A VARIATIONAL METHOD FOR $\Phi^4_3$

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ABSTRACT. We introduce an explicit description of the $\Phi^4_3$ measure on a bounded domain. Our starting point is the interpretation of its Laplace transform as the value function of a stochastic optimal control problem along the flow of a scale regularization parameter. Once small scale singularities have been renormalized by the standard counterterms, $\Gamma$-convergence allows to extend the variational characterization to the unregularized model.

1. INTRODUCTION

The $\Phi^4_d$ Gibbs measure on the $d$-dimensional torus $\Lambda = \Lambda_L = T^d_L = (\mathbb{R} / (2\pi L \mathbb{Z}))^d$ is the probability measure $\nu$ obtained as the weak limit for $T \to \infty$ of the family $(\nu_T)_{T > 0}$ given by

$$\nu_T(d\phi) = \frac{\exp[-V_T(\phi_T)]}{Z_T} \vartheta(d\phi),$$

where

$$V_T(\phi) := \lambda \int_{\Lambda} \left( |\phi(\xi)|^4 - a_T |\phi(\xi)|^2 - b_T \right) d\xi,$$

$$Z_T := \int e^{-V_T(\phi_T)} \vartheta(d\phi).$$

Here $\lambda \geq 0$ is a fixed constant, $\Delta$ is the Laplacian on $\Lambda$, $\vartheta$ is the centered Gaussian measure with covariance $(1 - \Delta)^{-1}$, $Z_T$ is a normalization factor, $a_T, b_T$ given constants and $\phi_T = \rho_T * \phi$ with $\rho_T$ some appropriate smooth and compactly supported cutoff function such that $\rho_T \to \delta$ as $T \to \infty$. The measures $\vartheta$ and $\nu_T$ are realized as probability measures on $\mathcal{S}'(\Lambda)$, the space of tempered distributions on $\Lambda$. They are supported on the H"older–Besov space $C((2-d)/2 - \kappa)(\Lambda)$ for all small $\kappa > 0$. The existence of the limit $\nu$ is conditioned on the choice of a suitable sequence of renormalization constants $(a_T, b_T)_{T > 0}$. The constant $b_T$ is not necessary, but is useful to decouple the behavior of the numerator from that of the denominator in eq. (1).

The aim of this paper is to give a proof of convergence using a variational formula for the partition function $Z_T$ and for the generating function of the measure $\nu_T$. As a byproduct we obtain also a variational description for the generating function of the limiting measure $\nu$ via $\Gamma$-convergence of the variational problem. Let us remark that, to our knowledge, it is the first time that such explicit description of the unregulated $\Phi^4_3$ measure is available.

Our work can be seen as an alternative realization of Wilson’s [49] and Polchinski’s [45] continuous renormalization group (RG) method. This method has been made rigorous by Brydges, Slade et al. [11, 18, 10] and as such witnesses a lot of progress and successes [14, 15, 4, 16, 17, 18]. The key idea is the nonperturbative study of a certain infinite dimensional Hamilton–Jacobi–Bellman equation [13] describing the effective, scale dependent, action of the theory. Here we avoid the analysis involved by the direct study of the PDE by going to the equivalent stochastic control formulation, well established and understood in finite dimensions [22]. The time parameter of the evolution corresponds to an increasing amount of small scale fluctuations of the Euclidean field and our main tool is a variational representation formula, introduced by Boué and Dupuis [7], for the logarithm of the partition function interpreted as the value function of the control problem. See also the related papers of "Ust"unel [48] and Zhang [50] where extensions and further results on the

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The variational formula has been used by Lehec \cite{38} to prove some Gaussian functional inequalities, following the work of Borell \cite{6}. In this representation we can avoid the analysis of an infinite dimensional second order operator and concentrate more on pathwise properties of the Euclidean interacting fields. We are able to leverage techniques developed for singular SPDEs, in particular the paracontrolled calculus developed in \cite{28}, to perform the renormalization of various non-linear quantities and show uniform bounds in the $T \to \infty$ limit.

Define the normalized free energy $\mathcal{W}_T$ for the cutoff $\Phi_3^d$ measure, as

$$
\mathcal{W}_T(f) := -\frac{1}{|\Lambda|} \log \int_{\mathcal{S}_d(\Lambda)} \exp[-|\Lambda| f(\phi) - V_T(\phi_T)] \vartheta(d\phi)
$$

where $f \in C(\mathcal{S}_d(\Lambda); \mathbb{R})$ is a given function. The main result of the paper is the following

**Theorem 1.** Let $d = 3$ and take a small $\kappa > 0$. There exist renormalization constants $a_T, b_T$ (which depend polynomially on $\lambda$) such that the limit

$$
\mathcal{W}(f) := \lim_{T \to \infty} \mathcal{W}_T(f),
$$

exists for every $f \in C(\mathcal{S}^{\epsilon/2-\kappa}; \mathbb{R})$ with linear growth. Moreover the functional $\mathcal{W}(f)$ has the variational form

$$
\mathcal{W}(f) = \inf_{u \in \mathbb{H}^{\epsilon/2-\kappa}} \mathbb{E} \left[ f(W_\infty + Z_\infty(u)) + \Psi_\infty(u) + \lambda \|Z_\infty(u)\|^4_{L^4} + \frac{1}{2} \|l(u)\|^2_{L^2([0,\infty) \times \Lambda)} \right]
$$

where

- $\mathbb{E}$ denotes expectations on the Wiener space of a cylindrical Brownian motion $(X_t)_{t \geq 0}$ on $L^2(\Lambda)$ with law $\mathbb{P}$;
- $\mathbb{W}$ a collection of polynomial functions of the Brownian motion $(X_t)_{t \geq 0}$ comprising a Gaussian process $(W_t)_{t \geq 0}$ such that $\text{Law}_\mathbb{P}(W_t) = \text{Law}_\phi(\phi_t)$;
- $\mathbb{H}^{\epsilon/2-\kappa}$ is the space of predictable processes (wrt. the Brownian filtration) in $L^2(\mathbb{R}_+; H^{\epsilon/2-\kappa})$;
- $(Z_t(u), l_t(u))_{t \geq 0}$ are explicit (non-random) functions of $u \in \mathbb{H}^{\epsilon/2-\kappa}$ and $\mathbb{W}$;
- $\Psi_\infty(u)$ a nice polynomial (non-random) functional of $(\mathbb{W}, u)$, independent of $f$.

See Section 4 and in particular Theorem 3 for precise definitions of the various objects and a more detailed statement of this result. With respect to the notations in Lemma 8 observe that

$$
f(W_\infty + Z_\infty(u)) + \Psi_\infty(u) = \Phi_\infty(\mathbb{W}, Z(u), K(u)),
$$

where $K(u)$ is another functional of $(\mathbb{W}, u)$.

Theorem 1 implies directly the convergence of $(\nu_T)_T$ to a limit measure $\nu$ on $\mathcal{S}_d(\Lambda)$. Taking $f$ in the linear dual of $\mathcal{S}^{\epsilon/2-\kappa}$ it also gives the following formula for the Laplace transform of $\nu$:

$$
\int_{\mathcal{S}_d(\Lambda)} \exp(-f(\phi)) \nu(d\phi) = \exp(-|\Lambda| (\mathcal{W}(f/|\Lambda|) - \mathcal{W}(0))).
$$

To our knowledge this is the first such explicit description (i.e. without making reference of the limiting procedure). The difficulty is linked to the conjectured singularity of the $\Phi_3^d$ measure with respect to the reference Gaussian measure. Another possible approach to an explicit description goes via integration by parts (IBP) formulas, see \cite{2} for an early proof and a discussion of this approach. More recently \cite{29} gives a self-contained proof of the IBP formula for any accumulation point of the $\Phi_3^d$ in the full space. However is still not clear how to use these formulas directly to obtain uniqueness of the measure and/or other properties (either on the torus or on the more difficult situation of the full space). Therefore, while our approach here is limited to the finite volume situation, it could be used to prove additional results, like large deviations or weak universality very much like for SPDEs, see e.g. \cite{33, 34, 24}. 


The parameter $L$, which determines the size of the spatial domain $\Lambda = \Lambda_L$, will be kept fixed all along the paper and we will not attempt here to obtain the infinite volume limit $L \to \infty$. For this reason we will avoid to explicitly show the dependence of $W_T$ with $\Lambda$. However some care will be taken to obtain estimates uniform in the volume $|\Lambda|$.

An easy consequence of the estimates needed to establish the main theorem is the following corollary (well known in the literature, see e.g. [5]):

**Corollary 1.** There exists functions $E_+(\lambda), E_-(\lambda)$ not depending on $|\Lambda|$, such that

$$\lim_{\lambda \to 0^+} \frac{E_\pm(\lambda)}{\lambda^3} = 0,$$

and, for any $\lambda > 0$,

$$E_-(\lambda) \leq W_T(0) \leq E_+(\lambda).$$

A similar statement for $d = 2$ will be sketched below in order to introduce some of the ideas on which the $d = 3$ proof is based.

The construction of the $\Phi^4_{d,3}$ measure in finite volume is basic problem of constructive quantum field theory to which many works have been devoted, especially in the $d = 2$ case. It is not our aim to provide here a comprehensive review of this literature. As far as the $d = 3$ case is concerned, let us just mention some of the results that, to different extent, prove the existence of the limit as the ultraviolet (small scale) regularization is removed. After the early work on Glimm and Jaffe [25, 26], in part performed in the Hamiltonian formalism, all the subsequent research has been formulated in the Euclidean setting: i.e. as the problem of construction and study of the probability measures $\nu$ on a space of distributions. Feldman [21], Park [44], Benfatto et al. [5], Magnén and Sénéor [39] and finally Brydges et al. [12] obtained the main results we are aware of. Recent advances in the analysis of singular SPDEs put forward by the invention of regularity structures by M. Hairer [32] and related approaches [28, 19, 43] or even RG–inspired ones [37], have allowed to pursue the stochastic quantization program to a point where now can be used to prove directly the existence of the finite volume $\Phi^4_{d,3}$ measure in two different ways [40, 1]. Uniqueness by these methods requires additional efforts but seems at reach. Some results on the existence of the infinite volume measure [29] and dynamics [27] have been obtained recently. For an overview of the status of the constructive program wrt. the analysis of the $\Phi^4_{d,3}$ models the reader can consult the introduction to [1] and [29].

This paper is organized as follows. In Section 2 we set up our main tool, the Boué–Dupuis variational formula of Theorem 2. Then, as a warmup exercise, we use the formula to show bounds and existence of the $\Phi^4_2$ measure in Section 3. We then pass to the more involved situation of three dimensions in Section 4 where we introduce the renormalized variational problem. In Section 5 we establish uniform bounds for this new problem and in Section 6 we prove Theorem 1. Section 7 and Section 8 are concerned with some details of the analytic and probabilistic estimates needed throughout the paper. Appendix A gather background material on functional spaces, paraproducts and related functional analytic background material.

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**Conventions.** Let us fix some notations and objects. Let $a = (1 + a^2)^{1/2}$. Denote with $\mathcal{S}(\Lambda)$ the space of Schwartz functions on $\Lambda$ and with $\mathcal{S}'(\Lambda)$ the dual space of tempered distributions. The notation $\hat{f}$ or $\mathcal{F} f$ stands for the space Fourier transform of $f$. In order to easily keep track
of the volume dependence of various objects we normalize the Lebesgue measure on $\Lambda$ to have unit mass. We denote the normalized integral and measure by

$$\int f := \frac{1}{|\Lambda|} \int_{\Lambda} f, \quad d^q x = \frac{1}{|\Lambda|} dx$$

where $|\Lambda|$ is the volume of $\Lambda$. Norms in all the related functional spaces (Lebesgue, Sobolev and Besov spaces) are understood similarly normalized unless stated otherwise. The various constants appearing in the estimates will be understood uniform in $|\Lambda|$, unless otherwise stated. The constant $\kappa > 0$ represents a small positive number which can be different from line to line. The reader is referred to the Appendix for an overview of the functional spaces and the additional notations used in the paper.

2. A stochastic control problem

We begin by constructing a probability space $\mathbb{P}$ endowed with a process $(Y_t)_{t \in [0, \infty]}$ belonging to $C([0, \infty], \mathcal{C}^{(2-d)/2-\kappa}(\Lambda))$ and such that $\text{Law}\varphi (\phi_T) = \text{Law}\varphi (Y_T)$ for all $T \geq 0$ and $\text{Law}(Y_{\infty}) = \vartheta$.

Fix $\alpha < -d/2$ and let $\Omega := C(\mathbb{R}^+; H^{-\alpha})$, $(\xi_t)_{t \geq 0}$ the canonical process on $\Omega$ and $\mathcal{B}$ the Borel $\sigma$-algebra of $\Omega$. On $(\Omega, \mathcal{B})$ consider the probability measure $\mathbb{P}$ which makes the canonical process $X$ a cylindrical Brownian motion in $L^2(\Lambda)$. In the following $\mathbb{E}$ without any qualifiers will denote expectations wrt. $\mathbb{P}$ and $\mathbb{E}_Q$ will denote expectations wrt. some other measure $Q$. On the measure space $(\Omega, \mathcal{B}, \mathbb{P})$ there exists a collection $(B^n_t)_{n \in (L-1)^d}$ of complex (2-dimensional) Brownian motions, such that $B^n_t = B^n_{t-n}, B^n_t B^n_t$ independent for $m \neq \pm n$ and $X_t = |\Lambda|^{-1/2} \sum_{n \in (L-1)^d} e^{i(n,\cdot)} B^n_t$.

Note that $X$ has a.s. trajectories in $C(\mathbb{R}^+, \mathcal{C}^{(2-d)/2-\kappa}(\Lambda))$ for any $\varepsilon > 0$ by standard arguments.

Fix some $\rho \in C_c^\infty(\mathbb{R}_+, \mathbb{R}_+)$ such that $\rho(0) = 1$. Let $\rho_t(x) := \rho(x/t)$ and

$$\sigma_t(x) := (2\rho_t(x)\sigma_t(x))^{1/2} = (-2(x/t)\rho(x/t)\rho'(x/t))/t^{1/2},$$

where $\rho_t$ is the partial derivative of $\rho_t$ with respect to $t$. Consider the process $(Y_t)_{t \geq 0}$ defined by

$$Y_t := \frac{1}{|\Lambda|^{1/2}} \sum_{n \in (L-1)^d} \int_0^t \frac{\sigma_s(n)}{\langle n \rangle} e^{i(n,\cdot)} dB^n_s, \quad t \geq 0. \tag{4}$$

It is a centered Gaussian process with covariance

$$\mathbb{E}[\langle Y_t, \varphi \rangle \langle Y_s, \psi \rangle] = \frac{1}{|\Lambda|} \sum_{n,m \in (L-1)^d} \mathbb{E} \left[ \int_0^t \frac{\sigma_s(n)}{\langle n \rangle} dB^n_s \varphi(n) \int_0^s \frac{\sigma_s(m)}{\langle m \rangle} dB^n_s \psi(m) \right] = \frac{1}{|\Lambda|} \sum_{n \in (L-1)^d} \frac{\rho^2_{\min(s,t)}(n)}{\langle n \rangle^2} \varphi(n) \psi(n),$$

for any $\varphi, \psi \in \mathcal{S}(\Lambda)$ and $t, s \geq 0$, by Fubini theorem and Ito isometry. By dominated convergence

$$\lim_{t \to \infty} \mathbb{E}[\langle Y_t, \varphi \rangle \langle Y_t, \psi \rangle] = |\Lambda|^{-1} \sum_{n \in (L-1)^d} \langle n \rangle^{-2} \varphi(n) \psi(n)$$

for any $\varphi, \psi \in L^2(\Lambda)$.

Note that up to any finite time $T$ the r.v. $Y_T$ has a bounded spectral support and the stopped process $Y^{T}_{t} = Y_{t \wedge T}$ for any fixed $T > 0$, is in $C(\mathbb{R}_+, \mathbb{W}^{k,2}(\Lambda))$ for any $k \in \mathbb{N}$. Furthermore $(Y^{T}_{t})_t$ only depends on a finite subset of the Brownian motions $(B^n_t)_n$.

We will usually write $g(D)$ for the Fourier multiplier operator with symbol $g$. With this convention we can compactly denote

$$Y_t = \int_0^t J_s dX_s, \quad t \geq 0, \tag{5}$$

where $J_s := (D)^{-1} \sigma_s(D)$. We observe that $Y_t$ has a distribution given by the pushforward $(\rho_t(D))_s \vartheta$ of $\vartheta$ through $\rho_t(D)$. We write the measure $\nu_T$ in (11) in terms of expectations over $\mathbb{P}$ as

$$\int g(\varphi)\nu_T(d\varphi) = \frac{\mathbb{E}[g(Y_T)e^{-1_T(Y_T)}]}{2_T}, \tag{6}$$
for any bounded measurable $g : \mathcal{Y}(\Lambda) \to \mathbb{R}$.

For fixed $T$ the polynomial appearing in the expression for $V_T(Y_T)$ is bounded below (since $\lambda > 0$) and $\mathcal{Y}$ is well defined and also bounded away from zero (this follows easily from Jensen’s inequality). However as $T \to \infty$ we tend to lose both these properties due to the fact that we will be obliged to take $a_T \to +\infty$ to renormalize the non–linear terms. To obtain uniform upper and lower bounds we need a more detailed analysis and we proceed as follows.

Denote by $\mathbb{H}_a$ the space of progressively measurable processes which are $\mathbb{P}$–almost surely in $\mathcal{H} := L^2(\mathbb{R}_+ \times \Lambda)$. We say that an element $v$ of $\mathbb{H}_a$ is a drift. Below we will need also drifts belonging to $\mathcal{H}^\alpha := L^2(\mathbb{R}_+; H^\alpha(\Lambda))$ for some $\alpha \in \mathbb{R}$, we denote the corresponding space with $\mathbb{H}_a^\alpha$. Consider the measure $Q_T$ on $(\Omega, \mathcal{B})$ whose Radon–Nyikodim derivative wrt. $\mathbb{P}$ is given by

$$\frac{dQ_T}{d\mathbb{P}} = \frac{e^{-V_T(Y_T)}}{\mathcal{Z}_T}.$$ 

Since $Y_T$ depends on finitely many Brownian motions $(B^n)_n$, then it is well known \cite{46, 23} that any $\mathbb{P}$–absolutely continuous probability can be expressed via Girsanov transform. In particular, by the Brownian martingale representation theorem there exists a drift $u^T \in \mathbb{H}_a$ such that

$$\frac{dQ_T}{d\mathbb{P}} = \exp\left(\int_0^\infty u^t dX_s - \frac{|\Lambda|}{2} \int_0^\infty \|u_s\|^2 ds\right),$$

(recall that we normalized the $L^2(\Lambda)$ norm) and the entropy of $Q_T$ wrt. $\mathbb{P}$ is given by

$$H(Q_T|\mathbb{P}) = E_{Q_T}\left[\log \frac{dQ_T}{d\mathbb{P}}\right] = \frac{|\Lambda|}{2} E_{Q_T} \left[\int_0^\infty \|u_s\|^2 ds\right].$$

Here equality holds also if one of the two quantities is $+\infty$. By Girsanov theorem, the canonical process $X$ is a semimartingale under $Q_T$ with decomposition

$$X_t = \tilde{X}_t + \int_0^t u^s_t ds, \quad t \geq 0,$$

where $(\tilde{X}_t)_t$ is a cylindrical $Q_T$–Brownian motion in $L^2(\Lambda)$. Under $Q_T$ the process $(Y_t)_t$ has the semimartingale decomposition $Y_t = W_t + U_t$ with

$$W_t := \int_0^t \int_0^\infty J_s dX_s, \quad U_t = I_t(u^T),$$

where for any drift $v \in \mathbb{H}_a$ we define

$$I_t(v) := \int_0^t J_s v_s ds.$$

The integral in the density can be restricted to $[0, T]$ since $u^T_t = 0$ if $t > T$. Now

$$\log \mathcal{Z}_T = - \log \left(\frac{dQ_T}{d\mathbb{P}}\right)^{-1} = V_T(Y_T) + \int_0^\infty u^t dX_s - \frac{|\Lambda|}{2} \int_0^\infty \|u_s\|^2 ds,$$

and taking expectation of (7) wrt $Q_T$ we get

$$\log \mathcal{Z}_T = E_{Q_T}\left[V_T(W_T + I_T(u^T)) + \frac{|\Lambda|}{2} \int_0^\infty \|u_s\|^2 ds\right].$$

For any $v \in \mathbb{H}_a$ define the measure $Q^v$ by

$$\frac{dQ^v}{d\mathbb{P}} = \exp\left(\int_0^\infty v_s dX_s - \frac{|\Lambda|}{2} \int_0^\infty \|v_s\|^2 ds\right).$$

Denote with $\mathbb{H}_c \subseteq \mathbb{H}_a$ the set of drifts $v \in \mathbb{H}_a$ for which $Q^v(\Omega) = 1$, in particular $u^T \in \mathbb{H}_c$. By Jensen’s inequality and Girsanov transformation we have

$$\log \mathcal{Z}_T = - \log E_{\mathbb{P}}[e^{-V_T(Y_T)}] = - \log E^v\left[e^{-V_T(Y_T)} - \int_0^\infty v_s dX_s + \frac{|\Lambda|}{2} \int_0^\infty \|v_s\|^2 ds\right]$$

for any bounded measurable $g : \mathcal{Y}(\Lambda) \to \mathbb{R}$. The integral in the density can be restricted to $[0, T]$ since $u^T_t = 0$ if $t > T$. Now

$$\log \mathcal{Z}_T = - \log \left(\frac{dQ_T}{d\mathbb{P}}\right)^{-1} = V_T(Y_T) + \int_0^\infty u^t dX_s - \frac{|\Lambda|}{2} \int_0^\infty \|u_s\|^2 ds,$$

and taking expectation of (7) wrt $Q_T$ we get

$$\log \mathcal{Z}_T = E_{Q_T}\left[V_T(W_T + I_T(u^T)) + \frac{|\Lambda|}{2} \int_0^\infty \|u_s\|^2 ds\right].$$

For any $v \in \mathbb{H}_c$, define the measure $Q^v$ by

$$\frac{dQ^v}{d\mathbb{P}} = \exp\left(\int_0^\infty v_s dX_s - \frac{|\Lambda|}{2} \int_0^\infty \|v_s\|^2 ds\right).$$

Denote with $\mathbb{H}_c \subseteq \mathbb{H}_a$ the set of drifts $v \in \mathbb{H}_a$ for which $Q^v(\Omega) = 1$, in particular $u^T \in \mathbb{H}_c$. By Jensen’s inequality and Girsanov transformation we have

$$\log \mathcal{Z}_T = - \log E_{\mathbb{P}}[e^{-V_T(Y_T)}] = - \log E^v\left[e^{-V_T(Y_T)} - \int_0^\infty v_s dX_s + \frac{|\Lambda|}{2} \int_0^\infty \|v_s\|^2 ds\right]$$
Some observations on these variational formulas. Remark 1. Setting the formula can be proved using the result of Üstünel \[48\] by observing that tame functional, according to his definitions. Namely, for some Proof. The original proof can be found in Boué–Dupuis \[7\] for functionals bounded above. In our uniformly in the scale parameter up to \(\infty\). The bound is saturated when \(v = u^T\). We record this result in the following lemma which is a precursor of our main tool to obtain bounds on the partition function and related objects.

**Lemma 1.** The following variational formula for the free energy holds:

\[
W_T(f) = -\frac{1}{|\Lambda|} \log \mathbb{E}[e^{-V^f_T(Y_T)}] = \min_{v \in \mathbb{H}_c} \mathbb{E}^v \left[ \frac{1}{|\Lambda|} V^f_T(W^v_T + I_T(v)) + \frac{1}{2} \int_0^\infty \|v_s\|_2^2 ds \right].
\]

where \(V^f_T := |\Lambda|f + V_T\).

This formula is nice and easy to prove but somewhat inconvenient for certain manipulations since the space \(\mathbb{H}_c\) is indirectly defined and the reference measure \(\mathbb{E}^v\) depends on the drift \(v\). A more straightforward formula has been found by Boué–Dupuis \[7\] which involves the fixed canonical measure \(\mathbb{P}\) and a general adapted drift \(u \in \mathbb{H}_a\). This formula will be our main tool in the following.

**Theorem 2.** The Boué–Dupuis (BD) variational formula for the free energy holds:

\[
W_T(f) = -\frac{1}{|\Lambda|} \log \mathbb{E}[e^{-V^f_T(Y_T)}] = \inf_{v \in \mathbb{H}_c} \mathbb{E} \left[ \frac{1}{|\Lambda|} V^f_T(Y_T + I_T(v)) + \frac{1}{2} \int_0^\infty \|v_s\|_2^2 ds \right].
\]

where the expectation is taken wrt to the measure \(\mathbb{P}\) on \(\Omega\).

**Proof.** The original proof can be found in Boué–Dupuis \[7\] for functionals bounded above. In our setting the formula can be proved using the result of Üstünel \[48\] by observing that \(V^f_T(Y_T)\) is a tame functional, according to his definitions. Namely, for some \(p, q \geq 1\) such that \(1/p + 1/q = 1\) we have

\[
\mathbb{E}[|V^f_T(Y_T)|^p] + \mathbb{E}[e^{-qV^f_T(Y_T)}] < +\infty.
\]

**Remark 1.** Some observations on these variational formulas.

a) They originates directly from the variational formula for the free energy of a statistical mechanical systems: \(V^f\) playing the role of the internal energy and the quadratic term playing the role of the entropy.

b) The infimum could not be attained in Theorem 2 while it is attained in Lemma 1.

c) The drift generated by absolutely continuous perturbations of the Wiener measure has been introduced and studied by Föllmer \[23\].

d) They are a non–Markovian and infinite dimensional extension of the well known stochastic control problem representation of the Hamilton–Jacobi–Bellman equation in finite dimensions \[22\].

e) The BD formula is easier to use than the formula in Lemma 1 since the probability do not depend on the drift \(v\). Going from one formulation to the other requires proving that certain SDEs with functional drift admits strong solutions and that one is able to approximate unbounded functionals \(V_f\) by bounded ones. See Üstünel \[48\] and Lehec \[38\] for a streamlined proof of the BD formula and for applications of the formula to functional inequalities on Gaussian measures. For example, from this formula is not difficult to prove integrability of functionals which are Lipshitz in the Cameron–Martin directions.

