Optimal approximation order of piecewise constants on convex partitions

Oleg Davyдов ∗  Oleksandr Kozyненко †  Dmytro Skorokhodov ‡

December 3, 2019

Abstract

We prove that the error of the best nonlinear $L_p$-approximation by piecewise constants on convex partitions is $O(N^{-\frac{d+1}{d+2}})$, where $N$ is the number of cells, for all functions in the Sobolev space $W^2_q(\Omega)$ on a cube $\Omega \subset \mathbb{R}^d$, $d \geq 2$, as soon as $\frac{2}{d+1} + \frac{1}{p} - \frac{1}{q} \geq 0$. The approximation order $O(N^{-\frac{d+1}{d+2}})$ is achieved on a polyhedral partition obtained by anisotropic refinement of an adaptive dyadic partition. Further estimates of the approximation order from the above and below are given for various Sobolev and Sobolev-Slobodeckij spaces $W^r_q(\Omega)$ embedded in $L^p(\Omega)$, some of which also improve the standard estimate $O(N^{-\frac{d}{d+1}})$ known to be optimal on isotropic partitions.

1 Introduction

Nonlinear approximation with piecewise polynomials seeks a better approximation in comparison to linear methods by adapting the spline to the local behavior of the data assuming it stems from a function in a smoothness space. Initial multivariate results of this type (Birman and Solomyak [3]) were obtained by adjusting partitions to the local Sobolev energy of the function, but more recent research focuses on the $n$-term approximation with appropriate wavelet-like bases [7, 13] and adaptive algorithms that recursively reduce the error function [6]. Very little beyond the results of [3] is known in the multivariate “free partition” setting, see e.g. [7, Section 6.5].

In this paper we continue investigations in [4, 5] and study approximation properties of piecewise constants on arbitrary convex partitions that are freely adjusted to functions in Sobolev spaces. This adjustment, and notably the use of anisotropic convex polyhedral partitions allows to nearly double the approximation order in comparison to “isotropic” partitions employed in [3]. Interestingly, these results do not extend to higher order splines, where introducing “anisotropy” to a partition does not bring any improvement in the order of approximation [5] (see also [2]), and a promising approach is to consider piecewise polynomials on several overlaid polyhedral partitions [6].

∗Department of Mathematics, Justus Liebig University, Arndtstrasse 2, 35392 Giessen, Germany
†Department of Mechanics and Mathematics, Oles Honchar Dnipro National University, pr. Gagarina 72, 49010 Dnipro, Ukraine
‡Department of Mechanics and Mathematics, Oles Honchar Dnipro National University, pr. Gagarina 72, 49010 Dnipro, Ukraine
In comparison to [5] we obtain improved convergence orders on much wider classes of Sobolev spaces in the setting typical for nonlinear approximation where the $L_p$-metric error is considered for $L_q$-metric classes of functions with $q < p$. To achieve this we generalize the cell counting techniques introduced in [3, Theorem 2.1].

Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a bounded domain. A finite collection $\Delta$ of subdomains $\omega \subset \Omega$, referred to as cells throughout the paper, is called a partition of $\Omega$ provided that $\omega \cap \omega' = \emptyset$, for every $\omega, \omega' \in \Delta$, $\omega \neq \omega'$, and $\sum_{\omega \in \Delta} |\omega| = |\Omega|$, where $|\cdot|$ stands for the Lebesgue measure in $\mathbb{R}^d$. We call a partition $\Delta$ of $\Omega$ convex if every cell $\omega \in \Delta$ is convex. For $N \in \mathbb{N}$, denote by $\mathcal{D}_N$ the set of all convex partitions of $\Omega$ comprising at most $N$ cells. For simplicity, we assume that $\Omega$ is a cube in $\mathbb{R}^d$, even if the results of the paper may be naturally extended to more general domains by for example splitting them into a finite number of affine images of cubes.

For $1 \leq q \leq \infty$ and $r \in \mathbb{N}$, let $W^r_q(\Omega)$ be the Sobolev space of measurable functions $f : \Omega \to \mathbb{R}$ endowed with the standard norm and seminorm

$$
\|f\|_{W^r_q(\Omega)} := \sum_{k \in \mathbb{Z}_+^d : |k| \leq r} \|D^k f\|_{L^q(\Omega)} \quad \text{and} \quad |f|_{W^r_q(\Omega)} := \sum_{k \in \mathbb{Z}_+^d : |k| = r} \|D^k f\|_{L^q(\Omega)},
$$

where $k = (k_1, \ldots, k_d) \in \mathbb{Z}_+^d$ is non-negative integer multi-index, $|k| := k_1 + \cdots + k_d$, and $D^k f$ denotes the (generalized) partial derivative of order $k$. We set $W^0_q(\Omega) := L^q(\Omega)$. For $f \in W^1_q(\Omega)$, we denote by $\nabla f$ the gradient of $f$.

For $1 \leq q < \infty$ and $r \in \mathbb{N}_+ \setminus \mathbb{Z}_+$, we consider the Sobolev-Slobodeckij space [12]:

$$
W^r_q(\Omega) := \left\{ f \in W^1_q(\Omega) : |f|_{W^r_q(\Omega)} < \infty \right\}, \quad \|f\|_{W^r_q(\Omega)} := \|f\|_{W^1_q(\Omega)} + |f|_{W^r_q(\Omega)},
$$

where

$$
|f|_{W^r_q(\Omega)} := \sup_{k \in \mathbb{Z}_+^d : |k| = \lfloor r \rfloor} \left( \int_{\Omega} \int_{\Omega} \frac{|D^k f(x) - D^k f(y)|^q}{|x - y|^{(r)q+d}} \, dx \, dy \right)^{\frac{1}{q}};
$$

$\lfloor r \rfloor$ and $\{ r \}$, respectively, are the integer and fractional parts of $r$, and $|x|$ stands for the Euclidean norm of $x \in \mathbb{R}^d$. Spaces $W^r_q(\Omega)$ coincide with the Besov spaces $B^r_{q,q}(\Omega)$ (see [14] p. 323).

For a partition $\Delta$ of $\Omega$, we denote by $\mathcal{S}_0(\Delta)$ the space of piecewise constant functions $s : \Omega \to \mathbb{R}$ that are constant on every cell $\omega \in \Delta$. For $1 \leq p \leq \infty$, we define the error of the best $L_p$-approximation of a function $f \in L^p(\Omega)$ by piecewise constant functions on partitions from $\mathcal{D}_N$:

$$
E_N(f)_p := \inf_{\Delta \in \mathcal{D}_N} \inf_{s \in \mathcal{S}_0(\Delta)} \|f - s\|_{L^p(\Omega)}.
$$

Since the partition $\Delta$ is allowed to depend on $f$, the optimal or near-optimal approximations $s$ do not belong to a linear space of functions when $N$ is fixed, and the methods of constructing them are referred to as nonlinear approximation [7]. There are very few results in the literature in this “free partition” setting, see [7] Section 6.5 and [4]. In what follows we will consider estimates for the order of approximation of functions $f \in W^r_q(\Omega)$ with $q < \infty$ only. Although results obtained in this paper hold true for functions $f \in W^r_\infty(\Omega)$, they also follow immediately from the results obtained in [5]. This is the main motivation for us to avoid consideration of the case $q = \infty$.

Birman and Solomyak’s result on piecewise polynomial approximation [3, Theorem 3.2] implies that

$$
E_N(f)_p = O(N^{-\frac{5}{q}}), \quad f \in W^r_q(\Omega), \quad 0 < r \leq 1,
$$

(1)
as soon as $q$ satisfies
\[
\frac{r}{d} + \frac{1}{p} - \frac{1}{q} > 0 \quad \text{and} \quad q < \infty,
\] (2)
where in the case $p = \infty$ we set $\frac{1}{p} := 0$. Note that the construction of a sequence of partitions $\{\triangle_N\}_{N=1}^{\infty}$ that attains the order in (1) for $q < p$ is based on adaptively refined dyadic subdivisions of $\Omega$. Considering $r > 1$, it follows from the embedding $W^r_q(\Omega) \subset W^{1,q'}(\Omega)$, with $\frac{1}{q'} = \frac{r}{d} + \frac{1}{q}$ (see [14, p. 328]) that
\[
E_N(f)_p = O\left(N^{-\frac{1}{d}}\right), \quad f \in W^r_q(\Omega), \quad r \geq 1,
\] (3)
under the same restrictions (2).

