \( \text{Ex.} C(Z, d) \) via a combinatorial optimization problem on the matroid of \( Z \) (see definition 5.20). Using this number, \( \text{Ex.} C(Z, d) \), we obtain the result below.

**Theorem 5.27.** Let \( Z \subseteq \mathbb{P}^n \) be a finite set of points, and suppose that \( D_0(\mathcal{A}_Z) \) has splitting type \( (a_1, \ldots, a_n) \). Then for a fixed integer \( d \),

\[
\sum_{i=1}^n \max\{0, d - a_i\} \leq nd + 1 - \text{Ex.} C(Z, d)
\]

and the inequality is strict if and only if \( Z \) admits very unexpected hypersurfaces in degree \( d \).

In the case the points of \( Z \) are not too concentrated on some proper subspace we obtain the following result which mimics theorem 5.1.
**Theorem 5.31.** Let \( Z \subseteq \mathbb{P}^n \) and let \( (a_1, \ldots, a_n) \) be the splitting type of \( D_0(A_Z) \), where \( a_i \leq a_{i+1} \). Suppose for all positive dimensional linear subspaces \( H \subseteq \mathbb{P}^n \), we have that
\[
\frac{|Z \cap H| - 1}{\dim H} \leq \frac{|Z| - 1}{n}.
\]
Then for an integer \( d \) the following are equivalent:

(a) \( Z \) admits very unexpected hypersurfaces in degree \( d \).
(b) \( Z \) admits unexpected hypersurfaces in degree \( d \).
(c) \( a_1 < d < a_n \).

Moreover we show in proposition 5.36 that this condition holds if an irreducible reflection group \( G \subseteq \mathbb{PGL}(K, 2) \) which acts on \( Z \).

After discussing the theory of Unexpected Hypersurfaces in general, we apply this duality between \( I(Z) \) and \( D_0(A_Z) \) to establish some structural results about both very unexpected \( Q \)-hypersurfaces and the module of derivations \( D_0(A_Z) \). Unlike the first part of the paper where there are few dimension and field constraints, these results focus on unexpected curves in \( \mathbb{P}^2_C \). In particular, we establish the following bound sharp for all \( d \geq 1 \).

**Theorem 7.6.** Let \( Z \subseteq \mathbb{P}^2_C \) and suppose that \( |Z| \) admits an unexpected curve in degree \( d \geq 1 \), then \( |Z| \leq 3d - 3 \).

We note that if \( d \) is the smallest such degree in which \( Z \) admits unexpected curves then it follows from theorem 5.1 that \( 2d + 1 \leq |Z| \). Consequently, no \( Z \subseteq \mathbb{P}^2_C \) admits unexpected curves in degree 3 or lower.

Additionally, we show the splitting type of \( D_0(A_Z) \) can be easily determined using only the initial degree of \( D_0(A_Z) \). We note that the splitting type is determined by the initial degree of the restriction of \( D_0(A_Z) \) to a general line.

**Theorem 6.9.** Let \( Z \) be a finite set of points in \( \mathbb{P}^2_C \) and let \( \alpha(D_0(A_Z)) \) denote the initial degree of \( D_0(A_Z) \). Define \( a = \min \{ \alpha(D_0(A_Z)), \left\lfloor \frac{|Z| - 1}{2} \right\rfloor \} \) then \( D_0(A_Z) \) has splitting type \( (a, |Z| - a - 1) \).

The paper proceeds as follows. After defining some notation in section 2. We discuss some needed background on the module of logarithmic derivations of a hyperplane arrangement in section 3. The reader familiar with Hyperplane Arrangements can likely skip this section with perhaps the exception of some non-standard notation found in definition 3.7 and definition 3.14.

We proceed in section 4, expanding on the Faenzi-Vallés duality between the module of derivations \( D_0(A_Z) \) of a hyperplane arrangement and certain elements of the ideal \( I(Z) \) of points dual to \( A_Z \). The results of this section are not wholly original as much of this is implicit in the first section of [13]. Our approach however, is much more explicit and amenable to computation it also has the advantage of working in arbitrary characteristic. We state two versions of this correspondence, the first (theorem 4.8) applies to the module, \( D_0(A_Z) \) itself, and we do not believe it has been stated before in this form. The second correspondence (theorem 4.14) applies to the restriction of \( D_0(A_Z) \) to a general line, generalizes the duality found in [13]. We note that despite the similarities in results, our method of proof and presentation is quite different from the one given in [13]. Additionally, the results here are not dependent on the characteristic of the ground field \( K \).

Section 5 introduces our definition of an very unexpected \( Q \)-hypersurface (see definition 5.7) for \( Q \) a generic subspace. We then look at the case when \( Q \) has codimension 2, establishing in theorem 5.21 that the degrees in which \( Z \) admits very unexpected \( Q \)-hypersurfaces depends only on the splitting type of \( D_0(A_Z) \) and a combinatorial optimization problem involving \( Z \).

Section 6 starts by establishing a lifting criterion for the restriction of \( D_0(A_Z) \) to a general line (see proposition 6.2). We then recall some results on vector bundles on \( \mathbb{P}^2_C \) and apply these to show that proposition 6.2 has especially strong consequences in \( \mathbb{P}^2_C \) (see theorem 6.8 and corollary 6.10).

In Section 7 we give strong combinatorial constraints on the sets of points \( Z \subseteq \mathbb{P}^2_C \) which can admit unexpected curves. In particular, if \( Z \subseteq \mathbb{P}^2_C \) admits an unexpected curve in degree \( d \), then theorem 7.9 shows that no more than \( d + 1 \) points of \( Z \) are in linearly general position and theorem 7.6 establishes a sharp bound on the number of points in \( Z \) showing that \( |Z| \leq 3d - 3 \).
In section 8 we show that if \( Z \subseteq \mathbb{P}^2_C \) admits unexpected curves in degree \( d \), then \( Z \) imposes independent condition on \((d-1)\) forms. We then briefly discuss generalizations to higher dimensions and some consequences.

We close with section 9, which discusses a few applications of these results to the field of Hyperplane arrangements. We focus on Terao’s Freeness Conjecture mainly in \( \mathbb{P}^2_C \). In particular, we look at the conjecture for real line arrangements and connect it to the Weak Dirac Conjecture on real point configurations. We have attempted to keep this paper as self contained and elementary as possible. This is largely true for the first 5 sections. However, in later sections we do apply some results from the theory of Vector Bundles and from the combinatorics of line arrangements in \( \mathbb{P}^n_C \).

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2. Notation and Conventions

Throughout this paper \( \mathbb{K} \) will denote an algebraically closed of arbitrary characteristic, unless specified otherwise. However, most of these results hold as long as \( \mathbb{K} \) is infinite. \( V \) and \( W \) will be dual \( \mathbb{K} \)-vector spaces. That is we suppose that there is a non-degenerate bilinear pairing \( B(\ , \ ) : V \times W \rightarrow \mathbb{K} \), inducing isomorphisms \( V \cong W^* \) and \( W \cong V^* \).

If \( V \) is a \( \mathbb{K} \) vector space, then \( V^* \) will denote the dual vector space of linear maps \( V \rightarrow \mathbb{K} \). Our pairing gives isomorphisms \( V \cong W^* \) and \( W \cong V^* \), we denote these isomorphisms \( v \mapsto \ell_v \) and \( w \mapsto \ell_w \), respectively. Here \( \ell_v(w) = \ell_w(v) = B(v, w) \). If \( H \subseteq V \) is a linear subspace, then \( H^\perp = \{ w \in W \mid \ell_w(H) = \{0\}\} \).

We similarly define \( L^\perp \subseteq V \), for \( L \subseteq W \).

Sym\((V^*)\) will denote the graded \( \mathbb{K} \)-algebra of symmetric tensors. Given a choice of basis \( \{Y_0, Y_1, .., Y_n\} \) of \( V^* \), Sym\((V^*)\), is naturally isomorphic to the polynomial algebra \( \mathbb{K}[Y_0, .., Y_n] \).

Moreover, the graded ring \( R = \text{Sym}(V^*) \) is naturally identifiable with the projective coordinate ring of \( \mathbb{P}(V) \). Dually, \( S = \text{Sym}(W^*) \) is the projective coordinate ring of \( \mathbb{P}(W) \).

The goal of this paper, is to relate properties of a finite set of points in \( \mathbb{P}(V) \) to their dual hyperplanes in \( \mathbb{P}(W) \).

3. Derivations of Hyperplane Arrangements

In this section we recall some facts about the module of logarithmic derivations \( D(\mathcal{A}) \) of a hyperplane arrangement \( \mathcal{A} \). In particular we state a few different known criteria for a general \( S \) derivation to lie in \( D(\mathcal{A}) \). We also give the definition (definition 3.18) of the splitting type of \( D(\mathcal{A}) \) which is used heavily in the sequel.

Definition 3.1. A (central) Subspace Arrangement, \( \mathcal{A} \), is a finite collection of linear subspaces \( \{H_0, .., H_s\} \) of a vector space \( W \).

If each \( H_i \) is a hyperplane, we say that \( \mathcal{A} \) is a Hyperplane Arrangement. We say \( \mathcal{A} \) is essential if the only subspace contained in all the hyperplanes in \( \mathcal{A} \) is the 0-subspace.

Remark 3.2. All subspace arrangements in this paper will be central. We make this restriction in order to identify a subspace arrangement \( \mathcal{A} \) in \( W \) with it’s image in \( \mathbb{P}(W) \), something we will do freely and often without comment.

A hyperplane arrangement is often defined in terms of a defining polynomial \( Q_{\mathcal{A}} = \prod_{H \in \mathcal{A}} \ell_H \). This is the product of linear forms each one defining a unique hyperplane in \( \mathcal{A} \).

Definition 3.3. If \( S \) is our graded polynomial ring and \( M \) is a graded \( S \)-module, then a \( \mathbb{K} \)-derivation of \( S \) into \( M \) is a graded \( \mathbb{K} \)-linear map \( \theta : S \rightarrow M \) which satisfies the Leibniz product rule. Namely for \( f, g \in S \)

\[
\theta(f \cdot g) = \theta(f) \cdot g + f \cdot \theta(g)
\]

These form a graded \( S \)-module, denoted Der\((S, M)\), obtained by setting \((f \cdot \theta)(g) = f(\theta(g))\).
We grade \( \text{Der}(S, M) \) by the \textit{polynomial degree}, namely, we set \( \deg \theta = \deg(\theta(\ell)) \), where \( \ell \in [S]_1 \).

In the case that \( M = S \), we set \( \text{Der}(S) := \text{Der}(S, S) \). In this paper our module \( M \) will either be \( S \) or a quotient ring of \( S \).

**Definition 3.4.** If \( \mathcal{A} \subseteq \mathbb{P}(W) \) is a Hyperplane Arrangement, we define the \textit{module of \( \mathcal{A} \)-derivations}, denoted \( D(\mathcal{A}) \), as submodule of \( \text{Der}(S) \) via

\[
D(\mathcal{A}) := \{ \theta \in \text{Der}(S) \mid \theta(I(H)) \subseteq I(H) \text{ for all } H \in \mathcal{A} \}
\]

**Remark 3.5.** Each element \( \alpha \in W \) defines a \( \mathbb{K} \)-derivation, \( \theta_\alpha \), of \( S = \text{Sym}(W^*) \). Namely, for \( \ell \in [S]_1 \) we set \( \theta_\alpha(\ell) = \ell(\alpha) \) and extended to all of \( S \) via the Leibniz product rule.

**Proposition 3.6.** Let \( S = \text{Sym}(W^*) \) and let \( M \) be a graded \( S \)-module, then there's an isomorphism of graded \( S \)-modules \( \text{Der}(S, M) \cong M \otimes_{\mathbb{K}} W \). Here the grading on \( M \otimes_{\mathbb{K}} W \) is given by that of \( M \).

Consequently, there's an isomorphism \( \text{Der}(S) \otimes_SM \cong \text{Der}(S, M) \).

**Proof.** Picking a basis \( Y_0, \ldots, Y_n \) for \( W^* \), we have \( S \cong \mathbb{K}[Y_0, \ldots, Y_n] \). Let \( \theta \in \text{Der}(S, M) \), let \( g_i = \theta(Y_i) \), by linearity and the Leibniz product rule we get these \( g_i \) completely determine \( \theta \). It follows that \( \theta \) is equal to the derivation \( \sum_{i=0}^n g_i \frac{\partial}{\partial Y_i} \).

Hence if \( W_0, \ldots, W_n \) is a basis of \( W \) dual to \( Y_0, \ldots, Y_n \), meaning \( Y_i(W_j) = \delta_{i,j} \). Then \( \theta = \sum_{i=0}^n g_i \otimes W_i \in M \otimes_{\mathbb{K}} W \) establishing the first result.

The second statement follows from the isomorphisms

\[
M \otimes_S \text{Der}(S) \cong M \otimes_S (S \otimes_{\mathbb{K}} W) \cong M \otimes_{\mathbb{K}} W
\]

\[\square\]

**Definition 3.7.** Let \( S = \text{Sym}(W^*) \), and fix a basis \( Y_0, Y_1, \ldots, Y_n \) of \( W^* \), so \( S \cong \mathbb{K}[Y_0, \ldots, Y_n] \). Also take \( W_0, \ldots, W_n \) to be the dual basis of \( W \). Given \( \lambda = \sum_i f_i \otimes W_i \in S \otimes W \), the preceding proposition shows \( \lambda \) defines a derivation \( \theta_\lambda \in \text{Der}(S) \). Namely,

\[
\theta_\lambda(g) = \sum_i f_i \frac{\partial g}{\partial Y_i}.
\]

Moreover, \( \lambda \) defines a polynomial map \( \rho_\lambda : W \to W \), or equivalently a rational map \( \mathbb{P}(W) \to \mathbb{P}(W) \), via

\[
\rho_\lambda(w) = \sum_{i=0}^n f_i(w) W_i = (f_0(w) : f_1(w) : \ldots : f_n(w))
\]

Finally, it defines a pairing \( \langle \cdot, \cdot \rangle_\lambda : W \times W^* \to \mathbb{K} \), linear only in \( W^* \), where for \( (s, \ell) \in W \times W^* \)

\[
\langle s, \ell \rangle_\lambda := \sum_{i=0}^n (f_i(s))(\ell(W_i))
\]

or in coordinates;

\[
\langle a_0, \ldots, a_n \rangle, c_0 Y_0 + \ldots + c_n Y_n \rangle_\lambda := \sum_{i=0}^n f_i(a_0, \ldots, a_n)c_i
\]

We extended this definition to a pairing \( \langle \cdot, \cdot \rangle_\lambda : W \times V \to \mathbb{K} \) via

\[
\langle s, t \rangle_\lambda := \sum_i f_i(s)(B(t, u_i))
\]

The following is immediate from the definitions

**Lemma 3.8.** For \( (s, t) \in W \times V \) and \( \lambda \in S \otimes W \)

\[
[\theta_\lambda(\ell_i)](s) = \langle s, \ell \rangle_\lambda = \ell_t(\rho_\lambda(s)).
\]

This proposition is essentially due to Stanley, though the presentation is our own.
Proposition 3.9. Let \( \lambda \in S \otimes W \), and \( \mathcal{A} \subseteq W \) a hyperplane arrangement with \( Q_\mathcal{A} = \prod_H \ell_H \). Then the following are equivalent:

(i) \( \theta_\lambda \in D(\mathcal{A}) \)
(ii) \( \theta_\lambda(\ell_H) \subseteq I(H) \) for all \( H \in \mathcal{A} \)
(iii) \( p_\lambda(H) \subseteq H \) for all \( H \in \mathcal{A} \)
(iv) For all \( H \in \mathcal{A} \), the restriction of \( \langle -, - \rangle_\lambda \) to \( H \times H^\perp \subseteq W \times V \) is identically 0.

Proof. \([i] \iff (ii)\) The implication \((i) \implies (ii)\) follows from the definition. For the converse note that \( I(H) \) is generated by \( \ell_H \), so every element \( f \in I(H) \) may be written \( f = g \ell_H \). Applying the Leibniz product rule we get
\[
\theta_\lambda(f) = \theta_\lambda(a_i) \ell_H + a_i \theta_\lambda(\ell_H).
\]
The first term is necessarily in \( I(H) \), and so if \( \theta_\lambda(\ell_H) \in I(H) \) then we conclude that the second sum is in \( I(H) \) as well, establishing the result.

\[\begin{align*}
(iii) \iff (iv) \iff (ii) \end{align*}\]

\( (ii) \) can be rephrased as follows: “for all \( \ell \in [I(H)]_1 \) and all \( p \in L \), \( [\theta_\lambda(\ell)](p) = 0^n \).”

Now using the fact that \([I(H)]_1 \) is naturally isomorphic to \( H^\perp \) under our isomorphism \( V \cong W^* \), we conclude by applying 3.8. \( \square \)

Definition 3.10. Under the characterization above, the identity map on \( W \) corresponds to a derivation known as the Euler Derivation which we denote \( \theta_e \). In coordinates, if \( S = \mathbb{K}[Y_0, \ldots, Y_n] \), then
\[
\theta_e = Y_0 \frac{\partial}{\partial Y_0} + Y_1 \frac{\partial}{\partial Y_1} + \ldots + Y_n \frac{\partial}{\partial Y_n}.
\]

The Euler Derivation can be alternatively characterized as the unique derivation where \( \theta_e(f) = \deg(f) f \) for all homogeneous \( f \), an identity originally due to Euler.

Definition 3.11. (Reduced Module of Derivations) Let \( \theta_e \) denote the Euler Derivation we define the Reduced Module of Derivations, denoted \( D_0(\mathcal{A}) \), as the quotient
\[
D_0(\mathcal{A}) := D(\mathcal{A})/(S\theta_e).
\]

By convention, we set \( D(\emptyset) = \text{Der}(S) \) and \( D_0(\emptyset) = \text{Der}(S)/(S\theta_e) \).

Definition 3.12. Let \( \mathcal{A} \subseteq \mathbb{P}(W) \) be a hyperplane arrangement, then \( D(\mathcal{A}) \) defines a reflexive sheaf, \( \hat{D}(\mathcal{A}) \) on \( \mathbb{P}(W) \) of rank \( \dim W \).

If \( L \subseteq \mathbb{P}(W) \) is a line we may tensor \( \hat{D}(\mathcal{A}) \) with the structure sheaf \( \mathcal{O}_L \). This may equivalently be viewed as a sheaf of \( \mathcal{O}_{\mathbb{P}(W)} \) modules, or the restriction of \( \hat{D}(\mathcal{A}) \) to \( L \). We let \( D(\mathcal{A})|_L \) denote the corresponding graded module, that is
\[
[D(\mathcal{A})|_L]_d = H^0(\hat{D}(\mathcal{A}) \otimes \mathcal{O}_L(-d), L)
\]

We may similarly define \( D_0(\mathcal{A})|_L \).

The above object, \( D(\mathcal{A})|_L \) has an equivalent algebraic definition, at least if the line \( L \) is general which we state now.

Proposition 3.13. For a general line \( L \subseteq \mathbb{P}(W) \), and for \( \ell \in S \) let \( \tilde{\ell} \) denote the image of \( \ell \) in \( S/I(L) \), then
\[
D(\mathcal{A})|_L = \{ \theta \in \text{Der}(S, S/I(L)) \mid \theta(\ell) \in (\tilde{\ell}) \text{ for all } \ell \text{ dividing } Q(\mathcal{A}) \}
\]
and similarly for \( D_0(\mathcal{A}) \).

Proof. First, for any \( f \in S \) we let \( \tilde{f} \) denote the image of \( f \) in \( S/I(L) \), and similarly if \( \theta = \sum_{i=0}^n f_i \frac{\partial}{\partial Y_i} \) we let \( \tilde{\theta} = \sum_{i=0}^n \tilde{f}_i \frac{\partial}{\partial Y_i} \in \text{Der}(S, S/I(L)) \).

Note that \( D(\theta)|_L \) is isomorphic to \( \text{Der}(S, S/I(L)) \), and so \( D(\mathcal{A})|_L \) is isomorphic to a submodule of \( \text{Der}(S, S/I(L)) \).

Now consider the case that \( \mathcal{A} \) consists of a single hyperplane \( H \). Choosing our coordinates \( Y_0, \ldots, Y_n \) so that \( H = (Y_0 = 0) \), then \( D(\mathcal{A}) \) is free on generators \( \{ Y_0 \frac{\partial}{\partial Y_0}, \frac{\partial}{\partial Y_1}, \ldots, \frac{\partial}{\partial Y_n} \} \). Then \( D(\mathcal{A})|_L \) is a free \( S/I(L) \)}
module with basis \( \{ \frac{\partial}{\partial x_0}, \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n} \} \). Yet these are also precisely the derivations \( \theta \in \text{Der}(S, S/I(L)) \) where \( \theta(X_0) \in (X_0) \), so the result follows in this case.

More generally, if \( A = \{ H_0, H_1, \ldots, H_k \} \) and \( L \) is any line not contained in a hyperplane in \( A \). Then for all \( i \neq j \) we have that \( L \cap H_i \) and \( L \cap H_j \) consist of distinct points. Consequently, letting \( U_i \) denote the complement of \( A \setminus \{ H_i \} \), we have that \( \{ U_i \cap L \}_{i=0,\ldots,k} \) is an open cover of \( L \). Therefore, for a section \( \sigma \in H^0(\text{Der}(S) \otimes \mathcal{O}_L(-d), L) \), we have that \( \sigma \in D(A) \mid_L \) if and only if

\[
\sigma \mid_{U_i} \in H^0(D(A) \otimes \mathcal{O}_L(-d), U_i \cap L) = H^0(D(H_i) \otimes \mathcal{O}_L(-d), U_i \cap L)
\]

for all \( j = 1, \ldots, k \).

Finally, note that for \( i \neq j \) we have that \( H^0(D(H_i) \otimes \mathcal{O}_L(-d), U_j \cap L) = \sigma \in H^0(\text{Der}(S) \otimes \mathcal{O}_L(-d), U_j \cap L) \).

Therefore, it follows that for \( \sigma \in H^0(\text{Der}(S) \otimes \mathcal{O}_L(-d), L) \), that we have the following string of equivalences

\[
\sigma \in H^0(D(A) \otimes \mathcal{O}_L(-d), L) \iff \sigma \in H^0(D(H_i) \otimes \mathcal{O}_L(-d), L \cap U_i) \iff \sigma \in H^0(D(H_i) \otimes \mathcal{O}_L(-d), L)
\]

for all \( i \in \{1, \ldots, k\} \).

The result now follows from the previous case. □

We can emulate the constructions from 3.7 for the module \( D_0(A) \mid_L \), to achieve a characterization similar to 3.9.

**Definition 3.14.** Let \( L \subseteq \mathbb{P}(W) \) be a line, for \( \gamma = \sum_j f_j \otimes w_j \in S/I(L) \otimes W \), we obtain a pairing \( \langle \ , \rangle_\gamma : L \times V \rightarrow \mathbb{K} \), defined by

\[
\langle p, v \rangle_\gamma := \sum_j f_j(p)(\ell_v(w_j))
\]

Similarly, define the polynomial map \( p_\gamma : L \rightarrow W \) and \( \theta_\gamma \in \text{Der}(S, S/I(L)) \).

**Proposition 3.15.** Let \( L \subseteq \mathbb{P}(W) \) be a general line, and \( A \subseteq \mathbb{P}^n \) a hyperplane arrangement, then for \( \gamma \in S/I(L) \otimes \mathbb{K} W \), the following are equivalent.

(i) \( \theta_\gamma \in D_0(A) \mid_L \)

(ii) \( p(L \cap H) \subseteq H \) for all \( H \in A \)

(iii) The restriction of \( \langle \ , \rangle_\gamma \) to \( (H \cap L) \times H^\perp \) is identically 0.

**Proof.** The proof is essentially identical to that of proposition 3.9 so we omit it. Note that in particular, we still have an analogue of lemma 3.8 for \( (p, q) \in L \times V \) that

\[
\langle p, q \rangle_\gamma = \theta_\gamma(\ell_q(p)) = \ell_q(\theta_\gamma(p))
\]

\( \square \)

If \( M \) is a finite reflexive graded module over \( \mathbb{K}[Y_0, \ldots, Y_n] \) defining a reflexive sheaf \( \tilde{M} \) on \( \mathbb{P}^n_\mathbb{K} \). Then the restriction of \( \tilde{M} \) to a general line \( L \subseteq \mathbb{P}^n_\mathbb{K} \), defines a vector bundle over \( L \). By the well known theorem of Birkhoff and Grothendieck, there exist integers \( k_0 \leq k_1 \leq k_2 \leq \ldots \leq k_m \) where

\[
\tilde{M} \mid_L \cong \oplus_{i=0}^m \mathcal{O}_L(-k_i).
\]

If \( A \subseteq \mathbb{P}^n_\mathbb{K} \) is a hyperplane arrangement, then \( D(A) \) can be naturally identified with the first syzygy module of the ideal \( J = (Q_A, \frac{\partial}{\partial y_0} Q_A, \frac{\partial}{\partial y_1} Q_A, \ldots, \frac{\partial}{\partial y_n} Q_A) \). This ensure that \( D(A) \) is reflexive.

