Notes on analytical study of holographic superconductors with Lifshitz scaling in external magnetic field

Zixu Zhao\textsuperscript{1,2}, Qiyuan Pan\textsuperscript{1,2}\textsuperscript{*} and Jiliang Jing\textsuperscript{1,2}\textsuperscript{†}
\textsuperscript{1}Institute of Physics and Department of Physics, Hunan Normal University, Changsha, Hunan 410081, China and 
\textsuperscript{2}Key Laboratory of Low Dimensional Quantum Structures and Quantum Control of Ministry of Education, Hunan Normal University, Changsha, Hunan 410081, China

Abstract

We employ the matching method to analytically investigate the holographic superconductors with Lifshitz scaling in an external magnetic field. We discuss systematically the restricted conditions for the matching method and find that this analytic method is not always powerful to explore the effect of external magnetic field on the holographic superconductors unless the matching point is chosen in an appropriate range and the dynamical exponent $z$ satisfies the relation $z = d - 1$ or $z = d - 2$. From the analytic treatment, we observe that Lifshitz scaling can hinder the condensation to be formed, which can be used to back up the numerical results. Moreover, we study the effect of Lifshitz scaling on the upper critical magnetic field and reproduce the well-known relation obtained from Ginzburg-Landau theory.

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\textsuperscript{*} panqiyuan@126.com
\textsuperscript{†} jjjing@hunnu.edu.cn
I. INTRODUCTION

As the most successful realization of the holographic principle, Maldacena first proposed the Anti-de Sitter/Conformal Field Theory (AdS/CFT) correspondence [1], which has been a powerful tool to deal with strongly coupled systems. In recent years, the AdS/CFT correspondence has been applied to study the condensed matter physics in order to understand the physics of high $T_c$ superconductors from the gravitational dual. Gubser first suggested that black hole horizons could exhibit spontaneous breaking of an Abelian gauge symmetry if the gravity was coupled to an appropriate matter lagrangian, including a charged scalar that condenses near the horizon [2]. Then Hartnoll, Herzog and Horowitz built the first holographic superconductor model and reproduced the properties of a $(2 + 1)$-dimensional superconductor in this simple model [3]. The pioneering work on this topic has led to many investigations concerning the condensation in bulk AdS spacetime, for reviews, see Refs. [4–6] and references therein.

From the AdS/CFT correspondence, the AdS black hole geometry corresponds to a relativistic CFT at finite temperature. However, many condensed matter systems do not have relativistic symmetry. Thus, Bu used the nonrelativistic AdS/CFT correspondence to study the holographic superconductors in the Lifshitz black hole geometry for $z = 2$ in order to explore the effects of the dynamical exponent and distinguish some universal properties of holographic superconductors [7]. It is found that the Lifshitz black hole geometry results in different asymptotic behaviors of temporal and spatial components of gauge fields than those in the Schwarzschild-AdS black hole, which brings some new features of holographic superconductor models. More recently, Lu et al. discussed the effects of the Lifshitz dynamical exponent $z$ on holographic superconductors and gave some different results from the Schwarzschild-AdS background [8]. To this day, there have attracted considerable interest to generalize the holographic superconducting models to nonrelativistic situations [9–16].

On the other hand, according to the Ginzburg-Landau theory, it should be noted that the upper critical magnetic field has the well-known relation $B_c \propto (1 - T/T_c)$ [17]. Using the semi-analytic method, Ge et al. reproduced this relation in the holographic superconductor model [18]. As an important step towards a realistic implementation of superconductivity through holography, Domènech et al. discussed the critical magnetic fields in the holographic superconductor and analyzed the effect of the dynamical magnetic field on the critical magnetic field [19]. Cai et al. studied the magnetic field effect on the holographic insulator/superconductor phase transition and found that the presence of the magnetic field causes the phase transition hard [20]. Along
this line, a number of attempts have been made in order to investigate the effects of applying an external magnetic field to holographic dual models \[21\]–\[32\]. All these papers to study the effect of external magnetic field on holographic dual models are made in relativistic situations. It is therefore very natural to consider the nonrelativistic situations, such as Lifshitz black hole.

In this work, we will use the matching method, which was first proposed in \[33\] and later refined in \[34\], to analytically investigate the effect of external magnetic field on holographic superconductors with Lifshitz scaling. We want to know whether the relation $B_c \propto (1 - T/T_c)$ can be reproduced in holographic superconductor with Lifshitz scaling, and discuss the effect of Lifshitz dynamical exponent $z$ on critical temperature as well as the upper critical magnetic field. Furthermore, it is of interest to analyze the restricted conditions for the matching method since this issue has not been discussed systematically, and examine whether the matching method is still valid to explore the effect of the external magnetic field on the holographic superconductor since we consider the totally different nonrelativistic situations. We will concentrate on the probe limit to avoid the complex computation in order to extract the main physics.

