Geometrical representation of Euclidean general relativity in the canonical formalism

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Abstract

We give an $SU(2)$ covariant representation of the constraints of Euclidean general relativity in the Ashtekar variables. The guiding principle is the use of triads to transform all free spatial indices into $SU(2)$ indices. A central role is played by a special covariant derivative. The Gauss, diffeomorphism and Hamiltonian constraints become purely algebraic restrictions on the curvature and the torsion associated with this connection. We introduce coordinates on the jet space of the dynamical fields which cleanly separate the constraint and gauge directions from the true physical directions. This leads to a classification of all local diffeomorphism and Gauss invariant charges.

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1 Introduction

The recent progress in non-perturbative quantum gravity using Ashtekar’s formulation of general relativity is due, in part, to the application to gravity of techniques used for studying Yang-Mills theory non-perturbatively [1]. The constraints of Euclidean or Lorenzian general relativity are appealing in this formulation because they are polynomials of order at most four in the basic variables. An example of the progress is a complete non-perturbative quantization [2] of the Husain-Kuchař model. This model consists of a four-dimensional generally covariant $SU(2)$ gauge theory, which happens to be perturbatively non-renormalizable, and has a phase space like that of general relativity, except that the Hamiltonian constraint vanishes identically [3]. More recently, a possibly complete non-perturbative quantization of general relativity has been given by Thiemann [4].

An important aspect of the approach is the use of the $SU(2)$ invariant Wilson loops as elementary classical variables of the theory. There is a non-countable number of such elementary variables, since they are labelled by the (inequivalent) loops around which the holonomies are evaluated. In order to control this configuration space, it is given as the projective limit of finite dimensional spaces associated with a finite number of inequivalent loops.

The aim of this paper is a better understanding of the reduced phase space of Euclidean general relativity in the Ashtekar variables. As a first step in this direction, we give a complete classification of all $SU(2)$ and diffeomorphism invariant local quantities. At the same time, this corresponds to a complete classification of the local conservation laws of the Husain-Kuchař model. The characterization “local” comes from the fact that we work in the context of jet-spaces, which provide an appealing (countable) projective family for analytic sections.

During the analysis, we are naturally led to a special covariant derivative, given by the $SU(2)$ covariant derivative, where the spacetime indices are converted into $su(2)$ indices using the inverse triads. A first result, of considerable interest in itself, is that the constraints can be rewritten in a purely geometrical way. The diffeomorphism constraint corresponds to the vanishing of the trace of the curvature of this covariant derivative, and the Gauss constraint corresponds to the vanishing of the trace of its torsion. The Hamiltonian constraint corresponds to an additional algebraic restriction on the curvature.
The paper is organized as follows. In the next section, we fix the notations and define the models. We then introduce the special covariant derivative, and give the covariant representation of the constraints and their algebra. The following two sections are devoted to explaining how this representation may be arrived at in a constructive way.

In the third section we give some ideas about jet-spaces as applied to $SU(2)$ Yang-Mills theory. A detailed analysis of the orbit space and Wilson loops in this context serves both as a warm up before discussing the more complicated case of diffeomorphism invariance, and as a possible bridge for comparison with the quantization using Wilson loops as fundamental variables. We first present a change of coordinates which allows the separation of coordinates purely along the gauge orbits from coordinates containing the gauge invariant information. We then show how the local information contained in the Wilson loops can be expressed in terms of these latter coordinates.

In the fourth section, we apply these ideas to the diffeomorphism and Gauss constraints of Euclidean general relativity in Ashtekar’s variables to obtain the geometrical representation of the constraint surface. The classification of all the local conservation laws of the Husain-Kuchař model [3], or equivalently, all local diffeomorphism and $SU(2)$ invariant quantities is done in section 5 and a corresponding appendix, containing the local BRST cohomology of the model.

Finally, we consider models with the Hamiltonian constraint function as the integrand of an action. In three dimensions such an action is topological in the sense that its field equations require both the curvature and the torsion of the covariant derivative to vanish. In four dimensions, the covariant field equations give an identically vanishing Hamiltonian constraint.

### 2 Geometrical representation of constraints

Let us first fix the conventions. The indices $i, j, k, \ldots$ denote the $SU(2)$ indices which are raised and lowered with the Euclidean metric, while the indices $a, b, c, \ldots$ denote three dimensional space indices. Let $\tilde{\eta}^{abc}$ be the alternating symbol in space, $\tilde{\epsilon} = \frac{1}{3!} \tilde{\eta}^{abc} e^i_a e^j_b e^k_c \epsilon_{ijk}$ the determinant of the triad $e^i_a$, $\tilde{E}^a_i = \tilde{\epsilon} e^a_i$ the density weighted cotriad and $A^i_a$ the $SU(2)$ connection. The generator of $SU(2)$ rotations is denoted by $\delta_i$, so for any $SU(2)$ vector
\[ \omega^j, \delta_i \omega^j = \epsilon^j_{\ i} \omega^i. \] The SU(2) covariant derivative is defined by \( D_a = \partial_a + A^i_a \delta_i \). The corresponding curvature is \( F^{ij}_a = \partial_{[a} A^i_{b]} + \epsilon^i_{\ jk} A^j_a A^k_b \), where the square brackets denote antisymmetrization without the factor \( \frac{1}{2} \). Let us also introduce, for later purposes

\[ T^{i}_{\ ab} := D_{a} \epsilon^{i}_{\ b}. \quad (2.1) \]

The constraints of Euclidean general relativity in Ashtekar’s variables are

\[ \tilde{G}_i \equiv -D_a \tilde{E}^a_i = 0 \]
\[ \tilde{H}_a \equiv \partial_a A^b_j \tilde{E}^b_i - A^b_a \partial_b \tilde{E}^b_i = 0 \]
\[ \tilde{C} \equiv F^{i}_{ab} \tilde{E}^a_i \epsilon^i_{\ jk} = 0. \quad (2.4) \]

One often replaces the diffeomorphism constraint by the vector constraint

\[ \tilde{V}_a \equiv F^{i}_{ab} \tilde{E}^b_i = \tilde{H}_a - A^i_a \tilde{G}_i. \quad (2.5) \]

which is an intermediate step in our redefinition of the constraint surface.

