Quantization via Star Products

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We study quantization via star products. We investigate a quantization scheme in which a quantum theory is described entirely in terms of the function space without reference to a Hilbert space, unlike the formulation employing the Wigner functions. The associative law plays an essential role in excluding the unwanted solutions to the stargen-value equation. This is demonstrated explicitly with the $D$-dimensional harmonic oscillator.
1 Introduction

In 1949, Moyal discovered an interesting description of quantum mechanics,[1] called Moyal quantization, in which the Wigner function[2] is used as the phase space distribution function, similar to that used in statistical mechanics. (See Ref.[3] for an early work on quantization in phase space.) The transformation function connecting the distribution functions at different times satisfies a differential equation similar to the classical equations of motion, where the Poisson bracket is replaced by the Moyal bracket.

After several decades, an abstract scheme of the Moyal quantization called deformation quantization was proposed.[4] This scheme is based on the general star product satisfying some axioms and on a deformation of the symplectic structure of the phase space in the classical theory, and it is formulated without a Hilbert space. Because one gets the Poisson bracket in the limit $\hbar \to 0$, this approach makes it possible to understand the connection between the classical and quantum theories from a new perspective. This scheme is believed to be equivalent to ordinary quantum mechanics and seems to effectively include functions that are written in the form of the (off-diagonal) Wigner functions or their linear combinations. There has been much progress in this field, especially with regard to its mathematical aspects.[5, 6, 7]

In the present paper we set up a general framework for quantization, which is slightly different from the deformation quantization. In this formulation, the state function is not necessarily the Wigner function that is constructed from vectors of the Hilbert space. We construct this scheme without resort to the Hilbert space. In our scheme, the associativity of the star product is used in defining the basic function space. This contrasts with the situation regarding the deformation quantization, where the associativity is a guiding principle in deforming the classical symplectic structure of the phase space. In our scheme, the functions describing physical states do not necessarily have counterparts in ordinary quantum mechanics. For example, unwanted states, such as those with negative energies, can appear if we do not impose the associativity of the star product. These state functions are excluded by associativity. For this reason, our theory is conjectured to be equivalent to quantum mechanics. This equivalence is demonstrated explicitly for the $D$-dimensional harmonic oscillator.

2 Assumptions and function space

Let us consider a dynamical system described by real coordinates, $q_i (i = 1, \ldots, N)$, and their canonical momenta, $p_i$. We make the following assumptions (1)-(6).

(1) There exist a function space $S$ of complex valued and integrable functions of $(q, p)$ and a map $\star : S \times S \to S$ satisfying the distributive and associative laws, the Hermiticity property, $(f_1 \star f_2)^\ast = f_2 \star f_1^\ast$ for $f_i \in S$, and the property $1 \star f = f \star 1 = f$ for $f \in S$. We postulate that the pair $(\star, S)$ consisting of this operator (which refer to as the star product hereafter) and
this function space is given, and physical states and observables are described by functions in \( S \).

(2) There exists a set of observables, \( \mathcal{O} = \{ O_1, O_2, \ldots, O_r \} \), that are mutually commuting with respect to the star product. Here \( r \) may or may not be finite.

We can define the eigenvalue problem associated with the star product as in the theory of vector spaces. For functions \( g^i \in S \) the function \( f_{\{\lambda \}\{\lambda'\}} \in S \) satisfying

\[
\begin{align*}
g^i \ast f_{\{\lambda \}\{\lambda'\}} &= \lambda^i f_{\{\lambda \}\{\lambda'\}}, \\
f_{\{\lambda \}\{\lambda'\}} \ast g^i &= \lambda'^i f_{\{\lambda \}\{\lambda'\}}
\end{align*}
\]

is called the phasespace eigenfunction[1] of \( g^i \), and \( \{ \lambda \} (\{ \lambda' \}) \) is called the set of left (right) eigenvalues. The above equations are sometimes called the stargen-value equations with respect to \( g^i \). For observables \( O^i \) and an observable or a state function \( \phi \in S \), let us define

\[
\phi_{\{\lambda \}\{\lambda'\}}^{(O)} := \int \frac{dq dp}{(2\pi\hbar)^N} \phi(q,p) \ast O_{\{\lambda'\}\{\lambda\}}(q,p),
\]

where \( O_{\{\lambda'\}\{\lambda\}} \) is an eigenfunction of \( O^i \). We call this quantity the matrix element of \( \phi \) with respect to the observables \( O^i \). Note the order of the subscripts in the above equation.

