Spatially overlapped partners in quantum field theory

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Abstract
In quantum field theory particles are physically defined as what Unruh–DeWitt particle detectors observe. By detecting a particle mode $A$, a reduced density operator for a quantum state of $A$ is constructed. Even if the entire quantum state of the quantum field is pure, the state of $A$ is not pure but mixed due to entanglement between other subsystems. The partner mode $B$ of the field is defined as a purification partner of $A$ such that the $AB$ system in a pure state. We show that, without any fine-tuning of the particle detector design of $A$, the weighting function of partner $B$ has spatial overlap of that of $A$. We show a general formula of partner $B$ associated with arbitrarily fixed $A$ of a free field in a general Gaussian state. We demonstrate an example of memory effects in an expanding Freedman–Robertson–Walker universe.

Keywords: quantum entanglement, spatially overlapped partners, quantum field theory, quantum memory, black hole

(Some figures may appear in colour only in the online journal)

1. Introduction

A quantum field is capable of playing a role of quantum information storage. After a quantum operation dependent on unknown parameters is performed to the field, the quantum state stores the memory of the parameters. In what kind of form does the field keep the information? There exist a lot of options. For instance, a two-body subsystem in a pure entangled state is able to keep the information. The two-body system is referred to as an entangled partner [1]. Since a field in the vacuum state has an infinite number of partners due to the ultraviolet divergence, it is well known that the entanglement entropy diverges to infinity. By use of the huge entanglement, a quantum field may attain large information capacity. From this point of view, the entanglement of the partners can be expected to provide relevant applications for future quantum information technology, such as entanglement harvesting [2, 3].

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Besides, the notion of entangled partner has shed light on fundamental physics like the black hole information loss problem [4]. In [1], the partner mode corresponding to a Hawking mode of a free field is explicitly identified. It turns out that the partner is a local zero-point fluctuation of the field. This may avoid a serious flaw of the information recovery scenario at the last burst of a black hole so as to maintain the unitarity of the process. It is widely argued that evaporating black hole energy of the order of the Planck scale is too small to emit the whole inside information to outside [5]. Since the amount of information is not elementary particle size but astrophysical size, the information carriers seem to request a huge number of highly excited states, and much larger energy than the Planck energy. However, as pointed out in [6] and [1], the zero-point fluctuation emitted at the last burst is able to retrieve the whole inside information because the fluctuation flow requires zero energy cost.

In quantum field theory particles are physically defined as what Unruh–De Witt particle detectors observe [7, 8]. Measuring a particle mode \( A \) by the detectors is capable of identifying a reduced density operator for a quantum state of \( A \) via quantum state tomography protocols. Even if the entire quantum state of the quantum field is pure, the quantum state of a subsystem is not generally pure but becomes mixed due to entanglement between other subsystems. The partner mode \( B \) associated to \( A \) in the field is defined as a purification partner of \( A \) such that the \( AB \) system is in a pure state. In [1], a special type of Unruh–De Witt detector for a Hawking particle succeeded in capturing the partner of a Hawking particle and clarifying its interesting properties. The mode of \( A \) is fixed by operators consisting linear combination of a field operator and its conjugate momentum operator with some weighting functions localized in a spatial region. The partner mode \( B \) associated with \( A \) is determined in a similar way by a linear combination of the field operator and its conjugate momentum. The weighting functions of \( B \) has no overlap of spatial support with that of \( A \). This means that the Hawking particle has a spatially separated partner (SSP) in [1].

In this paper, we elaborate a more general class of partners of a free scalar field in an arbitrary Gaussian state. We show a general formula of partner \( B \) associated with an arbitrarily fixed \( A \) of a free field in a general Gaussian state. It turns out that, without any fine-tuning of the choice of \( A \) mode, the spatial support of weighting functions of \( B \) mode has nonzero overlap with that of \( A \) mode. This implies that a particle observed by a general Unruh–De Witt detectors is accompanied by a spatially overlapped partner (SOP) for purification of the particle. Though the spatial overlap of \( A \) and \( B \) happens, it is possible to consider quantum entanglement between \( A \) and \( B \) since the operators of each system commute to each other and establish locality of \( A \) and \( B \) for the definition of the entanglement. In the similar way of usual SSP cases in [1], the pure states of SOP are also able to play a role of quantum memory devices about unknown parameters by imprinting them via parameter-dependent dynamical processes. In order to demonstrate that explicitly, we consider a simple example of SOP of a scalar field in an expanding universe with an expansion rate parameter \( \rho \). We show that there exists the \( \rho \)-dependence of entanglement entropy between a localized particle mode \( A \) and its SOP mode \( B \). Such an analysis of SOP may allow us to construct a more sensitive model for checking cosmological Bell inequality breaking in cosmic microwave back ground [11]. The aim of this paper is to stress a new concept of information storage by SOP in quantum field theory, which have not been pointed out to date. Though it is significant to analyze SOP information storage in black hole evaporation as well as SSP, it requires a more complicated calculation, that is outside of the reach of the present paper. It is also worthwhile to stress that a partner exists for an arbitrarily fixed particle mode in a quantum field in a general pure state, as will be mentioned in section 2, and the partner is expected to be an SOP in typical cases. Thus SOP may be applied to a wide class of physics issues including the black hole.
information loss problem. One might be afraid that the spatial overlap of $A$ and $B$ disturbs extraction of the imprinted information in SOP. As depicted in figure 1, it is difficult to read out the information of SOP by using two spatially separate detectors. However, by using a special quantum swapping device as depicted in figure 2 and possesses two independent intrinsic degrees of freedom associated with $A$ and $B$, the information as well as the entanglement can be extracted and read out perfectly [9].

In section 2, we prove existence of partner $B$ for an arbitrarily fixed mode $A$ of a field in a general state. In section 3, the partner formula for the vacuum state for a free scalar field is derived. Without any fine-tuning, the partner becomes an SOP. In section 4, we derive a general expression for partner formula, which is applicable to any Gaussian state and any complete set of canonical operators. In section 5, we demonstrate how SOPs store information about parameters of dynamical evolution of the field. As a simple example, an expanding universe with an expansion rate parameter $\rho$ is considered. There actually exists $\rho$-dependence of entanglement entropy between a localized particle mode and its SOP. In section 6, conclusions are presented.

Throughout this paper, scalar field theory is treated as the continuum limit of harmonic oscillator chain. We do not discuss any subtle problems regarding the continuum limit. Our results are applicable if the limit can be taken properly.

In this paper, the natural unit is adopted: $c = \hbar = 1$. 

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**Figure 1.** A schematic picture of information extraction using spatially separated detectors. The rectangle with red (resp. blue) line pattern denotes a detector for mode $A$ (resp. mode $B$). Since they do not have any spatial overlap, it is difficult to extract whole quantum information imprinted in SOPs.

**Figure 2.** A schematic picture of information extraction using a detector with two independent intrinsic degrees of freedom. The rectangle with red and blue line pattern denotes the detector. With such a special device, it is possible to extract whole quantum information imprinted in SOPs.
2. Correlation function definition of purification partner

In this section, we start with a definition of a partner for an arbitrary mode $A$ of a free quantum field in a general state $|\Psi\rangle$ by using a Unruh–De Witt detector and measurable correlation functions of $A$. This definition provides a significant advantage which provides direct methods to verify the partner of $A$ by realistic physical experiments. The mode $A$ is defined by what an extended Unruh–De Witt particle detector observes. Let $\hat{\varphi}(x)$ and $\Pi(x)$ be a scalar field and its conjugate momentum in a $(d + 1)$-dimensional curved spacetime. Let us consider a continuous-variable Unruh–De Witt detector with a measurement interaction as

$$\hat{H}_{\text{meas}}(t) = \lambda(t) f (\hat{q}_A(t), \hat{p}_A(t)) \hat{P}_D(t),$$

where $\lambda(t)$ is a time-dependent coupling between the field and the detector, and $\hat{P}_D(t)$ is a momentum operator conjugate to a pointer position operator $\hat{Q}_D(t)$ of the detector. Also $f (q, p)$ is a real function of $q$ and $p$, and $\hat{q}_A(t)$ and $\hat{p}_A(t)$ are Heisenberg operators associated with linear combination of the field operators as

$$\hat{q}_A = \int d^d x \left( x_A(x) \hat{\varphi}(x) + y_A(x) \hat{\Pi}(x) \right), \quad (1)$$

$$\hat{p}_A = \int d^d x \left( z_A(x) \hat{\varphi}(x) + w_A(x) \hat{\Pi}(x) \right). \quad (2)$$

which satisfy $[\hat{q}_A, \hat{p}_A] = i$, and the weighting functions of $A$, $x_A(x)$, $y_A(x)$, $z_A(x)$ and $w_A(x)$, are real functions localized in a spatial region. By varying $f (q, p) = \hat{a}_A^\dagger \hat{a}_A + \hat{a}_A^\dagger \hat{a}_A$ with non-negative integers $m$ and $n$, the detector is capable of measuring multipoint correlation functions $\langle \Psi | \langle \hat{q}_A^m \hat{p}_A^n | \Psi \rangle \rangle$ of $\hat{q}_A$ and $\hat{p}_A$. The entire measurement result of the correlation functions for all $n$ and $m$ can be summarized as a generating function $\chi_A (x_A, v_A) = \langle \Psi | e^{i (\hat{q}_A^m \hat{p}_A^n - x_A v_A)} | \Psi \rangle$ of the correlation functions. The components of the reduced state $\hat{\rho}_A$ of $A$ in the position basis can be determined by the measured function $\chi_A (x_A, v_A)$ as follows:

$$\langle \hat{x}_A | \hat{\rho}_A | \hat{x}_A \rangle = \frac{1}{(2\pi)^2} \int \chi_A (x_A - \bar{x}_A, v_A) e^{-\frac{i}{2} \theta(x_A + \bar{x}_A)} d v_A.$$  

The proof that $\hat{\rho}_A$ actually becomes non-negative Hermitian operator satisfying normalization condition, $\text{Tr} [\hat{\rho}_A] = 1$ is given in Appendix A. Because the mode $A$ is coupled to other modes in general, $\hat{\rho}_A$ usually becomes a mixed state. Since the entire field is in a pure state, there exists a purification partner mode $B$ of $A$, and the $AB$ system is in a pure entangled state $|\psi\rangle_{AB}$. Taking the partial trace of $B$, the reduced state $\hat{\rho}_A$ is reproduced as $\hat{\rho}_A = \text{Tr}_B [ |\psi\rangle_{AB} \langle \psi|_{AB} ]$. Now, we propose a generalized definition of partner mode. The partner mode $B$ of $A$ is characterized by a set of operators $(\hat{q}_B, \hat{p}_B)$ satisfying the following conditions (i), (ii) and (iii):

(i) Commutation relation: $[\hat{q}_B, \hat{p}_B] = i$.

(ii) Locality: $[\hat{q}_A, \hat{q}_B] = 0$, $[\hat{q}_A, \hat{p}_B] = 0$, $[\hat{p}_A, \hat{q}_B] = 0$, and $[\hat{p}_A, \hat{p}_B] = 0$.

(iii) Purification condition: the correlation space state $\hat{\rho}_{AB}$ whose components in the position basis are given by

$$\langle \hat{x}_A, \hat{x}_B | \hat{\rho}_{AB} | \hat{x}_A, \hat{x}_B \rangle = \frac{1}{(2\pi)^2} \int d v_A d v_B \chi \left( x_A - \bar{x}_A, v_A, x_B - \bar{x}_B, v_B \right) e^{-\frac{i}{2} \theta(x_A + \bar{x}_A + x_B + \bar{x}_B)}$$

is pure. Here, we have used the Wigner characteristic function defined by
\[ \chi(x_A, v_A, x_B, v_B) \equiv \langle \Psi | e^{i(x_A q_A - x_B q_B)} e^{i(x_B p_A - x_A p_B)} | \Psi \rangle \]

for the pure state \( | \Psi \rangle \) of system.

