On the locus of non-rigid hypersurfaces

Thomas Eckl and Aleksandr Pukhlikov

We show that the Zariski closure of the set of hypersurfaces of degree $M$ in $\mathbb{P}^M$, where $M \geq 5$, which are either not factorial or not birationally superrigid, is of codimension at least $\left(\frac{M-3}{2}\right) + 1$ in the parameter space.

Bibliography: 21 titles.

1. Formulation of the main result and scheme of the proof. Let $\mathbb{P}^M$, where $M \geq 5$, be the complex projective space, $\mathcal{F} = \mathbb{P}(H^0(\mathbb{P}^M, \mathcal{O}_{\mathbb{P}^M}(M)))$ the space parametrizing hypersurfaces of degree $M$. There are Zariski open subsets $\mathcal{F}_{\text{reg}} \subset \mathcal{F}_{\text{sm}} \subset \mathcal{F}$, consisting of hypersurfaces, regular in the sense of [14], and smooth, respectively. The well known theorem proven in [14] claims that every regular hypersurface $V \in \mathcal{F}_{\text{reg}}$ is birationally superrigid. Let $\mathcal{F}_{\text{srigid}} \subset \mathcal{F}$ be the set of (possibly singular) hypersurfaces that are factorial and birationally superrigid. The aim of this note is to show the following claim.

**Theorem 1.** The Zariski closure $\mathcal{F}\setminus\mathcal{F}_{\text{srigid}}$ of the complement is of codimension at least $\left(\frac{M-3}{2}\right) + 1$ in $\mathcal{F}$.

Note that we do not discuss the question of whether $\mathcal{F}_{\text{srigid}}$ is open or not.

We prove Theorem 1, directly constructing a set in $\mathcal{F}$, every point of which corresponds to a factorial and birationally superrigid hypersurface, with the Zariski closure of its complement of codimension at least $\left(\frac{M-3}{2}\right) + 1$. More precisely, let $\mathcal{F}_{\text{qsing} \geq 5}$ be the set of hypersurfaces, every point of which is either smooth or a quadratic singularity of rank at least $r$. We do not assume that singularities are isolated, but it is obvious that for $V \in \mathcal{F}_{\text{qsing} \geq 5}$ the following estimate holds:

$$\text{codim Sing } V \geq r - 1.$$ 

In particular, by the famous Grothendieck theorem ([7, XI.Corr.3.14], [1]) any $V \in \mathcal{F}_{\text{qsing} \geq 5}$ is a factorial variety, therefore a Fano variety of index 1:

$$\text{Pic } V = \mathbb{Z}K_V, \quad K_V = -H,$$

where $H \in \text{Pic } V$ is the class of a hyperplane section.

It is easy to see (Proposition 2) that $\text{codim}(\mathcal{F}\setminus\mathcal{F}_{\text{qsing} \geq 5}) \geq \left(\frac{M-3}{2}\right) + 1$.

Denote by $\mathcal{F}_{\text{reg,qsing} \geq 5} \subset \mathcal{F}_{\text{qsing} \geq 5}$ the subset, consisting of such Fano hypersurfaces $V \in \mathcal{F}$ that:

1. at every smooth point the regularity condition of [14] is satisfied;
2. through every singular point there are only finitely many lines on $V$.

We obtain Theorem 1 from the following two facts.
**Theorem 2.** The codimension of the complement of $F_{\text{reg}, \text{qsing} \geq 5}$ in $F$ is at least $(M - 3) + 1$ if $M \geq 5$.

**Theorem 3.** Every hypersurface $V \in F_{\text{reg}, \text{qsing} \geq 5}$ is birationally superrigid.

**Proof of Theorem 2** is straightforward and follows the arguments of [14, 16]; it is given in Section 2.

**Proof of Theorem 3** starts in the usual way [14, 16, 19]: take a mobile linear system $\Sigma \subset |nH|$ on a hypersurface $V \in F_{\text{reg}, \text{qsing} \geq 5}$. Assume that for a generic $D \in \Sigma$ the pair $(V, \frac{1}{n}D)$ is not canonical, that is, the system $\Sigma$ has a maximal singularity $E \subset V^+$, where $\varphi: V^+ \to V$ is a birational morphism, $V^+$ a smooth projective variety, $E$ a $\varphi$-exceptional divisor and the Noether-Fano inequality

$$\text{ord}_E \varphi^* \Sigma > na(E)$$

is satisfied (see [19] for definitions and details). We need to get a contradiction, which would immediately imply birational superrigidity and complete the proof of Theorem 3.

We proceed in the standard way.

Let $D_1, D_2 \in \Sigma$ be generic divisors and $Z = (D_1 \cap D_2)$ the self-intersection of the system $\Sigma$. Further, let $B = \varphi(E)$ be the centre of the maximal singularity $E$. If $\text{codim}_V B = 2$, then

$$\text{codim}_B (B \cap \text{Sing} V) \geq 2,$$

so we can take any curve $C \subset B, C \cap \text{Sing} V = \emptyset$, and applying [14, Sec.3], conclude that

$$\text{mult}_C \Sigma \leq n.$$

As $\text{mult}_B \Sigma > n$, we get a contradiction. So we may assume that $\text{codim}_V B \geq 3$.

**Proposition 1 (the $4n^2$-inequality).** The following estimate holds:

$$\text{mult}_B Z > 4n^2.$$

If $B \not\subset \text{Sing} V$, then the $4n^2$-inequality is a well known fact going back to the paper on the quartic three-fold [10], so in this case no proof is needed, see [19, Ch. 2] for details. Therefore we assume that $B \subset \text{Sing} V$. In that case Proposition 1 is a non-trivial new result, proved below in Sec. 3. The proof makes use of the fact that the condition of having at most quadratic singularities of rank $\geq r$ is stable with respect to blow ups, in some a bit subtle way. That fact is shown in Sec. 4.

Now we complete the proof of Theorem 3, repeating word for word the arguments of [14]. Namely, we choose an irreducible component $Y$ of the effective cycle $Z$, satisfying the inequality

$$\frac{\text{mult}_o Y}{\deg Y} > \frac{4}{M},$$

where $o \in B$ is a point of general position. Applying the technique of hypertangent divisors in precisely the same way as it is done in [14] (see also [19, Ch. 3]), we construct a curve $C \subset Y$, satisfying the inequality $\text{mult}_o C > \deg C$, which is
impossible. It is here that we need the regularity conditions. This contradiction completes the proof of Theorem 3.

Remark 1. (i) $4n^2$-inequality is not true for a quadratic singularity of rank $\leq 4$: the non-degenerate quadratic point of a three fold shows that $2n^2$ is the best we can achieve.

(ii) Birational superrigidity of Fano hypersurfaces with non-degenerate quadratic singularities was shown in [18]. Birational (super)rigidity of Fano hypersurfaces with isolated singular points of higher multiplicities $3 \leq m \leq M - 2$ was proved in [17], but the argument is really hard. These two results show that the estimate for the codimension of the non-rigid locus could most probably be considerably sharpened.

