Matrix models for stationary Gromov-Witten invariants of the Riemann sphere

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Abstract

Inspired by recent formulæ of Dubrovin, Yang, and Zagier, we interpret the tau function enumerating stationary Gromov-Witten invariants of \( \mathbb{P}^1 \) as an isomonodromic tau function associated with a difference equation. As a byproduct we obtain an analogue of the Kontsevich matrix model for this tau function. A connection with the Charlier ensemble is also considered.

Contents

1 Introduction and results 1
1.1 Connections with matrix models . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 3
1.2 Outline of the proof of Thm. 1.1 . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 6
2 Asymptotic analysis of the matrix difference equation 9
3 Proof of Prop. 1.6 12
4 The limiting Riemann-Hilbert problem 15
5 Proof of Prop. 1.7 16
A Proof of Lemma 2.2 18

1 Introduction and results

The well known conjecture by Witten [37] and subsequent proof by Kontsevich [28] says that if we consider the following generating function of intersection numbers on \( \overline{\mathcal{M}}_{g,n} \)

\[
F(t_0, t_1, t_2, \ldots) := \sum_{g,n \geq 0} \frac{t_0^{g-2+n}}{n!} \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{k_1} \cdots \psi_n^{k_n} \psi_1 = \frac{t_0^3}{6} + \frac{t_1}{24} + \frac{t_0^2 t_1}{24} + \frac{t_0 t_2}{24} + \frac{t_1^2}{24} + \ldots
\]

then \( \exp F \) is a tau function of the KdV hierarchy, termed Kontsevich–Witten tau function. In (1.1), \( \overline{\mathcal{M}}_{g,n} \) is the Deligne-Mumford-Knudsen compactification of the moduli space of Riemann surfaces with \( n \) marked points and \( \psi_i \) is the Chern class of the cotangent line bundle at the \( i \)th puncture, \( i = 1, \ldots, n. \)

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The dimensional constraint $k_1 + \ldots + k_n = 3g - 3 + n$ allows to read off the corresponding genus $g$ for every coefficient of the generating function (1.1).

The Kontsevich–Witten tau function is closely related to the Kontsevich matrix model

$$Z_N^{Kont}(z_1, \ldots, z_N) := \frac{\int_{H_N} \exp \operatorname{tr} (A M^2 - Z M^2) dM}{\int_{H_N} \exp (-Z M^2) dM} = \frac{\det \left( \sqrt{4\pi} \sum \Lambda(z)^{k-1} \right)_{j,k=1}^N}{\Delta(z_1, \ldots, z_N)}$$

where $H_N$ is the space of $N \times N$ hermitian matrices, $Z = \text{diag}(z_1, \ldots, z_N)$ is an $N \times N$ diagonal matrix, and $\Lambda(z) = \frac{1}{\pi} \int_0^{+\infty} \cos(\Theta + \xi x) d\xi$ is the standard Airy function. Here and elsewhere we denote

$$\Delta(x_1, \ldots, x_m) := \det (x^{k-1})_{j,k=1}^{1 \leq j < k \leq m}$$

the Vandermonde determinant. The proof of the equality in (1.2) uses standard techniques of matrix integration, and is completely analogous to the arguments which we will use to prove (1.11) below. The connection between (1.1) and (1.2) goes as follows. The Airy function admits an asymptotic expansion

$$\sqrt{4\pi} \sum \Lambda(z)^{k-1} \sim \sum_{j \geq 0} \frac{(0j-1)!!}{(1j-1)!!} \frac{1}{z^{2j+1/2}}$$

for $z \to \infty$ within $|\arg z| < \frac{\pi}{2}$, and so (1.2) admits an asymptotic expansion of the form $1 + O(z^{-1})$ which is a symmetric formal power series in $z_1^{-1}, \ldots, z_N^{-1}$.

Consider this expansion for (1.2) in terms of the scaled $M$ihwa times $t_k := -(2k-1)! \operatorname{tr} \left( \sqrt{2\pi} (-2k-1) \right)$, which is a formal power series in $t_0, t_1, t_2, \ldots$. Setting $d g(t) := 2k + 1$, it can be shown that terms up to degree $D$ in this expansion for $Z_N^{Kont}(z_1, \ldots, z_N)$ do not depend on $N$ as soon as $N > D$, in other terms they stabilize as $N \to \infty$; moreover, coefficients in front of monomials involving $t_k$’s with non-integer indexes vanish [25, 20]. Finally, the logarithm of this limiting expansion coincides with the generating function (1.1) [28].

Generalizations of this result in various directions have been considered; in particular to $r$-spin intersection numbers [38, 2], to open intersection numbers [19, 5], and to Gromov-Witten (GW) theory [29, 8].

One of the first important examples of the last case is the stationary GW theory of $P^1$. In this case, the generating function (1.1) is replaced by

$$F_{P^1}(t_0, t_1, t_2, \ldots; \epsilon) := \sum_{g, n \geq 0} \sum_{k_1, \ldots, k_n \geq 0} \frac{t_{k_1} \cdots t_{k_n} \epsilon^{2g-2}}{n!} \int_{[\mathcal{M}_{g,n}(P^1 ; d)]} \psi_1 \cdots \psi_n \epsilon^g \omega \cdot \epsilon^g \omega

= \left( \frac{1}{\epsilon^2} - \frac{1}{24} \right) t_0 + \frac{t_0^2}{2\epsilon^2} + \frac{t_0^3}{6\epsilon^4} + \left( \frac{1}{4\epsilon^2} + \frac{1}{24} + \frac{7\epsilon^2}{5760} \right) t_2 + \ldots$$

where $\mathcal{M}_{g,n}(P^1 ; d)$ denotes the moduli stack of degree $d$ stable maps from Riemann surfaces of genus $g$ with $n$ marked points to $P^1$; $[\mathcal{M}_{g,n}(P^1 ; d)]$ is the virtual fundamental class [7], which allows integration of characteristic classes, in this case the psi-classes $\psi_i$ as above (pulled back via the forgetful map $\mathcal{M}_{g,n}(P^1 ; d) \to \mathcal{M}_{g,n}$) and the classes $\epsilon^g \omega$ (pullback of the normalized Kähler class $\omega \in H^2(P^1 ; \mathbb{Z})$, $F_{P^1} \omega = 1$, via the evaluation maps $ev_i : \mathcal{M}_{g,n}(P^1 ; d) \to P^1$ at the $i$th marked point).

The dimensional constraint $k_1 + \ldots + k_n = 2(g - 1 + d)$ allows to recover the degree $d$ for every coefficient of the generating function (1.4). The exponential $\exp F_{P^1}$ is a tau function of the Toda hierarchy [34, 23].

Introduce the following entire function $f(z; \epsilon)$ of the complex variable $z$, depending on a parameter $\epsilon$ which will be assumed real positive for the rest of this work, $\epsilon > 0$;

$$f(z; \epsilon) := \frac{1}{\sqrt{2\pi \epsilon}} \int_{C_1} \exp \left( \frac{1}{\epsilon} \left( x - \frac{1}{x} \right) \right) \left( z + \frac{3}{2} \right) \log x \, dx.$$

The contour $C_1$ starts from 0 with $|\arg x| < \frac{\pi}{2}$ and arrives at $\infty$ with $\frac{\pi}{2} < |\arg x| < \pi$, see Fig. 1 below. The function $f$ is a Hankel function of the argument, see Rem. 2.1 and Sec. 2. Define

$$Z_N(z_1, \ldots, z_N) := \frac{\det \left( \frac{1}{\epsilon^{2j}} \left( \frac{z_j}{\epsilon^j} \right)^{-z_j} f(z_j + k - 1; \epsilon) \right)_{j,k=1}^N}{\Delta(z_1, \ldots, z_N)}$$
We will explain the connection of the function $Z_N$ to a suitable matrix model below (see (1.18)).

The function $f(z; \epsilon)$ satisfies the second order difference equation

$$f(z + 1; \epsilon) + f(z - 1; \epsilon) = \epsilon \left( z + \frac{1}{2} \right) f(z; \epsilon)$$  \hspace{1cm} (1.7)

and admits an asymptotic expansion within the sector $|\arg z| \leq \frac{\pi}{2} - \delta$, $\delta > 0$, of the form

$$\left( \frac{\epsilon z}{e} \right)^{-z} f(z; \epsilon) \sim 1 + \frac{24 - \epsilon^2}{24 \epsilon^2 z} + \frac{e^4 + 528 e^2 + 576}{1152 e^4 z^2} + \frac{1003 e^6 + 95400 e^4 + 406080 e^2 + 69120}{414720 e^6 z^3} + \ldots$$  \hspace{1cm} (1.8)

where the coefficients can be computed either by a steepest descent analysis or by the difference equation (1.7); these statements are proven below in Sec. 2. Therefore, within the same sector we also have

$$\left( \frac{\epsilon z}{e} \right)^{-z} f(z + k; \epsilon) \sim (\epsilon z)^k (1 + O(z^{-1})).$$  \hspace{1cm} (1.9)

for all $k = 0, 1, 2, \ldots$. This implies that the ratio (1.6) admits, in the same sector, an asymptotic expansion of the form $1 + O(z^{-1})$; this expansion for (1.6) is a symmetric formal power series in $z_1^{-1}, \ldots, z_N^{-1}$, which stabilizes once expressed in terms of the scaled Miwa variables

$$t_k := k! \left( \frac{1}{z_{k+1}} + \cdots + \frac{1}{z_N} \right).$$  \hspace{1cm} (1.10)

Namely, setting $\deg t_k := k + 1$, terms of degree $D$ in the expansion of $Z_N(z_1, \ldots, z_N)$ do not depend on $N$ as soon as $N > D$; the proof is exactly the same as for the Kontsevich model [25, 20].

