A New Finitely Controllable Class of Tuple Generating Dependencies: The Triangularly-Guarded Class

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Abstract
In this paper we introduce a new class of tuple-generating dependencies (TGDs) called triangularly-guarded (TG) TGDs. We show that conjunctive query answering under this new class of TGDs is decidable since this new class of TGDs also satisfies the finite controllability (FC) property. We further show that this new class strictly contains some other decidable classes such as weak-acyclic, guarded, sticky and shy. In this sense, the class TG provides a unified representation of all these aforementioned classes of TGDs.

Introduction
In the classical database management systems (DBMS) setting, a query Q is evaluated against a database D. However, it has come to the attention of the database community the necessity to also include ontological reasoning and description logics (DLs) along with standard database techniques (Calvanese et al. 2007). As such, the ontological database management systems (ODBMS) has arisen. In ODBMS, the classical database is enhanced with an ontology (Baader et al. 2016) in the form of logical assertions that generate new intensional knowledge. An expressive form of such logical assertions is the so-called tuple-generating dependencies (TGDs), i.e., Horn rules extended by allowing existential quantifiers to appear in the rule heads (Cabibbo 1998; Patel-Schneider and Horrocks 2007; Cali, Gottlob, and Lukasiewicz 2009).

Queries are evaluated against a database D and set of TGDs Σ (i.e., D ∪ Σ) rather than just D, as in the classical setting. Since for a given database D, a set Σ of TGDs, and a conjunctive query Q, the problem of determining if D ∪ Σ |= Q, i.e., the conjunctive query answering (CQ-Ans) problem, is undecidable in general (Beeri and Vardi 1981; Baget et al. 2011; Rosati 2011; Cali, Gottlob, and Kifer 2013), a major research effort has been put forth to identifying syntactic conditions on TGDs for which CQ-Ans is decidable. Through these efforts, we get the decidable syntactic classes: weakly-acyclic (WA) (Pagin et al. 2005), acyclic graph of rule dependencies (aGRD) (Baget et al. 2011), linear, multi-linear, guarded, weakly-guarded (W-GUARDed) (Rosati 2006; Cali, Gottlob, and Kifer 2013), sticky, weakly-sticky-join (WSJ) (Cali, Gottlob, and Pieris 2012; Gogacz and Marcinkowski 2017), shy (SHY) (Leone et al. 2012) and weakly-recursive (WR) (Civil and Rosati 2012). The weakly-recursive class is only defined for the simple TGDs, which are TGDs where the variables are only allowed to occur once in each atom and each atom do not mention constants (Civil and Rosati 2012).

Another research direction that sprangs up from those previously identified classes is the possibility of obtaining more expressive languages by a direct combination (i.e., union) of those classes, e.g., see (Krötzsch and Rudolph 2011; Cali, Gottlob, and Pieris 2012; Grau et al. 2013; Gottlob, Manna, and Pieris 2013). A major challenge then in this direction is that the union of two decidable classes is not necessarily decidable (Baget et al. 2011), e.g., it has been shown in (Gottlob, Manna, and Pieris 2013) that the union of the classes linear and sticky TGDs is undecidable.

At a model theoretic level, the results in (Rosati 2006) and (Bárany, Gottlob, and Otto 2010) had respectively shown that the finite controllability property holds for the linear and guarded fragments of TGDs. Here, a set of TGDs Σ is said to have the finite controllability (FC) property if for any database D and conjunctive query Q, we have that the first-order (FO) theory D ∪ Σ |= {Q} has the finite model property. It is folklore that a class of FO theories C is said to have the finite model (FM) property if φ ∈ C satisfiable iff φ has a finite model. The recent work in (Gogacz and Marcinkowski 2017) has further extended the result in (Rosati 2006) for linear TGDs into the sticky-join TGDs, while a more recent work by (Amendola, Leone, and Manna 2017) shows that the FC property also for the shy class (Leone et al. 2012). As will be revealed from this paper, our work further generalizes these previous results.

Despite these efforts, there are still some examples of simple TGDs that do not fall under the aforementioned classes.

Example 1. Let Σ₁ be a set of TGDs comprising of the fol-
Then it can be checked that $\Sigma_1$ does not fall into any of the classes previously mentioned above, and neither is it \textit{glut-guarded (G-guarded)} (Krotzsch and Rudolph 2011) nor \textit{tame (TAME)} (Gottlob, Mannia, and Pieris 2013). On the other hand, because none of the head atoms “$t(Y, Z)$” and “$u(X, Y)$” of $\sigma_1$ mentions the two cyclically-affected body variables “$X$” and “$Z$” together (which is under some pattern that we will generalize in Section 3), it then can be shown that for any database $D$ and query $Q$, it is sufficient to only consider a finite number of \textit{labeled nulls} in $\text{chase}(D, \Sigma_1)$ to determine if $\text{chase}(D, \Sigma_1) \models Q$. Actually, $\Sigma_1$ falls under a new class of TGDs we call \textit{triangularly-guarded}, which strictly contains several of the main syntactic classes, including WA, W-guarded, WSJ, G-guarded, SHY, TAME and WR.

The rest of the paper is structured into four main parts as follows: Section 2 provides background notions and definitions about databases, TGDs and the problem of (boolean) conjunctive query answering; Section 3 introduces the triangularly-guarded (TG) class of TGDs; while Section 4 presents the main results and shows that TG is both decidable and strictly contains some of the main syntactic classes mentioned above; Finally, Section 5 concludes the paper with some remarks.

**Preliminaries**

**Basic notions and notations**

We assume three countably infinite pairwise disjoint sets $\Gamma_V$, $\Gamma_C$ and $\Gamma_N$ of \textit{variables}, \textit{constants} and \textit{labeled nulls}, respectively. We further assume that $\Gamma_V$ is partitioned into two disjoint sets $\Gamma^1_V$ and $\Gamma^2_V$ (i.e., $\Gamma_V = \Gamma^1_V \cup \Gamma^2_V$), where $\Gamma^1_V$ and $\Gamma^2_V$ denote the sets of \textit{universally} $(\forall)$ and \textit{existentially} $(\exists)$ quantified variables, respectively. We also assume that the set of labeled nulls $\Gamma_N$ contains elements of the form $\{n_i | i \in \mathbb{N}\}$, where $\mathbb{N}$ is the set of natural numbers. Intuitively, $\Gamma_N$ is the set of "fresh" Skolem terms that are disjoint from the set of constants $\Gamma_C$.

A relational schema $\mathcal{R}$ (or just schema) is a set of \textit{relational symbols} (or predicates), where each is associated with some number $n \geq 0$ called its \textit{arity}. We denote by $r/n$ as the relational symbol $r \in \mathcal{R}$ whose arity is $n$, and by $|r|$ as the arity of $r$, i.e., $|r| = n$. We further denote by $r[i]$ as the $i$-th argument (or attribute) of $r$ where $i \in \{0, \ldots, |r|\}$. We denote by $\text{ARG}(r)$ as the set of arguments $\{a = r[i] | i \in \{0, \ldots, |r|\}\}$ of $r$. We extend this notion to the set of relational symbols $\mathcal{R}$, i.e., $\text{ARG}(\mathcal{R}) = \bigcup_{r \in \mathcal{R}} \text{ARG}(r)$.

A term $t$ is any element from the set $\Gamma_V \cup \Gamma_C \cup \Gamma_N$. Then an atom $a$ is a construct of the form $(t_1, \ldots, t_n)$ such that: (1) $r \in \mathcal{R}$; (2) $n = |r|$; and (3) $t_i$ (for $i \in \{1, \ldots, n\}$) is a term. We denote tuples of atoms by $\langle a \rangle$, e.g., $\langle a_1 \ldots a_n \rangle$. \textit{b} = $a_1 \ldots a_m$, \textit{c} = $c_1 \ldots c_n$, etc., and its length by $|\textit{b}|$.

