Algebraic Formalization of System Design Based on a Purposeful Approach

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Abstract. The article shows that the modern purposeful approach to the design of systems, which has a clear physical since and logical substantiation, allows to obtain a convenient mathematical description using modern algebra. The purposeful approach not only ensures the implementation of the system goals, but also allows you to formally describe the rules of optimization of the structure at the content level. The methods of decomposition and synthesis of the structure used in this approach to the design of systems are quite simply formalized and justified.

1. Introduction

The method of modern algebra allows obtaining new results in solving numerous problems of system theory [1-10]. Previously, algebraic formalization was used to solve the problems of observation, regulation, control of systems [1, 5, 7-10]. System design problems are difficult to formalize. They are usually solved at the descriptive level. All designed systems belong to the class of target systems, that is, there is always a certain purpose for the design, and the formulation of this purpose is usually strict and definite. Modern resources, on which basis systems are created, have reached a high level of development, allowing the use of an appropriate target approach (top-to-down approach) to the design of systems, that is, allowing to proceed from the goals of the system, and not from the possibilities of resources. The approach ensures the implementation of the system goals, allows justifying logically and formalizing the rules of system design. The essence of the approach is described in [11] and briefly explained below.

Within the framework of the approach, all the basic concepts used in the design of any system are strictly defined and logically linked: concept, technical appearance, technical requirements, system structure, system architecture, resources, efficiency criteria, design principles, restrictions. The main task in the design is to optimize the structure, based on the purpose (goals) of the system. The purposeful approach allows simplifying and determining the process of solving this problem by its natural decomposition.

The structure comprehensively characterizes the method and form of organization of parts of the system as a whole, that is, such an organization of resources that form the system, through which the implementation of system goals is achieved. Resources are understood as a material, energy, information or other form of ensuring the creation, existence and functioning of the system in accordance with its intended purpose. The existence and operation of any system is linked to the unavoidable consumption of resources. The system itself does not reimburse resources and they can...
be replenished only from the outside. The concept of "resources" is based on the principle of inevitability and irreversibility of their consumption.

Analysis of the entire set of resources involved in the creation of the system shows the possibility of their division into independent, non-overlapping classes. This division is inherent in the very nature of resources and is observed in practice. The presence of a finite set of independent resource classes determines the corresponding number of independent, orthogonal points of view on the system and its structure. Each class of resources strictly corresponds to its own, derived architecture, as an orthogonal projection of the system structure on this class of resources.

The structure in this approach is fully characterized by a strictly defined number of architectures, describing it from various independent, orthogonal points of view. Architecture is some orthogonal "slice", "layer" or 'stratum' of the structure. With a targeted approach, the architectures are designed according to the "top-to-down" principle: from the functional architecture reflecting the goals and purpose of the system to the architecture (topology) of the computer network and further to the physical (hardware) architecture and the architecture of the energy supply. A subset of the design choices for the respective classes of resources are evaluated according to the criteria of local selection. Generation of variants of architectures is carried out taking into account the axiomatic principles of design assigned in advance (for example, modularity, openness, etc.) and resource constraints. The combination of the best architecture choices selected in the each class of resources, forms the structure of the system.

The structure is checked for compliance with global selection criteria (e.g. weight, dimensions, energy consumption, etc.). If the criteria are met, the system design process is considered completed. If not satisfied, either the resource constraints are relaxed or the terms of reference (technical appearance and related goals and criteria) are adjusted. The process is iteratively repeated until the required goals are achieved.

It is obvious that the procedure for selecting the best variant of the structure on a set of criteria is the procedure for optimizing the structure. Thus, the purposeful approach not only ensures the implementation of the system goals, but also allows you to logically justify and formally describe the rules of optimization of the structure at the content level. The methods of decomposition and synthesis of the structure used in this approach to the design of systems are quite simply formalized and justified in the language of modern algebra.

The apparatus of modern algebra even in solving traditional problems of system theory allows to obtain new results. Previously, algebraic formalization was used to solve the problems of observation, regulation, control of systems [12-15]. System design problems are difficult to formalize. They are usually solved at the descriptive level. The article shows that the modern purposeful approach to the design of systems, which has a clear meaning and logical substantiation, allows to obtain a convenient mathematical description using modern algebra.

