On the Entanglement Properties of Two-Rebits Systems.

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Abstract

Following the recent work of Caves, Fuchs, and Rungta [Found. of Phys. Lett. \textbf{14} (2001) 199], we discuss some entanglement properties of two-rebits systems. We pay particular attention to the relationship between entanglement and purity. In particular, we determine (i) the probability densities for finding pure and mixed states with a given amount of entanglement, and (ii) the mean entanglement of two-rebits states as a function of the participation ratio.

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I. INTRODUCTION

It has been recently pointed out by Caves, Fuchs, and Rungta [1] that real quantum mechanics (that is, quantum mechanics defined over real vector spaces [2–3]) provides an interesting foil theory whose study may shed some light on which aspects of quantum entanglement are unique to standard quantum theory, and which ones are more generic over other physical theories endowed with the phenomenon of entanglement.

Nowadays there is general consensus on the fact that the phenomenon of entanglement is one of the most fundamental and non-classical features exhibited by quantum systems [6]. Quantum entanglement is the basic resource required to implement several of the most important processes studied by quantum information theory [6–11], such as quantum teleportation [12], and superdense coding [13]. A state of a composite quantum system constituted by subsystems A and B is called “entangled” if it can not be represented as a convex linear combination of product states. In other words, the density matrix $\rho^{AB}$ represents an entangled state if it can not be expressed as

$$\rho^{AB} = \sum_k p_k \rho_k^A \otimes \rho_k^B,$$

(1)

with $0 \leq p_k \leq 1$ and $\sum_k p_k = 1$. On the contrary, states of the form (1) are called separable. The above definition is physically meaningful because entangled states (unlike separable states) cannot be prepared locally by acting on each subsystem individually [14]. The entanglement of formation provides a natural quantitative measure of entanglement with a clear physical motivation [15,16].

In standard quantum mechanics the simplest systems exhibiting the phenomenon of entanglement are two-qubits systems. They play a fundamental role in Quantum Information Theory. It should be stressed that the concomitant space of (mixed) two-qubits states is 15-dimensional and its properties are not trivial. An explicit expression for the entanglement of formation of a two-qubits state $\rho$ has been found by Wootters [16]. Wootters’ celebrated formula has allowed for a systematic survey of the entanglement properties of the space of two-qubits states [17,20].
Within quantum mechanics defined over real vector spaces, the most basic kind of composite systems are two-rebits systems. Rebits are systems whose (pure) states are described by normalized vectors in a two dimensional real vector space. A rebit may be regarded as the simplest possible quantum object [5]. An explicit expression for the entanglement of formation of arbitrary states of two-rebits has been obtained by Caves, Fuchs and Rungta [1].

The aim of the present work is to explore numerically the entanglement properties of two-rebits systems. We pay particular attention to the relationship between entanglement and purity.

The paper is organized as follows. In section II we review the CFR expression for the entanglement of formation of arbitrary two-rebits states, and discuss some of its immediate consequences. The relationships, for two-rebits systems, between the amount of entanglement and the degree of mixture are investigated in sections III. Finally, some conclusions are drawn in section IV.

II. THE CFR FORMULA AND SOME OF ITS CONSEQUENCES

Caves, Fuchs, and Rungta formula for the entanglement of formation of a two-rebits state $\rho$ reads [1]

$$E[\rho] = h \left( \frac{1 + \sqrt{1 - C^2}}{2} \right),$$

(2)

where

$$h(x) = -x \log_2 x - (1 - x) \log_2 (1 - x),$$

(3)

and the concurrence $C$ is given by

$$C[\rho] = |\text{tr}(\tau)| = |\text{tr}(\rho \sigma_y \otimes \sigma_y)|.$$

(4)

The above expression has to be evaluated by recourse to the matrix elements of $\rho$ computed with respect to the product basis, $|i,j\rangle = |i\rangle |j\rangle$, $i,j = 0,1$. 

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We are also going to need a quantitative measure of mixedness. There are several measures of the degree of mixture that can be useful within the present context. The von Neumann measure

\[ S = -\text{Tr} (\rho \ln \rho), \tag{5} \]

is important because of its relationship with the thermodynamic entropy. On the other hand, the so called participation ratio,

\[ R(\rho) = \frac{1}{\text{Tr}(\rho^2)}, \tag{6} \]

is particularly convenient for calculations \cite{17,19}.

