Relations between quark and lepton mixing angles and matrices

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Abstract. We discuss the relations between the mixing angles and the mixing matrices of quarks and leptons. With Raidal’s numerical relations, we parameterize the lepton mixing (PMNS) matrix with the parameters of the quark mixing (CKM) matrix, and calculate the products of $V_{\text{CKM}}U_{\text{PMNS}}$ and $U_{\text{PMNS}}V_{\text{CKM}}$. Also, under the conjectures $V_{\text{CKM}}U_{\text{PMNS}} = U_{\text{bimax}}$ or $U_{\text{PMNS}}V_{\text{CKM}} = U_{\text{bimax}}$, we get the PMNS matrix naturally, and test Raidal’s relations in these two different versions. The similarities and the differences between the different versions are discussed in detail.

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1 Introduction

The mixing of quarks and leptons is one of the fundamental problems in particle physics. But its origin is still unknown, and the mixing is described phenomenologically by the mixing matrices, i.e., the Cabibbo-Kobayashi-Maskawa (CKM) matrix for quark mixing and the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) matrix for lepton mixing. To understand the mixing problem, two aspects should be considered. One is the mixing matrix, and the other is the mixing angle. However, these mixing angles cannot be determined by the standard model (SM) itself, and can only be fixed by the experimental data. So the mixing angles are taken as free parameters, and are not correlated with each other. Furthermore, the quark and lepton mixing matrices, which are composed of the mixing angles, are also independent of each other. If we can find the relation between these mixing angles or the relation between the mixing matrices, it will be helpful for our understanding of the inner essence of the SM and for the model construction of the grand unified theory.

In this paper, we discuss the relations between the mixing angles and the mixing matrices of quarks and leptons, respectively. First, for the mixing angles, Raidal has suggested some numerical relations

\[ \theta^\text{CKM}_1 + \theta^\text{PMNS}_1(\theta_{\text{atm}}) = \frac{\pi}{4}, \]
\[ \theta^\text{CKM}_2 \sim \theta^\text{PMNS}_2(\theta_{\text{cha}}) \sim \mathcal{O}(\lambda^3), \]
\[ \theta^\text{CKM}_3(\theta_C) + \theta^\text{PMNS}_3(\theta_{\text{sol}}) = \frac{\pi}{4}, \]

(1)

where \( \theta_i \) are the mixing angles of the CKM and the PMNS matrices. With these relations, we can link the elements of the CKM and the PMNS matrices together, and then can express the CKM and the PMNS matrices in a unified way. Furthermore, we can find the relation between these two mixing matrices.

Second, for the mixing matrices, we discuss the products of the CKM and the PMNS matrices. Both \( V^\text{CKM}U^\text{PMNS} \) and \( U^\text{PMNS}V^\text{CKM} \) are calculated in detail. We find that the product of the CKM and the PMNS matrices is rather near the bimaximal mixing pattern. So we can get the PMNS matrix in terms of the CKM matrix and the bimaximal mixing matrix. The PMNS matrix can be parameterized by the parameters of the CKM matrix, and the relations between the mixing angles are deduced naturally.

In Sects. 2 and 3, we discuss the quark and lepton mixing matrices, and the mixing angles and the parameterizations of quark and lepton mixing matrices, and show their relations. In Sect. 4, with the numerical relations between the quark and lepton mixing angles, we discuss the relation between the quark and lepton mixing matrices, and point out the similarities and the differences of different versions. In Sect. 5, we discuss the relations between the mixing angles under the conjecture that the product of quark and lepton mixing matrices is the bimaximal mixing pattern. Some conclusions are given in Sect. 6.

2 The quark and lepton mixing matrices

To see the generation of the quark mixing matrix, let us consider the charge-changing weak current

\[ j = 2 \sum_{\alpha' = u', c', t'} \bar{u}_{\alpha'} \gamma_{\rho} d_{\alpha'}, \]

(2)

where the u-type and d-type quark fields \( u_{\alpha'} \) and \( d_{\alpha'} \) do not have definite masses, but are the linear combinations of the massive quark fields \( u_\alpha \) and \( d_\alpha \),

\[ u_{\alpha'} = \sum_{\alpha = u, c, t} V^\alpha_{\alpha'} u_\alpha, \quad d_{\alpha'} = \sum_{\alpha = d, s, b} V^\alpha_{\alpha'} d_\alpha, \]

(3)

where \( V_u \) and \( V_d \) are unitary matrices, which can diagonalize the quark mass matrices. Substituting Eq. (3) into Eq. (2), we have

\[ j = 2 \sum_{\alpha', \alpha, \beta} \bar{u}_\alpha \gamma_{\rho} V^\alpha_{\alpha'} V^\beta_{\alpha'} d_\beta \]
\[ = 2 \sum_{\alpha, \beta} \bar{u}_\alpha \gamma_{\rho} V_{\text{CKM}}^\alpha_{\beta} d_\beta, \]

where

\[ V_{\text{CKM}} = V_u^\dagger V_d. \]

(4)
$V_{\text{CKM}}$ is the quark mixing (CKM) matrix, which links the flavor eigenstates to the mass eigenstates of quarks.

