On string theory on $\text{AdS}_3 \times S^3 \times T^4$
with mixed 3-form flux: tree-level S-matrix

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Abstract

We consider superstring theory on $\text{AdS}_3 \times S^3 \times T^4$ supported by a combination of RR and NSNS 3-form fluxes (with parameter of the NSNS 3-form $q$). This theory interpolates between the pure RR flux model ($q = 0$) whose spectrum is expected to be described by a (thermodynamic) Bethe ansatz and the pure NSNS flux model ($q = 1$) which is described by the supersymmetric extension of the $SL(2, R) \times SU(2)$ WZW model. As a first step towards the solution of this integrable theory for generic value of $q$ we compute the corresponding tree-level S-matrix for massive BMN-type excitations. We find that this S-matrix has a surprisingly simple dependence on $q$: the diagonal amplitudes have exactly the same structure as in the $q = 0$ case but with the BMN dispersion relation $e^2 = p^2 + 1$ replaced by the one with shifted momentum and mass, $e^2 = (p \pm q)^2 + 1 - q^2$. The off-diagonal amplitudes are then determined from the classical Yang-Baxter equation. We also construct the Pohlmeyer reduced model corresponding to this superstring theory and find that it depends on $q$ only through the rescaled mass parameter, $\mu \to \sqrt{1 - q^2} \mu$, implying that its relativistic S-matrix is $q$-independent.
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1 Introduction

The subject of this paper is the superstring theory on $AdS_3 \times S^3 \times T^4$ space-time supported by a combination of RR and NSNS 3-form fluxes. The corresponding type IIB supergravity background is the near-horizon limit of the mixed NS5-NS1 + D5-D1 solution that is the basis for an interesting example of $AdS$/CFT duality [1, 2]. In the near-horizon limit the dilaton is constant, the 3-form fluxes through $AdS_3$ and $S^3$ have related coefficients and the radii of the two spaces are equal.

The S-duality symmetry of type IIB supergravity transforms the NSNS 3-form into the RR 3-form, so that if the coefficients of the NSNS and RR fluxes are chosen as $q$ and $q'$, respectively, then they enter symmetrically into the supergravity equations, e.g., as $q^2 + q'^2 = 1$ (we set the curvature radius to 1). The perturbative fundamental superstring theory is not invariant under the S-duality and should thus depend non-trivially on the parameter $q$.\(^1\)

We shall assume that $0 \leq q \leq 1$, with $q = 0$ corresponding to the $AdS_3 \times S^3 \times T^4$ theory with pure RR flux and $q = 1$ to the $AdS_3 \times S^3 \times T^4$ theory with pure NSNS flux. The NSNS flux theory is given by the superstring generalization of the $SL(2) \times SU(2)$ WZW model while the RR flux theory has a Green-Schwarz (GS) formulation [3, 4] similar to the one [5] in the $AdS_5 \times S^5$ case. The “mixed” theory for generic value of $q$ (first discussed in GS formulation in [3]) has a nice $PSU(1,1|2) × PSU(1,1|2)$ supercoset formulation [6], exposing its classical integrability and UV finiteness.\(^2\)

The free string spectrum of the NSNS ($q = 1$) theory can be found using the chiral decomposition property of the WZW model [7] while the apparently more complicated spectral problem of the RR ($q = 0$) theory is expected to be solved, as in the $AdS_5 \times S^5$ case [8], by a thermodynamic Bethe ansatz (for recent progress towards its construction see [9, 10, 11, 12, 13, 14, 15, 16, 17]).

Solving the “interpolating” theory with $0 < q < 1$ is thus a very interesting problem as that may help to understand the relation between the more standard CFT approach in the $q = 1$ case and the integrability-based TBA approach in the $q \neq 1$ case.\(^3\)

The first step towards constructing the $q \neq 0$ generalization of the Bethe ansatz for the string spectrum is to find the corresponding S-matrix for the elementary (BMN-like) massive excitations. Our aim here will be to determine the string tree-level term in this S-matrix following the approach used in the $AdS_5 \times S^5$ case in [23, 24, 25, 26, 27], i.e. computing it directly from the gauge-fixed string action.

We shall start in section 2 with an explicit description of the bosonic part of the string action.

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\(^1\)The (leading-order) conformal invariance (or $\kappa$-symmetry) conditions of the superstring theory, being equivalent to the supergravity field equations, will still depend on the coefficients of the two types of fluxes through the S-duality invariant combination $q^2 + q'^2$, relating it to the square of the radius of the $AdS_3$ and $S^3$ (here set to 1).

\(^2\)In our notation the parameters of the GS action in [6] are $\chi = q$, $\kappa = \sqrt{1 - q^2}$.

\(^3\)It should be noted that the $SU(2)$ principal chiral model with a WZ term was studied in the past using the Bethe ansatz approach [18, 19, 20, 21] but this quantum solution is not directly relevant for the $AdS_3 \times S^3 \times T^4$ superstring case where the presence of fermions makes the world-sheet theory UV finite: there is thus no RG flow (or dynamical mass generation in the $q = 0$ case) while a fiducial (BMN-type [22]) mass scale is introduced by a “vacuum” choice or a gauge fixing.
in the sector corresponding to the string moving on $R \times S^3$. In conformal gauge it is described by the $SU(2)$ principal chiral model with a WZ term (with coefficient proportional to $q$). Fixing a gauge corresponding the BMN vacuum in which the center of mass of the string moves along a circle in $S^3$ we expand the action to quartic order in the fields, sufficient to compute the two-particle tree-level S-matrix.

The computation of this S-matrix for the bosonic string on $S^3$ with $B$-field flux is the subject of section 3. This S-matrix has a direct generalization to the full bosonic $AdS_3 \times S^3$ sector found by using the expression for the relevant gauge-fixed action given in Appendix A.1.

The simplicity of the bosonic result and the requirements of integrability (symmetry factorization and the Yang-Baxter equation) suggest a natural generalization to the fermionic sector, i.e. leading to the full tree-level S-matrix of the GS superstring theory on $AdS_3 \times S^3 \times T^4$ with mixed RR-NSNS flux, which we present in the section 4.

In Appendix A.2 we explain how the quadratic fermionic action that reproduces the non-trivial $BBFF$ part of this S-matrix should follow from the $AdS_3 \times S^3 \times T^4$ superstring action upon light-cone gauge fixing and expansion in powers of bosons. A candidate for the symmetry algebra of this S-matrix is discussed in Appendix B where we also comment on the symmetry of the S-matrix in the $q = 0$ case. Section 5 contains some concluding remarks.

Imposing the conformal gauge, one may solve the Virasoro conditions explicitly and reformulate the classical string theory in terms of current field variables. One way to do this is the Pohlmeyer reduction and another is the Faddeev-Reshetikhin construction (that applies in the bosonic $SU(2)$ case). In Appendix C we construct the Faddeev-Reshetikhin model corresponding to the bosonic string on $R \times S^3$ with $B$-flux, generalizing the discussion in [28], and present the $q \neq 0$ expression for the corresponding tree-level S-matrix (which is different from the bosonic string sigma model one).

In Appendix D we give a detailed construction of the Pohlmeyer-reduced model corresponding to the superstring theory on $AdS_3 \times S^3 \times T^4$ with mixed 3-form flux. Somewhat unexpectedly, we find that the reduced action is the same as in the $q = 0$ case [29, 30] but has the rescaled mass scale parameter, $\mu \rightarrow \sqrt{1-q^2}\mu$. As a result, the corresponding relativistic S-matrix does not depend on $q$.

## 2 Bosonic string action on $R \times S^3$ with $B$-flux

Let us start with some basic definitions. In general, the bosonic string sigma model action is

$$S = \frac{1}{2\pi\alpha'} \int d\tau d\sigma \; L , \quad L = -\frac{1}{2} \left[ \sqrt{g} g^{ab} G_{mn}(x) + \epsilon^{ab} B_{mn}(x) \right] \partial_a x^m \partial_b x^n . \quad (2.1)$$

The action of the $AdS_3 \times S^3 \times T^4$ superstring theory with mixed 3-form flux has the following structure:

$$S_{\text{tot}} = S_{AdS} + S_{S} + \text{fermionic terms} , \quad (2.2)$$

where the first two terms are given by the principal chiral models with an extra WZ term for the groups $SL(2,R)$ and $SU(2)$ respectively. The WZ term represents the NSNS 3-form flux

\[WZ\]
Both the NSNS 3-form coupling and the RR 3-form coupling (proportional to $\sqrt{1 - q^2}$) appear in the fermionic terms. In general, the string moving on a group space with a $B$-flux is described, in the conformal gauge, by the action of a principal chiral model with a WZ term

$$S = \frac{1}{2}h \left[ \int d^2\sigma \frac{1}{2} \text{tr}(J^a J_a) + q \int d^3\sigma \frac{1}{3} \epsilon^{abc} \text{tr}(J_a J_b J_c) \right], \quad (2.3)$$

$$J_a = g^{-1} \partial_a g, \quad h = \frac{R^2}{2\pi\alpha'} = \frac{\sqrt{\lambda}}{2\pi}. \quad (2.4)$$

Here $h$ is the effective coupling (string tension). The quantized coefficient of the WZ term is $k = \sqrt{\lambda} q$. $q = 1$ corresponds to the case of the WZW model. The corresponding classical equations of motion can be written as

$$(1 - q)\partial^+ J^- + (1 + q)\partial^- J^+ = 0, \quad \partial^+ J^- - \partial^- J^+ + [J^+, J^-] = 0, \quad (2.5)$$

or as

$$\partial^+ J^- + \frac{1}{2}(1 + q) [J^+, J^-] = 0, \quad \partial^- J^+ - \frac{1}{2}(1 - q) [J^+, J^-] = 0, \quad (2.6)$$

implying the existence of the Lax pair, i.e. the classical integrability of this model.\textsuperscript{6} Explicitly, the Lax pair is given by

$$L_\pm = \frac{1}{2} (1 \pm q + z^{\pm 1} \sqrt{1 - q^2}) J_\pm, \quad (2.7)$$

where $z$ is the spectral parameter.\textsuperscript{7}

In this section we shall concentrate on the sector of the full superstring theory when the string is moving on $R \times S^3$, i.e. when the Lagrangian is $L = -\frac{1}{2} \partial^+ t \partial^- t + L_S$ where $L_S$ is given by (2.3) with $g \in SU(2)$.

### 2.1 Explicit form of the Lagrangian

Using the familiar parametrization of $g \in SU(2)$ the Lagrangian $L_S$ may be written as

$$L_S = \frac{1}{2} \left[ \partial_+ \theta \partial^- \theta + \sin^2 \theta \partial_+ \phi_1 \partial^- \phi_1 + \cos^2 \theta \partial_+ \phi_2 \partial^- \phi_2 ight. \\
\left. + q \sin^2 \theta \left( \partial_+ \phi_1 \partial^- \phi_2 - \partial_+ \phi_2 \partial^- \phi_1 \right) \right]. \quad (2.8)$$

\textsuperscript{5}As already mentioned, we shall always assume that $0 \leq q \leq 1$.

\textsuperscript{6}Let us note that these equations may be written also as $\partial^a L_a = 0$, $L_a \equiv J_a + q e_{ab} J^b$, where $J_a = g^{-1} \partial_a g$ or as $\partial^a R_a = 0$, $R_a \equiv K_a - q e_{ab} K^b$, where $K_a = \partial_a g g^{-1}$.

\textsuperscript{7}Local and non-local conserved charges in such model were discussed in [31, 32].
Below we shall use also an alternative parametrization of $S^3$ in terms of an angle $\varphi$ and two “cartesian” coordinates $y_s$ ($s = 1, 2$).

\[ L_S = -\frac{1}{2} \left[ G(y) \partial^a \varphi \partial_a \varphi + F(y) \partial^a y_s \partial_a y_s + 2B_s(y) \epsilon^{ab} \partial_a y_s \partial_b \varphi \right], \]  

(2.9)

\[ G = \frac{(1 - \frac{1}{2} y^2)^2}{(1 + \frac{1}{2} y^2)^2} = 1 - \frac{1}{4} y^2 F, \quad F = \frac{1}{1 + \frac{1}{4} y^2} F, \]  

(2.10)

\[ B_s = qF(y) \dot{y}_s, \quad \dot{y}_s = \epsilon_{rs} y_r. \]  

(2.11)

Note that (2.9) can be written also as

\[ L_S = -\frac{1}{2} \left[ 1 - (1 - \frac{1}{2} y^2) y^2 F(y) \right] \partial^a \varphi \partial_a \varphi - \frac{1}{2} F(y) \partial^a y_s + q\dot{y}_s \epsilon^{ab} \partial_b \varphi)^2. \]  

(2.12)

In particular, for $q = 1$ (i.e. in the WZW case) eq. (2.12) becomes

\[ L_S(q = 1) \equiv L_{WZW} = \frac{1}{2} \partial_+ \varphi \partial_- \varphi + \frac{1}{2} F(y) (\partial_+ y_s + \dot{y}_s \partial_+ \varphi)(\partial_- y_s - \dot{y}_s \partial_- \varphi). \]  

(2.13)

Applying T-duality in $\varphi$ to (2.13) gives the T-dual sigma model which has no WZ term

\[ \tilde{L}_{WZW} = \frac{1}{2} G^{-1} (\partial_+ \tilde{\varphi} - F \dot{y}_s \partial_+ y_s)(\partial_- \varphi - F \dot{y}_s \partial_- y_s) + \frac{1}{2} F \partial_+ y_s \partial_- y_s \]
\[ = \frac{1}{2} \partial_+ \tilde{\varphi} \partial_- \varphi + \frac{1}{2} \tilde{F}(y) (\partial_+ y_s - \dot{y}_s \partial_+ \tilde{\varphi})(\partial_- y_s - \dot{y}_s \partial_- \tilde{\varphi}) - \frac{1}{8} \tilde{F}(y) F(y) \partial_+ y^2 \partial_- y^2 \]  

(2.14)

\[ \tilde{F} \equiv G^{-1} F = \frac{1}{1 - \frac{1}{4} y^2} F. \]  

(2.15)

The angle $\tilde{\varphi}$ can be completely decoupled (by a local rotation of $y_s$ with phase $\tilde{\varphi}$), resulting in the familiar relation [33] to a free field plus a 2d sigma model representing the $SU(2)/U(1)$ coset

\[ \tilde{L}_{WZW} = \frac{1}{2} \partial_+ \tilde{\varphi} \partial_- \varphi + \frac{1}{2} \tilde{F}(y) [\partial_+ \dot{y}_s \partial_- \tilde{\varphi} - \frac{1}{2} F(\tilde{y}) \partial_+ \tilde{y}^2 \partial_- \tilde{\varphi}]. \]  

(2.16)

This observation explains why we will find a relativistic expression when expanding the corresponding Nambu action near $\tilde{\varphi} = \sigma$ as discussed below. Note also that this T-dual of the WZW action is parity-invariant, which will also be reflected in the corresponding S-matrix.

### 2.2 Dispersion relation

The presence of the WZ term does not change the simplest BMN geodesic solution which in the conformal gauge is given by

\[ t = J \tau , \quad \varphi = J \tau , \quad y_s = 0. \]  

(2.17)

---

8 A similar parametrization of $S^5$ (and $AdS_5$) was used, e.g., in [26]. It is related to the one in (2.8) as follows: if we set $y_1 + iy_2 = ye^{i\vartheta}$, $y = 2 \tan \frac{\vartheta}{2}$ and $\varphi = \varphi_2$ then $Gd\varphi^2 + Fd\dot{y}_s dy_s = d\theta^2 + \sin^2 \theta d\phi_1^2 + \cos^2 \theta d\phi_2^2$ and $B_s dy_s = qF \epsilon_{rs} y_r dy_s = q \sin^2 \theta d\phi_1$.

9 This form reveals the chiral structure of the model as after a local rotation of $y_s$ it is proportional to $\partial_+ \varphi \partial_- \varphi + F(\tilde{y})(\partial_+ \dot{y}_s)(\partial_- y_s - 2\epsilon_{rs} \dot{y}_r \partial_- \varphi)$ and, e.g., the equation for $\varphi$ is readily integrated.

10 Setting $y_1 + iy_2 = ye^{i\vartheta}$ we find the corresponding metric to be $ds^2 = F(y)dy_2^2 + \tilde{F}(y) \dot{y}_2^2 (d\theta + d\varphi)^2 + d\varphi^2 = dx^2 + \tan^2 x d\theta^2 + d\varphi^2$, where $\theta = \theta + \varphi$, etc.
Here $\mathcal{J}$ is proportional to the $S^3$ angular momentum of the string. Our aim will be to study the 2-particle scattering of small $y_s$ excitations around this solution. To find the spectrum of quadratic fluctuations it is sufficient to set $\varphi = \mathcal{J} \tau$ in (2.9) and expand in $y_s$:

$$L = \frac{1}{2}(\dot{y}_r^2 - y'_r y'_r - \mathcal{J}^2 y_r^2) + q\mathcal{J} \epsilon_{sr} y_s y'_r + O(y^4)$$

$$= \frac{1}{2}(\dot{y}_r^2 - (y'_r - \mathcal{J} q \epsilon_{sr} y_s)^2 - \mathcal{J}^2 (1 - q^2) y_r^2) + O(y^4).$$ (2.18)

After a local $\sigma$-dependent rotation of $y_s$ (which shifts the spatial momentum by $q \mathcal{J}$) one finds two massive modes with $m^2 = \mathcal{J}^2 (1 - q^2)$. If $\sigma$ is $2\pi$-periodic as required for the closed-string world sheet, this local rotation of $y_s$ is not possible in general. Indeed, the momentum-space dispersion relation that follows directly from (2.18) is

$$e^2 - (p \pm \mathcal{J} q)^2 = (1 - q^2) \mathcal{J}^2, \quad e \equiv p_0, \quad p \equiv p_1,$$

$$e = \pm \sqrt{p^2 \pm 2 \mathcal{J} q p + \mathcal{J}^2},$$ (2.19) (2.20)

where $p$ takes integer values $p = n = 0, 1, 2, \ldots$. One can therefore shift $p$ and get the standard massive relativistic dispersion relation only if $\mathcal{J} q$ is integer. Note that in the WZW case we get a massless dispersion relation:

$$q = 1: \quad e = \pm p \pm \mathcal{J}. \quad (2.21)$$

The formal redefinition of $y_s$ or shift of the spatial momentum is allowed, however, in the discussion of the S-matrix we are interested here, for which we consider the limit $\mathcal{J} \gg 1, \quad p = n \gg 1$, and thus effectively decompactify the $\sigma$ direction. Explicitly, we may rescale $e \rightarrow \mathcal{J} e, \quad p \rightarrow \mathcal{J} p$ so that the new momentum $p = \frac{\hat{p}}{\mathcal{J}}$ takes continuous values. We then find (for the particle $e > 0$ branch)

$$e = \sqrt{\hat{p}^2 + 1 - q^2}, \quad \hat{p} = p \pm q.$$ (2.22)

The minimal energy states correspond to $\hat{p} = 0$ and fluctuations near this vacuum have the non-relativistic massive dispersion relation,

$$e = m + \frac{\hat{p}^2}{2m} + \ldots, \quad m = \sqrt{1 - q^2}.$$ (2.23)

Therefore, as long as all finite size effects are ignored and $p$ is treated is continuous, it can be shifted by $\pm q$ and we end up with the standard massive magnon dispersion relation with $q$-dependent mass $m = \sqrt{1 - q^2}$.

This suggests that for $0 < q < 1$ the corresponding spin chain interpretation (e.g., via the connection to the Landau-Lifshitz model [34, 35]) of this near-BMN expansion should be based again on a picture of magnon scattering near a ferromagnetic vacuum just as for the $q = 0$ case. As we shall see below (in Appendix D), this conclusion is also supported by the analysis of the corresponding Pohlmeyer-reduced theory which is given again by the complex sine-Gordon model but with the rescaled mass parameter $\mu \rightarrow \mu \sqrt{1 - q^2}$.

A similar analysis applies to other massive $AdS_3 \times S^3 \times T^4$ superstring modes, which turn out to have the same mass $\sqrt{1 - q^2}$, in agreement with what was found directly in the BMN limit in [22, 36, 6] (see also Appendix A.2).
2.3 Gauge fixing and expansion of the action to quartic order

A systematic way to compute the S-matrix for the above elementary massive excitations is to fix a gauge where in addition to $t \sim \tau$ one sets the momentum density corresponding to the angle $\varphi$ of $S^3$ to be constant.\(^\text{11}\) This can be done as in [24, 26] (following [35, 38]) by first applying T-duality in the $\varphi$ direction and then fixing the static gauge $t = J \tau$, $\tilde{\varphi} = J \sigma$.\(^\text{12}\)

To compare to the $AdS_5 \times S^5$ case [26] it is useful to consider a one-parameter family of gauges which includes also the uniform light-cone gauge [41, 23]. We shall thus start with the Polyakov action for string in $R_t \times S^3$, set

$$t = u - b \varphi , \quad b \equiv \frac{a}{1 - a} ,$$

where $u$ is a new coordinate and $a$ is a gauge parameter, and then perform the T-duality transformation in the $\varphi$ direction. The resulting dual Lagrangian is (cf. (2.9), (2.10), (2.11))

$$\tilde{L} = -\sqrt{g} g^{cd} h_{cd} - b P c^{cd} \partial_{d} u (\partial_{c} \tilde{\varphi} - B_s \partial_{c} y_s) ,$$

$$h_{cd} = -Q \partial_{c} u \partial_{d} u + P (\partial_{c} \tilde{\varphi} - B_s \partial_{c} y_s) (\partial_{d} \tilde{\varphi} - B_s \partial_{d} y_s) + F \partial_{c} y_s \partial_{d} y_s ,$$

$$Q = 1 + b^2 P , \quad P = (G - b^2)^{-1} .$$

Finally, we shall fix the following gauge:

$$u = c \tau , \quad \tilde{\varphi} = c J \sigma , \quad c \equiv \frac{1}{1 - a} .$$

Here $a = 0$ is the analog of the “static” gauge and $a = \frac{1}{2}$ is the uniform light-cone gauge (when $u = t + \varphi = 2 \tau$, cf. (2.17)). We shall also rescale $\sigma$ to absorb the $J = \frac{1}{\sqrt{\lambda}}$ factor so that the cylinder is decompactified in the $J \gg 1$ limit.\(^\text{13}\) The resulting dispersion relation then takes the form (2.22).

