Simultaneous Inference for Time Series Functional Linear Regression

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Abstract

We consider the problem of joint simultaneous confidence band (JSCB) construction for regression coefficient functions of time series scalar-on-function linear regression when the regression model is estimated by roughness penalization approach with flexible choices of orthonormal basis functions. A simple and unified multiplier bootstrap methodology is proposed for the JSCB construction which is shown to achieve the correct coverage probability asymptotically. Furthermore, the JSCB is asymptotically robust to inconsistently estimated standard deviations of the model. The proposed methodology is applied to a time series data set of electricity market to visually investigate and formally test the overall regression relationship as well as perform model validation.

Keywords: Convex Gaussian approximation; Functional time series; Joint simultaneous confidence band; Multiplier bootstrap; Roughness penalization.

1 Introduction

It is increasingly common to encounter time series that are densely observed over multiple oscillation periods or natural consecutive time intervals. To address statistical issues with respect to data structures such as the aforementioned, functional (or curve) time series analysis has undergone unprecedented development over the last two decades. See [27] and [6] for excellent book-length treatments of the topic. We also refer the readers to [26], [2, 3], [43], [42] and [16] among many others for articles that address various modelling, estimation, forecasting and inference aspects of functional time series analysis from both time and spectral domain perspectives.

The main purpose of this article is to perform simultaneous statistical inference for time series scalar-on-function linear regression. Specifically, consider the following time series
functional linear model (FLM):

$$Y_i = \beta_0 + \sum_{j=1}^{p} \int_{0}^{1} \beta_j(t)X_{ij}(t)dt + \epsilon_i, \quad i = 1, \ldots, n,$$

(1)

where \(\{X_i(t) := (X_{i1}(t), \ldots, X_{ip}(t))^T\}_{i=1}^{n}\) is a \(p\)-variate stationary time series of known functional predictors observed on \([0, 1]\), \(\{Y_i\}_{i=1}^{n}\) is a univariate stationary time series of responses, and \(\{\epsilon_i\}_{i=1}^{n}\) is a centered stationary time series of regression errors satisfying \(\mathbb{E}[X_{ij}(t)\epsilon_i] = 0\) for all \(t \in [0, 1]\) and \(j = 1, 2, \ldots, p\). Observe that \(X_i(t)\) and \(\epsilon_i\) could be dependent and \(\beta_0 + \sum_{j=1}^{p} \int_{0}^{1} \beta_j(t)X_{ij}(t)dt\) can be viewed as the best linear forecast of \(Y_i\) based on \(X_i(t)\). We are interested in constructing asymptotically correct joint simultaneous confidence bands (JSCB) for the regression coefficients \(\beta(t) := (\beta_1(t), \ldots, \beta_p(t))^T\); that is, we aim to find random functions \(L_{n,j}(t)\) and \(U_{n,j}(t)\), \(j = 1, 2, \ldots, p\), such that

$$\lim_{n \to \infty} \mathbb{P}\left(L_{n,j}(t) \leq \beta_j(t) \leq U_{n,j}(t), \text{ for all } t \in [0, 1] \text{ and } j = 1, 2, \ldots, p\right) = 1 - \alpha$$

for a pre-specified coverage probability \(1 - \alpha\). The need for JSCB arises in many situations when one wants to, for instance, rigorously investigate the overall magnitude and pattern of the regression coefficient functions, test various assumptions on the regression relationship and perform diagnostic checking and model validation of (1) without multiple hypothesis testing problems.

To date, results for the time series scalar-on-function regression (1) are scarce and are mainly on the consistency of functional principle component (FPC) based estimators ([26], [25]). To our knowledge, there is no literature on asymptotically correct JSCB construction for \(\beta(t)\) under time series dependence. On the other hand, there is a wealth of statistics literature dealing with estimation, convergence rate investigation, prediction, and application of FLM (1) when the data \(\{(Y_i, X_i(t))\}_{i=1}^{n}\) are independent and identically distributed (i.i.d.). See, for instance, [8], [15], [34], [7] and [31] for a far from exhaustive list of references. We also refer the readers to [9], [40], [56] and [46] for excellent recent reviews of the topic and more references. Meanwhile, the last two decades also witnessed an increase in statistics literature on the inference of FLM (1) for independent data. Since a JSCB is primarily an inferential tool, we shall review this literature in more detail. The main body of the aforementioned literature consists of results related to \(L^2\)-type tests on whether \(\beta(t) = 0\) or a fixed known function; see for instance [24], [33], [32] and [52], among others. Other contributions include confidence interval construction for the conditional mean and hypothesis testing for functional contrasts ([50]) and goodness of fit tests for (1) versus possibly nonlinear alternatives ([28], [21], [38]). On the other hand, however, to our knowledge for independent observations there are few results discussing confidence band construction for \(\beta(t)\). [30] proposed a simple methodology to construct a conservative confidence band for the slope function of scalar-on-function linear regression.
for independent data which covers “most” of points; Recently, [17] constructed an asymptotically correct simultaneous confidence band for function-on-function linear regression of independent data and the authors also discussed the scalar-on-function case briefly. The implementation of the latter paper requires estimating the convergence rate of the regression which could be a relatively difficult task in moderate samples.

There are two major challenges involved with JSCB construction of $β(t)$ for the time series FLM (1). Firstly, estimation of model (1) is related to an ill-posed inverse problem ([9], Section 2.2) and often estimators of $β(t)$ are not tight on $[0, 1]$. As a result, it has been a difficult problem to investigate the large sample distributional behavior of estimators of $β(t)$ uniformly across $t$. Secondly, the rates of convergence for various estimators of $β(t)$ depend sensitively on the smoothness of $X_p(t)$ and $β(t)$, and the penalization parameter used in the regression. Consequently, in practice it is difficult to determine the latter rates of convergence and hence the appropriate normalizing constants for the uniform inference of $β(t)$. In this article, we address the aforementioned challenges by proposing a simple and unified multiplier bootstrap methodology for the JSCB construction. For the roughness penalized estimators of $β(t)$, the multiplier bootstrap will be shown to well approximate their weighted maximum deviations on $[0, 1]$ uniformly across all quantiles and a wide class of smooth weight functions under quite general conditions in large samples. As a result, the bootstrap allows to construct asymptotically correct JSCB for $β(t)$ in a simple and unified way without having to derive the uniform distributional behavior or rates of convergence. Furthermore, the JSCB constructed by the multiplier bootstrap remains asymptotically correct when the weight function is estimated inconsistently under some mild conditions, adding another layer of robustness to the methodology. Theoretically, validation of the bootstrap depends critically on uniform Gaussian approximation and comparison results over all Euclidean convex sets for sums of stationary and weakly dependent time series of moderately high dimensions which we will establish in this paper. These results extend the corresponding findings for independent or $m$-dependent data investigated in [5], [20] and [19] among others.

The multiplier/weighted bootstrap technique has attracted much attention recently. Among others, [10] derived asymptotic consistency of the generalized bootstrap technique for estimating equations. [37] established the validity of the weighted bootstrap technique based on a weighted $M$-estimation framework. Non-asymptotic results on the multiplier bootstrap validity with applications to high-dimensional inference of independent data were established in [12]. Later, [51] considered a multiplier bootstrap procedure in the construction of likelihood-based confidence sets under possible model misspecifications.

The paper is organized as follows. In Section 2, we propose the methodology of the JSCB construction based on roughness penalization approach. The theoretical result on the multiplier bootstrap for estimators of $β(t)$ by roughness penalization estimation is discussed in Section 3. In particular, we introduce in Section 3.1 a general class of stationary
functional time series models based on orthonormal basis expansions and nonlinear system theory. Section 4 investigates finite sample accuracy of the bootstrap methodology for various basis functions and weighting schemes using Monte Carlo experiments. We analyze a time series dataset on electricity demand curves and daily total demand in Spain in Section 5 and conclude the article in Section 6 with discussions on several issues including the effects of pre-smoothing, practical choices of the basis functions and extensions to regression models with functional response. Additional simulations, examples, theoretical results and the proofs of all theoretical results are deferred to the supplemental material.

2 Methodology

Hereafter, for simplicity we shall assume that $Y_i$ and $X_i(t)$ are centered and hence $\beta_0 = 0$. Let $H = \mathcal{L}^2([0,1])$ be the Hilbert space of all square integrable functions on $[0,1]$ with inner product $\langle x, y \rangle = \int_0^1 x(t)y(t)dt$. We also denote by $C^d([0,1])$ the collection of functions that are $d$-times continuously differentiable with absolutely continuous $d$-th derivative on $[0,1]$.

2.1 Roughness Penalization Estimation

In order to facilitate the formulation of roughness penalization estimation, we first prepare some notations. Throughout this paper, we assume that $X_{ij}(t)$ for $j = 1, \ldots, p$, $i = 1, \ldots, n$ is continuous on $[0,1]$ a.s. and hence admits the following expansion

$$X_{ij}(t) = \sum_{k=1}^{\infty} f_{jk} x_{ij,k} \alpha_k(t),$$

where $f_{jk} = \text{Std}(\tilde{x}_{ij,k})$ with Std denoting standard deviation and $x_{ij,k} = \tilde{x}_{ij,k}/f_{jk}$ if $f_{jk} \neq 0$. Set $x_{ij,k} = 0$ if $f_{jk} = 0$. Notice that $f_{jk}$ captures the decay speed of $\tilde{x}_{ij,k}$ as $k$ increases and the random coefficient $\{x_{ij,k}\}_{k=1}^{\infty}$ remains at the same magnitude with variance 1 as $k$ increases if $f_{jk} \neq 0$. Similarly, write $\beta_j(t) = \sum_{k=1}^{\infty} \beta_{jk} \alpha_k(t)$. The following assumption restricts the decay speed of the basis expansion coefficients of $\{X_i(t)\}$ and $\{\beta(t)\}$.

**Assumption 2.1.** For some non-negative integers $d_1$ and $d_2$ with $d_1 \geq d_2$, assume that $\beta_j(t) \in C^{d_1}([0,1])$ and $X_{ij}(t) \in C^{d_2}([0,1])$ a.s.. Suppose that $|\beta_{jk}| \leq C_1 k^{-(d_1+1)}$, $\forall j = 1, \ldots, p$ and $C_1 > 0$ is some finite constant, the random coefficient $\tilde{x}_{ij,k} = O_{a.s.}(k^{-(d_2+1)})$ for $i = 1, \ldots, n$, $j = 1, \ldots, p$.

Let $d \geq 0$ be an integer. It is well-known that for a general $C^d([0,1])$ function the fastest decay rate for its $k$-th basis expansion coefficient is $O(k^{-d-1})$ for a wide class
of basis functions ([11]). For instance, the Fourier basis (for periodic functions), the weighted Chebyshev polynomials ([53]) and the orthogonal wavelets with degree \( m \geq d \) ([39]) admit the latter decay rate under some extra mild assumptions on the behavior of the function’s \( d \)-th derivative. Hence Assumption 2.1 essentially requires that the basis expansion coefficients of \( \beta_j(t) \) and \( X_{ij}(t) \) decay at the fastest rate. On the other hand, we remark that the basis expansion coefficients may decay at slower speeds for some orthonormal bases. An example is the Legendre polynomials where the coefficients decay at an \( O(k^{-d-1/2}) \) speed ([55]). For basis functions whose coefficients decay at slower rates, following the proofs of this paper it is obvious to see that the multiplier bootstrap method for the JSCB construction is still asymptotically valid under the corresponding restrictions on the tuning parameters. However, in this case the estimates of \( \beta_j(t) \) will converge at a slower speed and the bootstrap approximation may be less accurate. For the sake of brevity we shall stick to the fastest decay Assumption 2.1 for our theoretical investigations throughout this paper.

The roughness penalization approach to the FLM (1) solves the following penalized least squares problem

\[
\tilde{\beta}(t) = \arg \min_{\beta(t)} \left\{ \frac{1}{n} \sum_{i=1}^{n} \left[ Y_i - \sum_{j=1}^{p} \int_0^1 \beta_j(t)X_{ij}(t)dt \right]^2 + \lambda \sum_{j=1}^{p} \int_0^1 [\beta_j''(t)]^2 dt \right\}, \tag{3}
\]

see [50] and references therein. On the other hand, in functional data analysis, researchers usually work on the optimization problem on a finite-dimensional subspace ([34]), which makes the procedure easily implementable. Following the method proposed by [45], we truncate the coefficient functions to finite (but diverging) dimensional spans of a priori set of basis functions \( \{\alpha_k(t)\}_{k=1}^{\infty} \) while involving a smoothness penalty. Specifically, assume the truncation number of \( \beta_j(t) \) equals \( c_j \) \( (c_j \to \infty) \). Then the right hand side of (3) can be approximated by

\[
\frac{1}{n} \sum_{i=1}^{n} \left[ Y_i - \sum_{j=1}^{p} \int_0^1 \beta_{j,c_j}(t)X_{ij}(t)dt \right]^2 + \lambda \sum_{j=1}^{p} \int_0^1 [\beta_{j,c_j}''(t)]^2 dt,
\]

where \( \beta_{j,c_j}(t) = \sum_{k=1}^{c_j} \beta_{jk}\alpha_k(t) \). Simplifying the above expression, the estimation can be achieved by minimizing the following penalized least squares criterion function

\[
\frac{1}{n} \sum_{i=1}^{n} \left[ Y_i - \sum_{j=1}^{p} \sum_{k=1}^{c_j} \beta_{jk}\tilde{x}_{ij,k} \right]^2 + \lambda \sum_{j=1}^{p} \sum_{k,l=1}^{c_j} \beta_{jk}\beta_{jl}\tilde{R}_j(k,l)
\]

\[
= \frac{1}{n} [Y - X_{c}\theta_c]^T[Y - X_{c}\theta_c] + \theta_c^T R(\lambda) \theta_c, \tag{4}
\]

where \( \lambda \) \( (\lambda \to 0) \) is a common smoothing parameter that measures the rate of exchange between fit to the data and smoothness of the estimator, as measured by the residual sum
of squares in the first term, and variability of the functions \( \beta_{j,c}(t) \) in the second term. Here we impose a roughness penalty associated with the same smoothing parameter \( \lambda \) for every \( j \), which is commonly used in practice, see [18].

In (4), \( c = c(n) = \sum_{j=1}^{p} c_j \) and \( \theta_c \) is a \( c \)-dimensional block vector where \( \theta^c_j = (\theta_{j1}, ..., \theta_{jc})^\top \) is the \( j \)-th block with \( \theta_{jk} = f_{jk}\beta_{jk} \) for \( k = 1, ..., c_j \). Furthermore, \( X_c \) is the \( n \times c \) block design matrix, \( R(\lambda) \) is a \( c \times c \) block diagonal matrix with \( \lambda R_1 \) as its diagonals and \( R_j \) is a \( c_j \times c_j \) matrix with elements \( R_j(k,l) = \int_0^1 \alpha_k^n(t)\alpha_l^n(t)dt/(f_{jk}f_{jl}) \). Then the penalized least squares estimator turns out to be

\[
\hat{\theta}_c = \left[ \frac{X_c^\top X_c}{n} + R(\lambda) \right]^{-1} \frac{X_c^\top Y}{n}.
\]

Consequently, it is easy to find the matrix representation of the roughness penalized estimator \( \hat{\beta}(t) = C_j(t)\hat{\theta}_c \), where \( C_j(t) = (C_1^\top(t), ..., C_p^\top(t)) \) is a \( p \times c \) matrix and \( C_j(t) \) is a \( c \)-dimensional block vector with \( (\alpha_1(t)/f_{j1}, ..., \alpha_c(t)/f_{jc})^\top \) as its \( j \)-th block and other elements being 0.

### 2.2 JSCB Construction

JSCB construction of \( \beta(t) \) boils down to evaluating the distributional behavior of the weighted maximum deviation

\[
\Xi_{n,g_n} := \sqrt{n} \sup_{t \in [0,1]} |\hat{\beta}(t) - \beta(t)|_{g_n(t)},
\]

where \( |V(t)|_{g(t)} = \max_{1 \leq j \leq p} |V_j(t)/g_j(t)| \) for any function \( V(t) = (V_1(t), ..., V_p(t))^\top \) and weight function \( g(t) = (g_1(t), ..., g_p(t))^\top \). The weight function \( g_n(t) \) is assumed to belong to a class \( \mathcal{G} \) with

\[
\mathcal{G} = \{ f(t) : [0,1] \to \mathbb{R}^p, f = (f_1, ..., f_p)^\top \text{ is a continuous } p-\text{dimensional vector function satisfying } \inf_{t \in [0,1]} \min_{1 \leq j \leq p} f_j(t) \geq \kappa \text{ for some constant } \kappa > 0 \}.
\]

For a given \( \alpha \in (0,1) \), denote by \( \xi_{n,g_n}(\alpha) \) the \((1 - \alpha)\)-th quantile of \( \Xi_{n,g_n} \). Then a JSCB with coverage probability \( 1 - \alpha \) can be constructed as \( \beta_{j,t} = \beta_{j,n}(\alpha)|_{g_n(t)}/\sqrt{n} \), \( t \in [0,1] \), \( j = 1, 2, ..., p \).

Observe that the width of the JSCB is proportional to the weight function \( g_n(t) \). In practice one could simply choose some fixed weight functions such as \( g_n(t) \equiv 1 \) which yields equal JSCB width at each \( t \) and \( j \). Alternatively, when the sample size is sufficiently large and temporal dependence is weak or moderately strong, we recommend choosing \( g_n(t) = \text{Std}(\beta_{j,t})/\int_0^1 \text{Std}(\beta_j(s))ds \), \( j = 1, ..., p \). There are two advantages for this data-driven choice of weights. First, the resulting width of the JSCB reflects the standard deviation of \( \hat{\beta}(t) \) which gives direct visual information on the estimation uncertainty at
every \( t \in [0, 1] \) and \( j = 1, \ldots, p \). Second, this choice of weight function yields much smaller average width of the JSCB compared to some fixed choices such as \( g_n(t) \equiv 1 \); see our simulations in Section 4 for a finite-sample illustration. On the other hand, \( \text{Std}(\tilde{\beta}_j(t)) \) has to be estimated in practice. Later in this article, we shall discuss its estimation and also asymptotic robustness of our multiplier bootstrap methodology when the weight function is inconsistently estimated.

To motivate the multiplier bootstrap, denote \( \hat{\Sigma}_c(\lambda) = X_c^T X_c/n + R(\lambda) \). Note that \( \hat{\Sigma}_c(\lambda) \approx E(X_c^T X_c)/n + R(\lambda) := \Sigma_c(\lambda) \) under mild conditions. For simplicity, we assume that each \( \beta_j(t) \) has the same degree of smoothness, here we let each \( c_j \) have the same rate of divergence. By elementary calculations and basis expansions, we have

\[
\tilde{\beta}(t) - \beta(t) = C_f(t) \Sigma_c^{-1}(\lambda) \frac{X_c^T \varepsilon}{n} - C_f(t) \hat{\Sigma}_c^{-1}(\lambda) R(\lambda) \theta_c + O(c^{-d_1}),
\]

(5)

where \( \varepsilon = (\epsilon_1 + O_p(c^{-(d_1+d_2+1)}), \ldots, \epsilon_n + O_p(c^{-(d_1+d_2+1)}))^T \). Hence if \( c \) is sufficiently large and \( \lambda \) is relatively small such that \( \tilde{\beta}(t) \) is under-smoothed, i.e., the standard deviation of the estimation (captured by the first term on the right hand side of (5)) dominates the bias asymptotically, Eq.(5) reveals that the maximum deviation of \( \sqrt{n} \tilde{\beta}(t) \) on \([0, 1]\) is determined by the uniform probabilistic behavior of \( Q_n^x(t, \lambda) := C_f(t) \Sigma_c^{-1}(\lambda) Z_n^x \), where \( Z_n^x := \frac{1}{\sqrt{n}} \sum_{i=1}^n z_{ci} \) and \( z_{ci} = x_{ci} \varepsilon_i \) with \( x_{ci} \) being the \( i \)-th column of \( X_c^T \).

There are two major difficulties in the investigation of \( Q_n^x(t, \lambda) \) uniformly in \( t \). Firstly, \( \{z_{ci}\}_{i \in \mathbb{Z}} \) is typically a moderately high-dimensional time series whose dimensionality \( c \) diverges slowly with \( n \) and \( Q_n^x(t, \lambda) \) is not a tight sequence of stochastic processes on \([0, 1]\). Consequently, deriving the explicit limiting distribution of the maximum deviation of \( Q_n^x(t, \lambda) \) is a difficult task. Second, the convergence rate of \( \sup_{t \in [0, 1]} |Q_n^x(t, \lambda)|_{g_n(t)} \) depends on many nuisance parameters such as the smoothness of \( X(t) \) and \( \beta(t) \), and the diverging rate of the truncation parameters \( c_j \), which are difficult to estimate in practice. To circumvent the aforementioned difficulties, one possibility is to utilize certain bootstrap methods to avoid deriving and estimating the limiting distributions and nuisance parameters explicitly. In this article, we resort to the multiplier/wild/weighted bootstrap ([57]) to mimic the probabilistic behavior of the process \( Q_n^x(t, \lambda) \) uniformly over \( t \). In the literature, the multiplier bootstrap has been used for high dimensional inference on hyper-rectangles and certain classes of simple convex sets that can be well approximated by (possibly higher dimensional) hyper-rectangles after linear transformations; see for instance [12] and [14] for independent data and [64] for functional time series. The inference of \( \sup_{t \in [0, 1]} |Q_n^x(t, \lambda)|_{g_n(t)} \) uniformly over all quantiles and weight functions in \( \mathcal{G} \) can be transformed into investigating the probabilistic behavior of \( Z_n^x \) over a large class of moderately high-dimensional convex sets. However, these convex sets have complex geometric structures for which results that are based on approximations on hyper-rectangles and their linear transformations are not directly applicable. As a result, in this article we shall extend the uniform Gaussian approximation and comparison results over all high-
dimensional convex sets for sums of independent and \( m \)-dependent data established in, for instance, [5], [20] and [19] to sums of stationary and short memory time series in order to validate the multiplier bootstrap. These results may be of wider applicability in other moderately high-dimensional time series problems.

To be more specific, we will consider the bootstrapped sum given a block size \( m \):

\[
U_n^{\text{boots}} = \frac{1}{\sqrt{n-m+1}} \sum_{j=1}^{n-m+1} \left( \frac{1}{\sqrt{m}} \sum_{i=j}^{j+m-1} z_{ci} \right) u_j,
\]

where \( \{u_j\}_{j=1}^{n-m+1} \) is a sequence of i.i.d.

standard normal random variables which is independent of \( Z_n^u := \{z_{c1}, \ldots, z_{cn}\} \). Define \( Q_n^{\text{boots}}(t, \lambda) = C_f(t) \tilde{\Sigma}_c^{-1}(\lambda) U_n^{\text{boots}} \), then we will show that, conditional on the data, \( U_n^{\text{boots}} \)

approximates \( Z_n^c \) in distribution in large samples with high probability. Naturally, we can use the conditional distribution of \( \sup_{t \in [0,1]} \left| Q_n^{\text{boots}}(t, \lambda) \right|_{g_n(t)} \) to approximate the law of \( \sup_{t \in [0,1]} \left| Q_n^z(t, \lambda) \right|_{g_n(t)} \) uniformly over all quantiles and weight functions in \( \mathcal{G} \).

