**Abstract.** Berglund-Hübsch duality is an example of mirror symmetry between orbifold Landau-Ginzburg models. In this paper we study a D-module-theoretic variant of Borisov’s proof of Berglund-Hübsch duality. In the $p$-adic case, the D-module approach makes it possible to endow the orbifold chiral rings with the action of a non-trivial Frobenius endomorphism. Our main result is that the Frobenius endomorphism commutes with Berglund-Hübsch duality up to an explicit diagonal operator.

1. Introduction

Berglund-Hübsch duality was originally introduced [BH] as a generalization of the Greene-Plesser construction [GP] of mirror pairs. Let $W(x) \in \mathbb{C}[x] = \mathbb{C}[x_1, \ldots, x_n]$ be an invertible polynomial defining a Calabi-Yau hypersurface $X$ and let $G \subset (\mathbb{C}^*)^n$ be a group fixing $W$. Then the Berglund-Hübsch dual of the orbifold of $X$ by $G$ is the hypersurface $X^T$, defined by the “transpose” invertible polynomial $W^T(x) \in \mathbb{C}[x]$, orbifolded by an explicitly constructed group $G^T \subset (\mathbb{C}^*)^n$ fixing $W^T$. As shown in [K2] and [K], the construction of Berglund and Hübsch can be further generalized to Landau-Ginzburg models with invertible potentials (not necessarily of Calabi-Yau type) as follows. For any invertible polynomial $W(x)$ the bigraded chiral ring of the orbifold Landau-Ginzburg model $(W(x), G)$ is isomorphic to the (twisted) chiral ring of the orbifold LG model $(W^T(x), G^T)$.

In the context of the vertex-algebra approach to mirror symmetry [B], Borisov [B2] has shown that, as an isomorphism of bigraded vector spaces (that is, disregarding the multiplicative structure), Berglund-Hübsch duality can be lifted to the level of chains. Let $\mathbb{C}[x, y_0]$ be the quotient of $\mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n]$ by the ideal $\langle x_1y_1, \ldots, x_ny_n \rangle$ and let $\wedge(\mathbb{C}^n)$ be the standard exterior representation of the Clifford algebra with generators $e_i, e_i^\vee$ and relations $e_ie_j^\vee + e_j^\vee e_i = \delta_{ij}$ for all $i, j = 1, \ldots, n$. Borisov’s construction hinges on the differential

$$\delta_\infty = \sum_{i=1}^n x_i \partial_{x_i} W(x) \otimes e_i + \sum_{i=1}^n y_i \otimes e_i^\vee$$

acting on $\mathbb{C}[x, y_0] \otimes \wedge(\mathbb{C}^n)$. As shown in [B2], $(\mathbb{C}[x, y_0] \otimes \wedge(\mathbb{C}^n), \delta_\infty)$ contains a copy of the standard Koszul resolution of the Milnor ring $\mathbb{C}[x]/dW$ in such a way that the inclusion is a quasi-isomorphism. The starting point for this paper is to deform $\delta_\infty$ to

$$\delta_x = \sum_{i=1}^n (x_i \partial_{x_i} + \pi x_i \partial_{x_i} W(x)) \otimes e_i + \sum_{i=1}^n (y_i \partial_{y_i} + \pi y_i) \otimes e_i^\vee,$$

where $\pi \in \mathbb{C}^*$ is an arbitrary constant. As it turns out, the complex $(\mathbb{C}[x, y_0] \otimes \wedge(\mathbb{C}^n), \delta_x)$ contains a copy of the de Rham complex of the D-module $\mathbb{C}[x] e^{\pi W(x)}$. 

2. Deformed Berglund-Hübsch Duality

In this section we study a deformed version of Berglund-Hübsch duality. Let $\pi \in \mathbb{C}^*$ be an arbitrary constant. We consider the D-module $\mathbb{C}[x] e^{\pi W(x)}$, which is endowed with the action of the differential $\delta_x$.

3. Frobenius Endomorphism

In this section we study the Frobenius endomorphism of the orbifold chiral rings. We show that the Frobenius endomorphism commutes with Berglund-Hübsch duality up to an explicit diagonal operator.
The quasi-isomorphism (see e.g. [M]) between the latter and the Milnor ring allows us to provide an alternate chain-level realization of Berglund-Hübsch duality. More precisely, our method yields a chain-level proof of the “total unprojected” (in the terminology of [K]) Berglund-Hübsch duality, from which the usual “projected” duality of [B2] easily follows by restriction to the invariant sectors as in [K].

The key difference between our construction and [B2] emerges if one replaces \( \mathbb{C}[x] \) with the ring \( \mathbb{C}_p[[x]] \) of \( p \)-adic overconvergent power series. While the de Rham cohomology of the D-module \( \mathbb{C}_p[[x]]e^{\pi W(x)} \) (where now \( \pi \) is a fixed \((p-1)\)-th root of \( -p \)) is still isomorphic to the \( p \)-adic Milnor ring, the de Rham chain model has extra structure: a non-trivial Frobenius endomorphism which descends to cohomology. In this paper we show that the Frobenius endomorphisms extends naturally to a chain map \( Fr \) acting on the full chain complex \( \mathbb{C}_p[[x, y]]0 \otimes \bigwedge(\mathbb{C}_p^n) \). It is then natural to ask how the Frobenius endomorphism interacts with the Berglund-Hübsch duality quasi-isomorphism \( \Delta \). Our main result is that, at the level of cohomology, \( \Delta \) and \( Fr \) commute up to an explicit diagonal operator whose entries are non-negative integer powers of \( p \).

The interplay between the cohomological Frobenius and Berglund-Hübsch duality was first noticed in [P] and used to explore some arithmetic consequences of Berglund-Hübsch duality in the spirit of [W]. The present work originated as an attempt to understand the results of [P] at the level of chains. We hope further investigate the arithmetic implications of our construction in future work.

This paper is organized as follows. In Section 2 we review some basic facts about invertible polynomials \( W_A(x) \) over a field \( \mathbb{F} \) defined by a matrix \( A \). In Section 3 and Section 4 we introduce our “de Rham” modification \( \mathcal{B}_A(\mathbb{F}) \) of Borisov’s complex. In Section 5 we point out that \( \mathcal{B}_A(\mathbb{F}) \) is the total complex of a \( \mathbb{Z} \times \mathbb{Z} \)-bigraded bicomplex. In Section 6 we show that \( \mathcal{B}_A(\mathbb{F}) \) is quasi isomorphic to the de Rham cohomology of a certain D-module. To do this we follow the analogous argument given by Borisov in [B2]. However, the bigrading of [B2] is no longer preserved by our differentials and this is why we need the bigrading introduced in Section 3 instead. In Section 7 we prove that \( \mathcal{B}_A(\mathbb{F}) \) is quasi-isomorphic to a subcomplex \( \mathcal{C}_A(\mathbb{F}) \) which is in turn canonically isomorphic to \( \mathcal{C}_{AT}(\mathbb{F}) \). Together with the results of Section 5, this proves unprojected Berglund-Hübsch duality. In Sections 8 and 9 we specialize to the \( p \)-adic case and observe that the constructions of the previous sections can be extended by replacing polynomials with overconvergent \( p \)-adic power series. While not changing cohomology, this allows for the extra room needed in order to define a natural chain-level Frobenius endomorphism \( Fr \) à la Dwork (see e.g. [M], [SS]) whose compatibility with Berglund-Hübsch duality is then addressed. Finally, in Section 10 we illustrate our constructions by working out two simple examples.