We show that \( D_0(A) \) is reflexive for any nonempty arrangements \( A \subseteq \mathbb{P}^n_\mathbb{K} \). If \( |A| \neq 0 \) mod char\(\mathbb{K} \), this is well known as \( J = \text{Jac}(Q_A) = (\frac{\partial}{\partial y_0} Q_A, \frac{\partial}{\partial y_1} Q_A, \ldots, \frac{\partial}{\partial y_n} Q_A) \) and \( D_0(A) \) can be identified with the syzygy module of \( \text{Jac}(Q_A) \).

We establish this more generally, our proof requires the following reflexive criterion. We refer to [3] (see Lemma 15.23.5) for a proof.
Proposition 3.16 (Reflexive Criterion). Suppose

\[ 0 \rightarrow M \rightarrow L \rightarrow K \]

is an exact sequence of finite modules, over a commutative noetherian domain R. Then if L is reflexive and K is torsion free, then M is reflexive.

With this criteria we can establish our claim. This is well known for arrangements over \( \mathbb{C} \) and likely in general we include it for completeness.

Proposition 3.17. If \( \mathcal{A} \subseteq \mathbb{P}^n = \text{Proj}(K[Y_0,..,Y_n]) \) is a nonempty hyperplane arrangement, then \( D_0(\mathcal{A}) \) is a reflexive module.

Proof. The proof is by induction on the number of hyperplanes in \( \mathcal{A} \). First we consider the case \( |\mathcal{A}| = 1 \) or \( |\mathcal{A}| = 2 \), in these cases we can choose coordinates so that \( Q_{\mathcal{A}} = Y_0 \) or \( Q_{\mathcal{A}} = Y_0 Y_1 \) respectively. It can now be checked by direct computation that \( D_0(\mathcal{A}) \) is free on generators \( \{ \frac{\partial}{\partial Y_1}, \ldots, \frac{\partial}{\partial Y_n} \} \) and \( \{ Y_1 \frac{\partial}{\partial Y_1}, \ldots, \frac{\partial}{\partial Y_n} \} \) respectively.

For the general case if \( \mathcal{A}' \) is a hyperplane arrangement with \( k > 2 \) hyperplanes pick two distinct hyperplane \( L \) and \( H \) in \( \mathcal{A} \). Let \( \mathcal{A} = \mathcal{A}' \setminus \{ H \} \) and let \( \mathcal{B} \) denote the hyperplane arrangement \( \{ L, H \} \). Then we have the following exact sequence

\[ 0 \rightarrow D_0(\mathcal{A}') \rightarrow D_0(\mathcal{A}) \oplus D_0(\mathcal{B}) \rightarrow D_0(\{ L \}) . \]

As \( D_0(\{ L \}) \) is free it is in particular torsion free. Furthermore by inductive hypothesis \( D_0(\mathcal{A}) \) and \( D_0(\mathcal{B}) \) are both reflexive so we conclude by applying the preceding proposition. \( \square \)

Definition 3.18 (Splitting Type). If \( \mathcal{A} \subseteq \mathbb{P}^n \) is a hyperplane arrangement (resp. nonempty hyperplane arrangement), then there exists tuple of integers \( (a_0, a_1, \ldots, a_n) \), (resp. \( (a_1, \ldots, a_n) \)) referred to as the **Splitting Type** of \( D(\mathcal{A}) \) (resp. \( D_0(\mathcal{A}) \)).

This is the unique tuple satisfying \( 0 \leq a_0 \leq a_1 \leq \ldots \leq a_n \), so that if \( L \) is a general line then there’s an isomorphism

\[ D(\mathcal{A}) \mid_L \cong \bigoplus_{i=0}^n S/I(L)(-a_i) \]

(resp. \( D_0(\mathcal{A}) \mid_L \cong \bigoplus_{i=1}^n S/I(L)(-a_i) \)).

4. Derivation Bundle of Hyperplane Arrangements and the Ideals of Dual Points

In this section we introduce our duality and establish a relationship between \( D_0(\mathcal{A}_Z) \) and \( I(Z) \). We can summarize this relationship as follows: Given a set of points \( Z \subseteq \mathbb{P}(W) \) with dual hyperplane arrangement \( \mathcal{A}_Z \subseteq \mathbb{P}(V) \), we consider a ring \( T = R \otimes_K \mathbb{K}[\text{Gr}(n-2, V)] \) that contains naturally isomorphic copies of \( R = \text{Sym}(V^*) \) and \( S = \text{Sym}(W^*) \). We then show in theorem 4.8 that \( D_0(\mathcal{A}) \) is isomorphic to an \( S \)-submodule of the extended ideal \( I(Z)T \). This is analogous to the standard construction used in [13]. In theorem 4.10 we then give a novel interpretation of the restriction of this \( S \)-submodule to a general line.

Definition 4.1. Let \( \wedge^* V \), denote the exterior algebra of \( V \). This is the graded \( K \)-algebra generated in degree 1 by \( V \), subject to the relation \( v^2 = v \wedge v = 0 \) for all \( v \in V \).

Definition 4.2. Let \( \text{Gr}(k, V) \) denote the \( k \)-th grassmanian of \( V \) as a projective subvariety of \( \mathbb{P}(\wedge^k V) \). The projective coordinate ring of \( \text{Gr}(k, V) \) as a quotient of the polynomial ring of the ambient space is the **Plücker Algebra**, \( \text{PL}(k, V) \).

Fix a set of coordinates \( X_0, \ldots, X_n \) on \( V \), so that \( \text{Sym}(V^*) \cong K[X_0, \ldots, X_n] \). Extend these to coordinates on \( V^\otimes k \) for some 1 \( \leq k \leq n \), by letting \( A_{i_0, \ldots, i_{k-1}} \) denote an isomorphic copy of \( X_0, \ldots, X_n \), for each \( i \in \{0, \ldots, k-1\} \). We organize these into a \( k \times n + 1 \) matrix \( A \) with entries \( (A)_{i,j} = A_{i,j} \).

Let \( c(\text{Gr}(k, V)) \) denote the affine cone of \( \text{Gr}(k, V) \) as a subvariety of \( \wedge^k V \). Then the multiplication map \( \wedge : V^\otimes k \rightarrow c(\text{Gr}(k, V)) \subseteq \wedge^k V \), identifies the Plücker algebra \( \text{PL}(k, V) \) with the \( K \) algebra generated by the maximal \( (k \times k) \) minors of \( A \).
Restricting to the case where \( k = n \), multiplication in \( \Lambda V \) gives a non-degenerate pairing \( \wedge : V \times \Lambda^n V \to \Lambda^{n+1} V \). Choosing an isomorphism \( \Lambda^{n+1} V \cong K \) gives an isomorphism \( \Lambda^n V \cong V^* \), natural up to a \( K \)-scalar. We fix one of these isomorphisms and let \( \tau \) denote the induced isomorphism of polynomial rings \( \tau : \text{Sym}(W^*) \cong \text{Sym}(\Lambda^n V^*) \). As \( n = \dim V - 1 \), then \( \Lambda^n V^* = \text{Gr}(n, V) \), and we can identify \( \text{PL}(n, V) \) with \( \text{Sym}(\Lambda^n V^*) \cong \text{Sym}(W^*) \). We further describe \( \tau \) in coordinates below.

**Definition 4.3.** Taking the definitions of \( X_i \) and \( A_{j, \ell} \) from above, further require that \( A_{0, i} = X_i \). Define

\[
\mathbb{K}[\mathbb{A}] := \mathbb{K}[A_{i, j} \mid 0 \leq i \leq n - 1, 0 \leq j \leq n] := \mathbb{K}[X_0, \ldots, X_n][A_{i, j} \mid 1 \leq i \leq n - 1, 0 \leq j \leq n]
\]

Let \( \text{PL}(n) \) be the subalgebra of \( \mathbb{K}[\mathbb{A}] \) generated by the determinants

\[
M_i := \begin{vmatrix}
0 & \ldots & 0 & 1 & 0 & \ldots & 0 \\
A_{0,0} & \ldots & A_{0, i-1} & A_{0, i} & A_{0, i+1} & \ldots & A_{0, n} \\
A_{1,0} & \ldots & A_{1, i} & A_{1, i+1} & \ldots & A_{1, n} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
A_{n-1,0} & \ldots & A_{n-1, i} & \ldots & \ldots & \ldots & A_{n-1, n}
\end{vmatrix}
\]

Finally, taking \( Y_i \) to be a dual basis of \( X_i \) we define \( \tau : \mathbb{K}[Y_0, \ldots, Y_n] \to \text{PL}(n) \) via \( \tau(Y_i) = M_i \).

The preceding conversation shows that \( \text{PL}(n) \) is a polynomial algebra in the generators \( M_i \). The lemma below shows that our definition of \( \tau \) above matches the construction from the preceding remark.

**Lemma 4.4.** Let \( v \in V = \text{Spec}(\mathbb{K}[X_0, \ldots, X_n]) \) and let \( \ell_v = \sum_{i=0}^n c_i Y_i \in W^* \) be the corresponding linear form. Then as a polynomial in \( X_0, \ldots, X_n \) the linear form \( \tau(\ell_v) = \sum_i c_i M_i \) vanishes on \( v \).

**Proof.** Following definition 4.3 we see that \( \tau(\ell_v) = \sum_{i=0}^n c_i M_i \) is the Laplace expansion along the first row of the determinant of the matrix \( \begin{bmatrix} \overline{\nabla} & A \end{bmatrix} \), where \( \overline{\nabla} = [c_0 \ c_1 \ \ldots \ c_n] \). If we then evaluate \( X_0, \ldots, X_n \) at \( v \), so that \( X_i \mapsto c_i \), the matrix is singular as two rows are identical hence the determinant vanishes. \( \square \)

**Definition 4.5.** If \( X_0, \ldots, X_n \) form a basis of \( V^* \) and \( Y_0, \ldots, Y_n \) are dual coordinates on \( W^* \). Then for any \( \lambda = \sum_{i=0}^n f_i(Y_0, \ldots, Y_n) \ell_{X_i} \in \text{Sym}(W^*) \otimes W \), we define a polynomial \( F_\lambda \in \mathbb{K}[\mathbb{A}] \) via

\[
F_\lambda := \sum_{i=0}^n X_i \tau(f_i) = \sum_{i=0}^n X_i f_i(M_0, \ldots, M_n)
\]

**Definition 4.6.** Let \( J \subseteq \text{Sym}(W^*) = \mathbb{K}[X_0, \ldots, X_n] \) be any homogeneous ideal. We define a graded module denoted \( J^{\geq} \) over \( \text{PL}(n) \), thought of as a polynomial ring in the minors \( M_0, \ldots, M_n \).

First, if \( \mathfrak{m} = (X_0, X_1, \ldots, X_n) \) is the maximal ideal we define \( \mathfrak{m}^{\geq} \) as the \( \text{PL}(n) \)-submodule of \( \mathbb{K}[\mathbb{A}] \) generated by \( (X_0, \ldots, X_n) \). We grade both \( \text{PL}(n) \) and \( \mathfrak{m}^{\geq} \) by the \( X \)-degree, meaning \( \deg(M_i) = \deg(X_i) = 1 \). Equivalently, the \( d \)-th graded component of \( \mathfrak{m}^{\geq} \) is generated over \( \mathbb{K} \) by all terms of the form \( X_i M_0^{e_0} M_1^{e_1} \ldots M_n^{e_n} \) where \( \sum_{i=0}^n e_i = d - 1 \).

More generally for any homogeneous ideal \( J \), we set \( J^{\geq} = (J \mathbb{K}[\mathbb{A}]) \cap \mathfrak{m}^{\geq} \).

The following example and proposition that follows are the main motivation for the definition of \( J^{\geq} \).

**Example 4.7.** If \( P \) is the \( 0 \)-th coordinate point in \( \mathbb{P}^2 = \text{Proj}(\mathbb{K}[X_0, X_1, X_2]) \), then \( I(P) = (X_1, X_2) \). Given \( \sum_{i=0}^2 G_i X_i \in \mathfrak{m}^{\geq} \) where \( G_i \) is a polynomial in the maximal minors \( M_0, M_1, M_2 \) of the matrix

\[
\begin{bmatrix}
X_0 & X_1 & X_2 \\
0 & A_1 & A_2 \\
A_0 & A_1 & A_2
\end{bmatrix}
\]

It’s not hard to see that \( \sum_{i=0}^2 G_i X_i \in I(P)^{\geq} \) if and only if \( G_0 \in I(P) \mathbb{K}[\mathbb{A}] \).

Treating \( G_0 \) as a polynomial in \( X_0, \ldots, X_n \) with coefficients in the ring \( \mathbb{K}[A_0, A_1, A_2] \) and consider the evaluation map \( e_P : \mathbb{K}[\mathbb{A}] \to \mathbb{K}[A_0, A_1, A_2] \) obtained by evaluating at \( P \), we note that \( G_0 \in I(P) \mathbb{K}[\mathbb{A}] \) if and only if \( e_P(G_0) = 0 \). Furthermore, as \( e_P \) sends \( M_0 \mapsto 0 \), \( M_1 \mapsto -A_2 \) and \( M_2 \mapsto A_1 \) then \( G_0(P) = 0 \) if and only if \( M_0 \) divides \( G_0 \). From this it follows that \( I(P)^{\geq} \) is generated by \( \{X_0 M_0, X_1, X_2\} \).
In fact this generating set is redundant as we have the nontrivial relation $X_0M_0 + X_1M_1 + X_2M_2 = 0$, and so a minimal generating set for $I(P)^\gg$ is given by \{ $X_1, X_2$ \}.

The preceding definition is motivated by the following proposition.

**Theorem 4.8.** Let $Z \subseteq \mathbb{P}(V) = \text{Proj}(\mathbb{K}[X_0, \ldots, X_n])$ be a finite set of points, and let $\mathcal{A}_Z \subseteq \mathbb{P}(W)$ denote the dual hyperplane arrangement. Then for $\lambda \in S \otimes W$ the following are equivalent:

(i) $\theta_\lambda \in D(\mathcal{A}_Z)$

(ii) $F_\lambda \in I(Z) \cdot \mathbb{K}[A]$ Moreover, $F_\lambda = 0$ if and only if there exists $g \in S$ so that $\theta_\lambda = g \theta_e$, where $\theta_e = \sum_{i=0}^{n} Y_i \frac{\partial}{\partial Y_i}$ is the Euler derivation.

In essence there's an isomorphism $\eta : \text{PL}(n, V) \otimes S D_0(\mathcal{A}_Z)(-1) \rightarrow I^\gg(Z)$ given by

$$\eta \left( \sum_{i=0}^{n} f_i(Y_0, \ldots, Y_n) \frac{\partial}{\partial Y_i} \right) = \sum_{i=0}^{n} f_i(M_0, \ldots, M_n)X_i$$

The above theorem is a consequence of the following lemma which is useful in its own right.

**Lemma 4.9.** Fix $\alpha = (\alpha_1, \ldots, \alpha_{n-1})$ a tuple of $(n-1)$ linearly independent vectors in $V$. Letting $\alpha_i := (\alpha_{i,0}, \alpha_{i,1}, \ldots, \alpha_{i,n})$ in our chosen set of coordinates. We define the partial evaluation map

$$\varepsilon_\alpha : \mathbb{K}[A] \rightarrow \mathbb{K}[X_0, \ldots, X_n]$$

via $\varepsilon_\alpha(A_{i,j}) = \alpha_{i,j}$ for $1 \leq i \leq n-1$. Let $\lambda = \sum_{i=0}^{n} f_iX_i \in [S \otimes W]_d = \text{Sym}^d(W^*) \otimes W$, for any nonzero $w \in W$ where $\ell_w \in V^*$ vanishes on $\text{Span}(\alpha)$, there exists some nonzero linear form $h$ vanishing on $\text{Span}(\alpha)$ so that

$$\varepsilon_\alpha(F_\lambda) \equiv h^d \ell_{\rho_\lambda(w)} = h^d \left( \sum_{i=0}^{n} X_i f_i(\rho_\lambda(w)) \right) \mod (\ell_w)$$

**Proof.** Take $w, \alpha$ and $\lambda$ as stated above. Let $\lambda = \sum_{i=0}^{n} f_i(Y_0, \ldots, Y_n) \otimes X_i \in [S \otimes W]_d$, then $F_\lambda = \sum_{i=0}^{n} X_i f_i(M_0, \ldots, M_n)$.

Write $\ell_w = \sum_{i=0}^{n} c_iX_i$ and assume without loss of generality that $c_0 \neq 0$. Fix some index $j \in \{0, \ldots, n-1\}$ and let $\ell_w = c_n Y_j - c_j Y_n \in [S]_1 = W^*$. Noting that $\ell_w(\alpha) = \ell_w(\alpha_{j+1}) = 0$, we may write $\varepsilon_\alpha(\tau(\ell_w))$ as the determinant of the matrix

$$\varepsilon_\alpha(\tau(\ell_w)) = \begin{vmatrix} 0 & \ldots & 0 & c_n & 0 & \ldots & -c_j \\ X_0 & \ldots & X_{j-1} & X_j & X_{j+1} & \ldots & X_n \\ \alpha_{1,0} & \ldots & \alpha_{1,j} & \alpha_{1,j+1} & \ldots & \alpha_{1,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \alpha_{n-1,0} & \ldots & \alpha_{n-1,j} & \ldots & \alpha_{n-1,n-1} & \ldots & \alpha_{n-1,n} \end{vmatrix}$$

As $u \in \ker \ell_w$ and $\text{Span}(\alpha) \subseteq \ker w$, then either $\varepsilon_\alpha(\tau(\ell_u)) = 0$ and $u \in \text{Span}(\alpha)$, or $\varepsilon_\alpha(\tau(\ell_u)) \neq 0$ and $\text{Span}(\alpha, u) = \ker \ell_w$ which implies there's a scalar $r \in \mathbb{K}$ so $\varepsilon_\alpha(\tau(\ell_u)) = r \ell_w$. In either case we have $c_n M_i - c_i M_n \equiv 0 \mod (\ell_w)$. We conclude with the equalities below where here $M_i = \varepsilon_\alpha(M_i)$,

$$\varepsilon_\alpha(F_\lambda) = \sum_{i=0}^{n} f_i(M_0, M_1, \ldots, M_n)X_i$$

$$\equiv \sum_{i=0}^{n} f_i \left( \frac{c_0}{c_n} M_n, \frac{c_1}{c_n} M_n, \ldots, \frac{c_{n-1}}{c_n} M_n, M_n \right) X_i \mod (\ell_w)$$

$$\equiv \left( \frac{M_n}{c_n} \right)^d \sum_{i=0}^{n} f_i(c_0, \ldots, c_n) X_i \mod (\ell_w)$$

$$\equiv \left( \frac{M_n}{c_n} \right)^d \rho_\lambda(w) \mod (\ell_w)$$
Noting that because \( c_n \neq 0 \), we must have that \( E_n = (0 : \ldots : 0 : 1) \not\subseteq \text{Span}(\alpha) \subseteq \ker \ell_w \). Therefore, \( \varepsilon_\alpha(M_n) \neq 0 \) as it is the determinant of a non-singular matrix thereby establishing the result. \( \square \)

**Proof of theorem 4.8.** Let \( \lambda = \sum_i f_i \otimes w_i \subseteq S \otimes W \). We note that since \( D(A_Z) = \bigcap_{P \in Z} D(H_P) \) and \( I(Z) = \bigcap_{P \in Z} I(P) \), it suffices to establish the equivalence in consider the case \( Z \) consists of a single point \( P \). Furthermore, to establish the case for a single point it suffices to show that \( \varepsilon_\alpha(F_\lambda) \) is in \( I(P) \) for every (or even for general) \( \alpha \). This is because \( \theta \in \text{Der}(S) \) is in \( D(A_Z) \) if and only if the restriction of \( \theta \) to \( L \) is in \( D(A) \big|_L \) for general \( L \), and similarly \( F_\lambda \) vanishes at \( P \) if and only if \( \varepsilon_\alpha(F_\lambda) \) vanishes on \( P \) for general \( \alpha \).

Continuing, assume that \( \alpha \) is sufficiently general and let \( \ell_Q \) denote the linear form vanishing on \( \alpha \) and \( P \). We consider \( \varepsilon_\alpha(F_\lambda) \mod (\ell_Q) \). By lemma 4.9, we get that \( \varepsilon_\alpha(F_\lambda) \equiv h^d \rho_\lambda(\ell_Q) = h^d \rho_\lambda(\ell_Q) \). Yet for general \( \alpha \) we see that \( h(P) \neq 0 \) so \( F_\lambda \) vanishes on \( P \) if and only if \( \rho_\lambda(\ell_Q) \) vanishes on \( P \). Now for any linear form \( \ell_L \) recall that \( \ell_L(P) = 0 \) if and only if the corresponding \( L \in W \) lies on \( P^\perp = H_P \). Hence, we apply proposition 3.9 and conclude the proof of the first statement with the following chain of equivalences:

\[
F_\lambda \in I(P) \iff \text{for general } \alpha, \varepsilon_\alpha(F_\lambda) \in I(P) \iff \text{for general } \alpha, \rho_\lambda(\ell_Q) \in I(P) \text{ where } \ell_Q \text{ vanishes on } \text{Span}(\alpha, P) \iff \text{for general } Q \in H_P, \rho_\lambda(Q) \in H_P \iff \theta_\lambda \in D(H_P)
\]

To finish the proof, we must establish the claim about the kernel of \( \eta \). We see that \( F_\lambda = 0 \) if and only if for general \( \alpha \) and arbitrary \( \ell_H \) vanishing on \( \alpha \) that \( \varepsilon_\alpha(F_\lambda) \equiv 0 \) mod \( I(\ell_H) \). By lemma 4.9 the later condition occurs precisely when \( \rho_\lambda(\ell_H) \in (\ell_H) \) for every linear form \( \ell_H \). If this occurs we conclude for all \( H \in W \) that \( \rho_\lambda(H) = r_H H \) for some scalar \( r_H \). It immediately follows that as a rational map on \( \mathbb{P}(W) \), \( \rho_\lambda \) can be extended to the identity, allowing us to conclude that \( \theta_\lambda = f \theta_e \) where \( \theta_e \) is the Euler derivation. \( \square \)

We note that the proof above also establishes the following.

**Theorem 4.10.** Let \( Z \subseteq \mathbb{P}(V) \) and let \( A_Z \subseteq \mathbb{P}(W) \) be the dual hyperplane arrangement. Let \( L \subseteq \mathbb{P}(W) \) be a general line, and \( Q = L^\perp \subseteq \mathbb{P}(V) \) the dual linear subspace. Then there’s an isomorphism of vector spaces

\[
[I(Z) \cap I(Q)^m]_{m+1} \cong [D_0(A_Z) \big|_L]_m.
\]

We can in fact, prove a slightly stronger statement. Namely, the above isomorphism corresponds to an isomorphism of modules over naturally isomorphic (up to scalar) rings, we give this proof after 4.14. In order to make this stronger statement and to aid with the exposition for the rest of the paper, we introduce some new notation.

**Definition 4.11.** For \( Q \subseteq \mathbb{P}^n \) a codimension 2 subspace we define a ring \( \mathcal{F}_Q \) via

\[
\mathcal{F}_Q := \text{Sym}_K([I(Q)]_1).
\]

We note that if \( L_0, L_1 \) are linear forms which generate \( I(Q) \), then \( \mathcal{F}_Q \) is a polynomial ring in the generators \( L_0, L_1 \).

Fix \( Q \) and let \( \alpha \) be any basis of \( Q \). The following proposition shows that \( \mathcal{F}_Q \) can be viewed yet another way, as the image of the map \( \varepsilon_\alpha : \text{Sym}(W^*) \to \text{Sym}(V^*) \).

**Proposition 4.12.** Let \( Q = \text{Span}(\alpha), \) and \( L = Q^\perp \), then the map \( \varepsilon_\alpha : S = \text{Sym}(W^*) \to R = \text{Sym}(V^*) \), induces an isomorphism of \( K \)-algebras

\[
\tau_\alpha : S/I(L) \to \mathcal{F}_Q.
\]

**Proof.** First consider the restriction of \( \tau_\alpha \) as a map \( [S]_1 \to [R]_1 \). By lemma 4.4, \( \varepsilon_\alpha(\ell) \) must vanish on all points of \( Q = \text{Span}(\alpha) \), hence \( \varepsilon_\alpha(\ell) \in I(Q) \). In fact, given \( P \in \mathbb{P}(V) \setminus Q \), we see again by lemma 4.4, that \( \varepsilon_\alpha(\ell_P) \) defines the hyperplane \( \text{Span}(Q, P) \). It follows that \( \tau_\alpha \) induces an isomorphism of vector spaces \( [\text{Sym}(W^*)/I(L)]_1 \cong [\mathcal{F}_Q]_1 \).

As \( \text{Sym}(W^*) \) and \( \mathcal{F}_Q \) are both symmetric algebras generated over \( K \) in degree 1. The isomorphism \( \tau_\alpha : \text{Sym}(W^*) \to \mathcal{F}_Q \) follows. \( \square \)
Definition 4.13. Let $Q \subseteq \mathbb{P}(V)$ be a codimension 2 subspace, and let $J \subseteq \text{Sym}(V^*)$ be any homogeneous ideal. We define a graded $\mathcal{F}_Q$-module, $J_Q^{\gg}$, as the $\mathcal{F}_Q$-submodule of $\text{Sym}(V^*)$ whose $d$-th graded component is given by

$$[I_Q^{\gg}(Z)]_d := [I(Z) \cap I(Q)^{d-1}]_d.$$ 

We state the full version of this duality.

