Entanglement of Four-Qubit Rank-2 Mixed States

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Abstract

It is known that there are three maximally entangled states $|\Phi_1\rangle = (|0000\rangle + |1111\rangle)/\sqrt{2}$, $|\Phi_2\rangle = (\sqrt{2}|1111\rangle + |1000\rangle + |0100\rangle + |0010\rangle + |0001\rangle)/\sqrt{6}$, and $|\Phi_3\rangle = (|1111\rangle + |1100\rangle + |0010\rangle + |0001\rangle)/2$ in four-qubit system. It is also known that there are three independent measures $F_j^{(4)}$ ($j = 1, 2, 3$) for true four-way quantum entanglement in the same system. In this paper we compute $F_j^{(4)}$ and their corresponding linear monotones $G_j^{(4)}$ for three rank-two mixed states $\rho_j = p|\Phi_j\rangle\langle\Phi_j| + (1 - p)|W_4\rangle\langle W_4|$, where $|W_4\rangle = (|0111\rangle + |1011\rangle + |1101\rangle + |1110\rangle)/2$. We discuss the possible applications of our results briefly.
I. INTRODUCTION

Recently, much attention is being paid to quantum information theory (QIT) and quantum technology (QT)[1]. Most important notion in QIT and QT is a quantum correlation, which is usually termed by entanglement[2] of given quantum states. As shown for last two decades it plays a central role in quantum teleportation[3], superdense coding[4], quantum cloning[5], and quantum cryptography[6, 7]. It is also quantum entanglement, which makes the quantum computer\(^1\) outperform the classical one[9]. Thus, it is very important to understand how to quantify and how to characterize the entanglement.

A. entanglement measures

For bipartite quantum system many entanglement measures were constructed before such as distillable entanglement[10], entanglement of formation (EOF)[10], and relative entropy of entanglement (REE)[11, 12].

The distillable entanglement is defined to quantify how many maximally entangled states can be constructed from the copies of the given quantum state in the asymptotic region. Thus, in order to compute the distillable entanglement we should find the optimal purification (or distillation) protocol. If, for example, the optimal protocol generates \(n\) maximally entangled states from \(m\) copies of the quantum state \(\rho\), the distillation entanglement for \(\rho\) is given by

\[
D(\rho) = \lim_{m \to \infty} \frac{n}{m}. \quad (1.1)
\]

Although the distillable entanglement is well-defined, its analytical calculation is very difficult because it is highly non-trivial task to find the optimal purification protocol except very rare cases[13].

REE of a given quantum state \(\rho\) is defined as

\[
E_R(\rho) = \min_{\sigma \in \mathcal{D}} S(\rho||\sigma), \quad (1.2)
\]

where \(\mathcal{D}\) is a set of separable states and \(S(\rho||\sigma)\) is a quantum relative entropy; that is \(S(\rho||\sigma) = \text{tr}(\rho \ln \rho - \rho \ln \sigma)\). It is known that \(E_R(\rho)\) is an upper bound of the distillable

\(^1\) The current status of quantum computer technology was reviewed in Ref.[8].
entanglement. However, for REE it is also highly non-trivial task to find the closest separable state $\sigma$ of the given quantum state $\rho$. Still, therefore, we do not know how to compute REE analytically even in the two-qubit system except rare cases [14].

The EOF for bipartite pure states is defined as a von Neumann entropy of each party, which is derived by tracing out other party. For mixed state it is defined via a convex-roof method [10, 15];

$$E_F(\rho) = \min \sum_j p_j E_F(\psi_j),$$

where minimum is taken over all possible pure state decompositions, i.e. $\rho = \sum_j p_j |\psi_j\rangle \langle \psi_j|$, with $0 \leq p_j \leq 1$. The decomposition which minimizes $\sum_j p_j E_F(\psi_j)$ is called the optimal decomposition. For two-qubit system, EOF is expressed as [16]

$$E_F(C) = h \left(\frac{1 + \sqrt{1 - C^2}}{2}\right),$$

where $h(x)$ is a binary entropy function $h(x) = -x \ln x - (1 - x) \ln(1 - x)$ and $C$ is called the concurrence. For two-qubit pure state $|\psi\rangle = |\psi_{ij}\rangle |ij\rangle$ with $(i, j = 0, 1)$, $C$ is given by

$$C = |\epsilon_{i_1i_2} \epsilon_{j_1j_2} \psi_{i_1j_1} \psi_{i_2j_2}| = 2 |\psi_{00} \psi_{11} - \psi_{01} \psi_{10}|,$$

where the Einstein convention is understood and $\epsilon_{\mu\nu}$ is an antisymmetric tensor. For two-qubit mixed state $\rho$ the concurrence $C(\rho)$ can be computed by $C = \max(\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4, 0)$, where $\{\lambda_1^2, \lambda_2^2, \lambda_3^2, \lambda_4^2\}$ are eigenvalues of $\rho(\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y)$ with decreasing order. Thus, one can compute the EOF for all two-qubit states in principle.

B. Classification of Entanglement

Although quantification of the entanglement is important, it is equally important to classify the entanglement, i.e., to classify the quantum states into the different type of entanglement. The most popular classification scheme is a classification through a stochastic local operation and classical communication (SLOCC) [17]. If $|\psi\rangle$ and $|\phi\rangle$ are in same SLOCC class, this means that $|\psi\rangle$ and $|\phi\rangle$ can be used to implement same task of quantum information process although the probability of success for this task is different. Mathematically, if two $n$-party states $|\psi\rangle$ and $|\phi\rangle$ are in the same SLOCC class, they are related to each other by $|\psi\rangle = A_1 \otimes A_2 \otimes \cdots \otimes A_n |\phi\rangle$ with $\{A_j\}$ being arbitrary invertible lo-
cal operators. Moreover, it is more useful to restrict ourselves to SLOCC transformation where all \( \{ A_j \} \) belong to SL(2, \( \mathbb{C} \)), the group of \( 2 \times 2 \) complex matrices having determinant equal to 1. In the three-qubit pure-state system it was shown\[18\] that there are six different SLOCC classes, fully-separable, three bi-separable, W, and Greenberger-Horne-Zeilinger (GHZ) classes. Subsequently, the classification was extended to the three-qubit mixed-state system\[19\].

