On eigenvalue problems related to the laplacian in a class of doubly connected domains

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Abstract
We study eigenvalue problems in some specific class of doubly connected domains. In particular, we prove the following.
1. Let $B_1$ be an open ball in $\mathbb{R}^n$, $n > 2$ and $B_0$ be an open ball contained in $B_1$. Then the first eigenvalue of the problem

\[
\Delta u = 0 \quad \text{in} \quad B_1 \setminus B_0, \\
u = 0 \quad \text{on} \quad \partial B_0, \\
\frac{\partial u}{\partial \nu} = \tau u \quad \text{on} \quad \partial B_1,
\]

attains its maximum if and only if $B_0$ and $B_1$ are concentric. Here $\nu$ is the outward unit normal on $\partial B_1$ and $\tau$ is a real number.

2. Let $B_0 \subset M$ be a geodesic ball of radius $r$ centered at a point $p \in M$, where $M$ denote either a non-compact rank-1 symmetric space $(\mathbb{M}, ds^2)$ with curvature $-4 \leq K_M \leq -1$ or $M = \mathbb{R}^m$. Let $D \subset M$ be a domain of fixed volume which is geodesically symmetric with respect to the point $p \in M$ such that $B_0 \subset D$. Then the first non-zero eigenvalue of

\[
\Delta u = \mu u \quad \text{in} \quad D \setminus B_0, \\
\frac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad \partial(D \setminus B_0),
\]

attains its maximum if and only if $D$ is a geodesic ball centered at $p$. Here $\nu$ represents the outward unit normal on $\partial(D \setminus B_0)$ and $\mu$ is a real number.

Keywords Laplacian · Neumann eigenvalue problem · Steklov–Dirichlet eigenvalue problem · Doubly connected domain · Non-compact rank-1 symmetric space · Geodesically symmetric domain

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1 Introduction

In the last few years, the study of an eigenvalue problem on a punctured domain has been a topic of interest. Several interesting results have been proved in this area by considering various boundary conditions on a punctured domain. In [9], Ramm and Shivakumar considered Dirichlet boundary condition on a punctured ball in $\mathbb{R}^2$ (a ball of smaller radius is removed from a ball) and proved that the first eigenvalue of this problem attains its maximum if and only if the balls are concentric. In [8], Kesavan proved the above result for higher dimensions. Later this result was extended to a wider class of domains [1,4–7]. In [3], Banuelos et al. proved some classical inequalities between the eigenvalues of the mixed Steklov–Dirichlet and mixed Steklov–Neumann eigenvalue problems on a bounded domain in $\mathbb{R}^n$.

In this paper, we consider the mixed Steklov–Dirichlet problem and Neumann eigenvalue problem on some specific class of punctured domains and prove that the first non-zero eigenvalue of both problems is maximal for annular domain. In between, we also give the characterization of the first non-zero eigenvalue of the Neumann problem on an annulus. To the best of our knowledge, this is the first attempt to consider these problems on such domains. Now we state the main result related to the mixed Steklov–Dirichlet problem.

For $n > 2$, let $B_1$ be an open ball in $\mathbb{R}^n$ of radius $R_1$ and $B'_2$ be an open ball in $\mathbb{R}^n$ of radius $R_2$ such that $B_1 \subset B'_2$, where $0 < R_1 < R_2$. Consider the class $O_{R_1,R_2} = \{ B'_2 \setminus B_1 \}$ of domains in $\mathbb{R}^n$. Then we prove the following result.

Theorem 1.1 Consider the eigenvalue problem

$$\begin{align*}
\Delta u &= 0 \quad \text{in } B'_2 \setminus B_1, \\
u &= 0 \quad \text{on } \partial B_1, \\
\frac{\partial u}{\partial v} &= \tau u \quad \text{on } \partial B'_2,
\end{align*}$$

where $\nu$ is the outward unit normal on $\partial B'_2$. Then annular domains (concentric balls) maximize the first eigenvalue of (1) in the class $O_{R_1,R_2}$.

Definition 1.1 A domain $D$ in a Riemannian manifold $M$ is said to be geodesically symmetric with respect to a point $p \in M$ if $D = \exp_p(N_0)$, where $N_0$ is a symmetric neighborhood of the origin in $T_p M$.

Next we state our result for the first non-zero eigenvalue of Neumann problem on doubly connected domains in $\mathbb{R}^m$ and non-compact rank-1 symmetric space $\mathbb{M}$ with curvature $-4 \leq K_{\mathbb{M}} \leq -1$.

Let $(\mathbb{M}, d^2)$ be a non-compact rank-1 symmetric space with dim $\mathbb{M} = m = kn$, where $k = \dim_{\mathbb{R}} K$; $K = \mathbb{R}, \mathbb{C}, \mathbb{H}$ or $\mathbb{C}a$, and metric $d^2$ is such that $-4 \leq K_{\mathbb{M}} \leq -1$. Let $M = \mathbb{M}$ or $\mathbb{R}^m$. Let $B_0 \subset M$ be a geodesic ball of radius $r_1$ centered at a point $p \in M$, and $D \subset M$ be a domain of fixed volume which is geodesically symmetric with respect to the point $p$ such that $\overline{B}_0 \subset D$. We consider the following eigenvalue problem on a domain $\Omega$ in the class $\mathcal{F} = \{ D \setminus B_0 \}$.
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\[ \Delta u = \mu u \quad \text{in} \quad \Omega, \]
\[ \frac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad \partial \Omega, \quad (2) \]

where \( \nu \) is the outward unit normal on \( \partial \Omega \).

**Theorem 1.2** Among all domains \( \Omega \subset M \) in the class \( \mathcal{F} \), annular domain (concentric ball) maximizes the first non-zero eigenvalue of (2).

The key idea to prove above results is to construct a suitable test function for the variational characterization of first non-zero eigenvalue for the corresponding eigenvalue problem.

This article is organized as follows. In Sect. 2, we consider the mixed Steklov–Dirichlet problem on an annulus and calculate a first eigenfunction by separation of variable technique. We also prove some inequalities which are important to prove Theorem 1.1 and then provide a proof of this theorem. In Sect. 3, first we study the Neumann eigenvalue problem on annulus and then prove Theorem 1.2.

**2 The mixed Steklov–Dirichlet problem**

We begin with the study of mixed Steklov–Dirichlet eigenvalue problem on an annular domain.

**2.1 The mixed Steklov–Dirichlet problem on an annular domain**

Let \( 0 < R_1 < R_2 \) and \( \Omega_0 = \{ x \in \mathbb{R}^n \mid R_1 < \| x \| < R_2 \} \) be an annular domain in \( \mathbb{R}^n \). Let \( B_1 \) and \( B_2 \) be open balls in \( \mathbb{R}^n \) centered at the origin and of radius \( R_1 \) and \( R_2 \), respectively. Consider the eigenvalue problem

\[ \Delta u = 0 \quad \text{in} \quad \Omega_0, \]
\[ u = 0 \quad \text{on} \quad \partial B_1, \]
\[ \frac{\partial u}{\partial r} = \tau u \quad \text{on} \quad \partial B_2. \quad (3) \]

Let \( \tau_1(\Omega_0) \) denote the first eigenvalue of (3). We compute the first eigenvalue and corresponding eigenfunction of (3) by using the separation of variable technique.

