THE NUMBER OF REPRESENTATIONS OF $n$ AS A GROWING NUMBER OF SQUARES

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Abstract. Let $r_k(n)$ denote the number of representations of the integer $n$ as a sum of $k$ squares. In this paper, we give an asymptotic for $r_k(n)$ when $n$ grows linearly with $k$. As a special case, we find that

$$r_n(n) \sim \frac{B \cdot A^n}{\sqrt{n}},$$

with $B \approx 0.2821$ and $A \approx 4.133$.

1. Introduction and Statement of Results

The problem of how many ways a positive integer can be written as a sum of $k$ squares dates back more than 300 years. In 1640, Fermat stated (in a letter to Mersenne) that a positive integer $n$ can be written as the sum of two squares if and only if in the prime factorization of $n$, the exponents on all primes $p \equiv 3 \pmod{4}$ are even. In 1770, Lagrange proved that every positive integer is the sum of four squares. In 1834, Jacobi strengthened Lagrange’s theorem and gave a formula for $r_4(n)$, the number of ways that $n$ can be written as a sum of four squares. Jacobi’s result states that

$$r_4(n) = \begin{cases} 8 \sum_{d | n, d \text{ odd}} d & \text{if } n \text{ is odd} \\ 24 \sum_{d | n, d \text{ odd}} d & \text{if } n \text{ is even.} \end{cases}$$

Jacobi derived this result by considering the Jacobi theta function

$$\theta(z) = \sum_{n=-\infty}^{\infty} q^n^{2}, \quad q = e^{2\pi i z}$$

and deriving the relation

$$\theta(z)^4 = \sum_{n=0}^{\infty} r_4(n) q^n = 1 + 8 \sum_{n=1}^{\infty} \frac{q^n}{(1-q^n)^2} - 32 \sum_{n=1}^{\infty} \frac{q^{4n}}{(1-q^{4n})^2},$$

which is equivalent to the formula for $r_4(n)$ stated above.

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Let $r_k(n)$ denote the number of representations of $n$ as the sum of $k$ squares. Jacobi found formulas for $r_k(n)$ expressed in terms of divisor functions for $k \in \{2, 4, 6, 8\}$ but for values of $k$, other functions (the so-called cusp forms) are needed. In 1907, Glaisher gave such formulas for even $k \leq 18$ (see [3]). Shortly thereafter, Mordell [6] and Hardy [4] applied the circle method to determining asymptotics for $r_k(n)$, and decomposed $r_k(n) = \rho_k(n) + R_k(n)$ as the sum of the “singular series” and an error term. The singular series has size approximately $n^{(k/2) - 1}$ (at least if $k > 4$) and the error term (when $k$ is even) is $O(d(n)n^{(k/4) - 1/2})$ by deep work of Deligne. Here $d(n)$ is the number of divisors of $n$.

In 2012, the second author determined the implied constant in the estimate $r_k(n) = \rho_k(n) + O(d(n)n^{(k/4) - 1/2})$. The main result of [8] is the following.

**Theorem.** Suppose that $k$ is a multiple of 4. If either $k/4$ is odd or $n$ is odd, then we have

$$|r_k(n) - \rho_k(n)| \leq \left(2k + \frac{k(-1)^{k/4}}{(2k/2 - 1)B_{k/2}}\right)d(n)n^{\frac{k}{4} - \frac{1}{2}}.$$

Here $B_n$ is the usual Bernoulli number defined by

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} \frac{B_n x^n}{n!}.$$

This result gives a strong bound on $r_k(n)$ provided that $n$ is sufficiently large in terms of $k$. In particular, the main term is larger than the error term above if $n$ is larger than about $\frac{k^2}{\sqrt{\pi}e}$. In light of this, it is natural to consider the problem of finding an asymptotic for $r_k(n)$ when $k$ and $n$ both grow, but $n$ is much smaller than $k^2$. The main result of this paper is an asymptotic for when $n$ grows linearly with $k$.

