DIMENSION-FREE HARNACK INEQUALITIES FOR CONJUGATE HEAT EQUATIONS AND THEIR APPLICATIONS TO GEOMETRIC FLOWS

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ABSTRACT. Let $M$ be a differentiable manifold endowed with a family of complete Riemannian metrics $g(t)$ evolving under a geometric flow over the time interval $[0,T]$. In this article, we give a probabilistic representation for the derivative of the corresponding conjugate semigroup on $M$ which is generated by a Schrödinger type operator. With the help of this derivative formula, we derive fundamental Harnack type inequalities in the setting of evolving Riemannian manifolds. In particular, we establish a dimension-free Harnack inequality and show how it can be used to achieve heat kernel upper bounds in the setting of moving metrics. Moreover, by means of the supercontractivity of the conjugate semigroup, we obtain a family of canonical log-Sobolev inequalities. We discuss and apply these results both in the case of the so-called modified Ricci flow and in the case of general geometric flows.

1. Introduction

Let $M$ be a differentiable manifold endowed with a $C^1$ family of complete Riemannian metrics $g(t)$ indexed by the real interval $[0,T]$ where $T \in [0,\infty]$. The family $g(t)$ describes the evolution of the manifold $M$ under a geometric flow where $T$ is the first time where possibly a blow-up of the curvature occurs. This type of singularities is not excluded in our setting.

More specifically, we consider geometric flows of the following type:
\[
\partial_t g(t) = -2h(t), \quad \text{on } M \times [0,T],
\]
where $h(t)$ is a general time-dependent symmetric $(0,2)$-tensor. For fixed $t$, with respect to the metric $g(t)$, let $\mathcal{H}_t = \text{tr}(g(t)) h(t)$ be the metric trace of the tensor $h(t)$ and $\Delta_t$ the Laplace-Beltrami operator acting on functions on $M$. In practice, the geometric flow deforms the geometry of $M$ and smoothens out irregularities in the metric to give it a nicer and more symmetric form which provides geometric and topological information on the manifold.

Consider operators of the form $L_t = \Delta_t - \nabla^t \phi_t$ where $\phi_t$ is a time-dependent function on $M$. We also use the notation $g_t = g(t)$ and $h_t = h(t)$. In this paper we study the (minimal) fundamental solution to heat equations of the type
\[
(L_t - \partial_t)u(t,x) = 0, \quad \text{resp.} \quad (L_t + \partial_t - q_t)u(t,x) = 0,
\]
where $q_t = \partial_t \phi_t + \mathcal{H}_t$. The first equation is the classical heat equation, the second one appears naturally as conjugate heat equation. More precisely, we have the following relationship.

Remark 1.1. Set $d\mu_t = e^{-\phi_t} d\text{vol}_t$ where $\text{vol}_t$ denotes the Riemannian volume to the metric $g(t)$. Let $\Box = L_t - \partial_t$ be the standard heat operator and $\Box^*$ its formal adjoint with respect to the measure $d\mu_t$. In this paper we consider geometric flows $\partial_t g(t)$ evolving under a geometric flow over the time interval $[0,T]$. In particular, we will be concerned with geometric flows of the form
\[
\partial_t g(t) = -2h(t), \quad \text{on } M \times [0,T],
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\( \mu_t \otimes dt \). Thus,
\[
\int_0^T \int_M V \Box U \, d\mu_t \, dt = \int_0^T \int_M U \Box^* V \, d\mu_t \, dt
\]
for smooth functions \( U, V: M \times [0, T[ \rightarrow [0, \infty[ \). From this relation it is immediate that \( \Box^* = L_t + \partial_t - \varrho_t \).

**Example 1.2.** A typical situation covered by this setting is solving a geometric flow equation (e.g. Ricci flow) forward in time and the associated conjugate heat equation backward in time. In the case of the Ricci flow \( \partial_t g(t) = -2\text{Ric}_t \) and \( L_t = \Delta_t \), the conjugate heat equation reads as \( \Box^* u = (\Delta_t + \partial_t - R)u = 0 \) where \( R = \text{tr} \text{ric} \) denotes the (time-dependent) scalar curvature.

It should be mentioned that an important ingredient in the proof of the Poincaré conjecture by Perelman is a differential Harnack inequality which is related to a gradient estimate for solutions to the conjugate heat equation under forward Ricci flow on a compact manifold \([16]\). This relation has been one of our motivations to investigate solutions to conjugate heat equations and their associated semigroups also by methods of stochastic analysis.

Let \( X_t \) be the diffusion process generated by \( L_t = \Delta_t - \nabla^t \varphi_t \) (called \( L_t \)-diffusion process) which we assume to be non-explosive up to time \( T \). We consider the two-parameter semigroup associated to \( L_t \),
\[
P_{s,t}f(x) := \mathbb{E}[f(X_t) | X_s = x], \quad s \leq t,
\]
which satisfies the heat equation
\[
\begin{align*}
\frac{\partial}{\partial s} P_{s,t}f &= -L_s P_{s,t}f, \\
\lim_{s \rightarrow t} P_{s,t}f &= f.
\end{align*}
\]

In previous work, we already studied properties of heat equations under geometric flows, like properties of the semigroup \( P_{s,t} \) generated by \( L_t \), by adopting probabilistic methods. In \([8]\), for instance, the first author gave functional inequalities equivalent to a lower bound of \( \text{Ric}_t - \frac{1}{4} \partial_t \varphi_t \).

In \([9, 11]\) we established characterizations of upper and lower bounds for \( \text{Ric}_t - \frac{1}{2} \partial_t \varphi_t \) in terms of functional inequalities on the path space over \( M \).

On a more probabilistic side, in \([10]\) the authors studied existence and uniqueness of so-called evolution systems of measures related to the semigroup \( P_{s,t} \). Using such systems as reference measures, contractivity of the semigroup, as well as log-Sobolev inequality, have been investigated.

Although the evolution system of measures is helpful to shed light on properties of solutions to the heat equation, it is still difficult to obtain a full picture of this measure system, like its relation to the system of volume measures. It has been observed that if one uses volume measures as reference measures, many questions will be related to the conjugate heat equation and not the usual heat equation, see e.g. \([1, 7]\).

Recall that \( \mu_t(dx) = e^{-\varphi(x)} \, d\text{vol}_t \) where \( \text{vol}_t \) is the volume measure with respect to the metric \( g(t) \). Let
\[
P^g_{s,t}f(x) = \mathbb{E}\left[ \exp\left( - \int_s^t \varrho(r, X_r) \, dr \right) f(X_t) | X_s = x \right],
\]
where \( \varrho(t, x) = \varrho_t(x) \) is given by
\[
\frac{\partial}{\partial t} \mu_t(dx) = - (\partial_t \varphi_t + \mathcal{H})(t, x) \mu_t(dx) = - \varrho(t, x) \mu_t(dx).
\]
According to the Feynman-Kac formula, $P_{s,t}^0 f$ represents the solution to the equation

$$\frac{\partial}{\partial s} \varphi_s = -(L_s - \varphi_s) \varphi_s, \quad \varphi_t = f,$$

on $[0,t] \times M$ where $t < T$. We note that this equation is conjugate to the heat equation

$$\frac{\partial}{\partial s} u(s,x) = L_s u(s,\cdot)(x).$$

In this paper, we first give probabilistic formulae and estimates for $dP_{s,t}^0 f$ from where we then derive a dimension-free Harnack inequality. It is interesting to note that by combining the dimension-free Harnack inequalities for $P_{s,t}$ and $P_{s,t}^0$, one can obtain new upper bounds for the heat kernel to $L_t$ with respect to $\mu_t$, see Theorem 5.2. We apply this result then (see Corollary 5.4 below) to the following modified geometric flow for $g_t$ combined with the conjugate heat equation for $\phi_t$, i.e.,

$$\begin{align*}
\partial_t g_t &= -2(h + \text{Hess}(\phi))_t; \\
\partial_t \phi_t &= -\Delta_t \phi_t - \mathcal{H}_t.
\end{align*} \tag{1.1}$$

As is well-known [16], when $h_t = \text{Ric}_t$, this flow gives the gradient flow to Perelman’s entropy functional $\mathcal{F}$.

We observe that

$$\mu_s(P_{s,t}^0 f) = \mu_t(f) \tag{1.2}$$

which means that the family of measures $\mu_s$ plays for the semigroup $P_{s,t}^0$ a similar role as the invariant measure for the one-parameter semigroup $P_t$ on the static Riemannian case. Log-Sobolev inequalities with respect to the invariant measure are well established under certain curvature conditions on static Riemannian manifolds; they are related to other functional inequalities for $P_t$ and have many applications, see for instance [3, 14, 19, 21]. This leads to the natural question whether one can prove log-Sobolev inequalities with respect to $\mu_t$ in a similar way through functional inequalities for $P_{s,t}^0$. In Section 6 we discuss the relation between contraction properties of the semigroup and log-Sobolev inequalities with respect to $\mu_t$. Using the dimension-free Harnack inequality for $P_{s,t}^0$, we give a sufficient condition for supercontractivity of $P_{s,t}^0$ which we then use to establish existence of the (defective) log-Sobolev inequality for $\mu_t$. It is well-known that the log-Sobolev inequality or Sobolev inequality is an important tool to prove an upper bound of the heat kernel, see [2, 5, 23].

We prove the following result about system (1.1). Denote by $\rho_t \equiv \rho_t(o, \cdot)$ the distance function to a given base point $o$ in $M$ with respect to the metric $g_t$. Suppose that the geometric flow $g_t$ and the function $\phi_t$ satisfy (1.1). Furthermore, assume that $\text{Ric}_t + \text{Hess}_t(\phi_t) - \frac{1}{2}\partial_t g_t \geq K(t)$ and $\mu_t(\exp(A \rho_t^2)) < \infty$ for all $A > 0$ and $t \in [0,T]$, then there exists a function $\beta$ such that

$$\mu_t(f^2 \log f^2) \leq \mu_t(|\nabla f|^2) + \beta_t(r), \quad r > 0,$$

for $f \in C^0([0,T[ \times M)$ and $\mu_t(f^2) = 1$.

