DEGREE OF THE GENERALIZED PLÜCKER EMBEDDING OF A QUOT SCHEME AND QUANTUM COHOMOLOGY

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Abstract

We compute the degree of the generalized Plücker embedding $\kappa$ of a Quot scheme $X$ over $\mathbb{P}^1$. The space $X$ can also be considered as a compactification of the space of algebraic maps of a fixed degree from $\mathbb{P}^1$ to the Grassmannian $\text{Grass}(m,n)$. Then the degree of the embedded variety $\kappa(X)$ can be interpreted as an intersection product of pullbacks of cohomology classes from $\text{Grass}(m,n)$ through the map $\psi$ that evaluates a map from $\mathbb{P}^1$ at a point $x \in \mathbb{P}^1$. We show that our formula for the degree verifies the formula for these intersection products predicted by physicists through Quantum cohomology [Vaf92] [Int91] [Wit93]. We arrive at the degree by proving a version of the classical Pieri’s formula on the variety $X$, using a cell decomposition of a space that lies in between $X$ and $\kappa(X)$.

Let $X'$ be the space of all algebraic maps, of a fixed degree $q$, from the projective line $\mathbb{P}^1$ to $\text{Grass}(m,n)$, the Grassmannian of all $m$-dimensional subspaces of a fixed $n$-dimensional vector space $V$. In various papers, physicists have been discussing so-called correlation functions on this space ([Int91] [Vaf92] [Wit93]) which specify the intersection products on $X'$ of pullbacks of cohomology classes from the Grassmannian (a precise formulation can be found in Section 4). Further, based on certain physical arguments, there have been some conjectured formulas for these correlation functions (actually these conjectures deal with the more general case of maps from any Riemann surface to Grass($m, n$), but we will restrict our attention to maps from $\mathbb{P}^1$). In [BDW93] the authors give a mathematically rigorous proof of the conjectured formula in the case of maps from a Riemann surface of genus one to Grass($2, n$).

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this paper, we show that the conjecture holds for a certain class of intersection products on the space of maps from $\mathbb{P}^1$ into Grass$(m,n)$ for any $m$ and $n$ \cite{[EL]}. Some intriguing connections with quantum cohomology and Floer cohomology can be found in \cite{[GRK],[S],[AS],[Pin],[Pin94]}. 

We shall be working on a natural compactification of $X'$, namely a Quot scheme defined as follows: Let $V_{\mathbb{P}^1} = V \otimes O_{\mathbb{P}^1}$ and let $f$ be the polynomial $f(l) = p(l+1) + q$, where $p = n - m$. As explained in \cite{[Str]}, the quotient scheme $X = \text{Quot}_{\mathbb{P}^1} f$ that parameterizes all quotient sheaves $B$ of $V_{\mathbb{P}^1}$ such that the Hilbert polynomial $\chi(B(l)) = f(l)$, can be considered as a compactification of $X'$. Some technical points on the choice of this compactification and the independence of the physical predictions on this choice can be found in Section 4. The space $X$ can be concretely described as follows: each point in $X$ can be considered as an equivalence class of matrices $M = (M_{ij}(s,t))_{1 \leq i \leq m, 1 \leq j \leq n}$ where each $M_{ij}(s,t)$ is a homogeneous polynomial of degree $q_i$ and $\sum q_i = q$. Two such matrices $M$ and $M'$ are considered equivalent if, after rearranging the rows if necessary, the row degrees of $M$ and $M'$ are the same and there exists an $m \times m$ matrix $U$ such that its entry $U_{ij}(s,t)$ is a homogeneous polynomial of degree $q_j - q_i$, $\det U$ is a non-zero constant and $M' = UM$. In the sequel we will find it convenient to switch back and forth between this and the description of the points in $X$ as quotient sheaves.

Given a sheaf $B \in X$, let

$$0 \to A \to V_{\mathbb{P}^1} \to B \to 0$$

be the exact sequence defining $B$. If we choose an isomorphism $A \simeq \bigoplus_{i=1}^m O_{\mathbb{P}^1}(-q_i)$ and choose basis for $V$, then the quotient sheaf $B$ can be identified with the matrix $M$ of the map $A \to V_{\mathbb{P}^1}$.

There is a natural map from $X$ into a projective space, namely the generalized Plücker map $\kappa$ defined as follows: to each matrix $M \in X$, $\kappa$ assigns the $m \times m$ minors of $M$, considered as a point in

$$\mathbb{P} = \mathbb{P}(\wedge^m V \otimes H^0(\mathbb{P}^1, O_{\mathbb{P}^1}(q))).$$

The map $\kappa$ is not an embedding and the image, which we denote by $K_{m,p}^q$, is a singular variety. In this paper we compute the degree of $K_{m,p}^q$ and that of certain subvarieties of this space that can be thought of as pullbacks of Schubert varieties through the evaluation map from $X$ to Grass$(m,n)$ (see Section 4 for a precise formulation). These degrees are a particular case of the conjectures from physics and we show that our formulas agree with the predicted degrees. Since the degree of $K_{m,p}^q$ is also equal to the degree of the pole placement map in the critical dimension we derive in this paper also an important result in Systems theory.

Our methods are quite elementary. We generalize the classical methods used to compute the degree of the Grassmannian. We use an intermediate space, $A_{m,p}^q$ such that the map $\kappa$ from $X$ to $\mathbb{P}$ factors as

$$X \xrightarrow{\phi} A_{m,p}^q \xrightarrow{\pi} \mathbb{P}. \quad (0.1)$$
The map $\phi$ sends a matrix $M(s,t) \in X$ to $M(s,1)$ and the map $\pi$ sends $M(s,1)$ to its $m \times m$ minors homogenized to polynomials of total degree $q$. The key ingredient for our work is a cellular decomposition of $A^q_{m,p}$ described in [WR94] that parallels the standard cellular decomposition of a Grassmannian, though this situation is more complicated. This cellular decomposition enables us to use Bézout’s theorem on $\mathbb{P}$ to prove an analogue of Pieri’s formula (Proposition 3.2) on $K^q_{m,p}$. This reduces the computation of the degree to a combinatorial problem.