### 2.3 Tuning Parameter Selection and The Implementation Algorithm

In this subsection, we will first discuss the issue of tuning parameter selection for the roughness penalization regression. We have three parameters to choose, that is the auxiliary truncation parameters \( c_j \), the smoothing parameter \( \lambda \) and the window size \( m \).

We recommend choosing \( c_j = 2d_j \) where \( d_j \) can be selected via the cumulative percentage of total variance (CPV) criterion. Specifically, we choose \( d_j \) such that the quantity

\[
\frac{\rho_j}{\sum_{k=1}^{\infty} \rho_k}
\]

exceeds a pre-determined high percentage value (e.g., 85% used in the simulations), where \( \{\rho_k\}_{k=1}^{\infty} \) are the eigenvalues of \( \text{Cov}(X_{ij}(s), X_{ij}(t)) \). The rationale is that with the aid of the roughness penalization, \( c_j \) can be chosen at a relatively large value to reduce the sieve approximation bias without blowing up the variance of the estimation.

In addition, the generalized cross validation (GCV) method can be used to choose \( \lambda \), see for examples [8, 47]. To be more specific, the GCV criterion for the smoothing parameter is defined as

\[
\text{GCV}(\lambda) = \frac{1}{n} \sum_{i=1}^{n} \left[ \frac{Y_i - \widehat{\gamma}_i}{1 - \text{Trace}(H)/n} \right]^2,
\]

where \( H = X_c[X_{c}\top X_{c}/n + R(\lambda)]^{-1}X_{c}\top/n \) and \( \widehat{\gamma}_i \) is the \( i \)-th element of the vector \( \widehat{\gamma} = HY \). Thus one can select \( \lambda \) over a range by minimizing the above function.

For the window size \( m \), we suggest using the minimum volatility (MV) method, which was proposed by [44]. Denote the estimated conditional covariance matrix

\[
\widehat{\Sigma}^c = \widehat{\Sigma}^c(m) = \frac{1}{(n-m+1)m} \sum_{j=1}^{n-m+1} \left( \sum_{i=j}^{j+m-1} \tilde{z}_{ci} \right) \left( \sum_{i=j}^{j+m-1} \tilde{z}_{ci}' \right).
\]

The rationale behind the MV method is that the estimator \( \widehat{\Sigma}^c(m) \) becomes stable as a function of \( m \) when \( m \) is in an appropriate range. Let the grid of candidate window sizes be \( \{m_1, \ldots, m_{M_1}\} \). The MV criterion selects window size \( m_{j_0} \) such that it minimizes the function

\[
L(m_j) := \text{SE} \left( \left\{ \widehat{\Sigma}^c(m_{j+k}) \right\}_{k=-2}^{2} \right),
\]

where \( \text{SE} \) denotes the standard error

\[
\text{SE} \left( \left\{ \widehat{\Sigma}^c(m_{j+k}) \right\}_{k=-2}^{2} \right) = \left[ \frac{1}{4} \sum_{k=-2}^{2} \left( \widehat{\Sigma}^c(m_{j+k}) - \Sigma^c(m_j) \right)_F^2 \right]^{1/2},
\]

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with \( \Xi_j(m_j) = \sum_{k=-2}^{2} \Xi_j(m_{j+k})/5 \).

Next, we will describe the detailed steps of the multiplier bootstrap procedure for JSCB construction when the weight function \( g_{n_j}(t) = \text{Std}(Q_{n_j}^z(t, \lambda))/\int_0^1 \text{Std}(Q_{n_j}^z(s, \lambda)) \, ds \), \( j = 1, \ldots, p \). Note that \( \text{Std}(\sqrt{n} \hat{\beta}_j(t))/\text{Std}(Q_{n_j}^z(t, \lambda)) \to 1 \) in probability under some mild conditions.

(a) Select the window size \( m \), such that \( m \to \infty \), \( m = o(n) \).

(b) Choose the number of basis expansion \( c_j \) for each \( j = 1, \ldots, p \) and choose the smoothing parameter \( \lambda \).

(c) Estimate FLM (1) and obtain the residuals \( \hat{\epsilon}_i = Y_i - \sum_{j=1}^{p} \sum_{k=1}^{c_j} \hat{\theta}_{jk} x_{ij,k} \).

(d) Generate \( B \) (say 1000) sets of i.i.d. standard normal random variables \( \{u_j^{(r)}\}_{j=1}^{n-m+1}, r = 1, 2, \ldots, B \). For each \( r = 1, 2, \ldots, B \), calculate \( \hat{Q}_{n,r}^{\text{boots}}(t, \lambda) = \hat{C}(t)\hat{\Sigma}^{-1}(\lambda)\hat{U}_{n,r}^{\text{boots}} \), where \( \hat{U}_{n,r}^{\text{boots}} = \frac{1}{\sqrt{n-m+1}} \sum_{j=1}^{n-m+1} \left( \frac{1}{\sqrt{m}} \sum_{i=j}^{j+m-1} \hat{z}_{ci} \right) u_j^{(r)} \) with \( \hat{z}_{ci} = x_c \hat{\epsilon}_i \), \( \hat{C}(t), \hat{\Sigma}(\lambda) \) have similar definitions to \( C_f(t) \) and \( \hat{\Sigma} \) with \( f_{jk} \) replaced by its estimate \( \hat{f}_{jk} \).

(e) Estimate the bootstrap sample standard deviation of \( \{\hat{Q}_{n,r}^{\text{boots}}(t, \lambda)\}_{r=1}^{B} \), \( \text{Std}(\hat{Q}_{n,r}^{\text{boots}}(t, \lambda)) \), where \( \hat{Q}_{n,r}^{\text{boots}}(t, \lambda) \) denotes the \( j \)-th component of \( \hat{Q}_{n,r}^{\text{boots}}(t, \lambda), j = 1, \ldots, p \). Calculate an estimator of \( g_{n_j}(t) \) as \( \hat{g}_{n_j}(t) = \text{Std}(\hat{Q}_{n,r}^{\text{boots}}(t, \lambda))/\int_0^1 \text{Std}(\hat{Q}_{n,r}^{\text{boots}}(s, \lambda)) \, ds \) and obtain \( M_r = \sup_{t \in [0,1]} |\hat{Q}_{n,r}^{\text{boots}}(t, \lambda)| \hat{g}_{n_j}(t), r = 1, 2, \ldots, B \) with \( \hat{g}_{n}(t) = (\hat{g}_{n1}(t), \ldots, \hat{g}_{np}(t))^{\top} \).

(f) For a given level \( \alpha \in (0,1) \), let the \( (1-\alpha) \)-th sample quantile of the sequence \( \{M_r\}_{r=1}^{B} \) be \( q_{n,1-\alpha} \). Then the JSCB of \( \beta(t) \) can be constructed as \( \hat{\beta}_j(t) = \hat{g}_{n_j}(t)\hat{q}_{n,1-\alpha}/\sqrt{n} \) for \( t \in [0,1], j = 1, 2, \ldots, p \).

In the rare case where \( \hat{g}_{n_j}(t_0) \) is close to 0 at some \( t_0 \), one can lift up \( \hat{g}_{n_j}(t_0) \) to a certain threshold (say, \( \max_{t \in [0,1]} \text{Std}(\hat{Q}_{n,r}^{\text{boots}}(t, \lambda))/[100 \int_0^1 \text{Std}(\hat{Q}_{n,r}^{\text{boots}}(s, \lambda)) \, ds] \)) while keeping the weight function continuous such that \( \hat{g}_{n_j}(t) \in \mathcal{G} \). As we will show in Section 4, the above manipulations do not influence the asymptotic validity of the bootstrap.

If one is interested in constructing JSCB for a group of parameter functions, say \( \beta_{i_1}(t), \ldots, \beta_{i_k}(t) \), then one just need to focus on the \( i_1 \)-th, \( i_2 \)-th, \ldots, \( i_k \)-th elements of the bootstrap process \( \hat{Q}_{n,r}^{\text{boots}}(t, \lambda), [\hat{Q}_{n1,r}^{\text{boots}}(t, \lambda), \hat{Q}_{n2,r}^{\text{boots}}(t, \lambda), \ldots, \hat{Q}_{nk,r}^{\text{boots}}(t, \lambda)]^{\top} \), to conduct simultaneous inference of those parameter functions. The implementation procedure is very similar to the above and we shall omit the details.

### 3 Theoretical Results

In this section, we first model the functional time series \( X_i(t) \) from a basis expansion and nonlinear system ([58]) point of view and then investigate the multiplier bootstrap theory.
3.1 Functional Time Series Models

Based on the basis expansion (2), we aim at utilizing a general time series model for \( \{x_{ij,k}\}_{k=1}^{\infty} \) from a nonlinear system point of view as follows, which will serve as a preliminary for our theoretical investigations.

**Definition 1.** Assume that \( \{x_{ij,k}\}_{k=1}^{\infty} \) satisfy \( \|x_{ij,k}\|_q < \infty, q > 9 \), where \( \|Z\|_q := \mathbb{E}[\|Z\|^q]^{1/q} \) for a random variable \( Z \). We say that \( \{X_i(t)\} \) admits a physical representation if for each fixed \( j \) and \( k \), the stationary time series \( \{x_{ij,k}\}_{i=-\infty}^{\infty} \) can be written as \( x_{ij,k} = G_{jk}(F_i) \), where \( G_{jk} \) is a measurable function and \( F_i = (\ldots, \eta_{i-1}, \eta_i) \) with \( \eta_i \) being i.i.d. random elements. For \( l \geq 0 \), define the \( l \)-th physical dependence measure for the functional time series \( \{X_i(t)\} \) with respect to the basis \( \{\alpha_k(t)\}_{k=1}^{\infty} \) and moment \( q \) as
\[
\delta_x(l, q) = \sup_{1 \leq j \leq p, 1 \leq k < \infty} \|G_{jk}(F_i) - G_{jk}(F_{i,l})\|_q,
\]
where \( F_{i,l} = (F_{i-l-1}, \eta^*_{i-l}, \eta_{i-l+1}, \ldots, \eta_i) \) with \( \eta^*_{i-l} \) being an i.i.d. copy of \( \eta_{i-l} \).

Note that in the above definition \( \delta_x(l, q) \) does not depend on \( i \). The above formulation of time series \( x_{ij,k} \) can be viewed as a physical system where functions \( G_{jk} \) are the underlying data generating mechanisms and \( \{\eta_i\} \) are the shocks or innovations that drive the system. Meanwhile, \( \delta_x(l, q) \) measures the temporal dependence of \( \{X_i(t)\} \) by quantifying the corresponding changes in the system’s output uniformly across all basis expansion coefficients when the shock of the system \( l \) steps ahead is changed to an i.i.d. copy. We refer to [58] for more discussions of the physical dependence measures with examples on how to calculate them for a wide range of linear and nonlinear time series models.

Definition 1 is related to the class of functional time series formulated in [64]. The difference is that in (2) we separate the standard deviation \( f_{jk} \) from \( \tilde{x}_{ij,k} \) and the functional time series model in [64] is formulated without this extra step. Standardization of the basis expansion coefficients is needed in the fitting of the FLM (1) to avoid near singularity of the design matrix. Furthermore, Definition 1 is also related to the concept of \( m \)-approximable functional time series introduced in [26] as both formulations utilize the concepts of Bernoulli shifts and coupling. The difference lies in our adaptation of the basis expansion similar to that in [64] which separates the functional index \( t \) and time index \( i \) and hence makes it easier technically to investigate the behavior of various estimators of \( \beta(t) \) uniformly over \( t \). Now, we impose an assumption on the speed of decay for the dependence measures \( \delta_x(l, q) \).

**Assumption 3.1.** There exists some constant \( \tau > 5 \) such that for some finite constant \( C_2 > 0 \), the physical dependence measure satisfies
\[
\delta_x(l, q) \leq C_2(l+1)^{-\tau}, \quad l \geq 0.
\]

Assumption 3.1 is a mild short-range dependence assumption which asserts that the temporal dependence of the functional time series \( X_i(t) \) decays at a sufficiently fast polynomial rate. For independent functional data, the condition \( \delta_x(l, q) \leq C_2(l+1)^{-\tau} \) is automatically satisfied as there is no temporal dependence (\( \delta_x(l, q) = 0 \)). In Section C
of the online supplemental material, we will provide two examples on how to calculate \( \delta_x(l, q) \) for a class of functional MA(\( \infty \)) and functional AR(1) processes, respectively.

### 3.2 Validating The Multiplier Bootstrap

Adapting the nonlinear system point of view ([58]), we model the stationary time series of errors \( \{\epsilon_i\}_{i=1}^n \) as \( \epsilon_i = G(F_i) \) for some measurable function \( G \). Observe that both \( X_i(t) \) and \( \epsilon_i \) are generated by the same set of shocks \( \{\eta_j\}_{j=\mathbb{Z}} \) and hence they could be statistically dependent. To establish the main results, we need the following conditions:

**Assumption 3.2.** Suppose that the smallest eigenvalue of \( \Sigma_c(\lambda) \) is greater or equal to some constant \( \sigma > 0 \).

**Assumption 3.3.** The stationary process \( \{\epsilon_i\}_{i=1}^n \) satisfies \( \|\epsilon_i\|_q < \infty, \ q > 9 \) and there exist constants \( C_3 > 0 \) and \( \tau > 5 \) such that its physical dependence measure achieves \( \delta_x(l, q) \leq C_3(l + 1)^{-\tau}, \ l \geq 0 \).

**Assumption 3.4.** \( \max_{1 \leq j \leq c} \mathbb{E}|z_{ci,j}|^q \leq C_q < \infty \) and some constant \( C_q > 0 \).

The above assumptions are mild and are needed for establishing a Gaussian approximation and comparison theory for roughness-penalized estimators. Assumption 3.2 ensures positive definiteness of the design matrix in order to avoid multi-colinearity. Assumption 3.3 is a short range dependent condition on \( \{\epsilon_i\}_{i=1}^n \) in accordance with Assumption 3.1. Finally, Assumption 3.4 puts some moment restrictions on the random variable \( z_{ci,j} \).

Let \( \tilde{R}_j \) be a \( c_j \times c_j \) matrix with its \((k, l)\) element \( \tilde{R}_j(k, l) = \int_0^1 \alpha_k''(t)\alpha_l''(t)dt \). The following additional assumptions are needed for the validation of the multiplier bootstrap.

**Assumption 3.5.** For \( j = 1, \cdots, p \), we assume \( |\tilde{R}_j| = c^{2\gamma} \), where \( \gamma \) is a positive constant depending on the basis function, \( |A| \) is the spectral norm (largest singular value) of a matrix \( A \).

**Assumption 3.6.** For each \( k \geq 1 \), there exists some constant \( \psi \geq 0 \) such that \( \|\alpha_k(t)\|_\infty \leq C_4k^\psi \) for some positive constant \( C_4 \), where \( \|\cdot\|_\infty \) denotes the uniform norm of a bounded function, i.e, if \( f : \mathcal{X} \rightarrow \mathbb{R} \) then \( |f|_\infty = \sup_{x \in \mathcal{X}} |f(x)| \). In addition, for any \( t_1, t_2 \in [0, 1] \) and \( k \geq 1 \), there exists a nonnegative constant \( \phi \) and some finite constant \( C_5 \) such that \( |\alpha_k(t_1) - \alpha_k(t_2)| \leq C_5k^{\phi}|t_1 - t_2| \).

**Assumption 3.7.** For sufficiently large \( k \) and \( j = 1, \cdots, p \), \( |f_{jk}| \geq C_6k^{-(d_2+1)} \), where \( C_6 > 0 \) is some finite constant.

Assumption 3.5 is mild and can be easily checked for many frequently-used basis functions, such as the Fourier basis (\( \gamma = 2 \)) and the Legendre polynomial basis (\( \gamma = 4 \)).
Meanwhile, Assumption 3.6 is satisfied by most frequently-used sieve bases. For instance, the pair $(\psi, \phi) = (0, 1)$ for the trigonometric polynomial series, $(\psi, \phi) = (1/2, 0)$ for the polynomial spline basis functions and $(\psi, \phi) = (1, 5/2)$ for the normalized Legendre polynomial basis. We refer to Section D.2 of the supplementary material for a detailed discussion of the above claims on Assumptions 3.5 and 3.6. Assumption 3.7 is frequently adopted in the FLM literature, for example [22], which imposes a lower bound on the decay rate of $f_{jk}$. Similar to our discussion of Assumption 2.1, for $C^{d_2}[0, 1]$ functions the fastest decay speed of their basis expansion coefficients is of the order $O(k^{-(d_2+1)})$ for a wide class of basis functions. Hence Assumption 3.7 is mild. We remark that Assumptions 3.6–3.7 are required for controlling the bias of $\hat{\beta}(t)$ and are not needed for the Gaussian approximation and comparison results.

Define $\Xi^c := \frac{1}{n} \mathbb{E} \left( \sum_{i=1}^{n} z_{ci} \right) \left( \sum_{i=1}^{n} z_{ci}^2 \right)$. Recall the definition of $\mathbf{\hat{g}}_n(t) = (\hat{g}_{n1}(t), \ldots, \hat{g}_{np}(t))^\top$ in Step (e) of Section 2.3. Denote the following Kolmogorov distance

$$
\hat{K}(\mathbf{U}_n^{\text{boots}}, \mathbf{Z}_n^c) = \sup_{\mathbf{g}_n \in \mathcal{G}, x \in \mathbb{R}} \left| \mathbb{P}\left( \sup_{t \in [0,1]} |\hat{Q}_n^{\text{boots}}(t, \lambda)|_{\mathbf{g}_n} \leq x \right) - \mathbb{P}\left( \sup_{t \in [0,1]} |\hat{Q}_n^c(t, \lambda)|_{\mathbf{g}_n} \leq x \right) \right|
$$

where $\hat{Q}_n^c(t, \lambda) = \hat{C}_j(t) \hat{\Sigma}_n^{-1}(\lambda) \mathbf{Z}_n^c$. Then we have the following theorem on the consistency of the proposed multiplier bootstrap method.

**Theorem 1.** Suppose Assumptions 2.1–3.7 hold true, the smallest eigenvalue of $\Xi^c$ is bounded below by some constant $\hat{b} > 0$ and $m = O(n^{1/3})$. Define $\mathcal{B}_n^c = \{ \omega : \Delta_n(\omega) := |\hat{\Xi}^c - \Xi^c|_F \leq C_7 n^{-1/3} h_n \}$, where $\omega$ represents the element in the probability space, $| \cdot |_F$ indicates Frobenius norm, $h_n$ diverges to infinity at an arbitrarily slow rate and $C_7 > 0$ is a finite constant, then $\mathbb{P}(\mathcal{B}_n^c) = 1 - o(1)$. Under the event $\mathcal{B}_n^c$, we have

$$
\hat{K}(\mathbf{U}_n^{\text{boots}}, \mathbf{Z}_n^c) 
\lesssim C_8 \left( c^2 n^{-\frac{1}{2} + \frac{2}{d_2} + \frac{2}{r_1} + \lambda \frac{2}{2(\gamma + d_2 - 2) - 2}} \log^{3/2}(n) + c^5 n^{-\frac{1}{2} h_n^{1/2}} + c^9 n^{-\frac{1}{4} + \frac{1}{2} h_n^{1/2}} \right),
$$

where $C_8 > 0$ is some finite constant. Further assume

(i) $c \gg (n / \log(n))^{\frac{1}{2(\gamma + d_2 - 2) - 2}}$ and $\lambda \ll (\log(n) / n)^{\frac{2(\gamma + d_2 - 1)}{2(\gamma + d_2 - 2) - 2}}$,

(ii) $\mathbf{\hat{g}}_n(t) \in \mathcal{G}$ almost surely,

(iii) $c^2 n^{-\frac{1}{2} + \frac{2}{d_2} + \frac{2}{r_1} + \lambda \frac{2}{2(\gamma + d_2 - 2) - 2}} \log^{3/2}(n) + c^5 n^{-\frac{1}{2} h_n^{1/2}} + c^9 n^{-\frac{1}{4} + \frac{1}{2} h_n^{1/2}} \to 0$ as $n \to \infty$,

then the JSCB achieves

$$
\lim_{n \to \infty} \lim_{B \to \infty} \mathbb{P}\left( \beta_{jt} \in \left[ \frac{\tilde{\beta}_{jt} - \hat{Q}_{n1-\alpha}^{\text{boots}}(t)}{\sqrt{n}}, \frac{\hat{Q}_{n1-\alpha}^{\text{boots}}(t)}{\sqrt{n}} \right] \right) = 1 - \alpha.
$$

(7)
Theorem 1 states that the JSCB achieves the correct coverage probability asymptotically under the corresponding regularity conditions. In particular, (6) establishes the rate of the bootstrap approximation to the weighted maximum deviation of $Z_n^c$. More specifically, the first term on the right hand side of (6) captures the magnitude of Gaussian approximation error, the second term represents the multiplier bootstrap approximation error and the last term is related to the estimation error. Condition (i) in Theorem 1 imposes a lower bound on $c$ and upper bound on $\lambda$ in order to obtain an under-smoothed estimator (hence the estimation bias is asymptotically negligible). The constraints on $c$ and $\lambda$ in Condition (i) are mild. For example, if $X_{i} \in \mathcal{C}^1$, $\gamma = 4$ based on the normalized Legendre polynomials and $q, \tau \rightarrow \infty$, the parameter should be chosen as $(n/\log(n))^{1/2} \ll c \ll n^{1/3}$ and $\lambda \ll (\log(n)/n)^{12/5}$ such that the approximation error goes to 0. Condition (ii) is a mild assumption on the weight function. The rate $m = O(n^{1/3})$ is the optimal one that balances the bias and variance of the bootstrapped covariance matrices. Thanks to the fact that Theorem 1 in Section B of the supplemental material and the above Theorem 1 are established uniformly over all weight functions in $\mathcal{G}$, the JSCB achieves asymptotically correct coverage probability without assuming that $\hat{g}_{n,j}(t)$ is a uniformly consistent estimator of $\text{Std}(\hat{\beta}_{j}(t))/\int_0^1 \text{Std}(\hat{\beta}_{j}(s)) \, ds$ as long as $\hat{g}_{n}(t) \in \mathcal{G}$ almost surely. Hence the multiplier bootstrap is asymptotically robust to inconsistently estimated weight functions. The price one has to pay for inconsistently estimated weight functions is that the average width of the JSCB may be inflated.

