Perturbation Theory for Core and Core-EP Inverses of Tensor via Einstein Product

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Abstract. In this paper, for given tensors $A, E$ and $B = A + E$, we investigate the perturbation bounds for the core inverse $A^\ast$ and core-EP inverse $A^\circ$ under some conditions via Einstein product.

1. Introduction

There are several papers on the core inverse and core-EP inverse [1–5, 7–9]. Recently, there are recent monographs [10–12] on the generalized inverse.

For convenience, we first adopt some of the terminologies which will be used in this paper. For a positive integer $N$, let $[N] = \{1, \ldots, N\}$. An order $N$ tensor $A = (A_{i_1,j_1,\ldots,j_N})_{1\leq i_1\leq I_1, \ldots, 1\leq i_N\leq I_N}$ is a multidimensional array with $I_1 I_2 \cdots I_N$ entries. Let $\mathbb{C}^{I_1 \times \cdots \times I_N}$ and $\mathbb{R}^{I_1 \times \cdots \times I_N}$ be the sets of the order $N$ dimension $I_1 \times \cdots \times I_N$ tensors over the complex field $\mathbb{C}$ and the real field $\mathbb{R}$, respectively. Each entry of $A$ is denoted by $a_{i_1,i_2,\ldots,i_N}$.

For a tensor $A = (a_{i_1,i_2,\ldots,i_N}) \in \mathbb{C}^{I_1 \times \cdots \times I_N}$, let $B = (b_{i_1,i_2,\ldots,i_N}) \in \mathbb{C}^{I_1 \times \cdots \times I_N}$ be the conjugate transpose of $A$, where $b_{i_1,i_2,\ldots,i_N} = \overline{a_{i_1,i_2,\ldots,i_N}}$. The tensor $B$ is denoted by $A^\ast$. When $b_{i_1,i_2,\ldots,i_N} = \overline{a_{i_1,i_2,\ldots,i_N}}$, $B$ is the transpose of $A$, and is denoted by $A^\trans$. A tensor $D = (d_{i_1,i_2,\ldots,i_N}) \in \mathbb{C}^{I_1 \times \cdots \times I_N}$ is called a diagonal tensor if all its entries are zero except for $d_{i_1,i_2,\ldots,i_N}$. In case of all the diagonal entries $d_{i_1,i_2,\ldots,i_N} = 1$, we call $D$ as a unit tensor and is denoted by $I$. Similarly, $0$ denotes the zero tensor in case of all the entries are zero.

The Einstein product of tensors is defined in [13] by the operation $*$ via

$$ (A * B)_{i_1,i_2,\ldots,i_N} = \sum_{k_1,k_2,\ldots,k_N} A_{i_1,k_1,\ldots,k_N} B_{k_1,i_2,\ldots,i_N} \quad (1) $$

where $\mathcal{A} \in \mathbb{C}^{I_1 \times \cdots \times I_K \times K_{1} \times \cdots \times K_{N}}$, $B \in \mathbb{C}^{I_1 \times \cdots \times I_K}$, and $\mathcal{A} * B \in \mathbb{C}^{I_1 \times \cdots \times I_K \times I_{1} \times \cdots \times I_{N}}$. The associative law of this tensor product holds. In the above formula, when $B \in \mathbb{C}^{K_{1} \times \cdots \times K_{N}}$, then

$$ (A * B)_{i_2,\ldots,i_N} = \sum_{k_1,k_2,\ldots,k_N} A_{i_1,k_1,\ldots,k_N} B_{k_1,i_2,\ldots,i_N} $$

where $\mathcal{A} * B \in \mathbb{C}^{I_1 \times \cdots \times I_K \times K_{1} \times \cdots \times K_{N}}$. For convenience, we denote $\mathbb{C}^{I_1 \times \cdots \times I_K} \times I_{1} \times \cdots \times I_{N}$ simply by $\mathbb{C}^{(I_{1} \times \cdots \times I_{N})}$. 

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Definition 1.1. [16] For $\mathcal{A} \in \mathbb{C}^{(N)\times K(N)}$, the range $\mathcal{R}(\mathcal{A})$ and the null space $\mathcal{N}(\mathcal{A})$ of $\mathcal{A}$ are defined by

\[
\mathcal{R}(\mathcal{A}) = \{ y \in \mathbb{C}^{I \times I} : y = \mathcal{A} \ast_N X, X \in \mathbb{C}^{K \times \cdot} \}
\]

\[
\mathcal{N}(\mathcal{A}) = \{ X \in \mathbb{C}^{K \times \cdot} : \mathcal{A} \ast_N X = O \},
\]

where $O$ is an appropriate order zero tensor.