The next lemma provides a deterministic regularity result for \(I(v)\) which will be useful below. It says that the drift \(v\) generates shifts of the Gaussian free field in directions which belong to \(H^1\) uniformly in the scale parameter up to \(\infty\). The space \(H^1\) is the Cameron–Martin space of the free field \[36\].
Lemma 2. Let $\alpha \in \mathbb{R}$. For any $v \in L^2([0, \infty), H^\alpha)$ we have
\[
\sup_{0 \leq t \leq T} \|I_t(v)\|_{H^{\alpha+1}}^2 + \sup_{0 \leq s < t \leq T} \|I_t(v) - I_s(v)\|_{H^{\alpha+1}}^2 \lesssim \int_0^T \|v_r\|^2_{H^\alpha} dr.
\]

Proof. Using the fact that $\sigma_s(D)$ is diagonal in Fourier space, and denoting with $(e_k)_{k \in \mathbb{Z}}$ the basis of trigonometric polynomials, we have
\[
\left\| \int_r^t \sigma_s(D)v_s ds \right\|_{H^\alpha}^2 = \frac{1}{|\Lambda|} \sum_k \langle k \rangle^{2\alpha} \left| \int_r^t \langle \sigma_s(D)e_k, v_s \rangle ds \right|^2 \\
\leq \frac{1}{|\Lambda|} \sum_k \langle k \rangle^{2\alpha} \left( \int_r^t |\langle \sigma_s(D) e_k, e_k \rangle|^2 ds \right) \left( \int_r^t |\langle e_k, v_s \rangle|^2 ds \right) \\
\leq \frac{1}{|\Lambda|} \int_r^t \|v_s\|_{H^\alpha}^2 ds \sup_k \int_r^t |\langle e_k, \sigma_s(D) e_k \rangle|^2 ds \\
\leq \frac{1}{|\Lambda|} \int_r^t \|v_s\|_{H^\alpha}^2 ds \sup_k \langle e_k, \rho^T(D) e_k \rangle \lesssim \int_0^T \|v_s\|_{H^\alpha}^2 ds.
\]

On the other hand $\sigma_s(D)$ is a smooth Fourier multiplier and using Proposition 9 we have the estimate $\|\sigma_s(D)f\|_{H^\alpha} \lesssim \|f\|_{H^\alpha}/(s^{1/2})$ uniformly in $s \geq 0$, therefore, for all $0 \leq r \leq t \leq T$, we have
\[
\left\| \int_r^t \sigma_s(D)v_s ds \right\|_{H^\alpha}^2 \lesssim \left( \int_r^t \|\sigma_s(D)v_s\|_{H^\alpha} ds \right)^2 \lesssim (t-r) \left( \int_0^T \|v_s\|_{H^\alpha}^2 ds \right) .
\]

We conclude that
\[
\|I_t(v) - I_s(v)\|_{H^{\alpha+1}}^2 \lesssim \left\| \int_r^t \sigma_s(D)v_s ds \right\|_{H^\alpha}^2 \lesssim [1 \wedge (t-r)] \int_0^T \|v_s\|_{H^\alpha}^2 ds.
\]

Notation 1. In the estimates below the symbol $E(\lambda)$ will denote a generic positive deterministic quantity, not depending on $|\Lambda|$ and such that $E(\lambda)/\lambda^3 \rightarrow 0$ as $\lambda \rightarrow 0$. Moreover the symbol $Q_T$ will denote a generic random variable measurable wrt. $\sigma((W_t)_{t \in [0,T]})$ and belonging to $L^1(\mathbb{P})$ uniformly in $T$ and $|\Lambda|$.

3. Two dimensions

As a warm up we consider here the case $d = 2$ setting $f = 0$ for simplicity. From Theorem 2 we see that the relevant quantity to bound is of the form
\[
F_T(u) := E \left( \frac{1}{|\Lambda|} V_T(W_T + I_T(u)) + \frac{1}{2} \|u\|^2_U \right),
\]
for $u \in \mathbb{H}_2$. Let $Z_t = I_t(u)$ and write $W_t = Y_t$ as a mnemonic of the fact that under $\mathbb{P}$ the process $W$ is a martingale. From now on we leave implicit the integration variable over the spatial domain $\Lambda$. Choosing
\[
a_T = 6E[W_T(0)^2], \quad b_T = 3E[W_T(0)^2],
\]
we have
\[
V_T(W_T + Z_T) = \lambda T \left[ \|W_T^4\| + 4\lambda T \|W_T^2\|^2 \right] + 6\lambda \left[ \|W_T^2\|^2 \right] Z_T^2 + 4\lambda \left[ W_T^2 \right] Z_T^2 + \lambda T \left[ Z_T^4 \right],
\]
where
\[
\|W_T^4\| := W_T^4 - 6E[W_T^2 W_T^2] + 3E[W_T^2]^2,
\]
\[
\|W_T^2\| := W_T^2 - 3E[W_T^2]|W_T|,
\]
\[
\|W_T^2\| := W_T^2 - E[W_T^2],
\]
denote the Wick powers of the Gaussian r.v. $W_T$ [36]. These polynomials, when seen as stochastic processes in $T$, are $\mathbb{P}$–martingales wrt. the filtration of $(W_t)_t$. In particular they have an expression
as iterated stochastic integrals wrt. the Brownian motions \((B_t^n)_{t,n}\) introduced in eq. (4). Using Theorem 2 with \(u = 0\) we readily have an upper bound for the free energy:
\[
-\frac{1}{|A|} \log Z_T \leq \lambda \mathbb{E} \left[ f \left[ W_T^3 \right] \right] = 0.
\]
For a lower bound we need to estimate from below the average under \(\mathbb{P}\) of the variational expression
\[
\lambda \left[ W_T^3 \right]_T + 4\lambda \left[ W_T^2 \right]_T + 6\lambda \left[ \frac{W_T^2}{2} \right] Z_T + 4\lambda \left[ \frac{W_T^3}{2} \right] Z_T + \lambda \left[ \frac{W_T^5}{3} \right] + \frac{1}{2} \|u\|^2_2.
\]
The strategy we adopt is to bound pathwise, and for a generic drift \(u\), the contributions
\[
\Phi_T(Z) := 4\lambda \left[ W_T^3 \right] Z_T + 6\lambda \left[ W_T^2 \right] Z_T + 4\lambda \left[ \frac{W_T^3}{2} \right] Z_T,
\]
in terms of quantities involving only the Wick powers of \(W\) which we can control in expectation and the last two positive terms
\[
\frac{1}{2} \|u\|^2_2 + \lambda \left[ \frac{W_T^5}{3} \right].
\]
Any residual positive contribution depending on \(u\) can be dropped in the lower bound making the dependence on the drift disappear. To control term I we see that by duality and Young’s inequality, for any \(\delta > 0\),
\[
\|\langle D \rangle^3(fg)\|_{L^p} \leq \|\langle D \rangle^{\alpha+\delta} f\|_{L^{p_1}} \|\langle D \rangle^{-\alpha} g\|_{L^{p_2}} + \|\langle D \rangle^{\alpha+\delta} g\|_{L^{p_1}} \|\langle D \rangle^{-\alpha} f\|_{L^{p_2}}.
\]

Proof. See [31].

Using Proposition 1 we get, for any \(\delta > 0\),
\[
6\lambda \left[ W_T^2 \right] Z_T^2 \leq \lambda \|\|W_T^2\|_{W^{\epsilon,5}}\|Z_T^2\|_{W^{\delta,4}} \leq \lambda \|\|W_T^2\|_{W^{\epsilon,5}}\|Z_T\|_{L^4} \|Z_T\|_{L^4} \leq \frac{C_{\alpha}^2}{28} \|\|W_T^2\|_{W^{\epsilon,5}}^4 + \frac{4}{17} \|Z_T\|_{L^4}^2 + \frac{14}{17} \|Z_T\|_{L^4}^4.
\]
In order to bound the term III we observe the following:

Lemma 3. Let \(f \in W^{-1/2-\epsilon,p}\) for every \(1 \leq p < \infty\) and \(\epsilon > 0\). We claim that for any \(\delta > 0\) there exists a constant \(C = C(\delta, d)\), and an exponent \(K < \infty\) such that for any \(g \in W^{1,2}\)
\[
\lambda \left| \int f g^3 \right| \leq E(\lambda) \|f\|_{W^{-1/2-\epsilon,p}}^K + \delta(\|g\|_{W^{1-\epsilon,2}}^2 + \lambda \|g\|_{L^4}^4).
\]

Proof. By duality \(\|f g^3\| \leq \|f\|_{W^{-1/2-\epsilon,p}} \|g^3\|_{W^{1/2+\epsilon,p'}}\). Applying again Proposition 1 and Proposition 11, we get
\[
\|g^3\|_{W^{1/2+\epsilon,14/13}} \leq \|\langle D \rangle^{1/2+\epsilon} g^3\|_{L^{14/13}} \leq \|\langle D \rangle^{5/8} g\|_{L^{14/6}} \|g\|_{L^4}^2 \leq \|\langle D \rangle^{5/7} g\|_{L^4}^{17/7} \|g\|_{L^4}^4.
\]
So
\[
\lambda \left| \int f g^3 \right| \leq \lambda \|f\|_{W^{-1/2-\epsilon,14}} \|\langle D \rangle^{5/7} g\|_{L^4}^{17/7} \|g\|_{L^4}^4 + \lambda \|g\|_{L^4}^4 \leq \lambda \|f\|_{W^{-1/2-\epsilon,14}} \|\langle D \rangle^{5/7} g\|_{L^4}^{17/7} \|g\|_{L^4}^4 \leq \lambda \|f\|_{W^{-1/2-\epsilon,14}} \|\langle D \rangle^{5/7} g\|_{L^4}^{17/7} \|g\|_{L^4}^4 \leq \lambda \left| \int f g^3 \right| \leq E(\lambda) \|f\|_{W^{-1/2-\epsilon,p}}^K + \delta(\|g\|_{W^{1-\epsilon,2}}^2 + \lambda \|g\|_{L^4}^4).
\]
**Remark 2.** For the $d = 2$ case it would have been enough to estimate $\|g^3\|_{W^{\epsilon, p}}$. The stronger estimate will be useful below for $d = 3$ since there we will only have $W_T \in W^{-1/2 - \epsilon, p}$ for any large $p$.

Then
\begin{equation}
4\lambda \int W_T Z^3_T \leq CE(\lambda)\|W_T\|^K_{W^{-1/2 - \epsilon, p}} + \delta(\|Z_T\|^2_{W^{1 - \epsilon, 2}} + \lambda\|Z_T\|^4_L).
\end{equation}

Using eqs. (12), (13) and (14) we obtain, for $\delta$ small enough,
\begin{equation}
|\Phi_T(Z)| \leq Q_T + \delta \left[ \frac{1}{2} \|u\|^2_{H^1} + \lambda \int Z^4_T \right],
\end{equation}
where
\begin{equation*}
Q_T = O(\lambda^2)(1 + \|W_T^2\|_{H^{-1}}^2 + \|W_T^2\|^4_{W^{-\epsilon, 5}} + \|W_T\|^K_{W^{-1/2 - \epsilon, p}}).
\end{equation*}
Therefore
\begin{equation*}
F_T(u) \geq -\mathbb{E}[Q_T] + (1 - \delta) \left[ \frac{1}{2} \|u\|^2_{H^1} + \lambda \int Z^4_T \right] \geq -\mathbb{E}[Q_T].
\end{equation*}
This last average do not depends anymore on the drift and we are only left to show that
\begin{equation*}
\sup_T \mathbb{E}[Q_T] < \infty.
\end{equation*}

However, it is well known that the Wick powers of the two dimensional Gaussian free field are distributions belonging to $L^p(\Omega, W^{-\epsilon, b})$ for any $a \geq 1$ and $b \geq 1$ and hypercontractivity plus an easy argument gives the uniform boundedness of the above averages, see e.g. [12]. We have established:

**Theorem 3.** For any $\lambda > 0$ we have
\begin{equation}
\sup_T \frac{1}{|\Lambda|} |\log Z_T| \lesssim O(\lambda^2),
\end{equation}
where the constant in the r.h.s. is independent of $\Lambda$.

**Remark 3.** Observe that the argument above remains valid upon replacing $\lambda$ with $\lambda p$ with $p \geq 1$. This implies that $e^{-V_T(Y_T)}$ is in all the $L^p$ spaces wrt. the measure $\mathbb{P}$ uniformly in $T$ and for any $p \geq 1$.

### 4. Three dimensions

In three dimensions the strategy we used in two dimensions fails. Indeed here the Wick products are less regular: $[W^3_T] \in S^{-1 - \kappa}$ uniformly in $T$ for any small $\kappa > 0$ and $[W^3_T]$ does not even converge to a well-defined random distribution. This implies that there is no straightforward approach to control the terms
\begin{equation}
\int \|W^3_T\| Z_T, \quad \text{and} \quad \int \|W^3_T\| Z^3_T,
\end{equation}
like we did in Section 3. The only apriori estimate on the regularity of $Z_T = I_T(u)$ is in $H^1$, coming from Lemma 2 and the quadratic term in the variational functional $F_T(u)$. It is also well known that in three dimensions there are further divergences beyond the Wick ordering which have to be subtracted in order for the limiting measure to be non-trivial. For these reasons we introduce in the energy $V_T$ further scale dependent renormalization constants $\gamma_T, \delta_T$ beyond Wick ordering to have
\begin{equation}
\frac{1}{|\Lambda|} V^f_T(Y_T) = f(Y_T) + \int (\lambda [Y^3_T] - \lambda^2 \gamma_T [Y^2_T] - \delta_T).
\end{equation}
Repeating the computation from Section 3 we arrive at
\begin{equation}
F_T(u) = \mathbb{E} \left[ f(W_T + Z_T) + \lambda \int W^3_T Z_T^2 + \lambda^2 \int W^2_T Z^3_T + 4\lambda \int W^3_T Z^3_T \right]
- \mathbb{E} \left[ 2\lambda^2 \gamma_T f W_T Z_T^2 + \lambda^2 \gamma_T f Z^4_T + \lambda^2 \delta_T \right] + \mathbb{E} \left[ \lambda f Z^3_T + \frac{1}{2} \|u\|^2_{H^1} \right].
\end{equation}
where we introduced the convenient notations
\[ W_t^1 := 4[W_t^3], \quad W_t^2 := 12[W_t^2], \quad t \geq 0, \]
and we recall that \( f \) is a fixed function belonging to \( C(\mathcal{C}^{-1/2-\kappa}; \mathbb{R}) \) with linear growth.

This form of the functional is not very useful in the limit \( T \to \infty \) since some of the terms, taken individually, are not expected to behave well. We will perform a change of variables in the variational functional in order to obtain some explicit cancellations which will leave well behaved quantities of \( T \). The main drawback is that the functional will have a less compact and canonical form.

Some care has to be taken in order for the resulting quantities to be still controlled by the coercive terms. We will need some regularization which will make compatible Fourier cutoffs with \( L^4 \) estimates. To introduce such a regularization fix a smooth function \( \theta : \mathbb{R}_+ \to \mathbb{R}_+ \) such that \( \theta(\xi) = 1 \) if \( \xi \leq 1/4 \) and \( \theta(\xi) = 0 \) if \( \xi \geq 1/2 \). Then
\[ \theta_t(\xi) \sigma_s(\xi) = 0 \quad \text{for } s \geq t, \]
\[ \theta_t(\xi) = 1 \quad \text{for } \xi \leq ct \quad \text{for some } c > 0. \]

By the Mihlin-Hörmander theorem we deduce that the operator \( \theta_t = \theta_t(D) \) is bounded on \( L^p \) for any \( 1 < p < \infty \), see Proposition [9]. In the following, for any \( f \in C([0, \infty], \mathcal{F}^t(\Lambda)) \) we define \( f_t^\gamma := \theta_t f_t \)

\[ Z_t^\gamma = \theta_t Z_t = \int_0^t \theta_t(D)^{-1} \sigma_s(D) u_s ds = \int_0^t \theta_t(D)^{-1} \sigma_s(D) u_s ds = \theta_t Z_T. \]

In this way we have \( \|Z_t^\gamma\|_{L^p} \lesssim \|Z_t\|_{L^p} \) for all \( t \leq T \). The renormalized functional will depend on some specific renormalized combinations of the martingales \( ([W_t^3]_{t,k})_{t,k} \). Therefore it will be also convenient to introduce a collective notation for all the stochastic objects appearing in the functionals and specify the topologies in which they are expected to be well behaved. Let
\[ \mathcal{W} := (\mathcal{W}^1, \mathcal{W}^2, \mathcal{W}^{(3)}, \mathcal{W}^{[3]}\circ 1, \mathcal{W}^{2\circ[3]}, \mathcal{W}^{(2)\circ(2)}), \]
with \( \mathcal{W}^1 := W \),
\[ \mathcal{W}^{(3)} := J_t \mathcal{W}^3, \quad \mathcal{W}^{[3]} := \int_0^t J_s \mathcal{W}^{(3)}_s ds, \quad \mathcal{W}^{3\circ 1} := \mathcal{W}^1 \circ \mathcal{W}^{[3]}, \]
\[ \mathcal{W}^{2\circ[3]} := \mathcal{W}^2 \circ \mathcal{W}^{[3]} + 2\gamma \mathcal{W}^1, \quad \mathcal{W}^{(2)\circ(2)} := (J_t \mathcal{W}^2_t) \circ (J_s \mathcal{W}^2_s) + 2\gamma t. \]

We not need to include \( \mathcal{W}^{[3]} \) since it can be obtained as a function of \( \mathcal{W}^{(3)} \) thanks to the bound
\[ \|\mathcal{W}^{(3)}_t - \mathcal{W}^{[3]}_s\|_{\mathcal{C}^{-1/2-2\kappa}} \leq \int_s^t \|J_r \mathcal{W}^{(3)}_r\|_{\mathcal{C}^{-1/2-2\kappa}} dr \leq \left[ \int_s^t \|J_r \mathcal{W}^{(3)}_r\|^2_{\mathcal{C}^{-1/2-2\kappa}} dr \right]^{1/2} |t - s|^{1/2}, \]
valid for all \( 0 \leq s \leq t \leq T \) which shows that the deterministic map \( \mathcal{W}^{(3)} \mapsto \mathcal{W}^{[3]} \) is continuous from \( C([0, \infty], \mathcal{C}^{-1/2-\kappa}) \) to \( C^{1/2}([0, \infty], \mathcal{C}^{-1/2-2\kappa}) \). The pathwise regularity of all the other stochastic objects follows from the next Lemma, provided the function \( \gamma \) is chosen appropriately.

**Lemma 4.** There exists a function \( \gamma \in C^1(\mathbb{R}_+, \mathbb{R}) \) such that
\[ |\gamma_t| + (t)|\gamma_t| \lesssim \log(t), \quad t \geq 0, \]
and such that the vector \( \mathcal{W} \) is almost surely in \( \mathcal{G} \) where \( \mathcal{G} \) is the Banach space
\[ \mathcal{G} = C([0, \infty], \mathcal{W}) \cap \left\{ \mathcal{W}^{(3)} \in L^2(\mathbb{R}_+, \mathcal{C}^{-1/2-\kappa}), \mathcal{W}^{(2)\circ(2)} \in L^1(\mathbb{R}_+, \mathcal{C}^{-2\kappa}) \right\} \]
with
\[ \mathcal{W}_k := \mathcal{C}^{-1/2-\kappa} \times \mathcal{C}^{-1-\kappa} \times \mathcal{C}^{-1-2\kappa} \times \mathcal{C}^{-2\kappa}, \]
and equipped with the norm
\[ \|\mathbb{W}\|_S := \|\mathbb{W}\|_{C([0,\infty],\mathbb{M})} + \|\mathbb{W}^{(3)}\|_{L^2(\mathbb{R}_+,\mathcal{E}^{-1/2-\kappa})} + \|\mathbb{W}^{(2)\diamond 2}\|_{L^1(\mathbb{R}_+,\mathcal{E}^{-\kappa})}. \]

The norm \(\|\mathbb{W}\|_S\) belongs to all \(L^p\) spaces. Moreover the averages of the Besov norms \(B^\alpha_{p,r}\) of the components of \(\mathbb{W}\) of regularity \(\alpha\) are uniformly bounded in the volume \(|\Lambda|\) if \(p < \infty\).

**Proof.** The proof is based on the observation that one can choose \(\gamma\) in such a way that every component \(\mathbb{W}^{(i)}\) of the vector \(\mathbb{W}\) is such that \((\Delta_q \mathbb{W}^{(i)}_t(x))_{t \geq 0}\) for \(q \geq -1\) and \(x \in \Lambda\) is a martingale wrt. the Brownian filtration. The reader can find details for this statement in the proof of Lemma 24 below, in particular eq. (51) and (61) give the stochastic integral representations of the most difficult terms which require renormalization by \(\gamma\). The quadratic variation of these martingales can be controlled uniformly up to \(t = \infty\). As a consequence, there exists a version of \((\Delta_q \mathbb{W}^{(i)}_t(x))_{t \geq 0}\) which is continuous in \(t \in [0,\infty]\). Since each Littlewood–Paley block is a smooth function of \(x\) is not difficult from this to deduce that \((t,x) \mapsto \Delta_q \mathbb{W}^{(i)}_t(x)\) is \(C([0,\infty] \times \Lambda; \mathbb{R})\). By Burkholder–David–Gundy (BDG) estimates and \(L^p\) estimates on the the quadratic variations one can conclude that \(\|\mathbb{W}\|_S\) is in all \(L^p\) spaces provided we can control the appropriate moments of the norm of \(\mathbb{W}^T\) uniformly in \(T\). This is achieved in Lemma 24 for the more difficult resonant products where we show also the claimed uniformly of the finite \(r\) Besov norms \(B^\alpha_{p,r}\). □

For convenience of the reader we summarize the probabilistic estimates in Table 1.

| \(\mathbb{W}^1\) | \(\mathbb{W}^2\) | \(\mathbb{W}^{(3)}\) | \(\mathbb{W}^{[3]}\) | \(\mathbb{W}^{[3]\diamond 1}\) | \(\mathbb{W}^{2\diamond [3]}\) | \(\mathbb{W}^{(2)\diamond 2}\) |
|---|---|---|---|---|---|---|
| \(C^\epsilon^{1/2-}\) | \(C^\epsilon^{-1/2-}\) | \(C^\epsilon^{12}\) | \(C^\epsilon^{12-1/2-}\) | \(C^\epsilon^{12}\diamond 1\) | \(C^\epsilon^{12}\diamond 2\) | \(C^\epsilon^{12-1/2-}\) |

Table 1. Regularities of the various stochastic objects, the domain of the time variable is understood to be \([0,\infty]\). Estimates in these norms holds a.s. and in \(L^p(\mathbb{P})\) for all \(p \geq 1\) (see Lemma 4).

**Remark 4.** The requirement that \(\mathbb{W}^{(3)} \in L^2(\mathbb{C}^{12})\) will be used in Section 6 to establish equicoercivity and to relax the variational problem to a suitable space of measures.

We are now ready to perform a change of variables which renormalizes the variational functional.

**Lemma 5.** Define \(l = \tilde{l}^T(u) \in \mathbb{H}_a\), \(Z = Z(u) \in C([0,\infty], H^{1/2-\kappa})\), \(K = K(u) \in C([0,\infty], H^{1-\kappa})\) such that
\[ \tilde{l}^T(u) := ut + \lambda \int_{r \leq T} \mathbb{W}^{(3)}_t + \lambda \int_{r \leq T} J_t(\mathbb{W}^2_t > Z^2_t), \]
\[ Z_t(u) := I_t(u), \quad K_t(u) := I_t(u), \quad wt := -\lambda \int_{r \leq T} J_t(\mathbb{W}^2_t > Z^2_t) + l_t, \quad t \geq 0, \]
\[ (21) \]
Then the functional \(F_T(u)\) defined in eq. (18) has the form
\[ F_T(u) = \mathbb{E} \left[ \Phi_T(\mathbb{W}, Z(u), K(u)) + \lambda \int (Z_T(u))^4 + \frac{1}{2} \|\tilde{l}^T(u)\|^2_H \right], \]
where
\[ \Phi_T(\mathbb{W}, Z, K) := f(W_T + Z_T) + \sum_{i=1}^6 Y^{(i)}, \]
\[ \Upsilon_T^{(1)} := -\frac{\lambda}{2} R_2(W^2_T, K_T, K_T) + \frac{\lambda}{2} \int (W^2_T < K_T)K_T - \lambda^2 \int (W^2_T < W^{[3]}_T)K_T \]

\[ \Upsilon_T^{(2)} := \lambda \int (W^2_T > (Z_T - Z^i_T))K_T \]

\[ \Upsilon_T^{(3)} := \lambda \int_0^T \int (\mathcal{W}_i^2 > \dot{Z}_i^t)K_i dt \]

\[ \Upsilon_T^{(4)} := 4\lambda \int W_T K^3_T - 12\lambda^2 \int W_T W^{[3]}_T K^2_T + 12\lambda^3 \int W_T (W^{[3]}_T)^2 K_T \]

\[ \Upsilon_T^{(5)} := -\lambda^2 \int \gamma_T Z^i_T (Z_T - Z^i_T) - \lambda^2 \int \gamma_T (Z_T - Z^i_T)^2 - 2\lambda^2 \int_0^T \int_0^T \gamma_i (Z^i_T) \dot{Z}^i_t dt \]

\[ \Upsilon_T^{(6)} := -\lambda^2 \int W^{2[3]}_T K_T - \frac{\lambda^2}{2} \int_0^T \int W^{[2] \circ [2]}_i (Z^i_T)^2 dt - \frac{\lambda^2}{2} \int_0^T R_3,4(W^2_T, W^2_T, Z^i_T, Z^i_T) dt \]

where \( R_2 \) and \( R_3,4 \) are linear forms defined in the Appendix (and recalled in the proof below) and we have chosen

\[ \delta_T := \frac{\lambda^2}{2} \mathbb{E} \int_0^T \int (W^{[3]}_t)^2 dt + \frac{\lambda^3}{2} \mathbb{E} \int W^2_T (W^{[3]}_T)^2 \]

\[ + 2\lambda^3 \gamma_T \mathbb{E} \int W_T W^{[3]}_T - 4\lambda^3 \mathbb{E} \int W_T (W^{[3]}_T)^3. \]

**Proof.** Step 1. We are going to absorb the mixed terms (16) via the quadratic cost function. To do so we develop them along the flow of the scale parameter via Ito formula. For the first we have

\[ \lambda \int W^{[3]}_T Z_T = \lambda \int_0^T \int W^{[3]}_T \dot{Z}_t dt + \text{martingale}, \]

and we can cancel the first term on the r.h.s. by making the change of variables

\[ w_t := u_t + \lambda t = \mathcal{E}_{t \in T} W^{[3]}_t, \quad t \geq 0, \]

into the cost functional to get

\[ \lambda \int W^{[3]}_T Z_T + \frac{1}{2} \int_0^\infty \| u_s \|^2_{L^2} ds = -\frac{\lambda^2}{2} \int_0^T \int (W^{[3]}_t)^2 dt + \frac{1}{2} \int_0^\infty \| u_s \|^2_{L^2} ds + \text{martingale}, \]

where we used that \( J_t \) is self-adjoint. The divergent term \( \int W^{[3]}_T Z_T \) has been replaced with a divergent but purely stochastic term \( \int_0^T \int (W^{[3]}_t)^2 dt \) which does not affect anymore the variational problem and can be explicitly removed by adding its average to \( \delta_T \). As a consequence, we are not able to control \( (Z_t)_t \) in \( H^1 \) anymore and we should rely on the relation (21) and on a control over the \( H^1 \) norm of \( (K_t)_t \) coming from the residual quadratic term \( \| u \|^2_{L^2} \).