In terms of the above quantities, we make the following assumptions (3)-(5) concerning measurements.

(3) An ideal measurement of a set of observables \( O_i \) results in a specific set of values, and we assume each of these values to be one of left-right eigenvalues of \( O_i \):

\[
O_i \ast O_{\{\lambda \}\{\lambda\}} = O_{\{\lambda \}\{\lambda\}} \ast O = \lambda_i O_{\{\lambda \}\{\lambda\}}.
\]

(4) If the state is described by \( \Psi \), the probability of obtaining a particular set of values \( \lambda \) as a result of the measurement of observables \( O^i \in S \) is

\[
P(\Psi; O; \{ \lambda \}) = \Psi_{\{\lambda \}\{\lambda \}}^{(O)},
\]

where the r.h.s. is defined by (2.3). We assume that the state function \( \Psi(q,p) \) and the eigenfunction \( O_{\{\lambda \}\{\lambda\}} \in S \) are normalized as

\[
\int \frac{dq dp}{(2\pi\hbar)^N} O_{\{\lambda \}\{\lambda'\}}(q,p) = \delta_{\{\lambda \}\{\lambda'\}},
\]

\[
\int \frac{dq dp}{(2\pi\hbar)^N} \Psi(q,p) = 1,
\]

where the r.h.s. of (2.6) is the Kronecker delta or the delta function in a discrete or continuous spectrum.

(5) After the measurement of an observable, the state reduces to the state described by a stargen-value function of the observable with equal left and right stargen-values. In general, this function is not a linear combination of left-right stargen functions of unobserved quantities.
For a non-ideal observation, in which the observed value lies in a certain range, the probability is the sum of the r.h.s. of Eq. (2.5) over that range. Thanks to the Hermiticity of the star product, the probability (2.5) is a real number. Furthermore, it can be made positive definite by a proper choice of the star product.

Now, the physical state described by a state function $\Psi \in S$ that satisfies the condition

$$\Psi \star \Psi = \Psi$$

is called a pure state. A linear combination of state functions describing pure states is another state function. Because the probability (2.5) is linear with respect to the state function, a linear combination does not respect ‘quantum mechanical’ coherence, and for this reason, we call it an incoherent mixture of these states. Presumably, there may exist other state functions that are neither pure states nor incoherent mixtures and have no counterparts in ordinary quantum mechanics. In the $D$-dimensional harmonic oscillator treated in Section 3, some solutions to the star-generating equation correspond to neither pure states nor mixtures and violate the associative law of the star product. Because of this violation these states do not describe physical states.

The time development of our abstract system is determined by the following assumption.

(6) A physical state function $\Psi$ satisfies the Schrödinger-Moyal equation,

$$\frac{\partial}{\partial t} \Psi = -\{\Psi, H\}_M,$$  (2.9)

where

$$\{A, B\}_M := \frac{1}{i\hbar} (A \star B - B \star A)$$

is the Moyal bracket of $A$ and $B$, and $H \in S$ is called the Hamiltonian of the system. The observables are assumed to be time independent.

These assumptions define the scheme of our star quantization. The concrete procedure of obtaining the physical predictions is as follows.

(i) Define a suitable star product operation in the function space of well-behaved functions of the phasespace, which may have some singularities.

(ii) Define the maximal set of observables, including the Hamiltonian, which are functions of the phasespace.

(iii) Solve the star-generating equations for the maximal set of observables, which may have some singularities but should satisfy the normalization conditions.

(iv) In the function space spanned by the solutions obtained in (iii) and the functions of all the observables, determine the maximal function space, $S$, in which any function satisfies the assumptions for the star product.

(v) A measurement of an observable determines the initial state function through the assumption (5).

(vi) Solve the Schrödinger-Moyal equation to get the state function at a later time.
The probability of getting a specific value of an observable at that time is obtained by Eq. (2.5).

