On Ramsey numbers of hedgehogs

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Abstract
The hedgehog \( H_t \) is a 3-uniform hypergraph on vertices 1, \ldots, \( t + \binom{t}{2} \) such that, for any pair \((i, j)\) with 1 \( \leq i < j \leq t \), there exists a unique vertex \( k > t \) such that \([i, j, k]\) is an edge. Conlon, Fox and Rödl proved that the two-colour Ramsey number of the hedgehog grows polynomially in the number of its vertices, while the four-colour Ramsey number grows exponentially in the square root of the number of vertices. They asked whether the two-colour Ramsey number of the hedgehog \( H_t \) is nearly linear in the number of its vertices. We answer this question affirmatively, proving that \( r(H_t) = O(t^2 \ln t) \).

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1. Introduction
For a \( k \)-uniform hypergraph \( H \), the Ramsey number \( r(H) \) is the smallest \( n \) such that any two-colouring of \( K_n^{(k)} \), the complete \( k \)-uniform hypergraph on \( n \) vertices, contains a monochromatic copy of \( H \). Let \( r(H; q) \) denote the analogous Ramsey number for \( q \)-colourings, so that \( r(H) = r(H; 2) \).

It is a major open problem to determine the growth of \( r(K_t^{(3)}) \), the Ramsey number of the complete 3-uniform hypergraph on \( t \) vertices. It is known [6, 7] that there are constants \( c, c' > 0 \) such that

\[
2^{ct^2} \leq r(K_t^{(3)}) \leq 2^{2^{ct^2}}.
\]

Erdős conjectured that \( r(K_t^{(3)}) = 2^{2^{O(t)}} \), that is, the upper bound is closer to the truth. Erdős and Hajnal gave some evidence that this conjecture is true by showing that \( r_3(K_t^{(3)}; 4) \geq 2^{2^{ct^2}} \), that is, the four-colour Ramsey number of \( K_t^{(3)} \) is double-exponential in \( t \) (see e.g. [9]).

Definition. The hedgehog \( H_t \) is a 3-uniform hypergraph on \( t + \binom{t}{2} \) vertices 1, \ldots, \( t + \binom{t}{2} \) such that, for each 1 \( \leq i < j \leq t \), there exists a unique vertex \( k > t \) such that \([i, j, k]\) is an edge, and there are no additional edges.
We sometimes refer to the first \( t \) vertices as the \textit{body} of the hedgehog. For any \( k \geq 4 \), one can also define a \( k \)-uniform hedgehog \( H_t^{(k)} \) on \( t + \binom{t}{k-1} \), with a body of size \( t \) and a unique hyperedge for every \( k-1 \)-sized subset of the body. In this notation, we have \( H_t = H_t^{(3)} \).

Hedgehogs are interesting because their two-colour Ramsey number \( r(H_t; 2) \) is polynomial in \( t \), while their four-colour Ramsey number \( r(H_t; 4) \) is exponentially large in \( t \) \([3, 10]\). This suggests that the bound \( r(K_t^{(3)}; 4) \geq 2^{2^{\Omega(t)}} \) by Erdős and Hajnal may not be such strong evidence that \( r(K_t^{(3)}; 4) \leq 22^{\Theta_1(t)} \).

Hedgehogs are also interesting because they are a natural family of hypergraphs with \textit{degeneracy} 1. Degeneracy is a notion of sparseness for graphs and hypergraphs. For graphs, the degeneracy is defined as the minimum \( d \) such that every subgraph induced by a set of vertices has a vertex of degree at most \( d \). The Burr–Erdős conjecture \([2]\) states that there exists a constant \( c(d) \) depending only on \( d \) such that the Ramsey number of any \( d \)-degenerate graph \( G \) on \( n \) vertices satisfies \( r(G) \leq c(d) \cdot n \). Building on the work of Kostochka and Sudakov \([11]\) and Fox and Sudakov \([8]\), Lee \([12]\) recently proved this conjecture. We can similarly define the degeneracy of a hypergraph as the minimum \( d \) such that every sub-hypergraph induced by a subset of vertices has a vertex of degree at most \( d \). Under this definition, Conlon, Fox and Rödl \([3]\) observe that the 4-uniform analogue of the Burr–Erdős conjecture is false: the 4-uniform hedgehog \( H_t^{(4)} \), which is 1-degenerate, satisfies \( r(H_t^{(4)}; 3) \geq 2^{ct} \). They also observe that the 3-uniform analogue of the Burr–Erdős conjecture is false for three or more colours: the 3-uniform hedgehog, which is 1-degenerate, satisfies \( r(H_t; 3) \geq \Omega(t^3 \log^6 t) \).

However, the analogue of the Burr–Erdős conjecture for 3-uniform hypergraphs and two colours remains open. In particular, it was not known whether the Ramsey number of the hedgehog \( H_t \) is linear, or even near-linear, in the number of vertices, \( t + \binom{t}{2} \). Conlon, Fox and Rödl \([3]\) show \( r(H_t; 2) \leq 4t^3 \), and, with the above in mind, ask if \( r(H_t; 2) = t^{2+o(1)} \). We answer this question affirmatively.

**Theorem 1.1.** If \( t \geq 10 \) and \( n \geq 200t^2 \ln t + 400t^2 \), then every two-colouring of the complete 3-uniform hypergraph on vertices contains a monochromatic copy of the hedgehog \( H_t \). That is,

\[
r(H_t) < 200t^2 \ln t + 400t^2 + 1.
\]

We make no attempt to optimize the absolute constants here.

### 2. Ramsey number of hedgehogs

Throughout this section we assume \( t \geq 10 \), and that we have a fixed two-colouring of the edges of a complete 3-uniform hypergraph \( \mathcal{H} \) on vertex set \( V \) with \( n \geq 200t^2 \ln t + 400t^2 \) vertices. Let

\[
m_{\max} := 2t + \binom{t}{2}.
\]

Let \( \binom{S}{2} \) denote the set of pairs of elements of \( S \). For integer \( a \), let \( [a] = \{1, 2, \ldots, a\} \). For vertices \( u \) and \( v \) of \( \mathcal{H} \), we write \( uv \) as an abbreviation for the unordered pair \( \{u, v\} \).