Solving for the 2d metric $g_{cd}$ (and setting from now on $J = 1$ in (2.28) as discussed above) the corresponding Nambu Lagrangian in the gauge (2.24) takes the form

$$\tilde{L} = -\sqrt{h} + b c P (c - B_s y'_s) ,$$

$$h = \left[ c^2 Q - P (B_s y'_s)^2 - F y'_s \right] \left[ P (c - B_s y'_s)^2 + F y'_s \right] + \left[ P B_s y'_s (c - B_s y'_s) - F y'_s y'_s \right]^2 .$$

\(^\text{11}\)Note that in general one cannot fix the conformal gauge and in addition fix the fluctuation of $\varphi$ to be zero: the residual conformal transformations are parametrized only by the two functions $f(\sigma^+)$ and $f(\sigma^-)$ but $\varphi$ will not satisfy a free massless equation. This is still possible at quadratic order in the expansion in $y_s$ as discussed above.

\(^\text{12}\)Use of T-duality here is a formal trick to choose a gauge where the momentum conjugate to $\varphi$ is fixed; we are not interested in T-dual theory as such. Alternative ways of doing near-BMN expansion were discussed in [39, 40, 41].

\(^\text{13}\)As already mentioned, all finite size effects will be ignored as we will be interested only in the S-matrix.
Let us quote also the part, which can be obtained from the tree S-matrix will not be linear in to apply to the corresponding once again to the dispersion relation \((2.22)\).

The quadratic part \(L_2\) here is the same as in \((2.18)\) leading to the massive dispersion relation \((2.22)\). For \(q = 0\) the quartic part \(L_4\) agrees with eq. (4.16) in [26].\(^{14}\) Let us quote also the explicit expressions for \(a = 0, \frac{1}{2}, 1\):

\[
L_4(a = 0) = \frac{1}{2} \left[ y_s^2 y_r^2 - \frac{1}{4} (y_s^2)^2 + \frac{1}{4} (y_s^2 + y_r^2)^2 - (y_s' y_r')^2 \right] + q \left[ y_r y_s' \epsilon_{sp} y_s y_p - \frac{1}{2} (y_r^2 + y_r'^2) \epsilon_{sp} y_s y_p' \right],
\]

\[
L_4(a = \frac{1}{2}) = \frac{1}{2} y_s^2 y_r^2 + \frac{1}{2} q \left[ y_r y_s' \epsilon_{sp} y_s y_p - \frac{1}{2} (y_r^2 + y_r'^2 + y_s^2) \epsilon_{sp} y_s y_p' \right],
\]

\[
L_4(a = 1) = \frac{1}{2} y_s^2 y_r^2 + \frac{1}{4} (y_s^2)^2 - \frac{1}{4} (y_s^2 + y_r^2)^2 + (y_s' y_r')^2 - \frac{1}{2} y_r^2 \epsilon_{sp} y_s y_p' .
\]

The gauge dependence of the resulting S-matrix implies that it is not a directly observable quantity; in the corresponding Bethe ansatz it reflects the ambiguity in the choice of the corresponding spin chain length [23, 26, 27].

The above discussion admits a straightforward generalization to the case of bosonic string theory on \(AdS_3 \times S^3\): the Lagrangian \((2.9)\) picks up three extra terms representing the \(AdS_3\) part, which can be obtained from the \(S^3\) terms by a formal analytic continuation: \(\varphi \to t, y_s \to i z_s\), and then reversing the overall sign. We present the combined action in Appendix A where we also include the massless \(T^4\) directions.

The quadratic Lagrangian \(L_2\) in \((2.30)\) can be “diagonalized” by a \(\sigma\)-dependent rotation of \(y_s\), i.e. by setting

\[
y = y_1 + iy_2 = e^{iq\sigma} v, \quad y^* = y_1 - iy_2 = e^{-iq\sigma} v^* .
\]

We then find the standard massive Lagrangian for the complex scalar \(v\)

\[
L_2 = \frac{1}{2} \left[ \dot{y} y^* - (y' - i q y)(y'^* + i q y^*) - (1 - q^2) y y^* \right] = \frac{1}{2} \left[ \dot{v} v^* - v' v'^* - (1 - q^2) v v^* \right],
\]

corresponding once again to the dispersion relation \((2.22)\) with “shifted” momentum \((\hat{p} \to p)\).

\(^{14}\)As was observed there, the quartic part simplifies in the “light-cone” gauge \(a = \frac{1}{2}\). This does not appear to apply to the \(q\)-dependent terms. Note also that to quartic order the Lagrangian is linear in \(q\). However, the tree S-matrix will not be linear in \(q\) as \(q\) appears also in the dispersion relation.
Applying the field redefinition (2.35) to \( L_4 \) in (2.31) we get\(^{15}\)

\[
L_4 = \frac{1}{8} \left\{ 2q^2 (1 - q^2) v^2 v'^2 + 3iq (1 - q^2) vv^* (vv'^* - v^* v') + 4(1 - q^2) vv^* v'^* \\
+ q^2 [v'^2 (v'^2 - \dot{v}'^2) + v'^2 (v'^2 - \dot{v}'^2)] - iq [vv' (v'^2 - \dot{v}'^2) - v^* v'^* (v'^2 - \dot{v}'^2)] \right\}
\]

\[
+ \frac{1}{4} (a - \frac{1}{2}) \left\{ (1 - q^2)^2 v'^2 v'^2 - iq (1 - q^2) vv^* (vv'^* - v^* v') \\
- iq [vv' (v'^2 - \dot{v}'^2) - v^* v'^* (v'^2 - \dot{v}'^2)] - (v'^2 - \dot{v}'^2)(v'^2 - \dot{v}'^2) \right\} . \tag{2.37}
\]

For \( q = 0 \) and \( q = 1 \) we have

\[
L_4(q = 0) = \frac{1}{2} vv^* vv'^* + \frac{1}{4} (a - \frac{1}{2}) \left[ v^2 v'^2 - (v'^2 - \dot{v}'^2)(v'^2 - \dot{v}'^2) \right] , \tag{2.38}
\]

\[
L_4(q = 1) = \frac{1}{8} \left[ v^2 (v'^2 - \dot{v}'^2) + v'^2 (v'^2 - \dot{v}'^2) - iq [vv' (v'^2 - \dot{v}'^2) - v^* v'^* (v'^2 - \dot{v}'^2)] \right] \\
+ \frac{1}{4} (a - \frac{1}{2}) \left[ - i [vv' (v'^2 - \dot{v}'^2) - v^* v'^* (v'^2 - \dot{v}'^2)] - (v'^2 - \dot{v}'^2)(v'^2 - \dot{v}'^2) \right] . \tag{2.39}
\]

Note that for \( q = 1 \), i.e. at the WZW point, the Lorentz invariance and also the parity invariance of the quartic Lagrangian is restored in the \( a = 0 \) gauge:

\[
L_4(q = 1, a = 0) = \frac{1}{8} \left[ v^2 (v'^2 - \dot{v}'^2) + v'^2 (v'^2 - \dot{v}'^2) + (v'^2 - \dot{v}'^2)(v'^2 - \dot{v}'^2) \right] \\
= \frac{1}{8} \left[ - (v^2 \partial_+ v^* \partial_- v^* + v^2 \partial_+ v \partial_- v) + \partial_+ v \partial_- v \partial_+ v^* \partial_- v^* \right] . \tag{2.40}
\]

The reason for this was previously mentioned below (2.16).\(^{16}\)

## 3 Tree-level S-matrix of bosonic string on \( R \times S^3 \) with \( B \)-flux

Starting with the Lagrangian (2.30),(2.31) or (2.36),(2.37) it is straightforward (following [24, 25, 26]) to compute the corresponding tree-level 2-particle S-matrix for the elementary massive excitations of the bosonic string on \( S^3 \) with non-zero \( B \)-field flux.

Despite the apparently complicated dependence of \( L_4 \) on \( q \) (cf. eq. (2.37)) we shall find that the resulting S-matrix has a very simple structure: its expression for \( q \neq 0 \) can be found from its \( q = 0 \) limit by replacing \( e(p) = \sqrt{p^2 + 1} \) with the modified dispersion relation \( e(p) = \sqrt{(p \pm q)^2 + 1 - q^2} \). Here the plus sign corresponds to, e.g., boson \( \gamma \) and the minus sign to \( \gamma^* \) in (2.36): we assume that \( y \sim e^{-i e \tau - i p \sigma} \) as appropriate for ingoing fields in a scattering amplitude.

\(^{15}\)All \( \sigma \)-dependence cancels out of course due to the global \( U(1) \) invariance. Therefore the transformation simply amounts to shifts of the spatial derivatives \( y' = \dot{v}' + iqv, y'^* = \dot{v}'^* - iqv^* \), i.e. shifts of the corresponding spatial momenta.

\(^{16}\)Note that for \( q = 1 \) we have massless excitations, i.e. \( v = u_+ + u_- \) where \( \partial_- u_+ = 0, \partial_+ u_- = 0 \) are formal “in”-fields. Then integrating by parts we get

\[
L_4(q = 1, a = 0) = \frac{1}{8} (u_+^* u_-^* \partial_+ u_+ \partial_- u_- + c.c.) + \frac{1}{8} \partial_+ u_+ \partial_+ u_+ u_-^* \partial_- u_- u^* .
\]

While the meaning of the corresponding S-matrix is not immediately clear, there are no LL or RR, only LR scattering processes (see also section 3.2 below).
The S-matrix may be written as

$$S = I + i\hbar^{-1} T, \quad \hbar^{-1} \equiv \frac{2\pi}{\sqrt{\lambda}},$$

(3.1)

where $I$ is the identity operator, and $T$ is the interaction part of the scattering operator. As the theory under consideration is an integrable one we should have particle-number conservation and thus we can represent $T$ as a map from two-particle states to two-particle states ($p$ and $p'$ are spatial momenta)

$$T |y_m(p)y_n(p')\rangle = T^{rs}_{mn}(p,p') |y_r(p)y_s(p')\rangle.$$  

(3.2)

The expression for the S-matrix of the $R \times S^3$ theory in the case of the vanishing NSNS flux (for $q = 0$) is a truncation [25] of the $S^3$ part of the $AdS_5 \times S^5$ result in [26]. Let us start with summarizing this expression and then present our result for non-zero $q$.

### 3.1 Vanishing $B$-flux

Considering only the $S^3$ sector (i.e. two massive states) we may simply take the $S^5$ expression for the S-matrix in [26] and restrict the $SO(4)$ index to an $SO(2)$ index (the same result follows of course directly from (2.31) with $q = 0$)

$$T^{rs}_{mn}(p,p') = \left[-\frac{p^2 + p'^2}{2(e'p - ep')} + (a - \frac{1}{2})(ep' - e'p)\right] \delta^r_m \delta^s_n - \frac{pp'}{e'p - ep'} \epsilon^r_m \epsilon^s_n,$$

$$e = \sqrt{p^2 + 1}, \quad e' = \sqrt{p'^2 + 1}.$$  

(3.3)

Here $\epsilon_{rs} = -\epsilon_{sr}$ and $\epsilon^r_m \epsilon^s_n = \delta^r_m \delta^s_n - \delta^r_n \delta^s_m$. It is useful to change the basis of fields from the real $y_s = (y_1, y_2)$ to the complex $(y, y^*) = y_1 + iy_2$ one. We shall use labels $+$ for $y$ and $-$ for $y^*$. In this basis one finds that $T^{++}_{++} = T^{--}_{--} \neq 0$, $T^{+-}_{+-} = T^{-+}_{-+} \neq 0$, with all other amplitudes vanishing. Explicitly,

$$T^{++}_{++}(p,p') = \frac{(p + p')^2}{2(e'p - ep')} + (a - \frac{1}{2})(ep' - e'p) = \frac{1}{2}(p + p') (ep' + e'p) + (a - \frac{1}{2})(ep' - e'p),$$

$$T^{+-}_{+-}(p,p') = \frac{(p - p')^2}{2(e'p - ep')} + (a - \frac{1}{2})(ep' - e'p) = \frac{1}{2}(p - p') (ep' + e'p) + (a - \frac{1}{2})(ep' - e'p),$$

$$T^{++}_{+-}(p,p') = 0, \quad e = \sqrt{p^2 + 1}, \quad e' = \sqrt{p'^2 + 1},$$

(3.4)

where we have simplified the expressions using the identity

$$(e'p + ep')(e'p - ep') = (p + p')(p - p').$$

(3.5)

The above tree amplitude $T \equiv T^{++}_{++}$ matches [24, 25, 26] the expansion of the AFS [42] spin chain S-matrix in the $SU(2)$ sector.

Let us recall for completeness the expressions for leading-order (tree-level) S-matrices in several similar models. In the Landau-Lifshitz (LL) model [28] (here we use $\kappa = \hbar^{-1}$ as an
effective coupling)

\[ S_{LL}(p, p') = \frac{1 + \frac{i\kappa pp'}{p-p'} + \frac{2i\kappa pp'}{p-p'}}{1 - \frac{i\kappa pp'}{p-p'}} = 1 + \frac{2i\kappa pp'}{p-p'} + ... , \]  

where \( T = \frac{2i\kappa pp'}{p-p'} \) matches the expansion of the magnon S-matrix in the XXX spin chain Bethe ansatz. In the Faddeev-Reshetikhin (FR) model \cite{28}

\[ S_{FR}(p, p') = \frac{1 + \frac{2i\kappa x' - x}{x' - x} + \frac{4i\kappa}{x' - x} + ...}{1 - \frac{2i\kappa x' - x}{x' - x}} = 1 + \frac{4i\kappa}{x' - x} + ... , \]  

\[ x(p) = \frac{1}{p} [e(p) + 1] , \quad e(p) = \sqrt{p^2 + 1} , \]  

i.e.

\[ S_{FR}(p, p') = 1 + \frac{4i\kappa pp'}{p(e' + 1) - p'(e + 1)} + ... = 1 + \frac{2i\kappa [pp' - (e-1)(e'-1)]}{pe' - p'e} + ... . \]  

This agrees with (3.6) for small momenta. The FR S-matrix (3.9) is somewhat similar to the expansion of the BDS \cite{43} spin chain S-matrix given by \cite{24}

\[ S_{BDS}(p, p') = \frac{1 + \frac{i\kappa}{u - \bar{u}}}{1 - \frac{i\kappa}{u - \bar{u}}} = 1 + \frac{2i\kappa}{u - \bar{u}} + ... = 1 + \frac{2i\kappa pp'}{pe' - p'e} + ... , \]  

\[ u(p) = \frac{1}{2} \cot \frac{p}{2} \sqrt{1 + \frac{\lambda}{\pi} \sin^2 \frac{p}{2}} , \quad u_{p \to 0, \lambda p^2 = \text{fixed}} \to \frac{1}{p} e(p) , \quad \bar{p} = \frac{\sqrt{\lambda}}{4\pi} p . \]  

The asymptotic Bethe ansatz S-matrix \cite{8} contains, in addition to the BDS factor, the dressing phase, given at strong coupling by the AFS expression. The limit that allows to compare to the string world-sheet sigma model S-matrix is \( p \to 0, \lambda \gg 1 \) with \( \lambda p^2 = \text{fixed} \). In terms of the fixed (rescaled or string sigma model) momenta one then finds for the corresponding leading (string tree-level) part of the spin chain S-matrix \cite{24, 25, 26}

\[ S_{AFS}(p, p') = 1 + \frac{2i\kappa F(p, p')}{pe' - p'e} + ... , \quad e = \sqrt{p^2 + 1} , \]  

\[ \frac{2F(p, p')}{pe' - p'e} = \frac{2pp'}{pe' - p'e} + \theta_{AFS}(p, p') , \]  

\[ \theta_{AFS}(p, p') = \frac{(p - p')^2}{2(pe' - p'e)} + \frac{1}{2}(pe' - p'e) - (p - p') , \]  

\[ F(p, p') = pp' - (e-1)(e'-1) + \frac{1}{4} [pp' - (e-1)(e'-1)]^2 . \]  

A useful equivalent form is

\[ \frac{2F(p, p')}{pe' - p'e} = \frac{(p + p')^2}{2(e'p - ep')} + \frac{1}{2}(e'p - ep') - (p - p') . \]
The expression (3.16) should be compared to the one for $T_{++}^{++}$ in (3.4) taken in the $a = 0$ gauge (where the BMN charge $J$ plays the role of the spin chain length)

$$T_{++}^{++}(p, p') = \frac{(p + p')^2}{2(e'p - ep')} + \frac{1}{2}(e'p - ep') = \frac{e'p^2 + e'p'^2}{p - p'}.$$ (3.17)

They match up to the last term in (3.16), which is linear in momentum and hence cancels in the Bethe ansatz equations [26].

### 3.2 Non-vanishing B-flux

To find the $q \neq 0$ generalization of (3.4) let us start with the action (2.36), (2.37) written in terms of the complex scalar $v$ (2.35). The corresponding particle dispersion relation is the standard massive one, $e(p) = \sqrt{p^2 + 1 - q^2}$. Labelling again the excitations associated to $v$ and $v^*$ with the index + and − the tree-level S-matrix that follows directly from the quartic action (2.37) is found to be\(^\text{17}\)

$$T^{v \pm \pm}(p, p') = \frac{p + p' \mp 2q}{2(ep' - ep)} \left[ (1 - q^2)(p + p' \mp q) \mp q(e'p - ep') \right]$$

$$+ \left( a - \frac{1}{2} \right) \left[ (ep' - e'p) \mp q(p + p') (ee' - pp') - (1 - q^2) \right],$$

$$T^{v \pm \mp}(p, p') = \frac{p - p' \mp 2q}{2(ep' - ep)} \left[ (1 - q^2)(p - p' \mp q) \pm q(e'p - ep') \right]$$

$$+ \left( a - \frac{1}{2} \right) \left[ (ep' - e'p) \pm q(p - p') (ee' - pp') + (1 - q^2) \right],$$

$$T^{v \pm \pm}(p, p') = 0,$$ \hspace{1cm} $e = \sqrt{p^2 + 1 - q^2}, \quad e' = \sqrt{p'^2 + 1 - q^2}. \hspace{1cm} (3.18)$

The dependence on $q$ is explicit as well as implicit through the energies $e, e'$. These expressions of course agree with (3.4) for $q = 0$.

Using the generalization of the identity (3.5) to non-zero $q\(^\text{18}\)

$$(e'p + ep')(e'p - ep') = (1 - q^2)(p + p')(p - p'),$$ (3.19)

the scattering amplitudes (3.18) can be rewritten in the following simple form

$$T^{v \pm \pm}(p, p') = \frac{p + p' \pm 2q}{2(p - p')} \left[ e(p' \mp q) + e'(p \pm q) \right] + \left( a - \frac{1}{2} \right) \left[ e(p' \mp q) - e'(p \mp q) \right],$$

$$T^{v \pm \mp}(p, p') = \frac{p - p' \pm 2q}{2(p - p')} \left[ e(p' \pm q) + e'(p \mp q) \right] + \left( a - \frac{1}{2} \right) \left[ e'(p' \pm q) - e'(p \pm q) \right],$$

$$T^{v \pm \pm}(p, p') = 0,$$ \hspace{1cm} $e = \sqrt{p^2 + 1 - q^2}, \quad e' = \sqrt{p'^2 + 1 - q^2}. \hspace{1cm} (3.20)$

\(^\text{17}\)As usual, the factor $\frac{1}{4(e'p - ep')}$ arises when solving the two-dimensional delta-function constraint imposing the energy conservation. The superscript $v$ on $T$ indicates that we are scattering excitations associated to $v$ and $v^*$. Later we will return to the S-matrix for $y$ and $y^*$.

\(^\text{18}\)Note also that $\frac{e'p - ep'}{e'p - ep'} = \frac{1}{p - p'}$ and $\frac{e'p - ep' - (1 - q^2)^2}{e'p - ep'} = \frac{e'p - ep'}{p - p'}$.\(^\text{18}\)
We thus discover that the dependence of the S-matrix on \( q \) is remarkably simple: it can be absorbed into the shifts of the momentum of the particle \( v \) by \(-q\) and of the momentum of the antiparticle \( v^* \) by \(+q\) (the denominators are invariant under such shifts). Note that the shifts apply only to the spatial momenta, not to the energies \( e, e' \).

