1 Introduction

A non-negative function $f(x)$ defined on an interval $(a, b)$ is said to be logarithmic concave (log-concave) if for every $x, y \in (a, b)$ and every $0 < \lambda < 1$, we have

$$f(\lambda x + (1 - \lambda)y) \geq [f(x)]^\lambda [f(y)]^{1-\lambda}. \hspace{1cm} (1)$$

If the inequality in $(1)$ is reversed, the function $f$ is said to be log-convex.

An equivalent definition of log-concavity (resp. log-convexity) is that the product $f(x) \cdot f(y)$ decreases (resp. increases) in $|x - y|$, holding $x + y$ fixed. Likewise, a non-negative sequence $\{a_i\}_{i=0}^n$ is said to be log-concave if the product $a_i a_j$ decreases with $|i - j|$, holding $i + j$ fixed. It is known that log-concave functions and sequences are closed under multiplication, integration, and convolution [12].

Proschan shows that log-convex functions are closed under addition or arbitrary mixture [13]. However, the same is not true for log-concave functions, see examples in Barlow and Proschan [3]. Except for the work of Lynch [10] and Block et al. [4], little is known about general conditions that guarantee the mixture of log-concave functions to also be log-concave.

In this paper, we prove the following theorems:

**Theorem 1.** If $M > 1$ and $\alpha(s)$ is a log-concave function on $(0, M)$, then the function

$$f(x) = \int_0^M \alpha(s) \left(\frac{M}{s}\right)^{\alpha(s)} (1 - x)^s x^{M-s} ds \hspace{1cm} (2)$$

*The author thanks Scott D. Kominers and Kareen Rozen for comments and suggestions.*
is log-concave in $x$ on $(0,1)$, where $\binom{M}{s}$ denotes $\frac{\Gamma(M+1)}{\Gamma(s+1)\Gamma(M-s+1)} = \frac{1}{(M+1)\beta(s+1, M-s+1)}$.

Here $\Gamma(\cdot)$ denotes the Gamma function, and $\beta(\cdot, \cdot)$ denotes the Beta function.

**Theorem 2.** If $M$ is a positive integer and $\{\alpha_i\}_{i=0}^M$ is a log-concave sequence, then the function

$$g(x) = \sum_{i=0}^M \alpha_i \binom{M}{i} (1-x)^i x^{M-i}$$

(3)

is log-concave in $x$ on the interval $(0,1)$.

A direct corollary of these theorems is the following:

**Corollary.** If $M > 1$, then any log-concave mixture of distributions

$$\{\text{Beta}(M - s + 1, s + 1)\}_{0 < s < M}$$

has a log-concave density. Furthermore if $M$ is an integer, then any discrete log-concave mixture of distributions

$$\{\text{Beta}(M - i + 1, i + 1)\}_{i=0}^M$$

also has a log-concave density.

While similar to the conditions given by Lynch in [10], this result is not implied by those of Lynch because Beta densities are not jointly log-concave in the argument $x$ and the parameter $s$.

## 2 Proof of the Theorems

Before proceeding to the proofs, we recall two technical lemmata that will be of use:

**Lemma 1.** Let $a(q), b(q), u(q), v(q)$ be non-negative continuous functions defined on the interval $[0, m]$, such that $a(q)$ is decreasing in $q$. Suppose further that for
each \( q \), \( a(q) \geq b(q) \geq 0 \), and \( \int_{r=0}^{q} u(r) \, dr \geq \int_{r=0}^{q} v(r) \, dr \). Then we have

\[
\int_{q=0}^{m} a(q) u(q) \, dq \geq \int_{q=0}^{m} b(q) v(q) \, dq.
\]

**Proof.** Define \( U(q) = \int_{r=0}^{q} u(r) \, dt \) and \( V(q) = \int_{r=0}^{q} v(r) \, dr \). Using integration by parts, we have

\[
\hat{m}_{q=0} a(q) U(q) = \int_{q=0}^{m} a(q) dU(q) = a(m) U(m) + \int_{q=0}^{m} U(q) d(-a(q)).
\]

Since \( -a(q) \) is increasing and \( U(q) \geq V(q) \) pointwise, the Stieltjes integral \( \int_{q=0}^{m} U(q) d(-a(q)) \) is greater than or equal to \( \int_{q=0}^{m} V(q) d(-a(q)) \). Another use of integration by parts shows that

\[
\int_{q=0}^{m} a(q) u(q) \, dq \geq a(m) V(m) + \int_{q=0}^{m} V(q) d(-a(q)) = \int_{q=0}^{m} a(q) v(q) \, dq \geq \int_{q=0}^{m} b(q) v(q) \, dq.
\]

In the literature, Lemma 1 is often called the “majorization trick” (see for instance [11]).