(iii) There are a few other papers where various classes of singular Fano varieties were studied from the viewpoint of their birational rigidity. The most popular object was three-dimensional quartics [13, 5, 11, 21]. Other families were investigated in [2, 3]. A family of Fano varieties (Fano double spaces of index one) with a higher dimensional singular locus was recently proven to be birationally superrigid in [12].

(iv) A recent preprint of de Fernex [6] proves birational superrigidity of a class of Fano hypersurfaces of degree $M$ in $\mathbb{P}^M$ with not necessarily isolated singularities without assuming regularity. But the dimension of the singularity locus is bounded by $\frac{1}{2}M - 4$, and no estimate of the codimension of the complement of this class is given.

2. The estimates for the codimension. Let us prove Theorem 2.

First we discuss the regularity conditions in more details. Let $x$ be a smooth point on a hypersurface $V$ of degree $M$ in $\mathbb{P}^M$. Choose homogeneous coordinates $(X_0 : \ldots : X_M)$ on $\mathbb{P}^M$ such that $x = (1 : 0 : \ldots : 0)$. Then $V \cap \{X_0 \neq 0\}$ is the vanishing locus of a polynomial

$$q_1 + \cdots + q_M$$

where each $q_i$ is a homogeneous polynomial of degree $i$ in $M$ variables $X_1, \ldots, X_M$. The regularity condition of [14] states that $q_1, \ldots, q_{M-1}$ is a regular sequence in $\mathbb{C}[x_1, \ldots, x_M]$. In particular,

$$\text{codim}_{\mathbb{A}^M}(\{q_1 = \ldots = q_{M-1} = 0\}) = 1.$$ 

Since all the vanishing loci $\{q_i = 0\}$ are cones with vertex in $x$, the set $\{q_1 = \ldots = q_{M-1} = 0\}$ must consist of a finite number of lines through $x$. Hence there also is only a finite number of lines on $V$ through $x$.

If $x$ is a singular point on $V$ then $q_1 \equiv 0$. The regularity condition (2) is equivalent to

$$\text{codim}_{\mathbb{A}^M}(\{q_2 = \ldots = q_M = 0\}) = 1.$$ 

since $\{q_2 = \ldots = q_M = 0\} \subset V$, and because of homogeneity every line through $x$ on $V$ also lies in $\{q_i = 0\}$.

It is not known whether the set $\mathcal{F}_{\text{reg}}$ is Zariski-open in $\mathcal{F}$, but it certainly contains a Zariski-open subset of $\mathcal{F}$. The codimension in $\mathcal{F}$ of its complement $\mathcal{F} \setminus \mathcal{F}_{\text{reg}}$ is
defined as the codimension of the Zariski closure of the complement. On the other hand, $F_{\text{qsing} \geq 5}$ is certainly Zariski-open, hence $F \setminus F_{\text{qsing} \geq 5}$ is Zariski-closed. We have

$$\text{codim}_F(F \setminus F_{\text{reg}, \text{qsing} \geq 5}) = \min(\text{codim}_F(F \setminus F_{\text{reg}}), \text{codim}_F(F \setminus F_{\text{qsing} \geq 5})).$$

Hence the estimate of Theorem 2 follows from the following two propositions:

**Proposition 2.** The codimension of the complement of $F \setminus F_{\text{qsing} \geq 5}$ in $F$ is at least $\left(\frac{M-3}{2}\right) + 1$ if $M \geq 5$.

**Proposition 3.** The codimension of the (Zariski closure of the) complement of $F \setminus F_{\text{reg}}$ in $F$ is at least $M(M-5)/2 + 4$ if $M \geq 5$.

**Proof of Proposition 2.** Let $S_M := \mathbb{P}^{(M+1)/2} - 1$ be the projectivized space of all symmetric $M \times M$-matrices with complex entries. Let $S_{M,r}$ be the projectivized algebraic subset of $M \times M$ symmetric matrices of rank $\leq r$. The locus $Q_r(P)$ of hypersurfaces $H \in F$ with $P \in H$ a singularity that is at least a quadratic point of rank at most $r$ has codimension in $F$ equal to

$$\text{codim}_F Q_r(P) = 1 + M + \text{codim}_{S_M} S_{M,r} = 1 + M + \dim S_M - \dim S_{M,r} = M + \left(\frac{M+1}{2}\right) - \dim S_{M,r}.$$

Let $G(M - r, M)$ be the Grassmann variety parametrizing $(M - r)$-dimensional subspaces of $\mathbb{C}^M$. To calculate $\dim S_{M,r}$ we consider the incidence correspondence (see [8, Ex.12.4])

$$\Phi := \{(A, \Lambda) : \Lambda^T \cdot A = A \cdot \Lambda = 0\} \subset S_M \times G(M - r, M).$$

Since the fibers of the natural projection $\pi_2 : \Phi \to G(M - r, M)$ is given by a linear subspace of $S_M$ of dimension $\left(\frac{r+1}{2}\right) - 1$, the variety $\Phi$ is irreducible of

$$\dim \Phi = \left(\frac{r+1}{2}\right) - 1 + r(M - r).$$

Since on the other hand the natural projection $\pi_1 : \Phi \to S_M$ is generically $1 : 1$ onto $S_{M,r}$, $\dim \Phi = \dim S_{M,r}$.

Consequently, since the $Q_r(P)$ cover $Q_r$ and $P$ varies in $\mathbb{P}^M$,

$$\text{codim}_F Q_r \geq \text{codim}_F Q_r(P) - M = \left(\frac{M - r + 1}{2}\right) + 1.$$

This completes the proof of Proposition 2.

For $r = 4$, we have $\text{codim}_F Q_4 \geq M$ if

$$\text{codim}_F Q_4 - M \geq \left(\frac{M - 3}{2}\right) + 1 - M = \frac{(M - 2)(M - 7)}{2} \geq 0,$$

hence if $M \geq 7$. 