In Section 1 we define the space $A^q_{m,p}$ and recall its basic properties and its cell decomposition (Proposition 1.4). In Section 2 we define the subvarieties of $K^q_{m,p}$ (Definition 2.3) that play a role analogous to that of the Schubert subvarieties of a Grassmannian. In Section 3 we apply Bézout’s theorem to the intersection of these subvarieties by appropriate hyperplanes of $\mathbb{P}$. We then give a combinatorial description for the degree of $K^q_{m,p}$ and its subvarieties. In Section 4 we give a precise formulation of the Physics conjecture and interpret the degrees we compute in terms of this conjecture. We then show that our computation of the degrees agrees with the formula conjectured by the physicists.

We started this paper with a view of understanding the space $X$, considered as a compactification of the space of all $m$-input, $p$-output transfer functions of McMillan degree $q$ ([RR94], [Ros94]) in Systems theory. A substantial portion of this paper has been in circulation for some time under the title “Degree of the Generalized Plücker embedding of a Quot scheme”.

1 The cellular decomposition of $A^q_{m,p}$

In this section we summarize those results obtained in [WR94] which we will use in this paper. The space $A^q_{m,p}$ consists of equivalence classes of polynomial matrices $M(s) = (M_{ij}(s))_{1 \leq i \leq m, 1 \leq j \leq n}$ such that the degree of any $m \times m$ minor of $M$ is at most $q$ and at least one of these minors is a non-zero polynomial. Two such matrices $M(s)$ and $M'(s)$ are considered equivalent, if there exists a unimodular polynomial matrix $U(s)$ such that the matrix $M'(s) = U(s)M(s)$.

Definition 1.1 Given any $m \times n$ polynomial matrix $M(s)$, there exist unique $\nu = (\nu_1, \ldots, \nu_m)$ with $\nu_1 \leq \cdots \leq \nu_m$ and

$$\sum_{i=1}^{m} \nu_i = \text{maximum degree of } m \times m \text{ minors of } M(s)$$

and an $m \times m$ unimodular matrix $U(s)$ such that the matrix $M'(s) = U(s)M(s)$ has row degrees $\nu_1 \leq \cdots \leq \nu_m$. The numbers $\nu = (\nu_1, \ldots, \nu_m)$ are called the ordered Kronecker indices of the equivalence class of $M(s)$.

A matrix $M(s)$ is called row reduced if the ordered Kronecker indices are equal to the degrees of the rows of $M(s)$. $M(s)$ is row reduced if and only if the high order
coefficient matrix of $M(s)$ has full rank, where the high order coefficient matrix of a polynomial matrix $M(s)$ is a matrix whose entries of the $i$-th row are the coefficients of $s^{\nu_i}$ of the $i$-th row of $M(s)$ where $\nu_i$ is the highest power of $s$ in the $i$-th row of $M(s)$.

**Definition 1.2** ([For75][WR94]) Given an $m \times n$ row reduced polynomial matrix $M(s)$ with $M_h$ the high order coefficient matrix of $M(s)$, the $i$-th pivot index $\mu'_i$ is the largest integer such that the submatrix of $M_h$ formed from the intersection of columns $\mu'_1, \ldots, \mu'_i$ with the rows corresponding to indices $\leq \nu_i$ has rank $i$. The ordered pivot indices $\mu = (\mu_1, \mu_2, \ldots, \mu_m)$ of the equivalence class of $M(s)$ are the indices obtained from $(\mu'_1, \mu'_2, \ldots, \mu'_m)$ by reordering such that $\mu_i < \mu_{i+1}$ if $\nu_i = \nu_{i+1}$.

Let $\tilde{I}(m, p)$ be the set of $m$-tuple of integers defined by

$$
\tilde{I}(m, p) = \{ \alpha = (\alpha_1, \ldots, \alpha_m) | 1 \leq \alpha_1 < \cdots < \alpha_m, \: \alpha_k \neq \alpha_l \bmod n \text{ for } k \neq l \}.
$$

**Definition 1.3** For each equivalence class $M \in \mathcal{A}_{q_{m,p}}$ with Kronecker index $\nu = (\nu_1, \ldots, \nu_m)$ and ordered pivot index $\mu = (\mu_1, \ldots, \mu_m)$ we define a new index $\alpha = (\alpha_1, \ldots, \alpha_m) \in \tilde{I}(m, p)$ by $\alpha_l := n \nu_l + \mu_l$.

Further for each index $\alpha \in \tilde{I}(m, p)$ we define

$$
|\alpha| = \sum_{l=1}^{m} (\alpha_l - l) - \sum_{l=2}^{m} \sum_{k=1}^{l-1} \left\lfloor \frac{\alpha_l - \alpha_k}{n} \right\rfloor
$$

where $[r]$ is the largest integer less than or equal to $r$. We also define a partial order on $\tilde{I}(m, p)$ as follows: first associate with each $\alpha = (\alpha_1, \ldots, \alpha_m)$ an infinite sequence:

$$
f(\alpha) = (f_1(\alpha), f_2(\alpha), \cdots)
$$

where

$$
\{f_l(\alpha)\} = \{\alpha_j + k(n) \mid j = 1, \ldots, m \text{ and } k = 0, 1, 2, \ldots, \}
$$

and order the set such that

$$
f_1(\alpha) < f_2(\alpha) < \cdots,
$$

and then define the partial order on $\tilde{I}(m, p)$ by

$$
\alpha \leq \beta \text{ if and only if } f_l(\alpha) \leq f_l(\beta) \text{ for all } l.
$$

(1.2)

Next we define a topology on $\mathcal{A}_{q_{m,p}}$. Let $\mathcal{P}_{q_{m,p}}$ be the set of all $m \times n$ full rank polynomial matrices of degree at most $q$ and denote with $\mathcal{P}_{q,r_{m,p}}$ the subset of $\mathcal{P}_{q_{m,p}}$ formed by all matrices whose entries are polynomials of degree at most $r$. Then

$$
\mathcal{P}_{q,0_{m,p}} \subset \mathcal{P}_{q,1_{m,p}} \subset \mathcal{P}_{q,2_{m,p}} \subset \cdots
$$
with the union
\[ \mathcal{P}_{m,p}^q = \bigcup_{r=0}^{\infty} \mathcal{P}_{m,p}^{q,r}. \]