**Lemma 2.** Let \( M > 1, q > 0 \) and \( n > -2 \) be fixed parameters. Define the sets:

\[
A = \{ s : |s - (n-s)| \leq q, 0 \leq s, n-s \leq M \};
\]

\[
B = \{ s : |s - (n+1-s)| \leq q, 0 \leq s, n+1-s \leq M \};
\]

\[
C = \{ s : |s - (n-s)| \leq q, 0 \leq s+1, n+1-s \leq M \}.
\]

Then the following inequalities hold:

\[
\int_{A} \left( \frac{M-1}{s} \right) \left( \frac{M-1}{n-s} \right) \, ds \geq \int_{A} \left( \frac{M}{s} \right) \left( \frac{M-2}{n-s} \right) \, ds; \quad (4)
\]

\[
\int_{B} \left( \frac{M-1}{s} \right) \left( \frac{M-1}{n-s} \right) \, ds \leq \int_{B} \left( \frac{M}{s} \right) \left( \frac{M-2}{n-s} \right) \, ds; \quad (5)
\]

\[
\int_{C} \left( \frac{M-1}{s} \right) \left( \frac{M-1}{n-s} \right) \, ds \geq \int_{C} \left( \frac{M}{s+1} \right) \left( \frac{M-2}{n-s-1} \right) \, ds. \quad (6)
\]

**Proof.** We only give a proof for (4), as (5) and (6) follow along similar lines. Using properties of the \( \Gamma \) function—in particular \( \Gamma(M) = (M-1) \cdot \Gamma(M-1) \)—we see
that the generalized binomial coefficients satisfy

\[
\binom{M}{s} = \binom{M}{M-s} > 0, \forall -1 < s < M + 1;
\]
\[
\binom{M-1}{s} = \frac{M-s}{M} \binom{M}{s}, \quad \binom{M-1}{s-1} = \frac{s}{M} \binom{M}{s};
\]
\[
\binom{M}{s} = \binom{M-1}{s} + \binom{M-1}{s-1}.
\] (7)

To prove (4), first note that we can assume \( n > 0 \) and \( q \leq \min \{n, 2M-n\} \), which is the maximum difference between \( s \) and \( n-s \) when \( s \in A \). Writing \( k = \frac{n-q}{2} \), we have \( A = [k, n-k] \subset [0, M] \). Using (7) to write \( \binom{M-1}{n-s} = \binom{M-2}{n-s} + \binom{M-2}{n-s-1} \) and \( \binom{M}{n} = \binom{M-1}{s} + \binom{M-1}{s-1} \), we can calculate the difference between the two sides of (4) as:

\[
\int_{s=k}^{n-k} \left( \binom{M-1}{s} \binom{M-2}{n-s-1} - \binom{M-1}{s-1} \binom{M-2}{n-s} \right) ds
\]
\[
= \int_{s=k-1}^{k} \left( \binom{M-1}{n-s-1} \binom{M-2}{s} - \binom{M-1}{s} \binom{M-2}{n-s-1} \right) ds
\] (8)
\[
= \int_{s=k-1}^{k} \frac{n-2s-1}{M-1} \binom{M-1}{s} \binom{M-1}{n-s-1} ds.
\]

By (7), the two binomial coefficients above are non-negative in the range of integration. When \( k \leq \frac{n-k}{2} \), the term \( n-2s-1 \) is always non-negative, so is the last line of (8). When \( k > \frac{n-k}{2} \), we can write the last line of (8) as

\[
\int_{s=k-1}^{n-k-1} \frac{n-2s-1}{M-1} \binom{M-1}{s} \binom{M-1}{n-s-1} ds + \int_{s=n-k-1}^{k} \frac{n-2s-1}{M-1} \binom{M-1}{s} \binom{M-1}{n-s-1} ds.
\] (9)

Note that the integrand \( \frac{n-2s-1}{M-1} \binom{M-1}{s} \binom{M-1}{n-s-1} \) is non-negative when \( s \leq n-k-1 \leq \frac{n-k}{2} \), and it is an odd function with respect to \( s = \frac{n-k}{2} \). Thus the first integral in (9) is non-negative, while the second evaluates to zero. (8) follows, so does the lemma.

