High order nonlocal symmetries and exact solutions of variable coefficient KdV equation

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Abstract

In this paper, nonlocal symmetries and exact solutions of variable coefficient Korteweg-de Vries (KdV) equation are studied for the first time. Using pseudo-potential, high order nonlocal symmetries of time-dependent coefficient KdV equation are obtained. In order to construct new exact analytic solutions, new variables are introduced, which can transform nonlocal symmetries into Lie point symmetries. Furthermore, using the Lie point symmetries of closed system, we give symmetry reduction and some exact analytic solutions. For some interesting solutions, such as interaction solutions among solitons and other complicated waves are discussed in detail, and the corresponding images are given to illustrate their dynamic behavior.

Keywords: Nonlocal symmetry; Variable coefficient KdV equation; Exact solution;

1. Introduction

The theory of Lie group [1] was proposed by Norwegian mathematician Sophus Lie in the 19th century. In the 20th century, the Lie group theory developed rapidly[2–4]. Lie group theory was not only used to solve differential equations, but also established the relationships with many disciplines, such as nonlinear theory, integrable system, etc. Nowadays, Lie group theory has been widely applied to many fields, such as, mathematics, physics, numerical analysis, quantum mechanics, fluid dynamics system, etc.

As a generalization of the symmetry, a lot of studies have been devoted to seeking the generalized Lie point symmetry. P.J.Olver[2] construct a new type of symmetry by using recursion operator which was called nonlocal symmetry in the 80s of the last century. G.W. Bluman et al.[5, 6] presented many methods to find nonlocal symmetries of partial differential equations (PDEs) by using potential systems. F. Galas [7] obtained the nonlocal symmetries by using the pseudo-potentials of PDEs, and construct exact solutions by the obtained nonlocal symmetries firstly. Recently, Lou et al.[8–14] found that Painlevé analysis can also be used to construct nonlocal symmetries which was called residual symmetries.

Seeking nonlocal symmetry of nonlinear differential equations with variable coefficients is a meaningful and difficult task. Constructing high order nonlocal symmetry is also a very difficult work, because the higher order of symmetry can reflect the fundamental invariance of the equation. To construct high order nonlocal symmetry, the equation should have high order Lax pair, or high order pseudo-potential firstly. Many scholars have made researches on high order pseudo-potential. M.C. Nucci gave a way to construct the pseudo-potential of the nonlinear differential equation and some high order pseudo-potential of several nonlinear systems are given in [15, 16]. Here we consider the variable coefficient KdV equation, with the help of pseudo-potential, the high order nonlocal symmetries of this equation are obtained. Finally, by introducing new variables variables, the nonlocal symmetry is transformed into local symmetry and exact solutions are constructed by using the lie group theory.

This paper is arranged as follows: In Sec.2, the nonlocal symmetries were constructed by using the pseudopotential of variable coefficient KdV equation. In Sec.3, the process of transforming from nonlocal symmetries to local symmetries was introduced in detail. The finite symmetry transformation can be obtained by solving the initial value problem, and new exact solutions were constructed by using known solutions. In Sec.4 some symmetry reductions and exact solutions of the KdV equation were obtained by using the Lie point symmetry of closed system. Finally, some conclusions and discussions are given in Sec.5.

2. Nonlocal symmetries of variable coefficient KdV equation

The time-dependent coefficient KdV equation[17] reads

$$u_t = u_{xxx} + 6uu_x + Gu_x.$$  (1)

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where \( u = u(x,t) \) are the real functions, \( G = G(t) \) is a real function of \( t \). When \( G = 0 \), Eq.(1) reduce to the well-known KdV equation, and this equation can be derived from the similarity reductions of the Kadomtsev-Petviashvili equation. Pseudo-potentials, Lax pairs and Bäcklund transformations have been studied in [17]. The soliton solutions of (1) with time varying boundary conditions have been also studied by Chan and Li[18]using inverse scattering mathod. To our knowledge, nonlocal symmetries for Eq.(1) have not been obtained and discussed, which will be the goal of this paper.

