Learning by random walks in the weight space of the Ising perceptron

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Abstract. Several variants of a stochastic local search process for constructing the synaptic weights of an Ising perceptron are studied. In this process, binary patterns are sequentially presented to the Ising perceptron and are then learned as the synaptic weight configuration is modified through a chain of single- or double-weight flips within the compatible weight configuration space of the earlier learned patterns. This process is able to reach a storage capacity of $\alpha \approx 0.63$ for pattern length $N = 101$ and $\alpha \approx 0.41$ for $N = 1001$. If in addition a relearning process is exploited, the learning performance is further improved to a storage capacity of $\alpha \approx 0.80$ for $N = 101$ and $\alpha \approx 0.42$ for $N = 1001$. We found that, for a given learning task, the solutions constructed by the random walk learning process are separated by a typical Hamming distance, which decreases with the constraint density $\alpha$ of the learning task; at a fixed value of $\alpha$, the width of the Hamming distance distribution decreases with $N$.

Keywords: disordered systems (theory), neuronal networks (theory), analysis of algorithms, stochastic search

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1. Introduction

A single-layered feed-forward network of neurons, referred to as a perceptron, is an elementary building block of complex neural networks. It is also one of the basic structures for learning and memory [1]. In a perceptron, \( N \) input neurons (units) are connected to a single output unit by synapses of continuous or discrete-valued synaptic weights. The learning task is to set the weight values for these \( N \) synapses such that an extensive number \( M = \alpha N \) of input patterns are correctly classified (see figure 1(a)). The parameter \( \alpha \equiv M/N \) is called the constraint density. An assignment of these weights is referred to as a solution if the perceptron correctly classifies all the input patterns with this weight assignment. Compared with perceptrons with real-valued synaptic weights, Ising perceptrons, whose synaptic weights are binary, are much simpler for large-scale electronic implementations and more robust against noise. An Ising perceptron is also relevant in real neural systems, as the synaptic weight between two neurons actually takes bounded values and has a limited number of synaptic states [2, 3]. On the other hand, training a real-valued perceptron is easy (e.g., the Minover algorithm [4] and the AdaTron algorithm [5]) but training an Ising perceptron is known to be an NP-complete problem [6]. Given \( \alpha N \) input patterns, the computation time needed to find a solution may grow exponentially with the number of weights \( N \) in the worst case. A complete enumeration of all possible weight states is only feasible for small systems up to \( N = 25 \) [7]–[10]. In recent years research on efficient heuristic algorithms has been rather active [6], [11]–[17].

If the number \( M \) of input patterns is too large, a perceptron will be unable to correctly classify all of them, no matter how the synaptic weights are modified. This is a phase transition phenomenon of the solution space of the perceptron. In the case that the \( M \) input binary patterns are sampled uniformly and randomly from the set of all binary patterns, the maximal value \( \alpha_s \) of the constraint density \( \alpha \), the storage capacity at which a solution still exists, has been calculated by statistical physics methods. For the continuous perceptron subject to the spherical constraint, Gardner and Derrida found that \( \alpha_s = 2 \) [18, 19]. At the thermodynamic limit of \( N \to \infty \), the continuous perceptron is impossible to correctly classify more than \( 2N \) random input patterns. When the synaptic weight is restricted to binary values, \( \alpha_s \) was predicted to be 0.83 by Krauth and Mézard.
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Figure 1. The sketch of the Ising perceptron and the single-weight random walking process in the corresponding weight space. (a) $N$ input units (open circles) feed directly to a single output unit (solid circle). A binary input pattern $(\xi_1^\mu, \xi_2^\mu, \ldots, \xi_N^\mu)$ of length $N$ is mapped through a sign function to a binary output $\sigma^\mu$, i.e., $\sigma^\mu = \text{sgn} \left( \sum_{i=1}^{N} J_i \xi_i^\mu \right)$. The set of $N$ binary synaptic weights $\{J_i\}$ is regarded as a solution of the perceptron problem if the output $\sigma^\mu = \sigma_0^\mu$ for each of the $M = \alpha N$ input patterns $\mu \in [1, M]$, where $\sigma_0^\mu$ is a preset binary value. (b) A solution space random walking path (indicated by arrows). An open circle represents a configuration that satisfies the first $m + 1$ input patterns, while a black circle and a gray circle represents, respectively, a configuration that satisfies the first $m$ and the first $m - 1$ input patterns. An edge between two configurations means that these two configurations are related by a single-weight flip.

using the first-step replica-symmetry broken spin-glass theory [20]. This prediction was confirmed by numerical simulations of small size systems (plus an extrapolation to large $N$) [7, 9, 21].

