Incompleteness of measurement apparatuses

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Abstract

A complete apparatus is defined as reacting to every state of the measured system. Standard quantum mechanics of indistinguishable particles is shown to imply that apparatuses must be incomplete or else they would be drowned out by noise. Each quantum observable is then an abstract representation of many measurement apparatuses. Moreover, the measured systems must be prepared in a state that is sufficiently different from the states of particles in the environment. This is the main purpose of preparations. A system so prepared was said to have a “separation status”. A new, more satisfactory definition of separations status is given than that proposed in previous papers. Conditions are specified under which the particles in the environment may be ignored as is usually done in the theory of measurement.
1 Introduction

In quantum mechanics, systems of the same type (such as all electrons or all hydrogen atoms) are absolutely indistinguishable:

Any [registration] performed on the [composite] quantum system treats all [indistinguishable] subsystems in the same way, and it is indifferent to a permutation of the label that we attribute to the individual subsystems for computational purposes.

(Peres [1], p. 126). A difficulty was also mentioned (p. 128 of [1]) that then arose: measurements on quantum system $S$ can be disturbed by remote particles that are of the same type as $S$. A solution to the problem based on Cluster Separability Principle was suggested.

The suggestion was developed into a theory in [2, 3]. There, local kind of quantum observables was defined, similar to that introduced in [4, 5] for different purposes and the notion of separation status based on the local observables was introduced.

However, the local observables and the corresponding separation status solve only a part of the Peres’ problem. The present paper delivers a clearer definition of the problem and proposes a better solution to it. Instead of introducing new complicated kind of observables, we leave the observables as they are but allow measuring apparatuses to be incomplete. We can then give a more general, simpler and more satisfactory definition of separation status.

Next, the fact will be explained that no disturbance by the indistinguishable particles of the environment is observed in quantum experiments and that successful theories of these experiments can serenely ignore these particles. The conditions under which such method works will result from the explanation.

2 Born rule

One of the basic relations of quantum mechanics is the Born rule (see, e.g., [1], p. 54): Let $O$ be an observable of quantum system $S$ with a discrete non-degenerate spectrum $\{o_k\}$ and eigenstates $|k\rangle$,

$$O|k\rangle = o_k|k\rangle .$$

Let $S$ be in a state $|\psi\rangle$ and let the decomposition of the state into the eigenstates be

$$|\psi\rangle = \sum_k c_k|k\rangle .$$

Then the probability of measuring eigenvalue $o_k$ on the state $|\psi\rangle$ is $|c_k|^2$. 

In fact, this measurement of $O$ must be done by some apparatus, or meter, $M$, say. We are going to ask questions such as: Which part of the Born rule holds true for the measurements of $O$ by $M$? That is, for which states the rule is true?

We are going to work with more general observables and states. Let $S$ be a quantum system with Hilbert space $H$. A general state of $S$ is a positive, trace-1 operator $T : H \mapsto H$ (often called “density matrix”), and we denote the convex set of all state operators by $T(H)$. A general observable is an $n$-tuple $\{O_1, \ldots, O_n\}$ of commuting self-adjoint operators $O_k : H \mapsto H$, $k = 1, \ldots, n$. Let $\sigma \subset \mathbb{R}^n$ be the spectrum of $\{O_1, \ldots, O_n\}$, $B(\mathbb{R}^n)$ the set of Borel subsets of $\mathbb{R}^n$ and let $\Pi(X)$, $X \in B(\mathbb{R}^n)$, describe the spectral measure of $\{O_1, \ldots, O_n\}$ (see, e.g., [6]). In particular, $\Pi(X)$ is an orthogonal projection on $H$ for each $X \in B(\mathbb{R}^n)$,

$$\Pi(X)^{\dagger} = \Pi(X), \quad \Pi(X)^2 = \Pi(X),$$

and satisfies the normalisation condition,

$$\Pi(\mathbb{R}^n) = 1. \quad (1)$$

We shall also adopt the nomenclature of [7]: every quantum measurement can be split into preparation and registration. Then the generalized Born rule can be formulated as follows.

**Rule 1** The probability $P$ that a value of observable $\{O_1, \ldots, O_n\}$ within $X \in B(\mathbb{R}^n)$ will be obtained by a registration on state $T$ is

$$P = tr(T \Pi(X)). \quad (2)$$

In practice, the Born rule means that the relative frequencies of the values obtained by many registration by the same meter $M$ on the same state $T$ must tend to the probabilities given by the Born rule if the number or the registration increases.