Though \((\hat{q}_A, \hat{p}_A)\) are assumed to be a linear combination of \(\hat{\phi}(x)\) and \(\hat{\Pi}(x)\), \((\hat{q}_B, \hat{p}_B)\) are not. The partner operators \((\hat{q}_B, \hat{p}_B)\) can include non-linear terms like \(\hat{\phi}(x)^n\) and \(\hat{\Pi}(x)^m\) in general. The condition (ii) ensures the locality necessary to introduce the notion of entanglement, while (iii) gives the condition that the partner \((\hat{q}_B, \hat{p}_B)\) purifies \((\hat{q}_A, \hat{p}_A)\). As well as \(\hat{\rho}_A, \hat{\rho}_{AB}\) is a quantum state, i.e. a unit trace positive-semidefinite Hermitian operator. In order to introduce the concept of entanglement, \(\hat{\rho}_{AB}\) is determined by the correlation functions of local operators of \(A\) and \(B\). The Wigner characteristic function \(\chi(x_A, v_A, x_B, v_B)\) actually satisfies this postulate and yields all the correlation functions. Thus, our definition works well. Another necessary condition for \(\hat{\rho}_{AB}\) is the following: for a state \(\hat{\rho}\) of two harmonic oscillator system, \(\hat{\rho}_{AB} = \hat{\rho}\) must hold. Equation (3) actually obeys this condition which can be confirmed by using the Fourier transformation and its inverse transformation simultaneously. If we have a partner candidate \(B\) with \((\hat{q}_B, \hat{p}_B)\), experimental measurements of the correlation functions of \((\hat{q}_A, \hat{p}_A, \hat{q}_B, \hat{p}_B)\) allow us to corroborate the partner of \(A\) in principle.

Since for a Gaussian state, the Wigner characteristic function is fully characterized by a \(4 \times 4\) matrix called the covariance matrix, the condition (iii) gets simplified as explained in the next section.

By using the result on a pair of partners \(A\) and \(B\) for the Gaussian vacuum state \(|0\rangle\) of a field, nontrivial examples of partners for non-Gaussian states can be easily constructed. Let us consider a general unitary operation \(\hat{U}\) generated by a non-linear interaction Hamiltonian consist of \(\hat{\phi}\) and \(\hat{\Pi}\). The post-operated state \(|\Psi\rangle = \hat{U}|0\rangle\) is non-Gaussian. In the state, we have partners which are defined as \((\hat{q}'_A, \hat{p}'_A) = \left( \hat{U} \hat{q}_A \hat{U}^\dagger, \hat{U} \hat{p}_A \hat{U}^\dagger \right)\) and \((\hat{q}'_B, \hat{p}'_B) = \left( \hat{U} \hat{q}_B \hat{U}^\dagger, \hat{U} \hat{p}_B \hat{U}^\dagger \right)\). The characteristic function becomes the same as that of the corresponding Gaussian partners:

\[ \text{Tr} \left( |\Psi\rangle \langle 0| e^{i(x_A q_A - x_B q_B)} e^{i(x_B p_A - x_A p_B)} \right) = \text{Tr} \left( |0\rangle \langle 0| e^{i(x_A q_A - x_B q_B)} e^{i(x_B p_A - x_A p_B)} \right) . \]

Thus, \((\hat{q}'_A, \hat{p}'_A)\) and \((\hat{q}'_B, \hat{p}'_B)\) provide partners for a quantum field in a pure non-Gaussian state \(|\Psi\rangle\). From the viewpoint of pure mathematics, the example is merely a unitary-equivalent one to partners in Gaussian states. However it should be stressed that the above example is nontrivial in a physical sense. The above particle modes in the non-Gaussian state are physically detected by realistic particle detectors which fix what operators can be observed.

Beyond the above example, a natural question arises: if we fix an arbitrary mode \(A\) of a field in a general state, does its partner always exist? Interestingly the answer is ‘yes’ when we consider \(N\) coupled harmonic oscillators as a \(1 + 1\) dimensional discretized scalar quantum field in a general pure state \(|\Psi\rangle_{1, \ldots, N}\). Let us define a particle mode \(A\) as a linear combination:

\[ \hat{q}_A \equiv \sum_{n=1}^{N} (x_A(n) \hat{q}_n + y_A(n) \hat{p}_n), \quad \hat{p}_A \equiv \sum_{n=1}^{N} (z_A(n) \hat{q}_n + w_A(n) \hat{p}_n), \]

(4)

where \((\hat{q}_n, \hat{p}_n)\) denote the canonical operators for \(n\)th harmonic oscillator. Imposing the condition \([\hat{q}_A, \hat{p}_A] = i\), we have a constraint on the coefficients:

\[ \sum_{n=1}^{N} (x_A(n) w_A(n) - z_A(n) y_A(n)) = 1. \]
The Stone–von Neumann theorem [10] guarantees that there exists an unitary operator \( \hat{V}_N \) such that \( V_N \hat{q}_A \hat{V}_N^\dagger = \hat{q}_1 \) and \( V_N \hat{p}_A \hat{V}_N^\dagger = \hat{p}_1 \). The transformed state is given by \( |\Psi\rangle_{1, \ldots, N} \equiv \hat{V} |\Psi\rangle_{1, \ldots, N} \) and remains pure. Let us consider the Schmidt decomposition of \( |\Psi\rangle \) as

\[
|\Psi\rangle_{1, \ldots, N} = \sum_{n=0}^{\infty} \sqrt{p_n} |a_n\rangle_1 |\psi_n\rangle_{2, \ldots, N},
\]

where \( \{p_n\}_{n=0}^\infty \) is a probability distribution. Here we assume that the reduced state \( \hat{\rho}_1 \) of the first mode, which is defined as

\[
\hat{\rho}_1 = \text{Tr}_{2, \ldots, N} |\Psi\rangle \langle \Psi|,
\]

has a spectral decomposition in terms of a discrete basis \( \{|a_n\rangle : n = 0, 1, 2, \ldots \} \) of the sub-Hilbert space as

\[
\hat{\rho}_1 = \sum_{n=0}^{\infty} p_n |a_n\rangle_1 \langle a_n|_1.
\]

This may be not an essential constraint, and if the continuum spectrum emerges, a small modification and generalization of this argument is expected to yield the same conclusions.

To obtain the partner mode, let us consider the following creation and annihilation operators:

\[
\hat{b}^\dagger = \sum_{n=0}^{\infty} \sqrt{n+1} \sum_{i=1}^{\infty} |\psi_n^{(i)}\rangle_{2, \ldots, N} \langle \psi_n^{(i)}|_{2, \ldots, N}, \quad \hat{b} = \sum_{n=0}^{\infty} \sqrt{n+1} \sum_{i=1}^{\infty} |\psi_n^{(i)}\rangle_{2, \ldots, N} \langle \psi_n^{(i)}|_{2, \ldots, N},
\]

where we have introduced an orthonormal basis \( \{|\psi_n^{(i)}\rangle\} \), satisfying \( |\psi_n^{(i)}\rangle = |\psi_n\rangle \) for all \( n \), and \( \langle \psi_n^{(i)}| \psi_m^{(j)} \rangle = \delta_{nm} \delta_{ij} \). These operators satisfy \( [\hat{b}, \hat{b}^\dagger] = 1 \). Then, the conditions (i), (ii) and (iii) are satisfied for \( (\hat{q}_B, \hat{p}_B) \) defined by

\[
\hat{q}_B \equiv \hat{V}_N^\dagger \left( I \otimes \frac{1}{\sqrt{2}} \left( \hat{b} + \hat{b}^\dagger \right) \right) \hat{V}_N, \quad \hat{p}_B \equiv \hat{V}_N^\dagger \left( I \otimes \frac{1}{\sqrt{2}} \left( \hat{b} - \hat{b}^\dagger \right) \right) \hat{V}_N.
\]

Regarding this construction of the partner, the following three points should be noted. First, such a partner is non-unique if \( \hat{\rho}_1 \) is not full rank. For example, if \( p_0 = 0 \), \( |\psi_0\rangle_{2, \ldots, N} \) can be an arbitrary normalized vector orthogonal to \( |\psi_n\rangle_{2, \ldots, N} (n = 1, 2, \ldots) \). Furthermore, even when \( \hat{\rho}_1 \) is full rank, \( \{|\psi_n^{(i)}\rangle \}_{i=2}^{N-1} \) can be an arbitrary set of orthonormal vectors as long as \( \{|\psi_n^{(i)}\rangle \}_{i=1}^{N-1} \) forms an orthonormal basis. Second, \( (\hat{q}_B, \hat{p}_B) \) may not be linear combinations of \( \hat{q}_n \) and \( \hat{p}_n \). Third, the continuum limit to reproduce the original field remains subtle and requires further delicate analysis. Nevertheless, surprisingly, it is shown that for Gaussian states, there exists the unique partner whose canonical operators are given by linear combination of \( \hat{q}_n \) and \( \hat{p}_n \). A closed formula to obtain the partner mode is presented in the following section. In this case, it is possible to take the continuum limit, i.e. we have the unique partner for the free scalar field in Gaussian states.

### 3. Partner mode in the Gaussian vacuum states

In this section, we derive the partner formula for a Gaussian vacuum state of a free scalar field. The extension of the formula for an excited Gaussian state is given in the following section. We first derive the partner formula for a discretized scalar quantum field theory in a flat \((1 + 1)\)-dimensional spacetime. Let us impose a periodic boundary condition on the field:
\[ \hat{\phi}(t, x + L) = \hat{\phi}(t, x), \]

where \( L \) denotes the entire space length. The free Hamiltonian of the system is given by

\[ \hat{H} = \frac{1}{2} \int_{-L/2}^{L/2} dx : \hat{\Pi}(x)^2 : + \frac{1}{2} \int_{-L/2}^{L/2} dx : (\partial_x \hat{\phi}(x))^2 : + \frac{m^2}{2} \int_{-L/2}^{L/2} dx : \hat{\phi}(x)^2 :, \]

where \( : \hat{O} : \) is the normal ordering of a linear operator \( \hat{O} \), and \( \hat{\Pi}(x) \) is the canonical momentum of the field \( \hat{\phi}(x) \) satisfying

\[ [\hat{\phi}(x), \hat{\Pi}(x')] = i \delta(x - x'). \]

In order to obtain the partner formula, consider a corresponding discretized model with lattice spacing \( \epsilon \). The field operator \( \hat{\phi}(x) \) and its conjugate momentum \( \hat{\Pi}(x) \) correspond to

\[ \hat{\phi}(x) \rightarrow \frac{q_n}{\sqrt{m \epsilon}}, \quad \hat{\Pi}(x) \rightarrow \sqrt{\frac{m}{\epsilon}} p_n. \quad (6) \]

Introducing new variables \( N \equiv L/\epsilon \) and \( \eta \equiv 1/(m \epsilon)^2 \) reproduces the discretized Hamiltonian of the coupled harmonic oscillators:

\[ \hat{H} = \frac{1}{2} \sum_{n=1}^{N-1} \hat{p}_n^2 : + \left( \frac{1}{2} + \eta \right) \sum_{n=1}^{N} \hat{q}_n^2 : - \eta \sum_{n=1}^{N} \hat{q}_n \hat{q}_{n+1} : \quad (7) \]

where \( \hat{O} : \) means normal ordered operator of \( \hat{O} \) with respect to creation and annihilation operators, \( \hat{q}_n \) and \( \hat{p}_n \) satisfy the canonical commutation relations \( [\hat{q}_m, \hat{p}_n] = i \delta_{mn} \). The Hamiltonian generates the evolution with respect to a new time coordinate \( \tau \equiv m \epsilon t \). By using the mode functions

\[ u_k(n) \equiv \frac{1}{\sqrt{N}} \exp \left( 2\pi i k \frac{n}{N} \right), \]

the canonical operators are expanded as

\[ \hat{q}_n = \sum_{k=0}^{N-1} \frac{1}{\sqrt{2Nk}} \left( \hat{a}_k u_k(n) + \hat{a}_k^\dagger u_k(n)^* \right), \quad \hat{p}_n = \frac{1}{i} \sum_{k=0}^{N-1} \sqrt{\frac{\omega_k}{2}} \left( \hat{a}_k u_k(n) - \hat{a}_k^\dagger u_k(n)^* \right), \quad (8) \]

where the dispersion relationship is given by

\[ \omega_k^2 = 1 + 2\eta \left( 1 - \cos \left( \frac{2\pi k}{N} \right) \right). \]

Since the canonical commutation relation \( [\hat{q}_m, \hat{p}_n] = i \delta_{mn} \) yields \( \hat{a}_k, \hat{a}_k^\dagger = \delta_{kk'}, \hat{a}_k^\dagger \) and \( \hat{a}_k \) are creation and annihilation operator for a mode \( k \). The vacuum state \( |0\rangle \) is defined as a unit vector annihilated by \( \hat{a}_k \) for all \( k = 0, 1, \cdots N - 1 \). Hereafter, \( \langle \hat{O} \rangle \equiv \langle 0 | \hat{O} | 0 \rangle \) for a linear operator \( \hat{O} \).

