Eisenstein Series and String Thresholds

N.A. Obers\textsuperscript{a†} and B. Pioline\textsuperscript{b‡}

\textsuperscript{a}Nordita and Niels Bohr Institute,
Blegdamsvej 17, DK-2100 Copenhagen, Danmark

\textsuperscript{b}Centre de Physique Théorique,
Ecole Polytechnique\textsuperscript{*}, F-91128 Palaiseau, France

We investigate the relevance of Eisenstein series for representing certain $G(\mathbb{Z})$-invariant string theory amplitudes which receive corrections from BPS states only. $G(\mathbb{Z})$ may stand for any of the mapping class, T-duality and U-duality groups $SL(d, \mathbb{Z}), SO(d, d, \mathbb{Z})$ or $E_{d+1(d+1)}(\mathbb{Z})$ respectively. Using $G(\mathbb{Z})$-invariant mass formulae, we construct invariant modular functions on the symmetric space $K\backslash G(\mathbb{R})$ of non-compact type, with $K$ the maximal compact subgroup of $G(\mathbb{R})$, that generalize the standard non-holomorphic Eisenstein series arising in harmonic analysis on the fundamental domain of the Poincaré upper half-plane. Comparing the asymptotics and eigenvalues of the Eisenstein series under second order differential operators with quantities arising in one- and $g$-loop string amplitudes, we obtain a manifestly T-duality invariant representation of the latter, conjecture their non-perturbative U-duality invariant extension, and analyze the resulting non-perturbative effects. This includes the $R^4$ and $R^4H^{4g-4}$ couplings in toroidal compactifications of M-theory to any dimension $D \geq 4$ and $D \geq 6$ respectively.
# Contents

1 Introduction 3

2 Toroidal compactification and $SL(d, \mathbb{Z})$ Eisenstein series 7
   2.1 Moduli space and Iwasawa gauge .......................... 7
   2.2 Fundamental and antifundamental Eisenstein series .......... 8
   2.3 Higher representations and constrained Eisenstein series .... 9
   2.4 Decompactification and analyticity .......................... 10
   2.5 Partial Iwasawa decomposition ............................... 12

3 $SO(d,d,\mathbb{Z})$ Eisenstein series and one-loop thresholds 13
   3.1 Moduli space and Iwasawa gauge ............................ 13
   3.2 Spinor and vector Eisenstein series ........................ 14
   3.3 One-loop modular integral and method of orbits .......... 17
   3.4 A new second order differential operator .................... 19
   3.5 Large volume behaviour ..................................... 20
   3.6 Asymmetric thresholds and elliptic genus .................. 22
   3.7 Boundary term and symmetry enhancement .................... 24

4 U-duality and non-perturbative $R^4$ thresholds 25
   4.1 $R^4$ couplings and non-renormalization .................... 25
   4.2 String multiplet and non-perturbative $R^4$ couplings ...... 26
   4.3 Strings, particles and membranes ............................ 28
   4.4 Weak coupling expansion and instanton effects ............ 31

5 Higher genus integrals and higher derivative couplings 32
   5.1 Genus $g$ modular integral ................................. 32
   5.2 $N = 4$ topological string and higher derivative terms ...... 34
   5.3 Non-perturbative $R^4 H^{4g-4}$ couplings .................. 36

6 Conclusions 38

A $Gl(d)$, $SL(d)$, $SO(d,d)$ and $Sp(g)$ Laplacians 40
   A.1 Laplacian on the $SO(d)\backslash Gl(d,\mathbb{R})$ and $SO(d)\backslash SL(d,\mathbb{R})$ symmetric spaces . 40
   A.2 Laplacian on the $[SO(d) \times SO(d)]\backslash SO(d,d,\mathbb{R})$ symmetric space ........ 41
   A.3 Laplacian on the $U(g)\backslash Sp(g,\mathbb{R})$ symmetric space ............ 42
   A.4 Laplacian on the $K\backslash E_{d+1(d+1)}(\mathbb{R})$ symmetric space ............ 43
1 Introduction

While worldsheet modular invariance has played a major role in the context of perturbative string theory since its early days, the advent of target space and non-perturbative dualities has brought into play yet another branch of the mathematics of automorphic forms invariant under infinite discrete groups. Indeed, physical amplitudes should depend on scalar fields usually taking values (in theories with many supersymmetries) in a symmetric space $K\backslash G(\mathbb{R})$, where $K$ is the maximal compact subgroup of $G$, while duality identifies points in $K\backslash G(\mathbb{R})$ differing by the right action of an infinite discrete subgroup $G(\mathbb{Z})$ of $G(\mathbb{R})$. This includes in particular the mapping class group $SL(d,\mathbb{Z})$ in the case of toroidal compactifications of diffeomorphism-invariant theories, the T-duality group $SO(d,d,\mathbb{Z})$ in toroidal compactifications of string theories, as well as the non-perturbative U-duality group $E_{d+1(d+1)}(\mathbb{Z})$ in maximally supersymmetric compactified M-theory [1, 2, 3] (see for instance [4, 5] for reviews and exhaustive list of references). Moreover, supersymmetry constrains certain “BPS saturated” amplitudes to be eigenmodes of second order differential operators [6, 7, 8], so that harmonic analysis on such spaces provides a powerful tool for understanding these quantities. In the most favorable case, it can be used to determine exact non-perturbative results not obtainable otherwise, which can then be analyzed at weak coupling [9, 10]. Other exact results can also be obtained from string-string duality, although in a much less general way, since one needs to be able to control the result on one side of the duality map. This approach was taken for vacua with 16 supersymmetries in [11, 12, 13, 14, 15, 16, 17, 18]. In both cases, one generically obtains a few perturba-
tive leading terms which can in principle be checked against a loop computation, whereas
the non-perturbative contributions correspond to instantonic saddle points of the unknown
string field theory. A number of hints for the rules of semi-classical calculus in string
theory have been extracted from these results [19, 20] and reproduced in Matrix models
[21, 22, 23], but a complete prescription is still lacking. A better understanding of such
effects would be very welcome, as it would for instance allow quantitative computations of
perturbatively-forbidden processes in cases of more immediate physical relevance.

The prototypical example was proposed by Green and Gutperle, who conjectured that
the $R^4$ couplings in ten-dimensional type IIB theory were exactly given by an S-duality
invariant result [9]

$$f^{IIB}_{R^4} = \frac{1}{l_P^2} \sum_{(m,n) \neq 0} \left[ \frac{\tau_2}{|m + n\tau|^2} \right]^{3/2} = \frac{\zeta(3)}{l_s^8} \sum_{(p,q) = 1} \frac{1}{T^3_{(p,q)}}$$

(1.1)

where in the first expression $\tau = a + i/g_s = \tau_1 + i\tau_2$ is the complexified string coupling
transforming as a modular parameter under $SL(2,\mathbb{Z})_S$ and $l_P = g_s^{1/4}l_s$ the S-duality invari-
ant ten-dimensional Planck length. This result is interpreted in the second expression as a
sum over the solitonic $(p, q)$ strings of tension $T_{(p,q)} = |p + q\tau|/l_s^3$. In particular, the scaling
dimension $-8 + 3 \times 2$ is appropriate for an $R^4$ coupling in ten dimensions. The invariant
function in (1.1) is a particular case $s = 3/2$ of a set of non-holomorphic automorphic forms

$$\mathcal{E}_{2,s}^{SL(2,\mathbb{Z})} = \sum_{(m,n) \neq 0} \left[ \frac{\tau_2}{|m + n\tau|^2} \right]^s$$

(1.2)

also known as Eisenstein series, which together with a discrete set of cusp forms generate
the spectrum of the Laplacian on the fundamental domain of the upper half-plane, within
the set of modular functions increasing at most polynomially as $\tau_2 \to \infty$ (see [24] for an
elementary introduction):

$$\Delta_{U(1)\backslash SL(2)} \mathcal{E}_{2,s}^{SL(2,\mathbb{Z})} = \frac{s(s-1)}{2} \mathcal{E}_{2,s}^{SL(2,\mathbb{Z})}, \quad \Delta_{U(1)\backslash SL(2)} = \frac{1}{2} \tau_2^2 (\partial^2_{\tau_1} + \partial^2_{\tau_2}) .$$

(1.3)

Cusp forms are exponentially suppressed at large $\tau_2$, and lie at discrete values along the
$s = 1/2 + i\mathbb{R}$ axis, although no explicit form is known for them. Eisenstein series on the
other hand can be expanded at weak coupling (large $\tau_2$) by Poisson resummation on the
integer $m$ (see Appendix C for useful formulae):

$$\mathcal{E}_{2,s}^{SL(2,\mathbb{Z})} = 2\zeta(2s)\tau_2^s + 2\sqrt{\pi} \tau_2^{1-s} \frac{\Gamma(s-1/2)}{\Gamma(s)} \zeta(2s-1)$$

$$+ \frac{2\pi^s \sqrt{\tau_2}}{\Gamma(s)} \sum_{m \neq 0} \sum_{n \neq 0} \left| \frac{m}{n} \right|^{s-1/2} K_{s-1/2} (2\pi \tau_2 |mn|) e^{2\pi i mn \tau_1}$$

(1.4)
For \( s = 3/2 \), this exhibits a tree-level and one-loop term which can be checked against a perturbative computation, together with an infinite series of instantonic effects, from the saddle point expansion (C.3) of the modified Bessel function \( K_1 \):

\[
\mathcal{E}^{SL(2,\mathbb{Z})}_{2,s=3/2} = 2\zeta(3)e^{-3\phi/2} + \frac{2\pi^2}{3}e^{\phi/2} + 4\pi \sum_{N \neq 0} \sum_{n|N} \sqrt{N} \left[ e^{-2\pi N(e^{-\phi}+i\omega)} + e^{-2\pi N(e^{-\phi}-i\omega)} \right] + \ldots
\]

(1.5)

where \( e^{\phi} = g_s \) denotes the type IIB coupling. These effects can be interpreted as arising from D-instantons and anti-D-instantons [9]. As suggested in [25], one can in fact prove that the 32 supersymmetries of type IIB imply that the exact \( R^4 \) coupling should be an eigenmode of the Laplacian on the moduli space \( U(1)\backslash SL(2,\mathbb{R}) \), with a definite eigenvalue (3/8 in the conventions of the present paper) [26, 6, 8], which uniquely selects out the \( s = 3/2 \) Eisenstein series. In particular, it rules out contributions from cusp forms, which on the basis of the leading perturbative terms alone would have been acceptable [6].

Whereas harmonic analysis on the fundamental domain of the upper half-plane \( U(1)\backslash SL(2,\mathbb{R})/SL(2,\mathbb{Z}) \) is rather well understood, it is not so for the more general symmetric spaces of interest in string theory (see however [27, 28]). It is the purpose of this work to generalize these considerations to more elaborate cases, corresponding to a larger moduli space and discrete symmetry group. Such situations arise in compactifications with 16 or 32 supersymmetries, where supersymmetry prevents corrections to the scalar manifold. As we mentioned, this is the case of toroidal compactifications of string theories, with (part of the) moduli space \([SO(d) \times SO(d)]\backslash SO(d,d,\mathbb{R})/SO(d,d,\mathbb{Z}), or of M-theory, with moduli space \( K\backslash E_{d+1(d+1)}(\mathbb{R})/E_{d+1(d+1)}(\mathbb{Z}) \). This also happens in more complicated cases, such as type IIA on \( K_3 \), with moduli space \( \mathbb{R}^+ \times [SO(4) \times SO(20)]\backslash SO(4,20,\mathbb{R}) \) identified by the \( SO(4,20,\mathbb{Z}) \) (perturbative) mirror symmetry, or type IIB on \( K_3 \), with moduli space \( [SO(5) \times SO(21)]\backslash SO(5,21,\mathbb{R}) \) identified by the non-perturbative symmetry \( SO(5,21,\mathbb{Z}) \). It is even possible to have uncorrected tree-level scalar manifolds in theories with 8 supersymmetries, as in the FHSV model [29], with a moduli space \( K\backslash[SL(2,\mathbb{R}) \times SO(2,10,\mathbb{R}) \times SO(4,12,\mathbb{R})] \) where \( K \) is the obvious maximal compact subgroup. Note that in this case the duality group is broken to a subgroup of \( SO(2,10,\mathbb{Z}) \times SO(4,12,\mathbb{Z}) \), due to the effect of the freely acting orbifold. Usually however, in cases with 8 supersymmetries, amplitudes are given by sections of a symplectic bundle on the corrected moduli space, and our methods will not carry over in a straightforward way.

This generalization was in fact started in Ref. [11], where toroidal compactifications of type IIB string theory down to 7 or 8 dimensions were considered. It was demonstrated there that the straightforward extension of the order 3/2 Eisenstein series (1.2) to the U-duality groups \( SL(5,\mathbb{Z}) \) and \( SL(3,\mathbb{Z}) \) reproduces the tree-level and one-loop \( R^4 \) thresholds,
together with \((p, q)\)-string instantons. A generalization to lower dimensional cases was also proposed, and a distinct route using successive T-dualities was taken in Ref. \[30\] to obtain the contribution of the \(O(e^{-1/g_s})\) D-brane instantons in the toroidally compactified type IIA and IIB theories; it was also pointed out that S-duality suggests extra \(O(e^{-1/g_s^2})\) contributions yet to be understood. In the present work, we will take a more general approach, and investigate the properties and utility of the generalized Eisenstein series, that we define for any symmetric space \(K\backslash G(\mathbb{R})\) and any representation \(\mathcal{R}\) of \(G\) by

\[
\mathcal{E}_{\mathcal{R},s}^{G(\mathbb{Z})}(g) = \sum_{m \in \Lambda_{\mathcal{R}} \setminus \{0\}} \delta(m \wedge m) \left[ m \cdot \mathcal{R}^t \mathcal{R}(g) \cdot m \right]^{-s}.
\]

Here, \(g\) denotes an element in the coset \(K\backslash G(\mathbb{R})\), \(m\) a vector in an integer lattice \(\Lambda_{\mathcal{R}}\) transforming in the representation \(\mathcal{R}\). \(\wedge\) is an integer-valued product on the lattice, such that the condition \(m \wedge m = 0\) projects the symmetric tensor product \(\mathcal{R} \otimes \mathcal{R}\) onto its highest irreducible component, thus keeping only the “completely symmetric” part. This definition is to be contrasted with the one used in the mathematical literature \[28\]:

\[
\mathcal{E}_{\{w_i\}}^{G(\mathbb{Z})}(g) = \sum_{h \in G(\mathbb{Z})/N} \prod_{i=1}^r a_i(gh)^{-w_i}
\]

where \(w_i\) is now an arbitrary \(r\)-dimensional vector in weight space, and \(a(g)\) is the Abelian component of \(g\) in the Iwasawa decomposition of the rank-\(r\) non-compact group \(G(\mathbb{R}) = K \cdot A \cdot N\) into maximal compact \(K\), Abelian \(A\) and nilpotent \(N\) subgroups. Note that this definition is manifestly \(K\)-invariant on the left and \(G(\mathbb{Z})\)-invariant on the right. Choosing \(w\) along a highest-weight vector \(\lambda_{\mathcal{R}}\) associated to a representation \(\mathcal{R}\) reduces (1.7) to (1.6) where \(w = s\lambda_{\mathcal{R}}\), up to an \(s\)-dependent factor. This generalizes the equality in (1.1) to higher rank groups. The definition (1.6), albeit less general, has a clearer physical meaning: the lattice \(\Lambda_{\mathcal{R}}\) labels the set of BPS states in the representation \(\mathcal{R}\) of the duality group, \(\mathcal{M}^2 = m \cdot \mathcal{R}^t \mathcal{R}(g) \cdot m\) gives their mass squared (or tension), and \(m \wedge m = 0\) imposes the half-BPS condition; this will be shown to be a necessary requirement for the eigenmode condition \(\Delta_{K\backslash G} \mathcal{E}_{\mathcal{R},s}^{G(\mathbb{Z})} \propto \mathcal{E}_{\mathcal{R},s}^{G(\mathbb{Z})}\), but could be dropped if one were to address non–half-BPS saturated amplitudes.

The outline of this paper is as follows. In Section 2, we will discuss the simplest case of \(SL(d, \mathbb{Z})\) Eisenstein series, where most of the features arise without the complications in the parametrization of the moduli space. In Section 3, we will turn to \(SO(d, d, \mathbb{Z})\) Eisenstein series, and discuss their applications for the computation of T-duality invariant one-loop thresholds of string theories compactified on a torus \(T^d\). In Section 4, we covariantize this expression to obtain exact non-perturbative \(R^4\) couplings in toroidal compactifications of M-theory to \(D \geq 4\). In Section 5, we apply the same techniques to the \(g\)-loop threshold,
and use it to deduce $R^4 H^{4g-4}$ exact couplings in the same theory. Computational details will be relegated to the Appendices. This work appeared on the archive simultaneously with Ref. [31], which uses similar techniques, albeit with a different motivation.

## 2 Toroidal compactification and $SL(d, \mathbb{Z})$ Eisenstein series

### 2.1 Moduli space and Iwasawa gauge

Infinite discrete symmetries appear in the simplest setting in compactifications of a diffeomorphism invariant field theory on a torus $T^d$. Specifying the internal manifold requires a flat metric on the torus, that is a positive definite metric $g$. Equivalently we may specify a vielbein $e \in GL(d, \mathbb{R})$ such that $g = e^t e$, defined up to orthogonal rotations $SO(d, \mathbb{R})$ acting on the left, which leave $g$ invariant. This gauge invariance can be fixed thanks to the Iwasawa decomposition

$$GL(d, \mathbb{R}) = SO(d, \mathbb{R}) \times (\mathbb{R}^+)^d \times N_d,$$

where $(\mathbb{R}^+)^d$ denotes the Abelian group of diagonal $d \times d$ matrices with positive non zero entries, and $N_d$ the nilpotent group of upper triangular matrices with unit diagonal, by choosing $e$ in the last two factors, i.e. in an upper triangular form. The Abelian part corresponds to the radii of the torus, whereas $N_d$ parametrizes the Wilson lines $A_i^j$ of the Kaluza–Klein gauge field $g_{\mu i}$.

By general covariance, the Kaluza–Klein reduction of the field theory on the torus only involves contractions with the metric $g_{ij}$, so that the reduced theory is invariant under a symmetry $h \in GL(d, \mathbb{R})$ which transforms $g$ in the representation $g \rightarrow h^t gh$. The vielbein on the other hand is acted upon on the right, $e \rightarrow eh$, which has to be compensated by a field-dependent $SO(d, \mathbb{R})$ gauge transformation $e \rightarrow \omega(e, h)e$ to preserve the upper triangular form. Transforming by an element $h \propto 1$ in the center of $GL(d, \mathbb{R})$ corresponds to changing the volume, whereas an $SL(d, \mathbb{R})$ transformation affects the torus shape. This change is not always physical however, since an $SL(d, \mathbb{Z})$ rotation can be compensated by a global diffeomorphism of the torus, i.e. an element of the mapping class group. The toroidal compactification is therefore parametrized by the symmetric space

$$\mathbb{R}^+ \times [SO(d, \mathbb{R}) \backslash SL(d, \mathbb{R}) / SL(d, \mathbb{Z})]$$

and all physical amplitudes should be invariant under $SL(d, \mathbb{Z})$. In particular, the effective action including the massive Kaluza–Klein modes will only be invariant under $SL(d, \mathbb{Z})$, and not $GL(d, \mathbb{R})$. 

7
2.2 Fundamental and antifundamental Eisenstein series

Keeping the above in mind, it is now straightforward to generalize the $SL(2,\mathbb{Z})$ Eisenstein series \( (1.2) \) to the fundamental representation of $SL(d,\mathbb{Z})$ as

$$ E_{d,s}^{SL(d,\mathbb{Z})} = \sum_{m^i} \left[ m^ig_{ij}m^j \right]^{-s}, \quad (2.3) $$

where the subscript $d$ stands for the representation in which the integers $m^i, i = 1 \ldots d$ transform. In fact, the above form is really a $Gl(d,\mathbb{Z})$ Eisenstein series since we did not restrict $g$ to have unit determinant, but the dependence on $V_d = \sqrt{\det g}$ is trivial so we shall keep with this abuse of language. The $SL(d,\mathbb{Z})$-invariant form in \( (2.3) \) is easily seen to be an eigenmode of the Laplacian $\Delta_{Gl(d)}$ on the scalar manifold \((2.2)\):

$$ \Delta_{Gl(d)} E_{d,s}^{SL(d,\mathbb{Z})} = \frac{s(2s-d+1)}{2} E_{d,s}^{SL(d,\mathbb{Z})} \quad (2.4a) $$

$$ \Delta_{Gl(d)} = \frac{1}{4} g_{ik}g_{jl} \frac{\partial}{\partial g_{ij}} \frac{\partial}{\partial g_{kl}} + \frac{d+1}{4} g_{ij} \frac{\partial}{\partial g_{ij}} \quad (2.4b) $$

In fact, it is more appropriate to restrict to the $SO(d,\mathbb{R}) \backslash SL(d,\mathbb{R})$ moduli, in terms of which

$$ \Delta_{SL(d)} E_{d,s}^{SL(d,\mathbb{Z})} = \frac{s(d-1)(2s-d)}{2d} E_{d,s}^{SL(d,\mathbb{Z})} \quad (2.5a) $$

$$ \Delta_{SL(d)} = \frac{1}{4} g_{ik}g_{jl} \frac{\partial}{\partial g_{ij}} \frac{\partial}{\partial g_{kl}} - \frac{1}{4d} \left( g_{ij} \frac{\partial}{\partial g_{ij}} \right)^2 + \frac{d+1}{4} g_{ij} \frac{\partial}{\partial g_{ij}} \quad (2.5b) $$

Here we may wonder why we should choose the integers $m$ to lie in the fundamental representation $d$ of $SL(d)$. Choosing $m$ to transform in the contragredient representation

$$ E_{d,s}^{SL(d,\mathbb{Z})} = \sum_{m^i} \left[ m^ig_{ij}m^j \right]^{-s}, \quad (2.6) $$

does not bring much novelty, since a Poisson resummation over all integers $m_i$ brings us back to Eq. \((2.3)\), albeit with a transformed order $s \rightarrow d/2 - s$:

$$ E_{d,s}^{SL(d,\mathbb{Z})} = V_d\pi^s\Gamma(\frac{d}{2} - s) \frac{\pi^{\frac{d-4}{2}}\Gamma(s)}{\pi^{\frac{d-4}{2}}\Gamma(s)} E_{d,s}^{SL(d,\mathbb{Z})} \quad (2.7) $$

Note that the two series $E_{d,s}^{SL(d,\mathbb{Z})}$ and $E_{d,s}^{SL(d,\mathbb{Z})}$ have the same eigenvalue under $\Delta_{SL(d)}$, but different eigenvalues under $\Delta_{Gl(d)}$. This simply stems from their different dependence on the volume, and is not sufficient to lift their degeneracy under the $SL(d)$ Laplacian.

