Particle resonance in the Dirac equation in the presence of a delta interaction and a perturbative hyperbolic potential

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Abstract. In the present article we show that the energy spectrum of the one-dimensional Dirac equation, in the presence of an attractive vectorial delta potential, exhibits a resonant behavior when one includes an asymptotically spatially vanishing weak electric field associated with a hyperbolic tangent potential. We solve the Dirac equation in terms of Gauss hyper-geometric functions and show explicitly how the resonant behavior depends on the strength of the electric field evaluated at the support of the point interaction. We derive an approximate expression for the value of the resonances and compare the results calculated for the hyperbolic potential with those obtained for a linear perturbative potential. Finally, we characterize the resonances with the help of the phase shift and the Wigner delay time.

PACS. 03.65.Pm Relativistic wave equations – 03.65.Ge Solutions of wave equations: bound states

1 Introduction

Spontaneous particle production in the presence of strong electromagnetic fields is undoubtedly one of the most interesting phenomena associated with the quantum vacuum \[1\]. The study quantum effects associated with relativistic particles in strong fields dates back to the pioneering work of Klein \[2\] where he studied the reflection and penetration of electron waves on a potential barrier, obtaining the result that, for very strong potentials a large number of electrons with negative energy penetrate into the wall. This effect is called the Klein paradox. Sauter \[3\] found essentially the same results in the most general case when the barrier is smoothed. Particle creation by strong infinitely extended constant electric field, which is closely related to the Klein paradox, was predicted by Heisenberg and Euler \[4\].

The study of supercritical effects and resonant particle production by strong Coulomb-like potentials dates back to the pioneering works of Pieper and Greiner \[5\] and Gershtein and Zeldovich \[6\] where it was shown that spontaneous positron production was possible when two heavy bare nuclei with total charge larger than some critical value \(Z_c\) collided with each other. The critical \(Z_c = 1/\alpha = 137\) is the value for which the 1S state of the hydrogen-like atom with potential \(V = -Ze/r\) has energy \(E = -mc^2\). Supercriticality effects are based on spontaneous positron emission induced by the presence of very strong attractive electric potentials. The energy level of an unoccupied bound state sinks into the negative energy continuum. An electron of the Dirac sea is trapped by the potential, leaving a positron that escapes to infinity. The electric field responsible for supercritical effects has a strength of \(E \sim m_e^2c^3/(\epsilon h)\) and, in the region where the field acts, it makes a work larger than \(2m_e^2c^2\) which is the value of the gap between the negative and positive energy continua. Such strong electric fields could be produced in super-heavy nuclei, heavy-ion collisions \[7\] and in astrophysical phenomena.

In order to get a deeper understanding of the mechanism responsible for resonant peaks appearing in the energy spectrum when supercritical fields are present, and the role played by perturbative fields in the shape of resonances, we proceed to work with a vector point interaction in the presence of a homogeneous, asymptotically vanishing electric field \[8\]. Point interactions potentials permit us to tackle, in a simple way, more complex short-ranged potentials. Among the advantages of working with confining delta vector potentials we have the fact that, they only possess a single bound state and the treatment of the interaction reduces to a boundary condition. Bound states of the relativistic wave equation in the presence of point interactions have been carefully discussed in the literature \[11,12,13,14\]. The one-dimensional Dirac equation in the presence of a vector point delta interaction has also been a subject of study in the search for supercritical effects induced by attractive potentials. In this case, we see that a vectorial delta potential is strong enough to pull the bound state into the negative energy continuum \(E = -mc^2\) \[15,16\], nevertheless this supercritical state does not evolve to a real resonant state. Since we are inter-
ested in studying the mechanism of positron production by supercritical fields, we proceed to analyze the resonant behavior of the energy when a bound state dives into the negative continuum. This resonant behavior is associated with the appearance of simple poles of the resolvent on the unphysical sheet at a position very near the real axis [17], which can be identified as positron states with a short mean life.

