Some new inequalities for convex functions via Riemann-Liouville fractional integrals

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Abstract
Fractional analysis has evolved considerably over the last decades and has become popular in many technical and scientific fields. Many integral operators which allow us to integrate from fractional orders have been generated. Each of them provides different properties such as the semigroup property, singularity problems etc. In this paper, firstly, we obtained a new kernel, then some new integral inequalities which are valid for integrals of fractional orders by using Riemann-Liouville fractional integral. To do this, we used some well-known inequalities such as Hölder’s inequality or power mean inequality. Our results generalize some inequalities exist in the literature.

Keywords: Riemann-Liouville, Fractional calculus, Convexity

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1 Introduction

It is a well known fact that inequalities have important role in the studies of inequality theory, linear programming, extremum problems, optimization, error estimates and game theory. Over the years, only integer real order integrals were taken into account while handling new results about integral inequalities. However, in the recent years fractional integral operator have been considered by many scientists (see \([1]\)–\([12]\)) and the references therein. There are some inequalities in the literature that accelerates studies on integral inequalities. In the following part, the Hermite-Hadamard inequality which is one of the most famous and practical inequality in the literature is given:

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Theorem 1. Let \( f \) be defined from interval \( I \) (a nonempty subset of \( \mathbb{R} \)) to \( \mathbb{R} \) be a convex function on \( I \) and \( m, n \in I \) with \( m < n \). Then the double inequality given in the following holds:
\[
f \left( \frac{m+n}{2} \right) \leq \frac{1}{n-m} \int_m^n f(x) \, dx \leq \frac{f(m) + f(n)}{2}.
\]

Now we will mention about Riemann-Liouville fractional integration operator (see [6]) which able to integrate functions on fractional orders.

Definition 1. Let \( f \in L_1[m,n] \). \( J_m^\alpha f \) and \( J_n^\alpha f \) which are called left-sided and right-sided Riemann-Liouville integrals of order \( \alpha > 0 \) with \( 0 \leq m \leq x \leq n \) are defined by
\[
J_m^\alpha f = \frac{1}{\Gamma(\alpha)} \int_m^x (x-t)^{\alpha-1} f(t) \, dt, \quad x > m
\]
and
\[
J_n^\alpha f = \frac{1}{\Gamma(\alpha)} \int_x^n (t-x)^{\alpha-1} f(t) \, dt, \quad x < n
\]
respectively where \( \Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} \, dt \). Here \( J_m^\alpha f(x) = J_n^\alpha f(x) = f(x) \).

The results have been put forward inspiring from the following kernel obtained in [9].

Lemma 2. Let \( f : I \subseteq \mathbb{R} \rightarrow \mathbb{R} \) be a differentiable mapping on \( I^0 \) and \( m, n \in I \) with \( m < n \). If \( f' \in L_1([m,n]) \), \( \lambda, \mu \in \mathbb{R} \), and \( \xi \in [0,1] \), then
\[
\frac{\lambda f(m) + \mu f(n)}{2} + \frac{(2-\lambda-\mu)}{2} f(\xi m + (1-\xi)n) - \frac{1}{n-m} \int_m^n f(x) \, dx
\]
\[
= \frac{n-m}{2} \left[ (1-\xi) \int_0^1 (2(1-\xi)t-\lambda) f'(t(\xi m + (1-\xi)n) + (1-t)m) \, dt \\
+ \xi \int_0^1 (\mu-2\xi t) f'(t(\xi m + (1-\xi)n) + (1-t)n) \, dt \right].
\]

Now firstly, we will give a new lemma including Riemann-Liouville fractional integral operator, then we will obtain new inequalities for convex functions.