**Step 2.** From (23) we have the relation

\[ Z_T = -\lambda W^{[3]}_T + K_T, \]

which can be used to expand the second mixed divergent term in (16) as

\[ \frac{\lambda}{2} \int W^2_T Z_T = \frac{\lambda^3}{2} \int W^2_T (W^{[3]}_T)^2 - \lambda^2 \int W^2_T W^{[3]}_T K_T + \frac{\lambda}{2} \int W^2_T K^2_T. \]

Again, the first term on the r.h.s. a purely stochastic object and will give a contribution independent of the drift \( u \) absorbed in \( \delta_T \). We are still not done since this operation has left two new divergent terms on the r.h.s. of eq. (21): the \( H^1 \) regularity of \( K_T \) is not enough to control the products with \( W^2 \) which has regularity \( \mathcal{C}^{-1-\alpha} \), a bit below \(-1\). In order to proceed further we will isolate the divergent parts of these products via a paraproduct decomposition (see Appendix A for details)
and expand
\[ -\lambda^2 \int \mathcal{W}_T^2 \mathcal{W}_T^{[3]} K_T + \frac{\lambda}{2} \int \mathcal{W}_T^2 K_T^2 = \lambda \int (\mathcal{W}_T^2 \circ Z_T) K_T - \lambda^2 \int (\mathcal{W}_T^2 \circ \mathcal{W}_T^{[3]} ) K_T - \lambda^2 \int (\mathcal{W}_T^2 \circ Z_T^2) K_T + \frac{\lambda}{2} \int (\mathcal{W}_T^2 \circ K_T) K_T + \frac{\lambda}{2} \left( \int (\mathcal{W}_T^2 \circ K_T) K_T - \int (\mathcal{W}_T^2 \circ K_T) K_T \right). \]

The first two terms will require renormalizations which we put in place in Step 3 below. All the other terms will be well behaved and we collect them in \( \Upsilon^{(1)} \). In particular we observe that the last one can be rewritten as
\[ \frac{\lambda}{2} \left( \int (\mathcal{W}_T^2 \circ K_T) K_T - \int (\mathcal{W}_T^2 \circ K_T) K_T \right) = -\frac{\lambda}{2} \mathfrak{R}_2(\mathcal{W}_T^2, K_T, K_T) \]
using the trilinear form \( \mathfrak{R}_2 \) defined in Proposition 7.

**Step 3.** As we anticipated, the resonant term \( \mathcal{W}_T^2 \circ \mathcal{W}_T^{[3]} \) needs renormalization. In the expression of \( F_T \) in \( (18) \) we have the counterterm \(-2\lambda^2 \gamma_T \int W_T Z_T \) available, which we put in use now by writing
\[ -\lambda^2 \int (\mathcal{W}_T^2 \circ \mathcal{W}_T^{[3]} ) K_T - 2\lambda^2 \gamma_T \int W_T Z_T = -\lambda^2 \int \left( \frac{\mathcal{W}_T^2 \circ \mathcal{W}_T^{[3]} + 2 \gamma_T W_T}{2} \right) K_T + 2\lambda^2 \gamma_T \int W_T Z_T. \]

The first contribution is collected in \( \Upsilon^{(6)} \) and the expectation of the second will contribute to \( \delta_T \).

As far as the term \( \lambda \int (\mathcal{W}_T^2 \circ Z_T) K_T \) is concerned, we want to absorb it into \( \int \|w_s\|^2 \)ds like we did with the linear term in Step 2. Before we can do this we must be sure that, after applying Ito’s formula, it will be still possible to use \( \int Z_T^2 \) to control some of the growth of this term. Indeed the quadratic dependence in \( K_T \) (via \( Z_T \)) cannot be fully taken care of by the quadratic cost \( \int \|w_s\|^2 \)ds.

We decompose
\[ \lambda \int (\mathcal{W}_T^2 \circ Z_T) K_T = \lambda \int (\mathcal{W}_T^2 \circ Z_T) K_T + \lambda \int (\mathcal{W}_T^2 \circ (Z_T - Z_T^2)) K_T \]
and using the fact that the functions \( Z_T - Z_T^2 \) and \( K_T - K_T^2 \) are spectrally supported outside of a ball or radius \( cT \) we will be able to show that the second term is nice enough as \( T \to \infty \) to not require further analysis and we collect it in \( \Upsilon^{(2)} \). For the first we apply Ito’s formula to decompose it along the flow of scales as
\[ \lambda \int (\mathcal{W}_T^2 \circ Z_T) K_T = \lambda \int_0^T \int (\mathcal{W}_T^2 \circ Z_T^2) K_t dt + \lambda \int_0^T \int (\mathcal{W}_T^2 \circ Z_T^2) K_t dt + \text{martingale}. \]

The second term will be fine and we collect it in \( \Upsilon^{(4)} \).

**Step 4.** We are left with the singular term \( \int_0^T \int (\mathcal{W}_T^2 \circ Z_T^2) K_t dt \). Using eq. \( (21) \) and expanding \( w \) in the residual quadratic cost function obtained in Step 1, we compute
\[ \lambda \int_0^T \int (\mathcal{W}_T^2 \circ Z_T^2) K_t dt + \frac{1}{2} \int_0^\infty \|w_t\|^2_{L^2} dt = -\frac{\lambda^2}{2} \int_0^T \int (J_t(\mathcal{W}_T^2 \circ Z_T^2))^2 dt + \frac{1}{2} \int_0^\infty \|u_t\|^2_{L^2} dt \]
\[ = -\frac{\lambda^2}{2} \int_0^T \int (J_t(\mathcal{W}_T^2 \circ Z_T^2))(J_t(\mathcal{W}_T^2 \circ Z_T^2)) dt + \frac{1}{2} \|I\|^2_{H^1} \]
(25)

To renormalize the first term on the r.h.s. we observe that the remaining counterterm can be rewritten as
\[ \frac{\lambda^2}{2} \gamma_T \int Z_T^2 = -\lambda^2 \gamma_T \int (Z_T^2)^2 - \lambda^2 \gamma_T \int Z_T^2 (Z_T - Z_T^2) - \lambda^2 \gamma_T \int (Z_T - Z_T^2)^2. \]

(26)
Differentiating in $T$ the first term in the r.h.s. of eq. (26) we get
\begin{equation}
-\lambda^2\gamma_T\int (Z_T^2)^2 = -\lambda^2 \int_{0}^{T} \gamma_t(Z_t^2)^2 dt - 2\lambda^2 \int_{0}^{T} \gamma_t Z_t^2 Z_t^2 dt.
\end{equation}

The last term in eq. (26) and the last two contributions in (27) are collected in $\Upsilon^{(5)}$. The first contribution in eq. (28) has the right form to be used as a counterterm for the resonant product in (26). Using the commutator $\mathcal{K}_{3,t}$ introduced in Proposition $5$ we have
\begin{equation}
-\frac{\lambda^2}{2} \int_{0}^{T} \frac{1}{2} \int (J_t(W_t^2 > Z_t^2)) (J_t(W_t^2 > Z_t^2) + 2\gamma_t(Z_t^2)^2) dt = -\frac{\lambda^2}{2} \int_{0}^{T} \frac{1}{2} \int \mathcal{K}_{3,t}(W_t^2, W_t^2, Z_t^2, Z_t^2) dt
\end{equation}
and collect both terms in $\Upsilon^{(6)}$.

**Step 5.** We are now left with the cubic term which we rewrite as
\begin{equation}
4\lambda^2 \int W_T Z_T^3 = -4\lambda^4 \int W_T(W_T^3)^3 + 12\lambda^3 \int W_T(W_T^3)^2 K_T - 12\lambda^2 \int W_T W_T^3 K_T^2 + 4\lambda \int W_T K_T^3.
\end{equation}
The average of the first term is collected in $\delta_T$ while all the remaining terms in $\Upsilon^{(4)}$. At last we have established the claimed decomposition since the residual cost functional, from eq. (25) has the form $\|l\|^2_H$.

5. Bounds

The aim of this section is to give upper and lower bounds on $\mathcal{W}_T(f)$ uniformly on $T$ and $|A|$. In particular we will prove the bounds of Corollary $1$ taking the explicit dependence on the coupling constant $\lambda$ into account.

**Lemma 6.** There exists a finite constant $C$, which does not depend on $|A|$, such that
\begin{equation}
\sup_T |\mathcal{W}_T(f)| \leq C.
\end{equation}

**Proof.** Observe that from Lemma $5$ and Section $7$ we have that
\begin{equation}
\Phi_T(W, Z, K) \leq Q_T + \epsilon \left( \|Z_T\|_{L^4}^4 + \int_0^\infty \|l_t\|_{L^2}^2 dt \right),
\end{equation}
which immediately gives
\begin{equation}
-\mathbb{E}[Q_T] \leq -\mathbb{E}[Q_T] + (1 - \epsilon)\mathbb{E} \left( \|Z_T\|_{L^4}^4 + \int_0^\infty \|l_t\|_{L^2}^2 dt \right) \leq \mathcal{W}_T(f).
\end{equation}
On the other hand for any suitable drift $\tilde{u} \in \mathbb{H}_a$ we get the bound
\begin{equation}
\mathcal{W}_T(f) \leq \mathbb{E}[Q_T] + (1 + \epsilon)\mathbb{E} \left( \|I_T(\tilde{u})\|_{L^4}^4 + \int_0^\infty \|l_T^2(\tilde{u})\|_{L^2}^2 dt \right),
\end{equation}
where
\begin{equation}
l_T^2(\tilde{u}) = \tilde{u}_t + \lambda W_t \circ J_t(W_t^3 + W_t^2 > J_t(\tilde{u}))^3.
\end{equation}
Therefore it remains to produce an appropriate drift $\tilde{u}$ for which the r.h.s. in eq. (29) is finite (and so uniformly in $|A|$ and of $o(\lambda^3)$).

One possible strategy is to try and choose $\tilde{u}$ such that $l(\tilde{u}) = 0$, however this fails since estimates on this $u$ via Gronwall’s inequality would rely on the Hölder norm of $W_t^2$ for which we do not have uniform control in the volume. In order to overcome this problem we decompose $W^2$ and use weighted estimates similarly as done in [27].
Consider the decomposition
\[ W^2_s = U_s W^2_s + U_{<s} W^2_s, \]
where the random field \( U_s W^2_s \) is constructed as follows. Let \( \varphi \) be smooth function, positive and supported on \([-2, 2]^3\) and such that \( \sum_{m \in \Lambda \cap 2^d} \varphi^2_m \circ m = 1 \). Denote by \( \varphi_m = \varphi(\bullet - m) \). Let \( \tilde{\chi} \) be a smooth function supported in \( B(0, 1) \). Denote by \( \mathcal{X}_{\chi} \) the Fourier multiplier operator \( \mathcal{F}^{-1} \tilde{\chi}(k/N) \mathcal{F} \) and similarly \( \mathcal{X}_{\leq s} \mathcal{F} = \mathcal{F}^{-1}(1 - \tilde{\chi}(k/N)) \mathcal{F} \). Set \( L_m(s) = (1 + ||\varphi_m W^2_s||_{C^1_{1-s}}) \) and let
\[ U_{>s} W^2_s := \sum_{m \in \Lambda \cap 2^d} \varphi_m \mathcal{X}_{>L_m(s)}(\varphi_m W^2_s), \]
and
\[ U_{<s} W^2_s := \sum_{m \in \Lambda \cap 2^d} \varphi_m \mathcal{X}_{\leq L_m(s)}(\varphi_m W^2_s). \]
(with slight abuse of notation we drop the time dependence of the operators \( U_{<s}, U_{>s} \)).

Observe that the laws of both \( U_{>s} W^2_s \) and \( U_{<s} W^2_s \) are translation invariant w.r.t. to translations by \( m \in \Lambda \cap 2^d \). By [47], Theorem 2.4.7 and Bernstein inequality
\[ ||U_{>s} W^2_s||_{C^{1-\delta}} \lesssim \sup_m ||\mathcal{X}_{>L_m(s)}(\varphi_m W^2_s)||_{C^{1-\delta}} \lesssim 1 \]
Furthermore for a weight \( \rho \) (see Appendix A for precision on the weighted spaces \( L^p(\rho), C^\alpha(\rho) \) and \( B_{p,q}(\rho) \) below)
\[ ||U_{<s} W^2_s||_{C^{1-\delta}(\rho^2)} \lesssim \sup_m ||\varphi_m U_{<s} W^2_s||_{C^{1-\delta}(\rho^2)} \lesssim \sup_m \left( 1 + ||\varphi_m W^2_s||_{C^{1-\delta}(\rho)} \right) \lesssim \sup_m \rho(m) \left( 1 + ||\varphi_m W^2_s||_{C^{1-\delta}(\rho)} \right) \lesssim 1 + ||W^2_s||_{C^{1-\delta}(\rho)}, \]
where we used the possibility to compare weighted and unweighted norms once localized via \( \varphi_m \).

We now let \( \hat{u} \) be the solution to the (integral) equation
\[ \hat{u}_t = -\lambda J_{t \leq T} W^3_s + J_t U_{>s} W^2_s \geq \theta_t(I_t(\hat{u})), \quad t \geq 0, \]
which can be solved globally. For \( 3\delta < 1/2 \) and \( p \geq 1 \), we have, for \( t \in [0, T] \),
\[ ||I_t(\hat{u})||_{B_{p,\infty}^{1-\delta}(\rho)} \lesssim \lambda \int_0^T ||J_s W^2_s||_{B_{p,\infty}^{1-\delta}(\rho)} + \lambda ||J_s U_{>s} W^2_s \geq \theta_s(I_s(\hat{u}))||_{B_{p,\infty}^{1-\delta}(\rho)} ds \]
\[ \lesssim \lambda \int_0^T \frac{ds}{(s)^{1/2+\delta}} ||J_s W^2_s||_{B_{p,\infty}^{1-\delta}(\rho)} + \lambda \int_0^T \frac{ds}{(s)^{1/2+\delta}} ||U_{>s} W^2_s \geq \theta_s(I_s(\hat{u}))||_{B_{p,\infty}^{1-\delta}(\rho)} ds. \]

Therefore Gronwall’s lemma implies that, for \( t \in [0, T] \):
\[ ||I_t(\hat{u})||_{B_{p,\infty}^{1/2-\delta}(\rho)} \lesssim \left( \lambda \int_0^T \frac{ds}{(s)^{1/2+\delta}} ||J_s W^2_s||_{B_{p,\infty}^{1/2-\delta}(\rho)} \right) \exp \left( \lambda \int_0^T \frac{ds}{(s)^{1/2+\delta}} ||J_s W^3_s||_{B_{p,\infty}^{1/2-\delta}(\rho)} ds \right) \]
\[ \lesssim \left( \lambda \int_0^T \frac{ds}{(s)^{1/2+\delta}} ||J_s W^3_s||_{B_{p,\infty}^{1/2-\delta}(\rho)} \right). \]
Taking \( \rho = \frac{\rho_0}{\lambda} \) and using Besov embedding we deduce from (33):
\[ \sup_T ||I_T(\hat{u})||_{L^4} \lesssim \lambda^4 \left( \int_0^\infty \frac{ds}{(s)^{1/2+\delta}} ||J_s W^3_s||_{B_{p,\infty}^{1/2-\delta}} \right)^4 \lesssim \lambda^4. \]
Now computing \( l^T(\hat{u}) \) from eq. (30) and (32), we obtain
\[ l^T(\hat{u}) = \lambda W_{t \leq T} J_t U_{<s} W^2_s \geq \theta_t(I_t(\hat{u})), \quad t \geq 0. \]
It now remains to prove that \( \mathbb{E}[\|T^{\delta}(\bar{u})\|_{\mathcal{H}}^2] \lesssim O(\lambda^3) \) uniformly in \( T > 0 \). Note that, for \( s \in [0, T] \),
\[
\|J_s \mathcal{U}_s \mathbb{W}_s^2 \geq \theta_s(I_s(\bar{u}))\|_{L^2(\rho^2)} \lesssim \frac{1}{(s_{1/2})^{1/3+\delta}} \|J_s \mathcal{U}_s \mathbb{W}_s^2 \geq \theta_s(I_s(\bar{u}))\|_{B_{2,2}^{1+\delta/2}(\rho^2)} \\
\lesssim \frac{1}{(s_{1/2})^{1/3+\delta}} \|J_s \mathcal{U}_s \mathbb{W}_s^2\|_{\mathcal{L}^{-1+\delta/2}(\rho)} \|I_s(\bar{u})\|_{B_{2,2}^{1+\delta/2}(\rho)}.
\]

We know that the distribution of \( \bar{u} \) is invariant under translation by \( m \in \Lambda^3 \cap \mathbb{Z}^d \). Recalling that \( \sum_{m \in \Lambda^3 \cap \mathbb{Z}^d} \varphi^2(\bullet - m) = 1 \) and letting \( \rho \) be a polynomial weight with sufficient decay and such that \( \rho^3 \geq \varphi^2 \), we have
\[
\mathbb{E}[\|T^{\delta}(\bar{u})\|_{\mathcal{H}}^2] = \lambda^2 \mathbb{E}\|s \mapsto \mathcal{U}_s \mathcal{U}_s \mathbb{W}_s^2 \geq \theta_s(I_s(\bar{u}))\|_{\mathcal{H}}^2 \\
\lesssim \lambda^2 \sum_{m \in \Lambda^3 \cap \mathbb{Z}^d} \mathbb{E}\|s \mapsto \varphi^2(\bullet - m)J_s \mathcal{U}_s \mathbb{W}_s^2 \geq \theta_s(I_s(\bar{u}))\|_{\mathcal{H}}^2 \\
\text{(by trans. inv.)} \lesssim \lambda^2 \Lambda^3 \mathbb{E}\|s \mapsto \varphi^2 \mathcal{U}_s \mathcal{U}_s \mathbb{W}_s^2 \geq \theta_s(I_s(\bar{u}))\|_{\mathcal{H}}^2 \\
\text{(using } \rho^3 \geq \varphi^2 \text{)} \lesssim \lambda^2 \int_0^T \mathbb{E}\|s \mapsto \mathcal{U}_s \mathcal{U}_s \mathbb{W}_s^2 \geq \theta_s(I_s(\bar{u}))\|_{L^2(\rho^3)}^2 \\
\text{(by eq. (33))} \lesssim \lambda^2 \int_0^T \frac{\mathbb{E}\|s \mapsto \mathcal{U}_s \mathcal{U}_s \mathbb{W}_s^2 \geq \theta_s(I_s(\bar{u}))\|_{L^2(\rho^3)}^2}{(s_{1/2})^{1/3+\delta}} \\
\lesssim \lambda^4 \int_0^T \frac{\mathbb{E}\|s \mapsto \mathcal{U}_s \mathcal{U}_s \mathbb{W}_s^2 \geq \theta_s(I_s(\bar{u}))\|_{L^2(\rho^3)}^2}{(s_{1/2})^{1/3+\delta}} [1 + \mathbb{E}\|s \mapsto \mathbb{W}_s^2\|_{\mathcal{L}^{-1-\delta/2}(\rho)}^4 + \mathbb{E}\|s \mapsto \mathbb{W}_s^3\|_{L^2(\mathbb{R}_+, B_{p,p}^{-1-\delta}(\rho))}^4].
\]

This last quantities are bounded since standard arguments allow to bound, for \( p \) sufficiently large,
\[
\left[ \mathbb{E}\|s \mapsto \mathbb{W}_s^2\|_{\mathcal{L}^{-1-\delta/2}(\rho)}^p \right]^{p/8} \leq \mathbb{E}\|s \mapsto \mathbb{W}_s^2\|_{\mathcal{L}^{-1-\delta/2}(\rho)}^p \leq \mathbb{E}\|s \mapsto \mathbb{W}_s^2\|_{L^p(\mathbb{R}_+, B_{p,p}^{-1-\delta}(\rho))}^p \\
= \sum_{i \geq 1} 2^{(-1-\delta/2)p} \int_{\Lambda} d\rho(x) \mathbb{E}|\Delta_i \mathbb{W}_s^2(x)|^p \lesssim \sum_{i \geq 1} 2^{(-1-\delta/2)p} \mathbb{E}|\Delta_i \mathbb{W}_s^2(0)|^p \lesssim 1
\]
uniformly in \( s \geq 0 \). Similarly, we have
\[
\left[ \mathbb{E}\|s \mapsto \mathbb{W}_s^3\|_{L^p(\mathbb{R}_+, B_{p,p}^{-1-\delta}(\rho))}^4 \right]^{p/4} \leq \mathbb{E}\|s \mapsto \mathbb{W}_s^3\|_{L^p(\mathbb{R}_+, B_{p,p}^{-1-\delta}(\rho))}^p = \mathbb{E}\|s \mapsto \mathbb{W}_s^3(0)|^p,
\]
Now
\[
\sup_{s \in [0,T]} \mathbb{E}|\mathbb{W}_s^3(0)|^p \lesssim \sup_{s \in [0,T]} (\mathbb{E}|\mathbb{W}_s^2(0)|^2)^{p/2} \lesssim T^{3p/2}
\]
and
\[
\|\mathbb{W}_s^3\|_{L^2(\mathbb{R}_+, B_{p,p}^{-1-\delta}(\rho))} \lesssim \int_0^T \|J_s \mathbb{W}_s^3\|_{B_{p,p}^{-1-\delta}(\rho)} \\
\|\mathbb{W}_s^3\|_{L^2(\mathbb{R}_+, B_{p,p}^{-1-\delta}(\rho))} \lesssim \int_0^T \|J_s \mathbb{W}_s^3\|_{B_{p,p}^{-1-\delta}(\rho)} \ d s \\
\lesssim \int_0^T \left\| \sigma_t(D) \mathbb{W}_s^2 \right\|_{B_{p,p}^{-1-\delta}(\rho)}^2 \ d s \\
\lesssim \int_0^T \|\sigma_t(D)\mathbb{W}_s^2\|_{L^p(\mathbb{R}_+, B_{p,p}^{-1-\delta}(\rho))}^{2m} \ d s \\
\lesssim \int_0^\infty \langle t \rangle^{-1-\delta} \langle t \rangle^{-3} \|\mathbb{W}_s^2\|_{L^p(\mathbb{R}_+, B_{p,p}^{-1-\delta}(\rho))}^2 \ d s.
\]
This concludes the proof of the lemma.
Theorem 4. The decomposition of the noise is similar to the one given in [27] but differs in the fact that we choose the frequency cutoff dependent on the size of the noise instead of the point, to preserve translation invariance. The price to pay is that the decomposition is nonlinear, however this does not present any inconvenience in our context.

6. Gamma convergence

In this section we establish the $\Gamma$-convergence of the variational functional obtained in Lemma 5 as $T \to \infty$. $\Gamma$-convergence is a notion of convergence introduced by De Giorgi which is well suited for the study of variational problems. The book [8] is a nice introduction to $\Gamma$-convergence in the context of the calculus of variations. For the convenience of the reader we recall here the basic definitions and results.

Definition 1. Let $T$ be a topological space and let $F, F_n : T \to (-\infty, \infty]$. We say that the sequence of functionals $(F_n)_n$ $\Gamma$-converges to $F$ iff

i. For every sequence $x_n \to x$ in $T$

$$F(x) \leq \liminf_{n \to \infty} F_n(x_n);$$

ii. For every point $x$ there exists a sequence $x_n \to x$ (called a recovery sequence) such that

$$F(x) \geq \limsup_{n \to \infty} F_n(x_n).$$

Definition 2. A sequence of functionals $F_n : T \to (-\infty, \infty]$ is called equicoercive if there exists a compact set $K \subseteq T$ such that for all $n \in \mathbb{N}$

$$\inf_{x \in K} F_n(x) = \inf_{x \in T} F_n(x).$$

A fundamental consequence of $\Gamma$-convergence is the convergence of minima.

Theorem 4. If $(F_n)_n$ $\Gamma$-converges to $F$ and $(F_n)_n$ is equicoercive, then $F$ admits a minimum and

$$\min_T F = \lim_{n \to \infty} \inf_T F_n.$$ 

For a proof see [20].

In this section we allow all constants to depend on the volume $|\Lambda|$: this is not critical since, at this point, the aim is to obtain explicit formulas at fixed $\Lambda$.

We denote

$$\mathcal{H}^{\alpha,p} := L^2([0, \infty); W^{\alpha,p}), \quad \alpha \in \mathbb{R},$$

and by $\mathcal{H}_w^{\alpha,p}$ the reflexive Banach space $\mathcal{H}^{\alpha,p}$ endowed with the weak topology. We will write $\mathcal{H}^\alpha$ for $\mathcal{H}^{\alpha,2}$, and $\mathcal{H}$ for $\mathcal{H}^0$ and let $\mathcal{L} := \mathcal{H}^{-1/2-\kappa,3}$, this space will be useful as it gives sufficient control over $Z$:

Lemma 7. For $\kappa$ small enough $u \mapsto Z(u)$ is a compact map $\mathcal{L} \to C([0, \infty], L^4)$.

Proof. By definition of $Z$ we have for any $0 < \varepsilon < 1/8 - \kappa/2$,

$$\|Z_{t_2}(u) - Z_{t_1}(u)\|_{W^{4,\varepsilon}} = \left\| \int_{t_1}^{t_2} J_s u_s ds \right\|_{W^{4,\varepsilon}} \leq \int_{t_1}^{t_2} \left\| \sigma_s(D) u_s \right\|_{W^{4,\varepsilon}} ds$$

$$\lesssim \int_{t_1}^{t_2} \frac{1}{(s)^{1/2 + \varepsilon}} \|D\|^{1-1+\varepsilon} u_s \|_{W^{4,\varepsilon}} ds$$

$$\lesssim \int_{t_1}^{t_2} \frac{1}{(s)^{1/2 + \varepsilon}} \|D\|^{1-1+\varepsilon} u_s \|_{W^{1/4+\varepsilon,3}} ds$$

$$\lesssim \int_{t_1}^{t_2} \frac{1}{(s)^{1/2 + \varepsilon}} ds \int \| u_s \|_{W^{1/2-\kappa,3}}^2 ds \lesssim \| u \|_{\mathcal{L}}^2 \int_{t_1}^{t_2} \frac{1}{(s)^{1/2 + \varepsilon}} ds,$$
where we have used a Sobolev embedding in the second to last line. Since

$$\lim_{t_1 \to t_2} \int_{t_1}^{t_2} \frac{1}{(s)^{1+2\kappa}} \, ds = 0, \quad \int_{0}^{\infty} \frac{1}{(s)^{1+2\kappa}} \, ds < \infty,$$

for any $t_2 \in [0, \infty]$, we can conclude by the Rellich–Kondrachov embedding theorem and the Arzelà–Ascoli theorem, that bounded sets in $L$ are mapped to compact sets in $C([0, \infty], L^4)$, proving the claim. \qed

We will need the following lemma, which establishes pointwise convergence for the functional $\Phi_T$ defined in Lemma [3]. In the sequel, by an abuse of notation, we will denote both a generic element of $\mathcal{G}$ and the canonical random variable on $\mathcal{G}$ by

$$X = (X^1, X^2, X^{(3)}, X^{[3]}_{\circ \circ}, X^{(2) \circ (2)}, X^{[2]}_{\circ \circ}).$$

**Lemma 8.** Define $l^\infty(u) = l^\infty(X, u) \in \mathbb{H}_u$ such that

$$l^\infty(u) = u_t + \lambda W_i^\infty + \lambda J_1(\mathcal{W}_i^2) > Z_i^\infty, \quad t \geq 0.$$  (35)

For any sequence $(u^T, X^T)_{[T]}$ such that $u^T \to u$ in $L_w$, $l^T = l^T(X^T, u^T) \to l = l^\infty(X, u)$ in $\mathcal{H}_w$ and

$$X^T = (X^{T,1}, X^{T,2}, X^{T,(3)}, X^{T,[3]}_{\circ \circ}, X^{T,(2) \circ (2)}, X^{T,[2]}_{\circ \circ})$$

in $\mathcal{G}$ we have

$$\lim_{T \to \infty} \Phi_T(X^T, Z(u^T), K(u^T)) = \Phi_\infty(X, Z(u), K(u)),$$

where $\Phi_\infty$ is defined by

$$\Phi_\infty(X, Z(u), K(u)) := f(X^{1}_\infty + Z_\infty(u)) + \sum_{i=1}^{6} \mathcal{Y}_{\infty}^{(i)}(X, Z(u), K(u)),$$

with $\mathcal{Y}_{\infty}^{(i)}(X, Z, K) = \mathcal{Y}_{\infty}^{(i)}$ given by

- $\mathcal{Y}_{\infty}^{(1)} := \frac{\lambda}{2} \mathcal{R}_2(X^{2}_{\infty}, K, K_{\infty}) + \frac{\lambda}{2} \int (X^{2}_{\infty} < K_{\infty}) K_{\infty} - \lambda^2 \int (X^{2}_{\infty} < X^{[3]}_{\infty}) K_{\infty},$
- $\mathcal{Y}_{\infty}^{(2)} := 0$,
- $\mathcal{Y}_{\infty}^{(3)} := \lambda \int_{0}^{\infty} \int (X^{2}_{s} > \tilde{Z}^{2}_{s}) K_{s} \, dt,$
- $\mathcal{Y}_{\infty}^{(4)} := 4\lambda \int X^{1}_{\infty} K^{3}_{\infty} - 12 \lambda^2 \int (X^{1}_{\infty} X^{[3]}_{\infty}) K_{\infty} + 12 \lambda^2 \int X^{1}_{\infty} (X^{[3]}_{\infty})^2 K_{\infty},$
- $\mathcal{Y}_{\infty}^{(5)} := -2\lambda^2 \int_{0}^{\infty} \int \gamma \tilde{Z}^{2}_{s} \tilde{Z}^{2}_{s} \, dt,$
- $\mathcal{Y}_{\infty}^{(6)} := -\lambda^2 \int X^{2}_{\infty} [K_{\infty} - \lambda \int_{0}^{\infty} \int X^{(2) \circ (2)} (Z^{2}_{s})^2 \, dt + \frac{\lambda^2}{2} \int_{0}^{\infty} \mathcal{R}_{3,i}(X^{2}_{s}, X^{2}_{s}, \tilde{Z}^{2}_{s}) \, dt,$

where, with abuse of notation, we set

$$X^{1}_{\infty} X^{[3]}_{\infty} := X^{1}_{\infty} X^{[3]}_{\infty} + X^{1}_{\infty} < X^{[3]}_{\infty} + X^{[3]}_{\circ \circ},$$
$$X^{1}_{\infty} [X^{[3]}_{\infty}]^2 := X^{1}_{\infty} (X^{[3]}_{\infty} \circ X^{[3]}_{\infty}) + 2 X^{[3]}_{\circ \circ} X^{[3]}_{\infty} + 2 \mathcal{R}_1(X^{[3]}_{\infty}, X^{[3]}_{\infty}, X^{1}_{\infty}),$$

and where $\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3$ are the trilinear forms defined in Proposition [7], Proposition [7] and Proposition [8] respectively.
Proof. Lemma \textbf{7} implies that for any \( u^T \to u \) in \( \mathcal{L}_w \) we have \( Z(u^T) \to Z(u) \) in \( C([0, \infty], L^4) \) and by the convergence of \( t^T \to l \) in \( \mathcal{H}_w \) we have also \( K(u^T) \to K(u) \) in \( C([0, \infty], H^{1-\kappa}) \). The products \( X^T \times X^T \) and \( X^T \times X^T \) can be decomposed using paraproducts and, after replacing the resonant products by the corresponding stochastic objects in \( X^T \), we obtain the finite \( T \) analogs of the expressions in eq. \textbf{[37]}). After this preprocessing, it is easy to see by continuity that we have \( X^T \times X^T \to X^T \times X^T \) and \( X^T \times X^T \) in \( \mathcal{H}^{1/2-\kappa} \). For \( \Upsilon^{(1)} \) and \( \Upsilon^{(4)} \) the first term of \( \Upsilon \) the statement follows from uniform bounds for \( (X^T, Z(u^T), K(u^T)) \) on \( \mathcal{S} \times C([0, \infty], H^{1/2-\kappa}) \times C([0, \infty], H^{1-\kappa}) \) and multilinearity. For \( \Upsilon^{(2)} \) and the first two terms of \( \Upsilon \) convergence to 0 follows from the bounds established in Lemma \textbf{19} and Lemma \textbf{22} for \( \Upsilon^{(3)} \), the last term of \( \Upsilon^{(5)} \) and the last two terms of \( \Upsilon \) we can again use uniform bounds and multilinearity as well as dominated convergence, thanks to Proposition \textbf{8}.

