Scalable Kernelization for Maximum Independent Sets

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Abstract. The most efficient algorithms for finding maximum independent sets in both theory and practice use reduction rules to obtain a much smaller problem instance called a \textit{kernel}. The kernel can then be solved quickly using exact or heuristic algorithms—or by repeatedly kernelizing recursively in the branch-and-reduce paradigm. It is of critical importance for these algorithms that kernelization is fast and returns a small kernel. Current algorithms are either slow but produce a small kernel, or fast and give a large kernel. We attempt to accomplish both of these goals simultaneously, by giving an efficient parallel kernelization algorithm based on graph partitioning and parallel bipartite maximum matching. We combine our parallelization techniques with two techniques to accelerate kernelization further: dependency checking that prunes reductions that cannot be applied, and reduction tracking that allows us to stop kernelization when reductions become less fruitful. Our algorithm produces kernels that are orders of magnitude smaller than the fastest kernelization methods, while having a similar execution time. Furthermore, our algorithm is able to compute kernels with size comparable to the smallest known kernels, but up to two orders of magnitude faster than previously possible. Finally, we show that our kernelization algorithm can be used to accelerate existing state-of-the-art heuristic algorithms, allowing us to find larger independent sets faster on large real-world networks and synthetic instances.

1 Introduction

The maximum independent set problem is a classic NP-hard problem [21] with applications spanning many fields, such as computer vision [18], computer graphics [37], map labeling [22] and routing in road networks [29]. Given a graph $G = (V,E)$, our goal is to compute a maximum cardinality set of vertices $I \subseteq V$ such that no vertices in $I$ are adjacent to one another. Such a set is called a maximum independent set (MIS).

One of the most powerful techniques for solving the MIS problem in practice is kernelization—reducing the input to its most difficult part, the kernel. A kernel (for the MIS problem) of a graph $G$ is a smaller graph $r(G)$ obtained by applying a specified polynomial time algorithm to $G$ that reduces its size while preserving the information required to find an MIS in $G$. The algorithm is often composed of a set of algorithms (so called \textit{reduction} rules), which are applied exhaustively. After finding an MIS in $r(G)$ we “undo” the kernelization to find an MIS of $G$. Fixed-parameter tractable algorithms for the MIS problem are exponential in the size of the kernel, and therefore the MIS problem is considered “hard” for a particular instance when its kernel size is large [39]. Thus, it is often desirable to apply many different reduction rules to reduce the input size as much as possible when solving the problem exactly.

In practice, kernelization is used as a preprocessing step to other algorithms [10,14,15,34,39,42], where speeding up kernelization directly speeds up the algorithm. However, kernelization may also be applied repeatedly as part of an algorithm [2,12,30]. In either case, the smallest kernels (or seemingly equivalently, the most varied reductions) give the best chance at finding solutions. For instance, the reductions used by

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Akiba and Iwata [2] are the only ones known to compute an exact MIS on certain large-scale graphs, and are further successful in computing exact solutions in an evolutionary approach [30]. However, it is not always beneficial to compute the smallest kernel possible. Fast and simple reductions can compute kernels that are “small enough” for local search to quickly find high-quality, and even exact, solutions much faster than the reductions used to find the smallest kernels [12,15]. Fast and simple reductions can even be used to solve many large-scale instances exactly [39] just as quickly as the algorithm by Akiba and Iwata [2].

Thus, for kernelization, there is a trade-off between kernel size and kernelization time. The smallest kernels are necessary to solve the most instances to (near-)optimality, but the fastest reductions have just enough power to solve many instances quickly. Thus, to be effective for a majority of applications, kernelization routines should compute a kernel that is as small as possible as quickly as possible.

**Our Results.** To this end, we develop an efficient shared-memory parallel kernelization algorithm based on graph partitioning and parallel bipartite maximum matching. We combine our parallelization with *dependency checking*—a strategy for pruning inapplicable reductions—as well as *reduction tracking* that allows us to stop kernelization when reductions become less fruitful. These pruning techniques achieve large additional speedups over the kernelization Akiba and Iwata [2], which computes similarly sized kernels. Our experimental evaluation shows that on average our algorithm finds kernels that are a factor 7 smaller than the algorithms of Chang et al. [12], while having similar a running time. At the same time our algorithms are a factor 41 faster on average than other algorithms that are able to find kernels of similar size. In further experiments we apply our kernelization algorithm to state-of-the-art heuristic maximum independent set algorithms and find that our kernels can be used to find larger independent sets faster in large real-world networks and synthetic instances.

### 2 Related Work

The *maximum clique* and *minimum vertex cover* problems are equivalent to the maximum independent set problem: a maximum clique in the complement graph $\overline{G}$ is a maximum independent set in $G$, and a minimum vertex cover $C$ in $G$ is the complement of a maximum independent set $V \setminus C$ in $G$. Thus, an algorithm that solves one of these problems can be used to solve the others. Many branch-and-bound algorithms have been developed for the maximum clique problem [35,36,41], which use vertex reordering and pruning techniques based on approximate graph coloring [41] or MaxSAT [31], and can be further sped up by applying local search to obtain an initial solution of high quality [7].

A common theme among algorithms for these (and other) NP-hard problems is that of *kernelization*—reducing the input to a smaller instance that, when solved optimally, optimally solves the original instance. Rules that are used to reduce the graph while retaining the ability to compute an optimal solution are called *reductions*. Reductions and kernelization have long been used in algorithms for the minimum vertex cover and maximum independent set problems [11,13,20,40], for efficient exact algorithms and heuristics alike.