2 Results Via Riemann-Liouville Fractional Integrals

Lemma 3. Let \( f : I \subseteq \mathbb{R} \rightarrow \mathbb{R} \) be a differentiable mapping on \( I^0 \) and \( m, n \in I \) with \( m < n \). If \( f' \in L_1([m,n]) \), \( \lambda, \mu \in \mathbb{R} \), \( \alpha > 0 \) and \( \xi \in [0,1] \) then
\[
\frac{\lambda f(m) + \mu f(n)}{2} + \frac{(2-\lambda-\mu)}{2} f(\xi m + (1-\xi)n) - \frac{1}{n-m} \int_m^n f(x) \, dx
\]
\[
= \frac{n-m}{2} \left[ (1-\xi) \int_0^1 (2(1-\xi)^{\alpha} t^\alpha - \lambda) f'(t(\xi m + (1-\xi)n) + (1-t)m) \, dt \\
+ \xi \int_0^1 (\mu-2\xi t^\alpha) f'(t(\xi m + (1-\xi)n) + (1-t)n) \, dt \right]
\]
where \( \Gamma(.) \) is the gamma function.
Proof. Integrating by part and changing variables of integration $x = t(\xi m + (1 - \xi) n) + (1 - t) m$ yield

$$
\int_0^1 (2(1 - \xi) t^\alpha - \lambda) f'(t(\xi m + (1 - \xi) n) + (1-t)m) dt
= (2(1 - \xi) t^\alpha - \lambda) \frac{f(t(\xi m + (1 - \xi) n) + (1-t)m)}{(1 - \xi)(n - m)} \bigg|_0^1
- \frac{2\alpha}{(n - m)} \int_0^1 t^{\alpha - 1} f(t(\xi m + (1 - \xi) n) + (1-t)m) dt
= \frac{[2(1 - \xi) - \lambda] f(\xi m + (1 - \xi) n) + \lambda f(m)}{(1 - \xi)(n - m)}
- \frac{2\alpha}{(n - m)} \int_0^1 t^{\alpha - 1} f(t(\xi m + (1 - \xi) n) + (1-t)m) dt
= \frac{[2(1 - \xi) - \lambda] f(\xi m + (1 - \xi) n) + \lambda f(m)}{(1 - \xi)(n - m)}
- \frac{2\alpha}{(n - m)} \int_m^{\xi + (1 - \xi) n} (x - m)^{\alpha - 1} f(x) dx
= \frac{[2(1 - \xi) - \lambda] f(\xi m + (1 - \xi) n) + \lambda f(m)}{(1 - \xi)(n - m)}
- \frac{2\Gamma(\alpha + 1)}{(1 - \xi)^\alpha(n - m)^{\alpha + 1}} \int_{[\xi m + (1 - \xi) n]^1}^f f(m). (4)
$$

On the other hand, with similar way and changing variables of integration $x = t(\xi m + (1 - \xi) n) + (1 - t) n$ yield

$$
\int_0^1 (\mu - 2\xi t^\alpha) f'(t(\xi m + (1 - \xi) n) + (1-t)n) dt
= (\mu - 2\xi t^\alpha) \frac{f(t(\xi m + (1 - \xi) n) + (1-t)n)}{(\xi (m - n)} \bigg|_0^1
- \frac{2\alpha}{(n - m)} \int_0^1 t^{\alpha - 1} f(t(\xi m + (1 - \xi) n) + (1-t)n) dt
= \frac{(2\xi - \mu) f(\xi m + (1 - \xi) n) + \mu f(n)}{\xi(n - m)}
- \frac{2\alpha}{(n - m)} \int_0^1 t^{\alpha - 1} f(t(\xi m + (1 - \xi) n) + (1-t)n) dt
= \frac{(2\xi - \mu) f(\xi m + (1 - \xi) n) + \mu f(n)}{\xi(n - m)}
- \frac{2\alpha}{\xi^\alpha(n - m)^{\alpha + 1}} \int_{\xi m + (1 - \xi) n}^n (n - x)^{\alpha - 1} f(x) dx
= \frac{(2\xi - \mu) f(\xi m + (1 - \xi) n) + \mu f(n)}{\xi(n - m)}
- \frac{2\Gamma(\alpha + 1)}{\xi^\alpha(n - m)^{\alpha + 1}} \int_{[\xi m + (1 - \xi) n]}^f f(n). (5)
$$

By multiplying 4 with $(1 - \xi)$, 5 with $\xi$, summing them side by side and multiplying the equality with $\frac{\alpha - m}{\alpha}$ we get the desired result.

