CONNECTION FORMULAE FOR TRIVARIATE $q$-POLYNOMIALS

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Abstract. In this short paper, we establish connection formulae for trivariate $q$-polynomials.

1. Introduction, Notations and Definitions

The $q$-analysis can be traced back to the earlier works of L. J. Rogers [5]. It has wideranging applications in the analytic number theory and $q$-deformation of well-known functions [6] as well as in the study of solvable models in statistical mechanics [7]. During the 80's the interest on this analysis increased with quantum groups theory which models of $q$-deformed oscillators have been developed [8]. The $q$-analogs of boson operators have been defined in [9] where the corresponding wavefunctions were constructed in terms of the continuous $q$-Hermite polynomials of Rogers and other polynomials. Actually, known models of $q$-oscillators are closely related with $q$-orthogonal polynomials.

In this paper, we adopt the common notation and terminology for basic hypergeometric series as in Refs. [3, 4]. Throughout this paper, we assume that $q$ is a fixed nonzero real or complex number and $|q| < 1$. The $q$-shifted factorial and its compact factorial are defined [3, 4], respectively by:

$$(a; q)_0 := 1, \quad (a; q)_n = \prod_{k=0}^{n-1} (1 - aq^k), \quad (a; q)_\infty := \prod_{k=0}^{\infty} (1 - aq^k) \quad (1.1)$$

and $(a_1, a_2, \ldots, a_r; q)_m = (a_1; q)_m(a_2; q)_m \cdots (a_r; q)_m$, $m \in \{0, 1, 2 \cdots\}$.

Here, in our present investigation, we are mainly concerned with the Jackson-Hahn-Cigler (JHC) $q$-addition

$$(x \ominus_q y)^n := \sum_{k=0}^{n} \binom{n}{k}_q q^{k} x^{n-k} y^k = x^n - \Phi_{q^n} \left( \frac{y}{x} ; q \right) = P_n(x, -y), \quad n = 0, 1, 2, \cdots, \quad (1.2)$$

where the Cauchy polynomials $P_n(x, y)$ as given below (see [2] and [3]):

$$P_n(x, y) := (x - y)(x - qy) \cdots (x - q^{n-1}y) = (y/x; q)_n x^n := (x \ominus_q y)^n \quad (1.3)$$

which has the following generating function [2]

$$\sum_{n=0}^{\infty} P_n(x, y) \frac{t^n}{(q; q)_n} = \frac{(yt; q)_\infty}{(xt; q)_\infty}. \quad (1.4)$$

The generating function (1.4) is also the homogeneous version of the Cauchy identity or the $q$-binomial theorem given by [3]

$$\sum_{k=0}^{\infty} \left( \frac{a; q}{(q; q)_k} \right)_k z^k = \Phi_0 \left[ \begin{array}{c} a; q; z \\ -; - \end{array} ; q; z \right] = \left( \frac{a_z; q}_\infty \right)_z, \quad |z| < 1, \quad (1.5)$$

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where the basic or $q$-hypergeometric function in the variable $z$ (see Slater [15, Chap. 3], Srivastava and Karlsson [16, p. 347, Eq. (272)]) for details) is defined as:

$$r\Phi_s \left[\begin{array}{c} a_1, a_2, \ldots, a_r; \\ b_1, b_2, \ldots, b_s; \\ q; z \end{array}\right] = \sum_{n=0}^{\infty} \frac{(-1)^n q^{(\ell)} z^n}{(b_1, b_2, \ldots, b_s; q)_n (q; q)_n}$$

when $r > s + 1$. Note that, for $r = s + 1$, we have:

$$r+1\Phi_r \left[\begin{array}{c} a_1, a_2, \ldots, a_{r+1}; \\ b_1, b_2, \ldots, b_r; \\ q; z \end{array}\right] = \sum_{n=0}^{\infty} \frac{(a_1, a_2, \ldots, a_{r+1}; q)_n}{(b_1, b_2, \ldots, b_r; q)_n} z^n.$$ 