Let

\[ F^{i}_{jk} = F^{i}_{ab} \epsilon^{a}_{\ j} \epsilon^{b}_{\ k}, \quad F_i = F^j_{ij}, \quad F = \epsilon_{ijk} F^{ijk} \]
\[ T^{i}_{jk} = T^{i}_{ab} \epsilon^{a}_{\ j} \epsilon^{b}_{\ k}, \quad T_i = T^j_{ij}, \quad T = \epsilon_{ijk} T^{ijk}. \quad (2.6) \]

Consider the covariant derivative

\[ D_i = \epsilon^a_i D^a. \quad (2.7) \]

Its curvature \( F^{i}_{jk} \) and torsion \( T^{i}_{jk} \) are given by

\[ [D_i, D_j] = F^k_{ij} \delta_k - T^k_{ij} D_k \]

The Bianchi identities following from \([D_k, [D_i, D_j]] + \text{cyclic} \ (k, i, j) = 0\) are

\[ D_k F^j_{mn} - F^j_{ki} T^i_{mn} + \text{cyclic} \ (k, m, n) = 0 \]
\[ D_k T^j_{mn} - T^j_{ki} T^i_{mn} + \epsilon^i_{\ ki} F^i_{mn} + \text{cyclic} \ (k, m, n) = 0. \quad (2.10) \]
The constraint surface defined by the equations (2.2)-(2.4) may equivalently be represented by the equations

\[
T_i = 0 \quad (2.11)
\]
\[
F_i = 0 \quad (2.12)
\]
\[
F = 0. \quad (2.13)
\]

Indeed, the first equation is just the Gauss constraint divided by \(\tilde{e}\), the second equation is the vector constraint divided by \(\tilde{e}\) and contracted with \(e^a_i\), while the last equation is the Hamiltonian constraint divided by \((\tilde{e})^2\).

Let \(\vec{\lambda}(x), \vec{\mu}(x)\) be space dependent \(SU(2)\) vectors with \([\vec{\lambda}, \vec{\mu}]^i = \epsilon^i_{jk}\lambda^j\mu^k\), let \(\rho(x), \sigma(x)\) be space dependent scalars and let the smeared version of the constraint be defined by

\[
\mathcal{T}[\vec{\lambda}] = \int d^3x \, \tilde{e}T_i\lambda^i \quad (2.14)
\]
\[
\mathcal{F}[\vec{\mu}] = \int d^3x \, \tilde{e}F_i\mu^i \quad (2.15)
\]
\[
\mathcal{C}[\rho] = \int d^3x \, \tilde{e}F\rho. \quad (2.16)
\]

A direct computation using the first of the Bianchi identities (2.9) gives for the constraint algebra

\[
\{\mathcal{T}[\vec{\lambda}], \mathcal{T}[\vec{\mu}]\} = \mathcal{T}[[\vec{\lambda}, \vec{\mu}]] \quad (2.17)
\]
\[
\{\mathcal{T}[\vec{\lambda}], \mathcal{F}[\vec{\mu}]\} = \mathcal{F}[[\vec{\lambda}, \vec{\mu}]] \quad (2.18)
\]
\[
\{\mathcal{T}[\vec{\lambda}], \mathcal{C}[\rho]\} = 0 \quad (2.19)
\]
\[
\{\mathcal{F}[\vec{\lambda}], \mathcal{F}[\vec{\mu}]\} = \mathcal{F}[-\vec{F}_{jk}\lambda^j\mu^k + \frac{3}{2}F_i(\lambda^i\vec{\mu} - \mu^i\vec{\lambda}) + \mathcal{F}[\vec{T}_{jk}\lambda^j\mu^k] \quad (2.20)
\]
\[
\equiv \mathcal{T}[-\vec{F}_{jk}\lambda^j\mu^k] + \mathcal{F}[\vec{T}_{jk}\lambda^j\mu^k - \frac{3}{2}T_i(\lambda^i\vec{\mu} - \mu^i\vec{\lambda})]
\]
\[
\{\mathcal{C}[\rho], \mathcal{F}[\vec{\lambda}]\} = \mathcal{T}[-\frac{1}{2}F\rho\vec{\lambda} + 2\rho\tilde{e}^m_iF^i_{km}\lambda^k + 2\rho[\vec{\lambda}, \vec{F}] + \mathcal{F}[-\frac{1}{2}\rho T\vec{\lambda} + 2[\vec{D}\rho, \vec{\lambda}] + \rho\vec{T}_{ij}\epsilon^{ij}_{\ k}\lambda^k] \quad (2.21)
\]
\[
\equiv \mathcal{C}[-4F^i(\rho D_i\sigma - \sigma D_i\rho)] \quad (2.22)
\]
\[
\equiv \mathcal{F}[-4F(\rho \vec{D}\sigma - \sigma \vec{D}\rho)] \quad (2.23)
\]
Note that the algebra of the modified vector constraints contains structure functions, but that these relations contain no derivatives of the smearing functions. This is contrary to what happens for the usual representation (2.3). All the structure functions are $SU(2)$ tensors and contain no space indices.

3 Orbit space of $SU(2)$ Yang-Mills theory in the jet-bundle approach and Wilson loops.

3.1 Gauge orbits

Let us take for simplicity Euclidean space $\mathbb{R}^3$ as the base space of the trivial principal bundle $\pi : \mathbb{R}^3 \times SU(2) \rightarrow \mathbb{R}^3$. An analytic connection $A^i_a$ is a section from $\mathbb{R}^3$ to $su(2)$ which can be represented by giving all its partial derivatives at a point $x_0$. Let us denote by $V^k$ the space with coordinates

$$(A^i_a, \partial^{b_1} A^i_a, \cdots, \partial^{b_k} \cdots \partial^{b_1} A^i_a).$$

Using a multi-index notation, denote coordinates on $V^k$ collectively by $\partial^B A^i_a$, where the order $|B|$ of the multiindex is less than $k$. The bundle $\pi : \mathbb{R}^3 \times V^k \rightarrow \mathbb{R}^3$ is called the $k$-th order jet-bundle and denoted by $J^k$. A point $\tau$ in $J^k$ has coordinates

$$\tau = (x, A^i_a, \partial^{b_1} A^i_a, \cdots, \partial^{b_k} \cdots \partial^{b_1} A^i_a).$$

The spaces $V^k$, and the bundles $J^k$, for $k \in \mathbb{N}$, form a projective family. (For more details see for example [5].)

A local function $f$ of the connection is by definition a smooth, space dependent function, which depends only on a finite number of derivatives of the connection. Hence it belongs to $C^\infty(J^k)$ for some $k$; $f = f(\tau)$.

Gauge transformations of the connection are characterized by giving a group element $g(x_0)$ at every point $x_0$. If $\tau_i$ are the Pauli matrices, then $T_j = -\frac{i}{2} \tau_j$ are generators of $SU(2)$, and gauge transformations act on $A = A^i_a dx^a$ as $A_g = g^{-1} A g + g^{-1} dg$. If $g$ is of the form $g = \exp(\epsilon^i T_i)$ with space dependent $\epsilon^i$, the corresponding infinitesimal gauge transformation are $\delta_\epsilon A^i_a = D^i_a \epsilon^i$.  

