Non-existence of ground states and gap of variational problems for combined power-type nonlinear scalar field equations involving the Sobolev critical exponent in three space dimensions

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Abstract

In this paper, we consider minimization problems related to the combined power-type nonlinear scalar field equations involving the Sobolev critical exponent in three space dimensions. In four and higher space dimensions, it is known that for any frequency and any power of the subcritical nonlinearity, there exists a ground state. In contrast to those cases, when the space dimension is three and the subcritical power is three or less, we can show that there exists a threshold frequency, above which no ground state exists, and below which the ground state exists (see Theorems 1.1 and 1.2). Furthermore, we prove the difference between two typical variational problems used to characterize the ground states (see Theorem 1.3).

1 Introduction

In this paper, we consider the existence of minimizer for the following minimization problem:

\[ m_N^\omega := \inf \{ S_\omega(u) : u \in H^1(\mathbb{R}^d) \setminus \{0\}, \ N_\omega(u) = 0 \}, \tag{1.1} \]

where \( S_\omega \) and \( N_\omega \) are the functionals on \( H^1(\mathbb{R}^d) \) defined by

\[ S_\omega(u) := \frac{1}{2} \| \nabla u \|^2_{L^2} + \frac{\omega}{2} \| u \|^2_{L^2} - \frac{1}{p+1} \| u \|^{p+1}_{L^{p+1}} - \frac{d-2}{2d} \| u \|^{\frac{2d}{d-2}}_{L^{\frac{2d}{d-2}}}, \tag{1.2} \]

\[ N_\omega(u) := \| \nabla u \|^2_{L^2} + \omega \| u \|^2_{L^2} - \| u \|^{p+1}_{L^{p+1}} - \| u \|^{\frac{2d}{d-2}}_{L^{\frac{2d}{d-2}}}, \tag{1.3} \]

with \( 1 < p < \frac{d+2}{d-2} \). We refer \( S_\omega \) and \( N_\omega \) as action and Nehari’s functional, respectively.

The main reason why we consider the minimization problem (1.1) is that when a minimizer exists, it solves the following Sobolev-critical scalar field equation:

\[ -\Delta u + \omega u - |u|^{p-1}u - |u|^{\frac{4}{d-2}}u = 0 \quad \text{in} \ H^{-1}(\mathbb{R}^d). \tag{1.4} \]

We note that the equation (1.4) is related to the following nonlinear Schrödinger equation

\[ i \frac{\partial \psi}{\partial t} + \Delta \psi + |\psi|^{p-1} \psi + |\psi|^{\frac{4}{d-2}} \psi = 0 \quad \text{in} \ \mathbb{R}^d \times \mathbb{R}. \tag{1.5} \]
When we look for solutions to (1.5) of the form $\psi(t, x) = e^{i\omega t}u(x) \ (\omega > 0)$, we see that $u$ satisfies (1.4).

In this paper, we call a minimizer for $m^N_\omega$ a “ground state of $m^N_\omega$.” The definition of ground state seems to vary depending on papers. Indeed, in our previous papers [3, 1], a ground state is defined as a minimizer for the following problem:

$$m^S_\omega := \inf \{ S_\omega(v) : v \in H^1(\mathbb{R}^d) \text{ is a nontrivial (complex-valued) solution to (1.4)} \}. \quad (1.6)$$

Clearly, if $u \in H^1(\mathbb{R}^d)$ is a solution to (1.4), then $u \in H^1(\mathbb{R}^d)$ satisfies $N_\omega(u) = 0$, so that $m^S_\omega \leq m^S_\omega$. Furthermore, as mentioned, a ground state (minimizer for $m^N_\omega$) satisfies (1.4), so that $m^N_\omega = m^S_\omega$ as long as a ground state exists.

The aim of this paper is to study the existence and non-existence of ground state (minimizer for $m^N_\omega$) in $\mathbb{R}^3$. To this end, we introduce a supplementary variational value $m_\infty$ as

$$m_\infty := \inf \{ \mathcal{H}^t(u) : u \in \dot{H}^1(\mathbb{R}^d) \setminus \{0\}, \mathcal{N}^t(u) = 0 \}, \quad (1.7)$$

where

$$\mathcal{H}^t(u) := \frac{1}{2} \| \nabla u \|_{L^2}^2 - \frac{d}{2} \| u \|_{\frac{2d}{d-2}}^{\frac{2d}{d-2}}, \quad (1.8)$$

$$\mathcal{N}^t(u) := \| \nabla u \|_{L^2}^2 - \| u \|_{\frac{2d}{d-2}}^{\frac{2d}{d-2}}. \quad (1.9)$$

Putting here$
\sigma := \inf \{ \| \nabla u \|_{L^2}^2 : u \in \dot{H}^1(\mathbb{R}^d) \text{ with } \| u \|_{L^{\frac{2d}{d-2}}} = 1 \}, \quad (1.10)$

we can verify the following (see Section A for the proof):

$$m_\infty = \frac{1}{d} \sigma \frac{d}{d-2}. \quad (1.11)$$

Notice that the value $\sigma$ gives us the best constant of Sobolev’s inequality in $\mathbb{R}^d$ (see [5, 23]):

$$\sigma \| u \|_{L^{\frac{2d}{d-2}}}^2 \leq \| \nabla u \|_{L^2}^2 \quad (1.12)$$

with equality if and only if

$$u(x) = z\lambda_0^{-1}W(\lambda_0^{\frac{1}{d-2}} x + x_0) \quad \text{for some } z \in \mathbb{C}, \lambda_0 > 0 \text{ and } x_0 \in \mathbb{R}^d, \quad (1.13)$$

where $W$ is the Talenti function on $\mathbb{R}^d$ with $W(0) = 1$, namely,

$$W(x) := \left( 1 + \frac{|x|^2}{d(d-2)} \right)^{-\frac{d-2}{2}}. \quad (1.14)$$

It is well known that for any $d \geq 3$, we have

$$\| \nabla W \|_{L^2}^2 = \| W \|_{L^{\frac{2d}{d-2}}}^{\frac{2d}{d-2}} = \sigma \frac{d}{d-2}. \quad (1.15)$$

Moreover, the values $m_\infty$ is related to a sufficient condition for the existence of ground state (cf, [8]). Indeed, we have the following Brezis-Nirenberg type proposition:
Proposition 1.1. Assume $d \geq 3$ and $1 < p < \frac{d+2}{d-2}$. Then, for any $\omega > 0$ with $m_{\omega}^{N} < m_{\infty}$, a ground state (minimizer for $m_{\omega}^{N}$) exists.

Remark 1.1. It is known (see [13] and Proposition 2.1 of [3]) that for any ground state $Q_{\omega}$, there exist $\theta \in [0, 2\pi)$, $y \in \mathbb{R}^{d}$ and a positive radial ground state $\Phi_{\omega}$ such that $Q_{\omega} = e^{i\theta}\Phi_{\omega}(\cdot - y)$. Thus, we may assume that ground states are positive radial solutions to \[(1.4)\].

We give a proof of Proposition 1.1 in Appendix B. We can also find that $m_{\omega}^{N}$ satisfies the following:

Proposition 1.2. Assume $d \geq 3$ and $1 < p < \frac{d+2}{d-2}$. Then, the value $m_{\omega}^{N}$ is nondecreasing with respect to $\omega$ on $(0, \infty)$.

We will prove Proposition 1.2 in Section C. Observe from Proposition 1.2 that the set \{$\omega > 0$: $m_{\omega}^{N} < m_{\infty}$\} is connected. This fact together with Proposition 1.1 motivates us to introduce the following value:

$$\omega_{c}^{N} := \sup \{ \omega > 0: m_{\omega}^{N} < m_{\infty} \} . \quad (1.16)$$

Remark 1.2. (i) It follows from a result in [24] that $m_{\omega}^{N} < m_{\infty}$ for any $3 < p < 5$ and any $\omega > 0$. Hence, when $3 < p < 5$, $\omega_{c}^{N} = \infty$ (see also [1, 3]).

(ii) The authors of the present paper showed in [3] that for any $1 < p \leq 3$, there exists $\omega_{0} > 0$ such that if $\omega \in (0, \omega_{0})$, then $m_{\omega}^{N} < m_{\infty}$. Hence, when $1 < p \leq 3$, $\omega_{c}^{N} > 0$.

Now, we state main results of this paper (recall that by a ground state, we mean a minimizer for $m_{\omega}^{N}$):

Theorem 1.1. Assume $d = 3$ and $1 < p \leq 3$. Then, the following holds:

(i) $0 < \omega_{c}^{N} < \infty$.

(ii) If $0 < \omega < \omega_{c}^{N}$, then a ground state exists; and if $\omega > \omega_{c}^{N}$, then there is no ground state.

Theorem 1.2. Assume $d = 3$ and $1 < p < 3$. If $\omega = \omega_{c}^{N}$, then a ground state exists.

Remark 1.3. When $p = 3$ and $\omega = \omega_{c}^{N}$, we have not obtained any information about the existence of ground state.

Here, we would like to emphasize that the significance of the study for the existence and non-existence of ground state in $\mathbb{R}^{3}$. When $d \geq 4$, it is known that for any $\omega > 0$ and any $1 < p < \frac{d+2}{d-2}$, a ground state exists (see Proposition 1.1 of [1]). On the other hand, Theorem 1.1 tells us that when $d = 3$, we cannot always find a solution to (1.4) (or a standing wave of (1.5)) through the minimization problem for $m_{\omega}^{N}$.

Note here that, as mentioned before, we have $m_{\omega}^{N} \leq m_{\omega}^{S}$ for any $\omega > 0$, in particular, $m_{\omega}^{N} = m_{\omega}^{S}$ if a ground state exists. It would be an interesting problem to reveal whether or not we always have $m_{\omega}^{N} = m_{\omega}^{S}$. Concerning this, we can obtain the following result:

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Theorem 1.3. Assume $d = 3$. If $1 < p < 3$, then $m^N_\omega < m^S_\omega$ for any $\omega > \omega^N_\omega$; and if $p = 3$, then there exists $\omega_* \geq \omega^N_\omega$ such that $m^N_\omega < m^S_\omega$ for any $\omega > \omega_*$.

Remark 1.4. By Proposition 1.1 and the definition of $\omega^N_\omega$, we see that if $d \geq 3$, $1 < p < d+2$ and $\omega < \omega^N_\omega$, then we have $m^N_\omega = m^S_\omega$. We will prove Theorem 1.3 in Section 4.

We make a comment on a result in [2]. The same authors of the present paper considered the following minimization problem in [2]:

$$m^K_\omega := \inf \left\{ S_\omega(u) : u \in H^1(\mathbb{R}^d) \setminus \{0\}, \ K(u) = 0 \right\},$$

where

$$K(u) := \|\nabla u\|_{L^2}^2 - \frac{d(p - 1)}{2(p + 1)} \|u\|_{L^{p+1}}^{p+1} - \|u\|_{L^{\frac{2d}{d-2}}}^{\frac{2d}{d-2}}.$$ (1.17)

In Theorem 1.2 of [2], the authors claimed that if $d = 3$, $1 + \frac{4}{3} < p < 5$ and $\omega$ is sufficiently large, then there is no minimizer for $m^K_\omega$. However, the claim is false; indeed, we can show that for any $3 < p < 5$ and any $\omega > 0$, there exists a minimizer for $m^K_\omega$ in a way similar to Proposition 1.1 of [1]. In the present paper, we correct the mistake. To this end, we introduce the following value $\omega^K_\omega$, as well as $\omega^N_\omega$:

$$\omega^K_\omega := \sup \left\{ \omega > 0 : m^K_\omega < m_\infty \right\}.$$ (1.18)

Then, we can obtain Theorem 1.4 below, in which the third claim (iii) gives us the correct range of the exponent $p$ concerning the existence of minimizer for $m^K_\omega$:

Theorem 1.4. Assume $d = 3$ and $\frac{7}{3} < p < 5$. Then, all of the following holds:

(i) $\omega^N_\omega = \omega^K_\omega$.

(ii) $m^N_\omega = m^K_\omega$ for any $\omega > 0$.

(iii) Assume $1 + \frac{4}{3} < p \leq 3$ ($p = 3$ is excluded). If $0 < \omega < \omega^N_\omega (= \omega^K_\omega)$, then a minimizer for $m^K_\omega$ exists; and if $\omega > \omega^N_\omega (= \omega^K_\omega)$, then there is no minimizer for $m^K_\omega$.

(iv) Assume $1 + \frac{4}{3} < p < 3$. If $\omega = \omega^N_\omega (= \omega^K_\omega)$, then a minimizer for $m^K_\omega$ exists.

Remark 1.5. We can verify that any ground state (minimizer for $m^N_\omega$) becomes a minimizer for $m^K_\omega$, and vice versa.

We give a proof of Theorem 1.4 in Section 6.

In what follows, we will always fix $d = 3$ unless otherwise noted. We introduce the notation used in this paper:

Notation and auxiliary results.

(i) We define the operators $L_+$ and $L_-$ by

$$L_+ := -\Delta + V_+ \quad \text{with} \quad V_+ := -5W^4,$$

$$L_- := -\Delta + V_- \quad \text{with} \quad V_- := -W^4.$$ (1.20)

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The operators $L_+$ and $L_-$ are self-adjoint on $L^2(\mathbb{R}^3)$ with the domain $H^2(\mathbb{R}^3)$. For the information about the spectrum of $L_+$, see Lemma 7.1 below.

Furthermore, we define the operator $H$ by

$$Hu := L_+\Re[u] + iL_-\Im[u].$$

We refer $H$ as the linearized operator around the Talenti function $W$.

(ii) $J(u) := \left(\frac{1}{2} - \frac{1}{p+1}\right)\|u\|^{p+1}_{L^{p+1}} + \frac{1}{3}\|u\|^6_{L^6}.$

(iii) For $d \geq 3$, we define the operator $\Lambda : \dot{H}^1(\mathbb{R}^3) \to \dot{H}^1(\mathbb{R}^3)$ by

$$\Lambda u := \frac{1}{2}u + x \cdot \nabla u.$$ (1.24)

Notice that $\Lambda W \notin L^2(\mathbb{R}^3)$, and $\Lambda W$ is a zero energy resonance of $L_+$. By direct computations, we can verify that

$$V_+ \partial_j W = \Delta \partial_j W \quad \text{for} \quad 1 \leq j \leq 3, \quad V_+ \Lambda W = \Delta \Lambda W, \quad V_- W = \Delta W.$$ (1.25)

(iv) We use the symbol $\langle \cdot, \cdot \rangle$ to denote the scalar product in $L^2(\mathbb{R}^3)$, namely, if $f, g \in L^2(\mathbb{R}^3)$, then

$$\langle f, g \rangle = \Re \int_{\mathbb{R}^3} f(x)\overline{g(x)} \, dx.$$ (1.26)

We also use the same symbol to denote the pairing between $H^1(\mathbb{R}^3)$ and $H^{-1}(\mathbb{R}^3)$, namely if $f \in H^1(\mathbb{R}^3)$ and $g \in H^{-1}(\mathbb{R}^3)$, then

$$\langle f, g \rangle = \Re \int_{\mathbb{R}^3} \langle \nabla f(x) \langle \nabla \rangle^{-1} g(x) \rangle \, dx.$$ (1.27)

(v) For $\nu > 0$, we define the scaling operator $T_\nu$ to be that for any function $u$ on $\mathbb{R}^3$,

$$T_\nu u(x) := \nu^{-1} u(\nu^{-\frac{2}{3}}x).$$ (1.28)

Notice that $\|\nabla T_\nu u\|_{L^2} = \|\nabla u\|_{L^2}$ and $\|T_\nu u\|_{L^6} = \|\nabla u\|_{L^6}$.

(vi) For any $(y, \nu, \theta) \in \mathbb{R}^3 \times (0, \infty) \times [-\pi, \pi)$, we define the transformation $T_{(y, \nu, \theta)}$ by

$$T_{(y, \nu, \theta)}[u](x) := \nu^{-1} e^{i\theta} u(\nu^{-\frac{2}{3}}x + y).$$ (1.29)

(vii) We put

$$b_0 = (0, 1, 0) \in \mathbb{R}^3 \times (0, \infty) \times [-\pi, \pi).$$

Moreover, for $\delta > 0$ and $r > 0$, we define

$$B_{H^1}(W, \delta) := \{ u \in \dot{H}^1(\mathbb{R}^3) : \|u - W\|_{H^1} \leq \delta \},$$

$$B(b_0, r) := \{ (y, \nu, \theta) \in \mathbb{R}^3 \times \mathbb{R} \times [-\pi, \pi) : |y| + |\nu - 1| + |\theta| < r \}.$$
2 Unrealizable sequence

We will employ the contradiction argument in order to prove Theorems 1.1 through 1.3. The following theorem plays a crucial role to prove these theorems:

**Theorem 2.1.** Assume $1 < p \leq 3$. Then, there is no sequence \( \{(Q_n, \omega_n)\} \) in \( H^1(\mathbb{R}^3) \times (0, \infty) \) satisfying all of the following conditions:

(i) \( Q_n \) is a solution to \( (1.4) \) with \( \omega = \omega_n \).

(ii) \( \lim \inf_{n \to \infty} \omega_n > 0 \) if \( 1 < p < 3 \), and \( \lim \sup_{n \to \infty} \omega_n = \infty \) if \( p = 3 \).

(iii) \( \lim_{n \to \infty} \|Q_n\|_{L^\infty} = \infty \), and \( \lim_{n \to \infty} \omega_n \|Q_n\|_{L^\infty}^{-1} = 0 \).

(iv) Define \( \tilde{Q}_n \) by \( \tilde{Q}_n(x) := \|Q_n\|_{L^\infty}^{-1}Q_n(\|Q_n\|_{L^\infty}^{-2}x) \). Then, \( \lim_{n \to \infty} \tilde{Q}_n = W \) strongly in \( \dot{H}^1(\mathbb{R}^3) \).

We will give the proof of Theorem 2.1 in Section 2.2 below.

2.1 Preliminaries

In order to prove Theorem 2.1, we need some preparations.

It is well known (see, e.g., Theorem 6.23 of [19]) that for any \( \alpha > 0 \) and any function \( u \) on \( \mathbb{R}^3 \),

\[
(-\Delta + \alpha)^{-1}u(x) = \int_{\mathbb{R}^3} \frac{e^{-\sqrt{\alpha}|x-y|}}{4\pi|x-y|}u(y)dy. \tag{2.1}
\]

Moreover,

\[
(-\Delta)^{-1}f(x) := \int_{\mathbb{R}^3} \frac{\Gamma(\frac{3}{2})}{12\pi^3}|x-y|^{-1}f(y)dy, \tag{2.2}
\]

where \( \Gamma \) denotes the Gamma function, namely

\[
\Gamma(\nu) = \int_0^\infty e^{-t}t^{\nu-1}dt \quad \text{for } \nu > 0.
\]

For our convenience, we record the Hardy- Littlewood-Sobolev inequality: For \( 0 < s < 3 \), and \( 1 < q < r < \infty \) with \( s - \frac{3}{q} = \frac{3}{r} \),

\[
\left\| \int_{\mathbb{R}^3} \frac{f(y)}{|x-y|^{3-s}}dy \right\|_{L^r} \lesssim \|f\|_{L^q}. \tag{2.3}
\]

See, e.g. Theorem 2.6 of [20].

The following lemma holds from the Young and the weak Young inequalities (see, e.g., (9) in Section 4.3 of [19])

**Lemma 2.1.** Let \( \alpha > 0 \), and let \( 1 \leq s \leq q \leq \infty \) and \( 3(\frac{1}{s} - \frac{1}{q}) < 2 \). Then, we have

\[
\|(-\Delta + \alpha)^{-1}\|_{L^s(\mathbb{R}^3) \to L^q(\mathbb{R}^3)} \lesssim \alpha^{\frac{3}{2}(\frac{1}{s} - \frac{1}{q})^{-1}}, \tag{2.4}
\]

where the implicit constant depends only on \( d, s, q \) and \( \alpha \). Furthermore, if \( 1 < s \leq q < \infty \) and \( 3(\frac{1}{s} - \frac{1}{q}) < 2 \), then,

\[
\|(-\Delta + \alpha)^{-1}\|_{L^{s,\text{weak}}(\mathbb{R}^3) \to L^q(\mathbb{R}^3)} \lesssim \alpha^{\frac{3}{2}(\frac{1}{s} - \frac{1}{q})^{-1}}, \tag{2.5}
\]
where the implicit constant depends only on $d$, $s$ and $q$.

Lemma 2.1 does not deal with the case where $3\left(\frac{1}{s} - \frac{1}{q}\right) = 2$. In that case, we use the following:

**Lemma 2.2.** Let $\alpha > 0$, and let $3 < q < \infty$. Then, we have

\[ \|(-\Delta + \alpha)^{-1}\|_{L^{\frac{3q}{3+2q}}(\mathbb{R}^3) \to L^q(\mathbb{R}^3)} \lesssim 1, \tag{2.6} \]

where the implicit constant depends only on $s$ and $q$.

**Proof of Lemma 2.2.** It follows from the first resolvent equation that for any $\alpha > 0$ and $\alpha_0 \geq 0$,

\[ (-\Delta + \alpha)^{-1} - (-\Delta + \alpha_0)^{-1} = -(\alpha - \alpha_0)(-\Delta + \alpha)^{-1}(-\Delta + \alpha_0)^{-1}. \tag{2.7} \]

We see from (2.7) with $\alpha_0 = 0$, Lemma 2.1 and the Hardy-Littlewood-Sobolev inequality (2.3) (or Sobolev’s embedding) that

\[ \|f\|_{L^q} \leq \|\{(\alpha - \alpha_0)(-\Delta + \alpha)^{-1}(-\Delta + \alpha_0)^{-1}\}f\|_{L^q} + \|\{(\Delta)^{-1}f\|_{L^q} \lesssim \|f\|_{L^{\frac{3q}{3+2q}}}. \tag{2.8} \]

Note that if $3 < q$, then $\frac{3q}{3+2q} > 1$. Hence, the desired estimate (2.4) holds. \hfill \square

In addition to the estimates above, we will use the following result given in Lemma 2.7 of [9]:

**Lemma 2.3.** Let $\alpha > 0$. Then, we have

\[ \alpha^\frac{1}{2} \langle W, (-\Delta + \alpha)^{-1}V_+ \Lambda W \rangle = 6\pi + O(\alpha^\frac{1}{2}). \tag{2.9} \]

Moreover, for any $2 < p < 5$ and any $0 < \delta_1 < \min\{1, p - 2\}$, we have

\[ \langle W^p, (-\Delta + \alpha)^{-1}V_+ \Lambda W \rangle = -\frac{5 - p}{10(p + 1)}\|W\|^p_{L^{p+1}} + O(\alpha^\frac{\delta_1}{2}). \tag{2.10} \]

The restriction about $p$ in Lemma 2.3 comes from the fact that $W \not\in L^{p+1}(\mathbb{R}^3)$ for $1 \leq p \leq 2$. When $1 < p \leq 2$, we use the following estimate, instead of (2.10) in Lemma 2.3:

**Lemma 2.4.** Let $\alpha > 0$. Then, for any $1 < p \leq 2$ and any $0 < \delta < \frac{p}{2}$, we have

\[ \left| \langle W^p, (-\Delta + \alpha)^{-1}V_+ \Lambda W \rangle \right| \lesssim \alpha^{-\frac{2p}{2} - \delta}, \tag{2.11} \]

where the implicit constant depends only on $p$ and $\delta$. 

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Proof of Lemma 2.4. Let $1 < p \leq 2$ and $0 < \delta < \frac{p}{2}$. Furthermore, put $\delta' := \frac{6\delta}{p-2\delta}$ and $q' := \frac{3+\delta'}{p}$. Notice that $\frac{3}{2} < q' < \infty$. Hence, it follows from (2.3) in Lemma 2.1 with $s = q'$ and $q = \infty$, and $W \in L^{3+\delta'}(\mathbb{R}^3)$ that

\[
\left|\langle(-\Delta + \alpha)^{-1}V_+AW, W^p\rangle\right| = \left|\langle V_+\Lambda W, (-\Delta + \alpha)^{-1}W^p\rangle\right|
\leq \|V_+\Lambda W\|_{L^1} \|(-\Delta + \alpha)^{-1}W^p\|_{L^\infty} \lesssim \|(-\Delta + \alpha)^{-1}W^p\|_{L^{3+\delta'}} \lesssim \alpha^{-\frac{p}{2} - \delta},
\]
where the implicit constants depend only on $p$ and $\delta$. Hence, we have proved the lemma.

For $\alpha > 0$, we define the operators $G(\alpha)$ and $G(\alpha)^*$ by

\[
G(\alpha)f := \left\{1 + (-\Delta + \alpha)^{-1}V_+\right\}Re[f] + i\left\{1 + (-\Delta + \alpha)^{-1}V_-\right\}Im[f], \quad (2.13)
\]
\[
G(\alpha)^*f := \left\{1 + V_+(-\Delta + \alpha)^{-1}\right\}Re[f] - i\left\{1 + V_-(-\Delta + \alpha)^{-1}\right\}Im[f]. \quad (2.14)
\]

Notice that

\[
H + \alpha = (-\Delta + \alpha)G(\alpha). \quad (2.15)
\]

Theorem 2.2. Assume $6 < r < \infty$. Then, there exists $\alpha_* > 0$ depending only on $r$ such that for any $0 < \alpha < \alpha_*$ the inverse of $G(\alpha)$ exists as bounded operator from $L^r(\mathbb{R}^3)$ to itself, and for any $f \in L^r(\mathbb{R}^3)$,

\[
\|G(\alpha)^{-1}f\|_{L^r} \lesssim \alpha^{-\frac{1}{2}}\|f\|_{L^r}, \quad (2.16)
\]
where the implicit constant depends only on $d$ and $r$. Furthermore, if $f \in L^r(\mathbb{R}^3)$ satisfies

\[
\langle f, V_+\nabla W \rangle = 0, \quad \langle f, V_+\Lambda W \rangle = 0, \quad \langle f, iV_-\Lambda W \rangle = 0, \quad (2.17)
\]
then

\[
\|G(\alpha)^{-1}f\|_{L^r} \lesssim \|f\|_{L^r}, \quad (2.18)
\]
where the implicit constant depends only on $r$.

We give a proof of Theorem 2.2 in Section 3.

In order to ensure the orthogonalities in (2.17), we need the following geometric decomposition:

Theorem 2.3. There exist $0 < \delta_{geo} < 1$ and $0 < r_{geo} < \frac{1}{2}$ such that for any $0 < \alpha < 1$, there exists a unique continuous mapping $b_\alpha = (y_\alpha, \nu_\alpha, \theta_\alpha): B_{H^1}(W, \delta_{geo}) \rightarrow B(b_0, r_{geo})$ with the following properties:

\[
b_\alpha(W) = (y_\alpha(W), \nu_\alpha(W), \theta_\alpha(W)) = (0, 1, 0) = b_0, \quad (2.19)
\]
and if $u \in B_{H^1}(W, \delta_{geo})$, then

\[
\langle T_{b_\alpha(u)}[u] - W, G(\alpha)^*V_+\partial_jW \rangle = 0 \quad \text{for } 1 \leq j \leq 3, \quad (2.20)
\]
\[
\langle T_{b_\alpha(u)}[u] - W, G(\alpha)^*V_+\Lambda W \rangle = 0, \quad (2.21)
\]
\[
\langle T_{b_\alpha(u)}[u] - W, iG(\alpha)^*V_-W \rangle = 0. \quad (2.22)
\]
Remark 2.1. Since the conditions (2.20) through (2.22) contain the parameter \( \alpha \), Proposition 2.3 does not follow directly from the implicit function theorem; The point is that \( \delta_{\text{geo}} \) is independent of \( \alpha \).