Theorem 4.14. Let $Z \subseteq \mathbb{P}(V) = \text{Proj}(R)$ be a finite set of points and $\mathcal{A}_Z \subseteq \mathbb{P}(W) = \text{Proj}(S)$ the dual hyperplane arrangement. Let $L \subseteq \mathbb{P}(W)$ be a general line, then the isomorphism of $\mathbb{K}$ algebras $\tau : S/I(L) \cong \mathcal{F}_Q = \text{Sym}([I(Q)]_1)$, induces an isomorphism of graded modules $I_Q^{\gg}(Z)(-1) \cong D_0(\mathcal{A}_Z) |_L \otimes S/I(L) \mathcal{F}_Q$ via the map

$$
\eta_Q : D_0(\mathcal{A}_Z) |_L \otimes \mathcal{F}_Q \cong I_Q^{\gg}(Z)(-1)
$$

$$\eta_Q \left( \sum_{i=0}^{n} f_i \frac{\partial}{\partial Y_i} \right) = \sum_i \tau(f_i)X_i$$

Here $\{Y_i\}_{i \in [n+1]}$ and $\{X_i\}_{i \in [n+1]}$ are dual bases of $W^*$ and $V^*$ respectively.

Proof: This proof is very similar to the proof of theorem 4.8. We make note of some of the differences. Given $\gamma \in S/I(L) \otimes W$, we get both a rational map $\rho_\gamma : L \rightarrow \mathbb{P}(W)$ and a derivation $\theta_\gamma$ of $\text{Sym}(W^*)$ into $\text{Sym}(W^*)/I(L)$.

Similarly, we get a polynomial $F_\gamma \in I_Q^{\gg}(\emptyset)$ uniquely determined up to scalar. Now it again follows that $\theta_\gamma \in D_0(\mathcal{A}) |_L$ if and only if $\rho_\gamma(H \cap L) \subseteq H$ for all $H \in \mathcal{A}$. Additionally, we have that for any $\ell \in [I(Q)]_1$, that $F_\gamma = \ell_{Q}^{\gg} \rho_\gamma(\ell) \mod \ell$.

The proof now continues as in theorem 4.8.

Applying this isomorphism of modules, we note that the splitting type of of $\mathcal{A}_Z$ determines the dimension of $I_Q^{\gg}(Z)$.

Corollary 4.15. If $D_0(\mathcal{A}_Z)$ has splitting type $(a_1, a_2, \ldots, a_n)$, then for a general codimension 2 linear subspace,

$$\dim[I_Q^{\gg}(Z)]_d = \sum_{i=1}^{n} \max\{0, d - a_i\}$$

Example 4.16 (Ceva Arrangements). Fix $m \geq 3$ and suppose that $\mathbb{K} = \mathbb{C}$ or more generally that char($\mathbb{K}$) is coprime from $m$. $C_m$ denote the Ceva Arrangement of Hyperplanes in $\mathbb{P}_m^m = \mathbb{P}(W)$. Taking $\text{Sym}(W^*) = \mathbb{C}[Y_0, Y_1, \ldots, Y_n]$ this is the arrangement of $n + 1 + m\binom{n+1}{2}$ hyperplanes with defining polynomial

$$Q_{C_m} = Y_0Y_1 \ldots Y_n \prod_{0 \leq i < j \leq n} (Y_i^m - Y_j^m).$$

It is shown in [24] that $D(C_m)$ is free with a basis given by

$$\left\{ Y_0^{\ell m + 1} \frac{\partial}{\partial Y_0} + Y_1^{\ell m + 1} \frac{\partial}{\partial Y_1} + \ldots Y_n^{\ell m + 1} \frac{\partial}{\partial Y_n} \mid \text{for } 0 \leq \ell \leq n \right\}.$$ 

Let $Z_m$ denote the dual set of points inside the dual projective space $\mathbb{P}(V)$, and supposing $\text{Sym}(V^*) = \mathbb{K}[X_0, \ldots, X_n]$ with $\{X_i\}$ forming a basis dual to $\{Y_i\}$. Then by theorem 4.8 we get that $I^{\gg}(Z_m)$ is free with basis

$$\left\{ M_0^{\ell m + 1}X_0 + M_1^{\ell m + 1}X_1 + \ldots M_n^{\ell m + 1}X_n \mid \text{for } 1 \leq \ell \leq n \right\},$$

where the $M_i$ are defined as in definition 4.3.

For both basis we verify that each of these elements lies in their respective modules to help illustrate theorem 4.8. First given $\theta_\ell = \sum_{i=0}^{n} Y_i^{\ell m + 1} \frac{\partial}{\partial Y_i}$, we note any factor of $Q_{C_m}$ is of the form $Y_i$ or $Y_j - \zeta Y_k$ where $\zeta$ is a principle $m$-th root of unity. As $\theta_\ell(Y_i) = Y_i^{\ell m + 1}$ it follows that $\theta_\ell \in D(\{Y_i = 0\})$. Additionally, $\theta_\ell(Y_j - \zeta Y_k) = Y_j^{\ell m + 1} - \zeta Y_k^{\ell m + 1}$, yet as $\zeta^{\ell m} = 1$ this is the same as $\theta_\ell(Y_j - \zeta Y_k) = Y_j^{\ell m + 1} - (\zeta Y_k)^{\ell m + 1}$. The identity $a^d - b^d = (a - b) \sum_{i=0}^{d-1} a^i b^{d-1-i}$ now establishes that $\theta_\ell \in D(C_m)$.
Proposition 4.17. The following diagram commutes for all codimension 2 subspaces $Q$,
\[
\begin{array}{ccc}
D_0(A_Z) & \xrightarrow{\text{res}_{Q^\perp}} & D_0(A_Z) |_{Q^\perp} \\
\downarrow \eta & & \downarrow \eta_Q \\
I^{\gg}(Z) & \xrightarrow{\varepsilon_Q} & I^{\gg}_Q(Z)
\end{array}
\]
with the sides isomorphisms for general $Q$.

Proof. First note, that by proposition 4.12 we have a commuting diagram of commutative $\mathbb{K}$-algebras
\[
\begin{array}{ccc}
\text{Sym}(W^*) & \xrightarrow{\tau^*} & \text{Sym}(W^*)/I(Q^\perp) \\
\downarrow \varepsilon & & \downarrow \tau_Q \\
\text{PL}(n) & \xrightarrow{\varepsilon_Q} & F_Q
\end{array}
\]
where the top map sends $f \in \text{Sym}(W^*)$ to its coset $\bar{f} \in \text{Sym}(W^*)/I(Q^\perp)$.

Working in coordinates given $\theta = \sum_{i=0}^{n} F_i \partial_{Y_i}$, we have that
\[
\varepsilon_Q \eta_Q(\theta) = \sum_{i=0}^{n} \varepsilon_Q(\tau(F_i))X_i = \sum_{i=0}^{n} \tau_Q(\tilde{F}_i)X_i = \eta_Q(\text{res}_{Q^\perp}(\theta))
\]
establishing the result. \hfill $\Box$ 

5. Unexpected Hypersurfaces

In [7], the authors gave a characterization of the degrees $d$, in which a finite set of points $Z \subseteq \mathbb{P}_C^2$ admits unexpected curves in the specific case when $m = d - 1$. In this section we introduce the concept of very unexpected hypersurfaces (definition 5.7) and study them using the duality of section 4. Namely in theorem 5.31, we achieve a higher dimensional generalization of the main result of [7] which we recall below.

Theorem 5.1 ([7]). For a finite set of points $Z \subseteq \mathbb{P}^2$, let $A_Z$ denote the dual line arrangement, and let $\alpha_1, \alpha_2$ denote the splitting type of the bundle defined by $D_0(A_Z)$. Then exactly one of the following statements holds:

(i) There is some line $L \subseteq \mathbb{P}^2$ with $|L \cap Z| > \alpha_1 + 1$, in which case $|L \cap Z| = \alpha_2 + 1$ and $Z$ never admits unexpected curves.

(ii) $Z$ admits unexpected curves in degree $d$ for precisely those $d$ with $\alpha_1 < d < \alpha_2$. 

Similarly let $F_\ell = \sum_{i=0}^{n} M_i^{m+1} X_i$ to show $F_\ell \in I^{\gg}(Z)$ it suffices to show that $F_\ell$ is 0 when evaluated at each point in $Z_m$. For each $P \in Z_n$ we let $ev_P : \mathbb{K}[A] \to \mathbb{K}[A]$ denote this evaluation map. Every point in $Z_m$ is represented by $E_i$ or $E_j - \zeta \epsilon E_k$ for $E_i$ the standard basis vectors of $V$. Then $ev_{E_i}(F_\ell) = ev_{E_i}(M_i^{m+1})$ and $ev_{E_j - \zeta \epsilon E_k}(F_\ell) = ev_{E_i}(M_j^{m+1} - \zeta \epsilon M_k^{m+1})$. As similarly $M_i | M_i^{m+1}$ and $M_j - \zeta \epsilon M_k | M_j^{m+1} - \zeta \epsilon M_k^{m+1}$ we can conclude by lemma 4.4.

The previous results about the module $I^{\gg}(Z)$ and its relationship with $D_0(A_Z)$ can be summed up as stating that the isomorphism of projective spaces $\mathbb{P}(W) \cong \mathbb{P}(\wedge V)$ extends to an isomorphism of sheaves $D_0(A_Z) \cong I^{\gg}(Z)(-1)$. This sheaf $I^{\gg}(Z)$ and its relationship with $D_0(A_Z)$ is implicit in [13]. The relationship of $I^{\gg}_Q(Z)$ with a general codimension 2 subspace however was only made explicit in the case $Z \subseteq \mathbb{P}^2$.

As we have committed to working algebraically we state and prove one more result which is a simple corollary of the fact that previously stated isomorphisms correspond to an underlying isomorphism of sheaves.

For a finite set of points $Z \subseteq \mathbb{P}^2$, let $A_Z$ denote the dual line arrangement, and let $(\alpha_1, \alpha_2)$ denote the splitting type of the bundle defined by $D_0(A_Z)$. Then exactly one of the following statements holds:

(i) There is some line $L \subseteq \mathbb{P}^2$ with $|L \cap Z| > \alpha_1 + 1$, in which case $|L \cap Z| = \alpha_2 + 1$ and $Z$ never admits unexpected curves.

(ii) $Z$ admits unexpected curves in degree $d$ for precisely those $d$ with $\alpha_1 < d < \alpha_2$. 

Namely in theorem 5.31, we achieve a higher dimensional generalization of the main result of [7] which we recall below.
The most striking part of this characterization is that it does not depend directly on \(\dim[I(Z)]_d\) or \(\dim[I(Z) \cap I(Q)]_d\), a feature also present in our generalization. As some papers have already introduced a notion of unexpected hypersurface we recall this definition below, before discussing why it is inadequate for our needs.

**Definition 5.2.** If \(Z \subseteq \mathbb{P}^n = \text{Proj}(R)\) is a set of points and \(Q\) is some general linear subspace, we say \(Z\) admits unexpected \(mQ\)-hypersurfaces in degree \(d\) if

\[
\dim[I(Z) \cap I(Q)^m]_d > \max \left\{ 0, \left( \frac{n + d}{m} \right) - \dim[R/I(Z)]_d - \dim[R/I(Q)]_d \right\}
\]

or equivalently,

\[
\dim[I(Z) \cap I(Q)^m]_d > \max \left\{ 0, -\dim[R/I(Z)]_d + \sum_{j=0}^{d} \left( \frac{\dim Q + d - j}{\dim Q} \right) \left( \frac{\text{codim} Q - 1 + j}{\text{codim} Q - 1} \right) \right\}
\]

In particular, if \(m = d - 1\) and \(\dim Q = n - 2\) the inequality becomes

\[
\dim[I(Z) \cap I(Q)^{d-1}]_d > \max\{0, nd + 1 - \dim[R/I(Z)]_d\}
\]

Despite the ease in which the above definition can be stated, it has a few shortcomings. The first shortcoming is of a semantic nature, namely there are sets of points which by definition admit unexpected \(mQ\)-hypersurfaces, but where we believe the difference in dimension is unsurprising. The second issue is somewhat larger if we hope to generalize theorem 5.1, namely it cannot be determined from \(D_0(A_Z)\) whether or not \(Z\) admits ostensibly unexpected \((d - 1)Q\)-hypersurfaces in degree \(d\).

Both of these issues are illustrated by the following example.

**Example 5.4.** Let \(H \subseteq \mathbb{P}^3\) be any plane, and let \(U\) consist of 10 of points on \(H\). Now take two general points \(P_0\) and \(P_1\) not on \(H\). Let \(Z = P_0 + P_1 + U\), and \(Q \subseteq \mathbb{P}^2\) a generic line, if we let \(\ell_H\) be a linear form defining \(H\), and \(\ell_0, \ell_1\) be linear forms defining \(\text{Span}(Q, P_0)\) and \(\text{Span}(Q, P_1)\) respectively. Then taking \(f = \ell_H \ell_0 \ell_1\), we get that \(f\) lies in \([I(Z + 2Q)]_3\). If the points in \(U\) are general points on \(H\), then \(h_U(3) = \min\{ \binom{3^2 + 3}{3}, |U| \} = 10\), \(h_Z(3) = 12\), and \(h_{2Q}(3) = 4 + (2)(3) = 10\), in which case \(Z\) admits an unexpected hypersurface in degree 3.

However, taking \(U'\) to be 10 points lying on a smooth conic in \(H\) and letting \(Z' = P_0 + P_1 + U'\), then \(h_{U'}(3) = 7\) and \(h_{Z'}(3) = 9\) so \(Z'\) does not admit unexpected hypersurfaces in degree 3.
Note that there is an isomorphism of intersection lattices $L_{A_2} \cong L_{A_2'}$, and that both $D_0(A_2)$ and $D_0(A_2')$ have splitting type $(2,4,5)$.

In the above example, the “unexpectedness” is explained by the fact that most of the points of $Z$ lie on the plane $H$. This gives us a lower bound on \(\dim[I(Z + 2Q)]_d\) since

\[
\dim[I(Z + 2Q)]_3 > \dim[I(H + 2Q) \cap I(P_0 + P_1)]_3 \geq \dim[I(H + 2Q)]_3 - 2.
\]

Furthermore, there is no reason to expect equality in the inequality

\[
\dim[I(H) \cap I(Q)^2]_3 \leq \max \{0, \dim[I(Q)^2]_3 - \dim[R/I(H)]_3\}
\]

since $Q$ and $H$ have nonempty intersection. This situation is elaborated on further by the following proposition which computes the dimension of $[I_Q^p(H)]_d = [I(H) \cap I(Q)^{d-1}]_d$ can impose on $I(Q)^{d-1}$.

**Proposition 5.5.** Let $H, Q \subseteq \mathbb{P}(V)$ be nonempty linear subspaces, with $Q$ general of codimension 2. Then

\[
\dim[I_Q^p(H)]_d = \dim[I(H) \cap I(Q)^{d-1}]_d = d(\text{codim} \, H).
\]

As a consequence, if $Z \subseteq H$, then $\dim[I_Q^p(Z)]_d \geq d(\text{codim} \, H)$.

**Proof.** Let $h = \dim H$. We may choose a basis $\{X_0, \ldots, X_n\}$ of $V^*$ so that $I(H) = (X_{h+1}, \ldots, X_n)$. Moreover, let $\ell_i := \varepsilon_Q(M_i) \in [F_Q]_1$, denote the linear form vanishing on $Q$ and the $i$-th coordinate point.

We proceed by induction on $h$, establishing that $I_Q^p(H)$ is free with basis $\{X_{d+1}, \ldots, X_n\}$. First consider the case $h = 0$, so that $H$ is the 0-th coordinate point. For each $f \in [I_Q^p(H)_Q]_d$, we may write $f = \sum_{i=0}^n f_iX_i$ with each $f_i \in [F_Q]_{d-1}$. Evaluating $f$ at $H$ shows that $f_0(H) = 0$. As $F_Q$ is a polynomial ring in two variables, we conclude that $\ell_0$ divides $f_0$. Using the identity $\sum_{i=0}^n X_i\ell_i = 0$, and letting $g_i = f_i - \ell_i f_0/\ell_0$ we get $f = \sum_{i=1}^n g_iX_i$. It follows that $I_Q^p(Q)$ is a free $F_Q$-module with basis $X_1, \ldots, X_n$.

Now when $h \geq 1$, let $H_0 \subseteq H$ be the coordinate subspace, with defining ideal $I(H_0) = (X_{h}, \ldots, X_n) \supseteq I(H)$. We get by inductive hypothesis that every element $f \in I_Q^p(H_0)$ may be written in the form $f = \sum_{i=h}^n f_iX_i$. As $X_j \in I(H)$ for $j > h$, we see that $f \in I(H)$ if and only if $f_0 \in I(H) \cap F_Q$. However, as $K$ is infinite and $h > 0$ we have for general $Q$ that there is no finite collection of hyperplanes through $Q$ which vanish on $H$, and consequently we must have $I(H) \cap F_Q = 0$. Hence $\sum_{i=h}^n f_iX_i \in I_Q^p(H)$ if and only if $f_0 = 0$, and so $I_Q^p(H)$ is free with basis $X_{h+1}, \ldots, X_n$ as claimed.

Noting that $\dim[F_Q]_{t-1} = t$, it follows that $\dim[I_Q^p(H)]_d = (\text{codim} \, H)(\dim[F_Q]_{d-1}) = d(\text{codim} \, H)$ as desired. \(\square\)

**Example 5.6.** In view of the preceding lemma, we see that example 5.4 can be generalized. Namely, for $n > 2$ we let $H \subseteq \mathbb{P}^n$ be a proper linear subspace of dimension $d > 1$. Fix a degree $t > 1$ and let $Z$ consist of $\binom{n+d}{t}$ general points on $H$, so that $\dim[R/I(Z)]_s = \min\{\binom{s+d}{s}, |Z|\}$. Then the prior lemma shows $\dim[I_Q^p(Z)]_s = \max\{(n-d)s, ns + 1 - |Z|\}$, and hence that $Z$ admits unexpected $Q$-hypersurfaces in all degrees $2 \leq s \leq t$.

With this discussion in mind we introduce our definition of very unexpected hypersurface.

**Definition 5.7.** Let $Z \subseteq \mathbb{P}(V)$ be a finite set of points and $R = \text{Sym}(V^*)$ the projective coordinate ring. For $Q$ a generic linear subspace, we say that $Z$ admits very unexpected $mQ$-hypersurfaces in degree $d$, if there is a subset $W \subseteq Z$ satisfying the following conditions:

1. $[I(Z) \cap I(Q)^m]_d = [I(W) \cap I(Q)^m]_d$

2. For all irreducible subvarieties $X \subseteq \mathbb{P}(V)$,

$$|W \cap X| \leq \dim[I(Q)^m/(I(X) \cap I(Q)^m)]_d$$

3. $W$ imposes less condition on $[I(Q)^m]_d$ than on $[R]_d$, that is

$$\dim[R/I(W)]_d > \dim[I(Q)^m/(I(W) \cap I(Q)^m)]_d$$

**Remark 5.8.** We note that condition (II) only needs to be checked on positive dimensional irreducible subvarieties.
Example 5.9. To illustrate the difference between definition 5.7 and definition 5.2, we revisit example 5.4. Letting $Z$ and $Z'$ denote the points sets from that example, we show neither set admits very unexpected hypersurfaces where $m = d - 1$.

Note that $Z$ does not admit very unexpected hypersurfaces in particular because it does not admit unexpected hypersurfaces. Namely for any subset $W \subseteq Z$, we have
\[
\dim[I(Q)^{d-1}/(I(W) \cap I(Q)^{d-1})]_d \geq \dim[R/I(Z)]_d \geq \dim[R/I(W)]_d,
\]
and so $W$ does not satisfy (III).

For $Z'$, if $W \subseteq Z'$ is a subset satisfying (II) for some $d > 0$ then $|W \cap H| \leq \dim[I(Q)^{d-1}/(I(H) \cap I(Q)^{d-1})]_d = 2d+1$ in particular if $d = 3$ we have $|W \cap H| \leq 7$. If $W$ also satisfies (I) then $\dim[I(W)/W]_3 = 1$ and so in particular $\dim[I(Q)^m/(I(W) \cap I(Q)^m)]_d = 9$ as $\dim[R/I(W)]_d \leq |W| \leq |W \cap H| + 2 = 9$ we see any such $W$ cannot satisfy condition (III).

**Remark 5.10.** It’s possible that there are other definitions that are preferable in some ways. One change that might be useful is to require condition (ii) in the case where $X$ is not necessarily irreducible, or if we allow $Z$ to be nonreduced perhaps take $X$ to be a positive dimensional subscheme. We use the above definition for now as it is strong enough for our purposes while still being relatively easy to check.

In this paper we will be focusing on the case where $\text{codim} Q = 2$ and $m = d - 1$. We introduce this definition in general because we think it is a natural and potentially useful modification given our discussion in example 5.4.

**Remark 5.11.** Despite the fact the above definition is strictly stronger than definition 5.2, the two definitions agree in $\mathbb{P}^2$. This is a consequence of the fact that the only positive dimensional subvarieties that are needed to check in condition (ii) are hypersurfaces. More generally, if $Z \subseteq \mathbb{P}^n$ is a finite set of points contained in a hypersurface defined by $(f = 0)$ and $Q \in \mathbb{P}^n$ is the generic point. Then applying the dimension count from 5.3, that
\[
\dim[I(Q)^m/(f) \cap I(Q)^m]_d = \dim[I(Q)^m/(fI(Q)^m)]_d = \dim[I(Q)^m]_d - \dim[I(Q)^m]_d - \deg f = \max \left\{ 0, \left( \frac{n + d}{n} - \frac{n + d - \deg f}{n} \right) \right\}
\]
It follows for all $m$ and $d$ that
\[
\dim[R/I(Z)]_d \leq \dim[R/(f)]_d = \left( \frac{n + d}{n} - \frac{n + d - \deg(f)}{n} \right) \leq \dim[I(Q)^m/((f) \cap I(Q)^m)]_d
\]
Establishing that condition (iii) could never be satisfied under these conditions.

A similar argument shows that $Z$ can never admit very unexpected $mQ$-hypersurfaces if $Q$ is a hyperplane.

One potential issue with 5.7 is that condition (II) seems difficult to verify, given that naively there is a potentially infinite number of irreducible varieties we must check. However, we make a few observations showing that it is easier to verify than it may seem, and can be reduced to a finite number of subvarieties.

Suppose that $Z \subseteq \mathbb{P}^n$ admits unexpected $mQ$-hypersurfaces in degree $d$ and furthermore, that there’s no $P \in Z$ where $[I(Z) \cap I(Q)^m]_d \subseteq [I(Z - P) \cap I(Q)^m]_d$. This is a relatively harmless assumption since if such a $P$ does exist, then $Z \setminus P$ still admits unexpected hypersurfaces in degree $d$.

Now if there is some positive dimensional variety $X_1 \subseteq \mathbb{P}^n$ so that $|Z \cap X_1| > \dim[I(Q)^m/(I(X_1) \cap I(Q)^m)]_d$. Then $Z \cap X_1$ imposes less than $|Z \cap X_1|$ conditions on $I(Q)^m$ and so we may find a subset $U_1 \subseteq Z \cap X_1$ with $|U_1| = \dim[I(Q)^m/(I(X_1) \cap I(Q)^m)]_d$ and $[I(Q)^m \cap I(U_1)]_d = [I(Q)^m \cap I(X \cap Z)]_d$. Setting $Z_1 = (Z \subseteq U_1) \cup U_1$ we make two observations both of which follow readily:
(A) $[I(Z) \cap I(Q)^m]_d = [I(Z \setminus X) \cap I(X \cap Z) \cap I(Q)^m]_d = [I(Z_1) \cap I(Q)^m]_d$
(B) If there’s a strict containment $[I(X_1) \cap I(Q)^m]_d \subseteq [I(U_1) \cap I(Q)]_d$, then $Z_1$ admits unexpected hypersurfaces if and only if $Z$ does.

We may continue in this way stopping when we find a subset $Z_k \subseteq Z_{k-1} \subseteq ... \subseteq Z$, where either
1. $Z_k$ does not admit unexpected hypersurfaces; or
2. $W = Z_k$ satisfies the conditions (I), (II) and (III) of definition 5.7.

If $Z_k$ does not admit unexpected hypersurfaces then by observation (B), we must have $[I(X_k) \cap I(Q)^m]_d = [I(U_k) \cap I(Q)^m]_d$. Then

$$[I(Z) \cap I(Q)^m]_d \subseteq [I(U_k) \cap I(Q)^m]_d = [I(X_k) \cap I(Q)^m]_d.$$ 

Hence, the polynomials in $[I(Z) \cap I(Q)^m]$ vanish on the positive dimensional variety $X_k$.

From the preceding discussion we can conclude the following proposition.

**Proposition 5.12.** Let $Z \subseteq \mathbb{P}^n$ and suppose $Z$ admits unexpected $mQ$-hypersurfaces in degree $d$. Then there exists $W \subseteq Z$, so that $W$ satisfies conditions I and II of definition 5.7 and $Z$ admits very unexpected hypersurfaces if and only if $W$ admits unexpected hypersurfaces.