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The total derivative $d_a$ of a function $f(\tau)$ is
\[
d_a f = \partial_a f + \partial_B A^i_a \frac{\partial f}{\partial (\partial_B A^i_a)}.
\] (3.3)

Under infinitesimal gauge transformations $\delta \epsilon A^i_a = D_a \epsilon^i$, and $f(\tau)$ changes according to
\[
\delta \epsilon f = D_a \epsilon^i \frac{\partial f}{\partial A^i_a} + \cdots + \partial_c \cdots \partial_c (D_a \epsilon^i) \frac{\partial f}{\partial (\partial_c \cdots \partial_c A^i_a)}.
\] (3.4)

Thus in the jet-bundle $J^k$, infinitesimal gauge transformations are generated by the family of vector fields
\[
\vec{X}_\epsilon = \partial_B (D_a \epsilon^i) \frac{\partial}{\partial (\partial_B A^i_a)},
\] (3.5)

which are tangent to the fibers $V^k$, and are parametrized by the functions $\epsilon^i$. The vector fields on $J^k$ form a module over the algebra of local functions $C^\infty(J^k)$, and a generating set for the above family is obtained from (3.5) by choosing the following values for the functions $\epsilon^i$ and their derivatives at the point $x_0$:
\[
\epsilon^i = \delta^i_j, \quad \partial_a \epsilon^i = 0, \cdots, \partial_{a_1 \cdots a_{k+1}} \epsilon^i = 0 \quad \leftrightarrow \quad \vec{X}_j
\]
\[
\epsilon^i = 0, \quad \partial_a \epsilon^i = \delta^a_b \delta^i_j, \cdots, \partial_{a_1 \cdots a_{k+1}} \epsilon^i = 0 \quad \leftrightarrow \quad \vec{X}_j^b
\]
\[
\cdots
\]
\[
\epsilon^i = 0, \quad \partial_a \epsilon^i = 0, \cdots, \partial_{a_1 \cdots a_{k+1}} \epsilon^i = \delta^{(b_1 \cdots b_{k+1})}_{(a_1 \cdots a_{k+1})} \delta^i_j \quad \leftrightarrow \quad \vec{X}_j^{(b_1 \cdots b_{k+1})}.
\] (3.6)

In other words, an element of the family of vector fields (3.5) is obtained by fixing a point
\[(x, \epsilon^i, \partial_{a_1} \epsilon^i, \cdots, \partial_{a_k} \cdots \partial_{a_{k+1}} \epsilon^i)\] (3.7)

in the jet-bundle $J^{k+1}$ associated with the sections of the bundle $\pi : \mathbb{R}^3 \times su(2) \longrightarrow \mathbb{R}^3$. The above generating set corresponds to fixing the points defined by the vectors tangent to each of the coordinate lines in the fibre $V^{k+1}$ of $J^{k+1}$.

The vector fields
\[
\vec{X}_j^b, \cdots, \vec{X}_j^{(b_1 \cdots b_{k+1})}
\] (3.8)
are in involution. The commutation rules for the entire set $\vec{X}_j$, $\vec{X}_j^b$, $\ldots$, $\vec{X}_j^{(b_1\cdots b_{k+1})}$ are summarized in the relation

$$[\vec{X}_\epsilon, \vec{X}_\eta] = \partial_B[D_a(\epsilon^i_{jk}\epsilon^j\eta^k)] \frac{\partial}{\partial(\partial_B A_i^a)}.$$  

(3.9)

The involution property is deduced from this by choosing the canonical values (3.6) for $\epsilon^i_j$, $\eta^k$ and their derivatives.

By Frobenius theorem, the set of vector fields $\vec{X}_j$, $\vec{X}_j^b$, $\ldots$, $\vec{X}_j^{(b_1\cdots b_{k+1})}$ is integrable, and hence tangent to finite dimensional integral submanifolds of the fibers $V^k$. These submanifolds are just the gauge orbits $\mathcal{G}^k$. The collection of maximal dimensional gauge orbits defines a foliation of the fibers $V^k$; the gauge orbits are the leaves of this foliation.

### 3.2 Orbit space

Let us now investigate the linear independence of the vector fields (3.8) in order to study the structure of the space of orbits $V^k/\mathcal{G}^k$. Consider the following functions on $V^k$:

$$A_i^a, \partial_{(b_1} A_i^{a_1)}, \ldots, \partial_{(b_k} \cdots \partial_{b_1} A_i^{a_1)};$$  

(3.10)

$$F_{b_1 a}, D_{(b_2} F_{b_1)^a}, \cdots, D_{(b_k} \cdots D_{b_2} F_{b_1)^a};$$  

(3.11)

These functions can be taken as new coordinates on $V^k$ [6, 7]. In the Abelian case, this change of coordinates corresponds exactly to separating the old coordinates $\partial_{b_k} \cdots \partial_{b_1} A_a$ (3.1) into pieces symmetrized and anti-symmetrized on $a$ and $b_i$, for any $1 \leq i \leq k$. In the non-Abelian case the basic idea is the same, although the details are different due to the commutator in $F_{ab}^i$.

In the new coordinates, the family of vector fields (3.3) is

$$\vec{X}_\epsilon = \sum_{l=0}^k [\partial_{(b_l} \cdots \partial_{b_1} D_{a)} \epsilon^i_{j\cdots k}] \frac{\partial}{\partial(\partial_{(b_l} \cdots \partial_{b_1} A_i^{a_1})} + \epsilon^i_{j\cdots k} D_{(b_l} \cdots D_{b_2} F_{b_1)^a} \epsilon^k_{j\cdots k} \frac{\partial}{\partial(D_{(b_l} \cdots D_{b_2} F_{b_1)^a})}] .$$  

(3.12)

It is then straightforward to check that an equivalent generating set to (3.4) is obtained by making the following choice for the gauge parameters $\epsilon_1$ and
their derivatives:

\[ \epsilon^i = \delta_j^i, \quad D_a \epsilon^i = 0, \quad \ldots, \quad \partial_{(a_1 \ldots a_k} D_{a_{k+1})} \epsilon^i = 0 \quad \leftrightarrow \quad \tilde{Y}_j, \]

\[ \epsilon^i = 0, \quad D_a \epsilon^i = \delta^b_a \delta_j^i, \quad \ldots, \quad \partial_{(a_1 \ldots a_k} D_{a_{k+1})} \epsilon^i = 0 \quad \leftrightarrow \quad \frac{\partial}{\partial A_b^j}, \]

\[ \epsilon^i = 0, \quad D_a \epsilon^i = 0, \quad \ldots, \quad \partial_{(a_1 \ldots a_k} D_{a_{k+1})} \epsilon^i = \delta_{(a_1 \ldots a_{k+1})}^i \delta_j^i \quad \leftrightarrow \quad \frac{\partial}{\partial \partial_{(b_1 \ldots b_k} A_{b_{k+1})}^j}, \]

where

\[ \tilde{Y}_k = \sum_{i=0}^{k} [\epsilon^i_{jk} D_{(b_l} \ldots D_{b_2} F_{b_1 a)}^j \partial_{D_{(b_l} \ldots D_{b_2} F_{b_1 a)}}^{\partial}]. \quad (3.13) \]

This new choice of values for the gauge parameters corresponds to the following situation. We have the Whitney sum bundle \( J^k \oplus J^{k+1} \), whose fiber consists of \( V^k \oplus V^{k+1} \). In this direct sum, the new choice of gauge parameters corresponds to a change of coordinates in the second factor, from the old coordinates \((3.7)\) to the new ones

\[ (x, \epsilon^i, D_{a_1} \epsilon^i, \ldots, \partial_{(a_1} \ldots \partial_{a_k} D_{a_{k+1})} \epsilon^i). \quad (3.14) \]

The associated generating set for the gauge orbits is obtained by fixing, in the second factor of the sum, those points which are determined by the vectors tangent to the new coordinate lines.