The remaining part of the paper is organized as follows. In Section 3 a probabilistic formula for the derivative of the conjugate heat semigroup is given and used to derive a gradient estimate for $P_{s,t}^0$ under suitable curvature conditions. In Section 4 we derive two versions of dimension-free Harnack inequalities from the mentioned gradient inequality for $P_{s,t}^0$ which are then applied in Section 5.1 to gain a sufficient and necessary condition for supercontractivity of $P_{s,t}^0$. In this section we also clarify the relation between the supercontractivity of $P_{s,t}^0$ and the log-Sobolev inequality with respect to $\mu_t$. These results are applied to system (1.1) of the modified geometric flow under conjugate heat equation.
2. Brownian motion with respect to evolving manifolds

Let \((M, g_t)_{t \in I}\) be an evolving manifold indexed by \(I = [0, T]\). Let \(\nabla^t\) be the Levi-Civita connection with respect to \(g_t\). Denote by \(\mathcal{M} := M \times I\) space-time and consider the bundle

\[ \mathcal{T}M \xrightarrow{\pi} \mathcal{M} \]

where \(\pi\) is the projection. As observed by Hamilton \([15]\) there exists a natural space-time connection \(\nabla\) on \(T \mathcal{M}\) considered as bundle over space-time \(\mathcal{M}\) such that

\[
\begin{aligned}
\nabla_v X &= \nabla^t_v X, \\
\nabla_{\partial_t} X &= \partial_t X + \frac{1}{2} (\partial_t g_t)(X, \cdot)_{g_t},
\end{aligned}
\]

for all \(v \in (T_s, M, g_t)\) and all time-dependent vector fields \(X\) on \(M\). This connection is compatible with the metric, i.e.

\[
\frac{d}{dt} |X_{g_t}|^2 = 2 \langle X, \nabla_{\partial_t} X \rangle_{g_t}.
\]

**Remark 2.1.** Let \(G = O(n)\) where \(n = \dim M\) and consider the \(G\)-principal bundle \(\mathcal{F} \xrightarrow{\pi} \mathcal{M}\) of orthonormal frames with fibres

\[ \mathcal{F}(s, t) = \{ u: \mathbb{R}^n \to (T_s, M, g_t) \mid u \text{ isometry} \}. \]

As usual, \(a \in G\) acts on \(\mathcal{F}\) from the right via composition. The connection \(\nabla\) gives rise to a \(G\)-invariant splitting of the sequence

\[
\begin{array}{c}
0 \longrightarrow \ker d\pi \longrightarrow T\mathcal{F} \xrightarrow{d\pi} \pi^* T\mathcal{M} \longrightarrow 0,
\end{array}
\]

which induces a decomposition of \(T\mathcal{F}\) as \(T\mathcal{F} = V \oplus H := \ker d\pi \oplus h(\pi^* T\mathcal{M})\). For \(u \in \mathcal{F}\), the space \(H_u\) is the horizontal space at \(u\) and \(V_u = \{ v \in T_u \mathcal{F} \mid (d\pi)_v = 0 \}\) the vertical space at \(u\). The bundle isomorphism

\[
h: \pi^* T\mathcal{M} \xrightarrow{\sim} H \hookrightarrow T\mathcal{F}, \quad h_u: T_{\pi(u)} \mathcal{M} \xrightarrow{\sim} H_u, \quad u \in \mathcal{F}, \tag{2.1}
\]

is the horizontal lift of the \(G\)-connection.

**Corollary 2.2.** To each \(aX + b\partial_t \in T_{(s, t)} \mathcal{M}\) and each frame \(u \in \mathcal{F}(s, t)\), there exists a unique “horizontal lift” \(aX^* + bD_t \in H_u\) of \(aX + b\partial_t\) such that

\[
\pi_u(aX^* + bD_t) = aX + b\partial_t.
\]

In explicit terms, \(X^*\) is the horizontal lift of \(X\) with respect to the metric \(g_t\), and \(D_t = \frac{d}{ds} \big|_{s=0} u_s\) where \(u_s\) is the horizontal lift based at \(u\) of the curve \(s \mapsto (x, t + s)\).

We consider curves in \(\mathcal{M}\) of the form

\[
\gamma_t = (x_t, \ell_t), \quad t \in [0, T[
\]

where \(\ell_t\) is a monotone differentiable transformation on \([0, T[. The horizontal lift of such a curve \(\gamma_t\) in \(\mathcal{M}\) is a curve \(u_t\) in \(\mathcal{F}\) such that \(\pi u_t = \gamma_t\) and \(\nabla^\gamma_e (u_t e) = 0\) for each \(e \in \mathbb{R}^n\). Then

\[
\gamma(t) := \int_0^t u_s \, ds^{-1}: (T_x M, g_{\ell_t}) \to (T_x M, g_{\ell_t}), \quad 0 \leq r \leq s < T,
\]

gives parallel transport along \(\gamma_t\). In the following we consider the special case \(\ell_t = t\).
Remark 2.3. Vector fields and differential forms on \( M \) can be seen as time-dependent vector fields and differential forms on \( M \). It is convenient to write objects on \( M \) as \( G \)-equivariant functions on \( \mathcal{F} \). In particular, then

1) functions \( f \in C^\infty(\mathcal{M}) \) read as \( \tilde{f} \in C^\infty(\mathcal{F}) \) via \( \tilde{f} := f \circ \pi; \)
2) time-dependent vector fields \( Y \) on \( M \) read as \( \tilde{Y} : \mathcal{F} \to \mathbb{R}^n \) via \( \tilde{Y}(u) := u^{-1} Y_{\pi(u)}; \)
3) time-dependent differential forms \( \alpha \) on \( M \) as \( \tilde{\alpha} : \mathcal{F} \to (\mathbb{R}^n)^* \) via \( \tilde{\alpha}(u) = \alpha_{\pi(u)}(u \cdot) \).

The following formulas hold:

\[
\nabla_X f = X^* \tilde{f}, \quad \tilde{\partial}_t f = D_t \tilde{f}, \quad \nabla_X Y = X^* \tilde{Y}, \quad \nabla_{\tilde{\partial}_i} Y = D_t \tilde{Y},
\]

\[
\nabla_X \alpha = X^* \tilde{\alpha}, \quad \nabla_{\tilde{\partial}_i} \alpha = D_t \tilde{\alpha}.
\]

The proofs are straightforward. For instance, to check the last equality, let \( u_t \) be a horizontal curve such that \( \pi u_t = \gamma_t = (x, t) \) where \( x \in M \) is fixed. Then

\[
(D_t \tilde{\alpha})(u_t) = \frac{d}{dt}_{|_{t=s}} \tilde{\alpha}(u_t) = \frac{d}{dt}_{|_{t=s}} \alpha_{\pi(u)}(u_t \cdot)
\]

\[
= \frac{d}{dr}_{|_{r=0}} \alpha_{(\pi,s+\tau)}((/s,s+\tau)\pi u_s, \cdot)
\]

\[
= (\nabla_{\tilde{\partial}_i} \alpha)(u_s, \cdot) = (\nabla_{\tilde{\partial}_i} \alpha)(u_s).
\]

Remark 2.4. The vector fields

\[
H_i \in \Gamma(T^\tau \mathcal{F}), \quad H_i(u) = (ue_i)^* \equiv h_u(ue_i), \quad i = 1, \ldots, n,
\]

where \( (e_1, \ldots, e_n) \) denotes the standard basis of \( \mathbb{R}^n \), are the standard-horizontal vector fields on \( \mathcal{F} \).

The operator

\[
\Delta_{\mathrm{hor}} = \sum_{i=1}^n H_i^2
\]

is called Bochner’s horizontal Laplacian on \( \mathcal{F} \). Note that

\[
\nabla \tilde{f} = \Delta_{\mathrm{hor}} \tilde{f} \quad \text{and} \quad \nabla_{\tilde{\partial}_i} \alpha = \Delta_{\mathrm{hor}} \tilde{\alpha}
\]

(2.3)

where \( \Delta_{\mathrm{rough}} = \text{tr}(\nabla^2) \) is the rough Laplacian on differential one-forms. Recall that, for fixed \( t \in I \),

\[
d\Delta_t f = \text{tr}(\nabla^2)df - \text{Ric}(df, \cdot)
\]

(2.4)

by the Weitzenböck formula.

Let \( \pi : \mathcal{F} \to \mathcal{M} \) denote the canonical projection. For \( \phi \in C^{1,2}(\mathcal{M}) \), we define a vector field on \( \mathcal{F} \) by

\[
H^\phi \in \Gamma(T^\tau \mathcal{F}), \quad H^\phi(u) = h_u(\nabla^\tau \phi(t, \cdot)_x), \quad \pi(u) = (x, t).
\]

Consider the following Stratonovich SDE on \( \mathcal{F} \):

\[
\begin{cases}
    dU = D_t(U) dt + \sum_{i=1}^n H_i(U) \circ dB^i - H^\phi(U) dt, \\
    U_s = u_s, \quad \pi(u_s) = (x, s), \quad s \in [0, T].
\end{cases}
\]

(2.5)

Here \( B \) denotes standard Brownian motion on \( \mathbb{R}^n \) (speeded up by the factor 2, i.e., \( dB^i dB^j = 2 \delta_{ij} dt \)) with generator \( \Delta_{\mathrm{ge}} \). Eq. (2.5) has a unique solution up to its lifetime \( \zeta := \lim_{k \to \infty} \zeta_k \) where

\[
\zeta_k := \inf \{ t \in [s, T] : \rho_t(\pi(U_s), \pi(U_t)) \geq k \}, \quad n \geq 1, \quad \inf \emptyset := T.
\]

(2.6)
and where \( \rho_t \) stands for the Riemannian distance induced by the metric \( g(t) \). We note that if \( U \) solves (2.5) then
\[
\pi(U_t) = (X_t, t)
\]
where \( X \) is a diffusion process on \( M \) generated by \( L_t = \Delta_t - \nabla' \phi_t \). In case of \( \phi = 0 \) we call \( X \) a \((g_t)\)-Brownian motion on \( M \).

More precisely, we have the following result.

**Proposition 2.5.** Let \( U \) be a solution to the SDE (2.5). Then

1. For any \( C^2 \)-function \( F : \mathcal{F} \to \mathbb{R} \),
\[
d(F(U)) = (D_tF(U)) dt + \sum_{i=1}^n (H_iF(U)) dB^i + (\Delta_{hor} F)(U) dt - (H^\phi F)(U) dt;
\]
2. For any \( C^2 \)-function \( f : \mathcal{M} \to \mathbb{R} \), we have
\[
d(f(X)) = (\partial_t f)(X) dt + \sum_{i=1}^n (U e_i f) dB^i + (L_t f)(X) dt.
\]

Let \( U \) be a solution to the SDE (2.5) and \( \pi(U_t) = (X_t, t) \). Furthermore let
\[
\|/_{s,t} := U_t^{-1} : (T_{s, M}, g_s) \to (T_{s, M}, g_t), \quad s \leq t < T,
\]
be the induced parallel transport along \( X_t \) (which by construction consists of isometries). We use the notation
\[
X_t = X^{(s, x)}_t, \quad t \geq s,
\]
if \( X_s = x \). Note that \( X_t = X^{(s, x)}_t \) solves the equation
\[
dx^{(s, x)}_t = U_t \circ dB_t - \nabla' \phi_t(X^{(s, x)}_t) dt, \quad X^{(s, x)}_s = x.
\]

For any \( f \in C^2_0(M) \),
\[
f(X^{(s, x)}_t) - f(x) - \int_s^t L_r f(X^{(s, x)}_r) dr = \int_s^t \langle /_{s,r}^{-1} \nabla' f(X^{(s, x)}_r), U_r dB_r \rangle, \quad t \in [s, T],
\]
is a martingale up to the lifetime \( \zeta \). In the case \( s = 0 \), we write again \( X^t \) instead of \( X^{(0, x)}_t \).

