A Ginsparg-Wilson Approach to Lattice Chern-Simons Theory

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The concept of lattice modified symmetry formulations is adapted to the parity symmetry of gauge fields and applied to the pure Abelian Chern-Simons action in three dimensions. We derive an analogue of the Ginsparg-Wilson relation for the parity anti-symmetry, which is motivated from the perfect lattice action, and which we denote as the Chern-Simons-Ginsparg-Wilson relation (CSGWR). In addition to the overlap type solutions, we construct explicitly simple and local polynomial solutions to the CSGWR. We show that these actions are exactly invariant under a lattice modified parity transformation. That transformation is local as well, and it turns into the standard parity transformation in the continuum limit.
1 Introduction

In odd space-time dimensions, there is a possibility of adding a gauge invariant topological Chern-Simons (CS) term to the gauge field action. The CS term breaks both, the parity and the time-reversal symmetry, and — when combined with a Maxwell or Yang-Mills term — it leads to massive gauge excitations [1]. For an Abelian model in three space-time dimensions, the pure CS Lagrangian takes the form

$$L_{CS}[A] = \frac{\kappa}{2} A_\mu \epsilon_{\mu\nu\sigma} \partial \nu A_\sigma, \quad (1.1)$$

where $\kappa$ is a dimensionless coupling constant, and $\epsilon$ is the anti-symmetric unit tensor.

The pure CS theory is a topological field theory [2]. Being dominant at large distances, the CS action may be used as a low energy effective field theory for condensed matter systems [3].

On the lattice the CS theory has been studied by using the Hamiltonian formalism [4], by introducing a mixed CS action with two gauge fields of opposite parity [3], or by means of two gauge fields living on the links of two dual lattices [5].

While in the continuum the pure CS theory is exactly soluble, the situation is quite different on the lattice: the kernel defining the naive CS action exhibits a set of zeros which are not due to gauge invariance and the theory is not integrable even after gauge fixing [6]. Since the Lagrangian (1.1) is of first order in the derivative, the appearance of extra zeros in its naive lattice formulation is reminiscent of the doubling of fermions on the lattice. It has been recently shown [7] that the non-integrability of the CS kernel is a general feature of any gauge invariant, local, parity-odd and cubically symmetric gauge theory on an infinite Euclidean lattice, provided that the non-compact link variables are parity-odd,

$$A_\mu(\vec{x}) \to -A_\mu(-\vec{x}) \quad \text{and} \quad A_\mu(\vec{p}) \to -A_\mu(-\vec{p}). \quad (1.2)$$

Here $\vec{x}$ are the link centers and $\vec{p}$ are the corresponding momenta. Since an additional Maxwell term regularizes the CS action [7], the presence of $\pm$Throughout this paper, we define locality on the lattice such that the couplings decay at least exponentially in the distance, which corresponds to analyticity in momentum space.
extra zeros did not cause problems in previous investigations of the Maxwell CS action on the lattice. Adding a Maxwell term to the CS action opens a gap in the spectrum with a mechanism similar to Wilson fermions, where a gap is opened and the unphysical poles at the corners of the Brillouin zone are shifted to large energies. However, in the case of lattice fermions it is possible to keep track of the chiral symmetry much more closely than it is done in the case of Wilson fermions, by requiring the anti-commutator of the inverse Dirac operator with $\gamma_5$ to be local. This is known as the Ginsparg-Wilson relation (GWR) [8]; it excludes additive mass renormalization [9] and it preserves chiral symmetry in an exact, though lattice modified form [10].

Inspired by this analogy with fermionic models, we adapt the Ginsparg-Wilson approach to CS theories. This amounts to modifying the standard definition (1.2) of parity on the lattice. Such an approach has been followed in a interesting paper by Fosco and López [11]; however, their modified parity transformation is non-local, which is problematic in view of its continuum limit. ² In this paper we construct a Ginsparg-Wilson approach to lattice CS gauge theory in which the modified parity transformation, as well as the action itself, is manifestly local. ³

For fermions in odd dimensions, a lattice modified parity symmetry was worked out already in the framework of the GWR [13]. The so-called parity anomaly is reproduced correctly in this way, i.e. different types of 3d Ginsparg-Wilson fermions cover all parity universality classes. However, in that formulation the parity transform of the gauge field is standard, so in the case of a pure gauge action a new transformation for the gauge fields is needed.

