SOBOLEV MAPPING OF SOME HOLOMORPHIC PROJECTIONS

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Abstract. Sobolev irregularity of the Bergman projection on a family of domains containing the Hartogs triangle is shown. On the Hartogs triangle itself, a sub-Bergman projection is shown to satisfy better Sobolev norm estimates than its Bergman projection.

1. Introduction

If $Ω \subset \mathbb{C}^n$ is an open set, $1 < p < \infty$, and $k \in \mathbb{Z}^+$, let $L^p_k(Ω)$ denote the usual $L^p$ Sobolev space of order $k$: the measurable functions $f$ such that

$$\|f\|_{L^p_k(Ω)} = \left( \sum_{|α| \leq k} \int_Ω |∂^α f|^p dV \right)^{\frac{1}{p}}$$

is finite, where derivatives are interpreted in the distributional sense.

This paper continues investigations from [19], [20] by demonstrating irregularity in the $L^p$ Sobolev spaces for the Bergman projection associated to the domains defined in (1.2). These generalize the Hartogs triangle, which is $H_1$ in (1.2).

The Bergman projection, $B = B_{Ω}$, orthogonally projects $L^2(Ω)$ onto the closed subspace $O(Ω) \cap L^2(Ω)$, $O(Ω)$ denoting holomorphic functions. On $L^2(Ω) = L^2_0(Ω)$, $B$ is represented as an integral operator

$$B f(z) = \int_Ω B_{Ω}(z, w) f(w) dV(w), \quad f \in L^2(Ω),$$

where $dV$ denotes Lebesgue measure and $B_{Ω}(z, w) \in O(Ω) \times \overline{O(Ω)}$ is the Bergman kernel. If $f \notin L^2(Ω)$ let (1.1) define $B f$, whenever the integral converges. For many classes of pseudoconvex domains, precise pointwise estimates on $B_{Ω}(z, w)$ were obtained and shown to imply $\|B f\|_{L^p_k(Ω)} \leq C \|f\|_{L^p_k(Ω)}$ for all $1 < p < \infty$ and $k \in \mathbb{Z}^+$. See [12, 23, 25, 26, 27, 30, 31]. Thus $B$ is $L^p_k$-regular in these cases. In the special case $p = 2$, regularity for all $k \in \mathbb{Z}^+$ was shown in [8] whenever $Ω$ has a plurisubharmonic defining function, without establishing pointwise estimates on $B_{Ω}(z, w)$. This result was generalized in [9, 24]. On the other hand, $L^2_k$ regularity does not always hold. Irregularity of $B$ in $L^2_k(Ω)$, for certain $Ω$, was discovered in connection to Condition $R$ of Bell-Ligocka [5, 6]. It is shown in [4] that $B$ is irregular on $L_k^2(W)$ for large $k$ on the pseudoconvex “worm” domains $W$ given in [16].

The irregularity of the Bergman projection demonstrated in [19, 20] is of a different kind. It occurs on the Lebesgue spaces $L^p(Ω) = L^p_0(Ω)$ for certain $p \neq 2$ and does not involve derivatives. For $γ > 0$, define

$$(1.2) \quad \mathbb{H}_γ = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^γ < |z_2| < 1\}.$$  

It is shown in [20] that the Bergman projection on $\mathbb{H}_γ$ (for any $γ$) is a degenerate $L^p$ operator, bounded only for $p$ in a proper subinterval of $(1, \infty)$. In particular, the situation on $\mathbb{H}_1$ is that $B : L^p(\mathbb{H}_1) \rightarrow L^p(\mathbb{H}_1)$ boundedly if and only if $p \in \left(\frac{4}{3}, 4\right)$. (See Theorem 3.1 below for the general statement.) This limited range of $L^p$ boundedness has consequences

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for approximation and duality theory in $O(\mathbb{H})$, see [10]. Similar consequences hold when irregularity can be characterized on norm scales other than $L^p$.

It turns out that $B$ is even more degenerate as an $L^p$ Sobolev map.

**Theorem 1.3.** Let $\mathbb{H}_\gamma$ be given by (1.2) and $B$ denote the Bergman projection on $\mathbb{H}_\gamma$.

1. Let $\gamma = 1$. Then $B : L^p_k(\mathbb{H}_1) \rightarrow L^p_k(\mathbb{H}_1)$ boundedly if and only if $p \in \left(\frac{1}{2}, 2\right)$. Additionally, if $k \geq 2$ is an integer, $B$ is unbounded on $L^p_k(\mathbb{H}_1)$ for all $p \in (1, \infty)$.

2. Let $\gamma \neq 1$ be any other positive number. Then if $k \geq 1$ is an integer, $B$ is unbounded on $L^p_k(\mathbb{H}_\gamma)$ for all $p \in (1, \infty)$.

There are other papers showing Bergman irregularity on $L^p_0(\Omega)$, for specific pseudoconvex $\Omega$: [1, 11, 13, 34]. A unifying result, explaining irregularity in these cases and [19, 20], is lacking. A weighted regularity result on $L^p_k(\mathbb{H}_\gamma)$ is often called the Forelli-Rudin lemma; see [21], [32], or [35] for the ‘standard’ proof, based on asymptotics of the gamma function. Different proofs are given in [19], [20], [13], which also address $\beta \neq 0$.

A proof of Theorem 1.3 requires handling how derivatives commute past the Bergman projection. An initial difficulty is that $\mathbb{H}_\gamma$ is not smoothly bounded, so Stokes’ theorem cannot be applied in the usual way, e.g., as in [28, Lemma 3], [29, Proposition 3.3], or [30, Lemma 5.1]. This is circumvented by applying Stokes theorem on appropriately chosen discs and annuli intersecting $\mathbb{H}_\gamma$.

Proofs of the regularity and irregularity statements in Theorem 1.3 proceed differently and are presented separately. Irregularity is proved first, as Proposition 3.6. After developing some general tools, regularity of $B$ on $L^p_k(\mathbb{H}_1)$ is proved as Corollary 4.23 in Section 4. To partially repair the irregularity described in Theorem 1.3, substitute operators related to $B$ are considered in section 5.

When $X$ and $Y$ are expressions involving several variables, write $X \lesssim Y$ to mean $X \leq CY$ for a constant $C$ independent of certain of these variables. The independence of which variables is specified in use. $X \approx Y$ means $X \lesssim Y \lesssim X$ holds.

**2. Sobolev Regularity in One Variable**

Let $D \subset \mathbb{C}$ denote the unit disc. The Bergman projection $B_D$ is bounded from $L^p_k(D) \rightarrow L^p_k(D)$ for all $1 < p < \infty$ and $k \in \mathbb{Z}^+$. This is well-known when $k = 0$, apparently first proved in [33] using singular integral operator theory; see [17, Chapter 2]. For any $k \in \mathbb{Z}^+$, a proof modeled on arguments in [28] is given below. This serves as a template for the proof of Theorem 1.3, part (1).

The Bergman kernel of $D$ is

\[
(2.1) \quad B_D(z, w) = \frac{1}{\pi} \sum_{j=0}^{\infty} (j + 1)(z \bar{w})^j = \frac{1}{\pi} \frac{1}{(1 - z \bar{w})^2}.
\]

Note $B_D(z, w)$ can be viewed as a function of $s = z \bar{w}$.