The SLOCC transformation enables us to construct the entanglement measures for the multipartite states. As Ref.\[20\] showed, any linearly homogeneous positive function of a pure state that is invariant under determinant 1 SLOCC operations is an entanglement monotone. One can show that the concurrence \( C \) in Eq. (1.5) is such an entanglement monotone as follows. Let \( |\psi\rangle = \psi_{ij} |ij\rangle \) with \( i, j = 0, 1 \). Then, \( |\tilde{\psi}\rangle \equiv (A \otimes B)|\psi\rangle = \tilde{\psi}_{ij} |ij\rangle \), where \( \tilde{\psi}_{ij} = \psi_{\alpha\beta} A_{i\alpha} B_{j\beta} \). Using \( \epsilon_{i_1i_2} \epsilon_{j_1j_2} \epsilon_{k_1k_2} \epsilon_{l_1l_2} \epsilon_{m_1m_2} \epsilon_{n_1n_2} \epsilon_{p_1p_2} \epsilon_{q_1q_2} \epsilon_{r_1r_2} \epsilon_{s_1s_2} \epsilon_{t_1t_2} \epsilon_{u_1u_2} \epsilon_{v_1v_2} \epsilon_{w_1w_2} \epsilon_{x_1x_2} \epsilon_{y_1y_2} \epsilon_{z_1z_2} \) for arbitrary matrix \( M \), it is easy to show

\[
\tau_3 = \left| 2 \epsilon_{i_1i_2} \epsilon_{i_3i_4} \epsilon_{j_1j_2} \epsilon_{j_3j_4} \epsilon_{k_1k_2} \epsilon_{k_3k_4} \phi_{i_1j_1k_1} \phi_{i_2j_2k_2} \phi_{i_3j_3k_3} \phi_{i_4j_4k_4} \right|^{1/2}.
\] (1.6)

This is exactly the same with a square root of the residual entanglement\(^3\) introduced in Ref.\[21\]. The three-tangle (1.6) has following properties. If \( |\psi\rangle \) is a fully-separable or a partly-separable state, its three-tangle completely vanishes. Thus, \( \tau_3 \) measures the true three-way entanglement. It also gives \( \tau_3(\text{GHZ}_3) = 1 \) and \( \tau_3(\text{W}_3) = 0 \) to the three-way entangled states, where

\[
|\text{GHZ}_3\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle) \quad \text{and} \quad |\text{W}_3\rangle = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle).
\] (1.7)

For mixed state quantification of the entanglement is usually defined via a convex-roof method\[10, 15\]. Although the concurrence for an arbitrary two-qubit mixed state can be, in principle, computed following the procedure introduced in Ref.\[16\], still we do not know how

\(^2\) For complete proof on the connection between SLOCC and local operations see Appendix A of Ref.\[18\].

\(^3\) In this paper we will call \( \tau_3 \) three-tangle and \( \tau^2_3 \) residual entanglement.
to compute the three-tangle (or residual entanglement) for an arbitrary three-qubit mixed state. However, the residual entanglement for several special mixtures were computed in Ref.[22]. More recently, the three-tangle for all GHZ-symmetric states[23] was computed analytically[24].

It is also possible to construct the SLOCC-invariant monotones in the higher-qubit systems. In the higher-qubit systems, however, there are many independent monotones, because the number of independent SLOCC-invariant monotones is equal to the degrees of freedom of pure quantum state minus the degrees of freedom induced by the determinant 1 SLOCC operations. For example, there are $2(2^n - 1) - 6n$ independent monotones in $n$-qubit system. Thus, in four-qubit system there are six invariant monotones. Among them, it was shown in Ref.[25] by making use of the antilinearity[15] that there are following three independent monotones which measure the true four-way entanglement:

$$\mathcal{F}_1^{(4)} = (\sigma_\mu \sigma_\nu \sigma_2 \sigma_2) \cdot (\sigma_\mu \sigma_2 \sigma_\lambda \sigma_2) \cdot (\sigma_2 \sigma_\nu \sigma_\lambda \sigma_2)$$

$$\mathcal{F}_2^{(4)} = (\sigma_\mu \sigma_\nu \sigma_2 \sigma_2) \cdot (\sigma_\mu \sigma_2 \sigma_\lambda \sigma_2) \cdot (\sigma_2 \sigma_\nu \sigma_\lambda \sigma_2) \cdot (\sigma_2 \sigma_\lambda \sigma_\lambda \sigma_2)$$

$$\mathcal{F}_3^{(4)} = \frac{1}{2} (\sigma_\mu \sigma_\nu \sigma_2 \sigma_2) \cdot (\sigma_\mu \sigma_2 \sigma_\nu \sigma_2) \cdot (\sigma_\nu \sigma_2 \sigma_\mu \sigma_2) \cdot (\sigma_\nu \sigma_2 \sigma_\lambda \sigma_2) \cdot (\sigma_\nu \sigma_2 \sigma_\lambda \sigma_2)$$

where $\sigma_1 = 1_2$, $\sigma_1 = \sigma_2$, $\sigma_2 = \sigma_2$, $\sigma_3 = \sigma_2$, and the Einstein convention is introduced with a metric $g^{\mu \nu} = \text{diag}\{-1, 1, 0, 1\}$. Furthermore, it was shown in Ref.[26] that there are following three maximally entangled states in four-qubit system:

$$|\Phi_1\rangle = \frac{1}{\sqrt{2}} (|0000\rangle + |1111\rangle)$$

$$|\Phi_2\rangle = \frac{1}{\sqrt{6}} \left(\sqrt{2} |1111\rangle + |1000\rangle + |0100\rangle + |0010\rangle + |0001\rangle\right)$$

$$|\Phi_3\rangle = \frac{1}{2} (|1111\rangle + |1100\rangle + |0010\rangle + |0001\rangle)$$

|       | $\mathcal{F}_1^{(4)}$ | $\mathcal{F}_2^{(4)}$ | $\mathcal{F}_3^{(4)}$ |
|-------|----------------|----------------|----------------|
| $|\Phi_1\rangle$ | 1 | 1 | $\frac{1}{2}$ |
| $|\Phi_2\rangle$ | $\frac{8}{9}$ | 0 | 0 |
| $|\Phi_3\rangle$ | 0 | 0 | 1 |
| $|W_4\rangle$ | 0 | 0 | 0 |

Table I: $\mathcal{F}_1^{(4)}$, $\mathcal{F}_2^{(4)}$, and $\mathcal{F}_3^{(4)}$ of the maximally entangled and $W_4$ states.
The measures $F_1^{(4)}$, $F_2^{(4)}$, and $F_3^{(4)}$ of $|\Phi_1\rangle$, $|\Phi_2\rangle$, $|\Phi_3\rangle$, and

$$|W_4\rangle = \frac{1}{2}(|0111\rangle + |1011\rangle + |1101\rangle + |1110\rangle)$$  

(1.10)

are summarized in Table I. As Table I shows, $|\Phi_1\rangle$ is detected by all measures while $|\Phi_2\rangle$ (or $|\Phi_3\rangle$) is detected by only $F_1^{(4)}$ (or $F_3^{(4)}$). As three-qubit system, $|W_4\rangle$ is not detected by all measures.

C. Physical Motivations

As states earlier, W and GHZ classes represent the true 3-way entanglement in three-qubit system. However, the three-tangle $\tau_3$ and residual entanglement $\tau_3^2$ cannot detect the entanglement of W class, but yield a maximal value to GHZ class. Then, it is natural to ask how much entanglement is detected by $\tau_3$ and $\tau_3^2$ for the rank-2 mixture $\rho(p) = p|\text{GHZ}_3\rangle\langle\text{GHZ}_3| + (1 - p)|W_3\rangle\langle W_3|$. This was explored in the first reference of Ref.[22], whose residual entanglement is

$$\tau_3^2(\rho(p)) = \begin{cases} 
0 & \text{for } 0 \leq p \leq p_0 \\
g_I(p) & \text{for } p_0 \leq p \leq p_1 \\
g_{II}(p) & \text{for } p_1 \leq p \leq 1 
\end{cases}$$  

(1.11)

where

$$g_I(p) = p^2 - \frac{8\sqrt{6}}{9}\sqrt{p(1-p)^3} \quad g_{II}(p) = 1 - (1-p)\left(\frac{3}{2} + \frac{1}{18\sqrt{465}}\right)$$  

(1.12)

$$p_0 = \frac{4\sqrt{2}}{3 + 4\sqrt{2}} \sim 0.6269 \quad p_1 = \frac{1}{2} + \frac{3}{310\sqrt{465}} \sim 0.7087.$$  

Thus, one can say that the influence of W class is dominant at $0 \leq p \leq p_0$ while influence of GHZ class is dominant at $p_1 \leq p \leq 1$. In the intermediate region $p_0 \leq p \leq p_1$ two classes seem to compete with each other. For three-tangle similar method can be applied and the result is

$$\tau_3(\rho_p) = \begin{cases} 
0 & \text{for } 0 \leq p \leq p_0 \\
\frac{p}{1-p_0} & \text{for } p_0 \leq p \leq 1 
\end{cases}$$  

(1.13)

The expression of $\tau_3$ is much simpler than that of $\tau_3^2$. It is mainly due to the fact that $\tau_3$ is a linear invariant under the SLOCC transformation.
One can ask the same question to four-qubit system by choosing a rank-2 state. However, situation is much more complicated because the four-qubit system has nine different SLOCC classes\cite{27}. The nine classes and their representative states are summarized in Table II. Thus, there are too many combination to choose the rank-2 states. Motivated by the three-qubit case we choose two classes $G_{abcd}$ and $L_{ab3}$ and construct the rank-2 state $\rho_j = p|\Phi_j\rangle\langle\Phi_j| + (1-p)|W_4\rangle\langle W_4| (j = 1, 2, 3)$. The purpose of this paper is to compute $F^{(4)}_j$ and $G^{(4)}_j$ $(j = 1, 2, 3)$, where $G^{(4)}_j$ is a linear entanglement monotone defined as

$$G^{(4)}_1 = \left( F^{(4)}_1 \right)^{1/3} \quad G^{(4)}_2 = \left( F^{(4)}_2 \right)^{1/4} \quad G^{(4)}_3 = \left( F^{(4)}_3 \right)^{1/6}. \quad (1.14)$$

We will show that as in the three-qubit case the influence of $L_{ab3}$ class is strong at $0 \leq p \leq p_0$, where $p_0$ is dependent on $|\Phi_j\rangle$ and is larger than the corresponding 3-qubit value 0.6269 for most cases. Of course, with increasing $p$ the influence of $G_{abcd}$ becomes stronger gradually.