Going back to our particular setting recall that from Lemma \textbf{5} we learned that

\[
W_T(f) = \inf_{u \in \mathbb{H}_a} F_T(u),
\]

with

\[
F_T(u) = \mathbb{E} \left[ \Phi_T(W, Z(u), K(u)) + \lambda \| Z_T(u) \|_{L^4}^4 + \frac{1}{2} \| t^T(u) \|_{\mathcal{H}}^2 \right],
\]

where \( t^T(u), Z(u), K(u) \) are functions of \( u \) according to eq. \textbf{21}). This form of the functional is appropriate to analyze the limit \( T \to \infty \) and obtain the main result of the paper, stated precisely in the following theorem which is a simple restatement of the basic consequence of Theorem \textbf{5} below.

**Theorem 5.** We have

\[
\lim_{T \to \infty} W_T(f) = W(f) := \inf_{u \in \mathbb{H}_a} F_\infty(u),
\]

where

\[
F_\infty(u) = \mathbb{E} \left[ \Phi_\infty(W, Z(u), K(u)) + \lambda \| Z_\infty(u) \|_{L^4}^4 + \frac{1}{2} \| t_\infty(u) \|_{\mathcal{H}}^2 \right],
\]

and where \( \Phi_\infty \) and \( l_\infty \) are defined in Lemma \textbf{8}.

In order to use \( \Gamma \)-convergence, we need to properly modify the variational setting in order to guarantee enough compactness and continuity uniformly as \( T \to \infty \).

The analytic estimates contained in Section \textbf{7} below allow to infer that there exists a small \( \delta \in (0, 1) \), and a finite constant \( Q_T > 0 \) uniformly bounded in \( T \) such that

\[
-Q_T + (1 - \delta) \mathbb{E} \left[ \lambda \| Z_T \|_{L^4}^4 + \frac{1}{2} \| t^T(u) \|_{\mathcal{H}}^2 \right] \leq F_T(u),
\]

and

\[
F_T(u) \leq Q_T + (1 + \delta) \mathbb{E} \left[ \lambda \| Z_T \|_{L^4}^4 + \frac{1}{2} \| t^T(u) \|_{\mathcal{H}}^2 \right].
\]

As long as \( T \) is finite, the original potential \( V_T \) is bounded below so in particular we have

\[
-C_T + \mathbb{E} \left[ \frac{1}{2} \| u \|_{\mathcal{H}}^2 \right] \leq F_T(u).
\]

From this we conclude that we can relax the optimization problem and ask that \( u \in \mathbb{L}_a \), once we have established Lemma \textbf{10} where \( \mathbb{L}_a \) is the space of predictable processes in \( \mathcal{L} \):

\[
W_T(f) = \inf_{u \in \mathbb{L}_a} F_T(u).
\]

The reason of this relaxation lies in the fact that the cost terms \( \| t^T(u) \|_{\mathcal{H}} \) and \( \| Z_T \|_{L^4} \) control the \( \mathcal{L} \) norm of \( u \) uniformly in \( T \), modulo constants depending only on \( \| \Phi \|_{\mathcal{S}} \) which are bounded in average uniformly in \( T \).
Note that eq. (37) implies that for any sequence \((u^T)_T\) such that \(F_T(u^T)\) remains bounded we must have that also
\[
\sup_T \mathbb{E}[\|l^T(u^T)\|_2^2] < \infty.
\]
To prove \(\Gamma\)-convergence we need to find a space with a topology which, on the one hand is strong enough to enable to prove the \(\Gamma\)-liminf inequality, and on the other hand allows to obtain enough compactness from \(F_T\). Almost sure convergence on \(\mathcal{S} \times \mathcal{L}\) would allow for the former but is too strong for the latter. For this reason we need a setting based on convergence in law as precised in the following definition.

**Definition 3.** Denote by \((\mathcal{X}, u)\) be the canonical variables on \(\mathcal{S} \times \mathcal{L}\) and consider the space of probability measures
\[
\mathcal{X} := \{ \mu \in \mathcal{P}(\mathcal{S} \times \mathcal{L}) | \mu = \text{Law}_\mathcal{P}(\mathcal{W}, u) \text{ for some } u \in \mathcal{L}_w \text{ with } \mathbb{E}_\mu[\|u\|_2^2] < \infty \}.
\]
Equip \(\mathcal{X}\) with the following topology:

1. \(\mu_n\) converges to \(\mu\) weakly on \(\mathcal{S} \times \mathcal{L}_w\).
2. \(\sup_n \mathbb{E}_{\mu_n}[\|u\|_2^2] < \infty\).

Denote by \(\overline{\mathcal{X}}\) the closure of \(\mathcal{X}\) in the space of probability measures \(\mu\) on \(\mathcal{S} \times \mathcal{L}_w\) such that \(\mathbb{E}_\mu[\|u\|_2^2] < \infty\).

Condition (b) allows to exclude pathological points in \(\overline{\mathcal{X}}\) and makes possible Lemma 13 below. Then
\[
(41) \quad \mathcal{W}_T(f) = \inf_{\mu \in \overline{\mathcal{X}}} \tilde{F}_T(\mu),
\]
where
\[
\tilde{F}_T(\mu) := \mathbb{E}_\mu \left[ \Phi_T(\mathcal{X}, Z(u), K(u)) + \lambda \|Z_T(u)\|_{L^4}^4 + \frac{1}{2} \|l^T(u)\|_{\mathcal{H}}^2 \right]
\]
and where \(\mathbb{E}_\mu\) denotes the expectation on \(\mathcal{S} \times \mathcal{L}_w\) wrt. the probability measure \(\mu\). Our first aim will be to prove that the family \((\tilde{F}_T)_T\) is indeed equicoercive on \(\overline{\mathcal{X}}\).

**Lemma 9.** Let \((\mu^T)_T\) be a family of measures on \(\mathcal{S} \times \mathcal{L}\) such that \(\sup_T \mathbb{E}_{\mu^T}[\|u\|_2^2] < \infty\). Then \((\mu^T)_T\) is tight on \(\mathcal{S} \times \mathcal{L}_w\), in particular there exists a subsequence converging in \(\overline{\mathcal{X}}\).

**Proof.** Observe that \(\text{Law}(\mathcal{W})\) on \(\mathcal{S}\) is tight since \(\mathcal{S}\) is a separable metric space, so for any \(\varepsilon > 0\), we can find a compact set \(K^1_\varepsilon \subset \mathcal{S}\) such that \(\mu((\mathcal{S}\setminus K^1_\varepsilon) \times \mathcal{L}) < \varepsilon/2\). Now let \(K^2_\varepsilon := K^1_\varepsilon \times B(0, C) \subset \mathcal{S} \times \mathcal{L}_w\), for some large \(C\) to be chosen later. Then \(K^2_\varepsilon\) is a compact subset of \(\mathcal{S} \times \mathcal{L}_w\) and
\[
\text{Pr}_\mu[(\mathcal{X}, u) \notin K^2_\varepsilon] \leq \varepsilon + \frac{1}{C} \mathbb{E}_{\mu^T}[\|u\|_2^2].
\]
Choosing \(C > \sup_T 2\mathbb{E}_{\mu^T}[\|u\|_2^2]/\varepsilon\) gives tightness. \(\square\)

**Lemma 10.** There exists a constant \(C\), depending on \(\kappa\) and \(\lambda\), such that
\[
\mathbb{E}_{\mu^T}[\|u\|_2^2] \leq C + 2\lambda \mathbb{E}_{\mu^T}[\|Z_T(u)\|_{L^4}^4] + \mathbb{E}_{\mu^T}[\|l^T(u)\|_{\mathcal{H}}^2].
\]

**Proof.** We use \(\|l^T(u)\|_{\mathcal{H}} \leq \|l^T(u)\|_{\mathcal{H}}\) in the bound
\[
\mathbb{E}_{\mu^T}[\|u\|_2^2] \leq \lambda \mathbb{E}_{\mu^T}[\|X(3)\|_2^2] + \lambda \mathbb{E}_{\mu^T}[\|s \mapsto J_s(X(3) > \theta_s Z_T(u))\|_2^2] + \mathbb{E}_{\mu^T}[\|l^T(u)\|_{\mathcal{H}}^2]
\]
\[
\leq \lambda \mathbb{E}_{\mu^T}[\|X(3)\|_2^2] + \lambda \mathbb{E}_{\mu^T} \left[ \int_0^\infty \frac{\|X(3)^4\|_{L^4}(s-\lambda s)^{1+\lambda} \|Z_T(u)\|_{L^4}^2}{(s+\epsilon)^{1+\lambda}} ds \right] + \mathbb{E}_{\mu^T}[\|l^T(u)\|_{\mathcal{H}}^2]
\]
\[
\leq \lambda \mathbb{E}_{\mu^T}[\|X(3)\|_2^2] + 2\lambda \mathbb{E}_{\mu^T}[\|Z_T(u)\|_{L^4}^4] + 2 \mathbb{E}_{\mu^T}[\|l^T(u)\|_{\mathcal{H}}^2].
\]
\(\square\)
**Corollary 2.** The family $(\tilde{F}_T)_T$ is equicoercive on $\overline{\mathcal{X}}$.

**Proof.** Define for some $K > 0$ large enough
\[
\mathcal{K} = \left\{ \mu : \lambda E_{\mu}[\| Z_T(u) \|_{L^1}^2] + \frac{1}{2} E_{\mu}[\| t^T(u) \|_{H^1}^2] \leq K \right\}
\]
From eq. (37) we have
\[
\lambda E_{\mu}[\| Z_T(u) \|_{L^1}^2] + \frac{1}{2} E_{\mu}[\| t^T(u) \|_{H^1}^2] \leq C + \tilde{F}_T(\mu),
\]
so
\[
\inf_{\mu \in \mathcal{K}} \tilde{F}_T(\mu) \geq K - C
\]
on the other hand from (38) it follows that
\[
\sup_{T} \inf_{\mu \in \overline{\mathcal{X}}} \tilde{F}_T(\mu) < \infty.
\]
So for $K$ large enough
\[
\inf_{\mu \in \overline{\mathcal{X}}} \tilde{F}_T(\mu) = \inf_{\mu \in \mathcal{K}} \tilde{F}_T(\mu)
\]
And by Lemma 10 and Lemma 9 $\mathcal{K}$ is a compact set. $\square$

To be able to use the equicoercivity we will need to show that we can extend the infimum in (41) to $\overline{\mathcal{X}}$. For this we will first need some properties of the space $\overline{\mathcal{X}}$. In particular we will need to show that measures with sufficiently high moments are dense in $\overline{\mathcal{X}}$ in a way which behaves well with respect to $\tilde{F}_T$. For this we introduce some approximations which will be useful in the sequel.

**Definition 4.** Let $u \in \mathcal{L}$, $N \in \mathbb{N}$, and $(\eta_{\varepsilon})_{\varepsilon > 0}$ be a smooth Dirac sequence on $\Lambda$ and $(\varphi_{\varepsilon})_{\varepsilon > 0}$ be another smooth Dirac sequence compactly supported on $\mathbb{R}_+ \times \Lambda$. Denote by $\ast_{\Lambda}$ the convolution only wrt the space variable, and by $\ast$ the space-time convolution. Define the following approximations of the identity:
\[
\begin{align*}
\text{reg}_{x,\varepsilon}(u) & := u \ast_{\Lambda} \eta_{\varepsilon}, \\
\text{reg}_{t,x,\varepsilon}(u)(t) & := e^{-\varepsilon t} u \ast \varphi_{\varepsilon}(t) = e^{-\varepsilon t} \int_0^t u(t-s) \ast_{\Lambda} \varphi_{\varepsilon}(s)ds.
\end{align*}
\]
Denote by
\[
\tilde{F}_T^N(u) := \inf \left\{ t \geq 0 : \int_0^t \| u(s) \|_{W^{-1/2,-3}}^2 ds \geq N \right\},
\]
and
\[
\text{cut}_N(u)(t) := u(t) \mathbb{1}_{\{ t \leq \tilde{F}_T^N(u) \}}.
\]
Observe the following properties of these maps:
- reg$_{x,\varepsilon}$ is a continuous map $\mathcal{L}_w \to \mathcal{H}_w$ and $\mathcal{L} \to \mathcal{H}$;
- reg$_{t,x,\varepsilon}$ is a continuous map $\mathcal{L}_w \to \mathcal{H}$;
- cut$_N$ is continuous as a map $\mathcal{L} \to B(0,N) \subset \mathcal{L}$;
- if $u$ is a predictable process then reg$_{x,\varepsilon}(u)$, reg$_{t,x,\varepsilon}(u)$, cut$_N(u)$ will also be predictable.
Furthermore we have the bounds
\[
\|\text{reg}_{x,\varepsilon}(u)\|_L, \|\text{reg}_{t,x,\varepsilon}(u)\|_L, \|\text{cut}_N(u)\|_L \leq \|u\|_L.
\]
uniformly in $\varepsilon, N$, and for every $u \in \mathcal{L},$
\[
\lim_{\varepsilon \to 0} \|\text{reg}_{x,\varepsilon}(u) - u\|_L = \lim_{\varepsilon \to 0} \|\text{reg}_{t,x,\varepsilon}(u) - u\|_L = \lim_{N \to \infty} \|\text{cut}_N(u) - u\|_L = 0.
\]
With abuse of notation, for $\mu \in \mathcal{P}(\mathcal{S} \times \mathcal{L})$ and $f : \mathcal{L} \to \mathcal{L}$, we let
\[
f_\ast \mu = (\text{Id}, f)_\ast \mu = \text{Law}_\mu(\mathcal{X}, f(u)).
\]
**Remark 6.** Let us briefly comment on the uses of these approximation in the sequel. \( \text{reg}_{t,x,\varepsilon} \) will be used when one wants to obtain a sequence of weakly convergent measures on \( \mathcal{S} \times \mathcal{H} \) or \( \mathcal{S} \times \mathcal{L} \) from a sequence of measures weakly convergent on \( \mathcal{S} \times \mathcal{L}_w \). \( \text{reg}_{x,\varepsilon} \) will be used when one wants to obtain a measure on \( \mathcal{S} \times \mathcal{H} \) from one on \( \mathcal{S} \times \mathcal{L} \), while preserving the estimates on the moments of \( Z(u) \) since \( Z(u \ast \eta_{\varepsilon}) = Z(u) \ast \eta_{\varepsilon} \).

**Lemma 12.** Let \( \mu \in \mathcal{X} \). Then there exist \( \{\mu_n\}_n \) in \( \mathcal{X} \) such that \( \mu_n \to \mu \) on \( \mathcal{S} \times \mathcal{L} \) (now with the norm topology) and \( \sup_n \mathbb{E}_{\mu_n}[\|u\|^2_{\mathcal{L}}] < \infty \).

**Proof.** By definition of \( \mathcal{X} \) of there exists \( \mu_n \to \mu \) weakly on \( \mathcal{S} \times \mathcal{L}_w \). Then \( \{\text{reg}_{t,x,\varepsilon}\}_n \mu_n \to (\text{reg}_{t,x,\varepsilon})_n \mu \) on \( \mathcal{S} \times \mathcal{L} \) as \( n \to \infty \), and since \( (\text{reg}_{t,x,\varepsilon})_n \mu \to \mu \) weakly on \( \mathcal{S} \times \mathcal{L} \) as \( \varepsilon \to 0 \), we obtain the statement by taking a diagonal sequence. \( \square \)

**Lemma 11.** Let \( \mu_n \to \mu \) on \( \mathcal{S} \times \mathcal{L} \), such that \( \sup_n \mathbb{E}_{\mu_n}[\|u\|^2_{\mathcal{L}}] < \infty \). Then

1. for every Lipschitz function \( f \) on \( \mathcal{L} \), \( \mathbb{E}_{\mu_n}[f(u)] \to \mathbb{E}_{\mu}[f(u)] \);
2. for every Lipschitz function \( f \) on \( C([0,\infty], L^4) \) we have \( \mathbb{E}_{\mu_n}[f(Z(u))] \to \mathbb{E}_{\mu}[f(Z(u))] \).

**Proof.** Let \( f \) be a Lipschitz function on \( \mathcal{L} \) with Lipschitz constant \( L \). Let \( \eta \in C(\mathbb{R}, \mathbb{R}) \) be supported on \( B(0, 2) \) with \( \eta = 1 \) on \( B(0, 1) \), and \( \eta_N(x) = \eta(x/N) \). Then \( u \mapsto (f(u)\eta_N(\|u\|_\mathcal{L})) \) is bounded, and

\[
\lim_{n \to \infty} \mathbb{E}_{\mu_n}[f(u)\eta_N(\|u\|_\mathcal{L})] = \mathbb{E}_{\mu}[f(u)\eta_N(\|u\|_\mathcal{L})],
\]

and

\[
\mathbb{E}_{\mu_n}[f(u)\eta_N(\|u\|_\mathcal{L})] - \mathbb{E}_{\mu}[f(u)] = \mathbb{E}_{\mu_n}[f(u)\eta_N(\|u\|_\mathcal{L}) - f(u)] 1_{\{\|u\|_\mathcal{L} \geq N\}} \\
\leq \mathbb{E}_{\mu_n}[2L\|u\|_\mathcal{L} 1_{\{\|u\|_\mathcal{L} \geq N\}}] \\
\leq 2L\mathbb{E}_{\mu_n}[\|u\|^2_{\mathcal{L}}]^{1/2} \mu_n(\|u\|_\mathcal{L} \geq N) \\
\leq \frac{2L}{N} \mathbb{E}_{\mu_n}[\|u\|^2_{\mathcal{L}}].
\]

Using that \( \sup_n \mathbb{E}_{\mu_n}[\|u\|^2_{\mathcal{L}}] < \infty \) we have

\[
\lim_{n \to \infty} \mathbb{E}_{\mu_n}[f(u)] - \mathbb{E}_{\mu}[f(u)] \leq \lim_{n \to \infty} \mathbb{E}_{\mu_n}[f(u)\eta_N(\|u\|^2_{\mathcal{L}})] - \mathbb{E}_{\mu}[f(u)\eta_N(\|u\|^2_{\mathcal{L}})] \\
+ \sup_n \mathbb{E}_{\mu_n}[f(u)\eta_N(\|u\|^2_{\mathcal{L}})] - \mathbb{E}_{\mu_n}[f(u)] \\
+ \sup_n \mathbb{E}_{\mu}[f(u)\eta_N(\|u\|^2_{\mathcal{L}})] - \mathbb{E}_{\mu}[f(u)] \\
\leq \frac{4L}{N} \sup_n \mathbb{E}_{\mu_n}[\|u\|^2_{\mathcal{L}}] \lesssim N^{-1},
\]

and sending \( N \to \infty \) gives the statement. The second statement follows from the first and Lemma 7. \( \square \)

From the definition of \( \mathcal{X} \) we have that

\[
\mathcal{W}_T(f) = \inf_{\mu \in \mathcal{X}} \mathcal{F}_T(\mu),
\]

where

\[
\mathcal{F}_T(\mu) := \mathbb{E}_\mu \left[ \Phi_T(\mathcal{X}, Z(u), K(u)) + \lambda \|Z(u)\|^4_{\mathcal{L}} + \frac{1}{2} \|f^T(u)\|^2_{\mathcal{L}} \right].
\]

The space \( \mathcal{X} \) is not necessarily closed w.r.t weak convergence of measures. To be able to use an argument involving equicoercivity we need to show that we can pass to the closure \( \mathcal{X} \) of \( \mathcal{X} \) in the infimum. To do this we first need to prove that we can approximate measures in \( \mathcal{X} \) by measures with bounded support in the second marginal which are still in \( \mathcal{X} \). This is the content of the following
Lemma 13. Let $\mu \in \mathcal{X}$ such that $E_{\mu}[\|Z_T(u)\|_{L^4}] + E_{\mu}[\|u\|_2^2] < \infty$. For any $L > 0$ there exists $\mu_L \in \mathcal{X}$ such that $\|u\|_{L} \leq L$, $\mu_L$-almost surely, $\mu_L \to \mu$ weakly on $\mathcal{S} \times \mathcal{L}$ as $L \to \infty$,$$
abla E_{\mu_L}[\|Z_T(u)\|_{L^4}] \to E_{\mu}[\|Z_T(u)\|_{L^4}^4], \quad \text{and} \quad E_{\mu_L}[\|u\|_2^2] \to E_{\mu}[\|u\|_2^2].$$Furthermore for any $\mu_L$ there exists $(\mu_L,n) \in \mathcal{X}$ such that $\|u\|_{L} \leq L$, $\mu_L,n$-almost surely and $\mu_L,n \to \mu_L$ weakly on $\mathcal{S} \times \mathcal{L}$.

Proof.

Step 1 First let us see how to approximate $\mu$ with $\mu_L$ which are defined such that $\|Z_T(u)\|_{L^4} \leq L$, $\mu_L$ almost surely. As $\mu \in \mathcal{X}$, there exists $(\mu_n) \subset \mathcal{X}$ such that $\mu_n \to \mu$ on $\mathcal{S} \times \mathcal{L}$ and $\sup_n E_{\mu_n}[\|u\|_2^2] < \infty$. Since $\mu_n \in \mathcal{X}$ there exist $(u^n)_n$ adapted such that $\mu_n = \text{Law}(\mathcal{W}, u^n)$. Define $
abla \tilde{Z}_n := E\left[\int_0^T J_t u^n_t dt | \mathcal{F}_s \right] = E\left[\int_0^T J_t E[u^n_t | \mathcal{F}_s] dt \right]$. Then $\tilde{Z}$ is a martingale with continuous paths in $\mathcal{L}^4(\Lambda)$. Define the stopping time $T_{L,n} = \inf\{t \in [0, T] : \|\tilde{Z}_t^n\|_{L^4} \geq L\}$ where the inf is equal to $T$ if the set is empty. Observe that $\tilde{Z}_{T_{L,n}} = \int_0^T J_t E[u^n_t | \mathcal{F}_{T_{L,n}}] dt = Z_T(u^n)$ with $u^n_{L,n} := E[u^n_t | \mathcal{F}_{T_{L,n}}]$ adapted, by optional sampling, and almost surely $\|\tilde{Z}_{T_{L,n}}\|_{L^4} \leq L$. Now set $\tilde{\mu}_{L,n} := \text{Law}_\mathcal{F}(\mathcal{W}, u^{L,n})$.

Step 1.1 (Tightness) The next goal is to show that for fixed $L$, we can select a suitable convergent subsequence from $(\tilde{\mu}_{L,n})_n$. For this we first show that $(\tilde{\mu}_{L,n})_n$ is tight on $\mathcal{S} \times \mathcal{L}$. From the definition of $\mathcal{X}$ we have that $\sup_n E_{\mu_n}[\|u\|_2^2] < \infty$, and by construction
\[ \sup_n E_{\tilde{\mu}_{L,n}}[\|u\|_2^2] \leq \sup_n E_{\mathcal{F}}[\|E[u^n_t | \mathcal{F}_{T_{L,n}}]\|_2^2] \leq \sup_n E_{\mathcal{F}}[\|u^n\|_2^2] = \sup_n E_{\mu_n}[\|u\|_2^2] < \infty, \]
which gives tightness according to Lemma 9. We can then select a subsequence which converges on $\mathcal{L}$.w.

