Normalized non-redundant vector tomographic portraits of spin states

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Abstract

Non-redundant and normalized four-component vector tomographic portrait fully describing the states of spin 1/2 quantum particles was introduced. Dequantizer and quantizer for such portrait were found, and generalization to the case of spin \((2^N - 1)/2\) was done \((N\) is a natural number). It was shown that such a portrait is completely defined by a triple of non-complanar vectors with the lengths equal or less then unity. A clear geometric interpretation of the choice of parameters for finding normalized dequantizers and quantizers is presented and numerical examples of such dequantizers and quantizers for spin 1/2 are given.

Keywords: Tomographic representation, quantum tomography, spin tomogram, non-negative vector portrait of state.

1 Introduction

As is known, in the traditional formulation of quantum mechanics the pure states of a quantum particle with spin \(s\) are described by \((2s + 1)\)-component complex spinors \((\psi_1, \psi_2, ..., \psi_{2s+1})\). Mixed states of such particles are represented by \((2s + 1) \times (2s + 1)\)-density matrices whose off-diagonal elements, in general, are also complex. On the other hand, the tomographic approach (see [1, 2, 3]) makes it possible to portray the states of quantum systems by real nonnegative quantities. In [4] the tomographic distribution for rotated spin variables was constructed, but the method suggested there is inconvenient because of redundant data containing in the tomogram.

The tomographic description is closely connected with the state reconstruction problem, to the solution of which for the spin states the following papers were devoted: [5, 6, 7, 8, 9, 10, 11, 12]. The scope of this problem is finding of the transformation procedure of recovering of the density matrix from the set of expectation values of observables constituting a quorum. In the state reconstruction problem the superfluous amount of data for obtaining of the density matrix is possible not to consider as deficiency. Often the extra data can enable to carry out more exact accounts. On the contrary, in the tomographic formulation of quantum mechanics the redundant data in the tomogram for application of the inverse map is an essential inconvenience.

The tomographic description of systems with spin was also evolved in [13, 14] and in other papers. In [15, 16] it was introduced the positive non-redundant vector tomographic portrait fully describing the states of quantum particles including both spatial and spin information. In the case of consideration of only spin subspace the essence of this approach is based on the inverse problem studied in [7, 9, 10]. But we will follow the dequantizer – quantizer terminology used in [15, 16], where the states of quantum particles with spin \(s\) are described as the \((2s + 1)^2\)-component vector tomographic portrait \(\mathbf{w} = (w_1, w_2, ..., w_{(2s+1)^2})\),

\[
\mathbf{w} = \text{Tr}\{\hat{\rho} \mathbf{U}\},
\]

(1)

and \(\hat{\mathbf{U}} = (\hat{u}_1, \hat{u}_2, ..., \hat{u}_{(2s+1)^2})\) is \((2s + 1)^2\)-component dequantizer vector with components \(\hat{u}_k\) that are projectors onto the \((2s + 1)^2\) pure and/or mixed spin states, which are chosen so that the matrix of linear transformation \(\hat{\rho} \rightarrow \mathbf{w}\) would be reversible.
To find the inverse transformation of (1) the \((2s + 1)^2\)-component quantizer vector \(\hat{D}\) is used, whose components \(\hat{D}_1, \hat{D}_2, \ldots, \hat{D}_{(2s+1)^2}\) are the \((2s + 1) \times (2s + 1)\)-matrixes satisfying the conditions

\[
\text{Tr} \left\{ \hat{U}_j \hat{D}_{j'} \right\} = \sum_{k,l=1}^{2s+1} U_{j(kl)} D_{j'(lk)} = \delta_{jj'}; \quad \sum_{j=1}^{(2s+1)^2} U_{j(kl)} D_{j'(lk')} = \delta_{kk'} \delta_{ll'}.
\]

Here letters \(j, j' = 1, (2s + 1)^2\) are the indexes corresponding to the numbers of the components of the tomographic vector \(w\), and letters in parentheses \((kl)\) or \((k'l')\) are the spin indexes of \((2s + 1) \times (2s + 1)\)-matrices.

Using the quantizer \(\hat{D}\) the inverse transformation of (1) is written as the scalar product of two vectors

\[
\hat{\rho} = w \hat{D}.
\]

The components of the vector \(\hat{U}\) or \(\hat{D}\) form the basis in the space of \((2s + 1) \times (2s + 1)\)-matrices, and any \((2s + 1) \times (2s + 1)\)-density matrix is a convex sum of the components of \(D\).

