Nonperturbative fluctuations and metastability in a simple model: from observables to microscopic theory and back

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Abstract. Slow dynamics in glassy systems is often interpreted as due to thermally activated events between ‘metastable’ states. This emphasises the role of nonperturbative fluctuations, which is especially dramatic when these fluctuations destroy a putative phase transition predicted at the mean-field level. To gain insight into such hard problems, we consider the implementation of a generic back-and-forth process, between microscopic theory and observable behaviour via effective theories, in a toy model that is simple enough to allow for a thorough investigation: the one-dimensional $\varphi^4$ theory at low temperature. We consider two ways of restricting the extent of the fluctuations, which both lead to a nonconvex effective potential (or free energy): either through a finite-size system or by means of a running infrared cutoff within the nonperturbative renormalisation group formalism. We discuss the physical insight one can get and the ways to treat strongly nonperturbative fluctuations in this context.

Keywords: finite-size scaling, renormalisation group, metastable states
I. Introduction

Glass-forming liquids are systems whose salient physical properties appear controlled by ‘nonperturbative’ phenomena. At least in the deeply supercooled regime, when approaching the glass transition, the dynamics is best described as activated and cooperative [1–3]. The presence of such thermally activated processes is a prototypical example where the role of the fluctuations must be treated in a nonperturbative way. This is already well known in the case of standard homogeneous nucleation, for instance when a supersaturated vapour transforms into a stable liquid. The theoretical
treatment of this problem involves rare, localised events, which are described as nucleation droplets in a phenomenological approach [4] and instantonic solutions of some free-energy functional at a field-theoretical level [5].

Activation is connected to ‘metastability’. In simple cases, the starting point of a theoretical description is a mean-field Landau free-energy functional, or a classical effective action in quantum field theory, which has several minima and is therefore non-convex. This nonconvexity results from the absence of fluctuations in the mean-field description, and introducing fluctuations in the theory leads to the exact free-energy or effective action with the needed convexity property. At the mean-field level, the deepest minimum is the stable thermodynamic state and the higher free-energy minima the metastable states (or false vacua in quantum field theory). The return to convexity has been theoretically described by explicitly accounting for excitations that are nonuniform in space, e.g. spin waves for systems with a continuous symmetry and droplets for Ising-like models, which encode the nonperturbative nature of the phenomenon. In the Ising-like case, the nucleation (and growth) of a droplet then describes the escape of the system from a metastable state to reach the stable state and leads to activated dynamics [5].

In more involved situations encountered in systems undergoing a so-called fluctuation-induced first-order transition [6, 7], the mean-field theory does not provide a proper starting point as the relevant metastable states are themselves generated by fluctuations. One must therefore find a way to include fluctuations with wavelengths up to some finite scale in order to produce metastability and then study the localised excitations of such an effective theory [8, 9].

The difficulty in the case of glass-forming systems is even stronger. The nature of metastability and of the metastable states is much more elusive [10–12] and the effective or coarse-grained landscape of minima and saddle-points is expected to be very complex, with a number of minima that is exponentially large in the system size. In this case, the ‘bottom–up’ approach, deriving the behaviour of macroscopic observables starting from the microscopic theory, i.e. interacting particles in the continuum, is just too difficult. Even computer simulations are of limited use for providing a full resolution because of the very fast growth of the equilibration time as one approaches the glass transition.

A reasonable starting point would then be an effective theory that encodes the main physical ingredients while leaving out the inessential ones. From a renormalisation group perspective, one would like to integrate out short length-scale fluctuations to obtain an effective theory that governs the long-range cooperative fluctuations. Numerical simulations can be particularly useful in this respect since they take into account, virtually exactly but for systems of limited size, all fluctuations. Provided one has some intuition about the nature of the effective theory and of the associated local order parameter, one can then try to extract the parameters of this theory from the exactly computed behaviour of finite-size systems. This ‘top–down’ approach would be instrumental in validating and establishing the proper effective theory.

Recently, there have been several numerical studies aimed at measuring the so-called Franz–Parisi potential $V(q)$ [13] in models of glass-forming liquids [14–17]. This potential plays the role of a Landau free-energy where the order parameter is taken as the similarity or overlap between liquid configurations. It is at the root of recent
approaches that map the physics of supercooled liquids on effective theories of the random-field and random-bond Ising type [18–22]. Studying for an ensemble of reference liquid configurations the average value of \( V(q) \) and its fluctuations in finite-size systems should provide a way to access some of the parameters of the effective magnetic-like theory, as discussed in [22]. Once the parameters are known, predictions of the effective theory on long length scales can be further checked against other numerical or experimental observations.

We think that this back-and-forth process, between microscopic theory and observable behaviour via effective theories, is a key to solving the glass transition problem. It, however, requires a better understanding of the role played by nonperturbative fluctuations and the development of a theoretical approach able to capture them at all scales. The aim of this work is to study this problem on a toy model that is simple enough to be thoroughly investigated, and to use this as a benchmark for future work on glassy systems.

II. Model and main issues

The model that we focus on is the one-dimensional \( \varphi^4 \) scalar field theory defined by the Hamiltonian

\[
H_L[\varphi] = \int_0^L dx \left[ \frac{c}{2} (\partial_x \varphi)^2 + V(\varphi(x)) \right]
\]

where \( L \) is the system size, \( c > 0 \), and the local potential \( V(\varphi) \) has a double-well form:

\[
V(\varphi) = \frac{r}{4} (\varphi^2 - 1)^2
\]

with \( r > 0 \). We are interested in the low-temperature regime. There, the physics of the model is governed by strong nonperturbative fluctuations: these are kinks or domain walls between the positively magnetised and negatively magnetised phases (to use the language of magnetic systems). These spatially localised defects have a finite cost and their density is always finite, albeit very small at low temperature (it follows a Boltzmann distribution). However, by redistributing the positions of these kinks the system can gain entropy. It is therefore their presence that destroys the phase transition which is predicted at the mean-field level and remains in a perturbative treatment.

Even though the present analysis is motivated by the simplicity of the model, which allows a detailed study, this is more than an academic problem. The one-dimensional \( \varphi^4 \) field theory is actually central to several fields in statistical physics and condensed-matter physics where it appears \textit{mutatis mutandis} in quite different problems: a Langevin dynamics in a double well [23], quantum double wells [24–28], quantum-impurity problems [29, 30] are all different incarnations of this very same model (with sometimes extra difficulties and decorations).

Let us now illustrate the main issues we are going to address in this work.
The first issue concerns what we called the ‘top–down’ approach. Mirroring the current situation in glasses, the problem we are considering is one in which we want to infer the parameters of a theory that we conjecture to be of the Ising/\phi^4 type from the results of simulations and to further check that the effective microscopic theory we have in mind is the correct one. The input from simulations and other essentially exact computations that we consider as available knowledge is the probability of observing a given average value \phi of the field in a system of finite size L, or, more precisely, its logarithm,

\[ U_L(\phi) = -\frac{1}{\beta L} \ln P_L(\phi) \]  

where \( P_L(\phi) \) is the probability density to observe \( \frac{1}{L} \int dx \phi(x) \) equal to \( \phi \) in a system of size \( L \) with periodic boundary conditions and \( \beta = 1/(k_B T) \). We will call \( U_L(\phi) \) the finite-size effective potential since it takes into account all fluctuations exactly, up to the length-scale \( L \). In the \( L \rightarrow 0 \) limit it coincides with the bare potential \( V(\phi) \), whereas in the thermodynamic limit it is equal to the exact effective potential (Gibbs free energy) as a function of \( \phi \). This function \( U_L(\phi) \) is also known in quantum field theory as the ‘constraint effective potential’ [31].

Of course, in the case of the one-dimensional \( \phi^4 \) field theory we know from the start that the proper effective theory is just that given in equation (1). Nevertheless, the problem of inferring the bare parameters of the theory, \( c, r \), and the energy of a kink from the behaviour of finite-size systems is not straightforward. Moreover, understanding in detail the evolution of \( U_L(\phi) \), i.e. how the change in shape of the finite-size effective potential is related to the progressive integration of nonperturbative fluctuations is also very instructive. The knowledge gained in the case of this simple problem will likely be useful for tackling more difficult and still unsolved ones.

The second issue is the development of a ‘bottom–up’ approach that progressively takes into account fluctuations, including the nonperturbative ones, and allows one to eventually describe the macroscopic behaviour. As explained before, numerical simulations are not helpful in this respect since by construction they can be performed on finite-size systems only. This applies more specifically to glassy systems where the time scales needed to relax large systems close to the glass transition are unreachable even with the best available computers. Extrapolations to obtain the thermodynamic limit are then often dangerous and quite unrealistic. Needless to say this is of course not true for the one-dimensional theory studied here. But as already stressed, the model is nonetheless used as a benchmark.

The theoretical method of choice for progressively bridging the gap from microscopic to macroscopic physics is the renormalisation group (RG) [32]. The perturbative RG has been fully developed and understood since the 1970s and 1980s. The nonperturbative RG, on the other hand, has been the focus of intense research only since the 1990s: for reviews, see [9, 33]. (At this point, we should acknowledge that there is always an ambiguity when using the adjectives ‘perturbative’ and ‘nonperturbative’. The former usually refers to an expansion in a few coupling constants and/or an expansion around the mean-field (Gaussian) theory in powers of the difference between the spatial dimension and the upper critical dimension. The nonperturbative RG avoids such expansions.
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and is based on quite different approximation schemes that can potentially describe strong-coupling physics and, a key point for us here, the effect of nonperturbative fluctuations.

The nonperturbative (NP) RG method that is currently more used is the one introduced by Wetterich [9, 35]. It has been successfully applied to a variety of problems in high and low-energy physics [9, 33]. In the field of statistical physics it has led to nontrivial solutions of long-standing problems in frustrated magnets [36], disordered systems such as the random-field Ising model [37, 38] and out-of-equilibrium dynamical phenomena [39]. The NPRG starts from an exact flow equation for the running effective action, \( \Gamma_k[\phi] \), which is essentially the Legendre transform of the free-energy functional computed for an infinite system in which only fluctuations on length-scales less than \( 1/k \) have been integrated out. This running effective action at (length) scale \( 1/k \) coincides in the ultraviolet or microscopic limit, \( k \to \Lambda \) (where \( 1/\Lambda \) is the microscopic length scale as, e.g. a lattice spacing), with the bare Hamiltonian and in the infrared or macroscopic limit, \( k \to 0 \), with the exact effective action (Gibbs free energy) as a function of the field \( \phi(x) \).

The success of this NPRG relies on determining a simple yet rich enough truncation of the exact NPRG equation. There are cases in which this strategy has been instrumental in tackling problems in which nonperturbative fluctuations are present: the XY model in 2 dimensions [40], the random-field Ising model [37, 38], or the return to convexity in the case of a first-order transition [9, 41, 42]. However, the case we are considering in this work is somehow more difficult. There are strong nonlinear effects due to the interplay between the sharp changes in the field value on small length scales, which are associated with the kinks, and the long-wavelength field variations on a scale of the order of the distance between the kinks. We find that this one-dimensional physics, which one of course knows how to solve by other techniques, is harder to access via the NPRG and remains an unsolved problem within this approach. We will comment in the Conclusion about the (more favourable) situation in higher dimensions.

In both of the above situations, i.e. either in a system of finite size \( L \) or within the NPRG in a system in the thermodynamic limit but in the presence of an infrared cutoff on fluctuations of wavelengths larger than \( 1/k \), the fluctuations are limited. As a result, the relevant potential, be it the finite-size one \( U_L(\phi) \) or the running effective one \( U_k(\phi) = \Gamma_k(\phi)/(\beta L) \), need not be convex. Just like in the mean-field limit where no fluctuations are taken into account, which in the present case leads to a Landau potential equal to the bare \( V(\phi) \), metastability can thus be present. As the length scale over which fluctuations are allowed increases, i.e. with increasing \( L \) or decreasing \( k \), the metastability should become less pronounced and in the macroscopic limit, \( L \to \infty \) or \( k \to 0 \), both \( U_L(\phi) \) and \( U_k(\phi) \) should converge to the convex exact effective potential.