Remark 1. If we choose $\alpha = 1$ in Lemma 3, we gel Lemma 2.
Theorem 4. Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $I$ and $m, n \in I$ with $m < n$. If $|f'|$ is convex on $I$, $f' \in L_1([m, n])$, $\lambda, \mu \in \mathbb{R}^+$, $\alpha > 0$ and $\xi \in [0, 1]$ then
\[
\|\begin{aligned}
&\left\| \frac{\lambda f(m) + \mu f(n)}{2} + \frac{(2 - \lambda - \mu) f(\xi m + (1 - \xi) n)}{2} \right\| \\
&\Gamma \left( \frac{\alpha + 1}{n-m} \right) \left| \left( 1 - \xi \right)^{1-a} j_a^{\alpha} \left[ \xi m + (1 - \xi) n \right] f(m) + \xi^{1-a} j_a^{\alpha} \left[ \xi m + (1 - \xi) n \right] f(n) \right| \\
&\leq \frac{n - m}{2} \left| \left( 1 - \xi \right)^{1-a} j_a^{\alpha} \left[ \xi m + (1 - \xi) n \right] f(m) + \xi^{1-a} j_a^{\alpha} \left[ \xi m + (1 - \xi) n \right] f(n) \right|
\end{aligned}\]

where $\Gamma(\cdot)$ is the gamma function and
\[
\rho_1 = \frac{\lambda^{\frac{2-a}{\alpha}}}{2^{\frac{2-a}{\alpha}} (1 - \xi)^{\frac{1}{\alpha}}} - \frac{\lambda^{\frac{2-a}{\alpha}}}{2^{\frac{2-a}{\alpha}} (1 - \xi)^{\frac{1}{\alpha}} (2 + \alpha)} + \frac{2 (1 - \xi)}{2 + \alpha} - \frac{\lambda}{2},
\]
\[
\rho_2 = \frac{\lambda - 2 (1 - \xi)}{2 + \alpha},
\]
\[
\Psi_1 = \frac{\lambda^{\frac{1-a}{\alpha}}}{2^{\frac{1-a}{\alpha}} (1 - \xi)^{\frac{1}{\alpha}}} - \frac{\lambda^{\frac{1-a}{\alpha}}}{2^{\frac{1-a}{\alpha}} (1 + \alpha) (1 - \xi)^{\frac{1}{\alpha}}} + \frac{2 (1 - \xi)}{1 + \alpha} - \lambda - \rho_1,
\]
\[
\Psi_2 = \frac{\lambda - 2 (1 - \xi)}{1 + \alpha} + \frac{2 (1 - \xi)}{2 + \alpha},
\]
\[
\gamma_1 = \frac{\mu^{\frac{2-a}{\alpha}}}{2^{\frac{2-a}{\alpha}} \xi^{\frac{1}{\alpha}}} - \frac{\mu^{\frac{2-a}{\alpha}}}{2^{\frac{2-a}{\alpha}} (2 + \alpha) \xi^{\frac{1}{\alpha}}} - \frac{\mu}{2 + \alpha} - \frac{\mu}{2},
\]
\[
\gamma_2 = \frac{\mu - \frac{2 \xi}{2 + \alpha}}{2 + \alpha},
\]
\[
\delta_1 = \frac{\mu^{\frac{1-a}{\alpha}}}{2^{\frac{1-a}{\alpha}} \xi^{\frac{1}{\alpha}}} - \frac{\mu^{\frac{1-a}{\alpha}}}{2^{\frac{1-a}{\alpha}} (1 + \alpha) \xi^{\frac{1}{\alpha}}} + \frac{2 \xi}{1 + \alpha} - \mu - \gamma_1,
\]
\[
\delta_2 = \frac{\mu - \frac{2 \xi}{1 + \alpha}}{2 + \alpha} + \frac{2 \xi}{2 + \alpha}.
\]