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2. Invertible Polynomials

Let $\mathbb{F}$ be a field and consider the map
\[
W: \text{GL}_n(\mathbb{Z}_{\geq 0}) \to \mathbb{F}[x] = \mathbb{F}[x_1, \ldots, x_n]
\]
defined by
\[
A \mapsto W_A(x) = \sum_{i=1}^{n} x^{e_i A},
\]
where $\{e_i\}_{1 \leq i \leq n}$ is the standard basis of $\mathbb{Z}^n$, and for $v = (v_1, \ldots, v_n) \in \mathbb{Z}^n_{\geq 0}$, we write $x^v = x_1^{v_1} \cdots x_n^{v_n}$. For simplicity, we assume that $\text{char} \mathbb{F} = 0$ or $\text{char} \mathbb{F} > \det A$. A matrix $A \in \text{GL}_n(\mathbb{Z}_{\geq 0})$ is Berglund-Hübsch over $\mathbb{F}$ if $W_A(x)$ is an invertible polynomial, i.e., $W_A(x)$ is quasi-homogeneous and $(\partial_1 W_A(x), \ldots, \partial_n W_A(x))$ is a regular sequence in $\mathbb{F}[x]$. For each $n \in \mathbb{Z}_{\geq 0}$ we let
\[
\text{BH}(\mathbb{F}) = \bigcup_n \text{BH}_n(\mathbb{F}),
\]
where
\[
\text{BH}_n(\mathbb{F}) = \{ A \in \text{GL}_n(\mathbb{Z}_{\geq 0}) \mid A \text{ is Berglund-Hübsch over } \mathbb{F} \}.
\]

**Remark 2.1.** Berglund-Hübsch matrices satisfy the following properties.

1. If $A \in \text{BH}_n(\mathbb{F})$ and $B \in \text{BH}_m(\mathbb{F})$, then $A \oplus B \in \text{BH}_{n+m}(\mathbb{F})$.
2. If
\[
A = \begin{bmatrix} A & B \\ 0 & C \end{bmatrix} \in \text{BH}(\mathbb{F}),
\]
then $C \in \text{BH}(\mathbb{F})$. We call $A \in \text{BH}_n(\mathbb{F})$ irreducible if it cannot be written as $B \oplus C$ with $B, C \in \bigcup_{m \leq n} \text{BH}_m(\mathbb{F})$.

3. Let $W_n \subseteq \text{GL}_n(\mathbb{Z}_{\geq 0})$ be the Weyl group. Given $S \in W_n$ and $A \in \text{BH}_n(\mathbb{F})$, then $S A, A S \in \text{BH}_n(\mathbb{F})$. Moreover,
\[
W_{S A}(x) = W_A(x) \quad \text{and} \quad W_{A S}(x) = W_A(x) \cdot S,
\]
where $\cdot$ denotes the right action of $W_n$ on $\mathbb{F}[x]$ by permutation of the variables.

**Remark 2.2.** Define the group of scaling symmetries of $A \in \text{BH}_n(\mathbb{F})$ to be $G_A = \mathbb{Z}^n / \mathbb{Z}^n A^T$. If $\mathbb{F}$ contains a primitive $(\det A)$-th root of unity $\zeta$, one can consider the $\mathbb{Z}_{\geq 0}^n$-action on $\mathbb{F}[x]$ defined for $\lambda \in \mathbb{Z}_{\geq 0}^n$ by
\[
\lambda \cdot x^\gamma = \zeta^{\lambda^T \gamma} x^\gamma. \tag{2.1}
\]
Under this action $\lambda \cdot W_A(x) = W_A(x)$ if and only if $\lambda A^T \in (\det A)\mathbb{Z}^n$, which provides a canonical identification between $G_A$ and the stabilizer of $W_A(x)$ under the action (2.1). Unless otherwise stated, we represent equivalence classes in $G_A$ by their canonical representatives in $\mathbb{Z}^n \cap ([0, 1]n A^T)$. Using this identification, for each $\lambda \in G_A$ we introduce a vector $J^\lambda \in \mathbb{Z}^n$ defined by
\[
(J^\lambda)_i = \begin{cases} 
0 & \text{if } \zeta^{\lambda_i} = 1; \\
1 & \text{otherwise,}
\end{cases}
\]
and the submatrix $A^\lambda$ of $A$ such that $W_{A^\lambda}(x)$ is obtained from $W_A(x)$ by setting $x_i = 0$ whenever $(J^\lambda)_i = 1$. 
Proposition 2.3 ([K2]). Let $A \in BH_n(F)$ be irreducible. Then there exists $S \in W_n$ such that $W_{AS}(x)$ is in one of the following canonical forms:

1. a loop,
   \[x_1^{a_1}x_2 + x_2^{a_2}x_3 + \ldots + x_{n-1}^{a_{n-1}}x_n + x_n^{a_n}x_1,\]
2. a chain,
   \[x_1^{a_1}x_2 + x_2^{a_2}x_3 + \ldots + x_{n-1}^{a_{n-1}}x_n + x_n^{a_n}.\]

Corollary 2.4. Let $A \in BH_n(F)$. Then

1. $A^T \in BH_n(F)$,
2. for each $\lambda \in G_A$, we have $A^\lambda \in BH_{n-\sum |J^\lambda|}(F)$, and
3. the matrix defined by
   \[A^{\text{orb}} := \bigoplus_{\lambda \in G_A} A^\lambda\]
   is in $BH_{n|G_A|-\sum |J^\lambda|}(F)$.

Corollary 2.5. Let $A \in BH_n(F)$ and let $\beta \in \mathbb{Z}^n$ such that $\beta A^{-1}$ is in one of the following canonical forms:

1. if $A$ is a chain, then $(\beta A^{-1})_{ij} \in \mathbb{Q} \setminus \mathbb{Z}$ for all $1 \leq j \leq i \leq k \leq n$.
2. if $A$ is a loop, then $(\beta A^{-1})_{ij} \in \mathbb{Q} \setminus \mathbb{Z}$ for all $1 \leq j, k \leq n$.

Proof. Both statements follow from
   \[A^T_i(\beta A^{-T})_i + (\beta A^{-T})_{i+1} = \beta_i = (\beta A^{-1})_{i-1} + A_{ii}(\beta A^{-1})_i,\]
where $i$ is considered modulo $n$ in the case of loops.

3. Exterior Operators

Let $e_1, \ldots, e_n$ be the standard generators of $F^n$. We denote by $\bigwedge(F^n)$ the exterior algebra $\bigwedge(Fe_1 \oplus \ldots \oplus Fe_n)$ viewed as a representation of the Clifford algebra $Cl_n(F)$ with generators $e_i$ (multiplication) and $e_i^\gamma$ (contraction), and (odd) commutators $[e_i, e_j] = \delta_{ij}$ for all $1 \leq i, j \leq n$. As an $F$-module, $\bigwedge(F^n)$ is generated by monomials $e^I = e_i^1 \cdots e_i^n$, where $I = (I_1, \ldots, I_n) \in \mathbb{Z}_0^n$. In particular, $e^I = 0$ if and only if $I_i \geq 2$ for some $i$. Given $A \in BH_n(F)$ and $\pi \in F^*$, for $1 \leq i \leq n$ we also consider
   \[E_{A,i}^\pi = \frac{1}{\pi} \sum_{j=1}^n e_j A_{ji}^T \quad \text{and} \quad E_{A,i}^\pi = \frac{1}{\pi} \sum_{j=1}^n e_j^\pi (A^{-1})_{ji},\]
so that
   \[[E_{A,i}^\pi E_{A,j}^\pi] = \sum_{k,m} A_{ki}^T (A^{-1})_{mj} e_k^\pi e_m^\pi = \sum_k A_{ik} (A^{-1})_{kj} = \delta_{ij}.\]

Lemma 3.1. If $\ast^A \in GL(\bigwedge(F^n))$ is defined by
   \[\ast^A(e_{i_1}, \ldots, e_{i_k}) = E_{A,1}^{\pi_{i_1}} E_{A,2}^{\pi_{i_2}} \cdots E_{A,n}^{\pi_{i_k}} (E_{A,1} E_{A,2} \cdots E_{A,n}) ,\]
then

1. $\ast^A E_{A,i} = e_i^\pi \ast^A$, $\ast^A E_{A,i} = e_i \ast^A$, and
2. $\ast^A \ast^A$ commutes with the action of $Cl_n(F)$ on $\bigwedge(F^n)$. 