### 3.3 Data-Driven Basis Functions Based on Functional Principal Components

A popular data-driven orthonormal basis in functional data analysis is the FPC. Observe that FPCs have to be estimated from the data which inevitably causes estimation errors. When one employs data-driven basis functions such as the FPCs to fit model (1), the additional estimation error must be taken into account. Note that $f_{jk}^2 = \mathbb{E}(\tilde{\tau}_{ij,k}^2)$ are the eigenvalues of the corresponding covariance operator. Throughout this subsection, for any given $j$, we assume that $f_{j1} > f_{j2} > \ldots > f_{jc_j} > 0$. This assumption implies that the first $c_j$ eigenvalues are separated, which is commonly used in the theoretical investigation for FPC-based methods. The next proposition establishes the asymptotic validity of the bootstrap for the FPC basis functions under some extra conditions.

**Proposition 1.** Under Assumptions 2.1–3.7, the multiplier bootstrap result (6) hold true for the FPC basis functions. Further assume that

$$\lambda^{-\frac{2d_2+3}{4(d_2+2)_1}} > c^{4(d_2+1)-d_1}/\sqrt{n} \quad (8)$$

and Conditions (i)–(iii) in Theorem 1 hold true, then (7) holds for the FPC basis.
This proposition imposes an extra constraint (8) on the smoothness of $\beta(t)$ and $X_i(t)$ to make the additional bias term resulted from FPC estimation negligible compared to the standard deviation.

4 Simulation Studies

Throughout this section, we focus on the case where $p = 1$. Three basis functions will be considered, i.e., the Fourier bases (Fou.), the Legendre polynomial bases (Leg.) and the functional principal components (FPC). Due to page constraints, we refer the readers to Section A of the supplementary material for more numerical studies for $p > 1$ and statistical powers for $p = 1$.

Recall model (1) and restate the basis expansions as $\beta(t) = \sum_{k=1}^{\infty} \beta_k \alpha_k(t)$, $X_i(t) = \sum_{k=1}^{\infty} \tilde{x}_{ik} \alpha_k(t)$ when $p = 1$. Next, denote $\tilde{x}_i = (\tilde{x}_{i1}, \tilde{x}_{i2}, \ldots)^\top$, $\eta_i = (\eta_{i1}, \eta_{i2}, \ldots)^\top$ and $D$ is an infinite-dimensional tridiagonal coefficient matrix with 1 on the diagonal and $1/5$ on the offdiagonal. We will investigate the following models:

- **FMA(1) model.** $\tilde{x}_i = D (\eta_i + \phi_1 \eta_{i-1})$, we choose the MA coefficient $\phi_1 = 0.5$ or 1. The entries $\{\eta_{ik}\}_{k=1}^{\infty}$ of $\eta_i$ are independent $\mathcal{N}(0, k^{-2})$ random variables.
- **FAR(1) model.** $\tilde{x}_i = \phi_2 D \tilde{x}_{i-1} + \eta_i$ ($\phi_2 \in [0, 0.723]$), we choose the AR coefficient $\phi_2 = 0.2$ or 0.5 to represent weak to moderately strong dependencies. The entries $\{\eta_{ik}\}_{k=1}^{\infty}$ of $\eta_i$ are chosen as independent $\mathcal{N}(0, e^{-(k-1)})$ random variables.

The following basis expansion coefficients of $\beta(t)$ and the error process $\{\epsilon_i\}_{i=1}^{n}$ in model (1) are considered:

(a). $\beta_1 = 0.8$, $\beta_2 = 0.5$, $\beta_3 = -0.3$ and $\beta_k = k^{-3}$ for $k \geq 4$.
(b). $\{\epsilon_i\}_{i=1}^{n}$ are dependent on $X_i(t)$. Let $\{s_i\}_{i=1}^{n}$ follow an AR(1) process $s_i = 0.2 s_{i-1} + e_i$ where $e_i$ is i.i.d. $\sqrt{3/4 t_k}$-distributed and set $\epsilon_i = 0.5 s_i \tilde{x}_{i1}$ where $\tilde{x}_{i1}$ is the first FPC score of $X_i(t)$.
(c). $\beta_1 = 0.8$, $\beta_2 = 0.5$, $\beta_3 = -0.3$ and $\beta_k = e^{-k}$ for $k \geq 4$.
(d). $\{\epsilon_i\}_{i=1}^{n}$ are independent of $X_i(t)$. Let $\{\epsilon_i\}_{i=1}^{n}$ follow an AR(1) process $\epsilon_i = 0.2 \epsilon_{i-1} + e_i$ where $e_i$ is i.i.d. standard normally distributed.

For the case based on FPC expansions, we also use the above settings except that we will modify the component-wise dependence structure of $\tilde{x}_i$ in the sense that $D$ is diagonal with all diagonal elements being 1. This guarantees the true FPCs of $X_i(t)$ are $\{\alpha_k(t)\}_{k=1}^{\infty}$.

In the simulation studies, the bootstrap procedures discussed in Section 2.3 are employed with $B = 1000$ to find the critical values $\hat{q}_{n,1-\alpha}$ at levels $\alpha = 0.05$ and 0.1. The simulation results are based on 1000 Monte Carlo experiments. Tables 1–2 reports simulated coverage probabilities and average JSCB widths with aforementioned three types of basis functions and two types of weight functions; i.e., $g_{nj}(t) \equiv 1$ and $\hat{g}_{nj}(t) = \hat{\text{Std}}(\hat{Q}_{nj}^z(t, \lambda))/\int_0^1 \hat{\text{Std}}(\hat{Q}_{nj}^z(s, \lambda))ds$.

From Tables 1–2, we find that most of the results for $n = 800$ are close to the nominal
Table 1: Simulated coverage probabilities. Average JSCB widths are in parentheses. $X_i(t) \sim \text{FMA}(1)$, $\{\beta_k\}$ follows scenario (a) and $\{\epsilon_i\}$ follows scenario (b).

| $\hat{g}_{nj}$ | Basis | $\phi_1 = 0.5$ | $\phi_1 = 0.5$ | $\phi_1 = 0.5$ | $\phi_1 = 0.5$ |
|----------------|-------|----------------|----------------|----------------|----------------|
|                |       | 1              | 1              | 1              | 1              |
| 1              | Fou.  | 0.943(1.46)    | 0.936(1.46)    | 0.884(1.28)    | 0.886(1.29)    |
|                | Leg.  | 0.952(2.95)    | 0.945(2.86)    | 0.907(2.55)    | 0.895(2.48)    |
|                | FPC   | 0.960(1.65)    | 0.932(1.69)    | 0.903(1.45)    | 0.886(1.49)    |
| Std            | Fou.  | 0.936(1.18)    | 0.936(1.22)    | 0.875(1.07)    | 0.879(1.10)    |
|                | Leg.  | 0.947(1.34)    | 0.940(1.33)    | 0.884(1.21)    | 0.885(1.21)    |
|                | FPC   | 0.938(1.37)    | 0.924(1.40)    | 0.870(1.25)    | 0.858(1.27)    |
|                |       |                |                |                |                |
| 1              | Fou.  | 0.957(1.16)    | 0.958(1.12)    | 0.903(1.01)    | 0.914(0.99)    |
|                | Leg.  | 0.953(2.20)    | 0.951(2.08)    | 0.900(1.90)    | 0.890(1.81)    |
|                | FPC   | 0.945(1.17)    | 0.948(1.19)    | 0.893(1.04)    | 0.894(1.06)    |
| Std            | Fou.  | 0.949(0.92)    | 0.940(0.94)    | 0.883(0.84)    | 0.903(0.86)    |
|                | Leg.  | 0.943(0.98)    | 0.941(0.97)    | 0.887(0.89)    | 0.888(0.88)    |
|                | FPC   | 0.943(1.00)    | 0.934(1.01)    | 0.874(0.90)    | 0.871(0.92)    |

levels. When $n = 400$ and the data-adaptive weights are used, the coverage probabilities are reasonably close to the nominal levels for most cases under weaker dependence ($\phi_1 = 0.5$, $\phi_2 = 0.2$). However, under stronger dependence the performances of the three bases weaken slightly when $n = 400$. The decrease in estimation accuracy and coverage probability in finite samples under stronger temporal dependence is well-known in time series analysis. This decrease seems to be universal across various inferential tools (such as subsampling, block bootstrap, multiplier bootstrap, and self-normalization) though some methods may be less sensitive to stronger dependence. One explanation is that the variances of the estimators tend to be higher under stronger dependence which leads to less accurate estimators. This reduced accuracy then results in deteriorated coverage probabilities in small to moderately large samples.

Furthermore, we find that the performances of the JSCB for dependent predictors and errors are similar to those for independent case, which supports our theoretical result that the multiplier bootstrap is robust to dependence between the predictors and errors. We observe from the simulation results that the JSCB is narrower when the weights are selected proportional to the standard deviations of the estimators. Therefore we would like to recommend such weights provided that there is no sacrifice in the coverage accuracy.
Table 2: Simulated coverage probabilities. Average JSCB widths are in parentheses. 
\( X_i(t) \sim \text{FAR}(1), \{\beta_k\} \) follows scenario (c) and \( \{\epsilon_i\} \) follows scenario (d).

| \( \tilde{g}_{nj} \) Basis | \( n = 400 \) | \( 1 - \alpha = 0.95 \) | \( 1 - \alpha = 0.90 \) |
|--------------------------|-------------------------------|-----------------------------|
|                          | \( \phi_2 = 0 \) | \( \phi_2 = 0 \) |
| Fou. | 0.945(1.77) | 0.898(1.55) | 0.891(1.08) |
| Leg. | 0.942(3.10) | 0.901(2.67) | 0.909(1.12) |
| FPC | 0.939(1.91) | 0.893(1.67) | 0.890(0.93) |

| \( \tilde{g}_{nj} \) Basis | \( n = 800 \) | \( 1 - \alpha = 0.95 \) | \( 1 - \alpha = 0.90 \) |
|--------------------------|-------------------------------|-----------------------------|
|                          | \( \phi_2 = 0 \) | \( \phi_2 = 0 \) |
| Fou. | 0.960(1.25) | 0.903(1.10) | 0.909(1.14) |
| Leg. | 0.959(2.12) | 0.908(1.83) | 0.914(1.10) |
| FPC | 0.952(1.28) | 0.909(1.12) | 0.871(0.91) |

5 Empirical Illustrations

We consider the daily curves of electricity real demand (MWh) in Spain from January 1st 2015 to December 31st 2017. These data can be obtained from the Red Eléctrica de España system operator (https://www.esios.ree.es/en). Since the daily electricity demand on weekdays and weekends have different behaviors, in this paper we focus on the weekday curves (from Monday to Friday) with \( n = 782 \) days. The hourly records of electricity demand in year 2011–2012 have been investigated in [1].

The original dataset are recorded by 10-minute intervals from 00:00–23:50 on each day, which consists of 144 observations. We consider the daily log-transformed real demand curves by smoothing and rescaling them to a continuous interval \([0, 1]\). The plot of the smoothed functional time series is shown in Fig. 1. The stationarity test of [29] is also implemented and it turns out that the test does not reject the stationarity hypothesis at 5% level during the considered period. Next, we aim to investigate the relationship between daily electricity real demand curves and future daily total demand. To this end
we explore the FLM:

$$Y_{i+1} = \beta_0 + \sum_{j=1}^{3} \int_0^1 \beta_j(t) X_{i+1-j}(t) dt + \epsilon_{i+1}, \ i = 3, 4, ..., 781$$

with $Y_{i+1}$ being the daily total real demand on the $(i+1)$-th weekday.

The Legendre polynomial and FPC bases are used in this example. Firstly, we select $d_j = 3$ by CVP criterion to explain at least 95% of the variability of the data and set $c_j = 2d_j = 6, \ j = 1, 2, 3$. The block size is chosen as $m = 16$ by MV method proposed in Section 2.3 for aforementioned bases, the smoothing parameter $\lambda$ is selected as $8.5 \times 10^{-12}$ based on Legendre polynomial bases and $1.7 \times 10^{-11}$ based on FPC bases according to GCV criterion. The JSCBs are constructed based on 10000 bootstrap replications.

Figure 2: Estimated coefficient functions (black solid curve), 95% JSCBs (blue dotdashed) and 95% point-wise confidence bands (red dashed) with data-driven weights.
We exhibit the plots of JSCBs and point-wise confidence bands for \( \beta_1(t) \), \( \beta_2(t) \) and \( \beta_3(t) \) in Fig. 2. From it, one finds that both \( \beta_1(t) \) and \( \beta_2(t) \) are significantly non-zero, \( \beta_3(t) \) is insignificant under the JSCB construction for both bases. In particular, for both bases \( \beta_1(t) \) is significantly positive at the morning and evening peak times, significantly negative at off-peak time. While \( \beta_2(t) \) is only significantly positive in the afternoon period and significantly negative in the late evening period, contributing to a slightly weaker impact on the response compared to \( \beta_1(t) \). On the other hand, the point-wise confidence bands for \( \beta_3(t) \) based on both bases do not fully cover the horizontal axis \( \beta_3(t) = 0 \), which will lead to a spurious significance. These findings illustrate that, the electricity demand behavior over the last two weekdays (especially the peak time from the last weekday) is highly correlated with the total demand of the next weekday. Moreover according to the JSCB fluctuations for the coefficients \( \beta_1(t) \) and \( \beta_2(t) \), it turns out apparently that the constancy and linearity coefficient hypotheses are rejected.

For a further investigation, we fit the model

\[
Y_{i+1} = \beta_0 + \sum_{j=1}^{2} \int_0^1 \beta_j(t)X_{i+1-j}(t)dt + \epsilon_{i+1}
\]

and find that the \( R^2 \) of the regression is 0.8032 (0.8044) for the Legendre polynomial (FPC) bases. Therefore, we conclude that the majority of the variability of the total weekday electricity demand in Spain can be explained/predicted by the demand curves of the past two weekdays.

6 Discussions

We would like to conclude this article by discussing some issues related to the practical implementation of the regression as well as possible extensions. Firstly, the predictors \( X_i(t) \) are typically only discretely observed with noises in practice and hence pre-smoothing is required to transfer the discretely observed predictors into continuous curves which inevitably produces some smoothing errors. In this article, for the purposes of brevity and keeping the discussions on the main focus, we assume that the smooth curves of \( X_i(t) \) are observed. It can be seen from the proofs that the results of the paper hold under the densely observed functional data scenario as long as the smoothing error is of the order \( O_p(\log n/\sqrt{n}) \) uniformly. The smoothing effects for time series or densely observed functional data have been intensively investigated in the literature; see for instance [23], [61] and [59], among many others. Though the aforementioned references are not exactly intended for functional time series, their results can be extended to the functional time series setting which we will pursue in a separate future work. On the other hand, we do not expect that our theory and methodology will directly carry over to the sparsely observed functional data setting ([60]) and the corresponding investigations are beyond the scope of the current paper.

Secondly, we note that the choice of the basis function is a non-trivial task in practice. In the literature, there are several discussions with respect to the choices of basis functions.
or methods for the regression (1) in general. See for instance Section 6.1 in [46] and the references therein. Here we shall add some additional notes. For functional time series whose observation curves are clearly periodic such as the yearly temperature curves, the Fourier basis is a natural choice. Similar choices can be made based on prior knowledge of the shapes of the observation curves in various scenarios. From our limited simulation studies and data analysis in Sections 4 and 5, it is found that many popular classes of basis functions produce similar estimates of the regression curves and inference results, which demonstrates a certain level of robustness towards the basis choices.

Finally, in some real data applications the response time series may be function-valued as well. One prominent example is the functional auto-regression [6]. We hope that our multiplier bootstrap methodology as well as the underlying Gaussian approximation and comparison results will shed light on the simultaneous inference problem for FLM with functional responses. Indeed, using the basis decomposition technique, FLM with functional responses could be viewed as multiple FLMs with scalar responses at the basis expansion coefficient level where the regression errors may be cross-correlated and the set of regressors are identical. We will investigate this direction in a future research endeavor.

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Supplemental Material for “Simultaneous Inference for Time Series Functional Linear Regression”

Abstract

Section A of this supplemental material provides additional simulation results under various number of predictors and data generating mechanisms. Section B establishes a Gaussian approximation theory for the roughness penalization estimation method, which may be of independent interest. Examples of physical dependence measure calculation for a class of functional MA(\(p\)) models and a class of functional AR(1) models are presented in Section C. Theoretical results and proofs to all theoretical results of the main article can be found in Section D.

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A Additional simulation results

In this section we would like to conduct some additional simulation studies that complement those in Section 4 of the main paper. The significance levels are set at \(\alpha = 0.05\) and 0.1. We choose the bootstrap replications \(B = 1000\) based on 1000 simulations throughout this section.
A.1 Simulation studies with order $p > 1$

In the simulation studies of the paper, we only consider the case $p = 1$. In this subsection we shall consider $p = 2, 3$ with the following parameter setups:

- For $j = 1, 2, 3$, $\beta_{j1} = 0.8$, $\beta_{j2} = 0.5$, $\beta_{j3} = -0.3$ and $\beta_{jk} = k^{-4}$ for $k \geq 4$;
- For every $j = 1, 2, 3$, consider FMA(1) model $\tilde{x}_{ij} = D(\eta_j + \phi_3 \eta_{j-1})$ where $D$ is defined in Section 4 of the paper and the MA coefficient $\phi_3 = 0.2$. The random vectors $\tilde{x}_{ij}$ are independent across $j$. The innovation entries $\{\eta_{ij,k}\}_{k=1}^{\infty}$ of $\eta_{ij}$ are independent $k^{-1.2}N(0, 1)$ random variables.
- The error process $\{\epsilon_i\}_{i=1}^{n}$ are independent and identically distributed (i.i.d.) with standard normal distribution.

Table 1: Simulated coverage probabilities over different sample sizes $n$ and orders $p$, where the weight function $\tilde{g}_{nj}(t) = \text{Std}(\tilde{Q}_{nj}^z(t, \lambda))/\int_0^t \text{Std}(\tilde{Q}_{nj}^z(s, \lambda))ds$.

|       | Basis 400 | Basis 800 |
|-------|-----------|-----------|
| $p = 1$ | 1 - $\alpha = 0.95$ | 1 - $\alpha = 0.90$ |
| Fou.  | 0.935(1.78) | 0.946(1.29) |
| Leg.  | 0.930(1.72) | 0.946(1.24) |
| FPC   | 0.932(1.76) | 0.940(1.26) |
| $p = 2$ | 1 - $\alpha = 0.95$ | 1 - $\alpha = 0.90$ |
| Fou.  | 0.928(1.95) | 0.942(1.41) |
| Leg.  | 0.928(1.88) | 0.941(1.36) |
| FPC   | 0.931(1.92) | 0.935(1.37) |
| $p = 3$ | 1 - $\alpha = 0.95$ | 1 - $\alpha = 0.90$ |
| Fou.  | 0.920(2.03) | 0.949(1.48) |
| Leg.  | 0.923(1.97) | 0.936(1.42) |
| FPC   | 0.925(2.01) | 0.935(1.43) |

We list the simulated results for various orders $p$ in Tables 1. From it, we find that for all three basis functions, the coverage probabilities for $p = 1, 2, 3$ are similar although the coverage probabilities for $p = 2$ and 3 tend to be slightly smaller than those of $p = 1$. In particular, all coverage probabilities for $p = 1, 2$ and 3 are reasonably close to the nominal levels when $n = 800$. 

2
A.2 Statistical power of the JSCB test

Next, we shall perform simulation studies with \( p = 1 \) to investigate the accuracy and power of the JSCB when it is used as a test. Specifically, we shall perform the significance test \( \beta(t) \equiv 0 \) versus \( \beta(t) \neq 0 \). Consider the following setting under scenario:

- \( \beta_k = \delta(-1)^k k^{-1}, \ k \geq 1 \) and \( \delta \) ranges from 0 to 0.5 with sample size \( n = 800 \).
- \( \tilde{x}_i = \Phi \tilde{x}_{i-1} + \eta_i \), where \( \Phi \) is an infinite-dimensional tridiagonal matrix with 1/5 on the diagonal and 1/15 on the off-diagonal, \( \eta_{lk} \sim 2k^{-1.2}N(0, 1) \) for \( k \geq 1 \).
- Let the error process \( \{\epsilon_i\}_{i=1}^n \) follow an AR(1) process \( \epsilon_i = 0.2\epsilon_{i-1} + e_i \) where \( e_i \) is i.i.d. standard normally distributed. Moreover \( \{\epsilon_i\} \) are independent of \( X(t) \).

Fig. 1 shows the simulated rejection probabilities for the test with three types of basis functions at nominal levels \( \alpha = 0.05, 0.1 \). From it, we observe that the power performances of the three basis functions are quite similar with data-driven weight functions in the sense that as \( \delta \) increases, the simulated power increases fast. On the other hand, the power curves of the constant weight function increase slower than those of the adaptive weight function. This is consistent with our simulation results that the JSCB is narrower on average when the weights are proportional to the standard deviation of the estimators.

![Figure 1: Simulated rejection probabilities at nominal levels \( \alpha = 0.05 \) (Left) and \( \alpha = 0.1 \) (right) with fixed constant weight function 1 (dashed) and data-driven weight function (solid).](image)

B Gaussian approximation theory

Throughout the supplemental material, we will consistently use the following notations. For a random variable \( Z \), denote \( \|Z\|_q := (E|Z|^q)^{1/q} \) as its \( L^q \) norm. For a square integrable random function \( X(t) \in L^2[0, 1] \), we use \( |X(t)|_{L^2} := (\int_0^1 X^2(t)dt)^{1/2} \) to stand for its \( L^2 \) norm. Furthermore, we denote \( |\cdot| \) as the spectral norm (largest singular value) for a matrix or the Euclidean norm for a random vector. The notations \( |\cdot|_F \) and \( \|\cdot\|_\Psi \) indicate the Frobenius norm and Orlicz norm respectively, the notation \( |\cdot|_{\max} \) signifies the largest...
element of a matrix. We define $|f(x)|_\infty := \sup_{x \in \mathcal{X}} |f(x)|$ to state the supremum norm of $f(x)$ and the symbol $C$ denotes a generic finite constant whose value may vary from place to place.

In this section we shall establish a Gaussian approximation theory for the weighted maximum deviations of $Q_n^z(t, \lambda)$ uniformly over all quantiles and a wide class of weight functions. As we mentioned in the main article, this result is based on uniform Gaussian approximation results over all Euclidean convex sets for sums of stationary and weakly dependent time series of moderately high dimensions which we will establish in Section D of this supplemental material when we prove the results of this section. The result extends the corresponding findings for independent and $m$-dependent data established in [2], [5] and [4] among others, which may be of separate interest.

Define $U^c_n = \frac{1}{\sqrt{n}} \sum_{i=1}^n u_{ci}$ where $\{u_{ci}\}_{i=1}^n$ is a sequence of $c$-dimensional Gaussian random vectors which is independent of $\{z_{ci}\}_{i=1}^n$ and preserves the covariance structure of $\{z_{ci}\}_{i=1}^n$. Further denote $Q_n^u(t, \lambda) = C f(t) \Sigma^{-1}(\lambda) U^c_n$ and define the distance of interest as

$$K(Z^c_n, U^c_n) = \sup_{g_n \in \mathcal{G}} \sup_{x \in \mathbb{R}} \left| \mathbb{P} \left( \sup_{t \in [0,1]} |Q^z_n(t, \lambda)|_{g_n(t)} \leq x \right) - \mathbb{P} \left( \sup_{t \in [0,1]} |Q^u_n(t, \lambda)|_{g_n(t)} \leq x \right) \right|.$$

Now, we state the Gaussian approximation result for the roughness penalization estimator.