We will prove Theorem 2.3 in Section 7.4.

2.2 Proof of Theorem 2.1

We give a proof of Theorem 2.1:

Proof of Theorem 2.1. Suppose for contradiction that there exists a sequence \( \{(Q_n, \omega_n)\} \) in \( H^1(\mathbb{R}^3) \times (0, \infty) \) satisfying the conditions (i) through (iv) in Theorem 2.1: hence \( Q_n \) is a solution to (1.4) with \( \omega_n = \omega_n \); and
\[
\lim \inf_{n \to \infty} \omega_n > 0 \quad \text{if} \quad 1 < p < 3, \quad \text{and} \quad \lim \sup_{n \to \infty} \omega_n = \infty \quad \text{if} \quad p = 3,
\]
(2.23)
\[
\lim_{n \to \infty} M_n = \infty, \quad \lim_{n \to \infty} \alpha_n = 0,
\]
(2.24)
\[
\lim_{n \to \infty} \tilde{Q}_n = W \quad \text{strongly in} \quad \dot{H}^1(\mathbb{R}^3),
\]
(2.25)
where
\[
M_n := \|Q_n\|_{L^\infty}, \quad \alpha_n := \omega_n M_n^{-4}, \quad \tilde{Q}_n := T_{M_n} Q_n = M_n^{-1} Q_n(M_n^{-2}).
\]
(2.26)

We also introduce \( \beta_n := M_n^{p-5} \). Note that \( \tilde{Q}_n \) is the normalization of \( Q_n \) in \( L^\infty(\mathbb{R}^3) \), namely
\[
\|\tilde{Q}_n\|_{L^\infty} = 1.
\]
(2.27)

We shall derive a contradiction. Let \( \delta_{\text{geo}} \) and \( r_{\text{geo}} \) be the constants given in Theorem 2.2. By (2.24) and (2.25), passing to a subsequence, we may assume that \( 0 < \alpha_n < 1 \) and \( \tilde{Q}_n \in B_{H^1}(W, \delta_{\text{geo}}) \) for all \( n \). Hence, for any \( n \), we can apply Theorem 2.3 to \( \tilde{Q}_n \), and obtain the sequence \( \{(y_n, \nu_n, \theta_n)\} \) in \( B(b_0, r_{\text{geo}}) \) such that:
\[
\lim_{n \to \infty} |y_n| = 0, \quad \lim_{n \to \infty} \nu_n = 1, \quad \lim_{n \to \infty} \theta_n = 0,
\]
(2.28)
\[
\langle T_{(y_n, \nu_n, \theta_n)}[Q_n] - W, G(\alpha_n) V_+ \partial_j W \rangle = 0, \quad 1 \leq j \leq 3,
\]
(2.29)
\[
\langle T_{(y_n, \nu_n, \theta_n)}[\tilde{Q}_n] - W, G(\alpha_n) V_+ \Lambda W \rangle = 0,
\]
(2.30)
\[
\langle T_{(y_n, \nu_n, \theta_n)}[\tilde{Q}_n] - W, G(\alpha_n) V_- (iW) \rangle = 0,
\]
(2.31)
\[
\lim_{n \to \infty} \|T_{\nu_n} [\tilde{Q}_n (\cdot + y_n)] - W\|_{H^1} = 0.
\]
(2.32)

Then, we define
\[
u[n] := T_{(y_n, \nu_n, \theta_n)}[Q_n],
\]
(2.33)
\[
\eta[n] := \nu[n] - W = T_{(y_n, \nu_n, \theta_n)}[\tilde{Q}_n] - W.
\]
(2.34)
Observe that $u_{[n]}$ satisfies
\[ -\Delta u_{[n]} + \alpha_{[n]} u_{[n]} - \beta_{[n]} |u_{[n]}|^{p-1} u_{[n]} - |u_{[n]}|^4 u_{[n]} = 0, \]
(2.35)
\[ \alpha_{[n]} := \alpha_n \nu_n^{-4}, \]
(2.36)
\[ \beta_{[n]} := \beta_n \nu_n^{-5}. \]
(2.37)

Note that
\[ \beta_{[n]} = \omega_n^{\frac{5}{p-2}} \alpha_{[n]}^{\frac{5}{p-2}}. \]
(2.38)

Furthermore, plugging (2.34) into (2.35), one can see that $\eta_{[n]}$ satisfies
\[ (H + \alpha_{[n]}) \eta_{[n]} = F_{[n]} := -\alpha_{[n]} W + \beta_{[n]} W^p + N_{[n]}, \]
(2.39)
where $H$ is the linearized operator around $W$ (see (1.22)), and
\[ N_{[n]} := |W + \eta_{[n]}|^4 (W + \eta_{[n]}) - W^5 - 5 W^4 \Re \eta_{[n]} - i W^4 \Im \eta_{[n,\nu]} \]
\[ + \beta_{[n]} \{ |W + \eta_{[n]}|^{p-1} (W + \eta_{[n]}) - W^p \}. \]
(2.40)

Using the factorization (2.15), we can rewrite the equation (2.39) as
\[ G(\alpha_{[n]}) \eta_{[n]} = (-\Delta + \alpha_{[n]})^{-1} F_{[n]}. \]
(2.41)

By the orthogonalities (2.29) through (2.31), and (2.41), one can see that
\[ \langle (-\Delta + \alpha_{[n]})^{-1} F_{[n]}, V_{+} \partial_j W \rangle = \langle \eta_{[n]}, G(\alpha_{[n]})^* V_{+} \partial_j W \rangle = 0 \quad \text{for } 1 \leq j \leq 3, \]
(2.42)
\[ \langle (-\Delta + \alpha_{[n]})^{-1} F_{[n]}, V_{+} \Lambda W \rangle = \langle \eta_{[n]}, G(\alpha_{[n]})^* V_{+} \Lambda W \rangle = 0, \]
(2.43)
\[ \langle (-\Delta + \alpha_{[n]})^{-1} F_{[n]}, i V_{-} W \rangle = \langle \eta_{[n]}, G(\alpha_{[n]})^* V_{-} (i W) \rangle = 0. \]
(2.44)

Here, we claim the following:

**Lemma 2.5.** For any $1 < p \leq 3$, any $0 < \varepsilon < \frac{p-1}{20}$, and any sufficiently large $n$, the following estimate holds:
\[ \| \eta_{[n]} \|_{L^h} \lesssim \alpha_{[n]}^{-\varepsilon}, \]
(2.45)
where the implicit constant depends only on $p$, $\varepsilon$ and the quantity $\lim_{n \to \infty} \omega_n$ (if finite).

**Proof of Lemma 2.5.** We shall divide the proof into 2 steps.

**Step 1.** Let $0 < \varepsilon < \frac{p-1}{20}$. Then, by (2.41), (2.39), Theorem (2.2) and (2.42) through (2.44), one can see that
\[ \| \eta_{[n]} \|_{L^h} \lesssim \| F_{[n]} \|_{L^h} \leq \alpha_{[n]} \| (-\Delta + \alpha_{[n]})^{-1} W \|_{L^h} + \beta_{[n]} \| (-\Delta + \alpha_{[n]})^{-1} W^p \|_{L^h} \]
\[ + \| (-\Delta + \alpha_{[n]})^{-1} N_{[n]} \|_{L^h}. \]
(2.46)
We consider the first two terms on the right-hand side of (2.46). It follows from Lemma 2.1 and (2.38) that

$$
\alpha_a[(\Delta + a_n)^{-1}aW\|_L^{1/2} + \beta_a[(\Delta + a_n)^{-1}aW\|_L^{1/2}
\lesssim \alpha_a[\alpha_a]^{-1}\leq 2\varepsilon\|W\|_L^{1/2} + \|\omega_n\alpha_n\|_a\|a\|_a\|W\|_L^{1/2} + \|\omega_n\alpha_n\|_a\|a\|_a\|W\|_L^{1/2} (2.47)
\lesssim \alpha_a[\alpha_a]^{-1}\leq 2\varepsilon + \|\omega_n\|_a\|a\|_a\|W\|_L^{1/2},
$$

where the implicit constants depend only on $p$, $\varepsilon$ and the quantity $\liminf_{n \to \infty} \omega_n$ (if finite).

Next, we consider the last term on the right-hand side of (2.46). Our aim here is to show that

$$
\|((\Delta + a_n)^{-1}aN_\eta[\|_L^{1/2} \leq o_n(\|\eta[\|_L^{1/2}. (2.48)
$$

Notice that by (2.48), the last term on the right-hand side of (2.46) can be absorbed into the left-hand side. Hence, the claim (2.45) follows from (2.46), (2.47) and (2.48).

**Step 2.** It remains to prove (2.48). It follows from (2.46) and the triangle inequality that

$$
\|((\Delta + a_n)^{-1}aN_\eta[\|_L^{1/2}
\leq \|((\Delta + a_n)^{-1}a\{W + \eta[|a^4(W + \eta[| - W^5 - 5W^4\Re[\eta[| - iW^4\Im[\eta[| \} \|_L^{1/2}
+ \beta_a[\|((\Delta + a_n)^{-1}a\{W + \eta[|p^{-1}(W + \eta[| - W^p\} \|_L^{1/2}. (2.49)
$$

We consider the first term on the right-hand side of (2.49). By Lemma 2.2 elementary computations, Hölder’s inequality and (2.32), one can see that

$$
\|W + \eta[|^4(W + \eta[| - W^5 - 5W^4\Re[\eta[| - iW^4\Im[\eta[| \|_L^{1/2}
\lesssim \|W + \eta[|^4(W + \eta[| - W^5 - 5W^4\Re[\eta[| - iW^4\Im[\eta[| \|_L^{1/2}
\lesssim \|W^3|\eta[|^2\|_L^{1/2} + ||\eta[|^5\|_L^{1/2} (2.50)
\lesssim \|W^3|\eta[|\|_L^{1/2} + \|\eta[|\|_L^{1/2} \leq o_n(1)\|\eta[|\|_L^{1/2},
$$

where the implicit constants depend only on $\varepsilon$. Next, we consider the second term on the right-hand side of (2.49). It follows from the fundamental theorem of calculus and
Lemma 2.1 that

\[ \beta_{[n]}\|(-\Delta + \alpha_{[n]})^{-1}\left\{ |W + \eta_{[n]}|^{p-1}(W + \eta_{[n]}) - W^p \right\} \|_{L^+} \]

\[ \leq \beta_{[n]}\|(-\Delta + \alpha_{[n]})^{-1}\left\{ \frac{p+1}{2} \int_0^1 |W + \theta \eta_{[n]}|^{p-1} d\theta \|_{L^+} \right\} \]

\[ + \beta_{[n]}\|(-\Delta + \alpha_{[n]})^{-1}\left\{ \frac{p-1}{2} \int_0^1 |W + \theta \eta_{[n]}|^{p-3}(W + \theta \eta_{[n]})^2 d\theta \|_{L^+} \right\} \]

\[ \lesssim \beta_{[n]}\|(-\Delta + \alpha_{[n]})^{-1}\left\{ W^{p-1}\eta_{[n]} \right\} \|_{L^+} + \beta_{[n]}\|(-\Delta + \alpha_{[n]})^{-1}\left\{ W^{p-1}\eta_{[n]} \right\} \|_{L^+} \]

\[ + \beta_{[n]}\|(-\Delta + \alpha_{[n]})^{-1}\left\{ \int_0^1 \left\{ |W + \theta \eta_{[n]}|^{p-1} - W^{p-1} \right\} d\theta \|_{L^+} \right\} \]

\[ + \beta_{[n]}\|(-\Delta + \alpha_{[n]})^{-1}\left\{ \int_0^1 \left\{ |W + \theta \eta_{[n]}|^{p-3}(W + \theta \eta_{[n]})^2 - W^{p-1} \right\} d\theta \|_{L^+} \right\} \]

\[ \lesssim \beta_{[n]}^{\frac{p-5}{\alpha}} \|W^{p-1}\eta_{[n]}\|_{L^{\frac{6}{p-1+\varepsilon}}} \]

\[ + \beta_{[n]}^{\frac{p-5}{\alpha}} \sup_{0 \leq \theta \leq 1} \left\| \left\{ |W + \theta \eta_{[n]}|^{p-1} - W^{p-1} \right\} \eta_{[n]} \right\|_{L^{\frac{6}{p-1+\varepsilon}}} \]

\[ + \beta_{[n]}^{\frac{p-5}{\alpha}} \sup_{0 \leq \theta \leq 1} \left\| \left\{ |W + \theta \eta_{[n]}|^{p-3}(W + \theta \eta_{[n]})^2 - W^{p-1} \right\} \eta_{[n]} \right\|_{L^{\frac{6}{p-1+\varepsilon}}} \]

\[ \leq \beta_{[n]}^{\frac{p-5}{\alpha}} \sup_{0 \leq \theta \leq 1} \left\| \left\{ |W + \theta \eta_{[n]}|^{p-1} - W^{p-1} \right\} \eta_{[n]} \right\|_{L^{\frac{6}{p-1+\varepsilon}}} \]

\[ + \beta_{[n]}^{\frac{p-5}{\alpha}} \sup_{0 \leq \theta \leq 1} \left\| \left\{ |W + \theta \eta_{[n]}|^{p-3}(W + \theta \eta_{[n]})^2 - W^{p-1} \right\} \eta_{[n]} \right\|_{L^{\frac{6}{p-1+\varepsilon}}} \]

\[ \lesssim \sup_{0 \leq \theta \leq 1} \|\theta \eta_{[n]}\|_{L^{\frac{6}{p-1+\varepsilon}}} \|\eta_{[n]}\|_{L^+} \leq \|\eta_{[n]}\|_{L^{\frac{6}{p-1+\varepsilon}}} \|\eta_{[n]}\|_{L^+} = \alpha_n(1) \|\eta_{[n]}\|_{L^+}, \]

where the implicit constants depend only on $p$ and $\varepsilon$. Consider the first term on the right-hand side of (2.51). By Hölder’s inequality, (2.38) and (2.23), one can see that

\[ \beta_{[n]}^{\frac{p-5}{\alpha}} \|W^{p-1}\eta_{[n]}\|_{L^{\frac{6}{p-1+\varepsilon}}} \leq \beta_{[n]}^{\frac{p-5}{\alpha}} \|W^{p-1}\|_{L^{\frac{6}{p-1+\varepsilon}}} \|\eta_{[n]}\|_{L^+} \lesssim \alpha_{\varepsilon}^{\frac{p-3}{\alpha}} \|\eta_{[n]}\|_{L^+}, \]

where the implicit constant depends only on $p$, $\varepsilon$ and the quantity $\lim \inf_n \omega_n$ (if finite). Next, consider the second and third terms on the right-hand side of (2.51). When $1 < p \leq 2$, it follows from (2.23), (2.38), Hölder’s inequality and the convexity, that

\[ \beta_{[n]}^{\frac{p-5}{\alpha}} \sup_{0 \leq \theta \leq 1} \left\| \left\{ |W + \theta \eta_{[n]}|^{p-1} - W^{p-1} \right\} \eta_{[n]} \right\|_{L^{\frac{6}{p-1+\varepsilon}}} \]

\[ + \beta_{[n]}^{\frac{p-5}{\alpha}} \sup_{0 \leq \theta \leq 1} \left\| \left\{ |W + \theta \eta_{[n]}|^{p-3}(W + \theta \eta_{[n]})^2 - W^{p-1} \right\} \eta_{[n]} \right\|_{L^{\frac{6}{p-1+\varepsilon}}} \]

\[ \lesssim \sup_{0 \leq \theta \leq 1} \|\theta \eta_{[n]}\|_{L^{\frac{6}{p-1+\varepsilon}}} \|\eta_{[n]}\|_{L^+} \leq \|\eta_{[n]}\|_{L^{\frac{6}{p-1+\varepsilon}}} \|\eta_{[n]}\|_{L^+} = \alpha_n(1) \|\eta_{[n]}\|_{L^+}, \]

where the implicit constant depends only on $p$, $\varepsilon$ and the quantity $\lim \inf_{n \to \infty} \omega_n$ (if
finite), while when \(2 < p \leq 3\), by \((2.48)\), \((2.49)\), Hölder’s inequality and \((2.42)\),
\[
\beta_{[n]}^{p-5} \sup_{0 \leq \theta \leq 1} \| \{ |W + \theta \eta_{[n]}|^p - W^{p-1} \} \eta_{[n]} \|_{L^{\frac{6}{p-6}}} \\
+ \beta_{[n]}^{\frac{p-5}{2}} \sup_{0 \leq \theta \leq 1} \| \{ |W + \theta \eta_{[n]}|^p - W + \theta \eta_{[n]} \}^2 - W^{p-1} \} \eta_{[n]} \|_{L^{\frac{6}{p-6}}}
\lesssim \| \{ W^{p-2} + |\eta_{[n]}|^p \} |\eta_{[n]}| \|_{L^{\frac{6}{p-6}}} \| \eta_{[n]} \|_{L^\frac{1}{2}}
\lesssim \| W \|_{L^\frac{p}{p-2}}^p + \| \eta_{[n]} \|_{L^p} \| \eta_{[n]} \|_{L^\frac{1}{2}} = o_n(1) \| \eta_{[n]} \|_{L^\frac{1}{2}},
\]
where the implicit constant depends only on \(p, \varepsilon\) and the quantity \(\liminf_{n \to \infty} \omega_n\) (if finite).

Putting the estimates \((2.49)\) through \((2.54)\) together, we obtain the desired estimate \((2.48)\).

Now, observe from \((2.39)\) and \((2.43)\) that for any \(n\),
\[
- \alpha_{[n]} \langle W, (-\Delta + \alpha_{[n]})^{-1} V_+ \Lambda W \rangle \\
\leq \beta_{[n]} \langle W^p, (-\Delta + \alpha_{[n]})^{-1} V_+ \Lambda W \rangle + \langle N_{[n]}, (-\Delta + \alpha_{[n]})^{-1} V_+ \Lambda W \rangle.
\]
Furthermore, by Lemma 2.3 the left-hand side of \((2.55)\) can be written as
\[
- \alpha_{[n]} \langle W, (-\Delta + \alpha_{[n]})^{-1} V_+ \Lambda W \rangle = 6\pi \alpha_{[n]}^\frac{1}{2} + O(\alpha_{[n]}).
\]
On the other hand, by Lemma 2.3 (if \(2 < p \leq 3\)) and Lemma 2.4 with \(\delta = \frac{p-1}{8}\) (if \(1 < p \leq 2\)), one can verify that the first term on the right-hand side of \((2.55)\) is estimated as follows:
\[
\beta_{[n]} \langle W^p, (-\Delta + \alpha_{[n]})^{-1} V_+ \Lambda W \rangle \lesssim \begin{cases} \beta_{[n]} & \text{if } 2 < p \leq 3, \\
\beta_{[n]} \alpha_{[n]}^{-\frac{2p-8}{p-6}} & \text{if } 1 < p \leq 2. \end{cases}
\]
We consider the second term on the right-hand side of \((2.55)\), we claim the following:

**Lemma 2.6.** Assume \(1 < p \leq 3\) and \(0 < \varepsilon < \frac{p-1}{20}\). Then, the following estimate holds for any sufficiently large number \(n\):
\[
\| (-\Delta + \alpha_{[n]})^{-1} N_{[n]} \|_{L^\frac{1}{2}} \lesssim \alpha_{[n]}^{\frac{1}{2} + \varepsilon},
\]
where the implicit constant depends only on \(p, \varepsilon\) and the quantity \(\liminf_{n \to \infty} \omega_n\) (if finite).

**Proof of Lemma 2.6.** As in \((2.49)\), we see that
\[
\| (-\Delta + \alpha_{[n]})^{-1} N_{[n]} \|_{L^\frac{1}{2}} \\
\leq \| (-\Delta + \alpha_{[n]})^{-1} \left\{ |W + \eta_{[n]}|^4 (W + \eta_{[n]}) - W^5 - 5W^4 \Re \eta_{[n]} - iW^4 \Im \eta_{[n]} \right\} \|_{L^\frac{1}{2}} \\
+ \beta_{[n]} \| (-\Delta + \alpha_{[n]})^{-1} \left\{ |W + \eta_{[n]}|^{p-1} (W + \eta_{[n]}) - W^p \right\} \|_{L^\frac{1}{2}}.
\]

From (2.1), one can verify that
\[
|(-\Delta + \alpha[n])^{-1} \left\{ |W + \eta[n]|^4 (W + \eta[n]) - W^5 - 5W^4 \Re[\eta[n]] - iW^5 \Im[\eta[n]] \right\} | \\
\lesssim (-\Delta + \alpha[n])^{-1} \{ W^3 |\eta[n]|^2 + |\eta[n]|^5 \},
\] (2.60)
and
\[
|(-\Delta + \alpha[n])^{-1} \left\{ |W + \eta[n]|^{p-1} (W + \eta[n]) - W^p \right\} | \\
\lesssim (-\Delta + \alpha[n])^{-1} \{ |W|^{p-1} + |\eta[n]|^{p-1} \} |\eta[n]|.
\] (2.61)

We consider the first term on the right-hand side of (2.59). By (2.60), Lemma 2.2, Hölder’s inequality and Lemma 2.5, one can see that
\[
\|\|(-\Delta + \alpha[n])^{-1} \{ |W + \eta[n]|^4 (W + \eta[n]) - W^5 \Re[\eta[n]] - iW^5 \Im[\eta[n]] \}∥_{L^\frac{3}{2}} + \|(-\Delta + \alpha[n])^{-1} |\eta[n]|^5 \|_{L^\frac{3}{2}} \leq |\| W^2 |\eta[n]|^2 \|_{L^\frac{3}{2}} + \|(-\Delta + \alpha[n])^{-1} |\eta[n]|^5 \|_{L^\frac{3}{2}} \leq \| W^3 |\eta[n]|^2 \|_{L^\frac{3}{2}} + \alpha^{-1+\epsilon} \| |\eta[n]|^5 \|_{L^\frac{3}{2}} \leq \alpha^{-1+\epsilon} + \alpha^{-1+\epsilon} \leq \alpha^{-1+\epsilon} \alpha^{-1+\epsilon},
\] (2.62)
where the implicit constants depend only on \( p \), \( \varepsilon \) and the quantity \( \lim \inf_{n \to \infty} \omega_n \) (if finite).

Next, we consider the second term on the right-hand side of (2.59). It follows from (2.61), Lemma 2.1, 2.23, Hölder’s inequality and Lemma 2.5 that if \( \varepsilon \) is sufficiently small depending only on \( p \),
\[
\beta[n]\|(-\Delta + \alpha[n])^{-1} \{ |W + \eta[n]|^{p-1} (W + \eta[n]) - W^p \}∥_{L^\frac{3}{2}} \leq \beta[n]\|(-\Delta + \alpha[n])^{-1} \{ W^{p-1} |\eta[n]| \}∥_{L^\frac{3}{2}} \leq \beta[n]\|(-\Delta + \alpha[n])^{-1} |\eta[n]|^p \|_{L^\frac{3}{2}} \leq \beta[n] \alpha^{-1+\epsilon} \| |\eta[n]|^p \|_{L^\frac{3}{2}} \leq \alpha^{-1+\epsilon} + \alpha^{-1+\epsilon} \leq \alpha^{-1+\epsilon},
\] (2.63)
where the implicit constants depend only on \( p \), \( \varepsilon \) and the quantity \( \lim \inf_{n \to \infty} \omega_n \) (if finite).

Putting (2.59), (2.62) and (2.63) together, we obtain the desired estimate (2.58).

Now, we shall finish the proof of Theorem 2.1. By Hölder’s inequality and Lemma 2.6, one can estimate the last term on the right-hand side of (2.59) as follows: for any \( 0 < \varepsilon < \frac{p-1}{20} \),
\[
|\langle \mathcal{N}_n, (-\Delta + \alpha[n])^{-1} V_+ AW \rangle | \leq \| V_+ AW \|_{L^\frac{1}{1+\varepsilon}} \| (-\Delta + \alpha[n])^{-1} \mathcal{N}_n \|_{L^\frac{3}{2}} \leq \alpha^{-1+\varepsilon},
\] (2.64)
where the implicit constants depend only on $p$, $\varepsilon$ and the quantity $\lim_{n \to \infty} \omega_n$. Then, plugging (2.56), (2.67), and (2.64) into (2.55), we see that for any $0 < \varepsilon < \frac{p-1}{20}$,

$$\frac{1}{\alpha[n]} \leq \begin{cases} \beta[n] + \alpha[n]^{\frac{1}{2}+\varepsilon} & \text{if } 2 < p \leq 3, \\ \beta[n] + \alpha[n]^{\frac{2-p}{2} - \frac{p-1}{8}} + \alpha[n]^\varepsilon & \text{if } 1 < p \leq 2, \end{cases}$$

(2.65)

where the implicit constant depends only on $p$, $\varepsilon$, and the quantity $\lim_{n \to \infty} \omega_n$. Furthermore, transposing the second term on the right-hand side of (2.65) to the left, and dividing both sides by $\alpha[n]^{\frac{1}{2}}$, we obtain

$$1 \leq \begin{cases} \frac{1}{\alpha[n]} \beta[n] & \text{if } 2 < p \leq 3, \\ \frac{1}{\alpha[n]} \beta[n] \alpha[n]^{\frac{2-p}{2} - \frac{p-1}{8}} & \text{if } 1 < p \leq 2, \end{cases}$$

(2.66)

Assume $2 < p \leq 3$. Then, it follows from (2.55), (2.23), (2.24) and $\lim_{n \to \infty} \nu_n = 1$ that

$$\lim_{n \to \infty} \omega_n = \omega_0 = 0.$$ 

(2.67)

However, (2.67) contradicts (2.66). Similarly, when $1 < p \leq 2$, from (2.56), (2.23), (2.24) and $\lim_{n \to \infty} \nu_n = 1$, we can verify that

$$\lim_{n \to \infty} \omega_n = \omega_0 = 0.$$ 

(2.68)

This also contradicts (2.66). Thus, we have completed the proof of Theorem 2.1. 

3 Proof of Theorem 1.1

In this section, we prove Theorem 1.1. To this end, we will employ Proposition 1.2. The following result immediately follows from results in [1] and Theorem 2.1.

Proposition 3.1. Assume $1 < p \leq 3$. Then, there exists $\omega_1 > 0$ such that for any $\omega > \omega_1$, there is no ground state (no minimizer for $m_\omega^N$).

Proof of Proposition 3.1 We prove by contradiction: If the claim of Proposition 3.1 was false, then there existed a sequence $\{(\Phi_n, \omega_n)\}$ in $H^1(\mathbb{R}^3) \times (0, \infty)$ with the following properties:

(i) $\Phi_n$ is a ground state for $m_{\omega_n}$; we may assume that $\Phi_n$ is a positive radial solution to (1.4) with $\omega = \omega_n$ (see Remark 1.1).