**Definition 1** Given state $T$ and $X \in B(\mathbb{R}^n)$, let us denote by $\omega[M, T](X)$ the frequencies of finding values of observable $\{O_1, \ldots, O_n\}$ within $X$ obtained from many registrations by meter $M$ on state $T$. Let

$$\omega[M, T](X) \mapsto tr(T \Pi(X)) \quad (3)$$

for some states $T$ and all $X \in B(\mathbb{R}^n)$. Then we say that meter $M$ measures $\{O_1, \ldots, O_n\}$.

Let us consider some examples.
1. The position $\vec{x}$ of particle $S$ with Hilbert space $H$ is a triple of self-adjoint operators and its spectral measure is described by (in $Q$-representation)

$$\Pi(X) = \chi_X(\vec{x}) ,$$

where $\chi_X(\vec{x})$ is the characteristic function of $X \in \mathcal{B}(\mathbb{R}^3)$. The spectrum $\sigma_{\vec{x}}$ is $\mathbb{R}^3$. Usually, the position is registered by some detector with active volume $D$ (see, e.g., [8]). If the detector gives a response (clicks) then we conclude that a particle has been detected inside $D$. If the wave function of the detected particle is $\psi(\vec{x})$ then the probability that the particle will be found inside the detector is

$$P(D) = \int_{\mathbb{R}^3} d^3x \chi_D(\vec{x}) |\psi(\vec{x})|^2 ,$$

Hence, $P(D) = 0$ if $\text{supp} \psi(\vec{x}) \cap D = \emptyset$. The integral on the right-hand side represents the the trace [2] with $T = |\psi\rangle\langle\psi|$. A better meter $M$ registering position is composed of several such sub-detectors, $M_1, \ldots, M_n$ with disjoint active volumes $D_1, \ldots, D_n$ and the frequency $\omega[M, \psi]_k$ that the particle will be found inside $D_k$ then satisfies

$$\omega[M, \psi]_k \mapsto \int_{\mathbb{R}^3} d^3x \chi_{D_k}(\vec{x}) |\psi(\vec{x})|^2 .$$

The meter does not register the whole spectrum but only the part $\sigma'_{\vec{x}} \subset \sigma_{\vec{x}}$ defined by

$$\sigma'_{\vec{x}} = \bigcup_{k=1}^n D_k .$$

Hence, for all states $\psi$ such that

$$\text{supp} \psi(\vec{x}) \subset \sigma'_{\vec{x}}$$

the detector satisfies the Born rule for all $X \in \mathcal{B}(\mathbb{R}^3)$ because zero probability for $S$ being outside of $\sigma'_{\vec{x}}$ results from both the Born rule and the registrations by $M$. However, for the states that do not satisfy Eq. (4), the meter still gives zero probability for $S$ being outside of $\sigma'_{\vec{x}}$ contradicting the Born rule.

2. Next, consider a meter that can register energy (a proportional counter, say). It reacts to a particle only if the particle energy is larger than some threshold. Again, such a meter will not react to some states, here to those whose wave function in momentum representation has a support that lies under the threshold. This example shows that the problem need not be caused just by the geometric arrangement of the experiment as in point 1.
3. The Stern-Gerlach meter (see, e.g., [1], p. 14) can register the spin observable only if the particle arriving at it can pass through the opening between the magnets within a narrow range of directions. Thus, it does not react to a number of states. This example shows that the problem can arise even for a meter that registers the whole spectrum.

Let us compare this with the well-known cases (see, e.g., [2]) of meters that do not register the whole spectra. For instance, a real meter can only discriminate between sufficiently different values of an observable \(O\) with a continuous spectrum that is, it registers only some coarse-grained version of the spectrum. Thus, one introduces a finite partition the space \(\mathbb{R}^n\),

\[
\mathbb{R}^n = \bigcup_{i=1}^n X_i,
\]

and defines a new observable with spectrum \(\{1, 2, \ldots, n\}\) that is easily constructed from \(O\) (for details, see [2], p. 35). Notice that the idea is to modify the observable so that the correspondence between observable and meter via the Born rule is improved. The meter then does react to all states of the system and satisfies the Born rule corresponding to the corrected observable.

The possibility that a meter may control only a (sometimes rather small) proper subset of the whole Hilbert space, as the meters of the above examples do, does not seem to be ever mentioned. This might be due to the belief that, as in most cases of non-ideal real circumstances, the shortcoming of real meters is a natural way of practical things which just must be taken properly into account in each particular instance and that some real meters might be arbitrarily close to the ideal or, at least, that continuous improvement of techniques will make meters better. The main aim of the next section is to show that standard quantum mechanics of indistinguishable particles sets a theoretical limit to this: a meter that were ideal in this sense would be unable to register its observable at all.

