Ill-posedness for the two component Degasperis-Procesi equation in critical Besov space

Jinlu Li¹, Min Li²,*and Weipeng Zhu³

¹ School of Mathematics and Computer Sciences, Gannan Normal University, Ganzhou 341000, China
² Department of mathematics, Jiangxi University Of Finance and Economics, Nanchang 330032, China
³ School of Mathematics and Big Data, Foshan University, Foshan, Guangdong 528000, China

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Abstract: In this paper, we study the Cauchy problem for the two component Degasperis-Procesi equation in critical Besov space $B^{1}_{\infty,1}(\mathbb{R})$. By presenting a new construction of initial data, we proved the norm inflation of the corresponding solutions in $B^{1}_{\infty,1}(\mathbb{R})$ and hence ill-posedness. This is quite different from the local well-posedness result [27] for the Degasperis–Procesi equation in critical Besov space $B^{1}_{\infty,1}(\mathbb{R})$ due to the coupled structure of density function.

Keywords: two component Degasperis-Procesi equation; Ill-posedness; Critical Besov space.

MSC (2010): 35Q53, 37K10.

1 Introduction

In this paper, we consider the Cauchy problem for the following two component Degasperis-Procesi equation [2, 24]

\[
\begin{aligned}
\partial_t m + 3mu_x + m_x u + k_3 \rho_x &= 0, & (t, x) &\in \mathbb{R}^+ \times \mathbb{R}, \\
\partial_t \rho + k_2 u \rho_x + (k_1 + k_2) u_x \rho &= 0, & (t, x) &\in \mathbb{R}^+ \times \mathbb{R}, \\
m &= u - u_{xx}, & (t, x) &\in \mathbb{R}^+ \times \mathbb{R}, \\
u(0, x) &= u_0(x), & \rho(0, x) &= \rho_0(x), & x &\in \mathbb{R},
\end{aligned}
\]

(1.1)

here $(k_1, k_2, k_3) = (1, 1, c)$ or $(c, 1, 0)$ and $c$ is an arbitrary constant. System (1.1) first appeared in [2] as the Hamiltonian extension of the Degasperis–Procesi equation. It is worth mentioning that, in the case $(k_1, k_2, k_3) = (c, 1, 0)$, system (1.1) is no more coupled and the equation on $\rho$ becomes

*E-mail: lijinlu@gnnu.edu.cn; limin@jxufe.edu.cn(Corresponding author); mathzwp2010@163.com
linear. Therefore, we only consider the first case \((k_1, k_2, k_3) = (1, 1, c)\). For simplicity, we might as well assume that \(c = 1\), which makes system (1.1) equivalent to the following quasi-linear evolution equation of hyperbolic type:

\[
\begin{cases}
    \partial_t u + uu_x = -\partial_x(1 - \partial_x^2)^{-1}\left(\frac{3}{2}u^2 + \frac{1}{2}\rho^2\right), & (t, x) \in \mathbb{R}^+ \times \mathbb{R}, \\
    \partial_t \rho + u\rho_x = -2u_x\rho, & (t, x) \in \mathbb{R}^+ \times \mathbb{R}, \\
    u(0, x) = u_0(x), \quad \rho(0, x) = \rho_0(x), & x \in \mathbb{R}.
\end{cases}
\]

In particular, for \(\rho_0 \equiv 0\), system (1.2) reduces to the Degasperis–Procesi (DP) equation\[22\]. The DP equation can be regarded as a model for nonlinear shallow water dynamics and its asymptotic accuracy is the same as for the well known Camassa-Holm equation\[3\]. The DP equation was also proved formally integrable by constructing a Lax pair \[13\] and the direct and inverse scattering approach to pursue it can be seen in \[23\]. Moreover, they also presented \[13\] that the DP equation has a bi-Hamiltonian structure and an infinite number of conservation laws, and admits travelling wave and exact peakon solutions which are analogous to the Camassa–Holm peakons\[4, 6\]. The Cauchy problem of the DP equation is locally well-posed in certain Sobolev and Besov spaces \[5, 7, 8\], also the non-uniform dependence and ill-posedness problem have recently been studied in \[31, 28\]. It has global strong solutions \[16, 8\], the finite-time blow-up solutions \[10, 11\] and global weak solutions \[12, 10, 9\]. Different form the Camassa-Holm equation, the DP equation has not only peakon solutions \[13\], periodic peakon solutions \[9\], but also shock peakons\[15\] and the periodic shock waves \[11\]. Both Camassa-Holm and DP equation admit cusped travelling waves \[17, 4\]. This type of weak solutions are known to arise as solutions for the governing equations for waves in a channel or along a sloped beach \[18, 19\], and also as solutions for the governing equations for equatorial ocean waves \[20, 21\].

