ROBUST REGULARIZATION OF TOPOLOGY OPTIMIZATION PROBLEMS WITH APOSTERIORI ERROR ESTIMATORS

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ABSTRACT. Topological optimization finds a material density distribution minimizing a functional of the solution of a partial differential equation (PDE), subject to a set of constraints (typically, a bound on the volume or mass of the material).

Using a finite elements discretization (FEM) of the PDE and functional we obtain an integer programming problem. Due to approximation error of FEM discretization, optimization problem becomes mesh-dependent and possesses false, physically inadequate optima, while functional value heavily depends on fineness of discretization scheme used to compute it. To alleviate this problem, we propose regularization of given functional by error estimate of FEM discretization. This regularization provides robustness of solutions and improves obtained functional values as well.

While the idea is broadly applicable, in this paper we apply our method to the heat conduction optimization. This type of problems are of practical importance in design of heat conduction channels, heat sinks and other types of heat guides.

Keywords: Topological optimization, greedy methods, finite element methods, error estimators, regularization.

1. Introduction

Topology optimization methods aim to find a material distribution in a volume that minimizes a functional depending on the solution of a PDE describing a physical process, e.g., elastic deformation, heat distribution or electrical current flow.

As in general these problems are nonconvex, and even non-smooth, most methods can be viewed as heuristics and do not guarantee reaching a global or local optimum. The solutions produced by commonly used techniques may exhibit substantial mesh dependence with significant difference in functional values obtained for different mesh resolutions, often due to non-physical solutions (“checkerboards”). These solutions are eliminated by different types of regularization techniques, penalizing non-smoothness of the solutions.

While some mesh dependence is unavoidable for discretized problems, we observe that undesirable mesh-dependent solutions are the ones for which the FEM discretization does not yield an accurate estimate of the functional. Informally, the less “smooth”, in the sense of presence of rapid oscillations, the discrete solutions is, the larger is the range of functional values that can be attained by functions agreeing with the discrete solution at sample points.

This view suggests the following idea for regularization: instead of choosing a measure of smoothness a priori, we directly include an estimate of the range of possible functional values in the regularized functional, by adding a term corresponding to the FEM error. While the approximation error can not be measured
directly, computable upper bounds are typically available. Thus, instead of mini-
mizing the value of the discretized functional, we minimize the upper bound of the 
true functional.

More specifically, we use *a posteriori estimates* to modify the original functionals. 
This allows us to perform topological optimization using standard sensitivity-based 
greedy algorithms using a fixed (non-adaptive) discretization, which is important 
for efficient implementation of the method (including much simpler parallelization).

To summarize, our approach consists of the following elements:

- We propose using *a posteriori* error estimators as reliable regularizers for 
topological optimization.
- We obtain a weak FEM problem formulation to compute the sensitivity of 
our regularized functionals which can be solved efficiently using a standard 
FEM solver;
- We describe a greedy optimization algorithm based on the computed sen-
tsitivity to compute the optimal topology.
- For two model problems, we obtained new configurations that have sig-
ificantly lower functional values than the configurations reported in the 
literature.

Although we focus on a particular model problem in heat transfer, our approach 
can be easily generalized to other topology optimization settings.

We start with the description of the method (Sections 2-5) and then compare it 
to closely related methods and discuss its relationship to most common topology 
optimization techniques (Section 6).

2. CONTINUOUS PROBLEM

As a model problem in this paper we consider the steady heat conduction prob-
lem:

\[ -\nabla \cdot (w \nabla u) = f, \quad u_{\partial \Omega} = 0, \]

where \( \Omega \) is a bounded region with a sufficiently smooth boundary \( \partial \Omega \), \( w \) is in the 
set of piecewise-continuous functions, which we denote \( PC \), \( f \in L^2(\Omega) \), and we 
search for \( u \in V \), where \( V \) is the Sobolev space \( W^{1}_{2}(\Omega) \) of functions with zero trace 
(due to the boundary conditions).

We solve a two-material optimization problem, i.e., \( w \) is allowed to take only two 
values: 1 (corresponding to the main material) and \( \varepsilon \ll 1 \) (corresponding to the 
filler material), with \( \varepsilon \) being the ratio of conductivity of filler material to that of 
main material. Our model topology optimization problem is formulated as follows: find a material distribution \( w \) such that

\[ \min_{w \in PC} F(w) = \int_{\Omega} u(x)f(x) \, dx = (u, f), \]

s. t. \( \int w(x) \, dx \geq c, \quad u \) is solution of \( 1 \).