The set of all \( m \times n \) polynomial matrices whose entries are polynomials of degree at most \( r \) is an affine space \( \mathcal{Q}^{mn(r+1)} \) and the conditions that the degrees of the \( m \times m \) minors are at most \( q \) are polynomial conditions on \( \mathcal{Q}^{mn(r+1)} \) which defines an algebraic set. \( \mathcal{P}_{m,p}^{q,r} \) is a Zariski open set of this algebraic set. Take the Zariski topology on \( \mathcal{P}_{m,p}^{q,r} \). The direct limit of the topologies on \( \mathcal{P}_{m,p}^{q,r} \) for \( r = 0,1,\ldots \) defines a topology on \( \mathcal{P}_{m,p}^q \). In other words, a subset of \( \mathcal{P}_{m,p}^q \) is open if and only if its intersection with \( \mathcal{P}_{m,p}^{q,r} \) is open as a subset of \( \mathcal{P}_{m,p}^{q,r} \) for each \( r \).

The topology which we take on \( A_{m,p}^q \) is the quotient topology under row equivalence, i.e. a subset \( U \) of \( A_{m,p}^q \) is open if, and only if the subset \( V \) of \( \mathcal{P}_{m,p}^q \) formed by all the polynomial matrices in the equivalence classes of \( U \) is open. Since the minors of an \( m \times n \) matrix are polynomials of its entries, any polynomial condition on \( K_{m,p}^q \) induces a polynomial condition on \( \mathcal{P}_{m,p}^q \). So the map \( \pi \) defined in (0.1) is continuous under the topology we defined on \( A_{m,p}^q \).

Let \( C_\alpha \) be the subset of \( A_{m,p}^q \) consisting all the elements with index \( \alpha \). The main result given in [WR94] can then be summarized in the following proposition:

**Proposition 1.4** [WR94]

1) \( C_\alpha \) is an open cell of dimension \( |\alpha| \).

2) \( C_\alpha \cap C_\beta = \emptyset \) if \( \alpha \neq \beta \).

3) \( \overline{C_\alpha} = \bigcup_{\beta \in I(m,p)} C_\beta \).

**2 Generalized Schubert Subvarieties of** \( K_{m,p}^q \)

We fix the coordinates of \( K_{m,p}^q \) first. For each \( i = (i_1, \ldots, i_m) \), \( 0 \leq i_1 < \cdots < i_m \leq n \), let
\[
z_{(i;0)}t^q + z_{(i;1)}t^{q-1}s + \cdots + z_{(i;q)}s^q
\]
be the \( m \times m \) minor of an \( M(s,t) \in X \) consisting of the \( i_1 \)th through \( i_m \)th columns. Then
\[
z = (z_{(i;d)})_{0 \leq i_1 < \cdots < i_m \leq n, 0 \leq d \leq q}
\]
is the homogeneous coordinate of the image of \( M \) in \( K_{m,p}^q \).

Let \( I(m) \) be the set of \( m \)tuple of integers defined by
\[
I(m) = \{ i = (i_1, \ldots, i_m) \mid 1 \leq i_1 < \cdots < i_m \}.
\]
Define
\[ |i| = \sum_{l=1}^{m} (i_l - l) \]  
and the partial order
\[ (i_1, \ldots, i_m) \leq (j_1, \ldots, j_m) \text{ if and only if } i_l \leq j_l \text{ for all } l \] on \( I(m) \).

Let \( e_l \) be the vector whose \( l \)-th component is 1 and all the other components are zero and \( F_l = \sp{e_1, \ldots, e_l} \).

For any \( i = (i_1, \ldots, i_m) \in I(m) \), \( i \leq (p+1, \ldots, n) \), let \( S_i \) be the Schubert variety \( S(F_{i_1}, \ldots, F_{i_m}) \) under the standard flag \( F_{i_1} \subset \cdots \subset F_{i_m} \); i.e.
\[ S_i = S(F_{i_1}, \ldots, F_{i_m}) = \{ x \in \Grass(m,n) \mid \dim x \cap F_{i_l} \geq l \}. \] (2.3)

Then \( S_i \) is a subvariety of dimension \( |i| \) defined by
\[ S_i = \{ x \in \Grass(m,n) \mid x_j = 0 \text{ for all } j \not\leq i \} \] (2.4)
where \((x_i)\) are the Plücker coordinates of a point \( x \in \Grass(m,n) \).

We would like to generalize the Schubert varieties in \( \Grass(m,n) = K_{m,p}^0 \) to \( K_{m,p}^q \).

For each \( d \leq q \), the subset \( \{ z \in K_{m,p}^q \mid z_{(i;0)} = 0 \text{ for all } l > d \text{ and } i \} \) can be identified naturally with \( K_{m,p}^d \). By abuse of notation we will denote this subset as \( K_{m,p}^d \subseteq K_{m,p}^q \).

Let \( \psi_d \) be the projection of \( z \in K_{m,p}^d \subseteq K_{m,p}^q \) on its components \( (z_{(i;0)}) \); i.e.
\[ \psi_d(z) = \{ z_{(i;0)} \mid 1 \leq i_1 < \cdots < i_m \leq n \}. \] (2.5)

Then \( \psi_d \) is a rational map (it is in fact a central projection) and
\[ \psi_d(K_{m,p}^d) = \Grass(m,n). \]

A first attempt to generalize Schubert varieties could be to take \( \psi_d^{-1}(S_i) \) in \( K_{m,p}^d \subseteq K_{m,p}^q \) for each \( d \leq q \) and for each Schubert variety \( S_i \) in \( \Grass(m,n) \). Unfortunately this is not a closed set. A better definition comes from the cell decomposition of \( A_{m,p}^q \). But we first need the following definition.