The typical evolution with \( L \) of \( U_L(\phi) \) is shown in figure 1 and that of the running effective potential \( U_k(\phi) \) is plotted in figure 2. (In the latter case, we obtained the result by using the so-called local potential approximation (LPA) [9] of the exact NPRG equation.) The progressive disappearance of metastability is clearly observed in the two

\[ \text{In fact, the Wilson-Fisher } \epsilon \text{-expansion is perturbative both in the coupling constant and in the distance from the upper critical dimension. There are cases studied in the context of disordered systems where the RG is perturbative in the distance from the upper critical dimension but functional, i.e. it retains an infinite number of coupling constants: see, e.g. [34]. In these cases, the } \epsilon \text{-expansion does allow one to take into account nonperturbative fluctuations.} \]
The question we want to address in the former case is as follows: say we are given some numerical data in the form of figure 1; how can one extract information about the corresponding effective theory and its parameters? On the other hand, in the latter case we would like to develop an approximation to the NPRG that is able to reproduce the main features associated with the nonperturbative fluctuations in the present model, for instance the known fact that the curvature of the potential in $\phi = 0$ at low temperature is positive but very small as it behaves asymptotically as $\exp(-\beta S^*)$, where $S^*$ is the energy cost of a kink.

The rest of the paper is organised in two main sections, a first one where we address the top–down approach from finite-size studies and a second one where we discuss the bottom–up one through the NPRG. To avoid disrupting the flow of the presentation, some technical details are relegated to the appendices.

### III. The finite-size effective potential

In this section we study the behaviour of the finite-size effective potential $U_L(\phi)$ and its evolution with the system size $L$. We first describe intuitively the shape of $U_L(\phi)$ and explain the ideas on how to extract the relevant quantities, such as the correlation length $\xi$ and the surface tension $\gamma$ from the evolution of $U_L(\phi)$, by focusing in particular on the behaviour of two quantities: the curvature of $U_L(\phi)$ in $\phi = 0$, $\kappa_L = U''_L(0)$, and the height of the barrier between the potential in $\phi = 0$ and the minima in $\phi_{\text{min}} \approx \pm 1$ (when present), $\Delta_L = U_L(0) - U_L(\phi_{\text{min}})$. We then present detailed analytic results for $U_L(\phi)$ in the limit of zero temperature, which are obtained through the instanton technique, and use them as a benchmark to check the validity of our recipes for extracting the correlation length and the surface tension (see [44] for a general introduction to these two quantities). Finally, we numerically determine the behaviour of $U_L(\phi)$ at finite temperature, i.e. at finite (but large) $\xi$. To this aim, we have combined Monte Carlo (MC) simulations and perturbation expansions based on real-space RG and transfer-matrix treatments. We then apply again the recipes for extracting the temperature dependence.

Figure 1. One-dimensional $\varphi^4$ scalar field theory. Plot of the finite-size effective potential $U_L(\phi)$ as a function of $\phi$ for different values of $L$, as obtained by Monte Carlo simulation: $L = 4, 8, 16, 32, 128$ for model parameters $r = 2, c = 4, \beta = 1$. 

| $L$ | $U_L(\phi)$ |
|-----|-------------|
| 4   | 0.8         |
| 8   | 0.6         |
| 16  | 0.4         |
| 32  | 0.2         |
| 128 | 0.1         |
of $\xi$ and $\gamma$ and compare the output with the direct numerical computation of these quantities.

### III.A. The shape of $U_L(\phi)$ and its evolution with $L$

At any given finite temperature, i.e. at any given finite correlation length $\xi$, if the system size goes to infinity, then the magnetisation distribution goes to a Gaussian centered at $\phi = 0$, due to the central limit theorem, and eventually converges to a Dirac delta function. (We use in this section the language of magnetic systems and call $\phi$ the magnetisation, or, more properly, the magnetisation density.) Thus, in the thermodynamic limit, the finite-size effective potential displays a unique (parabolic) minimum in $\phi = 0$. On the other hand, at any given finite system size, as the temperature goes to zero and the correlation length goes to infinity, the magnetisation goes to either plus or minus one with probability one. For $\xi \gg L$ the finite-size effective potential is given by two symmetric minima centered in $\pm 1$. As a result, a nontrivial distribution of $P_L(\phi)$ and a nontrivial shape of $U_L(\phi)$ arise between the two opposite limits considered above.

In order to figure out intuitively the evolution of $P_L(\phi)$ and $U_L(\phi)$ with the system size, let us first focus on the typical configurations of the field $\varphi(x)$ which, at least at low enough temperature, dominate the Gibbs measure. These are the configurations associated with the ground states of the system, corresponding to constant positive or negative magnetisation profiles $\varphi(x) = \pm 1$, and the lowest excitations above them, involving domain walls (i.e. kinks and anti-kinks), which correspond to instantons that minimise the Hamiltonian and connect positively and negatively magnetised regions. At a low enough temperature (large enough correlation length), the typical configurations of the field are thus well described by regions with almost constant $\pm 1$ magnetisation separated by narrow domain walls. The width $\sigma$ of a domain wall is the typical size of an interface. The energy of a domain wall, $S^*$, is, by definition, proportional to
the microscopic surface tension $\gamma$ of the model, which is defined as the energy cost associated with the creation of an interface between two regions with opposite magnetisation. It is easy to show (see below for more detail) that the typical distance between domain walls is of the order of the correlation length $\xi$, which is proportional to $e^{3S\sigma}$.

If $L$ is smaller than $2\sigma$, no domain walls can be present in the system. (We consider periodic boundary conditions, so that the number of domain walls must be even.) Thus, $P_L(\phi) \approx e^{-\beta V_0(\phi)}$ and $U_L(\phi) \approx V(\phi)$ (see figure 3(a)). Therefore, from equation (2), $\Delta \approx 7/4$ and $\kappa \approx -1 - r$.

For $L > 2\sigma$, but still much smaller than the correlation length $\xi$, the probability of finding a domain wall is very small. The typical field configurations are then approximately constant $\pm 1$ magnetisation profiles plus some small thermal fluctuations, whose amplitude depends on $V''(\phi = \pm 1) = 2r$. On the other hand, configurations with zero magnetisation correspond to field profiles with $2n$ domains walls, with $n \in \mathbb{N}^*$, that are suitably placed between 0 and $L$ (obviously, $2\sigma \ll L$). The thermodynamic weight of these configurations is proportional to $e^{-2n\beta S^*}$ and the probability of having $\phi = 0$ is obtained as

$$P_L(\phi = 0) \propto \frac{L}{2} \left[ e^{-2\beta S^*} + \frac{(L - 4\sigma)^2}{8} e^{-4\beta S^*} + \frac{(L - 6\sigma)^4}{192} e^{-6\beta S^*} + \ldots \right],$$

where the terms $(L - 4\sigma)^2/8, (L - 6\sigma)^4/192$ etc, correspond to the combinatorial factors accounting for the number of field configurations with 4, 6, etc, domain walls between 0 and $L$ that have zero magnetisation (see the next section for more detail). As long as $2\sigma \ll L \ll \xi$, all configurations with more than a single kink/anti-kink pair are highly suppressed and their contribution can be neglected. Therefore, $P_L(\phi = 0)$ is dominated by field profiles with only two domains walls. Since all such profiles have the same combinatorial factor (and thus the same probability), independently of the distance between the kink and the anti-kink, all intermediate magnetisation values sufficiently away from $\pm 1$ occur with approximately the same probability. As a result, for $2\sigma \ll L \ll \xi$ the finite-size effective potential $U_L(\phi)$ is given by two deep narrow symmetric minima around $\phi_{\min} \approx \pm 1$ (whose curvature is simply given by $2\sigma/L$) that are separated by a central region where $U_L(\phi)$ is approximatively constant. This is sketched in figure 3(b). The barrier height $\Delta_L = U_L(0) - U_L(\phi_{\min})$ is then given by $2S^*/L$, and the curvature in $\phi = 0$ is $k_L = U''_L(0) \approx 0$.

The qualitative shape of $U_L(\phi)$ does not show any significant change until $L \lesssim \xi$. At this point, the terms of equation (4) corresponding to field configurations with more than two domain walls start to give a significant contribution to $P_L(\phi)$. Since there are exponentially more configurations of the domain walls corresponding to zero magnetisation with respect to configurations yielding positive or negative magnetisation, $P_L(\phi)$ starts to develop a secondary maximum around $\phi = 0$ as a result of this entropic effect. Correspondingly, $U_L(\phi)$ develops a secondary minimum in zero, as sketched in figure 3(c). In this regime the behaviour of the barrier height $\Delta_L$ and of the curvature $k_L$ are model dependent and cannot be determined by simple heuristic arguments: they must be computed in some explicit way, as we do in the following sections.

For $L \approx \xi$ the barrier $\Delta_L$ is expected to disappear as the value of the potential in $\phi = 0$ crosses that in $\phi \approx \pm 1$ (see figure 3(d)). As $L$ further increases the minima in $\pm 1$
become higher and eventually disappear. However, the potential may still remain non-convex, as illustrated in figure 3(e). Full convexity is recovered only for $L \to \infty$. It is then easy to show that the finite-size effective potential coincides with the Gibbs free-energy density (or exact effective potential) $U(\phi)$ of the system:

$$U_L(\phi) = U(\phi) + o\left(\frac{1}{L}\right),$$

where $U(\phi)$ is defined as the Legendre transform of the Helmholtz free energy,

$$U(\phi) = \beta^{-1}f(\beta, h) + h\phi,$$

where $h$ is the external magnetic field and $\langle \varphi \rangle = -\theta f(\beta, h)/\theta(\beta h) = \phi$ (we have again used the magnetic language). As a consequence, for $L \gg \xi$ the finite-size effective potential is a convex function of $\phi$ and presents a unique minimum in $\phi = 0$: see figure 3(f).

In the thermodynamic limit the curvature $\kappa_L$ approaches $\kappa_\infty = U''(0) = \chi^{-1}$, where $\chi$ is the magnetic susceptibility defined as $\chi = \partial(\varphi)/\partial(\beta h)|_{h=0} = L(\langle \varphi^2 \rangle - \langle \varphi \rangle^2)|_{h=0}$.

Based on the arguments discussed above, we can qualitatively determine the behaviour of the quantities of interest for us, $\kappa_L$ and $\Delta_L$, as a function of $L$. They are schematically represented in figure 4. On very short length scales, $L < 2\sigma$, the curvature is negative, $\kappa_L \simeq -\sigma$. Then, for $2\sigma \ll L \ll \xi$, $\kappa_L$ is approximately zero. For $L \ll \xi$, $\kappa_L$ starts to grow and for $L \to \infty$ it approaches $1/\chi$ as $1/L$. In turn, the barrier height $\Delta_L$ behaves roughly as $2\gamma/L$ for $2\sigma \ll L \ll \xi$ and rapidly vanishes for $L \gg \xi$.

We can therefore extract the important physical quantities by focusing on the behaviour of $\kappa_L$ and $\Delta_L$. For instance, one possible recipe is to try to collapse the curves of $\kappa_L$ versus $L$ obtained at different temperatures onto a master curve by rescaling the axes by adjustable parameters. The parameters that provide the best collapse should then be $\chi(T)^{-1}$ for $\kappa_L$ (vertical axis) and $\xi(T)$ for $L$ (horizontal axis). Another possibility would be to plot $L\Delta_L$ as a function of $L$ and, knowing the correlation length $\xi(T)$ from the previous operation, to look for a plateau or a region of weak dependence on $L$ for $L < \xi$: at a low enough temperature, the height of the plateau should then be twice the

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**Figure 3.** Sketch of the evolution of the shape of the finite-size effective potential $U_L(\phi)$ for different system sizes $L$: (a) $L < 2\sigma$; (b) $2\sigma \ll L \ll \xi$; (c) $L \lesssim \xi$; (d) $L \simeq \xi$; (e) $L \gtrsim \xi$; (f) $L \gg \xi$. 

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Surface tension $\gamma(T)$. (Alternatively, one could do a log–log plot as in figure 4(b)). In the next sections we will implement and check these ideas in a quantitative way.