A physical interpretation of this optimization problem is the design of optimal 
heat-conducting device, producing least amount of heat when amount of high-
conductivity material is limited. The same mathematical formulations appears 
not only in heat conduction problems, but also in electrostatic problems [1] or 
modelling of amoeboid organism growing towards food sources [2].
Note that in general, the problem does not have an optimal solution in $V$; regularized versions of the problem however, may have sufficiently smooth solutions. The analysis of existence and smoothness of solutions of the optimization problem is beyond the scope of this paper.

3. Discretization

A common way to solve problem (1) is to introduce in $\Omega$ a mesh $\Omega_h$, consisting of elements $\Omega_i^h$, and replace $w$ by its projection $w_h$ to the finite-dimensional space of functions $\theta_i$, which are piecewise constant on the elements:

$$w_h(x) = \sum_{i=1}^{n} \eta_i^h \theta_i(x), \eta_i^h \in \{\varepsilon, 1\}, \theta_i(x) \in \{0, 1\},$$

where $n$ is number of mesh cells and $\theta_i(x)$ is 1 on $i$-th cell, and zero otherwise. Problem (1) can be written in the following variational form:

$$B(u, v) = l(v), \quad \forall v \in V,$$

where $B(u, v)$ is a bilinear form on $V \times V$ and $l(v)$ is a linear functional on dual space:

$$B(u, v) = \int_\Omega w \nabla u \nabla v \, d\Omega, \quad l(v) = \int_\Omega f v \, d\Omega.$$

We obtain finite element discretization by replacing $V$ in (3) by a suitable finite-dimensional space $V_h$. This leads us to the following system of linear equations for $u_h \in V_h$:

$$B(u_h, v_h) = l(v_h), \quad \forall v_h \in V_h,$$

From now on, we will assume $f$ is a function for which $f_h = f$.

The optimization problem (2) is then approximated by the following integer programming problem:

$$\min_{\eta_i^h \in \{\varepsilon, 1\}^n} F_h(w_h) = (u_h, f_h), \quad \text{s.t.} \sum_{i=1}^{n} \eta_i^h \geq c, \quad u_h \text{ is solution of (4)}.$$

The solutions to the discretized problem may exhibit spurious behavior, becoming increasingly oscillatory, with non-physical “checkerboard” patterns emerging, with oscillations on the scale of mesh elements. The emergence of these patterns in the case of elasticity is attributed to the fact that on the scale of the mesh, the discretization exhibit artificial stiffness resulting in a significant underestimation of the functional value. We propose a new way of tackling this type of problems by introducing an additional term into the functional that penalizes $\eta$ that give rises to large errors in the solution.

4. A posteriori error estimators as regularizers

Transition from the initial optimization problem (2) to its discrete counterpart (5) introduces a discretization error, which can be estimated from above as

$$(u, f) - (u_h, f_h) \leq \|u_h - u\| \|f\|.$$
where the regularization term $\Theta_h(\eta)$ is an upper bound for the error of solution given by $\eta$. We assume that for some $\alpha > \alpha_0$ it provides an upper bound for the true functional:

$$F(w_h) \leq F_h(\alpha, w_h).$$

Our idea is to minimize the upper bound $F_h(\alpha, w_h)$, instead of the discretization of the functional itself. While the value of $F_h(w_h)$ for the resulting solution may be larger, one can guarantee that $F(w_h)$ is not arbitrarily large, unlike the case of standard greedy methods, which may lead to solutions for which $F(w_h)$ can be significantly larger than $F_h(w_h)$.

There are many ways to introduce the regularizer term based on the error estimation; we present a simple error estimator bounding the squared error:

$$\|u - u_h\|^2 \leq \Theta_h(w_h)$$

to ensure that the regularized functional is differentiable everywhere. We obtain a simple regularizer from the following variational form [3]. Let $e = u - u_h$, then

$$B(e, q) = l(q) - B(u_h, q), \forall q \in V.$$ 

For Galerkin discretizations, the error is orthogonal to $V_h$, and hence we need to choose subspace $V_h' \neq V_h$ to solve for the error:

$$B(\tilde{e}_h, q_h) = l(q_h) - B(u_h, q_h), \forall q_h \in V_h'.$$

Choosing $\Theta_h(w_h) = (\tilde{e}_h, \tilde{e}_h)$ leads us to the regularized version of (5):

$$\min_{\eta \in \{0, 1\}^n} F_h(\alpha, w_h) = (u_h, f_h) + \alpha(\tilde{e}_h, \tilde{e}_h)$$ 

s. t. $\sum_i \eta_h^i \geq c$, $u_h$ is a solution of (4), $\tilde{e}_h$ is solution of (7).