The search for an integrable CS kernel on the lattice is motivated by the attempt to compute topological invariants in a lattice gauge theory. There are, for instance, potential applications to polymer physics [14].

In Section 2 we discuss the perfect CS lattice action, which motivates the formulation of the Chern-Simons-Ginsparg-Wilson relation (CSGWR) as a criterion for a lattice modified parity anti-symmetry. In Section 3 we solve the CSGWR along the lines of the overlap formula. However, we then concentrate

²To be explicit: the lattice action in Ref. [11] is local, but the lattice modified parity transformation includes in leading order of the momentum a non-local term $p_\mu p_\nu / p^2$.

³A synopsis of this work was anticipated in Ref. [12].
on a simple polynomial solution, which is constructed in Section 4. This solution does not have a fermionic analogue. In Section 5 we derive the corresponding locally modified parity transformation. Section 6 is devoted to our conclusion and an outlook regarding the topological properties of such lattice formulations.

2 A perfect Chern-Simons lattice action

The GWR for lattice fermions was first observed for the perfect action of free fermions, and then adapted as a general condition for a lattice modified but exact chirality in even dimensions [8, 9, 10]. For fermions in odd dimensions it provides a lattice modified parity symmetry [13]. In both cases, the anomalies are reproduced correctly due to a non-trivial transformation of the fermionic measure under the lattice modified chiral resp. parity transformation.

We are going to carry out the analogous step for the pure Chern-Simons action. For comparison, we first briefly review the perfect action for free lattice fermions. We follow the steps presented in detail in Refs. [15].

2.1 Review of the motivation for the fermionic Ginsparg-Wilson relation

We consider a free, massless fermion in $d$ dimensional Euclidean space. Its action in the continuum,

$$s[\bar{\psi}, \psi] = \frac{1}{(2\pi)^d} \int d^d p \, \bar{\psi}(-p) i \gamma_\mu p_\mu \psi(p) ,$$  \hspace{1cm} (2.1)

is chirally symmetric in even dimensions, and parity odd in odd dimensions.

We perform a block variable renormalization group transformation (RGT) onto a lattice of unit spacing. The blocking factor is infinite in this case, so the lattice fermion fields $\bar{\Psi}_x, \Psi_x$ are related to the continuum fields as

$$\bar{\Psi}_x \sim \int_{C_x} d^d y \, \bar{\psi}(y) , \hspace{1cm} \Psi_x \sim \int_{C_x} d^d y \, \psi(y) ,$$ \hspace{1cm} (2.2)

where $C_x$ is a unit hypercube with center $x \in \mathbb{Z}^d$. If we use a “$\delta$ function blocking”, then the above relations are equations. However, we keep the
RGT more general — for good reasons — so our statement is just that the lattice variables and the block integrals are related.

To be explicit, the perfect lattice action $S[\bar{\Psi}, \Psi]$ is given by

$$e^{-S[\bar{\Psi}, \Psi]} = \int D\bar{\psi} D\psi D\bar{\eta} D\eta \exp \left\{ -s[\bar{\psi}, \psi] + \sum_{x \in \mathbb{Z}^d} \left( \bar{\Psi}_x - \int_{C_x} \bar{\psi}(y) \right) \eta_x 
+ \bar{\eta}_x \left( \Psi_x - \int_{C_x} \psi(y) \right) + \sum_{y \in \mathbb{Z}^d} \bar{\eta}_x R_{xy} \eta_y \right\} .$$

Here $\bar{\eta}, \eta$ are auxiliary Grassmann lattice fields. In the limit $R \to 0$ the blocking (2.2) is implemented by a $\delta$ function, but we keep the term $R$ more general. Performing the integrals over $\bar{\eta}, \eta$ and $\bar{\psi}, \psi$ we arrive at the perfect lattice action [15]

$$S[\bar{\Psi}, \Psi] = \frac{1}{(2\pi)^d} \int_{B} d^d p \bar{\Psi}(-p) \Delta^{-1}(p) \Psi(p) ,$$

$$\Delta(p) = \sum_{l \in \mathbb{Z}^d} \frac{\Pi^2(p + 2\pi l)}{\bar{\gamma}_\mu(p_\mu + 2\pi l_\mu)} + R(p) ,$$

$$B := ] - \pi, \pi]^3 , \quad \Pi(p) := \prod_{\mu=1}^{d} \frac{\hat{p}_\mu}{p_\mu} , \quad \hat{p}_\mu := 2 \sin \frac{p_\mu}{2} .$$