**Theorem 1.** Let $a$ be a positive integer and $b$ be any integer. Then there are constants $A$ (depending only on $a$) and $B$ (depending on $a$ and $b$) so that

$$r_n(an + b) \sim \frac{B \cdot A^n}{\sqrt{n}}.$$

Here $f(n) \sim g(n)$ means that $\lim_{n \to \infty} \frac{f(n)}{g(n)} = 1$.

When $a = 1$ and $b = 0$, the proof produces a value for $A \approx 4.132731376$ and $B \approx 0.28209420367$. Here is a table of values of $r_n(n)$ compared with $\frac{B \cdot A^n}{\sqrt{n}}$. 


Next, we give a summary of the method we use to prove Theorem 1. We can extract the coefficient $r_n(an + b)$ via

$$r_n(an + b) = \int_{-1/2}^{1/2} q^{-an-b} \theta(x + iy)^n \, dx.$$ 

To derive asymptotics for this integral, we use the saddle point method. The value of the integral above does not depend on $y$, and we choose $y$ so that $q^{-an-b} \theta(x + iy)^n$ has a saddle point when $x = 0$, which is also the place where the absolute value of the integrand is maximized.

In Section 2, we review relevant background and prove a Lemma that gives an asymptotic for integrals of the type given above. In Section 3, we prove Theorem 1 by verifying the hypotheses of the lemma.

## 2. Background

It is not hard to show that for $n \in \mathbb{Z}$,

$$\int_{-1/2}^{1/2} q^n \, dx = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{if } n \neq 0. \end{cases}$$

This provides a convenient way to extract the coefficient of $q^n$ from a generating function. In particular, if $\theta(z) = \sum_{n \in \mathbb{Z}} q^n$, then $\theta^n(z) = \sum_{m=0}^{\infty} r_n(m)q^m$. Assuming we are able to switch the infinite sum and integral, we obtain

$$r_n(an + b) = \int_{-1/2}^{1/2} q^{-an-b} \theta^n(z) \, dx.$$ 

Next, we justify interchanging the sum and integral. In fact we will also need that for any non-negative integer $k$, if $f_m(z) = r_n(m)q^m$, then

$$\frac{d^k}{dz^k} \theta^n(z) = \sum_{m=0}^{\infty} \frac{d^k}{dz^k} f_m(z).$$
Because \( \frac{d^k}{dz^k}f_m(z) = r_n(m)(2\pi imz)^ke^{2\pi imz} \), \( r_n(m) \) is bounded by a polynomial in \( m \) (say of degree \( r \)), and
\[
\sum_{m=1}^{\infty} m^{r+n}e^{-2\pi my}
\]
converges absolutely for any \( y > 0 \), it follows that the sequence of partial sums \( \sum_{m=0}^{N} \frac{d^k}{dz^k}f_m(z) \) converges uniformly. Combining this with the result (see for example Theorem 25.2 on page 185 of [7]) that if a sequence of functions \( g_n \) converges uniformly to \( g \), then \( \lim_{n \to \infty} \int_a^b g_n(x) \, dx = \int_a^b g(x) \, dx \). This yields the desired result.

The method of steepest descent (also known as the saddle point method) is a procedure for obtaining asymptotics for integrals of the form
\[
\int_a^b f(x + iy)g(x + iy)^n \, dx \sim f(iy)g(iy)^n \sqrt{\frac{2\pi g(iy)}{-ng''(iy)}}.
\]
Results of this type have appeared in the literature many times before (see for example Section 5.7, pages 87-89 of [2]). To keep the paper self-contained, we provide a complete proof.