It follows from (3) that the normalization condition of \(\hat{\rho}\) can be written in terms of \(w\):

\[
\text{Tr} \hat{\rho} = \left( \text{Tr} \hat{D} \right) w.
\]

The components of \(w\) satisfy the conditions \(0 \leq w_i \leq 1\). As for the normalization of the vector \(w\), then its existence depends on the choice of the projectors \(\{\hat{U}_k\}_{k=1}^{(2s+1)^2}\). It is obvious from (1) that

\[
\sum_{j=1}^{(2s+1)^2} w_j = \sum_{j=1}^{(2s+1)^2} \text{Tr} \{\hat{\rho} \hat{U}_{j} \} = \text{Tr} \left\{ \hat{\rho} \sum_{j=1}^{(2s+1)^2} \hat{U}_{j} \right\}.
\]

Therefore, \(w\), in general, is not normalized to a constant number. In [15] we have constructed an example of such a dequantizer for the spin \(1/2\), for which only the third and the fourth components of \(w\) are related by the normalization condition \(w_3 + w_4 = 1\).

The absence of a constant normalization of the tomographic vector leads to inconveniences in numerical calculations and limits the range of use of this approach in practical applications. Therefore, the question of finding of the projecting states, which ensure the fulfillment of the equality

\[
\sum_{j=1}^{(2s+1)^2} w_j = \text{const},
\]

is topical.

The aim of this work is the construction of normalized and non-redundant vector tomographic portraits of spin states.

The paper is organized as follows. In Sections 2 and 3 the normalized four-component dequantizer vector for the spin \(1/2\) without redundancy is found in general case, the corresponding quantizer vector is calculated, and their properties are investigated. In Section 4 a graphic geometric interpretation of the choice of parameters for finding of realizations of such dequantizers is presented, and two examples of dequantizers and quantizers are given, whose components are pure and mixed states respectively. In Section 5 the procedure for finding the quantizers and dequantizers is generalized for normalized and non-redundant vector tomographic portraits of spin \(s = (2^N - 1)/2\) states, where \(N\) is a natural number.

The conclusion and prospects are presented in 6.
Normalized non-redundant dequantizer and its properties

From (5) it is obvious that for the fulfillment of (6) for any normalized state \( \hat{\rho} \) it is necessary and sufficient that the relation
\[
\sum_{j=1}^{(2s+1)^2} \hat{U}_j = \text{const} \times \hat{E}
\]
must be satisfied, where \( \hat{E} \) is the unit \((2s+1) \times (2s+1)\)-matrix. Since the projections \( \hat{U}_j \) are normalized by the condition \( \text{Tr} \hat{U}_j = 1 \), then
\[
\sum_{j=1}^{(2s+1)^2} \hat{U}_j(\hat{k}\hat{k}) = 2s+1, \quad k = 1, 2s+1.
\]
Therefore, in (6) we have the equality \( \text{const} = 2s+1 \), i.e., the normalization condition for the vector \( w \) has the form
\[
(2s+1)^2 \sum_{j=1}^{(2s+1)^2} w_j = 2s+1, \quad (7)
\]
and for the components of the matrix vector \( \hat{U} \) the following relation is fulfilled:
\[
\sum_{j=1}^{(2s+1)^2} \hat{U}_j = (2s+1) \times \hat{E}. \quad (8)
\]

However, if we multiply the matrices \( \hat{U}_j \) by some weight factor, we can obtain any preassigned \( \text{const} \).

We can also construct a weighted tomographic scheme if instead of (6) we introduce the requirement for the normalization of the vector \( w \) with the set of weights \( \eta_k \) as follows:
\[
\sum_{j=1}^{(2s+1)^2} \eta_j \hat{U}_j = \hat{E}. \quad \text{Then relation (8) will take the form:} \quad \sum_{j=1}^{(2s+1)^2} \eta_j \hat{U}_j = \hat{E}. \quad \text{But in the present paper we restrict ourselves to the case (7).}
\]

Let us find matrices \( \hat{U}_k \) satisfying (8) for spin \( s = 1/2 \) and explore their properties.

The real 4-component vector \( w \) represents a state if and only if the density matrix being received from (3) is Hermitian, non-negative, and normalized. So, the following relations must be valid:
\[
\sum_{j=1}^{4} D_{j(11)} w_j > 0, \quad \sum_{j=1}^{4} D_{j(22)} w_j > 0 \quad \text{(one of these two sums can also be equal zero)}, \quad (9)
\]
and
\[
\left( \sum_{i=1}^{4} D_{i(11)} w_i \right) \times \left( \sum_{k=1}^{4} D_{k(22)} w_k \right) - \left| \sum_{l=1}^{4} D_{l(12)} w_l \right|^2 \geq 0, \quad (10)
\]
\[
\sum_{k=1}^{4} \text{Tr}\{\hat{D}_k\} w_k = 1. \quad (11)
\]

Hermiticity of \( \hat{\rho} \) is automatically provided owing to the hermiticity of \( \{\hat{D}_j\}_{j=1}^{13} \).