For \( u, v \in V \), let

\[
d_{uv}^{(r)} := |\{w : \{u, v, w\} \text{ red}\}|,
\]

\[
d_{uv}^{(b)} := |\{w : \{u, v, w\} \text{ blue}\}|.
\]
For a set of pairs $F \subset \binom{V}{2}$, let

$$N^{(b)}(F) := \{ w : \exists uv \in F \text{ s.t. } \{u, v, w\} \text{ blue} \},$$

$$N^{(r)}(F) := \{ w : \exists uv \in F \text{ s.t. } \{u, v, w\} \text{ red} \}.$$

Here, and throughout, we use $b$ and $r$ to refer to the colours blue and red, respectively. For a vertex $v$ and set $X$, let

$$U^{(b)}_{\leq m}(v, X) = \{ u \in X : d^{(r)}_{uv} \leq m \},$$

$$U^{(r)}_{\leq m}(v, X) = \{ u \in X : d^{(r)}_{uv} \leq m \}.$$

If $X$ is omitted, take $X = V$. We define $U^{(b)}_{\leq m}(v, X)$ to be sets of $u$ such that $d^{(r)}_{uv}$ is small, rather than those such that $d^{(b)}_{uv}$ is small, because we wish to think of the $U^{(b)}$ as sets helpful in finding a blue hedgehog. Similarly, we think of the $U^{(r)}$ as sets helpful in finding a red hedgehog.

**Lemma 2.1.** For any $0 \leq m < |V|/2 − 1$, and $v \in V$,

$$\min(|U^{(b)}_{\leq m}(v)|, |U^{(r)}_{\leq m}(v)|) \leq 2m.$$

**Proof.** Fix $m$ and $v$. For convenience, let $A = U^{(b)}_{\leq m}(v)$ and $B = U^{(r)}_{\leq m}(v)$. Assume for contradiction that $|A|, |B| \geq 2m + 1$. For every $u$, we have $d^{(r)}_{uv} + d^{(b)}_{uv} = |V| − 2 > 2m$, so $A$ and $B$ are disjoint. Consider the set $E'$ of edges of $H$ containing $v$, one element of $A$, and one element of $B$. On one hand, $|E'| = |A| \cdot |B|$. On the other hand, for every $u \in A$, the pair $uv$ is in at most $m$ such red triples, so the number of red triples of $E'$ is at most $|A| \cdot m$. Additionally, for every $u \in B$, the pair $uv$ is in at most $m$ such blue triples, so the number of blue triples of $E'$ is at most $|B| \cdot m$. Hence, $(|A| + |B|) \cdot m \leq |E'| = |A| \cdot |B|$, a contradiction to $|A|, |B| \geq 2m + 1$.

The following ‘matching condition’ for hedgehogs is useful.

**Lemma 2.2.** Let $S \subset V$ be a set of $t$ vertices. If, for all non-empty sets $F \subset \binom{V}{2}$, we have $|N^{(b)}(F)| \geq |F| + t$, then there exists a blue hedgehog with body $S$. Similarly, if, for all non-empty sets $F \subset \binom{V}{2}$, we have $|N^{(r)}(F)| \geq |F| + t$, then there exists a red hedgehog with body $S$.

**Proof.** By symmetry, it suffices to prove the first part. Consider the bipartite graph $G$ between pairs in $\binom{S}{2}$ and vertices of $V \setminus S$, where $uv \in \binom{S}{2}$ is connected with $w \in V \setminus S$ if and only if triple $\{u, v, w\}$ is blue. If, for all non-empty $F \subset \binom{S}{2}$, we have $|N^{(b)}(F)| \geq |F| + t$, then any such $F$ has at least $|F| + t − |S| = |F|$ neighbours in $G$. By Hall’s marriage lemma on $G$, there exists a matching in $G$ using every element of $\binom{S}{2}$. Taking triples $\{u, v, w\}$, where $uv \in \binom{S}{2}$ and $w \in V \setminus S$ is the vertex matched with pair $uv$, gives a blue hedgehog with body $S$.

**2.1 Special cases**

We start by finding monochromatic hedgehogs in two specific classes of colourings on $H$. We base our proof of Theorem 1.1 on the argument for the first class of colourings, which we call simple colourings. We use the result for the second class of colourings, which we call balanced colourings, as a specific case in the general argument.
2.1.1 Simple colourings
Consider hypergraphs that are coloured in the following way.

1. Start with a graph $G$ on $[n]$.
2. Colour a complete hypergraph $H$ on $[n]$ by colouring the triple $\{u, v, w\}$ blue if at least one of $uv, uw, vw$ is in $G$, and red otherwise.

Lemma 2.3. If $n \geq t^2 + t$, any hypergraph coloured as above has a monochromatic $H_t$.

Proof. Set $X = V(G)$. For $i = t - 1, t - 2, \ldots, 0$, pick a vertex $v_i \in X$ whose degree in $G$ is at least $i$ and let $\hat{U}(v_i) \subset X$ be an arbitrary set of $i$ neighbours of $v_i$. Remove $v_i \cup \hat{U}(v_i)$ from $X$. We call this the peeling step of $v_i$. Figure 1 shows the first three peeling steps of this process for $t = 5$. If this process succeeds, we have found a set $S = \{v_{t-1}, \ldots, v_0\}$ of $t$ vertices and disjoint sets of vertices $\hat{U}(v_0), \ldots, \hat{U}(v_{t-1})$ also disjoint from $S$, from which we can greedily embed a blue hedgehog in $H$ with body $\{v_0, \ldots, v_{t-1}\}$: for each $v_iv_j$ with $i < j$, pick an arbitrary unused element of $\hat{U}(v_j)$ for the third vertex of the hedgehog’s edge containing $v_iv_j$.

