Swampland conjecture in $f(R)$ gravity by the Noether Symmetry Approach

Micol Benetti,$^{1,2}$ Salvatore Capozziello,$^{1,2,3,4}$† and Leila Lobato Graef‡

$^1$Dipartimento di Fisica “E. Pancini”, Università di Napoli “Federico II”, Via Cintia, I-80126, Napoli, Italy,
$^2$Istituto Nazionale di Fisica Nucleare (INFN), sez. di Napoli, Via Cintia 9, I-80126 Napoli, Italy,
$^3$Gran Sasso Science Institute, Via F. Crispi 7, I-67100, L’Aquila, Italy,
$^4$Laboratory for Theoretical Cosmology, Tomsk State University of Control Systems and Radioelectronics (TUSUR), 634050 Tomsk, Russia,
$^5$Instituto de Física, Universidade Federal Fluminense,
Av. Gal. Milton Tavares de Souza s/n, Gragoatá, 24210-346 Niterói, Rio de Janeiro, Brazil.

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Swampland conjecture has been recently proposed to connect early time cosmological models with the string landscape, and then to understand if related scalar fields and potentials can come from some fundamental theory in the high energy regime. In this paper, we discuss swampland criteria for $f(R)$ gravity considering models where duality symmetry is present. In this perspective, specific $f(R)$ models can naturally belong to the string landscape. In particular, it is possible to show that duality is a Noether symmetry emerging from dynamics. The selected $f(R)$ models, satisfying the swampland conjecture, are consistent, in principle, with both early and late-time cosmological behaviors.

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I. INTRODUCTION

It has been argued that string theory landscape of vacua is vast and populated by low-energy effective field theories, surrounded by an even more vast swampland where field theories are incompatible with quantum gravity [1–4]. The swampland can be defined as the set of (apparently) consistent effective field theories which cannot be completed into any quantum gravity in the high energy regime. Since there can be nothing manifestly wrong with the effective theory, inconsistencies would manifest if one try to complete it in the ultraviolet regime [5]. Consistently, embedding effective field theories into a general quantum theory involving gravity, in particular, in the context of string theory, requires distinguishing consistent low energy effective field theories coupled to gravity from inconsistent counterparts. Establishing the correct criteria to identify the boundary between the landscape and the swampland lead in a series of conjectures known as weak gravity [6] and swampland conjectures [7], motivated by black hole physics [8] and string compactification [9]. Recently, it was proposed that an effective field theory, to be consistently embedded into quantum gravity, must satisfy two specific criteria [10]. Based on them, it was argued that if string theory should be the ultimate quantum gravity theory, there are evidences that exact de Sitter solutions, with a positive cosmological constant, cannot describe the fate of the late-time universe [7,11,13]. On the other hand, some models with varying equation of state can still be consistent with the conjecture [12]. Models with curvaton-like mechanisms [14] or also with electroweak axion potential energy [15] were studied showing that other mechanisms are still at stake (see also [16–20] for recent works on the topic). In addition, the two swampland criteria practically rule out simple single field slow-roll models of inflation [10,22], while more complex models like, among others, multi-field inflation [23] and warm inflation [24,25] are still allowed. The possibility that string theory does not allow for de Sitter vacua is not new [21], however, it has recently gained impetus by new evidences on the instability of a de Sitter phase (see e.g. [11]), and especially by the recent quantitative proposal for a more detailed constraint on potentials that are not in the swampland region [7].

Swampland criteria can be investigated also for alternative theories of gravity [26,27]. It is particularly interesting because some modified theories of gravity are a useful paradigm to cure shortcomings of General Relativity at ultraviolet and infrared scales, due to the lack of a full quantum gravity theory [28]. In particular, some of them are, in principle, capable of successfully address phenomenology ranging from inflation to the accelerated behavior of present universe [29,37]. In [38] an interesting analysis of modified gravity theories in the context of the swampland criteria was presented. Here we suggest a different perspective on these theories. We present a new interpretation based on the Noether Symmetry Approach from which duality, one of the main feature of string theory, naturally emerge. Our further conjecture is that models possessing duality can belong to the string landscape and, viceversa, this feature is connected to the presence of Noether symmetries. In our analysis, the important aspect is that some of these models can be framed into fundamental theories [39–47]. In other words, one of the main characteristics of string-dilaton theory, the
scale factor duality, holds also for some classes of \( f(R) \) models presenting Noether symmetries \[35\] [18]. In general, it can be demonstrated that string duality is related to Noether symmetries for several alternative theories of gravity \[39\] [51]. According to this result, it is possible to ask for the validity of swampland criteria for \( f(R) \) gravity as a natural class of theories emerging from the string landscape \[35\]. On the other hand, since \( f(R) \) gravity is also working at late epochs to address the accelerated expansion \[52\], it is realistic searching for models addressing swampland criteria, inflation and dark energy issues under the same standard. In this work, we want to discuss \( f(R) \) models related to string landscape that can be of interest also at late time for dark energy behavior.