III.B. $U_L(\phi)$ in the $T \to 0 \ (\xi \to \infty)$ limit

As explained above, at very small temperature the Gibbs measure is dominated by the ground state of the system and the lowest excitations above it. The ground states of equation (1) correspond to constant field configurations $\varphi(x) = \pm 1$. The lowest excitations above the ground states correspond to nonuniform kink and anti-kink profiles that are obtained by minimising the Hamiltonian:

$$\left. \frac{\delta H_L[\varphi(x)]}{\delta \varphi(x)} \right|_{\varphi^*} = 0 \Rightarrow c \left. \frac{\partial^2 \varphi^*(x)}{\partial x^2} = \frac{\partial V(\varphi(x))}{\partial \varphi(x)} \right|_{\varphi^*},$$

with the boundary conditions $\varphi^*(x \to -\infty) = \mp 1$ and $\varphi^*(x \to +\infty) = \pm 1$. This differential equation can be solved exactly for the $\varphi^4$ theory in $d = 1$, yielding

$$\varphi^*(x) = \pm \tanh(x/\sigma),$$

with $\sigma = \sqrt{2c/r}$. The energy cost associated with these domain walls can be obtained by plugging equation (8) into equation (1) (momentarily, we have redefined the system from $-L/2$ to $L/2$ and taken the instanton centre far away from the edges); this gives

$$S^* = \int H_L[\varphi^*(x)] \, dx \approx \sqrt{8rc/9}. \text{ Note that we can now make more precise the notion of}$$

Figure 4. Schematic plot of the curvature $\kappa_L = U_L''(0)$ (top) and of the barrier height $\Delta_L = U_L(0) - U_L(\phi_{\text{min}})$ times the system size $L$ (bottom) as a function of $L$. The barrier height is shown on a log–log plot. The labels (a)–(f) correspond to the shapes of $U_L(\phi)$ in figure 3. Note that the behaviour of $L\Delta_L$ at small $L \lesssim 2\sigma$ is not universal and depends on the bare parameters.
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‘low-temperature regime’: it is obtained by letting either $\beta$ or $r$ be large, such that the Boltzmann factor associated with the presence of a domain wall, $\exp(-\beta \sqrt{8rc}/9)$, is much smaller than one.

Small fluctuations in the field around the instantonic profile can be easily taken into account at a Gaussian level. After expanding the Hamiltonian around the instantonic solution up to second order in $\delta \varphi(x) = \varphi(x) - \varphi^*(x)$, the thermodynamic weight of a single instanton is expressed as

$$Z_1 \simeq e^{-\beta S^*} \int D\varphi \ e^{-\frac{\beta}{2} \int dx \ dy \ \frac{\delta^2 H_0}{\delta \varphi(x) \delta \varphi(y)} \mid \delta \varphi(x) \delta \varphi(y)}.$$  \hspace{1cm} (9)

In order to compute the functional integral above one thus need to diagonalise the operator corresponding to the kernel

$$M(x, y) = \beta \left[ -c \frac{\partial^2}{\partial x^2} + V''(\varphi^*(x)) \right] \delta(x - y),$$  \hspace{1cm} (10)

which yields

$$Z_1 \simeq e^{-\beta \tilde{S}^*} = e^{-\beta S^* + \frac{1}{2} \ln(2\pi/\det M)}.$$  \hspace{1cm} (11)

In the present low-temperature limit, it can be shown, following the arguments of, e.g. \cite{46}, that $\det M \sim \beta r$ and, thus, $\tilde{S}^* \simeq S^* - (1/2) \beta^{-1} \ln(\beta r) + O(\beta^{-1})$.

At very low temperature the typical configurations of the field are therefore described by a dilute gas of domain walls separated by regions with constant $\varphi = \pm 1$. The partition function of the system can thus be written as a sum over the number $n$ of kink/anti-kink pairs (as discussed above, the number of domain walls must be even to be compatible with the periodic boundary conditions) weighted by the energy cost $e^{-2k\tilde{S}^*}$ times an appropriate combinatorial coefficient $I_{2n}$ accounting for all the possible configurations of the positions of $2n$ domain walls between 0 and $L$:

$$Z_L = \sum_{n=0}^{[L/2\sigma]} I_{2n}(L) e^{-2n\beta \tilde{S}^*},$$  \hspace{1cm} (12)

where $[x]$ denotes the integer part of $x$. Note that, since the instantons have a finite width $\sigma$, we cannot place more than $(L/2\sigma)$ kink/anti-kink pairs between 0 and $L$. The problem of determining the combinatorial coefficients $I_{2n}$ is equivalent to computing the entropy of a gas of $2n$ hard spheres of size $\sigma$ on a ring of length $L$ (see appendix A). The resulting expression is

$$I_{2n}(L) = \frac{1}{n} \frac{L}{(2n - 1)!} (L - 2n\sigma)^{2n-1}.$$  \hspace{1cm} (13)

Two length scales thus naturally emerge from the calculation: $\sigma$, the typical size of an interface, and $e^{\beta S^*}$ (here and in the following we have set the microscopic length $1/\Lambda$ to unity), which corresponds to the typical distance between two consecutive instantons. Following arguments similar to those in \cite{47} the latter can easily be shown to be (up to a trivial proportionality constant) the correlation length $\xi$ of the system as obtained from the exponential decay of the two-point connected correlation function.
After introducing the rescaled variables $\tilde{\zeta} = L/e^S$ and $\alpha = \sigma/L$, the partition function finally reads

$$Z_L(\tilde{\zeta}, \alpha) = 2 \sum_{n=0}^{\lfloor L/(2\alpha) \rfloor} \frac{\tilde{\zeta}^{2n}}{(2n)!} (1 - 2n\alpha)^{2n-1}. \quad (14)$$

The computation of the magnetisation probability distribution $P_L(\phi)$ in the $T \to 0$ limit can be carried out in a similar way. Note that an analogous computation has already been done for the Ising model in $d = 1$ [47] (see also below).

For each given instantonic configuration with $2n$ alternate kinks and anti-kinks we define $x_i$, $i = 1, \ldots, 2n$, as the lengths of the regions with constant $\phi = \pm 1$. In terms of these variables, the extensive magnetisation $M$ reads

$$M = \int_0^L \phi(x) \, dx = \pm \sum_{i=1}^n (x_{2i-1} - x_{2i}). \quad (15)$$

Note that thanks to the translational invariance, one can choose without loss of generality to place the first domain wall at $x = 0$. The sign of $M$ in front of the sum thus depends on whether the first instanton is from $\phi = -1$ to $\phi = +1$, or vice versa. Since each domain wall has a width $\sigma$, we also have that

$$\sum_{i=1}^{2n} x_i = L - 2n\sigma. \quad (16)$$

In consequence, the extensive magnetisation is bounded as $|M| \leq L - 2n\sigma$. When enforcing the constraints given by equations (15) and (16) one obtains

$$P_L(M) = \frac{1}{Z_L} \left[ \delta(M - L) + \delta(M + L) + 2 \sum_{n=1}^{[L-1/M]/2\alpha} J_{2n}(M, L) e^{-2nS^*} \right], \quad (17)$$

where $Z_L$ is defined in equation (12). Again, the combinatorial factors $J_{2n}(M, L)$ can be computed exactly (see appendix A). After introducing the rescaled variables $\zeta$, $\alpha$ defined above and the magnetisation density $\phi = M/L$ and using the fact that $\delta(L\phi) = (1/L)\delta(\phi)$ and $P_L(L\phi) = (1/L)P_L(\phi)$, we finally obtain:

$$P_L(\phi) = \frac{1}{Z_L(\zeta, \alpha)} \left[ \delta(\phi - 1) + \delta(\phi + 1) + 2 \sum_{n=1}^{[1-(\phi)/2\alpha]} (\zeta/2)^{2n} \left[ (1 - 2n\alpha)^2 - \phi^2 \right]^{n-1} \right], \quad (18)$$

where $Z_L(\zeta, \alpha)$ is given in equation (14). It is easily checked that $P_L(\phi)$ is properly normalised, $\int_{-1}^1 P_L(\phi) \, d\phi = 1$. One also finds that in the limit $\sigma \to 0$, i.e. when the domain walls become infinitely sharp, and for $S^* = 2J$, equations (14) and (18) give back the exact results derived for the one-dimensional Ising model [47]. These calculations are explicitly done in appendix A.

The finite-size effective potential and its evolution with the system size can be now explicitly determined in the $T \to 0$ limit from the relation in equation (3). $U_L(\phi)$ behaves as anticipated in the previous section: it presents two narrow minima in
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$\phi = \pm 1$, corresponding to the $\delta$-functions and a secondary minimum in $\phi = 0$ due to the entropic term in equation (18). As $L$ increases (i.e. $\alpha$ decreases) the minimum in $\phi = 0$ becomes deeper and deeper, as the sum over $n$ in equation (18) grows exponentially fast with $\zeta / \alpha$. For $L \approx \xi$ the value of the minimum in $\phi = 0$ crosses that of the two symmetric minima in $\phi = \pm 1$ (strictly speaking the value of the minima in $\phi = \pm 1$ is defined only for a nonzero temperature; otherwise, one has to consider the weight of the delta peaks). Nevertheless, at any finite $L$ the potential remains nonconvex due to the vestiges of the two minima in $\pm 1$. It is only in the thermodynamic limit that $U_{L \to \infty}(\phi) = U(\phi)$ recovers full convexity.

From equations (3), (14) and (18), one can compute all the desired characteristics of $U_{L}(\phi)$, such as the curvature in $\phi = 0$, $\kappa_{L}$, and the barrier height (when present), $\Delta_{L}$. Following the ideas presented in the previous section, we plot in figure 5 the curvature $\kappa_{L}$ multiplied by the magnetic susceptibility $\chi$ as a function of the system size $L$ divided by the correlation length $\xi$, for different temperatures. We have set $\beta = 1$, $c = 2r$, and a range of $r$ from 2 to 6 (as discussed above, the low-temperature limit here means that $\exp(-\beta \sqrt{8rc/9}) = \exp(-4r/3) \ll 1$). The curves show a perfect collapse, as expected. Via the instanton calculation we indeed have access to all the physical quantities of the present simple model. This is a consistency check of the recipe discussed above and an illustration of the range of temperatures where the asymptotic results apply. We show in figure 6 the evolution of the barrier height $\Delta_{L}$ multiplied by the system size $L$ and divided by twice the domain-wall (free) energy $\tilde{S}^{*}$ as a function of $L/\xi$ for $\beta = 1$, $c = 2r$, and for $r$ varying from 4 to 10. This log–log plot is very similar to the sketch in figure 4(b). The value $L\Delta_{L}/(2\tilde{S}^{*}) \approx 1$ (implying $\gamma \approx \tilde{S}^{*}$ as expected) is observed for $L \sim 2\sigma$, which corresponds to very small values of $\xi/L$, especially at low temperature.

4 Note that at small but finite temperature, the two $\delta$-functions in $\phi = \pm 1$ acquire a finite width due to the small gaussian thermal fluctuations of the fields around $\phi(x) = \pm 1$. Locally, the amplitude of such thermal fluctuations is simply related to $\nabla^{2}(\phi = \pm 1) = 2r$. Therefore the curvature of the two minima of $U_{L}(\phi)$ in $\phi = \pm 1$ is $2\beta r/L$.

5 See footnote 4.

6 Note that in the $T \to 0$ limit, the correlation function has a simple exponential form, $G_{i}^{(2)}(r) \approx e^{-r/\xi}$. As a consequence, the magnetic susceptibility becomes $\chi = 2 \int G_{i}^{(2)}(r) \propto \xi$ and $\kappa_{\infty} = 1/(2\xi)$. This result can be obtained analytically, as shown in appendix B.
There is then a broad regime, up to $\xi/L \lesssim 1$, where one observes a small decay, by less than a factor of 10. Finally, for $\xi/L \gtrsim \xi$ there is a fast (exponential) decay.

These plots validate at a quantitative level the proposed ways of extracting the parameters of the theory from the behaviour of the finite-size effective potential. We now turn to the same exercise but in the finite temperature regime where the analytical solution via the instantons is no longer a sufficient description.

III.C. $U_L(\phi)/L$ for finite but large $\xi$

In this section we apply and test the empirical recipes to extract $\xi$, $\chi$ and $\gamma$ proposed above on a system at large but finite correlation length (corresponding to a low but finite temperature). We obtain a numerical estimate of the finite-size effective potential $U_L(\phi)$ for a range of values of $L$ through several methods.

First, we have performed MC simulations. To this aim, we first discretise the continuum field theory of equation (1) by replacing the gradient by its discrete lattice version. The Hamiltonian thus becomes

$$H_c(\{\varphi_i\}) = a \sum_{i=1}^{L} \left[ \frac{c}{2a^2} (\varphi_i - \varphi_{i+1})^2 + V(\varphi_i) \right].$$

We set the lattice spacing $a$ to 1 (note that for $c = 2r$ the width of a domain wall is then $\sigma = 2 > a$) and we consider the periodic boundary conditions: $\varphi_{L+1} = \varphi_{1}$.

