Fixed points of a random restricted growth sequence

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Abstract

We call $i$ a fixed point of a given sequence if the value of that sequence at the $i$-th position coincides with $i$. Here, we enumerate fixed points in the class of restricted growth sequences. The counting process is conducted by calculation of generating functions and leveraging a probabilistic sampling method.

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1 Introduction

For any given sequence $\pi$ of length $n$, $i \in [n] := \{1, \cdots, n\}$ is a fixed point of $\pi$ if the $i$-th entry $\pi_i$ of the sequence $\pi$ is equal to $i$. We denote the number of fixed points of the sequence $\pi$ by $F_{\pi}$. The term fixed point is motivated naturally by the same concept in the class of permutations, where it represents any point that is not moved by a permutation. Fixed points and derangements of permutations are well studied due to their importance in various branches of mathematics including algebra, probability, and combinatorics; see for instance [2, 4, 6, 10, 11, 13, 19, 20] for a few examples. Recently, a new line of research toward extending results with regards to fixed points in other classes of discrete sequences has emerged; for instance, Archibald, Blecher, and Knopfmacher [1] considered fixed points in compositions and words over the alphabet $[k]$. Inspired by them, in this note, we further investigate fixed points for another important class of sequences in combinatorics; namely, restricted growth sequences. These sequences are of interest in connection with set partitions [15], $q$-analogues [7], certain combinatorial matrices [12], and Gray codes [9].

Before we state our results, a few definitions are in order. Throughout this note, we use $\mathbb{N}$ as the set of all natural numbers. A sequence of natural numbers $\pi = \pi_1 \pi_2 \cdots \pi_n \in \mathbb{N}^n$ is called a restricted growth sequence if

$$\pi_1 = 1 \quad \text{and} \quad \pi_{j+1} \leq 1 + \max\{\pi_1, \cdots, \pi_j\} \quad \text{for all} \ 1 \leq j < n.$$

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There is a bijective connection between these sequences and canonical set partitions. A partition of a set $A$ is a collection of non-empty, mutually disjoint subsets, called blocks, whose union is the set $A$. A partition $\Pi$ with $k$ blocks is called a $k$-partition and denoted by $\Pi = A_1|A_2|\cdots|A_k$. A $k$-partition $A_1|A_2|\cdots|A_k$ is said to be in the standard form if the blocks $A_i$ are labeled in such a way that
\[
\min A_1 < \min A_2 < \cdots < \min A_k.
\]
The partition can be represented equivalently by the canonical sequential form $\pi_1\pi_2\cdots\pi_n$, where $\pi_i \in [n]$ and $i \in A_{\pi_i}$ for all $i$ [15]. In words, $\pi_i$ is the label of the partition block that contains $i$. It is easy to verify that a word $\pi \in [k]^n$ is a canonical representation of a $k$-partition of $[n]$ in the standard form if and only if it is a restricted growth sequence [15].

Throughout this note, we use the terms restricted growth sequence and set partition interchangeably. As it becomes clear, in the context of restricted growth sequences, the fixed points are also closely related to the records. We remind the reader that the $i$-th entry $\pi_i$ in the sequence $\pi$ is a record if $\pi_i > \pi_j$ for all $j \in [i-1]$. Clearly, for any given fixed point $i$ in a restricted growth sequence $\pi$, $i$ is a record. In addition, each $j \in [i-1]$ is a fixed point and hence a record. Therefore, $F_\pi$ is precisely the length of the maximal prefix of $\pi$ whose elements are records. We refer the reader to [6, 14, 16] for a few discussions around records for restricted growth sequences.

We denote by $\mathcal{R}_n$ the set of all restricted growth sequences of length $n$, and denote by $\mathcal{R}_{n,k}$ the set of all restricted growth sequences of length $n$ with maximal letter $k$.

Let $S_{n,k}$ be a Stirling number of the second kind and $B_n$ be the $n$-th Bell number [15]. It is well-known that the cardinality of the set $\mathcal{R}_n$ is $B_n$. In addition, the cardinality of the set $\mathcal{R}_{n,k}$ is $S_{n,k}$ with the exponential generating function $e^{y(e^x-1)}$, where $y$ counts the number of blocks. The sequence of Bell numbers $(B_n)_{n \geq 0}$ can be then defined, for instance, through the formula
\[
B_n = \sum_{k=0}^{n} S_{n,k},
\]
or, recursively via the formula $B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k$ with $B_0 = 1$, or through Dobinski’s formula [8]
\[
B_n = \frac{1}{e} \sum_{m=0}^{\infty} \frac{m^n}{m!}, \quad n \geq 0.
\]

