Exact solutions for pseudo plane flows of first and second kind for couple Stress fluid flows

Subin P. Joseph

Abstract
Two distinct families of exact solutions for Newtonian fluid flows and couple stress fluid flows are derived in this paper. The partial differential equations governing the motion of such flows are highly nonlinear and are in higher orders. Due to this difficulty, the available exact solutions for couple stress flows are very less and most of them are available in the case of one dimensional flows or planar flows. We derive the exact solutions as pseudo plane flows of first kind and second kind.

Keywords
Newtonian flows, Couple stress flows, Exact solutions, Pseudo plane flows.

AMS Subject Classification
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1 Department of Mathematics, Government Engineering College, Wayanad-670644, Kerala, India.
*Corresponding author: subinpj@gecwyd.ac.in

1. Introduction

Exact solutions to nonlinear partial differential equations are very difficult to obtain. Since several physical phenomenons are formulated in terms of nonlinear partial differential equations, finding solutions to these equations play a major role in mathematical physics. Most of these equations can be solved only in specific cases. So there have been developed several numerical and approximate methods to solve such nonlinear partial differential equations. But exact solutions are needed to check the degree of correctness of these methods. In some cases exact solutions are very rare and it is difficult to check the correctness of approximate methods.

The Navier-Stokes equation of motion of an incompressible viscous fluid flow is one of the most important nonlinear partial differential equations in mathematical physics. The available exact solutions of these equations are very few in the literature and are mostly in the cases of two dimensional or axisymmetric cases. Converting this equation to an ordinary differential equations by means of suitable assumptions and transformations have lead to solutions for specified flows by solving the resulting ODE by available methods[3, 6, 7, 12, 17, 18].

Newtonian fluid flows governed by the Navier-Stokes equations are suitable for analyzing only limited flows in real applications. There are several models proposed to represent non-Newtonian fluid flows. Some of these flows appearing in the applications are more suitably represented by couple stress fluid flows. Several fluids appearing in scientific and industrial applications such as paints slurries and body fluid flow can be studied using the theory of couple stress fluid flows. Since the corresponding partial differential equations are in higher orders, the exact solutions are much more difficult to obtain even in specific flows compared to Newtonian fluid flows. Hence the available solutions are very rare in the literature and these known solutions are mostly in the case of unidirectional or plane flows[1, 2, 4, 5, 8–11, 16].

In this paper, we solve the governing equation of motion of an incompressible Newtonian fluid flow and couple stress fluid flow under conservative body forces. Due to the non linearity and appearance of higher order derivative we assume particular forms for the required solutions and then solve the corresponding equations to obtain new solutions for Newtonian flows and couple stress flows. The exact solutions
2. Equations of motion

The Navier-Stokes equations of motion of an incompressible fluid flow under conservative body forces are given by

\[
\frac{\partial v}{\partial t} + (v \cdot \nabla)v = -\frac{\nabla P}{\rho} + \nabla \phi + v \nabla^2 v
\]  

(2.1)

with

\[
\nabla \cdot v = 0
\]  

(2.2)

where \( v \) is the velocity field, \( P \) is the pressure, \( \rho \) is the density, \( \nabla \phi \) is the conservative body force and \( v \) is the coefficient of kinematic viscosity.

The equation of motion governing a couple stress fluid flow is initially proposed by Stokes[16]. For an incompressible couple stress fluid flow with conservative body forces and without body couples, the governing equations are given by

\[
\nabla \cdot v = 0
\]  

(2.3)

and

\[
\frac{\partial v}{\partial t} + (v \cdot \nabla)v = -\frac{\nabla P}{\rho} + \nabla \phi + v \nabla^2 v - \eta \nabla^4 v,
\]  

(2.4)

where \( \eta \) is the parameter due to couple stress. Taking the curl of the above equations the vorticity equations are obtained.