The organization of the work is as follows. In Sec. II, we will review the asymptotic Lifshitz black holes and study the holographic superconductors with Lifshitz scaling. In Sec. III we investigate the properties of the holographic superconductors with Lifshitz scaling in an external magnetic field. We will conclude in the last section of our main results.

II. HOLOGRAPHIC SUPERCONDUCTOR MODELS WITH LIFSHITZ SCALING

In order to study holographic superconductor with Lifshitz scaling, we will present background for the gravity dual of the Lifshitz fixed point. It is well known that there exist field theories with anisotropic scaling symmetry between the temporal and spatial coordinates $t \rightarrow \lambda^z t$, $x^i \rightarrow \lambda x^i$, which can be found in some condensed matter systems near the critical point. From the generalized gauge/gravity correspondence, one can attain scaling symmetry. The metric can be written as $ds^2 = L^2 \left( -r^{2z} dt^2 + r^2 \sum_{i=1}^d dx_i^2 + dr^2 / r^2 \right)$, where $0 < r < \infty$ and $L$ is the radius of curvature of the geometry. Kachru et al. first proposed this geometry \[35\], in which the action sourcing this geometry was also given. The scale transformation is as follows

$$ t \rightarrow \lambda^z t, \quad x^i \rightarrow \lambda x^i, \quad r \rightarrow \frac{r}{\lambda}, \quad (1) $$

where $z$ is called the dynamical exponent. When $z = 1$, the above geometry reduces to the usual $AdS_{d+2}$ spacetime.
The Lifshitz black holes can be constructed as in Ref. [36] via the action

\[ S = \frac{1}{16\pi G_{d+2}} \int d^{d+2}x \sqrt{-g} \left( R - 2\Lambda - \frac{1}{2} \partial_{\mu} \Phi \partial^{\mu} \Phi - \frac{1}{4} \epsilon^{\lambda \mu \nu} F_{\mu \nu} F^{\mu \nu} \right), \]  

(2)

where \( \Lambda = -(z + d - 1)(z + d)/(2L^2) \) is the cosmological constant, \( \Phi \) is a massless scalar and \( F_{\mu \nu} \) is an abelian gauge field strength. The geometry background is as follows

\[ ds^2 = L^2 \left[ -r^{2z} f(r) dt^2 + r^2 \sum_{i=1}^{d} dx_i^2 + \frac{dr^2}{r^2 f(r)} \right], \]  

(3)

with \( f(r) = 1 - r_{+}^{z+d}/r^{z+d} \). For convenience, we set \( L = 1 \) in the following discussion. Therefore the Hawking temperature of the black hole is given by

\[ T = \frac{z + d}{4\pi r_{+}^z}, \]  

(4)

where \( r_{+} \) is the radius of the event horizon.

In the probe limit, we will consider a Maxwell field and a charged complex scalar field coupled via the action

\[ S = \int d^{d+2}x \sqrt{-g} \left( -\frac{1}{4} F_{\mu \nu} F^{\mu \nu} - |\nabla \psi - iA\psi|^2 - m^2 |\psi|^2 \right). \]  

(5)

Taking the ansatz of the matter fields as \( \psi = \psi(r) \) and \( A = \phi(r) dt \), we can obtain the equations of motion from the action (5) for the scalar field \( \psi \) and gauge field \( \phi \)

\[ \psi''(r) + \left[ \frac{f'(r)}{f(r)} + \frac{d + z + 1}{r} \right] \psi'(r) + \left[ \frac{\phi(r)^2}{r^2 f(r)} - \frac{m^2}{r^2 f(r)} \right] \psi(r) = 0, \]  

(6)

\[ \phi''(r) + \frac{d - z + 1}{r} \phi'(r) - \frac{2\psi(r)^2}{r^2 f(r)} \phi(r) = 0. \]  

(7)

Introducing \( u = r_{+}/r \), we therefore have

\[ \psi''(u) + \left[ \frac{f'(u)}{f(u)} + \frac{1 - d - z}{u} \right] \psi'(u) + \left[ \frac{u^{2z-2} \phi(u)^2}{r_{+}^2 f(u)^2} - \frac{m^2}{u^2 f(u)} \right] \psi(u) = 0, \]  

(8)

\[ \phi''(u) + \frac{z - d + 1}{u} \phi'(u) - \frac{2\psi(u)^2}{u^2 f(u)} \phi(u) = 0. \]  

(9)