(ii) $\lim_{n \to \infty} \omega_n = \infty$.

Furthermore, defining $M_n := \|\Phi_n\|_{L^\infty}$, $\alpha_n := \omega_n M_n^{-4}$, $\beta_n := M_n^{p-5}$ and $\tilde{\Phi}_n := T_{M_n} \Phi_n$,

we can verify that $\tilde{\Phi}_n := T_{M_n} \Phi_n$ obeys

$$-\Delta \tilde{\Phi}_n + \alpha_n \tilde{\Phi}_n - \beta_n |\tilde{\Phi}_n|^{p-1} \tilde{\Phi}_n - |\tilde{\Phi}_n|^4 \tilde{\Phi}_n = 0,$$

(3.1)
and
(iii) \( \lim_{n \to \infty} M_n = \infty, \) \( \lim_{n \to \infty} a_n = 0 \) (see Lemma 2.3 of [1]).
(iv) \( \lim_{n \to \infty} \Phi_n = W \) strongly in \( H^1(\mathbb{R}^3) \) (see Proposition 2.1 of [1]).

However, when \( 1 < p \leq 3 \), Theorem 2.1 forbids such a sequence to exist. Thus, the claim of Proposition 3.1 is true.

From Proposition 3.1 we see that \( \omega_c^N < \infty \). For \( \omega \geq \omega_c^N \), we obtain the following result:

**Proposition 3.2.** Assume \( \omega > 0 \) and \( 1 < p \leq 3 \). Then, for any \( \omega \geq \omega_c^N \), we have \( m_{\omega}^N = m_{\infty} \).

For a proof of Proposition 3.2, we need the following characterization of \( m_{\omega}^N \):

**Lemma 3.3.** Assume \( \omega > 0 \) and \( 1 < p < 5 \). Then,

\[
m_{\omega}^N = \inf \{ J(u) : u \in H^1(\mathbb{R}^3) \setminus \{0\}, \mathcal{N}_\omega(u) \leq 0 \},
\]

where \( J \) is the functional defined by (1.23).

**Proof of Lemma 3.3.** Observe that for any \( \omega > 0 \),

\[
\mathcal{S}_\omega(u) - \frac{1}{2} \mathcal{N}_\omega(u) = \left( \frac{1}{2} - \frac{1}{p+1} \right) \| u \|_{L^{p+1}}^{p+1} + \frac{1}{3} \| u \|^3_{L^6} = J(u).
\]

This yields that

\[
m_{\omega}^N = \inf \{ J(u) : u \in H^1(\mathbb{R}^3) \setminus \{0\}, \mathcal{N}_\omega(u) = 0 \}.
\]

Hence, it suffices to show that if \( \mathcal{N}_\omega(u) < 0 \), then \( m_{\omega}^N < J(u) \). Suppose that \( \mathcal{N}_\omega(u) < 0 \). Then, we can take \( t_0 \in (0, 1) \) such that \( \mathcal{N}_\omega(t_0 u) = 0 \). Furthermore, this together with (3.4) yields that

\[
m_{\omega}^N \leq J(t_0 u) < J(u).
\]

Since \( u \) is arbitrary, we find from (3.4) that the claim (3.2) is true.

Now, we give a proof of Proposition 3.2.

**Proof of Proposition 3.2** We divide the proof into two parts:

**Step 1.** We shall show that if \( \omega > \omega_c^N \), then we have \( m_{\omega}^N = m_{\infty} \).

First, observe from the definition of \( \omega_c^N \) (see (1.15) and Proposition 1.2) that if \( \omega > \omega_c^N \), then we see that \( m_{\omega}^N \geq m_{\infty} \).

We prove the claim by contradiction. Hence, suppose to the contrary that there exists \( \omega_1 > \omega_c^N \) such that \( m_{\omega_1}^N > m_{\infty} \). Put \( \mu_1 := m_{\omega_1}^N - m_{\infty} > 0 \). Moreover, for \( \varepsilon > 0 \), we define the functions \( W_\varepsilon \) and \( V_\varepsilon \) by

\[
W_\varepsilon(x) := T_\varepsilon W, \quad V_\varepsilon(x) := \varchi(|x|)W_\varepsilon(x),
\]

where \( \varchi \) is a cut-off function. Then, we have

\[
m_{\omega_1}^N = \inf \{ J(u) : u \in H^1(\mathbb{R}^3) \setminus \{0\}, \mathcal{N}_{\omega_1}(u) \leq 0 \},
\]

but

\[
m_{\infty} = \inf \{ J(u) : u \in H^1(\mathbb{R}^3) \setminus \{0\}, \mathcal{N}_{\infty}(u) \leq 0 \},
\]

and

\[
\mathcal{N}_{\omega_1}(u) < \mathcal{N}_{\infty}(u) \quad \text{for all } u \in H^1(\mathbb{R}^3) \setminus \{0\}.
\]

This yields that

\[
m_{\omega_1}^N < m_{\infty},
\]

contradicting our assumption. Therefore, we have \( m_{\omega_1}^N = m_{\infty} \).
where $\chi$ is a smooth cut-off function satisfying $\chi(r) \equiv 1$ for $0 \leq r \leq 1$, $\chi(r) \equiv 0$ for $r \geq 2$ and $|\chi'(r)| \leq 10$. Then, using (1.15), we can verify that

$$\|\nabla V_\varepsilon\|_{L^2}^2 = \sigma \varepsilon^2 + O(\varepsilon^2), \quad (3.7)$$

$$\|V_\varepsilon\|_{L^6}^6 = \sigma \varepsilon^3 + O(\varepsilon^6), \quad (3.8)$$

$$\|V_\varepsilon\|_{L^{p+1}}^{p+1} = \begin{cases} O(\varepsilon^{p+1}) & \text{if } 1 \leq p < 2, \\ O(\varepsilon^3 |\log \varepsilon|) & \text{if } p = 2, \\ O(\varepsilon^{p-5}) & \text{if } 2 < p < 5. \end{cases} \quad (3.9)$$

We see from (3.7) through (3.9) that for any $\varepsilon > 0$ and $|\omega| > \omega$, however, this is a contradiction. Thus, we have proved that if $\omega > \omega_0$, we have

$$N_{\omega_0}(t_\varepsilon V_\varepsilon) = 0, \quad t_\varepsilon = 1 + O(\varepsilon^2). \quad (3.11)$$

In particular, we find that for any sufficiently small $\varepsilon > 0$, there exists $t_\varepsilon > 0$ such that

$$m_{\omega_0} \leq J(t_\varepsilon V_\varepsilon) = \left(1 + \frac{1}{p + 1}\right) t_\varepsilon^{p+1} \|V_\varepsilon\|_{L^{p+1}}^{p+1} + \frac{\varepsilon^3}{3} \|V_\varepsilon\|_{L^6}^6 \quad (3.12)$$

Furthermore, taking $\varepsilon \ll \mu_1$ in (3.12), and recalling the definition of $\mu_1$ and the hypothesis $m_{\omega_0} > m_\infty$, we find that

$$m_{\omega_0} \leq m_\infty + \frac{1}{2} \mu_1 = \frac{1}{2} m_{\omega_0} + \frac{1}{2} m_\infty < m_{\omega_0}. \quad (3.13)$$

However, this is a contradiction. Thus, we have proved that if $\omega > \omega_c^N$, then $m_{\omega} = m_\infty$.

**Step 2.** We shall show that if $\omega = \omega_c^N$, then $m_{\omega} = m_\infty$.

Having proved that $m_{\omega} = m_\infty$ for any $\omega > \omega_c^N$, we see from Proposition 1.12 that $m_{\omega_c^N} \leq m_\infty$. Then, it follows from Proposition 1.1 that a ground state $\Phi$ for $\omega_c^N$ exists. In particular, $N_{\omega_c^N}(\Phi) = 0$ and $S_{\omega_c^N}(\Phi) = m_{\omega_c^N}$. Furthermore, we may assume that $\Phi$ is a positive radial solution to (1.4) with $\omega = \omega_c^N$ (see Remark 1.1). Put $\mu_c^N := m_\infty - m_{\omega_c^N}$ ($\mu_c^N > 0$ by the hypothesis). Moreover, let $\{\omega_n\}$ be a sequence in $(0, \infty)$ such that $\lim_{n \to \infty} \omega_n = \omega_c^N$, and $\omega_n > \omega_c^N$ for all $n \geq 1$. We have already proved that for any $n \geq 1$,

$$m_{\omega_n} = m_\infty. \quad (3.14)$$

We also see from $N_{\omega_n}(\Phi) = 0$ that

$$\lim_{n \to \infty} N_{\omega_n}(\Phi) = \lim_{n \to \infty} (\omega_n - \omega_c^N) \|\Phi\|_{L^2}^2 = 0.
This implies that there exists a sequence \( \{ t_n \} \) in \((0, \infty)\) such that
\[
N_{\omega_n}(t_n \Phi) = 0 \quad \text{for all } n \geq 1, \quad (3.15)
\]
\[
\lim_{n \to \infty} t_n = 1. \quad (3.16)
\]
Furthermore, we can take a number \( n_c \) such that for any \( n \geq n_c \),
\[
|S_{\omega_n}(t_n \Phi) - S_{\omega_N}(t_n \Phi)| = \frac{t_n^2}{2} |\omega_c^N - \omega_n| \| \Phi \|_{L^2}^2 < \frac{1}{4} \mu_c^N. \quad (3.17)
\]
Observe from (3.17) and \( S_{\omega_N}(\Phi) = m_{\omega_N}^N \) that
\[
S_{\omega_N}(t_n \Phi) \leq S_{\omega_N}(t_n \Phi) + |S_{\omega_N}(t_n \Phi) - S_{\omega_N}(\Phi)|
\leq S_{\omega_N}(\Phi) + \frac{1}{4} \mu_c^N = m_{\omega_N}^N + \frac{1}{4} \mu_c^N. \quad (3.18)
\]
Using (3.14), the definition of \( m_{\omega}^N \), (3.15), (3.17), (3.18) and \( \mu_c^N = m_{\infty} - m_{\omega_N}^N \), we find that for any \( n \geq n_c \),
\[
m_{\infty} = m_{\omega_N}^N \leq S_{\omega_N}(t_n \Phi) < S_{\omega_N}(t_n \Phi) + \frac{1}{4} \mu_c^N < m_{\omega_N}^N + \frac{1}{2} \mu_c^N = m_{\infty} - \frac{1}{2} \mu_c^N.
\]
However, this is a contradiction. Thus, we have completed the proof.

Now, we shall prove Theorem 1.1.

**Proof of Theorem 1.1.** The claim (i) follows immediately from Remark 1.2 (ii) and Proposition 3.1.

We move on to the proof of the claim (ii). We find from Proposition 1.1 and Proposition 1.2 that if \( \omega < \omega_N^c \), then a ground state exists. It remains to show that for any \( \omega > \omega_N^c \), there is no ground state (no minimizer for \( m_{\omega}^N \)). We prove this by contradiction. Hence, suppose to the contrary that there exists \( \omega_* > \omega_N^c \) for which a ground state \( u_* \) exists. Since \( N_{\omega_*}(u_*) = 0 \) and \( \omega_* > \omega_N^c \), we see that \( N_{\omega_N}(u_*) < 0 \). Furthermore, it is easy to see that there exists \( t_c \in (0, 1) \) such that \( N_{\omega_N}(t_c u_*) = 0 \). Hence, we find from Proposition 3.2 and the definition of \( m_{\omega}^N \) (see (1.1)) that
\[
m_{\infty} = m_{\omega_N}^N \leq S_{\omega_N}(t_c u_*) - \frac{1}{2} N_{\omega_N}(t_c u_*) = \left( \frac{1}{2} - \frac{1}{p+1} \right) \| t_c u_* \|_{L^{p+1}}^{p+1} + \frac{1}{3} \| t_c u_* \|_{L^6}^6
\leq \left( \frac{1}{2} - \frac{1}{p+1} \right) \| u_* \|_{L^{p+1}}^{p+1} + \frac{1}{3} \| u_* \|_{L^6}^6 = S_{\omega_*}(u_*) - \frac{1}{2} N_{\omega_*}(u_*)
= S_{\omega_*}(u_*) = m_{\omega_N}^N = m_{\infty}.
\]
However, this is a contradiction. Thus, we have proved the claim (ii) and completed the proof of Theorem 1.1.
4 Proof of Theorem 1.3

Our aim in this section is to prove Theorem 1.3. First, we recall the following fact:

Lemma 4.1. Let $1 < p < 5, \omega > 0$ and $u_\omega \in H^1(\mathbb{R}^3)$ be a solution to (1.4). Then, there exists $(x_0, \theta_0) \in \mathbb{R}^3 \times [-\pi, \pi)$ such that $\| u_\omega \|_{L^\infty} = e^{i\theta_0} u_\omega(x_0)$.

Proof. It is known that $u_\omega \in W^{2,q}_{\text{loc}}(\mathbb{R}^3)$ for all $2 < q < \infty$ and $\lim_{|x| \to \infty} u_\omega(x) = 0$ (see e.g. Brezis and Lieb [7, Theorem 2.3]). Furthermore, using Schauder’s estimate, we can verify that $u_\omega \in C^{2}(\mathbb{R}^3)$ (see e.g. Gilberg and Trudinger [14, Theorems 6.2 and 6.6], Struwe [22, Appendix B]). Therefore, there exists $(x_0, \theta_0) \in \mathbb{R}^3 \times [-\pi, \pi)$ such that $e^{i\theta_0} u_\omega(x_0) = \| u_\omega \|_{L^\infty} < \infty$.

The key to the proof of Theorem 1.3 is the following proposition:

Proposition 4.2. Assume $1 < p \leq 3$ and $\omega > \omega_c^N$. Suppose that

$$m_\omega^S = m_\omega^N.$$  \hfill (4.1)

Then, for any minimizing sequence $\{ u_n \}$ for $m_\omega^S$, and there exists a sequence $\{ (x_n, \theta_n) \}$ in $\mathbb{R}^3 \times [-\pi, \pi)$ and a subsequence of $\{ u_n \}$ (still denoted by the same symbol) such that putting $\alpha_n := \omega \| u_n \|_{L^\infty}^{-1}$ and $\tilde{u}_n := T\| u_n \|_{L^\infty} [e^{i\theta_n} u_n(\cdot + x_n)]$, we have: $\| u_n \|_{K^\infty} = e^{i\theta_n} u_n(x_n),$

$$\lim_{n \to \infty} \alpha_n = 0,$$ \hfill (4.3)

$$\lim_{n \to \infty} \tilde{u}_n = W \quad \text{strongly in } \dot{H}^1(\mathbb{R}^3).$$ \hfill (4.4)

For the proof of Proposition 4.2 we also use the following fact (Lemma 4.3 through Lemma 4.6):

Lemma 4.3. Assume $1 < p < 5$. Then, there exists a constant $C_\omega > 0$ depending only on $p$ and $\omega > 0$ such that

$$m_\omega^N \geq C_\omega.$$ \hfill (4.5)

Proof of Lemma 4.3. For any $\varepsilon > 0$, Lemma 3.3 enables us to take a function $u_\varepsilon \in H^1(\mathbb{R}^3) \setminus \{0\}$ such that $\mathcal{N}_\omega(u_\varepsilon) \leq 0$ and $\mathcal{J}(u_\varepsilon) \leq m_\omega^N + \varepsilon$. Then, it follows from $\mathcal{N}_\omega(u_\varepsilon) \leq 0$ and elementary computations that

$$\| \nabla u_\varepsilon \|_{L^2}^2 + \omega \| u_\varepsilon \|_{L^2}^2 \leq \| u_\varepsilon \|_{L^{p+1}}^{p+1} + \| u_\varepsilon \|_{L^6}^6 \leq \frac{\omega^2}{2} \| u_\varepsilon \|_{L^2}^2 + C(\omega) \| u_\varepsilon \|_{L^6}^6,$$ \hfill (4.6)

where $C(\omega) > 0$ is some constant depending only on $p$ and $\omega$. Notice that the first term on the right-hand side of (4.6) can be absorbed into the second term on the left-hand side. Hence, we see from Sobolev’s inequality (1.12) and (4.6) that

$$\sigma \| u_\varepsilon \|_{L^6}^2 \leq \| \nabla u_\varepsilon \|_{L^2}^2 \leq C(\omega) \| u_\varepsilon \|_{L^6}^6.$$ \hfill (4.7)
This together with $J(u_\varepsilon) \leq m_N^\varepsilon + \varepsilon$ implies
\[
\frac{1}{d} \left( \frac{\sigma}{C(\omega)} \right)^{\frac{2}{d}} \leq \frac{1}{3} ||u_\varepsilon||_{L^6}^6 \leq J(u_\varepsilon) \leq m_N^\varepsilon + \varepsilon. \tag{4.8}
\]
Taking $\varepsilon \rightarrow 0$ in (4.8) yields (4.5). \hfill \square

Furthermore, we will employ the following lemmas (Lemma 4.4 through Lemma 4.6):

Lemma 4.4 (Fröhlich, Lieb and Loss [12]). Let $d \geq 1$, $1 < p_1 < p_2 < p_3$, and let $C_1, C_2, C_3$ be positive constants. Then, there exist constants $C > 0$ and $\eta > 0$ such that for any measurable function $u$ on $\mathbb{R}^d$ satisfying
\[
\|u\|_{L^{p_1}} \leq C_1, \tag{4.9}
\]
\[
C_2 \leq \|u\|_{L^{p_2}}, \tag{4.10}
\]
\[
\|u\|_{L^{p_3}} \leq C_3, \tag{4.11}
\]
we have
\[
|\{x \in \mathbb{R}^d : |u(x)| > \eta\}| \geq C. \tag{4.12}
\]

Lemma 4.5 (Lieb [18]). Let $d \geq 1$, and let $\{u_n\}$ be a bounded sequence in $H^1(\mathbb{R}^d)$. Assume that there exist $C > 0$ and $\eta > 0$ such that
\[
\inf_{n \in \mathbb{N}} |\{x \in \mathbb{R}^d : |u_n(x)| > \eta\}| \geq C. \tag{4.13}
\]
Then, there exist a subsequence of $\{u_n\}$ (still denoted by the same symbol $\{u_n\}$), a non-trivial function $u_\infty \in H^1(\mathbb{R}^d)$ and a sequence $\{y_n\}$ in $\mathbb{R}^d$ such that
\[
\lim_{n \rightarrow \infty} u_n(\cdot + y_n) = u_\infty \quad \text{weakly in } H^1(\mathbb{R}^d). \tag{4.14}
\]

Lemma 4.6 (Brezis and Lieb [7]). Let $d \geq 1$, $1 \leq r < \infty$, $u_\infty \in H^1(\mathbb{R}^d)$, and let $\{u_n\}$ be a bounded sequence in $H^1(\mathbb{R}^d)$ such that
\[
\lim_{n \rightarrow \infty} u_n = u_\infty \quad \text{weakly in } \dot{H}^1(\mathbb{R}^d), \quad \text{and almost everywhere in } \mathbb{R}^d, \tag{4.15}
\]
\[
\sup_{n \geq 1} \|u_n\|_{L^r} < \infty. \tag{4.16}
\]
Then, we have:
\[
\lim_{n \rightarrow \infty} \int_{\mathbb{R}^d} \left\{ |\nabla u_n|^2 - |\nabla \{u_n - u_\infty\}|^2 - |\nabla u_\infty|^2 \right\} \, dx = 0, \tag{4.17}
\]
and
\[
\lim_{n \rightarrow \infty} \int_{\mathbb{R}^d} \left| |u_n|^r - |u_n - u_\infty|^r - |u_\infty|^r \right| \, dx = 0. \tag{4.18}
\]

Now, we shall give a proof of Proposition 4.2.
Proof of Proposition 4.2. Let $\omega > \omega_c^N$, and assume $m^S_\omega = m^N_\omega$. Furthermore, let $\{u_n\}$ be a minimizing sequence for $m^S_\omega$ (hence each $u_n$ is a solution to (1.4)). Note that for any $n \in \mathbb{N}$,

$$\mathcal{N}_\omega(u_n) = 0.$$  \hspace{1cm} (4.19)

It follows from Proposition 3.2 and (1.11) that

$$\lim_{n \to \infty} S_\omega(u_n) = m^S_\omega = m^N_\omega = m_\infty = \frac{1}{3} \sigma^2.$$  \hspace{1cm} (4.20)

We divide the proof into several steps:

**Step 1.** We shall show that the sequence $\{u_n\}$ is bounded in $H^1(\mathbb{R}^3)$; more precisely, there exists a number $N_1$ such that for any $n \geq N_1$,

$$\|\nabla u_n\|_2^2 + \omega \|u_n\|_2^2 \leq \frac{4(p+1)}{p-1} m_\infty.$$  \hspace{1cm} (4.21)

We see from (4.20) and (4.19) that there exists a number $N_1$ such that for any $n \geq N_1$,

$$2m_\infty \geq S_\omega(u_n) - S_\omega(u_n) - \frac{1}{p+1} \mathcal{N}_\omega(u_n)$$

$$= \left(\frac{1}{2} - \frac{1}{p+1}\right) \left\{\|\nabla u_n\|_2^2 + \omega \|u_n\|_2^2\right\} + \left(\frac{1}{p+1} - \frac{1}{6}\right) \|u_n\|_6^6.$$  \hspace{1cm} (4.22)

This gives us the desired estimate (4.21).

**Step 2.** We shall show that

$$\lim_{n \to \infty} \|u_n\|_{L^{p+1}} = 0.$$  \hspace{1cm} (4.23)

Suppose to the contrary that the claim (4.23) was false. Then, passing to some subsequence, we may assume that there exists $C > 0$ such that

$$\inf_{n \in \mathbb{N}} \|u_n\|_{L^{p+1}} \geq C.$$  \hspace{1cm} (4.24)

Furthermore, we see from (4.21), (4.19), Lemma 4.4 and Lemma 4.5 that there exist a nontrivial function $u_\infty \in H^1(\mathbb{R}^3)$ and a sequence $\{y_n\}$ in $\mathbb{R}^3$ such that, passing to some subsequence, we have

$$\lim_{n \to \infty} u_n(\cdot + y_n) = u_\infty \text{ weakly in } H^1(\mathbb{R}^3), \text{ and almost everywhere in } \mathbb{R}^3.$$  \hspace{1cm} (4.25)

Here, Lemma 4.6 together with (4.21), (4.25) and (4.19) shows that

$$\lim_{n \to \infty} \mathcal{N}_\omega(u_n(\cdot + y_n) - u_\infty) = -\mathcal{N}_\omega(u_\infty).$$  \hspace{1cm} (4.26)

Moreover, Lemma 4.6 together with (4.25), (3.3), $\mathcal{J} = S_\omega - \frac{1}{2} \mathcal{N}_\omega$, (1.19) and $u_\infty \neq 0$ shows that

$$0 < \mathcal{J}(u_\infty) = m^N_\omega - \mathcal{J}(u_n(\cdot + y_n) - u_\infty) + o_n(1).$$  \hspace{1cm} (4.27)

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Now, suppose for contradiction that $\mathcal{N}_\omega(u_\infty) > 0$. Then, it follows from (4.20) that for any sufficiently large $n$,

$$\mathcal{N}_\omega(u_n(\cdot + y_n) - u_\infty) < 0.$$  

(4.28)

Furthermore, Lemma 3.3 together with (4.28) shows that for any sufficiently large $n$,

$$m_\omega^N \leq \mathcal{J}(u_n(\cdot + y_n) - u_\infty).$$  

(4.29)

However, plugging (4.29) into (4.27) reaches a contradiction. Thus, we have $\mathcal{N}_\omega(u_\infty) \leq 0$.

Then, we see from Lemma 3.3, the lower semicontinuity of the weak limit, (4.20) and (4.19) that

$$m_\omega^N \leq \mathcal{J}(u_\infty) \leq \liminf_{n \to \infty} \mathcal{J}(u_n) = \liminf_{n \to \infty} \{ S_\omega(u_n) - \frac{1}{p+1} \mathcal{N}_\omega(u_n) \} = m_\omega^N,$$

so that

$$\mathcal{J}(u_\infty) = m_\omega^N.$$  

(4.30)

Furthermore, we have $\mathcal{N}_\omega(u_\infty) = 0$; otherwise (namely when $\mathcal{N}_\omega(u_\infty) < 0$) $u_\infty$ must obey $m_\omega^N < \mathcal{J}(u_\infty)$ by the same argument as the proof of Lemma 3.3. Thus, $u_\infty$ is a minimizer for $m_\omega^N$ (ground state). However, since $\omega > \omega_c^N$, this contradicts Theorem 1.1 (ii). Hence, the claim (4.23) must hold.

**Step 3.** We shall show that

$$\lim_{n \to \infty} \|u_n\|_{L^2} = 0.$$  

(4.31)

Since $u_n$ is an $H^1$-solution to (1.4), we have $\mathcal{K}(u_n) = 0$ (see (1.18)) and $\mathcal{N}(u_n) = 0$. Furthermore, the difference between the equations $\mathcal{K}(u_n) = 0$ and $\mathcal{N}(u_n) = 0$ yields the following identity (Pohozaev’s identity):

$$\omega \|u_n\|_{L^2}^2 = \frac{5 - p}{2(p + 1)} \|u_n\|_{L^{p+1}}^{p+1}. $$  

(4.32)

This together with (4.23) shows (4.31).

**Step 4.** We shall show that

$$\lim_{n \to \infty} \|u_n\|_{L^6}^6 = 3m_\infty.$$  

(4.33)

We see from (4.20), (4.19) $\mathcal{J} = S_\omega - \frac{1}{2} \mathcal{N}_\omega$ and (4.23) that

$$m_\infty = \lim_{n \to \infty} \mathcal{J}(u_n) = \frac{1}{3} \lim_{n \to \infty} \|u_n\|_{L^6}^6,$$

(4.34)

so that (4.33) holds.

**Step 5.** We shall show that

$$\lim_{n \to \infty} \|u_n\|_{L^\infty} = \infty.$$  

(4.35)

Suppose to the contrary that $\liminf_{n \to \infty} \|u_n\|_{L^\infty} < \infty$. Then, passing to some subsequence, we have $\sup_{n \geq 1} \|u_n\|_{L^\infty} < \infty$. Furthermore, we see from $m_\infty = m_\omega^N$ (see (4.19)), (4.31) and (4.33) that the subsequence satisfies

$$3m_\infty = \lim_{n \to \infty} \|u_n\|_{L^6}^6 \leq \sup_{n \geq 1} \|u_n\|_{L^\infty} \lim_{n \to \infty} \|u_n\|_{L^2}^2 = 0.$$  

(4.36)
However, this contradicts Lemma 4.3. Thus, (4.35) holds.