4
Proof of Proposition 3. Let $\Phi = \{(x, H) : x \in H\} \subset \mathbb{P}^M \times \mathcal{F}$ be the incidence variety of hypersurfaces of degree $M$ in $\mathbb{P}^M$. Let $\Phi_{\text{reg}}$ be the subset of pairs $(x, H)$ satisfying the regularity conditions. Note that the Zariski closure $\Phi \setminus \Phi_{\text{reg}}$ in $\Phi$ maps onto the Zariski closure $\overline{\mathcal{F} \setminus \mathcal{F}_{\text{reg}}}$ in $\mathcal{F}$. The fiber of $\Phi_{\text{reg}}$ over a point $x \in \mathbb{P}^M$ under the natural projection $\pi_1 : \mathbb{P}^M \times \mathcal{F} \to \mathbb{P}^M$ can be described as

$$\Phi_{\text{reg}}(x) := \{H : x \in H \text{ satisfies the regularity conditions}\} \subset \mathcal{F}.$$ 

Choosing homogeneous coordinates $(X_0 : \ldots : X_M)$ on $\mathbb{P}^M$ such that $x = (1:0:\ldots:0)$ we can write $\mathcal{F} = \mathbb{P}H^0(\mathbb{P}^M, \mathcal{O}_{\mathbb{P}^M}(M))$ as a projectivized product

$$\mathcal{F} = \mathbb{P}(\bigoplus_{i=0}^M \mathcal{P}_{i,M} \cdot X_0^{M-i}),$$

where the $\mathcal{P}_{i,M}$ are the vector spaces of homogeneous polynomials in $X_1, \ldots, X_M$ of degree $i$. In particular, the $\pi_1$-fiber $\Phi(x)$ of $\Phi$ over $x$ is $\mathbb{P}(\bigoplus_{i=1}^M \mathcal{P}_{i,M} \cdot X_0^{M-i})$.

For another point $x' \in \mathbb{P}^M$ also choose homogeneous coordinates $(X'_0 : \ldots : X'_M)$ on $\mathbb{P}^M$ such that in these new coordinates $x' = (1:0:\ldots:0)$. Then the projective-linear automorphism on $\mathbb{P}^M$ given by the coordinate change from $(X_0 : \ldots : X_M)$ to $(X'_0 : \ldots : X'_M)$ maps a polynomial $F(X_0, \ldots, X_M)$ to the polynomial $F(X'_0, \ldots, X'_M)$. In particular, the induced linear automorphism on the affine cone $H^0(\mathbb{P}^M, \mathcal{O}_{\mathbb{P}^M}(M))$ over $\mathcal{F}$ maps the product structure $\prod_{i=0}^M \mathcal{P}_{i,M} \cdot X_0^{M-i}$ onto the product structure $\prod_{i=0}^M \mathcal{P}_{i,M}' \cdot (X'_0)^{M-i}$. Hence the induced projective-linear automorphism on $\mathcal{F}$ maps $\Phi(x)$ onto $\Phi(x')$ and $\Phi_{\text{reg}}(x)$ to $\Phi_{\text{reg}}(x')$ because the regularity conditions only depend on these product structures.

Consequently, the $\pi_1$-fibers of the Zariski closure $\overline{\Phi \setminus \Phi_{\text{reg}}}$ are the Zariski closure $\overline{\Phi(x) \setminus \Phi_{\text{reg}}(x)}$, hence

$$\dim \overline{\Phi \setminus \Phi_{\text{reg}}} = \dim \overline{\Phi(x) \setminus \Phi_{\text{reg}}(x)} + M.$$ 

Since $\dim \overline{\mathcal{F} \setminus \mathcal{F}_{\text{reg}}} \leq \dim \overline{\Phi \setminus \Phi_{\text{reg}}}$ we conclude

$$\text{codim}_{\mathcal{F}} \overline{\mathcal{F} \setminus \mathcal{F}_{\text{reg}}} \geq \dim \mathcal{F} - \dim \overline{\Phi \setminus \Phi_{\text{reg}}} = \text{codim}_{\mathcal{F}} \overline{\Phi(x) \setminus \Phi_{\text{reg}}(x)} - M = \text{codim}_{\mathcal{F}} \Phi(x) \setminus \Phi_{\text{reg}}(x) - (M - 1).$$

Let $\tilde{\Phi}(x) = \prod_{i=1}^M \mathcal{P}_{i,M}$ and $\tilde{\Phi}_{\text{reg}}(x)$ be the preimages of $\Phi(x)$, $\Phi_{\text{reg}}(x)$ in the affine cone $H^0(\mathbb{P}^M, \mathcal{O}_{\mathbb{P}^M}(M)) = \prod_{i=0}^M \mathcal{P}_{i,M}$ over $\mathcal{F}$. Obviously,

$$\text{codim}_{\mathcal{F}} \overline{\Phi(x) \setminus \Phi_{\text{reg}}(x)} = \text{codim}_{\mathcal{F}} \Phi(x) \setminus \Phi_{\text{reg}}(x).$$

$\Phi(x) \setminus \Phi_{\text{reg}}(x)$ consists of a subset $S_1$ Zariski-closed in $\mathcal{P}_{1,M} \times \prod_{i=2}^M \mathcal{P}_{i,M}$ of polynomials $q_1 + \cdots + q_M$ not satisfying regularity condition (1), where $\mathcal{P}_{1,M} = \mathcal{P}_{1,M} \setminus \{0\}$, and a Zariski-closed subset $S_2$ of $\{0\} \times \prod_{i=2}^M \mathcal{P}_{i,M}$ of polynomials $q_2 + \cdots + q_M$ not satisfying
regularity condition (2). Hence $\Phi(x) \setminus \Phi_{\text{reg}}(x)$ is the union of the Zariski closure of $S_1$ in $\Phi(x)$ and $S_2$. Consequently,

$$\text{codim}_{\Phi(x)}(\Phi(x) \setminus \Phi_{\text{reg}}(x)) = \min(\text{codim}_{P_{1,M} \times \prod_{i=2}^M P_{i,M}} S_1, \text{codim}_{\Phi(x)}(S_2)).$$

For $1 \leq j < i \leq M$ let $\pi_{i,j} : P_{1,M} \times \prod_{k=2}^i P_{k,M} \rightarrow P_{1,M} \times \prod_{k=2}^j P_{k,M}$ be the natural projection. Following the notations in [14] we set for $k = 2, \ldots, M - 1$

$$Y_k := \{(q_1, \ldots, q_k) \in P_{1,M} \times \prod_{i=2}^k P_{i,M} : \text{codim}_{\Phi(M)} \{q_1 = \ldots = q_k = 0\} < k\},$$

$$R_k := (P_{1,M} \times \prod_{i=2}^k P_{i,M}) \setminus Y_k,$$

$$\mu_k := \min_{(q_1, \ldots, q_{k-1}) \in R_{k-1}} \text{codim}_{\pi^{-1}_{k,k-1}(q_1, \ldots, q_{k-1})} (\pi^{-1}_{k,k-1}(q_1, \ldots, q_{k-1} \cap Y_{k-1})).$$

$S_1$ can be stratified into disjoint subsets

$$S_1 = \bigcup_{i=2}^{M-1} \pi^{-1}_{M,i}(Y_i) \cap \pi^{-1}_{M,i-1}(R_{i-1}).$$

Each stratum $\pi^{-1}_{M,i}(Y_i) \cap \pi^{-1}_{M,i-1}(R_{i-1})$ is Zariski-closed in $\pi^{-1}_{M,i-1}(R_{i-1})$. Hence

$$\text{codim}_{P_{1,M} \times \prod_{i=2}^M P_{i,M}} S_1 = \min_{2 \leq i \leq M-1} \text{codim}_{\pi^{-1}_{M,i-1}(R_{i-1})} (\pi^{-1}_{M,i}(Y_i) \cap \pi^{-1}_{M,i-1}(R_{i-1})) \geq \min_{2 \leq i \leq M-1} \mu_i.$$ 