No improvement of the order $N^{-\frac{1}{d}}$ is possible for any smooth non-constant function $f$ on “isotropic” partitions (see [4]). Recall that a sequence of convex partitions $\{\triangle_N\}_{N=1}^{\infty}$ is said to be isotropic if there exists $\gamma > 0$ such that $\text{diam}(\omega) \leq \gamma \rho(\omega)$ for all $\omega \in \bigcup_{N \in \mathbb{N}} \triangle_N$, where $\text{diam}(\omega)$ is the diameter of the cell $\omega$ and $\rho(\omega)$ is the diameter of the largest $d$-dimensional ball contained in $\omega$. If this condition is not satisfied, then we say that the partitions in the sequence are “anisotropic”. It was shown in [4, 5] (and earlier in [9] for $d = 2$) that on a wider set of all convex partitions $\mathfrak{Q}_N$ significantly better order of approximation can be achieved. More precisely, for $1 \leq p \leq 2$, we have
\[
E_N(f)_p = \mathcal{O}\left(N^{-\frac{2}{d}+1}\right), \quad f \in W^2_p(\Omega),
\] which almost doubles the approximation order in comparison to $N^{-\frac{1}{d}}$ when $d$ is large. This improvement in order is obtained on a sequence of anisotropic partitions constructed in two steps by first subdividing $\Omega$ uniformly into smaller sub-cubes and then splitting each of the sub-cubes with the help of equidistant hyperplanes orthogonal to the average gradient of $f$ on the sub-cube. Moreover, it is shown in [5] that $N^{-\frac{2}{d}+1}$ is the saturation order of piecewise constant approximation on convex partitions in the following sense: if $E_N(f)_{\infty} = \mathcal{O}(N^{-\frac{2}{d}+1})$ as $N \to \infty$ for a twice continuously differentiable function $f : \Omega \to \mathbb{R}$, then the Hessian of $f$ cannot be either positive definite or negative definite at any point in $\Omega$. This result has been extended to $E_N(f)_p$ for all $1 \leq p < \infty$ in [10].

In this paper we are interested in determining Sobolev and Sobolev-Slobodeckij spaces $W^r_q(\Omega)$ for which the order of $E_N(f)_p$ is $\mathcal{O}(N^{-\frac{2}{d}+1})$ or at least $\mathcal{O}(N^{-\frac{1}{2}})$, $N \to \infty$, such that anisotropic partitions provide superior convergence. Our main result is the estimate (Theorem 3.2)
\[
E_N(f)_p \leq C N^{-\frac{2}{d}+1} \|f\|_{W^2_p(\Omega)}, \quad f \in W^2_p(\Omega),
\] (4)
for all $q < \infty$ satisfying
\[
\frac{2}{d+1} + \frac{1}{p} - \frac{1}{q} \geq 0,
\]
where $C$ depends only on $d, p, q$. The optimal order $N^{-\frac{2}{d}+1}$ is achieved on a sequence of adaptively refined dyadic subdivisions that subsequently undergo anisotropic refinement (see Algorithm 3.1). The proof of (4) relies on a generalization of Birman-Solomyak’s cell counting techniques [3, Theorem 2.1] to the case when the dyadic cells may carry varying numbers of degrees of freedom (see Section 2), which seems to be of independent interest.
Since the spaces \(W^2_q(\Omega)\) are embedded into \(L^p(\Omega)\) whenever \(\frac{2}{d} + \frac{1}{p} - \frac{1}{q} \geq 0\) when \(p < \infty\) or \(\frac{2}{d} - \frac{1}{q} > 0\) when \(p = \infty\) [1, Theorem 4.12], this leaves open the question on the order of \(E_N(f)_p\) as \(N \to \infty\) for functions \(f \in W^2_q(\Omega)\) when the parameters

\[1 \leq p \leq \infty\text{ and } 1 \leq q < \infty\]

satisfy

\[
\frac{2}{d} + \frac{1}{p} < \frac{1}{q} \leq \frac{2}{d} + \frac{1}{p}, \quad \text{if } p < \infty,
\]

\[
\frac{2}{d} + \frac{1}{p} < \frac{1}{q} < \frac{2}{d} + \frac{1}{p}, \quad \text{if } p = \infty.
\]

We partially fill this gap in Theorem 3.3 by showing that under these conditions

\[E_N(f)_p = O\left(N^{-\frac{d}{2}}\right), \quad f \in W^2_q(\Omega).\]

This proves in particular that

\[E_N(f)_p = o\left(N^{-\frac{d}{2}}\right), \quad N \to \infty, \quad f \in W^2_q(\Omega),\]

as soon as

\[
\frac{2}{d} + \frac{1}{p} - \frac{1}{q} > \frac{1}{d^2}.
\]

For the remaining range of parameters \(p\) and \(q\) the estimate \(E_N(f)_p = O\left(N^{-\frac{d}{2}}\right)\) that follows directly from (3) remains the best known for \(f \in W^2_q(\Omega)\). The above results extend to Sobolev and Sobolev-Slobodeckij spaces \(W^r_q(\Omega)\) with \(r > 2\) by embedding, see Corollary 3.4.

Finally, we consider the question whether the estimates (1) and (3) can be improved for \(r < 2\). Theorem 4.1 shows that for all \(1 \leq q < \infty\) and \(\frac{4}{q} < r < 2\),

\[
\sup_{\|f\|_{W^r_q(\Omega)} \leq 1} E_N(f)_\infty \neq O\left(N^{-\frac{2d}{d+1}}\right), \quad N \to \infty.
\]

(Recall that \(r > \frac{4}{q}\) ensures that \(W^r_q(\Omega)\) is embedded into \(L_{\infty}(\Omega)\).) This shows in particular that (1) cannot be improved for \(p = \infty\). Another consequence is

\[
\sup_{\|f\|_{W^r_q(\Omega)} \leq 1} E_N(f)_\infty \neq O\left(N^{-\frac{4}{d+1}}\right) \quad \text{if } r < \frac{2d}{d+1}.
\]

Hence an estimate of the type \(E_N(f)_\infty \leq C N^{-\frac{4}{d+1}}\|f\|_{W^r_q(\Omega)}\) cannot hold for all \(f \in W^r_q(\Omega)\) when \(r < \frac{2d}{d+1}\), leaving only a small gap of \(\frac{2d}{d+1} \leq r < 2\) open for the possibility of extending (4) to the spaces \(W^r_q(\Omega)\) with \(r < 2\) in the case \(p = \infty\).

The paper is organized as follows. We devote Section 2 to auxiliary results that generalize Theorem 2.1 in Birman-Solomyak’s paper [3]. The main results about the order of the best nonlinear approximation by piecewise constants are presented in Section 3 (estimates from above) and Section 4 (estimates from below).

2 Cell counting

Let \(\Omega\) be a cube \((a_1, a_1 + h) \times \cdots \times (a_d, a_d + h)\) in \(\mathbb{R}^d, d \geq 1\), with side length \(h\). We follow [3] to define dyadic subdivisions of \(\Omega\). Let \(\Box\) be a partition of \(\Omega\) into a finite
number of open cubes. A partition $\square'$ of $\Omega$ is an elementary extension of $\square$ if it can be obtained from $\square$ by uniformly splitting some of its cubes into $2^d$ equal open cubes with halved side length. We call a partition $\square$ of $\Omega$ a dyadic subdivision if it is obtained from the singleton partition $\{\Omega\}$ with the help of a finite number of elementary extensions.

A crucial tool in establishing the main results of [3] including the estimate (1) is provided by Theorem 2.1 in [3] that gives an upper bound on the number of cells in a specially constructed dyadic subdivision of $\Omega$.

Similarly, the main ingredient for obtaining our estimate (4) is an upper bound on the number of cells in a specially constructed convex partition $\triangle$ of $\Omega$. The partition $\triangle$ is built in two steps by first constructing a dyadic subdivision $\square$ of $\Omega$ and then splitting every cube $\omega \in \square$ anisotropically with the help of parallel hyperplanes orthogonal to the average gradient of $f$ on $\omega$, where the number of hyperplanes depends on the size of $\omega$ relative to $\Omega$, see Algorithm 3.1. In order to obtain this bound we extend the techniques of [3] to the case when each dyadic subcube in $\square$ is assigned a different number of degrees of freedom depending on its size.