Sometimes for convenience and when clear from the context, we refer to an argument as lower-case letters (e.g., “a”, “b”, etc.) rather than “$r[i]$”. We denote by $\text{REL}(a)$, $\text{TERMS}(a)$, $\text{VAR}(a)$, $\text{CONST}(a)$ and $\text{NULLS}(a)$ as the relational symbol, the set of terms, variables, constants and labeled nulls mentioned in atom $a$, respectively. We extend these notions to a set or tuples of atoms $S$ such that $\text{TERMS}(S)$, $\text{VAR}(S)$, $\text{CONST}(S)$ and $\text{NULLS}(S)$ denote the sets $\bigcup_{a \in S} \text{TERMS}(a)$, $\bigcup_{a \in S} \text{VAR}(a)$, $\bigcup_{a \in S} \text{CONST}(a)$ and $\bigcup_{a \in S} \text{NULLS}(a)$, respectively. We say that a tuple of atoms $\langle a \rangle$ is \textit{connected} if either: (1) $\langle a \rangle$ is an atom (i.e., a singleton), or (2) $\text{TERMS}(a) \cap \text{TERMS}(a+1) \neq \emptyset$ holds, for each $i \in \{1, \ldots, l-1\}$. More specifically, we further say that $\langle a \rangle$ is \textit{NULLS-connected} if either: (1) $\langle a \rangle$ is an atom, or $\text{NULLS}(a) \cap \text{NULLS}(a+1) \neq \emptyset$ holds, for each $i \in \{1, \ldots, l-1\}$.

An instance $I$ is any set (can be infinite) of atoms such that $\text{VAR}(I) = \emptyset$, i.e., contains no variables. A database $D$ is a finite set of ground atoms $\text{VAR}(D) = \emptyset$ and $\text{NULLS}(D) = \emptyset$.

For an atom $a = r(t_1, \ldots, t_n)$, we denote by $\text{ARG}(a) |_{X}$ (resp. $\text{VAR}(a) |_{A}$) as the set of arguments (resp. variables) $\{r[i] | i \in \{1, \ldots, n\}\}$ (resp. $\{X | t_i = X \land r[i] \in A\}$) but \textit{restricted} to those variables (resp. argument positions) mentioning $X$ (resp. from $A$).

Given two sets of terms $T_1$ and $T_2$, a homomorphism $\theta : T_1 \rightarrow T_2$ is a function from $T_1$ onto $T_2$ such that $t \in (T_1 \cap \Gamma_C)$ implies $\theta(t) = t$, i.e., identity for the constants $\Gamma_C$. Then for a given atom $a = r(t_1, \ldots, t_n)$, a set of terms $T$ and a homomorphism $\theta : \text{VAR}(a) \rightarrow T$, a \textit{substitution} of $a$ under $\theta$ (or just $\theta$-substitution for convenience), denoted $a_{\theta}$ (or sometimes $\theta(a)$), is the atom such that $a_{\theta} = r(\theta(t_1), \ldots, \theta(t_n))$. We naturally extend to conjunctions of atoms $a_1 \land \ldots \land a_n$, so that $\theta(a_1 \land \ldots \land a_n) = a_1 \land \ldots \land a_n_{\theta}$. Given two homomorphisms $\theta_1 : T_1 \rightarrow T_2$ and $\theta_2 : T_2 \rightarrow T_3$, we denote by $\theta_2 \circ \theta_1$ as the \textit{composition} of $\theta_1$ with $\theta_2$ such that $\theta_2 \circ \theta_1 : T_1 \rightarrow T_3$ and $(\theta_2 \circ \theta_1)(t) = \theta_2(\theta_1(t))$ for all $t \in T_1$. Then lastly, given again a homomorphism $\theta : T_1 \rightarrow T_2$ and some set of terms $T' \subseteq T_2$, we denote by $\theta|_{T'}$ as the \textit{restriction} of the homomorphism $\theta$ to the domain $T' \subseteq T_2$. Given a pair of atoms $\langle a_1, a_2 \rangle$ and corresponding pair of homomorphisms $\theta_1 : \text{TERMS}(a_1) \rightarrow \Gamma_C \cup \Gamma_V$ and $\theta_2 : \text{TERMS}(a_2) \rightarrow \Gamma_C \cup \Gamma_V$, we say that $\theta_1$ and $\theta_2$ is a \textit{most general unifier (MGU)} of the pairs $\langle a_1, a_2 \rangle$ if it unifies $a_1$ and $a_2$ (i.e., $a_1 \theta_1 = a_2 \theta_2$), and for any unifiers $\theta_1$ and $\theta_2$ of $a_1$ and $a_2$ (i.e., $a_1 \theta_1 = a_2 \theta_2$), there are unifiers $\theta'_1$ and $\theta'_2$ such that $\theta'_1 \circ \theta'_1 = \theta'_2 \circ \theta'_2$.

**TGDs, BCQ-Ans and Chase**

A \textit{tupling generating dependency} (TGD) rule $\sigma$ of schema $\mathcal{R}$ is a first-order (FO) formula of the form:
\begin{equation}
\forall \mathbf{X} \mathbf{Y} (\Phi(\mathbf{X}, \mathbf{Y}) \rightarrow \exists \mathbf{Z} \Psi(\mathbf{Y}, \mathbf{Z})),
\end{equation}
where:
1. $\mathbf{X} = X_1 \ldots X_k$, $\mathbf{Y} = Y_1 \ldots Y_l$ and $\mathbf{Z} = Z_1 \ldots Z_m$ are pairwise disjoint tuple of variables;
2. $\Phi(\mathbf{X}, \mathbf{Y}) = b_1(V_1) \land \ldots \land b_n(V_n)$ is a conjunction of atoms such that $V_i \subseteq \mathbf{X}$ and $b_i \in \mathcal{R}$, for $i \in \{1, \ldots, n\}$;
3. $\Psi(\mathbf{Y}, \mathbf{Z}) = r_1(W_1) \land \ldots \land r_m(W_m)$ is a conjunction of atoms where $W_i \subseteq \mathbf{Y}$ and $r_i \in \mathcal{R}$, for $i \in \{1, \ldots, m\}$.1
For a given TGD $\sigma$ of the form (3), we denote by $BD(\sigma)$ the set of atoms $\{b_1(V_1), \ldots, b_n(V_n)\}$, which we also refer to as the body of $\sigma$. Similarly, by $HD(\sigma)$ we denote the set of atoms $\{r_1(W_1), \ldots, r_m(W_m)\}$, which we also refer to as the head of $\sigma$. For convenience, when it is clear from the context, we simply drop the quantifiers in (3) such that a TGD rule $\sigma$ of the form (3) can simply be referred to as: $\Phi(X, Y) \Rightarrow \Psi(Y, Z)$. Then, for a given set of TGDs $\Sigma$, we denote by $ATOMS(\Sigma)$ the set of all atoms occurring in $\Sigma$ such that $ATOMS(\Sigma) = \bigcup_{\sigma \in \Sigma} (BD(\sigma) \cup HD(\sigma))$, and by $REL(\Sigma)$ as the set of all relational symbols mentioned in $\Sigma$. Then lastly, for a given rule $\sigma$ of the form (3) and atom $a \in HD(\sigma)$, we denote by $\forall$-VAR($a$) and $\exists$-VAR($a$) as the set of variables $VAR(a) \cap Y$ and $VAR(a) \cap Z$, respectively, i.e., the set of all the universally ($\exists$) and existentially ($\forall$) quantified variables of $a$, respectively. We extend this notion to the TGD rule $\sigma$ of the form (3) so that we set $VAR(\sigma) = XYZ$, $\forall$-VAR($\sigma$) = $XY$ and $\exists$-VAR($\sigma$) = $Z$.

A boolean conjunctive query (BCQ) $Q$ is a FO formula $\exists X \varphi(X) \rightarrow q$ such that $\varphi(X) = r_1(Y_1) \land \ldots \land r_l(Y_l)$, where $r_i \in R$ and $Y_i \subseteq X$, for each $i \in \{1, \ldots, l\}$, and where we set $BD(Q) = \{r_1(Y_1), \ldots, r_l(Y_l)\}$. Given a database $D$ and a set of TGDs $\Sigma$, we say that $D \cup \Sigma$ entails $Q$, denoted $D \cup \Sigma \models Q$, if $D \cup \Sigma \models \exists X \varphi(X)$. The central problem tackled in this work is the boolean conjunctive query answering (BCQ-Ans): given a database $D$, a set of TGDs $\Sigma$ and BCQ $Q$, does $D \cup \Sigma \models Q$? It is well known that BCQ-Ans is undecidable in general (Beeri and Vardi 1981).