At the horizon \( u = 1 \), the regularity gives the boundary conditions

\[ \psi'(1) = -\frac{m^2}{z + d} \psi(1), \quad \phi(1) = 0. \]  

(10)

Near the boundary \( (u \to 0) \), the solutions behave as

\[ \psi(u) = J_- u^\Delta_- + J_+ u^\Delta_+ , \quad \phi(u) = \mu - \rho \left( \frac{u}{r_{+}} \right)^{d-z} , \quad (1 \leq z < d), \]  

(11)
where the scaling dimension $\Delta_{\pm}$ of the scalar operator dual to the bulk scalar $\psi$ is given by $\Delta_{\pm} = [(z + d) \pm \sqrt{(z + d)^2 + 4\rho^2}]/2$, $\mu$ and $\rho$ are interpreted as the chemical potential and the charge density in the dual field theory respectively. We impose boundary condition $J_\pm = 0$ in the following discussion. For clarity, we set $J = J_+$ and $\Delta = \Delta_+$ in this work.

It should be noted that $\phi(u) = \rho - \mu \log u$ for the case $z = d$. For simplicity, we will not consider this case in the analytical studies, just as in Ref. [8].

Near $u = 1$, the solution of $\psi(u)$ and $\phi(u)$ behave as

$$\psi(u) = \psi(1) - \psi'(1)(1 - u) + \frac{1}{2} \psi''(1)(1 - u)^2 + \ldots \quad (12)$$

$$\phi(u) = \phi(1) - \phi'(1)(1 - u) + \frac{1}{2} \phi''(1)(1 - u)^2 + \ldots \quad (13)$$

From (8), (9) and (10), we obtain

$$\psi''(1) = \frac{m^2}{z + d} \left[ 1 + \frac{m^2}{2(z + d)} \right] \psi(1) - \frac{\phi'(1)^2}{2r_+^2(z + d)^2} \psi(1), \quad (14)$$

$$\phi''(1) = -\phi'(1) \left[ (z - d + 1) + \frac{2\psi(1)^2}{z + d} \right]. \quad (15)$$

Substituting (10), (14) and (15) into (12) and (13) respectively, we have

$$\psi(u) = \left( 1 + \frac{m^2}{z + d} \right) \psi(1) - \frac{m^2}{z + d} \psi(1)u + \frac{1}{4(z + d)^2} \left[ m^4 + 2(z + d)m^2 - \frac{\phi'(1)^2}{r_+^2} \right] \psi(1)(1 - u)^2, \quad (16)$$

$$\phi(u) = -\phi'(1)(1 - u) - \frac{1}{2} \left[ (z - d + 1) + \frac{2\psi(1)^2}{z + d} \right] \phi'(1)(1 - u)^2. \quad (17)$$

We need the following conditions so as to match the asymptotic solutions at some intermediate point $u = u_m$

$$J\psi_m^\Delta = \left( 1 + \frac{m^2}{z + d} \right) \psi(1) - \frac{m^2}{z + d} \psi(1)u_m + \frac{1}{4(z + d)^2} \left[ m^4 + 2(z + d)m^2 - \frac{\phi'(1)^2}{r_+^2} \right] \psi(1)(1 - u_m)^2, \quad (18)$$

$$J\Delta u_m^{\Delta - 1} = -\frac{m^2}{z + d} \psi(1) - \frac{1}{2(z + d)^2} \left[ m^4 + 2(z + d)m^2 - \frac{\phi'(1)^2}{r_+^2} \right] \psi(1)(1 - u_m), \quad (19)$$

$$\mu - \rho \left( \frac{u_m}{r_+} \right)^{d-z} = -\phi'(1)(1 - u_m) - \frac{1}{2} \left[ (z - d + 1) + \frac{2\psi(1)^2}{z + d} \right] \phi'(1)(1 - u_m)^2, \quad (20)$$

$$- \rho(d-z) \frac{1}{r_+^{d-z}} u_m^{d-z-1} = \phi'(1) + \left[ (z - d + 1) + \frac{2\psi(1)^2}{z + d} \right] \phi'(1)(1 - u_m). \quad (21)$$

Using (15) and (19), we have

$$J = \frac{u_m^{1-\Delta} \left[ m^2(u_m - 1) - 2(z + d) \right]}{(z + d)(\Delta - 2)(u_m - 1)} \psi(1), \quad (22)$$
\[ \phi'(1) = -r_+^z \alpha \]
\[ = -r_+^z \sqrt{m^4 + 2(z + d)m^2 \left( \frac{2 - u_m}{1 - u_m} \right) + \frac{2 \Delta(z + d)[m^2(u_m - 1) - 2(z + d)]}{(1 - u_m)[(\Delta - 2)u_m - \Delta]}}. \]  
(23)