The corresponding pseudo-potential has been obtained in[17],

\[
\begin{align*}
\phi_{xt} &= -u\phi, \\
\phi_t &= (G + 2u)\phi_x - u\phi. 
\end{align*}
\]

To seek the nonlocal symmetries of variable coefficient KdV equation(1), one must solve the following linearized equations,

\[
\begin{align*}
\sigma^1_t - \sigma^1_{xxx} - 6\sigma^1 u_x - 6u\sigma^1_x - G\sigma^1_x &= 0, \\
\sigma^1, \sigma^2 &\text{ are symmetries of } u \text{ and } G, \text{ which means Eqs.(1) is form invariant under the transformations,}
\end{align*}
\]

\[
\begin{align*}
u &\to v + \epsilon\sigma^1, \\
G &\to G + \epsilon\sigma^2,
\end{align*}
\]

with the infinitesimal parameter \( \epsilon \). Be different from Lie point symmetries, we assume nonlocal symmetries of the Eq.(1)have the following form,

\[
\begin{align*}
\sigma^1 &= Xu_t + T u_x - U, \\
\sigma^2 &= T G_t - \Delta,
\end{align*}
\]

where \( X, T, U, \Delta \) are functions of \( [x, t, u, G, \phi, \psi] \). By substituting Eq.(5) into Eq.(3) and eliminating \( u_t, \phi_{xx}, \psi \) in terms of Eq.(1) and pseudo-potential Eqs.(3), it yields a system of determining equations for the functions \( X, T, U, \Delta \), solving these determining equations can obtain,

\[
\begin{align*}
X &= \frac{1}{3}x\tilde{F}_1 + F_2, \\
T &= \tilde{F}_1, \\
U &= \tilde{c}_2 u + \frac{2}{3}\tilde{c}_1\phi + \tilde{c}_3\phi \phi_x + \tilde{c}_5\phi^2 + \tilde{c}_4, \\
\Delta &= \frac{1}{3}x\tilde{F}_1 t - \frac{1}{3}(2G + 12u)\tilde{F}_1 t - \tilde{F}_2 t - (3\tilde{c}_2\phi^2 + 6\tilde{c}_1)u - 6\tilde{c}_5\phi^3 - 6\tilde{c}_4,
\end{align*}
\]

where \( \tilde{c}_i (i = 1, ..., 5) \) are five arbitrary constants and \( \tilde{F}_1, \tilde{F}_2 \) are arbitrary functions of \( t \).

**Remark 1**: It is show that the results(6) are high order nonlocal symmetries of variable coefficient KdV equation when \( \tilde{c}_2 \) or \( \tilde{c}_3 \neq 0 \), and they are local symmetries when \( \tilde{c}_2 = \tilde{c}_3 = \tilde{c}_5 = 0 \).

Nonlocal symmetries need to be transformed into local ones[8, 9] before construct exact solutions. Hence, We need to construct a new system called closed system, and Lie symmetries of the closed system contain the nonlocal symmetries obtained above.

3. Localization of the nonlocal symmetry

For simplicity, letting \( \tilde{c}_1 = \tilde{c}_2 = \tilde{c}_4 = \tilde{c}_5 = 0, \tilde{c}_3 = 1, \tilde{F}_1(t) = \tilde{F}_2(t) = 0 \) and introducing an auxiliary variable \( \psi = \phi_x \) in formula (6) i.e.,

\[
\sigma^1 = -\phi\psi, \\
\sigma^2 = 0.
\]

To localize the nonlocal symmetry (7), we have to solve the following linearized equations,

\[
\begin{align*}
\sigma^1_t + \sigma^1 \phi + u\sigma^3 &= 0, \\
\sigma^1_t - \sigma^2 \psi - G\sigma^3 - 2\sigma^1 \psi - 2u\sigma^3 + \sigma^1 \phi + u_x\sigma^3 &= 0, \\
\sigma^3 &= \sigma^3_x
\end{align*}
\]

which is form invariant under the following transformation,

\[
\begin{align*}
\phi \to \phi + \epsilon\sigma^1, \\
\psi \to \psi + \epsilon\sigma^3,
\end{align*}
\]

with the infinitesimal parameter \( \epsilon \), and \( \sigma^1, \sigma^2 \) given by (7). It is not difficult to verify that the solutions of (8) have the following forms,

\[
\begin{align*}
\sigma^3 &= \phi f, \\
\sigma^4 &= \phi_x f + \phi f_x,
\end{align*}
\]
where \( f \) satisfies the following equations,

\[
\begin{align*}
    f_x &= \frac{\phi}{4}, \\
    f_t &= \frac{\phi^2 G}{2} - \frac{\phi u}{2} - \psi^2,
\end{align*}
\]