In the present article, applying the idea developed by Titchmarsh [15] and Barut [19], we compute the energy spectrum of the one-dimensional Dirac equation in the presence of a vector Dirac delta interaction and a weak electric field associated with a linear and an hyperbolic potential. We find that, in both cases the energy spectrum exhibits a resonant behavior. This problem can be considered the relativistic extension of the one-dimensional Stark effect [21,22,23].

The paper is structured as follows: In section 2, we solve the one-dimensional Dirac equation in the presence of an attractive vector potential and a hyperbolic tangent vector potential. In section 3, we compute the energy resonances and show how they depend on the electric field strength evaluated at the delta interaction point. We also derive an approximate analytic expression for the energy resonances. In section 4, we characterize the resonance by computing the phase shift \( \phi \) and the Wigner time delay \( \tau \). Finally, in section 5 we summarize our conclusions.

2 The one-dimensional Dirac equation

The energy spectrum of the Dirac equation in the presence of a delta potential has been studied by different authors in the literature. The attractive vectorial delta interaction is only able to support a bound state which becomes a zero-momentum resonance for \( E = -m \) but it is not strong enough to sink the bound state into the Dirac sea to create a time-decaying resonant state. In this section we proceed to solve the 1+1 Dirac equation in the presence of an attractive vector point delta interaction and a potential \( V(x) \) which we will consider perturbative. The Dirac equation, expressed in natural units \( (\hbar = c = 1) \) takes the form [20]

\[
\left( i \gamma^\mu \left( \frac{\partial}{\partial x^\mu} - ieA_\mu \right) - m \right) \psi = 0, \tag{1}
\]

where \( A_\mu \) is the vector potential that in our case takes the form \( eA_\mu = -g \delta(x) + V(x) \delta_0 \), \( e \) is the charge, and \( m \) is the mass of the electron. The Dirac matrices \( \gamma^\mu \) satisfy the commutation relation \( \{ \gamma^\mu, \gamma^\nu \} = 2\eta^{\mu\nu} \) with \( \eta^{\mu\nu} = \text{diag}(1, -1) \). Since we are working in 1+1 dimensions, we choose to work in a two-dimensional representation of the Dirac matrices

\[
\gamma^0 = \sigma_3, \gamma^1 = -i\sigma_2. \tag{2}
\]

Substituting the matrix representation (2) into equation (1), and taking into account that the potential interaction does not depend on time, we obtain

\[
\{-i\sigma_1 \frac{d}{dx} + (V(x) - E) + m\sigma_3\}X(x) = 0, \tag{3}
\]

with \( \psi = \sigma_3X \), and

\[
X(x) = \left( \begin{array}{c} X_1 \\ X_2 \end{array} \right), \tag{4}
\]

where \( E \) is the energy eigenvalue. The components of Eq. (4) satisfy the boundary conditions:

\[
X_1(0^+) = X_1(0^-) \cos g - iX_2(0^-) \sin g, \tag{5}
\]

\[
X_2(0^+) = -iX_1(0^-) \sin g + X_2(0^-) \cos g. \tag{6}
\]

The relations given by Eq. (5) describe an attractive delta potential and a linear perturbative interaction of strength \( g \) [14].

Taking into account the gamma matrices representation (2), we obtain the system of equations

\[
(m + V(x) - E) X_1 - i \frac{dX_2}{dx} = 0, \tag{6}
\]

\[
i \frac{dX_1}{dx} + (m - V(x) + E) X_2 = 0. \tag{7}
\]

Introducing the functions \( \Omega_1 \) and \( \Omega_2 \)

\[
X_1 = \Omega_1 + i \Omega_2, \quad X_2 = \Omega_1 - i \Omega_2, \tag{8}
\]

we see that the system of equations given by Eqs. (6)-(7) reduces to

\[
\frac{d\Omega_1}{dx} + i(V(x) - E) \Omega_1 - m\Omega_2 = 0, \tag{9}
\]

\[
\frac{d\Omega_2}{dx} - i(V(x) - E) \Omega_2 - m\Omega_1 = 0. \tag{10}
\]

which is more tractable in the search of exact solutions. Using Eq. (9) and Eq. (10) we obtain that \( \Omega_1(x) \) satisfies the second order differential equation

\[
\frac{d^2}{dx^2} \Omega_1(x) + \{i \frac{dV(x)}{dx} + (V(x) - E)^2 - m^2 \} \Omega_1(x) = 0. \tag{11}
\]