The question of linear independence is now reduced to the investigation of the linear independence of the three vector fields \( \vec{Y}_j \), since the other vector fields, being tangent to different coordinate lines, are obviously independent. Alternatively, one sees that the coordinates \((3.10)\) are coordinates purely along the gauge orbits, while the remaining coordinates transform under the adjoint action of the group. This is reminiscent of what happens if one considers holonomies around closed loops as the basic variables of the theory, which also transform under the adjoint action. This analogy will be made more precise in the last part of this section.

As an example consider the space \( V^1 \). The coordinates on \( V^1 \) are

\[ (A_a^i, \partial_{(b_1} A_a^i), F_{b_1 a}). \quad (3.15) \]
In two spacetime dimensions the three vector fields $\vec{Y}_i$ are

$$\vec{Y}_k = \epsilon^i_{jk} F^i_{01} \frac{\partial}{\partial F^i_{01}}. \quad (3.16)$$

These are the field space analogs of the usual angular momentum generators (for one particle) in three dimensions $\epsilon_{ij}^k x^i \partial_k$. On the origin $F_{01} = 0$, and all three vector fields vanish, while for $F_{01} \neq 0$, there is one relation between them. Their orbits are the 2-dimensional spheres centered at the origin in $\mathbb{R}^3$ with coordinates $F^i_{01}$.

In more than two spacetime dimensions, or for $V^k$ with $k > 1$, the three vector fields $\vec{Y}_k$ are of the form

$$\vec{Y}_k = \epsilon^i_{jk} x^j_S \frac{\partial}{\partial x^i_S}, \quad (3.17)$$

where we have used $x^i_S$ to denote the coordinates $D_{(b_1} \cdots D_{b_2} F^i_{b_1)\alpha}$. The range $N$ of the index $S = 1, 2, \cdots, N$ depends on the spacetime dimension and $k$. Thus $\vec{Y}_k$ looks like the sum of the angular momentum generators of $N$ particles: $\vec{Y}_k = \vec{Y}_k^{(1)} + \cdots + \vec{Y}_k^{(N)}$. In the generic situation, all of the $N$ particles will not lie on a line through the origin, and therefore the orbits of the three $Y_k$ will be three-dimensional.

### 3.3 Wilson loops

Any gauge invariant polynomial or formal power series on $J^k$ can be written as a power series in the $x^i_S$, where all the internal indices are tied up with the invariant tensors $\delta_{ij}$ and $\epsilon_{ijk}$. This follows from an analysis of the BRST cohomology (see for instance [3]). On the other hand, it is well known that Wilson loops are non-local gauge invariant objects, and that their knowledge, for all loops, fixes the gauge potentials up to a gauge transformation [8]. The object of the following is to show that analytic Wilson loops can be written as a formal power series of invariant monomials in the coordinates $x^i_S$.

First of all, it is straightforward to see that holonomies can be described as a formal power series on $J^\infty$. Consider a path $\gamma$ in $\mathbb{R}^3$, with base point $x_0$. Divide $\gamma$ into $n + 1$ segments given by displacement vectors $\Delta x^i_k, 0 \leq k \leq n$. Then the discretized holonomy is

$$H^D_\gamma[A] = [1 - A^i_0(x_0) \tau^i \Delta x^0_0] [1 - A^i_1(x_1) \tau^i \Delta x^1_0] \cdots [1 - A^i_n(x_n) \tau^i \Delta x^a_n]$$
\[
\prod_{k=0}^{n} \left[ 1 - A^i_a(x_k) \tau^i \Delta x^a_k \right],
\]
with the continuum limit given by
\[
H_\gamma[A] = \lim_{\Delta x^a \to 0} H_{\gamma}^D[A].
\] (3.19)

To rewrite \(H_{\gamma}^D[A]\) as a polynomial on \(J^n\), we have to express each \(A^i_a(x_k)\) as a function of derivatives of \(A^i_a\) evaluated at the base point \(x_0\). The answer is simply
\[
A^i_a(x_1) = A^i_a(x_0) + (\partial_b A^i_a)(x_0) \Delta x^b_1
\]
\[
A^i_a(x_2) = A^i_a(x_0) + (\partial_b A^i_a)(x_0) (\Delta x^b_1 + \Delta x^b_2) + (\partial_{b_2} \partial_{b_1} A^i_a)(x_0) \Delta x^b_1 \Delta x^b_2
\]
etc. (3.20)

Each term in (3.18) may now be rewritten in the new coordinates (3.10)-(3.11). In the continuum limit, we get a formal power series on \(J^\infty\).

We now want to show that, for closed loops, the coordinates (3.10) do not appear and that, if one takes the trace to obtain a gauge invariant functional, the Wilson loop, the remaining coordinates (3.11) are contracted on their internal indices with invariant tensors. This can be deduced as follows. The gauge invariance of the Wilson loop implies the gauge invariance of its discretized version, which can be described, as seen above, as a polynomial on \(J^n\). Since this polynomial is gauge invariant, it must be an invariant polynomial in the \(x^i_S\) \[6\]. Alternatively, we can give the following constructive proof.

Let us adopt the conventions of the non abelian Stokes theorem \[9\], i.e., take a surface \(\Sigma\) in \(\mathbb{R}^3\) defined by analytic functions \(x^a = f^a(s, t)\) with \(0 \leq s, t \leq 1\) and \(f'^a = \partial f^a / \partial s, f^a = \partial f^a / \partial t\). Let \(h(s, t)\) be the holonomy along the curve \(f^a(s', t), 0 \leq s' \leq s\) at fixed \(t\) and \(g(s, t)\) the holonomy along the curve \(f^a(s, t'), 0 \leq t' \leq t\) at fixed \(s\). Note that in our conventions, the holonomy around a path \(\gamma\) is defined by \(H_\gamma = P \exp(\int_\gamma - A^i_a T_i dx^a)\). Following \[9\], we divide the square \([0, 1] \times [0, 1]\) into \(nm\) rectangles with sides \(\frac{1}{n}, \frac{1}{m}\). The holonomy around the boundary \(\partial \Sigma\) is given by
\[
H(\partial \Sigma) = \lim_{n,m \to \infty} H_{n,m}(\partial \Sigma)
\] (3.21)
with

\[ H_{n,m}(\partial \Sigma) = \mathcal{P}_{s,t} \prod_{l,k=0}^{n-1,m-1} Sp(l,k). \]  