Putting $a = 0$, the relation (1.5) becomes Euler’s identity [3]

$$\sum_{k=0}^{\infty} \frac{z^k}{(q; q)_k} = \frac{1}{(z; q)_\infty}, \quad |z| < 1 \quad (1.6)$$

and its inverse relation [3]

$$\sum_{k=0}^{\infty} \frac{(-1)^k q^{(\ell)} z^k}{(q; q)_k} = (z; q)_\infty. \quad (1.7)$$

The relations (1.6) and (1.7) satisfy

$$e_q(a)E_q(b) = e_q(a \oplus_q b), \quad e_q(a)E_q(-a) = 1. \quad (1.8)$$

In [1], Mohammed introduced the trivariate $q$-polynomials as

$$F_n(x, y, z; q) = (-1)^n q^{(\ell)n} \sum_{k=0}^{n} \binom{n}{k}_q (-1)^k q^{(\ell)k} P_{n-k}(y, x) z^k \quad (1.9)$$

with the following generating function [1, Theorem 2.6]

$$\sum_{n=0}^{\infty} F_n(x, y, z; q) \frac{(-1)^n q^{(\ell)n} t^n}{(q; q)_n} = \frac{(xt, zt, q)_\infty}{(yt; q)_\infty}. \quad (1.10)$$

In this paper, we shall establish a connection formulae for trivariate $q$-polynomials. The connection formulae for orthogonal polynomials are useful in mathematical analysis and also have applications in quantum mechanics, such as finding the relationships between the wavefunctions of some potential functions used for describing physical and chemical properties in atomic and molecular systems [10].

In the rest, we establish some connection formulae and discuss some particular cases.

2. MAIN RESULTS

In this section, we give the following fundamental theorem.

**Theorem 1.** Let $f(x, y, z)$ be a three-variable analytic function in a neighborhood of $(x, y, z) = (0, 0, 0) \in \mathbb{C}^3$. Then, $f(x, y, z)$ can be expanded in terms of $F_n(x, y, z; q)$ if and only if $f$ satisfies the following $q$-difference equation:

$$(q^{-1} x - y) [f(x, y, z) - f(x, y, qz)] = z [f(q^{-1} x, y, qz) - z f(x, qy, qz)]. \quad (2.1)$$

To determine if a given function is an analytic function in several complex variables, we often use the following Hartogs’s Theorem. For more information, please refer to Taylor [12, p. 28] and Liu [13, Theorem 1.8].

**Lemma 2.** [Hartogs’s Theorem [11, p.15]] If a complex-valued function is holomorphic (analytic) in each variable separately in an open domain $D \subset \mathbb{C}^2$, then it is holomorphic (analytic) in $D$. 
Lemma 3. [14] p. 5 Proposition 1] If \(f(x_1, x_2, \ldots, x_k)\) is analytic at the origin \((0, 0, \ldots, 0) \in \mathbb{C}^k\), then, \(f\) can be expanded in an absolutely convergent power series

\[
f(x_1, x_2, \ldots, x_k) = \sum_{n_1, n_2, \ldots, n_k=0}^{\infty} \alpha_{n_1, n_2, \ldots, n_k} x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k}. \tag{2.2}
\]

Proof of Theorem 7] From the Hartogs’s Theorem and the theory of several complex variables (see Lemmas 2 and 3), we assume that

\[
f(x, y, z) = \sum_{n=0}^{\infty} A_n(x, y)z^n. \tag{2.3}
\]

Then, substituting the above equation into (2.1), we have:

\[
(q^{-1}x - y) \sum_{n=0}^{\infty} (1 - q^n)A_n(ax, y)z^n = \sum_{n=0}^{\infty} q^n[A_n(q^{-1}x, y) - A_n(x, qy)]z^{n+1} \tag{2.4}
\]