For the sake of completeness we state below the discrete analogs of the preceding two lemmata, which will be used in the proof of Theorem 2. We omit the proofs because they are the same.

**Lemma 1'**. Let \( a_1 \geq a_2 \geq \cdots \geq a_m \geq 0, b_1, b_2, \ldots, b_m \geq 0 \) be two sequences of real numbers satisfying \( a_q \geq b_q \) for each \( q \). Consider two more sequences of non-negative
real numbers \( u_1, u_2, \ldots, u_n \) and \( v_1, v_2, \ldots, v_n \), such that \( \sum_{r=1}^{q} u_r \geq \sum_{r=1}^{q} v_r \) for each \( q \). Then,

\[
\sum_{q=1}^{m} a_q u_q \geq \sum_{q=1}^{m} b_q v_q.
\]

Lemma 2': If \( M \) is a positive integer, \( n \) is a non-negative integer and \( k \) is an integer such that \( k \leq \frac{n+1}{2} \), then the following inequalities hold:

\[
\sum_{i=k}^{n-k} \binom{M-1}{i} \binom{M-1}{n-i} \geq \sum_{i=k}^{n-k} \binom{M}{i} \binom{M-2}{n-i};
\]

\[
\sum_{i=k}^{n-k+1} \binom{M-1}{i} \binom{M-1}{n-i} \leq \sum_{i=k}^{n-k+1} \binom{M}{i} \binom{M-2}{n-i};
\]

\[
\sum_{i=k}^{n-k} \binom{M-1}{i} \binom{M-1}{n-i} \geq \sum_{i=k}^{n-k} \binom{M}{i+1} \binom{M-2}{n-i-1};
\]

where as usual we define \( \binom{M}{i} = 0 \) when \( i < 0 \) or \( i > M \).

Proof of Theorem 1: We first make some preliminary simplifications. When \( \alpha(s) \) is identically zero, the result is trivial. Otherwise

\[
f(x) = \int_{s=0}^{M} \frac{s}{x} (1 - x)^{s} x^{M-s} ds
\]

is strictly positive for \( x \in (0, 1) \). Thus \( \log f \) is well-defined on the open interval. Its derivative is \( f' \) and its second derivative is \( f'' \). It thus suffices to show that \( f'(x)^2 \geq f(x) \cdot f''(x) \).

We will show that the following stronger inequality holds:

\[
\frac{M-1}{M} f'(x)^2 \geq f(x) \cdot f''(x) \text{ for every } x \in (0, 1).
\] (10)

Using (7), the derivative of \( \binom{M}{s} (1 - x)^{s} x^{M-s} \) is

\[
\binom{M}{s} (-s)(1-x)^{s-1} x^{M-s} + \binom{M}{s} (1-x)^{s} x^{M-s-1}
\]

\[
=M \left( \binom{M-1}{s-1} (1-x)^{s-1} x^{M-s} + \binom{M-1}{s} (1-x)^{s} x^{M-s-1} \right).
\]
It follows that
\[
\begin{align*}
f'(x) &= M \left( \int_{s=-1}^{M} [\alpha(s) - \alpha(s+1)] \left( \frac{M-1}{s} \right) (1-x)^s x^{M-s-1} ds \right);
\end{align*}
\]
\[
\begin{align*}
f''(x) &= M(M-1) \left( \int_{s=-2}^{M} [\alpha(s) - 2\alpha(s+1) + \alpha(s+2)] \left( \frac{M-2}{s} \right) (1-x)^s x^{M-s-2} ds \right);
\end{align*}
\]
where we define \( \alpha(s) = 0 \) whenever \( s < 0 \) or \( s > M \).