With this discussion in mind we introduce the following definition.

**Definition 5.13.** Fix positive integers $m, n, c$ and $d$. If $Z \subseteq \mathbb{P}^n$ is a finite set of points we set

$$B. \text{loc}_d(Z, m, c) := \bigcap_Q V([I(Z) \cap I(Q)^m]_d).$$

Where $Q$ is over all linear subspaces of dimension $c$. Moreover, we set $B. \text{loc}_d(Z) := B. \text{loc}_d(Z, d - 1, n - 2)$ as this is the case we will focus on.

If $m = d - 1$ and $c = n - 2$, we also define $B. \text{loc}_d(M)$ for a submodule $M \subseteq m^{\geq}$ via

$$B. \text{loc}_d(M) = \bigcap_{F \in [M]_d} \bigcap_{Q \in Gr(n - 2, n)} V(\varepsilon_Q(F)).$$

That is $B. \text{loc}(F_\delta)$ is the intersection of all the hypersurfaces defined by $\varepsilon_Q(F_\delta)$ as $Q$ varies.

From the discussion proceeding this definition, we may conclude the following

**Proposition 5.14.** Fix $m, n, c$ and $d$ as above. For $Z \subseteq \mathbb{P}^n = \text{Proj}(R)$, and $Q$ the generic $c$-dimensional linear subspace, we have $Z$ admits very unexpected $mQ$-hypersurfaces if and only if there’s a subset $W \subseteq Z$ satisfying

(I) $[I(Z) \cap I(Q)^m]_d = [I(W) \cap I(Q)^m]_d$

(II') For all irreducible components $X$ of $B. \text{loc}_d(Z, m, c)$

$$|W \cap X| \leq \dim[I(Q)^m / (I(X) \cap I(Q)^m)]_d$$

(III) $W$ imposes less condition on $[I(Q)^m]_d$ than on $[R]_d$, that is

$$\dim[R/I(W)]_d > \dim[I(Q)^m / (I(W) \cap I(Q)^m)]_d$$

Consequently, if $\dim B. \text{loc}_d(Z, m, c) = 0$ then $Z$ admits very unexpected hypersurfaces if and only if $Z$ admits unexpected hypersurfaces.

**Remark 5.15.** Note that $W \subseteq Z$ may satisfy (II') without satisfying (II). For instance the points $Z = C_5$ dual to the Ceva Arrangement $A_{C_5} \subseteq \mathbb{P}^2_\mathbb{C}$ consist of 18 points which admit unexpected curves in all degrees $d$ with $6 < d < 11$. Taking $d = 7$ we note that $W = Z$ does not satisfy condition (II) of definition 5.7, since taking $X = \mathbb{P}^2$ we see that $|W \cap X| = 18 > 15 = \dim[I(Q)^6 / (0)]_7$.

More generally, if $H$ is a 2-dimensional linear subspace in $\mathbb{P}^3$ then taking $Z \subseteq H$ it follows from proposition 5.43 that $W = Z$ does not satisfy condition (II) in degree $d = 7$. However, in either case $W = Z$ satisfies condition (II') above.

**Example 5.16.** It should be noted here that $B. \text{loc}_d(Z)$ and $B. \text{loc}_d(\mathbb{P}^d(\mathbb{C}))$ are not necessarily the same. For instance, if $Z \subseteq \mathbb{P}^2_\mathbb{C}$ is 5 general points, then a computation shows that $[I(\mathbb{P}^2(\mathbb{C}))]_3 = 0$, and so $B. \text{loc}_3(\mathbb{P}^d(\mathbb{C})) = \mathbb{P}^2$. Yet a direct computation shows that $\dim[I(Z) \cap I(Q)^2]_3 = 2$ and $B. \text{loc}_3(Z) = Z$. It is true, however, that $B. \text{loc}_d(Z) \subseteq B. \text{loc}_d(\mathbb{P}^d(\mathbb{C}))$.

**Remark 5.17.** From here on we restrict the view of the paper, to the case where $c = n - 2$ and $m = d - 1$ that is we study $[I(Z) \cap I(Q)^{d-1}]_d$. 


The following proposition provides a classification of those varieties that can appear in $B.\text{loc}_d(Z)$.

**Proposition 5.18.** For any submodule $M \subseteq \mathfrak{m}^\infty$, (resp. $Z \subseteq \mathbb{P}^n$) the base locus $B.\text{loc}_d(M)$ (resp. $B.\text{loc}_d(Z)$) is a union of linear subspaces.

**Proof.** We prove both statements in parallel, let $B = B.\text{loc}_d(M)$ or $B = B.\text{loc}_d(Z)$.

Let $C$ be a positive dimensional irreducible subvariety which is contained in $B$ and not a linear subspace. We establish that $\text{Span}(C) \subseteq B$ from which the result follows.

First we show for a general hyperplane $H$, that $\text{Span}(H \cap C) = H \cap \text{Span}(C)$. Note that $\text{Span}(H \cap C) \subseteq H \cap \text{Span}(C)$ and so it suffices to show they have the same dimension. To do this take $c_1, ..., c_t \in C$ to be $t = \dim \text{Span}(C)$ linearly independent points, and let $L$ be any hyperplane containing $\text{Span}(c_1, ..., c_t)$, but with $C \not\subseteq L$. Then

$$\dim \text{Span}(L \cap C) = \dim \text{Span}(C) - 1 = \dim (L \cap \text{Span}(C)).$$

It now follows that $\dim \text{Span}(H \cap C) \geq -1 + \dim \text{Span}(C)$ for a general hyperplane $H \subseteq \text{Span}(C)$, since among hyperplanes $H$ which properly intersect $C$, the quantity $\dim \text{Span}(H \cap C)$ is lower semi-continuous. So in particular,

$$\dim \text{Span}(C) - 1 = \dim \text{Span}(L \cap C) \leq \dim \text{Span}(H \cap C) \leq \dim H \cap \text{Span}(C) = \dim \text{Span}(C) - 1.$$

Thus establishing the claim.

Proceeding let $Q \subseteq \mathbb{P}^n$ be a general codimension 2 subspace, and let $\ell$ a general linear form vanishing on $Q$. As $Q$ is a hypersurface considered as a subvariety of $(\ell = 0)$, we get for any $f \in \mathbb{E}_Q([M]_d)$ (resp. any $f \in [I_Q^\infty(Z)]_d$) that there exist linear forms $r \in [R]_1$ and $\ell_Q \in \mathbb{I}(Q)$ so that

$$f = (\ell_Q)^{d-1}r \mod (\ell).$$

Note that as $\ell$ is general, we may assume that $f \neq 0 \mod (\ell)$. Since $Q$ is general we can assume that for every positive dimensional component $C$ of $B$, that $C \not\subseteq Q$ and furthermore that $Q$ contains no component of $C \cap (\ell = 0)$. As $r$ is linear, it vanishes on $\text{Span}((\ell = 0) \cap B) = \text{Span}(B) \cap (\ell = 0)$. It follows that for any component $C$ of $B$ that $f$ vanishes on a general hyperplane section of $\text{Span}(C)$. As $\text{Span}(C)$ is irreducible we conclude that if $\dim(C) > 0$ then $f$ vanishes on $\text{Span}(C)$ as desired. □

**Remark 5.19.** Given $B \subseteq \mathbb{P}^n$ as in the proof above, we note the proof shows that in fact for a general hyperplane $H$, that $\text{Span}(B \cap H) = n - 2$. This places further constraints on $B$. With some more work it can be shown that $B = B.\text{loc}_d(I^\infty(Z))$ for some $Z$ and some $d$ if and only if $B$ is a union of linear subspaces $H_0, ..., H_s$ which satisfy

$$\sum_{i \in J} \dim H_i < \dim \text{Span} \left( \bigcup_{i \in J} H_i \right) \text{ for all } J \subseteq \{0, ..., s\}, \text{ with } |J| > 2.$$

In particular it follows the $H_i$ are disjoint.

Combining the 2 preceding proposition with proposition 5.3, it follows that conditions (I) and (II) of proposition 5.14 can be checked by looking at the combinatorics of linear subspaces spanned by subsets of $Z$. With this in mind we state one of the main theorems of this section though we postpone the proof until after theorem 5.24.

**Definition 5.20.** Given a finite set of points $Z \subseteq \mathbb{P}^n$ and a real number $d \in \mathbb{R}$ we define the modified expected number of conditions, as the integer $\text{Ex.} \ C(Z, d)$, which is the solution to optimization problem

$$\text{Ex.} \ C(Z, d) = \min \left\{ \sum_{i = 0}^s (d \dim(H_i) + 1) \mid \{H_0, ..., H_s\} \text{ are nonempty linear subspaces with } Z \subseteq \bigcup_{i = 0}^s H_i \right\}.$$

**Theorem 5.21.** Let $Z \subseteq \mathbb{P}(V)$ be a finite set of points. Then $Z$ admits unexpected $(d - 1)Q$-hypersurfaces in degree $d$ if and only if

$$\dim[I(Q)^{d-1}]_d - \dim[I(Q)^{d-1} \cap I(Z)]_d < \text{Ex.} \ C(Z, d)$$
or in the notation of section 4,
\[ \dim[I(Q)^{d-1}]_d - \dim[I_Q^\geq(Z)]_d = \dim[I(Q)^{d-1}/I_Q^\geq(Z)]_d < \text{Ex. C}(Z, d) \]

It turns out that the linear program defined in definition 5.20 can be studied via combinatorial objects known as Matroids. We recall the definition of a Matroid now for convenience.

**Definition 5.22.** A matroid is a finite set \( M \) along with a rank function \( \text{rk}_M : 2^M \to \mathbb{Z} \), which satisfies the following 3 conditions

1. **(RK 1)** \( 0 \leq \text{rk}_M(A) \leq |A| \)
2. **(RK 2)** If \( A \supseteq B \), then \( \text{rk}_M(A) \leq \text{rk}_M(B) \)
3. **(RK 3)** \( \text{rk}_M(A) + \text{rk}_M(B) \geq \text{rk}_M(A \cup B) + \text{rk}_M(A \cap B) \)

We note a function satisfying only **(RK 3)** is a submodular function.

Equivalently, it may be defined as a nonempty collection of subsets \( \mathcal{I} \) of \( M \), which satisfy

1. **(IND 1)** If \( A \in \mathcal{I} \) and \( B \subseteq A \) then \( B \in \mathcal{I} \)
2. **(IND 2)** If \( A, B \in \mathcal{I} \) and \( |A| < |B| \), then there exists some \( b \in B \) so that \( A \cup \{b\} \in \mathcal{I} \).

We now recall a few more pieces of related terminology. We refer to [25] for definitions.

- A subset \( I \subseteq M \) is independent if and only if \( |I| = \text{rk}_M(I) \). Conversely for \( A \subseteq M \), \( \text{rk}_M(A) \) is equal to the largest size of an independent \( I \subseteq A \).
- A maximal independent set is a basis of \( M \). Every basis has the same size namely \( \text{rk}_M(M) \), and every independent subset is contained in some basis.
- A flat of rank \( r \) is a subset \( F \subseteq M \), which is maximal among subsets of \( M \) with rank \( r \). Every subset \( A \subseteq M \) is contained in a unique flat, \( F \), with \( \text{rk}_M(A) = \text{rk}_M(F) \), this flat \( F \) is often denoted \( \text{Cl}_M(A) \).
- Let \( f : M \to \mathbb{R} \) be any increasing submodular function (meaning it satisfies only conditions \( ii \) and \( iii \) of definition 5.22) defines a matroid \( M_f \). Its independent sets are those \( I \subseteq M \), which satisfy \( |A| < f(A) \) for all nonempty \( A \subseteq I \).

**Example 5.23.** Every finite set of points \( Z \subseteq \mathbb{P}(V) \) defines a matroid, \( M(Z) \) namely for every nonempty \( A \subseteq Z \), we set
\[
\text{rk}_{M(Z)}(A) = 1 + \dim \text{Span}(A)
\]
A flat of this matroid, is the intersection of \( Z \) with a linear subspace \( L \subseteq \mathbb{P}^n \).

Matroids of this type are referred to as representable matroids, and are in some sense the prototypical example of a matroid.

One result which we will need later on relates information about the matroid determined by \( Z \) to information about the ideal \( I(Z) \). The result below is shown in section 4 of [22], where a further generalization is given to schemes of fat points.

**Theorem 5.24 ([22]).** Let \( Z \subseteq \mathbb{P}^n \) be a nonempty finite set of points satisfying
\[
|Z \cap L| \leq d(\dim L) + 1
\]
for all nonempty linear subspaces \( L \subseteq \mathbb{P}^n \). Then \( Z \) imposes independent conditions on \( d \) forms meaning
\[
\dim[\text{Sym}(V^*)/I(Z)]_d = |Z|
\]

We are now ready to state and prove the main result of this section.

**Proof of Theorem 5.21.** Fix a integer \( d \), note that it follows from the definition that
\[
\text{Ex. C}(Z, d) = \min \left\{ \sum_{i=0}^s (d \dim \text{Span}(A_i) + 1) \mid \{A_0, ..., A_s\} \text{ are nonempty subsets of } Z \text{ with } Z = \bigcup_{i=0}^s A_i \right\}.
\]

Before proving either direction of the equivalence. We establish the claim below.

**Claim 5.25.** \( \text{Ex. C}(Z, d) \) is equal to the largest size of a subset \( B \subseteq Z \) which satisfies the following 3 conditions
(C1) \([I_Q^{Z}(B)]_d = [I_Q^{Z}(Z)]_d\)
(C2) For all linear subspaces \(L\), \(|B \cap L| \leq \dim[I(Q)^{d-1}/I_Q^{Z}(L)]_d = d(\dim B) + 1\)
(C3) \(B\) imposes independent conditions on \(d\) forms.

**proof of claim.** Applying results from [11] (see theorem (8) and comment (16)), we may define a matroid \(M_d\) on the set \(Z\) whose independent sets are precisely those \(I \subseteq Z\) where \(|A| \leq d \dim(\text{Span} A) - 1\) for all nonempty \(A \subseteq I\). The linear programming duality given in [11], now states that

\[
\text{rk}(M_d) = \text{Ex.}\ C(Z,d).
\]

From this we can conclude that \(\text{Ex.}\ C(Z,d)\) is equal to the largest size of a subset which satisfies condition (C2), namely any basis of \(M_d\) works. To finish the proof of the claim we find a basis of \(M_d\) satisfying (C1) and (C3).

By proposition 5.12, there is some \(W \subseteq Z\) so that \(W\) satisfies conditions (C1) and (C2). As \(W\) satisfies (C2) it is independent in \(M_d\) and we can therefore extend it to a basis \(W \subseteq B\) of \(M_d\). Now as \(W \subseteq B \subseteq Z\) we have that \([I^{Z}(B)]_d = [I^{Z}(Z)]_d\), and therefore \(B\) satisfies (C1).

Lastly, we note that theorem 5.24 ensures that \(B\) since \(B\) satisfies (C2) it necessarily imposes independent conditions on \(d\) forms, thereby establishing condition (C3) and the claim. \(\square\)

Now continuing with the proof of the equivalence. If \(\dim[I(Q)^{d-1}]_d - \dim[I_Q^{Z}(Z)]_d < \text{Ex.}\ C(Z,d)\) we can find some \(B \subseteq Z\) so that \(|B| = \text{Ex.}\ C(Z,d)\) and \(B\) satisfies conditions (C1), (C2) and (C3). Then we have

\[
\dim[I(Q)^{d-1}]_d - \dim[I(Q)^{d-1} \cap I(Z)]_d < \text{Ex.}\ C(Z,d) = |B| = \dim[\text{Sym}(V^*)/I(B)]_d.
\]

Letting \(W = B\), we see that \(W\) satisfies the necessary criteria of definition 5.7, and so \(Z\) admits very unexpected hypersurfaces.

Conversely, suppose that \(Z\) admits very unexpected hypersurfaces. Then by definition there exists \(U \subseteq Z\) so that for general \(Q\),

(I) \([I_Q^{Z}(U)]_d = [I_Q^{Z}(Z)]_d\)
(II) For all linear subspaces \(L\), we have \(|U \cap L| \leq \dim[I(Q)^{d-1}/I_Q^{Z}(L)]_d = d(\dim L) + 1\)
(III) \(\dim[R/I(U)]_d > \dim[I(Q)^{d-1}/I_Q^{Z}(U)]_d\)

Finding a subset \(W \subseteq U\) so that \([I(U)]_d = [I(W)]_d\) and \(W\) imposes independent conditions on \(d\) forms. We get by the claim above that \(|W| \leq \text{Ex.}\ C(Z,d)\) and so

\[
\dim[I(Q)^{d-1}/I_Q^{Z}(U)]_d < \dim[R/I(U)]_d = |W| < \text{Ex.}\ C(Z,d).
\]

\(\square\)

**Remark 5.26.** Let \(L \subseteq \mathbb{P}^n\) be a nonempty linear subspace. We note that the above proof relies on a somewhat remarkable agreement between the dimension \(\dim[I(Q)^{d-1}/(I(L) \cap I(Q)^{d-1})]_d\) and the quantity \(d \dim L + 1\) appearing in the inequality from theorem 5.24. This is even more remarkable considering that the proof of theorem 5.24 is almost entirely combinatorial relying on a generalization of Edmonds Matroid Partition Theorem.

Combining the above result with theorem 4.14, we obtain the following as a corollary.

**Theorem 5.27.** Let \(Z \subseteq \mathbb{P}^n\) be a finite set of points, and suppose that \(D_0(A_Z)\) has splitting type \((a_1, \ldots, a_n)\). Then for a fixed integer \(d\),

\[
\sum_{i=1}^{n} \max\{0, d - a_i\} \leq nd + 1 - \text{Ex.}\ C(Z,d)
\]

and the inequality is strict if and only if \(Z\) admits very unexpected hypersurfaces in degree \(d\).

**Remark 5.28.** Note one consequence of this is if \(Z\) admits very unexpected hypersurfaces in degree \(d\), then \(a_1 < d < a_n\).
Proof. Let $H_1, ..., H_s$ be any collection of linear subspaces covering $Z$. Note that $I_Q^\geq(Z) \supseteq \bigcap_{i=1}^s I_Q^\geq(H_i)$ and that $\bigcap_{i=1}^s I_Q^\geq(H_i)$ is the kernel of the canonical map $[I(Q)^{d-1}]_d \to \bigoplus_{i=1}^s [I(Q)^{d-1}/I_Q^\geq(H_i)]_d$. We have by dimension counting that for a fixed $d$

$$\dim[I_Q^\geq(Z)]_d \geq \dim\bigcap_{i=1}^s [I_Q^\geq(H_i)]_d \geq nd + 1 - \left(\sum_{i=1}^s d\dim(H_i) + 1\right).$$

Taking $H_1, ..., H_s$ so $\sum_{i=1}^s d\dim(H_i) + 1 = \text{Ex. } C(Z, d)$, the rest follows directly from theorem 5.21 and corollary 4.15.

The final consequence follows since if $d \leq a_1$ then $I_Q^\geq(Z) = 0$, if $d \geq a_n$ then note that $\text{Ex. } C(Z, d) \leq |Z|$, and so

$$nd - (|Z| - 1) = \sum_{i=1}^n \max\{0, d - a_i\} \geq nd + 1 - \text{Ex. } C(Z, d) \geq nd + 1 - |Z|$$

Establishing that $\text{Ex. } C(Z, d) = |Z|$ and that the middle inequality is an equality.

The following lemma, shows that the inequality in the preceding corollary above may be replaced by

$$\sum_{i=1}^n \max\{0, a_i - d\} \geq |Z| - \text{Ex. } C(Z, d) \geq 0.$$

**Lemma 5.29.** Let $Z \subseteq \mathbb{P}^n$ be a finite set of points and suppose that $(a_1, ..., a_n)$ is the splitting type of $D_0(A_Z)$. Then for all real numbers $c$ and $d$

$$\sum_{i=1}^n \max\{0, d - a_i\} \geq nd + 1 - c \iff \sum_{i=1}^n \max\{0, a_i - d\} \geq |Z| - c$$

**Proof.** Using that $\sum_{i=1}^n a_i = |Z| - 1$ we obtain

$$\sum_{i=1}^n \max\{0, d - a_i\} \geq nd + 1 - c \iff$$

$$\left(\sum_{i=1}^n d - a_i\right) - \left(\sum_{j : a_j \geq d} d - a_j\right) \geq nd + 1 - c \iff$$

$$nd - (|Z| - 1) + \sum_{j : a_j \geq d} (a_j - d) \geq nd + 1 - c \iff$$

$$\sum_{i=1}^n \max\{0, a_i - d\} \geq |Z| - c$$

□

We now conclude this section by discussing a few conditions on $Z$ which makes it easier to determine if $Z$ has very unexpected hypersurfaces in some degree $d$. The first is a consequence of the preceding lemma and theorem 5.21.

**Corollary 5.30.** Let $Z \subseteq \mathbb{P}^n$ be a finite set of points, with $(a_1, a_2, ..., a_n)$ the splitting type of $D_0(A_Z)$. Suppose we have for a fixed integer $d \geq 0$ that

$$\text{Ex. } C(Z, d) = \min\{|Z|, nd + 1\}.$$

Then the following are equivalent:

(a) $Z$ admits very unexpected hypersurfaces in degree $d$

(b) $Z$ admits unexpected hypersurfaces in degree $d$
(c) \( a_1 < d < a_n \)

**Proof.** Note first that by definition \( \text{Ex.} \ C(Z, d) \leq \min\{nd + 1, |Z|\} \), since \( \text{Ex.} \ C(Z, d) \leq |Z| \) and \( \text{Ex.} \ C(Z, d) \leq d \dim(\mathbb{P}^n) + 1 \).

\([\text{(a) } \iff \text{(c)}] \) First, as mentioned after theorem 5.27 we have that \( (a) \implies (c) \). For the reverse direction assume that \( a_1 < d < a_n \). First in the case that \( \text{Ex.} \ C(Z, d) = nd + 1 \) we see that \( Z \) admits unexpected hypersurfaces in degree \( d \) as \( d > a_1 \), and so the inequality in theorem 5.27 is strict. For the case when \( \text{Ex.} \ C(Z, d) = |Z| \), we similarly conclude by applying lemma 5.29 and using that \( d < a_n \).

\([\text{(a) } \iff \text{(b)}] \) The forward direction is by definition. For the reverse we use the equivalence of (a) and (c), and note it suffices to show that \( Z \) cannot admit unexpected hypersurfaces in degree \( d \) if \( d \leq a_1 \) or \( d \geq a_n \). If \( d \leq a_1 \), we note this is impossible as \( [I^\gg_Q(Z)]_d = 0 \). If \( d \geq a_n \), then

\[
\dim[I^\gg(Z)]_d = \sum_{i=1}^n \max\{0, d - a_n\} = nd - \sum_{i=1}^n = nd - (|Z| - 1) = nd + 1 - |Z|.
\]

As \( \dim[I(Q)^d]_d = nd + 1 \) we conclude that \( Z \) imposes independent conditions on \( [I(Z)]_d \), and so \( Z \) cannot admit unexpected hypersurfaces. \( \square \)

In the case that the points of \( Z \) are not too concentrated on one or more proper subspaces, it turns out that \( \text{Ex.} \ C(Z, d) = \max\{nd + 1, |Z|\} \) holds for all \( d \) and we obtain the following result.

**Theorem 5.31.** Let \( Z \subseteq \mathbb{P}^n \) and let \((a_1, .., a_n)\) be the splitting type of \( D_0(A_Z) \), where \( a_i \leq a_{i+1} \). Suppose for all positive dimensional linear subspaces \( H \subseteq \mathbb{P}^n \), we have that

\[
\frac{|Z \cap H| - 1}{\dim H} \leq \frac{|Z| - 1}{n}.
\]

Then for an integer \( d \) the following are equivalent:

(a) \( Z \) admits very unexpected hypersurfaces in degree \( d \).

(b) \( Z \) admits unexpected hypersurfaces in degree \( d \).

(c) \( a_1 < d < a_n \).

**Proof.** By corollary 5.30, it suffices to show that \( \text{Ex.} \ C(Z, d) = \max\{nd + 1, |Z|\} \). Let \( \mathcal{H} = \{H_1, .., H_s\} \) be a collection of positive dimensional linear subspaces, so that setting \( W = Z \setminus \bigcup_{i=1}^s H_i \) we have

\[
|W| + \sum_{i=1}^s d \dim(H_i) + 1 = \text{Ex.} \ C(Z, d).
\]

As \( |W| + \sum_{i=1}^s d \dim(H_i) + 1 \) is at a minimum, we make the following observations:

**Ob. 1** \( d \dim(H_i) + 1 \leq |H_j \cap Z| \).

**Ob. 2** For all \( H \subseteq \mathcal{H} \) we have \( \sum_{H_j \cap H} d \dim(H_j) + 1 \leq d \dim \text{Span} \big( \bigcup_{H_j \cap H} H_j \big) + 1 \).

