QUANTITATIVE RESULTS FOR NON-NORMAL MATRICES SUBJECT TO
RANDOM AND DETERMINISTIC PERTURBATIONS

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Abstract. We consider the eigenvalue distribution of a fixed matrix subject to a small perturbation. In particular, we prove quantitative comparison results which show that the logarithmic potential is stable under perturbations with small norm or low rank, provided the smallest and largest singular values are not too extreme. We also establish a quantitative version of the Tao–Vu replacement principle. As an application of our results, we study the spectral distribution of banded Toeplitz matrices subject to small random perturbations, including the case when the bandwidth grows with the dimension. For this model, we obtain a rate of convergence in Wasserstein distance of the empirical spectral measure to its limiting distribution. We give a number of examples of our methods including random multiplicative perturbations and large classes of deterministic perturbations that behave similar to random perturbations.

1. Introduction

Due to the spectral theorem, the eigenvalues of Hermitian matrices are stable under small perturbations. For example, when $A$ and $B$ are $n \times n$ Hermitian matrices, Weyl’s perturbation theorem (see [12, Corollary III.2.6]) guarantees that

$$
\max_{1 \leq j \leq n} |\lambda_j(A) - \lambda_j(B)| \leq \|A - B\|,
$$

where $\lambda_1(M) \geq \cdots \geq \lambda_n(M)$ are the ordered eigenvalues of the $n \times n$ Hermitian matrix $M$ and $\|M\|$ is its spectral norm. In contrast, the spectrum of a non-normal matrix can be extremely sensitive to small perturbations if there is pseudospectrum present [70]. Consider the case of the $n \times n$ matrix

$$
T := \begin{pmatrix}
0 & 1 & 0 & 0 & \cdots & 0 \\
0 & 0 & 1 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & \cdots & \cdots & 0 \\
0 & 0 & \cdots & \cdots & \cdots & 0
\end{pmatrix}
$$

with ones on the super-diagonal and zeros everywhere else. If $E$ is the $n \times n$ matrix with $\varepsilon > 0$ in the $(n, 1)$-entry and zeros everywhere else, then the eigenvalues of $T + E$ lie on the circle $\{z \in \mathbb{C} : |z| = \varepsilon^{1/n}\}$ in the complex plane, while all eigenvalues of $T$ are zero.

In this paper, we consider the eigenvalues of perturbed non-normal matrices. Our main results are non-asymptotic, and allow us to locate the eigenvalues on both the macroscopic and mesoscopic scales. In Section 1.1, we present our main theoretical tools and results for working with perturbations of non-normal matrices. In Section 1.2, we apply our results to study the model $A + E$, where $A$ is a banded Toeplitz matrix (such as $T$ above) and $E$ is a random or deterministic perturbation with small spectral norm. Our results apply to the case when the bandwidth of $A$ grows...
with the dimension, and we also obtain a rate of convergence for the empirical spectral measure in Wasserstein distance. We give some other examples and applications of our results in Section 1.3. We discuss several closely related works throughout the article and provide further references and comparisons in Section 1.4. The introduction concludes with a description of our notation in Section 1.5 and an outline of our methods in Section 1.6.

1.1. Theoretical tools and results. For a probability measure $\mu$ on $\mathbb{C}$ that integrates $\log | \cdot |$ in a neighborhood of infinity, its logarithmic potential $\mathcal{L}_\mu$ is the function $\mathcal{L}_\mu : \mathbb{C} \to [-\infty, +\infty)$ given by

$$\mathcal{L}_\mu(z) := \int_{\mathbb{C}} \log |w - z| \, d\mu(w).$$

It follows from Fubini’s theorem that the logarithmic potential is finite almost everywhere. Here and in the sequel, “almost everywhere” and “almost all” will be with respect to the Lebesgue measure on $\mathbb{C}$.

For an $n \times n$ matrix $A$, we let $\lambda_1(A), \ldots, \lambda_n(A) \in \mathbb{C}$ denote the eigenvalues of $A$ (counted with algebraic multiplicity) and $\mu_A$ be its empirical spectral measure, defined by

$$\mu_A := \frac{1}{n} \sum_{j=1}^{n} \delta_{\lambda_j(A)}, \quad (1.3)$$

where $\delta_z$ denotes the point mass at $z$. The logarithmic potential of $\mu_A$ is given by

$$\mathcal{L}_A(z) \equiv \mathcal{L}_{\mu_A}(z) := \int_{\mathbb{C}} \log |w - z| \, d\mu_A(w) = \frac{1}{n} \sum_{j=1}^{n} \log |\lambda_j(A) - z| = \frac{1}{n} \log |\det(A - zI)|, \quad (1.4)$$

where $\det(A - zI)$ is the determinant of $A - zI$, $z \in \mathbb{C}$, and $I$ is the identity matrix.

In many cases the convergence of the logarithmic potential for almost every $z \in \mathbb{C}$ is enough to guarantee the convergence of the empirical spectral measure; see, for instance, [64, Theorem 2.8.3] or [38, 58, 68]. The following replacement principle due to Tao and Vu [68] compares the empirical spectral measures of two random matrices. For a matrix $A$, we let $\|A\|_2$ denote its Frobenius norm given by

$$\|A\|_2 := \sqrt{\text{tr}(AA^*)} = \sqrt{\text{tr}(A^*A)}, \quad (1.5)$$

where $A^*$ is the conjugate transpose of $A$.

**Theorem 1.1** (Replacement principle [68]). Suppose for each $n$ that $M_1$ and $M_2$ are $n \times n$ ensembles of random matrices. Assume that

(i) the expression $\frac{1}{n} \|M_1\|_2^2 + \frac{1}{n} \|M_2\|_2^2$ is bounded in probability (resp., almost surely); and

(ii) for almost all complex numbers $z$,

$$\mathcal{L}_{M_1}(z) - \mathcal{L}_{M_2}(z)$$

converges in probability (resp., almost surely) to zero as $n \to \infty$ and, in particular, for each fixed $z$, these logarithmic potentials are finite with probability tending to 1 as $n$ tends to infinity (resp., almost surely nonzero for all but finitely many $n$).

Then, $\mu_{M_1} - \mu_{M_2}$ converges in probability (resp., almost surely) to zero as $n \to \infty$.

Our first main result is a non-asymptotic version of Theorem 1.1, which captures how close $\mu_{M_1}$ is to $\mu_{M_2}$. Let $C^\infty_c(\mathbb{C})$ be the set of smooth, compactly supported functions $\varphi : \mathbb{C} \to \mathbb{C}$, and let $\text{supp}(\varphi)$ denote the support of $\varphi$. $\|M\|$ denotes the spectral norm of the matrix $M$.

**Theorem 1.2** (Non-asymptotic replacement principle). Let $M_1$ and $M_2$ be two $n \times n$ random matrices (not necessarily independent). Let $\varepsilon, \eta > 0$, $T > 2$ (all possibly depending on $n$), and take $\varphi \in C^\infty_c(\mathbb{C})$. Assume the following:
(1) (Norm bound) \( \mathbb{P}(\|M_1\| + \|M_2\| \geq T) \leq \varepsilon. \)

(2) (Concentration of log determinants) For \( Z \) uniformly distributed on \( \text{supp}(\Delta \varphi) \), independent of \( M_1 \) and \( M_2 \), we have
\[
\mathbb{P}(|L_{M_1}(Z) - L_{M_2}(Z)| \geq \eta) \leq \varepsilon.
\]

Then there exists a constant \( C_{1.2} > 0 \) depending only on \( \varphi \) such that for every integer \( m \geq 1 \)
\[
\left| \int_{\mathbb{C}} \varphi \, d\mu_{M_1} - \int_{\mathbb{C}} \varphi \, d\mu_{M_2} \right| \leq C_{1.2} \left( \eta + \frac{\log T}{m\sqrt{\varepsilon}} \right)
\]
with probability at least \( 1 - 2(m + 1)\varepsilon \). The constant \( C_{1.2} \) is described explicitly in (2.15) and is usually relatively simple, for example, if \( \text{supp} \varphi \) is contained in the unit disk, then one may take \( C_{1.2} = 14\sqrt{\pi} \|\Delta \varphi\|_\infty \), where \( \| \cdot \|_\infty \) denotes the \( L_\infty \)-norm.

Remark 1.3. We have formulated Theorem 1.2 in probabilistic terms. However, if the matrices \( M_1 \) and \( M_2 \) are deterministic and we can take \( \varepsilon \) arbitrarily small, the result can be reformulated. Indeed, taking \( \varepsilon = m^{-3/2} \) and letting \( m \) approach infinity gives
\[
\left| \int_{\mathbb{C}} \varphi \, d\mu_{M_1} - \int_{\mathbb{C}} \varphi \, d\mu_{M_2} \right| \leq C_{1.2}\eta
\]
provided
\[
|L_{M_1}(z) - L_{M_2}(z)| < \eta
\]
for almost every \( z \in \text{supp}(\varphi) \).

Since
\[
\int_{\mathbb{C}} \varphi \, d\mu_{M_i} = \frac{1}{n} \sum_{j=1}^{n} \varphi(\lambda_j(M_i))
\]
for \( i = 1, 2 \), it is often useful to let \( \varphi \) approximate an indicator function. In particular, allowing \( \varphi \) to depend on \( n \), one can use Theorem 1.2 (along with the explicit formula for \( C_{1.2} \) given in (2.15)) to obtain local laws which describe the mesoscopic behavior of the eigenvalues; such local laws have been established in the random matrix theory literature for a variety of ensembles, see \cite{1, 2, 3, 6, 11, 13, 19, 20, 26, 27, 28, 30, 36, 37, 40, 43, 45, 47, 49, 51, 67, 69, 75, 76, 77} and references therein for a partial list of such results. We will use Theorem 1.2 to establish a rate of convergence for the empirical spectral measure of Toeplitz matrices subject to small random perturbations in Section 1.2.

In order to use Theorem 1.2, we need to be able to control the difference \( L_{M_1}(z) - L_{M_2}(z) \) for a dense enough collection of \( z \in \mathbb{C} \). Our next two main results provide such bounds. Recall that the singular values of the \( n \times n \) matrix \( A \) are the eigenvalues of \( \sqrt{AA^*} \), where \( A^* \) is the conjugate transpose of \( A \). We let \( \sigma_1(A) \geq \cdots \geq \sigma_n(A) \) denote the ordered singular values of \( A \) and \( \nu_A \) be the empirical measure constructed from the singular values of \( A \):
\[
\nu_A := \frac{1}{n} \sum_{j=1}^{n} \delta_{\sigma_j(A)}.
\]

We will often write
\[
\sigma_{\min}(A) := \sigma_n(A)
\]
to denote the smallest singular value of \( A \).

Our next result can be compared to Weyl’s perturbation theorem (see (1.1)), which implies that the eigenvalues of Hermitian matrices are stable under small perturbations. Theorem 1.4 below shows that the logarithmic determinant (and hence the empirical spectral measure by Theorem 1.2) of an arbitrary matrix is also stable under small perturbations, provided the smallest and largest singular values of the matrix are not too extreme.
**Theorem 1.4** (Norm comparison principle). Let $M_1$ and $M_2$ be $n \times n$ matrices. Take $z \in \mathbb{C}$ and $\varepsilon \in (0, 1/2)$, and assume that
\[
\sigma_{\text{min}} := \min\{\sigma_{\text{min}}(M_i - zI) : i = 1, 2\} > 0
\] (1.8)
and $\|M_1 - M_2\| < \varepsilon/2$. Then
\[
|L_{M_1}(z) - L_{M_2}(z)| \leq 6 (|\log(\varepsilon/2)| + |\log \sigma_{\text{min}}|) \nu_{M_2-zI}(0, \varepsilon) + 2 \frac{\varepsilon}{\varepsilon/2} \|M_1 - M_2\|. \tag{1.9}
\]

A few comments concerning Theorem 1.4 are in order. First, the right-hand side of (1.9) importantly depends only on $\nu_{M_2-zI}$ and not on $\nu_{M_1-zI}$; this means one only needs to control the number of small singular values for one of the matrices. Second, Theorem 1.4 makes no assumptions on the randomness of the matrices $M_1$ and $M_2$. In particular, one can take the matrix $M_2$ to be entirely deterministic, and in some cases, the eigenvalues can be explicitly computed. We give examples of this in Sections 1.2 and 1.3 below.

We now turn to our final result of the subsection. Recall the following interlacing result for the eigenvalues of Hermitian matrices (see [12, Exercise III.2.4]). If $A$ is an $n \times n$ Hermitian matrix with eigenvalues $\lambda_1(A) \geq \cdots \geq \lambda_n(A)$ and $E$ is a positive semi-definite Hermitian matrix of rank one, then the eigenvalues $\lambda_1(A+E) \geq \cdots \geq \lambda_n(A+E)$ of $A+E$ interlace with the eigenvalues of $A$:
\[
\lambda_i(A+E) \geq \lambda_i(A) \geq \lambda_{i+1}(A+E)
\] (1.10)
for $1 \leq i \leq n-1$. In many cases, (1.10) implies that low rank perturbations of Hermitian matrices do not change the spectrum significantly. Our next main result captures a similar behavior for the logarithmic potential (and hence the empirical spectral measure by Theorem 1.2) of arbitrary matrices whose smallest and largest singular values are not too extreme.

**Theorem 1.5** (Rank comparison principle). Let $M_1$ and $M_2$ be $n \times n$ matrices. Take $z \in \mathbb{C}$ and $\varepsilon \in (0, 1/2)$, and assume that
\[
\sigma_{\text{min}} := \min\{\sigma_{\text{min}}(M_i - zI) : i = 1, 2\} > 0.
\]

Similarly, define
\[
\sigma_{\text{max}} := \max\{|M_i - zI| : i = 1, 2\}.
\]

Then
\[
|L_{M_1}(z) - L_{M_2}(z)| \leq 2 (|\log \sigma_{\text{min}}| + |\log \sigma_{\text{max}}|) \frac{\text{rank}(M_1 - M_2)}{n}. \tag{1.10}
\]

Theorems 1.4 and 1.5 are closely related to a number of techniques used in the random matrix theory literature to study non-Hermitian matrices, including those found in [5,7,24,65]. The authors are not aware of any works where these results are stated in the deterministic forms given above.

1.2. **Applications to perturbations of banded Toeplitz matrices.** Using our main results from the previous section, we now study the eigenvalues of the model $A + n^{-\gamma}E$, where $A$ is a banded Toeplitz matrix and $E$ is a random or deterministic perturbation. Here, $\gamma > 0$ is chosen so that
\[
n^{-\gamma}\|E\| = o(1). \tag{1.11}
\]

We begin by defining the class of banded Toeplitz matrices we will study.

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1Here and throughout the article, we use asymptotic notation under the assumption that $n$ tends to infinity. See Section 1.5 for a complete description of our asymptotic notation.
Definition 1.6 (Banded Toeplitz matrix). Let \( \{a_j\}_{j \in \mathbb{Z}} \) be a sequence of complex numbers, indexed by the integers, and let \( k \geq 0 \) be an integer. We say that \( A = (A_{ij})_{i,j=1}^{n} \) is an \( n \times n \) Toeplitz matrix with symbol \( \{a_j\}_{j \in \mathbb{Z}} \) truncated at \( k \) if

\[
A_{ij} = \begin{cases} 
  a_{i-j} & \text{if } |i-j| \leq k \\
  0 & \text{if } |i-j| > k.
\end{cases}
\]

That is, the matrix \( A \) has the form

\[
A = \begin{bmatrix}
  a_0 & a_{-1} & a_{-2} & \cdots & a_{-k} & 0 & \cdots & 0 \\
  a_1 & a_0 & a_{-1} & \cdots & \cdots & \cdots & \cdots & 0 \\
  a_2 & a_1 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
  \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
  a_k & \vdots & \vdots & \cdots & a_{-k} & \cdots & \cdots & \cdots \\
  0 & \vdots & \vdots & \cdots & a_{-2} & \cdots & \cdots & \cdots \\
  \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
  0 & \cdots & 0 & a_k & \cdots & a_2 & a_1 & a_0
\end{bmatrix}.
\]

We refer the reader to [18] and references therein for further details about the spectral properties of banded Toeplitz matrices.

Given the sequence \( \{a_j\}_{j \in \mathbb{Z}} \) and an integer \( k \geq 0 \), we use the following convention for summations:

\[
\sum_{|j| \leq k} f(a_j) = \sum_{j \in \mathbb{Z}} \sum_{|j| \leq k} f(a_j)
\]

for any function \( f : \mathbb{C} \to \mathbb{C} \).

1.2.1. Convergence of the empirical spectral measure. We begin with our most general result for small perturbations of Toeplitz matrices. \( S^1 \) denotes the unit circle \( \{z \in \mathbb{C} : |z| = 1\} \) in the complex plane centered at the origin.

Theorem 1.7. Let \( \{a_j\}_{j \in \mathbb{Z}} \) be a sequence of complex numbers, indexed by the integers, so that

\[
\sum_{j \in \mathbb{Z}} |a_j|^2 < \infty \quad \text{and} \quad \sum_{j \in \mathbb{Z}} |ja_j| < \infty. \tag{1.13}
\]

Let \( k_n \) be a sequence of non-negative integers that converges to \( k \in [0, \infty] \) as \( n \to \infty \) and satisfies

\[
k_n = o \left( \frac{n}{\log n} \right). \tag{1.14}
\]

Let \( A \) be an \( n \times n \) Toeplitz matrix with symbol \( \{a_j\}_{j \in \mathbb{Z}} \) truncated at \( k_n \), and take \( \gamma > 0 \). Let \( E \) be an \( n \times n \) random matrix which satisfies:

1. there exists \( \alpha \geq 0 \) so that

\[
||E|| = O(n^\alpha) \tag{1.15}
\]

with probability \( 1 - o(1) \); and

2. for almost every \( z \in \mathbb{C} \), there exists \( \kappa_z > 0 \) so that

\[
\mathbb{P}(\sigma_{\min}(A + n^{-\alpha-\gamma}E - zI) \leq n^{-\kappa_z}) = o(1). \tag{1.16}
\]
Then there exists a (deterministic) probability measure $\mu$ on $\mathbb{C}$ so that the empirical spectral measure $\mu_{A+n^{-\alpha-\gamma}E}$ of $A + n^{-\alpha-\gamma}E$ converges weakly in probability to $\mu$. Moreover, $\mu$ is the distribution of

$$
\sum_{|j| \leq k} a_j U^j,
$$

where $U$ is a random variable uniformly distributed on $S^1$. Here, we use the convention that if $k = \infty$,

$$
\sum_{|j| \leq k} a_j U^j = \sum_{j \in \mathbb{Z}} a_j U^j.
$$

The assumptions on $\{a_j\}_{j \in \mathbb{Z}}$ in (1.13) and the condition on $k_n$ in (1.14) are likely artifacts of our proof, and we believe these conditions can be relaxed. In fact, our methods show that these conditions can be relaxed for specific choices of $A$. The assumptions on $E$ in Theorem 1.7 are very general and apply to a wide range of random matrix ensembles including matrices with independent and identically distributed (iid) light-tailed entries, matrices with iid heavy-tailed entries, elliptic random matrices, and random unitary matrices. In the following corollary we specialize Theorem 1.7 to a few of these cases. See Figure 1 for eigenvalue plots in some example cases.