### 3 Incomplete apparatuses

Let us now simplify things by considering observables described by a single operator \((n = 1)\). The foregoing section motivates the following definition.

**Definition 2** Let \(S\) be a quantum system with Hilbert space \(H\) and let observable \(O\) be a s.a. operator on \(H\) with spectrum \(\sigma\) and spectral measure \(\Pi(X), X \in \mathcal{B}(\mathbb{R})\). Let meter \(M\) register observable \(O\). We say that \(M\) is complete if Equation (5) holds true for all \(T \in \mathcal{T}(H)\) and \(X \in \mathcal{B}(\mathbb{R})\).
If there is any state \( T \) for which the frequency \( \omega[M, T][R] \) of registering any value by \( M \) is zero, meter \( M \) is called incomplete. Let the subset of states for which this is the case be denoted by \( T(H)_{M_0} \) and the subset of states for which Equation (3) holds by \( T(H)_M \). The convex set \( T(H)_M \) is called the domain of \( M \).

Thus, the three examples in the foregoing section describe incomplete apparatuses. It seems that most textbooks assume that apparatuses are complete (at least implicitly, for example, [7, 1], or explicitly by the “probability reproducibility condition” of [9], p. 29). We now prove that a complete meter cannot work.

Let registrations by meter \( M \) be performed on a system \( S \) with Hilbert space \( H \). Suppose that meter \( M \) registers observable \( O \) with spectral measure \( \Pi(X) \), \( X \in \mathcal{B}(\mathbb{R}) \) and is complete. Then, because of the normalisation condition (1), we must have

\[
tr(T\Pi(\mathbb{R})) = 1
\]

for any state \( T \). This means that any registration on \( T \) by \( M \) must give some result.

Then, according to the theory of indistinguishable systems, \( M \) must also register some values on any state \( T' \) of any system \( S' \) of the same type as \( S \). Clearly, this is a difficulty: the measurement of observable \( O \) of \( S \) by \( M \) is disturbed by the existence of a system of the same type as \( S \) anywhere else in the world, even if it is localised arbitrarily far away from \( S \) because it cannot be distinguished from \( S \) by \( M \). In fact, for most microsystems \( S \), the world contains a huge number of systems of the same type so that a horrible noise must disturb any registration by a complete meter.

To show the problem in more detail, let us consider two distant laboratories, A and B. Let \( O \) be a non-degenerate discrete observable of \( S \) with eigenstates \( |k\rangle \) and eigenvalues \( o_k \). Let state \( |k\rangle \) be prepared in A and \( |l\rangle \) in B so that \( k \neq l \) and let \( O \) is registered in laboratory A by complete meter \( M \). Using Fock space formalism, we have

\[
O = \sum_n o_n a_k^\dagger a_n , \tag{5}
\]

where \( a_k \) is an annihilation operator of state \( |k\rangle \) (see, e.g., [1], p. 137). Such an observable perfectly expresses the fact that the meter cannot distinguish particles of the same type. The state prepared by the two laboratories is

\[
a_k^\dagger a_l^\dagger |0\rangle . \tag{6}
\]

For the average \( \langle O \rangle \) of (5) in state (6), the standard theory of measurement gives

\[
\langle O \rangle = \langle 0 | a_l a_k \left( \sum_n a_n a_n^\dagger a_n \right) a_k^\dagger a_l^\dagger |0 \rangle .
\]
Using the relation
\[ a_r^\dagger a_s^\dagger = \eta a_s^\dagger a_r + \delta_{rs} , \]
where \( \eta = 1 \) for bosons and \( \eta = -1 \) for fermions, we can bring all annihilation operators to the right and all creation ones to the left obtaining
\[ \langle A \rangle = a_k + a_l . \]

The result is independent of the distance between the laboratories. Thus, the measurement in A by any complete meter depends on what is done in B.

Let us next suppose that \( \mathcal{M} \) is incomplete in such a way that the state of any system of the same type as \( S \) that may occur in the environment of \( S \) lies in \( T(H)_\mathcal{M} \). Apparently, such an assumption can be checked experimentally by looking at the level of noise of the meter. Then, if we prepare a copy of system \( S \) in a state that lies within \( T(H)_\mathcal{M} \) the registration of \( S \) by \( \mathcal{M} \) cannot be disturbed by the systems in the environment. In fact, this must be the way of how all quantum measurement are carried out. We can say that objective properties of our environment require certain kind of incompleteness of registration apparatus \( \mathcal{M} \) in order that \( \mathcal{M} \) can work in this environment.