The numerical simulations are performed with a Metropolis algorithm: at each time step we pick a site $i$ at random and attempt to change the value of $\varphi_i$ by a random quantity, $\delta \varphi_i$, extracted from a Gaussian distribution with zero mean and variance $\sigma_i$. We then compute the energy difference $\Delta H = c \delta \varphi_i (\delta \varphi_i + 2 \varphi_i - \varphi_{i+1} - \varphi_i) + V(\varphi_i + \delta \varphi_i) - V(\varphi_i)$ and accept the move with the Metropolis probability $p = \min\{1, e^{-\beta \Delta H_c}\}$. Time is advanced by $1/L$. The typical width of the field shifts, $\sigma_\varphi$, is optimised recursively during the dynamics by enforcing that the acceptance rate of the moves (averaged over the last 100 MC steps) is approximately equal to 0.3.
We start from a given initial condition (for instance $\phi_i = +1 \forall i$) and let the system evolve and equilibrate. The equilibration time $\tau$, which of course depends on $\beta$, $r$ and $L$, can be extracted from the exponential decay of dynamical correlation functions such as $(1/L) \sum_i \langle \phi_i(t) \phi_i(t') \rangle \simeq \langle \phi^2 \rangle e^{-t/\tau}$. In order to compute the magnetisation probability distribution, $P_L(\phi)$, we measure the instantaneous magnetisation $\phi(t) = (1/L) \sum_i \phi_i(t)$ at regular time intervals corresponding to several times the equilibration time, say $10\tau$. This allows us to make sure that the values of $\phi(t)$ measured during the dynamics are statistically independent. In this way we construct a histogram of the magnetisations, which gives an estimate of $P_L(\phi)$ and, from equation (3), we obtain $U_L(\phi)$. The results for $\beta = 1$, $r = 2$, $c = 2r$ and $L$ varying from 4 to 128 are shown in figure 1.

Note that in order to obtain an accurate enough estimate of $P_L(\phi)$ and of $U_L(\phi)$, we need to sample rare events which take place with an exponentially small probability in the system size. As a consequence, the number of measurements of the instantaneous magnetisation must scale exponentially with $L$. Since the computational time of a single MC step scales linearly with the system size, this implies that the total computational time of our MC simulations scales as $\tau L e^L$. Therefore, the MC results are limited to not too large values of $L$, typically $L \lesssim 10^2$.

In order to overcome this limitation and study larger system sizes, we have used a $1/L$ perturbation expansion combined with an exact computation of the (Helmholtz) free-energy of the model through both a real-space RG approach and a transfer-matrix technique.

Let us start with the definition of the magnetisation probability distribution,

$$P_L(\phi) = \frac{\text{Tr}_{\{\phi\}} \delta(L\phi - \sum_i \phi_i)e^{-\beta H_L}}{\text{Tr}_{\{\phi\}} e^{-\beta H_L}}, \tag{20}$$

where $\text{Tr}_{\{\phi\}} \equiv \int \prod_i d\phi_i$. By using the integral representation of the $\delta$-function, one easily obtains

$$P_L(\phi) = e^{L f_L(\beta, 0)} \int_{-\infty}^{\infty} d\mu \ e^{-U_L(\beta, \mu) + \mu \phi}, \tag{21}$$

and

$$U_L(\phi) = -\frac{1}{\beta L} \ln \int_{-\infty}^{\infty} d\mu \ e^{-U_L(\beta, \mu) + \mu \phi} - f_L(\beta, 0), \tag{22}$$

where $f_L(\beta, \mu)$ is the Helmholtz free-energy density of a system of size $L$ in the presence of an external uniform magnetic field $\mu/\beta$:

$$f_L(\beta, \mu) = -\frac{1}{\beta L} \ln \text{Tr}_{\{\phi\}} e^{-\beta H_L + \mu \sum_i \phi_i}. \tag{23}$$

For large enough $L$ the integral in equations (21) and (22) is dominated by the maximum in $\mu = \mu^*$, which is given by

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7 Since the integrand in equations (21) and (22) is an analytic function, we can modify the contour in the complex plane.
Expanding the argument of the exponential around \(\mu^*\) leads to
\[
\beta f_L(\beta, \mu) + \mu \phi = \beta f_L(\beta, \mu^*) + \mu^* \phi + \frac{\beta}{2} f_{i*}^{(2)}(\delta \mu)^2 + \frac{\beta}{3!} f_{i*}^{(3)}(\delta \mu)^3 + \frac{\beta}{4!} f_{i*}^{(4)}(\delta \mu)^4 + \ldots,
\]
where \(f_{i*}^{(n)} = \frac{\partial^n f_L(\beta, \mu)}{\partial \mu^n}_{|_{\mu^*}}\) and \(\delta \mu = \mu - \mu^*\). One can thus treat all terms beyond the Gaussian level in a perturbative way and obtain a systematic expansion of \(U_L(\phi)\) and \(P_L(\phi)\) in powers of \(1/L\). From a straightforward calculation, one finds up to the order \(1/L^2\):
\[
U_L(\phi) \simeq f_L(\beta, \mu^*) + \mu^* \phi - f_L(\beta, 0) - \frac{1}{\beta L} \ln \left[ \frac{2\pi}{\beta |f_{i*}^{(2)}|} \right] + \frac{1}{\beta L^2} \left[ -f_{i*}^{(4)} + \frac{5}{24 \beta |f_{i*}^{(2)}|^2} \right].
\]

The above equation deserves some comments:

1. As already mentioned, in the thermodynamic limit, \(U_L(\phi)\) converges to the Gibbs free-energy density \(U(\phi)\), which is defined as the Legendre transform of the Helmholtz free-energy density \(f_L(\beta,h)\) [see equation (6)] and is therefore a convex function of the magnetisation \(\phi\).

2. Equation (25) is actually an expansion in powers of \(\xi/L\). The successive derivatives of the Helmholtz free energy with respect to the external field \(\mu\) yield the \(n\)-points connected correlation functions, \(\beta f_{i*}^{(n)} = (1/L) \sum_{i_1, \ldots, i_n} \langle \varphi_{i_1} \cdots \varphi_{i_n} \rangle_{\text{cont}},\) which thus behave as \(\xi^{n-1}\). As a result, the expansion of equation (25) does not converge for \(L/\xi < 1\) (even if \(L\) is large) and is expected to behave poorly compared to the numerical simulations in this regime. On the other hand, it should provide a good description of the finite-size effective potential for \(L/\xi > 1\).

3. In order to make some use of equation (25) we need to know the expression of the Helmholtz free energy of the model on a ring of \(L\) sites, at temperature \(\beta\) and in the presence of an external uniform magnetic field \(\mu/\beta\).

The calculation of \(f_L(\beta, \mu)\) can be done exactly by using a real-space RG approach, called the Migdal–Kadanoff (MK) scheme. It consists in integrating out iteratively half of the sites of the systems (say the odd sites) at each decimation step, and computing recursively the effective pair interaction potential, \(W_n(\varphi, \varphi')\), among the remaining sites. Consider, for instance, three consecutive sites, \(i, i+1\) and \(i+2\), at the \(p\)th step of the renormalisation procedure. After integrating out the field on the site \(i+1\), one finds the following exact recursive equation:
\[
W_{p+1}(\varphi_i, \varphi_{i+2}) = -\frac{1}{\beta} \ln \int_{-\infty}^{+\infty} d\varphi_{i+1} e^{-\beta [W_p(\varphi_i, \varphi_{i+1}) + W_p(\varphi_{i+1}, \varphi_{i+2})]},
\]
with the initial condition

\[
\text{初始条件}
\]
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\[ W_0(\varphi, \varphi') = \frac{c}{2a^2}(\varphi - \varphi')^2 + \frac{1}{2} \left[ V(\varphi) + V(\varphi') - \frac{\mu}{\beta}(\varphi + \varphi') \right]. \]  

(27)

For a system of size \( L = 2^p \), after \( p - 1 \) decimation steps, there are only two sites left and the Helmholtz free-energy density can be obtained as a simple integration:

\[ f_L(\beta, \mu) = -\frac{1}{2p\beta} \ln \int_{-\infty}^{+\infty} d\varphi \, d\varphi' \, e^{-2\beta W_{2p}(\varphi, \varphi')} . \]  

(28)

This procedure allows one to obtain very accurate numerical values of \( f_L(\beta, \mu) \) and of its derivatives, provided that the size of the system is an integer power of 2. In order to access other values of the system size \( L \), we have complemented the RG calculation by a transfer–matrix (TM) approach.

Indeed, the partition function of the system can be written as

\[ Z_L(\beta, \mu) = \text{Tr}_{(\varphi)} T_{\varphi_1 \varphi_2} T_{\varphi_2 \varphi_3} \cdots T_{\varphi_p \varphi_1} = \text{Tr} T^L = \lambda_1^L + \lambda_2^L + \ldots , \]  

(29)

where the TM operator is such that \( T_{\varphi, \varphi'} = \exp(-\beta W_0(\varphi, \varphi')) \) with \( W_0 \) given by equation (27). One can then numerically diagonalise the operator by discretising the values of the fields \( \varphi \) and \( \varphi' \) and compute its eigenvalues, \( \lambda_1, \lambda_2, \ldots \), which leads to an approximate expression for the Helmholtz free-energy density,

\[ f_L(\beta, \mu) \simeq -\frac{1}{\beta} \ln \lambda_1 - \frac{1}{\beta L} e^{L \ln(\lambda_2/\lambda_1)} + o[(\lambda_3/\lambda_2)^L] . \]  

(30)

Since the correlation length of the system is given by

\[ \xi^{-1} = -\ln(\lambda_2/\lambda_1) , \]  

(31)

Equation (30) provides a good approximation for \( f_L(\beta, \mu) \) only for \( L \gg \xi \).

The finite-size effective potential \( U_L(\phi) \) is then obtained from equation (25). The numerical results for the curvature \( \kappa_L \) in \( \phi = 0 \) and for the barrier height \( \Delta_L \) at small but finite temperature (or rather, correlation length) are displayed in figures 7 and 9. In figure 7(a), we plot \( \kappa_L \) versus \( L \) for several temperatures (actually, several values of \( r \) as we fix \( \beta = 1 \)) and in figure 7(b) we show the best data collapse on a mastercurve after rescaling both the curvature and the system size by temperature-dependent adjustable parameters. (Note that the curvature \( \kappa_L \) is obtained from the 1/L expansion only as the numerical accuracy of our MC data is not high enough to allow a good determination of the curvature.) In figure 8 we plot the best-fit parameter \( \xi_{\text{fit}} \) versus \( r \) and compare it to a direct determination of the correlation length through MC simulations and the instanton technique: we find very good agreement between the two sets of data. The same agreement is obtained for \( \chi_{\text{fit}} \), which is found to be proportional to the correlation length, \( \xi_{\text{fit}} \) or \( \xi \), as expected in one dimension.

In figure 9(a), we display a log–log plot of \( L \Delta_L \) versus \( L/\xi_{\text{fit}} \) where \( \xi_{\text{fit}} \) is obtained from the previous data collapse in figure 7. (As could be anticipated, the 1/L expansion fails completely for \( L/\xi < 1 \) and is not shown here.) Figure 9(b) then shows the same data with \( L \Delta_L \) divided by a temperature-dependent adjustable parameter \( \gamma_{\text{fit}} \) that ensures the best collapse of all curves for \( L/\xi < 1 \) (this parameter is determined up to
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1. a multiplicative constant. When plotted as a function of $r$, we find that this best-fit parameter $\gamma_{\text{fit}}$ matches very well the $r$ dependence of the direct estimate of the surface tension of the model through the instanton technique: see figure 10. Here, $\beta = 1$ and $c = 2r$, so that $\beta S^* = 2r/3 - (1/2)\ln r$ for large enough $r$. We have arbitrarily adjusted the unknown constant in $\gamma_{\text{fit}}$ so that the latter is roughly equal to $2\beta S^*$: the plot in figure 9(b) is shown with this choice of constant (which merely shifts all curves by a constant amount on the log scale).