**Corollary 1.8.** Let $\{a_j\}_{j \in \mathbb{Z}}$ be a sequence of complex numbers, indexed by the integers, so that (1.13) holds. Let $k_n$ be a sequence of non-negative integers that converges to $k \in [0, \infty]$ as $n \to \infty$ and satisfies (1.14). Let $A$ be an $n \times n$ Toeplitz matrix with symbol $\{a_j\}_{j \in \mathbb{Z}}$ truncated at $k_n$. Assume one of the following conditions on the $n \times n$ matrix $E$ and the parameter $\gamma > 0$:

1. $E$ is an $n \times n$ random matrix whose entries are iid copies of a random variable with mean zero, unit variance, and finite fourth moment, and $\gamma > 1/2$.
2. $E$ is an $n \times n$ random matrix whose entries are iid copies of a random variable with mean zero and unit variance, and $\gamma > 1$.
3. $E$ is an $n \times n$ random matrix uniformly distributed on the unitary group $U(n)$, and $\gamma > 0$.

Then there exists a (deterministic) probability measure $\mu$ on $\mathbb{C}$ so that the empirical spectral measure $\mu_{A+n^{-\gamma}E}$ of $A + n^{-\gamma}E$ converges weakly in probability to $\mu$. Moreover, in all three cases $\mu$ is the distribution of the random variable given in (1.17), where $U$ is a random variable uniformly distributed on $S^1$.

**Proof.** In order to utilize Theorem 1.7, we only need to verify that $E$ satisfies (1.15) and (1.16) in each of the cases above. For the first case, these bounds (with $\alpha = 1/2$) follow from [4, Theorem 5.8] and [65, Theorem 2.1]. The second case follows (with $\alpha = 1$) from [65, Theorem 2.1] and the fact that

$$
\frac{1}{n^2} \|E\|^2 \leq \frac{1}{n^2} \|E\|_2^2 \to 1
$$

almost surely by the law of large numbers. In the third case, $\|E\| = 1$ since $E$ is unitary. In this last case, the least singular value bound in (1.16) follows from [56, Theorem 1.1].

The choice of $\gamma$ in each of the three cases given in Corollary 1.8 is so that (1.11) is satisfied. Theorem 1.7 and its corollary are similar to several recent works concerning fixed matrices perturbed by random matrices [7,8,10,29,60,61,62,73]. The case when $E$ contains independent Gaussian entries was investigated in [7,29,60,61,62]. Since Theorem 1.7 applies to a large class of perturbations $E$, it is more closely related to the results in [8,73]. Our assumptions on $E$ differ from these previous works, allowing us to apply our results to a slightly different classes of perturbations, including deterministic matrices (see Section 1.2.3). In addition, our results allow the bandwidth of $A$ to grow with $n$, which differs from the results in [8]. See Figure 2 for an example with growing bandwidth. Our method of proof also differs from the methods used in [8,73] and suggests a concrete way to accurately approximate the eigenvalues for reasonably small $n$ by comparing them to the
Eigenvalues of a deterministic circulant matrix; see Section 3 and Conjecture 3.6 for further details about this approximation. Some numerical simulations are presented in Figure 3. We also note that Theorem 1.7 allows us to consider multiplicative perturbations (see also Figure 1(3)).
Figure 2. Above are plots of the eigenvalues of a perturbed Toeplitz matrix $M$ truncated at $k_n$, where $k_n = \lfloor n^{1/3} \rfloor - 1$ (which tends to infinity with $n$) and where the symbol for $M$ is defined by $a_{-j} = 1/(j + 1)^{2.1}$ for $j \geq 0$ and and $a_j = (0.999)^j$ for $j \geq 6$ and $a_j = 0$ for $1 \leq j \leq 5$. The eigenvalues of $M + n^{-\gamma}U$, where $U$ is a Haar uniform random unitary matrix, are plotted as blue $\times$ symbols, and the eigenvalues of the corresponding finite approximation are plotted as a black circles ($\odot$). By Theorem 1.7 (see also Corollary 1.8 part (3)), the empirical spectral measure of $M + n^{-\gamma}U$ converges weakly in probability to a deterministic measure, which is also the limit of the finite approximations. The finite approximation is the eigenvalues of a circulant matrix $C$ as described in Section 3, defined by $c_{n-j} = a_{-j}$ for $1 \leq j \leq \lfloor n^{1/3} \rfloor - 1$ and $c_j = a_j$ for $0 \leq j \leq \lfloor n^{1/3} \rfloor$ and all other $c_j$ are equal to zero. The finite approximation appears to be reasonably close to the random eigenvalues in this case, even though the finite approximation is still visibly changing for the values of $n$ shown above.
Figure 3. Above is a series of plots of the eigenvalues of a perturbed Toeplitz matrix $M$ with symbol $\{a_{-2} = 2, a_{-1} = 0, a_0 = 0, a_1 = 0, a_2 = 0, a_3 = -1\}$ and all other $a_j$ equal to zero. The plots are for values of $n$ ranging from 100 to 4000 and also show two different approximations of the eigenvalues. The eigenvalues of $M + n^{-2}G$, where $G$ is an iid real Gaussian perturbation, are plotted as blue $\times$ symbols, and the eigenvalues of the corresponding finite approximation are plotted as a black circles ($\circ$). The finite approximation comes from the circulant matrix $C$ (with $c_{n-j} = a_{-j}$ for $1 \leq j \leq n-1$ and $c_j = a_j$ for $0 \leq j \leq n-1$) described in Section 3. By Theorem 1.7 and Lemma 3.1, we know that the spectral measure $\mu_{M + n^{-2}G}$ is approximated by the spectral measure $\mu_C$ as $n$ increases. Interestingly, the scaled circulant matrix described in Conjecture 3.6, whose eigenvalues are plotted as red squares ($\Box$), appears to be a very good match for the eigenvalues of $M + n^{-2}G$ even for small $n$.

Corollary 1.9 (Multiplicative perturbation). Let $\{a_j\}_{j \in \mathbb{Z}}$ be a sequence of complex numbers, indexed by the integers, so that (1.13) holds. Let $k_n$ be a sequence of non-negative integers that converges to $k \in [0, \infty]$ as $n \to \infty$ and satisfies (1.14). Let $A$ be an $n \times n$ Toeplitz matrix with symbol $\{a_j\}_{j \in \mathbb{Z}}$ truncated at $k_n$. Let $E'$ be an $n \times n$ random matrix whose entries are iid copies of a real standard normal random variable, and take $\gamma > 1$. Then there exists a probability measure $\mu$ on $\mathbb{C}$ so that the empirical spectral measure $\mu_{A(I + n^{-\gamma-1/2}E')}^{A(I + n^{-\gamma-1/2}E')}$ converges weakly
in probability to \( \mu \). Moreover, \( \mu \) is the distribution given in (1.17), where \( U \) is a random variable uniformly distributed on \( S^1 \).

**Proof.** Since \( A(I + n^{-\gamma-1/2}E') = A + n^{-\gamma-1/2}AE' \), we will apply Theorem 1.7 with \( E := AE' \). It remains to check that \( E \) satisfies (1.15) and (1.16). The bound in (1.15) with \( \alpha = 1/2 \) follows from Proposition B.2 (which provides an upper bound on \( \|A\| \)) and [71, Corollary 5.35] (which provides an upper bound on \( \|E'\| \)). The least singular value bound in (1.16) follows from Proposition B.3.

\[ \square \]

### 1.2.2. Rate of convergence

Our next result provides a rate of convergence, in Wasserstein distance, for the empirical spectral distribution of \( A + n^{-\gamma}E \) to its limiting distribution. Recall that for two probability measures \( \mu \) and \( \nu \) on \( \mathbb{C} \), the \( L^1 \)-Wasserstein distance between \( \mu \) and \( \nu \) is given by

\[
W_1(\mu, \nu) := \inf_{\pi} \int |x - y| d\pi(x, y),
\]

where the infimum is over all probability measures \( \pi \) on \( \mathbb{C} \times \mathbb{C} \) with marginals \( \mu \) and \( \nu \).

For \( z \in \mathbb{C} \) and \( r_1, r_2 > 0 \), we define the closed rectangular box

\[
\mathcal{R}(z, r_1, r_2) := \{ w \in \mathbb{C} : |\text{Re}(w) - \text{Re}(z)| \leq r_1, |\text{Im}(w) - \text{Im}(z)| \leq r_2 \}
\]

in the complex plane. For a finite set \( S \), let \( |S| \) denote the cardinality of \( S \).

**Theorem 1.10.** Let \( k \geq 0 \) be a fixed integer, and let \( \{a_j\}_{j \in \mathbb{Z}} \) be a sequence of complex numbers indexed by the integers. Let \( A \) be the \( n \times n \) Toeplitz matrix with symbol \( \{a_j\}_{j \in \mathbb{Z}} \) truncated at \( k \). Define the function \( f : S^1 \to \mathbb{C} \) as

\[
f(\omega) = \sum_{|j| \leq k} a_j \omega^j,
\]

and assume there exists \( \varepsilon_0 > 0 \) and \( c_0 \geq 1 \) so that for any \( 0 < \varepsilon' \leq \varepsilon_0 \),

\[
\sup_{z \in \mathbb{C}} \{ 0 \leq j \leq n - 1 : f(\omega_n^j) \in \mathcal{R}(z, n^{-\varepsilon'}, n^{-c_0\varepsilon'}) \cup \mathcal{R}(z, n^{-c_0\varepsilon'}, n^{-\varepsilon'}) \} = O_{\varepsilon'}(n^{1-3\varepsilon'}),
\]

where \( \omega_n := \exp \left( \frac{2\pi \sqrt{1}}{n} \right) \) (1.21)
is a primitive \( n \)-th root of unity\(^2\). Let \( \gamma, \delta > 0 \), and let \( E \) be an \( n \times n \) random matrix so that

(i) there exists \( M > 0 \) and \( \alpha \geq 0 \) so that

\[
\|E\| \leq Mn^\alpha
\]

with probability \( 1 - O(n^{-\delta}) \).

(ii) there exists \( \kappa > 0 \) so that

\[
\sup_{z \in \mathbb{C}, |z| \leq \beta} \mathbb{P}(\sigma_{\min}(A + n^{-\alpha-\gamma}E - zI) \leq n^{-\kappa}) = O \left( n^{-\beta} \right)
\]

for \( \beta := \sqrt{2} \left( \sum_{|j| \leq k} |a_j| + M + 1 \right) \).

Then there exists \( C, \varepsilon > 0 \) (depending on \( k, f, \{a_j\}_{|j| \leq k}, \varepsilon_0, c_0, \gamma, \delta \), and the constants from assumptions (i), (ii), and (1.20)) so that

\[
W_1(\mu_{A+n^{-\gamma-\alpha}E}, \mu) \leq Cn^{-\varepsilon}
\]

with probability \( 1 - o(1) \), where \( \mu \) is the distribution of \( f(U) \) and \( U \) is a random variable uniformly distributed on \( S^1 \).

---

\(^2\)Here, we use \( \sqrt{-1} \) to denote the imaginary unit and reserve i as an index.
Condition (1.20) is technical and requires that the points $f(\omega_j^2)$ (and hence the curve $f(S^1)$ itself) not concentrate in any small region in the plane. We have chosen to use rectangles as this matches the geometric construction given in the proof, but other shapes could also be used with appropriate modifications to the proof. If $A$ is Hermitian then $f$ is real-valued and will fail to satisfy (1.20). However, the eigenvalues can be rotated by a phase (i.e., by considering $e^{\alpha+i\theta} (A - n^{-\gamma}E)$ for an appropriate choice of $\theta \in [0, 2\pi]$), so that Theorem 1.10 is applicable. The assumptions on $E$ in Theorem 1.10 are general and apply to a variety of random matrix ensembles. We give a few examples of Theorem 1.10 below.

**Example 1.11.** Consider the $n \times n$ matrix $T$ given in (1.2). $T := T$ is a Toeplitz matrix with $a_{-1} = 1$ and $a_j = 0$ for all $j \neq -1$. Thus,

$$f(\omega) = \frac{1}{\omega} = \bar{\omega}.$$ 

It is easy to check that $f(\omega_j^2)$, $0 \leq j \leq n - 1$ are uniformly spaced on $S^1$, and it follows that condition (1.20) is satisfied with $c_0 = 6$ and any $\varepsilon > 0$. Let $E$ be an $n \times n$ random matrix whose entries are iid copies of a random variable with mean zero, unit variance, and finite fourth moment. It follows from Proposition B.1 that $\|E\| \leq n^\alpha$ with probability at least $1 - O(n^{1/2-\alpha})$ for any $\alpha > 1/2$. The least singular value bound in (1.22) follows from [65, Theorem 2.1]. Therefore, for any $\gamma > 0$, Theorem 1.10 can be applied (where we take $\alpha = 1/2 + \gamma/2$ and $\gamma$ in Theorem 1.10 is taken to be $\gamma/2$) to obtain

$$W_1(\mu_{T+n^{-1/2-\gamma}E}, \mu) \leq Cn^{-\varepsilon}$$

with probability $1 - o(1)$ for $C, \varepsilon > 0$, where $\mu$ is the uniform probability measure on the unit circle $S^1$. 

**Example 1.12.** Theorem 1.10 also applies to matrices with heavy-tailed entries. For example, let $A := T$ be the same matrix as in Example 1.11. Let $E$ be an $n \times n$ random matrix whose entries are iid copies of a non-constant random variable $\xi$ with $\mathbb{E}|\xi|^\eta < \infty$ for some $\eta > 0$ (and no other moment assumptions). A bound on the norm of $E$ follows from the arguments in [21, Lemma 56]. The least singular value bound in (1.22) follows from [21, Theorem 32] (alternatively, see [14, Lemma A.1]). Therefore, we conclude from Theorem 1.10 that there exists $\gamma_0 > 0$ (depending only on $\xi$) so that for any $\gamma > \gamma_0$, there exists $C, \varepsilon > 0$ with

$$W_1(\mu_{T+n^{-1/2-\gamma}E}, \mu) \leq Cn^{-\varepsilon},$$

where $\mu$ is the uniform probability measure on the unit circle $S^1$.

**Example 1.13.** Let $A$ be the Toeplitz matrix in (1.12) with $a_{-1} = 4$, $a_1 = 1$, and $a_j = 0$ for all other $j \in \mathbb{Z}$. Then

$$f(\omega) = \frac{4}{\omega} + \omega$$

so that $f(S^1)$ is an ellipse. It can be checked that $f$ satisfies (1.20) for $\varepsilon_0$ sufficiently small and $c_0 = 6$. Let $E$ be an $n \times n$ random matrix uniformly distributed on the unitary group $U(n)$. Then $\|E\| = 1$ (with probability 1), and for any $\gamma > 0$, the least singular value bound in (1.22) follows from [56, Theorem 1.1]. Therefore, Theorem 1.10 implies that there exists $C, \varepsilon > 0$ so that

$$W_1(\mu_{A+n^{-\gamma}E}, \mu) \leq Cn^{-\varepsilon}$$

with probability $1 - o(1)$, where $\mu$ is the distribution of $f(U)$ and $U$ is a random variable uniformly distributed on $S^1$.

We conjecture that, under certain conditions such as when the entries of $E$ are iid standard normal random variables, the optimal rate of convergence for the Wasserstein distance in Theorem
1.10 is $O(\log n/n)$. Our method only allows us to conclude a bound of the from $O(n^{-\gamma})$ for small values of $\varepsilon > 0$, depending upon $\gamma$ and the other parameters specified in Theorem 1.10.

1.2.3. Deterministic, non-asymptotic results. Our main results can also be applied to the case when $E$ is deterministic as the following proposition shows. Below, we derive a non-asymptotic bound for a matrix that is a single Jordan block. There are many asymptotic results in the literature for perturbed Jordan canonical form matrices, for example [7, 8, 29, 39, 74], and we consider more general matrices in Jordan canonical form in Section 1.3.

Proposition 1.14. Let $\gamma \geq 5$ be a real number, and assume that $n \geq \gamma^2$ is an integer. Let $T$ be the $n$ by $n$ matrix given in (1.2) with all entries on the super diagonal equal to 1 and all other entries equal to zero. Let $R$ be an arbitrary deterministic $n$ by $n$ matrix where each entry is $\pm n^{-\gamma}$. Then, for any smooth function $\varphi : \mathbb{C} \to \mathbb{C}$ with support contained in $\{z \in \mathbb{C} : |z| < 1/4\}$, we have that

$$\left| \sum_{i=1}^{n} \varphi(\lambda_i(T + R)) \right| \leq 6\|\Delta \varphi\|_{\infty} \gamma \log n,$$

where $\|\Delta \varphi\|_{\infty}$ is the $L^\infty$-norm of $\Delta \varphi$. Furthermore, for any $\varepsilon > 0$ at most $O_\varepsilon(\log n)$ eigenvalues of $T + R$ can lie inside the disk of radius $1/4 - \varepsilon$.

Proposition 1.14 guarantees the same eigenvalue behavior for $T + R$, regardless of the choice of signs for the entries of $R$. In this way, Proposition 1.14 can be viewed as describing an adversarial model in which no matter what signs the adversary chooses for the matrix $R$, the eigenvalues always have the same non-asymptotic behavior. The restriction of the support of $\varphi$ to the disk of radius $1/4$ in Proposition 1.14 is for simplicity and can likely be extended to any disk of radius less than one using similar methods. More generally, we conjecture that the eigenvalues of $T + R$, regardless of the choice of signs for the entries of $R$, will behave in a similar fashion as the eigenvalues of $T + n^{-\gamma}E$ when $E$ is a random matrix with iid standard normal entries.

1.3. Additional examples. Below are two cases where Theorem 1.4 can be applied to matrices in Jordan canonical form to derive non-asymptotic results. Asymptotic results that cover similar types of matrices with general types of random perturbations have appeared in [8, 73, 74] and references therein.

Example 1.15 (A has blocks of size $o(\log n)$). Let $\gamma > 0$ and suppose that $A$ is an $n$ by $n$ matrix in Jordan canonical form with all eigenvalues equal to zero for simplicity, and where each block has size at most $m := \log(n)/t_n \geq 1$ for all $n \geq 1$, where $t_n \to \infty$ as $n \to \infty$ and $1 \leq t_n \leq \log n$ for all $n$. Let $M_1$ be an $n$ by $n$ matrix with iid random entries bounded uniformly in $n$ and having mean zero and variance $1/n$, and let $M_2$ be the $n$ by $n$ zero matrix. Letting $C$ and $c$ denote unspecified constants below, we will apply Theorem 1.2 to show that the eigenvalue measures of $A + n^{-\gamma}M_1$ and $A + M_2$ are close to each other when integrated against a smooth test function $\varphi$.

Because Theorem 1.2 samples $Z$ uniformly at random from $\text{supp} \Delta \varphi$, we can ignore cases where $Z$ is within $e^{-\gamma t_n/10}$ of 1 or of 0, which happens with reasonably small probability. This lets us apply Lemma A.2 to show that the smallest singular value of $A - ZI$ is at least $n^{-\gamma/4}$ with probability at least $1 - ce^{-3\gamma t_n/10}$. Also, note that the difference between the two matrices is $n^{-\gamma}M_1$ which has norm at most $cn^{-\gamma}$ by [72, Theorem 4.4.5] with probability at least $1 - 2e^{-n}$. Thus, we can combine the lower bound on the smallest singular value of $A - ZI$ with Weyl’s inequality (see Theorem 1.17) to show that $\sigma_{\text{min}} \geq cn^{-\gamma/4}$ with probability at least $1 - ce^{-3\gamma t_n/10} - 2e^{-n}$, where $\sigma_{\text{min}}$ is the minimum of all singular values of $A + M_2 - zI$ and $A + n^{-\gamma}M_1 - zI$. We can now apply Theorem 1.4 to show that the difference in log potentials is sufficiently small, namely,

$$|\mathcal{L}_{A+M_1}(z) - \mathcal{L}_{A+M_2}(z)| \leq \frac{2}{\varepsilon_{1.4}^{12}} \left| n^{-\gamma}M_2 \right| \leq \frac{2cn^{-\gamma}}{2cn^{-\gamma/2}} = n^{-\gamma/2}$$

(1.23)
with probability at least \(1 - ce^{-3\gamma t_n/10}\) for \(n\) sufficiently large. Finally, we can now choose parameters and apply Theorem 1.2 to show

\[
\left| \int_{\mathbb{C}} \varphi \, d\mu_A - \int_{\mathbb{C}} \varphi \, d\mu_{A+n^{-\gamma}M_1}\right| \leq C \left(n^{-\gamma/2} + e^{-\gamma t_n/20}\right)
\]

with probability at least \(1 - ce^{-\gamma t_n/10}\), where \(C\) and \(c\) are constants and \(n\) is sufficiently large depending on the various parameters.

