Higher dimensional realizations of activated dynamic scaling at random quantum transitions

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Abstract

We show that many of the unusual properties of the one-dimensional random quantum Ising model are shared also by dilute quantum Ising systems in the vicinity of a certain quantum transition in any dimension $d > 1$. Thus, while these properties are not an artifact of $d = 1$, they do require special circumstances in higher dimensions.

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There is considerable current interest in the properties of phase transitions in random quantum systems. Experimentally accessible quantum transitions, such as the transition from an insulator to a metal [1] or a superconductor [2], and various magnetic-nonmagnetic transitions in heavy fermion compounds [3], high-$T_c$ cuprates [4], or insulating dipolar Ising magnets [5], often occur in situations with strong randomness, and are only poorly understood. Theoretically many authors [6–8] have analyzed the random Ising chain in a transverse field - perhaps the simplest random quantum system. In particular, Fisher [8] used a real space renormalization group (RG) approach to obtain a detailed description of the thermodynamics and static correlation functions in the vicinity of the critical point. The properties of the system were found to be very unusual as compared to conventional quantum critical points. Specifically, length scales were found to diverge logarithmically with energy scales, implying a dynamic critical exponent $z = \infty$. The transition was shown to be flanked on either side by “Griffiths” regions with a susceptibility diverging due to contributions from statistically rare fluctuations. There are very few reliable results on other random quantum transitions, especially in finite dimensions $d > 1$; thus it is important to understand if the anomalous properties of the random quantum Ising chain are a specialty of $d = 1$, or if there exist higher dimensional quantum transitions which share these properties. Numerical work on higher dimensional transverse field Ising spin glasses has found evidence for the presence of Griffiths regions, but the dynamic scaling at the critical point seems conventional [9].

In this paper, we provide a simple example of such anomalous scaling in higher dimensional random quantum Ising systems. We consider bond or site diluted Ising models with short range interactions. As was first suggested by Harris [10], and as we argue below, these models have two quantum transitions (See Fig [1]): At low dilution, below the percolation threshold, there is a phase transition when the long range ferromagnetic order is destroyed by increasing the transverse field. This is expected to be in the universality class of the generic random bond quantum Ising transition. Right at the percolation threshold, there is a finite range of transverse field strengths at which the system remains critical. There is thus another quantum transition, across the percolation threshold, at low but non-zero
transverse field strengths, which is potentially in a different universality class. These two critical lines meet at a multicritical point \((M)\). The properties of the second transition, at the percolation threshold, are determined largely by the statistics and geometry of the percolating clusters about which much information is available. This permits us to make definitive statements about the scaling properties of this transition in any dimension. We show that the dynamic scaling is activated, with length scales diverging logarithmically with energy scales. We also demonstrate the existence of Griffiths phases on either side with diverging susceptibility. Our approach shows clearly the connection between these behaviors, and \(T = 0\) phase transitions at which quantum fluctuations are “dangerously irrelevant” \([8,11]\); indeed, various exponents of the transition are given by those of the classical percolation transition, and are hence known either exactly or numerically. We also obtain bounds on exponents characterizing the multicritical point.

For concreteness, we consider bond-diluted Ising models defined by the Hamiltonian

\[ H = - \sum_{\langle ij \rangle} J_{ij} \sigma_i^z \sigma_j^z - \sum_i h \sigma_i^x \]  

where \(\langle ij \rangle\) labels the nearest-neighbor sites of a \(d\)-dimensional lattice \((d > 1)\), and \(J_{ij}\) equals 0 with probability \(p\) and equals \(J > 0\), with probability \(1 - p\). The transverse field \(h\) is taken to be non-random, although our results remain valid for weakly random distributions as well. At \(p = 0\), as \(h\) is increased, there is a zero temperature phase transition from an ordered ground state to a disordered one (See Fig 1). On the other axis, when \(h = 0\), so that the system is classical, there is a percolation transition at \(p = p_c\). For \(p < p_c\), there is a thermodynamically large connected cluster, while for \(p > p_c\), there only are finite connected clusters. For \(p < p_c\), for small enough \(h\), the system retains long range order. This is ultimately destroyed for some \(h > h_c(p)\), with \(h_c(p)\) expected to be a monotonically decreasing function of \(p\). On the other hand, if \(p > p_c\), there is no long range order for any \(h\).