Definition 1.2. [16] The inner product on $\mathbb{C}^{N \times \cdot \times N_c}$ is defined by

\[
\langle X, Y \rangle = \sum_{n \in [N], r \in [K]} \bar{x}_{n_1 n_2 \cdots n_r} y_{n_1 n_2 \cdots n_r}, \quad \forall X, Y \in \mathbb{C}^{N \times \cdot \times N_c},
\]

and the spectral norm $\| \cdot \|_2$ is defined as [17, Lemma 2.1]

\[
\| X \|_2 = \sqrt{\lambda_{\text{max}}(X^* \ast_N X)},
\]

where $\lambda_{\text{max}}(X^* \ast_N X)$ denotes the largest eigenvalue of $X^* \ast_N X$.

Definition 1.3. [14] Let $\mathcal{A} \in \mathbb{C}^{(N)\times K(N)}$. The tensor $X \in \mathbb{C}^{K(N)\times I(N)}$ which satisfies

1. $\mathcal{A} \ast_N X \ast_N \mathcal{A} = \mathcal{A}$;
2. $X \ast_N \mathcal{A} \ast_N X = X$;
3. $(\mathcal{A} \ast_N X)^* = \mathcal{A} \ast_N X$;
4. $(X \ast_N \mathcal{A})^* = X \ast_N \mathcal{A}$

is called the Moore-Penrose inverse of $\mathcal{A}$, abbreviated by M-P inverse, denoted by $\mathcal{A}^+$. If the equation (i) of the above equations (1) – (4) holds, then $X$ is called an (i)-inverse of $\mathcal{A}$, denoted by $\mathcal{A}^{(i)}$.

Definition 1.4. [17] Assume that $\mathcal{A} \in \mathbb{C}^{(N)\times I(N)}$. Define

\[
\mathcal{A}^0 = I \quad \text{and} \quad \mathcal{A}^p = \mathcal{A}^{p-1} \ast_N \mathcal{A}, \quad \text{for } p \geq 2.
\]

It is easily seen that

\[
[0] \subseteq \cdots \subseteq \mathcal{R}(\mathcal{A}^{p+1}) \subseteq \mathcal{R}(\mathcal{A}^p) \subseteq \cdots \subseteq \mathcal{R}(\mathcal{A}) \subseteq \mathcal{R}(I) = \mathbb{C}^{I \times \cdot \times I_0}
\]

and

\[
[0] = \mathcal{N}(I) \subseteq \mathcal{N}(\mathcal{A}) \subseteq \mathcal{N}(\mathcal{A}^2) \subseteq \cdots \subseteq \mathcal{N}(\mathcal{A}^p) \subseteq \mathcal{N}(\mathcal{A}^{p+1}) \subseteq \cdots \subseteq \mathbb{C}^{I \times \cdot \times I_0}.
\]

The smallest non-negative integer $p$ such that $\mathcal{R}(\mathcal{A}^{p+1}) = \mathcal{R}(\mathcal{A}^p)$ (or $\mathcal{N}(\mathcal{A}^{p+1}) = \mathcal{N}(\mathcal{A}^p)$), denoted by $\text{Ind}(\mathcal{A})$, is called the index of $\mathcal{A}$.

Definition 1.5. [17] Let $\mathcal{A} \in \mathbb{C}^{(N)\times I(N)}$. The tensor $X \in \mathbb{C}^{(N)\times I(N)}$ which satisfies

1. $X \ast_N \mathcal{A} \ast_N X = X$;
2. $\mathcal{A} \ast_N X \ast_N \mathcal{A} = \mathcal{A}$;
3. $(\mathcal{A} \ast_N X)^* = \mathcal{A} \ast_N X$;
4. $(X \ast_N \mathcal{A})^* = X \ast_N \mathcal{A}$

is called the Drazin inverse of $\mathcal{A}$, denoted by $\mathcal{A}^d$. Especially, if $\text{Ind}(\mathcal{A}) = 1$, $X$ is called the group inverse of $\mathcal{A}$, denoted by $\mathcal{A}^g$.