This paper is organized as follows. In Sec. II we present a summary of \( f(R) \) gravity and cosmology. Sec. III is devoted to \( f(R) \) models presenting duality as generated from the presence of a Noether symmetry in the Lagrangian. Such models can be, in principle, related to the string landscape. After recalling conformal transformations, swampland criteria for \( f(R) \) are derived in Sec. IV. Models connecting early and late cosmological behaviors are discussed, in light of the swampland criteria, in Sec. V while discussion and conclusions are reported in Sec. VI.

The Noether Symmetry Approach adopted to relate duality with conserved quantities is outlined in Appendix A.

II. \( f(R) \) Gravity and Cosmology

Let us start recalling the field equations for \( f(R) \) gravity, as well as the related Friedmann cosmology for this theory. A general action describing \( f(R) \) gravity in four dimensions, adopting physical units \( 8\pi G_N = c = \hbar = 1 \), is

\[
A = \int d^4x\sqrt{-g}[f(R) + \mathcal{L}_m],
\]

where \( f(R) \) is a function of the Ricci scalar \( R \) and \( \mathcal{L}_m \) is the standard matter Lagrangian density. The Einstein field equations can be written in the form

\[
G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R = T^{\text{curv}}_{\alpha\beta} + T^m_{\alpha\beta},
\]

where

\[
T^{\text{curv}}_{\alpha\beta} = \frac{1}{2f(R)}g_{\alpha\beta}[f(R) - Rf(R)]
\]

\[
+ \frac{f(R)^{\alpha\beta}}{f(R)}(g_{\alpha\mu}g_{\beta\nu} - g_{\alpha\nu}g_{\beta\mu}),
\]

and

\[
T^m_{\alpha\beta} = \frac{1}{f(R)}T^m_{\alpha\beta}
\]

is the stress-energy tensor of matter taking into account the non-trivial coupling to geometry. The standard perfect-fluid stress-energy is

\[
T^m_{\alpha\beta} = (\rho_m + p_m)u_\alpha u_\beta - p_m g_{\alpha\beta},
\]

where \( \rho_m \) and \( p_m \) are the matter-energy density and pressure. Furthermore, the lower index \( m \) means derivative with respect to the Ricci scalar and, as soon as \( f(R) = R \), the curvature contribution \( T^{\text{curv}}_{\alpha\beta} \) is zero and standard General Relativity is recovered.

Assuming a Friedmann-Robertson-Walker (FRW) metric, from the curvature-stress-energy tensor, we can define a curvature pressure

\[
p_{\text{curv}} = \frac{1}{f(R)}\left\{\frac{1}{2}[f(R) - Rf(R)] - 3 \left(\frac{\dot{a}}{a}\right) \dot{R}f(R)R \right\},
\]

and a curvature density

\[
\rho_{\text{curv}} = \frac{1}{f(R)}\left\{\frac{1}{2}[f(R) - Rf(R)] - 3 \left(\frac{\dot{a}}{a}\right) \dot{R}f(R)R \right\},
\]

where dot indicates derivative with respect to the cosmic time, and the effective equation of state can be related to the curvature contributions (see Ref. [52] for details).

It is straightforward to deduce a point-like Lagrangian for \( f(R) \) gravity in FRW metric [53], that is

\[
\mathcal{L} = \mathcal{L}_{\text{curv}} + \mathcal{L}_m = a^3[f(R) - Rf(R)] + 6a\dot{a}^2f(R)
\]

\[
+ 6a^2\dot{R}f(R)R - 6a\dot{f}(R)R + a^3p_m,
\]

where \( p_m \) is the standard matter pressure. Then the Euler-Lagrange equations are

\[
2 \left(\frac{\dot{a}}{a}\right) + \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = -p_{\text{tot}}
\]

and

\[
f(R) \left[R + 6 \left(\frac{\dot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2}\right)\right] = 0,
\]

where \( p_{\text{tot}} = p_{\text{curv}} + p_m \). From the energy condition, we have

\[
\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = \frac{1}{3}p_{\text{tot}},
\]

with \( p_{\text{tot}} = p_{\text{curv}} + p_{\text{matter}} \). Combining Eqs. [5] and [7] we obtain the standard Friedmann equation

\[
\frac{\dot{a}}{a} = -\frac{1}{6}(p_{\text{tot}} + 3p_{\text{tot}}),
\]

where the source is improved by the curvature contributions. The Euler-Lagrange Eq. [4] gives the curvature constrain

\[
R = -6 \left[\frac{\dot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2\right]
\]
assuming a null spatial curvature. In this picture, the form of the function \( f(R) \) can give rise to accelerated/decelerated behaviors \[31\] addressing cosmic dynamics other than the standard matter dominated. This form can be achieved asking for first principle as Noether symmetries as we will see below.