**Step 6.** We shall finish the proof of Proposition 4.2.

By Lemma 4.1, we can take a sequence \( \{(x_n, \theta_n)\} \) in \( \mathbb{R}^3 \times [-\pi, \pi) \) such that \( e^{i\theta_n} u_n(x_n) = \|u_n\|_{L^\infty} \). Then, define \( M_n \) and \( \tilde{u}_n \) by \( M_n := \|u_n\|_{L^\infty} \), and \( \tilde{u}_n := T_{M_n}[e^{i\theta_n} u_n(\cdot + x_n)] = M_n^{-1} e^{i\theta_n} u_n(M_n^{-2} \cdot + x_n) \). We can verify that \( \tilde{u}_n \) satisfies
\[
-\Delta \tilde{u}_n + \alpha_n \tilde{u}_n - \beta_n |\tilde{u}_n|^{p-1} \tilde{u}_n - |\tilde{u}_n|^4 \tilde{u}_n = 0, \tag{4.37}
\]
\[
\|\tilde{u}_n\|_{L^\infty} = 1, \tag{4.38}
\]
where
\[
\alpha_n := \omega M_n^{-4}, \quad \beta_n := M_n^{p-5}. \tag{4.39}
\]

Observe from (4.35) that
\[
\lim_{n \to \infty} \alpha_n = \lim_{n \to \infty} \beta_n = 0, \tag{4.40}
\]
and from (4.21) that for any \( n \geq N_1 \),
\[
\|\nabla \tilde{u}_n\|_{L^2}^2 \leq \frac{4(p+1)}{p-1} m_n. \tag{4.41}
\]

Moreover, multiplying the equation (4.37) by the complex conjugate of \( \tilde{u}_n \), and integrating the resulting equation over \( \mathbb{R}^3 \), we obtain
\[
\|\nabla \tilde{u}_n\|_{L^2}^2 = \|\tilde{u}_n\|_{L^6}^6 - \alpha_n \|\tilde{u}_n\|_{L^2}^2 + \beta_n \|\tilde{u}_n\|_{L^{p+1}}^{p+1}. \tag{4.42}
\]

Next, use (4.41), \( \|\tilde{u}_n\|_{L^\infty} = 1 \), the \( W_{loc}^{2,q} \) estimate, and Schauder’s estimate (see [14]) to verify that there exists \( u_\infty \in \dot{H}^1(\mathbb{R}^3) \) such that, passing to some subsequence,
\[
\lim_{n \to \infty} \tilde{u}_n = u_\infty \quad \text{weakly in } \dot{H}^1(\mathbb{R}^3), \quad \text{and strongly in } C_{loc}^2(\mathbb{R}^3). \tag{4.43}
\]

Furthermore, Lemma 4.6 together with (4.41) and (4.43) shows that
\[
\lim_{n \to \infty} \left\{ \mathcal{H}^\dagger(\tilde{u}_n) - \mathcal{H}^\dagger(u_\infty - u_\infty) \right\} = 0, \tag{4.44}
\]
\[
\lim_{n \to \infty} \left\{ \mathcal{N}^\dagger(\tilde{u}_n) - \mathcal{N}^\dagger(u_\infty - u_\infty) \right\} = 0, \tag{4.45}
\]
where \( \mathcal{H}^\dagger \) and \( \mathcal{N}^\dagger \) are the functional defined by (1.8) and (1.9), respectively.

We see from (4.37), (4.43), (4.40) and (4.38) that the limit \( u_\infty \) satisfies
\[
\begin{aligned}
-\Delta u_\infty &= |u_\infty|^4 u_\infty \quad \text{in } \mathbb{R}^3, \\
u_\infty(0) &= 1.
\end{aligned} \tag{4.46}
\]

Moreover, it is easy to verify that
\[
\mathcal{N}^\dagger(u_\infty) = \|\nabla u_\infty\|_{L^2}^2 - \|u_\infty\|_{L^6}^6 = 0. \tag{4.47}
\]
Now, assume that \( u_\infty \) attains the equality in Sobolev’s inequality \((1.12)\):
\[
\| \nabla u_\infty \|_{L^2}^2 = \sigma \| u_\infty \|_{L^6}^6.
\] (4.48)

Then, by the results in \([5, 23]\), \( u_\infty = z_0\lambda_0^{-1} W(\lambda_0^{-2} \cdot + x_0) \) for some \( \lambda_0 > 0 \), \( x_0 \in \mathbb{R}^3 \) and \( z_0 \in \mathbb{C} \). Furthermore, it follows from \( \| u_\infty \|_{L^\infty} = 1 \), \((4.47)\) and \( 1 = W(0) > W(x) \) for all \( x \in \mathbb{R}^3 \setminus \{0\} \) that \( \lambda_0 = z_0 = 1 \) and \( x_0 = 0 \). Thus, \((4.48)\) implies that \( u_\infty = W \).

We shall prove \((4.48)\). We see from Sobolev’s inequality \((1.12)\) and \((4.47)\) that
\[
\sigma \| u_\infty \|_{L^6}^6 \leq \| u_\infty \|_{L^6}^6.
\] (4.49)

By \((4.47), (4.49)\) and \((1.11)\), we see that
\[
\mathcal{H}^t(u_\infty) = \frac{1}{2} \| \nabla u_\infty \|_{L^2}^2 - \frac{1}{6} \| u_\infty \|_{L^6}^6 = \frac{1}{3} \| u_\infty \|_{L^6}^6 \geq \frac{1}{3} \sigma \| u_\infty \|_{L^6}^6 = m_\infty.
\] (4.50)

On the other hand, we see from \((4.42), (4.39), (4.31), (4.23)\) and \((4.33)\) that
\[
\lim_{n \to \infty} \mathcal{H}^t(\tilde{u}_n) = \lim_{n \to \infty} \left\{ \frac{1}{3} \| \tilde{u}_n \|_{L^6}^6 - \frac{\alpha_n}{2} \| \tilde{u}_n \|_{L^2}^2 + \frac{\beta_n}{2} \| \tilde{u}_n \|_{L^{p+1}}^{p+1} \right\}
=
\lim_{n \to \infty} \left\{ \frac{1}{3} \| u_n \|_{L^6}^6 - \frac{\omega_n}{2} \| u_n \|_{L^2}^2 + \frac{1}{2} \| u_n \|_{L^{p+1}}^{p+1} \right\} = m_\infty.
\] (4.51)

Similarly, we can verify that
\[
\lim_{n \to \infty} \mathcal{N}^t(\tilde{u}_n) = \lim_{n \to \infty} \left\{ -\alpha_n \| \tilde{u}_n \|_{L^2}^2 + \beta_n \| \tilde{u}_n \|_{L^{p+1}}^{p+1} \right\} = 0.
\] (4.52)

Putting \((4.51), (4.52)\) and \((4.47)\) together, we obtain
\[
\lim_{n \to \infty} \mathcal{N}^t(\tilde{u}_n - u_\infty) = \lim_{n \to \infty} \left\{ \| \nabla(\tilde{u}_n - u_\infty) \|_{L^2}^2 - \| \tilde{u}_n - u_\infty \|_{L^6}^6 \right\} = 0.
\] (4.53)

Furthermore, we see from \((4.29)\) that
\[
\mathcal{H}^t(\tilde{u}_n - u_\infty) = \frac{1}{3} \| \nabla(\tilde{u}_n - u_\infty) \|_{L^2}^2 + o_n(1).
\] (4.54)

Now, we find from \((4.50), (4.44), (4.51)\) that
\[
m_\infty \leq \mathcal{H}^t(u_\infty) = \mathcal{H}^t(\tilde{u}_n) - \mathcal{H}^t(\tilde{u}_n - u_\infty) + o_n(1)
\leq \mathcal{H}^t(\tilde{u}_n) + o_n(1) = m_\infty + o_n(1).
\] (4.55)

Using \((4.37)\), and taking \( n \to \infty \) in \((4.55)\), we see that
\[
\frac{1}{3} \| u_\infty \|_{L^6}^6 = \mathcal{H}^t(u_\infty) = m_\infty.
\] (4.56)

Furthermore, this together with \((4.47)\) and \((1.11)\) implies that
\[
\| \nabla u_\infty \|_{L^2}^2 = \| u_\infty \|_{L^6}^6 = \sigma \| u_\infty \|_{L^6}^6.
\] (4.57)

Then, the claim \((4.48)\) follows from \((4.57)\), and therefore \( u_\infty = W \). In addition, it follows from \((4.44), (1.51), (1.51)\) and \((4.56)\) that
\[
\frac{1}{3} \| \nabla (\tilde{u}_n - u_\infty) \|_{L^2}^2 = o_n(1),
\] (4.58)

so that \( \lim_{n \to \infty} \tilde{u}_n = u_\infty = W \) strongly in \( \dot{H}^1(\mathbb{R}^3) \). Thus, we have completed the proof. \( \square \)
We are now in a position to give a proof of Theorem 1.3.

Proof of Theorem 1.3. Suppose to the contrary that the claim of Theorem 1.3 is false. Then, one sees that: if $1 < p < 3$, then there exists $\omega_2 > \omega_c^N$ such that $m_{\omega_2}^S = m_{\omega_2}^N$; and if $p = 3$, then there exists a sequence $\{\omega_k\}$ such that $m_{\omega_k}^S = m_{\omega_k}^N$, and $\lim_{k \to \infty} \omega_k = \infty$. When $1 < p < 3$, we define $\omega_k \equiv \omega_2$, so that $m_{\omega_k}^S = m_{\omega_k}^N$, and $\liminf_{k \to \infty} \omega_k = \omega_2 > \omega_c^N > 0$ (see Theorem 1.1).

Now, for each $k$, let $\{u_n^k\}$ be a minimizing sequence for $m_{\omega_k}^S$. Notice that $u_n^k$ is a solution to (1.4) with $\omega = \omega_k$. Then, applying Proposition 1.2 to the sequence $\{u_n^k\}$ and taking a number $n_k$ such that $\omega_k\|u_n^k\|_{L^\infty}^{-4} \leq \frac{1}{k}$, one can obtain sequences $\{(u_n, \omega_n)\}$ and $\{(x_n, \theta_n)\}$ such that: $u_n$ is a solution to (1.4) with $\omega = \omega_n$; $\liminf_{n \to \infty} \omega_n = \omega_2 > 0$ if $1 < p < 3$, and $\lim_{n \to \infty} \omega_n = \infty$ if $p = 3$; and putting $M_n := \|u_n\|_{L^\infty}$, $\alpha_n := \omega_n M_n^{-4}$, and $\tilde{u}_n := T_{M_n}(e^{i\theta_n} u_n(\cdot + x_n))$, we have

\[
\lim_{n \to \infty} M_n = \infty, \quad \lim_{n \to \infty} \alpha_n = 0, \quad (4.59)
\]
\[
\lim_{n \to \infty} \tilde{u}_n = W \text{ \ strongly in } \dot{H}^1(\mathbb{R}^3). \quad (4.60)
\]

However, Theorem 2.1 forbids such a sequence $\{(\tilde{u}_n, \omega_n)\}$ to exist. Thus, the claim of Theorem 1.3 must be true.

5 Proof of Theorem 1.2

In this section, we shall give a proof of Theorem 1.2. For this purpose, we prepare the following lemma:

Lemma 5.1. Assume $1 < p < 5$. Then, $m_{\omega}^N$ is continuous at $\omega = \omega_c^N$, namely for any sequence $\{\omega_n\}$ in $(0, \omega_c^N)$ with $\lim_{n \to \infty} \omega_n = \omega_c^N$, we have $\lim_{n \to \infty} m_{\omega_n}^N = m_{\omega_c}^N$.

Proof. By Proposition 5.2, it suffices to show the left-continuity.

Let $\{\omega_n\}$ be a sequence in $(\omega_c^N, \omega_c^N)$ such that $\lim_{n \to \infty} \omega_n = \omega_c^N$. It follows from Proposition 1.2 and the definition of $\omega_c^N$ that

\[
m_{\omega_n}^N < m_\infty \quad \text{for all } n \geq 1. \quad (5.1)
\]

By (5.1) and Proposition 1.1 we see that there exists a positive radial ground state $\Phi_{\omega_n}$ for $m_{\omega_n}^N$. Note that $\mathcal{S}_{\omega_n}(\Phi_{\omega_n}) = m_{\omega_n}^N < m_\infty$ and $\mathcal{N}_{\omega_n}(\Phi_{\omega_n}) = 0$. Here, we may assume that $\frac{1}{2} \omega_c \leq \omega_n < \omega_c$ for all $n$. Then, a computation similar to (4.22) shows that there exists $C_1 > 0$ depending only on $p$ such that for any $n$,

\[
\|\Phi_{\omega_n}\|_{\dot{H}^1} \leq C_1, \quad (5.2)
\]

Here, suppose to the contrary that there existed a subsequence of $\{\omega_n\}$ (still denoted by the same symbol) such that $\lim_{n \to \infty} m_{\omega_n}^N < m_\infty$. Put $\mu_c^N := m_\infty - \lim_{n \to \infty} m_{\omega_n}^N > 0$,
We see from $\mathcal{N}_{\omega_n}(\Phi_{\omega_n}) = 0$ and (5.2) that
\[
\lim_{n \to \infty} \mathcal{N}_{\omega_n}(\Phi_{\omega_n}) = \lim_{n \to \infty} (\omega_n - \omega_c^N)\|\Phi_{\omega_n}\|_{L^2}^2 = 0.
\]
This implies that there exists a sequence $\{t_n\}$ in $(0, \infty)$ such that
\[
\mathcal{N}_{\omega_n}(t_n \Phi_{\omega_n}) = 0 \quad \text{for all } n \geq 1,
\]
(5.3)
\[
\lim_{n \to \infty} t_n = 1.
\]
(5.4)
From (5.2) and (5.3), we can take a number $n_c$ such that for any $n \geq n_c$,
\[
|S_{\omega_n}(t_n \Phi_{\omega_n}) - S_{\omega_n}(t_m \Phi_{\omega_n})| = \frac{t_n^2}{2}|\omega_c^N - \omega_n|\|\Phi_{\omega_n}\|_{L^2}^2 < \frac{1}{4} \mu_c^N,
\]
(5.5)
\[
|S_{\omega_n}(t_n \Phi_{\omega_n}) - S_{\omega_n}(\Phi_{\omega_n})| < \frac{1}{4} \mu_c^N.
\]
(5.6)
Observe from the definition of $m_c^N, m_{c-c}^N = m_c^N$, (5.6), $S_{\omega_n}(\Phi_n) = m_{\omega_n}^N$ and (5.6) that for $n \geq n_c$,
\[
m_c^N \leq S_{\omega_n}(t_n \Phi_{\omega_n}) \leq S_{\omega_n}(\Phi_{\omega_n}) + |S_{\omega_n}(t_n \Phi_{\omega_n}) - S_{\omega_n}(\Phi_{\omega_n})| \\
+ |S_{\omega_n}(\Phi_{\omega_n}) - S_{\omega_n}^{\infty}(t_n \Phi_{\omega_n})| \\
< m_{\omega_n}^N + \frac{1}{2} \mu_c^N.
\]
(5.7)
Since $\mu_c^N = m_c^N - \lim_{n \to \infty} m_{\omega_n}^N$, we have
\[
m_c^N \leq \liminf_{n \to \infty} m_{\omega_n}^N + \frac{1}{2} \mu_c^N = m_c^N - \frac{1}{2} \mu_c^N.
\]
(5.8)
However, this is a contradiction. Thus, we have completed the proof. \qed

We are now in a position to prove Theorem 1.2.

**Proof of Theorem 1.2** Suppose to the contrary that there is no ground state for $m_{\omega_c}^N$. Let $\{(\Phi_{\omega_n}, \omega_n)\}$ be a sequence in $H^1(\mathbb{R}^3) \times (0, \omega_c^N)$ with the following properties:

(i) $\Phi_n$ is a positive radial ground state for $m_{\omega_n}^N$,

(ii) $\lim_{n \to \infty} \omega_n = \omega_c^N > 0$.

Note that $\mathcal{N}_{\omega_n}(\Phi_{\omega_n}) = 0$. From Lemma 5.1 and Proposition 3.2, we see that
\[
\lim_{n \to \infty} S_{\omega_n}(\Phi_{\omega_n}) = \lim_{n \to \infty} m_{\omega_c}^N = m_{\omega_c}^N = m_c^N.
\]
(5.9)
Furthermore, define $M_n := \|\Phi_n\|_{L^\infty}$ and $\tilde{\Phi}_n := T_{M_n} \Phi_n = M_n^{-1} \Phi_n (M_n^{-2})$. We emphasize that (5.9) corresponds to (4.24). Furthermore, the hypothesis that there is no ground state for $m_{\omega_c}^N$ leads us to $\lim_{n \to \infty} \|\Phi_{\omega_n}\|_{L^{p+1}} = 0$, as well as (4.24). Then, we can verify that $\tilde{\Phi}_n$ obeys
\[
- \Delta \tilde{\Phi}_n + \alpha_n \tilde{\Phi}_n - \beta_n |\tilde{\Phi}_n|^{p-1} \tilde{\Phi}_n - |\tilde{\Phi}_n|^4 \tilde{\Phi}_n = 0.
\]
(5.10)
where $\alpha_n := \omega_n M_n^{-4}$, and $\beta_n := M_n^{p-5}$.

Now, we suppose for contradiction that there is no ground state (no minimizer) for $m_{\omega_c}^N$. Then, by a similar argument to the proof of Proposition 4.2 we can verify the following:

(iii) $\lim_{n \to \infty} M_n = \infty$, $\lim_{n \to \infty} \alpha_n = 0$,

(iv) $\lim_{n \to \infty} \tilde{\Phi}_n = W$ strongly in $H^1(\mathbb{R}^3)$.

However, Theorem 2.1 does not allow such a sequence to exist. Thus, there exists a ground state for $m_{\omega_c}^N$.

6 Proof of Theorem 1.4

In this section, we give a proof of Theorem 1.4. The following lemma follows from the facts that the definition of $K$ does not include $\omega$, and $S_{\omega_1} < S_{\omega_2}$ for all $\omega_1 < \omega_2$:

Lemma 6.1. Assume $d \geq 3$ and $1 + \frac{4}{d} < p < \frac{d+2}{d-2}$. Then, the value $m_K$ is nondecreasing with respect to $\omega$ on $(0, \infty)$.

First, notice that by the definition of $\omega_K$ (see (1.19)) and Lemma 6.1.

Next, recall the following result obtained in [2]:

Proposition 6.2 (Proposition 2.1 in [2]). Assume $d \geq 3$, $1 + \frac{4}{d} < p < \frac{d+2}{d-2}$, and $\omega > 0$. If $m_{\omega}^K < m_\infty$, then a minimizer for $m_{\omega}^K$ exists.

Now, observe from Lemma 6.1 that the set $\{\omega > 0 : m_{\omega}^K < m_\infty\}$ is connected, so that

$$m_{\omega}^K < m_\infty \text{ for any } 0 < \omega < \omega_K^c.$$  \hfill (6.1)

Moreover, we can prove the following proposition by the same argument as in Section 2 of [4]:

Proposition 6.3. Assume $d \geq 3$ and $1 + \frac{4}{d} < p < \frac{d+2}{d-2}$, and let $\omega > 0$. Then, any minimizer for $m_{\omega}^K$ becomes a solution to (1.4).

Proposition 6.4. Assume $d = 3$ and $\frac{7}{3} < p < 5$. Then, for any $\omega \geq \omega_K^c$, we have $m_{\omega}^K = m_\infty$.

Proof. First, consider the case where $\omega > \omega_K^c$. We see from Lemma 6.1 and the definition of $\omega_K^c$ (see (1.19)) that $m_{\omega}^K \geq m_{\omega_2}^K \geq m_\infty$ for any $\omega > \omega_K^c$. Here, suppose to the contrary that there exists $\omega_2 > \omega_K^c$ such that $m_{\omega_2}^K > m_\infty$. Put $\mu_2 := m_{\omega_2}^K - m_\infty(> 0)$. Then, as well as (3.10) in the proof of Proposition 3.2 we can verify that for any $\varepsilon > 0$ and any $t > 0$,

$$t^{-2}K(tV_\varepsilon) = \sigma^3 + O(\varepsilon^2) - t^p O(\varepsilon^5 - p) - t^4 \sigma^2 + t^4 O(\varepsilon^6),$$  \hfill (6.2)
where \( V_\varepsilon \) is the same function as in \((3.6)\). Furthermore, we find from \((6.2)\) that for any sufficiently small \( \varepsilon > 0 \), there exists \( t_\varepsilon > 0 \) such that

\[
K(t_\varepsilon V_\varepsilon) = 0, \quad t_\varepsilon = 1 + O(\varepsilon^2). \tag{6.3}
\]

Hence, we see from \((6.3)\), the definition of \( m^K_{\omega_2} \) (see \((1.17)\)), \((6.2)\), \((6.3)\), \((6.4)\), \((6.5)\), \( p \leq 2 \), and the hypothesis \( m^K_{\omega_2} > m_\infty \) that if \( \varepsilon^2 \ll \mu_2 \), then one has

\[
m^K_{\omega_2} \leq S_{\omega_2}(t_\varepsilon V_\varepsilon) = S_{\omega_2}(t_\varepsilon V_\varepsilon) - \frac{1}{2} K(t_\varepsilon V_\varepsilon)
= \frac{\omega_2}{2} t_\varepsilon^2 \|V_\varepsilon\|_2^2 + \left( \frac{3(p-1)}{4(p+1)} - \frac{1}{p+1} \right) t_\varepsilon^{p+1} \|V_\varepsilon\|_{L^{p+1}}^{p+1} + \frac{1}{3} t_\varepsilon^6 \|V_\varepsilon\|_6^6
= t_\varepsilon^2 O(\varepsilon^2) + t_\varepsilon^{p+1} O(\varepsilon^{5-p}) + t_\varepsilon^6 m_\infty + O(\varepsilon^6)
= m_\infty + O(\varepsilon^2) \leq m_\infty + \frac{1}{2} \mu_2 = \frac{1}{2} m^K_{\omega_2} + \frac{1}{2} m_\infty < m^K_{\omega_2}, \tag{6.4}
\]

which is a contradiction. Thus, we have proved that \( m^K_{\omega} = m_\infty \) for all \( \omega > \omega_c^N \). Next, we consider the case where \( \omega = \omega_c^N \). We see from Lemma \((6.1)\) and the definition of \( \omega_c^N \) that \( m^K_{\omega_c^N} \leq m_\infty \). Suppose to the contrary that \( m^K_{\omega_c^N} < m_\infty \). Then, applying Propositions \((6.2)\) and \((6.3)\) we find that a minimizer \( \Phi^K_{\omega_c^N} \) for \( m^K_{\omega_c^N} \) exists and satisfies \((1.4)\). Note that \( S_{\omega_c^N}(\Phi_{\omega_c^N}) = m^K_{\omega_c^N} \) and \( \mathcal{N}_{\omega_c^N}(\Phi_{\omega_c^N}) = \langle S'_{\omega_c^N}(\Phi_{\omega_c^N}), \Phi_{\omega_c^N} \rangle = 0 \). Put \( \mu_c^K := m_\infty - m^K_{\omega_c^N} \). Moreover, let \( \{\omega_n\} \) be a sequence of positive numbers such that \( \lim_{n \to \infty} \omega_n = \omega_c^N \), and \( \omega_n > \omega_c^N \) for all \( n \geq 1 \). We have already proved that for any \( n \geq 1 \),

\[
m^K_{\omega_n} = m_\infty. \tag{6.5}
\]

Furthermore, we can take a number \( n_c \) such that for any \( n \geq n_c \),

\[
|S_{\omega_n}(\Phi_{\omega_c^N}) - S_{\omega_c^N}(\Phi_{\omega_c^N})| = \frac{1}{2} |\omega_c^N - \omega_n| \|\Phi_{\omega_c^N}\|_{L^2}^2 < \frac{1}{2} \mu_c^K, \tag{6.6}
\]

Using \((6.5)\), the definition of \( m^K_{\omega} \), \( K(\Phi_{\omega_c^N}) = 0 \), \((6.6)\) and \( \mu_c^K = m_\infty - m_{\omega_c^N} = m_\infty - S_{\omega_c^N}(\Phi_{\omega_c^N}) \), we find that for any \( n \geq n_c \),

\[
m_\infty = m^K_{\omega_n} \leq S_{\omega_n}(\Phi_{\omega_c^N}) < S_{\omega_c^N}(\Phi_{\omega_c^N}) + \frac{1}{2} \mu_c^K = m_\infty - \frac{1}{2} \mu_c^K. \tag{6.7}
\]

However, this is a contradiction. Thus, we have completed the proof.

Now, we are in a position to prove Theorem \((1.4)\).

**Proof of Theorem** \((1.4)\) We shall prove the claim (i).

We shall show that \( \omega_c^K \leq \omega_c^N \). Suppose to the contrary that \( \omega_c^K > \omega_c^N \). Then, we can take \( \omega_2 \) such that \( \omega_c^N < \omega_2 < \omega_c^K \), and Proposition \((6.2)\) together with \((6.1)\) shows that a minimizer \( Q \) for \( m^K_{\omega_2} \) exists. Note that \( S_{\omega_2}(Q) = m^K_{\omega_2} \) and \( K(Q) = 0 \). Furthermore, it follows from Proposition \((6.3)\) that \( \mathcal{N}_{\omega_2}(Q) = \langle S'_{\omega}(Q), Q \rangle = 0 \).
Now, we find from $\omega^N_c < \omega_2$, Proposition 3.2, $\mathcal{N}_{\omega_2}(Q) = 0$, the definition of $m^N_\omega$ (see (1.1)), $\mathcal{S}_{\omega_2}(Q) = m^K_\omega$, $\omega_2 < \omega^K_c$ and (6.1) that

$$m_\infty = m^N_{\omega_2} \leq \mathcal{S}_{\omega_2}(Q) = m^K_{\omega_2} < m_\infty,$$

which is a contradiction. Thus, we have proved the claim.

Suppose to the contrary that $\omega^K_c < \omega^N_c$. Let $\omega \in (\omega^K_c, \omega^N_c)$. Then, we see from Proposition 1.1 that there exists a positive radial ground state $\Phi^N_\omega$ of $m^N_\omega$. Then, we can prove in a way similar to Lemma 2 of [6] that $\Phi^N_\omega$ and the first and the second derivatives decays exponentially. In particular, we have $x \cdot \nabla \Phi^N_\omega \in H^1(\mathbb{R}^3)$. Hence, we see that

$$\mathcal{K}(\Phi^N_\omega) = \left\langle \mathcal{S}'_\omega(\Phi^N_\omega), \frac{3}{2} \Phi^N_\omega + x \cdot \nabla \Phi^N_\omega \right\rangle = 0. \quad (6.9)$$

It follows from the definition of $m^K_\omega$, $\omega > \omega^K_c$ and Proposition 6.4 that

$$m_\infty = m^K_\omega \leq \mathcal{S}_\omega(\Phi^N_\omega). \quad (6.10)$$

On the other hand, we see from $\omega < \omega^N_c$ and the definition of $\omega^N_c$ that

$$\mathcal{S}_\omega(\Phi^N_\omega) < m_\infty, \quad (6.11)$$

which contradicts (6.10). Thus, we have $\omega^N_c = \omega^K_c$.