According to Hartwig and Spindelböck decomposition [18] of tensors, every tensor $\mathcal{A} \in \mathbb{C}^{(N)\times I(N)}$ of rank $r$ can be represented by

\[
\mathcal{A} = U \ast_N \left( \Sigma \ast_N \mathcal{K} \begin{bmatrix} 0 & \Sigma \ast_N \mathcal{L} \end{bmatrix} \right) \ast_N U^*,
\]
where \( \Sigma \in \mathbb{C}^{R(N) \times R(N)} \) is a diagonal tensor of singular values of \( \mathcal{A} \), and the tensors \( \mathcal{K} \in \mathbb{C}^{R(N) \times R(N)}, \mathcal{L} \in \mathbb{C}^{R(N) \times (G_2 \times R_3)} \) satisfy

\[
\mathcal{K} *_N \mathcal{K}' + \mathcal{L} *_N \mathcal{L}' = \mathcal{I}
\]  

(3)

It follows from (2) that the Moore-Penrose inverse of \( \mathcal{A} \) is given as follows:

\[
\mathcal{A}^+ = \mathcal{U} *_N \left( \begin{array}{cc}
\mathcal{K}' *_N \Sigma^{-1} & O \\
\mathcal{L}' *_N \Sigma^{-1} & O 
\end{array} \right) *_N \mathcal{U}'.
\]

If \( \text{Ind}(\mathcal{A}) \leq 1 \), then the group inverse of \( \mathcal{A} \) is

\[
\mathcal{A}^g = \mathcal{U} *_N \left( \begin{array}{cc}
\mathcal{K}^{-1} *_N \Sigma^{-1} & \mathcal{K}^{-1} *_N \Sigma^{-1} *_N \mathcal{K}^{-1} *_N \mathcal{L}
\end{array} \right) *_N \mathcal{U}'.
\]

**Lemma 1.6.** [15] Let \( \mathcal{E} \in \mathbb{C}^{(R) \times (R)} \) be a tensor of index \( k \). If \( \| \mathcal{E} \|_2 < 1 \), then \( \mathcal{I} + \mathcal{E} \) is nonsingular and

\[
\|(\mathcal{I} + \mathcal{E})^{-1}\|_2 \leq \frac{1}{1 - \| \mathcal{E} \|_2}.
\]

**Lemma 1.7.** [15] Let \( \mathcal{E} \in \mathbb{C}^{(R) \times (R)} \). If \( \| \mathcal{E} \|_2 < 1 \), then

\[
(\mathcal{I} - \mathcal{E})^{-1} = \sum_{n=0}^{\infty} \mathcal{E}^n,
\]

(4)

\[
\|(\mathcal{I} - \mathcal{E})^{-1} - \mathcal{I}\|_2 \leq \frac{\| \mathcal{E} \|_2}{1 - \| \mathcal{E} \|_2}.
\]

(5)

The recent results on the core inverse of tensor can be found in [19, 20].

**Definition 1.8.** [19, 20] Let \( \mathcal{A} \in \mathbb{C}^{(R) \times (R)} \) be a given core tensor. A tensor \( \mathcal{X} \in \mathbb{C}^{(R) \times (R)} \) satisfying

\[
*_{\mathcal{A}} \mathcal{X}^2 = \mathcal{X}; \quad \mathcal{A} *_{\mathcal{A}} \mathcal{X}^2 = \mathcal{X}; \quad (\mathcal{A} *_{\mathcal{A}} \mathcal{X})' = \mathcal{A} *_{\mathcal{A}} \mathcal{X}
\]

is called the core inverse of \( \mathcal{A} \) and denoted by \( \mathcal{A}^\circ \).

**Lemma 1.9.** [19, 20] Let \( \mathcal{A} \in \mathbb{C}^{(R) \times (R)} \) be given. Then \( \mathcal{A}^\circ \) satisfies equations (1) and (2) in Definition 1.3.

By the definition of core inverse, we have the following lemma.

**Lemma 1.10.** [19, 20] Let \( \mathcal{A} \in \mathbb{C}^{(R) \times (R)} \) be of the form (2) and \( \text{Ind}(\mathcal{A}) \leq 1 \). Then

\[
\mathcal{A}^\circ = \mathcal{U} *_N \left( \begin{array}{cc}
*_{\mathcal{A}} \mathcal{K}^{-1} & O \\
O & O
\end{array} \right) *_N \mathcal{U}'.
\]

Another important generalized inverse is the core-EP inverse.

**Definition 1.11.** [8, 19] Let \( \mathcal{A} \in \mathbb{C}^{(R) \times (R)} \) and \( \text{Ind}(\mathcal{A}) = k \). A tensor \( \mathcal{X} \in \mathbb{C}^{(R) \times (R)} \) satisfying

\[
*_{\mathcal{A}} \mathcal{X}^k = \mathcal{X}; \quad \mathcal{A} *_{\mathcal{A}} \mathcal{X}^k = \mathcal{X}; \quad (\mathcal{A} *_{\mathcal{A}} \mathcal{X})' = \mathcal{A} *_{\mathcal{A}} \mathcal{X}
\]

is called core-EP inverse of \( \mathcal{A} \) and it is denoted as \( \mathcal{A}^\bar{\circ} \).