**Lemma 2.** Suppose that \( f(z) \) and \( g(z) \) are holomorphic functions and \( y \in \mathbb{R} \). Suppose that \( a \) and \( b \) are real numbers with \( a < 0 < b \). Suppose that for \( x \in [a, b] \), \( |g(x + iy)| \leq |g(iy)| \) with equality if and only if \( x = 0 \), \( g(iy) \) is real and positive, \( f(iy) \neq 0 \), \( g'(iy) = 0 \), and \( g''(iy) \) is a negative real number. Then we have
\[
\int_a^b f(x + iy)g(x + iy)^n \, dx \sim f(iy)g(iy)^n \sqrt{\frac{2\pi g(iy)}{-ng''(iy)}}.
\]

**Proof.** The function \( h(z) = \ln(g(z)) \) will be holomorphic in a neighborhood of \( z_0 = iy \) since \( g(iy) > 0 \). We consider the Taylor expansion of \( h(z) \) in a neighborhood of \( z = z_0 \),
\[
h(z) = h(z_0) + h'(z_0)(z - z_0) + \frac{h''(z_0)(z - z_0)^2}{2!} + E(z).
\]
We have \( g'(z_0) = \frac{g'(iy)}{g(iy)} = 0 \). Moreover, Theorem 8 on page 125 of [1] gives the formula
\[
E(z) = \frac{(z - z_0)^3}{2\pi i} \int_{\Gamma} \frac{g(w) \, dw}{(w - z_0)^3(w - z)},
\]
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where $\Gamma$ is any simple closed curve contained in the region in which $h(z)$ is holomorphic that contains $z_0$ and $z$. We see then that

$$|E(z)| \leq |z - z_0|^3 \cdot (\text{length of } \Gamma) \cdot \max_{w \in \Gamma} \frac{1}{|w - z_0|^3} \cdot \max_{w \in \Gamma} |h(w)|.$$  

If we require that $|z - z_0| < \delta$, we may choose $\Gamma$ to be a circle of radius $2\delta$ and it follows that there is a constant $C$ (depending on $\delta$) so that $|E(z)| \leq C|z - z_0|^3$.

We split up the integral into the contribution near $z_0$ (say the interval $I_1 = [-1/n^{2/5}, 1/n^{2/5}]$), and the contribution $I_2$ away from $z_0$.

For the contribution away from $z_0$, once $n$ is large enough, the maximum value of $f(x + iy)g(x + iy)^n$ occurs at either $-1/n^{2/5}$ or $1/n^{2/5}$. The contribution away from $z_0$ is hence at most

$$\max\{|e^{nh(-n^{-2/5} + iy)}f(-1/n^4)|, |e^{nh(n^{-2/5} + iy)}f(1/n^4)|\}.$$  

Using the bound on $E(z)$ above, we see that

$$nh(\pm n^{-2/5} + iy) = n \ln(g(iy)) + \frac{h''(iy)n^{1/5}}{2!} + O\left(\frac{1}{n^{1/5}}\right).$$  

It follows that as $n$ tends to infinity,

$$\left|\int_{I_2} f(x + iy)g(x + iy)^n dx\right| \leq C_2 f(iy)g(iy)^n \cdot e^{h''(iy)n^{1/5}/2}$$  

for some constant $C_2$. Since

$$h''(iy) = \frac{g''(iy)g(iy) - [g'(iy)]^2}{g(iy)^2} = \frac{g''(iy)}{g(iy)} < 0,$$  

as $n \to \infty$ this contribution is exponentially smaller than the main contribution.

For the contribution close to $z_0$, we consider

$$\int_{-n^{-2/5}}^{n^{-2/5}} f(x + iy)g(x + iy)^n dx,$$

make the change of variables $u = \sqrt{n}x dx$, and get

$$\int_{-n^{1/10}}^{n^{1/10}} f(u/\sqrt{n} + iy)g(u/\sqrt{n} + iy)^n \frac{du}{\sqrt{n}}$$

$$= \frac{g(iy)^n}{\sqrt{n}} \int_{-n^{1/10}}^{n^{1/10}} f(u/\sqrt{n} + iy)e^{u^2g''(iy)/2g(iy)} \cdot e^{nE(u/\sqrt{n} + iy)} du.$$
Fix $\epsilon > 0$. Because $f$ is continuous, it is uniformly continuous on a short interval surrounding $z_0$ and so there is an $N_1$ for that all $n \geq N_1$, and all $u \in [-n^{1/10}, n^{1/10}]$, $|f(u/\sqrt{n} + iy) - f(iy)| < \epsilon/2$. Also, for $u \in [-n^{1/10}, n^{1/10}]$ we have

$$|nE(u/\sqrt{n} + iy)| \leq nC|u/\sqrt{n}|^3 \leq C|u|^3 n^{-1/2} \leq Cn^{3/10}n^{-1/2} = Cn^{-2/5}.$$ 