III. \( f(R) \) GRAVITY IN STRING LANDSCAPE

Several \( f(R) \) gravity models can be connected to string-equivalent models with the following considerations. Let us define the tree-level dilaton-graviton string effective action

\[
\mathcal{A} = \int d^D x \sqrt{-G} e^{-2\phi} [R + 4 \nabla_\mu \phi \nabla^\mu \phi + \Lambda],
\]

(10)

that is obtained in the low-energy limit considering only the scalar dilaton and the graviton \[39-47\]. \( D = n + 1 \) are the spatial + time dimensions, \( G \) is the determinant of the \( D \)-dimensional spacetime metric, and \( \Lambda \) is a string charge mimicking the cosmological constant. In this picture, the effective string action reduce to a scalar-tensor theory. In general, this theory shows the symmetry

\[
\phi \rightarrow \tilde{\phi} = \phi - \frac{1}{2} \ln(g_{00}),
\]

(11)

that, in the case of a spatially flat, homogeneous and isotropic metric,

\[
ds^2 = dt^2 - a^2(t) dx_i^2,
\]

(12)

reduces to

\[
a \rightarrow \tilde{a} = a^{-1}, \quad \phi \rightarrow \tilde{\phi} = \phi - n \ln(a),
\]

(13)

where \( a(t) \) is the cosmic scale factor. This is the duality symmetry of the scale factor \( a \) for the string-dilaton cosmology. The relations of Eqs. (13), between the cosmological solutions allow to construct the pre-Big Bang cosmological models \[15\].

From the action Eq. (10) we can derive the field equations by varying with respect to the metric tensor and the dilaton field. We obtain Einstein a Klein-Gordon field equations. In 4D, we have

\[
G_{\mu\nu} = \frac{1}{2} \Lambda g_{\mu\nu} + 2 g_{\mu\nu} \Box \phi - 2 g_{\mu\nu} \nabla_\mu \phi \nabla^\nu \phi - 2 \nabla_\mu \nabla_\nu \phi, \quad (14)
\]

and

\[
\Box \phi = \nabla_\mu \phi \nabla^\mu \phi - \frac{1}{4} (R + \Lambda).
\]

(15)

The dilaton solution \( \phi \) can be achieved from the trace of Eq. (14). It is

\[
\Box \phi = -\frac{R}{6} + \frac{4}{3} \nabla_\mu \phi \nabla^\mu \phi - \frac{\Lambda}{3}.
\]

(16)

Comparing with Eq. (15), we get

\[
\Box \phi = -\frac{1}{2} R,
\]

(17)

and also

\[
\nabla_\mu \phi \nabla^\mu \phi = \frac{1}{4} (\Lambda - R).
\]

(18)

With these simple considerations in mind, let us now interpret the further mode coming from the dilaton \( \phi \) under the standard of \( f(R) \) gravity. Let us assume the transformation

\[
g_{\mu\nu}(x) \rightarrow \tilde{g}_{\mu\nu}(x) = \Omega^2(x) g_{\mu\nu}(x).
\]

(19)

The string-dilaton and \( f(R) \) actions can be mapped into each other as

\[
\sqrt{-\tilde{g}} e^{-2\phi} (R + 4 \nabla_\mu \phi \nabla^\mu \phi + \Lambda) = \sqrt{-g} f(\tilde{R}),
\]

(20)

which, using Eq. (19), can be recast as

\[
e^{-2\phi} (R + 4 \nabla_\mu \phi \nabla^\mu \phi + \Lambda) = \Omega^4 f(\tilde{R}).
\]

(21)

The latter becomes

\[
\tilde{f}(\tilde{R}) = 2 \Lambda \Omega^{-4} e^{-2\phi},
\]

(22)

which, by choosing \( \Omega = e^{-\phi} \) according to the dilaton coupling, assumes the form

\[
f(\tilde{R}) = 2 \Lambda e^{2\phi},
\]

and then the scale factor duality \( a \rightarrow 1/a \) is recovered for \( f(R) \) gravity. It is straightforward to show that a general point-like \( f(R) \) Lagrangian exhibiting duality is \[18\]

\[
\mathcal{L} = [f(R) - f'(R) R] + 12 \left( \frac{\dot{a}}{a} \right)^2 f'(R).
\]

(23)

General duality transformations can be achieved asking for Noether symmetries (see App. A). Assuming

\[
R = A e^{-\phi}, \quad f(R) = e^{-2\phi} F(\phi),
\]

(24)

it is be possible to generalize the Lagrangian \[23\] to the form

\[
\mathcal{L} = [e^{-2\phi} (F'(\phi) - F(\phi))] - 12 A^{-1} e^{-\phi} \left( \frac{\dot{a}}{a} \right)^2 [F'(\phi) - 2 F(\phi)].
\]

(25)

From Eq. (25), specific \( f(R) \) showing duality can be obtained. In particular,

\[
f(R) = \frac{1}{A} \left( \xi R - \frac{\gamma}{A} R^2 \right),
\]

(26)

where \( A, \xi, \) and \( \gamma \) are constants, is the Starobinsky model for inflation \[54\]. By the same Noether Symmetry Approach, one can find out other models as \( f(R) = R^{1/2} \).
f(R) = R^{3/2} [32], f(R) = R^2 [35], and f(R) = R - R^{-4/3} that exhibit duality. As discussed in Ref. [34], some of these models are in good agreement with dark energy behavior so they can be, in principle, suitable to represent early and late time cosmology starting from first principles. In the next section we will discuss swampland criteria for f(R) gravity. In Appendix A details on Noether symmetries related to duality are reported.