The theoretically predicted storage capacity $\alpha_s$ represents the upper limit of achievable constraint density $\alpha$ by any learning strategies. As the constraint density $\alpha$ increases, it is expected that the solution space of the Ising perceptron breaks into a huge number of disjoint ergodic components [22]. Solutions from different components are significantly different. One can define a connected component of the weight space as a cluster of solutions in which any two solutions are connected by a path of successive single-weight flips [23, 24]. These solution clusters are separated by weight configurations that only correctly classify a subset of the input patterns. These partial solutions act as dynamical traps for local search algorithms and make the learning task hard. An adaptive genetic algorithm was suggested by Köhler in 1990, which could reach $\alpha \simeq 0.7$ for systems of $N = 255$ [11]. Simulated annealing techniques were used by Horner [22] but critical slowing down of the search process was observed, due to the very rugged energy landscape of the problem. Simulated annealing was also used to study the statistical structure of the energy landscape for the Ising perceptron. The analysis of the distribution of distances between global minima obtained by simulated annealing for small $\alpha$ indicated that the distance distribution becomes a delta function in the thermodynamic limit [25]. Making use of the advantage that efficient algorithms exist for the real-valued perceptron, an alternative approach was to clip the trained real-valued weights of the continuous perceptron into binary values [13], [26]–[29]. Not all synaptic weights can be correctly specified by clipping, however, and for those uncertain weights, complete enumeration was
then adopted. A message-passing algorithm was developed by Braunstein and Zecchina for the Ising perceptron [15], which was able to reach $\alpha \simeq 0.7$ for $N \geq 1000$. The efficiency of this belief-propagation algorithm was later conjectured to be due to the existence of a sub-exponential number of large solution clusters in the weight space [24]. An on-line learning algorithm inspired from this belief-propagation algorithm was also studied [16], in which hidden discrete internal states are added to the synaptic weights.

In real neural systems, the microscopic mechanism of perceptronal learning is the Hebbian rule of synaptic modification (spiking-time-dependent synaptic plasticity may be exploited, see, e.g., [30,31]). The learning processes in biological perceptronal systems are expected to be much simpler than the various sophisticated learning processes of artificial perceptrons. Two other important aspects of biological perceptron systems are (i) the patterns to be classified are usually read into the system in a sequential order, so they are being learned one by one, and (ii) when a new pattern is being learned, there are biological mechanisms which reactivate old learned patterns; such recalling processes help to prevent old patterns from being forgotten as new patterns are learned (see, e.g., the experimental investigation of [32]). Motivated by these biological considerations, we investigate in this paper a simple sequential learning mechanism, namely synaptic weight space random walking. In this random walking mechanism, the $\alpha N$ patterns are introduced into the system in a randomly permuted sequential order, and random walk of single- or double-weight flips is performed until each newly added pattern is correctly classified (learned). The previously learned patterns are not allowed to be misclassified in later stages of the learning process. We perform extensive numerical simulations on several variants of this simple sequential local learning rule and find that this mechanism has good performance on systems of $N \sim 10^3$ neurons or less.

The paper is organized as follows. The Ising perceptron learning is defined in more detail in section 2. Several strategies of learning by random walks are presented in section 3. In section 4, experimental study of learning algorithms is carried out. The overlap distribution of solutions as well as performances of different local search algorithms is reported. A summary and discussion are given in section 5.

Sequential random walk search algorithms were recently investigated in various combinatorial satisfaction problems (see, e.g., [33]–[35]). The present work adds evidence that the solution space random walking mechanism, although very simple and easy to implement, is able to solve many nontrivial problem instances of a given complex learning or constraint satisfaction problem.

