New approach to the Sivers effect in the collinear twist-3 formalism

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The single-transverse spin asymmetry (SSA) for hadron production in transversely polarized proton scattering receives major contribution from Sivers effect, which can be systematically described within the collinear twist-3 factorization framework in various processes. Conventional method in the evaluation of Sivers effect known as pole calculation is technically quite different from the non-pole method used in evaluating final state high twist effect. In this paper, we extend the non-pole technique to Sivers effect, and show the consistence with the conventional method through explicit calculation of $\mathcal{O}(\alpha_s)$ correction in semi-inclusive deep inelastic scattering. As a result, we clarify that the conventional pole calculation is implicitly using the equation of motion and the Lorentz invariant relations whose importance became widely known in the non-pole calculation, we also demonstrate the advantages in using the new non-pole method.

I. INTRODUCTION

The origin of the single transverse-spin asymmetries (SSAs) in high-energy hadron scatterings has been a long-standing mystery over 40 years since the strikingly large asymmetries were observed in mid-1970s [1, 2]. RHIC experiment has provided many data of the SSAs for various hadron productions in the last decade [3–7] and motivated a lot of theoretical work on the development of the perturbative QCD framework. Much theoretical effort has been devoted to develop a reliable QCD-based theory that can deal with those experimental data and finally the twist-3 framework in the collinear factorization approach was established as a possible new framework which can provide a systematic description of the large SSAs.

It is commonly known that there are two effects that lead to the single transverse-spin asymmetries observed in experiment, i.e., initial state Sivers effect and final state Collins effect. The Sivers effect is essentially twist-3 contribution generated from the transversely polarized initial state. Started from the pioneering work by Efremov and Teryaev [8], more systematic techniques were developed in a series of work done around ’00 [9–12]. A solid foundation of the twist-3 calculation for the Sivers effect was finally provided in Ref. [12]. We will show the calculation technique in detail in next section and here we just give a brief introduction. The twist-3 effect of the transversely polarized proton can be expressed by the dynamical twist-3 function $T_{q,F} \simeq F_3 \langle pS_\perp | \psi gF^+ \bar{\psi} | pS_\perp \rangle$ and the cross section in deep inelastic scattering (DIS) can be derived as

$$d\sigma = iT_{q,F} \otimes D \otimes d\hat{s},$$

where $D$ represents the usual twist-2 fragmentation function and $d\hat{s}$ is a hard partonic cross section. Because all nonperturbative functions are real in this equation, the partonic cross section have to give an imaginary contribution in order to cancel $i$ in the coefficient. This imaginary contribution can be given by the pole part of a propagator in the partons scattering. In the quantum field theory, the propagator is given by the time-ordered product of two fields and it has $i\epsilon$-term in the denominator. The imaginary contribution can emerge from a residue of contour integration. This is a basic mechanism of the pole calculation for the Sivers type contribution. Next we turn to the twist-3 fragmentation effect of the spin-0 particle like the pion which is known as Collins effect. The calculation for the twist-3 fragmentation contribution in proton-proton collision was completed in Ref. [13] and the calculation for DIS was done in Ref. [14] in a formal way. The dynamical twist-3 fragmentation function can be defined as $\tilde{D}_{q,F} \simeq F_3 \langle 0 | gF^+ \bar{\psi} hX \rangle \langle hX | \psi | 0 \rangle$ and the cross section is expressed by the same form as in Eq. (1) with $T_{q,F}$ replaced by $\tilde{D}_{q,F}$, and $D$ replace by the usual twist-2 parton distribution function. The main difference is that the fragmentation function $\tilde{D}_{q,F}$ is complex and therefore the hard part doesn’t have to give the imaginary contribution. This is called non-pole contribution, which has significant differences from the pole calculation. The cross section for the pole contribution only depends on the dynamical function, while the result for the non-pole contribution is expressed in terms of three types of the nonperturbative functions, the dynamical, intrinsic and kinematical functions. In general, the hard cross section for

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each nonperturbative function is not gauge- and Lorentz-invariant, because they are not physical observables, only their sum leads to physical result as measured by experiment. This problem is solved by using two types of the relations among the nonperturbative functions which are called equation of motion relation and Lorentz invariant relation [13].

As discussed above, the calculations for the pole contribution and the nonpole contribution are technically different from each other. Both techniques construct an important foundation of the higher twist calculation but not so many people are familiar with both these two because of the technical differences. In this paper, we reexamine the result of the pole calculation from the viewpoint of the nonpole calculation in order to understand two calculations in a unified way. We show that the nonpole calculation has several technical advantages, thus should be extended to various channels and higher twist calculations.

The remainder of the paper is organized as follows: in Sec. II we introduce the notation and review the conventional pole calculation in detail. In Sec. III we show the nonpole calculation method for the twist-3 contribution in order to reexamine the pole contributions. Finally, in Sec. IV we summarize the achievements in this paper and make some comments on possible applications of the new non-pole method.

II. CONVENTIONAL POLE CALCULATION AT TWIST-3

The conventional collinear expansion framework at twist-3 has been developed in Refs. [8-17]. We review here the pole calculation for semi-inclusive deep inelastic scattering (SIDIS) in order to show the difference with the new method of non-pole calculation that we propose in next section.

We consider the process of polarized SIDIS

\[ e(l) + p^1(p, S_\perp) \rightarrow e(l') + h(P_h) + X. \]  

(2)

where the initial proton is transversely polarized. \( l \) and \( l' \) are, respectively, the momenta of the incoming and outgoing electrons. \( p \) and \( S_\perp \) are the momentum and transverse spin of the beam proton, \( P_h \) is the momentum of the final state hadron. In this paper, we focus on one-photon exchange process with the photon invariant mass \( q^2 = (l - l')^2 = -Q^2 \), the extension to charged current interaction is straightforward. The polarized cross section for SIDIS is given by

\[ \frac{d^4 \Delta \sigma}{dx_B dy dz_h dp_{h\perp}} = \frac{\alpha_{em}^2}{32\pi^2 z_h x_B^2 S^2_{ep} Q^2} L^\mu\nu W_{\mu\nu}. \]  

(3)

where the standard Lorentz invariant variables in SIDIS are defined as

\[ S_{ep} = (p + l)^2, \quad x_B = \frac{Q^2}{2p \cdot q}, \quad z_h = \frac{p \cdot P_h}{p \cdot q}, \quad y = \frac{p \cdot q}{p \cdot l}. \]  

(4)

The leptonic tensor is defined as follows

\[ L^\mu\nu = 2 (\not{l}^\mu + l^\mu l_\nu - l_\mu l^\nu g^\mu\nu). \]  

(5)

In order to simplify the discussion, we will only consider the metric part \( L^\mu\nu \rightarrow -Q^2 g^{\mu\nu} \). The SSA in SIDIS can be generated by both the initial state and final state twist-3 process. We focus in this paper the contribution from initial state twist-3 distribution functions of the transversely polarized proton, then the polarized differential cross section can be written as

\[ \frac{d^4 \Delta \sigma}{dx_B dy dz_h dp_{h\perp}} = \frac{\alpha_{em}^2}{32\pi^2 z_h x_B^2 S^2_{ep} Q^2} \sum_i \int \frac{dz}{z^2} W_i Dr_{\to h}(z) \]  

(6)

where \( Dr_{\to h}(z) \) is the usual twist-2 unpolarized fragmentation function. The hadronic part \( W_i \) describes a scattering of the virtual photon on a transversely polarized proton, with the leptonic metric part contracted. We will make the subscript \( i \) implicit in the rest part of this paper for simplicity.