3. **Derivative formula**

Let \( L_t = \Delta_t - \nabla' \phi_t \) where \( \phi \) is \( C^{1,2}([0, T[\times M) \). Throughout this section, we assume that the diffusion \( (X_t) \) generated by \( L_t \) is non-explosive up to time \( T \). Recall that \( \mu_t(dx) = e^{-\phi_t(x)} d\text{vol}_t \) and
\[
\frac{\partial}{\partial t} \mu_t(dx) = - (\partial_t \phi + \mathcal{H})(t, x) \mu_t(dx) = - g(t, x) \mu_t(dx)
\]
with
\[
g(t, x) \equiv g_t(x) = (\partial_t \phi + \mathcal{H})(t, x).
\]

For each \( t \), we assume that \( g_t \) is bounded from below by a constant depending on \( t \).

For \( f \in C_b(M) \) let
\[
P_{s,t}^e f(x) = \mathbb{E} \left[ \exp \left( - \int_s^t g(r, X_r) dr \right) f(X_t) \bigg| X_s = x \right]. \quad (3.1)
\]

Then
\[
\frac{\partial}{\partial s} P_{s,t}^e f = -(L_s - g_s) P_{s,t}^e f, \quad P_{t,t}^e = f. \quad (3.2)
\]
That $u(s, x) := P_{s,t}^0 f(x)$ represents the solution to (3.2) is easily seen from the fact that
\[ \exp\left(-\int_s^t g(a, X_a) \, da\right) P_{r,t}^0 f(X_r^{(s,x)}) \quad r \in [s, t], \]
is a martingale under given assumptions.

Our first step is to establish a derivative formula for $P_{s,t}^0$. When the metric is static, the derivative formula for $P_s f$ is known as Bismut formula [4, 13]. The more general versions in [17] have been adapted to Feynman-Kac semigroups in [18].

We fix $s \in [0, T]$ and consider the random family $Q_{s,t} \in \text{Aut}(T_x M)$, $0 \leq s \leq t < T$, given as solution to the following ordinary differential equation:
\[ \frac{dQ_{s,t}}{dt} = -\left(\text{Ric} - \frac{1}{2} \partial_x + \text{Hess}(\phi)\right)_{|_{s,t}} Q_{s,t}, \quad Q_{s,s} = \text{id}, \] (3.3)
where
\[ \left(\text{Ric} - \frac{1}{2} \partial_x + \text{Hess}(\phi)\right)_{|_{s,t}} := \|_{s,t}^{-1} \circ \left(\text{Ric} - \frac{1}{2} \partial_x + \text{Hess}(\phi)\right)(X_t) \circ \|_{s,t}. \]

**Theorem 3.1.** Let $L_t = \Delta_t - \nabla^\phi \partial_t$ as above. For each $t$, assume that both $\varphi_t$ and
\[ (\text{Ric} + \text{Hess}(\phi) - \frac{1}{2} \partial_x)(t, \cdot) \]
are bounded below and that $|d\varphi_t|$ is bounded. Then, for $v \in T_x M$ and $f \in C^1_b(M)$,
\[ (dP_{s,t}^0 f)(v) = \mathbb{E}^{(s,x)} \left[ \exp\left(-\int_s^t \varphi_t(X_r) \, dr\right) \times \left( df(\|_{s,t}^{-1} Q_{s,t} v)(X_t) - f(X_t) \int_s^t d\varphi_t(\|_{s,t}^{-1} Q_{s,t} v) \, dr\right) \right]. \] (3.4)

**Proof.** By the definition of $Q$ as the solution to (3.3), we have
\[ d(U_{s,t}^{-1} Q_{s,t}) = -U_{s,t}^{-1} (\text{Ric} + \text{Hess}(\phi))_{|_{s,t}} Q_{s,t} \] (3.5)
Set
\[ N_r(v) = dP_{r,t}^0(f(\|_{s,t}^{-1} Q_{s,t} v)), \quad s \leq r \leq t. \]
Write $N_r(v) = F(U_r, \varphi_r(v))$ where
\[ F(u, w) := (dP_{r,t}^0 f)_x(uw), \quad \pi(u) = (x, r) \text{ and } w \in \mathbb{R}^n, \]
\[ q_r(v) := U_{s,t}^{-1} Q_{s,t} v. \]
By means of Proposition 2.5 we have
\[ d(F(U_r, w)) \equiv (D_r F)(U_r, w) \, dr + (\Delta_{\text{bon}} F)(U_r, w) \, dr - (H^\phi F)(U_r, w) \, dr \] (3.6)
where
\[ (D_r F)(u, w) = \partial_u(dP_{r,t}^0 f)_x(uw) + \frac{1}{2} (\partial_u g_{r,t})(dP_{r,t}^0 f)_x(uw) \]
and where $\equiv$ stands for equality modulo the differential of a local martingale. Using the Weitzenböck formula we observe that
\[ \partial_u(dP_{r,t}^0 f) = -d(L_r - \varphi_r) P_{r,t}^0 f \]
\[ = -d(\Delta_r - \nabla^\phi \varphi_r - \varphi_r) P_{r,t}^0 f \]
\[ = -\text{tr}(\nabla^\phi)^2 dP_{r,t}^0 f + dP_{r,t}^0 f \left( (\text{Hess}\varphi_r)^{\nabla^\phi} \right) + dP_{r,t}^0 f (\text{Ric}_{r,t}^{\nabla^\phi}) + d\left(\varphi_r P_{r,t}^0 f\right)_{X_r}. \]
Taking \((2.3)\) and \((2.2)\) into account, we have 

\[(\Delta_{\text{hor}} F)(U_r, w) = \text{tr}(\nabla)\nabla^2 dP_{r,t}^\phi f(U_r, w)\]

and 

\[(H^\phi F)(U_r, w) = \text{Hess}(\phi_t)\left(dP_{r,t}^\phi f\right)^{\phi_t} \nabla \cdot dP_{r,t}^\phi f(U_r, w)\]

Thus \(\text{Eq. (3.6)}\) rewrites as 

\[d(F(U_r, w)) = dP_{r,t}^\phi f (\text{Ric}^\phi_t) U_r, w + \frac{1}{2} \left(\partial_t g_t\right) \left(dP_{r,t}^\phi f\right)^{\phi_t} \nabla \cdot dP_{r,t}^\phi f(U_r, w)\]  \(\text{(3.7)}\)

Combining Eqs. \((3.7)\) and \((3.5)\) we thus obtain 

\[dN_r(v) = d(F(U_r, \cdot))(q_r(v)) + F(U_r, \partial_t \phi_t(v)) dr\]

\[\equiv d\left(q_r p_{r,t}^\phi f\right)_{X_r} \text{ for } Q_{s+r,v} dr\]

Hence we get 

\[d\left(\exp\left(\int_s^r q_u(X_u) du\right) N_r(v)\right) = d\left(\exp\left(\int_s^r q_u(X_u) du\right) p_{s,t}^\phi f(X_r)\right) \text{ for } Q_{s+r,v} dr\]

Integrating this equality from \(s\) to \(t\) and taking expectation gives Eq. \((3.4)\).

\[\square\]

**Corollary 3.2.** Suppose that \(q_r\) is bounded below for each \(t\), and assume that there are functions \(\kappa, K \in C([0,T])\) such that \(|dq_r| \leq \kappa(t)\), respectively 

\[\text{Ric}_t - \frac{1}{2} \partial_t g_t + \text{Hess}_t (\phi_t) \geq K(t)\]

Then, for \(f \in C^\infty_0(M)\) and \(f \geq 0\),

\[|\nabla s p_{s,t}^\phi f| \leq \exp\left(\int_s^r K(r) dr\right) p_{s,t}^\phi |\nabla s f| + p_{s,t}^\phi f \int_s^r \kappa(r) \exp\left(\int_s^r K(u) du\right) dr\]

**Proof.** The condition \(\text{Ric}_t - \frac{1}{2} \partial_t g_t + \text{Hess}_t (\phi_t) \geq K(t)\) implies that

\[||Q_{s,t}|| \leq \exp\left(\int_s^t K(r) dr\right)\]

The inequality follows thus from the bound \(|dq_r| \leq \kappa(t)\). \(\square\)

### 4. Dimension-free Harnack Inequalities

We first derive two Harnack type inequalities for \(p_{s,t}^\phi\). Such dimension-free Harnack inequalities have been studied first by Wang, see e.g. \([22]\) for more results in this direction. We denote by \(\mathcal{B}_b(M)\) the space of bounded measurable functions on \(M\).

**Theorem 4.1.** Suppose that \(q_r\) is bounded below, \(|dq_r| \leq \kappa(t)\), and

\[\text{Ric}_t - \frac{1}{2} \partial_t g_t + \text{Hess}_t (\phi_t) \geq K(t)\]

The following two Harnack type inequalities hold for any \(p > 1\).
(i) For $0 \leq s \leq t < T$ and any non-negative function $f \in \mathcal{B}_0(M)$,

\[
(P^0_{s,t,f})^p(x) \leq (P^0_{s,t,f})^p(y)
\]

\[
\times \exp \left( (p-1) \int_s^t \sup_g e^{-t} \, dt + \frac{pp^2(x,y)}{4(p-1)v(s,t)} + \frac{p\eta(s,t)\rho_{s}(x,y)}{v(s,t)} \right),
\]

where

\[
\alpha(s,t) := \int_s^t \exp \left( \frac{1}{2} \int_s^u K(u) \, du \right) \, du;
\]

\[
\eta(s,t) := \int_s^t \int_s^u \kappa(r) \exp \left( \frac{1}{2} \int_s^u K(u) \, du - \int_s^r K(u) \, du \right) \, dr \, dv.
\]

(ii) For $0 \leq s \leq t < T$ and any non-negative function $f \in \mathcal{B}_0(M)$,

\[
(P^0_{s,t,f})^p(x) \leq (P^0_{s,t,f})^p(y) \mathbb{E}^\gamma \left[ \exp \left( -(p-1) \int_s^t \varphi_t(X_r) \, dr \right) \right]
\]

\[
\times \exp \left( \frac{pp^2(x,y)}{4(p-1)v(s,t)} + \frac{2p\eta(s,t)\rho_{s}(x,y)}{v(s,t)} \right).
\]

Proof. To facilitate the notion we restrict ourselves to the case $s = 0$. By approximation and the monotone class theorem, we may assume that $f \in C^2(M)$ of compact support and $\inf_M f > 0$. Given $x \neq y$ and $t > 0$, let $\gamma: [0, t] \rightarrow M$ be the constant speed $g_0$-geodesics from $x$ to $y$ of length $\rho_0(x,y)$. Let $\nu_s = d\gamma_s / ds$. Thus we have $|\nu_s| = \rho_0(x,y)/t$. Let

\[
h(s) = \int_0^s \exp \left( \frac{1}{2} \int_u^s K(u) \, du \right) \, du.
\]

Then $h(0) = 0$ and $h(t) = t$. Let $y_s = \gamma_{h(s)}$ and

\[
\varphi(s) = \log \mathbb{E}^{\gamma_s} \left[ \exp \left( \int_0^s \varphi_t(X_r) \, dr \right) P^0_{s,t,f}(X_s) \right]^p
\]

\[
= \log P^0_{0,s}(P^0_{0,t,f})^p(y_s), \quad s \in [0, t].
\]