**Lemma 1.12.** [8, 19] Let \( \mathcal{A} \in \mathbb{C}^{(R) \times (R)} \) and \( \text{Ind}(\mathcal{A}) = k \). There is a Schur form of \( \mathcal{A} \),

\[
\mathcal{A} = \mathcal{U} *_N \left( \begin{array}{cc}
T_1 & T_2 \\
O & T_3
\end{array} \right) *_N \mathcal{U}',
\]

(6)

where \( \mathcal{U} \in \mathbb{C}^{(R) \times (R)} \) is a unitary tensor, \( T_1 \) is a upper triangular tensor and \( T_3 \) is a nilpotent tensor with \( \text{Ind}(T_3) = k \).
Theorem 2.1. Let \( \mathcal{A} \) be of the form (2) and \( \text{Ind}(\mathcal{A}) \leq 1 \), \( \mathcal{B} = \mathcal{A} + \mathcal{E} \). If the perturbation \( \mathcal{E} \) satisfies \( \mathcal{A} * \mathcal{A} = \mathcal{B} * \mathcal{B} \), then
\[
(\mathcal{A} * \mathcal{A})^{-1} = \mathcal{B}^{-1}.
\]

Definition 1.13. [15, 22] Let \( I_1, \ldots, I_M, K_1, \ldots, K_N \) be given integers and \( \mathcal{A}, \mathcal{B} \) are the integers defined as
\[
\mathcal{A} = I_1 I_2 \cdots I_M, \quad \mathcal{B} = K_1 K_2 \cdots K_N.
\]

The reshaping operation
\[
\text{rsh} : \mathbb{C}^{(M)\times(K)} \rightarrow \mathbb{C}^{(M)\times(K)}
\]
transforms a tensor \( \mathcal{A} \in \mathbb{C}^{(M)\times(K)} \) into the matrix \( A \in \mathbb{C}^{(M)\times(K)} \) using the Matlab function \texttt{reshape} as follows:
\[
\text{rsh}(\mathcal{A}) = A = \text{reshape}(\mathcal{A}, \mathcal{I}, \mathcal{R}), \quad \mathcal{A} \in \mathbb{C}^{(M)\times(K)}, \quad A \in \mathbb{C}^{(M)\times(K)}.
\]

The inverse reshaping \( A \in \mathbb{C}^{(M)\times(K)} \) is the tensor \( \mathcal{A} \in \mathbb{C}^{(M)\times(K)} \) defined by
\[
\text{rsh}^{-1}(A) = \mathcal{A} = \text{reshape}(A, I_1, \ldots, I_M, K_1, \ldots, K_N).
\]

Also, an appropriate definition of the tensor rank, arising from the reshaping operation, was proposed in [22].

Definition 1.14. [15, 22] Let \( \mathcal{A} \in \mathbb{C}^{(M)\times(K)} \) and \( A = \text{reshape}(\mathcal{A}, \mathcal{I}, \mathcal{R}) = \text{rsh}(\mathcal{A}) \in \mathbb{C}^{(M)\times(K)} \). Then the tensor rank of \( \mathcal{A} \), denoted by \( \text{rshrank}(\mathcal{A}) \), is defined by \( \text{rshrank}(\mathcal{A}) = \text{rank}(A) \).

2. Perturbation for core inverse

In this section, we present the optimal perturbations for the core inverse of tensors via Einstein product under two-sided and one-sided conditions.

Theorem 2.1. Let \( \mathcal{A}, \mathcal{E} \in \mathbb{C}^{(M)\times(K)} \) be of the form (2) and \( \text{Ind}(\mathcal{A}) \leq 1 \), \( \mathcal{B} = \mathcal{A} + \mathcal{E} \). If the perturbation \( \mathcal{E} \) satisfies \( \mathcal{A} * \mathcal{A} = \mathcal{B} * \mathcal{B} \), then
\[
\mathcal{B} = (I + \mathcal{A} * \mathcal{E})^{-1} * \mathcal{A} * \mathcal{E} = \mathcal{A} * (I + \mathcal{E} * \mathcal{A})^{-1},
\]
and
\[
\mathcal{B} * \mathcal{B} = (I + \mathcal{A} * \mathcal{E})^{-1} * \mathcal{A} * \mathcal{A} + (I + \mathcal{E} * \mathcal{A})^{-1} * \mathcal{A} * \mathcal{E} * (I - \mathcal{E} * \mathcal{A}).
\]