If we choose the unit vector \( e_k = (\alpha_k, \beta_k, \gamma_k) \) whose components are normalized as
\[
|e_k| = \alpha_k^2 + \beta_k^2 + \gamma_k^2 = 1, \quad (12)
\]
then the wave vector of the state with the spin projection along \( e_k \), reliably equal to \( 1/2 \), is found from equation \( e_k \hat{s} \psi_k = \psi_k/2 \), where \( \hat{s} \) is the spin operator. With the accuracy up to a phase factor we have
\[
\psi_k = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{\gamma_k + 1} \\ \alpha_k + i\beta_k \\ \sqrt{\gamma_k + 1} \end{pmatrix}. \quad (13)
\]

Taking the matrix product \( \psi_k \psi_k^\dagger \) we find the projector corresponding to this state
\[
\hat{U}_k = \psi_k \psi_k^\dagger = \frac{1}{2} \begin{bmatrix} \gamma_k + 1 & \alpha_k - i\beta_k \\ \alpha_k + i\beta_k & 1 - \gamma_k \end{bmatrix}. \quad (14)
\]
By replacing the indexes 2

\[ u_{k(2)} = \frac{1}{2} \pm \frac{1}{2} \sqrt{\alpha_k^2 + 2\beta_k + \gamma_k^2}, \quad \text{det} \hat{U}_k = u_{k(1)}u_{k(2)} = \left(1 - \alpha_k^2 - \beta_k^2 - \gamma_k^2\right)/4. \]  

(15)

By direct calculation we also find \( \text{Tr}\{\hat{U}_k\hat{U}_k\} \) and \( \text{Tr}\{\hat{U}_j\hat{U}_k\} \)

\[ \text{Tr}\{\hat{U}_k\hat{U}_k\} = u_{k(1)}^2 + u_{k(2)}^2 = \left(1 + \alpha_k^2 + \beta_k^2 + \gamma_k^2\right)/2, \]

(16)

\[ \text{Tr}\{\hat{U}_j\hat{U}_k\} = (1 + \alpha_j\alpha_k + \beta_j\beta_k + \gamma_j\gamma_k)/2. \]

(17)

Since the matrices \( \hat{U}_k \) correspond to pure states \( \{\hat{e}_k\} \), for which the vectors \( \{e_k\} \) are normalized by condition (12), then, as it should be, \( u_{k(1)} = 0 \), \( u_{k(2)} = 1 \), \text{det} \( \hat{U}_k = 0 \), and \( \text{Tr}\{\hat{U}_k\hat{U}_k\} = 1 \).

However, with the help of sets of values \( \{\alpha_k, \beta_k, \gamma_k\}\) \( k = 1, 4 \) we can also parameterize the mixed states \( \{\hat{U}_k\}_k \). For this, the matrices \( \hat{U}_k \) must be positive definite, i.e., the following inequalities must be satisfied:

\[ u_{k(1)} > 0, \quad u_{k(11)} > 0, \quad u_{k(22)} > 0, \quad \text{det} \hat{U}_k > 0, \quad \text{Tr}\{\hat{U}_k^2\} < 1. \]

(18)

From formulas (14,15,16) it is obvious that (18) is fulfilled if

\[ \alpha_k^2 + \beta_k^2 + \gamma_k^2 < 1. \]

(19)

Note also that from (16) the estimate \( \text{Tr}\{\hat{U}_2\hat{U}_k\} > 1/2 \) follows, and since \( \hat{U}_k \) can be any density matrix, then this inequality is true for any state \( \hat{\rho} \) of spin 1/2, i.e.,

\[ \text{Tr}\{\hat{\rho}^2\} > 1/2. \]

(20)

Further we will specifically indicate whether we use pure or mixed states \( \hat{U}_k \), i.e., when the equalities (12) or inequalities (19) are fulfilled respectively, and in the absence of such an indication we will assume that our reasoning is true in the general case for both pure and mixed states.

Matrix equation (8) is obviously equivalent to the following vector equation:

\[ e_1 + e_2 + e_3 + e_4 = 0. \]

(21)

The traces of the products of the matrices \( \hat{U}_j\hat{U}_k \) satisfy some additional conditions. To derive them, we write down formula (8) as \( \hat{u}_1 + \hat{u}_2 = 2\hat{E} - \hat{U}_3 - \hat{U}_4 \), lift the left and right sides of this equality to a square, and take the \( \text{Tr}\{\cdot\} \) operation. Since \( 4\text{Tr}\hat{E} - 4\text{Tr}\hat{U}_3 + 4\text{Tr}\hat{U}_4 = 0 \), then

\[ 2\text{Tr}\{\hat{U}_1\hat{U}_2\} + \text{Tr}\{\hat{U}_1\hat{U}_1\} + \text{Tr}\{\hat{U}_2\hat{U}_2\} = 2\text{Tr}\{\hat{U}_3\hat{U}_4\} + \text{Tr}\{\hat{U}_3\hat{U}_3\} + \text{Tr}\{\hat{U}_4\hat{U}_4\}. \]