To determine $\varphi'(s)$, we first note that

\[
d(P^0_{s,t,f}(X_s))^p = dM_s + \left( (L_s + \partial_s)(P^0_{s,t,f})^p(X_s) \right) ds
\]

\[
= dM_s + p(p-1)(P^0_{s,t,f})^{p-2}(X_s)|\nabla^2 P^0_{s,t,f}|^2(X_s) ds
\]

\[
+ p\rho_{s}(X_s)(P^0_{s,t,f})^p(X_s) ds, \quad s \leq \zeta_k,
\]

where $M_s$ is the local martingale part of $(P^0_{s,t,f})^p(X_s)$. This implies

\[
\mathbb{E}^\gamma \left[ \exp \left( -\int_0^s \varphi_t(X_r) \, dr \right) P^0_{s,t,f}(X_{s,\zeta_k}) \right]^p - (P^0_{0,t,f})^p(x)
\]

\[
\geq p(p-1) \mathbb{E}^{\gamma} \left[ \int_0^s \exp \left( -p \int_0^u \varphi_t(X_r) \, dr \right) (P^0_{u,t,f})^{p-2}(X_u) |\nabla^u P^0_{u,t,f}|^2(X_u) du \right].
\]

Since $\inf_M f > 0$, by letting $k \rightarrow \infty$, we deduce that

\[
\mathbb{E}^\gamma \left[ \exp \left( -\int_0^s \varphi_t(X_r) \, dr \right) P^0_{s,t,f}(X_s) \right]^p - (P^0_{0,t,f})^p(x)
\]

\[
\geq p(p-1) \int_0^s \mathbb{E}^\gamma \left[ \exp \left( -p \int_0^u \varphi_t(X_r) \, dr \right) (P^0_{u,t,f})^{p-2}|\nabla^u P^0_{u,t,f}|^2(X_u) \right] du.
\]
Moreover, by adopting Corollary 3.2 for $P_{s,t}^0$, i.e.

$$|\nabla^s P_{s,t}^0 f|_s \leq \exp \left( - \int_s^t K(r) \, dr \right) P_{s,t}^0 |\nabla^s f|_s + p P_{s,t}^0 f \int_s^t \kappa(r) \exp \left( - \int_s^r K(u) \, du \right) \, dr,$$

we thus obtain for $s \in [0, t]$,

$$\frac{d\varphi(s)}{ds} \geq \left\{ \frac{1}{p \rho_0(x, y)} \left( p(p-1) P_{0,s}^0 (P_{s,t}^0)^p |\nabla^s \log P_{s,t}^0 f|_s^2 \right) \right\} (y_s)$$

$$\geq \left\{ \frac{1}{p \rho_0(x, y)} \left( p(p-1) P_{0,s}^0 (P_{s,t}^0)^p |\nabla^s \log P_{s,t}^0 f|_s^2 \right) \right\} (y_s)$$

$$\geq \left\{ \frac{1}{p \rho_0(x, y)} \left( p(p-1) P_{0,s}^0 (P_{s,t}^0)^p |\nabla^s \log P_{s,t}^0 f|_s^2 \right) \right\} (y_s)$$

where the last inequality follows from the simple fact that

$$aA^2 + bA \geq -\frac{b^2}{4a}, \quad a > 0.$$

Since

$$h'(s) = \frac{t \exp \left( 2 \int_0^s K(u) \, du \right)}{\exp \left( 2 \int_0^r K(u) \, du \right)} dr,$$
This proves part (i) of the theorem. In addition, by adopting in (4.3) the estimate
\[ p \exp \left( \int_0^s 2K(u) \, du \right) \] we thus arrive at
\[ \frac{d\varphi(s)}{ds} \geq - \frac{p \exp \left( \int_0^s 2K(u) \, du \right)}{4(p-1) \exp \left( 2 \int_0^s K(u) \, du \right) dr} \rho_0(x,y)^2 \]
\[ - \frac{p \exp \left( 2 \int_0^s K(u) \, du \right) \int_0^s \kappa(r) \exp \left( - \int_0^r K(u) \, du \right) dr \, ds}{\int_0^s \exp \left( 2 \int_0^r K(u) \, du \right) dr} \rho_0(x,y), \quad s \in [0, t]. \]

Integrating over \( s \) from 0 and \( t \), we get
\[ (P_{0,t}^y f)^p(x) \leq \mathbb{E}^y \left[ \exp \left( - \int_0^t \varrho_r(X_r) \, dr \right) f(X_t)^p \right] \]
\[ \times \exp \left( \frac{pp_0(x,y)^2}{4(p-1) \exp \left( 2 \int_0^t K(u) \, du \right) dr} \rho_0(x,y)^2 \right) \]
\[ + \frac{p \int_0^t \int_0^s \kappa(r) \exp \left( 2 \int_0^s K(u) \, du - \int_0^r K(u) \, du \right) dr \, ds}{\int_0^t \exp \left( 2 \int_0^r K(u) \, du \right) dr} \rho_0(x,y) \]
\[ \leq \mathbb{E}^y \left[ \exp \left( - \int_0^t \varrho_r(X_r) \, dr \right) f(X_t)^p \right] \exp \left( (p-1) \int_0^t \sup \varrho_r \, dr \right) \]
\[ \times \exp \left( \frac{pp_0(x,y)^2}{4(p-1) \exp \left( 2 \int_0^t K(u) \, du \right) dr} \rho_0(x,y)^2 \right) \]
\[ + \frac{p \int_0^t \int_0^s \kappa(r) \exp \left( 2 \int_0^s K(u) \, du - \int_0^r K(u) \, du \right) dr \, ds}{\int_0^t \exp \left( 2 \int_0^r K(u) \, du \right) dr} \rho_0(x,y) \]. \quad (4.3)

This proves part (i) of the theorem. In addition, by adopting in (4.3) the estimate
\[ \mathbb{E}^y \left[ \exp \left( - \int_0^t \varrho_r(X_r) \, dr \right) f(X_t)^p \right] \leq (P_{0,t}^y f^p)(y) \mathbb{E}^y \left[ \exp \left( -(p-1) \int_0^t \varrho_r(X_r) \, dr \right) \right], \]
part (ii) of the theorem follows as well. \( \square \)

5. Heat kernel estimate

Let \( p \) be the fundamental solution of \( L_t = \Delta_t - \nabla^t \phi_t \) in the sense that
\[ \begin{cases}
\partial_t p(s, x; t, y) = -\Delta_t \varphi_t p(s, x; t, y), \\
\lim_{s \to t} p(s, x; t, y) = \delta_y(x).
\end{cases} \]
Note that $p$ is the density of $P_{s,t}(x, dy)$ with respect to $\mu_t(dy)$, i.e.

$$P_{s,t}f(x) = \int p(s, x; t, y) f(y) \mu_t(dy) = \int p(s, x; t, y) e^{-\phi_t(y)} \mu_t(dy),$$

where $\mu_t$ denotes the volume measure with respect to $g_t$. In \[8\] the following Harnack inequality for the 2-parameter semigroup $P_{s,t}$ has been derived; it can be seen as a special case to Theorem \ref{thm5.1}.

**Theorem 5.1.** Suppose that

$$\text{Ric}_t - \frac{1}{2} \partial_t g_t + \text{Hess}_t(\phi_t) \geq K(t).$$

Then for any non-negative function $f \in \mathcal{B}_p(M)$ and $0 \leq s \leq t < T$,

$$(P_{s,t}f)^p(x) \leq P_{s,t}f^p(y) \exp \left( \frac{p}{4(p-1)\alpha(s,t)} p_s^2(x, y) \right)$$

where

$$\alpha(s, t) := \int_s^t \exp \left( 2 \int_s^r K(u) du \right) dr.$$

It is interesting to observe that Theorem 5.1 for $P_{s,t}$, along with the Harnack inequality (4.1) for $P^\rho_{s,t}$, will allow us to attain upper bounds for the heat kernel $p$.

**Theorem 5.2.** Suppose that, for all $t \in [0, T]$,

$$\text{Ric}_t - \frac{1}{2} \partial_t g_t + \text{Hess}_t(\phi_t) \geq K_1(t), \quad \text{Ric}_t + \frac{1}{2} \partial_t g_t + \text{Hess}_t(\phi_t) \geq K_2(t).$$

Furthermore suppose that $|d\nu_t| \leq \kappa(t) < \infty$ and $\sup g_t^\gamma < \infty$ for each $t \in [0, T]$. Then the following heat kernel upper bound holds:

$$p(0, x; t, y) \leq \frac{\exp \left( \frac{1}{2} \int_0^r \sup g_u^\gamma du + \frac{t}{4\alpha_1(0, t/2)} + \frac{t + 4\eta_2(t/2, t)}{4\alpha_2(t/2, t)} \sqrt{t} \right)}{\sqrt{\mu_0 \left(B_0(x, \sqrt{t})\right)} \mu_t \left(B_t(y, \sqrt{t})\right)},$$

where

$$\alpha_1(0, t/2) := \int_0^{t/2} \exp \left( 2 \int_0^r K_1(u) du \right) dr;$$

$$\alpha_2(t/2, t) := \int_{t/2}^t \exp \left( 2 \int_{t/2}^{r} K_2(u) du \right) dr; \quad (5.1)$$

$$\eta_2(t/2, t) := \int_{t/2}^t \int_{t/2}^r K(r) \exp \left( 2 \int_{t/2}^{r} K_2(u) du - \int_{t/2}^{r} K_2(u) du \right) dr dv.$$

**Proof.** We first observe that

$$p(0, x; t, y) = \int_M p(0, x; t/2, z) p(t/2, z; t, y) \mu_{t/2}(dz) \leq \left( \int_M p(0, x; t/2, z)^2 \mu_{t/2}(dz) \right)^{1/2} \left( \int_M p(t/2, z; t, y)^2 \mu_{t/2}(dz) \right)^{1/2}.$$
Hence we are left with the task to estimate the following two terms:

\[ I_1 = \int_M p(0, x; t/2, z)^2 \mu_{t/2}(dz), \]

\[ I_2 = \int_M p(t/2, z; t, y)^2 \mu_{t/2}(dz). \]

In order to estimate \( I_1 \) we proceed with Theorem 5.1. As

\[ \text{Ric}_t - \frac{1}{2} \partial_t g_t + \text{Hess}(\phi_t) \geq K_1(t) \]

for some \( K_1 \in C([0, T]) \), we get

\[
(P_{s,t} f)^2(x) \mu_s(B_s(x, \sqrt{2(t-s)})) \exp\left(-\frac{t-s}{\int_s^t \exp\left(\frac{t-s}{\int_s^t K_1(u) \, du} \right) \, dr}\right) \mu_s(dy)
\]

\[
\leq \int_M (P_{s,t} f)^2(x) \exp\left(-\frac{\rho^2(x,y)}{2 \int_s^t \exp\left(\frac{t-s}{\int_s^t K_1(u) \, du} \right) \, dr}\right) \mu_s(dy)
\]

\[
\leq \int_M (P_{s,t} f^2(y) \mu_s(dy)
\]

\[
\leq \exp\left(\int_s^t \sup \varrho^+(u, \cdot) \, du\right) \int_M f(y)^2 \mu_s(dy).
\]

