FRAMED MOTIVES OF SMOOTH AFFINE PAIRS.

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ABSTRACT. The theory of framed motives by Garkusha and Panin gives computations in the stable motivic homotopy category $\text{SH}(k)$ in terms of Voevodsky’s framed correspondences. In particular the motivically fibrant $\Omega$-resolution in positive degrees of the motivic suspension spectrum $\Sigma^\infty_+ X_+$, where $X_+ = X \amalg*$, for a smooth scheme $X \in \text{Sm}_k$ over an infinite perfect field $k$, is computed.

The computation by Garkusha, Neshitov and Panin of the framed motives of relative motivic spheres $(A_1 \times X, (A_1 - 0) \times X)$, $X \in \text{Sm}_k$, is one of ingredients in the theory. In the article we extend this result to the case of a pair $(X, U)$ given by a smooth affine variety $X$ over $k$ and an open subscheme $U \subset X$.

The result gives the explicit motivically fibrant $\Omega$-resolution in positive degrees for the motivic suspension spectrum $\Sigma^\infty_{\mathbb{P}^1} (X_+/U_+)$ of the factor-sheaf $X_+/U_+$.

1. INTRODUCTION

In the unpublished notes [1] Voevodsky had suggested the computational approach to the Morel-Voevodsky stable motivic homotopy category $\text{SH}(k)$ [2], [3], [4] over a perfect base field. Realising this idea Garkusha and Panin constructed the theory of framed motives over an infinite perfect field $k$, see [5] and [6], [7], [8]. The aim of the present article is to extend the computations [5] th. 4.1, 11.1] of stable motivic fibrant resolutions of the motives of smooth schemes $X$ to the case of open pair factor-sheaf $X/U$ with $X$ smooth affine.

In [1] for any smooth scheme $X$ and open $U \subset X$ Voevodsky had introduced pointed sheaves $\text{Fr}(-, X/U) \in \text{Shv}_\bullet$ of so-called stable Voevodsky’s framed correspondences. According to the definition [5] def. 2.8] and fundamental Voevodsky’s lemma [5] lm. 3.5, prop. 3.2]

$$\text{Fr}(-, X/U) = \lim_{\longrightarrow} \text{Fr}_n(-, X/U),$$

$$\text{Fr}_n(-, X/U) = \text{Hom}_{\text{Shv}_\bullet}(- \wedge (\mathbb{P}^1/\infty)^{\wedge n}, X/U \wedge (\mathbb{A}^1/G_m)^{\wedge n}),$$

where $\text{Shv}_\bullet$ denotes the pointed Nisnevich sheaves, and sections of $\text{Fr}_n(-, X/U)$ have precise geometrical description, see def. [1 or [5] def. 2.5].

In [5] theorem 11.7] Garkusha and Panin computed the functor

$$\Omega^\infty \Sigma^\infty \Sigma^\infty : \text{H}_\bullet(k) \rightarrow \text{SH}_{S^1}(k).$$

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from the unstable pointed motivic homotopy category to the $S^1$-stable one as the functor
\[ M_{fr}: sShv_{\bullet} \to \text{Spec}_{S^1}(sShv_{\bullet}) \]

from the category of pointed simplicial sheaves to the motivic $S^1$-spectra, such that by \cite[ths. 10.1(2), 7.6]{5} the level injective local fibrant replacement $M_{fr}(-)_{fr}$ lands in motivically fibrant $\Omega_{S^1}$-spectra in positive degrees.

For a smooth scheme $X$ over $k$ the spectrum $M_{fr}(X)$ is equal to
\[
(C^*\text{Fr}(-, X), C^*\text{Fr}(-, X \wedge S^1), \ldots C^*\text{Fr}(-, X \wedge S^i) \ldots),
\]
where the endo-functor $C^*: F(-) \mapsto F(\Delta^* \times -)$ on $sShv_{\bullet}$ is corepresented by the co-simplicial scheme $\Delta^*$, $\Delta^n = \{(x_0, \ldots, x_n) \in \mathbb{A}^{n+1}, x_1 + \ldots x_n = 1\}$, and the pointed simplicial sheaf $\text{Fr}(-, Y)$, for a simplicial scheme $Y$, is defined by the pointed sheaves $\text{Fr}(-, Y)$, for $Y \in \text{Sm}_k$.

In the article we extend so-called Cone Theorem by Garkusha, Nesbitov and Panin \cite{8}, and prove that for a smooth affine $X$ over $k$ and open $U \subset X$, the spectrum $M_{fr}(X_+/U_+)_f$, where $X_+/U_+$ is the pointed factor-sheaf, is schemewise simplicially weak equivalent to
\[
(C^*\text{Fr}(-, X/U)_f, C^*\text{Fr}(-, X/U \wedge S^1)_f, \ldots C^*\text{Fr}(-, X/U \wedge S^i)_f \ldots),
\]
where $(-)_f$ denotes the fibrant replacement within the (level) injective local model structure. The result of \cite{8} covers the case of sheaves $(\mathbb{A}^n/(\mathbb{A}^n - 0)) \times X$, $X \in \text{Sm}_k$.

As a consequence this proves the results of \cite[th. 4.1, 11.1, 11.7]{5} for the case of affine pairs.

**Theorem 1** (corollaries \cite[5][6][7]). For a smooth affine scheme $X$ over an infinite perfect field $k$, and an open subscheme $U \subset X$, the following holds:

1) The canonical morphism of $\mathbb{P}^1$-spectra of pointed simplicial sheaves
\[
\Sigma_{\mathbb{P}^1}^\infty (X/U) \to M_{\mathbb{P}^1}(X,U),
\]
where the spectrum $M_{\mathbb{P}^1}(X,U)$ is given by
\[
(C^*\text{Fr}(-, X/U), C^*\text{Fr}(-, X/U \wedge T), \ldots C^*\text{Fr}(-, X/U \wedge T^\wedge i))
\]
where $T = (\mathbb{A}^1, \mathbb{G}_m)$, is a stable motivic weak equivalence in $\text{SH}(k)$, and the Nisnevich local fibrant replacement $M_{\mathbb{P}^1}(X,U)_f$ is a motivically fibrant $\Omega_{\mathbb{P}^1}$-spectrum in positive degrees.

2) The $S^1$-spectrum of pointed simplicial sheaves $M_{fr}(X,U)_f$ given by \cite{3} is a motivically fibrant $\Omega$-spectrum in positive degrees and has the homotopy type of the spectrum $\Sigma_{\mathbb{G}_m}^\infty \Sigma_{\mathbb{G}_m}^\infty \Sigma_{S^1}^\infty (X/U)$ in $\text{SH}_{S^1}(k)$.

3) Let $M_{fr}^{G_m}(X,U)_f$ be a $(S^1, \mathbb{G}_m^{\wedge 1})$-bi-spectrum given by
\[
(M_{fr}(X,U)_f, M_{fr}((X,U) \wedge \mathbb{G}_m^{\wedge 1})_f, \ldots M_{fr}((X,U) \wedge \mathbb{G}_m^{\wedge i})_f \ldots).
\]
Then the canonical morphism of bi-spectra
\[
\Sigma_{\mathbb{G}_m}^\infty \Sigma_{S^1}^\infty (X/U) \to M_{fr}^{G_m}(X,U)_f
\]
is a stable motivic weak equivalence, and $M^{fr}_f(X, U)$ is a motivically fibrant $\Omega$-bi-spectrum in $S^1$-positive degrees.

By the definition it follows that $M_{fr}(X/U) = M_{fr}(X//U)$, where $X//U$ is the simplicial cone of the open immersion $j: U \hookrightarrow X$, that is the colimit in the category of pointed simplicial schemes

(6) \[ X//U = \text{colim}(X \xrightarrow{j} U \xrightarrow{i_0} \Delta^1 \times U \xrightarrow{i_0} U \to *), \]

where $\Delta^1$ is the simplicial interval, $i_0, i_1: U \to \Delta^1 \times U$ are the unit and zero faces, and $*$ is a rational point, that is the base point of $X//U$. So the mentioned Cone Theorem is equivalent to the level Nisnevich local equivalence

(7) \[ M_{fr}(X//U) \cong_{\text{nis}} M_{fr}(X, U) \in \text{Spec}_{S^1}(\text{sShv}_\bullet), \]

where $M_{fr}(X, U)$ is the motivic spectrum with terms $C^* Fr(-, (X/U) \wedge S^1)$.

In distinct to [8] our arguments for (7) are unstable, so we prove the equivalence (7) levelwise independently.

**Theorem 2** (Cone Theorem, Theorem [4]). For a smooth affine pair $(X, U)$ over an infinite field $k$ let $X//U$ denote the simplicial cone of the morphism $U \to X$, see (6) for the case $U \neq \emptyset$.

Then the canonical morphism of simplicial pointed sheaves

$\epsilon: Fr(-, X//U) \to Fr(-, X/U),$

is a motivic equivalence.

**Remark 1.** Note that the pointed simplicial sheaf $Fr(-, X//U)$ is simplicially equivalent to $Fr(-, X)/Fr(-, U)$, and the morphism

$Fr(-, X)/Fr(-, U) \to Fr(-, X/U)$

is not a simplicial equivalence, but is a motivic one by the above.

The result is given by the composition of motivic equivalences

(8) \[ Fr(-, X//U) \to Fr^{qf}(-, X/U) \to Fr(-, X/U), \]

where the first morphism is a Nisnevich local equivalence, the second one is an $A^1$-homotopy-equivalence. The mid-term $Fr^{qf}$ is the subsheaf of $Fr$ given by so-called quasi-finite framed correspondences, introduced in [8], see definition [8] in section [5] in our text.

The theory of framed motives was revisited by the team of five authors EHKSY in [9] with the technique of $\infty$-categories; this allowed to obtain clear $\infty$-categorical picture of the theory and obtain the universal properties of the constructed categories.

