Angular Momentum Generation from Holographic Chern-Simons Models

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Abstract

We study parity-violating effects, particularly the generation of angular momentum density and its relation to the parity-odd and dissipationless transport coefficient Hall viscosity, in strongly-coupled quantum fluid systems in 2+1 dimensions using holographic method. We employ a class of 3+1-dimensional holographic models of Einstein-Maxwell system with gauge and gravitational Chern-Simons terms coupled to a dynamical scalar field. The scalar can condensate and break the parity spontaneously. We find that when the scalar condensates, a non-vanishing angular momentum density and an associated edge current are generated, and they receive contributions from both gauge and gravitational Chern-Simons terms. On the other hand, the emergence of Hall viscosity is a consequence of the gravitational Chern-Simons term alone. We present both general analytic results and numeric results which take back-reactions into account. The ratio between Hall viscosity and angular momentum density resulting from the gravitational Chern-Simons term has in general a deviation from the universal 1/2 value obtained from field theory and condensed matter physics.
1 Introduction

When parity is broken, additional transport phenomena can take place and reveal interesting underlying dynamical and topological structures of the systems. Quantum Hall effect is a well known example. In fact, in 2+1 dimensions, when parity is broken, in addition to Hall conductivity, a few other parity-odd transport coefficients can also arise. These transport coefficients had been systematically studied in [1] for relativistic fluids and recently in [2] for non-relativistic fluids. Among them, Hall viscosity is the dissipationless and parity-odd cousin of shear viscosity, just like Hall conductivity can be viewed similarly compared to ordinary (longitudinal) conductivity. The effect of Hall viscosity can be interpreted as a Lorentz-type force (sometimes called the “Lorentz shear force”) acting perpendicular to the shear flow. Hall viscosity was first studied for various quantum Hall states [3–9] and then for chiral superfluid states [7,9] and topological insulators [10,11]. It was also studied using general approaches such as linear response theory [12], effective field theories [13–16],
viscoelastic-electromagnetism [17] and quantum hydrodynamics of vortex flow [18–20]. It was first noticed in [7,9] and later re-derived using more general methods in [12,14,16] that Hall viscosity is equal to a half of the orbital spin density of the systems. In the absence of mechanical rotation or spin-orbit coupling, this is the total angular momentum density of the system. This reveals another interesting effect when parity is broken – the generation of angular momentum density and edge current. The microscopic origin of such an angular momentum density varies for different systems, but the common feature is the formation of vortices. For quantum Hall states, this is from the cyclotron motion of the electrons or quasi-particles in magnetic field. For chiral superfluids, this is due to the relative orbital angular momentum of the two paired electrons in Cooper pairs [21–23]. The effect of the non-vanishing angular momentum density and Hall viscosity is to accumulate momentum and charges on the boundaries [18] and to induce an edge current.

Over the last decade, the gauge/gravity correspondence [24–26] has often offered new insights to the understanding of strongly-coupled quantum systems, such as quark-gluon plasma, superconductors, superfluids, quantum Hall effects and topological insulators, just to name a few. In this paper we are trying to understand the generation of angular momentum density and its relation to Hall viscosity in 2+1-dimensional strongly-coupled systems from the holographic point of view. In [27] a holographic model with dynamical gravitational Chern-Simons term was first used to calculate Hall viscosity. This model was further upgraded and numerically computed in [28,29]. Recently [30,31] studied both Hall viscosity and Curl viscosity using similar holographic models with Chern-Simons terms. Spontaneous generation of angular momentum from holographic models with gauge and gravitational Chern-Simons terms was also studied in [32]. The common feature of all these studies is that in their holographic actions, there are Chern-Simons terms (gauge [33,34] or gravitational [35], or both) coupled to a dynamical axion scalar field, which break parity when the axion condensates. However, none of them reported to find both Hall viscosity and angular momentum density at the same time, thus does not yield a unified picture of them as that from the studies using non-holographic approaches [7,9,13,14,16]. Recently, [36] studied a different class of model – the holographic $p_x + ip_y$ model of [37]. They found both non-vanishing Hall viscosity and angular momentum density in the superfluid phase, and showed that the ratio between Hall viscosity and angular momentum density is a constant, at least near the critical regime. This suggests an agreement with previous results from [7,9,13,14,16]. In fact, the holographic $p_x + ip_y$ model can be viewed as a dual description to chiral superfluid states, like those studied in [21–23], and it was from computing Hall viscosity for such states (among others) that [7] first pointed out the relation between Hall viscosity and angular momentum density. The holographic $p_x + ip_y$ model is different from those Chern-Simons models in [27–30,32] that it does not contain Chern-Simons terms in the action, so the action is perfectly parity-preserving. But the ground state breaks spatial parity by locking it to non-Abelian gauge parity, which is broken by the appearance of non-Abelian gauge connection, and this is the only source for the emergence of Hall viscosity, angular momentum density, and Hall conductivity (studied numerically earlier in [38]). Thus in holographic $p_x + ip_y$ model all the parity-odd transport coefficients and angular momentum are generated in a unified way.

In this paper, we go back to the holographic gauge and gravitational Chern-Simons models, and compute the angular momentum density using the method proposed in [36]. We
will show that there is indeed a generation of angular momentum density accompanying the emergence of Hall viscosity, and it receives contributions from both the gauge and gravitational Chern-Simons terms. Their ratio resulting from the gravitational Chern-Simons term is not exactly a fixed constant, but remains more or less unchanged as temperature is varied, except at the very low temperature regime. The ratio depends on conformal dimension of the condensate that breaks parity spontaneously.

The paper is organized as following. In section 2, we give the general formalism of the holographic Chern-Simons models we are using and the ground state ansatz. In section 3 and 4, we compute vector mode and tensor mode fluctuations to obtain angular momentum density and Hall viscosity, respectively. In section 5, we present numeric results, first in the probe limit and then take back-reactions into account. Conclusions and comments follow in section 6. Through out this paper, we will work in 3+1 spacetime dimensions.

Note added: We notice that [39] appeared on the same day when the original version of this paper was submitted for arXiv publication, and it overlaps with this paper on the analytic calculation of angular momentum density. Our methods for computing angular momentum density are not the same, but we eventually reach the same analytic result. We thank the authors of [39] for email communications and pointing out the subtlety of vector mode boundary conditions in the original version of this paper.

2 Holographic Chern-Simons Models

For a general review on Chern-Simons modified gravity theory, we refer readers to [40]. In this section, we will only list key ingredients relevant to the calculation of Hall viscosity and angular momentum.

2.1 Bulk and Boundary Actions

The bulk action of our holographic Chern-Simons model in 3+1 dimension is:

\[ S_{\text{bulk}} = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left( R - 2\Lambda - \frac{1}{4} F_{\mu\nu}^2 \right) + S_\vartheta + S_{\text{CS}} + S_{\text{A-CS}}, \tag{2.1} \]

where \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \) is the field strength of Maxwell field. The cosmological constant \( \Lambda = -3/L^2 \) and \( L \) is the AdS radius. The real (pseudo) scalar \( \vartheta \)'s action is

\[ S_\vartheta = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left\{ -\frac{1}{2} \left( \partial \vartheta \right)^2 - V [\vartheta] \right\}, \tag{2.2} \]

We choose the potential to be

\[ V [\vartheta] = \frac{1}{2} m^2 \vartheta^2 + \frac{1}{4} c_4 \vartheta^4, \tag{2.3} \]

though in the actual calculation we will try to keep \( V [\vartheta] \) general and not to implement this form until we have to.
Abelian gauge Chern-Simons term is

\[ S_{\text{CS}}^A = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left\{ \frac{\lambda_A}{4} \Theta_A[\vartheta] *F \right\}, \]  

(2.4)

where \( \lambda_A \) is the coupling constant,

\[ *F = F^{\mu\nu} F_{\mu\nu}, \]  

(2.5)

and the dual field strength is

\[ *F^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta}. \]  

(2.6)

\( \Theta_A[\vartheta] \) is a general functional of \( \vartheta \).

