Spectral Gap and Exponential Decay of Correlations

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We study the relation between the spectral gap above the ground state and the decay of the correlations in the ground state in quantum spin and fermion systems with short-range interactions on a wide class of lattices. We prove that, if two observables anticommute with each other at large distance, then the nonvanishing spectral gap implies exponential decay of the corresponding correlation. When two observables commute with each other at large distance, the connected correlation function decays exponentially under the gap assumption. If the observables behave as a vector under the $U(1)$ rotation of a global symmetry of the system, we use previous results on the large distance decay of the correlation function to show the stronger statement that the correlation function itself, rather than just the connected correlation function, decays exponentially under the gap assumption on a lattice with a certain self-similarity in (fractal) dimensions $D < 2$. In particular, if the system is translationally invariant in one of the spatial directions, then this self-similarity condition is automatically satisfied. We also treat systems with long-range, power-law decaying interactions.
1 Introduction

In non-relativistic quantum many-body systems, a folk theorem states that a nonvanishing spectral gap above the ground state implies exponentially decaying correlations in the ground state. Perhaps this has been the most popular folk theorem in this field since Haldane [1] predicted a “massive phase” in low dimensional, isotropic quantum systems. Quite recently, this statement was partially proved [2] for quantum lattice systems with a global U(1) symmetry in (fractal) dimensions $D < 2$. More precisely, a bound which decays to zero at large distance was obtained for correlation functions whose observables behave as a vector under the U(1)-rotation. Unfortunately, the bound is weaker than the expected exponential decay. On the other hand, exponential clustering of the correlations was also proved recently [3, 4] for quantum many-body lattice systems under the gap assumption. This is a non-relativistic version of Fredenhagen’s theorem [5, 6] of relativistic quantum field theory. Clearly the following natural question arises: can this clustering property be combined with the above bound for the decay of the correlations to yield the tighter, exponentially decaying bound for the correlation functions themselves, rather than just for the connected correlation functions? We emphasize that these are different statements; given clustering, the decay of the correlation functions requires also that certain matrix elements vanish in the ground state sector.

In this paper, we address this problem and reexamine the above folk theorem by relying on the exponential clustering of the correlations. Our first step is to provide a rigorous proof of the exponential clustering. We extend the previous results in this case to treat long-range interactions including both power-law and exponentially decaying interactions. In the former case, all the upper bounds for the correlations become power-law bounds. We then prove that ground state correlation functions of observables which transform as vectors under a U(1) symmetry decay exponentially or with a power law, depending on the form of the interaction, given an additional assumption on a certain self-similarity. In particular, if the system is translationally invariant in one of the spatial directions, this self-similarity condition is automatically satisfied. Therefore the corresponding correlation functions decay exponentially for translationally invariant systems on one-dimensional regular lattices. As a byproduct, we also prove that, if two observables anticommute with each other at large distance, then the corresponding correlation in the ground state decays exponentially under the gap assumption for a wide class of lattice fermion systems with exponentially decaying interactions in any dimensions. In this case, we do not need any other assumption except for those on the interactions and the spectral gap.

This paper is organized as follows: In the next section, we give the precise definitions of the models, and describe our main results. In Section 3, we prove the clustering of generic correlation functions under the gap assumption, and obtain the upper decaying bound for the fermionic correlations. The decay of the bosonic correlations are treated in Section 4. Appendix A is devoted to the proof of the Lieb-Robinson bound for the group velocity of the information propagation in the models with a long-range interaction decaying by power law.
2 Models and main results

We consider quantum systems on generic lattices [7]. Let \( \Lambda_s \) be a set of the sites, \( x, y, z, w, \ldots \), and \( \Lambda_b \) a set of the bonds, i.e., pairs of sites, \( \{x, y\}, \{z, w\}, \ldots \). We call the pair, \( \Lambda := (\Lambda_s, \Lambda_b) \), the lattice. If a sequence of sites, \( x_0, x_1, x_2, \ldots, x_n \), satisfies \( \{x_{j-1}, x_j\} \in \Lambda_b \) for \( j = 1, 2, \ldots, n \), then we say that the path, \( \{x_0, x_1, x_2, \ldots, x_n\} \), has length \( n \) and connects \( x_0 \) to \( x_n \). We denote by \( \text{dist}(x, y) \) the graph-theoretic distance which is defined to be the shortest path length that one needs to connect \( x \) to \( y \). We denote by \( |X| \) the cardinality of the finite set \( X \). The Hamiltonian \( H_\Lambda \) is defined on the tensor product \( \bigotimes_{x \in \Lambda_s} H_x \) of a finite dimensional Hilbert space \( H_x \) at each site \( x \). We assume \( \sup_{\Lambda_s} \sup_x \dim H_x \leq N < \infty \). For a lattice fermion system, we consider the Fock space.

Consider the Hamiltonian of the form,

\[ H_\Lambda = \sum_{X \subseteq \Lambda_s} h_X, \quad (2.1) \]

where \( h_X \) is the local Hamiltonian of the compact support \( X \). We consider both power-law and exponentially decaying interactions \( h_X \).

For the power-law decaying interactions \( h_X \), we require the following conditions:

**Assumption 2.1** The interaction \( h_X \) satisfies

\[ \sum_{X \ni x, y} \|h_X\| \leq \frac{\lambda_0}{[1 + \text{dist}(x, y)]^\eta} \quad (2.2) \]

with positive constants, \( \lambda_0 \) and \( \eta \), and the lattice \( \Lambda \) equipped with the metric satisfies

\[ \sum_{z \in \Lambda_s} \frac{1}{[1 + \text{dist}(x, z)]^\eta} \times \frac{1}{[1 + \text{dist}(z, y)]^\eta} \leq \frac{p_0}{[1 + \text{dist}(x, y)]^\eta} \quad (2.3) \]

with a positive constant \( p_0 \).

**Remark:** If

\[ \sup_{\Lambda_s} \sup_x \sum_{y \in \Lambda_s} \frac{1}{[1 + \text{dist}(x, y)]^\eta} < \infty, \quad (2.4) \]

then the inequality (2.3) holds as follows:

\[
\begin{align*}
\sum_{z \in \Lambda_s} \frac{1}{[1 + \text{dist}(x, z)]^\eta} \times \frac{1}{[1 + \text{dist}(z, y)]^\eta} &= \frac{1}{[1 + \text{dist}(x, y)]^\eta} \sum_{z \in \Lambda_s} \frac{[1 + \text{dist}(x, y)]^\eta}{[1 + \text{dist}(x, z)]^\eta[1 + \text{dist}(z, y)]^\eta} \\
&\leq \frac{1}{[1 + \text{dist}(x, y)]^\eta} \sum_{z \in \Lambda_s} 2^\eta \left\{ \frac{1}{[1 + \text{dist}(x, z)]^\eta} + \frac{1}{[1 + \text{dist}(z, y)]^\eta} \right\}.
\end{align*}
\]  

(2.5)
where we have used the inequality, \([1 + \text{dist}(x, y)]^\eta \leq 2^\eta([1 + \text{dist}(x, z)]^\eta + [1 + \text{dist}(z, y)]^\eta)\).

From the assumption (2.2) and the condition (2.4), one has

\[
\sup_x \sum_{X \ni x} \|h_X\| |X| \leq s_0 < \infty, \tag{2.6}
\]

where \(s_0\) is a positive constant which is independent of the volume of \(|\Lambda_s|\).

Instead of these conditions, we can also require:

**Assumption 2.2** The interaction \(h_X\) satisfies

\[
\sup_x \sum_{X \ni x} \|h_X\| |X|[1 + \text{diam}(X)]^\eta \leq s_1 < \infty, \tag{2.7}
\]

where \(\eta\) is a positive constant, \(\text{diam}(X)\) is the diameter of the set \(X\), i.e., \(\text{diam}(X) = \max\{\text{dist}(x, y) | x, y \in X\}\), and \(s_1\) is a positive constant which is independent of the volume of \(|\Lambda_s|\).

For exponentially decaying interactions \(h_X\), we require one of the following two assumptions:

**Assumption 2.3** There exists a positive \(\eta\) satisfying the condition (2.4). The interaction \(h_X\) satisfies

\[
\sum_{X \ni x, y} \|h_X\| \exp[-(\mu + \varepsilon) \text{dist}(x, y)] \leq \lambda_0 \exp[-\mu \text{diam}(X)] \tag{2.8}
\]

with some positive constants, \(\lambda_0, \mu\) and \(\varepsilon\).

**Remark:** From the conditions, we have

\[
\exp[-(\mu + \varepsilon) \text{dist}(x, y)] \leq \frac{\lambda_0^\eta \exp[-\mu \text{dist}(x, y)]}{[1 + \text{dist}(x, y)]^\eta} \tag{2.9}
\]

with a positive constant \(\lambda_0^\eta\), and

\[
\sum_{z \in \Lambda_s} \frac{\exp[-\mu \text{dist}(x, z)]}{[1 + \text{dist}(x, z)]^\eta} \times \frac{\exp[-\mu \text{dist}(z, y)]}{[1 + \text{dist}(z, y)]^\eta} \leq \frac{p_0 \exp[-\mu \text{dist}(x, y)]}{[1 + \text{dist}(x, y)]^\eta} \tag{2.10}
\]

with a positive constant \(p_0\) in the same way as in the preceding remark.

**Assumption 2.4** The interaction \(h_X\) satisfies

\[
\sup_x \sum_{X \ni x} \|h_X\| |X| \exp[\mu \text{diam}(X)] \leq s_1 < \infty, \tag{2.11}
\]

where \(\mu\) is a positive constant, and \(s_1\) is a positive constant which is independent of the volume of \(|\Lambda_s|\).
Remark: This assumption is milder than that in [6] by the absence of the factor $N^2|X|$ in the summand.