As in [3] we say that a non-negative function $\Phi$ of subcubes in $\Omega$ is subadditive if

$$\sum_{\omega' \in \square} \Phi(\omega') \leq \Phi(\omega)$$

whenever $\square$ is a dyadic subdivision of a subcube $\omega$ of $\Omega$. For a cube $\omega \subset \Omega$, we set

$$g_\alpha(\omega) := |\omega|^{\alpha} \Phi(\omega) \quad \text{and} \quad N_\gamma(\omega) := \left\lfloor \frac{|\Omega|}{|\omega|} \right\rfloor^\gamma, \quad \alpha, \gamma \geq 0,$$

and, for a dyadic subdivision $\square$ of $\Omega$, let

$$G_\alpha(\square) := \max_{\omega \in \square} g_\alpha(\omega).$$

Lemma 2.1. Let $\Omega$ be a cube in $\mathbb{R}^d$ and $\Phi$ be a non-negative subadditive function defined on subcubes in $\Omega$, and let $0 \leq \gamma \leq \alpha$. Assume that a sequence of dyadic subdivisions $\{\square_k\}_{k=0}^\infty$ of $\Omega$ is obtained recursively as follows: set $\square_0 := \{\Omega\}$ and, for $k \in \mathbb{N}$, construct an elementary extension $\square_k$ of $\square_{k-1}$ by subdividing all cubes $\omega \in \square_{k-1}$ satisfying

$$g_\alpha(\omega) \geq 2^{-d \alpha} G_\alpha(\square_{k-1})$$

into $2^d$ equal cubes. Then, for $k \in \mathbb{N}$,

$$G_\alpha(\square_k) \leq C_1(d, \gamma, \alpha) N_k^{\frac{\alpha+1}{\gamma+1}} |\Omega|^{\alpha} \Phi(\Omega),$$

where

$$N_k := \sum_{\omega \in \square_k} N_\gamma(\omega),$$

and the constant $C_1(d, \gamma, \alpha)$ depends only on $d$, $\gamma$ and $\alpha$. 

Remark 1. Lemma 2.1 in the case $\gamma = 0$ coincides with Theorem 2.1 in [3].

Proof. Let $k \in \mathbb{N}$. Denote by $S_k$ the set of all cubes from $\square_{k-1}$ that are subdivided to obtain $\square_k$, and set $t_k := \sum_{\omega \in S_k} N_\gamma(\omega)$. By the definition of $N_\gamma(\omega)$, we have a trivial estimate

$$t_k \leq |\Omega|^{1-\gamma} \sum_{\omega \in S_k} |\omega|^{-\gamma}.$$
By the construction of $\square_k$ it is also clear that
\[ 2^{-da} G_\alpha(\square_{k-1}) \leq \min_{\omega \in S_k} g_\alpha(\omega). \]
Combining the above inequalities and applying the definition of $g_\alpha$ we obtain
\[
(2^{-da} G_\alpha(\square_{k-1}))^{\frac{\alpha+1}{\alpha+1}} \leq \min_{\omega \in S_k} (g_\alpha(\omega))^{\frac{\alpha+1}{\alpha+1}} = \min_{\omega \in S_k} |\omega|^{a(\gamma+1)} \Phi^{\frac{\alpha+1}{\alpha+1}}(\omega)
\]
\[
\leq \left( \sum_{\omega \in S_k} N_\gamma(\omega) \right)^{-1} \sum_{\omega \in S_k} N_\gamma(\omega) |\omega|^{a(\gamma+1)} \Phi^{\frac{\alpha+1}{\alpha+1}}(\omega)
\]
\[
\leq \frac{|\Omega|^\gamma}{t_k} \sum_{\omega \in S_k} |\omega|^{\frac{\alpha+1}{\alpha+1}} \Phi^{\frac{\alpha+1}{\alpha+1}}(\omega).
\]
Applying the Hölder inequality with parameters $\frac{\alpha+1}{\alpha-\gamma}$ and $\frac{\alpha+1}{\gamma+1}$, and using subadditivity of $\Phi$, we have
\[
(2^{-da} G_\alpha(\square_{k-1}))^{\frac{\alpha+1}{\alpha+1}} \leq \frac{|\Omega|^\gamma}{t_k} \left( \sum_{\omega \in S_k} |\omega|^{\frac{\alpha+1}{\alpha-\gamma}} \right)^{\frac{\alpha-\gamma}{\alpha+1}} \left( \sum_{\omega \in S_k} \Phi(\omega) \right)^{\frac{\alpha+1}{\alpha+1}} \leq \frac{1}{t_k} |\Omega|^{\frac{\alpha+1}{\alpha+1}} \Phi^{\frac{\alpha+1}{\alpha+1}}(\Omega).
\]
The latter inequality can be rewritten as
\[
t_k \leq 2^{a(\gamma+1)\frac{\alpha}{\alpha+1}} (G_\alpha(\square_{k-1}))^{-\frac{\alpha}{\alpha+1}} |\Omega|^{\frac{\alpha+1}{\alpha+1}} \Phi^{\frac{\alpha+1}{\alpha+1}}(\Omega). \tag{9}
\]
Note that to obtain (9) in the case $\alpha = \gamma$, instead of the Hölder inequality we only need to use non-negativity and subadditivity of function $\Phi$: $\sum_{\omega \in S_k} \Phi(\omega) \leq \sum_{\omega \in \square_{k-1}} \Phi(\omega) \leq \Phi(\Omega)$.

Next, by the construction of dyadic subdivisions $\square_j$, $j \in \mathbb{N}$, we have
\[
G_\alpha(\square_j) \leq \max \left\{ 2^{-da} G_\alpha(\square_{j-1}); \max_{\omega \in \square_j \setminus \square_{j-1}} |\omega|^{a} \Phi(\omega') \right\} \tag{10}
\]
\[
\leq \max \left\{ 2^{-da} G_\alpha(\square_{j-1}); \max_{\omega \in \square_j} 2^{-da} |\omega|^{a} \Phi(\omega) \right\} \leq 2^{-da} G_\alpha(\square_{j-1}).
\]
Applying (10) recursively we obtain that for every $k, j \in \mathbb{N}$, $j \leq k$,
\[
G_\alpha(\square_{k-1}) \leq 2^{-da(k-j)} G_\alpha(\square_{j-1}). \tag{11}
\]
In addition, for every $j \in \mathbb{N}$, we have
\[
N_j - N_{j-1} = \sum_{\omega \in S_j} \sum_{\hat{\omega} \in \square_j : \hat{\omega} \subseteq \omega} (N_\gamma(\hat{\omega}) - N_\gamma(\omega))
\]
\[
= \sum_{\omega \in S_j} \sum_{\hat{\omega} \in \square_j : \hat{\omega} \subseteq \omega} \left( \left\lceil \frac{|\Omega|^{\gamma}}{|\omega|} \right\rceil - \left\lceil \frac{|\Omega|}{|\omega|} \right\rceil \right)
\]
\[
= \sum_{\omega \in S_j} \left( 2^d \left\lceil 2^d \left( \frac{|\Omega|}{|\omega|} \right)^\gamma \right\rceil - \left\lceil \frac{|\Omega|}{|\omega|} \right\rceil \right)
\]
\[
\leq (2^{d(\gamma+2)} - 1) \sum_{\omega \in S_j} N_\gamma(\omega) = (2^{d(\gamma+2)} - 1) t_j.
\]
Summing up the above inequalities for \( j = 1, \ldots, k \) and taking into account that
\( t_1 = N_0 = 1 \), we obtain
\[
N_k - N_0 \leq (2^{d(\gamma + 2)} - 1) \sum_{j=1}^{k} t_j \leq 2^{d(\gamma+2)} \sum_{j=1}^{k} t_j - N_0.
\]