It follows that there is some $N_2$ so that for $n \geq N_2$ $1 - \frac{\epsilon}{2} \leq e^{nE(u/\sqrt{n}+iy)} \leq 1 + \frac{\epsilon}{2}$ for all $u \in [-n^{1/10}, n^{1/10}]$. The triangle inequality then shows that for $n \geq \max\{N_1, N_2\}$,

$$\left| \int_{-n^{-2/5}}^{n^{2/5}} f(x + iy)g(x + iy)^n \, dx - \frac{f(iy)g(iy)^n}{\sqrt{n}} \int_{-n^{1/10}}^{n^{1/10}} e^{u^2g''(iy)/(2g(iy))} \, du \right| < \epsilon \frac{|f(iy)|g(iy)^n}{\sqrt{n}} \int_{-n^{1/10}}^{n^{1/10}} e^{u^2g''(iy)/(2g(iy))} \, du.$$

The desired result now follows from making the change of variables $v = u\sqrt{-g''(iy)/g(iy)}$ and

$$\lim_{n \to \infty} \int_{-n^{1/10}}^{n^{1/10}} e^{u^2g''(iy)/(2g(iy))} \, du = \frac{\sqrt{2g(iy)}}{\sqrt{-g''(iy)}} \int_{-\infty}^{\infty} e^{-v^2} \, dv = \frac{\sqrt{2\pi g(iy)}}{\sqrt{-g''(iy)}}.$$

3. Proof of the main result

In this section, we will prove Theorem 1 by verifying the hypotheses of the Lemma 2. We will show that there is a positive real number $y$ that makes $g'(iy) = 0$. We let $f(z) = q^{-b}$ and $g(z) = q^{-a}\theta(z)$. It is clear that $f(z)$ is holomorphic, and (as mentioned in Section 2), since we can differentiate $\theta(z) = 1 + 2 \sum_{n=1}^{\infty} q^n$ termwise, it follows that $\theta(z)$ is holomorphic as well. Thus, $g(z)$ is holomorphic.

Next, we will show that $|g(x + iy)| \leq |g(iy)|$. We have

$$|g(x + iy)| = \sum_{n=-\infty}^{\infty} e^{2\pi i(x(n^2-a))} e^{2\pi i(y)(n^2-a)} \leq \sum_{n=-\infty}^{\infty} |e^{2\pi i(x(n^2-a))} e^{2\pi i(y)(n^2-a)}|$$

$$= \sum_{n=-\infty}^{\infty} |e^{2\pi i(x(n^2-a))}| |e^{2\pi i(y)(n^2-a)}|.$$

We know $|e^{2\pi i(x(n^2-a))}| = 1$ to be true. Thus, we have

$$|g(x + iy)| \leq \sum_{n=-\infty}^{\infty} |e^{2\pi i(y)(n^2-a)}| = \sum_{n=-\infty}^{\infty} e^{-2\pi y(n^2-a)} = |g(iy)| = g(iy),$$

□
as desired.

Now, we will prove that for $-1/2 \leq x \leq 1/2$, if $|g(x + iy)| = |g(iy)|$, then $x = 0$. Assume that $|g(x + iy)| = |g(iy)|$ and fix integers $n_1$ and $n_2$. Using the triangle inequality, we have