The CKM matrix is measured by different experiments to a good degree of accuracy [5], and the elements of the modulus of the CKM matrix are summarized as

$$
\begin{pmatrix}
0.9739 - 0.9751 & 0.221 - 0.227 & 0.0029 - 0.0045 \\
0.221 - 0.227 & 0.9730 - 0.9744 & 0.039 - 0.044 \\
0.0048 - 0.014 & 0.037 - 0.043 & 0.9990 - 0.9992 \\
\end{pmatrix}.
$$

We can see that the CKM matrix is very near the unit matrix, and it can be parameterized by the Wolfenstein parametrization [6]

$$
V_{\text{CKM}} = \begin{pmatrix}
1 - \frac{1}{2} \lambda^2 & \frac{\lambda}{2} & A\lambda^3(\rho - i\eta) \\
-\frac{\lambda}{2} & 1 - \frac{1}{2} \lambda^2 & A\lambda^2 \\
A\lambda^3(1 - \rho - i\eta) & -A\lambda^2 & 1 \\
\end{pmatrix},
$$

where $\lambda$ measures the strength of the deviation of $V_{\text{CKM}}$ from the unit matrix ($\lambda = \sin \theta_C = 0.2243 \pm 0.0016$, $\theta_C$ is the Cabibbo mixing angle), and $A$, $\rho$ and $\eta$ are the other three parameters, with the best fit values $A = 0.82$, $\rho = 0.20$ and $\eta = 0.33$ [5].

Similarly, the lepton mixing (PMNS) matrix can be written as

$$
U_{\text{PMNS}} = U_1^\dagger U_\nu.
$$

where $U_1$ and $U_\nu$ are unitary matrices, which can diagonalize the charged-lepton and the neutrino mass matrices, and $U_{\text{PMNS}}$ links the flavor eigenstates to the mass eigenstates of leptons.

The elements of the modulus of the PMNS matrix are summarized as [7]

$$
\begin{pmatrix}
0.77 - 0.88 & 0.47 - 0.61 & 0.20 - 0.53 \\
0.19 - 0.52 & 0.42 - 0.73 & 0.58 - 0.82 \\
0.20 - 0.53 & 0.44 - 0.74 & 0.56 - 0.81 \\
\end{pmatrix}.
$$

We can see that the PMNS matrix deviates from the unit matrix very much, but is quite near the bimaximal mixing pattern, which reads

$$
U_{\text{bimax}} = \begin{pmatrix}
\sqrt{2}/2 & \sqrt{2}/2 & 0 \\
-1/2 & 1/2 & \sqrt{2}/2 \\
1/2 & -1/2 & \sqrt{2}/2 \\
\end{pmatrix}.
$$

Since the CKM matrix is quite near the unit matrix, and the PMNS matrix is quite near the bimaximal matrix, we may assume that the deviation of the PMNS matrix from the bimaximal can just be described by the CKM matrix, that is

$$
U_{\text{PMNS}} V_{\text{CKM}} = U_{\text{bimax}},
$$

or

$$
V_{\text{CKM}} U_{\text{PMNS}} = U_{\text{bimax}}.
$$

So we can get

$$
U_{\text{PMNS}} = U_{\text{bimax}} V_{\text{CKM}}^\dagger,
$$

or

$$
U_{\text{PMNS}} = V_{\text{CKM}}^\dagger U_{\text{bimax}}.
$$

Eq. (9) and Eq. (10) have both been pointed out by Minakata and Smirnov [8], and the similar results have also been discussed in the literature [9]. Thus, the PMNS matrix can be expressed thoroughly by the CKM matrix, and can be parameterized by the Wolfenstein parameters of the CKM matrix. So we can get the relations between the mixing angles of quarks and leptons. We will discuss these two cases in Sec. 5.
3 The mixing angles of the quark and lepton mixing matrices

Both the CKM matrix and the PMNS matrix can be written as

\[
\begin{pmatrix}
-c_2c_3 & c_2s_3 & s_2e^{-i\delta} \\
-c_1s_3 - s_1s_2c_3e^{i\delta} & c_1c_3 - s_1s_2s_3e^{i\delta} & s_1c_2 \\
-s_1s_3 + c_1s_2c_3e^{i\delta} & c_1c_3 + c_1s_2s_3e^{i\delta} & c_1c_2
\end{pmatrix}
\] (13)

where \(s_i = \sin \theta_i\), \(c_i = \cos \theta_i\) (for \(i = 1, 2, 3\)), which describe the mixings between 2nd and 3rd, 3rd and 1st, and 1st and 2nd generations of quarks or leptons, and \(\delta\) is the Dirac CP-violating phase. Altogether there are eight (four for quark sector and four for lepton sector) parameters in the mixing matrices, describing both the real and the imaginary parts of the mixing matrices. If neutrinos are of Majorana type, it is always feasible to parameterize the neutrino mixing matrix as a product of Eq. (13) and a diagonal phase matrix with two unremovable phase angles \(\text{diag}(1, e^{i\alpha}, e^{i\beta})\) [10], where \(\alpha, \beta\) are the Majorana CP-violating phases.

For quark sector, these angles are measured to a good degree of accuracy (for example, see [1]). The best fit values of the three mixing angles are \(\theta_1^{\text{CKM}} = 24.2^\circ\), \(\theta_2^{\text{CKM}} = 0.2^\circ\), and \(\theta_3^{\text{CKM}}(\theta_C) = 12.9^\circ\).

