Fast quasi-adiabatic dynamics

S. Martínez-Garaot,1 A. Ruschhaupt,2 J. Gillet,3 Th. Busch,3 and J. G. Muga1,4

1Departamento de Química Física, UPV/EHU, Apdo. 644, 48080 Bilbao, Spain
2Department of Physics, University College Cork, Cork, Ireland
3OIST Graduate University, Onna, Okinawa 904-0111, Japan
4Department of Physics, Shanghai University, 200444 Shanghai, People’s Republic of China
(Dated: November 24, 2014)

We put forward a fast quasi-adiabatic approach to speed up slow adiabatic manipulations of quantum systems by driving a control parameter as near to the adiabatic limit as possible over the entire protocol duration. The underlying theory and several examples for internal or translational dynamics of atomic systems are worked out. Specifically, we show that the population inversion in a two-level system, the splitting and cotunneling of two-interacting bosons, and the stirring of a Tonks-Girardeau gas on a ring to achieve mesoscopic superpositions of many-body rotating and non-rotating states, can be significantly speeded up.

PACS numbers: 32.80.Xx, 37.10.Gh, 67.85.-d

Developing technologies based on delicate quantum coherences of atomic systems is a major scientific and technical challenge due to pervasive noise-induced and manipulation errors. Shortening the process duration below characteristic decoherence times provides a way out to avoid the effects of noise, but the protocol (time dependence of control parameters) should still be robust with respect to offsets of the external driving parameters. Shortcuts to adiabaticity (STA) are a set of techniques to shorten the times of slow adiabatic processes, minimizing noise effects while keeping or enhancing robustness [1–3]. There are different approaches but they are not always easy to implement in practice, because of the need to control many variables, or the difficulty to realize certain terms added to the original Hamiltonian (the one that sets the adiabatic dynamics to be speeded up). Here we put forward a simple, but effective, fast quasi-adiabatic (FAQQUAD) approach that engineers the time dependence of a single control parameter \( \lambda(t) \), without changing the structure of the original Hamiltonian, to perform a process as quickly as possible while making it as adiabatic as possible at all times. The two goals are contradictory so a compromise is needed. We impose that the standard adiabaticity parameter [4] is constant throughout the process, and consistent with the boundary conditions (BC) of \( \lambda(t) \) at \( t = 0 \) and \( t = t_f \).

In the simplest scenario we assume that the adiabatic process of reference involves a passage through at least one avoided crossing by a monotonous change of \( \lambda(t) \). While the system is in general multilevel, only the two quasi-crossing levels (say \( E_1, E_2 \)) in the instantaneous basis \( \{|\phi_j(t)\rangle\} \) are considered under the adiabaticity condition

\[
\hbar \left| \frac{\dot{\phi}_1(t)\phi_2(t)}{E_1(t) - E_2(t)} \right| \ll 1.
\]

(1)

More levels can be taken into account as shown in the final example.) We impose

\[
\hbar \left| \frac{\dot{\phi}_1(t)\phi_2(t)}{E_1(t) - E_2(t)} \right| = \hbar \left| \frac{\dot{\phi}_1(t)\phi_2(t)}{E_1(t) - E_2(t)} \right| = c.
\]

(2)

As \( \lambda = \lambda(t) \) and \( t = t(\lambda) \) we apply the chain rule to write

\[
\dot{\lambda} = \mp \frac{c}{\hbar} \left| \frac{E_1(\lambda) - E_2(\lambda)}{\phi_1(\lambda)|\partial_{\lambda}\phi_2(\lambda)} \right| = \mp \frac{c}{\hbar} \left| \frac{E_1(\lambda) - E_2(\lambda)}{\phi_1(\lambda)\frac{\partial H}{\partial \lambda} \phi_2(\lambda)} \right|,
\]

where the overdot is time derivative and \( \mp \) applies to a monotonous decrease/increase of \( \lambda(t) \). Eq. (2) must be solved with the BC \( \lambda(0), \lambda(t_f) \). This fixes both \( c \) and the integration constant. The corresponding solution, \( \lambda_F(t) \), changes fast when the transitions are unlikely and slowly otherwise. Related approaches are analyzed and compared in the final discussion [6].