2. The random classification problem

For the Ising perceptron depicted schematically in figure 1(a), $N$ input units are connected to a single output unit by $N$ synapses of weight $J_i = \pm 1$ ($i=1,2,\ldots,N$). The perceptron tries to learn $M = \alpha N$ associations \{\(\xi^\mu, \sigma^\mu_0\)\} ($\mu=1,2,\ldots,M$), where \(\xi^\mu \equiv (\xi_1^\mu, \xi_2^\mu, \ldots, \xi_N^\mu)\) is an input pattern with $\xi_i^\mu = \pm 1$, and $\sigma^\mu_0 = \pm 1$ is the desired classification of the input pattern $\mu$. Given the input pattern $\xi^\mu$, the actual output $\sigma^\mu$ of the perceptron is

\[\sigma^\mu = \text{sgn} \left( \sum_{i=1}^{N} J_i \xi_i^\mu \right).\]
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The perceptron can modify its synaptic weight configuration \( \{ J_i \} \equiv (J_1, J_2, \ldots, J_N) \) to achieve complete classification, i.e., \( \sigma^\mu = \sigma_0^\mu \) for each of the \( M \) input patterns. The solution space of the Ising perceptron is composed of all the weight configurations \( \{ J_i \} \) that satisfy \( \sigma_0^\mu \sum_i J_i \xi_i^\mu > 0 \) for \( \mu = 1, 2, \ldots, M \).

For the random Ising perceptron problem studied in this paper, each of the \( M \) input binary patterns \( \xi^\mu \) is sampled uniformly and randomly from the set of all \( 2^N \) binary patterns of length \( N \), and the classification \( \sigma_0^\mu \) is equal to \( \pm 1 \) with equal probability. For \( N \) sufficiently large, the solution space of such a model system is non-empty as long as \( \alpha < 0.83 \) \cite{20}. To construct such a solution configuration \( \{ J_i \} \), however, is quite a nontrivial task.

A more stringent learning problem is to find a weight configuration \( \{ J_i \} \) such that, for each input pattern \( \xi^\mu \),

\[
\sigma_0^\mu \sum_i J_i \xi_i^\mu \geq \kappa, (2)
\]

where \( \kappa > 0 \) is a preset parameter \cite{20}. The most efficient way of solving this constraint satisfaction problem appears to be the message-passing algorithm of \cite{15,16}.

One can perform a gauge transform of \( \xi_i^\mu \rightarrow \xi_i^\mu \sigma_0^\mu \) to each input pattern. Under this gauge transform, each desired output is transformed to \( \sigma_0^\mu = 1 \). Without loss of generality, in the remaining part of this paper we will assume \( \sigma_0^\mu = 1 \) for any input pattern \( \mu \). Consider the case of \( N \) being odd, we define the stability field of a pattern \( \mu \) as

\[
h^\mu = \sum_{i=1}^N J_i \xi_i^\mu. (3)
\]

To ensure the local stability of input pattern \( \mu \) under changes of weight configuration \( \{ J_i \} \), in analogy to equation (2), we introduce a stability parameter \( \Delta \geq 1 \) and require that \( h^\mu \geq \Delta \) for each \( \mu \). Input patterns with \( h^\mu \geq 3 \) are stable against a single-weight flip. For the single-weight flipping processes of the next section, the input patterns with \( h^\mu = 1 \) are referred to as barely learned patterns, as these patterns may become misclassified after the weight configuration makes a single flip. Similarly, for the double-weight flipping process of the next section, the input patterns with \( h^\mu = 1 \) or \( h^\mu = 3 \) are referred to as barely learned patterns.

3. Learning by random walks

Random walk processes were used in a series of works \cite{33,34}, \cite{36}–\cite{39} to find solutions for constraint satisfaction problems. They were also used as tools to study the solution space structure of these constraint satisfaction problems \cite{33,34,40}. Various local search strategies have been developed to improve the performance of random walk stochastic searching \cite{41}–\cite{43}.

The random walk learning strategies of this work follow the SEQSAT algorithm of \cite{34}. An initial weight configuration \( \{ J_1^{(0)}, J_2^{(0)}, \ldots, J_N^{(0)} \} \) is randomly generated at time \( t = 0 \). The first pattern \( \xi^1 \) is applied to the Ising perceptron. If this pattern is correctly classified under the initial weight configuration (i.e., \( h^1 > 0 \)), then the second pattern \( \xi^2 \) is applied; otherwise the weight configuration is adjusted by a sequence of elementary local changes.
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until $\xi^1$ is correctly classified. The algorithm then proceeds with the second pattern $\xi^2$, the third pattern $\xi^3$, etc, in a sequential order. An elementary local change of weight configuration is achieved either by a single-weight flip (SWF) or by a double-weight flip (DWF).