**Exact Algorithms.** Butenko et al. [10] and Butenko and Trukhanov [11] were able to find exact maximum independent sets in graphs with thousands of vertices by first applying reductions. Further works have introduced reductions to more quickly solve the maximum clique problem [34,42] and enumerate $k$-plexes [14]. Though these works apply reduction techniques as a preprocessing step, many works apply reductions as a natural step of the algorithm. Reductions were originally used to reduce the running time of the brute force $O(n^2 2^n)$ algorithm to the $O(2^{n/3})$ time algorithm of Tarjan and Trojanowski [40], and to give the current best polynomial space algorithm with running time of $O^*(1.1996^n)$ by Xiao and Nagamochi [45]. These algorithms apply reductions during recursion, only branching when the graph can no longer be reduced [19]—known as the *branch-and-reduce* method.
Akiba and Iwata [2] were the first to show the effectiveness of the branch-and-reduce method for solving the minimum vertex cover problem in practice for large sparse real-world graphs. Using a large collection of reductions, they solve graphs with millions of vertices within seconds. In contrast, the vast majority of instances can not be solved by the MCS clique solver [41] within a 24-hour time limit [2]. However, as later shown by Strash [39], many of these same instances can be solved just as quickly by first kernelizing with two simple standard reductions and then running MCS.

**Heuristic Algorithms.** Kernelization and reductions play an important role in heuristic algorithms too. Lamm et al. [30] showed that including reductions in a branch-and-reduce inspired evolutionary algorithm enables finding exact solutions much faster than provably exact algorithms. Dahlum et al. [15] find that standard kernelization techniques are too slow to be effective for local search and show that applying simple reductions in an online fashion improves the speed of local search. Chang et al. [12] improved on this result, by implementing reduction rules to reduce the lead time for kernelization for local search. They introduce two kernelization techniques: a reduction rule to collapse maximal degree-two paths in a single shot, resulting in a fast linear-time kernelization algorithm (LinearTime), and a near linear-time algorithm (NearLinear) that uses triangle counting to detect when the domination reduction can be applied. NearLinear has running time $O(\Delta m)$ where $\Delta$ is the maximum degree of the graph. They further introduce “reducing–peeling” to find a large initial solution for local search. This technique can be viewed as computing one path through the search space of a branch-and-reduce algorithm: they repeatedly exclude high-degree vertices and kernelize the graph until it is empty, then take the independent set found as an initial solution for local search.

### 3 Preliminaries

**Basic Concepts.** Let $G = (V, E)$ be an undirected graph on $n = |V|$ nodes and $m = |E|$ edges. We assume that $V = \{0, \ldots, n-1\}$, and to eliminate ambiguity, we at times denote by $V[G]$ and $E[G]$ the sets $V$ and $E$, respectively, for a particular graph $G$. Throughout this paper, we assume that $G$ is simple: it has no multi-edges or self loops. The set $N(v) = \{ u \mid \{ v, u \} \in E \}$ denotes the open neighborhood (also simply called the neighborhood) of $v$. We further define the open neighborhood of a set of nodes $U \subseteq V$ to be $N(U) = \bigcup_{v \in U} N(v)$. We similarly define the closed neighborhood as $N[v] = N(v) \cup \{ v \}$ and $N[U] = N(U) \cup U$. We sometimes use $N_G$ to denote the neighborhood in a particular graph $G$. A graph $H = (V_H, E_H)$ is said to be a subgraph of $G = (V, E)$ if $V_H \subseteq V$ and $E_H \subseteq E$. We call $H$ an induced subgraph when $E_H = \{ \{ u, v \} \in E \mid u, v \in V_H \}$. For a set of nodes $U \subseteq V$, $G[U]$ denotes the subgraph induced by $U$. A set $I \subseteq V$ of vertices, is said to be an independent set if all nodes in $I$ are pairwise nonadjacent; that is, $E[G[I]] = \emptyset$. The maximum independent set problem is that of finding a maximum cardinality independent set which is called a maximum independent set (MIS).

The graph partitioning problem is to partition $V$ into $k$ blocks $V_1 \cup \cdots \cup V_k = V$ with $V_i \cap V_j = \emptyset$, $\forall i \neq j$ while optimizing a given cost function—typically the number of edges with end vertices in different blocks. Additionally, a balance constraint is applied, which demands that the blocks have approximately equal size with respect to the number of vertices or, alternatively, the sum of weights associated with the vertices. Boundary vertices are adjacent to vertices in other blocks and cut edges cross block boundaries.

#### 3.1 Reductions

We now briefly describe the reduction rules that we consider. Each reduction allows us to choose vertices that are in some MIS by following simple rules. If an MIS is found on the kernel graph $K$, then each reduction may be undone, producing an MIS in the original graph.
The Linear Time Algorithm by Chang et al. [12]. Chang et al. [12] present a kernelization algorithm \textsc{LinearTime} that runs in time $O(m)$. It removes vertices of degree zero and one and uses a reduction rule using maximal paths of degree two. They split the rule into five cases depending on the length of the maximal

\section*{Reductions of Akiba and Iwata [2].} Akiba and Iwata [2] use a full suite of advanced reduction rules, which can efficiently solve the minimum vertex cover problem for a variety of instances. Here, we briefly describe the reductions we use, for the maximum independent set problem.

\textit{Vertex folding [13]:} For a vertex $v$ with degree two whose neighbors $u$ and $w$ are not adjacent, either $v$ is in some MIS, or both $u$ and $w$ are in some MIS. Therefore, we can contract $u$, $v$, and $w$ to a single vertex $v'$ and decide which vertices are in the MIS later. If $v'$ is in the computed MIS, then $u$ and $w$ are added to the independent set, otherwise $v$ is added. Thus, a vertex fold contributes a vertex to an independent set.

\textit{Linear programming relaxation [33]:} A well-known linear programming relaxation for the MIS problem with a half-integral solution (i.e., using only values 0, 1/2, and 1) can be solved using bipartite matching: maximize $\sum_{v \in V} x_v$ such that $\forall (u, v) \in E$, $x_u + x_v \leq 1$ and $\forall v \in V$, $x_v \geq 0$. Vertices with value 1 must be in the MIS and can thus be removed from $G$ along with their neighbors. We use a version [26] that computes a solution whose half-integral part is minimal using maximum bipartite matching.