The chase procedure (or just chase) (Maier, Mendelzon, and Sagiv 1979; Johnson and Klug 1984; Abiteboul, Hull, and Vianu 1995; Fagin et al. 2005; Deutsch, Nash, and Remmel 2008; Zhang, Zhang, and You 2015) is a main algorithmic tool proposed for checking implication dependencies (Maier, Mendelzon, and Sagiv 1979). For a given $I$, homomorphism $\eta$ and TGD $\sigma = \Phi(X, Y) \rightarrow \Psi(Y, Z)$, we have that $I^{\sigma, \eta}$, $I'$ defines a single chase step as follows: $I' = I \cup \{\eta'((\Psi(Y, Z))) \mid \text{such that:} (1) \eta: XY \rightarrow \Gamma_C \cup \Gamma_N$ and $\eta((\Phi(X, Y))) \subseteq I; \text{and (2)} \eta': \text{XYZ} \rightarrow \Gamma_C \cup \Gamma_N$ and $\eta'_{[XY]} = \eta$. As in the literature, we further assume here that each labeled nulls used to eliminate the $\exists$-quantified variables in $Z$ follows lexicographically all the previous ones, i.e., follows the order $n_1, n_1+1, n_1+2, \ldots$. A chase sequence of a database $D$ w.r.t. to a set of TGDs $\Sigma$ is a sequence of chase steps $I^{{\sigma_1}, \eta_1}$, $I^{{\sigma_2}, \eta_2}$, $I^{{\sigma_3}, \eta_3}$, $I^{{\sigma_4}, \eta_4}$, $I^{{\sigma_{i+1}}, \eta_{i+1}}$, where $i \geq 0$, $I_0 = D$ and $\sigma_i \in \Sigma$. An infinite chase sequence $I^{{\sigma_1}, \eta_1}$, $I^{{\sigma_2}, \eta_2}$, $I^{{\sigma_3}, \eta_3}$, $I^{{\sigma_4}, \eta_4}$, $I^{{\sigma_{i+1}}, \eta_{i+1}}$, $\forall i \geq 0$, is fair if $\eta((\Phi(X, Y))) \subseteq I$, for some $\eta: \text{XY} \rightarrow \Gamma_C \cup \Gamma_N$ and $\sigma = \Phi(X, Y) \rightarrow \Psi(Y, Z) \in \Sigma$, implies $\exists \eta': \text{XYZ} \rightarrow \Gamma_C \cup \Gamma_N$, where $\eta'_{[XY]} = \eta$, such that $\eta'((\Psi(Y, Z))) \subseteq I_k$ and $k > i$. Then finally, we let $\text{chase}(D, \Sigma) = \bigcup_{i=0}^{\infty} I_i$.

**Theorem 1.** ([Deutsch, Nash, and Remmel 2008; Cali, Gottlob, and Pieris 2012]) Given a database $D$, a set of TGDs $\Sigma$ and BCQ $Q$, $D \cup \Sigma \models Q$ iff $\text{chase}(D, \Sigma) \models Q$. 

**Cyclically-affected arguments**

As observed in (Leone et al. 2012), the notion of affected arguments in (Cali, Gottlob, and Kifer 2013) can sometimes consider arguments that may not actually admit a “firing” mapping $\forall$-variables into nulls. For this reason, it was introduced in (Leone et al. 2012) the notion of a “null-set.” Given a set of TGDs $\Sigma$, let $a \in ATOMS(\Sigma)$, $a \in ARG(a)$ and $X = \text{VAR}(a)_{\sigma(a)}$. Then the null-set of $a$ in a under $\Sigma$, denoted as $\text{NULLSET}(a, \sigma, \Sigma)$ (or just NULLSET($a, \sigma$) if clear from the context), is defined inductively as follows: If $a \in HD(\sigma)$, for some $\sigma \in \Sigma$, then (1) $\text{NULLSET}(a, \sigma) = \{n_1^a\}$, if $\text{VAR}(a)_{\sigma(a)} = X \in \exists$-VAR($a$), or (2) $\text{NULLSET}(a, \sigma)$ is the intersection of all null-sets $\text{NULLSET}(b, \sigma)$ such that $b \in BD(\sigma)$, $b \in \exists$-VAR($b$) and $\text{VAR}(b)_{\sigma(b)} = X$. Otherwise, if $a \in BD(\sigma)$, for some $\sigma \in \Sigma$, then $\text{NULLSET}(a, \sigma) = \text{null}$ is the union of all $\text{NULLSET}(a, \sigma')$ such that $REL(\sigma') = REL(a)$ and $\sigma' \in HD(\sigma')$, where $\sigma' \in \Sigma$.

Borrowing similar notions from (Krötzsch and Rudolph 2011) used in identifying the so-called glut variables, the existential dependency graph $\mathcal{G}_E(\Sigma)$ is a graph $(N, E)$, whose nodes $N$ is the union of all $\text{NULLSET}(a, \sigma)$, where $a \in ATOMS(\Sigma)$ and $a \in ARG(a)$, and edges:

$$E = \{ (n_1^a, n_2^a) \mid \exists \sigma \in \Sigma \text{ of form (3)}, \exists Y \subseteq Y, n_1^a \in \bigcap \text{NULLSET}(Y, \sigma, \Sigma) \text{ and NULLSET}(a, \sigma) = \{n_1^a\},$$

for some $a \in HD(\sigma)$ and $Z \subseteq Z$.

where $\bigcap \text{NULLSET}(Y, \sigma, \Sigma)$ denotes the intersection of all null-sets $(b, \sigma)$ such that $b \in BD(\sigma)$, $b \in \exists$-VAR($b$) and $\text{VAR}(b)_{\sigma(b)} = X$. We note that our definition of a dependency graph here generalizes the existential dependency graph in (Krötzsch and Rudolph 2011) by combining the notion of null-sets in (Leone et al. 2012), then with the graph $\mathcal{G}_E(\Sigma) = (N, E)$ as defined above, we denote by $\text{CYC-NULL}(\Sigma)$ as the smallest subset of $N$ such that $n_1^a \in \text{CYC-NULL}(\Sigma)$ iff either: (1) $n_1^a$ is in a cycle in $\mathcal{G}_E(\Sigma)$, or (2) $n_1^a$ is reachable from some other node $n_2^a \in \text{CYC-NULL}(\Sigma)$, where $n_2^a$, is in a cycle in $\mathcal{G}_E(\Sigma)$.

**Triangularly-Guarded (TG) TGDs**

This section now introduces the triangularly-guarded class of TGDs, which is the focus of this paper. We begin with an instance of a BCQ-Ans problem that corresponds to a need of an infinite number of nulls in the underlying chase derivation.

**Example 2 (Unbounded nulls).** Let $\Sigma_2 = \{\sigma_{11}, \sigma_{12}\}$ be the set of TGDs obtained from $\Sigma_1 = \{\sigma_{11}, \sigma_{12}\}$ of Example 7 by just changing the rule $\sigma_{12}$ into the rule $\sigma_{12}'$ such that:

$$\sigma_{12}' = t(X, Y) \land \forall(Y, Z) \rightarrow t(X, Z) \land \forall(X, Y).$$

Then we have that $\sigma_{12}'$ is obtained from $\sigma_{12}$ of $\Sigma_1$ by changing the variable $"Y"$ in the head atom $"t(Y, Z)"$ of $\sigma_{12}$ into $"X"$, i.e., to obtain $"t(X, Z)"$. Intuitively, this allows

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2We assume that $\sigma \neq \sigma'$ or $X \neq X'$ implies $n_1^X \neq n_1^{X'}$, for each pair of elements $(n_1^X, n_1^{X'})$ of all null-sets (Leone et al. 2012).
In order to avoid undesirable clashes in variables, we assume that each \( \langle B_1, h_1 \rangle \in \Sigma^1 \) and \( \langle B_2, h_2 \rangle \in \Sigma^1 \) mentioned in (7) have the variables renamed so that \( \forall (B_1 \cup \{ h_1 \}) \cap \forall (B_2 \cup \{ h_2 \}) = \emptyset \). Then we set \( \Sigma^+ = \Sigma^\infty \) as the fixpoint of \( \Sigma^+ \). We note that even though \( \Sigma^+ \) can be infinite in general, it follows from Theorem 2 that it is enough to consider just a finite number of iterations \( \Sigma^i \) to determine “recursive triangular-components” (as will be defined exactly in Definition 2).