2. Description of the research

In the most general form, any system \( \Sigma \) can be represented by a triple

\[ \Sigma = (X, s, Y), \]

where \( s \) - is the structure of the system that images the set of inputs \( X \) to the set of outputs \( Y \), i.e.

\[ s : X \rightarrow Y. \]

At the design stage, the system is formalized in the form of a triple

\[ \Sigma = (R, s, C), \]

in which as a set of inputs \( R \) is considered a set of resources on the basis of which the system is expected to implement, and as a set of outputs \( C \) is considered a set of goals and related functions that the system must have in accordance with the requirements of technical specification to develop it. Then the structure of \( s \) is an image of \( s : R \rightarrow C \).

Along with the representation of the system in the form of a structure-imaging the set of resources \( R \) onto the set of goals \( C \), the system \( \Sigma \) can be defined as a purposeful set \( \omega_{s} \), which is a binary relationship between the sets of resources \( R \) and goals \( C \) (see Fig. 1), that is \( \omega_{s} \subset R \times C \) and
\[ \omega_s = R \times C = \{(r, c) \in \omega_s; \ r \mapsto s(r) = c\}, \text{ and } R \subset R, \ C \subset C. \]

Let there exist a partition \( R/\sim = R/\omega_s \) of the set \( R \) of resources into equivalence classes with respect to \( \omega_s \). Then (see [16]), as shown by the commutative diagram in Fig. 2, the structure of \( s: R \to C \) admits factorization \( S = \bar{S} \cdot P \), where \( P \) - is surjective, \( \bar{S} \) - is injective.

Physically, the factorization of the structure \( s \) means stratification of system \( \Sigma=(R,s,C) \) into independent (non-overlapping) levels of description-strata corresponding to the independent classes of resources identified during factorization and the corresponding classes of targets.

**Figure 1.** The system, as a binary relation.

**Figure 2.** Factorization of system structure.

**Definition.** Stratification of system \( \Sigma=(R,s,C) \) is the set of \( R/\omega_s \) classes of equivalence of its resources with respect to \( \omega_s \).

**Statement.** Stratification \( R/\omega_s \) induces in the structure \( s \) of system \( \Sigma \) functionally-oriented “non-overlapping” architectures \( a \in \bar{S} \).

Indeed, injectivity \( \bar{S} \) means that each stratum resources \( R_a \subset R/\omega_s \) through architecture \( a \in \bar{S} \) is imaged onto the certain subset of purposes \( C_a \subset C \) so that \( \bigcup_a C_a = C \), \( \bigcap_a C_a = \emptyset \),

that is, architecture \( a: R_a \to C_a \) is the image of some class of resources onto relevant class of purposes.

**Theorem (on factorization of the system structure).** Factorization (decomposition) of structure \( S = \bar{S}P \) into the composition of surjective \( P \) and injective \( \bar{S} \) imaging allows to implement the full set of purposes \( C \) of system \( \Sigma \) for independent designed architectures \( a \in \bar{S} \).

**Proving.** Truly:

a) for identified stratification of resources \( R/\omega_s \) image \( \bar{S} \) is the only one;

b) if \( s \) is surjective (and it was originally due to the requirements of the technical specification, since the structure needs to cover all the set of purposes specified with the task for the development of the system), then \( \bar{S} \) is surjective as well, which proves the statement of the theorem.

The following consequence follows from the theorem.

**Consequence.** The image (structure) \( S = \bar{S}P \) is bijective.
Truly:

a) from the limitedness of set $R$ of involved resources and surjectivity of image $p$ onto itself follows the bijectivity of $p$;

b) from the injectivity and surjectivity of image $\overline{S}$ follows the bijectivity of $\overline{S}$;

c) the composition $S = \overline{Sp}$ of the bijective image $p: R \rightarrow R/\omega_b$ and $\overline{S}: R/\omega_b \rightarrow C$ is bijective, and $s^{-1} = (\overline{Sp})^{-1} = p^{-1} \overline{s}^{-1}$. The existence of an inverse image of $s^{-1}$ follows from the bijectivity of $s$.

Thus, the system can be designed following the principle "down-to-top" from resources to goals, and Vice versa, following the principle "top – to-down" - from goals to resources. That is, from a formal mathematical point of view, both approaches are absolutely equivalent and legitimate. At the same time, in modern conditions, according to a number of additional criteria, a purposeful approach – from goals to resources – is more preferable, since it allows to significantly simplify and to vastly formalize the creative stage of generating options for the structures of the designed system. At the same time, such an approach ensures the implementation of the entire set of goals prescribed by the purpose of the system, and the set of resources involved in the implementation of the goals of the system are considered as limitations, which in modern conditions are significantly softer than before, and for some classes of resources are sometimes completely removed.