**Example 1.16** (\(A\) has blocks of size \(\omega(\log n)\)). Suppose \(A\) is a matrix in Jordan canonical form with blocks of size at least \(m = t_n \log(n)\), where \(t_n \to \infty\) as \(n \to \infty\) and \(1 \leq t_n \leq n/\log n\) for all \(n\). We will assume all eigenvalues of \(A\) are zero for simplicity. Then \(A\) has at most \(n/(t_n \log(n))\) blocks. Let \(E\) be an \(n\) by \(n\) zero-one matrix so that each block in \(A + E\) has a 1 in the lower left corner (thus, turning each block into a circulant matrix as defined in Section 3). Let \(M_1\) be an \(n\) by \(n\) with iid random entries bounded uniformly in \(n\) and having mean zero and variance \(1/n\). As in Example 1.15, we will apply Theorem 1.2 and Theorem 1.4 to show that the eigenvalue measures of \(A + n^{-\gamma}M_1\) and \(A + n^{-\gamma}E\) are close to each other when integrated against a smooth test function \(\varphi\). Throughout this example, we will use \(C, c\) to denote unspecified positive constants that may change from line to line; however, we note that in specific cases for the sequence \(t_n\), the function \(\varphi\), and the type of random entries in \(M_1\), specific values for the constants may be computed.

As in Example 1.15, we can show that \(\sigma_{\min}(A + n^{-\gamma}E - ZI) \geq C n^{-\gamma}\) with probability at least \(1 - cn^{-\gamma/4}\). Also, by [65, Theorem 2.1] (rescaled as in [74, Theorem 9]), we have \(\sigma_{\min}(A + n^{-\gamma}M_1 - ZI) \geq n^{C-\gamma}\) with probability at least \(1 - cn^{-\gamma/4}\). Putting this together with Theorem 1.4, we have

\[
\left| \mathcal{L}_{A+n^{-\gamma}M_1}(Z) - \mathcal{L}_{A+n^{-\gamma}E}(Z) \right| \leq C \left(\frac{1}{t_n} + n^{-\gamma/2}\right)
\]

with probability at least \(1 - cn^{-\gamma/4}\). Finally, we can choose parameters and apply Theorem 1.2 to show that

\[
\left| \int_{\mathbb{C}} \varphi \, d\mu_{A+n^{-\gamma}M_1} - \int_{\mathbb{C}} \varphi \, d\mu_{A+n^{-\gamma}E}\right| \leq C \left(\frac{1}{t_n} + n^{-c\gamma}\right)
\]

with probability at least \(1 - cn^{-\gamma/4}\), where \(C\) and \(c\) are constants and \(n\) is sufficiently large depending on various constants.

### 1.4. Related works

The use of the logarithmic potential to study the eigenvalues of non-Hermitian random matrices has a long history in the field, including in the investigation of the famous circular law; we refer the reader to the works [5, 31, 32, 33, 34, 35, 38, 64, 65, 68] and references therein as well as the survey [14] for further historical details. Our main methods for studying the logarithmic potential of the empirical spectral measure are similar to several works in the random matrix theory literature including [5, 7, 65, 68]. Many of the methods introduced in these works are comparison methods—which show how the spectrum of one matrix can be compared to another in order to compute the limiting eigenvalue distribution. For instance, the results in [65, 68] show conditions under which small perturbations (or low rank perturbations) of random matrices do not change the limiting distributions of the eigenvalues. The replacement principle (Theorem 1.1), which was a major inspiration for the current article, is another example appearing in [68]. The non-asymptotic replacement principle introduced above was motivated by similar methods used by Tao and Vu to study roots of random polynomials [66].

In recent years, a number of results have exploited the logarithmic potential to understand the local behavior of the eigenvalues of non-Hermitian random matrices, including local laws and related rates of convergence for the empirical spectral measure. These results are too numerous to list in entirety but include [3, 19, 20, 23, 30, 50, 51, 53, 76, 77] and references therein.
Perturbations of the matrix $T$ (defined in (1.2)) were investigated by Davies and Hager in [24]. Similar to our results, they investigated both random perturbations having small norm as well as low rank perturbations using a relevant Grushin problem. The pseudospectrum of Toeplitz matrices was investigated in [54]. Random and deterministic perturbations of Toeplitz matrices have also been considered in [16] and [17]. The limiting distribution of the eigenvalues for Gaussian perturbations of non-normal matrices was investigated by Śniady [63] and later generalized by Guionnet, Zeitouni, and the second author [39] by analyzing the logarithmic potential and using tools from free probability theory.

Extensions of these results have appeared for the cases of Gaussian perturbations [7,29,60,61,62] as well as more general perturbations [8,73]. In particular, the methods in [61] are strong enough to handle general Toeplitz matrices. We conjecture that our results should also hold for such a general class of Toeplitz matrices, but our present methods require the restriction to banded Toeplitz matrices. We also emphasize the work of Vogel and Zeitouni in [73]. Similar to our results, Vogel and Zeitouni provide a (nearly) deterministic comparison result for deterministic matrices subject to small random perturbations. In specific cases, our main results can recover their bounds. Our non-asymptotic results (such as Theorem 1.10) extend their results in certain cases by providing a rate of convergence in Wasserstein distance. Very recently, the eigenvectors of random perturbations of Toeplitz matrices were investigated and shown to be localized by Basak, Vogel, and Zeitouni [9].

Localized random perturbations of infinite banded Laurent matrices were studied in [15], where it is shown that the spectrum can be approximated by perturbed circulant matrices. Our methods (discussed below in Section 1.6) similarly use a connection between the eigenvalues of perturbed Toeplitz matrices and those of circulant matrices.

1.5. Notation. For a vector $x \in \mathbb{C}^n$, $\|x\|$ denotes its Euclidean norm. For a matrix $A$, $\|A\|$ is its spectral norm and $\|A\|_2$ is its Frobenius norm, defined in (1.5). $A^T$ is the transpose of $A$ and $A^*$ denotes the conjugate transpose of $A$. All matrices under consideration in this article are assumed to have complex entries, unless otherwise indicated. $I_n$ denotes the $n \times n$ identity matrix; often we will write $I$ when its size can be deduced from context.

We let $\lambda_1(A), \ldots, \lambda_n(A)$ be the eigenvalues of the $n \times n$ matrix $A$ (counted with algebraic multiplicity). The empirical spectral measure $\mu_A$ of $A$ is defined in (1.3). The logarithmic potential of $\mu_A$ is denoted $\mathcal{L}_A$ and is defined in (1.4).

The singular values of the $n \times n$ matrix $A$ are the eigenvalues of $\sqrt{AA^T}$. We let $\sigma_1(A) \geq \cdots \geq \sigma_n(A)$ denote the ordered singular values of $A$ and $\nu_A$ be the empirical measure constructed from the singular values of $A$ as defined in (1.7). We often use $\sigma_{\text{min}}(A) := \sigma_n(A)$ to denote the smallest singular value of $A$, which can be computed by the variational characterization

$$\sigma_{\text{min}}(A) = \min_{x \in \mathbb{C}^n : \|x\| = 1} \|Ax\|. \quad (1.24)$$

For two probability measures $\mu$ and $\nu$ on $\mathbb{C}$, the $L^1$-Wasserstein distance between $\mu$ and $\nu$ is denoted as $W_1(\mu, \nu)$ and defined in (1.18).

We use $\log(x)$ to denote the natural logarithm of $x$. $\sqrt{-1}$ will denote the imaginary unit, and we reserve $i$ as an index. $[n]$ denotes the discrete interval $\{1, \ldots, n\}$. For a finite set $S$, $|S|$ is the cardinality of $S$.

Let $C_c^\infty(\mathbb{C})$ be the set of smooth, compactly supported functions $\varphi : \mathbb{C} \to \mathbb{C}$. $\text{supp}(\varphi)$ denotes the support of $\varphi$ and $\|\varphi\|_\infty$ is its $L^\infty$-norm.

Let

$$B(z, r) := \{w \in \mathbb{C} : |z - w| < r\}$$

be the open ball of radius $r > 0$ centered at $z \in \mathbb{C}$ in the complex plane. $\mathbb{R}(z_1, z_2)$ denotes the rectangle in the complex plane defined in (1.19). $S^1$ denotes the unit circle $\{z \in \mathbb{C} : |z| = 1\}$ in the
complex plane centered at the origin. The quantifiers “almost everywhere” and “almost all” will be with respect to the Lebesgue measure on $\mathbb{C}$.

Asymptotic notation is used under the assumption that $n$ tends to infinity. We use $X = O(Y)$, $Y = \Omega(X)$, $X \ll Y$, or $Y \gg X$ to denote the estimate $|X| \leq CY$ for some constant $C > 0$, independent of $n$, and all $n \geq C$. If $C$ depends on other parameters, e.g. $C = C_{k_1,k_2,...,k_p}$, we indicate this with subscripts, e.g. $X = O_{k_1,k_2,...,k_p}(Y)$. The notation $X = o(Y)$ denotes the estimate $|X| \leq c_n Y$ for some sequence $(c_n)$ that converges to zero as $n \to \infty$. $X = \omega(Y)$ means $|X| \geq c_n Y$ for some sequence $(c_n)$ that converges to infinity as $n \to \infty$. We write $X = \Theta(Y)$ if $X \ll Y \ll X$.

1.6. Overview and outline of the methods. In this section, we outline our main methods used in the proofs. Recall from (1.4), that the logarithmic potential $L_M$ of the empirical spectral measure $\mu_M$ of an $n \times n$ matrix $M$ is given by

$$L_M(z) = \frac{1}{n} \log |\det(M - zI)| = \frac{1}{n} \sum_{j=1}^{n} \log \sigma_j(M - zI),$$

where we used the fact that the absolute value of the determinant is given as the product of singular values. This technique, which allows us to focus on the singular values rather than the eigenvalues, is at the heart of Girko’s Hermitianization technique in random matrix theory, see, for instance, [5, 14, 31, 32, 33, 34, 35, 65, 68] and references therein. Our proofs of Theorems 1.4 and 1.5 utilize the fact that the singular values of $M - zI$ are stable under small perturbations (as well as low rank perturbations). Heuristically, if none of the singular values of $M - zI$ are too extreme (as to avoid the poles of $|\cdot|$ at zero and infinity), the logarithmic potential $L_M$ would also be stable. Our proof uses Weyl’s perturbation theorem for singular values (cf. (1.1)), see Theorem 1.3 in [22] or Problem III.6.13 in [12].

**Theorem 1.17** (Weyl’s perturbation theorem for singular values). *If $M_1$ and $M_2$ are two $n \times n$ matrices, then*

$$\max_{1 \leq j \leq n} |\sigma_j(M_1) - \sigma_j(M_2)| \leq \|M_1 - M_2\|.$$

For studying perturbations $A + n^{-a-\gamma}E$ of a Toeplitz matrix $A$, we rely on a comparison method to compare the eigenvalues with those of a circulant matrix. For instance, if $A$ is a banded Toeplitz matrix, we show the existence of a low rank matrix $A'$ so that $A + A'$ is a deterministic circulant matrix. Since the eigenvalues of circulant matrices are well-known and easy to compute (see Lemma 3.1), we can give an explicit description of the limiting empirical spectral measure of $\mu_{A + A'}$. Our method can be summarized by the following sequence of comparisons:

$$L_{A + n^{-a-\gamma}E} \approx L_{A + A' + n^{-a-\gamma}E} \approx L_{A + A'},$$

where the first approximation utilizes Theorem 1.5 (since $A'$ has low rank) and the second uses Theorem 1.4 (since $n^{-a-\gamma}E$ has small norm). We can then pass from an approximation of the logarithmic potentials to an approximation involving the empirical spectral measures using Theorem 1.2.

This paper is organized as follows. We prove Theorems 1.2, 1.4, and 1.5 in Section 2. In Section 3, we present some preliminary results on circulant matrices and provide the proof of Theorem 1.7. The proof of Theorem 1.10 is presented in Section 4. We prove Proposition 1.14 in Section 5. The two appendices contain auxiliary results: Appendix A contains some deterministic results used in the proof of Proposition 1.14 and Appendix B contains some bounds relating to Corollary 1.9.
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2. PROOF OF RESULTS IN SECTION 1.1

This section is devoted to the proofs of Theorems 1.2, 1.4, and 1.5.

2.1. Proof of Theorem 1.4. We begin with the proof of Theorem 1.4. We will need the following lemma.

**Lemma 2.1** (Lemma 4.3 from [44]). For any probability measure $\mu$ and $\nu$ on $\mathbb{R}$ and any $0 < a < b$,

$$
\left| \int_a^b \log(x) d\mu(x) - \int_a^b \log(x) d\nu(x) \right| \leq 2 \left| \log b - \log a \right| \| \mu - \nu \|_{[a,b]},
$$

where

$$
\| \mu - \nu \|_{[a,b]} := \sup_{x \in [a,b]} |\mu([a,x]) - \nu([a,x])|.
$$

We now turn to the proof of Theorem 1.4. For simplicity we define

$$
D := \max\{1, \|M_1 - zI\|, \|M_2 - zI\|\}.
$$

By writing the logarithmic potential in terms of the logarithm of the determinant, we find

$$
\mathcal{L}_{M_i}(z) = \int_0^\infty \log(x) d\nu_{M_i-zI}(x)
= \int_{\sigma_{\min}}^D \log(x) d\nu_{M_i-zI}(x)
= \int_{[\sigma_{\min},\varepsilon/2]} \log(x) d\nu_{M_i-zI}(x) + \int_{(\varepsilon/2,D]} \log(x) d\nu_{M_i-zI}(x).
$$

Using Weyl’s inequality for singular values (Theorem 1.17) and the assumption that $\|M_1 - M_2\| < \varepsilon/2$, we have

$$
\nu_{M_1-zI}([0,\varepsilon/2]) \leq \nu_{M_2-zI}([0,\varepsilon]),
$$

and so by the triangle inequality

$$
\|\nu_{M_1-zI} - \nu_{M_2-zI}\|_{[0,\varepsilon/2]} := \sup_{x \in [0,\varepsilon/2]} |\nu_{M_1-zI}([0,x]) - \nu_{M_2-zI}([0,x])| \leq 2\nu_{M_2-zI}([0,\varepsilon]).
$$

Thus, by Lemma 2.1, we conclude that

$$
\left| \int_{[\sigma_{\min},\varepsilon/2]} \log(x) d\nu_{M_i-zI}(x) - \int_{[\sigma_{\min},\varepsilon/2]} \log(x) d\nu_{M_2-zI}(x) \right|
\leq 4 \left( \log(\varepsilon/2) + \log \sigma_{\min} \right) \nu_{M_2-zI}([0,\varepsilon]).
$$

It remains to bound

$$
\left| \int_{(\varepsilon/2,D]} \log(x) d\nu_{M_1-zI}(x) - \int_{(\varepsilon/2,D]} \log(x) d\nu_{M_2-zI}(x) \right|.
$$

Recall that $\sigma_1(M_i - zI) \geq \cdots \geq \sigma_n(M_i - zI)$ denote the ordered singular values of $M_i - zI$ (which are all contained in the interval $[0,D]$). We will show that most singular values can be paired so that the difference of the integrals above is not too large, and the singular values that
cannot be paired will also contribute a well-controlled error. Let \( j_1 \) be the number of singular values of \( M_1 - zI \) in \((\varepsilon/2, D]\), and let \( j_2 \) be the number of singular values of \( M_2 - zI \) in \((\varepsilon/2, D]\). Then

\[
\left| \int_{\varepsilon/2}^{D} \log(x) d\nu_{M_1 - zI}(x) - \int_{\varepsilon/2}^{D} \log(x) d\nu_{M_2 - zI}(x) \right| \leq \frac{1}{n} \left| \sum_{j=1}^{j_1} \log(\sigma_j(M_1 - zI)) - \sum_{j=1}^{j_2} \log(\sigma_j(M_2 - zI)) \right| \quad (2.3)
\]

where we use the convention that if \( j_1 = 0 \) or \( j_2 = 0 \) then the respective sums in (2.3) are zero. Thus, we obtain

\[
\left| \int_{\varepsilon/2}^{D} \log(x) d\nu_{M_1 - zI}(x) - \int_{\varepsilon/2}^{D} \log(x) d\nu_{M_2 - zI}(x) \right| \leq \frac{1}{n} \sum_{j=1}^{\min\{j_1, j_2\}} |\log(\sigma_j(M_1 - zI)) - \log(\sigma_j(M_2 - zI))| \quad (2.4)
\]

\[
+ \frac{1}{n} \sum_{j=\max\{j_1, j_2\}+1}^{\max\{j_1, j_2\}} |\log(\sigma_j(M_K - zI))| \quad (2.5)
\]

where \( K = 1 \) if \( j_1 > j_2 \), \( K = 2 \) if \( j_2 > j_1 \) and the value of \( K \) is irrelevant if \( j_1 = j_2 \), since then the sum (2.5) is empty. Here, we again use the convention that if \( \min\{j_1, j_2\} = 0 \), then the sum in (2.4) is zero. When \( j_1 \neq j_2 \), we will show that (2.5) is still small. Indeed, note that each singular value that appears in the sum (2.5) must have a corresponding singular value from the other matrix that lies in the interval \([0, \varepsilon/2]\). Thus, we know the number of terms in the sum (2.5) is at most

\[
\max\{n\nu_{M_1 - zI}([0, \varepsilon/2]), n\nu_{M_2 - zI}([0, \varepsilon/2])\} \leq 2n\nu_{M_2 - zI}([0, \varepsilon])
\]

(using (2.1) for the inequality). Furthermore, by Weyl’s perturbation theorem for singular values (Theorem 1.17), we know each singular value that appears as a summand in (2.5) is bounded above by \( \varepsilon < 1 \), which means each summand is at most \( |\log(\varepsilon/2)| \). Thus, we have that

\[
\frac{1}{n} \sum_{j=\min\{j_1, j_2\}+1}^{\max\{j_1, j_2\}} |\log(\sigma_j(M_K - zI))| \leq 2 |\log(\varepsilon/2)| \nu_{M_2 - zI}([0, \varepsilon]).
\]

Note that the function \( \log(\cdot) \) is \( \left(\frac{2}{n}\right)\)-Lipschitz continuous on \((\varepsilon/2, D]\), and so bounding the sums in (2.4) and (2.5), we see that

\[
\left| \int_{\varepsilon/2}^{D} \log(x) d\nu_{M_1 - zI}(x) - \int_{\varepsilon/2}^{D} \log(x) d\nu_{M_2 - zI}(x) \right| \leq 2 |\log(\varepsilon/2)| \nu_{M_2 - zI}([0, \varepsilon]) + \frac{2}{n\varepsilon} \sum_{j=1}^{n} |\sigma_j(M_1 - zI) - \sigma_j(M_2 - zI)|.
\]

Applying Weyl’s perturbation theorem for singular values (Theorem 1.17), we see that

\[
\frac{1}{n} \sum_{j=1}^{n} |\sigma_j(M_1 - zI) - \sigma_j(M_2 - zI)| \leq ||M_1 - M_2||.
\]
Combining the last two inequalities above with (2.2) gives us a final bound of
\[
|\mathcal{L}_{M_1}(z) - \mathcal{L}_{M_2}(z)| \leq (6|\log(\varepsilon/2)| + 4|\log \sigma_{\text{min}}|) \nu_{M_2-zI}([0,\varepsilon]) + \frac{2}{\varepsilon} \|M_1 - M_2\| \tag{2.6}
\]
which completes the proof.