Step 1.2 (Bounds) Let $\tilde{\mu}_L$ be the limit of the sequence constructed in Step 1.1. In this step we prove bounds on the relevant moments of $\tilde{\mu}_L$. Let $f_1^M, f_2^M$ be sequences of functions on $\mathbb{R}$ which are Lipschitz, convex and monotone for every $N$, while for every $x \in \mathbb{R}$
\begin{align*}
0 \leq f_1^M(x) \leq x^2, \quad &\lim_{M \to \infty} f_1^M(x) = x^2, \\
0 \leq f_2^M(x) \leq x^4, \quad &\lim_{M \to \infty} f_2^M(x) = x^4.
\end{align*}
Then $f_1^M(\|u\|_{\mathcal{L}})$ is a lower-semi continuous positive function on $\mathcal{L}$ so by the Portmanteau lemma we have $E_{\tilde{\mu}_L}[f_1^M(\|u\|_{\mathcal{L}})] \leq \liminf_{n \to \infty} E_{\tilde{\mu}_{L,n}}[f_1^M(\|u\|_{\mathcal{L}})]$, and since it is also Lipschitz continuous and convex we have
\begin{align*}
\liminf_{n \to \infty} E_{\tilde{\mu}_{L,n}}[f_1^M(\|u\|_{\mathcal{L}})] &= \liminf_{n \to \infty} E_{\mathcal{F}}[f_1^M(\|E[u^n | \mathcal{F}_{T_{L,n}}]\|_{\mathcal{L}})] \\
&\leq \liminf_{n \to \infty} E_{\mathcal{F}}[f_1^M(\|u^n\|_{\mathcal{L}})] = E_{\mu}[f_1^M(\|u\|_{\mathcal{L}})].
\end{align*}
Therefore
\begin{align*}
E_{\tilde{\mu}_L}[\|u\|_2^2] &= \lim_{M \to \infty} E_{\tilde{\mu}_L}[f_1^M(\|u\|_{\mathcal{L}})] \\
&\leq \lim_{M \to \infty} E_{\mu}[f_1^M(\|u\|_{\mathcal{L}})] = E_{\mu}[\|u\|_2^2].
\end{align*}
Proceeding similarly for $Z$, we see that $f_2^N(\|Z_T\|_{L^4})$ is a continuous function on $\mathcal{L}$ bounded below, and Lipschitz-continuous and convex on $\mathcal{L}$ so we again can estimate
\begin{align*}
E_{\tilde{\mu}_L}[f_2^N(\|Z_T\|_{L^4})] &= \lim_{n \to \infty} E_{\tilde{\mu}_{L,n}}[f_2^M(\|Z_T\|_{L^4})], \\
E_{\tilde{\mu}_L}[f_2^N(\|Z_T\|_{L^4})] &= \lim_{n \to \infty} E_{\tilde{\mu}_{L,n}}[f_2^M(\|Z_T\|_{L^4})] \\
&= \lim_{n \to \infty} E_{\mathcal{F}}[f_2^M(\|E[Z_T(u^n) | \mathcal{F}_{T_{L,n}}]\|_{L^4})] \\
&\leq \lim_{n \to \infty} E_{\mathcal{F}}[f_2^M(\|Z_T(u^n)\|_{L^4})] = E_{\mu}[f_2^M(\|Z_T(u^n)\|_{L^4})].
\end{align*}
and, taking $N \to \infty$, obtain

$$\mathbb{E}_{\tilde{\mu}_L} [\|Z_T\|_{L^1}] \leq \mathbb{E}_\mu [\|Z_T\|_{L^1}].$$

**Step 1.3** (Weak convergence) Now we prove weak convergence of $\tilde{\mu}_L$ to $\mu$ on $\mathcal{S} \times \mathcal{L}$. Let $f : \mathcal{S} \times \mathcal{L} \to \mathbb{R}$ be bounded and continuous. By dominated convergence and continuity of $f$, $\lim_{\varepsilon \to 0} \mathbb{E}_{\tilde{\mu}_L} [f(\mathcal{X}, \text{reg}_{t,x,c}(u))] = \mathbb{E}_{\tilde{\mu}_L} [f(\mathcal{X}, u)]$. Using furthermore that $(\mathcal{X}, u) \mapsto f(\mathcal{X}, \text{reg}_{t,x,c}(u))$ is continuous on $\mathcal{S} \times \mathcal{L}_w$ and Lemma 7 in the 5th line below, we can estimate

$$\lim_{L \to \infty} \left| \mathbb{E}_\mu f(\mathcal{X}, u) - \mathbb{E}_{\tilde{\mu}_L} f(\mathcal{X}, u) \right| = 0.$$

**Step 2** In this step we improve the approximation to have bounded support. Let $\mu_n \to \mu$ be the subsequence selected in Step 1. Recall that $\mu_n = \text{Law}(\mathcal{W}, u^n)$ with adapted $u^n$. Define $\tilde{Z}_{t,n} := \mathbb{E} [Z_T (\text{cut}_N(u)) \mid \mathcal{F}_t]$, and similarly to Step 1, $T_{n,n} := \inf \{ t > 0 : \| \tilde{Z}_{t,n} \|_{\mathcal{L}} \geq L \}$, set $u_{n,n,L} := \mathbb{E} [\text{cut}_N(u) \mid \mathcal{F}_{T_{n,n}}]$. Then $\| u_{n,n,L} \|_{\mathcal{L}} \leq \| u \|_{\mathcal{L}}$ uniformly in $n$ and $\mathbb{P}$-almost surely, so $\mu_{n,n,L} = \text{Law}(\mathcal{W}, u_{n,n,L})$ is tight on $\mathcal{S} \times \mathcal{L}_w$ and we can select a weakly convergent subsequence. Denote the limit by $\mu_{L,n}$. Now we follow the strategy from Step 1.

**Step 2.1** (Bounds) Let $f^1_M$ be defined like in Step 1.2. Then again we have

$$\liminf_{n \to \infty} \mathbb{E}_{\mu_{n,n,L}} [f^1_M (\|u\|_{\mathcal{L}})] = \liminf_{n \to \infty} \mathbb{E}_\mu [f^1_M (\|u\|_{\mathcal{L}})] \leq \lim_{n \to \infty} \mathbb{E}_\mu [f^1_M (\|u\|_{\mathcal{L}})] \leq \mathbb{E}_{\mu} [f^1_M (\|u\|_{\mathcal{L}})].$$

It follows that

$$\mathbb{E}_{\mu_{L,n}} [\|u\|_{\mathcal{L}}^2] = \lim_{M \to \infty} \mathbb{E}_{\mu_{L,n}} [f^1_M (\|u\|_{\mathcal{L}})] \leq \lim_{M \to \infty} \liminf_{n \to \infty} \mathbb{E}_{\mu_{n,n,L}} [f^1_M (\|u\|_{\mathcal{L}})] \leq \lim_{M \to \infty} \mathbb{E}_{\mu} [f^1_M (\|u\|_{\mathcal{L}})] = \mathbb{E}_\mu [\|u\|_{\mathcal{L}}^2].$$

**Step 2.1** (Weak convergence) Now we prove that $\mu_{L,n} \to \tilde{\mu}_L$ weakly on $\mathcal{L}$. Let $f : \mathcal{S} \times \mathcal{L} \to \mathbb{R}$ be bounded and continuous. By dominated convergence and continuity of $f$,

$$\lim_{\varepsilon \to 0} \mathbb{E} \tilde{\mu}_L [f(\mathcal{X}, \text{reg}_{t,x,c}(u))] = \mathbb{E} \tilde{\mu}_L [f(\mathcal{X}, u)],$$
and furthermore since \( f(\mathcal{X},\text{reg}_{t;\varepsilon}(u)) \) is continuous on \( \mathfrak{S} \times \mathcal{L} \), we have

\[
\lim_{N \to \infty} \left[ \mathbb{E}_{\mu_L} [f(\mathcal{X}, u)] - \mathbb{E}_{\mu_{N,L}} [f(\mathcal{X}, u)] \right]
= \lim_{N \to \infty} \lim_{\varepsilon \to 0} \left[ \mathbb{E}_{\mu_{N,L}} [f(\mathcal{X}, \text{reg}_{t;\varepsilon}(u))] - \mathbb{E}_{\mu_{N,L,N}} [f(\mathcal{X}, \text{reg}_{t;\varepsilon}(u))] \right]
= \lim_{N \to \infty} \lim_{\varepsilon \to 0} \left[ \mathbb{E}_{\varepsilon} [f(\mathcal{W}, \mathbb{E} \left[ \text{reg}_{t;\varepsilon}(u) \right], \mathcal{F}_{T_L})] - f(\mathcal{W}, \mathbb{E} \left[ \text{reg}_{t;\varepsilon}(u) \right], \mathcal{F}_{T_{N,L,N}}) \right]
\leq \lim_{N \to \infty} \sup_{\varepsilon} \left( \sup_{\mathfrak{S} \times \mathcal{L}} |f| \right) \lim_{N \to \infty} \mathbb{E}_{\mu_N} \left[ \sum_{n} \|u^n\|^2_{\mathcal{L}} \right] / N^2
\leq 0
\]

**Step 3.** We now put everything together. Since all \( \mu_{L,N} \) are supported on the set \( \{u : \|Z_T(u)\|_{L^4} \leq L\} \), weak convergence and Lemma 7 imply

\[
\lim_{N \to \infty} \mathbb{E}_{\mu_{N,L}} \left[ \|Z_T(u)\|_{L^4}^2 \right] = \mathbb{E}_{\mu_L} \left[ \|Z_T(u)\|_{L^4}^2 \right].
\]

By the Portmanteau lemma,

\[
\lim_{N \to \infty} \inf \mathbb{E}_{\mu_{N,L}} \left[ \|u\|_{\mathcal{L}}^2 \right] \geq \mathbb{E}_{\mu_L} \left[ \|u\|_{\mathcal{L}}^2 \right],
\]

and

\[
\lim_{L \to \infty} \inf \mathbb{E}_{\mu_L} \left[ \|u\|_{\mathcal{L}}^2 \right] \geq \mathbb{E}_{\mu} \left[ \|u\|_{\mathcal{L}}^2 \right]
\]

which together with Step 1.2 imply \( \lim_{L \to \infty} \mathbb{E}_{\mu_L} \left[ \|u\|_{\mathcal{L}}^2 \right] = \mathbb{E}_{\mu} \left[ \|u\|_{\mathcal{L}}^2 \right] \), and by the same argument \( \lim_{L \to \infty} \mathbb{E}_{\mu_L} \left[ \|Z_T(u)\|_{L^4}^2 \right] = \mathbb{E}_{\mu} \left[ \|Z_T(u)\|_{L^4}^2 \right] \). For any \( \delta > 0 \) we can choose a \( \mu_L \) such that

\[
\left| \mathbb{E}_{\mu_L} \left[ \|Z_T(u)\|_{L^4}^2 \right] - \mathbb{E}_{\mu} \left[ \|Z_T(u)\|_{L^4}^2 \right] \right| + \mathbb{E}_{\mu_L} \left[ \|u\|_{\mathcal{L}}^2 \right] - \mathbb{E}_{\mu} \left[ \|u\|_{\mathcal{L}}^2 \right] \leq \delta.
\]

Since then by (42)

\[
\mathbb{E}_{\mu} \left[ \|u\|_{\mathcal{L}}^2 \right] \geq \mathbb{E}_{\mu_{N,L}} \left[ \|u\|_{\mathcal{L}}^2 \right] \geq \mathbb{E}_{\mu} \left[ \|u\|_{\mathcal{L}}^2 \right] - \delta
\]

we can choose \( N \) large enough so that

\[
\left| \mathbb{E}_{\mu_{N,L}} \left[ \|Z_T(u)\|_{L^4}^2 \right] - \mathbb{E}_{\mu} \left[ \|Z_T(u)\|_{L^4}^2 \right] \right| + \mathbb{E}_{\mu_{N,L}} \left[ \|u\|_{\mathcal{L}}^2 \right] - \mathbb{E}_{\mu} \left[ \|u\|_{\mathcal{L}}^2 \right] \leq \delta
\]

which implies the statement of the theorem. \( \square \)

**Lemma 14.** If \( T < \infty \) we have

\[
\mathcal{W}_T(f) = \inf_{\mu \in \mathcal{X}} \hat{F}_T(\mu).
\]

**Proof.** To prove the claim it is enough to show that for any \( \mu \in \mathcal{X} \), for any \( \alpha > 0 \), there exists a sequence \( \mu_n \in \mathcal{X} \) such that \( \limsup_{n \to \infty} \hat{F}_T(\mu_n) \leq \hat{F}_T(\mu) + \alpha \). W.l.o.g we can assume that \( \hat{F}_T(\mu) < \infty \). Observe that, as long as \( T < \infty \) we can also express

\[
\hat{F}_T(\mu) = \mathbb{E}_{\mu} \left[ \frac{1}{|A|} V_T(X_T + Z_T(u)) + \frac{1}{2} \|u\|_{\hat{H}}^2 \right],
\]

and deduce that \( \mathbb{E}_{\mu} \left[ \|u\|_{\hat{H}}^2 \right] < \infty \) since \( V_T \) is bounded below at fixed \( T \). By Lemma 13 there exists a sequence \( \langle \mu_L \rangle \subset \mathcal{X} \), such that \( \mu_L, \|u\|_{\mathcal{L}} \leq L \) almost surely, \( \mu_L \to \mu \) on \( \mathfrak{S} \times \mathcal{L} \) and

\[
\mathbb{E}_{\mu_L} \left[ \|Z_T(u)\|_{L^4}^2 \right] \to \mathbb{E}_{\mu} \left[ \|Z_T(u)\|_{L^4}^2 \right], \quad \mathbb{E}_{\mu_L} \left[ \|u\|_{\mathcal{L}}^2 \right] \to \mathbb{E}_{\mu} \left[ \|u\|_{\mathcal{L}}^2 \right].
\]

First we have to improve the regularity of \( \mu_L \) to get convergence on \( \mathfrak{S} \times \mathcal{L}_w \), but without affecting our control on the moments of \( Z_T \), so let \( \mu_L^\varepsilon := (\text{reg}_{x;\varepsilon}) \cdot \mu_L \) and \( \mu^\varepsilon := (\text{reg}_{x;\varepsilon}) \cdot \mu \). Then

\[
\mathbb{E}_{\mu_L^\varepsilon} \left[ \|Z_T(u)\|_{L^4}^2 \right] \to \mathbb{E}_{\mu^\varepsilon} \left[ \|Z_T(u)\|_{L^4}^2 \right], \quad \mathbb{E}_{\mu_L^\varepsilon} \left[ \|u\|_{\hat{H}}^2 \right] \to \mathbb{E}_{\mu^\varepsilon} \left[ \|u\|_{\hat{H}}^2 \right],
\]

\[
\mathbb{E}_{\mu_L^\varepsilon} \left[ \|Z_T(u)\|_{L^4}^2 \right] \to \mathbb{E}_{\mu^\varepsilon} \left[ \|Z_T(u)\|_{L^4}^2 \right], \quad \mathbb{E}_{\mu_L^\varepsilon} \left[ \|u\|_{\mathcal{L}}^2 \right] \to \mathbb{E}_{\mu^\varepsilon} \left[ \|u\|_{\mathcal{L}}^2 \right].
\]
and $\mu^\varepsilon_L \to \mu^\varepsilon$ on $\mathcal{G} \times \mathcal{H}$. By continuity of $\tilde{F}_T$ and the bound (33), $\tilde{F}_T(\mu^\varepsilon_L) \to \tilde{F}_T(\mu^\varepsilon)$ as $L \to \infty$ and $\tilde{F}_T(\mu^\varepsilon) \to \tilde{F}_T(\mu)$ as $\varepsilon \to 0$. In particular we can find $L$ and $\varepsilon$ such that $|\tilde{F}_T(\mu^\varepsilon_L) - \tilde{F}_T(\mu^\varepsilon)| < \alpha/2$. By Lemma 13 there exists a sequence $(\mu^{\varepsilon,L}_n)_{n,L}$ such that each measure $\mu^{\varepsilon,L}_n$ is supported on $\mathcal{G} \times B(0,L)$ and $\mu^{\varepsilon,L}_n \to \mu^\varepsilon$ weakly on $\mathcal{G} \times \mathcal{H}_w$. Setting $\mu^{\varepsilon,\delta,L}_n := (\text{reg}_{t,x,\delta})_*(\text{reg}_{\varepsilon,x,\delta})_* \mu^{\varepsilon,L}_n$ and $\mu^{\varepsilon,\delta}_L := (\text{reg}_{t,x,\delta})_* \mu^\varepsilon_L$, we use $\mu^{\varepsilon,\delta}_L \to \mu^{\varepsilon,\delta}_L$ on $\mathcal{G} \times \mathcal{H}$ with norm topology. Then, for some $\chi \in C(\mathbb{R}, \mathbb{R})$, $\chi = 1$ on $B(0,1)$, supported on $B(0,2)$ and for any $N \in \mathbb{N}$, $V_T(X_T^1 + Z_T(u))\chi(\|X\|_\mathcal{G}/N), \|u\|_\mathcal{H}$ are continuous bounded functions on the common support of $\mu^{\varepsilon,\delta}_L$ and

$$
\lim_{n \to \infty} \left| \mathbb{E}_{\mu^{\varepsilon,\delta,L}_n} \left[ \frac{1}{\Lambda} V_T(X_T^1 + Z_T(u)) + \frac{1}{2} \|u\|_\mathcal{H}^2 \right] - \mathbb{E}_{\mu^{\varepsilon,\delta}_L} \left[ \frac{1}{\Lambda} V_T(X_T^1 + Z_T(u)) + \frac{1}{2} \|u\|_\mathcal{H}^2 \right] \right| 
\leq \lim_{n \to \infty} \left| \mathbb{E}_{\mu^{\varepsilon,\delta,L}_n} \left[ \frac{1}{\Lambda} \chi(\|X\|_\mathcal{G}/N) V_T(X_T^1 + Z_T(u)) + \frac{1}{2} \|u\|_\mathcal{H}^2 \right] - \mathbb{E}_{\mu^{\varepsilon,\delta}_L} \left[ \frac{1}{\Lambda} \chi(\|X\|_\mathcal{G}/N) V_T(X_T^1 + Z_T(u)) + \frac{1}{2} \|u\|_\mathcal{H}^2 \right] \right| 
+ \sup_n \left| \mathbb{E}_{\mu^{\varepsilon,\delta,L}_n} \left[ \frac{1}{\Lambda} (1 - \chi) (\|X\|_\mathcal{G}/N) V_T(X_T^1 + Z_T(u)) \right] \right| 
+ \sup_n \left| \mathbb{E}_{\mu^{\varepsilon,\delta}_L} \left[ \frac{1}{\Lambda} (1 - \chi) (\|X\|_\mathcal{G}/N) V_T(X_T^1 + Z_T(u)) \right] \right| 
\leq \sup_n \left| \mathbb{E}_{\mu^{\varepsilon,\delta,L}_n} [1(\|X\|_\mathcal{G} \geq N)] V_T(X_T^1 + Z_T(u)) \right| 
+ \left| \mathbb{E}_{\mu^{\varepsilon,\delta}_L} [1(\|X\|_\mathcal{G} \geq N)] V_T(X_T^1 + Z_T(u)) \right| 
\leq 2 \sup_n \left| [\|X\|_\mathcal{G} \geq N] \right| \mathbb{E}_{\mu^{\varepsilon,\delta,L}_n} \|X\|_\mathcal{G}^p + L^8 \right| 
\leq 2 \mathbb{E}_{\mu^{\varepsilon,\delta}_L} \|X\|_\mathcal{G}^p + L^8 \right| 
\leq 2 \frac{\mathbb{E}_{\mu^{\varepsilon,\delta}_L} \|X\|_\mathcal{G}^p + L^8 \right|}{N}
$$

and by dominated convergence (since $\mu^{\varepsilon,\delta}_L$ is supported on $\mathcal{G} \times B(0,L)$) we can find a $\delta$ such that $|\tilde{F}_T(\mu^{\varepsilon,\delta}_L) - \tilde{F}_T(\mu^{\varepsilon,\delta}_L)| < \alpha/2$ which proves the statement. \hfill \Box

The proof of Lemma 14 does not apply when $T = \infty$. However also in this case the functional

$$
\tilde{F}_\infty(\mu) := \mathbb{E}_\mu \left[ \Phi_\infty(X, Z(u), K(u)) + \lambda \|Z_\infty(u)\|_{\mathcal{L}^4}^4 + \frac{1}{2} \|\mathcal{L}^\infty(u)\|_{\mathcal{H}}^2 \right],
$$

has a well defined meaning, so we can investigate the relation between the two variational problems (on $\mathcal{X}$ and on $\tilde{\mathcal{X}}$) when the cutoff is removed. An additional difficulty derives from the fact that approximating the drift $u$ we might destroy the regularity of $\mathcal{L}^\infty(u)$, since now $\mathcal{L}^\infty(u)$ needs to be more regular than $u$, contrary to the finite $T$ case. To resolve this problem we need to be able to smooth out the remainder without destroying the bound on $Z_T(u)$. To do so smoothing $\mathcal{L}^\infty(u)$ directly, and constructing a corresponding new $u$ will not work, since $\mathcal{L}^\infty(u)$ by itself does not give enough control on $u$ and $Z(u)$. However we are still able to prove the following lemma by regularizing an “augmented” version of $\mathcal{L}^\infty(u)$. 

Lemma 15. There exists a family of continuous functions \(\rem\) : \(\mathcal{L} \mapsto \mathcal{L}\), which are also continuous \(\mathcal{L}_w \mapsto \mathcal{L}_w\), such that for any \(T \in [0, \infty)\),
\[
\|\rem(u)\|_\mathcal{L} \leq \|X\|_\mathcal{E} + \|u\|_\mathcal{L},
\]
\[
\|Z_T(\rem(u))\|_{L^4} \leq \|X\|_\mathcal{E} + \|Z_T(u)\|_{L^4},
\]
\[
\|\|\inf (\rem(X, u))\|_\mathcal{H} \| \leq C (1 + \|X\|_\mathcal{E})^4 + \|Z_\infty(u)\|_{L^4}^4 + \|u\|_\mathcal{L}^2,
\]
and \(\|\|\inf (\rem(X, u))\|_\mathcal{H}\) depends continuously on \((X, u) \in \mathcal{G} \times \mathcal{L}\). Furthermore
\[
\rem(X, u) \rightarrow u \text{ in } \mathcal{L},
\]
and if \(\|\inf (u)\| \in \mathcal{H}\)
\[
\|\inf (\rem(X, u))\| \rightarrow \|\inf (u)\| \text{ in } \mathcal{H} \text{ as } \varepsilon \to 0.
\]

Proof. Let \(X^2 = U_{\mathcal{E}}X^2 + U_{\mathcal{E}2}X^2\) be the decomposition introduced in Section 5 and observe that for any \(c > 0\) we can easily modify it to ensure that \(\|U_{\mathcal{E}}X^2\|_{\mathcal{E}^{-1-\varepsilon}} < c\), almost surely for any \(\mu \in \mathcal{X}\) and for any \(1 < p < \infty\), \(\mathbb{E}_\mu [\|U_{\mathcal{E}}X^2\|^p_{\mathcal{E}^{-1-\varepsilon}}] \leq C\) where \(C\) depends on \(|\mu|, \kappa, c, p\). Now set \(\tilde{I}_t(u) = -\lambda J_s(U_{\mathcal{E}}X^2_\mathcal{E}) + Z^\theta_s(u) + \|\inf (u)\|_{L^2}\). Then \(u\) satisfies
\[
u_s = -\lambda X_s^{(3)} - \lambda J_s(U_{\mathcal{E}}X^2_\mathcal{E}) + Z^\theta_s(u) + \tilde{I}_s(u).
\]
From this equation we can see that, like in Section 5,
\[
\|u\|_{L^2} \leq \lambda \|X_s^{(3)}\|_\mathcal{L} + \|\tilde{I}_s(u)\|_{L^2},
\]
and choosing, \(c\) small enough we get
\[
\|u\|_\mathcal{L} \leq \lambda \|X_s^{(3)}\|_\mathcal{L} + \|\tilde{I}_s(u)\|_{L^2}.
\]
And similarly we observe that
\[
Z_T(u) = -\lambda X_T^{(3)} - \lambda \int_0^T J_s(U_{\mathcal{E}}X^2_\mathcal{E}) + Z^\theta_s(u)ds + Z_T(\tilde{I}(u)),
\]
so again with \(c\) small enough and since \(Z^\theta_s = \theta_sZ_T\) for \(s \leq T\):
\[
\|Z_T(u)\|_{L^4} \leq \lambda \|X_T^{(3)}\|_{L^4} + \|Z_T(\tilde{I}(u))\|_{L^4}.
\]
Conversely, it is not hard to see that we have the inequalities
\[
\|Z_T(\tilde{I}(u))\|_{L^4} \leq \lambda \|X_T^{(3)}\|_{L^4} + \|Z_T(u)\|_{L^4},
\]
and
\[
\|\tilde{I}(u)\|_\mathcal{L} \leq \lambda \|X_s^{(3)}\|_\mathcal{L} + \|u\|_\mathcal{L}.
\]
Clearly the map \((X, u) \mapsto (\tilde{X}, \tilde{I}(u))\) is continuous as a map \(\mathcal{G} \times \mathcal{L} \rightarrow \mathcal{L}\) and using Lemma 5 also as a map \(\mathcal{G} \times \mathcal{L}_w \rightarrow \mathcal{G} \times \mathcal{L}_w\), and the inverse is clearly continuous \(\mathcal{G} \times \mathcal{L} \rightarrow \mathcal{G} \times \mathcal{L}\). We now show that it is also continuous as a map \(\mathcal{G} \times \mathcal{L}_w \rightarrow \mathcal{G} \times \mathcal{L}_w\). Assume that \(\tilde{I}(u) \rightarrow I(u)\) weakly, since then \(\|\tilde{I}(u)\|_{L^2}\) bounded, this implies by (14) that also \(\|u\|^2_{\mathcal{L}}\) is bounded, and so we can select a weakly convergent subsequence, converging to \(u^*\). Then \(u^*\) solves the equation
\[
u_s^* = -\lambda X_s^{(3)} - \lambda J_s(U_{\mathcal{E}}X^2_\mathcal{E}) + Z^\theta_s(u^*) + \tilde{I}_s(u),
\]
(which can be seen for example by testing with some \(h \in \mathcal{L}^*\)) which implies that \(u^* = u\) (e.g. by Gronwall). Now define \(\rem\) (\(u\)) to be the solution to the equation
\[
\rem(u) = -\lambda X_s^{(3)} - \lambda J_s(U_{\mathcal{E}}X^2_\mathcal{E}) + Z^\theta_s(\rem(u)) + \reg_{\mathcal{L}, \varepsilon}(\tilde{I}_s(u)).
\]
Then by the properties discussed above \(u \mapsto \rem(u)\) is continuous in both the weak and the norm topology, we also have from (14) and (17) that
\[
\|\rem(u)\|_{L^2} \leq \lambda \|X_s^{(3)}\|_\mathcal{L} + \|u\|_\mathcal{L},
\]
from \([44]\), we have
\[
\|Z_T(\text{rem}_e(u))\|_{L^4} \lesssim \lambda \|X_T^{(3)}\|_{L^4} + \|Z_T(u)\|_{L^4},
\]
and by definition of \(\text{rem}_e(u)\)
\[
\|\tilde{l}(\text{rem}_e(u))\|_{\mathcal{H}} = \|\text{reg}_{e,\tilde{l}}(\tilde{l}(u))\|_{\mathcal{H}}
\]
\[
\lesssim \varepsilon \lambda \|X^{(3)}\|_{\mathcal{L}} + \|u\|_{\mathcal{L}}.
\]
(48)

Now observe that
\[
\|l^{\infty}(\text{rem}_e(u))\|_{\mathcal{H}}^2 \lesssim \|s \to \lambda \int \frac{1}{(s)^{1+\kappa}} \|\mathcal{U}_s X^{2}_s \times Z_s(\text{rem}_e(u))\|_{\mathcal{H}} + \|\tilde{l}(\text{rem}_e(u))\|_{\mathcal{H}}^2
\]
\[
\lesssim \varepsilon \lambda \int \frac{1}{(s)^{1+\kappa}} \|\mathcal{U}_s X^{2}_s \times Z_s(\text{rem}_e(u))\|_{\mathcal{H}}^2 + \|\tilde{l}(\text{rem}_e(u))\|_{\mathcal{H}}^2
\]
\[
\lesssim \lambda (1 + \|X\|_{\mathcal{E}})^4 + \|Z^{\infty}(\text{rem}_e(u))\|_{\mathcal{H}}^2 + \|u\|_{\mathcal{L}}^2
\]
\[
\lesssim \lambda (1 + \|X\|_{\mathcal{E}})^4 + \|Z^{\infty}(u)\|_{\mathcal{L}}^2 + \|u\|_{\mathcal{L}}^2.
\]

Observing that also \(\|\lambda \int (\mathcal{U}_s X^{2}_s \times Z_s(\text{rem}_e(u)))\|_{\mathcal{H}}\) depends continuously on \((X, u)\) (both in the weak and strong topology on \(L\)) gives the statement.