Further we assume the existence of a “uniform gap” above the ground state sector of the Hamiltonian $H_\Lambda$. The precise definition of the “uniform gap” is:

**Definition 2.5 (Uniform gap):** We say that there is a uniform gap above the ground state sector if the spectrum $\sigma(H_\Lambda)$ of the Hamiltonian $H_\Lambda$ satisfies the following conditions: The ground state of the Hamiltonian $H_\Lambda$ is $q$-fold (quasi)degenerate in the sense that there are $q$ eigenvalues, $E_{0,1}, \ldots, E_{0,q}$, in the ground state sector at the bottom of the spectrum of $H_\Lambda$ such that

$$\Delta E := \max_{\mu, \mu'} \{|E_{0,\mu} - E_{0,\mu'}|\} \to 0 \quad \text{as} \quad |\Lambda_s| \to \infty. \quad (2.12)$$

Further the distance between the spectrum, $\{E_{0,1}, \ldots, E_{0,q}\}$, of the ground state and the rest of the spectrum is larger than a positive constant $\Delta E$ which is independent of the volume $|\Lambda_s|$. Namely there is a spectral gap $\Delta E$ above the ground state sector.

Let $A_X, B_Y$ be observables with the support $X, Y \subset \Lambda_s$, respectively. We say that the pair of two observables, $A_X$ and $B_Y$, is fermionic if they satisfy the anticommutation relation, $\{A_X, B_Y\} = 0$ for $X \cap Y = \emptyset$. If they satisfy the commutation relation, then we call the pair bosonic.

Define the ground-state expectation as

$$\langle \cdots \rangle_{0, \Lambda} := \frac{1}{q} \text{Tr} (\cdots) P_{0, \Lambda}, \quad (2.13)$$

where $P_{0, \Lambda}$ is the projection onto the ground state sector. For the infinite volume,

$$\langle \cdots \rangle_{0} := \lim_{|\Lambda_s| \to \infty} \langle \cdots \rangle_{0, \Lambda}, \quad (2.14)$$

where we take a suitable subsequence of finite lattices $\Lambda$ going to the infinite volume so that the expectation converges to a linear functional for a set of quasi-local observables. Although the ground-state expectation thus constructed depends on the subsequence of the lattices and on the observables, our results below hold for any ground-state expectation thus constructed. Further, we denote by

$$\omega(\cdots) := \lim_{|\Lambda_s| \to \infty} \langle \Phi_\Lambda, (\cdots) \Phi_\Lambda \rangle \quad (2.15)$$

the ground-state expectation in the infinite volume for a normalized vector $\Phi_\Lambda$ in the sector of the ground state for finite lattice $\Lambda$.

**Theorem 2.6 (Clustering of fermionic correlations):** Let $A_X, B_Y$ be fermionic observables with a compact support. Assume that there exists a uniform spectral gap $\Delta E > 0$ above the ground state sector in the spectrum of the Hamiltonian $H_\Lambda$ in the sense of Definition 2.5. Let $\omega$ be a ground-state expectation (2.15) in the infinite volume limit. Then the following bound is valid:

$$\left| \omega(A_X B_Y) - \frac{1}{2} \left[ \omega(A_X P_0 B_Y) - \omega(B_Y P_0 A_X) \right] \right| \leq \text{Const.} \times \begin{cases} [1 + \text{dist}(X, Y)]^{-\tilde{\eta}}, & \text{for power-law decaying } h_X; \\ \exp[-\tilde{\mu} \text{dist}(X, Y)], & \text{for exponentially decaying } h_X, \end{cases} \quad (2.16)$$
where $P_0$ is the projection onto the sector of the infinite-volume ground state, and

$$\tilde{\eta} = \frac{\eta}{1 + 2v_\eta/\Delta E} \quad \text{and} \quad \tilde{\mu} = \frac{\mu}{1 + 2v_\mu/\Delta E}. \quad (2.17)$$

Here $v_\eta$ and $v_\mu$ are, respectively, an increasing function of $\eta$ and $\mu$, and give an upper bound of the group velocity of the information propagation.

**Remark:** Clearly there exists a maximum $\mu_{\text{max}}$ such that the bound (2.8) holds for any $\mu \leq \mu_{\text{max}}$. Combining this observation with (2.17), there exists a maximum $\tilde{\mu} = \max_{\mu \leq \mu_{\text{max}}} \{\mu/(1 + 2v_\mu/\Delta E)\}$ which gives the optimal decay bound. When the interaction $h_X$ is of finite range, one can take any large $\mu$. But the upper bound $v_\mu$ of the group velocity exponentially increases as $\mu$ increases because $v_\mu$ depends on $\lambda_0$ of (2.8). In consequence, a finite $\tilde{\mu}$ gives the optimal bound.

Formally applying the identity, $\langle A_X P_0 B_Y \rangle_0 = \langle B_Y P_0 A_X \rangle_0$, for the bound (2.16), we have the following decay bound for the correlation:

**Corollary 2.7** Let $A_X, B_Y$ be fermionic observables with a compact support. Assume that there exists a uniform spectral gap $\Delta E > 0$ above the ground state sector in the spectrum of the Hamiltonian $H_\Lambda$ in the sense of Definition 2.5. Then the following bound is valid:

$$|\langle A_X B_Y \rangle_0| \leq \text{Const.} \times \left\{ \begin{array}{ll}
[1 + \text{dist}(X, Y)]^{-\tilde{\eta}}, & \text{for power-law decaying } h_X; \\
\exp[-\tilde{\mu} \text{dist}(X, Y)], & \text{for exponentially decaying } h_X,
\end{array} \right. \quad (2.18)$$

in the infinite volume limit, where $\tilde{\eta}, \tilde{\mu}$ are as defined above.

**Theorem 2.8 (Clustering of bosonic correlations):** Let $A_X, B_Y$ be bosonic observables with a compact support. Assume that there exists a uniform spectral gap $\Delta E > 0$ above the ground state sector in the spectrum of the Hamiltonian $H_\Lambda$ in the sense of Definition 2.5. Let $\omega$ be a ground-state expectation (2.15) in the infinite volume limit. Then the following bound is valid:

$$\left| \omega(A_X B_Y) - \frac{1}{2} [\omega(A_X P_0 B_Y) + \omega(B_Y P_0 A_X)] \right| \leq \text{Const.} \times \left\{ \begin{array}{ll}
[1 + \text{dist}(X, Y)]^{-\tilde{\eta}}, & \text{for power-law decaying } h_X; \\
\exp[-\tilde{\mu} \text{dist}(X, Y)], & \text{for exponentially decaying } h_X,
\end{array} \right. \quad (2.19)$$

where $\tilde{\eta}, \tilde{\mu}$ are as defined above.

**Remark:** Theorem 2.8 is a clustering bound for the connected correlation functions. We now make some additional definitions that will enable us, in certain cases, to prove the decay of $[\omega(A_X P_0 B_Y) + \omega(B_Y P_0 A_X)]/2$ so that Theorem 2.8 can be replaced with a stronger bound below, Theorem 2.10.

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$^1$ $\omega(\cdots P_0 \cdots)$ is also defined as a bilinear functional for a set of quasilocal observables in the weak$^*$ limit.

$^2$ See Section 3 for details.
Definition 2.9 (Self-similarity): Write \( m = q^2 \) with the degeneracy \( q \) of the ground state sector. We say that the system has self-similarity if the following conditions are satisfied: For any observable \( A \) of compact support and any given large \( L > 0 \), there exist transformations, \( R_1, R_2, \ldots, R_m \), and observables, \( B^{(1)}, B^{(2)}, \ldots, B^{(m)} \), such that the Hamiltonian \( H_\Lambda \) is invariant under the transformations, i.e., \( R_j(H_\Lambda) = H_\Lambda \) for any lattice \( \Lambda \) with sufficiently large \( |\Lambda_s| \), and that the observables satisfy the following conditions:

\[
B^{(j)} = R_j(A) \quad \text{and} \quad \left(B^{(j)}\right)^\dagger = R_j(A^\dagger) \quad \text{for} \quad j = 1, 2, \ldots, m, \tag{2.20}
\]

and

\[
dist(supp A, supp B^{(j)}) \geq L \quad \text{for} \quad j = 1, 2, \ldots, m, \tag{2.21}
\]

and

\[
dist(supp B^{(j)}, supp B^{(k)}) \geq L \quad \text{for} \quad j \neq k. \tag{2.22}
\]

In Section 4, we will discuss other conditions similar to this self-similarity condition.

Theorem 2.10 Assume that the degeneracy \( q \) of the ground state sector of the Hamiltonian \( H_\Lambda \) is finite in the infinite volume limit, and that there exists a uniform spectral gap \( \Delta E > 0 \) above the ground state sector in the spectrum of the Hamiltonian \( H_\Lambda \) in the sense of Definition 2.5. Further assume that the system has self-similarity in the sense of Definition 2.9, and that there exists a subset \( \mathcal{A}_b^s \) of bosonic observables with a compact support such that \( R_j(\mathcal{A}_b^s) \subset \mathcal{A}_b^s = (\mathcal{A}_b^s)^\dagger \) for \( j = 1, 2, \ldots, m \), and that \( \langle A_X^s, B_Y^s \rangle_0 \rightarrow 0 \) as \( dist(X, Y) \rightarrow \infty \) for any pair of bosonic observables, \( A_X^s, B_Y^s \in \mathcal{A}_b^s \). Let \( \omega \) be a ground-state expectation (2.15) in the infinite volume limit, and let \( A_X, B_Y \) be a pair of bosonic observables satisfying \( A_X \in \mathcal{A}_b^s \). Then the following bound is valid:

\[
|\omega(A_X B_Y)| \leq \text{Const.} \times \left\{ \begin{array}{ll}
[1 + \text{dist}(X, Y)]^{-\eta}, & \text{for power-law decaying } h_X; \\
\exp[-\mu \text{dist}(X, Y)], & \text{for exponentially decaying } h_X,
\end{array} \right.
\tag{2.23}
\]

where \( \eta, \mu \) are as defined above.