(3.22)

In this equation \( \mathcal{P}_{s,t} \) denotes the ordering which puts a matrix with the large value of the first argument to the right and, for identical first arguments it puts the one with the smaller second argument to the right, while \( Sp(l,k) \) is the holonomy around the spoon loop with bowl based at \( f^n(l,k) \):

\[ Sp(l,k) \equiv h^{-1}(\frac{l}{n},0) g^{-1}(\frac{l}{n}, \frac{k-1}{m}) [I + \frac{1}{nm} T_i f^{\alpha}_{a\beta} f^b(f(\frac{l}{n}, \frac{k}{m})) + o(\frac{1}{nm})] g(\frac{l}{n}, k) h(\frac{l}{n}, 0). \]  

(3.23)

Following the reasoning in [10], we find that this holonomy reduces to

\[ Sp(l,k) = h^{-1}(\frac{l}{n},0) [I + \frac{1}{nm} T_i f^{\alpha}_{a\beta} f^b(f(\frac{l}{n}, \frac{k}{m})) \{1 + \frac{1}{m} \hat{f}^c D_c \} F^i_{ab}(0,0) + o(\frac{1}{nm})] h(\frac{l}{n},0). \]  

(3.24)

Injecting this result into formula (3.22), and using the fact that \( \text{Tr}(T_i T_j \ldots T_k) \) is an invariant tensor under the adjoint action of \( \text{su}(2) \), (it is a linear combination of \( \delta_{ij}, \epsilon_{ijk} \) and their contractions), we find indeed that the Wilson loop \( \text{Tr} H(\partial \Sigma) \) can be written as a power series depending on the field
strengths and all their symmetrized covariant derivatives $D_{(b_1} \cdots D_{b_k} F_{i)}^i$, $k = 1, \cdots, \infty$ evaluated at the base point of the loop. There are contractions on the group indices with invariant $su(2)$ tensors, and on the spatial indices with coefficients characterizing the loop $\partial \Sigma$. It also follows from this derivation that, at every finite level of approximation, the Wilson loop can be described as a local function, depending on invariant monomials in the $x^i_S$, and it is only when one takes the continuum limit $n, m \to \infty$ that it becomes a function involving an infinite number of derivatives.

4 Construction of the geometrical representation of the constraint surface

So far we have considered the jet space associated with $SU(2)$ Yang-Mills theory and considered an alternative set of coordinates on this space. This set of coordinates was defined in such a way as to isolate pure gauge directions. In this section we describe a similar change of coordinates on the jet space of Euclidean canonical general relativity in Ashtekar’s variables. This is again designed to isolate pure gauge directions, for the gauge orbits generated by the kinematical constraints. This leads to the geometrical representation of the constraint surface. The general strategy and theorems on how to do this are explained in Ref. [11], and have already been used in the context of Lorentzian tetrad gravity in Ref. [12]. Similar ideas for gravity in Ashtekar’s variables in the Lagrangian approach have been discussed in Ref. [13].

The field content of the theory is given by the $SU(2)$ connection $A_a^i$ and the dreibein $e_a^i$. In addition, there are the gauge parameters for the Gauss and diffeomorphism constraints $\eta^i$ and $\eta^a$. We must consider the jet-space of all these fields. Since we will not consider the gauge orbits generated by the Hamiltonian constraint, we do not concern ourselves with the associated gauge parameter.

The smeared constraints

$$\int d^3x \ (\tilde{G}_i \eta^i + \tilde{H}_a \eta^a)$$

(4.1)

generate the gauge transformations

$$\gamma A_a^i = D_a \eta^i + L_\eta A_a^i$$

$$\gamma e_a^i = -\eta^k \delta_k e_a^i + L_\eta e_a^i$$

(4.2)
where the Lie derivative $L_\eta$ is given by $L_\eta A^i_a = \eta^c \partial_c A^i_a + A^i_a \partial_a \eta^c$, and similarly for $e^i_a$. The gauge parameters $\eta^i, \eta^a$ are taken to be commuting in this section, but in the BRST context, they are replaced by anticommuting “ghosts”.

Following Sec. 3 of [12], we consider the set of coordinates

$$\partial(a_1 \cdots \partial a_i e^i_b), \partial(a_1 \cdots \partial a_i A^i_a)$$

(4.3)

$$D(i_1 \cdots D_i k^k_{i_1 i_2}), D(i_1 \cdots D_i k^k_{i_1 j})$$

(4.4)

$$\hat{C}^i = \eta^i + \eta^a A^i_a, \hat{\xi}^i = e^i_a \eta^a$$

(4.5)

$$\partial(a_1 \cdots \partial a_2 K^i_{a_1}), \partial(a_1 \cdots \partial a_2 L^i_{a_1})$$

(4.6)

where $l = 0, \ldots, k$. The $K^i_a$ and $L^i_a$ are gauge parameters replacing the derivatives of $\eta^i$ and $\eta^a$, and are defined by the combinations appearing on the r.h.s of the gauge transformations (1.2): $K^i_a \equiv \gamma e^i_a, L^i_a \equiv \gamma A^i_a$.

By following the same reasoning as in the previous section, that is, rewriting the vector fields generating the gauge transformations in the new coordinate system, and then showing that a generating set is obtained by giving canonical values to the combinations of gauge parameters (1.3) and (4.4), we can see that the coordinates (4.3) are purely along the gauge orbits. Indeed, giving canonical values to the parameters (4.6), one finds that the generating set contains the vector fields tangent to these coordinate lines.

An alternative way to see this is the following: If, for example, $f = f(\partial(a e^i_b), \partial(a A^i_a))$, then

$$f + \gamma f = f + \frac{\partial f}{\partial (a e^i_b)} \partial(a K^i_b) + \frac{\partial f}{\partial (a A^i_a)} \partial(a L^i_a) = f(\partial(a e^i_b) + \partial(a K^i_a), \partial(a A^i_a) + \partial(a L^i_a))$$

(4.7)

for infinitesimal transformations, which shows that the gauge transformations are just translations by (4.6) along the coordinate lines of $\partial(a e^i_b)$ and $\partial(a A^i_a)$.

The coordinates (4.4) are therefore the only ones that are partly transversal to the gauge orbits. Denoting these collectively by $T^r$, we see that they transform among themselves with the parameters (4.5) alone according to

$$\gamma T^r = -\hat{C}^k \delta_k T^r + \hat{\xi}^k D_k T^r.$$  

(4.8)

If one now expresses the sum of the smeared constraints (4.1) in the new coordinate systems, one finds the expression

$$\int d^3 x \tilde{e}(F_i \hat{\xi}^i - T_i \hat{C}^i).$$  

(4.9)
This implies the geometric representation (2.11)-(2.12) of the constraint surface in terms of the $\mathcal{T}^r$ alone, in agreement with the general theorem of [11].