Next, we shall prove the claim (ii). Put $\omega_c := \omega^N_c = \omega^K_c$, and let $\omega < \omega_c$. Then, it follows from Propositions 1.1 and 6.2 that there exist a minimizer $\Phi^N_\omega$ of $m^N_\omega$ and a minimizer $\Phi^K_\omega$ of $m^K_\omega$. Note that $\mathcal{N}_\omega(\Phi^K_\omega) = 0$ and $\mathcal{K}(\Phi^N_\omega) = 0$ (see (6.9)). Hence, we have

$$m^N_\omega \leq \mathcal{S}_\omega(\Phi^K_\omega) = m^K_\omega, \quad (6.12)$$

and

$$m^K_\omega \leq \mathcal{S}_\omega(\Phi^N_\omega) = m^N_\omega. \quad (6.13)$$

Note that $\omega_c = \infty$ for $3 < p < 5$. Hence, we may assume that $\frac{7}{3} < p \leq 3$. By (6.12) and (6.13), we have $m^N_\omega = m^K_\omega$ for any $\omega < \omega_c$. Then, for any $\omega \geq \omega_c$, it follows from Propositions 3.2 and 6.4 that

$$m^N_\omega = m^K_\omega = m_\infty. \quad (6.14)$$

Thus, we obtain $m^N_\omega = m^K_\omega$ for any $\omega > 0$.

Next, we shall prove the claim (iii). From Proposition 6.2 there exists a minimizer $\Phi^K_\omega$ of $m^K_\omega$ for $\omega \in (0, \omega_c)$. We shall show that there exists no minimizer of $m^K_\omega$ for $\omega \in (\omega_c, \infty)$. Suppose to the contrary that there exists a $\omega_* \in (\omega^K_c, \infty)$ such that $m^K_{\omega_*}$ has a minimizer $\Phi^K_{\omega_*}$. By $\omega_* > \omega^K_c$, $\Phi^K_{\omega_*} \neq 0$, Lemma 6.1 and Proposition 6.4 we have

$$m_\infty = \mathcal{S}_{\omega_*}(\Phi^K_{\omega_*}) > \mathcal{S}_{\omega_*}(\Phi^K_{\omega_*}) \geq m^K_{\omega_*} = m_\infty,$$

which is a contradiction.
Finally, we will give a proof of (iv). Assume \( \frac{7}{3} < p < 3 \). It follows from Theorem 1.2 and \( m^N_{\omega} = m^K_{\omega} \) (see Claim (ii)) that there exists a minimizer \( \Phi^N_{\omega} \) such that \( S^N_{\omega}(\Phi^N_{\omega}) = m_{\infty} \). Since \( \Phi^N_{\omega} \) satisfies \( K(\Phi^N_{\omega}) = 0 \), one has, by Proposition 6.4, that
\[
m_{\infty} = m^K_{\omega} \leq S^N_{\omega}(\Phi^N_{\omega}) = m_{\infty}.
\]
(6.16)
Thus, we see that \( \Phi^N_{\omega} \) is also a minimizer for \( m^K_{\omega} \). This completes the proof. \( \square \)

7 Proofs of Theorems 2.2 and 2.3

In this section, we give a proof of Theorems 2.2 and 2.3.

7.1 Preliminaries

First, let us recall the information about the spectrum of \( L_+ \) (see, e.g., Section 5.2 of [11], and Proposition 2.2 of [10]):

Lemma 7.1. Assume \( d \geq 3 \). Then, there exists \( e_0 > 0 \) such that \( -e_0 \) is the only one negative eigenvalue of the operator \( L_+ := -\Delta + V_+ \). Furthermore, the spectrum \( \sigma(L_+) \) of \( L_+ \) is contained in \( \{-e_0\} \cup [0, \infty) \), and
\[
\{ u \in \dot{H}^1(\mathbb{R}^d) \colon L_+ u = 0 \text{ in the distribution sense} \} = \text{span}_C\{ AW, \partial_1 W, \ldots, \partial_d W \}, \quad (7.1)
\]
where \( \partial_j W \) means the partial derivative of \( W \) with respect to the \( j \)-th variable for each \( 1 \leq j \leq d \).

We can also derive the following information about the spectrum of \( L_- \):

Lemma 7.2. Assume \( d \geq 3 \). Then, the spectrum \( \sigma(L_-) \) of \( L_- \) equals \([0, \infty)\), and
\[
\{ u \in \dot{H}^1(\mathbb{R}^d) \colon L_- u = 0 \text{ in the distribution sense} \} = \text{span}_C\{ W \colon (y, \nu, \theta) \in \mathbb{R}^d \times (0, \infty) \times [-\pi, \pi) \}, \quad (7.2)
\]
Proof of Lemma 7.2. It is well known that the essential spectrum of \( L_- \) equals \([0, \infty)\). Let \( u \in H^2(\mathbb{R}^3) \) be a function such that \( L_- u = -\mu u \) for some \( \mu > 0 \). Then, it follows from Hölder’s inequality, (1.15) and Sobolev’s inequality (1.12) that
\[
\|\nabla u\|^2_{L^2} = \langle -\Delta u, u \rangle = \langle -V_- u, u \rangle - \langle \mu u, u \rangle
\]
\[
\leq \|W^4\|^2_{L^\frac{1}{4}}\|u^2\|^2_{L^\frac{4}{1}} - \mu \|u\|^2_{L^2} = \sigma \|u\|^2_{L^6} - \mu \|u\|^2_{L^2} < \|\nabla u\|^2_{L^2}, \quad (7.3)
\]
which is a contradiction. In particular, there is no eigenvalue of \( L_- \) for \( \mu > 0 \). When \( \mu = 0 \), (7.3) means that \( u \) attains the equality in Sobolev’s inequality (1.12), which together with (1.13) and \( L_- u = 0 \) implies that \( u \) must be of the form in (1.13), that is, \( u = z W \) for some \( z \in \mathbb{C} \). \( \square \)
For $r \geq 1$, we define
\begin{equation}
X_r := \{ f \in L^r(\mathbb{R}^3) : \langle f, V_+ \nabla W \rangle = 0, \langle f, V_+ AW \rangle = 0, \langle f, iV_- W \rangle = 0 \}.
\end{equation}
Moreover, we define the operators $G_0$ and $G$ by
\begin{align}
G_0f &:= (-\Delta)^{-1}V_+ \Re[f] + i(-\Delta)^{-1}V_- \Im[f], \quad (7.5) \\
Gf &:= (1 + G_0)f = \{1 + (-\Delta)^{-1}V_+\} \Re[f] + i\{1 + (-\Delta)^{-1}V_-\} \Im[f]. \quad (7.6)
\end{align}
Notice that by (2.7),
\begin{equation}
G(\alpha) = G - \alpha(-\Delta + \alpha)^{-1}G_0,
\end{equation}
where $G(\alpha)$ is the operator defined by (2.13).

It follows from the Hardy-Littlewood-Sobolev inequality (with $s = 2$) that for $3 < r < \infty$ and any $f \in L^r(\mathbb{R}^3)$,
\begin{align}
\|G_0f\|_{L^r} &\lesssim \|V_+ \Re[f]\|_{L^{\frac{6r}{6r-3}}} + \|V_- \Im[f]\|_{L^r} \\
&\leq \|V_+\|_{L^\infty} \|\Re[f]\|_{L^r} + \|V_-\|_{L^\infty} \|\Im[f]\|_{L^r} \lesssim \|f\|_{L^r},
\end{align}
where the implicit constants depend only on $r$. Hence, or any $3 < r < \infty$, $G_0$ is a bounded operator from $L^r(\mathbb{R}^3)$ to itself. Furthermore, we have:

**Lemma 7.3.** Let $3 < r < \infty$. Then, $G_0$ maps $X_r$ into itself.

**Proof of Lemma 7.3.** Let $f \in X_r$. By (7.8), it suffices to show that $G_0f$ satisfies the orthogonality conditions of $X_r$. Since $V_+ \partial_j W = \Delta \partial_j W$ ($1 \leq j \leq 3$) (see Lemma 7.1) and $\langle f, V_+ \partial_j W \rangle = 0$, one sees that
\begin{equation}
\langle G_0f, V_+ \partial_j W \rangle = \langle (-\Delta)^{-1}V_+ \Re[f], V_+ \partial_j W \rangle = \langle V_+ f, \partial_j W \rangle = 0. \quad (7.9)
\end{equation}
Similarly, one can verify that
\begin{equation}
\langle G_0f, V_+ AW \rangle = \langle G_0f, iV_- W \rangle = 0. \quad (7.10)
\end{equation}
Thus, we have proved that $G_0f \in X_r$. \qed

**Lemma 7.4.** For any $6 < r < \infty$, $G_0$ is a compact operator from $X_r$ to itself.

**Proof of Lemma 7.4.** It follows from (2.22) that $|V_+(x)| \sim |V_-(x)| \sim (1 + |x|)^{-4}$ and the Hardy-Littlewood-Sobolev inequality (with $s = 1$) that for any $f \in L^r(\mathbb{R}^3)$,
\begin{align}
\|\nabla G_0f\|_{L^r} &\leq \|\nabla (-\Delta)^{-1}V_+ \Re[f]\|_{L^r} + \|\nabla (-\Delta)^{-1}V_- \Im[f]\|_{L^r} \\
&\lesssim \int_{\mathbb{R}^3} |x - y|^{-2}(1 + |y|)^{-4}|f(y)| \, dy \|L^r \\
&\lesssim \|(1 + |y|)^{-4}f\|_{L^{\frac{6r}{6r-3}}} \leq \|(1 + |y|)^{-4}\|_{L^\infty} \|f\|_{L^r} \lesssim \|f\|_{L^r}, \quad (7.11)
\end{align}
where the implicit constants depend only on $r$. Let $\{f_n\}$ be a bounded sequence in $X_r$. Then, (7.8) shows that $\{G_0f_n\}$ is also a bounded sequence in $X_r$. Moreover, by the fundamental theorem of calculus and (7.11), one can see that for any $y \in \mathbb{R}^3$,

$$
\|G_0f_n - G_0f_n(x + y)\|_{L^r}
\leq \|(-\Delta)^{-1}V_+\mathbb{R}[f_n] - (-\Delta)^{-1}V_+\mathbb{R}[f_n(x + y)]\|_{L^r}
+ \|(-\Delta)^{-1}V_-\mathbb{S}[f_n] - (-\Delta)^{-1}V_-\mathbb{S}[f_n(x + y)]\|_{L^r}
= \|\int_0^1 \frac{3}{3r}(-\Delta)^{-1}V_+\mathbb{R}[f_n(\cdot + \theta y)] d\theta\|_{L^r}
\leq |y|\|\nabla V_+\mathbb{R}[f_n]\|_{L^r} + |y|\|\nabla V_-\mathbb{S}[f_n]\|_{L^r}
\lesssim |y|\|\nabla G_0f_n\|_{L^r}
\lesssim |y| \sup_{n \geq 1} \|f_n\|_{L^r},
$$

which implies the equicontinuity of $\{G_0f_n\}$ in $X_r$.

From the point of view of Lemma D.1 it suffices to show the tightness of $\{G_0f_n\}$ in $L^r(\mathbb{R}^3)$. Notice that if $|x - y| \leq \frac{1}{2}|x|$, then $|y| \geq |x| - |x - y| \geq \frac{1}{2}|x|$. Hence, we see from (2.2), the Hardy-Littlewood-Sobolev inequality (2.3) with $s = 2$, Hölder’s inequality and $|V(x)| \sim |V_+(x)| \lesssim (1 + |x|)^{-4}$ that for any $R \geq 1$,

$$
\int_{|x| \geq R} |G_0f_n(x)|^r \, dx
\lesssim \int_{|x| \geq R} \left| (-\Delta)^{-1}V_+\mathbb{R}[f_n(x)] \right|^r \, dx + \int_{|x| \geq R} \left| (-\Delta)^{-1}V_-\mathbb{S}[f_n(x)] \right|^r \, dx
\lesssim \int_{|x| \geq R} \left\{ \int_{\frac{1}{2}|x| \leq |x - y|} |x - y|^{-1} (1 + |y|)^{-4} |f_n(y)| \, dy \right\}^r \, dx
+ \int_{|x| \geq R} \left\{ \int_{|x - y| \leq \frac{1}{2}|x|} |x - y|^{-1} (1 + |y|)^{-4} |f_n(y)| \, dy \right\}^r \, dx
\lesssim R^{-\frac{r}{2}} \left\| \int_{\mathbb{R}^3} |x - y|^{-\frac{r}{4}} (1 + |y|)^{-4} |f_n(y)| \, dy \right\|_{L^r}^r
+ R^{-\frac{r}{2}} \left\| \int_{\mathbb{R}^3} |x - y|^{-1} |y|^{\frac{r}{2}} (1 + |y|)^{-4} |f_n(y)| \, dy \right\|_{L^r}^r
\lesssim R^{-\frac{r}{2}} \| (1 + |x|)^{-4} f_n \|_{L^{\frac{6}{r}}}^r + R^{-\frac{r}{2}} \| |x|^{\frac{r}{2}} (1 + |x|)^{-4} |f_n| \|_{L^{\frac{6}{r}}}^r
\leq R^{-\frac{r}{2}} \| (1 + |x|)^{-4} f_n \|_{L^r}^r + R^{-\frac{r}{2}} \| |x|^{\frac{r}{2}} (1 + |x|)^{-4} \|_{L^2}^r \| f_n \|_{L^r}
\lesssim R^{-\frac{r}{2}} \| f_n \|_{L^r}^r,
$$

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Hence, it follows from Lemma 7.1, Lemma 7.2 and (7.14) that

\[ \langle Hu, \phi \rangle = \langle -(\Delta)Gu, \phi \rangle = \langle Gu, -\Delta \phi \rangle = 0. \]  

(7.14)

Hence, it follows from Lemma 7.4 that \( X \) is an eigenvalue of \( \Delta \) exists as a bounded operator from \( X \) to itself, namely \( -1 \in \rho(G_0) \). Suppose that \( u \in X \) satisfies \( Gu = 0 \). Then, for any \( \phi \in C_c^\infty(\mathbb{R}^3) \),

\[ \langle Hu, \phi \rangle = \langle -(\Delta)Gu, \phi \rangle = \langle Gu, -\Delta \phi \rangle = 0. \]

(7.14)

**Lemma 7.5.** Assume \( 6 < r < \infty \). Then, the inverse of \( G \) exists as a bounded operator from \( X_r \) to itself.

**Proof of Lemma 7.5.** It follows from Lemma 7.4 that \( G_0 \) is a compact operator from \( X_r \) to itself. Hence, it follows from the Riesz-Schauder theorem (see, e.g., Theorem VI.15 of [21]) that if \( -1 \) is not an eigenvalue of the operator \( G_0 : X_r \to X_r \), then the inverse \( G^{-1} = (1 + G_0)^{-1} \) exists as a bounded operator from \( X_r \) to itself, namely \( -1 \in \rho(G_0) \).

Now, for \( 1 \leq r \leq \infty \), we introduce the operators \( P \) and \( \Pi \) on \( L^r(\mathbb{R}^3) \) as

\[ Pf := \sum_{j=1}^{3} \frac{\langle f, V_+ \partial_j W \rangle}{\langle \partial_j W, V_+ \partial_j W \rangle} \partial_j W + \frac{\langle f, V_\Lambda W \rangle}{\langle \Lambda W, V_+ \Lambda W \rangle} \Lambda W + \frac{\langle f, iV_- W \rangle}{\langle iW, iV_- W \rangle} (iW), \]

(7.17)

\[ \Pi f := \sum_{j=1}^{3} \frac{\langle f, V_+ \partial_j W \rangle}{\|V_+ \partial_j W\|^2_{L^2}} V_+ \partial_j W + \frac{\langle f, V_\Lambda W \rangle}{\|V_+ \Lambda W\|^2_{L^2}} \Lambda W + \frac{\langle f, iV_- W \rangle}{\|V_- W\|^2_{L^2}} (iV_- W). \]

(7.18)

Observe that \( P^2 = P \), \( (1-P)^2 = (1-P) \), and for any \( 3 \leq r \leq \infty \) and any \( f \in L^r(\mathbb{R}^3) \),

\[ \|Pf\|_{L^r} + \|(1-P)f\|_{L^r} + \|\Pi f\|_{L^r} + \|(1-\Pi)f\|_{L^r} \lesssim \|f\|_{L^r} \lesssim \|f\|_{L^r}, \]

(7.19)

where the implicit constant depends only on \( r \). Moreover, it follows from [1.25] that

\[ GP = \Pi G = 0. \]

(7.20)

**Lemma 7.6.** Let \( 3 < r \leq \infty \). Then,

\[ (1-P)L^r(\mathbb{R}^3) := \{(1-P)f : f \in L^r(\mathbb{R}^3)\} = X_r, \]

(7.21)

\[ (1-\Pi)L^r(\mathbb{R}^3) := \{(1-\Pi)f : f \in L^r(\mathbb{R}^3)\} = X_r. \]

(7.22)

Furthermore, \( (1-P)f = (1-\Pi)f = f \) for all \( f \in X_r \).
Proof of Lemma 7.6. Let $g \in (1 - P)L^r(\mathbb{R}^3)$. Then, there exists $f \in L^r(\mathbb{R}^3)$ such that $g = (1 - P)f$. By (7.19), $g = (1 - P)f \in L^r(\mathbb{R}^3)$. Furthermore, it is not difficult to see that
\[(1 - P)f, V_+ \partial_j W = 0, \quad (1 - P)f, V_+ \Lambda W = 0, \quad (1 - P)f, iV_- W = 0. \quad (7.23)\]
Hence, $(1 - P)L^r(\mathbb{R}^3) \subset X_r$. On the other hand, let $f \in X_r$. Then, it is obvious that $f = Pf + (1 - P)f = (1 - P)f$. Thus, $X_r \subset (1 - P)L^r(\mathbb{R}^3)$. Similarly, we can prove that $(1 - \Pi)L^r(\mathbb{R}^3) = X_r$. □

7.2 Estimates of scalar products

Using the fundamental theorem of calculus and (1.14), one can rewrite the Talenti function $W$ as
\[W(x) = \left(\frac{|x|^2}{3}\right)^{-\frac{1}{2}} - \frac{1}{2} \int_0^1 \left(\theta + \frac{|x|^2}{3}\right)^{-\frac{3}{2}} \, d\theta. \quad (7.24)\]
Furthermore, it is known (see, e.g., Corollary 5.10 of [19]) that
\[\mathcal{F} \left[ |x|^{-1} \right] (\xi) = \frac{4\pi^{\frac{3}{2}}}{\Gamma(\frac{3}{2})} |\xi|^{-2}, \quad (7.25)\]
where $\Gamma$ denotes the gamma function.

Proposition 7.7. There exist $A_+ > 0$ and $A_- > 0$ such that for any $0 < \alpha < 1$, $2 \leq q < \infty$ and $g \in L^q(\mathbb{R}^3)$, the following estimates hold:
\[\left| \langle (\Delta + \alpha)^{-1} V_+ g, \Lambda W \rangle + A_+ \mathcal{F}[V_+ g](0) \alpha^{-\frac{1}{2}} \right| \lesssim \min \{ \|g\|_{L^q}, \|(1 + |x|)^{-1}g\|_{L^2} \}, \quad (7.26)\]
\[\left| \langle (\Delta + \alpha)^{-1} V_- g, iW \rangle - A_- \mathcal{F}[V_- g](0) \alpha^{-\frac{1}{2}} \right| \lesssim \min \{ \|g\|_{L^q}, \|(1 + |x|)^{-1}g\|_{L^2} \}, \quad (7.27)\]
where the implicit constants depend only on $q$.

We will prove Proposition 7.7 shortly. Here, it is worthwhile noting that for any $q \geq 2$,
\[\mathcal{F}[V_+ g](0) \leq \|V_+ g\|_{L^1} \lesssim \min \{ \|g\|_{L^q}, \|(1 + |x|)^{-1}g\|_{L^2} \}. \quad (7.28)\]
Hence, Proposition 7.7 together with (7.28) yields:

Corollary 7.1. It holds that for any $0 < \alpha < 1$, $2 \leq q < \infty$ and $g \in L^q(\mathbb{R}^3)$,
\[\left| \langle (\Delta + \alpha)^{-1} V_+ g, \Lambda W \rangle \right| \lesssim \alpha^{-\frac{1}{2}} \min \{ \|g\|_{L^q}, \|(1 + |x|)^{-1}g\|_{L^2} \}, \quad (7.29)\]
\[\left| \langle (\Delta + \alpha)^{-1} V_- g, iW \rangle \right| \lesssim \alpha^{-\frac{1}{2}} \min \{ \|g\|_{L^q}, \|(1 + |x|)^{-1}g\|_{L^2} \}, \quad (7.30)\]
where the implicit constants depend only on $q$.

Now, we give a proof of Proposition 7.7.
Proof of Proposition 7.7. Observe that we can rewrite $\Lambda W$ as

$$\Lambda W(x) = \left(1 + \frac{|x|^2}{3}\right)^{-\frac{3}{2}} \left\{ \frac{1}{2} \left(1 + \frac{|x|^2}{3}\right) - \frac{|x|^2}{3} \right\}$$

$$= \left(1 + \frac{|x|^2}{3}\right)^{-\frac{3}{2}} \left\{ \frac{1}{2} - \frac{|x|^2}{6} \right\}$$

$$= -\frac{|x|^2}{6} \left(\frac{|x|^2}{3}\right)^{-\frac{3}{2}} + \frac{1}{2} \left(1 + \frac{|x|^2}{3}\right)^{-\frac{3}{2}} + Y(x),$$

where

$$Y(x) := -\frac{|x|^2}{6} \left(1 + \frac{|x|^2}{3}\right)^{-\frac{3}{2}} + \frac{|x|^2}{6} \left(\frac{|x|^2}{3}\right)^{-\frac{3}{2}}. \tag{7.32}$$

Notice that by the fundamental theorem of calculus, one has

$$\int_0^1 \left(\theta + \frac{|x|^2}{3}\right)^{-\frac{3}{2}} d\theta \leq \int_0^1 \left(\theta + \frac{|x|^2}{3}\right)^{-\frac{3}{2}} d\theta. \tag{7.33}$$

By (7.24), (7.31) and (7.33), we are convinced that the proofs of (7.26) and (7.27) are similar. Hence, we omit the proof of (7.27).

We shall prove (7.26) by an argument similar to the proof of Lemma 2.5 of [9]. According to (7.31), one has

$$\langle (-\Delta + \alpha)^{-1} V + g, \Lambda W \rangle = -\frac{3^2}{6} \langle (-\Delta + \alpha)^{-1} V + g, |x|^{-1} \rangle$$

$$+ \frac{1}{2} \langle (-\Delta + \alpha)^{-1} V + g, \left(1 + \frac{|x|^2}{3}\right)^{-\frac{3}{2}} \rangle$$

$$+ \langle (-\Delta + \alpha)^{-1} V + g, Y \rangle. \tag{7.34}$$

In what follows, let $2 \leq q < \infty$, and let $q'$ denote the Hölder conjugate of $q$, namely $q' = \frac{q}{q-1}$. Furthermore, we allow the implicit constants to depend on $q$.

**Step 1.** We consider the first term on the right-hand side of (7.34). Using Parseval’s identity, (7.24) and the formula $(-\Delta + \alpha)^{-1} = F^{-1}([|\xi|^2 + \alpha]^{-1} F$, one can see that

$$-\frac{3^2}{6} \langle (-\Delta + \alpha)^{-1} V + g, |x|^{-1} \rangle$$

$$= -C_1 \langle [|\xi|^2 + \alpha]^{-1} F[V + g], |\xi|^{-2} \rangle$$

$$= -C_1 \Re \int_{|\xi| \leq 1} \frac{F[V + g](0)}{(|\xi|^2 + \alpha)|\xi|^2} d\xi - C_1 \Re \int_{|\xi| \leq 1} \frac{F[V + g](\xi) - F[V + g](0)}{(|\xi|^2 + \alpha)|\xi|^2} d\xi \tag{7.35}$$

$$- C_1 \Re \int_{|\xi| \leq 1} \frac{F[V + g](0)}{(|\xi|^2 + \alpha)|\xi|^2} d\xi,$$

where $C_1$ is some positive constant.