A remarkable property of two-rebits states, which transpires immediately from the CFR expressions (2-4), is that their square concurrence (and, consequently, their entanglement of formation) are completely determined by the expectation value of one single observable, namely, \( \sigma_y \otimes \sigma_y \). On the contrary, it has been recently proved that there is no observable (not even for pure states) whose sole expectation value constitutes enough information to determine the entanglement of a two-qubits state \cite{22}. The operator \( \sigma_y \otimes \sigma_y \) has eigenvalues 1 and \(-1\), both two-fold degenerated. Let us denote by \( |\phi_{1,2}\rangle \) the pair of eigenvectors with eigenvalue 1, and \( |\phi_{3,4}\rangle \) the eigenvectors with eigenvalue \(-1\), so that

\[ \sigma_y \otimes \sigma_y = \sum_{i=1}^{2} |\phi_i\rangle \langle \phi_i| - \sum_{i=3}^{4} |\phi_i\rangle \langle \phi_i|. \tag{7} \]

A notable consequence of the CFR expressions (2-4) is that there are mixed states of two rebits with maximum entanglement (that is, with \( C^2 = 1 \)). For instance all states of the form

\[ \rho = p |\phi_1\rangle \langle \phi_1| + (1-p) |\phi_2\rangle \langle \phi_2|, \tag{8} \]

with \( 0 \leq p \leq 1 \), are maximally entangled. Hence, for any participation rate within the range \( 1 \leq R \leq 2 \) there exist two-rebits states with maximum entanglement. We shall return to this point later, when we discuss the distribution of general two-rebits states in the \((R, C^2)\)-plane.
III. ENTANGLEMENT VS. PURITY FOR ARBITRARY TWO-REBITS STATES.

A. Measure on the Two-Rebits State Space

In order to explore numerically the properties of arbitrary two-rebits states, it is necessary to introduce an appropriate measure \( \mu \) on the space \( S_R \) of general two-rebits states. Such a measure is needed to compute volumes within the space \( S_R \), as well as to determine what is to be understood by a uniform distribution of states on \( S_R \). In order to find a natural measure on \( S \) we are going to follow a line of reasoning akin to the one pursued by Zyczkowski et al. in the case of two-qubits states.

An arbitrary (pure and mixed) state \( \rho \) of a (real) quantum system described by an \( N \)-dimensional real Hilbert space can always be expressed as the product of three matrices,

\[
\rho = RD[\{\lambda_i\}]R^T. \tag{9}
\]

Here \( R \) is an \( N \times N \) orthogonal matrix and \( D[\{\lambda_i\}] \) is an \( N \times N \) diagonal matrix whose diagonal elements are \( \{\lambda_1, \ldots, \lambda_N\} \), with \( 1 \geq \lambda_i \geq 0 \), and \( \sum_i \lambda_i = 1 \). The group of orthogonal matrices \( O(N) \) is endowed with a unique, uniform measure \( \nu \) [21]. On the other hand, the simplex \( \Delta \), consisting of all the real \( N \)-uples \( \{\lambda_1, \ldots, \lambda_N\} \) appearing in (9), is a subset of a \((N-1)\)-dimensional hyperplane of \( \mathbb{R}^N \). Consequently, the standard normalized Lebesgue measure \( \mathcal{L}_{N-1} \) on \( \mathbb{R}^{N-1} \) provides a natural measure for \( \Delta \). The aforementioned measures on \( O(N) \) and \( \Delta \) lead then to a natural measure \( \mu \) on the set \( S_R \) of all the states of our (real) quantum system, namely,

\[
\mu = \nu \times \mathcal{L}_{N-1}. \tag{10}
\]

We are going to consider the set of states of a two-rebits system. Consequently, our system will have \( N = 4 \). All our present considerations are based on the assumption that the uniform distribution of states of a two-rebit system is the one determined by the measure (10). Thus, in our numerical computations we are going to randomly generate states of a two-qubits system according to the measure (10).
B. Entanglement and Degree of Mixture.

The relationship between the amount of entanglement and the purity of quantum states of composite systems has been recently discussed in the literature [17–20]. The amount of entanglement and the purity of quantum states of composite systems exhibit a dualistic relationship. As the degree of mixture increases, quantum states tend to have a smaller amount of entanglement. In the case of two-qubits systems, states with a large enough degree of mixture are always separable [17]. To study the relationship between entanglement and mixture we need quantitative measures for these two quantities. As already mentioned, the entanglement of formation provides a natural quantitative measure of entanglement with a clear physical motivation [15,16].