\textit{Unconfined [44]:} Though there are several definitions of unconfined vertex in the literature, we use the simple one from Akiba and Iwata [2]. A vertex $v$ is unconfined when determined by the following simple algorithm. First, initialize $S = \{v\}$. Then find a $u \in N(S)$ such that $|N(u) \cap S| = 1$ and $|N(u) \setminus N[S]|$ is minimized. If there is no such vertex, then $v$ is confined. If $N(u) \setminus N[S] = \emptyset$, then $v$ is unconfined. If $N(u) \setminus N[S]$ is a single vertex $w$, then add $w$ to $S$ and repeat the algorithm. Otherwise, $v$ is confined. Unconfined vertices can be removed from the graph, since there always exists a MIS that contains no unconfined vertices.

\textit{Diamond:} Although not mentioned in their paper, Akiba and Iwata [2] extend the unconfined reduction in their implementation [25]. Let $S$ be the set constructed in the unconfined reduction for a vertex $v$ that is not unconfined. If there are nonadjacent vertices $u_1, u_2$ in $N(S)$ such that $N(u_1) \setminus N(S) = N(u_2) \setminus N(S) = \{v_1, v_2\}$, then we can remove $v$ from the graph because there always exists a MIS that does not contain $v$. Note that this implies that $\{v_1, v_2\} \subseteq S$.

\textit{Twin [44]:} Let $u$ and $v$ be vertices of degree three with $N(u) = N(v)$. If $G[N(u)]$ has edges, then add $u$ and $v$ to $I$ and remove $u$, $v$, $N(u)$, $N(v)$ from $G$. Otherwise, some vertices in $N(u)$ may belong to some MIS $I$. We still remove $u$, $v$, $N(u)$ and $N(v)$ from $G$, and add a new gadget vertex $w$ to $G$ with edges to $u$’s two-neighborhood (vertices at a distance 2 from $u$). If $w$ is in the computed MIS, then none of $u$’s two-neighbors are in $I$, and therefore $N(u) \subseteq I$. Otherwise, if $w$ is not in the computed MIS, then some of $u$’s two-neighbors are in $I$, and therefore $u$ and $w$ are added to $I$.

\section*{The Reduction of Butenko et al. [9].} We describe one reduction that was not included in the algorithm by Akiba and Iwata [2], but was shown by Butenko et al. [9] to be highly effective on medium-sized graphs derived from error-correcting codes.

\textit{Isolated Vertex Removal [9]:} If a vertex $v$ forms a single clique $C$ with all its neighbors, then $v$ is called isolated (simplicial is also used in the literature) and is always contained in some MIS. To see this, at most one vertex from $C$ may be in any MIS. Either it is $v$ or, if a neighbor of $v$ is in an MIS, then we select $v$ instead. Note that this reduction rule is completely contained in the unconfined reduction rule as every neighbor $u \in N(v)$ of a simplicial vertex $v$ is unconfined, leaving only $v$ without any neighbors in the graph. As it can be implemented more efficiently than the unconfined reduction, we apply the reduction by isolated vertex removal before removing unconfined vertices.
degree two path and the endpoints of the path. The full description can be found in their paper [12]. The
degree two path rule is a specialization of the vertex folding rule explained above, and does not cover
the case of a vertex with two neighbors of degree higher than two. However, in contrast to the vertex folding
rule, it has linear time complexity. This algorithm often removes a large fraction of a graphs vertices in
very little time; however, it still leaves the possibility to apply more powerful, but time consuming reduction
rules. We therefore run LinearTime as a preprocessing step of our algorithm.

4 Parallel Kernelization

As current machines usually have more than one processor and kernelization can run for hours on large
instances, parallelization is a promising way to make larger graphs feasible for maximum independent set
algorithms. In this section, we describe how we parallelize kernelization: we partition the graph into blocks
so that “local” reductions can be run on blocks in parallel, and perform parallel maximum bipartite matching
for the “global” reduction by linear programming. Our algorithm first applies the reductions parallelized by
partitioning exhaustively. We then apply the reduction by linear programming. These steps are repeated until
no more vertices can be removed from the graph. (See pseudocode in Appendix A.)

4.1 Blockwise Reductions

Many reductions have an element of locality. In particular, we call a reduction local if it is applied one
vertex at a time, if determining that the reduction can be applied is based on local graph structure (for ex-
ample, by its neighborhood or by neighbors of neighbors), and the reduction itself modifies only local graph
structure. A challenge in parallelizing local reductions is in how to apply them simultaneously. Fine-grained
parallelism would require locks, since attempting to simultaneously remove or contract (near-)neighboring
vertices in the graph results in a race condition: these (near-)neighbors may both be mistakenly added to
the independent set or the graph may be modified incorrectly. However, with locks, local reductions become
more expensive—reductions must wait if they overlap other reductions in progress.

To avoid locks altogether, we partition the graph into vertex-disjoint blocks and perform local reductions
on each block in parallel (i.e., blockwise). Note that the only way for two blockwise reductions to simultane-
ously (mistakenly) reduce neighbors is if they are incident to a cut edge. We therefore avoid race conditions
by restricting reductions to only read and write to vertices and neighborhoods within a single block. By
using a high-quality partitioning that minimizes the number of cut edges, we expect the number of vertices
excluded from these local reductions to be small. To avoid race conditions when removing boundary ver-
tices from the graph, we leave the adjacency lists of neighboring vertices unchanged. We only change the
adjacency list of vertices when performing vertex contractions.

We now explain how to apply each local reduction in our parallel framework. Let $V_i$ be the block in
which we are applying the reduction. Further, let $B_k$ denote the set of vertices of distance at most $k$ from
some boundary vertex in our partitioning. Note that $B_0$ is the set of boundary vertices and $B_k = N[B_{k-1}]$.