**2.1. $L^p_0$ boundedness.** A family of integral estimates will be used. When $\beta = 0$, the result is often called the Forelli-Rudin lemma; see [21], [32], or [35] for the ‘standard’ proof, based on asymptotics of the gamma function. Different proofs are given in [19], [20], [13], which also address $\beta \neq 0$.

**Lemma 2.2.** Let $D \subset \mathbb{C}$ be the unit disc, $\epsilon \in (0, 1)$ and $\beta < 2$. Then for $z \in D$,

\[
\int_D \frac{(1 - |w|^2)^{-\epsilon}}{|1 - z \bar{w}|^2} |w|^{-\beta} dV(w) \lesssim (1 - |z|^2)^{-\epsilon},
\]

for a constant $C = C(\beta, \epsilon)$ independent of $z$. 

Lemma 2.2 suffices to show \( L^p_0(D) \) boundedness of \( B_D \).

**Corollary 2.3.** The Bergman projection \( B_D \) maps \( L^p(D) \) to \( L^p(D) \) for all \( 1 < p < \infty \).

In fact, the operator whose kernel is \( |B_D(z, w)| \) is bounded on \( L^p(D) \) for \( 1 < p < \infty \).

**Proof.** This can be proved using the standard form of Schur’s Lemma, demonstrated, e.g., on page 184 of [28].

Alternately, Lemma 4.1 below can be used (note \( B_D(z, w) \) is conjugate symmetric). Let \( K(z, w) = |B_D(z, w)| \) and take \( h(w) = 1 - |w|^2 \) as the auxiliary function. Lemma 2.2 shows that estimate (4.2) in Lemma 4.1 holds for all \( 0 < \epsilon < 1 \). Lemma 4.1 then gives the claimed boundedness by setting \( \beta = 1 \) and sending \( \alpha \to 0^+ \). \( \square \)

### 2.2. Integration by parts

Define the vector field

\[
T_w = \bar{w} \frac{\partial}{\partial \bar{w}} - w \frac{\partial}{\partial w},
\]

and write \( T^k_w \) to mean \( T_w \circ \cdots \circ T_w \) composed \( k \) times. If \( f \in L^p_k(D) \), clearly \( T^k_w f \in L^p(D) \). If, in addition, \( f \in \mathcal{O}(D) \), a partial converse holds: if \( D_\delta = \{ z \in D : |z| > \delta \} \), then

\[
\| f \|_{L^p_k(D_{\delta})} \lesssim \| T^k_w f \|_{L^p(D_{\delta})}
\]

for a constant independent of \( f \). This holds since any first derivative can be written as a linear combination \( A T_w + B \frac{\partial}{\partial w} \) on \( D_\delta \), for bounded functions \( A \) and \( B \).

The crucial property \( T_w \) satisfies is

**Proposition 2.5.** \( T_w \) annihilates \( C^1 \) radial functions of \( w \in \mathbb{C} \).

**Proof.** A \( C^1 \) radial function \( g \) can be written as \( g(w) = f(|w|^2) \), where \( f \in C^1([0, \infty)) \). Therefore

\[
T_w g = \bar{w} f'(|w|^2) \cdot w - w f'(|w|^2) \cdot \bar{w} = 0.
\]

\( \square \)

Recall that \( r : \mathbb{C} \to \mathbb{R} \) is a defining function for \( \Omega \) if \( \{ r < 0 \} = \Omega \) and \( |\nabla r(w)| \neq 0 \) when \( r = 0 \). Proposition 2.5 implies, in particular, that \( T_w \) annihilates defining functions of discs and annuli centered at the origin along their boundaries. An integration by parts result follows:

**Proposition 2.6.** Let \( \Omega \subset \mathbb{C} \) be either a disc or an annulus centered at the origin. Then if \( f, g \in L^1(\Omega) \cap C(\overline{\Omega}) \),

\[
\int_{\Omega} T_w f \cdot g \, dV = - \int_{\Omega} f \cdot T_w g \, dV.
\]

\( ^1 \)A version of this also holds in several variables. See, e.g., [7, 3, 22] for a statement of the result, as well as elementary proofs for \( p = 2 \). For general \( p \), see [15].
Proof. Choose a defining function for $\Omega$ with $|\nabla r(w)| = 1$ for all $w \in b\Omega$. Stokes’ theorem yields
\[
\int_{\Omega} T_w f \cdot g \, dV = \int_{\Omega} \frac{\partial f}{\partial w}(w) \cdot \bar{w}g(w) \, dV(w) - \int_{\Omega} \frac{\partial f}{\partial w}(w) \cdot w g(w) \, dV(w)
\]
\[
= -\int_{\Omega} f \cdot \frac{\partial}{\partial w}(\bar{w}g(w)) \, dV + \int_{\partial \Omega} f \cdot \bar{w} \cdot \frac{\partial r}{\partial w} \, dS
\]
\[
+ \int_{\Omega} f \cdot \frac{\partial}{\partial w}(wg(w)) \, dV - \int_{\partial \Omega} f w g \cdot \frac{\partial r}{\partial w} \, dS
\]
\[
= -\int_{\Omega} f \cdot \left[ \frac{\partial}{\partial w}(\bar{w}g(w)) - \frac{\partial}{\partial w}(wg(w)) \right] \, dV + \int_{\partial \Omega} f g \cdot T_w r \, dS
\]
\[
= -\int_{\Omega} f \cdot T_w \bar{g} \, dV.
\]
Here $dS$ denotes induced surface measure on $b\Omega$. The last boundary integral vanishes since $T_w r = 0$ on $b\Omega$.

2.3. $L^p_k$ boundedness for $k > 0$.

**Theorem 2.7.** The Bergman projection $B_D$ is a bounded operator from $L^p_k(D) \to L^p_k(D)$ for all $k \in \mathbb{Z}^+$ and $1 < p < \infty$.

**Proof.** Fix $k, p$ and let $f \in L^p_k(D)$. Since $B_D f \in \mathcal{O}(\Omega)$, only holomorphic derivatives need to be estimated. For $z \neq 0$,
\[
\frac{\partial^k}{\partial z^k} B_D f(z) = \frac{\partial^k}{\partial z^k} \int_D B_D(z, w) f(w) \, dV(w)
\]
\[
= \int_D \frac{\partial^k}{\partial z^k} (B_D(z, w)) f(w) \, dV(w)
\]
\[
= \frac{1}{z^k} \int_D \bar{w}^k \frac{\partial^k}{\partial \bar{w}^k} (B_D(z, w)) f(w) \, dV(w).
\]
(2.8)

The last equality follows because $B_D(z, w)$ can be viewed as a function of the variable $s = z\bar{w}$. Define a new kernel $K_k(z, w)$, obtained by subtracting away the $(k-1)$-jet of $B_D(z, w)$ in the $s$ variable, i.e.,
\[
K_k(z, w) := \frac{1}{\pi^2} \left[ \frac{1}{(1-s)^2} - \sum_{j=0}^{k-1} (j+1) s^j \right] = \frac{1}{\pi^2} \frac{\partial}{\partial s} \left[ \sum_{j=k}^{\infty} s^{j+1} \right]
\]
\[
= \frac{(k+1)s^k - ks^{k+1}}{(1-s)^2}.
\]
(2.9)