2.2. \textbf{Proof of Theorem 1.5.} We now turn to the proof of Theorem 1.5. Since
\[
\mathcal{L}_{M_i}(z) = \int_0^{\infty} \log(x) d\nu_{M_i-zI}(x) = \int_{\sigma_{\text{max}}}^{\sigma_{\text{min}}} \log(x) d\nu_{M_i-zI}(x)
\]
for \(i = 1, 2\), it follows from Lemma 2.1 that
\[
|\mathcal{L}_{M_1}(z) - \mathcal{L}_{M_2}(z)| \leq 2(\|\log \sigma_{\text{min}}\| + \|\log \sigma_{\text{max}}\|) \|\nu_{M_1-zI} - \nu_{M_2-zI}\|_{[\sigma_{\text{min}}, \sigma_{\text{max}}]},
\]
where
\[
\|\mu - \nu\|_{[a,b]} = \sup_{x \in [a,b]} |\mu([a,x]) - \nu([a,x])|
\]
for any real numbers \(a \leq b\) and any two probability measures \(\mu\) and \(\nu\).

It remains to show
\[
\|\nu'_{M_1-zI} - \nu'_{M_2-zI}\|_{[\sigma_{\text{min}}, \sigma_{\text{max}}]} \leq \frac{1}{n} \text{rank}(M_1 - M_2). \tag{2.7}
\]
To this end, let \(\nu'_{M_i-zI}\) be the empirical measure constructed from the squared singular values of \(M_i - zI\) for \(i = 1, 2\), i.e.,
\[
\nu'_{M_i-zI} = \frac{1}{n} \sum_{j=1}^{n} \delta_{s_j^2(M_i-zI)}, \quad i = 1, 2.
\]
Then for any \(x \geq 0\),
\[
\nu'_{M_i-zI}((- \infty, x^2]) = \nu_{M_i-zI}((- \infty, x])
\]
for \(i = 1, 2\). Thus, by definition of \(\sigma_{\text{min}}\) and \(\sigma_{\text{max}}\), we have
\[
\|\nu_{M_1-zI} - \nu_{M_2-zI}\|_{[\sigma_{\text{min}}, \sigma_{\text{max}}]} = \sup_{x \geq 0} |\nu_{M_1-zI}((- \infty, x]) - \nu_{M_2-zI}((- \infty, x])|
\]
\[
= \sup_{x \geq 0} |\nu'_{M_1-zI}((- \infty, x^2]) - \nu'_{M_2-zI}((- \infty, x^2])|
\]
\[
\leq \sup_{x \in \mathbb{R}} |\nu'_{M_1-zI}((- \infty, x]) - \nu'_{M_2-zI}((- \infty, x])|.
\]
It follows from Theorem A.44 in [4] that
\[
\sup_{x \in \mathbb{R}} |\nu'_{M_1-zI}((- \infty, x]) - \nu'_{M_2-zI}((- \infty, x])| \leq \frac{1}{n} \text{rank}(M_1 - M_2),
\]
which when combined with the bounds above yields (2.7). The proof of the theorem is complete.

2.3. \textbf{Proof of Theorem 1.2.} We conclude this section with the proof of Theorem 1.2. Define
\[
f_i(z) := \frac{1}{n} \log |\det(zI - M_i)| \tag{2.8}
\]
for \(i = 1, 2\). For \(r > 0\), we define
\[
B(r) := \{z \in \mathbb{C} : |z| < r\}
\]
to be the ball of radius \(r\) centered at the origin. We will let \(d^2z\) denote integration with respect to the Lebesgue measure on \(\mathbb{C}\), i.e., \(\int g(z) d^2z\). We will need the following results.
Lemma 2.2. Under the assumptions of Theorem 1.2, there exists a constant \( C_{2.2} > 0 \) (depending only on \( \varphi \)) so that
\[
\max_{i=1,2} \int_{C} |\Delta \varphi(z)|^2 |f_i(z)|^2 \, d^2 z \leq C_{2.2} \log^2 T
\]
with probability at least \( 1 - \varepsilon \), where we make take the constant \( C_{2.2} \) to be \( \|\Delta \varphi\|^2_{\infty} D_{2.2} \) where \( D_{2.2} \) is a constant depending only on the diameter and distance from the origin of \( \text{supp}(\Delta \varphi) \) (see (2.9)).

Proof. Since the result is trivially true when \( \varphi \equiv 0 \), we assume \( \varphi \) is nonzero. By bounding \( |\Delta \varphi(z)|^2 \) using the \( L^\infty \)-norm \( \|\Delta \varphi\|^2_{\infty} \), it suffices to show that
\[
\max_{i=1,2} \int_{\text{supp}(\Delta \varphi)} |f_i(z)|^2 \, d^2 z \leq D_{2.2} \log^2 T
\]
with probability at least \( 1 - \varepsilon \). We will prove this bound on the event where \( \|M_1\| + \|M_2\| \leq T \) (which by supposition holds with probability at least \( 1 - \varepsilon \)).

By the Cauchy–Schwarz inequality,
\[
|f_i(z)|^2 = \frac{1}{n^2} \left( \sum_{j=1}^{n} \log |z - \lambda_j(M_i)| \right)^2 \leq \frac{1}{n} \sum_{j=1}^{n} \log^2 |z - \lambda_j(M_i)|,
\]
and hence
\[
\max_{i=1,2} \int_{\text{supp}(\Delta \varphi)} |f_i(z)|^2 \, d^2 z \leq \max_{i=1,2} \max_{1 \leq j \leq n} \int_{\text{supp}(\Delta \varphi)} \log^2 |z - \lambda_j(M_i)| \, d^2 z
\]
\[
\leq \sup_{0 < |\lambda| \leq T} \int_{\text{supp}(\Delta \varphi)} \log^2 |z - \lambda| \, d^2 z
\]
since \( \|M_1\| + \|M_2\| \leq T \).

Let \( D \) be the diameter of \( \text{supp}(\Delta \varphi) \). We will bound the integral \( \int_{\text{supp}(\Delta \varphi)} \log^2 |z - \lambda| \, d^2 z \) for an arbitrary \( \lambda \) satisfying \( |\lambda| \leq T \) by considering two cases.

For the first case, assume that there is some element of the set \( \text{supp}(\Delta \varphi) - \lambda \) that is within distance 1 of the origin. Then we have that the set \( \text{supp}(\Delta \varphi) - \lambda \) is contained in \( B(D+1) \), so that
\[
\int_{\text{supp}(\Delta \varphi)} \log^2 |z - \lambda| \, d^2 z = \int_{B(D+1)} \log^2 |z - \lambda| \, d^2 z \leq \int_{B(D+1,0)} \log^2 |z| \, d^2 z.
\]
We can now convert to polar coordinates and use integration by parts twice to get that
\[
\int_{B(D+1,0)} \log^2 |z| \, d^2 z = \int_{0}^{2\pi} \int_{0}^{D+1} \log^2(r) \, dr \, d\theta = 2\pi \int_{0}^{D+1} \log^2(r) \, dr
\]
\[
= 2\pi \left( \frac{(D+1)^2}{2} \log^2(D+1) - \int_{0}^{D+1} r \log(r) \, dr \right)
\]
\[
= \pi(D+1)^2 \left( \log^2(D+1) - \log(D+1) + \frac{1}{2} \right).
\]

For second case, assume conversely that all elements of the set \( \text{supp}(\Delta \varphi) - \lambda \) are distance greater than 1 from the origin. Then, letting \( \mathcal{D} \) be the distance from \( \text{supp}(\Delta \varphi) \) to the origin, we have
\[
\int_{\text{supp}(\Delta \varphi)} \log^2 |z - \lambda| \, d^2 z = \int_{\text{supp}(\Delta \varphi) - \lambda} \log^2 |z| \, d^2 z
\]
\[
\leq \int_{\text{supp}(\Delta \varphi) - \lambda} \log^2 |\mathcal{D} + D + T| \, d^2 z
\]
\[
\leq \pi D^2 \log^2(T + \mathcal{D} + D).
\]
Combining the bounds from the two cases, we have shown that for any \( \lambda \) satisfying \(|\lambda| \leq T\) we have
\[
\int_{\text{supp}(\Delta \varphi)} \log^2 |z - \lambda| d^2 z \leq \log^2 (T + \Delta + D) \pi (D + 1)^2 \max \left\{ \log^2 (D + 1) - \log (D + 1) + \frac{1}{2}, 1 \right\}.
\]

To complete the proof of the lemma, we note that \( \log^2 (T + \Delta + D) \leq \log^2 (T) \log^2 (2 \Delta + D) \) (since \( \log (T + \Delta + D) / \log (T) \) is decreasing in \( T \) on the interval \( T \in [2, \infty) \) when \( \Delta + D \geq 0 \)). Thus, for the statement of the lemma, we can set the constant \( C_{2.2} = \| \Delta \varphi(z) \|_\infty^{2} D_{2.2} \), where we define the constant \( D_{2.2} \) by
\[
D_{2.2} = \pi (D + 1)^2 \left( \frac{\log^2 (2 \Delta + D)}{\log^2} \right) \max \left\{ \log^2 (D + 1) - \log (D + 1) + \frac{1}{2}, 1 \right\},
\]
which we note depends only on the diameter of \( \text{supp}(\Delta \varphi) \) and its distance from the origin. \( \square \)

**Lemma 2.3** (Monte Carlo sampling lemma; Lemma 6.1 from [66]). Let \((X, \rho)\) be a probability space, and let \( F : X \to \mathbb{C} \) be a square integrable function. Let \( m \geq 1 \), let \( Z_1, \ldots, Z_m \) be drawn independently at random from \( X \) with distribution \( \rho \), and let \( S \) be the empirical average
\[
S := \frac{1}{m} (F(Z_1) + \cdots + F(Z_m)).
\]
Then, for any \( \delta > 0 \), one has the bound
\[
\left| S - \int_X F d\rho \right| \leq \frac{1}{\sqrt{m}\delta} \left( \int_X \left| F - \int_X F d\rho \right|^2 d\rho \right)^{1/2}
\]
with probability at least \( 1 - \delta \).

**Lemma 2.4.** Assume the conditions of Theorem 1.2 and define \( f_i \) as in (2.8). Define
\[
F(z) := \Delta \varphi(z)(f_1(z) - f_2(z)).
\]
Let \( Z_1, \ldots, Z_m \) be uniformly distributed on \( K := \text{supp}(\Delta \varphi) \), independent of \( M_1 \) and \( M_2 \). Then
\[
\left| \int_K F(z) d^2 z - \frac{|K|}{m} \sum_{j=1}^{m} F(Z_j) \right| \leq 4\sqrt{|K| C_{2.2} \log T m \sqrt{\varepsilon}}
\]
with probability at least \( 1 - (m + 1)\varepsilon \), where \( C_{2.2} \) is the constant from Lemma 2.2. Here, \( |K| \) denotes the Lebesgue measure of \( K \).

**Proof.** Assume \( \varphi \) is nonzero as the result is trivial when \( \varphi \equiv 0 \). Let \( \rho \) denote the uniform probability distribution on \( K \). By Lemma 2.2, it follows that
\[
\int_K |F|^2 d\rho = \frac{1}{|K|} \int_K |\Delta \varphi|^2 \left| (f_1(z) - f_2(z)) \right|^2 d^2 z
\]
\[
\leq \frac{2}{|K|} \int_K |\Delta \varphi|^2 \left| (f_1(z)) \right|^2 + \left| f_2(z) \right|^2 d^2 z \quad \text{(Cauchy–Schwarz)}
\]
\[
\leq \frac{4C_{2.2}}{|K|} \log^2 T
\]
with probability at least \( 1 - \varepsilon \). Here the constant \( C_{2.2} > 0 \) is from Lemma 2.2. Take
\[
S := \frac{1}{m} \sum_{j=1}^{m} F(Z_j).
\]
By Lemma 2.3, with probability at least $1 - m\varepsilon$ (with respect to $Z_1, \ldots, Z_m$),
\[
\left| S - \int F \, d\rho \right| \leq \frac{2}{m\sqrt{\varepsilon}} \left( \int |F|^2 \, d\rho \right)^{1/2}.
\] (2.12)

Combining (2.12) with (2.11), we find that, with probability at least $1 - (m + 1)\varepsilon$,
\[
\left| S - \int F \, d\rho \right| \leq \frac{4\sqrt{C_{2,2}}}{m\sqrt{\varepsilon}|K|} \log T.
\] (2.13)

Since
\[
\int F \, d\rho = \frac{1}{|K|} \int_K F(z) \, d^2z = \frac{1}{|K|} \int_\mathbb{C} F(z) \, d^2z,
\]
the conclusion follows by scaling (2.13) by $|K|$. □

We are now in a position to prove Theorem 1.2.

**Proof of Theorem 1.2.** By Green's formula (see, for instance, Section 2.4.1 in [42]), we can write
\[
\frac{1}{n} \sum_{j=1}^{n} \varphi(\lambda_j(M_i)) = \frac{1}{2\pi} \int_\mathbb{C} \Delta \varphi(z) f_i(z) \, d^2z
\]
for $i = 1, 2$. Thus,
\[
\frac{1}{n} \sum_{j=1}^{n} \varphi(\lambda_j(M_1)) - \frac{1}{n} \sum_{j=1}^{n} \varphi(\lambda_j(M_2)) = \frac{1}{2\pi} \int_\mathbb{C} F(z) \, d^2z,
\]
where $F$ is defined in (2.10). By Lemma 2.4, it follows that
\[
\left| \frac{1}{n} \sum_{j=1}^{n} \varphi(\lambda_j(M_1)) - \frac{1}{n} \sum_{j=1}^{n} \varphi(\lambda_j(M_2)) \right| \leq \frac{|K||S|}{2\pi} + \frac{4\sqrt{|K|C_{2,2}}}{2\pi m\sqrt{\varepsilon}} \log T
\] (2.14)
with probability at least $1 - (m + 1)\varepsilon$, where
\[
S := \frac{1}{m} \sum_{j=1}^{m} F(Z_j)
\]
and $Z_1, \ldots, Z_m$ are iid random variables, independent of $M_1$ and $M_2$, uniformly distributed on $K := \text{supp}(\Delta \varphi)$. Here, $|K|$ denotes the Lebesgue measure of $K$.

By (1.6) and the union bound,
\[
\sup_{1 \leq j \leq m} |F(Z_j)| \leq \sup_{1 \leq j \leq m} |\Delta \varphi(Z_j)||f_1(Z_j) - f_2(Z_j)|
\]
\[
\leq \sup_{1 \leq j \leq m} |\Delta \varphi(Z_j)| \left| \frac{1}{n} \log |\det(Z_jI - M_1)| - \frac{1}{n} \log |\det(Z_jI - M_2)| \right|
\]
\[
\leq \|\Delta \varphi\|_{\infty} \eta
\]
with probability at least $1 - m\varepsilon$, where $\|\Delta \varphi\|_{\infty}$ is the $L^\infty$-norm of $\Delta \varphi$. On this same event, $|S| \leq \|\Delta \varphi\|_{\infty} \eta$. Therefore, combining the bounds above with (2.14), we find
3.1. **Circulant matrices.** A circulant matrix is an $n \times n$ matrix of the form

$$
C = 
\begin{bmatrix}
    c_0 & c_{n-1} & \cdots & c_2 & c_1 \\
    c_1 & c_0 & c_{n-1} & \cdots & c_2 \\
    \vdots & c_1 & c_0 & \ddots & \vdots \\
    c_{n-2} & \ddots & \cdots & c_{n-1} \\
    c_{n-1} & c_{n-2} & \cdots & c_1 & c_0
\end{bmatrix},
$$

where $c_0, c_1, \ldots, c_{n-1} \in \mathbb{C}$. In this case, we say $c_0, c_1, \ldots, c_n$ generate the matrix $C$.

**Lemma 3.1** (Properties of circulant matrices). Let $C$ be the $n \times n$ circulant matrix in (3.1) generated by $c_0, c_1, \ldots, c_{n-1} \in \mathbb{C}$.

(i) The eigenvalues of $C$ are given by

$$
\lambda_j = c_0 + c_{n-1} \omega_n^j + c_{n-2} \omega_n^{2j} + \cdots + c_1 \omega_n^{(n-1)j}, \quad j = 0, 1, \ldots, n - 1, \tag{3.2}
$$

where $\omega_n$ is a primitive $n$-th root of unity as in (1.21). The corresponding normalized eigenvectors are given by

$$
v_j := \frac{1}{\sqrt{n}}(1, \omega_n^j, \omega_n^{2j}, \ldots, \omega_n^{(n-1)j})^T, \quad j = 0, 1, \ldots, n - 1.
$$

(ii) $C$ is a normal matrix.

(iii) The singular values of $C$ are given by

$$
\sigma_j := |\lambda_j|, \quad j = 0, 1, \ldots, n - 1,
$$

where $\lambda_0, \lambda_1, \ldots, \lambda_{n-1}$ are the eigenvalues of $C$ given in (3.2).

3. **Convergence of the empirical spectral measure**

This section is devoted to the proof of Theorem 1.7. We begin with some preliminary details about circulant matrices.
Proof. A simple computation shows that $C$ has eigenvalue $\lambda_j$ with corresponding eigenvector $v_j$ for $j = 0, 1, \ldots, n - 1$. In addition, since the eigenvectors $v_0, \ldots, v_{n-1}$ are orthonormal, it follows from Theorem 2.5.3 in \cite{41} that $C$ is normal. Part (iii) follows since the singular values of any normal matrix are given by the absolute values of its eigenvalues; see 2.6.P15 in \cite{41}. \hfill \Box

### 3.2. Proof of Theorem 1.7.

We begin the proof of Theorem 1.7 with a number of lemmata.

**Lemma 3.2.** Let $\{a_j\}_{j \in \mathbb{Z}}$ be a sequence of complex numbers, indexed by the integers, so that

$$\sum_{j \in \mathbb{Z}} |ja_j| < \infty. \quad (3.3)$$

For each $n \geq 1$, let $k_n$ be a non-negative integer, and define a sequence of functions $f_n : S^1 \to \mathbb{C}$ by

$$f_n(\omega) = \sum_{|j| \leq k_n} a_j w^j.$$

In addition, let the function $f : S^1 \to \mathbb{C}$ be given by

$$f(\omega) = \sum_{j \in \mathbb{Z}} a_j w^j. \quad (3.4)$$

Then the following properties hold:

(i) the image $f(S^1)$ has Lebesgue measure zero in $\mathbb{C}$;

(ii) the set of images $\bigcup_{n=1}^{\infty} f_n(S^1)$ has Lebesgue measure zero in $\mathbb{C}$; and

(iii) if $k_n \to \infty$ as $n \to \infty$, then, for each $z \in \mathbb{C} \setminus f(S^1)$, there exists $\varepsilon > 0$ and $N \in \mathbb{N}$ so that $z$ is at least distance $\varepsilon$ from $\bigcup_{n=N}^{\infty} f_n(S^1)$.