$$
|g(x + iy)| = \sum_{n=-\infty}^{\infty} e^{2\pi i x(n^2 - a)} e^{2\pi i (iy)(n^2 - a)} \\
= \left| e^{2\pi i x(n_1^2 - a)} e^{-2\pi y(n_1^2 - a)} + e^{2\pi i x(n_2^2 - a)} e^{-2\pi y(n_2^2 - a)} + \sum_{n \neq n_1, n_2} e^{2\pi i x(n^2 - a)} e^{-2\pi y(n^2 - a)} \right| \\
\leq \left| e^{2\pi i x(n_1^2 - a)} e^{-2\pi y(n_1^2 - a)} \right| + \left| e^{2\pi i x(n_2^2 - a)} e^{-2\pi y(n_2^2 - a)} \right| + \sum_{n \neq n_1, n_2} \left| e^{2\pi i x(n^2 - a)} e^{-2\pi y(n^2 - a)} \right| \\
= \sum_{n=-\infty}^{\infty} \left| \frac{e^{-2\pi y(n^2 - a)}}{e^{-2\pi y(n_1^2 - a)}} \right| = |g(iy)|.
$$

Because of the assumption that $|g(x + iy)| = |g(iy)|$, the left hand side and right hand side of the above inequality are equal, and this forces all intermediate terms to be equal. Letting $z = e^{2\pi i x(n_1^2 - a)} e^{-2\pi y(n_1^2 - a)}$ and $w = e^{2\pi i x(n_2^2 - a)} e^{-2\pi y(n_2^2 - a)}$, we have that $|z + w| = |z| + |w|$, and a straightforward calculation shows that $|z + w| = |z| + |w|$ forces $z$ and $w$ to have the same phase. This implies that for any two integers $n_1$ and $n_2$, $e^{2\pi i x(n_1^2 - a)} = e^{2\pi i x(n_2^2 - a)}$. Setting $n_1 = 0$ and $n_2 = 1$ yields $e^{2\pi i x} = 1$, and this forces $x = 0$.

It is straightforward to see that $f(iy) \neq 0$ since $f(iy) = e^{2\pi y b} \neq 0$.

Next, we will show that there is a $y$ so that $g'(iy) = 0$. We have that

$$
g'(iy) = \sum_{n=-\infty}^{\infty} e^{-2\pi y(n^2 - a)} (-2\pi (n^2 - a)).
$$

We rewrite the right hand side as

$$
\sum_{n^2 - a < 0} (-2\pi (n^2 - a)) e^{-2\pi y(n^2 - a)} + \sum_{n^2 - a \geq 0} (-2\pi (n^2 - a)) e^{-2\pi y(n^2 - a)}.
$$

In the first sum, there are finitely many positive terms, all of which tend to $\infty$ as $y \to \infty$. As $y \to 0$, we obtain $\sum_{n^2 - a < 0} (-2\pi (n^2 - a)) > 0$. 
In the second sum, the terms are negative and decreasing. It is easy to see that
\[
\frac{-2\pi e^{2\pi y}}{(e^{2\pi y} - 1)^2} = \sum_{r=1}^{\infty} -2\pi r e^{-2\pi y r} < \sum_{r=1}^{\infty} (-2\pi (n^2 - a)) e^{-2\pi y (n^2 - a)} < 0.
\]
Thus, the second sum in (2) tends to zero as \(y \to \infty\) and tends to \(-\infty\) as \(y \to 0\) (since choosing \(y\) very small can make the term \((-2\pi (n^2 - a)) e^{-2\pi y (n^2 - a)}\) arbitrarily close to \(-2\pi (n^2 - a))\). It follows that \(g'(iy) \to -\infty\) as \(y \to 0\), and \(g'(iy) \to \infty\) as \(y \to 0\). Since \(g'(iy)\) is continuous, by the intermediate value theorem, there is some positive real number \(y\) for which \(g'(iy) = 0\).

Lastly, one can easily see that
\[
g''(iy) = (-2\pi i (n^2 - a))^2 \sum_{n=-\infty}^{\infty} e^{-2\pi y (n^2 - a)} < 0.
\]
For the special case that \(a = 1\) and \(b = 0\), we have \(f(z) = 1\) and \(g(z) = \sum_{n=-\infty}^{\infty} q^{n^2 - 1}\). We find that the value of \(y\) that makes \(g'(iy) = 0\) is \(y \approx 0.07957745473668\), and this leads to \(A = g(iy) \approx 4.133\) and \(B = \sqrt{-\frac{2\pi A}{-g'(iy)}} \approx 0.2821\).

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