Our main result is that (1.6) is the correct analogue of the Kontsevich model for the stationary GW theory of $\mathbb{P}^1$.

**Theorem 1.1.** The expansion of $\log Z_N(z_1, \ldots, z_N)$, expressed in terms of the scaled Miwa variables (1.10), stabilizes as $N \to \infty$ to the generating function (1.4) of stationary GW invariants of $\mathbb{P}^1$.

For example, using the first terms of the expansion in (1.8) we can compute $Z_{N=4}(z_1, \ldots, z_4)$ up to terms of order $3$ in $z_1^{-1}, \ldots, z_4^{-1}$ as

$$Z_{N=4}(z_1, \ldots, z_4) = 1 + \frac{24 z_1^2}{24 z_1} \left( \frac{1}{z_2} + \cdots + \frac{1}{z_4} \right) + \left( \frac{1}{z_2} + \frac{1}{z_3} + \frac{1}{z_4} \right) \left( \frac{1}{z_1} + \frac{1}{z_2} + \cdots + \frac{1}{z_4} \right) + \frac{1003}{1152 z_1} + \frac{265}{41472} \left( \frac{1}{z_1} + \cdots + \frac{1}{z_3} \right) + \frac{47}{2} \frac{1}{z_1 z_3} + \frac{1}{z_1 z_4} \left( \frac{1}{z_2} + \cdots + \frac{1}{z_4} \right) + \frac{1}{z_1} \frac{1}{z_2} + \frac{1}{z_1} \frac{1}{z_3} + \frac{1}{z_1} \frac{1}{z_4} + \frac{1}{z_2} \frac{1}{z_3} + \frac{1}{z_2} \frac{1}{z_4} + \frac{1}{z_3} \frac{1}{z_4} \right)
$$

and then, in view of the relations

$$t_0 = \frac{1}{z_1} + \cdots + \frac{1}{z_4}, \hspace{1cm} t_0^2 = \frac{1}{z_1^2} + \cdots + \frac{1}{z_4^2}, \hspace{1cm} t_0^3 = \frac{1}{z_1^3} + \cdots + \frac{1}{z_4^3}, \hspace{1cm} t_0^4 = \frac{1}{z_1^4} + \cdots + \frac{1}{z_4^4}, \hspace{1cm} t_0^5 = \frac{1}{z_1^5} + \cdots + \frac{1}{z_4^5}, \hspace{1cm} t_0^6 = \frac{1}{z_1^6} + \cdots + \frac{1}{z_4^6},\hspace{1cm} \frac{e^2 t_2}{6} = \frac{1}{z_1 z_2} + \cdots + \frac{1}{z_3 z_4} + \frac{2}{z_1 z_2 z_3} + \cdots + \frac{2}{z_2 z_3 z_4},$$

the expansion for $\log Z_{N=4}(z_1, \ldots, z_4)$ correctly reproduces the terms up to degree $3$ given by example in (1.4).

### 1.1 Connections with matrix models

**Matrix models with external source.** A cosh-potential for a Kontsevich matrix model for stationary GW invariants of $\mathbb{P}^1$ has been proposed in [3]; in the naive interpretation with a flat hermitian measure

...
this matrix model reads

\[
\int_{H_N} \exp \text{tr} \left( MZ - \frac{2}{\epsilon} \cosh M \right) dM = \pi^{N(N-1)/2} \frac{\det \left( \int_{\mathbb{R}} z^k e^{\frac{1}{2} \cosh x} dx \right)^N}{\Delta(z_1, \ldots, z_N)}. \tag{1.11}
\]

The equality above can be derived as follows. First we decompose integration in eigenvalues and angular variables

\[
\int_{H_N} \exp \text{tr} \left( MZ - \frac{2}{\epsilon} \cosh M \right) dM = \pi^{N(N-1)/2} \prod_{i=1}^{N} \int_{\mathbb{R}^N} \left( \int_{U_N} e^{\text{tr} (UXU^{\dagger})Z} dU \right) \Delta^2(X) e^{-\frac{1}{2} \text{tr} \cosh(X)} dX.
\tag{1.12}
\]

denoting \(dU\) the normalized Haar measure over the unitary group \(U_N\) of \(N \times N\) matrices, \(dX = dx_1 \ldots dx_N\), and \(\Delta(X) := \Delta(x_1, \ldots, x_N)\). Then we use Harish-Chandra–Itzykson–Zuber formula

\[
\int_{U_N} e^{\text{tr} (UXU^{\dagger})Z} dU = \left( \prod_{i=1}^{N-1} i! \right) \frac{\det (e^{x_i z_j - \frac{1}{2} \cosh x_i})}{\Delta(X) \Delta(Z)}. \tag{1.13}
\]

to rewrite the previous expression as

\[
\pi^{N(N-1)/2} \frac{1}{N!} \Delta(Z) \int_{\mathbb{R}^N} \Delta(X) \det \left( e^{x_i z_j - \frac{1}{2} \cosh x_i} \right)_{i,j=1}^N dX. \tag{1.14}
\]

and finally the equality in (1.11) is a consequence of the Andreief identity

\[
\int_{\mathbb{R}^N} \det (\phi_i(x_j))_{i,j=1}^N \det (\psi_i(x_j))_{i,j=1}^N dx_1 \ldots dx_N = N! \det \left( \int_{\mathbb{R}} \phi_i(x) \psi_j(x) dx \right)_{i,j=1}^N \tag{1.15}
\]

with \(\phi_i := x^{i-1}\) and \(\psi_i(x) := e^{x z_i - \frac{1}{2} \cosh x}.\) Noting now that

\[
\int_{\mathbb{R}} e^{x z_i - \frac{1}{2} \cosh x} dx = 2K_z \left( \frac{2}{\epsilon} \right) \tag{1.16}
\]

where \(K_z(\cdot)\) is the modified Bessel function of second kind of order \(\nu\) and argument \(\zeta\) [1], the matrix integral (1.11) can be alternatively expressed as

\[
2^N \pi^{N(N-1)/2} \frac{\det (\hat{\delta}_{j=1}^{N-1} K_{z_j} \left( \frac{2}{\epsilon} \right))}{\Delta(z_1, \ldots, z_N)} = 2^N \pi^{N(N-1)/2} \frac{\det (\hat{\delta}_{j=1}^{N-1} K_{z_j} \left( \frac{2}{\epsilon} \right))}{\Delta(z_1, \ldots, z_N)}. \tag{1.17}
\]

The main difference with the model (1.6) considered in this work is the presence of derivatives instead of integral shifts. We observe that the following modification of (1.11)

\[
\int_{H_N} \exp \text{tr} \left( ZM - \frac{2}{\epsilon} \cosh M \right) dM = 2^N \pi^{N(N-1)/2} \frac{\det (K_{z_j+k-1} \left( \frac{2}{\epsilon} \right))}{\Delta(z_1, \ldots, z_N)}. \tag{1.18}
\]

(which coincides with (1.11) for \(N = 1\) only) produces a result which is closer to the model (1.6) under consideration in this work; indeed, from (2.3) and the formula \(K_{\nu}(\zeta) = \frac{\pi^{\nu+1} H^{(1)}_{\nu}(i\zeta)}{2^{\nu+1}}\), one concludes that the transformations \(\epsilon \mapsto i\epsilon, z_j \mapsto z_j - \frac{1}{2}\) essentially convert (1.6) to (1.18). As above, \(\Delta(A)\) denotes the discriminant of the characteristic polynomial of the matrix \(A\). The equality in (1.18)

\(^3\)More precisely, in [2] the integrand of the matrix model partition function is identified as \(\exp \sum_{g,q} \frac{1}{2} (MA - e^{-M} - qe^{-M})\). Up to minor modifications, the parameters \(g,q\) can be combined into a single parameter \(q = q^{-\frac{1}{2}}g\); then the integrand of (1.11) is recovered by the identification \(\Lambda = gZ\).

\(^4\)However in [3] the measure considered is not identified as the flat measure \(dM\). We thank Prof. A. Alexandrov for pointing this out.
is proven by the same arguments above, noting that after the angular integration using the Harish-Chandra–Itzykson–Zuber formula the left side is written as

\[
\frac{\pi^{N(N-1)/2}}{N!} \frac{1}{\Delta(Z)} \int_{\mathbb{R}^N} \Delta(e^X) \det \left(e^{x_iz_j - \frac{1}{2} \cosh x_i} \right)^N_{i,j=1} \, dX
\]  

and now the equality follows again from (1.15), this time with \( \phi_i(x) := e^{(i-1)x} \).

Finally let us note that (1.18) admits the alternative expression

\[
\int_{H_N^+} \exp \left( Z \log M - \frac{1}{\epsilon} (M + M^{-1}) \right) \frac{\Delta(\log M)}{\Delta(M)} \, dM
\]

where \( H_N^+ \) is the cone of positive definite \( N \times N \) hermitian matrices.

**Remark 1.2.** This matrix model has been obtained by completely independent means in the recent paper [4] from the free-fermion description of the Gromov-Witten theory of \( \mathbb{P}^1 \) [35]. Moreover, in [4] it is shown that a simple modification also describes the stationary sector of the Gromov-Witten theory of \( \mathbb{P}^1 \) relative to one point.