Recently, a lot of literature was devoted to studying the well-posedness (especially the ill-posedness) problem of the Camassa-Holm type equations (CH, DP and Novikov equation etc.) in the critical Besov spaces. For example, Guo et al. \[14\] prove norm inflation and hence ill-posedness for the Camassa-Holm type equations in the critical Sobolev space \(H^\frac{3}{2}\) and even in the Besov space \(B^{1+\frac{1}{p}}_{p,r}(\mathbb{R})\) with \(p \in [1, \infty], r \in (1, \infty]\), which implies \(B^{1+\frac{1}{p}}_{p,1}(\mathbb{R})\) is the critical Besov space for these Camassa-Holm type equations. Li et al. \[29, 30\] demonstrated the non-continuity and then sharp ill-posedness of the Camassa-Holm type equations in \(B^{\sigma}_{p,\infty}(\mathbb{R})\) with \(\sigma > \max\{\frac{3}{2}, 1 + \frac{1}{p}\}\). Later, it has been proved the local well-posedness for the Cauchy problem of the Camassa-Holm type equations in \(B^{1+\frac{1}{p}}_{p,1}(\mathbb{R})\) with \(p \in [1, \infty]\) by adopt the compactness argument and Lagrangian coordinate transformation\[25\]. In the remaining case \(p = \infty\), Guo et al. \[26\] prove the ill-posedness for the Camassa-Holm equation in \(B^{1}_{\infty,1}\).

Different from the CH equation, the authors in \[27\] have proved the well-posedness for the Cauchy problem of the DP equation in critical Besov space \(B^{1}_{\infty,1}\). For the two component DP equation\[1.2\], the local well-posedness problem in certain Sobolev and Besov spaces has been studied by Yan et al.in \[24\], here we mainly focus on the well-posedness problem in critical Besov space \(B^{1}_{\infty,1}(\mathbb{R})\). As the structure is changed from the DP equation due to the coupled density...
function $\rho$, here we have a different result for the two component DP equation (1.2) in this paper. Now, let us state our main theorem of this paper.

**Theorem 1.1.** For any $n \in \mathbb{Z}^+$ large enough there exists $(u_0, \rho_0)$ with

$$\|u_0\|_{B_{\infty,1}^1} + \|\rho_0\|_{B_{\infty,1}^0} \leq \frac{1}{\log \log n},$$

such that if we denote by $(u, \rho) \in C([0,1];H^3 \times H^2)$, the solution of the two component Degasperis-Procesi with initial data $(u_0, \rho_0)$, then

$$\|u(t_0)\|_{B_{\infty,1}^1} + \|\rho(t_0)\|_{B_{\infty,1}^0} \geq \log \log n,$$

with $t_0 \in (0, \frac{1}{\log n}]$.

**Remark 1.1.** This implies the ill-posedness of the two component DP equation (1.2) in Besov space $B_{\infty,1}^1(\mathbb{R})$. In fact, by Theorem 1.1 we have construct the norm inflation in space $B_{\infty,1}^1(\mathbb{R})$, thus the data-to-solution map is not continuous at origin in this space, in the sense of Hadamard this implies the ill-posedness.

## 2 Littlewood-Paley analysis

Next, we will recall some facts about the Littlewood-Paley decomposition, the nonhomogeneous Besov spaces and their some useful properties (see [1] for more details).