The TGD extension \( \Sigma^+ \) of \( \Sigma \) contains as members pairs of sets of atoms and head atoms of the form \( \langle \{ B, h \}, \rangle \), respectively. Loosely speaking, the set \( B \) represents the possible union of body atom of some TGD in \( \Sigma \) while \( h \) the head atom of some TGD that can be linked (transitively) through the repeated applications of the steps in (6)-(9) (which is done until a fixpoint is reached). The base case \( \Sigma^0 \) in (6) first considers the pairs \( \langle \{ B, h \}, \rangle \) where \( B = BD(\sigma) \) and \( h \in H \), for each \( \sigma \in \Sigma \). Inductively, assuming we have already computed \( \Sigma^i \), we have that \( \Sigma^{i+1} \) is obtained by adding the previous step \( \Sigma^i \) as well as adding the set as defined through (6)-(9).

More specifically, using similar ideas to the TGD expansion in (Calì, Gottlob, and Pieris 2012) that was used in identifying the sticky-join class of TGDs and tame reachability in (Gottlob, Manna, and Pieris 2013) used for the tame class, the set (6)-(9) considers the other head types that can (transitively) reached from some originating TGD. Indeed, as described in (6)-(9), for \( \langle \{ B_1, h_1 \} \in \Sigma^i \) and \( \langle B_2, h_2 \rangle \in \Sigma^i \) (i.e., as in (7)), we add the pair \( \langle B_1 h_1 \wedge B^*, h_2 \rangle \) into \( \Sigma^{i+1} \). Intuitively, with the homomorphism “\( \eta_2 \)” as described in (6)-(9), the aforementioned pair \( \langle B_1 h_1 \wedge B^*, h_2 \rangle \) encodes the possibility that “\( B_2 h_2 \)” can be derived transitively from the union of bodies \( B_1 h_1 \) and \( B^* = B_2 h_2 \setminus b h_2 \). Here, we consider the union of \( B_1 h_1 \) and \( B^* = B_2 h_2 \setminus b h_2 \) (instead of just \( B_1 h_1 \)) because it may take a combination of these atoms to reveal a possible “recursive triangular-component” \( \Sigma^+ \). We note that because \( \eta_1 \) only maps variables from \( \forall \{ B_1 \cup \{ h_1 \} \) onto \( \forall \{ B_1 \cup \{ h_1 \} \) for the condition \( h_1 h_1 = b h_2 \) (i.e., as in (5)), then we can track some of the originating variables from \( B_1 \) all the way through the head “\( b h_2 \)” and which can be retained through iterative applications of the criterion given in (6)-(9). Importantly, we note that the connection between \( h_2 \) and \( h_2 \) is inferred with \( h_2 h_1 = b h_2 \) (where \( b \in B_2 h_2 \) corresponding to the most general unifier (MGU) \( h_1 b \) and please see Condition (2) of set (6)-(9)).

Example 3. Let \( \Sigma_3 \) be the following set of TGD rules:

\[ \sigma_31 : t(X, Y) \rightarrow \exists Z t(Y, Z), \]
\[ \sigma_32 : t(X, Y) \rightarrow s(X) \wedge s(Y), \]
\[ \sigma_33 : t(X_1, V) \wedge s(V) \wedge t(W, Z_1) \rightarrow u(X_1, V, W, Z_1), \]
\[ \sigma_34 : \forall(X_2, Y, Z_2) \rightarrow \forall(X_2, Z_2), \]
\[ \sigma_35 : \forall(X_3, Z_3) \rightarrow t(X_3, Z_3). \]

Then from the rules \( \sigma_33, \sigma_34, and \sigma_35 \), we get the three pairs \( p_{o1} = \{ t(X_1, V), s(V), t(W, Z_1), u(X_1, V, W, Z_1) \}, p_{o2} = \{ t(X_1, V, Y, Z_2), t(X_2, Z_2), u(X_1, V, Y, Z_2) \}, p_{o3} = \{ t(X_3, Z_3), t(X_3, Z_3) \} \) in \( \Sigma^0 \), respectively. Then through the unification of head atom “\( \forall(X_2, Z_2) \)” of the pair \( p_{o2} \) and the body atom

\[ \forall(X_1, V, W, Z_1) \]
"v(X_3, Z_3)" of the pair \( p_{03} \), then we get the pair \( p_{11} = \langle \{ u(X_2, Y, Z_2), f(X, Z_1) \} \rangle \) in \( \Sigma_2^+ \). Then finally, through the unification of the head atom "\'(X_1, V, W, Z_1)" of the pair \( p_{03} \) and the body atom "\'(X_2, Y, Y, Z_2)" of the pair \( p_{11} \), then we further get the pair \( p_{21} = \langle \{ f(X_1, V), s(V), f(V, Z_1) \}, f(X_1, Z_1) \rangle \) in \( \Sigma_3^+ \).

As will be seen in Definition 2, the last pair \( p_{21} \) in Example 3 corresponds to what we will call a "recursive triangular-component" that will be defined precisely in Definition 2 in the following section.

**Triangularly-guarded TGDs**

In this section, we introduce the key notion of triangularly-guarded TGDs, which are so-called triangular-components.

We first introduce the notion of cyclically-affecteonly variables in the body (i.e., set \( B \) of some pair \( \langle B, h \rangle \in \Sigma^+ \) where \( \Sigma \) is a set of TGDs. So towards this purpose, for a given pair \( \langle B, h \rangle \in \Sigma^+ \), we define \( \text{VAR}(\Sigma, B) \) (i.e., "\( \text{VAR} \)"

Loosely speaking, in the aforementioned variable markup, we can think of \( a \) as corresponding to some "body atom" while \( c \) and \( a' \) as "head atoms" that are reachable through the TGD extension \( \Sigma^+ \) (see Definition 1). Intuitively, the marked variables represent element positions that may fail the sticky-join property, i.e., disappear in the recursion of the rules. Intuitively, the sticky-join property insures decidability because only a finite number of elements can circulate among the recursive application of the rules. As will be revealed in Definition 2, we further note that we only consider marked variables in terms of the triple \( \langle a, c, a' \rangle \) because we only consider them for "recursive triangular-components."

**Definition 2 (Recursive triangular-components).** Let \( \Sigma \) be a set of TGDs and \( \Sigma^+ \) its extension as defined in Definition 1. Then a recursive triangular-component (RTC) \( T \) is a tuple

\[
\langle (B, h), (a, b, c), (X, Z), a' \rangle
\]

where

1.) \( (B, h) \in \Sigma^+ \); 2.) \( \{ a, b \} \subseteq B, a \neq b \) and \( c \in h; 3.) a' is an atom and there exists a homomorphism \( \theta \) : \( \text{VAR}(a) \rightarrow \text{VAR}(a') \) such that \( a\theta = a' \) and either one of the following holds:

(a) \( c = a' \), or

(b) there exists \( \theta' \in \Sigma^+ \) and function \( \eta : \text{VAR}(B') \cup \{h\}' \rightarrow \Gamma \), where \( \theta' \) is just a renaming substitution such that \( \text{VAR}(B'\theta' \cup \{h\}' \eta) \cap \text{VAR}(B \cup \{h\}) = \emptyset \), and \( c \in B' \theta' \cup \{h\}' \eta \) and \( a' \eta \theta' \).  