**Theorem 1.** Under Assumptions 1–5 of the main article and suppose the smallest eigenvalue of $\Xi^c$ is bounded below by some constant $b > 0$, there exists a constant $C > 0$ such that

$$K(Z^c_n, U^c_n) \leq C \left( c^{7/4} n^{-1/2} + \frac{1}{2} + \frac{2}{\gamma + 1} + \lambda \frac{\log(3/2 - \nu)}{\log(n)} \right). \quad (1)$$

**Proof.** See Section D.1 for the proof. \qed

The above theorem shows that when $\lambda$, $q$ and $\tau$ are sufficiently large and $c$ is sufficiently small, the distribution of $\sup_{t \in [0,1]} |Q^z_n(t, \lambda)|_{g_n(t)}$ can be well approximated by that of $\sup_{t \in [0,1]} |Q^u_n(t, \lambda)|_{g_n(t)}$ uniformly over all quantiles and weight functions in $\mathcal{G}$. The constraints on $c$ and $\lambda$ are also mild. For example, if $X_i(t), \beta(t) \in \mathcal{C}^1$ and $\gamma = 4$ based on normalized Legendre polynomial bases, Theorem 1 in the paper shows that $\hat{\beta}(t)$ is an under-smoothed estimator as long as $c \gg (n/\log(n))^{1/2}$ and $\lambda \ll (\log(n)/n)^{12/5}$. Hence $\hat{\beta}(t)$ is under-smoothed and at the same time (1) goes to 0 for a relatively wide range of $c$ and $\lambda$. 

4
C Calculating physical dependence measures for two classes of functional time series models

Here, we show two examples on how to calculate \( \delta_x(l, q) \) for a class of functional MA(\( \infty \)) processes and functional AR(1) processes, respectively.

C.1 FMA(\( \infty \)) model

**Example 1 (Functional MA(\( \infty \)) model).** Let \( \eta_i(s) \) be i.i.d. centered and continuous Gaussian random functions with \( \sup_{s \in [0,1]} \mathbb{E}[\|\eta_i(s)\|_2] < \infty \). For each integer \( m \geq 0 \), let \( B_m(t, s) = a_m B^*_m(t, s) \) where \( \{a_m\} \) is a positive deterministic sequence with \( \sum_{m=0}^{\infty} a_m < \infty \) and \( B^*_m(\cdot, \cdot) \) is a \( C([0,1]^2) \) deterministic function such that \( |B^*_m(t, s)| \leq C \) for all \( t, s \) and \( m \) and some finite constant \( C \). Consider the functional MA(\( \infty \)) model,

\[
X_i(t) = \sum_{m=0}^{\infty} \int_0^1 B_m(t, s) \eta_{i-m}(s) ds. \tag{2}
\]

Let \( C(t, s) := \mathbb{E}(\eta_i(t) \eta_i(s)) \) be the covariance function of \( \eta_i(t) \). Let \( v_1(t), v_2(t), \ldots \) and the corresponding \( u_1 \geq u_2 \geq \cdots \) be the eigenfunctions and eigenvalues of \( C(t, s) \). By the basis expansion method, we can write \( X_i(t) = \sum_{j=1}^{\infty} \tilde{x}_{i,j} \alpha_j(t) \), \( B^*_m(t, s) = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} b^m_{j,k} \alpha_j(t) v_k(s) \) and \( \eta_i(s) = \sum_{j=1}^{\infty} \eta_{i,j} v_j(s) \). Next, by substituting the above into (2), we obtain

\[
\tilde{x}_{i,j} = \sum_{m=0}^{\infty} a_m \left( \sum_{k=1}^{\infty} b^m_{j,k} \eta_{i-m,k} \right) =: f^j_i x_{i,j}.
\]

Here \( f^2_j = \sum_{m=0}^{\infty} a_m^2 \theta_{jm} \) is the variance of the random coefficient \( \tilde{x}_{i,j} \), where \( \theta_{jm} := \sum_{k=1}^{\infty} (b^m_{j,k})^2 u_k \). Let \( \eta_i = (\eta_{ij})_{j \geq 1} \). Then we see that \( x_{ij,k} \) can be written in the form of physical dependence measure in Definition 1 of the paper. The following lemma bounds the physical dependence measures for (2).

**Lemma 1.** The dependence measures \( \delta_x(l, q) = O(a_l) \) for any given \( q \geq 2 \) if

\[
\sum_{k=0}^{\infty} a_k^2 \theta_{jk} \geq C \theta_{jm} \tag{3}
\]

for sufficiently large \( j \) and \( m \) and some positive constant \( C \) that does not depend on \( j \) or \( m \).

Assumption (3) is a mild condition in general. For instance, it is easy to see that (3) holds if \( \{B^*_m(t, s)\}_{m=0}^{\infty} \) is finitely generated; that is, for each non-negative integer \( m \), \( B^*_m(t, s) \) can only choose from \( r \) candidate functions \( \{\hat{B}_j(t, s), j = 1, 2, \cdots, r\} \) for some \( r < \infty \). Note that functional MA(\( r \)) models belong to the finitely generated category.
when \( r \) is finite. For another example, (3) holds if \( B_m^*(t, s) \) admits the decomposition
\[
B_m^*(t, s) = \gamma_m(t)\kappa_m(s)
\]
for some uniformly bounded and continuous functions \( \gamma_m(\cdot) \) and \( \kappa_m(\cdot) \). We refer to Lemma 2 in the following for the proof. We make a further note that another sufficient condition for (3) is that, for some non-negative integer \( k_0 \), \( \theta_{jk_0} \geq C_0\theta_{jk} \) for sufficiently large \( j \) and \( k \) and some positive constant \( C_0 \) that does not depend on \( j \) or \( k \). For many frequently used basis functions such as the Fourier, wavelet and orthogonal polynomial bases, the decay speed of \( \theta_{jk} \) with respect to \( j \) is determined by the smoothness of \( B_k^*(t, s) \) in \( t \) and \( \theta_{jk_0} \geq C_0\theta_{jk} \) is satisfied when there exists a \( k_0 \) such that \( B_k^*(t, s) \) is at most as smooth as \( B_k^*(t, s) \) in \( t \) for all sufficiently large \( k \) under some extra mild basis-specific assumptions.

**Proof of Lemma 1.** Note that \( \theta_{jm} = \mathbb{E}[\int_0^1 \int_0^1 B_m^*(t, s)\alpha_j(t)\eta_l(s)dt\,ds]^2 \leq C \) for some finite constant \( C \) that does not depend on \( j \) or \( m \). Hence inequality (3) holds for all \( j \) and sufficiently large \( m \) if it holds for sufficiently large \( j \) and \( m \). Observe that \( \tilde{x}_{i,j} \) has the following MA(\( \infty \)) representation
\[
\tilde{x}_{i,j} = \sum_{m=0}^{\infty} a_m\eta_{i,j}^{(m)},
\]
where \( \eta_{i,j}^{(m)} = \sum_{k=1}^{\infty} b_{j,k}^m\eta_{i-m,k} \). Therefore a direct application of the definition of the physical dependence measure yields that
\[
\delta_x(l, q) \leq a_l \sup_j \left[ \| \sum_{k=1}^{\infty} b_{j,k}^l(\eta_{i-l,k} - \eta_{i-l,k}^*) \|_q / f_j \right]
\]
provided \( f_j \neq 0 \), where \( \eta_{i-l,k}^* \) is an i.i.d. copy of \( \eta_{i-l,k} \). If \( f_j = 0 \), then it is clear that \( \tilde{x}_{i,j} = 0 \) almost surely and the temporal dependence at the \( j \)-th basis expansion level is uniformly 0. Now observe that \( \eta_{i-l,k} \) are independent Gaussian random variables across \( k \). Hence \( \sum_{k=1}^{\infty} b_{j,k}^l(\eta_{i-l,k} - \eta_{i-l,k}^*) \) is normally distributed with mean 0 and variance \( 2\theta_{jl} \). Furthermore, the \( L^2 \) norm of a centered Gaussian random variable is proportional to its standard deviation. Therefore
\[
\| \sum_{k=1}^{\infty} b_{j,k}^l(\eta_{i-l,k} - \eta_{i-l,k}^*) \|_q = C_q\theta_{jl}^{1/2}
\]
for some constant \( C_q \). Hence \( \delta_x(l, q) = O(a_l) \) by inequality (3) since \( f_j^2 = \sum_{m=0}^{\infty} a_m^2\theta_{jm} \).

**Lemma 2.** Inequality (3) holds if for each \( m \), \( B_m^*(t, s) \) admits the decomposition \( B_m^*(t, s) = \gamma_m(t)\kappa_m(s) \) for some uniformly bounded and continuous functions \( \gamma_m(\cdot) \) and \( \kappa_m(\cdot) \).

**Proof.** Write \( \gamma_m(t) = \sum_{i=1}^{\infty} \gamma_i^m\alpha_i(t) \) and \( \kappa_m(s) = \sum_{i=1}^{\infty} \kappa_i^mv_i(s) \). Then \( b_{j,k}^m = \gamma_j^m\kappa_k^m \). Therefore inequality (3) holds if
\[
\sum_{k=0}^{\infty} a_k^2\tilde{\theta}_k \geq C_1\tilde{\theta}_m
\]
for sufficiently large $m$, where $\tilde{\theta}_k = \sum_{r=1}^{\infty}(\kappa_k)^2 u_r$. Observe that $\tilde{\theta}_k = \mathbb{E}[\int_0^1 \kappa_k(s)\eta_0(s)\,ds]^2 \leq C$ for some constant $C$ that does not depend on $k$ since $\kappa_k(s)$ is uniformly bounded by assumption. Therefore the right hand side of (4) is bounded above by $C_1 C$. Therefore (4) holds unless all $\theta_k$ are 0. But if all $\theta_k$ are 0, (4) is trivial. □

C.2 FAR(1) model

In this paper, we focus on the discussion when the physical dependence measure is of polynomial decay, that is, Assumption 2 of the main article holds true. However, all our results can be extended to the case when it is of exponential decay

$$\delta_x(l,q) \leq C \rho^l, 0 < \rho < 1. \quad (5)$$

Next, we will demonstrate an example of FAR(1) model to verify this exponential decay (5) of the dependence measures.

Example 2 (Functional AR(1) model). Let $\epsilon_i(t)$ be i.i.d. centered and continuous Gaussian random functions with $\sup_{t \in [0,1]} \mathbb{E}\|\epsilon_i(t)\|_2 < \infty$. Consider the following model

$$X_i(t) = \int_0^1 B(t,s)X_{i-1}(s)\,ds + \epsilon_i(t), \quad (6)$$

where $B(t,s) : [0,1]^2 \to \mathbb{R}$ is a continuous, symmetric function satisfying $\int_0^1 \int_0^1 B^2(t,s)\,ds\,dt < \infty$ and $\int_0^1 \int_0^1 B(t,s)x(t)x(s)\,ds\,dt \geq 0$ with any random function $x(t) \in \mathcal{L}^2([0,1])$. Thus $B(t,s)$ is called a symmetric and positive-definite kernel on $[0,1]^2$. Define $C(t,s) := \mathbb{E}(\epsilon_i(t)\epsilon_j(s))$ as the covariance function of $\epsilon_i(t)$. Let $v_1(t), v_2(t), \cdots$ and the corresponding $u_1 \geq u_2 \geq \cdots$ be the eigenfunctions and eigenvalues of $C(t,s)$. By the basis expansion method, we can write $X_i(t) = \sum_{k=1}^{\infty} \tilde{x}_{i,k}v_k(t)$, $B(t,s) = \sum_{k=1}^{\infty} b_kv_k(t)v_k(s)$ and $\epsilon_i(t) = \sum_{k=1}^{\infty} \epsilon_{i,k}v_k(t)$. Now, we can rewrite (6) as

$$\tilde{x}_{i,k} = b_k \tilde{x}_{i-1,k} + \epsilon_{i,k} = \sum_{m=0}^{\infty} b_k^m \epsilon_{i-m,k}. \quad (7)$$

Here denote $f_k^2 = \text{Var}(\tilde{x}_{i,k}) = (\sum_{m=0}^{\infty} b_k^{2m}) u_k$. If we let $\rho := \sup_k |b_k| \in (0,1)$, then $f_k^2 \geq u_k$. Further observe that $\epsilon_{i,k}$ are independent Gaussian random variables across $k$, hence $(\epsilon_{i-1,k} - \epsilon_{i,k})$ is normally distributed with mean 0 and variance $2u_k$, then we have

$$\|x_{i,k} - x_{i,k}^*\|_q = \|b_k^l(\epsilon_{i-1,k} - \epsilon_{i-1,k}^*) / f_k\|_q \leq C_q \sqrt{2u_k b_k^l} / \sqrt{u_k} \leq C \rho^l.$$
D Theoretical results and corresponding proofs

D.1 Proof of the results in Section B

Proof of Theorem 1. On the outset, recall the matrix $\Sigma_c^{-1}(\lambda) = \left( \mathbb{E} \frac{X_c^T X_c}{n} + R(\lambda) \right)^{-1}$. For ease of proof, we assume that $R(\lambda)$ is $c \times c$ diagonal block matrix with its $j$th block being $R_j(\lambda) \in \mathbb{R}^{c_j \times c_j}$ for $j = 1, ..., p$. Denote the eigenvalues of $R_j(\lambda)$ and $\Sigma_c^{-1}(\lambda)$ as $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_c$ and $\rho_1 \geq \rho_2 \geq \cdots \geq \rho_c$. Since $\mathbb{E} \frac{X_c^T X_c}{n}$ is positive semi-definite, we have $\rho_i \leq \mathcal{C} \lambda_i^{-1}$. The diagonal entries $R_{j,ii}(\lambda)$ increases as $i$ increases, then the eigenvalues of $\Sigma_c^{-1}(\lambda)$ will approximate to zero at some truncation number. More specifically, define the truncated matrix denoted by $\hat{R}_j^{-1}(\lambda)$, which is similar to $R_j^{-1}(\lambda)$ for $j = 1, ..., p$ with its diagonal components at locations larger than $(k_0, k_0)$ being zeros and $k_0 = \lfloor \lambda^{-1} \mathcal{C} \lambda_i^{-1} \rfloor$. Consequently, we can construct the truncated version of $\Sigma^{-1}(\lambda)$ as

$$\Sigma_c^{-1}(\lambda) = \hat{R}^{-1}(\lambda) \left[ \left( \mathbb{E} \frac{X_c^T X_c}{n} \right)^{-1} + \hat{R}^{-1}(\lambda) \right]^{-1} \left( \mathbb{E} \frac{X_c^T X_c}{n} \right)^{-1}.$$

Next, we turn to consider the Kolmogorov distance between these two quantities $Q_n^2(t, \lambda)$ and $Q_n^c(t, \lambda) := C_f(t) \Sigma_c^{-1}(\lambda) U_n^c$. Denote

$$A_{x}^{gn} = \left\{ S \in \mathbb{R}^c : \sup_{t \in [0,1]} \max_{1 \leq j \leq p} \left| \mathbb{E}_j^T C_f(t) \Sigma_c^{-1}(\lambda) S / g_n(t) \right| \leq x \right\}$$

where $E_j \in \mathbb{R}^p$ contains 1 at the $j$th location and 0 at others, further let $\mathcal{A}_n = \{ A_{x}^{gn} : x \in \mathbb{R}, g_n(t) \in G \}$. Since it is easy to check that $A_{x}^{gn}$ is a convex set and $\mathcal{A}_n$ is a collection of convex sets, then we will write the Kolmogorov distance between $Z_n^c$ and $U_n^c$ on $\mathcal{A}_n$ as

$$\hat{K}(Z_n^c, U_n^c) = \sup_{g_n \in G} \sup_{x \in \mathbb{R}} \left| \mathbb{P}(Z_n^c \in A_x^{gn}) - \mathbb{P}(U_n^c \in A_x^{gn}) \right| = \sup_{A \in \mathcal{A}_n} \left| \mathbb{P}(Z_n^c \in A) - \mathbb{P}(U_n^c \in A) \right|,$$

where $A$ is also a convex set. Our aim is to apply the idea of [4] to obtain Gaussian approximation. First, we follow the smoothing technique of [2]. For $A \in \mathcal{A}_n$, let $h_A(x) = 1_A(x)$ where $1_A(x)$ stands for the indicator function of event $A$, and define the smoothed function

$$h_{A, \varepsilon_1}(\omega) = \psi \left( \frac{\text{disc}(\omega, A)}{\varepsilon_1} \right),$$

where $\text{disc}(\omega, A) = \inf_{\nu \in A} |\omega - \nu|$ and

$$\psi(x) = \begin{cases} 1, & x < 0, \\ 1 - 2x^2, & 0 \leq x < \frac{1}{2}, \\ 2(1 - x)^2, & \frac{1}{2} \leq x < 1, \\ 0, & x \geq 1. \end{cases}$$
From Lemma 2.3 (iv) of [2], we have $|\nabla h_{A, \epsilon_1}| \leq 2\epsilon_1^{-1}$ for all $\omega \in \mathbb{R}^c$. Then, we recall the following main results in the literature.

**Lemma 3** (Lemma 4.2 of [5]). For any $d$-dimensional random vector $W$, 

$$\mathcal{K}(W, Z) \leq 4d^{1/2} \epsilon_1 + \sup_{A \in A_n} |\mathbb{E}[h_{A, \epsilon_1}(W) - h_{A, \epsilon_1}(Z)]|,$$

where $Z$ is a $d$-dimensional standard Gaussian vector.

**Lemma 4** (Remark 2.2 of [4]). Let $W = \sum_{i=1}^{n} X_i$ be a sum of $d$-dimensional random vectors such that $\mathbb{E}(X_i) = 0$ and $\text{Cov}(W) = \Sigma_w$. Suppose $W$ can be decomposed as follows:

1. $\forall i \in [n], \exists i \in N_i \subset [n]$, such that $W - X_{N_i}$ is independent of $X_i$, where $[n] = \{1, \ldots, n\}$.
2. $\forall i \in [n], j \in N_i$, $\exists N_i \subset N_{ij} \subset [n]$, such that $W - X_{N_{ij}}$ is independent of $\{X_i, X_j\}$.
3. $\forall i \in [n], j \in N_i, k \in N_{ij}$, $\exists N_{ij} \subset N_{ijk} \subset [n]$ such that $W - X_{N_{ijk}}$ is independent of $\{X_i, X_j, X_k\}$.

Suppose further that for each $i \in [n], j \in N_i, k \in N_{ij}$, $|X_i| \leq \beta, |N_i| \leq n_1, |N_{ij}| \leq n_2, |N_{ijk}| \leq n_3$. Then there exists a universal constant $C$ such that 

$$\mathcal{K}(W, \Sigma_w^{1/2} Z) \leq C d^{1/4} n_1 \Sigma_w^{-1/2} \beta^3 n_1 (n_2 + n_3),$$

where $Z$ is a $d$-dimensional standard Gaussian random vector.

Recall $z_{ci} = x_{ci} \epsilon_i$, since $\{x_{ij,k}\}$ and $\{\epsilon_i\}$ are both stationary processes, we can rewrite $z_{ci}$ into a physical representation of stationary multivariate time series, i.e.,

$$z_{ci} = H(\mathcal{F}_i),$$

where $H = (H_1, \ldots, H_c)^T$ is a measurable vector function. Here, we define the dependence measure on the element $\{z_{ci,j}\}_{j=1}^{c}$ of the process $\{z_{ci}\}_{i=1}^{n}$ as 

$$\delta_2(l, q) := \max_{1 \leq k \leq c} \|H_k(\mathcal{F}_i) - H_k(\mathcal{F}_{i,l})\|_q,$$

where $\mathcal{F}_{i,l}$ is defined in Definition 1 of the main article. Under Assumptions 2 and 4 in the paper, we can deduce that 

$$\delta_2(l, q) \leq \max_{1 \leq j \leq p} \max_{1 \leq k \leq c} \|x_{ij,k} - x_{i,j,k}^*\|_q \epsilon_i \|2q + \max_{1 \leq j \leq p} \max_{1 \leq k \leq c} \|x_{ij,k}^*\|_q \epsilon_i - \epsilon_i^* \|_q$$

$$\leq C(l + 1)^{-\tau}, \quad \tau > 5,$$

where $x_{ij,k}^*$ and $\epsilon_i^*$ are i.i.d. copies of $x_{ij,k}$ and $\epsilon_i$, respectively. To deduce the error bound of Gaussian approximation, we need to make use of truncation approximation, $m$-dependent
approximation techniques and finally apply the aforementioned two lemmas. Now define the truncated version of $z_{ci}$ as

$$
\bar{z}_{ci} = \begin{cases} 
z_{ci}, & |z_{ci}| \leq c^\frac{1}{2} n^{\frac{3}{4}}, \\
0, & \text{otherwise}. 
\end{cases}
$$

Given a large constant $M = M(n)$, let the $M$-dependent approximation of $\bar{z}_{ci}$ be

$$
\bar{z}_{ci}^M = \mathbb{E}(\bar{z}_{ci}|\eta_i, \cdots, \eta_n), \ i = 1, \cdots, n.
$$

Consequently, we will define the partial sum of the truncated series as $\bar{Z}^c_n = \sum_{i=1}^{n} \bar{z}_{ci}/\sqrt{n}$ and the $M$-dependent version as $\bar{Z}_{n}^M = \sum_{i=1}^{n} \bar{z}_{ci}^M/\sqrt{n}$. Further let $\bar{Z}^*_n = \bar{Z}^c_n - \mathbb{E}\bar{Z}_n^c$, $\bar{Z}_{n}^M = \bar{Z}^M_n - \mathbb{E}\bar{Z}_n^c$ and let $\bar{U}_{n}^M$ be a Gaussian random vector preserving the covariance structure of $\bar{Z}_n^M$. With Lemma 3 and the fact $|\nabla h_{A,\epsilon_1}| \leq 2\epsilon_1^{-1}$, we have

$$
\mathbb{K}(\bar{Z}^*_n, \bar{U}_{n}^c) \leq 4c^\frac{1}{4} \epsilon_1 + \sup_{A \in \mathcal{A}_n} \left| \mathbb{E}\left[h_{A,\epsilon_1}(\bar{Z}^c_n) - h_{A,\epsilon_1}(\bar{U}_{n}^c)\right]\right|
$$

$$
\leq 4c^\frac{1}{4} \epsilon_1 + \sup_{A \in \mathcal{A}_n} \left| \mathbb{E}\left[h_{A,\epsilon_1}(\bar{Z}^c_n) - h_{A,\epsilon_1}(\bar{Z}^*_n)\right]\right| + \sup_{A \in \mathcal{A}_n} \left| \mathbb{E}\left[h_{A,\epsilon_1}(\bar{Z}^*_n) - h_{A,\epsilon_1}(\bar{Z}_{n}^M)\right]\right|
$$

$$
+ \sup_{A \in \mathcal{A}_n} \left| \mathbb{E}\left[h_{A,\epsilon_1}(\bar{Z}_{n}^M) - h_{A,\epsilon_1}(\bar{U}_{n}^M)\right]\right| + \sup_{A \in \mathcal{A}_n} \left| \mathbb{E}\left[h_{A,\epsilon_1}(\bar{U}_{n}^M) - h_{A,\epsilon_1}(\bar{U}_{n}^c)\right]\right|
$$

$$
\leq 4c^\frac{1}{4} \epsilon_1 + \frac{C}{\epsilon_1} \mathbb{E}|\bar{Z}^c_n - \bar{Z}^*_n| + \frac{C}{\epsilon_1} \mathbb{E}|\bar{Z}^*_n - \bar{Z}_{n}^M|
$$

$$
+ \sup_{A \in \mathcal{A}_n} \left| \mathbb{E}\left[h_{A,\epsilon_1}(\bar{Z}_{n}^M) - h_{A,\epsilon_1}(\bar{U}_{n}^M)\right]\right| + \frac{C}{\epsilon_1} \mathbb{E}|\bar{U}_{n}^M - \bar{U}_{n}^c|
$$

$$
:= 4c^\frac{1}{4} \epsilon_1 + \frac{C \epsilon_2}{\epsilon_1} + \frac{C \epsilon_3}{\epsilon_1} + \frac{C \epsilon_4}{\epsilon_1} + \sup_{A \in \mathcal{A}_n} \left| \mathbb{E}\left[h_{A,\epsilon_1}(\bar{Z}_{n}^M) - h_{A,\epsilon_1}(\bar{U}_{n}^M)\right]\right|, \quad (8)
$$

where $\epsilon_2 := \mathbb{E}|\bar{Z}^c_n - \bar{Z}^*_n|$, $\epsilon_3 := \mathbb{E}|\bar{Z}^*_n - \bar{Z}_{n}^M|$ and $\epsilon_4 := \mathbb{E}|\bar{U}_{n}^M - \bar{U}_{n}^c|$.