Since $K_k(z, w)$ and $B_D(z, w)$ differ by terms annihilated by $\frac{\partial^k}{\partial \bar{w}^k}$ and $K_k(z, w)$ is antiholomorphic in $w$,

\[
= \frac{1}{z^k} \int_D \bar{w}^k \frac{\partial^k}{\partial \bar{w}^k} (K_k(z, w)) f(w) \, dV(w)
\]
\[
= \frac{1}{z^k} \int_D \bar{K}_w^k (K_k(z, w)) f(w) \, dV(w),
\]
\[
= \frac{(-1)^k}{z^k} \int_D K_k(z, w) T_w^k f(w) \, dV(w).
\]

The last equality follows from Proposition 2.6.
The modified kernel $K_k(z,w)$ satisfies a stronger estimate than $B_D(z,w)$. Indeed, equation (2.10) shows
\[ |K_k(z,w)| \lesssim \frac{|z|^k|w|^k}{|1-z\bar{w}|^2}, \]
for a constant independent of $z,w \in D$. This can be used to counteract the factor $\frac{1}{z^k}$ appearing in (2.8). Thus
\[ \left| \frac{\partial^k}{\partial z^k} B_D f(z) \right| \lesssim \int_D \frac{|w|^k}{|1-z\bar{w}|^2} T_w^k f(w) dV(w) \leq \int_D \frac{1}{|1-z\bar{w}|^2} T_w^k f(w) dV(w) \]
(2.10)
\[ \approx \int_D |B_D(z,w)| T_w^k f(w) dV(w). \]
Since $T_w^k f \in L^p(D)$, Corollary 2.3 says that (2.10) defines an $L^p(D)$ function. This implies $\frac{\partial^k}{\partial z^k} B_D f(z) \in L^p(D)$.

For any positive integer $l \leq k$, the same argument – but for the modified kernel $K_l(z,w)$ – shows $\frac{\partial^l}{\partial z^l} B_D f(z) \in L^p(D)$. Thus $B_D f \in L^p_k(D)$. \hfill \square

3. Sobolev irregularity

The starting point is the characterization of $L^p_\gamma$ boundedness of the Bergman projection on $\mathbb{H}_\gamma$.

**Theorem 3.1** ([20]). Let $\mathbb{H}_\gamma$ be defined in (1.2), $B$ denote the Bergman projection on $\mathbb{H}_\gamma$, and $1 < p < \infty$.

1. Let $\gamma = \frac{m}{n} \in \mathbb{Q}^+$, with $\text{gcd}(m,n) = 1$.
   Then $B : L^p(\mathbb{H}_\gamma) \to L^p(\mathbb{H}_\gamma)$ is bounded if and only if $p \in \left(\frac{2m+2n}{m+n+1}, \frac{2m+2n}{m+n-1}\right)$.

2. Let $\gamma > 0$ be irrational.
   Then $B : L^p(\mathbb{H}_\gamma) \to L^p(\mathbb{H}_\gamma)$ is bounded if and only if $p = 2$.

Let $(\lambda(m,n), \rho(m,n)) = \left(\frac{2m+2n}{m+n+1}, \frac{2m+2n}{m+n-1}\right)$ denote the interval of $L^p$ boundedness in (1) above. When $\mathbb{H}_{m/n}$ is fixed, denote this also as $I_{m/n}^p$.

Some ingredients in the proof of Theorem 3.1 are used to prove the irregularity statements in Theorem 1.3. Focus on $\mathbb{H}_{m/n}$ and let $B = B_{m/n}$. An index $(\alpha_1, \alpha_2) \in \mathbb{Z}^+ \times \mathbb{Z}$ is $L^p$-allowable if the monomial $z_1^{\alpha_1} z_2^{\alpha_2} \in L^p(\mathbb{H}_{m/n})$, where $z_2$ is either $z_2$ or $\bar{z}_2$. This set can be characterized:

**Lemma 3.2** ([20], eq. (3.3)). Let $p \in [1,\infty)$. The $L^p$-allowable indices are
\[ \mathcal{S}(\mathbb{H}_{m/n}, L^p) = \left\{ \alpha = (\alpha_1, \alpha_2) : \alpha_1 \geq 0, \; n\alpha_1 + m\alpha_2 \geq \left\lfloor -\frac{2}{p} (m+n) + 1 \right\rfloor \right\}. \]

See also Lemma 4.4 in [10]. Here $[x]$ is the greatest integer $\leq x$. In particular, the $L^2$ monomials are
\[ \mathcal{S}(\mathbb{H}_{m/n}, L^2) = \{ (\alpha_1, \alpha_2) : \alpha_1 \geq 0, \; n\alpha_1 + m\alpha_2 \geq -m - n + 1 \}. \]

As notation for the ray bounding the sets $\mathcal{S}(\mathbb{H}_{m/n}, L^p)$, let
\[ \ell(\mathbb{H}_{m/n}, L^p) = \left\{ (x,y) \in \mathbb{R}^2 : x \geq 0, \; nx + my = \left\lfloor -\frac{2}{p} (m+n) + 1 \right\rfloor \right\}. \]
An elementary consequence of orthogonality is also used.
Lemma 3.4 ([20, Proposition 5.1]). If both \((\beta_1, \beta_2), (\beta_1, -\beta_2) \in S (\mathbb{H}_{m/n}, L^2)\), then

\[
B \left( z_1^{\beta_1} z_2^{\beta_2} \right) = C z_1^{\beta_1} z_2^{-\beta_2},
\]

for a constant \(C > 0\).

The unboundedness statements in Theorem 3.1 for \(p \notin I_0^p\) are proved as follows. Let \(p \geq \rho(m,n)\).

(A): Choose \((\beta_1, \beta_2) \in \mathbb{Z}^+ \times \mathbb{Z}^+\) with \((\beta_1, -\beta_2) \in \ell (\mathbb{H}_{m/n}, L^2)\).

(B): Lemma 3.2 implies \(z_1^{\beta_1} z_2^{-\beta_2} \notin S (\mathbb{H}_{m/n}, L^{\rho(m,n)})\).

(C): Let \(f(z_1, z_2) = z_1^{\beta_1} z_2^{\beta_2}\); Lemma 3.4 says \(B f = z_1^{\beta_1} z_2^{-\beta_2}\). Thus \(\|f\|_{L^p} < \infty\), while \(\|B f\|_{L^p} = \infty\).

Duality implies the same conclusion if \(p \leq \lambda(m,n)\).

For \(\mathbb{H}_{\gamma}\) with \(\gamma\) irrational, the same kind of test functions \(f(z) = z_1^{\beta_1} z_2^{\beta_2}\), with \((\beta_1, \beta_2) \in \mathbb{Z}^+ \times \mathbb{Z}^+\), are used. An additional argument is needed, which adjusts \(\beta_1, \beta_2\) according to the size of \(|p - 2|\), in order to conclude \(B f \notin L^p (\mathbb{H}_{\gamma})\) for \(p \neq 2\); see Section 6 in [20]. A further computation also shows \(\frac{\partial}{\partial z_j} B f \notin L^2 (\mathbb{H}_{\gamma})\), \(j = 1, 2\).