Let us construct a set of canonical variables \( \{ \hat{q}_A, \hat{p}_A \} \) in the previous section. For the derivation of the partner formula, we will use the covariance matrix. For a review of its properties, see appendix B. The covariance matrix associated to the canonical variables \( \{ \hat{q}_A, \hat{p}_A \} \) is given by

\[ m_A = \begin{pmatrix} \langle \hat{q}_A^2 \rangle & \text{Re} \langle \hat{q}_A \hat{p}_A \rangle \\ \text{Re} \langle \hat{p}_A \hat{q}_A \rangle & \langle \hat{p}_A^2 \rangle \end{pmatrix}. \]
Through a local symplectic transformation

\[
\begin{pmatrix}
\hat{Q}_A \\
\hat{P}_A
\end{pmatrix} = S_A \begin{pmatrix}
\hat{q}_A \\
\hat{p}_A
\end{pmatrix} = \begin{pmatrix}
\cos \theta'_A & \sin \theta'_A \\
-\sin \theta'_A & \cos \theta'_A
\end{pmatrix} \begin{pmatrix}
e^{\theta_A} & 0 \\
0 & e^{-\theta_A}
\end{pmatrix} \begin{pmatrix}
\cos \theta_A & \sin \theta_A \\
-\sin \theta_A & \cos \theta_A
\end{pmatrix} \begin{pmatrix}
\hat{q}_A \\
\hat{p}_A
\end{pmatrix}
\]

(9)

with \(\theta'_A, \sigma_A, \theta_A \in \mathbb{R}\), it is possible to bring the covariance matrix \(M_A\) for new canonical variables \((\hat{Q}_A, \hat{P}_A)\) to the following standard form:

\[
M_A = \begin{pmatrix}
\langle \hat{Q}_A^2 \rangle & \text{Re} \left( \langle \hat{Q}_A \hat{P}_A \rangle \right) \\
\text{Re} \left( \langle \hat{P}_A \hat{Q}_A \rangle \right) & \langle \hat{P}_A^2 \rangle
\end{pmatrix} = \frac{\sqrt{1+g^2}}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},
\]

(10)

where \(g\) is a non-negative parameter. As is easily seen, we can take \(\theta'_A = 0\) without loss of generality. Solving equation (10), \(\sigma_A\) and \(\theta_A\) are fixed. Note that this \(g\) is uniquely determined by the elements of \(m_A\) since

\[
\frac{1}{4} (1 + g^2) = \det M_A = \det m_A = \langle \hat{q}_A^2 \rangle \langle \hat{p}_A^2 \rangle - \frac{1}{4} (\langle \hat{q}_A \hat{p}_A \rangle + \langle \hat{p}_A \hat{q}_A \rangle)^2,
\]

(11)

where we have used \(\det S_A \det S_A^T = 1\). When \(g = 0\), the mode \(A\) is in a pure state. If \(g \neq 0\), the mode \(A\) is in a mixed state, meaning that there is a purification partner mode \(B\). Hereafter, we assume \(g \neq 0\), i.e. the mode \(A\) is in a mixed state. In the following, we will construct a set of canonical variables \((\hat{Q}_B, \hat{P}_B)\) that represents the purification partner of mode \(A\) such that the composite system \(AB\) is in a pure state. This purification partner \(B\) of \(A\) is characterized by a set of canonical variables

\[
\hat{Q}_B = \sum_{n=1}^{N} (X_B(n) \hat{q}_n + Y_B(n) \hat{p}_n), \quad \hat{P}_B = \sum_{n=1}^{N} (Z_B(n) \hat{q}_n + W_B(n) \hat{p}_n)
\]

(12)

which must satisfy the following:

(i) **Commutation relation:** \([\hat{Q}_B, \hat{P}_B] = i\)

(ii) **Locality:** \([\hat{Q}_A, \hat{Q}_B] = 0, [\hat{Q}_A, \hat{P}_B] = 0, [\hat{P}_A, \hat{Q}_B] = 0, \text{ and } [\hat{P}_A, \hat{P}_B] = 0\)

(iii) **Purification condition:** the covariance matrix takes the following form:

\[
M_{AB} = \begin{pmatrix}
\langle \hat{Q}_A^2 \rangle & \text{Re} \left( \langle \hat{Q}_A \hat{P}_A \rangle \right) & \langle \hat{Q}_A \hat{Q}_B \rangle & \langle \hat{Q}_A \hat{P}_B \rangle \\ 
\text{Re} \left( \langle \hat{P}_A \hat{Q}_A \rangle \right) & \langle \hat{P}_A^2 \rangle & \langle \hat{P}_A \hat{Q}_B \rangle & \langle \hat{P}_A \hat{P}_B \rangle \\ 
\langle \hat{Q}_B \hat{Q}_A \rangle & \langle \hat{Q}_B \hat{P}_A \rangle & \langle \hat{Q}_B^2 \rangle & \text{Re} \left( \langle \hat{Q}_B \hat{P}_B \rangle \right) \\ 
\langle \hat{P}_B \hat{Q}_A \rangle & \langle \hat{P}_B \hat{P}_A \rangle & \text{Re} \left( \langle \hat{P}_B \hat{Q}_B \rangle \right) & \langle \hat{P}_B^2 \rangle
\end{pmatrix}
\]

\[
= \begin{pmatrix}
\frac{1}{2} \sqrt{1 + g^2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} \sqrt{1 + g^2} & 0 & -\frac{g}{2} \\ 0 & 0 & \frac{1}{2} \sqrt{1 + g^2} & 0 \\ \frac{-g}{2} & 0 & 0 & \frac{1}{2} \sqrt{1 + g^2}
\end{pmatrix}
\]

(13)

such that the state of the composite system \(AB\) is in a pure state.

The condition (iii) in section 2 is now simplified to a condition on the covariance matrix for a two-mode Gaussian state. Equation (13) is what is called the standard form of the covariance.
matrix for a pure Gaussian state \[12, 13\]. More details about the covariance matrix can be found in appendix B. As we will see, the purification condition plays a crucial role to obtain the partner formula.

Since the operators \((\hat{Q}_A, \hat{P}_A, \hat{Q}_B, \hat{P}_B)\) are constructed as a linear combination of \(\{(\hat{q}_n, \hat{p}_n)\}_{n=1}^N\), any expectation value of a product of operators \((\hat{Q}_A, \hat{P}_A, \hat{Q}_B, \hat{P}_B)\) for a Gaussian state is calculated by using the Wick’s theorem. Thus, the covariance matrix \(M_{AB}\) characterizes arbitrary observable on the two-mode system \(AB\), meaning that it gives a reduced state in the correlation space.

On the other hand, the locality conditions imply that no operation \(\hat{U}_B(\hat{Q}_B, \hat{P}_B)\) on mode \(B\) generated by \((\hat{Q}_B, \hat{P}_B)\) affects the reduced state of mode \(A\), and vice-versa. Therefore, in the correlation space spanned by \((\hat{Q}_A, \hat{P}_A, \hat{Q}_B, \hat{P}_B)\) \[14\] \[15\], \(A\) and \(B\) are locally independent.

Since locality of \(A\) and \(B\) can be introduced, quantum entanglement among \(A\) and \(B\) is well defined. The entanglement entropy \(S_{EE}\) between mode \(A\) and its partner \(B\) depends on the positive parameter \(g\) as follows \[16\]:

\[
S_{EE} = \sqrt{1 + g^2} \ln \left(\frac{1}{g} \sqrt{1 + g^2 + 1}\right) + \ln \left(\frac{g}{2}\right).
\] (14)

The purification condition on equation (13) can be summarized as follows:

\[
\langle \hat{Q}_A \hat{P}_B \rangle = \langle \hat{P}_A \hat{Q}_B \rangle = 0,
\] (15)
\[
\langle \hat{Q}_A \hat{Q}_B \rangle = -\langle \hat{P}_A \hat{P}_B \rangle = \frac{g}{2},
\] (16)
\[
\text{Re} \left(\langle \hat{Q}_B \hat{P}_B \rangle\right) = 0,
\] (17)
\[
\langle \hat{Q}_B^2 \rangle = \langle \hat{P}_B^2 \rangle = \frac{\sqrt{1 + g^2}}{2}.
\] (18)

In addition, the commutation relation \([\hat{Q}_B, \hat{P}_B] = i\) gives us:

\[
\langle \hat{Q}_B \hat{P}_B - \hat{P}_B \hat{Q}_B \rangle = i.
\] (19)

To obtain the solution of the above equations, let us expand \((\hat{Q}_A, \hat{P}_A, \hat{Q}_B, \hat{P}_B)\) in terms of \(\hat{a}_k\) and \(\hat{a}_k^\dagger\) as follows:

\[
\hat{Q}_A = \left(\frac{\sqrt{1 + g^2}}{2}\right)^{1/2} \sum_{k=0}^{N-1} \left( Q_A(k) \hat{a}_k + Q_A(k) \hat{a}_k^\dagger \right),
\] (20)
\[
\hat{P}_A = \left(\frac{\sqrt{1 + g^2}}{2}\right)^{1/2} \sum_{k=0}^{N-1} \left( P_A(k) \hat{a}_k + P_A(k) \hat{a}_k^\dagger \right),
\] (21)
\[
\hat{Q}_B = \left(\frac{\sqrt{1 + g^2}}{2}\right)^{1/2} \sum_{k=0}^{N-1} \left( Q_B(k) \hat{a}_k + Q_B(k) \hat{a}_k^\dagger \right),
\] (22)
\[
\hat{P}_B = \left( \frac{\sqrt{1 + g^2}}{2} \right)^{1/2} \sum_{k=0}^{N-1} \left( P_B^*(k) \hat{a}_k + P_B(k) \hat{a}_k^\dagger \right),
\]

where we have factored out \( \left( \frac{\sqrt{1 + g^2}}{2} \right)^{1/2} \) for future convenience.

For Hermite operators \( \hat{O}_1 \) and \( \hat{O}_2 \) defined as linear combinations of \( a_k \) and \( a_k^\dagger \) such as

\[
\hat{O}_i = \left( \frac{\sqrt{1 + g^2}}{2} \right)^{1/2} \sum_{k=0}^{N-1} \left( O_i(k) \hat{a}_k + O_i(k) \hat{a}_k^\dagger \right),
\]

we get

\[
\langle \hat{O}_1 \hat{O}_2 \rangle = \frac{\sqrt{1 + g^2}}{2} \langle O_1, O_2 \rangle,
\]

where we have defined the standard inner product in \( \mathbb{C}^N \):

\[
\langle O_1, O_2 \rangle \equiv \sum_{k=0}^{N-1} O_i(k) O_2(k).
\]

Equations (15)–(19) are expressed as the followings:

\[
\begin{pmatrix}
\langle Q_A, Q_A \rangle & \langle Q_A, P_A \rangle & \langle Q_A, Q_B \rangle & \langle Q_A, P_B \rangle \\
\langle P_A, Q_A \rangle & \langle P_A, P_A \rangle & \langle P_A, Q_B \rangle & \langle P_A, P_B \rangle \\
\langle Q_B, Q_A \rangle & \langle Q_B, P_A \rangle & \langle Q_B, Q_B \rangle & \langle Q_B, P_B \rangle \\
\langle P_B, Q_A \rangle & \langle P_B, P_A \rangle & \langle P_B, Q_B \rangle & \langle P_B, P_B \rangle \\
\end{pmatrix}
= \begin{pmatrix}
1 & \frac{i}{\sqrt{1 + g^2}} & \frac{g}{\sqrt{1 + g^2}} & 0 \\
\frac{i}{\sqrt{1 + g^2}} & 1 & 0 & -\frac{g}{\sqrt{1 + g^2}} \\
\frac{g}{\sqrt{1 + g^2}} & 0 & 1 & \frac{i}{\sqrt{1 + g^2}} \\
0 & -\frac{g}{\sqrt{1 + g^2}} & \frac{i}{\sqrt{1 + g^2}} & 1 \\
\end{pmatrix}.
\]

Since \( Q_A \) and \( P_B \) are orthonormal, \( |\langle Q_A, Q_B \rangle|^2 + |\langle P_B, Q_B \rangle|^2 = 1 \) and \( |\langle Q_B, Q_B \rangle|^2 = 1 \) imply that

\[
Q_B(k) = \langle Q_A, Q_B \rangle Q_A(k) + \langle P_B, Q_B \rangle P_B(k)
= \frac{g}{\sqrt{1 + g^2}} Q_A(k) - \frac{i}{\sqrt{1 + g^2}} P_B(k).
\]