Here we follow the original approach [5], since it is more explicit geometrical, and closure to the precise computations. The main content and novelty of the present work relates to the geometrical arguments in the proof of the moving lemma, section [4], providing the second equivalence of (8).
In the same time let us point that the definition of the tangentially framed correspondences introduced in [9] and its extend to the case of pairs $(X,U)$ is useful for the proof of proposition [1] and corollary [1] because the correspondences of this type are $\infty$-commutative monoids already on the simplicial level (without motivic localisation). Let us point that originally in [8] the first equivalence in (8) was proven for linear sheaves $ZF$ and $ZF^{df}$ instead of $Fr$ and $Fr^{df}$, because the linear ones are commutative monoids, but $Fr$ are $Fr^{df}$ are not.

1.1. Remarks on the generality of the base field. Originally the theory of framed motives [5] was written with the exceptional case of $\text{char } k = 2$, but was extended to this case by [10]. That is way we use the results of the theory [5] in all characteristics.

The theory of famed motives was extended to the case of finite fields in [11] and [9]. The same arguments extend our result for the case of finite fields as well, but we don’t touch this here.

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1.3. Notation. $\text{Sm}_k$ denotes the category of smooth separated schemes of finite type over the base field $k$. $\text{AffSm}_k \subset \text{Sm}_k$ denotes the subcategory of affine smooth schemes.

For a scheme $X$ and a vector of regular functions $\varphi = (\varphi_i), \varphi \in \mathcal{O}(X)$, we denote by $Z(\varphi)$ the common vanishing locus of functions $\varphi_i$.

For an $S$-scheme $X$ we write $\dim_S X = r$ iff the Krull dimension of $X \times_S \sigma$ is equal to $r$ for any point $\sigma \in S$. We write $\dim_S X \geq r$ (dim$_S X \leq r$) iff dim$(X \times_S \sigma) \geq r$ (dim$(X \times_S \sigma) \leq r$) for any $\sigma \in S$. For a morphism of $S$-schemes $X \rightarrow Y$ we denote $\dim_S(X/Y) = \dim_S X - \dim_S Y$, and $\text{codim}_S(X/Y) = \dim_S Y - \dim_S X$ if $X$ and $Y$ are of constant dimension over $S$.

2. Framed correspondences and motives.

In the section we recall some definitions and results from [5] on the theory of framed correspondences and framed motives.

The aim of the theory is to make computations in the stable motivic homotopy category $\text{SH}(k)$, in particular to compute hom-groups $[S,X/U]_{\text{SH}(k)}$ for $S,X \in \text{Sm}_k$ and an open subscheme $U \subset X$. The fundamental Voevodksy’s idea, the theory is based on, is to use for this aim the computation of hom-sets $[S,X/U]_{\text{Shv}(k)}$ and $[S \wedge (\mathbb{P}^1/\infty)^n, X/U \wedge (\mathbb{A}^1/\mathbb{G}_m)^n]_{\text{Shv}(k)}$ in the categories of (pointed) Nisnevich sheaves $\text{Shv}(k)$ and $\text{Shv}_*(k)$. This computation can be done precisely and it is called as Voevodsky’s lemma [5] lm. 3.5, prop. 3.2; the the answer is called as the framed correspondences [5, def. 2.5], and see def. [1] that follows.
At the end of the section we recall one of the main results of [5] that gives the computation of stable motivically fibrant resolutions in positive degrees in \(SH(k)\) of \(\Sigma^{\infty}_{\mathbb{P}}X, X \in \text{Sm}_k\).

**Definition 1.** (i) For any schemes \(S, X\) and closed \(Y \subset X\) an *explicit framed correspondence of a level \(n\) from \(S\) to the pair \((X, X - Y)\) is a set \((Z, V, \varphi, g)\), where \(Z\) is reduced closed subscheme in \(\mathbb{A}^n_S\) finite over \(S\), \((V, Z) \to (\mathbb{A}^n_S, Z)\) is Nisnevich neighbourhood, \(\varphi = (\varphi_i)_{i=1,...,n}: V \to \mathbb{A}^n\) and \(g: V \to X\) are regular maps such that \(Z = (\varphi, g)^{-1}(0 \times Y)\) is finite over \(S\). So we get the diagram

\[
\begin{array}{ccc}
\mathbb{A}^n_S & \xrightarrow{(\varphi, g)} & \mathbb{A}^n_X \\
\downarrow c & & \uparrow \\
S & \xrightarrow{f} & 0 \times Y \\
\end{array}
\]

where \(f\) is finite, \(e\) is etale, and \(c\) is a closed embedding.

(ii) Define the set \(\text{Fr}_n(S, X/(X - Y))\) of *level \(n\) framed correspondences* as the set of equivalent classes of explicit frame correspondences from \(S\) to \((X, X - Y)\) under the equivalence relation given by shrinking of the neighbourhood \(V\) of \(Z\). So \(\varphi_1 = (Z_1, V_1, \varphi_1, g_1), \varphi_2 = (Z_2, V_2, \varphi_2, g_2)\) are equivalent whenever \(Z_1 = Z_2 = Z\) and there is an explicit framed correspondence \(\varphi = (Z, V, \varphi, g)\) with morphisms of Nisnevich neighbourhoods \(w_i: V \to V_i\) for \(i = 1, 2\) such that \(w_i\) acts identically on \(Z_i, \varphi = w_i^*(\varphi_i), g = w_i^*(g)\).

(iii) Define morphisms

\[
\begin{align*}
\text{Fr}_n(S, X/(X - Y)) & \to \text{Fr}_{n+1}(S, X/(X - Y)) \\
(Z, V, \varphi, g) & \mapsto (Z', V', \varphi', g'),
\end{align*}
\]

where \(Z' = Z \times 0 \subset \mathbb{A}^{n+1}_S, V' = V \times \mathbb{A}^1, \varphi_{n+1} = t_{n+1}, \varphi' = \varphi_i \circ pr, g' = g \circ pr\)

where \(t_{n+1}\) is the last coordinate function on \(\mathbb{A}^{n+1}_S\), and \(pr: V \times \mathbb{A}^1 \to V\) is the canonical projection.

Define \(\text{Fr}(S, X/(X - Y)) = \lim_n \text{Fr}_n(S, X/(X - Y))\).

Define \(\text{Fr}_n(S, X) = \text{Fr}_n(S, X/\emptyset), \text{Fr}(S, X) = \text{Fr}(S, X/\emptyset)\).

Define a category of *smooth open pairs* \(\text{Sm}_k^\text{pair}\) with objects being pairs \((X, U)\) given by a smooth scheme \(X\) and open subscheme \(U \subset X\), and with morphisms \((X, U) \to (X', U')\) given by regular maps \(f: X \to X', f^{-1}(U') \supset U\). We denote an object \((X, U) \in \text{Sm}_k^\text{pair}\) also as \(X/U\).

**Definition 2.** For a smooth pair \((X, U) \in \text{Sm}_k^\text{pair}\) we denote by \(\text{Fr}(X/U)\) the pointed sheaf \(\text{Fr}(-, X/U)\) pointed at the framed correspondence with empty support. We consider \(\text{Fr}(X/U)\) as a pointed simplicial sheaf constant in the simplicial direction.

**Definition 3.** Define the functor \(C^*: s\text{Shv}_k \to s\text{Shv}_k: F \mapsto F(- \times \Delta^*)\), where the right side is considered as the simplicial sheaf in view of the totalisation functor form the
category of bi-simplicial sheaves to sShv•. Here Δ• is the standard affine co-simplicial scheme with \[ Δ^n = \{ (x_0, \ldots, x_n) \in \mathbb{A}^{n+1} | x_1 + \ldots + x_n = 1 \} \].

Denote by \( T \) the pair \((\mathbb{A}^1, \mathbb{G}_m)\) ∈ Smpair•. Define the smash-product functor

\[
∧: \text{Sm}^\text{pair}_k × \text{Sm}^\text{pair}_k \to \text{Sm}^\text{pair}_k
\]

\[
((X_1, U_1), (X_2, U_2)) \mapsto (X_1 × X_2, X_1 × U_2 ∪ X_2 × U_1).
\]

We write \( T^n \) for \( T ∧ n \), so \( T^n = (\mathbb{A}^n, \mathbb{A}^n - 0) \) ∈ Smpair•.

**Definition 4.** Define the \( \mathbb{P}^1 \)-spectra

\[
M_{\text{fr}}^{\mathbb{P}^1} (X, U) = (\text{Fr}(X/U), \text{Fr}((X/U) ∧ T), \ldots \text{Fr}((X/U) ∧ T^i), \ldots)
\]

\[
M_{\text{fr}}^{\mathbb{P}^1} (X, U) = C^* (M_{\text{fr}}^{\mathbb{P}^1} (X, U)),
\]

see definition 20 in Appendix for the structure maps of the spectra \( M_{\text{fr}}^{\mathbb{P}^1} (X, U) \). Define \( M_{\text{fr}}^{\mathbb{P}^1} (X) = M_{\text{fr}}^{\mathbb{P}^1} (X, \emptyset) \), \( M_{\mathbb{P}^1} (X) = M_{\mathbb{P}^1} (X, \emptyset) \).

**Remark 2.** Writing \((X/U) ∧ T\) in (10) we mean \((X, U) ∧ T\) in sense of (9).

Denote by \((-)_{\text{fr}}\) the the Nisnevich local resolution endo-functor on sShv•. Now we can formulate one of the main results of [5].

**Theorem 3** (Theorem 4.1 [5]). Let \( X \) be a smooth scheme over a infinite perfect field \( k \). Then the canonical morphism

\[
Σ_{\mathbb{P}^1}^\infty (X) \to M_{\mathbb{P}^1} (X)_{\text{fr}}
\]

is stable motivic weak equivalence and the right side is a motivically fibrant \( \Omega_{\mathbb{P}^1} \)-spectrum in positive degrees.