Now suppose this process finds vertices $v_{t-1}, v_{t-2}, \ldots, v_{i+1}$ but fails to find $v_i$ for some $i \leq t - 1$. After picking $v_i$, we remove $v_j$ and $j$ of its neighbours from $X$, for a total of $j + 1$ vertices. Then we have removed exactly

$$t + (t - 1) + \cdots + (i + 2) = \binom{t + 1}{2} - \binom{i + 2}{2}$$

vertices from $X$. Hence,

$$|X| \geq \left( t^2 + t \right) - \binom{t + 1}{2} + \binom{i + 2}{2} = \binom{t + 1}{2} + \binom{i + 2}{2} > \frac{t^2 + i^2}{2} \geq t,$$

and every vertex has degree at most $i - 1$ in the subgraph of $G$ induced $X$. Thus, there exists an independent set $S \subset X$ in $G$ of size at least $|X|/i \geq t$. Furthermore, any vertex has at most $i - 1$ neighbours in $X$, so any two vertices $u, v \in S$ share at least

$$|X| - 2i \geq \binom{t}{2} + \binom{i + 2}{2} - 2i > t + \binom{t}{2}$$

red triples in the sub-hypergraph of $H$ induced by $X$, so we can greedily find a red hedgehog with body $S$.

2.1.2 Balanced colourings
In this section we consider the case where our colouring is ‘balanced’. Lemma 2.1 tells us that, for every vertex $v$ and every non-negative integer $m$ less than $|V|/2 - 1$, one of

$$|U^{(b)}_{\leq m}(v)| = \#\{u : d^{(r)}_{uv} \leq m\} \quad \text{and} \quad |U^{(r)}_{\leq m}(v)| = \#\{u : d^{(b)}_{uv} \leq m\}$$

is at most $2m$. In ‘balanced’ colourings, we assume, for all $v \in V$ and all $2t \leq m \leq m_{\max} := 2t + \binom{t}{2}$, both of $|U^{(b)}_{\leq m}(v)|$ and $|U^{(r)}_{\leq m}(v)|$ are $O(m)$. We show, in this case, there is a monochromatic hedgehog. The proof is by choosing a random subset of approximately $4t$ vertices, and showing that, with positive probability, we can remove vertices so that the remaining set of $t$ vertices is the body of some red hedgehog.

Lemma 2.4. Let $c \geq 1$. Consider a two-coloured hypergraph $H = (V, E)$ on $n \geq 40ct^2$ vertices. Suppose that, for all $2t \leq m \leq m_{\max}$ and all $v \in V$, we have

$$|U^{(b)}_{\leq m}(v)| \leq cm.$$ (2.1)

Then $H$ has a red hedgehog $H_t$. 

Proof. It suffices to prove for $n = 40ct^2$, so assume without loss of generality that $n = 40ct^2$. Pick a random set $S$ by including each vertex of $V$ in $S$ independently with probability $4t/n$. By the Chernoff bound, $\mathbb{P}[|S| \leq 3t] \leq e^{-t/8}$.

Fix $m$ such that $2t \leq m \leq m_{\text{max}}$ and $m$ is a multiple of $t$. Let $e_1, \ldots, e_p$ be the pairs such that $d_{e_\ell} \leq m$ for all $\ell \in [p]$, and let $X_1, \ldots, X_p$ the indicator random variables for these pairs being in $(S)$. Let $X = X_1 + \cdots + X_p$. By (2.1), we have $p \leq cmn/2$. Each $X_\ell$ for $\ell \in [p]$ is a Bernoulli$(16t^2/n^2)$ random variable. Consider a graph on $[p]$ where $\ell$ and $\ell'$ are adjacent (written $\ell \sim \ell'$) if $e_\ell$ and $e_{\ell'}$ share a vertex. This is a valid dependency graph for $\{X_\ell\}$ as $X_\ell$ is independent of all $X_{\ell'}$ such that $e_{\ell'}$ is vertex-disjoint from $e_\ell$. Furthermore, by the condition (2.1), each endpoint of any pair $e_\ell$ is in at most $cm$ pairs, so each $\ell \in [p]$ has degree at most $2cm$ in the dependency graph, and the total number of pairs $(\ell, \ell')$ such that $\ell \sim \ell'$ is at most $2cm p$. We have

$$\mathbb{E}[X] = \frac{16t^2p}{n^2} = \frac{2p}{5cn} \leq \frac{m}{5} < \frac{3m}{4} - t, \quad (2.2)$$

$$\text{Var}[X] = \sum_{\ell, \ell' \in [p]} \mathbb{E}[X_\ell X_{\ell'}] - \mathbb{E}[X_\ell] \mathbb{E}[X_{\ell'}]$$

$$= \sum_{\ell \sim \ell'} \mathbb{E}[X_\ell X_{\ell'}] - \mathbb{E}[X_\ell] \mathbb{E}[X_{\ell'}]$$

$$\leq 2cm \cdot \left(\left(\frac{4t}{n}\right)^3 - \left(\frac{4t}{n}\right)^4\right)$$

$$< \frac{128t^3 cm}{n^3}$$

$$< \frac{64t^3 c^2 m^2}{n^2}$$

$$= \frac{m^2}{25t}.$$
Hence,
\[
\mathbb{P}\left[ \left\{ uv \in \binom{S}{2} : d^{(r)}_{uv} \leq m \right\} > m - t \right] = \mathbb{P}[X > m - t] \\
= \mathbb{P}[X - \mathbb{E}[X] \geq m - t - \mathbb{E}[X]] \\
\leq \mathbb{P}[X - \mathbb{E}[X] \geq m/4] \\
\leq \frac{\text{Var}[X]}{(m/4)^2} \\
< \frac{16}{25t}.
\]

The first inequality is by (2.2) and the second is by Chebyshev’s inequality. By the union bound over the multiples of \( t \) in \([2t, m_{\text{max}}]\), of which there are less than \( t \), the probability that there exists some \( m \in [2t, m_{\text{max}}] \) a multiple of \( t \) with
\[
\# \left\{ uv \in \binom{S}{2} : d^{(r)}_{uv} \leq m \right\} \leq m - t
\]
is less than \( t \cdot 16/(25t) = 16/25 \). Again by the union bound, with probability more than \( 1 - (16/25 + e^{-t/8}) > 0 \) over the randomness of \( S \), we have (i) \(|S| \geq 3t \), and (ii) for all \( m \) a multiple of \( t \) in \([2t, m_{\text{max}}] \), (2.3) holds. Hence, there exists an \( S \) such that (i) and (ii) hold, so consider such an \( S \). Remove \(|S| - t \geq 2t \) vertices from \( S \), at least one from each of the \( 2t \) pairs with smallest \( d^{(r)}_{uv} \), to obtain a set of \( t \) vertices \( T \) such that, for all \( m \) a multiple of \( t \) in \([2t, m_{\text{max}}] \), we have
\[
\# \left\{ uv \in \binom{T}{2} : d^{(r)}_{uv} \leq m \right\} \leq \max\left(0, \# \left\{ uv \in \binom{S}{2} : d^{(r)}_{uv} \leq m \right\} - 2t \right) \leq \max(0, m - 3t).
\]