4.) \( X \) and \( Z \) are two distinct variables where \( X, Z \subseteq \text{VAR}(a) \); \( Z \subseteq \text{VAR}(b) \); \( X \subseteq \text{VAR}(a') \); and lastly, 5.) there exists a tuple of distinct atoms \( d_1, \ldots, d_m \subseteq B \) such that:

(a) \( a = d_1 \) and \( d_m = b \), and for each \( i \in \{1, \ldots, m - 1\} \), there exists \( Y_i \in \text{LINK}(B, d_i, d_{i+1}) \);

(b) for some \( i \in \{1, \ldots, m-1\} \), there exists \( Y' \in \text{LINK}(B, d_i, d_{i+1}) \) such that \( Y' \subseteq \text{ARG}(a') \) implies all occurrences of variables in positions \( \text{ARG}(a') \) in the atom \( a \) are in \( \text{M-VAR}(a, c, a') \).

Generally, a recursive triangular-component (RTC) \( T \) of the form (10) (see Definition 2 and Figure 1), can possibly enforce an infinite cycle of labeled nulls being "pulled" together into a relation in the chase derivation. We explain this by using again the TGDs \( \Sigma_2 = \{\sigma_{11}, \sigma_{12}\} \) and database \( D_2 \) of Example 2. Here, let us assume that \( B = BD(\sigma_{12}) \) and \( h \in HD(\sigma_{12}) \) such that \( (B, h) \) is the pair mentioned in (10). Then with the body atoms \( t(X, Y) \), \( u(Y, Z) \in BD(\sigma_{12}) \) and head atom \( t(X, Z) = h \in HD(\sigma_{12}) \) also standing for the atoms \( a, b \) and \( c = h \) in (10), respectively, then we can form the RTC \( T \):

\[
\langle (B, h), \langle t(X, Y), u(Y, Z), t(X, Z) \rangle, (X, Z), t(X, Z) \rangle.
\]

We note here from Condition 3.) of Definition 2 that the atom \( c' \) in (10) is also the head atom "\( (X, Z) \)" i.e., the choice (a) \( c = a' \) of Condition 3.) holds in this case. For simplicity, we note that one example RTC \( T \) above retains the names of the variables "\( X \)" and "\( Y \)" mentioned in the first example.
Figure 1: Recursive triangular-component (RTC).

Figure 2: RTC $T_1$ of $\Sigma_3$ with $\langle B, h \rangle = p_{21}$ of Example 3 in (10). Loosely speaking, for two atoms $t(n_1, n_2), t(n_1, n_k) \in \text{chase}(D_2, \Sigma_2)$, we have that rule $\sigma^{12}$ and its head atom “$t(X, Z)$” would combine the two nulls “$n_i$” and “$n_j$” into a relation “$t(n_1, n_2)$” in $\text{chase}(D_2, \Sigma_2)$. Since the variable “$X$” is retained in each RTC cycle via Condition $A_4$ (see Figure 1), this makes possible that nulls held by “$X$” in each cycle (in some substitution) to be pulled together into some other nulls held by “$Z$” as derived through the head atom “$t(X, Z)$”.

We further note that the connecting variable “$Y$” between the two body atoms “$t(X, Y)$” and “$u(Y, Z)$” corresponds to the variables $Y_i \in \text{LINK}(B, d_i, d_{i+1})$ of point $a$ of Condition $A_5$, and for some $i$, some $Y' \in \text{LINK}(B, d_i, d_{i+1}) \setminus \{X, Z\}$ also appears as marked (i.e., $Y' \in \text{VAR}(a, c, a')$) in point $b$ of Condition $A_5$ with respect to the atom $a'$. Intuitively, we require in $b$ of Condition $A_5$ that some of these variables $Y'$ occur as marked (w.r.t. $a'$) so that labeled nulls of some link variables have a chance to disappear in the RTC cycle, otherwise, they can only link and combine a bounded number of labeled nulls due to the sticky-join property (Call, Gottlob, and Pieris 2012).

Example 4. Consider again the pair $p_{21} = \langle \{t(X_1, V), s(V), t(V, Z_1)\}, \{t(X_1, Z_1)\}\rangle \in \Sigma_3^3$ in Example 2. Then with the pair $p_{21}$ standing for $\langle B, h \rangle$ in (10), the atoms “$t(X_1, V)$”, “$t(V, Z_1)$”, “$t(X_1, Z_1)$” and “$t(X_1, Z_1)$” for the atoms $a$, $b$, and $c$ in (10), respectively, and variables $\{X_1, Z_1\}$ for the variables $\{X, Z\}$ in (10), then we can get a corresponding RTC $T_1 = \langle p_{21}, \{t(X_1, V), t(V, Z_1), t(X_1, Z_1)\}, \{X_1, Z_1\}\rangle$ as illustrated in Figure 2.

Definition 3 (Triangularly-guarded TGDs). A set of TGDs $\Sigma$ is triangularly-guarded (TG) iff for each RTC $T$ of the form (10) (see Definition 2 and Figure 1), we have that there exists some atom $d \in B$ such that $\{X, Z\} \subseteq \text{VAR}(d)$.

For convenience, we denote by TG as the class of all the triangularly-guarded TGDs.

Example 5. Consider again the TGDs $\Sigma_1$ in Example 1 containing rules $\sigma_{11}$ and $\sigma_{12}$. Then because there cannot be any pair $\langle B, h \rangle \in \Sigma_1^+$ that would combine the two variables “$X$” and “$Z$” of rule $\sigma_{12}$ into a single head atom in $\Sigma_1^+$, then it follows that $\Sigma_1$ cannot have any RTC. Therefore, it trivially follows from Definition 3 that $\Sigma_1$ is in the class TG.

Main Results

We now examine the important properties of the TG class of TGDs. In particular, we show that membership decision and BCQ-Ans under this new class TG of TGDs are both decidable, with the latter because TG satisfies the FC property.

As we mentioned within the paragraph right after Definition 1 it is actually sufficient to only consider a finite number of iterations of the TGD expansion $\Sigma'$ to determine for the existence of RTC (otherwise, it can be the case that determining membership for the class TG is undecidable). So towards this purpose, for a set of TGDs $\Sigma$ of schema $\mathcal{R}$, we define the following number “$B(\Sigma)$” as follows:

$$B(\Sigma) = \max \{ |\mathcal{R}| \cdot \text{BELL} \left( \max \{ |\mathcal{R}|, \text{BELL} \left( \text{MAX} + \text{const}(\Sigma) \right) \right) \}$$

where: (1) $\text{BELL} = \max \{ |\mathcal{R}| \cdot \text{BELL} \left( \text{MAX} + \text{const}(\Sigma) \right) \}$, the largest size of body atoms mentioned in a rule of $\Sigma$; (2) $\text{MAX} = \max \{ r \mid r \in \mathcal{R} \}$, the maximum arity of a relation $r \in \mathcal{R}$; (3) TERMS$(\Sigma)$ is the set of terms mentioned in $\Sigma$; and (4) for a number $n \in \mathbb{N}$, $\text{BELL}(n)$ denotes the bell number of a set of size $n$, (which is a number exponential to $n$). More specifically, the bell number $\text{BELL}(n)$ can be defined through the recursive definition $\text{BELL}(n+1) = \sum_{k=0}^{n} \binom{n}{k} \cdot \text{BELL}(k)$, where $\text{BELL}(0) = 1$.

Intuitively, what we aim to achieve with the number $B(\Sigma)$ is to set a bound on the iterations of $\Sigma'$ (where we recall that $\Sigma^+ = \Sigma'^\omega$) so that we can be sure that there exists an RTC $T \in \Sigma^+$ iff there also exists one in $T' \in B(\Sigma)$. The main idea of why it is sufficient to consider the number $B(\Sigma)$ as the bound for the iterations of $\Sigma'$ is related to the ideas of equivalent atom types in (Chen et al. 2011). Thus, borrowing ideas from (Chen et al. 2011), given two atoms $a_1$ and $a_2$, we say that $a_1$ and $a_2$ are type-equivalent, denoted $a_1 \sim a_2$, if there exists an isomorphism (i.e., bijective mapping) $h : \text{TERMS}(a_1) \rightarrow \text{TERMS}(a_2)$, where the following conditions are satisfied: (1) $\text{REL}(a_1) = \text{REL}(a_2)$; (2) $h(a) = h^{-1}(a)$, for each $a \in \{\text{CONST}(a_1) \cup \text{CONST}(a_2)\}$; and (3) $h(X) = h^{-1}(X)$, for each $X \in \text{VAR}(a_1) \cap \text{VAR}(a_2)$.