**Connection with the Charlier ensemble.** Introduce a discrete measure

\[
\mu_a := \sum_{n \geq 0} e^{-a} a^n \frac{\delta_{n+\frac{1}{2}}}{n!}
\]

supported on \( \{\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots\} \); here \( \delta_\xi \) is the Dirac delta measure supported at \( \xi \in \mathbb{R} \) and \( a > 0 \) is a parameter. The monic discrete orthogonal polynomials \( \pi_\ell(x; a) = x^\ell + \cdots \) relative to the measure (1.21) are known to be the (suitably scaled) **Charlier polynomials**;

\[
\pi_\ell(x; a) := (-a)^{\ell/2} _2F_0 \left( -\ell, \frac{1}{2} - x; -\frac{1}{a} \right),
\]

\[
\int_{\mathbb{R}} \pi_\ell(x; a) \pi_\ell'(x; a) \, d\mu_a(x) = \sum_{n \geq 0} \pi_\ell \left( n + \frac{1}{2}; a \right) \pi_\ell' \left( n + \frac{1}{2}; a \right) \frac{e^{-a} a^n}{n!} = a^\ell \ell! \delta_{\ell, \ell'}.
\]

The following result concerning a scaling limit of these orthogonal polynomials has been communicated to us by P. Lazag.

**Lemma 1.3 ([30]).** For all \( \zeta \in \mathbb{R} \) and \( \ell \in \mathbb{Z} \) we have

\[
\lim_{L \to +\infty} \frac{\pi_{L+\ell} \left( L + \frac{\zeta}{2} \right)}{\Gamma(L + \frac{\zeta}{2} + \frac{1}{2})} = e^{\ell-\frac{1}{2}} J_{\ell-\frac{1}{2}} \left( \frac{2}{\epsilon} \right).
\]

Consider now a matrix model of \( L \times L \) hermitian matrices with spectrum distributed according to the discrete measure (1.21) (**Charlier ensemble**). In particular, the probability distribution of the eigenvalues is given by

\[
\frac{1}{Z_{L,a}} \Delta^2(x_1, \ldots, x_L) d\mu_a^\otimes L(x_1, \ldots, x_L), \quad Z_{L,a} := \int_{\mathbb{R}^L} \Delta^2(x_1, \ldots, x_L) \, d\mu_a(x_1) \cdots d\mu_a(x_L).
\]

According to general results [18, 6] the expectation value of a product of characteristic polynomials admits the following expression

\[
\left\langle \prod_{i=1}^N \det (u_i 1 - M) \right\rangle_{L,a} = \frac{\det (\pi_{L+k-1}^\otimes (u_j))_{j,k=1}^N}{\Delta(u_1, \ldots, u_N)}
\]

in terms of the monic orthogonal polynomials \( \pi_0, \pi_1, \ldots \); here the expectation value is taken according to the distribution (1.23).

Combining (1.24) with Lemma 1.3 we obtain the following interpretation of (1.6).
Remark 1.5. Since for all $n = 0, 1, 2, \ldots$

$$\text{res}_{z=n+\frac{1}{2}} \Gamma \left( \frac{1}{2} - z \right)(\log a)^n = -\sqrt{\frac{\pi}{2}} a^n n!$$

we observe that the partition function for the Charlier ensemble of $L \times L$ hermitian matrices introduced above admits the following alternative expression;

$$\int_{H_L(C)} \exp tr \ V(M) \, dM, \quad V(z) := \log \Gamma \left( \frac{1}{2} - z \right) + z (\log a + i\pi)$$

where $C$ is a contour from $\infty + i\delta$ to $\infty - i\delta$ ($\delta > 0$) surrounding the positive real axis, and $H_L(C)$ is the set of unitarily diagonalizable $L \times L$ matrices with spectrum on $C$. This is seen as the integral (1.28) localizes at the simple poles of the Gamma function and is therefore expressed as a sum of the relative residues (1.27).

Stirling approximation of the logarithm of the Gamma function in (1.28) seems to hint at a connection with the one-matrix model with logarithmic potential which was proposed in [24] (see also [32]) to describe the Gromov–Witten theory of $\mathbb{P}^1$ (and in particular, its stationary sector). Further speculation about this connection is beyond the scope of this work and is deferred to future investigation.

1.2 Outline of the proof of Thm. 1.1

In this section we describe the steps in the proof of Thm. 1.1 and the organization of the paper.

Dubrovin–Yang–Zagier formulae. Crucial to the proof of this result are the explicit formulae for stationary GW invariants of $\mathbb{P}^1$, conjectured by Dubrovin and Yang in [21] and proven together with Zagier in [22] (and independently proven in [31] within the framework of Topological Recursion). This result can be summarized as follows.

Introduce the $2 \times 2$ matrix valued formal series

$$R(z; \epsilon) := \frac{\pi}{\epsilon \cos(\pi z)} \begin{pmatrix} J_{z-\frac{1}{2}} \left( \frac{\pi}{\epsilon} \right) & J_{z+\frac{1}{2}} \left( \frac{\pi}{\epsilon} \right) \\ J_{z-\frac{1}{2}} \left( \frac{\pi}{\epsilon} \right) & J_{z+\frac{1}{2}} \left( \frac{\pi}{\epsilon} \right) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \mathcal{O}(z^{-1})$$

(1.29)
where $J_n(z)$ are the Bessel functions of the first kind, identified with their formal expansions as $z \to +\infty$ [1]. Introduce also the expressions

$$S_1 = \frac{1}{\epsilon} \left( \frac{\pi}{\epsilon \cos(\pi \epsilon)} \right) \left( J_{\epsilon - \frac{1}{2}} \left( \frac{z}{\epsilon} \right) - J_{\epsilon + \frac{1}{2}} \left( \frac{z}{\epsilon} \right) \right) + \log(\epsilon z),$$

$$S_n = -\frac{1}{n} \sum_{\sigma \in \mathcal{S}_n} \text{tr} \left( R(z_{\sigma(1)}; \epsilon) \cdots R(z_{\sigma(n)}; \epsilon) \right) \frac{\partial_z J_{\epsilon - \frac{1}{2}} \left( \frac{z}{\epsilon} \right)}{\partial_{z^2} J_{\epsilon + \frac{1}{2}} \left( \frac{z}{\epsilon} \right)} + \frac{\delta_{n, 2}}{(z_1 - z_2)^2}$$

understood as formal series in $z_1^{-1}, \ldots, z_n^{-1}$; note that (1.31) is well defined in this sense, as it is regular along the diagonals $z_i = z_j$.

The main result conjectured in [21] and proven in [22] is that for the stationary GW invariants of $\mathbb{P}^1$

$$\langle \tau_{k_1} \cdots \tau_{k_n} \rangle_{\mathbb{P}^1, d} := \sum_{g \geq 0} \epsilon^{2g-2} \int_{[M_{g,n}(\mathbb{P}^1, d)]} \psi_1^{k_1} \cdots \psi_n^{k_n} \text{ev}_1^* \omega \cdots \text{ev}_n^* \omega$$

entering the generating function (1.4), we have an expression in terms of formal residues, namely for all $n \geq 1$, $k_1, \ldots, k_n \geq 0$ the following identity holds true;

$$\langle \tau_{k_1} \cdots \tau_{k_n} \rangle_{\mathbb{P}^1, d} = (-1)^n \text{res}_{z_1 = \infty} \cdots \text{res}_{z_n = \infty} S_n(z_1, \ldots, z_n) \prod_{j=1}^n \epsilon^{k_j+1} \frac{dz_j}{(k_j + 1)!}.$$ (1.33)

In the case $n = 1$, (1.33) reproduces the explicit formula for one-point stationary GW invariants of $\mathbb{P}^1$ due to Pandharipande [36].

The strategy of the proof of Thm. 1.1 can be summarized as follows; the logic is completely parallel to the one employed in [11, 15, 14].

1. We identify the right hand side of (1.33) as logarithmic derivatives of a tau function of isomonodromic type. Indeed, as we shall recall in Sec. 4 (e.g. see Lemma 5.2), logarithmic derivatives of arbitrary order of such tau functions can be expressed in terms of formal residues as in the right hand side of (1.33).

2. We identify logarithmic derivatives of the tau function of isomonodromic type with the limiting coefficients in the aforementioned expansion of $Z_N(z_1, \ldots, z_N)$. To accomplish such identification, we also interpret $Z_N(z_1, \ldots, z_N)$, for any $N$, as the expansion in every sector of a tau function of isomonodromic type (see Sec. 3).

We now outline this approach with more details.

**Tau functions of isomonodromic type.** Isomonodromic tau function have been originally introduced in the context of isomonodromic deformations, where a matrix linear ODE $\frac{d}{dz} \Psi(z; s) = A(z; s) \Psi(z; s)$, with $A$ rational in $z$, is assumed to depend analytically on parameters $s$ in such a way that its generalized monodromy data is constant in $s$ [27]. Location of poles of $A(z)$ are part of the parameters $s$.

The definition of isomonodromic tau function in loc. cit. was later rephrased (and generalized) in [9] for a general matrix Riemann–Hilbert problem (RHP)

$$\Gamma_+(z) = \Gamma_-(z) J(z), \quad z \in \Sigma, \quad \Gamma(\infty) = 1$$

posed on some piecewise smooth oriented contour $\Sigma$ in the complex $z$-plane, with a jump matrix $J$ defined on $\Sigma$. Concretely, assume that $J(z) = J(z; s)$ depends analytically on parameters $s$, and is actually the restriction to $\Sigma$ of one (or more) analytic function(s) of $z$. The Malgrange differential $\Omega$ is then defined as the following one-form, in the open set in the parameter space $\{s\}$ where the RHP $\Gamma_+(z; s) = \Gamma_-(z; s) J(z; s)$ has a solution;

$$\Omega := \sum_j \Omega_j \partial s_j, \quad \Omega_j := \int_{\Sigma} \text{tr} \left( \Gamma^{-1} \frac{\partial \Gamma}{\partial z} \frac{\partial J}{\partial s_j} J^{-1} \right) \frac{dz}{2\pi i}.$$ (1.35)

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5We denote $\mathcal{S}_n$ the symmetric group over $\{1, 2, \ldots, n\}$. 