**Remark 3.** In other words the last statement of the theorem above means that for each \( l > 0 \) the pointed sheaf \( C^* \text{Fr}(X ∧ T^l)_{\text{fr}} \) is motivically fibrant within the Morel-Voevodsky model structure on sShv•, and the canonical morphism

\[
C^* \text{Fr}(X ∧ T^l)_{\text{fr}} \to \Omega_{\mathbb{P}^1} (C^* \text{Fr}(X ∧ T^{l+1})_{\text{fr}}),
\]

see def. 21 is a schemewise simplicial weak equivalence.

### 3. QUASI-FINITE FRAMED CORRESPONDENCES

In the section we recall the definition of quasi-finite framed correspondences \( \text{Fr}^\text{qf} (-, X/U) \) introduced in [8], that is a useful tool in studying of framed correspondences.

Main result of the section is the Nisnevich local equivalence \( \text{Fr}(X//U) ≃ \text{Fr}^\text{qf}(X/U) \), corollary 1, where \( X//U \) is simplicial cone. This generalises [8, corollary 5.3]. We apply this to prove the Mayer-Vietoris property for the framed correspondences functor \( \text{Fr}: \text{Sm}_k → \text{Shv}_• \), corollary 2 and see defs. 1-2 for the functor \( \text{Fr} \).

**Definition 5.** Let \( \text{Fr}^\text{qf}_n (-, X/(X - Y)) \) be a subpresheaf of \( \text{Fr}_n (-, X/(X - Y)) \) that consists of framed correspondences \( a = (V, Z, \varphi, g) \) such that \( \varphi^{-1}(0) \) is quasi-finite over \( U \). Set \( \text{Fr}^\text{qf} (-, X/(X - Y)) = \lim_{→} \text{Fr}^\text{qf}_n (-, X/(X - Y)) \).
Proposition 1. Let $X \in \text{Sm}_k$ and $U \subset X$ be an open subscheme. Then the natural morphism of pointed Nisnevich sheaves
\[ \text{Fr}_n(X)/\text{Fr}_n(U) \to \text{Fr}_n^qf(X/U) \]
is a Nisnevich local equivalence.

Proof. To prove the claim we consider a henselian $k$-schemes $S$ and construct the inverse morphism for the canonical morphism of germs
\[ c: \text{Fr}(S,X)/\text{Fr}(S,U) \to \text{Fr}_n^qf(S,X/U) \]
\[ (Z,V,\varphi,g) \mapsto (Z \times_{g,X,i}(X \setminus U), V, \varphi, g). \]

Let $(Z,V,\varphi,g) \in \text{Fr}_n^qf(S,X/U)$. Then $g: V \to X$ and $Z = V \times_{(\varphi,g),X,i}(X \setminus U)$, where $i: X \setminus U \to X$ is the canonical closed immersion. Since $S$ is local henselian it follows by lemma 1 that $Z(\varphi) = W_f \amalg W_{qf}$, such that $Z \subset W_f$ and $W_f$ is finite over $S$. Define the morphism of pointed sets
\[ r: \text{Fr}_n^qf(S,X/U) \to \text{Fr}(S,X)/\text{Fr}(S,U) \]
\[ (Z,V,\varphi,g) \mapsto (W_f, V - W_{qf}, \varphi, g). \]

We need to check that the morphism $r$ is well defined. To do this we need to compare the set of etale neighbourhoods of the subscheme $Z$ and the subscheme $W_f$. The sets are equal that follows form lemma 2.

Now we see that the composition $r \circ c$ is identity immediately by definitions. The composition $c \circ r$ is identity since by definition of $\text{Fr}_n^qf$ the element given by $(Z,V,\varphi,g)$ is equal to the element given by $(Z,V - W_{qf},\varphi,g)$. \square

Lemma 1. Let $S$ be a henselian local scheme and $W$ be a quasi-finite scheme over $S$. Then $W = W_f \amalg W_{qf}$, where $W_f$ is finite over $S$ and the closed fibre of $W_{qf}$ is empty.

Proof. By Zariski’s main [12, Theorem 8.12.6] the morphism $W \to S$ can be passed throw $W \to \overline{W} \to S$ with $W$ being finite over $S$. Then the closed fibre of $\overline{W}$ splits
\[ \overline{W} \times_S \sigma = W \times_S \sigma \amalg (\overline{W} \setminus W) \times_S \sigma. \]
Hence since $U$ is local henselian the scheme $\overline{W}$ splits into the union of clopen subschemes
\[ \overline{W} = W_f \amalg \overline{W}_{qf}, W_f \times_S \sigma = W \times_S \sigma, \overline{W}_{qf} \times_S \sigma = (\overline{W} \setminus W) \times_S \sigma. \]
Thus we get the required splitting
\[ W = W_f \amalg W_{qf}, W_f \times_S \sigma = W, W_{qf} \times_S \sigma = \emptyset. \]
\square

Lemma 2. Let $W \to S$ be a finite morphism, $S$ be a local henselian scheme, $\sigma \in S$ be the closed point. Let $W' \to W$ be en etale morphism such that $W' \times_S \sigma \simeq W \times_S \sigma$. Then $W' \simeq W$.

Proof. Since $S$ is local henselian and $W \to S$ is finite, it follows that $W$ is semilocal henselian. So the claim follows. \square
Definition 6. Let \( Y \hookrightarrow X \) be a morphism of smooth schemes. The *simplicial cone* \( X//Y \) of the morphism \( Y \to X \) is a simplicial scheme defined as follows.

If \( U = \emptyset \) then \( X//Y = X \). If \( U \neq \emptyset \), then \( X//Y \) is the colimit of the diagram in the category of simplicial schemes

\[
X//U = \lim\limits_{\leftarrow} (X \leftarrow U \hookrightarrow \Delta^1 \times X \leftarrow \Delta^1 \to *) \in \Delta^{\text{op}}_{\text{Sm}} k,
\]

where \( \Delta^1 \) is the simplicial interval, \( i_0, i_1 : U \to \Delta^1 \times X \) are the unit and zero faces, and \( * \) is a rational point that is the base point of \( X//U \).

Corollary 1. Let \( X \in \text{Sm}_k \) and \( U \subset X \) be an open subscheme. Then the natural morphism of simplicial pointed sheaves

\[
\text{Fr}_n(X//U) \to \text{Fr}_n^q(X/U)
\]

is a Nisnevich local equivalence, see def. 6 and def. 14 for \( \text{Fr}_n(X//U) \).

Proof. The claim follows by proposition 1 and the simplicial equivalence of pointed simplicial presheaves \( \text{Fr}_n(X//U) \simeq \text{Fr}_n(X)/\text{Fr}_n(U) \).

Corollary 2. Let \( X \in \text{Sm}_k, U, V \subset X \) be open subschemes, \( X = U \cup V \). Then the canonical morphisms of pointed simplicial Nisnevich sheaves

\[
\text{Fr}(V//((U \cap V)) \to \text{Fr}(X/U), \text{Fr}(V \amalg_{(U \cap V)} U) \to \text{Fr}(U \cup V)
\]

are Nisnevich local equivalences; see def. 4 and def. 14 for \( \text{Fr}(V \amalg_{(U \cap V)} U) \).

Proof. It is easy to see from the definition that \( \text{Fr}_n^q(V//((U \cap V)) \to \text{Fr}_n^q(X/U) \) Then by corollary 1 we have the Nisnevich local equivalence \( \text{Fr}(V//((U \cap V)) \to \text{Fr}(X//U). \) Then the equivalence \( \text{Fr}(V \amalg_{(U \cap V)} U) \to \text{Fr}(X) \) follows. \)

Definition 7. Let \( f_1 : U \to X \) and \( f_2 : U \to Y \) be morphisms in \( \text{Sm}_k \). Define a simplicial scheme \( V \amalg_{(U \cap V)} U \) as the colimit in the category of simplicial schemes

\[
V \amalg_{(U \cap V)} U = \lim\limits_{\leftarrow} (X \leftarrow f_1 U \hookrightarrow U \times \Delta^1 \leftarrow \Delta^1 \to Y) \in \Delta^{\text{op}}_{\text{Sm}} k
\]

where \( \Delta^1 \) denotes the simplicial interval, and, and the morphisms \( i_0, i_1 : U \to U \times \Delta^1 \) are the face morphisms.

Remark 4. The simplicial scheme \( V \amalg_{(U \cap V)} U \) represents the homotopy co-equaliser \( \text{hoceq}(U \to (X \amalg_{(U \cap V)} Y)) \) in the homotopy category of simplicial presheaves.

4. Moving Lemma.

In this section we prove a moving lemma contracting the sheaf \( \text{Fr}(-, X/(X - Y)) \) to the subsheaf \( \text{Fr}_n^q(-, X/(X - Y)) \) by an exhaustive family of partly defined \( \Delta^1 \)-homotopies. Throw out the section we work with given smooth affine \( X \in \text{AffSm}_k \) over an infinite field \( k \), and a closed subscheme \( Y \subset X \).

The main results of the section are proposition 2 and proposition 3. Before we formulate the result let us give the following definition.
Definition 8. Given a scheme $Y$ and integers $n$ and $i$, we put
\[ \Gamma_d = \{ s = (s_i) \in \Gamma(\mathbb{P}^n \times X, \mathcal{O}(d)^n) \mid |s_i|_{N \times Y} = x_i \cdot x_{\infty}^d \}, \]
where $\mathcal{N} = \text{Spec} \, k[\mathbb{A}^n]/(x_1, \ldots, x_n)^2 \subset Y \times \mathbb{P}^n$.