Proof. By using Lemma 3 and using properties of absolute value we have
\[
\left| \frac{\lambda f(m) + \mu f(n)}{2} + \frac{(2 - \lambda - \mu) f(\xi m + (1 - \xi) n)}{2} \right| - \Gamma \left( \frac{\alpha + 1}{n-m} \right) \left| \left( 1 - \xi \right)^{1-a} j_a^{\alpha} \left[ \xi m + (1 - \xi) n \right] f(m) + \xi^{1-a} j_a^{\alpha} \left[ \xi m + (1 - \xi) n \right] f(n) \right|
\]
\[
\leq \frac{n - m}{2} \left| \left( 1 - \xi \right)^{1-a} j_a^{\alpha} \left[ \xi m + (1 - \xi) n \right] f(m) + \xi^{1-a} j_a^{\alpha} \left[ \xi m + (1 - \xi) n \right] f(n) \right|
\]
Using convexity of $|f'|$ we have
\[
\frac{1}{2} \left| \frac{\lambda f(m) + \mu f(n)}{2} + \frac{(2 - \lambda - \mu) f(\xi m + (1 - \xi) n)}{2} \right| \\
- \frac{\Gamma(\alpha + 1)}{(n-m)^\alpha} \left[ (1 - \xi)^{1-\alpha} f^\alpha_{\xi m+(1-\xi)n} f(m) + \xi^{1-\alpha} f^\alpha_{\xi m+(1-\xi)n} f(n) \right] \\
\leq \frac{n-m}{2} \left[ (1 - \xi) \int_0^1 |2 (1 - \xi) t^\alpha - \lambda| \left[ t |f'|(\xi m + (1 - \xi) n)| + (1 - t) |f'| (m)| \right] dt \\
+ \xi \int_0^1 |\mu - 2 \xi t^\alpha| \left[ t |f'|(\xi m + (1 - \xi) n)| + (1 - t) |f'| (n)| \right] dt \right].
\]

With simple calculations it can be seen
\[
\int_0^1 t |2 (1 - \xi) t^\alpha - \lambda| dt = \begin{cases} \rho_1, & \lambda < 2 (1 - \xi) \\ \rho_2, & \lambda \geq 2 (1 - \xi) \end{cases} \\
\int_0^1 (1 - t) |2 (1 - \xi) t^\alpha - \lambda| dt = \begin{cases} \Psi_1, & \lambda < 2 (1 - \xi) \\ \Psi_2, & \lambda \geq 2 (1 - \xi) \end{cases} \\
\int_0^1 t |\mu - 2 \xi t^\alpha| dt = \begin{cases} \gamma_1, & \mu < 2 \xi \\ \gamma_2, & \mu \geq 2 \xi \end{cases} \\
\int_0^1 (1 - t) |\mu - 2 \xi t^\alpha| dt = \begin{cases} \delta_1, & \mu < 2 \xi \\ \delta_2, & \mu \geq 2 \xi \end{cases}.
\]

By using necessary coefficients in (6), the proof is completed.