Now consider \(p = p_c\). Again there is zero magnetization and no long range order for any \(h\). However the system stays critical for \(h < h_M = h_c(p_c)\) (Fig 1). To see this, note that,
although there is no thermodynamically large connected cluster at $p_c$, there still is an infinite connected cluster with a fractal dimension smaller than the spatial dimension. The spins on this cluster align together at $h = 0$. A small but non-zero $h$ is not sufficient to destroy this order on the critical cluster. In fact, two spins on any sufficiently large finite cluster remain strongly correlated with each other for small $h$. The critical cluster eventually loses order when $h$ reaches $h_M$. To make this more precise consider the disorder averaged, equal time ($\tau$), two point spin correlation function. Spins at points 0 and $x$ are correlated only if they belong to the same cluster; thus $G(x) \equiv \langle \sigma^z(x, \tau = 0)\sigma^z(0, 0) \rangle - \langle \sigma^z(x, 0) \rangle \langle \sigma^z(0, 0) \rangle = \int dCC\mathcal{P}(0$ and $x$ belong to the same cluster) $\mathcal{P}(0$ and $x$ belong to the same cluster, they have correlation $C$), where angular brackets are averages over quantum/thermal fluctuations, square brackets represent disorder averages, and $\mathcal{P}(E)$ is the probability of the event $E$.

At $p = p_c$, the first probability $\sim x^{-d+2-\eta_p}$ for large $x$ [12]. The arguments above imply that the integral over $C$ is non-zero and independent of $x$ for large $x$ for $h < h_M$. Thus $G(x, p = p_c) \sim x^{-d+2-\eta_p}$ for large $x$ for $h < h_M$, and there is a critical line for $h < h_M$ at $p = p_c$.

In this paper, we will primarily focus attention on the transition across this critical line. We show that at $T = 0$, the properties of this transition are strikingly similar to the $d = 1$ results for the random quantum Ising chain [8]. First consider the equal-time two point function away from $p_c$. Arguing as in the previous paragraph, we get $G(x) \sim \mathcal{P}(0$ and $x$ belong to the same cluster) $\sim x^{-d+2-\eta_p}$ for large $x$ where the correlation length $\xi \sim |p - p_c|^{-\nu_p}$. Similarly, for $p < p_c$ the mean magnetization $\langle \sigma^z(x, 0) \rangle \sim \mathcal{P}(\text{any given site belongs to infinite cluster}) \simeq (p_c - p)\beta_p$. The exponents $\beta_p, \eta_p, \nu_p$ are those of classical percolation theory, and are known exactly in $d = 2$ and for $d > 6$. For any $d < 6$, they satisfy $\beta_p = \nu_p(d - 2 + \eta_p)/2$.

Now consider dynamic correlations. The imaginary part of the local dynamic susceptibility $[\chi''_L(\omega)] = \Sigma_N\mathcal{P}(\text{given site } i \text{ belongs to cluster of } N \text{ sites})\int ydy\mathcal{P}(\text{if site } i \text{ belongs to cluster of } N \text{ sites}, \chi''_I(\omega) = y)$. The energy levels of a cluster of $N$ sites can be described for $h \ll J$ as follows: The two lowest levels correspond to the states of a single effective Ising
spin with magnetic moment $\sim N$ in an effective transverse field $h_{\text{eff},N}$. For large $N$, $h_{\text{eff},N}$ can be estimated in $N^{\text{th}}$ order perturbation theory to be $\tilde{h} \exp(-cN)$. (In general, there would also be a prefactor that varies as a power of $N$; we will ignore this as it is subdominant to the exponential in $N$. When necessary, we will indicate the modifications induced by keeping this prefactor). The quantities $\tilde{h}$ and $c$ are of order $h$ and $\ln(J/h)$ respectively but their precise values depend on the particular cluster being considered. As the distribution of $\tilde{h}$ and $c$ is not expected to become very broad near the transition, we will replace them by their typical values $h_0$ and $c_0$ respectively. Apart from these two lowest levels, there are other levels separated from these by energies $\sim J$. These can be ignored for the low energy physics, and for small $\omega \ll h$, we only need to consider large clusters.