5. Optimization problem

5.1. Greedy optimization. To solve (5) we use the approach that has been successfully used in the topology optimization: the greedy “hard-kill” methods introduced in [4, 5] and [6], and used more recently in [7]). Those methods remove/add a whole patch of material on each step, in contrast to the “soft-kill” methods which evolve a smooth material density function which is discretized in the final step by thresholding and/or filtering. One can easily see that both continuous and discrete functionals $F(w)$ and $F_h(w_h)$ are monotonous with respect to changes in $w$, so “removing material” (i.e., replacing 1 by $\varepsilon$) always increases it, because boundary conditions are chosen such that $u$ to be non-negative. The greedy approach to solving this problem removes the material from an element that results in least change to the functional. To estimate the change in the functional, we use its sensitivity, i.e., the derivative of the functional with respect to the coefficient $\eta_h^i$ of the material distribution corresponding to the $i$-th element:

$$S_i = \frac{\partial F_h}{\partial \eta_h^i}, \quad i = 1, \ldots, n,$$

An efficient way to compute sensitivities will be described in Section 5.2 Algorithm 1 summarizes the complete method.

For a given problem we can vary the number of elements $n$ and mesh connectivity, which affects the material distribution and functional value. We will discuss the effects of those parameters in numerical experiments section 6.2.1.
Algorithm 1: Greedy method

Data: Total material bound $c$, number of patches to remove at each step $p_s$, mesh $\mathcal{M}$, low material density $\varepsilon$, regularization parameter $\alpha$

Result: Vector of the material distribution coefficients $\eta^h$ and the value of $F^h$

1. Fill the whole mesh $\mathcal{M}$ by setting $\eta^i = 1, i = 1, \ldots, n$;
2. while $\sum_i \eta^i > c$ do
3.   Calculate sensitivities of regularized functional $\partial F^h/\partial \eta^i$;
4.   Choose up to $p_s$ elements with smallest sensitivities;
5.   Set those elements density to $\varepsilon$;
6.   Calculate $F^h$;
7. return $F^h, \eta$

5.2. Computing sensitivity. The derivatives of the regularized functional consist of two terms, corresponding to the original functional and the regularization term. The expression for the former is well-known (we provide the derivation for the sake of completeness); we derive the expression for the regularizer.

We represent the form $B$ in finite-dimensional spaces $V$ and $V'$ by symmetric stiffness matrices $K$ and $M$: $B(u^h, v^h) = (Ku^h, v^h)$, and $B(u'_h, v'_h) = (Mu'_h, v'_h)$, which leads us to the following equations for $u^h$ and $u'_h$:

\begin{align*}
Ku^h &= f^h, \\
Mu'_h &= f'_h.
\end{align*}

We use the same notation $u^h$ for functions and their coefficient vectors, as the difference is clear from the context (the coefficient vectors are used only in matrix equations).

Matrices $P$ and $P'$ project a function in $V_h$ to a function in $V'_h$ and vice versa. To simplify derivation, for the rest of the paper, we assume that $f'_h = f^h = f$ and $V_h \subset V'_h$, for example $V'_h$ being generated by the same basis function family, but on finer mesh, hence $P$ is the inclusion operator.

The discretized unregularized functional can be written as $F_h = u^T_h f_h$; Using $K u^h = f_h$, where $f_h$ is independent of $\eta$, we obtain

\begin{equation}
\frac{\partial F_h}{\partial \eta^i_h} = \left( \frac{\partial K^{-1}}{\partial \eta^i_h} f_h, f_h \right),
\end{equation}

From $KK^{-1} = I$ it follows

\begin{equation}
\frac{\partial K}{\partial \eta^i_h} K^{-1} + \frac{\partial K^{-1}}{\partial \eta^i_h} K = 0.
\end{equation}

We conclude that

\begin{equation}
\frac{\partial F_h}{\partial \eta^i_h} = - \left( \frac{\partial K}{\partial \eta^i_h} u^h, u^h \right).
\end{equation}

5.2.1. Sensitivity of the regularization term. Now, we have two natural choices for error estimate $\hat{e}_h$ from (8). First, $e_h = u_h - P u_h$, and second, $e'_h = P u_h - u'_h$.