In the same way as for $S_1$ we obtain

$$\text{codim}_{\prod_{i=2}^M P_{i,M}} S_2 \geq \min_{2 \leq i \leq M} \nu_i,$$

where

$$\nu_k := \min_{(q_2, \ldots, q_{k-1}) \in Q_{k-1}} \text{codim}_{\pi^{-1}_{k,k-1}(q_2, \ldots, q_{k-1})} (\sigma^{-1}_{k,k-1}(q_2, \ldots, q_{k-1} \cap Z_k),$$

$$Q_k := \prod_{i=2}^k P_{i,M} \setminus Z_k,$$

$$Z_k := \{(q_2, \ldots, q_k) \in \prod_{i=2}^k P_{i,M} : \text{codim}_{\Phi(M)} \{q_2 = \ldots = q_k = 0\} < k - 1\}$$

and $\sigma_{k,k-1} : \prod_{i=2}^k P_{i,M} \rightarrow \prod_{i=2}^{k-1} P_{i,M}$ is the natural projection. Consequently,

$$\text{codim}_{\prod_{i=1}^M P_{i,M}} S_2 \geq \min_{2 \leq i \leq M} \nu_i + M,$$
because \( \dim P_{1,M} = M \). Using the technique of \([13]\),

\[
\mu_i \geq \binom{M}{i}, \quad i = 2, \ldots, M - 1, \quad \text{and} \quad \nu_j \geq \binom{M + 1}{j}, \quad j = 2, \ldots, M.
\]

Unfortunately these estimates are too weak for our purposes if \( i = M - 1 \) and \( j = M \).

Using the technique of \([16]\) we obtain a better estimate for

\[
\text{codim}_{\pi_{M-1}(M-2)(R_{M-2})} (R_{M-2}) \cap \pi_{M-1}(Y_{M-1}) = \text{codim}_{\pi_{M-1,1}(M-2)(R_{M-2})} (R_{M-2}) \cap Y_{M-1}.
\]

First of all, \( \pi_{M-1,1}(M-2)(R_{M-2}) \cap Y_{M-1} \) fibers over \( P_{1,M} = R_1 \), hence the codimension is at least the minimal codimension in a fiber. So we can fix a \( q_1 \in R_1 \) and choose affine coordinates \( x_1, \ldots, x_M \) such that \( q_1 = x_1 \). Restricting the \( q_2, \ldots, q_{M-1} \) to \( \{x_1 = 0\} \cong \mathbb{A}^{M-1} \) we obtain homogeneous polynomials in the variables \( x_2, \ldots, x_M \).

Hence their vanishing sets can be projectivized in \( \mathbb{P}^{M-2} \), and setting

\[
R'_{M-3} := \{(q_2, \ldots, q_{M-2}) : \text{codim}_{\mathbb{P}^{M-2}}(\{q_2 = \ldots = q_{M-2} = 0\}) = M-3\} \subset \prod_{i=2}^{M-2} P'_{i,M-1},
\]

\[
Y'_{M-2} := \{(q_2, \ldots, q_{M-1}) : \text{codim}_{\mathbb{P}^{M-2}}(\{q_2 = \ldots = q_{M-1} = 0\}) < M-2\} \subset \prod_{i=2}^{M-1} P'_{i,M-1}
\]

we want to determine a lower bound for

\[
\text{codim}_{(\pi'_{M-1,M-2})^{-1}(R'_{M-3})(\pi'_{M-1,M-2})^{-1}(R_{M-3})} \cap Y'_{M-2}.
\]

Here, \( P'_{i,M-1} \) is the space of homogeneous polynomials of degree \( i \) in \( M-1 \) variables \( x_2, \ldots, x_M \) and \( \pi'_{M-1,1,M-2} : \prod_{i=1}^{M-1} P'_{i,M-1} \to \prod_{i=1}^{M-2} P'_{i,M-1} \) is the natural projection.

For each tuple \((q_2, \ldots, q_{M-2}) \in R'_{M-3} \), integers \( 2 \leq b \leq M-2 \) and \( 2 \leq i_1 < \ldots < i_{b-1} \leq M-2 \), there exists a \( b \)-dimensional linear subspace \( L_b \subset \mathbb{P}^{M-2} \) such that \( \{q_1 = \ldots = q_{i_{b-1}} = 0\} \cap L_b \subset \mathbb{P}^{M-2} \) has only 1-dimensional components. Vice versa, a tuple \((q_2, \ldots, q_{M-1}) \) lies in \((\pi'_{M-1,M-2})^{-1}(R'_{M-3}) \cap Y'_{M-2} \) if for each 1-dimensional irreducible component \( B \subset \{q_2 = \ldots = q_{M-2} = 0\} \) spanning the linear subspace \( \langle B \rangle \subset \mathbb{P}^{M-2} \) of dimension \( b \) there exist integers \( 2 \leq i_1 < \ldots < i_{b-1} \leq M-2 \) such that \( \{q_1 = \ldots = q_{i_{b-1}} = 0\} \cap \langle B \rangle \) contains \( B \) as a 1-dimensional irreducible component and \( q_i|B \equiv 0 \) for all \( i \in \{2, \ldots, M-1\} \setminus \{i_1, \ldots, i_{b-1}\} \) (hence for all \( 2 \leq i \leq M-1 \))

In the terminology of \([16]\) \( q_1, \ldots, q_{b-1} \) is called a \textit{good sequence} for \( B \subset \langle B \rangle \). Its existence can be shown inductively, using the regularity condition defining \( R'_{M-3} \).

If \( b = 1 \), the \( i_1, \ldots, i_{b-1} \) do not exist, and the condition restricts to

\[
q_2|B \equiv \ldots \equiv q_{M-1}|B \equiv 0
\]
on the line \( B = \langle B \rangle \).
We can cover \((\pi'_{M-1,M-2})^{-1}(R'_{M-3}) \cap Y'_{M-2}\) by subsets \(Z(b; i_1, \ldots, i_{b-1}; L_b)\) consisting of all tuples \((q_2, \ldots, q_{M-1}) \in (\pi'_{M-1,M-2})^{-1}(R'_{M-3})\) such that

\[
\dim \{q_{i_1} = \ldots = q_{i_{b-1}} = 0\} \cap L_b = 1,
\]

\(\{q_1 = \ldots = q_{b-1} = 0\} \cap L_b\) contains irreducible components linearly spanning \(L_b\) and \(q_i \equiv 0\) on such a component, for each \(2 \leq i \leq M - 1\). Here, \(1 \leq b \leq M - 2\), \(2 \leq i_1 < \ldots < i_{b-1} \leq M - 2\), and the \(L_b\) are parametrized by the (projective) Grassmann variety \(\mathbb{G}(b, M - 2)\) of \(b\)-dimensional linear subspaces \(L_b \subset \mathbb{P}^{M-2}\). For \(b = 1\),

\[
Z(1; L) := \{(q_2, \ldots, q_{M-1}) : q_{i|L} \equiv 0, 2 \leq i \leq M - 1\}.
\]

All these subsets are Zariski-closed in varying Zariski-open subsets of \((\pi'_{M-1,M-2})^{-1}(R'_{M-3})\).