**Ob. 3** \( \sum_{i=1}^s \dim(H_i) \leq \dim \text{Span} \big( \bigcup_{i=1}^s H_i \big) < n \).

(Ob. 1) and (Ob. 2) must hold since otherwise we could find a set of points \( W' \) and a collection of subspaces \( \mathcal{H}' = \{H'_1, .., H'_k\} \) with \( Z \subseteq W' \cup \bigcup_{H'_j \in \mathcal{H}'} H'_j \) and \( \sum_{H'_j \in \mathcal{H}'} \dim H'_j + 1 < \text{Ex.} \ C(Z, d) \). For instance, in (Ob. 1) we would consider \( W' = W \cup (Z \cap H_i) \) and \( \mathcal{H}' = \mathcal{H} \setminus \{H_i\} \). (Ob. 3) is a consequence of (Ob. 2).

Note that (Ob. 1) implies that \( \text{Ex.} \ C(Z, d) = |Z| \) for all \( d \geq \frac{|Z| - 1}{n} \), so suppose that \( nd + 1 < |Z| \). Let \( g_i = |Z \cap H_i| - (d \dim(H_i) + 1) \geq 0 \), and note that by hypothesis \( \frac{g_i}{\dim H_i} = \frac{|Z \cap H_i| - 1}{\dim H_i} - d \leq \frac{|Z| - nd - 1}{n} \).
Combining this with our formula for Ex. C(Z, d), we obtain the following

\[
\text{Ex. } C(Z, d) = |W| + \sum_{i=1}^{s} (d \dim(H_i) + 1)
\]

\[
= |Z| - \sum_{i=1}^{s} g_i \geq |Z| - \sum_{i=1}^{s} (\dim H_i) \left( \frac{|Z| - nd - 1}{n} \right)
\]

\[
\geq |Z| - (|Z| - nd - 1) \left( \frac{\sum_{i=1}^{s} \dim H_i}{n} \right)
\]

Now as \( \sum \dim H_i \leq n \) by (Ob. 3) we obtain \( \text{Ex. } C(Z, d) \geq |Z| - (|Z| - nd - 1) = nd + 1 \). As it’s always true that \( \text{Ex. } C(Z, d) \leq nd + 1 \), the result now follows. \( \square \)

Remark 5.32. To close we spell out the connection between theorem 5.31 and the original theorem 5.1 from [7]

**proof of Theorem 5.1.** Let \( Z \subseteq \mathbb{P}^2 \) and let \((a_1, a_2)\) be the splitting type of \( D_0(\mathcal{A}_Z) \). First consider the case where there exists some \( L \subseteq \mathbb{P}^2 \) so that \(|L \cap Z| > a_1 + 1\). Let \( Q \in \mathbb{P}^2 \) be a general point, and take \( f \in [I_Q(Z)]_{a_1+1} \) be a minimal generator of \( I^\cap(Z) \), then applying Bezout’s Theorem we see that \( L \) must be a component of the variety \( f = 0 \). Hence, \( f \) factors as \( f = \ell g \) where \( \ell \) is the linear form defining \( L \) and \( g \in [F_Q]_{a_1} = [I(Q)^{a_1}] \) is a product of linear forms. As \( f \) is a minimal generator and \( Q \) is general it follows each linear form in \( g \) vanishes at precisely one point of \( |Z \setminus L| \). Therefore, as \( a_1 = \deg g = |Z \setminus L| \), we conclude that \(|Z \cap L| = |Z| - |Z \setminus L| = a_1 + 1\).

Now noting that if \( \text{Ex. } C(Z, d) = \sum_{i=1}^{k} d \dim H_i + 1 \) for linear subspaces, \( H_i \) that we must have \( d \dim(H_1 + H_2) + 1 \geq d \dim H_1 + d \dim H_2 + 2 \). It follows that \( \dim(H_1 + H_2) > \dim H_1 + \dim H_2 \), in the case that \( Z \subseteq \mathbb{P}^2 \), this implies that there is at most one line or plane among the \( H_i \). Therefore, we conclude that \( \text{Ex. } C(Z, d) = \min\{2d + 1, (d \dim L + 1) + |Z \setminus L|, |Z|\} \) or equivalently

\[
\text{Ex. } C(Z, d) = \begin{cases} 
2d + 1 & \text{if } d \leq a_1 \\
\quad d + 1 + a_1 & \text{if } a_1 \leq d \leq a_2 \\
\quad a_1 + a_2 + 1 & \text{if } a_2 \leq d
\end{cases}
\]

Applying theorem 4.14 and a direct comparison now shows that \( Z \) admits no unexpected curves, establishing this case.

For the other case we have \(|L \cap Z| \leq a_1 + 1\) for all \( L \subseteq \mathbb{P}^2 \). Then for all lines \( L \subseteq \mathbb{P}^2 \) we have the inequality \(|Z \cap L| \leq a_1 + 1 \leq \left\lceil \frac{|Z| - 1}{2} \right\rceil + 1\). Subtracting through by 1 gives

\[
|Z \cap L| - 1 \leq a_1 \leq \frac{|Z| - 1}{2}
\]

allowing us to conclude by theorem 5.31. \( \square \)

5.1. **Computations and Examples of Unexpected Hypersurfaces.** Combinatorial Optimization problems similar to the linear program, Ex. \( C(Z, d) \), from in definition 5.20 have been studied before. One notable instance of this is in the paper [23]. In [23] the author fixed a submodular function \( \mu : S \to \mathbb{R} \) and a real parameter \( \lambda \), and studied the optimization problem

\[
\min \left\{ \sum_{i=0}^{t} \mu(S_i) - \lambda \Bigg| \{S_0, ..., S_t\} \text{ is a partition of } S \right\}.
\]

It was shown in section 3 of [23] that for a fixed \( \mu \) and \( \lambda \) that there is a unique **finest** and a unique **coarsest** partition of \( S \) achieving this minimum. Here we say a partition \( \pi \) if **finer** than the partition \( \tau \) (or equivalently that \( \tau \) is **coarser** than \( \pi \)) and write \( \pi \leq \tau \), if every block of \( \pi \) is contained in a block of \( \tau \). In section 4 of [23] an algorithm was given which solves this problem for a fixed \( \mu \). It was shown in particular that minimum is a piecewise linear function of \( \lambda \).
We note that \( \text{Ex. } C(Z, d) \) is equivalent to
\[
d \min \left\{ \sum_{i=0}^{t} \left( \text{rk}_{M(Z)}(A_i) - \frac{d - 1}{d} \right) \left| \{A_0, \ldots, A_t\} \text{ is a partition of } Z \right. \right\}
\]
and so the algorithm given in [23] can be used to solve \( \text{Ex. } C(Z, d) \).

**Definition 5.33.** Let \( Z \subseteq \mathbb{P}^n \) be a finite set of points, for each \( d \geq 0 \), we define the *modified expected base locus*, which we denote \( \text{Ex. Bl}(Z, d) \) to be the coarsest partition in the partition order which satisfies
\[
\sum_{B \in \text{Ex. Bl}(Z,d)} (d \dim \text{Span}(B) + 1) = \text{Ex. } C(Z, d).
\]
Meaning that if \( \Pi \) is any other partition with \( \sum_{P \in \Pi} (d \dim \text{Span}(P) + 1) = \text{Ex. } C(Z, d) \), then for every \( P \in \Pi \) there is some \( B \in \text{Ex. Bl}(Z,d) \) so that \( P \subseteq B \).

Section 3 of [23] establishes not only that \( \text{Ex. Bl}(Z, d) \) exists, but also that in the partition order \( \text{Ex. Bl}(Z, d) \geq \text{Ex. Bl}(Z, d + 1) \). We now make a few observations about \( \text{Ex. Bl}(Z, d) \) and \( \text{Ex. } C(Z, d) \) in order to compute \( \text{Ex. } C(Z, d) \) more easily. These results are heavily influenced by the results and techniques in [23]. However, our results are stronger in some cases as we can take advantage of the fact that \( \text{rk}_{M(Z)} \) is the rank function of a matroid, and not merely a submodular function.

**Lemma 5.34.** For any real number \( d > 0 \), and for distinct blocks \( B_1, \ldots, B_k \in \text{Ex. Bl}(Z, d) \) we have
\[
\sum_{i=1}^{k} (d \dim \text{Span}(B_i) + 1) < d \dim \text{Span} \left( \bigcup_{i=1}^{k} B_i \right) + 1
\]
In particular, for each pair of distinct blocks \( B_1 \) and \( B_2 \), \( \text{Span}(B_1) \) and \( \text{Span}(B_2) \) are disjoint subspaces.
Similarly, if \( C_1 \sqcup \ldots \sqcup C_{\ell} \) is a partition of a block \( B_\ell \in \text{Ex. Bl}(Z, d) \) into nonempty subsets, then
\[
\sum_{j=1}^{\ell} (d \dim \text{Span}(C_j) + 1) \geq d \dim \text{Span}(B) + 1
\]

**Proof.** If \( \text{Ex. Bl}(Z, d) = \{B_1, \ldots, B_m\} \) then let \( A = \left\{ \bigcup_{i=1}^{k} B_i, B_{k+1}, \ldots, B_m \right\} \). As \( A \) is coarser than \( \text{Ex. Bl}(Z, d) \), we get
\[
\sum_{a \in A} d \dim \text{Span}(a) + 1 > \sum_{b \in \text{Ex. Bl}(Z,d)} d \dim \text{Span}(b) + 1.
\]
Subtracting away the shared terms now gives the desired inequality.

The proof of the second claim follows similarly, since we must have
\[
\left( \sum_{j=1}^{\ell} d \dim \text{Span}(C_j) + 1 \right) + \left( \sum_{B \in \text{Ex. Bl}(Z,d); B \neq B_\ell} d \dim \text{Span}(B) + 1 \right) \geq \sum_{B \in \text{Ex. Bl}(Z,d)} d \dim \text{Span}(B) + 1
\]

It can be somewhat laborious to determine if a given set of points satisfies the combinatorial condition in theorem 5.31. Furthermore, most of the observed configurations of points \( Z \) which admit unexpected curves possess certain kinds of symmetry, namely their dual arrangements \( A_Z \) are reflection arrangements. We designed this next proposition with these examples in mind.

**Definition 5.35.** A *psuedoreflection* is a matrix \( R \in \mathbb{GL}(n, \mathbb{K}) \) so that \( R^k = I_n \) for some \( k > 1 \) and the set of points in \( \mathbb{K}^n \), which are fixed by \( R \), denoted \( \text{Fix}_R \), form a hyperplane. A *reflection group* is a subgroup, \( G \), of \( \mathbb{GL}(n, \mathbb{K}) \), which is generated by psuedoreflections. \( G \) is an *irreducible reflection group* if there no nontrivial \( G \)-invariant subspace of \( \mathbb{K}^n \).
Proposition 5.36. If \( Z \subseteq \mathbb{P}^n_k \) is a finite set of points, and there is an irreducible reflection group \( G \subseteq \mathbb{PGL}(k, n) \) acting on \( Z \). Then for all positive dimensional linear subspaces \( H \subseteq \mathbb{P}^n \) we have
\[
\frac{|Z \cap H| - 1}{\dim H} \leq \frac{|Z| - 1}{n}.
\]
Consequently by corollary 5.30, \( Z \) admits very unexpected hypersurfaces in degree \( d \) for precisely those \( d \) with \( a_1 < d < a_n \), where \( (a_1, ..., a_n) \) is the splitting type of \( D_0(A_Z) \).

We first note a useful criterion, which is used in the proof of the above proposition.

Claim 5.37. Let \( Z \subseteq \mathbb{P}^n \), then the following are equivalent:

1. For all positive dimensional subspaces \( H \subseteq \mathbb{P}^n \),
\[
\frac{|Z \cap H| - 1}{\dim H} \leq \frac{|Z| - 1}{n}.
\]

2. \( \text{Ex. } C(Z, q) = \min\{qn + 1, |Z|\} \) for all \( q \in \mathbb{Q} \).

Proof of claim. The forward direction is established in theorem 5.31. For the reverse direction, we prove the contrapositive. Namely, suppose that there is some \( H \subseteq \mathbb{P}^n \) with \( \frac{|Z \cap H| - 1}{\dim H} > \frac{|Z| - 1}{n} \). Then choose any \( q \) with
\[
\frac{|Z \cap H| - 1}{\dim H} > q > \frac{|Z| - 1}{n}.
\]
Note then that \( |Z \cap H| > q \dim H + 1 \) and that \( qn + 1 > |Z| \), hence we have that
\[
\text{Ex. } C(Z, d) \leq q \dim H + 1 + |Z \setminus H| < |Z| < qn + 1
\]
establishing the result.

Proof of proposition 5.36. First, note that if \( G \) is any group acting on \( Z \) then this action extends to the lattice of partitions of \( Z \). Furthermore, if \( \Pi \) is any partition of \( Z \), then for any \( g \in G \) we have \( \sum_{P \in \Pi} d \dim \text{Span}(P) + 1 = \sum_{P \in \Pi} d \dim \text{Span}(gP) + 1 \). From this it follows that \( \text{Ex. } \text{Bl}(Z, d) \) is fixed by the \( G \) action, in the sense that blocks of \( \text{Ex. } \text{Bl}(Z, d) \) are taken to other blocks of \( \text{Ex. } \text{Bl}(Z, d) \).

Now we continue to establishing the proposition. By the preceding claim it suffices to show that for rational \( q \), \( \text{Ex. } \text{Bl}(Z, q) \) is either the discrete or the indiscrete partition. Suppose that \( B \in \text{Ex. } \text{Bl}(Z, d) \) is a block with \( |B| \geq 2 \), let \( r \in G \) be a psuedoreflection and \( H_r = \text{Fix}_r \) the hyperplane of the points fixed by \( r \). As \( \dim \text{Span}(B) \geq 1 \) then consequently \( H_r \cap \text{Span}(B) \) and hence \( \text{Span}(rB) \cap \text{Span}(B) \) are both nonempty. Applying lemma 5.34, we see that we must have \( rB = B \) and so \( r \text{Span}(B) = \text{Span}(B) \). Therefore, \( \text{Span}(B) \) is a nonzero \( G \)-invariant subspace of \( \mathbb{P}^n \). As \( G \) is an irreducible reflection group we must have that \( \text{Span}(B) = \mathbb{P}^n \) and so \( B = Z \) by lemma 5.34.

For a set of points \( Z \subseteq \mathbb{P}^n \), if the splitting type of \( D_0(A_Z) \) is known, then determining when \( Z \) admits very unexpected hypersurfaces comes down to computing \( \text{Ex. } C(Z, d) \). The following two propositions can be useful in determining \( \text{Ex. } C(Z, d) \). The first places bounds on how \( \text{Ex. } C(Z, d) \) can change between degrees.

Lemma 5.38. For \( Z \subseteq \mathbb{P}(V) \), the sequence of forward differences \( \delta_d = \text{Ex. } C(Z, d + 1) - \text{Ex. } C(Z, d) \) is nonincreasing. Furthermore, we have
\[
\sum_{A \in \text{Ex. } \text{Bl}(Z, d)} \dim(A) \geq \delta_d \geq \sum_{B \in \text{Ex. } \text{Bl}(Z, d+1)} \dim(B).
\]
Let \( A \in \text{Ex.Bl}(Z,d) \)

\[
\sum_{A \in \text{Ex.Bl}(Z,d)} \dim \text{Span}(A) = \sum_{A \in \text{Ex.Bl}(Z,d)} \left[ (d+1) \dim \text{Span}(A) + 1 \right] - \sum_{A \in \text{Ex.Bl}(Z,d)} \left[ d \dim \text{Span}(A) + 1 \right]
\]

\[
\geq \sum_{A \in \text{Ex.Bl}(Z,d)} \left[ (d+1) \dim \text{Span}(A) + 1 \right] - \sum_{B \in \text{Ex.Bl}(Z,d+1)} \left[ d \dim \text{Span}(B) + 1 \right]
\]

\[
\geq \sum_{B \in \text{Ex.Bl}(Z,d+1)} \left[ (d+1) \dim \text{Span}(B) + 1 \right] - \sum_{B \in \text{Ex.Bl}(Z,d+1)} \left[ d \dim \text{Span}(B) + 1 \right]
\]

\[
= \sum_{B \in \text{Ex.Bl}(Z,d+1)} \dim \text{Span}(B).
\]

Now noting that \( \delta_d = \sum_{A \in \text{Ex.Bl}(Z,d+2)} \left[ (d+1) \dim \text{Span}(A) + 1 \right] - \sum_{B \in \text{Ex.Bl}(Z,d+2)} \left[ d \dim \text{Span}(B) + 1 \right] \) establishes the result. \( \square \)

The following proposition shows that if the splitting type \( (a_1, \ldots, a_n) \) is known it suffices to check if \( Z \) admits very unexpected hypersurfaces by only looking around the degrees in the splitting type.

**Proposition 5.39.** Let \( Z \subseteq \mathbb{P}^n \) and let \( (a_1, a_2, \ldots, a_n) \) denote the splitting type of \( D_0(A_Z) \). If \( Z \) does not admit very unexpected hypersurfaces in degree \( d \), but does admit them in either degree \( d-1 \) or degree \( d+1 \), then \( d = a_i \) for some \( i \).

**Proof.** Let \( d \) be an index satisfying the hypothesis. Define indexes \( j \) and \( \ell \) so that \( a_k < d \) for all \( k \leq j \), and \( a_k < d+1 \) for all \( k \leq \ell \). The proposition is established if we show \( \ell > j \).

Applying the inequality from theorem 5.27 in degrees \( d-1, d \) and \( d+1 \) we obtain the following three equations

(\text{Eq. 1}) Ex. C(Z, d-1) + \sum_{k=1}^{j} (d-1-a_k) \geq n(d-1) + 1

(\text{Eq. 2}) Ex. C(Z, d) + \sum_{k=1}^{\ell} (d-a_k) = nd + 1; and

(\text{Eq. 3}) Ex. C(Z, d+1) + \sum_{k=1}^{\ell} (d+1-a_k) \geq n(d+1) + 1.

Subtracting (\text{Eq. 1}) from (\text{Eq. 2}) and (\text{Eq. 2}) from (\text{Eq. 3}) gives (\text{Eq. 4}) and (\text{Eq. 5}) below.

(\text{Eq. 4}) \( \delta_d-1 + j = \text{Ex. C}(Z, d) - \text{Ex. C}(Z, d-1) + j \geq n \)

(\text{Eq. 5}) \( \delta_d + \ell = \text{Ex. C}(Z, d+1) - \text{Ex. C}(Z, d) + \ell \geq n \)

By the preceding lemma \( \delta_d \leq \delta_d-1 \), with this and (\text{Eqs. 4 & 5}) we have

\[ \delta_d + j \leq \delta_d-1 + j \leq n \leq \delta_d + \ell. \]

We may conclude that \( j < \ell \), if either (\text{Eq. 4}) or (\text{Eq. 5}) is strict. Yet this happens precisely when \( Z \) admits very unexpected hypersurfaces in degree \( d-1 \) or \( d+1 \). \( \square \)

**Example 5.40.** Let \( \mathbb{F}_q \) be the finite field with \( q = p^e \) elements, and \( \mathbb{K} \) an infinite field containing \( \mathbb{F}_q \). Let \( \mathbb{P}^n_{\mathbb{F}_q} \subseteq \mathbb{P}^n_{\mathbb{K}} \) consist of those points which in homogeneous coordinates can be written as \( (\alpha_0 : \alpha_1 : \ldots : \alpha_n) \) with \( \alpha_i \in \mathbb{F}_q \). It is well known that \( |\mathbb{P}^n_{\mathbb{F}_q}| = \frac{q^{n+1}-1}{q-1} = q^n + q^{n-1} + \ldots + q + 1 \), and that \( A_{\mathbb{P}^n_{\mathbb{F}_q}} \) is free with exponents \( (1,q,q^2,\ldots,q^n) \). The generator in degree \( q^i \) is of the form

\[
\sum_{j=0}^{n} Y_j^{q^i} \frac{\partial}{\partial Y_j}
\]

and so the corresponding generator of \( I^> \left( \mathbb{P}^n_{\mathbb{F}_q} \right) \) is

\[
\sum_{j=0}^{n} M_{ij} X_i = \begin{bmatrix}
X_0 & X_1 & \ldots & X_n \\
X_0^{q^i} & X_1^{q^i} & \ldots & X_n^{q^i} \\
A_{1,0}^{q^i} & A_{1,1}^{q^i} & \ldots & A_{1,n}^{q^i} \\
\vdots & \ddots & \ddots & \vdots \\
A_{n-1,0}^{q^i} & \ldots & \ldots & A_{n-1,n}^{q^i}
\end{bmatrix}
\]
Furthermore, note that $\mathbb{P}^n_q$ is acted on by the group $\text{GL}(n, \mathbb{F}_q)$. This in particular contains the irreducible reflection group consisting of the permutation matrices, so by proposition 5.36 $\mathbb{P}^n_q$ admits very unexpected hypersurfaces in all degrees $d$ with $q < d < q^n$.

**Example 5.41.** Fix some primitive $m$-th root of unity $\zeta \in \mathbb{C}$, for $m \geq 2$. Define a configuration of points $F_m \subseteq \mathbb{P}^n_C$ as consisting of the $m(\begin{pmatrix} m+1 \\ 2 \end{pmatrix})$ points whose $i$-th coordinate is $-1$ and $j$-th coordinate is $\zeta^k$ for all $0 \leq k \leq d - 1$ and all pairs $0 \leq i < j \leq n$. Let $C_m = F_m \cup \{E_0, E_1, \ldots, E_m\}$ here $E_i$ is the $i$-th coordinate point.

Then $A_{C_m}$ is an Extended Ceva Arrangement, it is a reflection arrangement corresponding to the reflection group $G(m, 1, n+1) \subseteq \text{PGL}(\mathbb{C}, n)$. The splitting type of $D_0(A_{C_m})$ is $(m+1, 2m+1, \ldots, nm+1)$ (see [24] for details). As $A_{C_m}$ is a reflection arrangement, we again apply proposition 5.36 to conclude that $A_{C_m}$ admits very unexpected hypersurfaces in all degrees $d$ with $m+1 < d < nm+1$.

Both of our classes of examples come from reflection arrangements, more generally proposition 5.36 gives a good criterion for determining if the points dual to a given reflection arrangement admit unexpected hypersurfaces. We note that reflection arrangements have been classified and that their exponents and hence their splitting type can be found in the appendix of [24].

Our final example shows that the degrees in which a set of points $Z$ admits very unexpected hypersurfaces do not need to be consecutive. This is in contrast with the situation in the plane as shown in theorem 5.1. Before outlining the example we state a useful proposition and definition.

**Definition 5.42.** Let $V_1$ and $V_2$ be finite dimensional $\mathbb{K}$-vector spaces, and suppose we have finite sets of points $Z_1 \subseteq \mathbb{P}(V_1)$, $Z_2 \subseteq \mathbb{P}(V_2)$. There are inclusion maps $\iota_i : \mathbb{P}(V_i) \to \mathbb{P}(V_1 \oplus V_2)$ for $i = 1, 2$. We then define $Z_1 \oplus Z_2 \subseteq \mathbb{P}(V_1 \oplus V_2)$ as the set of points

$$Z_1 \oplus Z_2 := \iota_1(Z_1) \cup \iota_2(Z_2).$$

**Proposition 5.43.** $Z_1 \oplus Z_2$ admits very unexpected curves in degree $d \geq 1$ if and only if $Z_1$ or $Z_2$ admits unexpected curves in degree $d$.

**Proof.** First, note that for hyperplane arrangements $A_1 \subseteq \mathbb{P}(W_1)$ and $A_2 \subseteq \mathbb{P}(W_2)$ there is an arrangement $A_1 \times A_2 \subseteq \mathbb{P}(W_1 \oplus W_2)$ induced by the projections $p_i : \mathbb{P}(W_1 \oplus W_2) \to \mathbb{P}(W_i)$. Namely, $A_1 \times A_2$ is formed by taking all hyperplanes of the form $\pi_i^{-1}(H)$ for $H \in A_i$.

We now note two facts:

**(Fact 1)** $A_{Z_1 \oplus Z_2} = A_{Z_1} \times A_{Z_2}$;

**(Fact 2)** If $S$ is the projective coordinate ring of $\mathbb{P}(W_1 \oplus W_2)$ there is an isomorphism of $S$-modules,

$$D(A_{Z_1} \times A_{Z_2}) \cong (S \otimes D(A_{Z_2})) \oplus (S \otimes D(A_{Z_1})).$$

The first can be seen by following each the constructions through the duality. We omit a proof of the second referring to [24] for details.

One consequence of fact 2 is that if $D_0(A_1)$ has splitting type $(a_1, \ldots, a_n)$ and $D_0(A_2)$ has splitting type $(b_1, \ldots, b_m)$, then $D_0(A_1 \times A_2)$ has a splitting type (up to reordering) of $(1, a_1, \ldots, a_n, b_1, \ldots, b_m)$. Applying theorem 5.27, now yields the inequalities valid for any $d \geq 1$. Each inequality strict if and only if the corresponding set of points admits very unexpected hypersurfaces

**(Ineq. 1)** $\sum_{i=1}^{m} \max\{0, d - a_i\} \geq nd + 1 - \text{Ex. } C(Z_1, d)$

**(Ineq. 2)** $\sum_{j=1}^{m} \max\{0, d - b_j\} \geq md + 1 - \text{Ex. } C(Z_2, d)$

**(Ineq. 3)** $d - 1 + \left(\sum_{i=1}^{n} \max\{0, d - a_i\}\right) + \left(\sum_{j=1}^{m} \max\{0, d - b_j\}\right) \geq (n + m + 1)d + 1 - \text{Ex. } C(Z_1 \oplus Z_2, d)$

We now claim that $\text{Ex. } C(Z_1 \oplus Z_2, d) = \text{Ex. } C(Z_1, d) + \text{Ex. } C(Z_2, d)$. First note that if we assume this claim and subtract $d - 1$ from both sides of (Ineq. 3), then the resulting inequality may be written as the sum of (Ineq. 1) and (Ineq. 1). From this it follows that (Ineq. 3) is strict if and only if either (Ineq. 3) or (Ineq. 1) is strict and the proposition follows.