In order to avoid a breakdown of the matching method, i.e., to ensure that \( \phi'(1) \) is real, it is interesting to observe that, for different masses of the scalar field, from (23) the matching point \( u_m \) has a range

\[
\begin{cases}
0 < u_m < 1, & \text{for } -(3 - \sqrt{5})(z + d) \leq m^2 \leq 0, \\
\ u_{md} < u_m < 1, & \text{for } -\frac{(z+d)^2}{4} \leq m^2 < -(3 - \sqrt{5})(z + d),
\end{cases}
\]  
(24)

where we have defined the divergent point

\[
u_{md} = \frac{\Delta[m^4 + 6m^2(z + d) + 4(z + d)^2]}{m^2(\Delta - 1)(m^2 + 4z + 4d) - \sqrt{m^4(2m^2 + 4z + 4d)^2 - 8\Delta(\Delta - 2)(z + d)^3}}.
\]  
(25)

This shows that the matching point is not truly arbitrary except in the case of \( -(3 - \sqrt{5})(z + d) \leq m^2 \leq 0 \), which is reminiscent of that seen for the Gauss-Bonnet holographic superconductors \( [34] \). Thus, if the mass of the scalar field satisfies the inequality \( -\frac{(z+d)^2}{4} \leq m^2 < -(3 - \sqrt{5})(z + d) \), the matching point has to be in an appropriate range of values which depend on Lifshitz scaling, spacetime dimension and scalar mass.

From (21) we obtain

\[
\psi(1)^2 = \frac{z + d}{2(1 - u_m)} \left[ -\frac{(d - z)u_{m}^{d-z-1}\rho}{\phi'(1)^{d-z}} - (z - d + 1)(1 - u_m) - 1 \right].
\]  
(26)

Considering (4), from (26) we have

\[
\psi(1)^2 = \frac{(z + d)[1 + (z - d + 1)(1 - u_m)]}{2(1 - u_m)} \left( \frac{T_c}{T} \right)^{\frac{d}{d-z}} \left[ 1 - \left( \frac{T}{T_c} \right)^{\frac{d}{d-z}} \right],
\]  
(27)

where the critical temperature \( T_c \) is given by

\[
T_c = \frac{z + d}{4\pi} \left[ \frac{(d - z)u_{m}^{d-z-1}\rho}{\alpha[1 + (z - d + 1)(1 - u_m)]} \right]^{\frac{d}{d-z}}.
\]  
(28)

It should be noted that, besides the constraint condition (24), in order to ensure the accuracy and correctness of the calculations for \( T_c \) we require another constraint

\[
\frac{d - (z + 2)}{d - (z + 1)} < u_m < 1,
\]  
(29)

which depends only on Lifshitz scaling \( z \) and spacetime dimension \( d \). According to (27), the constraint (29) can be used to ensure that \( \psi(1) \) is real. Considering the possibility \( u_{md} < \frac{d-(z+2)}{d-(z+1)} \), for the matching point \( u_m \) we arrive at

\[
\max \left[ 0, u_{md}, \frac{d - (z + 2)}{d - (z + 1)} \right] < u_m < 1,
\]  
(30)
which will lead to the correct critical temperature $T_c$. Thus, the matching point $u_m$, which depends on Lifshitz scaling $z$, spacetime dimension $d$ and scalar mass $m$, is not truly arbitrary and must obey the constraint (30).

This means that, from Eqs. (16) and (17), the asymptotic solutions $\psi(u)$ and $\phi(u)$ both are physical solutions.

For the case of $z = 1$ and $d = 2$, it is to be noted that we can easily obtain $0 < u_m < 1$ for all the scalar masses.

For concreteness, we choose $z = 1$ and $z = 2$ with $d = 3$, $m^2 = -3$ and $u_m = 1/2$ which satisfies the range given in (30) and get

$$T_c(z = 1) = \frac{1}{\pi} \left(\frac{1}{2}\right)^{\frac{1}{2}} \left(\frac{5}{309}\right)^{\frac{1}{2}} \rho^{\frac{1}{2}} \approx 0.202 \rho^{\frac{1}{2}},$$

(31)

$$T_c(z = 2) = \frac{5}{4\pi} \left(\frac{79 - 20\sqrt{13}}{1041}\right)^{\frac{1}{2}} \rho^{\frac{1}{2}} \approx 0.0747 \rho^{\frac{1}{2}}.$$

(32)

Obviously, $T_c(z = 2)$ is smaller than $T_c(z = 1)$, which means that the larger dynamical exponent $z$ makes the condensation harder to form. This tendency is the same as found in Ref. [8]. For $z = 1, d = 2, m^2 = -2$ and $u_m = 1/2$, it is to be noted that our result reduces to $T_c = \frac{3\sqrt{\pi}}{4\pi\sqrt{2\sqrt{3}}}$, which is obtained in Refs. [18, 33].