(1) Truncation approximation.

We shall first control the truncation error $\epsilon_2$. Note that $\mathbb{E}\bar{z}_i = 0$, then we have

$$
|\bar{Z}^c_n - \bar{Z}^*_n| = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} |z_{ci} - \mathbb{E}z_{ci} - \bar{z}_{ci} + \mathbb{E}\bar{z}_{ci}|
$$

$$
\leq \frac{1}{\sqrt{n}} \sum_{i=1}^{n} |z_{ci} - \bar{z}_{ci}| + \frac{1}{\sqrt{n}} \sum_{i=1}^{n} |\mathbb{E}(z_{ci} - \bar{z}_{ci})|
$$

$$
:= I + II.
$$

For I, notice that for any $i = 1, \cdots, n,$
It is readily seen that \( \{\bar{z}_{ci} \} \) is a sequence of martingale differences, then we have
\[
\mathbb{P}\left( |z_{ci}| > c^{1/2} n^{3/2q} \right) = \mathbb{E}\left[ 1\{|z_{ci}| > c^{1/2} n^{3/2q} \} \right]
\]
\[
\leq \mathbb{E}\left[ \left( \frac{|z_{ci}|}{c^{3/2} n^{3/2q}} \right)^q \right] = c^{-q/2} n^{-3/2} \mathbb{E}|z_{ci}|^q
\]
\[
= c^{-q/2} n^{-3/2} \sum_{j=1}^{c} z_{ci,j}^2 \leq c^{-q/2} n^{-3/2} c^{q/2-1} \sum_{j=1}^{c} \mathbb{E}|z_{ci,j}|^q \leq C_q n^{-3/2},
\]
where the second to last inequality follows from the inequality \( \mathbb{E}|X_1 + \cdots + X_c|^q/2 \leq c^{q/2-1} \sum_{j=1}^{c} \mathbb{E}|X_j|^q/2 \) for random variables \( \{X_j\}_{j=1}^{c} \) and Assumption 5 of the main paper. Hence, we have \( \mathbb{P}\left( \left| \sum_{i=1}^{n} (z_{ci} - \bar{z}_{ci}) \right| > n^{-2} \right) \to 0 \). As a result, \( I = o_p(n^{-2}) \).

For II, since for any \( i = 1, \ldots, n, \)
\[
\mathbb{E}(z_{ci} - \bar{z}_{ci}) \leq \mathbb{E}\left[ |z_{ci}| 1\{|z_{ci}| > c^{1/2} n^{3/2q} \} \right]
\]
\[
\leq \mathbb{E}\left[ |z_{ci}| \left( \frac{|z_{ci}|}{c^{3/2} n^{3/2q}} \right)^{q-1} \right]
\]
\[
= c^{-q/2} n^{-3/2 + 3/2q} \mathbb{E}|z_{ci}|^q
\]
\[
\leq C_q c^{1/2} n^{-3/2 + 3/2q},
\]
where the second inequality uses the fact that for the nonnegative random variable \( y \) and some number \( a > 0 \), the inequality \( y 1\{y \geq a\} \leq y \left( \frac{a}{y} \right)^p \) for any \( p > 0 \) holds true. Consequently, \( II = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \mathbb{E}(z_{ci} - \bar{z}_{ci}) = \mathcal{O}(c^{1/2} n^{-1+3/2q}) \). Now, by choosing \( \beta = c^{1/2} n^{-1+3/2q} \), then \( \epsilon_2 = \mathcal{O}(c^{1/2} n^{-1+3/2q}) \).

(2) \( M \)-dependence approximation.

Next we will deduce the approximation rate between our original process and its \( M \)-dependent sequence, i.e., control \( \epsilon_3 \) in (8). Recall the physical dependence measure \( \delta_{z}(l, q) \) of \( z_{ci,j} \) and denote \( \Theta_{M,q} = \sum_{l=M}^{\infty} \delta_{z}(l, q) \). Let
\[
Z_n^c = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (z_{ci} - \bar{z}_{ci}^M) =: \frac{1}{\sqrt{n}} \sum_{i=1}^{n} z_{i\Delta}.
\]
It is readily seen that \( \{z_{i\Delta}^M, i = 1, \ldots, n \} \) is a sequence of martingale differences, then we have
\[ \left\| \bar{Z}_n^* - \tilde{Z}_n^M \right\|_q = \left\| \bar{Z}_n^c - \tilde{Z}_n^M \right\|_q \]
\[ = \left\| \frac{1}{\sqrt{n}} \sum_{i=1}^n \hat{z}_{i,j} \right\|_q^2 = \left\{ \mathbb{E} \left[ \sum_{j=1}^c \left( \frac{1}{\sqrt{n}} \sum_{i=1}^n \hat{z}_{i,j} \right)^2 \right] \right\}^{2/q} \]
\[ \leq \left\{ c^{q/2-1} \sum_{j=1}^c \mathbb{E} \left[ \frac{1}{\sqrt{n}} \sum_{i=1}^n \hat{z}_{i,j}^\Delta \right]^q \right\}^{2/q} \]
\[ \leq c \sup_{1 \leq j \leq c} \left\| \frac{1}{\sqrt{n}} \sum_{i=1}^n \hat{z}_{i,j}^\Delta \right\|_q^2 \leq Cc\Theta_{M,q}^2, \]

where \( \hat{z}_{i,j}^\Delta \) is the entrywise of vector \( \hat{z}_{i,j}^\Delta \), the first inequality is due to the fact that \( \mathbb{E}|X_1 + \cdots + X_c|^q \leq c^{q/2-1} \sum_{j=1}^c \mathbb{E}|X_j|^q / 2 \) with \( X_j \) being random variables. The second inequality above is followed by Lemma A.1 of [8] using Burkholder’s inequality. As a result, \( \left\| \bar{Z}_n^* - \tilde{Z}_n^M \right\|_q \leq Cc^{1/2}M^{-\tau+1} \).

Therefore, when we choose \( M \) appropriately to satisfy
\[ \frac{M^{-\tau+1}}{n^{-1+\frac{\tau}{2}}} \to 0, \]
for example \( M = O(n^{-1}) \) for \( \tau > 5 \), then we have \( |\bar{Z}_n^* - \tilde{Z}_n^M| = o_p(c^{1/2}n^{-1}) \). Consequently, \( \epsilon_3 = o(\epsilon_2) \).

(3) Using Lemma 4 to deduce the final result.

At last, we will employ Lemma 4 so as to deal with the last term of Eq. (8). Let \( \hat{Z}_n^M = \sum_{i=1}^n \hat{z}_i^\top / \sqrt{n} \), then by the truncation and \( m \) dependence approximation techniques, we have \( |\hat{z}_i^\top / \sqrt{n}| \leq \beta \) and \( \mathbb{E}\hat{z}_i^\top / \sqrt{n} = 0 \). Recall \( \Xi^c = \mathbb{E} \left[ \sum_{i=1}^n \hat{z}_i^\top \right] / n \), denote \( \Xi_{M}^c := \text{Cov}(\hat{Z}_n^M) = \mathbb{E} \left[ \sum_{i=1}^n (\hat{z}_i^M - \mathbb{E}\hat{z}_i^M) \left( \sum_{i=1}^n \hat{z}_i^M \right) \right] / n \). Next we will find out the difference of covariance matrix between \( \hat{Z}_n^M \) and \( \hat{Z}_n^c \) based on Frobenius norm turns to be

\[ \left\| \Xi - \Xi_{M}^c \right\|_F \]
\[ \leq \frac{1}{n} \left\{ \left\| \mathbb{E} \left[ \sum_{i=1}^n (\hat{z}_i^M - \mathbb{E}\hat{z}_i^M) \left( \sum_{i=1}^n \hat{z}_i^\top \right) \right] \right\|_F + \left\| \mathbb{E} \left[ \sum_{i=1}^n \hat{z}_i^M \right] \left[ \sum_{i=1}^n (\hat{z}_i^M - \mathbb{E}\hat{z}_i^M) \right]^\top \right\|_F \]
\[ + \left\| \sum_{i=1}^n \mathbb{E}(\hat{z}_i - \hat{z}_i^\Delta) \sum_{i=1}^n \mathbb{E}(\hat{z}_i - \hat{z}_i^\Delta)^\top \right\|_F \}
\[ \leq C(cn^{-2} + c^3M^{-\tau+1} + c^2n^{-2+3/q}) = O(c^{3/2}n^{-1}), \]
where the last inequality follows by the error bound for II and Eq. (19) and Cauchy Schwarz inequality. Further denote $\tilde{U}_{nj}^M$ and $U_{nj}^c$ are components of vector $\tilde{U}_n^M$ and $U_n^c$, respectively. Then,

$$
\epsilon_4 = \mathbb{E}\left[\sum_{j=1}^c (\tilde{U}_{nj}^M - U_{nj}^c)^2\right] \leq \mathbb{E}\left[\sum_{j=1}^c (\tilde{U}_{nj}^M - U_{nj}^c)^2\right] = \mathcal{O}\left(c^{3/2}n^{-1}\right).
$$

Since we assume that the smallest eigenvalue of $\Xi_n^c$ is bounded below by some constant $b > 0$, we can deduce that

$$
\lambda_{\min}(\Xi_n^c) \geq \lambda_{\min}(\Xi_n^c - \Xi_M^c) + \lambda_{\min}(\Xi_M^c)
$$

$$
= -\lambda_{\max}(\Xi_n^c - \Xi_M^c) + \lambda_{\min}(\Xi_M^c)
$$

$$
\geq b - Cc^{3/2}n^{-1} > 0.
$$

Hence, the small eigenvalue of $\Xi_n^c$ is also bounded below by some positive constant, then $|\Xi_n^c| \leq C$. Further note that $n_1 = M$, $n_2 = 2M$, $n_3 = 3M$ and together with the Eq. (4.24) in [4], we have

$$
\mathcal{K}(\tilde{Z}_n^M, \tilde{U}_n^M) \leq 4c^{1/2}\epsilon_1 + 2C\beta^3 M^2 \frac{1}{\epsilon_1} + (\epsilon_1 + 3M\beta) + \mathcal{K}(\tilde{Z}_n^M, \tilde{U}_n^M). \tag{9}
$$

By substituting $\beta = c^{1/2}n^{-1/2} + \frac{2}{\pi}$ and optimizing $\epsilon_1$, we can choose $\epsilon_1 = \mathcal{O}(c^{2/3}n^{-1/2} + \frac{2}{\pi}M^2)$. Then the second term in Eq. (8) turns out to be $\mathcal{O}(c^{-1}n^{-1/2} + \frac{3}{2}M^{-2})$ and the fourth term is $\mathcal{O}(n^{-1/2} + \frac{3}{2}M^{-2})$. By substituting $M = \mathcal{O}(n^{-1/4})$, we have

$$
\mathcal{K}(Z_n^c, U_n^c) = \mathcal{O}(c^{7/4}n^{-1/2} + \frac{2}{\pi}n^{1/4} + \frac{2}{\pi}).
$$

Finally, we aim to control the Kolmogorov distance between $Q_n^c(t, \lambda)$ and $Q_n^u(t, \lambda)$. Denote $\Delta(\lambda) = \Sigma^{-1}(\lambda) - \Sigma^{-1}(\lambda)$, similarly using the foregoing Gaussian approximation, we can obtain

$$
\sup_{x \in \mathbb{R}, g_n \in \mathcal{G}} \mathbb{P}\left(\sup_{t \in [0,1]} |C_f(t)\Delta(\lambda)Z_n^c|_{g_n} \leq x\right) - \mathbb{P}\left(\sup_{t \in [0,1]} |C_f(t)\Delta(\lambda)U_n^c|_{g_n} \leq x\right) = \mathcal{O}(c^{7/4}n^{-1/2} + \frac{2}{\pi}n + \frac{2}{\pi})
$$

Therefore, it suffices to control the weighted maximum deviation term involving Gaussian vector $U_n^c$. We will employ the chaining technique. Let the sampling time points be $\{t_i\}^{n}_{i=0}$ where $t_i = i/r_n$ and $r_n = \mathcal{O}(n^{\nu})$, $\nu > 0$ is a positive integer that diverges to infinity. Furthermore, denote $\Pi(t, \lambda) := C_f(t)\Delta(\lambda)U_n^c$, then for any $j = 1, ..., p$, we
can calculate the following Orlicz norm of weighted maximum deviation with $\Psi(x) \equiv \exp(x^2) - 1$ over discrete time points as

$$
\left\| \max_{0 \leq i \leq r_n} \| \Pi(t_{i,n}, \lambda) \|_{g_n(t_{i,n})} \right\|_{\Psi} 
= \max_{0 \leq i \leq r_n} \max_{1 \leq j \leq p} \left\| E_j^T C_f(t_{i,n}) \Delta(\lambda) U_n^c \right\|_{g_{nj}(t_{i,n})} 
\leq C \sqrt{\log(p n)} \max_{0 \leq i \leq r_n} \max_{1 \leq j \leq p} \left\| E_j^T C_f(t_{i,n}) \Delta(\lambda) U_n^c \right\|_{\Psi} 
\leq C \sqrt{\log(n)} \max_{0 \leq i \leq r_n} \left\| E_j^T C_f(t_{i,n}) \right\| \left\| \Sigma_c^{-1}(\lambda) - \Sigma_c^{-1}(\lambda) \right\| U_n^c \right\|_{\Psi} 
\leq C \sqrt{\log(n)} \left\| E_j^T C_f(t_{i,n}) \right\| \left\| R^{-1}(\lambda) - \bar{R}^{-1}(\lambda) \right\| \left\| U_n^c \right\|_2 
+ \max_{0 \leq i \leq r_n} \max_{1 \leq j \leq p} \left\| E_j^T C_f(t_{i,n}) \right\| \left\| R^{-1}(\lambda) \right\| \left\| \left( I_c + \mathbb{E} \frac{X_c X_c^T}{n} R^{-1}(\lambda) \right)^{-1} - \left( I_c + \mathbb{E} \frac{X_c X_c^T}{n} R^{-1}(\lambda) \right)^{-1} \right\| U_n^c \right\|_2 
\leq C \sqrt{\log(n)} \max_{0 \leq i \leq r_n} \left\| E_j^T C_f(t_{i,n}) \right\| \left\| R^{-1}(\lambda) - \bar{R}^{-1}(\lambda) \right\| U_n^c \right\|_2 
= C \sqrt{\log(n)} \max_{0 \leq i \leq r_n} \left\| E_j^T C_f(t_{i,n}) \right\| \left\| R^{-1}(\lambda) - \bar{R}^{-1}(\lambda) \right\| U_n^c \right\|_2 
\leq C \sqrt{\log(n)} \max_{0 \leq i \leq r_n} \left\| E_j^T C_f(t_{i,n}) \right\| \left\| \Sigma_c^{-1}(\lambda) - \Sigma_c^{-1}(\lambda) \right\| U_n^c \right\|_2 
\leq C \lambda^{2(2\gamma + d_2 + \phi)} \sqrt{\log n}.
$$

where the first inequality above follows by the definition $\inf_{t \in [0,1]} \min_{1 \leq j \leq p} g_{nj} \geq \kappa > 0$, together with the maximum inequality $\max_{1 \leq i \leq r_n} X_i \|_{\Psi} \leq C_{\Psi} \psi^{-1}(\eta) \max_i \| X_i \|_{\Psi}$, where $C_{\Psi}$ is a constant depending only on function $\Psi$. The second inequality uses the fact that the Orlicz norm of some Gaussian random variable is proportional to its standard deviation. Elementary calculations and the relationship $\Sigma_c^{-1}(\lambda) = R^{-1}(\lambda) \left[ I_c + \mathbb{E} \frac{X_c X_c^T}{n} R^{-1}(\lambda) \right]^{-1}$ with $c \times c$ identity matrix $I_c$ are carried out for the third inequality. Finally by the fact $\text{Tr}(AB) \leq \text{Tr}(A) \| B \|$ for positive semi-definite matrix $A$ and arbitrary matrix $B$ and additional calculations, we obtain the last two inequalities.

On the other hand, for any fixed $i$ and $j$,

$$
\left\| \sup_{s \in [t_{i+1,n}, t_{i+1,n}]} \| \Pi(s, \lambda) - \Pi(t_{i,n}, \lambda) \|_{g_n(s)} \right\|_{\Psi} 
= \left\| \left. \sup_{s \in [t_{i+1,n}, t_{i+1,n}]} \left. \max_{1 \leq j \leq p} \frac{E_j^T [C_f(s) - C_f(t_{i,n})] \Delta(\lambda) U_n^c}{g_{nj}(s)} \right\| \right\|_{\Psi} 
\leq C \int_{t_{i,n}}^{t_{i+1,n}} \left\| E_j^T C_f(s) \right\| \left\| \Delta(\lambda) \right\| \left\| U_n^c \right\|_2 \| ds 
\leq C \sup_{s \in [0,1]} \left\| E_j^T C_f(s) \right\| \left\| \Delta(\lambda) \right\| \left\| U_n^c \right\|_2 / r_n \leq C \lambda^{\frac{d_2 + \psi + 2}{d_1 + d_2 - \psi}} e^{\phi + d_2^2} / r_n,
$$

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where $C'_f(s)$ is the first derivative over $s$. Similarly by the maximum inequality, we have for any $j = 1, \ldots, p$,

$$
\left\| \max_{0 \leq i \leq r_n - 1} \sup_{s \in [t_{i,n}, t_{i+1,n}]} \left| \Pi(s) - \Pi(t_{i,n}, \lambda) \right|_{g_n(s)} \right\|_{\Phi} \leq C \lambda^{d_2 + d_2 + 2} c^{d_2 + 2} \sqrt{\log(r_n)/r_n}.
$$

Further denote $\bar{Q}_n^u(t, \lambda) := C_f(t) \Sigma_\gamma^{-1}(\lambda) U_n^c$ with entries $\{\bar{Q}_{n,j}^u(t, \lambda)\}_{j=1}^p$. Therefore, it yields that

$$
\sup_{t \in [0,1]} \left\| \Pi(t, \lambda) \right\|_{g_n(t)} \leq \max \left\{ \max_{0 \leq i \leq r_n} \left\| \Pi(t_{i,n}, \lambda) \right\|_{g_n(t_{i,n}, \lambda)} \right\} \sup_{0 \leq i \leq r_n - 1} \sup_{s \in [t_{i,n}, t_{i+1,n}]} \left| \bar{Q}_n^u(s, \lambda) - \bar{Q}_n^u(t_{i,n}, \lambda) \right|_{g_n(s)} \right\} \leq C \lambda^{\gamma + d_2 - \delta} \sqrt{\log n}.
$$

As a result, we conclude $\sup_{t \in [0,1]} |C_f(t) \Delta(\lambda) Z_n^c| \leq C \lambda^{\gamma + d_2 - \delta} \sqrt{\log n}$.