We consider the first term on the right-hand side of (7.35). Using the spherical coordinates, one can verify that

$$\int_{|\xi| \leq 1} \frac{F[V + g](0)}{(|\xi|^2 + \alpha)|\xi|^2} d\xi = C_2 F[V + g](0) \alpha^{-\frac{3}{2}} \arctan (\alpha^{-\frac{1}{2}}), \tag{7.36}$$
where $C_2$ is some positive constant. Note here that
\begin{equation}
\left| \frac{\pi}{2} - \arctan \left( \alpha^{- \frac{1}{2}} \right) \right| = \int_{\alpha^{- \frac{1}{2}}}^{\infty} \frac{1}{1 + t^2} \, dt \leq \int_{\alpha^{- \frac{1}{2}}}^{\infty} t^{-2} \, dt \leq \alpha^{\frac{1}{2}}. \tag{7.37}
\end{equation}

We consider the second term on the right-hand side of (7.35). First, observe that
\begin{equation}
\int_{|\xi| \leq 1} \frac{\xi}{(|\xi|^2 + \alpha)|\xi|^2} \, d\xi = 0. \tag{7.38}
\end{equation}

Then, it follows from $d = 3$, (7.38) and elementary computations that
\begin{equation}
\left| \int_{|\xi| \leq 1} \frac{\mathcal{F}[V_+ g](\xi) - \mathcal{F}[V_+ g](0)}{(|\xi|^2 + \alpha)|\xi|^2} \, d\xi \right|
\leq \int_{|\xi| \leq 1} \frac{1}{(|\xi|^2 + \alpha)|\xi|^2} \int_{\mathbb{R}^3} |e^{-ix \cdot \xi} - 1 \cdot (-ix) |V_+(x)g(x)| \, dx \, d\xi \tag{7.39}
\end{equation}
\begin{equation}
\leq \int_{|\xi| \leq 1} \frac{1}{(|\xi|^2 + \alpha)|\xi|^2} \int_{\mathbb{R}^3} \min \{|x|, |x|^2 \} |V_+(x)g(x)| \, dx \, d\xi.
\end{equation}

Let $1 < q_1 \leq 2$ be a fixed number depending only on $q$ such that $q_1 - 1 < \frac{3}{q}$ (hence $(4 - q_1)q' > 3$). Then, it follows from (7.39), $|V_+(x)| \lesssim (1 + |x|)^{-4}$ and Hölder’s inequality that
\begin{equation}
\left| \int_{|\xi| \leq 1} \frac{\mathcal{F}[V_+ g](\xi) - \mathcal{F}[V_+ g](0)}{(|\xi|^2 + \alpha)|\xi|^2} \, d\xi \right|
\leq \int_{|\xi| \leq 1} \frac{|\xi|^{q_1}}{(|\xi|^2 + \alpha)|\xi|^2} \int_{\mathbb{R}^3} |x|^{q_1} |V_+(x)g(x)| \, dx \tag{7.40}
\end{equation}
\begin{equation}
\leq \int_{|\xi| \leq 1} |\xi|^{q_1 - 4} \, d\xi \| (1 + |x|)^{-4 + q_1} \|_{L^q} \|g\|_{L^q} \lesssim \|g\|_{L^q}.
\end{equation}

Similarly, one can verify that
\begin{equation}
\left| \int_{|\xi| \leq 1} \frac{\mathcal{F}[V_+ g](\xi) - \mathcal{F}[V_+ g](0)}{(|\xi|^2 + \alpha)|\xi|^2} \, d\xi \right|
\leq \int_{|\xi| \leq 1} \frac{|\xi|^{q_1}}{(|\xi|^2 + \alpha)|\xi|^2} \int_{\mathbb{R}^3} |x|^{q_1} |V_+(x)g(x)| \, dx \tag{7.41}
\end{equation}
\begin{equation}
\lesssim \|x|^{\frac{3}{q}} (1 + |x|)^{-3} \|_{L^2} \int_{|\xi| \leq 1} |\xi|^{-\frac{11}{q'}} \, d\xi \| (1 + |x|)^{-1} \|g\|_{L^2} \lesssim \| (1 + |x|)^{-1} \|g\|_{L^2}.
\end{equation}

Putting the estimates (7.40) and (7.41) together, we find that for any $2 \leq q < \infty$,
\begin{equation}
\left| \int_{|\xi| \leq 1} \frac{\mathcal{F}[V_+ g](\xi) - \mathcal{F}[V_+ g](0)}{(|\xi|^2 + \alpha)|\xi|^2} \, d\xi \right| \lesssim \min \left\{ \|g\|_{L^q}, \| (1 + |x|)^{-1} \|g\|_{L^2} \right\}. \tag{7.42}
\end{equation}
We consider the last term on the right-hand side of (7.35). By Hölder’s inequality, Plancherel’s theorem and \(|V_+(x)| \lesssim (1 + |x|)^{-4}\), one can see that
\[
\left| \int_{1 \leq |\xi|} \frac{\mathcal{F}[V_+g](\xi)}{|\xi|^2} d\xi \right| \leq \| |\xi|^{-4} \|_{L^2(|\xi| \geq 1)} \| \mathcal{F}[V_+g] \|_{L^2} \lesssim \| V_+g \|_{L^2} \leq \min \{ \| g \|_{L^q}, \| (1 + |x|)^{-1}g \|_{L^2} \}. \tag{7.43}
\]

Now, we conclude from (7.35) through (7.37), (7.42) and (7.43) that there exists a positive constant \(A_+\) such that for any \(2 \leq q < \infty\),
\[
\left| -\frac{3}{6} \langle (-\Delta + \alpha)^{-1}V_+g, (1 + \frac{|x|^2}{3})^{-\frac{2}{3}} \rangle \right| \lesssim \min \{ \| g \|_{L^q}, \| (1 + |x|)^{-1}g \|_{L^2} \}. \tag{7.44}
\]

**Step 2.** We shall derive an estimate for the second term on the right-hand side of (7.34). It follows from Hölder’s inequality, Lemma 2.2 that for any \(2 \leq q < \infty\),
\[
\left| \langle (-\Delta + \alpha)^{-1}V_+g, (1 + \frac{|x|^2}{3})^{-\frac{2}{3}} \rangle \right| \lesssim \langle (-\Delta + \alpha)^{-1}V_+g \rangle \| (1 + \frac{|x|^2}{3})^{-\frac{2}{3}} \|_{L^\infty} \tag{7.45}
\]
\[
\lesssim \|V_+g\|_{L^\frac{q}{2}} \lesssim \min \{ \| g \|_{L^q}, \| (1 + |x|)^{-1}g \|_{L^2} \}.
\]

**Step 3.** Finally, we consider the last term on the right-hand side of (7.34) and finish the proof. It follows from (7.33) that
\[
\|Y\|_{L^\frac{q}{2}} \lesssim \left\| \int_{0}^{1} \left( \theta + \frac{|x|^2}{3} \right)^{-\frac{2}{3}} d\theta \right\|_{L^\frac{q}{2}} \lesssim \int_{0}^{1} \left\| \theta^{-\frac{2}{3}} |x|^{-d+\frac{2}{3}} \right\|_{L^\frac{q}{2}(|x| \leq 1)} d\theta + \| |x|^{-d} \|_{L^\frac{q}{2}(|x| \leq 1)} \lesssim 1. \tag{7.46}
\]

Hence, by the same computation as (7.45), and (7.46), one can see that for any \(2 \leq q < \infty\),
\[
\left| \langle (-\Delta + \alpha)^{-1}V_+g, Y \rangle \right| \lesssim \|((-\Delta + \alpha)^{-1}V_+g)\|_{L^\infty} \| Y \|_{L^\frac{q}{2}} \lesssim \min \{ \| g \|_{L^q}, \| (1 + |x|)^{-1}g \|_{L^2} \}. \tag{7.47}
\]

Putting (7.31), (7.44), (7.45) and (7.47) together, we obtain the desired estimate (7.28).

**Proposition 7.8.** The following estimate holds for any \(0 < \alpha < 1\), \(2 \leq q < \infty\) and any \(g \in L^q(\mathbb{R}^3)\):
\[
\left| \langle (-\Delta + \alpha)^{-1}V_+g, \nabla W \rangle \right| \lesssim \min \{ \| g \|_{L^q}, \| (1 + |x|)^{-1}g \|_{L^2} \}, \tag{7.48}
\]
where the implicit constant depends only on \(q\).
Proof of Proposition 7.8. The general symbol $d$ to make the dependence on the dimension clear. Using the integration by parts and (7.24), one can see that for any $1 \leq j \leq 3$,

$$
\langle (\Delta + \alpha)^{-1} V g, \partial_j W \rangle
$$

$$
= -\frac{1}{3} \langle (\Delta + \alpha)^{-1} \partial_j (V g), |x|^{-1} \rangle
$$

$$
+ \langle (\Delta + \alpha)^{-1} \partial_j (V g), \frac{1}{2} \int_0^1 \left( \theta + \frac{|x|^2}{3} \right)^{-\frac{3}{2}} d\theta \rangle .
$$

(7.49)

Throughout the proof, we allow the implicit constants to depend on $q$.

**Step 1.** We consider the first term on the right-hand side of (7.49). First, note that

$$
\int_{|\xi| \leq 1} \frac{\xi_j}{(|\xi|^2 + \alpha)|\xi|^2} d\xi = 0.
$$

(7.50)

Then, by the same computation as (7.35), and (7.30), one sees that

$$
-\frac{1}{3} \langle (\Delta + \alpha)^{-1} \partial_j (V g), |x|^{-1} \rangle
$$

$$
= -C_1 \Re \int_{|\xi| \leq 1} \frac{i \xi_j \{ F[V g](\xi) - F[V g](0) \}}{(|\xi|^2 + \alpha)|\xi|^2} d\xi
$$

$$
+ C_1 \Re \int_{1 \leq |\xi|} \frac{i \xi_j F[V g](\xi)}{(|\xi|^2 + \alpha)|\xi|^2} d\xi,
$$

(7.51)

where $C_1$ is some positive constant.

We consider the first term on the right hand side of (7.51). By elementary computations and $|V_+ (x)| \sim (1 + |x|)^{-4}$, one can verify that

$$
|F[V_+ g](\xi) - F[V_+ g](0)| = | \int_{\mathbb{R}^3} \{ e^{-ix \cdot \xi} - 1 \} (V_+ g)(x) dx |
$$

$$
\lesssim \int_{\mathbb{R}^3} \min \{ 1, |\xi||x| \} (1 + |x|)^{-4} |g(x)| \, dx
$$

$$
\lesssim |\xi| \min \{ \|g\|_{L^\infty}, (1 + |x|)^{-1}\|g\|_{L^2} \} .
$$

(7.52)

Using (7.32), one can find that

$$
\left| \int_{|\xi| \leq 1} \frac{i \xi_j \{ F[V_+ g](\xi) - F[V_+ g](0) \}}{(|\xi|^2 + \alpha)|\xi|^2} d\xi \right| \lesssim \min \{ \|g\|_{L^\infty}, (1 + |x|)^{-1}\|g\|_{L^2} \} .
$$

(7.53)

We consider the second term on the right-hand side of (7.51). It is easy to verify that if $q \geq 2$, then

$$
\left| \int_{1 \leq |\xi|} \frac{i \xi_j F[V_+ g](\xi)}{(|\xi|^2 + \alpha)|\xi|^2} d\xi \right| \leq \int_{1 \leq |\xi|} \frac{|F[V_+ g](\xi)|}{|\xi|^3} d\xi
$$

$$
\lesssim \|\xi|^{-3}\|_{L^q(|\xi| \geq 1)} \|V_+ g\|_{L^2} \lesssim \min \{ \|g\|_{L^\infty}, (1 + |x|)^{-1}\|g\|_{L^2} \} .
$$

(7.54)
Thus, putting (7.51), (7.52), and (7.54) together, we find that for any \( q \geq 2 \), the first term on the right-hand side of (7.49) is estimated as follows:

\[
| \langle (-\Delta + \alpha)^{-1} \partial_j (V + g), |x|^{-1} \rangle | \lesssim \min \{ \| g \|_{L^q}, \| (1 + |x|)^{-1} g \|_{L^2} \}. \tag{7.55}
\]

**Step 2.** We move on to the second term on the right-hand side of (7.49), and finish the proof. By Hölder’s inequality, one has

\[
| \langle (-\Delta + \alpha)^{-1} \partial_j (V + g), \frac{1}{2} \int_0^1 (\theta + \frac{|x|^2}{3})^{-\frac{3}{2}} d\theta \rangle | \leq \| (-\Delta + \alpha)^{-1} \partial_j (V + g) \|_{L^q} \frac{1}{2} \int_0^1 (\theta + \frac{|x|^2}{3})^{-\frac{3}{2}} d\theta \|_{L^{\frac{6}{5}}} . \tag{7.56}
\]

Using the first resolvent equation (2.7) with \( \alpha_0 = 0 \), Sobolev’s inequality, and (2.3) in Lemma 2.1, we see that for any \( 2 \leq q < \infty \),

\[
\| (-\Delta + \alpha)^{-1} \partial_j (V + g) \|_{L^q}
\leq \| (-\Delta)^{-1} \partial_j (V + g) \|_{L^q} + \| (\alpha - \alpha)^{-1} (-\Delta)^{-1} \partial_j (V + g) \|_{L^q}
\lesssim \| \nabla \|^{-1} \partial_j (V + g) \|_{L^q} + \| \nabla \|^{-1} (\alpha - \alpha)^{-1} (V + g) \|_{L^q}
\lesssim \| V + g \|_{L^2} + \| (\Delta + \alpha)^{-1} \partial_j (V + g) \|_{L^q}
\lesssim \| V + g \|_{L^q} \lesssim \min \{ \| g \|_{L^q}, \| (1 + |x|)^{-1} g \|_{L^2} \} . \tag{7.57}
\]

Moreover, it follows that

\[
\| \int_0^1 (\theta + \frac{|x|^2}{3})^{-\frac{3}{2}} d\theta \|_{L^{\frac{6}{5}}} \lesssim \int_0^1 \| \theta^{-\frac{3}{4}} |x|^{-\frac{3}{2}} \|_{L^{\frac{6}{5}}(\{1 \leq x \leq 1\})} d\theta + \| |x|^{-3} \|_{L^{\frac{6}{5}}(\{1 \leq x \leq 1\})} \lesssim 1 . \tag{7.58}
\]

Plugging (7.57) and (7.58) into (7.59), we find that for any \( 2 \leq q < \infty \),

\[
| \langle (-\Delta + \alpha)^{-1} \partial_j (V + g), \frac{1}{2} \int_0^1 (\theta + \frac{|x|^2}{3})^{-\frac{3}{2}} d\theta \rangle | \lesssim \min \{ \| g \|_{L^q}, \| (1 + |x|)^{-1} g \|_{L^2} \} . \tag{7.59}
\]

Now, we find from (7.49), (7.55), and (7.59) that the desired estimate (7.58) holds. \( \square \)

In the special case \( g = \partial_j W \) in Proposition 7.8, we can obtain a detailed estimate:

**Lemma 7.9.** It holds that for any \( \alpha > 0 \) and \( 1 \leq j \leq 3 \),

\[
| \langle (-\Delta + \alpha)^{-1} V + \partial_j W, \partial_j W \rangle + \| \partial_j W \|_{L^2}^2 | \lesssim \alpha^\frac{3}{4} . \tag{7.60}
\]
Proof of Lemma 7.9. It follows from the first resolvent equation (2.7) with \( \alpha_0 = 0 \), and (2.5) in Lemma 2.1 that
\[
\left| \langle (-\Delta + \alpha)^{-1}V_+\partial_j W, \partial_j W \rangle - \langle (-\Delta)^{-1}V_+\partial_j W, \partial_j W \rangle \right| \\
= \alpha \left| \langle (-\Delta + \alpha)^{-1}(-\Delta)^{-1}V_+\partial_j W, \partial_j W \rangle \right| \\
\lesssim \alpha \|(-\Delta)^{-1}V_+\partial_j W\|_{L^2} \|(-\Delta + \alpha)^{-1}\partial_j W\|_{L^2} \\
= \alpha \|\partial_j W\|_{L^2} \alpha^{-\frac{3}{2}} \|\partial_j W\|^{\frac{3}{2}} \lesssim \alpha^{\frac{1}{2}}.
\]

Moreover, by (1.25), one can see that
\[
\langle (-\Delta)^{-1}V_+\partial_j W, \partial_j W \rangle = \langle (-\Delta)^{-1}\Delta\partial_j W, \partial_j W \rangle = -\|\partial_j W\|_{L^2}^2.
\]

Putting (7.61) and (7.62) together, we obtain (7.60). \( \square \)

### 7.3 Resolvent estimate

For \( 3 < r \leq \infty \), we introduce the linear subspaces \( X_r \) and \( Y_r \) of \( L^r(\mathbb{R}^3) \times L^r(\mathbb{R}^3) \) equipped with the norms \( \| \cdot \|_{X_r} \) and \( \| \cdot \|_{Y_r} \):
\[
X_r := X_r \times P L^r(\mathbb{R}^3), \quad \| (f_1, f_2) \|_{X_r} := \| f_1 \|_{L^r} + \| f_2 \|_{L^r},
\]
\[
Y_r := X_r \times \Pi L^r(\mathbb{R}^3), \quad \| (f_1, f_2) \|_{Y_r} := \| f_1 \|_{L^r} + \| f_2 \|_{L^r},
\]
where \( P L^r(\mathbb{R}^3) := \{ Pf : f \in L^r(\mathbb{R}^3) \} \) and \( \Pi L^r(\mathbb{R}^3) := \{ \Pi f : f \in L^r(\mathbb{R}^3) \} \) (see (7.17) and (7.18)).

Next, for any \( \varepsilon > 0 \), we define the linear operator \( B_\varepsilon : X_r \to L^r(\mathbb{R}^3) \) by
\[
B_\varepsilon f = B_\varepsilon (f_1, f_2) := \varepsilon f_1 + f_2 \quad \text{for} \quad f = (f_1, f_2) \in X_r.
\]

We also define the linear operator \( C : L^r(\mathbb{R}^3) \to Y_r \) by
\[
C f := ((1 - P)f, \Pi f) \quad \text{for} \quad f \in L^r(\mathbb{R}^3).
\]

Note here that by Lemma 7.10 \( (1 - P)L^r(\mathbb{R}^3) = X_r \).

**Lemma 7.10.** Let \( 3 < r \leq \infty \) and \( \varepsilon > 0 \). Then, \( B_\varepsilon \) is surjective. Moreover, \( C \) is injective.

**Proof of Lemma 7.10.** Let \( g \in L^r(\mathbb{R}^3) \), and put \( f := (\varepsilon^{-1}(1-P)g, Pg) \). Then, by Lemma 7.6 \( f \in X_r \). Moreover, it is obvious that \( B_\varepsilon f = g \). Hence, \( B_\varepsilon : X_r \to L^r(\mathbb{R}^3) \) is surjective.

Next, suppose \( C f = 0 \). Then, \( (1 - P)f = 0 \) and \( \Pi f = 0 \). In particular,
\[
f = Pf + (1 - P)f = Pf = \sum_{j=1}^{3} c_j \partial_j W + a\Lambda W + ibW
\]
for some \( c_1, c_2, c_3, a, b \in \mathbb{R} \). Furthermore, this together with \( \Pi f = 0 \) implies that \( c_1 = c_2 = c_d = a = b = 0 \). Thus, \( f = 0 \) and the linear operator \( C : L^r(\mathbb{R}^3) \to Y_r \) is injective. \( \square \)
Lemma 7.11. Assume $3 < r < \infty$. Let $G_0$ be the operator defined by (7.5). Then, the following estimates hold for all $\alpha > 0$:

\[
\| \alpha (\Delta + \alpha)^{-1} G_0 P \|_{L^r \to L^r} \lesssim \alpha^{\frac{3}{2} - \frac{3}{2r}}, \quad (7.68)
\]

\[
\| \alpha \Pi (\Delta + \alpha)^{-1} G_0 \|_{L^r \to \Pi L^r} \lesssim \alpha^{-\frac{3}{2}} \alpha, \quad (7.69)
\]

where the implicit constants depend only on $r$.

Proof of Lemma 7.11. First, we shall prove (7.68). It follows from (1.25) and Lemma 2.1 that for any $f \in L^r(\mathbb{R}^3)$,

\[
\| \alpha (\Delta + \alpha)^{-1} G_0 Pf \|_{L^r}
\]

\[
\leq \| \alpha (\Delta + \alpha)^{-1} (-\Delta)^{-1} V_+ \Re [Pf] \|_{L^r} + \| \alpha (\Delta + \alpha)^{-1} (-\Delta)^{-1} V_- \Im [Pf] \|_{L^r}
\]

\[
\leq \alpha \sum_{j=1}^{3} \| \frac{\langle f, V_+ \partial_j W \rangle}{\| (\partial_j W, V_+ \partial_j W) \|} \| (-\Delta + \alpha)^{-1} (-\Delta)^{-1} V_+ \partial_j W \|_{L^r}
\]

\[
+ \alpha \| \frac{\langle f, V_+ \Lambda W \rangle}{\| (\Lambda W, V_+ \Lambda W) \|} \| (-\Delta + \alpha)^{-1} (-\Delta)^{-1} V_+ \Lambda W \|_{L^r}
\]

\[
+ \alpha \| \frac{\langle f, V_- \Lambda W \rangle}{\| (W, V_- W) \|} \| (-\Delta + \alpha)^{-1} (-\Delta)^{-1} V_- \Lambda W \|_{L^r}
\]

\[
\lesssim \alpha \| f \|_{L^r} \sum_{j=1}^{3} \| (-\Delta + \alpha)^{-1} \partial_j W \|_{L^r}
\]

\[
+ \alpha \| f \|_{L^r} \| (-\Delta + \alpha)^{-1} \Lambda W \|_{L^r} + \alpha \| f \|_{L^r} \| (-\Delta + \alpha)^{-1} W \|_{L^r}
\]

\[
\lesssim \alpha^{\frac{3}{2} \left( \frac{3}{2} - \frac{3}{2r} \right)} \| f \|_{L^r} \left\{ \sum_{j=1}^{3} \| \partial_j W \|_{L^2_{\text{weak}}} + \| \Lambda W \|_{L^2_{\text{weak}}} + \| W \|_{L^2_{\text{weak}}} \right\}
\]

\[
\lesssim \alpha^{\frac{3}{2} - \frac{3}{2r}} \| f \|_{L^r},
\]

where the implicit constant depends only on $r$. Thus, the desired estimate (7.84) holds.

Next, we shall prove (7.69). It follows from (1.25), Corollary 7.1 and Proposition 7.8

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that for any \( f \in L^r(\mathbb{R}^3) \),
\[
\| \alpha \Pi(-\Delta + \alpha)^{-1}G_0 f \|_{L^r} \leq \| \alpha \Pi(-\Delta + \alpha)^{-1}(\Delta - 1) V_+ \Re[f] \|_{L^r} + \| \alpha \Pi(-\Delta + \alpha)^{-1}(\Delta - 1) V_+ \Re[f] \|_{L^r} 
\]
\[
\leq \alpha \sum_{j=1}^{\infty} \frac{|\langle (-\Delta)^{-1}(\Delta + \alpha)^{-1} V_+ \Re[f], (\Delta) \partial_j W \rangle|}{\| V_+ \partial_j W \|_{L^2}^2} \| V_+ \partial_j W \|_{L^r} 
\]
\[
+ \alpha \frac{|\langle (-\Delta)^{-1}(\Delta + \alpha)^{-1} V_+ \Re[f], (\Delta) \Delta W \rangle|}{\| V_+ \Delta W \|_{L^2}^2} \| V_+ \Delta W \|_{L^r} 
\]
\[
+ \alpha \frac{|\langle (-\Delta)^{-1}(\Delta + \alpha)^{-1} V_- \Im[f], (\Delta) W \rangle|}{\| V_- W \|_{L^2}^2} \| V_- W \|_{L^r} 
\]
\[
\leq \alpha \frac{1}{r} \| f \|_{L^r}.
\]

Thus, we have completed the proof. \(\square\)

Now, we are in a position to prove Theorem 2.2.

**Proof of Theorem 2.2** Let \( 0 < \alpha < c_0 \), where \(-c_0\) is the only negative eigenvalue of \( L_+ = -\Delta + V_+ \). Assume \( 6 < r < \infty \). Furthermore, let \( \varepsilon > 0 \) be a small constant to be chosen later, dependently on \( r \). Note here that by Lemma 2.2, the operator \( G(\alpha) \) defined by (2.13) maps \( L^r(\mathbb{R}^3) \) to itself. Then, we define the operator \( A_\varepsilon(\alpha) \) from \( X_r \) to \( Y_r \) by
\[
A_\varepsilon(\alpha) := CG(\alpha)B_\varepsilon,
\]
where \( G(\alpha) \) is the operator defined by (2.13). Observe that for any \( (f_1, f_2) \in X_r \),
\[
A_\varepsilon(\alpha)(f_1, f_2) = \left( \begin{array}{cc} A_{11} f_1 + A_{12} f_2 \\ A_{21} f_1 + A_{22} f_2 \end{array} \right) = \left( \begin{array}{cc} A_{11} & A_{12} \\ A_{21} & A_{22} \end{array} \right) \left( \begin{array}{c} f_1 \\ f_2 \end{array} \right),
\]
where we do not care about the distinction between column vectors and row ones, and
\[
A_{11} := \varepsilon(1 - P)G(\alpha)|_{X_r}, \quad A_{12} := (1 - P)G(\alpha)P, \quad A_{21} := \varepsilon\Pi G(\alpha)|_{X_r}, \quad A_{22} := \Pi G(\alpha)P.
\]

Now, we claim that the inverse of \( A_\varepsilon(\alpha) : X_r \to Y_r \) exists. Notice that by Lemma 7.10 and Lemma 3.12 of [15], this claim implies that \( G(\alpha) : L^r(\mathbb{R}^3) \to L^r(\mathbb{R}^3) \) has the inverse and
\[
G(\alpha)^{-1} = B_\varepsilon A_\varepsilon(\alpha)^{-1} C.
\]

We shall prove the claim in several steps. Throughout the proof, we allow the implicit constants to depend on \( r \).

**Step 1.** We consider the operator \( A_{11} : X_r \to X_r \).
Recall from Lemma 7.5 that $G|_{X_r} : X_r \to X_r$ has the inverse, say $K_1$, such that

$$\|K_1\|_{X_r \to X_r} \lesssim 1. \quad (7.77)$$

Note that Lemma 7.6 shows that

$$(1 - P)G|_{X_r} = G|_{X_r}. \quad (7.78)$$

Then, using (7.78) and (7.7), one can rewrite $A_{11}$ as

$$A_{11} = \varepsilon G|_{X_r} + \varepsilon S_{11} = \varepsilon G|_{X_r} (1 + K_1 S_{11}), \quad (7.79)$$

where

$$S_{11} := -\alpha (1 - P)(-\Delta + \alpha)^{-1} G_0|_{X_r}. \quad (7.80)$$

By (7.19), Lemma 2.2 and the Hardy-Littlewood-Sobolev inequality ((2.3) with $s = 2$), we see that for any $f \in L^r(\mathbb{R}^3)$, Notice that by (7.77) and Lemma 7.11,

$$\| S_{11} f \|_{L^r} \lesssim \alpha \|(-\Delta + \alpha)^{-1} G_0 f \|_{L^r}$$

$$\lesssim \alpha^{\frac{3}{2}} \left\{ \|(-\Delta)^{-1} V_\infty [f] \|_{L^4} + \|(-\Delta)^{-1} V_\infty \mathcal{S}[f] \|_{L^4} \right\}$$

$$\lesssim \alpha^{\frac{3}{2}} \frac{3}{4} \| (1 + |x|)^{-4} f \|_{L^{\frac{4}{3}}}$$

$$\lesssim \alpha^{\frac{3}{2}} \frac{3}{4} \| f \|_{L^r}. \quad (7.81)$$

Observe from (7.81) and Lemma 7.6 that $S_{11}$ maps $X_r$ into itself. Furthermore, it follows from (7.77), (7.83) and the theory of Neumann’s series that for any sufficiently small $\alpha > 0$, the operator $1 + K_1 S_{11}$ has the inverse, say $K_2$, such that

$$\|K_2\|_{X_r \to X_r} \leq (1 - \|K_1 S_{11}\|_{X_r \to X_r})^{-1} \lesssim 1. \quad (7.82)$$

We summarize:

$$A_{11}^{-1} = \varepsilon^{-1} K_2 K_1, \quad \|A_{11}^{-1}\|_{X_r \to X_r} \lesssim \varepsilon^{-1}. \quad (7.83)$$

**Step 2.** We consider the operators $A_{12} : PL^{r'}(\mathbb{R}^3) \to X_r$ and $A_{21} : X_r \to \Pi L^{r'}(\mathbb{R}^3)$. It follows from (7.77), (7.20), (7.19) and Lemma 7.11 that for any $f \in L^{r'}(\mathbb{R}^3)$,

$$\|(1 - P)G(\alpha)Pf\|_{L^r} = \alpha \|(1 - P)(-\Delta + \alpha)^{-1} G_0 Pf\|_{L^r}$$

$$\lesssim \alpha \|(-\Delta + \alpha)^{-1} G_0 Pf\|_{L^r} \lesssim \alpha^{\frac{3}{2}} \frac{3}{4} \| f \|_{L^r}, \quad (7.84)$$

so that

$$\|A_{12}\|_{PL^{r'} \to X_r} \lesssim \alpha^{\frac{3}{2}} \frac{3}{4}. \quad (7.85)$$

Similarly, one can verify that

$$\|A_{21}\|_{X_r \to \Pi L^{r'}} \lesssim \varepsilon \alpha^\frac{3}{4}. \quad (7.86)$$
Step 3. We consider $A_{22}: PL^r(\mathbb{R}^3) \rightarrow \Pi L^r(\mathbb{R}^3)$. By (1.17) and (1.20), we see that for any $f \in L^r(\mathbb{R}^3)$,

$$A_{22}f = -\alpha \Pi(-\Delta + \alpha)^{-1} G_0 Pf.$$  \hfill (7.87)