Remark: 1. If the finite system is translationally invariant in one of the spatial directions with a periodic boundary condition, then the self-similarity condition of Definition 2.9 is automatically satisfied by taking the translation as the transformation \( R_j \). Thus we do not need an additional assumption for such systems.

2. Theorem 2.10 can be extended to a system having infinite degeneracy of the ground state sector in the infinite volume limit if the degeneracy for finite volume is sufficiently small compared to the volume of the system. See Theorem 4.1 in Section 4 for details.

In order to apply this theorem, we need to be able to show that \( \langle A_X B_Y \rangle_0 \rightarrow 0 \) as \( dist(X, Y) \rightarrow \infty \) in the infinite volume for any pair of bosonic observables, \( A_X, B_Y \in \mathcal{A}_b^s \). However, this was proven [2] for quantum spin or fermion systems with a global U(1) symmetry on a class of lattices with (fractal) dimension \( D < 2 \) as defined in (2.25) below, so long as the observables behave as a vector under the U(1) rotation.

We first define the dimension for these lattices. The “sphere”, \( S_r(x) \), centered at \( x \in \Lambda_s \) with the radius \( r \) is defined as

\[
S_r(x) := \{ y \in \Lambda_s | \text{dist}(y, x) = r \}. \tag{2.24}
\]
Assume that there exists a “(fractal) dimension” $D \geq 1$ of the lattice $\Lambda$ such that the number $|S_r(x)|$ of the sites in the sphere satisfies

$$\sup_{x \in \Lambda_s} |S_r(x)| \leq C_0 r^{D-1}$$

(2.25)

with some positive constant $C_0$. This class of the lattices is the same as in [8].

Consider spin or fermion systems with a global $U(1)$ symmetry on the lattice $\Lambda$ with (fractal) dimension $1 \leq D < 2$, and require the existence of a uniform gap above the ground state sector of the Hamiltonian $H_\Lambda$ in the sense of Definition 2.5. Although the method of [2] can be applied to a wide class of such systems, we consider only two important examples, the Heisenberg and the Hubbard models. We take the set $A_b^s$ to be the bosonic observables which behave as a vector under the $U(1)$ rotation. In the rest of this section we use the results of [2] to show as in (2.29,2.31) that the correlation function for this class of observables in these models does decay to zero as $\text{dist}(X, Y) \to \infty$. The bounds (2.29,2.31) provide only a slow bound on the decay. However, this slow bound on the decay suffices, in conjunction with the self-similarity condition of Definition 2.9 to apply Theorem 2.10. Thus, under the self-similarity assumption as well as the gap assumption, all the upper bounds below (2.29,2.31) are replaced with exponentially decaying bounds by Theorem 2.10. In particular, a system with a translational invariance automatically satisfies the self-similarity condition as mentioned above. Therefore the corresponding correlations show exponential decay for translationally invariant systems on one-dimensional regular lattices.

**XXZ Heisenberg model:** The Hamiltonian $H_\Lambda$ is given by

$$H_\Lambda = H^{XY}_\Lambda + V_\Lambda(\{S^{(3)}_x\})$$

(2.26)

with

$$H^{XY}_\Lambda = 2 \sum_{\{x,y\} \in \Lambda_b} J^{XY}_{x,y} \left[ S^{(1)}_x S^{(1)}_y + S^{(2)}_x S^{(2)}_y \right],$$

(2.27)

where $S_x = (S^{(1)}_x, S^{(2)}_x, S^{(3)}_x)$ is the spin operator at the site $x \in \Lambda_s$ with the spin $S = 1/2, 1, 3/2, \ldots$, and $J^{XY}_{x,y}$ are real coupling constants; $V_\Lambda(\{S^{(3)}_x\})$ is a real function of the $z$-components, $\{S^{(3)}_x\}_{x \in \Lambda_s}$, of the spins. For simplicity, we take

$$V_\Lambda(\{S^{(3)}_x\}) = \sum_{\{x,y\} \in \Lambda_b} J^Z_{x,y} S^{(3)}_x S^{(3)}_y$$

(2.28)

with real coupling constants $J^Z_{x,y}$. Assume that there are positive constants, $J^{XY}_{\text{max}}$ and $J^Z_{\text{max}}$, which satisfy $|J^{XY}_{x,y}| \leq J^{XY}_{\text{max}}$ and $|J^Z_{x,y}| \leq J^Z_{\text{max}}$ for any bond $\{x, y\} \in \Lambda_b$.

Consider the transverse spin-spin correlation, $\langle S^+_x S^-_y \rangle_0$, where $S^+_x := S^{(1)}_x \pm i S^{(2)}_x$.

**Theorem 2.11** Assume that the fractal dimension $D$ of (2.25) satisfies $1 \leq D < 2$, and that there exists a uniform spectral gap $\Delta E > 0$ above the ground state sector in the spectrum of the Hamiltonian $H_\Lambda$ of (2.26) in the sense of Definition 2.5. Then there exists a positive constant $\gamma$ such that the transverse spin-spin correlation satisfies the bound,

$$|\langle S^+_x S^-_y \rangle_0| \leq \text{Const.} \exp \left[ -\gamma \{\text{dist}(x, y)\}^{1-D/2} \right],$$

(2.29)

in the thermodynamic limit $|\Lambda_s| \to \infty$. 
The proof is given in [2], and we remark that the result can be extended to more complicated correlations such as the multispin correlation, \( \langle S_{x_1}^+ \cdots S_{x_j}^+ S_{y_1}^- \cdots S_{y_l}^- \rangle \). If the system satisfies the self-similarity condition of Definition 2.9, then the upper bound, (2.29), can be replaced with a stronger exponentially decaying one by Theorem 2.10.

**Hubbard model** [9, 10]: The Hamiltonian on the lattice \( \Lambda \) is given by

\[
H_\Lambda = - \sum_{\{x,y\} \in \Lambda_s} \sum_{\alpha = \uparrow, \downarrow} \left( t_{x,y} c_{x,\alpha}^\dagger c_{y,\alpha} + t_{x,y}^* c_{y,\alpha}^\dagger c_{x,\alpha} \right) + V(\{n_{x,\alpha}\}) + \sum_{x \in \Lambda_s} B_x \cdot S_x, \tag{2.30}
\]

where \( c_{x,\alpha}^\dagger, c_{x,\alpha} \) are, respectively, the electron creation and annihilation operators with the z component of the spin \( \mu = \uparrow, \downarrow \), \( n_{x,\alpha} = c_{x,\alpha}^\dagger c_{x,\alpha} \) is the corresponding number operator, and \( S_x = (S_x^{(1)}, S_x^{(2)}, S_x^{(3)}) \) are the spin operator given by \( S_x^{(a)} = \sum_{\alpha, \beta = \uparrow, \downarrow} c_{x,\alpha}^\dagger \sigma_{\alpha,\beta}^{(a)} c_{x,\beta} \) with the Pauli spin matrix \( \sigma_{\alpha,\beta}^{(a)} \) for \( a = 1, 2, 3 \); \( t_{i,j} \in \mathbb{C} \) are the hopping amplitude, \( V(\{n_{x,\alpha}\}) \) is a real function of the number operators, and \( B_x = (B_x^{(1)}, B_x^{(2)}, B_x^{(3)}) \in \mathbb{R}^3 \) are local magnetic fields. Assume that the interaction \( V(\{n_{x,\alpha}\}) \) is of finite range in the sense of the graph theoretic distance.

**Theorem 2.12** Assume that the fractal dimension \( D \) of (2.25) satisfies \( 1 \leq D < 2 \), and that there exists a uniform spectral gap \( \Delta E > 0 \) above the ground state sector in the spectrum of the Hamiltonian \( H_\Lambda \) of (2.30) in the sense of Definition 2.5. Then the following bound is valid:

\[
\left| \langle c_{x,\uparrow}^\dagger c_{x,\downarrow}^\dagger c_{y,\uparrow} c_{y,\downarrow} \rangle_0 \right| \leq \text{Const. exp} \left[ -\gamma \{\text{dist}(x,y)\}^{1-D/2} \right] \tag{2.31}
\]

with some constant \( \gamma \) in the thermodynamic limit \( |\Lambda_s| \to \infty \). If the local magnetic field has the form \( B_x = (0, 0, B_z) \), then we further have

\[
\left| \langle S_x^+ S_y^- \rangle_0 \right| \leq \text{Const. exp} \left[ -\gamma' \{\text{dist}(x,y)\}^{1-D/2} \right] \tag{2.32}
\]

with some constant \( \gamma' \).

The proof is given in [2]. Clearly the Hamiltonian \( H_\Lambda \) of (2.30) commutes with the total number operator \( N_\Lambda = \sum_{x \in \Lambda_s} \sum_{\mu = \uparrow, \downarrow} n_{x,\mu} \) for a finite volume \( |\Lambda_s| < \infty \). We denote by \( H_{\Lambda,N} \) the restriction of \( H_\Lambda \) onto the eigenspace of \( N_\Lambda \) with the eigenvalue \( N \). Let \( P_{0,\Lambda,N} \) be the projection onto the ground state sector of \( H_{\Lambda,N} \), and we denote the ground-state expectation by

\[
\langle \cdots \rangle_{0,\nu} = \text{weak}^* - \lim_{|\Lambda_s| \to \infty} \frac{1}{q_N} \text{Tr} (\cdots) P_{0,\Lambda,N}, \tag{2.33}
\]

where \( q_N \) is the degeneracy of the ground state, and \( \nu \) is the limit of the filling factor \( N/|\Lambda_s| \) of the electrons. Since the operators \( S_x^\pm \) do not connect the sectors with the different eigenvalues \( N \), we have

**Theorem 2.13** Assume that the fractal dimension \( D \) of (2.25) satisfies \( 1 \leq D < 2 \), and that there exists a uniform spectral gap \( \Delta E > 0 \) above the ground state sector in the
the ground state sector. The ground state expectation of the commutator is written as 
\[ \left\langle S_x^+ S_y^- \right\rangle_{0,\nu} \leq \text{Const. exp} \left[ -\gamma' \{\text{dist}(x, y)\}^{1-D/2} \right] \] (2.34)
with some constant \(\gamma'\) in the infinite volume limit.