The above framework resembles that of the deformation quantization scheme in formal appearance. The difference between the two schemes lies in the definition of the physical states. In both schemes, the physical spectrum is assumed to be determined by the left-right stargen-values of observables consisting of a maximal commuting set. The function describing the state after a measurement that consists of a subset of the maximal set of commuting observables is a left-right stargen-value function of the observable that is actually measured. In the deformation quantization scheme, however, this function is assumed to be a linear combination of the left-right stargen-value functions of all observables, some of which may not have been measured. Contrastingly, our scheme requires that the function be a left-right stargen-value function of only those observables that are actually measured. The function space of our scheme is larger than that of the deformation quantization, and our function space may contain functions possessing singularities, as long as they satisfy the normalization conditions. In particular, we can consider functions that are not (linear combinations of) the off-diagonal Wigner functions. Therefore it is not obvious whether the associativity of the star product applied to those functions holds (see below).

An example of the star product is that defined by Groenewold:[1, 8]

\[
A \star B(q, p) = e^{i\frac{\hbar}{2}\sum_i \left( \frac{\partial A}{\partial q_i} \frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i} \frac{\partial B}{\partial q_i} \right)} A(q, p) B(q', p') \bigg|_{q=q', p=p'} .
\] (2.11)

In what follows we do not consider other choices for the star product, dealing only with (2.11).

Although our framework is self-contained and requires no reference to the Hilbert space of ordinary quantization, we review, for the sake of comparison, some relations between the Hilbert space and the function space of the phasespace. For an operator \( \hat{A} \) of the Hilbert space a function of the phasespace is defined by the Weyl correspondence:[9]

\[
A_W(q, p) = \int du \, e^{-i\frac{\pi}{\hbar}u} \langle q + \frac{u}{2} | \hat{A} | q - \frac{u}{2} \rangle .
\] (2.12)

Conversely, we see

\[
\langle q | \hat{A} | q' \rangle = \int \frac{dp}{(2\pi\hbar)^N} \, e^{i\pi(p-q')/\hbar} A_W \left( \frac{q + q'}{2}, p \right) .
\] (2.13)

These relations, however, do not always define a one to one map between the operators of the Hilbert space and functions of the phasespace. For example, the integral (2.13) for the function \( A_W = (q + ip)^{-1} \) has a singularity along \( q + q' = 0 \), which makes it impossible to define the corresponding operator in the Hilbert space. (For the case of systems with constraints, see Ref.[10].)

The Weyl correspondence is used in deriving various relations between quantities in ordinary quantum mechanics and those in the Moyal scheme. For example, the phasespace
eigenfunction of an observable \( A \) is \( |n⟩⟨m| \), where \( |n⟩ \) is the eigenvector of \( \hat{A} \), and the Wigner function \( \Psi(q,p) \) is defined in terms of the state vector \( |ψ⟩ \) as \( \Psi(q,p) = ⟨ψ|ψ⟩ \). The correspondence between operator multiplication and the star product is given by the relation

\[
(\hat{A}\hat{B})_W = A_W \star B_W. \tag{2.14}
\]

Finally, we comment on the associativity of the star product. The associative law of the star product would seem to be guaranteed by the corresponding operator relation of the quantum mechanics through the relation (2.14). However, this is not always the case, as there might exist functions of the phase space that have no operator counterparts.

### 3 Application to the harmonic oscillator

The Schrödinger equation for the one-dimensional harmonic oscillator described by the Hamiltonian \( H_1 = (p^2 + q^2)/2 \) has solutions with positive energy eigenvalues expressed in terms of Hermite polynomials. There also exist solutions with negative energies, but they cannot be normalized and therefore are excluded. The eigenfunctions \( w_{nm} = |n⟩⟨m| \) corresponding to solutions with positive energies, i.e., those for which \( n, m \geq 0 \), were calculated long ago [8] and found to be expressed in terms of the associated Laguerre polynomials:

\[
w_{nm} \propto a^{m-n} f_0(z)L_n^{(m-n)}(z), \quad f_0(z) = 2e^{-z/2}, \tag{3.1}
\]

where \( a = (q + ip)/\sqrt{2} \), \( z = 4H_1/\hbar \). These functions are solutions to the stargen-value equations

\[
H_1 \star w_{nm} = E_n w_{nm}, \quad w_{nm} \star H_1 = E_m w_{nm}, \tag{3.2}
\]

with \( E_n = (n + 1/2)\hbar \). (For calculations of the Wigner functions in various models, see Ref.[11].)

