CONSTRUCTION OF MASAS
ALMOST ORTHOGONAL TO A GIVEN SUBALGEBRA

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Abstract. Given an arbitrary countably generated $\text{II}_1$ factor $M$, an irreducible subfactor with infinite index $Q \subset M$ and $\varepsilon > 0$, we construct an irreducible hyperfinite subfactor $R \subset M$ such that $R \perp_{\varepsilon} Q$. We derive from this the existence of singular and semiregular MASAs $A \subset M$ such that $A \perp_{\varepsilon} Q$.

1. Introduction

It has been shown in [P10] that given any irreducible inclusion of $\text{II}_1$ factors with infinite Jones index, $Q \subset M$, any “finite structure” that can be simulated in $M$ can also be simulated in $M \ominus Q$. Using ultrapower formalism, this amounts to the following: given any countably generated subalgebra $B \subset M^\omega$ there exists a unitary element $u \in M^\omega$ such that $uBu^* \perp Q^\omega$ (orthogonality in the sense of [P1]). It shows in particular that $M$ contains copies of any finite dimensional algebra $B$ that are almost orthogonal to $Q$. Thus, one can find inside $M$ copies of the left regular representation of any finite group $G$ that are arbitrarily orthogonal to $Q$.

On the other hand, an iterative technique for constructing embeddings of approximately finite dimensional (AFD) von Neumann algebras $B$ into $\text{II}_1$ factors $M$, with various required properties being satisfied, has been developed in ([P1, 3, 9]). This method allowed for instance to show that any irreducible subfactor $P$ of $M$ contains copies of the hyperfinite $\text{II}_1$ factor $R$ that are irreducible in $M$ ([P1]), as well as copies of the diffuse von Neumann algebra $L^\infty([0,1])$ that are maximal abelian in $M$ ([P1], [P3]).

In this paper we combine these two techniques to show that, given any irreducible inclusion of $\text{II}_1$ factors with infinite index, $Q \subset M$, any tracial AFD algebra $B$ can
be embedded into $M$ so that to be almost perpendicular to $Q$, while at the same time satisfying various prescribed properties (such as being contained in an irreducible subfactor $P \subset M$, have trivial centralizer, etc). More precisely, we prove:

**1.1. Theorem.** Let $M$ be a countably generated II$_1$ factor, $Q \subset M$ a von Neumann subalgebra and $P \subset M$ an irreducible subfactor satisfying $P \not \prec M Q$. Given any $\varepsilon > 0$, there exists a hyperfinite subfactor $R \subset P$ such that $R^\prime \cap M = \mathbb{C}1$ and $R \perp \varepsilon Q$.

The condition $P \not \prec M Q$ for two subalgebras of the II$_1$ factor $M$ is in the sense of (Definition 2.4 in [P6]) and requires that the Hilbert bimodule $pL^2M_Q$ contains no sub-bimodule $pH_Q$ with $\dim(H_Q) < \infty$. This condition automatically implies that $Q$ has uniform infinite index in $M$, i.e., given any non-zero projection $p \in Q^\prime \cap M$, the [PP]-index of the inclusion $Qp \subset pMp$ is infinite. When $Q$ is an irreducible subfactor of $M$, this simply means that $Q \subset M$ has infinite Jones index [J].

Since by ([P1,3]) any irreducible subfactor $R \subset M$ contains a maximal abelian *-subalgebra (MASA) of $M$ that’s singular (respectively semiregular) in $M$, the above result implies existence of singular (resp. semiregular) MASAs of $M$ that are almost orthogonal to $Q$. Also, since any tracial AFD algebra can be embedded into $R$ and any countable amenable group $G$ gives rise to a tracial AFD von Neumann algebra ([C]), the above result implies existence of copies of the left regular representation of $G$ that are almost orthogonal to $Q$.

**1.2. Corollary.** With same hypothesis and notations as in the above theorem, given any $\varepsilon > 0$ we have:

1° There exists a singular (respectively a semiregular) MASA $A$ of $M$ which is contained in $P$ and satisfies $A \perp \varepsilon Q$.

2° If $G$ is a countable amenable group, then there exists a copy $\{u_g\}_{g \in G} \subset P$ of the left regular representation of $G$ such that $\|E_Q(u_g)\|_2 \leq \varepsilon$, $\forall g \in G \setminus \{e\}$.

The $\varepsilon$-orthogonality between subalgebras in these statements is with respect to the Hilbert structure given by the (unique) trace state $\tau$ on the ambient factor $M$. Thus, $B \perp \varepsilon Q$ for subalgebras $B, Q \subset M$ means that $\|E_Q(b)\|_2 \leq \varepsilon\|b\|_2$ for all $b \in B \otimes \mathbb{C} \overset{def}{=} \{b \in B \mid \tau(b) = 0\}$, where as usual $E_Q$ denotes the trace preserving expectation onto $Q$. We will in fact prove $\varepsilon$-perpendicularity in a stronger sense, with the hyperfinite II$_1$ factor $R \subset P$ in the above theorem constructed so that $\|E_Q(b)\|_q \leq \varepsilon\|b\|_q$ for all $b \in R \otimes \mathbb{C}1$ and all $1 \leq q \leq 2$ (see Theorem 4.2). This is equivalent to the condition $\|E_R(x)\|_p \leq \varepsilon\|x\|_p$, $\forall x \in Q \otimes \mathbb{C}1$, $\forall 2 \leq p \leq \infty$, so in particular $R$ satisfies $\|E_R(x)\| \leq \varepsilon\|x\|$, $\forall x \in Q \otimes \mathbb{C}1$. It is an open question whether $R$ can be constructed so that $\|E_Q(b)\| \leq \varepsilon\|b\|$, $\forall b \in R \otimes \mathbb{C}1$, as well.
2. SOME PRELIMINARIES

We use in this paper the same notations as in ([P7,8,9]) and refer to [AP] for basics in II₁ factors theory. One notation that’s frequently used is \((\mathcal{B})_r\) for the ball of radius \(r > 0\) of a given Banach space \(\mathcal{B}\) (the space and its norm being clear from the context).

We will often use the ultrapower formalism: If \(M\) is a II₁ factor and \(\omega\) is a free (or non-principal) ultrafilter on \(\mathbb{N}\), then \(M^\omega\) denotes its \(\omega\)-ultrapower II₁ factor (see e.g. [C], [AP] or 1.6 in [P7] for complete definitions). Thus, \(M^\omega\) is endowed with the ultrapower trace \(\tau((x_n)_n) = \lim_{n \to \omega} \tau(x_n), \forall (x_n)_n \in M^\omega\).