The proof is given in [2]. If the system satisfies the self-similarity condition of Definition 2.9, then these three upper bounds, (2.31), (2.32) and (2.34), can be replaced with a stronger exponentially decaying one by Theorem 2.10.

### 3 Clustering of correlations

In order to prove the power-law and the exponential clustering, Theorems 2.6 and 2.8, we follow the method [3]. The key tools of the proof are Lemma 3.1 below and the Lieb-Robinson bound [6, 11] for the group velocity of the information propagation. The sketch of the proof is that the static correlation function can be derived from the time-dependent correlation function by the lemma, and the large-distance behavior of the time-dependent correlation function is estimated by the Lieb-Robinson bound. As a byproduct, we obtain the decay bound (2.18) for fermionic observables.

Consider first the case of the bosonic observables. Let \(A_X, B_Y\) be bosonic observables with compact supports \(X, Y \subset \Lambda_s\), respectively, and let \(A_X(t) = e^{itH_A}a_X e^{-itH_A}\), where \(t \in \mathbb{R}\) and \(H_A\) is the Hamiltonian for finite volume. Let \(\Phi\) be a normalized vector in the ground state sector. The ground state expectation of the commutator is written as
\[
\langle \Phi, [A_X(t), B_Y]\Phi \rangle = \langle \Phi, A_X(t)(1 - P_{0,\Lambda})B_Y\Phi \rangle - \langle \Phi, B_Y(1 - P_{0,\Lambda})A_X(t)\Phi \rangle + \langle \Phi, A_X(t)P_{0,\Lambda}B_Y\Phi \rangle - \langle \Phi, B_Y P_{0,\Lambda}A_X(t)\Phi \rangle. \tag{3.1}
\]
In terms of the ground state vectors \(\Phi_{0,\nu}, \nu = 1, 2, \ldots, q\), with the energy eigenvalues, \(E_{0,\nu}\), and the excited state vectors \(\Phi_n\) with \(E_n, n = 1, 2, \ldots, \), one has
\[
\langle \Phi, A_X(t)(1 - P_{0,\Lambda})B_Y\Phi \rangle = \sum_{\nu,\nu'} \sum_{n \neq 0} a_\nu^* a_{\nu'} \langle \Phi_{0,\nu}, A_X\Phi_n \rangle \langle \Phi_n, B_Y\Phi_{0,\nu'} \rangle e^{-it(E_n - E_{0,\nu})}, \tag{3.2}
\]
\[
\langle \Phi, B_Y(1 - P_{0,\Lambda})A_X(t)\Phi \rangle = \sum_{\nu,\nu'} \sum_{n \neq 0} a_\nu^* a_{\nu'} \langle \Phi_{0,\nu}, B_Y\Phi_n \rangle \langle \Phi_n, A_X\Phi_{0,\nu'} \rangle e^{it(E_n - E_{0,\nu'})}, \tag{3.3}
\]
\[
\langle \Phi, A_X(t)P_{0,\Lambda}B_Y\Phi \rangle = \sum_{\nu,\nu'} \sum_{\mu} a_\nu^* a_{\nu'} \langle \Phi_{0,\nu}, A_X\Phi_{0,\mu} \rangle \langle \Phi_{0,\mu}, B_Y\Phi_{0,\nu'} \rangle e^{-it(E_{0,\mu} - E_{0,\nu'})}, \tag{3.4}
\]
and
\[
\langle \Phi, B_Y P_{0,\Lambda}A_X(t)\Phi \rangle = \sum_{\nu,\nu'} \sum_{\mu} a_\nu^* a_{\nu'} \langle \Phi_{0,\nu}, B_Y\Phi_{0,\mu} \rangle \langle \Phi_{0,\mu}, A_X\Phi_{0,\nu'} \rangle e^{it(E_{0,\mu} - E_{0,\nu'})}, \tag{3.5}
\]
where we have written
\[
\Phi = \sum_{\nu = 1}^q a_\nu \Phi_{0,\nu}. \tag{3.6}
\]
In order to get the bound for \(\langle \Phi, A_X(t = 0)B_Y\Phi \rangle\), we want to extract only the “negative frequency part” (3.2) from the time-dependent correlation functions (3.1). For this purpose, we use the following lemma [3]:
Lemma 3.1 Let \( E \in \mathbb{R} \), and \( \alpha > 0 \). Then

\[
\lim_{T \to \infty} \lim_{\epsilon \downarrow 0} \frac{i}{2\pi} \int_{-T}^{T} e^{-iEt} e^{-\alpha t^2} dt = \frac{1}{2\pi} \sqrt{\frac{\pi}{\alpha}} \int_{-\infty}^{\infty} d\omega \exp[-(\omega + E)^2/(4\alpha)]
= \left\{ \begin{array}{ll}
1 + \mathcal{O}(\exp[-\Delta E^2/(4\alpha)]) & \text{for } E \geq \Delta E; \\
\mathcal{O}(\exp[-\Delta E^2/(4\alpha)]) & \text{for } E \leq -\Delta E.
\end{array} \right. \tag{3.7}
\]

Proof: Write

\[
I(E) = \frac{i}{2\pi} \int_{-T}^{T} e^{-iEt} e^{-\alpha t^2} dt. \tag{3.8}
\]

Using the Fourier transformation,

\[
e^{-iEt} e^{-\alpha t^2} = \frac{1}{2\pi} \sqrt{\frac{\pi}{\alpha}} \int_{-\infty}^{\infty} \exp[-(\omega + E)^2/(4\alpha)] e^{it\omega} d\omega, \tag{3.9}
\]

we decompose the integral \( I(E) \) into three parts as

\[
I(E) = I_-(E) + I_0(E) + I_+(E), \tag{3.10}
\]

where

\[
I_-(E) = \frac{i}{2\pi} \frac{1}{2\pi} \sqrt{\frac{\pi}{\alpha}} \int_{-T}^{T} dt \frac{1}{t + i\epsilon} \int_{-\infty}^{-\Delta \omega} d\omega \exp[-(\omega + E)^2/(4\alpha)] e^{i\omega t}, \tag{3.11}
\]

\[
I_0(E) = \frac{i}{2\pi} \frac{1}{2\pi} \sqrt{\frac{\pi}{\alpha}} \int_{-T}^{T} dt \frac{1}{t + i\epsilon} \int_{-\Delta \omega}^{\Delta \omega} d\omega \exp[-(\omega + E)^2/(4\alpha)] e^{i\omega t}, \tag{3.12}
\]

and

\[
I_+(E) = \frac{i}{2\pi} \frac{1}{2\pi} \sqrt{\frac{\pi}{\alpha}} \int_{-T}^{T} dt \frac{1}{t + i\epsilon} \int_{\Delta \omega}^{\infty} d\omega \exp[-(\omega + E)^2/(4\alpha)] e^{i\omega t}, \tag{3.13}
\]

where we choose \( \Delta \omega = bT^{-1/2} \) with some positive constant \( b \).

First let us estimate \( I_0(E) \). Note that

\[
\frac{1}{t + i\epsilon} = \frac{t + i\epsilon}{t^2 + \epsilon^2}. \tag{3.14}
\]

Using this identity, one has

\[
I_0(E) = \frac{i}{2\pi} \frac{1}{2\pi} \sqrt{\frac{\pi}{\alpha}} \int_{-\Delta \omega}^{\Delta \omega} d\omega \exp[-(\omega + E)^2/(4\alpha)] \int_{-T}^{T} dt \left[ \frac{t \sin \omega t}{t^2 + \epsilon^2} - \frac{i\epsilon \cos \omega t}{t^2 + \epsilon^2} \right], \tag{3.15}
\]

where we have interchanged the order of the double integral by relying on \(|t| \leq T < \infty\).

Since the integral about \( t \) can be bounded by some constant, one obtains

\[
|I_0(E)| \leq \text{Const.} \times \alpha^{-1/2} \Delta \omega \leq \text{Const.} \times \alpha^{-1/2} T^{-1/2}. \tag{3.16}
\]

Therefore the corresponding contribution is vanishing in the limit \( T \uparrow \infty \).

Note that

\[
\frac{i}{2\pi} \int_{-T}^{T} dt \frac{e^{i\omega t}}{t + i\epsilon} = \left\{ \begin{array}{ll}
\mathcal{O}(\omega^{-1}T^{-1}) & \text{for } \omega > 0; \\
e^{i\omega} + \mathcal{O}(\omega^{-1}T^{-1}) & \text{for } \omega < 0.
\end{array} \right. \tag{3.17}
\]
Using this, the function $I_+(E)$ of (3.13) can be evaluated as
\[ |I_+(E)| \leq \text{Const.} \times T^{-1/2}. \] (3.18)
This is also vanishing in the limit.

Thus it is enough to consider only the integral $I_-(E)$. In the same way as the above, one has
\[ I_-(E) = \frac{1}{2\pi} \sqrt{\frac{\pi}{\alpha}} \int_{-\infty}^{-\Delta \omega} d\omega \exp[ -(\omega + E)^2/(4\alpha)] e^{i\omega} + O(T^{-1/2}). \] (3.19)
Since $e^{i\omega} \leq 1$ for $\omega < 0$, one has
\[ \lim_{T \to \infty} \lim_{\alpha \to 0} I_-(E) = \frac{1}{2\pi} \sqrt{\frac{\pi}{\alpha}} \int_{-\infty}^{0} d\omega \exp[ -(\omega + E)^2/(4\alpha)]. \] (3.20)
Note that, for $E \leq -\Delta E$,
\[ \frac{1}{2\pi} \sqrt{\frac{\pi}{\alpha}} \int_{-\infty}^{0} d\omega \exp[ -(\omega + E)^2/(4\alpha)] \leq \frac{1}{2} \exp[-\Delta E^2/(4\alpha)], \] (3.21)
and, for $E \geq \Delta E$,
\[ \frac{1}{2\pi} \sqrt{\frac{\pi}{\alpha}} \int_{-\infty}^{0} d\omega \exp[ -(\omega + E)^2/(4\alpha)] = \frac{1}{2\pi} \sqrt{\frac{\pi}{\alpha}} \int_{-\infty}^{\infty} d\omega \exp[ -(\omega + E)^2/(4\alpha)] - \frac{1}{2\pi} \sqrt{\frac{\pi}{\alpha}} \int_{0}^{\infty} d\omega \exp[ -(\omega + E)^2/(4\alpha)] = 1 + O(\exp[-\Delta E^2/(4\alpha)]). \] (3.22)
Clearly these imply (3.7).