These momentum shifts are precisely the same as those done in eq. (2.35), which put the action (2.31) into the form (2.37) with a standard massive dispersion relation. Thus undoing these shifts simply amounts to going back to the “unrotated” basis of fields \( y = y_1 + iy_2, \ y^* = y_1 - iy_2 \). This leads to the following final very simple expression for the S-matrix and dispersion relations for the \((y, y^*)\) in the \(0 \leq q \leq 1\) case:

\[
T^{\pm\pm}_{\pm\pm}(p, p') = \frac{p + p'}{2(p - p')}(e_{\pm} p' + e'_{\pm} p) + (a - \frac{1}{2})(e_{\pm} p' - e'_{\pm} p),
\]

\[
T^{\pm\mp}_{\pm\pm}(p, p') = \frac{p - p'}{2(p + p')}(e_{\pm} p' + e'_{\pm} p) + (a - \frac{1}{2})(e_{\pm} p' - e'_{\pm} p),
\]

\[
T^{\pm\pm}_{\pm\pm}(p, p') = 0, \quad e_{\pm} = \sqrt{(p \pm q)^2 + 1 - q^2}, \quad e'_{\pm} = \sqrt{(p' \pm q)^2 + 1 - q^2}, \quad (3.21)
\]

where the \( \pm \) indices on \( e \) indicate that for states with positive/negative charge we take \( p \pm q \) in the dispersion relation.

The form of the S-matrix is thus formally the same as in (3.4) but with different \( e(p) \): all the dependence on \( q \) enters through \( q \)-dependence of the energy \( e(p) \) in (3.21). To summarize, the S-matrix admits two equivalent representations, depending on whether we scatter the rotated fields \( v \) or the original fields \( y \) which are symbolically (here \( e(p, m) \equiv \sqrt{p^2 + m^2} \) and \( p_i = (p, p') \)):

\[
T^v(p_i \pm q, e(p_i, 1 - q^2)) \quad \text{and} \quad T(p_i, e(p_i \pm q, 1 - q^2)).
\]

This remarkable property of the S-matrix, which is by no means obvious from the action (2.31) (having a non-trivial dependence on \( q \)) should be a consequence of some symmetry of the underlying integrable model.\(^1\)

For the natural gauge choice \( a = 0 \), for which comparison with the standard spin chain is most direct (length is \( J \)), we find the following \( q \)-generalization of (3.17)

\[
T^{++}_{++}(p, p')_{a=0} = \frac{e p^2 + e' p'^2}{p - p'}, \quad e = \sqrt{p^2 + 2qp + 1}, \quad e' = \sqrt{p'^2 + 2qp' + 1}. \quad (3.22)
\]

Let us now comment on some properties of the tree-level S-matrix (3.20). First, it satisfies the following identities

\[
T^{v}_{\pm\pm}(p, p') = T^{v}_{\pm\pm}(p', p)|_{e' = -\sqrt{p'^2 + 1 - q^2}},
\]

\[
T^{v}_{\pm\pm}(p, p') + T^{v}_{\pm\pm}(p', p) = 0, \quad T^{v}_{\pm\pm}(p, p') + T^{v}_{\pm\pm}(p', p) = 0,
\]

\[
[T^{v}_{\pm\pm}(p, p')]^* + T^{v}_{\pm\pm}(p', p) = 0, \quad [T^{v}_{\pm\pm}(p, p')]^* + T^{v}_{\pm\pm}(p', p) = 0, \quad p, p' \in \mathbb{R}. \quad (3.23)
\]

\(^1\)Let us note also that the corresponding Pohlmeyer-reduced theory depends on \( q \) only via the rescaling of the mass parameter by \((1 - q^2)^{1/2} \). Therefore, its (relativistic) S-matrix takes the same form as in the \( q = 0 \) case, see Appendix D.
These are the crossing symmetry, braiding unitarity and hermitian analyticity relations respectively. The latter two combined imply the expected QFT unitarity of the S-matrix. Furthermore, as the T-matrix is diagonal \( (T^v_{\pm\pm}(p,p') = 0) \), it trivially satisfies the classical Yang-Baxter equation.

In the special case of \( q = 1 \) when the world-sheet action is given by the \( SU(2) \) WZW model we get massless excitations (see (2.21)): there are left- and right-moving modes, for which \( p = -e = -\frac{1}{2}p_L \) and \( p = e = \frac{1}{2}p_R \) respectively. To consider the \( q \to 1 \) limit in the S-matrix (3.20) let us first multiply it by the Jacobian factor \( e'p - ep' \). We may then compute different possible scattering amplitudes, i.e. left-left (LL), right-right (RR), left-right (LR) and left-right (RL), by simply substituting in the on-shell relations into (2.39) as appropriate. The LL and RR amplitudes vanish, while for the LR scattering processes we find

\[
\hat{T}^{v\pm\pm}(p_L,p'_R) = 2p_Lp'_R \left[ (1 - 2a)p_Lp'_R - 1 \pm a(p_L + p'_R) \right], \\
\hat{T}^{v\pm\mp}(p_L,p'_R) = 2p_Lp'_R \left[ (1 - 2a)p_Lp'_R + 1 \mp a(p'_L - p'_R) \right], \\
\hat{T}^{v\mp\pm}(p_L,p'_R) = 0,
\]

and the RL scattering amplitudes immediately follow from these. For the gauge choice \( a = 0 \) this tree-level S-matrix is relativistically invariant as expected (see (2.40)).

The simple structure of the S-matrix (3.21) in the \( R \times S^3 \) sector we have found above has a direct analog in the case of strings moving in \( AdS_3 \times S^1 \) (one needs just to reverse sign of the first term in \( T^{\mp\mp} \), etc.). Using the Lagrangian of Appendix A we have also computed the full bosonic S-matrix in the \( AdS_3 \times S^3 \) sector and again the results for \( q = 0 \) and \( 0 < q \leq 1 \) are found to be related by momentum shifts in the energy as described above.\(^\text{22}\) This pattern also suggests a generalization to the full \( AdS_3 \times S^3 \times T^4 \) superstring with \( 0 \leq q \leq 1 \), which is discussed in the next section.

### 4 Tree-level S-matrix of \( AdS_3 \times S^3 \times T^4 \) superstring with mixed flux

The bosonic-sector results of the previous section suggest a natural generalization to the full tree-level world-sheet S-matrix for the massive BMN modes of superstring theory on \( AdS_3 \times S^3 \times T^4 \) with a mixed RR-NSNS flux.

#### 4.1 Vanishing B-flux

Let us start with the \( q = 0 \) (pure RR flux) case. The corresponding massive tree-level S-matrix can be found from its known \( AdS_5 \times S^5 \) counterpart \(^\text{26}\) by a suitable truncation. Below

\(^{20}\)One can easily see that for massless scattering states the inverse of this factor can be divergent and as such it should be regularized properly. We shall avoid this issue by working with the standard QFT amplitudes, i.e. coefficients of \( \delta^{(2)}(p_1 + p_2 + p_3 + p_4) \).

\(^{21}\)Note that if the in-state consists of a left mode and a right mode, then simple 2-d kinematical considerations show that the out-state must also consist of a left mode and a right mode. Furthermore, the momentum of the ingoing left mode should equal the momentum of the outgoing left mode and similarly for the right mode.

\(^{22}\)In Appendix A we comment also on the S-matrix including the massless \( T^4 \) modes.
we present the resulting $\text{AdS}_3 \times S^3$ S-matrix in the basis where the massive excitations are represented by two complex bosonic $(y_{\pm}, z_{\pm})$ \(^{23}\) and two complex fermionic $(\zeta_{\pm}, \chi_{\pm})$ fields: \(^{24}\)

**Boson-Boson**

\[
\begin{align*}
T \ket{y_{\pm}y_{\pm}'} &= (l_1 + c) \ket{y_{\pm}y_{\pm}'} \\
T \ket{z_{\pm}z_{\pm}'} &= (-l_1 + c) \ket{z_{\pm}z_{\pm}'} \\
T \ket{y_{\pm}z_{\pm}'} &= (l_3 + c) \ket{y_{\pm}z_{\pm}'} + l_5 \ket{\zeta_{\pm}z_{\pm}'} - l_5 \ket{\chi_{\pm}z_{\pm}'} \\
T \ket{z_{\pm}y_{\pm}'} &= (-l_3 + c) \ket{z_{\pm}y_{\pm}'} - l_5 \ket{\chi_{\pm}z_{\pm}'} + l_5 \ket{\zeta_{\pm}z_{\pm}'} \\
\end{align*}
\]

**Fermion-Fermion**

\[
\begin{align*}
T \ket{\zeta_{\pm}\zeta_{\pm}'} &= c \ket{\zeta_{\pm}\zeta_{\pm}'} \\
T \ket{\chi_{\pm}\chi_{\pm}'} &= c \ket{\chi_{\pm}\chi_{\pm}'} \\
T \ket{\zeta_{\pm}\chi_{\pm}'} &= l_5 \ket{y_{\pm}z_{\pm}'} + l_5 \ket{z_{\pm}y_{\pm}'} \\
T \ket{\chi_{\pm}\zeta_{\pm}'} &= -l_5 \ket{z_{\pm}y_{\pm}'} - l_5 \ket{y_{\pm}z_{\pm}'} \\
\end{align*}
\]

**Boson-Fermion**

\[
\begin{align*}
T \ket{y_{\pm}\zeta_{\pm}'} &= (l_6 + c) \ket{y_{\pm}\zeta_{\pm}'} - l_5 \ket{\zeta_{\pm}y_{\pm}'} \\
T \ket{\zeta_{\pm}y_{\pm}'} &= (l_8 + c) \ket{\zeta_{\pm}y_{\pm}'} - l_5 \ket{y_{\pm}\zeta_{\pm}'} \\
T \ket{y_{\pm}\chi_{\pm}'} &= (l_6 + c) \ket{y_{\pm}\chi_{\pm}'} - l_5 \ket{\chi_{\pm}y_{\pm}'} \\
T \ket{\chi_{\pm}y_{\pm}'} &= (l_8 + c) \ket{\chi_{\pm}y_{\pm}'} - l_5 \ket{y_{\pm}\chi_{\pm}'} \\
T \ket{z_{\pm}\zeta_{\pm}'} &= (-l_6 + c) \ket{z_{\pm}\zeta_{\pm}'} + l_5 \ket{\zeta_{\pm}z_{\pm}'} \\
T \ket{\zeta_{\pm}z_{\pm}'} &= (-l_8 + c) \ket{\zeta_{\pm}z_{\pm}'} + l_5 \ket{z_{\pm}\zeta_{\pm}'} \\
T \ket{z_{\pm}\chi_{\pm}'} &= (-l_6 + c) \ket{z_{\pm}\chi_{\pm}'} + l_5 \ket{\chi_{\pm}z_{\pm}'} \\
T \ket{\chi_{\pm}z_{\pm}'} &= (-l_8 + c) \ket{\chi_{\pm}z_{\pm}'} + l_5 \ket{z_{\pm}\chi_{\pm}'} \\
\end{align*}
\]

Here the functions $l_i$ and $c$ depending on momenta are given by

\[
\begin{align*}
l_1(p, p') &= \frac{(p + p')^2}{2(e'p - ep')}, & c(p, p') &= \frac{(a - \frac{1}{2})}{2}(ep' - e'p), \\
l_2(p, p') &= \frac{(p - p')^2}{2(e'p - ep')}, & l_3(p, p') &= \frac{-(p - p')(p + p')}{2(e'p - ep')}, \\
l_4(p, p') &= -pp' \left[\sqrt{(e + p)(e' - p') - \sqrt{(e - p)(e' + p')}}\right], \\
l_5(p, p') &= -pp' \left[\sqrt{(e + p)(e' - p')} + \sqrt{(e - p)(e' + p')}\right].
\end{align*}
\]

\(^{23}\)Here $y_{\pm}, y_{\mp}$ are bosonic $S^3$ excitations denoted by $(y, y^*) = y_1 \pm iy_2$ above. $z_{\pm}, z_{\mp}$ are the counterparts $(z, z^*) = z_1 \pm iz_2$ in the $\text{AdS}_3$ sector (see Appendix A). A prime on a field indicates that it has momentum $p'$. \(^{24}\)This ansatz is only valid at tree level: at higher loop orders additional scattering processes will appear. For example, the $(y_{\pm}y_{\pm}') \rightarrow |z_{\pm}z_{\pm}'\rangle$, $|z_{\pm}z_{\mp}'\rangle \rightarrow |y_{\pm}y_{\pm}'\rangle$, $|y_{\pm}z_{\pm}'\rangle \rightarrow |z_{\pm}z_{\pm}'\rangle$, $|z_{\pm}z_{\pm}'\rangle \rightarrow |y_{\pm}z_{\pm}'\rangle$ amplitudes, and similar amplitudes involving the fermions, should all be non-zero at 1-loop. Their vanishing at the tree level is actually a requirement of symmetry factorization – see discussion below.
\[ l_6(p, p') = \frac{(p + p')p'}{2(e'p - ep')}, \quad l_7(p, p') = -\frac{(p - p')p'}{2(e'p - ep')} \]
\[ l_8(p, p') = \frac{(p + p')p}{2(e'p - ep')}, \quad l_9(p, p') = \frac{(p - p')p}{2(e'p - ep')} \]
\[ e = \sqrt{p^2 + 1}, \quad e' = \sqrt{p'^2 + 1}. \] (4.2)

Using the identity (3.5) the functions \( l_i \) can be simplified as follows:
\[ l_1(p, p') = \frac{(p + p')(e'p + ep')}{2(p - p')}, \quad l_2(p, p') = \frac{(p - p')(e'p + ep')}{2(p + p')} \]
\[ l_3(p, p') = -\frac{1}{2}(e'p + ep'), \quad l_4(p, p') = -\frac{pp'}{2(p + p')}\left[\sqrt{(e + p)(e' + p') - \sqrt{(e - p)(e' - p')}}\right] \]
\[ l_5(p, p') = -\frac{pp'}{2(p - p')}\left[\sqrt{(e + p)(e' + p') + \sqrt{(e - p)(e' - p')}}\right] \]
\[ l_6(p, p') = -\frac{p'(e'p + ep')}{2(p - p')}, \quad l_7(p, p') = \frac{p'(e'p + ep')}{2(p + p')} \]
\[ l_8(p, p') = \frac{pp'(e'p + ep')}{2(p - p')}, \quad l_9(p, p') = \frac{pp'(e'p + ep')}{2(p + p')} \] (4.3)

This S-matrix is invariant under a \( U(1)^3 \) symmetry with the fields \( \{y_\pm, z_\pm, \zeta_\pm, \chi_\pm\} \) charged as follows
\[ \pm \{\alpha_1 + \alpha_2, \alpha_1 - \alpha_2, \alpha_1 + \alpha_3, \alpha_1 - \alpha_3\}. \] (4.4)

The index \( \pm \) that the fields carry indicates the charge under the \( U(1) \) symmetry with parameter \( \alpha_1 \). Crucially, with respect to this \( U(1) \) there are no reflection processes. More precisely, the association of the momenta to the \( U(1) \) charge of the states is preserved by the scattering process.

One may wonder a priori why the above truncation of the \( AdS_5 \times S^5 \) S-matrix should indeed represent the massive sector of the S-matrix of the \( AdS_3 \times S^3 \times T^4 \) theory with pure RR flux. This is to large extent fixed by the correspondence of the spectra and bosonic sectors and by the expected supersymmetries. The symmetry algebra of the \( AdS_5 \times S^5 \) S-matrix is known to be \( (\mathfrak{psu}(2|2) \oplus \mathfrak{psu}(2|2)) \times \mathbb{R}^3 \cong \mathfrak{psu}(2|2)^2 \times \mathbb{R}^3 \), while we find that the corresponding symmetry in the \( AdS_3 \times S^3 \times T^4 \) theory is \( [\mathfrak{ps}(u(1)^2)]^2 \times \mathbb{R}^3 \) with the two central elements in the first factor identified (see Appendix B for details and notation).\(^{25}\)

\(^{25}\)There appears to be some confusion in the literature regarding the symmetry preserved by the BMN vacuum in this theory (of course, as in the \( AdS_5 \times S^5 \) case, not all of the symmetry of the S-matrix may be visible in the off-shell Lagrangian, see, e.g., [27]). In the discussion of the giant magnons in the \( SU(2) \) sector of the \( AdS_3 \times S^3 \times T^4 \) theory in [10] the remaining symmetry was claimed to be \( \mathfrak{su}(1|1)^2 \), but it was implicitly assumed that two copies of this algebra should appear as a symmetry when discussing the S-matrix (cf. eq. (3.18) in the first paper in [10]). In [13] the symmetry of the S-matrix for the \( AdS_3 \times S^5 \times S^1 \) theory was assumed to be \( \mathfrak{su}(1|1) \) while the corresponding symmetry for \( AdS_3 \times S^3 \times T^4 \) case was doubled: \( \mathfrak{su}(1|1)^2 \). In [14] the symmetry of the S-matrix of the \( AdS_3 \times S^3 \times S^3 \times S^1 \) theory was taken as \( (u(1) \in \mathfrak{su}(1|1)^2) \times \mathbb{R}^2 \), which is consistent with
Indeed, $\text{psu}(2|2)$ has a $\text{ps}(u(1|1)^2)$ subgroup preserved by the truncation. The resulting $[\text{ps}(u(1|1)^2)]^2 \times \mathbb{R}^3$ symmetry matches (modulo the quantum deformation and central elements) the symmetry of the S-matrix of the corresponding Pohlmeyer-reduced theory [50]. While the limiting $AdS_3 \times S^3 \times T^4$ case was not explicitly discussed in [14] (which considered the $AdS_3 \times S^3 \times S^3 \times S^1$ theory), we believe the massive sector of the corresponding string tree-level (strong-coupling) S-matrix should match the above result (4.1),(4.2),(4.3). It should also be in agreement with the tree-level part of the expression in [17] found by direct string-theory computation and claimed to be in agreement with [14].

Indeed, as we shall explain in Appendix A.2, the quadratic fermionic action that reproduces all of the above amplitudes involving two fermions and two bosons is the same as the $AdS_3 \times S^3 \times T^4$ limit of the action found in [12, 17] directly from the $AdS_3 \times S^3 \times S^3 \times S^1$ superstring action. This explicitly confirms that the S-matrix (4.1) is in agreement with the world-sheet theory.

### 4.2 Non-vanishing $B$-flux

The above observations combined with the $q \neq 0$ bosonic sector results and the requirements of integrability allow us to conjecture the expression for the tree-level S-matrix for the massive states of the superstring theory with a non-zero NSNS flux ($q \neq 0$). From the direct computation of the $yy \rightarrow yy$ amplitudes in the $S^3$ sector in the previous section and of the $yy \rightarrow zz$ and $zz \rightarrow zz$ amplitudes, which follow from the quartic Lagrangian in Appendix A, we know that to find the $q \neq 0$ generalization of the functions $l_{1,2,3}$ and $c$ one should take the $q = 0$ expressions given in (4.3) and modify the dispersion relation

\begin{equation}
    e \rightarrow e_\pm = \sqrt{(p \pm q)^2 + 1 - q^2}, \quad e' \rightarrow e'_\pm = \sqrt{(p' \pm q)^2 + 1 - q^2},
\end{equation}

where, as in (3.21), for states with positive/negative charge we use $e_+/e_-$, i.e. we shift the momentum in the dispersion relation by $\pm q$.

The functions $l_{6,7,8,9}$ are then constrained by the first requirement of integrability – the symmetry factorization property of the S-matrix. In the $q = 0$ case the symmetry algebra of the S-matrix takes the form of a direct sum with central extension $s$ – see Appendix B. Combined with integrability this implies the S-matrix should factorize under this structure. In the theory with mixed RR and NSNS flux, i.e. $q \neq 0$, the global symmetry of the string Lagrangian is unaltered and the symmetry algebra of the S-matrix should be unchanged – $q$ should appear in the particular representation used. Therefore, we expect the S-matrix will still factorize in the same way.

Let us first briefly review the factorization in the $q = 0$ case. Formally defining the following the above symmetry algebra $[\text{ps}(u(1|1)^2)]^2 \times \mathbb{R}^3$ (with the two central elements in the first factor identified) for the $AdS_3 \times S^3 \times T^4$ case, taking into account that the symmetry should be doubled and the central extensions identified in this limit.

Let us mention also a remark in the first paper in [11] that the proposed Bethe ansatz agrees in the $AdS_3 \times S^3 \times T^4$ limit with the standard $AdS_5 \times S^5$ one in the $\text{psu}(1,1|2)$ sector. This is again consistent with the S-matrix truncation picture.

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26 Let us mention also a remark in the first paper in [11] that the proposed Bethe ansatz agrees in the $AdS_3 \times S^3 \times T^4$ limit with the standard $AdS_5 \times S^5$ one in the $\text{psu}(1,1|2)$ sector. This is again consistent with the S-matrix truncation picture.
The Yang-Baxter equation

\[ \text{generalization procedure as was found for the functions factorize for any value of } q \text{ scattering processes, the requirement that the tree-level S-matrix of the is a map from the space } | \Omega_1 \rangle \rightarrow \Omega_2 \text{ Yang-Baxter operator. Using these relations it is simple to find the appropriate classical Yang-Baxter equation (i.e. with the required minus signs) that our tree-level S-matrix is required to satisfy.} \]

The factorization property does not constrain \( \{ \Phi, \psi \} \) have charges \{1, 0\} and \{0, 1\} respectively.