In terms of \( s = t/t_f \), we define \( \lambda(s) = \lambda(st_f) \) so that \( \dot{\lambda}(t) = \dot{\lambda}/t_f \), where the prime is the derivative with respect to \( s \). From Eq. (2)

\[
\dot{\lambda}' = \mp \frac{\tilde{c}}{\hbar} \left| \frac{E_1 - E_2}{\phi_1(\partial_{s}\phi_2)} \right|, \tag{3}
\]

with \( \tilde{c} = ct_f = \mp \hbar \int_{\lambda(0)}^{\lambda(s)} \frac{d\lambda}{\left| \frac{E_1 - E_2}{\phi_1(\partial_{s}\phi_2)} \right|} \). It is thus enough to solve the FAQQUAD protocol once, i.e. using Eq. (3) we get \( \lambda_F(s) \) and \( \tilde{c} \) to satisfy \( \lambda(s = 0) \) and \( \dot{\lambda}(s = 1) \), and then adapt (scale) the result for each \( t_f, \) as \( \lambda_F(t = st_f) = \lambda_F(s) \), and \( c = \tilde{c}/t_f \). Similarly, the gap \( \omega_{12}(t) = |E_1(t) - E_2(t)|/\hbar \) is given in terms of a universal gap function \( \tilde{\omega}_{12}(\lambda_F(s)) \) as \( \omega_{12}(t) = \tilde{\omega}_{12}(\lambda_F(t/t_f)) \). Depending on \( \tilde{c} \), a large time \( t_f \) might be necessary to make the process fully adiabatic (i.e., with a small enough \( c \) but, surprisingly, much shorter times for which the process is not fully adiabatic also lead to the desired results. Since the system is nearly adiabatic this is explained by adiabatic perturbation theory. In the adiabatic basis the wave function is expanded as

\[
|\Psi(t)\rangle = \sum_{n} g_n(t) e^{i\beta_n(t)} |\phi_n(t)\rangle, \tag{4}
\]

where \( \beta_n(t) = -\frac{\hbar}{\lambda} \int_{0}^{t} E_n(t')dt' + i \int_{0}^{t} \phi_n(t') \phi_n(t')dt' \) and \( |\phi_n(t)\rangle \) is an instantaneous eigenstate of the Hamiltonian. From \( i\hbar |\dot{\Psi}(t)\rangle = H_0(t) |\Psi(t)\rangle \) we get, choosing \( |\phi_n(t)\rangle |\dot{\phi}_n(t)\rangle \) to
be real (in particular $\langle \phi_n(t) | \phi_n(t) \rangle = 0$),
\begin{equation}
\dot{g}_n(t) = -\sum_{k \neq n} e^{iW_{nk}(t)} \langle \phi_n(t) | \phi_k(t) \rangle g_k(t),
\end{equation}
where $W_{nk}(t) = \int_0^t \omega_{nk}(t') dt'$ is a dynamical-gap phase and $\omega_{nk}(t) := [E_n(t) - E_k(t)]/\hbar$. Integrating Eq. (4),
\begin{equation}
g_n(t) - g_n(0) = -\sum_{k \neq n} \int_0^t e^{iW_{nk}(t')} \langle \phi_n(t') | \phi_k(t') \rangle g_k(t') dt',
\end{equation}
which is still exact. Assuming that the initial state is $|\phi_m(0)\rangle$ and approximating $g_k(t') = \delta_{km}$ one finds to first order, for $n \neq m$,
\begin{equation}
g_n^{(1)}(t) = -\int_0^t \langle \phi_n(t') | \dot{\phi}_m(t') \rangle e^{iW_{nm}(t')} dt',
\end{equation}
which should satisfy $|g_n(t)| \ll 1$ for an adiabatic evolution. In FAQUAD $\langle \phi_n(t) | \phi_n(t) \rangle = cr\omega_m(t)$, with $r = \text{sgn}(\langle \phi_n(t) | \phi_m(t) \rangle)$, which is also a good estimate of the minimal time for a complete transition. The upper envelope for the $n$-th state probability is $4c^2/t_f^2$.