From Lemma 3.1 and the expression (3.1) of the correlation function with (3.2) and (3.3), one has
\[ \lim_{T \to \infty} \lim_{\epsilon \to 0} \int_{-T}^{T} dt \frac{1}{t + i\epsilon} \langle \Phi, [A_X(t), B_Y] \Phi \rangle e^{-\alpha t^2} \]
\[ = \langle \Phi, A_X(1 - P_{0,\Lambda}) B_Y \Phi \rangle + O(\exp[-\Delta E^2/(4\alpha)]) \]
\[ + \lim_{T \to \infty} \lim_{\epsilon \to 0} \int_{-T}^{T} dt \frac{1}{t + i\epsilon} \langle [\Phi, A_X(t) P_{0,\Lambda} B_Y \Phi] - \langle \Phi, B_Y P_{0,\Lambda} A_X(t) \Phi \rangle \rangle e^{-\alpha t^2} \] (3.23)
for finite volume.

In the following, we treat only the power-law decaying interaction $h_X$ because one can treat the exponentially decaying interactions in the same way. See also refs. [3, 4] in which the exponential clustering of the correlations is proved for finite-range interactions under the gap assumption along the same line as below.

In order to estimate the left-hand side, we recall the Lieb-Robinson estimate (A.1) in Appendix A,
\[ \left\| \frac{1}{t} [A_X(t), B_Y] \right\| \leq \text{Const.} \times \frac{1}{(1 + r)^{\eta}} \frac{e^{\eta |t|} - 1}{|t|}, \] (3.24)
for \( r > 0 \), where we have written \( r = \text{dist}(X, Y) \). Using this estimate, the integral can be evaluated as

\[
\left| \int_{-T}^{T} dt \frac{\langle \Phi, [A_X(t), B_Y] \Phi \rangle}{t + i\epsilon} e^{-\alpha t^2} \right|
\leq \int_{|t| \leq \ell} dt \frac{\langle \Phi, [A_X(t), B_Y] \Phi \rangle}{t + i\epsilon} e^{-\alpha t^2} + \int_{|t| > \epsilon\ell} dt \frac{\langle \Phi, [A_X(t), B_Y] \Phi \rangle}{t + i\epsilon} e^{-\alpha t^2}
\leq \text{Const.} \times \frac{1}{(1 + r)^{\eta - \epsilon_0}} + \frac{\text{Const.}}{\sqrt{\alpha \ell}} \exp[-\alpha c^2 \ell^2],
\]

where \( c \) is a positive, small parameter, and \( \ell = \log(1 + r) \), and we have used

\[
\int_{|t| \leq \ell} e^{\psi|t|} - \frac{1}{|t|} dt \leq 2e^{c\epsilon \ell}.
\]

In order to estimate the integral in the right-hand side of (3.23), we consider the matrix element \( \langle \Phi_{0,\nu}, A_X(t) P_{0,\Lambda} B_Y \Phi_{0,\nu} \rangle \) because the other matrix elements in the ground state can be treated in the same way. Using Lemma 3.1, one has

\[
\lim_{T \to \infty} \lim_{\epsilon \downarrow 0} \frac{i}{2\pi} \int_{-T}^{T} dt \frac{1}{t + i\epsilon} \langle \Phi_{0,\nu}, A_X(t) P_{0,\Lambda} B_Y \Phi_{0,\nu} \rangle e^{-\alpha t^2} = \sum_{\mu=1}^{q} \langle \Phi_{0,\nu}, A_X \Phi_{0,\mu} \rangle \langle \Phi_{0,\mu}, B_Y \Phi_{0,\nu} \rangle \frac{1}{2\pi} \sqrt{\frac{\pi}{\alpha}} \int_{-\infty}^{0} d\omega \exp[-(\omega + \Delta E_{\mu,\nu})^2/(4\alpha)],
\]

where \( \Delta E_{\mu,\nu} = E_{0,\mu} - E_{0,\nu} \). Using the assumption (2.12) and the dominated convergence theorem, we have that, for any given \( \epsilon > 0 \), there exists a sufficiently large volume of the lattice \( \Lambda_s \) such that

\[
\left| \lim_{T \to \infty} \lim_{\epsilon \downarrow 0} \frac{i}{2\pi} \int_{-T}^{T} dt \frac{e^{-\alpha t^2}}{t + i\epsilon} \langle \Phi_{0,\nu}, A_X(t) P_{0,\Lambda} B_Y \Phi_{0,\nu} \rangle - \frac{1}{2} \langle \Phi_{0,\nu}, A_X P_{0,\Lambda} B_Y \Phi_{0,\nu} \rangle \right| \leq \epsilon.
\]

Combining this observation, (3.23) and (3.25), and choosing \( \alpha = \Delta E/(2c\ell) \), one obtains

\[
\left| \omega(A_X B_Y) - \frac{1}{2} [\omega(A_X P_0 B_Y) + \omega(B_Y P_0 A_X)] \right| \leq \text{Const.} \times \frac{1}{(1 + r)^{\eta - \epsilon_0}} + \text{Const.} \times \exp\left[\frac{-c\Delta E}{2}\right]
\]

in the infinite volume limit, where the ground-state expectation \( \omega \) is given by (2.15). Choosing \( c = \eta/(\nu + \Delta E/2) \), we have

\[
\left| \omega(A_X B_Y) - \frac{1}{2} [\omega(A_X P_0 B_Y) + \omega(B_Y P_0 A_X)] \right| \leq \frac{\text{Const.}}{[1 + \text{dist}(X, Y)]^{\tilde{\eta}}},
\]

with \( \tilde{\eta} = \eta/(1 + 2\nu/\Delta E) \). In the same way, we have

\[
\left| \omega(A_X B_Y) - \frac{1}{2} [\omega(A_X P_0 B_Y) + \omega(B_Y P_0 A_X)] \right| \leq \text{Const.} \times \exp[-\tilde{\mu} \text{dist}(X, Y)]
\]

for \( \mu \).
for the exponentially decaying interaction $h_X$, where $\tilde{\mu} = \mu/(1 + 2v/\Delta E)$. This proves Theorem 2.8. The corresponding bound for finite-range interactions was already obtained in [4]. Using the definition (2.13) of the expectation $\langle \cdot \cdot \cdot \rangle_{0,\Lambda}$ and the identity,

$$\langle A_X P_0 B_Y \rangle_{0,\Lambda} = \langle B_Y P_0 A_X \rangle_{0,\Lambda}, \quad (3.32)$$

for the integral in the right-hand side of (3.23), we obtain

$$\left| \langle A_X B_Y \rangle_{0,\Lambda} - \langle A_X P_0 A_X \rangle_{0,\Lambda} \right| \leq \text{Const.} \times \begin{cases} [1 + \text{dist}(X,Y)]^{-\tilde{\eta}}, & \text{for power-law decaying } h_X; \\ \exp[-\tilde{\mu} \text{dist}(X,Y)], & \text{for exponentially decaying } h_X \end{cases} \quad (3.33)$$

for any finite lattice $\Lambda \supset X, Y$ in the same way as in the above.

Next consider the case that the pair, $A_X, B_Y$, is fermionic. Note that

$$\langle \Phi_{0,\nu}, \{ A_X(t), B_Y \} \Phi_{0,\nu'} \rangle = \langle \Phi_{0,\nu}, A_X(t)(1 - P_{0,\Lambda}) B_Y \Phi_{0,\nu'} \rangle + \langle \Phi_{0,\nu}, B_Y(1 - P_{0,\Lambda}) A_X(t) \Phi_{0,\nu'} \rangle + \langle \Phi_{0,\nu}, A_X(t) P_{0,\Lambda} B_Y \Phi_{0,\nu'} \rangle + \langle \Phi_{0,\nu}, B_Y P_{0,\Lambda} A_X(t) \Phi_{0,\nu'} \rangle. \quad (3.34)$$

Since the difference between bosonic and fermionic observables is in the signs of some terms, one has

$$\left| \omega(A_X B_Y) - \frac{1}{2} [\omega(A_X P_0 B_Y) - \omega(B_Y P_0 A_X)] \right| \leq \text{Const.} \times \begin{cases} [1 + \text{dist}(X,Y)]^{-\tilde{\eta}}, & \text{for power-law decaying } h_X; \\ \exp[-\tilde{\mu} \text{dist}(X,Y)], & \text{for exponentially decaying } h_X. \end{cases} \quad (3.35)$$

In particular, thanks to the identity (3.32), we obtain

$$\left| \langle A_X B_Y \rangle_0 \right| \leq \text{Const.} \times \begin{cases} [1 + \text{dist}(X,Y)]^{-\tilde{\eta}}, & \text{for power-law decaying } h_X; \\ \exp[-\tilde{\mu} \text{dist}(X,Y)], & \text{for exponentially decaying } h_X. \end{cases} \quad (3.36)$$

This is nothing but the desired bound. We stress that, for infinite degeneracy of the infinite-volume ground state, this upper bound is also justified in the same argument with the dominated convergence theorem.