Armed with Assumption 4 of the main article, we can analogously calculate

$$
\left\| \max_{0 \leq i \leq r_n - 1} \sup_{s \in [t_{i,n}, t_{i+1,n}]} \left| Q_n^u(s, \lambda) - Q_n^u(t_{i,n}, \lambda) \right|_{g_n(s)} \right\|_{\Phi} \leq C c^{d_2 + 2} \sqrt{\log(n)/r_n}
$$

Next define $\delta_1, n = \lambda^{\gamma + d_2 - \delta} \log(n)$ and $\delta_2, n = c^{d_2 + 2} \log(n)/r_n$, we can derive that, for any $x \in \mathbb{R}$,

$$
P \left( \sup_{t \in [0,1]} \left| Q_n^u(t, \lambda) \right|_{g_n(t)} \leq x \right) - P \left( \sup_{t \in [0,1]} \left| Q_n^u(t, \lambda) \right|_{g_n(t)} \leq x \right) \leq \left[ P \left( \sup_{t \in [0,1]} \left| Q_n^u(t, \lambda) \right|_{g_n(t)} \leq x + \delta_1, n \right) + P \left( \sup_{t \in [0,1]} \left| Q_n^u(t, \lambda) - Q_n^u(t_{i,n}, \lambda) \right|_{g_n(t)} \geq \delta_1, n \right) \right]
$$

$$
- P \left( \sup_{t \in [0,1]} \left| Q_n^u(t, \lambda) \right|_{g_n(t)} \leq x \right) \leq \left[ P \left( \sup_{t \in [0,1]} \left| Q_n^u(t, \lambda) \right|_{g_n(t)} \leq x + \delta_1, n \right) - P \left( \sup_{t \in [0,1]} \left| Q_n^u(t, \lambda) \right|_{g_n(t)} \leq x + \delta_1, n \right) \right] + \log^{-1}(n)
$$

$$
+ P \left( \sup_{t \in [0,1]} \left| Q_n^u(t, \lambda) \right|_{g_n(t)} \leq x + \delta_1, n \right) - P \left( \sup_{t \in [0,1]} \left| Q_n^u(t, \lambda) \right|_{g_n(t)} \leq x \right) \leq C c^{\gamma} n^{1/2 + \eta/2} + \log^{-1}(n) + P \left( \max_{0 \leq i \leq r_n} \left| Q_n^u(t_{i,n}, \lambda) \right|_{g_n(t_{i,n})} \leq x + \delta_1, n + \delta_2, n \right)
$$

$$
+ P \left( \max_{0 \leq i \leq r_n - 1} \sup_{s \in [t_{i,n}, t_{i+1,n}]} \left| Q_n^u(s, \lambda) - Q_n^u(t_{i,n}, \lambda) \right|_{g_n(s)} \geq \delta_2, n \right)$$
\[ + \mathbb{P} \left( \sup_{t \in [0,1]} |\tilde{Q}_n^u(t, \lambda) - Q_n^u(t, \lambda)|_{g_n(t)} \geq \delta_{1,n} \right) \]

\[ + \mathbb{P} \left( \max_{0 \leq i \leq r_n - 1} \sup_{s \in [t_i, t_{i+1}, n]} |Q_n^u(s, \lambda) - Q_n^u(t_{i,n}, \lambda)|_{g_n(s)} \geq \delta_{2,n} \right) \]

\[ - \mathbb{P} \left( \max_{0 \leq i \leq r_n - 1} |\tilde{Q}_n^u(t_{i,n}, \lambda)|_{g_n(t_{i,n})} \leq x - \delta_{1,n} - \delta_{2,n} \right) \]

\[ \leq C \left( c_4 n^{-\frac{1}{2} + \frac{9}{2} \tau_1 + \frac{2}{\tau_1} + \log^{-1}(n) + (\delta_{1,n} + \delta_{2,n}) \sqrt{\log(r_n)} \right) \]

\[ \leq C \left( c_4 n^{-\frac{1}{2} + \frac{9}{2} \tau_1 + \frac{2}{\tau_1} + \lambda^{\frac{1}{n(2\tau_2 - \tau_2)}} \log^{3/2}(n) + \log^{-1}(n) \right) \] (10)

where the second to the last inequality follows by the anti-concentration inequality in [3, Theorem 3] and Markov inequality. Similarly, we have the negative lower bound with same magnitude of (10). In conclusion, the final approximation error turns out to be

\[ K(Z_n', U_n') = \mathcal{O} \left( c_4 n^{-\frac{1}{2} + \frac{9}{2} \tau_1 + \frac{2}{\tau_1} + \lambda^{\frac{1}{n(2\tau_2 - \tau_2)}} \log^{3/2}(n) \right). \]

### D.2 Discussion of the claims on Assumptions 6–7 of the paper

At the outset, we present several examples of Assumption 6 in Section 3.2.

**Example 3** (Integral of the second derivative).

1. **Fourier bases.** Consider \( \alpha_k(t) = \{1, \sqrt{2} \cos(2k\pi t), \sqrt{2} \sin(2k\pi t), \cdots \} \) for \( t \in [0,1] \).
   
   Its second derivative is

   \[ \alpha''_k(t) = \begin{cases} 
   0, & k = 1, \\
   -4\sqrt{2}k^2\pi^2 \cos(2k\pi t), & k = 2, 4, \cdots, \\
   -4\sqrt{2}k^2\pi^2 \sin(2k\pi t), & k = 3, 5, \cdots. 
   \end{cases} \]

   Then, we have

   \[ \int_0^1 [\alpha_k''(t)]^2 dt = \begin{cases} 
   0, & k = 1, \\
   16k^4\pi^4, & k \geq 2. 
   \end{cases} \]

   As a result, one can obtain \( |\tilde{R}_j| = c^4 \).

2. **Normalized Legendre polynomials.** The Legendre polynomial of degree \( n \) can be obtained using Rodrigue’s formula

   \[ P_n(t) = \frac{1}{2^n n!} \frac{d^n}{dt^n} (t^2 - 1)^n, \quad -1 \leq t \leq 1. \]

   For \( t \in [0,1] \), the normalized Legendre polynomials turn out to be
\[ \alpha_k(t) = \begin{cases} 1, & k = 0, \\ \sqrt{2k + 1}P_k(2t - 1), & k > 0. \end{cases} \]

To derive the spectral norm of \( \tilde{R}_j \), first recall the Bonnet's recursion formula for Legendre polynomials,

\[ (n + 1)P_{n+1}(x) = (2n + 1)xP_n(x) - nP_{n-1}(x). \]

By elementary calculations, we have

\[ \frac{d}{dx} P_{n+1}(x) = (2n + 1)P_n(x) + [2(n - 2) + 1]P_{n-2}(x) + [2(n - 4) + 1]P_{n-4}(x) + \cdots. \]

Consequently, due to the fact that

\[ \sum_{i=1}^{n} i^2 = \frac{n(n + 1)(2n + 1)}{6}, \quad \sum_{i=1}^{n} i^5 = \frac{n^6}{6} + \frac{n^5}{2} + \frac{5}{12n^4} - \frac{1}{12n^2}, \]

we can deduce that

\[ P''_n(t) = \sum_{j=1}^{[(n+1)/2]} \sum_{k=1}^{[(n-2j-1)/2]} [2(n - 2j) + 3][2(n - 2j - 2k) + 5]P_{n-2j-2k+2}(t). \]

Next, we can calculate

\[
\begin{align*}
\int_{-1}^{1} [P''_n(t)]^2 dt &= \int_{-1}^{1} \left( \sum_{j=1}^{[(n+1)/2]} \sum_{k=1}^{[(n-2j-1)/2]} [2(n - 2j) + 3][2(n - 2j - 2k) + 5]P_{n-2j-2k+2}(t) \right)^2 dt \\
&+ \sum_{j_1 \neq j_2} \left( \sum_{k_1=1}^{[(n-2j_1-1)/2]} [2(n - 2j_1) + 3][2(n - 2j_1 - 2k_1) + 5]P_{n-2j_1-2k_1+2}(t) \right) \\
&\times \left( \sum_{k_2=1}^{[(n-2j_2-1)/2]} [2(n - 2j_2) + 3][2(n - 2j_2 - 2k_2) + 5]P_{n-2j_2-2k_2+2}(t) \right) dt = n^7.
\end{align*}
\]

Therefore, it yields that \( \int_{0}^{1} [\alpha''_k(t)]^2 dt = (\sqrt{k})^2k^7 = k^8 \) and \( |\tilde{R}_j| = c^8 \) for \( j = 1, \ldots, p \).

In the following, we will show some examples on Assumption 7 in Section 3.2. Recall \( \mathbf{\alpha}(t) = (\alpha_1(t), \alpha_2(t), \ldots, \alpha_k(t))^\top \) and define \( \zeta_k := \sup_{t \in [0,1]} |\mathbf{\alpha}(t)| \) whose upper bound has been discussed in many excellent works, see [9], [1] and references therein. For example, \( \zeta_k \leq \sqrt{k} \) for tensor-products of univariate polynomial spline, trigonometric polynomial or
wavelet bases and $\zeta_k \leq k$ for tensor-products of power series or orthogonal polynomial bases. With the above result and the relationship $|\alpha_k(t)|_\infty \leq \zeta_k$, then the statement of the first part of Assumption 7 will be verified easily. As for the second part, we list some commonly used basis functions as follows:

**Example 4** (Supremum of the first derivative).

1. **Fourier bases.** Consider the Fourier bases with the same representation in Example 3 and their first derivatives are

   $$\alpha_k'(t) = \begin{cases} 0, & k = 1, \\ -2\sqrt{2}k\pi \sin(2k\pi t), & k = 2, 4, \ldots, \\ 2\sqrt{2}k\pi \cos(2k\pi t), & k = 3, 5, \ldots. \end{cases}$$

   Then, we have $\sup_{t\in[0,1]} |\alpha_k'(t)| = \mathcal{O}(k)$.

2. **Univariate spline series of order 3.** With a finite number of equally spaced knots $l_1, \ldots, l_{k-4}$ in $[0, 1]$, $\alpha_k(t) = \{1, t, t^2, (t - l_1)^3, \ldots, (t - l_{k-4})^3\}$. Then the first derivative of spline basis function can be given as

   $$\alpha_k'(t) = \{0, 1, 2t, 3t^2, 3[(t - l_1)^3]^{2/3}, \ldots\}.$$  

   Therefore, we conclude $\sup_{t\in[0,1]} |\alpha_k'(t)| = \mathcal{O}(1)$.

3. **Normalized Legendre polynomials.** For $t \in [0, 1]$, recall

   $$\alpha_k(t) = \{1, \sqrt{3}t, \sqrt{5/4}(3t^2 - 1), \ldots\}.$$  

   By the discussion in the proof of Example 2, we have

   $$\alpha_k'(t) \leq 2\sqrt{2k + 1} \sup_{x\in[-1,1]} |P_k'(2t - 1)| \sum_{m=1}^{k} (2m - 1) = \mathcal{O}(k^{5/2}),$$

   where $P_k(\cdot)$ is the Legendre polynomial basis function on $[-1,1]$.

   Consequently, the Assumption 7 of the paper is mild and can be satisfied by most basis functions.

### D.3 Proof of the results in Section 3.2 of the paper

To prove Theorem 1 in Section 3.2 of the paper, we need two intermediate results on the theoretical bootstrap approximation and consistency of estimators, respectively.

- **Theoretical bootstrap approximation.**
First define the conditional variance of $U_{n}^{\text{boots}}$ and the corresponding Kolmogorov distance as

\[
\Xi^c := \frac{1}{(n-m+1)m} \sum_{j=1}^{n-m+1} \left( \sum_{i=j}^{j+m-1} z_{ci} \right) \left( \sum_{i=j}^{j+m-1} \bar{z}_{ci} \right),
\]

\[
\mathcal{K}(U_{n}^{\text{boots}}, Z_n^c) = \sup_{g_n \in \mathcal{G}, x \in \mathbb{R}} \left\{ \mathbb{P} \left( \sup_{t \in [0,1]} |Q_n^{\text{boots}}(t, \lambda)|_{g_n(t)} \leq x \right) \right\} - \mathbb{P} \left( \sup_{t \in [0,1]} |Q_n(t, \lambda)|_{g_n(t)} \leq x \right),
\]

then we will obtain the following proposition which establishes the rate of the bootstrap approximation to the weighted maximum deviation of $Z_n^c$.

\textbf{Proposition 1.} Assume that the smallest eigenvalue of $\Xi^c$ is bounded below by some constant $b > 0$ and $m = O(n^{1/3})$. For some finite constant $C > 0$, define $B_n^c = \{ \omega : \Delta_n^c(\omega) := \Xi^c - \Xi^c \leq Ccn^{-1/3}h_n \}$, where $\omega$ represents the element in the probability space, $h_n$ diverges to infinity at an arbitrarily slow rate, then $\mathbb{P}(B_n^c) = 1 - o(1)$.

Under Assumptions 1–8 of the main article, on the event $B_n^c$, we have

\[
\mathcal{K}'(U_{n}^{\text{boots}}, Z_n^c) \leq C \left( c^{7/4}n^{-\frac{1}{2} + \frac{9}{2r} + \frac{2}{r-1}} + \lambda \frac{1}{2(2d_2 + \psi)} \log^{3/2}(n) + c^{5/8}n^{-1/6}h_n^{1/2} + c^{d_2 + \psi + 5/2} \log^2(n)/\sqrt{n} \right),
\]

(11)

where $C < \infty$. Further suppose

(i) $c \gg (n/\log(n))^{\frac{1}{2(2d_2 + \psi)r + 1}}$ and $\lambda \ll (\log(n)/n)^{\frac{2(2d_2 + \psi) + 1}{2(2d_2 + \psi)r + 1}}$,

(ii) $\tilde{g}_n(t) \in \mathcal{G}$ almost surely,

(iii) $c^{7/4}n^{-\frac{1}{2} + \frac{9}{2r} + \frac{2}{r-1}} + \lambda \frac{1}{2(2d_2 + \psi)} \log^{3/2}(n) + c^{5/8}n^{-1/6}h_n^{1/2} + c^{d_2 + \psi + 5/2} \log^2(n)/\sqrt{n} \to 0$ as $n \to \infty$,

then the JSCB based on the roughness penalization approach achieves

\[
\lim_{n \to \infty} \lim_{B \to \infty} \mathbb{P} \left( \beta_j(t) \in \left[ \tilde{\beta}_j(t) - \hat{q}_{n,1-\alpha}\tilde{g}_{nj}(t)/\sqrt{n}, \tilde{\beta}_j(t) + \hat{q}_{n,1-\alpha}\tilde{g}_{nj}(t)/\sqrt{n} \right] \right)
\]

for $\forall t \in [0,1]$ and $j = 1, ..., p = 1 - \alpha.$

(12)

From the above Proposition 1, we conclude that with Condition (iii), Gaussian approximation rate and the bootstrap approximation rate in (11) both converge to 0. In particular, if $X_i(t) \in \mathcal{C}^1$, $\beta(t) \in \mathcal{C}^2$, $\gamma = 2$ based on Fourier bases and $q$ and $\tau$ go to infinity, the above theorem shows that $\tilde{\beta}(t)$ is an under-smoothed estimator as long as $c \gg (n/\log(n))^\frac{1}{8}$ as well as $\lambda \ll (\log(n)/n)^\frac{3}{8}$. Further with the constraint $c \ll n^{\frac{1}{4}}$, the right hand side of (11) goes to 0.
Proof. Recall \( \tilde{\Sigma}_n^c = \mathbb{E}[U_n^{\text{boots}}(U_n^{\text{boots}})^\top | Z_1^n] \) and \( \Xi_n^c = \mathbb{E}[U_n^c(U_n^c)^\top] \). By Lemma 9 in Section D.5, we have \( \| \tilde{\Sigma}_{jk}^c - \Xi_{jk}^c \|_2 = O \left( \frac{1}{m} + \sqrt{\frac{m}{n}} \right) \). By choosing \( m = O(n^{1/3}) \), \( | \tilde{\Sigma}_n^c - \Xi_n^c |_F = O(cn^{-1/3}) = o(cn^{-1/3} \ln n) \), then \( \mathbb{P}(B_n^c) = 1 - o(1) \). Next, we will investigate the approximation errors resulted from \( \tilde{\Sigma}_n^c(\lambda) \). By Lemma 8 and elementary calculations, we have

\[
\left\| \sup_{t \in [0,1]} \left| C_f(t)[\Sigma_n^c(\lambda) - \tilde{\Sigma}_n^c(\lambda)]U_n^\text{boots} \right|_{g_n(t)} \psi \right\| \leq \left\| \sup_{t \in [0,1]} \max_{1 \leq j \leq p} | E_j C_f(t)[\Sigma_n^c(\lambda) - \tilde{\Sigma}_n^c(\lambda)]U_n^\text{boots} \right|_{g_n(t)} \psi \right\| \leq Cc \log n \sqrt{n} \sup_{t \in [0,1]} \left\| E_j C_f(t)U_n^\text{boots} \right\|_2 \leq Cc^{d_2 + \psi/2} \log n / \sqrt{n}.
\]

On the other hand, we can also calculate

\[
\left\| \max_{0 \leq t \leq r_n} \sup_{x \in [t_n, t_{n+1}]} \left| C_f(t)[\Sigma_n^c(\lambda)U_n^\text{boots} - C_f(t_n)\Sigma_n^c(\lambda)U_n^\text{boots}] \right|_{g_n(s)} \right\| \leq Cc^{d_2 + \psi/2} \sqrt{\log n / r_n}.
\]

Denote \( \delta_{3,n} = c^{d_2 + \psi/3} \log^{3/2} n / \sqrt{n} \) and \( \delta_{4,n} = c^{d_2 + \psi/2} \log n / r_n \), by the inequality \( \mathbb{P}(A \cap B) \leq \mathbb{P}(A) + \mathbb{P}(B^c) \) for two events \( A \) and \( B \), we can obtain

\[
\mathbb{K}(U_n^\text{boots}, Z_n^c) = \sup_{x \in \mathbb{R}} \sup_{g_n \in \mathcal{G}} \left| \mathbb{P} \left( \sup_{t \in [0,1]} \left| C_f(t)\tilde{\Sigma}_n^c(\lambda)U_n^\text{boots} \right|_{g_n(t)} \leq x | Z_1^n \right) - \mathbb{P} \left( \sup_{t \in [0,1]} \left| C_f(t)\Sigma_n^c(\lambda)Z_n^c \right|_{g_n(t)} \leq x \right) \right| \leq \sup_{x \in \mathbb{R}} \sup_{g_n \in \mathcal{G}} \left| \mathbb{P} \left( \sup_{t \in [0,1]} \left| C_f(t)\Sigma_n^c(\lambda)U_n^\text{boots} \right|_{g_n(t)} \leq x + \delta_{3,n} | Z_1^n \right) \right|
\]

\[
\leq \sup_{x \in \mathbb{R}} \sup_{g_n \in \mathcal{G}} \left| \mathbb{P} \left( \max_{0 \leq t \leq r_n} \left| C_f(t_n)\Sigma_n^c(\lambda)U_n^\text{boots} \right|_{g_n(t)} \leq x + \delta_{3,n} + \delta_{4,n} | Z_1^n \right) \right| - \mathbb{P} \left( \sup_{t \in [0,1]} \left| C_f(t)\Sigma_n^c(\lambda)Z_n^c \right|_{g_n(t)} \leq x \right) \right| \leq \sup_{x \in \mathbb{R}} \sup_{g_n \in \mathcal{G}} \left| \mathbb{P} \left( \max_{0 \leq t \leq r_n} \left| C_f(t_n)\Sigma_n^c(\lambda)U_n^\text{boots} \right|_{g_n(t)} \leq x - \delta_{4,n} | Z_1^n \right) \right| + 3 \log^{-1}(n)
\]

\[
+ \sup_{x \in \mathbb{R}} \sup_{g_n \in \mathcal{G}} \left| \mathbb{P} \left( \sup_{t \in [0,1]} \left| C_f(t)\Sigma_n^c(\lambda)U_n^\text{boots} \right|_{g_n(t)} \leq x | Z_1^n \right) \right| - \mathbb{P} \left( \sup_{t \in [0,1]} \left| C_f(t)\Sigma_n^c(\lambda)Z_n^c \right|_{g_n(t)} \leq x \right) \right| \leq C(\delta_{3,n} + \delta_{4,n}) \sqrt{\log n + 3 \log^{-1}(n)}
\]

\[
+ \sup_{x \in \mathbb{R}} \sup_{g_n \in \mathcal{G}} \left| \mathbb{P} \left( \sup_{t \in [0,1]} \left| C_f(t)\Sigma_n^c(\lambda)U_n^\text{boots} \right|_{g_n(t)} \leq x | Z_1^n \right) \right| - \mathbb{P} \left( \sup_{t \in [0,1]} \left| C_f(t)\Sigma_n^c(\lambda)U_n^c \right|_{g_n(t)} \leq x \right) \right| \leq C(\delta_{3,n} + \delta_{4,n}) \sqrt{\log n + 3 \log^{-1}(n)}
\]

\[
+ \sup_{x \in \mathbb{R}} \sup_{g_n \in \mathcal{G}} \left| \mathbb{P} \left( \sup_{t \in [0,1]} \left| C_f(t)\Sigma_n^c(\lambda)U_n^\text{boots} \right|_{g_n(t)} \leq x | Z_1^n \right) \right| - \mathbb{P} \left( \sup_{t \in [0,1]} \left| C_f(t)\Sigma_n^c(\lambda)Z_n^c \right|_{g_n(t)} \leq x \right) \right| \leq C(\delta_{3,n} + \delta_{4,n}) \sqrt{\log n + 3 \log^{-1}(n)}
\]
= P_1 + P_2 + \mathcal{O} \left( c^{d_2+\psi+5/2} \log^2(n)/\sqrt{n} \right).  \tag{13}

By Gaussian approximation result established in Section B, the second term in Eq. (13) turns out to be

\[ P_2 = \sup_{A \in \mathcal{A}} |\mathbb{P}(Z_n^c \in A) - \mathbb{P}(U_n^c \in A)| \]
\[ \leq Cc^2 n^{-\frac{1}{2} + \frac{2}{8\psi} + \frac{2}{3} + \frac{1}{2(2\psi + d_2 - \psi)}} \log^{3/2}(n). \]

Next, we will use the result in Lemma 3 to derive the error bound for \( P_1 \). For any \( \epsilon_1 > 0 \), we have

\[ P_1 \leq 4c^{1/4}\epsilon_1 + \sup_{A \in \mathcal{A}} \mathbb{E}[h_{A,\epsilon_1}(U_n^{\text{boots}}) - h_{A,\epsilon_1}(U_n^c)] \]
\[ \leq 4c^{1/4}\epsilon_1 + \frac{2}{\epsilon_1} \mathbb{E}[U_n^{\text{boots}} - U_n^c] \]
\[ \leq 4c^{1/4}\epsilon_1 + \frac{C}{\epsilon_1} \left| (\tilde{\Sigma}^c)^{1/2} - (\Sigma^c)^{1/2} \right|_F \]
\[ \leq 4c^{1/4}\epsilon_1 + \frac{C}{\epsilon_1} \left| \tilde{\Sigma}^c - \Sigma^c \right|_F \]
\[ \leq 4c^{1/4}\epsilon_1 + \frac{Cc}{\epsilon_1} n^{-1/3} \ln n, \]

where the third inequality is due to the fact \( \mathbb{E}[U_n^{\text{boots}} - U_n^c] \leq \sqrt{\text{Tr}((\tilde{\Sigma}^c)^{1/2} - (\Sigma^c)^{1/2})^2} = \left| (\tilde{\Sigma}^c)^{1/2} - (\Sigma^c)^{1/2} \right|_F \). The fourth inequality uses the inequality \( |R_1^{1/2} - R_2^{1/2}|_F \leq C|R_1 - R_2|_F \) for any positive definite matrices \( R_1 \) and \( R_2 \) (see Lemma 2.2 in [10] for more details).

By choosing \( \epsilon_1 = \mathcal{O}(c^{3/8} n^{-1/6} \sqrt{\ln n}) \), we are able to derive that

\[ P_1 \leq Cc^{5/8} n^{-1/6} \ln^{1/2}. \tag{14} \]

In conclusion, we combine the three error bounds and obtain the convergence order in (11).

Now, we will construct JSCB in (12) by an under-smoothed estimator \( \tilde{\beta}_j(t) \), which means the uniform convergence rate for the standard deviation term dominates those for bias terms.