---

\[1\] In all expressions for the Laplacians in the main text of the paper we employ the convention that $\partial/\partial \tilde{g}_{ij}$ is taken with respect to the diagonally rescaled metric $\tilde{g}_{ij} = (1 - \delta_{ij}/2)g_{ij}$, and for simplicity of notation we omit the tilde. As explained in Appendix $A$ this redefinition has the advantage that unrestricted sums can be used.

\[2\] We denote the contragredient representations of $\mathcal{R}$ by $\bar{\mathcal{R}}$, not to be confused by complex conjugation (all finite dimensional representations considered in this paper are real).
2.3 Higher representations and constrained Eisenstein series

We may also choose \( m \) to transform in a higher dimensional representation, i.e. as a tensor \( m_{ij...} \) with prescribed symmetry properties. In order to determine whether we still get an eigenmode, it is useful to take a more algebraic approach. We consider acting with the Laplacian on the integral representation

\[
\left[m^t M m\right]^{-s} = \frac{\pi^s}{\Gamma(s)} \int_0^\infty \frac{dt}{t^{1+s}} \exp\left(-\frac{\pi}{t} m^t M m\right)
\]  

where \( M = R^t R \) denotes the mass matrix in the representation \( R \). Deriving only once in the exponential yields the action of the Laplacian on \( M \), which transforms as a symmetric tensor product \( R \otimes_s R \). In order to get an eigenmode, this tensor product should be irreducible when contracted with the charges \( m \). This puts a quadratic constraint on \( m \), which we generically denote \( m \wedge m = 0 \). In other words, \( m \wedge m = 0 \) projects onto the highest irreducible component of the symmetric tensor product \( R \otimes_s R \). One may want to drop the quadratic constraint, and still impose higher cubic and quartic constraints, in order to obtain candidates for quarter-BPS amplitudes, but we will not pursue this line here. Assuming this constraint is fulfilled, we therefore get an insertion of \(-\pi Q[R \otimes R]/4t\) in the integral, where \( Q[S] \) is the Casimir \((T^i)^2\) in the representation \( S \) (we normalize the Laplacian such that \( \Delta = \frac{1}{4}(T^i)^2 \)). The other term with two derivatives acting in the exponential gives a contribution \( \pi^2 \tilde{Q}[R \otimes_s R]/4t^2 \), where \( \tilde{Q}[S] \) denotes the operator \( T^i \otimes T^i \) acting on the symmetric tensor product \( S \otimes_s S \). By developing the square in \((T^i \otimes 1 + 1 \otimes T^i)^2\), we find that \( Q[S \otimes_s S] = 2Q[S] + 2\tilde{Q}[S] \), so that all in all

\[
e^{\pi m^t M m/t} \Delta e^{-\pi m^t M m/t} = \frac{\pi^2}{8t^2} (Q[R \otimes_s^4] - 2Q[R \otimes_s^4]) (m^t M m)^2 - \frac{\pi}{4t} Q[R \otimes_s^2] (m^t M m)
\]  

We now use the expression for the Casimir of the \( p \)-th symmetric power of a representation of highest weight \( \lambda \),

\[
Q[R_\lambda^\otimes_p] = (p\lambda, p\lambda + 2\rho)
\]  

where \( \rho \) is the Weyl vector, i.e. the sum of all the fundamental weights, and \((\cdot, \cdot)\) the inner product on the weight space with the length of the roots normalized to 2 (since we restrict to simply laced Lie groups of ADE type). Using formula (B.2) to integrate by part in (2.8), we thus find

**Proposition 1** The constrained Eisenstein series (1.6) associated to the representation of highest weight \( \lambda \) is an eigenmode of the Laplacian with eigenvalue

\[
\Delta_{K \setminus G} \mathcal{E}_{R_\lambda^s}^{G(Z)} = s(\lambda, s\lambda - \rho) \mathcal{E}_{R_\lambda^s}^{G(Z)}
\]  

where \( G = K \setminus G \) is the quotient of the group by the center.\]
This result reproduces the eigenvalue \((2.5a)\) for the fundamental representation of \(SL(d, \mathbb{Z})\) but will be applied for many other situations in the following. It implies in particular that Eisenstein series associated to representations related by outer automorphisms, i.e. symmetries of the Dynkin diagram, are degenerate under \(\Delta_{K\setminus G}\), as well as two Eisenstein series of same representation but order \(s\) and \(\frac{[\lambda, \rho]}{[\lambda, \lambda]} - s\). We also note that \((2.11)\) can be obtained more quickly by noting that \(M^{-2s} = (m^t M m)^{-s}\) transforms as the symmetric power of order \(-2s\) of \(\mathcal{R}\), and substituting \(p = -2s\) in \((2.10)\). Finally, we note that Eisenstein series are in fact eigenmodes of the complete algebra of invariant differential operators \([28]\).

In some cases, it may happen that the constraint \(m \wedge m = 0\) can be solved in terms of a lower dimensional representation. This is for instance the case of \(p\)-th symmetric tensors of \(SL(d, \mathbb{R})\), where the constraint implies that the integers \(m^{ijkl...}\) themselves are, up to an integer \(r\), the symmetric power of a fundamental representation \(n^i\):

\[
m^{ijkl...} = r \ n^i n^j n^k n^l ... \ . \quad (2.12)
\]

The summation over \(r\) can then be carried out explicitly, and the result is proportional to the Eisenstein function in the fundamental representation, with a redefined order \(s \to ps\). This, however, does not happen for antisymmetric tensors. Since the antisymmetric representations are associated to the nodes of the Dynkin diagram, we thus see that a subset of eigenmodes of the Laplacian is in general provided by the Eisenstein series associated to the nodes of the Dynkin diagram, up to cusp forms.

2.4 Decompactification and analyticity

Our definition of Eisenstein series has so far remained rather formal: the infinite sums appearing in \((2.3), (2.6)\) are absolutely convergent for \(s > d/2\) only, and need to be analytically continued for other values of \(s\) in the complex plane\(^\dagger\). It turns out that the analyticity properties can be determined by induction on \(d\), which corresponds to the physical process of decompactification. We thus assume the torus \(T^{d+1}\) to factorize into a circle of radius \(R\) times a torus \(T^d\) with metric \(g_{ab}\) and use the integral representation \((2.8)\) of the Eisenstein series, say in the fundamental representation,

\[
\mathcal{E}^{SL(d+1, \mathbb{Z})}_{d+1, s} = \frac{\pi^s}{\Gamma(s)} \int_0^\infty \frac{dt}{t^{1+s}} \sum_{m,n} \exp \left( -\frac{\pi}{t} \left[ n^a g_{ab} n^b + R^2 m^2 \right] \right) \quad (2.13)
\]

where \(m\) denotes the first component of \(n^i\). The leading term as \(R \to \infty\) corresponds to the \(m = 0\) contribution, which reduces to the \(SL(d, \mathbb{Z})\) Eisenstein series. Subleading

\(^\dagger\)Other regularization methods have also been discussed in Ref. [10]
contributions arise by Poisson resumming on the unrestricted (since now \( m \neq 0 \)) integers \( n_a \):

\[
E_{d+1,s}^{\text{SL}(d,\mathbb{Z})} = E_{d,s}^{\text{SL}(d,\mathbb{Z})} + \frac{\pi^s}{V_d \Gamma(s)} \int_0^\infty dt \sum_{n_a} \sum_m \exp \left( -\pi t n_a g^{ab} n_b - \frac{\pi}{t} R^2 m^2 \right) \tag{2.14}
\]

where \( V_d \) stands for the volume \( \sqrt{\det g} \) of the torus \( T^d \). Separating the \( n_a = 0 \) contribution from the still subleading \( n_a \neq 0 \) one, we get

\[
E_{d+1,s}^{\text{SL}(d+1,\mathbb{Z})} = E_{d,s}^{\text{SL}(d,\mathbb{Z})} + \frac{2\pi^s \Gamma(s - d/2) \zeta(2s - d)}{\pi^{d/2} \Gamma(s) R^{2s-d} V_d} + \frac{2\pi^s}{\Gamma(s) R^{2s-d-2}} \sum_{m} \sum_{n_a} \left| n_a g^{ab} n_b \right| K_{s-d/2} \left( 2\pi |m| R \sqrt{n_a g^{ab} n_b} \right) \tag{2.15}
\]

Using the asymptotic behaviour of the Bessel function \([C23]\), we see that the last term is exponentially suppressed of order \( O(e^{-R}) \), and the sum is absolutely convergent and thus analytic in \( s \). For \( d = 1 \), the \( SL(d,\mathbb{Z}) \) Eisenstein series reduces to \( 2\zeta(2s) R^{-2s} \) and has a simple pole at \( s = 1/2 \). For \( d > 1 \), induction shows that the pole at \( s = d/2 \) from the second term cancels the one in the \( SL(d,\mathbb{Z}) \) Eisenstein series, leaving the pole at \( s = (d+1)/2 \) from the zeta function in \((2.15)\). We thus have

**Proposition 2** The \( SL(d,\mathbb{Z}) \) Eisenstein series of order \( s \) in the fundamental representation can be analytically continued to the \( s \)-plane with \( s = d/2 \) excluded, where it has a single pole with residue

\[
E_{d,s}^{\text{SL}(d,\mathbb{Z})} \approx \frac{\pi^{d/2}}{V_d \Gamma(d/2)} \frac{1}{s - d/2} \tag{2.16}
\]

This result is well known in the mathematical literature \([27]\). Of course, the same holds for the antifundamental representation by replacing \( V_d \) by its inverse. Let us mention in passing that, together with the functional relation \([27]\), this implies a relation which generalizes \( 2\zeta(0) = -1 \):

\[
E_{d,s=0}^{\text{SL}(d,\mathbb{Z})} = E_{d,s=0}^{\text{SL}(\overline{d},\mathbb{Z})} = -1 \tag{2.17}
\]

We also note that the pole at \( s = d/2 \) coincides with the vanishing of the eigenvalue of the Eisenstein series under the Laplacian \( \Delta_{\text{SL}(d)} \). This is so because the residue is moduli independent. An invariant modular form can still be obtained by subtracting the pole, in which case the eigenmode equation gets a harmonic anomaly:

\[
\Delta_{\text{SL}(d)} \hat{E}_{d,s=d/2}^{\text{SL}(d,\mathbb{Z})} = \frac{\pi^{d/2}(d - 1)}{2\Gamma(d/2)V_d} \tag{2.18}
\]
The case $d = 2$, particularly relevant in the sequel, corresponds to Kronecker’s first limit formula (see e.g. [24]),

$$\hat{E}^{SL(2,\mathbb{Z})}_{2,s=1} = -\pi \log \left(4e^{-4\gamma\tau_2} |\eta(\tau)|^4 \right)$$  \hspace{1cm} (2.19)

where $\eta(\tau)$ denotes the usual Dedekind function and $\gamma$ is Euler’s constant.

This computation can unfortunately not be made for constrained Eisenstein series, since the constraint prevents a simple Poisson resummation. We shall come back to this problem in the next section for the $SO(d,d,\mathbb{Z})$ case. We can however conjecture the analytic structure from a simple argument: the divergences arise from the large $m$ region, where the integers can be approximated by $N = \dim \mathcal{R}$ continuous variables. The $N_c$ quadratic constraints restrict the phase space to $\mathbb{R}^{N-N_c}$, while inserting an extra factor $r^{-N_c}$ in spherical coordinates, from $\delta(r^2) = \delta(r)/2r$. We are therefore led to the integral

$$\int_{r^{-N_c}r^{-N_c-1}r^{-2s}dr},$$

which converges for $s > (N-2N_c)/2$. We therefore expect a simple pole at $s = (N-2N_c)/2$ for an Eisenstein series of an $N$-dimensional representation with $N_c$ independent constraints.

2.5 Partial Iwasawa decomposition

In determining the decompactification behaviour, we assumed the torus $T^{d+1}$ to factorize into $T^d \times S_1$. This may be too restrictive, as for instance in M-theory applications, where we are interested in the perturbative type II limit corresponding to a vanishingly small circle of radius $R_a = g_s l_s$ but still want to retain the effect of the off-diagonal metric, i.e. the Ramond one-form $A$. It is then convenient to take the Kaluza–Klein ansatz

$$dx^i g_{ij} dx^j = R^2(dx^1 + A_a dx^a)^2 + dx^a \hat{g}_{ab} dx^b$$  \hspace{1cm} (2.20)

which is nothing but a partial Iwasawa decomposition. This breaks the higher dimensional symmetry $SL(d+1, \mathbb{R})$ to a subgroup $SL(d, \mathbb{R})$, together with a nilpotent group of constant shifts $A_a \to A_a + \Lambda_a$, which is what remains from the Kaluza–Klein gauge invariance on a flat torus. In terms of these variables, the Laplacian takes the form (See Appendix [A.5] for details on the derivation.)

$$\Delta_{Gl(d+1)} = \Delta_{Gl(d)} - \frac{1}{4} \hat{g}_{ab} \frac{\partial}{\partial g_{ab}} + \frac{1}{4} \left(R \frac{\partial}{\partial R} \right)^2 + \frac{d}{4} R \frac{\partial}{\partial R} + \frac{\hat{g}_{ab}}{2R^2} \frac{\partial}{\partial A_a} \frac{\partial}{\partial A_b}$$  \hspace{1cm} (2.21)

One can then check that each term in (2.15) – upon reinstating the dependence on $A_a$ – is an eigenmode of the Laplacian with the correct eigenvalue.
3 $SO(d,d,\mathbb{Z})$ Eisenstein series and one-loop thresholds

In this section we turn to the construction of Eisenstein series for $SO(d,d,\mathbb{Z})$ and its application to one-loop thresholds in type II string theory. Higher genus contributions are also amenable to an Eisenstein series representation, and will be addressed in Section 5.

3.1 Moduli space and Iwasawa gauge

Owing to the occurrence of winding states charged under the 2-form $B_{\mu\nu}$, any closed string theory on a torus $T^d$ exhibits a larger symmetry $O(d,d,\mathbb{Z})$, a discrete subgroup of the $O(d,d,\mathbb{R})$ symmetry of the massless degrees of freedom. The symmetry is actually reduced to $SO(d,d,\mathbb{Z})$ in type II theories, where the elements in $O(d,d,\mathbb{Z})$ with determinant $-1$ map type IIA to type IIB. This T-duality is valid to all orders in perturbation theory, and postulated to hold non-perturbatively as well. It contains the mapping class group $SL(d,\mathbb{Z})$ of the torus as a subgroup, as well as generators that are non-perturbative from a world-sheet point of view. The moduli space includes a symmetric subspace

$$[SO(d,\mathbb{R}) \times SO(d,\mathbb{R})] \backslash SO(d,d,\mathbb{R})/SO(d,d,\mathbb{Z})$$

(3.1)

describing the metric of the torus and the two-form background, which can again be parametrized using the Iwasawa decomposition

$$SO(d,d,\mathbb{R}) = [SO(d,\mathbb{R}) \times SO(d,\mathbb{R})] \times (\mathbb{R}^+)^d \times N_{2d}^{SO},$$

(3.2)

More precisely, the Abelian part $(\mathbb{R}^+)^d$ corresponds to the $d$ radii (and $d$ inverse radii) of the torus and the nilpotent part $N_{2d}^{SO}$ parametrizes the Wilson lines $A^j_i$ of the Kaluza–Klein gauge field and the antisymmetric tensor $B_{ij}$. In particular, in the basis where the $SO(d,d,\mathbb{Z})$ invariant tensor is $\eta = \begin{pmatrix} 0 & 1_d \\ 1_d & 0 \end{pmatrix}$, the gauge-fixed vielbein $e$ can be chosen as

$$e = \begin{pmatrix} \frac{1}{R_1} & \frac{1}{R_2} \\ R_1 & R_2 \end{pmatrix} \begin{pmatrix} 1 \\ -A_2^1 \\ \vdots \\ 1 \\ A_2^1 \end{pmatrix} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \begin{pmatrix} B_{11} \\ B_{12} \\ \vdots \\ B_{22} \end{pmatrix}$$

(3.3)

and right symmetry transformations by an $SO(d,d,\mathbb{R})$ element have to be compensated by left $SO(d,\mathbb{R}) \times SO(d,\mathbb{R})$ gauge transformations. The $SL(d,\mathbb{R})$ subgroup corresponds to block diagonal elements $\begin{pmatrix} g^{-1} \\ g \end{pmatrix}$. In analogy with the $SO(d) \backslash SL(d,\mathbb{R})$ case, we can
trade the vielbein $e$ for the gauge invariant moduli matrix

$$M(V) = e^t e = \begin{pmatrix} g^{-1} & g^{-1}B \\ -Bg^{-1} & g^{-1}Bg^{-1} \end{pmatrix}$$

(3.4)

which provides the mass matrix for BPS states in the vector representation $V$ of $SO(d,d)$, namely momentum and winding states. D-branes on the other hand transform as (conjugate) spinor representations $S(C)$, and their mass matrix is given accordingly by $R(e)^t R(e)$ where $R(e)$ is the spinor or conjugate spinor representation of the group element $e$. We can therefore build $SO(d,d,\mathbb{Z})$ invariant functions by summing the BPS mass or tension over all BPS states, which we do now.

3.2 Spinor and vector Eisenstein series

In order to define these T-duality invariant functions, we need to be more explicit about the mass matrix and BPS conditions of these states. These have been reviewed in [5] (see [32] for a résumé) so we shall be brief in recalling them. The mass in the vector representation in terms of the KK momenta and winding numbers $m_i, m^i (i = 1 \ldots d)$, reads

$$M^2(V) = m \cdot e^t e \cdot m = \tilde{m}_i g^{ij} \tilde{m}_j + m^i g_{ij} m^j,$$

$$k = m_i m^i = 0,$$

(3.5a)

(3.5b)

where the last equation records the (quadratic) half-BPS condition $m \wedge m = k = 0$. Integer shifts of $B \rightarrow B + b$ induce a spectral flow $m_i \rightarrow m_i - b_{ij} n^j$ on the lattice of BPS states, leaving the dressed charge $\tilde{m}_i = m_i + B_{ij} m^j$ invariant and preserving the condition $m \wedge m = 0$. For the spinor representation with $2^{d-1}$ charges $(m^{[1]}, m^{[3]}, m^{[5]}, \ldots )$ describing the wrapping numbers along the odd cycles of $T^d$, the charges can be encapsulated in a differential form $m = m^i dx_i + \frac{1}{3!} m^{ijk} dx_i \wedge dx_j \wedge dx_k + \ldots$ and the effect of the $B$-field is to boost the charges as $\tilde{m} = \exp \left( \frac{1}{2} B_{ij} dx^i \wedge dx^j \right) m$, where $\cdot$ denotes the inner product.}

$^{14}$Integer subscripts or superscripts in square brackets denote the number of antisymmetric $SL(d)$ indices. When separated by a semi-colon as in (3.7), they stand for groups of antisymmetric indices with no mutual symmetry property. The upper or lower position of the indices denotes a gradient or contragradient representation of $SL(d)$. 

14
\[ \mathcal{M}^2(S) = \frac{1}{V_d} \left[ (\tilde{m}^i)^2 + \frac{1}{3!} (\tilde{m}^{ijk})^2 + \frac{1}{5!} (\tilde{m}^{ijklm})^2 + \ldots \right] \]  

(3.6a)

\[ \tilde{m}^i = m^i + \frac{1}{2} m^{jki} B_{jk} + \frac{1}{8} m^{jklmi} B_{jk} B_{lm} + \ldots \]  

(3.6b)

\[ \tilde{m}^{ijk} = m^{ijk} + \frac{1}{2} m^{lmijk} B_{lm} + \ldots \]  

(3.6c)

\[ \tilde{m}^{ijklm} = m^{ijklm} + \ldots \]  

(3.6d)

gives, up to a power \( V_d/(g_s^{2/8}) = l_p^{d-8} \) of the T-duality invariant Planck length and subject to the half-BPS conditions

\[ k^{ijkl} = m^{[ij} m^{kl]} = 0 \]  

(3.7a)

\[ k^{ij[kl} m^{m]n} = m^{ij[kl} m^{m]n} = 0 \]  

(3.7b)

\[ k^{ijklmn} = m^{ij[kl} m^{m]n} = 0 , \]  

(3.7c)

the mass of type IIB D-branes wrapped on an odd-dimensional cycle, or the tension of type IIA D-branes wrapped on an odd-dimensional cycle. Here we made explicit the constraints up to \( d = 6 \) only. In particular, we note that the first occurrence of the quadratic constraints is for \( d = 4 \), in which case they reduce to a singlet. For \( d = 5 \) they form a vector 5, while for \( d = 6 \) they transform in an antisymmetric representation 66 of the T-duality group \( SO(6,6;\mathbb{Z}) \). More generally, one should require the representation \( \mathcal{R} \otimes_s \mathcal{R} \) to be irreducible. Similarly, for the conjugate spinor representation with wrapping numbers \( m = (m, m^{[2]}, m^{[4]}, \ldots) \) around the even cycles of \( T^d \), we have

\[ \mathcal{M}^2(C) = \frac{1}{V_d} \left[ \tilde{m}^2 + \frac{1}{2} (\tilde{m}^{ij})^2 + \frac{1}{4!} (\tilde{m}^{ijkl})^2 + \ldots \right] \]  

(3.8a)

\[ \tilde{m} = m + \frac{1}{2} m^{ij} B_{ij} + \frac{1}{8} m^{ijkl} B_{ij} B_{kl} + \ldots \]  

(3.8b)

\[ \tilde{m}^{ij} = m^{ij} + \frac{1}{2} m^{klij} B_{kl} + \ldots \]  

(3.8c)

\[ \tilde{m}^{ijkl} = m^{ijkl} + \ldots \]  

(3.8d)

with half-BPS conditions

\[ k^{ijkl} = m^{[ij} m^{kl]} + m m^{ijkl} = 0 \]  

(3.9a)

\[ k^{ij[kl} m^{m]n} = m^{ij[kl} m^{m]n} = 0 \]  

(3.9b)

\[ k^{ij[kl} m^{m]npq} = n^{ij[n} m^{m]npq} + n^{ij[kl} n^{m]npq} = 0 \]  

(3.9c)
This describes the tension of type IIB D-branes wrapped on an even-dimensional cycle, or the mass of type IIA D-branes wrapped on an even-dimensional cycle.

With these T-duality invariant building blocks in hand, we may now define the Eisenstein series for each of these three representations as

$$\mathcal{E}_{R,s}^{SO(d,d;\mathbb{Z})} = \sum_m \delta(m \wedge m) [\mathcal{M}^2(R)]^{-s}$$  \hspace{1cm} (3.10)

where have used the labels $R = V, S, C$ for the vector, spinor and conjugate spinor representations. Here $\delta(m \wedge m)$ stands for the quadratic constraints (3.5b), (3.7), (3.9), and $\mathcal{M}^2(R)$ are the mass formulae given in (3.5a), (3.6), (3.8). Not surprisingly, an explicit computation (see Appendix B) shows that these Eisenstein series are indeed eigenmodes of the Laplacian on the scalar manifold (3.1),

$$\Delta_{SO(d,d)} e^{-m t M(V)m / t} = \Delta(R, s) \mathcal{E}_{R,s}^{SO(d,d;\mathbb{Z})}$$  \hspace{1cm} (3.11a)

where the eigenvalues are given by

$$\Delta(V, s) = s(s - d + 1), \quad \Delta(S, s) = \Delta(C, s) = \frac{sd(s - d + 1)}{4},$$  \hspace{1cm} (3.12)

in agreement with Eq. (2.11). The degeneracy of the spinor and conjugate spinor (as well as the vector for $d = 4$) is a consequence of the outer automorphism which relates the two (or the three for $d = 4$, due to triality). We emphasize that the derivation shows that the quadratic 1/2 BPS constraints are essential for these Eisenstein series to be eigenmodes. For instance, in the case of the vector representation, the analogue of (2.9) is

$$\Delta_{SO(d,d)} e^{-m t M(V)m / t} = \left[\left(\frac{m^t M(V)m}{t^2} - 4(m \wedge m)^2 \right) - \frac{d m^t M(V)m}{t} \right] e^{-m t M(V)m / t}$$  \hspace{1cm} (3.13)

where $m \wedge m = m_i m^i$ vanishes on half-BPS states only. For low dimensional cases however, the constraints drop or can be solved, so that we are back to ordinary Eisenstein series. This includes the $d = 1$ vector series,

$$\mathcal{E}_{V,s}^{SO(1,1;\mathbb{Z})} = 2\zeta(2s) \left(R^{2s} + R^{-2s}\right)$$  \hspace{1cm} (3.14)

or the $d < 4$ spinor series,

$$\mathcal{E}_{S,s}^{SO(1,1;\mathbb{Z})} = 2\zeta(2s) R^{-s}, \quad \mathcal{E}_{C,s}^{SO(1,1;\mathbb{Z})} = 2\zeta(2s) R^s$$  \hspace{1cm} (3.15a)

$$\mathcal{E}_{S,s}^{SO(2,2;\mathbb{Z})} = \mathcal{E}_{2,s}^{SL(2;\mathbb{Z})}(U), \quad \mathcal{E}_{C,s}^{SO(2,2;\mathbb{Z})} = \mathcal{E}_{2,s}^{SL(2;\mathbb{Z})}(T)$$  \hspace{1cm} (3.15b)

$$\mathcal{E}_{S,s}^{SO(3,3;\mathbb{Z})} = \mathcal{E}_{4,s}^{SL(4;\mathbb{Z})}, \quad \mathcal{E}_{C,s}^{SO(3,3;\mathbb{Z})} = \mathcal{E}_{4,s}^{SL(4;\mathbb{Z})}$$  \hspace{1cm} (3.15c)
where the identities in the last two lines follow from the local isomorphisms $SO(2, 2, \mathbb{R}) = SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$ ($U$ and $T$ denote the standard complex moduli $U = (g_{12} + i V_2)/g_{11}$, $T = B_{12} + i V_2$ ) and $SO(3, 3, \mathbb{R}) = SL(4, \mathbb{R})$.