Comparing coefficients of \(z^n\), \(n \geq 1\), we readily find that

\[
(q^{-1}x - y)(1 - q^n)A_n(x, y) = q^n(1 - aq^{n+1})[A_{n-1}(q^{-1}x, y) - A_{n-1}(x, qy)]. \tag{2.5}
\]

After simplification, we get

\[
A_n(x, y) = \frac{q^{n-1}}{1 - q^n} \theta_{xy}[A_{n-1}(x, y)]. \tag{2.6}
\]

By iteration, we gain

\[
A_n(x, y) = \frac{q^2 (q; q)_n}{(q; q)_n} A_0(x, y). \tag{2.7}
\]

Letting \(f(x, y, 0) = A_0(x, y) = \sum_{n=0}^{\infty} \mu_n p_n(y, x)\), we have

\[
A_k(a, x) = \frac{q^{k-1}}{(q; q)_k} \sum_{n=0}^{\infty} \mu_n (q; q)_n p_{n-k}(y, x). \tag{2.8}
\]

Replacing (2.8) in (2.3), we have:

\[
f(a, x, y) = \sum_{k=0}^{\infty} (-1)^k \frac{q^k}{(q; q)_k} \sum_{n=0}^{\infty} \mu_n (q; q)_n p_{n-k}(y, x) z^k
\]

\[
= \sum_{n=0}^{\infty} \mu_n \sum_{k=0}^{n} \left[\frac{n}{q} \right]_{q} (-1)^k q^k p_{n-k}(y, x) z^k. \tag{2.9}
\]

On the other hand, if \(f(x, y, z)\) can be expanded in term of \(F_n(x, y, z)\), we can verify that \(f(x, y, z)\) satisfies (2.1). The proof of the assertion (2.1) of Theorem 1 is now completed.

Theorem 4. Let \(f(x, y, z)\) be a three-variable analytic function in a neighborhood of \((x, y, z) = (0, 0, 0) \in \mathbb{C}^3\).

If \(f(x, y, z)\) satisfies the \(q\)-difference equation

\[
(q^{-1}x - y)[f(x, y, z) - f(x, y, qz)] = z[f(q^{-1}x, y, qz) - f(x, qy, qz)] \tag{2.10}
\]

then we have:

\[
f(x, y, z) = L(z\theta_{xy})\{f(x, y, 0)\}. \tag{2.11}
\]
Proof of Theorem 4. From the theory of several complex variables \[14\], we begin to solve the \( q \)-difference equation (2.10). First we may assume that

\[
f(x, y, z) = \sum_{n=0}^{\infty} A_n(x, y) z^n.
\]  

(2.12)

Then substituting the above equation into (2.10), we have:

\[
(q^{-1} x - y) \sum_{n=0}^{\infty} (1 - q^n) B_n(a, x, y) z^n = \sum_{n=0}^{\infty} q^n [A_n(q^{-1} x, y) - A_n(x, qy)] z^{n+1}
\]  

(2.13)

Comparing coefficients of \( z^n \), \( n \geq 1 \), we readily find that

\[
(q^{-1} x - y)(1 - q^n) B_n(a, x, y) = q^{n-1}(1 - aq^{n-1}) [B_{n-1}(a, q^{-1} x, y) - B_{n-1}(a, x, qy)].
\]  

(2.14)

After simplification, we get

\[
A_n(x, y) = \frac{q^{n-1}}{1 - q^n} \sum_{r=0}^{n} \begin{pmatrix} n \cr r \end{pmatrix} [A_0(x, y)].
\]  

(2.15)

By iteration, we gain

\[
A_n(x, y) = \frac{q^n}{(q; q)_n} b_{xy} [A_0(x, y)].
\]  

(2.16)

Now we return to calculate \( A_0(x, y) \). Just taking \( z = 0 \) in (2.12), we immediately obtain \( A_0(x, y) = f(x, y, 0) \).

The proof of the assertion \( (2.11) \) of Theorem 4 is now completed by substituting (2.16) back into (2.12). \( \square \)

3. Summation formula for trivariate \( q \)-polynomials

In this section, we give the summation formula for trivariate \( q \)-polynomials by direct computation and derive summation formulae for the second Hahn polynomials \( \psi_n^{(0)}(x, y|q) \).