Consider a smooth function on \( \Omega_0 \) given by \( h(r, u) = f(r)g(u) \), where \( f \) and \( g \) are real valued functions defined on \([ R_1, R_2 ]\) and \( S^{n-1} \), respectively. Let \( S(r) \) be the sphere of radius \( r \) centered at the origin and \( g(u) \) is an eigenfunction of \( \Delta_{S(r)} \) with eigenvalue \( \lambda(S(r)) \). Then

\[ \Delta h(r, u) = g(u) \left[ -f''(r) - \frac{(n-1)}{r} f'(r) \right] + f(r) \Delta_{S(r)} g(u) \]
\[ = g(u) \left[ -f''(r) - \frac{(n-1)}{r} f'(r) + \lambda(S(r)) f(r) \right]. \]
Assume that \( h(r, u) \) is a solution of (3). Then the function \( f \) satisfies

\[
-f''(r) - \frac{(n-1)}{r} f'(r) + \lambda(S(r)) f(r) = 0,
\]
\[
f(R_1) = 0, \quad f'(R_2) = \tau f(R_2).
\]  

(4)

Let \( 0 = \lambda_0(S(r)) < \lambda_1(S(r)) \leq \lambda_2(S(r)) \ldots \not\to \infty \) be the eigenvalues of \( \Delta_S(r) \).

We will now prove that if \( \lambda(S(r)) = \lambda_0(S(r)) \) and \( \tau_0 \) is the corresponding eigenvalue of (4), then \( \tau_0 \) is the first eigenvalue of (3).

For \( i \geq 0 \), let \( f_i \) be functions satisfying

\[
-f_i'''(r) - \frac{(n-1)}{r} f_i'(r) + \lambda_i(S(r)) f_i(r) = 0,
\]
\[
f_i(R_1) = 0, \quad f_i'(R_2) = \tau_i f_i(R_2),
\]

(5)

where \( \tau_i \) is the first eigenvalue of (5) and \( f_i \) is an eigenfunction corresponds to \( \tau_i \). Thus \( f_i \) will not change sign in \([R_1, R_2]\). We may assume that \( f_i \)'s are positive functions for \( i \geq 0 \). The above equation is equivalent to

\[
r f_i'''(r) + (n-1) f_i'(r) - r \lambda_i(S(r)) f_i(r) = 0,
\]
\[
f_i(R_1) = 0, \quad f_i'(R_2) = \tau_i f_i(R_2).
\]

We will now prove that \( \tau_i \leq \tau_{i+1} \) for \( i \geq 0 \).

Let’s fix \( i \). Multiply the equation

\[
r f_i''' + (n-1) f_i' - r \lambda_i(S(r)) f_i = 0
\]

by \( f_{i+1} \) and the equation

\[
r f_{i+1}''' + (n-1) f_{i+1}' - r \lambda_{i+1}(S(r)) f_{i+1} = 0
\]

by \( f_i \). By subtracting the second equation from the first and using the fact \( \lambda_i(S(r)) \leq \lambda_{i+1}(S(r)) \), we get

\[
0 = r^{n-2} \left( f_{i+1} \left( r f_i''' + (n-1) f_i' - r \lambda_i(S(r)) f_i \right) - f_i \left( r f_{i+1}''' + (n-1) f_{i+1}' - r \lambda_{i+1}(S(r)) f_{i+1} \right) \right)
\]

\[
= r^{n-2} \left( r \left( f_i'' f_{i+1} - f_i'' f_{i+1} f_i \right) + (n-1) \left( f_i' f_{i+1} - f_i' f_{i+1} f_i \right) - r f_i f_{i+1} (\lambda_i(S(r)) - \lambda_{i+1}(S(r))) \right)
\]

\[
\geq r^{n-1} \left( f_i'' f_{i+1} - f_i'' f_{i+1} f_i \right) + (n-1) r^{n-2} \left( f_i' f_{i+1} - f_i' f_{i+1} f_i \right)
\]

\[
= \left( r^{n-1} \left( f_i' f_{i+1} - f_i' f_{i+1} f_i \right) \right)'.
\]

The condition \( \left( r^{n-1} \left( f_i' f_{i+1} - f_i' f_{i+1} f_i \right) \right)' \leq 0 \) shows that \( \left( r^{n-1} \left( f_i' f_{i+1} - f_i' f_{i+1} f_i \right) \right) \) is a decreasing function of \( r \). As a result,

\[
R_2^{n-1} \left( f_i'(R_2) f_{i+1}(R_2) - f_i'(R_2) f_i(R_2) \right)
\]

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\[
\leq R_1^{n-1} \left( f_i'(R_1) f_{i+1}(R_1) - f_{i+1}'(R_1) R_1 f_i(R_1) \right).
\]

Since \( f_i(R_1) = f_{i+1}(R_1) = 0 \), we have \( R_1^{n-1} \left( f_i'(R_2) f_{i+1}(R_2) - f_{i+1}'(R_2) f_i(R_2) \right) \leq 0 \). This gives \( f_i'(R_2) f_{i+1}(R_2) \leq f_{i+1}'(R_2) f_i(R_2) \). As a consequence, we have

\[
\tau_i = \frac{f_i'(R_2)}{f_i(R_2)} \leq \frac{f_{i+1}'(R_2)}{f_{i+1}(R_2)} = \tau_{i+1}.
\]

Thus \( \tau_0 \) is the smallest eigenvalue of (4) with eigenfunction \( f_0 \) and \( \lambda(S(r)) = \lambda_0(S(r)) \). Hence \( \tau_1(\Omega_0) = \tau_0 \).

Next we compute the explicit form of \( f_0 \). Since \( \lambda_0(S(r)) = 0 \), \( f_0 \) satisfies the equation \( r f_0''(r) + (n-1) f_0'(r) = 0 \). This can also be written as \( (r^{n-1} f_0'(r))' = 0 \). Using the fact \( f_0(R_1) = 0 \), we obtain

\[
f_0(r) = \int_{R_1}^r \frac{c}{r^{n-1}} dr
\]

for some constant \( c \).

**Remark 2.1** Since constant function is an eigenfunction of \( \Delta S(r) \) with eigenvalue zero, without loss of generality, we may assume it to be 1. Then the function \( f_0 \), the first eigenfunction of the eigenvalue problem (3), is given by

\[
f_0(r) = \begin{cases} 
\ln r - \ln R_1, & \text{for } n = 2, \\
\left( \frac{1}{R_1^{n-2}} - \frac{1}{r^{n-2}} \right), & \text{for } n \geq 3.
\end{cases}
\]

(6)

Note that the first eigenfunction of (3) is radial, positive and increasing function on \( \Omega_0 \).

### 2.2 Computations towards the proof of Theorem 1.1

In this section, we first prove some results needed to prove Theorem 1.1 and then present a proof of the theorem.

We fix some notations used in this subsection. To prove Theorem 1.1, without loss of generality, we may assume that \( B_1 \) is an open ball in \( \mathbb{R}^n \) of radius \( R_1 \) centered at the origin, and \( B'_2 \) is an open ball of radius \( R_2 \) such that \( 0 < R_1 < R_2 \) and \( \overline{B_1} \subseteq B'_2 \). In particular, we consider all domains \( D = B'_2 \setminus \overline{B_1} \) by fixing the ball \( B_1 \) and shifting the outer ball such that \( \overline{B_1} \subseteq B'_2 \). Then the variational characterization for the first eigenvalue of (1) is given by

\[
\tau_1 = \inf_{u \in H^1(D)} \left\{ \frac{\int_D \| \nabla u \|^2 dV}{\int_{\partial B_2'} u^2 d\nu} \mid u = 0 \text{ on } \partial B_1 \right\}.
\]

(7)
Since the Laplacian is invariant under the rotations and reflections, if the outer ball moves so that its center lies on a sphere centered at the origin, the first eigenvalue of (1) must remain same. Thus it is enough to consider that the center of outer ball moves along the x-axis. Let \( B'_2 \) and \( B_2 \) be the balls of radius \( R_2 \) in \( \mathbb{R}^n \) centered at \((x, 0, 0, \ldots, 0)\) and the origin, respectively and \( 0 < x \leq (R_2 - R_1) \). We denote \( \partial B'_2 \) by \( S'(R_2) \). The co-ordinates of \( S'(R_2) \) are given by
\[
x_1 = R_2 \cos(\phi_1) + x, \quad x_2 = R_2 \sin(\phi_1) \cos(\phi_2), \quad x_3 = R_2 \sin(\phi_1) \sin(\phi_2) \cos(\phi_3), \\
\vdots, \quad x_n = R_2 \sin(\phi_1) \sin(\phi_2) \cdots \sin(\phi_{n-2}) \cos(\phi_{n-1}),
\]
where \( \phi_1, \phi_2, \ldots, \phi_{n-2} \in [0, \pi] \) and \( \phi_{n-1} \in [0, 2\pi] \).