### 3.3 One-loop modular integral and method of orbits

Under toroidal compactification on a torus $T^d$, any string theory exhibits the T-duality symmetry $SO(d, d, \mathbb{Z})$, and all amplitudes should be expressible in terms of modular forms of this group. For half-BPS saturated couplings, the one-loop amplitude often reduces to an integral of a lattice partition function over the fundamental domain $\mathcal{F}$ of the moduli space of genus-1 Riemann surfaces,

$$I_d = 2\pi \int_{\mathcal{F}} \frac{d^2 \tau}{\tau_2} Z_{d,d}(g, B; \tau)$$  \hspace{1cm} (3.16)

where $Z_{d,d}$ is the partition function (or theta function) of the even self-dual lattice describing the toroidal compactification,

$$Z_{d,d} = V_d \sum_{m_i, n_i} e^{-\pi \tau_2 (m_i^2 + \tau n_i^2)(g_{ij} + B_{ij})(m_j^2 + \tau n_j^2)} = (\tau_2)^{d/2} \sum_{m_i, n_i} e^{-\pi \tau_2 M^2(V) - 2\pi i \tau_1 m \cdot m}$$  \hspace{1cm} (3.17a)

This is for instance the case for $R^4$ couplings in type II strings on $T^d$, or $R^2$ or $F^2$ couplings in type II on $K_3 \times T^2$. In the above formula, a Poisson resummation on the integers $m_i$ takes from the Lagrangian representation, manifestly invariant under the genus 1 modular group, to the Hamiltonian representation, manifestly invariant under T-duality.

It is natural to expect a connection between this one-loop modular integral and the $SO(d, d, \mathbb{Z})$ Eisenstein series defined above. As is well known, the $\tau$-integral can be carried out by the method of orbits, which corresponds to a large volume expansion of the integral. This was first carried out in [33] and extended to higher dimensional tori in [10, 14]. We will briefly review these results for later comparison with the Eisenstein series.

In order to carry out the integral on the fundamental domain of the upper half plane, one uses the fact that an $SL(2, \mathbb{Z})$ modular transformation on $\tau$ can be reabsorbed by an $SL(2, \mathbb{Z})$ action on the doublet $(m^i, n^i)$: one can thus restrict the sum over $(m^i, n^i)$ to a sum over their $SL(2, \mathbb{Z})$ orbits, while unfolding the integration to a larger domain depending on the centralizer of the orbit. The orbits can be classified by defining the sub-determinants,

---

This integral is divergent in the infrared region $\tau_2 \rightarrow \infty$ when $d \geq 2$. It may be regulated in many different ways, see e.g. [33] for $d = 2$. Different regulation schemes lead to results differing by an additive, moduli independent constant, which we ignore in our analysis.
\[ d^{ij} = \frac{1}{2}(m^i n^j - m^j n^i), \] so that \( d^{ij} \) is a \( d \times d \), rank 2 antisymmetric matrix. We then have the trivial orbit, \( m^i = n^i = 0 \), with a contribution

\[ I_{tr}^d = 2\pi V_d \int_{\mathcal{F}} \frac{d^2 \tau}{\tau_2^2} = \frac{2\pi^2}{3} V_d ; \]

the degenerate orbits, with all \( d \)'s being zero: in this case we can set \( n^i = 0 \), unfold the integration domain \( \mathcal{F} \) onto the strip \( \tau_1 \in [-\frac{1}{2}, \frac{1}{2}] \), \( \tau_2 \in \mathbb{R}^+ \), and carry out the integrals:

\[ I_d^d = 2 V_d \sum_{m^i} \frac{1}{m^i g_{ij} m^j} = 2 V_d \mathcal{E}_{s=1}^{SL(2,\mathbb{Z})} (g_{ij}) ; \]

the non-degenerate orbits, where at least one of the \( d^{ij} \) is non-zero. The \( SL(2,\mathbb{Z}) \) modular action can be completely fixed in order to unfold the integration domain to twice the upper-half plane. After Gaussian integration on \( \tau \), we obtain:

\[ I_{n.d}^d = 4\pi V_d \sum_{m^i, n^i} \exp \left[ -2\pi \sqrt{(m \cdot g \cdot m)(n \cdot g \cdot n) - (m \cdot g \cdot n)^2 + 2\pi i B_{ij} m^i n^j} \right] / \sqrt{(m \cdot g \cdot m)(n \cdot g \cdot n) - (m \cdot g \cdot n)^2} \]

The summation is performed over all sets of \( 2n \) integers, having at least one non-zero \( d^{ij} \), modded out by the \( SL(2,\mathbb{Z}) \) modular action (for \( d = 1 \), this is \( m > 0, 0 \leq n < m \)). These terms are all exponentially suppressed at large \( V_d \), albeit not in a uniform way.

For low-dimensional cases, the sum can be further simplified, and yields the well-known results:

\[ I_1 = \frac{2\pi^2}{3} \left( R + \frac{1}{R} \right) , \quad I_2 = -2\pi \log(T_2 U_2)|\eta(T)\eta(U)|^4 \]

It is remarkable that these results can be rewritten in terms of \( SO(d, d, \mathbb{Z}) \) Eisenstein series. Indeed, using the properties (3.15) and (2.19), we find

**Proposition 3** For \( d = 1, 2 \), the one-loop integral \( I_d \) in (3.16) can be rewritten as the sum of the \( SO(d, d, \mathbb{Z}) \) Eisenstein series of order 1 in the spinor and conjugate spinor representations:

\[ I_d = 2 \mathcal{E}_{s=1}^{SO(d,d,\mathbb{Z})} + 2 \mathcal{E}_{C,s=1}^{SO(d,d,\mathbb{Z})} \]

In particularly, the result is manifestly invariant under the extended T-duality \( O(d, d, \mathbb{Z}) \), where the extra generator exchanges the two spinors. We shall now substantiate a similar claim for \( d > 2 \), by showing that the two sides are eigenmodes of second order differential operators with the same eigenvalues, and that they also agree in various limits. At this point, we note that the fact that the two spinor representations contribute is in agreement with the invariance of the modular integral under the extended group \( O(d, d, \mathbb{Z}) \) which exchanges the two spinors. Besides, for \( d = 1 \) the vector Eisenstein series \( -\frac{\pi^2}{3} \mathcal{E}_{V,s=-1/2}^{SO(1,1,\mathbb{Z})} \) is an equally valid candidate.
3.4 A new second order differential operator

Given that our Eisenstein series are eigenmodes of the $SO(d, d)$ Laplacian \[3.11b\], we should ask about its action on the modular integral \[3.16\]. An explicit computation of the action of $\Delta_{SO(d,d)}$ on the integrand shows that the lattice sum satisfies the differential equation

$$\left[\Delta_{SO(d,d)} - 2\Delta_{SL(2)} + \frac{d(d-2)}{4}\right] Z_{d,d} = 0 \quad (3.23)$$

where $\Delta_{SL(2)} = \frac{1}{\tau^2} \left( \frac{\partial^2}{\partial \tau^2} + \frac{\partial^2}{\partial \nu^2} \right)$ is the Laplacian on the upper-half plane. Upon integrating by parts the second term, we get a boundary term which vanishes, so that the modular integral itself is an eigenmode of $\Delta_{SO(d,d)}$:

$$\Delta_{SO(d,d)} I_d = \frac{d(2-d)}{4} I_d \quad (3.24)$$

The modular integral $I_d$ is therefore degenerate with the $SO(d, d, \mathbb{Z})$ Eisenstein series

$$E_{SO(d,d,\mathbb{Z})}^{V,s=d/2-1}, \quad E_{SO(d,d,\mathbb{Z})}^{S,s=1}, \quad E_{SO(d,d,\mathbb{Z})}^{C,s=1}, \quad (3.25)$$
or their “duals”

$$E_{SO(d,d,\mathbb{Z})}^{V,s=d/2}, \quad E_{SO(d,d,\mathbb{Z})}^{S,s=d-2}, \quad E_{SO(d,d,\mathbb{Z})}^{C,s=d-2}, \quad (3.26)$$

We expect the functions in \(3.26\) to be related to the ones in \(3.25\) by a duality transformation analogous to \(2.7\), although we cannot prove this statement at present due to the presence of constraints.

Less expected however is the existence of a second differential operator $\Box_d$, involving only the metric, which also annihilates the integrand up to a total derivative:

$$\Box_d = \Delta_{Gl(d)} - \frac{1}{8} \left( g_{ij} \frac{\partial}{\partial g_{ij}} \right)^2 = \Delta_{SL(d)} + \frac{2-d}{8d} \left( g_{ij} \frac{\partial}{\partial g_{ij}} \right)^2 \quad (3.27)$$

where the $Gl(d)$ and $SL(d)$ Laplacians are given in \(2.4b\), \(2.5b\). Indeed an explicit computation shows that

$$\left[ \Box_d - \Delta_{SL(2)} + \frac{d(d-2)}{8} \right] Z_{d,d} = 0, \quad \text{i.e.} \quad \Box_d I_d = \frac{d(2-d)}{8} I_d. \quad (3.28)$$

The operator $\Box_d$ is non-invariant under $SO(d, d)$, but is invariant under complete inversion of the metric. \[16\]

\[16\] In Ref.\[34\] it was shown that the one-loop integral is an eigenfunction under a non-invariant second order operator $\Delta$ that involves both $g$ and $B$. The relation with $\Box_d$ in \(3.27\) is $\Box_d = \Delta_{SO} - \frac{1}{2} \Delta + \frac{d(d-2)}{8}$. The equation \(3.23\) involving $\Delta_{SO}$ was also given in Ref. \[14\].
This last property gives a strong constraint for the identification of the modular integral \( I_d \) with Eisenstein series. Indeed, one can show that the spinor Eisenstein series are eigenmodes of \( \Box_d \) for \( s = 1 \) only, whereas the vector is always an eigenmode:

\[
\Box_d \mathcal{E}^{SO(d,d,\mathbb{Z})}_{V,s} = \frac{s(s-d+1)}{2} \mathcal{E}^{SO(d,d,\mathbb{Z})}_{V,s} \quad (3.29a)
\]

\[
\Box_d \mathcal{E}^{SO(d,d,\mathbb{Z})}_{S,s=1} = \frac{d(2-d)}{8} \mathcal{E}^{SO(d,d,\mathbb{Z})}_{S,s=1} \quad (3.29b)
\]

\[
\Box_d \mathcal{E}^{SO(d,d,\mathbb{Z})}_{C,s=1} = \frac{d(2-d)}{8} \mathcal{E}^{SO(d,d,\mathbb{Z})}_{C,s=1} \quad (3.29c)
\]

In particular, we see that the spinor Eisenstein series of order \( s = 1 \) and \( s = d - 2 \) are distinct, even though they are degenerate under \( \Delta_{SO(d,d)} \). A peculiarity occurs for \( d = 4 \), where the spinor Eisenstein series is an eigenmode for all \( s \), whereas the conjugate spinor is an eigenmode for \( s = 1 \) only:

\[
\Box_4(V,s) = \Box_4(S,s) = \frac{s(s-3)}{2}, \quad \Box_4(C,s = 1) = -1 \quad (3.30)
\]

The three \( SO(4,4,\mathbb{Z}) \) Eisenstein series at \( s = 1 \) are therefore degenerate under both \( \Delta_{SO} \) and \( \Box_4 \), and we conjecture that they are actually equated by triality. The degeneracy is however lifted at \( s \neq 1 \).

Summarizing the results in this section, we see that the only candidates for representing the modular integral \((3.16)\) are the order \( s = 1 \) spinor and conjugate spinor series, together with the order \( s = d/2 - 1 \) vector series and their duals. In order to sort out these possibilities, we need to determine the behaviour of these invariant functions in various limits.

### 3.5 Large volume behaviour

The large volume limit of the modular integral \( I_d \) has already been obtained from the orbit decomposition. The behaviour of the Eisenstein series on the other hand can be obtained by Poisson resummation techniques similar to the ones described in Section 2, with the complication of the constraints. The actual computation is deferred to Appendix C and we present only the results, specializing to the relevant value of \( s \). In the case of the vector
representation, we are able to determine the complete large volume expansion:

\[
\mathcal{E}_{\mathbf{V}, s = \frac{d}{2} - 1}^{SO(d, d, \mathbb{Z})} = \frac{\pi^{\frac{d}{2} - 2}}{\Gamma\left(\frac{d}{2} - 1\right)} \left(V_d \, \mathcal{E}_{d, s = 1}^{SL(d, \mathbb{Z})}(g_{ij}) + \frac{\pi^2}{3} V_d + \right.

+ 2\pi V_d \sum_{m^i, n^i} \exp\left(-2\pi \sqrt{|(m \cdot g \cdot m)(n \cdot g \cdot n) - (m \cdot g \cdot n)^2| + 2\pi i B_{ij} m^i n^j}\right) \right) (3.31)
\]

Here the sum runs over non-degenerate \(SL(2, \mathbb{Z})\) orbits of \((m^i, n^i)\). Comparing with the expansion \(I_{\mathbf{tr}}^d + I_{d}^d + I_{d}^{n.d.}\) of the modular integral (3.16) in Equations (3.18)-(3.20), we see a complete matching and thus obtain the theorem

**Theorem 4** The integral \(I_d\) (3.16) of the \((d, d)\) lattice partition function on the fundamental domain of the moduli space of genus 1 Riemann surfaces is given for \(d \geq 3\) by the \(SO(d, d, \mathbb{Z})\) Eisenstein series of order \(s = d/2 - 1\) in the vector representation

\[
I_d = 2 \frac{\Gamma\left(\frac{d}{2} - 1\right)}{\pi^{\frac{d}{2} - 2}} \mathcal{E}_{\mathbf{V}, s = \frac{d}{2} - 1}^{SO(d, d, \mathbb{Z})} (3.32)
\]

This provides a convenient representation of the one-loop integral \(I_d\), manifestly invariant under T-duality. The Eisenstein series of order \(s = d/2\) in the vector representation is degenerate with the one above, but singular, so we ignore it here.

In the case of the Eisenstein series of order 1 in the spinor representations, the determination of the asymptotic behaviour is complicated by the presence of the constraints, and we have to content ourselves with the partial results

\[
\mathcal{E}_{\mathbf{S}, s = 1}^{SO(d, d, \mathbb{Z})} = V_d \, \mathcal{E}_{d, s = 1}^{SL(d, \mathbb{Z})}(g_{ij}) + \frac{\pi^2}{3} V_d + \ldots \quad (3.33a)
\]

\[
\mathcal{E}_{\mathbf{C}, s = 1}^{SO(d, d, \mathbb{Z})} = \frac{\pi^2}{3} V_d + \pi V_d \, \mathcal{E}_{d, s = 1}^{SL(d, \mathbb{Z})}(g_{ij}) + \ldots \quad (3.33b)
\]

to be compared with

\[
I_d = \frac{2\pi^2}{3} V_d + 2V_d \, \mathcal{E}_{d, s = 1}^{SL(d, \mathbb{Z})}(g_{ij}) + \ldots \quad (3.34)
\]

The second term in (3.33a) is correct for \(d \leq 3\), but we are not able to prove it explicitly for \(d > 3\), due to the presence of the constraints; there are also exponentially suppressed corrections that we did not write. The second term in (3.33b) denotes the Eisenstein series of \(SL(d, \mathbb{Z})\) in the antisymmetric representation, and appears only when \(d \geq 2\). For the particular order \(s = 1/2\), it is easy to check from (3.14) that this series has the same eigenvalue as the Eisenstein series of order 1 in the fundamental representation. For \(d = 2, 3\) we have also explicitly checked the equality of the two Eisenstein series, so that we are led to assert
Conjecture 5 For any $d$, the Eisenstein series of $SL(d,\mathbb{Z})$ in the antisymmetric 2-tensor representation [2] at the particular order $s = 1/2$ coincides with the Eisenstein series of order 1 in the fundamental representation:

$$E^{SL(d,\mathbb{Z})}[2]_{s=1/2} = \frac{1}{\pi} E^{SL(d,\mathbb{Z})}_{d,s=1}$$

(3.35)

Assuming this is true, we can now formulate our second claim for the one-loop threshold:

Conjecture 6 The integral (3.16) of the $(d, d)$ lattice partition function on the fundamental domain of the moduli space of genus 1 Riemann surfaces is given for $d \geq 3$ by the $SO(d, d, \mathbb{Z})$ Eisenstein series of order $s = 1$ in any of the two spinor representations:

$$I_d = 2 E^{SO(d,d,\mathbb{Z})}_{S; s=1} = 2 E^{SO(d,d,\mathbb{Z})}_{C; s=1}$$

(3.36)

This is to be contrasted with the $d = 1, 2$ case (3.22), where the two spinors contribute in order to enforce the $O(d, d, \mathbb{Z})$ invariance of the integral (3.16). When $d > 2$, we conjecture that the two Eisenstein series are equal for the particular order $s = 1$, so that a single series is sufficient to reproduce the threshold. For $d = 3$, this conjecture is actually a theorem, as follows from the computation of $R^4$ couplings in 7 dimensions [10]. For $d = 4$, the conjecture (3.36) together with the theorem (3.32) implies that the one-loop integral (3.16) is invariant under $SO(4, 4)$ triality, a fact not obvious from its representation as a theta function.

3.6 Asymmetric thresholds and elliptic genus

So far, we focused on symmetric thresholds of the type (3.16), which often appear for half-BPS saturated couplings in type II strings, and showed how they could be expressed in terms of Eisenstein series of the $SO(d, d, \mathbb{Z})$ duality group. For heterotic strings however, the BPS condition constrains only the left-movers to be in their ground states, and the amplitude usually involves all excitations of the right-moving oscillators. Here we want to investigate the possible relevance of Eisenstein series for these quantities. Even though the negative outcome can already be anticipated due to the issue of symmetry enhancement, this will allow us to establish some identities that may become useful in later studies.

One-loop BPS-saturated couplings for toroidal compactifications of the heterotic string can usually be written as the modular integral

$$T^{\text{het}} = \int_{\mathcal{F}} \frac{d^2 \tau}{\tau_2^2} Z_{d,d}(g, B; \tau) \mathcal{A}(F, R, \tau)$$

(3.37)
where the insertion $\mathcal{A}$ is an almost holomorphic modular form of weight 0, depending on the background gauge-field $F$ and curvature $R$ in the uncompact dimensions. By almost holomorphic, we mean that $\mathcal{A}$ can be expanded as a finite polynomial in $1/\tau^2$

$$\mathcal{A}(F, R, \tau) = \sum_{\nu=0}^{\nu_{\text{max}}} \frac{1}{\tau^{2\nu}} A^{(\nu)}(F, R, \tau)$$  \hspace{1cm} (3.38)

with $A^{(\nu)}(F, R, \tau)$ a meromorphic function in $q = e^{2\pi i \tau}$. The non-holomorphic contributions $\nu \geq 1$ come from back-reaction effects, or equivalently from contact terms at the boundary of moduli space. In all string applications, the coefficients $A^{(\nu)}$ have Laurent expansions with at most a simple pole in $q$, arising from the left-moving tachyon.

When the elliptic genus does not depend on the gauge fields, it is actually possible to switch on Wilson lines $Y$, giving an $SO(d, d+k, \mathbb{Z})$-invariant threshold

$$I_{d,k} = \int_{\mathcal{F}} \frac{d^2 \tau}{\tau^2} Z_{d,d+k}(g, B, Y; \tau) A_k(R, \tau)$$  \hspace{1cm} (3.39)

where

$$Z_{d,d+k} = (\tau_2)^{d/2} \sum_{m_i, p_I, n^I} \eta \left[ M^2(V) - 2\pi i \tau_1 v^I \right]$$  \hspace{1cm} (3.40a)

$$M^2(V) = v^I M_{d,k}(V) v, \quad v = (m_i, p_I, n^I), \quad i = 1 \ldots d, \quad I = 1 \ldots k$$  \hspace{1cm} (3.40b)

and $A_k$ is now an almost holomorphic modular form of weight $-k/2$. We can derive, also in this case, a set of second order partial differential equations satisfied by the lattice partition function $Z_{d,d+k}$. It is convenient to choose the following Iwasawa gauge, in the basis where the $SO(d, d+k)$ invariant tensor reads $\eta = \begin{pmatrix} 1_d & 1_k \\ 1_k & 1_d \end{pmatrix}$:

$$M_{d,k}(V) = \begin{pmatrix} 1 \\ Y^t & 1 \\ C^t & -Y & 1 \end{pmatrix} \cdot \begin{pmatrix} g^{-1} & 1_k \\ 1_k & g \end{pmatrix} \cdot \begin{pmatrix} 1 & Y & C \\ 1 & -Y^t \end{pmatrix}$$  \hspace{1cm} (3.41)

with $C = B - YY^t/2$ and $B$ antisymmetric, as a result of the $SO(d, d+k)$ constraint $M^t_{d,k} \eta M_{d,k} = \eta$. The right action by the $SO(d, d+k)$ elements

$$\begin{pmatrix} 1 & y & -yy^t/2 \\ 1 & -y^t \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & b \\ 1 & 1 \end{pmatrix}$$  \hspace{1cm} (3.42)

preserving the Iwasawa gauge generates a set of continuous Borel symmetries

$$Y \rightarrow Y + y, \quad B \rightarrow B + \frac{1}{2}(yY^t - Yy^t) \quad \text{or} \quad B \rightarrow B + b$$  \hspace{1cm} (3.43)
which reduces to a discrete subgroup at the quantum level. The Laplacian then takes the form
\[
\Delta_{SO(d,d+k)} = \frac{1}{4} g_{ik} g_{jl} \left[ \frac{\partial}{\partial g_{ij}} \frac{\partial}{\partial g_{kl}} + \frac{\partial}{\partial B_{ij}} \frac{\partial}{\partial B_{kl}} \right] + \frac{2 - k}{4} g_{ij} \frac{\partial}{\partial g_{ij}} + \Delta_Y \tag{3.44}
\]
where
\[
\Delta_Y = \frac{1}{2} g_{ij} \delta^{IJ} \left[ \frac{\partial}{\partial Y_{iI}} - \frac{1}{2} Y_{iI} \frac{\partial}{\partial B_{ij}} \right] \left[ \frac{\partial}{\partial Y_{jJ}} - \frac{1}{2} Y_{jJ} \frac{\partial}{\partial B_{ij}} \right] \tag{3.45}
\]
We may then show that the lattice partition function satisfies the following identities, generalizing (3.23) and (3.28),
\[
\left[ \Delta_{SO(d,d+k)} + \frac{d(d+k-2)}{4} - D \right] Z_{d,d+k} = 0 \tag{3.46a}
\]
\[
\left[ \Box_{d,k} + \frac{d(d+k-2)}{8} - \frac{1}{2} D \right] Z_{d,d+k} = 0 \tag{3.46b}
\]
where \(D\) is the modular-covariant second order differential operator acting on modular forms of weight \((k/2,0)\), and \(\Box_{d,k}\) generalizes the non-invariant operator (3.27):
\[
D = 4 \tau_2^2 \partial_\tau D_\tau, \quad D_\tau = \partial_\tau - i \frac{k}{4 \tau_2} \tag{3.47a}
\]
\[
\Box_{d,k} = \frac{1}{4} g_{ik} g_{jl} \frac{\partial}{\partial g_{ij}} \frac{\partial}{\partial g_{kl}} - \frac{1}{8} \left( g_{ij} \frac{\partial}{\partial g_{ij}} \right)^2 + \frac{d + 1 - k/2}{4} g_{ij} \frac{\partial}{\partial g_{ij}} + \frac{1}{2} \Delta_Y \tag{3.47b}
\]

3.7 Boundary term and symmetry enhancement

A quick glance at the partial differential equations (3.46) may lead us to the conclusion that the \(SO(d, d + k)\)-invariant one-loop integral (3.39) should again be an eigenmode of the operators \(\Delta_{SO(d,d+k)}\) and \(\Box_{d,k}\). This is wrong however, due to the presence in \(\mathcal{A}\) of the tachyonic pole in \(1/q\). This pole is usually killed by the integration on the strip \(\tau_1 \in [-1,1]\) (at large \(\tau_2\)), except at special points of the moduli space where the lattice contains a length 2 vector: the contribution \(q^l q^0\) from the lattice sum cancels the pole, which signals an enhancement of the gauge symmetry in space-time. In particular, using the identity (3.46) requires particular care for the boundary term
\[
- \int F \frac{d^2 \tau}{\tau_2} \frac{\partial}{\partial \tau} \left[ \frac{1}{\tau_2^\nu} A_k^{(\nu)}(F, R) \frac{\partial}{\partial \tau} (Z_{d,d+k}) \right] = \lim_{\tau_2 \to \infty} (\tau_2)^{d/2 - \nu - 1} \left[ A_k^{(\nu)}(F, R) \frac{Z_{d,d+k}}{\tau_2^{d/2}} \right] q^l q^0 \tag{3.48}
\]
The contribution from the constant term in \(\mathcal{A}\) and the ground state of \(Z_{d,d+k}/(\tau_2)^{d/2}\) yields a moduli-independent divergent (for \(d/2 - \nu - 1 > 0\)) term which implies a harmless non-harmonicity, whereas the pole term in \(\mathcal{A}\) generates a harmonic anomaly localized at enhanced symmetry points in the moduli space, clearly not captured by any candidate
Eisenstein series. Finally, the term integrated by parts now involves the descendant of the elliptic genus, as explained in [14]. For $d = 2$, the answer to this problem is well known: the threshold involves the automorphic form of $SO(2, 2 + k)$ constructed by Borcherds [35, 36] (see also [37, 14] in the physics literature). The evaluation of the modular integral (3.39) by the method of orbits gives a presentation of this form as an infinite product over a sublattice, and each term vanishes on a particular divisor of $[SO(2) \times SO(k)] \backslash SO(2, 2 + k)$ where the gauge symmetry is enhanced. It would be interesting to construct the generalization of these objects to $d > 2$, where the complex structure is not present (and find the analogue of the generalized prepotentials obtained in Ref.[14]) but we will not pursue this line here.