In the conventional twist expansion framework, one needs to consider diagrams as shown in Fig. 1 in which the hadronic part reads

\[ W_{\text{pole}} = \int \frac{d^4k_1}{(2\pi)^4} \int \frac{d^4k_2}{(2\pi)^4} \int d^4y_1 \int d^4y_2 e^{ik_1 \cdot y_1} e^{i(k_2 - k_1) \cdot y_2} (pS_\perp |\psi_l(0)gA^a(y_2)\psi_l(y_1)|pS_\perp) H^{\text{pole}}_{ji,a}(k_1, k_2). \]  

(7)

The twist-3 contribution is generated by the pole terms, which come from the imaginary part of the propagator

\[ \frac{1}{k^2 + i\epsilon} = P \left( \frac{1}{k^2} \right) - i\pi \delta(k^2). \]  

(8)
Thus only $-i\pi\delta(k^2)$ is considered in conventional pole method. When we consider $O(\alpha_s)$ contribution, there are three types of pole contributions: soft-gluon-pole contribution(SGP, $x_2 - x_1 = 0$), soft-fermion-pole (SFP, $x_1 = 0$ or $x_2 = 0$, $x_1 \neq x_2$) and hard-pole (HP, $x_1 = x_B$, $x_2 \neq x_B$ or $x_2 = x_B, x_1 \neq x_B$). Representative diagrams for each pole contribution are shown in Fig. 2. It’s known that there are other contributions given by the diagrams including the two quark lines in the same side of the cut [18]. However, we don’t consider those contributions because it’s easy to generalize our result to those cases. One can factor out the $\delta$-function in $H_{ji,\alpha}^{\text{pole}}(k_1, k_2)$ for the three pole contributions

$$
H_{ji,\alpha}^{\text{pole}}(k_1, k_2) = H_{ji,\alpha}^{\text{SGP}}(k_1, k_2) \left\{ -i\pi\delta\left[ (p_c - (k_2 - k_1))^2 \right] \right\} (2\pi)\delta\left[ (k_2 + q - p_c)^2 \right] + H_{ji,\alpha}^{\text{SFP}}(k_1, k_2) \left\{ -i\pi\delta\left[ (p_c - k_2 + k_1 - q)^2 \right] \right\} (2\pi)\delta\left[ (k_2 + q - p_c)^2 \right] + H_{ji,\alpha}^{\text{HP}}(k_1, k_2) \left\{ -i\pi\delta\left[ (k_1 + q)^2 \right] \right\} (2\pi)\delta\left[ (k_2 + q - p_c)^2 \right] + \text{(complex conjugate diagrams)},
$$

(9)

where the factor $(2\pi)\delta\left[ (k_2 + q - p_c)^2 \right]$ representing the on-shell condition of the unobserved parton, $p_c$ is the four-momentum of the final state fragmenting parton. We can show the Ward-Takahashi identity(WTI) for the pole contributions,

$$(k_2 - k_1)^\alpha H_{ji,\alpha}^{\text{pole}}(k_1, k_2) = 0.
$$

(10)

Considering $k_1$ and $k_2$ derivatives, we can derive the relations,

$$(x_2 - x_1) \frac{\partial}{\partial k_1^\beta} H_{ji,\alpha}^{\text{pole}}(k_1, k_2) \bigg|_{k_1 = x_1 p} = H_{ji,\beta}^{\text{pole}}(x_1 p, x_2 p),
$$

$$(x_2 - x_1) \frac{\partial}{\partial k_2^\beta} H_{ji,\alpha}^{\text{pole}}(k_1, k_2) \bigg|_{k_1 = x_1 p} = -H_{ji,\beta}^{\text{pole}}(x_1 p, x_2 p),
$$

(11)

where $H_{ji,\alpha}^{\text{pole}}(k_1, k_2) = p^\beta H_{ji,\alpha}^{\text{pole}}(k_1, k_2)$. Thus we can derive the following useful relations for SFP and HP

$$
\frac{\partial}{\partial k_1^\beta} H_{ji,\alpha}^{\text{SFP}(HP)}(k_1, k_2) \bigg|_{k_1 = x_1 p} = \frac{1}{x_2 - x_1} H_{ji,\beta}^{\text{SFP}(HP)}(x_1 p, x_2 p),
$$

$$
\frac{\partial}{\partial k_2^\beta} H_{ji,\alpha}^{\text{SFP}(HP)}(k_1, k_2) \bigg|_{k_1 = x_1 p} = -\frac{1}{x_2 - x_1} H_{ji,\beta}^{\text{SFP}(HP)\mu\nu}(x_1 p, x_2 p),
$$

(12)

which lead to

$$
-\frac{\partial}{\partial k_1^\beta} H_{ji,\alpha}^{\text{SFP}(HP)}(k_1, k_2) \bigg|_{k_1 = x_1 p} = \frac{\partial}{\partial k_2^\beta} H_{ji,\alpha}^{\text{SFP}(HP)}(k_1, k_2) \bigg|_{k_1 = x_1 p}.
$$

(13)

However, we cannot use WTI for the SGP diagrams to derive similar equation as shown above because the SGP term corresponds to $\delta(x_2 - x_1)$. So far, the only way to derive this is to calculate all the relevant diagrams explicitly, which is annoying in high order perturbative QCD calculations. In SIDIS at $O(\alpha_s)$, the authors of Ref. [12] have checked explicitly that the above relation also hold true for $H_{ji,\alpha}^{\text{SGP}}$,

$$
\frac{\partial}{\partial k_1^\beta} H_{ji,\alpha}^{\text{SGP}}(k_1, k_2) \bigg|_{k_1 = x_1 p} = -\frac{\partial}{\partial k_2^\beta} H_{ji,\alpha}^{\text{SGP}}(k_1, k_2) \bigg|_{k_1 = x_1 p}.
$$

(14)
Now we can perform collinear expansion of the hard part
\[ H_{ji,\rho}^{\text{pole}}(k_1, k_2) \]
\[ \simeq H_{ji,\rho}^{\text{pole}}(x_1 p, x_2 p) + \frac{1}{k_1} H_{ji,\rho}^{\text{pole}}(k_1, k_2) \bigg|_{k_i = x_i p} \omega_{\beta}^\alpha k_1^\beta + \frac{1}{k_2} H_{ji,\rho}^{\text{pole}}(k_1, k_2) \bigg|_{k_i = x_i p} \omega_{\beta}^\alpha k_2^\beta \]
\[ = H_{ji,\rho}^{\text{pole}}(x_1 p, x_2 p) + \frac{\partial}{\partial k_2^\alpha} \bigg|_{k_i = x_i p} \omega_{\beta}^\alpha (k_2 - k_1)^\beta, \]
\[
(15)
\]
where the transverse projection operator is defined as \( \omega_{\beta}^\alpha = g_{\beta}^\alpha - \bar{n}\alpha n_\beta \), with the unit vectors taken as \( \bar{n} = [1, 0, 0] \), \( n = [0, 1, 0] \). We work in the hadron frame and \( p^\mu = p^+ \bar{n}^\mu \).

The next step is to decompose the gluon field \( A^\alpha \) into longitudinal and transverse components
\[ A^\alpha = \frac{A^\alpha}{p^+} p^\alpha + \omega_{\beta}^\alpha A^\beta, \]
\[
(16)
\]
where \( A^n = A \cdot n \). Substitute Eqs. (15, 16) into Eq. (7), we can extract the twist-3 contribution

\[
W_{\text{pole}} = \int \frac{d^4k_1}{(2\pi)^4} \int \frac{d^4k_2}{(2\pi)^4} \int d^4y_1 \int d^4y_2 \, e^{ik_1 \cdot y_1} e^{i(k_2-k_1) \cdot y_2} \langle pS_\perp | \bar{\psi}(0) g A^n(y_2) \psi(y_1) | pS_\perp \rangle \\
\times \frac{1}{p^+} \frac{\partial}{\partial k_2^+} H_{ji,p}^{\text{pole}}(k_1, k_2)|_{k_i = (k_i - n)_p} \omega^n_{\beta}(k_2 - k_1)^eta \\
+ \omega^n_{\alpha} \int \frac{d^4k_1}{(2\pi)^4} \int \frac{d^4k_2}{(2\pi)^4} \int d^4y_1 \int d^4y_2 \, e^{ik_1 \cdot y_1} e^{i(k_2-k_1) \cdot y_2} \langle pS_\perp | \bar{\psi}(0) g A^\beta(y_2) \psi(y_1) | pS_\perp \rangle \\
\times H_{ji,\alpha}^{\text{pole}}(x_1 p, x_2 p)
\]

\[
= i \omega^n_{\beta} p^+ \int dx_1 \int dx_2 \int \frac{dy_{1-\perp}}{2\pi} \int \frac{dy_{2-\perp}}{2\pi} \, e^{ix_1 p^+ y_{1-\perp}} e^{i(x_2-x_1) p^+ y_{2-\perp}} \langle pS_\perp | \bar{\psi}(0) g \left[ \partial^\beta A^n(y_2) - \partial^\alpha A^\beta(y_2) \right] \psi(y_1) | pS_\perp \rangle \\
\times \frac{\partial}{\partial k_2^+} H_{ji,p}^{\text{pole}}(k_1, k_2)|_{k_i = x_1 p} + H_{ji,\alpha}^{\text{pole}}(x_1 p, x_2 p),
\]

The last term in Eq. (17) can be eliminated by using the relation Eq. (11), and the first term can be rewritten as

\[
W_{\text{pole}} = i \omega^n_{\alpha} \int dx_1 \int dx_2 \text{Tr} \left[ M^\alpha_F(x_1, x_2) \frac{\partial}{\partial k_2^+} H_{ji,p}^{\text{pole}}(k_1, k_2)|_{k_i = x_1 p} \right],
\]

(18)