The gravitational Chern-Simons term is

\[ S_{\text{CS}} = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left\{ -\frac{\lambda}{4} \Theta[\vartheta] *RR \right\}, \]  

(2.7)

where \( \lambda \) is the coupling constant and the Pontryagin density is defined as

\[ *RR = R^{\mu\nu\rho\sigma} R_{\nu\mu\rho\sigma}, \]  

(2.8)

and the dual Riemann tensor is

\[ *R^{\mu\nu\rho\sigma} = \frac{1}{2} \epsilon^{\rho\sigma\eta\kappa} R_{\mu\nu}^{\eta\kappa}, \]  

(2.9)

where \( \epsilon^{\rho\sigma\eta\kappa} \) is the Levi-Civita tensor. We choose the convention\(^1\) \( \epsilon_{txyz} = \sqrt{-g} \). \( \Theta[\vartheta] \) is a general functional of \( \vartheta \). Again we will try to keep its form general for as long as possible in our calculation. A more detailed discussion of its form will be presented in Section 5.

The boundary terms include the Gibbons-Hawking term

\[ S_{\text{GH}} = \frac{1}{\kappa^2} \int_{\text{boundary}} d^3x \sqrt{-\gamma} K \]  

(2.10)

and a counter term

\[ S_{\text{ct}} = -\frac{2}{\kappa^2 R} \int_{\text{boundary}} d^3x \sqrt{-\gamma}, \]  

(2.11)

where \( \hat{n}_\mu \) is the outgoing unit normal 1-form of the boundary and \( \gamma_{\mu\nu} = g_{\mu\nu} - \hat{n}_\mu \hat{n}_\nu \) is the induced metric on the boundary. The extrinsic curvatures are \( K_{\mu\nu} = \gamma_\rho^{\mu} \gamma_\nu^{\rho} \nabla_\rho \hat{n}_\sigma \) and \( K = K_{\mu}^\mu \). There is also a Chern-Simons boundary term, analog to Gibbons-Hawking term, added such that the Dirichlet boundary value problem is well posed:

\[ S_{\partial\text{CS}} = \frac{1}{2\kappa^2} \int_{\text{boundary}} d^3x \sqrt{-\gamma} \left\{ -\lambda \Theta[\vartheta] \hat{n}_\mu \epsilon^{\mu\sigma\gamma\delta} K_{\sigma}^{\nu} \nabla_{\gamma} K_{\delta\eta} \right\}. \]  

(2.12)

\(^1\)Our convention for \( \epsilon^{\rho\sigma\eta\kappa} \) differs from that of [27,36] by a sign, thus the corresponding terms in both the Kubo formula and 2-point functions differ by a sign, but the final expression for Hall viscosity remains the same because these two signs cancel.
2.2 Perturbative Expansion of Bulk Action

To compute 2-point functions, we perturbatively expand the on-shell actions around the background up to second order in field fluctuations. The background and fluctuations are

\[
g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}, \\
A_\mu = \bar{A}_\mu + a_\mu, \\
\vartheta = \bar{\vartheta} + \delta \vartheta,
\]

where \( \bar{g}_{\mu\nu}, \bar{A}_\mu \) and \( \bar{\vartheta} \) are the background and \( h_{\mu\nu}, a_\mu \) and \( \delta \vartheta \) are fluctuations. To fully consider the back-reactions of the gauge fields on the metric, we assume \( h_{\mu\nu}, a_\mu \) and \( \delta \vartheta \) are of the same order. We also define the short-hand notations:

\[
\Theta[\vartheta] = \bar{\Theta} + \delta \Theta, \quad \text{where} \quad \bar{\Theta} = \Theta[\bar{\vartheta}], \quad \delta \Theta = \left. \frac{\delta \Theta[\vartheta]}{\delta \vartheta} \right|_{\vartheta = \bar{\vartheta}},
\]

and similarly for \( \Theta_A[\vartheta] \). The first order on-shell action which is linear in fluctuations is

\[
S_{\text{bulk}}^{(1)} = \frac{1}{2\kappa^2} \int d^4x \partial_\mu \left\{ \sqrt{-g} \left[ \bar{\nabla}_\nu h^{\mu\nu} - \bar{\nabla}_\mu h - \bar{F}^{\mu\nu} a_\nu + \lambda_4 \bar{\Theta} \hat{F}^{\mu\nu} a_\nu \right. \right. \\
- \left. \left. \delta \vartheta \bar{\nabla}_\mu \bar{\vartheta} - \lambda \tilde{\Theta} \hat{F}^{\mu\nu} \bar{\nabla}_\nu h_{\alpha\beta} + \lambda h_{\alpha\beta} \hat{F}^{\mu\nu} \bar{\nabla}_\nu \Theta \right\} \right\}.
\]

The second order on-shell action quadratic in fluctuations is

\[
S_{\text{bulk}}^{(2)} = \frac{1}{4\kappa^2} \int d^4x \partial_\mu \left\{ \sqrt{-g} \left[ \frac{1}{2} h \bar{\nabla}_\nu h^{\mu\nu} + \frac{3}{2} h^{\mu\nu} \bar{\nabla}_\nu h - h^{\rho\sigma} \bar{\nabla}_\rho h^{\mu}_{\sigma} - 2h^{\mu\rho} \bar{\nabla}_\sigma h_{\rho\sigma} \right. \right. \\
+ \frac{3}{2} h^{\rho\sigma} \bar{\nabla}_\rho h_{\rho\sigma} - \frac{1}{2} h \bar{\nabla}^\mu h - a_\nu \left( \frac{1}{2} \bar{F}^{\mu\nu} h + \bar{F}^{[\mu}_{\rho} h^{\nu]\rho} + F^{(1)\mu\nu} \right) \\
+ \left( h^{\mu\nu} - \frac{1}{2} g^{\mu\nu} h \right) \delta \Theta \bar{\nabla}_\nu \bar{\vartheta} - \delta \Theta \bar{\nabla}^\mu \delta \vartheta \right\} \right\} \\
+ \frac{\lambda}{4\kappa^2} \int d^4x \partial_\mu \left\{ \sqrt{-g} \left[ \hat{\Theta} \hat{R}^{\rho\mu\beta\gamma} h_{\alpha} \left( \bar{\nabla}_\beta h_{\gamma\delta} + \bar{\nabla}_{[\gamma} h_{\delta\beta]} \right) + \hat{\Theta} \hat{R}^{\rho\mu\beta\gamma} h_{\alpha} \bar{\nabla}_\beta h_{\gamma\delta} \\
- h_{\alpha\beta} h_\rho \hat{R}^{\rho\mu\beta\gamma} \bar{\nabla}_\nu \Theta - \frac{1}{2} \Theta \hat{R}^{\rho\mu\beta\gamma} \bar{\nabla}_\nu \Theta \left( \bar{\nabla}_\gamma \bar{\nabla}_\delta h^{\mu\nu} + \bar{\nabla}_\gamma \bar{\nabla}^{[\alpha\beta]} h_{\delta} \right) \bar{\nabla}_\nu h_{\alpha\beta} \right. \right. \\
+ \frac{1}{2} \hat{\Theta} \hat{R}^{\rho\mu\beta\gamma} h_{\alpha} \left( \bar{\nabla}_\gamma \bar{\nabla}_\delta h^{\mu\nu} + \bar{\nabla}_\gamma \bar{\nabla}^{[\alpha\beta]} h_{\delta} \right) - \hat{\Theta} \hat{R}^{\rho\mu\beta\gamma} \delta \Theta \bar{\nabla}_\nu h_{\alpha\beta} \\
+ \hat{\Theta} \hat{R}^{\rho\mu\beta\gamma} h_{\alpha} \bar{\nabla}_\nu \delta \Theta \right\} \right\} \\
+ \frac{\lambda A}{2\kappa^2} \int d^4x \partial_\mu \left\{ \sqrt{-g} \left[ \delta \Theta_A \hat{F}^{\mu\nu} a_\nu + \bar{\Theta}_A \hat{F}^{(1)\mu\nu} a_\nu \right] \right\}.
\]

Here all co-variant derivatives \( \bar{\nabla} \) and raising and lowering indices are with respect to the background metric \( \bar{g}_{\mu\nu} \), with \( h \equiv h_{\mu}^{\mu} \) and \( F^{(1)}_{\mu\nu} \equiv \bar{\nabla}_{[\mu} a_{\nu]} \). These actions are written as integrals of total derivatives, which means they are boundary terms.