4 U-duality and non-perturbative $R^4$ thresholds

While Eisenstein series provide a nice way to rewrite one-loop integrals such as (3.16), their utility becomes even more apparent when trying to extend the perturbative computation into a non-perturbatively exact result. Indeed, a prospective exact threshold should reduce in a weak coupling expansion to a sum of T-duality invariant Eisenstein-like perturbative terms, plus exponentially suppressed contributions, and Eisenstein series of the larger non-perturbative duality symmetry are natural candidates in that respect. This approach was taken in [10] for $R^4$ couplings in type II theories toroidally compactified to 7, 8, 9 dimensions, where the technology of $SO(d, d, \mathbb{Z})$ Eisenstein series was hardly needed; here we would like to extend it to lower dimensional compactifications, in an attempt to understand non-perturbative effects in these cases as well.

4.1 $R^4$ couplings and non-renormalization

Four graviton $R^4$ couplings in maximally supersymmetric theories have been argued in dimension $D \geq 8$ to receive no perturbative corrections beyond the tree-level and one-loop terms, and we shall assume that this holds in lower dimensions as well. The tree level term is simply obtained by dimensional reduction of the ten-dimensional $2\zeta(3)e^{-2\phi}$ term found in [38], while the one-loop term was explicitly shown to be given by the modular integral (3.16), after cancellation of the bosonic and fermionic oscillators, so that

$$f_{R^4} = 2\zeta(3)\frac{V_d}{g_s^2} + I_d + \text{non pert.} \quad (4.1)$$

While the Ramond scalars are decoupled from the perturbative expansion by Peccei-Quinn symmetries, the full non-perturbative result should depend on all the scalars in the symmetric space $K \backslash E_{d+1(d+1)}(\mathbb{R})$, where $E_{d+1(d+1)}(\mathbb{R})$ is the maximally non-compact real form
(also known as the normal real form) of the series of classical simply laced Lie groups $E_2 = SL(2)$, $E_3 = SL(3) \times SL(2)$, $E_4 = SL(5)$, $E_5 = SO(5, 5)$ and exceptional Lie groups $E_6$, $E_7$, $E_8$ [39, 40]. It should furthermore be invariant under the discrete symmetry group $E_{d+1(d+1)}(\mathbb{Z})$ also known as the U-duality group [41], which arises from the T-duality $SO(d, d, \mathbb{Z})$ by adjoining the exchange of the eleventh M-theory direction with any perturbative direction. The moduli space $K \setminus E_{d+1(d+1)}(\mathbb{R})$ has the structure of a bundle on the manifold $[SO(d) \times SO(d)] \setminus SO(d, d, \mathbb{R})$ on which the Neveu-Schwarz scalars $\phi, g, B$ live, with a fibre transforming as a spinor representation of $SO(d, d, \mathbb{R})$ in which the Ramond scalars live. For $D \geq 8$, it was shown [6, 8] that the exact threshold is an eigenmode of the Laplacian on the full scalar manifold as a consequence of supersymmetry, and we shall also assume that this persists in lower dimensions.

As shown in conjecture 1, the one-loop contribution can be written as the order $s = 1$ $SO(d, d, \mathbb{Z})$ Eisenstein series in the spinor representation. On the other hand, the tree-level term can itself be represented as an Eisenstein series in the singlet representation, using the property

$$E_{1,s}^{G(\mathbb{Z})} = 2\zeta(2s)$$

valid for any $G$, which provides a natural representation for Apery’s transcendental number $\zeta(3)$. In analogy with the $D \geq 8$ case, we do not expect any further perturbative contribution. For $2 < d < 8$, the $R^4$ threshold should thus be an automorphic form of $E_{d+1(d+1)}(\mathbb{Z})$ with asymptotic behavior

$$f_{R^4} = \frac{V_d}{g_s^2} E_{1,s=3/2}^{SO(d,d,\mathbb{Z})} + 2 E_{S,s=1}^{SO(d,d,\mathbb{Z})} + \text{non pert.}$$

4.2 String multiplet and non-perturbative $R^4$ couplings

In order to propose a non-perturbative extension of this result, we therefore need to unify the singlet and spinor representations of $SO(d, d, \mathbb{Z})$ into a representation of $E_{d+1(d+1)}(\mathbb{Z})$. Remarkably, there is one, namely the string multiplet, corresponding to the leftmost node in the Dynkin diagram

$$\begin{array}{cccccccc}
\frac{1}{P_M} & & & & & & & \\
R_1 & - & R_2 & - & R_3 & - & R_4 & \cdots - \frac{1}{R_d+1} \\
\frac{1}{P_M} & & & & & & & \\
R_1 & - & R_2 & - & R_3 & - & R_4 & \cdots - \frac{1}{R_d+1} \\
\frac{1}{P_M} & & & & & & & \\
R_1 & - & R_2 & - & R_3 & - & R_4 & \cdots - \frac{1}{R_d+1}
\end{array}$$

where each node is labelled by the tension of the states transforming in the corresponding representation [42, 5]. The string multiplet is described by a collection of charges
\[ m^{[1]}, m^{[4]}, m^{[1;6]} \] describing the wrappings of membranes, five-branes and Kaluza–Klein monopoles respectively\(^\text{‡}\) with a BPS mass given by

\[ \mathcal{T}^2 = \frac{1}{l_M} (\tilde{m}^{[1]})^2 + \frac{1}{l^2_M} (\tilde{m}^{[4]})^2 + \frac{1}{l^3_M} (\tilde{m}^{[1;6]})^2. \] \hspace{1cm} (4.5)

The dressed charges are given by

\[
\begin{align*}
\tilde{m}^{[1]} &= m^{[1]} + C_3 m^{[4]} + (C_3 C_6 + \mathcal{E}_6) m^{[1;6]} \\
\tilde{m}^{[4]} &= m^{[4]} + C_3 m^{[1;6]} \\
\tilde{m}^{[1;6]} &= m^{[1;6]}
\end{align*}
\] \hspace{1cm} (4.6)

where \( C_3 \) and \( C_6 \) are the expectation value of the M-theory three-form and its dual, to be supplemented with an extra \( \mathcal{K}_1 \mathcal{K}_8 \) form in \( D = 3 \). See Ref. \[43\] for the \( d \leq 4 \) case and \[44, 5\] for the general \( d \) case. This amounts to an explicit partial Iwasawa decomposition of the symmetric spaces \( K \backslash E_{d+1(d+1)}(\mathbb{R}) \). The corresponding state preserves half the supersymmetries provided the following conditions are obeyed \[43, 5\]:

\[
\begin{align*}
k^{[5]} &= m^{[1]} m^{[4]} = 0 \hspace{1cm} (4.7a) \\
k^{[2;6]} &= m^{[1]} m^{[1;6]} + m^{[4]} m^{[4]} = 0 \hspace{1cm} (4.7b) \\
k^{[5;6]} &= m^{[4]} m^{[1;6]} = 0 \hspace{1cm} (4.7c)
\end{align*}
\]

The above constraints in turn transform as a U-duality multiplet, namely the three-brane multiplet \[5\]. For completeness, Table 4.1 lists the U-duality groups and string multiplets for any \( d \leq 6 \).

The decomposition of this \( E_{d+1(d+1)}(\mathbb{Z}) \) irreducible representation into \( SO(d, d, \mathbb{Z}) \) representations was carried out in \[44, 5\], and indeed gives a singlet \( m = m^s \), a spinor

\[ ^\text{‡} \text{For simplicity we restrict ourselves to the case } d \leq 6, \text{ i.e. } D \geq 4. \]

| \( D \) | \( d + 1 \) | U-duality group | irrep | \( SL(d + 1) \) content | \( SO(d, d) \) content |
|---|---|---|---|---|---|
| 10 | 1 | 1 | 1 | 1 |
| 9 | 2 | \( SL(2, \mathbb{Z}) \) | 2 | 2 | \( 1 + 1 \) |
| 8 | 3 | \( SL(3, \mathbb{Z}) \times SL(2, \mathbb{Z}) \) | (3, 1) | 3 | \( 1 + 2 \) |
| 7 | 4 | \( SL(5, \mathbb{Z}) \) | 5 | \( 4 + 1 \) | \( 1 + 4 \) |
| 6 | 5 | \( SO(5, 5, \mathbb{Z}) \) | 10 | \( 5 + 5 \) | \( 1 + 8_8 + 1 \) |
| 5 | 6 | \( E_6(6)(\mathbb{Z}) \) | \( 27 \) | \( 6 + 15 + \bar{6} \) | \( 1 + 16 + 10 \) |
| 4 | 7 | \( E_7(7)(\mathbb{Z}) \) | \( 133 \) | \( 7 + 35 + 28 + \ldots \) | \( 1 + 32 + (1 + 66) + \ldots \) |

Table 4.1: String multiplets of \( E_{d+1(d+1)}(\mathbb{Z}) \).
\[ S = (m', m^{sij}, m^s, m^{ijklm}), \]  
plus some other multiplets \( O \) when \( d \geq 4 \). In particular, for \( d = 4 \), there is an extra singlet \( O = m^{ijkl} \) of \( SO(4,4,\mathbb{Z}) \), and for \( d = 5 \) a vector \( O = (m^{ijkl}, m^{ijklmn}) \) of \( SO(5,5,\mathbb{Z}) \). The mass formula (4.5) is easily rewritten, for vanishing RR backgrounds, in terms of T-duality quantities, using the relations \( l_M^3 = g_s l_s^3 \), \( R_s = g_s l_s \):  
\[ T^2 = m^2 + \frac{V_d}{g_s^2} M^2(S) + \frac{V_d^2}{g_s^4} M^2(O) \]  
(4.8)

where we set \( l_s = 1 \) and \( M^2(O) \) is the usual T-duality invariant mass for a singlet \( (d = 4) \) or a vector \( (d = 5) \). Given this group theory fact, it is therefore quite tempting to consider the following non-perturbative generalization of (4.1):  

**Conjecture 7** The exact four-graviton \( R^4 \) coupling in toroidal compactifications of type II theory on \( T^d \), or equivalently M-theory on \( T^{d+1} \), is given, up to a factor of Newton’s constant, by the Eisenstein series of the U-duality group \( E_{d+1}(d+1)(\mathbb{Z}) \) in the string multiplet representation, with order \( s = 3/2 \):  
\[ f_{R^4} = \frac{V_{d+1}}{l_M^9} \xi_{\text{string}, s=3/2}^{E_{d+1}(d+1)(\mathbb{Z})} \]  
(4.9)

Here \( l_M \) is the eleven-dimensional Planck length, \( V_{d+1} = R_s V_d \) the volume of the M-theory torus \( T^{d+1} \). The quantity \( V_{d+1}/l_M^9 = l_P^{d-8} \) is the U-duality invariant gravitational constant in dimension \( D = 10 - d \). As an immediate check, the proposal has the appropriate scaling dimension \( d + 1 - 9 + 3 \times 2 \) for an \( R^4 \) coupling in dimension \( D = 10 - d \).  

### 4.3 Strings, particles and membranes

Before showing how this conjecture reproduces the tree-level and one-loop terms, a few remarks are in order. Firstly, our claim reduces to the Green-Gutperle conjecture (1.1) in the \( d = 1 \) case of M-theory on \( T^2 \), or equivalently \( D=10 \) type IIB; it also contains the \( D = 7, 8 \) extension of [10] where the string multiplet transforms as a \((3, 1)\) and \(5\) of \( SL(3, \mathbb{Z}) \times SL(2, \mathbb{Z}) \) and \( SL(5, \mathbb{Z}) \) respectively, as well as the \( D = 6 \) proposal in [10], although in a refined way, since it is now a *constrained* Eisenstein series that is involved. This is needed to obtain an eigenmode of the Laplacian on the scalar manifold \( K \setminus E_{d+1}(d+1)(\mathbb{R}) \). Although such a requirement was strictly proved in \( D \geq 8 \) \([6, 8]\), it should very plausibly hold in lower dimensions. Using the general formula (2.11), we can compute the eigenvalue of the \( E_{d+1}(d+1)(\mathbb{Z}) \) Eisenstein series in the string, particle and membrane representations. These representations correspond to the leftmost, rightmost and upmost nodes in the Dynkin diagram (4.4) and can be labelled by \( SL(d + 1) \) charges as follows: The charges of the string multiplet are given in (4.5) while the particle and membrane multiplet have charges
m(1), m[2], m[5], m[1:7]... and m, m[3], m[1:5]... respectively and are listed in Tables 4.2 and 4.3. Using the weights given in [5] we can compute the eigenvalues under the Laplacian:

\[
\Delta_{E_{d+1}(d+1)} E_{\text{string};s} = \frac{s(4s - d^2 + d - 4)}{8 - d} E_{\text{string};s}
\]

\[
\Delta_{E_{d+1}(d+1)} E_{\text{particle};s} = \frac{s(2(9 - d)s + d^2 - 17d + 12)}{2(8 - d)} E_{\text{particle};s}
\]

\[
\Delta_{E_{d+1}(d+1)} E_{\text{membrane};s} = \frac{(d + 1)s(2s - 3d + 4)}{2(8 - d)} E_{\text{membrane};s}
\]

(See Appendix A.4, Eq. (A.21) for the explicit form of the Laplacian on the \( K\backslash E_{d+1}(d+1)(\mathbb{R}) \) scalar manifold). Substituting \( s = 3/2 \) in (4.10a) and noting that the U-duality invariant factor \( V_{d+1}/l_M^d = l_P^{d-8} \) is inert under the Laplacian, we obtain

**Corollary 8** \( R^4 \) couplings in M-theory compactified on a torus \( T^{d+1}, d \leq 7 \) and \( d \neq 2 \), are eigenmodes of the Laplacian on the symmetric space \( K\backslash E_{d+1}(d+1)(\mathbb{R}) \), with eigenvalue

\[
\Delta_{E_{d+1}(d+1)} f_{R^4} = \frac{3(d + 1)(2 - d)}{2(8 - d)} f_{R^4}.
\]

For \( d = 2 \), this formula does not apply, due to the harmonic anomaly (2.18). Property (4.11) could in principle be proved from supersymmetry arguments along the lines of [6, 8], and holds order by order in the the weak coupling expansion. In particular, the tree level contribution \( V_d/g_s^2 l_P^2 = e^{\frac{126}{d-2}} / l_P^2 \), albeit not U-duality invariant, is an eigenmode of \( \Delta_{E_{d+1}(d+1)} \) with the same eigenvalue as above, see Appendix A.4, Eq. (A.24).

| \( d + 1 \) | U-duality group | \( SL(d+1) \) content | \( SO(d, d) \) content |
|---|---|---|---|
| 10 | 1 | 1 | 1 |
| 9 | 2 | \( SL(2, \mathbb{Z}) \) | 3 | 2+ 1 |
| 8 | 3 | \( SL(3, \mathbb{Z}) \times SL(2, \mathbb{Z}) \) | (3, 2) | \( 3+ 3 \) |
| 7 | 4 | \( SL(5, \mathbb{Z}) \) | 10 | \( 4+ 6 \) |
| 6 | 5 | \( SO(5, 5, \mathbb{Z}) \) | 16 | \( 5+ 10+ 1 \) |
| 5 | 6 | \( E_{6(6)}(\mathbb{Z}) \) | 27 | \( 6+ 15+ 6 \) |
| 4 | 7 | \( E_{7(7)}(\mathbb{Z}) \) | 56 | \( 7+ 21+ 21+ 7 \) |

Table 4.2: Particle multiplets of \( E_{d+1}(d+1)(\mathbb{Z}) \)

Secondly, we assumed according to conjecture 1 that the Eisenstein series in the spinor of \( SO(d, d, \mathbb{Z}) \) reproduces the one-loop threshold; for \( d = 1, 2 \), this is incorrect, since we need also the conjugate spinor. However, the two contribute to two different kinematic
The Eisenstein series of Conjecture 9 representations of $SO(3,3)$ expansion, this would start as a one-loop term $f_\text{tree}$ are all degenerate with the $+$ structure, while the $-$ is given at one-loop only by the $SO(d, d, \mathbb{Z})$ Eisenstein series of order $s = 1$ in the conjugate spinor representation, and is U-duality invariant by itself.

Thirdly, we could have considered the representation (3.32) of the one-loop threshold structures $(t_8 t_8 \pm \epsilon_8 \epsilon_8/4) R^4$, and (4.9) is only concerned with the $+$ structure, while the $-$ is given at one-loop only by the $SO(d, d, \mathbb{Z})$ Eisenstein series of order $s = 1$ in the particle representation, yielding the correct one-loop term plus an extra (presumably tree-level) perturbative contribution. Indeed, it is easy to check that the Eisenstein series

\begin{align}
E^{E_{d+1(d+1)}(\mathbb{Z})}_{\text{string}}; s = 3/2 ; & E^{E_{d+1(d+1)}(\mathbb{Z})}_{\text{particle}}; s = d/2 - 1 ; & E^{E_{d+1(d+1)}(\mathbb{Z})}_{\text{membrane}}; s = 1 ; \\
E^{E_{d+1(d+1)}(\mathbb{Z})}_{\text{string}}; (d+1)(d-2)/4 ; & E^{E_{d+1(d+1)}(\mathbb{Z})}_{\text{particle}}; s = 3(d+1)/(9-d) ; & E^{E_{d+1(d+1)}(\mathbb{Z})}_{\text{membrane}}; s = 3(d-2)/2 ;
\end{align}

are all degenerate with $f_{R^4}$ under the Laplacian. It is thus quite tempting to conjecture

**Conjecture 9** The Eisenstein series of $E_{d+1(d+1)}(\mathbb{Z})$, $d > 2$, in the string multiplet representation at the particular order $s = 3/2$ is equal to the one in the particle multiplet of order $s = d/2 - 1$, and to the one in the membrane multiplet of order $s = 1$, up to numerical

| $D$ | $d + 1$ | U-duality group | irrep | $SL(d + 1)$ content | $SO(d, d)$ content |
|-----|---------|-----------------|-------|---------------------|--------------------|
| 10  | 1       | 1               | 1     | 1                   | 1                  |
| 9   | 2       | $SL(2, \mathbb{Z})$ | 1     | 1                   | 1                  |
| 8   | 3       | $SL(3, \mathbb{Z}) \times SL(2, \mathbb{Z})$ | (1, 2) | 1 + 1               | 2                  |
| 7   | 4       | $SL(5, \mathbb{Z})$ | 5     | 1 + 4               | 4 + 1              |
| 6   | 5       | $SO(5, 5, \mathbb{Z})$ | 16    | 1 + 10 + 5          | 8_C + 8_V          |
| 5   | 6       | $E_{6(6)}(\mathbb{Z})$ | 78    | 1 + 20 + 36 + ...   | 16 + (1+45)+16     |
| 4   | 7       | $E_{7(7)}(\mathbb{Z})$ | 912   | 1 + 35 + ...        | 32 + ...           |

Table 4.3: Membrane multiplets of $E_{d+1(d+1)}(\mathbb{Z})$
coefficients and powers of Newton’s constant:

\[
\frac{V_{d+1}}{l_M^9} \mathcal{E}_{\text{string}, s=3/2} = \frac{\Gamma(d/2 - 1)}{\pi^{d/2 - 2}} \mathcal{E}_{\text{particle}, s=d/2 - 1} = \frac{V_{d+1}}{l_M^9} \mathcal{E}_{\text{membrane}, s=1}
\]

(4.13)

Again, it is easy to check that the scaling dimensions match. Note that the restriction \(d > 2\) applies because we are making use of (3.36). For \(d = 3, 4\), this conjecture nicely checks with (2.7), (3.35), (3.36):

\[
\begin{align*}
\mathcal{E}_{5, s=3/2} &= \pi \mathcal{E}_{10, s=1/2} = \mathcal{E}_{5, s=1} \\
\mathcal{E}_{10, s=3/2} &= \mathcal{E}_{16, s=1} = \mathcal{E}_{10, s=1}
\end{align*}
\]

(4.14a)

up to factors of Newton’s constant, whereas \(d > 4\) gives new identities. The automorphic forms in (4.13) should give three different representations of the same \(R^4\) threshold in M-theory on \(T^{d+1}\).