**Definition 2.1** For each \((i;d), i = (i_1, \ldots, i_m), 1 \leq i_1 < \cdots < i_m \leq n, \text{ and } d = km + r, 0 \leq r < m, \text{ let } \alpha = (i;d) \text{ where } \alpha = (\alpha_1, \ldots, \alpha_m) \text{ is defined through:} \)
\[ \alpha_l = \begin{cases} k(n) + i_{l+r} & \text{ for } l = 1, 2, \ldots, m-r \\ (k+1)(n) + i_{m-r} & \text{ for } l = m-r+1, \ldots, m. \end{cases} \] (2.6)
Then \( C_\alpha \) with \( \alpha = (\alpha_1, \ldots, \alpha_m) = (i; d) \) is the “thickest” cell among all the cells \( C_\beta \) in \( \pi^{-1}(K^d_{m,p}) \) such that

\[
\psi_d(\pi(C_\beta)) \subset S_i.
\]

It can be proved that \( \pi(C_\alpha) \) is a subvariety (see Lemma 2.3 and Proposition 2.4) and \( \psi_d(\pi(C_\alpha)) = S_i \). So \( \pi(C_\alpha) \) is the variety we want. In order to give a definition similar to (2.4) we re-index the coordinates of \( I_P \supset K_{m,p}^q \), as \( z(i; d) = z_\alpha \).

We set

\[
I(m, p) = \{ \alpha = (\alpha_1, \ldots, \alpha_m)| 1 \leq \alpha_1 < \cdots < \alpha_m, \ \alpha_m - \alpha_1 < n \}.
\]

If \( i \in I(m) \) and \( i \leq (p + 1, \ldots, n) \) then \( \alpha = (i; d) \in I(m, p) \) and for each \( \alpha \in I(m, p) \) there exists unique \( i \in I(m), i \leq (p + 1, \ldots, n) \), and \( d \) such that \( \alpha = (i; d) \). Further, for \( I(m, p) \) the notion of the partial order (1.2) agrees with the partial order (2.3) and the notion of \( |\alpha| \) defined in (1.1) reduces to (2.1). So \( I(m, p) \) is a subset of \( \tilde{I}(m, p) \) as well as \( I(m) \).

**Remark 2.2** In a partially ordered set, an element \( \alpha \) is said to cover another element \( \beta \) if \( \beta < \alpha \) and there exists no \( \gamma \) such that \( \beta < \gamma < \alpha \) [Kri86]. From the definition of the partial order one can see that for any \( \alpha = (i; d) \) and \( \beta = (j; b) \) in \( I(m, p) \), \( \alpha \) covers \( \beta \) if and only if either

a. \( b = d \) and \( i \) covers \( j \) or

b. \( b = d - 1, i = (1, i_2, \ldots, i_m) \) and \( j = (i_2, \ldots, i_m, n) \) for some \( 1 < i_2 < \cdots < i_m < n \).

**Definition 2.3** For any \( \alpha \in I(m, p) \), \( \alpha = (i; d) \leq (p + 1, \ldots n; q) \), let \( Z_\alpha \), or \( Z_i^d \), be the closed subset of \( K_{m,p}^q \) defined by

\[
Z_\alpha = Z_i^d = \{ z \in K_{m,p}^q | z_\beta = 0 \text{ for all } \beta \not\leq \alpha \}
\]

and \( O_\alpha \), or \( O_i^d \), be the open set of \( Z_\alpha \) defined by

\[
O_\alpha = O_i^d = \{ z \in Z_\alpha | z_\alpha \neq 0 \}
\]

**Proposition 2.4**

\[
Z_\alpha = \bigcup_{\beta \in I(m, p), \beta \leq \alpha} O_\beta.
\]

**Proof:** Follows from the definition. \( \square \)

**Lemma 2.5** Let \( \alpha = (i; d) \). Then
1) \( O_\alpha = \bigcup \pi(C_\beta), \) where the union is over all cells in \( A^q_{m,p} \) with Kronecker indices \( \nu = (\nu_1, \ldots, \nu_m), \sum \nu_l = d \) and pivot \( \mu = (\mu_1, \ldots, \mu_m) \) such that \( \{\mu_1, \ldots, \mu_m\} = \{i_1, \ldots, i_m\} \) as unordered sets.

2) \[ Z_\alpha = \bigcup_{\beta \leq \alpha} \pi(C_\beta) = \pi(\overline{C}_\alpha). \] (2.7)

Proof: Using the echelon form of the elements in \( C_\alpha \) (see \cite{WR94}) one can see that \( \pi(C_\alpha) \subset Z_\alpha \). Since \( C_\beta \subset \overline{C}_\alpha \) for all \( \beta \) with Kronecker indices \( \nu, \sum \nu_l = d \) and pivot indices as above and since \( \pi \) is continuous, \( \bigcup \pi(C_\beta) \subset Z_\alpha \). Furthermore, \( z_\alpha \neq 0 \) for all the points in \( \pi(C_\beta) \). So \( \bigcup \pi(C_\beta) \subset O_\alpha \).

On the other hand, for any \( z \in O_\alpha \) let \( M(s) \in \pi^{-1}(z) \) be row reduced. Then by looking at the high order coefficient matrix \( \overline{\text{For75, WR94}} \) of \( M(s) \) one concludes immediately that \( M \) must have Kronecker index \( \nu, \sum \nu_l = d \) and pivot index \( \mu \) such that \( \{\mu_1, \ldots, \mu_m\} = \{i_1, \ldots, i_m\} \).

Therefore \( \bigcup \pi(C_\beta) = O_\alpha \) and (2.7) follows because of Proposition 1.4 and Proposition 2.4. \( \square \)

**Proposition 2.6** \( Z_\alpha \) is an irreducible subvariety of \( K^q_{m,p} \) of dimension \( |\alpha| \).

Proof: Since \( \overline{C}_\alpha \) is irreducible and \( \pi \) is one to one on the open set consisting of all matrices whose maximal minors do not have any common factors, \( Z_\alpha = \pi(\overline{C}_\alpha) \) is irreducible and \( \dim Z_\alpha = \dim \overline{C}_\alpha = |\alpha| \). \( \square \)

**Remark 2.7** From Proposition 2.4 and Remark 2.2 one can see immediately that \( \psi_d(Z^d_i) = S_i \)

where \( S_i \) is the Schubert variety of \( \text{Grass}(m, n) \) defined by (2.4). In the next section we shall show that \( Z_\alpha \) is birationally equivalent to the Schubert variety \( S_\alpha \subset \text{Grass}(m, n(q + 1)) \) (see Proposition 3.1). This is one of the reasons why we use two kinds of indices to label the variety. When \( d = 0 \), \( Z^0_i \) reduces to the Schubert variety \( S_i \).