(11)

it is easy to obtain the following result,

\[
    \sigma^2 = f^2,
\]

(12)

which is form invariant under,

\[
    f \rightarrow f + \epsilon \sigma^5.
\]

(13)

One can see that the nonlocal symmetry (7) in the original space \( \{x, t, u, G, \phi, \psi, f\} \) has been successfully localized to a Lie point symmetry in the enlarged space \( \{x, t, u, G, \phi, \psi, f\} \). It is not difficult to verify that the auxiliary dependent variable \( f \) just satisfies the Schwartzian form of the variable coefficient KdV equation,

\[
    \frac{f_x}{f_t} = \{f; x\} - G = 0
\]

(14)

where \( \{f; x\} = \left( \frac{f_x}{f_t} \right)_x - \frac{1}{4} \left( \frac{f_t}{f_x} \right)^2 \) is the Schwartzian derivative.

After we successfully transform the nonlocal symmetries (7) into local symmetries. New exact solutions can be constructed naturally by Lie group theory. With the Lie point symmetry (7), (10), (12), by solving the following initial value problem,

\[
\begin{align*}
    \frac{d\tilde{u}(\epsilon)}{d\epsilon} &= \tilde{\phi}(\epsilon)\tilde{\phi}(\epsilon), \\
    \tilde{u}(0) &= u, \\
    \frac{d\tilde{\phi}(\epsilon)}{d\epsilon} &= 0, \\
    \tilde{G}(0) &= G, \\
    \frac{d\tilde{f}(\epsilon)}{d\epsilon} &= -\tilde{\phi}(\epsilon)\tilde{\phi}(\epsilon), \\
    \tilde{\phi}(0) &= \phi, \\
    \frac{d\tilde{G}(\epsilon)}{d\epsilon} &= -\frac{1}{\epsilon^2}\tilde{\phi}^2(\epsilon) - \tilde{f}(\epsilon)\tilde{\phi}(\epsilon), \\
    \tilde{\phi}(0) &= \psi, \\
    \frac{d\tilde{f}(\epsilon)}{d\epsilon} &= -\frac{1}{\epsilon^2}\tilde{\phi}^2(\epsilon), \\
    \tilde{f}(0) &= f,
\end{align*}
\]

(15)

where \( \epsilon \) is the group parameter, we arrive at the symmetry group theorem as follows:

**Theorem 1.** If \( \{u, G, \phi, \psi, f\} \) is the solution of the prolonged system (1)(2) and (11), so is \( \{\tilde{u}, \tilde{G}, \tilde{\phi}, \tilde{\psi}, \tilde{f}\} \)

\[
\begin{align*}
    \tilde{u}(\epsilon) &= \frac{8\epsilon^2 f u - \epsilon^2 \phi^4 + 8\epsilon^2 f \phi - 16\epsilon(u - 8\epsilon \phi + 8u)}{8\epsilon^2 f^2 - 2\epsilon f + 1}, \\
    \tilde{G}(\epsilon) &= G, \\
    \tilde{\phi}(\epsilon) &= \frac{\phi}{\epsilon f}, \\
    \tilde{\psi}(\epsilon) &= \frac{\psi}{\epsilon f}, \quad \tilde{f}(\epsilon) = \frac{f}{\epsilon f},
\end{align*}
\]

(16)

with \( \epsilon \) is an arbitrary group parameter.