These plots therefore confirm that the ideas and recipes we have proposed to extract the correlation length $\xi$, the susceptibility $\chi$ and the surface tension $\gamma$ (or alternatively the amplitude of the gradient term $c$) from finite-size numerical data for the effective potential work nicely. Without a priori knowledge one can empirically determine the

Figure 7. (a): Log–log plot of the curvature $K_L = U''_L(0)$ as a function of $L$ for $\beta = 1$, $r = 4, 6, 8, 10, 12$ and $c = 2r$. (b): the same data with a rescaling of the horizontal and vertical axes, as $L/\xi_{\text{fit}}$ and $\chi_{\text{fit}}K_L$, respectively, to provide the best collapse to a mastercurve.

Figure 8. Plot of the scaling parameter $\xi_{\text{fit}}$ versus $r$ and a comparison with the direct computation of the correlation length through MC and instanton techniques. Recall that having set $\beta = 1$, the temperature dependence is controlled by $r$. (Since $c = 2r$, the instanton width $\sigma = \sqrt{2c/r} = 2$.) The error bars associated with the fitting procedure are very small in this case and of the order of the symbol size.
IV. Nonperturbative RG

In this section we take a different approach from the above one. It is more of a 'bottom–up' approach, where we start from a known microscopic (or to the least effective) theory and try to include the fluctuations, in particular possibly strongly non-perturbative ones, to describe the observed macroscopic behaviour. As already stated, relevant parameters of the underlying (effective) theory from observations on finite-size systems.
the method of choice to achieve this is the RG, more precisely the nonperturbative RG (NPRG).

The NPRG emerged from Wilson’s work [32] in the early 1970s and has since been formulated into several alternative approaches [35, 48, 49]. All of them have the common denominator of treating and summing up fluctuations in a continuous way. In this work we focus on the formalism that has been originally developed by Wetterich and coworkers since the 1990s [9, 35]. In a nutshell, the idea is to start with a given field-theoretical model described by a microscopic (bare) action $S[\varphi]^{}\!$, and to add to it an infrared (IR) regulator in the form of a mass term:

$$S[\varphi] \rightarrow S[\varphi] = S[\varphi] + \frac{1}{2} \int q \varphi (q) R_k(q) \varphi (-q)$$

where $\int q \equiv \int d^D q/(2\pi)^D$ with $D$ the space dimension and $k$ is a running (momentum) scale. The IR cutoff function $R_k(q)$ goes to zero when $k \rightarrow 0$ and provides a mass to the small-momentum modes, $R_k(q) \sim b_k k^2$ when $q \rightarrow 0$, but is otherwise arbitrary. One also has that $k \ll \Lambda$, where $\Lambda^{-1} \ll 1$ is the shortest wavelength on which the field $\varphi$ can fluctuate; $\Lambda$ is the ultraviolet (UV) cutoff scale where the continuum theory meets the microscopic details.

From the regularised action $S_k[\varphi]$ one can define a cutoff-dependent generating functional of the connected correlation function (the analogue of a Helmholtz free-energy functional at the scale $k$),

$$W_k[J] = \ln \int D\varphi \exp \left[ -\beta S_k[\varphi] + \int x J(x) \varphi(x) \right],$$

where $\int_x \equiv \int d^D x$, and its Legendre transform,

$$\Gamma_k[\phi] + W_k[J] = \int_x J(x) \phi(x) - \frac{\beta}{2} \int_q \phi(q) R_k(q) \phi(-q),$$

where for convenience one subtracts the contribution from the regulator in the definition of $\Gamma_k[\phi]$. The latter is called the effective average action or the running effective action. In the above transformation, $J(x)$ is fixed by the condition that $\phi(x) = \langle \varphi(x) \rangle_k$, and the average is taken by using the modified action $S_k[\varphi]$.

The running effective action $\Gamma_k[\phi]$ continuously interpolates between the bare action at the UV scale$^9$ and the exact effective action (or Gibbs free-energy functional) $\Gamma[\phi]$, which is the generating functional of the 1PI correlation function, when $k \rightarrow 0$. Its evolution with $k$ is described by an exact RG flow equation [35]

$$\frac{\partial \Gamma_k[\phi]}{\partial k} = \frac{\beta}{2} \int_{sy} R_k(x-y) \left[ (\Gamma_k^{(2)} + \beta R_k)^{-1} \right]_{xy}$$

with the initial condition $\Gamma_\Lambda[\phi] = \beta S[\phi]$ and $\Gamma_k^{(n)}(x_1, \ldots, x_n) \equiv \delta^n \Gamma_k/\delta \phi(x_1) \ldots \delta \phi(x_n)$. By differentiation, this functional flow equation is equivalent to an infinite hierarchy of coupled flow equations for the running effective potential, $U_k(\phi) = \Gamma_k[\phi]/L^D$, and the running 1PI correlation functions (then all evaluated for uniform field configurations).

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$^8$ The NPRG can also be formulated by starting with a microscopic theory defined on a lattice; see [50].

$^9$ Strictly speaking, this is true if one chooses the cutoff function so that it diverges when $k \rightarrow \Lambda$. 

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Finding the exact solution of the functional integro-differential equation in equation (35) is an impossible task in general and one needs to develop approximation, or closure, schemes that basically replace equation (35) by a finite set of coupled equations for functions. This has been systematically and successfully pursued for a series of problems in both high and low-energy physics [9].

Applications of the NPRG formalism to the one-dimensional $\varphi^4$ theory with $S[\varphi]$ given by equation (1) have been previously considered [25–28]. In these studies it was found that simple approximation schemes fail to recover the low-temperature physics of the model, in particular the activated scaling form of the correlation length ($\xi \propto e^{\beta S}$). Here, we will show the underlying reason for this failure. To this end, we will first derive the exact asymptotic low-temperature form of the running effective action $\Gamma_k[\phi]$ by using the instanton approach and mapping to the one-dimensional Ising model.

IV.A. Running effective action and instantons in the limit $T \to 0$ ($\xi \to \infty$)

We first compute the expression of the running effective potential, $U_k(\phi) = \Gamma_k[\phi]/L$ with $\phi$ a uniform field, by using the instanton technique in the low-temperature regime where the correlation length is large (see also above).

For the one-dimensional $\varphi^4$ theory under study, the $k$-dependent regularised action reads

$$S_k[\varphi] = \int_0^L dx \left[ \frac{\xi}{2} (\partial_x \varphi)^2 + V(\varphi(x)) \right] + \frac{1}{2} \int_0^L dx \int_0^L dy \phi(x)R_k(x-y)\phi(y)$$

$$- L \left[ V(\phi_{0,k}) + \frac{1}{2} R_k(0)\phi_{0,k}^2 \right]$$

(36)

where $V(\varphi)$ is given in equation (2),

$$\phi_{0,k} = \text{argmin}_\varphi \left[ V(\varphi) + \frac{1}{2} R_k(0)\varphi^2 \right].$$

(37)

and we have added the last term in equation (36) for convenience, so that $S_k[\phi_{0,k}] = 0$. Contrary to the previous section on the finite-size effective potential, we take here the thermodynamic limit and let $L \to \infty$. The restriction to the spatial extent of the fluctuations is now provided by the IR regulator $R_d(q)$.

A very simple regulator is the Callan–Symanzik one, $R_k(q) = k^2$, which amounts to adding a conventional mass term to the bare action. (Note that in this case the running effective action is equal to the bare action only in the limit $\Lambda \to \infty$, but this has no consequences for the physics at intermediate and small momentum scales.) It is then easy to see that there exists a threshold $k = k_c$ such that $\phi_{0,k} \equiv 0$ for all $k \leq k_c$. This threshold corresponds to the moment along the RG flow where the running modified potential $V(\varphi) + \frac{1}{2} k^2 \varphi^2$ develops two minima and has a double-well shape. One can expect that this qualitative evolution is very general and does not depend on the details of the regulator. The precise form of $R_d(q)$ changes only the point $k_c$ where the double-well shape first appears.

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For \( k < k_c \) we can thus evaluate the probability of finding a particular magnetisation in the system by using the instanton method, much like in section III.B. This probability is given by

\[
P_k(\phi = M/L) = N \sum_{n \geq 1} e^{-2n \beta S^*_k} \int_0^{\infty} \left( \prod_{i=1}^{2n} \frac{dz_i}{2} \right) \delta \left( \sum_{i=1}^{2n} z_i - L - 2n \sigma_k \right) \delta \left( \sum_{i=1}^{n} (z_{2i-1} - z_{2i}) - \frac{M}{\phi_{0,k}} \right)
\]

where \( z_i \) is the length of the \( i \)th interval separated by two domain walls, \( N \) is a normalisation constant, \( S^*_k \) is the action evaluated on a single instanton profile and \( \sigma_k \) is the instanton width. By exponentiating the Dirac delta functions and passing from a discrete sum to a continuum one so that \( \alpha L = 2n \), we get

\[
P_k(\phi) = N' \int_0^{\infty} d\alpha e^{-\alpha L \beta S_k} \int_{-\infty}^{\infty} d\mu d\nu \int_0^{\infty} \left( \prod_{i=1}^{n} \frac{dz_i}{2} \right) \exp \left[ \mu L (1 + \alpha \sigma_k) + L \nu \frac{\phi}{\phi_{0,k}} - \sum_{i=1}^{n} \left( (\mu + \nu) z_{2i-1} + (\mu - \nu) z_{2i} \right) \right]
\]

where \( N' \) is another normalisation constant. Integration over the variables \( z_i \) then leads to

\[
P_k(\phi) = N' \int_0^{\infty} d\alpha e^{-\alpha L \beta S_k} \int_{-\infty}^{\infty} d\mu d\nu \exp \left[ L \left( \mu + \alpha \mu \sigma_k + \nu \frac{\phi}{\phi_{0,k}} - \frac{\alpha}{2} \ln(\mu^2 - \nu^2) \right) \right].
\]

For large \( L \) we can use a saddle point evaluation of the integrals over \( \mu \) and \( \nu \), which gives

\[
P_k(\phi) = N' \int_0^{\infty} d\alpha \exp \left[ -\alpha L \beta S_k + L \left( \alpha - \alpha \ln \alpha + \frac{\alpha}{2} \left( 1 + \alpha \sigma_k \right)^2 - \left( \frac{\phi}{\phi_{0,k}} \right)^2 \right) \right].
\]

Finally the integral over \( \alpha \) can also be evaluated with the saddle point method and the value of \( \alpha \) at the saddle point is found to be

\[
\alpha = \sqrt{1 - \frac{\phi^2}{\phi_{0,k}^2}} e^{-\beta S^*_k}
\]

where the range of magnetisations is limited to \( \phi < \phi_{0,k} \). The low-temperature regime corresponds to \( \beta S^*_k \) large and the system is then described by a dilute instanton gas. The running effective potential \( U_k \) is just \(-1/(\beta L)\) times \( \ln P_k(\phi) \), to which one subtracts the contribution of the IR regulator, and it is given by

\[
U_k(\phi) = -\frac{1}{\beta L} \ln P_k(\phi) - \frac{1}{2} R_k(0)(\phi^2 - \phi_{0,k}^2) + V(\phi_{0,k})
\]

\[
= -\frac{1}{\beta} \sqrt{1 - \frac{\phi^2}{\phi_{0,k}^2}} e^{-\beta S^*_k} + \frac{1}{2} R_k(0)(\phi^2 - \phi_{0,k}^2) + V(\phi_{0,k}),
\]

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which is valid for \( \phi < \phi_{0,k} \) and is asymptotically exact when the temperature goes to zero. The associated flow equation reads

\[
\frac{\partial U_k(\phi)}{\partial k} = \frac{\partial}{\partial k} \left[ -\frac{1}{\beta} \sqrt{1 - \frac{\phi^2}{\phi_{0,k}^2} e^{-\beta S_k^*}} \right] + \frac{1}{2} \frac{\partial R_k(0)}{\partial k} (\phi_{0,k}^2 - \phi^2) \tag{44}
\]

To go further and find the asymptotic low-temperature expressions for the 1PI vertices at scale \( k \), we use a short-cut provided by an approximate mapping when \( T \to 0 \) between the \( \phi^4 \) theory and the Ising model. The latter is described by the Hamiltonian

\[
H[\{\sigma_i\}] = -J \sum_{i=1}^{L} \sigma_i \sigma_{i+1} - h \sum_{i=1}^{L} \sigma_i \tag{45}
\]

where \( \sigma_{i+1} = \sigma_i \) if periodic boundary conditions are used and we have set the lattice spacing to one. The thermodynamic limit with \( L \to \infty \) is considered here. This calculation for the one-dimensional Ising model is rather standard \cite{51} and the details are given in appendix B.