Furthermore,
\[
\frac{\|\mathcal{A}\|_2}{1 + \|\mathcal{A} * \mathcal{E}\|_2} \leq \|\mathcal{B}\|_2 \leq \frac{\|\mathcal{A}\|_2}{1 - \|\mathcal{A} * \mathcal{E}\|_2}
\]
and
\[
\frac{\|\mathcal{B} * \mathcal{B} - \mathcal{A} * \mathcal{A}\|_2}{\|\mathcal{A} * \mathcal{A}\|_2} \leq \frac{\|\mathcal{A} * \mathcal{E}\|_2}{1 - \|\mathcal{A} * \mathcal{E}\|_2}.
\]

Proof. We assume that the perturbation \( \mathcal{E} \) is partitioned by
\[
\mathcal{E} = U * \begin{pmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{pmatrix} * V^T.
\]

Using the fact that \( \mathcal{A} * \mathcal{A} * \mathcal{E} = \mathcal{E} * \mathcal{A} * \mathcal{A} = \mathcal{E} \), together with
\[
\mathcal{A} * \mathcal{A} = U * \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} * V^T,
\]
From Definition 1.11 and (6), we can obtain that
\[
\mathcal{A}^\# = U * \begin{pmatrix} T^{-1} & 0 \\ 0 & 0 \end{pmatrix} * V^T.
\]
implies $E_{12} = O$, $E_{21} = O$, $E_{22} = O$. It is easy to see that the perturbation $E$ has the form

$$E = U^* N \begin{pmatrix} E_{11} & 0 \\ O & O \end{pmatrix} U^*. $$

Furthermore, we obtain

$$B = A + E = U^* N \begin{pmatrix} \Sigma_n K + E_{11} & \Sigma_n L \\ O & O \end{pmatrix} U^*. $$

In view of Lemma 1.6, since $\|A^* N E\|_2 < 1$, then $I + A^* N E$ is invertible and

$$\|(I + A^* N E)^{-1}\|_2 \leq \frac{1}{1 - \|A^* N E\|_2}. $$

Moreover,

$$I + A^* N E = U^* N \begin{pmatrix} I + (\Sigma_n K)^{-1} E_{11} & 0 \\ O & I \end{pmatrix} U^*, $$

and

$$(I + (\Sigma_n K)^{-1} E_{11})^{-1} = (\Sigma_n K)^{-1} (\Sigma_n K + E_{11})^{-1} = (\Sigma_n K + E_{11})^{-1} (\Sigma_n K)^{-1}.$$ This implies that $(\Sigma_n K + E_{11})^{-1}$ exists.

Then the core inverse of $B$ exists and has the following expression,

$$B^* = U^* N \begin{pmatrix} (\Sigma_n K + E_{11})^{-1} & 0 \\ O & O \end{pmatrix} U^*
\begin{pmatrix} (I + (\Sigma_n K)^{-1} E_{11})^{-1} & 0 \\ O & O \end{pmatrix} U^*$$

By using (4) of Lemma 1.7, direct computation shows that

$$(I + A^* N E)^{-1} A^* N (I + E N A^*)^{-1}.$$ Next, the perturbation bounds of core inverse are estimated. It is easy to verify that

$$B^* N B - A^* N A
= U^* N \begin{pmatrix} I + (\Sigma_n K)^{-1} E_{11} & 0 \\ O & O \end{pmatrix} U^* [I + (\Sigma_n K)^{-1} E_{11} - I] \begin{pmatrix} K^{-1} & 0 \\ 0 & K^{-1} \end{pmatrix} U^*$$

Taking forms of both sides, we obtain

$$\|B^* N B - A^* N A\|_2 \leq \|(I + A^* N E)^{-1}\|_2 \|A^* N E\|_2 \|I - A^* N A\|_2 = \|(I + A^* N E)^{-1}\|_2 \|A^* N E\|_2 \|A^* N A\|_2.$$
That is
\[
\frac{\|\mathcal{B}^* \mathcal{E} - \mathcal{A}^* \mathcal{E}\|_2}{\|\mathcal{A}^* \mathcal{E}\|_2} \leq \frac{\|\mathcal{A}^* \mathcal{E}\|_2}{1 - \|\mathcal{A}^* \mathcal{E}\|_2}
\]
The proof is complete. \(\square\)

Next, we provide a perturbation bound for the core inverse under one-sided condition.