Let $\mathcal{B} := \{\xi \in \mathbb{R} : |\xi| \leq 4/3\}$ and $\mathcal{C} := \{\xi \in \mathbb{R} : 3/4 \leq |\xi| \leq 8/3\}$. Choose a radial, non-negative, smooth function $\chi : \mathbb{R} \mapsto [0,1]$ such that it is supported in $\mathcal{B}$ and $\chi \equiv 1$ for $|\xi| \leq 3/4$. Setting $\varphi(\xi) := \chi(\xi/2) - \chi(\xi)$, then we deduce that $\varphi$ is supported in $\mathcal{C}$. Moreover,

$$\chi(\xi) + \sum_{j \geq 0} \varphi(2^{-j}\xi) = 1 \quad \text{for any } \xi \in \mathbb{R}.$$ 

We should emphasize that the fact $\varphi(\xi) \equiv 1$ for $4/3 \leq |\xi| \leq 3/2$ will be used in the sequel.

For every $u \in \mathcal{S}'(\mathbb{R})$, the inhomogeneous dyadic blocks $\Delta_j$ are defined as follows

$$\Delta_j u = \begin{cases} 0, & \text{if } j \leq -2; \\ \chi(D)u = \mathcal{F}^{-1}(\chi \mathcal{F}u), & \text{if } j = -1; \\ \varphi(2^{-j}D)u = \mathcal{F}^{-1}(\varphi(2^{-j}\cdot)\mathcal{F}u), & \text{if } j \geq 0. \end{cases}$$

In the inhomogeneous case, the following Littlewood-Paley decomposition makes sense

$$u = \sum_{j \geq -1} \Delta_j u \quad \text{for any } u \in \mathcal{S}'(\mathbb{R}).$$
Definition 2.1 (see [1]). Let \( s \in \mathbb{R} \) and \( (p, r) \in [1, \infty]^2 \). The nonhomogeneous Besov space \( B^s_{p,r}(\mathbb{R}) \) is defined by

\[
B^s_{p,r}(\mathbb{R}) := \left\{ f \in \mathcal{S}'(\mathbb{R}) : \|f\|_{B^s_{p,r}(\mathbb{R})} < \infty \right\},
\]

where

\[
\|f\|_{B^s_{p,r}(\mathbb{R})} = \begin{cases} 
\left( \sum_{j \geq -1} 2^{sjr} \|\Delta_j f\|_{L^p(\mathbb{R})} \right)^{\frac{1}{r}}, & \text{if } 1 \leq r < \infty, \\
\sup_{j \geq -1} 2^{sj} \|\Delta_j f\|_{L^p(\mathbb{R})}, & \text{if } r = \infty.
\end{cases}
\]

The following Bernstein’s inequalities will be used in the sequel.

Lemma 2.1 (see Lemma 2.1 in [1]). Let \( \mathcal{B} \) be a Ball and \( \mathcal{C} \) be an annulus. There exist constants \( C > 0 \) such that for all \( k \in \mathbb{N} \cup \{0\} \), any positive real number \( \lambda \) and any function \( f \in L^p \) with \( 1 \leq p \leq q \leq \infty \), we have

\[
\sup \hat{f} \subset \lambda \mathcal{B} \Rightarrow \|\partial_x^k f\|_{L^q} \leq C^{k+1} \lambda^{k+\left(\frac{1}{p} - \frac{1}{q}\right)} \|f\|_{L^p},
\]

\[
\sup \hat{f} \subset \lambda \mathcal{C} \Rightarrow C^{-k-1} \lambda^k \|f\|_{L^p} \leq \|\partial_x^k f\|_{L^q} \leq C^{k+1} \lambda^k \|f\|_{L^p}.
\]

Lemma 2.2 (see Lemma 2.100 in [1]). Let \( 1 \leq r \leq \infty, 1 \leq p \leq p_1 \leq \infty \) and \( \frac{1}{p_2} = \frac{1}{p} - \frac{1}{p_1} \). There exists a constant \( C \) depending continuously on \( p, p_1 \), such that

\[
\left\| \left(2^j \|\Delta_j, v\partial_x\|_{L^r}(\mathcal{R})\right)_j \right\|_{L^r} \leq C \left( \|\partial_x v\|_{L^\infty} \|f\|_{B^1_{p_1,r}} + \|\partial_x f\|_{L^{p_2}} \|\partial_x v\|_{B^0_{p_1,r}} \right).
\]