Observe from (7.87), (1.25), Proposition 7.7 and Lemma 7.9 that

$$A_{22}f = -\alpha \sum_{j=1}^{3} \frac{\langle f, V_+ \partial_j W \rangle}{\|\partial_j W, V_+ \partial_j W\|_{L^2}^2} (-\Delta + \alpha)^{-1} G_0 \partial_j W, V_+ \partial_j W \rangle V_+ \partial_j W$$

$$- \alpha \frac{\langle f, V_+ AW \rangle}{\|AW, V_+ AW\|_{L^2}^2} (-\Delta + \alpha)^{-1} G_0 AW, V_+ AW \rangle V_+ AW$$

$$- \alpha \frac{\langle f, iV_- W \rangle}{\|iW, iV_- W\|_{L^2}^2} (-\Delta + \alpha)^{-1} G_0(iW), iV_- W \rangle V_-(iW)$$

$$= \alpha \sum_{j=1}^{3} c_j(\alpha) \langle f, V_+ \partial_j W \rangle V_+ \partial_j W$$

$$+ \alpha \{b_+ \alpha^{-\frac{7}{2}} + c_+(\alpha)\} \langle f, V_+ AW \rangle V_+ AW$$

$$+ \alpha \{b_- \alpha^{-\frac{9}{2}} + c_- (\alpha)\} \langle f, iV_- AW \rangle V_-(iW)$$

where $b_+, b_- \neq 0$ are some constants depending only on $c_1(\alpha), c_2(\alpha), c_3(\alpha), c_+(\alpha), c_-(\alpha)$ are some constants such that

$$\sum_{j=1}^{3} |c_j(\alpha)| + |c_+(\alpha)| + |c_-(\alpha)| \leq 1.$$  \hfill (7.89)

Here, using the above constants $c_1(\alpha), c_2(d), c_3(\alpha)$ and $b_+, b_-$, we define an operator $Q: PL^r(\mathbb{R}^3) \rightarrow \Pi L^r(\mathbb{R}^3)$ by

$$Qf := \alpha \sum_{j=1}^{3} c_j(\alpha) \langle f, V_+ \partial_j W \rangle V_+ \partial_j W$$

$$+ b_+ \alpha^{-\frac{7}{2}} \langle f, V_+ AW \rangle V_+ AW + b_- \alpha^{-\frac{9}{2}} \langle f, iV_- W \rangle V_-(iW).$$  \hfill (7.90)

Furthermore, we define $S_{22}: PL^r(\mathbb{R}^3) \rightarrow \Pi L^r(\mathbb{R}^3)$ by

$$S_{22}f := \alpha c_+(\alpha) \langle f, V_+ AW \rangle V_+ AW + \alpha c_- (\alpha) \langle f, iV_- W \rangle V_-(iW).$$  \hfill (7.91)
Then, one sees that
\[ A_{22} = Q + S_{22}. \] (7.92)
Observe that the inverse of \( Q : PL'(\mathbb{R}^3) \rightarrow \Pi L'(\mathbb{R}^3) \) is given by
\[ Q^{-1} = \alpha^{-1} \sum_{j=1}^{3} \frac{\partial_j W}{c_j(\alpha)} \langle \partial_j W, V_+ \partial_j W \rangle^2 \langle \cdot, \partial_j W \rangle \]
\[ + \alpha^{-\frac{1}{2}} \frac{\Lambda W}{b_+ \langle \Lambda W, V_+ \Lambda W \rangle^2} \langle \cdot, \Lambda W \rangle \]
\[ + \alpha^{-\frac{1}{2}} \frac{i W}{b_- \langle i W', i V_- W \rangle^2} \langle \cdot, i W \rangle. \] (7.93)
Observe that \( \|Q^{-1}\|_{PL' \rightarrow \Pi L'} \lesssim \alpha^{-\frac{1}{2}} \). Rewrite \( A_{22} \) as
\[ A_{22} = Q(1 + Q^{-1} S_{22}). \] (7.94)
Then, it follows from \( \langle V_+ \Lambda W, \partial_j W \rangle = \langle V_- (i W), \partial_j W \rangle = 0 \) (1 \( \leq j \leq 3 \)), \( \tag{7.89} \) and \( \tag{7.81} \) that for any \( f \in PL'(\mathbb{R}^3) \),
\[ \|Q^{-1} S_{22} f\|_{L'} \lesssim \alpha^{-\frac{1}{2}} \|\langle S_{22} f, \Lambda W \rangle\| + \alpha^{-\frac{1}{2}} \|\langle S_{22} f, i W \rangle\| \lesssim \alpha^{-\frac{1}{2}} \|f\|_{L'}. \] (7.95)
Furthermore, it follows from the theory of Neumann’s series that for any sufficiently small \( \alpha > 0 \), the operator \( 1 + Q^{-1} S_{22} \) has the inverse, say \( L \), such that
\[ \|L\|_{PL' \rightarrow PL'} \leq (1 - \|Q^{-1} S_{22}\|_{PL' \rightarrow PL'}) \lesssim 1. \] (7.96)
We summarize:
\[ A_{22}^{-1} = L Q^{-1}, \quad \|A_{22}^{-1}\|_{\Pi L' \rightarrow PL'} \lesssim \alpha^{-\frac{1}{2}}. \] (7.97)
**Step 4.** We shall finish the proof. Put
\[ \tilde{A}_\varepsilon := \begin{pmatrix} 1 & A_{11}^{-1} A_{12} \\ A_{22}^{-1} A_{21} & 1 \end{pmatrix}, \] (7.98)
so that
\[ A_\varepsilon(\alpha) = \begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix} \tilde{A}_\varepsilon. \] (7.99)
We see from \( \tag{7.83} \), \( \tag{7.81} \), \( \tag{7.83} \) and \( \tag{7.86} \) that
\[ \|1 - \tilde{A}_\varepsilon\|_{X' \rightarrow X'} \leq \|A_{11}^{-1} A_{12}\|_{PL' \rightarrow X'} + \|A_{22}^{-1} A_{21}\|_{X' \rightarrow PL'} \]
\[ \lesssim \varepsilon^{-1} \|A_{12}\|_{PL' \rightarrow X'} + \alpha^{-\frac{1}{2}} \|A_{21}\|_{X' \rightarrow \Pi L'} \]
\[ \lesssim \varepsilon^{-1} \alpha^{-\frac{1}{2}} + \varepsilon. \] (7.100)
Hence, we find that there exist $\alpha_* \in (0, \varepsilon_0)$ and $\varepsilon_0 \in (0, 1)$, both depending only on $r$, such that for any $0 < \alpha < \alpha_*$,

$$\|1 - \tilde{A}_{\varepsilon_0}\|_{X_r \to X_r} \leq \frac{1}{2}$$  \hspace{1cm} (7.101)

Furthermore, the theory of Neumann’s series shows that $\tilde{A}_{\varepsilon_0}$ has the inverse $D := \tilde{A}_{\varepsilon_0}^{-1}$ satisfying

$$\|D\|_{X_r \to X_r} \leq 2.$$  \hspace{1cm} (7.102)

Then, for any $0 < \alpha < \alpha_*$, the inverse of $A_{\varepsilon_0}(\alpha)$ exists and is given by

$$A_{\varepsilon_0}(\alpha)^{-1} = \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix} \begin{pmatrix} A_{11}^{-1} & 0 \\ 0 & A_{22}^{-1} \end{pmatrix},$$  \hspace{1cm} (7.103)

where $D_{jk}$ is the $(j, k)$-entry of $D$. Here, we may assume that $\varepsilon_0^{-1} \leq \alpha_*^{-\frac{1}{2}}$. Then, we see from (7.83), (7.97) and (7.102) that for any $0 < \alpha < \alpha_*$,

$$\|A_{\varepsilon_0}(\alpha)^{-1}\|_{Y_r \to X_r} \lesssim \|A_{11}^{-1}\|_{X_r \to X_r} + \|A_{22}^{-1}\|_{\Pi L^r \to \Pi L^r} \lesssim \alpha^{-\frac{1}{2}}.$$  \hspace{1cm} (7.104)

Thus, as mentioned in (7.76), for any $0 < \alpha < \alpha_*$, the inverse of $G(\alpha)$ exists as an operator from $L^r(\mathbb{R}^3)$ to itself and

$$G(\alpha)^{-1} = B_{\varepsilon_0} A_{\varepsilon_0}(\alpha)^{-1} C.$$  \hspace{1cm} (7.105)

Observe from (7.65), (7.66), (7.19) and $\varepsilon_0 < 1$ that

$$\|B_{\varepsilon_0}\|_{X_r \to L^r} \lesssim 1, \quad \|C\|_{L^r \to Y_r} \lesssim 1.$$  \hspace{1cm} (7.106)

Hence, we see from (7.104) through (7.106) that for any $0 < \alpha < \alpha_*$,

$$\|G(\alpha)^{-1}\|_{L^r \to L^r} \lesssim \alpha^{-\frac{1}{2}},$$  \hspace{1cm} (7.107)

which proves the estimate (2.16). In order to prove (2.18), we assume $f \in X_r$. Then $\Pi f = 0$ and therefore

$$G(\alpha)^{-1} f = B_{\varepsilon_0} \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix} \begin{pmatrix} A_{11}^{-1} & 0 \\ 0 & A_{22}^{-1} \end{pmatrix} \begin{pmatrix} (1 - P)f \\ 0 \end{pmatrix}$$  \hspace{1cm} (7.108)

$$= D_{11} A_{11}^{-1} (1 - P)f + D_{21} A_{11}^{-1} (1 - P)f.$$  \hspace{1cm} (7.109)

Hence, by (7.108), (7.83) with $\varepsilon = \varepsilon_0$, and (7.102), we can obtain the desired estimate (2.18). Thus, we have completed the proof.  \hfill $\square$
7.4 Proof of Theorem 2.3

In this section, we prove Theorem 2.3. For \( \alpha > 0 \), we introduce the notation

\[
Z_{\alpha,j} := \begin{cases} 
G(\alpha)^* V_+ \partial_j W & \text{if } 1 \leq j \leq 3, \\
G(\alpha)^* V_+ \Lambda W & \text{if } j = 4, \\
G(\alpha)^* (iV_- W) & \text{if } j = 5.
\end{cases}
\] (7.109)

It follows from the definition of \( G(\alpha)^* \) (see (2.14)), \( \Im [V_+ \nabla W] = 0 \) and (1.25) that if \( 1 \leq j \leq 3 \), then

\[
Z_{\alpha,j} = (1 + V_+ (-\Delta + \alpha)^{-1}) V_+ \partial_j W
\]

\[
= V_+ \partial_j W + V_+ (-\Delta + \alpha)^{-1} \{-(-\Delta + \alpha) + \alpha\} \partial_j W
\]

\[
= \alpha V_+ (-\Delta + \alpha)^{-1} \partial_j W.
\] (7.110)

Similarly, one can verify that

\[
Z_{\alpha,4} := \alpha V_+ (-\Delta + \alpha)^{-1} \Lambda W,
\] (7.111)

\[
Z_{\alpha,5} := -\alpha V_- (-\Delta + \alpha)^{-1} (iW).
\] (7.112)

For the proof of Theorem 2.3 we will use the following fact:

**Lemma 7.12.** Let \( \alpha > 0 \). Then, we have

\[
\langle \partial_j W, Z_{\alpha,j} \rangle = O(\alpha) \quad \text{for any } 1 \leq j \leq 3,
\] (7.113)

\[
\langle \Lambda W, Z_{\alpha,4} \rangle = O(\alpha^{\frac{1}{2}}),
\] (7.114)

\[
\langle iW, Z_{\alpha,5} \rangle = O(\alpha^{\frac{1}{2}}).
\] (7.115)

**Proof of Lemma 7.12.** We shall prove the claim (7.113). By (7.110), one can see that if \( 1 \leq j \leq 3 \), then

\[
\langle \partial_j W, Z_{\alpha,j} \rangle = \langle \partial_j W, \alpha V_+ (-\Delta + \alpha)^{-1} \partial_j W \rangle = \alpha \langle (-\Delta + \alpha)^{-1} V_+ \partial_j W, \partial_j W \rangle.
\] (7.116)

This together Lemma 7.9 shows (7.113).

Next, we shall prove (7.114). By (7.111), one sees that

\[
\langle \Lambda W, Z_{\alpha,4} \rangle = \alpha \langle (-\Delta + \alpha)^{-1} V_+ \Lambda W, \Lambda W \rangle.
\] (7.117)

From this together with Corollary 7.1, we obtain the desired result (7.114). Similarly, we can prove (7.115).

Now, we give a proof of Theorem 2.3.
Proof of Theorem 2.3. The proof basically follows the standard proof of the implicit function theorem. The main difference is to show that the radii $\delta_{\text{geo}}$ and $r_{\text{geo}}$ are independent of the parameter $\alpha$.

First, we define the functions $\mathcal{H}_{\alpha,1}, \ldots, \mathcal{H}_{\alpha,5}$ from $\dot{H}^1(\mathbb{R}^3) \times \mathbb{R}^3 \times (0, \infty) \times (-\pi, \pi)$ to $\mathbb{R}$ by

$$
\mathcal{H}_{\alpha,j}(u, y, \nu, \theta) = \langle T_{(y, \nu, \theta)}[u] - W, Z_{\alpha,j} \rangle \quad \text{for } 1 \leq j \leq 5.
$$

(7.118)

To make the representation simple, we introduce the following notation:

$$
B := \mathbb{R}^3 \times (0, \infty) \times (\pi, \pi), \quad b := (y, \nu, \theta), \quad b_0 := (0, 1, 0).
$$

(7.119)

Then, define the function $\mathcal{H}_{\alpha} : \dot{H}^1(\mathbb{R}^3) \times B \to \mathbb{R}^5$ by

$$
\mathcal{H}_{\alpha}(u, b) := \left( \mathcal{H}_{\alpha,1}(u, b), \ldots, \mathcal{H}_{\alpha,5}(u, b) \right).
$$

(7.120)

Observe that for any $(u, b) = (u, y, \nu, \theta) \in \dot{H}^1(\mathbb{R}^3) \times B$ with $y = (y_1, y_2, y_3)$, and any $1 \leq j \leq 5$:

$$
\partial_y \mathcal{H}_{\alpha,j}(W, b_0) = \langle \partial_j W, Z_{\alpha,j} \rangle = O(\alpha) \quad \text{for } 1 \leq j \leq 3,
$$

(7.121)

$$
\partial_{\nu} \mathcal{H}_{\alpha,j}(W, b_0) = -\nu^{-1} \langle T_{\nu}[u], Z_{\alpha,j} \rangle - 2\nu^{-1} \langle e^{i\theta} \nu^{-2} x \cdot \nabla u(\nu^{-2} x + y), Z_{\alpha,j} \rangle,
$$

(7.122)

$$
\partial_{\theta} \mathcal{H}_{\alpha,j}(W, b_0) = \langle iW, Z_{\alpha,j} \rangle = O(\alpha^{1/2}).
$$

(7.123)

In particular, by (7.124), (7.125), (7.126) and Lemma 7.12, one sees that

$$
\partial_{y_1} \mathcal{H}_{\alpha,j}(W, b_0) = \langle \partial_{y_1} W, Z_{\alpha,j} \rangle = O(\alpha) \quad \text{for } 1 \leq j \leq 3,
$$

(7.124)

$$
\partial_{\nu} \mathcal{H}_{\alpha,4}(W, b_0) = -2\langle \Lambda W, Z_{\alpha,4} \rangle = O(\alpha^{1/2}),
$$

(7.125)

$$
\partial_{\theta} \mathcal{H}_{\alpha,5}(W, b_0) = \langle iW, Z_{\alpha,5} \rangle = O(\alpha^{1/2}).
$$

(7.126)

Moreover, by (7.110) through (7.112), we can verify that the off-diagonal components of $\partial_b \mathcal{H}_{\alpha}(W, b_0)$ vanish, namely, for any $1 \leq j \leq 5$,

$$
\partial_{y_k} \mathcal{H}_{\alpha,j}(W, b_0) = \langle \partial_k W, Z_{\alpha,j} \rangle = 0 \quad \text{if } j \neq k \quad (1 \leq k \leq 3),
$$

(7.127)

$$
\partial_{\nu} \mathcal{H}_{\alpha,j}(W, b_0) = -2\langle \Lambda W, Z_{\alpha,j} \rangle = 0 \quad \text{if } j \neq 4,
$$

(7.128)

$$
\partial_{\theta} \mathcal{H}_{\alpha,j}(W, b_0) = \langle iW, Z_{\alpha,j} \rangle = 0 \quad \text{if } j \neq 5.
$$

(7.129)

Put

$$
A_{\alpha} := \partial_b \mathcal{H}_{\alpha}(W, b_0).
$$

(7.130)

Then, it follows from (7.124) through (7.129) that the differential mapping $A_{\alpha} : \mathbb{R}^5 \to \mathbb{R}^5$
is bijective, and the inverse $A^{-1}_\alpha$ is of the form
\[
A^{-1}_\alpha = \begin{bmatrix}
O(\alpha^{-1}) & 0 & 0 & 0 & 0 \\
0 & O(\alpha^{-1}) & 0 & 0 & 0 \\
0 & 0 & O(\alpha^{-1}) & 0 & 0 \\
0 & 0 & 0 & O(\alpha^{-2}) & 0 \\
0 & 0 & 0 & 0 & O(\alpha^{-2})
\end{bmatrix}.
\] (7.131)

As well as the standard proof of the implicit function theorem, we consider the mapping $G_\alpha: H^1(\mathbb{R}^3) \times B \to \mathbb{R}^5$ defined by
\[
G_\alpha(u, b) := b - A^{-1}_\alpha H_\alpha(u, b).
\] (7.132)

Note that $b = G_\alpha(u, b)$ implies that $H_\alpha(u, b) = 0$. Hence, for the desired result, it suffices to prove that: there exist $0 < \delta_{geo} < 1$ and $0 < r_{geo} < \frac{1}{2}$ such that for any $0 < \alpha < 1$, there exists a unique continuous mapping $b_\alpha: B_{H^1}(W, \delta_{geo}) \to B(b_0, r_{geo})$ such that
\[
b_\alpha(W) = b_0,
\] (7.133)
\[
b_\alpha(u) = G_\alpha(u, b_\alpha(u)).
\] (7.134)

We shall prove this claim in several steps. Throughout the proof, all of the implicit constants depend only on the fixed dimension $d = 3$.

**Step 1.** Let $0 < \delta < 1$ and $0 < r < \frac{1}{2}$. Furthermore, let $u \in B_{H^1}(W, \delta)$ and $b = (y, \nu, \theta) \in B(b_0, r)$. Then, we shall prove the following estimates:
\[
\sup_{1 \leq k \leq 3} \|T_b[\partial_k u] - \partial_k W\|_{L^2} \lesssim \delta + r,
\] (7.135)
\[
\|T_b[u] - W\|_{L^6} \lesssim \delta + r,
\] (7.136)
\[
\|(1 + |x|)^{-1} \{\nu^{-1} e^{i\theta_x} \nu^{-2} x \cdot \nabla u(\nu^{-2} x + y) - x \cdot \nabla W\}\|_{L^2} \lesssim \delta + r.
\] (7.137)

Let us begin by proving (7.135). It follows from elementary computations, the fun-
damental theorem of calculus, \(|y| + |\nu - 1| + |\theta| < r < \frac{1}{2}\) and \(0 < \delta < 1\) that

\[
\|T_b[\partial_k u] - \partial_k W\|_{L^2} \leq \|e^{i\theta}\nu^{-1}\partial_k u(\nu^{-2} \cdot + y) - e^{i\theta}\nu^{-1}\partial_k W (\nu^{-2} \cdot + y)\|_{L^2} \\
+ \|e^{i\theta}\nu^{-1}\partial_k W (\nu^{-2} \cdot + y) - e^{i\theta}\nu^{-1}\partial_k W\|_{L^2} \\
+ \|e^{i\theta}\partial_k W - \partial_k W\|_{L^2} \\
\leq \nu^2 \|\partial_k u - \partial_k W\|_{L^2} + \nu^2 \|\int_0^1 y \cdot \nabla \partial_k W (\cdot + \kappa y) \, d\kappa\|_{L^2} \\
+ \|\int_1^\nu 2\lambda^{-1} T_\lambda [\Lambda \partial_k W] \, d\lambda\|_{L^2} + \|\int_0^\theta e^{it} \partial_k W\|_{L^2} \\
\leq 4\delta + 4r \|\nabla \partial_k W\|_{L^2} + 4r \|\Lambda \partial_k W\|_{L^2} + r \|\partial_k W\|_{L^2} \lesssim \delta + r. \tag{7.138}
\]

Thus, we have obtained the estimate (7.135). Similarly, we can prove (7.136). It remains to prove (7.137). By the triangle inequality, one has

\[
\|(1 + |x|)^{-1}\{\nu^{-1} e^{i\theta}\nu^{-2} x \cdot \nabla u(\nu^{-2} x + y) - x \cdot \nabla W\}\|_{L^2} \\
\leq |\nu^{-1} - 1||| (1 + |x|)^{-1}\nu^{-2} x \cdot \nabla u(\nu^{-2} x + y)\|_{L^2} \\
+ \||(1 + |x|)^{-1}\{e^{i\theta}\nu^{-2} x \cdot \nabla u(\nu^{-2} x + y) - x \cdot \nabla W\}\|_{L^2}. \tag{7.139}
\]

We consider the first term on the right-hand side of (7.139). By \(|\nu - 1| < r < \frac{1}{2}\), one can verify that for any \(x \in \mathbb{R}^3\),

\[
|\nu^{-1} - 1||| (1 + |x|)^{-1}\nu^{-2} x \cdot \nabla u(\nu^{-2} x + y)\|_{L^2} \\
= |\nu^{-1} - 1|\nu^3||(1 + |\nu^2 x|)^{-1} (x \cdot \nabla u(x + y))\|_{L^2} \tag{7.140} \\
\lesssim |\nu - 1| \|\nabla u\|_{L^2} \lesssim r \|\nabla u - \nabla W\|_{L^2} + r \|\nabla W\|_{L^2} \leq r\delta + r \leq \delta + r.
\]

Next, we consider the second term on the right-hand side of (7.139). By computations
We shall show that there exist $0 < \delta < \delta_{\text{geo}} < 1$ and $|\nu - 1| < r$, one can see that

$$
\| (1 + |x|)^{-1} \left\{ e^{it\theta} \nu^{-2} x \cdot \nabla u(\nu^{-2} x + y) - x \cdot \nabla W \right\} \|_{L^2} 
\lesssim \| (1 + |x|)^{-1} e^{it\theta} \nu^{-2} x \cdot \{ \nabla u(\nu^{-2} x + y) - \nabla W(\nu^{-2} x + y) \} \|_{L^2} 
\lesssim \| (1 + |x|)^{-1} \nu^{-2} x \cdot \{ \nabla W(\nu^{-2} x) - \nabla W(\nu^{-2} x) \} \|_{L^2} 
\lesssim \| (1 + |x|)^{-1}(e^{it\theta} - 1) x \cdot \nabla W \|_{L^2} 
\lesssim \| \nabla u - \nabla W \|_{L^2} 
\lesssim \delta + r.
$$

(7.141)

Putting (7.139), (7.140) and (7.141) together, we obtain the desired estimate (7.137).

**Step 2.** We shall show that there exist $0 < \delta_{\text{geo}} < 1$ and $0 < r_{\text{geo}} < \frac{1}{2}$ such that for any $0 < \alpha < 1$, any $u \in B_{\dot{H}^1}(W, \delta_{\text{geo}})$, and any $b, c \in B(b_0, r_{\text{geo}})$,

$$
|S_\alpha(u, b) - S_\alpha(u, c)| \leq \frac{1}{4} |b - c|, 
$$

(7.142)

$$
|S_\alpha(u, b) - b| \leq \frac{r_{\text{geo}}}{2}. 
$$

(7.143)

Let $\alpha > 0$, $u \in \dot{H}^1(\mathbb{R}^3)$ and $b, c \in B := \mathbb{R}^3 \times (0, \infty) \times (-\frac{\pi}{2}, \frac{\pi}{2})$. Then, observe from (7.134) that

$$
S_\alpha(u, b) - S_\alpha(u, c) 
= b - c - A_\alpha^{-1} \left\{ \mathcal{H}_\alpha(u, b) - \mathcal{H}_\alpha(u, c) \right\} 
= A_\alpha^{-1} A_\alpha(b - c) - A_\alpha^{-1} \int_0^1 \partial_h \mathcal{H}_\alpha(u, c + t(b - c))(b - c) dt 
= A_\alpha^{-1} \int_0^1 \left\{ \partial_h \mathcal{H}_\alpha(W, b_0) - \partial_h \mathcal{H}_\alpha(u, c + t(b - c)) \right\} (b - c) dt.
$$

(7.144)

Let $0 < \delta < 1$ and $0 < r < \frac{1}{2}$ be constants to be specified later. Then, observe from
that for any \( u \in B_{\hat{H}}(W, \delta) \) and \( b = (y, \nu, \theta) \in B(b_0, r) \), we have

\[
\| A_{\alpha}^{-1} \{ \partial_b \mathcal{K}_\alpha(W, b_0) - \partial_b \mathcal{K}_\alpha(u, b) \} \|_{\mathbb{R}^5} \rightarrow \mathbb{R}^5 \approx \sum_{1 \leq j \leq 3} \sum_{1 \leq k \leq 3} \alpha^{-1} |\partial_{y_k} \mathcal{K}_{\alpha,j}(W, b_0) - \partial_{y_k} \mathcal{K}_{\alpha,j}(u, b)|
\]

(7.145)

Step 2.1. Consider the first term on the right-hand side of (7.145). By (7.121), (7.124), (7.127), (7.110), Lemma 7.8 and (7.135), one can see that

\[
\sum_{1 \leq j \leq 3} \sum_{1 \leq k \leq 3} \alpha^{-1} |\partial_{y_k} \mathcal{K}_{\alpha,j}(W, b_0) - \partial_{y_k} \mathcal{K}_{\alpha,j}(u, b)|
\]

(7.146)

Step 2.2. Consider the second term on the right-hand side of (7.145). Observe from (7.110) that

\[
(W, Z_{\alpha,j}) = 0 \quad \text{for all } 1 \leq j \leq 3.
\]

(7.147)

By (7.122), (7.124), (7.128), (7.110), Lemma 7.8 \(|\nu^{-1} - 1| \lesssim |\nu - 1| < r\), (7.136) and
one can see that
\[
\sum_{1 \leq j \leq 3} \alpha^{-1} |\partial_\nu \mathcal{H}_{\alpha,j}(W, b_0) - \partial_\nu \mathcal{H}_{\alpha,j}(u, b)| \\
\leq 2 \sum_{1 \leq j \leq 3} \alpha^{-1} |\nu^{-1} T_b[u] - W, Z_{\alpha,j}| \\
+ 2 \sum_{1 \leq j \leq 3} \alpha^{-1} |\nu^{-1} e^{i\theta} \nu^{-2} x \cdot \nabla u(\nu^{-2} x + y), Z_{\alpha,j})| \\
+ 2 \sum_{1 \leq j \leq 3} \alpha^{-1} |\nu^{-1} e^{i\theta} \nu^{-2} x \cdot \nabla u(\nu^{-2} x + y) - \Lambda W, Z_{\alpha,j})| \\
\lesssim \|\nu^{-1} \{T_b[u] - W\}\|_{L^6} + \|(\nu^{-1} - 1)W\|_{L^6} \\
+ |\nu^{-1} - 1| \|(1 + |x|)^{-1} \nu^{-2} |x| \nabla u(\nu^{-2} x + y)\|_{L^2} \\
+ \|(1 + |x|)^{-1} \nu^{-1} e^{i\theta} \nu^{-2} x \cdot \nabla u(\nu^{-2} x + y) - x \cdot \nabla W\|_{L^2} \\
\lesssim \delta + r. 
\] (7.148)