Intuitively, $a_1 \sim a_2$ means that $a_1$ and $a_2$ are isomorphic and agrees on the (argument) positions mentioning the constants and shared variables between $a_1$ and $a_2$. In relation to the number $B(\Sigma)$ described in (11), the exponent “$|\mathcal{R}| \cdot \text{BELL} \left( \text{MAX} + \text{const}(\Sigma) \right)$” considers the size of all type-equivalent atoms since “$B(\Sigma)$” only considers the possible type-equivalent atoms taking the constants into account, while factor “$|\mathcal{R}| \cdot \text{BELL}(\Sigma)$” further considers those atom types for each of the possible relation symbols and as mentioned per each rule of $\sigma$. Finally, we raise

4Recall that “$\text{BELL}(0)$” is the base case of $\Sigma'$ in Definition 1.
“max” to the power of the exponent (i.e., “$|\Sigma|^0 \cdot (|R| \cdot \text{Bell}(\text{maxa} + \text{const}(\Sigma)))$”) since we have to consider the derivation for each possible body atom in the rules.

**Theorem 2 (Decidable class membership).** Let $\Sigma$ be a set of TGDs and $B(\Sigma)$ the number as defined in (7) above. Then there exists an RTC $T \in \Sigma^+$ (see Definition 1) if and only if there exists an RT $T' \in \Sigma^{B(\Sigma)}$.

**Proof (Sketch).** We prove the contraposition: “there is no RTC $T \in \Sigma^+$ if there is no RTC $T' \in \Sigma^{B(\Sigma)}$.”

“(=>$) Clearly, because $\Sigma^{B(\Sigma)} \subseteq \Sigma^\infty$ (see Definition 1), then it follows from Definition 5 that there does not exist an RTC $T \in \Sigma^+$ implies there does not exist any RTC $T' \in \Sigma^{B(\Sigma)}$ as well.

(“<=$)” Assume that there does not exist an RTC $T' \in \Sigma^{B(\Sigma)}$. Now let us call a path of length $k \in \Sigma^0$ (i.e., the initial step of $\Sigma^\infty$), as a sequence of pairs: $(\langle B_1, h_1 \rangle, \ldots, \langle B_k, h_k \rangle)$, where for each $i \in \{1, \ldots, k - 1\}$, there exists some atom $b_{i+1} \in B_{i+1}$ such that $h_i$ and $b_{i+1}$ are unifiable. Then have that such a path actually corresponds to the derivation sequence of the sequence of steps in computing $\Sigma^*$. Intuitively, at each step $\Sigma'$ (as described through Definition 1), what we actually do is “unfold” these atoms $b_{i+1} \in B_{i+1}$ to reveal if they can possibly lead to an RTC. On the other hand, because the number $B(\Sigma)$ would consider path of lengths $B(\Sigma)$ in $\Sigma^{B(\Sigma)}$, then we can be sure that we had considered (i.e., unfolded) all the possible atom types that will be encountered along the path up to type-equivalence, “$\sim$.” Therefore, since $\Sigma^{B(\Sigma)}$ does not have an RTC, then it follows that $\Sigma^+$ cannot have an RTC as well. □

**Theorem 3 (Membership complexity).** Deciding whether $\Sigma \in TG$ is in 2-EXPTime but is PSPACE-hard in general.

**Proof.** ("upper-bound") Based on the number $B(\Sigma)$ as defined in (1) above, the upper-bound follows by only selecting the pairs $(B, h) \in \Sigma'$ at each step in (7-9) (see Definition 1) for which their corresponding path has length $k \leq B(\Sigma)$ (see proof of Theorem 2 for the notion of "path"). Then through this restriction, we can get that $|\Sigma^{B(\Sigma)}|$ will be within order $O(2^{\text{const}(\Sigma)} \cdot R \cdot \text{maxa \-maxa}(\Sigma))$.

("lower-bound") Now we show PSPACE-hardness from “first-principles” as follows. Let $L$ be an arbitrary decision problem in PSPACE. Then from the definition of complexity class PSPACE [Papadimitriou 1994], there exists some deterministic Turing machine $M$ such that for any string $s$, $s \in L$ if $M$ accepts $s$ using at most $p(|s|)$ steps for some polynomial $p(n)$. Consider a Turing machine $M$ to be the tuple $(Q, \Gamma, \Sigma, \delta, q_0, F)$, where (1) $Q \neq \emptyset$ is a finite set of states; (2) $\Gamma \neq \emptyset$ is a finite set of alphabet symbols; (3) $\Sigma \subseteq \Gamma$ is the "blank" symbol; (4) $\Sigma \subseteq \Gamma$ is the "left-end-marker" symbol; (5) $\Sigma \subseteq \Gamma \setminus \{\emptyset, \|\}$ is the set of input symbols; (6) $\delta : (Q \setminus F) \times (\Gamma \setminus \{\|\}) \rightarrow Q \times F$ is the transition function; (7) $q_0 \in Q$ is the initial state; and (8) $F = (F_{\text{accept}} \cup F_{\text{reject}})$ is the set of final states such that $F_{\text{accept}} \cap F_{\text{reject}} = \emptyset$ and $F_{\text{accept}} \cup F_{\text{reject}}$ is the accepting (rejecting) states. Then given an input string $s = c_1 \ldots c_l$ we define the TGDs $\Sigma_S(s), \Sigma_R(s), \Sigma_L(s)$ and $\Sigma_A(s)$ as follows:

$$\Sigma_S(s) = \left\{ (T(X, Y) \land T(Y, Z)) \rightarrow cf(X, Y, Z, q_0, c_1, \ldots, c_l, \ldots, 0) \right\};$$

$$\Sigma_R(s) = \left\{ (cf(X, Y, Z, q, T_1, \ldots, T_k, a, T_{k+1}, \ldots, T_{p(|s|)}, k) \rightarrow cf(X, Y, Z, q', T_1, \ldots, T_{k-1}, b, T_{k+1}, \ldots, T_{p(|s|)}, k - 1), k + 1) \right\};$$

$$\Sigma_L(s) = \left\{ (cf(X, Y, Z, q, T_1, \ldots, T_k, a, T_{k+1}, \ldots, T_{p(|s|)}, k) \rightarrow cf(X, Y, Z, q', T_1, \ldots, T_{k-1}, b, T_{k+1}, \ldots, T_{p(|s|)}, k - 1), k - 1) \right\};$$

$$\Sigma_A(s) = \left\{ (cf(X, Y, Z, q, T_1, \ldots, T_k, T_{k+1}, \ldots, T_{p(|s|)}, k) \rightarrow t(X, Z) | q \in F \land k \leq p(|s|)) \right\};$$

Here, we have that “$\Sigma_S(s)$” corresponds to the starting configuration, “$\Sigma_R(s)$” the right-transition movements $\delta(q, a) = (q', b, R)$, “$\Sigma_L(s)$” the left-transition movements $\delta(q, a) = (q', b, L)$ and “$\Sigma_A(s)$” the accepting states of the machine $M$ under the input string $s$.

Then with $\Sigma = \left\{ (t(X, Y) \rightarrow \exists \Xi t(Y, Z)) \lor \Sigma_S(s) \cup \Sigma_R(s) \cup \Sigma_L(s) \cup \Sigma_A(s) \right\}$, it follows that $M$ accepts $s$ if $\Sigma \in TG$. We note here that we added the TGD rule “$(t(X, Y) \rightarrow \exists \Xi t(Y, Z))$” to $\Sigma$ so that $\Sigma$ is not weak acyclic (WA).