Taking

\[ f(y) := (k \wedge p(s, x; t, y)), \quad y \in M, \quad k \in \mathbb{N}, \]

we obtain

\[
\int_M (k \wedge p(s, x; t, y))^2 \mu_t(dy) \leq \frac{\exp\left(\int_s^t \sup \varrho^+(u, \cdot) \, du + \alpha_1(s, t)^{-1}(t-s)\right)}{\mu_s(B_s(x, \sqrt{2(t-s)}))},
\]

where

\[ \alpha_1(s, t) = \int_s^t \exp\left(2 \int_s^{t'} K_1(u) \, du\right) \, dr. \]

Letting \( k \to \infty \), we arrive at

\[
\int_M p(s, x; t, y)^2 \mu_t(dy) \leq \frac{\exp\left(\int_s^t \sup \varrho^+(u, \cdot) \, du + \frac{t-s}{\alpha_1(s, t)}\right)}{\mu_s(B_s(x, \sqrt{2(t-s)}))}.
\]

This shows that

\[
I_1 \leq \frac{\exp\left(\int_0^{t/2} \sup \varrho^+(u, \cdot) \, du + \frac{t}{2\alpha_1(0, t/2)}\right)}{\mu_0(B_0(x, \sqrt{t}))} \tag{5.2}
\]

To estimate the second term \( I_2 \), we write

\[
\int_M p(t/2, z; t, y)^2 \mu_{t/2}(dz) = \int_M p_t^* (0, y; t/2, z)^2 \mu_{t/2}(dz)
\]
where \( p_1^\ast(s, x; r, y) := p^\ast(t - s, x; t - r, y) \) and where \( p^\ast \) denotes the adjoint heat kernel to \( p \). Recall that
\[
\begin{align*}
\partial_s p_1^\ast(s, x; r, y) &= -(L_{t-s} + q_{t-s}) p_1^\ast(s, \cdot; r, y)(x), \\
\lim_{s \to r} p_1^\ast(s, x; r, y) &= \delta_y(x).
\end{align*}
\]

Let \( (\tilde{P}^\ast_{t, s})_{0 \leq t \leq s \leq T} \) be the semigroup generated by the operator \( L_{t-} + q_{t-} \). By Theorem 4.1, this time using relying on the assumption,
\[ \text{Ric}_t + \frac{1}{2} \partial_t g_t + \text{Hess}_t(\phi_t) \geq K_2(t), \]
we have
\[
(\tilde{P}^\ast_{0, t/2} f)^2(x) \leq (\tilde{P}^\ast_{0, t/2} f^2)(y) \exp\left(\int_0^{t/2} \sup \phi^+(t - s, \cdot) \, ds + \frac{\rho_1^2(x, y)}{2\alpha_2(t/2, t)} + \frac{2\eta_2(t/2, t)p_1(x, y)}{\alpha_2(t/2, t)}\right),
\]
where
\[
\begin{align*}
\alpha_2(t/2, t) &= \int_0^{t/2} \exp\left(2 \int_0^u K_2(t - u) \, du\right) \, dr = \int_0^{t/2} \exp\left(2 \int_1^{t/2} K_2(u) \, du\right) \, dr, \\
\eta_2(t/2, t) &= \int_0^{t/2} \int_0^\infty \kappa(r) \exp\left(2 \int_0^r K_2(t - u) \, du\right) \, dr \, dv \int_0^{t/2} \int_0^\infty \kappa(r) \exp\left(2 \int_0^r K_2(u) \, du - \int_1^{t/2} K_2(u) \, du\right) \, dr \, dv.
\end{align*}
\]
By means of this formula, we can proceed as above to obtain
\[
(\tilde{P}^\ast_{0, t/2} f)^2(x) \mu_t(B_t \sqrt{\eta_t}) \exp\left(\int_0^{t/2} \sup \phi^+(t - s, \cdot) \, ds - \frac{t}{2\alpha_2(t/2, t)} - \frac{2\eta_2(t/2, t)\sqrt{t_1}}{\alpha_2(t/2, t)}\right)
\leq \int_M (\tilde{P}^\ast_{0, t/2} f^2)(y) \mu_t(dy) = \int_M f(y)^2 \mu_{t/2}(dy).
\]
Thus taking \( f(z) := p_1^\ast(0, y; t/2, z) \wedge k \) and letting \( k \to \infty \), we obtain
\[
\int_M p_1^\ast(0, y; t/2, z)^2 \mu_{t/2}(dz) \leq \frac{\exp\left(\int_0^{t/2} \sup \phi^+(u, \cdot) \, du + \frac{t}{2\alpha_2(t/2, t)} + \frac{2\eta_2(t/2, t)\sqrt{t_1}}{\alpha_2(t/2, t)}\right)}{\mu_t(B_t \sqrt{\eta_t})}.
\]
Finally, combining (5.2) and (5.3) we obtain
\[
p(0, x; t, y) \leq \sqrt{I_1 I_2} \leq \frac{\exp\left(\frac{1}{2} \int_0^{t/2} \sup \phi^+(u, \cdot) \, du + \frac{t}{4\alpha_1(0, t/2)} + \frac{2\eta_2(t/2, t)\sqrt{t_1}}{4\alpha_2(t/2, t)}\right)}{\sqrt{\mu_0(B_0(x, \sqrt{\eta_1}) \mu_t(B_t \sqrt{\eta_t})}}
\]
where the functions \( \alpha_1, \alpha_2, \eta_2 \) are defined by (5.1).

\[ \Box \]

**Remark 5.3.** In [12], the author used a horizontal coupling of curves to obtain a dimension-free Harnack inequality for \( P_{s,t}^\ast \) generated by \( L_s \), first introduced by Wang, and applied it then for an upper bound of the heat kernel. A major difference to our approach when \( \phi = 0 \) is that we use the Harnack inequality again to deal with the term \( I_2 \), while in [12] a comparison result for \( P_{s,t}^\ast \) and \( P^\ast_{s,t} \) is used so that the condition there includes both upper and lower bounds of \( q \).
Gaussian upper bounds for the heat kernel on evolving manifolds have recently also obtained by Buzano and Yudowitz [6]. Their bounds depend on the behaviour of the distance function along the flow rather than directly involving curvature bounds.

We now specify our estimates in some specific situations. First we consider the modified geometric flow for $g_t$, combined with the conjugate heat equation for $\phi$, i.e.,

$$
\begin{align*}
\frac{\partial}{\partial t} g_t(x, t) &= -2(h + \text{Hess}(\phi))(x, t); \\
\frac{\partial}{\partial t} \phi_t(x) &= -\Delta \phi_t(x) - \mathcal{H}_t(x, t).
\end{align*}
$$

(5.4)

For this modified geometric flow, we have the following result.

**Corollary 5.4.** Suppose that $(g_t, \phi_t)$ evolve by Eq. (5.4) and that

$$
\text{Ric}_t - \frac{1}{2} \frac{\partial}{\partial t} g_t + \text{Hess}_t(\phi_t) \geq K_1(t), \quad \text{Ric}_t + \frac{1}{2} \frac{\partial}{\partial t} g_t + \text{Hess}_t(\phi_t) \geq K_2(t).
$$

Then

$$
p(0, x; t, y) \leq \exp \left( \frac{t}{4\alpha_1(0, t/2)} + \frac{t}{4\alpha_2(2, t/2)} \right) \frac{1}{\sqrt{\mu_0(B_0(x, \sqrt{t})) \mu_t(B_t(y, \sqrt{t}))}},
$$

where

$$
\alpha_1(0, t/2) := \int_0^{t/2} \exp \left( 2 \int_0^r K_1(u) \, du \right) \, dr;
$$

$$
\alpha_2(t/2, t) := \int_{t/2}^t \exp \left( 2 \int_{t/2}^r K_2(u) \, du \right) \, dr.
$$

**Proof.** It is immediate from the definition of $\varrho$ that

$$
\varrho_t = \frac{\partial}{\partial t} \phi_t + \text{tr}_g_t(h_t + \text{Hess}_t(\phi_t)) = \frac{\partial}{\partial t} \phi_t + \Delta \phi_t + \mathcal{H}_t = 0.
$$

The proof is hence completed by applying Theorem 5.2. \hfill \Box

We next consider the standard geometric flow for the evolution of the metric $g$, i.e.,

$$
\begin{align*}
\frac{\partial}{\partial t} g_t(x, t) &= -2h(x, t); \\
\phi_t(x) &= 0.
\end{align*}
$$

(5.5)

For this geometric flow, we have $\varrho = \mathcal{H}$ and we thus obtain the following result.

**Corollary 5.5.** Suppose that $g_t$ evolves by Eq. (5.5). Furthermore assume that

$$
\text{Ric}_t - \frac{1}{2} \frac{\partial}{\partial t} g_t \geq K_1(t), \quad \text{Ric}_t + \frac{1}{2} \frac{\partial}{\partial t} g_t \geq K_2(t),
$$

as well as $|\partial \mathcal{H}_t| \leq \kappa(t) < \infty$ and $\sup \mathcal{H}_t^+ < \infty$ for each $t \in [0, T]$. Then

$$
p(0, x; t, y) \leq \exp \left( \frac{1}{2} \int_0^t \sup \mathcal{H}_u^+ \, du + \frac{t}{4\alpha_1(0, t/2)} + \frac{t + 4\eta_2(t/2, t)}{4\alpha_2(2, t/2)} \sqrt{t} \right) \frac{1}{\sqrt{\mu_0(B_0(x, \sqrt{t})) \mu_t(B_t(y, \sqrt{t}))}},
$$

where

$$
\alpha_1(0, t/2) := \int_0^{t/2} \exp \left( 2 \int_0^r K_1(u) \, du \right) \, dr.
$$
\[ \alpha_2(t/2, t) := \int_{t/2}^{t} \exp \left( 2 \int_{t/2}^{r} \kappa(r) du \right) dr; \]
\[ \eta_2(t/2, t) := \int_{t/2}^{t} \int_{t/2}^{r} \kappa(r) \exp \left( 2 \int_{t/2}^{r} \kappa(r) du - \int_{t/2}^{r} K_2(u) du \right) dr dv. \]

### 6. Super log-Sobolev inequalities

The semigroup \( P_t^e \) is called supercontractive if it maps \( L^p(M, \mu_t) \) into \( L^q(M, \mu_t) \), i.e.,
\[ \| P_t^e \|_{(p, t) \rightarrow (q, s)} < \infty, \]
for any \( 1 < p < q < \infty \) and \( 0 \leq s \leq t < T \). In the following section, we investigate the relation between supercontractivity of \( P_t^e \) and a log-Sobolev inequality with respect to \( \mu_t \).

#### 6.1. Supercontractivity of \( P_t^e \) and Super log-Sobolev inequality.

We state first the main result of this section.