\(^2\)In this paper we define the symmetrization \( A_{(\mu} B_{\nu)} \equiv A_\mu B_\nu + A_\nu B_\mu \) and the anti-symmetrization \( A_{[\mu} B_{\nu]} \equiv A_\mu B_\nu - A_\nu B_\mu \) without the factor of \( \frac{1}{2} \).
2.3 Equations of Motion and Background

The EOMs are

\[
R_{\mu \nu} - \frac{1}{2} R g_{\mu \nu} + \Lambda g_{\mu \nu} + \lambda C_{\mu \nu} = \frac{1}{2} T_{\mu \nu}, \quad (2.17)
\]

\[
\nabla_\mu \left( F^{\mu \nu} - \lambda A \bar{\theta} A^* F^{\mu \nu} \right) = 0, \quad (2.18)
\]

\[
\nabla^2 \vartheta - \frac{\delta V}{\delta \vartheta} - \lambda \frac{\delta \Theta}{\delta \vartheta} * RR - \lambda A \frac{\delta \Theta}{\delta \vartheta} * FF = 0, \quad (2.19)
\]

where

\[
C_{\mu \nu} = \frac{1}{2} \nabla^\alpha \nabla_\beta \left( \Theta * R_{\alpha \mu \beta \nu} + \Theta * R_{\alpha \nu \beta \mu} \right)
= \frac{1}{2} \left[ \left( \nabla^\alpha \nabla^\beta \Theta \right) * R_{\alpha \mu \beta \nu} + \left( \nabla^\beta \Theta \right) \epsilon_{\beta \mu \gamma \delta} \left( \nabla^\gamma R^\delta_{\nu} \right) \right] + (\mu \leftrightarrow \nu) \quad (2.20)
\]

and

\[
T_{\mu \nu} = \left[ F_{\mu \rho} F_{\nu}^{\rho} - \frac{1}{4} g_{\mu \nu} F^2 \right] + \left[ \left( \nabla_\mu \vartheta \right) \left( \nabla_\nu \vartheta \right) - \frac{1}{2} g_{\mu \nu} \left( \nabla \vartheta \right)^2 - g_{\mu \nu} V[\vartheta] \right]. \quad (2.21)
\]

We choose the background ansatz to be

\[
d\bar{s}^2 = -F(z) d\tilde{t}^2 + \frac{dz^2}{F(z)} + r(z)^2 \left( dx^2 + dy^2 \right),
\]

\[
\bar{A} = \Phi(z) dt, \quad \bar{\theta} = \Theta(z) \quad (2.22)
\]

The temperature is given by

\[
F(z) = 4 \pi T(z - z_H) + O \left((z - z_H)^2\right) \text{ and } \Phi(z) = O(z - z_H) \text{ near the horizon } z = z_H. \text{ The boundary is at } z = \infty.
\]

The background EOMs are

\[
\frac{d^2 r(z)}{dz^2} + \frac{1}{4} r(z) \left( \frac{d\vartheta(z)}{dz} \right)^2 = 0, \quad (2.23)
\]

\[
\frac{d^2 F(z)}{dz^2} - \frac{2F(z)}{r(z)^2} \left( \frac{dr(z)}{dz} \right)^2 - \left( \frac{d\Phi(z)}{dz} \right)^2 + \frac{F(z)}{2} \left( \frac{d\vartheta(z)}{dz} \right)^2 = 0, \quad (2.24)
\]

\[
\frac{d^2 \Phi(z)}{dz^2} + \frac{2}{r(z)} \frac{dr(z)}{dz} \frac{d\Phi(z)}{dz} = 0, \quad (2.25)
\]

\[
\frac{d}{dz} \left[ r(z)^2 F(z) \left( \frac{d}{dz} \vartheta(z) \right) \right] - r(z)^2 \frac{\delta V}{\delta \vartheta} = 0. \quad (2.26)
\]

and a constraint equation from the trace of Einstein equation:

\[
\frac{d^2 F(z)}{dz^2} + 4 \frac{F(z)}{r(z)} \frac{d^2 r(z)}{dz^2} + 2 \frac{F(z)}{r(z)^2} \left( \frac{dr(z)}{dz} \right)^2 + 4 \frac{F(z)}{r(z)} \frac{dF(z)}{dz} \frac{dr(z)}{dz} + \frac{1}{2} F(z) \left( \frac{d\vartheta(z)}{dz} \right)^2 + 2 V[\vartheta] = \frac{12}{L^2}. \quad (2.27)
\]

7
3 Vector Mode Fluctuations and Angular Momentum Density

3.1 Formula for Angular Momentum Density

In this section, we follow the method proposed in [36] to calculate the angular momentum density. The gauge conditions are chosen to be $h_{\mu z} = a_z = 0$. We need to study only the static case, so all fluctuations are time-independent. For completeness, we first review the derivation of the formula used to compute angular momentum density. The metric at the boundary is $\gamma_{\alpha\beta} = \eta_{\alpha\beta} + \delta\gamma_{\alpha\beta}$, where $\eta_{\alpha\beta} = (-1, 1, 1)$ is the flat Lorentzian metric and $\delta\gamma_{\alpha\beta} = \tilde{h}_{\alpha\beta}$ is the metric fluctuation at the boundary. $\alpha, \beta = t, x, y$ and $i, j, k = x, y$. The energy-stress tensor is defined as

$$T^{\alpha\beta} = \frac{2}{\sqrt{-\gamma}} \frac{\delta S}{\delta\gamma_{\alpha\beta}} \bigg|_{\gamma_{\alpha\beta} = \eta_{\alpha\beta}}. \tag{3.1}$$

Then the linearized (first order) on-shell action we calculate from holography will in general takes the form

$$S^{(1)} = \frac{1}{2} \int d^3x \tilde{h}_{\alpha\beta}(x) T^{\alpha\beta}(x). \tag{3.2}$$

The $\beta = t$ component of the conservation law $\nabla_\alpha T^{\alpha\beta} = 0$ in the static case and flat background reads

$$\partial_i T^{ti}(\vec{x}) = 0, \tag{3.3}$$

which has a general solution

$$T^{ti}(\vec{x}) = \frac{1}{2} \epsilon^{ij} \partial_j \ell(\vec{x}), \tag{3.4}$$

where $\epsilon^{xy} = -\epsilon^{yx} = 1$, $\epsilon^{xx} = \epsilon^{yy} = 0$ and $\ell(\vec{x})$ is an arbitrary function. It is straightforward to see that $\ell(\vec{x})$ is the angular momentum density by definition:

$$L = \int d^2 x \epsilon_{ij} x^i T^{ij}(\vec{x}) = \frac{1}{2} \int d^2 x \epsilon_{ij} x^i \epsilon^{jk} \partial_k \ell(\vec{x}) = \int d^2 \vec{x} \ell(\vec{x}), \tag{3.5}$$

where in the last step we have integrated by parts, assumed that all fields are vanishing at spatial infinity, and used $\epsilon_{ij} \epsilon^{jk} = -\delta^k_i$. Plug (3.4) into (3.2), turn on only $h_{ti}(\vec{x})$ fluctuation and integrate by parts, we get

$$S^{(1)} = \frac{1}{2} \int d^3 x \ell(\vec{x}) \epsilon^{ij} \partial_i \tilde{h}_{ij}(\vec{x}). \tag{3.6}$$