The paper is organized as follows. In sections II we derive the entanglement of $\rho_1$, $\rho_2$, and $\rho_3$ analytically. We also derive the optimal decompositions explicitly for each range in $p$. To check the correctness of our results we use the criterion discussed in Ref.\cite{28}, i.e. *entanglement should be a convex hull of the minimum of the characteristic curves*. In section III we discuss the possible applications of our results. In the same section a brief conclusion is given.
II. ENTANGLEMENT OF $\rho_j$ ($j = 1, 2, 3$)

In this section we will compute the entanglement of $\rho_j = p|\Phi_j\rangle\langle\Phi_j|+(1-p)|W_4\rangle\langle W_4|$. Before explicit calculation it is convenient to discuss the general method of our calculation briefly. First, we define a pure state

$$|Z_j(p, \varphi)\rangle = \sqrt{p}|\Phi_j\rangle - e^{i\varphi}\sqrt{1-p}|W_4\rangle.$$  \hspace{1cm} (2.1)

Since $|Z_j(p, \varphi)\rangle$ is a pure state, one can compute $F_i^{(4)}$ ($i = 1, 2, 3$) of it by making use of Eq. (1.8). As shown below, $F_i^{(4)}$ has a nontrivial zero at $p = p_0$ for most cases. This is due to the fact that $F_i^{(4)}$ cannot detect the entanglement of $|W_4\rangle$. Exploiting this fact we construct the optimal decomposition, which yields $F_i^{(4)}(\rho_j) = 0$ at $0 \leq p \leq p_0$. At $p_0 \leq p \leq 1$ region we conjecture the optimal decomposition and corresponding $F_i^{(4)}(\rho_j)$ by making use of the continuity of entanglement with respect to $p$ and convex condition. Same procedure can be applied to the computation of $G_i^{(4)}(\rho_j)$ ($i, j = 1, 2, 3$). Finally, we adopt a numerical method, which guarantees the correctness of our guess.

A. Case $\rho_1$

In this subsection we will compute the entanglement of $\rho_1 = p|\Phi_1\rangle\langle\Phi_1|+(1-p)|W_4\rangle\langle W_4|$. One can show

$$F_1^{(4)}[Z_1(p, \varphi)] = 3p^2 - 3(1-p)^2 e^{4i\varphi},$$

$$F_2^{(4)}[Z_1(p, \varphi)] = p^2 - 4(1-p)^2 e^{4i\varphi},$$

$$F_3^{(4)}[Z_1(p, \varphi)] = \frac{p^6}{2},$$ \hspace{1cm} (2.2)

where $|Z_1(p, \varphi)\rangle$ is defined in Eq. (2.1) with $j = 1$.

1. $F_1^{(4)}(\rho_1)$ and $G_1^{(4)}(\rho_1)$

From Eq. (2.2) one can show that $F_1^{(4)}[Z_1(p, \varphi)]$ has a nontrivial zero ($\varphi = 0$)

$$p_0 = \frac{\sqrt{3}}{\sqrt{3} + 1} \approx 0.634.$$ \hspace{1cm} (2.3)
The existence of finite \( p_0 \) guarantees that \( F_1^{(4)}(\rho_1) \) should vanish at \( 0 \leq p \leq p_0 \). At \( p = p_0 \) this fact can be verified because we have the optimal decomposition

\[
\rho_1(p_0) = \frac{1}{4} \left[ |Z_1(p_0, 0)\rangle\langle Z_1(p_0, 0)| + |Z_1(p_0, \frac{\pi}{2})\rangle\langle Z_1(p_0, \frac{\pi}{2})| + |Z_1(p_0, \pi)\rangle\langle Z_1(p_0, \pi)| + |Z_1(p_0, 3\frac{\pi}{2})\rangle\langle Z_1(p_0, 3\frac{\pi}{2})| \right].
\]  

At the region \( 0 \leq p < p_0 \), \( F_1^{(4)}(\rho_1) \) should vanish too because one can find the following optimal decomposition

\[
\rho_1(p) = \frac{p}{p_0} \rho_1(p_0) + \left(1 - \frac{p}{p_0}\right) |W_4\rangle\langle W_4|.
\]  

Combining these facts, one can conclude that \( F_1^{(4)}(\rho_1) = 0 \) at \( 0 \leq p \leq p_0 \).

Next, we consider the \( p_0 \leq p \leq 1 \) region. Eq. (2.4) at \( p = p_0 \) strongly suggests that the optimal decomposition at this region is

\[
\rho_1(p) = \frac{1}{4} \left[ |Z_1(p, 0)\rangle\langle Z_1(p, 0)| + |Z_1(p, \frac{\pi}{2})\rangle\langle Z_1(p, \frac{\pi}{2})| + |Z_1(p, \pi)\rangle\langle Z_1(p, \pi)| + |Z_1(p, 3\frac{\pi}{2})\rangle\langle Z_1(p, 3\frac{\pi}{2})| \right].
\]  

If Eq. (2.6) is a correct optimal decomposition in this region, \( F_1^{(4)}(\rho_1) \) reduces to

\[
F_1^{(4)}(\rho_1) = p(6p - 2p^2 - 3).
\]

Since the right-hand side of Eq. (2.7) is convex, our conjecture (Eq. (2.6)) seems to be right. In conclusion, we can write

\[
F_1^{(4)}(\rho_1) = \theta(p - p_0)p(6p - 2p^2 - 3),
\]

where \( \theta(x) \) is a step function defined as

\[
\theta(x) = \begin{cases} 
1 & x \geq 0 \\
0 & x < 0.
\end{cases}
\]

However, if our choice Eq. (2.6) is incorrect, Eq. (2.8) is merely an upper bound of \( F_1^{(4)}(\rho_1) \). Thus, we need to prove that Eq. (2.8) is really optimal value. To prove this we should examine, in principle, all possible decompositions of \( \rho_1 \), i.e. \( \rho_1 = \sum_i p_i |\psi_i\rangle\langle \psi_i| \), and minimize the corresponding value \( \sum_i p_i F_1^{(4)}(\psi_i) \). However, it is impossible because \( \rho_1 \) has infinite number of decomposition.
In order to escape this difficulty to some extent one may rely on Caratheodory’s theorem for convex hull [29], which states that for four-qubit rank-2 states five vector decomposition is sufficient to minimize $\mathcal{F}_1^{(4)}(\rho_1)$. Thus, we need to investigate decompositions with 2, 3, 4, or 5 vectors. This method was used in the first reference of Ref.[22] to minimize the residual entanglement of three-qubit rank-2 mixture. Still, however, it is difficult, at least for us, to parametrize all decompositions with first few vectors.