For lepton sector, with the help of various experimental data from KamLAND [11], SNO [12], K2K [13], Super-Kamiokande [14] and CHOOZ [15] experiments, we now have a much better understanding of these mixing angles,

\[
\sin^2 2\theta_{\text{atm}} = 1.00 \pm 0.05, \\
\sin^2 2\theta_{\text{cha}} = 0 \pm 0.065, \\
\tan^2 \theta_{\text{sol}} = 0.41 \pm 0.05,
\]

where \(\theta_{\text{atm}}, \theta_{\text{cha}},\) and \(\theta_{\text{sol}}\) are the mixing angles of atmospheric, CHOOZ and solar neutrino oscillations, and we have \(\theta_{\text{atm}} = \theta_1^{\text{PMNS}} = 45.0^\circ \pm 6.5^\circ\), \(\theta_{\text{cha}} = \theta_2^{\text{PMNS}} = 0^\circ \pm 7.4^\circ\), and \(\theta_{\text{sol}} = \theta_3^{\text{PMNS}} = 32.6^\circ \pm 1.6^\circ\) [3].

An interesting numerical relation between the third mixing angles of quarks and leptons was pointed out by Smirnov [16],

\[
\theta_3^{\text{CKM}}(\theta_C) + \theta_3^{\text{PMNS}}(\theta_{\text{sol}}) = \frac{\pi}{4}.
\] (14)

And this relation is called the quark-lepton complementarity (QLC) [8].

Raidal extended this relation to three generations [3].

\[
\theta_1^{\text{CKM}} + \theta_1^{\text{PMNS}}(\theta_{\text{atm}}) = \frac{\pi}{4}, \\
\theta_2^{\text{CKM}} \sim \theta_2^{\text{PMNS}}(\theta_{\text{cha}}) \sim O(\lambda^3), \\
\theta_3^{\text{CKM}}(\theta_C) + \theta_3^{\text{PMNS}}(\theta_{\text{sol}}) = \frac{\pi}{4}.
\]

With these relations, we can find that the mixing angles of quarks and leptons are not independent of each other. And thus we can get the trigonometric functions of the mixing angles of leptons in terms of those of quarks, and link the parameters of the PMNS matrix with those of the CKM matrix. Therefore, we can parameterize the PMNS matrix and the CKM matrix in a same framework [3]. And then we can test the product relations in Eq. (9) and Eq. (10). We will discuss these cases in Sec. 4.

4 The relations between the mixing angles

In Wolfenstein parametrization of the CKM matrix, we have (to the order of \(\lambda^3\))

\[
\sin \theta_1^{\text{CKM}} = A\lambda^2, \quad \cos \theta_1^{\text{CKM}} = 1, \\
\sin \theta_2^{\text{CKM}} e^{-i\delta} = A\lambda^3(\rho - i\eta), \quad \cos \theta_2^{\text{CKM}} = 1, \\
\sin \theta_3^{\text{CKM}} = \lambda, \quad \cos \theta_3^{\text{CKM}} = 1 - \frac{1}{2} \lambda^2.
\] (15)

Using Eq. (11), we can get the trigonometric functions of the mixing angles of leptons (to the order of \(\lambda^3\))

\[
\sin \theta_1^{\text{PMNS}} = \sin\left(\frac{\pi}{4} - \theta_1^{\text{CKM}}\right) = \frac{\sqrt{2}}{2}(1 - A\lambda^2), \\
\cos \theta_1^{\text{PMNS}} = \frac{\sqrt{2}}{2}(1 + A\lambda^2),
\]
\[
\sin \theta_2^{PMNS} e^{-i\delta} = A \lambda^3 (\zeta - i\xi),
\]
\[
\cos \theta_2^{PMNS} = 1,
\]
\[
\sin \theta_3^{PMNS} = \frac{\sqrt{2}}{2} (1 - \lambda - \frac{1}{2} \lambda^2),
\]
\[
\cos \theta_3^{PMNS} = \frac{\sqrt{2}}{2} (1 + \lambda - \frac{1}{2} \lambda^2),
\] (16)

where \(A\) and \(\lambda\) are the Wolfenstein parameters of the CKM matrix. So the CKM and the PMNS matrices have only one set of parameters with Raidal’s numerical relations. Because there are in total four angles in the mixing matrix (three mixing angles and one \(CP\)-violating phase angle), and only two precise numerical relations are known, we have to introduce another two new parameters \(\zeta\) and \(\xi\) to describe the PMNS matrix fully.

In Eq. (16), we set \(\sin \theta_2^{PMNS} e^{-i\delta} = A \lambda^3 (\zeta - i\xi)\). Because of the inaccurate experimental data of neutrino oscillations, we have not fixed the value of \(|U_{e3}^{PMNS}|\), and only known its upper bound \([7]\). Therefore, we may also set \(\sin \theta_2^{PMNS} e^{-i\delta} = A \lambda^2 (\zeta - i\xi)\). Choosing which of them is to be determined by the future experimental data, and we discuss these two cases here, respectively.

**Case 1:** \(\sin \theta_2^{PMNS} e^{-i\delta} = A \lambda^3 (\zeta - i\xi)\).

Substituting Eq. (16) into Eq. (13), we can get the PMNS matrix as

\[
U_{PMNS} = \begin{pmatrix}
\frac{\sqrt{2}}{2} (1 + \lambda - \frac{1}{2} \lambda^2) & \frac{\sqrt{2}}{2} (1 - \lambda - \frac{1}{2} \lambda^2) & A \lambda^3 (\zeta - i\xi) \\
-\frac{1}{2} (1 - \lambda + (A - \frac{1}{2}) \lambda^2 + A \lambda^3 (1 - \zeta - i\xi)) & \frac{1}{2} (1 + \lambda + (A - \frac{1}{2}) \lambda^2 + A \lambda^3 (1 - \zeta - i\xi)) & \frac{\sqrt{2}}{2} (1 - A \lambda^2) \\
\frac{1}{2} (1 - \lambda - (A + \frac{1}{2}) \lambda^2 + A \lambda^3 (1 - \zeta - i\xi)) & -\frac{1}{2} (1 + \lambda - (A + \frac{1}{2}) \lambda^2 - A \lambda^3 (1 - \zeta - i\xi)) & \frac{\sqrt{2}}{2} (1 + A \lambda^2)
\end{pmatrix}
\]

(17)

We can see from Eq. (17) the followings.