Following the AdS/CFT dictionary, near the critical temperature $T \sim T_c$ we can express the relation for the condensation operator $\langle O \rangle = Jr^\Delta$ as

$$\langle O \rangle^{\frac{1}{\Delta}} = \left(\frac{4\pi T_c}{z + d}\right)^{\frac{1}{\Delta}} \left\{ \frac{u_m^{1-\Delta} [m^2(u_m - 1) - 2(z + d)]}{(\Delta - 2)u_m - \Delta} \right\}^{\frac{1}{\Delta}} \left[ 1 + \frac{(z - d + 1)(1 - u_m)}{2(z + d)(1 - u_m)} \right]^{\frac{1}{\Delta}} \left[ 1 - \left(\frac{T}{T_c}\right)^{\frac{1}{\Delta}} \right]^{\frac{1}{\Delta}}.$$

(33)

The analytic result supports the numerical computation [8] that the phase transition of holographic superconductors with Lifshitz scaling belongs to the second order and the critical exponent of the system takes the mean-field value 1/2. The Lifshitz scaling and spacetime dimension will not influence the result.

Fixing $d = 3$ and $u_m = 1/2$, in Fig. 1 we present the condensate of the scalar operator $\langle O \rangle$ as a function of temperature with different dynamical exponent $z$ for the mass of the scalar field $m^2 = -3$. From Fig. 1 we see that the gap becomes smaller as $z$ increases, which corresponds to the lower critical temperature. This agrees with the numerical results obtained in [8]. It implies that the matching method is still powerful to study the holographic superconductors in Lifshitz black hole.
III. EFFECT OF EXTERNAL MAGNETIC FIELD ON SUPERCONDUCTOR MODELS WITH LIFSHITZ SCALING

Now we are in a position to study the effect of external magnetic filed on the holographic superconductors with Lifshitz scaling. From the gauge/gravity correspondence, the asymptotic value of the magnetic field corresponds to a magnetic field added to the boundary field theory. Near the upper critical magnetic field $B_c$, the scalar field $\psi$ can be regarded as a perturbation. Following [37], we set the ansatz

$$A = \phi(u)dt + Bxdy, \quad \psi = \psi(x,u). \quad (34)$$

We therefore obtain the scalar field equation for $\psi$

$$\psi''(x,u) + \left[\frac{f'(u)}{f(u)} + \frac{1 - d - z}{u}\right] \psi'(x,u) + \left[\frac{u^{2z-2}\phi(u)^2}{r^2 z f(u)^2} - \frac{m^2}{u^2 f(u)}\right] \psi(x,u) + \frac{1}{r^2 z f(u)} (\partial_x^2 - B^2 x^2) \psi(x,u) = 0. \quad (35)$$

Eq. (35) can be solved by separating the variables separable form

$$\psi(x,u) = X(x)R(u). \quad (36)$$

Substituting (36) into (35), we can get

$$r^2_z f(u) \left\{ \frac{R''(u)}{R(u)} + \left[\frac{f'(u)}{f(u)} + \frac{1 - d - z}{u}\right] \frac{R'(u)}{R(u)} + \left[\frac{u^{2z-2}\phi(u)^2}{r^2 z f(u)^2} - \frac{m^2}{u^2 f(u)}\right] \right\} - \left[\frac{X''(x)}{X(x)} + B^2 x^2\right] = 0. \quad (37)$$

The equation for $X(x)$ can be considered as the Schrödinger equation in one dimension with frequency determined by $B$ [37]

$$-X''(x) + B^2 x^2 X(x) = \lambda_n BX(x), \quad (38)$$
where \( \lambda_n = 2n + 1 \) denotes the separation constant. We consider the lowest mode \( (n = 0) \) solution, which is the first to condensate and the most stable solution after condensation \[37\]. Thus, we can express the equation of \( R(u) \) as

\[
R''(u) + \left[ \frac{f'(u)}{f(u)} + \frac{1 - d - z}{u} \right] R'(u) + \left[ \frac{u^{2z-2}\phi(u)^2}{r^2 f(u)^2} - \frac{m^2}{u^2 f(u)} - \frac{B}{r^2 f(u)} \right] R(u) = 0.
\]

(39)

At the horizon \( (u = 1) \), from Eq. \[39\], we have

\[
R'(1) = -\frac{1}{z + d} \left( m^2 + \frac{B}{r^2} \right) R(1).
\]

(40)

The asymptotic behavior \( (u \to 0) \) for \[39\] can be expressed as

\[
R(u) = J_- u^\Delta_- + J_+ u^\Delta_+.
\]

(41)

In the following calculation we still let \( J_- = 0 \) and set \( J = J_+ \) and \( \Delta = \Delta_+ \) just as discussed in the previous section.