### 4 Vanishing of the matrix elements in the ground state

The aim of this section is to prove the bound (2.23) for the correlation and discuss an extension of Theorem 2.10 to a system having infinite degeneracy of infinite-volume ground state. The latter result is summarized as Theorem 4.1 below. We will give only the proof of Theorem 4.1 because Theorem 2.10 is proved in the same way. By the clustering bounds (3.30) and (3.31), it is sufficient to show that all the matrix elements, $\langle \Phi_{0,\nu}, A_X \Phi_{0,\nu} \rangle$, in
the sector of the ground state are vanishing. The key idea of the proof is to estimate the absolute values of the matrix elements by using the self-similarity condition and the decay bound (4.2) below of the correlations at a sufficiently large distance.

We denote by \( q_\Lambda \) the degeneracy of the sector of the ground state for the finite lattice \( \Lambda \), and we allow \( q_\Lambda \to \infty \) as \( |\Lambda_s| \uparrow \infty \). We write \( m = q_\Lambda^2 \) for short. To begin with, we write the bound (3.33) as

\[
\left| \langle A_X B_Y \rangle_{0,\Lambda} - \langle A_X P_{0,\Lambda} B_Y \rangle_{0,\Lambda} \right| \leq G_0(\text{dist}(X,Y)),
\]

(4.1)

where we have written the upper bound of the right-hand side by the function \( G_0 \) of the distance. We assume that the following bound holds:

\[
\left| \langle A_X B_Y \rangle_{0,\Lambda} \right| \leq G_1(\text{dist}(X,Y)) \quad \text{(4.2)}
\]

with an upper bound \( G_1 \) which is vanishing at the infinite distance. Further we define \( \tilde{G}_\Lambda \) as

\[
\tilde{G}_\Lambda(A_X, B_Y) := \max \{ G_0(\text{dist}(X,Y)), G_1(\text{dist}(X,Y)) \}. \quad \text{(4.3)}
\]

**Theorem 4.1** Let \( \omega \) be a ground-state expectation (2.15) in the infinite volume limit, and let \( A_X, B_Y \) be a pair of bosonic observable with compact supports \( X,Y \). Assume that there exists a uniform spectral gap \( \Delta E > 0 \) above the ground state sector in the spectrum of the Hamiltonian \( H_\Lambda \) in the sense of Definition 2.5. Suppose that, for any given \( \epsilon > 0 \), there exists \( M_0 > 0 \) such that, for any large lattice \( \Lambda \) satisfying \( |\Lambda_s| \geq M_0 \), there exists a set of observables, \( B^{(j)} \), \( j = 1,2, \ldots, m \), and a set of transformations, \( R_j \), \( j = 1,2, \ldots, m \), satisfying the following conditions: Any pair of the observables, \( A_X, B^{(1)}, \ldots, B^{(m)} \), is bosonic,

\[
B^{(j)} = R_j(A), \quad (B^{(j)})^\dagger = R_j(A_X^\dagger) \quad \text{and} \quad R_j(H_\Lambda) = H_\Lambda, \quad \text{(4.4)}
\]

and

\[
q_\Lambda^3 \max_{i,j \in \{0,1, \ldots, m\}, \ i \neq j} \left\{ \tilde{G}_\Lambda \left( \left( B^{(i)} \right)^\dagger, B^{(j)} \right) \right\} < \epsilon, \quad \text{(4.5)}
\]

where we have written \( B^{(0)} = A_X^\dagger \). Then we have the bound,

\[
|\omega(A_X B_Y)| \leq \text{Const.} \times \left\{ \begin{array}{ll}
[1 + \text{dist}(X,Y)]^{-\tilde{q}}, & \text{for power-law decaying } h_X; \\
\exp[-\tilde{\mu} \text{dist}(X,Y)], & \text{for exponentially decaying } h_X,
\end{array} \right. \quad \text{(4.6)}
\]

in the infinite volume limit.

**Proof:** From the bound (3.30) or (3.31) and the Schwarz inequality,

\[
|\omega(A_X P_0 B_Y)|^2 \leq \omega(A_X P_0 A_X^\dagger) \omega(B_Y^\dagger P_0 B_Y), \quad \text{(4.7)}
\]

it is sufficient to show \( \omega(A_X P_0 A_X^\dagger) = \omega(B_Y^\dagger P_0 B_Y) = 0 \). Further, we have

\[
\left\langle \Phi, A P_{0,\Lambda} A^\dagger \Phi \right\rangle \leq q_\Lambda \left\langle A P_{0,\Lambda} A^\dagger \right\rangle_{0,\Lambda} = q_\Lambda \left\langle A^\dagger P_{0,\Lambda} A \right\rangle_{0,\Lambda} \quad \text{(4.8)}
\]
for any ground state vector $\Phi$ with norm one and any observable $A$ on the finite lattice $\Lambda$. Therefore we estimate $q_\Lambda \langle A_X P_{0,\Lambda} A_X^\dagger \rangle_{0,\Lambda}$.

Note that, from the clustering bound (4.1), (4.2) and (4.3), we have

$$
\left| \langle A_X P_{0,\Lambda} B_Y \rangle_{0,\Lambda} \right| \leq \left| \langle A_X B_Y \rangle_{0,\Lambda} \right| + \left| \langle A_X B_Y \rangle_{0,\Lambda} - \langle A_X P_{0,\Lambda} B_Y \rangle_{0,\Lambda} \right|
\leq 2\tilde{G}_\Lambda(A_X, B_Y).
$$

We define

$$
B^{(j)}_i := \langle \Phi_{0,\nu'}, B^{(j)} \Phi_{0,\nu} \rangle
$$

for $j = 0, 1, \ldots, m$ and for the finite lattice $\Lambda$, where we have written $i = (\nu', \nu)$ with $i = 1, 2, \ldots, m$ for short. Since $B^{(j)}_1, B^{(j)}_2, \ldots, B^{(j)}_m$ is an $m$-dimensional vector, there exist complex numbers, $C_j, j = 0, 1, \ldots, m$, such that, at least, one of $C_j$ is nonvanishing and that

$$
\sum_{j=0}^m C_j B^{(j)}_i = 0.
$$

Let $\ell$ be the index which satisfies $|C_\ell| = \max\{|C_0|, |C_1|, \ldots, |C_m|\}$. Clearly, we have

$$
B^{(\ell)}_i = -\sum_{j \neq \ell} \frac{C_j}{C_\ell} B^{(j)}_i.
$$

Therefore

$$
\langle (B^{(\ell)}_i)^\dagger P_{0,\Lambda} B^{(\ell)}_i \rangle_{0,\Lambda} = \frac{1}{q_\Lambda} \sum_{i=1}^m \left| B^{(\ell)}_i \right|^2 = -\sum_{j \neq \ell} \frac{C_j}{C_\ell} q_\Lambda \sum_{i=1}^m (B^{(\ell)}_i)^* B^{(j)}_i
\leq m \max_{j \neq \ell} \left\{ \left| \langle (B^{(\ell)}_i)^\dagger P_{0,\Lambda} B^{(j)}_i \rangle_{0,\Lambda} \right| \right\}
\leq 2q_\Lambda^2 \max_{j \neq \ell} \left\{ \tilde{G}_\Lambda \left( (B^{(\ell)}_i)^\dagger, B^{(j)}_i \right) \right\},
$$

where we have used the inequality (4.9) for getting the last bound. When $\ell = 0$, we obtain

$$
q_\Lambda \langle A_X P_{0,\Lambda} A_X^\dagger \rangle_{0,\Lambda} \leq 2\epsilon
$$

from $B^{(0)} = A_X^\dagger$ and the assumption (4.5). When $\ell \neq 0$, we reach the same conclusion by using the relation,

$$
\langle A_X^\dagger P_{0,\Lambda} A_X \rangle_{0,\Lambda} = \langle R_\ell(A_X^\dagger) P_{0,\Lambda} R_\ell(A) \rangle_{0,\Lambda} = \langle (B^{(\ell)}_i)^\dagger P_{0,\Lambda} B^{(\ell)}_i \rangle_{0,\Lambda},
$$

which is derived from the assumption (4.4). \(\blacksquare\)

**Remark:** 1. The advantage of Theorem 4.1 is that it is easier to find $B^{(j)}$ and $R_j$ because of the finiteness of the lattice. Actually one can construct $B^{(j)}$, $R_j$ and $\Lambda$ satisfying the requirement by connecting $m$ copies of a small, finite lattice to each other at their boundaries. But, if the degeneracy $q_\Lambda$ exceeds $\sqrt{|\Lambda|}$, we cannot find the observables, $B^{(j)}$, and the transformations, $R_j$. Therefore our argument does not work in such cases.
2. Under the weaker assumption,

\[ q^2_{\Lambda} \max_{i,j \in \{0,1,\ldots,m\} : i \neq j} \{ \tilde{G}_{\Lambda} \left( (B^{(i)})^\dagger, B^{(j)} \right) \} < \epsilon, \quad (4.16) \]

than (4.5), we can obtain the bound,

\[ |\langle A_X B_Y \rangle_0| \leq \text{Const.} \times \begin{cases} [1 + \text{dist}(X,Y)]^{-\bar{n}}, & \text{for power-law decaying } h_X; \\ \exp[-\bar{\mu} \text{dist}(X,Y)], & \text{for exponentially decaying } h_X, \end{cases} \quad (4.17) \]

in the infinite volume limit.

3. Consider the situation of the above Remark 2 or the case with a finite degeneracy of the infinite-volume ground state. Then, instead of introducing the transformations \( R_j \), we can directly require

\[ \langle A_X^\dagger P_0 A_X \rangle_0 = \langle (B^{(j)})^\dagger P_0 B^{(j)} \rangle_0 \quad \text{for } j = 1, 2, \ldots, m, \quad (4.18) \]

in the infinite volume limit, and at infinite distance between the observables \( A_X \) and \( B^{(j)} \).