The factorization relies on the following identities

\[ l_0 = \frac{1}{2}(l_1 + l_3), \quad l_8 = \frac{1}{2}(l_1 - l_3), \quad l_7 = \frac{1}{2}(l_2 + l_3), \quad l_9 = \frac{1}{2}(l_2 - l_3). \]

Observing that the \( ++ \rightarrow ++ \) scattering amplitudes in (4.1) are only built from the \( ++ \rightarrow ++ \) amplitudes in (4.8) and the same is true for the \( + - \rightarrow +- \), \( -+ \rightarrow -+ \) and \( -- \rightarrow -- \) scattering processes, the requirement that the tree-level S-matrix of the \( q \neq 0 \) theory should factorize for any value of \( q \) implies that the functions \( l_{0,7,8,9} \) should be given by the same generalization procedure as was found for the functions \( l_{1,2,3} \) and \( c \), i.e. \( q \neq 0 \) form should be the same as in (4.3) with the dispersion relation modified as in (4.5).

The factorization property does not constrain \( l_4 \) and \( l_5 \). To fix \( l_4 \) and \( l_5 \) we use a second requirement of integrability – the classical Yang-Baxter equation.\(^{27}\) The Yang-Baxter equation

\[ \hat{T} | \phi_{\pm} \phi_{\pm}' \rangle = \frac{1}{2}(l_1 + c) | \phi_{\pm} \phi_{\pm}' \rangle, \quad \hat{T} | \phi_{\pm} \phi_{\pm}' \rangle = \frac{1}{2}(l_2 + c) | \phi_{\pm} \phi_{\pm}' \rangle + l_4 | \psi_{\pm} \psi_{\pm}' \rangle, \]

\[ \text{where the minus sign in the second case comes from moving two fermions past each other. In the } q = 0 \text{ case the factorized tree-level S-matrix for } \{ \phi, \psi \} \text{ is then given by} \]

\[ \hat{T} | \phi_{\pm} \phi_{\pm}' \rangle = \frac{1}{2}(l_1 + c) | \phi_{\pm} \phi_{\pm}' \rangle, \quad \hat{T} | \phi_{\pm} \phi_{\pm}' \rangle = \frac{1}{2}(l_2 + c) | \phi_{\pm} \phi_{\pm}' \rangle + l_4 | \psi_{\pm} \psi_{\pm}' \rangle, \]

\[ \hat{T} | \phi_{\pm} \phi_{\pm}' \rangle = \frac{1}{2}(l_3 + c) | \phi_{\pm} \phi_{\pm}' \rangle, \quad \hat{T} | \phi_{\pm} \phi_{\pm}' \rangle = \frac{1}{2}(l_4 + c) | \phi_{\pm} \phi_{\pm}' \rangle - l_5 | \phi_{\pm} \psi_{\pm}' \rangle, \quad \hat{T} | \psi_{\pm} \phi_{\pm}' \rangle = \frac{1}{2}(l_5 + c) | \phi_{\pm} \psi_{\pm}' \rangle - l_5 | \phi_{\pm} \phi_{\pm}' \rangle. \]

It is worth noting that this factorized S-matrix (4.8) has a \( U(1)^2 \) symmetry under which \( \{ \phi, \psi \} \) have charges \{1, 0\} and \{0, 1\} respectively.

\(^{27}\)We have defined the S-matrix (5) as a map from the space \( | \Phi(p) \rangle \) to \( | \Phi(p) \rangle \), which at leading order is just the identity (3.1),(3.2). However, the S-matrix (5) that naturally satisfies the Yang-Baxter equation (for example, through the use of ZF operators)

\[ \hat{S}_{12}(p', p'') \hat{S}_{23}(p, p'') \hat{S}_{12}(p, p') = \hat{S}_{23}(p, p') \hat{S}_{12}(p, p'') \hat{S}_{23}(p', p''), \]

is a map from the space \( | \Phi(p) \rangle \) to \( | \Phi(p) \rangle \), i.e. the momenta are interchanged. This S-matrix is related to ours by composing with the permutation operator \( (S = PS) \), which flips the two outgoing states picking up a minus sign when they are both fermions. Therefore, at leading order it is given by the permutation operator. Using these relations it is simple to find the appropriate classical Yang-Baxter equation (i.e. with the required minus signs) that our tree-level S-matrix is required to satisfy.
should apply to the massive subsector of the S-matrix of the $AdS_3 \times S^3 \times T^4$ theory.\footnote{In a general integrable theory the truncated S-matrix for the scattering of all particles of a given mass should satisfy the Yang-Baxter equation in its own right. This is a consequence of the fact that the scattering of particles of different mass should be diagonal in the space of masses (there can still be a non-trivial S-matrix in flavour space) by the conservation of higher charges (see also comments in Appendix A).} We then find that the functions $l_4$ and $l_5$ should depend on $q$ not only through $e, e'$ but also explicitly and are generalized to $q \neq 0$ in slightly different ways depending on the charges of the excitations being scattered:

\begin{align}
&l_4^{\pm \to \pm \pm}(p, p') = -\frac{pp'}{2(p + p')} \left[ \sqrt{(e_\pm + p \pm q)(e'_\mp + p' \mp q)} - \sqrt{(e_\pm - p \mp q)(e'_\mp - p' \pm q)} \right], \\
&l_5^{\pm \to \pm \pm}(p, p') = -\frac{pp'}{2(p - p')} \left[ \sqrt{(e_\pm + p \pm q)(e'_\mp + p' \mp q)} + \sqrt{(e_\pm - p \mp q)(e'_\mp - p' \pm q)} \right].
\end{align}

(4.10)

Here the superscripts label the different scattering processes and $e_\pm$ and $e'_\pm$ are given by their $q$-dependent expressions in (4.5).

In summary, our result for the tree-level S-matrix for the massive states of the mixed-flux $AdS_3 \times S^3 \times T^4$ superstring theory is given by (4.1) with the parametrizing functions $l_{1,2,3,6,7,8,9}$ and $c$ given by (4.3) and the functions $l_{4,5}$ given by (4.10) with the dispersion relation modified as in (4.5). Alternatively, the S-matrix can be represented as a tensor product of two copies of the factorized S-matrix (4.8) with the corresponding parametrizing functions $l_{1,2,3}, c$ and $l_{4,5}$ generalized to $q \neq 0$ as described above.

In Appendix A.2 we shall present the action quadratic in both fermions and bosons that reproduces the corresponding amplitudes of the $q \neq 0$ generalization of the S-matrix described above and explain how this action should follow from the gauge-fixed superstring action. This providing a strong indication that this S-matrix is consistent with the $AdS_3 \times S^3 \times T^4$ world-sheet theory for $q \neq 0$.

The factorized S-matrix (4.8) has a non-local supersymmetry – see Appendix B – (the non-locality appears in the form of a braiding in the coproduct [37, 26, 27]). It is therefore natural to expect that the $q \neq 0$ generalization of this S-matrix will be invariant under a modified supersymmetry algebra with structure constants depending on $q$. It would be interesting to determine this algebra and relate its central extension to a $q$-modified dispersion relation. We shall discuss a candidate for this symmetry algebra in Appendix B.

A natural generalization of the standard magnon dispersion relation might be the following: for the positively/negatively charged states we should have

\begin{equation}
\label{eq:energy2}
e_\pm^2 = 1 - q^2 + \left(2h \sin\frac{p}{2} \right)^2q^2,
\end{equation}

where $h$ should be identified with the string tension, $h = \frac{\sqrt{3}}{2\pi} = \frac{R^2}{2\pi \alpha'}$.\footnote{It is unclear whether the expression for $h$ is renormalized for $q \neq 0$. In the $q = 0$ case it appears that it is not. There is though a 1-loop shift in $h(\lambda)$ in the case of another 1-parameter deformation – the $AdS_3 \times S^3 \times S^3 \times S^1$ theory [15, 16]. To investigate this question by a direct 1-loop computation is an open problem.} Note that for $q = 1$ eq. 11 is valid for the positively/negatively charged states.
(4.11) reduces to \( e_{\pm} = 2h \sin \frac{p}{2} \pm 1 \) leading to a massless dispersion relation for small \( p \).

This relation may be viewed as a “lattice” generalization of the above string world-sheet dispersion relation (4.5) with “\( \frac{p}{2} \to \sin \frac{p}{2} \)”. Indeed, in the BMN limit when \( h \) is large and the momentum \( p \) is small, scaling as \( h^{-1} \), we find, after redefining \( p \to h^{-1}p \)

\[
e_{\pm}^2 = 1 - q^2 + (p \pm q)^2 + \mathcal{O}(h^{-2}) .
\]

(4.12)

This matches the result (2.22),(3.21) we obtained directly from the string perturbation theory. In the semiclassical (“giant magnon” [44]) limit when \( h \) is large and \( p \) stays finite we get from (4.11)

\[
e_{\pm} = 2h \sin \frac{p}{2} \pm q + \mathcal{O}(h^{-1}) .
\]

(4.13)

This suggests that the leading classical energy (minus the angular momentum) of the giant magnon solution should be unaltered by the \( q \)-deformation. The expansion (4.13) predicts the presence of a string 1-loop correction to the giant magnon energy proportional to \( q \) (there was no 1-loop correction in the \( q = 0 \) case [45, 10]). It would be interesting to derive this correction by a direct string-theory computation and thus check the conjectured form of the dispersion relation (4.11).

5 Concluding remarks

In this paper we have found the generalization of the tree-level S-matrix for massive BMN-type excitations of the \( AdS_3 \times S^3 \times T^4 \) superstring theory in the case of non-zero NSNS 3-form flux (parametrized by \( q \in (0, 1) \)). We have directly computed the S-matrix in the bosonic sector discovering its very simple dependence on \( q \) via the modified dispersion relation. Using the requirements of integrability (factorization and Yang-Baxter properties of the S-matrix) we then suggested its generalization to the full superstring case.

While we have little doubt (on the basis of integrability and arguments in Appendix A.2) in the correctness of the proposed fermionic part of the \( q \neq 0 \) S-matrix, it would be useful to check it in full (including 4-fermion part) by starting directly with the component form of the corresponding superstring action.

One straightforward extension of the computations presented in this paper is to the similar case of the \( AdS_3 \times S^3 \times S^3 \times S^1 \) theory with mixed 3-form flux [9, 6], generalizing the recent discussion of the \( q = 0 \) case in [17]. It would be important also to generalize the semiclassical 1-loop computations done in the \( AdS_3 \times S^3 \times T^4 \) theory with pure RR flux [10, 12, 51, 15, 16, 52, 17] to the \( q \neq 0 \) case in order to check non-renormalization of the effective string tension \( h \) and determine the corresponding 1-loop dressing phases.

The next obvious step is to use the information about the S-matrix to conjecture the corresponding \( q \neq 0 \) generalization of the asymptotic Bethe ansatz which for \( q = 0 \) was first conjectured (by analogy with the \( AdS_5 \times S^5 \) case) in [9].\(^{30}\) It is natural to expect that as long as \( q < 1 \) (i.e. away from the WZW point or the case of pure NSNS flux) the underlying

\(^{30}\)The full structure of the asymptotic Bethe ansatz still remains to be understood already in the \( q = 0 \) case of the \( AdS_3 \times S^3 \times T^4 \) superstring, cf. [11, 13, 14, 16, 17].

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integrable system should be similar to the one describing the pure RR case \((q = 0)\), i.e. there should be a “ferromagnetic” BPS vacuum with standard massive BMN excitations (with a candidate dispersion relation suggested above in (4.11)).

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Appendix A: Expansion of the action of \(AdS_3 \times S^3 \times T^4\) string theory with \(B\)-flux

Here we shall discuss quadratic and quartic terms in the expansion of \(AdS_3 \times S^3 \times T^4\) string action that are relevant for the derivation of the S-matrix in 4.

A.1 Bosonic sector

The generalization of the Lagrangian (2.9) to the case when the string moves on \(AdS_3 \times S^3 \times T^4\) can be written as

\[
L = -\frac{1}{2} \left\{ -\hat{G}(z) \partial^a t \partial_a t + \hat{F}(z) \partial^a z_s \partial_a z_s + 2 \epsilon^{ab} \hat{B}_s(z) \partial_a z_s \partial_a t \\
+ G(y) \partial^a \varphi \partial_a \varphi + F(y) \partial^a y_s \partial_a y_s + 2 \epsilon^{ab} B_s(y) \partial_a y_s \partial_b \varphi + \partial^a x_k \partial_a x_k \right\},
\]

\[(A.1)\]

\[
G = \frac{(1 - \frac{1}{4} y^2)^2}{(1 + \frac{1}{4} y^2)^2} = 1 - y^2 F, \quad F = \frac{1}{(1 + \frac{1}{4} y^2)^2},
\]

\[(A.2)\]

\[
\hat{G} = \frac{(1 + \frac{1}{4} z^2)^2}{(1 - \frac{1}{4} z^2)^2} = 1 + z^2 \hat{F}, \quad \hat{F} = \frac{1}{(1 - \frac{1}{4} z^2)^2},
\]

\[(A.3)\]

\[
B_s(y) = q F(y) \epsilon_{rs} y_r, \quad \hat{B}_s(z) = q \hat{F}(z) \epsilon_{rs} z_r.
\]

\[(A.4)\]

\[31\]It should be emphasized again that the superstring theory case considered here, where there are no UV divergences and the mass scale is introduced by the choice of a vacuum or a gauge, is very different from the case of the quantum bosonic \(SU(2)\) principal chiral model with a WZ term [18, 20, 21]. There, in the absence of the WZ term (for \(q = 0\)), there is a dynamical mass generation and thus a massive S-matrix [48]. However, for \(q \neq 0\) there is an RG flow (of \(h\) and thus of \(q\) in (2.3)) between the trivial UV \((q = 0)\) and non-trivial WZW \((q = 1)\) fixed points, so that there is no mass generation and the underlying S-matrix is massless.
Here $x_k$ are the “massless” $T^4$ fields. The $AdS_3$ and $S^3$ parts are related by the formal analytic continuation $t \to \varphi$, $z_r \to iy_r$, and then reversing the overall sign of the Lagrangian.\(^{32}\)

Following the discussion in section 2, let us consider the redefinition of $t$ as in (2.24) and then T-dualize in the $\varphi$ direction. This leads to the following generalization of (2.25),(2.26),(2.27):

\[
\bar{L} = -\sqrt{g} g^{cd} h_{cd} - \rho c^d \partial_d u \left[ b \hat{G}(\partial_c \varphi - B_\sigma \partial_c y_\sigma) + G \hat{B}_s \partial_c z_s \right],
\]

\[
h_{cd} = -Q \partial_c u \partial_d u + P (\partial_c \varphi - B_\sigma \partial_c y_\sigma + b \hat{B}_s \partial_c z_s) (\partial_d \varphi - B_\sigma \partial_d y_\sigma + b \hat{B}_s \partial_d z_s)
+ F \partial_c y_\sigma \partial_d y_\sigma + \hat{F} \partial_c z_s \partial_d z_s + \partial_c x_k \partial_d x_k,
\]

\[
Q = G \hat{G} P, \quad P = (G - \hat{G})^{-1}.
\]

Fixing the gauge in (2.28) (with $J = 1$) we get the following generalization of (2.29)

\[
\bar{L} = -\sqrt{h} + c P \left[ b \hat{G}(c - B_s y^s) + G \hat{B}_s z^s \right],
\]

\[
h = \left[ c^2 Q - P (B_s y_s - b \hat{B}_s z_s)^2 - \hat{F} y_s^2 - \hat{F} z_s^2 - \dot{x}_k^2 \right]
\times \left[ P (c - B_r y^r + b \hat{B}_r z^r)^2 + F y^r_2 + F z^r_2 + \dot{x}_k^2 \right]
+ \left[ P (B_s y_s - b \hat{B}_s z_s) (c - B_r y^r + b \hat{B}_r z^r) - F y^r y^r_1 - \hat{F} z^r_1 - \dot{x}_k x^r_1 \right]^2.
\]

Expanding $\bar{L}$ in powers of $y_s$ and $z_s$ we get the following generalization of (2.30),(2.31)

\[
\bar{L} = L_2 + L_4 + ...,
\]

\[
L_2 = \frac{1}{2} \left( \dot{z}_s^2 - \dot{z}_r^2 - \dot{z}_s^2 \right) + q \epsilon_{sr} z_s z^r + \frac{1}{2} (y_s^2 - y^r_2 - y^r_s) + q \epsilon_{sr} y^r s_1 + \frac{1}{2} (\dot{x}_k^2 - x_k^2),
\]

\[
L_4 = \frac{1}{4} \left[ y_s^2 (2y^2_2 + \dot{z}_r^2 + z^r_2 - \dot{z}_s^2 (2z^2_2 + \dot{y}_r^2 + y^2_r) + (y^r_2 - z^2_r)(\dot{x}_k^2 + x_k^2) \right]
+ q \left[ \frac{1}{2} (y_r y^r_1 + \dot{z}_r z^r_1 + \dot{x}_k x^r_1) \epsilon_{sp}(y_s y^r p - z_s z^r p) \right.
\]

\[
- \frac{1}{4} (y^r_2 + y^r_1 + \dot{z}_r^2 + z^r_1 + \dot{x}_k x^r_1) \epsilon_{sp}(y_s y^r p - z_s z^r p) - \frac{1}{4} (y^r_2 - z^2_r) \epsilon_{sp}(y_s y^r p + z_s z^r p) \right]
\]

\[
+ (a - \frac{1}{2}) \left\{ \frac{1}{4} (y^2_2 + \dot{z}_s^2) - \frac{1}{2} (y^r_2 + y^r_1 + \dot{z}_r^2 + z^r_1 + \dot{x}_k x^r_1)^2 \right\}
\]

\[
+ q \left[ - (y_r y^r_1 + \dot{z}_r z^r_1 + \dot{x}_k x^r_1) \epsilon_{sp}(y_s y^r p + z_s z^r p) \right.
\]

\[
+ \frac{1}{2} (y^r_2 + y^r_1 - \dot{z}_r^2 - z^r_1 - \dot{x}_k x^r_1) \epsilon_{sp}(y_s y^r p + z_s z^r p) \right\}.\]

The kinetic terms for $y_s$ and $z_s$ are the same, implying the same massive dispersion relation (2.22) while $x_k$ are massless.

As there are no cubic terms in the bosonic Lagrangian, and there cannot be a boson—boson—fermion cubic term when fermions are included, the above quartic Lagrangian (ignoring the massless fields $x_k$) is sufficient to compute the tree-level $S$-matrix for the four massive bosons, $y_s$ and $z_s$. The result of this computation was summarized in sections 3 and 4.

---

\(^{32}\) The sign of coefficient in $\hat{B}_s(z)$ can be opposite to that of in $B_s(y)$ but the two cases are related, e.g., by coordinate redefinition $t \to -t$. The expansion of the action below in the case of $\hat{B}_s(z) = -q \hat{F}(z) \epsilon_{rs} z_r$ can be found by reversing the sign of $\epsilon_{rs} z_r$ in all terms where it appears.
Furthermore, the tree-level S-matrix truncated to just the massive mode sector (including all massive bosons and fermions) should satisfy the classical Yang-Baxter equation in its own right as by integrability (i.e. as a consequence of the existence of higher conserved charges) the scattering of particles of different mass should be diagonal in the space of masses (there can still be a non-trivial S-matrix in flavour space). It would be of interest to compute the S-matrix for the scattering of both the massless excitations with the massive excitations and amongst themselves. At tree level the allowed scattering processes following from (A.11) are (here $x_L$ and $x_R$ stand for the left- and right-moving components of $x_k$)

$\begin{align*}
\begin{array}{c}
\uparrow x_L \\
y/z \\
x_L
\end{array} & \quad \begin{array}{c}
\uparrow x_R \\
y/z \\
x_R
\end{array} & \quad \begin{array}{c}
\uparrow x_L \\
y/z \\
x_R
\end{array} & \quad \begin{array}{c}
\uparrow x_R \\
y/z \\
x_L
\end{array}
\end{align*}$

where along each line the spatial momentum is unchanged.