For \(\gamma\) rational or irrational, an extension of Theorem 3.1 results:

Corollary 3.5 (of the proof of Theorem 3.1). \(B\) fails to map \(L_k^p (\mathbb{H}_{m/n})\) to \(L^p (\mathbb{H}_{m/n})\) if \(p \notin I_k^p\) for any \(k \in \mathbb{Z}^+\).

If \(\gamma\) is irrational, \(B\) fails to map \(L_k^p (\mathbb{H}_{\gamma})\) to \(L^p (\mathbb{H}_{\gamma})\) for any \(p \in (1, \infty), k \in \mathbb{Z}^+\).

Proof. Notice that \((\beta_1, \beta_2) \in \mathbb{Z}^+ \times \mathbb{Z}^+\) implies \(f \in C^\infty (\mathbb{H}_{m/n})\), for \(f\) defined above. Consequently \(\|f\|_{L_k^p} < \infty\) for all \(k \in \mathbb{Z}^+\).

Corollary 3.5 establishes the irregularity statements in Theorem 1.3 for irrational \(\gamma\). For rational \(\mathbb{H}_{m/n}\), it reduces irregularity to determining which \(p \in I_0^p\) satisfy \(\|DB f\|_{L^p} \lesssim \|f\|_{L_k^p}\) for an arbitrary first derivative \(D\). Determining this requires an additional argument. A lattice point diagram – a notion introduced and used in [20] – illustrates the argument.
Four $\mathbb{H}_\gamma$ diagrams are shown, corresponding to $\gamma = \frac{m}{n} = \frac{1}{2}, 1, 2, 3$. The indices $\alpha \in S(\mathbb{H}_{m/n}, L^2)$ are exactly those lattice points on and above the line labeled $L^2$ for the corresponding $\gamma$. The dotted lines, labeled $L^{\lambda(m,n)}$, are lines parallel to their corresponding $L^2$ lines but passing through the lattice points in $\ell(\mathbb{H}_{m/n}, L^{\lambda(m,n)})$.

Thus any lattice point strictly below the dotted lines – in other words, below the shaded region – correspond to monomials $\notin L^{\lambda(m,n)}$ for the given $\mathbb{H}_{m/n}$.

Notice that (up to a constant) $z_1$ derivatives of fourth quadrant monomials are represented by a shift left and $z_2$ derivatives by a shift down in the lattice point diagram. These operations are labeled $\partial_1, \partial_2$ in the diagram. The important observation is clear from the lattice point diagram: unless $\gamma = 1$, a monomial on the $L^2$ line is driven below $L^{\lambda(m,n)}$ line by a single application of $\partial_1$ or $\partial_2$.

The irregularity statements in Theorem 1.3 now follow.

**Proposition 3.6.** Let $1 < p < \infty$.

(i) Let $\gamma \neq 1$. Then $B$ fails to map $L^p_k(\mathbb{H}_\gamma) \to L^p_k(\mathbb{H}_{\gamma})$ for any $k \in \mathbb{Z}^+$.

(ii) On $\mathbb{H}_1$, $B$ fails to map $L^p_k(\mathbb{H}_{1}) \to L^p_k(\mathbb{H}_{1})$ for any $k = 2, 3, \ldots$.

**Proof.** Consider case (i). If $\gamma \notin \mathbb{Q}$, Corollary 3.3 gives the result. If $\gamma = \frac{m}{n} \in \mathbb{Q}$, consider the holomorphic monomial $\mu(z_1, z_2) = z_1^{\beta_1} z_2^{\beta_2}$ with $(\beta_1, -\beta_2) \in \ell(L^2, \mathbb{H}_{m/n})$. Since $\gamma \neq 1$, either $\frac{\partial \mu}{\partial z_1}$ or $\frac{\partial \mu}{\partial z_2} \notin L^{\lambda(m,n)}(\mathbb{H}_{m/n})$. Let $J^p_k = \{ p : \| D^k \mu \|_p \leq \| \mu \|_p, \forall \text{ order } k \text{ derivatives } D^k \}$.

It follows that $I^p_k \cap J^p_k = \emptyset$. Thus $B$ fails to map $L^p_k(\mathbb{H}_{m/n}) \to L^p_k(\mathbb{H}_{m/n})$.

The result holds with even more force for $k \in \mathbb{Z}^+, k \geq 2$, since $J^p_k \subset J^p_1$ and $\mu = Bf$ for $f \in C^\infty(\mathbb{H}_{m/n})$ by Lemma 3.4.

For $\mathbb{H}_1$, take the same monomial $\mu$. Notice that two derivatives of $\mu$ give a monomial with exponent $\notin I^p_k$, i.e. $\partial^2 \mu, \partial^2 \mu \notin L^{4/3}(\mathbb{H}_1)$. Thus case (ii) is proved. \hfill \Box

**Remark 3.7.** A precise non-isotropic version of irregularity is useful in other contexts. The two derivative operations $\partial_1, \partial_2$ are not symmetric with respect to how they drive monomials out of the boundedness interval $I^p_k$, depending on whether $\gamma > 1$ or $\gamma < 1$. The lattice point diagram makes this plain: if $\gamma > 1$ (a “fat Hartogs triangle” in the terminology of [18]) more $\partial_1$ derivatives are allowed, while if $\gamma < 1$ (a “thin Hartogs triangle”) more $\partial_2$ derivatives are allowed.

Various statements about such partial Sobolev mapping can be made, mostly left to the interested reader. One statement is

**Proposition 3.8.** Let $\gamma = \frac{m}{n} \in \mathbb{Q}$.

1. $\partial_1^2 \circ B_\gamma$ fails to map $L^\infty(\mathbb{H}_{\gamma}) \to L^p(\mathbb{H}_{\gamma})$, for any $p \geq \frac{2m+2n}{m+2n-1}$.

2. $\partial_2^2 \circ B_\gamma$ fails to map $L^\infty(\mathbb{H}_{\gamma}) \to L^p(\mathbb{H}_{\gamma})$, for any $p \geq \frac{2m+2n}{2m+2n-1}$.

Note that the bounds on $p$ in Proposition 3.8 are different, and that neither is equal to $\lambda(m,n)$ or $\rho(m,n)$.

4. **Sobolev regularity**

A general version of Schur’s Lemma will be used. The next result extends Lemma 2.4 from [19].