3 Proof of Theorem 1.1

For localization, we first introduce the following bump function in the frequency space. let \( \hat{\theta} \in \mathcal{C}_0^\infty(\mathbb{R}) \) be a smooth and even function with values in \([0, 1]\) which satisfies

\[
\hat{\theta}(\xi) = \begin{cases} 
1, & \text{if } |\xi| \leq \frac{1}{200}, \\
0, & \text{if } |\xi| \geq \frac{1}{100}.
\end{cases}
\]

Then, shift the frequency of the above function by letting \( \phi(x) = \theta(x) \sin(\frac{17}{24}x) \). For the convenience of notation, hereafter

\[
n \in 16\mathbb{N}^+ = \{16, 32, 48, \cdots\}, \quad N(n) = \{k \in \mathbb{N}^+ : \frac{n}{4} \leq k \leq \frac{n}{2}\},
\]

and define the norm \( B^0_{\infty,1}(\mathbb{N}(n)) \) by

\[
\|u\|_{B^0_{\infty,1}(\mathbb{N}(n))} = \sum_{j \in \mathbb{N}(n)} \|\Delta_j u\|_{L^\infty}.
\]

Now, we construct the initial data \((u_0, \rho_0)\) as following:

\[
u_0(x) = 0, \quad \rho_0(x) = n^{-\frac{1}{2}} \log n \cdot \sin(\frac{17}{12}2^n x) \cdot \sum_{\ell \in \mathbb{N}(n)} \phi(2^\ell (x - 2^{2n+\ell})).
\]
3.1 Estimation of initial data

First of all, we give the estimates for $\rho_0$.

**Lemma 3.1.** There exists a positive constant $C$ independent of $n$ such that

$$
\|\rho_0\|_{L^\infty} \leq C n^{-\frac{1}{2}} \log n,
$$

$$
\|\rho_0\|_{B_{\infty,1}^{0}} \leq C \|\rho_0\|_{L^\infty} \leq C n^{-\frac{1}{2}} \log n.
$$

**Proof.** By the definition of $\rho_0$ and using convolution expressions, it’s easy to verify that the support set of $\rho_0$ satisfies

$$
\text{supp} \hat{\rho}_0 \subset \left\{ \xi \in \mathbb{R} : \frac{17}{12} 2^n - \frac{17}{12} 2^\ell - \frac{1}{2} \leq |\xi| \leq \frac{17}{12} 2^n + \frac{17}{12} 2^\ell + \frac{1}{2} \right\}.
$$

(3.3)

Since $\phi$ and $\tilde{\chi}$ are Schwartz functions, we have

$$
|\phi(x)| + |\tilde{\chi}(x)| \leq C (1 + |x|)^{-M}, \quad M \geq 100.
$$

(3.4)

then, by a direct computation, we have

$$
\|\rho_0\|_{L^\infty} \leq C n^{-\frac{1}{2}} \log n \left\| \sum_{\ell \in \mathbb{N}(n)} \phi(2^\ell (x - 2^{2n+\ell})) \right\|_{L^\infty}
$$

$$
\leq C n^{-\frac{1}{2}} \log n \left\| \sum_{\ell \in \mathbb{N}(n)} \frac{1}{(1 + |2^\ell x - 2^{2n+2\ell}|)^M} \right\|_{L^\infty} \leq C n^{-\frac{1}{2}} \log n.
$$

Due to the fact of (3.3) and the definition of $\Delta_j$, it’s easy to check that $\Delta_j \rho_0 = \rho_0$ if $j = n$ and $\Delta_j u_0 = 0$ if $j \neq n$, we deduce that

$$
\|\rho_0\|_{B_{\infty,1}^{0}} \leq C \|\rho_0\|_{L^\infty} \leq C n^{-\frac{1}{2}} \log n.
$$

This completes the proof of Lemma 3.1. \qed

The following lower bound estimate of the squared term is crucial to our proof.