Suppose at time $t$ the weight configuration is $\{J^{(t)}\} \equiv (J_1^{(t)}, J_2^{(t)}, \ldots, J_N^{(t)})$, and suppose this configuration correctly classifies the first $m$ input patterns $\xi^\mu (\mu = 1, \ldots, m)$ but not the $(m+1)$th pattern $\xi^{m+1}$. The configuration $\{J_t\}$ will keep wandering in the solution space of the first $m$ patterns until a configuration that correctly classifies $\xi^{m+1}$ is reached (see figure 1(b)). In the SWF protocol, a set $A(t)$ of allowed single-weight flips is constructed based on the current configuration $\{J^{(t)}\}$ and the $m$ learned patterns. $A(t)$ includes all integer positions $j \in [1, N]$ with the property that the single-weight flip of $J_j^{(t)} \rightarrow -J_j^{(t)}$ does not render any barely learned patterns $\mu \in [1, m]$ (whose $h^\mu = 1$) being misclassified. At time $t' = t + 1/N$ an integer position $j$ is chosen uniformly and randomly from set $A(t)$ and the weight configuration is changed to $\{J^{(t')}\}$ such that $J_i^{(t')} = J_i^{(t)}$ if $i \neq j$ and $J_j^{(t')} = -J_j^{(t)}$. It is obvious that the new configuration $\{J^{(t')}\}$ also satisfies all the first $m$ patterns.

The DWF protocol is very similar to the SWF protocol, with the only difference that the allowed set $A(t)$ at time $t$ contains ordered pairs of integer positions $(i, j)$ with $i < j$. This set of ordered pairs can also be easily constructed. If, with respect to configuration $\{J^{(t)}\}$, there are no barely learned patterns (whose stability field $h^\mu = 1 \text{ or } 3$) among the first $m$ learned patterns, then $A(t)$ contains all the $N(N-1)/2$ ordered pairs of integers $(i, j)$ with $1 \leq i < j \leq N$. Otherwise, randomly choose a barely learned pattern, say $m_1 \in [1, m]$, and for each integer $i \in [1, N]$ with the property that $J_i^{(t)} \xi^m_1 < 0$, do the following: (1) if $J_i^{(t)} \xi^\mu_i < 0$ for all the other barely learned patterns, then add all the ordered pairs $(i, j)$ with $j \in [i + 1, N]$ into the set $A(t)$; (2) otherwise, add all the ordered pairs $(i, j)$ into the set $A(t)$, with the property that the integer $j \in [i + 1, N]$ satisfies $J_j^{(t)} \xi^\mu_j < 0$ for all those barely learned patterns $\mu \in [1, m]$ with $J_i^{(t)} \xi^\mu_i > 0$.

The waiting time $\Delta t_{m+1}$ of satisfying the $(m+1)$th pattern is defined as the total elapsed time from first satisfying the $m$th pattern to first satisfying the $(m+1)$th pattern. And the total time $T_{m+1}$ of satisfying the first $(m+1)$ patterns is simply $T_{m+1} = \sum_{\mu=1}^{m+1} \Delta t_{\mu}$. One time unit corresponds to $N$ elementary local changes of the weight configuration. The random walk searching process stops if all the $M$ input patterns have been correctly classified, or if the last visited weight configuration becomes an isolated point (i.e., the set $A(t)$ becomes empty after a new pattern is included into the set of learned patterns), or if the last waiting time $\Delta t_{m+1}$ exceeds a preset maximal time value $\Delta t_{\text{max}}$, which is equal to $\Delta t_{\text{max}} = 1000$ in the present work.

The SWF and DWF random walk processes as mentioned above are very simple to implement and they do not overcome any barriers in the energy landscape of the perceptron learning problem. However, as we demonstrate in the next section, their performances are quite remarkable for problem instances with pattern length $N \leq 10^3$.