Let \( p \) be a point on \( S'(R_2) \) and \( r \) be the distance of \( p \) from the origin. Then
\[
r = \sqrt{R_2^2 + x^2 + 2R_2x \cos(\phi_1)}.
\]

**Definition 2.1** For \( a, b, c, z \in \mathbb{C} \), the hypergeometric function \( \, _2F_1(a, b; c; z) \) is defined as follows.

**Power series form:** If \( c \) is not a non-positive integer, and either \(|z| < 1\) or \(|z| = 1\) with \( \text{Re}(c - a - b) > 0 \). Then
\[
\, _2F_1(a, b; c; z) := \sum_{m=0}^{\infty} \frac{(a)_m (b)_m}{(c)_m} \frac{z^m}{m!},
\]
where \((q)_m\) is the (rising) Pochhammer symbol defined by
\[
(q)_m = \begin{cases} 1 & m = 0, \\ q(q+1) \cdots (q+m-1) & m > 0. \end{cases}
\]

**Integral representation:** If \( \text{Re}(c) > \text{Re}(b) > 0 \) and \( z \) is not a real number greater than or equal to 1,
\[
\, _2F_1(a, b; c; z) := \frac{\Gamma(c)}{\Gamma(b) \Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a} dt,
\]
where \( \Gamma(s) = \int_0^\infty e^{-t} t^{s-1} dt \).

We use MATHEMATICA \([10]\) in Lemmas 2.1 and 2.2 to compute some integrals and sum of some series.

**Lemma 2.1** Let \( x \) and \( R_2 \) be as above. Then for all \( k \in \mathbb{N} \),
\[
\int_{-1}^1 \left( \frac{(1-s^2)^k}{(R_2^2 + x^2 + 2R_2xs)^{2k-1}} \right) ds \geq \left( \frac{2k - 2}{2k - 1} \frac{2k - 4}{2k - 3} \cdots \frac{4}{3} \right) \left( \frac{1}{R_2^{4k-2}} \right). \quad (8)
\]

**Proof** By using MATHEMATICA, we get
\[
\int_{-1}^1 \left( \frac{(1-s^2)^k}{(R_2^2 + x^2 + 2R_2xs)^{2k-1}} \right) ds = \frac{\sqrt{\pi} \Gamma(k)}{(R_2^2 + x^2)^{2k-1} \Gamma(k + \frac{1}{2})}.
\]
\[ 2F_1 \left( k - \frac{1}{2}, k; k + \frac{1}{2}; \frac{4R_2^2x^2}{(R_2^2 + x^2)^2} \right). \] (9)

By substituting power series form of \( 2F_1 \left( k - \frac{1}{2}, k; k + \frac{1}{2}; \frac{4R_2^2x^2}{(R_2^2 + x^2)^2} \right) \) in the above expression, we obtain

\[
\int_{-1}^{1} \left( \frac{(1 - s^2)^{k-1}}{(R_2^2 + x^2 + 2R_2xs)^{2k-1}} \right) ds
\]

\[
= \frac{\sqrt{\pi}}{(R_2^2 + x^2)^{2k-1}} \frac{\Gamma(k)\Gamma(k + \frac{1}{2})}{\Gamma(k + 1)} \sum_{m=0}^{\infty} \frac{(k - \frac{1}{2})_m(k)_m}{(k + \frac{1}{2})_m m!} \left( \frac{4R_2^2x^2}{(R_2^2 + x^2)^2} \right)^m
\]

\[
= \frac{\sqrt{\pi}}{R_2^{2k-2}} \frac{\Gamma(k)\Gamma(k + \frac{1}{2})}{\Gamma(k + 1)} \sum_{m,l=0}^{\infty} \frac{4^m \Gamma(k + m)(k - \frac{1}{2})_m (-1)^l \Gamma(2m + 2k - 1 + l)}{\Gamma(k)(k - \frac{1}{2} + m) m! l!} \frac{\Gamma(i + 2k - 1 + l)}{\Gamma(2i + 2k - 1)}
\]

\[
\left( \frac{x}{R_2} \right)^{2l+2m}.
\] (10)

Then the coefficient of \( \frac{x^{2l}}{R_2^{2l + 4k - 2}} \) for \( t \geq 0 \) is given by

\[
\frac{\sqrt{\pi}}{\Gamma(k + \frac{1}{2})} \sum_{i=0}^{t} \frac{4^i \Gamma(k + i)(k - \frac{1}{2})_i (-1)^{i-t} \Gamma(i + 2k - 1 + t)}{\Gamma(k)(k - \frac{1}{2} + i) i!(t - i)!} \frac{\Gamma(i + 2k - 1 + t)}{\Gamma(2i + 2k - 1)}.
\]

Using MATHEMATICA, we have

\[
\sum_{i=0}^{t} \frac{4^i \Gamma(k + i)(k - \frac{1}{2})_i (-1)^{i-t} \Gamma(i + 2k - 1 + t)}{\Gamma(k)(k - \frac{1}{2} + i) i!(t - i)!} \frac{\Gamma(i + 2k - 1 + t)}{\Gamma(2i + 2k - 1)}
\]

\[
= \frac{(2k - 1) \Gamma(2k + t - 1)}{(2k + 2t - 1) \Gamma(2k - 1) \Gamma(t + 1)}.
\]

We denote the coefficient of \( \frac{x^{2l}}{R_2^{2l + 4k - 2}} \) by \( \alpha(t) \). Then (10) can be written as

\[
\int_{-1}^{1} \left( \frac{(1 - s^2)^{k-1}}{(R_2^2 + x^2 + 2R_2xs)^{2k-1}} \right) ds = \sum_{t=0}^{\infty} \alpha(t) \frac{x^{2t}}{R_2^{2t + 4k - 2}}.
\]
Note that \( x, R_2 > 0 \) and \( \alpha(t) \) is positive for all \( t \geq 0 \). Thus

\[
\int_{-1}^{1} \left( \frac{(1 - s^2)^{k-1}}{\left(R_2^2 + x^2 + 2R_2xs\right)^{2k-1}} \right) ds \geq \alpha(0) \left( \frac{1}{R_2^{4k-2}} \right),
\]

where

\[
\alpha(0) = \frac{\sqrt{\pi}}{\Gamma(k)} \frac{\Gamma(k + \frac{1}{2})}{\Gamma(2k - 1) \Gamma(2k - 1)} = \left( \frac{2k - 2}{2k - 3} \frac{2k - 4}{2k - 3} \cdots \frac{4}{3} \right).
\]

This proves the lemma. \( \Box \)

**Lemma 2.2** Let \( x \) and \( R_2 \) be as above. Then for all \( k \in \mathbb{N} \),

\[
\int_{-1}^{1} \left( \frac{(1 - s^2)^{k-1}}{\left(R_2^2 + x^2 + 2R_2xs\right)^{2k-1}} \right) ds = \left( \frac{2k - 2}{2k - 1} \frac{2k - 4}{2k - 3} \cdots \frac{4}{3} \right) \left( \frac{1}{R_2^{4k-1}} \right). \tag{11}
\]

**Proof** By using MATHEMATICA, we get

\[
\int_{-1}^{1} \left( \frac{(1 - s^2)^{k-1}}{\left(R_2^2 + x^2 + 2R_2xs\right)^{2k-1}} \right) ds = \frac{\sqrt{\pi}}{\Gamma(k)} \frac{\Gamma(k)}{(R_2^2 + x^2)^{2k-1}} \Gamma(k + \frac{1}{2})
\]

\[
\sum_{m=0}^{\infty} \frac{\Gamma\left(\frac{2k-1}{4} + m\right) \Gamma\left(\frac{2k+1}{4} + m\right) \Gamma\left(k + \frac{1}{2} + m\right) \Gamma\left(k + \frac{1}{2} + m\right)\Gamma\left(k + \frac{1}{2} + m\right)}{R_2^{2m+2k-1}}.
\tag{12}
\]