**Lemma 4.1.** Let $\Omega \subset \mathbb{C}^n$ be a domain and $K : \Omega \times \Omega \to [0, \infty)$ a kernel function. Suppose there is an auxiliary function $h : \Omega \to [0, \infty)$ and numbers $0 \leq \alpha < \beta, 0 \leq \gamma < \delta$ such that the following two estimates hold: For all $\epsilon \in [\alpha, \beta]$,

\[
\int_{\Omega} K(z, w)h(w)^{-\epsilon} dV(w) \lesssim h(z)^{-\epsilon},
\]
and for all $\epsilon \in [\gamma, \delta]$,
\begin{equation}
\int_{\Omega} K(z, w)h(z)^{-\epsilon} \, dV(z) \lesssim h(w)^{-\epsilon}.
\end{equation}

Then the operator $K$, $K(f)(z) := \int_{\Omega} K(z, w)f(w) \, dV(w)$, maps $L^p(\Omega) \to L^p(\Omega)$ for all $p$ in the range
\begin{equation}
\frac{\gamma}{\beta} + 1 < p < \frac{\delta}{\alpha} + 1.
\end{equation}

Proof. Let $\frac{1}{p} + \frac{1}{q} = 1$, $g \in L^p(\Omega)$ and $s \in [\alpha, \beta]$. Then
\begin{equation}
|K(f)(z)|^p \leq \left( \int_{\Omega} |K(z, w)|^p |f(w)| h(w)^{\frac{mp}{n}} \, dV(w) \right) \left( \int_{\Omega} K(z, w) h(w)^{-s} \, dV(w) \right)^{\frac{q}{n}}
\lesssim \left( \int_{\Omega} |K(z, w)|^p |f(w)| h(w)^{\frac{mp}{n}} \, dV(w) \right) h(z)^{-\frac{mp}{n}}.
\end{equation}

The first inequality follows from Hölder’s inequality, the second from (4.2). Now
\begin{equation}
\int_{\Omega} |K(f)(z)|^p \, dV(z) \lesssim \int_{\Omega} \left( \int_{\Omega} |K(z, w)|^p |f(w)| h(w)^{\frac{mp}{n}} \, dV(w) \right) h(z)^{-\frac{mp}{n}} \, dV(z)
= \int_{\Omega} |f(w)|^p h(w)^{\frac{mp}{n}} \left( \int_{\Omega} K(z, w) h(z)^{-\frac{mp}{n}} \, dV(z) \right) \, dV(w).
\end{equation}

When $s \in [\alpha, \beta]$ may chosen so that also $\frac{mp}{n} \in [\gamma, \delta)$, estimate (4.3) implies
\begin{equation}
(4.5) \lesssim \int_{\Omega} |f(z)|^p \, dV(z),
\end{equation}
and thus $K : L^p(\Omega) \to L^p(\Omega)$ boundedly. The existence of such an $s$ is equivalent to saying both the inequalities $\frac{mp}{n} \gamma < \beta$ and $\alpha < \frac{mp}{n} \delta$ hold. This is equivalent to saying (4.4) holds, as claimed. □

A class of kernels on the domains $\mathbb{H}_{m/n}$, containing the Bergman kernel $B_{m/n}(z, w)$ and its derivatives, can be analyzed via Lemma 4.1. The following lemma generalizes Proposition 4.2 of [20], which required $c = d$.

**Lemma 4.6.** Let $K : \mathbb{H}_{m/n} \times \mathbb{H}_{m/n} \to \mathbb{C}$ be an integral kernel satisfying
\begin{equation}
|K(z_1, z_2, w_1, w_2)| \lesssim \frac{|z_2|^c|w_2|^d}{|1 - z_2\bar{w}_2|^2 |z_2^m \bar{w}_2^n - z_1^m \bar{w}_1^n|^2},
\end{equation}
and let $K$ be the operator defined by $K(f)(z) := \int_{\mathbb{H}_{m/n}} K(z, w) f(w) \, dV(w)$.

Then $K : L^p(\mathbb{H}_{m/n}) \to L^p(\mathbb{H}_{m/n})$ is bounded operator for all $p \in (1, \infty)$ satisfying
\begin{equation}
\frac{2m + 2n}{2m + 2n + dm - 2mn} < p < \frac{2m + 2n}{2mn - cm}.
\end{equation}

**Remark 4.9.** If the exponent $c \geq 2n$, the upper bound in (4.8) can be taken to be $\infty$. This follows since $|z_2|^c \leq |z_2|^{2n}$ for all $z = (z_1, z_2) \in \mathbb{H}_{m/n}$. Similarly if $d \geq 2n$, the lower bound in (4.8) is 1.

The range in (4.8) implicitly says the following three conditions must hold in order to have a non-degenerate range of $p$:
\begin{align*}
c > 2n \left(1 - \frac{1}{m}\right) - 2, \quad d > 2n \left(1 - \frac{1}{m}\right) - 2, \quad c + d > 2n \left(2 - \frac{1}{m}\right) - 2.
\end{align*}
Proof of Lemma 4.6. Apply Lemma 4.1 with \( h(w) = (|w_2|^{2n} - |w_1|^{2m})(1 - |w_2|^2) \) as the auxiliary function and \( \epsilon \in (0, 1) \) to be chosen. It follows that

\[
(4.2) = \int_{D_{m/n}} |K(z, w)| h(w)^{-\epsilon} \, dV(w)
\]

\[
\lesssim \int_{D_{m/n}} \frac{|z_2|^c |w_2|^d (|w_2|^{2n} - |w_1|^{2m})^{-\epsilon} (1 - |w_2|^2)^{-\epsilon}}{|1 - z_2 \overline{w_2}|^2 |z_1^m w_1|^2} \, dV(w)
\]

\[
(4.10) = \int_{D^*} \frac{|z_2|^c |w_2|^d (1 - |w_2|^2)^{-\epsilon}}{|1 - z_2 \overline{w_2}|^2} \left[ \int_{W} \frac{(|w_2|^{2n} - |w_1|^{2m})^{-\epsilon}}{|z_2^m w_2 + z_1^m w_1|^2} \, dV(w_1) \right] \, dV(w_2),
\]

where \( D^* \) is the punctured unit disc and the integral in brackets is taken over the region \( W = \{ w_1 : |w_1| < |w_2|^{n/m} \} \). Denote this inner integral by \( I \).

\[
I = \int_{D_{m/n}} \frac{1}{|z_2|^{2n} |w_2|^{2n+2m}} \int_{W} \left( 1 - \frac{|w_1|^m}{|w_2|^m} \right)^2 \left| 1 - \left( \frac{z_1^m}{z_2^m} \right) \left( \frac{w_1^m}{w_2^m} \right) \right|^2 \, dV(w_1)
\]

\[
= \frac{|w_2|^{2n/m - 2n - 2m}}{|z_2|^{2n}} \int_{D} \frac{(1 - |u|^2)^{-\epsilon}}{|1 - z_2^m \overline{w_2}|^2} |u|^{2/m - 2} \, dV(u),
\]

after the \( m \)-to-1 integral transformation \( u = \frac{w_1^m}{w_2^m} \). Lemma 2.2 yields the estimate

\[
(4.11) \lesssim \frac{|w_2|^{2n/m - 2n - 2m}}{|z_2|^{2n}} \left( 1 - \frac{z_1^m}{z_2^m} \right)^{2} \left( 1 - \frac{|z_1|^2}{|z_2|^2} \right)^{-\epsilon}
\]

\[
(4.12) = |z_2|^{2n - 2n} |w_2|^{2n/m - 2n - 2m} (|z_2|^{2n} - |z_1|^{2m})^{-\epsilon}.
\]