In what follows, we denote a random restricted growth sequence, sampled uniformly from $\mathcal{R}_{n,k}$ (resp. $\mathcal{R}_n$) by $\pi_{(n,k)}$ (resp. $\pi_{(n)}$). That is,
\[
P(\pi_{(n,k)} = \pi) = \frac{1}{S_{n,k}} \quad \text{for all} \quad \pi \in \mathcal{R}_{n,k},
\]
and
\[
P(\pi_{(n)} = \pi) = \frac{1}{B_n} \quad \text{for all} \quad \pi \in \mathcal{R}_n.
\]

We use $E(\cdot)$ to refer to the expectation with respect to the probability distribution $P(\cdot)$. We denote by $F_n := F_{\pi_{(n)}}$ the number of fixed points in a uniformly sampled random restricted growth sequence $\pi_{(n)}$. Fig. 1 shows the empirical distributions of $F_n$ over 20000 independently sampled instances of $\pi_{(n)}$ for $n = 50$ and $n = 200$. 2
1.1 Statement of results

For any $k \in \mathbb{N}$, we define $Q_k(x; q)$ to be the ordinary generating function enumerating fixed points over the set of restricted growth sequences in $\bigcup_{n=k}^{\infty} R_{n,k}$; that is,

$$Q_k(x, q) := \sum_{n=k}^{\infty} x^n S_{n,k} E(F_n | \pi(n) \in R_{n,k}) = \sum_{n=k}^{\infty} \sum_{\pi \in R_{n,k}} x^n q^{F_{\pi}}; \quad x, q \in \mathbb{C},$$  \hspace{1cm} (2)

where $\mathbb{C}$ is the set of all complex numbers. Knowing an explicit form of (2), would in principle give us the distribution of $F_n$ in full details for all $n \in \mathbb{N}$. However, finding the explicit form of $Q_k(x, q)$ is a daunting task. Hence, we instead study the following exponential generating function:

$$R(x, y; q) := \sum_{n \geq 0} ([x^n]Q(x, y; q)) \frac{x^n}{n!} = \sum_{k \geq 1} y^k \sum_{n \geq 0} \frac{x^n}{n!} B_n E(F_n | \pi(n) \in R_{n,k}),$$  \hspace{1cm} (3)

where $[x^n]f$ denotes the coefficient of $x^n$ in $f$ and $Q(x, y; q)$ is the ordinary generating function for $Q_k(x; q)$; that is,

$$Q(x, y; q) := \sum_{k \geq 0} Q_k(x; q)y^k.$$

Our first result states that

**Theorem 1.1.** The exponential generating function $R(x, y; q)$ is given by

$$R(x, y; q) = e^{y(e^x-1)} + \int_{0}^{x} (q-1)ye^{x-t+(1+q)ye^{x-t}} dt.$$

The proof of this theorem is given in Section 2. Note, by Theorem 1.1 $R(x, y; q)$ reduces to $e^{y(e^x-1)}$ when $q = 1$, which is exactly the exponential generating function for the size of $R_{n,k}$ (see [15]). Moreover, for $q = 0$, it implies

$$R(x, y; 0) = e^{y(e^x-1)} - \int_{0}^{x} ye^{y(e^x-1)+t} dt = e^{y(e^x-1)} - e^{y(e^x-1)} + 1 = 1,$$

which coincides with the fact that the only restricted growth sequence with no fixed point is the null sequence.

Figure 1: Empirical distributions of $F_{50}$ (left) and $F_{200}$ (right) based on 20000 samples.
Recall that $R_{n,0}$ is an empty set for $n \geq 0$. Let $R_m(x, y)$ be the exponential generating function for the number of sequences in $R_{n,k}$ with exactly $m$ fixed points; that is,

$$R_m(x, y) := \sum_{n \geq 0} \sum_{k=0}^{\infty} \sum_{\pi \in R_{n,k}} x^n y^k,$$

with $R_0(x, y) = 1$. Then, Theorem 1.1 implies that

**Corollary 1.2.** For all $m \geq 1$,

$$R_m(x, y) = \frac{1}{(m-1)!} \int_0^x y^m t^{m-1} e^{y(e^x-t-1)+m(x-t)} dt - \frac{1}{m!} \int_0^x y^{m+1} t^m e^{y(e^x-t-1)+(m+1)(x-t)} dt.$$