The continuous line in Fig. 1 depicts the behavior of the mean entanglement of formation $\langle E \rangle$ of real density matrices, as given by the CFR formula, as a function of the participation ratio $R$. The dashed line in Fig. 1 corresponds to the mean entanglement of the same matrices, as given by Wootters’ formula. In other words, in Fig. 1 the continuous line describes the mean entanglement of formation of the real density matrices when regarded as defined on a real vector space, while the dashed line describes the entanglement of formation of these same matrices when they are considered in the context of the standard complex vector space. We see that the CFR formula always gives, for the mean entanglement of formation, a value larger than the one obtained by recourse of the Wootters’ expression. In this respect, our numerical results are fully consistent with the general arguments provided in [1].

The continuous line in Fig. 2 illustrates the behavior of the mean entanglement of formation $\langle E \rangle$ of real density matrices (given by the CFR expression) as a function of the participation ratio $R$. The dashed line in Fig. 2 shows the behavior of the mean entanglement of formation $\langle E \rangle$ of complex density matrices (given by Wootters’ formula) as a function of the participation ratio $R$.

The largest eigenvalue $\lambda_m$ of the density matrix constitutes a legitimate measure of
mixture, in the sense that states with larger values of $\lambda_m$ can be regarded as less mixed. Its extreme values correspond to (i) pure states (with $\lambda_m = 1$) and (ii) the density matrix $\frac{1}{4} I$ (with $\lambda_m = 1/4$). In Fig. 3 we depict the mean entanglement $\langle E \rangle$ of all the two-rebits states with a given value of their maximum eigenvalue $\lambda_m$, as a function of this last quantity. The upper line corresponds to the CFR expression and the lower line to Wootters formula. Notice that, in the case of Wootters formula, the mean entanglement vanishes for $\lambda_m \leq 1/3$.

We have also computed numerically the probability $P(E)$ of finding a two-rebits state endowed with an amount of entanglement $E$. In Fig. 4 we compare (i) the distributions associated with two-rebits states with (ii) the distributions associated with two-qubits states which were recently obtained by Zyczkowski et al. [17,23]. Fig. 4a depicts the probability $P(E)$ of finding two-qubits states endowed with a given entanglement $E$ (as computed with Wootters’ expression). The solid line correspond to arbitrary states and the dashed line to pure states. In a similar way, Fig. 4b exhibits a plot of the probability $P(E)$ of finding two-rebits states endowed with a given entanglement $E$ (as computed with the CFR formula). The solid line correspond to arbitrary states and the dashed line to pure states. Comparing Figs 4a and 4b we find that the distributions $P(E)$ describing arbitrary states (that is, both pure and mixed states) exhibit the same qualitative shape for both two-qubits and two-rebits states: in the two cases the distribution $P(E)$ is a decreasing function of $E$. On the contrary, the distribution $P(E)$ corresponding to pure two-rebits states differs considerably from the one associated with pure two-qubits states. The probability distribution $P(E)$ for pure states of two-rebits reaches its maximum value for separable states ($E = 0$), and it is a monotonous decreasing function of the entanglement of formation $E$. On the contrary, the distribution corresponding to pure states of two-qubits is an increasing function of $E$ for low values of the entanglement, and decreases with $E$ for large enough values of this variable. It adopts its maximum value for an intermediate value of $E$. The general conclusion that we may draw from Fig. 4 is that the two curves representing the distributions $P(E)$ associated with (i) pure states and (ii) arbitrary states do not differ, in the case of two-rebits states, as much as they do in the case of two-qubits states.
The distribution \( P(E) \) for pure two-rebits states can be obtained analytically. Let us write a pure two-rebits state in the form
\[
|\Psi\rangle = \sum_{i=1}^{4} c_i |\phi_i\rangle,
\]
where
\[
\sum_{i=1}^{4} c_i^2 = 1, \quad c_i \in \mathbb{R}.
\]
The states \(|\phi_i\rangle, \quad i = 1, \ldots, 4\) are the eigenstates of the operator \( \sigma_y \otimes \sigma_y \), in the same order as in equation (7). The four real numbers \( c_i \) constitute the coordinates of a point lying in the three dimensional unitary hyper-sphere \( S_3 \) (which is embedded in \( \mathbb{R}^4 \)). We now introduce on \( S_3 \) three angular coordinates, \( \phi_1, \phi_2, \) and \( \theta \), defined by
\[
\begin{align*}
c_1 &= \cos \theta \cos \phi_1, \\
c_2 &= \cos \theta \sin \phi_1, \\
c_3 &= \sin \theta \cos \phi_2, \\
c_4 &= \sin \theta \sin \phi_2, \quad 0 \leq \theta < \frac{\pi}{2}, \quad 0 \leq \phi_1, \phi_2 < 2\pi.
\end{align*}
\]
In terms of the above angular coordinates, the concurrence of the pure state \(|\Psi\rangle\) is given by
\[
C = |\langle \sigma_y \otimes \sigma_y \rangle| = |\cos 2\theta|.
\]
The element of volume on the three dimensional hyper-sphere is \( \sin \theta \cos \theta \, d\theta \, d\phi_1 \, d\phi_2 \). Thus, the total volume associated with a small interval \( d\theta \) is
\[
dv = 4\pi^2 \sin \theta \cos \theta \, d\theta.
\]
Inspection of equations (14) and (13) allows one to deduce that the probability density \( P(C) \) of finding a pure two-rebits state with concurrence \( C \) is
\[
P(C) = \frac{1}{\pi^2} \left| \frac{dv}{d\theta} \right| \left| \frac{d\theta}{dC} \right| = 1.
\]
The concomitant probability density \( P(E) \) of finding a pure state with entanglement of formation \( E \) is then equal to