Combining (9), (11) and (10) with the latter inequality, we conclude that
\[
N_k \leq 2^{d(\gamma+2)} \sum_{j=1}^{k} t_j \leq 2^{d(\gamma+2)} \sum_{j=1}^{k} \frac{2^{d(\gamma+1)\alpha+1}}{\alpha+1} (G_{\alpha}(\square_{j-1}))^{-\frac{\alpha+1}{\alpha+1}} |\Omega|^{-\frac{\alpha+1}{\alpha+1}} \Phi^{-\frac{\gamma+1}{\gamma+1}}(\Omega)
\]
\[
= 2^{d(\gamma+2) + \frac{2d(\gamma+1)\alpha+1}{\alpha+1}} |\Omega|^{-\frac{\alpha+1}{\alpha+1}} \Phi^{-\frac{\gamma+1}{\gamma+1}}(\Omega) \sum_{j=1}^{k} (G_{\alpha}(\square_{j-1}))^{-\frac{\alpha+1}{\alpha+1}}
\]
\[
\leq 2^{d(\gamma+2) + \frac{2d(\gamma+1)\alpha+1}{\alpha+1}} |\Omega|^{-\frac{\alpha+1}{\alpha+1}} \Phi^{-\frac{\gamma+1}{\gamma+1}}(\Omega) (G_{\alpha}(\square_{k-1}))^{-\frac{\alpha+1}{\alpha+1}} \sum_{j=1}^{k} 2^{-\frac{2d(\gamma+1)(k-j)}{\alpha+1}}
\]
\[
\leq 2^{d(\gamma+2)} \cdot \left(1 - 2^{-\frac{2d(\gamma+1)}{\alpha+1}}\right)^{-1} |\Omega|^{-\frac{\alpha+1}{\alpha+1}} \Phi^{-\frac{\gamma+1}{\gamma+1}}(\Omega) (G_{\alpha}(\square_{k}))^{-\frac{\alpha+1}{\alpha+1}},
\]
which finishes the proof of the lemma with
\[
C_1(d, \gamma, \alpha) = 2^{\frac{2d(\gamma+1)(\gamma+2)}{\gamma+1}} \left(1 - 2^{-\frac{2d(\gamma+1)}{\alpha+1}}\right)^{-\frac{\alpha+1}{\alpha+1}}.
\]

In addition, we will need a similar statement in the case when \( 0 < \alpha < \gamma \) and \( \Phi^{\gamma/\alpha} \) rather than \( \Phi \) is subadditive.

**Lemma 2.2.** Let \( \Omega \) be a cube in \( \mathbb{R}^d \), let \( 0 < \alpha < \gamma \), and let \( \Phi \) be a non-negative function defined on subcubes in \( \Omega \) such that \( \Phi^{\gamma/\alpha} \) is subadditive. Assume that a sequence of dyadic subdivisions \( \{\square_k\}_{k=0}^\infty \) of \( \Omega \) is constructed as in Lemma 2.1. Then, for \( k \in \mathbb{N} \),
\[
G_{\alpha}(\square_k) \leq C_2(\alpha, \gamma, \alpha) N_k^{-\frac{\alpha}{\gamma}} |\Omega|^\alpha \Phi(\Omega),
\]
where \( N_k \) is given by (8), and the constant \( C_2(\alpha, \gamma, \alpha) \) depends only on \( d, \gamma \) and \( \alpha \).

**Proof.** Using the same notations \( S_k \) and \( t_k \) as in the proof of Lemma 2.1 we have
\[
(1 - 2^{-d\gamma}) |\Omega|^\gamma \sum_{\omega \in S_k} |\omega|^{-\gamma} \leq t_k \leq |\Omega|^\gamma \sum_{\omega \in S_k} |\omega|^{-\gamma}, \quad k \in \mathbb{N},
\]
which is trivial when \( k = 1 \) and in the case \( k \geq 2 \) follows from the fact that \( |\Omega| \geq 2^d |\omega| \) for all \( \omega \in S_k \). Due to the construction of \( \square_k \) it is clear that
\[
2^{-d\alpha} G_{\alpha}(\square_{k-1}) \leq \min_{\omega \in S_k} g_{\alpha}(\omega).
\]
Combining the above inequalities and applying the definition of \( g_{\alpha} \) we infer
\[
2^{-d\alpha} G_{\alpha}(\square_{k-1}) \leq \min_{\omega \in S_k} g_{\alpha}(\omega) = \min_{\omega \in S_k} |\omega|^\alpha \Phi(\omega) \leq \frac{|\Omega|^\gamma}{t_k} \sum_{\omega \in S_k} |\omega|^{\alpha-\gamma} \Phi(\omega).
\]
Applying the Hölder inequality with parameters $\frac{\gamma}{d\alpha}$ and $\frac{\gamma}{d\alpha}$, and using the subadditivity of $\Phi \tilde{\gamma}$ we obtain:

$$2^{-d\alpha} G_\alpha(\square_{k-1}) \leq \frac{|\Omega|^\gamma}{t_k} \left( \sum_{\omega \in S_k} |\omega|^{-\gamma} \right)^{1-\frac{\alpha}{d\gamma}} \left( \sum_{\omega \in S_k} \Phi \tilde{\gamma}(\omega) \right)^{\frac{\alpha}{d\gamma}} \leq \frac{|\Omega|^{\alpha\gamma}}{(1 - 2^{-d\gamma})^{1-\frac{\alpha}{d\gamma}}} t_k^{1-\frac{\alpha}{d\gamma}} \Phi(\Omega) = \frac{|\Omega|^{\alpha\gamma} \Phi(\Omega)}{(1 - 2^{-d\gamma})^{1-\frac{\alpha}{d\gamma}}} t_k^{-\frac{\alpha}{d\gamma}}.$$

As a result, we have

$$t_k \leq \frac{2^{d\gamma} |\Omega|^\gamma}{(1 - 2^{-d\gamma})^{\frac{\alpha}{d\gamma}-1}} (G_\alpha(\square_{k-1}))^{-\frac{\alpha}{d\gamma}} \Phi(\Omega). \quad (12)$$

Repeating the arguments presented in the proof of Lemma 2.1 with inequality (12) being applied instead of (9), we obtain:

$$N_k \leq 2^{d(\gamma+1) \cdot (1 - 2^{-d\gamma})^{-\frac{\alpha}{d\gamma}}} |\Omega|^\gamma \Phi(\Omega) (G_\alpha(\square_{k-1}))^{-\frac{\alpha}{d\gamma}}.$$

Observing that inequality (10) also holds true in the considered case and applying it to the latter inequality, we complete the proof of the lemma, with

$$C_2(d, \gamma, \alpha) = 2^{\frac{d\alpha(\gamma+1)}{\gamma}} (1 - 2^{-d\gamma})^{-1}. \quad \Box$$

3 Upper estimates for $E_N(f)_p$

In this section we establish the upper estimates of the error of the best nonlinear $L_p$-approximation of functions from Sobolev space $W^2_q(\Omega)$ by piecewise constants.

We start with presenting an algorithm for constructing a convex partition of $\Omega$ and corresponding piecewise constant approximation for any function $f \in W^2_q(\Omega)$ whenever the conditions of embedding into $L^q_{m}(\Omega)$ are satisfied. To this end we will use the dyadic refinements defined in Lemma 2.1 for $\gamma = \frac{1}{d}$ and a specific $\alpha$ depending on $d, p, q$. In this case each $N_\gamma(\omega) = (|\Omega|/|\omega|)^{1/d}$ is a non-negative integer power of two, $N_{k+1} \leq 2^d N_k$ and the sequence $\{N_k\}_{k=0}^\infty$ is increasing.

**Algorithm 3.1.** Let $1 \leq p \leq \infty$ and $1 \leq q < \infty$ be such that $2/d + 1 - 1/p \geq 0$ when $p < \infty$ and $2/d - 1/q > 0$ when $p = \infty$, and let $f \in W^2_q(\Omega)$. Given any $N \in \mathbb{N}$, construct a sequence of dyadic subdivisions $\{\square_k\}_{k=0}^{m+1}$ of $\Omega$ following the algorithm in Lemma 2.1 with parameters: $\gamma = \frac{1}{d}$, $\Phi(\cdot) := |f|_{W^2_q(\cdot)}^q$ and

$$\alpha := q \left( \frac{2}{d} + \frac{1}{p} \left( 1 + \frac{1}{d} - \frac{1}{q} \right) \right), \quad (13)$$

where $m$ is such that $N_{m} \leq N < N_{m+1}$. The convex partition $\Delta_N$ is obtained by subdividing each $\omega \in \square_m$ into $N_{\gamma}(\omega)$ slices by equidistant hyperplanes orthogonal to the average gradient $h_\omega := |\omega|^{-1} \int_\omega \nabla f(x) \, dx$, and the piecewise constant approximant $s_N(f)$ of $f$ is defined by

$$s_N(f) := \sum_{\delta \in \Delta_N} f_\delta \cdot \chi_\delta,$$
where
\[ f_\delta := \frac{1}{|\delta|} \int_\delta f(x) \, dx, \quad \chi_\delta(x) = \begin{cases} 1, & x \in \delta, \\ 0, & x \in \Omega \setminus \delta. \end{cases} \]

A key tool for estimating the error \( \| f - s_N(f) \|_{L_p(\Omega)} \) is provided by Lemmas 2.1 and 2.2. Observe that Lemma 2.1 is applicable when \( \alpha \geq \frac{1}{d} \), which is equivalent to the inequality \( \frac{2}{d+1} + \frac{1}{p} - \frac{1}{q} \geq 0 \). This leads to the optimal order estimate in Theorem 3.2. In the case \( \alpha < \frac{1}{d} \) we apply Lemma 2.2 and obtain in Theorem 3.3 certain suboptimal order estimates that are nevertheless of interest whenever they improve the order \( O(N^{-\frac{2}{d}}) \) of isotropic piecewise constant approximation.