Vertex Folding: Let $v \in V_i$ be a vertex with neighborhood \{u, w\} \subset V_i. Contracting $v, u, w$ into $v'$ will
cause a race condition whenever $u, w \in V_i \cap B_0$, as their neighbors in some other block $V_j$ must have their
adjacency lists updated to include $v'$. We handle vertex folding with two cases. First, for $u, w \in V_i \setminus B_0$ (or
equivalently, $v \in V_i \setminus B_1$), we apply the reduction normally. Then $N(\{u, w\}) \subset V_i$ and there is no race
condition. Secondly, without loss of generality, if $u \in V_i \cap B_0$ and $w \in V_i \setminus B_0$ then we still apply vertex
folding, using $u$ as the new vertex $v'$. Neighborhoods of vertices in $N(u) \setminus \{v\}$ remain unchanged.
Isolated Vertex Removal: Let \( v \in V_i \setminus B_0 \) be an isolated vertex. Then we add \( v \) to \( \mathcal{I} \) and remove \( N[v] \subseteq V_i \) from the graph as usual. (See Figure [1])

Twin: Let \( u, v \in V_i \) such that \( N(u) = N(v) \subseteq V_i \), and note that \( u, v \) will not be boundary vertices as otherwise \( N(u) \not\subseteq V_i \). We have two cases:

\[ G[N(u)] \text{ has edges:} \] Since \( u, v \) are not boundary vertices, we add \( u, v \) to \( \mathcal{I} \) and remove \( \{u, v\} \cup N(u) \subseteq V_i \) from the graph.

\[ G[N(u)] \text{ has no edges:} \] We only apply this reduction when \( N[N[u]] \subseteq V_i \); we remove \( \{u, v\} \cup N(u) \), and create a new vertex \( w \in V_i \) with neighborhood \( N(w) = N[N[u]] \); otherwise we would modify the adjacency list of a vertex in a different block.

Unconfined: Unlike other blockwise reductions, every vertex \( v \in V_i \) is eligible for the unconfined reduction, including boundary vertices. If a vertex is unconfined, we mark it as excluded from the independent set and remove it from the graph (by setting a flag if \( v \) is a boundary vertex). However, the algorithm for finding unconfined vertices must be adapted—it does not simply rely on a (two-)neighborhood, but depends on an expanding set of vertices \( S \), which should be drawn from \( V_i \) in order to avoid a race condition. In particular, a vertex \( u \in N(S) \) can only be used if \( u \in V_i \) and \( S \subseteq V_i \) must hold. This way, we ensure all vertices that we classify as unconfined are truly unconfined and can be removed from the graph. We might, however, falsely classify some vertices as confined.

Diamond: As with the unconfined reduction, we can safely remove even boundary vertices from the graph by using the diamond reduction. However, since vertices in \( V \setminus V_i \) cannot be inserted into \( S \) during the blockwise unconfined reduction, there might be \( u_1, u_2 \) such that \( N(u_1) \setminus N(S) = N(u_2) \setminus N(S) = \{v_1, v_2\} \) and \( \{v_1, v_2\} \not\subseteq S \) because they are located in different blocks, so we have to check that \( v_1, v_2 \in S \). If not, they might be removed by another reduction which can lead to race conditions.

4.2 Parallel Linear Programming

Unlike the local reductions, the reduction by linear programming is not applied to single vertices and their (near-)neighbors. It instead relies on a global view of the graph to find a set of vertices that can be removed at once. Therefore our parallelization strategy for local reductions cannot be applied to the linear programming reduction. The computationally expensive part of this reduction is finding a maximum bipartite matching of the bi-double graph: \( B(G) = (L_V \cup R_V, E') \), where \( L_V = \{l_v \mid v \in V\} \), \( R_V = \{r_v \mid v \in V\} \), and \( E' = \{(l_u, r_v) \mid \{u, v\} \in E\} \).

Azad et al. [4] give a parallel augmenting path based algorithm for maximum bipartite matching. Their algorithm requires a maximal matching as input, which we first compute using the maximal matching algorithm by Karp and Sipser [28], which was parallelized by Azad et al. [5]. For better performance at repeated applications of the reduction, we reuse the parts of the previous matching which are still part of the graph. If the graph changed only slightly since the last application, this is still close to a maximum matching, which results in less work for the maximum matching algorithm. This technique is also used by Akiba and Iwata [2].

To obtain the half-integral result of the linear program, we use the set of vertices reachable by alternating paths starting from matched vertices in \( L_V \). To find these, we start a depth first search from each vertex \( v \in L_V \) in parallel and mark all reached vertices. We then obtain the result by iterating in parallel over all vertices in the original graph and checking whether their respective vertices in \( L_V \) and \( L_R \) are marked.
5 Pruning Reductions

5.1 Dependency Checking

Akiba and Iwata [2] apply their reductions \( r_1, \ldots, r_j \) by iterating over all reductions and trying to apply the current reduction \( r_1 \) to all vertices. If \( r_1 \) reduces at least one vertex, they restart with reduction \( r_1 \). When reduction \( r_j \) is executed, but does not reduce any vertex, all reductions have been applied exhaustively, and a kernel is found. Trying to apply every reduction to all vertices can be expensive in later stages of the algorithm where few reductions succeed. The algorithm may repeatedly attempt to apply the same reduction to a vertex even though the graph has not changed sufficiently to allow the reduction to succeed. For example, let \( G' \) be a graph obtained by applying reductions to a graph \( G \). If vertex \( v \) is not isolated in \( G \) and \( N_{G'}[v] = N_G[v] \), then \( v \) is still not isolated in \( G' \) and can be pruned from further attempts.

We define a scheme for checking dependencies between reductions, which allows us to avoid applying isolated vertex removal, vertex folding, and twin reductions when they will provably not succeed. After unsuccessfully trying to apply one of these reductions to a vertex \( v \), one only has to consider \( v \) again for reduction after its neighborhood has changed. We therefore keep a set \( D \) of viable candidate vertices: vertices whose neighborhood has changed and vertices that have never been considered for reductions. Initially we set \( D = V \). Then for each \( v \in D \), we remove \( v \) from \( D \) and try to apply our reductions to \( v \). If \( v \) is removed from the graph (or a new vertex \( w \) is inserted), we set \( D = D \cup N[v] \) (or \( D = D \cup N[w] \)). We repeat until \( D \) is empty. Figure 2 shows an example.

Note that this strategy does not support unconfined and diamond reductions, as they depend on a set \( S \) that can include vertices with large distances from the starting vertex. Thus a vertex can become unconfined due to a change in the graph outside of its neighborhood. Neither does it support the linear programming reduction, which operates on the entire graph instead of a single vertex. However, when performing these reductions we continue to add vertices whose neighborhoods have changed to \( D \), saving effort when next attempting isolated vertex removal, vertex folding, and twin reductions.