In this paper, therefore, we will adopt the alternative numerical method presented in Ref.[28]. We plot the $p$-dependence of $\mathcal{F}_1^{(4)}[Z_1(p, \varphi)]$ for various $\varphi$ (See solid lines of Fig. 1(a)). These curves have been referred as the characteristic curves. As Ref.[28] showed, $\mathcal{F}_1^{(4)}(\rho_1)$ is a convex hull of the minimum of the characteristic curves. Fig. 1(a) indicates that Eq.(2.8) (thick dashed line) is really the convex characteristic curve, which implies that Eq.(2.8) is really optimal. This method was also used in the third reference of Ref.[22] to minimize the residual entanglement of three-qubit rank-3 mixture.

Now, let us consider $\mathcal{G}_1^{(4)}(\rho_1)$. It is easy to show that $\mathcal{G}_1^{(4)}(\rho_1)$ vanishes at $0 \leq p \leq p_0$ due to the optimal decomposition Eq. (2.5). If one chooses Eq. (2.6) as an optimal decomposition at $p_0 \leq p \leq 1$, the resulting $\mathcal{G}_1^{(4)}(\rho_1)$ is not convex in the full range. Thus, we should adopt a technique introduced in Ref.[22]. In this case the optimal decomposition is

$$
\rho_1(p) = \frac{p - p_0}{1 - p_0} |\Phi_1\rangle\langle\Phi_1| + \frac{1 - p}{1 - p_0} \rho_1(p_0),
$$

(2.10)

which results in $\mathcal{G}_1^{(4)}(\rho_1) = (p - p_0)/(1 - p_0)$. Combining all these facts, one can conclude

$$
\mathcal{G}_1^{(4)}(\rho_1) = \theta(p - p_0) \frac{p - p_0}{1 - p_0}.
$$

(2.11)

To confirm that Eq. (2.11) is correct, we plot the characteristic curves $\mathcal{G}_1^{(4)}[Z_1(p, \varphi)]$ for various $\varphi$ as solid lines and Eq. (2.11) as thick dashed line in Fig. 1(c). This figure shows that Eq. (2.11) is convex hull of the minimum of the characteristic curves, which strongly supports the validity of Eq. (2.11).

2. $\mathcal{F}_2^{(4)}(\rho_1)$ and $\mathcal{G}_2^{(4)}(\rho_1)$

From Eq. (2.2) one can notice that $\mathcal{F}_2^{(4)}[Z_1(p, \varphi)]$ has a nontrivial zero ($\varphi = 0$)

$$
p_0 = \frac{2}{3} \approx 0.667.
$$

(2.12)
FIG. 1: (Color online) Plot of the $p$ dependence of (a) $F_1^{(4)}[Z_1(p, \varphi)]$, (b) $F_2^{(4)}[Z_1(p, \varphi)]$, and (c) $G_1^{(4)}[Z_1(p, \varphi)]$ in Eq. (2.2). We have chosen $\varphi$ from 0 to $2\pi$ as an interval 0.1. The thick dashed lines correspond to $F_1^{(4)}(\rho_1)$ in Eq. (2.8), $F_2^{(4)}(\rho_1)$ in Eq. (2.13) and $G_1^{(4)}(\rho_1)$ in Eq. (2.11). These figures indicate that Eq. (2.8), Eq. (2.13), and Eq. (2.11) are convex hull of the minimum of the characteristic curves.

Thus, Eq. (2.5) and Eq. (2.6) with $p_0 = 2/3$ can be the optimal decompositions for $F_2^{(4)}(\rho_1)$ at $0 \leq p \leq p_0$ and $p_0 \leq p \leq 1$, respectively. Then, the resulting $F_2^{(4)}(\rho_1)$ becomes

$$F_2^{(4)}(\rho_1) = \theta(p - p_0)p^2[p^2 - 4(1 - p)^2].$$

In order to confirm that our result (2.13) is correct, we plot the characteristic curves for various $\varphi$ (solid lines) and Eq. (2.13) (thick dashed line) in Fig. 1 (b). As Fig. 1(b) exhibits, our result (2.13) is convex hull of the minimum of the characteristic curves, which strongly supports that Eq. (2.13) is really optimal one.

Similarly, $G_2^{(4)}(\rho_1)$ becomes Eq. (2.11) with changing only $p_0$ to 2/3. The corresponding optimal decompositions are Eq. (2.5) at $0 \leq p \leq p_0$ and Eq. (2.10) at $p_0 \leq p \leq 1$, respectively.
respectively. Of course, we have to change \( p_0 \) to \( 2/3 \).

3. \( F_3^{(4)} (\rho_1) \) and \( G_3^{(4)} (\rho_1) \)

Eq. (2.2) shows that \( F_3^{(4)} [Z_1(p, \varphi)] \) do not have nontrivial zero. In addition, it is independent of the phase angle \( \varphi \). This fact may indicate that there are infinite number of optimal decompositions for \( F_3^{(4)} (\rho_1) \). The simplest one is

\[
\rho_1(p) = \frac{1}{2} |Z_1(p, 0)\rangle \langle Z_1(p, 0)| + \frac{1}{2} |Z_1(p, \pi)\rangle \langle Z_1(p, \pi)|, \tag{2.14}
\]

which gives \( F_3^{(4)} (\rho_1) = p^6/2 \). If one chooses Eq. (2.14) as an optimal decomposition for \( G_3^{(4)} (\rho_1) \), it generates \( G_3^{(4)} (\rho_1) = p^{21/6} \). Since it is not concave, we do not need to adopt a technique to make \( G_3^{(4)} (\rho_1) \) convex as we did previously. We summarize our results in Table III.