### A.2 Fermionic sector

A systematic derivation of the gauge-fixed superstring action to quartic order in both bosons and fermions goes beyond the scope of the present paper. Here we shall limit ourselves to a discussion of terms quadratic in fermions and quadratic in bosons that are relevant for the derivation of the most non-trivial parts of the S-matrix (4.1), i.e. scattering processes such as $FF \rightarrow BB$, $BB \rightarrow FF$ and $BF \rightarrow BF$, which, in particular, involve the functions $l_4$ and $l_5$ in (4.10).\footnote{Note that according to (4.1) the amplitudes $FF \rightarrow FF$ corresponding to quartic fermionic terms in the action vanish in the light-cone gauge $a = \frac{1}{2}$ where $c = 0$ (see (4.2)).}

We shall start with the general form of the quadratic term in the type IIB superstring action (thus avoiding gauge-fixing subtleties of truncating to the supercoset action \cite{9,6}), as was also done in the absence of $B$-flux in \cite{12,17}. The fermionic “kinetic” term in the IIB GS superstring action in a curved background is a direct generalization of its flat-space form:

\begin{align}
L_2 &= i(\eta^{ab}\delta_{IJ} - \epsilon^{ab}\rho_{3IJ})\partial_a x^m e^\hat{n}_m \bar{\theta}^I \Gamma_{\hat{n}} (D_b)^{JK} \theta^K , \\
D_a &= \partial_a + \frac{1}{4} \partial_a x^k e^k \left[ (\omega_{\hat{m}\hat{n}\hat{k}} - \frac{1}{2} \rho_3 H_{\hat{m}\hat{n}\hat{k}}) \Gamma_{\hat{m}\hat{n}\hat{l}} - \frac{1}{3!} \rho_1 F_{\hat{m}\hat{n}\hat{l}} \Gamma_{\hat{m}\hat{n}\hat{l}} \Gamma_{\hat{k}} \right] , \\
\rho_3 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \\
\rho_1 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} ,
\end{align}

where $\hat{m}$, etc., are tangent-space indices, $\theta^I$ are two real MW spinors and $\rho_\alpha$ act in the $I, J = 1, 2$ space. $D_a$ is the generalized covariant derivative that appears in the Killing spinor equation or gravitino transformation law in type IIB supergravity (we have included only NSNS and RR 3-form background field couplings). The tangent space components of the fluxes corresponding to the metric in (A.1) are\footnote{Here $\hat{r}'$, $s'$ denote the spatial $AdS_3$ directions $z_s$ and $\hat{r}$, $\hat{s}$ the $S^3$ transverse directions $y_s$.}

\begin{align}
H_{\hat{r}'\hat{r}'s'} &= -2q c_{\hat{r}'s'}, \\
H_{\hat{r}'\hat{s}} &= -2q c_{\hat{r}'s},
\end{align}
\[ F_{\hat{p}\hat{q}''} = -2\sqrt{1 - q^2} \epsilon_{\hat{p}\hat{q}''} , \quad F_{\hat{p}\hat{q}''} = -2\sqrt{1 - q^2} \epsilon_{\hat{p}\hat{q}''} . \]  

(A.16)

One may expand the action to quadratic order in bosonic fields using

\[ e_{\varphi}^2 = 1 - \frac{1}{2} y_{s}^2 + \ldots , \quad e_{s}^2 = 1 - \frac{1}{4} y_{s}^2 + \ldots , \quad \omega_{3s} = -y_{s}d\varphi + \ldots , \quad \omega_{s3} = y_{s}d\varphi + \ldots , \]  

and similar relations for the AdS$_3$ part. The leading term in the fermionic action in the light-cone gauge is found by setting

\[ t = \tau , \quad \varphi = \tau , \]  

(A.18)

i.e. choosing \( u \equiv t + \varphi = 2\tau \), \( v \equiv t - \varphi = 0 \) (cf. (2.17),(2.24)) and fixing the l.c. kappa-symmetry gauge \( \Gamma^{a} \theta^{l} = 0 \). One then finds as in [36] that the fermions split into two groups: 4 massive and 4 massless. The massive ones, denoted by \( \zeta_{R,L} \) and \( \chi_{R,L} \) (corresponding to \( \zeta_{+} \) and \( \chi_{+} \) in (4.1)), have the following action

\[ \mathcal{L}_2 = i\zeta^*_R (\partial_- + iq) \zeta_R + i\zeta^*_L (\partial_+ - iq) \zeta_L - \sqrt{1 - q^2} (\zeta^*_R \zeta_L + \zeta^*_L \zeta_R) + i\chi^*_R (\partial_- + iq) \chi_R + i\chi^*_L (\partial_+ - iq) \chi_L - \sqrt{1 - q^2} (\chi^*_R \chi_L + \chi^*_L \chi_R) . \]  

(A.19)

Here the \( q \)-dependent derivative shifts come from the NSNS flux (A.15) term and the mass terms – from the RR flux (A.16) term in D in (A.13). The equations of motion that follow from (A.19) are

\[ (\partial_- + iq) \zeta_R + i\sqrt{1 - q^2} \zeta_L = 0 , \quad (\partial_+ - iq) \zeta_L + i\sqrt{1 - q^2} \zeta_R = 0 , \]  

(A.20)

and similar ones for \( \chi \). Combining them gives the following second-order equation

\[ (\partial_+ - iq)(\partial_- + iq) \zeta_{R,L} + (1 - q^2) \zeta_{R,L} = 0 , \]  

(A.21)

which is the same as the free equation for the massive fields \( (y, y^*) \) and \( (z, z^*) \) in the bosonic sector following from (A.10) (note that \( (\partial_+ - iq)(\partial_- + iq) = \partial^2 - (\partial_- - iq)^2 \), cf. (2.36)). This implies that all massive bosons and fermions have the same dispersion relation (2.22) or (4.5).

The structure of (A.12) and the expansions (A.17) imply that there are no terms quadratic in fermions and linear in the “transverse” bosons \( (y_s, z_r) \). To find terms that are quadratic in both the fermions and the bosons one may follow the same procedure as used in the purely bosonic case in section 2.3 and the previous subsection, i.e. apply T-duality in the \( \varphi \) direction and then fix the gauge (2.28) where \( u = 2\tau \), \( \bar{\varphi} = 2\sigma \) (here we choose \( a = \frac{1}{2} \), \( J = 1 \)) together with \( \Gamma^{u} \theta^{l} = 0 \). An alternative is to follow the approach of [40] used for \( q = 0 \) in [12, 17].

Here we will not give a systematic derivation of the resulting Lagrangian and just indicate which types of terms one should expect to find. Let us focus on the terms involving \( y \) and \( \zeta \) only. As \( \partial \varphi \) terms will appear in the \( H_3 \) and \( F_3 \) parts of D in (A.13) the corresponding fermionic terms linear in \( \partial \varphi \) will enter the action in the same way as the bosonic WZ term \( b_a = \epsilon_{ab}\epsilon_{rs} y_{r} \partial_s y_{s} \) in (A.9),(A.11), i.e. there will be products of the connection terms \( q(\zeta^*_R \zeta_R - \zeta^*_L \zeta_L) \) and \( \sqrt{1 - q^2} (\zeta^*_R \zeta_L + \zeta^*_L \zeta_R) \) with the bosonic second-derivative terms \( \partial d y d y \). In addition, there will be also \( q \)-dependent terms where the fermionic kinetic term is multiplied by the bosonic \( b_a \).
To simplify using integrability constraints follows also directly from the

4.2 A.20 4.1 given in (4.2) with (4.5) and (4.10) is found to be

\[ \mathcal{L}_4 = -\frac{1}{2} \left[ \sqrt{1 - q^2} \zeta_R^* \zeta_R + \frac{1}{2}(\zeta_R^* \zeta_R - \zeta_L^* \zeta_L) \right] \partial_+ y^* \partial_+ y - \frac{1}{2} \zeta_R^* \zeta_L \partial_- y^* \partial_- y + \left[ \frac{i}{4}(\zeta_R^* \zeta_R - \zeta_L^* \zeta_L) y^* \partial_+ y + \frac{i}{4}(\zeta_R^* \zeta_R - \zeta_L^* \zeta_L) y^* y \right] \]

The structure of other terms, e.g., involving \( yy\chi\chi \), is very similar. We observe that the \( q \)-dependent terms here indeed have the structure expected from the T-duality based gauge-fixing procedure outlined above.

In the limit of \( q = 0 \) the Lagrangian (A.22) reduces to

\[ \mathcal{L}_4 = -\frac{1}{2} \zeta_R^* \zeta_R \partial_+ y^* \partial_+ y - \frac{1}{2} \zeta_R^* \zeta_L \partial_- y^* \partial_- y + \frac{i}{4}(\zeta_R^* \zeta_R - \zeta_L^* \zeta_L) y^* y \]

This Lagrangian matches the \( AdS_3 \times S^3 \times T^4 \) limit of the corresponding part of the \( AdS_3 \times S^3 \times S^1 \) Lagrangian found in [12, 17]. Indeed, relabelling \( y, \zeta_{R,L} \rightarrow y_2, \chi_{1,\pm} \) it is the same as the \( \alpha \rightarrow 1 \) limit of eq. E.1 in [17] (keeping only the fields \( y_2, \chi_{1} \) there), or equivalently, as

the limit \( \alpha \rightarrow 0 \) (keeping only the fields \( y_3, \chi_1 \) there).

The above discussion thus gives a strong indication that the expression for the \( q \)-dependent S-matrix found in section 4.2 using integrability constraints follows also directly from the \( AdS_3 \times S^3 \times T^4 \) superstring action.

Appendix B:

Comments on the symmetry algebra of the superstring S-matrix

Below we shall first describe the supersymmetry of the tree-level S-matrix of the massive modes in the \( AdS_3 \times S^3 \times T^4 \) theory with pure RR flux (\( q = 0 \)) which can be found by analyzing the algebra underlying the supercoset formulation of the theory. We shall then comment on possible \( q \neq 0 \) generalization of it.

---

35 Equivalently, the T-dualization procedure with the fermionic terms included should lead to \( q \)-dependent “cross-terms” similar to the ones in (A.11) (for \( a = \frac{1}{2} \)) with \( \partial x_k \partial x_k \) terms replaced by the fermionic kinetic terms.

36 To recall, in (4.1) we used the notation \((y_+, y_-) = (y, y^*) \) and \( \zeta_{R,L} = \zeta_{+,\pm} \).
Let us start with briefly reviewing the symmetry of the $AdS_5 \times S^5$ theory based on the superalgebra $\mathfrak{psu}(2,2|4)$, which we formulate in terms of $8 \times 8$ (traceless and supertraceless) supermatrices:

\begin{align}
\begin{array}{cccc}
\star & \star & \star & \star \\
\bullet & \bullet & \bullet & \bullet \\
\star & \star & \star & \star \\
\bullet & \bullet & \bullet & \bullet \\
\star & \star & \star & \star \\
\bullet & \bullet & \bullet & \bullet \\
\star & \star & \star & \star \\
\bullet & \bullet & \bullet & \bullet \\
\end{array}
\end{align}

(B.1)

Here the top left $4 \times 4$ block is the $\mathfrak{su}(2,2)$ bosonic subalgebra with signature $(+,+,−,−)$, while the bottom right $4 \times 4$ block is the $\mathfrak{su}(4)$ bosonic subalgebra. The top right and bottom left $4 \times 4$ blocks contain the Grassmann-odd parts of the algebra. The BMN geodesic can then be written as a supergroup-valued solution:

\[ f = \exp(\mathcal{J} \tau \text{diag}(i,i,−i,i,i,−i,i,−i)) \, , \]

which preserves the symmetry algebra $\mathfrak{psu}(2|2)^2$ plus central extensions. The $\mathfrak{psu}(2|2)^2$ algebra is denoted by the shaded regions in (B.1).\(^{37}\)

The truncation to the $AdS_3 \times S^3 \times T^4$ theory is found by taking the elements of $\mathfrak{psu}(2,2|4)$ marked $\star$ and $\bullet$ in (B.1). Each of these sets forms a $\mathfrak{u}(1,1|2)$ superalgebra; however, combined they give $\mathfrak{ps}(\mathfrak{u}(1,1|2)^2)$, where the $\mathfrak{ps}$ projections correspond to the vanishing of the overall trace and supertrace. Eq. (B.2) is still a solution here and this allows us to write down the symmetry algebra preserved by the BMN geodesic in the $AdS_3 \times S^3 \times T^4$ theory. It can be found by taking the “intersection” of the shaded areas and the $\star$ and $\bullet$ regions in (B.1). The resulting algebra is

\[ [\mathfrak{ps}(\mathfrak{u}(1|1)^2)]^2 = [\mathfrak{u}(1) \in \mathfrak{psu}(1,1)^2] \ltimes \mathfrak{u}(1)^2 \rightarrow [\mathfrak{u}(1) \in \mathfrak{psu}(1|1)^2] \ltimes \mathfrak{u}(1) \, , \]

(B.3)

where we have dropped one of the $\mathfrak{u}(1)$ central extensions ($\star = i$, $\bullet = −i$) as it acts trivially. This is in agreement with the fact that the global bosonic symmetry is $U(1)^3$ (cf. (4.4)).

The massive tree-level S-matrix (in the case of vanishing RR flux, i.e. $q = 0$) (4.1),(4.2) should therefore have symmetry given by (B.3) (up to central extensions encoding the energy and momentum), while the factorized S-matrix (4.8),(4.2) should only have half this symmetry.

Explicitly, the generators of the supersymmetry algebra of the factorized S-matrix (4.8) in the $q = 0$ case are: two $U(1)$ generators $\mathcal{R}$ and $\mathcal{L}$; four supercharges $\mathcal{Q}_{\pm\mp}$ and $\mathcal{G}_{\pm\mp}$ ($+$ and $-$ denote the charges under the $U(1) \times U(1)$ bosonic subalgebra) and three central extensions $\mathcal{C}$, $\mathcal{P}$ and $\mathcal{K}$. The commutation relations are given by

\[ [\mathcal{R}, \mathcal{R}] = 0 \, , \quad [\mathcal{L}, \mathcal{L}] = 0 \, , \quad [\mathcal{L}, \mathcal{Q}_{\pm\mp}] = \mp i\mathcal{Q}_{\pm\mp} \, , \]

\[ [\mathcal{L}, \mathcal{G}_{\pm\mp}] = \mp i\mathcal{G}_{\pm\mp} \, , \]

\(^{37}\)That is, this is the subalgebra of $\mathfrak{psu}(2,2|4)$ that commutes with the solution. We use the notation $\mathfrak{a}^2 \equiv \mathfrak{a} \oplus \mathfrak{a}$ for an algebra $\mathfrak{a}$. Below $\ltimes$ stands for central extension.
They are consistent with the following set of reality conditions

\[ R^i = -R^i, \quad L^i = -L^i, \quad N^i_{\pm+} = S_{\pm+}, \quad P^i = K, \quad C^i = C. \]  

\[ \tag{B.5} \]

This superalgebra is a centrally-extended semi-direct sum of \( u(1) \) (generated by \( R - L \)) with two copies of the superalgebra \( \text{psu}(1|1) \), i.e.

\[ [u(1) \in \text{psu}(1|1)^2] \ltimes u(1) \ltimes \mathbb{R}^3. \]  

\[ \tag{B.6} \]

The central extensions are \( R + L, C, P \) and \( K \).

Indeed, we expect \( R + L \) to be central as it has the same action on the tensor product states \((4.6)\) whether acting on the first or the second entry. As with the other three central extensions, \( C, P \) and \( K \), there is therefore only a single copy of this \( u(1) \) central extension when we consider the symmetry of the full S-matrix

\[ [u(1) \in \text{psu}(1|1)^2] \ltimes u(1) \ltimes \mathbb{R}^3. \]  

\[ \tag{B.7} \]

in agreement with \((B.3)\).

From this discussion it is natural to expect the symmetry algebra for \( q \neq 0 \) to be the same as in the \( q = 0 \) case. The dispersion relation \((4.11)\) should then follow from a modification of the representation. Indeed, the fact that \( R + L \) is central suggests its eigenvalue can be altered allowing the parameter \( q \) to be introduced. We leave the detailed study of the relevant representation in the \( q \neq 0 \) case for the future.

To conclude this appendix, we will briefly discuss the details of the invariance of the tree-level S-matrix under supersymmetry in the case \( q = 0 \) following the \( AdS_5 \times S^5 \) story \([37, 26, 27, 46, 47]\). The particular representation of interest to us consists of one complex boson and one complex fermion. The generators have the following action on the one-particle states

\[ R |\phi_\pm\rangle = \pm i |\phi_\pm\rangle, \quad \text{and} \quad R |\psi_\pm\rangle = 0, \]
\[ L |\phi_\pm\rangle = 0, \quad \text{and} \quad L |\psi_\pm\rangle = \pm i |\psi_\pm\rangle, \]
\[ N_{\pm+} |\phi_\pm\rangle = 0, \quad \text{and} \quad N_{\pm+} |\psi_\pm\rangle = a |\phi_\pm\rangle, \]
\[ N_{\pm+} |\phi_\pm\rangle = b |\psi_\pm\rangle, \quad \text{and} \quad N_{\pm+} |\psi_\pm\rangle = 0, \]
\[ S_{\pm+} |\phi_\pm\rangle = 0, \quad \text{and} \quad S_{\pm+} |\psi_\pm\rangle = c |\phi_\pm\rangle, \]
\[ S_{\pm+} |\phi_\pm\rangle = d |\psi_\pm\rangle, \quad \text{and} \quad S_{\pm+} |\psi_\pm\rangle = 0, \]
\[ C |\phi_\pm\rangle = C |\phi_\pm\rangle, \quad \text{and} \quad C |\psi_\pm\rangle = C |\psi_\pm\rangle, \]
\[ P |\phi_\pm\rangle = P |\phi_\pm\rangle, \quad \text{and} \quad P |\psi_\pm\rangle = P |\psi_\pm\rangle, \]
\[ K |\phi_\pm\rangle = K |\phi_\pm\rangle, \quad \text{and} \quad K |\psi_\pm\rangle = K |\psi_\pm\rangle. \]  

\[ \tag{B.8} \]
Here \( a, b, c, d, C, P \) and \( K \) are representation parameters that will eventually be functions of the energy and momentum of the state. For the supersymmetry algebra to close the following constraints should be satisfied

\[
ab = P, \quad cd = K, \quad ad = C + \frac{1}{2}, \quad bc = C - \frac{1}{2}.
\]

(B.9)

These can easily be seen to imply that

\[
C^2 = \frac{1}{4} + PK,
\]

(B.10)

which is just the shortening condition for this atypical representation. Physically, it will be interpreted as the dispersion relation, with \( C \) playing the rôle of the energy and \( P \) and \( K \) defined in terms of the momentum. The representation parameters are further constrained by the reality conditions (B.5)

\[
a^* = d, \quad b^* = c, \quad P^* = K, \quad C^* = C.
\]

(B.11)

We can solve the set of equations (B.9) for \( a, b, c, d \) in terms of \( C, P \) and \( K \):

\[
a = \gamma \sqrt{P} \left( \frac{2C + 1}{2C - 1} \right)^{\frac{1}{4}}, \quad b = \gamma^{-1} \sqrt{P} \left( \frac{2C - 1}{2C + 1} \right)^{\frac{1}{4}},
\]

\[
c = \gamma \sqrt{K} \left( \frac{2C - 1}{2C + 1} \right)^{\frac{1}{4}}, \quad d = \gamma^{-1} \sqrt{K} \left( \frac{2C + 1}{2C - 1} \right)^{\frac{1}{4}},
\]

(B.12)

where \( \gamma \) is a phase parametrizing the normalization of the fermionic states with respect to the bosonic ones and can therefore be set to one.

To define the action of the symmetry on the two-particle states we need to introduce the coproduct

\[
\Delta(\mathcal{R}) = \mathcal{R} \otimes I + I \otimes \mathcal{R}, \quad \Delta(\mathcal{L}) = \mathcal{L} \otimes I + I \otimes \mathcal{L},\n\]

\[
\Delta(\mathcal{Q}) = \mathcal{Q} \otimes I + U \otimes \mathcal{Q}, \quad \Delta(\mathcal{S}) = \mathcal{S} \otimes I + U^{-1} \otimes \mathcal{S},\n\]

\[
\Delta(\mathcal{P}) = \mathcal{P} \otimes I + U^2 \otimes \mathcal{P}, \quad \Delta(\mathcal{C}) = \mathcal{C} \otimes I + I \otimes \mathcal{C}, \quad \Delta(\mathcal{K}) = \mathcal{K} \otimes I + U^{-2} \otimes \mathcal{K},
\]

(B.13)

and the opposite coproduct, defined as

\[
\Delta^{op}(\mathcal{J}) = \mathcal{P}(\Delta(\mathcal{J})),
\]

(B.14)

where \( \mathcal{J} \) is an arbitrary generator and \( \mathcal{P} \) defines the graded permutation of the tensor product.

We have deformed the coproduct from the usual one via the introduction of the new abelian generator \( \mathcal{U} \) (\( \Delta(\mathcal{U}) = \mathcal{U} \otimes \mathcal{U} \)) [47]. This is done according to a \( \mathbb{Z} \)-grading of the algebra, whereby the charges \(-2, -1, 1, 2\) are associated to the generators \( \mathcal{K}, \mathcal{S}, \mathcal{Q}, \mathcal{P} \) and the remaining generators are uncharged. The action of \( \mathcal{U} \) on the single particle states is given by

\[
\mathcal{U} |\phi_\pm\rangle = U |\phi_\pm\rangle, \quad \mathcal{U} |\psi_\pm\rangle = U |\psi_\pm\rangle.
\]

(B.15)

This braiding allows for the existence of the non-trivial S-matrix.
The first consequence of this non-trivial braiding is found by requiring that for the central extensions the coproduct should equal its opposite, implying
\[ \mathcal{P} \propto (1 - U^2), \quad \mathcal{R} \propto (1 - U^{-2}). \quad (B.16) \]
We fix the normalization of \( \mathcal{P} \) relative to \( \mathcal{R} \) by taking both constants of proportionality to be
\[ \frac{\hbar}{2} = \frac{\sqrt{\lambda}}{4\pi}. \quad (B.17) \]
Acting on the single-particle states gives us the relations
\[ P = \frac{\hbar}{2} (1 - U^2), \quad K = \frac{\hbar}{2} (1 - U^{-2}), \quad (B.18) \]
where \( U \) should satisfy, as a consequence of \((B.11)\), the following reality condition
\[ U^* = U^{-1}. \quad (B.19) \]
Motivated by the well-known construction in the \( AdS_5 \times S^5 \) case (implying a similar one in the \( AdS_3 \times S^3 \times T^4 \) case with \( q = 0 \)) we identify \( C \) with the energy and define \( U \) in terms of the spatial momentum as
\[ C = \frac{e}{2}, \quad U = e^{-\frac{i}{2}p}. \quad (B.20) \]
Using \((B.18)\) and \((B.20)\) we can substitute in for \( C, P \) and \( K \) in terms of the energy and momentum in the shortening conditions \((B.10)\) to find the following dispersion relation
\[ e^2 = 1 + 4 \hbar^2 \sin^2 \frac{p}{2}. \quad (B.21) \]
In terms of the energy and momentum the representation parameters \( a, b, c \) and \( d \) \((B.12)\) are
\[ a = \sqrt{\frac{\hbar}{2} (1 - e^{-ip})(\frac{e + 1}{e - 1})^{\frac{1}{4}}}, \quad b = \sqrt{\frac{\hbar}{2} (1 - e^{-ip})(\frac{e - 1}{e + 1})^{\frac{1}{4}}}, \]
\[ c = \sqrt{\frac{\hbar}{2} (1 - e^{-ip})(\frac{e - 1}{e + 1})^{\frac{1}{4}}}, \quad d = \sqrt{\frac{\hbar}{2} (1 - e^{-ip})(\frac{e + 1}{e - 1})^{\frac{1}{4}}}. \quad (B.22) \]
Rescaling \( p \to \hbar^{-1}p \) and expanding the various representation parameters to the appropriate order in \( \hbar^{-1} \) we find that the factorized tree-level S-matrix \((4.8)\) of the theory with pure RR flux \((q = 0)\) co-commutes with this symmetry.