We briefly mention that targeted forms of dependency checking have been used before. Previous works, including Akiba and Iwata [2] and Chang et al. [12] perform so-called “iterated” reductions, which allow for the repeated application of successful reductions. These include, for example, iteratively removing degree-one and -zero vertices until none remain, and applying the domination reduction when triangle counts change [12]. Unlike these previous works, our focus is on eliminating reductions that cannot be applied, as is not targeted at any particular reduction, but a collection of reductions. Strash [39] implements similar dependency checking for isolated vertex and vertex folding reductions, though it is not mentioned in his paper. We are the first to introduce such a strategy that can be used with any collection of reductions.

5.2 Reduction Tracking: Counteracting Diminishing Returns

It is not always ideal to apply reductions exhaustively—for example if only few reductions will succeed and they are costly. We note that, during later stages of our algorithm, local reductions may lead to very few graph changes, while the linear programming reduction often significantly reduces the graph size. Therefore, it may be better to stop local reductions early before applying the linear programming reduction, as any remaining local reductions can still be applied afterwards. Furthermore, in our parallel algorithm, applying the local reductions exhaustively can take significantly longer for some blocks than for others. That is, the total graph size is not significantly reduced once the first threads finish their blocks.
We therefore implement reduction tracking to detect and stop local reductions when they are not quickly reducing the graph. Once the first thread finishes applying local reductions, we assign it to sample the current graph size at fixed time intervals. We then stop local reductions on all threads when the change in graph size becomes small relative to the rate of vertex removals and switch to the linear programming reduction. We continue local reductions afterwards. For the sequential case, we start sampling the current size immediately when starting the local reductions. Sampling is performed by an additional thread. Although the resulting graph is kernel-like, it may be possible to reduce it further. We call such a graph a quasi kernel.

6 Experimental Evaluation

Methodology. We implemented our algorithm using C++ and compiled all code using gcc 5.4.0 with full optimizations turned on (-O3 flag). For shared memory parallelization we use OpenMP 4.0. Our implementation includes the parallel application of reduction rules, the dependency checking scheme and the reduction tracking technique. For graph partitioning we use ParHIP [32], the parallel version of the KaHIP graph partitioner [38]. We compare against several existing sequential kernelization techniques. For fast reduction strategies, we compare against the kernelization routines LinearTime and NearLinear recently introduced by Chang et al. [12]. We use the authors’ original implementation, written in C. For extensive reduction strategies, we use the full reduction suite of Akiba and Iwata’s VCSolver [2]. We modified their code to stop execution after kernelization and output the kernel size. For all instances, we perform three independent runs of each algorithm. Their code was compiled and run sequentially with Java 1.8.0_102. All results are averages over three runs on a machine with 512 GB RAM and two Intel Xeon E5-2683 v4 processors with 16 cores running at 2.1 GHz each, running Ubuntu 14.04.5 LTS.

Algorithm Configuration. We run our algorithm with all reduction rules explained in this paper but restrict the isolated vertex removal reduction to cliques of size 3 or less. We use the ultrafast configuration of the parallel partitioner and default values for all other parameters. When running our algorithm in parallel on \( p \) threads, we partition the graph into \( p \) blocks. We stop applying local reduction rules when the reduction in graph size per time during the last time interval is less than 5% of the average size reduction per time since starting to apply local reductions (i.e., since the last application of the linear programming reduction). An experimental evaluation of this technique can be found in Appendix C.3. As the LinearTime algorithm by Chang et al. [12] has very low running times and reduces the initial graph size, we run it as a preprocessing step using the original implementation. We then partition the resulting kernel and process it with our parallel kernelization algorithm. Throughout this section, we will refer to sequential runs of our algorithm as FastKer and to parallel runs (32 threads, unless otherwise stated) as ParFastKer.

Instances. We perform experiments on large web [8] and road networks [6,17], random (hyper)-geometric graphs [24,43] and Delaunay triangulations [32]. Basic instance properties can be found in Appendix B.

6.1 Comparison with State-of-the-Art

We now compare our implementation to the implementations of VCSolver by Akiba and Iwata [2] and the LinearTime and NearLinear algorithm by Chang et al. [12]. Table [1] and Figure [3] give an overview. Figure [5] normalizes running time and kernel size on each instance by the result of VCSolver.

First note that LinearTime’s running time is almost negligible compared to that of VCSolver, almost never surpassing 1% of VCSolver’s time. LinearTime also decreases the graph size significantly for most graphs (except for the Delaunay triangulations, where LinearTime is not able to reduce the graph size at all), however, the LinearTime kernel is still orders of magnitude larger than VCSolver’s kernel. Due to fast running time and graph size reduction, we use LinearTime as a preprocessing step to our algorithm.
The NearLinear algorithm by Chang et al. [12] uses fewer reduction rules than our algorithm, so it finds larger kernels, often orders of magnitude larger than the kernels by VCSolver and our algorithms. The largest relative difference to the smallest kernel size of NearLinear is the 1 329 923-vertex kernel for europe.osm. This is 159 times larger than the smallest kernel and 94 times larger than the quasi kernel found by ParFastKer. For the Delaunay triangulations and the random geometric graph, the relative kernel size difference is comparatively low. This is because the kernel for these graphs is still very large compared to the input size, but we find quasi kernels much closer to the size found by VCSolver than NearLinear. LinearTime actually cannot remove any vertices from the Delaunay instances and only very few from the random geometric instance. In the geometric mean, LinearTime’s kernel is a factor 12 larger than ParFastKer’s quasi kernel and NearLinear’s a factor 7. Due to NearLinear’s fast worst-case running time, it runs faster than FastKer on 8 out of 12 instances and on 2 instances even faster than ParFastKer. As LinearTime is a preprocessing step of our algorithm, it is of course always faster.