\(\square\)

**Lemma 16.** For any \(\mu \in \mathcal{X}\) such that \(\tilde{F}_\infty(\mu) < \infty\) there exists a sequence of measures \(\mu_L \in \mathcal{X}\) such that

1. For any \(p < \infty\),
\[
\mathbb{E}_{\mu_L}[\|u\|_{L^p}^p] \leqslant \mathbb{E}_{\mu}[\|l^{\infty}(u)\|_{\mathcal{H}}^p] < \infty,
\]
(49)

2. \(\mu_L \to \mu\) weakly on \(\mathcal{G} \times \mathcal{L}\) and \(\text{Law}_{\mu_L}(l^{\infty}(u)) \to \text{Law}_\mu(l^{\infty}(u))\) weakly on \(\mathcal{H}\).

3. \(\lim_{L \to \infty} \tilde{F}_\infty(\mu_L) = \tilde{F}_\infty(\mu)\).

4. For any \(\mu_L\) there exists a sequence \(\mu_{n,L} \in \mathcal{X}\) such that
\[
\sup_n \mathbb{E}_{\mu_{n,L}}[\|u\|_{L^p}^p] < \infty,
\]
(50)

\(\mu_{n,L} \to \mu_L\) weakly on \(\mathcal{G} \times \mathcal{L}_w\) and \(\text{Law}_{\mu_{n,L}}(l^{\infty}(u)) \to \text{Law}_\mu(l^{\infty}(u))\) weakly on \(\mathcal{H}_w\).

**Proof.** By Lemma \([13]\) there exists a sequence \(\mu_L \to \mu\) weakly on \(\mathcal{G} \times \mathcal{L}\) such that
\[
\mathbb{E}_{\mu_L}[\|Z_T(u)\|_{L^4}] \to \mathbb{E}_{\mu}[\|Z_T(u)\|_{L^4}],
\]
\[
\mathbb{E}_{\mu_L}[\|u\|_{L^2}] \to \mathbb{E}_{\mu}[\|u\|_{L^2}],
\]
and \(\mu_L\) is supported on \(\mathcal{G} \times B(0, \tilde{L}) \subset \mathcal{G} \times \mathcal{L}\). Now set \(\mu_L^\varepsilon := (\text{rem}_e)_* \mu_L\). Then \(\mu_L^\varepsilon \to \mu\) := \(\text{rem}_e\)_* \mu on \(\mathcal{G} \times \mathcal{L}\) and by the bounds from Lemma \([15]\) also \(\mathbb{E}_{\mu_L^\varepsilon}[\|Z_T(u)\|_{L^4}] \to \mathbb{E}_{\mu}^{\varepsilon}[\|Z_T(u)\|_{L^4}]
\]
and \(\mathbb{E}_{\mu_L^\varepsilon}[\|l^{\infty}(u)\|_{\mathcal{H}}^p] \to \mathbb{E}_{\mu}^{\varepsilon}[\|l^{\infty}(u)\|_{\mathcal{H}}^p]\). The bounds from Lemma \([15]\) imply also \(\mathbb{E}_{\mu}^{\varepsilon}[\|Z_T(u)\|_{L^4}] \to \mathbb{E}_{\mu}^{\varepsilon}[\|Z_T(u)\|_{L^4}]
\]
\(\mathbb{E}_{\mu}^{\varepsilon}[\|l^{\infty}(u)\|_{\mathcal{H}}^p] \to \mathbb{E}_{\mu}^{\varepsilon}[\|l^{\infty}(u)\|_{\mathcal{H}}^p]\), and furthermore
\[
\mathbb{E}_{\mu_L^\varepsilon}[\|u\|_{L^p}] \lesssim \mathbb{E}_{\mu_L^\varepsilon}[\|X\|_{\mathcal{E}} + \|u\|_{L^p}]
\]
\[
\lesssim \mathbb{E}_{\mu_L^\varepsilon}[\|X\|_{\mathcal{E}} + \|u\|_{L^p}] + \tilde{L}^p,
\]
and similarly
\[
\mathbb{E}_{\mu_L^\varepsilon}[\|l^{\infty}(u)\|_{\mathcal{H}}^p] \lesssim \mathbb{E}_{\mu_L^\varepsilon}[\|X\|_{\mathcal{E}} + \|u\|_{L^p}]
\]
\[
\lesssim \mathbb{E}_{\mu_L^\varepsilon}[\|X\|_{\mathcal{E}} + \|u\|_{L^p}] + \tilde{L}^p,
\]
and by continuity of \(\tilde{F}_\infty\) and \([45]\) we are also able to deduce that we can find \(\varepsilon\) small enough that \(\tilde{F}_\infty(\mu^\varepsilon) - \tilde{F}_\infty(\mu)\) is smaller than \(1/2L\) and \(\tilde{F}_\infty(\mu^\varepsilon) - \tilde{F}_\infty(\mu^\varepsilon)\) is also smaller than \(1/2L\). Choosing \(\mu_L = \mu_L^\varepsilon\) we obtain the first three points of the Lemma. For the fourth point recall that from Lemma \([13]\) we have sequences \(\mu_{n,L} \to \mu_L\) weakly on \(\mathcal{G} \times \mathcal{L}_w\), and \(\mu_{n,L} \in \mathcal{X}\), which have support in \(\mathcal{G} \times B(0, \tilde{L})\) and since \(\text{rem}_e\) is continuous on \(\mathcal{G} \times \mathcal{L}_w\) we obtain the desired sequence.
Lemma 17. If $T = \infty$ we have
$$
\inf_{\mu \in \mathcal{X}} \bar{F}_{\infty}(\mu) = \inf_{\mu \in \mathcal{X}} F_{\infty}(\mu).
$$

Proof. One can now proceed very similarly to the proof of Lemma 13. Let $\mu \in \mathcal{X}$ such that $\bar{F}_{\infty}(\mu) < \infty$. By Lemma 13 for any $L, \mu \in \mathcal{X}$, there exists a $\mu_L$ such that $|\bar{F}_{\infty}(\mu) - \bar{F}_{\infty}(\mu_L)| < 1/L$, and a sequence $(\mu_{n,L})$ such that $\mu_{n,L} \to \mu_L$ weakly on $\mathcal{S} \times \mathcal{L}_w$, and such that (50) is satisfied. Define $\mu_{n,L} := \text{Law}(\mathcal{X}, \mathcal{R}_n, \mu_n(u))$, and observe that now $\mu_{n,L} \to \mu_L$ on $\mathcal{S} \times \mathcal{L}$, Law $\mu_{n,L}(\mathcal{X}, T^\infty(u)) \to \text{Law} \mu_{\infty,L}(\mathcal{X}, T^\infty(u))$ on $\mathcal{S} \times \mathcal{H}$, and that we have $\sup_n E_{\mu_{n,L}}[\|u\|^2] + E_{\mu_{n,L}}[\|T^\infty(u)\|^2_{\mathcal{H}}] < \infty$. Then for some $\chi \in C(\mathcal{R}, \mathcal{R})$, $\chi = 1$ on $B(0, 1)$ supported on $B(0, 2)$, for any $N \in \mathbb{N}$, the function
$$
\chi \left( \frac{\|\mathcal{X}\|_\mathcal{S} + \|u\|_\mathcal{L} + \|T^\infty(u)\|_\mathcal{H}}{N} \right) \left( \Phi_{\infty}(\mathcal{X}, Z(u), K(u)) + \lambda \|Z_{\infty}(u)\|^4_{L^4} + \frac{1}{2} \|T^\infty(u)\|^2_{\mathcal{H}} \right)
$$
is bounded and continuous on $\mathcal{S} \times \mathcal{L}$, and so by weak convergence
$$
\lim_{n \to \infty} \frac{\bar{F}_{\infty}(\mu_{n,L}) - \bar{F}_{\infty}(\mu_{\infty,L})}{\mu_{n,L}} 
\leq \lim_{n \to \infty} E_{\mu_{n,L}} \left[ \chi(\mathcal{X}, u) \left( \Phi_{\infty}(\mathcal{X}, Z(u), K(u)) + \lambda \|Z_{\infty}(u)\|^4_{L^4} + \frac{1}{2} \|T^\infty(u)\|^2_{\mathcal{H}} \right) \right] -
\sup_{n} E_{\mu_{n,L}} \left[ (1 - \chi(\mathcal{X}, u)) \left( \Phi_{\infty}(\mathcal{X}, Z(u), K(u)) + \lambda \|Z_{\infty}(u)\|^4_{L^4} + \frac{1}{2} \|T^\infty(u)\|^2_{\mathcal{H}} \right) \right] 
\leq 2 \sup_{n} E_{\mu_{n,L}} \left[ \mathbb{1}_{\{\|\mathcal{X}\|_\mathcal{S} + \|u\|_\mathcal{L} + \|T^\infty(u)\|_\mathcal{H} > N\}} \Phi_{\infty}(\mathcal{X}, Z(u), K(u)) + \lambda \|Z_{\infty}(u)\|^4_{L^4} + \frac{1}{2} \|T^\infty(u)\|^2_{\mathcal{H}} \right] 
\leq \sup_{n} \left[ \frac{1}{N} E_{\mu_{n,L}} \left[ \|\mathcal{X}\|_\mathcal{S} + \|u\|_\mathcal{L} + \|T^\infty(u)\|_\mathcal{H} \right] E_{\mu_{n,L}} \left[ \|\mathcal{X}\|_\mathcal{S} + \|u\|_\mathcal{L} + \|T^\infty(u)\|_\mathcal{H} \right] \right] 
\to 0 \text{ as } N \to \infty
$$
As we can find $\varepsilon, \delta$ such that $|\bar{F}_{\infty}(\mu_{n,L}) - \bar{F}_{\infty}(\mu_L)| < 1/L$ we can conclude. \hfill \Box

Finally we can state the key result of this section.

Theorem 6. The family $\{\bar{F}_T\}$ $\Gamma$-converges to $\bar{F}_\infty$ on $\mathcal{X}$. Therefore
$$
\lim_{T} W_T(f) = \lim_{T} \inf_{\mu \in \mathcal{X}} \bar{F}_T(\mu) = \inf_{\mu \in \mathcal{X}} \bar{F}_\infty(\mu) = W(f).
$$

Proof. In order to establish $\Gamma$-convergence consider a sequence $\mu^T \to \mu$ in $\mathcal{X}$. We need to prove that $\liminf_{T \to \infty} \bar{F}_T(\mu^T) \geq \bar{F}_\infty(\mu)$. It is enough to prove this statement for a subsequence, the full statement follows from the fact that every sequence has a subsequence satisfying the inequality. Take a subsequence (not relabeled) such that
$$
\sup_{T} \bar{F}_T(\mu^T) < \infty.
$$
If there is no such subsequence there is nothing to prove. Otherwise tightness for the subsequence follows like in the proof of equicoercivity. Then invoking the Skorokhod representation theorem
of (35) we can extract a subsequence (again, not relabeled) and find random variables \((\tilde{X}_T, \tilde{u}^T)_T\) and \((\tilde{X}, \tilde{u})\) on some probability space \((\Omega, \mathbb{P})\) such that \(\text{Law}_{\tilde{\mathbb{P}}}(\tilde{X}_T, \tilde{u}^T) = \mu^T\), \(\text{Law}_{\tilde{\mathbb{P}}}(\tilde{X}, \tilde{u}) = \mu\) and almost surely \(\tilde{X}_T \to \tilde{X}\) in \(\mathcal{S}\), \(\tilde{u}^T \to \tilde{u}\) in \(L_w\). Note that \(l_t := l^T(\tilde{X}_T, \tilde{u}^T) \to l := l^\infty(\tilde{X}, \tilde{u})\) in \(L_w\) and using (51) we deduce that the almost sure convergence \(l^T \to l\) in \(\mathcal{H}_w\), maybe modulo taking another subsequence, again not relabeled. Note that, by our analytic estimates (which hold pointwise on the probability space) we have

\[
\Phi_T(\tilde{X}_T, Z(\tilde{u}^T), K(\tilde{u}^T)) + \lambda \|Z_T(\tilde{u}^T)\|^4_{L^4} + \frac{1}{2} \|l^T(\tilde{u}^T)\|^2_{\mathcal{H}} + H(\tilde{X}_T) \geq 0,
\]

for some \(L^1(\tilde{\mathbb{P}})\) random variable \(H(\tilde{X}_T)\) such that \(\mathbb{E}_{\tilde{\mathbb{P}}}[H(\tilde{X}_T)] = \mathbb{E}[H(W)]\). Fatou’s lemma and Lemma 8 then give

\[
\liminf_{T \to \infty} F_T(\mu^T) = \liminf_{T \to \infty} \mathbb{E}_{\tilde{\mathbb{P}}} \left[ \Phi_T(\tilde{X}_T, Z(\tilde{u}^T), K(\tilde{u}^T)) + \lambda \|Z_T(\tilde{u}^T)\|^4_{L^4} + \frac{1}{2} \|l^T(\tilde{u}^T)\|^2_{\mathcal{H}} \right]
\]

\[
= \liminf_{T \to \infty} \mathbb{E}_{\tilde{\mathbb{P}}} \left[ \Phi_T(\tilde{X}_T, Z(\tilde{u}^T), K(\tilde{u}^T)) + \lambda \|Z_T(\tilde{u}^T)\|^4_{L^4} + \frac{1}{2} \|l^T(\tilde{u}^T)\|^2_{\mathcal{H}} + H(\tilde{X}_T) \right] - \mathbb{E}[H(W)]
\]

\[
\geq \mathbb{E}_{\tilde{\mathbb{P}}} \left[ \Phi_T(\tilde{X}_T, Z(\tilde{u}^T), K(\tilde{u}^T)) + \lambda \|Z_T(\tilde{u}^T)\|^4_{L^4} + \frac{1}{2} \|l^T(\tilde{u}^T)\|^2_{\mathcal{H}} + H(\tilde{X}_T) \right] - \mathbb{E}[H(W)]
\]

\[
\geq \mathbb{E}_{\tilde{\mathbb{P}}} \left[ \Phi_\infty(\tilde{X}, Z(\tilde{u}), K(\tilde{u}^T)) + \|Z_\infty(\tilde{u})\|^4_{L^4} + \frac{1}{2} \|l^\infty(\tilde{u})\|^2_{\mathcal{H}} \right] = F_\infty(\mu),
\]

which is the \(\Gamma\)-liminf inequality. Now all that remains is constructing a recovery sequence, for this we can again assume w.l.o.g that \(F_\infty(\mu) < \infty\). From Lemma 16 there is \(\mu_L\) such that \(|F_\infty(\mu) - F_\infty(\mu_L)| < \frac{1}{L}\) and (19) is satisfied. Then choosing \(\mu_L = \text{Law}_{\mu_L}(X, 1_{\{t \leq T\}}u_t)\) we obtain that \(l^T(1_{\{t \leq T\}}u_t) = 1_{\{t \leq T\}}f^T(u)\), so \(\|l^T(1_{\{t \leq T\}}u_t)\|_{\mathcal{H}} \leq \|f^T(u)\|_{\mathcal{H}}\), and \(\|Z_T(1_{\{t \leq T\}}u_t)\|^4_{L^4} = \|Z_T(u)\|^4 \leq \|u\|^4\), which is integrable by (19). By dominated convergence and Lemma 8 we obtain \(\lim_{T \to \infty} F_T(\mu_L^T) = F_\infty(\mu_L)\). Extracting a suitable diagonal sequence gives the recovery sequence. \(\square\)

7. Analytic estimates

In this section we collect a series of analytic estimate which together allow to establish the pointwise bounds (37) and (38) and the continuity required for Lemma 8. First of all note that

\[
\|K_t\|^2_{H^{1-\epsilon}} \lesssim \lambda^2 f_0^t \frac{1}{(1+|t|)^{\frac{1}{2}}} \|W^2_s\|^2_{B_{1,\infty}} ds \|Z_t\|^2_{L^4} + f_0^t \|l_s\|^2_{L^2} ds \lesssim \lambda^3 \left( f_0^t \frac{1}{(1+|t|)^{\frac{1}{2}}} \|W^2_s\|^2_{B_{1,\infty}} ds \right)^2 + \lambda \|Z_t\|^4_{L^4} + f_0^t \|l_s\|^2_{L^2} ds,
\]

which implies that quadratic functions of the norm \(\|K_t\|_{H^{1-\epsilon}}\) with small coefficient can always be controlled, uniformly in \([0, \infty]\), by the coercive term

\[
\lambda \int Z_t^4 + \frac{1}{2} \int_0^\infty \|l_s\|^2_{L^2} ds.
\]

**Lemma 18.** For any small \(\epsilon > 0\) there exists \(\delta > 0\) such that

\[
|\gamma_T^{(1)}| \leq C(\epsilon, \delta) E(\lambda) Q_T + \epsilon \|K_T\|^2_{H^{1-\epsilon}} + \epsilon \lambda \|Z_T\|^4_{L^4}.
\]

**Proof.** By Proposition 7

\[
\lambda \left| f(W^2_T \circ K_T) - K_T \right| \lesssim \lambda \|W^2_T\|^2_{B_{1,\infty}} \|K_T\|^2_{B^{2/3}_{7/8}} \lesssim \lambda \|W^2_T\|^2_{B_{1,\infty}} \|K_T\|^2_{B^{2/3}_{7/8}}
\]

\[
\lesssim \lambda \|W^2_T\|^2_{B_{1,\infty}} \|K_T\|^2_{H^{7/8}} \|K_T\|^2_{B^{4/7}_{7/8}} \lesssim \lambda \|W^2_T\|^2_{B_{1,\infty}} + \|K_T\|^2_{H^{7/8}} + \lambda \|K_T\|^4_{L^4},
\]

\[
\lambda \left| f(W^2_T \circ K_T) - f(W^2_T) \circ K_T \right| \lesssim \lambda \|W^2_T\|^2_{B_{1,\infty}} \|K_T\|^2_{H^{7/8}} \|K_T\|^2_{B^{4/7}_{7/8}} \lesssim \lambda \|W^2_T\|^2_{B_{1,\infty}} + \|K_T\|^2_{H^{7/8}} + \lambda \|K_T\|^4_{L^4}.
\]
By Proposition \[1\]
\[
\left| \lambda \int (\mathbb{W}^2_T \prec K_T) K_T \right| \lesssim \lambda \left\| \mathbb{W}_T^2 \right\|_{B_{1,\infty}}^2 \left\| K_T \right\|_{B^9_{7/3,2}}^2
\]
which is estimated in the same way and finally,
\[
\left| \lambda^2 \int (\mathbb{W}_T^2 \prec \mathbb{W}_T^{[3]}) K_T \right| \lesssim \lambda^2 \left\| \mathbb{W}_T^2 \right\|_{B^{-1,1/2}_{4,4}} \left\| \mathbb{W}_T^{[3]} \right\|_{B^{-1/2-\delta/2}_{4,4}} \left\| K_T \right\|_{H^{1/2+\delta}}
\]
\[
\leq C(\delta) \lambda^2 \left( \left\| \mathbb{W}_T^2 \right\|_{B^{-1,1/2}_{4,4}} \left\| \mathbb{W}_T^{[3]} \right\|_{B^{-1/2-\delta/2}_{4,4}} \right)^2 + \delta \left\| K_T \right\|_{H^{1/2+\delta}}^2.
\]
\[
\square
\]

**Lemma 19.** For any small \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that
\[
\left| Y_T^{(2)} \right| \leq T^{-\delta} (C(\varepsilon, \delta) E(\lambda) Q_T + \varepsilon \left\| K \right\|_{H^{1-\delta}} + \varepsilon \lambda \left\| Z_T \right\|_{L^4})
\]

**Proof.** Using the spectral support properties of the various terms we observe that
\[
\left\| \mathbb{W}_T^2 \right\|_{B^{-1+\delta}_{p,q}} \lesssim \left\| \mathbb{W}_T^2 \right\|_{B^{-1+\delta}_{p,q}} T^{2\delta},
\]
and
\[
T^{2\delta} \left( \left\| Z_T - Z_T^p \right\|_{L^2} \lesssim \left\| Z_T - Z_T^p \right\|_{H^{2\delta}} \lesssim \left\| Z_T - Z_T^p \right\|_{H^{1/2-\delta}} \lesssim \left\| Z_T - Z_T^p \right\|_{L^2}^{1/2-\delta} \lesssim \left\| Z_T \right\|_{H^{1/2-\delta}}^{1/2-\delta},
\]
where we used also interpolation and the \( L^2 \) bound \( \left\| Z_T \right\|_{L^2} \lesssim \left\| Z_T \right\|_{L^2} \). We recall also that (54)
\[
Z_T = K_T + \lambda \mathbb{W}_T^{[3]}.
\]
Therefore we estimate as follows
\[
\lambda \int (\mathbb{W}_T^2 \prec (Z_T - Z_T^p)) K_T = \lambda \int (\mathbb{W}_T^2 \prec (K_T - K_T^p)) K_T + \lambda^2 \int (\mathbb{W}_T^2 \prec (\mathbb{W}_T^{[3]} - \mathbb{W}_T^{[3]}) K_T.
\]
For the second term we can estimate
\[
\lambda^2 \int (\mathbb{W}_T^2 \prec (\mathbb{W}_T^{[3]} - \mathbb{W}_T^{[3]}) K_T \lesssim \lambda^2 \left\| \mathbb{W}_T^2 \right\|_{B^{-1+\delta}_{4,4}} \left\| \mathbb{W}_T^{[3]} \right\|_{B^{0}_{4,2}} \left\| K_T \right\|_{H^{1-\delta}}
\]
\[
\leq \lambda^2 \left\| W_T^2 \right\|_{B^{-1+\delta}_{4,4}} \left\| W_T^{[3]} \right\|_{B^{0}_{4,2}} \left\| K_T \right\|_{H^{1-\delta}}
\]
for the first term we get
\[
\lambda \int (\mathbb{W}_T^2 \prec (K_T - K_T^p) K_T \lesssim \lambda \left\| W_T^2 \right\|_{B^{-1/2-\delta}_{7,\infty}} \left\| K_T \right\|_{B^{2}_{7/3,2}} \left\| K_T \right\|_{B^{2+\delta}_{7/3,2}}
\]
\[
\leq \lambda \left\| W_T^2 \right\|_{B^{-1/2-\delta}_{7,\infty}} T^{1/2} T^{-1/2-\delta} \left\| K_T \right\|_{B^{2+\delta}_{7/3,2}} \left\| K_T \right\|_{B^{2}_{7/3,2}}
\]
\[
\leq \lambda T^{-\delta} \left\| W_T^2 \right\|_{B^{-1/2-\delta}_{7,\infty}} \left\| K_T \right\|_{B^{2}_{7/3,2}} \left\| K_T \right\|_{B^{2+\delta}_{7/3,2}}
\]
which we can again estimate like in Lemma \[18\]
\[
\square
\]

**Lemma 20.** For any small \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that
\[
\left| Y_T^{(3)} \right| \leq C(\varepsilon, \delta) E(\lambda) Q_T + \varepsilon \sup_{0 \leq t \leq T} \left\| K_t \right\|_{H^{1-\delta}}^2 + \varepsilon \lambda \left\| Z_T \right\|_{L^4}^4.
\]
We can estimate the first two terms by using Lemma 3. We establish that
\[
\|Z_t\|_{B_p^1} = \|\theta\left(\frac{(D)}{t}\right)\frac{(D)}{t^2}Z_t\|_{B_p^1} \leq C\|\theta\left(\frac{(D)}{t}\right)\frac{(D)}{t^{2+\theta}}Z_t\|_{B_p^1}.
\]
By Proposition 4 for any \(\varepsilon > 0\) there exists \(\delta > 0\) such that
\[
\lambda j_0^T \int (W_t^2 > Z_t^2) K_t dt \leq C j_0^T \|W_t^2\|_{B_6^0} \|Z_t\|_{B_6^2} \|K_t\|_{H^{1-\delta}} dt.
\]
While
\[
\lambda^{3/2} j_0^T \int_{0}^{T} \|K_t\|_{H^{1-\delta}} \|W_t^2\|_{B_6^0} \|W_t^2\|_{B_6^0} \|K_t\|_{H^{1-\delta}} \leq C \lambda^{11/3} j_0^T \int_{0}^{T} \|W_t^2\|_{B_6^0} \|W_t^2\|_{B_6^0} \|W_t^2\|_{B_6^0} \|K_t\|_{H^{1-\delta}} + \lambda \|Z_t\|_{L^4}.
\]

Lemma 21. For any small \(\varepsilon > 0\) there exists \(\delta > 0\) such that
\[
|T_T^{(4)}| \leq C(\varepsilon, \delta)E(\lambda)Q_T + \varepsilon\|K_T\|_{H^{1-\delta}}^2 + \varepsilon\lambda \|Z_T\|_{L^4}.
\]
Proof. Using Lemma 3 we establish that
\[
\lambda j_0^T W_T K_T^2 \leq E(\lambda) \|W_T\|_{H^{1-\delta}}^2 + \delta(\|K_T\|_{H^{1-\delta}}^2 + \lambda \|K_T\|_{L^4}^2).
\]
Next, we can write,
\[
\lambda^3 \int W_T(W_T^{[3]} K_T) \leq \lambda^3 \int W_T(W_T^{[3]} > W_T^{[3]} K_T) + \lambda^3 \|W_T\|_{B_6^0} \|W_T^{[3]}\|_{B_6^0} \|K_T\|_{H^{1-\delta}}^2.
\]
which can be easily estimated by Young’s inequality. Decomposing
\[
W_T(W_T^{[3]} > W_T^{[3]}) = W_T > W_T^{[3]} + W_T < W_T^{[3]} + W_T \circ (W_T^{[3]} > W_T^{[3]}),
\]
We can estimate the first two terms by
\[
\lambda^3 \int W_T > (W_T^{[3]} > W_T^{[3]}) K_T \leq \lambda^3 \|W_T\|_{B_6^0} \|W_T^{[3]}\|_{B_6^0} \|K_T\|_{H^{1-\delta}},
\]
and
\[
\lambda^3 \int W_T < (W_T^{[3]} > W_T^{[3]}) K_T \leq \lambda^3 \|W_T\|_{B_6^0} \|W_T^{[3]}\|_{B_6^0} \|K_T\|_{H^{1-\delta}}.
\]
Young’s inequality gives then the appropriate result. For the final term we use Proposition 18 to get
\[ \lambda^3 \left| \int W_T \circ (W_T^{[3]} \circ W_T^{[3]}) K_T \right| \lesssim \lambda^3 \left| \int W_T \circ (W_T^{[3]} \circ W_T^{[3]}) K_T \right| + \lambda^3 \|W_T\|_{B_2^3,1/2-\delta} \|W_T^{[3]}\|^2_{B_4^{1/2-\delta}} \|K_T\|_{H^{1-\delta}} \]
\[ \lesssim \lambda^3 \|W_T^{[3]}\|_{B_4^3,1/2-\delta} \|W_T^{[3]}\|_{B_4^{1/2-\delta}} \|K_T\|_{H^{1-\delta}} + \lambda^3 \|W_T\|_{B_2^3,1/2-\delta} \|W_T^{[3]}\|^2_{B_4^{1/2-\delta}} \|K_T\|_{H^{1-\delta}} \]
\[ \lesssim \lambda^3 C(\delta, \varepsilon) \|W_T^{[3]}\|_{B_4^3,1/2-\delta} \|W_T^{[3]}\|_{B_4^{1/2-\delta}} + \|W_T\|_{B_2^3,1/2-\delta} \|W_T^{[3]}\|^2_{B_4^{1/2-\delta}} \varepsilon + \varepsilon \|K_T\|^2_{H^{1-\delta}}. \]