**Proof.** Under condition (3.3), it follows from Theorem 6.28 in \cite{46} that the function $t \mapsto f(e^{\sqrt{-1}t})$ is differentiable and

$$\frac{d}{dt} f(e^{\sqrt{-1}t}) = \sum_{j \in \mathbb{Z}} j \sqrt{-1} a_j e^{\sqrt{-1}t} j.$$

In particular, the derivative satisfies the bound

$$\left| \frac{d}{dt} f(e^{\sqrt{-1}t}) \right| \leq \sum_{j \in \mathbb{Z}} |ja_j| < \infty$$

for all $t \in \mathbb{R}$. By Lemma 7.25 in \cite{57}, it follows that $f(S^1)$ has Lebesgue measure zero in $\mathbb{C}$.

Conclusion (ii) now follows from (i). Indeed, by taking $a_j = 0$ for all $|j| > k$, conclusion (i) also applies to functions of the form

$$f(\omega) = \sum_{|j| \leq k} a_j \omega^j$$

for any integer $k \geq 0$. Since the countable union of sets of Lebesgue measure zero has Lebesgue measure zero, the proof of (ii) is complete.

In order to prove (iii), let $z \in \mathbb{C} \setminus f(S^1)$. Since $f$ is differentiable, $f$ is continuous, and hence $f(S^1)$ is compact. Thus, there exists $\varepsilon > 0$ with the property that $z$ is distance $2\varepsilon$ from $f(S^1)$. Since $k_n \to \infty$, it follows that there exists $N \in \mathbb{N}$ so that

$$\sup_{\omega \in S^1} |f_n(\omega) - f(\omega)| \leq \sum_{|j| > k_n} |a_j| < \varepsilon$$

for all $n \geq N$. This implies that $z$ is at most distance $\varepsilon$ from $f_n(S^1)$ for each $n \geq N$. Indeed, if there exists $n \geq N$ and $\omega \in S^1$ so that $|z - f_n(\omega)| < \varepsilon$, then $|z - f(\omega)| < 2\varepsilon$, a contradiction. \hfill \Box

Recall the definition of the $L^1$-Wasserstein distance $W_1(\mu, \nu)$ between $\mu$ and $\nu$ given in (1.18).
Lemma 3.3. Let $k$ be a non-negative integer, and define $f : S^1 \to \mathbb{C}$ by

$$f(\omega) = \sum_{|j| \leq k} a_j \omega^j$$

for some complex numbers $a_j$ with $-k \leq j \leq k$. Let $\mu$ be distribution of $f(U)$, where $U$ is a random variable uniformly distributed on $S^1$, and let $\mu_n$ be the probability measure given by

$$\mu_n = \frac{1}{n} \sum_{i=0}^{n-1} \delta_{f(\omega_i)},$$

where $\omega_n$ is a primitive $n$-th root of unity as in (1.21). Then, there exists a constant $C > 0$ (depending only on $k$, $f$, and $\{a_j\}_{|j| \leq k}$) so that

$$W_1(\mu_n, \mu) \leq \frac{C}{n}.$$  \hfill (3.5)

Proof. Let $\theta$ be a random variable, uniform on $[0, 2\pi)$. Define another random variable $\phi_n$ as follows. Take $\phi_n = 2\pi i - \frac{1}{n}$ whenever $2\pi i - \frac{1}{n} \leq \theta < 2\pi i$ for some integer $1 \leq i \leq n$. Then $U := e^{\sqrt{-1}\theta}$ is uniformly distributed on $S^1$ and, taking $\psi_n := e^{\sqrt{-1}\phi_n}$, we see that $f(\psi_n)$ has distribution $\mu_n$. By construction, it follows that

$$|U - \psi_n| \leq \frac{2\pi}{n}$$  \hfill (3.6)

almost surely. We then see that

$$|f(U) - f(\psi_n)| \leq \sum_{|j| \leq k} |a_j||U^j - \psi_n^j| \leq C|U - \psi_n|$$

for some constant $C > 0$ depending only on $k$, $f$, and $\{a_j\}_{|j| \leq k}$. In view of (3.6), we conclude that

$$\mathbb{E}[f(U) - f(\psi_n)] \leq \frac{2\pi C}{n},$$

which in the language of measures (and after adjusting the constant $C$) gives (3.5). \hfill \Box

Lemma 3.4. Let $\{a_j\}_{j \in \mathbb{Z}}$ be a sequence of complex numbers, indexed by the integers, satisfying

$$\sum_{j \in \mathbb{Z}} |a_j| < \infty,$$  \hfill (3.7)

and let $k_n$ be a non-negative integer sequence tending to infinity. Define $f, f_n : S^1 \to \mathbb{C}$ by

$$f_n(\omega) = \sum_{|j| \leq k_n} a_j \omega^j \quad \text{and} \quad f(\omega) = \sum_{j \in \mathbb{Z}} a_j \omega^j.$$

Let $\mu_n$ to be the probability measure on $\mathbb{C}$ given by

$$\mu_n = \frac{1}{n} \sum_{i=0}^{n-1} \delta_{f_n(\omega_i)},$$

where $\omega_n$ is a primitive $n$-th root of unity as in (1.21). Let $\mu$ be the distribution of $f(U)$, where $U$ is a random variable, uniform on $S^1$. Then $\mu_n \to \mu$ weakly as $n \to \infty$.

Proof. Let $\psi_n$ be a random variable uniformly distributed on $1, \omega_n, \omega_n^2, \ldots, \omega_n^{n-1}$. Then $f_n(\psi_n)$ has distribution $\mu_n$. Since $k_n$ tends to infinity, it follows from (3.7) that

$$\sup_{\omega \in S^1} |f_n(\omega) - f(\omega)| \leq \sum_{|j| > k_n} |a_j| = o(1).$$
This implies that
\[ |f_n(\psi_n) - f(\psi_n)| = o(1) \]
almost surely.

It remains to show that \( f(\psi_n) \) converges in distribution to \( f(U) \). As a consequence of Lemma 3.3 (by taking the identity function for \( f \)), it follows that \( \psi_n \to U \) in distribution as \( n \to \infty \) (since convergence in Wasserstein distance implies convergence in distribution; see [25]). In addition, under condition (3.7), it follows from Theorem 6.27 in [46] that \( f \) is continuous. Thus, by the continuous mapping theorem it follows that \( f(\psi_n) \to f(U) \) in distribution as \( n \to \infty \). Combining the above, we see that \( f_n(\psi_n) \to f(U) \) in distribution, which in the language of measures means that \( \mu_n \to \mu \) weakly as \( n \to \infty \). □

**Lemma 3.5.** Let \( \{a_j\}_{j \in \mathbb{Z}} \) be a sequence of complex numbers, indexed by the integers, and let \( k < \frac{n}{2} \) be a non-negative integer. Define \( f : \mathbb{S}^1 \to \mathbb{C} \) by
\[
|f(\omega)| = \sum_{|j| \leq k} a_j w^j.
\]
Let \( A \) be the \( n \times n \) Toeplitz matrix with symbol \( \{a_j\}_{j \in \mathbb{Z}} \) truncated at \( k \). Then there exists an \( n \times n \) matrix \( A' \) with rank at most \( 2k \) so that \( A + A' \) is circulant with eigenvalues given by \( f(1), f(\omega_n), f(\omega_n^2), \ldots, f(\omega_n^{n-1}) \), where \( \omega_n \) is a primitive \( n \)-th root of unity as in (1.21). Moreover, \( A' \) can be chosen so that the Frobenius norm \( \|A'\|_2 \) satisfies
\[
\|A'\|_2^2 \leq n \sum_{|j| \leq k} |a_j|^2.
\] (3.8)

**Proof.** We define \( A' \) so that \( C = A + A' \), where \( C \) is the circulant matrix given in (3.1) with \( c_j = a_j \) for \( 0 \leq j \leq k \), \( c_{n-j} = a_{-j} \) for \( 1 \leq j \leq k \), and \( c_j = 0 \) otherwise. Then \( A' \) has at most \( 2k \) rows that are non-zero, and hence \( \text{rank}(A') \leq 2k \). In view of Lemma 3.1, the eigenvalues of \( C^T \) (and hence of \( C \)) are given by \( f(1), f(\omega_n), f(\omega_n^2), \ldots, f(\omega_n^{n-1}) \). The bound in (3.8) follows from the construction above. □

**Proof of Theorem 1.7.** If \( k_n \) converges to an integer \( k \), then it must be the case that \( k_n = k \) for all sufficiently large \( n \). In this case, we may assume that \( a_j = 0 \) for all \( |j| > k \). Therefore, without loss of generality, it suffices to assume that \( k_n \) tends to infinity and satisfies (1.14).

We begin with some notation. Define the functions \( f_n, f : \mathbb{S}^1 \to \mathbb{C} \) by
\[
f_n(\omega) = \sum_{|j| \leq k_n} a_j w^j \quad \text{and} \quad f(\omega) = \sum_{j \in \mathbb{Z}} a_j \omega^j.
\]
We recall that \( \|A\|_2 \) denotes the Frobenius norm of \( A \). By definition, it follows that
\[
\frac{1}{n} \|A\|_2^2 \leq \sum_{j \in \mathbb{Z}} |a_j|^2,
\] (3.9)
and so by (1.13)
\[
\|A\|_2^2 \leq n \sum_{j \in \mathbb{Z}} |a_j|^2 = O(n).
\] (3.10)
In addition, by Weyl's perturbation theorem (see Theorem 1.17) and (1.15)
\[
\frac{1}{n} \|A + n^{-\alpha-\gamma} E\|_2^2 \leq 2 \sum_{j \in \mathbb{Z}} |a_j|^2 + 1
\] (3.11)
with probability $1 - o(1)$. Let $A'$ be the matrix from Lemma 3.5 so that $C := A + A'$ is circulant, $\text{rank}(A') \leq 2k_n$, and
\[
\frac{1}{n} \|A'\|_2^2 \leq \sum_{j \in \mathbb{Z}} |a_j|^2. \tag{3.12}
\]
In addition, it follows from Lemma 3.5 that $f_n(1), f_n(\omega_n), f_n(\omega_n^2), \ldots, f_n(\omega_n^{n-1})$ are the eigenvalues of $C$, where $\omega_n$ is a primitive $n$-th root of unity as in (1.21). In view of (3.9) and (3.12), we have
\[
\frac{1}{n} \|C\|_2^2 \leq 4 \sum_{j \in \mathbb{Z}} |a_j|^2, \tag{3.13}
\]
and hence
\[
\|C\|_2^2 \leq 4n \sum_{j \in \mathbb{Z}} |a_j|^2 = O(n). \tag{3.14}
\]

Our goal is to apply the replacement principle, Theorem 1.1, to show that
\[
\mu_{A + n^{-\alpha - \gamma}E} - \mu_C \rightarrow 0
\]
weakly in probability as $n \to \infty$. This would complete the proof since Lemma 3.4 implies that $\mu_C \to \mu$ weakly as $n \to \infty$. The Frobenius norm condition of Theorem 1.1 follows immediately from (1.13), (3.11), and (3.13), so it remains to compare the logarithmic determinants of $A + n^{-\alpha - \gamma}E$ and $C$.

Fix $z \in \mathbb{C}$ with $z \notin f(S^1)$ and such that (1.16) holds. The set of $z \in \mathbb{C}$ which fail to satisfy these properties has Lebesgue measure zero by Lemma 3.2 and the assumptions on $E$. Lemma 3.2 and Lemma 3.1 imply that there exists $\varepsilon' > 0$ so
\[
\sigma_{\min}(C - zI) \geq \varepsilon'
\]
for all sufficiently large $n$. Thus, by Weyl’s perturbation theorem (see Theorem 1.17) and (1.15),
\[
\sigma_{\min}(C + n^{-\alpha - \gamma}E - zI) \geq \frac{\varepsilon'}{2}, \tag{3.16}
\]
with probability $1 - o(1)$. Applying Theorem 1.5 (using (1.15), (3.10) and (3.14) to bound the norms and (1.16) and (3.16) to bound the smallest singular values), we see that
\[
|\mathcal{L}_{A + n^{-\alpha - \gamma}E}(z) - \mathcal{L}_{C + n^{-\alpha - \gamma}E}(z)| = O\left(\frac{\text{rank}(A') \log n}{n}\right)
\]
with probability $1 - o(1)$. In view of (1.14) and the fact that $\text{rank}(A') \leq 2k_n$ we obtain
\[
|\mathcal{L}_{A + n^{-\alpha - \gamma}E}(z) - \mathcal{L}_{C + n^{-\alpha - \gamma}E}(z)| \rightarrow 0
\]
in probability.

Applying Theorem 1.4 (using (3.15) and (3.16) to bound the smallest singular values, (1.15) to bound the norm, and taking $\varepsilon$ in Theorem 1.4 to be $\varepsilon'/4$), we obtain
\[
|\mathcal{L}_{C + n^{-\gamma - \alpha}E}(z) - \mathcal{L}_C(z)| = O(n^{-\gamma})
\]
with probability $1 - o(1)$. Combining the bounds above, we conclude that
\[
|\mathcal{L}_{A + n^{-\alpha - \gamma}E}(z) - \mathcal{L}_C(z)| \rightarrow 0
\]
in probability as $n \to \infty$. This confirms the last condition in Theorem 1.1, and hence the proof of the theorem is complete. \(\square\)

We end this section with a conjecture that is suggested by the proof of Theorem 1.7.
Conjecture 3.6. Let $A$ satisfy the conditions of Theorem 1.7, and let $A'$ be the matrix from Lemma 3.5. Then, we conjecture that the empirical spectral measure $\mu_{A + n^{-\gamma}A'}$ converges weakly in probability to $\mu$ where $\mu$ is the distribution of $\sum_{|j| \leq k} a_j U^j$ where $U$ is a random variable uniformly distributed on $S^1$.

Numerical evidence supports Conjecture 3.6. In fact, numerical evidence suggests that the eigenvalues of $A + n^{-\gamma}A'$ better approximate the eigenvalues of $A + n^{-\alpha-\gamma}E$ than $A + A'$ (which we use in the proof Theorem 1.7).

4. Rate of convergence

The proof of Theorem 1.10 is based on the following result.

Theorem 4.1. Let $k \geq 0$ be a fixed integer, and let $\{a_j\}_{j \in \mathbb{Z}}$ be a sequence of complex numbers indexed by the integers. Let $A$ be the $n \times n$ Toeplitz matrix with symbol $\{a_j\}_{j \in \mathbb{Z}}$ truncated at $k$. Define the function $f : S^1 \rightarrow \mathbb{C}$ as

$$f(\omega) = \sum_{|j| \leq k} a_j \omega^j,$$

and assume there exists $\varepsilon_0 > 0$ and $c_0 \geq 1$ so that for any $0 < \varepsilon' \leq \varepsilon_0$,

$$\sup_{z \in \mathbb{C}} |\{0 \leq j \leq n - 1 : f(\omega^n_j) \in \mathfrak{R}(z, n^{-\varepsilon'}, n^{-c_0 \varepsilon'}) \cup \mathfrak{R}(z, n^{-\varepsilon'}, n^{-\varepsilon})\}| = O_{\varepsilon'}(n^{-3\varepsilon'}), \quad (4.1)$$

where $\omega_n$ is a primitive $n$-th root of unity as in (1.21). Let $\gamma, \delta > 0$, and let $E$ be an $n \times n$ random matrix so that

(i) there exists $M > 0$ and $\alpha \geq 0$ so that

$$\|E\| \leq Mn^\alpha$$

with probability $1 - O(n^{-\delta})$;

(ii) there exists $\kappa > 0$ so that

$$\sup_{z \in \mathbb{C}, |z| \leq \beta} \mathbb{P}(\sigma_{\min}(A + n^{-\alpha-\gamma}E - zI) \leq n^{-\kappa}) = O(n^{-\delta}) \quad (4.2)$$

for $\beta := \sqrt{2} \left( \sum_{|j| \leq k} |a_j| + M + 1 \right)$.

Then there exists $C, \varepsilon > 0$ (depending on $k, f, \{a_j\}_{|j| \leq k}, \varepsilon_0, c_0, \gamma, \delta$, and the constants from assumptions (i), (ii), and (4.1)) so that

$$W_1(\mu_{A + n^{-\alpha-\gamma}E}, \mu_n) \leq Cn^{-\varepsilon}$$

with probability $1 - o(1)$, where $\mu_n$ is the deterministic probability measure

$$\mu_n := \frac{1}{n} \sum_{j=0}^{n-1} \delta_{f(\omega^n_j)}.$$

Theorem 1.10 now follows immediate from Theorem 4.1, Lemma 3.3, and the triangle inequality for the Wasserstein metric.

The rest of this section is devoted to the proof of Theorem 4.1. Assume the setup and notation of Theorem 4.1. We allow the implicit constants in our asymptotic notation (such as $O(\cdot)$ and $\ll$) to depend on the parameters and constants of Theorem 4.1 (such as $k, f, \{a_j\}_{|j| \leq k}, \varepsilon_0, c_0, \gamma, \delta$, and the constants from assumptions (i), (ii), and (4.1)) without denoting this dependence.

Define the event

$$\Omega := \{\|E\| \leq Mn^\alpha\},$$
which by supposition holds with probability $1 - O(n^{-\delta})$. On $\Omega$, for $n$ sufficiently large,

$$\|A + n^{-\alpha - \gamma}E\| < \|A\| + M \leq \sum_{|j| \leq k} |a_j| + M, \quad (4.3)$$

where the bound for $\|A\|$ follows from Proposition B.2 in Appendix B.

By Lemma 3.5, there exists an $n \times n$ deterministic matrix $A'$ with rank at most $2k$ so that $A + A'$ is circulant with eigenvalues given by $f(1), f(\omega_n), f(\omega_n^2), \ldots, f(\omega_n^{n-1})$. In particular, this means that the empirical spectral measure of $A + A'$ is precisely $\mu_n$:

$$\mu_n = \mu_{A + A'}. \quad (4.4)$$

From Lemma 3.1, we see that the singular values of $A'$ are then $|f(1)|, |f(\omega_n)|, |f(\omega_n^2)|, \ldots, |f(\omega_n^{n-1})|$. This implies that

$$\|A + A'\| \leq \sup_{\omega \in S^1} |f(\omega)| \leq \sum_{|j| \leq k} |a_j|, \quad (4.5)$$

and hence

$$\|A + A' + n^{-\alpha - \gamma}E\| < \sum_{|j| \leq k} |a_j| + M \quad (4.6)$$

on the event $\Omega$.

Define the box $R$ in the complex plane by

$$R = \left\{ z \in \mathbb{C} : -\sum_{|j| \leq k} |a_j| - M - 1/2 \leq \text{Re}(z), \text{Im}(z) < \sum_{|j| \leq k} |a_j| + M + 1/2 \right\}.$$

Notice that

$$B \left(0, \sum_{|j| \leq k} |a_j| + M + 1/4 \right) \subset R \subset B(0, \beta). \quad (4.7)$$

We call a set $S$ in the complex plane of the form

$$S = \{ z \in \mathbb{C} : a \leq \text{Re}(z) < b, c \leq \text{Im}(z) < d \}$$

with $b - a = d - c > 0$ a square, and we say $b - a$ is the side length and $(a + b)/2 + \sqrt{1(c + d)/2}$ is the center of $S$. For example, $R$ is a square with center 0 and side length $2 \left( \sum_{|j| \leq k} |a_j| \right) + 2M + 1$.

Let $R_1, \ldots, R_L$ be a partition of $R$ into disjoint squares, all with equal side length of $\Theta(n^{-\alpha})$ for some $a > 0$ to be chosen later. In particular, we note that

$$\bigcup_{i=1}^{L} R_i = R.$$

We call a set $S$ in the complex plane of the form

$$S = \{ z \in \mathbb{C} : a \leq \text{Re}(z) < b, c \leq \text{Im}(z) < d \}$$

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with $b - a = d - c > 0$ a square, and we say $b - a$ is the side length and $(a + b)/2 + \sqrt{1(c + d)/2}$ is the center of $S$. For example, $R$ is a square with center 0 and side length $2 \left( \sum_{|j| \leq k} |a_j| \right) + 2M + 1$.