Also, if \(N\) is a von Neumann subalgebra of the II₁ factor \(M\), then we denote by \(e_N\) the orthogonal projection of \(L^2 M\) onto \(L^2 N\). Thus, if \(x \in M\) is viewed as the operator of left multiplication on \(L^2 M\), then \(e_Nxe_N = E_N(x)e_N\),forall \(x \in M\), implying that \(\text{sp} MeN M\) is a *-subalgebra in \(\mathcal{B}(L^2 M)\). Following Jones notations and terminology from ([J]), we denote by \(\langle M, e_N \rangle \subset \mathcal{B}(L^2 M)\) the basic construction algebra \(\text{sp} MeN M\) = \((JM\mathcal{N}J_M)'\), where \(J_M\) is the canonical conjugation on \(L^2 M\), \(J(\xi) = \xi^*\), \(\forall \xi \in L^2 M\). We have \(e_N(M, e_N)e_N = Ne_N\) and the semi-finite von Neumann algebra \(\langle M, e_N \rangle\) is endowed with the canonical normal faithful semi-finite trace \(Tr = Tr_{\langle M, e_N \rangle}\), satisfying the condition \(Tr(xy) = \tau(xy), \forall x, y \in M\).

When \(N \subset M\) is a von Neumann subalgebra and we consider the corresponding inclusion of ultrapower algebras \(N^\omega \subset M^\omega\), then it is useful to keep in mind that the canonical trace \(Tr\) of elements in the basic construction algebra \(\langle M^\omega, e_N^\omega \rangle\) is obtained as a limit of the trace \(Tr\) of elements in \(\langle M, e_N \rangle\):

2.1. Lemma. If \(x = (x_n)_n, y = (y_n)_n \in M^\omega\) then \(e_{N^\omega}xe_{N^\omega} = E_{N^\omega}(x)e_{N^\omega} = (E_N(x_n))_n e_{N^\omega}\) and

\[
Tr(xe_{N^\omega}ye_{N^\omega}) = \lim_{n \to \omega} Tr(xne_Nyn e_N) = \lim_{n \to \omega} \tau(x_n E_N(y_n)).
\]

Proof. Immediate by the definitions. □

As we mentioned in the introduction, if \(Q, P\) are von Neumann subalgebras of the II₁ factor \(M\), then the notation \(P \not\prec_M Q\) means that the Hilbert \(P - Q\) bimodule \(\rho L^2 Q\) does not contain any (non-zero) Hilbert sub-bimodule \(\rho \mathcal{H}_Q\) with \(\text{dim} (\mathcal{H}_P) < \infty\). It is equivalent to the condition: \(\forall F \subset M\) finite, \(\forall \varepsilon > 0, \exists u \in \mathcal{U}(P)\) such that \(\|E_Q(xuy)\|_2 \leq \varepsilon, \forall x, y \in F\) (see 2.1-2.4 in [P5] for all this; see also Section 1.3 in [P8]). This last condition readily implies that if \(P \not\prec_M Q\) then \(P^\omega \not\prec_{M^\omega} Q^\omega\) (see e.g., 2.1 in [P9]; N.B. the converse holds true as well).

It is trivial to see, by using the definitions, that if \(P \not\prec_M Q\) and \(B_0 \subset P\) is a finite dimensional *-subalgebra, then \((B_0' \cap P) \not\prec_M Q\) as well. When passing to ultra powers, this entails:
2.2. Lemma. With the above notations, if $B \subset P^\omega$ is a separable AFD von Neumann subalgebra, then $P \not\prec_M Q$ implies $(B' \cap P^\omega) \not\prec_M Q^\omega$.

Proof. Let $B_n \subset R$ be an increasing sequence of finite dimensional subalgebras that generates $B$. Let $F \subset M^\omega$ be a finite set and $\varepsilon > 0$.

Since $B_n \cap P^\omega \not\prec_M Q^\omega$, there exists a unitary element $u_n = (u_{n,k}) \in P^\omega$ such that $\|E_Q \cdot (xu_ny)\|_2 \leq \varepsilon/2$, for all $x, y \in F$. Thus, if $x = (x_k)_k, y = (y_k)_k$, $u_n = (u_{n,k})_k$, with $x_k, y_k \in (M)_1$ and $u_{n,k} \in U(P)$, then

$$\lim_{k \to \omega} \|E_Q(x_ku_nky_k)\|_2 \leq \varepsilon/2 < \varepsilon.$$ 

Denote by $V_n$ the set of all $k \in N$ such that $\|E_Q(x_ku_nky_k)\|_2 < \varepsilon$ for all $x, y \in F$. Note that $V_n$ corresponds to an open closed neighborhood of $\omega$ in $\Omega$, under the identification $\ell^\infty \mathbb{N} = C(\Omega)$. Let now $W_n \subset N$, $n \geq 0$, be defined recursively as follows: $W_0 = N$ and $W_{n+1} = W_n \cap V_{n+1} \cap \{k \in N \mid k > \min W_n\}$. Note that, with the same identification as before, $W_n$ is a strictly decreasing sequence of neighborhoods of $\omega$ in $\Omega$.

Define $u = (u'_m)_m$, where $u'_k = u_{m,k}$ for $k \in W_{m-1} \setminus W_m$. Then the above conditions show that $u$ is a unitary element in $P^\omega$ which satisfies $\|E_Q \cdot (xuy)\|_2 \leq \varepsilon$, $\forall x, y \in F$. □

Recall from [P6,7] that if $B_0, B \subset M$ are von Neumann algebras and $E \subset M \oplus B_0 \overset{def}{=} \{x \in M \mid E_{B_0}(x) = 0\}$ is a subset, then $B$ is $n$-independent to $E$ relative to $B_0$, if the expectation on $B_0$ of any word with alternating letters from $B_0, B \oplus \mathbb{C}$ and length at most $2n$ (so at most $n$ alternations) is equal to $0$.

2.3. Lemma. Assume $P \subset M$ is an irreducible inclusion of II$_1$ factors and $P_0 \subset P^\omega$ a finite dimensional factor. Given any finite set $E \subset M^\omega \subset P_0$ there exists a diffuse abelian subalgebra $A \subset P'_0 \cap P^\omega$ such that $A$ is free independent to $E$ relative to $P_0$. In particular, $A$ is 2-independent to $E$ relative to $P_0$ and thus, for any $1 \geq c > 0$ there exists a projection $q \in P'_0 \cap P^\omega$ such that $E_{P_0}(qz) = 0$ and $E_{P_0}(qzq^*) = \tau(q)^2 E_{P_0}(zz^*) = c^2 E_{P_0}(zz^*)$, $\forall z \in E$.

Proof. This is just a particular case of (Lemma 1.4 in [P4]) or (Theorem 4.3 in [P7]). □

Recall now that if $y \in M$ and $1 \leq p < \infty$, then one denotes $\|y\|_p = \tau(|y|^p)^{1/p}$. For a fixed $y$, the $L^p$-norms $\|y\|_p$ are increasing in $p$, with the limit $\lim_{p \to \infty} \|y\|_p$ equal to the operator norm $\|y\|$, which we also view as $\|y\|_\infty$. The completion $L^p(M)$ of $M$ in the norm $\|\cdot\|_p$ identifies naturally with the space of densely defined closed operators $Y$ on $L^2 M$ that are affiliated with $M$ and have the property that $|Y|$ has spectral decomposition $|Y| = \int \lambda d\tau(e_\lambda)$ satisfying $\int \lambda^p d\tau(e_\lambda) < \infty$. 