We now need the following results from percolation theory for the probability $P(N, p)$ that a given site belongs to a large cluster of $N$ sites. At $p = p_c$, $P(N, p_c) \sim N^{1-\tau}$ where $\tau$ equals $5/2$ for $d > 6$, and equals $1 + d/D$ for $d < 6$ where $D$ is the fractal dimension of the critical clusters. In particular $\tau = 187/91$ in $d = 2$, and is approximately $2.18$ in $d = 3$. Away from $p_c$, for $d < 6$ or $d > 8$, $P(N, p)$ satisfies the scaling form

$$P(N, p) \sim N^{1-\tau} g \left( N/\xi^D \right)$$

The scaling function $g(y)$ is universal but is different for $p < p_c$ and $p > p_c$; it approaches $1$ for $y \ll 1$, while for $y \gg 1$,

$$g(y, p > p_c) \sim y^{-\theta + \tau} e^{-c_+ y}$$
$$g(y, p < p_c) \sim y^{-\theta' + \tau} e^{-c_- y^{1/d}}$$

with $\theta = 1$, $3/2$ and $5/2$ for $d = 2, 3$ and $d > 8$ respectively, and $\theta' = 5/4$, $-1/9$ in $d = 2, 3$ respectively. The constants $c_{+, -}$ are of order unity. For $6 < d < 8$, $P(N, p)$ satisfies a more complicated two-variable scaling form. For simplicity, we shall not discuss this case here, though including it is straightforward.

Using these percolation results, we get for the dynamic susceptibility of the Ising model

$$[\chi''_L(\omega)] \sim \int \frac{dN}{N^{\tau-1}} g(N/\xi^D) \delta(\omega - h_0 e^{-c_0 N})$$
Note that the scaling variable is $\ln(1/\omega)/\xi^D$. This is a precise statement of the activated dynamic scaling mentioned earlier, which has thus been shown to occur in all $d > 1$ in the present model. Asymptotic forms in various limits can be obtained from the limiting behavior of $g(x)$ described above. For $p \geq p_c$, we get $\chi''_L \sim \omega^{-1+\alpha} (\ln(h_0/\omega))^{1-\tau}$ with $\alpha \sim \xi^{-D}$ so that at $p = p_c$, $\alpha = 0$ (Including the power-law prefactor in $h_{\text{eff},N}$ will only change the power of $\ln(1/\omega)$ in by $\sim \xi^{-D}$, and similarly for the prefactor in the expression for $p < p_c$ given below). Note that just on the disordered side, the system is gapless with a power-law density of states. The physical origin of this is, as usual, the presence of rare, large clusters with arbitrarily small energy gaps. For $p < p_c$, the presence of the infinite cluster (and the associated long range order) gives rise to a delta function at $\omega = 0$. For $\omega \neq 0$, $\chi''_L(\omega)$ is still determined by contributions from the finite clusters. Proceeding as before, we find $\chi''_L(\omega \neq 0) \sim (1/\omega) (\ln(h_0/\omega))^{1-\tau} \exp\left(-\kappa (\ln(h_0/\omega))^{1-1/d}\right)$ with $\kappa \sim \xi^{-D(1-1/d)}$. Again the system is gapless. The gaplessness of both the ordered and disordered phases in the vicinity of the transition is unlike quantum transitions in pure systems, but is probably generic to many random quantum transitions [8,15,9].