\( M^\alpha_F(x_1, x_2) \) is the F-type dynamical function which can be further expanded as

\[
M^\beta_{ij,F}(x_1, x_2) = p^+ \int \frac{dy_{1-\perp}}{2\pi} \int \frac{dy_{2-\perp}}{2\pi} \, e^{ix_1 p^+ y_{1-\perp}} e^{i(x_2-x_1) p^+ y_{2-\perp}} \langle pS_\perp | \bar{\psi}(0) g F^{\beta n}(y_2^-) \psi(y_1^-) | pS_\perp \rangle \\
= -\frac{M_N}{2} \epsilon^{\beta n \sigma_\perp} \langle \delta \rangle_{ij} T_{q,F}(x_1, x_2) + \cdots,
\]

(19)

with the nucleon mass \( M_N \) and the field strength tensor defined as \( F^{\beta n}(y_2^-) = \partial^\beta A^n(y_2^-) - \partial^\alpha A^\beta(y_2^-) \), notice that the nonlinear gluon term in the field strength tensor has been omitted because it comes from Feynman diagrams with linked gluons more than one, therefore does not show in Eq. (18). \( T_{q,F}(x_1, x_2) \) is the well-known Qiu-Sterman function, defined as follows:

\[
T_{q,F}(x_1, x_2) = \left( \frac{g}{2\pi M_N} \right) \int \frac{dy_{1-\perp}dy_{2-\perp}}{4\pi} \, e^{ix_1 p^+ y_{1-\perp}} e^{i(x_2-x_1) p^+ y_{2-\perp}} \langle pS_\perp | \bar{\psi}(0) g \epsilon^{\beta n \sigma_\perp} F^{\beta n}(y_2^-) \psi(y_1^-) | pS_\perp \rangle.
\]

(20)

Using Eq. (12), one can evaluate the derivative over the hard part in Eq. (18) for SFP and HP. For SGP, we need to rely on the master formula (19)

\[
\frac{\partial}{\partial k_2^+} H_{ji,p}^{\text{SGP}}(k_1, k_2)|_{k_i = x_1 p} = \frac{1}{2NC_F} \left[ i\pi \delta(x_2-x_1) \right] \left( \frac{\partial}{\partial p^e_+} - \frac{p_{col}^e}{p_+} \frac{\partial}{\partial p_-^e} \right) H_{ji}(x_1 p),
\]

(21)

where \( H_{ji}(x_1 p) \) is the \( g^*q \rightarrow qg \) scattering cross section without the gluon line with the momentum \((x_2-x_1)p\). Combine the three pole contributions together, we reproduce the final result based on the conventional pole calculation

\[
\frac{d^4\Delta\sigma}{dx_{B}dy_{D}dz_{H}dp_{H}^\perp} = \frac{M_N\alpha_s^2}{32\pi^2S_B^0S_A^0Q^2} \sum_q \int \frac{dz_{H}}{z_{H}} \frac{dz_{B}(z)}{z_{B}(z)} \int d^4x \left[ (xp + q - p_c)^2 \right] \left( (\delta + Q^2)e^{P_{col}p^e_{+}} + i\epsilon^{\eta n \sigma_\perp} \right) \\
\times \left[ x \frac{d}{dx} T_{q,F}(x, x) \hat{\sigma}_D + T_{q,F}(x, x) \hat{\sigma}_N + T_{q,F}(0, x) \hat{\sigma}_{SFP} + T_{q,F}(x, x) \hat{\sigma}_{HP} \right],
\]

(22)

1 We rescaled the function as \( T_{q,F}(x_1, x_2) \rightarrow (g/2\pi M_N)T_{q,F}(x_3, x_2) \) from the original definition in Ref. [4] for the convenience. Our definition of \( T_{q,F}(x_1, x_2) \) is the same with \( F_{\beta n}^{\alpha}(x_2, x_1) \) in Ref. [15].
where all hard cross sections are listed below,

\[ \hat{\sigma}_D = \frac{1}{2N} \frac{16Q^2}{\hat{s}\hat{t}\hat{u}} \left[ (\hat{s} + \hat{t})^2 + (\hat{t} + \hat{u})^2 \right], \]

\[ \hat{\sigma}_{ND} = \frac{1}{2N} \frac{16Q^2}{\hat{s}\hat{t}\hat{u}} \left[ Q^4 + 4Q^2\hat{t}\hat{u} + 4\hat{t}^3 + 2\hat{u}^3 + \hat{u}^4 \right], \]

\[ \hat{\sigma}_{SF P} = -\frac{1}{2N} \frac{16Q^2}{\hat{s}\hat{t}\hat{u}} \left[ \hat{t}^3 + 3\hat{u}^2 \right], \]

\[ \hat{\sigma}_{HP} = \left( \frac{1}{2N} \right) \left( \frac{1}{\hat{s} + \hat{q}} \right) \frac{16Q^2}{\hat{s}^2\hat{u}^2} \left[ Q^4 + 3Q^2\hat{s} + \hat{s}^3 + Q^2(3\hat{s}^2 + \hat{t}^2) \right]. \]

with the standard Mandelstam variables defined as

\[ \hat{s} = (xp + q)^2, \quad \hat{t} = (xp - pc)^2, \quad \hat{u} = (q - pc)^2. \]

In the next section, we show that the new non-pole method can reproduce these hard cross sections. We would like to make a comment on the relation for the SGP diagrams, this relation is required to construct the gauge-invariant matrix for the dynamical function. However, there is no simple way to prove this relation and we have to check it diagram by diagram. This is a frustrating point of the conventional pole calculation. We will show that the new method doesn’t have such difficulty and it’s a more flexible calculation technique.

### III. RESULT OF THE NEW NON-POLE CALCULATION

We introduce the new non-pole calculation method in this section. The main difference between the pole and non-pole methods is on the decomposition of the propagator shown in Eq. (8). In the new method we keep both the principle value part and the pole part until we perform the contour integrations in the end, while only the pole part is considered in the conventional method. In this sense, the new method can be regarded as a more rigorous approach of the twist-3 calculation.

#### A. General formalism

![Diagram](image-url)

FIG. 3. Diagrammatic description of Eq. (25).

In the new method we propose here, the hadronic part should be written as a sum of all the diagrams, i.e.,

\[ W = \sum_i W^{(i)} \],

where \( i \) denotes the number of gluon attachment. Let’s start with the diagram in Fig. 3 without any gluon attachment, which is given by

\[ W^{(0)} = \int \frac{d^4k}{(2\pi)^4} \int d^4y_1 e^{ik \cdot y_1} \langle pS_\perp | \bar{\psi}_j(0) \psi_i(y_1) | pS_\perp \rangle H_{ji}(k). \]

(25)

The twist-3 contribution from diagrams without gluon attachment can be obtained by performing collinear expansion of the hard part

\[ H_{ji}(k) \simeq H_{ji}(xp) + \left. \frac{\partial}{\partial k_{\alpha}} H_{ji}(k) \right|_{k=\infty} \omega_{\alpha} \beta kj. \]

(26)
Then Eq. (25) can be decomposed as two parts

\[ W^{(0)} = \int \frac{d^4k}{(2\pi)^4} \int d^4y_1 e^{i(k_1 - k_2) \cdot y_1} (pS_\perp | \bar{\psi}_j(0) \psi_i(y_1) | pS_\perp) \left( H_{ji}(xp) + \frac{\partial}{\partial k^\alpha} H_{ji}(k) \right)_{k=xp} \omega_\alpha k^\beta \]

\[ = p^+ \int dx \int \frac{dy_1}{2\pi} e^{iyp^+ y_1} (pS_\perp | \bar{\psi}_j(y_1) \psi_i(y_1) | pS_\perp) H_{ji}(xp) \]

\[ + i\omega_\alpha p^+ \int dx \int \frac{dy_1}{2\pi} e^{iyp^+ y_1} (pS_\perp | \bar{\psi}_j(0) \partial^\beta \psi_i(y_1) | pS_\perp) \left( \frac{\partial}{\partial k^\alpha} H_{ji}(k) \right)_{k=xp}. \] (27)

In general, the first term can give the twist-3 contribution when the hard part gives transverse component \( H_{ji}(xp) \sim (\gamma^+)_ji \). Next we consider the diagrams with one gluon attachment as shown in Fig. 4 which were also considered in the conventional method. Here we need to consider a set of diagrams \( H_{ji,\rho}(k_1, k_2) \) shown in Fig. 4 and their complex conjugate. We call them non-pole diagrams because we don’t separate the pole term from the propagators. The non-pole contribution to the hadronic part reads

\[ W^{(1)} = \int \frac{d^4k_1}{(2\pi)^4} \int \frac{d^4k_2}{(2\pi)^4} \int d^4y_1 \int d^4y_2 e^{ik_1 \cdot y_1} e^{i(k_2 - k_1) \cdot y_2} (pS_\perp | \bar{\psi}_j(0) \gamma^\rho(y_2) \gamma^\beta \psi_i(y_1) | pS_\perp) H_{ji,\rho}(k_1, k_2) \] (28)