Let $R_1, \ldots, R_L$ be a partition of $R$ into disjoint squares, all with equal side length of $\Theta(n^{-\alpha})$ for some $a > 0$ to be chosen later. In particular, we note that

$$\bigcup_{i=1}^{L} R_i = R.$$

A volume argument shows that

$$L = O(n^{2a}). \quad (4.8)$$

For $1 \leq i \leq L$, we define $X_i$ to be the number of eigenvalues of $A + n^{-\alpha - \gamma}E$ in $R_i$, and set $Y_i$ to be the number of eigenvalues of $A + A'$ in $R_i$. We will need the following result.

**Lemma 4.2.** Under the assumptions of Theorem 4.1, for any sufficiently small $a, \varepsilon > 0$, there exists a constant $C > 0$ so that

$$\sup_{1 \leq i \leq L} |X_i - Y_i| \leq C \left( n^{1-\varepsilon+6c_0a} + n^{1-3a} \right) \quad (4.9)$$

with probability $1 - o(1)$. (Recall that the squares $R_i$ have side length $\Theta(n^{-\alpha})$ and $c_0$ is the constant from (4.1).) Here the sufficient smallness of $a$ and $\varepsilon$ depends on $k$, $\{a_j\}_{|j| \leq k}$, $f$, $\varepsilon_0$, $c_0$, $\gamma$, $\delta$, and
the constants from assumptions (i), (ii), and (4.1) in Theorem 4.1; the constant $C$ depends on $a$ and $\varepsilon$ as well as these other parameters.

Before proving Lemma 4.2, we first complete the proof of Theorem 4.1. In view of (4.4), it suffices to show that

$$W_1(\mu_{A+n^{-a-\gamma}E}, \mu_{A+A'}) \leq C n^{-\varepsilon}$$

with probability $1 - o(1)$. Recall that $\lambda_1(B), \ldots, \lambda_n(B)$ denote the eigenvalues of the $n \times n$ matrix $B$. For convenience, we fix an arbitrary ordering of the eigenvalues (e.g., first ordering by magnitude and then by argument). Since, for any permutation $\sigma : [n] \to [n]$, we have

$$W_1(\mu_{A+n^{-a-\gamma}E}, \mu_{A+A'}) \leq \frac{1}{n} \sum_{i=1}^{n} |\lambda_i(A + n^{-a-\gamma}E) - \lambda_{\sigma(i)}(A + A')|,$$  \hspace{1cm} (4.10)

it suffices to construct an advantageous permutation $\sigma$. To do so, we work on the event

$$F := \left\{ \sup_{1 \leq i \leq L} |X_i - Y_i| \leq C \left(n^{1-\varepsilon+6\varepsilon_0a} + n^{1-3a}\right) \right\} \cap \Omega.$$

By Lemma 4.2 and assumption (i) from Theorem 4.1, $F$ holds with probability $1 - o(1)$ for $\varepsilon, a > 0$ sufficiently small (in particular we will take $\delta_0 a < \varepsilon$) and $C > 0$ sufficiently large.

Notice that the permutation $\sigma$ defines a pairing between the eigenvalues of $A + n^{-a-\gamma}E$ and the eigenvalues of $A + A'$. Thus, in order to define $\sigma$, we may equivalently construct such a pairing, i.e., we say $\lambda_i(A + n^{-a-\gamma})$ and $\lambda_j(A + A')$ are paired if and only if $\sigma(i) = j$.

We now construct $\sigma$ on the event $F$. Indeed, $\sigma$ itself will be random, so for each outcome in $F$, we construct a possibly different permutation $\sigma$. Fix an outcome in $F$, and observe from (4.3) and (4.5) that all the eigenvalues of both matrices are contained in $R$ (by (4.7) and the fact that $F \subset \Omega$). This means all the eigenvalues are contained in the squares $R_1, \ldots, R_L$. As we construct the permutation $\sigma$, we will say the index $i$ (or the pairing of $i$) is good if both $\lambda_i(A + n^{-a-\gamma}E)$ and $\lambda_{\sigma(i)}(A + A')$ are in the same square $R_k$, $1 \leq k \leq L$; otherwise we call the index $i$ (or pairing of $i$) bad. To start, arbitrarily choose $\min\{X_1, Y_1\}$ eigenvalues of $A + n^{-a-\gamma}E$ in $R_1$ and pair them arbitrarily with $\min\{X_1, Y_1\}$ eigenvalues of $A + A'$ in $R_1$. After this first step, there may remain some eigenvalues in $R_1$ that are unpaired; we will leave them unpaired until the last step. Next, repeat the procedure for $R_2$: arbitrarily choose $\min\{X_2, Y_2\}$ eigenvalues of $A + n^{-a-\gamma}E$ in $R_2$ and pair them arbitrarily with $\min\{X_2, Y_2\}$ eigenvalues of $A + A'$ in $R_2$. Continue in this way, choosing $\min\{X_k, Y_k\}$ eigenvalues of $A + n^{-a-\gamma}E$ in $R_k$ and pairing them arbitrarily with $\min\{X_k, Y_k\}$ eigenvalues of $A + A'$ in $R_k$ for all $1 \leq k \leq L$. So far, all the pairings we have made are good pairings. To complete the construction of $\sigma$, now pair all the remaining unpaired eigenvalues of $A + n^{-a-\gamma}E$ arbitrarily with the remaining unpaired eigenvalues of $A + A'$; all the pairings in this last step are bad pairings. This procedure constructs the random permutation $\sigma$ on the event $F$; we will only work with $\sigma$ on $F$, but $\sigma$ can easily be extended to the entire probability space by taking $\sigma$ to be the identity permutation on $F^c$.

In view of (4.10), we have

$$W_1(\mu_{A+n^{-a-\gamma}E}, \mu_{A+A'}) \leq \frac{1}{n} \sum_{i=1}^{n} |\lambda_i(A + n^{-a-\gamma}E) - \lambda_{\sigma(i)}(A + A')| \leq \frac{1}{n} \sum_{i \text{ is good}} |\lambda_i(A + n^{-a-\gamma}E) - \lambda_{\sigma(i)}(A + A')|$$

$$+ \frac{1}{n} \sum_{i \text{ is bad}} |\lambda_i(A + n^{-a-\gamma}E) - \lambda_{\sigma(i)}(A + A')|. \hspace{1cm} (4.10)$$
On the one hand, if \( i \) is good, then both \( \lambda_i(A + n^{-a-\gamma}E) \) and \( \lambda_{\sigma(i)}(A + A') \) lie in the same square, so the distance between them is at most the diameter of the square, which by construction is \( \Theta(n^{-a}) \). On the other hand, if \( i \) is bad, then both \( \lambda_i(A + n^{-a-\gamma}E) \) and \( \lambda_{\sigma(i)}(A + A') \) lie in \( R \) which has diameter less than \( 2\beta = O(1) \) (see (4.7)). However, we note that there cannot be too many bad indices. Indeed, after we make the good pairings, each square \( R_k \) has at most \( C(n^{1-\varepsilon} + 6c_0a) + n^{1-3a} \) unpaired eigenvalues remaining on the event \( F \). In view of (4.8) then, there are at most \( O(n^{1-\varepsilon} + 8c_0a + n^{1-a}) \) total bad indices. Therefore, using that there are at most \( n \) good indices, we conclude that

\[
W_1(\mu_{A+n^{-a-\gamma}E}, \mu_{A+A'}) \ll \frac{1}{n} \sum_{i \text{ is good}} n^{-a} + \frac{1}{n} \sum_{i \text{ is bad}} 2\beta
\]

\[
\ll n^{-a} + n^{-\varepsilon + 8c_0a}
\]

on the event \( F \). Choosing \( a, \varepsilon \) sufficiently small with \( 8c_0a < \varepsilon \) completes the proof of Theorem 4.1, and it only remains to prove Lemma 4.2.

**Proof of Lemma 4.2.** Let \( a, \varepsilon > 0 \) be sufficiently small to be chosen later, and recall that the squares \( R_i \) all have the same side length \( s = \Theta(n^{-a}) \). For \( 1 \leq i \leq L \), we let \( \bar{R}_i \) and \( \tilde{R}_i \) be the squares with the same center as \( R_i \) but with side lengths of \( s + n^{-c_0a} \) and \( s - n^{-c_0a} \), respectively, where \( c_0 \) is the constant from (4.1). For \( 1 \leq i \leq L \), define smooth functions \( \tilde{\varphi}_i, \varphi_i : \mathbb{C} \to [0,1] \) so that \( \tilde{\varphi}_i \) is supported on \( \bar{R}_i \) with \( \tilde{\varphi}_i(z) = 1 \) for \( z \in \bar{R}_i \) and \( \varphi_i \) is supported on \( \tilde{R}_i \) with \( \varphi_i(z) = 1 \) for \( z \in \tilde{R}_i \).

We can construct \( \varphi_i \) and \( \tilde{\varphi}_i \) using products of bump functions in such a way that

\[
\sup_{1 \leq i \leq L} (\| \Delta \tilde{\varphi}_i \|_{\infty} + \| \Delta \varphi_i \|_{\infty}) = O(n^{6c_0a})
\]

(4.11)

by construction of the squares \( R_i, \bar{R}_i, \) and \( \tilde{R}_i \).

By construction of these functions we find

\[
\sum_{j=1}^n \tilde{\varphi}_i(\lambda_j(A + n^{-a-\gamma}E)) \leq X_i \leq \sum_{j=1}^n \tilde{\varphi}_i(\lambda_j(A + n^{-a-\gamma}E))
\]

and

\[
\sum_{j=1}^n \varphi_i(\lambda_j(A + A')) \leq Y_i \leq \sum_{j=1}^n \varphi_i(\lambda_j(A + A'))
\]

(4.13)

for \( 1 \leq i \leq L \). By taking \( a \leq \varepsilon_0 \), we apply (4.1) to find that

\[
\sup_{1 \leq i \leq L} \left| \sum_{j=1}^n (\tilde{\varphi}_i(\lambda_j(A + A')) - \varphi_i(\lambda_j(A + A'))) \right| = O(n^{1-3a}).
\]

(4.12)

(No union bound is required here since the eigenvalues \( f(1), f(\omega_n), f(\omega_n^2), \ldots, f(\omega_n^{n-1}) \) of \( A + A' \) are deterministic.) Thus, we can write (4.13) as

\[
\sum_{j=1}^n \tilde{\varphi}_i(\lambda_j(A + A')) - O(n^{1-3a}) \leq Y_i \leq \sum_{j=1}^n \varphi_i(\lambda_j(A + A')) + O(n^{1-3a})
\]

(4.14)
uniformly for \(1 \leq i \leq L\). Subtracting (4.14) from (4.12) yields
\[
\sum_{j=1}^{n} \tilde{\varphi}_i(\lambda_j(A + n^{-\alpha-\gamma}E)) - \sum_{j=1}^{n} \tilde{\varphi}_i(\lambda_j(A + A')) - O(n^{1-3\alpha})
\]
\[
\leq X_i - Y_i \leq \sum_{j=1}^{n} \tilde{\varphi}_i(\lambda_j(A + n^{-\alpha-\gamma}E)) - \sum_{j=1}^{n} \tilde{\varphi}_i(\lambda_j(A + A')) + O(n^{1-3\alpha})
\]
uniformly for \(1 \leq i \leq L\). The goal is to compare \(X_i\) and \(Y_i\) by comparing the differences of the sums above using Theorem 1.2.

To this end, take \(b := 100c_0a\). For \(1 \leq i \leq L\), let \(\tilde{Z}_i\) and \(\tilde{Z}_i\) be random variables, independent of all other sources of randomness, uniform on the supports of \(\tilde{\varphi}_i\) and \(\varphi_i\), respectively. Let \(Q\) be all the points in the complex plane at distance at most \(n^{-b}\) from \(f(S^1)\). For \(n\) sufficiently large, \(Q \subset B(0, \beta)\). In addition, since \(f(S^1)\) has arc length \(O(1)\), it follows that \(Q\) has Lebesgue measure \(O(n^{-b})\).

Define the events
\[
\mathcal{E}_i := \{\sigma_{\min}(A + n^{-\alpha-\gamma}E - \tilde{Z}_iI) > n^{-\kappa}\} \cap \{\tilde{Z}_i \not\in Q\} \cap \{\sigma_{\min}(A + n^{-\alpha-\gamma}E - \tilde{Z}_iI) > n^{-\kappa}\} \cap \{\tilde{Z}_i \not\in Q\}
\]
for \(1 \leq i \leq L\). It follows from (4.2) (by conditioning on \(\tilde{Z}_i\)) and the bound on the Lebesgue measure of \(Q\) that
\[
\sup_{1 \leq i \leq L} \mathbb{P}(\mathcal{E}_i^c) = O(n^{-98c_0a})
\]
for \(a\) sufficiently small (in particular this requires \(98c_0a < \delta\)). Hence, by the union bound (see (4.8)), the event
\[
\mathcal{E} := \Omega \bigcap \left( \bigcap_{i=1}^{L} \mathcal{E}_i \right)
\]
holds with probability \(1 - O(n^{-96c_0a})\).

We now work on the event \(\mathcal{E}\). Indeed, since \(\mathcal{E} \subset \Omega\), it follows that the norm bounds in (4.3), (4.5), and (4.6) hold on the event \(\mathcal{E}\). Moreover, note that for \(z \not\in Q\), \(\sigma_{\min}(A + A' - zI) \geq n^{-b}\) as the singular values of \(A + A' - zI\) are the values \(|f(1) - z|, |f(\omega_n) - z|, |f(\omega_n^2) - z|, \ldots, |f(\omega_n^{n-1}) - z|\) by Lemma 3.1. By Weyl’s perturbation theorem (Theorem 1.17) taking \(a\) sufficiently small so that \(b < \gamma\), we see that
\[
\sigma_{\min}(A + A' + n^{-\alpha-\gamma}E - \tilde{Z}_iI) \geq \frac{n^{-b}}{2}, \quad 1 \leq i \leq L
\]
for \(n\) sufficiently large on the event \(\mathcal{E}\).

Therefore, by Theorem 1.5,
\[
\sup_{1 \leq i \leq n} \left| \mathcal{L}_{A+n^{-\alpha-\gamma}E}(\tilde{Z}_i) - \mathcal{L}_{A+A'+n^{-\alpha-\gamma}E}(\tilde{Z}_i) \right| = O\left( \frac{\log n}{n} \right)
\]
(4.16)
on the event \(\mathcal{E}\) since \(A'\) has rank at most \(2k = O(1)\).

We next apply Theorem 1.4. On the event \(\mathcal{E}\), \(\nu_{A+A'-\tilde{Z}_iI}([0, n^{-2b}]) = 0\) for all \(1 \leq i \leq L\) (from the discussion above), and hence Theorem 1.4 implies that
\[
\sup_{1 \leq i \leq L} \left| \mathcal{L}_{A+A'+n^{-\alpha-\gamma}E}(\tilde{Z}_i) - \mathcal{L}_{A+A'}(\tilde{Z}_i) \right| \ll n^{2b-\gamma}
\]
on the event \(\mathcal{E}\). Taking \(a\) (and hence \(b\)) sufficiently small yields
\[
\sup_{1 \leq i \leq L} \left| \mathcal{L}_{A+A'+n^{-\alpha-\gamma}E}(\tilde{Z}_i) - \mathcal{L}_{A+A'}(\tilde{Z}_i) \right| \ll n^{-\epsilon}
\]
(4.17)
for $\varepsilon < \gamma/2$. Combining (4.16) and (4.17) shows that

$$\sup_{1 \leq i \leq L} |\mathcal{L}_{A+n^{-\alpha}\gamma E}(\tilde{Z}_i) - \mathcal{L}_{A+A'}(\tilde{Z}_i)| = O(n^{-\varepsilon})$$

(4.18)

on the event $\mathcal{E}$ for any $\varepsilon < \min\{\gamma/2, 1/2\}$. By repeating the argument above with $\tilde{Z}_i$ taking the place of $\tilde{Y}_i$, we similarly find that

$$\sup_{1 \leq i \leq L} |\mathcal{L}_{A+n^{-\alpha}\gamma E}(\tilde{Z}_i) - \mathcal{L}_{A+A'}(\tilde{Z}_i)| = O(n^{-\varepsilon})$$

(4.19)

on the event $\mathcal{E}$.

Using (4.18) and (4.19), we now apply Theorem 1.2 (we continue to use the norm bounds in (4.3), (4.5), and (4.6) which all hold with probability $1 - O(n^{-\delta})$). Indeed, applying Theorem 1.2 (using the description of $C_{1,2}$ given in (2.15)) with $m = [n^{6c_{0a}}]$, (4.11), and the union bound, it follows that

$$\sup_{1 \leq i \leq L} \left| \sum_{j=1}^{n} \tilde{\phi}_i(\lambda_j(A+n^{-\alpha}\gamma E)) - \sum_{j=1}^{n} \tilde{\phi}_i(\lambda_j(A+A')) \right| \leq n^{1-\varepsilon+6c_{0a}} + n^{1-6c_{0a}}$$

(4.20)

with probability $1 - O(n^{-34c_{0a}})$. Repeating the argument for $\tilde{\phi}_i$, $1 \leq i \leq L$ (using (4.19)), we similarly obtain

$$\sup_{1 \leq i \leq L} \left| \sum_{j=1}^{n} \tilde{\phi}_i(\lambda_j(A+n^{-\alpha}\gamma E)) - \sum_{j=1}^{n} \tilde{\phi}_i(\lambda_j(A+A')) \right| \leq n^{1-\varepsilon+6c_{0a}} + n^{1-6c_{0a}}$$

(4.21)

with probability $1 - O(n^{-34c_{0a}})$.

Combining (4.20) and (4.21) with (4.15), we conclude that

$$\sup_{1 \leq i \leq L} |X_i - Y_i| \leq n^{1-\varepsilon+6c_{0a}} + n^{1-6c_{0a}} + n^{-3a} \leq n^{1-\varepsilon+6c_{0a}} + n^{-3a}$$

with probability $1 - O(n^{-34c_{0a}})$. This completes the proof of the lemma. \hfill \Box

5. PROOF OF PROPOSITION 1.14

To prove Proposition 1.14, we will combine Theorem 1.2 with Theorem 1.4 by comparing the perturbation $R$ with a perturbation by an $n$ by $n$ matrix $E$ that has all entries equal to zero except the $(n, 1)$ entry is equal to $n^{-\gamma}$.

In order to use Theorem 1.2, we will need to understand the smallest singular values $T + R - zI$ and $T + E - zI$ for any $z$ satisfying $|z| \leq 1/4$. The singular values of $T + E - zI$ are well explained by Lemma A.2, which shows (from the first case, which holds as long as $n \geq 9$) that there is one singular value of size $\frac{n^{-\gamma} \sqrt{1 - (1/4)^{\delta}}}{\sqrt{2n^{\gamma} + 1 + 8}} \geq 0.229 n^{-\gamma}$ (where the inequality holds as long as $\gamma \geq 5$ and $n \geq \gamma^2$), and the other singular values have constant size (by combining Lemma A.1 with Weyl’s perturbation theorem, see Theorem 1.17). The $n - 1$ largest singular values of $T + R - zI$ may be bounded in the same way: again, Lemma A.1 combined with Weyl’s perturbation theorem show that the $n - 1$ largest singular values are of constant size. It remains to bound the smallest singular value of $T + R - zI$, which we do next.