## 3 Degree of \( Z_\alpha \)

To prove our formula for the degree we shall apply Bézout’s theorem on the projective space \( \mathbb{P} \). By \cite{Ful84} Prop. 8.4, if \( Z \subset \mathbb{P} \) is a variety and \( H \) is a hyperplane such that \( Z \cap H = \bigcup Z_i \)

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where $Z_i$ are irreducible subvarieties with $\dim Z_i = \dim Z - 1$ then the degree of $Z$ is given through the formula $\deg Z = \sum m_i \deg Z_i$ where $m_i$ is the multiplicity of $Z$ and $H$ along $Z_i$. Furthermore, by [Ful84] Remark 8.2, $m_i = 1$ if $Z$ is generically non-singular along $Z_i$ and generically meets $H$ transversally along $Z_i$.

We first construct a rational map from a Schubert variety of the Grassmannian $\text{Grass}(m, n(q + 1))$ into $Z_\alpha$. Consider an $m \times n(q + 1)$ full rank matrix $Q \in \text{Grass}(m, n(q + 1))$.

Let the Plücker coordinate of $Q \in \text{Grass}(m, n(q + 1)) \subset \mathbb{P}$ be $x = (x_i)$ where $i = (i_1, \ldots, i_m)$ and $x_i$ is the $m \times m$ minor of $Q$ consisting of the $i_1$th through $i_m$th columns. Let

$$SC_i = \{x \in S_i | x_i \neq 0\}$$

where $S_i$ is the Schubert variety defined by (2.4) with $n$ replaced by $n(q + 1)$. Then $SC_i$ is a cell and

$$SC_i \cap SC_j = \emptyset, \text{ if } i \neq j \text{ and } \bigcup_{j \in I(m), j \leq i} SC_j.$$

Each $Q \in SC_i$ has a unique echelon form

$$Q = \begin{bmatrix}
  * & \cdots & * & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
  * & \cdots & * & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
  \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\
  * & \cdots & * & 0 & \cdots & 0 & * & \cdots & * & 1 & 0 & \cdots & 0
\end{bmatrix} \quad (3.1)$$

For

$$Q = [Q_0|Q_1|\cdots|Q_q] \in \text{Grass}(m, n(q + 1))$$

where $\{Q_i\}$ are $m \times n$ matrices, and

$$i = (i_1, \ldots, i_m), \ 1 \leq i_1 < \cdots < i_m \leq n,$$

let

$$z_{(i;0)} + z_{(i;1)s} + \cdots + z_{(i;q)s^q}$$

be the $m \times m$ minor formed by the $i_1$th through $i_m$th columns of the polynomial matrix

$$Q(s) = Q_0 + Q_1s + \cdots + Q_qs^q. \quad (3.2)$$

Define a rational map $\tau : \text{Grass}(m, n(q + 1)) \rightarrow \mathbb{P}$ by

$$\tau(Q) = (z_{(i;d)})_{1 \leq i_1 < \cdots < i_m \leq n, \ 0 \leq d \leq q}$$
for all points $Q \in \text{Grass}(m,n(q+1))$ for which the maximum degree of the minors of $Q(s)$ is at most $q$ and at least one minor is non-zero (these points form a locally closed subset). Since each $z_{(i;d)}$ is a linear combination of $m \times m$ minors of $Q$, which in turn are the coordinates of $\tilde{\mathbb{P}}$ there exists a linear subspace $\tilde{E} \subset \tilde{\mathbb{P}}$ such that $\tau$ is the restriction to $\text{Grass}(m,n(q+1))$ of the linear projection $\tau : \tilde{\mathbb{P}} - \tilde{E} \rightarrow \mathbb{P}$. We have

$$\tau(S_\alpha) \subset Z_\alpha.$$  \hfill (3.3)

For a fixed $\alpha = (\alpha_1, \ldots, \alpha_m) \in I(m,p)$, let

$$\tilde{U}_\alpha = \bigcup_{\beta=(\beta_1,\ldots,\beta_m) \in I(m,p), \beta \leq \alpha \text{ and } \beta_1 > \alpha_m - n} SC_{\beta} \subset \text{Grass}(m,n(q+1))$$

and

$$U_\alpha = \bigcup_{\beta=(\beta_1,\ldots,\beta_m) \in I(m,p), \beta \leq \alpha \text{ and } \beta_1 > \alpha_m - n} C_{\beta} \subset A^q_{m,p}.$$  

Then $\tilde{U}_\alpha$ and $U_\alpha$ are open sets of $S_\alpha$ and $\overline{U}_\alpha$, respectively.

Let $\phi$ be defined by

$$\phi(Q) = Q(s)$$

where $Q(s)$ is defined by (3.2). Then the following diagram commutes:

$$\begin{array}{ccc}
\tilde{U}_\alpha & \xrightarrow{\phi} & U_\alpha \\
\tau \searrow & & \swarrow \pi \\
& Z_\alpha & 
\end{array}$$

If $Q$ is in the echelon form of (3.1), then $Q(s) = \phi(Q)$ is in the echelon form defined in [WR94, Proposition 3.5]. Furthermore, if $T$ is the elementary unimodular row operation which add an $s^k$ multiple of the $l$th row of $Q(s)$ to the $r$th row, then $\phi^{-1}(T(Q(s)))$ is in

$$SC_{(\beta_1,\ldots,\beta_{r-1},\beta_r+1,\ldots,\beta_l+kn)},$$

i.e.

$$\phi^{-1}(T(Q(s))) \notin \tilde{U}_\alpha.$$  

Therefore $\phi : \tilde{U}_\alpha \rightarrow U_\alpha$ is one to one and onto.

**Proposition 3.1** $S_\alpha$ and $Z_\alpha$ are birationally equivalent under $\tau$.

**Proof:** Let the open set $U$ be defined by

$$U = \{Q \in \tilde{U}_\alpha | \text{the } m \times m \text{ minors of } \phi(Q) \text{ are relative prime} \}.$$  \hfill (3.4)

Since $\pi = \tau \circ \phi^{-1}$ and $\pi : \phi(U) \rightarrow Z_\alpha$ is one to one [For75], $\tau : U \rightarrow Z_\alpha$ is one to one, which means that $S_\alpha$ is birationally equivalent to $Z_\alpha$ (see [Har77], Chapter I, Corollary 4.5).