Now, let us examine the solutions that are not necessarily off-diagonal Wigner functions but satisfy the stargen-value equations with normalization conditions. For example, the functions \( w_{nm} \) with \( m < 0 \) satisfy the stargen-value equation. In the one-dimensional case, however, the observed energies are the right and left eigenvalues of the stargen-value equations, and the values \( m < 0 \) are excluded. Contrastingly, in the \( D \)-dimensional case with \( D \geq 2 \), the total energy can be positive even if the sub-energies for some dimensions are negative. This is a peculiar feature of our scheme, whose basic function space has no reference to a Hilbert space, which is absent in the case of the ordinary quantum mechanics and in the deformation quantization scheme.

The Hamiltonian of the \( D \)-dimensional harmonic oscillator is

\[
H = \frac{1}{2} \sum_{i=1}^{D} (\dot{p}_i^2 + q_i^2). \tag{3.3}
\]
The stargen-value equations for the Hamiltonian are $H \star f = f \star H = E f$. For later convenience, we introduce the creation and annihilation coordinates:

$$a_i = \frac{1}{\sqrt{2}}(q_i + ip_i), \quad a_i^* = \frac{1}{\sqrt{2}}(q_i - ip_i), \quad z_i = \frac{4}{\hbar} a_i a_i^*. \quad (3.4)$$

The star product is written in terms of these coordinates as

$$A \star B(a, a^*) = e^{\frac{\hbar}{2} \sum_i \left( \frac{\partial}{\partial a_i} \frac{\partial}{\partial a_i^*} - \frac{\partial}{\partial a_i^*} \frac{\partial}{\partial a_i} \right)} A(a, a^*) B(a', a'^*) \bigg|_{a = a', a^* = a'^*}. \quad (3.5)$$

The solutions to the stargen-value equations are the direct products of the one-dimensional solutions, and are expressed as

$$f(a_i, z_i) = \prod_i (a_i)^{k_i} e^{-z_i/2} F_i(z_i), \quad (3.6)$$

$$E = \left( n + \frac{D}{2} \right) \hbar, \quad n = \sum_{i=1}^D n_i, \quad \sum_{i=1}^D k_i = 0, \quad (3.7)$$

where each $F_i(z_i)$ satisfies

$$z_i F_i'' + (k_i + 1 - z_i) F_i' + n_i F_i = 0. \quad (3.8)$$

Here, the $n_i$ are the separation constants of the $D$-dimensional equation, and the $k_i$ are powers of Laurent expansions and are integers.

The quantity $n_i$ is the quantum number of the left stargen-value equation, $H_i \star f = (n_i + 1/2) \hbar f$, of the sub-Hamiltonian $H_i = (q_i^2 + p_i^2)/2$ of the $i$-th dimension, and the quantity $n_i + k_i$ is the quantum number of the right stargen-value equation, $f \star H_i = (n_i + k_i + 1/2) \hbar f$.

At this stage, the sub-energy of each dimension is not necessarily positive.

The admissible solutions, whose normalization integrals over the range with large values of the coordinates are not divergent are determined in the appendix, and we see that the solutions for each dimension are (apart from the normalization constant) given by

$$W_{\nu \kappa}(a, z) = a^n f_0(z) F_{\nu \kappa}(z), \quad f_0(z) = 2e^{-z/2}, \quad (3.9)$$

with integers $\nu$ and $\kappa$, where

$$F_{\nu \kappa}(z) = \begin{cases} \frac{L^{(\kappa)}(z)}{z^\kappa L^{(-\kappa)}(z)} & \text{for } \nu \geq 0, \\ z^{-\kappa} L^{(-\kappa)}_{\nu + \kappa}(z) & \text{for } 0 > \nu \geq -\kappa. \end{cases} \quad (3.10)$$

The energy eigenvalues of these solutions are

$$E_{\text{left}} = \left( \nu + \frac{1}{2} \right) \hbar, \quad E_{\text{right}} = \left( \nu + \kappa + \frac{1}{2} \right) \hbar. \quad (3.11)$$
Note that the solutions with negative $\nu$ (and $\nu + \kappa \geq 0$) have negative left energies, though they have positive right energies. (Here we consider stargen-value equations with sub-Hamiltonians.)