For the last term we estimate
\[ \left| \int \lambda^2 (W_T \circ W_T^{[3]}) K_T \right| \lesssim \lambda^2 \|W_T \circ W_T^{[3]}\|_{B_3^5,1/2-\delta} \|K_T\|^2_{B_3^{1/2-\delta}}, \]
which can be estimated like in Lemma 18 after we observe that
\[ \|W_T \circ W_T^{[3]}\|_{B_3^{1/2-\delta}} \leq \|W_T\|_{B_3^{1/2-\delta}} + \|W_T \circ W_T^{[3]}\|_{B_3^{1/2-\delta}} + \|W_T\|_{B_3^{1/2-\delta}} \]
\[ \lesssim \|W_T\|_{B_3^{1/2-\delta}} \|W_T^{[3]}\|_{B_3^{1/2-\delta}} + \|W_T^{[3]}\|_{B_3^{1/2-\delta}} \]
and use Lemma 24 to bound \( W_T^{[3]} \). □

**Lemma 22.** Assume that
\[ \sup_T \frac{|\gamma_T|}{(T)^{1/4}} + \int_0^T \frac{\|\gamma_t\|}{(t)^{5/4}} < \infty. \]
Then for any small \( \varepsilon > 0 \) there exists a \( \delta > 0 \) such that
\[ |Y^{(5)}_T| \leq C_2 E(\lambda) \left[ \frac{|\gamma_T|}{(T)^{1/4}} + \int_0^T \frac{\|\gamma_t\|}{(t)^{5/4}} \right]^2 + \varepsilon \|Z_T\|^2_{H^{1/2-\delta}} + \varepsilon \lambda \|Z_T\|^4_{L^4}. \]

**Proof.** We can estimate
\[ \lambda^2 \gamma_T \int Z_T^2 (Z_T - Z_T^2) \leq \lambda^2 |\gamma_T| \|Z_T\|_{L^2} \|Z_T - Z_T^2\|_{L^2} \leq \lambda^2 \frac{|\gamma_T|}{(T)^{1/4}} \|Z_T\|_{L^2} \|Z_T - Z_T^2\|_{H^{1/4}}, \]
and
\[ \lambda^2 \gamma_T \int (Z_T - Z_T^2)^2 \leq \lambda^2 |\gamma_T| \|Z_T - Z_T^2\|^2_{L^2} \leq \lambda^2 \frac{|\gamma_T|}{(T)^{1/4}} \|Z_T - Z_T^2\|_{L^2} \|Z_T - Z_T^2\|_{H^{1/4}}. \]
These bounds imply that both of them remain bounded provided \( \gamma_T \) does not grow too fast in \( T \) which is indeed insured by Lemma 24. For the last term we can apply the estimate
\[ \lambda^2 \int_0^T \int \gamma_t Z_T^2 Z_T^2 dt \leq \lambda^2 \int_0^T \|\gamma_t\|_{L^2} \|Z_T^2\|_{L^2} dt \leq \lambda^2 \|Z_T\|_{L^2} \|Z_T\|_{H^{1/4}} \int_0^T \frac{|\gamma_t|}{(t)^{5/4}} \]
Again, after we have fixed \( \gamma_t \) below, we will see this to be bounded. Collecting these bounds we get
\[ |Y^{(5)}_T| \leq C_2 \lambda^2 \left[ \frac{|\gamma_T|}{(T)^{1/4}} + \int_0^T \frac{\|\gamma_t\|}{(t)^{5/4}} \right]^2 + \varepsilon \|Z_T\|^4_{L^4} + \varepsilon \|Z_T\|^2_{H^{1/2-\delta}}. \]
□

**Lemma 23.** For any small \( \varepsilon > 0 \) there exists a \( \delta > 0 \) such that
\[ |Y^{(6)}_T| \leq C(\varepsilon, \delta) E(\lambda) Q_T + \varepsilon \|K_T\|^2_{H^{1-\delta}} + \varepsilon \lambda \|Z_T\|^4_{L^4}. \]

**Proof.** We start by observing that
\[ \lambda^2 \left| \int (W_T \circ W_T^{[3]} + 2 \gamma T W_T) K_T \right| \leq \lambda^2 \|W_T^{[3]}\|_{W^{-1/2-\varepsilon/2, \delta}} \|K_T\|_{W^{1/2+\varepsilon/2}}. \]
and using Lemma 24 and eq. 52 we have this term under control. Next split
\[
\frac{\lambda^2}{2} \left| \frac{1}{T} \int_0^T \left[ (J_t(W_t^2 > Z_t^2))^2 + 2 \gamma_{j_1}(Z_t^2)^2 \right] dt \right|
\]
\[
\leq \frac{\lambda^2}{2} \left( \frac{1}{T} \int_0^T \left( (J_t W_t^2 > Z_t^2) \right)^2 dt \right) + \frac{\lambda^2}{2} \left( \frac{1}{T} \int_0^T (J_t W_t^2 \circ J_t W_t^2)(Z_t^2)^2 dt \right)
\]
\[
\leq \lambda^2 \left( \sup_{t \in T} ||Z_t^2||_{B^{1/3}_6} ||Z_t^2||_{B^{1/3}_3} \right) \int_0^T \frac{||W_t^2||^2_{B^{1/3}_6}}{t^{1+\delta}} dt,
\]
\[
\leq \lambda^2 \left( \sup_{t \in T} ||Z_t^2||_{L^4} ||Z_t^2||_{H^{1/2-\delta}_6} \right) \int_0^T \frac{||W_t^2||^2_{B^{1/3}_6}}{t^{1+\delta}} dt.
\]
Which can be easily estimated by Young’s inequality. From Proposition 7 and Proposition 2
\[
\frac{\lambda^2}{2} \left( \sup_{t \in T} ||Z_t^2||_{L^4} ||Z_t^2||_{H^{1/2-\delta}_6} \right) \int_0^T \frac{||W_t^2||^2_{B^{1/3}_6}}{t^{1+\delta}} dt
\]
\[
\leq \lambda^2 \frac{||W_t^2||^2_{B^{1/3}_6}}{t^{1+\delta}} \leq \langle t \rangle^{-1/2} \frac{||W_t^2||^2_{B^{1/3}_6}}{t^{1+\delta}}.
\]
The integrability of this term in time follows from the inequality
\[
||J_t W_t^2||^2_{B^{1/3}_6} \leq \langle t \rangle^{-1/2} \frac{||W_t^2||^2_{B^{1/3}_6}}{t^{1+\delta}}.
\]
To prove this last bound, recall that \(t^{1/2} J_t\) is a Fourier multiplier with symbol
\[
\langle k \rangle^{-1} \langle k \rangle (\langle k \rangle/t)(\langle k \rangle/t)^{1/2} = \langle k \rangle^{-1} \eta(\langle k \rangle/t),
\]
where \(\eta\) is a smooth function supported in an annulus of radius 1. Using this observation and applying Proposition 5 gives the estimate. Applying Proposition 6 and Proposition 2 we get
\[
\lambda^2 \left( \frac{1}{T} \int_0^T \left( (J_t W_t^2 \circ J_t W_t^2)(Z_t^2)^2 \right) dt \right)
\]
\[
\leq \lambda^2 \left( \sup_{t \in T} ||Z_t^2||_{L^4} ||Z_t^2||_{H^{1/2-\delta}_6} \right) \int_0^T \frac{||W_t^2||^2_{B^{1/3}_6}}{t^{1+\delta}} dt
\]
\[
\leq \varepsilon \left( \frac{1}{2} \sup_{t \in T} ||Z_t^2||_{H^{1/2-\delta}_6} + \lambda ||Z_T||_{L^4}^4 \right) + C(\varepsilon, \delta) \lambda^7 \int_0^T \frac{||W_t^2||^2_{B^{1/3}_6}}{t^{1+\delta}} dt
\]
\[
\leq \varepsilon \left( \frac{1}{2} ||Z_T||_{H^{1/2-\delta}_6} + \lambda ||Z_T||_{L^4}^4 \right) + C(\varepsilon, \delta) \lambda^7 \int_0^T \frac{||W_t^2||^2_{B^{1/3}_6}}{t^{1+\delta}} dt.
\]
Finally we have
\[
\lambda^2 \left( \int_0^T \frac{||W_t^2||^2_{B^{1/3}_6}}{t^{1+\delta}} dt \right) \leq \lambda^2 \left( \int_0^T ||W_t^2||^2_{L^4} dt \right) ||Z_T||_{H^1} ||Z_T||_{L^4}.
Proof. We will concentrate in proving the bounds on the renormalized terms in eq. (56) and (57).

Moreover there exists a function \( \gamma_t \in C^1(\mathbb{R}^+, \mathbb{R}) \) such that for any \( \varepsilon > 0 \) and any \( p > 1 \),

\[
\sup_T \mathbb{E}[\|W_T \circ W_T^3\|_{B^{-\varepsilon}}^p] \leq C(\varepsilon, p, q),
\]

Moreover there exists a function \( \gamma_t \in C^1(\mathbb{R}^+, \mathbb{R}) \) such that for any \( \varepsilon > 0 \) and any \( p > 1 \),

\[
\sup_T \mathbb{E}[\|W_T^2 \circ W_T^3 - 2\gamma_T W_T\|_{B^{-\varepsilon}}^p] \leq C(\varepsilon, p, q),
\]

and

\[
\sup_T \mathbb{E}[\|J_t W_T^2 \circ J_t W_T^2 - 2\gamma_t\|_{B^{-\varepsilon}}] \leq C(\varepsilon, p, q)
\]

Furthermore \( \gamma \) is independent of \( \Lambda \). By Besov embedding Hölder norms of these objects are also uniformly bounded in \( T \) (but not uniformly in \( \Lambda \)).

Proof. We will concentrate in proving the bounds on the renormalized terms in eq. (56) and (57) and leave to the reader to fill the details for the easier term in eq. (55). Recall the representation of \( W_t = Y_t \) in terms of the family of Brownian motions \( (B^n_t)_{t,n} \) in eq. (4). Wick’s products of the Gaussian field \( W_T \) can be represented as iterated stochastic integrals w.r.t. \( (B^n_t)_{t,n} \). In particular, if we let \( dw_s(k) = (k)^{-1} \sigma_s(k) dB_s^k \), we have

\[
W_T^2(x) = 12[W_T^2](x) = 24 \sum_{k_1, k_2} e^{(k_1 + k_2) \cdot x} \int_0^T \int_0^{s_2} dw s_1(k_1) dw s_2(k_2),
\]

\[
W_T^3(x) = 24 \sum_{k_1, k_2, k_3} e^{ik(123) \cdot x} \int_0^T \int_0^{s_3} \int_0^{s_2} \left( \int_0^{s_3} \sigma_s^2(k(123)) \right) dw s_1(k_1) dw s_2(k_2) dw s_3(k_3),
\]

where \( k(123) := k_1 + k_2 + k_3 \). Now products of iterated integrals can be decomposed in sums of iterated integrals and we get

\[
\Delta_q(W_T^{20})(x) = \Delta_q(W_T^2 \circ W_T^3 - 2\gamma_T W_T)(x)
\]

\[
= \sum_{k_1, \ldots, k_5} \int_{A_T^5} G_{0,q}^{20}(\lambda, k) \int_0^{s_5} dw s_1(k_1) \cdots dw s_5(k_5)
\]

\[
+ \sum_{k_1, \ldots, k_3} \sum_{k_4} \int_{A_T^3} G_{1,q}^{20}(\lambda, k) \int_0^{s_3} dw s_1(k_1) \cdots dw s_3(k_3)
\]

\[
+ \sum_{k_1} \int_{A_T} G_{2,q}^{20}(\lambda, k) dw s_1(k_1),
\]
where \( A_T^n := \{ 0 \leq s_1 < \cdots < s_n \leq T \} \subseteq [0, T]^n \) and where the deterministic kernels are given by

\[
G_{0,q}^{203} ((s, k)_{1 \ldots 5}) := \left( 24^2 \right) K_q(k_{(1 \ldots 5)}) e^{i(k_{(1 \ldots 5)}):x} \sum_{\sigma \in \text{Sh}(2, 3)} \sum_{i < j} \times K_i(k_{(\sigma_1 \sigma_2)}) K_j(k_{(\sigma_3 \sigma_4 \sigma_5)}) \left( \int_{s_{(\sigma_5)}}^T \frac{\sigma_u(k_{(\sigma_3 \sigma_4 \sigma_5)})^2}{\langle k_{(\sigma_3 \sigma_4 \sigma_5)} \rangle^2} \, du \right),
\]

\[
G_{1,q}^{203} ((s, k)_{1 \ldots 3}) := \left( 24^2 \right) K_q(k_{(1 \ldots 3)}) e^{i(k_{(1 \ldots 3)}):x} \sum_{\sigma \in \text{Sh}(1, 2)} \sum_{i < j} \int_0^T \frac{\sigma_x(p)^2}{\langle p \rangle^2} \times K_i(k_{\sigma_1 + q}) K_j(k_{(\sigma_2 \sigma_3) - q}) \left( \int_{s_{(\sigma_3 \sigma_2 \sigma_3)}}^T \sigma_u(k_{(\sigma_2 \sigma_3)}) - p)^2}{\langle k_{(\sigma_2 \sigma_3)} \rangle - p)^2} \, du \right),
\]

\[
G_{2,q}^{203} ((s, k)_{1}) := \left( 24^2 \right) K_q(k_{1}) e^{ik_{1}:x} \sum_{i < j, p_1, p_2} \int_0^T \frac{\sigma_x(p_1)^2}{\langle p_1 \rangle^2} \frac{\sigma_x(p_2)^2}{\langle p_2 \rangle^2} \times K_i(p_1 + p_2) K_j(k_{1} - p_1 - p_2) \left( \int_{s_{(1 \cdots 1)}}^T \frac{\sigma_u(k_{1} - p_1 - p_2)^2}{\langle k_{1} - p_1 - p_2 \rangle^2} \, du \right),
\]

\[
G_{2,q}^{203} ((s, k)_{1}) := G_{2,q}^{203} ((s, k)_{1}) - 2 \gamma_T K_q(k_{1}) e^{ik_{1}:x},
\]

where \( \text{Sh}(k, l) \) is the set of permutations of \( \{1, \ldots, k + l\} \) keeping the orders \( \sigma(1) < \cdots < \sigma(k) \) and \( \sigma(k + 1) < \cdots < \sigma(k + l) \) and where, for any symbol \( z \), we denote with expression of the form \( z_{1 \cdots n} \) the vector \( (z_1, \ldots, z_n) \). Estimation of \( \Delta_q(W^2_T \circ W^3_T)(x) \) reduces then to estimate each of the three iterated integrals using BDG inequalities to get, for any \( p \geq 2 \),

\[
I_{0,q} = \left\{ \mathbb{E} \left[ \left| \sum_{k_{1 \ldots 5}, s_{1 \ldots 5}} \int_{A_T^n} G_{0,q}^{203} ((s, k)_{1 \ldots 5}) d w_{s_1}(k_1) \cdots d w_{s_5}(k_5) \right|^{2p} \right] \right\}^{1/p} \cong \mathbb{E} \left[ \sum_{k_{1 \ldots 5}, s_{1 \ldots 5}} \left| \int_{A_T^n} G_{0,q}^{203} ((s, k)_{1 \ldots 5}) d w_{s_1}(k_1) \cdots d w_{s_5}(k_5) \right|^2 \right] \leq \sum_{k_{1 \ldots 5}, s_{1 \ldots 5}} \left| \int_{A_T^n} G_{0,q}^{203} ((s, k)_{1 \ldots 5}) \right|^2 \sigma_{s_1}(k_1)^2 \frac{\sigma_{s_2}(k_2)}{(k_2)^2} \cdots \sigma_{s_5}(k_5)^2 \frac{\sigma_{s_5}(k_5)}{(k_5)^2} \, ds_1 \cdots ds_5.
\]

The kernel \( G_{0,q}^{203} ((s, k)_{1 \ldots 5}) \) being a symmetric function of its argument, we can simplify this expression into an integral over \( [0, T]^5 \):

\[
I_{0,q} \cong \sum_{k_{1 \ldots 5}, s_{1 \ldots 5}} \int_{[0, T]^5} \left| G_{0,q}^{203} ((s, k)_{1 \ldots 5}) \right|^2 \sigma_{s_1}(k_1)^2 \frac{\sigma_{s_2}(k_2)}{(k_2)^2} \cdots \sigma_{s_5}(k_5)^2 \frac{\sigma_{s_5}(k_5)}{(k_5)^2} \, ds_1 \cdots ds_5.
\]

Under the measure \( \frac{\sigma_{s_1}(k_1)^2}{(k_1)^2} \, ds_1 \cdots ds_5 \), we have

\[
\int_{s_{(\sigma_5)}}^T \frac{\sigma_u(k_{(\sigma_3 \sigma_4 \sigma_5)})^2}{\langle k_{(\sigma_3 \sigma_4 \sigma_5)} \rangle^2} \, du \lesssim \frac{1}{\langle k_{(\sigma_5)} \rangle^2}.
\]

Therefore with some standard estimates we can reduce us to consider

\[
I_{0,q} \leq \sum_{k_{1 \ldots 5}, s_{1 \ldots 5}} \int_{[0, T]^5} \frac{K_q(k_{(1 \ldots 5)})^2}{(k_5)^4} \rho_{k_{(12)}} \rho_{k_{(345)}} \frac{\sigma_{s_1}(k_1)^2}{(k_1)^2} \cdots \frac{\sigma_{s_5}(k_5)^2}{(k_5)^2} \, ds_1 \cdots ds_5
\]

\[
\lesssim \sum_{k_{1 \ldots 5}} \int_{[0, T]^5} \frac{K_q(k_{(1 \ldots 5)})^2}{(k_5)^4} \rho_{k_{(12)}} \rho_{k_{(345)}} \frac{\sigma_{s_1}(k_1)^2}{(k_1)^2} \cdots \frac{\sigma_{s_5}(k_5)^2}{(k_5)^2} \, ds_1 \cdots ds_5
\]
Now by similar reasoning we also have
\[ |G_{1,q}^{2o[3]}((s,k)_{1−3})| \lesssim \sum_{σ∈Sh(1,2)} \frac{|K_q(k_{1−3})|}{⟨k_σ⟩} \sum_p \sum_{i,j} \int_0^T dr_{r} \sigma_r(p)^2 |K_i(k_{σ1} + p)K_j(k_{σ2σ3} − p)| \left( \frac{p}{σ_r(p)^2} + 1 \right)^{2σ_r(p)^2} \lesssim \sum_r \frac{K_q(r)^2}{⟨r⟩^2} \lesssim q^2. \]

Finally, we note that the same strategy cannot be applied to the first chaos, since the kernel \( G_{2,q}^{2o[3]} \) cannot be uniformly bounded. We let
\[ A_T(s_1,k) := (24^2) \sum_{i,j} \sum_{q_1,q_2} \int_0^T dr_1 \int_0^T dr_2 \sigma_{r1}(q_1)^2 σ_{r2}(q_2)^2 \times K_i(q_1 + q_2)K_j(k_1 − q_1 − q_2) \left( \int_{r_1 ∨ r_2 ∨ s_1} \sigma_{r1}(q_1 − q_1 − q_2)^2 du \right), \]
so
\[ G_{2,q}^{2o[3]}((s,k)_1) = K_q(k_1)e^{ik_1·x}A_T(s_1,k_1) − 2γ_T. \]

Observe that
\[ A_T(0,0) = (12^2 · 2) \sum_{q_1,q_2} \int_0^T du \int_0^u dr_1 \int_0^u \sigma_{r1}(q_1)^2 σ_{r2}(q_2)^2 \times \int_{r_1 ∨ r_2} \sigma_{r1}(q_1 + q_2)^2 du \sum_{i,j} K_i(q_1 + q_2)K_j(−q_1 − q_2). \]

We choose \( γ_T \) as
\[ γ_T = A_T(0,0) = (12^2 · 2) \sum_{q_1,q_2} \int_0^T du \int_0^u dr_1 \int_0^u \sigma_{r1}(q_1)^2 σ_{r2}(q_2)^2 \frac{σ_{r1}(q_1 + q_2)^2}{(q_1 + q_2)^2}, \]
where we used the fact that for all \( q ∈ \mathbb{R}^d \) we have \( \sum_{i,j} K_i(q)K_j(q) = 1, \) since \( ∫ f ∘ g = ∫ fg. \) Note that, as claimed,
\[ |γ_T| \lesssim \sum_{q_1,q_2} \frac{1}{(q_1)^2(q_2)^2(q_1 + q_2)^2} \lesssim 1 + \log(T). \]

Now
\[ A_T(s_1,k_1) − 2γ_T = (24^2 · 6) \sum_{q_1,q_2} \int_0^T dr_1 dr_2 \sigma_{r1}(q_1)^2 σ_{r2}(q_2)^2 \sum_{i,j} K_i(q_1 + q_2) \times \]
All together these estimates imply that

\[
\times \left( K_j(k_1 - q_1 - q_2) \int_{s_1 \lor r_1 \lor r_2} \frac{\sigma^2_j(k_1 - q_1 - q_2)}{(k_1 - q_1 - q_2)^2} \, du - K_j(q_1 + q_2) \int_{r_1 \lor r_2} \frac{\sigma^2_j(q_1 + q_2)}{(q_1 + q_2)^2} \, du \right)
\]

so when \(|q_1 + q_2| \gg |k_1|\) the quantity in round brackets can be estimated by \(|k_1|\langle q_1 + q_2 \rangle^{-4}\) while when \(|q_1 + q_2| \lesssim |k_1|\) it is estimated by \(|q_1 + q_2|^{-2}\) so we have

\[
|A_T(s_1, k_1) - \gamma_T| \lesssim \sum_{q_1, q_2} \frac{1}{(q_1)^2} \frac{1}{(q_1 + q_2)^2} \left( \mathbf{1}_{|q_1 + q_2| \leq |k_1|} + \mathbf{1}_{|q_1 + q_2| \geq |k_1|} \right) \frac{|k_1|}{(q_1 + q_2)^2}
\]

\[
\lesssim 1 + \log \langle k_1 \rangle.
\]

And then with this choice of \(\gamma_T\) the kernel \(G_{2,q}^{2\circ[3]}\) stays uniformly bounded as \(T \to \infty\) and satisfies

\[
|G_{2,q}^{2\circ[3]}((s, k)_1)| \lesssim K_q(k_1) \log \langle k_1 \rangle.
\]

From this we easily deduce that

\[
I_{2,q} = \mathbb{E} \left[ \left( \sum_{k_1} \int_{A_T} G_{2,q}^{2\circ[3]}((s, k)_1) \, dy_{s_1}(k_1) \right)^{2p} \right]^{1/p} \lesssim q^{2q}, \quad q \geq -1.
\]

All together these estimates imply that

\[
\mathbb{E} \|\Delta_q \mathbb{W}_T^{2\circ[3]}\|_{L^{2p}}^{2p} \lesssim (q^{2q/2})^{2p}, \quad q \geq -1.
\]

Standard argument allows to deduce eq. (56). The analysis of the other renormalized product proceeds similarly. Let

\[
V(t) := \mathbb{W}_t^{(2)\circ(2)} = J_t \mathbb{W}_t^{2} \circ J_t \mathbb{W}_t^{2} - 2J_t, \quad t \geq 0.
\]

First note that by definition of Besov spaces we have

\[
\mathbb{E} \left[ \left( \int_0^\infty \|V(t)\|_{B_{r,r}^{\epsilon - d/r}} \, dt \right)^p \right] \lesssim \mathbb{E} \left[ \left( \int_0^\infty \left( \sum_q 2^{-qr(\epsilon + d/r)} \|\Delta_q V(t)\|_{L_r} \right)^{1/r} \, dt \right)^p \right].
\]

By Minkowski’s integral inequality this is bounded by

\[
\lesssim \left( \int_0^\infty dt \left\{ \mathbb{E} \left[ \left( \sum_q 2^{-qr(\epsilon + d/r)} \|\Delta_q V(t)\|_{L_r} \right)^{p/r} \right] \right\}^{1/p} \right)^p.
\]

When \(r \geq p\) Jensen’s inequality and Fubini’s theorem give

\[
\lesssim \left( \int_0^\infty dt \left\{ \sum_q 2^{-qr(\epsilon + d/r)} \left( \mathbb{E} \|\Delta_q V(t)(0)\|^2 \right)^{1/r} \right\}^p \right)^p.
\]

Finally hypercontractivity and stationarity allow to reduce this to bound

\[
\lesssim \left( \int_0^\infty dt \left\{ \sum_q 2^{-qr(\epsilon + d/r)} (\mathbb{E} \|\Delta_q V(t)(0)\|^2)^{r/2} \right\}^{1/r} \right)^p.
\]