IV. THE SWAMPLAND CRITERIA IN f(R) GRAVITY

The swampland criteria for f(R) gravity can be discussed adopting conformal transformations and recasting the theory from the Jordan to the Einstein frame [33]. The f(R) gravity action of Eq. (27) can be rewritten as

\[ \tilde{A} = \int d^4x \sqrt{-g} \left[ -\frac{1}{2} \tilde{R} + \frac{1}{2} \tilde{g}_{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) + \tilde{\mathcal{L}}_M \right], \]

specifying the above transformation (19) as

\[ \tilde{g}_{\mu\nu} = e^{-2k\phi} g_{\mu\nu}. \]

We have the effective scalar field

\[ \phi = \frac{1}{2k} \ln(f_R), \]

with k a generic constant and \( f_R = e^{2k\phi} \) (see [30] for details). In the action of Eq. (27), V(\phi) is the effective potential related to the conformal field,

\[ V = \frac{1}{2} \left( f - Rf_R \right). \]

Deriving this potential with respect to the field \( \phi \), we obtain

\[ \frac{\partial V}{\partial \phi} = \frac{2k}{f_R} \left[ -f + \frac{Rf_R}{2} + \frac{f_R}{2} \left( \frac{\partial f}{\partial f_R} \right) - f_R \left( \frac{\partial R}{\partial f_R} \right) \right], \]

and finally we can write the relation

\[ \frac{\nabla_\phi V}{V} = \left( \frac{4k}{f - Rf_R} \right) \left[ -f + \frac{Rf_R}{2} \right] + \frac{f_R}{2} \left( \frac{\partial f}{\partial f_R} \right) \]

\[ - \frac{f_R}{2} \left( \frac{\partial R}{\partial f_R} \right). \]

A quantitative proposal for constraints on potentials, that are not in the swampland, is reported in [7]. Having defined the general form for the effective potential in f(R) gravity, we can now discuss the Swampland Conjecture [10] in order to be consistent with string theory at the high energy regime. The criteria are the following:

- **Swampland Criterion 1**
  Any effective Lagrangian has a proper field range for \( |\Delta \phi| \leq \Delta \) where the expectation is that \( \Delta \approx \mathcal{O}(1) \). In particular if we go a large distance in field space, a tower of light modes appear. This criterion implies a limit to the quantity \( \Delta \phi \). The condition can be written as

\[ |\Delta \phi| = \left| \frac{1}{2k} \Delta \ln |f_R| \right| = \left| \frac{1}{2k} \frac{1}{f_R} f_{RR} \Delta R \right| < \Delta \approx \mathcal{O}(1). \]

In order to obtain, this expression for the first swampland criteria for f(R) models, we used Eq. (29). Clearly, in order to achieve the condition, the shape of f(R) function and its derivatives are extremely relevant. It is important to see that the ratio between the first and the second derivative in R has the main role in view of satisfying the criterion. In particular the shape of f(R) is important for convergence and stability of models approaching singularities [33].

- **Swampland Criterion 2**
  The effective potential V(\phi), for V > 0, has to satisfy the lower bound condition \( |\nabla_\phi V|/V \geq c \approx \mathcal{O}(1) \) in Planck units. For the specific case of f(R), this condition can be written as

\[ \frac{|\nabla_\phi V|}{V} = \left| \left( \frac{4k}{f - Rf_R} \right) \left[ -f + \frac{Rf_R}{2} + \frac{f_R}{2} \left( \frac{\partial f}{\partial f_R} \right) \right] \right| > c \approx \mathcal{O}(1). \]

This expression has been obtained by Eq. (32). For invertible f_R functions, the above equation simplifies to

\[ \frac{|\nabla_\phi V|}{V} = \left| \left( \frac{4k}{f - Rf_R} \right) \left[ \frac{Rf_R}{2} - f \right] \right| > c \approx \mathcal{O}(1). \]

As we will see below, this criterion sets strong constraint on the possible forms of f(R) satisfying the swampland conjecture and can be considered for selecting viable models at late times.

V. FROM EARLY TO LATE TIME ACCELERATION

In this section we will take into account some f(R) models satisfying the above swampland conditions. The perspective is to relate early and late cosmological eras [31]. It is important to stress that, according to Sec. III, the models below can be selected by the Noether Symmetry Approach that guarantees duality invariance. According to our prescription, they can be seen as effective models related to the string landscape. Specifically, we will choose power law models [52], [57], [58] and the Starobinsky model [59] connecting inflation and dark energy epochs. The first choice is related to the fact that we...
can study small deviations with respect to the Hilbert-Einstein action of General Relativity implying \( f(R) = R \). As we will see, these deviations can be suitably related to the swampland criteria. The second choice comes from the fact that Starobinsky model, according to the recent PLANCK release \[60, 61\], is one of the best candidate to fit inflationary behavior and, as discussed above, it can be recovered from the Noether Symmetry Approach in relation to duality (see Eq.\((36)\)).