We will need the Sobolev-Poincaré inequality in the following form. Let \( \omega \) be a cube in \( \mathbb{R}^d \), and let \( 1 \leq \xi \leq \infty \) and \( 1 \leq \eta < \infty \) be such that either \( \frac{1}{d} + \frac{1}{\xi} - \frac{1}{\eta} \geq 0 \) when \( \xi < \infty \), or \( \frac{1}{d} - \frac{1}{\eta} > 0 \) when \( \xi = \infty \). Then for every \( g \in W^{1,\eta}_\infty(\omega) \) with zero mean value over \( \omega \), there exists a constant \( C_{SP}(d, \xi, \eta) > 0 \) such that
\[
\| g \|_{L_\xi(\omega)} \leq C_{SP}(d, \xi, \eta) |\omega|^{\frac{1}{d} + \frac{1}{\xi} - \frac{1}{\eta}} \| \nabla g \|_{L_\eta(\omega)}, \tag{14}
\]
where, for any vector-function \( f = (f_1, \ldots, f_d) : \omega \to \mathbb{R}^d \) with components \( f_j \in L_\xi(\omega) \), \( j = 1, \ldots, d \), we set
\[
\| f \|_{L_\xi(\omega)} := \left\| (f_1^2 + \cdots + f_d^2)^{1/2} \right\|_{L_\xi(\omega)}, \quad 1 \leq \xi \leq \infty.
\]
A proof of (14) for the case of the unit cube can be found e.g. in [11, Theorem 8.12], and the general case follows by using an affine mapping between \([0, 1]^d \) and \( \omega \).

**Theorem 3.2.** Let \( d \geq 2 \), \( N \in \mathbb{N} \), and let \( 1 \leq p \leq \infty \) and \( 1 \leq q < \infty \) be such that
\[
\frac{2}{d+1} + \frac{1}{p} - \frac{1}{q} \geq 0. \tag{15}
\]
Then for any \( f \in W^{2,q}_\infty(\Omega) \) the error of the piecewise constant approximant \( s_N(f) \) generated by Algorithm 3.1 satisfies
\[
E_N(f)_p \leq \| f - s_N(f) \|_{L_p(\Omega)} \leq C_3(d, p, q, |\Omega|) N^{-\frac{2}{d+1}} \left( |f|^q_{W^{1,q}_\infty(\Omega)} + |f|^q_{W^{2,q}_\infty(\Omega)} \right)^{\frac{1}{q}},
\]
where the constant \( C_3(d, p, q, |\Omega|) \) depends only on \( d, p, q \) and \( |\Omega| \).

**Remark 2.** The assertion of Theorem 3.2 in the case \( q = p \) was proved in [5] and in the case \( q > p \) easily follows from that result.

**Proof.** For every cube \( \omega \in \square_m \), we denote by \( c_\omega \) the center of \( \omega \) and consider the linear approximant of \( f \) on \( \omega \) in the form (see [9]):
\[
\ell_\omega(x) := \frac{1}{|\omega|} \int_\omega f(x) \, dx + \langle h_\omega, x - c_\omega \rangle,
\]
where \( \langle \cdot, \cdot \rangle \) is the scalar product in \( \mathbb{R}^d \).

We start with estimating the \( L_p \)-error of approximation of \( f \) by the piecewise constant function
\[
\tilde{s}_N(\ell) := \sum_{\delta \in \Delta} \tilde{s}_\delta \cdot \chi_\delta,
\]
where
\[ \tilde{s}_\delta := |\delta|^{-1} \int_\delta \ell_\omega(x) \, dx, \quad \delta \in \Lambda_\omega := \{ \delta \in \Delta_N : \delta \subset \omega \}. \]

In view of triangle inequality, for every \( \omega \in \square_m \),
\[ \| f - \tilde{s}_N(\ell) \|_{L_p(\omega)} \leq \| f - \ell_\omega \|_{L_p(\omega)} + \| \ell_\omega - \tilde{s}_N(\ell) \|_{L_p(\omega)}. \]

Let us estimate \( \| f - \ell_\omega \|_{L_p(\omega)} \) by applying the Sobolev-Poincaré inequality (14) twice. To this end we choose \( 1 \leq \tau \leq \infty \) such that simultaneously \( \frac{1}{d} + \frac{1}{\tau} - \frac{1}{q} \geq 0 \) and \( \frac{1}{d} + \frac{1}{\tau} - \frac{1}{q} \geq 0 \), for example \( \tau = \frac{2pq}{p+q} \). Since \( \int_\omega (f(x) - \ell_\omega(x)) \, dx = 0 \), we can apply (14):
\[ \| f - \ell_\omega \|_{L_p(\omega)} \leq k_1 |\omega|^{\frac{1}{2} + \frac{1}{p} - \frac{1}{q}} \| \nabla f - h_\omega \|_{L_r(\omega)}, \]
where \( k_1 = C_{SP}(d, p, \tau) \). Using the triangle inequality and the Sobolev-Poincaré inequality for the second term, we obtain
\[
\| \nabla f - h_\omega \|_{L_r(\omega)} = \left\| \left( \sum_{j=1}^d \left( \frac{\partial f}{\partial x_j} - \frac{1}{|\omega|} \int_\omega \frac{\partial f}{\partial x_j}(x) \, dx \right)^2 \right)^{\frac{1}{2}} \right\|_{L_r(\omega)} \\
\leq \sum_{j=1}^d \left\| \frac{\partial f}{\partial x_j} - \frac{1}{|\omega|} \int_\omega \frac{\partial f}{\partial x_j}(x) \, dx \right\|_{L_r(\omega)} \\
\leq \sum_{j=1}^d k_2 |\omega|^{\frac{1}{2} + \frac{1}{p} - \frac{1}{q}} \| \nabla \left( \frac{\partial f}{\partial x_j} \right) \|_{L_q(\omega)} \leq k_2 |\omega|^{\frac{1}{2} + \frac{1}{p} - \frac{1}{q}} \| f \|_{W^2_q(\omega)},
\]
where \( k_2 = C_{SP}(d, \tau, q) \). Combining the last two inequalities we obtain
\[ \| f - \ell_\omega \|_{L_r(\omega)} \leq k_3 |\omega|^{\frac{1}{2} + \frac{1}{p} - \frac{1}{q}} \| f \|_{W^2_q(\omega)} , \quad (16) \]
where \( k_3 := k_1 k_2 \) depends only on \( d, p, \tau \) and \( q \).

Now we estimate the \( L_p \)-distance between \( \ell_\omega \) and \( \tilde{s}_N(\ell) \) on \( \omega \),
\[
\| \ell_\omega - \tilde{s}_N(\ell) \|_{L_p(\omega)}^p = \sum_{\delta \in \Lambda_\omega} \| \ell_\omega - \tilde{s}_\delta \|_{L_p(\delta)}^p = \sum_{\delta \in \Lambda_\omega} \left( \int_\delta \left| \langle h_\omega, \mathbf{x} - \frac{1}{|\delta|} \int_\delta u \, du \rangle \right|^p \, dx \right) \\
= \sum_{\delta \in \Lambda_\omega} \frac{1}{|\delta|^p} \int_\delta \left( \int_\delta \langle h_\omega, \mathbf{x} - u \rangle \, du \right|^p \, dx \\
\leq \sum_{\delta \in \Lambda_\omega} \frac{1}{|\delta|^p} \int_\delta \| h_\omega \|_2 \langle \mathbf{x} - u \rangle \, du \|^p \, dx,
\]
where \( (\mathbf{x} - u) h_\omega \) denotes the length of the projection of \( \mathbf{x} - u \) on a unit vector parallel to \( h_\omega \) and \( \| h_\omega \|_2 \) stands for the Euclidean norm of \( h_\omega \). By the Hölder inequality,
\[
\| h_\omega \|_2 = \left( \sum_{k=1}^d \left( \frac{1}{|\omega|} \int_\omega \frac{\partial f}{\partial x_k}(x) \, dx \right)^2 \right)^{\frac{1}{2}} \leq \frac{1}{|\omega|} \sum_{k=1}^d \int_\omega \left| \frac{\partial f}{\partial x_k}(x) \right| \, dx \\
\leq \frac{1}{|\omega|} \sum_{k=1}^d |\omega|^{-\frac{1}{2}} \left( \int_\omega \left| \frac{\partial f}{\partial x_k}(x) \right|^q \, dx \right)^{\frac{1}{q}} = |\omega|^{-\frac{1}{2}} \| f \|_{W^2_q(\omega)}.
\]
It is also clear that, for every $\delta \in A_\omega$ and $x, u \in \delta$, we have $(x - u)_{N_\omega} \leq \frac{\text{diam} (\omega)}{N_\gamma (\omega)}$.