We see that the perfect action is again chirally resp. parity symmetric in the usual sense if and only if $R = 0$. In this case, however, the Dirac operator $\Delta^{-1}$ is non-local, which avoids a contradiction with the Nielsen-Ninomiya No-Go theorem [16] resp. its counterpart in odd dimensions [13]. Therefore we better insert a non-vanishing, local term $R$. Then the lattice action becomes local [18, 15] and the exact symmetry ($\Delta + \Delta^\dagger = 0$) is replaced by the GWR

$$\Delta + \Delta^\dagger = 2R .$$

The lattice modifications of the symmetries avoids again a contradiction with the No-Go theorem. However, we know that this lattice formulation is related to the exact symmetries in the continuum solely by the renormalization group, hence we expect the symmetry to be still present in some way. This is confirmed by the invariance under a local \footnote{The locality of the transformation depends on the locality of the term $R$, as discussed for example in Ref. [17].} lattice modified chiral
resp. parity [13] transformation. We see in eq. (2.3) that a non-trivial term \( \hat{\eta}R\eta \) breaks the symmetry, but such a breaking in the transformation term is only superficial and ultimately harmless.

In fact, the GWR can be used as a criterion for a lattice modified but exact symmetry also in the interacting case [8, 9, 13]. Once this relation is established, one may also construct new solutions, which are not related to the perfect action any more [19].

2.2 A perfect lattice Chern-Simons term

We now want to repeat the blocking from the continuum for Abelian gauge fields, in order to construct a perfect Chern-Simons lattice action. This is new for itself; moreover, it will motivate the CSGWR and its (non-perfect) solutions to be constructed in Sections 3 and 4.

For the blocking of an Abelian continuum gauge field \( a_\mu \), we follow again Ref. [15]. The blocking scheme amounts to the integration over all straight connections between points in adjacent hypercubes, which are separated by the unit vector \( \hat{\mu} \). This implies the following relation to the lattice gauge field \( A_\mu \),

\[
A_{\mu,x} \sim \int_{C_{x-\hat{\mu}/2}} d^d y \left( 1 + y_\mu - x_\mu \right) a_\mu(y) + \int_{C_{x+\hat{\mu}/2}} d^d y \left( 1 - y_\mu + x_\mu \right) a_\mu(y), \tag{2.6}
\]

where \( x \) is now a link center (i.e. \( x_\mu \) is a half-integer, while the remaining components of \( x \) are integers). Note that a continuum gauge transformation

\[
a_\mu \rightarrow a_\mu + \partial_\mu \varphi \tag{2.7}
\]

just implies a lattice gauge transformation

\[
A_{\mu,x} \rightarrow A_{\mu,x} + \phi_{x+\hat{\mu}/2} - \phi_{x-\hat{\mu}/2}, \quad \phi_x = \int_{C_x} d^d y \varphi(y). \tag{2.8}
\]

In analogy to the fermionic RGT (2.3), we construct the perfect lattice Chern-Simons action \( S_{CS}[A] \) as