For \(b > 1\) they fiber surjectively onto \(\prod_{k=1}^{b-1} \mathcal{P}'_{i_k}\). Hence their codimension is estimated by a lower bound for each given \(q_{i_1}, \ldots, q_{i_{b-1}}\), of the codimension of all tuples of \(q_i, i \in \{1, \ldots, M - 1\} \setminus \{i_1, \ldots, i_{b-1}\}\), such that \(q_{i|L_b} \equiv 0\) on an irreducible curve \(B\) linearly spanning \(L_b\). To find such a lower bound choose homogeneous coordinates \((X_2 : \ldots : X_M)\) such that

\[
L_b = \{X_{b+3} = \ldots = X_M = 0\}.
\]

Then \(q_i \in \mathcal{P}'_{i,M-1}\) cannot vanish on an irreducible curve \(B\) linearly spanning all of \(L_b\) if \(q_{i|L_b}\) is of the form

\[
\prod_{k=1}^{i} (a_{k,2} X_2 + \cdots + a_{k,b+2} X_{b+2}).
\]

Consequently the codimension of all \(q_i \in \mathcal{P}'_{i,M-1}\) vanishing on such a curve \(B\) is at least the dimension of the space of polynomials in this form, that is \(b \cdot i + 1\). Here, \(b\) is the dimension of the space of hyperplanes in \(\mathbb{P}^{b}\). It follows that the codimension of \(Z(b; i_1, \ldots, i_{b-1}; L_b)\) in \((a\text{ Zariski-open subset of})\ (\pi'_{M-1,M-2})^{-1}(R'_{M-3})\) is at least

\[
\sum_{2 \leq i \leq M - 1 \atop i \neq i_1, \ldots, i_{b-1}} (b \cdot i + 1) \geq b \cdot (2 + \cdots + (M - 1 - b) + (M - 1)) + (M - 1 - b)
\]

\[= b \cdot \frac{(M - 1 - b)(M - b)}{2} + (b + 1)(M - 1) - 2b.\]

Similarly, the codimension of \(Z(1; L)\) in \((\pi'_{M-1,M-2})^{-1}(R'_{M-3})\) is at least

\[3 + \ldots + M = \frac{M(M + 1)}{2} - 3\]

because \(i + 1\) is the codimension of the set of polynomials \(q_i \in \mathcal{P}'_{i,M-1}\) vanishing on the line \(L \subset \mathbb{P}^{M-2}\).

Taking all these data together

\[
\text{codim}(\pi'_{M-1,M-2})^{-1}(R'_{M-3})(\pi'_{M-1,M-2})^{-1}(R'_{M-3}) \cap Y'_{M-2}
\]
must be at least the minimum of the numbers
\[
b \cdot \frac{(M - 1 - b)(M - b)}{2} + (b + 1)(M - 1) - 2b - (b + 1)(M - 2 - b)
\]
\[
= b \cdot \frac{(M - 1 - b)(M - b)}{2} + b^2 + 1, \quad 2 \leq b \leq M - 2,
\]
and
\[
\frac{M(M + 1)}{2} - 3 - 2(M - 3) = \frac{M(M - 3)}{2} + 3.
\]
Here, \((b + 1)(M - 2 - b)\) and \(2(M - 3)\) are the dimensions of the Grassmann varieties parametrizing the linear subspaces \(L_b\). An easy analysis of the derivative shows that the function
\[
F(b) = b \cdot \frac{(M - 1 - b)(M - b)}{2} + b^2 + 1
\]
is everywhere increasing for \(M \geq 5\), hence the minimum of \(F(b)\) is \((M - 2)(M - 3) + 5\) if \(2 \leq b \leq M - 2\). Hence the overall minimum is
\[
\frac{M(M - 3)}{2} + 3.
\]
Following the same line of arguments we also obtain a lower bound for
\[
\text{codim}_{\pi_{M-1}^{-1}(Q_{M-1})} \cap Z_M.
\]
First note that it is not necessary to fix \(q_1\) since linear terms do not occur. Hence \(q_2, \ldots, q_M\) are polynomials in \(X_1, \ldots, X_M\). Adapting the calculations above shows that the codimension is at least the minimum of the numbers
\[
b \cdot \frac{(M - b)(M + 1 - b)}{2} + b^2 + 1, \quad 2 \leq b \leq M
\]
and
\[
\frac{(M + 1)(M - 2)}{2} + 3,
\]
that is \(\frac{(M + 1)(M - 2)}{2} + 3\), arguing as before.

Finally, all these estimates imply that \(\text{codim}_{\overline{\mathcal{F} \setminus \mathcal{F}_{\text{reg}}}}\) is bounded from below by the minimum of the numbers
\[
\binom{M}{i} - (M - 1), 2 \leq i \leq M - 2, \quad \frac{M(M - 3)}{2} + 3 - (M - 1),
\]
\[
\binom{M + 1}{j} - (M - 1) + M, 2 \leq j \leq M, \quad \frac{(M + 1)(M - 2)}{2} + 3 - (M - 1) + M,
\]
that is
\[
\frac{M(M - 3)}{2} + 3 - (M - 1) = \frac{M(M - 5)}{2} + 4
\]
for $M \geq 5$.

3. **The $4n^2$-inequality.** Let us prove Proposition 1. We fix a mobile linear system $\Sigma$ on $V$ and a maximal singularity $E \subset V^+$ satisfying the Noether-Fano inequality ord$_E \varphi^*\Sigma > na(E)$. We assume the centre $B = \varphi(E)$ of $E$ on $V$ to be maximal, that is, $B$ is not contained in the centre of another maximal singularity of the system $\Sigma$. In other words, the pair $(V, \frac{1}{n}\Sigma)$ is canonical outside $B$ in a neighborhood of the generic point of $B$.

Further, we assume that $B \subset \text{Sing } V$ (otherwise the claim is well known), so that codim($B \subset V$) $\geq 4$. Let

$$
\varphi_{i,i-1} : V_i \to V_{i-1}
$$

$E_i \to B_{i-1}$

$i = 1, \ldots, K$, be the resolution of $E$, that is, $V_0 = V$, $B_0 = B$, $\varphi_{i,i-1}$ blows up $B_{i-1} = \text{centre}(E, V_{i-1})$, $E_i = \varphi_{i,i-1}^{-1}(B_{i-1})$ the exceptional divisor, and, finally, the divisorial valuations, determined by $E$ and $E_K$, coincide.