Proof. By definition,
\[ *^A e_i = E^\vee_{A^T,i} *^A \quad \text{and} \quad *^A e_i = E_{A^T,i} *^A. \]
Therefore,
\[ *^A E_{A,i} = *^A \pi \sum_j e_j A^T_{ji} = \pi \sum_j E^\vee_{A^T,j} A^T_{ji} *^A = \sum_{k,j} e_k (A^{-T})_{kj} A^T_{ji} *^A e_i *^A. \]
Similarly, \( *^A E^\vee_{A,i} = e_i *^A. \) This proves part (1). Part (2) follows from
\[ *^{A^T} *^A e_i = *^{A^T} E^\vee_{A^T,i} *^A = e_i *^{A^T} *^A \]
and
\[ *^{A^T} *^A e_i = *^{A^T} E_{A,i} *^A = e_i *^{A^T} *^A. \]

\[ \square \]

Remark 3.2. The operator
\[ \text{ext} = \sum_{i=1}^n e_i e_i^\vee = \sum_{i=1}^n E_{A,i} E^\vee_{A,i} \]
is diagonal on \( \mathcal{L}(\mathbb{F}^n) \). If \( \text{char} \mathbb{F} = 0 \), its eigenvalues count the total exterior degree. Moreover,
\[ *^A \text{ext} = \sum_{i=1}^n e_i^\vee e_i *^A = (n \text{Id} - \text{ext}) *^A. \]

4. The Basic Complex

Given a graded vector space \( V \) endowed with a differential \( d \) of degree 1, we denote by \( (V, d) \) the corresponding chain complex and by \( H(V, d) \) its cohomology. If \( V \) is bigraded and \( d, d' \) are graded commutative differentials of bidegree \((1,0)\) and \((0,1)\) respectively, we denote the corresponding bicomplex by \( (V, d, d') \) and by \( H(V, d, d') \) its total cohomology. If \( V \) is vector space acted upon by a collection of commuting endomorphisms \( \phi_1, \ldots, \phi_n \), we denote the corresponding Koszul complex by \( \text{Kos}(V, \phi_1, \ldots, \phi_n) \).

Given \( A \in \text{BH}_n(\mathbb{F}) \), consider the subring \( \widetilde{R}_A(\mathbb{F}) \) of \( \mathbb{F}[x_1, \ldots, x_n, y_1, \ldots, y_n] \) generated by monomials \( x^\gamma y^\lambda \) such that \( (\lambda A^{-T})_i \geq 0 \) for all \( 1 \leq i \leq n \). We define \( \mathcal{R}_A(\mathbb{F}) \) to be the quotient of \( \widetilde{R}_A(\mathbb{F}) \) by the ideal generated by monomials \( x^\gamma y^\lambda \) for which \( \gamma A^{-1} \lambda T > 0 \). Given \( \pi \in \mathbb{F}^* \), we define \( \theta_{A,i}, T^\gamma_{A,i}, \psi^\gamma_{A,i}, \varphi_{A,i}, \in \text{End}_x(\mathcal{R}_A(\mathbb{F})) \) by the formulas
\[
\theta_{A,i}(x^\gamma y^\lambda) = \gamma_i x^\gamma y^\lambda ;
\]
\[
T^\gamma_{A,i}(x^\gamma y^\lambda) = \pi^{-1}(\lambda A^{-T})_i x^\gamma y^\lambda ;
\]
\[
\psi^\gamma_{A,i}(x^\gamma y^\lambda) = x^\gamma y^{\lambda + c_i A^T} ;
\]
\[
\varphi_{A,i}(x^\gamma y^\lambda) = \pi (\theta_i W_A(x)) x^\gamma y^\lambda = \pi \sum_{j=1}^n A_{ji} x^{\gamma + c_j A} y^\lambda .
\]
We also define the odd linear endomorphisms of \( \mathcal{R}_A(\mathbb{F}) \otimes \mathcal{L}(\mathbb{F}^n) \)
\[ d_{A,i} = (\theta_{A,i} + \varphi_{A,i}) e_i , \quad d_A = \sum_{i=1}^n d_{A,i} \]
as well as
\[
\begin{align*}
    d'_{A,i} &= (T'_{A,i} + \psi'_{A,i})e_i^\vee, \quad d''_A = \sum_{i=1}^n d'_{A,i}.
\end{align*}
\]

**Lemma 4.1.** \(B_A(F) = (R_A(F) \otimes \bigwedge (F^n), d_A + d'_A)\) is a chain complex.

*Proof.* The morphism \(d_A\) is the Koszul differential for the sequence
\[
(\theta_{A,1} + \varphi_{A,1}, \theta_{A,2} + \varphi_{A,2}, \ldots, \theta_{A,n} + \varphi_{A,n})
\]
of commuting operators acting on \(R_A(F)\). Therefore, \([d_A, d_A] = 0\), and similarly \([d''_A, d''_A] = 0\).
Moreover, since \((\theta_{A,i} + \varphi_{A,i})\) and \((T'_{A,j} + \psi'_{A,j})\) commute,
\[
\begin{align*}
    [d_{A,i}, d'_{A,j}] &= [(\theta_{A,i} + \varphi_{A,i}) e_i, (T'_{A,j} + \psi'_{A,j}) e_j^\vee] \\
    &= (\theta_{A,i} + \varphi_{A,i}) (T'_{A,j} + \psi'_{A,j}) [e_i, e_j^\vee] \\
    &= (\theta_{A,i} + \varphi_{A,i}) (T'_{A,j} + \psi'_{A,j}) \delta_{ij}.
\end{align*}
\]

If \(0 \neq (\theta_{A,i} T'_{A,i})(x^\gamma y^\lambda) = \gamma_i (A^{-1} \lambda^T ; x^\gamma y^\lambda)\),
then \(x^\gamma y^\lambda = 0\) in \(R_A(F)\) and thus \((\theta_{A,i} + \varphi_{A,i}) (T'_{A,j} + \psi'_{A,j}) = 0\). For
\[
(\varphi_{A,i} T'_{A,i})(x^\gamma y^\lambda) = \sum_{j=1}^n A_{ji} (\lambda A^{-T})_i x^{\gamma + e_i^\vee} y^\lambda
\]
we note that if for some \(j\) we have \(A_{ji}, (\lambda A^{-T})_i > 0\), then
\[
(\gamma + e_j A)A^{-1} \lambda^T > (e_j A)(A^{-1} \lambda^T) = \sum_{m=1}^n A_{jm} (A^{-1} \lambda^T)_m > A_{ji} (A^{-1} \lambda^T)_i > 0
\]
and conclude as before that \(x^{\gamma + e_i^\vee} y^\lambda = 0\) in \(R_A(F)\). It is similarly shown that
\(\varphi_{A,i} \psi_{A,j} = 0\) and \(\theta_{A,i} \psi_{A,j} = 0\). Therefore, \([d_A, d'_A] = 0\). \(\square\)

**Remark 4.2.** Note that for any monomial \(x^\gamma y^\lambda\), the vector \(\lambda\) encodes a group element of \(G_A\) by \([21]\). For \(\lambda \in G_A\) we take \(0 \leq (\lambda A^{-T})_i < 1\) for each \(i\), so
\(\gamma A^{-1} \lambda^T = 0\) means that \(\gamma_i = 0\) if \(\lambda\) acts non-trivially on \(x_i\), that is, if \((J^\lambda)_i = 1\).

5. **Bigrading**

Let \(P'_{A,i} \in \text{End}_F (R_A(F))\) be given by
\[
P'_{A,i}(x^\gamma y^\lambda) = \begin{cases} 0, & \text{if } (\lambda A^{-T})_i = 0; \\ x^{\gamma} y^{\lambda}, & \text{otherwise}. \end{cases}
\]

**Lemma 5.1.** Let \(Q_{A,i}, Q'_{A,i}, Q_{A}\) and \(Q'_{A}\) be linear endomorphisms of \(R_A(F) \otimes \bigwedge (F^n)\) defined by
\[
Q'_{A,i} = P'_{A,i} e_i^\vee e_i \quad \text{and} \quad Q_{A,i} = e_i^\vee e_i + Q'_{A,i}
\]
as well as
\[
Q_A = \sum_{i=1}^n Q_{A,i} \quad \text{and} \quad Q'_A = \sum_{i=1}^n Q'_{A,i}.
\]

Then for each \(1 \leq i, j \leq n\),
\[
\begin{align*}
(1) \quad [Q_{A,i}, Q'_{A,j}] &= 0, \\
(2) \quad [Q'_{A,i}, d_{A,j}] &= 0 \quad \text{and} \quad [Q_{A,i}, d_{A,j}] = \delta_{ij} d_{A,j},
\end{align*}
\]
\(\langle 3 \rangle\) \([Q_{A,i}, d'_{A,j}] = 0\) and \([Q'_{A,i}, d'_{A,j}] = \delta_{ij}d'_{A,j}\).