Continuing to the proof of our claim, we first note that if $d = 1$ then for any set of points $\text{Ex. } \text{Bl}(Z, 1) = \{Z\}$. A direct computation establishes the claim in this case.
Now we may assume \( d \geq 2 \). Take a block \( B \in \text{Ex. Bl}(Z_1 \oplus Z_2, d) \) and define \( B_1 = B \cap Z_1 \) and \( B_2 = B \cap Z_2 \). We note that if \( B_1 \) and \( B_2 \) are nonempty, then lemma 5.34 states

\[
d(\dim \text{Span } B - \dim \text{Span } B_1 - \dim \text{Span } B_2) \leq 1.
\]

Yet as \( B_1 \) and \( B_2 \) are contained in disjoint subspaces, \( \dim \text{Span}(B) = \dim \text{Span}(B_1) + \dim \text{Span}(B_2) + 1 \) and the inequality becomes \( d \leq 1 \) giving a contradiction. Therefore, for each block \( B \) we have \( B \subseteq Z_1 \) or \( B \subseteq Z_2 \), and consequently \( \text{Ex. Bl}(Z_1 \oplus Z_2, d) = \Pi_1 \cup \Pi_2 \) for some partitions \( \Pi_1 \) and \( \Pi_2 \) of \( Z_1 \) and \( Z_2 \) respectively. From this it readily follows from the definition that \( \text{Ex. Bl}(Z_1 \oplus Z_2, d) = \text{Ex. Bl}(Z_1, d) \cup \text{Ex. Bl}(Z_2, d) \). This establishes the claim that \( \text{Ex. } C(Z_1 \oplus Z_2, d) = \text{Ex. } C(Z_1, d) + \text{Ex. } C(Z_2, d) \) and completes our proof. \( \square \)

**Example 5.44.** If \( C_2, C_7 \subseteq \mathbb{P}_C^2 \) are the configurations of points described in example 5.41, then \( C_2 \oplus C_7 \) is a configuration of 33 points in \( \mathbb{P}_C^2 \). The module of derivations, \( D_0(A_{C_2} \times A_{C_7}) \), has splitting type \((1, 3, 5, 8, 15)\). Using the computation from example 5.41 along with proposition 5.43, it follows that \( C_2 \oplus C_7 \) admits very unexpected hypersurfaces in degree \( d \) if and only if \( d = 4 \) or \( 8 < d < 15 \).

6. A Lifting Criterion and the Structure of Unexpected Curves in \( \mathbb{P}_C^2 \)

One feature of the theorems 4.8 and 4.14, is they allow us to view elements of reduced Module of Derivations as explicit polynomials. This permits us to use techniques such as unique factorization and polynomial division that are not as well developed for general modules. In this section we give a few applications of this viewpoint. First, we state a lifting criterion in proposition 6.2, this allows us under certain conditions to lift an element of the restricted module \( D_0(A_Z) \mid_2 \) to the module \( D_0(A_Z) \). This criterion has especially strong implications in \( \mathbb{P}_C^2 \), such as in theorem 6.8, where we show that for \( Z \subseteq \mathbb{P}^2 \) every polynomial defining an unexpected curve in \( I^\prec(Z) \) can be lifted to an element of \( I^\gg(Z) \).

This result ends up putting very strong conditions on the combinatorics of sets of points \( Z \) which admit unexpected curves, which we explore in the next section.

**Proposition 6.1.** Let \( Z \subseteq \mathbb{P}(V) \cong \mathbb{P}^n \). Consider \( G \in I^\gg(Z) \subseteq \mathbb{K}[A] \). If there is some \( F \in \mathbb{K}[A] \) so that for general \( \alpha \in \text{Gr}(n - 1, V) \) we have that \( \varepsilon_\alpha(F) \in I^\alpha(Z) \) and \( \varepsilon_\alpha(F) \mid \varepsilon_\alpha(G) \). Then \( F \) and \( G \) have a common divisor \( H \in I^\gg(Z) \).

**Proof.** For any prime ideal, \( I \), we set \( \nu_I(F) \) as the valuation \( \nu_I(F) := \sup\{m \geq 0 \mid F \in I^m\} \). Now we define two ideals of \( \mathbb{K}[A] \), \( X \) is the ideal \((X_0, ..., X_n)\) and we let \( M \) denote the ideal generated by the maximal minors of the matrix \( A \). Lastly, for \( \alpha \in \text{Gr}(n - 1, V) \), \( I(\alpha) \) is the ideal of \( \mathbb{K}[X_0, ..., X_n] \subseteq \mathbb{K}[A] \) defined by the subspace \( \alpha \).

Before continuing we note a few facts:

**Fact 1** For each of the 3 ideals, \( X \), \( I(\alpha) \) and \( M \), that we have defined we have \( I^k = I^{(k)} \).

**Fact 2** For any \( f \in \mathbb{K}[A] \) we have \( \nu_X(f) \geq \nu_M(f) \) and \( \nu_{I(\alpha)}(\varepsilon_\alpha(f)) = \nu_M(f) \) for general \( \alpha \).

**Fact 3** For any \( f \in \mathbb{K}[A] \), we have the inequality

\[
(\nu_X(f) - \nu_M(f)) + n^2\nu_M(f) \leq \deg(f).
\]

**Fact 4** If \( \nu_X(f) = \nu_M(f) + 1 \), then equality occurs in Fact 3 if and only if \( f \in \mathfrak{m}^\gg \).

The first fact follows for \( X \) and \( I(\alpha) \) since both are complete intersections, for \( M \) we refer to section 2.2 of [19]. The second fact follows since \( M \) is essentially \( I(\alpha) \) for \( \alpha \) the generic point. The third is a consequence of the first and that \( \deg(M_i) = n^2 \). Lastly, the fourth fact follows since if \( \nu_X(f) = \nu_M(f) \) then setting \( d = \nu_X(f) \) we have \( f \in [(X_0, ..., X_n)M^{d-1}]n^2(d-1)+1 \), but this is precisely \([\mathfrak{m}]^d \).

First we claim for general \( \alpha \), that any \( f \in I^\gg(\alpha,Z) \) factors into irreducible components as \( f = f_0 \prod_{i=0}^k \ell_i \) where each \( \ell_i \) is an element of the special fibre ring \( \mathcal{F}_\alpha = \text{Sym}(I(\alpha)) \). It suffices to show that if \( f = pq \), then either \( p \) or \( q \) is in \( \mathcal{F}_\alpha \). Noting that \( \nu_X(f) = 1 + \nu_{I(\alpha)}(f) \), and using the additive property of valuations, we obtain

\[
\nu_X(p) + \nu_X(q) = 1 + \nu_{I(\alpha)}(p) + \nu_{I(\alpha)}(q) \leq 1 + \nu_X(p) + \nu_X(q).
\]

Since all numbers above are integers, and \( \nu_{I(\alpha)}(h) \leq \nu_X(h) \) for every \( h \in \mathbb{K}[X_0, ..., X_n] \), we may assume without loss of generality that \( \nu_{I(\alpha)}(p) = \nu_X(p) \) and \( \nu_{I(\alpha)}(q) = \nu_X(q) + 1 \). It now follows that \( p \in [I(\alpha)^\nu_X(p)]_{\nu_X(p)} \subseteq \mathcal{F}_\alpha \), which establishes our claim.
Continuing with the proof of the proposition, we let \( \varepsilon_\mathfrak{g} \) denote the generic evaluation. In other words \( \varepsilon_\mathfrak{g} \) is the inclusion \( \varepsilon_\mathfrak{g} : \mathbb{K}[\mathcal{A}] \to \mathbb{F}[X_0, \ldots, X_n] \), for \( \mathbf{F} \) the function field \( \mathbb{F} := \mathbb{K}(A_{ij} \mid (i, j) \leq (n, n + 1)) \).

By assumption \( \varepsilon_\mathfrak{g}(F) \) divides \( \varepsilon_\mathfrak{g}(G) \), so there exists \( h \in \mathbb{K}[\mathcal{A}] \) and \( k \in \mathbb{K}[A_{ij} \mid (i, j) \leq (n, n + 1)] \) with \( h \) and \( k \) coprime so that \( \frac{h}{k} \varepsilon_\mathfrak{g}(F) = \varepsilon_\mathfrak{g}(G) \iff hF = kG \) where this last equality is in \( \mathbb{K}[\mathcal{A}] \). Now by unique factorization in the polynomial ring \( \mathbb{K}[\mathcal{A}] \), we get \( k \mid F \). Setting \( \tilde{F} = \frac{h}{k} F \in \mathbb{K}[\mathcal{A}] \), we have \( h\tilde{F} = G \) and \( k\tilde{F} = F \). Moreover, since \( \varepsilon_\mathfrak{g}(F) \notin \mathcal{F}_q \) it follows that \( \tilde{F} \notin \mathcal{F}_q \) and so \( h \in \mathcal{F}_q \). We finish the proof by establishing that \( \tilde{F} \in I^{\gg}(Z) \).

Since \( F \) differs from \( \tilde{F} \) only by \( \mathbb{F} \) scalar, and \( F \in I(Z) \) we have \( \tilde{F} \in I(Z) \) and so it suffices to show that \( \tilde{F} \in \mathfrak{m}^{\gg} \). Since \( \varepsilon_\mathfrak{g}(\tilde{F}) \in I^{\gg}_g(Z) \) and \( \varepsilon_\mathfrak{g}(h) \in \mathcal{F}_q \) we have the inequalities
\[
\nu_{M}(h) \leq \nu_{I(q)}(h) = \nu_{X}(\varepsilon_{\mathfrak{g}}(h)) = \nu_{X}(h); \quad \text{and} \quad \nu_{M}(\tilde{F}) \leq \nu_{I(q)}(\tilde{F}) = \nu_{X}(\varepsilon_{\mathfrak{g}}(\tilde{F})) - 1 = \nu_{X}(\tilde{F}) - 1.
\]

As \( h\tilde{F} \in I^{\gg}(Z) \), we have that \( \nu_{M}(h) + \nu_{M}(F) = \nu_{X}(h) + \nu_{X}(F) - 1 \) and so the above inequalities must be equality. Similarly, using the inequalities \( 1 + n^2 \nu_{M}(\tilde{F}) \leq \deg(\tilde{F}) \), \( n^2 \nu_{M}(h) \leq \deg(h) \) and \( \tilde{F} h = G \in I^{\gg}(Z) \) we have that
\[
1 + n^2 \nu_{M}(\tilde{F}) + n^2 \nu_{M}(h) \leq \deg(\tilde{F}) + \deg h = \deg G = 1 + n^2 \nu_{M}(\tilde{F}) h.
\]

Allowing us to conclude that \( 1 + n^2 \nu_{M}(\tilde{F}) = \deg(\tilde{F}) \) and \( n^2 \nu_{M}(h) = \deg(h) \) which completes the proof. \( \square \)

The preceding lemma when combined with the results of section 4 allows us under certain circumstances to lift elements of \( D_0(\mathcal{A}) \mid L \) to elements of \( D_0(\mathcal{A}) \). One example of this is illustrated in the following proposition.

**Proposition 6.2.** Let \( \mathcal{A} \subseteq \mathbb{P}^n_k \) and suppose that \( D_0(\mathcal{A}) \) has splitting type \( (a_1, a_2, \ldots, a_n) \) with \( a_1 < a_2 \leq a_3 \leq \ldots \leq a_n \). If \( \theta_1 \in D_0(\mathcal{A}) \) is a nonzero element of degree \( < a_2 \), then \( D_0(\mathcal{A}) \) has a minimal generator in degree \( a_1 \).

**Proof.** Using the translation given by theorem 4.8, there’s a nonzero \( F_\lambda \in [I^{\gg}(Z)]_d \) where \( d < a_2 + 1 \). If \( Q \) is the generic codimension 2 linear subspace, then by theorem 4.14 \( I^{\gg}_Q(Z) \) is free on generators \( f_1, \ldots, f_n \) with \( \deg f_i = a_i + 1 \). Hence, \( \varepsilon_Q(F_\lambda) = \sum_{i=1}^n g_i f_i \). Yet as \( \deg f_j > \deg \varepsilon_Q(F_\lambda) \) for all \( j \geq 2 \), we must have \( \varepsilon_Q(F_\lambda) = g_1 f_1 \). After clearing denominators we may lift \( f_1 \) to an element \( \tilde{f}_1 \) of \( \mathbb{K}[\mathcal{A}] \).

Now as \( \varepsilon_Q(\tilde{f}_1) \) divides \( \varepsilon_Q(F_\lambda) \) we see by the previous lemma that there exists \( F_1 \in I^{\gg}_Q(Z) \) which divides both \( \tilde{f}_1 \) and \( F_\lambda \). As \( F_1 \) divides \( \tilde{f}_1 \) we must have \( F_1 \in [I^{\gg}(Z)]_{a_1+1} \) and so by theorem 4.8 there’s a nonzero \( \theta_1 \in [D_0(\mathcal{A}Z)]_{a_1} \). \( \square \)

The previous two propositions will prove to be especially useful when our points (or line arrangements) are in the plane \( \mathbb{P}^2 \). We will establish this using some results on vector bundles on \( \mathbb{P}^2 \) which we recall now.

**Definition 6.3.** We say a vector bundle \( M \) on \( \mathbb{P}^n_\mathbb{C} \) is semistable, if for all proper subbundles \( N \subseteq M \), we have
\[
\frac{c_1(N)}{\text{rank } N} \leq \frac{c_1(M)}{\text{rank } M}
\]
where here \( c_1 \) is the first Chern class, and \( \text{rank } M \) is the dimension of a fibre.

If \( \text{rk}(M) = 2 \), semistability has a simpler characterization originally due to Hartshorne (see lemma 3.1 of [18]).

**Lemma 6.4.** Let \( M \) be a rank 2 bundle on \( \mathbb{P}^n_\mathbb{C} \), then if \( M \) is semistable if and only if letting \( c_1 = c_1(M) \)
\[
H^0 \left( \mathcal{M} \left( \left[ -\frac{c_1 - 1}{2} \right] \right) \right) = 0.
\]

In the case that our bundle \( M \) is the Derivation Bundle, \( \widetilde{D_0(A)} \) of a hyperplane arrangement, it was shown by Terao that \( c_1(M) = 1 - |\mathcal{A}| \), where \( |\mathcal{A}| \) is the number of hyperplanes in \( \mathcal{A} \). Using this together
with the previous lemma now allows us to characterize semistability of \( \overline{D_0(A)} \) for \( A \) a line arrangement in \( \mathbb{P}^2_C \).

**Proposition 6.5.** For \( A \subseteq \mathbb{P}^2_C \) define \( d = \left\lfloor \frac{|A|-2}{2} \right\rfloor \). Then the derivation bundle \( D_0(A) \) is semistable if and only if \( [D_0(A)]_d = 0 \). In particular, if \( D_0(A) \) is not semistable, then \( D_0(A) \) contains a nonzero derivation in degree \( \left\lfloor \frac{|A|}{2} \right\rfloor - 1 \).

One property of semistable bundles is the celebrated theorem of Grauert and M"{u}lich, which characterizes the splitting type of semistable bundles. In light of theorem 5.1, we see that if \( D_0(A_Z) \) is semistable then \( Z \subseteq \mathbb{P}^2_C \) admits no unexpected curves.

**Theorem 6.6** (Grauert-M"{u}lich). If \( B \) is a semistable bundle on \( \mathbb{P}^n_C \), with splitting type \( a_1 \leq a_2 \leq \ldots \leq a_k \), then for all \( 1 \leq i < k \), we have \( 0 \leq a_{i+1} - a_i \leq 1 \).

**Theorem 6.7.** [7] For \( Z \subseteq \mathbb{P}(V) \cong \mathbb{P}^2 \), if \( D_0(A_Z) \) is semistable then \( Z \) admits no unexpected curves.

This theorem in conjunction with 6.2 allows us to say that every unexpected curve in \( \mathbb{P}^2_C \) comes from a global section of the derivation bundle. More precisely a polynomial defining a degree \( d \) unexpected curve corresponds via the duality of theorem 4.8 to an element of \( [D_0(A)]_{d-1} \).

**Theorem 6.8.** Let \( Z \subseteq \mathbb{P}^2_C \) be a finite set of points. If \( D_0(A_Z) \) has splitting type \( (a_1, a_2) \) with \( a_2 - a_1 \geq 2 \) (in particular if \( Z \) admits unexpected curves), then \( D_0(A_Z) \) has a generator in degree \( a_1 \); Equivalently, \( I^\prec(Z) \) has a generator in degree \( a_1 + 1 \).

**Proof.** As \( a_2 - a_1 > 1 \), \( \overline{D_0(A_Z)} \) is not semistable by the Grauert-M"{u}lich theorem. Hence by proposition 6.5, there’s a nonzero \( \theta \in D_0(A) \) with \( \deg \theta \leq \left\lfloor \frac{|Z|^2}{2} \right\rfloor < a_2 \). Applying proposition 6.2 now yields the required generator of degree \( a_1 \).

Combining this with the Grauert-M"{u}lich Theorem, we obtain the following result.

**Theorem 6.9.** Let \( Z \) be a finite set of points in \( \mathbb{P}^2_C \) and let \( \alpha(D_0(A_Z)) \) denote the initial degree of \( D_0(A_Z) \). Define \( a = \min \{ \alpha(D_0(A_Z)), \left\lfloor \frac{|Z|^2}{2} \right\rfloor \} \) then \( D_0(A_Z) \) has splitting type \( (a, |Z| - a - 1) \).

Translating the above statement via the duality of theorem 4.8, we obtain the corollary below.

**Corollary 6.10.** For a finite set of points \( Z \subseteq \text{Proj}(\mathbb{C}[X_0, X_1, X_2]) \), suppose \( Z \) admits unexpected curves. If \( Q = (A_0 : A_1 : A_2) \) is the generic point, then \( I_Q^\prec(Z) \) is a free \( F_Q \)-module on generators \( f \) and \( g \) with \( \deg f < \deg g - 1 \) and \( f \) can be lifted to an element \( F \) of \( I^\prec(Z) \) with \( c_Q(F) = f \).

In particular, \( f \) can be written as

\[
    f = X_1 f_1 + X_2 f_2 + X_3 f_3,
\]

where \( f_i \) is a polynomial of degree \( \deg f - 1 \) in the maximal minors of

\[
    \begin{bmatrix}
    A_0 & A_1 & A_2 \\
    X_0 & X_1 & X_2
    \end{bmatrix}
\]

The above corollary states that the polynomials defining unexpected curves are “as simple as possible”, in the sense that they have the minimal possible degree as a polynomial in the coordinates of our general point \( Q \). This stands in stark contrast to most other sets of points where this is not the case. As an illustration, taking 8 randomly chosen points \( Z \subseteq \mathbb{P}^2_C \), a computation with Macaulay2 showed that \( I_Q^\prec(Z) \) has generators of \( X \)-degree 4 and 5. The first generator had an \( A \)-degree of 12 giving a total degree of 16, showing that the above result is far from expected. Similar computations with 6 points and 10 points gave minimal polynomials with \( A \)-degrees of 6 and 20, respectively.

Below we present a simpler example illustrating a similar point.

**Example 6.11.** If \( Z \subseteq \mathbb{P}^2_C \) consists of the 3 coordinate points and \((1 : 1 : 1)\). Then for generic \( Q \), \( I_Q^\prec(Z) \) has generators \( f_1 \) and \( f_2 \) of degrees 2 and 3 as polynomials in \( X \). Many different \( f_2 \) are possible. On the
other hand, if we require $f_1$ to be a polynomial of minimal degree in $\mathbb{K}[A]$, it is unique up to $\mathbb{C}$ scalar. The corresponding polynomial formula is

$$f_1 = (A_0 - A_1)M_1X_1 + (A_0 - A_2)M_2X_2 = (A_0 - A_1)A_2X_0X_1 - (A_0 - A_2)A_1X_0X_2 + (A_1 - A_2)A_0X_1X_2$$

where here $M_i$ is the minor of the $2 \times 3$ matrix from the matrix above and proposition 4.12. $f_1$ is irreducible, and defines the unique smooth conic through $Z$ and $Q$. As the $A$ degree and $X$ degree of $f_1$ are the same, we can see $f_1$ cannot be written in the form from corollary 6.10.

7. Combinatorial Constraints on Points Admitting Unexpected Curves

In this section we explore combinatorial constraints necessarily satisfied by sets of points admitting unexpected curves. Most of these constraints apply only when this unexpected curve is irreducible. Yet this turns out to be a fairly weak assumption, since if $Z$ admits a unique unexpected curve in degree $d$ there is always a subset $W \subseteq Z$ so $|Z \setminus W| = k$ and $W$ admits a unique irreducible unexpected curve in degree $d - k$.

We start by exploring the consequences of corollary 6.10. In the case the curve of degree $d$ is irreducible we show in lemma 7.5 that corollary 6.10 gives a bound on the number of distinct lines through a point $P \in Z$ and the remaining points of $Z$, showing that there are at most $d$ lines. As $|Z| \geq 2d + 1$ this is a very strong combinatorial condition which states that on average each line through a fixed point $P$ contains 3 or more points of $Z$. We are able to use this in theorem 7.6 to give a sharp bound on the number of points in $Z$, this bound is achieved by the Ceva type point configurations $C_d$ from example 5.41.

Furthermore, in proposition 7.13 we also give and upper bound on the number of lines spanned by points of $Z$. We then close the section by applying a theorem of Terao to state a combinatorial condition that guarantees that $Z$ will admit an unexpected curve.

We note that throughout this section, we often state theorems with the assumption that “there’s some nonzero (possibly irreducible) $f \in [I^\geq(Z)]_d$”. By theorem 6.8 perhaps the prototypical example for us are points $Z \subseteq \mathbb{P}^2_\mathbb{K}$ admitting unexpected curves in degree $d$. However, this also holds in other contexts for instance if $A_Z \subseteq \mathbb{P}^2_\mathbb{K}$ is free.

**Lemma 7.1.** For $Z \subseteq \mathbb{P}(V) \cong \mathbb{P}^2_\mathbb{K},$ consider $F(X_0, X_1, X_2; A_0, A_1, A_2) \in [I^\geq(Z)]_d$ then for every $P = (P_0 : P_1 : P_2) \in Z$,

$$\varepsilon_P(F) = F(X_0, X_1, X_2; P_0, P_1, P_2) \in I(P)^d.$$  

Moreover, $\varepsilon_P(F) = 0$ if and only if the linear form $\ell_P = P_0M_0 + P_1M_1 + P_2M_2$ divides $F$.

**Proof.** We may choose coordinates so $P = (1 : 0 : 0)$ and $\ell_P = M_0$. Writing $F$ as $F = F_0X_0 + F_1X_1 + F_2X_2$, then as $F \in I(P) = (X_1, X_2)$ we have

$$F(P_0, P_1, P_2, A_0, A_1, A_2) = F_0(1, 0, 0, A_0, A_1, A_2) = 0.$$  

As each $M_i$ is antisymmetric in $A$ and $X$, it follows that $\varepsilon_P(F_0) = 0$ and so $F_0 \in I(P^\perp) = M_0$ by proposition 4.12. Then applying the identity $M_0X_0 + M_1X_1 + M_2X_2 = 0$, we may write $F = f_1X_1 + f_2X_2$ where $f_i = \left(F_i - \frac{M_i}{M_0}F_0\right)$. Noting for arbitrary $Q \subseteq \mathbb{P}^2$, that $\varepsilon_Q(f_i) \in I(Q)^{d-1}$. It follows that

$$\varepsilon_P(F) = \varepsilon_P(f_1)X_1 + \varepsilon_P(f_2)X_2 \in (X_1, X_2)I(P)^{d-1} = I(P)^d,$$

which establishes the first statement.

Continuing with the proof of the second statement, assume that $\varepsilon_P(F) = \varepsilon_P(f_1)X_1 + \varepsilon_P(f_2)X_2 = 0$. Noting $\varepsilon_P(M_1) = X_2$ and $\varepsilon_P(M_2) = -X_1$, it follows that $\varepsilon_P(f_1M_2 - f_2M_1) = -\varepsilon_P(f_1)X_1 - \varepsilon_P(f_2)X_2 = 0$, so $f_1M_2 - f_2M_1 \in (M_0) = \ker \varepsilon_P$. Let $\tilde{f}_1, \tilde{f}_2 \in \mathbb{K}[M_1, M_2]$ so that $f_i = \tilde{f}_i \mod (M_0)$ for each $i \in \{1, 2\}$. Then $\tilde{f}_1M_2 - \tilde{f}_2M_1 \in (M_0) \cap \mathbb{K}[M_1, M_2] = 0$, so $\tilde{f}_1M_2 = \tilde{f}_2M_1$ and we get by unique factorization that there exists some $g \in \mathbb{K}[M_1, M_2]$ with $g = \frac{\tilde{f}_2}{M_2} = \frac{\tilde{f}_1}{M_1}$.