Next, we study the exponential generating function $T(x)$ for the total number of fixed points in $R_n$; that is,

$$T(x) := \frac{\partial}{\partial q} R(x, 1; q) \big|_{q=1},$$

(4)

where by an inductive argument, provided in Section 2, we conclude that

**Corollary 1.3.** The average number of fixed points over all the restricted growth sequences is

$$E(\mathcal{F}_n) = \frac{1}{B_n} \sum_{i=1}^{n} \left( \sum_{j=1}^{i} \sum_{\ell=j}^{\infty} \frac{(-1)^{\ell-j} i^n j^{\ell-j}}{\ell-j)! (i-\ell)!} \right).$$

(5)

Our final result, Theorem 1.4 provides a closed form expression for the probability distribution of $\mathcal{F}_n$ in terms of explicit polynomials of Bell numbers. We will use the following well-known extension of Dobinski’s identity (1) to express the result. Recall that, for any integers $n, t \geq 0$ we have:

$$\Theta_n(t) := \frac{1}{e} \sum_{m=0}^{\infty} \frac{m^n}{(m-t)!} = \frac{1}{e} \sum_{k=0}^{\infty} \frac{(k+t)^n}{k!} = \frac{1}{e} \sum_{\ell=0}^{n} \binom{n}{\ell} t^{n-\ell} \sum_{k=0}^{\infty} \frac{k^\ell}{k!}$$

$$= \sum_{\ell=0}^{n} \binom{n}{\ell} t^{n-\ell} B_\ell,$$

(6)

where for the last equality we applied the original formula (1). The theorem states

**Theorem 1.4.** For $n \in \mathbb{N}$, we have

$$P(\mathcal{F}_n = j) = \begin{cases} \frac{B_{n-1}}{B_n} j! \Theta_{n-j-1}(j) & \text{if } j = 1 \\ \frac{j! \Theta_{n-j-1}(j)}{B_n} & \text{if } 1 < j < n \\ \frac{1}{B_n} & \text{if } j = n \end{cases}$$

(7)

In particular, for $n \geq 2$, we have

$$E(\mathcal{F}_n) = \frac{n + \sum_{j=1}^{n-1} j^2 \Theta_{n-j-1}(j)}{B_n}. $$

4
The proof of Theorem 1.4 is given in Section 3. It is based on the sampling method devised in [18] and has been recently exploited for the enumeration of other complex quantities over $\mathbb{R}_n$. See for instance [17] where the authors enumerate horizontal visibility graphs of sequences in $\mathbb{R}_n$. We remark that the $j = 1$ and $j = n$ cases are obvious; if $\pi \in \mathbb{R}_n$ such that $F_\pi = 1$, then it must be that $\pi_2 = 1$ and clearly there are $B_{n-1}$ choices for such words. Also, there is exactly one word $\pi = 12\cdots n \in \mathbb{R}_n$ for which $F_\pi = n$. However, we include our brute-force probabilistic calculation in Section 3 for the sake of completeness.

We make a final remark that from (5) and (7), we obtain the following identity for $n \geq 2$ which may be of interest independently to the reader:

$$
\sum_{j=1}^{n-1} \sum_{\ell=0}^{n-j-1} \binom{n-j-1}{\ell} j^{n-j+1-\ell} B_\ell = \sum_{i=1}^{n} \left( \sum_{j=1}^{i} \sum_{\ell=j}^{i} \frac{(-1)^{\ell-i}(n-j)}{(\ell-j)! (i-\ell)!} \right) - n.
$$

2 Proof of Theorems 1.1 and Corollary 1.3

The proof is based on the observation that each restricted growth sequence $\pi$ with maximal value $k$ and $F_\pi = i$ fixed points falls into one of the following cases:

**Case $F_\pi = i = k$:** Here $\pi = 12\cdots k\pi^{(k)}$, where $\pi^{(k)}$ is a word over alphabet $[k]$.

**Case $F_\pi = i < k$:** Here $\pi = 12\cdots i\pi^{(i)}(i+1)\pi^{(i+1)}\cdots k\pi^{(k)}$, where $\pi^{(j)}$ is a word over alphabet $[j]$, for all $j = i, i+1, \ldots, k$, and $\pi^{(i)}$ is not the empty word.

By considering these cases we arrive at the following equation:

$$
Q_k(x; q) = \frac{x^k q^k}{1 - kx} + \sum_{i=1}^{k-1} \frac{ix^{k+1}q^i}{\prod_{j=i}^{k}(1 - jx)},
$$

which implies

$$
(1 - kx)Q_k(x; q) - xQ_{k-1}(x; q) = x^k q^{k-1}(q - 1),
$$

with $Q_0(x; q) = 1$ and $Q_1(x; q) = \frac{xq}{1-x}$.