4.4 Weak coupling expansion and instanton effects

Now, in order to justify the claim (4.9), we need to show that it reproduces the perturbative contributions in (4.1) in a weak coupling expansion. This is achieved as usual by a sequence of Poisson resummations on the integral representation

\[
f_{R^4} = \frac{V_d}{g_s^2} \frac{\pi^s}{\Gamma(s)} \int \int \frac{dt d\theta}{t^{1+s}} \sum \exp \left\{ -\frac{\pi}{t} \left[ m^2 + \frac{1}{g_s^2} \left( (\tilde{m}^i)^2 + V^2_d(m_i) \right) + \frac{V_d}{g_s^4} \tilde{m}^2 \right] + 2\pi i \theta (m\tilde{m} + m^i m_i) \right\}
\]

(4.15)

where the integral runs from 0 to \(+\infty\) for \(t\) and 0 to 1 for the Lagrange multiplier \(\theta\); the sum is on unrestricted integers, not vanishing all at the same time, and for definiteness we restricted to the \(d = 4\) case with vanishing RR fields, and defined \(m_i = \epsilon_{ijkl} m^{ijkl} / 3!\) and \(\tilde{m} = \epsilon_{ijkl} m^{ijkl} / 4!\). The leading contribution as \(g_s \to 0\) arises from the term \(m_i = m^i = \tilde{m} = 0\) with \(m \neq 0\), and reproduces the tree-level term in (4.11). After subtracting this term, the sum over \(m\) is now unrestricted, and we can Poisson resum on \(m\) using the formula (C.1):

\[
f_{R^4} = 2\zeta(2s) \frac{V_d}{g_s^2} + \frac{V_d}{g_s^2} \frac{\pi^s}{\Gamma(s)} \int \int \frac{dt d\theta}{t^{1+s}} \sum \exp \left\{ -\frac{\pi}{t} \left[ m^2 + \frac{1}{g_s^2} \left( (\tilde{m}^i)^2 + V^2_d(m_i) \right) + \frac{V_d}{g_s^4} \tilde{m}^2 \right] + 2\pi i \theta m^i m_i \right\}
\]

(4.16)

where we should substitute \(s = 3/2\). This now contains several contributions when \(\tilde{m} = 0\) (and therefore \(m_i, m^i\) not simultaneously zero): for \(m = 0\), we precisely recover the
Eisenstein series of order $s - 1/2 = 1$ in the spinor representation, whereas $m \neq 0$ contains non-perturbative $e^{-1/g_s}$ effects:

$$f_{R^4} = 2\zeta(2s)\frac{V_d}{g_s^2} + \left(\frac{V_d}{g_s^2}\right)^{\frac{3}{2}} \epsilon^{SO(d,d,\mathbb{Z})}_{S_{s-1/2}} \frac{g_s^2}{2\pi} \frac{2\pi^s}{\Gamma(s)} \sum_{m} \sum_{m',m_i} \delta(m_i m')$$

$$\left[ \frac{m^2 V_d}{(\tilde{m})^2 + V_d^2(m)^2} \right]^{\frac{2s-1}{4}} K_{s-\frac{1}{2}} \left( -\frac{2\pi|m|}{g_s} \sqrt{(\tilde{m})^2 + V_d^2(m)^2} \right) + \ldots \ (4.17)$$

Using the saddle point approximation (C.3) of the Bessel function at $s = 3/2$, we see that these non-perturbative terms can be interpreted as superposition of Euclidean D0 and D2-branes wrapped on a one-cycle $m^i$ or a three-cycle $\epsilon^{ijkl} m_l$ of $T^4$, preserving half of the supersymmetries ($m^i m_i = 0$) [30]. In addition to these terms, we have further contributions arising from $\bar{m} \neq 0$:

$$\left(\frac{V_d}{g_s^2}\right)^{\frac{3}{4}} \frac{2\pi^s}{\Gamma(s)} \int_0^1 d\theta \sum_{m} \sum_{m',m_i} \left[ \frac{m^2 g_s^2 V_d}{V_d^2 m^2 + g_s^2 (\tilde{m})^2 + g_s^2 V_d^2 (m)^2} \right]^{\frac{2s-1}{4}}$$

$$K_{s-\frac{1}{2}} \left( -\frac{2\pi|m| + \theta \tilde{m}|}{g_s} \sqrt{V_d^2 m^2 + g_s^2 (\tilde{m})^2 + g_s^2 V_d^2 (m)^2} \right) e^{2\pi i \theta m^i m_i} \ (4.18)$$

which behave superficially as $e^{-1/g_s^2}$. Such non-perturbative effects are certainly unexpected in toroidal compactifications to $D > 4$, since there are no half-BPS instanton configurations with this action (the NS5-brane does have a tension scaling as $1/g_s^2$, but it can only give rise to Euclidean configurations with finite actions when $D \leq 4$). Unfortunately, the infinite sum is not uniformly convergent ($|m + \theta n|$ can vanish at any rational value of $\theta$), so we cannot be positive about the existence of such effects at that stage. The matching of the tree-level and one-loop contributions together with the consistent interpretation of the D-brane contribution is however a strong support to our conjecture.

5 Higher genus integrals and higher derivative couplings

5.1 Genus $g$ modular integral

Having discussed the modular integrals arising in one-loop amplitudes, one may ask if our methods carry over to higher-loop amplitudes, which are notoriously difficult to evaluate. We shall not attempt to make any full-fledged higher-genus amplitude computation, but we

---

\footnote{One may carry out the Gaussian $\theta$ integration by summing over $m$ modulo $\tilde{m}$ only and then compute the sum over $m$, but this only takes us back to (4.15).}
will consider the higher-genus analogue of (3.16), namely the integral of a lattice partition function on the $3g-3$-dimensional moduli space of genus $g$ curves

$$I^g_d = \int_{\mathcal{M}_g} d\mu \, Z^g_{d,d}(g_{ij}, B_{ij}; \tau)$$

(5.1a)

$$Z^g_{d,d} = V^g_d \sum_{m_A, n_A \in \mathbb{Z}} \exp \left[ -\pi (g_{ij} + B_{ij})(m_A^i + \tau_{AB} n_A^i) \tau_2^{AC} (m_C^i + \tau_{CD} n_D^i) \right]$$

(5.1b)

Here, the integers $m_A^i, n_A^i$ denote the winding numbers along the cycles $\gamma_A$ and $\gamma^A$ of a symplectic basis of the homology lattice of the genus $g$ curve, and the period matrix $\tau_{AB}$, of positive definite imaginary part, describes the complex structure on the curve. $(m_A^i, n_A^i)$ transforms as a symplectic vector under $Sp(g, \mathbb{Z})$ which now plays the role of the modular group. $\mu$ is the modular invariant Weil-Peterson measure on the moduli space $\mathcal{M}_g$ of genus $g$ curves (see for instance [45] for a review). Except for $g = 1, 2$, $\tau_{AB}$ is a redundant parametrization of the Teichmüller space of dimension $(3g - 3)$, constrained by Schottky relations. Nonetheless, for our computation it will be convenient to consider it as a set of independent parameters living in the symmetric space $U(g) \backslash Sp(g, \mathbb{R})$, with partial Iwasawa decomposition

$$M(V) = \begin{pmatrix} \mathbb{I}_g & \mathbb{I}_g \\ \tau_1 & \mathbb{I}_g \end{pmatrix} \cdot \begin{pmatrix} \tau_2^{-1} \\ \mathbb{I}_g \end{pmatrix} \cdot \begin{pmatrix} \mathbb{I}_g & \tau_1 \\ \mathbb{I}_g & \mathbb{I}_g \end{pmatrix}$$

(5.2)

Note that the boost parameter $\tau_1$ is now symmetric, as imposed by the symplectic condition. From this it is straightforward (see Appendix A for the derivation) to determine an $Sp(g, \mathbb{R})$ invariant second order differential operator, namely the Laplacian on this manifold:

$$\Delta_{Sp(g)} = \frac{1}{4} \tau_2^{AC} \tau_2^{BD} \left( \frac{\partial}{\partial \tau_{1AB}} \frac{\partial}{\partial \tau_{1CD}} + \frac{\partial}{\partial \tau_{2AB}} \frac{\partial}{\partial \tau_{2CD}} \right)$$

(5.3)

which reduces to twice the $SL(2, \mathbb{R})$ Laplacian (1.3) for $g = 1$. An explicit computation along the same lines as before shows that the genus $g$ lattice sum continues to obey a partial differential equation

$$\left[ \Delta_{SO(d,d)} - \Delta_{Sp(g)} + \frac{dg(d-g-1)}{4} \right] Z^g_{d,d} = 0$$

(5.4)

The non-trivial step is now to integrate by parts the $\Delta_{Sp(g)}$ term. As we already emphasized, except in the $g = 1, 2$ case, the integration measure is not the $Sp(g, \mathbb{R})$-invariant measure on $\tau$-space, but its restriction to the solution of Schottky constraints. Nevertheless, we assume that the expression of $\Delta_{Sp(g)}$ in terms of the independent coordinates still yields

\footnote{Again, the derivatives w.r.t. to the symmetric matrices $\tau_1$ and $\tau_2$ are computed in terms of the diagonally rescaled matrices $(1 - \delta_{AB}/2)\tau_{1,2:AB}$.}
the appropriate Laplacian, and we can therefore integrate it by parts. Under this plausible assumption, we obtain

$$\Delta_{SO(d,d)} I_g^q = \frac{dg(g + 1 - d)}{4} I_d^g$$

(5.5)

Quite amazingly, comparison with (3.12) shows that this eigenvalue agrees with the order $s = g$ Eisenstein series in the spinor and conjugate spinor representation. We are therefore led to the

**Conjecture 10** The integral (5.1) of the $(d, d)$ lattice partition function on the fundamental domain of the moduli space of genus $g$ Riemann surfaces is given, up to an overall factor, by the $SO(d,d,\mathbb{Z})$ Eisenstein series of order $g$ in the spinor representation:

$$I_g^q \propto E^{SO(d,d,\mathbb{Z})}_{s=g} + E^{SO(d,d,\mathbb{Z})}_{C,s=g}$$

(5.6)

Note that the superposition of the two spinor representations is required by the $O(d,d,\mathbb{Z})$ invariance of the integrand. Normalizing (5.6) would require a knowledge of the Weil-Peterson volume of the moduli space of genus $g$ curves.

5.2 $N = 4$ topological string and higher derivative terms

The conjecture (5.6) is less substantiated than the 1-loop conjecture (3.36), since we do not have a second differential operator at our disposal, nor can we control the large volume limit of the lefthand side of (5.6). It is however strongly reminiscent of the genus $g$ partition function of the $N = 4$ topological string [46] on $T^2$, which was shown to be exactly given by the Eisenstein series of order $s = g$ in the spinor representation $E^{SL(2,\mathbb{Z})}_{s=g}(T)$ [47]. The precise result

$$F^q(u_L, u_R) \propto \sum_{(m,n)} |n + mT|^{2g-4} \left( \frac{u_L^+ u_R^+}{n + mT} + \frac{u_L^- u_R^-}{n + mT} \right)^{4g-4}$$

(5.7)

involves a set of harmonic variables $u$, with charge 1/2 under the R-symmetry $SO(2)$. This result was obtained from a set of first-order differential equations, which, loosely speaking, are nothing but the holomorphic half of our second-order differential equation (5.5). It was subsequently used to derive a set of higher derivative topological couplings $R^4 H^{4g-4}$ in type IIB string compactified over $T^2$ [48]. Our conjecture (5.6) suggests a natural generalization of these results to lower dimensions, which we shall now present.

The topological amplitude (5.7) can be identified with higher derivative couplings $R^4 H^{4g-4}$ in type IIB string theory on $T^2$ in the following way. The field-strength of
the Ramond two-forms $B_{\mu \nu}, D_{\mu \nu \rho \sigma}$ transform as a doublet $H_{RR}^1$ of $SL(2, \mathbb{R})$. Using the $SO(2) \backslash SL(2, \mathbb{R})$ two-bein $e_i^{\pm \pm}$, these two three-forms can be converted into an $SO(2)$ doublet $H_{RR}^{\pm \pm} = H_{RR}^1 e_i^{\pm \pm}$, and further contracted with the harmonic variables into an $SO(2)$ invariant $H_{RR}^1 = u_r^+ u_R^+ H_{RR} + u_{\tilde{r}}^+ u_{\tilde{R}}^+ H_{RR}$. Integrating (5.14) against $R^4 \hat{H}^{4g-4}$ in harmonic superspace yields the physical coupling

$$
\int d^8 x \sqrt{-\gamma} \sum_{p=2-2g} (-)^p R^4 (H_{RR}^{++})^{2g-2+p} (H_{RR}^{--})^{2g-2-p} \sum_{m,n} T^g \frac{\gamma}{(m+nT)^{g+p}(m+n\tilde{T})^{g-p}}
$$

Using the identity $(m+nT)H_{RR}^{++} - (m+n\tilde{T})H_{RR}^{--} = mH_{RR}^{2} - nH_{RR}^{1}$, we can rewrite the above result in the more suggestive way

$$
\int d^8 x \sqrt{-\gamma} \sum_{m,n} R^4 \frac{(m_i H_{RR}^{i})^{4g-4}}{(m_i M^{ij}(\mathbf{C}) m_j)^{3g-2}}
$$

where $M(\mathbf{C})$ is the mass matrix in the conjugate spinor representation $\mathbf{C}$ of $SO(2,2,\mathbb{Z})$. Indeed, $H^i$ transforms as a conjugate spinor under the T-duality group, while $m_i = (m, n)$ transforms in the dual way. More generally, in type IIB on $T^d$ the 2-form and 1-form potentials in the RR sector transform in the conjugate spinor and spinor representation of $SO(d,d)$ respectively, while in type IIA these two representations are interchanged.

Using the representation (5.9), the generalization of the $g$-loop $R^4 H^{4g-4}$ coupling to lower dimensions is then obvious: in type IIA variables,

**Conjecture 11** The $R^4 H^{4g-4}$ couplings between 4 gravitons and $4g-4$ Ramond three-form field-strengths in type IIA compactified on $T^d$, $d \leq 4$ are given at genus $g$ by the $SO(d,d,\mathbb{Z})$ constrained Eisenstein series in the spinor representation with insertions of $4g-4$ charges:

$$
I = \int d^{10-d} x \sqrt{-\gamma} \sum_{m} \delta(m \wedge m) e^{6(g-1)\phi} \frac{R^4 (m \cdot H_{RR}^{i})^{4g-4}}{(m \cdot M(\mathbf{S}) \cdot m)^{3g-2}}
$$

where $\phi$ is the T-duality invariant dilaton, related to the ten-dimensional coupling as $e^{-2\phi} = V_4 / g_s^2 l_s^4$, and we work in units of $l_s$. The restriction $d \leq 4$ is due to the fact that for $D = 5$ three-form field-strengths are Poincaré dual to two-form field-strengths, while for $D = 4$ they become part of the scalar manifold after dualization. A similar conjecture also holds for the coupling computed by the topological B-model [46],

**Conjecture 12** The $R^4 F^{4g-4}$ couplings between 4 gravitons and $4g-4$ Ramond two-form field-strengths in type IIA compactified on $T^d$, $d \leq 6$ are given at genus $g$ by the $SO(d,d,\mathbb{Z})$ parameters, and generalizes the usual $t_8 t_8 + \epsilon_8 \epsilon_8 / 4$ combination [48].

---

110 The precise contraction of the Lorentz indices is also obtained by dressing $\hat{H}_{RR}$ with Grassmann parameters, and generalizes the usual $t_s t_s + \epsilon_8 \epsilon_8 / 4$ combination [48].
constrained Eisenstein series in the conjugate spinor representation with insertions of $4g-4$ charges:

$$I = \int d^{10-d}x \sqrt{-\gamma} \sum_m \delta(m \wedge m) e^{6(g-1)\phi} \frac{R^4 (m \cdot F_{RR})^{4g-4}}{(m \cdot M(string) \cdot m)^{3g-2}}$$  \quad (5.11)

Here, the restriction $d \leq 6$ is due to the fact that for $D = 3$ two-forms field-strengths become part of the scalar manifold after Poincaré dualization. The relation between these two conjectures and the genus $g$ integral (5.6) is similar to the case of $(t_8t_8 \pm \epsilon_8\epsilon_8/4)R^4$ couplings in dimensions 8 or higher: the insertions of the vertex operators of the four gravitons and the $4g - 4$ two-forms $F_{RR}$ or three-forms $H_{RR}$ saturate the fermionic zero-modes and select one out of the two spinor contributions in the modular integral (5.6). The end results (5.10) and (5.11) involve covariant modular functions instead of invariant ones, but behave as Eisenstein series of order $3g-3$ up to a phase, that were also used in the context of non-perturbative type IIB string in [49, 50].

5.3 Non-perturbative $R^4H^{4g-4}$ couplings

Having put the $g$-loop amplitude in a manifestly T-duality invariant form (5.10), it is now straightforward to propose a non-perturbative completion, invariant under the full U-duality group. For that purpose, we note that the set of three-form field-strengths in M-theory compactified on $T^{d+1}$ fall into a representation of $E_{d+1(d+1)}$ dual to the string multiplet which already appeared in Section 4 (this is strictly speaking only correct for $D \geq 5$ as explained below (5.10)). The string multiplet decomposes under $SO(d, d, \mathbb{Z})$ into a singlet (the Neveu-Schwarz $H_{NS}$) a spinor (the Ramond three-forms obtained by reducing the M-theory four-form field-strength), as well as further terms for $d \geq 4$. It is therefore tempting to propose

**Conjecture 13** The $R^4H^{4g-4}$ couplings between 4 gravitons and $4g - 4$ three-form field-strengths in M-theory compactified on $T^{d+1}$, $d \leq 4$ are exactly given, up to a power of Newton’s constant, by the $E_{d+1(d+1)}(\mathbb{Z})$ constrained Eisenstein series in the string representation with insertions of $4g-4$ charges:

$$I = \frac{V_{d+1}}{l_s^d} \int d^{10-d}x \sqrt{-\gamma} \sum_m \delta(m \wedge m) \frac{R^4 (m \cdot H)^{4g-4}}{(m \cdot M(string) \cdot m)^{3g-2}}$$  \quad (5.12)

As an immediate check, we note that this proposal has the appropriate scaling dimension. The leading contribution arises by restricting the summation to $m^s \neq 0$ only, where $m^s$ is
the top charge in the string multiplet \( m \), contracted with the top three-form \( H^{NS} \):

\[
I = \frac{V_4}{g_s^{2/3}} \int d^{10-d}x \sqrt{-\gamma} \ 2\zeta (2g+1) \ R^4 \ H_N^{4g-4} + \ldots
\]  

(5.13)

corresponding to a tree-level interaction involving the Neveu-Schwarz three-form only. The next-to-leading contribution is obtained by Poisson resummation on the integer \( m \), and setting the dual integer to zero, as in our analysis of \( R^4 \) couplings. This has the effect of setting \( m \cdot H = m_{RR} \cdot H_{RR} \) (for vanishing value of the Ramond scalars) and shifting the order \( 3g - 3/2 \to 3g - 2 \). We thus reproduce the \( g \)-loop result (5.10). The analysis of non-perturbative effects is as in the \( R^4 \) case, and shows order \( e^{-1/g_s} \) D-brane effects as well as, for \( d \geq 4 \), contributions superficially of order \( e^{-1/g_s^2} \). More explicitly, in the simplest example of ten-dimensional type IIB theory, we obtain, in units of the 10D Planck length,

\[
\sum_{m,n} \left[ \frac{\tau_2}{m+n\tau_2} \right]^{3g-\frac{5}{2}} R^4 (mH_{NS} - nH_{RR})^{4g-4} = 2\zeta (2g+1) R^4 H_{NS}^{4g-4} \\
+ 2\sqrt{\pi\tau_2}^{3g-3} \frac{\Gamma(3g - 2)}{\Gamma(3g - \frac{5}{2})} \zeta (6g - 4) R^4 (H_{RR} - \tau_1 H_{NS})^{4g-4} + O(e^{-1/g_s})
\]  

(5.14)

Turning finally to the case of non-perturbative \( R^4 F^{4g-4} \) couplings, we note that the two-form field-strengths of M-theory compactified on \( T^{d+1} \) transform as the dual of the particle multiplet. The particle multiplet is the representation associated to the rightmost node in the Dynkin diagram (4.4) (see Ref. [5] for further details). It is thus quite natural to propose a non-perturbative completion as

\[
I = \int d^{10-d}x \sqrt{-\gamma} \sum_{m} \delta(m \wedge m) \frac{R^4 \ (m \cdot F)^{4g-4}}{(m \cdot M(\text{particle}) \cdot m)^{4g-5+\frac{d}{2}}}
\]  

(5.15)

where the power \( 4g - 5 + d/2 \) has been set by dimensional analysis. The particle multiplet decomposes as a vector and conjugate spinor of \( SO(d, d, \mathbb{Z}) \) in that order, so that this proposal implies a one-loop term given by the \( SO(d, d, \mathbb{Z}) \) Eisenstein series of order \( 2g - 3 + d/2 \) in the vector representation, plus a higher perturbative term which should reproduce the genus \( g \) term (5.11). Due to the presence of constraints, we are unfortunately not able to prove this statement at present. For \( g = 1 \), this conjecture is implied by the alternative form of the \( R^4 \) threshold in (4.13). Note that this proposal may in principle lift the difficulty raised by Berkovits and Vafa, who noted that in 8 dimensions the non-perturbative generalization of the genus \( g \) \( R^4 F^{4g-4} \) terms should include a mixing between the \( U(1) \backslash SL(2, \mathbb{R}) \) and \( SO(3) \backslash SL(3, \mathbb{R}) \) moduli [18]. Here the mixing is built-in since the particle multiplet transforms in the \( (3, 2) \) of \( SL(3, \mathbb{Z}) \times SL(2, \mathbb{Z}) \). Let us finally note that our techniques could also be used to generalize the conjectures about \( \nabla^{2k} R^4 \) and \( R^{3m+1} \) terms [51, 52], but the status of these is less clear.

37
6 Conclusions

Duality provides strong constraints on the non-perturbative extension of string theory. It is especially powerful in vacua with many supersymmetries, where physical amplitudes and low energy couplings have to be invariant under the symmetry group. For a restricted class of BPS saturated couplings, the supersymmetry constraints close into a set of partial differential equations, which together with perturbative boundary conditions allows to determine the result exactly. Such techniques have enabled us to obtain convenient representations of one-loop thresholds manifestly invariant under T-duality, to compute higher-genus amplitudes not tractable otherwise, and to propose an exact non-perturbative completion of $R^4 H^{4g-4}$ couplings in toroidal compactifications of M-theory. Upon expansion in weak coupling, these results reveal a tree-level and $g$-loop contribution, non-perturbative order $e^{-1/g}$ effects that can be attributed to Euclidean D-branes wrapped on various cycles of the internal torus, as well as further ill-understood non-perturbative effects superficially of order $e^{-1/g^2}$, appearing in dimension $D = 6$ and lower. It would be very interesting to ascertain the behaviour of these effects, and eventually give an instantonic interpretation for them. In $D = 4$ we expect such $e^{-1/g^2}$ effects from the Euclidean NS5-brane wrapped on $T^6$ which should be extracted from our conjecture (1.9). Finally, the generalization to $D \leq 2$ should involve Eisenstein-like series for affine Lie algebras and even hyperbolic Kac-Moody algebras.