Accordingly, the course of any successful measurement must be as follows. First, a registration apparatus \( \mathcal{M} \) for a system \( S \) with Hilbert space \( H \) is constructed and checked. In particular, the level of its noise must be sufficiently low. From the construction of the meter, we can infer some set \( T_\mathcal{M} \) of states on which the meter is able make registrations. \( T_\mathcal{M} \) might be smaller than the whole domain,
\[ T_\mathcal{M} \subset T(H)_\mathcal{M} \]
(the domain is often difficult to specify). Second, systems \( S \) is prepared in one of such states. The registration by \( \mathcal{M} \) will then not be disturbed and the probabilities of the results can be calculated theoretically by formula (3).

The above ideas also have some relevance to the meaning of preparation processes in quantum measurements:

**Assumption 1** Any preparation of a single microsystem must yield a state that is sufficiently different from states of all systems of the same type that occur in the environment. We shall refer to such a system as having a separation status.

The separation status that is required from a prepared state can be characterized by the set \( T_\mathcal{M} \) of the meter that is planned to make registration on it.

A separation status of a microsystem is a property that is uniquely determined by a preparation. Hence, it belongs to objective properties of quantum systems according to [11] [12]. But it is a property that is a necessary condition for any
other objective property because each preparation must create a separation status. Moreover, only a separation status makes a quantum system distinguishable from each other system in the environment and so to a physical object. Thus, a quantum physical object can come into being, namely in a preparation process, and can expire, viz. if it loses its separation status.

We can understand the role of incompleteness of meters better if we compare quantum apparatuses with classical ones. To this aim, we construct a simple model of an eye. Indeed, an eye is a classical registration device, either by itself or as a final part of other classical apparatuses.

Our model consists of an optically sensitive surface (retina) that can register visible light (i.e., with a wave-length between 0.4 and 0.75 µm). It can distinguish between some small intervals of the visible wave lengths and between small spots where the retina is hit by light.

The retina covers one side of a chamber that has walls keeping light away except for a small opening at the side opposite to the retina wall. The radius of the circular opening even if very small is much larger than the wave length of visible light so that this light waves suffer only a negligible bending as they pass the opening. The assumed insensitivity of the retina to smaller wave lengths is an incompleteness that helps to make the picture sharp.

Another aspect of incompleteness is that only the light that can pass the opening will be registered. Again, this is important: if the retina were exposed to all light that can reach it from the neighbourhood, only a smeared, more or less homogeneous signal would result. Because of the restrictions, a well-structured colour picture of the world in front of the eye will appear on the retina.

4 Observables

There are two ways of how one could react to the necessary incompleteness of registration apparatuses. First, one can try to modify the observable that is measured by an meter so that the results of the registrations and the probabilities calculated from the Born rule coincide, similarly as it has been done above for coarse-grained version of the spectrum. Second, one can leave the observables as they are and accept the fact that meters can register observables only partially. In our previous work ([2, 3]), we have tried the first way. It turned out, however, that the modification that was necessary for an observable to describe how a real meter worked was messy. The notion of observable became rather complicated and only some idealized kinds of meters could be captured in this way.

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1An eye with a small opening instead of a lens occurs in some animals such as nautilus.
The mentioned idealized kind of incomplete meter \( \mathcal{M} \) can be described as follows. Such an \( \mathcal{M} \) determines a closed linear subspace \( H_{ss} \) of \( H \) so that, instead of Equation (3), we have

\[
\omega[\mathcal{M}, \mathcal{T}](X) \mapsto tr\left( \left( \Pi_{ss} \mathcal{T} \Pi_{ss} \right) \Pi(X) \right)
\]

for all \( \mathcal{T} \in \mathcal{T}(H) \) and \( X \in \mathcal{B}(\mathbb{R}) \), where \( \Pi_{ss} \) is the orthogonal projection onto \( H_{ss} \). Then,

\[
\mathcal{T}(H)_{\mathcal{M}} = \mathcal{T}(H_{ss})
\]

and

\[
\mathcal{T}(H)_{\mathcal{M}0} = \mathcal{T}(\{1 - \Pi_{ss}\}H)
\]

because any element \( \mathcal{T} \) of \( \mathcal{T}(H_{ss}) \) satisfies

\[
\mathcal{T} = \Pi_{ss} \mathcal{T} \Pi_{ss}.
\]