Vorticity equation of the incompressible Navier-Stokes fluid flow is given by

\[
\frac{\partial \omega}{\partial t} + \nabla \times (\omega \times u) - v \nabla^2 \omega = 0
\]  

(2.5)

where \( \omega = \nabla \times u \) is the vorticity field. Similarly the vorticity equation of the incompressible couple stress fluid flow is given by

\[
\frac{\partial \omega}{\partial t} + \nabla \times (\omega \times v) - v \nabla^2 \omega + \eta \nabla^4 \omega = 0
\]  

(2.6)

When the couple stress parameter \( \eta \) vanishes then the equation (2.4) and (2.6) becomes the Navier-Stokes equation (2.2) and the corresponding vorticity equation (2.5) respectively. The beltrami flows are those flow for which velocity field and vorticity field are parallel and the condition becomes

\[
v \times \omega = 0
\]  

(2.7)

Beltrami flows are particular cases of the so called generalized beltrami flows defined by the condition

\[
\nabla \times (\omega \times v) = 0
\]  

(2.8)

Several authors have applied the condition of generalized beltrami flows in the equation (2.5) to obtain exact solution of Newtonian fluid flows, as in this case the equation (2.5) reduces into set of linear partial differential equations. Solutions of such flows have been discussed in[3, 6, 7, 12, 14, 15, 17, 18]. In the case of couple stress fluid flows such solutions are very rare[9, 10].

Several families of exact solution for Navier-Stokes flows and couple stress fluid flows are derived in coming sections by solving the equations (2.5) and (2.6) respectively. We assume solutions in the form of a vector potentials so that the equation (2.2) is always satisfied.

3. Pseudo Plane flow of second kind

Here, a solution in the form of pseudo plane flow of an incompressible couple stress fluid flow is derived. Before deriving such solutions, first we derive a planar solution generated by a stream function. This solution then will be extended to the required pseudo plane solution. Since the solution has to be extended, instead of stream function formulation, the corresponding vector potential is taken in the form

\[
A = (0, 0, \psi(x, y))
\]  

(3.1)

where the stream function is \( \psi(x, y) = e^{at} (c_1 f(x) + c_2 g(y)) \) and \( a_1 \) and \( a_2 \) are arbitrary real parameters. Then the velocity field becomes

\[
v = e^{at} (c_2 g'(y), -c_1 f'(x), 0)
\]  

(3.2)

Substituting this value of the velocity field in the equation (2.6) we get

\[
0, 0, -c_1 e^{2at} \left( f'''(x) g'(y) - g'''(y) f'(x) \right)
\]

\[
- e^{at} \left( c_1 f''(x) + f'(x) \right) - v f''(x)
\]

\[
+ c_2 \left( a g''(y) + f'(y) - v g'(y) \right)
\]  

(3.3)

This equation is satisfied if the set of ordinary differential equations

\[
f'''(x) g'(y) - g'''(y) f'(x) = 0,
\]  

(3.4)

\[a f''(x) + f'(x) = 0
\]  

(3.5)

\[a g''(y) + f'(y) - v g'(y) = 0
\]  

(3.6)

are simultaneously satisfied. The nontrivial solutions of the first differential equation (3.4) are either

\[
f(x) = a_1 \sin px + a_2 \cos px
\]  

(3.7)
where \( p \) and \( c_i \)'s are real parameters.

Now the solutions of the equation (3.5) are given by

\[
f(x) = b_1 e^{\alpha x} + b_2 e^{-\alpha x} + b_3 e^{\beta x} + b_4 e^{-\beta x}
\]

(3.11)

where

\[
\alpha = \sqrt{\frac{v - \sqrt{v^2 - 4a\mu}}{2\mu}}
\]

(3.12)

and

\[
\beta = \sqrt{\frac{v + \sqrt{v^2 - 4a\mu}}{2\mu}}
\]

(3.13)

and \( b_i \)'s are parameters. Similarly solutions of equation (3.6) are given by

\[
g(y) = b_5 e^{\alpha y} + b_6 e^{-\alpha y} + b_7 e^{\beta y} + b_8 e^{-\beta y}
\]

(3.14)