Theorem 1. Sensitivity of the functional $F_{h, \alpha}$ regularized by $e_h$ is

\begin{equation}
\frac{\partial F_{h, \alpha}}{\partial \eta^i} = - \left( \frac{\partial K}{\partial \eta^i} u^h, u^h \right) - 2\alpha \left( P' M^{-1} \frac{\partial M}{\partial \eta^i} u'_h, e_h \right) + 2\alpha \left( \frac{\partial K}{\partial \eta^i} u^h, K^{-1} e_h \right).
\end{equation}
Sensitivity of the functional $F_{h,\alpha}$ regularized by $e'_h$ is

\[ \frac{\partial F_{h,\alpha}}{\partial \eta_i} = -\left( \frac{\partial K}{\partial \eta_i} u_h, u_h \right) - 2\alpha \left( \frac{\partial M}{\partial \eta_i} u'_h, M^{-1} e'_h \right) + 2\alpha \left( PK^{-1} \frac{\partial K}{\partial \eta_i} u_h, e'_h \right). \]

**Proof.** The first part of the formulas is given by (12). For the second part, notice that from (10) and (11) follows:

\[ e_h = K^{-1} f_h - P' M^{-1} f'_h. \]

Then sensitivity of $e_h$ can be written as

\[ \frac{\partial e_h}{\partial \eta_i} = -K^{-1} \frac{\partial K}{\partial \eta_i} K^{-1} f_h + P' M^{-1} \frac{\partial M}{\partial \eta_i} M^{-1} f'_h, \]

and hence

\[ \frac{\partial (e_h, e'_h)}{\partial \eta_i} = -2\left( \frac{\partial K}{\partial \eta_i} u_h, K^{-1} e_h \right) + 2(P' M^{-1} \frac{\partial M}{\partial \eta_i} u'_h, e_h). \]

This, combined with (12) proves (13). Now, consider $e'_h$:

\[ e'_h = PK^{-1} f_h - M^{-1} f'_h. \]

Then sensitivity of $e'_h$ can be written as

\[ \frac{\partial e'_h}{\partial \eta_i} = -PK^{-1} \frac{\partial K}{\partial \eta_i} K^{-1} f_h + M^{-1} \frac{\partial M}{\partial \eta_i} M^{-1} f'_h, \]

and hence

\[ \frac{\partial (e'_h, e''_h)}{\partial \eta_i} = -2(PK^{-1} \frac{\partial K}{\partial \eta_i} u'_h, e'_h) + 2\left( \frac{\partial M}{\partial \eta_i} u'_h, M^{-1} e'_h \right). \]

This, combined with (12) proves (14).

\[ \square \]

### 5.3. Computational details.

To compute $(\partial K/\partial \eta_i u_h, u_h)$, $(\partial M/\partial \eta_i u'_h, M^{-1} e'_h)$ and $(\partial K/\partial \eta_i u_h, K^{-1} e_h)$ observe that

\[ \left( \frac{\partial K}{\partial \eta_i} u_h, v_h \right) = \int_{\Omega} \theta_i \nabla u_h \nabla v_h \, d\Omega, \ i = 1, \ldots, n, \]

\[ \left( \frac{\partial M}{\partial \eta_i} u'_h, v'_h \right) = \int_{\Omega} \theta_i \nabla u'_h \nabla v'_h \, d\Omega, \ i = 1, \ldots, n. \]

Using these expressions we compute sensitivities by solving for $u_h$ and $u'_h$ using a FEM solver, computing their gradients, and then compute integrals (15) and (16). If an optimal-complexity linear solver is used (e.g., multigrid) the overall cost of the computation is linear in the total number of elements in $V_h$ and $V'_h$.

### 6. Numerical experiments

#### 6.1. Test cases.

Although topology optimization by greedy methods has been studied extensively, there are relatively few papers considering topology optimization for heat conduction. We use [7] and [8] for comparison purposes.

We consider two model 2D problems, both of each have been studied in literature, which allows us to provide comparisons with other methods: problem A was considered in [8] and Problem B in [9] and [7].