By substituting the power series for the hypergeometric function in the above expression, we get

\[
\int_{-1}^{1} \left( \frac{(1 - s^2)^{k-1}}{\left(R_2^2 + x^2 + 2R_2xs\right)^{2k-1}} \right) ds
\]

\[
= \frac{\sqrt{\pi}}{\Gamma(k)} \frac{\Gamma\left(k + \frac{1}{2}\right)}{\Gamma\left(\frac{2k-1}{4}\right) \Gamma\left(\frac{2k+1}{4}\right) \Gamma\left(k + \frac{1}{2} + m\right) \Gamma\left(k + \frac{1}{2} + m\right)\Gamma\left(k + \frac{1}{2} + m\right)\Gamma\left(k + \frac{1}{2} + m\right)}\left(\frac{4R_2^2x^2}{(R_2^2 + x^2)^2}\right)^{m}
\]

\[
= \frac{\sqrt{\pi}}{\Gamma(k)} \frac{\Gamma\left(k + \frac{1}{2}\right)}{\Gamma\left(\frac{2k-1}{4}\right) \Gamma\left(\frac{2k+1}{4}\right) \Gamma\left(k + \frac{1}{2} + m\right) \Gamma\left(k + \frac{1}{2} + m\right)\Gamma\left(k + \frac{1}{2} + m\right)\Gamma\left(k + \frac{1}{2} + m\right)}\left(\frac{4R_2^2x^2}{(R_2^2 + x^2)^2}\right)^{m}
\]

\[
= \frac{\sqrt{\pi}}{\Gamma(k)} \frac{\Gamma\left(k + \frac{1}{2}\right)}{\Gamma\left(\frac{2k-1}{4}\right) \Gamma\left(\frac{2k+1}{4}\right) \Gamma\left(k + \frac{1}{2} + m\right) \Gamma\left(k + \frac{1}{2} + m\right)\Gamma\left(k + \frac{1}{2} + m\right)\Gamma\left(k + \frac{1}{2} + m\right)}\left(\frac{4R_2^2x^2}{(R_2^2 + x^2)^2}\right)^{m}
\]

\[
= \frac{\sqrt{\pi}}{\Gamma(k)} \frac{\Gamma\left(k + \frac{1}{2}\right)}{\Gamma\left(\frac{2k-1}{4}\right) \Gamma\left(\frac{2k+1}{4}\right) \Gamma\left(k + \frac{1}{2} + m\right) \Gamma\left(k + \frac{1}{2} + m\right)\Gamma\left(k + \frac{1}{2} + m\right)\Gamma\left(k + \frac{1}{2} + m\right)}\left(\frac{4R_2^2x^2}{(R_2^2 + x^2)^2}\right)^{m}
\]
\begin{align*}
\left(1 + \frac{x^2}{R_2^2}\right)^{-\frac{4m+2k-1}{2}} &= \left(\sum_{m,l=0}^{\infty} \frac{4^m \Gamma\left(\frac{2k-1}{4} + m\right) \Gamma\left(\frac{2k+1}{4} + m\right) \Gamma\left(k + \frac{1}{2}\right) \Gamma\left(k + \frac{1}{2} + m\right) (-1)^l \frac{1}{\Gamma\left(\frac{4m+2k-1}{2} + l\right)}}{\Gamma\left(\frac{2k-1}{4}\right) \Gamma\left(\frac{2k+1}{4}\right) \Gamma\left(k + \frac{1}{2} + m\right) m!l!} \right) \\
\times \frac{\sqrt{\pi} \Gamma(k)}{R_2^{2k-1} \Gamma\left(k + \frac{1}{2}\right)} &= \sum_{t=0}^{\infty} \beta(t) \frac{x^{2t}}{R_2^{2t+2k-1}},
\end{align*}

where \(\beta(t), t \geq 0\), is given by

\[\beta(t) = \begin{cases} 
\frac{\sqrt{\pi} \Gamma(k)}{\Gamma\left(k + \frac{1}{2}\right)} & \text{for } t = 0, \\
0 & \text{for } t \neq 0.
\end{cases}\]

We simplify the above expression using MATHEMATICA and get

\[\beta(t) = \begin{cases} 
\frac{\sqrt{\pi} \Gamma(k)}{\Gamma\left(k + \frac{1}{2}\right)} & \text{for } t = 0, \\
0 & \text{for } t \neq 0.
\end{cases}\]

Hence

\[\int_{-1}^{1} \left(\frac{(1 - s^2)^{k-1}}{R_2^{2k-1} + x^2 + 2 R_2 s x s^{\frac{2k-1}{2}}}\right) ds = \beta(0) \frac{1}{R_2^{2k-1}}
\]

\[= \left(\frac{2k - 2}{2k - 1} \frac{2k - 4}{2k - 3} \ldots \frac{4}{3}\right) \left(\frac{1}{R_2^{2k-1}}\right).\]

We denote the area element \(\sin^{2k-1}(\phi_1) \sin^{2k-2}(\phi_2) \ldots \sin(\phi_{2k-1}) d\phi_1 d\phi_2 \ldots d\phi_{2k}\) of \(S'(R_2)\) by \(d\phi\) in the following proposition.
Proposition 2.1 Let $R_1$ and $R_2$ be defined as above and $r(p)$ denotes the distance of a point $p$ in $S'(R_2)$ from the origin. For $n = 2k + 1, k \geq 1$, the following holds.

$$
\int_{S'(R_2)} \left( \frac{1}{R_1^{2k-1}} - \frac{1}{r^{2k-1}} \right)^2 dv \geq \int_{S(R_2)} \left( \frac{1}{R_1^{2k-1}} - \frac{1}{R_2^{2k-1}} \right)^2 dv. \tag{14}
$$

Proof By substituting $r = \sqrt{R_2^2 + x^2 + 2 R_2 x \cos(\phi_1)}$ and using the polar coordinate system, we have

$$
\int_{S'(R_2)} \left( \frac{1}{R_1^{2k-1}} - \frac{1}{r^{2k-1}} \right)^2 dv
= \int_0^{2\pi} \cdots \int_0^{\pi} \int_0^{\pi} \left( \frac{1}{R_1^{2k-1}} - \frac{1}{(R_2^2 + x^2 + 2 R_2 x \cos(\phi_1))^{2k-1}} \right)^2 R_2^{2k} d\phi
= 4\pi \frac{\pi}{2} \frac{4}{3} \cdots \left( \frac{2k - 3}{2k - 2} \frac{2k - 5}{2k - 4} \cdots \frac{\pi}{2} \right) R_2^{2k}
\times \sin^{2k-1}(\phi_1) d\phi_1
$$

and

$$
\int_{S(R_2)} \left( \frac{1}{R_1^{2k-1}} - \frac{1}{R_2^{2k-1}} \right)^2 dv
= \int_0^{2\pi} \cdots \int_0^{\pi} \int_0^{\pi} \int_0^{\pi} \left( \frac{1}{R_1^{2k-1}} - \frac{1}{R_2^{2k-1}} \right)^2 R_2^{2k} d\phi
= 4\pi \frac{\pi}{2} \frac{4}{3} \cdots \left( \frac{2k - 3}{2k - 2} \frac{2k - 5}{2k - 4} \cdots \frac{\pi}{2} \right) R_2^{2k}
\times \left( \frac{2k - 2}{2k - 1} \frac{2k - 4}{2k - 3} \cdots \frac{4}{3} \right) \left( \frac{1}{R_1^{2k-1}} - \frac{1}{R_2^{2k-1}} \right)^2.
$$