Then, for all \( m \) with \( 2t \leq m \leq m_{\text{max}} - t \), set \( m' \) to be the smallest multiple of \( t \) larger than \( m \), so that
\[
\# \left\{ uv \in \binom{T}{2} : d^{(r)}_{uv} \leq m \right\} \leq \# \left\{ uv \in \binom{T}{2} : d^{(r)}_{uv} \leq m' \right\} \leq \max(0, m' - 3t) \leq m - 2t.
\]

Now, we show that our matching condition holds. Setting \( m = 2t \) in (2.4), we have \( d^{(r)}_{uv} > 2t \) for all \( uv \in \binom{T}{2} \). Hence, for any non-empty subset \( F \subset \binom{T}{2} \) of size at most \( t \), any \( uv \in F \) satisfies \( d^{(r)}_{uv} > t + |F| \). If \( F \subset \binom{T}{2} \) has size greater than \( t \), then, by setting \( m = t + |F| \) in (2.4), we know that there are at most \( m - 2t = |F| - t \) pairs \( uv \in F \) such that \( d^{(r)}_{uv} \leq t + |F| \), so again there exists \( uv \in F \) such that \( d^{(r)}_{uv} > t + |F| \). We conclude that, for all non-empty subsets of pairs \( F \subset \binom{T}{2} \), there exists \( uv \in F \) such that \(|N^{(r)}(F)| \geq d^{(r)}_{uv} \geq t + |F| \). By Lemma 2.2, there exists a red hedgehog with body \( T \). \( \square \)

### 2.2 Proof of Theorem 1.1

#### 2.2.1 Proof outline

To prove Theorem 1.1, we follow the proof of Lemma 2.3. First, ‘peel off’ vertices \( v \) into a set \( S \) to try to find a blue or red hedgehog.\(^1\) If we succeed, we are done. If we fail, we end up with an induced two-coloured hypergraph that is ‘balanced’ in the sense of Lemma 2.4. In this case, we simply apply Lemma 2.4.

In the proof of Lemma 2.3, we started with \( X = V \) and iteratively removed from \( X \) a vertex \( v \) and a set \( \hat{U}(v) \) of size \( t \) such that, for all \( u \in \hat{U}(v) \), vertices \( u \) and \( v \) share many blue triples.

\(^1\) For technical reasons, we peel vertices to find both blue and red hedgehogs, as opposed to Lemma 2.3 where we only peeled vertices to find a blue hedgehog.
This deletes $O(t)$ vertices per round, which is small enough for the argument to succeed. For general hypergraphs, we peel off vertices $v$ with many ‘blue-heavy neighbours’, meaning there exists some $m$ such that $|U_{ \leq m}^{(b)}(v, X)| \geq 10m$. However, $m$ can be $\Theta(t^2)$, so if we simply deleted $v$ along with $10m$ of its blue-heavy neighbours $\tilde{U}^{(b)}(v) \subset U_{ \leq m}^{(b)}(v, X)$, we could delete $\Theta(t^2)$ vertices for every $v$, which is too many. Instead, when we peel off $v$, we delete $v$ from $X$, add a penalty of $t/m$ to each $u \in \tilde{U}^{(b)}(v)$, accumulated as $\alpha^{(b)}(u)$, and delete from $X$ every vertex $u$ with $\alpha^{(b)}(u) > 1/2$. With these penalties, we guarantee that, on average, we delete $O(t)$ vertices from $X$ per peeled vertex $v$.

However, we need more care. In Lemma 2.3, we can find a hedgehog with body $S$ because, for any peeled vertices $v, v' \in S$, the edges $\{u, v, v'\}$ are blue for every $u \in \tilde{U}(v)$. However, in our procedure, for a $v$ chosen with corresponding $\tilde{U}^{(b)}(v)$ of size $10m$, there are some vertices $w$ such that $\{u, v, w\}$ is blue for few (at most $4m$) vertices $u \in \tilde{U}^{(b)}(v)$. We denote this set of ‘bad’ vertices by $B^{(b)}(v)$. As much as possible, we wish to avoid choosing both $v$ and, at some later step, $w \in B^{(b)}(v)$ for the body $S^{(b)}$ of our blue hedgehog. Ideally, we simply delete all vertices $u \in B^{(b)}(v)$ in the step we peel off $v$. However, $B^{(b)}(v)$ can have $\Omega(m)$ vertices, which again could be too many if $m = \Theta(t^2)$. Instead, for each $w \in B^{(b)}(v)$ we add a penalty of $t/d^{(b)}_{wv}$, accumulated as $\beta^{(b)}(w)$, and delete from $X$ every vertex $w$ with $\beta^{(b)}(w) > 1/4$. We guarantee that, on average, we delete $O(t \ln t)$ vertices from $X$ per peeled vertex $v$ (Lemma 2.9). See Figure 2.

To finish the proof, we show that if our peeling produces a set $S^{(b)} = \{v_1, \ldots, v_t\}$ (where $v_i$ is chosen before $v_{i+1}$), then, because we track the penalties $\alpha^{(b)}(u)$ and $\beta^{(b)}(w)$ carefully, the matching condition of Lemma 2.2 holds. On the other hand, if the peeling procedure fails, the sub-hypergraph induced by $X$ is large and balanced, in which case we apply Lemma 2.4.

2.2.2 The peeling procedure

We now describe the procedure formally. Start with $S^{(b)} = S^{(r)} = \emptyset$, and $X = V$. For all $u \in V$, initialize $\alpha^{(r)}(u) = \alpha^{(b)}(u)$, $\beta^{(r)}(u) = \beta^{(b)}(u) = 0$. If, at any point, $S^{(b)}$ or $S^{(r)}$ has $t$ vertices, stop.