Finally, applying the identity $X_0M_0 + X_1M_1 + X_2M_2 = 0$ again we can write

$$F = f_1X_1 + f_2X_2 - g(M_0X_0 + M_1X_1 + M_2X_2) = (\tilde{g}M_0)X_0 + (f_1 - \tilde{g}M_1)X_1 + (f_2 - \tilde{g}M_2)X_2.$$
Noting that \( f_i - gM_i = f_i - \tilde{f}_i \equiv 0 \mod (M_0) \), we conclude that \( M_0 \) divides \( F \). This establishes the forward direction, the reverse direction follows as \( \varepsilon_P(\ell_P) = 0 \).

As we will see, the preceding lemmas imposes a very strong combinatorial condition on the configurations of points which can admit unexpected curves. Before we state the first of these conditions we introduce a new piece of notation.

**Definition 7.2.** Let \( Z \subseteq \mathbb{P}^2_K \) be a finite configuration of points, with \( |Z| \geq 2 \). For each \( P \in \mathbb{P}^2_K \), define a set of lines, \( L_P(Z) \), as follows

\[
L_P(Z) := \{ \text{Span}(Q_i, P) \mid Q_i \in Z \setminus \{ P \} \}.
\]

**Remark 7.3.** Note \( |L_P(Z)| \leq |Z \setminus \{ P \}| \) with equality and only if for distinct \( Q, Q' \in |Z \setminus \{ P \}| \) we have \( \text{Span}(Q, P) \neq \text{Span}(Q', P) \).

The number \( |L_P(Z)| \) defined above has an equivalent purely algebraic definition.

**Lemma 7.4.** Let \( Z \subseteq \mathbb{P}^2_K \) be a finite set of at least 2 points, then for any \( P \in \mathbb{P}^2_K \) we have

\[
|L_P(Z)| = \min\{d \mid |(I(Z) \cap I(P)^d)|_d \neq 0\}.
\]

**Proof.** Let \( m = \min\{d \mid |(I(Z) \cap I(P)^d)|_d \neq 0\} \). For any \( Q \in Z \setminus \{ P \} \), we get by Bezout’s Theorem that the line \( \text{Span}(P, Q) \) must be a component of the base locus of \( |(I(Z) \cap I(P)^d)|_d \). Hence, letting \( G_p \) denote the product of the linear forms defining the elements of \( L_P(Z) \), we have \( \deg G_p = |L_P(Z)| \) and \( |(I(Z) \cap I(P)^d)|_d = |(L_p(Z))|_d = [(G_p)]_d \) which completes the proof.

We introduce the first combinatorial constraint below, it occurs whenever \( I^\triangleright(Z) \) contains an irreducible element. As we will see this simple constraint ends up having a number of strong consequences.

**Lemma 7.5.** Let \( Z \subseteq \mathbb{P}^2_K \), with \( |Z| \geq 2 \) and suppose \( F \in [I^\triangleright(Z)]_d \) is an irreducible polynomial. Then for all \( P \in Z \) we have \( |L_P(Z)| \leq d \).

Consequently, if \( Z \subseteq \mathbb{P}^2_C \) admits an irreducible unexpected curve in degree \( d \), then \( |L_P(Z)| \leq d \) for all \( P \in Z \).

**Proof.** As \( F \) is irreducible, we know by lemma 7.1 that \( \varepsilon_P(F) \neq 0 \) and \( \varepsilon_P(F) \in [I(P)^d \cap I(Z)]_d \) for all \( P \in Z \). Hence, by lemma 7.4 we get \( |L_P(Z)| \leq d \).

The second statement follows from the first in light of corollary 6.10.

G. Dirac conjectured (see [9]) that for any set \( Z \) of noncollinear points in \( \mathbb{R}^2 \), there always exists some \( P \in Z \) with \( |L_P(Z)| \geq \left\lceil \frac{|Z|}{2} \right\rceil \). This turned out to be false. However, since then alternative conjectures have been proposed, one version of the conjecture was established in [15] for points in \( \mathbb{C}^2 \). This result allows us theorem 7.6 below. We explore possible further consequences of the conjectures in section 9.

**Theorem 7.6.** Let \( Z \subseteq \mathbb{P}^2_C \) and suppose that \( |Z| \) admits an unexpected curve in degree \( d \geq 1 \), then \( |Z| \leq 3d - 3 \).

**Proof.** This follows from 7.5 and Han’s improvement of the Dirac Conjecture [15], which states for a finite set of points \( Z \subseteq \mathbb{P}^2_C \) which span \( \mathbb{P}^2 \) there always exists some \( P \in Z \) so \( |L_P(Z)| \geq \left\lceil \frac{|Z|}{3} \right\rceil + 1 \).

Namely, suppose \( Z \) admits an irreducible curve in degree \( d \), then for all \( P \in Z \), \( |L_P(Z)| \leq d \). Now applying Han’s result, there exists \( P \in Z \) so

\[
d \geq |L_P(Z)| \geq \left\lceil \frac{|Z|}{3} \right\rceil + 1.
\]

Solving for \( |Z| \) now yields \( 3(d - 1) \geq |Z| \), the desired inequality.

**Remark 7.7.** It should be noted that the paper [15], is rather vague and states the result only for points in “the plane”. However, the proof works for complex line arrangements, as the main nonelementary tool is a Hirzebruch type inequality for complex line arrangements first proved in [6].
Equivalently, Langer’s Inequality [20], could replace and or rederive [15]’s result. Langer’s Inequality states that letting \( \ell_r = | \{ L \subseteq \mathbb{P}^2_\mathbb{C} : |L \cap Z| = r \} | \) we have that if \( \ell_r = 0 \) for \( r > \frac{2}{3} |Z| \) that
\[
\sum_{P \in Z} |L_P(Z)| = \sum_{r \geq 2} r \ell_r \geq \left( \frac{|Z|^2 + 3|Z|}{3} \right).
\]
For further discussion see the survey article [26], where the author first learned of these results.

In [7] it was shown that any set of points \( Z \subseteq \mathbb{P}^2_\mathbb{C} \) in linearly general position can never admit unexpected curves. This proposition provides a strengthening of that result, and extends it to an arbitrary field.

**Proposition 7.8.** Let \( Z \subseteq \mathbb{P}^2_\mathbb{K} \) suppose there’s a nonzero \( F \in [I^\geq(Z)]_d \) for \( 1 \leq d \leq |Z| - 2 \). Then no subset \( W \subseteq Z \) with \( |W| > d + 1 \) is in linearly general position. If \( F \) is irreducible and \( d \) is even this can be improved to say no subset \( W \subseteq Z \) with \( |W| > d \) is in linearly general position.

**Proof.** We first proceed in the special case that \( |L_P(Z)| \leq d \) for all \( P \in Z \), note that by lemma 7.5 this includes the case that \( F \) is irreducible. If \( W \subseteq Z \) is in linearly general position, then for all \( P \in W \) and all \( L \in L_P(W) \), we have that \( |L \cap (W \setminus \{P\})| \leq 1 \). Therefore \( |W| - 1 = |L_P(W)| \leq |L_P(Z)| \leq d \) implying
\[
|W| \leq |L_P(W)| + 1 \leq d + 1.
\]

If furthermore \( d \) is even, then suppose by contradiction that \( W \subseteq Z \) is in linearly general position with \( |W| = d + 1 \). As \( |W| = d + 1 \) we get that \( |L_P(W)| = |L_P(Z)| = d \) for all \( P \in W \). Now fix some \( Q \in Z \setminus W \) and define a partition \( \Pi_Q \) of \( W \), where \( P \in W \) is contained in the block \( \text{Span}(Q, P) \cap W \). Now as \( \text{Span}(Q, P) \subseteq L_P(Z) = L_P(W) \), we get \( |\text{Span}(Q, P) \cap W| = 2 \), therefore \( \Pi_Q \) is a partition where each block has size 2 contradicting the fact that \( |W| = d + 1 \) is odd.

Now continuing with the general case, let \( F \) be a nonzero possibly reducible polynomial. Let \( Z' \subseteq Z \) be the subset \( Z' = \{ P \in Z \mid \varepsilon_F(P) \neq 0 \} \), and let \( T = Z \setminus Z' \). Then by lemma 7.1, we see that \( F \) factors as \( F = C \prod_{P \in T} \ell_P \). Furthermore, \( G \in [I^\geq(Z')] \) and \( \varepsilon_F(G) \neq 0 \) for all \( P \in Z' \), so by the proof of lemma 7.5 we have \( |L_P(Z')| \leq \deg(G) = d - |T| \) for all \( P \in Z' \). If \( W \subseteq Z \) is in linearly general position, then so is \( W' = W \cap Z' \). Applying the result from our first case we see
\[
|W| \leq |W'| + |T| \leq d + 1
\]
establishing the result. \( \square \)

We immediately obtain the following corollary by applying corollary 6.10.

**Theorem 7.9.** If \( Z \subseteq \mathbb{P}^2_\mathbb{C} \) admits an unexpected curve in degree \( d \), then every subset \( W \subseteq Z \) of points in linearly general position has
\[
|W| \leq d + 1.
\]
Furthermore, we have \( |W| \leq d \) if \( d \) is even and the unexpected curve is irreducible.

**Remark 7.10.** The author suspects the bound of theorem 7.9 can be somewhat improved over \( \mathbb{C} \). Namely, given \( Z \subseteq \mathbb{P}^2_\mathbb{C} \), which admits an unexpected curve in degree \( d \), then every subset \( W \subseteq Z \) in linearly general position must have \( |W| \leq d \). It should be noted, however, that the bound given in proposition 7.8 is sharp in positive characteristic at least if \( d - 1 \) is a prime power.

Namely, let \( \mathbb{K} \) be a field of characteristic \( p > 0 \). Let \( q = p^e \) and take \( Z = \mathbb{P}^2_{\mathbb{F}_q} \subseteq \mathbb{P}^2_\mathbb{K} \). Then as shown in example 5.40 \( Z \) will have an unexpected curve in degree \( q + 1 \). A smooth conic such as \( X_1^2 = X_0X_2 \) will contain exactly \( q + 1 \) points of \( Z \) which form a subset in linearly general position. This achieves the bound from proposition 7.8 if \( q + 1 \) is even.

If \( q + 1 \) is odd, then \( \text{char}(\mathbb{K}) = 2 \) and for each smooth conic \( C \subseteq \mathbb{P}^2_\mathbb{K} \) there is a point \( N_C \in \mathbb{P}^2_\mathbb{K} \setminus C \) which is contained in every tangent line of \( C \). This is often referred to as the nucleus of \( C \). As an example, we can verify that \( C : X_1^2 = X_0X_2 \) has nucleus \( N = (0 : 1 : 0) \). In this case taking \( W \) to be \( (Z \cap C) \cup \{N\} \), gives a linearly general subset of size \( q + 2 \).

**Remark 7.11.** Proposition 7.5 can be applied to generalize an inductive technique, stated as Lemma 6.5 in [7], restricted to the case the bundle \( D_0(\mathcal{A}_Z) \) is semistable.
Proposition 7.12. Let $Z \subseteq \mathbb{P}^2_C$ and $P \in \mathbb{P}^2$ and suppose $I^\vartriangleright(Z)$ has splitting type \((a,b)\) with \(a \leq b\). If \(L_P(Z) > a\), then \(I^\vartriangleright(Z+P)\) has splitting type \((a+1,b)\) (or \((a,a+1)\) if \(a=b\)).

In particular, if \(Z\) does not admit unexpected curves, and \(L_P(Z) > \left\lceil \frac{|Z|}{2} \right\rceil\), then \(Z+P\) does not admit unexpected curves.

Proof. First suppose that \([I^\vartriangleright(Z)]_a = 0\), then by theorem 6.9 we have \((a,b) = \left(\left\lceil \frac{|Z|+1}{2} \right\rceil, \left\lceil \frac{|Z|+1}{2} \right\rceil\right)\), and can conclude that \(|Z|\) admits no unexpected curves. Now for every \(P \in \mathbb{P}^2 \setminus Z\) we have that \(Z+P\) does not admit unexpected curves. Since if it did we would have by theorem 6.8, that \([I^\vartriangleright_0(Z+P)]_d \neq 0\) for some \(d \leq a = \left\lceil \frac{|Z|+1}{2} \right\rceil\).

Now suppose that \([I^\vartriangleright(Z)]_a \neq 0\) and that \(|L_P(Z)| > a\). Note that \(\ell_p F \in [I^\vartriangleright(Z+P)]_{a+1}\) and it suffices to show that \([I^\vartriangleright(Z+P)]_a = 0\), since then by theorem 6.9 \(I^\vartriangleright(Z+P)\) has splitting type \((\alpha, |Z| - \alpha)\) where \(\alpha = \min \{a + 1, \left\lceil \frac{|Z|+2}{2} \right\rceil\}\).

For any \(F \in [I^\vartriangleright(Z+P)]_d\), we have that \(\varepsilon_p(F) \in [I(P)]_d \cap I(Z)]_d\). Yet as \(|L_P(Z)| > a\) we have by applying lemma 7.4 that \(\varepsilon_p(F)\) must be 0. By lemma 7.1 this means that \(\ell_p\) divides \(F\), however then \(F/\ell_p\) is a nonzero element of \([I^\vartriangleright(Z)]_a-1\) giving a contradiction. \(\Box\)

Proposition 7.13. Let \(Z \subseteq \mathbb{P}^2_C\), define
\[
\mathcal{L} = \{\text{Span}(P,Q) \mid P, Q \in Z \text{ are distinct points }\}
\]
and suppose that \(Z\) admits an irreducible unexpected curve in degree \(d\), then
\[
|\mathcal{L}| \leq d^2 - d + 1
\]

Proof. We prove the theorem under the slightly weaker assumption that there is some irreducible \(F_\lambda \in [I^\vartriangleright(Z)]_d\). Without loss of generality assume that \(E_0 = (1 : 0 : 0) \in Z\), so we may write \(F_\lambda = f_1 X_1 + f_2 X_2\) with \(f_1(M_0, M_1, M_2) \in \mathbb{K}[M_0, M_1, M_2]\). Recall the map \(\rho_\lambda : \mathbb{P}^2 \to \mathbb{P}^2\) from definition 3.7, in coordinates
\[
\rho_\lambda(a_0, a_1, a_2) = (0 : f_1(a_0, a_1, a_2) : f_2(a_0, a_1, a_2)).
\]
By theorem 4.8 we get that \(\rho_\lambda(H) \subseteq H\) for all \(H \in \mathcal{A}_Z\), namely all \(H\) of the form \(H = P^\perp\) for \(P \in Z\). Define \(\mathcal{P} = \{L \cap H \mid L \text{ and } H \text{ are distinct lines in } \mathcal{A}_Z\}\). By projective duality we have that \(\mathcal{L}^\perp = \mathcal{P}\) and so in particular \(|\mathcal{L}| = |\mathcal{P}|\). Now as \(\rho_\lambda(H) \subseteq H\) for all \(H \in \mathcal{A}_Z\), we get for any \(H \cap L = Q \in \mathcal{P}\) that
\[
\rho_\lambda(Q) = \rho_\lambda(H \cap L) \subseteq \rho_\lambda(H) \cap \rho_\lambda(L) \subseteq H \cap L = Q.
\]
Hence \(\mathcal{P}\) is contained in the vanishing locus of the minors of
\[
\begin{bmatrix}
Y_0 & Y_1 & Y_2 \\
0 & F_1 & F_2
\end{bmatrix}.
\]
So \(\mathcal{P}\) is contained in the solutions of the polynomial system
\[
\begin{align*}
Y_1 F_2 - Y_2 F_1 &= 0 \\
Y_0 F_2 &= 0 \\
Y_0 F_1 &= 0
\end{align*}
\]
(7.13.1)
To count solutions, let \(V\) denote the variety defined by this system, we look at solutions on the line \(Y_0 = 0\) and solutions on the subset \(Y_0 \neq 0\). On \(Y_0 = 0\) we get that the system (1), reduces to
\[
\begin{align*}
Y_1 F_2 - Y_2 F_1 &= 0 \\
Y_0 &= 0
\end{align*}
\]
from which we get by Bezout’s theorem that the number of solutions is at most \(\deg(Y_0) \deg(Y_1 F_2 - Y_2 F_1) = d\). On the subset \(Y_0 \neq 0\), the first equation in the system is redundant and the system reduces to
\[
\begin{align*}
F_1 &= 0 \\
F_2 &= 0.
\end{align*}
\]
As $F$ is irreducible $F_1$ and $F_2$ have no shared component so by Bezout’s Theorem this system has at most $\deg(F_1)\deg(F_2) = (d-1)^2$ solutions. Combining both results, we can conclude that

$$|\mathcal{L}| = |\mathcal{P}| \leq |V \cap (Y_0 = 0)| + |V \cap (Y_0 \neq 0)| \leq d + (d-1)^2 = d^2 - d + 1.$$ 

\[\square\]

**Example 7.14.** We note that the above bound is sharp in every degree. Namely, the point configuration $C_m \subseteq \mathbb{P}^2$ of example 5.41 achieves the bound. To see this write $C_m = \{E_0, E_1, E_2\} \cup F_m$, where $F_m = \{-E_i + \zeta^k E_j | 0 \leq i < j \leq 2\}$. Then the points in $F_m$ generate $m^2 + 3$ lines, the 3 coordinate lines $(X_i = 0)$, and all lines of the form $\text{Span}(-E_0 + \zeta^i E_1, -E_1 + \zeta^k E_2, -E_0 + \zeta^{i+k} E_2)$ with defining equation $\zeta^{i+k} X_1 + \zeta^k X_1 + X_2 = 0$. The only lines unaccounted for in $\mathcal{L}_{C_m}$ are those of the form $\text{Span}(E_i, -E_j + \zeta^t E_k)$ with $\{i, j, k\} = \{0, 1, 2\}$ of which there are $3m$.

Then $|\mathcal{L}_{C_m}| = m^2 + 3m + 3$, and $C_m$ admits a unique unexpected curve in degree $m + 2$. Noting that $m^2 + 3m + 3 = (m+2)^2 - (m+2) + 1$ we conclude that the above bound is sharp for all $d \geq 4$.

We close this section with a previously unnoticed combinatorial condition which guarantees the existence of unexpected curves.

**Proposition 7.15.** Let $Z \subseteq \mathbb{P}^2$ be a finite set of points. Further suppose that no line $L \subseteq \mathbb{P}^2$ has $|Z \cap L| \geq \frac{|Z|-1}{2}$. Define $\mathcal{L}$ as in proposition 7.13. Then for any integer $d \leq \frac{|Z|+1}{2}$, if

$$\left(\sum_{P \in Z} |L_P(Z)|\right) - |Z| - |\mathcal{L}| + 1 < (d-1)(|Z| - d)$$

then $|Z|$ admits unexpected curves in degree $d$.

**Proof.** Let $c_1(D_0(A_Z))$ denote the Chern polynomial of $D_0(A_Z)$. Then by theorem 2.5 of [27],

$$c_1(D_0(A_Z)) = 1 - (|Z| - 1)t + \left(\sum_{L \in \mathcal{L}} (|L \cap Z| - 1) - |Z| + 1\right) t^2.$$ 

Noting that

$$\sum_{L \in Z} |L \cap Z| = \sum_{L \in Z} \sum_{P \in (L \cap Z)} 1 = \sum_{L \in \mathcal{L}} \sum_{P \in Z} \sum_{L \in L_P(Z)} 1 = \sum_{P \in Z} |L_P(Z)|.$$

We get the following formula for $c_2(D_0(A_Z))$,

$$c_2(D_0(A_Z)) = \sum_{L \in \mathcal{L}} (|L \cap Z| - 1) - |Z| + 1
= \left(\sum_{P \in Z} |L_P(Z)|\right) - |\mathcal{L}| - |Z| + 1.$$ 

In particular, we see that our hypothesized inequality is equivalent to $c_2(D_0(A_Z)) < (d-1)(|Z| - d)$.

Now theorem B of [5] states that if $(a, b)$ denotes the splitting type of $D_0(A_Z)$ then $ab \leq c_2(D_0(A_Z))$. So if we satisfy $c_2(D_0(A_Z)) < (d-1)(|Z| - d)$, then $ab < (d-1)(|Z| - d)$. Letting $k = \frac{|Z| - 1}{2}$, $g_1 = k - (d-1)$ and $g_2 = k - a$, then this inequality becomes

$$k^2 - g_2^2 = (k - g_2)(k + g_2) = ab < (d-1)(|Z| - d) = (k - g_1)(k + g_1) = k^2 - g_1^2.$$ 

Therefore, we may conclude that $g_1 < g_2$ and so $a < d - 1$. Applying theorem 5.1 now establishes the result. \[\square\]
8. Regularity Bounds

In remark 3.8 of [7], it is claimed that the definition of unexpected curves

“... leaves open the possibility that the points of $Z$ do not impose independent conditions on curves of some degree $j + 1$, and ... a general fat point $jP$ fails to impose the expected number of conditions on the linear system defined by $[I_Z]_{j+1}$. Theorem 3.7 gives the surprising result that this is impossible.”

However, it appears to the author that theorem 3.7 of [7] is a weaker statement than the above quotation claims. Rather it establishes that $Z$ imposes independent conditions on a specific degree $t_z \geq j + 1$, if $Z$ admits an unexpected curve in degree $j + 1$.

In this section, we establish the full claim for points $Z \subseteq \mathbb{P}_2$. In fact we prove a stronger claim. Namely if $Z \subseteq \mathbb{P}_2$ admits unexpected curves in degree $d + 1$, then $Z$ imposes independent conditions on forms in degree $d$. This claim is false in general in positive characteristic, though it does hold for certain values of $d$ (see proposition 8.2).

Before proceeding we recall the definition of Castelnuovo-Mumford Regularity, this number determines when $Z$ imposes independent conditions on $d$ forms.

**Definition 8.1.** Given a finite set of points $Z \subseteq \mathbb{P}(V)$ the Castelnuovo-Mumford Regularity of $Z$, denoted $\text{reg}(Z)$, is the integer

$$\text{reg}(Z) := 1 + \min\{r \mid \dim_K[\text{Sym}(V^*)/I(Z)]_r = |Z|\}.$$ 

It should be noted that the above definition is highly nonstandard, and applies only to this specific situation. We refer to Exercise 4E.3 and theorem 4.2 of [12], for proofs that the definition given is equivalent the standard definitions for graded modules.

This result has some applications to Terao’s conjecture as well, which we explore in the last section.

**Proposition 8.2.** Let $\mathbb{K}$ be an infinite field, and let $A = \mathbb{K}[s, t]$ be a standard graded polynomial ring on 2 variables, let $\text{Pow}_d : [A]_1 \rightarrow [A]_d$ be the $d$-th power map, that is the map $\ell \mapsto \ell^d$.

Then the image of $\text{Pow}_d$ spans $[A]_d$ over $\mathbb{K}$ if and only if the pair $(\text{char}\mathbb{K}, d)$ satisfies one of the following

**Characteristic Hypothesis**

1. $(\text{char}\mathbb{K}, d) = (0, d)$; or
2. $(\text{char}\mathbb{K}, d) = (p, q(p^e) - 1)$ for some $e \geq 0$ and $q$, with $1 \leq q \leq p$.

**Proof.** This result is likely well known and consists of standard techniques so we only give a brief sketch.

Let $L$ be the $(d + 1) \times (d + 1)$ matrix whose $i$-th row is $(s + a_i t)^d$ in the standard monomial basis of $[A]_d$, also suppose that $a_i \neq a_j$ for $i \neq j$. Then $L$ can be seen to be a Vandermonde Matrix whose $j$ column has been scaled by $(d)_j$. Using the well known Vandermonde Determinant formula the matrix is nonsingular and hence the rows span $[A]_d$ if and only if $\prod_{i=0}^{d} \binom{d}{i}$ is nonzero as an element of $\mathbb{K}$. In particular, we may conclude if $\text{char}(\mathbb{K}) = 0$.

If $\text{char}(\mathbb{K}) = p > 0$, we recall Lucas’s theorem on Binomial coefficients which states $\binom{d}{i} \equiv 0 \mod p$ if and only if each digit of $i$ written in base $p$ does not exceed the corresponding digit of $d$. In base $p$, the only numbers $d$ where this criterion holds for all $0 \leq i \leq d$ are those $d$ where the non-leading digits are all $p - 1$. This happens precisely when $d = qp^e - 1$ for $1 \leq q \leq p$. $\square$

**Proposition 8.3.** Let $Z \subseteq \mathbb{P}_2$ be a finite set of points, suppose that $\dim B. \text{loc}_{d+1}(I^\oplus(Z)) = 0$ and that the pair $(\text{char}\mathbb{K}, d)$ satisfies the characteristic hypothesis of proposition 8.2. Then $\text{reg}(Z) \leq d + 1$.