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Remarkably, the differential of $\Omega$ depends on $J$ only, and not on the solution $\Gamma$; in many cases, $\Omega$ (or some simple modification) is closed, and we can accordingly introduce the tau function $\tau(s)$ by

$$\frac{\partial}{\partial s_j} \log \tau(s) = \Omega_j. \quad (1.36)$$

This recovers the original setting of [27] when the the RHP is associated with a matrix linear ODE with rational coefficients.

More concretely, in this work we shall consider the $2 \times 2$ matrix version of the difference equation (1.7)

$$\Psi(z + 1) = A(z) \Psi(z), \quad A(z) = \begin{pmatrix} \epsilon (z + \frac{1}{2}) & -1 \\ 1 & 0 \end{pmatrix}, \quad (1.37)$$

which has a unique formal solution in the form

$$(1 + O(z^{-1})) \left( \frac{\epsilon z}{e} \right)^{z \sigma_3}. \quad (1.38)$$

In Sec. 2 we study the Stokes phenomenon of (1.37), i.e. we construct sectors in the $z$-plane, cut along $z < 0$, and analytic solutions to (1.37) which have the same asymptotic expansion, given by the formal solution (1.38), in every sector. The connection matrices relating different analytic solutions constitute the monodromy data of the difference equation (1.37), and they essentially define jumps of a RHP $\Gamma_0(z) = \Gamma_{0-}(z) J_0(z)$, see Sec. 2.

Then we shall fix some $N \geq 1$ and we add dependence on parameters $z = (z_1, \ldots, z_N)$ to the RHP constructed above, by dressing the jump matrices $J_0(z)$ as in (3.1). Associated to this RHP we then have a tau function, $\tau_N(z)$, to be defined concretely in (3.12). Moreover, as it follows from the results of [26, 10] such tau function admits a representation in terms of the determinant of a characteristic matrix. To state this result, let us introduce the following generalization of (1.6)

$$\hat{Z}_N(z_1, \ldots, z_N) = \frac{\det \left( \frac{1}{\epsilon - 1} (\Gamma_0(z_j + k - 1))_{1,1} \right)_{j,k=1}^N}{\Delta(z_1, \ldots, z_N)} \quad (1.39)$$

where the piecewise analytic matrix $\Gamma_0$ mentioned above is defined in Sec. 2; we stress that, by construction, the $(1,1)$-entry of $\Gamma_0$ admits the same asymptotic expansion (1.8) in every sector, so that it is really a good generalization of (1.6) to every sector of the complex $z$-plane.

**Proposition 1.6.** The tau function $\tau_N(z)$ coincides with $\hat{Z}_N(z)$;

$$\tau_N(z) = \hat{Z}_N(z). \quad (1.40)$$

Next, we consider a limiting (in the sense $N \to \infty$) RHP, in terms of the standard Miwa times

$$T_k := \frac{1}{k} \left( \frac{1}{z_1^k} + \cdots + \frac{1}{z_N^k} \right), \quad k \geq 1 \quad (1.41)$$

related to the scaled Miwa times (1.10) by

$$t_k = \frac{(k+1)!}{\epsilon^k} T_{k+1}, \quad k \geq 0. \quad (1.42)$$

As before, this problem is constructed by dressing the jump matrices $J_0(z)$, this time in terms of parameters $T$ as prescribed by (4.2). We also associate a tau function $\tau(T)$ to this problem, see (4.10).

**Proposition 1.7.** Logarithmic derivatives of the tau function $\tau(T)$ of (4.10) are expressed in terms of the stationary GW invariants of $\mathbb{P}^1$, see (1.32);

$$\frac{\partial^n \log \tau(T)}{\partial T_{\ell_1} \cdots \partial T_{\ell_n}} = \frac{\ell_1! \cdots \ell_n!}{\ell_1 + \cdots + \ell_n - n} \langle \tau_{k_1} \cdots \tau_{k_n} \rangle_{\mathbb{P}^1, d}. \quad (1.43)$$

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We use the Pauli matrix $\sigma_3 = \text{diag}(1, -1)$.
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2 Asymptotic analysis of the matrix difference equation

In this section we study asymptotics of solutions to the difference equation (1.7), so to encode its general solution in a $2 \times 2$ matrix solution of (1.37), piecewise analytic in suitable sectors, and having the same asymptotic expansion (1.38) in every sector.

From now on we omit the dependence on the parameter $\epsilon > 0$, in the interest of clarity.

Solutions to the difference equation (1.7) can be expressed by Mellin contour integrals; in particular we choose

$$f(z) := \frac{1}{\sqrt{2\pi \epsilon}} \int_{C_1} \exp \left( \frac{1}{\epsilon} \left( x - \frac{1}{x} \right) - \left(z + \frac{3}{2}\right) \log x \right) dx,$$

$$g(z) := \frac{1}{i\sqrt{2\pi \epsilon}} \int_{C_2} \exp \left( \frac{1}{\epsilon} \left( x - \frac{1}{x} \right) - \left(z + \frac{3}{2}\right) \log x \right) dx,$$

(2.1)

where $C_1, C_2$ are contours in the $x$-plane with a branch cut along $x < 0$, $|\arg x| < \pi$, for the definition of $\log x$. More precisely

- $C_1$ starts from 0 with $|\arg x| < \frac{\pi}{2}$ and arrives at $\infty$ with $\frac{\pi}{2} < \arg x < \pi$, and
- $C_2$ starts from $\infty$ with $-\pi < \arg x < -\frac{\pi}{2}$ and arrives at $\infty$ with $\frac{\pi}{2} < \arg x < \pi$.

These contours are depicted in Fig. 1.

Figure 1: Contours $C_1, C_2$ in the $x$-plane; the dashed line represents the branch cut along $x < 0$ for the definition of $\log x$ in the integrand of (2.1).

Remark 2.1. $g$ can be expressed in terms of the Bessel function of first kind [1]

$$g(z) = \sqrt{\frac{2\pi}{\epsilon}} J_{z+\frac{1}{2}} \left( \frac{2}{\epsilon} \right)$$

(2.2)

while $f$ can be expressed in terms of the Bessel function of first and second kind, or equivalently in terms of the Hankel function $H^{(1)}$

$$f(z) = \frac{\pi}{2\epsilon} \left( i J_{z+\frac{1}{2}} \left( \frac{2}{\epsilon} \right) - Y_{z+\frac{1}{2}} \left( \frac{2}{\epsilon} \right) \right) = i \sqrt{\frac{\pi}{2\epsilon}} H^{(1)}_{z+\frac{1}{2}} \left( \frac{2}{\epsilon} \right).$$

(2.3)

Note that the $z$-dependence is in the order of the Bessel functions.

Lemma 2.2. The following asymptotic relations hold:

1. $f(z) \sim \left( \frac{\epsilon}{z} \right)^z (1 + O(z^{-1}))$, as $z \to \infty$ within $|\arg z| < \frac{\pi}{2} - \delta$, for all $\delta > 0$.

2. $g(z - 1) \sim \left( \frac{\epsilon}{z} \right)^{-z} (1 + O(z^{-1}))$, as $z \to \infty$ within $|\arg z| \leq \pi - \delta$, for all $\delta > 0$.

The proof is based on the steepest descent method; we defer it to App. A.

Let us fix angles $\alpha_1, \ldots, \alpha_4$ satisfying

$$-\pi < \alpha_1 < \frac{\pi}{2} < \alpha_2 < 0 < \alpha_3 < \frac{\pi}{2} < \alpha_4 < \pi$$

(2.4)
and corresponding sectors in the $z$-plane, with a branch cut along $z < 0, \ |\arg z\ | < \pi$;

$S_1 := \{ -\pi < \arg z < \alpha_1 \}, \ S_j := \{ \alpha_{j-1} < \arg z < \alpha_j \} \ (j = 2, 3, 4), \ S_5 := \{ \alpha_4 < \arg z < \pi \}$. (2.5)

Define a piecewise analytic $2 \times 2$ matrix $\Psi_0 = \Psi_0(z)$ as

\[
\Psi_0(z) := \begin{cases} 
\begin{pmatrix} e^{-i\pi z}g(-z-1) & -e^{i\pi z}f(-z-1) \\
e^{-i\pi z}g(-z) & e^{i\pi z}f(-z) \end{pmatrix}, & z \in S_1 \\
\begin{pmatrix} \frac{1}{2\cos(\pi z)}g(-z-1) & g(z) \\
-\frac{1}{2\cos(\pi z)}g(-z) & g(z-1) \end{pmatrix}, & z \in S_2 \\
\begin{pmatrix} f(z) & g(z) \\
f(z-1) & g(z-1) \end{pmatrix}, & z \in S_3 \\
\begin{pmatrix} \frac{1}{2\cos(\pi z)}g(-z-1) & g(z) \\
-\frac{1}{2\cos(\pi z)}g(-z) & g(z-1) \end{pmatrix}, & z \in S_4 \\
\begin{pmatrix} e^{i\pi z}g(-z-1) & -e^{-i\pi z}f(-z-1) \\
e^{-i\pi z}g(-z) & e^{i\pi z}f(-z) \end{pmatrix}, & z \in S_5
\end{cases}
\]

and define also

\[
\Gamma_0(z) := \Psi_0(z) \left( \frac{z}{e} \right)^{-z\sigma_3}. \tag{2.7}
\]