We have focused in this work on half-BPS saturated couplings in maximally super-symmetric theories. It would be interesting to extend our techniques to (i) couplings preserving a lesser amount of supersymmetry, and (ii) half BPS states in theories with less supersymmetry. Given that the quadratic half-BPS constraint imposes second order differential equations and that the quarter-BPS condition is cubic in the charges, one may envisage that quarter-BPS saturated couplings should be eigenmodes of a cubic Casimir operator, and expressable as generalized Eisenstein series. As for the second issue, one has to face situations where the gauge symmetry can be enhanced at a particular point in the moduli space, a case where Eisenstein series seem to be of little relevance. The differential equations (3.46) and the generalized prepotentials of [14] should prove useful for constructing automorphic forms with the required singularity structure, generalizing [53, 37, 36, 54]. Particularly interesting cases include the toroidal compactifications of the heterotic string, where five-brane instantons are little understood; type IIB compactified on $K_3$, where the moduli space unifies the dilaton with the other scalars in a simple form $[SO(5) \times SO(21)]/SO(5, 21)$ and where tensionless strings appear at singularities of $K_3$; the FHSV model [29], where the duality group is broken to a subgroup of $SO(2,10,\mathbb{Z})$ by the freely acting orbifold construction.
On a more mathematical level, our results provide a wealth of explicit examples of modular functions on symmetric spaces of non-compact type $K \backslash G$, with $G$ a real simply laced Lie group in the normal real form, that generalize the Eisenstein series on the fundamental domain of the upper half-plane. These functions can be associated to any fundamental representation of $G$, and are eigenmodes of the Laplacian with an easily computable eigenvalue. From analyzing their asymptotics and their behaviour under the Laplace operator as well as some other differential operator, we have been able to obtain a number of relations between Eisenstein series in various representations, although we had to content ourselves with conjectures rather than proofs in several cases. This has shown that Eisenstein series may become equal for certain values of the order $s$, the most useful example being the equality of the vector, spinor and conjugate spinor Eisenstein series of $SO(d, d, \mathbb{Z})$ at $s = d/2 - 1$, $s = 1$ and $s = 1$ respectively. On the other hand, two Eisenstein series with the same eigenvalue under the Laplacian may still be separated by an extra differential operator, like $\Box_d$ in the $SO(d, d)$ case. We have not addressed the question of the analyticity of Eisenstein series with respect to the order $s$: this would require an asymptotic expansion analogous to (1.4) or (2.15) with a uniformly suppressed general term. Unfortunately, it seems that the presence of constraints tends to give rise to ill-behaved expansions such as (3.31). This problem is the mathematical counterpart of the physical one raised above, namely understanding the instanton effects that are superficially of order $e^{-1/g_s^2}$. It would be interesting to understand more precisely what Eisenstein series are needed to generate the spectrum of the Laplace operator for any eigenvalue (note in that respect that the order $s$ is no longer a good parametrization, since the relation between the eigenvalue and $s$ depends on the representation). From a mathematical point of view however, Eisenstein series are the least interesting part of the spectrum on such manifolds, which should also include a discrete family of cusp forms. Perhaps string theory will provide an explicit example of these elusive objects.

Acknowledgements: We are grateful to N. Berkovits, J. Bernstein, R. Borcherds, G. W. Moore, M. Petropoulos, E. Verlinde, J-B. Zuber and G. Zwart for useful discussions or correspondence, and especially to E. Kiritsis for participation at an early stage of this project. B.P. thanks Nordita and both of the authors the CERN Theory Division for their kind hospitality and support during the completion of this work.

Note added (Jan. 2010): We have corrected misprints in Eqs. 2.9, 2.11, 2.19, 4.3, and an error in the derivation in Sec. C.1 and C.3 of the large volume expansion of the Eisenstein series in the vector representation of $SO(d, d)$, and in the spinor representations of $SO(4, 4)$. This mistake does not affect the result for $s = d/2 - 1$, which is the case relevant for
threshold integrals. We also removed an erroneous conjecture below Eq. 2.12, and added a comment (footnote 4) about the need to regulate the integral (3.16).

Appendices

A Gl(d), SL(d), SO(d, d) and Sp(g) Laplacians

In this appendix we give some details of the derivation of the Laplacians (2.4b), (2.5b), (3.11b) and (5.3) on the scalar manifolds for the four cases of Gl(d), SL(d), SO(d, d) and Sp(g) symmetry, as well as some useful alternative forms. The Laplacians are computed from the general expression

\[ \Delta = \frac{1}{\sqrt{\gamma}} \partial_\mu \sqrt{\gamma} \gamma^{\mu\nu} \partial_\nu, \quad ds^2 = \gamma_{\mu\nu} dx^\mu dx^\nu = -\frac{1}{2} \text{Tr} \left( dM dM^{-1} \right) \]  

where \( \gamma \) is the bi-invariant metric on the symmetric space \( K \backslash G \), parametrized by the symmetric matrix \( M \).

A.1 Laplacian on the SO(d)\Gl(d, \mathbb{R}) and SO(d)\SL(d, \mathbb{R}) symmetric spaces

For the \( SO(d)\backslash Gl(d, \mathbb{R}) \) case, we can choose \( M = g \) a symmetric positive definite matrix, and the metric \( ds^2 \) and volume element take the form

\[ ds^2 = g^{ik} g^{jl} dg_{ij} dg_{kl}, \quad \det(ds^2) = 2^{\frac{d(d-1)}{2}} (\det g)^{-(d+1)} \]  

Its inverse is easily computed by ordering the indices,

\[ ds^\text{inv} = \sum_{i,j} g_{ij} g_{ij} dg^{ij} + \frac{1}{2} \sum_{i<j; k<l} (g_{ik} g_{jl} + g_{il} g_{jk}) dg^{ij} dg^{kl} + 2 \sum_{i<k<l} g_{ik} g_{il} dg^{ij} dg^{kl} \]  

and using the relation

\[ \frac{\partial \det g}{\partial g_{ij}} = (2 - \delta_{ij}) g^{ij} \det g. \]  

We find, after some algebra,

\[ \Delta_{Gl(d)} = \sum_{i \leq j; k \leq l} \frac{\partial}{\partial g_{ij}} g_{ik} g_{jl} \frac{\partial}{\partial g_{kl}} - \frac{d + 1}{2} \sum_{i \leq j} g_{ij} \frac{\partial}{\partial g_{ij}} \]  

which can also be put in the form

\[ \Delta_{Gl(d)} = \sum_{i \leq j; k \leq l} g_{ik} g_{jl} \frac{\partial}{\partial g_{ij}} \frac{\partial}{\partial g_{kl}} + \frac{d + 1}{2} \sum_{i \leq j} g_{ij} \frac{\partial}{\partial g_{ij}} \]
In order to avoid the cumbersome sums over ordered indices, it is convenient to introduce the diagonally rescaled metric

\[ \tilde{g}_{ij} = (1 - \delta_{ij}/2) g_{ij}. \]  

(A.7)

This then satisfies the properties

\[ \frac{\partial g_{ij}}{\partial \tilde{g}_{kl}} = \delta_i^k \delta_j^l + \delta_i^l \delta_j^k, \quad \frac{\partial \det g}{\partial \tilde{g}_{ij}} = 2 g^{ij} \det g \]  

(A.8)

which allows to write the above Laplacian in the covariant form

\[ \Delta_{Gl(d)} = \frac{1}{4} g_{ik} g_{jl} \frac{\partial}{\partial g_{ij}} \frac{\partial}{\partial g_{kl}} + \frac{d + 1}{4} g_{ij} \frac{\partial}{\partial g_{ij}} \]  

(A.9)

where now repeated indices are summed over without further restrictions. This is the form given in (2.4b), where we omitted the tilde on the redefined metric as done throughout the text of the paper for simplicity of notation.

To compute the Laplacian on the \( SO(d) \backslash SL(d) \) symmetric space from this, we decompose the element \( g \) of \( Gl(d) \) as \( g = t\tilde{g} \), with \( \det \tilde{g} = 1 \). The metric then takes the form

\[ ds_{Gl}^2 = ds_{Sl}^2 + d \left( \frac{dt}{t} \right)^2, \quad \sum_{i<j} g_{ij} \frac{\partial}{\partial g_{ij}} = t \partial_t \]  

(A.10)

so that the Laplacian reads

\[ \Delta_{Gl(d)} = \Delta_{Sl(d)} + \frac{1}{d} t \partial_t t \partial_t = \Delta_{Sl(d)} + \frac{1}{4d} \left( g_{ij} \frac{\partial}{\partial g_{ij}} \right)^2 \]  

(A.11)

Together with (A.9), this yields the result (2.5b) for the \( SL(d) \) Laplacian.

A.2 Laplacian on the \([SO(d) \times SO(d)] \backslash SO(d, d, \mathbb{R})\) symmetric space

Next, we turn to the Laplacian on the symmetric space \([SO(d) \times SO(d)] \backslash SO(d, d)\) of dimension \( d^2 \). We can choose the symmetric moduli matrix \( M \) as in (3.4), so that the metric in (A.1) reads

\[ ds^2 = g^{ij} g^{kl} (dg_{ij} dg_{kl} + dB_{ij} dB_{kl}) \]  

(A.12)

This is a fibration on the coset \( SO(d) \backslash Gl(d) \), so we only need to compute the Laplacian on the fiber. We order the indices as

\[ ds_{B}^2 = 2 \sum_{i<j<k<l} \left( g^{ik} g^{jl} - g^{il} g^{jk} \right) dB_{ij} dB_{kl} \]  

(A.13)
The determinant of the metric on the fiber is \( \gamma_B = 1/(\det g)^{d-1} \), up to an irrelevant numerical factor. Using (A.2), the volume form on the total manifold is therefore \( \sqrt{\gamma} = (\det g)^{-d} \).

The inverse metric reads

\[
ds_{B, \text{inv}}^2 = \frac{1}{2} \sum_{i<j;k<l} (g_{ik}g_{jl} - g_{il}g_{jk}) dB_{ij} dB_{kl} \tag{A.14}
\]

so that the Laplacian on the fiber is given by

\[
\Delta_B = \frac{1}{2} \sum_{i<j;k<l} (g_{ik}g_{jl} - g_{il}g_{jk}) \frac{\partial}{\partial B_{ij}} \frac{\partial}{\partial B_{kl}} \tag{A.15}
\]

where we let \( \partial B_{ij}/\partial B_{kl} = \delta_i^k \delta_j^l - \delta_i^l \delta_j^k \). Putting this together with the Laplacian (A.6) on the base (with the appropriate volume element), we find

\[
\Delta_{SO(d,d)} = \sum_{i\leq j;k\leq l} g_{ik}g_{jl} \frac{\partial}{\partial g_{ij}} \frac{\partial}{\partial g_{kl}} + \sum_{i\leq j} g_{ij} \frac{\partial}{\partial g_{ij}} + \frac{1}{4} \sum_{ijkl} g_{ik}g_{jl} \frac{\partial}{\partial B_{ij}} \frac{\partial}{\partial B_{kl}} \tag{A.16}
\]

where the sum in the last term runs over unconstrained indices. An alternative form using the redefined metric (A.7) is

\[
\Delta_{SO(d,d)} = \frac{1}{4} g_{ik}g_{jl} \left[ \frac{\partial}{\partial g_{ij}} \frac{\partial}{\partial g_{kl}} + \frac{\partial}{\partial B_{ij}} \frac{\partial}{\partial B_{kl}} \right] + \frac{1}{2} g_{ij} \frac{\partial}{\partial g_{ij}} \tag{A.17}
\]

which is the one given in (3.11b). The \( SO(d,d+k) \) Laplacian (3.44) can be computed using similar techniques, but we will not give the details of this computation here.

### A.3 Laplacian on the \( U(g)\backslash Sp(g, \mathbb{R}) \) symmetric space

Next, we derive the Laplacian on the \( U(g)\backslash Sp(g) \) symmetric space, relevant for the genus \( g \) amplitude in (5.1). Using for \( M \) the moduli matrix (5.2), the metric in (A.1) takes the form

\[
ds^2 = \tau_2^{AC} \tau_2^{BD} (d\tau_{1AB}d\tau_{1CD} + d\tau_{2AB}d\tau_{2CD}) , \tag{A.18}
\]

This is again a fibration on the coset \( SO(g)\backslash Gl(g) \), so again we only need to compute the Laplacian on the fiber. The determinant of the metric on the fiber is \( \gamma|_{\tau_1} = 1/(\det \tau_2)^{g+1} \), up to an (irrelevant) numerical factor, so that, using (A.2) the volume form on the total manifold is \( \sqrt{\gamma} = (\det \tau_2)^{-(g+1)} \). With the known result (A.6) for the \( Gl(d) \) Laplacian, we then obtain

\[
\Delta_{Sp(g)} = \sum_{A \leq B; C \leq D} \frac{\partial}{\partial \tau_{2AB}} \tau_{2AC} \tau_{2BD} \frac{\partial}{\partial \tau_{2CD}} + \tau_{2AC} \tau_{2BD} \sum_{A \leq B; C \leq D} \frac{\partial}{\partial \tau_{1AB}} \frac{\partial}{\partial \tau_{1CD}} - (g+1) \sum_{A \leq B} \tau_{2AB} \frac{\partial}{\partial \tau_{2AB}} \tag{A.19}
\]
Diagonally rescaling $\tau_1$ and $\tau_2$ as before gives the more compact and covariant expression

$$\Delta_{Sp(g)} = \frac{1}{4} T_{2AC} T_{2BD} \left( \frac{\partial}{\partial \tau_{1AB}} \frac{\partial}{\partial \tau_{1CD}} + \frac{\partial}{\partial \tau_{2AB}} \frac{\partial}{\partial \tau_{2CD}} \right)$$  \hspace{1cm} (A.20)

which is the form given in (5.3) and reduces to half the usual Laplacian on the Poincaré upper half-plane for $g = 1$.

A.4 Laplacian on the $K\backslash E_{d+1}(d+1) \mathbb{R}$ symmetric space

We finally give here also the Laplacian on the the scalar manifold $K\backslash E_{d+1}(d+1) \mathbb{R}$ of eleven-dimensional supergravity on $T^{d+1}$ (equivalently type IIA string theory on $T^d$). In this case, the scalars are given by the metric $g_{IJ}$, $I = 1 \ldots d + 1$, a three-form $C_{IJK}$ and its dual $E_6$ (and for $D = 11 - (d + 1) \leq 3$ an extra $K_{1,8}$-form, which will not be included below). For $d \leq 6$, the corresponding Laplacian is given by

$$\Delta_{E_{d+1}(d+1)} = \frac{1}{4} g_{IK} g_{JL} \frac{\partial}{\partial g_{IJ}} \frac{\partial}{\partial g_{KL}} + \frac{(d + 7)(d - 4)}{4(d - 8)} g_{IJ} \frac{\partial}{\partial g_{IJ}} + \frac{1}{4(8 - d)} \left( g_{IJ} \frac{\partial}{\partial g_{IJ}} \right)^2 \hspace{1cm} (A.21)$$

The eigenvalues (4.10) of the Eisenstein series of the particle, string and membrane multiplet can be checked explicitly from this form using the mass formulae of these multiplets and the techniques employed in Appendix B. To this end it is important to express the 11D Planck length $l_M$, which is not invariant under the U-duality group $E_{d+1(d+1)}(\mathbb{Z})$, in terms of the invariant Planck length $l_P$ using the relation $V_{d+1}/l_M^d = l_P^{d-8}$.

Note also that, since for $d \leq 4$ the U-duality groups $E_{d+1(d+1)}(\mathbb{Z})$ are of the $Sl$ and $SO$ type the Laplacian above should reduce to the corresponding forms by appropriate redefinition of the scalars. For $d = 5, 6$, with U-duality group $E_6, E_7$ the above Laplacian is not contained in the previous results.

It is useful to determine the T-duality decomposition of the Laplacian (A.21). For that purpose, we compute the kinetic terms of the scalars in the Kaluza–Klein reduction of ten-dimensional type IIA theory. Going to the Einstein frame $g \rightarrow e^{4\phi/(8 - d)} g$, where
\[ e^\phi = g_s / \sqrt{V_d} \] is the invariant dilaton, we find
\[ S = \int d^{10-d}x \sqrt{-g} \left[ R + \frac{4}{8-d} \partial \phi \partial \phi - \frac{1}{4} \partial g \partial g^{-1} + \frac{1}{4} \partial g^{-1} \partial B g^{-1} + \frac{e^{2\phi}}{2} \partial \mathcal{R} \cdot M(S) \cdot \partial \mathcal{R} + \ldots \right] \] (A.22)

Here, \( \mathcal{R} \) denote the Ramond scalars transforming in the spinor representation of \( \text{SO}(d,d) \), and the dots stand for extra scalars which originate from dualizing the Kaluza–Klein one-form, Neveu-Schwarz two-form or Ramond forms in \( d \geq 5 \). From the property
\[ \sum_{k=\text{even}} k \binom{d}{k} = \sum_{k=\text{odd}} k \binom{d}{k} = d 2^{d-2} , \] (A.23)
it follows that the mass matrix \( M(S) \), like \( M(C) \), has unit determinant. The volume element is thus given by \( \sqrt{\gamma} = e^{2d-1\phi} \) (for \( d < 5 \) and in fact also \( d = 5 \)), and the Laplacian on the symmetric space \( K \setminus E_{d+1(d+1)}(\mathbb{R}) \) then reads, in variables appropriate for T-duality,
\[ \Delta_{E_{d+1(d+1)}} = \frac{8 - d}{16} (\partial_\phi^2 + 2^{d-1} \partial_\phi) + \Delta_{\text{SO}(d,d)} + \frac{e^{-2\phi}}{2} \partial \mathcal{R} \cdot M^{-1}(S) \cdot \partial \mathcal{R} + \ldots \] (A.24)

From this we can for example check that the Einstein-frame tree-level \( R^4 \) term \( e^{12\phi/(d-8)} \), or the one-loop term \( e^{2(d-2)\phi/(d-8)} \mathcal{E}_{\text{SO}(d,d,\mathbb{Z})}^{S,C,s=1} \) are eigenmodes of the U-duality invariant Laplacian as required by the conjecture (4.11).

### A.5 Decompactification of the Laplacians

We conclude by giving the decompactification formulae for the \( \text{Gl}(d) \) and \( \text{SO}(d,d) \) Laplacians. These are relevant for the study of the decompactification properties of the corresponding Eisenstein series.

We will consider only the \( \text{SO}(d,d) \) case, since the resulting formulae for \( \text{Gl}(d) \) and \( \text{SL}(d) \) can easily be obtained from this case. For the metric we take the \( U(1) \)-fibered form
\[ dx^i g_{ij} \ dx^i = R^2(dx^1 + A_a dx^a)^2 + dx^a \hat{g}_{ab} dx^b \] (A.25)
where \( a = 2, \ldots d \) and the original metric is \( g_{ij} \). We also define
\[ B_{1a} = B_a , \quad B_{ab} = \hat{B}_{ab} + \frac{1}{2} [A_a B_b - A_b B_a] \] (A.26)
In terms of these variables, T-duality takes the simple form
\[ R \leftrightarrow \frac{1}{R} , \quad e^\phi \leftrightarrow e^\hat{\phi} , \quad A_a \leftrightarrow B_a , \quad (\hat{g}_{ab}, \hat{B}_{ab}) \ \text{inv.} \] (A.27)
For the purpose of dimensional reduction it is, however, more convenient to introduce a modified $\tilde{B}_{ab}$ field invariant under gauge transformations of $A_a$ (but not under shifts of $B_a$):

$$B_{ab} = \tilde{B}_{ab} + A_a B_b - B_a A_b$$  \hspace{1cm} (A.28)

In the expressions below, we also use the diagonally rescaled metric (A.7) for $\hat{g}$ whenever it appears in derivatives. The Jacobian for the change of variables from $(g_{ij}, B_{ij})$ to $(R, A_a, \hat{g}_{ab}, B_a, \tilde{B}_{ab})$ is given by

$$\frac{\partial}{\partial g_{11}} = \frac{1}{2R} \frac{\partial}{\partial R} - \frac{A_a}{2R^2} \frac{\partial}{\partial A_a} + \frac{1}{2} \frac{A_a A_b}{2R^2} \frac{\partial}{\partial \hat{g}_{ab}} + \frac{A_a B_b}{R^2} \frac{\partial}{\partial B_{ab}}$$  \hspace{1cm} (A.29a)

$$\frac{\partial}{\partial g_{1a}} = \frac{1}{R^2} \frac{\partial}{\partial A_a} - \frac{A_a}{4R^2} \frac{\partial}{\partial \hat{g}_{ab}} - \frac{B_b}{R^2} \frac{\partial}{\partial B_{ab}}, \hspace{1cm} \frac{\partial}{\partial g_{ab}} = \frac{\partial}{\partial \hat{g}_{ab}}$$  \hspace{1cm} (A.29b)

$$\frac{\partial}{\partial B_{1a}} = \frac{\partial}{\partial B_a} + A_b \frac{\partial}{\partial \hat{g}_{ab}}, \hspace{1cm} \frac{\partial}{\partial B_{ab}} = \frac{\partial}{\partial \hat{B}_{ab}}$$  \hspace{1cm} (A.29c)

The Jacobian relevant for the $GL(d)$ Laplacian is simply obtained by ignoring the terms involving $B$. Then, we find for the $GL(d)$ Laplacian the decomposed result (2.21), while for $SO(d, d)$ we have after some algebra

$$\Delta_{SO(d+1,d+1)} = \Delta_{SO(d,d)} - \frac{1}{2} \hat{g}_{ab} \frac{\partial}{\partial \hat{g}_{ab}} + \frac{1}{4} \left(R \frac{\partial}{\partial R}\right)^2 + \frac{\hat{g}_{ab}}{2R^2} \frac{\partial}{\partial A_a} \frac{\partial}{\partial A_b}$$

$$+ \frac{R^2 \hat{g}_{ab}}{2} \frac{\partial}{\partial B_a} \frac{\partial}{\partial B_b} - \frac{1}{R^2} \hat{g}_{ab} B_c \frac{\partial}{\partial A_a} \frac{\partial}{\partial B_{bc}} + \frac{1}{2R^2} \hat{g}_{ab} B_c B_d \frac{\partial}{\partial B_{ab}} \frac{\partial}{\partial B_{cd}}$$  \hspace{1cm} (A.30)

We also note that the corresponding Jacobian for the change to the $(R, A_a, \hat{g}_{ab}, B_a, \tilde{B}_{ab})$ variables can be obtained from the one in (A.29) by substituting $\tilde{B} \rightarrow 2\tilde{B}$ except for the last equation. For completeness, we also give the decomposed $SO(d, d)$ Laplacian in these variables

$$\Delta_{SO(d+1,d+1)} = \Delta_{SO(d,d)} - \frac{1}{2} \hat{g}_{ab} \frac{\partial}{\partial \hat{g}_{ab}} + \frac{1}{4} \left(R \frac{\partial}{\partial R}\right)^2 + \frac{\hat{g}_{ab}}{2R^2} \frac{\partial}{\partial A_a} \frac{\partial}{\partial A_b} + \frac{R^2 \hat{g}_{ab}}{2} \frac{\partial}{\partial B_a} \frac{\partial}{\partial B_b}$$

$$+ \frac{1}{8} \hat{g}_{ac} \left(R^2 A_b A_d + \frac{B_b B_d}{R^2}\right) \frac{\partial}{\partial B_{ab}} \frac{\partial}{\partial B_{cd}} - \frac{1}{2} \hat{g}_{ab} \left(R^2 A_c \frac{\partial}{\partial B_a} + B_c \frac{\partial}{\partial A_a}\right) \frac{\partial}{\partial B_{bc}}$$  \hspace{1cm} (A.31)

which manifestly exhibits the T-duality symmetry (A.27).