From the comparison between the expressions of the pair correlation function obtained in the low-temperature limit, where the long-distance properties of both theories are described in the continuum, i.e. for the Ising case,

\[
G_k^{(2)}(r; m) \simeq (1 - m^2) e^{-r/\xi(m)} \tag{46}
\]

with the correlation length

\[
\xi(m) \simeq \frac{1}{2} \sqrt{1 - m^2} e^{2JL}, \tag{47}
\]

where \( m \) is the magnetisation per site, and for the (modified) \( \phi^4 \) theory (from the instanton calculation),

\[
G_k^{(2)}(r; \phi) \simeq (\phi_{0,k}^2 - \phi^2) e^{-r/\xi(\phi)} \tag{48}
\]

where

\[
\xi(\phi) \simeq \frac{1}{2} \sqrt{1 - \frac{\phi^2}{\phi_{0,k}^2} e^{2S_k^*}}, \tag{49}
\]

one can see that the two theories map onto each other with the following formal replacements when \( T \to 0 \):

\[
2J \to S_k^* \quad (1 - m^2) \to (\phi_{0,k}^2 - \phi^2) \tag{50}
\]

and \( S_k^* \), the instanton action, is identified with the domain–wall energy when \( T \to 0 \).

At low temperature, the two-point 1PI correlation function can be written as

\[
\frac{1}{\beta} \Gamma_k^{(2)}(p; \phi) \simeq \frac{1}{2\beta (\phi_{0,k}^2 - \phi^2)} \xi(\phi) \left[ p^2 + \xi^{-2}(\phi) \right] - R_k(0) \tag{51}
\]

where the last term is due to the definition of the running effective action in equation (34). This expression can be put in the form
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\[ \beta^{-1} \Gamma^{(2)}_k(p; \phi) = Z_k(\phi)p^2 + U''_k(\phi) \]

where \( Z_k(\phi) \) can be obtained as

\[ Z_k(\phi) = \lim_{p \to 0} \frac{1}{2\beta} \frac{\partial^2}{\partial p^2} \Gamma^{(2)}_k(p, \phi) = \frac{1}{2\beta(\phi_{0,k}^2 - \phi^2)} \xi(\phi). \]  

(53)

In addition, one can check that

\[ U''_k(\phi) + R_k(0) = \frac{1}{\beta} \frac{1}{\phi_{0,k}^2 - \phi^2} \frac{e^{-\beta S_k}}{\sqrt{1 - \phi^2/\phi_{0,k}^2}}, \]  

(54)

in complete agreement with equation (43).

By using the mapping with the one-dimensional Ising model, we can also obtain low-temperature expressions for the higher-order 1PI correlation functions, \( \Gamma^{(3)}_k, \Gamma^{(4)}_k, \ldots \), at the scale \( k \). With the results given in appendix C we obtain

\[ \Gamma^{(3)}_k(p_1, p_2, p_3; \phi) = (2\pi)\delta(p_1 + p_2 + p_3) \frac{c(\phi)s(\phi)^2}{2\xi(\phi)} \times [3 - \xi(\phi)^2(p_1p_2 + p_1p_3 + p_2p_3)] \]  

(55)

\[ \Gamma^{(4)}_k(p_1, p_2, p_3, p_4; \phi) = (2\pi)\delta(p_1 + p_2 + p_3 + p_4) \frac{1}{2\xi(\phi)s(\phi)^6} \]

\[ \times (-[c(\phi)^2 + s(\phi)^2]\xi(\phi)^4p_1p_2p_3p_4 - [3c(\phi)^2 + s(\phi)^2]) \]

\[ \times \xi(\phi)^2(p_1p_2 + p_1p_3 + p_1p_4 + p_2p_3 + p_2p_4 + p_3p_4) + 3[5c(\phi)^2 + s(\phi)^2]) \]  

(56)

where \( c(\phi) = \phi \) and \( c(\phi)^2 + s(\phi)^2 = \phi_{0,k}^2 \) (\( c \) should not be confused with the notation also used for the prefactor of the derivative term in the bare action). More generally, the running 1PI correlation function can be cast in the form

\[ \Gamma^{(n)}_k(p_1, \ldots, p_n; \phi) = (2\pi)\delta(p_1 + \cdots + p_n)\xi(\phi)^{-1}g_n(\xi(\phi)p_1, \ldots, \xi(\phi)p_n; \phi), \]

where the remaining \( \phi \) dependence in \( g_n \) does not contain exponential terms involving \( \exp(\beta S_k^\ast) \).

**IV.B. Approximation schemes**

Having obtained the exact expressions for the running effective potential and the running 1PI correlation functions in the low-temperature limit, we can now test approximation schemes for the exact NPRG equation in equation (35). Among the several approximation schemes so far proposed, we will focus first on the most popular one, the so-called derivative expansion. In this approximation, the running effective action at the scale \( k \) is expanded in gradients of the field,

\[ \Gamma_k[\phi] = \beta \int_x \left[ U_k(\phi(x)) + \frac{1}{2} Z_k(\phi(x)) \left( \frac{\partial \phi(x)}{\partial x} \right)^2 + \cdots \right] \]

(57)

where the higher-order terms involve 4, 6 etc, derivatives of the field. We will show that finite truncations of the derivative expansion are unable to reproduce the exact features of low-temperature physics.
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IV.B.1. LPA. The local potential approximation (LPA) is the lowest order of the derivative equation. It corresponds to

$$\Gamma_\text{LPA}[\phi] = \beta \int_x \left[ U_k(\phi(x)) + \frac{1}{2} \left( \frac{\partial \phi(x)}{\partial x} \right)^2 \right],$$

(58)

where the coefficient of the gradient term is constant and is not renormalised. Plugging this ansatz into equation (35), computing it for a uniform field \(\phi\) and choosing the simple regulator \(R_k(q) = b_k k^2\) with \(b_k\) constant (taken to 1) leads to the following differential equation for the running effective potential \(U_k(\phi)\):

$$\partial_k U_k(\phi) = \frac{1}{4\beta} \frac{1}{\sqrt{U_k''(\phi) + k^2}} \partial_k (k^2).$$

(59)

It is easily verified that the exact expression in equation (43) does not satisfy the above equation. The latter is actually unable to reproduce the correct scaling of the correlation length, with, e.g. \(U_k'' + k^2 \propto \exp(-\beta S^*)\) (see equation (54)).

In figure 2, we have plotted the running effective potential \(U_k(\phi)\) at several values of \(k\), as obtained from the LPA with a regulator of the form \(43\) \(R_k(p) = (k^2 - p^2) \Theta(k^2 - p^2)\). The curves illustrate the return to convexity of the potential. However, as is also known from previous attempts \([25, 27, 28]\), if the LPA provides a good description for values of \(T\) higher than the energy barrier of the double well, or more precisely than the instanton energy cost \(S^*\), they fail to reproduce the low-temperature result with a thermally activated dependence of the correlation length, \(\propto \exp(\beta S^*)\). For instance, the curvature of the effective potential in zero, \(\kappa_{k=0} = U_k''(0)\), which should vanish exponentially when \(T \to 0\) as \(\exp(-\beta S^*)\) (see also section III), is generically found to vanish as a power law of \(T\) instead. The nonperturbative regime associated with the rare localised events, which is captured by the instanton calculation, is therefore completely missed.

IV.B.2. Second order of the derivation expansion. The next order corresponds to the following ansatz

$$\Gamma_\text{LPA}[\phi] = \beta \int_x \left[ U_k(\phi(x)) + \frac{1}{2} Z_k(\phi) \left( \frac{\partial \phi(x)}{\partial x} \right)^2 \right].$$

(60)

When inserted into the exact RG flow equation, this ansatz leads to two coupled differential equations for the functions \(U_k(\phi)\) and \(Z_k(\phi)\) (the latter is obtained from the exact flow equation for the second vertex \(\Gamma_k^{(2)}\) with the use of the prescription given in the first equality of equation (53)):

$$\partial_k U_k(\phi) = \frac{1}{4\beta} \partial_k (b_k k^2) [Z_k(\phi)(U_k''(\phi) + b_k k^2)]^{-1/2}$$

(61)
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\[ \partial_t Z_0(\phi) = \frac{1}{\beta} \partial_t (b_0 k^2) \left[ -\frac{5}{64} U''_k(\phi)^2 Z_0(\phi)^{1/2} (U''_k(\phi) + b_0 k^2)^{-7/2} + \frac{9}{32} Z'_k(\phi) U'''_k(\phi) Z_0(\phi)^{-1/2} (U''_k(\phi) + b_0 k^2)^{-5/2} + \frac{7}{64} Z''_k(\phi) Z_0(\phi)^{-3/2} (U''_k(\phi) + b_0 k^2)^{-3/2} - \frac{1}{8} Z'''_k(\phi) Z_0(\phi)^{-1/2} (U''_k(\phi) + b_0 k^2)^{-3/2} \right] \]

where the IR cutoff function is of the same form as above (and a residual \( k \)-dependence is allowed in \( b_k \)).

When inserting the exact expression for \( U_k(\phi) \) and \( Z_k(\phi) \) given in equations (43), (53) and (54), one can see that equation (61) is now satisfied at leading order in \( \exp(-\beta S_k^*) \) but not equation (62). The exact expressions indeed generate terms of order \( \exp(2\beta S_k^*) \) on the right-hand side of equation (62), which do not cancel and have no counterparts on the left-hand side (which is itself essentially of order \( \exp(\beta S_k^*) \)). The problem found at the LPA level can be formally cured at the level of the effective average potential but at the expense of an inconsistency at the level of the function \( Z_k(\phi) \).

IV.B.3. Fourth order of the derivative expansion. To check whether the results found above correspond to a more systematic pattern, we have considered the fourth order, which corresponds to taking

\[ \Gamma_1[\phi] = \int dx \left[ \frac{1}{2} U_k(\phi(x)) + \frac{1}{2} Z_k(\phi(x)) (\partial \phi(x))^2 + \frac{1}{4!} Y_k(\phi(x)) (\partial \phi(x))^4 \right]. \]

The equation for the running effective potential in equation (61) is unchanged, but that for \( Z_k(\phi) \) is now obtained as

\[ \partial_t Z_k(\phi) = \frac{1}{\beta} \partial_t (b_0 k^2) \left[ -\frac{5}{64} U''_k(\phi)^2 Z_k(\phi)^{1/2} (U''_k(\phi) + b_0 k^2)^{-7/2} + \frac{9}{32} Z'_k(\phi) U'''_k(\phi) Z_k(\phi)^{-1/2} (U''_k(\phi) + b_0 k^2)^{-5/2} + \frac{7}{64} Z''_k(\phi) Z_k(\phi)^{-3/2} (U''_k(\phi) + b_0 k^2)^{-3/2} - \frac{1}{8} Z'''_k(\phi) Z_k(\phi)^{-1/2} (U''_k(\phi) + b_0 k^2)^{-3/2} \right] \times (U''_k(\phi) + b_0 k^2)^{-1/2}. \]

An equation for \( Y_k(\phi) \) is also derived by considering the flow of the 4-point 1PI vertex, but it is too long to be given here.