**Lemma 5.1.** Let $n$ be a positive integer, let $\gamma \geq 5$, let $|z| \leq 1/4$, let $n \geq \gamma^2$, and let $T$ and $R$ be defined as in Proposition 1.14. Then, the smallest singular value of $T + R - zI$ is at least $0.15 n^{-\gamma}$.

**Proof.** We follow an approach similar to the one outlined by Rudelson and Vershynin in [55]. In general, we will show that for any unit vector $x = (x_1, \ldots, x_n)^T$, we must have that $\|(T + R - zI)x\| \geq c n^{-\gamma}$ where $c$ is an absolute constant (in particular, we will take $c = 0.15$). We will consider two
cases: first where $x$ does not have entries approximating a geometric progression, and second where $x$ does have entries that approximate a geometric progression. The intuition for using these two cases is that the unit singular vector for $T - zI$ for the smallest singular value (which is zero) has the form

$$v_0(z) = (1, -z, (-z)^2, (-z)^3, \ldots, (-z)^{n-1}) \left( \frac{1 - |z|^2}{1 - |z|^{2n}} \right)^{1/2};$$

thus, one might expect that the unit singular vector for the smallest singular value of a perturbation of $T - zI$ would have a similar structure.

Let $(r_{ij}) = R$. For the first case, assume that there is an $i$ with $1 \leq i \leq n - 1$ such that $|zx_i + x_{i+1}| \geq 2n^{-\gamma+1}$. Note that $\left| \sum_{j=1}^{n} x_j r_{ij} \right| \leq \left( \sum_{j=1}^{n} r_{ij}^2 \right)^{1/2} = n^{-\gamma+1/2}$ by the Cauchy–Schwarz inequality. Thus, the $i$-th entry of $(T + R - zI)x$ has size at least $2n^{-\gamma+1} - n^{-\gamma+1/2} \geq n^{-\gamma+1}$ by the triangle inequality, which proves that $\| (T + R - zI)x \| \geq n^{-\gamma+1} > 0.15n^{-\gamma}$. Thus, it is sufficient to consider the second case below.

For the second case, the defining assumption is that for every $i$ with $1 \leq i \leq n - 1$, we have $|zx_i + x_{i+1}| < 2n^{-\gamma+1}$. Since we have assumed that $|z| \leq 1/4$, this means that $x$ has entries that approximate a rapidly decreasing geometric progression, as we will now show. Rearranging the previous inequalities using the triangle inequality, we see $-2n^{-\gamma+1} + |z||x_i| < |x_{i+1}| < 2n^{-\gamma+1} + |z||x_i|$ for $1 \leq i \leq n - 1$. We may use induction to show for $2 \leq i \leq n$ that

$$-2n^{-\gamma+1}(1 + |z| + \cdots + |z|^{i-2}) + |x_i| |z|^{i-1} \leq |x_i| \leq 2n^{-\gamma+1}(1 + |z| + \cdots + |z|^{i-2}) + |x_1| |z|^{i-1}. \tag{5.2}$$

Because $(1 + |z| + \cdots + |z|^{i-2}) \leq \frac{1}{|z|}\gamma$, this shows that $x$ has approximately the form of a geometric progression, with $x_i$ having absolute value close to $|x_1| |z|^{i-1}$.

Next, we show that $|x_1|$ is bounded below by something close to $\sqrt{1 - |z|^2}$. Since $(x_1, \ldots, x_n)$ is a unit vector, we have that

$$1 = \sum_{i=1}^{n} |x_i|^2 = |x_1|^2 + \sum_{i=1}^{n-1} |x_{i+1}|^2 < |x_1|^2 + \sum_{i=1}^{n-1} \left( 2n^{-\gamma+1} + |z||x_i| \right)^2$$

$$= |x_1|^2 + \sum_{i=1}^{n-1} \left( 4n^{-2\gamma+2} + 4n^{-\gamma+1}|z||x_i| + |z|^2|x_i|^2 \right)$$

$$\leq |x_1|^2 + 4n^{-2\gamma+3} + 4n^{-\gamma+1} |z| \left( \sum_{i=1}^{n-1} |x_i| |z|^{i-1} + \frac{2n^{-\gamma+1}}{1 - |z|} \right) + |z|^2$$

$$\leq |x_1|^2 + 4n^{-2\gamma+3} + \frac{8n^{-2\gamma+3}}{1 - |z|} + 4n^{-\gamma+1} \frac{|x_1| |z|}{1 - |z|} + |z|^2$$

$$\leq \left( |x_1| + \frac{2n^{-\gamma+1}|z|}{1 - |z|} \right)^2 + \frac{4n^{-2\gamma+3}(3 - |z|)}{1 - |z|} + |z|^2.$$
Proof. Consider the function
\[ |x_1| > \sqrt{1 - |z|^2} - \frac{4n^{-2\gamma+3}(3 - |z|)}{(1 - |z|)\sqrt{1 - |z|^2}} - \frac{2n^{-\gamma+1}|z|}{1 - |z|} \]
\[ \geq \sqrt{1 - |z|^2} - \frac{8n^{-\gamma+1}}{3} \left( \frac{3n^{-\gamma+2}}{\sqrt{15/4}} + \frac{1}{4} \right) \]
(since \( |z| \leq 1/4 \))
\[ \geq \sqrt{1 - |z|^2} - n^{-\gamma+1} \]
(since \( \gamma \geq 5 \) and \( n \geq 25 \)).

We can perform a similar computation to find an upper bound on \( |x_1| \), thus showing that \( |x_1| \) is close to \( \sqrt{1 - |z|^2} \). In particular,
\[ 1 = \sum_{i=1}^{n} |x_i|^2 = |x_1|^2 + \sum_{i=1}^{n-1} |x_{i+1}|^2 > |x_1|^2 + \sum_{i=1}^{n-1} (-2n^{-\gamma+1} + |z||x_i|)^2 \]
\[ = |x_1|^2 + 4n^{-2\gamma+2}(n - 1) - 4n^{-\gamma+1} \sum_{i=1}^{n-1} |z||x_i| + |z|^2(1 - |x_n|^2) \]
\[ \geq |x_1|^2 + |z|^2 - |z|^2|x_n|^2 + 4n^{-2\gamma+2}(n - 1) - 8n^{-2\gamma+2}(n - 1) \frac{|z|}{1 - |z|} - 4n^{-\gamma+1}|x_1| \frac{|z|}{1 - |z|} \]
\[ \geq \left( |x_1| - 2n^{-\gamma+1} \frac{|z|}{1 - |z|} \right)^2 - 4n^{-2\gamma+2}\frac{|z|^2}{(1 - |z|)^2} + |z|^2 - \left( |z|^n|x_1| + \frac{|z|2n^{-\gamma+1}}{1 - |z|} \right)^2 + 4n^{-2\gamma+2}(n - 1) \frac{1 - 3|z|}{1 - |z|} \]
where the last bound uses the assumptions that \( |z| \leq 1/4, \gamma \geq 5 \) and \( n \geq \gamma^2 \). To see why this last inequality is true, note that under these assumptions, the corresponding error term cancels out the term \( \left( |z|^n|x_1| + \frac{|z|2n^{-\gamma+1}}{1 - |z|} \right)^2 \). To show that this cancellation occurs, we note that the term \( \left( |z|^n|x_1| + \frac{|z|2n^{-\gamma+1}}{1 - |z|} \right)^2 \) has size at most \( \left( \frac{|z|3n^{-\gamma+1}}{1 - |z|} \right)^2 \) under the given assumptions, which follows using the following lemma.

**Lemma 5.2.** If \( |z| \leq 1/4, \gamma \geq 5 \) and \( n \geq \gamma^2 \), then \( |z|^{n-1} < n^{-\gamma} \).

**Proof.** Consider the function \( f(n) = n^\gamma |z|^{n-1} \). To prove the lemma, it is sufficient to show that \( f(n) < 1 \) for all \( n, \gamma, \) and \( z \) satisfying the hypotheses. Thus, taking a logarithm, it is sufficient to show that \( g(n) := (n - 1) \log |z| + \gamma \log n < 0 \). Note that \( g'(n) = \log |z| + \frac{\gamma}{n} < 0 \) whenever \( n, \gamma, \) and \( z \) satisfy the hypotheses of the lemma; thus
\[ g(n) < (\gamma^2 - 1) \log |z| + 2\gamma \log \gamma < \frac{\gamma^2}{2} \log |z| + 2\gamma \log \gamma \]
\[ = \frac{\gamma}{2} \left( \gamma \log |z| + 4 \log \gamma \right). \]
Note that the function \( h(\gamma) = \gamma \log |z| + 4 \log \gamma \) satisfies \( h'(\gamma) = \log |z| + \frac{4}{\gamma} < 0 \), and so given the hypotheses, we have that \( h(\gamma) < h(5) < 0 \). This proves that \( g(n) < 0 \), which completes the proof of the lemma. \( \square \)

We can now rearrange the last large sequence of inequalities to get that
\[ |x_1| < (1 - |z|^2)^{1/2} + 2n^{-\gamma+1} \frac{|z|}{1 - |z|} \]
\[ \text{if } \gamma \geq 5 \text{ and } n \geq 25. \]
We now have a good estimate on the size of $|x_1|$, which is close to $\sqrt{1 - |z|^2}$, and we also know that the entries in $x$ form an approximate geometric progression (from (5.2)). Together, this allows us to get a good upper bound on $\sum_{j=1}^{n} x_j r_{ij}$, namely
\[
\left| \sum_{j=1}^{n} x_j r_{ij} \right| \leq n^{-\gamma} \sum_{i=1}^{n} |x_i| \leq n^{-\gamma} \sum_{i=1}^{n} \left( |x_1| |z|^{i-1} + \frac{2n^{-\gamma+1}}{1 - |z|} \right) \\
\leq \frac{n^{-\gamma} |x_1|}{1 - |z|} + \frac{2n^{-2\gamma+1}}{1 - |z|} \leq \frac{n^{-\gamma} \sqrt{1 - |z|^2}}{1 - |z|} + \frac{2n^{-2\gamma+2}(|z| + (1 - |z|))}{(1 - |z|)^2} \\
= \frac{n^{-\gamma} \sqrt{1 - |z|^2}}{1 - |z|} + \frac{2n^{-2\gamma+2}}{(1 - |z|)^2} < 1.334n^{-\gamma},
\]
where the last inequality uses the assumptions that $|z| < 1/4$ and $\gamma \geq 5$ and $n \geq \gamma^2$.

We can complete the proof of Lemma 5.1 using two subcases. If $\sum_{j=1}^{n} x_j r_{ij} \geq 1.484n^{-\gamma}$ for any $1 \leq i \leq n - 1$ or if $\sum_{j=1}^{n} x_j r_{ij} \geq 1.484n^{-\gamma}$, then the size of the coordinate for the corresponding row is at least $1.484n^{-\gamma} - 1.334n^{-\gamma} = 0.15n^{-\gamma}$. Thus, in this first subcase, $\| (T + R - zI)x \|$ is at least $0.15n^{-\gamma}$, as desired.

We may now complete the proof of Lemma 5.1 by considering the second subcase, in which we assume that $\sum_{j=1}^{n} x_j r_{ij} < 1.484n^{-\gamma}$ for every $1 \leq i \leq n - 1$ and additionally that $\sum_{j=1}^{n} x_j r_{ij} < 1.484n^{-\gamma}$. We already know that $x$ has entries forming an approximate geometric progression, and the stronger assumption in this second subcase will allow us to reduce the error in the approximation by a factor of $n$. In particular, using the same reasoning as (5.2), we have
\[
-1.484n^{-\gamma}(1 + |z| + \cdots + |z|^{i-2}) + |x_1| |z|^{i-1} \leq |x_i| \leq 1.484n^{-\gamma}(1 + |z| + \cdots + |z|^{i-2}) + |x_1| |z|^{i-1},
\]
for $2 \leq i \leq n$.

We now consider the last entry of $(T + R - zI)x$, which has size at least
\[
\sum_{i=1}^{n} x_i r_{ni} - |z||x_n| \geq n^{-\gamma}|x_1| - n^{-\gamma} \sum_{i=2}^{n} |x_i| - |z||x_n| \\
\geq n^{-\gamma}|x_1| - n^{-\gamma} \sum_{i=2}^{n} \left( \frac{1.484n^{-\gamma}}{1 - |z|} + |x_1||z|^{i-1} \right) - |z| \left( \frac{1.484n^{-\gamma}}{1 - |z|} + |z||z|^{n-1}|x_1| \right) \\
\geq n^{-\gamma}|x_1| - \frac{1.484n^{-2\gamma+1}}{1 - |z|} - n^{-\gamma}|x_1||z| - \frac{1.484n^{-\gamma}|z|}{1 - |z|} - |z|^n|x_1| \\
> 0.15n^{-\gamma},
\]
where the last inequality uses the facts that $|z| \leq 1/4$ and $\gamma \geq 5$ and $n \geq \gamma^2$ along with $|x_1| \geq \sqrt{1 - |z|^2} - n^{-\gamma+1} \geq 0.968$.

Thus $\| (T + R - zI)x \| > 0.15n^{-\gamma}$. Because $x$ was an arbitrary unit vector, we have proven that the smallest singular value of $T + R - zI$ is at least $0.15n^{-\gamma}$. $\square$

5.1. Applying Theorems 1.2 and 1.4 to prove Proposition 1.14. With the bound on the smallest singular value from Lemma 5.1, we are now in a position where we can apply Theorem 1.4 and Theorem 1.2. We will compare the two matrices $T + R$ and $T + E$, where we recall that $E$ is the $n \times n$ matrix with entry $(n, 1)$ equal to $-n^{-\gamma}$ and all other entries equal to zero, and that $R$ is a deterministic matrix in which each entry is $-n^{-\gamma}$ or $n^{-\gamma}$. Recall that from Lemma A.2, we know that the smallest singular value of $T + E - zI$ is at least $\frac{n^{-\gamma}\sqrt{1 - (1/4)^2}}{\sqrt{2n^{-\gamma+2}+8}} \geq 0.229n^{-\gamma}$, where we used the assumptions from Proposition 1.14 that $\gamma \geq 5$ and $n \geq \gamma^2$. 

\[35\]
We will now apply Theorem 1.4. By the above and Lemma 5.1, we may bound \( \sigma_{\min} \) below by \( 0.15n^{-\gamma} \) and set \( \varepsilon = 1/3 \) (noting that \( \| R - E \| \leq n^{-\gamma}(1 + n) \leq 2n^{-\gamma + 1} < 1/6 \) for any \( n \geq 2 \), and we further note that \( \nu_{A + E - zI}([0, \varepsilon]) = 1/n \) by the proof of Lemma A.1 (in fact, all but one singular value must have size at least \( |z| - 1 \geq 3/4 \) since we are considering only \( z \) with \( |z| \leq 1/4 \). Thus, Theorem 1.4 (using the slightly improved bound in (2.6)) implies, for \( |z| \leq 1/4 \), that

\[
|L_{A + R}(z) - L_{A + E}(z)| \leq (6\log(\varepsilon/2) + 4|\log \sigma_{\min}|) \nu_{A + E - zI}([0, \varepsilon]) + \frac{2}{\varepsilon} \| R - E \|
\]

\[
\leq \left( 6\log(6) + 4(\gamma \log n + \log \left( \frac{100}{15} \right) ) \frac{1}{n} + 12n^{-\gamma + 1} \right) \gamma \log n
\]

\[
= \frac{\gamma \log n}{n} \left( 4 + \frac{6\log 6 + 4\log(100/15) + 12n^{-\gamma + 2}}{\gamma \log n} \right)
\]

where we used the assumption that \( \gamma \geq 5 \) and \( n \geq \gamma^2 \) in the last line.

We can now use the result above to apply Theorem 1.2. Note that \( \| T + R - zI \| + \| T + E - zI \| \leq 2(|z| + 1) + 2n^{-\gamma + 1} \leq 2.5 + 2n^{-\gamma + 1} \leq \varepsilon \) whenever \( \gamma \geq 5 \) and \( n \geq \gamma^2 \). Thus, we may take \( T = e \) to satisfy the first assumption of Theorem 1.2 with any positive \( \varepsilon \). Also, by the application of Theorem 1.4 in the previous paragraph, we may satisfy the second condition of Theorem 1.2 by taking \( \eta = \frac{\gamma \log n}{n} \left( 4 + \frac{6.688}{\log n} \right) \), again using any positive value for \( \varepsilon \). We also note that from the assumptions on \( \varphi \), we have that \( C_{1.2} \leq 2\log_2(\Delta \varphi) \| \Delta \varphi \|_\infty (\frac{\pi}{2}) \log_2(2.75)/\sqrt{\pi} < (1.095)\| \Delta \varphi \|_\infty \).

Because the bound in Theorem 1.2 works for any integer \( m \), and because we are able to satisfy the two assumptions with any positive \( \varepsilon \), we may follow Remark 1.3 and set \( \varepsilon = 1/m^{3/2} \) and take the limit as \( m \to \infty \), in which case \( \log \frac{T}{m^{3/2}} - 0 \) and \( 2(m + 1)\varepsilon \to 0 \), thus proving that

\[
\frac{1}{n} \left| \sum_{i=1}^{n} \varphi(\lambda_i(T + R)) \right| \leq C_{1.2} \eta < \left( 4.38 + \frac{4.02}{\log n} \right) \| \Delta \varphi \|_\infty \gamma \log n \frac{n}{n} < 6\| \Delta \varphi \|_\infty \gamma \log n,
\]

given the assumptions that \( \gamma \geq 5 \) and \( n \geq 25 \). Here, we used that, under the assumptions on \( \gamma \) and \( n \), none of the eigenvalues of \( T + E \) lie in \( \{ z \in \mathbb{C} : |z| \leq 1/4 \} \) and so \( \sum_{i=1}^{n} \varphi(\lambda_i(T + E)) = 0 \).

**Appendix A. Singular value computations for a Jordan block matrix**

Let \( M_n \) be an \( n \times n \) matrix composed of Jordan blocks \( B_i \) for \( i = 1, \ldots, \ell_n \). Each block \( B_i \) is an \( m_i \times m_i \) matrix with eigenvalue \( c_i \) on the diagonal, ones on the first superdiagonal, and all other entries equal to zero; thus,

\[
B_i = \begin{pmatrix}
  c_i & 1 & 0 & \cdots & 0 \\
  0 & c_i & 1 & \cdots & \vdots \\
  \vdots & \ddots & \ddots & \ddots & \vdots \\
  0 & \cdots & 0 & c_i & 1 \\
  0 & \cdots & 0 & 0 & c_i
\end{pmatrix}
\]

Let \( E_i \) be an \( m_i \times m_i \) matrix with the \((m_i, 1)\) entry equal to \( \varepsilon_i \) (which may be a function of \( m_i, c_i \), and \( z \)) and all other entries equal to zero.

**Lemma A.1.** Let \( i \) and \( m_i \) be positive integers, let \( z \) and \( c_i \) be complex constants, and let \( B_i \) be an \( m_i \times m_i \) Jordan block with eigenvalue \( c_i \). The matrix \( A = B_i - zI \) has \( m_i - 1 \) singular values bounded below by \( \| c_i - z \| - 1 \) and above by \( | c_i - z | + 1 \). Furthermore, if \( M \) is an \( m_i \times m_i \) matrix
with the first \( m_i - 1 \) rows equal to the first \( m_i - 1 \) rows of \( A \) (including the case \( M = A \)), then every singular value except the smallest satisfies \( \sigma_k(M) \geq |c_i - z| - 1 \) (for \( 1 \leq k \leq m_i - 1 \)). If \( |c_i - z| < 1 \), then \( A \) has smallest singular value of size at most \( |c_i - z|^n \), and if \( |c_i - z| > 1 \), all singular values of \( A \) have size at least \( \min\{|c_i - z| - 1, \sqrt{|c_i - z|^2 - |c_i - z|}\} \).