Thus (14) is equivalent to

$$
\int_0^{\pi} \left( \frac{1}{R_1^{2k-1}} - \frac{1}{(R_2^2 + x^2 + 2 R_2 x \cos(\phi_1))^{2k-1}} \right)^2 \sin^{2k-1}(\phi_1) d\phi_1
\geq \left( \frac{2k - 2}{2k - 1} \frac{2k - 4}{2k - 3} \cdots \frac{4}{3} \right) \left( \frac{1}{R_1^{2k-1}} - \frac{1}{R_2^{2k-1}} \right)^2. \tag{15}
$$
Let \( s = \cos(\phi_1) \), then the above inequality reduces to

\[
\int_{-1}^{1} \left( \frac{1}{R_1^{2k-1}} - \frac{1}{(R_2^2 + x^2 + 2R_2xs)^{2k-1}} \right)^2 (1-s^2)^{k-1} ds
\]

\[
\geq \left( \frac{2k-2}{2k-1} \frac{2k-4}{2k-3} \cdots \frac{4}{3} \right) \left( \frac{1}{R_1^{2k-1}} - \frac{1}{R_2^{2k-1}} \right)^2.
\]

This is equivalent to

\[
\int_{-1}^{1} \left( \frac{1}{(R_2^2 + x^2 + 2R_2xs)^{2k-1}} \right)^2 (1-s^2)^{k-1} ds \geq \left( \frac{2k-2}{2k-1} \frac{2k-4}{2k-3} \cdots \frac{4}{3} \right) \left( \frac{1}{R_2^{4k-2}} - \frac{2}{R_2^{2k-1}R_2^{2k-1}} \right),
\]

which follows from Lemmas 2.1 and 2.2.

**Lemma 2.3** Let \( x \) and \( R_2 \) be as above and \( k \in \mathbb{N}, k \geq 2 \). Then the following holds.

\[
\int_{-1}^{1} \left( \frac{1-s^2}{(R_2^2 + x^2 + 2R_2xs)^{2(k-1)}} \right)^2 ds \geq \left( \frac{2k-3}{2k-2} \frac{2k-5}{2k-4} \cdots \frac{2}{3} \right) \left( \frac{1}{R_2^{4k-4}} \right).
\]

**Proof** By using MATHEMATICA, we get

\[
\int_{-1}^{1} \left( \frac{1-s^2}{(R_2^2 + x^2 + 2R_2xs)^{2(k-1)}} \right)^2 ds = \left( \frac{\sqrt{\pi}}{(R_2^2 + x^2)^{2k-2}} \frac{\Gamma(k - \frac{1}{2})}{\Gamma(k)} \right)
\]

\[
\times {}_2F_1 \left( k - 1, k - \frac{1}{2}; k; \frac{4R_2^2x^2}{(R_2^2 + x^2)^2} \right),
\]

\[
= \sum_{t=0}^{\infty} \hat{a}(t) \frac{x^{2t}}{R_2^{2t+4k-4}},
\]

where \( \hat{a}(t), t \geq 0 \) is given by

\[
\hat{a}(t) = \frac{\sqrt{\pi} \Gamma(k - \frac{1}{2})}{\Gamma(k)} \sum_{i=0}^{t} \frac{4^i \Gamma(k + i - \frac{1}{2}) (k - 1) (-1)^i}{\Gamma(k - \frac{1}{2}) (k - 1 + i) i!(t-i)!} \Gamma(i + 2k - 2 + t)
\]

\[
= \frac{\sqrt{\pi} \Gamma(k - \frac{1}{2})}{\Gamma(k)} \frac{(k-1) \Gamma(2k-2+t)}{\Gamma(2k-2) \Gamma(1+t)(k+t-1)}.
\]
The sum of the above series is evaluated by using MATHEMATICA.

Observe that $R, x > 0$ and $\hat{\alpha}(t) > 0$ for all $t \geq 0$. Then it follows by (18) and (19) that

$$\int_{-1}^{1} \left( \frac{(1 - s^2)^{2k-3}}{(R_2^2 + x^2 + 2R_2x s)^{2(k-1)}} \right) ds \geq \hat{\alpha}(0) \frac{1}{R_2^{4k-4}}$$

$$= \left( \frac{2k - 3}{2k - 2} \frac{2k - 5}{2k - 4} \cdots \frac{\pi}{2} \right) \frac{1}{R_2^{4k-4}}.$$ 

This proves the lemma. □

**Lemma 2.4** Let $x$ and $R$ be as above and $k \in \mathbb{N}, k \geq 2$. Then the following holds.

$$\int_{-1}^{1} \left( \frac{(1 - s^2)^{2k-3}}{(R_2^2 + x^2 + 2R_2x s)^{k-1}} \right) ds = \left( \frac{2k - 3}{2k - 2} \frac{2k - 5}{2k - 4} \cdots \frac{\pi}{2} \right) \left( \frac{1}{R_2^{2(k-1)}} \right).$$  \tag{20}

**Proof** By proceeding the same way as in Lemma 2.3, we get

$$\int_{-1}^{1} \left( \frac{(1 - s^2)^{2k-3}}{(R_2^2 + x^2 + 2R_2x s)^{k-1}} \right) ds = \left( \frac{\sqrt{\pi}}{(R_2^2 + x^2)^{k-1}} \right) \frac{\Gamma(k - \frac{1}{2})}{\Gamma(k)}$$

$$\times \mathrm{2F}_1 \left( k - \frac{1}{2}, \frac{k}{2}; k; \frac{4R_2^2x^2}{(R_2^2 + x^2)^2} \right)$$

$$= \sum_{t=0}^{\infty} \hat{\beta}(t) \frac{x^{2t}}{R_2^{2t+2k-2}},$$  \tag{21}

where $\hat{\beta}(t)$ is given by

$$\hat{\beta}(t) = \frac{\sqrt{\pi}}{\Gamma(k - \frac{1}{2})} \sum_{i=0}^{t} \frac{4^i \Gamma\left( k - \frac{1}{2} + i \right) \Gamma\left( \frac{k}{2} + i \right) \Gamma\left( k \right)}{\Gamma\left( k - \frac{1}{2} \right) \Gamma\left( \frac{k}{2} + i \right) \Gamma\left( k + i \right) \Gamma(2i + k - 1)} \Gamma(i + k + t - 1)$$

$$\times \Gamma(2i + k - 1).$$

$$= \begin{cases} 
\frac{\sqrt{\pi}}{\Gamma(k)} \hat{\beta}(0), & \text{for } t = 0, \\
0, & \text{for } t \neq 0.
\end{cases} \tag{22}$$

Hence the lemma follows by substituting the values of $\hat{\beta}(t)$ from (22) into (21). □

**Proposition 2.2** Let $R_1$ and $R_2$ be defined as above and $r(p)$ be the distance of a point $p$ in $S'(R_2)$ from the origin. For $n = 2k, k \geq 2$ the following holds.

$$\int_{S'(R_2)} \left( \frac{1}{R_1^{2(k-1)}} - \frac{1}{r^{2(k-1)}} \right)^2 dv \geq \int_{S(R_2)} \left( \frac{1}{R_1^{2(k-1)}} - \frac{1}{R_2^{2(k-1)}} \right)^2 dv. \tag{23}$$
**Proof** By proceeding in the same way as in Proposition 2.1, inequality (23) is equivalent to
\[
\int_{-1}^{1} \left( \frac{1}{R_2^2 + x^2 + 2R_2xs} \right)^{2(k-1)} - \frac{2}{R_1^2(2(k-1)) \left( \frac{1}{R_2^2 + x^2 + 2R_2xs} \right)^{k-1}} \left( 1 - s^2 \right)^{\frac{2k-3}{2}} ds \geq \left( \frac{2k-3}{2k-5} \frac{2}{2k-4} \cdots \frac{\pi}{2} \right) \left( \frac{1}{R_2^{4(k-1)}} - \frac{2}{R_1^{2(k-1)} R_2^{2(k-1)}} \right).
\]
This follows from Lemmas 2.3 and 2.4.