Similar to the strategy in dealing with diagrams without gluon attachment, the first step to extract the twist-3 contribution from one gluon attachment diagrams is to perform collinear expansion of the hard part

\[ H_{ji,\rho}(k_1, k_2) = H_{ji,\rho}(xp, x_2p) + \frac{\partial H_{ji,\rho}(k_1, k_2)}{\partial k_1^\alpha} \bigg|_{k_1=xp} \omega_\beta k_1^\beta + \frac{\partial H_{ji,\rho}(k_1, k_2)}{\partial k_2^\alpha} \bigg|_{k_2=xp} \omega_\beta k_2^\beta. \] (29)

One also need to decompose the gluon field \( A^\rho \) into longitudinal and transverse components as in Eq. (16). Then Eq. (28) can be expanded as follows

\[ W^{(1)} = \int \frac{d^4k_1}{(2\pi)^4} \int \frac{d^4k_2}{(2\pi)^4} \int d^4y_1 \int d^4y_2 e^{ik_1 \cdot y_1} e^{i(k_2 - k_1) \cdot y_2} (pS_\perp | \bar{\psi}_j(0) A^\rho(y_2) \gamma^\beta \psi_i(y_1) | pS_\perp) H_{ji,\rho}(xp, x_2p) \]

\[ \times \frac{1}{p^+ \gamma^\rho} \left[ H_{ji,\rho}(xp, x_2p) + \frac{\partial}{\partial k_1^\alpha} H_{ji,\rho}(k_1, k_2) \bigg|_{k_1=xp} \omega_\beta k_1^\beta + \frac{\partial}{\partial k_2^\alpha} H_{ji,\rho}(k_1, k_2) \bigg|_{k_2=xp} \omega_\beta k_2^\beta \right] \]

\[ + \omega_\beta \int \frac{d^4k_1}{(2\pi)^4} \int \frac{d^4k_2}{(2\pi)^4} \int d^4y_1 \int d^4y_2 e^{ik_1 \cdot y_1} e^{i(k_2 - k_1) \cdot y_2} (pS_\perp | \bar{\psi}_j(0) A^\beta(y_2) \psi_i(y_1) | pS_\perp) H_{ji,\alpha}(xp, x_2p). \] (30)
notice that other terms in the combination of Eqs. (16) and (29) contribute to higher twist. The hard part shown in above equation can be further simplified by using the WTI relations, which is straightforward to derive in the non-pole calculation method [14].

\[(k_2 - k_1)^a H_{j_i,\alpha}(k_1, k_2) = H_{j_i}(k_2) - H_{j_i}(k_1),\]  
(31)

The non-pole hard part doesn’t have the delta function \(\delta(x_2 - x_1)\), therefore, we can derive the following useful relations

\[
H_{j_i,p}(x_1p, x_2p) = \int \frac{1}{x_2 - x_1 - i\epsilon} \left[ H_{j_i}(x_2p) - H_{j_i}(x_1p) \right],
\]

\[
\frac{\partial}{\partial k_1^\beta} H_{j_i,p}(k_1, k_2) \bigg|_{k_1=x_1p, k_2=x_2p} = \int \frac{1}{x_2 - x_1 - i\epsilon} \left[ H_{j_i,\beta}(x_1p, x_2p) - \frac{\partial}{\partial k_1^\beta} H_{j_i}(k_1) \right]_{k_1=x_1p},
\]

\[
\frac{\partial}{\partial k_2^\beta} H_{j_i,p}(k_1, k_2) \bigg|_{k_1=x_1p, k_2=x_2p} = -\int \frac{1}{x_2 - x_1 - i\epsilon} \left[ H_{j_i,\beta}(x_1p, x_2p) - \frac{\partial}{\partial k_2^\beta} H_{j_i}(k_2) \right]_{k_2=x_2p},
\]

(32)

where the sign of \(i\epsilon\) was determined by the fact that there is only the final state interaction in SIDIS. By using these useful relations derived from WTI, the hard part terms \(H_{j_i,\alpha}(k_1, k_2)\) and \(H_{j_i,p}(k_1, k_2)\) contained in Eq. (30) are respectively given by

\[
H_{j_i,p}(x_1p, x_2p) = (x_2 - x_1) \left[ \int \frac{1}{x_2 - x_1 - i\epsilon} \left[ H_{j_i}(x_2p) - H_{j_i}(x_1p) \right] - \frac{1}{x_2 - x_1 - i\epsilon} \frac{\partial}{\partial k_2^\beta} H_{j_i}(k_2) \bigg|_{k_2=x_2p} \right] + \frac{\partial}{\partial k_1^\beta} H_{j_i}(k_2) \bigg|_{k_2=x_2p},
\]

\[
H_{j_i,p}(x_1p, x_2p) + \frac{\partial}{\partial k_1^\beta} H_{j_i,p}(k_1, k_2) \bigg|_{k_1=x_1p} \omega_{\alpha}^\beta k_1^\beta + \frac{\partial}{\partial k_2^\beta} H_{j_i,p}(k_1, k_2) \bigg|_{k_1=x_1p} \omega_{\alpha}^\beta k_2^\beta
\]

\[
= \int \frac{1}{x_2 - x_1 - i\epsilon} \left[ H_{j_i}(x_2p) - H_{j_i}(x_1p) \right] - H_{j_i,\alpha}(x_1p, x_2p) \omega_{\alpha}^\beta (k_2 - k_1)^\beta + \frac{\partial}{\partial k_2^\beta} H_{j_i}(k_2) \bigg|_{k_2=x_2p} \omega_{\alpha}^\beta (k_2 - k_1)^\beta
\]

\[
+ \left[ \frac{\partial}{\partial k_2^\beta} H_{j_i}(k_2) \bigg|_{k_2=x_2p} - \frac{\partial}{\partial k_1^\alpha} H_{j_i}(k_1) \bigg|_{k_1=x_1p} \right] \omega_{\alpha}^\beta k_2^\beta \right\},
\]

(33)

Substitute Eq. (33) into Eq. (30), we obtain the final result

\[
W^{(1)} = p^+ \int dx \int \frac{dy_1}{2\pi} e^{i x p + y_1 \not{p} \not{s}_1 \not{p}_1} \langle 0 | \int_0^1 dy_2 A^\alpha(y_2) | \bar{\psi}_i(y_1^+) | p S_1 \rangle H_{j_i}(x p)
\]

\[
- i\omega_{\alpha}^\beta p^+ \int dx_1 \int dx_2 \int \frac{dy_2}{2\pi} e^{i x_2p + y_2 \not{p} \not{s}_1 \not{p}_1} \langle \bar{\psi}_i(y_1^+) 0 | \int_0^1 dy_2 A^\alpha(y_2) \psi_i(y_1^-) | p S_1 \rangle H_{j_i,\alpha}(x_1p, x_2p) \frac{\partial}{\partial k^\alpha} H_{j_i}(k) \bigg|_{k=x_2p},
\]

\[
+ i\omega_{\alpha}^\beta p^+ \int dx \int \frac{dy_2}{2\pi} e^{i x p + y_1 \not{p} \not{s}_1 \not{p}_1} \langle 0 | \int_0^1 dy_2 A^\alpha(y_2) \psi_i(y_1^-) | p S_1 \rangle \frac{\partial}{\partial k^\alpha} H_{j_i}(k) \bigg|_{k=x_2p},
\]

\[
+ i\omega_{\alpha}^\beta p^+ \int dx \int \frac{dy_2}{2\pi} e^{i x p + y_1 \not{p} \not{s}_1 \not{p}_1} \langle 0 | \int_0^1 dy_2 A^\alpha(y_2) \psi_i(y_1^-) | p S_1 \rangle \frac{\partial}{\partial k^\alpha} H_{j_i}(k) \bigg|_{k=x_2p},
\]

(34)

Summing over all twist-3 contributions in the diagrams in Figs. 1 and 3 represented by Eqs. (27) and (34), respectively, we can contract the gauge-invariant expression

\[
W = \int dx \text{Tr}[M(x)H(xp)] + i\omega_{\alpha}^\beta \int dx \text{Tr} \left[ M^\beta_\alpha(x) \frac{\partial}{\partial k^\alpha} H(k) \bigg|_{k=x_2p} \right]
\]

\[
- i\omega_{\alpha}^\beta \int dx_1 \int dx_2 \frac{1}{x_2 - x_1 - i\epsilon} \text{Tr}[M_F^\beta(x_1, x_2) H_\alpha(x_1p, x_2p)],
\]