**Proposition 2.3.** The following statements hold in all sectors $S_1, ..., S_5$:

1. The matrix $\Psi_0(z)$ solves the matrix difference equation (1.37), and

2. The matrix $\Gamma_0(z)$ admits an asymptotic expansion $\Gamma_0(z) \sim 1 + O(z^{-1})$.

**Proof.**

1. Integrating by parts, we have $(i = 1, 2)$

\[
0 = \int_{C_1} \partial_x \left( e^{\frac{i}{x} \log x} \right) dx = \int_{C_1} \left( 1 + \frac{1}{x^2} - \frac{z + \frac{1}{2}}{x} \right) e^{\frac{i}{x} \log x} dx
\]

\[
= \int_{C_1} \left( e^{\frac{i}{x} \log x} + \frac{1}{x} e^{\frac{i}{x} \log x} - \frac{z + \frac{1}{2}}{x} e^{\frac{i}{x} \log x} - \left( z + \frac{1}{2} \right) e^{\frac{i}{x} \log x} \right) dx
\]

which implies

\[
f(z-1) + f(z+1) - \epsilon \left( z + \frac{1}{2} \right) f(z) = 0 = g(z-1) + g(z+1) - \epsilon \left( z + \frac{1}{2} \right) g(z). \tag{2.8}
\]

Therefore the statement is true for the sector $S_3$. The statement in the remaining sectors is obtained noting that if $p(z)$ is any anti-periodic function $p(z+1) = -p(z)$, then $f(z) := p(z)f(-z-1)$ and $g(z) := p(z)g(-z-1)$ solve the same difference equation;

\[
\tilde{f}(z-1) + \tilde{f}(z+1) - \epsilon \left( z + \frac{1}{2} \right) \tilde{f}(z) = 0 = \tilde{g}(z-1) + \tilde{g}(z+1) - \epsilon \left( z + \frac{1}{2} \right) \tilde{g}(z). \tag{2.9}
\]

2. In the sector $S_3$ the statement follows directly from Lemma 2.2. For the sector $S_1$ we exploit the fact that $f, g$ are entire function and note that $0 < \arg(e^{i\pi z}) < \frac{\pi}{2}$, due to (2.4), so that can
apply Lemma 2.2 as

\[ e^{i\pi z} f(-z) = e^{i\pi z} f(e^{i\pi z}) \sim e^{i\pi z} \left( \frac{\epsilon z}{e} \right)^{-z} (1 + \mathcal{O}(z^{-1})) = \left( \frac{\epsilon z}{e} \right)^{-z} (1 + \mathcal{O}(z^{-1})) \]

\[ e^{-i\pi z} g(-z - 1) = e^{-i\pi z} g(e^{i\pi z} - 1) \sim e^{-i\pi z} \left( \frac{\epsilon z}{e} \right)^{z} (1 + \mathcal{O}(z^{-1})) = \left( \frac{\epsilon z}{e} \right)^{z} (1 + \mathcal{O}(z^{-1})). \]

The statement is proven likewise in the sectors \( S_2, S_4, S_5 \).

Denote

\[ \Sigma := e^{i\alpha_1 R^+} \cup \cdots \cup e^{i\alpha_4 R^+} \cup \mathbb{R}_- \]  

(\text{rays oriented outwards}) so that \( \Gamma, \Psi \) are analytic for \( z \in \mathbb{C} \setminus \Sigma = S_1 \cup \cdots \cup S_5 \)

\[ \begin{array}{c}
\begin{array}{c}
S_1 \\
S_2 \\
S_3 \\
S_4 \\
S_5
\end{array}
\end{array} \]

\[ \begin{array}{c}
\begin{array}{c}
+ \\
- \\
+ \\
- \\
+
\end{array}
\end{array} \]

Figure 2: Contour \( \Sigma \), sectors \( S_1, \ldots, S_5 \), and notation for the boundary values.

**Lemma 2.4.** \( \Psi_0(z) \) satisfies the jump condition

\[ \Psi_{0+}(z) = \Psi_{0-}(z) \hat{J}_0(z) \]  

(2.11)

where the boundary values are taken according to the orientation of \( \Sigma \) (see Fig. 2) and the matrix \( \hat{J}_0(z) \) is defined on \( \Sigma \) by

\[ \hat{J}_0(z) = \begin{cases} 
\hat{J}_0^{(1)}(z) = \begin{pmatrix} 1 & iq \\
0 & 1 + q^{-1} \end{pmatrix}, & z \in e^{i\alpha_1 \mathbb{R}^+} \\
\hat{J}_0^{(2)}(z) = \begin{pmatrix} 1 & 0 \\
-\frac{i}{1+q} & 1 \end{pmatrix}, & z \in e^{i\alpha_2 \mathbb{R}^+} \\
\hat{J}_0^{(3)}(z) = \begin{pmatrix} 1 & 0 \\
-\frac{i}{1+q} & 1 \end{pmatrix}, & z \in e^{i\alpha_3 \mathbb{R}^+} \\
\hat{J}_0^{(4)}(z) = \begin{pmatrix} 1 & i \\
0 & 1 + q \end{pmatrix}, & z \in e^{i\alpha_4 \mathbb{R}^+} \\
\hat{J}_0^{(5)}(z) = q^{-\sigma_3}, & z \in \mathbb{R}_-
\end{cases} \]  

(2.12)

where we denote

\[ q := e^{2\pi i z}. \]  

(2.13)

**Proof.** It is a computation based on the identity

\[ g(-z - 1) = 2 \cos(\pi z) f(z) - i e^{-i\pi z} g(z), \]  

(2.14)

11
which can be proven by performing the change of variable \( x \mapsto -\frac{1}{x} \) in the integral defining (2.1) and applying the Cauchy theorem. Alternatively, in view of Rem. 2.1, this identity follows from the known relation
\[
J_{-\nu}(\zeta) = i \sin(\pi \nu) H_{\nu}^{(1)}(\zeta) + e^{-i \pi \nu} J_{\nu}(\zeta)
\]  
(2.15)
of Hankel and Bessel functions [1].

It follows that
\[
\Gamma_{0+}(z) = \Gamma_{0-}(z) J_0(z), \quad J_0(z) := \left( \frac{e^{\pi \epsilon}}{\epsilon} \right)^{z \sigma_3} \tilde{J}_0(z) \left( \frac{e^{\pi \epsilon}}{\epsilon} \right)^{-z \sigma_3}
\]  
(2.16)
where the notation for the boundary values in the definition of \( J_0 \) is relevant only along \( z < 0 \).

The jump matrices \( \tilde{J}_0(z) \), \( J_0(z) \) satisfy the following properties.

1. \( J_0^{(1)}(z) \equiv 1 \), hence \( \Gamma_0 \) extends analytically across \( y < 0 \).

2. \( J_0 \) is exponentially close to the identity as \( z \to \infty \), i.e. \( J_0(z) = 1 + O(z^{-\infty}) \) as \( z \) approaches \( \infty \) along any of the rays \( e^{i \alpha_j} \mathbb{R}_+ \), \( j = 1,2,3,4 \).

3. The no-monodromy condition \( \tilde{J}_0(1) \cdots \tilde{J}_0(z) = 1 \) holds true.

4. The jump matrices have unit determinant, \( \det \tilde{J}_0^{(1)}(z) \equiv 1 \), \( i = 1,\ldots,5 \), \( \det J^{(i)}(z) \equiv 1 \), \( i = 1,\ldots,4 \).

Lemma 2.5. We have \( \det \Psi_0(z) \equiv 1 \equiv \det \Gamma_0(z) \) identically in \( z, \epsilon \).

Proof. As \( \det \tilde{J}_0(z) \) is identically 1 on \( \Sigma \), we infer that \( \Delta(z) := \det \Psi_0(z) \) is an entire function of \( z \). Moreover, \( \Delta \) is periodic, \( \Delta(z+1) = \Delta(z) \), as it follows from (1.37). Hence, \( \Delta(z) \equiv 1 \) everywhere. ■

Remark 2.6. The general results of Birkhoff [16] cannot be applied directly to the difference equation (1.37).

3 Proof of Prop. 1.6

Let us denote \( \Sigma' := e^{i \alpha_1} \mathbb{R}_+ \cup \cdots \cup e^{i \alpha_4} \mathbb{R}_+ \).

Fix \( N \geq 0 \), points \( z = (z_1,\ldots,z_N) \) in the complex plane, \( |\arg z_j| < \pi \); by the freedom in the choice of the angles \( \alpha_i \), compare with (2.4), we can assume that \( z_1,\ldots,z_N \in \mathbb{C} \setminus \Sigma' \). Associated with this data, introduce the jump matrix \( J_N(z;z) : \Sigma' \to \text{SL}_2(\mathbb{C}) \) by
\[
J_N(z;z) := D_N^{-1}(z;z) J_0(z) D_N(z;z), \quad D_N(z;z) := \begin{pmatrix} 1 & 0 \\ 0 & \prod_{j=1}^{N} \left( \frac{1}{1 - \frac{z}{z_j}} \right) \end{pmatrix}.
\]  
(3.1)

Riemann–Hilbert Problem 3.1. Find a \( 2 \times 2 \) matrix \( \Gamma_N(z;z) \), analytic in every sector of \( \mathbb{C} \setminus \Sigma' \), satisfying the following jump condition along \( \Sigma' \)
\[
\Gamma_{N+}(z;z) = \Gamma_{N-}(z;z) J_N(z;z),
\]  
(3.2)
and the following boundary condition at infinity
\[
\Gamma_N(z;z) \sim 1 + O(z^{-1}).
\]  
(3.3)

Remark 3.2. As in Lemma (2.5) it can be shown that \( \det \Gamma_N(z;z) \equiv 1 \) identically in \( z \), whenever \( \Gamma_N(z;z) \) exists. Hence, the solution to the RHP 3.1 is unique, if it exists.
Introduce the Jimbo-Miwa-Ueno differential \[ \Omega_N = \sum_{j=1}^{N} \Omega_{N,j} \, dz_j, \quad \Omega_{N,j} := \text{res}_{y=0} \text{tr} \left( \Gamma_N^{-1} \partial \Gamma_N \partial D_N \Gamma_N^{-1} \right) \, dz \] (3.4)

and the Malgrange differential \[ \hat{\Omega}_N = \sum_{j=1}^{N} \hat{\Omega}_{N,j} \, dz_j, \quad \hat{\Omega}_{N,j} := \int_{\Sigma^r} \text{tr} \left( \Gamma_N^{-1} \partial \Gamma_N \partial J_N \Gamma_N^{-1} \right) \, dz \, \frac{dz}{2\pi i}. \] (3.5)