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2 For technical reasons, we peel vertices $v$ in increasing order of the corresponding $m.$
Recall that $m_{\text{max}} = 2t + \binom{t}{2}$. For $m = 2t, 2t + 1, \ldots, m_{\text{max}}$, do the following, which we refer to as Stage($m$).

While there exists a vertex $v \in X$ and a colour $\chi \in \{b, r\}$ such that $|U_{\leq m}^{(\chi)}(v, X)| \geq 10m$:

(a) let $\hat{U}^{(\chi)}(v)$ be the set $U_{\leq m}^{(\chi)}(v, X)$ truncated to $10m$ vertices arbitrarily,
(b) let $B^{(\chi)}(v) = \{w : |u \in \hat{U}^{(\chi)}(v) : \{u, v, w\} \text{ is colour } \chi | \leq 4m\}$,
(c) add $v$ to $S^{(\chi)}$,
(d) for all $u \in \hat{U}^{(\chi)}(v)$, add $t/m$ to $a^{(\chi)}(u)$,
(e) for all $w \in B^{(\chi)}(v)$, add $(1/4, t/d^{(\chi)}_{vw})$ to $b^{(\chi)}(w)$,
(f) delete from $X$ all vertices $u$ with $a^{(\chi)}(u) \geq 1/2$ or $b^{(\chi)}(u) \geq 1/4$,
(g) delete $v$ from $X$.

Note that $B^{(\chi)}(v)$ and $\hat{U}^{(\chi)}(v)$ are only defined for $v \in S^{(\chi)}$. We refer to steps (a)–(g) as the peeling step for $v$, denoted Peel($v$). We let $m_v$ denote the value such that the peeling step for $v$ occurred during Stage($m_v$), and call $m_v$ the peeling parameter of $v$. Throughout the analysis, let $X_v$ denote the set $X$ immediately after Peel($v$). For any $m \in [2t, m_{\text{max}}]$, let $X_m$ denote the set $X$ immediately after Stage($m$), so that $X_m \supseteq X_{m_v}$.

The above process terminates in one of two ways. Either we ‘get stuck’, that is, we complete Stage($m_{\text{max}}$) and $|S^{(b)}| < t$ and $|S^{(r)}| < t$, or we ‘finish’, that is, we terminate earlier with $|S^{(b)}| = t$ or $|S^{(r)}| = t$. We show there is a monochromatic hedgehog in each case. In Section 2.2.5 we handle the case where we ‘get stuck’. In Section 2.2.6 we handle the case where we ‘finish’.

### 2.2.3 Basic facts about peeling

We first establish the following facts about the procedure.

**Lemma 2.5.** For any $m$ such that $2t \leq m \leq m_{\text{max}}$, for any time in the procedure after Stage($m$), the following holds: for all colours $\chi \in \{b, r\}$, for all $m' = 2t \leq m' \leq m$, and for all vertices $v \in X$, we have $|U_{\leq m'}^{(\chi)}(v, X)| < 10m'$.

**Proof.** Fix $m$ with $2t \leq m \leq m_{\text{max}}$. We have $|U_{\leq m}^{(\chi)}(v, X_m)| < 10m$ for all $v \in X_m$. If not, then there exists a vertex $v \in X_m$ with $|U_{\leq m}^{(\chi)}(v, X_m)| \geq 10m$, in which case we would have peeled vertex $v$ during Stage($m$), and we would have deleted $v$ from $X_m$ during Peel($v$), which is a contradiction. Throughout the procedure, $X$ is non-increasing. Thus, at any point in the procedure after Stage($m$), we have $X \subseteq X_m$, so for all $v \in X$, we have $v \in X_m$ and

$$|U_{\leq m}^{(\chi)}(v, X)| \leq |U_{\leq m}^{(\chi)}(v, X_m)| < 10m'. \quad \square$$

**Lemma 2.6.** For all colours $\chi \in \{b, r\}$ and all vertices $v \in S^{(\chi)}$, we have $|B^{(\chi)}(v)| \leq 2m_v$.

**Proof.** We prove this for $\chi = b$, and the case $\chi = r$ follows from symmetry. We double-count the number $Z$ of red triples $\{u, v, w\}$ such that $u \in \hat{U}^{(b)}(v)$ and $w \in B^{(b)}(v)$. On one hand, every $u \in \hat{U}^{(b)}(v)$ is in at most $m_v$ red triples because we chose $\hat{U}^{(b)}(v)$ as a subset of $U_{\leq m}^{(\chi)}(v, X_v)$, so the total number of red triples is at most $m_v \cdot |\hat{U}^{(b)}(v)| = 10m_v^2$. On the other hand, by definition of $B^{(b)}(v)$, each $w \in B^{(b)}(v)$ is in at least $|\hat{U}^{(b)}(v)| - 4m_v = 6m_v$ such red triples. Thus, the number of such triples is at least $|B^{(b)}(v)| \cdot 6m_v$. Hence, $10m_v^2 \geq Z \geq 6m_v |B^{(b)}(v)|$, so $|B^{(b)}(v)| \leq 2m_v$ as desired. \square
Lemma 2.7. For all colours $\chi \in \{b, r\}$ and all vertices $v, v' \in S(\chi)$, we have $d_{vv'}^{(\chi)} \geq 4t$.

**Proof.** Assume for the sake of contradiction that $d_{vv'}^{(\chi)} < 4t$. Without loss of generality, $v$ was added to $S(\chi)$ before $v'$. We have $d_{vv'}^{(\chi)} < 4t < 4m_v$, so during $\text{Peel}(v)$, vertex $v'$ is included in $B^{(\chi)}(v)$.

Hence, min $(1/4, t/d_{vv'}^{(\chi)}) = 1/4$ is added to $\beta^{(\chi)}(v')$ during step (e) of $\text{Peel}(v)$, so during step (f) of $\text{Peel}(v)$, vertex $v'$ is deleted from $X$ if it has not been deleted already. Thus, we could not have added $v'$ to $S(\chi)$ after $\text{Peel}(v)$, which is a contradiction, so $d_{vv'}^{(\chi)} \geq 4t$, as desired. \qed

2.2.4 Bounding the number of deleted vertices

**Lemma 2.8.** For all colours $\chi \in \{b, r\}$ and all vertices $v \in S(\chi)$, during $\text{Peel}(v)$, the total increase in $\alpha^{(\chi)}(u)$ over all $u \in V$ is exactly $10t$.