**B  Eigenmodes and eigenvalues of the Laplacians**

In this appendix we give some details on the explicit computation of the eigenvalues under the Laplacian and the non-invariant differential operator (3.27) of the various Eisenstein
series and modular integral considered in the main text. These computations are most easily done using the integral representation

\[ [\mathcal{M}^2]^{-s} = \frac{\pi^s}{\Gamma(s)} \int_0^\infty \frac{dt}{t^{1+s}} \exp \left( -\frac{\pi}{t} \mathcal{M}^2 \right) \]  

(B.1)
of the generic term in the Eisenstein series. The result of differentiation can be integrated by parts using

\[ \int_0^\infty \frac{dt}{t^{1+s}} \left[ \alpha C^2 t^2 + \beta C t \right] e^{-C/t} = s(\alpha s + \alpha + \beta) \int_0^\infty \frac{dt}{t^{1+s}} e^{-C/t} \]  

(B.2)

B.1 \hspace{1em} SL(d, \mathbb{Z}) Eisenstein series in the fundamental representation

We start with the fundamental representation of \( SL(d, \mathbb{Z}) \), for which the mass matrix reads \( \mathcal{M}^2(d) = m^i g^m = m^i g_{ij} m^j \), and obeys the identities

\[ \frac{\partial \mathcal{M}^2(d)}{\partial g_{ij}} = 2m^i m^j, \quad \frac{\partial^2 \mathcal{M}^2(d)}{\partial g_{ij} \partial g_{kl}} = 0 \]  

(B.3)

so that using the Laplacian (2.4b), we obtain, setting \( t' = t/\pi \),

\[ e^{\mathcal{M}^2(d)/t'} \Delta_{G(d)} e^{-\mathcal{M}^2(d)/t'} = \frac{1}{t'^2} g_{ik} g_{jl}(m^i m^j)(m^k m^l) - \frac{d + 1}{4t'} g_{ij} 2m^i m^j \]

\[ = \frac{1}{t'^2} \left[ \mathcal{M}^2(d) \right]^2 - \frac{d + 1}{2t'} \mathcal{M}^2(d) \]  

(B.4)

Then, using the identity (B.2) we immediately find the eigenvalue \( s(s + 1 - d/2) \) as given in (2.4a). The corresponding eigenvalue under the \( SL(d) \) Laplacian follows by subtracting the \( (t\partial/\partial t)^2/d \) contribution in (A.11),

\[ \frac{1}{4d} e^{\mathcal{M}^2(d)/t'} \left( g_{ij} \frac{\partial}{\partial g_{ij}} \right)^2 e^{-\mathcal{M}^2(d)/t'} = \frac{1}{d} \left( \frac{1}{t'^2} \left[ \mathcal{M}^2(d) \right]^2 - \frac{1}{t'} \mathcal{M}^2(d) \right) \]  

(B.5)

so that the eigenvalue is \( s(s + 1 - (d + 1)/2) - s^2/d \) as given in (2.5a).

B.2 \hspace{1em} SO(d, d, \mathbb{Z}) Eisenstein series in the vector representation

For the case of the vector representation of \( SO(d, d) \), the mass matrix now reads

\[ \mathcal{M}^2(V) = m^i M(V)m = \tilde{m}_i g^{ij} \tilde{m}_j + n^i g_{ij} n^j, \quad \tilde{m}_i = m_i + B_{ij} n^j \]  

(B.6)

and satisfies

\[ \frac{\partial \mathcal{M}^2(V)}{\partial g_{ij}} = 2 \left[ -\tilde{m}_i \tilde{m}_j + n^i n^j \right], \quad \frac{\partial \mathcal{M}^2(V)}{\partial B_{ij}} = 2 \left[ \tilde{m}_i n^j - \tilde{m}_j n^i \right] \]  

(B.7)
where \( \tilde{m}^i = g^{ij}\tilde{m}_j \). To compute the action of the Laplacian (3.11b) and of the operator \( \Box_d \) in (3.27) we need the quantities

\[
D_0 = \left( \frac{1}{2} g_{ij} \frac{\partial M^2(V)}{\partial g_{ij}} \right)^2 = (\tilde{m}^2)^2 + (n^2)^2 - 2\tilde{m}^2n^2 \tag{B.8a}
\]

\[
D_1 = \frac{1}{4} g_{ik} g_{jl} \frac{\partial M^2(V)}{\partial g_{ij}} \frac{\partial M^2(V)}{\partial g_{kl}} = (\tilde{m}^2)^2 + (n^2)^2 - 2(\tilde{m}n)^2 \tag{B.8b}
\]

\[
D_2 = \frac{1}{4} g_{ik} g_{jl} \frac{\partial M^2(V)}{\partial B_{ij}} \frac{\partial M^2(V)}{\partial B_{kl}} = 2[\tilde{m}^2n^2 - (\tilde{m}n)^2] \tag{B.8c}
\]

as well as

\[
C_1 = \frac{1}{4} g_{ij} \frac{\partial}{\partial g_{ij}} g_{kl} = (d+1)\tilde{m}^2, \quad C_2 = \frac{1}{4} g_{ik} g_{jl} \frac{\partial^2 M^2(V)}{\partial B_{ij} \partial B_{kl}} = (d-1)n^2 \tag{B.9a}
\]

\[
C_0 = \frac{1}{4} g_{ij} \frac{\partial}{\partial g_{ij}} g_{kl} = \tilde{m}^2 + n^2, \quad C = \frac{1}{2} g_{ij} \frac{\partial M^2(V)}{\partial g_{ij}} = -\tilde{m}^2 + n^2 \tag{B.9b}
\]

Using these data it is then easy to compute

\[
e^{M^2(V)/t} \Delta_{SO(d,d)} e^{-M^2(V)/t} = \frac{1}{t^2} \left[ D_1 + D_2 \right] - \frac{1}{t} \left[ C_1 + C_2 + C \right] \tag{B.10a}
\]

\[
e^{M^2(V)/t} \Box_d \exp e^{-M^2(V)/t} = \frac{1}{t^2} \left[ (M^2(V))^2 - 4(mn)^2 \right] - \frac{d}{t'} M^2(V) \tag{B.10b}
\]

\[
e^{M^2(V)/t} \Box_d \exp e^{-M^2(V)/t} = \frac{1}{2t^2} \left[ (M^2(V))^2 - 4(mn)^2 \right] - \frac{d}{2t'} M^2(V) \tag{B.10c}
\]

The eigenvalues \( s(s - d + 1) \) and \( \frac{s}{2}(s - d + 1) \) of the vector Eisenstein series under the Laplacian \( \Delta_{SO(d,d)} \) and the non-invariant operator \( \Box_d \) follow by using the identity (B.2), provided the half-BPS constraint \( \tilde{m}n = mn = 0 \) is satisfied. These are the values quoted in (3.12) and (3.29) respectively.

### B.3 SO\((d,d,\mathbb{Z})\) Eisenstein series in the spinor representations

The direct computation of the eigenvalues of the (conjugate) spinor Eisenstein under the \( SO(d,d) \) Laplacian is more involved and will not be given here. The general results in (3.12) have been checked directly for \( d \leq 4 \), showing also in this case explicitly the importance of imposing the half-BPS constraints (3.7a) and (3.9a) that occur for \( d = 4 \).
Finally, we turn to the action of the $\Box_d$ operator on the (conjugate) spinor representation (3.6), (3.8) of $SO(d,d)$, with mass

$$
\mathcal{M}^2(S) = \frac{1}{V_d} \sum_{p=\text{odd}} \frac{(\tilde{m}^{[p]})^2}{p!}, \quad \mathcal{M}^2(C) = \frac{1}{V_d} \sum_{p=\text{even}} \frac{(\tilde{m}^{[p]})^2}{p!}
$$

(B.11)

where $\tilde{m}^{[p]} = m^{[p]} + B_2 m^{[p-2]} + B_4 m^{[p-4]} + \ldots$ are the dressed charges and $(\tilde{m}^{[p]})^2$ denotes the invariant square obtained with $p$ powers of the metric. We need the derivative

$$
\frac{\partial \mathcal{M}^2(S)}{\partial g_{ij}} = \frac{1}{V_d} \sum_p 2p[(\tilde{m}^{[p]})^2]_{ij} - (\tilde{m}^{[p]})^2 g_{ij}
$$

(B.12)

where $[(\tilde{m}^{[p]})^2]_{ij}$ denotes the invariant square with one power of the metric taken out. The direct computation of the full $\Box_d$ along the same lines as the cases treated above is rather intricate. We therefore employ a method that uses the underlying group theory and the realization that $\Box_d$ contains the $SL(d)$ Laplacian, as well as the structural form (3.7) of the constraints.

We first note that each term in (B.11) represents a totally antisymmetric tensor of $SL(d)$ with $p$ indices. For an antisymmetric $p$-tensor of $SL(d)$, the Casimir of the $r$th symmetric power is given by

$$
Q([p]^{\otimes r}) = \frac{rp(d-p)(r+d)}{d}
$$

(B.13)

so that according to the general formula (2.11) the action of the $SL(d)$ Laplacian is

$$
\Delta_{SL(d)}[(m^{[p]})^2] = \frac{p(d-p)s(2s-d)}{2d}[(m^{[p]})^2]^{-s},
$$

(B.14)

Using the identity (B.2), we therefore have, up to cross terms which we neglect for the moment,

$$
e^{\mathcal{M}^2(S)/t'} \Delta_{SL(d)} e^{-\mathcal{M}^2(S)/t'} = \\
\frac{1}{t'^2} \left[ \sum_p \frac{p(d-p)}{d} \left( \frac{(m^{[p]})^2}{V_d p!} \right)^2 + \text{cross} \right] - \frac{1}{t'} \left[ \sum_p \frac{p(d-p)(d+2)}{2d} \frac{(m^{[p]})^2}{V_d p!} \right]
$$

(B.15)

where we emphasize that this result is only valid when enforcing the quadratic constraints on the charges. We also need

$$
e^{\mathcal{M}^2(S)/t'} \left( \frac{1}{2 g_{ij}} \frac{\partial}{\partial g_{ij}} \right)^2 e^{-\mathcal{M}^2(S)/t'} = \frac{1}{t'^2} \left[ \sum_p \left( p - \frac{d}{2} \right) \frac{(m^{[p]})^2}{V_d p!} \right]^2 - \frac{1}{t'} \left[ \sum_p \left( p - \frac{d}{2} \right)^2 \frac{(m^{[p]})^2}{V_d p!} \right]
$$

(B.16)
obtained by direct calculation using (B.12). Using the form of \( \Box_d \) in (3.27) we obtain from the two expressions in (B.15) and (B.16) that

\[
e^{\mathcal{M}^2(S)/t'} \Box_d e^{-\mathcal{M}^2(S)/t'} = \frac{1}{t'^2} \left[ \sum_p \left( \frac{p(d-p)}{2} - \frac{d(d-2)}{8} \right) \left( \frac{(m_{[p]})^2}{V_d p!} \right)^2 + \text{cross} \right] + \frac{1}{t'} \left[ \sum_p \left( -p(d-p) + \frac{d(d-2)}{8} \right) \left( \frac{(m_{[p]})^2}{V_d p!} \right) \right] \tag{B.17}\]

Using (B.2) we then deduce that

\[
\frac{\Box_d (\mathcal{M}^2(S))^{-s}}{(\mathcal{M}^2(S))^{-s+2}} = s \left\{ (s + 1) \left[ \sum_p \left( \frac{p(d-p)}{2} - \frac{d(d-2)}{8} \right) \left( \frac{(m_{[p]})^2}{V_d p!} \right)^2 + \text{cross} \right] + \mathcal{M}^2(S) \sum_p \left( -p(d-p) + \frac{d(d-2)}{8} \right) \left( \frac{(m_{[p]})^2}{V_d p!} \right) \right\} \tag{B.18}\]

Requiring the righthand side to be proportional to (the diagonal terms in) \((\mathcal{M}^2(S))^2\) then shows us that (for generic value of \(d\)) this is only possibly when \(s = 1\), in which case we find that

\[
\Box_d (\mathcal{M}^2(S))^{-s} = \frac{d(2-d)}{8} (\mathcal{M}^2(S))^{-s}, \quad s = 1 \tag{B.19}\]

so that the \(s = 1\) spinor and conjugate spinor Eisenstein series are eigenmodes of \(\Box_d\) as recorded in (3.29b), (3.29c).

A special feature arises for the spinor representation of \(SO(4, 4)\), in which case we have \(p(d-p) = p(4-p) = 3\) for both the relevant values \(p = 1\) and \(3\) so that the terms in (B.18) are proportional to \((\mathcal{M}^2(S))^2\) for all values of \(s\). As a result we find that the spinor Eisenstein series for \(SO(4, 4)\) is an eigenmode of \(\Box_d\) with eigenvalue \(s(s-3)/2\) as noted in (3.30)\(^\ddagger\). This is not the case for the conjugate spinor representation of \(SO(4, 4)\).

Finally, we wish to point out some further checks on the cross terms that have been neglected so far. First, we have explicitly checked the full result by direct computation for the spinor representation in the cases \(d \leq 4\). In particular, for \(d = 4\) one finds, as expected that the constraint \(m^{[1]} \wedge m^{[3]} = 0\) of (3.7a) is crucial for the eigenvalue condition. More generally, using the metric on weight space \(g_{[p][q]} = p(d-q)/d\) with \([p] \leq [q]\) two totally antisymmetric \(SL(d)\) representations, we know from the group theory arguments in (2.9)\(^\ddagger\)  

\(^{\ddagger}\text{In a similar way one can see that the spinor and conjugate spinor Eisenstein series of }SO(2, 2)\text{ are eigenvalues for all }s, \text{ but that was expected since in that case }\Box_d \text{ reduces to the }SL(2)\text{ Laplacian.}\)
that the cross terms in (B.15) can be incorporated by replacing the $1/t^2$ term by

$$
\frac{1}{t^2} \left[ \sum_{p \leq q} (2 - \delta_{pq}) \frac{p(d - q)}{d} \left( \frac{(m^{[p]})^2}{V_d p!} \right) \left( \frac{(m^{[q]})^2}{V_d q!} \right) \right]
$$

(B.20)

Together with the directly computed cross terms in (B.16), this changes the $1/t^2$ term in (B.17) to

$$
\frac{1}{t^2} \sum_{p \leq q} (2 - \delta_{pq}) \left( \frac{d(p + q) + 2(p - q) - 2pq}{4} - \frac{d(d - 2)}{8} \right) \left( \frac{(m^{[p]})^2}{V_d p!} \right) \left( \frac{(m^{[q]})^2}{V_d q!} \right)
$$

(B.21)

which will produce an analogous correction to (B.18). For $s = 1$ we then see that, taking into account the cross terms from the second term in (B.18), the $(p, q)$-dependent part is given by

$$
(d(p + q) + 2(p - q) - 2pq) - (p(d - p) + q(d - q)) = (p - q)(p - q + 2)
$$

(B.22)

which we see vanishes (besides the diagonal terms $p = q$) for the cross terms $q - p = 2$.

If $q - p > 2$, there are non-trivial effects from the constraints. A simple way to see them is to consider the two-form in the $d = 4$ conjugate spinor. The $1/t^2$ contribution to $\Delta_{sl}$ includes a term $(m^{[2]})^4$, where the contraction is the non-factorized one. By the Cayley–Hamilton theorem for $4 \times 4$ antisymmetric matrices $A$,

$$
A^4 - \frac{1}{2} (\text{Tr} A^2) A^2 + (\text{Pfaffian} A)^2 1 = 0
$$

we see that this is $[(m^{[2]})^2]^2/2$ up to a $(m^{[2]} \wedge m^{[2]})^2$ term, which by the half-BPS condition (3.9a) is equivalent to an extra cross term $(m \ m^{[4]})^2$ that was not taken into account previously and will cancel the deficit seen in (B.22).

### C Large volume expansions of Eisenstein series

Here, we derive the results (3.31), (3.33) by considering the large volume expansions of the vector, spinor and conjugate spinor Eisenstein series of $SO(d, d)$. In the computations below we shall repeatedly use the Poisson resummation formula

$$
\sum_m e^{-\pi (m + a)^t A (m + a) + 2 \pi i m b} = \frac{1}{\sqrt{\det A}} \sum_{\tilde{m}} e^{-\pi (\tilde{m} + b)^t A^{-1} (\tilde{m} + b) - 2 \pi i (\tilde{m} + b) a}
$$

(C.1)

Note that an insertion of $m$ on the lefthand side translates into an insertion of $-a + iA^{-1}(\tilde{m} + b)$ on the righthand side. We also recall the integral representation of the Bessel function

$$
\int_0^\infty \frac{dx}{x^{1+s}} e^{-b/x - cx} = 2 \left| \frac{c}{b} \right|^{s/2} K_s(2 \sqrt{|bc|})
$$

(C.2)
It is an even function in \( s \), and admits the asymptotic expansion at large \( x \)

\[
K_s(x) = \sqrt{\frac{\pi}{2x}} e^{-x} \left( 1 + \sum_{k=1}^{\infty} \frac{1}{(2x)^k} \frac{\Gamma(s + k + \frac{1}{2})}{k!\Gamma(s - k + \frac{1}{2})} \right). \quad (C.3)
\]

The expansion truncates when \( s \) is half-integer, and in particular, for \( s = 1/2 \) the saddle point approximation is exact:

\[
K_{1/2}(x) = \sqrt{\frac{\pi}{2x}} e^{-x} \quad (C.4)
\]

We also recall some useful facts about the Riemann Zeta and Gamma functions

\[
\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s} = \frac{\pi^{s/2} \Gamma(1-s/2)}{\Gamma(s/2)} \zeta(1-s) \quad (C.5a)
\]

\[
\zeta(-1) = -\frac{1}{12}, \quad \zeta(0) = -\frac{1}{2}, \quad \zeta(2) = \frac{\pi^2}{6}, \quad (C.5b)
\]

\[
\Gamma(s) = \int_0^\infty \frac{dt}{t^{1+s}} e^{-t} \quad \Gamma(s+1) = s\Gamma(s), \quad \Gamma(1) = 1, \quad \Gamma(1/2) = \sqrt{\pi}. \quad (C.5c)
\]

It is also useful to recall that \( \zeta(s) \) has a simple pole at \( s = 1 \), simple zeros at \( s = -2, -3, \ldots \) whereas \( \Gamma(s) \) has simple poles at \( s = 0, -1, -2, \ldots \):

\[
\zeta(1+\epsilon) = \frac{1}{\epsilon} + \gamma + O(\epsilon), \quad \Gamma(\epsilon) = \frac{1}{\epsilon} - \gamma + O(\epsilon) \quad (C.6)
\]

where \( \gamma \approx 0.577215 \ldots \) is the Euler constant.

### C.1 \( SO(d,d,\mathbb{Z}) \) vector Eisenstein series

We first consider the large volume expansion of the Eisenstein series in the vector representation of \( SO(d,d) \), for which we use the integral representation

\[
\mathcal{E}_{s,V}^{SO(d,d,\mathbb{Z})} = \frac{\pi^s}{\Gamma(s)} \int_0^\infty \frac{dt}{t^{1+s}} \int_0^1 d\theta \sum_{m_i,n^i} \exp \left( -\frac{\pi}{t} (m_i^j + B_{ij}n^j)^2 - \frac{\pi}{t} (n^i)^2 + 2\pi i \theta m_i n^i \right) \quad (C.7)
\]

Here the integration over \( \theta \) incorporates the constraint \( m_i n^i = 0 \) and the squares denote the invariant contraction with the metric or inverse metric depending on the position of the indices. We first extract the \( n^i = 0 \) piece, and in the remaining part Poisson resum on the integers \( m_i \) which are now unconstrained. Then \( \mathcal{E}_{s,V}^{SO(d,d,\mathbb{Z})} = J_1(V) + J_2(V) \) with

\[
J_1(V) = \sum_{m_i} \left[ \frac{1}{m_i g^{ij} m_j} \right]^s = \mathcal{E}_{d,s}^{SL(d,\mathbb{Z})} = V_d \pi^s \Gamma(\frac{d}{2} - s) \Gamma(s) \mathcal{E}_{d,s}^{SL(d,\mathbb{Z})} \quad (C.8)
\]

\[
J_2(V) = \frac{V_d \pi^s}{\Gamma(s)} \int_0^\infty \frac{dt}{t^{1+s/2}} \int_0^1 d\theta \sum_{m^i} \sum_{n^i} \exp \left( -\pi t (m^i + \theta n^i)^2 - \frac{\pi}{t} (n^i)^2 + 2\pi i B_{ij} m^i n^j \right)
\]
Here we have recognized the first term \((\text{C.8})\) as the Eisenstein series of the antifundamental of \(SL(d)\) and used the identity \((\text{2.7})\) in the last step. Continuing with the second term \((\text{C.9})\) we note that although the integration over \(\theta\) runs from 0 to 1 only, we can reabsorb a shift \(\theta \to \theta + 1\) into a spectral flow \(m_i \to m_i + n_i\). We therefore extend the integration range of \(\theta\) to \(N \to \infty\) but sum on \(m_i\) modulo \(n_i\) only. Then, after performing the Gaussian integration on \(\theta\), the second term becomes

\[
J_2(V) = \frac{V_d \pi^s}{\Gamma(s)} \int_0^\infty \frac{dt}{t^{1+s-d-\frac{1}{2}}} \sum_{m_i \mod n_i} \exp \left( -\frac{\pi t (m \cdot m)(n \cdot n) - (m \cdot n)^2}{n \cdot n} - \frac{\pi}{t} n \cdot n + 2\pi i B_{ij} n^i n^j \right) \sqrt{n^i g_{ij} n^j}
\]

We now extract the terms for which \((m \cdot n)^2 - (m \cdot m)(n \cdot n) = 0\). By Schwarz inequality, this is the case if and only if \(m_i = \lambda n_i\) for all \(i\), and therefore the phase factor in \((\text{C.9})\) is irrelevant. For a given vector \(n\), the number of parallel vectors \(m\) modulo the spectral flow is \(\gcd\{n_i\}\), so that we have \(J_2(V) = J_{2a}(V) + J_{2b}(V)\) with

\[
J_{2a}(V) = \frac{V_d \pi^{\frac{d-1}{2}} \Gamma(s - \frac{d-1}{2})}{\Gamma(s)} \sum_{\gcd(n^i)} \left[ \frac{1}{n^i g_{ij} n^j} \right]^{\frac{d-2}{2}} = \frac{V_d \pi^{\frac{d-1}{2}} \Gamma(s - \frac{d-1}{2}) \zeta(2s - d + 1)}{\Gamma(s) \zeta(2s - d + 2)} E_{d,s-\frac{1}{2}+1}^s
\]

Here we have split the integers \(n^i\) into coprime \(n'^i\)'s and greatest common divisor \(r\), carried out the \(r\)-summation, and rewritten the coprime integers in terms of integers again at the expense of yet another \(r\) summation. Finally, for the remaining non-degenerate terms \(J_{2b}(V)\) in \((\text{C.9})\) we can perform the integral on \(t\) using \((\text{C.2})\) so that

\[
J_{2b}(V) = \frac{4V_d \pi^s}{\Gamma(s)} \sum_{m^i \mod n^i} \frac{1}{\sqrt{n^i g_{ij} n^j}} \left[ \frac{(n \cdot n)^2}{(m \cdot m)(n \cdot n) - (m \cdot n)^2} \right]^{\frac{d-1}{2}} K_{s-\frac{d}{2}-1} \left( 2\pi \sqrt{|(m \cdot m)(n \cdot n) - (m \cdot n)^2|} \right) e^{2\pi i B_{ij} n^i n^j}
\]

As in the non-degenerate orbit contribution of the one-loop integral, it is convenient to decompose the set of integers \(m^i, n^i\) with \(m^i \neq 0\) into \(SL(2,Z)\) equivalence classes, and write

\[
\begin{pmatrix} m^i \\ n^i \end{pmatrix} = \gamma \cdot \begin{pmatrix} \bar{m}^i \\ \bar{n}^i \end{pmatrix}, \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma
\]

where \((\bar{m}^i, \bar{n}^i)\) is any coset representative in the equivalence class labelled by the rank 2 matrix \(d^{ij} = \frac{1}{2}(\bar{m}^i \bar{n}^j - \bar{n}^i \bar{m}^j)\). The equivalence relation \(m^i \equiv m^i + n^i\) means that \(\gamma\) must
run over $\Gamma_\infty \backslash \Gamma$ only. Thus, we can rewrite
\[ J_{2b} = \frac{4V_d \pi^s}{\Gamma(s)} \sum_{d^j \neq 0 \atop \text{rk}(d^j) = 2} \mathcal{E}(d^j) \left[ (d^j)^2 \right]^{\frac{d+1+2s}{4}} K_{s-\frac{d}{2}-1} \left( 2\pi \sqrt{(d^j)^2} \right) e^{2\pi i B_{ij} d^j}, \] (C.13)
where
\[ \mathcal{E}(d^j) = \sum_{(c,d)=1} \|c \bar{m} + d \bar{n}\|^{d-2-2s}. \] (C.14)
This is recognized as an Eisenstein series (2.3) for $d=2$, evaluated at the Gram matrix of $\bar{m}$ and $\bar{n}$,
\[ \mathcal{E}(d^j) = \frac{1}{2\zeta(2s+2-d)} \mathcal{E}_{2,s+1-\frac{d}{2}} \left( \bar{m} \cdot \bar{m}, \bar{n} \cdot \bar{n} \right). \] (C.15)
(by $SL(2,\mathbb{Z})$ invariance, the r.h.s. only depends on $d^j$). At the special value $s = \frac{d}{2} - 1$, collecting these results and using (2.16) we find that the Eisenstein series in the vector representation reduces to
\[ \mathcal{E}^{SO(d,d,\mathbb{Z})}_{V;s=\frac{d}{2}-1} = \frac{\pi^{\frac{d}{2}-2}}{\Gamma(\frac{d}{2}-1)} \left[ V_d \mathcal{E}^{SL(d,\mathbb{Z})}_{d,s=1} + \frac{\pi^2}{3} V_d + 2\pi V_d \sum_{m^i,n^i} \exp \left( -2\pi \sqrt{|(m \cdot m)(n \cdot n) - (m \cdot n)^2| + 2\pi i B_{ij} m^i n^j} \right) \right], \] (C.16)
where the sum now runs over non-degenerate $SL(2,\mathbb{Z})$ orbits of vectors $(m,n)$. Here, to simplify the second term (C.10) we have used (C.5b) and $\mathcal{E}_{d,s=0}^{SL(d,\mathbb{Z})} = -1$ (see (2.16)). For the third term (C.16) we have used (C.4) to express the Bessel function $K_{1/2}$ as an exponential. We have thus reproduced the announced result (3.31).