Substituting the above inequalities into (17) we obtain

\[
\| \ell_\omega - s_N (\ell) \|_{L_p (\omega)}^p \leq \sum_{\delta \in A_\omega} |\delta| \cdot |\omega|^{-\frac{2}{p}} \left( \frac{\text{diam} (\omega)}{N_\gamma (\omega)} \right)^p |f|_{W_d^1 (\omega)}^p = d^2 \cdot |\omega|^{1 + \frac{2}{p} - \frac{2}{q} |\Omega|^{-\frac{2}{q}}} |f|_{W_d^1 (\omega)}^p.
\]

(18)

Combining (16) and (18) and applying the Hölder inequality we obtain

\[
\| f - \tilde{s}_N (\ell) \|_{L_p (\omega)} \leq k_4 \cdot |\omega|^{\frac{3}{2} + \frac{1}{p} - \frac{1}{q}} \left( |f|_{W_d^1 (\omega)} + |f|_{W_d^2 (\omega)} \right) \leq k_5 \cdot |\omega|^{\frac{1}{2}} \left( |f|_{W_d^1 (\omega)} + |f|_{W_d^2 (\omega)} \right)^{\frac{1}{q}},
\]

(19)

where $k_4 := \max \left\{ k_3 \cdot \sqrt{d} |\Omega|^{-\frac{1}{q}} \right\}$ and $k_5 := 2^{1 - \frac{1}{q}} k_4$. Using definitions of $\alpha$ and functions $g_\alpha$ and $G_\alpha$ we rewrite (19) in the following way

\[
\| f - \tilde{s}_N (\ell) \|_{L_p (\omega)}^p \leq k_5^p \cdot |\omega|^{-\frac{1}{q}} (g_\alpha (\omega))^{\frac{p}{q}} \leq k_5^p |\omega|^{-\frac{1}{q}} (G_\alpha (\square_m))^\frac{p}{q}.
\]

Hence,

\[
\| f - \tilde{s}_N (\ell) \|_{L_p (\omega)}^p \leq k_5^p \sum_{\omega \in \square_m} |\omega|^{-\frac{1}{q}} G_\alpha^\frac{p}{q} (\square_m) = k_5^p |\omega|^{-\frac{1}{q}} N_m \cdot G_\alpha^\frac{p}{q} (\square_m).
\]

Combining this inequality with Lemma 2.1 we arrive at

\[
\| f - \tilde{s}_N (\ell) \|_{L_p (\omega)} \leq k_5 \cdot |\omega|^{-\frac{1}{q}} N_m G_\alpha^\frac{1}{q} (\square_m) \leq k_6 N_m \cdot G_\alpha^\frac{1}{q} (\square_m) \leq k_7 N^{-\frac{\alpha + 1}{\alpha (1 + q)}} \Phi^\frac{\alpha}{q} (\Omega),
\]

(20)

where $k_6 := k_5^p C^\frac{1}{q} (d, \gamma, \alpha) \cdot |\omega|^{-\frac{1}{q}}$ and $k_7 := 2^{-\frac{\alpha + 1}{\alpha (1 + q)}} k_6$, with $C_1$ from (17) and $\gamma, \alpha$ as in Algorithm 3.1.

Finally, for every $\delta \in \triangle_N$, it is easy to see that $\| f - f_\delta \|_{L_p (\delta)} \leq 2 \inf_{c \in \mathbb{R}} \| f - c \|_{L_p (\delta)}$, where $f_\delta$ was defined in Algorithm 3.1. Therefore

\[
\| f - s_N (f) \|_{L_p (\omega)} \leq 2 k_7 N^{-\frac{\alpha + 1}{\alpha (1 + q)}} \left( |f|_{W_d^1 (\omega)}^q + |f|_{W_d^2 (\omega)}^q \right)^{\frac{1}{q}},
\]

which proves the theorem with

\[
C_3 (d, p, q, |\Omega|) = 2^{\frac{\alpha + 1}{\alpha q}} C^\frac{1}{q} (d, \gamma, \alpha) |\omega|^{-\frac{1}{q}} \max \left\{ C_{SP} (d, p, \tau) C_{SP} (d, \tau, q); \sqrt{d} |\Omega|^{-\frac{1}{q}} \right\}.
\]

We now establish an upper estimate for the order of $\| f - s_N (f) \|_{L_p (\omega)}$ as $N \to \infty$ provided that $q$ does not satisfy (13). To this end we will apply Lemma 2.2 instead of Lemma 2.1.
Theorem 3.3. Let $d \geq 2$, $N \in \mathbb{N}$, and let $1 \leq p \leq \infty$ and $1 \leq q < \infty$ be such that

$$\frac{2}{d+1} + \frac{1}{p} - \frac{1}{q} < 0$$

and

$$\begin{align*}
\frac{2}{d+1} + \frac{1}{p} - \frac{1}{q} &> 0, \quad p < \infty, \\
\frac{2}{d+1} + \frac{1}{p} - \frac{1}{q} &> 0, \quad p = \infty.
\end{align*}$$

Then for any $f \in W_q^2(\Omega)$ the piecewise constant spline $s_N(f)$ generated by Algorithm 3.1 satisfies

$$E_N(f)_p \leq \|f - s_N(f)\|_{L^p(\Omega)} \leq C_4(d, p, q, |\Omega|) N^{-d(\frac{2}{d+1} - \frac{1}{p} + \frac{1}{q})} \left( |f|_{W_q^2(\Omega)} + |f|_{W_q^2(\Omega)} \right)^{\frac{1}{q}},$$

where the constant $C_4(d, p, q, |\Omega|)$ depends only on $d, p, q, |\Omega|$. In particular,

$$E_N(f)_p \leq \|f - s_N(f)\|_{L^p(\Omega)} = o(N^{-\frac{1}{d}}), \quad f \in W_q^2(\Omega),$$

as soon as

$$\frac{2}{d} + \frac{1}{p} - \frac{1}{q} > \frac{1}{d^2}. \quad (20)$$

Proof. Repeating the proof of Theorem 3.2 and applying Lemma 2.2 we obtain the estimate

$$\|f - \tilde{s}_N(\ell)\|_{L^p(\Omega)} \leq k_5 |\Omega|^{-\frac{1}{d+1}} N_m^\frac{1}{d} C_{\phi}^\frac{1}{d} (m) \leq k_6 N_m^\frac{1}{d} N_m^\frac{d}{d+1} \phi^\frac{1}{d}(\Omega) \leq k_7 N^{-d(\frac{2}{d+1} - \frac{1}{p} + \frac{1}{q})} \Phi^\frac{1}{d}(\Omega),$$

where $k_6 = k_6(d, p, \tau, q, |\Omega|) := k_5 C_2^\frac{1}{d} (d, \gamma, \alpha) |\Omega|^{\frac{d}{d+1}}$ and $k_7 := 2^{-\frac{2d}{d+1}} k_6$, with the constant $C_2(d, \gamma, \alpha)$ of Lemma 2.2. The proof is completed by the same arguments as in the proof of Theorem 3.2. The estimate is valid with

$$C_4(d, p, q, |\Omega|) = 2^\frac{2d}{d+1} C_2^\frac{1}{d} (d, \gamma, \alpha) |\Omega|^{\frac{d}{d+1}} \max \left\{ C_{SP}(d, p, \tau) C_{SP}(d, \tau, q); \sqrt[d]{|\Omega|}^{-\frac{1}{d}} \right\},$$

with $\gamma$ and $\alpha$ as in Algorithm 3.1 and $1 \leq \tau \leq \infty$ being any number such that

$$\frac{1}{d} + \frac{1}{p} - \frac{1}{q} \geq 0 \quad \text{and} \quad \frac{1}{d} + \frac{1}{p} - \frac{1}{q} \geq 0. \quad \Box$$

Using the embedding theorem $W_q^r(\Omega) \subset W_d^2(\Omega)$, where $\frac{1}{q} = \frac{r-2}{d} + \frac{1}{q}$ [14, p. 328], we deduce from Theorems 3.2 and 3.3 the following statement that summarizes the main results of this section.