Similarly,

\[
P_B(k) = -\frac{g}{\sqrt{1 + g^2}} P_A(k) + \frac{i}{\sqrt{1 + g^2}} Q_B(k).
\]

Combining equations (28) and (29), we finally get the unique solution:

\[
Q_B(k) = \frac{\sqrt{1 + g^2}}{g} Q_A(k) + \frac{i}{g} P_A(k), \quad P_B(k) = -\frac{\sqrt{1 + g^2}}{g} P_A(k) + \frac{i}{g} Q_A(k).
\]

It should be noted that the commutativity condition among \( \langle \hat{Q}_A, \hat{P}_A \rangle \) and \( \langle \hat{Q}_B, \hat{P}_B \rangle \) automatically satisfied since

\[
[\hat{O}_1, \hat{O}_2] = \frac{\sqrt{1 + g^2}}{2} (\langle O_1, O_2 \rangle - \langle O_2, O_1 \rangle).
\]
Therefore, the partner mode is written as

\[
\hat{Q}_B = \sqrt{1 + g^2} \hat{Q}_A - \frac{i}{g} \left( \sqrt{1 + g^2} \right) \frac{1}{2} \sum_{k=0}^{N-1} \left( P_A(k) \hat{a}_k - P_A(k) \hat{a}_k^\dagger \right),
\]

(32)

\[
\hat{P}_B = -\sqrt{1 + g^2} g \hat{P}_A - \frac{i}{g} \left( \sqrt{1 + g^2} \right) \frac{1}{2} \sum_{k=0}^{N-1} \left( Q_A(k) \hat{a}_k - Q_A(k) \hat{a}_k^\dagger \right).
\]

(33)

By re-writing the last equation in terms of the weighting functions:

\[
X_A(n) = \sum_{k=0}^{N-1} \sqrt{\frac{\omega_k}{N}} [Q_A^*(k) u_k^*(n) + Q_A(k) u_k(n)],
\]

(34)

\[
Y_A(n) = \sum_{k=0}^{N-1} \sqrt{\frac{\omega_k}{N}} [Q_A^*(k) u_k^*(n) - Q_A(k) u_k(n)],
\]

(35)

\[
Z_A(n) = \sum_{k=0}^{N-1} \sqrt{\frac{\omega_k}{N}} [P_A^*(k) u_k^*(n) + P_A(k) u_k(n)],
\]

(36)

\[
W_A(n) = \sum_{k=0}^{N-1} \sqrt{\frac{\omega_k}{N}} [P_A^*(k) u_k^*(n) - P_A(k) u_k(n)],
\]

(37)

and similarly for the partner B weighting functions, the partner B can be written in terms of the weighting functions of mode A as follows:

\[
\hat{Q}_B = \left( \frac{\sqrt{1 + g^2}}{2} \right)^{1/2} \sum_{n=1}^{N} (X_B(n) \hat{q}_n + Y_B(n) \hat{p}_n),
\]

(38)

\[
\hat{P}_B = \left( \frac{\sqrt{1 + g^2}}{2} \right)^{1/2} \sum_{n=1}^{N} (Z_B(n) \hat{q}_n + W_B(n) \hat{p}_n),
\]

(39)

where

\[
X_B(n) \equiv \sqrt{1 + g^2} g X_A(n) - \frac{2}{g} \sum_{n'=1}^{N} \Delta_p(n - n') W_A(n'),
\]

(40)

\[
Y_B(n) \equiv \sqrt{1 + g^2} g Y_A(n) + \frac{2}{g} \sum_{n'=1}^{N} \Delta_g(n - n') Z_A(n'),
\]

(41)

\[
Z_B(n) \equiv -\sqrt{1 + g^2} g Z_A(n) - \frac{2}{g} \sum_{n'=1}^{N} \Delta_g(n - n') Y_A(n'),
\]

(42)

\[
W_B(n) \equiv -\sqrt{1 + g^2} g W_A(n) + \frac{2}{g} \sum_{n'=1}^{N} \Delta_q(n - n') X_A(n'),
\]

(43)
with

\[ \Delta_q(n - n') \equiv \langle \hat{q}_n \hat{q}_{n'} \rangle = \frac{1}{N} \sum_{k=0}^{N-1} \frac{1}{2\omega_k} \exp \left( \frac{2\pi i k n - n'}{N} \right), \]  

(44)

\[ \Delta_p(n - n') \equiv \langle \hat{p}_n \hat{p}_{n'} \rangle = \frac{1}{N} \sum_{k=0}^{N-1} \omega_k \frac{1}{2} \exp \left( \frac{2\pi i k n - n'}{N} \right). \]  

(45)

This is the partner formula for the vacuum of the free lattice scalar field theory. Before taking the continuum limit, let us analyze our results. From our partner formula, two different kinds of partner can be defined: the spatially separated partner (SSP) and spatially overlapped partner (SOP) as follows:

**Definition.** If the weighting functions of mode \(B\): \(\{X_B(n), Y_B(n), Z_B(n), W_B(n)\}_n\) have any spatial overlap with \(\{X_A(n), Y_A(n), Z_A(n), W_A(n)\}_n\), we call the modes \(A\) and \(B\) spatially overlapped partners (SOP). If not, we call them spatially separated partners (SSP).

This definition is straightforwardly extended for an arbitrary Gaussian state in the scalar field theory. In [17, 18], SSPs have been constructed for a special case to investigate the spatial structure of entanglement in the vacuum state. By using our partner formula, it is possible to investigate not only SSPs but also SOPs. Thus, it provides a new way to extract and make use of information stored in a quantum field. Furthermore, since one can identify the partner mode \(B\) for arbitrary mode \(A\), it can be used to introduce a tensor product structure in the entire Hilbert space even when there is not a natural tensor product structure in advance.

So far, we have obtained the partner formula in a \((1 + 1)\)-dimensional lattice free field theory. The extension of the results into a \((d + 1)\)-dimensional spacetime is obtained in a straightforward way. First, let us extend our results to a \(d\)-dimensional lattice space. Let \(\mathbf{n}\) be a \(d\)-dimensional vector which characterize the spatial position of each oscillator degree of freedom \((\hat{q}_n, \hat{p}_n)\). The extension of equations (40)–(43) to a \(d\)-dimensional lattice space can be obtained by replacing \(n\) into \(\mathbf{n}\). Then, the continuum limit can be taken. The partner formula for a \((d + 1)\) dimensional quantum field is given by

\[ \hat{Q}_A = \left( \frac{\sqrt{1 + g^2}}{2} \right)^{1/2} \int d^d\mathbf{x} \left[ X_A(x) \hat{\phi}^S(x) + Y_A(x) \hat{\Pi}^S(x) \right] \]  

(46)

\[ \hat{P}_A = \left( \frac{\sqrt{1 + g^2}}{2} \right)^{1/2} \int d^d\mathbf{x} \left[ Z_A(x) \hat{\phi}^S(x) + W_A(x) \hat{\Pi}^S(x) \right] \]  

(47)

\[ \hat{Q}_B = \left( \frac{\sqrt{1 + g^2}}{2} \right)^{1/2} \int d^d\mathbf{x} \left[ X_B(x) \hat{\phi}^S(x) + Y_B(x) \hat{\Pi}^S(x) \right] \]  

(48)

\[ \hat{P}_B = \left( \frac{\sqrt{1 + g^2}}{2} \right)^{1/2} \int d^d\mathbf{x} \left[ Z_B(x) \hat{\phi}^S(x) + W_B(x) \hat{\Pi}^S(x) \right], \]  

(49)

with the weighting functions of the partner \(B\) written in terms of those of the mode \(A\) as follows:
X_B(x) \equiv \frac{\sqrt{1 + g^2}}{g} X_A(x) - \frac{2}{g} \int d^d x' \Delta_p(x - x') W_A(x'), 
\tag{50}

Y_B(x) \equiv \frac{\sqrt{1 + g^2}}{g} Y_A(x) + \frac{2}{g} \int d^d x' \Delta_q(x - x') Z_A(x'), 
\tag{51}

Z_B(x) \equiv -\frac{\sqrt{1 + g^2}}{g} Z_A(x) - \frac{2}{g} \int d^d x' \Delta_q(x - x') Y_A(x'), 
\tag{52}

W_B(x) \equiv -\frac{\sqrt{1 + g^2}}{g} W_A(x) + \frac{2}{g} \int d^d x' \Delta_q(x - x') X_A(x'), 
\tag{53}

where

\Delta_q(x - x') = \int \frac{d^d k}{(2\pi)^d} \frac{1}{2E_k} e^{i k \cdot (x - x')}, 
\tag{54}

\Delta_p(x - x') = \int \frac{d^d k}{(2\pi)^d} \frac{E_k}{2} e^{i k \cdot (x - x')} 
\tag{55}

with $E_k \equiv \sqrt{|k|^2 + m^2}$.

4. Partner mode in excited Gaussian states

Let us consider an $N$ harmonic oscillator system in a pure Gaussian state $|\Psi\rangle$. Here, we do not assume that $|\Psi\rangle$ is the ground state of the Hamiltonian of the system. It is known that there exists a second-order ‘Hamiltonian’ $H = \sum_{k=0}^{N-1} \omega_k b_k^{\dagger} b_k$ whose ground state is $|\Psi\rangle$, where $\omega_k > 0$, $b_k^{\dagger}$ and $b_k$ are creation and annihilation operators, and $\langle \Psi | b_k | \Psi \rangle = 0$ [21]. Thus, if we fix a mode $A$ by

$$\hat{q}_A = \sum_{n=1}^{N} (x(n)\hat{q}_n + y(n)\hat{p}_n') , \quad \hat{p}_A = \sum_{n=1}^{N} (z(n)\hat{q}_n' + w(n)\hat{p}_n') , \tag{56}$$

where

$$\hat{q}_n = \sum_{k=0}^{N-1} \frac{1}{\sqrt{2\omega_k}} \left( b_k u_k(n) + b_k^{\dagger} u_k(n)^* \right) , \quad \hat{p}_n = \frac{1}{i} \sum_{k=0}^{N-1} \sqrt{\frac{\omega_k}{2}} \left( b_k u_k(n) - b_k^{\dagger} u_k(n) \right) , \tag{57}$$

the procedure to identify the partner mode $B$ presented in the previous section is applicable in a direct way.

Now, let us derive a more general expression of the partner formula for an arbitrary Gaussian state $|\Psi\rangle$. Fix a complete set of canonical operators $\{ (\hat{q}_n, \hat{p}_n) \}_{n=1}^{N}$ satisfying $[\hat{q}_n, \hat{p}_m] = i\delta_{nm}$, which is not necessarily assumed to be the same as that defined in equations (8) or (57). Without loss of generality, it is possible to assume $\langle \Psi | \hat{q}_m | \Psi \rangle = 0$ and $\langle \Psi | \hat{p}_n | \Psi \rangle = 0$ hold for all $n$ by shifting

$$\hat{q}_n \rightarrow \hat{q}_n - \langle \Psi | \hat{q}_n | \Psi \rangle , \quad \hat{p}_n \rightarrow \hat{p}_n - \langle \Psi | \hat{p}_n | \Psi \rangle . \tag{58}$$

Let us fix a mode $A$ characterized by weighting functions $\{(x(n), y(n), z(n), w(n))\}_{n=1}^{N}$ defined as
\[ \hat{q}_A = \sum_{n=1}^{N} (x(n)\hat{q}_n + y(n)\hat{p}_n) \equiv \mathbf{v}_A^T \hat{r}, \]

\[ \hat{p}_A = \sum_{n=1}^{N} (z(n)\hat{q}_n + w(n)\hat{p}_n) \equiv \mathbf{u}_A^T \hat{r}, \]

where we have defined \( \hat{r} \equiv (\hat{q}_1, \hat{p}_1, \ldots, \hat{q}_N, \hat{p}_N)^T \) and \( (x(1), y(1), \ldots, x(N), y(N))^T, \mathbf{u}_A = (z(1), w(1), \ldots, z(N), w(N))^T \in \mathbb{R}^{2N} \). Imposing \( [\hat{q}_A, \hat{p}_A] = i \), the vectors must satisfy \( \mathbf{v}_A^T \Omega \mathbf{u}_A = 1 \), where \( \Omega = \bigoplus_{n=1}^{N} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \).