It is well known that first class constraints play a double role, the first as generators of gauge transformations, the second as the restrictions which give physically acceptable initial data. Having considered the first aspect, we now turn to the constraints (2.11)-(2.13) as restrictions.

To do this explicitly, it is necessary to further split the coordinates $\mathcal{T}^r$. We decompose the dual $\epsilon^{ijl}F^k_{ij}$ of $F^k_{ij}$ into a trace free symmetric part, a trace, and an antisymmetric part,

$$F^k_{ij} = \epsilon_{ijl}F^{(kl)}_T + \frac{1}{6}\epsilon_{ij}^kF + \frac{1}{2}\epsilon_{ijl}\epsilon^{klm}F_m,$$  \hspace{1cm} (4.10)

where $F^{(kl)}_T = \frac{1}{2}\epsilon^{ij(k}F^{l)}_{ij} - \frac{1}{6}\delta^{kl}F$. From this decomposition it is clear that the only non-gauge and non-constraint coordinates on the jet space are the first and second terms in (4.10), their corresponding symmetrized derivatives, together with analogous coordinates from the identical decomposition of $T^k_{ij}$. Note also that in this decomposition, the third term is just the diffeomorphism constraint. As we will see in the next section, these remaining coordinates turn out to be useful in classifying spatial-diffeomorphism and Gauss invariant observables.

5 Classification of local conservation laws

Consider the four-dimensional generally covariant $SU(2)$ gauge field theory with action

$$S = \int \text{Tr}(e \wedge e \wedge F).$$  \hspace{1cm} (5.1)

This action is identical in form to that for general relativity except for the gauge group, which is $SU(2)$ instead of $SL(2, \mathbb{C})$. The Hamiltonian description has an identically vanishing first class Hamiltonian [3], and two first class constraints, which are the Gauss and the diffeomorphism constraints (2.2)-(2.3), or equivalently, their covariant versions (2.11)-(2.12). The geometrical coordinates presented in the last section are therefore very useful in discussing the local conservation laws of this model.

Local conserved currents $j^a$ are vector densities constructed from local functions of the fields and their derivatives which satisfy $d_aj^a = 0$ when the
equations of motion hold, and where $d_a$ is defined as in (3.3) above, but includes all the fields in the theory. The dual description is in terms of horizontal forms, which are defined to be forms on spacetime, or on space, with coefficients that are local functions. On spacetime, local functions also involve time derivatives, whereas on space, they involve only spatial derivatives. The horizontal exterior derivative is defined by $d \equiv dx^a d_a$, where the index $a$ goes from 0 to 3 for spacetime, or from 1 to 3 for space. Thus, in $n$ spacetime dimensions, we define the $(n - 1)$-form $j_{a_1 \ldots a_{n-1}} := \epsilon_{a_1 \ldots a_n} j^{a_n}$, and the conservation equation becomes $dj = 0$, when the equations of motion are satisfied.

Let $\Sigma$ denote the surface defined by the equations of motion and their derivatives, and $\tilde{\Sigma}$ the surface defined by the constraints and their spatial derivatives. Then, for theories in $n$ spacetime dimensions, the vector space of local conservation laws is the equivalence classes of horizontal $(n - 1)$-forms $j$ on spacetime which satisfy $dj = 0$ on $\Sigma$, and where two such forms are considered equivalent if they differ on $\Sigma$ by the exterior horizontal derivative of a horizontal $n - 2$ form $k$: $j \sim j + dk$ on $\Sigma$. In what follows, and in the Appendix, we consider only “dynamical” conservation laws and cohomology groups, and not “topological” ones. The latter come from non-triviality of the triad manifold ($\tilde{e} \neq 0$). Following [12], one can easily generalize the subsequent considerations to include these additional conservation laws and cohomology groups.

One can prove (see Appendix) that the vector space of local conservation laws of a diffeomorphism invariant gauge field theory with vanishing Hamiltonian is isomorphic to the direct sum of the following two vector spaces: (i) the vector space of conservation laws in space associated with $\tilde{\Sigma}$, and (ii) the vector space of equivalence classes of horizontal $(n - 1)$-forms in space which are invariant on $\tilde{\Sigma}$ under the transformations generated by the constraints, up to exact $(n - 1)$-forms in space; the equivalence relation sets two such forms to be equal if they differ on $\tilde{\Sigma}$ by the horizontal exterior derivative of a horizontal $(n - 2)$-form in space. Let us call this last space $\mathcal{O}$.

In the present case, one can prove (Appendix) that the former space is trivial. To describe $\mathcal{O}$, we use the decomposition (4.10): The non-gauge coordinates (4.4) decompose into sets

$$D_{(i_i \ldots Di_i} \epsilon^{k} {i_j)} F, \ D_{(i_i \ldots Di_i} \epsilon^{k} {i_j)} T$$

$$\quad D_{(i_i \ldots Di_i} \epsilon_{i_j} jF_T^{(kl)}, \ D_{(i_i \ldots Di_i} \epsilon_{i_j} jT_T^{(kl)}.$$  

$$\quad (5.2)$$

$$\quad (5.3)$$

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The first two sets, denoted collectively by $\mathcal{T}^r$, do not vanish on the constraint surface, while the last obviously does. Consider functions $f(\mathcal{T})$ satisfying $\delta_i f(\mathcal{T}) = 0$. Denote by $L(\mathcal{T})$ the equivalence classes of all such functions under the equivalence relation $f \sim f + D_i M^i$, where $M^i(\mathcal{T})$ transforms like a vector under $SU(2)$ transformations.

With these notations, it follows from the Appendix that $\mathcal{O}$ is described by linear combinations of the Chern-Simons functional

$$\int Tr(AdA + \frac{2}{3}A^3)$$

and the functionals

$$\int d^3x \tilde{e}L(\mathcal{T}).$$

## 6 Models from the Hamiltonian constraint function

Consider the Hamiltonian constraint function in the three-dimensional action

$$S[A^i_a, e^b_j] = \int d^3x \tilde{e}F.$$  

The corresponding field equations are

$$\frac{\delta S}{\delta A^i_a(x)} \equiv \tilde{e} \epsilon^j_k(-T_k e^a_j - T^l jk e^a_l)(x) = 0$$

$$\frac{\delta S}{\delta e^a_i(x)} \equiv \tilde{e} (-\epsilon^i_a F + 2 \epsilon^i_l F^l_{ak})(x) = 0.$$  

Contracting both equations with $\epsilon^i_a$ yields $F = 0 = T$. Inserting the definitions of $F^i_{jk}$ and $T^r_{jk}$ in terms of their duals gives the final result $F^i_{jk} = 0 = T^r_{jk}$. This means that the field equations following from (6.1) require all the local gauge invariant quantities that can be built out of the connection $A^i_a$ and the triad $e^i_a$ to vanish. It is in this sense that the action (6.1) plays the same role for the theory based on the $A^i_a$ and $e^i_a$ with the covariant derivative $D_i$ as the Chern-Simons action plays for the theory based
on $A^i_a$ alone with the covariant derivative $D_a$. The theory given by (6.1) is in fact three-dimensional (Euclidean) Einstein gravity, better known in the form

$$S[A^i_a, e^j_b] = \int \text{Tr}(e \wedge F).$$

(6.4)