Introducing another dummy variable \( t \), we turn the desired inequality (10) into:
\[
\begin{align*}
\left( \int_{-1}^{M} [\alpha(s) - \alpha(s+1)] \left( \frac{M-1}{s} \right) (1-x)^s x^{M-s-1} ds \right) \left( \int_{-1}^{M} [\alpha(t) - \alpha(t+1)] \left( \frac{M-1}{t} \right) (1-x)^t x^{M-t-1} dt \right)
\geq \left( \int_{0}^{M} \alpha(s) \left( \frac{M}{s} \right) (1-x)^s x^{M-s} ds \right) \left( \int_{-1}^{M} [\alpha(t) - 2\alpha(t+1) + \alpha(t+2)] \left( \frac{M-2}{t} \right) (1-x)^t x^{M-t-2} dt \right)
\end{align*}
\]
(11)

By expanding the products on both sides of (11) and collecting terms that have \( (1-x)^n x^{2M-n-2} \) in common, it suffices to prove that for any \( n (-2 < n < 2M-2) \),
\[
\begin{align*}
\int_{s+t=n} [\alpha(s) - \alpha(s+1)][\alpha(t) - \alpha(t+1)] \left( \frac{M-1}{s} \right) \left( \frac{M-1}{t} \right) ds
\geq \int_{s+t=n} \alpha(s)[\alpha(t) - 2\alpha(t+1) + \alpha(t+2)] \left( \frac{M-2}{s} \right) \left( \frac{M-2}{t} \right) ds.
\end{align*}
\]
(12)

We show that the following three inequalities hold, which collectively imply (12):
\[
\begin{align*}
\int_{s+t=n} \alpha(s)\alpha(t) \left( \frac{M-1}{s} \right) \left( \frac{M-1}{t} \right) ds &\geq \int_{s+t=n} \alpha(s)\alpha(t) \left( \frac{M}{s} \right) \left( \frac{M-2}{t} \right) ds; \quad (13)
\end{align*}
\]
\[
\begin{align*}
\int_{s+t=n} \alpha(s)\alpha(t+1) \left( \frac{M-1}{s} \right) \left( \frac{M-1}{t} \right) ds &\leq \int_{s+t=n} \alpha(s)\alpha(t+1) \left( \frac{M}{s} \right) \left( \frac{M-2}{t} \right) ds;
\end{align*}
\]
\[
\begin{align*}
\int_{s+t=n} \alpha(s+1)\alpha(t+1) \left( \frac{M-1}{s} \right) \left( \frac{M-1}{t} \right) ds &\geq \int_{s+t=n} \alpha(s)\alpha(t+2) \left( \frac{M}{s} \right) \left( \frac{M-2}{t} \right) ds.
\end{align*}
\]
(14)

To prove (13), we use the “majorization trick” of Lemma 1.

Let \( a(q) = b(q) = \alpha \left( \frac{n-2}{2} \right) \cdot \alpha \left( \frac{n+2}{2} \right) \), then the log-concavity of \( \alpha(\cdot) \) ensures that
\( a(q) \) is non-negative and decreasing in \( q \). Moreover let

\[
u(q) = \frac{(M - 1)(M - 1)}{n + q} - \frac{M}{n + q} \approx \frac{M - 2}{n + q} + \frac{M}{n + q}.
\]

Inequality (13) becomes \( \int_{0}^{q} a(q) u(q) dq \geq \int_{0}^{q} b(q) v(q) dq \). It remains to check the majorization condition

\[
\int_{r=0}^{q} u(r) dr \geq \int_{r=0}^{q} v(r) dr \text{ for each } q.
\]

But this is exactly the inequality (4) in Lemma 2. So (13) is proved. In the same way, (14) and (15) reduce to (5) and (6) in Lemma 2. Theorem 1 follows.

We note that by (7), the function \( \left( \begin{array}{c} M \\ s \end{array} \right)(1-x)x^{M-s} \) is well defined and positive even for \(-1 < s < 0\) and \( M < s < M + 1 \). However, it fails to be log-concave on these intervals. Thus Theorem 1 would fail if we allow the range of integration in (2) to include these intervals.