The oscillation is due to a quantum interference: $g_n^{(1)}(t)$ results from the sum of paths where the jump at time $t'$ from $m$ to $n$ has an amplitude $c\omega_{nm}(t')$. $e^{iW_{nm}(t')}$ represents the dynamical phases before and after the jump, as $e^{iW_{nm}(t')} = e^{i\int_0^{t'} dt'' E_m(t'')} e^{i\int_0^{t'} dt'' E_n(t'')} e^{i\int_0^{t'} dt'' E_n(t'')}$, where the last exponential is a phase factor independent of $t'$.

To show the power of FAQUAD we will first apply it to the two-level model, as a paradigmatic testbed, and then to more complicated atomic systems for which other shortcut techniques are difficult or impossible to implement.

Population inversion. Consider first a two-mode model with a single avoided crossing. In the bare basis, $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $|2\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, the time-dependent state is $|\Psi(t)\rangle = b_1(t)|1\rangle + b_2(t)|2\rangle$ and $H_0 = \begin{pmatrix} 0 & -\sqrt{2}J \\ -\sqrt{2}J & U - \Delta \end{pmatrix}$, where the bias $\Delta = \Delta(t)$ is the control parameter, and $U > 0$, $J > 0$, are constant. The goal is to drive the eigenstate from $|\phi_1(0)\rangle = |2\rangle$ to $|\phi_1(t_f)\rangle = |1\rangle$. To design the reference adiabatic protocol we impose on $\Delta(t)$ the BC $\Delta(0) \gg U, J$ and $\Delta(t_f) = 0$. The FAQUAD protocol is shown in Fig. 1 (a) compared to a linear-in-time $\Delta(t)$ and a constant $\Delta = U$. The final ground state populations $|b_1(t_f)|^2$ versus dimensionless final time $\tau_f = Jt_f/\hbar$ are shown in Fig. 1 (b). Since the dressed states are essentially pure bare states at initial and final times their populations in bare and dressed state bases coincide at these times. For $\Delta = U$ between 0 and $t_f$, Rabi oscillations occur (see Fig. 1 (b)). The conditions for a $\pi$-pulse or multiple $\pi$-pulses are met periodically over $t_f$ alternated with times where the probability drops to zero because of destructive interference among two dressed states superposed with equal weights. By contrast the FAQUAD process is dominated by one dressed state and the influence of the transitions to the other one is minimized, because they are small in amplitude, and because at certain times they completely cancel each other out by destructive interference. The time interval between population maxima for FAQUAD is $2\pi/\Phi_{1,2}$, i.e., it is not governed by the Rabi frequency. The first maximum is at a small $t_f$ similar to the one for the $\pi$-pulse, but broader. Fig. 1 (b) also shows the poorer results of the linear ramp for $\Delta(t)$.

Interacting bosons in a double well. Pairs of interacting bosons in a double well potential may be manipulated to implement universal quantum logic gates for quantum computation or to observe fundamental phenomena such as cotunneling of two atoms [10]. We shall speed up two processes: the splitting of the two particles from one to the two separate wells, and cotunneling, see Fig. 2. The boson dynamics in a double well with tight lateral confinement is described by a two-site Bose-Hubbard Hamiltonian $\hat{\mathcal{H}}$. The Hamiltonian in the occupation number basis $|2,0\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, $|1,1\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ and $|0,2\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$, is $H_0 = \begin{pmatrix} 0 & \sqrt{2}J & -\sqrt{2}J & 0 \\ \sqrt{2}J & U + \Delta & -\sqrt{2}J & 0 \\ -\sqrt{2}J & 0 & U - \Delta & 0 \\ 0 & -\sqrt{2}J & 0 & U - \Delta \end{pmatrix}$, where the bias $\Delta = \Delta(t)$ is the control function, $J$ is the hopping energy and $U$ the interaction energy. We write the time-dependent states as $|\Psi(t)\rangle = c_1(t)|2,0\rangle + c_2(t)|1,1\rangle + c_3(t)|0,2\rangle$. Adiabatic processes that change $\Delta(t)$ slowly, keeping the $U/J$ ratio constant are possible to implement splitting or cotunnelling. Speeding them up by a “counterdiabatic” approach is not possible in practice because of the need to apply new terms in the Hamiltonian which
are difficult to implement. Alternative techniques could not be applied or are cumbersome because of the relatively large algebra involved. The FAQUAD approach provides a viable way out.