**Proof.** Fix $v \in S(\chi)$. We have $|\hat{U}^{(\chi)}(v)| = 10m_v$ by definition, and, for $u \in \hat{U}^{(\chi)}(v)$, each $\alpha^{(\chi)}(u)$ increases by exactly $t/m_v$, for a total increase of $10m_v \cdot (t/m_v) = 10t$. \qed

**Lemma 2.9.** For all colours $\chi \in \{b, r\}$ and all vertices $v \in S(\chi)$, during $\text{Peel}(v)$, the total increase in $\beta^{(\chi)}(w)$ over all $w \in V$ is at most $20t \ln t$.

**Proof.** By symmetry, it suffices to prove the lemma for $\chi = b$. Let $v \in S^{(b)}$. For $m = 0, \ldots, 4m_v$, let

$$a_m := |\{w \in X_v : d_{vw}^{(b)} = m\}|,$$

$$a_{\leq m} := a_0 + a_1 + \cdots + a_m = |U^{(r)}_{\leq m}(v, X_v)|.$$

$\text{Peel}(v)$ is after Stage$(m_v - 1)$. Hence, by Lemma 2.5, for $2t \leq m \leq m_v - 1$, we have $a_{\leq m} \leq 10m_v$. We know

$$|U^{(r)}_{\leq 4m_v}(v, X_v)| \geq |U^{(r)}_{\leq m_v}(v, X_v)| \geq 10m_v > 8m_v,$$

where the second inequality holds because $v$ was chosen to be peeled in Stage$(m_v)$. Hence, by Lemma 2.1,

$$a_{\leq 4m_v} = |U^{(r)}_{\leq 4m_v}(v, X_v)| \leq |U^{(r)}_{\leq 4m_v}(v)| \leq 8m_v.$$

As $a_{\leq m}$ is non-decreasing in $m$, we conclude $a_{\leq m} \leq 10m_v$ for $2t \leq m \leq 4m_v$.

For $m = 0, \ldots, 4m_v$, for any $w$ with $d_{vw}^{(b)} = m$, the peeling of $v$ increases $\beta^{(b)}(w)$ by exactly min $(1/4, t/m_v)$. Thus, for $a_m$ many $w$, the penalty $\beta^{(b)}(w)$ increases by min $(1/4, t/m_v)$. Furthermore, $\beta^{(b)}(w)$ increases only for $w \in B^{(b)}(v)$, which has at most $2m_v$ vertices by Lemma 2.6. For $2m_v - a_{\leq 4m_v}$ vertices $w$, $\beta^{(b)}(w)$ increases by less than $t/4m_v$, giving a total increase in $\beta^{(b)}(w)$ of less than $t$ from those vertices. The total increase in $\beta^{(b)}(w)$ is thus less than

$$\frac{1}{4}(a_0 + a_1 + \cdots + a_{4t}) + \frac{a_{4t+1}t}{4t+1} + \cdots + \frac{a_{4m_v}t}{4m_v} + t.$$  \hspace{1cm} (2.5)

The coefficients of $a_0, \ldots, a_{4m_v}$ in (2.5) are non-increasing, so (2.5) is $t$ plus a positive linear combination of $a_{\leq 4t}, a_{\leq 4t+1}, \ldots, a_{\leq 4m_v}$. Subject to $a_{\leq m} \leq 10m_v$ for $2t \leq m \leq 4m_v$, all of $a_{\leq 4t}, a_{\leq 4t+1}, \ldots, a_{\leq 4m_v}$ are simultaneously maximized if $a_0 = 0$ and $a_m = 10$ for $m = 1, \ldots, 4m_v$, so (2.5) is maximized there as well. Hence,

$$\text{total increase in } \beta^{(b)}(w) < \frac{1}{4}(a_0 + a_1 + \cdots + a_{4t}) + \frac{a_{4t+1}t}{4t+1} + \cdots + \frac{a_{4m_v}t}{4m_v} + t \leq t + \frac{1}{4} \cdot 40t + \frac{10t}{4t+1} + \frac{10t}{4t+2} + \cdots + \frac{10t}{4m_v}.$$
\[ \leq 11t + 10t \ln (4m_v/4t) \]
\[ < 20t \ln t, \]
where, for the last inequality, we used \( m_v \leq t^2 \) and \( t \geq 10 \). This is what we wanted to show. \qed

**Lemma 2.10.** The total number of vertices deleted from \( X \) in the peeling procedure is at most \( 200t^2 \ln t \).

**Proof.** A vertex is deleted either for being added to \( S^{(b)} \) or \( S^{(r)} \), having \( \alpha^{(b)}(\cdot) \) or \( \alpha^{(r)}(\cdot) \) at least \( 1/2 \), or having \( \beta^{(b)}(\cdot) \) or \( \beta^{(r)}(\cdot) \) at least \( 1/4 \). At the end of the procedure, we have the following inequalities. For all \( \chi \in \{b, r\} \) and all \( u \in V \), we have that \( \alpha^{(\chi)}(u) \) and \( \beta^{(\chi)}(u) \) are initially 0 and increase only during the peeling step of some vertex \( v \in S^{(\chi)} \). Hence, by Lemma 2.8, for \( \chi \in \{b, r\} \),

\[ \sum_{u \in V} \alpha^{(\chi)}(u) = 10t \cdot |S^{(\chi)}| \leq 10t^2. \]

Furthermore, by Lemma 2.9, for \( \chi \in \{b, r\} \),

\[ \sum_{u \in V} \beta^{(\chi)}(u) \leq 20t \ln t \cdot |S^{(\chi)}| \leq 20t^2 \ln t. \]

We conclude that, at the end of the procedure,

\[ \#\{\text{deleted } u\} \leq |S^{(b)}| + |S^{(r)}| + \#\{ u : \alpha^{(b)}(u) \geq 1/2 \} + \#\{ u : \alpha^{(r)}(u) \geq 1/2 \} \]
\[ + \#\{ u : \beta^{(b)}(u) \geq 1/4 \} + \#\{ u : \beta^{(r)}(u) \geq 1/4 \} \]
\[ < 2t + \sum_{u \in V} (2\alpha^{(b)}(u) + 2\alpha^{(r)}(u) + 4\beta^{(b)}(u) + 4\beta^{(r)}(u)) \]
\[ \leq 2t + 2 \cdot 10t^2 + 2 \cdot 10t^2 + 4 \cdot 20t^2 \ln t + 4 \cdot 20t^2 \ln t \]
\[ < 200t^2 \ln t. \]

\[\]

2.2.5 Case 1: Peeling procedure gets stuck

By Lemma 2.10, the number of vertices deleted in the peeling process is at most \( 200t^2 \ln t \), so, at the end of the peeling procedure, \( |X| \geq (200t^2 \ln t + 400t^2) - 200t^2 \ln t = 400t^2 \).