Now we show that the new TG class satisfies the finite-controllability (FC) property. So towards this purpose, we first introduce the notion of interchangeable nulls, which will play the key role in proving the FC property of the class TG of TGDs.

**Definition 4 (Interchangeable nulls).** Let $\overrightarrow{a} = a_1 \ldots a_l$ be a tuple of atoms where $\text{terms}(\overrightarrow{a}) \subseteq \Gamma_v$. A database, $\Sigma$ a set of TGDs and $n_i, n_j \in \Gamma_N (i, j \in \mathbb{N})$. Then we say that $n_i$ and $n_j$ are $\overrightarrow{a}$-interchangeable under $\text{chase}(D, \Sigma)$ if for each connected tuple of atoms $\overrightarrow{\alpha} \theta = \theta(a_1) \ldots \theta(a_l) \in \text{chase}(D, \Sigma)$, where $\theta$ is a bijective (renaming) substitution, we have that $\{n_i, n_j\} \subseteq \text{.Nulls}(\overrightarrow{\alpha} \theta)$ implies there exists a homomorphism $\theta' : \text{Nulls}(\overrightarrow{\alpha} \theta) \rightarrow \Gamma_N$ such that: (1) $\theta'(n_i) = \theta'(n_j)$; and (2) $\overrightarrow{a} \theta' \in \text{chase}(D, \Sigma)$.

Intuitively, with the tuple of atoms $\overrightarrow{a} = a_1 \ldots a_l$ as above, we have that $n_i$ and $n_j$ are “$\overrightarrow{a}$-interchangeable” under $\text{chase}(D, \Sigma)$ guarantees that if for some BCQ $Q = \exists \Xi \varphi(X) \rightarrow q$ we have that $\text{chase}(D, \Sigma) \models Q$, then if $\varphi = \theta(a_1) \ldots \theta(a_l)$ for some renaming substitution $\theta$ (i.e., $\overrightarrow{\alpha}$ is the same “type” as $\varphi$), then we have that simultaneously replacing all occurrences of $n_i$ by $n_j$ in $\text{chase}(D, \Sigma)$ would not affect the fact that $\text{chase}(D, \Sigma) \models Q$.

Before we present the following Theorem 4 it is necessary to firstly introduce the notion level in a chase that we define inductively as follows [Call, Gottlob, and Pieris 2012]: (1) for an atom $a \in D$, we set $\text{level}(a) = 0$; then inductively, (2) for an atom $a \in \text{chase}(D, \Sigma)$ obtained via some chase step $I_k \xrightarrow{a_i \rightarrow^n \alpha_{k+1}} I_{k+1}$, we set $\text{level}(a) = \max\{\text{level}(b) \mid b \in \text{bd}(\alpha_i)\} + 1$. Finally, for some given $k \in \mathbb{N}$, we set $\text{chase}(D, \Sigma) = \{a \mid \text{a} \in \text{chase}(D, \Sigma) \text{ and level}(a) \leq k\}$. Intuitively, $\text{chase}(D, \Sigma)$ is the instance containing atoms that can be derived in a fewer or equal to $k$ chase steps.
**Theorem 4 (Bounded nulls).** Let \( D \) be a database and \( \Sigma \in T_G \). Then for each tuple of atoms \( \overrightarrow{a} \), there exists numbers \( N, N' \in \mathbb{N} \), where \( N' > N \), and such that \( \forall k \in \mathbb{N} \), we have that \( n_k \in \text{nulls}(\overrightarrow{a}) \in \text{nulls}(\overrightarrow{a}) \) \( \iff \exists \theta \in \text{nulls}(\overrightarrow{a}) \implies \exists \theta \in \text{nulls}(\overrightarrow{a}) \end{align} \)

As the full proof is quite technical and tedious, for space reasons, we will only provide a sketch that outlines the main ideas of the proof.

Let us assume on the contrary that for some database \( D \) and \( \Sigma \in T_G \), we have that the following condition holds:

\[ \exists \overrightarrow{a}, \forall N, N' \in \mathbb{N}, \exists k \in \mathbb{N}, \forall n_j \in \Gamma, \forall n \in \Gamma, \text{ where}: \]

\[ n_j \in \text{nulls}(\overrightarrow{a}) \in \text{nulls}(\overrightarrow{a}) \]\n
\[ n_j \text{ } \overrightarrow{a} \text{-interchangeable under } \text{nulls}(\overrightarrow{a}) \text{.} \]

In particular, since this holds for all \( N, N' \in \mathbb{N} \), then we can set a particular \( N' \) to be the value: \( N' = N^N \), where:

\[ N = m^{[\text{dom}(D) + |\Sigma|] + |\text{var}(\overrightarrow{a})|} \]

and where we further have the number to be:

\[ m = d \cdot \max\{|\text{var}((B, h))| \mid (B, h) \in \Sigma^B(\Sigma)\} \cdot \max\{|r| \mid r \in R \} \cdot |\text{var}(\overrightarrow{a})| \]

where:

\[ d = |\Sigma|^d \cdot |R| \cdot \text{bell}(\max(a) + \text{const}(\Sigma)) \]

(i.e., the exponent of the number “\( B(\Sigma) \)” in (11)). Here, the value “\( m \)” considers the possible length of a sequence of chase derivation steps under \( \Sigma^B(\Sigma) \) and w.r.t. to the size of the database \( D \) as well as the tuple of atoms \( \overrightarrow{a} \).

Indeed, the number \( m \) is the maximum of possible rules that can be applied, while the exponent “\( |\text{dom}(D) + |\Sigma| + |\text{var}(\overrightarrow{a})| \)” (through which \( m \) is raised) considers all the possible instantiations under \( \text{dom}(D) \cup \text{const}(\Sigma) \). We also factor in the size of \( \text{var}(\overrightarrow{a}) \) as this will be needed in our arguments below.

Then given the condition in (12), we can build an infinite sequence of distinct labeled nulls: \( n_{k_0}, n_{k_1}, n_{k_2}, n_{k_3}, \ldots \), that satisfies the following conditions:

1. \( k_{j_0} < k_{j_1} < k_{j_2} < k_{j_3}, \ldots \)
2. \( N = j_0 < j_1 < j_2 < j_3, \ldots \)
3. for each \( i \in \{1, 2, 3, \ldots\} \), we have that:
   \[ n_{k_i} \in \text{nulls}(\overrightarrow{a}) \text{ and } n_{k_i} \in \text{nulls}(\overrightarrow{a}) \text{, and there exists a bijective substitution } \theta \text{ on } \overrightarrow{a} \text{ where:} \]
   
   \[ \begin{align*}
   (a) \quad & \overrightarrow{a} \theta \text{ is nulls-connected};
   (b) \quad & \overrightarrow{a} \theta \in \text{nulls}(\overrightarrow{a}) \text{;}
   (c) \quad & \emptyset \cup \{n_{k_0} \mid n_{k_i} \in \text{nulls}(\overrightarrow{a}) \theta \};
   (d) \quad & \text{there does not exist a homomorphism } \theta' : \text{nulls}(\overrightarrow{a}) \rightarrow \Gamma_N \text{ such that: (1) } \theta'(n_{k_0}) = \theta'(n_{k_i}) \text{; and (2) } \overrightarrow{a} (\theta' \circ \theta) \in \text{nulls}(\overrightarrow{a}) \theta. \]
through a disjoint-atom in the rule bodies would make \(n_{k_i}\) and \(n_{k_\ell}\) to be inter-changeable since this would result in some homomorphism \(\theta': \text{NULLS}(\vec{a}\theta_1) \rightarrow \Gamma_N\) such that \(\theta'(n_{k_\ell}) = \theta'(n_{k_i})\) and \(\vec{a}\theta'(\theta \circ \eta_1) \subseteq \text{chase}(D, \Sigma)\). We emphasize that it is also from this fact that nulls must "converge" into a single head atom in the chase derivation as well. Then because of our assumption that \(i > N' = N^N > (\Sigma^D(\Sigma'))\), it follows that there must be a recursion in the way the rules in \(\Sigma''\) were used in the generation of the sequence: \(\vec{a}_0, \vec{a}_1, \vec{a}_2, \vec{a}_3, \ldots, \vec{a}_{s-1}, \vec{a}_s\). Therefore, we get a contradiction because this implies the existence of an RTC in \(\Sigma \in \text{G-GUARDED}\). Indeed, it follows from the pigeonhole principle that because \(i > N' = N^N > (\Sigma^D(\Sigma'))\) (see again definition of \(N\) in (13) above), then even if all the rules in \(\Sigma''\) participate in the converging of nulls simultaneously (which can possibly cut-down the number of steps needed exponentially in making the nulls "meet" since more than one rule can be used in the process at the same time), then the fact that \(i > (\Sigma^D(\Sigma'))\) implies we have a recursion in the way the rules in \(\Sigma''\) were used in the generation of the sequence: \(\vec{a}_0, \vec{a}_1, \vec{a}_2, \vec{a}_3, \ldots, \vec{a}_{s-1}, \vec{a}_s\), above. \(\square\)