(35)
where the matrices are given by

\[
M_{ij}(x) = p^+ \int \frac{dy_i}{2\pi} e^{ixp^+y_i} \langle pS_\perp | \bar{\psi}_j(0) \psi_i(y_1^-) | pS_\perp \rangle,
\]

\[
M_{ij,\partial}(x) = p^+ \int \frac{dy_i}{2\pi} e^{ixp^+y_i} \langle pS_\perp | \bar{\psi}_j(0) \partial_\perp (y_1^-) \psi_i(y_1^-) | pS_\perp \rangle
+ p^+ \int \frac{dy_i}{2\pi} e^{ixp^+y_i} \langle pS_\perp | \bar{\psi}_j(0) i\gamma_5 \lim_{y_1^- \to \infty} d\bar{y}_2 F_{\gamma_5 n}^\partial(y_2^-) | \psi_i(y_1^-) | pS_\perp \rangle,
\]

where operator definition of \( f_{1T}^{(1)}(x) \) is

\[
f_{1T}^{(1)}(x) = \left( -\frac{i}{2M_N} \right) \int \frac{dy_i}{2\pi} e^{ixp^+y_i} \langle pS_\perp | \bar{\psi}_i(0) | e^{\gamma_5 \gamma_1} S_\perp \rangle \left( D_{\gamma_5}(y_1^-) + ig \int_{y_1^-}^{\infty} d\bar{y}_2 F_{\gamma_5 n}^\partial(y_2^-) \right) | \psi_i(y_1^-) | pS_\perp \rangle.
\]

The definition of \( M_{ij}(x_1, x_2) \) and its decomposition is defined in Eq. \( (35) \). In the present case, the first term in Eq. \( (35) \) can’t give a twist-3 contribution because the spin projection \( \gamma_\alpha \gamma_5 \gamma_1 \) is forbidden by PT-invariance. Therefore, we can eliminate the first term in Eq. \( (35) \) and rewrite the twist-3 hadronic part

\[
W = \frac{M_N}{2} e^{\gamma_5 \gamma_1} S_\perp \left\{ \int dx f_{1T}^{(1)}(x) \mathrm{Tr} \left[ \frac{\partial}{\partial k_\alpha} H(k) \right]_{k=xp} \right. \\
+ i \int dx_1 \int dx_2 T_{q,F}(x_1, x_2) \frac{1}{x_2 - x_1 - i\epsilon} \mathrm{Tr} \left[ \theta H_\alpha(x_1 p, x_2 p) \right] \right\}.
\]

In the new method presented above, we only needed the well-defined relations \( (32) \) to construct the gauge-invariant matrix elements. We find that the difficulty associated with the relation \( (14) \) in the conventional calculation was removed. This is one of the advantages in the new method. Another advantage is that, by using Eq. \( (39) \), we don’t need to calculate the derivative of the hard part from diagrams with one gluon attachment, this will significantly reduce the complexity of twist-3 calculation, in particular for high order calculations.

### B. SIDIS at \( \mathcal{O}(\alpha_s) \)

In this subsection, we show in detail the calculation of hadronic part for SIDIS at \( \mathcal{O}(\alpha_s) \). We factor out the on-shell \( \delta \)-function from the hard partonic part,

\[
H(k) = \tilde{H}(k)(2\pi)\delta \left[ (k + q - p_c)^2 \right] H_\alpha(x_1 p, x_2 p) = \tilde{H}_L^\alpha(x_1 p, x_2 p)(2\pi) \delta \left[ (x_2 p - q - p_c)^2 \right] + H_\alpha^R(x_1 p, x_2 p)(2\pi) \delta \left[ (x_1 p + q - p_c)^2 \right],
\]

where \( \tilde{H}_L^\alpha(x_1 p, x_2 p) \) is given by a sum of 12 diagrams in Fig. \( 4 \) and \( H_\alpha^R(x_1 p, x_2 p) \) is its complex conjugate. The derivative of \( H(k) \) over \( k \) can be converted to those over the standard Mandelstam variables \( \hat{s}, \hat{t}, \hat{u} \). For details, see Appendix B. Then we can calculate Eq. \( (35) \) as

\[
W = \pi M_N \int dx \delta \left[ (x_2 p - q - p_c)^2 \right] \left\{ \frac{df_{1T}^{(1)}(x)}{dx} \left( \epsilon \gamma_5 S_\perp - \epsilon \gamma_5 \gamma_1 S_\perp \right) \frac{2}{u} \tilde{\sigma}(\hat{s}, \hat{t}, \hat{u}) + f_{1T}^{(1)}(x) \right\}
\]

\[
\times \left\{ \frac{2}{u} \left( \frac{\partial}{\partial t} - \frac{\partial}{\partial s} \right) \tilde{\sigma}(\hat{s}, \hat{t}, \hat{u}) - 2 \left( \epsilon \gamma_5 \gamma_1 S_\perp - \epsilon \gamma_5 \gamma_1 \gamma_1 S_\perp \right) \frac{1}{u} \tilde{\sigma}_{q^* \to gg} - \epsilon \gamma_5 \gamma_1 S_\perp \mathrm{Tr} [\gamma_\alpha \tilde{H}(x)] \right\}
\]

\[
+i \epsilon \gamma_5 \gamma_1 S_\perp \int dx' T_{q,F}(x', x) \frac{1}{x - x' - i\epsilon} \mathrm{Tr} [\theta \tilde{H}_L^\alpha(x' p, xp)] - \frac{1}{x - x' + i\epsilon} \mathrm{Tr} [\theta \tilde{H}_R^\alpha(x, x' p)] \right\},
\]

where \( \tilde{\sigma}(\hat{s}, \hat{t}, \hat{u}) \) is the \( 2 \to 2 \) partonic cross section in SIDIS. For \( q^* \to gg \) channel, it reads

\[
\tilde{\sigma}_{q^* \to gg} = -8C_F Q^2 \frac{(\hat{s} + \hat{t})^2 + (\hat{t} + \hat{u})^2}{\hat{s}\hat{u}}.
\]
Notice that, for convenience, we have changed the notation $x_1 \to x', \ x_2 \to x$ in right-cut diagrams, and $x_1 \to x, \ x_2 \to x'$ in left-cut diagrams. We discuss the gauge- and Lorentz-invariances of the hard cross sections associated with \( f_{1T}(x) \). The hard cross section with the nonderivative function \( f_{1T}(x) \) is not apparently gauge-invariant because of the term \( e^{\alpha n_S} \Tr[\gamma_\alpha H(x)] \). The gauge-invariance requires the unpolarized spin projection \( x\bar{n} \) with \( H(x) \) like Eq. (33). On the other hand, the hard cross section associated with the derivative function \( \frac{d}{dx}f_{1T}(x) \) is not Lorentz-invariant. The vector \( n \) in the parametrization (19) satisfies \( \bar{n} \cdot n = 1 \) and \( n^2 = 0 \). These conditions are not enough to uniquely determine the choice of \( n \) and there are two possible choices in SIDIS,

\[
n^\alpha = \frac{p^+}{p \cdot p_c} p_c^\alpha \quad \text{or} \quad n^\alpha = \frac{p^+}{p \cdot p_c} p_c^\alpha + \frac{2p^+ p_c \cdot q}{2(p_c \cdot q)(p \cdot q) + Q^2(p \cdot p_c)} \left( q^\alpha - \frac{p \cdot q}{p \cdot p_c} p_c^\alpha \right).
\]

We can check that the coefficient \( (e^{\alpha n_S} - e^{\alpha \bar{n} n_S}) \) of \( \frac{d}{dx}f_{1T}(x) \) depends on the choice of \( n \). This ambiguity of the cross section is physically interpreted as the frame-dependence because the spatial components of \( n \) is determined so that it has the opposite direction of the momentum \( p \) as \( \bar{n} = -\bar{p}/p^+ \). From the requirement of the frame-independence, the cross section has to be proportional to the factor \( [(\hat{s} + Q^2)e^{\alpha n_S} + \hat{t}e^{\alpha \bar{n} n_S}] \) as already shown in the cross section (22) derived by the conventional pole method. We will show later that the gauge- and Lorentz-invariances of the cross section are guaranteed by using Eqs. (18, 19).

FIG. 5. Typical diagrams which have \( x' \)-dependent propagators. Calculation for the propagators (1)-(4) are shown in Eq. (45).