The considered consequence allows us to justify the possibility of a purposeful approach to system design:

– its acceptability is due to the bijectivity of structure $s$ and the existence of $s^{-1}$;

– its expediency is due to a simpler procedure of selecting "functional" structure $s^{-1}$, which is inverse to structure $s$, compared with the procedure for designing structure $s$ (usually based on combinatorial methods);

– the bijectivity of structure $s$ does not mean its uniqueness, but when a particular structure is fixed, the reverse structure $s^{-1}$ will be the only one.

Moreover, in favor of a purposeful approach says the fact that if you initially go from goals to resources, then the set of acceptable "rational" structures of $s^{-1}$ is significantly narrower than the set of "rational" structures of $s$ in the approach from resources to goals.

Thus, with the purposeful approach, the developer can proceed not from limited resources, but from a variety of tasks, goals, functions prescribed by the system.

With the purposeful approach, the main out of set $\mathcal{C} = \bigcup_a \mathcal{C}_a$ of subsets $\mathcal{C}_a$ of goals is a subset $\mathcal{C}_{af}$ and its corresponding functional architecture $a_f$, responsible for the implementation of the goals (functions) of the system for its purpose. The synthesis (selection) of the $a_f$ functional architecture is explained by the commutative diagrams shown in Fig. 3a, b.

![Figure 3](image_url)

**Figure 3.** The selection of a functional architecture of a system structure.

Allocation from the General structure and Autonomous consideration of any architecture, as already shown, are possible due to the factorization of the structure due to the presence of independent strata (classes) of resources and the induction of their respective independent architectures in the structure of the system.
Indeed, from the proven statements follows bijectivity \( \overline{p_i} \), that is, there is a one-to-one correspondence between the classes of resources \( R/\omega_0 \) and goals \( C/\alpha_{\omega,1} \), which allows to solve the problem of synthesis of the system separately for each resource class and the corresponding target class.

The further decomposition of the system structure synthesis procedure is related to the factorization of the image \( \overline{p_i} \) or the inverse image \( \overline{p_i}^{-1} \), the narrowing of which \( a_j \) for subset \( C_{af} \subset C/\omega_{\alpha,1} \) is shown in Fig. 3b.

Factorization of functional architecture \( a_j = \overline{a_j} \) means grouping of sub-goals and corresponding functional resources into equivalence classes at a lower level of detailing, which provides outlining of the basic functional subsystems of the designed system. Such a splitting \( C_{af}/\omega_{af} \) contains a subset of goals, providing solutions to the groups of system homogeneous tasks (such as information processing, control synthesis, etc.) and requires for its implementation a certain kind of resources from class (variety) of functional resources \( FcR \) (belonging to the entire set of resources \( R \)) involved to create system. This consistent and deeper factorization of target sets into nested non-overlapping subsets allows you to build a target tree and the corresponding resource tree. At the stage of system or, as it was called earlier, external design [17], the depth of detailing of the goals, as a rule, is limited to the construction of a set of independent architectures that form the structure of the system. Subsystems combined into these architectures are considered as indivisible elements with corresponding connections. Further detailing of architectures and their forming elements-subsystems is carried out at the stage of schematic or internal design [17].

Along with the satisfaction to an external purpose at the present time the main is the quality or rationality of structure \( \overline{s} \) of system \( \Sigma = (R/\omega_0, \overline{s}, C) \) and the components of the structure of architectures (see Fig. 3) characterized by qualitative indicators (properties) and numerical criteria. These properties and criteria are analyzed independently for any of architectures \( a \in s \) (by local properties and comparison criteria) and for structure \( s \) as a whole (by global properties and comparison criteria). It should also be borne in mind that in the structure of the system the depth of integration of elements in its constituent architectures may be different: due to the independence and uncorrelated architectures, some of them may be more integrated (strongly connected) than others.

3. Conclusion
Thus, it is shown that the purposeful approach, which has a clear meaningful sense and logical substantiation, allows both convenient mathematical description and substantiation using the language of modern algebra.

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