Both problems are formulated on a simple square domain of size 1, and a uniform heat distribution $f$ with unit density is assumed.
Problem A. For this problem, Dirichlet boundary conditions are specified on two boundary segments and Neumann conditions on the other two (Figure 1, left). The target volume fraction $c$ occupied by the material for this problem is set to 0.4, to be consistent with [8].

Problem B. For this problem, Dirichlet conditions are specified on the whole boundary (Figure 1), and the target volume fraction $c$ occupied by the material for this problem is set to 0.5, following [9] and [7].

Figure 1. Boundary conditions.

Our main comparison measure is the relative increase in the functional value

$$\hat{F}_h = \frac{F_h}{F_0},$$

where $F_0$ is the value of the functional corresponding to the whole domain filled with material, i.e., $\eta_h^i = 1$ for all elements. Since the functional always increases with the removal of the material, $\hat{F}_h \geq 1$ the smaller this number is, the better is the result.

6.2. Implementation. We implemented Algorithm 1 using FireDrake finite element solver ([10], [11], [12], [13]). We use the regularized functional (6) and expressions for sensitivities (15), (16). While for numerical experiments we consider only squares, the approach, obviously, works for arbitrary sufficiently regular geometry (the problem must be solvable with FEM, obviously) both in 2D and 3D.

For our experiments we choose $V_h$ to be the standard space of piecewise-linear continuous functions on triangular mesh; $w_h$ is represented by piecewise-constant functions on triangles. The space $V'_h$ needed for the error bound is also the space of piecewise-linear continuous functions on the mesh $\Omega'_h$ which is a uniformly refined version of original mesh $\Omega_h$. In the experiments we use three different meshes generated by GMSH [14] (Figure 2). Also, since we don’t need regularizer to be very precise for simplicity of implementation we used

$$\frac{\partial F_{h,\alpha}}{\partial \eta_i} = -\left( \frac{\partial K}{\partial \eta_i} u_h, u_h \right) - 2\alpha \left( \frac{\partial M}{\partial \eta_i} u'_h, M^{-1} e_h \right) + 2\alpha \left( \frac{\partial K}{\partial \eta_i} u_h, K^{-1} e_h \right),$$

as compromise between [13] and [14] which does not require projections.
6.2.1. Dependence on regularization parameter and mesh-dependence. We first explore the dependence of the final functional value on the regularization parameter $\alpha$, in the range from $10^{-5}$ until $10^{12}$. Simultaneously, we explore the mesh dependence of the method by running Algorithm 1 on all combinations of meshes and regularization parameters.

As evident from Figure 3, there is a noticeable dependence on the mesh choice at this resolution; At the same time, it is also clear that the method is relatively insensitive to the choice of $\alpha$, as long as it is sufficiently large: and the effects of the choice of $\alpha$ are not mesh-dependent, and are similar for both problems. Observe that the optimal values of $\alpha$ are large: effectively, minimization of the error needs to be prioritized over minimizing the discrete functional value for best results. Intuitively, one can interpret this as searching for optimal designs in the constrained space of functions minimizing the error on a finer mesh. As evident from Figure 4 functional converges with respect to mesh size very fast, with only one level of refinement sufficient.

6.3. Problem A. Based on the results shown in Figure 3, we use $\alpha = 10^8$ in the remaining experiments. The material distribution found by our algorithm can be
seen in Figure 5 (c). The obtained material distribution features thin heat-sink channels, similar to the ones from solutions of other authors.

A fundamental difference between our method and the method of [8] is that we solve the problem with discrete material values directly, while [8] solves a relaxation of the problem with a continuous material distribution. This approach has significant advantages as, due to convexity [15], such problems are easier to solve. In addition, by searching a larger space of material distributions, lower values of the functional can be obtained. However, as in practice creating continuous material variation is typically difficult or impossible, a final thresholding step needs to be applied, which typically leads to a significant increase in the functional.
Our solution appears to be less fractal than solution from [8]. Note that [8] considers continuous optimization problem, as material density allowed to be in $[\varepsilon, 1]$. Aggregated results for Problem A are shown in Table 1.

| Paper     | Elements | Solution type | $\hat{F}_h$ |
|-----------|----------|---------------|-------------|
| This      | 14694    | Discrete      | 3.53        |
| [8]       | 16384    | Continuous    | 2.718       |
| [8] with [16] | 16384 | Continuous    | 3.259       |

6.4. **Problem B.** Solution of this optimization problem features the same thin heat-sink channels as with problem A. Our solution looks different from solution of [7], which presents smaller number of larger channels with fractal-like border. On the other hand, our solution propose more thin channels with relatively smooth borders. The jigsaw borders of heat sinks are due to mesh effects (compare smoother borders of the channels in top and bottom versus rougher borders of the channels in left and right). The comparison results are shown in Table 2.