As explained in Sec. 4 below, for every $i = 0, \ldots, K - 1$ there is a Zariski open subset $U_i \subset V_i$ such that $U_i \cap B_i \neq \emptyset$ is smooth and either $V_i$ is smooth along $U_i \cap B_i$, or every point $p \in U_i \cap B_i$ is a quadratic singularity of $V_i$ of rank at least 5. In particular, the quasi-projective varieties $\varphi_{i,i-1}^{-1}(U_{i-1})$, $i = 1, \ldots, K$, are factorial and the exceptional divisor

$$
E^*_i = E_i \cap \varphi_{i,i-1}^{-1}(U_{i-1})
$$

is either a projective bundle over $U_{i-1} \cap B_{i-1}$ (in the non-singular case) or a fibration into quadrics of rank $\geq 5$ over $U_{i-1} \cap B_{i-1}$ (in the singular case). We may assume that $U_i \subset \varphi_{i,i-1}^{-1}(U_{i-1})$ for $i = 1, \ldots, K - 1$. The exceptional divisors $E^*_i$ are all irreducible.

As usual, we break the sequence of blow ups into the lower ($1 \leq i \leq L$) and upper ($L + 1 \leq i \leq K$) parts: codim $B_{i-1} \geq 3$ if and only if $1 \leq i \leq L$. It may occur that $L = K$ and the upper part is empty (see [15, 14, 19]). Set

$$
L_* = \max\{i = 1, \ldots, K \mid \text{mult}_{B_{i-1}} V_{i-1} = 2\}.
$$

Obviously, $L_* \leq L$. Set also

$$
\delta_i = \text{codim } B_{i-1} - 2 \quad \text{for} \quad 1 \leq i \leq L_*
$$

and

$$
\delta_i = \text{codim } B_{i-1} - 1 \quad \text{for} \quad L_* + 1 \leq i \leq K.
$$

We denote strict transforms on $V_i$ by adding the upper index $i$: say, $\Sigma^i$ means the strict transform of the system $\Sigma$ on $V_i$. Let $D \in \Sigma$ be a generic divisor. Obviously,

$$
D^i|_{U_i} = \varphi^*_i(D^{i-1}|_{U_{i-1}}) - \nu_i E^*_i,
$$
where the integer coefficients \( \nu_i = \frac{1}{2} \text{mult}_{B_{i-1}} \Sigma^{i-1} \) for \( i = 1, \ldots, L^* \) and \( \nu_i = \text{mult}_{B_{i-1}} \Sigma^{i-1} \) for \( i = L^* + 1, \ldots, K \).

Now the Noether-Fano inequality takes the traditional form

\[
\sum_{i=1}^{K} p_i \nu_i > n \left( \sum_{i=1}^{K} p_i \delta_i \right),
\]

where \( p_i \) is the number of paths from the top vertex \( E_K \) to the vertex \( E_i \) in the oriented graph \( \Gamma \) of the sequence of blow ups \( \varphi_{i-1} \), see [15, 14, 19] for details.

We may assume that \( \nu_1 < \sqrt{2n} \), otherwise for generic divisors \( D_1, D_2 \in \Sigma \) we have

\[
\text{mult}_B (D_1 \circ D_2) \geq 2\nu_1^2 > 4n^2
\]

and the \( 4n^2 \)-inequality is shown. We do not use the following claim, but nevertheless it is worth mentioning.

**Lemma 1.** The inequality \( \nu_1 > n \) holds.

**Proof.** Taking a point \( p \in B \) of general position and a generic complete intersection 3-germ \( Y \ni p \), we reduce to the case of a non log canonical singularity centered at a non-degenerate quadratic point, when the claim is well known, see [4, 20]. Q.E.D.

Obviously, the multiplicities \( \nu_i \) satisfy the inequalities

\[
\nu_1 \geq \cdots \geq \nu_{L^*}
\]

and, if \( K \geq L^* + 1 \), then

\[
2\nu_{L^*} \geq \nu_{L^*+1} \geq \cdots \geq \nu_K.
\]

Now let \( Z = (D_1 \circ D_2) \) be the self-intersection of the mobile system \( \Sigma \) and set \( m_i = \text{mult}_{B_{i-1}} \Sigma^{i-1} \) for \( 1 \leq i \leq L \). Applying the technique of counting multiplicities in word for word the same way as in [15, 14, 19], we obtain the estimate

\[
\sum_{i=1}^{L} p_i m_i \geq 2 \sum_{i=1}^{L^*} p_i \nu_i^2 + \sum_{i=L^*+1}^{K} p_i \nu_i^2.
\]

Denote the right hand side of this inequality by \( q(\nu_1, \ldots, \nu_K) \). We see that

\[
\sum_{i=1}^{L} p_i m_i > \mu,
\]

where \( \mu \) is the minimum of the positive definite quadratic form \( q(\nu_1, \ldots, \nu_K) \) on the compact convex polytope \( \Delta \) defined on the hyperplane

\[
\Pi = \left\{ \sum_{i=1}^{K} p_i \nu_i = n \left( \sum_{i=1}^{K} p_i \delta_i \right) \right\}
\]
by the inequalities (2.3). Let us estimate $\mu$.

We use the standard optimization technique in two steps. First, we minimize $q|_{\Pi}$ separately for the two groups of variables

$$\nu_1, \ldots, \nu_{L_*} \quad \text{and} \quad \nu_{L_*+1}, \ldots, \nu_K.$$ 

Easy computations show that the minimum is attained for

$$\nu_1 = \cdots = \nu_{L_*} = \theta_1 \quad \text{and} \quad \nu_{L_*+1} = \cdots = \nu_K = \theta_2,$$

satisfying the inequality $2\theta_1 \geq \theta_2$. Putting

$$\Sigma_* = \sum_{i=1}^{L_*} p_i \quad \text{and} \quad \Sigma_* = \sum_{i=L_*+1}^{K} p_i,$$

we get the extremal problem

$$\bar{q}(\theta_1, \theta_2) = 2\Sigma_* \theta_1^2 + \Sigma_* \theta_2^2 \to \min$$

on the ray, defined by the inequality $2\theta_1 \geq \theta_2$ on the line

$$\Lambda = \left\{ \Sigma_* \theta_1 + \Sigma_* \theta_2 = n \sum_{i=1}^{K} p_i \delta_i \right\}.$$ 

Now we make the second step, minimizing $\bar{q}|_{\Lambda}$. The minimum is attained for $\theta_1 = \theta$, $\theta_2 = 2\theta$ (so that the condition $2\theta_1 \geq \theta_2$ is satisfied and for that reason can be ignored), where $\theta$ is obtained from the equation of the line $\Lambda$:

$$\theta = \frac{n}{\Sigma_* + 2\Sigma_*} \sum_{i=1}^{K} p_i \delta_i.$$ 

Now set

$$\Sigma_l = \sum_{i=1}^{L} p_i, \quad \Sigma_l^* = \sum_{i=L_*+1}^{L} p_i, \quad \Sigma_u = \sum_{i=L_*+1}^{K} p_i$$