**Theorem 2.2.** Let \(\mathcal{A}, \mathcal{E} \in \mathbb{C}^{(N) \times (N)}\) be of the form (2) and \(\text{Ind}(\mathcal{A}) \leq 1, \mathcal{B} = \mathcal{A} + \mathcal{E}\). If the perturbation \(\mathcal{E}\) satisfies
\[
\mathcal{A}^* \mathcal{E} = \mathcal{E}
\]
and \(\mathcal{A}^* \mathcal{E}\) is partitioned by \(\mathcal{A}^* \mathcal{E} = \mathcal{E} = \mathcal{E}\) and \(\mathcal{A}^* \mathcal{E}\) is of the form (2) and if the perturbation \(\mathcal{E}\) satisfies
\[
\mathcal{A}^* \mathcal{E} = \mathcal{E} = \mathcal{E}
\]
then
\[
\mathcal{B}^* = (\mathcal{I} + \mathcal{A}^* \mathcal{E})^{-1} \mathcal{A}^* \mathcal{E} = \mathcal{A}^* \mathcal{E} = \mathcal{A}^* \mathcal{E} \]
and
\[
\mathcal{B} \mathcal{E} = \mathcal{A}^* \mathcal{E}, \quad \mathcal{B}^* \mathcal{B} = \mathcal{A}^* \mathcal{E} + (\mathcal{I} + \mathcal{A}^* \mathcal{E})^{-1} \mathcal{A}^* \mathcal{E} = \mathcal{A}^* \mathcal{E} \mathcal{A}^* \mathcal{E} \mathcal{A}^* \mathcal{E} \mathcal{A}^* \mathcal{E} \mathcal{A}^* \mathcal{E} \mathcal{A}^* \mathcal{E}
\]
Furthermore,
\[
\frac{\|\mathcal{A}\|_2}{1 + \|\mathcal{A}^* \mathcal{E}\|_2} \leq \|\mathcal{B}\|_2 \leq \frac{\|\mathcal{A}\|_2}{1 - \|\mathcal{A}^* \mathcal{E}\|_2}
\]
and
\[
\frac{\|\mathcal{B}^* \mathcal{B} - \mathcal{A}^* \mathcal{E}\|_2}{\|\mathcal{A}^* \mathcal{E}\|_2} \leq \frac{\|\mathcal{A}^* \mathcal{E}\|_2}{1 - \|\mathcal{A}^* \mathcal{E}\|_2}
\]

**Proof.** We assume that the perturbation \(\mathcal{E}\) is partitioned by
\[
\mathcal{E} = \mathcal{U}^* \begin{pmatrix} \mathcal{E}_{11} & \mathcal{E}_{12} \\ \mathcal{E}_{21} & \mathcal{E}_{22} \end{pmatrix} \mathcal{U}^*
\]
Using the fact that \(\mathcal{A}^* \mathcal{E} = \mathcal{E}\), it together with
\[
\mathcal{A}^* \mathcal{E} = \mathcal{U}^* \begin{pmatrix} \mathcal{I} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} \end{pmatrix} \mathcal{U}^*
\]
implies \(\mathcal{E}_{21} = \mathcal{O}, \mathcal{E}_{22} = \mathcal{O}\). It is easy to see that the perturbation \(\mathcal{E}\) has the form
\[
\mathcal{E} = \mathcal{U}^* \begin{pmatrix} \mathcal{E}_{11} & \mathcal{E}_{12} \\ \mathcal{O} & \mathcal{O} \end{pmatrix} \mathcal{U}^*
\]
Furthermore, we obtain
\[
\mathcal{B} = \mathcal{A} + \mathcal{E} = \mathcal{U}^* \begin{pmatrix} \mathcal{A}^* + \mathcal{E}_{11} & \mathcal{A}^* + \mathcal{E}_{12} \\ \mathcal{O} & \mathcal{O} \end{pmatrix} \mathcal{U}^*
\]
Since \(\|\mathcal{A}^* \mathcal{E}\|_2 < 1\) and \(\mathcal{A}^* \mathcal{E}\) is an orthogonal projection, we obtain
\[
\|\mathcal{A}^* \mathcal{E} \mathcal{A}^* \mathcal{E}\|_2 \leq \|\mathcal{A}^* \mathcal{E}\|_2 \|\mathcal{A}^* \mathcal{E}\|_2 = \|\mathcal{A}^* \mathcal{E}\|_2 < 1
\]
Then \(\mathcal{I} + \mathcal{A}^* \mathcal{E} \mathcal{A}^* \mathcal{E}\) is invertible, so \(\mathcal{I} + (\Sigma + \mathcal{K})^{-1} \mathcal{E}_{11}^{-1}\) exists. Then the core inverse of \(\mathcal{B}\) exists and has the following expression.
\[
\mathcal{B}^* = \mathcal{U}^* \begin{pmatrix} \mathcal{I} + (\Sigma + \mathcal{K})^{-1} \mathcal{E}_{11}^{-1} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} \end{pmatrix} \mathcal{U}^*
\]
and