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where \( dE/dC \) is to be computed from expressions (2) and (3). It can be verified from equation (17) that the limit value of \( P(E) \) associated with states of maximum entanglement is \( P(E = 1) = \ln 2 \). This value corresponds to the horizontal line in Figure 4b.

**C. Maximum Entanglement Compatible with a Given Degree of Mixture.**

We are now going to determine which is the maximum entanglement \( E_m \) that a two-rebits state with a given participation radio \( R \) may have. Since \( E \) is a monotonic increasing function of the concurrence \( C \), we shall find the maximum value of \( C \) compatible with a given value of \( R \). In order to solve the concomitant variational problem (and bearing in mind that \( C = |\langle \sigma_y \otimes \sigma_y \rangle| \) ), let us first find the state that extremizes \( \text{Tr}(\rho^2) \) under the constraints associated with a given value of \( \langle \sigma_y \otimes \sigma_y \rangle \), and the normalization of \( \rho \). This variational problem can be cast as

\[
\delta \left[ \text{Tr}(\rho^2) + \beta \langle \sigma_y \otimes \sigma_y \rangle - \alpha \text{Tr}(\rho) \right] = 0, \tag{18}
\]

where \( \alpha \) and \( \beta \) are appropriate Lagrange multipliers. The solution of the above variational equation is given by the density matrix

\[
\rho_m = \frac{1}{2} \left[ \alpha I - \beta(\sigma_y \otimes \sigma_y) \right]. \tag{19}
\]

The value of the Lagrange multiplier \( \alpha \) is immediately determined by the normalization requirement,

\[
\text{Tr}(\rho) = 1 \implies \alpha = \frac{1}{2}. \tag{20}
\]

Consequently, those two-rebits states yielding the extremum values of \( Tr(\rho^2) \) (and also the extremum values of \( R = 1/Tr(\rho^2) \)) compatible both with normalization and a given value of \( \langle \sigma_y \otimes \sigma_y \rangle \) are described by the density matrix.
\[ \rho_m = \frac{1}{4} I - \frac{1}{2} \beta (\sigma_y \otimes \sigma_y), \] (21)

with the Lagrange multiplier \( \beta \) lying in the interval

\[ \beta \in \left[ -\frac{1}{2}, \frac{1}{2} \right]. \] (22)

Density matrices of the form (21), corresponding to negative values of \( \beta \), have been considered in [1], although not in connection with the variational problem that we are discussing here.