**Theorem 6.1.** Assume that \( \varrho_t \) is bounded,
\[ \text{Ric}_t + \text{Hess}_t(\varrho_t) - \frac{1}{2} \partial_t \varrho_t \geq K(t), \quad |d \varrho_t| \leq \kappa(t) \quad \text{for } t \in [0, T[ , \]
and that
\[ \| P_t^e \|_{(p, t) \rightarrow (q, s)} < \infty, \quad \text{for } 1 < p < q \text{ and } 0 \leq s \leq t < T. \]
Then, for every \( f \in H^1(M, \mu_t) \) such that \( \| f \|_{2,t} = 1, t \in [0, T[ \), the following super log-Sobolev inequalities hold:
\[ \int f^2 \log f^2 \, d\mu_t \leq r \int |\nabla_t f|^2 \, d\mu_t + \beta_t(r), \quad r > 0, \]
where
\[ \beta_t(r) = \tilde{\beta}_t(\gamma_t^{-1}(r), t) \]
and
\[ \tilde{\beta}_t(s, t) = \frac{pq}{q - p} \log \left( \| P_t^e \|_{(p, t) \rightarrow (q, s)} \right) + \int_s^t \left( 2 \left( \int_r^s \kappa(r) \exp \left( -\int_r^s K(v) \, dv \right) \, dr \right)^2 + \sup \varrho_t^+ \right) \, dr, \]
\[ \gamma(s, t) = \frac{4(p(q - 1) - 1)}{q - p} \sup \left( \int_s^t \exp \left( -2 \int_r^s K(u) \, du \right) \, dr \right), \]
\[ \gamma_t^{-1}(r) = \inf \{ s \in [0, t]: \gamma(s, t) \leq r \}. \]

The log-Sobolev inequality has been shown to be equivalent to the Sobolev inequality and can hence be used to obtain an upper bound estimate of the heat kernel, see e.g. [23] for details. In order to prove Theorem 6.1, we first establish a log-Sobolev inequality with respect to the semigroup \( P_t^e \).

**Proposition 6.2.** Assume that
\[ \text{Ric}_t + \text{Hess}_t(\varrho_t) - \frac{1}{2} \partial_t \varrho_t \geq K(t), \quad \sup |\varrho_t| < \infty \text{ and } |d \varrho_t| \leq \kappa(t) \text{ for } t \in [0, T[. \]
Then, for \( 0 \leq s \leq t < T \) and \( f \in C_0^\infty(M) \),
\[ P_t^e(f^2 \log f^2) \leq 4 \left( \int_s^t \exp \left( -2 \int_r^s K(u) \, du \right) \, dr \right) P_t^e(\nabla_t f)^2 + P_t^e f^2 \log P_t^e f^2. \]
where \( M \) is a local martingale. On the other hand, by Corollary 3.2, we have the estimate,

\[
\begin{align*}
\sup_{\tau_k} \left| \nabla \left( P_{t,r}^\Theta f \right)^2 \right| &\leq \exp \left( -\int_r^\tau K(u) \, du \right) \left( P_{t,r}^\Theta \nabla f \right)^2 \left( \int_r^\tau \kappa(u) \exp \left( -\int_u^\tau K(v) \, dv \right) \, du \right) \\
&\leq 2 \exp \left( -\int_r^\tau K(u) \, du \right) \left( P_{t,r}^\Theta \left( f \right) \nabla f \right)^2 \left( \int_r^\tau \kappa(u) \exp \left( -\int_u^\tau K(v) \, dv \right) \, du \right) \\
&\leq 2 \exp \left( -\int_r^\tau K(u) \, du \right) \left( P_{t,r}^\Theta \left( f \right) \nabla f \right)^2 \left( \int_r^\tau \kappa(u) \exp \left( -\int_u^\tau K(v) \, dv \right) \, du \right)
\end{align*}
\]

which gives

\[
\left| \nabla \left( P_{t,r}^\Theta f \right)^2 \right|_r \leq 4 \exp \left( -2 \int_r^\tau K(u) \, du \right) \left( P_{t,r}^\Theta f \right)^2 \left( \int_r^\tau \kappa(u) \exp \left( -\int_u^\tau K(v) \, dv \right) \, du \right) \left( \int_r^\tau \kappa(u) \exp \left( -\int_u^\tau K(v) \, dv \right) \, du \right)^2.
\]

Substituting back into (6.3), we obtain

\[
\begin{align*}
\sup_{\tau_k} \left| \nabla \left( P_{t,r}^\Theta f \right)^2 \right|_r &\leq 4 \exp \left( -2 \int_r^\tau K(u) \, du \right) \left( P_{t,r}^\Theta f \right)^2 \left( \int_r^\tau \kappa(u) \exp \left( -\int_u^\tau K(v) \, dv \right) \, du \right) \left( \int_r^\tau \kappa(u) \exp \left( -\int_u^\tau K(v) \, dv \right) \, du \right)^2.
\end{align*}
\]

Integrating both sides from \( s \) to \( t \) and \( \tau_k \), taking expectation, and letting \( k \uparrow +\infty \), we obtain the dominated convergence,

\[
\begin{align*}
P_{s,r}^\Theta \left( f \right) \left( \int_r^\tau K(u) \, du \right)^2 &\leq 4 \int_s^\tau \exp \left( -2 \int_r^\tau K(u) \, du \right) \left( P_{s,r}^\Theta f \right)^2 \left( \int_r^\tau \kappa(u) \exp \left( -\int_u^\tau K(v) \, dv \right) \, du \right)^2 \left( \int_r^\tau \kappa(u) \exp \left( -\int_u^\tau K(v) \, dv \right) \, du \right)^2 \, dr \\
&\leq 4 \int_s^\tau \exp \left( -2 \int_r^\tau K(u) \, du \right) \left( P_{s,r}^\Theta f \right)^2 \left( \int_r^\tau \kappa(u) \exp \left( -\int_u^\tau K(v) \, dv \right) \, du \right)^2 \sup_{\tau_k} \left| \nabla \left( P_{t,r}^\Theta f \right)^2 \right|_r \, dr.
\end{align*}
\]
In other words,
\[
P_{s,t}^E \left( f^2 \log f^2 \right) \leq 4 \left( \int_s^t \exp \left( -2 \int_r^t K(u) \, du \right) \, dr \right) P_{s,t}^E \| \nabla^t f \|_2^2 + P_{s,t}^E \log P_{s,t}^E \log f^2 \\
+ \int_s^t \left[ 2 \left( \int_r^t \kappa(u) \exp \left( - \int_r^t K(v) \, dv \right) \, du \right) \sup \varrho_r^+ \right] \, dr \, P_{s,t}^E \log f^2.
\] (6.4)

**Proof of Theorem 6.1.** By means of the fact that
\[
\log^+ (P_{s,t}^E \log f^2) \leq P_{s,t}^E \log f^2 \leq \exp \left( \int_s^t \sup \varrho_u^+ \, du \right) \| f \|_{L^\infty},
\]
we can integrate both sides of the log-Sobolev inequality (6.4) with respect to \( \mu_s \). Taking (1.2) into account, we get
\[
\mu_t(f^2 \log f^2) \leq 4 \left( \int_s^t \exp \left( -2 \int_r^t K(u) \, du \right) \, dr \right) \mu_t(\| \nabla^t f \|_2^2) + \mu_t(P_{s,t}^E \log f^2)
+ \int_s^t \left[ 2 \left( \int_r^t \kappa(u) \exp \left( - \int_r^t K(v) \, dv \right) \, du \right) \sup \varrho_r^+ \right] \, dr \mu_t(f^2).
\] (6.5)

We deal first with the term \( \mu_t(P_{s,t}^E \log f^2) \). Let \( 1 < p < q \). For any \( h \in ]0,1\) let
\[
r_h = \frac{ph}{p-1} \in ]0,1[.
\]
By the Riesz-Thorin interpolation theorem, we have
\[
\| P_{s,t}^E \|_{q_h,s} \leq \left\| P_{s,t}^E \right\|_{(p,h) \rightarrow (q,s)} \left\| P_{s,t}^E \right\|_{(1,q_h) \rightarrow (1,s)} \| f \|_{p,q_h}, \quad f \in L^p(M,\mu_s),
\] (6.6)
where
\[
\frac{1}{p_h} = \frac{1-r_h}{1} + \frac{r_h}{p} \quad \text{and} \quad \frac{1}{q_h} = \frac{1-r_h}{1} + \frac{r_h}{q}.
\]

Thus
\[
p_h = \frac{1}{1-h} \quad \text{and} \quad q_h = \left( 1 - \frac{p(q-1)}{q(p-1)} \right)^{-1}.
\]

Since \( \| P_{s,t}^E \|_{(1,q) \rightarrow (1,s)} \leq 1 \), we get from Eq. (6.6) that
\[
\int (P_{s,t}^E |f|^{2(1-h)})^q_h \, d\mu_s \leq \| P_{s,t}^E \|_{(p,h) \rightarrow (q,s)} \| f \|_{2,f}^{q_h/p_h}.
\]

Then, for \( f \in C^\infty_0(M) \) satisfying \( \| f \|_{2,f} = 1 \), we have
\[
\frac{1}{h} \left( \int (P_{s,t}^E |f|^{2(1-h)})^q_h \, d\mu_s - \left( \int P_{s,t}^E |f|^2 \, d\mu_s \right)^{q_h/p_h} \right)
= \frac{1}{h} \left( \int (P_{s,t}^E |f|^{2(1-h)})^q_h \, d\mu_s - 1 \right)
\leq \frac{1}{h} (\| P_{s,t}^E \|_{(p,h) \rightarrow (q,s)} - 1).
\] (6.7)

Taking the limit as \( h \to 0 \) in (6.7), as
\[
\lim_{h \to 0} \frac{1}{h} (\| P_{s,t}^E \|_{(p,h) \rightarrow (q,s)} - 1) = \frac{p}{p-1} \log \| P_{s,t}^E \|_{(p,p) \rightarrow (q,s)}.
\]
we obtain by dominated convergence,
\[
\frac{p(q - 1)}{q(p - 1)} \int P_{s,t}^p f^2 \log P_{s,t}^p f^2 \, d\mu_s - \int P_{s,t}^p (f^2 \log f^2) \, d\mu_s \leq \frac{p}{p - 1} \log \|P_{s,t}^p\|_{(p,t) \to (q,s)},
\]
or equivalently,
\[
\mu_s(P_{s,t}^p f^2 \log P_{s,t}^p f^2) \leq \frac{q(p - 1)}{p(q - 1)} \mu_s(f^2 \log f^2) + \frac{q}{q - 1} \log \|P_{s,t}^p\|_{(p,t) \to (q,s)}.
\] (6.8)
Substituting (6.8) back into Eq. (6.5), we arrive at
\[
\mu_t(f^2 \log f^2) \leq \gamma(s,t) \mu_t(|\nabla^t f|_v^2) + \tilde{\beta}(s,t)
\] (6.9)
for every \( f \in C_0^\infty(M) \) satisfying \( \|f\|_{2,t} = 1 \), where
\[
\gamma(s,t) = \frac{4p(q - 1)}{q - p} \int_s^t \exp \left(-2 \int_r^t K(u) \, du\right) \, dr,
\]
\[
\tilde{\beta}(s,t) = \frac{pq}{q - p} \log \left(\|P_{s,t}^p\|_{(p,t) \to (q,s)}\right)
+ \frac{p(q - 1)}{q - p} \int_s^t \left( \int_r^t \kappa(u) \exp \left(-\int_r^u K(v) \, dv\right) \, du\right)^2 \sup \gamma_r^p \, dr.
\]
We complete the proof by letting
\[
\beta_t(r) = \tilde{\beta}(\gamma_t^{-1}(r), t).
\]