\[
e^{-S_{CS}[A]} = \int Da \ DE \ exp \left\{ - s[a] \right\}
+ \sum_x \left[ iE_{\mu,x} \left( A_{\mu,x} - \int_{C_{x-\hat{\mu}/2}} d^d y \left( 1 + y_\mu - x_\mu \right) a_\mu(y) \right) \right].
\]
\[
- \int_{C_{x+\mu/2}} d^4y \left( 1 - y_\mu + x_\mu \right) a_\mu(y) - \frac{1}{2} \sum_y E_{\mu,x} R_{\mu\nu,x\nu E_{\nu,y}} \right) ,
\]
\[
= \int Da DE \exp \left\{ - s[a] + \frac{i}{(2\pi)^3} \int d^3p \ E_\mu(-p) a_\mu(p) \Pi_\mu(p) \right.
\]
\[
- \frac{1}{(2\pi)^3} \int_B d^3p \left[ iE_\mu(-p) A_\mu(p) + \frac{1}{2} E_\mu(-p) R_{\mu\nu}(p) E_\nu(p) \right] ,
\]
where
\[
s[a] = \frac{\kappa}{2} \frac{1}{(2\pi)^3} \int d^3p \ a_\mu(-p) c_{\mu\nu}(p) a_\nu(p) , \quad c_{\mu\nu}(p) := -i\epsilon_{\mu\alpha\nu} p_\alpha ,
\]
\[
\Pi_\mu(p) := \frac{\hat{p}_\mu}{p_\mu} \prod_{\nu=1}^3 \frac{\hat{p}_\nu}{p_\nu} . \quad (2.9)
\]
\[E_\mu\] is an auxiliary lattice vector field, which is also defined on the link centers. The classical equation of motion for the continuum gauge field \(a_\mu\) reads
\[
a_{\mu,cl}(p) = \frac{i}{\kappa} c^{(inv)}_{\mu\nu}(p) E_\nu(p) \Pi_\nu(p) . \quad (2.10)
\]
Note that the matrix \(c_{\mu\nu}\) is anti-symmetric and therefore singular. Hence the symbol \(c^{(inv)}\) represents a \textit{regularized} inversion of \(c_{\mu\nu}\), which can be obtained for instance by adding a tiny kinetic term to \(s[a]\). We do not specify this regularization here, but just fix after regularization the property
\[
c^{\text{reg}}_{\mu\rho} c^{(inv)}_{\rho\nu} = \delta_{\mu\nu} . \quad (2.11)
\]
Inserting the classical continuum gauge field is equivalent to carrying out the Gaussian functional integral \(\int Da\). It leads to
\[
e^{-S[A]} = \int DE \exp \left\{ - \frac{1}{(2\pi)^3} \int d^3p \left[ iE_\mu(-p) A_\mu(p) \right.ight.
\]
\[
+ \left. \frac{1}{2} E_\mu(-p) C^{(inv)}_{\mu\nu}(p) E_\nu(p) \right] \right\} \quad (2.12)
\]
and the integration \(\int DE\) yields
\[
S[A] = \frac{1}{(2\pi)^3} \int_B d^3p A_\mu(-p) C_{\mu\nu}(p) A_\nu(p)
\]
\[
C^{(inv)}_{\mu\nu}(p) = R_{\mu\nu}(p) + \frac{2}{\kappa} \sum_{\ell \in \mathbb{Z}^3} c^{(inv)}_{\mu\nu}(p + 2\pi \ell) \Pi_\mu(p + 2\pi \ell) \Pi_\nu(p + 2\pi \ell) . \quad (2.13)
\]
At this stage, the perfect lattice kernel $C_{\mu\nu}$ can be extracted by removing the regularization incorporated in the term $c^{\text{reg}}_{\mu\nu}$. We assume $R$ to be even with respect to standard parity, $R_{\mu\nu}(p) = R_{\mu\nu}(-p)$, hence

$$
C^{(\text{inv})}_{\mu\nu}(p) + C^{(\text{inv})}_{\mu\nu}(-p) = 2R_{\mu\nu}(p) \quad \text{resp.}
$$

$$
C_{\mu\nu}(p) + C_{\mu\nu}(-p) = 2C_{\mu\rho}(-p)R_{\rho\sigma}(p)C_{\sigma\nu}(p),
$$

(2.14)

which we call the general CSGWR. A finite term $R_{\mu\nu}$ breaks the parity antisymmetry superficially in the lattice action $S_{\text{CS}}[A]$.

In particular, for the choice

$$
R_{\mu\nu}(p) = \frac{1}{2}\delta_{\mu\nu}
$$

(2.15)

we arrive at the CSGWR in the simple form that we are going to use below,

$$
C_{\mu\nu} + C^P_{\mu\nu} = C^P_{\mu\rho}C_{\rho\nu},
$$

(2.16)

where $C^P_{\mu\nu} := PC_{\mu\nu}P$, and $P$ is the standard parity transformation operator.