**Proof of Theorem 2:** We take the same steps as before to reduce the result to inequalities (13), (14) and (15), with the integral replaced by a discrete sum. Those inequalities are in turn implied by Lemma 1' and Lemma 2'. The theorem follows.

It is worth noting that the discrete analog of inequality (10) is tight. In fact, whenever the \( \alpha \) sequence is geometric, we have

\[
g(x) = \sum_{i=0}^{M} \alpha_i \left( \begin{array}{c} M \\ i \end{array} \right)(1-x)^{M-i} = c(1 + \lambda x)^M
\]

for some \( c > 0 \) and \( \lambda \). Thus \( g(x) \cdot g''(x) = \frac{M-1}{M}g'(x)^2 \), which is slightly less than \( g'(x)^2 \) for large \( M \). This suggests that improvement of Theorem 2 (likewise Theorem 1) would depend on a different type of conditions on the mixing sequence/function \( \{\alpha_i\}_{i=0}^{M} \), rather than its own log-concavity.

**Proof of the Corollary:** Recall that the Beta\((M-s+1, s+1)\) distribution has
Therefore a mixture Beta distribution has density given by the form in Theorems 1 and 2. From the theorems, this density is log-concave whenever the weights are log-concave.

3 Applications

A recent application of this paper is seen in [7], where the authors assume subpopulations of consumers having distinct attention capacity. With a total of $M$ markets, the probability that a particular consumer pays attention to a market takes the form of a Beta distribution function in the price charged. The expected amount of attention that the market receives is therefore a mixture of Beta distributions. Using Theorem 2, the authors characterize equilibrium strategies and establish that they are monotone [7]. Log-concavity is important elsewhere in economics, see Baghnnoli and Bergstrom [2].

In the statistical theory of reliability, one is interested in the failure rate of mixture systems. The techniques here thus complement earlier work by Lynch [10] and Block et al. [4]. Finally, log-concavity has powerful implications for truncated distributions [5], hypothesis testing (e.g. the Karlin-Rubin theorem [9]), and maximum likelihood estimation [8]. The recent surge of interest in mixture models renders it necessary to gain further understanding of the log-concavity of general mixture distributions [1], [6]. This paper is a step toward that understanding for the special family of Beta distributions.

References

[1] Achlioptas, D. and McSherry, F. (2005). On spectral learning of mixtures of distributions. Proceedings of the 18th annual conference on Learning Theory,
Bertinoro, Italy.

[2] Bagnoli, M. and Bergstrom, T. (2005). Log-concave probability and its applications. *Economic Theory, Springer*, 26(2), 445-469.

[3] Barlow, R. and Proschan, F. (1975). “Statistical Theory of Reliability and Life Testing Probability Models,” New York: Holt, Rinehart and Winston.

[4] Block, H. W., Li, Y. and Savits, T. H. (2003). Preservation of properties under mixtures. *Probability in Engineering and Information Sciences* 17, 205-212.

[5] Burdett, K. (1996). Truncated Means and Variances. *Economics Letters*, 52, 263-267.

[6] Chang, G. and Walther, G. (2007). Clustering with mixtures of log-concave distributions. *Computational Statistics and Data Analysis*, 51, 6242-6251.

[7] Clippel, G. D., Eliaz, K. and Rozen, K. (2013). Competing for consumer attention. *Preprint*.

[8] Dümbgen, L. and Rufibach, K. (2009). Maximum likelihood estimation of a log-concave density and its distribution function: Basic properties and uniform consistency. *Bernoulli*, 15(1), 40-68.

[9] Karlin, S. and Rubin, H. (1956). The theory of decision procedures for distributions with monotone likelihood ratio. *Ann. Math. Statist.* 27(2), 272-299.

[10] Lynch, J. D. (1999). On conditions for mixtures of increasing failure rate distributions to have an increasing failure rate. *Probability in Engineering and Information Sciences*, 13, 33-36.

[11] Marshall, A. W. and Olkin, I. (1983). “Inequalities: Theory of majorization and its applications,” New York: Academic Press.

[12] Prékopa, A. (1973). On Logarithmic Concave Measures and Functions. *Acta Sci. Math. (Szeged)*, 34, 335-343.

[13] Proschan, F. (1963). Theoretical explanation of observed decreasing failure rate. *Technometrics*, 5, 373-383.