| \( j \) | \( F_{j}^{(4)} \) | \( G_{j}^{(4)} \) | \( p_0 \) |
|---|---|---|---|
| \( j = 1 \) | \( p(6p - 2p^2 - 3)\theta(p - p_0) \) | \( \frac{p-p_0}{1-p_0} \theta(p - p_0) \) | \( \frac{\sqrt{3}}{\sqrt{3}+1} \approx 0.634 \) |
| \( j = 2 \) | \( p^2[p^2 - 4(1-p)^2]\theta(p - p_0) \) | \( \frac{p-p_0}{1-p_0} \theta(p - p_0) \) | \( \frac{2}{3} \approx 0.667 \) |
| \( j = 3 \) | \( \frac{p^6}{2} \) | \( \frac{p^6}{2\pi} \) | |

Table III: Summary of \( F_{j}^{(4)} \) and \( G_{j}^{(4)} \) for \( \rho_1 \)

B. Case \( \rho_2 \)

In this subsection we would like to quantify the entanglement of \( \rho_2 \). Above all, we should say that Table I implies

\[
F_2^{(4)} (\rho_2) = G_2^{(4)} (\rho_2) = F_3^{(4)} (\rho_2) = G_3^{(4)} (\rho_2) = 0, \tag{2.15}
\]

because \( \rho_2 = p|\Phi_2\rangle\langle \Phi_2|+(1-p)|W_4\rangle\langle W_4| \) itself is an optimal decomposition for those entanglement measures. This fact is due to the fact that \( F_2^{(4)} \) and \( F_3^{(4)} \) cannot detect both \( |\Phi_2\rangle \) and \( |W_4\rangle \).

Let us now compute \( F_1^{(4)} (\rho_2) \) and \( G_1^{(4)} (\rho_2) \). It is straightforward to show

\[
F_1^{(4)} [Z_2(p, \varphi)] = \frac{8}{9} p^{3/2} |p^{3/2} - 2 \sqrt{3}(1-p)^{3/2} e^{3i\varphi}|, \tag{2.16}
\]
FIG. 2: (Color online) Plot of the $p$ dependence of (a) $\mathcal{F}^{(4)}_1[Z_2(p, \varphi)]$ in Eq. (2.16) and (b) $\mathcal{F}^{(4)}_3[Z_3(p, \varphi)]$ in Eq. (2.29). We have chosen $\varphi$ from 0 to $2\pi$ as an interval 0.1. The thick dashed lines correspond to $\mathcal{F}^{(4)}_1(\rho_2)$ in Eq. (2.25) and $\mathcal{F}^{(4)}_3(\rho_3)$ in Eq. (2.34). These figures indicate that Eq. (2.25) and Eq. (2.34) are convex hull of the minimum of the characteristic curves.

where $|Z_2(p, \varphi)\rangle$ is given in Eq. (2.1). We notice that $\mathcal{F}^{(4)}_1[Z_2(p, \varphi)]$ has a nontrivial zero ($\varphi = 0$)

$$p_0 = \frac{(2\sqrt{6})^{2/3}}{1 + (2\sqrt{6})^{2/3}} \approx 0.743.$$  

(2.17)

Thus, $\mathcal{F}^{(4)}_1(\rho_2)$ vanishes at $0 \leq p \leq p_0$ because one can find the optimal decomposition

$$\rho_2(p) = \frac{p}{p_0} \rho_2(p_0) + \left(1 - \frac{p}{p_0}\right) |W_4\rangle\langle W_4|,$$  

(2.18)

where

$$\rho_2(p_0) = \frac{1}{3} \left[ |Z_2(p_0, 0)\rangle\langle Z_2(p_0, 0)| + |Z_2(p_0, 2\pi/3)\rangle\langle Z_2(p_0, 2\pi/3)| + |Z_2(p_0, 4\pi/3)\rangle\langle Z_2(p_0, 4\pi/3)| \right].$$  

(2.19)

As the previous cases, we adopt, as a trial, the optimal decomposition at $p_0 \leq p \leq 1$ as

$$\rho_2(p) = \frac{1}{3} \left[ |Z_2(p, 0)\rangle\langle Z_2(p, 0)| + |Z_2(p, 2\pi/3)\rangle\langle Z_2(p, 2\pi/3)| + |Z_2(p, 4\pi/3)\rangle\langle Z_2(p, 4\pi/3)| \right].$$  

(2.20)

Then $\mathcal{F}^{(4)}_1(\rho_2)$ becomes $g_I(p)$, where

$$g_I(p) = \frac{8}{9} p^{3/2} \left[ p^{3/2} - 2\sqrt{6}(1-p)^{3/2} \right].$$  

(2.21)

However, $g_I(p)$ is not convex at the region $p \geq p_\ast \approx 0.9196$. Thus, we should adopt the technique previously used again to make $g_I(p)$ convex at the large-$p$ region.
Now, we define $p_1$ such as $p_0 \leq p_1 \leq p_*$. The parameter $p_1$ will be determined later. At the region $p_1 \leq p \leq 1$ we adopt the optimal decomposition for $\mathcal{F}^{(4)}_1(\rho_2)$ as a following form:

$$\rho_2(p) = \frac{p - p_1}{1 - p_1} \phi_2 \langle \phi_2 | + \frac{1 - p}{1 - p_1} \rho_2(p_1),$$

(2.22)

where

$$\rho_2(p_1) = \frac{1}{3} \left[ |Z_2(p_1,0)\rangle \langle Z_2 (p_1,0) | + |Z_2 \left( p_1, \frac{2\pi}{3} \right) \rangle \langle Z_2 \left( p_1, \frac{2\pi}{3} \right) | + |Z_2 \left( p_1, \frac{4\pi}{3} \right) \rangle \langle Z_2 \left( p_1, \frac{4\pi}{3} \right) | \right].$$