The construction of the corresponding “generalized observable” is simple. First, we have

\[
tr\left( \left( \Pi_{ss} \mathcal{T} \Pi_{ss} \right) \Pi(X) \right) = tr\left( \mathcal{T} \left( \Pi_{ss} \Pi(X) \Pi_{ss} \right) \right).
\]

Next, consider operator \( \Pi_{ss} \Pi(X) \Pi_{ss} \). It is bounded by 1 and self-adjoint because \( \Pi_{ss} \) and \( \Pi(X) \) are. It is obviously positive. Thus, it is an effect (see [9, 10]). A collection of effects \( E(X), X \in \mathcal{B}(\mathbb{R}) \), with certain properties (including the normalisation condition \( E(\mathbb{R}) = 1 \)) is called a positive-operator valued measure (POVM) (for a definition see, e.g., [9]) and generalizes the notion of observable. The collection of effects \( \Pi_{ss} \Pi(X) \Pi_{ss} \) for all \( X \in \mathcal{B}(\mathbb{R}) \) is not a POVM, however, because we have, instead of the above normalisation condition,

\[
\Pi_{ss} \Pi(\mathbb{R}) \Pi_{ss} = \Pi_{ss}
\]

Such a quantity could be called “truncated POVM”. Thus, the notion of observable had to be changed from a self-adjoint operator to a truncated POVM.

However, the above model of incomplete meter is too simple. For instance, some of the examples listed in Section 2 cannot be described by it. Indeed, consider the Stern-Gerlach meter that is arranged in such a way that it can react to particles moving within a thin tube around the third axis of coordinates \( x_1, x_2, x_3 \). The particle that can be registered must thus arrive at the magnets only within some small subset of the (1,2)-plane, the third component of its momentum must satisfy

\[
p_3 \in (a_3, b_3),
\]
which can be large, and

\[ p_1 \in (-c_1, c_1), \quad p_2 \in (-c_2, c_2), \]

where \( c_k < \epsilon \) for \( k = 1, 2 \) and for sufficiently small \( \epsilon \). However, these conditions can be satisfied, by any wave packets, only approximately. Then, the Born rule will also be satisfied only approximately. Now, a linear superposition of such packets need not be again such a packet. The above conditions mean that the wave function (in \( Q \)- or \( P \)-representation) of the registered particle must satisfy inequalities of the form

\[ |\psi(\lambda)|^2 < \epsilon' \]

for some fixed values of \( \lambda \) determined by the arrangement, where \( \lambda \) stands either for \( \vec{x} \) or for \( \vec{p} \), and \( \epsilon' \) is a small positive number. Suppose that another wave function, \( \phi \), also satisfies the condition. Then it only follows, for all \( c \) and \( c' \) satisfying \(|c|^2 + |c'|^2 = 1\), that

\[ |c\psi(\lambda) + c'\phi(\lambda)| < 2\epsilon'. \]

Hence, the packets need not form a closed linear subspace of \( H \). However, the approach using incomplete meters works even if the domain of an meter does not satisfy Eq. (7). In fact, the knowledge of the whole domain \( T(H)_M \) of an meter is not necessary for the construction of a model of a registration by it because it is sufficient to know only those elements of \( T(H)_M \) that are prepared for the experiment.

These are the reasons why we adopt the second way in the present paper. Then, the standard notion of observable (a self-adjoint operator) makes the following sense. We can assume that the union of domains of all possible meters that can register a given observable covers, in an ideal case, the whole set of states \( T(H) \). For example, proportional counters can register energy of free particles while meters using scattering of photons can register energy of its bounded states, etc. Thus, a standard observable is not associated with some measurement meter but with all meters that can measure it. As each of these meters must be incomplete, a number of meters is needed for one observable.

Everything that has been said in this and the foregoing sections can easily be extended if the notion of observable is generalized from a self-adjoint operator to a POVM.

5 Tensor-product method

We have seen in the foregoing sections that the disturbance of measurement by environmental particles is not observed, if the measuring meter is suitably incomplete and the measured system is prepared in a state with the corresponding separation
status. Such explanation why measurements are possible is missing in all textbooks on measurement theory and the entanglement of the registered system with environmental particles of the same type are ignored in the mathematics of the measurement theory.

The present section is going to study this mathematics in more detail. In particular, we shall consider two ways of description of states. First, the tensor product of the state of the environmental systems and the prepared state of the individual system and second, the (anti-)symmerized state of the whole system as required by rules of the theory of indistinguishable systems. The second way of description is in any case the correct one and we shall ask when the first way is compatible with it. That is, we just shall have to show that the two descriptions lead to the same measurable results. Of course, the tensor product way is the only practically feasible one because we cannot know the state of the whole system.