**Theorem 6.3.** We keep the situation of Theorem 6.1 that is \( \varrho_t \) bounded and\n\[
\text{Ric}_t + \text{Hess}_t(\varrho_t) - \frac{1}{2} \partial_t \varrho_t \geq K(t), \quad |d \varrho_t| \leq \kappa(t) \quad \text{for} \quad t \in [0, T[.
\]
Suppose there exists a function \( \beta_t: \mathbb{R}^+ \to \mathbb{R}^+ \) such that
\[
\mu_t(f^2 \log f^2) \leq r \mu_t(|\nabla^t f|_v^2) + \beta_t(r), \quad \text{for all} \quad t \in [0, T[ \quad \text{and} \quad \|f\|_{2,t} = 1.
\] (6.10)

Then
\[
\|P_{s,t}^p\|_{(p,t) \to (q,s)} < \infty \quad \text{for} \quad 1 < p < q \quad \text{and} \quad 0 \leq s \leq t < T.
\]
**Proof.** Let \( 0 \leq s < t < T \) and \( f \in C_0^\infty(M) \) such that \( f \geq \delta > 0 \). To calculate the derivative of \( \mu_s(P_{s,t}^p f)^{q(s)} \) with respect to \( s \), we start with some preparatory calculations:
\[
(L_s + \partial_s)(P_{s,t}^p f)^{q(s)}
= L_s(P_{s,t}^p f)^{q(s)} + q(s) (P_{s,t}^p f)^{q(s)-1} (\partial_s P_{s,t}^p f) + q'(s) (P_{s,t}^p f)^{q(s)-1} \log P_{s,t}^p f
+ q'(s) (P_{s,t}^p f)^{q(s)-1} \left( |\nabla^t P_{s,t}^p f|^2\right).
\] (6.11)
By Corollary 3.2 there exist positive constants \( c_1(s,t) \) and \( c_2(s,t) \) such that
\[
\left\| |\nabla^s P_{s,t}^p f|^2\right\|_\infty \leq c_1(s,t) \|f\|^2_\infty + c_2(s,t) \|\nabla^t f\|^2_\infty.
\]
Moreover, \( \|P_{s,t}^p f\|_\infty \leq (P_{s,t}^p 1) \|f\|_\infty \) and
\[
(P_{s,t}^p f)^{q(s)} \log^+ (P_{s,t}^p f) \leq (P_{s,t}^p f)^{q(s)+1} \leq (P_{s,t}^p 1)^{q(s)+1} \|f\|^{q(s)+1}_\infty.
\]
Combining these estimates, we obtain
\[
\left\| (L_s + \partial_s)(P_{s,t}^p f)^{q(s)}\right\|_\infty < \infty.
Now, by Theorem 5.1 we see that

\[
\frac{d}{ds} \mu_s((P^g_{s,t,f})^{(s)}) = -\mu_s(Q_s((P^g_{s,t,f})^{(s)}) + \mu_s(\partial_s((P^g_{s,t,f})^{(s)})) = -\mu_s(Q_s((P^g_{s,t,f})^{(s)}) + \mu_s((L_s + \partial_s)(P^g_{s,t,f})^{(s)}) = q(s)(q(s) - 1) \mu_s((|\nabla s P^g_{s,t,f}|^2 + (P^g_{s,t,f})^{(s)} - q'(s) \mu_s((P^g_{s,t,f})^{(s)} log P^g_{s,t,f}) - (1 - q(s)) \mu_s(Q_s((P^g_{s,t,f})^{(s)}).
\]

For \( \|P_{s,t,f}\|_{q(s),s} \), since \( \|P^g_{s,t,f}\|_{q(s),s} = \left( \mu_s((P^g_{s,t,f})^{(s)})^{1/q(s)} \right)^{1/q(s)} \), we thus find

\[
\frac{d}{ds} \|P^g_{s,t,f}\|_{q(s),s} = (q(s) - 1)\|P^g_{s,t,f}\|_{q(s),s}(\mu_s((|\nabla s P^g_{s,t,f}|^2 + (P^g_{s,t,f})^{(s)} - q'(s) \mu_s((P^g_{s,t,f})^{(s)} log P^g_{s,t,f}) - \frac{q'(s)}{q(s)} \|P^g_{s,t,f}\|_{q(s),s} \mu_s(Q_s((P^g_{s,t,f})^{(s)})
\]

\[
\geq (q(s) - 1)\|P^g_{s,t,f}\|_{q(s),s} \mu_s((|\nabla s P^g_{s,t,f}|^2 + (P^g_{s,t,f})^{(s)} - q'(s) \mu_s((P^g_{s,t,f})^{(s)} log P^g_{s,t,f}) - \frac{q'(s)}{q(s)} \|P^g_{s,t,f}\|_{q(s),s} \mu_s(Q_s((P^g_{s,t,f})^{(s)})
\]

\[
- \frac{q'(s)}{q(s)} \|P^g_{s,t,f}\|_{q(s),s} \mu_s(Q_s((P^g_{s,t,f})^{(s)}) \leq (q(s) - 1) \sup \frac{g^{-}}{q(s)} \geq r^2 = T.
\]

On the other hand, passing from \( f \) to \( f^{p/2}/\|f^{p/2}\|_{L^2} \) in the log–Sobolev inequality (6.10), we obtain

\[
\int f^p \log \left( \frac{f^p}{\|f^{p/2}\|_{L^2}^2} \right) d\mu_s \leq r^2 \int f^{p-2} |\nabla s f|^2 d\mu_s + \beta_s(r) \|f^{p/2}\|_{L^2}^2.
\]

In this inequality, replacing \( f \) and \( p \) by \( P^g_{s,t,f} \) and \( q(s) \) respectively, we obtain

\[
\mu_s((P^g_{s,t,f})^{(s)} log(P^g_{s,t,f})) - \|P^g_{s,t,f}\|_{q(s),s} \mu_s((P^g_{s,t,f})^{(s)} log P^g_{s,t,f}) \leq r^2 \int (P^g_{s,t,f})^{q(s)-2} |\nabla s P^g_{s,t,f}|^2 d\mu_s + \beta_s(r) \|P^g_{s,t,f}\|_{q(s),s} \mu_s((P^g_{s,t,f})^{(s)} log P^g_{s,t,f}.
\]

We now let

\[
q(s) = e^{4e^{-1/(1-t)}}(p - 1) + 1, \quad q(t) = p.
\]

Note that \( q \) is a decreasing function and \( q'(s)r/4 + (q(s) - 1) = 0. \) Thus, combining Eq. (6.13) with Eq. (6.12), we arrive at

\[
\frac{d}{ds} \|P^g_{s,t,f}\|_{q(s),s} \geq \frac{\beta_s(r)q'(s)}{q(s)^2} - \frac{(q(s) - 1) \sup g^{-}}{q(s)} \geq r^2 \|P^g_{s,t,f}\|_{q(s),s}, \quad 0 \leq s \leq t < T.
\]

It follows that

\[
\|P^g_{s,t,f}\|_{q(s),s} \geq \exp \left( - \int_s^t \frac{\beta_u(r)q'(u)}{q(u)^2} du + \int_s^t \frac{(q(u) - 1) \sup g^{-}}{q(u)} du \right) \|f\|_{p,t}.
\]
If we impose that \( q(s) = q \), then \( r = 4(t - s) \left( \log \frac{q - 1}{p - 1} \right)^{-1} \). Substituting the value of \( r \) into Eq. (6.14) yields

\[
\|P^0_{s,t}f\|_{q,s} \leq \exp \left( - \int_s^t \frac{\beta_n(4(t - s) \left( \log(q - 1)/(p - 1) \right)^{-1}) q'(u)}{q(u)^2} \, du + \int_s^t \frac{(q(u) - 1) \sup q_u}{q(u)} \, du \right) \|f\|_{p,t}. \]

Using the dimension-free Harnack inequality, we are now in position to give a necessary and sufficient condition for supercontractivity of \( P^0_{s,t} \).

**Theorem 6.4.** We keep the situation of Theorem 6.1 that is \( \mathcal{G}_t \) bounded and

\[
\text{Ric}_t + \text{Hess}_t(\phi_t) - \frac{1}{2} \partial_t g_t \geq K(t), \quad |dg_t| \leq \kappa(t) \quad \text{for } t \in [0, T[.
\]

The condition

\[
\|P^0_{s,t}\|_{(p,t)\rightarrow(q,s)} < \infty, \quad \text{for all } 1 < p < q < \infty, \quad 0 \leq s \leq t < T,
\]

then holds if and only if

\[
\mu_t\left( \exp(\lambda \rho^2_t) \right) < \infty \quad \text{for all } \lambda > 0 \text{ and } t \in [0, T[.
\]

**Proof.** By means of the Harnack inequality (4.1), the theorem can be proved along the same lines as in [20] Theorem 5.7.3 or [10]. For the reader’s convenience, we include a proof here. First, for \( 0 \leq s \leq t < T, p > 1 \), and \( f \in C_0(M) \), it follows from the Harnack inequality (4.1) that

\[
\left| (P^0_{s,t})^p f(x) \right| \leq \left( P^0_{s,t} |f|^p(y) \right) \exp \left( (p - 1) \int_s^t \sup \rho^p_t \, dr + \frac{pp_s^2(x,y)}{4(p - 1)\alpha(s,t)} + \frac{\eta(s,t)p_s(x,y)}{\alpha(s,t)} \right).
\]

Thus, if \( \mu_t(|f|^p) = 1 \), then

\[
1 \geq |P^0_{s,t}f(x)|^p \int \exp \left( (1 - p) \int_s^t \sup \rho^p_t \, dr - \frac{pp_s^2(x,y)}{4(p - 1)\alpha(s,t)} - \frac{\eta(s,t)p_s(x,y)}{\alpha(s,t)} \right) \mu_s(dy) \geq |P^0_{s,t}f(x)|^p \mu_s(B_s(o,R)) \times \exp \left( (1 - p) \int_s^t \sup \rho^p_t \, dr - \frac{p(\rho_s(x) + R)^2}{4(p - 1)\alpha(s,t)} - \frac{\eta(s,t)(\rho_s(x) + R)}{\alpha(s,t)} \right), \tag{6.15}
\]

where \( B_s(o,R) = \{ y \in M : \rho_s(y) \leq R \} \) denotes the geodesic ball (with respect to the metric \( g(t) \)) of radius \( R \) about \( o \in M \) and where \( \rho_t(\cdot) = \rho_t(o, \cdot) \). Since \( \mu_t\left( \exp(\lambda \rho^2_t) \right) < \infty \), the system of measures \( (\mu_t) \) is compact, i.e., there exists \( R = R(s) > 0 \), possibly depending on \( s \), such that

\[
\mu_s(B_s(o,R(s))) = \mu_s(\{ x : \rho_s(x) \leq R(s) \}) \geq 1 - \frac{\mu_s(p^2_t)}{R(s)^2} \geq 2^{-p}
\]