The following proposition generalizes the classical Pieri formula and it is one of the main results of this paper.
Proposition 3.2 Let $H_\alpha$ be the hyperplane of $\mathbb{P}$ defined by setting $z_\alpha = 0$. Then

$$Z_\alpha \cap H_\alpha = \bigcup_{\beta \in I(m,p), \beta < \alpha, |\beta| = |\alpha| - 1} Z_\beta$$

and the multiplicity of $Z_\alpha$ and $H_\alpha$ along $Z_\beta$ is one.

Proof: (3.5) follows from Proposition 2.4. So the only thing we need to prove is that the multiplicity of the intersection along each $Z_\beta$ is one. Let

$$\tilde{H}_\alpha = \tau^{-1}(H_\alpha) = \tau^{-1}(H_\alpha) \cup \tilde{E}.$$

Then

$$\tau(\tilde{H}_\alpha) = H_\alpha.$$

$\tilde{H}_\alpha$ is a hyperplane in $\tilde{\mathbb{P}}$ defined by

$$0 = x_\alpha + \text{a linear combination of } x_i's \text{ with } i \not\leq \alpha.$$

So

$$S_\alpha \cap \tilde{H}_\alpha = \{x \in S_\alpha | x_\alpha = 0\} = \bigcup_{i \in I(m), i < \alpha, |i| = |\alpha| - 1} S_i.$$

$\tau$ restricted to the open set $U$ defined by (3.4) is an isomorphism into an open subset of $Z_\alpha$ and $U \cap S_\beta \neq \emptyset$. So $\tau$ is a birational isomorphism between $S_\alpha$ and $Z_\alpha$ and between $S_\beta$ and $Z_\beta$ respectively. Now, by Pieri’s formula [Kle76] applied on the Grassmannian, the intersection multiplicity of $S_\alpha$ and $\tilde{H}_\alpha$ along $S_\beta$ is one. By [Ful84] Remark 8.2 and 3.1, the intersection multiplicity of $Z_\alpha$ and $H_\alpha$ along $Z_\beta$ is also one. $\square$

By Bézout’s Theorem (see discussion following Prop. 8.4 [Ful84]) we have the following:

Lemma 3.3

$$\deg Z_\alpha = \sum_{\beta \in I(m,p), \beta < \alpha, |\beta| = |\alpha| - 1} \deg Z_\beta.$$  

Theorem 3.4 The degree of $Z_\alpha$ is equal to the number of maximal totally ordered subsets of $I_\alpha = \{\beta \in I(m,p) | \beta \leq \alpha\}$.  

Proof: We use induction on the dimension $|\alpha|$ of the variety $Z_\alpha$. When $|\alpha| = 0$, $Z_\alpha = Z_{(1,\ldots,p)}$ is a point and its degree is 1. Assume that the degree of $Z_\beta$ is equal to the number of maximal totally ordered subsets of $I_\beta$ for any $|\beta| = |\alpha| - 1$. Notice
that a set \( I \) is a maximal totally ordered subset of \( I_\beta \) for some \( \beta \) covered by \( \alpha \) if and only if \( I \cup \{ \alpha \} \) is a maximal totally ordered subset of \( I_\alpha \). By Lemma 3.3,

\[
\deg Z_\alpha = \sum_{\substack{\beta \in I(m,p), \\ \beta < \alpha, |\beta| = |\alpha| - 1}} \deg Z_\beta
\]

\[= \sum_{\substack{\beta \in I(m,p), \\ \beta < \alpha, |\beta| = |\alpha| - 1}} \text{# of totally ordered subsets of } I_\beta
\]

\[= \text{# of totally ordered subsets of } I_\alpha.
\]

\[\square\]

**Corollary 3.5** The degree of \( K_{m,p}^q \) is equal to the number of maximal totally ordered subsets of \( I_{(p+1,p+2,...,n;q)} \).

In the next section we shall show that the number of maximal totally ordered subsets is also measured by the formula conjectured in Physics. In order to show this equivalence, we find it useful to give an abstract characterization of a function \( d(\alpha_1, \ldots, \alpha_m) \) that measures the degree of \( Z_\alpha \) as follows:

**Corollary 3.6** The function

\[d(\alpha_1, \ldots, \alpha_m) := \deg Z_\alpha \quad (3.7)\]

is the unique solution of the partial recurrence relation

\[d(\alpha_1, \ldots, \alpha_m) = \sum_{l=1}^{m} d(\alpha_1, \ldots, \alpha_{l-1}, \ldots, \alpha_1, \ldots, \alpha_m) \quad (3.8)\]

subject to the boundary conditions

\[d(\ldots, k, k, \ldots) = 0, \quad (3.9)\]

\[d(k, \ldots, k + n) = 0. \quad (3.10)\]

and subject to the initial conditions

\[d(1, 2, \ldots, m) = 1, \quad (3.11)\]

\[d(0, \ldots, \alpha_m) = 0 \quad \text{for } \alpha_m < n. \quad (3.12)\]

In [RRW93] we use Theorem 3.4 and Remark 2.2 to give explicit formulas for these degrees.
4 The conjecture from Physics

In this section we shall first give a precise formulation of the conjecture. We then interpret the degrees of the subvarieties $Z^d_i$ that we have just computed through this conjecture. Finally, we prove that our formula for their degrees agrees with the conjecture. We shall not address the physics behind the conjecture at all. We refer the reader to [Wit93] [Vaf92] for a discussion of the physical aspects.