Consider the complete two-coloured sub-hypergraph \( \mathcal{H}' \) of \( \mathcal{H} \) induced by the vertex set \( X \). By Lemma 2.5, at the end of the procedure, for all \( m = 2t, 2t + 1, \ldots, m_{\text{max}} \) and all \( v \in X \),

\[ |U^{(b)}_{\leq m}(v, X)| < 10m, \quad |U^{(r)}_{\leq m}(v, X)| < 10m. \]

Applying Lemma 2.4 to \( \mathcal{H}' \) with \( \epsilon = 10 \), we conclude \( \mathcal{H}' \) (and hence \( \mathcal{H} \)) has a red hedgehog \( H_t \). \footnote{By the same reasoning, \( \mathcal{H}' \) also has a blue hedgehog.}

2.2.6 Case 2: Peeling procedure finishes

Suppose we finish with \( |S^{(b)}| = t \). The analysis for \( |S^{(r)}| = t \) is symmetrical. We try to find a blue hedgehog. For brevity, in the rest of this section, let \( S = S^{(b)} \). Let \( S = \{v_1, \ldots, v_t\} \), where the \( v_i \) were chosen in the order \( v_1, \ldots, v_t \). For \( i = 1, \ldots, t \), let \( m_i = m_{v_i} \) be the peeling parameter for \( v_i \), so that \( m_1 \leq m_2 \leq \cdots \leq m_t \).

**Definition.** Call a pair \( v_i v_j \in \binom{S}{2} \) with \( i < j \) bad if \( v_j \in B^{(b)}(v_i) \). Otherwise, call \( v_i v_j \in \binom{S}{2} \) good. Let \( E_{\text{bad}} \subseteq \binom{S}{2} \) be the set of all bad pairs and let \( E_{\text{good}} \subseteq \binom{S}{2} \) be the set of all good pairs, so that \( \binom{S}{2} = E_{\text{bad}} \cup E_{\text{good}} \) is a partition.
Lemma 2.11.

\[
\sum_{v_i v_j \in E_{\text{bad}}} \frac{1}{d_{v_i v_j}^{(b)}} < \frac{1}{4}.
\]

**Proof.** Fix \(2 \leq j \leq t\). Consider all bad pairs \(v_i v_j\) with \(i < j\). At the peeling of \(v_j\), \(\beta(v_j) < 1/4\), otherwise \(v_j\) would have been deleted from \(X\) and we could not have peeled \(v_j\). Hence, at the peeling of \(v_j\),

\[
\frac{1}{4} > \beta^{(b)}(v_j) = \sum_{i : i < j, v_j \in B^{(b)}(v_i)} \min\left(\frac{1}{4}, \frac{t}{d_{v_i v_j}^{(b)}}\right) = \sum_{i : i < j, v_i v_j \in E_{\text{bad}}} \min\left(\frac{1}{4}, \frac{t}{d_{v_i v_j}^{(b)}}\right) = \sum_{i : i < j, v_i v_j \in E_{\text{bad}}} \frac{t}{d_{v_i v_j}^{(b)}}.
\]

The first equality is by definition of \(\beta^{(b)}(v_j)\), the second is by definition of \(E_{\text{bad}}\), and the last is because \(d_{v_i v_j}^{(b)} \geq 4t\) for all \(i < j\) by Lemma 2.7. Thus,

\[
\sum_{v_i v_j \in E_{\text{bad}}} \frac{1}{d_{v_i v_j}^{(b)}} = \sum_{j = 2}^{t} \sum_{i : i < j, v_i v_j \in E_{\text{bad}}} \frac{1}{d_{v_i v_j}^{(b)}} \leq \sum_{j = 2}^{t} \frac{1}{4t} < \frac{1}{4}.
\]

We prove that there is a blue hedgehog with body \(S\), by showing that the matching condition of Lemma 2.2 holds. Consider an arbitrary \(F \subset (\hat{S})\). Partition \(F = F_{\text{bad}} \cup F_{\text{good}}\), where \(F_{\text{bad}} = F \cap E_{\text{bad}}\) and \(F_{\text{good}} = F \cap E_{\text{good}}\). We wish to show that \(N^{(b)}(F) \geq |F| + t\).

**Subcase 1.**

\(|F_{\text{bad}}| \geq |F_{\text{good}}|\). By Lemma 2.11,

\[
\frac{|F_{\text{bad}}|}{\max_{v_i v_j \in E_{\text{bad}}} d_{v_i v_j}^{(b)}} \leq \sum_{v_i v_j \in E_{\text{bad}}} \frac{1}{d_{v_i v_j}^{(b)}} \leq \sum_{v_i v_j \in E_{\text{bad}}} \frac{1}{d_{v_i v_j}^{(b)}} < \frac{1}{4}.
\]

Thus, there exists some \(v_i v_j \in F_{\text{bad}}\) such that \(d_{v_i v_j}^{(b)} > 4 |F_{\text{bad}}|\). Furthermore, this \(v_i v_j\) satisfies \(d_{v_i v_j}^{(b)} \geq 4t\) by Lemma 2.7, so \(d_{v_i v_j}^{(b)} \geq 2|F_{\text{bad}}| + 2t\). Hence,

\[
|N^{(b)}(F)| \geq d_{v_i v_j}^{(b)} \geq 2|F_{\text{bad}}| + 2t \geq |F_{\text{bad}}| + |F_{\text{good}}| + 2t > |F| + t,
\]

as desired. The first inequality is because the blue edges containing \(v_i v_j\) are all elements of \(N^{(b)}(F)\). The second inequality is because \(d_{v_i v_j}^{(b)}\) is at least \(4|F_{\text{bad}}|\) and at least \(4t\) by above. The third inequality is by the assumption \(|F_{\text{bad}}| \geq |F_{\text{good}}|\). The fourth inequality is because \(|F| = |F_{\text{bad}}| + |F_{\text{good}}|\) and \(2t > t\).