(if $L \geq L_* + 1$; otherwise set $\Sigma_l^* = 0$). Obviously, the relations

$$\Sigma_l = \Sigma_* + \Sigma_l^* \quad \text{and} \quad \Sigma_* = \Sigma_l^* + \Sigma_u$$

hold. Recall that, due to our assumptions on the singularities of $V_i$ we have $\delta_i \geq 2$ for $i \leq L$. Therefore,

$$\theta \geq \frac{2\Sigma_l + \Sigma_u}{\Sigma_* + 2\Sigma_*} n$$

and so

$$\mu \geq 2 \frac{(2\Sigma_l + \Sigma_u)^2}{\Sigma_* + 2\Sigma_*} n^2.$$ 

12
Since
\[ \Sigma l \text{ mult}_B Z \geq \sum_{i=1}^{l} p_i m_i, \]
we finally obtain the estimate
\[ \text{mult}_B Z > 2 \frac{(2\Sigma l + \Sigma u)^2}{\Sigma l (\Sigma s + 2\Sigma^*)} n^2. \]
Therefore, the $4n^2$-inequality follows from the estimate
\[ (2\Sigma l + \Sigma u)^2 \geq 2\Sigma l (\Sigma s + 2\Sigma^*). \]
Replacing in the right hand side $\Sigma s + 2\Sigma^*$ by
\[ \Sigma s + 2(\Sigma^*_l + \Sigma^*_u) = \Sigma^*_l + \Sigma^*_l + 2\Sigma^*_u, \]
we bring the required estimate to the following form:
\[ 2\Sigma^*_l + \Sigma^*_u \geq 2\Sigma^*_l \Sigma^*_l, \]
which is an obvious inequality. Proof of Proposition 1 is now complete. Q.E.D.

4. Stability of the quadratic singularities under blow ups. We start with the following essential

**Definition 1.** Let $X \subset Y$ be a subvariety of codimension 1 in a smooth quasi-projective complex variety $Y$ of dimension $n$. A point $P \in X$ is called a quadratic point of rank $r$ if there are analytic coordinates $z = (z_1, \ldots, z_n)$ of $Y$ around $P$ and a quadratic form $q_2(z)$ of rank $r$ such that the germ of $X$ in $P$ is given by
\[ (P \in X) \cong \{ q_2(z) + \text{terms of higher degree} = 0 \} \subset Y. \]

**Theorem 4.** Let $X \subset Y$ be a subvariety of codimension 1 in a smooth quasi-projective complex variety $Y$ of dimension $n$, with at most quadratic points of rank $\geq r$ as singularities. Let $B \subset X$ be an irreducible subvariety. Then there exists an open set $U \subset Y$ such that

(i) $B \cap U$ is smooth, and

(ii) the blow up $\tilde{X}_U$ of $X \cap U$ along $B \cap U$ has at most quadratic points of rank $\geq r$ as singularities.

**Proof.** The statement is obvious if $B \not\subset \text{Sing}(X)$. So we assume from now on that $B \subset \text{Sing}(X)$.

By restricting to a Zariski-open subset of $Y$ we may assume that $B \subset \text{Sing}(X)$ is a smooth subvariety. By assumption there exist analytic coordinates $z = (z_1, \ldots, z_n)$ around each point $P \in B \subset Y$ such that the germ
\[ (P \in X) \cong \{ f(z) = z_1^2 + \ldots + z_r^2 + \text{terms of higher degree} = 0 \} \subset Y. \]
Then the singular locus \( \text{Sing}(X) \) is contained in the vanishing locus of the partial derivatives of this equation, hence in
\[
\begin{align*}
\left\{ \frac{\partial f}{\partial z_1} &= \cdots = \frac{\partial f}{\partial z_r} = 0 \right\}.
\end{align*}
\]
Since
\[
\frac{\partial f}{\partial z_i} = 2z_i + \text{terms of higher degree}, \quad 1 \leq i \leq r,
\]
setting \( z_1' := \frac{1}{2} \frac{\partial f}{\partial z_1}, \ldots, z_r' := \frac{1}{2} \frac{\partial f}{\partial z_r}, z_i' := z_i \) for \( r + 1 \leq i \leq n \) yields new analytic coordinates
\[
z_1', \ldots, z_r', z_{r+1}', \ldots, z_n'
\]
of \( Y \) around \( P \). In these new coordinates the defining equation of \( X \) still is of the form
\[
z_1'^2 + \ldots + z_r'^2 + \text{terms of higher degree} = 0,
\]
and \( B \subset \{ z_1' = \ldots = z_r' = 0 \} \). Perhaps after a further coordinate change we can even assume that
\[
B = \{ z_1' = \ldots = z_k' = 0 \}, \quad k \geq r.
\]

Claim. \( (P \in X) \cong \{ z_1'^2 + \ldots + z_r'^2 + f_{\geq 3} = 0 \} \) where \( f_{\geq 3} \) consists of terms of degree \( \geq 3 \) and is an element of \( (z_1', \ldots, z_k')^2 \).

Proof of Claim. \( B \subset \text{Sing}(X) \) must be contained in \( \left\{ \frac{\partial f_{\geq 3}}{\partial z_j'} = 0 \right\} \), hence \( \frac{\partial f_{\geq 3}}{\partial z_j'} \in (z_1', \ldots, z_k') \) for all \( k + 1 \leq j \leq n \). This is only possible if \( f_{\geq 3} \in (z_1', \ldots, z_k') \). Write \( f_{\geq 3} = z_1'f_1' + \ldots + z_k'f_k' \). Then as before \( \frac{\partial f_{\geq 3}}{\partial z_i'} = f_i' + \sum_{1 \leq j \leq k, j \neq i} z_j' \frac{\partial f_j'}{\partial z_i'} \in (z_1', \ldots, z_k') \) for all \( 1 \leq i \leq k \). But this is only possible if \( f_i' \in (z_1', \ldots, z_k') \) for all \( 1 \leq i \leq k \). □

Using the coordinates \( z_1', \ldots, z_n' \) we can cover the blow up of \( Y \) along \( B \) over \( P \in Y \) by \( k \) charts with coordinates
\[
t_1^{(i)}, \ldots, z_i, \ldots, t_k^{(i)}, z_{k+1}, \ldots, z_n, 1 \leq i \leq k,
\]
where \( z_j' = t_j^{(i)} z_i \) for \( 1 \leq j \leq k, j \neq i \), \( z_i' = z_i \) and \( z_l' = z_l \) for \( k + 1 \leq l \leq n \). To prove the theorem we only need to check in each chart that along the fiber of the exceptional divisor over \( P \in B \) there are at most quadratic points of rank \( \geq r \) as singularities. We distinguish several cases:

Case 1. \( 1 \leq i \leq r \), say \( i = 1 \).