**Lemma 3.2.** There exists a positive constant $c$ independent of $n$ such that

$$
\left\| (\rho_0)^2 \right\|_{B_{\infty,1}^{0}(\mathbb{N}(n))} \geq c \log^2 n, \quad n \gg 1.
$$

**Proof.** It follows that for any $j \in \mathbb{N}(n)$,

$$
\Delta_j [(\rho_0)^2] = \frac{1}{2} n^{-1} \log^2 n \cdot \Delta_j U, \quad U := \left( \sum_{\ell \in \mathbb{N}(n)} \phi(2^\ell (x - 2^{2n+\ell})) \right)^2.
$$

(3.5)
We rewrite $U$ as
\[ U = \sum_{\ell \in \mathbb{N}(n)} \phi^2(2^\ell (x - 2^{2n+\ell})) + \sum_{\ell, m \in \mathbb{N}(n) \setminus \{\ell = m\}} \phi(2^\ell (x - 2^{2n+\ell}))\phi(2^m (x - 2^{2n+m})) := U_1 + U_2. \]

Then, we have $\Delta_j U = \Delta_j U_1 + \Delta_j U_2$. For $\Delta_j U_2$, we have
\[
\|\Delta_j U_2\|_{L^\infty} \leq C \|U_2\|_{L^\infty} \\
\leq C \sum_{\ell \in \mathbb{N}(n) \setminus \{\ell = m\}} \|\phi(2^\ell (x - 2^{2n+\ell}))\phi(2^m (x - 2^{2n+m}))\|_{L^\infty} \\
\leq C \sum_{\ell, m \in \mathbb{N}(n) \setminus \{\ell = m\}} \frac{1}{(1 + 2^\ell |x - 2^{2n+\ell}|)^M} \cdot \frac{1}{(1 + 2^m |x - 2^{2n+m}|)^M} \|L^\infty \] (3.6)

By direct computations, for large enough $n$, we have
\[
\Delta_j U_1 = \Delta_j \phi^2(2^j (x - 2^{2n+j})) + \Delta_j \sum_{j \neq \ell \in \mathbb{N}(n)} \phi^2(2^\ell (x - 2^{2n+\ell})) =: \Delta_j U_{1,1} + \Delta_j U_{1,2}.
\]

Let us introduce the set $B_j$ defined by $B_j = \{x : 2^j |x - 2^{2n+j}| \leq 1\}$.

We can show that
\[
\|\Delta_j U_1\|_{L^\infty(B_j)} \geq \|\Delta_j U_{1,1}\|_{L^\infty(B_j)} - \|\Delta_j U_{1,2}\|_{L^\infty(B_j)}
\]

Notice that
\[
\phi^2(x) = \frac{1}{2} \theta^2(x) - \frac{1}{2} \theta^2(x) \cos\left(\frac{17}{12} x\right) := \Phi_1(x) + \Phi_2(x),
\]
then we have
\[
\Delta_j U_{1,1} = \Delta_j \Phi_2(2^j (x - 2^{2n+j})) = \Phi_2(2^j (x - 2^{2n+j})).
\]

This implies
\[
\|\Delta_j U_{1,1}\|_{L^\infty(B_j)} \geq \frac{1}{2} \theta^2(0) := c. \quad (3.7)
\]

By (3.4), we have
\[
\|\Delta_j U_{1,2}\|_{L^\infty(B_j)} \leq C \sum_{j \neq \ell \in \mathbb{N}(n)} \left\| 2^j \int_{\mathbb{R}} \tilde{\chi}(2^j (x - y))\phi^2(2^\ell (y - 2^{2n+\ell}))dy \right\|_{L^\infty(B_j)} \\
\leq C \sum_{j \neq \ell \in \mathbb{N}(n)} \left\| 2^j \int_{\mathbb{R}} \frac{1}{(1 + 2^j |x - y|)^M} \cdot \frac{1}{(1 + 2^\ell |y - 2^{2n+\ell}|)^M}dy \right\|_{L^\infty(B_j)}.
\]