When the system is homogeneous, $\ell(\vec{x}) = \ell$ is a constant and can be factored out of the integral. Then we have

$$S^{(1)} = \frac{\ell}{2} \int d^3 x \left( \frac{\partial}{\partial x} \tilde{h}_{ty}(\vec{x}) - \frac{\partial}{\partial y} \tilde{h}_{tx}(\vec{x}) \right). \tag{3.7}$$

(3.6) and (3.7) are the template formulae for computing angular momentum density in holography.
3.2 First Order On-Shell Action

We now calculate the linearized on-shell action $S^{(1)} = S^{(1)}_{\text{bulk}} + S^{(1)}_{\text{GH}} + S^{(1)}_{\text{∂CS}} + S^{(1)}_{\text{ct}}$. The first part of the contribution is from the $z$-derivative term in $S^{(1)}_{\text{bulk}}$ plus the boundary terms:

$$
\frac{1}{2\kappa^2} \int_{z=\infty} d^3 x \left\{ 2r(z) \left( F(z) \frac{dr(z)}{dz} - r(z) \frac{F(z)}{L} \right) h_x^r(\vec{x}, z) + \frac{r(z)}{2} \frac{d^2 \Phi(z)}{dz^2} a_t(\vec{x}, z)
+ r(z) \left( F(z) \frac{dr(z)}{dz} + \frac{r(z) dF(z)}{2} - \frac{2r(z)}{L} \sqrt{F(z)} \right) \left( h_x^r(\vec{x}, z) + h_y^y(\vec{x}, z) \right)
- r(z)^2 F(z) \frac{d\theta(z)}{dz} \delta \theta(\vec{x}, z)
+ \frac{\lambda}{2} \bar{\Theta}(z) \frac{dr(z)}{dz} \left( 2F(z) \frac{dr(z)}{dz} - r(z) \frac{dF(z)}{dz} \right) \left( \frac{\partial}{\partial x} h_y^y(\vec{x}, z) - \frac{\partial}{\partial y} h_x^r(\vec{x}, z) \right)
+ \frac{\lambda}{4} \bar{\Theta}(z) r(z) \left( 2F(z) \frac{dr(z)}{dz} - r(z) \frac{dF(z)}{dz} \right) \left( \frac{\partial^2}{\partial z \partial x} h_y^y(\vec{x}, z) - \frac{\partial^2}{\partial z \partial y} h_x^r(\vec{x}, z) \right) \right\}.

(3.8)

Using the asymptotic behavior of the metric:

$$
\begin{align*}
\begin{cases}
    r(z) = \frac{z}{L} + O \left( \frac{1}{z^3} \right) \\
    F(z) = \left( \frac{z}{L} \right)^2 + \Gamma + O \left( \frac{1}{z^2} \right), \\
    \Phi(z) = \Phi_0 + \frac{\Phi_1}{z} + O \left( \frac{1}{z^2} \right)
\end{cases}
\end{align*}

(3.9)

the first two lines of the integrand in (3.8) are

$$
\frac{\Gamma}{2L^2} \left( 2h_x^r - \bar{h}_x^r - \bar{h}_y^y \right) - \frac{\Phi_1}{L^2} a_t.
$$

Since we do not turn on these boundary fields, they have no contribution. The axion has near-boundary behavior

$$
\theta(z) = \theta_0 z^{-\Delta_-} \ldots + \theta_1 z^{-\Delta_+} \ldots,
$$

(3.10)

where

$$
\Delta_\pm = \frac{3}{2} \pm \sqrt{\frac{9}{4} + m^2 L^2}.
$$

(3.11)

The first coefficient $\theta_0$ is equal to the source $J$ and the second coefficient $\theta_1$ equal to the condensate $\langle \mathcal{O} \rangle$. Since we are considering a sourceless case, we set $\theta_0 = 0$. The axion fluctuation $\delta \theta$ has similar near-boundary behavior:

$$
\delta \theta = \delta \tilde{\theta} z^{-\Delta_-} \ldots,
$$

then the third line in (3.8) becomes

$$
\frac{\Delta_\pm}{L^4} \theta_1 \delta \tilde{\theta},
$$
which has no contribution because $\delta \theta$ is turned off. Thus the first part’s contribution is only from the last two lines in (3.8):

$$\frac{\lambda}{4\kappa^2} \int_{z=\infty} d^4x \left\{ \Theta(z) \frac{dr(z)}{dz} \left( 2F(z) \frac{dr(z)}{dz} - r(z) \frac{dF(z)}{dz} \right) \left( \frac{\partial}{\partial x} h_t^y(\bar{x}, z) - \frac{\partial}{\partial y} h_t^x(\bar{x}, z) \right) + \frac{1}{2} \Theta(z) r(z) \left( 2F(z) \frac{dr(z)}{dz} - r(z) \frac{dF(z)}{dz} \right) \left( \frac{\partial^2}{\partial z \partial x} h_t^y(\bar{x}, z) - \frac{\partial^2}{\partial z \partial y} h_t^x(\bar{x}, z) \right) \right\}. \quad (3.12)$$

The second part of the contribution to $S^{(1)}$ is from the $x$- and $y$-derivative terms in $S_{\text{bulk}}^{(1)}$. The part involving scalar and tensor mode fluctuations are quadratic in spatial derivatives:

$$\frac{1}{2\kappa^2} \int d^4x \left\{ \left( \frac{\partial^2}{\partial x^2} \Theta(z) h_s^y(\bar{x}, z) - \frac{\partial^2}{\partial y^2} h_s^x(\bar{x}, z) - \frac{\partial^2}{\partial x^2} h_t^y(\bar{x}, z) + 2 \frac{\partial^2}{\partial x \partial y} h_t^x(\bar{x}, z) \right) \right\},$$

thus is of higher order. The part involving vector mode fluctuations will give the main contribution:

$$\frac{\lambda}{4\kappa^2} \int d^4x \left\{ \left( r(z)^2 \frac{d}{dz} \left( \frac{1}{r(z)} \frac{dr(z)}{dz} \frac{dF(z)}{dz} \right) - 4F(z) \frac{dr(z)}{dz} \frac{d^2r(z)}{dz^2} \right) \Theta(z) + r(z)^3 \left( \frac{d}{dz} \frac{F(z)}{r(z)^2} \right) \frac{d}{dz} \left( \frac{r(z)}{r(z)} \frac{dF(z)}{dz} \right) \left( \frac{\partial}{\partial x} h_t^y(\bar{x}, z) - \frac{\partial}{\partial y} h_t^x(\bar{x}, z) \right) + r(z) \Theta(z) \left[ r(z)^2 \frac{d}{dz} \left( \frac{1}{r(z)} \frac{dF(z)}{dz} \right) - 2F(z) \frac{d^2r(z)}{dz^2} \right] \left( \frac{\partial^2}{\partial z \partial x} h_t^y(\bar{x}, z) - \frac{\partial^2}{\partial z \partial y} h_t^x(\bar{x}, z) \right) + \frac{\lambda_A}{2\kappa^2} \int d^4x \left\{ \Theta_A(\bar{z}) \frac{d\Phi(z)}{dz} \left( \frac{\partial}{\partial x} a_y(\bar{x}, z) - \frac{\partial}{\partial y} a_x(\bar{x}, z) \right) \right\}. \right\}$$