When inserting the exact low-temperature expressions for \( U_k(\phi) \), \( Z_k(\phi) \) and \( Y_k(\phi) \) [the latter can be obtained from equations (55) and (56)] into the three flow equations corresponding to the present ansatz, one finds that both the equation for \( U_k \) and that for \( Z_k \) in equation (64) are satisfied. For the latter, the term involving \( Y_k(\phi) \) on the right-hand side of equation (64) now exactly cancels the term in \( \exp(2\beta S_k^*) \), which led to an inconsistency in the second-order approximation (see above). On the other hand, one can check that the approximate equation for \( Y_k(\phi) \) is not satisfied by the exact expression because of the presence of terms of order \( \exp(4\beta S_k^*) \) on the right-hand side [while \( Y_k \) itself behaves as \( \exp(3\beta S_k^*) \)].
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IV.B.4. General scheme and further approximations. Guided by the above results, it is now easy to infer the general pattern. The prefactors of the terms with $2l$ derivatives of the field in the derivative expansion of the running effective action $\Gamma_k[\phi]$ are of the order $\exp[(2l-1)\beta S_k^*]$ in the low-temperature regime (and $U_k''(\phi) + R_k(0)$ is itself of the order $\exp(-\beta S_k^*)$). This dominant behaviour when $\beta S_k^* \to \infty$ emerges from the exact NPRG hierarchy of equations for the 1PI vertices, because terms that would naively lead to a higher power in $\exp(\beta S_k^*)$ on the right-hand side of the equations (the ‘beta functions’) exactly cancel out. This cancellation effect is, however, lost if one truncates the expansion, whatever the order of the truncation. We conjecture that the appropriate ansatz of $\Gamma_k[\phi]$ that reproduces the low-temperature physics of the model is instead

$$\Gamma_k[\phi] = \int_x \left[ U_k(\phi(x)) + \sum_{l=1}^{\infty} \frac{1}{(2l)!} Y_{k,2l}(\phi(x))(\partial \phi(x))^{2l} \right]$$

(65)

with, to make contact with the previous notations, $Y_{k,2}(\phi) \equiv Z_k(\phi)$ and $Y_{k,4}(\phi) \equiv Y_k(\phi)$. Note that the above form of $\Gamma_k$ is not the most general one: in the derivative expansion, the term of order $\partial^2 \phi$ is actually a combination of terms involving $(\partial \phi)^{2l}, \partial^2 \phi (\partial \phi)^{2l-2}, \ldots, \partial^2 \phi$ which even after integration by part cannot in general be reduced to a single contribution as in equation (65). The specific form in equation (65) results from the rather simple momentum dependence of the 1PI correlation functions in the one-dimensional Ising model and $\varphi^4$ theory at low temperature.

The above finding allows us to discuss another approximation of the NPRG called BMW [52]. It corresponds to a closure of the exact NPRG hierarchy at the level of the equation for the 1PI two-point function $\Gamma_k^{(2)}(p, \phi)$:

$$\partial_k \Gamma_k^{(2)}(p, \phi) = \beta \int \frac{dq}{2\pi} \partial_k R_k(q) \left[ G_k(q, \phi)^2 G_k(p + q, \phi) \right.$$  

$$\times \Gamma_k^{(3)}(p, q, -p - q; \phi)^2 - \frac{1}{2} G_k(q, \phi)^2 \Gamma_k^{(4)}(p, -p, q, -q; \phi) \right]$$

(66)

where $G_k(p, \phi) = [\Gamma_k^{(2)}(p, \phi) + R_k(p)]^{-1}$. The BMW closure consists in setting to zero the internal momentum $q$ appearing in the 3- and 4-point vertices on the right-hand side. After using the consistency relations, $\Gamma_k^{(3)}(p, 0, -p; \phi) = \partial \Gamma_k^{(2)}(p; \phi)/\partial \phi$ and $\Gamma_k^{(4)}(p, -p, 0, 0; \phi) = \partial^2 \Gamma_k^{(2)}(p; \phi)/\partial \phi^2$, one obtains a closed equation,

$$\partial_k \Gamma_k^{(2)}(p, \phi) = \beta \int \frac{dq}{2\pi} \partial_k R_k(q) \left[ G_k(q, \phi)^2 G_k(p + q, \phi) \right.$$  

$$\times \left[ \frac{\partial \Gamma_k^{(2)}(p; \phi)}{\partial \phi} \right]^2 - \frac{1}{2} G_k(q, \phi)^2 \frac{\partial^2 \Gamma_k^{(2)}(p; \phi)}{\partial \phi^2} \right],$$

(67)

which can be combined with the equation for the running effective potential $U_k(\phi)$. It is easily checked that equation (67) is not compatible with the exact low-temperature expressions of $U_k$ and $\Gamma_k^{(2)}$ given in section IV.A: after scaling the momenta by $\xi(\phi)$ (see section IV.A), the left-hand side of equation (67) scales as $\xi^{-1}$, whereas the right-hand side has a term in $\xi^0$ that does not cancel out. Just like truncations of the derivative...
expansion, the BMW closure is therefore unable to properly describe the nonperturba-
tive physics of the one-dimensional $\phi^4$ at low temperature.

The alternative to the existing approximation schemes of the NPRG is to start from
the exact low-temperature ansatz in equation (65). This, however, leads to an infinite
set of differential equations that cannot be treated with standard methods. We have
tried another route which amounts to considering the running effective action as being
local in the two variables $\phi(x)$ and $\partial\phi(x)$, and introduce an auxiliary field $\hat{\phi}(x)$ to decou-
ple $\partial\phi(x)$ from $\phi(x)$. This procedure, however, is highly ambiguous. In addition, say we
end up with a running effective action of the form $\Gamma_4[\phi, \hat{\phi}] = \int \mathcal{V}_4(\phi(x), \hat{\phi}(x))$, it is not
clear that standard approximations on this ansatz will correctly capture the expected
low-temperature physics. Actually we have tried an LPA approximation at the level of
the two fields $\phi$ and $\hat{\phi}$ and it completely misses the nonperturbative regime. More work
is needed to possibly find a solution to this unsatisfactory theoretical situation.

V. Discussion and concluding remarks

In this work we have studied the $\phi^4$ theory at low temperature in the regime where
the behaviour of the system is completely dominated by nonperturbative instantonic
fluctuations. We have first discussed empirical recipes to extract the parameters of
the underlying microscopic or effective theory from the numerical study of finite-size
systems. This strategy could be very useful in the analysis of the finite-size numerical
simulation of glassy systems, but the application to this problem deserves further work.

Our study also illustrates the difficulty of describing the low-temperature nonper-
turbative physics of the one-dimensional $\phi^4$ theory through truncations of the NPRG.
In a sense, however, the one-dimensional case is harder than the situation in higher
dimensions. There, the transition associated with a spontaneous symmetry breaking is
not destroyed by the fluctuations and the return to convexity has been shown to be
properly described through simple approximations of the NPRG [9, 41, 42]. We now
discuss in more detail this higher-dimensional situation.

Consider, for instance, the three-dimensional $\phi^4$ theory. As far as the finite-size
effective potential $U_L(\phi)$ is concerned, one can repeat and adapt the qualitative argu-
ments developed in section III.A. At a low enough temperature, the bare potential has
two minima in, say, $\phi = \pm 1$ and the relevant excitations above the uniform ground
states are system-spanning domain walls or interfaces between regions of essentially
constant positive and negative magnetisation. When the system size $L$ becomes larger
than the interface width (which is of the order of the correlation length [45]), the
system can accommodate one system-spanning interface: $U_L(\phi)$ should then have, on
top of the two symmetric minima for $\phi \approx \pm 1$, a plateau for intermediate values of the
field; the height of the plateau compared to the bottom of the minima is given by $\Upsilon L$, where $\Upsilon$ is the surface tension. As $L$ increases, this height decreases and goes to zero
in the thermodynamic limit. The effective potential is convex with a flat intermediate
portion corresponding to phase coexistence. The evolution with $L$ of $U_L(\phi)$ is schemati-
cally depicted in figure 11. In this case, studying finite-size systems should allow one

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to extract two physical quantities, the surface tension and the correlation length which corresponds to the interface width.

This three-dimensional $\varphi^4$ theory in the symmetry-broken region\(^{10}\) has also been studied in detail within the NPRG framework [9, 41, 42]. Although influenced by domain walls as in one dimension, the long-distance physics is nonetheless different as these nonperturbative fluctuations are not strong enough to destroy the phase transition. As a result, simple approximation schemes of the running effective action properly capture the effect of these fluctuations. The running effective potential $U_k(\phi)$ evolves with decreasing $k$ from the bare double-well potential to a convex effective potential when $k = 0$: see the schematic plot in figure 11. Provided one chooses an appropriate class of IR regulator [9], the intermediate ‘inner’ part of $U_k(\phi)$ displays at small $k$ a parabolic shape $\propto k^2\phi^2$ that comes in addition to the two symmetric minima in $\phi = \pm \phi_{0,k}$. This parabolic dependence corresponds to the expected exact behaviour obtained by considering the nonuniform configurations of the field involving domain walls. The remarkable feature is that this behaviour is recovered by using approximations of the NPRG, such as the first orders of the derivative expansion, which only consider expansions about uniform fields [42]. This is in stark contrast with the situation encountered in one dimension.

The nature of the NPRG flow somehow changes when the running IR momentum scale $k$ crosses some value $k_*$ that roughly corresponds to the point at which $k^2$ becomes of the order of magnitude of the curvature of the running effective potential in $\phi = 0$: $k_*^2 \sim |U_k''(0)|$; this in turn corresponds to the point where $1/k$ becomes of the order of the width of the domain wall in the (nonuniform) field configurations that minimise the running effective action at this scale $k$ [9, 42]. Whereas this information can be included in an improved instantonic theory of nucleation in the cases where metastability is

\(^{10}\) The return to convexity of the running effective potential has also been studied in the $O(N)$ model with $N > 2$. In this case, the excitations above the ground state take the form of spin waves: see [42, 53].
Nonperturbative fluctuations and metastability in a simple model: from observables to microscopic theory and back present [9, 54] as, e.g. when applying a nonzero external source or magnetic field, its interpretation in terms of physical quantities of the actual macroscopic system remains unclear. In particular this length scale $1/k^*$ does not provide any direct information on one of the important length scales in nucleation problems, i.e. the size of the critical droplet or bubble.

In any case, we think that providing a generic solution to the problem posed by nonperturbative fluctuations in model systems such as the one studied here would be very profitable for tackling the harder situations encountered in glassy systems which involve activated dynamics in a complex landscape.

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Appendix A. Computation of the combinatorial factors for the gas of instantons

The combinatorial coefficients $I_{2n}(L)$ are configuration integrals of $2n$ domain walls of width $\sigma$ on a ring of size $L$. This problem is equivalent to the computation of the partition function of $2n$ (discernible) hard spheres of size $\sigma$ on a ring of size $L$ in $D=1$. As done in the main text, we define $x_i, i = 1, \ldots, 2n$, as the lengths of the regions with constant $\phi = \pm 1$ (i.e. the gaps between the spheres). These variables must satisfy the constraint $\sum_{i=1}^{2n} x_i + 2n\sigma = L$. One then has

$$I_{2n}(L) = \frac{L}{n} \int_0^{L-2n\sigma} dx_1 \int_0^{L-2n\sigma-x_1} dx_2 \cdots \int_0^{L-2n\sigma-(x_1+x_2+\cdots+x_{2n-2})} dx_{2n-1},$$

where the factor $L$ comes from translational invariance and the factor $1/n$ accounts for the number of ways one can choose the first kink/anti-kink pair. In the following, we will determine the expression of $I_{2n}$ by recurrence. In order to do this, it is convenient to introduce the functions

$$g_n(y) = \int_0^y dx_{n-1} \int_0^{y-x_{n-1}} dx_{n-2} \cdots \int_0^{y-(x_{n-2}+\cdots+x_1)} dx_1,$$

in terms of which the combinatorial factors can be expressed as

$$I_{2n}(L) = \frac{L}{n} g_{2n}(L-2n\sigma). \quad (A.2)$$

From the definition in equation (A.1), one can write $g_{n+1}(y)$ in terms of $g_n(y)$,

$$g_{n+1}(x) = \int_0^y dx_n g_n(y-x_n) = \int_0^y dx_n g_n(x_n), \quad (A.3)$$

which, from $g_2(y) = \int_0^y dx_1 = y$ and by recurrence, immediately leads to

$$g_n(y) = \frac{y^{n-1}}{(n-1)!}. \quad (A.4)$$

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Finally, after plugging equation (A.4) into (A.2), one obtains equation (13) of the main text.

In order to compute the combinatorial factors $J_{2n}(M, L)$ one has to impose that the gaps $x_i$ satisfy the two following constraints:

$$\begin{aligned}
\sum_{i=1}^{2n} x_i &= L - 2n\sigma, \\
\sum_{i=1}^{n} (x_{2i-1} - x_{2i}) &= M,
\end{aligned}$$

which can be rewritten as

$$\begin{aligned}
\sum_{i=1}^{n} x_{2i-1} &= \frac{L - 2k\sigma + M}{2}, \\
\sum_{i=1}^{n} x_{2i} &= \frac{L - 2k\sigma - M}{2}.
\end{aligned}$$

As a result, the integrals over the variables $x_i$ can be divided into separate integrations over even and odd gaps, which can be written in terms of the functions $g_n(y)$ defined above. This yields

$$J_{2n}(M, L) = \frac{L}{2n} g_n\left(\frac{L - 2n\sigma - M}{2}\right) \times g_n\left(\frac{L - 2n\sigma + M}{2}\right).$$

The extra $1/2$ factor comes from the fact that only half of the configurations, namely, those with the first domain wall joining $\varphi = -1$ to $\varphi = +1$, contribute to magnetisation $+M$, whereas the others contribute to $-M$. After using the exact expression in equation (A.1), one finally finds

$$J_{2n}(M, L) = \frac{L}{2n} \frac{[(L - 2n\sigma)^2 - M^2]^{n-1}}{2^{2(n-1)(n-1)!}},$$

which, with the help of the intensive variables $\phi = M/L$ and $\alpha = \sigma/L$, leads to equation (18) of the main text.