2.1 Delta interaction and linear potential \( \lambda x \)

We are interested in solving the Dirac equation (3) in the presence of an attractive delta and a linear perturbative potential \( V(x) = \lambda x \), which corresponds to a constant electric field \( eE_x = -\lambda \). The perturbation of the energy spectrum of a non-relativistic particle by a constant electric field has been extensively discussed in the literature, mainly because to its relation to the Stark effect and the presence of energy resonances [17].

The solution to Eq. (11), for \( \lambda < 0 \), satisfying the boundary condition of a growing oscillatory solution for \( x < 0 \) and a damping solution for \( x > 0 \) can be expressed in terms of the parabolic cylinder functions \( D_v(x) \) [21] as

\[
\Omega_1^+(x) = AD_v \left( \sqrt{\frac{2}{\lambda}} e^{\frac{\lambda}{2}} (\lambda x - E) \right), \quad x \geq 0. \tag{12}
\]

\[
\Omega_1^-(x) = BD_v \left( -\sqrt{\frac{2}{\lambda}} e^{\frac{-\lambda}{2}} (\lambda x - E) \right), \quad x \leq 0. \tag{13}
\]
where $D_{\rho}$ are the parabolic cylinder functions $[24]$, $\rho = \frac{2\pi}{\lambda}$. $A$ and $B$ are constants to be determined using the boundary condition given by Eq. (5). The solution $\Omega_{\pm}^\ast (x)$ belongs to $L^2[0, \infty]$ with $Im E > 0$. The solution $\Omega_{\pm}^\ast (x)$ behaves asymptotically as an outgoing wave, a condition that defines a Siegert state $[25]$

Inserting Eq. (12) into Eq. (9) and using the recurrence relations for the parabolic cylinder functions $[24]$, we obtain

$$\Omega_{\pm}^\ast (x) = A \frac{me^{i\pi/4}}{\sqrt{2\lambda}} D_{\rho-1} \left( \sqrt{\frac{2}{\lambda}} e^{i\frac{k}{\lambda}} (\lambda x - E) \right)$$

$$\Omega_{\pm}^\ast (x) = B \frac{me^{-i\pi/4}}{\sqrt{2\lambda}} D_{\rho+1} \left( -\sqrt{\frac{2}{\lambda}} e^{i\frac{k}{\lambda}} (\lambda x - E) \right)$$

Using the boundary condition $[5]$, we obtain that the energy resonances satisfy the equation

$$(X^\pm_1 (0) X^\pm_2 (0) - X^\pm_2 (0) X^\pm_1 (0)) \cos(g) + i(X^\pm_1 (0) X^\pm_2 (0) - X^\pm_2 (0) X^\pm_1 (0)) \sin(g) = 0,$$  

and with the help of the definition of $X_1$, and $X_2$ in terms of $\Omega_1$ and $\Omega_2$ $[5]$, we obtain the result that the eigenvalue equation (15) can be written as:

$$\Omega_{\pm}^\ast (0) \Omega_{\pm}^\ast (0) - e^{2ig} \Omega_{\pm}^\ast (0) \Omega_{\pm}^\ast (0) = 0$$  

which, after substituting the expressions for $\Omega_1$ and $\Omega_2$ in terms of the parabolic cylinder functions reduces to

$$D_{\rho} \left( -\sqrt{\frac{2}{\lambda}} e^{i\frac{k}{\lambda}} E \right) D_{\rho-1} \left( \sqrt{\frac{2}{\lambda}} e^{i\frac{k}{\lambda}} E \right) - i e^{-2ig} D_{\rho} \left( \sqrt{\frac{2}{\lambda}} e^{i\frac{k}{\lambda}} E \right) D_{\rho-1} \left( -\sqrt{\frac{2}{\lambda}} e^{i\frac{k}{\lambda}} E \right) = 0$$