(22)

By replacing the indexes 2 \( \leftrightarrow \) 3 or 1 \( \leftrightarrow \) 3 we obtain formulas for the other remaining products:

\[ 2\text{Tr}\{\hat{U}_1\hat{U}_3\} + \text{Tr}\{\hat{U}_1\hat{U}_1\} + \text{Tr}\{\hat{U}_3\hat{U}_3\} = 2\text{Tr}\{\hat{U}_2\hat{U}_4\} + \text{Tr}\{\hat{U}_2\hat{U}_2\} + \text{Tr}\{\hat{U}_4\hat{U}_4\}, \]

(23)

\[ 2\text{Tr}\{\hat{U}_2\hat{U}_3\} + \text{Tr}\{\hat{U}_2\hat{U}_2\} + \text{Tr}\{\hat{U}_3\hat{U}_3\} = 2\text{Tr}\{\hat{U}_1\hat{U}_4\} + \text{Tr}\{\hat{U}_1\hat{U}_1\} + \text{Tr}\{\hat{U}_4\hat{U}_4\}. \]

(24)

These formulas are true for both pure and mixed normalized projecting states \( \hat{U}_k \). The only condition for their fulfillment is the requirement (21), where each vector \( e_k \) can have its own normalization, less than or equal to 1.

If we consider only pure or only mixed projectors \( \hat{U}_k \), for which \( |e_1| = |e_2| = |e_3| = |e_4| \), then from (22 - 24) we get

\[ \text{Tr}\{\hat{U}_1\hat{U}_2\} = \text{Tr}\{\hat{U}_3\hat{U}_4\}, \quad \text{Tr}\{\hat{U}_1\hat{U}_3\} = \text{Tr}\{\hat{U}_2\hat{U}_4\}, \quad \text{Tr}\{\hat{U}_2\hat{U}_3\} = \text{Tr}\{\hat{U}_1\hat{U}_4\}. \]

(25)
3 The quantizer corresponding to the normalized dequantizer

We define the matrix $\hat{R}$ of the dequantizer components $\hat{U}$ and the matrix $\hat{J}$ of the quantizer components $\hat{D}$ as follows:

$$\hat{R} = \begin{pmatrix} U_{1(11)} & U_{1(21)} & U_{1(12)} & U_{1(22)} \\ U_{2(11)} & U_{2(21)} & U_{2(12)} & U_{2(22)} \\ U_{3(11)} & U_{3(21)} & U_{3(12)} & U_{3(22)} \\ U_{4(11)} & U_{4(21)} & U_{4(12)} & U_{4(22)} \end{pmatrix}, \quad \hat{J} = \begin{pmatrix} D_{1(11)} & D_{2(11)} & D_{3(11)} & D_{4(11)} \\ D_{1(12)} & D_{2(12)} & D_{3(12)} & D_{4(12)} \\ D_{1(21)} & D_{2(21)} & D_{3(21)} & D_{4(21)} \\ D_{1(22)} & D_{2(22)} & D_{3(22)} & D_{4(22)} \end{pmatrix}. \quad (26)$$

Then relations (2) obviously take on a simple form $\hat{R} \times \hat{J} = \hat{I}$, where $\hat{I}$ is the unit 4x4-matrix, i.e., the matrices $\hat{R}$ and $\hat{J}$ are mutually inverse, and $\hat{J} = \hat{R}^{-1}$. For the existence of an inverse matrix, the condition $\det \hat{R} \neq 0$ is necessary. The calculation of this determinant with allowance for (21) yields

$$\det \hat{R} = i \Delta_1 \neq 0, \quad \text{where} \quad \Delta_1 = \begin{vmatrix} \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \\ \alpha_4 & \beta_4 & \gamma_4 \end{vmatrix} \neq 0. \quad (27)$$

Since the numbering of the four vectors $\{e_k\}_{k=1}^4$ is chosen arbitrarily, then condition (27) means that for the invertibility of transformation (1), where the components of the dequantizer $\hat{U}$ are given by formula (14), it is necessary and sufficient that any triplex of these vectors must not be coplanar, i.e.,

$$\Delta_2 = \begin{vmatrix} \alpha_3 & \beta_3 & \gamma_3 \\ \alpha_4 & \beta_4 & \gamma_4 \\ \alpha_1 & \beta_1 & \gamma_1 \end{vmatrix} \neq 0, \quad \Delta_3 = \begin{vmatrix} \alpha_4 & \beta_4 & \gamma_4 \\ \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \end{vmatrix} \neq 0, \quad \Delta_4 = \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix} \neq 0, \quad (28)$$

and from (21) it follows that $\Delta_1 = -\Delta_2 = -\Delta_3 = -\Delta_4$. Using the Cramer rule and properties of determinants known from the linear algebra, taking into account (21) we find the components of the quantizer $\hat{D}$:

$$D_{1(11)} = -\frac{1}{4 \Delta_1} \begin{vmatrix} 1 & \alpha_2 & \beta_2 \\ 1 & \alpha_3 & \beta_3 \\ 1 & \alpha_4 & \beta_4 \end{vmatrix} + \frac{1}{4}, \quad D_{1(22)} = \frac{1}{4 \Delta_1} \begin{vmatrix} 1 & \alpha_2 & \beta_2 \\ 1 & \alpha_3 & \beta_3 \\ 1 & \alpha_4 & \beta_4 \end{vmatrix} + \frac{1}{4},$$

$$D_{1(12)} = -\frac{1}{4 \Delta_1} \begin{pmatrix} 1 & \beta_2 & \gamma_2 \\ 1 & \beta_3 & \gamma_3 \\ 1 & \beta_4 & \gamma_4 \end{pmatrix} + i \begin{pmatrix} 1 & \alpha_2 & \gamma_2 \\ 1 & \alpha_3 & \gamma_3 \\ 1 & \alpha_4 & \gamma_4 \end{pmatrix}, \quad D_{1(21)} = [D_{1(12)}]^*. \quad (29)$$

Similar formulas for $D_{2}, D_{3},$ and $D_{4}$ are obtained from (29) by cyclic permutation of indexes 1, 2, 3, 4 corresponding to the components of the tomographic vector $w$.

Let us study the properties of the matrices $\{\hat{D}_k\}_{k=1}^4$ obtained. From (29) it is clear that these matrices are Hermitian and normalized by the condition

$$\text{Tr} \hat{D}_k = 1/2, \quad k = 1, 4. \quad (30)$$

Adding the matrices $\hat{D}_k$ after calculations we get

$$\sum_{k=1}^4 \hat{D}_k = \hat{E}. \quad (31)$$

From the secular equation we easily find the eigenvalues $d_{1(1)}$ and $d_{1(2)}$ of the matrix $\hat{D}_1$

$$d_{1(1,2)} = \frac{1}{4} \pm \frac{1}{4 \Delta_1} \begin{vmatrix} 1 & \alpha_2 & \beta_2 \\ 1 & \alpha_3 & \beta_3 \\ 1 & \alpha_4 & \beta_4 \end{vmatrix}^2 + \begin{vmatrix} 1 & \beta_2 & \gamma_2 \\ 1 & \beta_3 & \gamma_3 \\ 1 & \beta_4 & \gamma_4 \end{vmatrix}^2 + \begin{vmatrix} 1 & \alpha_2 & \gamma_2 \\ 1 & \alpha_3 & \gamma_3 \\ 1 & \alpha_4 & \gamma_4 \end{vmatrix}^2 \quad (32)$$
The eigenvalues of the matrices \( \hat{D}_2, \hat{D}_3, \) and \( \hat{D}_4 \) are obtained from (32) by means of a cyclic permutation of the indexes 1, 2, 3, 4. Knowing the eigenvalues \( d_{k(1,2)} \), we find \( \det \hat{D}_k = d_{k(1)}d_{k(2)} \) and \( \text{Tr}\{\hat{D}_k\hat{D}_k\} = d_{k(1)}^2 + d_{k(2)}^2 \).

The matrices \( \hat{D}_k \) are negative definite; one of the eigenvalues \( d_{k(1,2)} \) is negative, and the other is positive, i.e., \( \det \hat{D}_k = d_{k(1)}d_{k(2)} < 0 \).

For example, we prove this statement for \( \hat{D}_1 \). Since according to (29) \( \hat{D}_1 \) is a Hermitian matrix, then with the aid of a unitary transformation we can reduce it to the diagonal form

\[
\hat{W}^{-1}\hat{D}_1\hat{W} = \begin{pmatrix} d_{1(1)} & 0 \\ 0 & d_{1(2)} \end{pmatrix}, \quad \hat{D}_1 = \hat{W} \begin{pmatrix} d_{1(1)} & 0 \\ 0 & d_{1(2)} \end{pmatrix} \hat{W}^{-1},
\]

where \( \hat{W} \) is a unitary matrix, whose columns are eigenvectors of the matrix \( \hat{D}_1 \). Substituting (33) into (2) and using the properties of the \( \text{Tr}\{\cdot\} \) operation we get

\[
\text{Tr}\{\hat{U}_k\hat{D}_1\} = \text{Tr}\left\{\hat{U}_k' \begin{pmatrix} d_{1(1)} & 0 \\ 0 & d_{1(2)} \end{pmatrix}\right\} = \text{Tr} \left( \hat{U}_k' d_{1(1)} + \hat{U}_{k(22)}d_{1(2)} \right) = \delta_{1k},
\]

where the notation \( \hat{U}_k' = \hat{W}^{-1}\hat{U}_k\hat{W} \) was introduced. Since \( \hat{U}_k \) is a non-negative definite normalized matrix, then \( \hat{U}_k' \) are also non-negative definite and normalized. Therefore \( \text{Tr} \hat{U}_{k(11)} \geq 0, \text{Tr} \hat{U}_{k(22)} \geq 0 \), and \( \text{Tr} \hat{U}_{k(11)} + \text{Tr} \hat{U}_{k(22)} = 1 \).