Now we show how to calculate the hard partonic part \( \bar{H}^L(x'p, xp) \). There are four types of \( x' \)-dependences in Feynman gauge. Fig. 5 shows typical diagrams which have \( x' \)-dependent propagators. Each propagator can be calculated as follows:

propagator (1):

\[
\frac{\not{p} - (x - x')\not{p}}{|p_c - (x - x')p|^2 + i\epsilon} = \frac{1}{t}x\not{p} + \frac{x}{x - x' - i\epsilon} \not{p}_c,
\]

propagator (2):

\[
\frac{\not{p} - (x - x')\not{p} - \not{q}}{|p_c - (x - x')p - q|^2 + i\epsilon} = \frac{1}{u}x\not{p} - \frac{x}{x' - i\epsilon} \not{p}_c + (x\not{p} + \not{q} - \not{p}_c),
\]

propagator (3):

\[
\frac{x'\not{p} + \not{q}}{|x'p + q|^2 + i\epsilon} = \frac{1}{\hat{s} + Q^2}x\not{p} + \frac{x}{x' - x_B + i\epsilon} \not{s} + Q^2 \not{x_B} + \not{q},
\]

propagator (4):

\[
\frac{V_{\alpha\rho\tau}(x - x')p, -xp - q + p_c, x'p + q - p_c}{[x'p + q - p_c]^2 + i\epsilon} = \frac{1}{u} (x\not{p}_\tau g_{\alpha\rho} + x\not{p}_\alpha g_{\rho\tau} - 2x\not{p}_\rho g_{\alpha\tau})
\]

\[
+ \frac{x}{x - x' - i\epsilon u} \left[ (x\not{p} + q - p_c)x\not{g}_{\alpha\rho} - 2(x\not{p} + q - p_c)x\not{g}_{\alpha\rho} + (x\not{p} + q - p_c)x\not{g}_{\alpha\rho} \right],
\]

where \( V_{\alpha\rho\tau} \) comes from the 3-gluon vertex. We can find that all \( x' \)-dependence appear only in three types of denominators, \( x - x' - i\epsilon, x' - i\epsilon \) and \( x' - x_B + i\epsilon \). Products of two denominators can be disentangled as

\[
\frac{x}{x - x' - i\epsilon} \frac{x}{x' - i\epsilon} = \frac{x}{x - x' - i\epsilon} + \frac{x}{x' - i\epsilon},
\]

\[
\frac{x}{x - x' - i\epsilon} \frac{x}{x' - x_B + i\epsilon} = \frac{\hat{s} + Q^2}{\hat{s}} \left( x - x' - i\epsilon \frac{x}{x' - x_B + i\epsilon} \right).
\]
From the above discussion, we can conclude that the cross section associated with $\vec{H}_F^L(x', p, xp)$ is given by a form of

$$\frac{1}{x} e^{\alpha n S_L} \int dx' T_{q,F}(x', x) \frac{1}{x - x' - i\epsilon} \text{Tr}[x\phi H_{\phi}^L(x', p, xp)] = \frac{1}{x} \int dx' T_{q,F}(x', x) \left[ \frac{1}{x - x' - i\epsilon} H_{F1} + \frac{x}{(x - x' - i\epsilon)^2} H_{F2} + \frac{1}{x' - i\epsilon} H_{F3} + \frac{1}{x' - x - i\epsilon} H_{F4} \right]. \tag{47}$$

All hard parts $H_{Fi}$ are independent of $x'$. We can do the same discussion on $\vec{H}_F^R(xp, x', p)$. Then we can calculate each hard partonic cross section and obtain the following result for the hadronic part

$$W = \pi M_N \int \frac{dx}{x} \delta \left[ (xp + q - p_c)^2 \right] \left[ \frac{d^2}{dx^2} \frac{(\epsilon^\alpha n S_L - \epsilon c^\alpha n S_L)}{\hat{s}^2 u^2} \right] \frac{2}{u} \hat{\sigma}(\hat{s}, \hat{t}, \hat{u}) + \left[ (\hat{s} + Q^2) e^{\alpha n S_L} + i \epsilon^\alpha n S_L \right] f_{1T}(x) \hat{\sigma}_{ND}(\hat{s}, \hat{t}, \hat{u}) + i \int dx' T_{q,F}(x', x) \left[ \left( \frac{1}{x - x' - i\epsilon} - \frac{1}{x - x' + i\epsilon} \right) H_{F1} + \left( \frac{x}{(x - x' - i\epsilon)^2} - \frac{x}{(x - x' + i\epsilon)^2} \right) H_{F2} + \left( \frac{1}{x' - i\epsilon} - \frac{1}{x' - x - i\epsilon} \right) H_{F3} + \left( \frac{1}{x' - x - i\epsilon} - \frac{1}{x' - x - B + i\epsilon} \right) H_{F4} \right], \tag{48}$$

where the hard cross sections are given by

$$\hat{\sigma}_{ND} = 16e^2 Q^2 \frac{\alpha}{\hat{s}^2 u^2} \frac{(\hat{s} + Q^2) e^{\alpha n S_L} + i \epsilon^\alpha n S_L}{\hat{s}^2 u^2},$$

$$H_{F1} = \left[ (\hat{s} + Q^2) e^{\alpha n S_L} + i \epsilon^\alpha n S_L \right] \left[ \left( \frac{1}{2} \hat{\sigma}_{ND} + \frac{1}{2} \hat{\sigma}_{ND'} \right) \right],$$

$$H_{F2} = \left[ (\hat{s} + Q^2) e^{\alpha n S_L} + i \epsilon^\alpha n S_L \right] \hat{\sigma}_{ND} - \left( \epsilon^\alpha n S_L - \epsilon c^\alpha n S_L \right) \frac{2}{u} \hat{\sigma}(\hat{s}, \hat{t}, \hat{u}),$$

$$H_{F3} = -\frac{1}{2} \left[ (\hat{s} + Q^2) e^{\alpha n S_L} + i \epsilon^\alpha n S_L \right] \hat{\sigma}_{SFP},$$

$$H_{F4} = \frac{1}{2} \left[ (\hat{s} + Q^2) e^{\alpha n S_L} + i \epsilon^\alpha n S_L \right] \hat{\sigma}_{HP}. \tag{49}$$

$\hat{\sigma}_{ND}, \hat{\sigma}_{ND'}, \hat{\sigma}_{SFP}, \hat{\sigma}_{HP}$ can be found in Eq. (22). Since $H_{Fi}$ are all independent of $x'$, the $x'$-integration only involves $T_{q,F}(x', x)$ and the propagators. Then we can perform $x'$-integration in Eq. (48) as

$$\int dx' \left( \frac{1}{x - x' - i\epsilon} - \frac{1}{x - x' + i\epsilon} \right) T_{q,F}(x', x) = 2\pi i T_{q,F}(x, x),$$

$$\int dx' \left( \frac{x}{(x - x' - i\epsilon)^2} - \frac{x}{(x - x' + i\epsilon)^2} \right) T_{q,F}(x', x) = -\pi i x \frac{d}{dx} T_{q,F}(x, x),$$

$$\int dx' \left( \frac{1}{x' - x - i\epsilon} - \frac{1}{x' + i\epsilon} \right) T_{q,F}(x', x) = 2\pi i T_{q,F}(0, x),$$

$$\int dx' \left( \frac{1}{x' - x - B + i\epsilon} - \frac{1}{x' - x - x - B + i\epsilon} \right) T_{q,F}(x', x) = -2\pi i T_{q,F}(x, B, x). \tag{50}$$

where we have used the symmetric property of Qiu-Sterman function $T_{q,F}(x', x) = T_{q,F}(x, x')$ in the integration of the double pole coefficient. Substituting these relations into Eq. (48) and using Eqs. (A18, A19), we can finally derive the transverse polarized cross section in SIDIS based on the new method as

$$\frac{d^2 \Delta \sigma}{dx_B dy_0 dP_{x_B}} = \frac{M_N^2 \alpha_{em}^2}{32 \pi s_h \alpha_s^2 S_{21}^2 Q^2} \sum_q \int \frac{dz}{z^2} D_{q-h}(z) \int \frac{dx}{x} \delta \left[ (xp + q - p_c)^2 \right] \left( \frac{1}{2} \hat{\sigma}_{ND} + \hat{\sigma}_{ND} + T_{q,F}(0, x) \hat{\sigma}_{SFP} + T_{q,F}(x, B) \hat{\sigma}_{HP} \right). \tag{51}$$

This is exactly the same with the result of the conventional calculation (22). We would like to emphasize that the cross section is never gauge- and Lorentz-invariant if the kinematical function $f_{1T}(x)$ and Qiu-Sterman function $T_{q,F}(x, x)$ are independent with each other. The relation between them is needed for the physically acceptable result.
IV. SUMMARY

We proposed the new nonpole calculation method for the Sivers effect in the twist-3 cross section and confirmed the consistency with the conventional pole calculation. We found out that the relation \( f_{T}^{(1)}(x) = \pi T_{q,F}(x,x) \) is very important to guarantee the gauge- and Lorentz-invariance of the final result. We reproduced this relation without introducing the definition of the TMD Sivers function. The importance of Eq. (A18) has been mainly discussed in the context of the matching between the TMD factorization and the collinear twist-3 factorization frameworks. Our calculation showed that this is also important for the gauge- and Lorentz-invariances of the twist-3 physical observables for the Sivers effect. This result provides a new perspective on the relation. Same technique can be also applied to the gluon Sivers function and the twist-3 gluon distribution functions. The relation between them is relatively nontrivial compared to the quark functions. From the requirement of the gauge- and Lorentz-invariances of the twist-3 cross section, we can derive a similar relation with Eq. (A18) for the gluon distribution functions.