Here we give a simple example, starting from a trivial solution of (1)

\[
    u = 0, G = 1,
\]

(17)

it’s not difficult to derive the special solutions for the variables \( \phi, \psi, f \) from(2) and (11),

\[
    \phi = \tilde{c}, \psi = 0, f = \frac{\tilde{c}^2}{4}(t + x).
\]

(18)

Using theorem 1, it’s not hard to verify

\[
    u = \frac{\tilde{c}^2 \phi}{\epsilon \psi}, \quad G = 1, \quad \phi = \frac{\tilde{c}}{\epsilon \psi}, \quad \psi = \frac{\tilde{c}^2}{\epsilon \phi}, \quad f = \frac{\tilde{c}^2(t + x)}{4}.
\]

(19)

are still the solutions to the system (1), (2) and (11), where \( \tilde{c} = \tilde{c}^2 (t + x) - 4, \tilde{c} \) is an arbitrary constant. One can get more solutions by repeating the theorem 1. These solutions cannot be obtained by traditional Lie group method, Darboux transformation method, etc. Therefore, a series of new exact solutions of (1) can be constructed.

To search for more similarity reductions and exact solutions of Eq.(1), we use classical Lie group method. Assume the symmetries of whole prolonged system have the vector form,

\[
    V = \tilde{X} \frac{\partial}{\partial x} + \tilde{T} \frac{\partial}{\partial t} + \tilde{U} \frac{\partial}{\partial u} + \tilde{G} \frac{\partial}{\partial G} + \tilde{P} \frac{\partial}{\partial \phi} + \tilde{F} \frac{\partial}{\partial f},
\]

(19)

where \( \tilde{X}, \tilde{T}, \tilde{U}, \tilde{G}, \tilde{P}, \tilde{F} \) are the functions with respect to \( x, t, u, G, \phi, \psi, f \), which means that the closed system is invariant under the transformations

\[
(x, t, u, v, G, \phi, \psi, f) \rightarrow (x + \epsilon \tilde{X}, t + \epsilon \tilde{T}, u + \epsilon \tilde{U}, G + \epsilon \tilde{G}, \phi + \epsilon \tilde{P}, f + \epsilon \tilde{F}),
\]

(20)
with a small parameter $\epsilon$. Symmetries in the vector form (19) can be assumed as

$$
\begin{align*}
\sigma^1 &= \bar{X}u_t + \bar{T}u_t - \bar{Q}, \\
\sigma^2 &= \bar{T}G_t - \bar{Q}, \\
\sigma^3 &= \bar{X}\phi_t + \bar{T}\phi_t - \bar{P}, \\
\sigma^4 &= \bar{X}f_t + \bar{T}f_t - \bar{F},
\end{align*}
$$

(21)

where $\bar{X}, \bar{T}, \bar{U}, \bar{\Delta}, \bar{P}, \bar{F}$ are the functions with respect to $\{x, t, u, G, \phi, \phi, f\}$. And $\sigma^i, (i = 1, ..., 4)$ satisfy the linearized equations of the prolonged system, i.e.,

$$
\begin{align*}
\sigma^1_x - \sigma^1_{u_2} - 6\sigma^1 u_t - 6uu_2 - c^2u_x - G\sigma^1 &= 0, \\
\sigma^2_x + \sigma^2\phi + u\sigma^2 &= 0, \\
\sigma^3_x - \sigma^2\phi - G\sigma^3 - 2\sigma^1\phi_t - 2u\sigma^3 + \sigma^3\phi + u_c\sigma^3 &= 0, \\
\sigma^4_x - \frac{1}{3}\sigma^3\phi &= 0, \\
\sigma^4_x + \sigma^3\phi u + \frac{1}{3}\sigma^3\phi^2 - \frac{1}{3}\sigma^3\phi G - \frac{1}{3}\sigma^2\phi^2 + 2\sigma^3\phi_x &= 0,
\end{align*}
$$

(22)

Substituting Eqs.(21) into Eqs.(22) and eliminating $u_t, \phi_t, \phi, f_t, f$ in terms of the closed system, determining equations for the functions $\bar{X}, \bar{T}, \bar{U}, \bar{\Delta}, \bar{P}, \bar{Q}, \bar{F}$ can be obtained, by solving these equations, one can get

$$
\begin{align*}
\bar{X} &= \frac{c_1}{c_3} + \bar{F}_3, \\
\bar{T} &= c_t + c_2, \bar{U} = -\frac{c_4}{c_3} + c_3\phi, \bar{\Delta} = -\frac{2c_4G}{c_3} - \bar{F}_3t, \\
\bar{P} &= -\phi(c_3f - c_4), \bar{F} = -c_3f^2 + \frac{c_4G}{c_3} + c_5,
\end{align*}
$$

(23)

where $c_i, (i = 1, 2, ..., 5)$ are arbitrary constants, $\bar{F}_3 = \bar{F}_3(t)$ is arbitrary function of $t$.