One possible argument for excluding such undesirable functions may be to show the divergence of the normalization integrals over the singularity at the origin of the phasespace. Since the functions considered here have different right and left eigenvalues of the sub-Hamiltonian, the normalization integrals should vanish. Writing $a = r e^{i\theta}$, the integrals are of the form $\int e^{i\kappa r} r^{1-\kappa} f(r) dr d\theta$, where $f(r)$ is a regular function at the origin. Although the integrals vanish if they are first integrated over $\theta$, they are not well-defined for $\kappa > 0$, except in the case $\kappa = 1, \nu = -1$. This fact shows that we cannot exclude at least the undesirable functions $W_{-1,1} = a f_0 F_{-1,1}$ by the requirement of the normalization condition.

Another possible argument for selecting admissible functions is the requirement of the associativity of the star product for such functions. It can be verified that the star product operation is well-defined on the function space spanned by positive powers of $a$ and $a^*$ and the solutions (3.10) with $\nu \geq 0$. It can also be shown that the associativity rule is satisfied. Now, for the $\nu < 0$ case, we see that, for example,

$$a^* \ast W_{-1,1} = 2 f_0, \quad f_0 \ast a^* = 0, \quad f_0 \ast f_0 = f_0, \quad (3.12)$$

where $W_{-1,1}$ has negative left eigenvalues of the sub-Hamiltonian. From (3.12) we see

$$(f_0 \ast a^*) \ast W_{-1,1} = 0, \quad f_0 \ast (a^* \ast W_{-1,1}) = 2 f_0, \quad (3.13)$$

i.e., the violation of associativity. The origin of this is seen from the inverse Weyl map, (2.13), of $W_{-1,1}$, which diverges along $q + q' = 0$. Therefore, the requirement of associativity excludes the undesirable solutions that have well-defined normalization integrals. Similarly, other solutions having negative energies are shown to violate associativity. These functions cannot be written as linear combinations of functions satisfying (2.8).

The admissible observable functions are also determined by the requirement of associativity. For example, we see

$$(f_0 \ast a^*) \ast a^{*-1} = 0, \quad f_0 \ast (a^* \ast a^{*-1}) = f_0. \quad (3.14)$$

Similarly, all negative powers of $a$ and $a^*$ are shown to violate associativity.

Thus the basic function space, $\mathcal{S}$, of the $D$-dimensional harmonic oscillator consists of the function space spanned by the positive energy solutions to the stargen-value equations, supplemented by the observable functions generated by the positive powers of $a_i$ and $a_i^*$. The energy spectrum coincides with that of ordinary quantum mechanics. This offers evidence suggesting the equivalence of the present scheme and quantum mechanics.
Appendix

In order to find the admissible solutions to Eq. (3.8), we study the solutions to the confluent hypergeometric equation in general. The parameters $n_i$ appearing there can be non-integers at this stage. The solutions to the confluent hypergeometric equation,

$$\frac{d^2 u}{dz^2} + (\gamma - z) \frac{du}{dz} - \alpha u = 0,$$

(A.1)

are written symbolically with the confluent $\tilde{P}$ function as

$$u = \tilde{P} \left\{ \begin{array}{c} 0 \\ \alpha \\ 1 \end{array} \begin{array}{c} 0 \\ \gamma - \alpha \\ 1 - \gamma \end{array} \right\}.$$

(A.2)

This shows that the four explicit solutions are given by either functions that are analytic (except for poles) around $z = 0$ or such functions around $z = \infty$, which are the singularities of the equation. At most two of them are independent.

The solutions around $z = 0$ are given by $u_1 = F(\alpha, \gamma; z)$ and $u_2 = z^{1-\gamma} F(\alpha - \gamma + 1, 2 - \gamma; z)$ for $\gamma \notin Z_+$, where $Z_+$ is the set of positive integers (and, similarly, $Z_-$ represents the negative integers). (The case $\gamma \in Z_+$ will be discussed shortly.) Here, the hypergeometric function of fluent type $F(\alpha, \gamma; z)$ is defined by

$$F(\alpha, \gamma; z) = \sum_{i=0}^{\infty} \frac{(\alpha)_n}{(\gamma)_n} \frac{z^n}{n!}.$$

(A.3)

where $(\alpha)_n = \alpha (\alpha + 1) \cdots (\alpha + n - 1)$. We restrict ourselves to the case of integer $\gamma$ for the application of the present paper. The corresponding solutions constitute the basis of the Laurent expansions as mentioned in the main text.