Then the strict transform of \( X \) is given by the equation
\[
1 + (t_2^{(1)})^2 + \cdots + (t_r^{(1)})^2 + z_1 \cdot F + Q(t_2^{(1)}, \ldots, t_k^{(1)}) : G = 0,
\]
where \( Q \) is a quadratic polynomial in \( t_2^{(1)}, \ldots, t_k^{(1)} \) and \( G \in (z_{k+1}, \ldots, z_n) \). On the fiber of the exceptional divisor over \( P \), \( \{ z_1 = z_{k+1} = \ldots = z_n = 0 \} \), the gradient of this function can only vanish when \( t_2^{(1)} = \ldots = t_r^{(1)} = 0 \). But this locus does not
intersect the strict transform, hence in this chart the strict transform is smooth along the fiber of the exceptional divisor over \( P \).

**Case 2.** \( r + 1 \leq i \leq k \), say \( i = k \).

Then the strict transform of \( X \) is given by the equation

\[
(t_1^{(k)})^2 + \cdots + (t_{r}^{(k)})^2 + z_k \cdot F + Q(t_1^{(k)}, \ldots, t_{k-1}^{(k)}) \cdot G = 0,
\]

\( Q \) and \( G \) as above. On the fiber of the exceptional divisor over \( P \), \( \{ z_k = z_{k+1} = \ldots = z_n = 0 \} \), the gradient of this function can only vanish when \( t_1^{(k)} = \ldots = t_{r}^{(k)} = 0 \). We first discuss the origin in these coordinates,

\[
(0, \ldots, 0) \in \{ t_1^{(k)} = \ldots = t_{r}^{(k)} = z_k = z_{k+1} = \ldots = z_n = 0 \}.
\]

If \( F \) has a constant term then the strict transform of \( X \) is smooth in \( (0, \ldots, 0) \).

If \( F \) has no constant terms but contains linear terms then the rank of the quadratic term in the defining equation is still \( \geq r \) because we only add quadratic monomials containing \( z_k \) to \( (t_1^{(k)})^2 + \cdots + (t_{r}^{(k)})^2 \). Hence \( (0, \ldots, 0) \) is a quadratic point of rank \( \geq r \).

Finally, if \( F \) is of degree \( \geq 2 \) the quadratic term in the defining equation is \( (t_1^{(k)})^2 + \cdots + (t_{r}^{(k)})^2 \). Hence \( (0, \ldots, 0) \) is a quadratic point of rank \( r \).

The affine coordinate change to

\[
t_1^{(k)}, \ldots, t_{r}^{(k)}, t_{r+1}^{(k)} - a_{r+1}, \ldots, t_{k-1}^{(k)} - a_{k-1}, z_k, z_{k+1}, \ldots, z_n
\]

leads to a defining equation of the strict transform around the point

\[
(0, \ldots, 0, a_{r+1}, \ldots, a_{k-1}, 0, 0, \ldots, 0) \in \{ t_1^{(k)} = \ldots = t_{r}^{(k)} = z_k = z_{k+1} = \ldots = z_n = 0 \}
\]

in one of the forms already discussed. Consequently, in this chart all points in the strict transform of \( X \) also lying on the fiber of the exceptional divisor over \( P \) are smooth or quadratic points of rank \( \geq r \).

**Remark 2.** Note that \( \tilde{X}_U \) is again a subvariety of codimension 1 in the smooth quasi-projective blow up of \( U \) along \( B \cap U \). The universal property of blow ups [9, Prop.II.7.14] and the calculations in the proof above tell us that the exceptional locus \( E_U \subset \tilde{X}_U \) is a Cartier divisor on \( \tilde{X}_U \) such that the morphism \( E_U \to B \cap U \) is a fibration into quadrics of rank \( \geq r \) in a \( \mathbb{P}^{\text{codim}_Y B} \)-bundle.

**References**

[1] Call F. and Lyubeznik G., A simple proof of Grothendieck’s theorem on the parafactoriality of local rings, Contemp. Math. 159 (1994), 15-18.

[2] Cheltsov I. A., A double space with a double line. Sbornik: Mathematics 195 (2004), No. 9-10, 1503-1544.
[3] Cheltsov I. A., On nodal sextic fivefold. Math. Nachr. 280 (2007), No. 12, 1344-1353.

[4] Corti A., Singularities of linear systems and 3-fold birational geometry. In: Explicit Birational Geometry of 3-folds. Cambridge Univ. Press, 2000, 259-312.

[5] Corti A. and Mella M., Birational geometry of terminal quartic 3-folds. I. Amer. J. Math. 126 (2004), No. 4, 739-761.

[6] De Fernex T., Birational geometry of singular Fano hypersurfaces, preprint, arXiv:1208.6073, 2012.

[7] Grothendieck A., Cohomologie locale des faisceaux cohérents et théorèmes de Lefschetz locaux et globaux (SGA 2). Documents Mathématiques (Paris) 4, Société Mathématique de France, 2005.

[8] Harris J., Algebraic Geometry. Graduate Texts in Math. 133, Springer 1992.

[9] Hartshorne R., Algebraic Geometry. Graduate Texts in Math. 52, Springer 1977.

[10] Iskovskikh V. A. and Manin Yu. I., Three-dimensional quartics and counterexamples to the Lüroth problem, Math. USSR Sb. 86 (1971), no. 1, 140-166.

[11] Mella M., Birational geometry of quartic 3-folds. II. Math. Ann. 330 (2004), No. 1, 107-126.

[12] Mullany R., Fano double spaces with a big singular locus, Math. Notes 87 (2010), no. 3, 444-448.

[13] Pukhlikov A. V., Birational automorphisms of a three-dimensional quartic with an elementary singularity, Math. USSR Sb. 63 (1989), 457-482.

[14] Pukhlikov A. V., Birational automorphisms of Fano hypersurfaces, Invent. Math. 134 (1998), no. 2, 401-426.

[15] Pukhlikov A. V., Essentials of the method of maximal singularities. In: Explicit Birational Geometry of 3-folds. Cambridge Univ. Press, 2000, 73-100.

[16] Pukhlikov A. V., Birationally rigid Fano complete intersections, Crelle J. für die reine und angew. Math. 541 (2001), 55-79.

[17] Pukhlikov A. V., Birationally rigid Fano hypersurfaces with isolated singularities, Sbornik: Mathematics 193 (2002), No. 3, 445-471.

[18] Pukhlikov A. V., Birationally rigid singular Fano hypersurfaces, J. Math. Sci. 115 (2003), No. 3, 2428-2436.

[19] Pukhlikov A. V., Birationally rigid varieties. I. Fano varieties. Russian Math. Surveys. 62 (2007), No. 5, 857-942.
[20] Pukhlikov A. V., Birational geometry of singular Fano varieties. Proc. Steklov Math. Inst. 264 (2009), 159-177.

[21] Shramov K., Birational automorphisms of nodal quartic threefolds, arXiv:0803.4348, 2008.