The cohomology ring of the Grassmannian is generated by the Chern classes $X_1, \ldots, X_m$ of the canonical subbundle $S$ on Grass($m, n$). The complex codimension of the class $X_i$ is $i$ and the cohomology class $X_i^p$ is Poincaré dual to the class of a point. Let us fix a point $x \in \mathbb{P}^1$. Then there is an evaluation map $\psi : X' \to \text{Grass}(m, n)$ that sends a map to its value at $x$. Now any cohomology class on Grass($m, n$) can be pulled back via $\psi$ to $X$. The conjecture predicts the intersection products of such classes. The first problem with this is that $X'$ is not a compact space, so one has to interpret these products on the compactification. But the introduction of the compactification introduces other questions, namely, do these intersection products depend on the compactification? In other words, does the boundary component added to $X'$ change the intersection product? These questions are somewhat subtle in the case of maps from Riemann surfaces of higher genus to the Grassmanian and have been dealt with in [BDW95]. In our particular case of maps from $\mathbb{P}^1$ to Grass($m, n$), the compactification chosen by us, namely the Quot scheme $X$, is a smooth, irreducible variety of dimension $mp + nq$ for all $m, n$ and $q$. Further, there is a universal bundle $\tilde{S}$ over the Quot scheme $X$ that extends the pullback $\psi^*(S)$ on $X'$ to all of $X$. Thus, as in ([BDW95], Section 5.1), for any set of integers $a_i$ such that $ia_i = mp + nq$ we can define the intersection products of $\psi^*(X_i)$ unambiguously through the following definition:

$$<\psi^*X_1^{a_1} \cdots \psi^*X_m^{a_m}> := <c_1^{a_1} \cdots c_m^{a_m}>$$

(4.1)

where $c_i$ is the $i^{th}$ Chern class of the bundle $\tilde{S}$ on $X$.

Let $q_1, \ldots, q_m$ be the Chern roots of the canonical subbundle $S$ above, so $X_i$ is the $i^{th}$ elementary symmetric function in the $q_j$. Let

$$W = \sum_{i=1}^{m} \left( \frac{q_i^{n+1}}{n+1} + (-1)^m q_i \right).$$

This is called the Landau-Ginzburg potential by physicists. Since $W$ is a symmetric polynomial in the $q_i$, it can be expressed as a polynomial $W = W(X_1, \ldots, X_m)$ in the elementary symmetric functions. Let

$$h(X_1, \ldots, X_m) := \det \left[ \frac{\partial^2 W}{\partial X_i \partial X_j} \right]$$

(4.2)

be the determinant of the Hessian of $W(X_1, \ldots, X_m)$. If $a$ is a multiindex as above then the intersection product $<\psi^*X_1^{a_1} \cdots \psi^*X_m^{a_m}>$ is an integer and the conjecture
says that this number is computed by the formula

$$<\psi^*X_1^{a_1}\cdots\psi^*X_m^{a_m}> = (-1)^{m(m-1)/2} \sum_{dW=0} X_1^{a_1}\cdots X_m^{a_m} \frac{1}{h}.$$  \hspace{1cm} (4.3)

The summation in the above formula is over the finite number of critical points of the function $W(X_1,\ldots,X_m)$.

We will now interpret the degrees that we have computed in terms of these intersection products. Recall that $\kappa$ is the map from $X$ to $\mathbb{P}$ defined by the line bundle $\wedge^m(\tilde{S})$ on $X$ whose Chern class is $c_1(\tilde{S})$. Also $\kappa(\psi^{-1}(S_i)) = Z_i^q$. Thus the degree of the subvariety $Z_i^q \subset \mathbb{P}$ is given by $<\psi^*(s_i) \cdot (\psi^*(X_1))^{\dim Z_i^q}>$, where $s_i(X_1,\ldots,X_m)$ is the Schubert cocycle Poincaré dual to $S_i$. Further, for $d \leq q$, by a similar argument, the degree of $Z_i^d = <\psi^*(s_i) \cdot (\psi^*(X_1))^{\dim Z_i^d}>$.

**Theorem 4.1** The degree of the subvariety $Z_i^d$ as given by Theorem 3.4 also equals:

$$\deg Z_i^d = <\psi^*(s_i) \cdot (\psi^*(X_1))^{\dim Z_i^d}> = (-1)^{m(m-1)/2} \sum_{dW=0} \frac{s_i}{h} X_1^{\dim Z_i^d}$$  \hspace{1cm} (4.4)

where $h$ is the Hessian of the Landau-Ginzburg potential and and $s_i$ is the Schubert cocycle of $S_i \subset \text{Grass}(m,n)$. In particular one has

$$\deg K_{m,p}^q = (-1)^{m(m-1)/2} \sum_{dW=0} X_1^{mp+nq} \frac{1}{h}$$  \hspace{1cm} (4.5)

In order to establish a proof we will first make some simplifications in the formulas and we will reformulate the theorem into an equivalent theorem dealing with properties of the ring of symmetric functions $\mathbb{Z}[q_1,\ldots,q_m]^{|S_m|}$ only.

Some of these simplifications can also be found in [Wit93], page 42. Observe that the Jacobian

$$\det \left[ \frac{\partial X_i}{\partial q_j} \right]$$

for the change of variables from $q_i$ to $X_j$ is given by the Vandermonde determinant $\Delta = \prod_{i<j} (q_j - q_k)$ and on the critical points the transformation of the Hessian is given through

$$\det \left[ \frac{\partial^2 W}{\partial q_i \partial q_j} \right]_{dW=0} = \det \left[ \frac{\partial^2 W}{\partial X_i \partial X_j} \right] \left( \det \left[ \frac{\partial X_i}{\partial q_j} \right] \right)^2.$$

Thus the polynomial $h$ in terms of the Chern roots $q_j$ is given by

$$h(q_1,\ldots,q_m) = \frac{n^m \cdot (q_1 \cdots q_m)^{n-1}}{\Delta^2}. $$
The Schubert cocycle \( s_i \) can be identified with a Schur symmetric function. For this consider the partition

\[ \mu := (p + 1 - i_1, p + 2 - i_2, \ldots, p + m - i_m). \]

Let

\[ |\mu| := \mu_1 + \cdots + \mu_m = mp - |i|. \] (4.6)

It is well known that the Schubert cocycle \( s_i \) can be identified with the Schur symmetric function \( s_\mu \) and \( s_\mu \) has a classical representation due to Jacobi (\( \sim 1835 \)) as a quotient of two alternating functions resulting in a symmetric function:

\[ s_\mu(q_1, \ldots, q_m) = \frac{\det[q_\mu^j + m-j]}{\det[q_m^j]}, \quad i, j = 1, \ldots, m. \] (4.7)