As VCSolver implements a larger set of reduction rules, adding the desk and funnel reductions by Xiao and Nagamochi [44] as well as Akiba and Iwata’s own packing reduction rule, it achieves smaller kernel sizes. In the geometric mean, ParFastKer’s quasi kernel’s are 20% larger than VCSolver’s (excluding rhg, which has an empty kernel). However, comparing the kernel sizes to the size of the input network, these differences in size are negligible. The largest obtained difference relative to the size of the input network is 2.9% on del24 (0.6% on sk-2005 when only considering the real-world instances). In addition, VCSolver only applies a scheme similar to our dependency checking for the removal of degree zero and one vertices, so our algorithm runs faster on all instances except rhg where FastKer takes 138 seconds for kernelization and VCSolver finishes after 124 seconds. ParFastKer, however, is faster than VCSolver on this instance, taking 65 seconds. On 11 out of 15 instances, FastKer is faster by a factor of over 6 than VCSolver and on 4 instances even by a factor of over 40. The largest speedup of FastKer over VCSolver is 142 on asia.osm and the geometric mean of the speedups is 12. As FastKer is the sequential version of our algorithm, this is a purely algorithmic speedup. Using parallelization, ParFastKer achieves speedups of 41 over VCSolver in the geometric mean, combining the algorithmic speedup with parallel speedup. On all instances, except for rhg, the speedup is over 10 and on 9 instances over 50. For exact running times and kernel sizes for all benchmark instances, we refer the reader to the appendix.

### 6.2 Scalability

Figure 4 shows the parallel speedup of our algorithm on the six hardest instances of our benchmark set (i.e. those with the longest sequential running time). The left plot shows the total speedup relative to two threads for all parts of our algorithm combined: LinearTime preprocessing, partitioning and parallel reductions with dependency checking and inexact reduction pruning. The center and right plots show the speedups for the reductions parallelized by partitioning and the reduction by linear programming, respectively. The
preprocessing step of our algorithm, the LinearTime algorithm by Chang et al., is sequential and thus limits the possible scalability of our parallelization, however running times are very short.

We observe that, due to the overhead caused by having to find a partition of the graph, the single threaded execution is on average 1.9 times faster than the parallelization using 2 threads. However, our algorithm scales well so that parallelization brings better performance for higher numbers of threads. Compared to the two-threaded case, our highest speedup is 46.5 for rgg26 on 32 threads. The main reason for this is that reductions on this graph are so slow that, for low thread counts, our inexact reduction pruning technique stops local reductions early, switching to a very long lasting reduction by linear programming. For the other graphs, the speedup on 32 threads compared to 2 threads is between 6 and 16.3 with 16 being perfect speedup. The speedup relative to the single threaded case is between 2.9 and 12 (42.5 for rgg26).

Figure 4 shows that local reductions parallelized by partitioning are faster single threaded than on two threads. This is caused by our inexact reduction pruning technique which starts after the first thread finishes reductions. When the number of threads is low, reductions might already have become too slow when the first thread finishes, causing longer times of slow size reduction. For higher thread counts, there is always a thread that finishes while other threads are still applying reductions fast and thus less time is wasted by slow reductions. After the drop at two threads, the speedup for 32 threads compared to 2 threads for these reductions is between 8 and 37 (between 5 and 31 compared to 1 thread). For some graphs, the reduction by linear programming, which we parallelized using the parallel maximum bipartite matching algorithm by Azad et al. [4], is a bottleneck of our algorithm.

6.3 Local Search on the Quasi Kernel

We now demonstrate the impact that quickly finding a small quasi kernel has on algorithms for finding large independent sets with local search. Currently, the algorithms with the best trade-off between speed and solution quality are LinearTime and NearLinear by Chang et al. [12]. In our previous experiments, we compared only against the LinearTime and NearLinear kernelization. However, we now run the full algorithm of Chang et al., which first kernelizes the graph, then invokes “reducing-peeling” to compute an initial solution for local search, and then runs local search [3] on the kernel. We compare their original algorithms against variants that first kernelize the graph with ParFastKer and then run on the quasi kernel. We use a time limit of 30 minutes, including kernelization and finding an initial solution. Figure 5 shows the size of the independent set found over time for the largest web graph, road network and generated graph.
from our benchmark set (excluding graphs that their implementation cannot process due to their 32-bit implementation). Results for the other graphs from our benchmark set can be found in Appendix C.4.

For web graphs we see that using ParFastKer’s quasi kernel, the independent set found is much larger (80 009 858 for webbase-2001) than the one found by LinearTime (18 286 vertices less) and NearLinear only converges to approximately the size found using the quasi kernel after several hundred seconds. On road networks, we observe interesting behavior: local search seemingly converges for all algorithms, but to different independent set sizes: the smaller the initial kernel size, the larger independent set size. On europe.osm, the final solution size is 25 633 238 for LinearTime, 84 more for NearLinear and 188 more for both versions that use ParFastKer’s quasi kernel. Also, using ParFastKer’s quasi kernel, the algorithm converges much faster. In particular, after an initial improvement over the starting solution, that takes about 0.1 seconds plus the time for kernelization and finding an initial solution (which is about 5 seconds), very few changes occur with ParFastKer’s quasi kernel. LinearTime and NearLinear, on the other hand, make an increase of several hundred vertices for the first 30 to 40 seconds. On the Delaunay triangulation graphs, the smaller quasi kernel enables local search to find larger independent sets.

7 Conclusion

We presented an efficient parallel kernelization algorithm based on graph partitioning and parallel bipartite maximum matching, vertex pruning as well as reduction tracking. On the one hand, our algorithm produces kernels that are orders of magnitude smaller than the fastest kernelization methods, while having a similar execution time. On the other hand, our algorithm is able to compute kernels with size comparable to the smallest known kernels, but up to two orders of magnitude faster that previously possible. Experiments with local search algorithms show that we find larger independent sets faster. In future work, we want to apply our parallel kernelization techniques in more MIS algorithms, such as exact branch-and-reduce [2], and transfer our techniques to other problems that use kernelization such as [10,27,23].