To obtain the exact solutions of couple stress fluid flows in the desired form, the parameters in the solutions given by equations (3.7), (3.8), (3.9), (3.10), (3.11) and (3.14) are to be properly chosen so that the ordinary differential equations (3.4), (3.5) and (3.6) are simultaneously satisfied. Applying the required compatibility conditions, we get two different values for \( a \). Using these values we get four families of new exact solutions to the couple stress fluid flow in terms of stream function, which are given by

\[
\psi_1(x,y) = e^{-p^2(x+y)} (c_1 \sin(py) + c_2 \cos(px))
\]

\[
+ c_3 \sin(py) + c_4 \cos(py)
\]

(3.15)

and

\[
\psi_2(x,y) = e^{-p^2(x+y)} \left( c_5 \sinh \left( \frac{v}{\mu} + p^2 x \right) + c_6 \cosh \left( \frac{v}{\mu} + p^2 x \right) + c_7 \sinh \left( \frac{v}{\mu} + p^2 y \right) + c_8 \cosh \left( \frac{v}{\mu} + p^2 y \right) \right)
\]

(3.16)

\[
\psi_3(x,y) = e^{-p^2(x+y)} (c_9 \sinh(px) + c_{10} \cosh(px))
\]

\[
+ c_{11} \sinh(py) + c_{12} \cosh(py)
\]

(3.17)

\[
\psi_4(x,y) = e^{-p^2(x+y)} \left( c_{13} \sin \left( \sqrt{\frac{p^2 - v}{\mu}} x \right) + c_{14} \cos \left( \sqrt{\frac{p^2 - v}{\mu}} x \right) + c_{15} \sin \left( \sqrt{\frac{p^2 - v}{\mu}} y \right) + c_{16} \cos \left( \sqrt{\frac{p^2 - v}{\mu}} y \right) \right)
\]

(3.18)

where \( c_i \)'s are arbitrary real parameters. Among the above exact solutions \( \psi_1 \) and \( \psi_4 \) are families of periodic exact solutions for couple stress fluid flows. Families of exact solutions for Newtonian fluid flow can also be obtained from above solutions by letting \( \mu = 0 \). Clearly we get only two families of exact solutions for such flows given by the stream functions \( \psi_1 \) and \( \psi_3 \).

Now we will proceed to obtain the required exact solutions for pseudo plane flow of couple stress fluids. We assume that required vector potential takes the following form. Let

\[
A = a \mu (b h(y), c k(x), d f(x) + e g(y))
\]

(3.19)

where \( a, b, c, d \) and \( e \) are arbitrary parameters and \( f(x), g(y), k(x) \) and \( h(y) \) are arbitrary functions to be determined. Then the velocity field will be

\[
v = e^{\alpha t} (e g(y), -d f'(x), c k'(x) - b h'(y))
\]

(3.20)

This vector field is a solution to couple stress fluid flow if it satisfies the equation (2.6). On substitution, equation (2.6) will be satisfied if the following set of ordinary differential equations are simultaneously satisfied.

\[
bd h^{(3)}(y)f'(x) + ce g''(y)k''(x) = 0
\]

(3.21)

\[
bd f'''(x)h''(y) + ce k^{(3)}(x)g'(y) = 0
\]

(3.22)

\[
f^{(3)}(x)g''(y) - g^{(3)}(y)f'(x) = 0
\]

(3.23)

\[
ah''(y) + \mu h^{(6)}(y) - \nu h^{(4)}(y) = 0
\]

(3.24)

\[
ak''(x) + \mu k^{(6)}(x) - \nu k^{(4)}(x) = 0
\]

(3.25)

\[
a f''(x) + \mu f^{(6)}(x) - \nu f^{(4)}(x) = 0
\]

(3.26)

and

\[
a g''(y) + \mu g^{(6)}(y) - \nu g^{(4)}(y) = 0
\]

(3.27)
Among these differential equations, the equations (3.23), (3.26) and (3.27) have been solved simultaneously in the first part (see equations (3.4), (3.5) and (3.6)). The family of exact solutions of these differential equations for \( f \) and \( g \) are given by equations (3.15), (3.16), (3.17) and (3.18). Now we have to solve the remaining differential equations with these solutions in hand. Solving all these equations simultaneously, we get the required solutions in terms of the vector potential (3.19) as follows.