The quantity in the first [...] is a total derivative, so this part can be integrated by parts, which gives

$$\frac{\lambda}{4\kappa^2} \int d^3x \Theta(z) \frac{dr(z)}{dz} \left( r(z) \frac{dF(z)}{dz} - 2F(z) \frac{dr(z)}{dz} \right) \left( \frac{\partial}{\partial x} h_t^y(\bar{x}, z) - \frac{\partial}{\partial y} h_t^x(\bar{x}, z) \right) \bigg|_{z=\infty}^{z=z_H}$$

$$+ \frac{\lambda}{4\kappa^2} \int d^4x \left\{ \Theta(z) \left[ - \frac{dr(z)}{dz} \left( r(z) \frac{dF(z)}{dz} - 2F(z) \frac{dr(z)}{dz} \right) + r(z)^3 \frac{d}{dz} \left( \frac{1}{r(z)} \frac{dF(z)}{dz} \right) \right] - 2r(z) F(z) \frac{d^2r(z)}{dz^2} \left( \frac{\partial^2}{\partial z \partial x} h_t^y(\bar{x}, z) - \frac{\partial^2}{\partial z \partial y} h_t^x(\bar{x}, z) \right) \right\}$$

$$+ \frac{\lambda_A}{2\kappa^2} \int d^4x \left\{ \Theta_A(\bar{z}) \frac{d\Phi(z)}{dz} \left( \frac{\partial}{\partial x} a_y(\bar{x}, z) - \frac{\partial}{\partial y} a_x(\bar{x}, z) \right) \right\}.$$

Combine this with (3.12), we have:

$$S^{(1)} = \frac{\lambda}{4\kappa^2} \int_{z=z_H} d^3x \Theta(z) \frac{dr(z)}{dz} \left( 2F(z) \frac{dr(z)}{dz} - r(z) \frac{dF(z)}{dz} \right) \left( \frac{\partial}{\partial x} h_t^y(\bar{x}, z) - \frac{\partial}{\partial y} h_t^x(\bar{x}, z) \right)$$
Equation (3.14) can be directly integrated out, which gives

\[
\frac{F(z)}{2r(z)^2} \frac{\partial}{\partial z} \left( \frac{r(z)^2}{F(z)} \partial h_i^z(\vec{x}, z) \right) = O\left(\partial^2\right),
\]

(3.14)

where \( \partial \) is understood, and the \( ti \)-component of the linearized Einstein equation and \( i \)-component of the linearized Maxwell equation

\[
\frac{d\Phi(z)}{dz} \partial_i h_i^z(\vec{x}, z) + \frac{F(z)}{r(z)^2} \frac{\partial}{\partial z} \left( \partial_i a_i(\vec{x}, z) \right) = 0,
\]

(3.15)

Equation (3.14) can be directly integrated out, which gives

\[
\frac{r(z)^2}{F(z)} \partial_i h_i^z(\vec{x}, z) = C_1(\vec{x}) + O\left(\partial^2\right),
\]

where \( C_1(\vec{x}) \) is an arbitrary function independent of \( z \). Since the right hand side is already independent of \( z \), the left hand side must also be independent of \( z \). To achieve this, the \( z \)-dependence of \( h_i^z(\vec{x}, z) \) must cancel the prefactor \( r(z)^2/F(z) \). Noticing that this factor goes to 1 at the boundary, and we want to normalize \( h_i^z(\vec{x}, z) \) at the boundary as \( h_i^z(\vec{x}, z \to \infty) = \bar{h}_{ti}(\vec{x}) \), we get the solution

\[
h_i^z(\vec{x}, z) = \frac{F(z)}{r(z)^2} \bar{h}_{ti}(\vec{x}) + O\left(\partial\right).
\]

(3.18)
Here indices of boundary fields $\tilde{h}_{ti}$ are raised and lowered by the 3-d flat Lorentzian metric $\eta_{\alpha\beta} = (-1, 1, 1)$. $C_1(\vec{x})$ is then determined accordingly. Plug this solution into equation (3.15) and integrate in out, we get

$$\partial_i \left( a_i(\vec{x}, z) + \Phi(z) \tilde{h}_{ti}(\vec{x}) \right) = C_2(\vec{x}) + O \left( \partial^2 \right),$$

where $C_2(\vec{x})$ is an arbitrary function independent of $z$. Same as before, we want the left hand side to be $z$-independent. Noticing that $\Phi(z \to \infty) = \Phi_0 + O \left( z^{-1} \right)$ and we want to normalize $a_i(\vec{x}, z)$ at the boundary as $a_i(\vec{x}, z \to \infty) = 0$, we get the solution

$$a_i(\vec{x}, z) = (\Phi_0 - \Phi(z)) \tilde{h}_{ti}(\vec{x}) + O \left( \partial \right). \quad (3.19)$$

Using background EOMs it is straightforward to check that the solutions (3.18) and (3.19) solve the second order equations (3.16) and (3.17) as well. Thus the vector mode fluctuations are completely solved at the leading order in derivative expansion.

Plug these solutions into (3.13), and integrate by parts the bulk integrals, we arrive at

$$S^{(1)} = -\frac{\lambda_A}{4\kappa^2} \int_{z=\infty} d^3x \tilde{\Theta}_A(z) (\Phi_0 - \Phi(z))^2 \left( \frac{\partial}{\partial x} \tilde{h}_{ty}(\vec{x}) - \frac{\partial}{\partial y} \tilde{h}_{tx}(\vec{x}) \right)$$

$$+ \frac{\lambda}{8\kappa^2} \int_{z=z_H} d^3x \tilde{\Theta}(z) \frac{dF(z)}{dz} \left( 2 \left( F(z) \frac{dF(z)}{dz} - \frac{dF(z)}{dz} \right) \right) \left( \frac{\partial}{\partial x} \tilde{h}_{ty}(\vec{x}) - \frac{\partial}{\partial y} \tilde{h}_{tx}(\vec{x}) \right)$$

$$+ \frac{\lambda_A}{4\kappa^2} \int_{z=z_H} d^3x \tilde{\Theta}_A(z) (\Phi_0 - \Phi(z))^2 \left( \frac{\partial}{\partial x} \tilde{h}_{ty}(\vec{x}) - \frac{\partial}{\partial y} \tilde{h}_{tx}(\vec{x}) \right)$$

$$- \frac{\lambda}{8\kappa^2} \int d^4x r(z)^4 \left[ \frac{d}{dz} \left( \frac{F(z)}{r(z)^2} \right) \right]^2 \frac{d\tilde{\Theta}(z)}{dz} \left( \frac{\partial}{\partial x} \tilde{h}_{ty}(\vec{x}) - \frac{\partial}{\partial y} \tilde{h}_{tx}(\vec{x}) \right)$$

$$+ \frac{\lambda_A}{4\kappa^2} \int d^4x (\Phi_0 - \Phi(z))^2 \frac{d\tilde{\Theta}_A(z)}{dz} \left( \frac{\partial}{\partial x} \tilde{h}_{ty}(\vec{x}) - \frac{\partial}{\partial y} \tilde{h}_{tx}(\vec{x}) \right).$$

Noticing $F(z) = 4\pi T (z - z_H) + O ((z - z_H)^2)$ and $\Phi(z) = O (z - z_H)$ near the horizon and $\Phi(z \to \infty) = \Phi_0 + O \left( z^{-1} \right)$ near the boundary, we get

$$S^{(1)} = \frac{1}{2\kappa^2} \left[ \frac{1}{2} \Phi_0^2 \lambda_A \tilde{\Theta}_A(z_H) - 4\pi^2 T^2 \lambda \tilde{\Theta}(z_H) \right] \int d^3x \left( \frac{\partial}{\partial x} \tilde{h}_{ty}(\vec{x}) - \frac{\partial}{\partial y} \tilde{h}_{tx}(\vec{x}) \right)$$

$$+ \frac{1}{2\kappa^2} \int_{z_H}^{\infty} dz \left[ \frac{\lambda_A}{2} (\Phi_0 - \Phi(z))^2 \frac{d\tilde{\Theta}_A(z)}{dz} \right] \int d^3x \left( \frac{\partial}{\partial x} \tilde{h}_{ty}(\vec{x}) - \frac{\partial}{\partial y} \tilde{h}_{tx}(\vec{x}) \right)$$

$$+ \frac{1}{2\kappa^2} \int_{z_H}^{\infty} dz \left\{ -\frac{\lambda}{4} r(z)^4 \left[ \frac{d}{dz} \left( \frac{F(z)}{r(z)^2} \right) \right]^2 \frac{d\tilde{\Theta}(z)}{dz} \right\} \int d^3x \left( \frac{\partial}{\partial x} \tilde{h}_{ty}(\vec{x}) - \frac{\partial}{\partial y} \tilde{h}_{tx}(\vec{x}) \right).$$