The magnetization in response to a uniform external applied magnetic field $H$ along the $\hat{z}$ direction can be calculated similarly. For small $H \ll h$, only large clusters contribute. The magnetization per site of a cluster of size $N$ is that of an Ising spin of magnetic moment $N$ in a transverse field $h_{\text{eff},N}$, and is therefore given by

$$M_N(H) = \frac{NH}{((NH)^2 + h_{\text{eff},N}^2)^{1/2}}$$

Thus the total magnetization per site (after subtracting the regular contribution of the infinite cluster for $p < p_c$) is

$$M(H) - M(H = 0) \sim \int dN \frac{1}{N^{\tau-1}} g(N/\xi^D) M_N(H)$$

The singular part therefore has the scaling form
\[ M_{\text{sing}}(H) \sim \frac{1}{(\ln(h_0/H))^{\tau - 2}} \Phi \left( c \frac{\ln(h_0/H)}{\xi^D} \right) \] (5)

with \( c \) a non-universal constant, and \( \Phi(y) \) a universal function which is related to \( g(y) \) by

\[ \Phi(y) = \int_1^\infty w^{1-\tau} dw g(wy) \]

Again note that the scaling variable is \( \ln(h_0/H)\xi^D \), which is unlike conventional critical behavior, but is similar to the \( d = 1 \) random quantum Ising chain [8].

We thus have the following asymptotic forms as \( H \to 0 \):

\[ M_{\text{sing}}(H) \sim \begin{cases} 
(\ln(h_0/H))^{2-\tau} & p = p_c \\
\xi^D(\ln(h_0/H))^{1-\theta}(H/h_0)^\gamma & p \geq p_c \\
\xi^D(1-1/d)(\ln(h_0/H))^{-\theta'+1-1/d}e^{-(\gamma' \ln(h_0/H))^{1-1/d}} & p \leq p_c 
\end{cases} \] (6)

with \( \gamma, \gamma' \sim \xi^{-D} \) (as with the dynamic susceptibility, including the power-law prefactor in \( h_{\text{eff},N} \) changes the power of \( \ln(1/H) \) for \( p > p_c \), and the prefactor for \( p < p_c \) by \( \sim \xi^{-D} \)). Note that the magnetization rises as a power of \( H \), with a continuously varying exponent which is smaller than 1 in a region of the disordered phase close to the transition. This is again similar to the \( d = 1 \) result, and gives rise to a divergent linear susceptibility throughout this region. In the ordered side \( dM/dH \sim 1/H \) with weak corrections. Thus the linear susceptibility diverges in the ordered side as well.

A further similarity with the \( d = 1 \) results can be found by generalizing the problem to systems with Potts symmetry. For the random \( q \)-state quantum Potts model in \( d = 1 \), it has been shown [16] that all the critical exponents and the scaling functions for the mean spatial correlations and magnetization are independent of \( q \), i.e., they are the same as for the Ising case. For the diluted models that we have considered here, it should be clear that a vertical critical line exists at the percolation threshold for all \( q \). The exponents and appropriate scaling functions of the corresponding quantum transitions are again independent of \( q \) as they are determined mainly by the geometric properties of the lattice near the percolation threshold. As in \( d = 1 \), all the \( q \) dependence is in non-universal quantities and in a high-energy cutoff limiting the regime of universal scaling behavior.
So far, we have focussed on the $T = 0$ scaling properties, and demonstrated the similarities with the $d = 1$ results. What about the finite $T$ properties? For the classical dilute Ising model at $p = p_c$, the correlation length at finite $T$ behaves as $\exp(\text{constant} / T)$ \[\text{(1)}\]. This is essentially due to the presence of one dimensional segments in the critical percolating clusters. For the quantum problem for $h < h_M$, these one dimensional segments would give rise to a thermal correlation length ($\xi_T$) with a similar exponential dependence on $1/T$, and a prefactor that is a power-law in $T$; this is the behavior in the “high $T$” region of Fig 2. In contrast, for the $d = 1$ random quantum Ising chain, the correlation length rises as a power of $\ln(1/T)$ at the critical couplings. Away from the critical point, the crossovers are as shown in Fig 2. The low $T$ behavior appears when $\xi \sim \xi_T$, or $T \sim \ln^{-1}(1/|p - p_c|)$. On the disordered side, the low $T$ system is described well as a collection of rigid Ising clusters with effective transverse fields and a size distribution as before; this leads, for instance, to a linear susceptibility $\chi_T \sim T^{-1+\kappa}$ (up to log corrections) with $\kappa \sim \xi^{-D}$. On the ordered side, there is a finite temperature phase transition; as in the classical case, as $p \nearrow p_c$, the transition temperature falls to zero as $T_c \sim \ln^{-1}(1/(p_c - p))$.