**Subcase 2.**

\(|F_{\text{bad}}| < |F_{\text{good}}|\). In particular, \(|F_{\text{good}}| > 0\), so \(|F|\) has some good pair \(v_i v_j\) with \(i < j\). This pair is in at least \(4m_t \geq 8t\) blue triples, so \(|N^{(b)}(F)| \geq 8t\).

Let \(I\) be the set of all indices \(i\) such that there exists \(j\) with \(i < j \leq t\) with \(v_i v_j \in E_{\text{good}}\). For each \(i\), there are less than \(t\) indices \(j\) such that \(i < j \leq t\), so

\[
|I| \cdot t > |F_{\text{good}}|.
\]

(2.6)

For each \(i \in I\), arbitrarily fix \(j_i > i\) such that \(v_i v_{j_i}\) is good. For \(i \in I\), define

\[
U^*_i := N^{(b)}(\{v_i v_{j_i}\}) \cap \hat{U}^{(b)}(v_i), \quad U^*_i := \bigcup_{i \in I} U^*_i.
\]
so that \( U^*_t \subset N^{(b)}(F) \). For all \( i \in I \), the pair \( v_iv_{ji} \) is good, so \( v_{ji} \notin B^{(b)}(v_i) \). Hence, by the definition of \( B^{(b)}(v_i) \), there are more than \( 4m_i \) vertices \( u \in U^{(b)}(v_i) \) such that \( \{u, v_i, v_{ji}\} \) is blue. Thus, for all \( i \in I \), the set \( U^*_t \) has at least \( 4m_i \) vertices. In the peeling of \( v_i \), the penalty \( \alpha^{(b)}(u) \) increases by \( t/m_i \) for each \( u \in U^*_t \). Hence, in peeling \( v_i \), the sum of penalties \( \sum_{u \in U^*_t} \alpha^{(b)}(u) \), increases by at least \( 4m_i \cdot t/m_i = 4t \). Thus,

\[
4t \cdot |I| \leq \sum_{u \in U^*_t} \alpha^{(b)}(u). \tag{2.7}
\]

On the other hand, the vertex \( u \) is deleted from \( X \) whenever \( \alpha^{(b)}(u) \geq 1/2 \), the penalty \( \alpha^{(b)}(u) \) increases by at most \( t/2t = 1/2 \) in any peeling step, and the penalty \( \alpha^{(b)}(u) \) never changes after \( u \) is deleted from \( X \). Thus, for all vertices \( u \in V \), we have

\[
\alpha^{(b)}(u) \leq 1. \tag{2.8}
\]

We conclude that

\[
2|F| \leq 4|F_{\text{good}}| \leq 4t |I| \leq \sum_{u \in U^*_t} \alpha^{(b)}(u) \leq \sum_{u \in U^*_t} 1 = |U^*_t| \leq |N^{(b)}(F_{\text{good}})| \leq |N^{(b)}(F)|.
\]

The first inequality is by the assumption \(|F_{\text{bad}}| < |F_{\text{good}}|\), the second is by (2.6), the third is by (2.7), the fourth is by (2.8), the fifth is by \( U^*_t \subset N^{(b)}(F_{\text{good}}) \), and the sixth is by \( F_{\text{good}} \subset F \). Combining with \(|N^{(b)}(F)| \geq 8t \), we conclude \(|N^{(b)}(F)| \geq |F| + t \), as desired.

This covers all subcases, so we have proved that, for any non-empty subset \( F \subset \binom{S}{2} \), we have \( N^{(b)}(F) \geq |F| + t \). Hence, the matching condition of Lemma 2.2 holds, so there is a blue hedgehog with body \( S \), as desired. This completes the proof of Theorem 1.1.

\[\square\]

References

[1] Alon, N., Krivelevich, M., and Sudakov, B. (2003) Turán numbers of bipartite graphs and related Ramsey-type questions. *Combin. Probab. Comput.* 12 477–494.

[2] Burr, S. A. and Erdős, P. (1975) On the magnitude of generalized Ramsey numbers for graphs. In *Infinite and Finite Sets, Vol. 1 (Keszthely, 1973)*, Vol. 10 of Colloq. Math. Soc. János Bolyai, pp. 214–240. North-Holland.

[3] Conlon, D., Fox, J. and Rödl, V. (2017) Hedgehogs are not colour blind. *J. Combin.* 8 475–485.

[4] Conlon, D., Fox, J. and Sudakov, B. (2010) Hypergraph Ramsey numbers. *J. Amer. Math. Soc.* 23 247–266.

[5] Conlon, D., Fox, J. and Sudakov, B. (2015) Recent developments in graph Ramsey theory. In *Survey in Combinatorics 2015*, Vol. 424 of London Mathematical Society Lecture Note Series, pp. 49–118, Cambridge University Press.

[6] Erdős, P., Hajnal, A. and Rado, R. (1965) Partition relations for cardinal numbers. *Acta Math. Acad. Sci. Hungar.* 16 93–196.

[7] Erdős, P. and Rado, R. (1952) Combinatorial theorems on classifications of subsets of a given set. *Proc. London Math. Soc.* 3 417–439.

[8] Fox, J. and Sudakov, B. (2009) Two remarks on the Burr–Erdős conjecture. *European J. Combin.* 30 1630–1645.

[9] Graham, R. L., Rothschild, B. L. and Spencer, J. H. (1990) *Ramsey Theory*, second edition, Wiley.

[10] Kostochka, A. V. and Rödl, V. (2006) On Ramsey numbers of uniform hypergraphs with given maximum degree. *J. Combin. Theory Ser. A* 113 1555–1564.

[11] Kostochka, A. V. and Sudakov, B. (2003) On Ramsey numbers of sparse graphs. *Combin. Probab. Comput.* 12 627–641.

[12] Lee, C. (2017) Ramsey numbers of degenerate graphs. *Ann. of Math.* 185 791–829.

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