(2.23)

Eq. (2.22) leads $\mathcal{F}^{(4)}_1 (\rho_2)$ to $g_{II}(p)$ at the large-$p$ region, where

$$g_{II}(p) = \frac{8}{9} \left[ \frac{p - p_1}{1 - p_1} + \frac{1 - p}{1 - p_1} \left\{ p_1^3 - 2\sqrt{6} p_1^{3/2} (1 - p_1)^{3/2} \right\} \right].$$

(2.24)

As expected $g_{II}(p)$ is convex at $p_1 \leq p \leq 1$. The parameter $p_1$ is determined by $\frac{\partial g_{II}}{\partial p_1} = 0$, which yields $p_1 \approx 0.861^4$. Thus, $\mathcal{F}^{(4)}_1 (\rho_2)$ can be summarized as

$$\mathcal{F}^{(4)}_1 (\rho_2) = \begin{cases} 0 & 0 \leq p \leq p_0 \\ g_{II}(p) & p_0 \leq p \leq p_1 \\ g_{II}(p) & p_1 \leq p \leq 1. \end{cases}$$

(2.25)

In order to confirm again that Eq. (2.25) is correct, we plot the $p$-dependence of the characteristic curves (solid lines) in Fig. 2(a) for various $\varphi$. Our result (2.25) is plotted as a thick dashed line. This figure shows that our result (2.25) is a convex characteristic curve, which strongly supports that our result (2.25) is correct.

Now, let us compute $G^{(4)}_1 (\rho_2)$. At $0 \leq p \leq p_0$, $G^{(4)}_1 (\rho_2)$ should be zero due to Eq. (2.18). If we adopt Eq. (2.20) as an optimal decomposition $G^{(4)}_1 (\rho_2) = g_{I}^{1/3}(p)$ is obtained. However, it is not convex in the full range. Therefore, we have to choose

$$\rho_2(p) = \frac{p - p_0}{1 - p_0} \phi_2 \langle \phi_2 | + \frac{1 - p}{1 - p_0} \rho_2(p_0)$$

(2.26)

as an optimal decomposition, which results in

$$G^{(4)}_1 (\rho_2) = \theta (p - p_0) \left( \frac{8}{9} \right)^{1/3} \frac{p - p_0}{1 - p_0}.$$ 

(2.27)

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$^4$ The parameter $p_1$ is obtained by an equation $6p_1(4p_1 - 3)^2 = (1 - p_1)(1 + 2p_1)^2$. 

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C. Case $\rho_3$

In this subsection we will compute the entanglement of $\rho_3 = p|\Phi_3\rangle\langle\Phi_3|+(1-p)|W_4\rangle\langle W_4|$. Since $F_1^{(4)}$ and $F_2^{(4)}$ cannot detect both $|\Phi_3\rangle$ and $|W_4\rangle$, it is easy to show

$$F_1^{(4)}(\rho_3) = G_1^{(4)}(\rho_3) = F_2^{(4)}(\rho_3) = G_2^{(4)}(\rho_3) = 0. \quad (2.28)$$

Now, let us compute $F_3^{(4)}(\rho_3)$ and $G_3^{(4)}(\rho_3)$. For $|Z_3(p, \varphi)\rangle$ in Eq. (2.1) it is possible to show that $F_3^{(4)}[Z_3(p, \varphi)]$ reduces to

$$F_3^{(4)}[Z_3(p, \varphi)] = p^5 \left| p - \frac{3}{2}(1 - p)e^{2i\varphi} \right|. \quad (2.29)$$

Eq. (2.29) implies that $F_3^{(4)}[Z_3(p, \varphi)]$ has a nontrivial zero ($\varphi = 0$) $p_0 = \frac{3}{5} = 0.6.$

Thus, $F_3^{(4)}(\rho_3)$ should be zero at the region $0 \leq p \leq p_0$ and its optimal decomposition is

$$\rho_3(p) = \frac{p}{p_0} \rho_3(p_0) + \left(1 - \frac{p}{p_0}\right) |W_4\rangle\langle W_4|, \quad (2.31)$$

where

$$\rho_3(p_0) = \frac{1}{2} \left[ |Z_3(p_0, 0)\rangle\langle Z_3(p_0, 0)| + |Z_3(p_0, \pi)\rangle\langle Z_3(p_0, \pi)| \right]. \quad (2.32)$$

If we adopt the optimal decomposition at $p_0 \leq p \leq 1$ as a form

$$\rho_3(p) = \frac{1}{2} \left[ |Z_3(p, 0)\rangle\langle Z_3(p, 0)| + |Z_3(p, \pi)\rangle\langle Z_3(p, \pi)| \right], \quad (2.33)$$

the resulting $F_3^{(4)}(\rho_3)$ becomes $\frac{5}{2}p^5 \left( p - \frac{3}{5} \right)$. Since this is convex, we conclude

$$F_3^{(4)}(\rho_3) = \theta(p - p_0) \frac{5}{2}p^5 \left( p - \frac{3}{5} \right). \quad (2.34)$$

In order to prove that Eq. (2.34) is correct we plot again the characteristic curves (solid lines) and our result (2.34) (thick dashed line) in Fig. 2(b), which supports that Eq. (2.34) is optimal one.

Finally, let us compute $G_3^{(4)}(\rho_3)$. If we take Eq. (2.33) as an optimal decomposition for $G_3^{(4)}(\rho_3)$ at $p_0 \leq p \leq 1$, the result is not convex in the full range of this region. Thus, we should choose

$$\rho_3(p) = \frac{p - p_0}{1 - p_0} |\Phi_3\rangle\langle\Phi_3| + \frac{1 - p}{1 - p_0} \rho_3(p_0) \quad (2.35)$$

as an optimal decomposition, which simply results in the right-hand side of Eq. (2.11) with $p_0 = 3/5$. 