From (34) we have four equations:

\[
\begin{align*}
\text{Tr} \hat{U}_{k(11)}d_{1(1)} + \text{Tr} \hat{U}_{k(22)}d_{1(2)} &= 1, \\
\text{Tr} \hat{U}_{k(11)}d_{1(1)} + \text{Tr} \hat{U}_{k(22)}d_{1(2)} &= 0,
\end{align*}
\]

\[
\begin{align*}
\text{Tr} \hat{U}_{k(11)}d_{1(1)} + \text{Tr} \hat{U}_{k(22)}d_{1(2)} &= 0, \\
\text{Tr} \hat{U}_{k(11)}d_{1(1)} + \text{Tr} \hat{U}_{k(22)}d_{1(2)} &= 0.
\end{align*}
\]

To satisfy these equations it is necessary that one of the eigenvalues of the matrix \( \hat{D}_1 \) be positive and the other be negative. Thus, \( \hat{D}_1 \) is negative definite. Similarly, negative definiteness is proved for the matrices \( \hat{D}_2, \hat{D}_3, \) and \( \hat{D}_4 \).

We also point out that since (31) is analogous to the equality (8) up to a coefficient of 2 and \( \text{Tr} \hat{U} = 2 \text{Tr} \hat{D} \), then the traces of the products \( \hat{D}_j\hat{D}_k \) satisfy the same equalities (22–24) as the traces of the products \( \hat{U}_j\hat{U}_k \).

### 4 Examples of dequantizers and quantizers

First of all, we indicate that relations (12) and/or (19), (21), and (27) or (28) for the components of the vectors \( e_k \), which determine the conditions for the existence of a reversible and normalized dequantizer, admit a simple geometric interpretation.

Let us construct an arbitrary quadrangle on the plane with sides less than or equal 1. Then let us choose the direction of the bypass of this quadrangle and determine at each side the beginning and the end in accordance with this direction. If you bend such a quadrangle along any of the diagonals, you get a triangular pyramid. Figure 1 shows examples of quadrangles, bending of which along the diagonals indicated by the dashed lines yields a pyramid. Carrying out the turns of this pyramid in space, we can orient it arbitrarily.

The four edges of this pyramid corresponding to the directed sides of the original quadrangle form a quadruple of vectors \{\( e_k \)\}_{k=1}^{4} satisfying (21). If some of these edges has a length equal to one, then it corresponds to a pure state \( \hat{U}_k \). The edges with lengths less than one correspond to mixed states. According to (27) or (28) the volume of our pyramid should not be zero, i.e., the pyramid should not be degenerate.
Figure 1: a), b), c) examples of initial quadrangles on a plane; d) a pyramid obtained by bending a quadrangle along a diagonal indicated by a dashed line.

Since there are an infinite number of such pyramids, then there are infinite number of possible normalized dequantizers, and having such a clear geometric interpretation, it is not difficult to find examples of them.

Example 1. Pure states. Choose the vectors \( \{e_k\} \) as follows: \( e_1 = (0, 4, 3)/5, \ e_2 = (4, 0, -3)/5, \ e_3 = (0, -4, 3)/5, \ e_4 = (-4, 0, -3)/5 \). This is the case of pure states \( \{\hat{U}_k\}_{k=1,4} \) because all four vectors are normalized to 1. With the help of (14) and (29) we find dequantizer \( \hat{U}^{(1)} \) and quantizer \( \hat{D}^{(1)} \) respectively

\[
\hat{U}^{(1)} = \frac{1}{5} \begin{pmatrix}
4 & -2i \\
2i & 1
\end{pmatrix}, \begin{pmatrix}
1 & 2 \\
2 & 4
\end{pmatrix}, \begin{pmatrix}
4 & 2i \\
-2i & 1
\end{pmatrix}, \begin{pmatrix}
1 & -2 \\
-2 & 4
\end{pmatrix},
\]

\[
\hat{D}^{(1)} = \frac{1}{2} \begin{pmatrix}
4/3 & -5i/4 \\
5i/4 & -1/3
\end{pmatrix}, \begin{pmatrix}
-1/3 & 5/4 \\
5/4 & 3/4
\end{pmatrix}, \begin{pmatrix}
4/3 & 5i/4 \\
-5i/4 & -1/3
\end{pmatrix}, \begin{pmatrix}
-1/3 & -5/4 \\
-5/4 & 4/3
\end{pmatrix}.
\]