**Proof.** The operators \(Q_{A,i}\) and \(Q'_{A,j}\) commute because they have monomials of the form \(x^\gamma y^\lambda e^f\) as a common basis of eigenvectors, which proves (1). For (2)

\[
[Q'_{A,i}, d'_{A,j}] = \left[ P'_{A,i}, e_i, (\theta_{A,j} + \varphi_{A,j}) e_j \right] = P'_{A,i} (\theta_{A,j} + \varphi_{A,j}) [e_i e_j, e_j] = \delta_{ij} P'_{A,i} (\theta_{A,j} + \varphi_{A,j}) e_j.
\]

The proof of Lemma (1) shows that \(P'_{A,i} \varphi_{A,i} = 0\). Similarly, \(P'_{A,i} \theta_{A,i} (x^\gamma y^\lambda) \neq 0\) implies that \(\gamma_i (A^{-1} X^T) \neq 0\) so that the corresponding term is 0 in \(\mathcal{R}_A(\mathbb{F})\). Therefore, \([Q'_{A,i}, d'_{A,j}] = 0\), which in turn implies that

\[
[Q_{A,i}, d'_{A,j}] = (\theta_{A,j} + \varphi_{A,j}) [e_i e_j, e_j] = \delta_{ij} d'_{A,j}.
\]

For part (3), if \(i \neq j\)

\[
[P'_{A,i}, T'_{A,j} + \psi_{A,j}] = 0 = [e_i e_j, e_j],
\]

and otherwise \(e_i e_j e_i = e_i e_j\), which means that

\[
[Q'_{A,i}, d'_{A,j}] = \delta_{ij} P'_{A,j} (T'_{A,j} + \psi_{A,j}) e_j = \delta_{ij} d'_{A,j}.
\]

Similarly,

\[
[Q_{A,i}, d'_{A,j}] = [e_i e_j, (T'_{A,j} + \psi_{A,j}) e_j] + \delta_{ij} d'_{A,j} = \delta_{ij} (T'_{A,j} + \psi_{A,j}) e_j e_j + \delta_{ij} d'_{A,j} = 0.
\]

\(\square\)

**Remark 5.2.** In particular, with respect to the Spec\((Q_A) \times\) Spec\((Q'_{A})\) bigrading, \(B_A(\mathbb{F})\) is the total complex of the bicomplex \((\mathcal{R}_A(\mathbb{F}) \otimes \wedge(\mathbb{F}^n), d_A, d'_{A})\).

### 6. Unprojected Orbifold de Rham Cohomology

Given \(A \in \mathbf{BH}_n(\mathbb{F})\) and \(\pi \in \mathbb{F}^*\), let

\[M_A(\mathbb{F}) = \mathbb{F}[x] e^{\pi W_A(x)}\]

be the module over the Weyl algebra \(\mathbb{F}[x_1, \ldots, x_n, \partial_1, \ldots, \partial_n]\) on which \(x_i\) acts by multiplication and

\[
\partial_i \cdot P(x) = \partial_i P(x) + \pi(\partial_i W_A(x)) P(x)
\]

for each \(1 \leq i \leq n\) and \(P(x) \in M_A(\mathbb{F})\). We denote by \(\text{DR}_A(\mathbb{F})\) the *de Rham complex* of \(M_A(\mathbb{F})\), which is by definition the Koszul complex

\[\text{Kos} (M_A(\mathbb{F}); \partial_1, \partial_2, \ldots, \partial_n),\]

where each \(\partial_i\) acts as in equation (6.1). Given \(\lambda \in \mathbb{Z}^n_{\geq 0}\) such that \((\lambda A^{-T})_i \geq 0\) for all \(1 \leq i \leq n\), let \(R^A(\mathbb{F}) \subseteq R_A(\mathbb{F})\) be generated by monomials of the form \(x^\gamma y^\lambda + \mu A^T\) for some \(\gamma, \mu \in \mathbb{Z}^n_{\geq 0}\). Then \(R^A(\mathbb{F}) \otimes \wedge(\mathbb{F}^n)\) is closed under \(d_A + d'_{A}\) and we denote by \(B^A(\mathbb{F}) \subseteq B_A(\mathbb{F})\) the corresponding subcomplex.

**Lemma 6.1.** If \(A \in \mathbf{BH}_n(\mathbb{F})\), then

(1) \(B_A(\mathbb{F}) \cong \bigoplus_{\lambda \in G_A} B^A_{\lambda}(\mathbb{F})\), and

(2) \(B^A(\mathbb{F})\) is quasi-isomorphic to \(B^0_{A^\ast}(\mathbb{F})\).
Proposition 6.4. The complex $B^0_A(F)$ is canonically quasi-isomorphic to $DR_A(F)$. Proof. The map $\Theta: M_A(F) \otimes \wedge(F^n) \to R^*_A(F) \otimes \wedge(F^n)$ defined by $\Theta(x^\gamma e^l) = x^{\gamma+1}e^l$ gives rise to an embedding $DR_A(F) \hookrightarrow B^0_A(F)$ of complexes. For each $\gamma \in \mathbb{Z}_{\geq 0}^n$, let $\Lambda_\gamma = \Lambda \left( \bigoplus_{i=0}^\infty F e_i \right)$. Then

$$\left( R^*_A(F) \otimes \wedge(F^n), d^\wedge \gamma \right) = \bigoplus_{\gamma,l} C_{\gamma,l},$$

where

$$C_{\gamma,l} = \left( x^{\gamma+1}F[y^e A^T]_{(\gamma+l),0} \otimes e^l \wedge_{(\gamma+l),0} \sum_{(\gamma+l),i=0} d^\vee_{\gamma,i} \right)$$

is a Koszul complex with cohomology $F x^{\gamma+1}e^l$. This implies that

$$H \left( \frac{R^*_A(F) \otimes \wedge(F^n)}{\text{Im } \Theta}, d^\wedge \gamma \right) = 0,$$

and using the spectral sequence of first quadrant bicomplexes we conclude that

$$H \left( B^0_A(F) / DR_A(F) \right) = 0.$$

Therefore, the inclusion $DR_A(F) \hookrightarrow B^0_A(F)$ is a quasi-isomorphism. □

Corollary 6.3. Let $A \in BH_n(F)$ and let $S(A)$ be the collection of monomials $x^\gamma e^l$ such that $|I| = 1$ and $1 \leq \gamma_i \leq a_i = A_i$ for all $i = 1, \ldots, n$. Then

1. $H(B^0_A(F))$ is isomorphic to the Milnor ring $F[x]/dW_A(x)$.
2. If $W_A(x)$ is a loop, then $H(B^0_A(F))$ is generated by monomials in $S(A)$.
3. If $W_A(x)$ is a chain, then $H(B^0_A(F))$ is generated by those monomials in $S(A)$ of the form

$$x^{a_1} x_2 x_3^{a_2} x_4 \ldots x^{a_{2m-1}} x_{2m} x_{2m+1} \ldots x_n$$

where $m \geq 0$ is such that $\gamma_{2m+1} < a_{2m+1}$.

Proof. Part (1) follows from Proposition 6.2 and the fact (see e.g. [M]) that there is a linear map from $F[x]/dW_A(x)$ to $H(DR_A(F))$ sending monomials to monomials. Comparison with the standard monomial basis for the Milnor ring of chains and loops (see e.g. [K2]) establishes (2) and (3). □

Proposition 6.4. The natural inclusion of $DR_{A_{orb}}(F)$ into $B_A(F)$ is a quasi-isomorphism.

Proof. The proposition follows from Lemma 6.1 and Proposition 6.2. □
7. Unprojected Duality

Given $A \in BH_n(\mathbb{F})$ and $\pi \in \mathbb{F}^*$, let $\psi_{A,i}, T_{A,i} \in \text{End}_\pi(\mathcal{R}_A(\mathbb{F}))$ for $1 \leq i \leq n$ be defined by

$$\psi_{A,i}(x^\gamma y^\lambda) = x^{\gamma + e_i} y^\lambda$$

and

$$T_{A,i}(x^\gamma y^\lambda) = \pi^{-1}(\gamma A^{-1})_i x^\gamma y^\lambda,$$

so that $d_A = \sum_{i=1}^n \hat{d}_{A,i}$, where

$$\hat{d}_{A,i} = (T_{A,i} + \psi_{A,i})E_{A,i}.$$

**Remark 7.1.** Since we are using logarithmic differentials, $e_i$ can be naturally interpreted as $dx_i/x_i$. One motivation for the change of basis to $E_{A,i}$ is the Shioda map $x^\gamma \mapsto z^{\gamma A^{-1} \det(A)}$ which sends $W_A$ to $z^{e_1 \det(A)} + \ldots + z^{e_n \det(A)}$. If we interpret $E_{A,i}$ as $dz_i/z_i$, its definition is simply the chain rule.