Compare with the formula (3.7), we get

$$\ell = \frac{\lambda_A}{2\kappa^2} \left( \Phi_0^2 \lambda_A \Theta(z_H) + \int_{z_H}^{\infty} dz (\Phi_0 - \Phi(z))^2 \frac{d\tilde{\Theta}_A[\theta(z)]}{dz} \right)$$

$$- \frac{\lambda}{4\kappa^2} \left( 16\pi^2 T^2 \Theta(z_H) + \int_{z_H}^{\infty} dz r(z)^4 \left[ \frac{d}{dz} \left( \frac{F(z)}{r(z)^2} \right) \right]^2 \frac{d\tilde{\Theta}[\theta(z)]}{dz} \right), \quad (3.20)$$
where $\Phi_0 = \Phi(z = \infty)$. We can see that in general both gauge and gravitational Chern-Simons terms contribute to the angular momentum density. The contributions are not just from the horizon area, as opposed to many transport coefficients such as the Hall viscosity. The fact that parts of the contribution are written as integrals from the horizon to the boundary suggests that the IR degrees of freedom interact non-trivially with the UV degrees of freedom to generate the angular momentum density. The above result is in agreement with that obtained in [39].

4 Tensor Mode Fluctuations and Hall Viscosity

The Hall viscosity for gravitational Chern-Simons model with an axion coupling has been computed in [27,28], but in a different form of the metric. Here, for completeness, we present the derivation again, appropriate for the background ansatz (2.22). The result we will derive here is also a generalization of the results in [27,28], since we have a generic Chern-Simons coupling function $\Theta[\vartheta]$ in (2.7). In this section, for computing viscosities, we only consider the homogeneous case where all the fluctuations are independent of spatial coordinates $x$ and $y$. In this case, the tensor mode fluctuations $h_{xy}$ and $h_{xx} - h_{yy}$ decouple from the rest.

4.1 Tensor Mode EOMs and Solutions

First we define:

$$h_{xy}(t, z) = r(z)^2 h_e(t, z), \quad \frac{1}{2} (h_{xx}(t, z) - h_{yy}(t, z)) = r(z)^2 h_o(t, z) ,$$

where the subscripts $e$ and $o$ mean even and odd under parity operation $x \leftrightarrow y$. We define the notations $\epsilon_{ij}$ ($i = e,o$) as following: $\epsilon_{eo} = -\epsilon_{oe} = 1$, $\epsilon_{ee} = \epsilon_{oo} = 0$. The the linearized Einstein equations for these fluctuations (in momentum space) are

$$\frac{d}{dz} \left[ r(z)^2 F(z) \left( \frac{d}{dz} h_i(\omega, z) \right) \right] + \omega^2 r(z)^2 F(z) h_i(\omega, z) = \Xi_i(\omega, z; \lambda) ,$$

where

$$\Xi_i(\omega, z; \lambda) = -i \frac{\omega}{2} \lambda \epsilon_{ij} \left\{ -2 \frac{d}{dz} \left[ r(z)^2 F(z) \frac{d\Theta(z)}{dz} \left( \frac{d}{dz} h_j(\omega, z) \right) \right] \right. + i \omega^3 \lambda \frac{r(z)^2}{F(z)} \frac{d\Theta(z)}{dz} \frac{dr(z)}{dz} \frac{d\Theta(z)}{dz} h_j(\omega, z) \bigg\}$$

and the repeated index $j$ is summed over $e$ and $o$. The incoming wave solution is

$$h_i(\omega, z) = \left( \frac{z - z_H}{z} \right)^{-i \frac{\omega}{2} \lambda} \left[ h_i + i \omega h_i^{(1)}(z) + O(\omega^2) \right] ,$$
where
\[
\begin{align*}
\tilde{h}_i^{(1)}(z) &= \tilde{h}_i \left[ \frac{1}{4\pi T} \ln \left( \frac{z - z_H}{z} \right) - r(z_H)^2 \int_{\infty}^{z} d\xi \frac{1}{r(\xi)^2 F(\xi)} \right] \\
&\quad + \lambda \epsilon_{ij} \tilde{h}_j \left\{ 2\pi T r(z_H)^2 \tilde{\Theta}'(z_H) \int_{\infty}^{z} d\xi \frac{1}{r(\xi)^2 F(\xi)} \right. \\
&\quad \left. - \frac{1}{2} \int_{\infty}^{z} d\xi \left[ \frac{d}{d\xi} \ln \left( \frac{F(\xi)}{r(\xi)^2} \right) \right] \frac{d\tilde{\Theta}(\xi)}{d\xi} \right\}. 
\end{align*}
\]

Because \( \theta(z) \) is sourceless near the boundary: \( \theta(z) \sim z^{-\Delta_+} \), for a general \( \Theta[\vartheta] = \vartheta^n \) \((n \geq 1)\) near the boundary, the last line in the above equation goes to zero faster than \( O(z^{-3}) \) near the boundary, so only the second line contributes to Hall viscosity.

### 4.2 2-Point Functions and Viscosities

The total second order on-shell action for the tensor mode, from (2.16) and the corresponding boundary terms, is
\[
S^{(2)} = \frac{1}{4\kappa^2} \int_{z=\infty} d^3x \sum_{i,j=e,o} \left\{ \frac{4r(z)^2}{L} \sqrt{F(z)} - \frac{d}{dz} \left( r(z)^2 F(z) \right) \right\} h_i^2 - r(z)^2 F(z) h_i \left( \frac{\partial}{\partial z} h_i \right) \\
+ \frac{\lambda}{2} \epsilon_{ij} \left[ r(z)^4 \left( \frac{d}{dz} F(z) \right) \frac{d\tilde{\Theta}(z)}{dz} h_i \left( \frac{\partial}{\partial t} h_j \right) + 2 r(z)^2 \tilde{\Theta}(z) \left( \frac{\partial^2}{\partial t^2} h_i \right) \left( \frac{\partial}{\partial t} h_j \right) \right. \\
+ 2 r(z) \frac{dr(z)}{dz} F(z) \tilde{\Theta}(z) \left( \frac{\partial}{\partial t} h_i \right) \left( \frac{\partial}{\partial z} h_j \right) \left. - 2 r(z)^3 F(z) \left( \frac{d}{dz} \tilde{\Theta}(z) \right) h_i \left( \frac{\partial^2}{\partial t \partial z} h_j \right) \right\}. 
\]

Following the holographic prescriptions of [41–43], we obtain the 2-point functions in momentum space:
\[
\begin{align*}
G_{ra}^{xy,xy}(\omega) &= -\frac{\Gamma}{2\kappa^2 L^2} - i\omega \frac{r(z_H)^2}{2\kappa^2} + O(\omega^2) , \\
G_{ra}^{xx-yy,xy}(\omega) &= -i\omega \frac{2\pi T \lambda}{\kappa^2} r(z_H)^2 \tilde{\Theta}'(z_H) + O(\omega^2) .
\end{align*}
\]

Comparing with Kubo formulae [36]:
\[
\begin{align*}
G_{ra}^{xy,xy}(\omega) &= p - i\omega \eta_H + O(\omega^2) , \\
G_{ra}^{xx-yy,xy}(\omega) &= 2i\omega \eta_H + O(\omega^2) ,
\end{align*}
\]
we get the shear viscosity
\[
\frac{\eta}{s} = \frac{1}{4\pi} ,
\]