Near the horizon \( u = 1 \), we can expand \( R(u) \) in a Taylor series as

\[
R(u) = R(1) - R'(1)(1 - u) + \frac{1}{2} R''(1)(1 - u)^2 + ....
\]

(42)

From \[39\], we have

\[
R''(1) = \frac{1}{(z + d)^2} \left[ m^2 \left( z + d + \frac{m^2}{2} \right) - \frac{\phi'(1)^2}{2r^2} + \frac{Bm^2}{r^2} + \frac{B^2}{2r^2} \right] R(1).
\]

(43)

Substituting \[40\] and \[43\] into \[42\], we get the approximate solution

\[
R(u) = R(1) + \frac{1}{z + d} \left( m^2 + \frac{B}{r^2} \right) R(1)(1 - u)
\]

\[
+ \frac{1}{2(z + d)^2} \left[ m^2 \left( z + d + \frac{m^2}{2} \right) - \frac{\phi'(1)^2}{2r^2} + \frac{Bm^2}{r^2} + \frac{B^2}{2r^2} \right] R(1)(1 - u)^2.
\]

(44)

Matching \[41\] and \[44\] for \( J_- = 0 \) at some intermediate point \( u = u_m \), we have the following two equations

\[
J u_m^\Delta = \left[ 1 + \frac{1}{z + d} \left( m^2 + \frac{B}{r^2} \right) \right] R(1) - \frac{1}{z + d} \left( m^2 + \frac{B}{r^2} \right) R(1) u_m
\]

\[
+ \frac{1}{2(z + d)^2} \left[ m^2 \left( z + d + \frac{m^2}{2} \right) - \frac{\phi'(1)^2}{2r^2} + \frac{Bm^2}{r^2} + \frac{B^2}{2r^2} \right] R(1)(1 - u_m)^2,
\]

(45)

\[
J \Delta u_m^{\Delta - 1} = -\frac{1}{z + d} \left( m^2 + \frac{B}{r^2} \right) R(1) - \frac{1}{(z + d)^2} \left[ m^2 \left( z + d + \frac{m^2}{2} \right) - \frac{\phi'(1)^2}{2r^2} + \frac{Bm^2}{r^2} + \frac{B^2}{2r^2} \right] R(1)(1 - u_m),
\]

(46)

which give a solution

\[
B = \frac{r^2}{(u_m - 1)(\Delta - 2)u_m - \Delta} \sqrt{\gamma + (u_m - 1)^2[\Delta - (\Delta - 2)u_m] \left( \frac{\phi'(1)}{r^2} \right)^2 - \beta},
\]

(47)
with
\[
\beta = 2u_m \left[(z + d) - m^2(u_m - 1)\right] - \Delta(u_m - 1) \left[2(z + d) - m^2(u_m - 1)\right],
\]
\[
\gamma = 2(z + d) \left\{2u_m^2(z + d) - m^2(u_m - 1)^2[\Delta - (\Delta - 2)u_m]\right\}.
\] (48)

When the external magnetic field is very close to the upper critical magnetic field \(B_c\), the condensation is so small that we can ignore all the quadratic terms in \(\psi\) and Eq. (9) reduces to
\[
\phi''(u) + \frac{z - d + 1}{u} \phi'(u) = 0.
\] (49)

We can obtain
\[
\phi(u) = \mu - \rho \left(\frac{u}{r_+}\right)^{d-z},
\] (50)
which results in
\[
\phi'(1) = -\frac{\rho}{r_+^{d-z}}(d-z).
\] (51)

Using (4), (28) and (51), we can express the critical magnetic field \(B_c\) as
\[
B_c = \left(\frac{4\pi T}{z + d}\right)^{\frac{d}{4}} \left(\frac{T_c}{T}\right)^{\frac{d}{4}} \frac{1}{(u_m - 1)(\Delta u_m - \Delta)}
\times \left\{\sqrt{(\beta^2 - \gamma)u_m^2(1+z-d)[1 + (1 + z - d)(1 - u_m)]^2 + \gamma \left(\frac{T}{T_c}\right)^{\frac{2d}{4}}} - \beta \left(\frac{T}{T_c}\right)^{\frac{d}{4}}\right\}.
\] (52)