## 3 Overlap-type solution to the CSGWR

First we remark that one can solve the CSGWR in analogy to the fermionic overlap formula. To this end, it is convenient to start from the formulation

$$
\Gamma^P_{\mu\rho}\Gamma_{\rho\nu} = \delta_{\mu\nu}, \quad \Gamma_{\mu\rho} := C_{\mu\rho} - \delta_{\mu\rho}.
$$

(3.1)

Let us take some lattice CS kernel $C_{\mu\nu}^{(0)}$ to start with, such as the Fröhlich-Marchetti formulation [6] (its form is given in footnote 5). We assume the correct continuum limit and the absence of doublers for $C_{\mu\nu}^{(0)}$, but we do not require any form of parity symmetry on the lattice. Hence the corresponding term $\Gamma_{\mu\nu}^{(0)} = C_{\mu\nu}^{(0)} - \delta_{\mu\nu}$ will in general not obey relation (3.1), but we can enforce the CSGWR by the transition to

$$
\Gamma_{\mu\nu} = \Gamma^{(0)}_{\mu\rho} \cdot \left(\Gamma^{(0)} \Gamma^{(0)^P} \right)^{-1/2}_{\rho\nu},
$$

(3.2)

which is a sort of gauge overlap formula [20]. This is similar to the formula introduced in Refs. [19] for the lattice Dirac operator $D$, where the Wilson
operator \( D_W \) was inserted as \( D_0 \); the use of kernels \( D_0 \) different from \( D_W \) was motivated in Refs. [21].

We do not discuss further properties of this type of solution here; instead we are going to show that simpler and more practical solutions exist, where further properties of interest — gauge invariance, locality of the action and of the corresponding modified parity transform — will be demonstrated.

4 A polynomial solution to the Chern-Simons-Ginsparg-Wilson relation

We start from a general form of a CS lattice action,

\[
S[A] = \frac{\kappa}{2} \frac{1}{(2\pi)^3} \int_B d^3p A_\mu(-p)C_{\mu\nu}(p)A_\nu(p),
\]

again on a 3d unit lattice in Euclidean space. The kernel \( C_{\mu\nu} \) is to be specified. For practical reasons we put the non-compact link variables \( A_\mu \) on the link centers again.

Our first concern is solving the CSGWR, eq. (2.16), which is equivalent to eq. (3.1). We make an explicit ansatz for the tensor \( \Gamma \) in momentum space,

\[
\Gamma_{\mu\nu}(p) = \delta_{\mu\nu}u(p) + L_{\mu\nu}(p)v(p) + M_{\mu\nu}(p)w(p),
\]

\[
L_{\mu\nu}(p) := -i\epsilon_{\mu\alpha\nu}\hat{p}_\alpha, \quad M_{\mu\nu}(p) := \hat{p}^2\delta_{\mu\nu} - \hat{p}_\mu\hat{p}_\nu, \quad \hat{p}^2 := \sum_{\alpha=1}^{3}\hat{p}_\alpha^2.
\]

The coefficients \( u, v, w \) in ansatz (4.2) are all assumed to be \textit{parity-even}, \textit{local} and symmetric under permutation of the axes. We denote the latter property as “\textit{lattice isotropy}”. Moreover, the continuum limit imposes

\[
u(p) = -1 + O(p^2), \quad v(p) = \frac{\kappa}{2} + O(p^2).
\]

The function \( w(p) \) does not have a constraint of that kind; it must just be non-vanishing. \footnote{As a special case, the Fröhlich-Marchetti kernel corresponds to \( u = -1, v = w = \kappa/2 \).}
The linear term $L_{\mu\nu}$ is parity-odd and it provides the naive discretization of the kernel $C_{\mu\nu}$. On the other hand, the Maxwell term $M_{\mu\nu}$ is parity-even, hence it breaks the odd parity symmetry of $C_{\mu\nu}$. We know from Ref. [7] that some parity-even ingredient in $C_{\mu\nu}$ is needed in order to avoid the doubling problem.

To compute the product $\Gamma^P\Gamma$, we make use of the identities
\[
L^P_{\mu\rho}(p)L_{\rho\nu}(p) = [\delta_{\mu\beta}\delta_{\alpha\nu} - \delta_{\mu\nu}\delta_{\alpha\beta}] \hat{p}_\alpha\hat{p}_\beta = -M_{\mu\nu}(p),
\]
\[
M^P_{\mu\rho}M_{\rho\nu} = (\hat{p}^2)^2\delta_{\mu\nu} - \hat{p}^2 \hat{p}_\mu\hat{p}_\nu = \hat{p}^2 M_{\mu\nu}. \tag{4.4}
\]
These are simple lattice analogues to the well-known relations in the continuum. They occur in this form because we have assumed lattice isotropy for the terms $v(p)$ and $w(p)$.