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It is well known that if \(1 \leq p < \infty\) then \((L^pM)^* \simeq L^qM\), where \(q = \frac{p}{p-1}\) (with the usual convention \(1/0 = \infty\)), the duality being given by \((\xi, \zeta) \mapsto \tau(\zeta \xi)\) for \(\xi \in L^pM, \zeta \in L^qM\), viewed as operators affiliated with \(M\). This also shows that if \(y \in M\) and \(1 \leq p, q \leq \infty\) with \(\frac{1}{p} + \frac{1}{q} = 1\), then \(\|y\|_p = \sup\{|\tau(yz)| : z \in (L^qM)_1\} \) and \(\|y\|_q = \sup\{|\tau(zy)| : y \in (L^pM)_1\}\).

It is also useful to recall that if \(x \in M \simeq M_{n \times n}(\mathbb{C})\) and \(1 \leq p \leq p' \leq \infty\) then
\[
\|x\|_{p'} \leq n^{\frac{1}{p} - \frac{1}{p'}} \|x\|_p \leq n\|x\|.
\]

If \(Q \subset M\) is a von Neumann algebra, then \(\tau(xy) = \tau(xE_Q(y))\), for all \(x \in Q, y \in M\). So the above formula for calculating \(\|\|_p\) shows that \(\|E_Q(y)\|_p \leq \|y\|_p\), \(\forall 1 \leq p \leq \infty\).

2.4. Notation. Let \(B, Q\) be von Neumann subalgebras of the II\(_1\) factor \(M\) and \(1 \leq p \leq \infty\). We denote \(c_p(B, Q) = \sup\{|\tau(bx)| : b \in B \ominus C, \|b\|_p \leq 1, x \in Q, \|x\|_q \leq 1\}\), where \(q = \frac{p}{p-1}\). We’ll also use the related constant \(c'_p(B, Q) = \sup\{|\tau(bx)| : b \in B \ominus C, \|b\|'_p \leq 1, x \in Q \ominus C, \|x\|_q \leq 1\}\), which clearly satisfies \(c'_p(B, Q) = c'_p(Q, B)\).

2.5. Lemma. With the above notations, we have

1° \(c'_p(B, Q) \leq c_p(B, Q) \leq 2c'_p(B, Q) = 2c'_p(Q, B) \leq 2c_p(Q, B)\).

2° If \(B \simeq M_{n \times n}(\mathbb{C})\) and \(1 \leq p \leq p' \leq \infty\), then \(c'_p(B, Q) \leq n^{\frac{1}{p} - \frac{1}{p'}} c'_p(Q, B)\).

3° If \(B_n \subset M\) is an increasing sequence of von Neumann algebras and \(B = \bigcup_n B_n\), then \(\lim_{n \to \infty} c_p(B_n, Q) = c_p(B, Q)\).

Proof. 1° If \(b \in B \ominus C\) and \(x \in Q\), then \(\tau(bx) = \tau(b(x - \tau(x)1))\) and \(\|x - \tau(x)\|_q \leq 2\|x\|_q\). Thus, if \(x \in (L^qQ)_1\) then \(\|x - \tau(x)1\|_q \leq 2\|x\|_q \leq 2\|x\|_2\) and so we have
\[
c_p(B, Q) = \sup\{|\tau(bx)| : b \in (L^pB \ominus C)_1, x \in (L^qQ)_1\} \leq \sup\{|\tau(by)| : b \in (L^pB \ominus C)_1, y \in (L^qQ \ominus C)_2\} = 2\sup\{|\tau(by)| : b \in (L^pB \ominus C)_1, y \in (L^qQ \ominus C)_1\} = 2c'_p(Q, B) = 2c'_p(Q, B).
\]

2° Let \(q = \frac{p}{p-1}\) and \(q' = \frac{p}{p'-1}\) and note that \(1 \leq q' \leq q \leq \infty\). Since the unit ball of \(L^pQ\) is included in the unit ball of \(L^pQ\), the until ball of \(L^qB\) is included in the ball of radius \(n^{\frac{1}{q'} - \frac{1}{q}}\) of \(L^qB\) and we have \(\frac{1}{q} - \frac{1}{q'} = \frac{1}{p} - \frac{1}{p'}\), it follows that
\[
c'_p(Q, B) = \sup\{|\tau(by)| : y \in (L^{p'}Q \ominus C)_1, b \in (L^qB)_1\} \leq n^{\frac{1}{q'} - \frac{1}{q'}} \sup\{|\tau(by)| : y \in (L^{p'}Q \ominus C)_1, b \in (L^qB)_1\} = n^{\frac{1}{p} - \frac{1}{p'}} c_p(Q, B).
\]

3° is straightforward and we leave its proof as an exercise. □
3. A technical lemma

In this section we prove a key technical result needed in the proof of Theorem 1.1. Its proof uses the incremental patching technique, in a manner similar to ([P4,6,7,9]).

3.1. Lemma. Let $M$ be a II$_1$ factor, $Q \subset M$ a von Neumann subalgebra, $P \subset M$ an irreducible subfactor such that $P \not\prec_M Q$ and $P_0 \subset P$ a finite dimensional subfactor. Given any finite set $F = F^* \subset (M \ominus P_0)_1$ and any $\delta_0 > 0$, there exists a unitary element $v_0 \in P_0' \cap P$ such that

$$\|E_Q(v_0xv^*_0)\|^2_2 \leq \delta_0, \forall x \in F;$$

$$|\tau(v_0x_1v^*_0x_2v_0x_3v^*_0x_4)| \leq \delta_0, \forall x_i \in F.$$

Proof. Let $\omega$ be a non-principal ultrafilter on $\mathbb{N}$ and denote by $M = \langle M^\omega, e_{Q^\omega} \rangle$ the semi finite von Neumann algebra associated with the basic construction for $Q^\omega \subset M^\omega$. Thus, $M = \text{sp}M^\omega e_{M^\omega} \subset \mathcal{B}(L^2M^\omega)$, where $e = e_{Q^\omega} \in \mathcal{B}(L^2M^\omega)$.

Fix $\delta > 0$ such that $\delta < \delta_0$. Denote by $\mathcal{W}$ the set of partial isometries $v \in P_0' \cap P^\omega = (P_0' \cap P)^\omega$ with the property that $vv^* = v^*v$ and which satisfy the conditions:

(a) $\|E_{Q^\omega}(vxv^*)\|^2_2 \leq \delta \tau(v^*v),$

(b) $|\tau(vx_1v^*x_2vx_3v^*x_4)| \leq \delta \tau(v^*v),$

(c) $E_{P_0}(vv^*F) = 0,$

(d) $E_{P_0}(FvF) = 0,$

(e) $E_{P_0}(vv^*FvF) = 0.$

We endow $\mathcal{W}$ with the order $\leq$ in which $v_1 \leq v_2$ if $v_1 = v_2v_1^*v_1$. $(\mathcal{W}, \leq)$ is then clearly inductively ordered and we let $v \in \mathcal{W}$ be a maximal element.