**Appendix C:**

**Faddeev-Reshetikhin model for the string on \( R \times S^3 \) with \( B \)-flux**

As already discussed in section 2, the motion of the bosonic string on \( S^3 \) with \( B \)-flux is described in conformal gauge by the \( SU(2) \) principal chiral model with a WZ term
\[ S = -\frac{1}{2} \hbar \left[ \int d^2 \sigma \frac{1}{2} tr(J_+J_-) - q \int d^3 \sigma \frac{1}{3} e^{abc} tr(J_aj_bj_c) \right], \quad J_a = g^{-1} \partial_a g. \quad (C.1) \]
Fixing the residual conformal diffeomorphism symmetry by choosing \( t = \mu \tau \), the conformal gauge (Virasoro) conditions are

\[
\text{tr}J_\pm^2 = -2\mu^2 ,
\]

while the first-order form of the equations of motion is as in (2.5),(2.6):

\[
\partial_+ J_- + \frac{1}{2}(1 + q)[J_+, J_-] = 0 , \quad \partial_- J_+ - \frac{1}{2}(1 - q)[J_+, J_-] = 0 .
\]

Note that the \( 1 \pm q \) factors here can be formally absorbed by a rescaling of either \( J_\pm \) or \( \sigma_\pm \). Let us write down the action that leads to these equations for the currents, generalizing the \( q = 0 \) case discussed in [19, 28]. \( ^{38} \) For \( g \in SU(2) \) we may solve the conditions (C.2) in terms of two unit 3-vector fields \( S^k_\pm \) (\( \hat{\sigma}^k \) are Pauli matrices and \( k = 1, 2, 3 \))

\[
J_\pm = i\mu S^k_\pm \hat{\sigma}^k , \quad S^k_\pm S^k_\mp = 1 .
\]

The equations of motion (C.3) then become

\[
\partial_+ S^i_- + \mu(1 + q)\epsilon^{ijk}S^j_+ S^k_- = 0 , \quad \partial_- S^i_+ - \mu(1 - q)\epsilon^{ijk}S^j_+ S^k_+ = 0 .
\]

The equations (C.5) follow from the following action, generalizing the action in the \( q = 0 \) case given in [28] (that leads to the FR Hamiltonian [19])

\[
S = \int d^2 \sigma \left[ (1 - q)C_+(S_-) + (1 + q)C_-(S_+) - \frac{1}{2}(1 - q^2)\mu^2 S^k_+ S^k_- \right] ,
\]

where\(^{39}\)

\[
C_\pm(S) \equiv -\frac{1}{2} \int_0^1 dx \, \epsilon^{ijk} S_i \partial_+ S_j \partial_\pm S_k , \quad \delta C_\pm = \frac{1}{2} \epsilon^{ijk} \delta S_i S_j \partial_\pm S_k .
\]

In what follows we shall rescale \( \tau, \sigma \) by \( \mu \), i.e. effectively set \( \mu = 1 \) and assume that \( \sigma \) is non-compact.

We observe that there is a simple way to relate the actions (C.6) with \( q = 0 \) and \( q \neq 0 \). Let us make the following conformal transformation

\[
\tilde{\sigma}^+ = (1 + q)\sigma^+ , \quad \tilde{\sigma}^- = (1 - q)\sigma^- , \quad \sigma^\pm = \frac{1}{2}(\tau \pm \sigma) .
\]

Since the action (C.6) is not conformally invariant it will change and become formally the same as at \( q = 0 \):

\[
\tilde{S} = \int d^2 \tilde{\sigma} \left[ \tilde{C}_+(S_-) + \tilde{C}_-(S_+) - \frac{1}{2} S^k_+ S^- \right] .
\]

\( ^{38} \)The Hamiltonian in the case of a non-zero coefficient of the WZ term was also discussed in [19] but our approach will be different.

\( ^{39} \)Note that \( C_\alpha \) enters the the \( SU(2) \) Landau-Lifshitz action, which can be written as \( \int d^2 \sigma \left[ C_\alpha(n) - \frac{1}{4} n_i^2 \right] \) where \( n_i \) is a unit vector with equations of motion \( \dot{n}_i = \epsilon_{ijk} n_j n_k^\prime \).
Let us note that the same transformation applied in the Pohlmeyer-reduced (PR) theory (which is also constructed by starting with first-order equations for the currents and solving the Virasoro conditions) will also remove the \( q \)-dependent factor \((1 - q^2)\) from the mass term and will therefore relate the \( q = 0 \) and \( q \neq 0 \) theories (see Appendix D).  

More explicitly, (C.8) implies that

\[
\tilde{\tau} = \tau + q\sigma , \quad \tilde{\sigma} = \sigma + q\tau , \quad (C.10)
\]

\[
e = \tilde{e} + q\tilde{p} , \quad p = \tilde{p} + q\tilde{e} , \quad \tilde{e} = \frac{e - qp}{1 - q^2} , \quad \tilde{p} = \frac{p - qe}{1 - q^2} , \quad (C.11)
\]

where \( p_a = (p_0, p_1) \equiv (e, p) \) is the 2-momentum conjugate to \( \sigma_a = (\tau, \sigma) \) (i.e. \( p_a\sigma^a = \tilde{p}_a\tilde{\sigma}^a \)). Since this is a conformal transformation rather than a Lorentz boost the mass gets rescaled:

\[
p^2_a = (1 - q^2)\tilde{p}^2_a .
\]

Let us now explicitly solve the unit-vector constraints in (C.4) by introducing two independent complex scalar fields as

\[
S^1_\pm = 2\sqrt{1 - |\phi_\pm|^2} \phi_\pm , \quad S^3_\pm = 1 - 2|\phi_\pm|^2 . \quad (C.12)
\]

Substituting into (C.6) we find the following first-order action for \( \phi^+ , \phi^- \) (generalizing the corresponding action \([28]\) in the \( q = 0 \) case)

\[
S = \int d^2\sigma \left\{ i(1 - q) \phi^*_- \partial_+ \phi^- + i(1 + q) \phi^*_+ \partial_- \phi^+ \\
- (1 - q^2) \left[ \sqrt{(1 - |\phi_+|^2)(1 - |\phi_-|^2)} (\phi^*_+ \phi^- + \phi^*_- \phi^+ ) \\
- |\phi_+|^2 - |\phi_-|^2 + 2|\phi_+|^2|\phi_-|^2 \right] \right\} . \quad (C.13)
\]

If we rescale \( \phi_\pm \) to have canonically normalized kinetic terms then \( q \) will enter the potential terms in a complicated way. However, the coordinate transformation (C.8),(C.10) provides a short-cut to determine the dependence on \( q \) by starting with the \( q = 0 \) expression.

Let us first look at quadratic terms in (C.13) near the trivial vacuum \( \phi_\pm = 0 \):  

\[
L_2 = i(1 - q)\phi^*_- \partial_+ \phi^- + i(1 + q)\phi^*_+ \partial_- \phi^+ + (1 - q^2) (\phi^+_+ \phi^-_- - \phi^+_- \phi^*_+) . \quad (C.14)
\]

For \( q = 0 \) the dispersion relation is \([28]\)

\[
(e + 1)^2 - p^2 = 1 , \quad (C.15)
\]

so that there is a particle (magnon or BMN) state which is light at small \( p \) and an antiparticle state that decouples at low momenta. For \( q \neq 0 \) we find\footnote{The same result is found using (C.11) to get the generalization of the dispersion relation at \( q = 0 \): \((\tilde{e} + 1)^2 - \tilde{p}^2 = 1 \).}

\[
(e + 1)^2 - (p + q)^2 = 1 - q^2 , \quad (C.16)
\]
which has a solution
\[ e = \sqrt{(p + q)^2 + 1 - q^2 - 1} . \] (C.17)

This is the same dispersion relation (up to an overall energy shift) as was found from the \( S^3 \) string sigma model in (2.20). If we allow for the overall shifts of the energy and the momentum then the dispersion relation becomes the standard massive one with \( q \)-dependent mass
\[ e^2 - p^2 = 1 - q^2 . \] (C.18)

Equivalently, starting with the \( q \neq 0 \) action in (C.13) and doing the \( U(1) \) redefinition
\[ \phi_\pm \to e^{i\bar{\tau}} \phi_\pm , \quad \bar{\tau} = \tau + q\sigma , \] (C.19)
one removes the \(|\phi_+|^2 + |\phi_-|^2\) term from the quadratic part of (C.13) (while the remaining terms stay invariant) thus ending up with the dispersion relation (C.18) with unshifted momentum and energy.

Next, let us discuss the S-matrix starting with the \( q = 0 \) case. Following [28] we redefine \( \phi_\pm \to e^{i\bar{\tau}} \phi_\pm \) in the \( q = 0 \) analog of (C.13) so that the \(|\phi_+|^2 + |\phi_-|^2\) quadratic terms are eliminated, or equivalently, shift \( e \) in (C.15) to get the standard relativistic dispersion relation \( e^2 - p^2 = 1 \). The interaction vertices and thus the S-matrix will still be non-relativistic.\(^{43}\)

After the field redefinition making the \( \phi_+, \phi_- \) propagator the standard Lorentz-invariant massive one (so that we have both positive and negative energy states) one is still to decide how to quantize the theory.\(^{44}\) This will not be important at the tree level we are interested in here as the tree-level S-matrix is given simply by the quartic vertices in the action.

The prescription of [28] gave the following quantum S-matrix\(^{45}\)
\[ S_{FR}(p, p') = \frac{x - x' - 2i\kappa}{x - x' + 2i\kappa} = 1 + \frac{4i\kappa}{x' - x} + \ldots , \] (C.20)
where \( x = x(p) \) (and similarly \( x' = x(p') \)) is related to the momentum \( p \) as
\[ x = \frac{1}{p}(\sqrt{p^2 + 1} + 1) , \quad p = \frac{2x}{x^2 - 1} , \quad e = \frac{x^2 + 1}{x^2 - 1} . \] (C.21)

To find the \( q \neq 0 \) generalization of this S-matrix we may use the coordinate or momentum transformation trick (C.10),(C.11): we first put tildes on \( p, p' \) in (C.20),(C.21) and then use

\(^{43}\)Recall that the origin of this non-invariance is in the Virasoro plus temporal gauge conditions that are solved by (C.4): originally the currents \( S_\pm \) should transform as vectors but the invariance is then broken by the unit-norm condition. One may formally ask for \( \phi_\pm \) in (C.13) to transform under Lorentz boosts as 2d Weyl fermions making their kinetic terms invariant but then the interaction terms in (C.13) will still fail to be invariant.

\(^{44}\)In [28] the “wrong” vacuum was chosen in which the negative energy states are all empty with a hidden motivation of getting a standard “ferromagnetic” type S-matrix. This amounts to the use of the retarded rather than the causal propagator (just as was the case in the LL model). Then it is straightforward to compute the corresponding two-particle S-matrix as it will be given simply by summing bubble graphs [28].

\(^{45}\)Here the energies are \( e = \sqrt{p^2 + 1} \) and \( e' = \sqrt{p'^2 + 1} \). Recall that we set the scale parameter or effective coupling \( \mu \) equal to 1. Below we also inserted as formal coupling constant \( \kappa \) as in (3.7).
(C.11) where now $e^2 - p^2 = 1 - q^2$, i.e.\(^{46}\)

\[
\tilde{x} = x(\tilde{p}) = \frac{1}{\tilde{p}}(\sqrt{\tilde{p}^2 + 1} + 1), \quad \tilde{p} = \frac{p - q\sqrt{p^2 + 1 - q^2}}{1 - q^2}, \quad \tilde{e} = \sqrt{\tilde{p}^2 + 1},
\]

with the corresponding $S$-matrix now being given by (cf. (C.20),(3.9))

\[
S(p, p') = \frac{\tilde{x} - \tilde{x}' - 2i\kappa}{\tilde{x} - \tilde{x}' + 2i\kappa} = 1 + \frac{4i\kappa}{\tilde{x} - \tilde{x}'} + \ldots = 1 + \frac{4i\kappa \tilde{p} \tilde{p}'}{\tilde{p}(\tilde{e} + 1) - \tilde{p}'(\tilde{e}' + 1)} + \ldots,
\]

where we have explicitly shown the tree-level part.

In contrast to what we observed in the sigma model case in section 3, here the $q$-dependence of the tree-level $S$-matrix cannot be found by just generalizing the dispersion relation as in (3.21). This may not be surprising given that the FR $S$-matrix (3.9) did not agree with the string sigma model $S$-matrix in (3.17) already in the $q = 0$ case.

Indeed, despite sharing the same classical integrable structure the $R \times S^3$ gauge-fixed string sigma model, its Faddeev-Reshetikhin formulation and its Pohlmeyer reduction all have different tree-level $S$-matrices. This may be attributed to the fact that these $S$-matrices are computed for different objects (and also are effectively gauge-dependent quantities).

**Appendix D:**

**Pohlmeyer-reduced theory for superstring on $AdS_3 \times S^3 \times T^4$ with mixed flux**

Pohlmeyer reduction (PR) for classical string theory on $AdS_3 \times S^3$ leads to a combination of complex sine-Gordon and complex sinh-Gordon models which admits a natural superstring generalization [29, 30]. The corresponding $S$-matrix near trivial (BMN-type) vacuum is relativistic and was studied in [49, 50]. Given that string supercoset sigma model and its PR counterpart are closely related (at least at the classical and 1-loop level) sharing, in particular, the same integrable structure, it is of interest to study how the PR model gets modified upon switching on non-zero NSNS 3-form flux, i.e. for $q \neq 0$. As we will show below, somewhat unexpectedly, the modification is remarkably simple: one just needs to replace the mass parameter $\mu$ of the PR theory (which is essentially the same as the BMN parameter $J$ in the string sigma model context setting the mass scale via the Virasoro condition) as

\[
\mu \rightarrow \sqrt{1 - q^2} \mu.
\]

As a result, the PR theory depends on $q$ only through the mass $\sqrt{1 - q^2} \mu$ of the elementary excitations which will thus have the same dispersion relation as “rotated” string excitations in (2.36) (i.e. with unshifted momentum $p = \hat{p}$ in (2.22)).

We shall first consider the bosonic theory in the conformal gauge where the starting equations are the same as in the FR case – eqs.(C.1),(C.2). Before turning to the group-theoretic formulation that naturally generalizes to the superstring (supercoset) case it is useful to describe the

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\(^{46}\)Note that compared to (C.16) here there are no shifts of $p_0$ and $p_1$ as we assumed that the original $q = 0$ kinetic term took a standard massive form due to the $\phi_\pm$ redefinition – see (4.2) and the surrounding discussion.
construction of the bosonic PR theory using explicit embedding coordinate parametrization of the sigma model.

D.1 Pohlmeyer reduction of bosonic $AdS_3 \times S^3$ string in embedding coordinates

D.1.1 PR model for string on $R \times S^3$

We shall start with the string action (2.3) in the conformal gauge written in terms of $\mathbb{R}^4$ embedding coordinates

$$S = \frac{\sqrt{\lambda}}{2\pi} \left( \frac{1}{2} \int d^2 \sigma \left[ \partial_+ X \cdot \partial_- X + \Lambda(X^2 - 1) \right] + \frac{1}{3} q \int d^3 \sigma \epsilon^{abc} \epsilon_{mnpq} \partial_a X_m \partial_b X_n \partial_p \partial_q X_2 \right). \quad (D.2)$$

The resulting equations of motion and the Virasoro constraints are

$$\partial_+ \partial_- X_m - \Lambda X_m + q K_m = 0, \quad X_m^2 = 1, \quad K_m = \epsilon_{mnp} \partial_+ X_n \partial_+ X_p \partial_- X_l, \quad (D.3)$$

$$(\partial_{\pm} X_m)^2 = \mu^2. \quad (D.4)$$

Note that here

$$X \cdot \partial_{\pm} X = 0, \quad \Lambda = -\partial_+ X \cdot \partial_- X. \quad (D.5)$$

Next, we introduce the fields $(\varphi, \chi)$ of the reduced theory

$$\partial_+ X \cdot \partial_- X = \mu^2 \cos 2\varphi, \quad K \cdot \partial_{\pm}^2 X = f_{\pm}(\varphi) \partial_{\pm} \chi, \quad (D.6)$$

where $f_{\pm}(\varphi)$ are to be determined. As the set of vectors $\{X, \partial_+ X, \partial_- X, K\}$ form a basis of $\mathbb{R}^4$ (assuming $\varphi$ is non-zero) we can write $\partial_{\pm}^2 X$ as linear combinations of them

$$\partial_{\pm}^2 X = -\mu^2 X + 2\partial_{\pm} \varphi \cot 2\varphi \partial_{\pm} X - 2\partial_{\pm} \varphi \csc 2\varphi \partial_{\pm} X + \frac{f_{\pm} \partial_{\pm} \chi}{\mu^4 \sin^2 2\varphi} K. \quad (D.7)$$

Taking the inner product we find the following equation of motion for $\varphi$

$$\partial_+ \partial_- \varphi + \frac{f_+ f_- \partial_+ \chi \partial_- \chi}{2\mu^6 \sin^3 2\varphi} + \frac{\mu^2 (1 - q^2)}{2} \sin 2\varphi = 0. \quad (D.8)$$

If we assume that $f_+ = -f_-$ then

$$f_+ = -f_- = A \sin^2 \varphi, \quad (D.9)$$

is a solution of

$$\partial_- \left( \frac{\partial_{\pm}^2 X \cdot K}{f_{\pm}} \right) - \partial_+ \left( \frac{\partial_{\pm}^2 X \cdot K}{f_{\pm}} \right) = 0. \quad (D.10)$$

---

47Note that when varying the WZ term the four vectors $\delta X$ and $\partial_a X$ are orthogonal to $X$ (as can be seen by varying or differentiating the sphere constraint, $X^2 = 1$). Therefore they only span a three-dimensional subspace of $\mathbb{R}^4$ so that $\epsilon_{mnpq} \delta X_m \partial_a X_n \partial_b X_p \partial_c X_q = 0$.

48Note that as the Virasoro constraints imply that $\partial_+ X$ and $\partial_- X$ are vectors with norm $\mu$, $\varphi$ is half the angle between them.
Another solution is

\[ f_+ = f_- = B \cos^2 \varphi \]  \hspace{1cm} (D.11)

Note that in solving (D.10) the \( q \)-dependence drops out and therefore these solutions are exactly the same as in the \( q = 0 \) case. Taking \( f_\pm \) to be given by (D.9) we get for \( \chi \)

\[ \partial_- (\tan^2 \varphi \partial_+ \chi) + \partial_+ (\tan^2 \varphi \partial_- \chi) = 0 \]  \hspace{1cm} (D.12)

Choosing \( A = 4\mu^3 \) the equation of motion for \( \varphi \) (D.8) is then given by

\[ \partial_+ \partial_- \varphi - \sec^2 \varphi \tan \varphi \partial_+ \chi \partial_- \chi + \frac{1}{2}(1 - q^2)\mu^2 \sin 2\varphi = 0 \]  \hspace{1cm} (D.13)

The equations (D.12) and (D.13) are those of the complex sine-Gordon model with mass-squared \( (1 - q^2)\mu^2 \), i.e. they can be found from the following Lagrangian

\[ L = \partial_+ \varphi \partial_- \varphi + \tan^2 \varphi \partial_- \chi \partial_+ \chi + \frac{1}{2}(1 - q^2)\mu^2 \cos 2\varphi \]  \hspace{1cm} (D.14)

Thus the only effect of \( q \) in the PR theory is to modify the mass parameter. In particular, at the WZW points \( q = \pm 1 \) we find, as might be expected, a massless theory (representing the \( SU(2)/U(1) \) gauged WZW model). In the above derivation the rôles of the two solutions (D.9), (D.11) can be interchanged. This modifies the reduced theory Lagrangian (D.14) by the replacement \( \tan^2 \varphi \rightarrow \cot^2 \varphi \).