Letting \(I_q(t) = \mathbb{E} [\|\Delta_q V(t)(0)\|^2]\) we have

\[
\mathbb{E} \left[ \left( \int_0^\infty \|\mathbb{W}_t^{(2)\circ(2)}\|_{B_{r,r}^{\epsilon - d/r}} \, dt \right)^p \right] \lesssim \left( \int_0^\infty dt \left\{ \sum_q 2^{-qr(\epsilon + d/r)} (I_q(t))^{r/2} \right\}^{1/r} \right)^p.
\]
Now we decompose the random field \( \Delta_q(\mathbb{W}_t^{(2)\circ(2)})(x) \) into homogeneous stochastic integral as above and obtain
\[
\Delta_q(\mathbb{W}_t^{(2)\circ(2)})(x) = \sum_{k_1, \ldots, k_4} \int_{A_t^4} G_{0,q}^{(2)\circ(2)}((s, k)_{1 \ldots 4}) \, dw_{s_1}(k_1) \cdots dw_{s_4}(k_4) \\
+ \sum_{k_1, k_2} \int_{A_t^2} G_{1,q}^{(2)\circ(2)}((s, k)_{12}) \, dw_{s_1}(k_1) \, dw_{s_2}(k_2) \\
+ G_{2,q}^{(2)\circ(2)}
\]
with
\[
G_{0,q}^{(2)\circ(2)}((s, k)_{1 \ldots 4}) = (24^2) K_q(k_{(1 \cdots 4)}) e^{\ell(k_{1 \cdots 4}) x} \times \sum_{i \sim j} \sum_{q} K_i(k_{(1 \sigma_2)} k_j(k_{(3 \sigma_4)}) \sigma_i(k_{(1 \sigma_2)}) \sigma_j(k_{(3 \sigma_4)}) \langle k_{(1 \sigma_2)} \rangle \langle k_{(3 \sigma_4)} \rangle \langle k \rangle.
\]
\[
G_{1,q}^{(2)\circ(2)}((s, k)_{12}) = (24^2) K_q(k_{(12)}) e^{\ell(k_{12}) x} \sum_{i \sim j} \sum_{q} \times \sum_{q_1, q_2} \langle q_1 \rangle \langle q_2 \rangle \langle q \rangle.
\]
\[
G_{2,q}^{(2)\circ(2)} = (24^2) \ll_{q=-1} \sum_{i \sim j} \sum_{q_1, q_2} \int_0^t \int_0^t \frac{\sigma_i(q_1)^2 \sigma_j(q_2)^2}{\langle q_1 \rangle^2 \langle q_2 \rangle^2} \, K_i(q_1 + q_2) K_j(-q_1 - q_2) \frac{\sigma_i(q_1 + q_2)}{\langle q_1 + q_2 \rangle^2}.
\]
Using our choice of \( \gamma_T \) in eq. (60) we have that
\[
\gamma_t = (12^2 \cdot 2) \sum_{q_1, q_2} \int_0^t dr_1 \int_0^t dr_2 \frac{\sigma_i(q_1)^2 \sigma_j(q_2)^2}{\langle q_1 \rangle^2 \langle q_2 \rangle^2} \, du,
\]
which implies also that
\[
G_{2,q}^{(2)\circ(2)} = 0, \quad \text{and} \quad |\gamma_t| \lesssim \frac{1}{\langle t \rangle}.
\]
as claimed. We pass now to estimate the other two chaoses. The technique is the same we used above. Consider first
\[
I_{0,q}(t) := \mathbb{E} \left[ \sum_{k_1, \ldots, k_4} \int_{A_t^4} G_{0,q}^{(2)\circ(2)}((s, k)_{1 \ldots 4}) \, dw_{s_1}(k_1) \cdots dw_{s_4}(k_4) \right]^2.
\]
\[
\lesssim \sum_{k_1, \ldots, k_4} \int_{[0, t]^4} |G_{0,q}^{(2)\circ(2)}((s, k)_{1 \ldots 4})|^2 \frac{\sigma_i(k_1)^2}{\langle k_1 \rangle^2} \cdots \frac{\sigma_i(k_4)^2}{\langle k_4 \rangle^2} \, ds_1 \cdots ds_4.
\]
\[
\lesssim \sum_{k_1, \ldots, k_4} K_q(k_{(1 \cdots 4)})^2 \int_{[0, t]^4} \frac{\sigma_i^2(k_{(12)})}{\langle k_{(12)} \rangle^2} \frac{\sigma_j^2(k_{(34)})}{\langle k_{(34)} \rangle^2} \frac{\sigma_i(k_1)^2}{\langle k_1 \rangle^2} \cdots \frac{\sigma_i(k_4)^2}{\langle k_4 \rangle^2} \, ds_1 \cdots ds_4.
\]
\[
\lesssim \sum_{k_1, \ldots, k_4} K_q(k_{(1 \cdots 4)})^2 \frac{\sigma_i^2(k_{(12)})}{\langle k_{(12)} \rangle^2} \frac{\sigma_j^2(k_{(34)})}{\langle k_{(34)} \rangle^2} \frac{1}{\langle k_1 \rangle^2} \cdots \frac{1}{\langle k_4 \rangle^2}.
\]
\[
\lesssim \frac{1}{\langle t \rangle^6} \sum_{k_1, \ldots, k_4} K_q(k_{(1 \cdots 4)})^2 \frac{\sigma_i^2(k_{(12)})}{\langle k_1 \rangle^2 \langle k_2 \rangle^2 \langle k_3 \rangle^2 \langle k_4 \rangle^2} \lesssim \frac{1}{\langle t \rangle^6} \frac{t^{2q}}{2^q}.
\]
where we used that $|\sigma_t(x)| \lesssim t^{-1/2} \mathbb{1}_{x \sim \cdot}$. Now taking $\varepsilon + d/r > 0$ we have

$$\int_0^\infty dt \left\{ \sum_q 2^{q(r(\varepsilon + d/r))} (I_{0,q}(t))^{r/2} \right\}^{1/r} \lesssim \int_0^\infty dt \left\{ \sum_q 2^{q(r(2\varepsilon - d/r))} \right\}^{1/r}$$

$$\lesssim \int_0^\infty \frac{dt}{\langle t \rangle^{1+\varepsilon + d/r}} \lesssim 1.$$ 

Taking into account that $|k_1|, |k_2| \lesssim t$ we can estimate

$$|G^{(2)\sigma(2)}_{1,q}((s,k)_{12})| \lesssim |K_q(k_{12})| \frac{1}{\langle p \rangle^2} \left( \frac{\sigma_t(k_1 + p) \sigma_t(k_2 - p)}{|k_1 + p|} \frac{|k_2 - p|}{\langle k_2 \rangle^2} \right) \lesssim |K_q(k_{12})| \langle t \rangle^{-2},$$

from which we deduce that

$$I_{1,q}(t) := \mathbb{E} \left[ \sum_{k_{1,2}} \int_{A_t^2} G^{(2)\sigma(2)}_{1,q}((s,k)_{12}) dw_{s_1} \langle k_1 \rangle dw_{s_2} \langle k_2 \rangle \right]^2$$

$$\lesssim \sum_{k_{1,2}} \int_{A_t^2} |G^{(2)\sigma(2)}_{1,q}((s,k)_{12})|^2 \frac{\sigma_s(k_1)^2 \sigma_s(k_2)^2}{\langle k_1 \rangle^2 \langle k_2 \rangle^2} ds_1 ds_2$$

$$\lesssim \langle t \rangle^{-4} \sum_{k_{1,2}} |K_q(k_{12})|^2 \int_{[0,t]^2} \frac{\sigma_s(k_1)^2 \sigma_s(k_2)^2}{\langle k_1 \rangle^2 \langle k_2 \rangle^2} ds_1 ds_2$$

$$\lesssim \langle t \rangle^{-4} \sum_{k_{1,2}} |K_q(k_{12})|^2 \frac{1}{\langle k_1 \rangle^2} \frac{1}{\langle k_2 \rangle^2} \lesssim \langle t \rangle^{-4} 2^{q} \mathbb{1}_{2q \leq t},$$

and then, as for $I_{0,q}$, we have

$$\int_0^\infty dt \left( \sum_q 2^{ q(r(\varepsilon + d/r))} (I_{1,q}(t))^{r/2} \right)^{1/r} \lesssim \int_0^\infty dt \left( \sum_q 2^{q(1 - \varepsilon - d/r)} \mathbb{1}_{2q \leq t} \right)^{1/r} \lesssim 1,$$

as claimed. From these estimates standard arguments give eq. (57). 

\textbf{Lemma 25.}

$$\mathbb{E}[\|\mathbb{W}^3_T\|_{L^p}]^{1/p} \lesssim T^{3/2}$$

This implies that $\mathbb{W}^3 \in C([0, \infty], B^{-1/2 - \kappa}_{p,p}) \cap L^2(\mathbb{R}^+, B^{-1/2 - \kappa}_{p,p})$ for any $p < \infty$ uniformly in the volume and $\mathbb{W}^3 \in C([0, \infty], \mathcal{C}^{-1/2 - \kappa}) \cap L^2(\mathbb{R}^+, \mathcal{C}^{-1/2 - \kappa})$

Estimating the cube

$$\mathbb{W}^3_T(x) = 12 \mathbb{W}^3_T(x) = 24 \sum_{k_{1,2,3}} e^{i(k_{1,2,3}) \cdot x} \int_0^T \int_0^{s_2} \int_0^{s_3} dw_{s_1} \langle k_1 \rangle dw_{s_2} \langle k_2 \rangle dw_{s_3} \langle k_3 \rangle$$

We get for any $p$, by space homogeneity,

$$\mathbb{E}[\|\mathbb{W}^3_T(x)\|_{L^p}]^{2p} = \mathbb{E}\left[ \sum_{k_{1,2,3}} \int_0^T \int_0^{s_2} \int_0^{s_3} dw_{s_1} \langle k_1 \rangle dw_{s_2} \langle k_2 \rangle dw_{s_3} \langle k_3 \rangle \right]^{2p}$$
Now and Hölder estimates follow by Besov-embedding. σ
Now the second properties follow by the fact that
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APPENDIX A. Besov spaces and paraproducts

In this section we will recall some well known results about Besov spaces, embeddings, Fourier multipliers and paraproducts. The reader can find full details and proofs in [27,28] and for weighted spaces in [27,41]. First recall the definition of Littlewood–Paley blocks. Let $\chi, \varphi$ be smooth radial functions $\mathbb{R}^d \to \mathbb{R}$ such that

- $\text{supp } \chi \subseteq B(0,R)$, $\text{supp } \varphi \subseteq B(0,2R) \setminus B(0,R)$;
- $0 \leq \chi, \varphi \leq 1$, $\chi(\xi) + \sum_{j \geq 0} \varphi(2^{-j}\xi) = 1$ for any $\xi \in \mathbb{R}^d$;
- $\text{supp } (2^{-j} \cdot) \cap \text{supp } (2^{-i} \cdot) = \emptyset$ if $|i - j| > 1$.

Introduce the notations $\varphi_{-1} = \chi$, $\varphi_j = \varphi(2^{-j} \cdot)$ for $j \geq 0$. For any $f \in \mathcal{S}'(\Lambda)$ we define the operators $\Delta_j f := \mathcal{F}^{-1}(\varphi_j(\xi)\hat{f}(\xi))$, $j \geq -1$.

Definition 5. We say a function $\rho : \mathbb{R}^d \to \mathbb{R}$ of the form $\rho(x) = (x)^{-\sigma}$ for $\sigma \geq 0$ is a weight.

Definition 6. For a weight $\rho$ let

$$
\|f\|_{L^p(\rho)} = \left( \int_{\mathbb{R}^d} |f(x)|^p \rho(x) dx \right)^{1/p}
$$

and by $L^p(\rho)$ the space of functions for which this norm is finite. For function defined on a torus in $\mathbb{R}^d$ we consider their periodic extensions on $\mathbb{R}^d$.

Definition 7. Let $s \in \mathbb{R}, p, q \in [1, \infty]$, and $\rho$ be a weight. For a Schwarz distribution $f \in \mathcal{S}'(\Lambda)$ define the norm

$$
\|f\|_{B^s_{p,q}(\rho)} = \|(2^j \|\Delta_j f\|_{L^p(\rho)})_{j \geq 1}\|_{l^q}.
$$

Then the space $B^s_{p,q}(\rho)$ is the set of functions in $\mathcal{S}'(\Lambda)$ such that this norm is finite. We denote $B^s_{p,q} = B^s_{p,q}(1/|\Lambda|)$ the normalized Besov space and $H^s = B^s_{2,2}$ the Sobolev spaces, and by $C^\alpha = B^\alpha_{\infty,\infty}$ the (unweighted) Hölder spaces.

Definition 8. Let $s \in \mathbb{R}$ and $\rho$ be a weight. Then we denote by

$$
\|f\|_{C^\alpha(\rho)} = \|(2^s \|\rho \Delta_j f\|_{L^\infty})_{j \geq 1}\|_{l^\infty}
$$

and by $C^\alpha(\rho)$ the space of Schwarz distributions such that this norm is finite.

Proposition 2. Let $\delta > 0$. We have for any $q_1, q_2 \in [1, \infty], q_1 < q_2$

$$
\|f\|_{B^s_{p,q_2}} \leq \|f\|_{B^s_{p,q_1}} \leq \|f\|_{B^{s+\delta}_{p,q_2}}.
$$

Furthermore, if we denote by $W^{s,p}$ the normalized fractional Sobolev spaces then for any $q \in [1, \infty]$

$$
\|f\|_{B^s_{p,q}} \leq \|f\|_{W^{s+\delta,p}} \leq \|f\|_{B^{s+\delta}_{p,q}}.
$$

Proposition 3. For any $s_1, s_2 \in \mathbb{R}$ such that $s_1 < s_2$, any $p, q \in [1, \infty]$ the Besov space $B^{s_1}_{p,q}$ is compactly embedded into $B^{s_2}_{p,q}$.

Definition 9. Let $f, g \in \mathcal{S}(\Lambda)$. We define the paraproducts and resonant product

$$
f \star g = g \ast f := \sum_{j < i - 1} \Delta_i f \Delta_j g, \quad \text{and} \quad f \circ g := \sum_{|i - j| \leq 1} \Delta_i f \Delta_j g.
$$

Then

$$fg = f \star g + f \circ g + f \rhd g.$$
Proposition 4. Let \( f, g \in \mathcal{S}(\Lambda) \). We define the paraproducts and resonant product by

\[
f \succ g = g \prec f := \sum_{j<i-1} \Delta_j f \Delta_j g, \quad \text{and} \quad f \circ g := \sum_{|i-j| \leq 1} \Delta_i f \Delta_j g.
\]

Then

\[
f g = f \prec g + f \circ g + f \succ g.
\]

Moreover for any weight \( \rho, \beta \leq 0, \alpha \in \mathbb{R} \) and \( p_1, p_2 \in [1, \infty] \), \( \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p} \) we have the estimates

\[
\| f \succ g \|_{B_{p,q}^{\alpha+\beta}(\rho)} \lesssim \| f \|_{B_{p_1,\infty}^{\alpha}(\rho)} \| g \|_{B_{p_2,\infty}^{\beta}(\rho)},
\]

and for any \( \alpha, \beta \in \mathbb{R} \) such that \( \alpha + \beta > 0 \) the estimates

\[
\| f \circ g \|_{B_{p,q}^{\alpha+\beta}(\rho)} \lesssim \| f \|_{B_{p_1,\infty}^{\alpha}(\rho)} \| g \|_{B_{p_2,\infty}^{\beta}(\rho)}.
\]

For a proof see Theorem 3.17 and Remark 3.18 in [41].

Proposition 5. For any weight \( \rho, \beta \leq 0, \alpha \in \mathbb{R} \) we have

\[
\| f \succ g \|_{B_{p,q}^{\alpha}(\rho^2)} \lesssim \| f \|_{C^{\alpha}(\rho)} \| g \|_{B_{p,q}^{\beta}(\rho)}.
\]

The proof is an easy modification of the proof of Theorem 3.17 in [41].

Proposition 6. Let \( \alpha \in (0,1) \), \( \beta, \gamma \in \mathbb{R} \) such that \( \beta + \gamma < 0, \alpha + \beta + \gamma > 0 \) and \( p_1, p_2, p_3, p \in [1, \infty] \) such that \( \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = \frac{1}{p} \). Then there exists a bounded trilinear form \( \tilde{R}_1(f,g,h) \) such that for any \( \delta > 0 \),

\[
\| \tilde{R}_1(f,g,h) \|_{B_{p_3,\infty}^{\alpha+\beta+\gamma}} \lesssim \| g \|_{B_{p_1,\infty}^{\alpha}} \| f \|_{B_{p_2,\infty}^{\beta}} \| h \|_{B_{p_3,\infty}},
\]

and when \( f, g, h \in \mathcal{S} \) we have

\[
\tilde{R}_1(f,g,h) = (f \succ g) \circ h - g(f \circ h).
\]

Proof. The proof is a slight modification of the one given in [28] Lemma 2.97 from [3] and interpolation implies that \( \| \Delta_j f g - \Delta_j (f g) \|_{L^p} \leq 2^{-j\alpha} \| f \|_{W^{\alpha,1}} \| g \|_{L^p} \). This in turn gives after some algebraic computations (see [28]) that

\[
\Delta_j (f \succ g) = (\Delta_j f) \succ g + R_j(f,g)
\]

with \( \| R_j(f,g) \|_{L^p} \lesssim 2^{-j(\alpha+\beta)} \| f \|_{B_{p_1,\infty}^{\alpha}} \| g \|_{B_{p_2,\infty}} \). Now to prove the statement of the proposition observe that for smooth \( f, g, h \) we have

\[
\tilde{R}_1(f,g,h) = \sum_{j,k \geq 1} \sum_{|i-j| \leq 1} \Delta_j (f \succ \Delta_k g) \Delta_k h - \Delta_k g \Delta_j f_i h
\]

Now observe that the term \( f \succ \Delta_k g \) has Fourier transform outside of \( 2^k B \) for some Ball \( B \) independent of \( k \), so choosing \( N \) large enough we can rewrite the sum as

\[
\tilde{R}_1(f,g,h) = \sum_{j,k \geq 1} \sum_{|i-j| \leq 1} 1_{k \leq i+N} (\Delta_j f \Delta_k g \Delta_i h + R_j(f, \Delta_k g)) - \Delta_k g \Delta_j f_i h
\]

\[
\sum_{j,k \geq 1} \sum_{|i-j| \leq 1} 1_{k \leq i+N} R_j(f, \Delta_k g) h - 1_{k \geq i+N} \Delta_k g \Delta_j f_i h
\]

Now we estimate the norm of the two terms separately. First note that for fixed \( j \)

\[
\sum_{k \geq 1} \sum_{|i-j| \leq 1} 1_{k \leq i+N} R_j(f, \Delta_k g)
\]
has a Fourier transform supported in $2^k B$. By Lemma 2.69 from \[3\] it is enough to get an estimate on $\sup_k \left| 2^{(\alpha+\beta+\gamma)j} \sum_{j \geq 1} \sum_{|i-j| \leq 1} 1_{k \in i+N} R_j(f, \Delta_k g, h) \right|_{L^p}$ to estimate it in $B_\infty^{\alpha+\beta+\gamma}$, so by Hölder inequality

$$\left\| \sum_{|i-j| \leq 1} \Delta_k g \right\|_{L^p} \lesssim \sum_{|i-j| \leq 1} 2^{-j(\alpha+\beta+\gamma)} \|f\|_{B^{\alpha\beta\gamma}_{1,\infty}} \|f\|_{B^{\alpha\beta}_{p_2,\infty}} \|h\|_{B^{\alpha\beta\gamma}_{p_3,\infty}}$$

For the second term observe that for fixed $k$ the Fourier transform of

$$\sum_{j \geq 1} \sum_{|i-j| \leq 1} 1_{k \in i+N} \Delta_k g \Delta_j f_i h$$

is supported in $2^k B$. Now we can estimate again by Hölder inequality

$$\lesssim \left\| \sum_{j \geq 1} \sum_{|i-j| \leq 1} 1_{k \in i+N} \Delta_k g \Delta_j f_i h \right\|_{L^p} \lesssim 2^{-\alpha k} \sum_{j \geq 1} \sum_{|i-j| \leq 1} 2^{-(\beta+\gamma)k} \|f\|_{B^{\alpha\beta\gamma}_{1,\infty}} \|f\|_{B^{\alpha\beta}_{p_2,\infty}} \|h\|_{B^{\alpha\beta\gamma}_{p_3,\infty}}$$

Proposition 7. Assume $\alpha \in (0, 1)$, $\beta, \gamma \in \mathbb{R}$ such that $\beta+\gamma < 0$, and $\alpha+\beta+\gamma = 0$, $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1$ and $\frac{1}{q_1} + \frac{1}{q_2} = 1$. Then there exists a bounded trilinear form $\mathcal{R}_2(f, g, h)$ for which

$$|\mathcal{R}_2(f, g, h)| \lesssim \|f\|_{B^{\alpha\beta\gamma}_{1,\infty}} \|g\|_{B^{\alpha\beta\gamma}_{p_2,\infty}} \|h\|_{B^{\alpha\beta\gamma}_{p_3,\infty}}$$

and

$$\mathcal{R}_2(f, g, h) = \frac{1}{|\Lambda|} \int_{[\Lambda]} [(f \ast g)h - (f \circ h)g]$$

for smooth functions.

Proof. This is modification of the proof of Lemma A.6 in \[30\]. Repeating an algebraic computation given in \[30\] in the proof of Lemma A.6, we get that for smooth $f, g, h$ we have

$$\mathcal{R}_2(f, g, h) = \left( \sum_{i \geq k-1, |j-k| \leq L} - \sum_{i \sim k, 1 < |j-k| \leq L} \right) \langle \Delta_i g, \Delta_j h \Delta_k f \rangle$$

Then we estimate

$$|\mathcal{R}_2(f, g, h)| \lesssim \left( \sum_{i \geq k-1, |j-k| \leq L} - \sum_{i \sim k, 1 < |j-k| \leq L} \right) \langle \Delta_i g, \Delta_j h \Delta_k f \rangle$$

$$\lesssim \sum_{i \geq k, j \sim k} |\langle \Delta_i g, \Delta_j h \Delta_k f \rangle|$$

$$\lesssim \sum_{i \geq k, j \sim k} \|\Delta_k f\|_{L^p_1} \|\Delta_i g\|_{L^p_2} \|\Delta_j h\|_{L^p_3}$$

$$\lesssim \sup_k \left( 2^{\alpha k} \|\Delta_k f\|_{L^p_1} \right) \sum_k \sum_{i \geq k, j \sim k} 2^{(\beta+\gamma)k} \|\Delta_i g\|_{L^p_2} \|\Delta_j h\|_{L^p_3}$$

$$\lesssim \|f\|_{B^{\alpha\beta\gamma}_{p_1,\infty}} \|g\|_{B^{\alpha\beta\gamma}_{p_2,\infty}} \|h\|_{B^{\alpha\beta\gamma}_{p_3,\infty}}$$
Proposition 8. There exists a family $(\mathcal{R}_{3,t})_{t \geq 0}$ of bounded multilinear forms on $\mathcal{C}^{-1-\kappa} \times \mathcal{C}^{-1-\kappa} \times H^{1/2-\delta} \times H^{1/2-\delta}$ such that for smooth $\varphi, \psi, g^{(1)}, g^{(2)}$ it holds

$$\mathcal{R}_{3,t}(\varphi, \psi, g^{(1)}, g^{(2)}) = \int [J_t(\varphi \ast g^{(1)})J_t(\psi \ast g^{(2)}) - (J_t\varphi \circ J_t\psi)g^{(1)}g^{(2)}],$$

and

$$|\mathcal{R}_{3,t}(\varphi, \psi, g^{(1)}, g^{(2)})| \lesssim \frac{1}{(t)^{1/2-\delta}} \|\varphi\|_{\mathcal{C}^{-1-\kappa}} \|\psi\|_{\mathcal{C}^{-1-\kappa}} \|g^{(1)}\|_{H^{1/2-\delta}} \|g^{(2)}\|_{H^{1/2-\delta}},$$

for some $\delta > 0$.

Proof. Note that $(t)^{1/2}J_t$ satisfies the assumptions of Proposition 10 and with $m = -1$, therefore using also Proposition 2

$$\|J_t(\varphi \ast g^{(1)}) - J_t(\varphi \ast g^{(1)})\|_{H^{1/2-2\kappa}} \lesssim (t)^{-1/2} \|\varphi\|_{H^{1/2-\kappa}} \|g^{(1)}\|_{H^{1/2-\delta}}$$

and therefore

$$\left|\int [J_t(\varphi \ast g^{(1)}) - (J_t\varphi \ast g^{(1)})J_t(\psi \ast g^{(2)})]\right| \lesssim \|J_t(\varphi \ast g^{(1)}) - J_t\varphi \ast g^{(1)}\|_{H^{1/2-2\kappa}} \|J_t(\psi \ast g^{(2)})\|_{H^{1/2-2\kappa}} \lesssim (t)^{-1/2} \|\varphi\|_{H^{1/2-\kappa}} \|g^{(1)}\|_{H^{1/2-\delta}} (t)^{-1/2} \|\psi\|_{H^{1/2-\kappa}} \|g^{(2)}\|_{H^{1/2-\delta}}$$

and by symmetry also

$$\left|\int [J_t(\varphi \ast g^{(1)})J_t(\psi \ast g^{(2)}) - (J_t\varphi \ast g^{(1)})(J_t\psi \ast g^{(2)})]\right| \lesssim (t)^{-1/2} \|\varphi\|_{H^{1/2-\kappa}} \|g^{(1)}\|_{H^{1/2-\delta}} (t)^{-1/2} \|\psi\|_{H^{1/2-\kappa}} \|g^{(2)}\|_{H^{1/2-\delta}}$$

Furthermore from Proposition 7 and for sufficiently small $\kappa, \delta$

$$\left|\int (J_t\varphi \ast g^{(1)})(J_t\psi \ast g^{(2)}) - \int ((J_t\varphi \ast g^{(1)}) \circ J_t\psi)g_t^1\right| \lesssim \|J_t\varphi\|_{H^{1/2-\kappa}} \|g^{(1)}\|_{H^{1/2-\delta}} \|J_t\psi\|_{H^{1/2-\kappa}} \|g^{(2)}\|_{H^{1/2-\delta}} \lesssim (t)^{-1/2} \|\varphi\|_{H^{1/2-\kappa}} \|g^{(1)}\|_{H^{1/2-\delta}} \|\psi\|_{H^{1/2-\kappa}} \|g^{(2)}\|_{H^{1/2-\delta}}$$

and applying Proposition 10

$$\|(J_t\varphi^{(1)} \ast g^{(1)}) \circ J_t\psi_t - (J_t\varphi_t \circ J_t\psi_t)(g^{(1)})\|_{H^{-1/2+\delta}} \lesssim \|J_t\varphi_t\|_{H^{1/2-\kappa}} \|g^{(1)}\|_{H^{1/2-\delta}} \|J_t\psi_t\|_{H^{-\kappa}} \lesssim (t)^{-1/2} \|\varphi\|_{H^{1/2-\kappa}} \|g^{(1)}\|_{H^{1/2-\delta}} \|\psi\|_{H^{1/2-\kappa}}$$

and putting things together gives the estimate.

Definition 10. A smooth function $\eta$ is said to be an $S^m$ multiplier if for every multiindex $\alpha$ there exists a constant $C_\alpha$ such that

$$|\frac{\partial^\alpha}{\partial \xi^\alpha} f(\xi)| \lesssim_\alpha (1 + |\xi|)^{m-|\alpha|}, \quad \xi \in \mathbb{R}^d.$$  

We say that a family $\eta_t$ is a uniformly $S^m$ multiplier if (62) is satisfied for every $t$ with $C_\alpha$ independent of $t$.

Proposition 9. Let $\eta$ be an $S^m$ multiplier, $s \in \mathbb{R}$, $p, q \in [1, \infty]$, and $f \in B^s_{p,q}(\mathbb{R}^d)$, then

$$\|\eta(D)f\|_{B^{s-m}_{p,q}} \lesssim \|f\|_{B^s_{p,q}}.$$  

Furthermore the constant depends only on $s, p, q, d$ and the constants $C_\alpha$ in (62).
Proposition 10. Assume \( m \leq 0, \alpha \in (0,1), \beta \in \mathbb{R} \). Let \( \eta \) be an \( S^m \) multiplier and \( q,p_1,p_2 \in [1,\infty] \),
\[
\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}, \ f \in B_{p_1,\infty}^\beta, \ g \in B_{p_1,\infty}^\alpha. \]
Then for any \( \delta > 0 \),
\[
\|\eta(D)(f \succ g) - (\eta(D)f \succ g)\|_{B_{p,q}^{\alpha+\beta-m-\delta}} \lesssim \|f\|_{B_{p_1,\infty}^\beta} \|g\|_{B_{p_1,\infty}^\alpha}.
\]
The constant depends only on \( \alpha, \beta, \delta \) and the constants in \( \mathbb{R}^2 \).

For a proof see [3] Lemma 2.78.

Proposition 11. Let \( \theta \ p, p_1, p_2 \) and \( s, s_1, s_2 \) be such that \( \frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2} \) and \( s = \theta s_1 + (1-\theta)s_2 \) and assume that \( f \in W^{s_1,p_1} \cap W^{s_2,p_2} \). Then
\[
\|f\|_{W^{s,p}} \leq \|f\|_{W^{s_1,p_1}}^{\theta} \|f\|_{W^{s_2,p_2}}^{1-\theta}.
\]
For a proof see [9].

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