Let $S_A(\mathbb{F}) \subseteq \mathcal{R}_A(\mathbb{F})$ be generated by monomials $x^\gamma y^\lambda$ such that $(\gamma A^{-1})_i \geq 0$ for all $1 \leq i \leq n$. Then $S_A(\mathbb{F}) \otimes \wedge(\mathbb{F}^n)$ is closed under $d_A + d_A'$. Let $C_A(\mathbb{F}) \subseteq B_A(\mathbb{F})$ denote the corresponding subcomplex.

**Lemma 7.2.** The inclusion $C_A(\mathbb{F}) \hookrightarrow B_A(\mathbb{F})$ is a quasi-isomorphism.

**Proof.** Consider the filtration

$$S_A(\mathbb{F}) \subseteq F^0 \subseteq F^{n-1} \subseteq \ldots \subseteq F^1 = \mathcal{R}_A(\mathbb{F}),$$

where $F^i$ is spanned by monomials $x^\gamma y^\lambda$ such that $(\gamma A^{-1})_j \geq 0$ for all $j < i$. In particular, $F^i/F^{i+1}$ is canonically identified with the space of monomials $x^\gamma y^\lambda$ such that $(\gamma A^{-1})_i < 0$. Consider the filtration $G^\bullet(\mathbb{F}) = F^\bullet \otimes \wedge(\mathbb{F}^n)$ of $\mathcal{R}_A(\mathbb{F}) \otimes \wedge(\mathbb{F}^n)$. Notice that

$$\left( \frac{G^i(\mathbb{F})}{G^{i+1}(\mathbb{F})} : d_A, d_A' \right)$$

is a bicomplex with respect to the Spec($Q_A$) $\times$ Spec($Q_A'$) bigrading, while

$$\left( \frac{G^i(\mathbb{F})}{G^{i+1}(\mathbb{F})} : \hat{d}_{A,i} = d_A - \hat{d}_{A,i} \right)$$

is a bicomplex with respect to the

Spec($E_{A,i} E_{A,i}'$) $\times$ (ext $- E_{A,i} E_{A,i}'$)

bigrading. Therefore, in order to prove that

$$H\left( \frac{G^i(\mathbb{F})}{G^{i+1}(\mathbb{F})} : d_A + d_A' \right) = 0,$$

(7.1)

it is sufficient to show that

$$H\left( \frac{G^i(\mathbb{F})}{G^{i+1}(\mathbb{F})} : \hat{d}_{A,i} \right) = 0.$$

(7.2)

If this is the case, the result then follows from the spectral sequence of the filtered complex $(B_A(\mathbb{F}), G^\bullet(\mathbb{F}))$. To prove (7.2), we distinguish two cases:

First, suppose that char $\mathbb{F} = 0$. In this case, $T_{A,i}$ acts by nonzero eigenvalues on $F^i/F^{i+1}$. By looking at the filtration of $F^i/F^{i+1}$ by Spec($T_{A,i}$), we conclude
that $T_{A,i} + \psi_{A,i}$ is injective. Therefore, $H \left( \frac{C_i(F)}{\hat{d}_{A,i}}, \hat{d}_{A,i} \right)$ is concentrated in top
$\text{Spec}(E_{A,i}E_{A,i}^\vee)$-degree and isomorphic to the quotient
$F^i / \left( F^{i+1} + \text{Im}(T_{A,i} + \psi_{A,i}) \right)$.

On the other hand, for each $f \in F^i$ there exists $N \in \mathbb{N}$ such that $\psi_{A,i}^n f \in F^{i+1}$, which implies (7.2). See Figure 1 for an illustration.

Second, suppose that $\text{char} \mathbb{F} > \text{det} A$. Let $\mathbb{K}$ be a field such that $\text{char} \mathbb{K} = 0$ and $\mathbb{F} = \mathbb{K}/p\mathbb{K}$. Consider the short exact sequence of complexes

$$0 \rightarrow \left( \frac{C_i(\mathbb{K})}{\hat{d}_{A,i}} \right) \rightarrow \left( \frac{C_i(\mathbb{K})}{\hat{d}_{A,i} + \psi_{A,i}} \right) \rightarrow 0.$$ 

Taking the long exact sequence and using the characteristic 0 case established above yields (7.2).

□

**Proposition 7.3.** Let $D_A^A : S_A(\mathbb{F}) \rightarrow S_{A^T}(\mathbb{F})$ be defined by $D_A^A(x^\gamma y^\lambda) = x^\lambda y^\gamma$. Then,

$$\Delta^A = D_A \otimes \star^A : S_A(\mathbb{F}) \otimes \wedge (\mathbb{F}^n) \rightarrow S_{A^T}(\mathbb{F}) \otimes \wedge (\mathbb{F}^n)$$

induces an isomorphism of complexes

$$C_A(\mathbb{F}) \xrightarrow{\cong} C_{A^T}(\mathbb{F}).$$

**Proof.** Since $D_A^T D_A^A = \text{Id}$ and $\star^A \in \text{GL}(\wedge(\mathbb{F}^n))$, we only need to prove that $\Delta^A$ is a chain map. Using

$$D_A^A \psi_{A,i} = \psi_{A^T,i}^\vee D_A^A \quad \text{and} \quad D_A^A T_{A,i} = T_{A^T,i}^\vee D_A^A,$$

it follows that

$$\Delta^A ((T_{A,i} + \psi_{A,i}) E_{A,i}) = \left( (T_{A^T,i}^\vee + \psi_{A^T,i}^\vee) \psi_{A,i}^\vee \right) \Delta^A$$

$\Box$
\[ \Delta^A \left( (T_{A,i} + \psi_{A,i})c^\gamma \right) = \left( (T_{A,i} + \psi_{A,i})E_{A,T,i} \right) \Delta^A. \]

\[ \square \]

**Theorem 7.4 (Unprojected Berglund-Hübsch Duality).** The complexes \( DR_{A,orb}(\mathbb{F}) \) and \( DR_{(AT),orb}(\mathbb{F}) \) are canonically quasi-isomorphic.

**Proof.** The theorem follows from Proposition 7.3, Lemma 7.2, and Proposition 6.4. \( \square \)

### 8. Overconvergent Power Series

Let \( p \in \mathbb{Z}_{\geq 0} \) be a prime, \( \mathbb{K} = \mathbb{C}_p, \mathbb{F} = \mathbb{K}/p\mathbb{K} \), \( A \in \text{BH}_n(\mathbb{F}) \), and \( \pi \in \mathbb{K} \) such that \( \pi^{p-1} = -1 \). Let \( \widetilde{R}_A(\mathbb{K}) \) be the ring of overconvergent power series

\[ \sum_{\gamma, \lambda \in \mathbb{Z}_{\geq 0}} a_{\gamma, \lambda} x^\gamma y^\lambda \]

such that \((\lambda A^{-T})_i \geq 0\) for all \( 1 \leq i < n \), and such that there exists \( M > 0 \) for which

\[ \text{ord}_p(a_{\gamma, \lambda}) \geq M(|\gamma| + |\lambda|) \quad (8.1) \]

for all but finitely many \( \gamma, \lambda \). Similarly, define \( \mathcal{R}_A^t(\mathbb{K}) \), \( \mathcal{S}_A^t(\mathbb{K}) \), \( \mathcal{B}_A^t(\mathbb{K}) \), and \( \mathcal{C}_A^t(\mathbb{K}) \) as before, by replacing polynomials with overconvergent power series.

**Lemma 8.1.**

- **The inclusions**
  
  \[ \mathcal{B}_A(\mathbb{K}) \longrightarrow \mathcal{B}_A^t(\mathbb{K}) \]
  
  \[ \uparrow \quad \uparrow \]
  
  \[ \mathcal{C}_A(\mathbb{K}) \longrightarrow \mathcal{C}_A^t(\mathbb{K}) \]

  are quasi-isomorphisms.