**Proof.** Let $\mathbb{P}^2 = \text{Proj}(R)$, where $R = \mathbb{K}[X_0, X_1, X_2]$. Take $\ell \in [R]_1$ to be a general linear form and consider the short exact sequence

$$0 \rightarrow [R/I(Z)]_{t-1} \overset{\ell}{\rightarrow} [R/I(Z)]_t \rightarrow [R/(I(Z) + (\ell))]_t \rightarrow 0$$

Letting $h_Z(t) := \dim[R/I(Z)]_t$ we conclude that for integers $t$ that

$$h_z(t) - h_z(t - 1) = \dim[R/(I(Z) + \ell)]_t.$$
Furthermore, as \( R/(I(Z) + \ell) \) is principally generated we can conclude that \( h_z(t) = h_z(t - 1) \) if and only if \( h_z(t - 1) = |Z| \). From this it follows from definition 8.1 that \( \text{reg}(Z) = \min\{ r \mid [R/(I(Z) + (\ell))]_r = 0 \} \) and it suffices to prove that \( [R/(I(Z) + (\ell))]_{d+1} = 0 \).

Fix \( F_\lambda \in [I^{Z\ell}(Z)]_{d+1} \) with \( \text{dim}\, \text{B.}\, \text{loc}(F_\lambda) = 0 \). For all points \( Q \) on the line \( \ell = 0 \), we have by lemma 4.9 that

\[
\varepsilon_Q(F_\lambda) = h_Q^{d} \rho_\lambda(\ell) \mod (\ell)
\]

for some linear form \( h_Q \) vanishing on \( Q \). Noting \( \varepsilon_Q(F_\lambda) \in I(Z) \), we get an inclusion of \( \mathbb{K} \)-vector spaces

\[
[(I(Z) + (\ell))/(\ell)]_{d+1} \supseteq \text{Span}\{h_Q^{d} \rho_\lambda(\ell) + (\ell) \mid Q \in \mathbb{P}^2 \text{ and } (\ell)(Q) = 0\}.
\]

By proposition 8.2 the set \( \{h_Q^{d} \mid Q \in (\ell) = 0\} \) spans \( [R/(\ell)]_d \) and so

\[
[I(Z) + (\ell)/(\ell)]_{d+1} \supseteq \rho_\lambda(\ell)[R/(\ell)]_d.
\]

Hence, \([I(Z) + (\ell)]_{d+1} \supseteq [(\ell, \rho_\lambda(\ell))]_{d+1} \). Let \( P \in \mathbb{P}^2 \) denote the point defined by the ideal \((\ell, \rho_\lambda(\ell))\), as \( B.\, \text{loc}(F_\lambda) \cap (\ell = 0) = \emptyset \) we can find some \( H \in \mathbb{P}^2 \) so that \( \varepsilon_H(F_\lambda) \in I(Z) \) and \( \varepsilon_H(F_\lambda) \notin I(P) = (\ell, \rho_\lambda(\ell)) \). Hence,

\[
[I(Z) + (\ell)]_{d+1} \supseteq [(\ell, \rho_\lambda(\ell))]_{d+1} + [\varepsilon_H(F_\lambda)]_{d+1} = [R]_{d+1}
\]

allowing us to conclude that \([R/(I(Z) + (\ell))]_{d+1} = 0 \) as desired. \( \square \)

Remark 8.4. It should be noted that the assumptions of the above theorem, may be relaxed in various ways to give slightly different bounds, which also require different proofs, and possibly stronger assumptions. We have chosen to give only the proof above for the sake of brevity, but will briefly comment on two of the possible changes now.

1. The condition \( 0 = \text{dim}\, \text{B.}\, \text{loc}_{d+1}(I^{Z\ell}(Z)) \) may be replaced with the weaker condition that \( 0 = \text{dim}\, \text{B.}\, \text{loc}_{d+1}(Z) \). However the bound then becomes \( \text{reg}(Z) \leq d + 2 \). The proof is similar to above, but \( \ell \) is replaced with a line through a general point \( Q \), which also vanishes on a point in \( Z \). It can then only be concluded that \( \text{dim}[R/(I(Z) + (\ell))]_{d+1} \leq 1 \). This worse bound of \( \text{reg}(Z) \leq d + 2 \) is in fact sharp, as can be illustrated by taking \( Z \) to be \( 2d + 3 \) points on a smooth conic.

Interestingly, this technique can be used to give a completely geometric proof of theorem 5.24 for points in \( \mathbb{P}^2 \), in contrast to the combinatorial proof given in [22].

2. A generalization to \( \mathbb{P}^n \), at least in characteristic 0, is possible however the proof becomes more involved and/or additional assumptions on \([I^{Z\ell}(Z)]_{d+1}(Z) \) are necessary. The main technique is still roughly the same except now induction is needed. After showing \([I(Z) + (\ell)]_{d+1} \supseteq (\ell, \rho_\lambda(\ell)) \) one proceeds as before showing

\[
[I(Z) + (\ell)]_{d+1} = [I(Z) + (\ell, \rho_\lambda(\ell))]_{d+1} \supseteq [(\ell, \rho_\lambda(\ell), \rho_\lambda(\ell)^2)]_{d+1} \supseteq ...
\]

If \((\ell, \rho_\lambda(\ell), ..., \rho_\lambda^{n-1}(\ell))\) is the ideal of a point we then proceed as in the proposition.

Combining the above from some results from earlier sections, we may obtain the following result

Theorem 8.5. If \( Z \subseteq \mathbb{P}^2 \) has an unexpected curve in degree \( d \), then \( \text{reg}(Z) \leq d \). In particular, \( Z \) imposes independent conditions on forms of degree \( d - 1 \).

Proof. Suppose that \( Z \) admits an unexpected curve in degree \( d \), without loss of generality we assume that \( Z \) does not admit an unexpected curve in degree \( d - 1 \). Then by theorem 5.1, we note that \( |Z| \geq 2d + 1 \) and that no line \( L \) contains more that \( d + 1 \) points of \( Z \). Additionally by theorem 6.8, there exists \( F \in [I^{Z\ell}(Z)]_d \) defining the curve. We claim that \( \text{dim}\, B.\, \text{loc}(F) = 0 \), which in light of proposition 8.3 establishes the claim.

Proceeding by contradiction assume that \( \text{dim}\, B.\, \text{loc}(F) = 1 \), applying proposition 5.18, we get \( B.\, \text{loc}(F) \) has a component which is a line. If \( \ell \in \mathbb{C}[X_0, X_1, X_2] \) is the linear form defining this line, \( L \), then viewing it as a polynomial in \( \mathbb{C}[X_0, X_1, X_2, A_0, A_1, A_2] \) and applying lemma 6.1, we get \( F = \ell h \) with \( h \in \mathbb{C}[M_0, M_1, M_2] \). Then as \( \varepsilon_Q(h) \in [I(Q)]_{\deg h} \) for all \( Q \in \mathbb{P}^2 \) we get that the variety \( V(\varepsilon_Q(F)) \) is a union of \( L \) and at most \( \deg h \) lines through \( Q \). For a general \( Q \in \mathbb{P}^2 \), each line in \( V(\varepsilon_Q(h)) \) contains at most one point of \( Z \). This forces us to conclude that \( |L \cap Z| \geq |Z| - \deg h \geq d + 2 \) giving us our desired contradiction. \( \square \)
We note that the theorem 8.5, gives a decent criteria purely algebraic criteria for establishing that a set of points $Z$ does not admit unexpected curves in a given degree. We explore this a bit in the next section in the context of Terao’s Conjecture.

9. Applications to Terao’s Conjecture in $\mathbb{P}^2$

A much studied problem in the theory of line arrangements is the freeness of the module of derivations $D(A_Z)$. One reason for this in particular is that if $D(A_Z)$ is free then many of the invariants of $D(A_Z)$ can be determined from combinatorics of the intersection lattice $L(A_Z)$ (or equivalently the matroid $M(Z)$).

A major open problem in the study of Hyperplane arrangements is Terao’s Freeness Conjecture

**Conjecture 9.1** (Terao’s Freeness Conjecture). Over $\mathbb{C}$ freeness of $D_0(A)$ can be determined by the intersection lattice $L(A_Z)$.

**Remark 9.2.** The above conjecture is usually stated for $D(A)$. However the two versions are equivalent because over $\mathbb{C}$, $D(A)$ splits as $(\text{Sym}(V^*))\theta_e \oplus D_0(A)$.

One natural question to ask given theorem 4.8, is what freeness of $D_0(A_Z)$ says about $I(Z)$. Namely, can $D_0(A_Z)$ be characterized in terms of $Z$? The following proposition (which is well known to experts) is helpful in addressing this question.

**Proposition 9.3.** $D_0(A)$ is free if and only if for a general line $L \subseteq \mathbb{P}(W)$, the restriction map $D_0(A) \to D_0(A)|_L$ is surjective.

**Proof.** The forward implication is clear. For the reverse implication, we apply a corollary of Saito’s Criterion which can be found as theorem 4.23 of [24]. Namely $D_0(A)$ is free if there exists $\theta_1, \ldots, \theta_n \in D_0(A_Z)$ which are linearly independent over the projective coordinate ring, $S$, of $\mathbb{P}(W)$, and where $\sum_{i=1}^n \deg(\theta_i) = |A| - 1$.

So suppose that $\text{res}_L : D_0(A) \to D_0(A_Z)|_L$ is surjective for a general line $L$. Let $\theta_1, \ldots, \theta_n$ be a $S/I(L)$-basis of $D_0(A_Z)|_L$, then for each $i$ we can find $\theta_i \in D_0(A)$ so that $\text{res}_L(\theta_i) = \theta_i$. As $\sum_{i=1}^n \deg(\theta_i) = \sum_{i=1}^n \deg(\theta_i) = |A| - 1$ it suffices to show that the $\theta_i$ are linearly independent over $S$. Yet if $\sum_{i=1}^n s_i \theta_i = 0$ for some $s_i \in S$ and some index $j$, then $\sum_{i=1}^n s_i \theta_i = 0$ in $D_0(A_Z)|_L$. As $L$ is general if $s_j \neq 0$ for some index $j$ then we can assume that $s_j \notin I(L)$ which gives a non-trivial relation among $\theta_1, \ldots, \theta_n$ and a contradiction.

**Corollary 9.4.** $A_Z$ is a free arrangement if and only if the evaluation map $\varepsilon_Q : I^{\gg}(Z) \to I^{\gg}_Q(Z)$ is surjective for general $Q$.

Theorem 6.8 has some applications to Terao’s conjecture, namely we give a new criterion for determining Freeness.

**Proposition 9.5.** Let $A \subseteq \mathbb{P}^2_\mathbb{C} = \text{Proj}(S)$ with splitting type $(a_1, a_2)$. If $a_2 - a_1 \geq 2$, then $D_0(A)$ is free if and only if it has a minimal generator in degree $a_2$.

**Proof.** We use the criterion from proposition 9.3. As $a_2 - a_1 > 2$, then we may apply theorem 6.8 to see there nonzero $\theta_1 \in [D_0(A)]_{a_1}$ and so the image of the restriction map contains the generator of $D_0(A)|_L$ in degree $a_1$. If $D_0(A)$ has a minimal generator $\theta_2$ in degree $a_2$, then $\theta_2 \neq f\theta_1$ for any $f \in S$. For a general line $L$, we still have $\text{res}_L(\theta_2) \notin S/I(L) \text{res}_L(\theta_1)$ and conclude that $\{\text{res}_L(\theta_1), \text{res}_L(\theta_2)\}$ is a generating set for $D_0(A)|_L$.

Additionally, it is well known that freeness can be determined from combinatorics and the splitting type. More precisely,

**Proposition 9.6.** Let $A$ and $B$ be hyperplane arrangements in $\mathbb{P}^n$, and suppose $A$ and $B$ have isomorphic intersection lattices. Suppose that $D_0(A)$ is free, then $D_0(B)$ is free if and only if it has the same splitting type as $A$.

**Proof.** By a theorem of Terao $c_2(D_0(A))$ is determined solely by $L_A$. The result is now a consequence of the criterion that $D_0(A)$ is free if and only if $c_2(D_0(A)) = a_1a_2$ where $(a_1, a_2)$ is the splitting type, see for instance [5].

\[\square\]
This characterization allows us to generalize a theorem of [27] which was stated only for balanced free arrangements in \( \mathbb{P}_k^2 \). Here balanced means free arrangements with splitting type \((a, a)\) or \((a, a + 1)\).

**Theorem 9.7.** For a finitely generated graded module \( M \), let \( \alpha(M) \) denote the initial degree of \( M \), that is
\[
\alpha(M) := \inf\{d \in \mathbb{Z} \mid [M]_d \neq 0\}
\]

Let \( \mathcal{A} \) and \( \mathcal{B} \) be combinatorially equivalent line arrangements \( \mathbb{P}_k^2 \). If \( D_0(\mathcal{A}) \) is free, then
\[
\alpha(D_0(\mathcal{B})) \leq \alpha(D_0(\mathcal{A}))
\]
with equality if and only if \( \mathcal{B} \) is free.

In particular, if \( \mathcal{A} \) is free with exponents \((1, a, b)\) and \( \mathcal{B} \) is not free, then \( D_0(\mathcal{B}) \) has a generator in degree \( < a \) and all other minimal generators are in degree \( > b \).

**Remark 9.8.** Note if \( \mathbb{K} \subseteq \mathbb{C} \), the following argument can be slightly simplified by applying theorem 6.8.

**Proof.** The reverse direction is immediate as in that case \( D_0(\mathcal{A}) \) and \( D_0(\mathcal{B}) \) are isomorphic.

To prove the forward implication, we apply the characterization given in 9.6. Hence, assume that \( D_0(\mathcal{A}) \) has splitting type \((a_1, a_2)\), and \( D_0(\mathcal{B}) \) has splitting type \((b_0, b_1)\), with \( (b_0, b_1) \neq (a_0, a_1) \). It suffices to show that \( \alpha(D_0(\mathcal{B})) < \alpha(D_0(\mathcal{A})) \).

By [28] freeness is an open property. Hence, if \( \mathcal{B} \) is not free it lies on closed subvariety of \( V_{\ell}(\mathcal{A}) \) the variety parameterizing arrangements with intersection lattice isomorphic to \( L(\mathcal{A}) \). We can view \( D(\mathcal{B}) \) as the kernel of the linear map \( \text{Der}(S) \to \prod_{H \in \mathcal{B}} S/(I(H)) \) which maps \( \theta \mapsto (\theta(\ell_H))_{H \in \mathcal{B}} \), so by lower semicontinuity of rank we may conclude that \( \dim[D_0(\mathcal{B})]_d \geq \dim[D_0(\mathcal{A})]_d \) for all \( d \). Applying the same argument to the restriction of \( D_0(\mathcal{B})_d \) to a general line, we see that \( b_0 < a_0 \leq a_1 < b_1 \). As \( \dim[D_0(\mathcal{B})]_a \geq \dim[D_0(\mathcal{A})]_a > 0 \), we can apply proposition 6.2 to get
\[
\alpha(D_0(\mathcal{B})) = b_0 < a_0 = \alpha(D_0(\mathcal{A})).
\]

The final sentence follows from this and proposition 9.5. \( \square \)

One corollary of the above theorem is an extension of a theorem of [13] over \( \mathbb{C} \), to positive characteristic.

**Corollary 9.9.** If \( \mathcal{A} \subseteq \mathbb{P}_k^2 \) is a free arrangement with splitting type \((a_1, a_2)\) and some point \( P \in \mathbb{P}_k^2 \) is incident to at least \( a_1 \) lines of \( \mathcal{A} \). Then any arrangement over \( \mathbb{P}_k^2 \) combinatorially equivalent to \( \mathcal{A} \) is also free.

**Proof.** Let \( \mathcal{B} \) be an arrangement that is combinatorially equivalent to \( \mathcal{A} \), and let \((b_1, b_2)\) denote it’s splitting type. Dualizing the problem statement, we see that there is a line \( H \) containing at least \( a_1 \) points of \( \mathcal{B}^{\perp} \) so by theorem 5.1 and the prior theorem we have \( a_1 - 1 \leq b_1 \leq a_1 \).

Let \( h \) denote the linear form defining this line. Furthermore for a general \( Q \), let \( g \) denote the product of linear forms through \( Q \) and each point not on \( H \), then \( hg \in [I^{\perp}(\mathcal{B}^{\perp})]_{a_2 + 2} \). If \( b_1 = a_1 - 1 \), then \( hg \) would correspond by theorem 4.8 to a minimal generator of \( D_0(\mathcal{B}) \) in degree \( b_2 = a_2 + 1 \). The prior theorem together with proposition 9.6 now gives a contradiction. \( \square \)

We now close by discussing connections between Terao’s conjecture and a conjecture due to Dirac. It was conjectured in [9] that for every finite set of points non collinear points \( Z \subseteq \mathbb{P}_k^2 \), that there is always some \( Q \in Z \) so that
\[
|L_Q(Z)| = |\{\text{Span}(Q, P) \mid P \in Z \setminus Q\}| \geq \frac{|Z|}{2}.
\]

However, some counterexamples have been found to the original formulation (see [14]). This has lead to two reformulations of the original conjecture which we reprint below.

**Conjecture 9.10 (Weak Dirac Problem).** Determine the smallest constant \( C \), so that for every finite set of noncollinear points \( Z \subseteq \mathbb{P}_k^2 \), there exists some \( Q \in Z \) where
\[
|L_Q(Z)| \geq \frac{|Z|}{C}
\]
Conjecture 9.11 (Strong Dirac Conjecture). There exists some constant $c_0 > 0$ so that for every set of finite noncollinear points $Z \subseteq \mathbb{R}^2$, there exists some $Q \in Z$ so that

$$|L_Q(Z)| \geq \frac{Z}{2} - c_0$$

Counterexamples have been found to Dirac’s Original Conjecture for every odd $n = |Z|$ with the exception of those $n$ of the form $n = 12k + 11$ with $k \geq 4$ (see [1]). Despite that the known counterexamples only barely break the original conjecture bound. Most satisfy the Strong Dirac Conjecture with $c_0 = 1/2$ and all but finitely many satisfy the conjecture with $c_0 = 3/2$.

We now show that any minimal counterexample to Terao’s Conjecture for real line arrangements must itself be a counterexample to the original Conjecture of G. Dirac, and must be extremal in the regards to the other two cases.

Theorem 9.12. Let $A$ and $B \subseteq \mathbb{P}^2$ be real (or complex) line arrangements, which form a counter example to Terao’s conjecture. Meaning $L_A \cong L_B$, but $D_0(A)$ is free with splitting type $(a_1, a_2)$ where as $D_0(B)$ is not free. Furthermore, suppose there is no pair of lines $(L, L') \in A \times B$ we can remove to get subarrangements $A' = A \setminus \{L\}$ and $B' = B \setminus \{L'\}$ forming a smaller counterexample.

Then letting $A^\perp$ be the set of points dual to $A$ we have

$$|L_P(Z)| \leq a_1 \leq \left\lfloor \frac{|Z| - 1}{2} \right\rfloor.$$

Our proof of the above theorem relies on the following proposition which seems useful in it’s own right. It is related to Terao’s well known Addition-Deletion Theorem

Proposition 9.13. Let $A_z \subseteq \mathbb{P}^2$ be a free line arrangement with splitting type $(a_1, a_2)$ and $Z$ the dual set of points. If there is some $P \in Z$ with $|L_P(Z)| > a_1 + 1$, then $|L_P(Z)| = a_2 + 1$ and $A_W$ is free where $W = Z \setminus \{P\}$.

Proof. By theorem B of [5], $c_2(D_0(A_Z)) \geq a_1a_2$, and $A$ is free if and only if equality holds. Furthermore, if $L_P(Z) > a_1 + 1$, then letting $F \in [I^>(Z)]$ it follows by lemma 7.5 that $\varepsilon_P(F) = 0$. Then by lemma 7.1 the linear form, $\ell_p$, defining the line dual to $P$ must divide $F$. However, then we necessarily have $F/\ell_P \in [I^>(W)]_{a_1} \cong [D_0(A)]_{a_1-1}$ so $A_W$ must have splitting type $(a_1 - 1, a_2)$.

We note that it suffices to show that $D_0(A_W)$ is free, since Terao’s Famous Addition-Deletion Formula then ensures that $L_P(W) = a_2 + 1$. Yet this follows since if $F$ and $G$ freely generate $D_0(A_Z)$, then $F/\ell_P$ and $G$ must generate $D_0(A_W)$. □

Proof of theorem 9.12. By the preceding proposition there exists no $P \in Z$ with $|L_P(Z)| > a_1 + 1$. Furthermore, by Terao’s Addition-Deletion formula there is no $P \in L_P(Z)$ with $|L_P(Z)| = a_1 + 1$, so that letting $A' = A \setminus \{\ell_p = 0\}$ we would get a smaller counterexample. Hence, for all $P \in Z$ we have

$$|L_P(Z)| \leq a_1 \leq \left\lfloor \frac{|Z| - 1}{2} \right\rfloor$$

establishing the result. □

References

[1] Jin Akiyama, Hiro Ito, Midori Kobayashi, and Gisaku Nakamura. Arrangements of n points whose incident-line-numbers are at most n/2. *Graphs Combin.*, 27(3):321–326, 2011.

[2] J. Alexander and A. Hirschowitz. Polynomial interpolation in several variables. *J. Algebraic Geom.*, 4(2):201–222, 1995.

[3] The Stacks Project Authors. *The Stacks Project*, 2020.

[4] Thomas Bauer, Grzegorz Malarz, Tomasz Szemberg, and Justyna Szpond. Quartic unexpected curves and surfaces. *Manuscripta mathematica*, 11 2018.

[5] Cristina Bertone and Margherita Roggero. Splitting type, global sections and Chern classes for torsion free sheaves on $\mathbb{P}^n$. *J. Korean Math. Soc.*, 47(6):1147–1165, 2010.

[6] R. Bojanowski. Zastosowania uogólnionej nierówności bogomolova-miyaoka-yau. master thesis (in polish). 2003.

[7] D. Cook, II, B. Harbourne, J. Migliore, and U. Nagel. Line arrangements and configurations of points with an unexpected geometric property. *Compos. Math.*, 154(10):2150–2194, 2018.
[8] Roberta Di Gennaro, Giovanna Ilardi, and Jean Vallès. Singular hypersurfaces characterizing the Lefschetz properties. *J. Lond. Math. Soc. (2)*, 89(1):194–212, 2014.

[9] G. A. Dirac. Collinearity properties of sets of points. *Quart. J. Math. Oxford Ser. (2)*, 2:221–227, 1951.

[10] Marcin Dumnicki, Lucja Farnik, Brian Harbourne, Grzegorz Malara, Justyna Szpond, and Halszka Tutaj-Gasinska. A matrixwise approach to unexpected hypersurfaces. 2019.

[11] J. Edmonds. *Combinatorial Optimization – Eureka, You Shrink*, volume 2570 of Lecture Notes in Computer Science, chapter Submodular Functions and Certain Polyhedra, pages 11–26. Springer-Verlag Berlin Heidelberg.

[12] D. Eisenbud. *The Geometry of Syzygies: a Second Course in Algebraic Geometry and Commutative Algebra*, volume 229 of Graduate Texts in Mathematics. Springer - New York, 2005.

[13] Daniele Faenzi and Jean Vallès. Logarithmic bundles and line arrangements, an approach via the standard construction. *J. Lond. Math. Soc. (2)*, 90(3):675–694, 2014.

[14] B Grünbaum. *Arrangements and Spreads*, volume 10 of CBMS Regional Conference Series in Mathematics. American Mathematical Society, 1972.

[15] Zeye Han. A note on the weak Dirac conjecture. *Electron. J. Combin.*, 24(1):Paper 1.63, 5, 2017.

[16] B. Harbourne, J. Migliore, U. Nagel, and Z. Teitler. Unexpected hypersurfaces and where to find them. *Michigan Math. J.*, page (to appear), 2018.

[17] B. Harbourne, J. Migliore, and H. Tutaj-Gasińska. New constructions of unexpected hypersurfaces in $\mathbb{P}^n$. 2019.

[18] Robin Hartshorne. Stable reflexive sheaves. *Math. Ann.*, 254(2):121–176, 1980.

[19] Melvin Hochster. Criteria for equality of ordinary and symbolic powers of primes. *Math. Z.*, 133:53–65, 1973.

[20] Adrian Langer. Logarithmic orbifold Euler numbers of surfaces with applications. *Proc. London Math. Soc. (3)*, 86(2):358–396, 2003.

[21] Juan C. Migliore. The geometry of the weak Lefschetz property and level sets of points. *Canad. J. Math.*, 60(2):391–411, 2008.

[22] U. Nagel and B. Trok. Segre’s Regularity Bound for Fat Point Schemes. *Ann. Sc. Norm. Super. Pisa Cl. Sci.*, page (to appear), March 2020.

[23] H. Narayanan. The principal lattice of partitions of a submodular function. *Linear Algebra Appl.*, 144:179–216, 1991.

[24] Peter Orlik and Hiroaki Terao. *Arrangements of hyperplanes*. Springer-Verlag, 1992.

[25] J. Oxley. *Matroid Theory*, volume 21 of Oxford Graduate Texts in Mathematics. Oxford University Press, 1992.

[26] P. Pokora. Hirzebruch-type inequalities viewed as tools in combinatorics. *ArXiv e-prints*, 2018.

[27] Henry K. Schenck. Elementary modifications and line configurations in $\mathbb{P}^2$. *Comment. Math. Helv.*, 78(3):447–462, 2003.

[28] Sergey Yuzvinsky. Free and locally free arrangements with a given intersection lattice. *Proc. Amer. Math. Soc.*, 118(3):745–752, 1993.

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