Next, we can calculate the \( \mathcal{L}^\infty \) rate of the bias term,

\[ \left| \mathbb{E}[\tilde{\beta}_j(t)] - \beta_j(t) \right|_\infty \]
\[ \leq Cc^{-d_1} + \sup_{t \in [0,1]} \left| \mathbb{E} \left\{ E_j^T C_f(t) \tilde{\Sigma}^{-1}(\lambda) \frac{X^\top \tilde{\epsilon}}{n} \right\} \right| + \sup_{t \in [0,1]} \left| \mathbb{E} \left\{ E_j^T C_f(t) \tilde{\Sigma}^{-1}(\lambda) R(\lambda) \theta_c \right\} \right| \]
\[ \leq C \left( c^{-d_1} + c^{d_2+\psi+3} \log n \right) n + \sup_{t \in [0,1]} \left| E_j^T C_f(t) \Sigma^{-1}(\lambda) R(\lambda) \theta_c \right| \]  \tag{15}
where the second inequality uses Cauchy-Schwarz inequality. Now, we turn to deal with the last term in (15). Notice that $\Sigma_c(\lambda) = \mathbb{E}X_n^\top X_n + R(\lambda)$, we decompose it as $\Sigma_c(\lambda) = D + P$ where $D$ is the diagonal matrix with elements $\{c_0 + \lambda k^{2(\gamma + d_2 + 1)}\}_{k=1}^c$ with $0 < c_0 < \lambda_{\min}(\mathbb{E}X_n^\top X_n/n)$ and $P := \mathbb{E}X_n^\top X_n/n - c_0I_c$ can be viewed as the perturbation matrix with the smallest eigenvalue bounded away from zero. Using the special case of Woodbury matrix identity $(D + P)^{-1} = D^{-1} - D^{-1}(DP^{-1} + I_c)^{-1}$, we can obtain

\[
\sup_{t \in [0,1]} \left| E_j^\top C_f(t)(D + P)^{-1}R(\lambda)\theta_c \right| 
\leq \sup_{t \in [0,1]} \left| E_j^\top C_f(t)D^{-1}R(\lambda)\theta_c \right| + \sup_{t \in [0,1]} \left| E_j^\top C_f(t)D^{-1}(DP^{-1} + I_c)^{-1}R(\lambda)\theta_c \right|
\]

\[
\leq \left[ \lambda - \frac{1}{2(\gamma + d_2 + 1)} \right] \sum_{k=1}^c k^{\psi - (d_1 + 1) + 2(\gamma + d_2 + 1)} + \sum_{k=\left[ \lambda - \frac{1}{2(\gamma + d_2 + 1)} \right]}^c \left( k^{\psi - (d_1 + 1)} \right)
\]

\[
+ \sup_{t \in [0,1]} \left| \text{Tr}((DP^{-1} + I_c)^{-1}R(\lambda)\theta_cE_j^\top C_f(t)D^{-1}) \right|
\]

\[
\leq C\lambda \frac{d_1 - \psi}{2(\gamma + d_2 + 1)} + \left| (DP^{-1} + I_c)^{-1} \right| \sup_{t \in [0,1]} \left| \text{Tr}(E_j^\top C_f(t)D^{-1}R(\lambda)\theta_c) \right|
\]

\[
\leq C\lambda \frac{d_1 - \psi}{2(\gamma + d_2 + 1)},
\]

where the last equality follows by $|\text{Tr}(AB)| \leq \lambda_{\max}(B)|\text{Tr}(A)|$ for positive semi-definite matrices $A, B$. As a result, the $L^\infty$ rate of the bias term turns to be

\[
\left| \mathbb{E}[\tilde{\beta}_j(t)] - \beta_j(t) \right|_\infty \leq C \left( c^{-d_1} + c^{d_2 + \psi + 3} \frac{\log n}{n} + \lambda \frac{d_1 - \psi}{2(\gamma + d_2 + 1)} \right).
\]

On the other hand, note that

\[
\sup_{t \in [0,1]} \left| \tilde{\beta}_j(t) - \mathbb{E}\tilde{\beta}_j(t) \right| = \sup_{t \in [0,1]} \left| E_j^\top C_f(t)\tilde{\Sigma}_c^{-1}(\lambda)Z_n^c/\sqrt{n} \right|
\]

With the Gaussian approximation result constructed in Section D.3, it suffices to derive the lower bound of $\sup_{t \in [0,1]} \left| E_j^\top C_f(t)\Sigma_c^{-1}(\lambda)U_n^c/\sqrt{n} \right|$. Again by the Orlicz norm, we can obtain for any $j = 1, ..., p$,

\[
\left| \sup_{t \in [0,1]} \left| E_j^\top C_f(t)\Sigma_c^{-1}(\lambda)U_n^c/\sqrt{n} \right| \right|_\Psi \geq \left( \max_{0 \leq t \leq r_n} \left| E_j^\top C_f(t)\Sigma_c^{-1}(\lambda)U_n^c/\sqrt{n} \right| \right)_\Psi
\]

\[
\geq C \sqrt{\frac{\log n}{n}} \max_{0 \leq t \leq r_n} \left| E_j^\top C_f(t)\Sigma_c^{-1}(\lambda)U_n^c \right|_\Psi.
\]
\[
\geq \sqrt{\log n} \max_{0 \leq i \leq r_n} \left\{ \text{Tr} \left( \Sigma_c^{-2}(\lambda) C_j^T(t_i,n) E_j E_j^T C_j(t_i,n) \right) \right\}^{1/2}
\]
\[
\geq C \sqrt{\log n} \left\{ \text{Tr} \left( \Sigma_c^{-2}(\lambda) D_j \right) \right\}^{1/2}
\]
\[
\geq C \sqrt{\log n} \left\{ \sum_{k=1}^{c} \rho_k(\Sigma_c^{-2}(\lambda)) \rho_{c-k+1}(D_j) \right\}
\]
\[
\geq \sqrt{\log n} \frac{1}{n} \left[ |\lambda|^{-2(\gamma+d_2+1)} \sum_{k=1}^{c} \lambda^{-2k-2(\gamma+d_2+1)+2(d_2+1)} + \sum_{k=\lambda^{-2(\gamma+d_2+1)}}^{\lambda^{-1}} k^{2(d_2+1)} \right]
\]
\[
\geq C \lambda^{-\frac{2d_2+3}{4(\gamma+d_2+1)}} \sqrt{\log n} \frac{1}{n},
\]

where \(\lambda_k(\cdot)\) denotes the \(k\)th eigenvalue of the matrix and \(D_j = \text{diag}(1/f_{j1}^2, \ldots, 1/f_{jc}^2)\). Now we comment on the above deductions. The first inequality uses chaining technique again and the second inequality follows by the fact \(\| \max_{1 \leq i \leq r_n} X_i \|_\Psi \geq C \Psi^{-1}(r_n) \max_{i} \| X_i \|_\Psi\) for Gaussian random variables \(\{X_i\}_{i=1}^{r_n}\). Due to the assumption \(\lambda_{\min}(\Sigma_c) \geq b > 0\) and the fact \(\text{Tr}(AB) = \text{Tr}(BA)\) for any matrices \(A, B\), the third inequality holds. The fourth inequality follows by \(\sup_{t \in [0,1]} \alpha_k^2(t) \geq \sum_{i=1}^{r_n} \alpha_k^2(t_i,n)/r_n\), the statement \(\text{Tr}(AB) \geq 0\) for the fifth inequality. Finally, the sixth inequality follows by the fact \(\rho_k(\Sigma_c(\lambda)) \leq \max\{C, \lambda k^{2(\gamma+d_2+1)}\}\) and by elementary calculations, we obtain the last inequality.

In summary, we have \(\sup_{t \in [0,1]} |E_j C_j(t)\Sigma_c^{-1}(\lambda) U_n/n|^{1/2} \geq C \lambda^{-\frac{2d_2+3}{4(\gamma+d_2+1)}} \sqrt{\log n} \frac{1}{n} \). By letting the above uniform rate larger than that of each bias term, we need to satisfy the Condition (i) \(c \gg (n/\log(n))^{2(d_1+d_2+\psi)+3}\) and \(\lambda \ll (\log(n)/n)^{2(\gamma+d_2+1)}\). Combing all these conditions, we complete the proof. □

- **Consistency properties of estimators.**

Next, we will show another result on the consistency of the estimated quantities. Here denote

\[
\epsilon_i = Y_i - \sum_{j=1}^{p} \sum_{k=1}^{c_j} \tilde{\theta}_{jk} \tilde{x}_{ij,k},
\]
\[
\hat{f}_{jk} = \frac{1}{n} \sum_{i=1}^{n} \left( \tilde{x}_{ij,k} - \frac{1}{n} \sum_{i=1}^{n} \tilde{x}_{ij,k} \right)^2,
\]

where \(\tilde{x}_{ij,k} = \tilde{x}_{ij,k}/\hat{f}_{jk}\).
Lemma 5. Under Assumptions 1–4 in the paper, we have
\[ \hat{f}_{jk}^2 = f_{jk}^2 \left( 1 + O_p \left( \sqrt{\frac{\log(n)}{n}} \right) \right) \text{ uniformly for } k = 1, \ldots, c_j, j = 1, \ldots, p, \]
\[ \max_{1 \leq i \leq n} \left\| \hat{c}_i - c_i \right\|_q = O \left( cn^{-1/2 + 1/q} \right) \text{ for } q > 2. \]

Proof. Without loss of generality, we assume \( \mathbb{E}(x_{ij,k}) = 0 \) and observe that \( f_{jk}^2 = \mathbb{E}(\tilde{x}_{ij,k}^2) \).

Note that
\[ \hat{f}_{jk}^2 = \frac{1}{n} \sum_{i=1}^{n} \left( \tilde{x}_{ij,k} - \frac{1}{n} \sum_{i=1}^{n} \tilde{x}_{ij,k} \right)^2 = f_{jk}^2 \frac{1}{n} \sum_{i=1}^{n} \left( x_{ij,k} - \frac{1}{n} \sum_{i=1}^{n} x_{ij,k} \right)^2. \]

By elementary calculation,
\[ \hat{f}_{jk}^2 = f_{jk}^2 \left( 1 + \frac{1}{n} \sum_{i=1}^{n}(x_{ij,k}^2 - 1) - \left( \frac{1}{n} \sum_{i=1}^{n} x_{ij,k} \right)^2 \right). \]

Let \( y_{ij,k} = x_{ij,k}^2 - 1 \), denote by \( \delta_y(l, \cdot) \) the physical dependence measure of \( y_{ij,k} \), then we have
\[ \delta_y(l, q/2) = \| y_{ij,k} - y_{ij,k}^* \|_{q/2} \leq \| x_{ij,k} + x_{ij,k}^* \|_{q} \| x_{ij,k} - x_{ij,k}^* \|_{q} \leq C(l + 1)^{-\tau}. \]

Using the Gaussian approximation result for the above partial sum process, we obtain that
\[ \max_{1 \leq j \leq p} \max_{1 \leq k \leq c_j} \left\| \frac{1}{n} \sum_{i=1}^{n} (x_{ij,k}^2 - 1) \right\| = O_p \left( \sqrt{\frac{\log(n)}{n}} \right), \]
\[ \max_{1 \leq j \leq p} \max_{1 \leq k \leq c_j} \left\| \left( \frac{1}{n} \sum_{i=1}^{n} x_{ij,k} \right)^2 \right\| = O_p \left( \frac{\log(n)}{n} \right). \]

Thus, \( \hat{f}_{jk}^2 = f_{jk}^2 \left( 1 + O_p \left( \sqrt{\frac{\log(n)}{n}} \right) \right) \) uniformly in \( j \) and \( k \).

On the other hand, note that \( \hat{e} = Y - \hat{X}_c \hat{\theta}_c \) where \( \hat{X}_c, \hat{\theta}_c \) have similar definitions to \( X_c \) and \( \theta_c \) with \( f_{jk} \) inside replaced by its estimate \( \hat{f}_{jk} \). Let \( E_i \) be an \( n \)-dimensional vector with ith element being 1 and others being 0, with the above result, we have for \( q > 2 \),
\[ \| \hat{e}_i - c_i \|_q \leq C c^{-(d_1 + d_2 + 1)} + \left\| E_i^T (X_c - \hat{X}_c) \theta_c - E_i^T \hat{X}_c \hat{\Sigma}^{-1} (\lambda) \hat{R}(\lambda) \theta_c - \right\| \]
\[ = O \left( c^{-(d_1 + d_2 + 1)} + \sqrt{c} (\log n/n)^{1/4} + \sqrt{c} \lambda^{d_1 + d_2 + 1} \right) = O \left( \frac{c}{\sqrt{n}} \right), \]
where the second inequality follows by the extension of Hölder’s inequality. Using the $L_q$ maximal inequality, it yields that

$$\left\| \max_{1 \leq i \leq n} |\hat{\epsilon}_i - \epsilon_i| \right\|_q = O(cn^{-1/2+1/q}).$$

\[\square\]

**Proof of Theorem 1 of the paper.** Recall the definitions of $\hat{\Xi}$ and $\hat{\Xi}^c$ and denote $\hat{\epsilon}_{jk}$ as the $(j, k)$th element of $\hat{\Xi}^c$. With the uniform consistency of $\hat{\epsilon}_i$ from Lemma 5, one can similarly calculate

$$\left| \hat{\Xi}^c - \hat{\Xi}^c \right|_F = O\left( c^2 n^{-1/2+1/q} \right).$$

Further armed with the result of Lemma 9 and the moving window chosen as $m = O(n^{1/3})$, for $\tau > 5$ and $q$ is large enough, we have $|\hat{\Xi}^c - \Xi^c|_F = O\left( c^2 n^{-1/2+1/q} + cn^{-1/3} \right) = o\left( c^2 n^{-1/2+1/q} h_n + cn^{-1/3} h_n \right)$, then $P(B_n^c) = 1 - o(1)$. Let $\hat{\mathcal{C}}_f(t)$ has similar definition of $C_f(t)$ with the standard deviation $f_{jk}$ replaced by its any estimates, denoted by $\tilde{f}_{jk}$. Furthermore, define the event as

$$\mathcal{D}' = \left\{ (\tilde{f}_{jk}, \Sigma) : \max_{1 \leq k \leq c} |\tilde{f}_{jk}^2 - f_{jk}^2| \leq \left( \frac{\log(n)}{n} \right)^{1/4}, |\Sigma(\lambda) - \Sigma_c(\lambda)| \leq \left( \frac{c \log(n)}{\sqrt{n}} \right)^{1/2} \right\}.$$

Similar to the proof of Theorem 1 in Section D.1 and by elementary calculation, we can derive

$$P\left( (\tilde{f}_{jk}, \Sigma(\lambda)) \notin \mathcal{D}' \right) \leq C \left( \frac{\sqrt{\log(n)}}{n} + c \log(n) e^{-c/\sqrt{n}} \right).$$

Together with the relation $P(A) \leq P(A \cap B) + P(B^c)$, we have

$$\hat{K}\left( \hat{U}^{\text{boots}}_n, Z^c_n \right) \leq \sup_{x \in \mathbb{R}} \sup_{g_n \in \mathcal{G}} P\left( \sup_{t \in [0, 1]} \left| \tilde{\mathcal{C}}_f(t) \Sigma^{-1}(\lambda) \hat{U}^{\text{boots}}_n \right|_{g_n(t)} \leq x \right) - P\left( \sup_{t \in [0, 1]} \left| \tilde{\mathcal{C}}_f(t) \Sigma^{-1}(\lambda) Z^c_n \right|_{g_n(t)} \leq x \right)$$

$$+ P\left( (\tilde{f}_{jk}, \Sigma(\lambda)) \notin \mathcal{D}' \right) \leq \sup_{x \in \mathbb{R}} \sup_{g_n \in \mathcal{G}} P\left( \sup_{t \in [0, 1]} \left| \tilde{\mathcal{C}}_f(t) \Sigma^{-1}(\lambda) \hat{U}^{\text{boots}}_n \right|_{g_n(t)} \leq x \right) - P\left( \sup_{t \in [0, 1]} \left| \tilde{\mathcal{C}}_f(t) \Sigma^{-1}(\lambda) U^c_n \right|_{g_n(t)} \leq x \right)$$

$$+ \sup_{x \in \mathbb{R}} \sup_{g_n \in \mathcal{G}} P\left( \sup_{t \in [0, 1]} \left| \tilde{\mathcal{C}}_f(t) \Sigma^{-1}(\lambda) U^c_n \right|_{g_n(t)} \leq x \right)$$

$$+ P\left( (\tilde{f}_{jk}, \Sigma(\lambda)) \notin \mathcal{D}' \right) =: P_3 + P_4 + O\left( \sqrt{\log(n)/n} + c \log(n) e^{-c/\sqrt{n}} \right).$$
Define a larger convex set

\[ \tilde{A}_n^{\beta_n} = \{ S \in \mathbb{R}^c : \sup_{t \in [0,1]} \max_{1 \leq j \leq p} |E_j^{\top} \tilde{C}_f(t) \Sigma^{-1} \lambda g_n(t) | \leq x \} \]

and similarly consider the collection \( \tilde{A}_n^{\beta_n} = \{ \tilde{A}_n^{\beta_n} : x \in \mathbb{R}, g_n(t) \in \mathcal{G}, (\tilde{f}_{jk}, \Sigma) \in \mathcal{D}_n \} \). Then the second term above will be controlled by Gaussian approximation result of Theorem 1, that is

\[ P_4 \leq C \frac{7}{12} n^{-\frac{1}{2}} + \frac{9}{2} n^{-\frac{1}{2}} + \frac{2}{1} n^{-\frac{1}{2}} \lambda^{\frac{1}{2}} n \log^{3/2}(n). \]

For the first distance, we use Lemma 3 again and obtain

\[ P_3 \leq C \max \left( \frac{9}{12} n^{-\frac{1}{2}} + \frac{1}{2} n^{-\frac{1}{2}} \lambda^{\frac{1}{2}} n \log^{3/2}(n) + \frac{9}{12} n^{-\frac{1}{2}} + \frac{1}{2} n^{-\frac{1}{2}} \lambda^{\frac{1}{2}} n \log^{3/2}(n) + \frac{5}{12} n^{-\frac{1}{2}} + \frac{1}{2} n^{-\frac{1}{2}} \lambda^{\frac{1}{2}} n \log^{3/2}(n) \right) \]

by choosing \( \epsilon_1 = \mathcal{O} \left( \max \left( \frac{9}{12} n^{-\frac{1}{2}} + \frac{1}{2} n^{-\frac{1}{2}} \lambda^{\frac{1}{2}} n \log^{3/2}(n) + \frac{9}{12} n^{-\frac{1}{2}} + \frac{1}{2} n^{-\frac{1}{2}} \lambda^{\frac{1}{2}} n \log^{3/2}(n) + \frac{5}{12} n^{-\frac{1}{2}} + \frac{1}{2} n^{-\frac{1}{2}} \lambda^{\frac{1}{2}} n \log^{3/2}(n) \right) \). In summary, we conclude

\[ \tilde{K}(\tilde{U}_n^{\text{boots}}, Z_n^c) \leq C \left( \frac{9}{12} n^{-\frac{1}{2}} + \frac{1}{2} n^{-\frac{1}{2}} \lambda^{\frac{1}{2}} n \log^{3/2}(n) + \frac{9}{12} n^{-\frac{1}{2}} + \frac{1}{2} n^{-\frac{1}{2}} \lambda^{\frac{1}{2}} n \log^{3/2}(n) + \frac{5}{12} n^{-\frac{1}{2}} + \frac{1}{2} n^{-\frac{1}{2}} \lambda^{\frac{1}{2}} n \log^{3/2}(n) \right). \]

\[ \square \]

### D.4 Proof of theoretical results in Section 3.3 of the paper

To prove Proposition 2 of the paper, here we consider the situation where FPCs are employed as basis functions, denoted by \{\tilde{\alpha}_k(t)\}_{k=1}^p. Now we let \( x_{ij,k} = \langle X_{ij}(t), \tilde{\alpha}_k(t) \rangle \), \( x_{ij,k}^* = x_{ij,k}/\tilde{f}_{jk} \) where \( \tilde{f}_{jk} = \text{Std}(x_{ij,k}) \) and \( \theta_{jk}^* = \beta_{jk}/\tilde{f}_{jk} \).

**Proof of Proposition 2.** Armed with FPCs, we denote the least squares estimator as \( \theta^*_c = [\widehat{X}_c^\top \widehat{X}_c/n + \tilde{R}(\lambda)]^{-1} \widehat{X}_c^\top \tilde{Y} / n \) where \( \widehat{X}_c \) and \( \tilde{R}(\lambda) \) have similar definitions to \( X_c, R(\lambda) \) with empirical FPCs as bases and \( f_{jk} \) to be estimated. Then we obtain

\[ \beta^*(t) = \tilde{C}_f(t) \theta^*_c, \]

where \( \tilde{C}_f(t) \) also has similar definition to \( C_f(t) \) with empirical FPCs and \( f_{jk} \) replaced by its estimate. In consequence, we have

\[ \left| \mathbb{E}[\beta^*(t)] - \beta(t) \right| \]

\[ \leq \left| [\tilde{C}_f(t) - C_f(t)] \theta^*_c \right| + \left| \tilde{C}_f(t) \left( \frac{\widehat{X}_c^\top \widehat{X}_c}{n} + \tilde{R}(\lambda) \right)^{-1} \tilde{R}(\lambda) \theta^*_c \right| \]

\[ + \left| \mathbb{E} \tilde{C}_f(t) \left( \frac{\widehat{X}_c^\top \widehat{X}_c}{n} + \tilde{R}(\lambda) \right)^{-1} \frac{\widehat{X}_c^\top \tilde{e}_n}{n} \right| + \mathcal{O}(c^{-d_1}). \]
The first term of Eq. (16) describes the bias from empirical estimation for eigenfunctions, the second term captures the standard deviation of the estimation and the last term denotes the truncation error in the basis expansion of $\beta(t)$.

First, for any $t, s \in [0, 1]$, let $\Gamma_j(t, s) = \text{cov}(X_{ij}(t), X_{ij}(s))$ be the covariance of $X_{ij}(\cdot)$ and $\Gamma_j(t, s)$ be its sample covariance. These two quantities can be written in the eigen-decomposition (also known as the Karhunen-Loève expansion)

$$
\Gamma_j(t, s) = \mathbb{E}[X_{ij}(t)X_{ij}(s)] = \sum_{k=1}^{\infty} f_{jk}^2 \alpha_k(t) \alpha_k(s),
$$

$$
\hat{\Gamma}_j(t, s) = \frac{1}{n} \sum_{i=1}^{n} X_{ij}(t)X_{ij}(s) = \sum_{k=1}^{\infty} f_{jk}^2 \tilde{\alpha}_k(t) \tilde{\alpha}_k(s),
$$

where the sequences $f_{1j}^2 > f_{2j}^2 > \cdots > f_{nj}^2 > 0$ and $\hat{f}_{1j}^2 \geq \hat{f}_{2j}^2 \geq \cdots$ are the population and sample eigenvalues, $(\alpha_k(t), k \geq 1)$ and $(\tilde{\alpha}_k(t), k \geq 1)$ are the corresponding eigenfunctions. Denote $\tilde{\delta}_k = \min_{1 \leq j \leq p}(f_{jk}^2 - f_{j,k+1}^2)$, with Assumptions 1 and 8 in the paper, we have $\tilde{\delta}_k \geq Ck^{-2(d_2+1)}$ for $k = 1, 2, \cdots$. Next, we need to prove the following statements:

- $\max_{1 \leq j \leq p} \sup_{t, s \in [0, 1]} \left| \hat{\Gamma}_j(t, s) - \Gamma_j(t, s) \right| = \mathcal{O}_p(1/\sqrt{n})$,
- $|\tilde{\alpha}_k(t) - \alpha_k(t)|_{L^2} = \mathcal{O}(k^{2(d_2+1)}/\sqrt{n})$.