Example 2. Mixed states. We take the four vectors \( e_1 = (0, -2, 1)/3, \ e_2 = (2, 0, -1)/3, \ e_3 = (0, 2, 1)/3, \ e_4 = (-2, 0, -1)/3 \) normalized to the same number \( \sqrt{5}/3 \), which is less than 1. Then, after calculations, we obtain the dequantizer \( \hat{U}^{(2)} \) with the components that are mixed states, and the corresponding quantizer \( \hat{D}^{(2)} \)

\[
\hat{U}^{(2)} = \frac{1}{3} \begin{pmatrix}
2 & i \\
-i & 1
\end{pmatrix}, \begin{pmatrix}
1 & 1 \\
1 & 2
\end{pmatrix}, \begin{pmatrix}
2 & -i \\
i & 1
\end{pmatrix}, \begin{pmatrix}
1 & -1 \\
-1 & 2
\end{pmatrix},
\]

\[
\hat{D}^{(2)} = \frac{1}{2} \begin{pmatrix}
2 & 3i/2 \\
-3i/2 & -1
\end{pmatrix}, \begin{pmatrix}
-1 & 3/2 \\
3/2 & 2
\end{pmatrix}, \begin{pmatrix}
2 & -3i/2 \\
3i/2 & -1
\end{pmatrix}, \begin{pmatrix}
-1 & -3/2 \\
-3/2 & 2
\end{pmatrix}.
\]

5 Generalization to the case of spin \( s = (2^N - 1)/2 \)

The problem of finding of normalized non-redundant dequantizers and corresponding quantizers for large values of spins in general case turns out to be nontrivial.

At the same time, for single spins \( s = (2^N - 1)/2 \), where \( N \) is a natural number, the application of the well known technique is possible. The point is that the \( (2^N) \times (2^N) \)-density matrices with the components \( \rho_{k,l} \) for such a quantum system can be treated as the \( 2 \times \ldots \times 2 \)-density matrices with the components \( \rho_{k_1,k_2,\ldots,k_N,l_1,l_2,\ldots,l_N} = \rho_{k,l} \) for the system of \( N \) spins 1/2 using some one-to-one correspondence \( g \) of sets of indexes

\[
\left\{ k \mid k = 1, (2s + 1) \right\} \overset{g^{-1}}{\rightarrow} \left\{ (k_1,k_2,\ldots,k_N) \mid k_1,k_2,\ldots,k_N = 1,2 \right\}.
\]
Note that this approach seems to be fruitful for the realization of quantum computations, whose algorithms can be modeled by the evolution of systems of qubits.

The components of the dequantizer $\hat{\mathbf{U}}$ for $p_{k_1,k_2,\ldots,k_N,l_1,l_2,\ldots,l_N}$ can be introduced as direct products of components of dequantizers $\hat{\mathbf{U}}^{(1)}, \hat{\mathbf{U}}^{(2)}, \ldots, \hat{\mathbf{U}}^{(N)}$ for spin 1/2, and these dequantizers can both be the same, $\hat{\mathbf{U}}^{(1)} = \hat{\mathbf{U}}^{(2)} = \ldots = \hat{\mathbf{U}}^{(N)}$, and be different,

$$\hat{\mathbf{U}}_{j_1,j_2,\ldots,j_N} = \hat{\mathbf{U}}^{(1)}_{j_1} \otimes \hat{\mathbf{U}}^{(2)}_{j_2} \otimes \ldots \otimes \hat{\mathbf{U}}^{(N)}_{j_N}, \quad j_1,j_2,\ldots,j_N = 1,4.$$  

(37)

The corresponding quantizer $\hat{\mathbf{D}}$ will have the following components:

$$\hat{\mathbf{D}}_{j_1,j_2,\ldots,j_N} = \hat{\mathbf{D}}^{(1)}_{j_1} \otimes \hat{\mathbf{D}}^{(2)}_{j_2} \otimes \ldots \otimes \hat{\mathbf{D}}^{(N)}_{j_N},$$

(38)

where to each dequantizer $\hat{\mathbf{U}}^{(i)}$ there corresponds its own quantizer $\hat{\mathbf{D}}^{(i)}$. The product of the components $\hat{\mathbf{U}}$ and $\hat{\mathbf{D}}$ will be defined as follows:

$$\hat{\mathbf{U}}_{j_1,j_2,\ldots,j_N} \hat{\mathbf{D}}_{k_1,k_2,\ldots,k_N} = \left( \hat{U}_{j_1} \hat{D}_{k_1} \right) \otimes \left( \hat{U}_{j_2} \hat{D}_{k_2} \right) \otimes \ldots \otimes \left( \hat{U}_{j_N} \hat{D}_{k_N} \right),$$