These two properties are crucial for the polynomial solution of the CS-GWR, because they imply that the right-hand side reproduces the same structure as the ansatz for $\Gamma$, up to some coefficients (which are momentum dependent, however).

One may compare the ansatz (4.2) with a Wilson fermion, where the Maxwell term plays the rôle of the Wilson term, which also breaks parity in odd dimensions. In both cases, one adds an $O(a)$ suppressed parity-even term that removes the doublers. However, the terms in the Wilson-Dirac operator do not have any properties analogous to the relations (4.4), hence in the framework of that structure one cannot find any solution of the (fermionic) GWR. \footnote{If one tries to insert the Wilson Dirac operator $D_W(x, y)$ into the GWR one arrives at a term $R$ which is non-local; for instance for the free fermion in $d = 4$ ($d = 2$) it decays only as $|x - y|^{-6}$ ($|x - y|^{-4}$) [17], which is not acceptable as a solution of the GWR.}

Based on the identities (4.4) we arrive at
\[
\Gamma^P_{\mu\rho}\Gamma_{\rho\nu} = u^2 \delta_{\mu\nu} + [2uw - v^2 + w^2 \hat{p}^2] M_{\mu\nu}. \tag{4.5}
\]
In view of condition (4.3), it is inevitable to set
\[
u = -1. \tag{4.6}
\]
Hence the breaking of the anti-parity of $C_{\mu\nu}$ is solely due to the Maxwell term. For the coefficients $v, w$ we have to require
\[
v^2 = -2w + w^2 \hat{p}^2. \tag{4.7}
\]
This condition allows for many solutions. It fixes, however, \( w(p = 0) = -\kappa^2/8 \). More generally, solving for \( w \) and requiring locality\(^7\) yields
\[
w = \frac{1}{\hat{p}^2} \left[ 1 - \sqrt{1 + \hat{p}^2 v^2} \right].
\]
(4.8)

At this point, we keep the maximal freedom in the choice of the functions \( v \) and \( w \), because we will need it later on, in order to fulfill a further criterion. We also see that the Fröhlich-Marchetti action — or any other action with constant coefficients \( v \) and \( w \) — fails to solve the CSGWR.

### 4.1 Gauge invariance

We now probe the effect of a general gauge transformation for the Abelian gauge field that we are considering. The definition requires first another lattice discretization of a derivative, which is actually ambiguous. However, in the spirit of the above formulation of the action, it is obvious to choose the symmetric nearest-neighbor formulation, as we did before in eq. (2.8),
\[
A_\mu(p) \to A'_\mu(p) = A_\mu(p) - i\hat{p}_\mu \phi(p).
\]
(4.9)

The difference between the action after and before this gauge transformation reads
\[
S[A'] - S[A] = \frac{\kappa}{2 (2\pi)^3} \int_B \, d^3p \left[ i\phi(-p)\hat{p}_\mu C_{\mu\nu}(p)A_\nu(p) - iA_\mu(-p)C_{\mu\nu}(p)\hat{p}_\nu \phi(p) - \phi(-p)\phi(p)\hat{p}_\mu C_{\mu\nu}(p)\hat{p}_\nu \right].
\]
(4.10)

Obviously, this difference vanishes if
\[
\hat{p}_\mu C_{\mu\nu}(p) = C_{\mu\nu}(p)\hat{p}_\nu = 0.
\]
(4.11)

It is easy to see that
\[
\hat{p}_\mu L_{\mu\nu}(p) = -i\epsilon_{\mu\nu\alpha\beta} \hat{p}_\alpha \hat{p}_\beta = 0 = L_{\mu\nu}(p)\hat{p}_\nu, \quad \text{and}
\]
\[
\hat{p}_\mu M_{\mu\nu}(p) = \hat{p}_\mu [\hat{p}_\nu \hat{p}^2 - \hat{p}_\nu \hat{p}_\nu] = \hat{p}_\nu \hat{p}_\mu \delta_{\mu\nu} - \hat{p}_\nu \hat{p}_\mu = 0 = M_{\mu\nu}(p)\hat{p}_\nu.
\]