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A High Level Algorithm Overview

Algorithm 1 Algorithm Overview

\[
G \leftarrow \text{input graph}
\]
\[
\{V_1, \ldots, V_k\} \leftarrow \text{partition}(G, k)
\]
\[\text{while } G \text{ changed in last iteration do}
\]
\[\text{for all blocks } V_i \text{ in parallel do}
\]
\[G \leftarrow \text{localReductions}(G, V_i)
\]
\[\text{end for}
\]
\[G \leftarrow \text{parallelLinearProgrammingReduction}(G)
\]
\[\text{end while}
\]

B Benchmark Properties

| name         | type     | # vertices | # edges  | source |
|--------------|----------|------------|----------|--------|
| uk-2002      | web      | 18.5 M     | 261.8 M  | [6]    |
| arabic-2005  | web      | 22.7 M     | 553.9 M  | [8]    |
| gsh-2015-tpd | web      | 30.8 M     | 489.7 M  | [8]    |
| uk-2005      | web      | 39.5 M     | 783.0 M  | [8]    |
| it-2004      | web      | 41.3 M     | 1 027.5 M| [8]    |
| sk-2005      | web      | 50.6 M     | 1 810.1 M| [8]    |
| uk-2007-05   | web      | 105.9 M    | 3 301.9 M| [8]    |
| webbase-2001 | web      | 118.1 M    | 854.8 M  | [8]    |
| asia.osm     | road     | 12.0 M     | 12.7 M   | [6]    |
| road_usa     | road     | 23.9 M     | 28.9 M   | [17]   |
| europe.osm   | road     | 50.9 M     | 54.1 M   | [6]    |
| rgg26        | rgg      | 67.1 M     | 574.6 M  | [6]    |
| rhg          | rhg      | 100.0 M    | 1 999.5 M| [43]   |
| del24        | delaunay | 16.8 M     | 50.3 M   | [6]    |
| del26        | delaunay | 67.1 M     | 201.3 M  | [32]   |

Table 2: Basic properties of the graphs used in our evaluation.

C Detailed Results

Here, we provide detailed results of our experiments. In addition to the time to reach a quasi kernel, we also provide the time it takes to reach a full kernel. We do this by first applying our algorithm as described throughout the paper to find a quasi kernel. We then apply the remaining reductions by running sequentially and disabling the inexact reduction pruning technique described in Section 5.2.

In the comparisons to the other algorithms, we also provide columns for a “same size comparison”. This is found by logging the current time and size throughout the algorithms. When comparing two algorithms with different kernel sizes, the time column of the same size comparison then reports the first time stamp at which the algorithm with the smaller kernel size logged a size smaller than (or equal to) the final size of the algorithm with the larger kernel.
The implementation by Chang et al. uses 32 bit integers as edge identifiers, so they cannot process graphs with large numbers of edges. Respective entries in the tables are marked with a star (*). As our algorithm uses their LinearTime implementation as a preprocessing steps, for graphs with too many edges, we use the original graph as input to our algorithm instead of the kernel found by LinearTime.

| Graph          | LinearTime | NearLinear | VCSolver | FastKer | ParFastKer |
|----------------|------------|------------|----------|---------|------------|
| name           | n          | [K] time   | [K] time | [K] time | [K] time   |
| uk-2002        | 19M        | 11.7M 1.5  | 4.0M 28.0 | 0.2M 336.9 | 0.3M 48.9  |
| arabic-2005    | 23M        | 15.6M 2.6  | 6.7M 246.1 | 0.6M 1033.2 | 0.6M 124.7 |
| gsh-2015-tpd   | 31M        | 2.0M 11.6  | 1.2M 97.4  | 0.4M 372.3  | 0.4M 54.0  |
| uk-2005        | 39M        | 28.2M 2.5  | 5.9M 60.5  | 0.8M 541.4  | 0.9M 116.2 |
| it-2004        | 41M        | 27.1M 3.3  | 11.3M 1544.6 | 1.6M 6749.0 | 1.7M 439.3 |
| sk-2005        | 51M        | * * * *    | * * * *   | 3.2M 1010.5 | 3.3M 2113.2 |
| uk-2007-05     | 106M       | * * * *    | * * * *   | 3.5M 18829.4 | 3.6M 17689.9 |
| webbase-2001   | 118M       | 51.7M 13.0 | 17.3M 121.1 | 0.7M 4207.8 | 0.8M 303.5 |
| asia.osm       | 12M        | 626.7K 0.8 | 594.4K 1.4 | 15.2K 204.7 | 34.9K 1.4 |
| road_usa       | 24M        | 2.5M 2.5   | 2.4M 4.1  | 0.2M 310.0  | 0.2M 6.8   |
| europe.osm     | 51M        | 1500.0K 4.1| 1329.9K 6.1| 8.4K 304.2  | 14.1K 5.4  |
| rgg26          | 67M        | 67.1M 1.0  | 51.3M 172.6 | 49.6M 9887.7 | 49.8M 6329.1 |
| rhg            | 100M       | * * * *    | * * * *   | 0 124.0  | 0 138.1  |
| del24          | 17M        | 16.8M 0.2  | 15.6M 12.7 | 12.4M 4789.5 | 12.9M 128.0 |
| del26          | 67M        | 67.1M 0.7  | 62.5M 53.3 | 49.9M 20728.7 | 51.7M 506.3 |

Table 3: Running times and kernel sizes ([K]) for all algorithms. The column “su” is the speedup of ParFastKer over VCSolver. Instances marked with a star (*) cannot be processed by the NearLinear and LinearTime implementations due to the 32-bit implementation. All times are in seconds. The fastest time and smallest kernel for each graph is displayed in bold and italic, respectively.