Note that

$$
\hat{\Gamma}_j(t, s) - \Gamma_j(t, s) = \frac{1}{n} \sum_{i=1}^{n} \left[ X_{ij}(t)X_{ij}(s) - \mathbb{E}X_{ij}(t)X_{ij}(s) \right]
$$

$$
= - \frac{1}{n} \sum_{i=1}^{n} \left( \sum_{k=1}^{\infty} \sum_{h=1}^{\infty} x_{ij,k}x_{ij,h}f_{jk}f_{jh}\alpha_k(t)\alpha_h(s) - \sum_{k=1}^{\infty} f_{jk}^2 \alpha_k(t) \alpha_k(s) \right)
$$

$$
= - \frac{1}{n} \sum_{i=1}^{n} \left( \sum_{k=1}^{\infty} b_{ij,k}^2 f_{jk}^2 \alpha_k(t) \alpha_k(s) + \sum_{k=1}^{\infty} \sum_{k=h=1, k \neq h}^{\infty} x_{ij,k}x_{ij,h}f_{jk}f_{jh}\alpha_k(t)\alpha_h(s) \right),
$$

where $b_{ij,k} = x_{ij,k}^2 - 1$. Further denote $\tilde{b}_{ij}^{kh} = x_{ij,k}x_{ij,h}$, then we can deduce the corresponding dependence measures as

$$
\delta_b(l, q/2) = \|b_{ij,k} - b_{ij,k}^*\|_{Q/2} \leq \|x_{ij,k} + x_{ij,k}^*\|_{Q} \|x_{ij,k} - x_{ij,k}^*\|_{Q} \leq C(l + 1)^{-\tau},
$$

$$
\delta_b(l, q/2) = \|\tilde{b}_{ij}^{kh} - (\tilde{b}_{ij}^{kh})^*\|_{Q/2} \leq \|x_{ij,k} - x_{ij,k}^*\|_{Q} \|x_{ij,h} + x_{ij,h}^*\|_{Q} \|x_{ij,k} - x_{ij,k}^*\|_{Q} \leq C(l + 1)^{-\tau},
$$

where $x_{ij,k}$ and $x_{ij,k}^*$ are i.i.d. copies of $x_{ij,k}, x_{ij,k}^*$. Furthermore let $B_{jk} = \sum_{i=1}^{n} b_{ij,k}/\sqrt{n}$ and $\tilde{B}_{jk} = \sum_{i=1}^{n} \tilde{b}_{ij}^{kh}/\sqrt{n}$, define the projection operator

$$
P_j(\cdot) = \mathbb{E}(\cdot | \mathcal{F}_j) - \mathbb{E}(\cdot | \mathcal{F}_{j-1}),$$

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with the result of Theorem 3 in [12], we can obtain that for any \( j, k, h \),
\[
\| B_{jk} \|_q \leq C \sum_{l=0}^{\infty} \| \mathcal{P}_0(b_{l,j,k}) \|_q \leq C \sum_{l=0}^{\infty} \| b_{l,j,k} - b_{l,j,k}^* \|_q \leq C,
\]
\[
\| \tilde{B}_{j,k,h} \|_q \leq C \sum_{l=0}^{\infty} \| \mathcal{P}_0(\tilde{b}_{lj}) \|_q \leq C.
\]

Consequently, we can deduce
\[
\left\| \max_{1 \leq j \leq p, t, s \in [0, 1]} \left| \tilde{\Gamma}_j(t, s) - \Gamma_j(t, s) \right| \right\|_q
\leq \sum_{k=1}^{\infty} \sup_{t, s \in [0, 1]} |\alpha_k(t)\alpha_k(s)| f_{jk}^2 \| B_{jk} \|_q / \sqrt{n} + \sum_{k=1}^{\infty} \sup_{t, s \in [0, 1]} |\alpha_k(t)| \sup_{s \in [0, 1]} |\alpha_k(s)| f_{jk} f_{jk} \| \tilde{B}_{j,k,h} \|_q / \sqrt{n}
\leq C / \sqrt{n},
\]
where the last inequality follows by the Assumption \( \sup_{t \in [0, 1]} |\alpha_k(t)| \leq C k^\psi \) for any \( k \geq 1 \) and \( f_{jk} \leq C k^{-(d_2 + 1)} \). Define \( |g(t, s)|_S := \left( \int_0^1 \int_0^1 g^2(t, s) ds dt \right)^{1/2} \) for some function \( g(t, s) \in \mathcal{L}([0, 1]^2) \), by Hall and Horowitz [6, Eq. (5.2)], we conclude for any \( k \),
\[
|\tilde{\alpha}_k(t) - \alpha_k(t)|_{L^2} \leq \tilde{\delta}_k^{-1} \left\| \tilde{\Gamma}_j - \Gamma_j \right\|_S = \mathcal{O} \left( \frac{k^{2(d_2 + 1)}}{\sqrt{n}} \right).
\]

Further by the result of Lemma 5 in Section D.3, we can deduce that
\[
\max_{1 \leq j \leq p, k \geq 1} \left| f_{jk}^2 - f_{jk} \right| = \mathcal{O}(\sqrt{\log(n) / n}).
\]

Followed by the proof of Theorem 3.6 (c) in [7] we have for any \( t \in [0, 1] \),
\[
\tilde{f}_{jk}^2 \tilde{\alpha}_k(t) - f_{jk}^2 \alpha_k(t)
= \int_0^1 \tilde{\Gamma}_j(t, s) \tilde{\alpha}_k(s) ds - \int_0^1 \tilde{\Gamma}_j(t, s) \alpha_k(s) ds
= \int_0^1 \left[ \tilde{\Gamma}_j(t, s) - \Gamma_j(t, s) \right] \alpha_k(s) ds + \int_0^1 \tilde{\Gamma}_j(t, s) \left[ \tilde{\alpha}_k(s) - \alpha_k(s) \right] ds
\]
By the Cauchy–Schwartz inequality, uniformly for all \( t \in [0, 1] \),
\[
\int_0^1 \tilde{\Gamma}_j(t, s) \left[ \tilde{\alpha}_k(s) - \alpha_k(s) \right] ds \leq C |\tilde{\alpha}_k(s) - \alpha_k(s)|_{L^2} \leq C k^{2(d_2 + 1) / \sqrt{n}}.
\]

On the other hand, since \( \max_{1 \leq j \leq p, t, s \in [0, 1]} \left| \tilde{\Gamma}_j(t, s) - \Gamma_j(t, s) \right| = \mathcal{O}(1 / \sqrt{n}) \), then we have
\[
\int_0^1 \left[ \tilde{\Gamma}_j(t, s) - \Gamma_j(t, s) \right] \alpha_k(s) ds \leq \frac{C}{\sqrt{n}}.
\]
In summary, \( |\tilde{f}_{jk}^2 \alpha_k(t) - f_{jk}^2 \alpha_k(t)| = \mathcal{O}_p(k^{2(d+1)}/\sqrt{n}) \). By the triangle inequality and the above results, for any \( 1 \leq j \leq p \) and \( t \in [0,1] \),

\[
\begin{align*}
& f_{jk}^2 |\tilde{\alpha}_k(t) - \alpha_k(t)| \\
\leq & |f_{jk}^2 \tilde{\alpha}_k(t) - f_{jk}^2 \alpha_k(t)| + \sup_{k \geq 1} |f_{jk}^2 - f_{jk}^2| \sup_{t \in [0,1]} |\tilde{\alpha}_k(t)| \\
\leq & Ck^{2(d+1)}/\sqrt{n}.
\end{align*}
\]

For any \( i = 1, \ldots, n, \ j = 1, \ldots, p \), since \( |f_{jk}^2| \geq Ck^{-2(d+1)} \) by Assumption 3 of the main paper, then we have for \( k \geq 1 \),

\[
|\tilde{\alpha}_k(t) - \alpha_k(t)| \leq Ck^{4(d+1)}/\sqrt{n}.
\]  

(18)

Next, we will identify the uniform convergence property for estimator \( \beta^*_j(t) \). With the above \( L^\infty \) convergence rate in (18), the bias resulted from estimated eigenfunctions at the first term of Eq. (16) can be obtained as

\[
\sup_{t \in [0,1]} \left| E^T_j \left[ \tilde{C}_j(t) - C_j(t) \right] \theta_c \right| \leq \sum_{k=1}^{c_j} |\tilde{\alpha}_k(t) - \alpha_k(t)| \leq Ck^{4(d+1)}/\sqrt{n}.
\]

where \( E_j \) is a \( p \)-dimensional vector with \( j \)th element being 1 and others being 0. For the other biases (the second term in Eq. (16) and the first term in Eq. (17)) together with the standard deviation terms, one can derive the same orders as those in the proof of Theorem 1. Hence with Assumptions in Theorem 1 and the extra condition

\[
\lambda^{-2d+3/4} > C^4(d+1)-d_1/\sqrt{n}
\]

holds to guarantee the standard deviation of the estimation dominates, then we finish the statement of Proposition 2.

\[\square\]

D.5 Additional results

In this subsection, we aim to derive the approximation of \( \tilde{\Sigma}_-^{-1}(\lambda) \) to \( \Sigma_c^{-1}(\lambda) \). Recall \( \Sigma_c(\lambda) = X^T_c X_c/n + R(\lambda) \), then we have the following lemma.

**Lemma 6.** Under Assumption 2 in the main paper, we have

\[
|\Sigma_c(\lambda) - \tilde{\Sigma}_c(\lambda)| = \mathcal{O}_p \left( \frac{c\log(n)}{\sqrt{n}} \right).
\]

**Proof.** Rewrite \( \tilde{\Sigma}_c(\lambda) = \frac{1}{n} \sum_{i=1}^{n} x_{ci}^T x_{ci} + R(\lambda) \), where \( x_{ci} \in \mathbb{R}^c \) is the \( i \)th column of \( X_c^T \). This proof mainly uses a Bernstein-type inequality for sums of random matrices to establish the convergence rate. For completeness, we present this inequality in the following lemma (\cite{11}).
Lemma 7. Let \( \{\Xi_i\}_{i=1}^n \) be a finite sequence of independent random matrices with dimensions \( d_1 \times d_2 \). Assume \( \mathbb{E}(\Xi_i) = 0 \) for each \( i \), \( \max_{1 \leq i \leq n} |\Xi_i| \leq R_n \) and define

\[
\sigma_n^2 = \max \left\{ \frac{1}{n} \sum_{i=1}^n \mathbb{E} \left( \Xi_i \Xi_i^\top \right), \frac{1}{n} \sum_{i=1}^n \mathbb{E} \left( \Xi_i^\top \Xi_i \right) \right\}.
\]

Then for all \( t > 0 \),

\[
\mathbb{P} \left( \sum_{i=1}^n |\Xi_i| \geq t \right) \leq (d_1 + d_2) \exp \left( \frac{-t^2/2}{\sigma_n^2 + R_n t/3} \right).
\]

To employ the above lemma, first we introduce the \( m \)-dependent approximation sequence to deal with the issue of independence. To be more specific, we denote \( x^m_{ci} = \mathbb{E}(x_{ci}|\eta_{-m}, \ldots, \eta_i), i = 1, \ldots, n \).

It is easy to find that \( x^m_{ci} \) and \( x^m_{cj} \) will be independent if \( |i - j| > m \). Denote \( \tilde{\Sigma}^m_c(\lambda) := \frac{1}{n} \sum_{i=1}^n x^m_{ci}(x^m_{ci})^\top + \mathbf{R}(\lambda), X_i = \text{vec}(x_{ci}x_{ci}^\top) \) and \( X^m = \text{vec}(x^m_{ci}(x^m_{ci})^\top) \). By Assumption 2 of the paper and the discussion of [13, Remark 2.3], we can derive that \( \Omega_{m,q} = O(c^{1/qm^{-\tau+1}}) \).

Consequently, we have

\[
\mathbb{P} \left( \left| \tilde{\Sigma}_c(\lambda) - \tilde{\Sigma}^m_c(\lambda) \right| \geq t \right) \leq \mathbb{P} \left( \left| c \left| \tilde{\Sigma}_c(\lambda) - \tilde{\Sigma}^m_c(\lambda) \right| \right|_{\max} \geq t \right)
\]

\[
\leq \mathbb{P} \left( \frac{1}{\sqrt{n}} \left| \sum_{i=1}^n (X_i - X^m_i) \right| \geq t \sqrt{n}/c \right)
\]

\[
\leq C \left( \frac{\log(c^2)}{\tau} \right)^{q/2} \left( c(m + 1)^{-q(\tau - 1)} \right) \frac{1}{(t\sqrt{n}/c)^q}. \tag{19}
\]

By choosing \( t\sqrt{n}/c = c^{1/qm^{-\tau+2}} \) as well as \( m \) sufficiently large, armed with Eq. (19), we have \( \mathbb{P} \left( \left| \tilde{\Sigma}_c(\lambda) - \tilde{\Sigma}^m_c(\lambda) \right| \geq t \right) = o(1) \). Furthermore, by Jensen’s inequality, we conclude that

\[
\left| \mathbb{E}(\tilde{\Sigma}_c(\lambda) - \tilde{\Sigma}^m_c(\lambda)) \right| \leq \mathbb{E}\left| \tilde{\Sigma}_c(\lambda) - \tilde{\Sigma}^m_c(\lambda) \right| \leq \frac{Cc^{1+1/qm^{-\tau+2}}}{\sqrt{n}} = o(1).
\]

Hence, we only need to control the \( m \)-dependence approximation sequence. First, we calculate some relative quantities as follows,

\[
R_m = \frac{1}{n} \sup_i \left| x^m_{ci}(x^m_{ci})^\top - \mathbb{E}x^m_{ci}(x^m_{ci})^\top \right|
\]

\[
\leq \frac{1}{n} \sup_i \left| x^m_{ci}(x^m_{ci})^\top \right| \leq \frac{Cc}{n}.
\]

Furthermore, we define \( k_0 = \left\lfloor \frac{m}{n} \right\rfloor \) and the index set sequences for \( i = 1, \ldots, m \) by

\[
\mathcal{I}_i = \begin{cases} 
\{i + km : k = 0, 1, \ldots, k_0\}, & \text{if } i + k_0 m \leq n, \\
\{i + km : k = 0, 1, \ldots, k_0 - 1\}, & \text{otherwise}.
\end{cases}
\]
Then we have $\sigma_m^2 \leq \frac{Cc^2}{mn}$. Consequently, we apply the above bounds to Bernstein-type inequality, i.e.,

$$P \left( \left| \tilde{\Sigma}^m_c (\lambda) - \mathbb{E} \tilde{\Sigma}^m_c (\lambda) \right| \geq t \right) \leq Cm \sup_i \mathbb{P} \left( \left| \frac{1}{n} \sum_{k \in I_i} x_i(x_i^m)^\top - \frac{1}{n} \sum_{k \in I_i} \mathbb{E} x_i(x_i^m)^\top \right| \geq \frac{t}{m} \right) \leq Cmc \exp \left( \frac{-t^2/(2m^2)}{\sigma_m^2 + R_m t / 3m} \right).$$

Now, by choosing $m = \mathcal{O}(\log(n))$, $t = \mathcal{O}(\frac{c \log(n)}{\sqrt{n}})$ and using triangle inequality, we conclude that

$$\left| \tilde{\Sigma}_c(\lambda) - \Sigma_c(\lambda) \right| \leq \left| \tilde{\Sigma}_c(\lambda) - \tilde{\Sigma}^m_c(\lambda) \right| + \left| \tilde{\Sigma}^m_c(\lambda) - \mathbb{E} \tilde{\Sigma}^m_c (\lambda) \right| + \left| \mathbb{E} \tilde{\Sigma}^m_c (\lambda) - \Sigma_c(\lambda) \right| = \mathcal{O}_p \left( \frac{c \log(n)}{\sqrt{n}} \right).$$

Next Lemma provides an approximation to the random term $\tilde{\Sigma}^{-1}_c(\lambda)$.

**Lemma 8.** Under Assumptions 2 and 3 in the main article, we will obtain

$$|\Sigma^{-1}_c(\lambda) - \tilde{\Sigma}^{-1}_c(\lambda)| = \mathcal{O}_p \left( \frac{c \log(n)}{\sqrt{n}} \right).$$

**Proof of Lemma 8.** Armed with the result of Lemma 6, we have

$$\left| \tilde{\Sigma}^{-1}_c(\lambda) - \Sigma^{-1}_c(\lambda) \right| = \left| \tilde{\Sigma}^{-1}_c(\tilde{\Sigma}_c(\lambda) - \Sigma_c(\lambda)) \Sigma^{-1}_c(\lambda) \right| \leq \left| \tilde{\Sigma}^{-1}_c(\lambda) \right| \left| \tilde{\Sigma}_c(\lambda) - \Sigma_c(\lambda) \right| \left| \Sigma^{-1}_c(\lambda) \right| \leq C \left( \frac{c \log(n)}{\sqrt{n}} \right).$$

**Lemma 9.** Suppose Assumptions 2 and 4 in the paper hold with $q = 4$ and $m \to \infty$ with $m/n \to 0$. Then we have

$$\left\| \tilde{\Xi}^{c}_{jk} - \Xi^{c}_{jk} \right\|_2 = \mathcal{O} \left( \frac{1}{m} + \sqrt{\frac{m}{n}} \right),$$

where $\tilde{\Xi}^{c}_{jk}$ and $\Xi^{c}_{jk}$ are $(j,k)$th elements of $\tilde{\Xi}^{c}$ and $\Xi^{c}$, respectively.
To prove the above lemma, we will follow the proof strategy of [14][Theorem 3] and following lemmas are needed. Denote $S_{i,m} = \sum_{j=i}^{i+m-1} z_{cj}$ and its $k$th entry by $S_{i,k,m}$. Recall
\[ \tilde{\Xi}^r = \frac{1}{(n - m + 1)m} \sum_{i=1}^{n-m+1} S_{i,m} S_{i,m}^T, \]
for $k \in \mathbb{Z}$, define the projection operator
\[ \mathcal{P}_k = \mathbb{E}(\cdot | \mathcal{F}_k) - \mathbb{E}(\cdot | \mathcal{F}_{k-1}). \]

**Lemma 10.** Under conditions of Lemma 9, for any $1 \leq j, k \leq c$, we have $\|\tilde{\Xi}_{jk} - \mathbb{E}[\tilde{\Xi}_{jk}]\|_2 = \mathcal{O}(\sqrt{m/n})$.

**Proof.** Since $S_{i,k,m}$ is $\mathcal{F}_{i+m-1}$ measurable, it can be written as $f(\mathcal{F}_{i+m-1})$. For $j,l \in \mathbb{Z}$, define
\[ S_{i,k,m}^l = f(\mathcal{F}_{i+m-1,(i-l)}), \quad \mathcal{F}_{j,(i-l)} = (\cdots, \eta_{i-l-1}, \eta'_{i-l}, \eta_{i-l+1}, \cdots, \eta_j), \]
where $\mathcal{F}_{j,(i-l)}$ is obtained by replacing $\eta_{i-j}$ in $\mathcal{F}_j$ by an i.i.d. copy $\eta'_{i-l}$ and $\mathcal{F}_{j,(i-l)} = \mathcal{F}_j$ if $i-l > j$. By [14][Lemma 6(i)], Assumptions 1 and 4 with $q = 4$, we have $\sup_{i,k} \|S_{i,k,m}^1\|_4 = \mathcal{O}(\sqrt{m})$. Since
\[ \|S_{i,k,m} - S_{i,k,m}^l\|_4 \leq \sum_{j=l}^{l+m-1} \delta_z(j,4), \]
we have
\[ \|S_{i,j,m} S_{i,k,m} - S_{i,j,m} S_{i,k,m}^l\|_2 \leq \|S_{i,j,m}\|_4 \|S_{i,k,m} - S_{i,k,m}^l\|_4 + \|S_{i,j,m} - S_{i,j,m}^l\|_4 \|S_{i,k,m}^l\|_4 = \mathcal{O}(\sqrt{m}) \sum_{i=l}^{l+m-1} \delta_z(i,4). \]

By [12][Theorem 1], $\|\mathcal{P}_{i-l}(S_{i,j,m} S_{i,k,m})\|_2 \leq \|S_{i,j,m} S_{i,k,m} - S_{i,j,m} S_{i,k,m}^l\|_2$. Further denote
\[ \Psi_{jk}^l = \frac{1}{(n - m + 1)m} \sum_{i=1}^{n-m+1} \mathcal{P}_{i-l}(S_{i,j,m} S_{i,k,m}). \]
Note that $\mathcal{P}_{i-l}(S_{i,j,m} S_{i,k,m})$ for $1 \leq i \leq n - m + 1$ are martingale differences, then we have $\|\Psi_{jk}\|_2^2 = \mathcal{O} \left( \frac{1}{m(n-m+1)} \left( \sum_{i=l}^{l+m-1} \delta_z(i,4) \right)^2 \right)$. Since $\tilde{\Xi}_{jk} - \mathbb{E}[\tilde{\Xi}_{jk}] = \sum_{i=0}^{\infty} \Psi_{jk}$ and $\sum_{i=0}^{\infty} \delta_z(i,4) < \infty$, we have $\|\tilde{\Xi}_{jk} - \mathbb{E}[\tilde{\Xi}_{jk}]\|_2 = \mathcal{O}(\sqrt{m/n})$. \hfill \qed

**Lemma 11.** Under conditions of Lemma 9, for any $1 \leq j, k \leq c$, we have
\[ |\mathbb{E}[\tilde{\Xi}_{jk}] - \Xi_{jk}| = \mathcal{O}(1/m). \]
Proof. Notice that $z_{ci,j} = H_j(F_i)$ is a stationary time series. Let $\Gamma_i(l)$ be its $l$th autocovariance, then we obtain that for any $1 \leq i \leq n - m + 1$, $|\Gamma_i(l)| \leq C(l + 1)^{-\tau}$ by Assumption 2 and 4 of the paper. Therefore,

\[
|E[\xi_j] - \eta_j| = \left| \frac{1}{(n - m + 1)m} \sum_{i=1}^{n-1} \left( E(S_{i,j,m}S_{i,k,m}) - \frac{m}{n} E\left( \sum_{i=1}^{n} z_{ci,j}\right) \left( \sum_{i=1}^{n} z_{ci,k}\right) \right) \right| \\
\leq \frac{2}{m} \max_{1 \leq i \leq n - m + 1} \left\{ \sum_{l=0}^{m-1} j |\Gamma_i(l)| + m \sum_{l \geq m} |\Gamma_i(l)| \right\} = O(1/m).
\]

Proof of Lemma 9. Combining the results of Lemma 10 and Lemma 11, we complete the proof.

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