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III. DISCUSSION AND CONCLUSIONS

We compute the three-kinds of true four-way entanglement measures $F_j^{(4)} (j = 1, 2, 3)$ and their corresponding linear entanglement monotones $G_j^{(4)} (j = 1, 2, 3)$ analytically for four-qubit rank-2 mixed states $\rho_j = p|\Phi_j\rangle\langle\Phi_j| + (1 - p)|W_4\rangle\langle W_4|$ ($j = 1, 2, 3$). All optimal decompositions consist of 2, 3, 4, and 5 vectors.

Our results can be used to find many different mixed states, which have vanishing entanglement. For example, let us consider $F_1^{(4)}$ with $p_0$ in Eq. (2.17). Let us represent, for simplicity, $|\Phi_2\rangle$ and $|W_4\rangle$ as

$$|\Phi_2\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |W_4\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \quad (3.1)$$

Imagine the two-dimensional space spanned by $|\Phi_2\rangle$ and $|W_4\rangle$ represented by a Bloch sphere. Then, the states in the Bloch sphere can be expressed as $\rho = \frac{1}{2}(1 + r \cdot \sigma)$, where $|r| = 1$ and $|r| < 1$ denote the pure and mixed states, respectively. In this representation the Bloch vectors of $|\Phi_2\rangle$, $|W_4\rangle$, and $|Z_2(p_0, \varphi)\rangle$ are

$$r(\Phi_2) = (0, 0, 1) \quad r(W_4) = (0, 0, -1) \quad (3.2)$$

$$r(Z_2(p_0, \varphi)) = (-2\sqrt{p_0(1-p_0)}, -2\sqrt{p_0(1-p_0)}\sin \varphi, 2p_0 - 1).$$

Thus, any states located in the tetrahedron, whose vertices are $(0,0,1)$, $(-2\sqrt{p_0(1-p_0)},0,2p_0-1)$, $(\sqrt{p_0(1-p_0)},-\sqrt{3p_0(1-p_0)},2p_0-1)$, and $(\sqrt{p_0(1-p_0)},\sqrt{3p_0(1-p_0)},2p_0-1)$ in the Bloch sphere representation, have vanishing $F_1^{(4)}$ and $G_1^{(4)}$.

| $\rho$ | $\mathcal{C}$ (concurrency) | $\tau$ (three-tangle) |
|-------|-----------------------------|----------------------|
| $\rho_1$ | $\frac{1}{2}(1 - 2\sqrt{p} - p) \theta(\alpha_1 - p)$, $(\alpha_1 = (\sqrt{2} - 1)^2)$ | 0 |
| $\rho_2$ | $\frac{3\sqrt{p} - \sqrt{3}}{3\sqrt{p}(3-p)} \theta(\alpha_2 - p)$, $(\alpha_2 = \frac{1}{3})$ | ? |
| $\rho_3$ | $\mathcal{C}_{AB} = \frac{1}{2}(1 - 2\sqrt{p} - p) \theta(\alpha_1 - p)$, $\mathcal{C}_{AC} = \mathcal{C}_{AD} = \mathcal{C}_{BC} = \mathcal{C}_{BD}$ | $\tau_{ACD} = \tau_{BCD} = 0$ |
| | $\mathcal{C}_{CD} = \frac{1}{2} \left\{ 1 - \sqrt{\frac{p}{2}} \left( \sqrt{1 + \sqrt{p}(2-p)} + \sqrt{1 - \sqrt{p}(2-p)} \right) \right\}$ | $\tau_{ABC} = \tau_{ABD} =$? |

Table IV: Entanglement for sub-states of $\rho_j$ ($j = 1, 2, 3$).
One can use our results to discuss the monogamy properties $^{[30]}$ of entanglement. For this purpose, however, we should compute the entanglement for the sub-states of $\rho_j$ ($j = 1, 2, 3$). The entanglement of the sub-states is summarized at Table IV. As this table shows, some three-tangle, at least for us, cannot be computed analytically. This is because still we do not have a closed formula for computing the three-tangles.

As mentioned above, there are nine SLOCC classes in the four-qubit system. Therefore, many rank-2 states can be constructed by choosing different classes. If, for example, $G_{abcd}$ and $L_{7\oplus \bar{1}}$ are chosen, one can construct the rank-2 state

$$\pi_j = p |\Phi_j\rangle \langle \Phi_j| + (1-p) |\xi\rangle \langle \xi|,$$  \hspace{1cm} (3.3)

where $|\xi\rangle = (|0000\rangle + |1011\rangle + |1101\rangle + |1110\rangle)/2$. Probably, our calculation procedure enable us to compute the entanglement of $\pi_2$ and $\pi_3$ although we have not checked it explicitly. For $\pi_1$, however, our procedure does not seem to work because of $\langle \xi | \Phi_1 \rangle \neq 0$. Using Table II one can construct many higher rank states. If, for example, $G_{abcd}$, $L_{7\oplus \bar{1}}$, and $L_{a_4}$ are chosen, one can construct the rank-3 state such as

$$\sigma_j = p |\Phi_j\rangle \langle \Phi_j| + q |\xi\rangle \langle \xi| + (1-p-q) |\eta\rangle \langle \eta|,$$  \hspace{1cm} (3.4)

where $|\eta\rangle = (|0001\rangle + |0110\rangle + |1000\rangle)/\sqrt{3}$. However, it seems to be highly difficult to compute the entanglement of higher rank states.

The most remarkable achievement and novelty of this paper is deriving the entanglement of four-qubit mixed states using an analytical approach. Thus, our result may serve as a quantitative reference for future studies of entanglement in quadripartite and/or multipartite mixed states.

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