Proposition 2. Let $U$ be a scheme of a finite Krull dimension, and $\varphi = (\mathcal{V}, \varphi, g) \in \text{Fr}_{\mathfrak{n}}(U, X/(X - Y))$. Then for any $l \in \mathbb{Z}$ for some $d \in \mathbb{Z}$ there is an open subset $U \subset \Gamma_d$ such that
\begin{enumerate}
  \item $\text{codim} \, (W/\Gamma_d) \geq l$, where $W = (\Gamma_d \setminus U) \subset \Gamma_d$ is the closed complement, and
  \item for any rational point $s = (s_i) \in U$ the scheme-theoretical preimage $(f \circ (\varphi, g))^{-1}(0)$ is quasi-finite over $U$, where $f = (f_i), f_i = s_i/x_{\infty}^d$.
\end{enumerate}
Moreover if $\varphi \in \text{Fr}_{\mathfrak{n}}(U, X/(X - Y))$ then there is $U \subset \Gamma_d$ as above and such that $U \ni (x_1 x_{d}^{d-1}, x_1 x_{d}^{d-1}, \ldots, x_n x_{d}^{d-1})$.

Proof. Set $\mathcal{N} = \text{Spec} \, k[\mathbb{A}^n]/(x_1, \ldots, x_n)^2$, and for any scheme $G$ define
\begin{equation}
\Gamma_{d,G} = \{ s = (s_1, \ldots, s_n) \in \Gamma(\mathbb{P}^n \times X \times G, \mathcal{O}(d)^n) \mid \text{codim}_{\mathcal{N} \times \mathcal{Y} \times G} (s_i) = x_i \cdot x_{\infty}^d \}.
\end{equation}
Consider the universal section $\tilde{s} = (\tilde{s}_i) \in \Gamma_{d,\Gamma_d}$, and a closed subscheme $S = (\tilde{f} \circ (\varphi, g))^{-1}(0) \subset \mathcal{V} \times \Gamma_d$, where $\tilde{f} = (\tilde{f}_i), \tilde{f}_i = \tilde{s}_i/x_{\infty}^d$. In other words
\[ S = \{ (p, s) \in \mathcal{V} \times \Gamma_d \mid s((\varphi, g)(p)) = 0 \}. \]
Let $B \subset U \times \Gamma_d$ be a closed subscheme such that $z = (u, s) \in B$ whenever the fibre $S \times_U z$ is not quasi-finite over $z$. Define $U = \Gamma_d \setminus \text{pr}_{\Gamma_d}(B)$, where $\text{pr}_{\Gamma_d}(B)$ denotes the closure of the image of $B$ under the projection $\text{pr}_{\Gamma_d}: U \to \Gamma_d$. The proposition follows from the lemma.

Lemma 3. For any $m \in \mathbb{Z}$ there is $M(m)$, for all $d > M(m)$, we have
\begin{equation}
\text{codim} \, ((\Gamma_d \setminus U)/\Gamma_d) \geq m - \text{dim}_k U,
\end{equation}
where $U \subset \Gamma_d$ is the open subscheme as above.

Let us deduce the proposition. Choose some $m > \text{dim} \, U + l$, and $d > M(m)$ as in the lemma above. Then point (1) of the proposition follows by (16). Point (2) follows by the definition of $B$, and moreover, it follows that if $\varphi \in \text{Fr}_{\mathfrak{n}}(U, X/(X - Y))$ then $B \ni (x_1 x_{d}^{d-1}, x_2 x_{d}^{d-1}, \ldots, x_n x_{d}^{d-1})$.

Briefly speaking we argue for lemma 3 that is the following. Consider the $m$-th power $S^m_{U \times \Gamma_d}$ of $S$ over $U \times \Gamma_d$. For large enough $d$ the closed subscheme $S \subset \mathcal{V} \times \Gamma_d$ is defined by $n$ independent equations, and $S^m_{U \times \Gamma_d} \subset \mathcal{V}^m \times \Gamma_d$ is defined by $nm$ independent equations. So $\dim (S^m_{U \times \Gamma_d}) = \dim (\mathcal{V}^m \times \Gamma_d) - nm = \dim U + \dim \Gamma_d$. On other side the relative dimension of $S$ over $B$ is at least one, and $\dim_B (S^m_{U \times \Gamma_d}) \geq m$. Then $\text{codim} \, (B/\Gamma_d) \geq m$, and the claim follows. We start the strict detailed proof.

Proof of lemma 3 Consider the schemes $\mathbb{P}^n \times X, \mathcal{E} = \mathbb{A}^n \times X \times U$ and $\overline{\mathcal{E}} = \mathbb{P}^n \times X \times U$. The inverse images of $\mathcal{O}(1)$ from $\mathbb{P}^n$ to $\mathbb{P}^n \times X$ and $\overline{\mathcal{E}}$ we denote these sheaves by the
same symbol. Let \([x_0 : x_1 : \ldots : x_n]\) denote coordinates on \(\mathbb{P}^n\), and their inverse images as well. Since \(X\) is affine, it follows that \(\mathcal{O}(1)\) on \(\mathbb{P}^n \times X\) is ample.

Next, define the regular map

\[
\psi = (\varphi, g, \text{pr}_U^V) : V \to \mathcal{E}, \quad \text{where } \text{pr}_U^V : V \to U.
\]

Since \(Z = \psi^{-1}(0 \times Y \times U)\) is finite over \(U\), it follows that \(\psi\) is quasi-finite over \(0 \times Y \times U\). Hence there is a Zariski neighbourhood \(V'\) of \(0 \times Y \times U\) in \(\mathcal{E}\) such that \(\psi\) is quasi-finite over \(V'\). Shrink \(V\) to the open subscheme \(V \times_{\mathcal{E}} V' \subset V\). Then new \(\psi\) is quasi-finite.

Let \(\psi^m : V_U^m \to \mathcal{E}_U^m\) be \(m\)-th power of \(\psi\), and define open subschemes

\[
\mathcal{E} = \{(p_1, \ldots, p_m) \in \mathcal{E}_U^m | p_i \neq p_j, \text{ for } i \neq j, \text{ and } p_i \not\in 0 \times Y \times U \text{ for all } i\} \subset \mathcal{E}_U^m,
\]

and

\[
\mathcal{V} = (\psi^m)^{-1}(\mathcal{E}) = \mathcal{V}_U^m - (\psi^m)^{-1}(\mathcal{E}_U^m \setminus \mathcal{E}) \subset \mathcal{V}_U^m.
\]

Define a closed subscheme

\[
\overline{S} \subset \{(p, s) \in \mathcal{V} \times \Gamma : s((\varphi, g)(p)) = 0\} \subset \mathcal{V} \times \Gamma.
\]

Note that \(\overline{S} = S_U^{m} \times \Gamma_d \times \mathcal{V}_U^m \subset S_U^{m} \times \Gamma_d\) is an open subscheme. Consider the commutative diagram of schemes over \(U\)

where the left bottom square is Cartesian.

It follows by sublemma \([3.1]\) which follows in the text, that \(\text{codim}_{\mathcal{V} \times \Gamma_d} \overline{S} = nm\), hence

\[
\dim_U(S) = \dim_U(\mathcal{V} \times \Gamma_d) - nm = \dim_k \Gamma_d,
\]

and thus

\[
\dim(S / (U \times \Gamma_d)) = 0.
\]

On other side the points of the subscheme \(B\) are given by the pairs \((u, s)\), \(s \in \Gamma(\mathcal{E}, \mathcal{O}(d^n))\), \(u \in U\), such that the vanishing locus \(Z((\varphi, g)^*(s)) = (f \circ (\varphi, g))^{-1}(0)\) is not quasi-finite over \((u, s)\). So for any \((u, s) \in B\) we have \(\dim_{(u, s)} Z((\varphi, g)^*(s)) \geq 1\), and thus \(\dim((S \times B) / B) \geq 1\). Then

\[
\text{codim}(BS / B) \geq m, \quad BS = \overline{S} \times (U \times \Gamma_d)B.
\]
Thus
\[ \dim \text{pr}(B) \leq \dim B \leq \dim BS - m \leq \dim S - m \leq \dim(U \times \Gamma_d) - m = \dim \Gamma_d - (m - \dim U). \]

**Sublemma 3.1.** Given a point \( p = (p_1, \ldots, p_m) \in \mathcal{E} \), a closed subscheme \( Q \subset \mathbb{P}^n \times X \times p \) is the union of graphs of \( p \)-points \( (p_i) : p \to \mathbb{P}^n \times X \). Let
\[ r_p : \Gamma_{d,p} \to \mathcal{O}(Q) \simeq k(p)^{m(m-n)} : (s_i) \mapsto \left((s_i/x_i^d)|_Q\right) \]
denotes the restriction homomorphism, see (15) for \( \Gamma_{d,p} \).

Then for any \( m \in \mathbb{Z} \) there is \( M(m) \in \mathbb{Z} \) such that, for all \( d > M(m) \), for any point \( p \in \mathcal{E} \), the homomorphism \( r_p \) is surjective.