**Theorem 5.** Let $f : \mathbb{I} \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on $I_0$ and $m, n \in I$ with $m < n$. If $|f'|^q$ is convex on $I$, $f' \in L_1 ([m,n]), \lambda, \mu \in \mathbb{R}^+, \alpha > 0, \xi \in [0,1]$ then
\[
\left| \frac{\lambda f(m) + \mu f(n)}{2} + \frac{(2 - \lambda - \mu) f(\xi m + (1 - \xi) n)}{2} \right| \\
- \frac{\Gamma(\alpha + 1)}{(n-m)^\alpha} \left[ (1 - \xi)^{1-\alpha} f^\alpha_{\xi m+(1-\xi)n} f(m) + \xi^{1-\alpha} f^\alpha_{\xi m+(1-\xi)n} f(n) \right] \\
\leq \frac{n-m}{2} \begin{cases} (1 - \xi) (\xi_1)^{1-\alpha} |\rho_1 |f'|(\xi m + (1 - \xi) n)|^q |\Psi_1 |f'| (m)|^{q\frac{1}{q}} \\
+ \xi (\eta_1)^{1-\alpha} |\gamma_1 |f'|(\xi m + (1 - \xi) n)|^q |\delta_1 |f'| (n)|^{q\frac{1}{q}}, \\
\text{for } \lambda < 2 (1 - \xi) \text{ and } \mu < 2 \xi; \\
(1 - \xi) (\xi_2)^{1-\alpha} |\rho_2 |f'|(\xi m + (1 - \xi) n)|^q |\Psi_2 |f'| (m)|^{q\frac{1}{q}} \\
+ \xi (\eta_2)^{1-\alpha} |\gamma_2 |f'|(\xi m + (1 - \xi) n)|^q |\delta_2 |f'| (n)|^{q\frac{1}{q}}, \\
\text{for } \lambda \geq 2 (1 - \xi) \text{ and } \mu < 2 \xi; \\
(1 - \xi) (\xi_3)^{1-\alpha} |\rho_2 |f'|(\xi m + (1 - \xi) n)|^q |\Psi_2 |f'| (m)|^{q\frac{1}{q}} \\
+ \xi (\eta_3)^{1-\alpha} |\gamma_2 |f'|(\xi m + (1 - \xi) n)|^q |\delta_2 |f'| (n)|^{q\frac{1}{q}}, \\
\text{for } \lambda \geq 2 (1 - \xi) \text{ and } \mu \geq 2 \xi \end{cases}.
\]
where \( q \geq 1 \), \( \Gamma ( \cdot ) \) is the gamma function and

\[
\zeta_1 = 2\lambda \sqrt{\frac{\lambda}{2(1-\xi)}} - \frac{4(1-\xi)}{(\alpha + 1)} \sqrt{\frac{\lambda}{2(1-\xi)}} \left( \frac{\lambda}{\alpha + 1} + 2 \right) - \frac{(1-\xi)^{\alpha+1}}{(\alpha + 1)}
\]

\[
\zeta_2 = \lambda - \frac{2(1-\xi)}{(\alpha + 1)}
\]

\[
\eta_1 = 2\mu \sqrt{\frac{\mu}{2\xi}} - \frac{4\xi}{(\alpha + 1)} \sqrt{\frac{\mu}{2\xi}} \left( \frac{\mu}{\alpha + 1} + \frac{2\xi}{\alpha + 1} - \mu \right)
\]

\eta_2 = \mu - \frac{2\xi}{(\alpha + 1)}

with \( \rho_1, \rho_2, \Psi_1, \Psi_2, \gamma_1, \gamma_2, \delta_1, \delta_2 \) described as in Theorem 4.

**Proof.** By using Lemma 3 and using properties of absolute value we have

\[
\left| \frac{\lambda f(m) + \mu f(n)}{2} + \frac{2 - \lambda - \mu f(\xi m + (1-\xi)n)}{2} \right|
\]

\[
- \Gamma(\alpha + 1) \left( \frac{\lambda f(m) + \mu f(n)}{2} + \frac{2 - \lambda - \mu f(\xi m + (1-\xi)n)}{2} \right)
\]

\[
\left( 1 - \xi \right)^{\alpha-1} f_{\alpha}^{\xi m + (1-\xi)n} f(m) + \xi^{\alpha-1} f_{\alpha}^{\xi m + (1-\xi)n} f(n)
\]

\[
\leq \frac{n-m}{2} \left[ \left( 1 - \xi \right) \left( \int_0^1 |2(1-\xi)t^\alpha - \lambda| |f'(t(\xi m + (1-\xi)n) + (1-t)m)| dt \right)^{\frac{1}{q}} + \xi \left( \int_0^1 |\mu - 2\xi t^\alpha| dt \right)^{\frac{1}{q}} \right]
\]