A. Power law Models

Let we start considering a power law model as

\[ f(R) \propto R^{1+\epsilon} \quad (36) \]

where \( \epsilon \) is a parameter that controls the magnitude of the corrections with respect to the Hilbert-Einstein action. Assuming small deviation with respect to GR, that is \(|\epsilon| << 1\), it is possible to write Eq. \((36)\) as

\[ F(R) = R^{1+\epsilon} \approx R + \epsilon R \log R + \mathcal{O}(\epsilon^2). \quad (37) \]

Such models can achieve the production of gravitational waves in the early Universe \[62\] as well as small deviation by the apsidal motion of eccentric binary stars \[63\]. Furthermore, these models have been tested to study null and timelike geodesics in the cases of Solar System \[64\] and for black hole solutions \[65, 66\]. Also if they are often considered nothing else that toy models, they indicate how deviations with respect to General relativity can affect dynamics.

By the conformal transformation of Eq.\((36)\), one obtains

\[ V(\phi) = \frac{\epsilon}{2(1+\epsilon)^2} \left[ e^{2\epsilon \phi} \right]^{1+\epsilon}, \quad (38) \]

by which it is possible to recover the cosmological constant as soon as \( \epsilon \rightarrow 1 \), while for \( \epsilon \rightarrow \infty \) it goes to an exponentially suppressed plateau.

If one considers the early universe, it has been argued that models of this form, with a non canonical kinetic term of inflaton, may be naturally obtained even if the original potential is not particularly flat \[67, 69\]. Specifically, such a potential can satisfy the swampland conditions taking into account that mechanisms in this scenario invalid the slow roll condition \[13\].

On the other hand, at late epochs, such \( f(R) \) model give rise to cosmological models which well fit SNeIa and Cosmic Microwave Background data for \( \epsilon \sim -0.6 \) and \( \epsilon \sim 0.4 \) \[77\]. However\(^1\) they have to be carefully considered to address the accelerated/decelerated transition necessary for structure formation. In order to discuss if some values of \( \epsilon \) satisfy the swampland criteria, let us recast Eq.\((34)\) for power-law models. This implies

\[ \left| \frac{V_{,\phi}}{V} \right| = \left| \left( \frac{2(1-\epsilon)}{\epsilon} \right) \right| > \mathcal{O}(1), \quad (39) \]

defined for \( \epsilon \neq 0 \). The \( \mathcal{O}(1) \) for the modulus in Eq. \((39)\) is satisfied in the regions \( \epsilon < 2/3 \) and \( \epsilon > 2 \), where the first is compatible with values required to describe an accelerated late expansion of the universe.

We note that the above swampland criteria is not defined for \( \epsilon \rightarrow 0 \), that is required in the solar system \[64\]. In some sense, this condition is obvious because standard General Relativity cannot be recovered in the string landscape if not equipped with further fields as dilaton eventually emerging for \( f(R) \neq R \) according to the above discussion.

Looking at the first swampland criteria, we can see that the condition \( |\Delta \phi| \lesssim \mathcal{O}(1) \) is also satisfied for the model considered. Indeed, using Eq.\((39)\) and the relation \( \Delta V = \frac{\partial V}{\partial \phi} \Delta \phi \), one can write

\[ |\Delta \phi| = \left| \frac{\Delta V}{V} \left( \frac{\epsilon}{2(1-\epsilon)} \right) \right| \lesssim \mathcal{O}(1). \quad (40) \]

Assuming the value \( \Delta \phi \sim 1 \) as the upper limit in the equation above, we can see that for the value \( \epsilon \sim 0.4 \) the first swampland condition is satisfied whenever \( |\Delta V| \lesssim 3V \). According to Ref. \[57\], this value can be in agreement with late time accelerated behavior. This means that the first swampland criteria is satisfied for effective field excursions such that the potential varies not much more than approximately two or three times its value.

If we consider, instead, the value \( \epsilon \sim -0.6 \), also suggested by observations, we obtain the less stringent limit \( |\Delta V| \lesssim 5.3V \). Clearly, these values of \( \epsilon \) can give rise to models matching only partially the cosmic evolution. However, they indicate that, from models coming from the string landscape lying in the string landscape, it is possible to fit the late time evolution.