**Proof of lemma 3.1.** For any \( p = (p_1, \ldots, p_m) \in \mathcal{E} \) and \( d \in \mathbb{Z} \), denote
\[ j_p : Q \amalg \mathcal{N} \times Y \to \mathbb{P}^n \times X \times p. \]
Then we have the restriction homomorphism of sheaves
\[ \mathcal{O}(d)^n \to j_p^*(j_p^*(\mathcal{O}(d)^n)) \]
on \( \mathbb{P}^n \times X \times p \) and the restriction homomorphism of global sections
\[ \Gamma(\mathcal{E}, \mathcal{O}(d))^n \to \Gamma(Q \amalg \mathcal{N} \times Y)^n. \]

Let
\[ j : G \to \mathcal{E}, \ G = \left( \coprod_{i=1}^{m} \Delta_i \right) \coprod \mathcal{N} \times Y \times \mathcal{E}, \]
and \( \Delta_i \subset \mathcal{E} \times_U \mathcal{E} \) denotes graph of the \( i \)-th projection \( \mathcal{E} \to \mathcal{E}. \)

The scheme \( \mathbb{P}^n \times X \times p \) is equal to \( \mathcal{E} \times_U p \), and \( j_p \) is the fibre of \( j \). Consider the universal restriction homomorphism of coherent sheaves over \( \mathcal{E} \)
\[ \rho^*: \mathcal{O}(d)^n \to j^*(j^*(\mathcal{O}(d)^n)), \]
that is the surjective homomorphism of coherent sheaves on the scheme \( \mathcal{E} \times_U \mathcal{E} \) that is open subscheme in \( \mathbb{P}^n \times U \times (\mathbb{A}^n \times X)^m \). Consider the direct image of \( \rho^* \)
\[ \text{pr}^*(\rho^*_p) : \text{pr}^*(\mathcal{O}(d)^n) \to \text{pr}^*(j^*(j^*(\mathcal{O}(d)^n))), \text{pr} : \mathcal{E} \times_U \mathcal{E} \to \mathcal{E}, \]
that is a homomorphism of coherent sheaves on \( \mathcal{E} \). It follows by the relative version of Serre’s theorem on ample bundles [13, Chapter III, Theorem 8.8] that for large enough \( d \) the homomorphism \( \text{pr}^*(\rho^*_p) \) is surjective.
The sheaf \( \text{pr}_*(\mathcal{O}(d^n)) \) is the constant sheaf on \( E \) defined by to the \( k \)-vector space \( \Gamma(\mathbb{P}^n \times X, \mathcal{O}(d))^n \), and the sheaf \( \text{pr}_*(j_*(\mathcal{O}(d^n))) \) is equal to

\[
\bigoplus_{i=1}^{m} \mathcal{O}(\Delta_i)^{n} \oplus \Gamma(\mathcal{N} \times Y, \mathcal{O}(d))^n \simeq \mathcal{O}(E)^{nm} \oplus \Gamma(\mathcal{N} \times Y, \mathcal{O}(d))^n;
\]

where the second summands denote the constant sheaf on \( E \) defined by the \( k \)-vector space \( \Gamma(\mathcal{N} \times Y, \mathcal{O}(d))^n \).

Hence since \( \text{pr}_*(ρ_Z) \) is surjective the fibre of \( \text{pr}_*(ρ_Z) \) at a point \( p = (p_1, \ldots, p_m) \in E \) is equal to the surjective homomorphism of \( k(p) \)-vector spaces

\[
\Gamma(\mathbb{P}^n \times X, \mathcal{O}(d))^n \otimes k(p) \rightarrow k(p)^{mn} \oplus \Gamma(\mathcal{N} \times Y, \mathcal{O}(d))^n \otimes k(p), \text{ for any } p \in E.
\]

Thus the homomorphism \( \Gamma_{d,p} = \Gamma_d \otimes k(p) \rightarrow k(p)^{nm} \) is surjective for any \( p \in E \), because of the diagram of pointed sets with Cartesian squares

\[
\begin{array}{c}
(\Gamma_{d,p}(x)) \\
\downarrow \\
(\Gamma(\mathbb{P}^n \times X \times p, \mathcal{O}(d))^n, (x)) \\
\downarrow \\
(\Gamma(\mathcal{N} \times Y, \mathcal{O}(d))_{k(p)}^n, (x))
\end{array}
\]

\[
\xrightarrow{(\bigoplus_{i=1}^{m} k(p_i), (x))}
\]

where \( \Gamma(\mathcal{N} \times Y, \mathcal{O}(d))_{k(p)}^n = \Gamma(\mathcal{N} \times Y, \mathcal{O}(d))^n \otimes k(p) \), and all sets of sections are pointed at the class of the vector-section \( (x) = (x_i \cdot x_\infty^{d-1}) \) given by coordinates on \( \mathbb{P}^n \).

\[\square\]

**Definition 9.** Suppose \( U, X \in \text{Sm}_k \), \( Y \subset X \) is a closed subset, \( ϕ = [(Z, \mathcal{V}, ϕ, g)] \in Fr_n(U, X/(X - Y)) \) is a framed correspondence, and \( s = (s_i)_{i=1}^n \in \Gamma(\mathbb{P}^n \times X, \mathcal{O}(d))^n \) is a section such that \( s_i|_{\mathcal{I}(0 \times Y)} = x_i \).

Denote by \( ϕ^s \in Fr_n(U, X/(X - Y)) \) the framed correspondence defined by the class of the set \( (Z, \mathcal{V}, ϕ, g) \), where \( ϕ : A^n \rightarrow A^n \) is the regular map defined by functions \( s_i/x_\infty^d \in k[A^n], \) \( i = 1 \ldots n \).

**Definition 10.** Suppose \( U, X, Y \) are as in def. [9] and \( ϕ = [(Z, \mathcal{V}, ϕ, g)] \in Fr_n(U \times A^1, X/(X - Y)) \) is a framed correspondence, and \( s = (s_i)_{i=1}^n \in \Gamma(A^1 \times \mathbb{P}^n \times X, \mathcal{O}(d))^n \) is a section such that \( s_i|_{\mathcal{I}(A^1 \times 0 \times Y)} = x_i \).

Denote by \( ϕ^s \in Fr_n(U \times A^1, X/(X - Y)) \) the framed correspondence given by \( (Z, \mathcal{V}, ϕ', g) \), where \( ϕ' = ϕ \circ ζ \circ φ' \), \( \mathcal{V} \rightarrow A^1 \) is the canonical projection, and for any \( \lambda \in A^1 \),
the map $\zeta: \mathbb{A}^n_X \to \mathbb{A}^n_X$ is the regular map defined by functions $(s_i|_{\Lambda_X^m})/x^d_\infty \in k[\Lambda_X^m]$, $i = 1 \ldots n$.

Proposition 3. Assume the field $k$ is infinite, and let $U \in \text{Sm}_k$. Then for any framed correspondence

$$(a) \varphi \in \text{Fr}_n(U, X/(X - Y)), \text{ or } (b) \varphi \in \text{Fr}_n^{\text{df}}(U, X/(X - Y)),$$

there is $d \in \mathbb{Z}_{>0}$ and a vector of sections $s = (s_i) \in \Gamma(P^n_X, \mathcal{O}(d))^n$ such that

$$(a) \varphi^s \in \text{Fr}_n^{\text{df}}(U, X/(X - Y)), \text{ or } (b) \varphi^{\lambda x + (1 - \lambda)x} \in \text{Fr}_n^{\text{df}}(U \times \mathbb{A}^1, X/(X - Y))$$

respectively, where $x = (x, x^d - 1)$. 

Proof. a) Since $k$ is infinite the claim follows immediate from proposition 2.

b) By proposition 2 for some sufficiently big $d$ there is an open subscheme $U \subset \Gamma_d$ such that $\text{codim} (\Gamma_d \setminus U) \geq 2$ and $\varphi^s \in \text{Fr}_n^{\text{df}}(U \times \mathbb{A}^1, X/(X - Y))$ for any $s \in U$. Hence we have $\varphi^{\lambda x + (1 - \lambda)x} \in \text{Fr}_n^{\text{df}}(U \times \mathbb{A}^1, X/(X - Y))$, for any $s \in \Gamma_d - \text{Cl}_d(pr_x^{-1}(pr_x(\Gamma_d \setminus U)))$, where $pr_x: \Gamma_d - x \to \mathbb{P}^{\text{dim} \Gamma_d - 1}$ is the linear projection with centre $x$.

Whence $\text{codim} \text{Cl}_d(pr_x^{-1}(pr_x(\Gamma_d \setminus U))) \geq \text{codim} (\Gamma_d \setminus U) - 1 \geq 1$, and so the claim follows. 

5. PROOF OF THE RESULT

In the section we prove the Cone Theorem announced in the introduction, theorem 4, and apply it to deduce the formula for the motivically fibrant resolution in $\text{SH}(k)$, corollary 5. Throw out the section the base field $k$ is assumed being infinite, and it is perfect starting form corollary 5.

Theorem 4. Let $X$ be affine smooth $k$-scheme, and $U \subset X$ be open subscheme. The natural morphism of pointed simplicial sheaves

$$\text{Fr}(X/\!\!/U) \to \text{Fr}(X/U)$$

is a motivic equivalence, see def. 6 for $X/\!\!/U$.

Proof. By proposition 4, that follows in the section, we have the simplicial scheme-wise equivalence of pointed simplicial sheaves $C^*\text{Fr}^{\text{df}}(X/U) \to C^*\text{Fr}(X/U)$. By corollary 1 we have the scheme-wise equivalence of pointed simplicial sheaves $\text{Fr}(X/\!\!/U)_f \to \text{Fr}^{\text{df}}(X/U)_f$. So the claim follows. 

Proposition 4. Let $X$ be affine smooth $k$-scheme, and $U \subset X$ be open subscheme. Then the natural morphism of sheaves of pointed sets

$$\text{Fr}^{\text{df}}(X/U) \to \text{Fr}(X/U)$$

is an $\mathbb{A}^1$-equivalence. 

**Definition 11.** Let $F$ be a sheaf of pointed sets. For a variety $X$ denote by $F \times X$ the (naive) schemewise product of sheaves, $(F \times X)(S) = F(S) \times \text{Map}(S,X), S \in \text{Sm}_k$. In particular denote by $F \times A^1$ the presheaf given by

$$(F \times A^1)(S) = F(S) \times \mathcal{O}(S), S \in \text{Sm}_k.$$

Let $F = F_\infty \supset \cdots \supset F_i \supset F_{i-1} \supset \cdots \supset F_0$ be a filtration of the presheaf $F$ and denote by $i_t: F_i \to F_{i+1}$ the canonical inclusion. Define a telescope of the filtration $F_*$ as the pointed presheaf

$$(19) \quad \text{Tel}(F_*) = \cdots \amalg (F_{i+1} \times A^1) \amalg_{F_i} (F_i \times A^1) \amalg_{F_{i-1}} F_i \cdots \amalg F_0 (F_0 \times A^1)$$

where the coproducts are defined with respect to the inclusions $i_t \times 1: F_i \times 1 \to F_{i+1} \times A^1$, $id_{F_i} \times 0: F_i \times 0 \to F_i \times A^1$.