- \( \Delta^A \) extends to an isomorphism of complexes

  \[ \mathcal{C}_A^t(\mathbb{K}) \xrightarrow{\cong} \mathcal{C}_{AT}^t(\mathbb{K}). \]

**Proof.** To prove (1), let \( f = \sum a_{\gamma, \lambda} x^\gamma y^\lambda \) be an overconvergent power series. Since \( \text{ord}_p(a_{\gamma, \lambda}) \geq 1 \) for all but finitely many \( \gamma, \lambda \), by reducing modulo \( p \) we obtain a polynomial \( \mathbf{f} \in \mathbb{F}[x_1, \ldots, x_n, y_1, \ldots, y_n] \). Therefore, \( R_A(\mathbb{K}) \) and \( R_A^t(\mathbb{K}) \) both reduce modulo \( p \) to \( R_A(\mathbb{F}) \). Since \( B_A(\mathbb{F}) \) and \( C_A(\mathbb{F}) \) decompose into subcomplexes with cohomology concentrated in top degree, the statement follows from the long exact sequences of the diagram

\[
\begin{array}{cccccc}
0 & \longrightarrow & B^t_A(\mathbb{K}) & \xrightarrow{P} & B^t_A(\mathbb{K}) & \longrightarrow & B_A(\mathbb{F}) & \longrightarrow & 0 \\
\uparrow & & \uparrow & & \uparrow & & \cong & & \\
0 & \longrightarrow & B_A(\mathbb{K}) & \xrightarrow{P} & B_A(\mathbb{K}) & \longrightarrow & B_A(\mathbb{F}) & \longrightarrow & 0 \\
\end{array}
\]

\[ 0 \quad 0 \quad 0 \]
as in [M Theorem 8.5]. Therefore \( B_A(\mathbb{K}) \rightarrow B^A_1(\mathbb{K}) \) is quasi-isomorphism. Similarly, \( C_A(\mathbb{K}) \rightarrow C^A_1(\mathbb{K}) \) is a quasi-isomorphism. The rest of the statement follows from Lemma 7.2.

Part (2) follows as in Proposition 7.3 after noticing that the overconvergence property is preserved by \( D^A \).

\[
9. \text{ The Frobenius Endomorphism}
\]

Let \( p^{Q_A}, p^{Q_A^\lambda} \in \text{End} \left( \mathcal{R}_A(\mathbb{K}) \otimes \wedge(\mathbb{K}^n) \right) \) be defined by

\[
p^{Q_A}(x^\gamma y^\lambda e^I) = p^\xi x^\gamma y^\lambda e^I \quad \text{and} \quad p^{Q_A^\lambda}(x^\gamma y^\lambda e^I) = p^\xi^\gamma x^\gamma y^\lambda e^I,
\]

where \( Q_A(x^\gamma y^\lambda e^I) = \xi x^\gamma y^\lambda e^I \) and \( Q_A^\lambda(x^\gamma y^\lambda e^I) = \xi^\gamma x^\gamma y^\lambda e^I \).

**Lemma 9.1.** If \( \Theta_A' : \mathcal{R}_A(\mathbb{K}) \rightarrow \mathcal{R}_A^p(\mathbb{K}) \) is defined by \( \Theta_A'(x^\gamma y^\lambda) = x^\nu y^\nu \), then

\[
\text{Fr}_A = \left( \Theta_A' \otimes \text{Id}_A(\mathbb{K}^n) \right) p^{Q_A}
\]
defines a chain map \( \mathcal{B}_A^1(\mathbb{K}) \rightarrow \mathcal{B}_A^1(\mathbb{K}) \).

**Proof.** This follows from

\[
\begin{align*}
\text{Fr}_A d_{A,i} &= \Theta_A'(\theta_{A,i} + \varphi_{A,i})e_i p^{Q_A+1} = d_{pA,i} \text{Fr}_A' ; \\
\text{Fr}_A' d_{A,i} &= \Theta_A'(T_{A,i}^\nu + \psi_{A,i}^\nu) e_i p^{Q_A} = d_{pA,i} \text{Fr}_A'.
\end{align*}
\]

\[
\square
\]

**Lemma 9.2.** If \( \Theta''_A : \mathcal{R}_{pA}^1(\mathbb{K}) \rightarrow \mathcal{R}_A^1(\mathbb{K}) \) is defined by

\[
\Theta''_A(x^\gamma y^\lambda) = Z_A(x)Z_A^\gamma(y)x^\gamma y^\lambda,
\]

where \( Z_A(x) = e^{\pi(W_{pA}(x) - W_A(x))} \), then

\[
\text{Fr}_A'' = \left( \Theta''_A \otimes \text{Id}_A(\mathbb{K}^n) \right) p^{Q_A^\nu}
\]
defines a chain map \( \mathcal{B}_A^1(\mathbb{K}) \rightarrow \mathcal{B}_A^1(\mathbb{K}) \).

**Proof.** It is well known (see e.g. [M]) that \( Z_A(x) \) satisfies (8.1). Therefore, \( \Theta''_A \) is well defined. We compute

\[
(\theta_{A,i} + \varphi_{A,i}) \Theta''_A = \theta_{A,i} (W_{pA}(x) - W_A(x)) \Theta''_A + \Theta''_A \theta_{pA,i} + \varphi_{A,i} \Theta''_A
\]

\[
= \Theta''_A \varphi_{pA,i} - \varphi_{A,i} \Theta''_A + \Theta''_A \theta_{pA,i} + \varphi_{A,i} \Theta''_A
\]

\[
= \Theta''_A (\theta_{pA,i} + \varphi_{pA,i}),
\]

from which we see that

\[
\text{Fr}_A'' d_{pA,i} = \Theta''_A (\theta_{pA,i} + \varphi_{pA,i}) e_i p^{Q_A^\nu} = d_{A,i} \text{Fr}_A''.
\]

On the other hand, for each \( 1 \leq i \leq n \)

\[
(T_{A,i}^\nu + \psi_{A,i}^\nu) \Theta_{A,p} = T_{A,i}^\nu (W_{pA}(y) - W_A^\nu(y)) \Theta''_A + \Theta''_A T_{A,i}^\nu + \psi_{A,i}^\nu \Theta''_A
\]

\[
= p \Theta''_A \psi_{pA,i} - \psi_{A,i} \Theta''_A + p \theta_{pA,i} T_{A,i}^\nu + \psi_{A,i} \Theta''_A
\]

\[
= p \Theta''_A (T_{pA,i}^\nu + \psi_{pA,i}^\nu).
\]

Therefore,

\[
\text{Fr}_A'' d_{pA,i} = \Theta''_A (T_{pA,i}^\nu + \psi_{pA,i}^\nu) e_i p^{Q_A^\nu-1} = d_{A,i} \text{Fr}_A''.
\]

\[
\square
\]
Lemma 9.3. Let $\hat{P}_{A,i} \in \text{End}_K \left( \mathcal{S}_A^i(K) \right)$ be defined by

$$
\hat{P}_{A,i}(x^\gamma y^\lambda) = 
\begin{cases}
0 & \text{if } (\gamma A^{-1})_i = 0; \\
\lambda x & \text{otherwise}.
\end{cases}
$$

(1) If we define

$$
\hat{Q}_{A,i} = \hat{P}_{A,i} E_{A,i} E^\vee_{A,i},
\hat{Q}_{A,i}^\vee = E^\vee_{A,i} E_{A,i} + Q_{A,i}
$$

then

$$
\Delta^A Q_{A,i}^\vee = \hat{Q}_{A^T,i}^\vee \Delta^A \quad \text{and} \quad \Delta^A Q_{A,i} = \hat{Q}_{A^T,i} \Delta^A.
$$

(2) If we define

$$
\hat{d}_{A,i} = (T_{A,i} + \psi_{A,i}) E_{A,i} \quad \text{and} \quad \hat{D}_{A,i}^\vee = (\theta_{A,i}^\vee + \psi_{A,i}^\vee) E^\vee_{A,i}
$$

then

$$
[\hat{Q}_{A,i}, \hat{d}_{A,j}] = 0 = [\hat{Q}_{A,i}^\vee, \hat{d}_{A,j}]
$$

and

$$
[\hat{Q}_{A,i}, \hat{d}_{A,j}] = \delta_{ij} \hat{d}_{A,j}; \quad [\hat{Q}_{A,i}^\vee, \hat{d}_{A,j}] = \delta_{ij} \hat{D}_{A,j}^\vee.
$$