### D.1.2 PR models for strings on \( AdS_3 \times S^1 \) and \( AdS_3 \)

The reduction for strings on \( AdS_3 \times S^1 \) works in much the same way as that for strings on \( R_t \times S^3 \). Here we start with the action

\[ S = \frac{\sqrt{\lambda}}{2\pi} \left( \frac{1}{2} \int d^2 \sigma \left[ \partial_+ Y \cdot \partial_- Y + \tilde{\Lambda}(Y^2 + 1) \right] + \frac{1}{3} q \int d^3 \sigma \epsilon^{abc} \epsilon_{\mu \nu \rho \sigma} Y^\mu \partial_a Y^\nu \partial_b Y^\rho \partial_c Y^\sigma \right) \]  \hspace{1cm} (D.15)

where \( Y^\mu \) are the coordinates on \( \mathbb{R}^{2,2} \) with signature \((-,-,+,+))\) and \( \tilde{\Lambda} \) is a Lagrange multiplier imposing the \( AdS_3 \) constraint. We fix the conformal gauge as well as the \( S^1 \) angle \( \varphi = \mu \tau \). The resulting equations of motion and the Virasoro constraints are then

\[ \partial_+ \partial_- Y_\mu - \tilde{\Lambda} Y_\mu + q\tilde{K}_\mu = 0 \]  \hspace{1cm} \( Y^2 = -1 \) \hspace{1cm} \( \tilde{K}^\mu = \epsilon^\nu_{\mu \nu \rho \sigma} Y^\nu \partial_+ Y^\rho \partial_- X^\sigma \)  \hspace{1cm} (D.16)

\[ (\partial_\pm Y)^2 = -\mu^2 \]  \hspace{1cm} (D.17)

Introducing the reduced theory fields \( \phi \) and \( \vartheta \)

\[ \partial_+ Y \cdot \partial_- Y = -\mu^2 \cosh 2\varphi \]  \hspace{1cm} \( \tilde{K} \cdot \partial^2_\pm Y = \pm 4\mu^3 \sinh^2 \varphi \partial_\pm \vartheta \)  \hspace{1cm} (D.18)

we find that they satisfy the following second-order equations

\[ \partial_+ \partial_- \phi - \text{sech}^2 \phi \tanh \phi \partial_+ \vartheta \partial_- \vartheta + \frac{1}{2}(1 - q^2)\mu^2 \sinh 2\varphi = 0 \]  \hspace{1cm} (D.19)
These equations are those of the complex sinh-Gordon model that follow from the Lagrangian \(^{49}\)

\[
L = \partial_+ \phi \partial_- \phi + \tanh^2 \phi \partial_- \vartheta \partial_+ \vartheta - \frac{1}{2}(1 - q^2)\mu^2 \cosh 2\phi .
\] (D.20)

Again, the only effect of \(q\) is to modify the mass parameter and \(q = \pm 1\) corresponds to a massless theory (\(SL(2, R)/U(1)\) gauged WZW model).

It is of interest to consider the case when the string moves just on \(AdS_3\) \(^{30, 55}\) that corresponds to the limit \(\mu \to 0\). To take the \(\mu \to 0\) limit of the Lagrangian (D.20) we should first generalize it by introducing the auxiliary field \(a_\pm\)

\[
L = \partial_+ \phi \partial_- \phi + \sinh^2 \phi \partial_- \vartheta \partial_+ \vartheta
- a_- \sinh^2 \phi \partial_- \vartheta - a_+ \sinh^2 \phi \partial_+ \vartheta + a_+ a_- \cosh^2 \phi - \frac{1}{2}(1 - q^2)\mu^2 \cosh 2\phi .
\] (D.21)

Integrating out \(a_\pm\) gives back (D.20). To get a finite and non-trivial \(\mu \to 0\) limit we shift \(\phi\) and rescale \(\vartheta\) and \(a_\pm\) as

\[
\{\phi, \vartheta, a_\pm\} \to \{\phi - \log \mu, \mu \vartheta, \mu a_\pm\}.
\] (D.22)

The resulting Lagrangian is then given by

\[
L = \partial_+ \phi \partial_- \phi + \frac{1}{4} e^{2\phi}(\partial_+ \vartheta - a_+)(\partial_- \vartheta - a_-) - \frac{1}{4}(1 - q^2)e^{2\phi} .
\] (D.23)

This can be written as

\[
L = \partial_+ \phi \partial_- \phi + \frac{1}{4} e^{2\phi} \partial_+ \xi \partial_- \xi
- \frac{1}{4}(1 - q^2)e^{2\phi} ,
\] (D.24)

\[
a_\pm \equiv \partial_\pm \tilde{\xi} , \quad \xi_\pm \equiv \vartheta - \tilde{\xi} .
\] (D.25)

One can alternatively get this Lagrangian through the reduction procedure for \(\mu = 0\), starting with the following definitions of the reduced-theory fields

\[
\partial_+ Y \cdot \partial_- Y = -\frac{1}{4} e^{2\phi} , \quad \tilde{K} \cdot \partial^2_\pm Y = \pm \frac{1}{4} e^{4\phi} \partial_\pm \xi_\pm .
\] (D.26)

Note that in the course of taking the \(\mu \to 0\) part of the diffeomorphism symmetry has been restored. This is a consequence of the fact that for \(\mu \to 0\) the conformal-gauge constraints (D.17) are invariant under conformal reparametrizations. The conformal reparametrizations acting on the reduced-theory fields are given by

\[
\sigma^\pm \to f_\pm(\sigma^\pm) , \quad \partial_\pm \to f'_\pm \partial_\pm , \quad e^{2\phi} \to f'_\pm e^{2\phi} , \quad \xi_\pm \to f^{-1}_\pm \xi_\pm .
\] (D.27)

To describe the physical degrees of freedom of the string this symmetry should be fixed. One way of doing this is to observe that the classical equations for \(\xi_\pm\) imply

\[
\partial_\pm U_\pm = 0 , \quad U_\pm \equiv e^{2\phi} \partial_\pm \xi_\pm .
\] (D.28)

where \(U_\pm\) transform under the conformal reparametrizations (D.27) as \(U_\pm \to f^{2\pm} U_\pm\). Therefore, these fields can be fixed to be equal to \(\gamma = \{+1, 0, -1\}\) depending on their sign. Then the Lagrangian (D.24) becomes

\[
L = \partial_+ \phi \partial_- \phi + \frac{1}{4} \gamma e^{-2\phi} - \frac{1}{4}(1 - q^2)e^{2\phi} .
\] (D.29)

As long as \(q \neq \pm 1\) we can shift \(\phi\) to find that this Lagrangian is equivalent to either the sinh-Gordon, Liouville or cosh-Gordon Lagrangian respectively. In the case of \(q = \pm 1\) we have either the Liouville Lagrangian, a free boson or the Liouville Lagrangian with the “wrong” sign of the potential.

\(^{49}\)An alternative Lagrangian with \(\tanh^2 \phi \to \coth^2 \phi\) is found by taking instead \(\tilde{K} \cdot \partial^2_\pm Y = 4\mu^3 \cosh^2 \phi \partial_\pm \vartheta\).
D.1.3 Comments on relation between classical solutions of string and PR models

Solutions of the reduced theory with $q \neq 0$ are formally related to solutions of the reduced theory with $q = 0$ through the following conformal rescaling of the 2d coordinates

$$
\sigma^\pm \rightarrow (1 \pm q)^{-1} \sigma^\pm, \quad \partial^\pm \rightarrow (1 \pm q) \partial^\pm. \quad (D.30)
$$

A similar observation was made in the discussion of the FR model (cf. (2.5),(C.8)). Indeed, if we formally consider the transformation (D.30) in (2.5) without letting $J^\pm$ transform then we recover the equations of the principal chiral model with $q = 0$. Since the fields of the PR theory are essentially the currents of the string sigma model, this explains why the transformation (D.30) maps between $q = 0$ and $q \neq 0$ cases of the reduced theory.

The standard prescription (for $q = 0$) of how to reconstruct string sigma model solutions from the solutions of the PR theory is to take a solution of the complex sine-Gordon equations $(\varphi_0, \chi_0)$ and solve the second-order linear equation

$$
\partial_+ \partial_- X_m + \mu^2 \cos 2\varphi_0 X_m = 0. \quad (D.31)
$$

For non-zero $q$ this equation is modified, implying that, while the solutions of the reduced theories with $q = 0$ and $q \neq 0$ are related simply by (D.1), the corresponding solutions of the string theory will in general have a more non-trivial relation.

Another general conclusion is that since the 1-loop string partition function should be equal to the 1-loop partition function of the PR theory [53] and the latter should depend on $q$ only via $\mu^2 \rightarrow (1-q^2)\mu^2$ the same should apply to the string partition function. Let us now consider some simple examples of solutions, introducing the following explicit coordinates on $S^3$

$$
X_1 + iX_2 = \sin \theta e^{i\phi_1}, \quad X_3 + iX_4 = \cos \theta e^{i\phi_2}, \quad (D.32)
$$
in terms of which the Lagrangian in (D.2) is given by (2.8), i.e.

$$
L = \partial_+ \theta \partial_- \theta + \sin^2 \theta \partial_+ \phi_1 \partial_- \phi_1 + \cos^2 \theta \partial_+ \phi_2 \partial_- \phi_2 + q \sin^2 \theta (\partial_+ \phi_1 \partial_- \phi_2 - \partial_+ \phi_2 \partial_- \phi_1). \quad (D.33)
$$

Let us first consider the analogue of the BMN solution

$$
X_1 + iX_2 = 0, \quad X_3 + iX_4 = e^{i\mu \tau}, \quad \theta = \phi_1 = 0, \quad \phi_2 = \mu \tau, \quad (D.34)
$$
for which the corresponding reduced theory solution is the vacuum one

$$
\varphi = 0, \quad \chi = 0. \quad (D.35)
$$

Expanding (D.33) around (D.34) to the leading order gives

$$
L = \partial_+ \theta \partial_- \theta + \theta^2 \partial_+ \phi_1 \partial_- \phi_2 - \mu^2 \theta^2 + q \mu \theta^2 (\partial_+ \phi_1 - \partial_- \phi_1) + \partial_+ \phi_2 \partial_- \phi_2. \quad (D.36)
$$

\footnote{Let us note that this transformation relating solutions to solutions is a symmetry for generic $q$ only if $\sigma$ is decompactified.}
Transforming \((\theta, \phi_1)\) to cartesian coordinates we find the following spectrum of fluctuation frequencies

\[
\pm \sqrt{(p - q\mu)^2 + (1 - q^2)\mu^2}, \quad \pm \sqrt{(p + q\mu)^2 + (1 - q^2)\mu^2}, \quad \pm p, \tag{D.37}
\]

where \(p\) is spatial momentum (which is integer if \(\sigma\) is 2\(\pi\) periodic). Expanding (D.14) around (D.35) in a similar way gives

\[
L = \partial_+ \varphi \partial_- \varphi + \varphi^2 \partial_+ \chi \partial_- \chi - (1 - q^2)\mu^2 \varphi^2. \tag{D.38}
\]

Transforming to cartesian coordinates we find the following spectrum of fluctuation frequencies

\[
2 \times \pm \sqrt{p^2 + (1 - q^2)\mu^2}. \tag{D.39}
\]

This is the same as the massive part of the string spectrum in (D.37) up to \(q\)-shifts in the spatial momentum.

Another simple explicit solution is the circular spinning string

\[
X_1 + iX_2 = \sqrt{\frac{\nu + q}{2\nu}} e^{i(\nu - q)\tau + i\sigma}, \quad X_3 + iX_4 = \sqrt{\frac{\nu - q}{2\nu}} e^{i(\nu + q)\tau - i\sigma},
\]

\[
\sin^2 \theta = \frac{\nu + q}{2\nu}, \quad \phi_1 = (\nu - q)\tau + \sigma, \quad \phi_2 = (\nu + q)\tau - \sigma, \tag{D.40}
\]

where

\[
\mu = \sqrt{\nu^2 + 1 - q^2}. \tag{D.41}
\]

Translated into the reduced theory this solution becomes\(^{51}\)

\[
\cos 2\phi = \frac{\nu^2 - 1 - q^2}{\nu^2 + 1 - q^2}, \quad \theta = \frac{(\nu^2 - q^2)(\tau - q\sigma)}{\sqrt{\nu^2 + 1 - q^2}}. \tag{D.42}
\]

Expanding (D.33) around (D.40) and (D.14) around (D.42) to quadratic order, we find the following characteristic equations respectively

\[
(\omega^2 - p^2) \left[ (\omega^2 - p^2) (\omega^2 - p^2 + 4(1 - q^2)) - 4\mu^2 (\omega + qp)^2 \right] = 0, \]

\[
(\omega^2 - p^2) (\omega^2 - p^2 + 4(1 - q^2)) - 4\mu^2 (\omega + qp)^2 = 0. \tag{D.43}
\]

\(^{51}\) One may wonder how the form of this solution is consistent with the claim that the PR solutions should depend on \(q\) only via \(\tilde{\mu} = \mu \sqrt{1 - q^2}\). The PR equations are invariant under the formal transformation: \(\sigma^\pm \rightarrow (1 + q)^{-1}\sigma^\pm\) and \(\mu \rightarrow (1 - q^2)^{1/2} \tilde{\mu}\) (see also the related transformation (D.30)). Performing this transformation on the \((\phi, \theta)\) solution above we find

\[
\cos 2\phi = 1 - 2\tilde{\mu}^{-2}, \quad \theta = (\tilde{\mu} - \tilde{\mu}^{-1})\tau, \quad \tilde{\mu} = \mu \sqrt{1 - q^2}.
\]

That is the solution can be put into a form such that it depends on \(q\) only via \(\tilde{\mu}\) and thus satisfies the PR equations. This transformation does not respect \(\sigma\)-periodicity. However, if and when the reduction procedure preserves the periodicity of a classical string solution is a subtle issue even for \(q = 0\) \([53, 54]\), which we will not address here.
Ignoring the trivial (longitudinal) string massless mode, we get the same characteristic equation, i.e. the same spectrum of fluctuation frequencies and the same 1-loop partition function.

Note that for \( q = 0 \) we get of course the same frequencies

\[
\pm \sqrt{p^2 + 2(\mu^2 - 1) \pm 2\sqrt{(\mu^2 - 1)^2 + \mu^2 p^2}} \tag{D.44}
\]

as found for the same solution in the \( AdS_5 \times S^5 \) case. At the WZW point, \( q = 1 \), we get

\[
\{-p, -p, p \pm 2\mu\}, \tag{D.45}
\]

i.e. the spectrum is massless up to a shift in the momentum.

### D.2 Pohlmeyer reduction in group-theoretic approach

To extend the Pohlmeyer reduction to include fermions we need first to formulate it in terms of group variable parametrization based on describing the principal chiral model for group \( G \) as the \( \frac{G \times G}{G_0} \) coset sigma model corresponding the symmetric coset space

\[
\frac{G_L \times G_R}{G_0}, \tag{D.46}
\]

where \( G_{L,R} \) are two copies of the group \( G = SU(2) \) and \( G_0 \) is the diagonal subgroup isomorphic to \( G \). Taking a group-valued field

\[
f = \begin{pmatrix} g_l & 0 \\ 0 & g_R \end{pmatrix} \in G_L \times G_R , \tag{D.47}
\]

we construct the left-invariant current

\[
J = f^{-1} df = \begin{pmatrix} \mathcal{J}_L = g_l^{-1} dg_L \\ 0 \\ \mathcal{J}_R = g_R^{-1} dg_R \end{pmatrix} \in \mathfrak{g}_L \oplus \mathfrak{g}_R . \tag{D.48}
\]

The algebra \( \mathfrak{g}_L \oplus \mathfrak{g}_R \) admits an \( \mathbb{Z}_2 \) automorphism

\[
\Omega \begin{pmatrix} a_L & 0 \\ 0 & a_R \end{pmatrix} = \begin{pmatrix} a_R & 0 \\ 0 & a_L \end{pmatrix} \tag{D.49}
\]

with the invariant subspace given by the diagonal subalgebra \( \mathfrak{g}_0 \). The trace is clearly invariant under this automorphism and hence we have the following orthogonal decomposition of the algebra

\[
\mathfrak{g}_L \oplus \mathfrak{g}_R = \mathfrak{g}_0 \oplus \mathfrak{p} . \tag{D.50}
\]

Decomposing the left-invariant current \( \mathcal{J} \) under the orthogonal decomposition \( \mathcal{J} \)

\[
\mathcal{J} = \mathcal{A} + \mathcal{P} , \quad \mathcal{A} = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} , \quad \mathcal{A} = \frac{1}{2}(\mathcal{J}_L + \mathcal{J}_R) , \tag{D.51}
\]

\[
\mathcal{P} = \begin{pmatrix} \mathcal{P} & 0 \\ 0 & -\mathcal{P} \end{pmatrix} , \quad \mathcal{P} = \frac{1}{2}(\mathcal{J}_L - \mathcal{J}_R) .
\]
the action (2.3) can be written as
\[ S = -\frac{\sqrt{\lambda}}{2\pi} \left[ \int d^2 x \, \frac{1}{2} \text{Tr}(\mathcal{P}_+ \mathcal{P}_-) - q \int d^3 x \, \frac{2}{3} \epsilon^{abc} \widetilde{\text{Tr}}(\mathcal{P}_a \mathcal{P}_b \mathcal{P}_c) \right]. \] (D.52)

Here \( \widetilde{\text{Tr}} \) is defined as
\[ \widetilde{\text{Tr}} \left( \begin{pmatrix} a_L & 0 \\ 0 & a_R \end{pmatrix} \right) = \text{Tr}(a_L) - \text{Tr}(a_R). \] (D.53)

\( \text{Tr} \) is normalized to -1 compared to \( \text{tr} \) which is normalized to -2. If the usual trace is used in the WZ term it vanishes as a consequence of the \( \mathbb{Z}_2 \) automorphism of the algebra.\(^{52}\)

To recover the action (2.3) from (D.52) we notice that the latter admits the following gauge symmetry
\[ f \to fg_0, \quad g_0 \in G_0, \quad \mathcal{A} \to g_0^{-1} \mathcal{A} g_0 + g_0^{-1} \varepsilon d g_0, \quad \mathcal{P} \to g_0^{-1} \mathcal{P} g_0, \] (D.54)

which follows from the cyclicity of both \( \text{Tr} \) and \( \widetilde{\text{Tr}} \). Using this symmetry to fix \( g_R = 1 \) we find that \( \mathcal{A} = \mathcal{P} \). Defining
\[ J = 2 \mathcal{A} = 2 \mathcal{P} \] (D.55)

and substituting into (D.52) we recover (2.3).

The equations of motion following from (D.52) can be projected onto \( \mathfrak{g}_L \)\(^{53}\)
\[ \mathcal{D}_- \mathcal{P}_+ + \mathcal{D}_+ \mathcal{P}_- - 2q[\mathcal{P}_-, \mathcal{P}_+] = 0, \quad \mathcal{D}_\pm = \partial_\pm + [\mathcal{A}_\pm,], \] (D.56)

while the conformal gauge Virasoro constraints are given by
\[ \text{Tr}(\mathcal{P}_\pm^2) = -\mu^2. \] (D.57)

Another condition is the flatness condition for the current \( J \)
\[ dJ + J \wedge J = 0, \] (D.58)

which can be decomposed under the orthogonal decomposition (D.50) and projected onto \( \mathfrak{g}_L \) to give
\[ d\mathcal{P} + \mathcal{A} \wedge \mathcal{P} + \mathcal{P} \wedge \mathcal{A} = 0, \quad d\mathcal{A} + \mathcal{A} \wedge \mathcal{A} + \mathcal{P} \wedge \mathcal{P} = 0. \] (D.59)

The equation of motion (D.56) and the first equation of (D.59) can be rewritten as
\[ \mathcal{D}_+ \mathcal{P}_- + q[\mathcal{P}_+, \mathcal{P}_-] = 0, \quad \mathcal{D}_- \mathcal{P}_+ - q[\mathcal{P}_-, \mathcal{P}_+] = 0. \] (D.60)

The Pohlmeyer reduction starts by introducing a constant matrix \( \tilde{T} = \begin{pmatrix} T & 0 \\ 0 & -T \end{pmatrix} \in \mathfrak{p} \) normalized as \( \text{Tr} \tilde{T}^2 = -1 \). We then solve the Virasoro conditions fixing the \( G_0 \) gauge symmetry as
\[ \mathcal{P}_+ = \mu T, \quad \mathcal{P}_- = \mu g^{-1} T g, \quad g \in G = SU(2). \] (D.61)

\(^{52}\)In particular, this action written in terms of \( \mathcal{P} \) agrees with the action in [6].

\(^{53}\)Or alternatively onto \( \mathfrak{g}_R \) — by construction, the equations are equivalent.
Substituting into (D.60) we find that these equations are solved by parametrizing
\[ A_+ = g^{-1} \partial_+ g + g^{-1} A_+ g - q \mu T , \quad A_- = A_- + q \mu g^{-1} T g , \]  
(D.62)
where \( A_\pm \) take values in the subalgebra of \( g = \mathfrak{su}(2) \) that commutes with \( T \) (we shall denote this subalgebra as \( \mathfrak{h} \)). This is a \( \mathfrak{u}(1) \) subalgebra that is spanned by \( T \) itself.