Now insert (4.12) into (4.10):

\[
(4.10) \lesssim |z_2|^{c + 2n\epsilon - 2n} (|z_2|^{2n} - |z_1|^{2m})^{-\epsilon} \int_{D^*} \frac{(1 - |w_2|^2)^{-\epsilon}}{|1 - z_2 \overline{w_2}|^2} |w_2|^A \, dV(w_2),
\]

where the exponent \( A = d + \frac{2n}{m} - 2n - 2m \) is required to be strictly greater than \(-2\) in order for the \( D^* \) integral to converge. This is equivalent to requiring

\[
(4.13) \quad \epsilon < \frac{1}{2n} \left( d + \frac{2n}{m} - 2n + 2 \right).
\]

When \( \epsilon \in (0, 1) \) can be chosen to satisfy this estimate, Lemma 2.2 shows

\[
\int_{H_{m/n}} |K(z, w)| h(w)^{-\epsilon} \, dV(w) \lesssim |z_2|^{c + 2n\epsilon - 2n} (|z_2|^{2n} - |z_1|^{2m})^{-\epsilon} (1 - |z_2|^2)^{-\epsilon} \leq h(z)^{-\epsilon},
\]

as long as the exponent \( c + 2n\epsilon - 2n \geq 0 \). But this is equivalent to saying

\[
(4.14) \quad \epsilon \geq 1 - \frac{c}{2n}.
\]

Inequalities (4.13) and (4.14) give the interval \([\alpha, \beta]\) in Lemma 4.1. Indeed, it suffices to take \( \alpha = 1 - \frac{2n}{m} \) and \( \beta = \frac{1}{2n} \left( d + \frac{2n}{m} - 2n + 2 \right) \).

To generate the interval \([\gamma, \delta]\) needed in Lemma 4.1, simply switch the roles of \( c \) and \( d \) in the argument above. This leads to taking \( \gamma = 1 - \frac{d}{2n} \) and \( \delta = \frac{1}{2n} (c + \frac{2n}{m} - 2n + 2) \). Lemma 4.1 now gives the claimed result. \( \square \)
4.1. Mapping of the differentiated projection. Boundedness of the Bergman projection associated to $\mathbb{H}_1$ on the Sobolev space $L^p_1(\mathbb{H}_1)$ can now be given. In [13], the Bergman kernel of $\mathbb{H}_{1/n} \ (n \in \mathbb{Z}^+)$, is computed as

$$B_{1/n}(z, w) = \frac{1}{\pi^2} \frac{z^n \bar{w}^n}{(1 - z \bar{w})^2(\bar{z}^n \bar{w}^n - z_1 \bar{w})^2}. \quad (4.15)$$

Throughout the section, subscripts on the projection $B_{1/n}$ and the kernel $B_{1/n}(z, w)$ are dropped.

**Theorem 4.16.** On $\mathbb{H}_{1/n} \ (n \in \mathbb{Z}^+)$, it holds that

1. $\frac{\partial}{\partial z_j} \circ B \ maps \ L^p_1(\mathbb{H}_{1/n}) \to L^p(\mathbb{H}_{1/n})$ for $p \in \left(1, \frac{2n}{2n+2}\right)$.
2. $\frac{\partial}{\partial z_2} \circ B \ maps \ L^p_1(\mathbb{H}_{1/n}) \to L^p(\mathbb{H}_{1/n})$ for $p \in \left(\frac{2n+2}{2n+3}, 2\right)$.

**Proof.** The spirit is similar to the proof of Theorem 2.7. Let $f \in L^1_1(\mathbb{H}_{1/n})$ for $1 < p < \infty$, and $j = 1, 2$.

$$\frac{\partial}{\partial z_j} Bf(z) = \frac{\partial}{\partial z_j} \int_{\mathbb{H}_{1/n}} B(z, w) f(w) \, dV(w) = \frac{1}{z_j} \int_{\mathbb{H}_{1/n}} \bar{w}_j \frac{\partial}{\partial w_j} (B(z, w)) f(w) \, dV(w)$$

$$= \frac{1}{z_j} \int_{\mathbb{H}_{1/n}} T_{w_j} (B(z, w)) f(w) \, dV(w), \quad (4.17)$$

since $B(z, w)$ is anti-holomorphic in $w$.

The $z_1$ and $z_2$ derivatives are handled slightly differently. Consider the $z_2$ derivative first. Equation (4.17) says

$$\frac{\partial}{\partial z_2} Bf(z) = \frac{1}{z_2} \int_{|w_1|=1} |w_1| \int_{|w_1|=0} \left\{ \int_{\mathcal{A}} T_{w_2} (B(z, w)) f(w) \, dV(w_2) \right\} \, dV(w_1), \quad (4.18)$$

where the inner integral is over $\mathcal{A} = \{w_2 : |w_1|^{1/m} < |w_2| < 1\}$ for each fixed $w_1$. Since $\mathcal{A}$ is an annulus centered at the origin, Proposition 2.6 transfers the vector field $T_{w_2} f(w)$ onto $f$ without picking up a boundary integral:

$$= \frac{1}{z_2} \int_{|w_1|=0} \left\{ \int_{\mathcal{A}} B(z, w) T_{w_2} f(w) \, dV(w_2) \right\} \, dV(w_1)$$

$$= \frac{1}{z_2} \int_{\mathbb{H}_{1/n}} B(z, w) T_{w_2} f(w) \, dV(w)$$

$$= \int_{\mathbb{H}_{1/n}} \frac{w_2}{z_2} B(z, w) \frac{\partial}{\partial w_2} (w) \, dV(w) - \int_{\mathbb{H}_{1/n}} \frac{\bar{w}_2}{z_2} B(z, w) \frac{\partial}{\partial \bar{w}_2} (w) \, dV(w), \quad (4.19)$$

derivatives interpreted distributionally. Since $f \in L^1_1(\mathbb{H}_{1/n}), \ \frac{\partial f}{\partial w_2}, \ \frac{\partial f}{\partial \bar{w}_2} \in L^p(\mathbb{H}_{1/n}).$

By (4.15), the integral kernels in (4.19) satisfy

$$\left| \frac{w_2}{z_2} B(z, w) \right| = \left| \frac{\bar{w}_2}{z_2} B(z, w) \right| \approx \frac{|z_2|^{n-1} |w_2|^{n+1}}{|1 - z_2 \bar{w}_2|^2 |z_2^2 \bar{w}_2^2 - z_1 \bar{w}|^2}.$$ 

Therefore Lemma 4.16 with $c = n - 1, \ d = n + 1,$ and $m = 1,$ shows

$$\left\| \frac{\partial}{\partial z_2} \circ Bf \right\|_{L^p(\mathbb{H}_{1/n})} \leq \left\| \frac{\partial f}{\partial w_2} \right\|_{L^p} + \left\| \frac{\partial f}{\partial \bar{w}_2} \right\|_{L^p} \leq \left\| f \right\|_{L^p(\mathbb{H}_{1/n})}$$

for $p \in \left(\frac{2n+2}{2n+3}, 2\right)$. This establishes part (2) of the theorem.
Consider the $z_1$ derivative. Equation (4.17) says
\[
\frac{\partial}{\partial z_1} Bf(z) = \frac{1}{z_1} \int_{|w_2|=1} f(z) \left\{ \int_\mathcal{D} T_{w_1}(B(z,w)) f(w) dV(w) \right\} dV(w_2),
\]
where the inner integral is taken over $\mathcal{D} = \{ w_1 : |w_1| < |w_2|^n \}$ for each fixed $w_2$. Estimating this term requires more care than was necessary for the $z_2$ derivative. As in the proof of Lemma 2.7, define a kernel by subtracting from $B(z,w)$ the term $B((0,z_2),(0,w_2))$.