Dividing the integral region in terms of $y$ into the following two parts to estimate:
\[
A_1 := \{y : |y - 2^{j+2n}| \leq 2^{2n}\},
A_2 := \{y : |y - 2^{j+2n}| \geq 2^{2n}\}.
\]
For $x \in B_j$ and $y \in A_1$, it is easy to check that
\[
|y - 2^{j+2n}| = |y - 2^j + 2^n + 2^{j+2n} - 2^{j+2n}|
\geq |2^{j+2n} - 2^j + 2^n| - |y - 2^j + 2^n| \geq 2^{2n}.
\]
For $x \in B_j$ and $y \in A_2$, it is easy to check that
\[
|x - y| = |x - 2^j + 2^n + 2^{j+2n} - y|
\geq |y - 2^j + 2^n| - |x - 2^j + 2^n| \geq 2^{2n} - 2^{-j} \geq 2^{2n-1}.
\]
Then, we have
\[
\left\| 2^j \int_R \frac{1}{(1 + 2^j |x - y|)^M} \frac{1}{(1 + 2^j |y - 2^2n + |)^{2M}} dy \right\|_{L^\infty(B_j)} \leq C 2^{j+2n-M} \leq C 2^{-n-M},
\]
which implies
\[
\| \Delta_j U_{1,2} \|_{L^\infty(B_j)} \leq C 2^{-n} \tag{3.8}
\]
Combining (3.6)-(3.8), we deduce that for $n \gg 1$,
\[
\| \Delta_j [(\rho_0)^2] \|_{L^\infty} \geq \| \Delta_j [(\rho_0)^2] \|_{L^\infty(B_j)}
\geq \| \Delta_j U_{1,1} \|_{L^\infty(B_j)} - \| \Delta_j U_{1,2} \|_{L^\infty(B_j)} - \| \Delta_j U_2 \|_{L^\infty}
\geq c n^{-1} \log^2 n.
\]
Therefore, by the definition of the Besov norm, we have
\[
\| (\rho_0)^2 \|_{B^0_{\infty,1}(\mathbb{R}(n))} = \sum_{j \in \mathbb{N}(n)} \| \Delta_j [(\rho_0)^2] \|_{L^\infty} \geq c \log^2 n.
\]
This completes the proof of lemma 3.2. \qed

### 3.2 Norm Inflation

Firstly, we define the Lagrangian flow-map $\psi$ associated to $u$ by solve the following ODE:
\[
\begin{align*}
\frac{d}{dt} \psi(t, x) &= u(t, \psi(t, x)), \\
\psi(0, x) &= x.
\end{align*}
\tag{3.9}
\]
Considering the equation
\[
\begin{align*}
\partial_t v + u \partial_x v &= P, & t \in [0, T), & x \in \mathbb{R}, \\
v(0, x) &= x, & x \in \mathbb{R}.
\end{align*}
\tag{3.10}
\]
Taking $\Delta_j$ on (3.10), we get
\[
\partial_t (\Delta_j v) + u \partial_x \Delta_j v = [u, \Delta_j] \partial_x v + \Delta_j P := R_j + \Delta_j P,
\]
Due to (3.9), then
\[
\frac{d}{dt} ((\Delta_j v) \circ \psi) = R_j \circ \psi + \Delta_j P \circ \psi,
\]
which is equivalent to the integral form
\[
(\Delta_j v) \circ \psi = \Delta_j v_0 + \int_0^t R_j \circ \psi d\tau + \int_0^t \Delta_j P \circ \psi d\tau. \tag{3.11}
\]
Now we can go back to the proof of the Theorem 1.1. For $n \gg 1$, it is easy to check that $(u, \rho) \in C([0, 1]; H^3 \times H^2)$ which satisfy for $t \in [0, 1]$
\[
\|u(t)\|_{W^{1,\infty}} + \|\rho(t)\|_{L^\infty} \leq (\|u_0\|_{W^{1,\infty}} + \|\rho_0\|_{L^\infty}) \exp \left( C \int_0^t \|\partial_x u(\tau), \rho(\tau)\|_{L^\infty} d\tau \right).
\]
For $n \gg 1$, we have for $t \in [0, 1]$
\[
\|u\|_{W^{1,\infty}} + \|\rho\|_{L^\infty} \leq C\|\rho_0\|_{L^\infty} \leq Cn^{-\frac{1}{2}} \log n. \tag{3.12}
\]
Moreover, we also have $t \in [0, 1]$
\[
\|u\|_{L^\infty} \leq C \int_0^t (\|u\|_{W^{1,\infty}}^2 + \|\rho\|_{L^\infty}^2) d\tau \leq Cn^{-1} \log^2 n. \tag{3.13}
\]
To prove Theorem 1.1, it suffices to show that there exists $t_0 \in (0, \frac{1}{\log n}]$ such that
\[
\|u(t_0, \cdot)\|_{B^1_{\infty, 1}} \geq \log \log n. \tag{3.14}
\]
We prove (3.14) by contraction. If (3.14) were not true, then
\[
\sup_{t \in (0, \frac{1}{\log n}]} \|u(t, \cdot)\|_{B^1_{\infty, 1}} < \log \log n. \tag{3.15}
\]
We divide the proof into two steps.