\[
A_1 = e^{-p^2(z+v+\mu p^2)} (c_4 \sin(py) - c_3 \cos(py), c_1 \cos(px) - c_2 \sin(px), (c_1 \sin(px) + c_2 \cos(px) + c_3 \sin(py) + c_4 \cos(py))),
\]

(3.28)

\[
A_2 = e^{-p^2(z+v+\mu p^2)} \left( -c_7 \sinh \left( \frac{y}{\mu} + p^2 \right) + c_6 \cosh \left( \frac{y}{\mu} + p^2 \right), c_1 \sinh \left( \frac{x}{\mu} + p^2 \right) + c_2 \cosh \left( \frac{x}{\mu} + p^2 \right) \right),
\]

(3.29)

\[
A_3 = e^{-p^2(z+\mu p^2-v)} (-c_{12} \sinh(py) - c_{11} \cosh(py), c_{10} \sinh(px) + c_9 \cosh(px), c_9 \sinh(px) + c_{10} \cosh(px))
\]

(3.30)

\[
A_4 = e^{-p^2(z+\mu p^2-v)} \left( c_{16} \sin \left( \sqrt{p^2 - \frac{V}{\mu}} \right), -c_{15} \cos \left( \sqrt{p^2 - \frac{V}{\mu}} \right), c_{13} \sin \left( \sqrt{p^2 - \frac{V}{\mu}} \right) \right),
\]

(3.31)

Taking curl of these equations we get the corresponding velocity vector fields. So the above four equations give the required four families of exact solutions for pseudo plane flows of couple stress fluid flows. The two periodic solutions given above are pure beltrami flows, but the other two solutions are generalized beltrami flows. Here also we get two families of exact solutions for pseudo plane flow of Newtonian fluids. These are obtained by putting \( \mu = 0 \) in the equations (3.28) and (3.30). The other two families of exact solutions are examples for couple stress fluid flows from which we are not able to derive exact solutions for Newtonian flows as particular cases.

### 4. Pseudo Plane flow of first kind

Here families of solutions in the form of pseudo plane flows of first kind of incompressible couple stress fluid flows are derived. Consider the vector potential in the form

\[
A = e^{at} \{ a_1 g(dz), a_2 h(dz), a_3 f(bx + cy) \}
\]

(4.1)

where \( a, a_1, a_2, a_3, b, c \) and \( d \) are arbitrary real parameters and \( f, g \) and \( h \) are arbitrary functions to be determined. Then the velocity field becomes

\[
v = e^{at} \left( a_3 c f'(bx + cy) - a_2 dh'(dz), a_1 dg'(dz) - a_3 b f'(bx + cy) \right)
\]

(4.2)

Substituting this value of the velocity field in the equation (2.6) it can be seen that this equation is satisfied if the functions satisfy the following differential equations

\[
f''(bx + cy) \left( a_2 bh''(dz) - a_1 cg''(dz) \right) = 0
\]

(4.3)

\[
f^{(3)}(bx + cy) \left( a_1 cg'(dz) - a_2 bh'(dz) \right) = 0
\]

(4.4)

\[
ag''(dz) + d^4 \mu g^{(6)}(dz) - d^2 v g^{(4)}(dz) = 0
\]

(4.5)

\[
ahr''(dz) + d^4 \mu h^{(6)}(dz) - d^2 vh^{(4)}(dz) = 0
\]

(4.6)

\[
(b^2 + c^2) \left( vf^{(4)}(bx + cy) - \mu \left( b^2 + c^2 \right) f^{(6)}(bx + cy) \right) - af''(bx + cy) = 0
\]

(4.7)

The first two equations are satisfied if choose \( h(dz) \) as constant multiple of \( g(dz) \) Then it is needed to solve the remaining differential equations for the functions \( f \) and \( g \). Solving the equation (4.5)

\[
g(dz) = b_1 e^{\alpha dz} + b_2 e^{-\alpha dz} + b_3 e^{\beta dz} + b_4 e^{-\beta dz}
\]