Assume $\tau(v^*v) < 1$ and denote $p = 1 - v^*v$. Notice that $p \in P_0' \cap P^\omega$ and $E_{P_0}(pF) = 0$. Since the uniqueness of trace preserving expectation onto $P_0$ implies that for a unitary element $u \in P_0' \cap M^\omega$ and $y \in M^\omega$ we have $E_{P_0}(y) = u^*E_{P_0}(uyu^*)u = E_{P_0}(uyu^*)$, it follows that for any $x \in F$ and $u \in \mathcal{U}(P_0' \cap P^\omega)$ we
have \( E_{P_0}(upx) = E_{P_0}(u(px)u^*) = E_{P_0}(pxu) \). By writing \( p \) as a linear combinations between \( u = 1 \) and \( u = 2p - 1 \) this implies that \( E_{P_0}(pF) = E_{P_0}(pFp) \) and thus \( E_{P_0}(pFp) = 0 \) as well.

Let \( w \) be a partial isometry in \( p(P_0' \cap P\omega)p \) with \( w^*w = ww^* \) and denote \( u = v + w \). Then \( u \) is a partial isometry in \( P_0' \cap P\omega \) with \( u^*u = uu^* \in P_0' \cap P\omega \). We will show that one can make an appropriate choice \( w \neq 0 \) such that \( u = v + w \) lies in \( \mathcal{M} \). This will contradict the maximality of \( v \), thus showing that \( v \) must be a unitary element. We will construct the partial isometry \( w \) by first choosing its support \( q = ww^* = w^*w \), then choosing the “phase \( w \)” above \( q \).

In order to get estimates on \( (a) \), note that by writing \( eux^*ueyux^* \) as \( e(v + w)x^*(v + w)^*e(v + w)x(v + w)^* \) and developing into the sum of 16 terms, we get

\[
(1a) \quad \|E_{Q\omega}(uxu^*)\|_2^2 = Tr(eux^*ueyux^*) \leq Tr(evx^*v^*evxv^*) + \Sigma_{1,a} + \Sigma_{2,a} + \Sigma_{3,a} + \Sigma_{4,a},
\]

where \( \Sigma_{i,a} \) denotes the sum of the absolute value of terms having \( i \) appearances of elements from \( \{w, w^*\} \), \( 1 \leq i \leq 4 \). Thus, there are four terms in \( \Sigma_{1,a} \), six in \( \Sigma_{2}(a) \), four in \( \Sigma_{3}(a) \), and one in \( \Sigma_{4}(a) \).

Similarly, in order to estimate \( (b) \), by developing \( \tau(ux_1u^*x_2ux_3u^*x_4) = \tau((v + w)x_1(v + w)^*x_2(v + w)x_3(v + w)^*x_4) \) into a sum of 16 terms we get

\[
(1b) \quad |\tau(ux_1u^*x_2ux_3u^*x_4)| \leq |\tau(vx_1v^*x_2vx_3v^*x_4)| + \sum_{i=1}^{4} \Sigma_{i,b}
\]

where \( \Sigma_{i,b} \) denotes the sum of absolute value of the terms \( \tau(y) \) with \( y \) having \( i \) appearances of elements from \( \{w, w^*\} \), \( 1 \leq i \leq 4 \).

Let us first take care of the terms \( |Tr(X)| \) in \( (1a) \) with \( X \) containing a pattern of the form \( ...ewxw^*e... \), or \( ...ewx^*w^*e... \), for a given \( x \in F \). There are seven such terms: the one in \( \Sigma_{4,a} \), all four in \( \Sigma_{3,a} \), and two in \( \Sigma_{2,a} \). We denote by \( \Sigma'_a \) the sum of these terms. Note that for each such \( X \) we have \( |Tr(X)| = |Tr(wxw^*ez)| \) for some \( z \in (M^\omega)_1 \). Thus, by applying the Cauchy-Schwartz inequality and taking into account the definition of \( Tr \), we get the estimate

\[
(2a) \quad |Tr(X)| = |Tr(....ewxw^*e...)| = |Tr(wxw^*ey)| \leq (Tr(ewx^*w^*wxw^*e))^{1/2}(Tr(qey^*yeq))^{1/2} \leq \|qx\|_2\|q\|_2,
\]

where the last inequality is due to the fact that \( Tr(qey^*yeq) \leq Tr(qeq) = \tau(q) \) and \( Tr(ewx^*w^*wxw^*e) = Tr(ewx^*qwx^*e) = \tau(wx^*qwx) = \tau(qx^*qeq) \).
Similarly, the seven terms $|\tau(y)|$ in (1b) with $y$ containing a pattern of the form $...w_{j}w^*...$, or $...w^*x_{i}w...$ (namely, the one in $\Sigma_{4,b}$, all four in $\Sigma_{3,b}$ and two in $\Sigma_{2,b}$) are majorized by

$$|\tau(y)| \leq \|qx_{j}q\|_{2} \|q\|_{2}.$$  \hfill (2b)

In addition, since $pFvFp \perp P_{0}p$, for the remaining two terms $y = wx_{1}v^*x_{2}wx_{3}v^*x_{4}$, $y = vx_{1}w^*x_{2}vx_{3}w^*x_{4}$ in $\Sigma_{2,b}$, we have

$$|\tau(y)| \leq \|q(x_{1}vx_{2})q\|_{2} \|q\|_{2}.$$  \hfill (2b')

By (Lemma 2.3; cf. 1.4 in [P4], or 4.3 in [P7]), since $pFp, pFvFp$ are perpendicular to $P_{0}p$, the subfactor $p(P_{0}^{I} \cap m\omega)_{p}$ of the $\Pi_{1}$ factor $pM^{\omega}_{p}$ contains a diffuse abelian subalgebra that’s 2-independent to $qP_{0}q$ and $qP_{0}q \perp P_{0}^{0}$, so that there exists a projection $q \in p(P_{0}^{I} \cap m\omega)_{p}$ of trace $\tau(q) = \varepsilon^{2}\tau(p)^{2}/12^4$ such that $E_{p_{P_{0}}}(q(pFp)) = 0$, $E_{p_{P_{0}}}(q(pFvFp)) = 0$ and $\|qzq\|_{2}^{2}/\tau(p) = (\tau(q)/\tau(p))^{2}\tau(\varepsilon^{*}z)/\tau(p)$ for all $z \in pFp \cup pFvFp$.