**Lemma 4.** For any pointed presheaf $F$ with the filtration (19) the canonical morphism $\text{Tel}(F) \to F$ is an $A^1$-homotopy equivalence.

**Proof.** Clearly for any presheaf $F$ the canonical morphism $F \times A^1 \to F$ is an $A^1$-equivalence. Denote $\text{Tel}^t(F) = (F_i \times A^1) \amalg_{F_{i-1}} (F_{i-1} \times A^1) \amalg_{F_{i-2}} F_i \cdots \amalg F_0 (F_0 \times A^1)$. It follows that the canonical morphism $\text{Tel}^t(F) \to F$ is an $A^1$-equivalence.

Since the injective limits preserves weak equivalences on simplicial sets, and the functor $\text{Hom}(\Delta^\bullet, -)$ commutes with the injective limits, it follows that injective limits preserve $A^1$-equivalences. Thus the claim follows since $F = \varinjlim_t F_t$ and $\text{Tel}(F_*) = \varinjlim_t \text{Tel}^t(F_*)$. \hfill $\Box$

We define the notion of the telescope for filtrations indexed by an arbitrary filtering ordered set.

Let $A$ be an ordered set and $F_\alpha \subset F$, $\alpha \in A$, be a filtration on $A$, i.e. $F = \bigcup_{\alpha \in A} F_\alpha$ and $F_\alpha \subset F_\beta$ for $\alpha < \beta$. Define a telescope of the filtration $F_*$ by the following.

Consider the ordered set $\mathcal{CA}$ of finite linearly ordered subsets of $A$, i.e. the category with objects being the sets $s = (\alpha_0 > \alpha_1 > \alpha_2 > \cdots > \alpha_n)$ and a unique morphism $s_1 \to s_2$ if $s_1$ is a subset of $s_2$. For any $n$-dimensional simplex $s = (\alpha_0 > \alpha_1 > \alpha_2 > \cdots > \alpha_n)$ in $\mathcal{CA}$ define $F_s = F_{\alpha_0}$. We get the functor $\mathcal{CA}$ to the category of subpresheaves of $F$.

Now we put

$$\text{Tel}(F_*) = \varinjlim_{s \in \mathcal{CA}} F_s \times \Delta^s,$$

where $\Delta^n$ denote the affine $n$-dimensional simplex, the morphisms $F_{s_1} \times \Delta^s \to F_{s_2} \times \Delta^s$ are given by the product of the morphisms $F_{s_1} \to F_{s_2}$ and $\Delta^s \to \Delta^s$, and the last one is the face map corresponding to the inclusion $s_1 \subset s_2$.

**Lemma 5.** For any presheaf $F$ with the filtration (19) indexed by the filtering set $A$ the canonical morphism $\text{Tel}(F) \to F$ is an $A^1$-homotopy equivalence.

**Proof.** Clearly for any presheaf $F$ the canonical morphism $F \times \Delta^n \to F$ is an $A^1$-equivalence. For any $\alpha \in A$ denote by

$$\text{Tel}^\alpha(F_*) = \varinjlim_{s \in \mathcal{CA} \subset \alpha} F_s \times \Delta^s,$$
where \(CA_{<\alpha}\) is the category of \(CA\) spanned by the objects \(s = (\alpha_0 > \alpha_1 > \ldots \alpha_n)\) with \(\alpha_0 \leq \alpha\). It follows from the above that the canonical morphism \(Tel^\alpha(F) \to F\) is an \(A^1\)-equivalence.

Since weak equivalences on simplicial sets are preserved under the injective limits, and \(\text{Hom}(\Delta^\bullet, -)\) commutes with the injective limits, it follows that injective limits preserves \(A^1\)-equivalences. Thus the claim follows from the commutative diagrams,

\[
\begin{array}{ccc}
Tel^\alpha(F) & \longrightarrow & F_\alpha \\
\downarrow & & \downarrow \\
Tel(F) & \longrightarrow & F
\end{array}
\]

since \(Tel(F) = \lim_{\to \alpha} Tel^\alpha(F)\), \(F = \lim_{\to \alpha} F_\alpha\). \(\square\)

**Lemma 6.** For any finite set of sections \(\alpha \subset Fr\) there is a section \(s \in \Gamma(\mathbb{P}^n_d, O(d))^n\) such that

\[
\alpha^s_\alpha \subset Fr^\alpha_n(S, X/U), \beta^{\lambda s_\alpha + (1-\lambda)x} \in Fr^\alpha_n(S \times A^1, X/U),
\]

where \(\beta = \alpha \cap Fr^\alpha_n(S, X/U)\), \(x = (x_i x^d_{\infty}^{-1})\), and see def. \([9]\) and def. \([10]\) for the maps \((-)^s\) and \((-)^{\lambda s_\alpha + (1-\lambda)x}\).

**Proof.** The claim follows from proposition \([3]\). \(\square\)

**Proof of proposition \([4]\).** Denote the pointed sheaves \(Fr = Fr_n(X/U)\) and \(Fr^\alpha = Fr^\alpha_n(X/U)\)

Consider the filtering set \(A\) of finite sets \(\alpha\) of sections \((\Phi_i)_{i \in \alpha}\) of sheaf the \(Fr\). So for each \(\alpha \in A\) we canonically have a set of correspondences

\[
\Phi_i \in Fr_n(S_i, X/U), S_i \in \text{Sm}_k, i \in \alpha.
\]

Note that \(A\) is a set, since the category \(\text{Sm}_k\) is small. Define \(Fr_\alpha \subset Fr\) as the smallest subpresheaves of \(Fr\) containing \(\alpha\). Define \(Fr^\alpha = Fr^\alpha_n \cap Fr_\alpha\). Then we have the pair of filtrations

\[
Fr = \lim_{\to \alpha} Fr_\alpha, Fr^\alpha = \lim_{\to \alpha} Fr^\alpha_\alpha.
\]

Lemma \([6]\) gives us the morphisms

\[
(-)^{s_\alpha} : Fr_\alpha \to Fr^\alpha,
\]

\[
(-)^{\lambda s_\alpha + (1-\lambda)x} : Fr_\alpha \times A^1 \to Fr, (-)^{\lambda s_\alpha + (1-\lambda)x} : Fr^\alpha_\alpha \times A^1 \to Fr^\alpha.
\]

Then we consider the telescopes of the filtrations on \(Fr\) and \(Fr^\alpha\) (see the discussion above for the definition), and extend \([5]\) to morphisms of pointed simplicial sheaves

\[
Tel(Fr_\alpha) \to Fr^\alpha_\alpha,
\]
\[
Tel(Fr_\alpha) \times A^1 \to Fr_\alpha, Tel(Fr^\alpha_\alpha) \times A^1 \to Fr^\alpha.
\]
To do this for an arbitrary linearly ordered subset \(\{\alpha_0, \ldots, \alpha_r\} \subset A\) we define morphisms

\[
Fr_\alpha \times \Delta^r \to Fr^{\text{qf}},
\]

\[
(c, (\lambda_0, \ldots, \lambda_r)) \mapsto c^s, \quad s = \sum_{i=0}^r \lambda_i s_{\alpha_i},
\]

and

\[
Fr_\alpha \times \mathbb{A}^1 \times \Delta^r \to Fr,
\]

\[
(c, \lambda, (\lambda_0, \ldots, \lambda_r)) \mapsto c^{\lambda s + (1-\lambda)x},
\]

\[
(c, \lambda, (\lambda_0, \ldots, \lambda_r)) \mapsto c^{\lambda s + (1-\lambda)x}.
\]

Then due to the homotopies (5) the compositions

\[
Tel(Fr) \to Fr_\ast \to Fr, \quad Tel(Fr^{\text{qf}}) \to Tel(Fr) \to Fr^{\text{qf}}
\]

are \(\mathbb{A}^1\)-homotopy equivalent to the canonical maps

\[
(20) \quad Tel(Fr) \to Fr_\ast \to Fr, \quad Tel(Fr^{\text{qf}}) \to Tel(Fr) \to Fr^{\text{qf}}
\]

Thus since by lemma 5 the canonical morphisms (20) are \(\mathbb{A}^1\)-equivalences, the \(\mathbb{A}^1\)-equivalence \(Fr^{\text{qf}} \simeq Fr\) follows. \(\square\)

Next we improve the Cone Theorem 4 to the case of products of pairs.

**Corollary 3.** Let \(X\) and \(Y\) be smooth affine \(k\)-schemes, and \(U \subset X\) and \(V \subset Y\) be open subschemes.

Then the canonical morphism

\[
Fr(X/U \wedge (Y//V)) \to Fr(X/U \wedge (Y,V))
\]

is a motivic equivalence of pointed simplicial Nisnevich sheaves. At the left side \((X,U) \times (Y,V)\) denotes the simplicial object in \(\text{Sm}_{\text{pair}}^\bullet(k)\) given by the open immersion of simplicial schemes \(U \times \mathcal{Y} \to X \times \mathcal{Y}\), and see def. 14.

**Proof.** The claim follows form three equalities:

1) By theorem 4 we have the motivic equivalence of pointed simplicial sheaves

\[
Fr(X \times Y//((U \times Y \cup X \times V))) \to Fr(X/U \wedge Y/U)
\]

2) By corollary 2 we have the Nisnevich local equivalence \(Fr(U \times Y \Pi_{(U \times V)} X \times V) \to Fr(U \times Y \cup X \times V),\) see def. 4 for \(\Pi^+.\) Hence

\[
Fr((X//U) \wedge (Y//U)) \to Fr(X \times Y//((U \times Y \cup X \times V)))
\]

is a Nisnevich local equivalence.

