Bloch vector, disclination and exotic quantum holonomy

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A topological formulation of the eigenspace anholonomy, where eigenspaces are interchanged by adiabatic cycles, is introduced. The anholonomy in two-level systems is identified with a disclination of the director (headless vector) of a Bloch vector, which characterizes eigenprojectors. The extensions of this formulation to nonadiabatic cycles and $N$-level systems are outlined.

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Introduction

An adiabatic cycle may induce a non-trivial geometric phase factor to the state vector of a quantum system [1]. This phenomena is called the phase holonomy, which reflects the underlying gauge structure [2, 3]. The adiabatic time evolution of a state vector, which initially is an eigenstate of a Hamiltonian, along a cycle is called a lifted path of the cycle [4]. The discrepancy between the initial and final points of the lifted path correspond to the phase holonomy. Aharonov and Anandan have extended the concept of the phase holonomy to nonadiabatic cycles [5]. This is accomplished by the use of a projection operator, instead of adiabatic parameters, to define the cycles. It has been found that the phase holonomy plays crucial role in a variety of physical contexts, such as quantum Hall effect, quantum information theory and quantum field theory [4, 6, 7].

After these developments on the studies of the quantum holonomy in phase factors, examples of an exotic kind of quantum holonomy have been reported [8–14]. Here, an adiabatic cycle may open up the corresponding trajectory of adiabatic (quasi-)eigenenergies as well as projection operators of adiabatic states. In other words, the adiabatic cycle may interchange eigenenergies and eigenspaces. Its gauge theoretical formulation has been proposed subsequently [10, 15, 16], in which multiple adiabatic states which are subject to the interchange are treated as a whole. An interpretation of the exotic quantum holonomy in terms of non-Hermitian quantum theory [17] has been also proposed [18, 19].

In this manuscript we develop a topological formulation for the eigenspace anholonomy, that enables to identify the quantity which predicts whether a given cycle exhibits the eigenspace anholonomy. It turns out that the present formulation naturally lead to a nonadiabatic extension of the exotic quantum holonomy. The first key concept in our approach is the ordered set of eigenprojectors. We regard the eigenspace anholonomy as a permutation, which is induced by an adiabatic cycle, among the elements of the ordered set of eigenprojectors. This may be considered as a counterpart to Simon's vector bundle formulation of the phase holonomy [2]. In a two-level system, we show that a permutation occurs only when the adiabatic cycle encloses a singular point odd times. The second key concept in our topological formulation is the cycles in a quantum dynamical variable that take the place of conventional cycles in adiabatic parameters. It is shown that the topological nature of the eigenspace anholonomy has a direct link with the homotopy classification of cycles [20]. For example, the singular point mentioned above may be called a disclination, or line defect [21]. Furthermore, in common with Aharonov-Anandan formulation, the new definition of the cycles allows us to extend the eigenspace anholonomy to nonadiabatic cycles.

Quantum kicked spin-$\frac{1}{2}$ with two adiabatic parameters

Throughout the presentation of our formulation, we assume that the systems are described either by Hermitian Hamiltonian or by unitary Floquet operator. We also assume that there is no spectral degeneracy in the adiabatic cycles. For concreteness, we employ a periodically driven spin-$\frac{1}{2}$ to illustrate our formulation, which is immediately

![FIG. 1. A disclination of eigenobjects of quantum kicked spin-$\frac{1}{2}$ (Eq. (1)) in the $(B_y,B_x)$-plane. (a) The Bloch vector $\mathbf{a}$ (Eq. (4)) at a circle $C (|B| = \pi)$. The Bloch vector is not well-defined at the origin, which reflects the multiple-valuedness of $\mathbf{a}$. A “branch cut” is depicted by a wavy line. Here $\mathbf{a}_0 = e_x$ is the normalized Bloch vector at the initial point $(\pi,0)$ of $C$. The adiabatic time evolution along $C$ induces a flip of $\mathbf{a}$. (b) The director (headless vector) $\mathbf{n}$ of $\mathbf{a}$ at the circle $C$. Since $\mathbf{n}$ is single-valued in the $(B_y,B_x)$-plane, no branch cut needs to be drawn. Still, the line defect remains at the origin.](http://researchmap.jp/tanaka-atushi/)
applicable to an arbitrary N-level systems is also to be shown.