Since $q \leq p$, it follows that for each such $x \in F$ one has

$$\|qxq\|_{2}^{2} = (\varepsilon^{4}\tau(p)^{2}/12^4)\tau(x^{*}x) \leq \varepsilon^{2}\tau(q)/12^{2}.$$  

Thus, $\|qxq\|_{2} \leq \varepsilon\tau(q)^{1/2}/12, \forall x \in F$. Hence, for this choice of $q$, the right hand side term in (2a) will be majorized by $\varepsilon\tau(q)/12$. By summing up over the seven terms in $\Sigma_{a}$, we get

$$\Sigma_{a}' \leq 7\varepsilon\tau(q)/12$$  \hfill (3a)

Similarly for the eleven terms in $\Sigma_{2,b}, \Sigma_{3,b}, \Sigma_{4,b}$ we get:

$$\Sigma_{2,b} + \Sigma_{3,b} + \Sigma_{4,b} \leq 11\varepsilon\tau(q)/12.$$  \hfill (3b)

We will now estimate the sum $\Sigma_{a}''$ of the terms $|Tr(X)|$ with $X$ running over the remaining four terms in $\Sigma_{2,a}$, the sum $\Sigma_{1,a}$ of the four terms $|Tr(X)|$ with $X$ having only one occurrence of $w, w^*$, the sum $\Sigma_{1,b}$ of the four terms $|\tau(y)|$ in (1b) with $y$ having only one occurrence of $w, w^*$, while at the same time taking care of the conditions $FwF \perp P_{0}, uw^*FwF \perp P_{0}$. We will do this by making an appropriate choice of the “phase $w$” above the support projection $q$ (which is fixed from now on).
Note that all elements entering in the sums $\Sigma_{1,a}$, $\Sigma_{1,b}$ are of the form $|\tau(wz)|$, where $z$ belongs to a finite set $E \subseteq (qM^\omega q)_1$. Let $\{e_{kl}\}_{k,l}$ be matrix units for $P_0$ and let $F \subseteq (qM^\omega q)_1$ denote the finite set $q(\cup_{k,l}F e_{kl}pF \cup E)q$. By results in ([P4], [P8]), there exists a hyperfinite subfactor $R \subseteq q(P_0' \cap P^\omega)q$ such that $E_{R_0 \cap qM^\omega q}(z') = E_{P_0q}(z')$, $\forall z' \in F$.

Since $P_0q$ and $R' \cap qM^\omega q$ are $\tau$-independent, if we denote $\tau_q$ the normalized trace $\tau(\cdot)/\tau(q)$ on $qM^\omega q$ then for each unitary element $w \in N := R' \cap q(P_0' \cap P^\omega)q$ and $z \in F$ we have

$$|\tau(wz)|/\tau(q) = |\tau_q(wz)| = |\tau_q(E_{R_0 \cap qM^\omega q}(wz))| = |\tau_q(wE_{R_0 \cap qM^\omega q}(qzq))| = |\tau_q(w)||\tau_q(E_{P_0q}(qzq))|/\tau(q).$$

Since $\|z\| \leq 1$, this implies that for any $w \in U(N)$ we have

$$|\tau(wz)| \leq |\tau(w)|, \forall z \in F,$$

(4)

$$\Sigma_{1,a} \leq 4|\tau(w)|,$$

(4a)

$$\Sigma_{1,b} \leq 4|\tau(w)|$$

(4b)

At this point, it is convenient to enumerate the elements in $F = \{x_1, ..., x_n\}$. For each $i = 1, 2, ..., n$ we have

$$\Sigma''_a = |\text{Tr}(ewx_i^*v^*evx_iw^*)| + |\text{Tr}(evx_i^*w^*ewx_i)^*v)|$$

$$+ |\text{Tr}(evx_i^*v^*ewx_iw^*)| + |\text{Tr}(evx_i^*w^*evx_iw^*)|$$

$$= |\text{Tr}(w^*ewY_{1,i})| + |\text{Tr}(w^*ewY_{2,i})|$$

$$+ |\text{Tr}(wY_{3,i}wY_{4,i})| + |\text{Tr}(w^*Y_{5,i}w^*Y_{6,i})|$$

where each one of the terms $Y_{j,i}$ depends on $x_i \in F$ and belongs to the set $S_0 := q((M^\omega)_1 e(M^\omega)_1)q \subset qL^2(\mathcal{M}, \text{Tr})q$.

Note that, as $i = 1, 2, ..., n$, the number of possible indices $(j, i)$ in (4a) is $4n$. There are $2n$ terms of the form $|\text{Tr}(w^*ewY)|$, $n$ terms of the form $|\text{Tr}(wXwY)|$ and $n$ terms of the form $|\text{Tr}(w^*Xw^*Y)|$, which by using the fact that $|\text{Tr}(w^*Xw^*Y)| = |\text{Tr}(wXwY)|$ we can view as $n$ additional terms of the form $|\text{Tr}(wXwY)|$. In all this, the elements $X, Y$ belong to $S_0 \subset qL^2(\mathcal{M}, \text{Tr})q$, and are thus bounded in
operator norm by 1 and are supported (from left and right) by projections of trace $Tr$ majorized by 1.

Recall that we are under the assumption $P \not\sim_M Q$. By Lemma 2.1, this implies $R' \cap (P'_0 \cap qP^\omega q) \not\sim_M Q^\omega$. Thus, $N' \cap qMq$ contains no finite non-zero projections of $M = \langle M^\omega, Q^\omega \rangle$.

To estimate the terms in $\Sigma^\prime_\alpha$ (and at the same time $\Sigma_{1,a}$, $\Sigma_{1,b}$), we will prove the following:

Fact 1. For any $\alpha > 0$ and any two $m$-tuples of elements $(Z_1, ..., Z_m)$, $(Z'_1, ..., Z'_m)$ in $S_0 \cap M_+$, there exists a unitary element $w \in N$ such that $|\tau(w)| \leq \alpha/4$ and

$$\Sigma^m_{i=1} Tr(w^*Z_iwZ'_i) \leq \alpha, \forall i.$$

To prove this, let $H$ denote the Hilbert space $L^2(qMq, Tr)^{\oplus m}$ and note that we have a unitary representation $U(N) \ni w \mapsto \pi(w) \in U(H)$, which on an $m$-tuple $X = (X_i)_{i=1}^m \in H$ acts by $\pi(w)(X) = (w^*X_iw)_i$.

Now note that this representation has no (non-zero) fixed point. Indeed, for if $X \in H$ satisfies $\pi(w)(X) = X$, $\forall w \in U(N)$, then on each component $X_i \in L^2(qMq, Tr)$ of $X$ we would have $w^*X_iw = X_i, \forall w$. Thus $X_iw = wX_i$ and since the unitaries of $N$ span linearly the algebra $N$, this would imply $X_i \in N' \cap L^2(qMq, Tr)$.

Hence, $X_i^*X_i \in N' \cap L^2(qMq, Tr)$ and therefore all spectral projections of $X_i^*X_i$ corresponding to intervals $[t, \infty)$ with $t > 0$ would be projections of finite trace in $N' \cap qMq$, forcing them all to be equal to 0. Thus, $X_i = 0$ for all $i$.

With this in mind, denote by $K_Z \subset H$ the weak closure of the convex hull of the set $\{\pi(w)(Z) \mid w \in U(N)\}$, where $Z = (Z_1, ..., Z_m)$ is viewed as an element in $H$. Since $K_Z$ is bounded and weakly closed, it is weakly compact, so it has a unique element $Z^0 \in K_Z$ of minimal norm $\|Z^0\|_{2,Tr}$. Since $K_Z$ is invariant to $\pi(w)$ and $\|\pi(w)(Z^0)\|_{2,Tr} = \|Z^0\|_{2,Tr}$, it follows that $\pi(w)(Z^0) = Z^0$. But we have shown that $\pi$ has no non-zero fixed points, and so $0 = Z^0 \in K_Z$.