After an appropriate local symplectic transformation, it is possible to bring the set of operators to the standard form \( (\hat{Q}_A, \hat{P}_A) \equiv (\mathbf{V}_A^T \hat{r}, \mathbf{U}_A^T \hat{r}) \), such that

\[
\begin{pmatrix}
\langle \Psi | \hat{Q}_A^2 | \Psi \rangle \\
\operatorname{Re} \left( \langle \Psi | \hat{Q}_A \hat{P}_A | \Psi \rangle \right)
\end{pmatrix}
= \frac{\sqrt{1 + g^2}}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\]

holds, where \( g \equiv \sqrt{\frac{4}{2} \left( \langle \Psi | \hat{q}_A^2 | \Psi \rangle \langle \Psi | \hat{p}_A^2 | \Psi \rangle - \operatorname{Re} \left( \langle \Psi | \hat{q}_A \hat{p}_A | \Psi \rangle \right) \right) - 1} \). This condition is equivalent to

\[
\mathbf{V}_A^T \mathbf{M} \mathbf{A} = \mathbf{U}_A^T \mathbf{M} \mathbf{A} = \frac{\sqrt{1 + g^2}}{2}, \quad \mathbf{V}_A^T \mathbf{M} \mathbf{A} = \mathbf{U}_A^T \mathbf{M} \mathbf{A} = 0,
\]

where we have defined the covariance matrix \( \mathbf{M} \equiv \operatorname{Re} \left( \langle \Psi | \hat{r} \hat{r}^T | \Psi \rangle \right) \). From \( \mathbf{v}_A^T \Omega \mathbf{u}_A = 1 \), we also have

\[
\mathbf{V}_A^T \Omega \mathbf{U}_A = 1.
\]

Now let us define another mode \( B \) by \( (\hat{Q}_B, \hat{P}_B) \equiv (\mathbf{V}_B^T \hat{r}, \mathbf{U}_B^T \hat{r}) \), where

\[
\mathbf{V}_B^T \Omega \mathbf{U}_B = 1
\]

is assumed to be satisfied. From the locality condition and the purification condition, the mode \( B \) is the partner of \( A \) if and only if

\[
\mathbf{V}_B^T \Omega \mathbf{V}_B = \mathbf{V}_B^T \Omega \mathbf{U}_B = \mathbf{U}_B^T \Omega \mathbf{V}_B = \mathbf{U}_B^T \Omega \mathbf{U}_B = 0
\]

and

\[
\mathbf{V}_B^T \mathbf{M} \mathbf{B} = -\mathbf{U}_B^T \mathbf{M} \mathbf{B} = \frac{g}{2}, \quad \mathbf{V}_B^T \mathbf{M} \mathbf{B} = \mathbf{U}_A^T \mathbf{M} \mathbf{B} = 0,
\]

\[
\mathbf{V}_B^T \mathbf{M} \mathbf{B} = \mathbf{U}_A^T \mathbf{M} \mathbf{B} = \frac{\sqrt{1 + g^2}}{2},
\]

\[
\mathbf{V}_B^T \mathbf{M} \mathbf{B} = 0
\]
hold. Since the partner mode is unique, if one could find \( V_B, U_B \in \mathbb{R}^N \) satisfying equations (65)–(70) under the constraints (63) and (64), then the mode \( B \) is the partner of \( A \). From equations (40)–(43), it is not hard to expect that

\[
V_B = \frac{\sqrt{1 + g^2}}{g} V_A - \frac{2}{g} \Omega M U_A, \quad U_B = -\frac{\sqrt{1 + g^2}}{g} U_A - \frac{2}{g} \Omega MV_A
\]

(71)
satisfy the requirements. In fact, it can straightforwardly be verified by using \( M \Omega M \equiv \frac{1}{2} \Omega \).

This identity always holds for pure Gaussian states \( |\Psi\rangle \), which follows from the fact that there exists a symplectic matrix \( S \) such that \( M = \frac{1}{2} SS^T \) [21]. In terms of weighting functions, equation (71) can be written as

\[
X_B(n) = \frac{\sqrt{1 + g^2}}{g} X_A(n) - \frac{2}{g} \sum_{m=1}^{N} (\text{Re} \langle \psi|\hat{p}_n\hat{q}_m|\psi\rangle) Z_A(m) + \langle \psi|\hat{p}_n\hat{p}_m|\psi\rangle W_A(m),
\]

(72)

\[
Y_B(n) = \frac{\sqrt{1 + g^2}}{g} Y_A(n) + \frac{2}{g} \sum_{m=1}^{N} (\langle \psi|\hat{q}_n\hat{q}_m|\psi\rangle Z_A(m) + \text{Re} \langle \psi|\hat{q}_n\hat{p}_m|\psi\rangle W_A(m),
\]

(73)

\[
Z_B(n) = -\frac{\sqrt{1 + g^2}}{g} Z_A(n) - \frac{2}{g} \sum_{m=1}^{N} (\text{Re} \langle \psi|\hat{q}_n\hat{p}_m|\psi\rangle X_A(m) + \langle \psi|\hat{p}_n\hat{p}_m|\psi\rangle Y_A(m),
\]

(74)

\[
W_B(n) = -\frac{\sqrt{1 + g^2}}{g} W_A(n) + \frac{2}{g} \sum_{m=1}^{N} (\langle \psi|\hat{q}_n\hat{q}_m|\psi\rangle X_A(m) + \text{Re} \langle \psi|\hat{q}_n\hat{p}_m|\psi\rangle Y_A(m).
\]

(75)

These are the general partner formula, which can be used for any Gaussian state and any complete set of canonical operators. As long as the continuum limit can be taken properly, we obtain the partner formula in the scalar field theory. Especially, the partner formula for weighting functions of the field \( \hat{\phi}(x) \) and its conjugate momentum \( \hat{\Pi}(x) \) is given by

\[
X_B(x) = \frac{\sqrt{1 + g^2}}{g} X_A(x) - \frac{2}{g} \int d^3y \left( \text{Re} \left( \langle \psi|\hat{\Pi}(x)\hat{\phi}(y)|\psi\rangle \right) Z_A(y) + \langle \psi|\hat{\Pi}(x)\hat{\Pi}(y)|\psi\rangle W_A(y) \right),
\]

(76)

\[
Y_B(x) = \frac{\sqrt{1 + g^2}}{g} Y_A(x) + \frac{2}{g} \int d^3y \left( \langle \psi|\hat{\phi}(x)\hat{\phi}(y)|\psi\rangle Z_A(y) + \text{Re} \left( \langle \psi|\hat{\phi}(x)\hat{\Pi}(y)|\psi\rangle \right) W_A(y) \right),
\]

(77)

\[
Z_B(x) = -\frac{\sqrt{1 + g^2}}{g} Z_A(x) - \frac{2}{g} \int d^3y \left( \text{Re} \left( \langle \psi|\hat{\Pi}(x)\hat{\phi}(y)|\psi\rangle \right) X_A(y) + \langle \psi|\hat{\Pi}(x)\hat{\Pi}(y)|\psi\rangle Y_A(y) \right),
\]

(78)

\[
W_B(x) = -\frac{\sqrt{1 + g^2}}{g} W_A(x) + \frac{2}{g} \int d^3y \left( \langle \psi|\hat{\phi}(x)\hat{\phi}(y)|\psi\rangle X_A(y) + \text{Re} \left( \langle \psi|\hat{\phi}(x)\hat{\Pi}(y)|\psi\rangle \right) Y_A(y) \right).
\]

(79)
These are the partner formula written in terms of two-point functions.

5. Partner mode in a curved spacetime

By using the result obtained in the previous section, let us investigate the memory effect in pairs of partners of free scalar field in a curved spacetime. The metric is denoted by $g_{\mu\nu}(x)$ whose signature is given by $(-, +, +, \cdots, +)$. Here $x$ denotes a point in the spacetime and Greek indices run over $0, 1, \cdots, d$. Assuming the spacetime is globally hyperbolic, it is possible to foliate the spacetime into a family of spatial slices $\Sigma_\tau$, where $\tau$ denotes a continuous parameter which can be regarded as time. For simplicity, we assume there are two regions, ‘in’ and ‘out’ region, where the spacetime becomes flat:

$$
\text{in the ‘in’ region},
\begin{align*}
\text{out' region},
\end{align*}
$$

Here, $(t, x)$ and $(\bar{t}, \bar{x})$ are the coordinate system in the ‘in’ and ‘out’ region, respectively. It should be stressed that we have imposed no constraint on the metric in the intermediate region between two flat regions as long as the spacetime is globally hyperbolic. An action for the free scalar field $\phi$ is given by

$$
S = \int d^{d+1}x \sqrt{-g(x)} \left( -g^{\mu\nu}\partial_\mu \phi \partial_\nu \phi - \left( m(x)^2 + \xi R(x) \right) \phi^2 \right),
$$

where $m(x)$ is the mass of the scalar field which may depend on the position $x$, $R(x)$ is the Ricci scalar of the spacetime and $\xi$ characterize the coupling between the scalar field and the gravitational field. Adopting the Heisenberg picture, the equation of motion is given by the Klein–Gordon equation $(\Box + m(x)^2 + \xi R(x)) \phi(x) = 0$, where $\Box \phi = \sqrt{-g} \partial_\mu \left( \sqrt{-g} g^{\mu\nu} \partial_\nu \phi \right)$. The conjugate momentum is given as

$$
\Pi(x) = -\sqrt{-g} g^{\tau\nu} \partial_\tau \phi(x) = \begin{cases}
\partial_\tau \phi(t, x) & \text{(in the ‘in’ region)}, \\
\partial_\tau \phi(\bar{t}, \bar{x}) & \text{(in the ‘out’ region)}.
\end{cases}
$$

In the flat region, the Ricci scalar vanishes. Let us assume $m(x)$ becomes constant in the flat regions as follows:

$$
m(x) = \begin{cases}
m & \text{(in the ‘in’ region),} \\
m & \text{(in the ‘out’ region).}
\end{cases}
$$

Then, there are two sets of solutions for the equation of motion which satisfy the following conditions:

$$
\begin{align*}
\phi_k(t, x) &= \frac{1}{\sqrt{(2\pi)^d 2E_k}} e^{i(k x - E_k t)} & \text{(in the ‘in’ region),} \\
\phi_{\bar{k}}(\bar{t}, \bar{x}) &= \frac{1}{\sqrt{(2\pi)^d 2E_{\bar{k}}}} e^{i(k \bar{x} - E_{\bar{k}} \bar{t})} & \text{(in the ‘out’ region),}
\end{align*}
$$

where $E_k \equiv \sqrt{k^2 + m^2}$ and $E_{\bar{k}} \equiv \sqrt{\bar{k}^2 + \bar{m}^2}$ are energies for the field with momentum $k$ in ‘in’ region and ‘out’ region, respectively. The normalization constants are chosen to satisfy $(\phi_k, \phi_{\bar{k}}) = (\phi_{\bar{k}}, \phi_k) = \delta^{(d)}(k - \bar{k})$, where we have introduced the inner product of functions $f_1, f_2$ as

$$
\langle f_1, f_2 \rangle = \int d^{d+1}x \sqrt{-g(x)} f_1(x) f_2(x).
$$
\( (f_1, f_2) \equiv -i \int_{\Sigma} d\Sigma^{\mu} \sqrt{g_{\Sigma}(x)} \partial_{\mu} f_1(x) \partial_{\mu} f_2(x) \equiv -i \int_{\Sigma} d\Sigma^{\mu} \sqrt{g_{\Sigma}(x)} (f_1(x) \partial_{\mu} f_2(x)^* - f_2(x)^* \partial_{\mu} f_1(x)) \).  