**Theorem 5 (Finite controllability (FC) property).** For any database \(D\), TGDs \(\Sigma \in \text{tg and BCQ} \ Q, \ D \cup \Sigma \cup \{\neg Q\}\) satisfies the finite model (FM) property.

**Proof (Sketch).** Let \(\vec{a} = a_1 \ldots a_i = \text{BD}(x)\), for some \(x \in \Sigma \cup \{Q\}\). Then by Theorem 4 each of the null \(n_j \in \{\text{NULLS}(\text{chase}^N(k+1)(\Sigma)) \setminus \text{NULLS}(\text{chase}^N(d, \Sigma))\}\) is always \(\vec{a}\)-interchangeable with some null \(n_i \in \{\text{NULLS}(\text{chase}^N(d, \Sigma))\}\). It then follows that \(\text{chase}(D, \Sigma)\) can be represented by a finite number of nulls from which the finite model property also follows for the theory \(D \cup \Sigma \cup \{\neg Q\}\) . \(\square\)

**Theorem 6 (Comparison with other syntactic classes).** For each class \(C \in \{\text{WA, W-GUARDED, WSJ, G-GUARDED, SHY, TAME, WR}\}\), we have that \(C \subseteq \text{tg}\).

**Proof (Sketch).** On the contrary, assume that for some class \(C \in \{\text{WA, W-GUARDED, WSJ, G-GUARDED, SHY, TAME, WR}\}\), we have that there exists a set of TGDs \(\Sigma \in \text{tg but } \Sigma \notin \text{tg}\). Then since \(\Sigma \notin \text{tg}\), it follows from Definition 3 that there exists an RTC: \(\mathcal{T} = (\langle B, h \rangle, \langle a, b, c, \langle X, Z \rangle, a', \rangle)\) (see Definition 2 and Figure 1), where there does not exist an atom \(d \in B\) such that \(\{X, Z\} \subseteq \text{VAR}(d)\). Then now let us consider the following cases:

**Case 1:** \(C = \text{WA}\): Then the fact that only the cyclically-affected variables are considered in Condition 4) of Definition 2 that \(\Sigma \in \text{WA}\).

**Case 2:** \(C = \text{W-GUARDED}\): Then the fact that variables occurring in some cyclically-affected arguments are un-guarded in \(\mathcal{T}\) (see Definition 3) contradicts the assumption that \(\Sigma \in \text{W-GUARDED}\).

**Case 3:** \(C = \text{WSJ}\): Then the fact that each pair \(\langle B, h \rangle \in \Sigma^+\) and where we have from Definition 1 that \(\Sigma^+\) also considers the "TGD Expansion" in (Calì, Gottlob, and Pieris 2012) used in the identification of the sticky-join class of TGDs, and because there exists marked variables \(Y' \in \text{LINK}(B, d_1, d_{i+1})\) such that \(Y' \in \text{VAR}(a')\) implies all occurrences of variables in positions \(\text{ARG}(a')\) in the atom \(a\) are in \(\text{M-VAR}(a, \vec{a'}, a')\), occurs in Condition 5(b) of Definition 2 contradicts the assumption that \(\Sigma \in \text{WSJ}\).

**Case 4:** \(C = \text{G-GUARDED}\): Similarly to Case 2 above, the fact that the variables "\(X\)" and "\(Z\)" in \(\mathcal{T}\) are such that \(\{X, Z\} \subseteq \text{VAR}(B)\) in Condition 4) of Definition 2, which are also the so-called "glut-variables" (Krotzsch and Rudolph 2011), then this contradicts that \(\Sigma \in \text{G-GUARDED}\).

**Case 5:** \(C = \text{SHY}\): Then the fact that the linking variables \(\text{LINK}(d_i, d_{i+1})\), for \(i \in \{1, \ldots, m-1\}\), are non-empty in Condition 5(a) of Definition 2 contradicts that \(\Sigma \in \text{SHY}\).

**Case 6:** \(C = \text{TAME}\): Since \(\langle B, h \rangle \in \Sigma^+\) and \(\Sigma^+\) also consider the so-called "tame reachability" in (Gottlob, Manna, and Pieris 2015) (see Definition 1), then it follows that some of the sticky rules "fed" a guard atom of some guarded rule in \(\Sigma\) as resulting for the unification procedure in Definition 1. This then contradicts that \(\Sigma \in \text{TAME}\).

**Case 7:** \(C = \text{WR}\): Then the fact that the variables "\(X\)" and "\(Z\)" are unguarded in the RTC \(\mathcal{T}\) by Definition 3, and where each \(\text{LINK}(d_i, d_{i+1})\) are non-empty by Condition 5(a) of Definition 2, implies that the "position graph" of \(\Sigma\) as described in (Civili and Rosati 2012), will have a cycle that passes through both an \(m\)-edge and an \(n\)-edge (i.e., the variables in \(\text{LINK}(d_i, d_{i+1})\)). Therefore, this contradicts the assumption that \(\Sigma \in \text{WR}\). \(\square\)

**Theorem 7 (BCQ-Ans complexity).** The BCQ-Ans combined complexity problem under the class TG is in \(\text{4-EXPTIME}\) but is \(\text{3-EXPTIME}\)-hard in general.

**Proof (Sketch).** ("upper-bound") Let \(D\) be a database, \(\Sigma \in \text{tg}\) a set of TGDs in \(\text{tg}\) and \(Q\) a BCQ. Then with the numbers \(N' \geq N^N\) and \(N\) as defined in the proof of Theorem 4 (i.e., see (13)), then we get that it is sufficient to only consider chase depths "\(\text{chase}^K(D, \Sigma)\)" where \(K\) is of the order \(O(2^{p(n)})\) (i.e., doubly-exponential to \(p(n)\)) and \(p(n)\) is a polynomial to \(|D|, |\Sigma|\) and \(|Q|\). It then follows from Theorem 5 in (Zhang, Zhang, and You 2015) that it is \((K-2)\)-\(\text{EXPTIME}\) to check if \(\text{chase}^K(D, \Sigma) \models Q\).

("lower-bound") Since it follows from (Krotzsch and Rudolph 2011) that BCQ-Ans under the G-GUARDED class is \(\text{3-EXPTIME}\)-complete, then because we have from Theorem 6 that our new TG class also contains G-GUARDED, then if follows that BCQ-Ans under TG is at least \(3\)-\(\text{EXPTIME}\)-hard. \(\square\)

**Concluding Remarks.** In this paper, we have introduced a new class of TGDs called triangularly-guarded TGDs (TG), for which BCQ-Ans is decidable as well as having the FC property (Theorem 5). We further showed that TG strictly contains the current main syntactic classes: WA, W-GUARDED, WSJ, G-GUARDED, SHY, TAME and WR (Theorem 6), which, to the
best of our knowledge, provides a unified representation of those aforementioned TGD classes. Since our new TG class of TGDs satisfies the FC property as well as strictly contains all the other important classes mentioned above, this work also provides an important and unifying explanation why those other important classes satisfies the FC property and thus, are decidable.

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