To develop the two descriptions, let us consider system $S$ and its environment $E$ with the system $E_S$ of all its subsystems that are indistinguishable from $S$. Let $\psi(\lambda)$ be the wave function of $S$, where $\lambda$ is a shorthand for four arguments, for example three components of position or momentum and one spin variable $m = -s, \ldots, +s$. In fact, $\psi(\lambda)$ can be any representation of state $\left| \psi \right\rangle$. Let $H$ be the Hilbert space of $S$ and let us assume that $E_S$ consists of $N$ subsystems so that the Hilbert space of $E_S$ is $H^N$. Here $\tau$ takes value $f$ for fermions and $b$ for bosons and as index in the expression $H^N_\tau$ it denotes the antisymmetrization (for $\tau = f$) or symmetrization (for $\tau = b$) of the tensor product of $N$ copies of $H$. If we are going to speak about both cases, we use the expression “$\tau$-symmetrization”. A wave function of $E_S$ has the form

$$\Psi(\lambda^{(1)}, \ldots, \lambda^{(N)}) \in H^N_\tau$$

and is $\tau$-symmetric in all arguments $\lambda^{(1)}, \ldots, \lambda^{(N)}$. Then the wave functions of the two descriptions are

$$\Psi(\lambda^{(1)}, \ldots, \lambda^{(N)})\psi(\lambda^{(N+1)})$$

and

$$\nu \Pi^{N+1}_\tau \left( \Psi(\lambda^{(1)}, \ldots, \lambda^{(N)})\psi(\lambda^{(N+1)}) \right),$$

where $\Pi^{N+1}_\tau : H^N_\tau \otimes H \mapsto H^{N+1}_\tau$ is the orthogonal projection onto the $\tau$-symmetrized subspace and $\nu$ is a suitable normalization factor.

To explain the projection $\Pi^{N+1}_\tau$ and the normalization factor $\nu$ (for details and general proofs, see e.g., [10]), we choose $N = 2$ and consider $\Pi_2^f : H^2 \mapsto H^2_f$. Let $\Psi(\lambda^{(1)}, \lambda^{(2)}) \in H^2$, then

$$\Pi_2^f \Psi(\lambda^{(1)}, \lambda^{(2)}) = \frac{1}{2} [\Psi(\lambda^{(1)}, \lambda^{(2)}) - \Psi(\lambda^{(2)}, \lambda^{(1)})],$$

$$\Pi_2^b \Psi(\lambda^{(1)}, \lambda^{(2)}) = \frac{1}{2} [\Psi(\lambda^{(1)}, \lambda^{(2)}) + \Psi(\lambda^{(2)}, \lambda^{(1)})].$$
Orthogonal projections do not preserve the normalization. Hence, the projection
must be followed by a normalization factor, which we will denote by \( \nu \) standing
before the projection symbol. Of course, \( \nu \) depends on the projection and the wave
function being projected, but we just write \( \nu \) instead of \( \nu(\Pi^2, \Psi) \) to keep equations
short.

Equation \((10)\) shows that we can recover the second description from the first
one, but if the two descriptions are to be equivalent in any sense, one had to recover
the first one from the second, too. For this aim, the separation status is necessary.
Let state \( \psi(\lambda) \) be prepared with separation status \( T(\mathbf{H})_M \) and let \( \Pi \psi = |\psi\rangle\langle\psi| \).

Now, we make use the fact that the operators on \( \mathbf{H} \) can act on different wave
function (elements of \( \mathbf{H} \)) in a product and that this action can be specified by the
argument of the function. For example, if we have a product \( \psi_1(\lambda(1))\psi_2(\lambda(2)) \) of two
functions and operator \( \mathbf{O} : \mathbf{H} \rightarrow \mathbf{H} \), operator \( \mathbf{O}^{(1)} : \mathbf{H} \otimes \mathbf{H} \rightarrow \mathbf{H} \otimes \mathbf{H} \) is defined by

\[
\mathbf{O}^{(1)} \left[ \psi_1(\lambda(1))\psi_2(\lambda(2)) \right] = (\mathbf{O}\psi_1)(\lambda(1))\psi_2(\lambda(2))
\]

while \( \mathbf{O}^{(2)} \) by

\[
\mathbf{O}^{(2)} \left[ \psi_1(\lambda(1))\psi_2(\lambda(2)) \right] = \psi_1(\lambda(1))(\mathbf{O}\psi_2)(\lambda(2)) .
\]