Note that these solutions play the role of probability distribution functions when multiplied by $f_0 = e^{-z/2}$. Therefore we should impose the condition that the solutions diverge more slowly than $e^{z/2}$ as $z \to \infty$, so that their integrations over phasespace are finite, which turns out to restrict the parameter region. A sufficient condition to realize this is the truncation of the power series of $z$, and it is in fact necessary also, because otherwise they diverge as $e^z$ in the limit $z \to \infty$. This restricts the parameters such that either $\alpha \in Z_- \cup \{0\}$ or $\alpha - \gamma + 1 \in Z_- \cup \{0\}$. Introducing $n = -\alpha$ and $k = \gamma - 1$ (in accordance with (3.8)), the solutions can be written in terms of the associated Laguerre polynomials, $L_n^{(k)}(z)$, as

$$u_1 = F(-n, k+1; z) = L_n^{(k)}(z), \quad \text{for} \quad n \geq 0,$$

(A.4)

$$u_2 = z^{-k} F(-(n+k), 1-k; z) = z^{-k} L_{n+k}^{(-k)}(z), \quad \text{for} \quad n \geq -k,$$

(A.5)

where $k \leq -1$. These two solutions appear to be different, but they are in fact identical in the range $n \geq -k \geq 1$ which is seen by the identity

$$z^{-k} L_{n+k}^{(-k)}(z) = (-1)^k \frac{n!}{(n+k)!} L_n^{(k)}(z), \quad \text{for} \quad n, \quad n+k \geq 0.$$

(A.6)
In the case $\gamma \in \mathbb{Z}_+$, the solutions are $\hat{u}_1 = F(\alpha, \gamma; z)$ and $\hat{u}_2$, where $\hat{u}_2 = u_1 \log z + z^{\gamma} F^*$ for $\alpha \in \mathbb{Z} \cup \{0\} \cup \{1\}$ or $\alpha - \gamma + 1 \in \mathbb{Z}_+ \cup \{0\}$, and $\hat{u}_2 = z^{\gamma} F(\alpha - \gamma + 1, 2 - \gamma; z)$ otherwise. Here $F^*$ is an analytic function around $z = 0$. The solution containing $F^*$ is excluded, since it diverges as $e^z$ in the limit $z \to \infty$. Therefore, the admissible solutions are

$$
\hat{u}_1 = L^{(k)}_n(z), \quad \text{for } n \geq 0, \quad \text{(A.7)}
$$

$$
\hat{u}_2 = z^{-k} L^{(-k)}_{n+k}(z), \quad \text{for } 1 > n \geq -k, \quad \text{(A.8)}
$$

where $k \geq 0$.

Other solutions represented by Eq. (A.2) are those around $z = \infty$, which might include solutions having essential singularities at $z = 0$. One of these diverges as $e^z$ in the limit $z \to \infty$, and therefore we exclude it. Another solution is obtained by setting $u = (1/z)^\alpha f(1/z)$. By solving the equation for $f(1/z)$, we obtain the polynomial solutions obtained above.

We now summarize the solutions to Eq. (3.8). The solutions for $n_i \geq 0$ are the associated Laguerre polynomials, $L^{(k_i)}_{n_i}(z_i)$, and those for $0 > n_i \geq -k_i$ are $z^{-k_i} L^{(-k_i)}_{n_i+k}(z_i)$. Defining

$$
\tilde{L}^{(k)}_n(z) = \left\{ \begin{array}{ll}
L^{(k)}_n(z) & \text{for } n \geq 0, \\
z^{-k} F^{(-k)}_{n+k}(z) & \text{for } 0 > n \geq -k,
\end{array} \right. \quad \text{(A.9)}
$$

the general solution of the left stargen-value equation, with total energy

$$
E_{left} = \left( n + \frac{D}{2} \right) \hbar, \quad \text{(A.10)}
$$

is written as a linear combination of functions of the form

$$
W(n_1, \ldots, n_D; k_1, \ldots, k_D) = f_0(z)(a_1)^{k_1} \cdots (a_D)^{k_D} \tilde{L}^{(k_1)}_{n_1}(z_1) \cdots \tilde{L}^{(k_D)}_{n_D}(z_D), \quad \text{(A.11)}
$$

where $k_i$ are arbitrary integers and $n_i$ are integers greater than or equal to $-k_i$, satisfying $\sum_{i=1}^D n_i = n$. (For the case $D = 1$ see Ref.[12].)

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