C.1 Sequential Results

| Graph  | quasi kernel | kernel |
|--------|--------------|--------|
| name   | [K] | LinearTime [s] | all reductions [s] | total [s] | [K] | time [s] | total [s] |
| uk-2002| 255 498 | 1.5 | 47.4 | 48.9 | 255 498 | 1.5 | 50.3 |
| arabic-2005 | 610 643 | 2.6 | 122.2 | 124.7 | 610 643 | 2.9 | 127.7 |
| gsh-2015-tpd | 425 736 | 11.6 | 42.4 | 54.0 | 425 736 | 1.8 | 55.8 |
| uk-2005 | 854 511 | 2.5 | 113.7 | 116.2 | 854 511 | 3.2 | 119.4 |
| it-2004 | 1650 982 | 3.3 | 436.0 | 439.3 | 1650 982 | 1.8 | 1069.1 |
| sk-2005 | 3264 231 | 2 | 3113.5 | 2111.3 | 3264 231 | 691.7 | 2804.9 |
| uk-2007-05 | 3631 632 | * | 1768.9 | 1768.9 | 3631 632 | 2566.3 | 4335.2 |
| webbase-2001 | 821 094 | 13.0 | 309.5 | 303.5 | 821 094 | 24.3 | 327.8 |
| asia.osm | 34 930 | 0.8 | 0.6 | 1.4 | 34 930 | 0.1 | 1.5 |
| road_usa | 247 395 | 2.5 | 4.3 | 6.8 | 247 395 | 0.3 | 7.1 |
| europe.osm | 14 066 | 4.1 | 1.3 | 5.4 | 14 066 | 0.1 | 5.5 |
| rgg26 | 49 843 889 | 1.0 | 6328.0 | 6329.1 | 49 837 688 | 919.4 | 7248.5 |
| rhg | 0 | * | 138.1 | 138.1 | 0 | 7.8 | 145.9 |
| del24 | 12 884 246 | 0.2 | 127.8 | 128.0 | 12 877 158 | 159.2 | 287.3 |
| del26 | 51 674 080 | 0.7 | 505.6 | 506.3 | 51 624 274 | 626.1 | 1132.5 |

Table 4: Kernel sizes and kernelization times for FastKer to reach a quasi kernel and to reach a full kernel by running our algorithm without stopping the reduction application on the quasi kernel.
C.2 Parallel Results

| graph       | \(|K|\) | LinearTime [s] | part. [s] | all reductions [s] | total [s] | \(|K|\) | time [s] | total [s] |
|-------------|--------|----------------|-----------|--------------------|-----------|--------|---------|-----------|
| uk-2002     | 266328 | 1.5            | 4.9       | 5.4                | 11.8      | 255594 | 4.8     | 16.6      |
| arabic-2005 | 628850 | 2.6            | 8.3       | 14.8               | 25.7      | 610288 | 21.0    | 46.6      |
| gsh-2015-tpd| 486328 | 11.6           | 13.5      | 6.6                | 31.7      | 425751 | 19.7    | 51.5      |
| uk-2005     | 901896 | 2.5            | 35.5      | 14.8               | 52.8      | 854383 | 18.1    | 70.9      |
| it-2004     | 1697934| 3.3            | 30.0      | 117.8              | 151.1     | 1645643| 148.2   | 299.3     |
| sk-2005     | 3504786| *              | 70.0      | 104.8              | 174.8     | 3256591| 211.7   | 386.6     |
| uk-2007-05  | 3735056| *              | 71.0      | 298.5              | 369.5     | 3629214| 510.0   | 879.5     |
| webbase-2001| 869443 | 13.0           | 18.3      | 23.6               | 54.9      | 821131 | 14.8    | 69.7      |
| asia.osm    | 34851  | 0.8            | 0.3       | 0.1                | 1.2       | 34823  | 0.1     | 1.3       |
| road_usa    | 246939 | 2.5            | 1.2       | 0.4                | 4.0       | 246939 | 0.3     | 4.3       |
| europe.osm  | 14152  | 4.1            | 0.7       | 0.2                | 5.0       | 14096  | 0.1     | 5.2       |
| rgg26       | 49847428| 1.0            | 44.2      | 103.7              | 148.9     | 4983810| 780.7   | 929.6     |
| rhg         | 16     | *              | 44.4      | 20.7               | 65.1      | 14096  | 0.1     | 5.2       |
| del24       | 12901142| 0.2            | 31.7      | 19.0               | 50.9      | 1287729| 135.0   | 185.8     |
| del26       | 51668286| 0.7            | 86.9      | 88.5               | 176.1     | 51624361| 611.0   | 787.0     |

Table 5: Kernel sizes and kernelization times for ParFastKer to reach a quasi kernel and to reach a full kernel by running our algorithm sequentially and without stopping the reduction application on the quasi kernel.

| graph       | VCSolver | ParFastKer |
|-------------|----------|------------|
| uk-2002     | 266328   | 266328     |
| arabic-2005 | 628850   | 628850     |
| gsh-2015-tpd| 486328   | 486328     |
| uk-2005     | 901896   | 901896     |
| it-2004     | 1697934  | 1697934    |
| sk-2005     | 3504786  | 3504786    |
| uk-2007-05  | 3735056  | 3735056    |
| webbase-2001| 869443  | 869443     |
| asia.osm    | 34851    | 34851      |
| road_usa    | 246939   | 246939     |
| europe.osm  | 14152    | 14152      |
| rgg26       | 49847428 | 49847428   |
| rhg         | 16       | 16         |
| del24       | 12901142 | 12901142   |
| del26       | 51668286 | 51668286   |

Table 6: Comparison between VCSolver and ParFastKer. "Same size comparison" compares the time that the algorithm with the smaller kernel size takes to reach the final size of the algorithm with the larger kernel.
C.3 Reduction Tracking: Counteracting Diminishing Returns

Table 7 shows the effect of our reduction tracking technique described in Section 5.2 on ParFastKer. It shows the algorithmic speedup achieved by enabling reduction tracking. Only graphs with speedup \( \geq 1.0 \) are shown. All other speedups are 1.0.

Our experiments show that stopping long lasting reductions early can lead to significant speedups with close to no penalty on the quasi kernel size. The quasi kernel size found with reduction tracking enabled is never more than 0.1% larger than without it. In fact, the quasi kernel is sometimes even slightly smaller. The reason for this is that different orders of reduction application can lead to different kernel sizes.

C.4 Local Search on the Quaski Kernel

| Graph        | Speedup |
|--------------|---------|
| uk-2002      | 1.1     |
| arabic-2005  | 2.2     |
| it-2004      | 1.8     |
| sk-2005      | 137.7   |
| uk-2007-05   | 2.4     |
| webbase-2001 | 1.3     |
| rgg26        | 1.2     |

Table 7: Overview of our reduction tracking technique for graphs with speedup \( \geq 1.0 \).