\footnote{The \( \eta_H \) term differs by a sign from [36] because our convention is \( \epsilon_{txy} = 1 \) in 3-d flat Minkowskian space (the boundary), which follows from \( \epsilon_{txyz} = \sqrt{-g} \) in 4-d bulk space.}
where the entropy density is \( s = \frac{2\pi}{\kappa} r(z_H)^2 \), and the Hall viscosity

\[
\eta_H = -\frac{\lambda}{4\kappa^2} \left\{ r(z)^4 \left[ \frac{d}{dz} \left( \frac{F(z)}{r(z)^2} \right) \right] \frac{d\Theta[\theta(z)]}{dz} \right\} \bigg|_{z=z_H} = -\frac{\pi T \lambda}{\kappa^2} r(z_H)^2 \bar{\Theta}'(z_H),
\]

where \( \bar{\Theta}'(z) \equiv \partial_z \Theta[\theta(z)] \). The middle part in the above equation is from the first term in the second line of (4.6) and we have used \( F(z) = 4\pi T(z - z_H) + O((z - z_H)^2) \) to go to right hand side. The Hall viscosity has a simple form which is expressed purely in terms of bulk quantities at the horizon. This is a generalization of results in [27,28], for a generic gravitational Chern-Simons term of form (2.7). The gauge Chern-Simons term (2.4) has no contribution. The reason is obvious: this term is totally independent of the metric; since Hall viscosity is a response to the metric perturbation, it is natural that the gauge Chern-Simons has no contribution. In contrary to the membrane paradigm of the Hall viscosity, the angular momentum density (3.20) has a more complicated form: part of it does have a membrane paradigm form while the rest is an non-trivial bulk integral. This difference indicates that the physics behind these two quantities are different (at least for the holographic Chern-Simons models studied in this paper and the dual field theories they describe), thus in general, we expect their ratio to have some non-trivial behavior. This is difficult to study analytically.

In the next section, we will present the numeric results.

5 Numeric Results for Holographic Chern-Simons Models

In this section, we solve the holographic gravitational Chern-Simons model numerically and present results of Hall viscosity, angular momentum density and their ratio in terms of physical parameters such as temperature \( T \) and charge density \( \rho \) (we will use Canonical ensemble in this section, where \( \rho \) is held fixed), first in the probe limit, then with full back-reactions. But before doing so, we would like to first discuss the choice of the general function \( \Theta[\theta] \) in (2.7). There is no unique choice for its form from the phenomenological model we write down here. When \( \Theta[\theta] = \text{constant} \), the gravitational Chern-Simons term (2.7) is a boundary term because the Pontryagin density is a total derivative. This is not the case we are interested here, because we want this term to be dynamical, at least to have a non-trivial \( z \)-profile to generate non-vanishing Hall viscosity. In [27–29], the authors chose \( \Theta[\theta] = \theta \), which results in the near-critical behavior of Hall viscosity to be \( \eta_H \sim (T_c - T)^{1/2} \), because the Hall viscosity is linear to the condensate \( \theta \sim \langle O \rangle \sim (T_c - T)^{1/2} \). Another form, \( \Theta[\theta] = \theta^2 \), is also interesting, because from (3.20) and (4.12), both Hall viscosity and angular momentum density are quadratic in order parameter \( \langle O \rangle \) now, so near critical regime they will scale as \( T_c - T \), instead of \( (T_c - T)^{1/2} \). This is in agreement with condensed matter theory arguments such as in \( p_x + ip_y \) paired states of BCS theory: since both Hall viscosity and angular momentum density have dimension \( 1/[\text{length}]^2 \), by dimensional analysis, they are proportional to square of the gap energy (i.e., the order parameter, which scales as \( (T_c - T)^{1/2} \)), thus scale as \( T_c - T \) near critical regime. Of course \( \Theta[\theta] \) can take other forms in general. For \( \Theta[\theta] = \theta^n \), Hall viscosity and angular momentum density will scale as \( (T_c - T)^{n/2} \) near critical regime and \( \eta_H/\ell \) will acquire a factor of \( n \). When \( \Theta[\theta] \) contains multiple terms of \( \theta \) with different
powers, the lowest power will dominate the near-critical behavior and the highest power the low temperature behavior. In this section we choose

$$\Theta[\vartheta] = \vartheta^2$$

so as to reproduce the $T_c - T$ scaling near the critical regime.

From (3.20) we see that angular momentum density receive contributions from both gauge and gravitational Chern-Simons terms. However, from (4.12), Hall viscosity is only determined by the gravitational Chern-Simons term. In general, the two Chern-Simons coupling functions $\lambda_A \Theta_A[\vartheta]$ and $\lambda \Theta[\vartheta]$ can be different and unrelated. To make our analysis simple, in most of this section, we will focus only on the gravitational Chern-Simons term, and turn off the gauge Chern-Simons term $\lambda_A = 0$. Only at the end of this section will we present the numeric result for the angular momentum density of in the gauge Chern-Simons model.

5.1 Gravitaional Chern-Simons Model: The Probe Limit

In this subsection we study the probe limit of the bulk theory, where the scalar field $\vartheta$ does not back-react on the metric and Maxwell field. This limit has been employed in [28,29]. The background now is the AdS-Reissner-Nordström black hole:

$$\begin{align*}
  r(z) &= \frac{z}{L}, \\
  F(z) &= \left(\frac{z}{L}\right)^2 - \left(1 + \frac{Q^2}{4}\right) \frac{z_H^3}{L^2 z} + \frac{Q^2 z_H^4}{4 L^2 z^2}, \\
  \Phi(z) &= \frac{Q z_H}{L} \left(1 - \frac{z_H}{z}\right),
\end{align*}$$

where $Q$ is the dimensionless charge. The temperature is

$$T = \frac{3 z_H}{4 \pi L^2} \left(1 - \frac{Q^2}{12}\right),$$

and to make the temperature non-negative, the charge has to satisfy $Q^2 \leq 12$. The near-boundary behavior of the scalar field is

$$\theta(z) \to \frac{\langle \mathcal{O} \rangle}{z^{\Delta_+}},$$

and that of the electric potential is

$$\Phi(z) = \mu - \frac{\rho}{z} + O\left(\frac{1}{z^2}\right),$$

where $\langle \mathcal{O} \rangle$ is the condensate, $\mu$ the chemical potential and $\rho$ the charge density (up to some factors of $\kappa$ and $L$). Mass $m$ is related to the conformal dimension $\Delta_+$ of the condensate operator $\mathcal{O}$ by (3.11). It has been shown in [28] that in this setup the black hole can develop a scalar hair only at very low temperature, where it is near extremal and the extremality factor $1 - Q^2/12$ is close to zero. Figure (1) shows numeric results for $c_4 L^2 = 0.5$ with
Figure 1: Condensate $\langle O \rangle$ (upper left), Hall viscosity $\eta_H$ (lower left), angular momentum density $\ell$ (lower right) and their ratio $\eta_H/\ell$ (upper right) as functions of $T/T_c$ in the probe limit, with $c_4L^2 = 0.5$. The red, brown and blue lines correspond to $m^2L^2 = -2.2$, $-2.1$ and $-2.05$, respectively. We have set $L = \lambda = \kappa = 1$, $\lambda_A = 0$.

Various values of $m$. It is interesting to notice that despite the seemingly different analytic expressions for Hall viscosity and angular momentum density, their ratio $\eta_H/\ell$ remains more or less unchanged for a vast range of temperature until one reaches the very low temperature regime. However, the value of the ratio is typically a huge number depending on the mass $m$ (or conformal dimension $\Delta+$) of the scalar condensate, and is far away from the 1/2 value found in condensed matter literature. We will see that this feature remains when full back-reactions are included.

The stability for a charged black hole with a neutral scalar condensation was discussed in [45] as well as in [28,29]. In this paper when talking about neutral scalar condensation, we always focus on a narrow window around $m^2L^2 = -2$, which is within the range discussed in these references and the black hole can develop a neutral scalar hair which condensate near the horizon.