(1) The bimaximal mixing pattern is deduced naturally as the leading-order approximation as long as we accept the numerical relations in Eq. (1).

(2) The leading and next-to-leading order terms are just the same as the expressions in the expansion of the PMNS matrix around the bimaximal mixing pattern by Rodejohann [17] and us [18].

(3) The Wolfenstein parameter \(\lambda\) can characterize both the deviation of the CKM matrix from the unit matrix (see Eq. (15)), and the deviation of the PMNS matrix from the exactly bimaximal mixing pattern (see the next-to-leading order term in Eq. (17)).

Since these two different kinds of deviations are characterized by only one parameter set, the product of the CKM matrix and the PMNS matrix may just be the exactly bimaximal mixing matrix (Eq. (17) and Eq. (1)). To see this clearly, we discuss these two versions of product, respectively.

(i) \(V_{CKM} U_{PMNS}\)

From Eq. (17) and Eq. (1), we have

\[
V_{CKM} U_{PMNS} = \begin{pmatrix}
\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\
\frac{1}{2} & -\frac{1}{2} & \frac{\sqrt{2}}{2} \\
-\frac{1}{2} & -\frac{1}{2} & \frac{-\sqrt{2}}{2}
\end{pmatrix} + \lambda \begin{pmatrix}
-\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\
-\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\
-\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0
\end{pmatrix} + \lambda^2 \begin{pmatrix}
-\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\
-\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\
-\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0
\end{pmatrix}
\]

(18)

We can see from Eq. (18) that the deviation of the product of the CKM matrix and the PMNS matrix from the exactly bimaximal mixing matrix is of order \(\lambda\).

(ii) \(U_{PMNS} V_{CKM}\)
Similarly, we have

\[
U_{\text{PMNS}} V_{\text{CKM}} = \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{1}{\sqrt{2}} & \frac{\sqrt{2}}{2} \end{pmatrix} + \lambda^2 \begin{pmatrix} 0 & 0 & \frac{\sqrt{2}}{2} A \\ -\frac{1}{2} A - \frac{1}{2} (\sqrt{2} - 1) A & -\frac{1}{2} (\sqrt{2} - 1) A & -\frac{1}{2} (\sqrt{2} - 1) A \\ -\frac{1}{2} A - \frac{1}{2} (\sqrt{2} - 1) A & -\frac{1}{2} (\sqrt{2} - 1) A & -\frac{1}{2} (\sqrt{2} - 1) A \end{pmatrix} \\
+ \lambda^3 \begin{pmatrix} 0 & 0 & \frac{\sqrt{2}}{2} A[(\zeta - i\xi) - \frac{\sqrt{2}}{2}(1 - \rho + in)] \\ -\frac{1}{2} A[\sqrt{2}(1 - \rho - in) - (\zeta + i\xi)] - \frac{1}{2} A(\zeta + i\xi) & -\frac{1}{2} A(1 - \rho + in) & -\frac{1}{2} A(\zeta + i\xi) \\ -\frac{1}{2} A[\sqrt{2}(1 - \rho - in) - (\zeta + i\xi)] - \frac{1}{2} A(\zeta + i\xi) & -\frac{1}{2} A(1 - \rho + in) & -\frac{1}{2} A(\zeta + i\xi) \end{pmatrix} + \cdots \quad (19)
\]

We can see from Eq. (19) that the deviation of the product of the PMNS matrix and the CKM matrix from the exactly bimaximal mixing matrix is smaller (to the order of \(\lambda^2\)) than the former one. So the conjecture in Eq. (10) is better than the conjecture in Eq. (10).

Case 2: \(\sin \theta_{\text{PMNS}} e^{-i\delta} = A\lambda^2(\zeta - i\xi)\).

Repeating the process, we get

\[
U_{\text{PMNS}} = \begin{pmatrix} \frac{\sqrt{2}}{2}(1 + \lambda - \frac{1}{2} \lambda^2) & \frac{\sqrt{2}}{2}(1 - \lambda - \frac{1}{2} \lambda^2) & A\lambda^2(\zeta - i\xi) \\ -\frac{1}{2} \{1 - \lambda - [\frac{1}{2} - A(1 - \zeta + i\xi)] \lambda^2\} & \frac{i}{2} \{1 - \lambda - [\frac{1}{2} - A(1 - \zeta - i\xi)] \lambda^2\} & \frac{i}{2} \{1 - \lambda - [\frac{1}{2} + A(1 - \zeta - i\xi)] \lambda^2\} \\ \frac{1}{2} \{1 - \lambda - [\frac{1}{2} + A(1 + \zeta + i\xi)] \lambda^2\} & -\frac{1}{2} \{1 - \lambda - [\frac{1}{2} + A(1 + \zeta - i\xi)] \lambda^2\} & -\frac{1}{2} \{1 - \lambda - [\frac{1}{2} + A(1 + \zeta + i\xi)] \lambda^2\} \end{pmatrix} \\
= \begin{pmatrix} \frac{\sqrt{2}}{2} & 0 & 0 \\ -\frac{i}{2\sqrt{2}} & 0 & 0 \\ -\frac{i}{2\sqrt{2}} & 0 & 0 \end{pmatrix} + \lambda \begin{pmatrix} \frac{\sqrt{2}}{2} & 0 & 0 \\ -\frac{i}{2\sqrt{2}} & 0 & 0 \\ -\frac{i}{2\sqrt{2}} & 0 & 0 \end{pmatrix} \\
+ \lambda^2 \begin{pmatrix} -\frac{\sqrt{2}}{2} & \frac{i}{2\sqrt{2} - 1} & A(\zeta - i\xi) \\ \frac{i}{2\sqrt{2} - 1} & -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} A(1 + \zeta + i\xi) \\ \frac{i}{2\sqrt{2} - 1} & \frac{i}{2\sqrt{2} - 1} & \frac{i}{2\sqrt{2} - 1} A(\zeta + i\xi) \end{pmatrix} + \cdots \quad (20)
\]