Let us suppose that the system is described by a time-periodic Hamiltonian [22] with adiabatic parameters $B$ and $\phi$: $\hat{H}(t) \equiv \frac{1}{2} B \cdot \hat{\sigma} + \frac{1}{2} \phi (1 - \hat{\sigma}) \sum_{m=-\infty}^{\infty} \delta(t - m)$, where $\hat{\sigma} \equiv e_x \hat{\sigma}_x + e_y \hat{\sigma}_y + e_z \hat{\sigma}_z$ is a unimodular linear combination of Pauli matrices $\hat{\sigma}_j$ ($j = x, y, z$). In the unperturbed part of $\hat{H}(t)$, a static magnetic field $B \equiv B e_\phi$ ($B \geq 0$) is applied, where $e_\rho \equiv e_x \cos \phi + e_y \sin \phi$. The Hamiltonian $\hat{H}(t)$ contains a periodically pulsed rank-1 perturbation [23] with strength $\phi$. We introduce a Floquet operator, which describes a unit time evolution generated by $\hat{H}(t)$:

$$\hat{U} \equiv \exp \left( -i \frac{1 - \hat{\sigma}_z}{2} \right) \exp \left( -i \frac{1}{2} B \cdot \hat{\sigma} \right). \quad (1)$$

It is straightforward to show that $\hat{U}$ is periodic in $\phi$ with the period $2\pi$ [9, 15]. Accordingly we identify the parameter space of the model with a two-dimensional plane $(B_x, B_y) \equiv (B \cos \phi, B \sin \phi)$.

We now diagonalize $\hat{U}$. First, $\hat{U}$ is expanded as $\hat{U} = e^{-i\phi/2} [\cos(\Delta/2) - i\hat{\sigma} \cdot \hat{a}]$, where

$$\Delta \equiv 2 \arccos \left( \frac{\cos \phi}{2} \frac{B}{2} \right),$$

$$\hat{a} \equiv \left( e_x \cos \frac{\phi}{2} - e_y \sin \frac{\phi}{2} \right) \frac{B}{2} - e_z \sin \frac{\phi}{2} \frac{B}{2}, \quad (3)$$

and $e_\phi \equiv e_y \cos \phi - e_x \sin \phi$. Because $\hat{a} \cdot \hat{a} = \sin^2(\Delta/2)$ holds, the eigenvalues of $\hat{U}$ become degenerate when $\sin(\Delta/2) = 0$ holds. Excluding the degeneracy points $B = 0, 2\pi e_x, 4\pi e_x, \ldots$, we can normalize $\hat{a}$:

$$\hat{a} \equiv \hat{a} / \sin(\Delta/2). \quad (4)$$

We obtain the spectral decomposition of $\hat{U}$ in the form

$$\hat{U} = z_+ \hat{P}(a) + z_- \hat{P}(-a), \quad (5)$$

where $z_{\pm} \equiv e^{-i(\phi \pm \frac{\Delta}{2})}$ are eigenvalues, and

$$\hat{P}(a) \equiv \frac{1 + a \cdot \hat{\sigma}}{2}, \quad (6)$$

is a projection operator parameterized by a unit vector $a$, which is called a (normalized) Bloch vector. Eq. (5) implies that $\hat{P}(\pm a)$ are the eigenprojectors of $\hat{U}$. In other words, for a given pair of $B_x$ and $B_y$, except at the degeneracy points, there are two normalized Bloch vectors $\pm a$, which correspond to two eigenprojectors. We note that the spectral decomposition (Eq. (5)) is applicable to an arbitrary unitary Floquet operator or Hermitian Hamiltonian as long as the corresponding two level system has no spectral degeneracy. Hence the following argument is applicable to two-level systems in general.