\[\square\]

### 2.3 Proof of Theorem 1.1

**Proof** For \( n > 2 \), since \( \left( \frac{1}{R_1^{n-2}} - \frac{1}{r^{n-2}} \right) = 0 \) on \( S(R_1) \), the function \( \left( \frac{1}{R_1^{n-2}} - \frac{1}{r^{n-2}} \right) \) satisfies the condition for the test function in (7). Thus by considering \( \left( \frac{1}{R_1^{n-2}} - \frac{1}{r^{n-2}} \right) \) as test function, we have
\[
\tau_1(D) \leq \frac{\int_D \left\| \nabla \left( \frac{1}{R_1^{n-2}} - \frac{1}{r^{n-2}} \right) \right\|^2 dV}{\int_{S'(R_2)} \left( \frac{1}{R_1^{n-2}} - \frac{1}{r^{n-2}} \right) dv}.
\]
By Proposition 2.1 and 2.2, we have
\[
\int_{S'(R_2)} \left( \frac{1}{R_1^{n-2}} - \frac{1}{r^{n-2}} \right)^2 dv \geq \int_{S(R_2)} \left( \frac{1}{R_1^{n-2}} - \frac{1}{r^{n-2}} \right)^2 dv,
\]
for all \( n > 2 \).

Next we estimate \( \int_D \left\| \nabla \left( \frac{1}{R_1^{n-2}} - \frac{1}{r^{n-2}} \right) \right\|^2 dV \).
\[
\int_D \left\| \nabla \left( \frac{1}{R_1^{n-2}} - \frac{1}{r^{n-2}} \right) \right\|^2 dV = \int_D \left( \frac{n-2}{r^{n-1}} \right)^2 dV
\]
\[
= \int_{D \cap \Omega_0} \left( \frac{n-2}{r^{n-1}} \right)^2 dV + \int_{D \setminus D \cap \Omega_0} \left( \frac{n-2}{r^{n-1}} \right)^2 dV.
\]
Since $r \geq R_2$ in $(D \setminus D \cap \Omega_0)$, we have
\[
\int_D \left\| \nabla \left( \frac{1}{R_1^{n-2}} - \frac{1}{r^{n-2}} \right) \right\|^2 dV \leq \int_{\Omega_0} \left( \frac{n-2}{r^{n-1}} \right)^2 dV - \int_{\Omega_0 \setminus D \cap \Omega_0} \left( \frac{n-2}{r^{n-1}} \right)^2 dV + \int_{D \setminus D \cap \Omega_0} \left( \frac{n-2}{R_2^{n-1}} \right)^2 dV.
\]

Since $\text{Vol}(\Omega_0 \setminus D \cap \Omega_0) = \text{Vol}(D \setminus D \cap \Omega_0)$, it follows that
\[
\int_D \left\| \nabla \left( \frac{1}{R_1^{n-2}} - \frac{1}{r^{n-2}} \right) \right\|^2 dV \leq \int_{\Omega_0} \left( \frac{n-2}{r^{n-1}} \right)^2 dV + \int_{\Omega_0 \setminus D \cap \Omega_0} \left[ \left( \frac{n-2}{R_2^{n-1}} \right)^2 - \left( \frac{n-2}{r^{n-1}} \right)^2 \right] dV.
\]

As $r \leq R_2$ in $(\Omega_0 \setminus D \cap \Omega_0)$, we get
\[
\int_D \left\| \nabla \left( \frac{1}{R_1^{n-2}} - \frac{1}{r^{n-2}} \right) \right\|^2 dV \leq \int_{\Omega_0} \left( \frac{n-2}{r^{n-1}} \right)^2 dV = \int_{\Omega_0} \left( \frac{1}{R_1^{n-2}} - \frac{1}{r^{n-2}} \right)^2 dV. \tag{25}
\]

By inequality (24) and (25), we have
\[
\frac{\int_D \left\| \nabla \left( \frac{1}{R_1^{n-2}} - \frac{1}{r^{n-2}} \right) \right\|^2 dV}{\int_{S(R_2)} \left( \frac{1}{R_1^{n-2}} - \frac{1}{r^{n-2}} \right)^2 d\nu} \leq \frac{\int_{\Omega_0} \left\| \nabla \left( \frac{1}{R_1^{n-2}} - \frac{1}{r^{n-2}} \right) \right\|^2 dV}{\int_{S(R_2)} \left( \frac{1}{R_1^{n-2}} - \frac{1}{r^{n-2}} \right)^2 d\nu},
\]

\[
\tau_1(D) \leq \tau_1(\Omega_0).
\]

Further, equality holds if and only if $\text{Vol}(\Omega_0 \setminus D \cap \Omega_0) = 0$. Since $\Omega_0$ and $D$ are open subsets of $\mathbb{R}^n$, $\text{Vol}(\Omega_0 \setminus D \cap \Omega_0) = 0$ holds if and only if $D = \Omega_0$. Hence the theorem follows.

\[ \square \]

**Remark 2.2** For $n = 2$, we failed to prove above theorem since we don’t have an analogue of (23) if we consider $\ln r - \ln R_1$ as a test function for the variational characterization of first eigenvalue.
3 Neumann eigenvalue problem

We prove Theorem 1.2 in this section. Throughout this section, \((\mathbb{M}, ds^2)\) denotes a non-compact rank-1 symmetric space with \(\dim \mathbb{M} = m = kn\) where \(k = \dim_{\mathbb{R}} K\); \(K = \mathbb{R}, \mathbb{C}, \mathbb{H}\) or \(\mathbb{C}a\), and metric \(ds^2\) is such that \(-4 \leq K_M \leq -1\). For a domain \(\Omega\), the variational characterization for the first non-zero eigenvalue of (2) is given by

\[
\mu_1 = \inf \left\{ \frac{\int_\Omega \|\nabla u\|^2 \, dv}{\int_\Omega |u|^2 \, dv} \mid \int_\Omega u \, dv = 0 \right\}. \tag{26}
\]

We start with the study of first non-constant eigenfunction of eigenvalue problem (2) on an annulus \(\Omega_0 \subset M\), where \(M = \mathbb{M}\) or \(\mathbb{R}^m\). Then we find test functions for a geodesically symmetric doubly connected domain \(\Omega \subset M\), and obtain an upper bound for the first non-zero eigenvalue of Neumann problem on \(\Omega\).

Let \(p \in M\). If \(\gamma(r)\) is a unit speed geodesic starting at \(p\), then the Riemannian volume density at \(\gamma(r)\) is

\[
J(r) = \begin{cases} \sinh^{kn-1} r \cosh^{k-1} r & \text{for } M = \mathbb{M}, \\ r^{m-1} & \text{for } M = \mathbb{R}^m, \end{cases}
\]

and the first eigenvalue \(\lambda_1(S(r))\) of the geodesic sphere \(S(p, r)\) is

\[
\lambda_1(S(r)) = \begin{cases} \left(\frac{kn-1}{\sinh^{2} r} - \frac{k-1}{\cosh^{2} r}\right) & \text{for } M = \mathbb{M}, \\ \frac{m-1}{r^2} & \text{for } M = \mathbb{R}^m. \end{cases}
\]

For a proof see [2].

3.1 Neumann eigenvalue problem on an annulus

Let \(\Omega_0 = \{ q \in M \mid r_1 < d(p, q) < r_2 \}\) be an annular domain centered at point \(p \in M\). Consider the following problem

\[
\begin{align*}
\Delta u &= \mu \, u \quad \text{in } \Omega_0, \\
\frac{\partial u}{\partial \nu} &= 0 \quad \text{on } \partial \Omega_0,
\end{align*} \tag{27}
\]

where \(\nu\) is the outward unit normal on \(\partial \Omega_0\). We will study the first non-zero eigenvalue \(\mu_1(\Omega_0)\) of (27).