One can also write down a four-dimensional action involving a function similar to the Hamiltonian constraint. Consider the following action made from an $SU(2)$ connection $A^i_\mu$, dreibein $e^{\mu i}$, and a scalar density $\tilde{\Phi}$:

$$S = \int d^4x \tilde{\Phi} e^{\mu i} e^{\nu j} F^k_{\mu \nu} \epsilon_{ijk}.$$  

(6.5)

The spacetime metric $g^{\mu \nu} = e^{i \mu} e^{j \nu}$ is degenerate, with degeneracy direction given by the 1-form $V_\mu = \tilde{\Phi} \epsilon^{\mu \alpha \beta \gamma} \epsilon_{ijk} e^{0 \alpha} e^{0 \beta}$; $V_\mu g^{\mu \nu} = 0$. The field equations are

$$e^{\mu i} e^{\nu j} F^k_{\mu \nu} \epsilon_{ijk} = 0,$$

$$\epsilon_{ijk} D_\mu (\tilde{\Phi} e^{\mu i} e^{\nu j}) = 0,$$

$$\tilde{\Phi} e^{\nu j} F^k_{\mu \nu} \epsilon_{ijk} = 0.$$  

(6.6)

It is clear that the first equation, which is like a Hamiltonian constraint, is identically satisfied as a consequence of the third equation. We suppose that $\tilde{\Phi}$ is different from zero everywhere. The dynamics is therefore determined entirely by the latter two equations. Both these equations have spatial projections which are the constraints of the theory. The standard 3+1 decomposition of the action reveals that the constraints are in fact just the Gauss and spatial-diffeomorphism constraints (2.2)-(2.3). Indeed, the 3+1 form of the action is

$$S = \int dt \int d^3x \left[ \Pi^a_i \dot{A}_i^a + A^i_0 D_a \Pi^{ai} + \tilde{\Phi} e^{ai} e^{bj} F^k_{ab} \epsilon_{ijk} \right]$$

(6.7)

where $\Pi^a_k = 2 \tilde{\Phi} e^0 i e^{aj} \epsilon_{ijk}$. We can rewrite $\tilde{\Phi} e^{ai} e^{bj} \epsilon_{ijk}$ entirely in terms of $\Pi^a_k$ and a Lagrange multiplier as follows:

$$\frac{1}{2\tilde{\Phi}} (e^{bl} e^0_l) \Pi^a_k = e^{bl} e^0_l e^a_i e^j_k$$

$$= e^{bl} (e^T_{il} + \frac{\delta_{il}}{3} e^0_m e^a_i) e^j_k$$

$$= e^{bl} e^T_{il} e^a_i e^j_k + \frac{1}{3} e^0_m e^a_i e^b_j e^a_i e^j_k.$$  

(6.8)
where $e^T_{ji}$ is the symmetric trace free part of $e^0_{ji}$. So finally

$$e^a_i e^b_j e^{ijk} = -\frac{3e^{bl}_0 e^0_m}{2\Phi e^0_m e^0_m} \Pi^{ak} + \frac{3}{e^0_m e^0_m} e^{bl}_0 e^T_{jl} e^a_j e^{ijk}. \quad (6.9)$$

Substituting this into the $3+1$ action, the second piece contracted with $F_{ab}^k$ vanishes, and we get

$$S = \int dt \int d^3x \left[ \Pi^a_i A^i_a + A^i_0 D^a_i \Pi^{ai} - N^b_k \Pi^a_k F_{ab}^k \right], \quad (6.10)$$

with the shift function $N^a$ defined by

$$N^a = \frac{3e^{al}_0 e^0_l}{2e^0_m e^0_m}. \quad (6.11)$$

The Hamiltonian constraint vanishes identically, a fact which is already clear from the first field equation. This theory is therefore locally equivalent to (5.1).

### 7 Conclusion

At the price of not using the canonical momenta given by the density weighted cotriad alone, but working instead with both the triads and the cotriads, we have shown that there is a natural covariant derivative acting on $su(2)$ tensors in canonical Euclidean general relativity in Ashtekar’s variables. The appealing feature of the associated tensor calculus is that the constraints become algebraic restrictions on the torsion and the curvature of this covariant derivative. This gives the canonical formulation a geometrical flavor analogous to the one of the original Lagrangian Einstein equations.

Furthermore, all $SU(2)$ and diffeomorphism invariant integrated local quantities can be classified. Their integrands are shown to be given either by the Chern-Simons Lagrangian or by the invariant volume element times $SU(2)$ invariant functions of covariant derivatives of the curvature and the torsion. This classification is achieved through the computation of the BRST cohomology of the Husain-Kuchař model.

These results help to address the following questions of (3):

(i) On the basis of our classification of local observables, one can look for a complete set of observables which are in involution to decide on the one
hand if the model is integrable and, on the other hand to try to quantize the model in a more traditional way, to be compared with the loop quantization of [2];

(ii) A complete computation of the local BRST cohomology including the Hamiltonian constraint would clearly show the difference the inclusion of this constraint makes on the level of local integrated observables. In fact it is not really necessary to do the computation, since we can use the results of [20], which state (in the context of metric gravity) that there are no local gauge invariant observables. Consequently, the inclusion of the Hamiltonian constraint as a generator for gauge symmetries is responsible for removing all local observables.

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Appendix: Local BRST cohomology

In this Appendix we give the computation of the local BRST cohomology associated with the theory given by the action (5.1). The analysis follows closely that of [12], where the local BRST cohomology of the Einstein-Yang-Mills theory is analysed.

Let us introduce besides the diffeomorphism and the $SU(2)$ ghosts $\eta^a$ and $\eta^i$ of (12), their canonically conjugate ghost momenta $P_a$ and $P_i$. The BRST charge [16] of the model (5.1) is given by

$$\Omega = \int d^3x \left( \tilde{G}_i \eta^i + \tilde{H}_a \eta^a - \frac{1}{2} P_k \epsilon_{ij} \eta^i \eta^j + P_i \eta^a \partial_a \eta^i + P_b \eta^a \partial_a \eta^b \right). \tag{A.1}$$

The nilpotent BRST transformations $s_\omega$ of the fields are generated by taking the Poisson bracket of the fields with $\Omega$. As in [12], it can then be verified that a new coordinate system for the jet-bundles associated to the fields $A^i_a, e^i_a$,
the ghosts $\eta^i, \eta^a$ and the ghost momenta $P_i, P_a$ is given by the coordinates \(1\) collectively denoted by $U^t$ and their BRST variations $V^t = s_\omega U^t$, the $T^r$, the $\hat{C}^i$, $\hat{\xi}^i$ and the $T^*_s \equiv D_{(i_1} \cdots D_{i_l)} \hat{C}_{i_1, \cdots, i_l}^s, D_{(i_1} \cdots D_{i_l)} \hat{\xi}_{i_1, \cdots, i_l}^s$, $l = 0, 1, \cdots$ with $\hat{C}_{i}^s = \frac{i}{\varepsilon} P_i$ and $\hat{\xi}_{i}^s = \frac{i}{\varepsilon} e_i^a (P_a - A^a_i P_k)$.