We examine the adiabatic time evolution of the eigenvector $\hat{P}(a)$ along a cycle $C$ in the $(B_x, B_y)$-plane. It is sufficient to examine the evolution of the normalized Bloch vector $\hat{a}$ instead of $\hat{P}(a)$ due to their equivalence. We depict the parametric evolution of $a$ in Fig. 1 (a). Let $a = a_0$ at the initial point $B_0$ on $C$. After a completion of the counterclockwise adiabatic rotation of $B$ along $C$, $a$ arrives at $-a_0$ (see, Fig. 1(a)), which implies that the final eigenprojector $\hat{P}(-a_0)$ is orthogonal to the initial one $\hat{P}(a_0)$. Hence $C$ induces the interchange of eigenprojectors $\hat{P}(\pm a_0)$ resulting in the realization the eigenspace anholonomy. This fact is stable against the deformation of the adiabatic cycle, as long as $C$ encloses the origin only once. Let us next examine the case that $C$ does not enclose the origin. The simplest case is the one where $C$ start from $B_0$ and keeps to stay $B_0$, i.e., $C$ is a trivial cycle. The direction of the Bloch vector at the final point of the cycle agrees with the one at the initial point. Namely, the eigenprojector returns to the original one after the completion of the adiabatic cycle. This remains correct as long as $C$ does not enclose the origin $O$ in the $(B_x, B_y)$-plane. Also, the initial and final Bloch vectors are the same when $C$ encloses the origin even times.

**Eigenspace anholonomy as an anholonomy of an ordered set of mutually orthogonal projection operators**

Here, we propose a novel interpretation of the normalized Bloch vector $a$ that allows the extension of our analysis to systems with an arbitrary number of levels. The central object is an ordered set of mutually orthogonal projection operators

$$p \equiv \langle \psi_0 | \psi_0 \rangle, \langle \psi_1 \rangle,$$

which can be specified by a normalized Bloch vector

$$p(a) = \langle \hat{P}(a), \hat{P}(-a) \rangle. \quad (8)$$

A given pair of $B_x$ and $B_y$, except at the degeneracy points, specifies two normalized Bloch vectors $\pm a$. One of them, say $a$, precisely determines $p$. Another normalized Bloch vector $-a$ correspond to another ordered set of projection operators $p(-a) = \langle \hat{P}(-a), \hat{P}(a) \rangle$, which is obtained by a permutation of the elements of $p(a)$. As for two-level systems, we can identify $p$ with a normalized Bloch vector $a$, which helps our geometric intuition. The $p$-space of two-level systems is equivalent to the sphere $S^2$.

In terms of the ordered set of projectors $p$, the eigenspace anholonomy is the permutation of the elements of $p$ induced by an adiabatic cycle. For example, let us start an adiabatic cycle $C$ that enclose the origin of $(B_x, B_y)$-plane in Fig. 1(a). After the completion of the cycle $C$, the elements of $p$ are interchanged. In other words, $C$ corresponds to a permutation of the elements of $p$. We remark that the formulation above resembles the fiber bundle interpretation of the phase holonomy [2], where a closed cycle corresponds to a geometric phase factor, which is an element of a holonomy group.

**Definition of cycles by quantum dynamical variables instead of c-number parameters**

So far, cycles are parameterized by the adiabatic parameters $(B_x, B_y)$. This
has been a common definition in the previous studies of the exotic quantum holonomy [8, 10]. Instead, we propose a way to define the cycles only in terms of quantum dynamical variables. The aim here is twofold. One is to complete a geometrical view of the eigenspace anholonomy. Another is to extend the exotic quantum holonomy into nonadiabatic cycles. To achieve this, we introduce a set of mutually orthogonal eigenprojectors
\[ b \equiv \{ |\psi_0 \rangle \langle \psi_0|, |\psi_1 \rangle \langle \psi_1| \}, \]  
where the order of the projector are disregarded.

As for the two level systems, we obtain a geometric interpretation of \( b \) with the help of a normalized Bloch vector \( \mathbf{a} \)
\[ b(\mathbf{a}) = \{ \hat{P}(\mathbf{a}), \hat{P}(-\mathbf{a}) \}, \]  
which agree with \( b(-\mathbf{a}) \), since the order of the elements in \( b \) is ignored. In other words, we identify \( \mathbf{a} \) and \(-\mathbf{a}\) in the specification of \( b \). In geometry, the identification of antipodal points on the sphere \( S^2 \) leads to the real projective plane \( \mathbb{R}P^2 \) [20, 24]. Hence we identify \( b \) with a point, which we denote as \( \mathbf{n} \), in the projective plane. In Fig. 1 (b), \( \mathbf{n} \) is depicted in the \((B_x, B_y)\)-plane. We note that \( \mathbf{n} \) is single-valued here. Still, we have a singularity at the origin \( O \), where the value of \( \mathbf{n} \) cannot be determined. This resembles a disclination of nematic liquid crystals [21]. In the studies of nematic liquid crystals, \( \mathbf{n} \) is called as a director, or a headless vector [21]. We use \( \mathbf{n} \), or equivalently \( b \), to define an adiabatic cycle \( C \). For the quantum kicked spin (Eq. (1)), we regard that the path \( C \) resides in the \( \mathbb{R}P^2 \)-space rather than in the \((B_x, B_y)\)-plane.