By multiplying (8) by $y^k$ and summing over $k \geq 2$, we obtain

$$
Q(x, y; q) - 1 - \frac{xyq}{1-x} = xy \frac{\partial}{\partial y} \left( Q(x, y; q) - 1 - \frac{xyq}{1-x} \right) - xy(Q(x, y; q) - 1) = \sum_{k \geq 2} x^k q^{k-1}(q - 1)y^k,
$$

which leads to

$$
(1 - xy)Q(x, y; q) - xy \frac{\partial}{\partial y} Q(x, y; q) = \frac{1 - xy}{1 - xyq}.
$$

5
Next, we translate (9) in terms of exponential generating function $R(x, y; q)$ defined by (3) and write

$$\frac{\partial}{\partial x} R(x, y; q) - y R(x, y; q) - y \frac{\partial}{\partial y} R(x, y; q) = y(q - 1)e^{xyq}.$$  

Solving this equation with the initial condition $R(0, y; q) = R(x, 0; q) = 1$ completes the proof of Theorem 1.1.

Recall (4). Theorem 1.1 shows that $T(x) = \int_0^x e^{(x-t)e^t+e^t+t-1}dt$. By induction on $m$, we have

$$\frac{d^m}{dx^m} T(x) = \sum_{i=1}^m a_{mi} e^{ix-1} + \int_0^x e^{(x-t)e^t+e^t+(m+1)t-1}dt,$$

(10)

where $a_{mi}$ are natural numbers satisfying the relations

$$\begin{align*}
a_{mm} &= 1 + a_{(m-1)(m-1)}, \\
a_{mi} &= a_{(m-1)(i-1)} + ia_{(m-1)i}, \quad i = 1, 2, \ldots, m-1,
\end{align*}$$

(11)

with $a_{11} = 1$. Let $a_i(x) = \sum_{m \geq i} a_{mi}x^m$. Then, by (11), we have

$$a_i(x) = \sum_{j=1}^i \frac{x^i}{(1-jx)(1-(j+1)x) \cdots (1-ix)}.$$

Note that, by a partial fraction decomposition, we have

$$\frac{1}{(1-jx)(1-(j+1)x) \cdots (1-ix)} = \sum_{\ell=j}^i \frac{(-1)^{\ell-i}(i-j)!(i-j)}{(i-j)!}.$$

Hence,

$$a_i(x) = \sum_{j=1}^i \sum_{\ell=j}^i \frac{(-1)^{\ell-i}(i-j)!}{(i-j)!} \frac{x^i}{(1-\ell x)}.$$

We inspect the coefficient of $x^m$ in $a_i(x)$ and derive

$$a_{mi} = \sum_{j=1}^i \sum_{\ell=j}^i \frac{(-1)^{\ell-i}(i-j)!}{(i-j)!} \frac{(m-j)}{(1-\ell x)}.$$

Therefore, (10) yields

$$\frac{d^m}{dx^m} T(x) = \sum_{i=1}^m \left( \sum_{j=1}^i \sum_{\ell=j}^i \frac{(-1)^{\ell-i}(i-j)!}{(i-j)!} \frac{(m-j)}{(1-\ell x)} \right) e^{ix-1} + \int_0^x e^{(x-t)e^t+e^t+(m+1)t-1}dt.$$

Finally, setting $x = 0$, we complete the proof of Corollary 1.3.
3 Proof of Theorem 1.4

The proof relies on the use of a generator of a uniformly random set partition of \([n]\) proposed by Stam [18]. We next describe Stam’s algorithm for a given \(n\).

1. For \(m \in \mathbb{N}\), let \(\mu_n(m) = \frac{m^n}{m! e^m}\). Dobinski’s formula (1) shows that \(\mu_n(\cdot)\) is a probability distribution on \(\mathbb{N}\).

   At time zero, choose a random \(M \in \mathbb{N}\) distributed according to \(\mu_n\), and arrange \(M\) empty and unlabeled boxes.

2. Arranges \(n\) balls labeled by integers from the set \([n]\).

   At time \(i \in [n]\), place the ball ‘\(i\)’ into one of the \(M\) boxes, chosen uniformly at random. Repeat until there are no balls remaining.

3. Label the boxes in the order that they cease to be empty. Once a box is labeled, the label does not change.

4. Form a set partition \(\pi\) of \([n]\) with \(i\) in the \(k\)-th block if and only if ball ‘\(i\)’ is in the \(k\)-th box.