(86)

Here, \( g_{\Sigma} \) is the determinant of the induced metric on the time slice \( \Sigma_t \), \( d\Sigma^{\mu} \equiv n^{\mu} d\Sigma \) with a unit normal vector \( n^{\mu} \) and the volume element \( d\Sigma \) of the spatial slice \( \Sigma_t \). It should be noted that for solutions \( f_1, f_2 \) of the equation of motion, it can be shown that the inner product is related with each other through

\[
\hat{\phi}^H(x) = \int d^d k \left( \hat{a}_k u_k(x) + \hat{\alpha}_k^\dagger u_k^*(x) \right) = \int d^d k \left( \hat{a}_k^\dagger u_k(x) + \hat{\alpha}_k u_k^*(x) \right),
\]

(87)

where the superscript \( H \) of \( \hat{\phi} \) is added to emphasize we adopt the Heisenberg picture. They are related with each other through

\[
\hat{a}_k = \langle \phi, u_k \rangle = \int d^d k' \left( \alpha_{k,k'} \hat{a}_{k'} + \beta_{k,k'} \hat{\alpha}_{k'}^\dagger \right),
\]

(88)

where the Bogoliubov coefficients are defined by

\[
\alpha_{k,k'} \equiv \langle \tilde{u}_{k'}, u_k \rangle, \quad \beta_{k,k'} \equiv \langle \tilde{u}_{k'}^*, u_k \rangle.
\]

(89)

The inverse transformation is given by

\[
\hat{a}_k = \int d^d k' \left( \alpha_{k,k'} \hat{a}_{k'} - \beta_{k,k'} \hat{\alpha}_{k'}^\dagger \right).
\]

(90)

Since the formula obtained in the previous section is applicable for any Gaussian state and any complete set of canonical operators, it is possible to obtain the partner even when we are working in the Heisenberg picture. As an example, at \( \tau = t \) in the ‘out’ region, let us consider a mode \( \hat{A} \) characterized by

\[
\hat{q}^H_A = \int d^d x \left( x_A(\bar{x}) \hat{\phi}^H(t, \bar{x}) + y_A(\bar{x}) \hat{\Pi}^H(t, \bar{x}) \right),
\]

(91)

\[
\hat{p}^H_A = \int d^d x \left( z_A(\bar{x}) \hat{\phi}^H(t, \bar{x}) + w_A(\bar{x}) \hat{\Pi}^H(t, \bar{x}) \right),
\]

(92)

satisfying \( [\hat{q}^H_A, \hat{p}^H_A] = i \). After an appropriate local symplectic transformation, the canonical operators reduce to the standard form

\[
\hat{Q}^H_A = \int d^d x \left( X_A(x) \hat{\phi}^H(t,x) + Y_A(x) \hat{\Pi}^H(t,x) \right),
\]

(93)

\[
\hat{P}^H_A = \int d^d x \left( Z_A(x) \hat{\phi}^H(t,x) + W_A(x) \hat{\Pi}^H(t,x) \right),
\]

(94)

which satisfy

\[
\left( \begin{array}{c} \langle \Psi | \hat{Q}^H_A | \Psi \rangle \\ \langle \Psi | \hat{P}^H_A | \Psi \rangle \\ \text{Re} \left( \langle \Psi | \hat{Q}^H_A \hat{P}^H_A | \Psi \rangle \right) \\ \text{Re} \left( \langle \Psi | \hat{P}^H_A \hat{Q}^H_A | \Psi \rangle \right) \end{array} \right) = \frac{\sqrt{1 + g^2}}{2} \left( \begin{array}{c} 1 \\ 0 \\ 1 \\ 0 \end{array} \right),
\]

(95)

where \( g \equiv \sqrt{4 \left( \langle \Psi | \hat{Q}^H_A | \Psi \rangle \langle \Psi | \hat{P}^H_A | \Psi \rangle \right)^2 \left( \langle \Psi | \hat{P}^H_A \hat{Q}^H_A | \Psi \rangle \right) - \left( \text{Re} \left( \langle \Psi | \hat{Q}^H_A \hat{P}^H_A | \Psi \rangle \right) \right) - 1 \) and \( |\Psi\rangle \) is a Gaussian state of the field, which is typically taken as the vacuum state in the ‘in’ region. The weighting functions for the partner \( \hat{B} \) are given by
\[ X_B(x) = \frac{\sqrt{1 + g^2}}{g} X_A(x) \]
\[ -\frac{2}{g} \int d^4y \left( \text{Re} \left( \langle \Psi | \hat{H}(\bar{t}, \bar{x}) \hat{\phi}(\bar{t}, \bar{y}) | \Psi \rangle \right) Z_A(y) + \langle \Psi | \hat{H}(\bar{t}, \bar{x}) \hat{H}(\bar{t}, \bar{y}) | \Psi \rangle W_A(y) \right). \]
\[ Y_B(x) = \frac{\sqrt{1 + g^2}}{g} Y_A(x) \]
\[ + \frac{2}{g} \int d^4y \left( \text{Re} \left( \langle \Psi | \hat{H}(\bar{t}, \bar{x}) \hat{\phi}(\bar{t}, \bar{y}) | \Psi \rangle \right) Z_A(y) + \langle \Psi | \hat{H}(\bar{t}, \bar{x}) \hat{H}(\bar{t}, \bar{y}) | \Psi \rangle Y_A(y) \right). \]
\[ Z_B(x) = -\frac{\sqrt{1 + g^2}}{2} Z_A(x) \]
\[ - \frac{2}{g} \int d^4y \left( \text{Re} \left( \langle \Psi | \hat{H}(\bar{t}, \bar{x}) \hat{\phi}(\bar{t}, \bar{y}) | \Psi \rangle \right) X_A(y) + \langle \Psi | \hat{H}(\bar{t}, \bar{x}) \hat{H}(\bar{t}, \bar{y}) | \Psi \rangle X_A(y) \right). \]
\[ W_B(x) = -\frac{\sqrt{1 + g^2}}{g} W_A(x) \]
\[ + \frac{2}{g} \int d^4y \left( \text{Re} \left( \langle \Psi | \hat{H}(\bar{t}, \bar{x}) \hat{\phi}(\bar{t}, \bar{y}) | \Psi \rangle X_A(y) + \langle \Psi | \hat{H}(\bar{t}, \bar{x}) \hat{H}(\bar{t}, \bar{y}) | \Psi \rangle Y_A(y) \right). \]

Now, consider a situation in which an experimenter prepares an Unruh–De Witt particle detector at \( t = t_{\text{obs}} \) in the ‘out’ region, which couples with a mode of the field to read out quantum information imprinted in the field. To perform such a protocol, one has to consider an interaction between the field and an external device. Therefore, it is useful to obtain a partner formula based on the Schrödinger picture. We want the partner of a mode \( A \) whose canonical variables are defined by

\[ \hat{\phi}_A^S = \int d^4 \bar{x} \left( \chi_A(\bar{x}) \hat{\phi}(\bar{x}) + y_A(\bar{x}) \hat{\Pi}(\bar{x}) \right), \]
\[ \hat{\Pi}_A^S = \int d^4 \bar{x} \left( z_A(\bar{x}) \hat{\phi}(\bar{x}) + w_A(\bar{x}) \hat{\Pi}(\bar{x}) \right), \]

where the superscript \( S \) of \( \hat{\phi} \) and \( \hat{\Pi} \) are added to emphasize that we adopt the Schrödinger picture. Since the pair of canonical variables (\( \hat{\phi}_A^S, \hat{\Pi}_A^S \)) must satisfy

\[ [\hat{\phi}_A^S, \hat{\Pi}_A^S] = i, \]

we have the following constraint:

\[ \int d^4 \bar{x} \left( \chi_A(\bar{x}) w_A(\bar{x}) - y_A(\bar{x}) z_A(\bar{x}) \right) = 1, \]

where we have used the canonical commutation relationship of the field and its conjugate momentum. Assuming the system is in the vacuum state \( |0 \rangle \) at \( \tau = t_0 \) in the ‘in’ region, it evolves into \( |\psi(\bar{t})\rangle = \hat{U}(\bar{t}, t_0)|0\rangle \) in the ‘out’ region, where

\[ \hat{U}(\bar{t}, t_0) \equiv \mathcal{T} \exp \left( -i \int_{t_0}^{\bar{t}} d\tau \int_{\Sigma_\tau} d^4x \hat{H} \right) \]

is the unitary evolution operator. Here, the Hamiltonian density \( \hat{H} \) is defined by
\[ \mathcal{H} \equiv \hat{\Pi} \partial_x \phi - \mathcal{L}(\phi) : \]  

The excited state \(|\psi(t)\rangle\) is a Gaussian state since the initial state \(|0\rangle\) is a Gaussian state and the Hamiltonian is bi-linear. Under the assumption that equation (104) is well defined, the Heisenberg operators and the Schrödinger operators are related through \(\hat{\Phi}^{\mathcal{H}}(t, \mathbf{x}) = U(t, t_0)^{-1} \hat{\Phi}^{\mathcal{S}}(x) U(t, t_0)\) and \(\hat{\Pi}^{\mathcal{H}}(t, \mathbf{x}) = U(t, t_0)^{-1} \hat{\Pi}^{\mathcal{S}}(x) U(t, t_0)\). Therefore, the partner mode \(B\) is characterized by

\[
\hat{Q}_B = \int d^d\mathbf{x} \left( X_B(\mathbf{x}) \hat{\phi}^5(\mathbf{x}) + Y_B(\mathbf{x}) \hat{\phi}^6(\mathbf{x}) \right),
\]

\[
\hat{P}_B^S = \int d^d\mathbf{x} \left( Z_B(\mathbf{x}) \hat{\phi}^5(\mathbf{x}) + W_B(\mathbf{x}) \hat{\phi}^6(\mathbf{x}) \right),
\]

where the weighting functions are defined in equations (96)–(99) with \(|\Psi\rangle = |0\rangle\). In terms of the Bogoliubov coefficients, the second moments of the field and its conjugate momentum are calculated as

\[
\langle 0 | \hat{\phi}^{\mathcal{H}}(t, \mathbf{x}) \hat{\phi}^{\mathcal{H}}(t, \mathbf{y}) |0\rangle
\]

\[
= \int d^d k d^d k' d^d p \left( \alpha_{kp}^* \alpha_{k'p}^* - \bar{\alpha}_{kp}^* \beta_{k'p}^* \right) \left( \bar{\alpha}_{k'p} (t, \mathbf{y}) \alpha_{kp}(t, \mathbf{y}) - \bar{\beta}_{k'p} (t, \mathbf{y}) \beta_{kp}(t, \mathbf{y}) \right).
\]

\[
\text{Re} \left( \langle 0 | \hat{\phi}^{\mathcal{H}}(t, \mathbf{x}) \hat{\Pi}^{\mathcal{H}}(t, \mathbf{y}) |0\rangle \right)
\]

\[
= \int d^d k d^d k' d^d p \text{Re} \left( \left( \alpha_{kp}^* \alpha_{k'p}^* - \bar{\alpha}_{kp}^* \beta_{k'p}^* \right) (t \mathbf{E}_\mathbf{k}^* \mathbf{E}_{-\mathbf{k}'}^*) \left( \bar{\alpha}_{k'p} (t, \mathbf{y}) \alpha_{kp}(t, \mathbf{y}) + \bar{\beta}_{k'p} (t, \mathbf{y}) \beta_{kp}(t, \mathbf{y}) \right) \right)
\]

\[
= \int d^d k d^d k' d^d p \text{Im} \left( \left( \alpha_{kp}^* \alpha_{k'p}^* - \bar{\alpha}_{kp}^* \beta_{k'p}^* \right) \mathbf{E}_k \left( \bar{\alpha}_{k'p} (t, \mathbf{y}) \alpha_{kp}(t, \mathbf{y}) + \bar{\beta}_{k'p} (t, \mathbf{y}) \beta_{kp}(t, \mathbf{y}) \right) \right)
\]

\[
\langle 0 | \hat{\Pi}^{\mathcal{H}}(t, \mathbf{x}) \hat{\Pi}^{\mathcal{H}}(t, \mathbf{y}) |0\rangle
\]

\[
= \int d^d k d^d k' d^d p \mathbf{E}_k \mathbf{E}_{-\mathbf{k}'} \left( \alpha_{kp}^* \alpha_{k'p}^* + \bar{\alpha}_{kp}^* \beta_{k'p}^* \right) \left( \bar{\alpha}_{k'p} (t, \mathbf{y}) \alpha_{kp}(t, \mathbf{y}) + \bar{\beta}_{k'p} (t, \mathbf{y}) \beta_{kp}(t, \mathbf{y}) \right).
\]

It should be noted that the second moments of the mode \(A\), which are needed to obtain \(g\), are calculated from the above moments. For example,

\[
\langle 0 | (\hat{\phi}_A^5)^2 |0\rangle
\]

\[
= \int d^d x d^d y \left( x_A(\mathbf{x}) \langle 0 | \hat{\phi}^{\mathcal{H}}(t, \mathbf{x}) |0\rangle x_A(\mathbf{y}) \right)
\]

\[
+ 2 x_A(\mathbf{x}) \text{Re} \left( \langle 0 | \hat{\phi}^{\mathcal{H}}(t, \mathbf{x}) \hat{\Pi}^{\mathcal{H}}(t, \mathbf{y}) |0\rangle y_A(\mathbf{y}) \right) + y_A(\mathbf{x}) \langle 0 | \hat{\Pi}^{\mathcal{H}}(t, \mathbf{x}) \hat{\Pi}^{\mathcal{H}}(t, \mathbf{y}) |0\rangle y_A(\mathbf{y}) \right).
\]

Even after the spacetime becomes static in the ‘out’ region, the partner has non-trivial dynamics in general. That is, if the experiment is performed at \(t = t_{\text{obs}} \neq t_{\text{obs}}\), the weighting functions of the partner \(B\) will be different from those at \(t = t_{\text{obs}}\). This is natural because the partner is capable of evolving in time due to the free evolution of the field. Our formula enables us to identify the unique partner mode \(B\), once \(t_{\text{obs}}\) is specified in the ‘out’ region.