Also, \( \Delta = 0 \) if \( q_i = q_j \) so the summation in (4.4) is over all subsets \( I \) consisting of \( m \) distinct roots of the polynomial \( z^n + (-1)^m \). On \( I \):

\[ (q_1 \cdots q_m)^n = 1, \]

so

\[ h(q_1, \ldots, q_m) = \frac{n^m}{(q_1 \cdots q_m)\Delta^2}. \] (4.8)

Thus to prove Theorem 1.1 it suffices to prove the following equivalent theorem:

**Theorem 4.2** The degree of the subvariety \( Z_i^d \) as given by Theorem 3.4 also equals:

\[ \deg Z_i^d = \frac{(-1)^{m(m-1)/2}}{n^m} \sum I (q_1 \cdots q_m)(q_1 + \cdots + q_m)^{mp-|\mu|+pq} \Delta^2 s_\mu. \] (4.9)

In particular one has

\[ \deg K_{m,p}^q = \frac{(-1)^{m(m+1)/2}}{n^m} \sum I (q_1 \cdots q_m) \left( \prod_{i<j}(q_i - q_j)^2 \right) (q_1 + \cdots + q_m)^{mp+pq}. \]

The proof we will present is purely combinatorial, and is just based on Proposition 3.2. In order to establish the proof we will show that (4.9) satisfies the recurrence relation (3.8), the boundary conditions (3.9) and (3.10) and the initial conditions (3.11), (3.12).

**Proof:** From Jacobi’s identity (4.7) and the fact that \( q_i^n = (-1)^{m+1} \) it is clear that formula (4.9) satisfies both boundary conditions (3.9) and (3.10). Note that if we let

\[ D(\alpha_1, \ldots, \alpha_m) := \det[q_i^\alpha_j], \]

then

\[ \det[q_i^{\mu_j + m-j}] = D(i_1, \ldots, i_m) = D(\alpha_1, \ldots, \alpha_m) \]
where $\alpha = (i; d)$ is defined by (3.6). So the recurrence property (3.8) follows from the Pieri type formula:

$$(q_1 + \ldots + q_m) \cdot D(\alpha_1, \ldots, \alpha_m) = \sum_j D(\alpha_1, \ldots, \alpha_j - 1, \ldots, \alpha_m).$$

In order to complete the proof it is therefore enough to show that the initial conditions (3.11) and (3.12) are satisfied. Equivalently we have to show that

$$(-1)^{m(m-1)/2} \frac{1}{n^m} \sum_I (q_1 \ldots q_m) \Delta^2 s_\mu = \begin{cases} 1 & \text{if } \mu = (p^m) \\ 0 & \text{if } \mu_1 > p \text{ and } |\mu| = mp \end{cases}.$$  

We will treat both cases simultaneously. Note that

$$\Delta = \prod_{j < k} (q_j - q_k) = \det \begin{bmatrix} 1 & \ldots & 1 \\ \vdots & \ddots & \vdots \\ q_m^{m-1} & \ldots & q_1^{m-1} \end{bmatrix}$$

and

$$s_\mu(q_1, \ldots, q_m) = \frac{\det[q_i^{\mu_j + m - j}]}{\Delta}.$$  

So we have

$$(-1)^{m(m-1)/2} \frac{1}{n^m} \sum_I (q_1 \ldots q_m) \Delta^2 s_\nu = \frac{1}{n^m} \sum_I \Delta \det[q_i^{\mu_j + m + 1 - j}]$$

$$= \frac{1}{n^m} \sum_I \det \begin{bmatrix} 1 & \ldots & 1 \\ \vdots & \ddots & \vdots \\ q_i^{m-1} & \ldots & q_m^{m-1} \end{bmatrix} \begin{bmatrix} q_1^{\mu_1 + m} & \ldots & q_1^{\mu_m + 1} \\ \vdots & \ddots & \vdots \\ q_m^{\mu_1 + m} & \ldots & q_m^{\mu_m + 1} \end{bmatrix}$$  

(4.10)

Since the summation over the index set $I$ involves all roots of the polynomial $z^n + (-1)^m$ we can view the right hand side of the last expression as a symmetric polynomial in $Z[y_1, \ldots, y_n]^S_n$, where $\{y_1, \ldots, y_n\}$ represent all roots of $z^n + (-1)^m$. This fact is most conveniently expressed by the Cauchy Binet formula, i.e. the expression in (4.10) is equal to

$$= \frac{1}{n^m} \det \begin{bmatrix} 1 & \ldots & 1 \\ \vdots & \ddots & \vdots \\ y_1^{m-1} & \ldots & y_n^{m-1} \end{bmatrix} \begin{bmatrix} y_1^{\mu_1 + m} & \ldots & y_1^{\mu_m + 1} \\ \vdots & \ddots & \vdots \\ y_n^{\mu_1 + m} & \ldots & y_n^{\mu_m + 1} \end{bmatrix}$$

$$= \frac{1}{n^m} \det \begin{bmatrix} p(\mu_1 + m) & p(\mu_1 + m - 1) & \ldots & p(\mu_1 + 1) \\ p(\mu_1 + m + 1) & p(\mu_2 + m) & \ldots & p(\mu_2 + 1) \\ \vdots & \vdots & \ddots & \vdots \\ p(\mu_1 + 2m - 1) & p(\mu_2 + 2m - 2) & \ldots & p(\mu_2 + m) \end{bmatrix}$$  

(4.11)
where \( p(k) := \sum_i y_i^k \) is the \( k \)th power symmetric function in \( \mathbb{Z}[y_1, \ldots, y_n]^{S_n} \).

Note that
\[
p(k) = \begin{cases} 
  n(-1)^{m-1} & \text{if } k = qn \\
  0 & \text{otherwise}
\end{cases}
\]

Assume now that \( \mu = (p^m) \). Then one verifies that the last expression evaluates to a diagonal matrix with all diagonal entries equal to \( (n)(-1)^{m-1} \). But this just means that (4.10) correctly evaluates to 1. On the other hand if \( \mu \neq (p^m) \) and \( |\mu| = mp \) we have \( \mu_m < p \). But then the last column in the matrix \( [p(\mu_j + m + i - j)] \) is zero what completes the proof.

\[\square\]

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