The BRST transformations in the new coordinate system act by convention from the right and are given by

$$
\begin{align*}
{s_\omega U^t} &= V^t, \quad {s_\omega V^t} = 0 \\
{s_\omega T^r} &= -\delta_k T^r \hat{C}^k + D_k T^r \hat{\xi}^k \\
{s_\omega \hat{C}^i} &= \frac{1}{2} \epsilon_{ijk} \hat{C}^j \hat{C}^k - \hat{F}^i, \quad s_\omega \hat{\xi}^i = -\delta_k \hat{C}^k \hat{\xi}^k - \hat{T}^i
\end{align*}
$$

(4.2) \text{ collectively denoted by } \hat{C}^i, \hat{\xi}^i. \text{ The part } \hat{C}_i^s \text{ belongs to the contactible part of the algebra and can be forgotten in the rest of Appendix E of that paper, because our theory does not fulfill the identities (2.9) and (2.10). Lemma 1 of [12], section 7. Apart from global considerations, the generators $U^t, V^t$ belong to the contactible part of the algebra and can be forgotten in the rest of the considerations. For the remaining generators, one first decomposes the ghosts $\eta^i, \eta^a$ into $s_\omega = s_0 + s_1 + s_2$. The part $s_0$ can be written as $s_0 = \delta + \gamma_G$, where the Koszul-Tate differential $\delta$ is defined by the first lines of (4.3) and (4.6) alone and $\gamma_G$ is defined by $\gamma_G Y = -\delta_k Y \hat{C}^k$ for $Y \in \hat{C}^i, T^r, T^*_s$ and $\gamma_G C^i = \frac{1}{2} \epsilon_{ijk} \hat{C}^j \hat{C}^k$. The part $s_1$ is given by $s_1 T^r = D_k T^r \hat{\xi}^k, s_1 T^*_s = D_k T^*_s \hat{\xi}^k$ and $s_1 \hat{\xi}^i = -\hat{T}^i$. Finally, the part $s_2$ acts as $s_2 \hat{C}^i = -F^i$, where the $s_i$’s, $i = 0, 1, 2$, vanish on the other generators.

The anticommutation relations between the $s_i$’s are the same as those in [12], where for the proof of Eq. (7.23) of [12], one has to use the Jacobi identities (2.3) and (2.10). Lemma 1 of [12] then stays true with $\omega_i (C)$ either given by a constant, or by $-\frac{2}{3} \text{Str} \hat{C}^3$, where Str denotes the symmetrized trace of the matrices.

We will now analyze equations (7.29), (7.30) of [12] directly and not follow entirely Appendix E of that paper, because our theory does not fulfill the
normality assumption needed in that approach.

By using the decomposition of the variables $T^r$ defined in section 5, we can first assume because of (7.30) of [12], that the invariant $\alpha_{i_l}$ only depends on the $T'_r$'s and the $\hat{\xi}$'s. Because $s_1$ commutes with the operator counting the generators (5.4), we then can take the equalities (7.29), (7.30) of [12] to be strong equalities and assume that $\beta_{i_{l-1}}$ is invariant and also only depends on the $T'_r$'s and the $\hat{\xi}$'s. The equations then become $s\alpha_{i_l} = 0$ and $\alpha_{i_l} = s\beta_{i_{l-1}} - 1$. From the descent equations argument of section 6 in [12], one concludes that, if $l < 3$, $\alpha_{i_l} = s\gamma_{i_l}$ for some $\gamma_{i_l}$ depending on $T'_r$, $\hat{\xi}$, $\hat{C}_{i_l}$. We can now use Appendix E of [12] starting from equation (E.4). It is at this stage, because we use Appendix C of [12] that we have to assume that our local functions depend polynomially on the $T'_r$ variables. Because we are in three dimensions and there are no abelian factors, we conclude that equation (E.21) of [12] holds with $P(\hat{F}) = 0 = q^* = G^*$ and a dependence on $T'$ rather than $T$. The same is true for equation (7.34) of [12].

Let $\hat{\theta} = \frac{1}{3!} \epsilon_{ijk} \hat{\xi}^i \hat{\xi}^j \hat{\xi}^k$. Let $q = -\frac{2}{3} \text{Str}\hat{C}^3 + \text{Str}\hat{C}\hat{F}$.

The final result is that the BRST cohomology $H^*(s_\omega)$ of the model is described by

$$\hat{\theta}(L_1(T') + L_2(T') \text{ Str}\hat{C}^3 + rq + s\beta, \ r \in \mathbb{R}).$$

(A.7)

So all the BRST cohomology is concentrated in ghost number 3 and 6. The local BRST cohomology in space $H^{*,*}(s_\omega|d)$ is obtained from $H^*(s_\omega)$ by replacing $\eta^a$ by $\eta^a + dx^a$ [11, 12]. Hence these groups can be described by

$$H^{0,3}(s_\omega|d) : d^3x \tilde{\epsilon} L_1(T') + r' \text{Str}(AF - \frac{1}{3} A^3),$$

(A.8)

$$H^{3,3}(s_\omega|d) : d^3x \tilde{\epsilon} L_2(T') \text{ Str}\hat{C}^3,$$

(A.9)

$$H^{1,2}(s_\omega|d) : dx^a \wedge dx^b \partial \frac{\partial}{\partial \eta^a \eta^b} q,$$

(A.10)

$$H^{2,1}(s_\omega|d) : dx^a \partial \frac{\partial}{\partial \eta^a} q$$

(A.11)

$$H^{3,0}(s_\omega|d) : q,$$

(A.12)

where the solutions involving $L_1(T')$, $L_2(T')$ are trivial if they are given by $D_i M^i(T')$. Note that there is no cohomology in ghost number $-1$ and form degree 3 which, by using the isomorphism of this group with the non trivial conservation laws of the constraint surface [14], excludes the latter.
It then follows from the relation between the local Hamiltonian BRST cohomology groups and the local Lagrangian BRST cohomology groups [15] and from the fact that the first class Hamiltonian is zero, that the Lagrangian local BRST cohomology groups can be entirely described by the local Hamiltonian BRST cohomology groups and in particular, $H^{-1,4}(s|d)$ in spacetime, with $s$ the BRST differential associated to the minimal solution of the Batalin-Vilkovisky master equation [17] for the Husain-Kuchař model, is isomorphic to $H^{0,3}(s_\omega|d)$. Using again the reasoning of [14], this last space also describes the local conservation laws of the latter model.

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