Note that there is a superconducting phase transition when \(B_c = 0\) at \(T = T_c\), we have
\[
\gamma = \beta^2,
\] (53)
which is related to Lifshitz scaling, spacetime dimension and the scalar mass, or
\[
u_m^{2(1+z-d)[1 + (1 + z - d)(1 - u_m)]^2} = 1,
\] (54)
which is related to Lifshitz scaling and spacetime dimension but independent of the scalar mass. For Eq. (53), the only root which is probably in the range \(0 < u_m < 1\) is \(u_m = u_{md}\) where the scalar mass satisfies\(-(z + d)^2/4 \leq m^2 < -(3 - \sqrt{5})(z + d)\). But from Eq. (30) we know that this fixed matching point \(u_{md}\) will cause a breakdown of the matching method. So we have to count on Eq. (54) instead of Eq. (53) to ensure the condition \(B_c = 0\) at \(T = T_c\). For Eq. (54), it is interesting to find that if
\[
z = d - 1, \quad \text{or} \quad z = d - 2,
\] (55)
the relation (54) always holds for all \( u_m \) selected in the range (30), which is shown in Fig. 2. That is to say, for the case (55), in an appropriate range (30) we can choose the matching point \( u_m \) arbitrarily. It should be noted that in this case the investigation of the critical magnetic field \( B_c \) does not bring new restriction on the selection of the matching point \( u_m \). From Fig. 2 we clearly find that the allowable range of the matching point \( u_m \) depends on Lifshitz scaling \( z \), spacetime dimension \( d \) and scalar mass \( m \). For the Breitenlohner-Freedman (BF) bound \( m^2 = -(z + d)^2/4 \) [38], it shows that the range of \( u_m \) becomes smaller as we amplify the value of \( z \) for the fixed \( d \) or increase the value of \( d \) for the fixed \( z \).

![FIG. 2: (color online) The allowable range of \( u_m \) for different masses if \( z = d - 1 \) and \( z = d - 2 \). In each panel, the region surrounded by the red and dashed line is determined by Eq. (24), and the cyan region corresponds to Eq. (30). In this case, the study of \( B_c \) does not bring new restriction on the selection of \( u_m \).](image)

![FIG. 3: (color online) The allowable range of \( u_m \) for different masses if \( z \neq d - 1 \) and \( z \neq d - 2 \). In each panel, the region surrounded by the red and dashed line is determined by Eq. (24), and the cyan region corresponds to Eq. (30). The blue line in each panel represents the fixed \( u_m \) given by Eq. (54) and is not in the cyan region, which indicates that the matching method is invalid in this case.](image)

However, when the constraint (55) can not be satisfied, for example, \( z = 1 \) and \( d = 4 \), if and only if \( u_m = 0.414 \), we have \( B_c = 0 \) at \( T = T_c \), which is reminiscent of that seen for the holographic superconductor in Gauss-Bonnet gravity with Born-Infeld electrodynamics [29]. But from Fig. 3 this fixed matching point
\( u_m \) is not in the allowable region for the correct critical temperature \( T_c \), which implies that we can not obtain the correct expression of the critical magnetic field \( B_c \). Obviously, we can observe the same phenomenon for the case \( z = 1 \) and \( d = 5 \), \( z = 2 \) and \( d = 5 \) in Fig. 3 and other cases when the constraint (55) can not be satisfied, i.e., \( z \neq d - 1 \) and \( z \neq d - 2 \). However, we find that the critical magnetic field \( B_c \) decreases as \( T/T_c \) goes up and vanishes at \( T = T_c \) from Fig. 4 where we use the numerical shooting method to solve Eq. (55) and obtain the critical magnetic field for different \( z \) with the fixed \( d = 3 \), \( d = 4 \) and \( d = 5 \). Thus, we argue that the matching method is not always valid to explore the effect of the external magnetic field on the holographic superconductor with Lifshitz scaling, for example, \( z \neq d - 1 \) and \( z \neq d - 2 \). In physics, this implies that we can not ensure the Ginzburg-Landau relation and the correctness of physical solutions \( \psi(u) \) and \( \phi(u) \) simultaneously in our analytic treatment for these cases.