![Figure 6](image)

(a) $\hat{F}_h = 8.256$, quadrilateral mesh with 6400 elements, $\alpha = 10^5$.
(b) $\hat{F}_h = 4.401$, mesh 2 with 2466 elements, $\alpha = 10^5$.
(c) $\hat{F}_h = 3.42$, mesh 3 with 4898 elements, $\alpha = 10^{12}$.

**Figure 6.** Left to right: material distribution from [7], [9] and ours for problem B.

| Paper     | Elements | Solution type | $\hat{F}_h$ |
|-----------|----------|---------------|-------------|
| This      | 14694    | Discrete      | 3.42        |
| [7]       | 6400     | Discrete      | 8.256       |

7. **Discussion and related work**

7.1. **Previous approaches to regularization.** Regularization techniques were used in topological optimization primarily to solve two problems: to avoid formation of spurious solutions (checkerboard patterns) and to remove mesh dependency. Regularization was applied in the context of all commonly used approaches to topology optimization: density-based, hard-kill and level set/phase-field methods.
Most regularization approaches can be roughly divided into two groups: filtering methods and constraint techniques [17].

There are several types of filtering methods: the two most widely used types are density filters which directly modify the solution ([18], [19]), and sensitivity filters ([20], [16]). Recently filtering methods based on the Helmholtz equation were applied to both sensitivity and material density ([21], [22]). Another recent development is the design of density filters which are based on projection schemes. Those methods operate on density field $\rho$, its filtered version $\tilde{\rho}$ and a projected field $\hat{\tilde{\rho}}$. Most filtering methods result in continuous density distributions, which need to be thresholded in the context of hard-kill methods, the projection methods reported to have good convergence and almost discrete designs ([23], [24], [25], [26], [27]).

Constraint methods modify optimization problem by imposing a constraint either as hard constraint, or via a penalty term in the objective functional. A common type constrained quantity is an integral of the form

$$\int_{\Omega} \|\nabla w\|_q d\Omega,$$

i.e., a $L_q$-norm of the gradient. E.g., $q = 1$ leads to the total variation constraint or penalty [28], [29], $q = 2$ results in the usual $L_2$ gradient norm (Dirichlet energy) and $q = \infty$ yields a pointwise constraint on the gradient magnitude [30], [31]. In [32], proposed a similar penalty based on density discrepancy. Regularization for level-set methods can be found, e.g., in [33], where the primary goal of regularization is to smooth the velocity field and maintain smoothness of the level set. Detailed reviews of relevant literature can be found in two excellent surveys: [17] and [34].

In contrast to most previous methods, our approach aims to use regularization to control the error of the FEM solution directly, and in this way, increase robustness of the method, by potentially increasing the value of the discretized functional for the solution, but ensuring that it does not deviate too much from the underlying smooth functional. Our experiments demonstrate however, that in the context of hard-kill methods, this regularization actually results in a decrease of the (unregularized) functional.
value. This happens because approximation error introduced by finite elements methods creates false minimums for objective functional. First, greedy methods will likely to get stuck in those minimums. Second those false minimums depend on mesh and result in growth of functional on different or refined mesh. Regularization removes those false minimums and mesh dependence, making functional value being closer to its true physical value.

![Figure 8. Solution and error estimation for problem B.](image)

**Figure 8.** Solution and error estimation for problem B.

### 8. Conclusions and future work

Greedy methods based on sensitivity analysis are popular due to their ease of implementation on top of existing FEM packages. On the downside, the integer programming problems resulted from discretization are complex, and the convergence behavior is often unpredictable.

In our experiments we have demonstrated that our regularization applied to “hard-kill” method allows it achieve results close to those of a density-based method from\cite{8}, and a significant improvement compared to a previously proposed “hard-kill” method\cite{7}, with the functional value two times less.

The estimator used to regularize our functional requires additional finer mesh and solution of additional problem on that mesh. Although it is very easy to implement, this incurs an additional cost. In the future, we will explore regularization with a more efficient (e.g., adaptive) error estimator, along with applications of approach proposed in this paper to topology optimization in the context of elasticity as well as to 3D problems.

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