An important application of this formalism gives

\[
\Pi^{(k)}\Psi((\lambda(1)), \ldots, (\lambda(N))) = 0
\]

for any \( k = 1, \ldots, N \). With this notation, we can achieve our aim: obviously,

\[
\Pi^{(N+1)}\nu \Psi(\lambda(1), \ldots, \lambda(N+1)) = \nu \Psi(\lambda(1), \ldots, \lambda(N)) \otimes \psi(\lambda(N+1)) ,
\]

where \( \nu_\psi \) is again a suitable normalization factor. Observe that this operation is
naturally described by the formalism of \( \tau \)-symmetrized wave functions rather than
by the Fock-space formalism. In fact, we can use \( \Pi^{(K)}_\psi \) for any fixed \( K = 1, \ldots, N \)
instead of \( \Pi^{(N+1)}_\psi \) and the result will again be the above tensor product with renamed
arguments.

The next point is to give an account of registration by an incomplete meter.
We construct two observables that represent the meter, each for one of the two
description ways, and show that the two ways of descriptions lead to the same
results. We work with a simple model to show the essential points; the general
situation can be dealt with in an analogous way.

Let meter \( \mathcal{M} \) register observable \( \mathbf{O} : \mathbf{H} \rightarrow \mathbf{H} \) that is additive, discrete and non-
degenerate. Let its eigenvalues be \( o_k \) and eigenvectors be \( \psi_k, k \in \mathbb{N} \). Let \( \mathcal{M} \) be
incomplete in the way that it reacts only to \( \psi_k \) if \( k = 1, \ldots, K \) for some \( K \in \mathbb{N} \).
Hence, the subspace $H_{ss}$ is spanned by vectors $\psi_k$, $k = 1, \ldots, K$, and the projection onto it is

$$\Pi_{ss} = \sum_{k=1}^{K} \Pi_k,$$

where

$$\Pi_k = |\psi_k\rangle\langle\psi_k|.$$

The action of the meter can now be described as follows. Let us prepare state $\psi$ with a separation status defined by meter $\mathcal{M}$. Hence, $\psi \in H_{ss}$ and its decomposition into the eigenstates of $O$ is

$$\psi = \sum_{k=1}^{K} c_k \psi_k$$

with $\sum_{k=1}^{K} |c_k|^2 = 1$. Then the probability $P_k$ of registering $o_k$ on $\psi$ is

$$P_k = |\langle\psi|\Pi_k|\psi\rangle|^2.$$

In this way,

$$P_k = |c_k|^2$$

for $k \leq K$ and $P_k = 0$ for $k > K$.

Let us start with the first way, Equation (9). We define the corresponding observable by restricting the action of $O$ or $\Pi_k$ to the second factor:

$$(1 \otimes \Pi_k) \left[ \Psi(\lambda^{(1)}, \ldots, \lambda^{(N)}) \otimes \psi(\lambda^{(N+1)}) \right] \equiv \Pi_k^{(N+1)} \left[ \Psi(\lambda^{(1)}, \ldots, \lambda^{(N)}) \psi(\lambda^{(N+1)}) \right] = c_k \Psi(\lambda^{(1)}, \ldots, \lambda^{(N)}) \psi_k(\lambda^{(N+1)}) \quad (12)$$

for $k \leq K$. Eq. (12) specifies the Born rule of the observable. Now, coming to the second way of description, Equation (10), we use the fact that the observable is additive. For example, it acts on product $\phi_1(\lambda^{(1)})\phi_2(\lambda^{(2)})$ as follows

$$(O^{(1)} + O^{(2)}) \left( \phi_1(\lambda^{(1)})\phi_2(\lambda^{(2)}) \right).$$

Then, to define the observable registered by $\mathcal{M}$, we need the action of its projection $\Pi'_k$ for eigenvalue $o_k$, $k \in \mathbb{N}$. Let us choose:

$$\Pi'_k = \sum_{l=1}^{N+1} (\Pi_k \Pi_{ss})^{(l)} \left[ \nu \Pi_r^{N+1} \left( \Psi(\lambda^{(1)}, \ldots, \lambda^{(N)}) \psi(\lambda^{(N+1)}) \right) \right]$$

(observe that operators $\Pi_k$ and $\Pi_{ss}$ commute). But projection $(\Pi_k \Pi_{ss})^{(l)}$ annihilates the state to the right if the argument $\lambda^{(l)}$ is in function $\Psi$ and gives $c_k \psi_k(\lambda^{(l)})$ if the argument is in $\psi$. The result of the projection is then

$$c_k \nu \Pi_r^{N+1} \left( \Psi(\lambda^{(1)}, \ldots, \lambda^{(N)}) \psi_k(\lambda^{(N+1)}) \right).$$
Thus, the Born rules for the observables of the two ways of description coincide.