The partner formula on equations (96)–(99) are simplified when the Bogoliubov coefficients satisfy the following conditions:

\[
\alpha_{kk'} = |r|^{-d/2} \delta_{k,k'} \delta(d) \left( k' - r^{-1} k \right), \quad \beta_{kk'} = |r|^{-d/2} \delta_{k,k'} \delta(d) \left( k' + r^{-1} k \right), \quad \alpha_{-k} = \alpha_k, \quad \beta_{-k} = \beta_k
\]
for a nonzero real number \( r \) with spatial homogeneity. The normalization condition

\[
\int d^d k' (\alpha_{kk'} \alpha_{kk'}^\ast - \beta_{kk'} \beta_{kk'}^\ast) = \delta^{(d)} (k - k')
\]  

(112)

is equivalent to

\[
|\alpha_k|^2 - |\beta_k|^2 = 1.
\]  

(113)

For this case, if an experiment is performed in the late time, i.e. \( t_{\text{obs}} \to \infty \), the entanglement entropy and the weighting functions of the partner \( B \) become independent of \( t_{\text{obs}} \). In addition, it can be shown that partner \( B \) only stores information related to particle creation effects \( |\beta_k|^2 \).

This fact implies that the entanglement partners contain long-lasting memory of the dynamics of evolution. For proof, let us show that the state itself becomes independent of \( t_{\text{obs}} \) and depend only on \( |\beta_k|^2 \) in the limit of \( t_{\text{obs}} \to \infty \). This properties can be checked form the following calculations on the elements of the covariant matrix in the limit of \( t \to \infty \):

\[
\langle 0 | \phi^H(i, \vec{x}) \bar{\phi}^H(i, \vec{y}) | 0 \rangle = \int d^d k d^d k' d^d p' \left( \bar{u}_k(i, \vec{x}) \alpha_{kp} - \bar{u}_k(i, \vec{x}) \beta_{kp} \right) \left( \bar{u}_{k'}(i, \vec{y}) \alpha_{kp'} - \bar{u}_{k'}(i, \vec{y}) \beta_{kp'} \right) \delta^{(d)}(p - p')
\]

\[
= \int d^d k \left( \bar{u}_k(i, \vec{x}) \alpha_{kk} - \bar{u}_{-k}(i, \vec{x}) \beta_{kk} \right) \left( \bar{u}_k(i, \vec{y}) \alpha_{kk} - \bar{u}_{-k}(i, \vec{y}) \beta_{kk} \right)
\]

\[
\to \int \frac{d^d k}{(2\pi)^d} \frac{1}{2E_k} \left( 1 + 2 |\beta_k|^2 \right) e^{ik(x - y)},
\]

(114)

\[
\text{Re} \left( \langle 0 | \phi^H(i, \vec{x}) \bar{\phi}^H(i, \vec{y}) | 0 \rangle \right) \to 0,
\]

(115)

\[
\langle 0 | \bar{\phi}^H(i, \vec{x}) \bar{\phi}^H(i, \vec{y}) | 0 \rangle \to \int \frac{d^d k}{(2\pi)^d} \frac{\bar{E}_k}{2} (1 + 2 |\beta_k|^2) e^{ik(x - y)},
\]

(116)

where we have used the Riemann–Lebesgue lemma, which claims that the Fourier coefficient will vanish for high frequency modes. More precisely, for an \( L^1 \) function \( f \), it holds that

\[
\tilde{f}(z) \equiv \int \frac{dx}{2\pi} f(x)e^{izx} \to 0 \quad (|z| \to \infty).
\]

(117)

A rough proof for the one-dimensional case is given by the integration by parts as follows:

\[
|\tilde{f}(z)| \leq - \int dx \frac{1}{iz} \frac{d}{dx} f(x) e^{izx} \leq \frac{1}{iz} \int dx \left| \frac{d}{dx} f(x) \right| \to 0 \quad (|z| \to \infty).
\]

(118)

As an example of our partner formula, let us consider a \((1 + 1)\)-dimensional expanding universe model [19, 20] whose metric is defined by

\[
dx^2 = (a + b \sinh(\rho \eta)) \left(-d\eta^2 + d\xi^2\right),
\]

where \( \eta \) is the conformal time. Since the mass is independent of the position in the spacetime in this model, \( m = \bar{m} \). There are two asymptotic regions \( \eta \to -\infty \) and \( \eta \to \infty \) where the spacetime becomes flat. We assume that the field is in the vacuum state in the ‘in’ region. Assuming the periodic boundary condition: \( \hat{\phi}(\eta, \xi + L) = \hat{\phi}(\eta, \xi) \), the unitary evolution matrix in equation (104) exists. In the limit of \( L \to \infty \), the dispersion relations are given by \( E_k = \bar{E}_k = \sqrt{k^2 + m^2} \). By using the result in [19, 20], the Bogoliubov coefficients are obtained as
where $r \equiv \sqrt{\frac{a+b}{a+b}}, \alpha_k \equiv \tilde{\alpha}_k \sqrt{\omega_k}$ and $\beta_k \equiv -\tilde{\beta}_k \sqrt{\omega_k}$ with

$$
\tilde{\alpha}_k \equiv \frac{\omega_k}{\omega_k} \frac{\Gamma(1 - i\omega_k/\rho) \Gamma(-i\omega_k/\rho)}{\omega_k \Gamma(1 - i\omega_k/\rho)}
$$

and

$$
\tilde{\beta}_k \equiv \frac{\omega_k}{\omega_k} \frac{\Gamma(1 + i\omega_k/\rho) \Gamma(i\omega_k/\rho)}{\omega_k \Gamma(1 + i\omega_k/\rho)}
$$

Here, we have defined $\omega_k \equiv \sqrt{k^2 + (a - b)m^2}$, $\bar{\omega}_k \equiv \sqrt{k^2 + (a + b)m^2}$ and $\omega_\pm \equiv \frac{1}{2} (\omega_k \pm \bar{\omega}_k)$. Equations (119) and (120) are good approximations for finite and large $L$, where the evolution is unitary. In this example, $r$ is related to the ratio between final and initial conformal factors with respect to the conformal time $\eta$. In the limit of $t_{obs} \rightarrow \infty$, the only contribution to the partner’s weighting functions comes from the particle creation rate $|\beta_k|^2$.

The parameter $g$ fixes the entanglement entropy as $S_{EE} = \sqrt{1 + g^2} \ln \left( \frac{1}{2} \left( \sqrt{1 + g^2} + 1 \right) \right) + \ln \left( \frac{1}{2} \right)$, where $g$ is determined by

$$
g^2 = 4 \left( \langle 0 | \hat{Q}_B^H \rangle^2 \langle 0 | \hat{P}_B^H \rangle^2 \langle 0 | - \Re \left( \langle 0 | \tilde{\alpha}_A^\dagger H^A \langle 0 \rangle \right) \right) - 1.
$$

The two-point functions of partner mode $B$ satisfy

$$
g^2 = 4 \langle 0 | \hat{Q}_B^H \rangle^2 \langle 0 | \hat{P}_B^H \rangle^2 | 0 \rangle - 1.
$$

Thus, the elements of the covariance matrix as such

$$
\langle 0 | \left( \hat{Q}_B^H \right)^2 | 0 \rangle = \int d^2 x d^2 y \left( \chi_0 (x) (0) | \tilde{\beta}_B (i, x) \tilde{\beta}_B^\dagger (i, y) | 0 \rangle \chi_0 (y) \right.
$$

$$
+ 2 \chi_0 (x) \Re \left( \langle 0 | \tilde{\beta}_B (i, x) \tilde{\Pi}_B^H (i, y) | 0 \rangle \chi_0 (y) + \chi_0 (x) \langle 0 | \tilde{\Pi}_B^H (i, x) \tilde{\Pi}_B^H (i, y) | 0 \rangle \chi_0 (y) \right)
$$

are integrable.

What follows are the results for when the original mode $A$ has some Gaussian weighting functions. We fixed the mass of the scalar field $m = 1$. The metric parameters $a$ and $b$ that determine the initial and final size of the universe were fixed to $a - b = 0.5$ and $a + b = 2.5$.

In addition, we consider the case in which $\chi_A (x) = \bar{\chi}_A (x) = 0$, that is no cross terms in $m_A$ appear. In figure 3, we show the mode $A$ weighting functions $X_A (x)$ and $W_A (x)$ after the symplectic transformation. In figure 4, we show the results for partner $B$ weighting functions $X_B (x)$ and $W_B (x)$ for the case in which there is no expansion $\rho = 0$. In figure 5, we show the results for partner $B$ weighting functions $X_B (x)$ and $W_B (x)$ for the case in which the expansion rate $\rho = 10$. Comparing with figure 4, a change of not only the amplitude of the weighting functions, but also in the width of the functions can be appreciated. As expected, the partner form is affected by the expansion rate. Finally, the entanglement entropy between mode $A$ and partner $B$ is shown in figure 6 as a function of the expansion rate $\rho$. It can be seen that the amount of entanglement between the modes tends to saturate for higher values of the universe expansion rate $\rho$. The reason is simple. For a large $\rho$, the scale factor is approximated by a step functional one as
where $\Theta(\eta)$ denotes the step function. The metric itself maintains an exponentially small amount of the information about $\rho$. Hence the entanglement of $A$ and $B$ cannot have high sensitivity of $\rho$ in this regime. Nevertheless, the entanglement between $A$ and $B$ stores the information of $\rho$.

Figure 3. Original mode $A$ with Gaussian weighting functions. The weighting functions $X_A(x)$ and $W_A(x)$ are obtained from the symplectic transformation of $x_A(x) = e^{-x^2}$ and $w_A(x) = \sqrt{\frac{1}{2\pi e^{3/2}}} e^{-(x-1)^2/2}$, where these functions satisfy the constraint coming from the canonical commutation relationship. For simplicity $y_A(x) = 0$ and $z_A(x) = 0$.

Figure 4. Partner mode $B$ associated to the Gaussian mode $A$ in figure 3 when there is no expansion of the universe ($\rho = 0$). The mass of the scalar field was taken to be $m = 1$.

Figure 5. Partner mode $B$ associated to the Gaussian mode $A$ in figure 3 when the expansion rate $\rho = 10$. In this model, the universe starts from a size of $(a - b = 0.5)$ in the remote past and ends with a size $(a + b = 2.5)$ in the remote future. The mass of the scalar field was taken to be $m = 1$.

$$\sqrt{a + b \tanh(\rho\eta)} = \sqrt{a - b + 2b\Theta(\eta)} + O(\exp(-\rho|\eta|)),$$

where $\Theta(\eta)$ denotes the step function. The metric itself maintains an exponentially small amount of the information about $\rho$. Hence the entanglement of $A$ and $B$ cannot have high sensitivity of $\rho$ in this regime. Nevertheless, the entanglement between $A$ and $B$ stores the information of $\rho$. 

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6. Summary and discussion

In this paper, we proposed a correlation function definition of purification partner in section 2 for a given particle mode \( A \) in an arbitrary state. This may be useful for verification experiments of the partner mode. We have also shown the existence of the partner for arbitrary mode \( A \) of a lattice field in a general state. For a general Gaussian state, the condition which identifies the partner is simplified. The entanglement entropy between the mode \( A \) and its partner \( B \) is evaluated by using equations (11) and (14). We showed the formula in equations (76)–(79) to obtain the partner in an arbitrary Gaussian state of scalar field theory. In addition, we provided a new class of partner: spatially overlapped partner (SOP). As is shown explicitly in an expanding universe model, the weighting functions of the partner contains information on the Bogoliubov coefficients, i.e. the partners play a role of a storage of dynamics information.