We now turn to the special point $M$ (Fig 4). Correlations at the point $M$ should decay faster than along the vertical critical line considered above. Thus, if $G_M(x) \sim x^{-\phi}$, then $\phi > d - 2 + \eta_p$. Similarly, as $p$ approaches $p_c$ from below at $h_M$, the magnetization should go to zero faster than for $h < h_M$. Thus $\beta_M > \beta_p$. A detailed theory of the scaling at this point we leave as an open question.

Finally it is interesting to ask if this vertical critical line at the percolation threshold continues to exist for systems with continuous symmetry, such as the $O(N)$ rotor models. For $N > 2$, the presence of one dimensional segments in the critical clusters implies rapidly decaying correlations in large clusters. Thus the vertical critical line will not be present for $O(N)$ models for $N > 2$. The $O(2)$ case is special as correlations only decay as a power-law in one dimension. Whether this is like the Ising or the $O(N > 2)$ system we leave as another open question.

To summarize, we have presented a simple example of a random quantum transition
in dimensions $d > 1$ which exhibits many of the properties of the transition in the $d = 1$ random quantum Ising chain. In particular, the dynamic scaling was activated, with $\ln(1/\text{energy scale}) \sim \xi^D$. Further, there were Griffiths regions on either side of the transition, with a singular density of states and a diverging susceptibility. It would be interesting to find experimental systems where this transition can be studied. (One possibility is the system $LiHo_xY_{1-x}F_4$ where, however, the presence of dipolar interactions may complicate the situation.) Theoretically, an important feature of this transition, as in the $d = 1$ system, is that it is controlled by a classical fixed point with quantum fluctuations being “dangerously irrelevant”. This feature is also found in quantum Ising models in a random longitudinal field, which undergo, for $d > 2$, a phase transition from an ordered to a paramagnetic phase at $T = 0$ \cite{17}. In fact, this fixed point also controls the non-zero $T$ phase transition, where it has been argued that the classical dynamic scaling is activated \cite{18}. Extending the argument to the quantum dynamics at the $T = 0$ transition, suggests that the quantum dynamic scaling would again be activated \cite{19}. Further, classical fixed points have also appeared in recent studies of the metal-insulator transition \cite{20}, and in the $T = 0$ onset of spin-glass order in metallic systems \cite{11}; whether, in these cases, the dynamic scaling is activated or conventional, due to the presence of itinerant fermions, is an interesting question, worthy of future investigation.

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FIG. 1. Phase diagram of the dilute Ising model in a transverse field ($h$) at zero temperature. The dilution probability is $p$. The multicritical point $M$ is at $p = p_c, h = h_M$. The quantum transition along the vertical phase boundary ($h < h_M, p = p_c$) is controlled by the classical percolation fixed point at $p = p_c, h = 0$; quantum effects (due to a non-zero $h$) are dangerously irrelevant, and lead to activated dynamic scaling near the $h < h_M, p = p_c$ line.
FIG. 2. Finite temperature phase diagram for $h < h_M$. The dashed lines ($T \sim 1/\ln(1/|p-p_c|)$) represent crossovers from the high $T$ regime, characterized by spin fluctuations on the critical infinite cluster, to the low $T$ regimes. The solid line for $p < p_c$ is the phase transition ($T = T_c \sim 1/\ln(1/(p_c-p))$) where long range order is destroyed.