One of the advantages in the new non-pole calculation method is that we don’t need to prove Eq. (14) for the SGP contribution as required in conventional pole method, which can be only checked through diagram by diagram calculation. It’s known that this relation may not be hold when the description of the fragmentation part is changed to other framework such as NRQCD for heavy quarkonium production. In the new method, we never separate the pole contributions and then no singularity arise from the relation associated with WTI. Our new method will extend the applicability of the collinear twist-3 framework.

In the new method, one does not need to perform derivatives over the initial parton’s transverse momentum in the calculation of Feynman diagrams with additional gluon attachment, this could significantly reduce the complexity of high twist calculation. We expect the new method present in this manuscript can be extended to higher-twist calculations, which become one of the standard method to investigate the nontrivial nuclear effect in heavy ion collisions. As we don’t need to perform derivatives over the initial parton’s transverse momentum in the new non-pole method, we expect the new approach will be of great use in performing next-to-leading order calculation at higher twist, in which the conventional collinear expansion caused ambiguity in setting up the initial parton’s kinematics, this ambiguity can be resolved in the new non-pole method.

Appendix A: Twist-3 quark-gluon correlation functions

1. Definition of the twist-3 functions

We introduce the definition of all related twist-3 functions for the transversely polarized proton.

**D-type dynamical function**

\[
M_{ij,D}^a(x_1, x_2) = (p^+)^2 \int \frac{dy^-_1}{2\pi} \int \frac{dy^-_2}{2\pi} e^{ix_1 p^+ y^-_1} e^{i(x_2-x_1) p^+ y^-_2} (pS_\perp | \bar{\psi}_j(0)[0, y^-_2] D^a_\perp(y^-_2)[y^-_2, y^-_1] \psi_i(y^-_1) | pS_\perp) \\
= -\frac{M_N}{2} \epsilon^{\alpha\beta n} S_{\perp} \langle \gamma \rangle_{ij} T_{q,D}(x_1, x_2) + \cdots ,
\]

\( D^a_\perp(y^-_2) = \partial^a_\perp - ig A^a_\perp(y^-_2) \), \([0, y^-_2]\) is the Wilson line

\[
[0, y^-_2] = P \exp \left( i g \int_{y^-_2}^{0} dy^- A^a(y^-) \right),
\]

The D-type function \( T_{q,D}(x_1, x_2) \) is real and antisymmetric \( T_{q,D}(x_1, x_2) = -T_{q,D}(x_2, x_1) \).
Then we can show

\[ M_{ij, \partial}(x) = p^+ \int \frac{dy_\perp}{2\pi} e^{ixp^+ y_i} (pS_\perp | \bar{\psi}_j(0) D^\beta_\perp(y_i^-) \psi_i(y_i^-) | pS_\perp) \]

\[ + p^+ \int \frac{dy_\perp}{2\pi} e^{ixp^+ y_i} (pS_\perp | \bar{\psi}_j(0) ig \left[ \int_{y_i^-}^{\infty} dy_2 F^{\beta n}(y_2^-) \right] \psi_i(y_i^-) | pS_\perp), \]

\[ = -i \frac{M_N}{2} e^{\beta n S_\perp}(\not{p})_{ij} f^{(1)}_{1T}(x) + \cdots. \quad (A3) \]

By using the translation invariance [13],

\[ \langle pS_\perp | \bar{\psi}_j(0) \tilde{D}^\alpha_\perp(0) [0, y_i^-] \psi_i(y_i^-) | pS_\perp \rangle + \langle pS_\perp | \bar{\psi}_j(0) [0, y_i^-] D^\alpha_\perp(y_i^-) \psi_i(y_i^-) | pS_\perp \rangle \]

\[ + \int_{y_i^-}^{0} \frac{dy_2}{2\pi} (pS_\perp | \bar{\psi}_j(0) [0, y_2^-] ig F^{\alpha n}(y_2^-) | pS_\perp \rangle = 0, \quad (A4) \]

we can show \( M_j^\alpha(x) = - M_j^\alpha(x) \) and therefore \( f^{(1)}_{1T}(x) \) is real function. The kinematical function \( f^{(1)}_{1T}(x) \) has another definition using the quark TMD correlator. Here we recall the definition of the quark Sivers function,

\[ M_{ij}(x, p_T) = \int \frac{dy^-}{2\pi} \frac{d^2 x}{2\pi} e^{ixp^+ y^-} e^{ip_T \cdot \xi_T} (p_{S_\perp} | \bar{\psi}_j(0) [0, \infty^-] [\infty^-, \infty^- + \xi_T] \]

\[ \times (\infty^- + \xi_T, y^- + \xi_T) \psi_i(y^- + \xi_T) | pS_\perp \rangle \]

\[ = - \frac{1}{2M_N} f^{(1)}_{1T}(x, p_T) e^{p_T \eta S_\perp \gamma_\mu + \cdots. \quad (A5) \]

We can find a relation between the first moment of \( M(x, p_T) \) and the correlator of the kinematical function \( M_j^\alpha(x) \),

\[ \int d^2 p_T p_T \cdot \partial M_{ij}(x, p_T) \]

\[ = \int \frac{dy^-}{2\pi} \left( -i \frac{\partial}{\partial \xi_T^\alpha} \right) e^{ixp^+ y^-} \left( p_{S_\perp} | \bar{\psi}_j(0) [0, \infty^-] [\infty^-, \infty^- + \xi_T] [\infty^- + \xi_T, y^- + \xi_T] \psi_i(y^- + \xi_T) | pS_\perp \rangle \]

\[ = i \int \frac{dy^-}{2\pi} e^{ixp^+ y^-} (p_{S_\perp} | \bar{\psi}_j(0) D^\alpha_\perp(y^-) \psi_i(y^-) | pS_\perp \rangle + i \int \frac{dy^-}{2\pi} e^{ixp^+ y^-} (p_{S_\perp} | \bar{\psi}_j(0) ig \left[ \int_{y_i^-}^{\infty} dy_2 F^{\alpha n}(y_2^-) \right] \psi_i(y^-) | pS_\perp \rangle \]

\[ = i \frac{1}{p^+} M_{ij, \partial}(x). \quad (A6) \]

Then \( f^{(1)}_{1T}(x) \) can be expressed by the first moment of the quark Sivers function [22, 26].

\[ f^{(1)}_{1T}(x) = \int d^2 p_T \frac{|p_T|^2}{2M_N} f_{1T}(x, p_T). \quad (A7) \]

F-type dynamical function

\[ M_{ij, F}(x_1, x_2) = p^+ \int \frac{dy^-}{2\pi} \int \frac{dy^-}{2\pi} e^{ix_2 p^+ y_i} e^{i(x_2 - x_1) p^+ y_2} (p_{S_\perp} | \bar{\psi}_j(0) g F^{\alpha n}(y_2^-) \psi_i(y_1^-) | pS_\perp \rangle \]

\[ = - \frac{M_N}{2} e^{\beta n S_\perp}(\not{p})_{ij} T_{q, F}(x_1, x_2) + \cdots, \quad (A8) \]

where the F-type function \( T_{q, F}(x_1, x_2) \) is real and symmetric \( T_{q, F}(x_1, x_2) = T_{q, F}(x_2, x_1) \).
2. Relation among the functions

In order to find a relation among the twist-3 functions, we use the identity for the $D_{\perp}^\alpha(y_2^-)[y_2^-, y_1^-]$ in $M_D^\alpha(x_1, x_2)$,

$$D_{\perp}^\alpha(y_2^-)[y_2^-, y_1^-] = [y_2^-, y_1^-]D_{\perp}^\alpha(y_1^-) + i \int_{y_1^-}^{y_2^-} dy_3^- [y_2^-, y_3^-] gF^\alpha(y_3^-)[y_3^-, y_1^-]$$

Combining $A11, A12, A13$, we can show $M_D^\alpha(x_1, x_2) = \frac{1}{x_1 - x_2 + i\epsilon} M_F^\alpha(x_1, x_2) + \delta(x_2 - x_1) M_F^\alpha(x_1)$ and then the relation among the twist-3 functions is given by

$$T_{q,F}(x_1, x_2) = \frac{1}{x_1 - x_2 + i\epsilon} T_{q,F}(x_1, x_2) + i\delta(x_2 - x_1) f_{1T}^{\perp(1)}(x_1).$$