Lemma 3.3. The following identity holds true:
\[ \Omega_N - \hat{\Omega}_N = \eta_N \] (3.6)

where
\[ \eta_N = \sum_{j=1}^{N} \eta_{N,j} \, dz_j, \quad \eta_{N,j} := \int_{\Sigma^r} \text{tr} \left( J_N^{-1} \partial J_N \partial D_N \Gamma_N^{-1} \right) \, dz \, \frac{dz}{2\pi i}. \] (3.7)

Proof. Let us denote \( \partial_j := \partial \frac{\partial}{\partial z_j} \). From (3.1) we have
\[ \partial_j J_N = \left[ J_N, \partial_j D_N \Gamma_N^{-1} \right] \]

and noting \( \Gamma_{N,+} = \Gamma_{N,-} J_N + \Gamma_{N,-} J_N \) we can rewrite the last expression as
\[ \text{tr} \left( \left( \left[ \Gamma_N^{-1} \Gamma_N^{-1} \right]^+ - J_N^{-1} J_N \right) \partial_j D_N \Gamma_N^{-1} \right) = \text{tr} \left( \left[ \Gamma_N^{-1} \Gamma_N^{-1} \partial_j D_N \Gamma_N^{-1} \right]^+ \right) - \text{tr} \left( J_N^{-1} J_N \partial_j D_N \Gamma_N^{-1} \right) \]

where \( [f]^+ := f_+ - f_- \). By Cauchy’s theorem
\[ \int_{\Sigma^r} \text{tr} \left[ \Gamma_N^{-1} \Gamma_N^{-1} \partial_j D_N \Gamma_N^{-1} \right]^+ \, dz = \sum_{j=1}^{N} \text{res}_{z=z_j} \text{tr} \left( \Gamma_N^{-1} \Gamma_N^{-1} \partial_j D_N \Gamma_N^{-1} \right) \, dz \]

and the proof is complete.

The following was proven originally in [27], in the slightly different context of isomonodromic deformations of a matrix linear ODE with rational coefficients. However, the proof applies equally well here.

Theorem 3.4. The Jimbo-Miwa-Ueno differential is closed;
\[ \frac{\partial}{\partial z_j} \Omega_{N,k} = \frac{\partial}{\partial z_k} \Omega_{N,j}. \] (3.11)

Hence we introduce the tau function \( \tau_N(z) \) as
\[ \Omega_{N,j} = \frac{\partial}{\partial z_j} \log \tau_N(z). \] (3.12)

From the theory of Schlesinger transformations [26, 10], we know that a tau function related to a rational dressing of jump matrices like (3.1), admits an explicit expression in terms of a finite size determinant. We recall this result applied to our setting.

Introduce the \( N \times N \) characteristic matrix \( G_N(z) \), with entries
\[ (G_N(z))_{j,k} = - \text{res}_{z=z_j} \left( \Gamma_0^{-1}(z_j) \Gamma_0(z) \right)_{2,2} \frac{z^{k-1} \, dz}{z-z_j}, \quad 1 \leq j, k \leq N. \] (3.13)

The following result follows from [10, App. B, Thm. 2.2], hence we omit the proof.
Theorem 3.5. We have
\[
\Omega_{N,j} = \frac{\partial}{\partial z_j} \log \prod_{1 \leq a < b \leq N} (z_b - z_a)^{N-1} \det G_N(z)
\] (3.14)
where the characteristic matrix \( G_N(z) \) is defined in (3.13).

Finally, Prop. 1.6 follows from the following computation of the determinant of the characteristic matrix (3.13).

Proposition 3.6. We have
\[
\det G_N(z) = \det \left( \frac{1}{e^{z-1}} (\Gamma_0(z) + k-1)_{1,1} \right)_{j,k=1}^N \quad (3.15)
\]
Proof. Introduce functions \( a(z), b(z), \) analytic in every sector \( S_1, \ldots, S_5, \) according to
\[
\Psi_0(z) = \begin{pmatrix} a(z) & b(z) \\ a(z-1) & b(z-1) \end{pmatrix} \quad (3.16)
\]
so that the entries (3.13) of the characteristic matrix are found as
\[
\left( \frac{e^{z_j}}{e} \right)^{-z_j} \left( \frac{e^{z_j}}{e} \right)^{z_j} \frac{\det H(z; z_j)}{z - z_j} = \sum_{k=1}^N (G_N(z))_{j,k} z^{-k} + O(z^{-N-1}) \quad (3.17)
\]
where we use \( \det \Gamma_0(z) \equiv 1 \) from Lemma 2.5. Introducing the matrix
\[
H(z; z_j) = \begin{pmatrix} a(z_j) & b(z) \\ a(z_j-1) & b(z-1) \end{pmatrix} \quad (3.18)
\]
we rewrite (3.17) as
\[
\left( \frac{e^{z_j}}{e} \right)^{-z_j} \left( \frac{e^{z_j}}{e} \right)^{z_j} \frac{\det H(z; z_j)}{z - z_j} = \sum_{k=1}^N (G_N(z))_{j,k} z^{-k} + O(z^{-N-1}). \quad (3.19)
\]
Recalling the difference equation (1.37) we have
\[
\begin{pmatrix} a(z_j+1) & b(z+1) + e(z_j - z)b(z) \\ a(z_j-1) & b(z-1) \end{pmatrix} = \begin{pmatrix} \epsilon (z_j + \frac{k}{2}) & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a(z_j) & b(z) \\ a(z_j-1) & b(z_j) \end{pmatrix} \quad (3.20)
\]
hence we get
\[
\det H(z + 1; z_j + 1) + \epsilon(z - z_j)a(z_j - 1)b(z) = \det H(z; z_j) \quad (3.21)
\]
from which we obtain
\[
\det H(z + N; z_j + N) + \epsilon(z - z_j) \sum_{\ell=0}^N b(z + \ell)a(z_j + \ell - 1) = \det H(z; z_j). \quad (3.22)
\]
Finally, from \( \epsilon(z_j)^{-\ell} b(z + \ell) = \frac{1}{(z_j)^{\ell}} \hat{b}_\ell(z), \hat{b}_\ell(z) = 1 + O(z^{-1}) \) we rewrite (3.19) as
\[
\left( \frac{e^{z_j}}{e} \right)^{-z_j} \sum_{\ell=1}^N a(z_j + \ell - 1) \hat{b}_\ell(z) = \sum_{k=1}^N (G_N(z))_{j,k} z^{-k} + O(z^{-N-1}) \quad (3.23)
\]
and so
\[
G_N(z) = \hat{G}_N(z)B_N \quad (3.24)
\]
where we write \( \hat{b}_\ell(z) = 1 + \sum_{j \geq 1} \hat{b}_j z^{-j} \) and
\[
B := \begin{pmatrix} 1 & \hat{b}_1 & \cdots & \hat{b}_1^{N-2} \\ 0 & 1 & \cdots & \hat{b}_2^{N-3} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}, \quad (\hat{G}_N(z))_{j,k} = \left( \frac{e^{z_j}}{e} \right)^{-z_j} a(z_j + k - 1) b(z) = \left( \frac{\Gamma_0(z_j + k - 1)}{e^{k-1}} \right)_{1,1} \quad (3.25)
\]
and the proof is complete by taking the determinant of identity (3.24), as \( \det B \equiv 1 \).

This completes the proof of Prop. 1.6.
4 The limiting Riemann-Hilbert problem

For all $N \geq 0$, we have the identity

$$D_N^{-1} = \begin{pmatrix} 1 & \frac{\sum_{j=1}^{N} T_j(z)z^\ell}{0} \\ 0 & z^{-\ell} \end{pmatrix}, \quad T_{\ell}(z) := \frac{1}{\ell} \sum_{j=1}^{N} z^{-\ell}.$$  \hspace{1cm} (4.1)

This identity is non-formal provided $\min_{j=1,...,N} |z_j| > |z|$.

This prompts to introduce an independent set of times $T_1, T_2, ..., T_\ell$ and to consider the following RHP.

Riemann–Hilbert Problem 4.1. Find a $2 \times 2$ matrix $\Gamma(z; T)$, analytic in every sector of $C \setminus \Gamma'$, satisfying the following jump condition along $\Sigma$

$$\Gamma_+(z; T) = \Gamma_-(z; T)J(z; T),$$  \hspace{1cm} (4.3)

and the following boundary condition at infinity

$$\Gamma(z; T) \sim 1 + O(z^{-1}).$$  \hspace{1cm} (4.4)

Analytic discussion of RHP 4.1, and limit of Problem 3.1 and of the tau function. For the sake of definiteness, in the RHP 4.1 one must first assume that for some $K \geq 1$ we have $T_{\ell} = 0$ whenever $\ell > K$. In principle, this assumption is in contradiction with the interpretation of the $T_j$'s as the standard Miwa variables of the $z_j$'s (1.41), appearing in (4.1); this interpretation is relevant in order to regard the RHP 4.1 as an analytic limit of the RHP 3.1. We now briefly address this issue.

Let us fix an arbitrary $K \geq 1$ and assume $T_{\ell} = 0$ whenever $\ell > K$. Under the assumption that

$$\text{Re} \left[ t_\ell e^{iK\alpha_\ell} \right] < 0, \quad j = 2, 3, \quad \text{Re} \left[ t_\ell e^{iK\alpha_\ell} \right] > 0, \quad j = 1, 4$$  \hspace{1cm} (4.5)

we conclude that $J(z; T) = 1 + O(z^{-\infty})$ as $z \to \infty$ along any ray of $\Sigma$. Hence, the solution to the RHP 4.1 exists and is unique for $T = (T_1, ..., T_K)$ in an open neighborhood of $T = 0$, with the argument of $T_K$ further restricted by (4.5); it defines a matrix function $\Gamma(z; T)$, its specifications to each sector of the $z$-plane being holomorphic in $T_1, ..., T_K$. Note that $\Gamma(T = 0) = \Gamma_0$ by construction.