Proof. We compute

$$
[\hat{Q}_{A,i}, \hat{d}_{A,k}] = [\hat{P}_{A,i} E_{A,i} E^\vee_{A,i}, (T_{A,j} + \psi_{A,j}) E_{A,j}]
$$

$$
= \delta_{ij} \hat{P}_{A,i} (T_{A,j} + \psi_{A,j}) E_{A,j}
$$

$$
= \delta_{ij} \hat{d}_{A,j},
$$

from which we see that

$$
[\hat{Q}_{A,i}, \hat{d}_{A,j}] = [E^\vee_{A,i} E_{A,i}, \hat{d}_{A,j}] + \delta_{ij} \hat{d}_{A,j} = \delta_{ij} \left( -\hat{d}_{A,j} + \hat{d}_{A,j} \right) = 0.
$$

Similarly,

$$
[\hat{Q}_{A,i}^\vee, \hat{d}_{A,j}] = \hat{P}_{A,i} \left( \theta_{A,i}^\vee + \psi_{A,i}^\vee \right) [E_{A,i} E^\vee_{A,i}, E_{A,j}]
$$

$$
= -\delta_{ij} \hat{P}_{A,i} \left( \theta_{A,i}^\vee + \psi_{A,i}^\vee \right) E^\vee_{A,i}.
$$

Since $\hat{P}_{A,i} \theta_{A,i}^\vee(x^\gamma y^\lambda) \neq 0$ implies $(\gamma A^{-1})_i (\lambda A^{-T})_i > 0$, then

$$
(\gamma A^{-1})_i (\lambda A^{-T})_i > 0
$$

and thus $\gamma A^{-1} \lambda^T > 0$. Therefore, $[\hat{Q}_{A,i}, \hat{d}_{A,j}] = 0$. As a consequence,

$$
[\hat{Q}_{A,i}^\vee, \hat{d}_{A,j}] = [E^\vee_{A,i} E_{A,i}, \hat{d}_{A,j}] = \delta_{ij} \hat{d}_{A,j},
$$

which concludes the proof of (2). For (1), we compute

$$
\Delta^A Q_{A,i}^\vee = \Delta^A P_{A,i} \otimes \star^A e_i^\vee e_i = \hat{P}_{A^T,i} \Delta^A \otimes \Delta^A E_{A^T,i} E^\vee_{A^T,i} \star^A = \hat{Q}_{A^T,i} \Delta^A ;
$$

$$
\Delta^A Q_{A,i} = \Delta (e_i^\vee e_i^\vee) = \left( E^\vee_{A^T,i} E_{A^T,i} + \hat{Q}_{A^T,i} \right) \Delta^A = \hat{Q}_{A^T,i} \Delta^A.
$$
Proposition 9.4. For each \( A \in \text{BH}_n(\mathbb{F}) \) the Frobenius endomorphism defined by
\[
\text{Fr}_A = ((\Theta'_A \Theta_A) \otimes \text{Id}_A(\kappa = 1)) p^{Q_A + Q_{\hat{A}}}
\]
is a chain map and
\[
\Delta^A \text{Fr}_A = \text{Fr}_{\text{Fr}_A^T} \Delta^A p^{2\text{ext} - n} p^{-2\hat{Q}_A} p^{2Q_{\hat{A}}}
\]

Proof. Since \( p^{Q_{\hat{A}}} (\Theta'_A \otimes \text{Id}) = (\Theta_A' \otimes \text{Id}) p^{Q_{\hat{A}}} \), then \( \text{Fr}_A = \text{Fr}_A' \text{Fr}_A' \) is a chain map. For the second statement, \( D^A \Theta_A' = \Theta'_A \Theta_A D^A \) implies
\[
\Delta^A \text{Fr}_A = \Delta^A (\Theta'_A \Theta_A \otimes \text{Id}) p^{Q_{\hat{A}} + Q_A}
\]
\[
= \text{Fr}_{\text{Fr}_A^T} p^{-Q_{\hat{A}} - Q_{\hat{A}}} \Delta^A p^{Q_{\hat{A}} + Q_A}
\]
\[
= \text{Fr}_{\text{Fr}_A^T} \Delta^A p^{-\hat{Q}_A - \hat{Q}_{\hat{A}}} p^{Q_{\hat{A}} + Q_A}
\]
\[
= \text{Fr}_{\text{Fr}_A^T} \Delta^A p^{2\text{ext} - n} p^{-2\hat{Q}_A} p^{2Q_{\hat{A}}}
\]
\[
\square
\]

Theorem 9.5. Let \( \#_A \) (respectively \( \#_{\hat{A}} \)) be the operator on \( S_\Lambda(\mathbb{K}) \) diagonalized by monomials and such that the eigenvalue of \( x^\gamma y^\lambda \) is the number of non-integer entries of \( \gamma A^{-1} \) (respectively \( \lambda A^{-T} \)). If \( \kappa \) is such that \( (\kappa \pi)^{p-1} = p \), then the twisted Frobenius endomorphism
\[
\text{TFr}_A = \text{Fr}_A (\kappa \pi)^{(p-1)(\#_A - \#_{\hat{A}})/2}
\]
is a chain map, and
\[
H(\Delta^A)H(\text{TFr}_A) = H(\text{TFr}_{\text{Fr}_A^T})H(\Delta^A).
\]

Proof. Since \( (\kappa \pi)^{(p-1)(\#_A - \#_{\hat{A}})/2} \) is diagonalized by monomials and acts trivially on \( \Lambda(\mathbb{K}^n) \), it commutes with \( d_A + d_{\hat{A}} \). Therefore, \( \text{TFr}_A \) is a chain map. Using Proposition 9.4 we calculate
\[
\Delta^A \text{TFr}_A = \Delta^A \text{Fr}_A (\kappa \pi)^{(p-1)(\#_A - \#_{\hat{A}})/2}
\]
\[
= \text{Fr}_{\text{Fr}_A^T} \Delta^A p^{2\text{ext} - n} p^{-2\hat{Q}_A} p^{2Q_{\hat{A}} (\kappa \pi)^{(p-1)(\#_A - \#_{\hat{A}})/2}}
\]
\[
= \text{TFr}_{\text{Fr}_A^T} \Delta^A p^{2\text{ext} - n} p^{-2\hat{Q}_A} p^{2Q_{\hat{A}} (\kappa \pi)^{(p-1)(\#_A - \#_{\hat{A}})/2}}
\]
where the last step follows from
\[
(\kappa \pi)^{(p-1)(\#_A - \#_{\hat{A}})/2} \Delta^A = \Delta^A (\kappa \pi)^{(p-1)(\#_A - \#_{\hat{A}})/2}.
\]
Therefore, the theorem is proven if the eigenvalues of
\[
2 \text{ext} - n - 2\hat{Q}_A + 2Q_{\hat{A}} \quad \text{and} \quad - (\#_A - \#_{\hat{A}})
\]
agree on a monomial basis \( x^\gamma y^\lambda e_I \) for \( H(B^\Lambda_\mathbb{A}(\mathbb{K})) \) for each \( \lambda \in G_A \). By Lemma 6.1 and Corollary 6.3 one can choose generators of the form \( x^\gamma y^\lambda e_I \), where \( |I| = n - |J_{\lambda}^\gamma| \) and \( 0 \leq (\lambda A^{-T})_i < 1 \) for all \( i = 1, \ldots, n \). In particular, the eigenvalue of
\[
2 \text{ext} - n + 2Q_{\hat{A}} - \#_{\hat{A}} \quad \text{on} \quad x^\gamma y^\lambda e_I \in S(A)
\]
is \( |I| \). On the other hand, inspection of the bases for the cohomology of chains and loops given in Corollary 6.3 shows that
\[
(2\hat{Q}_A - \#_A) = \text{ext} \text{ on } S(A),
\]
which concludes the proof. \( \square \)
10. Examples