In terms of the parameter \( \beta \), the expectation value \( \langle \sigma_y \otimes \sigma_y \rangle \), the concurrence squared \( C^2 \), and the participation ratio of the statistical operator \( \rho_m \) are given by

\[ \langle \sigma_y \otimes \sigma_y \rangle = -2\beta, \]
\[ C^2 = \langle \sigma_y \otimes \sigma_y \rangle^2 = 4\beta^2, \]
\[ R = \frac{4}{1 + 4\beta^2}. \] (23)

Hence, the maximum value of \( R \) compatible with a given value of \( C^2 \) is given by

\[ R_m(C^2) = \frac{4}{1 + C^2}. \] (24)

\( R_m(C^2) \) is a monotonic decreasing function of \( C^2 \) and adopts its values in the interval \( 2 \leq R \leq 4 \). This implies that, within this range of \( R \)-values, the maximum value of \( C^2 \) compatible with a given value of \( R \) is the one obtained when solving Eq. (24) for \( C^2 \), namely,

\[ C^2 = \frac{4}{R} - 1. \] (25)

On the other hand, for \( 1 \leq R \leq 2 \) there always exist density matrices of maximum entanglement (that is, with \( C^2 = 1 \)). As a consequence, the maximum value of \( C^2 \) compatible with a given value of \( R \) is given by

\[ C^2_m = \begin{cases} 
1 ; & 1 \leq R \leq 2 \\
\frac{4}{R} - 1 ; & 2 \leq R \leq 4 
\end{cases} \] (26)

In Fig. 5 we plot (in the \((R, C^2)\)-plane) one million numerically generated random two-rebits states. The solid line corresponds to the maximum concurrence squared \( C^2_m \), for a given value of the participation radio \( R \), as given by Eq. (26).
IV. CONCLUSIONS

We have explored numerically the entanglement properties of two-rebits systems. We paid particular attention to the relationship between entanglement and purity. We have computed numerically the mean entanglement of formation of two-rebits systems (as determined by the CFR formula (2)) as a function of the participation ratio $R$. We have also determined numerically the probability densities $P(E)$ of finding (i) pure two-rebits states and (ii) arbitrary two-rebits states, endowed with a given amount of entanglement $E$. Furthermore, we surveyed the distribution of general two-rebits states in the $(R, C^2)$-plane. In particular, we determined analytically the maximum possible value of the concurrence squared $C^2$ of two-rebits states compatible with a given value of the participation ratio. An interesting feature that deserves special mention is that, with regards to the probability of finding states with a given amount of entanglement, the difference between mixed states and pure states is much larger for qubits than for rebits.

It would be interesting to perform, for quaternionic quantum mechanics [24], an analysis similar as the one done here.

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FIGURE CAPTIONS

Fig. 1- Mean entanglement of formation $\langle E \rangle$ of real density matrices as a function of the participation ratio $R$. The continuous line corresponds to the CFR formula. The dashed line corresponds to the mean entanglement of the same matrices, as given by Wootters’ formula.

Fig. 2- The continuous line shows the behavior of the mean entanglement of formation $\langle E \rangle$ of real density matrices (given by the CFR expression) as a function of the participation ratio $R$. The dashed line shows the behavior of the mean entanglement of formation $\langle E \rangle$ of complex density matrices (given by Wootters formula) as a function of the participation ratio $R$.

Fig. 3- Mean entanglement $\langle E \rangle$ of all the two-rebits states with a given value of their maximum eigenvalue $\lambda_m$, as a function of this last quantity. The upper line corresponds to the CFR expression and the lower line to Wootters formula. Notice that, in the case of Wootters formula, the mean entanglement vanishes for $\lambda_m \leq 1/3$.

Fig. 4a- Plot of the probability $P(E)$ of finding two-qubits states endowed with a given entanglement $E$. The solid line correspond to arbitrary states and the dashed line to pure states.

Fig. 4b- Plot of the probability $P(E)$ of finding two-rebits states endowed with a given entanglement $E$. The solid line correspond to arbitrary states and the dashed line to pure states. The horizontal line corresponds to the limit value $P(E = 1) = \ln 2$ of the probability density associated with pure two-rebits states.

Fig. 5- Plot of in the $(R, E)$-plane of one million random numerically generated two-rebits states. The solid line corresponds to the maximum entanglement $E_m$, for a given value of the participation radio $R$. 
\[ R = \frac{1}{\text{Tr}(\rho^2)} \]
fig. 2

\[ \langle E \rangle = \frac{1}{\text{Tr}(\rho^2)} \]
fig. 3