## A Lieb-Robinson bound for group velocity

Quite recently, Nachtergaele and Sims [6] have extended the Lieb-Robinson bound [11] to a wide class of models with long-range, exponentially decaying interactions. In this appendix, we further extend the bound to the power-law decaying interactions. We also tighten the bound on the exponentially decaying case. (See Assumption 2.4 compared to that in [6]). However, in our proof, the time \( t \) must be real.

In the following, we treat only the case with bosonic observables and with the power-law decaying interaction \( h_X \) because the other cases including the previous results can be treated in the same way.

**Theorem A.1** Let \( A_X, B_Y \) be a pair of bosonic observables with the compact support, \( X, Y \), respectively. Assume that the system satisfies the conditions in Assumption 2.1 or 2.2. Then

\[ \| [A_X(t), B_Y] \| \leq C \| A_X \| \| B_Y \| \| X \| \| Y \| \frac{e^{v|t|} - 1}{[1 + \text{dist}(X,Y)]^q} \quad \text{for } \text{dist}(X,Y) > 0, \quad (A.1) \]

where the positive constants, \( C \) and \( v \), depend only on the interaction of the Hamiltonian and the metric of the lattice.

**Remark:** The same bound for fermionic observables is obtained by replacing the commutator with the anticommutator in the left-hand side.

For exponentially decaying interaction \( h_X \), the following bound is valid:
Theorem A.2  Let $A_X, B_Y$ be a pair of bosonic observables with the compact support, $X, Y$, respectively. Assume that the system satisfies the conditions in Assumption 2.3 or 2.4. Then

$$
\|[A_X(t), B_Y]\| \leq C \|A_X\| \|B_Y\| |X||Y| \exp[-\mu \operatorname{dist}(X, Y)] \left[ e^{\epsilon^2 t} - 1 \right] \quad \text{for } \operatorname{dist}(X, Y) > 0,
$$

(A.2)

where the positive constants, $C$ and $\nu$, depend only on the interaction of the Hamiltonian and the metric of the lattice.

Remark: For the proof under Assumption 2.3, we rely on the inequalities, (2.9) and (2.10). Assumption 2.4 is milder than that in ref. [6] as remarked in Section 2.

We assume that the volume $|\Lambda_s|$ of the lattice $\Lambda$ is finite. If it is necessary to consider the infinite volume limit, we take the limit after deriving the desired Lieb-Robinson bounds which hold uniformly in the size of the lattice. Let $A, B$ be observables supported by compact sets, $X, Y \subset \Lambda_s$, respectively. The time evolution of $A$ is given by

$$
A(t) = e^{itH_\Lambda} A e^{-itH_\Lambda}.
$$

First, let us derive the inequality (A.12) below for the commutator $[A(t), B]$.

We assume $t > 0$ because the negative $t$ can be treated in the same way. Let $\epsilon = t/N$ with a large positive integer $N$, and let

$$
t_n = \frac{t}{N} n \quad \text{for } n = 0, 1, \ldots, N.
$$

(A.3)

Then we have

$$
\|[A(t), B]\| - \|[A(0), B]\| = \sum_{i=0}^{N-1} \epsilon \times \frac{\|[A(t_{n+1}), B]\| - \|[A(t_n), B]\|}{\epsilon}.
$$

(A.4)

In order to obtain the bound (A.12) below, we want to estimate the summand in the right-hand side. To begin with, we note that the identity, $\|U^*OU\| = \|O\|$, holds for any observable $O$ and for any unitary operator $U$. Using this fact, we have

$$
\|[A(t_{n+1}), B]\| - \|[A(t_n), B]\| = \|[A(\epsilon), B(-t_n)]\| - \|[A, B(-t_n)]\|
\leq \|[A + i\epsilon[H_\Lambda, A], B(-t_n)]\| - \|[A, B(-t_n)]\| + O(\epsilon^2)
\leq \|[A + i\epsilon[I_X, A], B(-t_n)]\| - \|[A, B(-t_n)]\| + O(\epsilon^2)
$$

(A.5)

with

$$
I_X = \sum_{Z: Z \cap X \neq \emptyset} h_{Z},
$$

(A.6)

where we have used

$$
A(\epsilon) = A + i\epsilon[H_\Lambda, A] + O(\epsilon^2)
$$

(A.7)

and the triangle inequality. Further, by using

$$
A + i\epsilon[I_X, A] = e^{i\epsilon[I_X]} A e^{-i\epsilon[I_X]} + O(\epsilon^2),
$$

(A.8)
we have
\[
\|[A + i\epsilon [I_X, A], B(-t_n)]\| \leq \|[e^{i\epsilon t} A e^{-i\epsilon t} X, B(-t_n)]\| + \mathcal{O}(\epsilon^2)
\]
\[
= \|[A, e^{-i\epsilon t} X B(-t_n) e^{i\epsilon t} X] + \mathcal{O}(\epsilon^2)
\]
\[
\leq \|[A, B(-t_n)] + i\epsilon [I_X, B(-t_n)]\| + \mathcal{O}(\epsilon^2)
\]
\[
\leq \|[A, B(-t_n)]\| + \mathcal{O}(\epsilon^2) + \mathcal{O}(\epsilon^2). \quad (A.9)
\]

Substituting this into the right-hand side in the last line of (A.5), we obtain
\[
\|[A(t_n+1), B]\| - \|[A(t_n), B]\| \leq \epsilon \|[I_X, B(-t_n)]\| + \mathcal{O}(\epsilon^2)
\]
\[
\leq 2\epsilon \|A\| \|[I_X(t_n), B]\| + \mathcal{O}(\epsilon^2). \quad (A.10)
\]

Further, substituting this into the right-hand side of (A.4) and using (A.6), we have
\[
\|[A(t), B]\| - \|[A(0), B]\| \leq 2\|A\| \sum_{n=0}^{N-1} \epsilon \|[I_X(t_n), B]\| + \mathcal{O}(\epsilon)
\]
\[
\leq 2\|A\| \sum_{Z : Z \cap X \neq \emptyset} \sum_{n=0}^{N-1} \epsilon \|[h_Z(t_n), B]\| + \mathcal{O}(\epsilon). \quad (A.11)
\]

Since \(h_Z(t)\) is the continuous function of the time \(t\) for a finite volume, the sum in the
right-hand side converges to the integral in the limit \(\epsilon \downarrow 0 \ (N \uparrow \infty)\) for any fixed finite
lattice \(\Lambda\). In consequence, we obtain
\[
\|[A(t), B]\| - \|[A(0), B]\| \leq 2\|A\| \sum_{Z : Z \cap X \neq \emptyset} \int_0^{[t]} ds \|[h_Z(s), B]\|. \quad (A.12)
\]

We define
\[
C_B(X, t) := \sup_{A \in A_X} \frac{\|[A(t), B]\|}{\|A\|}, \quad (A.13)
\]
where \(A_X\) is the set of observables supported by the compact set \(X\). Then we have
\[
C_B(X, t) \leq C_B(X, 0) + 2 \sum_{Z : Z \cap X \neq \emptyset} \|h_Z\| \int_0^{[t]} ds C_B(Z, s) \quad (A.14)
\]
from the above bound (A.12).

We recall that the observables, \(A\) and \(B\), are, respectively, supported by the compact
sets, \(X, Y \subset \Lambda_x\). Assume dist\((X, Y) > 0\). Then we have \(C_B(X, 0) = 0\) from the definition
of \(C_B(X, t)\), and note that
\[
C_B(Z, 0) \leq \begin{cases} 2\|B\|, & \text{for } Z \cap Y \neq \emptyset; \\ 0, & \text{otherwise}. \end{cases} \quad (A.15)
\]

\(^3\)Since the local interaction \(h_Z\) with \(Z \subset X\) does not change the support \(X\) of \(A\) in the time evolution,
we can expect that the sum in the right-hand side of (A.14) can be restricted to the set \(Z\) satisfying
\(Z \cap X \neq \emptyset\) and \(Z \cap X \neq \emptyset\). However, this restriction does not affect the resulting Lieb-Robinson bound.
Therefore we omit the discussion.
Using these facts and the above bound (A.14) iteratively, we obtain

\[ C_B(X, t) \leq 2 \sum_{Z_1 : Z_1 \cap X \neq \emptyset} \|h_{Z_1}\| \int_{0}^{t} ds_1 C_B(Z_1, s_1) \]

\[ \leq 2 \sum_{Z_1 : Z_1 \cap X \neq \emptyset} \|h_{Z_1}\| \int_{0}^{t} ds_1 C_B(Z_1, 0) \]

\[ + 2^2 \sum_{Z_1 : Z_1 \cap X \neq \emptyset} \|h_{Z_1}\| \sum_{Z_2, Z_3 : Z_2 \cap Z_3 \neq \emptyset} \|h_{Z_2}\| \int_{0}^{t} ds_1 \int_{0}^{s_1} ds_2 C_B(Z_2, s_2) \]

\[ \leq 2 \|B\| (2t) \sum_{Z_1 : Z_1 \cap X \neq \emptyset, Z_1 \cap Y \neq \emptyset} \|h_{Z_1}\| \]

\[ + 2 \|B\| (2t)^2 \sum_{Z_1 : Z_1 \cap X \neq \emptyset} \|h_{Z_1}\| \sum_{Z_2, Z_3 : Z_2 \cap Z_3 \neq \emptyset, Z_2 \cap Y \neq \emptyset} \|h_{Z_2}\| \]

\[ + 2 \|B\| (2t)^3 \sum_{Z_1 : Z_1 \cap X \neq \emptyset} \|h_{Z_1}\| \sum_{Z_2, Z_3 : Z_2 \cap Z_3 \neq \emptyset} \|h_{Z_2}\| \sum_{Z_3, Z_4 : Z_3 \cap Z_4 \neq \emptyset, Z_3 \cap Y \neq \emptyset} \|h_{Z_3}\| + \cdots \]

(A.16)