Let us deduce from this that if $Z = (Z_i)_i, Z' = (Z'_i)_i$ are the two $m$-tuples of positive elements in $S_0$, then we can find $w \in U(N)$ such that (5) holds true. Indeed, for if there would exist $\alpha > 0$ such that $\Sigma_i Tr(\pi(w)(Z_i)Z'_i) \geq \alpha, \forall w \in U(N)$, then by taking convex combinations and weak closure, one would get $0 = \langle Z^0, Z' \rangle \geq \alpha$, a contradiction.

Note that by taking for one of the $i$ elements $Y_i, Y'_i$ to be equal to $e$, one can get $w \in U(N)$ to also satisfy $|\tau(w)|^2 \leq \alpha^2/16$. This finishes the proof of Fact 1.

We will now use this Fact 1 to prove:

Fact 2. Given any $m$-tuples $(X_i)_i, (Y_i)_i, (X'_i)_i, (Y'_i)_i \in S^m_0$ (not necessarily having positive operators as entries) and any $\alpha > 0$, there exists $w \in U(N)$ such that $|\tau(w)| \leq \alpha/4$, $\Sigma^m_{i=1} |Tr(w^*X_iwX'_i)| \leq \alpha$, $\Sigma^m_{i=1} |Tr(wY_iwY'_i)| \leq \alpha$, for all $i$. 

Indeed, because if we denote by $e_i$ the left support of $X_i'$ and $f_i$ the left support of $Y_i'$, then by the Cauchy-Schwartz inequality we simultaneously have for all $i$ the estimates

$$|Tr(w^*X_iwX_i')|^2 \leq Tr(w^*X_i^*X_iwX_i'X_i'^*)Tr(e_i) \leq Tr(w^*X_i^*X_iwX_i'X_i'^*),$$

and respectively

$$|Tr(wY_iwY_i')|^2 \leq Tr(w^*Y_i^*Y_iwY_i'Y_i'^*)Tr(f_i) \leq Tr(w^*Y_i^*Y_iwY_i'Y_i'^*).$$

Since all $X_i^*X_i, X_i'X_i'^*, Y_i^*Y_i, Y_i'Y_i'^*$ are positive elements in $S_0$, we can now apply the Fact 1 to deduce that there exist $w \in \mathcal{U}(N)$ such that $|\tau(w)| \leq \alpha/4$, $\sum_{i=1}^{m}|Tr(w^*X_iwX_i')| \leq \alpha$, $\sum_{i=1}^{m}|Tr(wY_iwY_i')| \leq \alpha$. This ends the proof of Fact 2.

For each $n \geq 1$, we apply Fact 2 to $\alpha = 2^{-n-1}$ to get a partial isometry $w_n \in P'_0 \cap P^\omega$ of support $w_nw_n^* = w_n^*w_n = q$ such that if we denote by $\Sigma_{1,a}(w_n), \Sigma_{1,b}(w_n)$, $\Sigma_{a}'(w_n)$ the values of $\Sigma_1(a), \Sigma_1(b)$, $\Sigma_{a}'$, obtained by plugging in $w_n$ for $w$ in the inequalities $(4a), (4b), (5a)$, respectively, then we have:

$$|\tau(w_nz)| \leq 2^{-n-1}, \forall z \in \mathcal{F};$$

$$\Sigma_{1,a}(w_n) \leq 2^{-n-1}, \Sigma_{1,b}(w_n) \leq 2^{-n-1}, \Sigma_{a}'(w_n) \leq 2^{-n-1}.$$  

Let $q = (q_n)_n$, with $q_n \in \mathcal{P}(P'_0 \cap P)$ and $w_n = (w_{n,k})_k$ with $w^*_{n,k}w_{n,k} = w_{n,k}^*w_{n,k} = q_k, \forall k$. For each $k \geq 1$, we denote $\mathcal{F}_k \subset (q_kMq_k)_1$ the set of all $k'$th entries of elements $z = (z_k)_k \in \mathcal{F} \subset (qM^\omega q)_1$. We also denote by $\Sigma_{1,a}(w_n)_k$ (resp. $\Sigma_{1,b}(w_n)_k$, $\Sigma_{a}'(w_n)_k$) the sum obtained at the $k'$th level of $\Sigma_{1,a}(w_n)$ (resp. $\Sigma_{1,b}(w_n)$, $\Sigma_{a}'(w_n)$). Thus, we have

$$\lim_{k \to \omega}|\tau(w_{n,k}z_k)| \leq 2^{-n-1}, \forall z_k \in \mathcal{F}_k;$$

$$\lim_{k \to \omega}\Sigma_{1,a}(w_n)_k \leq 2^{-n-1};$$

$$\lim_{k \to \omega}\Sigma_{1,b}(w_n)_k \leq 2^{-n-1};$$

$$\lim_{k \to \omega}\Sigma_{a}'(w_n)_k \leq 2^{-n-1}.$$  

Denote by $V_n$ the set of all $k \in \mathbb{N}$ such that $|\tau(w_{n,k}z_k)| < 2^{-n}, \forall z_k \in \mathcal{F}_k$, $\Sigma_{1,a}(w_n)_k < 2^{-n}, \Sigma_{1,b}(w_n)_k \leq 2^{-n}, \Sigma_{a}'(w_n)_k \leq 2^{-n}$. Note that $V_n$ corresponds to an open closed neighborhood of $\omega$ in $\Omega$, under the identification $\ell^\infty\mathbb{N} = C(\Omega)$. Let now $W_n$, $n \geq 0$, be defined recursively as follows: $W_0 = \mathbb{N}$ and $W_{n+1} = \mathbb{N} \setminus \{1,2,\ldots,n\}. \quad \Box$
$W_n \cap V_{n+1} \cap \{k \in \mathbb{N} \mid k > \min W_n\}$. Note that, with the same identification as before, $W_n$ is a strictly decreasing sequence of neighborhoods of $\omega$ in $\Omega$.

Define $w = (w'_m)_m$, where $w'_k = w_{m,k}$ for $k \in W_{m-1} \setminus W_m$. It is then easy to see that $w$ is a partial isometry in $P'_0 \cap P'^\omega$ with $ww^* = w^*w = q$ and that we have $\tau(wz) = 0, \forall z \in F, \Sigma_{1,a}(w) = 0, \Sigma_{1,b}(w) = 0, \Sigma''_a(w) = 0$. By taking into account the definition of $F$, it is easy to see that the first of these conditions implies that $FwF \perp P_0, uu^*FwF \perp P_0$, where $u = v + w$. By (1b), (3b) and $\Sigma_{1,b}(w) = 0$, it follows that $|\tau(uv^*_3u'v^*_4)| \leq \delta\tau(uu^*_q), \forall x \in F$. In turn, from (1a), (3a) and the fact that $\Sigma_{1,a}(w) = 0, \Sigma''_a(w) = 0$, it follows that $\|E_{Q^\omega}(uv^*_3uv^*_4)\| \leq \delta\tau(uu^*_q), \forall x \in F$. This shows that $u \in W$, while $u \leq v, u \neq v$, contradicting the maximality of $v$.