The first non-zero eigenvalue of (27) is, by separation of variables technique, either the second eigenvalue \(\tau_2\) of

\[
-\frac{1}{J(r)} \frac{\partial}{\partial r} \left( J(r) \frac{\partial f}{\partial r} \right) = \tau f, \tag{28}
\]
where \( f \) is defined on \([r_1, r_2]\) with \( f'(r_1) = 0 = f'(r_2) \) or the first eigenvalue \( \mu_1 \) of
\[
- \frac{1}{J(r)} \frac{\partial}{\partial r} \left( J(r) \frac{\partial}{\partial r} g \right) + \lambda_1(S(r)) g = \mu g,
\]
where \( \lambda_1(S(r)) \) is the first non-zero eigenvalue of Laplacian on \( S(r) \) corresponding to eigenfunctions \( \frac{x_i}{r} \), where \((x_1, x_2, \ldots, x_m)\) is the normal coordinate system centered at \( p \) and the function \( g \) is defined on \([r_1, r_2]\) with \( g'(r_1) = 0 = g'(r_2) \).

Let \( g \) be an eigenfunction corresponding to the first eigenvalue \( \mu_1 \) of (29). Then \( g \) does not change sign in \((r_1, r_2)\). We may assume that \( g \) is positive on \((r_1, r_2)\). Let \( f \) be an eigenfunction corresponding to the second eigenvalue \( \tau_2 \) of (28). Then the function \( f \) must change sign in \((r_1, r_2)\). Then \( f'(a) < 0 \). Let \( h \) be a non-trivial solution of
\[
- \frac{1}{J(r)} \frac{\partial}{\partial r} \left( J(r) \frac{\partial}{\partial r} h \right) = \mu_1 h.
\]
By differentiating above equation, we get
\[
- \frac{1}{J(r)} \frac{\partial}{\partial r} \left( J(r) \frac{\partial}{\partial r} h' \right) + \lambda_1(S(r)) h' = \mu_1 h'.
\]
Since \( h' \) and \( g \) satisfy (29) with the same eigenvalue, we may assume that \( h' = g \). Using (30) and the fact that \( f \) satisfies (28) with eigenvalue \( \tau_2 \), we get
\[
\frac{\partial}{\partial r} \left( J(r) \left( fh' - hf' \right) \right) = (\tau_2 - \mu_1) f h J(r).
\]
By integrating above equation and using the values of \( f \), we obtain
\[
\int_{r_1}^{a} (\tau_2 - \mu_1) f h J(r) = \left[ J(r) \left( fh' - hf' \right) \right]_{r_1}^{a} = -J(a)h(a) f'(a) - J(r_1) f(r_1) h'(r_1).
\]
Since
\[
\mu_1 h(r_2) = -g'(r_2) - \frac{J'(r_2)}{J(r_2)} g(r_2), \quad g'(r_2) = 0
\]
and \( g \) is positive in the interval \((r_1, r_2)\), we have \( h(r_2) < 0 \). As \( h' = g \) and \( g \) is a positive function, it follows that \( h \) is an increasing function. So \( h \leq 0 \) in \((r_1, r_2)\). Then from (32) it follows that \( \tau_2 > \mu_1 \). Thus \( \mu_1 = \mu_1(\Omega_0) \).

**Remark 3.1** Let \( g(r) \) be the solution of (29) and \((x_1, x_2, \ldots, x_m)\) be the geodesic normal coordinates centered at \( p \). Then the function \( g(r) \frac{x_i}{r} \), \( 1 \leq i \leq m \) are eigenfunctions of (27) corresponding to the first non-zero eigenvalue.
We now prove that \( g \) is an increasing function on \( (r_1, r_2) \).

**Lemma 3.1** \( g'(r) > 0 \) on \( (r_1, r_2) \) and \( \mu_1(\mathcal{O}_0) > \lambda_1(S(r_2)) \).

**Proof** The function \( g \) satisfies the equation

\[
\frac{\partial}{\partial r} \left( J(r) \frac{\partial}{\partial r} g \right) = (\lambda_1(S(r)) - \mu_1(\mathcal{O}_0)) g(r) J(r) \tag{33}
\]

with \( g'(r_1) = 0 = g'(r_2) \). Define \( \Psi(r) = J(r) g'(r) \). Then \( \Psi(r_1) = 0 = \Psi(r_2) \).

Since \( \Psi \) can not be identically zero, \( \Psi'(r) \) must change sign at least once in the interval \( (r_1, r_2) \). Hence \( (\lambda_1(S(r)) - \mu_1(\mathcal{O}_0)) \) must change sign at least once in the interval \( (r_1, r_2) \). Since \( \lambda_1(S(r)) \) is a strictly decreasing function, \( (\lambda_1(S(r_1)) - \mu_1(\mathcal{O}_0)) < 0 \) implies \( (\lambda_1(S(r)) - \mu_1(\mathcal{O}_0)) < 0 \) for all \( r \in (r_1, r_2) \). This contradicts the fact that \( (\lambda_1(S(r)) - \mu_1(\mathcal{O}_0)) \) change sign at least once in the interval \( (r_1, r_2) \). Thus \( (\lambda_1(S(r)) - \mu_1(\mathcal{O}_0)) \) must be positive on an interval \( (r_1, b) \) for some point \( b \in (r_1, r_2) \) and negative on \((b, r_2)\). Hence \( \mu_1(\mathcal{O}_0) > \lambda_1(S(r_2)) \).

Further, \( \Psi'(r) \) is positive on the interval \( (r_1, b) \) and negative on \((b, r_2)\). Since \( \Psi \) is an increasing function in \( (r_1, b) \) and a decreasing function in \((b, r_2)\), it follows that \( \Psi(r) \) is positive on \((r_1, r_2)\). Hence \( g'(r) > 0 \) on \((r_1, r_2)\). \( \square \)

### 3.2 Neumann eigenvalue problem on geodesically symmetric domain in \( \mathbb{R}^m \) and non-compact Rank-1 symmetric spaces

We prove Theorem 1.2 in this subsection. We begin with the following lemma, which is crucial to prove the theorem.

Let \( g \) be the first non-constant eigenfunction of \( (29) \) on \([r_1, r_2]\). Let \( G(r) = (g'(r))^2 + \lambda_1(S(r)) g^2(r) \).

**Lemma 3.2** \( G'(r) \leq 0 \) for \( r \in [r_1, r_2] \).

**Proof** We prove lemma for \( M = \mathbb{R}^m \) and \( M = \mathbb{M} \). For \( M = \mathbb{R}^m \), \( G(r) \) takes the form \( (g'(r))^2 + \frac{m-1}{r^2} g^2(r) \). Then

\[
G'(r) = 2 g'(r) \left[ \left( \frac{m-1}{r^2} - \mu_1(\mathcal{O}_0) \right) g(r) - \frac{m-1}{r} g'(r) \right] \\
+ 2 g(r) g'(r) \left( \frac{m-1}{r^2} \right) \\
- 2 g^2(r) \left( \frac{m-1}{r^3} \right),
\]

\[
\leq 4 \left( \frac{m-1}{r^2} \right) g(r) g'(r) - 2 \left( \frac{m-1}{r^3} \right) g^2(r) - 2 \left( \frac{m-1}{r} \right) g^2(r) \leq 0.
\]