Let \(N_i\) be the random number of nonempty boxes right after placing the \(i\)-th ball and \(X_i\) be the label of the box where the \(i\)-th ball was placed. Notice that if the \(i\)-th ball is dropped in an empty box, then \(X_i = N_{i-1} + 1\) and \(N_i = N_{i-1} + 1\). Otherwise, if the box was occupied previously, \(X_i = X_j\) where \(j < i\) is the first ball that was dropped in that box and \(N_i = N_{i-1}\). Then, \(X := X_1 \cdots X_n\) is the random set partition of \([n]\) produced by the algorithm.

We denote by \(P_m(\cdot)\) the conditional probability distribution \(P(\cdot | M = m)\). Clearly \(N_1 = 1\), \(N_i \leq i\), and

\[
P_m(N_{i+1} = t+1 | N_i = t) = \frac{m-t}{m} \quad \text{and} \quad P_m(N_{i+1} = t | N_i = t) = \frac{t}{m}.
\]

Let \(\alpha_{i,t}(m) := P_m(N_i = t)\). Then, taking into account that

\[
P_m(N_i = t) = P_m(N_i = t, N_{i-1} = t-1) + P_m(N_i = t, N_{i-1} = t),
\]

we obtain:

\[
\alpha_{i,t}(m) = \begin{cases} 
\frac{t}{m} \alpha_{i-1,t}(m) + \frac{m-t+1}{m} \alpha_{i-1,t-1}(m) & \text{if } 2 \leq t \leq m \text{ and } t \leq i \\
0 & \text{if } t > i \text{ or } t > m \\
\frac{1}{m^{i-1}} & \text{if } t = 1 \text{ and } 1 \leq i.
\end{cases}
\]

Recall that one can define the sequence of Stirling numbers of the second kind as the solution to the recursion

\[
S_{n,k} = kS_{n-1,k} + S_{n-1,k-1}, \quad n, k \in \mathbb{N}, \ k \leq n. \tag{12}
\]

A comparison with (12) reveals that for \(t \leq m\),

\[
P_m(N_i = t) = \frac{S_{i,t}}{m!} \frac{m!}{(m-t)!}.
\]
In addition,

\[ P_m(X_{i+1} = \ell|N_i = t) = \begin{cases} \frac{1}{m} & \text{if } \ell \leq t \\ \frac{m-t}{m} & \ell = t + 1 \\ 0 & \text{otherwise.} \end{cases} \]

Notice that some of the boxes may remain empty at the end of the algorithm’s run.

In order to obtain the probability distribution of \( F_n \), we make a simple observation that \( F_n \) has the same distribution as that of the random variable \( 1 \leq J \leq n \) defined as

\[ J := \min(n, \max\{ j | X_1 < \cdots < X_j \}) = \min(n, \max\{ j | N_j = j \}). \]

We will consider three cases;

**Case** \( j = 1 \): By the observation above

\[ P(F_n = j) = E\left( P_M(N_2 = 1|N_1 = 1)P_m(N_1 = 1) \right) = E\left( \frac{1}{M} \right) = \sum_{m=1}^{\infty} \frac{m^{n-1}}{m! eB_n} = \frac{B_{n-1}}{B_n}. \]

**Case** \( 2 \leq j < n \): We consider two possibilities such that either (i) \( m < j \), where

\[ P_m(N_j = j, N_{j+1} = j) = 0, \]

or (ii) \( 2 \leq j \leq m \), where

\[ P_m(N_j = j, N_{j+1} = j) = P_m(N_{j+1} = j|N_j = j) \prod_{s=1}^{j-1} P_m(N_{s+1} = s + 1|N_s = s) = \frac{j}{m} \prod_{s=1}^{j-1} \frac{m-s}{m}. \]

Thus, for \( 2 \leq j < n \), our observation implies

\[ P(F_n = j) = \sum_{m=j}^{\infty} \frac{m^n}{m! eB_n} \frac{j}{m} \prod_{s=1}^{j-1} \frac{m-s}{m} = \frac{j}{eB_n} \sum_{m=j}^{\infty} \frac{m^{n-j-1}}{(m-j)!} = \frac{j\Theta_{n-j-1}(j)}{B_n}, \]

where we used (6) for the last equality.

**Case** \( j = n \): We again investigate two possibilities such that either \( m < n \) or \( m \geq n \). Similar to the previous case, the latter is the only one contributing to the sum. Hence,

\[ P(F_n = n) = E(P_M(N_n = n)) = \sum_{m=n}^{\infty} \frac{1}{(m-n)! eB_n} = \frac{1}{B_n}. \]

Finally, equations (13), (14), and (15) yield (7).
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