- In a splitting process $\Delta(0) \gg U, J$ and $\Delta(\tau_f) = 0$, see Fig. 2 (a). The initial ground state is $|\phi_1\rangle = |0, 2\rangle$ and the final ground state $|\phi_1\rangle = |1, 1\rangle$. Figure 3 (a) shows the dependence of the three eigenenergies with $\Delta$. $\Delta_F(t)$ is very similar to the result for the two-level system in Fig 4 (a). The results of FAQUAD and the linear protocol are compared in Fig. 3 (b). The probability of the first peak for FAQUAD, 0.998 at $\tau_f = 1.2$, is achieved with the linear ramp for $\tau_f = 43$.

- Cotunneling. In a speeded-up cotunneling shown in Fig. 2 (b) the goal is to drive the system fast from $|\phi_1(0)\rangle = |0, 2\rangle$ to $|\phi_1(\tau_f)\rangle = |2, 0\rangle$ intermediated by $|1, 1\rangle$ (the Hamiltonian $H_0$ does not connect $|0, 2\rangle$ and $|0, 0\rangle$ directly). We impose $\Delta(0) \gg U, J$ and $\Delta(\tau_f) = -\Delta(0)$ to have $|0, 2\rangle$ and $|2, 0\rangle$ as the ground state at initial and final times respectively. The energy levels versus $\Delta$ are depicted in Fig. 3 (a) for repulsive interaction ($U > 0$). Figure 3 (a) shows the FAQUAD trajectory for $\Delta(t)$ for the repulsive strong-interaction regime, $U/J = 22.3$. Fig. 3 (b) depicts the final probabilities of the bare state $|2, 0\rangle$ for FAQUAD and a linear protocol that needs about $\tau_f = 65$ to achieve the value of the first peak of the FAQUAD method ($|c_1|^2 = 0.998$ at $\tau_f = 2.3$). The minima in the FAQUAD probability go in this case below the lower envelope $1 - 4\Delta^2/\Delta_f^2$ predicted by perturbation theory. The reason is a leak through the narrow avoided crossing at $\Delta = 0$ from the second to the third energy level, see Fig. 3 (a). The leak occurs at total process times in which the first avoided crossing produces a minimum of the ground state probability.

Superpositions of collective rotating and non-rotating atoms on a ring. Creating a macroscopic or mesoscopic superposition of a many-particle system is a difficult task and of interest for research in quantum information, quantum metrology, and fundamental aspects of quantum mechanics. However, it was recently proposed that a low-dimensional gas of interacting bosons in the Tonks-Girardeau (TG) limit placed on a ring can be perturbed in such a way, that a robust superposition of two angular momentum states can be achieved. This perturbation corresponds to the introduction of a narrow stirring potential, which is then accelerated to a certain value to spin up the gas.