4. Symmetry reduction and exact solutions of variable coefficient KdV equation

In this section, we will give two nontrivial similarity reductions and group invariant solutions of variable coefficient KdV equation(1)under consideration $c_3 \neq 0$. Without loss of generality, we let $c_1 = \bar{F}_3 = 0$. By solving the following characteristic equation,

$$
\frac{dx}{c_3} = \frac{dt}{c_3\phi} = \frac{du}{c_2} = \frac{dG}{c_4\phi - c_3f} = \frac{df}{c_3 + 2c_3f - c_3f^2},
$$

(24)

one can obtain

$$
\begin{align*}
G &= C, \quad f = \frac{\tan \theta + c_4}{c_3} \phi, \quad \bar{F}_3 = 2F_2(x) \sqrt{\theta} - 1, \\
u &= \frac{1}{c_3}c_3\bar{F}_2(x)F_1(x) \tanh^2(\theta) + \frac{1}{c_3^2}c_3\bar{F}_2(x)F_2(x)F_1(x) = \frac{c_3^2F_2(x)F_2(x)}{c_3F_2(x)F_1(x)F_3(x)},
\end{align*}
$$

(25)

where $\alpha = \sqrt{c_3c_5 + c_4^2}, \Theta = \frac{\alpha F_2(x)}{c_3}$.

Substituting Eqs.(25)into the prolonged system yields,

$$
\begin{align*}
F_2(x) &= \pm \frac{2c_3c_5F_1(x)}{c_3^2c_5 - c_3^2c_5F_1(x)}, \\
F_3(x) &= \frac{1}{2c_2F_2(x)} \left(8c_3c_5 + c_4^2\right)F_1^2 + 4c_3c_5 + c_4^2F_1^2 + c_4^2F_1^2 + 2c_3c_5F_1F_1 + 2c_3c_5F_1^2 - 2c_2^2F_1F_1F_1,
\end{align*}
$$

(26)

One can see that through the Eqs.(25)and (26), if we know the form of $F_1(x)$, then $u$ can be obtained directly. We known that auxiliary dependent variable $f$ satisfies the Schwartzian form, by substituting $f = \frac{\tan \theta + c_4}{c_3}$ into (14), one can get,

$$
2c_3^2FF_{xx} - 3c_3^2F_x^2 - 4(c_3c_5 + c_4^2)F^4 + 2c_3^2c_5F^2 - 2c_2^2F = 0,
$$

(27)

where $F = F(x) = F_1x$.

It is not difficult to verify that the above equation is equivalent to the following elliptic equation,

$$
F_x = \frac{\sqrt{4(c_3c_5 + c_4^2)F_2^2 + c_4^2c_5F_1^2 + 2c_3c_5c_6F_2^2 - c_2^2F}}{c_2}.
$$

(28)

It is known that the general solution of Eq.(28) can be written in terms of Jacobi elliptic functions. Hence, expression of solution (25) reflects the wave interaction between the soliton and the Elliptic function periodic wave. A simple solution of Eq.(28) is given as,

$$
F = a_0 + a_1 sn(x, n) + a_2 sn^2(x, n),
$$

(29)
where \( sn(x, n) \) is Jacobi elliptic function, substituting Eq.(29) into Eq.(28) yields following six solutions,