Corollary 3.4. Let $d \geq 2$, $r \geq 2$, $1 \leq p \leq \infty$, $1 \leq q < \infty$. Then

$$E_N(W_q^r(\Omega))_p = \mathcal{O}(N^{-\frac{2}{d+1}}) \quad \text{if} \quad \frac{r}{d} + \frac{1}{p} - \frac{1}{q} \geq \frac{2}{d(d+1)},$$

$$E_N(W_q^r(\Omega))_p = \mathcal{O}(N^{-\frac{1}{d}}) \quad \text{if} \quad \frac{r}{d} + \frac{1}{p} - \frac{1}{q} > \frac{1}{d^2}. \quad (21)$$

where

$$E_N(W_q^r(\Omega))_p := \sup_{f \in W_q^r(\Omega), E_N(f) < 1} E_N(f)_p.$$
4. An estimate from below

As the above results only apply to $W^r_q(\Omega)$ with $r \geq 2$, there remains the question whether the orders $E_N(f)_p = O(N^{-\frac{2}{d+1}})$ or at least $E_N(f)_p = o(N^{-\frac{2}{d+1}})$ can be achieved when $r < 2$. Leaving out the question of the validity of these bounds for individual functions in $W^r_q(\Omega)$, we provide an estimate from below for $E_N(W^r_q(\Omega))_\infty$ that shows in particular that the optimal order of (21) is not achieved when $p = \infty$ and $r < \frac{2d}{d+1}$.

**Theorem 4.1.** Let $d \geq 2$, $r \in (0, 2)$, $1 \leq q < \infty$, and $\frac{2}{r} - \frac{1}{q} > 0$. Then

$$
\lim_{N \to \infty} \sup_{f \in W^r_q(\Omega), \|f\|_{W^r_q(\Omega)} \leq 1} N^{\frac{2}{r}} E_N(f)_\infty > 0.
$$

(23)

In particular, $E_N(W^r_q(\Omega))_\infty \neq O(N^{-\frac{2}{d+1}})$ and

$$
E_N(W^r_q(\Omega))_\infty \neq O(N^{-\frac{2}{d+1}}) \quad \text{if} \quad r < \frac{2d}{d+1}.
$$

(24)

**Proof.** For simplicity, assume that $\Omega = (0, 1)^d$. For $m \in \mathbb{N}$, let $\square_m = \{\omega_i\}_{i \in \mathbb{N}^d}$ be the partition of $\Omega$ into $m^d$ subcubes

$$
\omega_i := \left(\frac{i_1 - 1}{m}, \frac{i_1}{m}\right) \times \ldots \times \left(\frac{i_d - 1}{m}, \frac{i_d}{m}\right), \quad i = (i_1, \ldots, i_d) \in \mathbb{N}^d
$$

where

$$
\mathbb{N}^d := \{i \in \mathbb{N}^d : 1 \leq i_k \leq m, k = 1, \ldots, d\}.
$$

We set $1 = (1, \ldots, 1) \in \mathbb{N}^d$ and $d(i, j) := \max_{k=1,\ldots,d} |i_k - j_k|$, $i, j \in \mathbb{N}^d$, and consider the infinitely differentiable compactly supported function $\varphi : \mathbb{R}^d \to \mathbb{R}$ defined by

$$
\varphi(x) := \begin{cases} 
\exp\left(-\frac{1}{|2x-1|^d}\right), & |2x-1| < 1, \\
0, & |2x-1| \geq 1,
\end{cases}
$$

where $|\cdot|$ stands for the Euclidean norm in $\mathbb{R}^d$. For $i \in \mathbb{N}^d$, we scale $\varphi$ into the cube $\omega_i$ as $\varphi_i(x) := \varphi(mx - 1 + 1)$, such that $\varphi_i(x) \neq 0$ on the set $\{x \in \Omega : |x - \frac{i - 1}{2m}| < \frac{1}{2m}\} \subset \omega_i$. Then

$$
|\varphi_i|_{W^r_q(\omega_i)} = m^{s - \frac{2}{r}} |\varphi|_{W^r_q(\Omega)}, \quad s \geq 0.
$$

(25)

We construct a function $f_m \in W^r_q(\Omega)$ as $f_m := \sum_{i \in \mathbb{N}^d} \varphi_i$, and claim that

$$
\|f_m\|_{W^r_q(\Omega)} \leq C_5 m^{r} \|\varphi\|_{W^r_q(\Omega)},
$$

(26)

where $C_5$ is a positive constant independent of $m$. By (25) we see that for any $k \in \mathbb{Z}_+$,

$$
|f_m|^{q}_{W^r_q(\Omega)} = m^d \cdot m^{kq - d} |\varphi|^q_{W^r_q(\Omega)} = m^{kq} |\varphi|^q_{W^r_q(\Omega)},
$$

which implies

$$
\|f_m\|_{W^r_q(\Omega)} \leq m^{\frac{r}{q}} \|\varphi\|_{W^r_q(\Omega)},
$$

(27)

and already shows (25) in the case $r = 1$.

It remains to give an upper estimate for the seminorm of $f_m$ in the Sobolev-Slobodeckij space $W^r_q(\Omega)$ when $r > 0$ and $r \neq 1$. We have

$$
|f_m|^{q}_{W^r_q(\Omega)} \leq \sum_{i \in \mathbb{N}^d} \sum_{j \in \mathbb{N}^d} \sup_{k \in \mathbb{Z}_+} \int_{\omega_i} \int_{\omega_j} \frac{|D^{k}\varphi_i(x) - D^{k}\varphi_j(y)|^q}{|x - y|^{q + d}} \, dx \, dy.
$$

(28)
We split the sum (23) into three pieces corresponding to \( i, j \) satisfying \( d(i, j) = 0 \), \( d(i, j) = 1 \) and \( d(i, j) \geq 2 \), respectively, and estimate each of them separately.

For the case \( d(i, j) = 0 \), that is \( i = j \), we have the following equality

\[
\sum_{k \in \mathbb{N}_m} \sup_{x \in \mathbb{X}^d_{(i,j)}} \int_{\omega_i} \int_{\omega_j} \frac{|D^k \varphi_i(x) - D^k \varphi_j(y)|^q}{|x - y|^{r+q-d}} \, dx \, dy = \sum_{k \in \mathbb{N}_m} |\varphi_i|^q_{W^r_q(\omega_i)} = m^r \cdot |\varphi_i|^q_{W^r_q(\Omega)}. \tag{29}
\]

If \( d(i, j) \geq 2 \), then we use the substitution \( y = u + \frac{1}{m} c^{ij} \), with \( c^{ij} := j - i \), to obtain

\[
\int_{\omega_i} \int_{\omega_j} \frac{|D^k \varphi_i(x) - D^k \varphi_j(y)|^q}{|x - y|^{r+q-d}} \, dx \, dy = \int_{\omega_i} \int_{\omega_j} |D^k \varphi_i(x) - D^k \varphi_i(u)|^q \, dx \, du.
\]

Since \( d(i, j) - 1 \geq \frac{1}{2} d(i, j) \) and \( |x_k - u_k| \leq \frac{1}{m}, \ k = 1, \ldots, d \), as long as \( x, u \in \omega_i \), we have

\[
|x - u - \frac{1}{m} c^{ij}| \geq \frac{1}{m} \cdot (d(i, j) - 1) \geq \frac{1}{2m} d(i, j),
\]

and hence

\[
\int_{\omega_i} \int_{\omega_j} \frac{|D^k \varphi_i(x) - D^k \varphi_i(u)|^q}{|x - u|^{r+q-d}} \, dx \, du \leq \left( \frac{2m}{d(i,j)} \right)^{r+q} \int_{\omega_i} \int_{\omega_j} |D^k \varphi_i(x) - D^k \varphi_i(u)|^q \, dx \, du \leq 2^{q+\{r\}q} m^{\{r\}q} |\varphi_i|^q_{W^r_q(\omega_i)} d(i,j)^{-\{r\}q-d}.
\]