Specifically, it is worth noticing that the value \( \epsilon \sim 0.4 \) give rise to \( f(R) \sim R^3/2 \). Such a \( f(R) \) function is important because it gives the only analytically invertible conformal model \[52\] described by

\[ \mathcal{L} = -\sqrt{-g} f_0 R^{3/2} \leftrightarrow \tilde{\mathcal{L}} = -\sqrt{-g} \left[ \frac{\tilde{R}}{2} + \frac{1}{2} \nabla \varphi \nabla \varphi - V_0 e^{\sqrt{3} \varphi} \right] \quad (41) \]

which corresponds to the so-called Liouville theory. It is exactly integrable and provides a model capable to match dark energy, matter and radiation epochs \[58, 70, 72\]. The general solution is

\[ a(t) = a_0 \left[ c_4 t^4 + c_3 t^3 + c_2 t^2 + c_1 t + c_0 \right]^{1/2}. \quad (42) \]

According to the values of the constants \( c_i \) which are combinations of the initial conditions. For example, \( c_4 \neq 0 \)

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\(^1\) Since the swampland criteria only establish an order of magnitude for the upper limit on the quantities of the theory, a detailed observational analysis is useless in this context.
gives a power law accelerated behavior while, a radiation dominated stage is obtained if $c_1$ prevails on the other terms. It is interesting to stress again that the form $f(R) \sim R^{3/2}$ is obtained by the Noether symmetries [52, 53] and it is compatible with string landscape thanks to duality invariance.

**B. The Starobinsky Model**

As we said above, a model like

$$f(R) \simeq R + \alpha R^2 \quad (43)$$
can be obtained considering string duality transformation gravity from Noether Symmetry Approach [43]. It can be related to the original Starobinsky model, working in early cosmology [53] and can improved allowing a description of both the early and late cosmological accelerated phases [59],

$$f(R) = R + \lambda R_0 \left[ \left( 1 + \frac{R^2}{R_0^2} \right)^{\frac{n}{2}} - 1 \right] + \frac{R^2}{6M^2}. \quad (44)$$

where $n$ and $\lambda$ are positive values, $M$ is the effective mass of the scalaron$^2$. The scalar curvature $R$ assume very large and positive values in the past while $R_0$ value is of the order of the current observed cosmological constant. The second term of the equation above is negligible in the early universe [59]. This yields a self-consistent cosmological model with a (quasi-)de Sitter stage in the early universe with slow-roll decay, that is a graceful exit to the subsequent radiation-dominated stage. At the same time, the last term of Eq. (44) is negligible in the recent universe [59], and then the model describes a successful late time cosmological acceleration satisfying cosmological [73, 74], Solar system and laboratory tests [59, 75].

According to these considerations, the Starobinsky model for early and late epochs can be related, from one side, to the string landscape satisfying the swampland criteria and, from the other side, to emerge in late acceler- ated epoch thanks to Eq. (44). In other words, the reported tension between the early de Sitter behavior and the swampland criteria [10, 23, 76] could be solved improving the Starobinsky model like in Eq. (44), that has been successfully tested against observations [77]. In particular, it is possible to found values of parameters $(n, \lambda)$ that can be constrained by data assuming a Chevallier-Polarski-Linder (CPL) [78, 79] equation of state of the form $w(z) = w_0 + w_a z/(1 + z)$. Specifically, three cases are been studied as compatible with the observations [77]: case 1) $n = 2$ and $\lambda = 0.95$; case 2) $n = 3$ and $\lambda = 0.73$; case 3) $n = 4$ and $\lambda = 0.61$. The best fit values constrained are $(w_0, w_a) = (-0.92, -0.23), (-0.94, -0.22)$ and $(-0.96, -0.21)$, respectively [77].

In order to analyze if the background dynamics of these three cases is compatible with the swampland conditions, we consider the ratio $R \slash R_0$ comparing the value of curvature $R$ with the effective cosmological constant $R_0$. According to the CPL parametrization, this ratio can be obtained as a function of the redshift.

Let us focus in the range $0 \leq z \leq 1$, restricting our consideration to the regime where quintessence dominates. In other words, we are far from the epoch where matter is dominating and the universe is decelerating. Considering the best fit values for the three cases [77], we can see that at $z = 0$, the ratio $R \slash R_0 \sim 1$ for the three models, while for $z = 1$ we get $R \slash R_0 \sim 1.3$.

In Fig. (1), the behavior of $|\nabla \phi V| \slash V$ for the Starobinsky model against the ratio $R \slash R_0$ for the three set of solutions is reported. The Swampland condition, $|\nabla \phi V| \slash V > \sim O(1)$ with $V > 0$, is not satisfied in late universe regime for the cosmological values used, although it is of the order of magnitude $O(1)$ at more higher red shift. For $z = 0$ ($R \slash R_0 \sim 1$), General Relativity is recovered and this model does not satisfy the swampland conditions. For $z = 1$ ($R \slash R_0 \sim 1.3$), in the case 3), shown in the blue line in Fig. (1), $|\nabla \phi V| \slash V$ is of the order of $O(1)$. This regime is recovered also for the others two cases at higher red shift. From these considerations, we can infer that the exit from the swampland regime coincides with the recovery of General Relativity.