(39)

from which the orthogonality and completeness conditions immediately follow,

$$\text{Tr} \left\{ \hat{\mathbf{U}}_{j_1,j_2,\ldots,j_N} \hat{\mathbf{D}}_{j_1',j_2',\ldots,j_N'} \right\} = \delta_{j_1,j_1'} \delta_{j_2,j_2'} \ldots \delta_{j_N,j_N'};$$

(40)

$$\sum_{j_1,j_2,\ldots,j_N=1}^{4} \hat{U}_{j_1(k_1l_1)j_2(k_2l_2)\ldots j_N(k_Nl_N)} \hat{D}_{j_1(k_1l_1')j_2(k_2l_2')\ldots j_N(k_Nl_N')} = \delta_{k_1k_1'} \delta_{l_1l_1'} \ldots \delta_{k_Nk_N'} \delta_{l_Nl_N'}. $$

(41)

Using the reverse renaming of indexes $(k_1, k_2, \ldots, k_N) \overset{g^{-1}}{\rightarrow} k$, $(l_1, l_2, \ldots, l_N) \overset{g^{-1}}{\rightarrow} l$ with the help of (36) and the re-designation of indexes $(j_1, j_2, \ldots, j_N) \overset{f}{\rightarrow} j$ with the help of a some one-to-one correspondence $f$

$$\left\{ (j_1, j_2, \ldots, j_N) \bigg| j_1, j_2, \ldots, j_N = 1,4 \right\} \overset{f}{\rightarrow} \left\{ j \bigg| j = 1, (2s+1)^2 \right\},$$

(42)

we can bring $\hat{\mathbf{U}}$ and $\hat{\mathbf{D}}$ to the form, in which they will represent $(2s+1)^2$-component vectors with components of $(2s+1) \times (2s+1)$-matrices

$$\hat{\mathbf{U}}_{j(kl)} = \hat{\mathbf{U}}_{j_1(k_1l_1)j_2(k_2l_2)\ldots j_N(k_Nl_N)},$$

(43)

$$\hat{\mathbf{D}}_{j(kl)} = \hat{\mathbf{D}}_{j_1(k_1l_1)j_2(k_2l_2)\ldots j_N(k_Nl_N)},$$

(44)

where $j = 1, (2s+1)^2$ is the index of the component of the vector $\hat{\mathbf{U}}$ or $\hat{\mathbf{D}}$, and $(kl)$ are the spin indexes of the $(2s+1) \times (2s+1)$-matrices, $k, l = 1, (2s+1)$.

If in (37) we now choose the dequantizers $\hat{\mathbf{U}}^{(1)}, \hat{\mathbf{U}}^{(2)}, \ldots, \hat{\mathbf{U}}^{(N)}$ so that their components satisfy (8), then $\hat{\mathbf{U}}$ and $\hat{\mathbf{D}}$ will automatically be normalized as follows:

$$\sum_{j=1}^{(2s+1)^2} \hat{\mathbf{U}}_{j(kl)} = 2^N \delta_{kl}, \quad \sum_{j=1}^{(2s+1)^2} \hat{\mathbf{D}}_{j(kl)} = \delta_{kl},$$

(45)

$$\text{Tr} \hat{\mathbf{U}}_j = 1, \quad \text{Tr} \hat{\mathbf{D}}_j = 1/2^N, \quad j = 1, (2s+1)^2.$$  

(46)

Thus, we have constructed the normalized dequantizer $\hat{\mathbf{U}}$ and quantizer $\hat{\mathbf{D}}$ satisfying orthogonality and completeness conditions (2) for density matrices of the order of $2^N \times 2^N$. By means of conversion (1) with use of $\hat{\mathbf{U}}$ such density matrices are transformed to $4^N$-component non-redundant tomographic vectors $\mathbf{w}$ with nonnegative components normalized by the condition (7), where $2s + 1 = 2^N$, and the inverse transformation are given by (3) with use of $\hat{\mathbf{D}}$.  

8
6 Conclusion

In conclusion, we point out the main results of the paper. The positive four-component non-redundant normalized vector tomographic portrait fully describing the states of spin-1/2 quantum particles was introduced and it was shown that such a portrait is defined by the thruple of non-complanar vectors with the lengths equal or less then unity.

The corresponding dequantizer and quantizer for spin 1/2 were found in general case and their properties were explored. In particular, it was shown that the vector-quantizer also turns out to be normalized.

A graphic geometric interpretation of the choice of parameters for finding of numerical realizations of four-component normalized vectors-dequantizers for the spin 1/2 was given and two examples of such dequantizers and quantizers were presented, whose components are pure and mixed states respectively.

It was also done the generalization of the procedure for finding of normalized and non-redundant dequantizers and quantizers for spin $s = (2^N - 1)/2$, where $N$ is a natural number.

The normalized tomographic portrait proposed in this paper is useful for constructing of a set of tomographic schemes and also for realizing of quantum calculations whose algorithms can be modeled by evolution processes of systems of qubits and qudits.

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