And similarly, we have

(i) \(V_{\text{CKM}} U_{\text{PMNS}}\)

\[
V_{\text{CKM}} U_{\text{PMNS}} = \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{1}{\sqrt{2}} & \frac{\sqrt{2}}{2} \end{pmatrix} + \lambda \begin{pmatrix} \frac{\sqrt{2} - 1}{2} & -\frac{\sqrt{2} - 1}{2} & \frac{\sqrt{2} - 1}{2} \\ -\frac{\sqrt{2} - 1}{2} & \frac{\sqrt{2} - 1}{2} & -\frac{\sqrt{2} - 1}{2} \\ -\frac{\sqrt{2} - 1}{2} & -\frac{\sqrt{2} - 1}{2} & \frac{\sqrt{2} - 1}{2} \end{pmatrix} \\
+ \lambda^2 \begin{pmatrix} -\frac{\sqrt{2} - 1}{2} \{\sqrt{2} - 1 + A(\zeta + i\xi)\} & \frac{\sqrt{2} - 1}{2} \{\sqrt{2} - 1 - A(\zeta + i\xi)\} & \frac{\sqrt{2} - 1}{2} \{\sqrt{2} - 1 + A(\zeta + i\xi)\} \\ \frac{\sqrt{2} - 1}{2} \{\sqrt{2} - 1 + A(\zeta + i\xi)\} & -\frac{\sqrt{2} - 1}{2} \{\sqrt{2} - 1 - A(\zeta + i\xi)\} & \frac{\sqrt{2} - 1}{2} \{\sqrt{2} - 1 + A(\zeta + i\xi)\} \\ -\frac{\sqrt{2} - 1}{2} \{\sqrt{2} - 1 + A(\zeta + i\xi)\} & \frac{\sqrt{2} - 1}{2} \{\sqrt{2} - 1 - A(\zeta + i\xi)\} & -\frac{\sqrt{2} - 1}{2} \{\sqrt{2} - 1 + A(\zeta + i\xi)\} \end{pmatrix} + \cdots \quad (21)
\]

and

(ii) \(U_{\text{PMNS}} V_{\text{CKM}}\)

\[
U_{\text{PMNS}} V_{\text{CKM}} = \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{1}{\sqrt{2}} & \frac{\sqrt{2}}{2} \end{pmatrix} + \lambda^2 \begin{pmatrix} 0 & 0 & 0 \\ -\frac{1}{2} A(1 + \zeta + i\xi) & -\frac{1}{2} A(\sqrt{2} - 1 + \zeta + i\xi) & -\frac{1}{2} A(\sqrt{2} - 1 + \zeta + i\xi) \\ -\frac{1}{2} A(1 + \zeta + i\xi) & -\frac{1}{2} A(\sqrt{2} - 1 + \zeta + i\xi) & -\frac{1}{2} A(\sqrt{2} - 1 + \zeta + i\xi) \end{pmatrix} \\
+ \cdots \quad (22)
\]

Again, we find that the deviation of \(U_{\text{PMNS}} V_{\text{CKM}}\) from the exactly bimaximal mixing matrix is rather small (to the order of \(\lambda^2\)), and that the deviation of \(V_{\text{CKM}} U_{\text{PMNS}}\) from the exactly bimaximal mixing matrix is larger (to the order of \(\lambda\)). So the former conjecture in Eq. (10) is still better than the conjecture in Eq. (10).

In summary, in both the cases of \(\theta_{\text{PMNS}} e^{-i\delta} = A\lambda^3(\zeta - i\xi)\) and \(\theta_{\text{PMNS}} e^{-i\delta} = A\lambda^2(\zeta - i\xi)\), the product of \(U_{\text{PMNS}} V_{\text{CKM}}\) is nearer to the exactly bimaximal mixing matrix than the product of \(V_{\text{CKM}} U_{\text{PMNS}}\).