For the single particle, such a situation is described by $i\hbar\partial_t \psi(x,t) = \left\{ -\frac{\hbar^2}{2M} \frac{\partial^2}{\partial x^2} + U_0 \delta[x-x_0(t)] \right\} \psi(x,t)$, where the stirrer is represented by a $\delta$-function of strength $U_0$, and periodic BC are assumed. Going to a moving frame by defining $y = x - x_0(t)$, the resulting Hamiltonian is $H = \frac{1}{2M} \left[ \hat{P}_y - \hbar \Omega(t)/L \right]^2 + U_0 \delta(y)$, where $L$ is the ring perimeter, $\hbar \Omega = M \dot{x}_0$, and $\hat{P}_y = -i\hbar \partial/\partial y$. The energy eigenvalues are $E(n) = \frac{2\hbar^2}{ML^2} \alpha_n^2$, and the $\alpha_n$ are solutions of $\frac{4\hbar^2}{ML^2} \alpha_n^2 = \cot(\pi \alpha_n/\Omega)/2 + \cot(\pi \alpha_n/\Omega + 2\pi)$. For $U_0 \rightarrow 0$, the $\alpha_n$ tend to $n - \Omega/(2\pi)$, with $n = 0, \pm 1, \pm 2, \ldots$, where the different signs correspond to clockwise or counterclockwise rotation in the lab frame and the $n$-th eigenstates are plane waves with momentum $n\hbar 2\pi/L$. For
0 < Ω < π the energies in the moving frame increase for states with \( n = 0 \) and increase for \( n > 0 \). For \( U_0 = 0 \) the spectrum shows degeneracies at \( Ω = 0, π \), which turn into avoided crossings once the stirrer couples different angular momentum eigenstates, as shown in Fig. 5(a) for \( U = 0 \). Adiabatically increasing the stirring frequency from \( Ω = 0 \) to \( π \) then allows to drive the system into a superposition of two angular momentum states and for a TG gas with an odd number of particles \( N \) it can be shown that the ground state at \( Ω = π \) corresponds to macroscopic superposition between the angular momentum states \( 0 \) and \( N - 1 \).

To design an optimal \( Ω(t) \) for the TG gas, one can first notice that all respective gaps at the avoided crossings at \( Ω = 0 \) and \( π \) are of the same size for \( U_0 \ll \epsilon_0 \) and therefore only consider

\[
\hbar \int |\langle \phi_1(t) | \partial_t \phi_2(t) \rangle |^2 \frac{E_1(t) - E_2(t)}{E_2(t) - E_3(t)} + \hbar |\langle \phi_2(t) | \partial_t \phi_3(t) \rangle |^2 \frac{E_2(t) - E_3(t)}{E_2(t) - E_3(t)} = c. \tag{7}
\]

The resulting \( Ω_F(s) \) is shown in Fig. 6(b) and the final state fidelities for a TG gas of \( N = 3 \) and \( N = 5 \) particles with respect to the exact ground state can be seen in Fig. 6. When compared to a linear ramp, a clear improvement is visible.

**Discussion.** The FAQUAD approach to speed up adiabatic manipulations of quantum systems achieves significant time shortening by distributing homogeneously the adiabaticity parameter along the process time while satisfying the boundary conditions of the control parameter. It is related to but different from other works that manipulate the time-dependence of a control parameter to delocalize in time the transition probability among adiabatic levels. In the parallel adiabatic transfer technique, the level gap is kept constant, but this cannot be applied to processes, such as the TG gas example, where initial and final gaps must be different. Refs. [1, 16] used a special case of Eq. 1 to implement fast trap expansions, and [19] to implement vibrational mode multiplexing, but no general theory was developed. In the “local adiabaticity” (LA) approach of refs. [1, 16] an equation similar to Eq. 2 was used without the factor \( |\langle \phi_1(\lambda) | \partial_\lambda \phi_2(\lambda) \rangle |^2 \). This leads to a different constant, \( c_{LA} \), and time dependence of the parameter, \( \lambda_{LA}(t) \), as well as to larger minimal times as illustrated below. Also, motivated by a comparison of “transition” and “relaxation” time scales in the Kibble-Zurek mechanism for phase transitions, Quan and Zurek developed “uniformly adiabatic” (UA) method. In our current notation \( \lambda = \pm \frac{c_{UA}}{\hbar} \left( \frac{E_2(\lambda) - E_3(\lambda)}{E_2(\lambda) - E_3(\lambda)} \right) \). FAQUAD is compared to the LA and UA approaches in Fig. 7 for the simple two-level system. It provides shortcuts at smaller process times (it achieves 0.9998 probability three times faster than LA) and an analytically predictable behavior via the perturbation theory analysis. We have demonstrated its applicability in different systems, in particular where other approaches are not available, and expect a broad range of applications, in quantum, optical and mechanical systems, due to the ubiquity of adiabatic methods.