The eigenvalue equation (17) permits us to obtain the energy solutions to the Dirac equation (11) in the presence of a vectorial delta potential and a linear potential $\lambda x$. For small values of $\lambda$ the linear potential plays the role of perturbing the Dirac hamiltonian with a vectorial delta interaction $-g\delta(x)$ creating resonant states that are supercritical when $g$ approaches $\pi$ and $E \rightarrow -m$. The solutions to Eq. (17) for small values of $\lambda$ should be obtained numerically. Fig. 1 shows $\Im$ versus $\Re$ for varying $\lambda$ showing a resonant behavior analogous to the one expected for the supercritical hydrogen atom.

2.2 Delta interaction and a hyperbolic potential

Now we proceed to solve the 1+1 Dirac equation in the presence of an attractive vector point interaction potential represented by $eV(x) = -g\delta(x)$, and an asymptotically vanishing electric field associated with the potential

$$V(x) = A \tanh(kx).$$

The potential [18] is asymptotically constant for large values of $x$ and has the linear potential as a limit for small values of $k$.

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**Fig. 1.** Behavior of $\Im(E)$ against $\Re(E)$ for the resonant energy states of a delta interaction with $g = -0.99k$ perturbed by a linear potential $\lambda x$ with $0 < \lambda < 0.03$. $\Im(E)$ and $-\Re(E)$ increase as the parameter $\lambda$ increases.

After substituting the potential [18] into Eq. (11), we obtain the second order differential equation

$$\frac{d^2 \Omega_1 (x)}{dx^2} + \left\{ ik \text{A} e^{ikx} + (A \tanh(kx) - E)^2 - m^2 \right\} \Omega_1 (x) = 0 \quad (19)$$

In order to be able to identify the solutions of (19) exhibiting a damping asymptotic behavior as $x \rightarrow +\infty$, we introduce, for $x > 0$, the new variable $z = -e^{-2kx}$. We see that $\Omega_1 (z)$ satisfies the second order differential equation

$$4k^2 z^2 \frac{d^2 \Omega_1 (z)}{dz^2} + 4k^2 z \frac{d \Omega_1 (z)}{dz} + \left[ -i \frac{4k^2}{(1-z)^2} + \left( A \frac{1+z}{1-z} - E \right)^2 - m^2 \right] \Omega_1 (z) = 0. \quad (20)$$

For $k > 0$ and $A < 0$, the solution of (20), vanishing as $x \rightarrow \infty$, with $\Re E \leq -m$ and $\Im E \geq 0$, can be expressed in terms of Gauss hyper-geometric functions $2F_1(a, b, c, z)$ $[24]$ as

$$\Omega_1^\ast (z) = C_1 (z - 1)^{-a} z^a 2F_1 (c, d; f; z), \quad (21)$$

where

$$a = \frac{\sqrt{m^2 - (E - A)^2}}{2k}, \quad b = \frac{\sqrt{m^2 - (E + A)^2}}{2k},$$

$$c = a - \frac{iA}{k}, \quad d = a + b(E, A, m, k) - \frac{A}{k},$$

$$f = 1 + 2a. \quad (22)$$

Using Eq. (19) we can obtain the expression for $\Omega_1 (z)$ in order to solve Eq. (19) for $x < 0$, we introduce the new variable $\bar{z} = e^{2kx}$, obtaining the following differential equation