For \( M = \mathbb{M} \), \( G(r) \) takes the form \( (g'(r))^2 + \left( \frac{kn-1}{\sinh^2 r} - \frac{k-1}{\cosh^2 r} \right) g^2(r) \). Then

\[
G'(r) = 2 g'(r) \left( (\lambda_1(S(r)) - \mu_1(\mathcal{O}_0)) g(r) - \frac{J'(r)}{J(r)} g'(r) \right)
\]
\[ +2 g(r) g'(r) \left( \frac{kn - 1}{\sinh^2 r} - \frac{k - 1}{\cosh^2 r} \right) \]
\[-2 g^2(r) \left( \frac{(kn - 1) \cosh r}{\sinh^3 r} - \frac{(k - 1) \sinh r}{\cosh^3 r} \right) \]
\[= 4 g(r) g'(r) \left( \frac{kn - 1}{\sinh^2 r} - \frac{k - 1}{\cosh^2 r} \right) \]
\[-2 g(r) g'(r) \mu_1(\Omega_0) - 2 (g'(r))^2 \left( \frac{(kn - 1) \cosh r}{\sinh r} \right) \]
\[+ \frac{(k - 1) \sinh r}{\cosh r} \right) - 2 g^2(r) \left( \frac{(kn - 1) \cosh r}{\sinh^3 r} - \frac{(k - 1) \sinh r}{\cosh^3 r} \right). \]

\[G'(r) \leq \frac{2(kn - 1)}{\sinh^3 r} \left( 2 g(r) g'(r) \sinh r - (g'(r))^2 \cosh r \sinh^2 r - g^2(r) \cosh r \right) \]
\[-\frac{2(k - 1)}{\cosh^3 r} \left( 2 g(r) g'(r) \cosh r + (g'(r))^2 \sinh r \cosh^2 r - g^2(r) \sinh r \right) \]
\[\leq -2(k - 1) \left( \frac{8 g(r) g'(r)}{\sinh^2 2r} + (g'(r))^2 \frac{2 \cosh 2r}{\sinh 2r} + g^2(r) \frac{8 \cosh 2r}{\sinh^3 2r} \right) \]
\[\leq -\frac{4(k - 1)}{\sinh^3 2r} \left( 2 g(r) - g'(r) \sinh 2r \right)^2 \]
\[\leq 0. \]

This proves the lemma. \(\square\)

**Proof (Proof of Theorem 1.2)** Let \(D\) be a domain in \(M\) which is geodesically symmetric with respect to a point \(p \in M\) and \(B_0\) be a ball of radius \(r_1\) contained in \(D\) with center at \(p\). Let \(B(r_2)\) be a ball in \(M\) of radius \(r_2\), centered at \(p\) such that \(\text{Vol}(D) = \text{Vol}(B(r_2))\). Define \(\Omega_0 = (B(r_2)) \setminus B_0\).
Define the function $h$ on $\Omega = D \setminus \overline{B_0}$ by

$$h(x) = \begin{cases} g(\|x\|), & \text{for } \|x\| \leq r_2 \\ g(r_2), & \text{for } \|x\| \geq r_2. \end{cases}$$

We observe that $h(x)$ is a radial function and denote it by $h(r)$. Since domain $D$ is geodesically symmetric with respect to the point $p$, it follows that $\int_{\Omega} h(r) \frac{x_i}{r} \, d\nu = 0$, where $(x_1, x_2, \ldots, x_m)$ is the normal coordinate system centered at point $p$. Since $h(r) \frac{x_i}{r}$ satisfies the property of the test function in (26), the variational formulation of $\mu_1(\Omega)$ gives

$$\mu_1(\Omega) \sum_{i=1}^m \int_{\Omega} \left( h(r) \frac{x_i}{r} \right)^2 \, d\nu \leq \sum_{i=1}^m \int_{\Omega} \| \nabla \left( h(r) \frac{x_i}{r} \right) \|^2 \, d\nu,$$

$$\mu_1(\Omega) \int_{\Omega} h^2(r) \, d\nu \leq \sum_{i=1}^m \int_{\Omega} \| \nabla \left( h(r) \frac{x_i}{r} \right) \|^2 \, d\nu. \quad (34)$$

Next we estimate $\int_{\Omega} h^2(r) \, d\nu$ and $\sum_{i=1}^m \int_{\Omega} \| \nabla \left( h(r) \frac{x_i}{r} \right) \|^2 \, d\nu$.

**Estimate for $\int_{\Omega} h^2(r) \, d\nu$:**

$$\int_{\Omega} h^2(r) \, d\nu = \int_{\Omega_0 \cap \Omega} h^2(r) \, d\nu + \int_{\Omega \setminus (\Omega_0 \cap \Omega)} h^2(r) \, d\nu$$

$$= \int_{\Omega_0} h^2(r) \, d\nu - \int_{\Omega_0 \setminus (\Omega_0 \cap \Omega)} h^2(r) \, d\nu + \int_{\Omega \setminus (\Omega_0 \cap \Omega)} h^2(r) \, d\nu$$

$$= \int_{\Omega_0} g^2(r) \, d\nu - \int_{\Omega_0 \setminus (\Omega_0 \cap \Omega)} g^2(r) \, d\nu + \int_{\Omega \setminus (\Omega_0 \cap \Omega)} g^2(r_2) \, d\nu.$$

Since $\text{Vol}(\Omega_0 \setminus (\Omega_0 \cap \Omega)) = \text{Vol}(\Omega \setminus (\Omega_0 \cap \Omega))$, it follows that

$$\int_{\Omega} h^2(r) \, d\nu = \int_{\Omega_0} g^2(r) \, d\nu + \int_{\Omega_0 \setminus (\Omega_0 \cap \Omega)} \left( g^2(r_2) - g^2(r) \right) \, d\nu$$

$$\geq \int_{\Omega_0} g^2(r) \, d\nu, \quad (35)$$

where the last inequality follows from the fact that $g$ is an increasing function on $[r_1, r_2]$.

**Estimate for $\sum_{i=1}^m \int_{\Omega} \| \nabla \left( h(r) \frac{x_i}{r} \right) \|^2 \, d\nu$:**

Note that

$$\sum_{i=1}^m \int_{\Omega} \| \nabla \left( h(r) \frac{x_i}{r} \right) \|^2 \, d\nu = \int_{\Omega} \left( (h'(r))^2 + \lambda_1(S(r))h^2(r) \right) \, d\nu. \quad (36)$$
Let \( G(r) = (h'(r))^2 + \lambda_1(S(r))h^2(r) \). Note that \( G(r) \) is a decreasing function on \([r_1, r_2]\). For \( r \geq r_2 \), \( h(r) \) is a constant function and \( \lambda_1 S(r) \) is a decreasing function of \( r \). This implies that \( G(r) \) is a decreasing function on \( r \geq r_2 \). Thus (36) can be written as

\[
\int_{\Omega} G(r) \, dv = \int_{\Omega_0} G(r) \, dv - \int_{\Omega_0 \setminus (\Omega_0 \cap \Omega)} G(r) \, dv + \int_{\Omega \setminus (\Omega_0 \cap \Omega)} G(r) \, dv.
\]

As \( G(r) \leq G(r_2) \) on \( \Omega \setminus (\Omega_0 \cap \Omega) \) and \( \text{Vol}(\Omega_0 \setminus \Omega_0 \cap \Omega) = \text{Vol}(\Omega \setminus \Omega_0 \cap \Omega) \), we get

\[
\int_{\Omega} G(r) \, dv \leq \int_{\Omega_0} G(r) \, dv + \int_{\Omega_0 \setminus (\Omega_0 \cap \Omega)} (G(r_2) - G(r)) \, dv.
\]

Since \( G(r) \geq G(r_2) \) on \( \Omega_0 \setminus (\Omega_0 \cap \Omega) \), it follows that

\[
\int_{\Omega} G(r) \, dv \leq \int_{\Omega_0} G(r) \, dv.
\]

By substituting above values in (34), we have

\[
\mu_1(\Omega) \leq \frac{\sum_{i=1}^m \int_{\Omega_0} \|\nabla \left(g(r) \frac{\xi_i}{r}\right)\|^2 \, dv}{\int_{\Omega_0} g^2(r) \, dv} = \mu_1(\Omega_0).
\]

Further, equality holds if and only if \( \text{Vol}(\Omega_0 \setminus \Omega_0 \cap \Omega) = 0 \), which is true if and only if \( \Omega = \Omega_0 \). \( \square \)

**Remark 3.2** In the case of compact rank-1 symmetric space, we don’t have an analogue of Lemma 3.2. Hence the above proof fails for the compact rank-1 symmetric spaces.

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