Equation (4.15) shows
\[
K(z,w) := B(z,w) - B((0,z_2),(0,w_2))
\]
\[
= \frac{1}{\pi^2} \left[ \frac{z_2^n \bar{w}_2^n}{(1 - z_2 \bar{w}_2)^2 (z_2^n \bar{w}_2^n - z_1 \bar{w}_1)^2} - \frac{1}{z_2^n \bar{w}_2^n (1 - z_2 \bar{w}_2)^2} \right]
\]
\[
= \frac{1}{\pi^2} \frac{2z_1 \bar{w}_1 z_2^n \bar{w}_2^n - z_1^2 \bar{w}_1^2}{z_2^n \bar{w}_2^n (1 - z_2 \bar{w}_2)^2 (z_2^n \bar{w}_2^n - z_1 \bar{w}_1)^2}.
\]

(4.21)

Since $B((0,z_2),(0,w_2))$ is independent of $w_1$ and $\bar{w}_1$, $K(z,w)$ may be substituted for $B(z,w)$ in equation (4.20). Since $\mathcal{D}$ is a disc centered at the origin, Proposition 2.6 applies:
\[
(4.20) = -\frac{1}{z_1} \int_{|w_2|=1} \left\{ \int_\mathcal{D} K(z,w) T_{w_1} f(w) dV(w_1) \right\} dV(w_2)
\]
\[
= -\frac{1}{z_1} \int_{\mathbb{H}_1/n} K(z,w) T_{w_1} f(w) dV(w)
\]
\[
= \int_{\mathbb{H}_1/n} \frac{w_1}{z_1} K(z,w) \frac{\partial f}{\partial w_1} (w) dV(w) - \int_{\mathbb{H}_1/n} \frac{\bar{w}_1}{z_1} K(z,w) \frac{\partial f}{\partial \bar{w}_1} (w) dV(w),
\]
(4.22) derivatives interpreted distributionally, as before. By hypothesis, the functions $\frac{\partial f}{\partial w_1}, \frac{\partial f}{\partial \bar{w}_1} \in L^p(\mathbb{H}_1/n)$.

From (4.21), the kernels in (4.22) satisfy
\[
\left| \frac{w_1}{z_1} K(z,w) \right| = \left| \frac{\bar{w}_1}{z_1} K(z,w) \right| \approx \left| \frac{w_1}{z_1} \right| \cdot \frac{\left| 2z_1 \bar{w}_1 z_2^n \bar{w}_2^n - z_1^2 \bar{w}_1^2 \right|}{\left| z_2^n |w_2|^n (1 - z_2 \bar{w}_2) |z_2^n \bar{w}_2^n - z_1 \bar{w}_1)^2 \right|}
\]
\[
\leq \left| \frac{w_1}{z_1} \right| \cdot \frac{\left| z_2^n |w_2|^n (1 - z_2 \bar{w}_2) |z_2^n \bar{w}_2^n - z_1 \bar{w}_1)^2 \right|}{\left| z_2^n |w_2|^n (1 - z_2 \bar{w}_2) |z_2^n \bar{w}_2^n - z_1 \bar{w}_1)^2 \right|}
\]
\[
\leq \left| \frac{w_2^{2n}}{z_2^n |w_2|^n (1 - z_2 \bar{w}_2) |z_2^n \bar{w}_2^n - z_1 \bar{w}_1)^2 \right|.
\]

The last two inequalities hold because $z, w \in \mathbb{H}_1/n$. Lemma 4.6 with $c = 0$, $d = 2n$, and $m = 1$, shows
\[
\left\| \frac{\partial}{\partial z_1} \circ Bf \right\|_{L^p(\mathbb{H}_1/n)} \lesssim \|f\|_{L^p(\mathbb{H}_1/n)}
\]
for $p \in \left(1, \frac{2n+2}{2n}\right)$, establishing part (1) of the theorem.

\[\square\]

**Corollary 4.23.** The Bergman projection $B$ is a bounded operator from $L^p(\mathbb{H}_1) \to L^p(\mathbb{H}_1)$, for all $\frac{4}{3} < p < 2$.

**Proof.** Set $n = 1$ in Theorem 4.16 and intersect the two intervals of $L^p$ boundedness. It follows that $D \circ B$ is $L^p$ bounded for $\frac{4}{3} < p < 2$ for any first derivative $D$. Since $B$ itself is $L^p$ bounded for $\frac{4}{3} < p < 4$ (Theorem 3.1), the result follows. \[\square\]
5. A SUBSTITUTE OPERATOR ON THE HARTOGS TRIANGLE

In light of Theorem [1.3] it is natural to seek operators related to $B$ which have better Sobolev mapping behavior than $B$ itself. Pursuing an idea in [10], a sub-Bergman operator will be constructed on $\mathbb{H}_1$ with such improved behavior. $\mathbb{H}_1$ is taken only for simplicity; the general pattern below extends to other domains.

Consider the set of bounded monomials on $\mathbb{H}_1$:

$$S(\mathbb{H}_1, L^\infty) = \{\alpha = (\alpha_1, \alpha_2) : \alpha_1 \geq 0, \alpha_1 + \alpha_2 \geq 0\}.$$  

Lemma 3.2 shows that $S(\mathbb{H}_1, L^\infty) = S(\mathbb{H}_1, L^p)$ for $p \geq 4$ and $S(\mathbb{H}_1, L^\infty) \not\subset S(\mathbb{H}_1, L^2)$. Following [10], define the $L^\infty$ sub-Bergman kernel

$$B^\infty(z, w) := \sum_{\alpha \in S(\mathbb{H}_1, L^\infty)} \frac{z^\alpha \bar{w}^\alpha}{\|z\|^2 L^2(\mathbb{H}_1)}.$$  

Notice the series in (5.1) is only part of the usual series that defines the Bergman kernel. The $L^\infty$ sub-Bergman projection is

$$B^\infty f(z) := \int_{\mathbb{H}_1} B^\infty(z, w) f(w) \, dV(w)$$

whenever the integral converges; $f$ is taken from certain $L^p(\mathbb{H}_1)$ classes below.