**Step 1: Lower bounds for $(\Delta_j u) \circ \psi$**

Now we consider the equation along the Lagrangian flow-map associated to $u$. Utilizing (3.11) to (1.2) yields
\[
(\Delta_j u) \circ \psi = \Delta_j u_0 + \int_0^t R_j^1 \circ \psi d\tau + \int_0^t \Delta_j F \circ \psi d\tau + \int_0^t (\Delta_j E \circ \psi - \Delta_j E_0) d\tau + t \Delta_j E_0,
\]
where
\[
R_j^1 = [u, \Delta_j] \partial_x u, \quad F = -\frac{3}{2} \partial_x (1 - \partial_x^2)^{-1}(u^2), \quad E = -\frac{1}{2} \partial_x (1 - \partial_x^2)^{-1}(\rho^2).
\]
Due to Lemma 3.2, we deduce
\[
\sum_{j \in \mathbb{N}(n)} 2^j \| \Delta_j E_0 \|_{L^\infty} \approx \sum_{j \in \mathbb{N}(n)} \| \Delta_j \partial_x E_0 \|_{L^\infty} \geq c \sum_{j \in \mathbb{N}(n)} \| \Delta_j (\rho_0)^2 \|_{L^\infty} \geq c \log^2 n. \tag{3.16}
\]

Notice that, for fixed \( t \) the Lagrangian flow-map \( \psi(t, \cdot) \) is a diffeomorphism of \( \mathbb{R} \), then we have for \( t \in (0, \frac{1}{\log n}] \),
\[
\| f(t, \psi(t, x)) \|_{L^\infty} = \| f(t, \cdot) \|_{L^\infty}.
\]

Then, using (3.12), (3.15) and the commutator estimate from Lemma 2.2, we have
\[
\sum_{j \geq -1} 2^j \| R_j^1 \circ \psi \|_{L^\infty} \leq C \sum_{j \geq -1} 2^j \| R_j^1 \|_{L^\infty} \leq C \| \partial_x u \|_{L^\infty} \| u \|_{B^1_{\infty,1}} \leq C n^{-\frac{1}{2}} \log n \cdot \log \log n. \tag{3.17}
\]

Also, by (3.13), we have
\[
2^j \| \Delta_j F \circ \psi \|_{L^\infty} \leq C 2^j \| \Delta_j F \|_{L^\infty} \leq C \| u \|_{L^\infty}^2 \leq C n^{-2} \log^4 n,
\]
which implies
\[
\sum_{j \in \mathbb{N}(n)} 2^j \| \Delta_j F \circ \psi \|_{L^p} \leq C n^{-1} \log^4 n. \tag{3.18}
\]

Combining (3.16)-(3.18) and using Lemmas 3.1-3.2 yields
\[
\sum_{j \in \mathbb{N}(n)} 2^j \| (\Delta_j u) \circ \psi \|_{L^\infty} \geq t \sum_{j \in \mathbb{N}(n)} 2^j \| \Delta_j E_0 \|_{L^\infty} - \sum_{j \in \mathbb{N}(n)} 2^j \| \Delta_j E \circ \psi - \Delta_j E_0 \|_{L^\infty} - C n^{-1} \log^4 n
\]
\[
\geq ct \log^2 n - \sum_{j \in \mathbb{N}(n)} 2^j \| \Delta_j E \circ \psi - \Delta_j E_0 \|_{L^\infty} - C n^{-1} \log^4 n.
\]