This shows that $v$ must be a unitary element. Thus, if we represent $v \in P'_0 \cap P'^\omega$ as a sequence of unitary elements $(v_n)_n$ in $P'_0 \cap P$, then we have

$$\lim_{n \to \omega} \|E_Q(v_nv^*_n)\| \leq \|E_Q(v^nv^*_n)\| \leq \delta < \delta_0,$$

$$\lim_{n \to \omega} |\tau(v_nv^*_n v^*_1 v^*_2 v^*_3 v^*_4)| \leq \delta < \delta_0,$$

for all $x, x_i \in F$. Thus, if we let $v_0 = v_n$ for some large enough $n$, then $v_0$ is a unitary element in $P'_0 \cap P$ that satisfies $\|E_Q(v_0v^*_0)\| \leq \delta_0$, for all $x \in F$, and $|\tau(v_0v^*_0 x_0 v^*_0 x_0 x_4)| \leq \delta_0$, for all $x_i \in F$.

□

4. Proof of the main result

4.1. Lemma. Let $M$ be a $\Pi_1$ factor, $Q \subset M$ a von Neumann subalgebra, $P \subset M$ an irreducible subfactor such that $P \not\subset Q$ and $P_0 \subset P$ a finite dimensional subfactor. Given any finite set $F_0 \subset M \ominus P_0, m_1 \geq 1$ and $\alpha > 0$, there exists a subfactor $P_1 \simeq M_{m_1 \times m_1}(\mathbb{C})$ in $P'_0 \cap P$ such that

(a) $c_q(P_0 \vee P_1, Q) \leq c_q(P_0, Q) + \alpha, \forall 1 \leq q \leq 2$;

(b) $|\tau(b_1x_2b_3x_4)| \leq \alpha, \forall b_1, b_3 \in (P_1 \ominus \mathbb{C}1)_1, \forall x_2, x_4 \in F_0$.

Proof. Note first that, since $\alpha$ can be taken arbitrarily small independently of the $\|\|$-size of the elements in $F_0$, it is sufficient to prove the statement in the case $F_0$ is contained in the until ball of $M$, an assumption that we will thus make for the rest of the proof.
Let $1 \in \mathcal{U}_0 \subset \mathcal{U}(P_0)$ be an orthonormal basis of $L^2P_0$ made up of unitary elements. Thus, $|\mathcal{U}_0| = m_0^2$, where $P_0 \simeq \mathbb{M}_{m_0 \times m_0}(\mathbb{C})$.

Let $\delta = \alpha^2/36m_0^2m_1$. Let $P_1 \subset P_0 \cap P$ be a type $I_{m_1}$ subfactor and $F_1 \subset (L^2(P_1^0) \otimes \mathbb{C})_1$ be a finite subset that’s $\delta \| \cdot \|_2$-dense in $(L^2(P_1^0) \otimes \mathbb{C})_1$, i.e., any $x \in (L^2(P_1^0) \otimes \mathbb{C})_1$ is $\delta$-close to $F_1$ in the norm $\| \cdot \|_2$.

Note that $F := \mathcal{U}_0(F_0 \cup F_1)\mathcal{U}_0$ is orthogonal to $P_0$. By applying Lemma 3.1 to this finite set $F$ and $\delta > 0$, we get a unitary element $v \in P_0' \cap P$ such that

\begin{equation}
\| E_Q(vxv^*) \|_2 \leq \sqrt{\delta}, \forall x \in F,
\end{equation}

\begin{equation}
|\tau(vx_1v^*x_2v x_3 v^* x_4)| \leq \delta, \forall x \in F
\end{equation}

By taking into account that $F_1$ is $\delta \| \cdot \|_2$-dense in $(L^2(P_1^0) \otimes \mathbb{C})_1$ and applying the triangle inequality in (1a) for $x \in F_1\mathcal{U}_0$, it follows that

\begin{equation}
\| E_Q(vx^*) \|_2 \leq 3\sqrt{\delta}, \forall x \in (L^2(P_1^0) \otimes \mathbb{C})_1\mathcal{U}_0.
\end{equation}

Denote $P_1 = vP_0v^*$ and note that $v((P_1^0 \otimes \mathbb{C})\mathcal{U}_0)v^* = (P_1 \otimes \mathbb{C})\mathcal{U}_0$. Thus, by applying (2a) to $x = by$ where $b \in F_1$ and $y \in \mathcal{U}_0$, it follows that

\[ \| E_Q(yu) \|_2 \leq 3\sqrt{\delta}, \forall y \in (L^2(P_1) \otimes \mathbb{C})_1, u \in \mathcal{U}_0. \]

or equivalently

\begin{equation}
\sup\{|\tau(yuz)| \mid z \in (L^2Q)_1 \} \leq 3\sqrt{\delta}, \forall y \in (L^2(P_1) \otimes \mathbb{C})_1, u \in \mathcal{U}_0.
\end{equation}

Since $\mathcal{U}_0$ is an orthonormal basis for $L^2(P_0)$ and $1 \in \mathcal{U}_0$, any element $x \in P_0 \lor P_1$ can be uniquely written as $\sum_{u \in \mathcal{U}_0} uy_u$, for some $y_u \in P_1$, with $\|x\|_2^2 = \sum_u \|y_u\|_2^2$ by Pythagoras Theorem. Moreover, one has $E_{P_0}(x) = \sum_u \tau(y_u)u$ with $\tau(x) = \tau(y_1)$.

Thus, if $x' = \sum_u uy_u$ lies in the unit ball of $L^2(P_0 \lor P_1)$ and has 0 expectation onto $P_0$, then $\tau(y_u) = 0, \forall u$. Moreover, if we denote as usual $p = \frac{q}{q-1} \leq 2$ and take into account that the unit ball of $L^pQ$ is contained in the unit ball of $L^2Q$, then from (3a) and the remarks in 2.4 we get the estimates

\begin{equation}
\| E_Q(x') \|_q = \sup\{|\tau(x'z)| \mid z \in (L^pQ)_1 \}
\leq \sup\{|\tau(x'z)| \mid z \in (L^2Q)_1 \}
\leq \sum_{u \in \mathcal{U}_0} \sup\{|\tau(uy_u z)| \mid z \in (L^2Q)_1 \}
\end{equation}
\[
\leq (\sum_{u \in l_0} \|y_u\|_2)^2 \leq (\sum_{u \in l_0} \|y_u\|_2^2) \|u_0\|^{1/2} \|u_0\|^{1/2} 3\sqrt{\delta}
\]
\[
\leq \|x\|_2 m_0 3\sqrt{\delta} \leq (m_0 m_1)^{1/2} \|x\|_q m_0 3\sqrt{\delta} \leq 3m_0^{3/2} m_1^{1/2} \sqrt{\delta} = \alpha/2,
\]
where for the last inequalities we have used the Cauchy-Schwarz inequality and the fact that, due to Lemma 2.5.2\(^\circ\), we have:

\[
\sum_u \|y_u\|^2_2 = \|x'\|^2_2 \leq \|x\|_2 \|x\|_q (\dim(P_0 \vee P_1))^{1/4} = (m_0 m_1)^{1/2} \|x\|_q.
\]