3) By theorem 4 again the morphism

\[
Fr(X/U \wedge (Y//V)) \to Fr((X//U) \wedge (Y//V))
\]

is a motivic equivalence, since morphisms

\[
Fr(X/U \wedge Y) \to Fr((X//U) \times Y),
\]

\[
Fr(X/U \wedge V) \to Fr((X//U) \times V)
\]

are of such type. \(\square\)
Now we apply the Cone Theorem (theorem 4 and corollary 3) to get the computational results in the stable motivic homotopy category announced in the introduction. Assume that the base field $k$ is perfect.

**Corollary 4.** Consider the canonical morphism of $S^1$-spectra of pointed simplicial Nisnevich sheaves

$$M_{fr}(X//U) \to M_{fr}(X,U),$$

see def. 23 for $M_{fr}$, and def. 6 for $X//U$. Then for any smooth affine $X$ and open $U$ the morphism (21) is a levelwise Nisnevich local equivalence in positive degrees.

**Proof.** Theorem 4 implies the motivic equivalence of pointed simplicial Nisnevich sheaves $Fr((X//U) \times K) \simeq Fr((X/U) \times K)$ for any simplicial set $K$. Hence we get the motivic equivalence $M_{fr}^{S^1}(X//U) \to M_{fr}^{S^1}(X,U)$, see def. ?? Then since $M_{fr} = C^*(M_{fr}^{S^1})$, the morphism (21) is a motivic equivalence. By proposition 5 it follows that Nisnevich local injective fibrant replacements $M_{fr}(X//U)$ and $M_{fr}(X,U)$ are levelwise motivically fibrant spectra in positive degrees. So the claim follows.

Now we assume in addition that the base field $k$ is perfect.

**Corollary 5.** The canonical morphism $\Sigma^\infty_{P^1}(X/U) \to M_{P^1}(X,U)_f$ is the stable motivic weak equivalence of $P^1$-spectra of pointed simplicial sheaves. The $P^1$-spectra $M_{P^1}(X,U)_f$ is a positively motivically fibrant $\Omega_{P^1}$-spectrum. See def. 23 for $M_{P^1}$.

**Proof.** By the first point of [5, Theorem 10.1] we have the stable motivic weak equivalences

$$\Sigma^\infty_{P^1}(X/U) \simeq \Sigma^\infty_{P^1}(X//U) \simeq M_{P^1}(X//U)_f$$

Namely [5, Theorem 10.1] provides the second equivalence in the sequence above, and the first one is induced by the motivic equivalence $X//U \simeq X/U$.

By corollary 4 we have the levelwise Nisnevich local equivalence of $(\mathbb{G}_m, S^1)$-bi-spectra

$$C^*(M_{fr}^{Gm,S^1}(X//U)) \simeq C^*(M_{fr}^{Gm,S^1}(X,U))$$

see def. 17 for $M_{fr}^{(Gm,S^1)}$. Hence we have the levelwise Nisnevich local equivalence of $(\mathbb{G}_m^{\Lambda^1} \land S^1)$-bi-spectra (see def. 18 for $M_{fr}^{(\mathbb{G}_m^{\Lambda^1} \land S^1)}$)

$$C^*(M_{fr}^{(\mathbb{G}_m^{\Lambda^1} \land S^1)}(X//U)) \simeq C^*(M_{fr}^{(\mathbb{G}_m^{\Lambda^1} \land S^1)}(X,U)),$$

since the above morphism of $(G_m^{\Lambda^1} \land S^1)$-spectra is the diagonal of bi-spectra morphism (23). Then due to the $\mathbb{A}^1$-local equivalences of simplicial schemes $\mathbb{A}^1//G_m \simeq (pt//G_m) \simeq G_m^{\Lambda^1} \land S^1$ by remark 7 we have the levelwise Nisnevich local equivalence

$$C^*(M_{fr}^{(\mathbb{A}^1//G_m)}(X//U)) \simeq C^*(M_{fr}^{(\mathbb{A}^1//G_m)}(X,U)),$$
see def. 19 for $M^{G^1/S_1}$. Now by corollary 3 and remark 8 we get the levelwise Nisnevich local equivalences of $T$-spectra and $\mathbb{P}^{A^1}$-spectra of pointed simplicial sheaves

\begin{equation}
C^*(M^T_{fr}(X//U)) \simeq C^*(M^T_{fr}(X,U)),
\end{equation}

\begin{equation}
C^*(M^{\mathbb{P}^{A^1}}_{fr}(X//U)) \simeq C^*(M^{\mathbb{P}^{A^1}}_{fr}(X,U)),
\end{equation}

see def. 20 and def. 21 or def. 4. Hence by def. 4 we get the levelwise schemewise simplicial equivalence of $\mathbb{P}^{A^1}$-spectra of pointed simplicial sheaves,

\begin{equation}
M_{\mathbb{P}^{A^1}}(X//U)_f \simeq M_{\mathbb{P}^{A^1}}(X,U)_f
\end{equation}

and combining with equivalence (22) we get the first claim of the corollary.

The second point of [5, Theorem 10.1] implies that $M_{\mathbb{P}^{A^1}}(X//U)_f$ is positively motivically fibrant $\Omega$-spectrum, hence by (25) the spectrum $M_{\mathbb{P}^{A^1}}(X,U)_f$ is positively motivically fibrant as well.

**Corollary 6.** The canonical morphism of bi-spectra

$$\Sigma^\infty_{G_m} \Sigma^\infty_{S_1}(X/U) \to M^G_{fr}(X,U)_f$$

is a stable motivic weak equivalence, and the spectrum $M^G_{fr}(X,U)_f$ is motivically fibrant. See def. 23 for $M^G_{fr}(X,U)$.

**Proof.** By the definition $M^G_{fr}(-) = C^*(M^{(G_m,S^1)}_{fr}(-))$. So by (23)

$$M^G_{fr}(X//U)_f \to M^G_{fr}(X,U)_f$$

is a levelwise schemewise simplicial equivalence of simplicial sheaves. So it is enough to prove the claim for the spectrum $M^G_{fr}(X//U)_f$.

By [5, Theorem 11.4] the canonical morphism

$$\Sigma^\infty_{G_m} \Sigma^\infty_{S_1}(X/U) \to M^G_{fr}(X//U)_f$$

is stable motivic weak equivalence.

It follows by [3, Corollary 7.5] that each term $C^*(Fr((X//U) \wedge G^1_m \wedge S^2))_f$, $j > 0$, of the spectrum $M^G_{fr}(X//U)_f$ is motivically fibrant.

The bi-spectra $M^G_{fr}(X)_f$ and $M^G_{fr}(U)_f$ are $\Omega_{S^1}$-bi-spectra by [3, Theorem 6.5] and are $\Omega_{G_m}$-bi-spectra by [6, Theorem A]. Then it follows that $M^G_{fr}(X//U)_f$ is an $\Omega_{(G_m,S^1)}$-bi-spectrum.

**Corollary 7.** The $S^1$-spectrum of pointed simplicial sheaves $M_{fr}(X,U)_f$ is a motivically fibrant $\Omega$-spectrum in positive degrees and has the homotopy type of $\Omega^\infty_{G_m} \Sigma^\infty_{G_m} \Sigma^\infty_{S^1}(X/U)$ in $\text{SH}^{S^1}(k)$.

**Proof.** The claim follows form corollary 6 since $M_{fr}(X,U)_f$ is the zero $G_m$-row of the spectrum $M^G_{fr}(X,U)_f$. 

□
6. Appendix. Framed motivic spectra.

In the appendix we summarise the definitions, constructions, and lemmas on framed correspondences and framed motivic spectra used in the text. Some of definitions recovers the ones from the section \[2\].

**Definition 12.** (i) We call by the category of **pointed smooth open pairs** \(\text{Sm}^\bullet_{\text{pair}}(k)\) the category with objects \((X, Z, U)\) given by \(X \in \text{Sm}_k\), closed subscheme \(Z \subset X\), and open \(U \subset X\); a morphism form \((X, Z, U)\) to \((X_2, Z_2, U_2)\) is given by \(f: X_1 \to X_2\), \(f^{-1}(Z_2) \supset Z_1\), \(f^{-1}(U_2) \supset U_1\).

Define the subcategory of **pointed schemes** \(\text{Sm}^\bullet_{\text{s}}(k)\) in \(\text{Sm}^\bullet_{\text{pair}}(k)\) spanned by objects of the type \((X, Z, \emptyset)\); and the subcategory of **open pairs** \(\text{Sm}^\bullet_{\text{pair}}\) spanned by objects of the type \((X, \emptyset, U)\).

(ii) Define the smash-product \(-\land-: \text{Sm}^\bullet_{\text{pair}}(k) \times \text{Sm}^\bullet_{\text{pair}}(k) \to \text{Sm}^\bullet_{\text{pair}}(k):\)

\[ ((X_1, Z_1, U_1), (X_2, Z_2, U_2)) \mapsto (X_1 \times X_2, Z_3, U_3), \]

\[ Z_3 = (Z_1 \times X_2) \cup (Z_2 \times X_1), \quad U_3 = (U_1 \times X_2) \cup (U_2 \times X_1). \]

**Definition 13.** Let \((X, Z, U) \in \text{Sm}^\bullet_{\text{pair}}(k)\). Define the pointed sheaf

\[ \text{Fr}((X, Z)/U) = \underset{\overset{\text{colim}}{\text{lim}}}{}((\text{Fr}(Z/(U \times_X Z)) \to \text{Fr}(X/U)) \in \text{Shv}_{\bullet}, \]

see def. 2 for \(\text{Fr}(X/U)\).