**VI. CONCLUSIONS**

In the last decades, there was a growing interest in connecting viable cosmological models with a quantum theory of gravity capable of discriminating among effective field models belonging to the so called string landscape. Swampland criteria have been recently established
in order to select effective field potentials. Specifically, considerable interest aroused because exact de Sitter solutions, with a positive cosmological constant, seem to be not compatible with the string landscape, i.e. they cannot be related to some fundamental theory in the high energy regime. In this paper, we improved the current discussion analyzing the viability of the $f(R)$ gravity in light of the swampland criteria. The $f(R)$ theories of gravity, considering the last observational releases (e.g. PLANCK), seem a realistic approach to both represent primordial universe emerging from inflation (the Starobinsky model is a paradigm in this sense) and to gives rise to models capable of addressing the late time accelerated expansion. It is worth noticing that several $f(R)$ models are invariant under string duality according to the presence of Noether symmetries. This feature allows to identify suitable $f(R)$ models naturally coming from the string landscape. In this sense, we can say that the presence of duality is a third swampland criterion.

We discussed power law $f(R)$ models pointing out values of the exponent where criteria are can be both in agreement with swampland conjecture and with late accelerated behavior. In this perspective, this could be a first step to relate effective models coming from some fundamental theory with late observed behavior of the cosmic flow passing through intermediate stages \cite{40, 41} if suitable distance indicators are selected. In particular, the exact general solution derived from $R^{3/2}$, seems useful to address several cosmic epochs according to the values of the integration parameters (see \cite{42} for a detailed discussion).

Furthermore, we considered the Starobinsky model that seems to connect early and late epochs. As discussed in \cite{48}, this model is duality invariant and well fit the PLANCK data \cite{60, 61} so it could be a realistic approach to trace the whole cosmic history. Taking into account a CPL parameterization for the equation of state, it is possible to represent the exit from the swampland regime towards the recovery of General Relativity.

In a forthcoming paper, this discussion will be improved considering a detailed matching with data of the above models.

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**Appendix A: The Noether Symmetry Approach**

The presence of Noether symmetries allows to reduce and then, in principle, to solve dynamics in a given dynamical system. In particular, dynamics related to the Lagrangian \cite{51} can be discussed by the Noether Symmetry Approach \cite{53, 55, 57, 72}. The conserved quantities are Noether symmetries that can be related to duality \cite{48}.

The approach can be outlined as follows. Let us take into account a canonical, non-degenerate point-like Lagrangian $\mathcal{L}(q^i, \dot{q}^i)$ where the conditions

$$\frac{\partial \mathcal{L}}{\partial \dot{q}^i} = 0, \quad \det H_{ij} \equiv \det \left| \frac{\partial^2 \mathcal{L}}{\partial \dot{q}^i \partial \dot{q}^j} \right| \neq 0,$$

hold. Here $H_{ij}$ is the Hessian matrix, the dot is the derivative with respect to the affine parameter $\lambda$. In general, $\mathcal{L}$ can be reduced to the standard mechanical form

$$\mathcal{L} = T(q, \dot{q}) - V(q),$$

where $T$ and $V$ are, respectively, the kinetic and potential terms. The energy function coming from $\mathcal{L}$ is

$$E_\mathcal{L} \equiv \frac{\partial \mathcal{L}}{\partial \dot{q}^i} \dot{q}^i - \mathcal{L},$$

which is a constant of motion, eventually equal to zero in cosmological context. Since cosmological problems have a finite number of degrees of freedom, one can take into account point transformations. Invertible coordinate transformations $Q^i = Q^i(q)$ induce transformations of the velocities, that is

$$\dot{Q}^i(q) = \frac{\partial Q^i}{\partial q^j} \dot{q}^j,$$

and the Jacobian of the transformation $J = \det \left| \frac{\partial Q^i}{\partial q^j} \right|$ is assumed to be non-zero so that the transformation is regular.

An infinitesimal point transformation is represented by a vector field

$$X = \alpha^i(q) \frac{\partial}{\partial q^i} + \left( \frac{d}{d\lambda} \alpha^i(q) \right) \frac{\partial}{\partial \dot{q}^i}.$$  \hspace{1cm} (A5)

$\mathcal{L}(q, \dot{q})$ is invariant under the transformation $X$ as soon as

$$L_X \mathcal{L} \equiv \alpha^i(q) \frac{\partial \mathcal{L}}{\partial q^i} + \left( \frac{d}{d\lambda} \alpha^i(q) \right) \frac{\partial \mathcal{L}}{\partial \dot{q}^i} = 0,$$

where $L_X \mathcal{L}$ is the Lie derivative of $\mathcal{L}$. In particular, the condition $L_X \mathcal{L} = 0$ means that the vector $X$ is a symmetry for the Lagrangian $\mathcal{L}$. Let us consider now a Lagrangian $\mathcal{L}$ and the related Euler-Lagrange equations

$$\frac{d}{d\lambda} \frac{\partial \mathcal{L}}{\partial \dot{q}^i} - \frac{\partial \mathcal{L}}{\partial q^i} = 0.$$  \hspace{1cm} (A7)
Considering the vector $X$ and contracting Eq. (A7), we obtain
\[ \alpha^i \left( \frac{d}{d\lambda} \frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} \right) = 0. \]  
(A8)