By writing any \(x \in L^2(P_0 \vee P_1) \otimes \mathbb{C}\) as a sum between its projection on \(P_0\) and respectively on \(P_1 \vee P_0 \odot P_0\), i.e., \(x = (x - E_{P_0}(x)) + E_{P_0}(x)\), and taking into account that \(E_{P_0}(x)\) \(\|q\| \leq \|x\|_q\) and \(\|x - E_{P_0}(x)\|_q \leq 2\|x\|_q\), by applying (4a) to \(x' = x - E_{P_0}(x)\) it follows that if such \(x\) satisfies \(\|x\|_q \leq 1\) then

(5a) \[
\|E_Q(x)\|_q \leq \|E_Q(E_{P_0}(x))\|_q + \|E_Q(x')\|_q \leq c_q(P_0, Q) + \alpha.
\]

In turn, if we apply (1b) to \(x_1, x_3 \in F_1\) and \(x_2, x_4 \in F_0 \subset (M \odot P_0)_1\) and taking into account that \(vF_1v^*\) is \(\delta\) \(\|\|_2\)-dense in \((L^2(P_1) \otimes \mathbb{C})_1\), by the triangle inequality (+ Cauchy-Schwarz) we obtain

(2b) \[
|\tau(b_1 x_2 b_3 x_4)| \leq 3\delta \leq \alpha, \forall b_1, b_3 \in (L^2(P_1) \otimes \mathbb{C})_1, x_2, x_4 \in F_0.
\]

But (5a), (2b) are just conditions (a), (b) required in the lemma. \(\square\)

4.2. Theorem. Let \(M\) be a countably generated \(\Pi_1\) factor, \(Q \subset M\) a von Neumann subalgebra and \(P \subset M\) an irreducible subfactor satisfying \(P \not\subset Q\). Given any \(\varepsilon > 0\), there exists a hyperfinite subfactor \(R \subset P\) that's irreducible in \(M\) and satisfies \(c_q(R, Q) \leq \varepsilon, c_p(Q, R) \leq \varepsilon, \forall 1 \leq q \leq 2 \leq p \leq \infty\).

Proof. Recall that a tracial von Neumann algebra \((M, \tau)\) is countably generated iff \(M\) is separable in the Hilbert norm \(\|\|_2\) implemented by the faithful normal trace state \(\tau\) (see e.g., [AP]).

So our \(\Pi_1\) factor \(M\) is \(\|\|_2\)-separable. Let \(\{x_k\}_k \subset M\) be a \(\|\|_2\)-dense sequence of elements. We construct recursively an increasing sequence of finite dimensional subfactors \(B_m \subset P\) such that \(B_0 = \mathbb{C}, B_{m-1}' \cap B_m\) is of type \(I_{2^m}\), \(\forall m \geq 1\), and such that if we denote \(F_m = \{x_k - E_{B_{m-1}}(x_k) \mid 1 \leq k \leq m\}\) then we have

(a) \[
c_q(B_m, Q) \leq \varepsilon(2^{-1} - 2^{-m-1}), \forall 1 \leq q \leq 2;
\]
Assume we have constructed the algebras $B_m$ up to $m = n$. By applying Lemma 4.1 to $P_0 = B_n$, $m_1 = 2^{n+1}$, $F = F_{n+1} \cup F_{n+1}^*$ and $\alpha = 2^{-n-2}$, we get a subfactor $P_1 \cong \mathbb{M}_{2^{n+1} \times 2^{n+1}}(\mathbb{C})$ inside $B'_n \cap P$ such that for all $1 \leq q \leq 2$ we have

$$c_q(B_n \vee P_1, Q) \leq c_q(B_n, Q) + \varepsilon 2^{-n-2}$$

$$\leq \varepsilon(2^{-1} - 2^{-n-1}) + \varepsilon 2^{-n-2} = \varepsilon(2^{-1} - 2^{-n-2}),$$

while for all $b_1, b_3 \in (P_1)_1$ and all $y_2, y_4 \in F_{n+1} \cup F_{n+1}^*$, we have

$$|\tau(b_1 y_2 b_3 y_4)| \leq 2^{-n-1}.$$

Thus, if we let $B_{n+1} = B_n \vee P_1$, then both conditions (a) and (b) are satisfied for $m = n + 1$.

We define $R = \overline{\cup_n B_n^w}$. Then $R$ is a copy of the hyperfinite II_1 factor inside $P$, which by (a) and Lemma 2.5.3° satisfies $c_q(R, Q) \leq \varepsilon/2$, $\forall 1 \leq q \leq 2$.

Assume $R' \cap M \neq \mathbb{C}1$. Let $x \in R' \cap M$ be an element of trace 0 and $\|x\| = 1$. Note that $E_R(x) = 0$. Denote $\alpha = \|x\|^2/2$. By the $\|\|_2$-density of $\{x_k\}_k$ in $M$, there exists $n \geq 3$ such that $\|x - x_n\|_2 \leq \alpha/3$. Thus, $y = x_n - E_{B_n}(x_n) \in F_n$ satisfies $\|x - y\|_2 \leq \alpha/3$ as well. By applying condition (b) above for $m = n$ and $y_2 = y$, $y_4 = y^*$, $b_1 = u \in \mathcal{U}(B'_{n-1} \cap B_n)$ and $b_3 = u^*$, it follows that $|\tau(uyu^*y^*)| \leq \alpha/3$. By using the triangle inequality and the fact that $u$ commute with $x$, this implies $\tau(x^*) = |\tau(uxu^*x^*)| \leq \alpha = \tau(x^*)/2$, a contradiction. Thus, $R$ must have trivial relative commutant in $M$.

So we have constructed a subfactor $R \subset P$ that’s irreducible in $M$ and satisfies $c_q(R, Q) \leq \varepsilon/2$, $\forall 1 \leq q \leq 2$. But then by Lemma 2.5.1°, we also have $c_p(Q, R) \leq \varepsilon$, $\forall 2 \leq p \leq \infty$.  

4.3. Remark. As we have already mentioned in the introduction, it is an open question whether for any irreducible subfactor with infinite index $Q$ of an arbitrary countably generated II_1 factor $M$ and any $\varepsilon > 0$, one can find a hyperfinite II_1 factor $R \subset M$ such that $c_p(R, Q) \leq \varepsilon$ for all $1 \leq p \leq \infty$ (uniformly in $p$). In particular, whether there exists an irreducible hyperfinite subfactor $R \subset M$ such that $\|E_Q(b)\| \leq \varepsilon\|b\|$, for all $b \in R \cap \mathbb{C}1$. Note that if true, this would show that for any countable amenable group $G$ there exists a copy $\{u_g\}_{g \in G}$ of the left regular representation of $G$ so that $\|E_Q(u_g)\| \leq \varepsilon$, $\forall g \in G \setminus \{e\}$. If in addition $Q$ contains a Cartan subalgebra $A$ of $M$, then it would be interesting to see whether there exist $\{u_g\}_{g}$ in the normalizer of $A$ and so that $E_Q(u_g) = 0$, $\forall g \in G \setminus \{e\}$. 

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