C.2 $SO(3,3,\mathbb{Z})$ spinor and conjugate spinor Eisenstein series

We start with the spinor representation of $SO(3,3)$ with Eisenstein series with integral representation,
\[ \mathcal{E}^{SO(3,3,\mathbb{Z})}_{s,s} = \frac{\pi^s}{\Gamma(s)} \int_0^\infty \frac{dt}{t^{1+s}} \sum_{m_i,n} \exp \left( -\frac{\pi}{V_3 t} (m_i + n B^i)^2 - \frac{V_3 \pi}{t} (n)^2 \right) \] (C.17)
where we have introduced the singlet charge $n = \frac{1}{3!} \epsilon_3 m^{[3]}$ dual to the three-form charge and $B^i = \frac{1}{2} \epsilon^{ijk} B_{jk}$ is the dual of the NS 2-form. We single out the contribution with $n = 0$ and for the remaining terms we Poisson resum on the (unconstrained) integer $m^i$ whose dual
charge is \( m_i \). The latter contribution splits up into a part with \( m_i = 0 \) and a remaining non-degenerate contribution, so that after some algebra we can write

\[
\mathcal{E}_{SO(3,3)}^{3,3}(Z; s) = \sum_{m^i} \left[ \frac{V_3}{m^ig^j} \right]^s + \frac{2\pi^{3/2}\Gamma(s - 3/2)\zeta(2s - 3)V_3^{2-s}}{\Gamma(s)}
\]

\[
+ \frac{2\pi^sV_3^{1/2}}{\Gamma(s)} \sum_{m_i} \sum_n \left( \frac{n^2}{m_ig^j} \right)^{3/2} K_{s-3/2}(2\pi V_3 |n| \sqrt{m_ig^j} m_j) e^{2\pi i m_i B^i}
\]  

(C.18)

In particular for \( s = 1 \) this becomes

\[
\mathcal{E}_{SO(3,3)}^{3,3}(Z; s=1) = V_3 \mathcal{E}_{3,3}^{SL(3,3); s=1} + \frac{\pi^2}{3} V_3 + \pi \sum_{m_i} \sum_n \exp(-2\pi V_3 |n| \sqrt{m_ig^j} m_j + 2\pi i m_i B^i) e^{2\pi i m_i B^i}
\]  

(C.19)

where we have used the definition (2.3) of the \( SL(d) \) Eisenstein series, (C.5b) and (C.4) in each of the three terms respectively. The two leading terms establish the claim in (3.33a), reproducing the trivial and degenerate orbit contribution of the 1-loop integral \( I_3 \) respectively. Moreover, exact agreement is also explicitly seen [10] between the third term and the non-degenerate orbit contribution (3.20).

For the conjugate spinor of \( SO(3,3) \) the integral representation of the Eisenstein series is

\[
\mathcal{E}_{C; s=1}^{SO(3,3)} = \frac{\pi s}{\Gamma(s)} \int_0^{\infty} dt \frac{1}{t^{1+s}} \sum_{m_i} \sum_n \exp(-\frac{\pi}{V_3 t} (n + m_i B^i)^2 - \frac{V_3}{t} m_ig^j m_j)
\]  

(C.20)

where in this case we have dualized the two-form into a one-form \( n_1 = \epsilon_3 m^{[2]} / 2 \), and the dual \( B \)-field is as above. In this case, we first separate the \( m_i = 0 \) contributions and for the remainder Poisson resum on the unconstrained integer \( n \), whereafter we distinguish between \( n = 0 \) and the rest. After some algebra we then have

\[
\mathcal{E}_{C; s}^{SO(3,3)} = 2\zeta(2s) V_3^s + \frac{\pi^{2s-2}\Gamma(2-s)}{\Gamma(s)} \sum_{m^i} \left[ \frac{V_3}{m^ig^j} \right]^{2-s}
\]

\[
+ \frac{2\pi^s\sqrt{V_3}}{\Gamma(s)} \sum_{m_i} \sum_n \left( \frac{n^2}{m_ig^j} \right)^{1/2} K_{s-1/2}(2\pi V_3 |n| \sqrt{m_ig^j} m_j) e^{2\pi i m_i B^i}
\]  

(C.21)

where we have also used the identity (2.7) to rewrite the second term in terms of the fundamental representation of \( SL(d) \). Setting \( s = 1 \) we find exactly the same result (C.19) as obtained for the spinor representation, with the first two terms interchanged as noted in (3.33b). The equality of the \( d = 3 \) spinor and conjugate series for \( s = 1 \) is obvious from the fact that the two representations have inverse mass matrices in this case and (since there
are no constraints on the charges) can hence be related by a complete Poisson resummation

\[ \mathcal{E}_{S;_{s}}^{SO(3,3,\mathbb{Z})} = \mathcal{E}_{C;_{2-s}}^{SO(3,3,\mathbb{Z})} \tag{C.22} \]

Equivalently, this identity follows from (3.15c) and (2.7).

### C.3 \(SO(4,4,\mathbb{Z})\) spinor and conjugate spinor Eisenstein series

Moving on to \(SO(4,4)\) we remark that from this case on, one needs to incorporate the non-trivial half-BPS constraints (3.7) and (3.9). The integral representation of the spinor Eisenstein series reads

\[ \mathcal{E}_{S;_{s}}^{SO(4,4,\mathbb{Z})} = \frac{\pi^{s}}{\Gamma(s)} \int_{0}^{\infty} dt \int_{0}^{1} d\theta \sum_{m^{i},n_{i}} \exp \left( -\frac{\pi}{V_{4} t} (m^{i} + B^{ij} n_{j})^{2} - \frac{\pi}{t} V_{4} n_{i}^{2} + 2\pi i \theta m^{i} n_{i} \right) \]

where we have dualized the three-form into a one-form \(n_{i} = \frac{1}{3} \epsilon_{ijkl} m^{jkl} \) and introduced the dual \(B\)-field \(B^{ij} = \frac{1}{2} \epsilon^{ijkl} B_{kl}\). The constraint \(m^{[1]} \wedge m^{[3]} = 0\) then becomes \(m^{i} n_{i} = 0\) and is incorporated due to the \(\theta\) integration. The evaluation of this integral proceeds in a way analogous to the \(SO(d,d)\) vector case, and omitting the details we record the final result

\[ \mathcal{E}_{S;_{s}}^{SO(4,4,\mathbb{Z})} = \sum_{m^{i}} \left[ \frac{V_{4}}{m^{i} g_{ij} m^{j}} \right]^{s} + \frac{V_{4}^{2-s} \pi^{3/2} \Gamma(s - \frac{3}{2}) \zeta(2) - 3}{\Gamma(s) \zeta(2s - 2)} \sum_{n_{i}} \left[ \frac{1}{n_{i} g^{ij} n_{j}} \right]^{s-1} + \]

\[ + \frac{4 \sqrt{4 \pi s}}{\Gamma(s)} \sum_{m^{i},n_{i}} \frac{1}{2 \zeta(2s - 2)} \mathcal{E}_{S;_{s}}^{SL(2,\mathbb{Z})} \begin{pmatrix} m \cdot m & m \cdot n \\ m \cdot n & n \cdot n \end{pmatrix} \frac{1}{\left[ (m \cdot m)(n \cdot n) - (m \cdot n)^{2} \right]^{2s-3}} K_{s-4} \left( 2\pi V_{4} \sqrt{|(m \cdot m)(n \cdot n) - (m \cdot n)^{2}|} \right) e^{2\pi i B^{ij} m_{i} n_{j}} \tag{C.24} \]

where all inner products are taken with the inverse metric, and the sum in the last term runs over non-degenerate \(SL(2,\mathbb{Z})\) orbits. In fact, this result can be obtained immediately from the result of the \(SO(4,4)\) vector representation (substitute \(d = 4\) in (C.8) + (C.10) + (C.13)), using the triality relation

\[ \mathcal{M}^{2}(S; g_{ij}, B_{ij}; m^{i}, n_{i}) = \mathcal{M}^{2}(V; V_{4} g^{ij}, B^{ij}; n_{i}, m^{i}) \tag{C.25} \]

between the \(SO(4,4)\) spinor and vector mass formulae. For use below we also note that the first two terms can be expressed in terms of \(SL(4)\) Eisenstein series,

\[ \mathcal{E}_{S;_{s}}^{SO(4,4,\mathbb{Z})} = V_{4}^{s} \mathcal{E}_{4;_{s}}^{SL(4,\mathbb{Z})} + \frac{V_{4}^{2-s} \pi^{3/2} \Gamma(s - \frac{3}{2}) \zeta(2s - 3)}{\Gamma(s) \zeta(2s - 2)} \mathcal{E}_{4;_{1-s}}^{SL(4,\mathbb{Z})} + \ldots \tag{C.26} \]
Evaluating this at $s = 1$ with the use of (C.5b) we reproduce the two leading terms (3.33a). Using (C.2) the non-degenerate contribution at $s = 1$ in (C.24) takes the form

$$2\pi \sum_{m_i, n_i} \exp \left( -2\pi V_4 \sqrt{[(m \cdot m)(n \cdot n) - (m \cdot n)^2] + 2\pi i B^{ij}m_i n_j} \right) \sqrt{[(m \cdot m)(n \cdot n) - (m \cdot n)^2]}$$

(C.27)

Although we have not been able to show it explicitly, this contribution should be equal the corresponding non-degenerate contribution in (C.16) for the $SO(4, 4)$ vector Eisenstein series at $s = 1$, and hence equal to the non-degenerate contribution of the 1-loop integral $I_4$.

C.4 $SO(d, d, \mathbb{Z})$ spinor and conjugate spinor Eisenstein series

More generally, we can compute for all $n$ the leading term for the spinor Eisenstein series, obtained by setting all charges $m^{[3]} = m^{[5]} = \ldots = 0$ except $m^{[1]}$, so that the constraints are trivial. This shows that

$$E_{S, s}^{SO(d,d,\mathbb{Z})} = \sum_{m^{[1]}} \left[ \frac{V_d}{m^{[1]} g_{ij} m^{[j]}} \right]^s + \ldots = V_d^s E_{d, s}^{SL(d,\mathbb{Z})} + \ldots$$

so that, for $s = 1$, we observe the leading term in (3.33a).

For the conjugate spinor, we can go even further and obtain the first two leading terms. Focusing on the contributions from $m^{[2]}$ and $m^{[2]}$ only and setting $m^{[4]} = m^{[6]} = \ldots = 0$ (so that the constraints can be ignored) we find that

$$E_{C, s}^{SO(d,d,\mathbb{Z})} = \sum_{m^{[2]}} \left[ \frac{V_d}{m^{[2]}^2} \right]^s + \frac{\pi^s \Gamma(s - \frac{1}{2})}{\pi^{s - \frac{1}{2}} \Gamma(s)} V_d^s \sum_{m^{[2]} m^{[2]}} \left[ m^{[2]} g_{ik} m^{[k]} \right]^{s - \frac{1}{2}} \delta(m^{[2]} \wedge m^{[2]}) + \ldots$$

$$= 2V_d^s \zeta(2s) + \frac{\pi^s \Gamma(s - \frac{1}{2})}{\pi^{s - \frac{1}{2}} \Gamma(s)} V_d^s E_{[2], s - \frac{1}{2}}^{SL(d,\mathbb{Z})} + \ldots$$

(C.29)

Here, the leading term is obtained from $m^{[2]} = 0$, while the second term follows after Poisson resummation on the unconstrained $m$ in the remainder and setting (the dual) $m = 0$. Substituting $s = 1$ we immediately recognize the leading term $\frac{\pi^2}{3} V_d$.

References

[1] C. M. Hull and P. K. Townsend, “Enhanced gauge symmetries in superstring theories,” Nucl. Phys. B451 (1995) 525–546, [hep-th/9505073]
[2] P. K. Townsend, “The eleven-dimensional supermembrane revisited,” *Phys. Lett.* B350 (1995) 184–187, [hep-th/9501063](http://arxiv.org/abs/hep-th/9501063).

[3] E. Witten, “String theory dynamics in various dimensions,” *Nucl. Phys.* B443 (1995) 85–126, [hep-th/9503124](http://arxiv.org/abs/hep-th/9503124).

[4] A. Giveon, M. Porrati, and E. Rabinovici, “Target space duality in string theory,” *Phys. Rept.* 244 (1994) 77–202, [hep-th/9401139](http://arxiv.org/abs/hep-th/9401139).

[5] N. A. Obers and B. Pioline, “U-duality and M-theory,” *Phys. Rept.* 318 (1999) 113–225, [hep-th/9809039](http://arxiv.org/abs/hep-th/9809039).

[6] B. Pioline, “A note on non-perturbative $R^4$ couplings,” *Phys. Lett.* B431 (1998) 73–76, [hep-th/9804023](http://arxiv.org/abs/hep-th/9804023).

[7] S. Paban, S. Sethi, and M. Stern, “Supersymmetry and higher derivative terms in the effective action of Yang-Mills theories,” *J. High Energy Phys.* 06 (1998) 012, [hep-th/9806028](http://arxiv.org/abs/hep-th/9806028).

[8] M. B. Green and S. Sethi, “Supersymmetry constraints on type IIB supergravity,” *Phys. Rev.* D59 (1999) 046006, [hep-th/9808061](http://arxiv.org/abs/hep-th/9808061).

[9] M. B. Green and M. Gutperle, “Effects of D instantons,” *Nucl. Phys.* B498 (1997) 195–227, [hep-th/9701093](http://arxiv.org/abs/hep-th/9701093).

[10] E. Kiritsis and B. Pioline, “On $R^4$ threshold corrections in IIB string theory and $(p, q)$ string instantons,” *Nucl. Phys.* B508 (1997) 509–534, [hep-th/9707018](http://arxiv.org/abs/hep-th/9707018).

[11] J. A. Harvey and G. Moore, “Five-brane instantons and $R^2$ couplings in $N = 4$ string theory,” *Phys. Rev.* D57 (1998) 2323–2328, [hep-th/9610237](http://arxiv.org/abs/hep-th/9610237).

[12] A. Gregori, E. Kiritsis, C. Kounnas, N. A. Obers, P. M. Petropoulos, and B. Pioline, “$R^2$ corrections and nonperturbative dualities of $N = 4$ string ground states,” *Nucl. Phys.* B510 (1998) 423–476, [hep-th/9708062](http://arxiv.org/abs/hep-th/9708062).

[13] C. Bachas, C. Fabre, E. Kiritsis, N. A. Obers, and P. Vanhove, “Heterotic/type I duality and D-brane instantons,” *Nucl. Phys.* B509 (1998) 33–52, [hep-th/9707126](http://arxiv.org/abs/hep-th/9707126).

[14] E. Kiritsis and N. A. Obers, “Heterotic type I duality in $D < 10$-dimensions, threshold corrections and D instantons,” *J. High Energy Phys.* 10 (1997) 004, [hep-th/9709058](http://arxiv.org/abs/hep-th/9709058).
[15] I. Antoniadis, B. Pioline, and T. R. Taylor, “Calculable $e^{-1/\lambda}$ effects,” *Nucl. Phys. B512* (1998) 61–78, hep-th/9707222.

[16] W. Lerche and S. Stieberger, “Prepotential, mirror map and F theory on K3,” *Adv. Theor. Math. Phys.* 2 (1998) 1105–1140, hep-th/9804176.

[17] W. Lerche, S. Stieberger, and N. P. Warner, “Quartic gauge couplings from $K_3$ geometry,” hep-th/9811228.

[18] K. Foerger and S. Stieberger, “Higher derivative couplings and heterotic type I duality in eight-dimensions,” hep-th/9901020.

[19] C. Bachas, “Heterotic versus type-I,” *Nucl. Phys. Proc. Suppl.* 68 (1998) 348, hep-th/9710102.

[20] M. B. Green and M. Gutperle, “D particle bound states and the D instanton measure,” *JHEP* 01 (1998) 005, hep-th/9711107.

[21] G. Moore, N. Nekrasov, and S. Shatashvili, “D particle bound states and generalized instantons,” hep-th/9803265.

[22] I. K. Kostov and P. Vanhove, “Matrix string partition functions,” *Phys. Lett. B444* (1998) 196, hep-th/9809130.

[23] E. Gava, A. Hammou, J. F. Morales, and K. S. Narain, “On the perturbative corrections around D string instantons,” *JHEP* 03 (1999) 023, hep-th/9902202.

[24] A. Terras, *Harmonic analysis on symmetric spaces and applications*, vol. I. Springer Verlag, 1985.

[25] M. B. Green and P. Vanhove, “D-instantons, strings and M theory,” *Phys. Lett. B408* (1997) 122–134, hep-th/9704145.

[26] N. Berkovits, “Construction of $R^4$ terms in $N = 2, D = 8$ superspace,” *Nucl. Phys. B514* (1998) 191–203, hep-th/9709116.

[27] A. Terras, *Harmonic analysis on symmetric spaces and applications*, vol. II. Springer Verlag, 1985.

[28] Harish-Chandra, *Automorphic Forms on Semisimple Lie Groups*. No. 62 in Lecture Notes in Mathematics. Springer Verlag, 1968.

[29] S. Ferrara, J. A. Harvey, A. Strominger, and C. Vafa, “Second quantized mirror symmetry,” *Phys. Lett. B361* (1995) 59–65, hep-th/9505162.
[30] B. Pioline and E. Kiritsis, “U duality and D-brane combinatorics,” Phys. Lett. B418 (1998) 61–69, hep-th/9710078

[31] O. Ganor, “Two conjectures on gauge theories, gravity, and infinite dimensional Kac-Moody groups,” hep-th/9903110

[32] N. A. Obers and B. Pioline, “U duality and M theory, an algebraic approach,” in 2nd Conference on Quantum Aspects of Gauge Theories, Supersymmetry and Unification, Corfu. Sept, 1998. hep-th/9812139

[33] L. J. Dixon, V. Kaplunovsky, and J. Louis, “Moduli dependence of string loop corrections to gauge coupling constants,” Nucl. Phys. B355 (1991) 649–688.

[34] E. Kiritsis and C. Kounnas, “Infrared behavior of closed superstrings in strong magnetic and gravitational fields,” Nucl. Phys. B456 (1995) 699–731, hep-th/9508078.

[35] R. E. Borcherds, “Automorphic forms on $O_{s+2,2}(R)$ and infinite products,” Invent. Math. 120 (1995) 161.

[36] R. E. Borcherds, “Automorphic forms with singularities on Grassmannians,” Invent. Math. 132 (1998) 491–562, alg-geom/9609022

[37] J. A. Harvey and G. Moore, “Algebras, BPS states, and strings,” Nucl. Phys. B463 (1996) 315–368, hep-th/9510182

[38] D. J. Gross and E. Witten, “Superstring modifications of Einstein’s equations,” Nucl. Phys. B277 (1986) 1.

[39] E. Cremmer and B. Julia, “The SO(8) supergravity,” Nucl. Phys. B159 (1979) 141.

[40] B. L. Julia, “Dualities in the classical supergravity limits: Dualizations, dualities and a detour via (4k+2)- dimensions,” in NATO Advanced Study Institute on Strings, Branes and Dualities, Cargese. May, 1997. hep-th/9805083

[41] C. M. Hull and P. K. Townsend, “Unity of superstring dualities,” Nucl. Phys. B438 (1995) 109–137, hep-th/9410167

[42] S. Elitzur, A. Giveon, D. Kutasov, and E. Rabinovici, “Algebraic aspects of matrix theory on $T^d$,” Nucl. Phys. B509 (1998) 122–144, hep-th/9707217

[43] R. Dijkgraaf, E. Verlinde, and H. Verlinde, “BPS spectrum of the five-brane and black hole entropy,” Nucl. Phys. B486 (1997) 77–88, hep-th/9603126.
[44] N. A. Obers, B. Pioline, and E. Rabinovici, “M theory and U duality on $T^d$ with gauge backgrounds,” *Nucl. Phys.* **B525** (1998) 163–181, [hep-th/9712084](https://arxiv.org/abs/hep-th/9712084).

[45] E. D’Hoker and D. H. Phong, “The geometry of string perturbation theory,” *Rev. Mod. Phys.* **60** (1988) 917.

[46] N. Berkovits and C. Vafa, “$N = 4$ topological strings,” *Nucl. Phys.* **B433** (1995) 123–180, [hep-th/9407190](https://arxiv.org/abs/hep-th/9407190).

[47] H. Ooguri and C. Vafa, “All loop $N = 2$ string amplitudes,” *Nucl. Phys.* **B451** (1995) 121–161, [hep-th/9505183](https://arxiv.org/abs/hep-th/9505183).

[48] N. Berkovits and C. Vafa, “Type IIB $R^4H^{4g-4}$ conjectures,” *Nucl. Phys.* **B533** (1998) 181–198, [hep-th/9803145](https://arxiv.org/abs/hep-th/9803145).

[49] A. Kehagias and H. Partouche, “The exact quartic effective action for the type IIB superstring,” *Phys. Lett.* **B422** (1998) 109–116, [hep-th/9710023](https://arxiv.org/abs/hep-th/9710023).

[50] M. B. Green, M. Gutperle, and H. Kwon, “Sixteen fermion and related terms in M theory on $T^2$,” *Phys. Lett.* **B421** (1998) 149–161, [hep-th/9710151](https://arxiv.org/abs/hep-th/9710151).

[51] J. G. Russo and A. A. Tseytlin, “One loop four graviton amplitude in eleven-dimensional supergravity,” *Nucl. Phys.* **B508** (1997) 245–259, [hep-th/9707134](https://arxiv.org/abs/hep-th/9707134).

[52] A. Kehagias and H. Partouche, “D instanton corrections as $(p,q)$ string effects and nonrenormalization theorems,” *Int. J. Mod. Phys.* **A13** (1998) 5075–5092, [hep-th/9712164](https://arxiv.org/abs/hep-th/9712164).

[53] P. Mayr and S. Stieberger, “Moduli dependence of one loop gauge couplings in $(0,2)$ compactifications,” *Phys. Lett.* **B355** (1995) 107–116, [hep-th/9504129](https://arxiv.org/abs/hep-th/9504129).

[54] P. Berglund, M. Henningson, and N. Wyllard, “Special geometry and automorphic forms,” *Nucl. Phys.* **B503** (1997) 256–276, [hep-th/9703195](https://arxiv.org/abs/hep-th/9703195).