$$4k^2 \bar{z}^2 \frac{d^2 \Omega_1 (\bar{z})}{d\bar{z}^2} + 4k^2 \bar{z} \frac{d \Omega_1 (\bar{z})}{d\bar{z}} + \left[ -i \frac{4k^2}{(1-\bar{z})^2} + \left( A \frac{1+\bar{z}}{1-\bar{z}} - E \right)^2 - m^2 \right] \Omega_1 (\bar{z}) = 0 \quad (23)$$
whose solution, exhibiting the behavior of an increasing reflected wave as \( x \to -\infty \), with \( \Re E \leq -m \) and \( \Im E \geq 0 \), is
\[
\Omega_1^-(z) = C_2(z - 1)^{-i\frac{\pi}{4}z} - b - 2 F_1 (g, h, p; z),
\]
with
\[
g = -b + a - \frac{iA}{k}, \quad h = -b - a - \frac{iA}{k}, \quad p = -1 - 2b.
\]
The energy eigenvalue equation corresponding to the Dirac equation Eq. (11), with the boundary conditions (5) can be obtained after substituting \( \Omega_1 \) and \( \Omega_2 \) into Eq. (16). When the potential \( V(x) \) vanishes at \( x = 0 \), such as in the linear potential or hyperbolic cases, the eigenvalue equation (10) can be written in terms of \( \Omega_1^-(x) \) and \( \Omega_1^+(x) \) as
\[
\frac{d\Omega_1^+(x)}{dx} \Omega_1^+(x) - e^{-2ig} \frac{d\Omega_1^-(x)}{dx} \Omega_1^-(x) - iE(1 - e^{-2ig}) \Omega_1^+(x) \Omega_1^+(x) = 0
\]
(26)

### 3 Approximate solutions

It is not straightforward to obtain an approximate expression for the energy eigenvalues of Eq. (17) in the vicinity of \( E = -m \) because the expansion parameter \( \lambda \) appears in Eq. (17) in the argument and order of the parabolic cylinder functions. It is also not possible to try to apply perturbation theory to find the complex energy eigenvalues. In this section, we derive an approximate solution to Eq. (3) in terms of Airy functions for the linear potential \( \lambda x \) and for the hyperbolic potential \( \lambda \tan(kx) \) for small \( \lambda \) and \( k \).

Since we are interested in studying Eq. (11) for very small values of \( \lambda \) and search solutions of Eq. (10) with respect to the resonant energy value \( E = -m \), for the linear potential \( V(x) = \lambda x \), we approximate in Eq. (11), \( \lambda x - E \sqrt{2} \) by \( -2 \lambda E x + E^2 \), obtaining in this way the approximate differential equation
\[
\frac{d^2}{dx^2} \Omega_1(x) + \{i\lambda - 2\lambda E x + E^2 - m^2\} \Omega_1(x) = 0.
\]
(27)

We proceed to solve Eq. (27) demanding the solutions to satisfy the resonance asymptotic conditions, that is, we choose damping solutions for \( x > 0 \) and diverging oscillating functions for \( x < 0 \). Those solutions are:
\[
\Omega_1^+(x) = A Ai \left( -\frac{i\lambda + E^2 - m^2 - 2E\lambda x}{(2E\lambda)^{2/3}} \right)
\]
(28)
\[
\Omega_1^-(x) = B Ci^+ \left( -\frac{i\lambda + E^2 - m^2 - 2E\lambda x}{(2E\lambda)^{2/3}} \right)
\]
(29)
where \( A \) and \( B \) are constants and \( Ai(x) \) and \( Ci^+(x) \) are the Airy functions [24].

Using Eq. (28) and Eq. (29) and the eigenvalue equation (26), we obtain that the approximate spectral equation takes the form
\[
(2E\lambda)^{1/3} i e^{-i\theta} + 2i Ai(\Xi) \left( i(2E\lambda)^{1/3} Ci^+(\Xi) + ECi^+(\Xi) \right) \sin g = 0
\]
(30)

where \( \Xi = m^2 E^2 + \lambda \).

For the hyperbolic potential \( \lambda \tan(kx) \) we also have that since the parameters \( \lambda \) and \( k \) are small, we can approximate the expression \( (\lambda \tan(kx) - E)^2 \) by \( -2\lambda k x E + E^2 \) in Eq. (19), obtaining in this way the approximate differential equation for \( \Omega_1 \),
\[
\frac{d^2}{dx^2} \Omega_1(x) + \{i\lambda k - 2\lambda k E x + E^2 - m^2\} \Omega_1(x) = 0
\]
(31)

which, after making the identification \( \lambda k \to \lambda \) reduces to Eq. (27), therefore the approximate eigenvalue equation is given by Eq. (30) with \( \lambda = \lambda k \).