The adiabatic path \( C \) induces trajectories of \( p \), or equivalently \( \mathbf{a} \). The induced trajectories are called as lifts of \( C \) [4] (Fig. 2(a)). For a given initial point \( \mathbf{n}_0 \) in \( C \), there are two possible normalized Bloch vectors \( \pm \mathbf{a}_0 \), each of which corresponds to a lift of \( C \). Suppose \( \mathbf{n}_0 \) is slightly deformed to \( \mathbf{n}_0 + \delta \mathbf{n} \). The corresponding Bloch vectors smoothly deformed from \( \pm \mathbf{a}_0 \) to \( \pm (\mathbf{a}_0 + \delta \mathbf{n}) \), respectively [25]. The repetition of this procedure determines the two lifts \( C_{\pm} \) of \( C \).

A lift \( \tilde{C} \) of \( C \) to the \( \mathbb{R}P^2 \)-space tells us how the eigenprojectors are interchanged by the cycle \( C \). When \( \tilde{C} \) is a closed path, each eigenprojector in \( \mathbb{R}P^2 \) draws a closed path, too. This implies the absence of the eigenspace anholonomy. On the other hand, when \( \tilde{C} \) is open, the initial and the final point of an eigenprojector are different, which is the case that the eigenspace anholonomy occurs.

Hence the problem of the eigenspace anholonomy is equivalent to the identification of a topological character of the lifts. As for two-level systems, this problem is further reduced to the investigation of a topological character of \( C \) in the \( \mathbb{R}P^2 \)-space. Namely, for a given cycle (i.e., closed path) \( C \), the lifted paths \( \tilde{C}_{\pm} \) may be open or closed, depending on a topological nature of \( C \). Here the homotopy classification of \( C \) [20, 24] comes into play. We say that \( C \) is homotopic to another cycle \( C' \), when \( C \) can be smoothly deformed to \( C' \) with the initial and final point \( \mathbf{n}_0 \) kept unchanged. As for the projective plane \( \mathbb{R}P^2 \), a cycle \( C \) is homotopic to either \( e \) or \( \gamma \) [20, 24], as is seen in Fig. 2 (b). The cycle \( e \) is homotopic to the constant cycle, which keeps to stay at \( \mathbf{n}_0 \). Imagine a cycle \( \gamma \) depicted in the northern unit hemisphere. \( \gamma \) starts from a point at the equator and arrives at the antipodal point of the initial point. This ensures that \( \gamma \) is closed. Furthermore, we suppose that \( \gamma \) does not touch the equator on the way. Hence \( \gamma \) is not homotopic to \( e \) (Fig. 2 (b)). On the other hand, \( \gamma^2 \), which is the repetition of \( \gamma \) twice, is homotopic to \( e \).

Thus we find that an adiabatic cycle \( C \) in the \( \mathbb{R}P^2 \)-space induces the eigenspace anholonomy if and only if \( C \) is homotopic to \( \gamma \), as for two-level systems. In hindsight, it is to be expected, from the topological nature of the problem, that the condition for the eigenspace anholonomy involves the homotopy of adiabatic cycles.