**Proof of Theorem A.1 under Assumption 2.1:** The first sum in the power series (A.16) is estimated as

\[ \sum_{Z_1 : Z_1 \cap X \neq \emptyset, Z_1 \cap Y \neq \emptyset} \|h_{Z_1}\| \leq \sum_{x \in X} \sum_{y \in Y} \sum_{z_1 \in x} \|h_{Z_1}\| \leq \frac{\lambda_0 |X||Y|}{[1 + \text{dist}(X, Y)]^\eta} \]  

(A.17)

from the assumption (2.2). The second, double sum is estimated as

\[ \sum_{Z_1 : Z_1 \cap X \neq \emptyset} \sum_{Z_2, Z_3 : Z_2 \cap Z_3 \neq \emptyset, Z_2 \cap Y \neq \emptyset} \|h_{Z_2}\| \]

\[ \leq \sum_{x \in X} \sum_{y \in Y} \sum_{z_{12} \in \Lambda_x} \sum_{z_{23} \in \Lambda_y} \|h_{Z_1}\| \sum_{z_{3} \in z_{12}} \|h_{Z_2}\| \]

\[ \leq \sum_{x \in X} \sum_{y \in Y} \sum_{z_{12} \in \Lambda_x} \lambda_0 \frac{\lambda_0}{[1 + \text{dist}(x, z_{12})[1 + \text{dist}(z_{12}, y)]^\eta} \]

\[ \leq \frac{\lambda_0^2 p_0 |X||Y|}{[1 + \text{dist}(X, Y)]^\eta} \]  

(A.18)

where we have used the assumptions (2.2) and (2.3). Similarly, the third, triple sum can be estimated as

\[ \sum_{Z_1 : Z_1 \cap X \neq \emptyset} \sum_{Z_2, Z_3 : Z_2 \cap Z_3 \neq \emptyset, Z_2 \cap Y \neq \emptyset} \sum_{Z_3, Z_4 : Z_3 \cap Z_4 \neq \emptyset, Z_3 \cap Y \neq \emptyset} \|h_{Z_3}\| \]
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\begin{equation}
\leq \sum_{x \in X} \sum_{y \in Y} \sum_{z_1 \in \Lambda_x, z_2 \in \Lambda_y} \frac{\lambda_0}{[1 + \text{dist}(x, z_1)]^\eta [1 + \text{dist}(z_1, z_2)]^\eta [1 + \text{dist}(z_2, y)]^\eta}
\end{equation}

\begin{equation}
\leq \sum_{x \in X} \sum_{y \in Y} \sum_{z_1 \in \Lambda_x} \frac{\lambda_0}{[1 + \text{dist}(x, z_1)]^\eta [1 + \text{dist}(z_1, y)]^\eta}
\end{equation}

\begin{equation}
\leq \frac{\lambda_0^2 p_0 | X || Y |}{[1 + \text{dist}(X, Y)]^\eta}.
\end{equation}

From these observations, we have

\begin{equation}
C_B(X, t) \leq \frac{2||B|| |X||Y|}{[1 + \text{dist}(X, Y)]^\eta} \left\{ 2|t|\lambda_0 + \frac{(2|t|)^2}{2!}\lambda_0^2 p_0 + \frac{(2|t|)^3}{3!}\lambda_0^3 p_0^2 + \cdots \right\}
\end{equation}

\begin{equation}
= \frac{2p_0^{-1}||B|| |X||Y|}{[1 + \text{dist}(X, Y)]^\eta} \{\exp[2\lambda_0 p_0 |t|] - 1\}.
\end{equation}

Consequently, we obtain

\begin{equation}
||[A(t), B]|| \leq \frac{2p_0^{-1}||A|| ||B|| |X||Y|}{[1 + \text{dist}(X, Y)]^\eta} \{\exp[2\lambda_0 p_0 |t|] - 1\}
\end{equation}

from (A.13). □

Proof of Theorem A.1 under Assumption 2.2: The first sum in the power series (A.16) is estimated as

\begin{equation}
\sum_{Z_1:Z_1 \cap X \neq \emptyset, Z_1 \cap Y \neq \emptyset} ||h_{Z_1}|| \leq \sum_{x \in X} \sum_{y \in Y} \sum_{Z_1 \ni x, y} ||h_{Z_1}||
\end{equation}

\begin{equation}
\leq \sum_{x \in X} \sum_{y \in Y} \sum_{Z_1 \ni x, y} ||h_{Z_1}||[1 + \text{dist}(x, y)]^{-\eta}[1 + \text{diam}(Z_1)]^\eta
\end{equation}

\begin{equation}
\leq [1 + \text{dist}(X, Y)]^{-\eta}|X||Y|s_0,
\end{equation}

where

\begin{equation}
s_0 = \sup_x \sum_{Z \ni x} ||h_{Z}||[1 + \text{diam}(Z)]^\eta.
\end{equation}

Clearly, this constant \(s_0\) is finite from the assumption (2.7). The second, double sum is estimated as

\begin{equation}
\sum_{Z_1:Z_1 \cap X \neq \emptyset} ||h_{Z_1}|| \sum_{Z_2:Z_2 \cap Z_1 \neq \emptyset, Z_2 \cap Y \neq \emptyset} ||h_{Z_2}||
\end{equation}

\begin{equation}
\leq \sum_{x \in X} \sum_{y \in Y} \sum_{Z_1 \ni x, z_1} \sum_{Z_2 \ni z_1, y} ||h_{Z_1}|| \sum_{Z_2 \ni z_1, y} ||h_{Z_2}||
\end{equation}

\begin{equation}
\leq \sum_{x \in X} \sum_{y \in Y} \sum_{Z_1 \ni x, z_1} [1 + \text{dist}(x, z_1)]^{-\eta}[1 + \text{dist}(z_1, y)]^{-\eta}
\end{equation}

\begin{equation}
\times \sum_{Z_2 \ni z_1, y} ||h_{Z_1}||[1 + \text{diam}(Z_1)]^\eta \sum_{Z_2 \ni z_1, y} ||h_{Z_2}||[1 + \text{diam}(Z_2)]^\eta
\end{equation}

\begin{equation}
\leq [1 + \text{dist}(X, Y)]^{-\eta} \sum_{x \in X} \sum_{y \in Y} \sum_{Z_1 \ni x, z_1} \sum_{Z_2 \ni z_1, y} ||h_{Z_1}||[1 + \text{diam}(Z_1)]^\eta
\end{equation}
× ∑_{Z_2 \ni z_{12}, y} \| h_{Z_2} \| [1 + \text{diam}(Z_2)]^\eta
\leq [1 + \text{dist}(X, Y)]^{-\eta} \sum_{x \in X} \sum_{y \in Y} \sum_{Z_1 \ni x} \| h_{Z_1} \| [1 + \text{diam}(Z_1)]^\eta
\times \sum_{Z_2 \ni y} \| h_{Z_2} \| [1 + \text{diam}(Z_2)]^\eta
\leq [1 + \text{dist}(X, Y)]^{-\eta} |X||Y| |s_0 s_1|, \quad (A.24)

where we have used the assumption (2.7) and the inequality,

\begin{align*}
[1 + \text{dist}(x, z)]^{-\eta} [1 + \text{dist}(z, y)]^{-\eta} &= [1 + \text{dist}(x, z) + \text{dist}(z, y) + \text{dist}(x, z) \text{dist}(z, y)]^{-\eta} \\
&\leq [1 + \text{dist}(x, y)]^{-\eta}, \quad (A.25)
\end{align*}

for any \( z \in \Lambda_s \). Similarly, the third, triple sum can be estimated as

\begin{align*}
&\sum_{Z_1: Z_1 \cap X \neq \emptyset} \| h_{Z_1} \| \sum_{Z_2: Z_2 \cap Z_1 \neq \emptyset} \| h_{Z_2} \| \sum_{Z_3: Z_3 \cap Z_2 \neq \emptyset, Z_3 \cap Y \neq \emptyset} \| h_{Z_3} \| \\
&\leq \sum_{x \in X} \sum_{y \in Y} \sum_{z_{12} \in \Lambda_s} \sum_{z_{23} \in \Lambda_s} \sum_{z_{12} \cap X \neq \emptyset} \sum_{z_{23} \cap Y \neq \emptyset} \| h_{Z_1} \| \sum_{Z_2: Z_2 \ni z_{12}} \| h_{Z_2} \| \sum_{Z_3: Z_3 \ni z_{23}} \| h_{Z_3} \| \\
&\leq [1 + \text{dist}(X, Y)]^{-\eta} \sum_{x \in X} \sum_{y \in Y} \sum_{z_{12} \in \Lambda_s} \sum_{z_{23} \in \Lambda_s} \sum_{Z_1 \ni z_{12}} \| h_{Z_1} \| [1 + \text{diam}(Z_1)]^\eta \\
&\times \sum_{Z_2 \ni z_{12}, z_{23}} \| h_{Z_2} \| [1 + \text{diam}(Z_2)]^\eta \sum_{Z_3 \ni z_{23}, y} \| h_{Z_3} \| [1 + \text{diam}(Z_3)]^\eta \\
&\leq [1 + \text{dist}(X, Y)]^{-\eta} |X||Y| |s_0 s_1|^2. \quad (A.26)
\end{align*}

From these observations, we have

\begin{align*}
C_B(X, t) &\leq \frac{2s_0 s_1^{-1} \| B \| |X||Y|}{[1 + \text{dist}(X, Y)]^\eta} \sum_{n=1}^{\infty} \frac{(2s_1 |t|)^n}{n!} \\
&= \frac{2s_0 s_1^{-1} \| B \| |X||Y|}{[1 + \text{dist}(X, Y)]^\eta} \{ \exp[2s_1 |t|] - 1 \}. \quad (A.27)
\end{align*}

As a result, we obtain

\begin{align*}
\| [A(t), B] \| &\leq \frac{2s_0 s_1^{-1} \| A \| \| B \| |X||Y|}{[1 + \text{dist}(X, Y)]^\eta} \{ \exp[2s_1 |t|] - 1 \} \quad (A.28)
\end{align*}

from (A.13). □

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