The SWF process, as a local search algorithm, will get stuck in one of the enormous metastable states when all the weights become frozen (here we identify a synaptic weight as being frozen if flipping its value causes at least one of the learned patterns to be misclassified), at a constraint density value much smaller than the theoretical threshold value of 0.83. The DWF process will also get jammed if the weight configuration becomes

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frozen with respect to any double-weight flips. To further improve the achievable storage capacity for the SWF and DWF learning processes, a simple relearning strategy is added to the random walk searching. The basic idea of the relearning strategy is: if some learned patterns are hindering the learning of new patterns very much, we first ignore them and proceed to learn a number of new patterns; after that, we learn the ignored patterns again and hope they can all be correctly classified.

In the present work, we implement the relearning strategy in the following way. Suppose that as the $m$th input pattern is presented to the Ising perceptron, the SWF or the DWF process is unable to learn it in a waiting time $\Delta t_m < \Delta t_{\text{max}}$. We then remove all the $k$ barely learned patterns $\mu \in [1, m - 1]$ with $h^{\mu} = 1$ from the list of learned patterns, and proceed to learn the patterns $\mu \in [m, m + k - 1]$ in a sequential manner (stage 1). If the SWF process or the DWF process succeeds in learning these $k$ patterns, we then return to learn the $k$ previously removed patterns again in a sequential manner (stage 2). If this relearning succeeds, we proceed with the patterns with index $\mu \geq m + k$. If this attempt fails either at stage 1 or at stage 2, we stop the whole random walk learning process or start with another trial by removing all the learned patterns. In practice, we find that the relearning process has a high probability to succeed in both stage 1 and stage 2 if $\alpha$ is not too large and the pattern length is of the order $10^3$ or less.

4. Results

Figure 2 demonstrates the simulation results for several random walk learning strategies. For each learning strategy, $N$ set of random input patterns $(\xi^1, \xi^2, \ldots, \xi^M)$ are generated. Each input pattern $\xi^\mu$ has length $N$. The random walk learning strategy is then applied to each set of patterns until it stops, at which point we record the number of correctly classified patterns $m$ and calculate the achieved storage capacity $\alpha = m/N$. The mean values of $\alpha$ are reported in figure 2. It appears that the storage capacity of all the four learning strategies decreases with $N$ roughly as a power law $\alpha \propto N^{-\gamma}$. At each value of $N$, the SWF strategy has the worst performance, while the DWF strategy with relearning has the best performance.

The SWF strategy is able to reach a storage capacity of $\alpha \approx 0.36$ for systems of $N = 101$ and $\alpha \approx 0.17$ for systems of $N = 1001$. These values are much less than the theoretical storage capacity of $\alpha \approx 0.83$. However, the DWF strategy performs much better, with a capacity of $\alpha \approx 0.63$ for $N = 101$ and $\alpha \approx 0.41$ for $N = 1001$. In real neural systems, perceptronal learning of elementary patterns probably does not involve too many neuronal cells and a value of $N \sim 10^2$ might be common. For perceptronal systems with $N \sim 10^2$–$10^3$, the SWF and DWF strategies can be regarded as efficient. It should be remarked that a multiple-weight flip (e.g., triple-weight flip) is believed to further improve the learning performance, but the computation cost increases correspondingly, especially for large problem sizes, and we do not attempt to adopt this scheme in this work.

If relearning is introduced into the random walk learning strategies, the performance can be further improved. This is thanks to the fact that a finite positive energy fluctuation only involving barely learned patterns is introduced when random walk searching processes get trapped. Once a finite number of barely learned patterns are relearned successfully, the capacity is boosted to a higher value. In our simulations, the capacity can even be increased more than two times; however, if the pattern length becomes large ($>10^3$),

\begin{align*}
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\end{align*}
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Figure 2. Comparison of the performances of several random walk search strategies. The achieved storage capacity $\alpha$ as averaged over many independent runs (100 for the smallest $N$ and 10 for the largest $N$) are shown as a function of the pattern length $N$. The solid lines are power-law fittings of the form $\alpha \propto N^{-\gamma}$, with $\gamma = 0.302, 0.347, 0.198, 0.241$ for SWF, SWF with relearning, DWF and DWF with relearning, respectively.