Proof. First, we show that the largest \( n - 1 \) singular values of \( B_i - zI \) have size \( \Theta(|c_i - z|) \). If \( A \) is an \( m \times m \) matrix and \( A' \) is an \( m - 1 \times m \) matrix consisting of the first \( m - 1 \) rows of \( A \), then Cauchy's interlacing law (see [68, Lemma A.1]) states that for any \( 1 \leq k \leq m - 1 \), we have \( \sigma_k(A) \geq \sigma_k(A') \geq \sigma_{k+1}(A) \), where \( \sigma_1(A) \geq \sigma_2(A) \geq \cdots \geq \sigma_m(A) \) are the singular values for \( A \), and similarly for \( A' \). If \( M \) shares the first \( m - 1 \) rows with \( A \), then we will similarly have \( \sigma_k(M) \geq \sigma_k(A') \geq \sigma_{k+1}(M) \) for any \( 1 \leq k \leq m - 1 \). If we set \( A = B_i - zI \), the singular values of \( A' \) are the square-roots of the eigenvalues of \( (A')^* \), which is a tridiagonal Toeplitz matrix with \( |c_i - z|^2 + 1 \) on the diagonal, \( \overline{z} - z \) on the superdiagonal, and \( c_i - z \) on the subdiagonal. The eigenvalues of a tridiagonal Toeplitz matrix can be computed explicitly (see, for example [52]), and for the matrix \( (A')^* \) are given by \( \lambda_k = \lambda_k(A'(A')^*) = |c_i - z|^2 + 2|c_i - z|\cos\left(\frac{k\pi}{m_i}\right) \) for \( k = 1, 2, \ldots, m_i - 1 \). Using the fact that \( -1 \leq \cos(\theta) \leq 1 \) for all real \( \theta \), we have that \( (|c_i - z| - 1)^2 \leq \lambda_k \leq (|c_i - z| + 1)^2 \). Thus, by Cauchy's interlacing law, we see for \( 2 \leq k \leq m_i - 1 \) that \( |c_i - z| - 1 \leq \sigma_k(A), \sigma_k(M) \leq |c_i - z| + 1 \), showing that for \( 2 \leq k \leq m_i - 1 \), the singular values are bounded above and below by constants depending only on \( c_i \) and \( z \). For \( k = 1 \), the explicit formula for the eigenvalues gives \( \sigma_1(A) \) is bounded below by \( (|c_i - z| + 1)^2 - \frac{|c_i - z|^2}{m_i^2} \). We can bound \( \sigma_1(A) \) from above by noting that for a unit column vector \( v = (v_1, \ldots, v_{m_i})^T \), we have \( \|Av\|^2 = |(c_i - z)v_{m_i}|^2 + \sum_{k=1}^{m_i-1} |v_k(c_i - z) + v_{k+1}|^2 \); thus, using the Cauchy–Schwarz inequality and the fact that \( v \) is a unit vector, we see that

\[
\|Av\|^2 \leq |c_i - z|^2 |v_{m_i}|^2 + \sum_{k=1}^{m_i-1} |v_k|^2 |c_i - z|^2 + 2 \sum_{k=1}^{m_i-1} |c_i - z| |v_k| |v_{k+1}| + \sum_{k=1}^{m_i-1} |v_{k+1}|^2 \\
\leq |c_i - z|^2 + 2|c_i - z| + 1 = (|c_i - z| + 1)^2,
\]

which proves that \( \sigma_1(A) \leq |c_i - z| + 1 \). We have now shown for \( k = 1, 2, \ldots, m_i - 1 \) that \( \sigma_k(A) \) is bounded above by \( |c_i - z| + 1 \) and bounded below by \( |c_i - z| - 1 \).

It remains to show an upper bound on the smallest singular value \( \sigma_m(A) \), which is easily done by noting that the column vector \( u = (1, (z - c_i), (z - c_i)^2, \ldots, (z - c_i)^{m_i-1})^T \) satisfies \( (B_i - zI)u = (0, 0, \ldots, 0, -(z - c_i)^m_i) \), and so \( \frac{\|(B_i - zI)u\|^2}{\|u\|^2} = \frac{|c_i - z|^{2m_i}(1 - |c_i - z|^2)}{1 - |c_i - z|^{2m_i}} \leq |c_i - z|^{2m_i} \), when \( |c_i - z| < 1 \). Thus, \( \sigma_m(A) \leq |c_i - z|m_i \).

The final assertion for the \( |c_i - z| > 1 \) case can be proven by noting that the matrix \( AA^* \) has all diagonal entries equal to \( |c_i - z|^2 + 1 \) or \( |c_i - z|^2 \), has all super diagonal entries equal to \( \bar{c}_i - \bar{z} \), has all subdiagonal entries equal to \( c_i - z \), and has all other entries equal to zero. By the Gershgorin Circle Theorem, the eigenvalues of \( AA^* \) are all contained in the disk with center \( |c_i - z|^2 + 1 \) and radius \( 2|c_i - z| \), or in the disk with center \( |c_i - z|^2 \) and radius \( |c_i - z| \). Thus, every eigenvalue of \( AA^* \) has size at least \( \min\{|c_i - z| - 1, |c_i - z|^2 - |c_i - z|\} \), which implies that the smallest singular value of \( A \) is at least \( \min\{|c_i - z| - 1, \sqrt{|c_i - z|^2 - |c_i - z|}\} \), which is positive due to the assumption that \( |c_i - z| > 1 \). \( \square \)

Lemma A.2. Let \( i \) and \( m_i \) be positive integers, let \( z \) and \( c_i \) be complex numbers (which may depend on \( m_i \)), let \( \epsilon \) be a non-negative real number (possibly depending on \( m_i, c_i \), and \( z \)), let \( E_i \) be the \( m_i \) by \( m_i \) matrix with entry \( (m_i, 1) \) equal to \( \epsilon \) and all other entries equal to zero, and let \( B_i \) be an \( m_i \) by \( m_i \) Jordan block with eigenvalue \( c_i \). Then the smallest singular value \( \sigma_n(A) \) for
that exists a unit column vector $x$ such that 

$$\sigma_n(A) = \begin{cases} 
\frac{\epsilon (1 - |z - c_i|^3/2)}{\sqrt{2m_i \epsilon^2 + 8}} & \text{if } |z - c_i| < 1 \text{ and } 2 |z - c_i|^m_i < \epsilon, \\
\frac{|z - c_i|^m_i (1 - |z - c_i|^{3/2})}{\sqrt{2m_i |z - c_i|^{2m_i} + 2}} & \text{if } |z - c_i| < 1 \text{ and } \epsilon < \frac{1}{2} |z - c_i|^m_i, \\
\frac{||z - c_i| - 1| |z - c_i|^{2m_i} - 1}{\sqrt{2m_i |z - c_i|^2 + 1}} & \text{if } |z - c_i| > 1 \text{ and } \epsilon < \frac{1}{2} |z - c_i|^m_i.
\end{cases}$$

Proof. Let $A = B_i - zI + E$, and assume that $A$ has smallest singular value $\sigma_n(A)$. Thus, there exists a unit column vector $x = (x_1, \ldots, x_n)^T$ such that $Ax = y$ and $\|Ax\| = \|y\| = \sigma_n(A)$.

For each case, we will show bounds on the $|x_k|$ that then translate into a bound on $\sigma_n(A)$. We will consider the first two cases together, assuming that that $|c_i - z| < 1$ and highlighting in the proof where we use the assumption $2 |z - c_i|^m_i < \epsilon$ for the first case, or, respectively, $\epsilon < \frac{1}{2} |z - c_i|^m_i$ for the second case. The equation $Ax = y$ may be written as the system of equations

$$\begin{align*}
x_2 &= y_1 + (z - c_i)x_1 \\
x_3 &= y_2 + (z - c_i)x_2 \\
\vdots \\
x_{m_i} &= y_{m_i-1} + (z - c_i)x_{m_i-1} \\
e x_1 &= y_{m_i} + (z - c_i)x_{m_i}.
\end{align*}$$

Successively plugging in the equations for $x_2, x_3, x_4, \ldots, x_n$, we may write each $x_k$ in terms of the $y_k$ and $x_1$ as follows:

$$\begin{align*}
x_2 &= y_1 + (z - c_i)x_1 \\
x_3 &= y_2 + (z - c_i)y_1 + (z - c_i)^2 x_1 \\
\vdots \\
x_{m_i} &= \left( \sum_{\ell=0}^{m_i-2} y_{m_i-1-\ell}(z - c_i)^\ell \right) + (z - c_i)^{m_i-1} x_1 \\
e x_1 &= \left( \sum_{\ell=0}^{m_i-1} y_{m_i-\ell}(z - c_i)^\ell \right) + (z - c_i)^{m_i} x_1.
\end{align*}$$

Using the fact that $|y_k| \leq \sigma_n(A)$ for all $1 \leq k \leq m_i$ and the assumptions that $|z - c_i| < 1$ and that $\epsilon \neq |z - c_i|^m_i$ (the latter of which follows from either the first or the second case assumption), we may solve the last equation for $x_1$ and take absolute values to arrive at

$$|x_1| \leq \frac{\sigma_n(A)}{(1 - |z - c_i|)|\epsilon - (z - c_i)^m_i|}. \quad (A.1)$$
By assumption $\|x\| = 1$, and so we have

$$1 = \|x\|^2 = \sum_{k=1}^{m_i} |x_k|^2 \leq |x_1|^2 + \sum_{k=2}^{m_i} \left( \left( \sigma_n(A) \sum_{\ell=0}^{k-2} |z - c_i|^\ell \right) + |z - c_i|^{k-1} \right)^2$$

$$\leq |x_1|^2 + \sum_{k=2}^{m_i} \left( \frac{2\sigma_n(A)^2}{(1 - |z - c_i|)^2} + 2 |z - c_i|^{2(k-1)} |x_1|^2 \right)$$

$$\leq \frac{2m_i \sigma_n(A)^2}{(1 - |z - c_i|)^2} + \frac{2 |x_1|^2}{1 - |z - c_i|}$$

$$\leq \frac{2\sigma_n(A)^2}{(1 - |z - c_i|)^3} \left( m_i + \frac{1}{|\epsilon - (z - c_i)^m_i|^2} \right),$$

where we plugged in (A.1) in the last inequality. Thus, we have shown that

$$1 \leq \frac{2\sigma_n(A)^2}{(1 - |z - c_i|)^3} \left( m_i + \frac{1}{|\epsilon - (z - c_i)^m_i|^2} \right) \quad (A.2)$$

Under the assumption that $2 |z - c_i|^{m_i} < \epsilon$ from the first case, (A.2) implies

$$\sigma_n(A) \geq \frac{\epsilon(1 - |z - c_i|)^{3/2}}{\sqrt{2m_i \epsilon^2 + 8}} \quad (A.3)$$

which proves the first case of the lemma.

Under the assumption that $\epsilon < \frac{1}{2} |z - c_i|^{m_i}$ from the second case, (A.2) can be rearranged to show

$$\sigma_n(A) \geq \frac{|z - c_i|^{m_i} (1 - |z - c_i|)^{3/2}}{\sqrt{2m_i |z - c_i|^{2m_i} + 8}} \quad (A.4)$$

which proves the second case of the lemma.

For the case where $|z - c_i| > 1$ and $\epsilon < \frac{1}{2} |z - c_i|^{m_i}$, we proceed similarly. From $Ax = y$, we have

$$x_1 = \frac{-y_1}{z - c_i} + \frac{x_2}{z - c_i}$$

$$x_2 = \frac{-y_2}{z - c_i} + \frac{x_3}{z - c_i}$$

$$\vdots$$

$$x_{m_i-1} = \frac{-y_{m_i-1}}{z - c_i} + \frac{x_{m_i}}{z - c_i}$$

$$x_{m_i} = \frac{-y_{m_i}}{z - c_i} + \frac{\epsilon x_1}{z - c_i}$$
Substituting in for \(x_m, x_{m-1}, \ldots, x_2\), we may write the right-hand side of each equation in terms of \(y_k, x_1, z, and c\) as follows:

\[
x_m = \frac{-ym_i}{z - c_i} + \frac{\epsilon x_1}{(z - c_i)^{1}}
\]

\[
\vdots
\]

\[
x_k = \left(\sum_{\ell=k}^{m_i} \frac{-y_{\ell}}{(z - c_i)^{m_i+1-k}}\right) + \frac{\epsilon x_1}{(z - c_i)^{m_i}}
\]

\[
\vdots
\]

\[
x_1 = \left(\sum_{\ell=1}^{m_i} \frac{-y_{\ell}}{(z - c_i)^{m_i}}\right) + \frac{\epsilon x_1}{(z - c_i)^{m_i}}
\]

From the last equation and the fact that \(|y_k| \leq \sigma_n(A)\) for all \(k\) and the assumption that \(\epsilon < \frac{1}{2} |z - c_i|^{m_i}\), we have

\[
|x_1| \leq \frac{\sigma_n(A)}{1 - |z - c_i|} \sum_{k=1}^{m_i} \frac{1}{|z - c_i|^k} \leq \frac{2\sigma_n(A)}{1 - |z - c_i|} \leq \frac{2\sigma_n(A)}{|z - c_i| - 1}. \tag{A.5}
\]

As before, we now compute a bound on \(\sigma_n(A)\) in terms of \(|x_1|\). By assumption \(||x|| = 1\), and so we have

\[
1 = ||x||^2 = \sum_{k=1}^{m_i} |x_k|^2
\]

\[
\leq \sum_{k=1}^{m_i} \left(\frac{\sigma_n(A)}{|z - c_i|} \left(\sum_{\ell=k}^{m_i} \frac{1}{|z - c_i|^\ell-k}\right) + \frac{\epsilon |x_1|}{|z - c_i|^{m_i+1-k}}\right)^2
\]

\[
\leq \sum_{k=1}^{m_i} \left[\frac{2\sigma_n(A)^2}{(|z - c_i| - 1)^2} + \frac{2\epsilon^2 |x_1|^2}{|z - c_i|^{2(m_i+1-k)}}\right]
\]

\[
\leq \frac{2\sigma_n(A)^2 m_i}{(|z - c_i| - 1)^2} + \frac{8\epsilon^2 \sigma_n(A)^2}{(|z - c_i|^2 - 1)(|z - c_i| - 1)^2},
\]

where we plugged in the bound from (A.5) for the last inequality. Rearranging this last inequality, we have

\[
\sigma_n(A) \geq \frac{|z - c_i| - 1}{\sqrt{|z - c_i|^2 - 1}} \frac{\sqrt{\frac{|z - c_i|^2 - 1}{2m_i(|z - c_i|^2 - 1) + 8\epsilon^2}}}{\sqrt{|z - c_i|^2 - 1}}. \tag{A.6}
\]

which proves the third case of the lemma.

Combining the inequalities in (A.3), (A.4), and (A.6) completes the proof. \(\square\)

**Appendix B. Multiplicative perturbations**

This section contains some relevant bounds required in the proof of Corollary 1.9.

**Proposition B.1** (Spectral norm bound). Let \(E\) be an \(n \times n\) random matrix whose entries are iid copies of a random variable \(\xi\) which has mean zero, unit variance, and finite fourth moment. Then there exists \(C > 0\) (depending only on \(\xi\)) so that, for any \(\alpha > 1/2\),

\[
P(||E|| \geq n^\alpha) \leq C n^{1/2 - \alpha}.
\]
Proof. It follows from [48, Theorem 2] that $E\|E\| \leq C\sqrt{n}$ for a constant $C > 0$ depending only on $\xi$. The claim now follows from Markov’s inequality. □

Our next result bounds the spectral norm of a banded Toeplitz matrix (see Definition 1.6).

Proposition B.2 (Spectral norm bound). Let $k \geq 0$ be an integer, and let $\{a_j\}_{j \in \mathbb{Z}}$ be a sequence of complex numbers indexed by the integers. Let $A$ be the $n \times n$ Toeplitz matrix with symbol $\{a_j\}_{j \in \mathbb{Z}}$ truncated at $k$ (as in Definition 1.6). Then

$$\|A\| \leq \sum_{|j| \leq k} |a_j|.$$ 

Proof. We decompose $A$ as

$$A = \sum_{j=0}^{k} a_j J^j + \sum_{j=1}^{k} a_{-j} (J^T)^j,$$

where $J$ is the $n \times n$ matrix whose entries are all zero except for ones along the sub-diagonal (i.e., $J$ is the transpose of the matrix $T$ given in (1.2)) and $J^0$ is the identity matrix. The claimed bound then follows from the triangle inequality since $\|J\| = 1$. □

Lastly, we will need the following least singular value bound.

Proposition B.3 (Least singular value bound). Let $A$ be a deterministic $n \times n$ matrix, and let $E$ be an $n \times n$ random matrix whose entries are iid copies of a real standard normal random variable. Then for any $z \in \mathbb{C}$ with $z \neq 0$ and any $\gamma > 1$, there exists $\kappa > 0$ (depending on $z$ and $\gamma$) so that

$$\mathbb{P}(\sigma_{\min}(A + n^{-1/2-\gamma}E) - z) \leq n^{-\kappa} = o(1).$$

Proof. Fix $z \in \mathbb{C}$ with $z \neq 0$ and $\gamma > 1$. It follows from [71, Corollary 5.35] that $\|E\| = O(\sqrt{n})$ with probability $1 - o(1)$. Thus, with probability $1 - o(1)$, $I + n^{-1/2-\gamma}E$ is invertible, and by utilizing a Neumann series

$$\|(I + n^{-1/2-\gamma}E)^{-1}\| \leq 2$$ (B.1)

and

$$\|(I + n^{-1/2-\gamma}E)^{-1} - (I - n^{-1/2-\gamma}E)\| = O(n^{-2\gamma})$$ (B.2)

with probability $1 - o(1)$. Therefore—using the bound $\sigma_{\min}(M_1 M_2) \geq \sigma_{\min}(M_1) \sigma_{\min}(M_2)$ for two $n \times n$ matrices $M_1$ and $M_2$ (which follows from the characterization given in (1.24))—we obtain

$$\sigma_{\min}(A(I + n^{-1/2-\gamma}E) - zI) \geq \sigma_{\min}(I + n^{-1/2-\gamma}E) \sigma_{\min}(A - z(I + n^{-1/2-\gamma}E)^{-1})$$

$$\geq \frac{1}{2} \sigma_{\min}(A - z(I + n^{-1/2-\gamma}E)^{-1})$$ (B.3)

by (B.1). By Weyl’s perturbation theorem (Theorem 1.17) and (B.2)

$$\sigma_{\min}(A - z(I + n^{-1/2-\gamma}E)^{-1}) \geq \sigma_{\min}(A - zI - zn^{-1/2-\gamma}E) - O(n^{-2\gamma})$$ (B.4)

with probability $1 - o(1)$. By a well-known result of Sankar, Spielman, and Teng [59] for the least singular value, we obtain, for any $\varepsilon > 0$,

$$\sigma_{\min}(A - zI - zn^{-1/2-\gamma}E) \geq |z|n^{-1-\gamma-\varepsilon}$$

with probability $1 - o(1)$. Using the fact that $\gamma > 1$, we can take $\varepsilon > 0$ sufficiently small so that the bound above and (B.4) imply

$$\sigma_{\min}(A - z(I + n^{-1/2-\gamma}E)^{-1}) \geq \frac{|z|}{2} n^{-1-\gamma-\varepsilon}$$

with probability $1 - o(1)$. Combined with (B.3), we conclude that

$$\sigma_{\min}(A(I + n^{-1/2-\gamma}E) - zI) > n^{-\kappa}$$
with probability \(1 - o(1)\) for a sufficiently large choice of \(\kappa > 0\).

\[\square\]

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