Example 10.1. Let \( n = 1 \) and \( A_{11} = 2 \). Then \( W_A(x) = W^T_A(x) = x_1^2 \) and \( G_A = G_{A^T} = \mathbb{Z}/2\mathbb{Z} \). The exterior operators are \( E_{A,1} = 2\pi c_1 \) and \( E^*_{A,1} = \frac{1}{2\pi} e_1^\prime \). Moreover, \( \mathcal{R}^0_A(\mathcal{F}) = \mathbb{F}[x_1] \oplus \mathbb{F}[y_1^2] \) and \( \mathcal{R}^1_A(\mathcal{F}) = y_1\mathbb{F}[y_1^2] \). The differentials are

\[
\begin{align*}
  d(x_1^{2^k}e_1) &= \gamma_1 x_1^{2^k+1}e_1 + 2\pi x_1^{2^k+2}e_1; \\
  d^\prime(y_1^{\lambda_1}e_1) &= \frac{1}{2\pi}\lambda_1 y_1^{\lambda_1} + y_1^{\lambda_1+2}.
\end{align*}
\]

It follows that \( H(\mathcal{B}_A^0(\mathcal{F})) = \mathbb{F}x_1e_1 \) and \( H(\mathcal{B}_A^1(\mathcal{F})) = \mathbb{F}y_1 \) are mapped one into the other by \( \Delta^A \). The relations in cohomology are

\[
\begin{align*}
  x_1^{2k+1}e_1 &= (-2\pi)^{-1}(2k-1)x_1^{2(k-1)+1}e_1 = \ldots = (-2\pi)^{-k}(2k-1)!x_1e_1; \\
  y_1^{2k+1} &= (-2\pi)^{-1}(2k-1)y_1^{2(k-1)+1} = \ldots = (-2\pi)^{-k}(2k-1)!y_1.
\end{align*}
\]

Let \( (c_m) \) be the sequence of rational numbers defined by

\[ e^{\pi(t^p - t)} = \sum_{m \geq 0} c_m (-\pi)^m t^m. \]

The action of the twisted Frobenius map in cohomology is thus

\[
H(\text{TFr}_A)(x_1e_1) = p(\kappa\pi)(p^{-1}/2)e^{\pi(x_1^{2^k} - x_1)}x_1^p e_1
\]

\[
= p(\kappa\pi)(p^{-1}/2) \sum_{m \geq 0} c_m (-\pi)^m x_1^{2(m + \frac{2}{2} - 1)}e_1
\]

\[
= p(\kappa\pi)(p^{-1}/2) \left( \sum_{m \geq 0} c_m (-\pi)^{-\frac{2}{2}} 2^{-2(m + \frac{2}{2})} (2(m - 1)!) \right) x_1e_1
\]

\[
= pk^{-1}/2 \left( \left( \frac{p - 1}{2} \right) + O(p) \right) x_1e_1.
\]

Similarly,

\[
H(\text{TFr}_A)(y_1) = p^2(\kappa\pi)(p^{-1}/2) \left( \sum_{m \geq 0} c_m (-\pi)^{-\frac{2}{2}} 2^{-2(m + \frac{2}{2})} (2(m - 1)!) \right) y_1
\]

\[
= pk^{-1}/2 \left( \left( \frac{p - 1}{2} \right) + O(p) \right) y_1.
\]

Comparison with the non-commutative Weil conjectures of Kontsevich [K] seems to suggest a further overall rescaling of \( \text{TFr}_A \). This is likely to be relevant for arithmetic applications. We hope to come back to this point in future work.

Example 10.2. Consider the dual chains \( W_A(x) = x_1^2 x_2 + x_2^2 \) and \( W_{A^T}(x) = x_1^2 + x_1 x_2^2 \). The elements of \( G_A \cong \mathbb{Z}^2/\mathbb{Z}^2 A \) and \( G_{A^T} \cong \mathbb{Z}^2/\mathbb{Z}^2 A \) are given in Table 1. We can find basis elements \( x^y y^\lambda e^I \) of \( \mathcal{C}_A \) and \( \mathcal{C}_{A^T} \) as described in the proof of Theorem 10.4. Each row of Table 2 contains a pair of elements dual under \( \Delta^A \) (up to constants), as well as the eigenvalues of

\[
Q_A + Q_A^\prime \quad \text{and} \quad (\#_A - \#_A^\prime)/2
\]

applied to \( x^y y^\lambda e^I \). Here we are using \( \star^A(e_1 e_2) = 1, \star^A(e_2) = -2\pi e_1 \) and \( \star^A(1) = E_{A^T,1} E_{A^T,2} = (2\pi e_1)(\pi e_1 + 3\pi e_2) = 6\pi^2 e_1 e_2 \).
Since for any $x^\gamma$
\[
(\theta_{A,1} + \varphi_{A,1})(x^{\gamma+e_1A}) = \gamma_1 x^\gamma + \pi \left(2x^{\gamma+e_1A}\right);
\]
\[
(\theta_{A,2} + \varphi_{A,2})(x^{\gamma+e_2A}) = \gamma_2 x^\gamma + \pi \left(x^{\gamma+e_1A} + 3x^{\gamma+e_2A}\right),
\]
in $H \left(B_\lambda^A(\mathbb{F})\right)$ we have the relation
\[
\gamma x^\gamma y^\lambda e^I = (-\pi)(x^{\gamma+e_1A} y^\lambda e^I, x^{\gamma+e_2A} y^\lambda e^I)A,
\]
which implies for $i = 1, 2$ that
\[
x^{\gamma+e_iA} y^\lambda e^I = (-\pi)^{-1} (\gamma A^{-1})_i x^\gamma y^\lambda e^I.
\]
Therefore, for $i = 1, 2$,
\[
x^{\gamma + k_i c_i A} y^\lambda e^I = (-\pi)^{-1} \left( (\gamma + (k_i - 1)c_i A^{-1}) x^{\gamma + (k_i - 1)c_i A} y^\lambda e^I \right) = (-\pi)^{-2} \left( (\gamma A^{-1})_i + (k_i - 1) \right) (\gamma A^{-1})_i + (k_i - 2) x^{\gamma + (k_i - 2)c_i A} y^\lambda e^I = (-\pi)^{-k_i} \left( (\gamma A^{-1})_i \right) (k_i) x^{\gamma y^\lambda e^I}. \tag{10.1}
\]

Take $x_1 x_2 e_1 e_2$ so that $\gamma = (1, 1)$ and $\gamma A^{-1} = (1/2, 1/6)$. Suppose that $p$ is a prime such that $6 | (p - 1)$. Then we can write
\[
(p, p) = (1, 1) + \left( \frac{p - 1}{2}, \frac{p - 1}{6} \right) A,
\]
which using equation (10.1) gives
\[
\text{TFr}_A(x_1 x_2 e_1 e_2) = p^2 (\kappa p)^{-1} x_1^p x_2^p Z_A(x) e_1 e_2
\]
\[
= p^3 x_1^p x_2^p \left( \sum_{k_1 \geq 0} (-\pi)^{k_1} c_{k_1} x^{k_1 c_1 A} \right) \left( \sum_{k_2 \geq 0} (-\pi)^{k_2} c_{k_2} x^{k_2 c_2 A} \right) e_1 e_2
\]
\[
= p^3 (-\pi)^{-2(p-1)} \left( \sum_{k_1, k_2 \geq 0} c_{k_1} c_{k_2} \left( \frac{1}{2} \right) \left( \frac{1}{6} \right) \right) x_1^p x_2^p e_1 e_2,
\]
where we have used the fact that $Z_A^T(y) = 1 + O(y_1, y_2)$. Next, consider
\[
\text{TFr}_A(x_2^p y_1 e_2) = p^3 e^{\pi (x_2^p y_2^p - x_2^p)} e^{\pi (y_1^p - y_2^p)} x_2^p y_1^p e_2.
\]

By equation (6.2), in cohomology we have the relation
\[
y^{\lambda + k_i c_i A^T} = (-\pi)^{-k_i} \left( (\lambda A^{-1})_i \right) y^{\lambda} = (-\pi)^{k_i} \left( \frac{3\lambda_1 - \lambda_2}{6} \right) (k_i) y^{\lambda},
\]
which if $6 | (p - 1)$ implies that
\[
\text{TFr}_A(x_2^p y_2 e_2) = p^3 (-\pi)^{-2(p-1)} \left( \sum_{k_1, k_2 \geq 0} c_{k_1} c_{k_2} \left( \frac{1}{2} \right) \left( \frac{2}{3} \right) \right) x_1^p y_1^p e_2.
\]

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