5 The relations between the mixing matrices

In the previous deductive process, we admit Raidal’s numerical relations between the mixing angles of quarks and leptons beforehand, and thus get the PMNS matrix in terms of the Wolfenstein parameters of the CKM matrix. Then
we calculate the product of $V_{\text{CKM}}U_{\text{PMNS}}$ and $U_{\text{PMNS}}V_{\text{CKM}}$, and compare their deviations from the exactly bimaximal mixing matrix. However, with the current experimental data, we can also make the conjectures $U_{\text{PMNS}}V_{\text{CKM}} = U_{\text{bimax}}$ or $V_{\text{CKM}}U_{\text{PMNS}} = U_{\text{bimax}}$ at first, and then get the PMNS matrix straightforward. Thereafter we can find whether Raidal’s relations hold well under these conjectures. We discuss the two different products, respectively. We have seen from Sec. 4 that $U_{\text{PMNS}}V_{\text{CKM}}$ is closer to the bimaximal mixing pattern (to the order of $\lambda^2$) than $V_{\text{CKM}}U_{\text{PMNS}}$ (to the order of $\lambda$), so this time we discuss the case $U_{\text{PMNS}}V_{\text{CKM}} = U_{\text{bimax}}$ first.

Case 1: $U_{\text{PMNS}}V_{\text{CKM}} = U_{\text{bimax}}$.

We suggest this product as a possibility for the relation between the quark and lepton mixing matrices. Although we have no theoretical fundamental for this suggestion, we can see that this product is consistent with Eq. 11 and Eq. 20 in Sec. 4. In the following deductive process, we can see that if we assume $U_{\text{PMNS}}V_{\text{CKM}} = U_{\text{bimax}}$, the Raidal’s relations can hold good, and the parametrization of the PMNS matrix can be deduced naturally.

Because $V_{\text{CKM}}$ is unitary, we can get $U_{\text{PMNS}}$ by multiplying $V_{\text{CKM}}^\dagger$ on the right side of $U_{\text{bimax}}$,

$$U_{\text{PMNS}} = U_{\text{bimax}} V_{\text{CKM}}^\dagger$$

$$= \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} & \frac{\sqrt{2}}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{\sqrt{2}}{2} \end{pmatrix} + \lambda \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 0 \end{pmatrix} + \lambda^2 \begin{pmatrix} -\frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \end{pmatrix} + \lambda^3 \begin{pmatrix} 0 & 0 & \frac{\sqrt{2}}{2}A(1 - \rho + i\eta) \\ \frac{\sqrt{2}}{2}A(\rho + i\eta) & 0 \end{pmatrix} + \cdots$$

(23)

We can see that the leading and the next-to-leading terms in Eq. 23 are just the same as those in Eq. 17 and Eq. 20. This indicates that Raidal’s relations (Eq. 11) and $U_{\text{PMNS}}V_{\text{CKM}} = U_{\text{bimax}}$ are in very good consistency with each other.

To see this more clearly, we can calculate the trigonometric functions of the mixing angles of the PMNS matrix, and then calculate the sums of the corresponding angles of quarks and leptons.

From Eq. 23, we have

$$c_{2}^{\text{PMNS}} s_{3}^{\text{PMNS}} = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} \lambda - \frac{\sqrt{2}}{4} \lambda^2,$$

$$c_{2}^{\text{PMNS}} c_{3}^{\text{PMNS}} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} \lambda - \frac{\sqrt{2}}{4} \lambda^2.$$  

(24)

From Eq. 21 we have (to the order of $\lambda^3$)

$$\tan \theta_{3}^{\text{PMNS}} = 1 - 2\lambda + 2\lambda^2 - 3\lambda^3.$$  

(25)

Thus, we can get (to the order of $\lambda^3$)

$$s_{3}^{\text{PMNS}} = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} \lambda - \frac{\sqrt{2}}{4} \lambda^2,$$

$$c_{3}^{\text{PMNS}} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} \lambda - \frac{\sqrt{2}}{4} \lambda^2.$$  

(26)

Similarly, we have

$$s_{1}^{\text{PMNS}} = \frac{\sqrt{2}}{2} - A\lambda^2 + A\lambda^3,$$

$$c_{1}^{\text{PMNS}} = \frac{\sqrt{2}}{2} + A\lambda^2 - A\lambda^3.$$  

(27)

Also, we have

$$s_{2}^{\text{PMNS}} e^{-i\delta} = -\frac{\sqrt{2}}{2}A\lambda^2 + \frac{\sqrt{2}}{2}(1 - \rho + i\eta)A\lambda^3,$$

(28)

and so

$$|s_{2}^{\text{PMNS}}|^2 = \frac{\sqrt{2}}{2}A\lambda^2 \sqrt{(\lambda - \lambda\rho - 1)^2 + (\lambda\eta)^2}.$$  

(29)
Substituting the best fit values of $A$, $\lambda$, $\rho$ and $\eta$, we have

$$|s_{2}^{\text{PMNS}}| = 0.48\lambda^{2},$$

and $c_{2}^{\text{PMNS}} = 1$ (to the order of $\lambda^{3}$).

Now we have got all the six the trigonometric functions of the mixing angles of leptons, and we can calculate the sums of the mixing angles of quarks and leptons.

Using Eq. (15) and Eq. (20), we have

$$\sin(\theta_{3}^{\text{CKM}} + \theta_{3}^{\text{PMNS}}) = s_{3}^{\text{CKM}} c_{3}^{\text{PMNS}} + c_{3}^{\text{CKM}} s_{3}^{\text{PMNS}} = \frac{\sqrt{2}}{2},$$

and thus

$$\theta_{3}^{\text{CKM}} + \theta_{3}^{\text{PMNS}} = \frac{\pi}{4}. \quad (31)$$

We can find that the QLC is satisfied precisely.