In particular, this allows to introduce the Jimbo-Miwa-Ueno and the Malgrange differentials as above, see (3.4)-(3.5):

$$\Omega = \sum_{\ell=1}^{K} \Omega_\ell dT_\ell, \quad \Omega_\ell := -\text{res}_{x=\infty} \left( \Gamma^{-1} \frac{\partial \Gamma}{\partial z} E_{22} \right) z^{\ell} dz,$$  \hspace{1cm} (4.6)

$$\hat{\Omega} = \sum_{\ell=1}^{K} \hat{\Omega}_\ell dT_\ell, \quad \hat{\Omega}_\ell := \int_{\Sigma'} \text{tr} \left( \Gamma^{-1} \frac{\partial \Gamma}{\partial z} \frac{\partial J^{-1}}{\partial T_\ell} \right) \frac{dz}{2\pi i}.$$  \hspace{1cm} (4.7)

Exactly as in Prop. 3.6, we establish the relation

$$\Omega = \hat{\Omega} + \eta, \quad \eta = \sum_{\ell=1}^{K} \eta_\ell dT_\ell, \quad \eta_\ell := \int_{\Sigma'} \text{tr} \left( J^{-1} \frac{\partial J}{\partial z} E_{22} \right) \frac{z^{\ell} dz}{2\pi i}.$$  \hspace{1cm} (4.8)

Here $E_{22}$ is the elementary matrix $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. 
Moreover, it can also be proven that $\Omega$ is closed [27]

$$\frac{\partial}{\partial T_j} \Omega_k = \frac{\partial}{\partial T_k} \Omega_j$$

and so we can introduce the tau function $\tau(T)$ as

$$\Omega_t = \frac{\partial}{\partial T_t} \log \tau(T).$$

It follows that $\tau(T)$ is an analytic function of $T_1, ..., T_K$ in an open neighborhood of $T = 0$ with $\arg T_K$ restricted by (4.5).

The main goal now is to identify the Taylor expansion of $\tau(T)$ at $T = 0$ with the formal limiting expansion of $Z_N(z_1, ..., z_N)$ as $N \to \infty$ in terms of the standard Miwa variables (1.41). This can be analytically achieved by the following argument. For fixed $T = (T_1, ..., T_K, 0, ...)$, introduce, for $N \geq 1$, the roots $Z^{(N)} = (z_1^{(N)}, ..., z_N^{(N)})$ of the Taylor polynomials of $e^\theta(z; T)$, i.e.

$$\exp \left( T_1 z + \cdots + T_K z^K \right) = (z - z_1^{(N)}) \cdots (z - z_N^{(N)}) + O(z^{N+1}).$$

Then one should check convergence (as $N \to \infty$, uniformly for $T_1, ..., T_K$ in compact sets) in all $L^p$ norms of the jump matrices for the RHP 3.1, defined in terms of $Z^{(N)}$, to those of the RHP 4.1. Then standard perturbation analysis of RHP permits to deduce convergence of $\Gamma_N(z; Z^{(N)})$ to $\Gamma(z; T_1, ..., T_K, 0, ...)$ and of tau functions $\tau_N(Z^{(N)}; T_1, ..., T_K, 0, ...)$, using the representations $\Omega_N = \hat{\Omega}_N - \eta N$ and $\Omega = \hat{\Omega} - \eta$ for logarithmic derivatives of the tau functions. Similar convergence of logarithmic derivatives of the tau functions can be deduced then by the fact the latter admit expressions in terms of the solution $\hat{\Gamma}$ to the RHP only, see e.g. Lemma 5.2.

For the main purpose of this work we are mostly concerned with the formal aspects of RHP 4.1, and the considerations above then play a minor role, so we refer to the detailed analysis for the Kontsevich–Witten tau function of [11], which is essentially similar to the case under consideration in this work.

5 Proof of Prop. 1.7

We first consider one-point intersection numbers, $n = 1$. To this end, applying definition (4.10), using the notation of (3.16) and denoting $':= \partial_z$, we compute

$$\frac{\partial}{\partial T_t} \log \tau(T) \bigg|_{T=0} = - \lim_{z \to \infty} \left( \begin{array}{cc} -a(z-1) & a(z) \\ b'(z) & b(z) \end{array} \right) \left( \begin{array}{c} b'(z) + b(z) \log(\varepsilon) \\ b'(z) - b(z) \log(\varepsilon) \end{array} \right) + \log(\varepsilon) \right) z\,dz \tag{5.1}$$

where we use the identity $(\frac{d}{dz})^n (\frac{\varepsilon}{z})^\gamma = \log(\varepsilon)$. Since

$$\det \Psi_0(z) = a(z)b(z-1) - a(z-1)b(z) \equiv 1$$

we can write

$$\frac{\partial}{\partial T_t} \log \tau(T) \bigg|_{T=0} = - \lim_{z \to \infty} \left( \begin{array}{cc} -a(z-1) & a(z) \\ \frac{b'(z)}{b'(z) - b(z)} \end{array} \right) + \log(\varepsilon) \right) z\,dz. \tag{5.3}$$

The formal residue is independent of the sector in which we let $z \to \infty$ by construction, as $\Gamma_0(z)$ has the same asymptotic expansion in every sector. E.g. we can assume, using the definition of $\Gamma_0(z)$ in the sector $S_1$, compare with (2.7), that

$$a(z) = f(z) = i \sqrt{\frac{\pi}{2\varepsilon} \frac{\alpha_{(1)}}{z + \frac{1}{2}}} \left( \frac{2}{\varepsilon} \right) \sim \sqrt{\frac{\pi}{2\varepsilon} \frac{1}{\cos(\pi z)}} \left( \frac{2}{\varepsilon} \right), \quad b(z) = g(z) = \sqrt{\frac{2\pi}{\varepsilon} \sin(\pi z)} \left( \frac{2}{\varepsilon} \right)$$

where we use the Hankel function $H^{(1)}_{\nu}(\zeta) = J_{\nu}(\zeta) + iY_{\nu}(\zeta)$, the identity

$$H^{(1)}_{\nu}(\zeta) = \frac{i}{\sin(\nu\pi)} (e^{-\nu\pi i}J_{\nu}(\zeta) - J_{-\nu}(\zeta)) \tag{5.5}$$
compare with (2.15) [1], and the fact that the term involving $J_{z+\frac{1}{2}}(\frac{z}{\epsilon})$ is sub-leading as $z \to +\infty$, hence inconsequential for the computation of the formal residue (5.3). Inserting (5.4) in (5.3) we obtain

$$\frac{\partial}{\partial T_t} \log \tau(T) \bigg|_{T=0} = - \text{res}_{z=\infty} \left( \frac{\pi}{\epsilon \cos(\pi z)} \begin{pmatrix} J_{z+\frac{1}{2}}(\frac{z}{\epsilon}) & J_{z-\frac{1}{2}}(\frac{z}{\epsilon}) \end{pmatrix} \begin{pmatrix} \frac{\partial z^2 J_{z+\frac{1}{2}}(\frac{z}{\epsilon})}{\partial T_t z} & \frac{\partial z^2 J_{z-\frac{1}{2}}(\frac{z}{\epsilon})}{\partial T_t z} \end{pmatrix} + \log(\epsilon z) \right) z^1 dz$$

$$= - \text{res}_{z=\infty} \epsilon S_1(z) z^1 dz = \frac{\ell!}{\epsilon^{\ell-1}} \langle \tau_{\ell-1} \rangle_{p_1,d}$$

where we used (1.30) and (1.33). This proves Prop. 1.7 for $n = 1$.

In order to proceed with higher order derivatives, we first note that we have a compatible system of ODEs of the form

$$\frac{\partial \Gamma}{\partial T_t} = M_t \Gamma - z^\ell \Gamma E_{22}, \quad \frac{\partial M_m}{\partial T_t} \frac{\partial M_t}{\partial T_m} = [M_t, M_m]$$

where $M_t = M_t(z; T)$ is a polynomial of degree $\ell$ in $z$

$$M_t(z) := \text{res}_{w=\infty} \frac{\Gamma(w; T) E_{22}^{-1}(w; T)}{w-z} w^\ell dw = \text{res}_{w=\infty} \frac{U(w; T)}{w-z} w^\ell dw, \quad \ell \geq 1$$

where

$$U(z; T) := \Gamma(z; T) E_{22}^{-1}(z; T).$$

This fact follows by a standard application of the Liouville theorem. The matrix $\Gamma e^{\ell E_{22}}$ is piecewise analytic in the complex $z$-plane and satisfies jump conditions independent of $T$ along $\Sigma'$. Hence the ratio $\frac{\partial}{\partial T_t} \left( \Gamma e^{\ell E_{22}} \right) \left( \Gamma e^{\ell E_{22}} \right)^{-1} =: M_t$ is analytic in $z$ everywhere and grows like a polynomial of degree $\ell$ at $z = \infty$. It follows that $M_t$ can be found as the polynomial part of the expansion at $z = \infty$, as in (10.1).