\[
\begin{align*}
[c_3] &= -\frac{c_1^2}{c_3^2}, \quad c_b = 4 - 2n^2, a_0 = \frac{1}{4n-4}, \quad a_1 = 0, \quad a_2 = \frac{a_3^2}{4n-4};
[c_3] &= -\frac{c_1^4}{c_3^2}, \quad c_b = 4n^2 - 2, \quad a_0 = \frac{1}{4n^2-4}, \quad a_1 = 0, \quad a_2 = \frac{-1}{4n^2-4};
[c_3] &= \frac{c_1^3(4n^2-2)+c_2^2}{c_3^2}, \quad c_b = -\frac{a_2^2}{12}, a_0 = \frac{a_3^2}{2n^2-2}, \quad a_1 = \frac{a_3^2}{2n^2-2}, \quad a_2 = 0; \\
[c_3] &= \frac{c_1^3(4n^2-2)+c_2^2}{c_3^2}, \quad c_b = \frac{a_2^2}{12}, a_0 = \frac{-a_3^2}{2n^2-2}, \quad a_1 = \frac{-a_3^2}{2n^2-2}, \quad a_2 = 0;
\end{align*}
\]

with \( c_2, c_4, c_5 \in R, 0 \leq n \leq 1 \).

Substituting Eqs.(30),(29) and \( F_1 = F \) into Eq.(26), one can obtain the solutions of \( u \). Because the expression is too prolix, it is omitted here. In order to study the properties of these solutions of KdV equation, we give some pictures of \( u \) as following.

**Case 1:** \( a_2 = 0 \)

In Fig.1, we plot the interaction solutions between solitary waves and elliptic function waves expressed by \( \{c_3 = \frac{c_1^3(4n^2-2)+c_2^2}{c_3^2}, \quad c_b = -\frac{a_2^2}{12}, a_0 = \frac{a_3^2}{2n^2-2}, \quad a_1 = \frac{a_3^2}{2n^2-2}, \quad a_2 = 0\} \) with parameters \( c_2 = 1, c_3 = 0.1, n = 0.8 \). We can see that the component \( u \) exhibits a soliton propagates on Jacobi elliptic sine function waves background for two cycles. In Fig.1, the first picture(a) shows that the height of the soliton is approximately \( 6 \) at \( t = -10 \). With the development of time, soliton produces elastic collisions with other waves, and the height of the soliton is changing continuously. Picture(e) shows that soliton reaches its highest height at \( t = -2 \). After the collision, the soliton reverts to the original height and continues to collide with the adjacent waves see the pictures (\( f \to i \)). The corresponding 3d image is given which exhibits a soliton propagating on period waves background.

**Case 2:** \( a_1 = 0 \)

In Figs.2, we plot another form of interaction solutions between solitary waves and elliptic function waves expressed by \( \{c_3 = \frac{c_1^3(4n^2-2)+c_2^2}{c_3^2}, \quad c_b = \frac{a_2^2}{12}, a_0 = \frac{-a_3^2}{2n^2-2}, \quad a_1 = \frac{-a_3^2}{2n^2-2}, \quad a_2 = 0\} \) with \( c_2 = c_3 = c_4 = c_5 = 1, \quad n = 0.8 \). The corresponding 3d image is given. For other types of solutions, we’re not going to give their figures here.

### 5. Summary and Discussion

In this paper, we have studied high order nonlocal symmetries and exact solutions of the variable coefficient KdV equation for the first time. First of all, starting from the known pseudo-potential of the variable coefficient equation, nonlocal symmetries are derived directly through a particular assumption. In order to take advantage of the nonlocal symmetries, auxiliary variables \( \psi, f \) are introduced. Then, the primary nonlocal symmetry is equivalent to a Lie point symmetry of a prolonged system. Applying the Lie group theorem to these local symmetries, the corresponding group invariant solutions are derived. Secondly, several classes of exact solutions are provided in the paper, including some special forms of exact solutions. For example, exact interaction solutions among soliton and other complicated waves. In fact, it is of interest to study these types of solutions, for example, in describing localized states in optically refractive index gratings. In the ocean, there are some typical nonlinear waves such as the solitary waves and the cnoidal periodic waves.

It is very meaningful to study the nonlocal symmetries and exact solutions of variable coefficient integrable models. However, there is still a lot of work to be done. For example, in a large number of nonlocal symmetries of an integrable model which one can be localized. Is it possible to apply the nonlocal symmetry theory of constant coefficient differential equation to the variable coefficient differential equation? Above topics will be discussed in the future series research works.

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