(4.8)
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where

\[ \alpha = \sqrt{\frac{v - \sqrt{v^2 - 4a\mu}}{2d^2\mu}} \]  \hspace{1cm} (4.9)

and

\[ \beta = \sqrt{\frac{v + \sqrt{v^2 - 4a\mu}}{2d^2\mu}} \]  \hspace{1cm} (4.10)

and \( b_i \)'s are parameters. To obtain periodic solutions in explicit form we solve equation (4.9) for \( a \) to obtain

\[ a = -d^2 (d^2\mu + v) \]  \hspace{1cm} (4.11)

Solving the equation (4.7)

\[ f(bx + cy) = b_5e^{\alpha(bx + cy)} + b_6e^{-\alpha(bx + cy)} + b_7e^{\beta(bx + cy)} + b_8e^{-\beta(bx + cy)} \]  \hspace{1cm} (4.12)

where

\[ \alpha = \sqrt{\frac{v - \sqrt{v^2 - 4a\mu}}{2\mu(b^2 + c^2)}} \]  \hspace{1cm} (4.13)

and

\[ \beta = \sqrt{\frac{v + \sqrt{v^2 - 4a\mu}}{2\mu(b^2 + c^2)}} \]  \hspace{1cm} (4.14)

and \( b_i \)'s are parameters. Again, to obtain periodic solutions in explicit form we solve equation (4.13) for \( a \) to obtain

\[ a = - \left( b^2 + c^2 \right) \left( \mu(b^2 + c^2) + v \right) \]  \hspace{1cm} (4.15)

Solving the equations (4.11) and (4.15) we get the value of the parameter

\[ d = \sqrt{b^2 + c^2} \]  \hspace{1cm} (4.16)

Applying all these solutions the required vector potential which generates the pseudo plane flow of first kind for couple stress fluid flow is given by

\[
e^{at} \left( b \left( A1 \cos \left( z\sqrt{b^2 + c^2} \right) + A2 \sin \left( z\sqrt{b^2 + c^2} \right) \right)
  + A3 \sinh \left( \frac{z\sqrt{\mu(b^2 + c^2) + v}}{\sqrt{\mu}} \right) \right)
  + A4 \cosh \left( \frac{z\sqrt{\mu(b^2 + c^2) + v}}{\sqrt{\mu}} \right) \),

\[ + \left( A1 \cos \left( 2\sqrt{\mu(b^2 + c^2) + 1(bx + cy)} \right) \right)
  + A6 \sinh \left( \frac{v}{\mu(b^2 + c^2) + 1(bx + cy)} \right) \)), \hspace{1cm} (4.17)

where

\[ a = - \left( b^2 + c^2 \right) \left( \mu(b^2 + c^2) + v \right) \]  \hspace{1cm} (4.18)

Taking the curl of this vector potential we obtain the required exact solutions for couple stress fluid flow. Also, by properly choosing the value of the parameters we can derive exact solutions for Newtonian fluid flows from these exact solutions.

5. Discussion

In this paper, we derived two types of new exact solutions for couple stress fluid flow and Newtonian fluid flow which are pseudo plane flows. To derive exact solution as a pseudo plane flow of second kind, we derive planar solutions using two dimensional vector potential and then this is extended to pseudo plane flows of second kind. Four sets of solutions are obtained using this method and these are given by equations (3.28), (3.29), (3.30) and (3.31). Among these solutions (3.28) and (3.31) are periodic solutions. Also it is clear that (3.28) and (3.29) generates exact solutions for Newtonian fluid flows. From other two solutions it is not possible to get any exact solutions for Newtonian fluid flow. As a second family of solutions, we get pseudo plane flow of first kind for couple stress fluid flow given by equation (4.17). From this solutions also we get certain exact solutions for Newtonian flows as particular solutions when \( \mu = 0 \) and choosing the parameters properly. So we have derived several exact solutions for couple stress fluid flows and Newtonian fluid flows, all of which are new exact solutions in the literature.
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