\[ B^r \ast N B = U^N \left( I + (\Sigma \ast N K)^{-1} \ast N E_{12} \right) \ast N U^r. \]

Now we can estimate

\[ B^r \ast N B - A^r \ast N A = U^N \left( I + (\Sigma \ast N K)^{-1} \ast N E_{12} \right) \ast N U^r. \]

Taking norms of both sides, we obtain

\[ \|B^r \ast N B - A^r \ast N A\| \leq \|(I + A^r \ast N E) - I\|\|A^r \ast N E\| \leq (1 - \|A^r \ast N E\|)\|I - A^r \ast N A\|. \]

This completes the proof of the theorem. \( \square \)

In a similar way, we obtain another one-sided perturbation formula.

**Theorem 2.3.** Let \( A, E \in \mathbb{C}^{N \times N} \) be of the form (2) and \( \text{Ind}(A) \leq 1, B = A + E. \) If the perturbation \( E \) satisfies \( A^r \ast N A \ast N E = E \) and \( \|A^r \ast N E\| < 1, \) then

\[ B^r = (I + A^r \ast N E)^{-1} \ast N A^r = A^r \ast N (I + E \ast N A^r)^{-1}. \]
3. Perturbation for core-EP inverse

In this section, we investigate the optimal perturbations for the core-EP inverse of tensors via Einstein product under one-sided conditions which extends the matrix case [9].

Theorem 3.1. Let \( A, E \in \mathbb{C}^{I(N) \times I(N)} \) be of the form (6) and \( \text{Ind}(A) = k \), \( B = A + E \in \mathbb{C}^{I(N) \times I(N)} \). If the perturbation \( E \) satisfies \( A^{*}N A^{*}N E = E \) and \( \|A^{*}N E\|_2 < 1 \), then

\[
B^{\#} = (I + A^{*}N E)^{-1} * N A^{*}N (I - A^{*}N A),
\]

and

\[
B^{\#} N B^{\#} = A^{*}N A^{*}N, \quad B^{\#} N B = A^{*}N A + (I + A^{*}N E)^{-1} * N A^{*}N E * N (I - A^{*}N A).
\]

Furthermore,

\[
\frac{\|A^{\#}\|_2}{1 + \|A^{*}N E\|_2} \leq \frac{\|B^{\#}\|_2}{1 - \|A^{*}N E\|_2} \leq \frac{\|A^{\#}\|_2}{1 - \|A^{*}N E\|_2},
\]

and

\[
\frac{\|B^{\#} N B - A^{*}N E \|_2}{\|A^{*}N E\|_2} \leq \frac{\|A^{*}N E\|_2}{1 - \|A^{*}N E\|_2}.
\]

Proof. We assume that the perturbation \( E \) is partitioned by

\[
E = U * N \begin{pmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{pmatrix} * N U^{*}.
\]

Since \( E \) satisfies \( A * N A * N E = E \), together with

\[
A * N A^{\#} = U * N \begin{pmatrix} I & O \\ O & O \end{pmatrix} * N U^{*},
\]

leads to \( E_{21} = 0, E_{22} = 0 \). It is straightforward to see that the perturbation \( E \) has the form

\[
E = U * N \begin{pmatrix} E_{11} & E_{12} \\ O & O \end{pmatrix} * N U^{*},
\]

and the tensor \( B \) keeps the Schur form

\[
B = A + E = U * N \begin{pmatrix} T_1 + E_{11} & T_2 + E_{12} \\ O & T_3 \end{pmatrix} * N U^{*}.
\]

Since \( \|A^{*}N E\|_2 < 1 \), and \( A * N A^{\#} \) is an orthogonal projection, we obtain

\[
\|A^{\#} N E * N A * N A^{\#}\|_2 \leq \|A^{*}N E\|_1 \|A * N A^{\#}\|_2 = \|A^{\#} N E\|_2 < 1.
\]