(after normalising \( \mu_s \) to a probability measure). Combining the last estimate with Eq. (6.15), we arrive at

\[
1 \geq |P^0_{s,t}f(x)|^p \ 2^{-p} \exp \left( (1 - p) \int_s^t \sup \rho^p_t \, dr - \frac{\eta(s,t)(\rho_s(x) + R)}{\alpha(s,t)} - \frac{p(\rho_s(x) + R)^2}{4(p - 1)\alpha(s,t)} \right)
\]

which further implies

\[
|P^0_{s,t}f(x)| \leq 2 \exp \left( \frac{p - 1}{p} \int_s^t \sup \rho^p_t \, dr + \frac{\eta(s,t)(\rho_s(x) + R)}{\alpha(s,t)} + \frac{(\rho_s(x) + R)^2}{4(p - 1)\alpha(s,t)} \right), \quad s < t. \tag{6.16}
\]
Therefore, we achieve
\[ \| P_{t,s}^c f \|_{p,s} \leq \left( \mu_s \left( \exp \left( q \left( c_1 + c_2 \rho_s^2 \right) \right) \right) \right)^{1/q} \]
for some positive constants \( c_1, c_2 \) depending on \( s \) and \( t \). Hence, if \( \mu_s(\exp(\lambda \rho_s^2)) < \infty \) for any \( \lambda > 0 \) and \( s \in [0, T] \), then \( P_{t,s}^c \) is supercontractive, i.e., for any \( 1 < p < q < \infty \),
\[ \| P_{t,s}^c \|_{(p,s)\rightarrow(q,s)} < \infty. \]

Conversely, if the semigroup \( P_{t,s}^c \) is supercontractive, by Theorem 6.1 the super log-Sobolev inequalities (6.10) holds. We first prove that \( \mu_s(\exp(\lambda \rho_s^2)) < \infty \) for \( s \in [0, T] \) and \( \lambda > 0 \). To this end, let \( \rho_{s,k} = \rho_s \wedge k \) and \( h_{s,k}(\lambda) = \mu_s(\exp(\lambda \rho_{s,k})) \). Taking \( \exp(\lambda \rho_{s,k}/2) \) in the super log-Sobolev inequality (6.1), we obtain
\[ \lambda h'_{s,k}(\lambda) - h_{s,k}(\lambda) \log h_{s,k}(\lambda) \leq h_{s,k}(\lambda) h^2 \left( \frac{r}{4} + \frac{\beta_s(r)}{\lambda^2} \right). \]
This implies
\[ \left( \frac{1}{\lambda} \log h_{s,k}(\lambda) \right)' = \frac{\lambda h'_{s,k}(\lambda) - h_{s,k}(\lambda) \log h_{s,k}(\lambda)}{\lambda^2 h_{s,k}(\lambda)} \leq \frac{r}{4} + \frac{\beta_s(r)}{\lambda^2}. \tag{6.17} \]
Integrating both sides of Eq. (6.17) from \( \lambda \) to \( 2\lambda \), we obtain
\[ h_{s,k}(2\lambda) \leq h_{s,k}(\lambda) \exp \left( \frac{r}{2} \lambda^2 + \beta_s(r) \right). \tag{6.18} \]
From this inequality, along with the fact that there exists a constant \( M_s \) such that
\[ \mu_s((\lambda \rho_s \geq M_s)) \leq \frac{1}{4} \exp \left( -\frac{r}{2} \lambda^2 - \beta_s(r) \right), \]
we get
\[ h_{s,k}(\lambda) = \int_{[\lambda \rho_s \geq M_s]} \exp(\lambda \rho_{s,k}) \, d\mu_s + \int_{[\lambda \rho_s < M_s]} \exp(\lambda \rho_{s,k}) \, d\mu_s \]
\[ \leq \mu_s((\lambda \rho_s \geq M_s))\mu_s(\exp(\lambda \rho_{s,k})^{1/2}) + e^M \mu_s((\lambda \rho_s < M_s)) \]
\[ \leq \frac{1}{4} \exp \left( -\frac{r}{2} \lambda^2 - \beta_s(r) \right) \left( \left( \frac{r}{4} + \frac{1}{2} \beta_s(r) \right) \right)^{1/2} \]
\[ \leq \frac{1}{2} h_{s,k}(\lambda) + e^M \mu_s((\lambda \rho_s < M_s)), \]
which implies \( h_{s,k}(\lambda) \leq 2e^M \mu_s((\lambda \rho_s < M_s)) \) for \( s \in [0, T[. \) As \( M_s \) is independent of \( k \), letting \( k \) tend to infinity, we arrive at
\[ \mu_s(\exp(\lambda \rho_s^2)) < \infty, \quad \text{for all } s \in [0, T]. \]

To prove that moreover \( \mu_s(\exp(\lambda \rho_s^2)) < \infty \) for \( s \in [0, T[ \) and \( \lambda > 0 \), we can follow the argument in [10] pp. 22-23. \( \square \)

As an application, we apply the results above to the modified flow \((M, g_t, \phi_t)\) evolving by
\[ \begin{cases} \partial_t g(x, \cdot)(t) = -2(h + \text{Hess}(\phi))(x, t); \\ \partial_t \phi_t = -\Delta \phi_t - \mathcal{H}_t. \end{cases} \]
For this system, we have \( \phi \equiv 0 \). Thus, by Theorem 6.1 we get the following result.
Corollary 6.5. Assume that \((g_t, \phi_t)\) follows the evolving equation \((5.5)\) and
\[
\text{Ric}_t + \text{Hess}_t(\phi_t) - \frac{1}{2} \partial_t g_t \geq K(t), \quad \text{for all } t \in [0, T[.
\]
Suppose that
\[
\|P_{s,t}\|_{(p,t)\rightarrow(q,s)} < \infty \quad \text{for } 1 < p < q \text{ and } 0 \leq s \leq t < T.
\]
Then the super log-Sobolev inequalities
\[
\int f^2 \log f^2 \, du \leq r \int |\nabla f|^2 \, du + \beta_t(r), \quad r > 0,
\]
hold for every \(f \in H^1(M, \mu_t)\) such that \(\|f\|_{L^2} = 1, \, t \in [0, T[\), where \(\beta_t(r) := \tilde{\beta}^{-1}(\gamma_t^{-1}(r), t)\) and
\[
\tilde{\beta}(s, t) = \frac{pq}{q-p} \log(\|P_{s,t}\|_{(q,s)\rightarrow(p,t)}),
\]
\[
\gamma(s, t) = \frac{4p (q-1)}{q-p} \int_s^t \exp \left( -2 \int_r^s K(u) \, du \right) \, dr,
\]
\[
\gamma_t^{-1} (r) = \inf \{ s \geq 0 : \gamma(s, t) \leq r \}.
\]

The following result is a consequence of Theorem 6.4.

Corollary 6.6. Assume that \((g_t, \phi_t)\) evolves according to \((5.5)\) and that
\[
\text{Ric}_t + \text{Hess}_t(\phi_t) - \frac{1}{2} \partial_t g_t \geq K(t), \quad t \in [0, T[,
\]
for some function \(K \in C([0, T[)\). Then
\[
\|P_{s,t}\|_{(p,t)\rightarrow(q,s)} < \infty, \quad \text{for } 1 < p < q < \infty \text{ and } 0 \leq s \leq t < T,
\]
if and only if
\[
\mu_t(\exp(\lambda \phi_t^2)) < \infty, \quad \text{for } \lambda > 0 \text{ and } t \in [0, T[.
\]

REFERENCES

1. Abimbola Abolarinwa, Differential Harnack estimates for conjugate heat equation under the Ricci flow, Asian-Eur. J. Math. 8 (2015), no. 4, 1550063, 17. MR 3424138
2. , Sobolev-type inequalities and heat kernel bounds along the geometric flow, Afr. Mat. 27 (2016), no. 1-2, 169–186. MR 3459325
3. Dominique Bakry, On Sobolev and logarithmic Sobolev inequalities for Markov semigroups, New trends in stochastic analysis (Charingworth, 1994), World Sci. Publ., River Edge, NJ, 1997, pp. 43–75. MR 1654503
4. Jean-Michel Bismut, Large deviations and the Malliavin calculus, Progress in Mathematics, vol. 45, Birkhäuser Boston, Inc., Boston, MA, 1985. 75001
5. Mihai Băileşteanu, Bounds on the heat kernel under the Ricci flow, Proc. Amer. Math. Soc. 140 (2012), no. 2, 691–700. MR 2846338
6. Reto Buzano and Louis Yudowitz, Gaussian upper bounds for the heat kernel on evolving manifolds, arXiv:2007.07112 (2020).
7. Xiaodong Cao, Hongxin Guo, and Hung Tran, Harnack estimates for conjugate heat kernel on evolving manifolds, Math. Z. 281 (2015), no. 1-2, 201–214. MR 3384867
8. Li-Juan Cheng, Diffusion semigroup on manifolds with time-dependent metrics, Forum Math. 29 (2017), no. 4, 775–798. MR 3669003
9. Li-Juan Cheng and Anton Thalmaier, Characterization of pinched Ricci curvature by functional inequalities, J. Geom. Anal. 28 (2018), no. 3, 2312–2345. MR 3833795
10. , Evolution systems of measures and semigroup properties on evolving manifolds, Electron. J. Probab. 23 (2018), Paper No. 20, 27. MR 3771757
11. , Spectral gap on Riemannian path space over static and evolving manifolds, J. Funct. Anal. 274 (2018), no. 4, 959–984. MR 3743188
12. Koléhé A. Coulibaly-Pasquier, *Heat kernel coupled with geometric flow and Ricci flow*, Séminaire de Probabilités L, Lecture Notes in Math., vol. 2252, Springer International Publishing, Cham, 2019, pp. 221–256.

13. Kenneth D. Elworthy and Xue-Mei. Li, *Formulae for the derivatives of heat semigroups*, J. Funct. Anal. **125** (1994), no. 1, 252–286. MR 1297021

14. Leonard Gross, *Logarithmic Sobolev inequalities*, Amer. J. Math. **97** (1975), no. 4, 1061–1083. MR 420249

15. Richard S. Hamilton, *The Harnack estimate for the Ricci flow*, J. Differential Geom. **37** (1993), no. 1, 225–243. MR 1198607

16. Grigori Perelman, *The entropy formula for the Ricci flow and its geometric applications*, arXiv:math/0211159v1 (2012).

17. Anton Thalmaier, *On the differentiation of heat semigroups and Poisson integrals*, Stochastics Stochastics Rep. **61** (1997), no. 3-4, 297–321. MR 1488139

18. James Thompson, *Derivatives of Feynman-Kac semigroups*, J. Theoret. Probab. **32** (2019), no. 2, 950–973. MR 3959634

19. Feng-Yu Wang, *Logarithmic Sobolev inequalities: conditions and counterexamples*, J. Operator Theory **46** (2001), no. 1, 183–197. MR 1862186

20. , *Functional inequalities, markov semigroups and spectral theory*, The Science Series of the Contemporary Elite Youth, Science Press, Beijing, 2005.

21. , *Log-Sobolev inequality on non-convex Riemannian manifolds*, Adv. Math. **222** (2009), no. 5, 1503–1520. MR 2555903

22. , *Analysis for diffusion processes on Riemannian manifolds*, Advanced Series on Statistical Science & Applied Probability, vol. 18, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2014. MR 3154951

23. Qi S. Zhang, *Some gradient estimates for the heat equation on domains and for an equation by Perelman*, Int. Math. Res. Not. (2006), Art. ID 92314, 39. MR 2250008

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