In order to show the accuracy of the approximations (31) and (27), we can compare the solutions (28) and (29) with the exact solutions given by Eq. (12). Fig. 2 shows that, for small values of \( \lambda \), the solutions (28) and (29) are a good approximation to the parabolic cylinder functions (21) and the Gauss functions (24) for small \( \lambda \) and \( k \). A comparison between the resonant energy values obtained after solving Eq. (29) and Eq. (17), shows that, for \( \lambda = -0.01 \), the eigenvalue equation (30) gives resonant energies with an error smaller than 0.1% . For \( \lambda = -0.01 \) the solution to Eq. (17) gives \( E = -1.016964 + 0.049316i \) and the approximate solution obtained after solving Eq. (30) is \( E = -1.016139 + 0.049965i \). Eq. (30) gives more accurate energy eigenvalues as \( \lambda \to 0 \). The advantage of using Eq. (30) for calculating the energy eigenvalues for small values of \( \lambda \), instead of solving Eq. (17), lies in the fact that, for large values of the index \( \rho \), the parabolic cylinder functions exhibit a divergent behavior that makes it troublesome to obtain accurate values for the energy spectrum. For values of \( |\lambda|^2 < 0.01 \) we made use of the approximate solutions in order calculate the resonance energies.

![Fig. 2. Comparison between the exact solutions for \( \Omega_1(x) \) (dashed line) and the approximate expressions (solid line) for a particle in an attractive delta perturbed by a linear potential \( \lambda x \) or \( \lambda \tan(kx) \) for small \( \lambda \), with \( \lambda = -0.1 \). Left and right figures depict respectively \(|\Omega_1(x)|\) on the left and right of the origin \( x = 0 \).](image-url)
The approximation (27) to Eq. (19) is valid when $\Lambda$ and $k$ are small with respect to $m$. For large values of $k$ the hyperbolic potential approaches a step potential, which is not able to sink the delta bound energy state negative energy continuum. Fig. 3 shows the dependence of the resonant energy as the slope parameter $k$ of the hyperbolic potential increases.

The solution to Eq. (19), behaving as wave incoming from the left is

$$\Omega_1^{-in}(z) = D_1(z - 1)^{-1/2} e^{-\frac{i\Lambda}{2} z} F_1(q, t, 1 + 2b; z),$$  \hspace{1cm} (35)

where

$$q = -a + b - \frac{i\Lambda}{k}$$ \hspace{1cm} (36)

$$t = -a + b - \frac{i\Lambda}{k}$$ \hspace{1cm} (37)

The component $\Omega_2^{-in}(z)$ can be obtained using Eq. (9). Figure 3 shows the behavior of the phase shift $\phi$ against the real part of the energy $E$ for $k = 0.1$ and $\Lambda = -0.1$. It can be observed that the phase shift increases and has a jump of $\pi$ as it approaches the value $\pi/2$ and $E$ is close to the real value of the resonance $E_{res} = -1.01837 + 0.05058i$. This result indicates that we are in the presence of a bound state that dissolves itself into the negative energy continuum.

$\text{Fig. 3.}$ Behavior of the resonant energy for a delta attractive interaction with $g = -0.99\pi$ perturbed by the potential $\Lambda \tanh(kx)$ with $\lambda = -0.1$ and $0 < k < 0.5$. For small values of $kE$ is close to $-1$ and $\Im(E)$ increases as $k$ increases.

4 Phase shifts.

Since the perturbative potential $\Lambda \tanh(kx)$ is asymptotically constant as $x \to \pm\infty$, and, the perturbative electric field $Ak\tanh(kx)^2/e$ vanishes for large values of $x$, we can use of the scattering formalism in order to compute the phase shift $\phi$, magnitude that will help us compute the influence of the perturbative hyperbolic interaction on the energy spectrum of the delta interaction. Analogous to the non-relativistic case [21], the particle is constrained to move in a half line and the reflection amplitude $r$ satisfies $|r| = 1$.