In Fig. 5, we present the critical magnetic field \( B_c \) as a function of \( T/T_c \) for different dynamical exponent \( z \). It is clearly shown that the critical magnetic field \( B_c \) decreases as we amplify \( z \), which is qualitatively in good agree-

\[
B_c \simeq \left( \frac{4\pi T_c}{z + d} \right)^{\frac{1}{2}} \left( \frac{1}{u_m - 1} \right) \left[ \sqrt{\beta^2 - \gamma} + \gamma \left( \frac{T}{T_c} \right)^{\frac{2d}{z}} - \beta \left( \frac{T}{T_c} \right)^{\frac{z}{2}} \right].
\]  

(56)

As an example, we choose \( z = 1 \) and \( z = 2 \) with \( d = 3 \), \( m^2 = -3 \) and \( u_m = 1/2 \) which satisfies the range (50) and constraint (55), and then have

\[
B_c(z = 1) \simeq \frac{1}{5} \pi T_c^{\frac{2}{3}} \left[ \sqrt{1545 + 856 \left( \frac{T}{T_c} \right)^{\frac{6}{3}}} - 49 \left( \frac{T}{T_c} \right)^{\frac{3}{3}} \right],
\]

(57)

\[
B_c(z = 2) \simeq \frac{9 - \sqrt{13}}{85} \pi T_c \left[ \sqrt{2} \left( 6053 + 1651\sqrt{13} + 10(221 + 27\sqrt{13}) \left( \frac{T}{T_c} \right)^{\frac{3}{3}} - (113 + 17\sqrt{13}) \left( \frac{T}{T_c} \right)^{\frac{2}{3}} \right] \right].
\]

(58)
ment with the numerical results shown in Fig. 4. This suggests that the dynamical exponent $z$ does have effects on the critical magnetic field. It should be noted that our result (56) can reduce to the case discussed in Ref. [18] when we take $z = 1$, $d = 2$, $m^2 = -2$ and $u_m = 1/2$, i.e., $B_c \simeq \frac{16\pi^2}{9} T_c^2 \left[ \sqrt{7} \sqrt{4 + 3 \left( \frac{T_c}{T} \right)^2} - 7 \left( \frac{T_c}{T} \right)^2 \right]$.

![FIG. 5: (color online) The critical magnetic field as a function of $T/T_c$ obtained by using the analytic matching method. We choose the mass of the scalar field by $m^2 = -3$ for the fixed $d = 3$ and $u_m = 1/2$. The top line corresponds to $z = 1$ (blue) and bottom one is $z = 2$ (red).](image)

With the range (30) and constraint (55), when $T \sim T_c$ we can have $B_c \propto (1 - T/T_c)$ for different Lifshitz scaling, spacetime dimension and scalar mass, which agrees well with the Ginzburg-Landau theory. Note that the relation is independent of $z$, which indicates that the dynamical exponent $z$ can not modify it. The result may be natural since we are working in the large $N$ limit. Thus, we can conclude that, for the case $1 \leq z < d$, the Ginzburg-Landau theory still holds in Lifshitz black hole.

**IV. CONCLUSIONS**

As a very good technique, the matching method can provide us an analytic understanding of the holographic superconductors in a straightforward way and help to confirm the numerical result. In this work, we have used the matching method to investigate the holographic superconductors with Lifshitz scaling and discussed the effectiveness of this analytic method. For the cases $1 \leq z < d$ considered here, we found that the critical temperature decreases with the increase of the dynamical exponent $z$, which shows that Lifshitz scaling makes the condensation harder to occur. Our analytic result can be used to back up the numerical computations in the holographic superconductors with Lifshitz scaling. Furthermore, we analytically studied the holographic superconductor with Lifshitz scaling in an external magnetic field. It is interesting to note that, for the case of $z = d - 1$ and $z = d - 2$, in order to avoid a breakdown of the matching method we have to choose the
matching point in an appropriate range which depends on Lifshitz scaling, spacetime dimension and scalar mass. We argued that the physical conditions lead to the matching range in the analytic treatment. In this case we observed that a larger \( z \) results in a smaller upper critical magnetic field, which is consistent with the numerical results. This shows that the dynamical exponent \( z \) does have effects on the upper critical magnetic field. We also reproduced the well-known relation \( B_c \propto (1-T/T_c) \) from the Ginzburg-Landau theory even in Lifshitz black hole, which shows that the Lifshitz scaling can not modify this relation. The result may be natural since we are working in the large \( N \) limit. However, for other cases, i.e., \( z \neq d-1 \) and \( z \neq d-2 \), the matching method can not ensure the Ginzburg-Landau relation and the correctness of physical solutions \( \psi(u) \) and \( \phi(u) \) simultaneously, and fails to give the correct expression of the critical magnetic field of the holographic superconductor with Lifshitz scaling. The fact implies that the matching method is not always powerful to explore the effect of the external magnetic field on the holographic superconductors. The extension of this work to the fully backreacted spacetime would be interesting. But since the backreacted solutions are usually not easy to master and the restricted conditions for the matching method should be reconsidered, we will leave it for further study.

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