Since
\[ \alpha^i \frac{d}{d\lambda} \frac{\partial L}{\partial \dot{q}^i} = \frac{d}{d\lambda} \left( \alpha^j \frac{\partial L}{\partial \dot{q}^j} \right) - \left( \frac{d\alpha^i}{d\lambda} \right) \frac{\partial L}{\partial q^i}, \]  
(A9)
from Eq. (A8), it follows
\[ \frac{d}{d\lambda} \left( \alpha^j \frac{\partial L}{\partial \dot{q}^j} \right) = L_X L. \]  
(A10)
The consequence is the Noether theorem, that is, if $L_X L = 0$, then the function
\[ \Sigma_0 = \alpha^k \frac{\partial L}{\partial q^k}, \]  
(A11)
is a constant of motion.

Considering the specific case which we are discussing, that is $f(R)$ gravity cosmology, the configuration space is $Q = \{a, R\}$, and the tangent space is $TQ = \{a, \dot{a}, R, \dot{R}\}$. The Lagrangian is an application
\[ L : TQ \rightarrow \mathbb{R}, \]  
(A12)
where $\mathbb{R}$ are the real numbers. The generator of symmetry is
\[ X = \alpha \frac{\partial}{\partial a} + \beta \frac{\partial}{\partial R} + \dot{a} \frac{\partial}{\partial \dot{a}} + \dot{R} \frac{\partial}{\partial \dot{R}}. \]  
(A13)

A symmetry exists if $L_X L = 0$ has solutions. Alternatively, a symmetry exists if at least one of the functions $\alpha$ or $\beta$ is different from zero. Going to our specific case, the Lagrangian \[4\], and setting to zero the coefficients of the terms $a^2$, $\dot{R}^2$, $\dot{a} \dot{R}$, we obtain the following system of equations
\[ f'(R) (\alpha + 2a \partial_a \alpha) + a f''(R) (\beta + a \partial_a \beta) = 0, \]  
(A14)
\[ a^2 f''(R) \partial_R \alpha = 0, \]  
(A15)
\[ 2f'(R) \partial_R \alpha + f''(R) (2\alpha + a \partial_a \alpha + a \partial_R \beta) + a \beta f'''(R) = 0, \]  
(A16)
and, finally, setting to zero the remnant terms, we obtain the constraint
\[ 3\alpha (f(R) - R f'(R)) - a R f''(R) - \frac{6k}{a^2} (\alpha f'(R) + a \beta f''(R)) = 0. \]  
(A17)
To solve the system (A14)-(A17), explicit forms of $\alpha$ and $\beta$ have to be found. We can say that if at least one of the functions $\alpha$ and $\beta$ are different from zero, a Noether symmetry exists. If $f''(R) \neq 0$, Eq. (A15) can be immediately solved being
\[ \alpha = \alpha(a). \]  
(A18)
The case $f''(R) = 0$ is trivial because it corresponds to General Relativity. Eqs. (A14) and (A16) can be written as
\[ f'(R) \left( \alpha + 2a \frac{d\alpha}{da} \right) + a f''(R) (\beta + a \partial_a \beta) = 0, \]  
(A19)
\[ f''(R) \left( 2\alpha + a \frac{d\alpha}{da} + a \partial_R \beta \right) + a \beta f'''(R) = 0. \]  
(A20)
Being the function $f = f(R)$, then $\partial f/\partial a = 0$. Eq. (A20) can be solved considering
\[ \partial_R (\beta f''(R)) = -f''(R) \left( 2\alpha + a \frac{d\alpha}{da} \right), \]  
(A21)
and, integrating, the solution is
\[ \beta = - \left[ 2\alpha + a \frac{d\alpha}{da} \right] \frac{f'(R)}{f''(R)} + \frac{h(a)}{f''(R)}. \]  
(A22)
Eq. (A19) gives
\[ f'(R) \left[ a - a^2 \frac{d^2 \alpha}{da^2} - a \frac{d\alpha}{da} \right] + \left[ h + \frac{dh}{da} \right] = 0, \]  
(A23)
with the solution
\[ \alpha = c_1 a + \frac{c_2}{a} \quad \text{and} \quad h = \frac{c}{a}, \]  
(A24)
where, $a$ being dimensionless, $c_1$ and $c_2$ have the same dimensions. Being $\alpha$ dimensionless, the dimensions of $\beta$ are $[\beta] = M^2$. Then also $[c] = M^2$, so it is:
\[ \beta = - \left[ 3c_1 + \frac{c_2}{a^2} \right] \frac{f'(R)}{f''(R)} + \frac{c}{af''(R)}. \]  
(A25)
This Noether symmetry implies the existence of a constant of motion. From Eq. (A11) and the Lagrangian \[4\] we obtain:
\[ \alpha \left( 6f''(R)a^2 \dot{R} + 12 f'(R)a\dot{a} \right) + \beta (6f''(R)a^2 \dot{a}) = \mu_0. \]  
(A26)
This constant of motion gives duality for $f(R)$ models \[5\].

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