Then \( I + A^{\#} * N E * N A * N A^{\#} \) is invertible, so \( (I + T_1^{-1} * N E_{11})^{-1} \) exists.
Then the core-EP inverse of $B$ exists, and it has the form as follows

$$B^o = U_1 (T_1 + E_{11})^{-1} \begin{pmatrix} O & O \\ O & O \end{pmatrix} U_2^*,$$

$$= U_1 (I + T_1^{-1} E_{11})^{-1} \begin{pmatrix} E_{11} & T_1^{-1} \\ O & O \end{pmatrix} U_2^*,$$

$$= (I + A^o E_N A_N A^o)^{-1} E_N A^o,$$

$$= (I + A^o E_N A^o)^{-1} E_N A^o,$$

and $B^o * N B$ possesses the following representation

$$B^o * N B = U_1 \begin{pmatrix} I & (T_1 + E_{11})^{-1} E_{11} (T_2 + E_{12}) \end{pmatrix} U_2^*.$$

Further,

$$A^o * N A = U_1 \begin{pmatrix} I & T_1^{-1} E_{11} \\ O & O \end{pmatrix} U_2^*.$$

Now we can estimate

$$B^o * N A = U_1 \begin{pmatrix} O & (T_1 + E_{11})^{-1} \end{pmatrix} U_2^*,$$

$$= U_1 \begin{pmatrix} O & (I + T_1^{-1} E_{11})^{-1} \end{pmatrix} U_2^*,$$

$$= U_1 \begin{pmatrix} O & (I + T_1^{-1} E_{11})^{-1} \end{pmatrix} U_2^*,$$

$$= (I + A^o E_N A_N A^o)^{-1} E_N A^o.$$

The proof is complete. $\Box$

In the same way, we obtain the similar perturbation formula.

**Theorem 3.2.** Let $A, E \in C^{(N)\times(N)}$ be of the form (6) and Ind($A$) = $k$, $B = A + E \in C^{(N)\times(N)}$. If the perturbation $E$ satisfies $A^o * N A_N A^o \neq 0$ and $\|A^o * N E_N\| < 1$, then

$$B^o = (I + A^o E_N)^{-1} = A^o * N (I + E_N A^o)^{-1},$$

$$B_N B^o = A_N A^o.$$
Theorem 3.3. Let \( A, E \in \mathbb{C}^{(N)\times(I)} \) be of the form (6) and \( \text{Ind}(A) = k, \ B = A + E \in \mathbb{C}^{(N)\times(I)} \). If the perturbation \( E \) satisfies \((I - A^{\ast} N A^\oplus) *_N E *_N A *_N A^\oplus = O \) and \( \text{rank}(A) = \text{rank}(B) \) with \( ||A^\oplus *_N E||_2 < 1 \), then

\[
B^\oplus = (I + A^\oplus *_N E)^{-1} *_N A^\oplus = A^\oplus *_N (I + E *_N A^\oplus)^{-1},
\]

\[
B *_N B^\oplus = A *_N A^\oplus.
\]

Proof. We assume that the perturbation \( E \) is partitioned by

\[
E = U *_N \begin{pmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{pmatrix} *_N U^\ast.
\]

Since \( E \) satisfies \((I - A^{\ast} N A^\oplus) *_N E *_N A *_N A^\oplus = O \), together with

\[
A *_N A^\oplus = U *_N \begin{pmatrix} I & O \\ O & O \end{pmatrix} *_N U^\ast
\]

implies \( E_{21} = O \), and then \( B \) has the following expression

\[
B = A + E = U *_N \begin{pmatrix} T_1 + E_{11} \\ O \\ O \\ T_2 + E_{12} \\ T_3 + E_{22} \end{pmatrix} *_N U^\ast.
\]

Now, from \( ||A^\oplus *_N E||_2 < 1 \) and \( \text{rank}(A) = \text{rank}(B^\oplus) \), we can obtain that \( T_1 + E_{11} \) is invertible and \( \text{rank}[(T_3 + E_{22})^\ast] = O, ((T_2 + E_{22})^\ast)^\ast = O \). Moreover,

\[
B^\oplus = U *_N \begin{pmatrix} (T_1 + E_{11})^{-1} & O \\ O & O \end{pmatrix} *_N U^\ast.
\]

Similar to the proof of Theorem 3.1, we obtain

\[
B^\oplus = (I + A^\oplus *_N E)^{-1} *_N A^\oplus = A^\oplus *_N (I + E *_N A^\oplus)^{-1},
\]

and

\[
B *_N B^\oplus = A *_N A^\oplus.
\]

The proof is complete. \( \square \)

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