The phase shift $\phi$ corresponds to the argument of $r$ as $\phi = -\arg(r)$.

With the help of the boundary condition (33) associated with the vectorial delta interaction $-g\delta(x)$, we proceed to calculate the reflection amplitude $t$

$$tX^1 = (X^1 + rX^1) \cos g - (X^2 + rX^2) \sin g$$ \hspace{1cm} (32)

$$tX^2 = -i (X^1 + rX^1) \sin g + (X^2 + rX^2) \cos g,$$ \hspace{1cm} (33)

where $t$ and $X^i$ are respectively the transmission coefficient and the transmitted wave. $X^i$ denotes the incident wave. From Eq. (32) and Eq. (33), we see that

$$r = \frac{X^1X^2 - X^1X^2}{(X^1X^2 - X^2X^2) \cos g + i (X^1X^1 - X^2X^2) \sin g}$$ \hspace{1cm} (34)

The slope of the curve shown in Figure 4 is related to the Wigner time delay $\tau$ and can be estimated using the Heisenberg principle [26]

$$\Gamma \approx \frac{\pi}{\frac{d\phi}{dE}|_{E=E_{res}}} = \frac{1}{\tau}$$  \hspace{1cm} (38)

The Wigner time $\tau$ permits us to evaluate how long the bound state lives before it decays into the negative energy continuum. For $\Lambda = -0.1$ and $k = 0.1$ we obtain that $\tau = 0.04622$, a value which is very close to $\Im(E_{res})$.

$\text{Fig. 4.}$ Behavior of the phase shift against the energy $E$ for an attractive delta perturbed by the potential $\Lambda \tanh(kx)$ with $\Lambda = -0.1$ and $k = 0.1$. The phase shift suffers a jump of $\pi$ close to the real part of the resonant energy $E_{res}$. 

The inverse of the imaginary part of the energy $E$ gives a measure of the mean of the life time associated with the energy $E$ \cite{20,27}, a state that can be interpreted as a positron, i.e a hole with a positive charge, created in the Dirac sea with a finite life-time.

5 Discussion of the results

The vector point interaction vectorial $-g\delta(x)$ described by the boundary conditions \cite{43} produces zero-momentum supercritical states for $g = \pi$ but they are not complex energy states associated with resonant states with a Breit-Wigner profile.

In this article we have analyzed the energy spectrum of the one-dimensional Dirac equation in the presence of an attractive vectorial delta point interaction when one introduces perturbative potentials of the form $\Lambda k x \delta(x)$ with small $\Lambda$ and $k$, corresponding respectively to a constant electric field and an an asymptotically vanishing electric field that reaches its maximal strength $\Lambda k$ at $x = 0$. The presence of the perturbative electric field induces the appearance of a resonant energy state in the one-dimensional Dirac equation. This phenomenon is analogous to the Stark effect \cite{17}.

Since the vector interaction $-g\delta(x)$ is not strong enough to produce a real resonance\cite{14,15,16}, we have that the presence of the perturbative potential plays a crucial role in the appearance of the resonant energy state. We have studied and discussed the presence of supercriticality in two different ways: first, studying the resonant behavior of the energy spectrum in the vicinity of $E = -m$, and second, analyzing the behaviour of the phase shift as a function of the energy.

With the help of the phase shift and the Wigner time-delay, we have shown that the mean life $\tau$ of the resonant bound state decreases as the strength of the perturbative potential increases, a result which is analogous to the one observed when a perturbative constant electric field is considered.

The perturbative potential $\Lambda k x \delta(x)$ diverges as $x \to \pm \infty$. This asymptotic behavior prevents us from obtaining asymptotically free solutions, therefore we cannot use standard scattering theory for obtaining the energy resonances \cite{27}. In this sense, the exact and approximate approaches presented in this article permits us to obtain the value of the resonances in cases where the potential does not go to zero at infinity.

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