Now

\[
\sum_{i, j \in \mathbb{N}_m, d(i,j) = 1} \int_{\omega_i} \int_{\omega_j} \frac{|D^k \varphi_i(x) - D^k \varphi_j(y)|^q}{|x - y|^{r+q-d}} \, dx \, dy \leq m^d \cdot \sum_{i \in \mathbb{N}_m, d(0, i) = 2} \int_{|x| \geq 1} \frac{dx}{|x|^{r+q-d}} = \frac{\mu_d m^d}{\{r\}q},
\]

where \( \mu_d \) is the volume of the \((d - 1)\)-dimensional unit sphere. (Remark that \( \{r\} \neq 0 \) as \( r \in \mathbb{N} \).) It follows that

\[
\sum_{i, j \in \mathbb{N}_m, d(i,j) \geq 2} \sup_{k \in \mathbb{N}_m, |k| \leq \{r\}} \int_{\omega_i} \int_{\omega_j} \frac{|D^k \varphi_i(x) - D^k \varphi_j(y)|^q}{|x - y|^{r+q-d}} \, dx \, dy \leq \frac{2^{q+\{r\}q} m^d \mu_d m^q}{\{r\}q} |\varphi|^q_{W^r_q(\Omega)}. \tag{30}
\]

It remains to consider the terms in (23) with \( d(i, j) = 1 \). In this case \( \omega_i \) and \( \omega_j \) have a common face of dimension between 0 and \( d - 1 \). We alter variables under the integral over \( \omega_j \) by substituting for \( y \) its reflection \( u \) with respect to \( \omega_i \cap \omega_j \). Evidently, \( \varphi_j(y) = \varphi_i(u) \) and \( |x - y| \geq |x - u| \), \( x \in \omega_i \). Hence, for all \( i, j \) with \( d(i, j) = 1 \),

\[
\int_{\omega_i} \int_{\omega_j} \frac{|D^k \varphi_i(x) - D^k \varphi_j(y)|^q}{|x - y|^{r+q-d}} \, dx \, dy \leq \int_{\omega_i} \int_{\omega_j} \frac{|D^k \varphi_i(x) - D^k \varphi_i(u)|^q}{|x - u|^{r+q-d}} \, dx \, du \leq m^{r-d} \cdot |\varphi|^q_{W^r_q(\Omega)}.
\]

Since

\[
\sum_{\ell = 0}^{d-1} \sum_{i \in \mathbb{N}_m, d(i,j) = \ell} m^{q-d} \leq \sum_{i \in \mathbb{N}_m} (3^d - 1) \cdot m^q = (3^d - 1) \cdot m^q,
\]
we arrive at
\[
\sum \sup_{\mathbf{1} \neq \mathbf{2}} \int_{\omega_1} \int_{\omega_2} \left| \frac{D^k \varphi_1(x) - D^k \varphi_1(y)}{|x-y|^{\tau q + d}} \right|^q \, dx \, dy \leq (3^d - 1) m^r q |\varphi|_{W^r_q(\Omega)}. \tag{31}
\]
Combining (29), (30) and (31) we obtain
\[
|f_m|_{W^r_q(\Omega)} \leq C_6 m^r,
\]
where \(C_6 := \left( 3^d |\varphi|_{W^r_q(\Omega)}^q + \frac{2^{q+\tau q + d + 1}}{3 q} |\varphi|_{W^r_q(\Omega)}^q \right)^{\frac{1}{q}} \) is independent of \(m\). Taking into account (27) we obtain (26).

Finally, we consider the function \(g_m := C_6^{-1} m^{-r} f_m\) and set \(N_m := m^d\) and \(\varepsilon_m := \|g_m\|_{L^\infty(\Omega)} = C_6^{-1} m^{-r} \|\varphi\|_{L^\infty(\Omega)}\). Let us show that
\[
\inf_{\Delta \in \mathfrak{D}_{Nm}} \inf_{s \in S_0(\Delta)} \|g_m - s\|_{L^\infty(\Omega)} > \frac{\varepsilon_m}{3}.
\]
Then (23) will follow in view of (26).

Let \(\Delta \in \mathfrak{D}_{Nm}\) be any convex partition comprising at most \(N_m\) cells. Assume to the contrary to our claim that \(\|g_m - s\|_{L^\infty(\Omega)} \leq \frac{\varepsilon_m}{3}\) for some \(s \in S_0(\Delta)\). Consider a cell \(\delta \in \Delta\) whose closure \(\bar{\delta}\) contains the center \(c\) of a cube \(\omega \in \square_m\). Then \(\bar{\delta}\) is located completely inside \(\omega\) as otherwise we can find a point \(c' \in \bar{\delta}\) sufficiently close to the boundary of \(\omega\) and a point \(c'' \in \bar{\delta}\) sufficiently close to \(c\), such that \(s(c') = s(c'')\) and \(g_m(c') < \min_{x \in \Omega} g_m(x) + \frac{\varepsilon_m}{12}\), and \(g_m(c'') > \max_{x \in \Omega} g_m(x) - \frac{\varepsilon_m}{12}\), and hence
\[
\|g_m - s\|_{L^\infty(\Omega)} \geq \max \left\{ \left| g_m(c') - s(c') \right|, \left| g_m(c'') - s(c'') \right| \right\}
\geq \frac{1}{2} \left( \max_{x \in \Omega} g_m(x) - \min_{x \in \Omega} g_m(x) \right) - \frac{\varepsilon_m}{6} = \frac{\varepsilon_m}{3},
\]
which contradicts the assumption. Therefore in each of \(m^d\) cubes \(\omega \in \square_m\) there is at least one cell \(\delta \in \Delta\) such that \(\bar{\delta}\) is located completely inside that cube. It follows that the total number of cells in \(\Delta\) is greater than \(m^d = N_m\), a contradiction to the choice of the partition. \(\Box\)

5 Acknowledgments

The work of Oleksandr Kozynenko was supported in part by a DAAD Scholarship in the programme Research Grants – Bi-nationally Supervised Doctoral Degrees, 2016/17 (57214225).

References

[1] Adams R. A., Fournier J. J. F., Sobolev spaces, 2nd ed, Amsterdam: Elsevier, 2003.
[2] Babenko V. F., Babenko Yu. V., Parfinovych N. V., Skorokhodov D. S., Exact Asymptotics of the Optimal \(L_p\)-error of Asymmetric Linear Spline Approximation, Jaen Journal on Approximation, V.6, No 1, (2014), P. 1–36.
[3] Birman M. S., Solomyak M. Z. Piecewise polynomial approximation of functions of the classes \(W^m_p\). Mat. Sb., 1967, 73 (115), no. 3, 331-355 (in Russian). English translation in Math. USSR-Sb., 1967, 2, no. 3, 295-317.
[4] Davydov O., Algorithms and error bounds for multivariate piecewise constant approximation, in "Approximation Algorithms for Complex Systems", (E. H. Georgoulis, A. Iske and J. Levesley, Eds.), Springer Proceedings in Mathematics, Vol. 3, Springer-Verlag, 2011, pp. 27–45.

[5] Davydov O., Approximation by piecewise constants on convex partitions. J. Approx. Theory, 164 (2012), 346-352.

[6] Davydov O., Rabarison F. Approximation by sums of piecewise linear polynomials. J. Approx. Theory, 185 (2014), 107-123.

[7] DeVore R.A., Nonlinear approximation, Acta Numer., vol. 7, 1998, pp. 51-150.

[8] DeVore R. A., Nonlinear approximation and its applications, in R. DeVore and A. Kunoth (eds.): Multiscale, Nonlinear and Adaptive Approximation - Dedicated to Wolfgang Dahmen on the Occasion of his 60th Birthday. Springer-Verlag, Berlin, 2009, 169–201.

[9] Kochurov A. S., Approximation by piecewise constant functions on the square, East J. Approx. 1, 1995, 463–478.

[10] Kozyrnenko O. V., Lower estimates on the saturation order of approximation of twice continuously differentiable functions by piecewise constants on convex partitions, Visn. Dnipro Univ., Ser. Mat., 26 (2018), 37 – 47.

[11] Lieb E. H., Loss M., Analysis, 2nd ed., Providence, RI: Amer. Math. Soc., 2001.

[12] Slobodeckiy L.N., Sobolev spaces of fractional order and their application to boundary-value problems for partial differential equations, Dokl. Akad. Nauk SSSR 118 (1958), 243-246.

[13] Temlyakov, V. Greedy Approximation. Cambridge University Press, Cambridge, 2011.

[14] Triebel, H., Interpolation Theory, Function Spaces, Differential Operators. Berlin, VEB Deutscher Verlag der Wissenschaften 1978. – 528 p.