Similarly,

$$\sin(\theta_{1}^{\text{CKM}} + \theta_{1}^{\text{PMNS}}) = \frac{\sqrt{2}}{2} + (\frac{\sqrt{2}}{2} - 1)A\lambda^{2} + A\lambda^{3},$$

and thus

$$\theta_{1}^{\text{CKM}} + \theta_{1}^{\text{PMNS}} = \frac{\pi}{4} - (\sqrt{2} - 1)A\lambda^{2} + \sqrt{2}A\lambda^{3}. \quad (32)$$

So Raidal’s relation violates a little (to the order of $\lambda^{2}$).

Also, for $s_{2}^{\text{PMNS}}$, we can find from Eq. (30) that $s_{2}^{\text{PMNS}} \sim \lambda^{2}$, this differs from Raidal’s relation slightly, and is consistent with the parametrization in Eq. (24).

In summary, if we assume that $U_{\text{PMNS}}V_{\text{CKM}} = U_{\text{bimax}}$, we can get the PMNS matrix with the bimaximal matrix and the CKM matrix, all the elements of the PMNS matrix can be expressed by the parameters of the CKM matrix. The QLC is satisfied perfectly, and Raidal’s relations can be deduced naturally (the deviation from Raidal’s relations is of the order of $\lambda^{2}$).

Case 2: $V_{\text{CKM}}U_{\text{PMNS}} = U_{\text{bimax}}$

This relation has been pointed out by Giunti and Tanimoto [19] and discussed by some other authors [20][21]. Giunti and Tanimoto [19] suggested that the deviation of $U_{\text{PMNS}}$ from the bimaximal mixing matrix is the CKM-like matrix, and Kang, Kim, and Lee [21] got this relation under the assumptions $Y_{u} = Y_{d}^{T}$, $Y_{u} = Y_{u}^{T}$ in SU(5) and $Y_{l} = Y_{u}$ in SO(10) grand unified theories.

Repeating the previous process, we can get the PMNS matrix as

$$U_{\text{PMNS}} = V_{\text{CKM}}^{\dagger}U_{\text{bimax}}$$

$$= \left( \begin{array}{ccc}
\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\
-\frac{1}{2} & \frac{1}{2} & \frac{\sqrt{2}}{2} \\
\frac{1}{2} & -\frac{1}{2} & \frac{\sqrt{2}}{2}
\end{array} \right) + \lambda \left( \begin{array}{ccc}
\frac{1}{2} & -\frac{1}{2} & -\frac{\sqrt{2}}{2} \\
0 & 0 & 0 \\
\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0
\end{array} \right) + \lambda^{2} \left( \begin{array}{ccc}
-\frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & 0 \\
\frac{1}{4} - \frac{1}{4} A & \frac{1}{4} + \frac{1}{4} A & -\frac{\sqrt{2}}{2} \\
\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & A
\end{array} \right) + \lambda^{3} \left( \begin{array}{ccc}
\frac{1}{2} A(1 - \rho + i\eta) & \frac{1}{2} A(1 - \rho + i\eta) & \frac{\sqrt{2}}{2} A(1 - \rho + i\eta) \\
0 & 0 & 0 \\
\frac{\sqrt{2}}{2} A(\rho + i\eta) & \frac{\sqrt{2}}{2} A(\rho + i\eta) & 0
\end{array} \right) + \cdots. \quad (33)$$

We can see that the leading term in Eq. (33) is the bimaximal mixing pattern as that in Eq. (17) and Eq. (20). However, from the next-to-leading term, there are differences between Eq. (33) and Eq. (17) and Eq. (20). This indicates that the degree of the breaking of Raidal’s relations (Eq. (11)) is larger than that of Case 1.

Similarly, we can get all the six the trigonometric functions of the mixing angles of leptons.

From Eq. (33), we have (to the order of $\lambda^{3}$)

$$s_{1}^{\text{PMNS}} = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}(A + \frac{1}{4})\lambda^{2},$$

$$c_{1}^{\text{PMNS}} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}(A + \frac{1}{4})\lambda^{2},$$
Comparing with Eq. (31) and Eq. (32), we can see that the difference is caused by the order of the product. If we \( U \) and thus \( \lambda = 0 \), so we can get

\[
|s_2^{PMNS}| = \frac{\sqrt{2}}{2} \lambda \sqrt{[\lambda^2(1 - \rho) - 1]^2 + (\lambda^2\rho)^2} = 0.68\lambda,
\]

\[
e_2^{PMNS} = 1 - 0.23\lambda^2,
\]

\[
s_3^{PMNS} = \frac{\sqrt{2}}{2} - \lambda - \frac{\sqrt{2}}{2} \lambda^2 + (A + \frac{1}{2})\lambda^3,
\]

\[
e_3^{PMNS} = \frac{\sqrt{2}}{2} + \lambda - \frac{\sqrt{2}}{2} \lambda^2 - (A + \frac{1}{2})\lambda^3.
\]  

(34)

And we can get the sums of mixing angles of quarks and leptons.

\[
\sin(\theta_1^{CKM} + \theta_1^{PMNS}) = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{8} \lambda^2,
\]

and thus

\[
\theta_1^{CKM} + \theta_1^{PMNS} = \frac{\pi}{4} - \frac{1}{4} \lambda^2.
\]  

(35)