\(^7\)This requirement singles out one of the two solutions for \( w(p) \) in terms of \( v(p) \).
Our kernel $C_{\mu\nu}(p)$ is a linear combination of $L_{\mu\nu}$ and $M_{\mu\nu}$,

$$C_{\mu\nu}(p) = L_{\mu\nu}(p)v(p) + M_{\mu\nu}(p)w(p) , \quad (4.12)$$

where the coefficients $v(p)$ and $w(p)$ are lattice isotropic. Therefore it obeys the property (4.11), and we conclude that our lattice CS action — given by eqs. (4.2), (4.6) and (4.8) — is indeed gauge invariant.

## 5 A lattice modified parity transformation

We start from a quite general ansatz for a lattice modified parity transformation of the gauge field, \(^8\)

$$A^P_\mu = P(\delta_{\mu\nu} + X_{\mu\nu})A_\nu , \quad (A^t_\mu)^P = A^t_\nu(\delta_{\nu\mu} + X_{\nu\mu})P , \quad (5.1)$$

where the tensor $X$ is of $O(a)$.

We want our lattice CS term to be odd with respect to such a modified parity transformation. \(^9\) This amounts to the requirement

$$C_{\mu\nu} = -[\delta_{\mu\rho} + X_{\mu\rho}] C^P_{\rho\sigma} [\delta_{\sigma\nu} + X_{\sigma\nu}] . \quad (5.2)$$

We insert $C_{\mu\nu} = L_{\mu\nu}v + M_{\mu\nu}w$ and arrive at the general condition

$$2M_{\mu\nu}w = X_{\mu\rho}[L_{\rho\mu}v - M_{\rho\mu}w] + [L_{\mu\rho}v - M_{\mu\rho}w]X_{\rho\nu} + X_{\mu\rho}[L_{\rho\sigma}v - M_{\rho\sigma}w]X_{\sigma\nu} . \quad (5.3)$$

To search for explicit solutions, we also make an ansatz for $X$. We assume it to have the same structure as $C$,

$$X_{\mu\nu} = L_{\mu\nu}x + M_{\mu\nu}y , \quad (5.4)$$

\(^8\)At this point, our notation distinguishes between the column vector field $A_\mu$ and its transpose $A^t_\mu$, in order to avoid confusion. In the rest of this paper we simply write $A_\mu$ in both cases, as it usually done.

\(^9\)We repeat that the two terms in $C = Lv + Mw$ are odd resp. even with respect to standard parity $P$. However, we require the entire term $C$ to be exactly odd under the modified parity, which we are going to construct.
the functions $x(p), y(p)$ being also even and lattice isotropic. In addition we have to arrange for them to be local, see below. Inserting this ansatz for $X$ into condition (5.3) leads to

$$2M_{\mu \nu}w = M_{\mu \nu} \left[ 2vx + \hat{p}^2(2vxy - 2wy - wx^2) - (\hat{p}^2)^2 wy^2 \right] + L_{\mu \nu}M_{\rho \nu} \left[ 2(vy - wx) + vx^2 + \hat{p}^2(vy^2 - 2wxy) \right].$$  \hspace{1cm} (5.5)

All in all, we have now three non-linear constraints for the four coefficients $v, w, x, y$:

\begin{align*}
v^2 + 2w - w^2 & = 0 \quad (5.6) \\
2(w - vx)(1 + y\hat{p}^2) + w\hat{p}^2(x^2 + \hat{p}^2 y^2) & = 0 \quad (5.7) \\
2(vy - wx) + vx^2 + y\hat{p}^2(vy - 2wx) & = 0 \quad (5.8)
\end{align*}

This system of equations is soluble in many ways, but we still have to worry about locality.