Using power mean inequality, it yields

\[
\left| \frac{\lambda f(m) + \mu f(n)}{2} + \frac{2 - \lambda - \mu f(\xi m + (1-\xi)n)}{2} \right|
\]

\[
- \Gamma(\alpha + 1) \left( \frac{\lambda f(m) + \mu f(n)}{2} + \frac{2 - \lambda - \mu f(\xi m + (1-\xi)n)}{2} \right)
\]

\[
\left( 1 - \xi \right)^{\alpha-1} f_{\alpha}^{\xi m + (1-\xi)n} f(m) + \xi^{\alpha-1} f_{\alpha}^{\xi m + (1-\xi)n} f(n)
\]

\[
\leq \frac{n-m}{2} \left[ \left( 1 - \xi \right) \left( \int_0^1 |2(1-\xi)t^\alpha - \lambda| |f'(t(\xi m + (1-\xi)n) + (1-t)m)| dt \right)^{\frac{1}{q}} + \xi \left( \int_0^1 |\mu - 2\xi t^\alpha| dt \right)^{\frac{1}{q}} \right]
\]

By taking into account convexity of \( |f'|^q \) we get

\[
\left| \frac{\lambda f(m) + \mu f(n)}{2} + \frac{2 - \lambda - \mu f(\xi m + (1-\xi)n)}{2} \right|
\]

\[
- \Gamma(\alpha + 1) \left( \frac{\lambda f(m) + \mu f(n)}{2} + \frac{2 - \lambda - \mu f(\xi m + (1-\xi)n)}{2} \right)
\]

\[
\left( 1 - \xi \right)^{\alpha-1} f_{\alpha}^{\xi m + (1-\xi)n} f(m) + \xi^{\alpha-1} f_{\alpha}^{\xi m + (1-\xi)n} f(n)
\]

\[
\leq \frac{n-m}{2} \left[ \left( 1 - \xi \right) \left( \int_0^1 |2(1-\xi)t^\alpha - \lambda| |f'(t(\xi m + (1-\xi)n) + (1-t)m)| dt \right)^{\frac{1}{q}} + \xi \left( \int_0^1 |\mu - 2\xi t^\alpha| dt \right)^{\frac{1}{q}} \right]
\]

\[
\times \left( \int_0^1 |\mu - 2\xi t^\alpha| dt \right)^{\frac{1}{q}} \left( \int_0^1 |t| f'(t(\xi m + (1-\xi)n) + (1-t)m)|^q \right)^{\frac{1}{q}} dt \right)^{\frac{1}{q}}
\]

\[
\times \left( \int_0^1 |\mu - 2\xi t^\alpha| dt \right)^{\frac{1}{q}} \left( \int_0^1 |t| f'(t(\xi m + (1-\xi)n) + (1-t)m)|^q \right)^{\frac{1}{q}} dt \right)^{\frac{1}{q}}
\]
By making necessary computations we have

\[
\int_0^1 |2(1-\xi)t^\alpha - \lambda| \, dt = \begin{cases} 
\zeta_1, & \lambda < 2(1-\xi) \\
\zeta_2, & \lambda \geq 2(1-\xi) 
\end{cases}
\]

\[
\int_0^1 |\mu - 2\xi t^\alpha| \, dt = \begin{cases} 
\eta_1, & \mu < 2\xi \\
\eta_2, & \mu \geq 2\xi 
\end{cases}
\]

3 Conclusions

A new Lemma was proved in this study. Using this lemma, new fractional type inequalities were obtained. New theorems for different types of convex functions can be obtained by using Lemma 3, and thus, new upper bounds can be obtained. Various applications for these inequalities can be revealed. Also Lemma 2 can be generalized and new integral inequalities can be obtained through different fractional integral operators.

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