And

\[
\sin(\theta_3^{CKM} + \theta_3^{PMNS}) = \frac{\sqrt{2}}{2} + (\frac{\sqrt{2}}{2} - 1)\lambda
\]

\[
+ (1 - 3\frac{\sqrt{2}}{4})\lambda^2 + (A + 1 - \frac{\sqrt{2}}{2})\lambda^3,
\]

and thus

\[
\theta_3^{CKM} + \theta_3^{PMNS} = \frac{\pi}{4} - (\sqrt{2} - 1)\lambda
\]

\[
+ (\sqrt{2} - \frac{3}{2})\lambda^2 + (\sqrt{2}A + \sqrt{2} - 1)\lambda^3.
\]  

(36)

We can see from Eq. (33) and Eq. (36) that both of the Raidal’s relations break down, and the QLC is broken to the order of \( \lambda \). This breaking has been pointed out by Minakata and Smirnov \( \text{[8]} \) and Kang, Kim, and Lee \( \text{[21]} \). Comparing with Eq. (33) and Eq. (32), we can see that the difference is caused by the order of the product. If we set \( V_{CKM}U_{PMNS} = U_{bimax} \), the deviations from Raidal’s relations are larger than the results if we set \( U_{PMNS}V_{CKM} = U_{bimax} \).

Also, from Eq. (31), we know \( |s_2^{PMNS}| = 0.68\lambda \), so we can get \( |U_{e3}^{PMNS}| = 0.68\lambda \). Substituting the best fit value \( \lambda = 0.2243 \) \( \text{[4]} \) into it, we have

\[
|U_{e3}^{PMNS}| = 0.15.
\]

This value is quite near the upper bound of \( |U_{e3}^{PMNS}| < 0.20 \). However, from Eq. (20), we know \( |s_2^{PMNS}| = 0.48\lambda^2 \), so we can get

\[
|U_{e3}^{PMNS}| = 0.48\lambda^2 = 0.024.
\]

We can see that this result is more consistent with the current experimental upper bound.

From the discussions above, we can see that there are the non-equivalence of Eq. (9) or Eq. (10) and Raidal’s numerical relations of the mixing angles, which means that we can not get Raidal’s numerical relations of the mixing angles exactly from Eq. (9) or Eq. (10), and vice versa. There are small deviations from the exact Raidal’s numerical relations of the mixing angles if we take Eq. (9) or Eq. (10) as precise results. (For example, see Eq. (32) and Eq. (36).)

Furthermore, we find that the product \( U_{PMNS}V_{CKM} = U_{bimax} \) is better than \( V_{CKM}U_{PMNS} = U_{bimax} \) from the viewpoints of both symmetric and phenomenological considerations. Of course, if the deviation of the PMNS matrix from the bimaxiaml mixing matrix is not exactly the CKM matrix, but is just the CKM-like matrix \( \text{[19,20,21]} \) (i.e., the elements of the matrix have the same hierarchy as the Wolfenstein parametrization, but with not exactly the same Wolfenstein parameters), Eq. (10) may still be satisfied. The two different cases can be further discriminated by future experiments.

If the relation \( U_{PMNS}V_{CKM} = U_{bimax} \) is supported by the future experimental data, using Eq. (4) and Eq. (6) we have

\[
U_{PMNS}V_{CKM} = U_{l1}^\dagger U_{\nu} V_{u1}^\dagger V_{d} = U_{bimax}.
\]
However, we know that $U_l$, $U_\nu$, $V_u$ and $V_d$ are not definite, and we can set $U_l$ and $V_d$ to be the unit matrix by redefining the quark and lepton fields, and thus we have

$$U_{\text{PMNS}} = U_\nu, \quad V_{\text{CKM}} = V_u^\dagger,$$

and thus

$$U_{\text{PMNS}} V_{\text{CKM}} = U_\nu V_u^\dagger = U_{\text{bimax}}.$$

So we can find that the relation between the CKM and the PMNS matrices can be transformed to the relation between $V_u$ and $U_\nu$, and we may regard this the complementarity of quark and lepton mixing matrices.

6 Conclusions

In this paper, we explore the relations between the mixing angles and mixing matrices of quarks and leptons. For the mixing angles, with Raidal’s relations, we can link the mixing angles of quarks and leptons in a same framework, and then express their mixing matrices in a unified way, i.e., we can parameterize the PMNS matrix with the Wolfenstein parameters of the CKM matrix [4]. With this unified parametrization, we discuss the relations between the quark and lepton mixing matrices. Both $V_{\text{CKM}} U_{\text{PMNS}}$ and $U_{\text{PMNS}} V_{\text{CKM}}$ are calculated in detail, and we can find that $U_{\text{PMNS}} V_{\text{CKM}}$ is more closer to the bimaximal mixing matrix than $V_{\text{CKM}} U_{\text{PMNS}}$.

Similarly, for the relation between the quark and lepton mixing matrices, if we have $V_{\text{CKM}} U_{\text{PMNS}} = U_{\text{bimax}}$, we can find that Raidal’s relations will violate, especially the elegant quark-lepton complementarity (QLC) will break (the degree of breaking is of order $\lambda^2$), and the QLC will be a precise relation exactly. Although $U_{\text{PMNS}} V_{\text{CKM}} = U_{\text{bimax}}$ is still a phenomenological suggestion, it is consistent with the experimental data and is supported by the analysis in Sec. 4. Future experimental discrimination between the two different cases of $V_{\text{CKM}} U_{\text{PMNS}} = U_{\text{bimax}}$ or $U_{\text{PMNS}} V_{\text{CKM}} = U_{\text{bimax}}$, will shed light on our understanding of the relation between the quark and lepton mixing matrices, and will be also helpful for the future model construction of the quark and lepton mixing matrices in a grand unified theory.

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