**Definition 14.** Denote by \(\text{op}\text{Sm}^\bullet_{\text{pair}}(k)\) the category of simplicial objects \(\text{Sm}^\bullet_{\text{pair}}(k)\). Define the functor

\[ \text{Fr}: \text{opSm}^\bullet_{\text{pair}}(k) \to \text{sShv}_{\bullet}, \]

as the functor induced by \(\text{Sm}^\bullet_{\text{pair}}(k) \to \text{Shv}_{\bullet}: (X, Z, U) \mapsto \text{Fr}((X, Z)/U)\).

**Definition 15.** A set of morphisms \(X \times \mathbb{A}^n \to X\) for all schemes \(X\) we call by **strong naive \(\mathbb{A}^1\)-local equivalences of schemes**.

A morphism of simplicial scheme \(f: \mathcal{X} \to \mathcal{Y}\) is called as **strong naive \(\mathbb{A}^1\)-local equivalence** iff \(f\) is termwise **strong naive \(\mathbb{A}^1\)-local equivalence of schemes**.

**Remark 5.** Any strong naive \(\mathbb{A}^1\)-local equivalence of simplicial smooth schemes \(\mathcal{X} \to \mathcal{Y}\) induces the \(\mathbb{A}^1\)-local equivalence \(\text{Fr}(\mathcal{X}) \to \text{Fr}(\mathcal{Y})\).

More generally a strong naive \(\mathbb{A}^1\)-local equivalence of simplicial smooth pairs \((\mathcal{X}_1, \mathcal{U}_1) \to (\mathcal{X}_2, \mathcal{U}_2)\) induces the the schemewise simplicial equivalence

\[ C^*(\text{Fr}(\mathcal{X}_1/\mathcal{U}_1)) \to C^*(\text{Fr}(\mathcal{X}_2/\mathcal{U}_2)). \]

**Definition 16.** Let \((X, U)\) be a pair given by smooth affine scheme \(X\) and open subscheme \(U \subset X\), or a simplicial pair. Define an \(S^1\)-spectrum of pointed simplicial sheaves

\[ M^S_{fr}(X, U) = (\text{Fr}(X/U), \text{Fr}((X/U) \wedge S^1), \ldots \text{Fr}((X/U) \wedge S^i), \ldots) \]
**Definition 17.** Let \((X, U)\) be as in def. 16. Define an \((S^1, \mathbb{G}^A_m)\)-spectrum of pointed simplicial sheaves \(M_{fr}^{(G_m, S^1)}(X, U)\) as the bi-spectrum with the terms
\[
\text{Fr}((X/U) \wedge S^i \wedge \mathbb{G}^A_m)
\]
where \(\mathbb{G}^A_m = \mathbb{G}_m/\{1\}\), see def. 6 and see def. 14 for \(\text{Fr}(\mathbb{G}^A_m)\). Define \(M_{fr}^{(G_m, S^1)}(X) = M_{fr}^{(G_m, S^1)}(X, \emptyset)\).

**Definition 18.** Let \((X, U)\) be as in def. 16. Define an \((\mathbb{G}^A_m \wedge S^1)\)-spectrum of pointed simplicial sheaves \(M_{fr}^{(G^A_m \wedge S^1)}(X, U)\) as
\[
\text{Fr}(X/U), \text{Fr}((X/U) \wedge \mathbb{G}^A_m \wedge S^1), \ldots \text{Fr}((X/U) \wedge \mathbb{G}^A_m \wedge S^i), \ldots).
\]

**Remark 6.** \(M_{fr}^{(G^A_m \wedge S^1)}(X, U)\) is the diagonal of the bi-spectrum \(M_{fr}^{(G_m, S^1)}(X, U)\).

**Definition 19.** Let \((X, U)\) be as in def. 16. Define an \((\mathbb{A}^1/(\mathbb{A}^1 - 0))\)-spectrum of pointed simplicial sheaves \(M_{fr}^{(\mathbb{A}^1/G_m)}(X, U)\) as
\[
\text{Fr}(X/U), \text{Fr}((X/U) \wedge (\mathbb{A}^1/(\mathbb{A}^1 - 0))), \ldots \text{Fr}((X/U) \wedge (\mathbb{A}^1/(\mathbb{A}^1 - 0)))^{\wedge i}, \ldots).
\]

**Remark 7.** It follows by remark 5 that the terms of spectra \(M_{fr}^{(G^A_m \wedge S^1)}(X, U)\) and \(M_{fr}^{(\mathbb{A}^1/G_m)}(X, U)\) are levelwise \(\mathbb{A}^1\)-local equivalent. Actually, we have a strong naive \(\mathbb{A}^1\)-local equivalence of pointed simplicial schemes \(\mathbb{A}^1/G_m \rightarrow \text{pt}/\mathbb{G}_m\), and an simplicial equivalence of pointed simplicial schemes \((\text{pt}/\mathbb{G}_m) \cong \mathbb{G}^A_m \wedge S^1\). Hence there is an \(\mathbb{A}^1\)-local equivalence of pointed simplicial presheaves \(\text{Fr}((X/U) \wedge (\mathbb{A}^1/G_m)) \rightarrow \text{Fr}((X/U) \wedge (\mathbb{G}^A_m \wedge S^i))\).

**Definition 20.** Let \((X, U)\) be as in def. 16. Define an \((\mathbb{A}^1/(\mathbb{A}^1 - 0))\)-spectrum of pointed simplicial sheaves
\[
M_{fr}^T(X, U) = (\text{Fr}(X/U), \text{Fr}((X/U) \wedge T), \ldots \text{Fr}((X/U) \wedge T^n), \ldots),
\]
where \(T\) denotes the open pair \((\mathbb{A}^1, \mathbb{A}^1 - 0)\), see 9 for the smash-product. the structure maps are given by
\[
\text{Fr}(S, (X/U) \wedge T^i) \rightarrow \text{Hom}_\bullet(\mathbb{A}^1/(\mathbb{A}^1 - 0) \times S, \text{Fr}((X/U) \wedge T^{i+1}))
\]
\[
(Z, V, \varphi_1, \ldots \varphi_n, g) \rightarrow (0 \times Z, \mathbb{A}^1 \times V, \varphi_1, \ldots \varphi_n, (t, g)),
\]
where \(\text{Hom}_\bullet\) denotes the pointed hom-set in \(\text{sShv}_\bullet\),
\[
(0 \times Z, \mathbb{A}^1 \times V, \varphi_1, \ldots \varphi_n, (t, g)) \in \text{Fr}(\mathbb{A}^1 \times S, (X/U) \wedge T^{i+1}),
\]
where the function \(t\) is the coordinate on \(\mathbb{A}^1\), and \(25\) is considered as morphism of simplicial sheaves \(\mathbb{A}^1 \times S \rightarrow \text{Fr}((X/U) \wedge T^{i+1})\), that passes throw \(\mathbb{A}^1/(\mathbb{A}^1 - 0) \times S\).

The canonical morphism of pointed sheaves \(\mathbb{P}^{\mathbb{A}^1} \rightarrow \mathbb{A}^1/(\mathbb{A}^1 - 0)\) induces the functor
\[
\nu: \text{Spec}_{\mathbb{A}^1/(\mathbb{A}^1 - 0)}(\text{sShv}_\bullet) \rightarrow \text{Spec}_{\mathbb{P}^{\mathbb{A}^1}}(\text{sShv}_\bullet)
\]
form the category of \((\mathbb{A}^1/(\mathbb{A}^1 - 0))\)-spectra to \(\mathbb{P}^{\mathbb{A}^1}\)-spectra of pointed simplicial sheaves.
Definition 21. Let \((X, U)\) be as in def. 16. Define \(\mathbb{P}^1\)-spectra
\[
M_{fr}^{\mathbb{P}^1}(X, U) = \nu(M_{fr}^T(X, U)), M_{fr}^{\mathbb{P}^1}(X) = \nu(M_{fr}^T(X)).
\]

Remark 8. It follows immediate form the definition that any (stable motivic) equivalence of \((\mathbb{A}^1/(\mathbb{A}^1 - 0))\)-spectra \(M_{fr}^T(X, U) \simeq M_{fr}^T(X', U')\), for some simplicial pairs of smooth schemes \((X, U), (X', U')\), induces the (stable motivic) equivalence of \(\mathbb{P}^1\)-spectra \(M_{fr}^{\mathbb{P}^1}(X, U) \simeq M_{fr}^{\mathbb{P}^1}(X', U')\).

Definition 22. For any \(M_{fr}^*\) from def. 16-21 set \(M_{fr}^*(X) = M_{fr}^*(X, \emptyset)\).

Definition 23. Define the following \(S^1\)-spectrum, \((\mathbb{G}_m, S^1)\)-bi-spectrum and \(\mathbb{P}^1\)-spectrum of pointed simplicial sheaves
\[
\begin{align*}
M_{fr}(-) &= C^*(M_{fr}^{S^1}(-)), M_{fr}^{\mathbb{G}_m}(-) = C^*(M_{fr}^{(\mathbb{G}_m, S^1)}(-)), \\
M_{fr}^{\mathbb{P}^1}(X, U) &= C^*(M_{fr}^{\mathbb{P}^1}(-))
\end{align*}
\]
respectively. The notation are agreed with [5].

Proposition 5. For any smooth scheme \(X\) over an infinite perfect field \(k\) and open subscheme \(U\) the spectrum \(M_{fr}(X, U)_f\) is a levelwise motivically fibrant \(\Omega_{S^1}\)-spectrum in positive degrees.

Proof. By [5] corollary 7.5] the spectrum \(M_{fr}((X, U) \wedge S^1)_f\) is levelwise motivically fibrant \(\Omega_{S^1}\)-spectrum. Hence the claim follows, since by construction \(M_{fr}((X, U) \wedge S^1)_f\) is equal to the shift of \(M_{fr}(X, U)_f\). \(\square\)

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