5.2 Gravitational Chern-Simons Model: Including Back-reactions

The probe limit usually works well in high temperature when the black hole is far from extremal, and the condensate is small and the back-reactions are weak. However, in the probe limit of the previous subsection, numerics shows that the scalar can only condensate when the black hole is near-extremal. But in this case the back-reactions play a very important
Figure 2: Condensate $\langle \mathcal{O} \rangle$ (upper left), Hall viscosity $\eta_H$ (lower left), angular momentum density $\ell$ (lower right) and their ratio $\eta_H/\ell$ (upper right) as functions of $T/T_c$, with full back-reactions included. Here $m^2L^2 = -2$, corresponding to conformal dimension $\Delta_+ = 2$. The black dashed line has $c_4L^2 = 0$, with an increment of 0.25 for each adjacent line toward the red one, which has $c_4L^2 = 1.75$. We have set $L = \lambda = \kappa = 1$, $\lambda_A = 0$.

role, thus the probe limit assumption is not consistent. Particularly, Hall viscosity is solely expressed in terms of quantities near the horizon and so is part of the angular momentum density, thus the accuracy of the numeric solutions near the horizon matters a lot. Even in high temperature, when the back-reactions are negligible near the boundary and not strong in most part of the bulk, they are still very important near the horizon. For example, $r(z)$ has significant deviation from its probe limit form near the horizon. To improve the accuracy of the numeric results, in this subsection we will take back-reactions into full account.

Figure (2) shows the numeric results when full back-reactions are included. The mass is chosen to be $m^2L^2 = -2$, corresponding to conformal dimension $\Delta_+ = 2$. The black dashed line is the linear case with $c_4 = 0$. The colored lines show non-linear effect, with bigger non-linear coefficient $c_4$ toward the red end. The non-linearity decreases the values of the condensate, Hall viscosity and angular momentum density. Numerically we find that below certain low temperature ($< 0.3T_c$) it is hard to find a condensate solution when $c_4 > 0$: that is the reason why all the colored curves terminate at some low temperature.

The ratio between Hall viscosity and angular momentum density $\eta_H/\ell$ remains more or less unchanged at high temperature, same as in the probe limit. It only starts to drop off dramatically once gets to low temperature regime where $T < 0.3T_c$. We believe the wiggles in
Figure 3: Hall viscosity to angular momentum density ratio $\eta_H/\ell$ near the critical regime $T \to T_c$ as a function of conformal dimension $\Delta_+$. $c_4 L^2 = \frac{1}{120}$. The dots are the numeric data and the red line is the fitting of (5.6). The vertical axis is shown in logarithmic scale.

The plot are possibly due to numeric noises. Non-linearity has almost no effect near the critical temperature, because here the condensate is close to zero and the non-linear term is of higher order. It will only show up when the temperature is lowered and the condensate becomes large enough such that the non-linear term is comparable to the other terms. The non-linearity does decrease the ratio, however, its effect to the ratio is much weaker compared to that to Hall viscosity and angular momentum density individually. The numeric plot suggests that the non-linear effect on the $\eta_H/\ell$ ratio is at its strongest at the mid-temperature regime where $T \approx 0.5T_c$. As the temperature is lowered further, the non-linear effect on the ratio shows a trend to become weaker as the colorful lines go closer. It is interesting to notice that as the temperature drops below $T \approx 0.25T_c$, the ratio decrease dramatically towards zero. Due to the difficulties of numeric calculation for extremely low temperature regime, we can only work out the dashed line ($c_4 = 0$) below $T \approx 0.25T_c$ and can not go beyond $T \lesssim 0.1T_c$. It will be interesting to see whether at zero temperature the ratio approaches some non-zero fixed value, for example, $1/2$ widely found in the study of field theory and condensed matter systems [7,9,13,14,16]. In this regime, the extremal black hole background will possibly be replaced by other soliton backgrounds with vanishing entropy density. We will leave this task of studying low and zero temperature regime to the future.

The ratio also depends on the mass $m$ (the conformal dimension $\Delta_+$) as in the probe limit. Since the non-linearity does not play an important role, to separate this effect, we can just study the ratio’s dependence on the mass at the critical temperature. Figure (3) shows the near-critical $\eta_H/\ell$ ratio as a function of conformal dimension, with a fitting of the following form:

$$\left.\frac{\eta_H}{\ell}\right|_{T \to T_c} = e^{-26.8\sqrt{\Delta_+(3-\Delta_+)+42.9}},$$

i.e. this near-critical ratio depends on mass $m$ exponentially.
Figure 4: Angular momentum density $\ell$ as a function of $T/T_c$, with full back-reactions included. Here $m^2L^2 = -2$, corresponding to conformal dimension $\Delta_+ = 2$. The black dashed line has $c_4L^2 = 0$, with an increment of 0.25 for each adjacent line toward the red one, which has $c_4L^2 = 1.75$. We have set $L = \lambda_A = \kappa = 1$, $\lambda = 0$.

5.3 Gauge Chern-Simons Model: Angular Momentum Density

At the end of this section, in Figure (4), we present the numeric result for the angular momentum density due to the gauge Chern-Simons term, i.e. the first line in (3.20). We turn off the gravitational Chern-Simons coupling: $\lambda = 0$ and choose $\Theta_A[\vartheta] = \vartheta^2$ as well. Qualitatively the plot is very similar to that of the gravitational Chern-Simons model.

6 Conclusions and Comments

We have shown that holographic models with gauge and/or gravitational Chern-Simons terms have a non-vanishing angular momentum density when parity is spontaneously broken by the condensation of the scalar coupled to the Chern-Simons terms. Unlike Hall viscosity, the angular momentum density (3.20) does not have a membrane paradigm form: it is not solely determined by the near-horizon behavior of the background fields; part of it is an integral over the whole bulk regime outside the horizon, which suggests that the UV and IR degrees of freedom that are responsible for the generation of angular momentum density interact non-trivially with each other and do not decouple. These results are in agreement with those obtained in [39]. The effect of this angular momentum density is to accumulate momentum at the 1-dimensional spatial boundary of the 2+1-dimensional system, inducing an edge current of momentum whose strength is proportional to the angular momentum density, as shown in [36] and [39].

We have presented numeric results focusing on the gravitational Chern-Simons model, both at the probe limit and with back-reactions fully included. Once the back-reactions are taken into account, the accuracy of numeric results is improved. Both Hall viscosity and angular momentum density are monotonically decreasing functions of temperature. Non-linearity of the scalar potential $V[\vartheta]$ plays little role in asymptotic behaviors near the critical regime, but decreases both Hall viscosity and angular momentum density below the critical temperature.
The Hall viscosity to angular momentum density ratio obtained numerically from the gravitational Chern-Simons term alone is not exactly a constant, but it does remain more or less unchanged for a vast range of temperature, except at the very low temperature regime. On the other hand, the ratio obtained from non-holographic approaches [7, 9, 13, 14, 16] is always 1/2. In fact, the apparently different forms of (4.12) and (3.20), and the facts that the former involves only gravitational Chern-Simons term and tensor mode metric fluctuations while the latter involves both gauge and gravitational Chern-Simons terms and only vector mode fluctuations, already suggest that the physical mechanisms of generating Hall viscosity and angular momentum density in holographic Chern-Simons models and the dual field theories they describe are quite different. Thus in general, a simple relationship between them would not be expected from these theories. How to understand the universal relation obtained from field theory and condensed matter theories and the non-universal results from holographic Chern-Simons models here (and what role the gauge Chern-Simons term plays regarding the relationship between Hall viscosity and angular momentum density) is still open questions to be answered in the future.

In this paper, we have obtained a generic analytic formula (3.20) for angular momentum density. But numerically we have only studied one possible ground state of the theory, that with a hairy AdS-Reissner-Nordström black hole. It is possible that at low temperature ($T < 0.3T_c$), where it is hard to be reached by our numeric computation, other ground states with lower entropy density might be thermodynamically favored and the system undergoes a phase transition. How the angular momentum density, Hall viscosity and their ratio behave in other phases, particularly gapped phases at low or zero temperature, is another interesting question that can be studied in the future.

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