**Step 2: Upper bounds for \( \Delta_j E \circ \psi - \Delta_j E_0 \)**

By easy computations, we can see that
\[
\partial_t E + u \partial_x E = - \partial_x (1 - \partial_x^2)^{-1}(\partial_t \rho) - \frac{1}{2} u \partial_x^2 (1 - \partial_x^2)^{-1}(\rho^2)
\]
\[
= - \partial_x (1 - \partial_x^2)^{-1} \left(- \frac{1}{2} \partial_x (u \rho^2) - \frac{3}{2} u_x \rho^2 - \frac{1}{2} u (1 - \partial_x^2)^{-1}(\rho^2) + \frac{1}{2} u \rho^2 \right)
\]
\[
= \frac{1}{2} \partial_x^2 (1 - \partial_x^2)^{-1}(u \rho^2) + \frac{3}{2} \partial_x (1 - \partial_x^2)^{-1}(u_x \rho^2) - \frac{1}{2} u (1 - \partial_x^2)^{-1}(\rho^2) + \frac{1}{2} u \rho^2
\]
\[
= \frac{1}{2} (1 - \partial_x^2)^{-1}(u \rho^2) - \frac{1}{2} u \rho^2 + \frac{3}{2} \partial_x (1 - \partial_x^2)^{-1}(u_x \rho^2) - \frac{1}{2} u (1 - \partial_x^2)^{-1}(\rho^2) + \frac{1}{2} u \rho^2
\]
\[
=: G,
\]

where
\[
G = - \frac{1}{2} u (1 - \partial_x^2)^{-1}(\rho^2) - \frac{1}{2} (1 - \partial_x^2)^{-1} (-u \rho^2 - \partial_x (3u_x \rho^2)).
\]
Utilizing (3.11) to (3.19) yields
\[
\Delta_j E \circ \psi - \Delta_j E_0 = \int_0^t [u, \Delta_j] \partial_x \circ \psi d\tau + \int_0^t \Delta_j G \circ \psi d\tau.
\]
Using the commutator estimate from Lemma 2.2, one has
\[
2^j \| [u, \Delta_j] \partial_x \circ E \|_{L^\infty} \leq C(\| \partial_x u \|_{L^\infty} \| E \|_{B_{\infty, \infty}^1} + \| \partial_x E \|_{L^\infty} \| u \|_{B_{\infty, \infty}^1}) \leq C(\| u \|_{W^{1, \infty}} + \| \rho \|_{L^\infty})^3
\]
and
\[
2^j \| \Delta_j G \|_{L^\infty} \leq C(\| u \|_{W^{1, \infty}} + \| \rho \|_{L^\infty})^3.
\]
Then, we have from (3.12)
\[
2^j \| \Delta_j E \circ \psi - \Delta_j E_0 \|_{L^\infty} \leq C(\| u \|_{W^{1, \infty}} + \| \rho \|_{L^\infty})^3 \leq Cn^{-\frac{3}{2}} \log^3 n,
\]
which leads to
\[
\sum_{j \in \mathbb{N}(n)} 2^j \| \Delta_j E \circ \psi - \Delta_j E_0 \|_{L^p} \leq Cn^{-\frac{1}{2}} \log^3 n.
\]
Combining Step 1 and Step 2, then for \( t = \frac{1}{\log n} \), we obtain for \( n \gg 1 \)
\[
\| u(t) \|_{B^1_{\infty, 1}} \geq \| u(t) \|_{B^1_{\infty, 1}(\mathbb{N}(n))} \geq C \sum_{j \in \mathbb{N}(n)} 2^j \| (\Delta_j u) \circ \psi \|_{L^\infty}
\]
\[
\geq ct \log^2 n - Cn^{-\frac{1}{2}} \log^3 n \geq \log \log n,
\]
which contradicts the hypothesis (3.15).

In conclusion, we obtain the norm inflation and hence the ill-posedness of the two component Degasperis-Procesi equation. Thus, Theorem 1.1 is proved.

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