We first consider the low momentum expansion. In agreement with eq. (4.3) we assume $v$ to take the form

$$v(p) = \frac{\kappa}{2} + v_1\hat{p}^2 + O(\hat{p}^2)^2, \quad (v_1 = \text{const.}) \hspace{1cm} (5.9)$$

which implies

\begin{align*}
w(p) &= \frac{-\kappa^2}{8} + \frac{\kappa}{2} \left( \frac{\kappa^3}{64} - v_1 \right) \hat{p}^2 + O(\hat{p}^2)^2, \\
x(p) &= \frac{-\kappa}{4} + \frac{1}{2} \left( \frac{\kappa^3}{64} - v_1 \right) \hat{p}^2 + O(\hat{p}^2)^2, \\
y(p) &= \frac{\kappa^2}{32} + \frac{\kappa}{8} \left( v_1 - \frac{5\kappa^3}{256} \right) \hat{p}^2 + O(\hat{p}^2)^2. \hspace{1cm} (5.10)
\end{align*}

The existence of this expansion around $p = 0$ indicates locality. A strict analogy to the fermionic case would suggest $X_{\mu \nu} = -\frac{1}{2}C_{\mu \nu}$, but we see from the expansion (5.10) that we have to deviate from this guess, since it solves condition (5.2) only up to $O(\hat{p}^2)$.

Multiplying condition (5.7) and (5.8) first by $v$ resp. $\hat{p}^2 w$, and second by $w$ resp. $v$ — along with the use of condition (5.6) — simplifies these quadratic forms to

\begin{align*}
x(1 + \hat{p}^2 y) &= -\frac{1}{2}v, \\
x^2 + 2y + \hat{p}^2 y^2 &= -w. \hspace{1cm} (5.11)
\end{align*}
To solve this system explicitly, we now implement the constraint (5.6) by inserting \( v \) and \( w \) as given in eqs. (5.11). If we find any solution of the resulting equation

\[
x^4 \hat{p}^2 - 2x^2 (1 + \hat{p}^2 y)^2 + \frac{1}{\hat{p}^2} \left[ (1 + \hat{p}^2 y)^4 - 1 \right] = 0 ,
\]

we obtain a complete solution, with \( v \) and \( w \) given by eqs. (5.11). In agreement with the low momentum expansion and with the requirement of locality, there are for instance unique expressions for \( v, w \) and \( y \) in terms of \( x, y = \sqrt{1 + \hat{p}^2 x^2} - 1 \).

\[
v = -2x \sqrt{1 + \hat{p}^2 x^2} \quad w = -2x^2 .
\]

This still represents the general set of local solutions.

To find explicit examples of solutions, we may choose \( x \) — or any of the other coefficients — so that the constraints of the low momentum expansion is respected, and insert it into eqs. (5.13). Then all the three constraints are guaranteed.

As a simple example we choose \( x \) to be constant,

\[
x = -\frac{\kappa}{4} , \quad y = \sqrt{1 + \frac{\kappa^2}{16} \hat{p}^2} - 1 ,
\]

\[
v = \frac{\kappa}{2} \sqrt{1 + \frac{\kappa^2}{16} \hat{p}^2} , \quad w = -\frac{\kappa^2}{8} .
\]

Thus we have finally arrived at a fully explicit solution of the CSGWR, which is exactly invariant under a lattice modified parity transformation. Both, the action and the transformation term, are manifestly local.

6 Conclusions

We constructed a modified parity symmetry for lattice gauge fields and applied it to the pure Abelian CS action in three dimensions. Based on the perfect action, we derived an analogue of the GWR for the parity anti-symmetry,
and we exhibited explicit, local solutions to this relation. It is interesting that for gauge fields simple polynomial solutions exist, in contrast to Ginsparg-Wilson fermions. Unlike an earlier approach along these lines [11], our lattice modified parity transformation is local as well. Hence this is a clean way to avoid the doubling problem of the pure CS lattice action, while keeping track of the parity anti-symmetry at finite lattice spacing.

It is well-known that some suitable modifications of the Fröhlich-Marchetti kernel are integrable, still preserving locality [4]. However, those ad hoc modifications impose a constraint on the kernel which is not compatible with the requirements of the CSGWR, and it is not odd under a locally modified parity transformation.

It is an open question if the lattice CS theories constructed in this work are topological on the lattice. It is certain that our polynomial solutions turn into a topological theory in the continuum limit, but it still requires further investigation to figure out if this property even holds on the lattice. A good check for this property will be an explicit computation of the topological invariant on the lattice.

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