Relearning fails either at stage 1 or at stage 2 with a high probability and the result is the same as that obtained by the pure SWF or DWF. For the DWF strategy with relearning, we find that the storage capacity is $\alpha \approx 0.80$ for $N = 101$ and $\alpha \approx 0.42$ for $N = 1001$. Relearning is indeed a biologically relevant strategy in perceptronal learning of real neural systems [32,44]. As a comparison, for problem instances of pattern length $N = 1001$, the belief-propagation inspired learning strategy of Baldassi et al [16] achieves $\alpha \approx 0.47$ when the number $K$ of internal states of their algorithm is set to $K = 40$. This storage capacity $\alpha$ decreases to $\alpha \approx 0.36$ at $K = 20$ and to $\alpha \approx 0.10$ at $K = 10$.

For the same set of input patterns $(\xi^1, \xi^2, \ldots, \xi^m)$, different runs of the SWF strategy or the DWF strategy lead to different solution configurations. The similarity between solutions can be measured by an overlap value $q$ as defined by

$$q = \frac{1}{N} \sum_{i=1}^{N} J_i J'_i,$$

where $(J_1, \ldots, J_N)$ and $(J'_1, \ldots, J'_N)$ are two solutions. The reduced Hamming distance $d_H$ between two solutions is related to the overlap $q$ by $d_H = (1 - q)/2$. The typical value of the overlap value at constraint density $\alpha \sim 0.83$ is predicted to be $q \approx 0.56$ according to the replica-symmetric calculation [20], suggesting that solutions are still far away from each other (with a reduced Hamming distance $d_H \approx 0.22$) as $\alpha$ approaches the theoretical storage capacity $\alpha_s$.

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Figure 3 shows the histogram $P(d_H)$ of reduced Hamming distances $d_H$ between different solutions found by the DWF strategy for a single problem instance with constraint density $\alpha$ and pattern length $N$. Different pattern lengths of $N = 101, 501, 1001$ are used, and 100 different solutions are constructed by repeated running of the DWF process. Other problem instances show similar properties. We notice from figure 3 that, at the same value of $\alpha$, the histograms $P(d_H)$ for different $N$ are peaked at almost the same $d_H$ value, but the width of $P(d_H)$ decreases as $N$ is enlarged. Actually the reduced Hamming distance is a self-averaging quantity and therefore the corresponding distribution is expected to exhibit less and less fluctuations around the typical value as the problem size increases. Such a behavior was observed earlier in [25] on a slightly modified Ising perceptron problem. The solutions obtained by the DWF strategy therefore have a typical level of similarity. Figure 3 also demonstrates that, as the constraint density increases, the histograms $P(d_H)$ shift to smaller $d_H$ values, suggesting that the level of similarity between the DWF-constructed solutions increases with $\alpha$. At $\alpha = 0.693$ the typical reduced Hamming distance is $d_H \approx 0.224$, compatible with the mean-field predictions [20]. Similar results are obtained for solutions found by the SWF strategy. In all our simulations, we do not observe double or multiple peaks for the histogram $P(d_H)$. The results of these and our other numerical simulations (not shown) are consistent with the proposal that, for a given problem instance, the solutions obtained by the random walking strategies are members of the same (large) solution cluster of the solution space [24, 25, 45]. Unlike the random $K$-satisfiability problem, the random $Q$-coloring problem, or some locked constraint satisfaction problems [46]–[48], the solution space organization of the Ising
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Figure 4. The learning time $T_{\alpha N}$ as a function of $\alpha$ for three problem instances of $N = 1001$.

The Ising perceptron problem is still not very clear. Kabashima et al [24] suggested that for $\alpha < 0.83$ the solution space of the Ising perceptron problem is equally dominated by exponentially many clusters of vanishing entropy and a sub-exponential number of large clusters. Our simulation results are compatible with this proposal, but more work needs to be done to clarify the solution space structure of the random Ising perceptron problem.