Then we compute second derivatives of $\log \tau(T)$, using the cyclic property of the trace and denoting $\tau := \frac{\partial \Gamma}{\partial T_t}$

$$\frac{\partial}{\partial T_t} \frac{\partial}{\partial T_t} \log \tau(T) = - \text{res}_{z=\infty} \text{tr} \left( \frac{\partial}{\partial T_t} (\Gamma^{-1}(z_1; T) \Gamma'(z_1; T) E_{22}) \right) z_1^\ell dz_1$$

$$\quad - \text{res}_{z_1=\infty} \text{tr} \left( \Gamma^{-1}(z_1; T) M'_t(z_1; T) E_{22} \Gamma(z_1; T) \right) z_1^\ell dz_1 + \text{res}_{z_1=\infty} \left( \ell_2 z_1^{\ell_2-1} \right) dz_1$$

$$\quad = \text{res}_{z_1=\infty} \text{res}_{z_2=\infty} \text{tr} \left( U(z_1; T) U(z_2; T) \right) - \frac{1}{(z_1 - z_2)^2} z_1^\ell z_2^\ell dz_1 dz_2.$$

**Lemma 5.1.** **In the sense of asymptotic expansions at $z = \infty$, we have**

$$U(z; T) \bigg|_{T=0} = \sigma_1 R(z) \sigma_1, \quad \sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

**Proof.** Using the notation of (3.16) we compute

$$U(z; T) \bigg|_{T=0} = \Gamma_0(z) E_{22} \Gamma_0^{-1}(z) = \begin{pmatrix} b(z) & a(z) \\ b(z-1) & a(z) \end{pmatrix}$$

and so the proof is complete by comparing with (5.4).

Comparing with (1.31) and (1.33) for $n = 2$ we conclude that Prop. 1.7 is also true for $n = 2$.

To complete the proof of Prop. 1.7 we state the next lemma. We omit its proof as it is based on algebraic manipulations by induction that have appeared several times in the literature; e.g. we refer the reader to [13, 15, 14].
Lemma 5.2. We have
\[
\frac{\partial^n \log \tau(T)}{\partial T_1 \cdots \partial T_n} = (-1)^n \text{res}_{z_1=\infty} \cdots \text{res}_{z_n=\infty} \left( \frac{1}{n} \sum_{\sigma \in \Theta_n} \text{tr} \left( U(z_{\sigma(1)}; T) \cdots U(z_{\sigma(n)}; T) \right) - \frac{\delta_{n,2}}{(z_1 - z_2)^2} \right) dz_1 \cdots dz_n.
\]

Finally, and the proof of Prop. 1.7 is complete by setting $T = 0$, applying Lemma 5.1, and comparing with (1.31) and (1.33).

A Proof of Lemma 2.2

It is convenient to introduce
\[
\hat{f}(z) := \int_{C_1} \exp \left( \frac{1}{\epsilon} \left( x - \frac{1}{x} \right) - \left( z + \frac{3}{2} \right) \log x \right) dx,
\]
\[
\hat{g}(z) := \int_{C_2} \exp \left( \frac{1}{\epsilon} \left( x - \frac{1}{x} \right) - \left( z + \frac{3}{2} \right) \log x \right) dx.
\]

Asymptotics for $\hat{g}$. Let us write $\xi := \epsilon \left( z + \frac{1}{2} \right)$ so that
\[
\hat{g}(z-1) = \int_{C_2} e^{\frac{i\pi}{4} \left( \epsilon - \frac{\xi}{2} \log x \right)} dx = \left| \xi \right| e^{-\frac{\xi}{2} \log |\xi|} \int_{C_2} e^{\frac{i\pi}{4} \left( x - \epsilon \log x \right)} e^{-\frac{\pi}{4} \epsilon x} dx (A.1)
\]
where $\xi = |\xi| e^{i\theta}$, $|\theta| < \pi$; in the second equality we performed the change of variable $x \mapsto x|\xi|$ and applied Cauchy theorem to deform the contour $|\xi|^{-1} C_0$ back to $C_0$. Since $C_0$ stays at a bounded distance from $x = 0$, we can apply Fubini theorem and write
\[
\int_{C_0} e^{\frac{i\pi}{4} \left( x - \epsilon \log x \right)} e^{-\frac{\pi}{4} \epsilon x} dx = \sum_{j \geq 0} \frac{(-1)^j}{j! \epsilon^j |\xi|^j} \int_{C_0} \frac{1}{x^j} e^{\frac{i\pi}{4} \left( x - \epsilon \log x \right)} dx. (A.2)
\]

We study each integral in the series in right hand side of (A.2) by the steepest descent method. The phase is $\varphi(x) := x - \epsilon \log x$, which has one saddle point at $x = e^{i\theta}$. Expanding $\varphi(x) = e^{i\theta} (1 - i\theta) + \frac{\epsilon}{2} \theta^2 (x - e^{i\theta})^2 + O((x - e^{i\theta})^3)$ we see that the steepest descent direction is $\frac{x - e^{i\theta}}{2}$.

For all $|\theta| < \pi$, the contour $C_0$ can be deformed to the steepest descent contour $\Im \varphi(x) = \Im e^{i\theta}$ in the vicinity of $x = e^{i\theta}$ in such a way that the main contribution to the integral for large $|\xi|$ comes from the neighborhood of the saddle point (see Fig. 3), and is computed by the gaussian integral;
\[
\hat{g}(z-1) \sim |\xi| e^{-\frac{z}{2} \log |\xi|} \sum_{j \geq 0} \frac{(-1)^j}{j! \epsilon^j |\xi|^j} \int_{e^{i\theta} + \epsilon \frac{\pi}{\sqrt{2}}} e^{\frac{i\pi}{2} \epsilon \frac{\pi}{\sqrt{2}} z} e^{-\frac{\pi}{4} \epsilon x} dx
\]
\[
= i \sqrt{2\pi \epsilon |\xi|} e^{-\frac{|\xi|}{2} \log |\xi|-1} (1 + O(|\xi|^{-1})).
\]

Finally, we recall $\xi = \epsilon \left( z + \frac{1}{2} \right)$ and so $\sqrt{2\pi \epsilon |\xi|} e^{-\frac{|\xi|}{2} \log |\xi|-1} \sim \left( \frac{\epsilon}{e} \right)^{-\frac{z}{2}}$.

This completes the proof of the asymptotic for $g(z-1)$.

Asymptotics for $\hat{f}$. Let us write $\xi := \epsilon \left( z + \frac{1}{2} \right)$ and divide the contour $C_1$ in $C_1^{in} := C_1 \cap \{|x| \leq 1\}$ and $C_1^{out} := C_1 \cap \{|x| \geq 1\}$. Performing two different scalings $x \mapsto x|\xi|^\frac{1}{2}$ we have
\[
\hat{f}(z) = \frac{\epsilon^{\frac{1}{2} \log |\xi|}}{|\xi|} \int_{|\xi| C_1^{in}} e^{\frac{i\pi}{4} \left( \frac{\xi}{2} + \epsilon \log x \right)} e^{-\frac{\pi}{4} \epsilon \sqrt{x}} dx + |\xi| e^{-\frac{z}{2} \log |\xi|} \int_{|\xi|^{-1} C_1^{out}} e^{\frac{i\pi}{4} \left( x - \epsilon \log x \right)} e^{-\frac{\pi}{4} \epsilon x} dx (A.3)
\]
where $\xi = |\xi| e^{i\theta}$, $|\theta| < \pi$. Applying Fubini theorem, the first integral is

$$\int_{|\xi|C^1_{n}} e^{-\frac{|\xi|^2}{2}(1+e^{i\theta} \log x)} e^{\frac{\pi i}{2} |\xi|} dX = \sum_{j \geq 0} \frac{1}{j! |\xi|^j} \int_{|\xi|C^1_{n}} x^j e^{-\frac{|\xi|^2}{2}(1+e^{i\theta} \log x)} dX$$  \hspace{1cm} (A.4)

and the second one is also written similarly as in (A.2).

We study each integral in the series in the right hand side of (A.4) by the steepest descend method. The phase is $\varphi(x) = \frac{1}{2} - e^{i\theta} \log x$, which has one saddle point at $x = e^{-i\theta}$. Expanding $\varphi(x) = e^{i\theta}(1 - i\theta) + \frac{\xi \theta}{2}(x - e^{-i\theta})^2 + O((x - e^{-i\theta})^3)$ we see that the steepest descent direction is $-\frac{3\theta}{2}$.

Let us restrict attention to $|\theta| < \frac{\pi}{2}$. (A.5)

The contour $C_1$ can be deformed so that $|\xi|C^1_{n}$ coincides with the steepest descent path in the vicinity of the saddle point $e^{-i\theta}$ (see Fig. 4), therefore giving the contribution

$$\frac{e^{\frac{\pi i}{2} (\log |\xi| - 1 + i\theta)}}{|\xi|} \sum_{j \geq 0} \frac{(e^{-i\theta})^j}{j! |\xi|^j} \int_{e^{-\theta} + e^{i\theta} \log R} e^{-\frac{|\xi|^2}{2}(x - e^{-i\theta})^2} dx = \frac{\sqrt{2\pi(\xi^2 + e^{\frac{\pi i}{2} (\log |\xi| - 1)}(1 + O(|\xi|^{-1}) (A.6)

where we recall that $\xi = \epsilon (z + \frac{3}{2})$ so that $\xi^{-\frac{1}{2}} e^{\frac{\pi i}{2} (\log |\xi| - 1)} \sim \left(\frac{\epsilon z}{2}\right)^{\frac{1}{2}}$. The contribution from the other term, relative to the contour $|\xi|^{-1}C^1_{out}$, is computed similarly as above for $g$ and is subleading with respect to (A.6), as long as we restrict to the range (A.5).

This completes the proof of the asymptotics for $f$.
When $\theta = -\frac{3}{4}\pi$ it is convenient to move the branch cut of $\log$. Level lines of the real part of the phase are also shown. In all cases it is clear how to deform $|\xi|^2$ to the steepest descent contour in the vicinity of the saddle point, so that the contributions from the tails at zero and infinity are exponentially smaller than the saddle point approximation.

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