As a future work, it is interesting to investigate the advantage of the identification for SOPs, especially in the context of the black hole information loss and the cosmological Bell inequality breaking in cosmic microwave background. As is presented in [9], the purification partners help to enhance the efficiency of entanglement harvesting.

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Appendix A. Proof of hermitianity, non-negativity, and normalization of \( \hat{\rho}_A \)

Let us confirm that \( \hat{\rho}_A \) is a quantum state, that is, a unit trace positive-semidefinite Hermitian operator. Since \( \chi (x_A, v_A)^* = \chi (-x_A, -v_A) \) holds, \( \langle \bar{x}_A | \hat{\rho}_A | x_A \rangle^* \) is computed as

![Figure 6. Entanglement entropy \( S_{EE} \) associated to the Gaussian mode \( A \) in figure 3 as a function of the universe expansion rate \( \rho \). The same values as in figure 5 are adopted for \( a, b \) and \( m \).](image)
\begin{align}
\langle x_A | \hat{\rho}_A | x_A \rangle^* & = \frac{1}{(2\pi)^2} \int \! d\nu_A \, d\chi \, \left( -\nu_A - \bar{\nu}_A \right) e^{\frac{i}{\hbar} q_A (\bar{x}_A + x_A)} \\
& = \frac{1}{(2\pi)^2} \int \! d\nu_A \, d\chi \, \left( -\nu_A - \bar{\nu}_A \right) e^{-\frac{i}{\hbar} \nu_A (x_A + \bar{x}_A)} \\
& = \langle x_A | \hat{\rho}_A | x_A \rangle. \quad (A.1)
\end{align}

Here, we have changed the sign of integration variables \( \nu_A \). Thus, \( \hat{\rho}_A \) is a Hermitian operator. By using \( \langle \Psi | \Psi \rangle = 1 \), the normalization condition of \( \hat{\rho}_A \) is directly checked as follows:

\begin{align}
\text{Tr}[\hat{\rho}_A] = & \frac{1}{(2\pi)^2} \int \! dx_A \, d\nu_A \, \chi (0, \nu_A) e^{-\nu_A x_A} \\
& = \frac{1}{(2\pi)^2} \int \! d\nu_A \, \text{Tr}(i \hbar \Phi) \int \! dx_A \, e^{-\nu_A x_A} \\
& = \int \! d\nu_A \langle \Psi | e^{\nu_A \hat{\phi}_A} e^{\nu_A \hat{\phi}_B} | \Psi \rangle \hat{\delta}(\bar{\nu}_A) \hat{\delta}(\nu_B) = \langle \Psi | \Psi \rangle = 1. \quad (A.2)
\end{align}

The operator \( \hat{\rho}_A \) is positive-semidefinite if and only if

\[
\int \! d\bar{x}_A \, dx_A \Phi(\bar{x}_A) \hat{\rho}_A \Phi(x_A) \geq 0
\]

holds for any complex function \( \Phi(x_A) \). Let us confirm this inequality. Substituting the definition of \( \langle \bar{x}_A | \hat{\rho}_A | x_A \rangle \) into the above equation, we get

\begin{align}
\int \! d\bar{x}_A \, dx_A \Phi(\bar{x}_A) \hat{\rho}_A \Phi(x_A) & = \frac{1}{(2\pi)^2} \int \! d\bar{x}_A \, dx_A \, d\nu_A \Phi(\bar{x}_A) \hat{\rho}_A \Phi(x_A) \\
& \times \chi (x_A - \bar{x}_A, \nu_A) e^{-\frac{i}{\hbar} \nu_A (x_A + \bar{x}_A)} \\
& = \int \! d\bar{x}_A \, dx_A \Phi(\bar{x}_A) \hat{\rho}_A \Phi(x_A) \\
& \times \langle \Psi | \int \! \frac{d\nu_A}{2\pi} e^{i (\nu_A \bar{x}_A - \bar{x}_A)} e^{-i ((x_A - \bar{x}_A) \bar{\nu}_A)} | \Psi \rangle,
\end{align}

where we have used the Baker–Campbell–Hausdorff formula. By using

\[
\int \! \frac{d\nu}{2\pi} e^{i (\nu (q - x))} = \delta (\bar{q} - x),
\]

and the spectrum decomposition of \( \hat{q}_A \):

\[
\hat{q}_A = \sum_\alpha \int \! dx_A x_A' \langle x_A' | x_A | \alpha \rangle \langle x_A' | \alpha \rangle,
\]

we get

\begin{align}
\int \! d\bar{x}_A \, dx_A \Phi(\bar{x}_A) \hat{\rho}_A \Phi(x_A) & = \sum_{\alpha, \beta} \int \! d\bar{x}_A \, dx_A \Phi(\bar{x}_A) \hat{\rho}_A \Phi(x_A) \\
& \times \langle x_A, \alpha | x_B, \beta \rangle e^{-i ((x_A - x_B) \bar{\nu}_A)} e^{-i ((x_A - x_B) \bar{\nu}_B)} \hat{\rho}_A | x_A, \alpha \rangle \langle x_B, \beta \rangle. \quad (A.4)
\end{align}
Since $\langle x_A, \alpha | e^{-i((x_A - \alpha)b_A)} | x_A, \alpha \rangle$ holds, the positive-semidefiniteness is finally proven as follows:

$$
\int dx_A d\xi_A \Phi(\xi_A)^* \langle \xi_A | \hat{\rho}_A | x_A \rangle \Phi(x_A)
= \sum_{\alpha, \beta} \int dx_A d\xi_A \Phi(\xi_A)^* \langle \xi_A, \alpha | \hat{\rho}_A | x_A, \alpha \rangle | \xi_A, \alpha \rangle | x_A, \beta \rangle \Phi(x_A) \geq 0.
$$

Therefore, $\hat{\rho}_A$ is a quantum state.

**Appendix B. Covariance matrix and its standard form**

Let us consider a system composed of $N(\geq 2)$ harmonic oscillators whose canonical variables are given by $(q_n, p_n)$ for $n = 1, \cdots, N$. By using

$$
\tilde{r} \equiv (\tilde{q}_1, \tilde{p}_1, \tilde{q}_2, \tilde{p}_2, \cdots, \tilde{q}_N, \tilde{p}_N),
$$

the commutation relationships are expressed as

$$
[\tilde{r}_\alpha, \tilde{r}_\beta] = i\Omega_{\alpha\beta},
$$

where $\Omega$ is defined by

$$
\Omega \equiv \bigoplus_{n=1}^{N} \begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix}.
$$

A Gaussian state $\hat{\rho}$ is fully characterized by the first and second moments of canonical variables. By locally shifting the canonical variable, it is always possible to make the first moments zero. Then, the state $\hat{\rho}$ is characterized by its $2N \times 2N$ covariance matrix:

$$
M \equiv \text{Re} \left( (\tilde{r}\tilde{r}^T) \right),
$$

where $\langle \hat{O} \rangle \equiv \text{Tr} \left( \hat{\rho} \hat{O} \right)$ denotes the expectation value for a linear operator $\hat{O}$. It should be noted that the reduced state for $n(< N)$ harmonic oscillators degree of freedom is also Gaussian when the total system is in a Gaussian state. Thus, for example, the reduced state for a subsystem composed of the first and the second harmonic oscillators is fully characterized by its covariance matrix defined by

$$
m_{12} \equiv \begin{pmatrix}
\langle \tilde{q}_1^2 \rangle & \text{Re} \left( \langle \tilde{q}_1 \tilde{p}_1 \rangle \right) & \langle \tilde{q}_1 \tilde{q}_2 \rangle & \langle \tilde{q}_1 \tilde{p}_2 \rangle \\
\text{Re} \left( \langle \tilde{p}_1 \tilde{q}_2 \rangle \right) & \langle \tilde{p}_1^2 \rangle & \langle \tilde{p}_1 \tilde{q}_2 \rangle & \langle \tilde{p}_1 \tilde{p}_2 \rangle \\
\langle \tilde{q}_2 \tilde{q}_1 \rangle & \text{Re} \left( \langle \tilde{q}_2 \tilde{p}_1 \rangle \right) & \langle \tilde{q}_2^2 \rangle & \langle \tilde{q}_2 \tilde{p}_2 \rangle \\
\langle \tilde{p}_2 \tilde{q}_1 \rangle & \text{Re} \left( \langle \tilde{p}_2 \tilde{p}_1 \rangle \right) & \langle \tilde{p}_2 \tilde{q}_2 \rangle & \langle \tilde{p}_2^2 \rangle
eq \Omega\text{.}
\end{pmatrix}
$$

In other words, by using the covariance matrix $m_{12}$, we can calculate the expectation value of any local operator composed of a product of $(\tilde{q}_1, \tilde{p}_1, \tilde{q}_2, \tilde{p}_2)$.

A linear transformation $S$ on the canonical variables $\tilde{r}$ is called symplectic if and only if $\tilde{R} \equiv S\tilde{r}$ satisfies the canonical commutation relationships. This condition is equivalent to $S\Omega S^T = \Omega$. The Gaussian state $\hat{\rho}$ is also characterized by the covariance matrix $M'$ for the new variable $\tilde{R}$, which is related with the original one via $M' = SMS^T$. 

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Consider a local symplectic transformation $S$ in the form of
\[
S = \begin{pmatrix}
S_1 & 0 & 0 \\
0 & S_2 & 0 \\
0 & 0 & I_{2(N-2)}
\end{pmatrix},
\]
where $S_1$ and $S_2$ are $2 \times 2$ symplectic matrices, and $I_{2(N-2)}$ is the $2(N-2) \times 2(N-2)$ identity matrix. By using this local transformation, it is known that the covariance matrix $m_{12}$ transforms into the following standard form \[12, 13\]:
\[
M_{12} \equiv \begin{pmatrix}
\langle \hat{Q}_1^2 \rangle & \text{Re} \left( \langle \hat{Q}_1 \hat{P}_1 \rangle \right) & \langle \hat{Q}_1 \hat{Q}_2 \rangle & \langle \hat{Q}_1 \hat{P}_2 \rangle \\
\text{Re} \left( \langle \hat{P}_1 \hat{Q}_1 \rangle \right) & \langle \hat{P}_1^2 \rangle & \langle \hat{P}_1 \hat{Q}_2 \rangle & \langle \hat{P}_1 \hat{P}_2 \rangle \\
\langle \hat{Q}_2 \hat{Q}_1 \rangle & \langle \hat{Q}_2 \hat{P}_1 \rangle & \langle \hat{Q}_2^2 \rangle & \text{Re} \left( \langle \hat{Q}_2 \hat{P}_2 \rangle \right) \\
\langle \hat{P}_2 \hat{Q}_1 \rangle & \langle \hat{P}_2 \hat{P}_1 \rangle & \text{Re} \left( \langle \hat{P}_2 \hat{Q}_2 \rangle \right) & \langle \hat{P}_2^2 \rangle 
\end{pmatrix}.
\]
(\ref{B.7})

where
\[
\begin{pmatrix}
\hat{Q}_i \\
\hat{P}_i
\end{pmatrix} \equiv S_i \begin{pmatrix}
\hat{q}_i \\
\hat{p}_i
\end{pmatrix}
\]
(\ref{B.8})
for $i = 1, 2$ and $a \geq 0$, $b \geq 0$ and $c_{\pm} \in \mathbb{R}$. The reduced state for the subsystem composed of the first and second oscillator is pure if and only if
\[
a = b, \quad c_+ = -c_-, \quad c_+ c_- = \frac{1}{4} - a^2
\]
(\ref{B.9})
hold \[12, 13\]. Therefore, the second harmonic oscillator purifies the first one if
\[
M_{12} = \begin{pmatrix}
\frac{1}{2} \sqrt{1 + g^2} & 0 & \frac{g}{2} & 0 \\
0 & \frac{1}{2} \sqrt{1 + g^2} & 0 & -\frac{g}{2} \\
g & 0 & \frac{1}{2} \sqrt{1 + g^2} & 0 \\
0 & -\frac{g}{2} & 0 & \frac{1}{2} \sqrt{1 + g^2}
\end{pmatrix}
\]
(\ref{B.10})
holds, where $g$ is a positive number. This condition plays a crucial role to obtain the partner formula. The factor $g$ is directly related with the entanglement entropy $S_{EE}$ between the first and second harmonic oscillator as follows \[16\]:
\[
S_{EE} = \sqrt{1 + g^2} \ln \left( \frac{1}{g} \left( \sqrt{1 + g^2} + 1 \right) \right) + \ln \left( \frac{g}{2} \right).
\]
(\ref{B.11})

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