Finally, to find the equation of motion of the reduced theory we substitute (D.61) and (D.62) into the second equation of (D.59) to give
\[ \partial_-(g^{-1} \partial_+ g + g^{-1} A_+ g) - \partial_+ A_- + [A_-, g^{-1} \partial_+ g + g^{-1} A_+ g] = (1 - q^2) \mu^2 [T, g^{-1} T g] . \]  
(D.63)
Again, as in the embedding coordinate parametrization of the previous section, we see that the only effect of the WZ term is to rescale the mass-squared parameter \( \mu^2 \) by \( (1 - q^2) \). We can then follow the final steps of the usual PR approach \([29, 30]\) to find a gauged WZW model for the coset \( G/H = SU(2)/U(1) \) plus an integrable potential with coefficient \( (1 - q^2) \mu^2 \). Choosing a particular parametrization of \( g \) and integrating out the gauge field \( A_\pm \) one recovers the complex sine-Gordon model in agreement with the embedding coordinate reduction approach.\(^{54}\) In particular as the gauge group is abelian the WZW model can be either vector or axially gauged. In the former case we find the complex sine-Gordon Lagrangian with \( \cot^2 \phi \) and in the latter case with \( \tan^2 \phi \) – see the comment below eq. (D.14). Note also that the reduced theory for \( q = 1 \), i.e. for the \( SU(2) \) WZW model, is given by the standard \( SU(2)/U(1) \) gauged WZW model.

A similar construction in the case of \( G = SL(2, R) \) will lead to the reduced theory given by the gauged WZW model for the coset \( G/H = SL(2, R)/U(1) \) plus an integrable potential with coefficient \( (1 - q^2) \mu^2 \), equivalent after gauge fixing to the complex sinh-Gordon model (in agreement with the discussion in section D.1.2).

In general, the above reduction procedure will work for any sigma model with a target space of the form (D.46) (times \( R_t \)) but for generic \( G \) there will be additional rank \( G - 1 \) massless modes (for any value of \( q \)).

Finally, let us give also the expression for the Lax connection corresponding to reduced theory equations. The set of sigma model equations (D.59) and (D.60) follow from a Lax connection with spectral parameter \( z \) (cf.(2.7))
\[ \mathcal{L}_\pm = A_\pm \pm q \mathcal{P}_\pm + z^{\pm 1} \sqrt{1 - q^2} \mathcal{P}_\pm . \]  
(D.64)
Substituting the change of variables (D.61), (D.62) we find the Lax connection of the reduced theory:
\[ \mathcal{L}_+ = g^{-1} \partial_+ g + g^{-1} A_+ g + z \sqrt{1 - q^2} \mu T , \quad \mathcal{L}_- = A_- + z^{-1} \sqrt{1 - q^2} \mu g^{-1} T g . \]  
(D.65)
Again, the dependence on \( q \) is only via the rescaling of \( \mu \) by \( \sqrt{1 - q^2} \).

\(^{54}\)Let us note again that while in the reduced theory solutions for zero and non-zero \( q \) are formally related by the simple transformation (D.30), the corresponding string solutions will have a more non-trivial relation. This can be seen from the change of variables (D.61) and (D.62), where the \( q \) does not enter via a rescaling of \( \mu \) by \( \sqrt{1 - q^2} \).
D.3 Pohlmeyer reduction for superstring on \( AdS_3 \times S^3 \) with mixed flux

Here we follow [29, 30] and start from the coset superspace

\[
PSU(1, 1|2)_L \times PSU(1, 1|2)_R \over SU(1, 1) \times SU(2),
\]

where denominator is the diagonal subgroup of the bosonic subgroup of the numerator. The algebra has \( \mathbb{Z}_4 \) orthogonal decomposition, which schematically takes the form

\[
\begin{pmatrix}
\frac{1}{2}(a + b) & 0 & 0 & 0 \\
0 & \frac{1}{2}(c + d) & 0 & 0 \\
0 & 0 & \frac{1}{2}(a + b) & 0 \\
0 & 0 & 0 & \frac{1}{2}(c + d)
\end{pmatrix}
+ \begin{pmatrix}
0 & \frac{1}{2}(a + i\beta) & 0 & 0 \\
0 & 0 & \frac{1}{2}(\delta - i\nu) & 0 \\
\frac{1}{2}(\nu + i\delta) & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{2}(\beta - i\alpha)
\end{pmatrix}
+ \begin{pmatrix}
0 & \frac{1}{2}(a - b) & 0 & 0 \\
0 & \frac{1}{2}(c - d) & 0 & 0 \\
0 & 0 & \frac{1}{2}(b - a) & 0 \\
0 & 0 & 0 & \frac{1}{2}(d - c)
\end{pmatrix}
+ \begin{pmatrix}
0 & 0 & \frac{1}{2}(\nu - i\beta) & 0 \\
0 & 0 & 0 & 0 \\
\frac{1}{2}(\nu - i\delta) & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{2}(\delta + i\nu)
\end{pmatrix}
\]

We decompose the left-invariant Maurer-Cartan one-form as

\[
J = g^{-1}dg = J_0 + J_1 + J_2 + J_3 , \quad dJ + J \wedge J = 0 .
\]

The resulting equations of motion can be written in terms of \( J \) projected onto one copy of \( PSU(1, 1|2) \)

\[
J \bigg|_{PSU(1, 1|2)_{L}} = J_0 + J_1 + J_2 + J_3 ,
\]

where \( J_{0,2} / J_{1,3} \) are elements of the Grassmann-even/odd subalgebra of \( PSU(1, 1|2) \).

The superstring action is \([6]^{55}\)

\[
S = \frac{\sqrt{\lambda}}{2\pi} \int d^2 x \ T \left[ j_2 + J_2 - \frac{1}{2} \sqrt{1 - q^2} (J_1, J_3) \right] - q \int d^3 x \ e^{abc} STr \left[ \frac{3}{2} j_{2a} j_{2b} j_{2c} + j_{1a} j_{3b} j_{2c} + j_{3a} j_{1b} j_{2c} \right] .
\]

The equations of motion are then given by \((D_\pm = \partial_\pm + [J_0_{\pm}] )\)

\[
D_- J_{2+} + D_+ J_{2-} + \sqrt{1 - q^2} ([J_1, J_3] - [J_3, J_1]) + q \left( 2 [J_{2+}, J_{2+}] + [J_{1-}, J_{3+}] + [J_{3-}, J_{1+}] \right) = 0 ,
\]

\[
[J_{2+}, J_{3-}] - \frac{1 + \sqrt{1 - q^2}}{q} [J_{1-}] + [J_{2-}, J_{3+}] + \frac{1 - \sqrt{1 - q^2}}{q} J_{1+} = 0 ,
\]

\[
[J_{2+}, J_{1-}] - \frac{1 - \sqrt{1 - q^2}}{q} [J_{3-}] + [J_{2-}, J_{1+}] + \frac{1 + \sqrt{1 - q^2}}{q} J_{3+} = 0 ,
\]

\[^{55}\text{This is equivalent to the action in [6] up to sign conventions. The action has } \kappa \text{-symmetry, is integrable, and reduces to the standard supercoset GS action in the limit } q \rightarrow 0.\]
We can then solve the first-order equations \( (\text{psu} \{ \text{decomposition of the algebra} \} \) with the usual ones at while combining \( (\text{Note that in the} \ q \text{\ sector of the algebra. Explicitly, we can write} \) solve the Virasoro constraints \( \text{Therefore, the Pohlmeyer reduction can be carried out in the same way as before, i.e. we first} \) \( (\text{To} \ \text{be supplemented by the Maurer-Cartan equations} \) \( (\text{D.80}) \) \( (\text{D.81}) \) \( (\text{D.82}) \) \( (\text{D.83}) \) \( (\text{D.84}) \) to be supplemented by the Maurer-Cartan equations

\[
\begin{align*}
\partial_- \mathcal{J}_{0+} - \partial_+ \mathcal{J}_{0-} + [\mathcal{J}_{0-}, \mathcal{J}_{0+}] + [\mathcal{J}_{2-}, \mathcal{J}_{2+}] + [\mathcal{J}_{1-}, \mathcal{J}_{3+}] + [\mathcal{J}_{3-}, \mathcal{J}_{1+}] &= 0, \\
\mathcal{D}_- \mathcal{J}_{2+} - \mathcal{D}_+ \mathcal{J}_{2-} + [\mathcal{J}_{1-}, \mathcal{J}_{1+}] + [\mathcal{J}_{3-}, \mathcal{J}_{3+}] &= 0, \\
\mathcal{D}_- \mathcal{J}_{1+} - \mathcal{D}_+ \mathcal{J}_{1-} + [\mathcal{J}_{2-}, \mathcal{J}_{3+}] + [\mathcal{J}_{3-}, \mathcal{J}_{2+}] &= 0, \\
\mathcal{D}_- \mathcal{J}_{3+} - \mathcal{D}_+ \mathcal{J}_{3-} + [\mathcal{J}_{2-}, \mathcal{J}_{1+}] + [\mathcal{J}_{1-}, \mathcal{J}_{2+}] &= 0,
\end{align*}
\]

and the Virasoro constraints

\( \text{STr}(\mathcal{J}_{2\pm}^2) = 0 \).

Let us introduce the parameter\(^{56}\)

\[
\zeta = \frac{1 - \sqrt{1 - q^2}}{q} = \frac{q}{1 + \sqrt{1 - q^2}}.
\]

Taking linear combinations of \((D.71)\) and \((D.72)\), they can be written as

\[
[\mathcal{J}_{2+}, \mathcal{J}_{1-} - \zeta \mathcal{J}_{3-}] = [\mathcal{J}_{2-}, \mathcal{J}_{3+} + \zeta \mathcal{J}_{1+}] = 0,
\]

while combining \((D.70)\) and \((D.74)\) we find the following first-order equations for \( \mathcal{J}_{2\pm} \)

\[
\begin{align*}
\mathcal{D}_+ \mathcal{J}_{2-} + q [\mathcal{J}_{2+}, \mathcal{J}_{2-}] - \frac{1}{2} q [\mathcal{J}_{1-} + \zeta^{-1} \mathcal{J}_{3-}, \mathcal{J}_{3+} + \zeta \mathcal{J}_{1+}] &= 0, \\
\mathcal{D}_- \mathcal{J}_{2+} - q [\mathcal{J}_{2-}, \mathcal{J}_{2+}] - \frac{1}{2} q [\mathcal{J}_{1-} - \zeta \mathcal{J}_{3-}, \mathcal{J}_{3+} - \zeta^{-1} \mathcal{J}_{1+}] &= 0.
\end{align*}
\]

Note that in the \( q \to 0 \) limit we have \( \zeta \to 0 \) and \( q \zeta^{-1} \to 2 \) and hence these equations agree with the usual ones at \( q = 0 \).

On the equations of motion (i.e. using \((D.79)\)) we can choose the following \( \kappa \)-symmetry gauge

\[
\mathcal{J}_{1-} = \zeta \mathcal{J}_{3-} \equiv \zeta \mathcal{Q}_{-}, \quad \mathcal{J}_{3+} = -\zeta \mathcal{J}_{1+} \equiv -\zeta \mathcal{Q}_{+}.
\]

Then \((D.80)\) and \((D.81)\) simplify to the form which is the same as in the bosonic case \((D.60)\).

Therefore, the Pohlmeyer reduction can be carried out in the same way as before, i.e. we first solve the Virasoro constraints \((D.77)\) using the \( G \)-gauge symmetry as

\[
\mathcal{J}_{2+} = \mu T, \quad \mathcal{J}_{2-} = \mu g^{-1} T g,
\]

where \( T \in \text{su}(1, 1) \oplus \text{su}(2) \subset \text{psu}(1, 1|2) \) satisfies \( \text{STr}(T^2) = 0 \), is non-zero in both the \( \text{su}(1, 1) \) and \( \text{su}(2) \) sectors of the algebra. Explicitly, we can write \( T \) as \( T = \mu T_1 + \mu T_2 \) where \( T_1 \in \text{su}(1, 1) \) and \( T_2 \in \text{su}(2) \) and \( \text{Tr}(T_1^2) = \text{Tr}(T_2^2) \). The matrix \( T \) defines an additional \( \mathbb{Z}_2 \) orthogonal decomposition of the algebra \( \text{psu}(1, 1|2) \):

\[
\text{psu}(1, 1|2) = \text{psu}(1, 1|2)^\perp \oplus \text{psu}(1, 1|2)^\parallel, \quad [T, \text{psu}(1, 1|2)^\perp] = 0, \quad \{T, \text{psu}(1, 1|2)^\parallel\} = 0.
\]

We can then solve the first-order equations \((D.80)\) and \((D.81)\) in the \( \kappa \)-symmetry gauge \((D.82)\):

\[
\mathcal{J}_{0+} = g^{-1} \partial_+ g + g^{-1} A_+ g - q \mu T, \quad \mathcal{J}_{0-} = A_- + q \mu g^{-1} T g.
\]

\(^{56}\text{Note that for the three special points } q = \{0, \pm 1\}, \text{ we have } \zeta = q.\)
Here \( A_{\pm} \in \mathfrak{psu}(1,1|2)^{\perp}_{\text{even}} \equiv \mathfrak{h} \), i.e. \( \mathfrak{h} \) is the subalgebra of \( \mathfrak{su}(1,1) \oplus \mathfrak{su}(2) \) that commutes with \( T \), i.e. it is \( \mathfrak{u}(1) \oplus \mathfrak{u}(1) \) generated by \( T_1 \) and \( T_2 \). The Pohlmeyer reduction proceeds by substituting the \( \kappa \)-symmetry fixing (D.82) in equations (D.75) and (D.76)

\[
\mathcal{D}_- \mathcal{Q}_+ - \zeta \mathcal{D}_+ \mathcal{Q}_- - \zeta [\mathcal{J}_{2-}, \mathcal{Q}_+] - [\mathcal{J}_{2+}, \mathcal{Q}_-] = 0, \\
\zeta \mathcal{D}_- \mathcal{Q}_+ + \mathcal{D}_+ \mathcal{Q}_- - [\mathcal{J}_{2-}, \mathcal{Q}_+] + \zeta [\mathcal{J}_{2+}, \mathcal{Q}_-] = 0.
\]

Taking linear combinations, and substituting in for \( \mathcal{J}_{2\pm} \) from (D.83) and \( \mathcal{J}_{0\pm} \) from (D.85) we find the following equations for \( \mathcal{Q}_\pm \) (\( D_{\pm} = \partial_{\pm} + [A_{\pm},] \))

\[
D_- \mathcal{Q}_+ - \mu \sqrt{1 - q^2} [T, g^{-1} (g \mathcal{Q}_- g^{-1}) g] = 0, \\
D_+(g \mathcal{Q}_- g^{-1}) - \mu \sqrt{1 - q^2} [T, g \mathcal{Q}_+ g^{-1}] = 0.
\]

Defining

\[
\mathcal{Q}_+^\parallel = c \Psi_R, \quad (g \mathcal{Q}_- g^{-1})^\parallel = c \Psi_L, \quad \mathcal{Q}_+^\perp = c \tilde{\Psi}_R, \quad (g \mathcal{Q}_- g^{-1})^\perp = c \tilde{\Psi}_L, \quad c = \sqrt{\frac{q^2}{2\mu}},
\]

and projecting onto the parallel and perpendicular subspaces (D.84) we find that \( \tilde{\Psi}_{L,R} \) satisfy

\[
D_- \tilde{\Psi}_R = D_+ \tilde{\Psi}_L = 0,
\]

and the residual \( \kappa \)-symmetry can be used to fix them to zero. The final system of equations describing the reduced theory is given by (D.73) and (D.87) after substituting in for the new set of variables \( \{g, A_{\pm}, \Psi_{R,L}\} \)

\[
\partial_- (g^{-1} \partial_+ g + g^{-1} A_+ g) - \partial_+ A_- + [A_-, g^{-1} \partial_+ g + g^{-1} A_+ g] \\
= (1 - q^2) \mu^2 [T, g^{-1} T g] + \sqrt{1 - q^2} \mu [\Psi_R, g^{-1} \Psi_L g],
\]

\[
D_- \Psi_R - \sqrt{1 - q^2} \mu [T, g^{-1} \Psi_L g] = 0, \quad D_+ \Psi_L - \sqrt{1 - q^2} \mu [T, g \Psi_R g^{-1}] = 0
\]

Therefore, we find the same reduced system of equations as in the \( q = 0 \) theory, but with \( \mu \to \sqrt{1 - q^2} \mu \). We can then follow the final steps of the usual PR approach [29, 30] to find the corresponding action of the PR model as that of the gauged WZW model for the coset \( \frac{SU(1,1) \times SU(2)}{U(1) \times U(1)} \) plus a potential and fermionic terms \( (k \) is the coupling of the PR model)

\[
S = \frac{k}{4\pi} \text{STr} \left[ \frac{1}{2} \int d^2 \sigma \ g^{-1} \partial_+ g \ g^{-1} \partial_- g - \frac{1}{3} \int d^3 \sigma \ \epsilon^{mnl} \ g^{-1} \partial_m g \ g^{-1} \partial_n g \ g^{-1} \partial_l g \\
+ \int d^2 x \ [A_+ \partial_- g g^{-1} - A_- \partial_+ g g^{-1} A_+ g A_- + A_+ A_- + (1 - q^2) \mu^2 g^{-1} T g T] \\
+ \int d^2 x \left( \Psi_L T D_+ \Psi_L + \Psi_R T D_- \Psi_R + \sqrt{1 - q^2} \mu g^{-1} \Psi_L g \Psi_R \right) \right].
\]

Let us note that as in the bosonic models of sections D.1 and D.2, the reduced theory solutions for zero and non-zero \( q \) are formally related by the simple transformation (D.30) along with

\[
\Psi_R \to \sqrt{1 + q} \Psi_R, \quad \Psi_L \to \sqrt{1 - q} \Psi_L.
\]
However, the corresponding string solutions will have a more non-trivial relation. This can be
seen from the change of variables used above: (D.83),(D.85),(D.82),(D.88), which is not just
given by rescaling $\mu$ by $\sqrt{1-q^2}$. The same is then true also for the corresponding reduced
theory action (D.92).

The set of supercoset sigma model equations (D.70)–(D.76) follow from the flatness condition
for the following Lax connection (with spectral parameter $z$)\footnote{Up to sign conventions this is equivalent to the Lax connection in [6] written in light-cone coordinates.}

\[
\begin{align*}
L_+ &= z^{-1}\tilde{q}(J_{3+} + \zeta J_{1+}) + J_{0+} + q J_{2+} + z \tilde{q}(J_{1+} - \zeta J_{3+}) + z^2 \sqrt{1-q^2} J_{2+}, \\
L_- &= z \tilde{q}(J_{1-} - \zeta J_{3-}) + J_{0-} - q J_{2-} + z^{-1} \tilde{q}(J_{3-} + \zeta J_{1-}) + z^{-2} \sqrt{1-q^2} J_{2-},
\end{align*}
\]

(D.94)

with

\[
\tilde{q}^2 = \frac{q \sqrt{1-q^2}}{2 \zeta}.
\]

(D.95)

On substituting here the change of variables used in the Pohlmeyer reduction we find

\[
\begin{align*}
L_+ &= g^{-1} \partial_+ g + g^{-1} A_+ g + z(\mu \sqrt{1-q^2})^{1/2} \Psi_R + z^2 \mu \sqrt{1-q^2} T, \\
L_- &= A_- + z^{-1}(\mu \sqrt{1-q^2})^{1/2} g^{-1} \Psi_L g + z^{-2} \mu \sqrt{1-q^2} g^{-1} T g.
\end{align*}
\]

(D.96)

This gives indeed the Lax connection of the PR theory. Note again that $q$ enters here only via
a simple rescaling of the mass parameter $\mu \to \sqrt{1-q^2} \mu$.

In the $q = 0$ case the PR model expanded near the trivial vacuum has the same massive
spectrum (with mass $\mu$) as the BMN-type spectrum of small fluctuations in the string sigma
model. The corresponding massive tree-level S-matrix of the Pohlmeyer reduction of the super-
string on $AdS_3 \times S^3$ is \textit{relativistically invariant} [49, 50]. It formally has the same structure as
the superstring S-matrix in (3.1),(4.1) with $\sqrt{\lambda} \to k$ being the coupling of the reduced theory
and with the functions of momenta $l_1,2,3,4,5$ given by (here the function $c = 0$)

\[
\begin{align*}
l_1 &= \coth \frac{\hat{\theta}}{2}, \quad l_2 = -\tanh \frac{\hat{\theta}}{2}, \quad l_3 = 0, \quad l_4 = -\frac{1}{2} \text{sech} \frac{\hat{\theta}}{2}, \quad l_5 = \frac{1}{2} \text{csch} \frac{\hat{\theta}}{2}.
\end{align*}
\]

(D.97)

Here $\hat{\theta}$ is the difference of the two rapidities

\[
\hat{\theta} = \theta - \theta', \quad p = \mu \sinh \theta, \quad e = \mu \cosh \theta, \quad p' = \mu \sinh \theta', \quad e' = \mu \cosh \theta'.
\]

(D.98)

The functions $l_6,7,8,9$ are then defined as in (4.9). The 1-loop result for the PR S-matrix and a
conjecture for its all-order expression based on supersymmetry was given in [50].

Since the generalization to the $q \neq 0$ case is found simply by replacing $\mu \to \sqrt{1-q^2} \mu$ in the
relativistic PR Lagrangian (D.92), the corresponding S-matrix thus remains the same as in the
$q = 0$ case.
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