The total time $T_{\alpha N}$ used by the DWF strategy to correctly classify the first $\alpha N$ patterns for a problem instance with $N = 1001$ is shown in figure 4 as a function of $\alpha$. The learning time grows almost linearly with $\alpha$ for $\alpha < 0.4$. As the constraint density $\alpha$ becomes large, different small connected solution components are expected to form in the solution space. Then the time needed for the random walk process to reach a solution may exceed the preset maximal waiting time of $\Delta t^{\text{max}} = 1000$ and the DWF process will then stop. If the DWF gets trapped in a small connected solution component where all configurations can not learn the current presented pattern, then the waiting time will saturate the preset maximal value. Note that figure 4 corresponds to a special case where the small connected component trapping DWF contains only one configuration which can not be moved to other configurations by flipping any pair of weights. The achieved storage capacity $\alpha$ can be increased to some extent if we make $\Delta t^{\text{max}}$ larger, but the search process will become more and more viscous as the solution space of the problem becomes more and more heterogeneous and complex [34]. We do not attempt to calculate the jamming point of the random walk searching processes [47].

5. Discussion

We proposed several stochastic learning strategies for the Ising perceptron problem based on the idea of solution space random walking [34]. Our simulation results in figure 2 doi:10.1088/1742-5468/2010/08/P08014
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demonstrated that the DWF strategy is able to correctly classify $\geq 0.4N$ random input patterns of length $N$ for $N \leq 1001$. If a simple relearning strategy is added to the DWF strategy, the learning performance is further improved. The learning time of the DWF strategy grows roughly linearly with the number of input patterns. This work suggested that learning by local and random changes of synaptic weights is efficient for perceptronal systems with $N \approx 10^2$–$10^3$ neurons. These local sequential learning strategies may be exploited in some biological perceptronal systems. In real neuronal systems, the number $N$ of involved neurons in an elementary pattern classification task may be of the order of $N \sim 10^1$–$10^3$.

The solutions obtained by the DWF strategy for a given perceptronal learning task are separated by a typical Hamming distance, which reduces as the number of input patterns increases (figure 3). However, solutions are still far away from each other, even near to the critical capacity. We suspected that for the problem instances studied in this paper, either the solution space of the problems is ergodic as a whole, or the solutions reached by the DWF strategies all belong to the same solution cluster of the solution space. In our random walking setting, once all weights are frozen, particularly for SWF, the current pattern with negative stability field will be no longer learned since the current weight configuration is isolated in the weight space (this weight configuration is denoted as the completely frozen solution); fortunately, DWF is able to go on even if all weights are frozen, since DWF incorporates the pairwise correlation among synaptic weights and therefore flipping certain pairs of weights is still permitted from the configuration where each single weight is not allowed to be flipped. If these flippable pairs of weights do not exist, DWF will get trapped, and the configuration is isolated once again. Actually, as the constraint density $\alpha$ increases, many such isolated solutions will show up, and SWF or DWF working by single- or double-weight flips is not capable of crossing energy barriers separating the isolated solutions from those connected ones, which can be bypassed to some extent using the relearning strategy which helps to escape from these small clusters and makes SWF or DWF keep on exploring the large cluster composed of exponentially many solutions. For small $\alpha$, a replica-symmetric ansatz is believed to give a good description of the solution space of the Ising perceptron [25]. Up to $\alpha_s$, point-like clusters will form, and searching for the compatible weights becomes more difficult [48]. It is desirable to have a theoretical understanding of the structural evolution of the solution space of the random Ising perceptron problem. How the dynamics of stochastic local search algorithms is influenced by the solution space structure of the random Ising perceptron is an important open issue.

In this work, we explore the weight space of the Ising perceptron using only single- and double-weight flips at the problem size of order $N \sim 10^1$–$10^3$. The learning performance is expected to be further improved if a multiple-weight flip (e.g., triple-weight flip) is adopted, despite the increasing computation time. However, the capacity decreases with increasing problem size, and yet DWF shows a lower slope than SWF (see figure 2). The large $N$ behavior of the algorithm and the scaling exponents $\gamma$ as a function of the number of weights selected for an elementary local change require further investigation and theoretical understanding.

Another interesting problem is the generalization problem where the input–output associations are no longer uncorrelated but the desired outputs are given by a teacher perceptron [17], [49]–[51]. The student perceptron tries to learn the rule provided by the
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teacher. After a sufficient number of examples is presented to the student perceptron, the student’s weights should match those of the teacher; at this point the network undergoes a first-order transition from poor to perfect generalization [49, 50]. It is worthwhile to extend the current random walk strategies to analyze the generalization problem in Ising perceptrons.

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