**FIG. 2.** (a) Lifting of a cycle \( C \) in \( \mathbb{R}P^2 \)-space to \( \mathbb{R}P^2 \)-space. The initial point of \( C \) is denoted by \( \mathbf{n}_0 \). An adiabatic time evolution along \( C \) induces a lift \( \tilde{C} \), whose initial point is \( p \). The lift \( \tilde{C} \) induces a permutation \( \phi_{[C]} \) of the elements of \( p \), where \( [C] \) denotes the class of paths that are homotopic to \( C \). (b) Example in two-level systems. In the bottom, two cycles (closed paths) \( e \) and \( \gamma \) in the \( \mathbb{R}P^2 \)-space, which is equivalent to \( \mathbb{R}P^2 \), are shown. Two filled circles corresponding to \( \mathbf{n}_0 \), which is the initial point of these cycles, are identical in \( \mathbb{R}P^2 \). The cycle \( e \) is homotopic to a zero-length cycle, and is not homotopic to \( \gamma \). In the top, the lifts of \( e \) and \( \gamma \) to the \( \mathbb{R}P^2 \)-space, which is equivalent to \( S^2 \), are shown. Because \( S^2 \) doubly covers \( \mathbb{R}P^2 \), there are two normalized Bloch vectors \( \pm \mathbf{a}_0 \) for a given director \( \mathbf{n}_0 \). Also, each cycle has two lifts (thick and dashed curves). The lifts \( \tilde{C}_{\pm} \) of \( e \) are closed, signifying the absence of eigenspace anholonomy. On the other hand, the lifts \( \tilde{C}_{\pm} \) are open. Along the adiabatic cycle \( \gamma \), the initial point \( \mathbf{a}_0 \) of \( \tilde{C}_{\pm} \) is transposed to \( -\mathbf{a}_0 \), which is the initial point of \( \tilde{C}_{\pm} \), and vice versa.

Nonadiabatic extension The time evolution along a cycle \( C \) in \( \mathbb{R}P^2 \)-space needs not to be adiabatic. We give an example. Suppose that the initial Bloch vector is prepared at \( \mathbf{a}_0 = e_{\mathbf{z}} \), which uniquely determines the initial \( b \). The time evolution induced by the Hamiltonian \( H = \pi \sigma_z / 2 \) for \( 0 \leq t \leq 1 \) flips the Bloch vector and \( \mathbf{a} \) arrives at \( -\mathbf{a}_0 \) at \( t = 1 \). The corresponding trajectory of \( b \), or equivalently \( \mathbf{n} \), forms a closed loop. Namely, this
time evolution induces a nonadiabatic cycle $C$, which is homotopic to $\gamma$ in Fig. 2 (b). Accordingly the cycle $C$ induces the exotic quantum holonomy on the Bloch vector $a$, or, equivalently on $p$ (Eq. (7)). This is nothing but a nonadiabatic extension of the eigenspace anholonomy. Our treatment here can be regarded as an extension of Aharonov and Anandan’s treatment of phase holonomy [5], to the exotic quantum holonomy.

$N$-dimensional extension We close our argument by outlining the extension to cases that involves an arbitrary number, say $N$, of levels. We also bring up several concepts of the topology residing behind the phenomena. First, it is straightforward to extend the definitions of the topology residing behind the phenomenon of the exotic quantum holonomy. In the present formulation, we have identified the exotic quantum holonomy and Kato’s exceptional point (EP), which is a branch point of the Riemann surface of eigenenergies, in non-Hermitian quantum theory [17, 32].

In a gauge theoretical approach introduced in Ref. [10], the eigenspace anholonomy and the off-diagonal geometric phase are treated together. These two concepts are disentangled and assigned to the different layers through the present formulation

A remark is due on the relationship between the exotic quantum holonomy and Kato’s exceptional point (EP), which is a branch point of the Riemann surface of eigenenergies, in non-Hermitian quantum theory [17, 32]. The adiabatic time evolution under the presence of the eigenspace anholonomy resembles a parametric evolution that encloses an EP, in the sense that these evolutions permute eigenspaces. An analytic continuation of adiabatic cycle in Hermitian Hamiltonian and unitary Floquet systems has enabled to interpret the exotic quantum holonomy as the result of parametric encirclement of EP in the complex plane [18, 19]. Although such a correspondence is valid only when an analytic continuation of the adiabatic cycle is available, the topological formulation is applicable regardless of the analytic continuation. Also, we do not know how the non-adiabatic extension of exotic quantum holonomy can be associated with EPs. On the other hand, we remind that the relationship between the phase holonomy and EPs is established through the analysis of the Riemann surface of (quasi-)eigenenergy [27]. Because the covering space structure naturally resides in the Riemann surfaces, an extension of the present approach to non-Hermitian systems should be interesting.

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