A rational expression for (5.1) can be extracted from [10, Proposition 4.33]:

$$B^\infty(z, w) = \frac{1}{\pi^2} \frac{2z_2 \bar{w}^2 - z_2^2 \bar{w}_1^2}{(z_2 \bar{w}_2 - z_1 \bar{w}_1)^2 (1 - z_2 \bar{w}_2)^2}.$$  

This immediately yields the bound

$$|B^\infty(z, w)| \leq \frac{|z_2|^2 |w_2|^2}{|z_2 \bar{w}_2 - z_1 \bar{w}_1|^2 |1 - z_2 \bar{w}_2|^2}.$$  

Lemma 4.6 with $m = n = 1$ and $c = d = 2$ shows for each fixed $1 < p < \infty$,

$$\left\|B^\infty f\right\|_{L^p(\mathbb{H}_1)} \leq \|f\|_{L^p(\mathbb{H}_1)}, \quad f \in L^p(\mathbb{H}_1).$$

Derivatives are now considered. Mapping properties of $\frac{\partial}{\partial z_2} \circ B^\infty$ may be obtained by following the proof of Theorem 4.16 with $B^\infty(z, w)$ replacing $B(z, w)$. The steps leading up to (4.19) show, for $f \in L^p(\mathbb{H}_1)$,

$$\frac{\partial}{\partial z_2} B^\infty f(z) = \int_{\mathbb{H}_1} \frac{w_2}{z_2} B^\infty(z, w) \frac{\partial f}{\partial w_2}(w) \, dV(w) - \int_{\mathbb{H}_1} \frac{\bar{w}_2}{z_2} B^\infty(z, w) \frac{\partial f}{\partial \bar{w}_2}(w) \, dV(w).$$

Thus the operator $\frac{\partial}{\partial z_2} \circ B^\infty$ is governed by the kernels

$$\left|\frac{w_2}{z_2} B^\infty(z, w)\right| \leq \left|\frac{\bar{w}_2}{z_2} B^\infty(z, w)\right| \leq \frac{|z_2| |w_2|^3}{|z_2 \bar{w}_2 - z_1 \bar{w}_1|^2 |1 - z_2 \bar{w}_2|^2}.$$  

Lemma 4.6 (and Remark 4.9) with $m = n = 1, c = 1, d = 3$, shows for each fixed $1 < p < 4$,

$$\left\|\frac{\partial}{\partial z_2} \circ B^\infty f\right\|_{L^p(\mathbb{H}_1)} \leq \left\|\frac{\partial f}{\partial w_2}\right\|_{L^p} + \left\|\frac{\partial f}{\partial \bar{w}_2}\right\|_{L^p} \leq \|f\|_{L^p(\mathbb{H}_1)}.$$
Mapping properties of \( \frac{\partial}{\partial z_1} \circ \tilde{B}^\infty \) may be obtained by considering
\[
\tilde{K}^\infty(z, w) := \tilde{B}^\infty(z, w) - \tilde{B}^\infty((0, z_2), (0, w_2))
\]
\[
= \frac{1}{\pi^2} \left[ \frac{2z_2^2 \bar{w}_2^3 - z_2^3 \bar{w}_2^3}{(1 - z_2 \bar{w}_2)^2(z_2 \bar{w}_2 - z_1 w_1)^2} - \frac{2z_2^2 \bar{w}_2 - z_2^3 \bar{w}_2}{z_2^2 (1 - z_2 \bar{w}_2)^2} \right]
\]
\[
= \frac{1}{\pi^2} \left( \frac{z_1 \bar{w}_1 (4z_2 \bar{w}_2 - 2z_2^2 \bar{w}_2 - 2z_1 \bar{w}_1 + z_1 \bar{w}_1 z_2 \bar{w}_2)}{(1 - z_2 \bar{w}_2)^2(z_2 \bar{w}_2 - z_1 w_1)^2} \right).
\]
Simple estimation shows \( \tilde{K}^\infty(z, w) \) satisfies a stronger estimate than (5.3):
\[
|\tilde{K}^\infty(z, w)| \lesssim \frac{|z_1||w_1||z_2||w_2|}{|1 - z_2 \bar{w}_2|^2|z_2 \bar{w}_2 - z_1 w_1|^2}.
\]
Repeating the steps from (4.20) through (4.22) – with \( \tilde{K}^\infty(z, w) \) replacing \( K(z, w) \) – shows, for \( f \in L^p_0(\mathbb{H}_1) \),
\[
\frac{\partial}{\partial z_1} \tilde{B}^\infty f(z) = \int_{\mathbb{H}_1} \frac{w_1}{z_1} \tilde{K}^\infty(z, w) \frac{\partial f}{\partial w_1}(w) dV(w) - \int_{\mathbb{H}_1} \frac{\bar{w}_1}{z_1} \tilde{K}^\infty(z, w) \frac{\partial f}{\partial \bar{w}_1}(w) dV(w).
\]
Thus the operator \( \frac{\partial}{\partial z_1} \circ \tilde{B}^\infty \) is governed by the kernels
\[
\left| \frac{w_1}{z_1} \tilde{K}^\infty(z, w) \right| = \frac{\bar{w}_1}{z_1} \tilde{K}^\infty(z, w) \lesssim \frac{|z_2||w_2|^3}{|1 - z_2 \bar{w}_2|^2|z_2 \bar{w}_2 - z_1 w_1|^2}.
\]
This bound is identical to the bound in (5.5). Consequently, for each fixed \( 1 < p < 4 \),
\[
(5.7) \quad \left\| \frac{\partial}{\partial z_1} \circ \tilde{B}^\infty f \right\|_{L^p_0(\mathbb{H}_1)} \lesssim \left\| \frac{\partial f}{\partial w_1} \right\|_{L^p} + \left\| \frac{\partial f}{\partial \bar{w}_1} \right\|_{L^p} \leq \|f\|_{L^p_0(\mathbb{H}_1)}.
\]
Combining (5.2), (5.6), and (5.7) proves the following
\begin{corollary}
\tilde{B}^\infty \text{ maps } L^p_1(\mathbb{H}_1) \to L^p_1(\mathbb{H}_1) \text{ boundedly for all } 1 < p < 4.
\end{corollary}

It is not difficult to verify that \( \tilde{B}^\infty \) fails to map \( L^p_1(\mathbb{H}_1) \to L^p_1(\mathbb{H}_1) \) for \( p \geq 4 \): take the monomial \( f(z) = z_1 z_2 \) and follow the arguments given in Section 3. The interested reader is also invited to extend Corollary 5.8 to higher order derivatives. The statements are
\begin{corollary}
\tilde{B}^\infty \text{ maps } L^p_2(\mathbb{H}_1) \to L^p_2(\mathbb{H}_1) \text{ boundedly for all } 1 < p < 2.
\end{corollary}

\begin{corollary}
\tilde{B}^\infty \text{ maps } L^p_0(\mathbb{H}_1) \to L^p_0(\mathbb{H}_1) \text{ boundedly for all } 1 < p < \frac{4}{3}.
\end{corollary}

\begin{remark}
Formulas (5.1) and (5.2) may be modified to define the \( L^\infty \) sub-Bergman kernel and projection on a general Reinhardt domain \( \mathcal{R} \). More generally, for fixed \( p \in [2, \infty) \), \( L^p \) sub-Bergman kernels and projections (\( \tilde{B}^\infty(z, w) \) and \( \tilde{B}^p \)) may be defined on \( \mathcal{R} \) by formulas analogous to (5.1), where the sum is taken over indices \( \alpha \in \mathcal{S}(\mathcal{R}, L^p) \) – see [10] Section 3.6.

In [10] Section 4.2.2, the \( \tilde{B}^p \) are constructed for each \( \mathbb{H}_m/n \) and shown to stabilize into \( m+n \) representatives. These operators are more regular on \( L^p_0 \) than \( B \) – see [10] Theorem 4.3. This improved regularity has important consequences for holomorphic duality and approximation – see [10] Section 4.4.

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