In general, the operator
\[ \Pi_k' = \sum_{l=1}^{N+1} (\Pi_k \Pi_{ss})^{(l)} \]
is not a projection because the product \((\Pi_k \Pi_{ss})^{(r)}(\Pi_k \Pi_{ss})^{(s)}\) does not in general vanish for \(r \neq s\) and then \((\Pi_k')^2 \neq \Pi_k'.\) However, on the subspace of \(\Pi_r^{N+1}(H_r^N \otimes H)\) with which we are working, the product is non-zero only if \(r = s\), so that it is a projection under these conditions.

The last question is whether the dynamical evolutions for the two ways of description are compatible. First, we define the corresponding Hamiltonians. Let \(H : H_r^N \otimes H \mapsto H_r^N \otimes H\) be a Hamiltonian for the first way of description and let us assume that
\[ H\Pi_r^{N+1} = \Pi_r^{N+1}H.\]

Such a Hamiltonian leaves the subspace \(\Pi_r^{N+1}(H_r^N \otimes H)\) invariant and can also be viewed as a Hamiltonian for the second way of description. Then, the two Schrödinger equations that we are going to compare are:
\begin{equation}
H[\Psi(\lambda^{(1)}, \ldots, \lambda^{(N)})\psi(\lambda^{(N+1)})] = i\hbar \frac{\partial}{\partial t}[\Psi(\lambda^{(1)}, \ldots, \lambda^{(N)})\psi(\lambda^{(N+1)})] \tag{13}
\end{equation}
for the first way of description and
\begin{equation}
H\Pi_r^{N+1}[\Psi(\lambda^{(1)}, \ldots, \lambda^{(N)})\psi(\lambda^{(N+1)})] = i\hbar \frac{\partial}{\partial t}\Pi_r^{N+1}[\Psi(\lambda^{(1)}, \ldots, \lambda^{(N)})\psi(\lambda^{(N+1)})] \tag{14}
\end{equation}
for the second way.

We are able to prove the compatibility only if the evolution preserves the separation status. Mathematically, this means that the Hamiltonian must commute with the projections defining the status:
\[ H\Pi_{ss}^{(k)} = \Pi_{ss}^{(k)}H \tag{15} \]
for all \(k = 1, \ldots, N+1.\) Then, the projections are conserved and their eigenspaces are stationary. In the case under study, this implies that the time derivative commutes with the projections, too:
\[ \frac{\partial}{\partial t}\Pi_{ss}^{(k)} = \Pi_{ss}^{(k)}\frac{\partial}{\partial t} \tag{16} \]
for all \(k = 1, \ldots, N + 1.\)

Now, the proof of the compatibility is very simple: applying projection \(\Pi_{ss}^{(N+1)}\) to both sides of equation (14) and using equations (11), (15), (16), we obtain equation (13).
For processes, in which e.g. a measured system loses its separation status, the two evolutions are not compatible and the second way equation must be used. Such processes occur during registration of which many examples have been given in [3]. We shall adapt the examples to the new definition of separation status in another paper.

6 Conclusion an outlook

From quantum mechanical theory of indistinguishable particles, a strong disturbance of measurement would follow for measurements by meters that were complete. Suitably incomplete apparatuses can measure, if the measured systems are prepared in states with the corresponding separation status. The incomplete apparatus gives probability zero to all values that are measured on states of the environment. Then, the environmental particles that are indistinguishable from the measured system can be ignored, in the practice of registrations and in their theoretical treatment, as it is usually done.

The environmental particles that are indistinguishable from the measured system cannot, however, be ignored in the Schrödinger equation if the evolution does not preserve the separation status.

The attempts to achieve a close relation between a quantum observable and its measurement apparatus is abandoned. The observables are defined as in the standard quantum theory. In this way, the simplicity and elegance of the standard theory of quantum observables is preserved. However, each observable represents a whole set of apparatuses, each registering only a part of it.

The new theory is logically consistent with the rest of quantum mechanics, agrees with the results of real measurements and our understanding of measurement apparatuses as well as that of preparation processes is improved.

A new definition of separation status is proposed that is different from that of [2, 3]. In such a way, some problems of the old definition are removed and the new notion of separation status is even more general and simpler to use than the old one.

The theory of objective properties [11, 12] that associates objectivity with preparation remains valid for states as objective properties of quantum systems because each preparation must create a separation status. The processes of separation-status change will find application in our theory of state reduction [13] similar to that described in [2].
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