Using the interchange symmetry $x_1 \leftrightarrow x_2$, we can rewrite the above relation as

$$0 = \left( \frac{1}{x_1 - x_2 + i\epsilon} - \frac{1}{x_1 - x_2 - i\epsilon} \right) T_{q,F}(x_1, x_2) + 2i\delta(x_2 - x_1) f_{1T}^{\perp(1)}(x_1).$$
Because the operator definition of $T_{q,F}(x_1, x_2)$ is including the factor $e^{ix_1 p^+ (y_1 - y_2)}$, we can perform $x_1$-integration,

$$
\int dx_1 \left( \frac{1}{x_1 - x_2 + i\epsilon} - \frac{1}{x_1 - x_2 - i\epsilon} \right) e^{ix_1 p^+ (y_1 - y_2)} = -2\pi i \left( \theta(y_2 - y_1) + \theta(y_1 - y_2) \right) e^{ix_2 p^+ (y_1 - y_2)} = -2\pi i e^{ix_2 p^+ (y_1 - y_2)},
$$

and then

$$
\int dx_1 \left( \frac{1}{x_1 - x_2 + i\epsilon} - \frac{1}{x_1 - x_2 - i\epsilon} \right) T_{q,F}(x_1, x_2) = -2\pi i T_{q,F}(x_2, x_2).
$$

(A16)

After the integration of (A15) with respect to $x_1$, we can derive the relation

$$
f_{1T}^{\perp}(x) = \pi T_{q,F}(x, x).
$$

(A17)

This is well known relation between the first moment of the Sivers function $f_{1T}^{\perp}(x)$ and the Qiu-Sterman function $T_{q,F}(x, x)$ [24, 27]. The same relation can be derived as we performed here in a simple way. We also show the relation given by the derivative of the (A18) with respect to $x$ as

$$
\frac{d}{dx} f_{1T}^{\perp}(x) = \pi \frac{d}{dx} T_{q,F}(x, x).
$$

(A19)

We will show that this relation plays a role of Lorenz-Invariant-Relation [15] which is required for the Lorentz-invariant form of the twist-3 cross section.

**Appendix B: Calculation of the derivative term $\frac{\partial}{\partial k^a} H(k)|_{k^a=0}$**

We show how to calculate the hard part $\frac{\partial}{\partial k^a} H(k)|_{k^a=0}$ in (39) without direct operation of the $k$-derivative. We can calculate the part of the kinematical function as

$$
i \omega^{\alpha}_{\beta} \int dx \text{Tr}[M^{\beta}_{\alpha}(x) \frac{\partial}{\partial k^\alpha} H(k)|_{k^a=0}] = \frac{M_N}{2} \int dx \frac{\partial}{\partial k^\alpha} f_{1T}^{\perp}(x) \text{Tr}[x p^a \frac{\partial}{\partial k^\alpha} H(k)|_{k^a=0}]
$$

$$
= \frac{M_N}{2} \int dx \frac{\partial}{\partial k^\alpha} f_{1T}^{\perp}(x) \left\{ \text{Tr}[k H(k)]|_{k^a=0} - \text{Tr}[\gamma^a H(xp)] \right\}.
$$

(B1)

We focus on the first term in the parenthesis. Because $H(k)$ carries the information about $k$, $q$ and $p_c$, it can be written by all possible Lorentz invariant variables,

$$
\text{Tr}[k H(k)] = \hat{\sigma}(k^2, \tilde{s}, \tilde{t}, \tilde{u}, Q^2)(2\pi)\delta(\tilde{s} + \tilde{t} + \tilde{u} + Q^2 - k^2),
$$

(B2)

where we defined the variables,

$$
\tilde{s} = (k + q)^2, \quad \tilde{t} = (k - p_c)^2.
$$

(B3)

We can set $k^2 = 0$ because $\frac{\partial}{\partial k^2} f_{1T}^{\perp}(x)|_{k^a=0} = 2 xp^a \frac{\partial}{\partial k^a}$ is canceled with $e^{i\alpha n S_\perp}$. We find that $\hat{\sigma}(\tilde{s}, \tilde{t}, \tilde{u}, Q^2)$ coincides with $\hat{\sigma}_{q g \rightarrow gg}$ in (13) in the collinear limit $k = xp$. The $k$-derivative is converted into $\tilde{s}$- and $\tilde{t}$- derivatives,

$$
\frac{\partial}{\partial k^a} = 2 q^a \frac{\partial}{\partial \tilde{s}} - 2 p_c^a \frac{\partial}{\partial \tilde{t}},
$$

(B4)

where we used the fact that $\tilde{s}(\tilde{t})$- and $\tilde{s}(\tilde{t})$-derivatives are identical. We calculate the $k$-derivative term in (B1) as

$$
\frac{M_N}{2} \int dx \frac{\partial}{\partial k^\alpha} f_{1T}^{\perp}(x) \text{Tr}[k H(k)]|_{k^a=0} = \pi M_N \int dx \frac{\partial}{\partial k^\alpha} f_{1T}^{\perp}(x) \left( 2 q^a \frac{\partial}{\partial \tilde{s}} - 2 p_c^a \frac{\partial}{\partial \tilde{t}} \right) \hat{\sigma}_{q g \rightarrow gg} \delta(\tilde{s} + \tilde{t} + \tilde{u} + Q^2)
$$

$$
= \pi M_N \int dx \frac{\partial}{\partial k^\alpha} f_{1T}^{\perp}(x) \left( \delta(\tilde{s} + \tilde{t} + \tilde{u} + Q^2) \left( 2 q^a \frac{\partial}{\partial \tilde{s}} - 2 p_c^a \frac{\partial}{\partial \tilde{t}} \right) \hat{\sigma}_{q g \rightarrow gg} + \left( 2 q^a - 2 p_c^a \right) \hat{\sigma}_{q g \rightarrow gg} \frac{\partial}{\partial x} \right) \delta(\tilde{s} + \tilde{t} + \tilde{u} + Q^2)
$$

$$
= \pi M_N \int dx \frac{\partial}{\partial \tilde{s}} \left( \delta(\tilde{s} + \tilde{t} + \tilde{u} + Q^2) \left( 2 q^a \frac{\partial}{\partial \tilde{s}} - 2 p_c^a \frac{\partial}{\partial \tilde{t}} \right) \hat{\sigma}_{q g \rightarrow gg} + \left( 2 q^a - 2 p_c^a \right) \hat{\sigma}_{q g \rightarrow gg} \frac{\partial}{\partial x} \right) \delta(\tilde{s} + \tilde{t} + \tilde{u} + Q^2)
$$

$$
+ \left( 2 q^a - 2 p_c^a \right) \hat{\sigma}_{q g \rightarrow gg} \frac{\partial}{\partial x} \delta(\tilde{s} + \tilde{t} + \tilde{u} + Q^2)
$$

(B5)
We can calculate $x$-derivative of $\hat{\sigma}_{qg^*\to qg}$ as

$$x \frac{\partial}{\partial x} \hat{\sigma}_{qg^*\to qg} = \left( (\hat{s} + Q^2) \frac{\partial}{\partial \hat{s}} + \hat{t} \frac{\partial}{\partial \hat{t}} \right) \hat{\sigma}_{qg^*\to qg}. \quad (B6)$$

Finally we obtain the result in (42),

$$\pi M_N \int \frac{dx}{x} \delta(\hat{s} + \hat{t} + u + Q^2) \left\{ \frac{\partial}{\partial x} (x \hat{\sigma}_{qg^*\to qg}) + \left( (\hat{s} + Q^2) e^{\alpha \bar{n} n S_\perp} + \hat{t} e^{\alpha \bar{n} n S_\perp} \right) \frac{\partial}{\partial \hat{t}} \hat{\sigma}_{qg^*\to qg} - \begin{array}{c}
\left( 2e^{\alpha \bar{n} n S_\perp} - 2e^{\alpha \bar{n} n S_\perp} \right) \frac{1}{\hat{u}} \hat{\sigma}_{qg^*\to qg} - e^{\alpha \bar{n} n S_\perp} \text{Tr} [\gamma_\alpha \hat{H}(x p)]
\end{array} \right\}. \quad (B7)$$

The derivatives of the Mandelstam variables can be done after the calculation of the diagrams, which is much easier than the direct $k$-derivative.

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[25] D. Boer, P. J. Mulders and F. Pijlman, Nucl. Phys. B 667, 201 (2003) doi:10.1016/S0550-3213(03)00527-3 [hep-ph/0303034].

[26] Z. B. Kang, J. W. Qiu, W. Vogelsang and F. Yuan, Phys. Rev. D 83, 094001 (2011) doi:10.1103/PhysRevD.83.094001 [arXiv:1103.1591 [hep-ph]].

[27] J. P. Ma and Q. Wang, Eur. Phys. J. C 37, 293 (2004) doi:10.1140/epjc/s2004-02009-x [hep-ph/0310245].

[28] Z. B. Kang, E. Wang, X. N. Wang and H. Xing, Phys. Rev. D 94, no. 11, 114024 (2016) doi:10.1103/PhysRevD.94.114024 [arXiv:1409.1315 [hep-ph]].

[29] Z. B. Kang, J. W. Qiu, X. N. Wang and H. Xing, Phys. Rev. D 94, no. 7, 074038 (2016) doi:10.1103/PhysRevD.94.074038 [arXiv:1605.07175 [hep-ph]].