**Step 2.3.** Consider the third term on the right-hand side of (7.145). By (7.123), (7.129), (7.110), Proposition 7.8 and (7.136) that
\[
\sum_{1 \leq j \leq 3} \alpha^{-1} |\partial_y \mathcal{H}_{\alpha,j}(u, b_0) - \partial_y \mathcal{H}_{\alpha,j}(u, b)| = \sum_{1 \leq j \leq 3} \alpha^{-1} |\langle T_b[iu] - iW, Z_{\alpha,j})\rangle| \\
= \sum_{1 \leq j \leq 3} \left|\langle (-\Delta + \alpha)^{-1} V_+ \{T_b[u] - W\}, \partial_j W\rangle\right| \lesssim \|T_b[u] - W\|_{L^6} \lesssim \delta + r. 
\] (7.149)

**Step 2.4.** Consider the fourth term on the right-hand side of (7.145). By (7.121), (7.127), (7.111), (7.112), Corollary 7.1 and (7.135), one can see that
\[
\sum_{j=4,5} \sum_{1 \leq k \leq 3} \alpha^{-\frac{1}{2}} |\partial_{y_k} \mathcal{H}_{\alpha,j}(u, b_0) - \partial_{y_k} \mathcal{H}_{\alpha,j}(W, b)| \\
\lesssim \sup_{1 \leq k \leq 3} \|T_b[\partial_k u] - \partial_k W\|_{L^2} \lesssim \delta + r. 
\] (7.150)

**Step 2.5.** Consider the fifth term on the right-hand side of (7.145). By (7.122), \(\Lambda W = \frac{1}{2} W + x \cdot \nabla W\), (7.123), (7.128), (7.111), (7.112), Corollary 7.1, \(\nu^{-1} - 1 \lesssim r\), (7.136) and
\[ \sum_{j=4,5} \alpha^{-\frac{1}{2}} |\partial_{b}\mathcal{H}_{a,j}(W, b) - \partial_{b}\mathcal{H}_{a,j}(u, b)| \]

\[ \lesssim \sum_{j=4,5} \alpha^{-\frac{1}{2}} |\langle \nu^{-1}T_{b}[u] - W, Z_{a,j} \rangle| + \sum_{j=4,5} \alpha^{-\frac{1}{2}} |\langle \nu^{-1}e^{i\theta}\nu^{-2}x \cdot \nabla u(\nu^{-2}y) - x \cdot \nabla W, Z_{a,j} \rangle| \]

\[ + \sum_{j=4,5} \alpha^{-\frac{1}{2}} |\nu^{-1} - 1| |\langle \nu^{-1}e^{i\theta}\nu^{-2}x \cdot \nabla u(\nu^{-2}y), Z_{a,j} \rangle| \]

\[ \lesssim \|\nu^{-1}T_{b}[u] - W\|_{L^6} + \|(1 + |x|)^{-1}\nu^{-1}T_{b}[x \cdot \nabla u] - x \cdot \nabla W\|_{L^2} \]

\[ + |\nu^{-1} - 1| |(1 + |x|)^{-1}\nu^{-2}|x| \nabla u(\nu^{-2}y)\|_{L^2} \]

\[ \lesssim \delta + r. \]  

(7.151)

**Step 2.6.** Consider the last term on the right-hand side of (7.145). By (7.123), (7.126), (7.129), (7.111), (7.112), Corollary 7.1 and (7.136),

\[ \sum \alpha^{-\frac{1}{2}} |\partial_{b}\mathcal{H}_{a,j}(W, b) - \partial_{b}\mathcal{H}_{a,j}(u, b)| \]

\[ = \sum \alpha^{-\frac{1}{2}} |T_{b}[iu] - iW, Z_{a,j}| \lesssim \|T_{b}[u] - W\|_{L^6} \lesssim \delta + r. \]  

(7.152)

**Step 2.7.** We shall derive the estimates (7.142) and (7.143). Plugging the estimates (7.146) though (7.152) into (7.145), one see that for any \( b \in B(b_0, r) \),

\[ \| A_{\alpha}^{-1} \left\{ \partial_{b}\mathcal{H}_{a}(W, b) - \partial_{b}\mathcal{H}_{a}(u, b) \right\} \|_{\mathbb{R}^5 \rightarrow \mathbb{R}^5} \lesssim \delta + r. \]  

(7.153)

Moreover, it is easy to verify that if \( b, c \in B(b_0, r) \), then \( c + t(b - c) \in B(b_0, r) \). Hence, it follows from (7.141) and (7.153) that for any \( u \in B_{H^1}(W, \delta) \) and \( b, c \in B(b_0, r) \),

\[ |S_{\alpha}(u, b) - S_{\alpha}(u, c)| \]

\[ \leq \sup_{0 \leq t \leq 1} \| A_{\alpha}^{-1} \left\{ \partial_{b}\mathcal{H}_{a}(W, b_0) - \partial_{b}\mathcal{H}_{a}(u, c + t(b - c)) \right\} \|_{\mathbb{R}^5 \rightarrow \mathbb{R}^5} |b - c| \]

\[ \lesssim (\delta + r)|b - c|, \]  

(7.154)

so that if \( \delta \ll 1 \) and \( r \ll 1 \), then

\[ |S_{\alpha}(u, b) - S_{\alpha}(u, c)| \leq \frac{1}{4}|b - c|. \]  

(7.155)

This proves (7.142).

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On the other hand, by (7.118), (7.110) through (7.112), Proposition 7.8, Corollary 7.1 and Sobolev’s inequality, one can see that for any 
\[ u \in B^{\dot{H}}_1(W, \delta), \]
\[
\left| G_\alpha(u, b_0) - b_0 \right| = \left| A_\alpha^{-1} G_\alpha(u, b_0) \right| \\
\leq \sum_{1 \leq j \leq 3} \alpha^{-1} \left| \langle u - W, \alpha V_+ (-\Delta + \alpha)^{-1} \partial_j W \rangle \right| \\
+ \alpha^{-\frac{1}{2}} \left| \langle u - W, \alpha V_+ (-\Delta + \alpha)^{-1} \Lambda W \rangle \right| + \alpha^{-\frac{1}{2}} \left| \langle u - W, \alpha V_- (-\Delta + \alpha)^{-1} (i W) \rangle \right| \\
\lesssim \| u - W \|_{L^6} \lesssim \| \nabla \{ u - W \} \|_{L^2} \leq \delta, \] (7.156)
so that if \( \delta \ll r \), then
\[
\left| G_\alpha(u, b_0) - b_0 \right| \leq r \frac{\delta}{4}. \] (7.157)
Furthermore, it follows from (7.155) and (7.157) that if \( \delta \ll r \ll 1 \), then for any \( u \in B_{H^1}(W, \delta) \) and \( b \in B(b_0, r) \),
\[
\left| G_\alpha(u, b) - b_0 \right| \leq \left| G_\alpha(u, b) - G_\alpha(u, b_0) \right| + \left| G_\alpha(u, b_0) - b_0 \right| \\
\leq \frac{1}{4} |b - b_0| + r \frac{\delta}{4} \leq r. \] (7.158)
Thus, we have proved (7.153).

**Step 3.** We shall finish the proof of the proposition. Let \( 0 < \delta_{geo} < 1 \) and \( 0 < r_{geo} < \frac{1}{2} \) be constants given in the previous step, and let \( 0 < \alpha < 1 \). Recall that it suffices to prove that for any \( 0 < \alpha < 1 \), there exists a unique continuous mapping \( b_\alpha : B_{H^1}(W, \delta_{geo}) \to B(b_0, r_{geo}) \) satisfying (7.133) and (7.134). To this end, we define the sequence \( \{ b_{\alpha,n} \} \) of mappings from \( B_{H^1}(W, \delta_{geo}) \) to \( \mathbb{R}^3 \) to be that for any \( u \in B_{H^1}(W, \delta_{geo}) \),
\[
b_{\alpha,1}(u) := G_\alpha(u, b_0), \] (7.159)
\[
b_{\alpha,n+1}(u) := G_\alpha(u, b_{\alpha,n}(u)) \quad \text{for } n \geq 1. \] (7.160)

**Step 3.1.** We shall show that
\[
\left| b_{\alpha,n}(u) - b_0 \right| \leq \frac{r_{geo}}{2} \quad \text{for any } n \geq 1 \text{ and } u \in B_{H^1}(W, \delta_{geo}). \] (7.161)
It follows from (7.133) that (7.161) is true for \( n = 1 \). Furthermore, if (7.161) holds for some \( n \geq 2 \) (hence \( b_{\alpha,n}(u) \in B(b_0, r_{geo}) \)), then by (7.133),
\[
\left| b_{\alpha,n+1}(u) - b_0 \right| \leq \left| G_\alpha(u, b_{\alpha,n}(u)) - b_0 \right| \leq r_{geo} \frac{\delta_{geo}}{2}. \] (7.162)
Thus, by induction, (7.161) must hold.

**Step 3.2.** We shall show that
\[
b_{\alpha,n}(W) = b_0 \quad \text{for any } n \geq 1. \] (7.163)
Thus, we have obtained a mapping $b_{\alpha,1}(W) = b_0$. Suppose that $b_{\alpha,n}(W) = b_0$ for some $n \geq 2$. Then, one can verify that

$$b_{\alpha,n+1}(W) = G_\alpha(u, b_{\alpha,n}(W)) = G_\alpha(W, b_0) = b_0.$$  \hspace{1cm} \text{(7.164)}

Thus, we have proved (7.163).

**Step 3.3.** We prove that for any $0 < \alpha < 1$, there exists a unique continuous mapping $b_\alpha : B_{H^2}(W, \delta_{geo}) \to B(b_0, r_{geo})$ satisfying (7.133) and (7.134), which completes the proof of the lemma.

Let $n \geq 1$ and $u \in B_{H^2}(W, \delta_{geo})$. Notice that by (7.161), $b_{\alpha,n}(u) \in B(b_0, r_{geo})$. Hence, it follows from (7.142) that

$$|b_{\alpha,n+1}(u) - b_{\alpha,n}(u)| = |G_\alpha(u, b_{\alpha,n}(u)) - G_\alpha(u, b_{\alpha,n-1}(u))|$$

$$\leq \frac{1}{2}|b_{\alpha,n}(u) - b_{\alpha,n-1}(u)| = \frac{1}{2}|G_\alpha(u, b_{\alpha,n-1}(u)) - G_\alpha(u, b_{\alpha,n-2}(u))|$$

$$\leq \left(\frac{1}{2}\right)^2 |b_{\alpha,n-1}(u) - b_{\alpha,n-2}(u)|$$

$$\leq \cdots \leq \left(\frac{1}{2}\right)^n |b_{\alpha,2}(u) - b_{\alpha,1}(u)| = \left(\frac{1}{2}\right)^n |G_\alpha(u, b_{\alpha,1}(u)) - G_\alpha(u, b_0)|$$

$$\leq \left(\frac{1}{2}\right)^n |b_{\alpha,1}(u) - b_0| \leq \left(\frac{1}{2}\right)^{n+2 + 2} r_{geo} \leq \left(\frac{1}{2}\right)^{n+2}.$$  \hspace{1cm} \text{(7.165)}

Furthermore, by the triangle inequality and (7.165), one can verify that

$$|b_{\alpha,n}(u) - b_{\alpha,m}(u)| \leq \sum_{l=m+1}^{n} |b_{\alpha,l}(u) - b_{\alpha,l-1}(u)| \leq \sum_{l=m+1}^{n} \left(\frac{1}{2}\right)^{l+2}$$

$$= \left(\frac{1}{2}\right)^{m+3} - \left(\frac{1}{2}\right)^{n+2} \to 0 \text{ as } m, n \to \infty,$$  \hspace{1cm} \text{(7.166)}

so that $\{b_{\alpha,n}(u)\}$ is Cauchy sequence in $\mathbb{R}^5$. Thus, for any $u \in B_{H^2}(W, \delta_{geo})$, the limit of $\{b_{\alpha,n}(u)\}$ exists. Let $u \in B_{H^2}(W, \delta_{geo})$, and put $b_\alpha(u) := \lim_{n \to \infty} b_{\alpha,n}(u)$. Then, using (7.161), one can verify that

$$b_\alpha(u) \in B(b_0, r_{geo}).$$  \hspace{1cm} \text{(7.167)}

Moreover, it follows from (7.163) that

$$b_\alpha(W) = \lim_{n \to \infty} b_{\alpha,n}(W) = b_0.$$  \hspace{1cm} \text{(7.168)}

Furthermore, by the continuity of the mapping $b \in B(b_0, r_{geo}) \to G_\alpha(u, b) \in \mathbb{R}^5$ (see (7.142)), one sees that

$$b_\alpha(u) = \lim_{n \to \infty} b_{\alpha,n+1}(u) = \lim_{n \to \infty} G_\alpha(u, b_{\alpha,n}(u)) = G_\alpha(u, b_\alpha(u)).$$  \hspace{1cm} \text{(7.169)}

Thus, we have obtained a mapping $b_\alpha(\cdot) : B_{H^2}(W, \delta_{geo}) \to B(b_0, r_{geo})$ satisfying (7.133) and (7.134).
Then, it follows from (7.142) that for any \( u, v \),

\[
|b_\alpha(u) - b_\alpha(v)| = |G_\alpha(u, b_\alpha(u)) - G_\alpha(v, b_\alpha(v))| 
\leq |G_\alpha(u, b_\alpha(u)) - G_\alpha(u, b_\alpha(v))| + |G_\alpha(u, b_\alpha(v)) - G_\alpha(v, b_\alpha(v))| 
\leq \frac{1}{4} |b_\alpha(u) - b_\alpha(v)| + |A_\alpha^{-1}\{\mathcal{H}_\alpha(u, b_\alpha(v)) - \mathcal{H}_\alpha(v, b_\alpha(v))\}|.  
\]

Furthermore, by (7.170), (7.171) through (7.172), Proposition 7.8, Corollary 7.1 and Sobolev’s inequality, one can verify that for any \( u, v \in B_{\dot{H}^1}(W, \delta_{\text{geo}}) \),

\[
|b_\alpha(u) - b_\alpha(v)| \lesssim |A_\alpha^{-1}\{\mathcal{H}_\alpha(u, b_\alpha(v)) - \mathcal{H}_\alpha(v, b_\alpha(v))\}| \lesssim \sum_{j=1}^{3} \alpha^{-1} |\langle T_{\alpha}(u - v), \alpha V_{\alpha}(-\Delta + \alpha)^{-1} \partial_j W \rangle| 
+ \alpha^{-\frac{1}{2}} |\langle T_{\alpha}(u - v), \alpha V_{\alpha}(-\Delta + \alpha)^{-1} \Delta W \rangle| 
+ \alpha^{-\frac{1}{2}} |\langle T_{\alpha}(u - v), \alpha V_{\alpha}(-\Delta + \alpha)^{-1} (iW) \rangle| 
\lesssim \|T_{\alpha}(u - v)\|_{L^6} = \|u - v\|_{L^6} \lesssim \|u - v\|_{\dot{H}^1}.  
\]

Thus, we have proved the continuity of \( b_\alpha(\cdot) \).

It remains to prove the uniqueness. Suppose that there exists a mapping \( c_\alpha : B_{\dot{H}^1}(W, \delta_{\text{geo}}) \to B(b_0, r_{\text{geo}}) \) such that \( c_\alpha(W) = b_0 \) and \( c_\alpha(u) = G_\alpha(u, c_\alpha(u)) \) for any \( u \in B_{\dot{H}^1}(W, \delta_{\text{geo}}) \).

Then, it follows from (7.172) that for any \( u \in B_{\dot{H}^1}(W, \delta_{\text{geo}}) \),

\[
|b_\alpha(u) - c_\alpha(u)| = |G_\alpha(u, b_\alpha(u)) - G_\alpha(u, c_\alpha(u))| \leq \frac{1}{4} |b_\alpha(u) - c_\alpha(u)|.  
\]

This implies that \( b_\alpha = c_\alpha \). Thus, we have completed the proof.

\[\square\]

**A Proof of (1.11)**

In this section, we give a proof of (1.11).

Let \( u \in \dot{H}^1(\mathbb{R}^d) \) satisfy \( \mathcal{N}_\omega(u) = 0 \). It follows from (1.12) and \( \mathcal{N}_\omega(u) = 0 \) that

\[
\|\nabla u\|_{L^2}^2 = \|u\|_{L^2}^{\frac{2d}{d-2}} \leq \sigma^{-\frac{d}{d-2}} \|\nabla u\|_{L^2}^{\frac{2d}{d-2}}.  
\]

This yields that

\[
\sigma^{\frac{d}{2}} \leq \|\nabla u\|_{L^2}^2.  
\]

For any \( u \in \dot{H}^1(\mathbb{R}^d) \) with \( \mathcal{N}_\omega(u) = 0 \), we obtain

\[
\mathcal{H}^1(u) = \frac{1}{d} \|\nabla u\|_{L^2}^2 \geq \frac{1}{d} \sigma^{\frac{d}{2}}.  
\]

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Taking an infimum on $u \in \dot{H}^1(\mathbb{R}^d)$ with $\mathcal{N}_\omega(u) = 0$, we see that
\[
m_\infty \geq \frac{1}{d} \sigma^{\frac{d}{2}}.
\tag{A.1}
\]

Let $u \in \dot{H}^1(\mathbb{R}^d)$ satisfy $\|u\|_{L^{\frac{2d}{d-2}}} = 1$. Putting $w = \|\nabla u\|_{L^\frac{2d}{d-2}} u$, one has
\[
\mathcal{N}_\omega(w) = \|\nabla u\|_{L^d}^d - \|\nabla u\|_{L^2}^d = 0.
\]
This implies that
\[
m_\infty \leq m_{\infty}(w) = \frac{1}{2} \|\nabla u\|_{L^2}^2 - \frac{d-2}{2d} \|\nabla u\|_{L^2}^d = \frac{1}{d} \|\nabla u\|_{L^2}^d.
\]

Taking an infimum on $u \in \dot{H}^1(\mathbb{R}^d)$ with $\|u\|_{L^{\frac{2d}{d-2}}} = 1$, we obtain
\[
m_\infty \leq \frac{1}{d} \sigma^{\frac{d}{2}}.
\tag{A.2}
\]

From (A.1) and (A.2), we infer that (1.11) holds.

## B Proof of Proposition 1.1

In this appendix, following [1, Proposition 1.1], we shall give a proof of Proposition 1.1.

**Proof of Proposition 1.1**. From Lemma 3.3, it suffices to prove the existence of minimizer for
\[
\inf \{ J(u) : u \in H^1(\mathbb{R}^3) \setminus \{0\}, \mathcal{N}_\omega(u) \leq 0 \}.
\tag{B.1}
\]

To this end, we consider a minimizing sequence $\{u_n\}$ for (B.1). We denote the Schwarz symmetrization of $u_n$ by $u_n^\star$. Note that $\|\nabla u_n^\star\|_{L^2} \leq \|\nabla u_n\|_{L^2}$ and $\|u_n^\star\|_{L^q} = \|u_n\|_{L^q}$ hold for each $q \in [2, 2^*)$ (see e.g. [19]). From these properties and Lemma 3.3 we have
\[
\mathcal{N}_\omega(u_n^\star) \leq 0 \quad \text{for any } n \in \mathbb{N},
\tag{B.2}
\]
\[
\lim_{n \to \infty} J(u_n^\star) = m_\omega^N.
\tag{B.3}
\]

It follows from (B.2) and (B.3) that
\[
\|\nabla u_n^\star\|_{L^2}^2 + \omega \|u_n^\star\|_{L^2}^2 \leq \|u_n\|_{L^{p+1}}^{p+1} + \|u_n\|_{L^6}^6 \lesssim 1.
\tag{B.4}
\]
Thus, we see that $\{u_n^\star\}$ is bounded in $H^1(\mathbb{R}^d)$. Since $\{u_n^\star\}$ is radially symmetric and bounded in $H^1(\mathbb{R}^d)$, there exists a radially symmetric function $Q_\omega \in H^1(\mathbb{R}^d)$ such that, passing to some subsequence,
\[
\lim_{n \to \infty} u_n^\star = Q_\omega \quad \text{weakly in } H^1(\mathbb{R}^d) \text{ and strongly in } L^{p+1}(\mathbb{R}^d).
\tag{B.5}
\]

We shall show that $Q_\omega$ becomes a minimizer for $m_\omega^N$. 58
We first claim that $Q_\omega \not\equiv 0$. Suppose the contrary that $Q_\omega \equiv 0$. Then, it follows from (B.2) and (B.5) that, passing to some subsequence,
\[
0 \geq \lim_{n \to \infty} \mathcal{N}_\omega(u_n^*) \geq \lim_{n \to \infty} \left\{ \|\nabla u_n^*\|_{L^2}^2 - \|u_n^*\|_{L^6}^6 \right\}. \tag{B.6}
\]
If $\|\nabla u_n^*\|_{L^2} \to 0$, then it follows from the boundedness of $\{u_n^*\}$ in $H^1(\mathbb{R}^3)$ and Hölder’s inequality that $\|u_n^*\|_{L^6} \to 0$ for all $2 < q \leq 6$. This together with (B.3) yields that
\[
m_\omega^N = \lim_{n \to \infty} J(u_n^*) = 0. \tag{B.7}
\]
However, this contradicts (4.5). Therefore, by taking a subsequence, we may assume $\lim_{n \to \infty} \|\nabla u_n^*\|_{L^2} > 0$.

Now, (B.6) with the definition of $\sigma$ gives us
\[
\lim_{n \to \infty} \|\nabla u_n^*\|_{L^2}^2 \geq \frac{2}{p+1} \|u_n^*\|_{L^{2(p+1)}}^2 + \frac{1}{3} \|u_n^*\|_{L^6}^6 \geq \sigma \lim_{n \to \infty} \|\nabla u_n^*\|_{L^2}^2. \tag{B.8}
\]
From this together with $\lim_{n \to \infty} \|\nabla u_n^*\|_{L^2} > 0$ and (B.6), we have
\[
\frac{2}{p+1} \|u_n^*\|_{L^{2(p+1)}}^2 \leq \lim_{n \to \infty} \|u_n^*\|_{L^6}^6. \tag{B.9}
\]
Hence, we see from (1.23), (B.3) and (B.8) that
\[
m_\omega^N = \lim_{n \to \infty} J(u_n^*)
\geq \lim_{n \to \infty} \left\{ \frac{p-1}{2(p+1)} \|u_n^*\|_{L^{p+1}}^p + \frac{1}{3} \|u_n^*\|_{L^6}^6 \right\}
\geq \frac{1}{3} \lim_{n \to \infty} \|u_n^*\|_{L^6}^6 \geq m_\infty.
\] However, this contradicts the assumption. Thus, $Q_\omega \not\equiv 0$.

Using Lemma 4.6, we have
\[
J(u_n^*) - J(u_n^* - Q_\omega) - J(Q_\omega) = o_n(1), \tag{B.10}
\]
\[
\mathcal{N}_\omega(u_n^*) - \mathcal{N}_\omega(u_n^* - Q_\omega) - \mathcal{N}_\omega(Q_\omega) = o_n(1). \tag{B.11}
\]
Furthermore, (B.10) together with (B.3) and the positivity of $J$ implies that
\[
J(Q_\omega) \leq m_\omega^N. \tag{B.12}
\]

Next, we shall show that $\mathcal{N}_\omega(Q_\omega) \leq 0$. Suppose that $\mathcal{N}_\omega(Q_\omega) > 0$. Then, it follows from (B.2) and (B.11) that $\mathcal{N}_\omega(u_n^* - Q_\omega) < 0$ for sufficiently large $n$. Hence, we can take $\lambda_n \in (0, 1)$ such that $\mathcal{N}_\omega(\lambda_n(u_n^* - Q_\omega)) = 0$. Furthermore, we see from $0 < \lambda_n < 1$, (B.3), (B.10) and $Q_\omega \not\equiv 0$ that
\[
m_\omega^N \leq J(\lambda_n(u_n^* - Q_\omega)) < J(u_n^* - Q_\omega) = J(u_n^*) - J(Q_\omega) + o_n(1)
= m_\omega^N - J(Q_\omega) + o_n(1) < m_\omega^N. \tag{B.13}
\]
for sufficiently large $n \in \mathbb{N}$, which is a contradiction. Thus, $\mathcal{N}_\omega(Q_\omega) \leq 0$.

Since $Q_\omega \neq 0$ and $\mathcal{N}_\omega(Q_\omega) \leq 0$, it follows from Lemma 3.3 that

$$m^N_\omega \leq \mathcal{J}(Q_\omega).$$

(B.14)

Moreover, it follows from the weak lower semicontinuity that

$$\mathcal{J}(Q_\omega) \leq \liminf_{n \to \infty} \mathcal{J}(u^*_n) \leq m^N_\omega.$$  

(B.15)

Combining (B.14) and (B.15), we obtain $\mathcal{J}(Q_\omega) = m^N_\omega$. Thus, we have proved that $Q_\omega$ is a minimizer for $m^N_\omega$. \square

C  Proofs of Proposition 1.2

In this section, we give a proof of Proposition 1.2.

**Proof of Proposition 1.2.** Let $\omega' > 0$. We see from Lemma 3.3 that for any $\varepsilon > 0$, there exists a nontrivial function $u_\varepsilon \in H^1(\mathbb{R}^3)$ such that $\mathcal{N}_\omega(u_\varepsilon) \leq 0$ and $\mathcal{J}(u_\varepsilon) \leq m_\omega^N + \varepsilon$.

Observe that $\mathcal{N}_\omega(u_\varepsilon) < \mathcal{N}_{\omega'}(u_\varepsilon)$ for any $0 < \omega < \omega'$. Hence, Lemma 3.3 together with the properties of $u_\varepsilon$ shows that for any $0 < \omega < \omega'$,

$$m_\omega^N \leq \mathcal{J}(u_\varepsilon) \leq m_{\omega'}^N + \varepsilon.$$  

(C.1)

Taking $\varepsilon \to 0$ in (C.1) yields $m_\omega^N \leq m_{\omega'}^N$. This proves the proposition. \square

D  Compactness in $L^p$

We record a compactness theorem in $L^p$ (see Proposition A.1 of [17]):

**Lemma D.1.** Let $d \geq 1$ and $1 \leq p < \infty$. A sequence $\{f_n\}$ in $L^p(\mathbb{R}^d)$ has a convergent subsequence if and only if it satisfies the following properties:

(i) There exists $A > 0$ such that

$$\|f_n\|_{L^p} \leq A$$

(D.1)

for all $n \geq 1$.

(ii) For any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\int_{\mathbb{R}^d} |f_n(x) - f_n(x + y)| \, dx < \varepsilon$$

(D.2)

for all $n \geq 1$ and any $|y| < \delta$.

(iii) For any $\varepsilon > 0$ there exists $R > 0$ such that

$$\int_{|x| \geq R} |f_n(x)|^p \, dx < \varepsilon$$

(D.3)

for all $n \geq 1$. 

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