We thank M. Palmero, X. Chen, and S. Ibáñez for helpful discussions. This work was supported by the Basque Country Government (Grant No. IT472-10), Ministerio de Economía y Competitividad (Grant No. FIS2012-36673-C03-01), the program UFI 11/55 of UPV/EHU, and by the Okinawa Institute of Science and Technology Graduate University. This publication has emanated from research conducted with the financial support of Science Foundation Ireland under the International Strategic Cooperation Award Grant Number SFI/13/ISCA/2845. S. M.-G. acknowledges a fellowship by UPV/EHU.

[1] X. Chen, A. Ruschhaupt, S. Schmidt, A. del Campo, D. Guéry-Odelin and J. G. Muga, Phys. Rev. Lett. **104**, 063002 (2010).

[2] A. Ruschhaupt, X. Chen, D. Alonso, and J. G. Muga, New J. Phys. **14**, 093040 (2012).
[3] E. Torrontegui, S. Ibáñez, S. Martínez-Garaot, M. Modugno, A. del Campo, D. Guéry-Odelin, A. Ruschhaupt, X. Chen, and J. G. Muga, Adv. At. Mol. Opt. Phys. 62, 117 (2013).
[4] L. I. Schiff, *Quantum mechanics* (McGraw-Hill, New York, 1981).
[5] J. Roland and N. J. Cerf, Phys. Rev. A 65, 042308 (2002).
[6] P. Richerme, C. Senko, J. Smith, A. Lee, S. Korenblit, and C. Monroe, Phys. Rev. A 88, 012334 (2013).
[7] H. T. Quan and W. H. Zurek, N. J. Phys. 12, 093025 (2010).
[8] S. Ibáñez and J. G. Muga, Phys. Rev. A 89, 033403 (2014).
[9] S. Fölling, S. Trotzky, P. Cheinet, M. Feld, R. Saers, A. Widera, T. Müller and I. Bloch, Nature 448, 1029 (2007).
[10] M. Anderlini, P. J. Lee, B. L. Brown, J. Sebby-Strabley, W. D. Phillips, and J. V. Porto, Nature 448, 452 (2007).
[11] T. Opatrný and K. Mølmer, New J. Phys. 16, 015025 (2014).
[12] E. Torrontegui, S. Martínez-Garaot and J. G. Muga, Phys. Rev. A 89, 043408 (2014).
[13] S. Martínez-Garaot, E. Torrontegui, X. Chen and J. G. Muga, Phys. Rev. A 89, 053408 (2014).
[14] M. Girardeau, J. Math. Phys. 1, 516 (1960).
[15] D. W. Hallwood, T. Ernst, and J. Brand, Phys. Rev. A 82, 063623 (2010).
[16] A. Kastberg, W. D. Phillips, S. L. Rolston, R. J. C. Spreeuw, and P. S. Jessen, Phys. Rev. Lett. 74, 1542 (1995).
[17] E. Torrontegui, X. Chen, M. Modugno, A. Ruschhaupt, D. Guéry-Odelin, and J. G. Muga, Phys. Rev. A 85, 033605 (2012).
[18] R. Bowler, J. Gaebler, Y. Lin, T. R. Tan, D. Hanneke, J. D. Jost, J. P. Home, D. Leibfried and D. J. Wineland, Phys. Rev. Lett. 109, 080502 (2012).
[19] S. Martínez-Garaot, E. Torrontegui, X. Chen, M. Modugno, D. Guéry-Odelin, S.-Y. Tseng, and J. G. Muga, Phys. Rev. Lett. 111, 213001 (2013).
[20] S. Guérin, S. Thomas, and H. R. Jauslin, Phys. Rev. A 65, 023409 (2002).
[21] S. Guérin, V. Hakobyan, and H. R Jauslin, Phys. Rev. A 84, 013423 (2011).