In the following we show that $P_1(\phi)$ given in equation (18) is properly normalised to 1. We start by computing the integrals over $\phi$ of the terms of the sum separately. By changing variable to $x = \phi/(1 - 2n\alpha)$ one gets

$$\begin{aligned}
\mathcal{Y}_{2n}(L) &= 2 \left(\frac{\zeta}{2}\right)^{2n} \int_{-(1-2n\alpha)^2}^{(1-2n\alpha)^2} d\phi \frac{[(1 - 2n\alpha)^2 - \phi^2]^{n-1}}{n! (n-1)!} \\
&= 2 \left(\frac{\zeta}{2}\right)^{2n} \frac{n!}{n! (n-1)!} (1 - 2n\alpha)^{2(n-1)} (1 - 2n\alpha) \int_{-1}^{1} dx (1 - x^2)^{n-1}.
\end{aligned}$$

The integral over $dx$ can be computed as

$$\int_{-1}^{1} dx (1 - x^2)^{n-1} = 2 \int_{0}^{\pi/2} d\theta (\cos \theta)^{2n-1} = \sqrt{\pi} \frac{\Gamma(n)}{\Gamma(n + 1/2)},$$

where $\Gamma(n + 1/2) = 2^{-n}(2n - 1)!! \sqrt{\pi}$. By using the fact that

$$(2n - 1)!! = \frac{(2n)!}{(2n)!!} = \frac{(2n)!}{2^n n!},$$

we have

$$\mathcal{Y}_{2n}(L) = 2 \left(\frac{\zeta}{2}\right)^{2n} \frac{n!}{n! (n-1)!} (1 - 2n\alpha)^{2(n-1)} (1 - 2n\alpha) \sqrt{\pi} \frac{\Gamma(n)}{\Gamma(n + 1/2)}.$$
one then finds
\[ \Upsilon_2(L) = 2\zeta^2 \frac{(1 - 2n\alpha)^{n-1}}{(2n)!}. \] (A.6)

From equations (18), (14) and (A.6), one ends up with
\[
\int_{-1}^{+1} d\phi P_L(\phi) = \frac{1}{Z_L(\zeta, \alpha)} \left[ \int_{-1}^{+1} d\phi (\delta(\phi - 1) + \delta(\phi + 1)) + \sum_{n=1}^{1/(2\alpha)} 2\zeta^{2n} \frac{(1 - 2n\alpha)^{n-1}}{(2n)!} \right]
\]
\[ = \frac{1}{Z_L(\zeta, \alpha)} \sum_{n=0}^{1/(2\alpha)} 2\zeta^{2n} \frac{(1 - 2n\alpha)^{n-1}}{(2n)!} = 1. \] (A.7)

The expression of \( P_L(\phi) \) for the one-dimensional Ising model [47] can be recovered as a particular case of equation (18) in the limit \( \sigma \to 0 \) (i.e. for infinitely sharp domain walls) and for \( S^* = 2J \). In particular one then has
\[ Z_L(\zeta, \alpha = 0) = 2 \sum_{n=0}^{\infty} \zeta^{2n} (2n)! = 2 \cosh \zeta, \]
and
\[ P_L(\phi) = \frac{1}{2 \cosh \zeta} \left[ (\delta(\phi - 1) + \delta(\phi + 1)) + \sum_{n=1}^{\infty} 2(\zeta/2)^{2n} \frac{(1 - \phi^2)^{n-1}}{n!(n-1)!} \right]. \]

These expressions coincide with the results of [47].

**Appendix B. Instanton calculation for the one-dimensional Ising model**

As stated in the main text, we consider the one-dimensional Ising model with a periodic boundary condition, which is described by the Hamiltonian
\[ H[\{\sigma_i\}] = -J \sum_{i=1}^{L} \sigma_i \sigma_{i+1} - h \sum_{i=1}^{L} \sigma_i \] (B.1)
where \( \sigma_{i+1} \equiv \sigma_i \), the lattice spacing which is as unity, and, contrary to the case studied in section III, only the thermodynamic limit \( L \to \infty \) is considered.

We summarise the main (known) results about the model. The partition function can be computed using the transfer matrix method [51]. The transfer matrix is given by
\[ V = \begin{pmatrix} e^{h+j} & e^{-j} \\ e^{-j} & e^{-h-j} \end{pmatrix}. \] (B.2)
where \( \hbar = \beta h \) and \( J = \beta J \) and the partition function is obtained as \( Z_N = \text{Tr} V^N \). One can easily diagonalise the transfer matrix with the rotation
\[ U = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \] (B.3)
where \( 1/\tan(2\theta) = e^{2J} \sinh \hbar \). The eigenvalues are given by

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\[ \lambda_{\pm} = e^J \cosh \bar{h} \pm e^{2J} \cosh^2 \bar{h} - 2 \sinh(2\bar{J}) \].

(B.4)

The average magnetisation \( m = \langle \sigma_i \rangle \) is then

\[ m = \frac{e^J \sinh \bar{h}}{\sqrt{e^{2J} \cosh^2 \bar{h} - 2 \sinh(2\bar{J})}} \]

and the two-point connected correlation function

\[ G_c^{(2)}(|i - j|) = \langle \sigma_i \sigma_j \rangle - m^2 = \sin^2(2\theta) \left( \frac{\lambda_-}{\lambda_+} \right)^{|i-j|} \],

(B.5)

which can be rewritten as

\[ G_c^{(2)}(r) = \sin^2(2\theta) e^{-r/\xi} \]

(B.6)

where \( \xi = \left[ \ln \left( \frac{\lambda_+}{\lambda_-} \right) \right]^{-1} \). These expressions provide the magnetisation and the two-point function at fixed external magnetic field \( h \). However, we would like to have the magnetisation instead of the external field as the primary variable since we want to work with the effective action defined by the Legendre transform (34).

One can thus invert the relation in equation (B.5) to obtain

\[ \frac{\lambda_+}{\lambda_-} = \frac{\sqrt{1 - m^2(1 - e^{-4J})} + e^{-2J}}{\sqrt{1 - m^2(1 - e^{-4J})} - e^{-2J}} \],

(B.7)

so that, in the limit of very low temperature, one gets

\[ \xi(m)^{-1} \approx \frac{2e^{-2J}}{\sqrt{1 - m^2}} \]

(B.8)

and

\[ G_c^{(2)}(r) = (1 - m^2) e^{-r/\xi(m)} \].

(B.9)

These results are used in section IV.A.

Appendix C. Derivation of \( \Gamma_k^{(3)} \) and \( \Gamma_k^{(4)} \) from the one-dimensional Ising model

We start from the calculation of the three-point correlation function in the Ising model. We need to compute

\[ \langle \sigma_i \sigma_{i+r_1} \sigma_{i+r_1+r_2} \rangle \].

(C.1)

It is given by

\[ \langle \sigma_i \sigma_{i+r_1} \sigma_{i+r_1+r_2} \rangle = \frac{1}{\lambda_+^N + \lambda_-^N} \text{Tr} [S V^{r_1} S V^{r_2} S V^{N-r_1-r_2}] \]

(C.2)

where

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\[ S = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \]  

(C.3)

After diagonalising the transfer matrix (see appendix B) and using that

\[ U^{-1}SU = \begin{pmatrix} \cos(2\theta) & -\sin(2\theta) \\ -\sin(2\theta) & -\cos(2\theta) \end{pmatrix}, \]  

(C.4)

one obtains in the thermodynamic limit

\[ \langle \sigma_i \sigma_{i+r} \sigma_{i+r_1+r_2} \rangle = c^3 + cs^2 \times \left( \left( \frac{\lambda_-}{\lambda_+} \right)^{r_1} + \left( \frac{\lambda_-}{\lambda_+} \right)^{r_2} - \left( \frac{\lambda_-}{\lambda_+} \right)^{r_1+r_2} \right) \]  

(C.5)

where \( c = \cos(2\theta) = m \) and \( s = \sin(2\theta) = 1 - m^2 \) (\( c \) should not be confused with the notation also used for the prefactor of the derivative term in the bare action). The connected three-point correlation function is then given by

\[ \langle \sigma_i \sigma_j \sigma_{i+r_1+r_2} \rangle = \langle (\sigma_i - \langle \sigma_i \rangle)(\sigma_{i+r_1} - \langle \sigma_{i+r_1} \rangle) \times (\sigma_{i+r_1+r_2} - \langle \sigma_{i+r_1+r_2} \rangle) \rangle = -2cs^2 \left( \frac{\lambda_-}{\lambda_+} \right)^{r_1+r_2}. \]  

(C.6)

If we call \( x_1 = i, x_2 = i + r_1, x_3 = i + r_1 + r_2 \), the above result translates into

\[ W^{(3)}(x_1 < x_2 < x_3) = -2cs^2 \left( \frac{\lambda_-}{\lambda_+} \right)^{x_3-x_1} \]  

(C.7)

so that the three-point connected correlation function for generic arguments can be written as

\[ W^{(3)}(x_1, x_2, x_3) = -2cs^2 \left[ \left( \frac{\lambda_-}{\lambda_+} \right)^{x_3-x_1} \Theta(x_3-x_2)\Theta(x_2-x_1) + \left( \frac{\lambda_-}{\lambda_+} \right)^{x_1-x_3} \Theta(x_1-x_2)\Theta(x_2-x_3) \right. \\
+ \left( \frac{\lambda_-}{\lambda_+} \right)^{x_2-x_1} \Theta(x_2-x_3)\Theta(x_3-x_1) + \left( \frac{\lambda_-}{\lambda_+} \right)^{x_1-x_2} \Theta(x_1-x_3)\Theta(x_3-x_2) \right. \\
\left. + \left( \frac{\lambda_-}{\lambda_+} \right)^{x_2-x_3} \Theta(x_2-x_1)\Theta(x_1-x_3) + \left( \frac{\lambda_-}{\lambda_+} \right)^{x_1-x_2} \Theta(x_1-x_2)\Theta(x_2-x_3) \right], \]  

(C.8)

where \( \Theta(x) \) is the Heaviside step function. Neglecting the underlying lattice and performing the Fourier transform leads to

\[ W^{(3)}(p_1, p_2, p_3) = (2\pi)\delta(p_1 + p_2 + p_3) \frac{4cs^2\xi^{-2}((p_1p_2 + p_1p_3 + p_2p_3) - 3\xi^{-2})}{(p_1^2 + \xi^{-2})(p_2^2 + \xi^{-2})(p_3^2 + \xi^{-2})}. \]  

(C.9)

We now use the mapping between the Ising model and the \( \varphi^4 \) theory at low temperature and the relation between the connected three-point correlation function and the 1PI three-point vertex [46]. We finally obtain

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\[ \Gamma_k^3(p_1, p_2, p_3) = -\Gamma_k^{(2)}(p_1)\Gamma_k^{(2)}(p_2)\Gamma_k^{(2)}(p_3) W^{(3)}(p_1, p_2, p_3) = (2\pi)\delta(p_1 + p_2 + p_3) \frac{c^2}{2} \xi^2(\phi) \]
\[ \times \left[ 3\xi^{-2}(\phi) - (p_1 p_2 + p_1 p_3 + p_2 p_3) \right] \]  
where \( c = \phi \) and \( s^2 = \phi_{k,k}^2 - \phi^2 \). Note that it is \( \tilde{\Gamma}_k^{(2)}(p) \equiv G_{c,k}^{(2)}(p)^{-1} \), obtained from 
\[ \tilde{\Gamma}_k[\phi] = \Gamma[\phi] + \frac{\beta}{2} \int dq R_q(\phi) \phi(\phi(-q)), \]
which appears in equation (C.10) and not \( \Gamma_k^{(2)}(p) \). The calculation of \( \Gamma_k^{(4)} \) can be done in an analogous way and leads to equation (56). Although a cumbersome derivation, the higher orders can also be obtained along the same lines.

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