Theorem 5. Let the trivariate \( q \)-polynomials \( F_n(x, y, z; q) \) be defined as in (1.10)

\[
F_k(x, \xi, \zeta; q) = \sum_{n=0}^{l} \sum_{r=0}^{l} \begin{pmatrix} k \cr n \end{pmatrix} \begin{pmatrix} l \cr r \end{pmatrix} (1)^{n+r} q^{-(\xi)^r} q^{-r(n+l+1)-(k+l)(n+r)} P_{n+r}(\xi \Theta_q y, \zeta \Theta_q z|q) F_k(x, y, z; q).
\]  

(3.1)

and

\[
F_l(x, \xi, \zeta; q) = \sum_{r=0}^{l} \begin{pmatrix} l \cr r \end{pmatrix} (1)^{r} q^{-r(2l+1)} P_r(\xi \Theta_q y, \zeta \Theta_q z|q) F_{l-r}(x, y, z; q).
\]  

(3.2)

Putting \( \zeta = z \) in the last equation we establish the following corollary.

Corollary 6.

\[
F_k(x, \xi, z; q) = \sum_{n=0}^{l} \sum_{r=0}^{l} \begin{pmatrix} k \cr n \end{pmatrix} \begin{pmatrix} l \cr r \end{pmatrix} (1)^{n+r} q^{-(\xi)^r} q^{-r(n+l+1)-(k+l)(n+r)} (\xi \Theta_q y)^{n+r} F_{k+l-n-r}(x, y, z; q).
\]  

(3.3)

For \( k = 0 \), we have:

\[
F_l(x, \xi, z; q) = \sum_{r=0}^{l} \begin{pmatrix} l \cr r \end{pmatrix} (1)^{r} q^{-r(2l+1)} (\xi \Theta_q y)^{r} F_{l-r}(x, y, z; q).
\]  

(3.4)
Proof. By replacing \( t \) by \( u \oplus_q t \) in (1.10) we get
\[
e_q \left[ y \left( u \oplus_q t \right) \right] E_q \left[ -x \left( u \oplus_q t \right) \right] E_q \left[ -z \left( u \oplus_q t \right) \right] = \sum_{n=0}^{\infty} F_n(x, y, z; q) \frac{(-1)^n q^{\binom{n}{2}} (u \oplus_q t)^n}{(q; q)_n}. \tag{3.5}
\]
We now apply to the r.h.s of (3.5) the identity
\[
\sum_{j=0}^{\infty} F(j) \frac{(x \oplus_q y)^j}{(q; q)_j} = \sum_{j=0}^{\infty} F(j + s) \frac{q^{\binom{j}{2}} x^j y^s}{(q; q)_j (q; q)_s}.
\tag{3.6}
\]
satisfied by the JHC \( q \)-addition. So that (3.5) becomes
\[
E_q \left[ -x \left( u \oplus_q t \right) \right] = e_q \left[ y \left( u \oplus_q t \right) \right] E_q \left[ -z \left( u \oplus_q t \right) \right] \sum_{j=0}^{\infty} \frac{(-1)^j u^j}{(q; q)_j} q^{\binom{j}{2} + \binom{1}{2} j} F_{k+l}(x, y, z; q). \tag{3.7}
\]
Note that the r.h.s of (3.7) is independent of variables \( y \) and \( z \) so that we can write for any two variables \( \xi, \zeta \) the following equality
\[
\sum_{k,l=0}^{\infty} \frac{(-1)^{k+l} u^{k+l}}{(q; q)_k(q; q)_l} q^{\binom{k+l}{2} + \binom{1}{2} (k+l)} F_{k+l}(x, \xi, \zeta; q) = \lambda_{u,t} (\xi, \zeta; y, z; q) \sum_{k,l=0}^{\infty} \frac{(-1)^{k+l} u^{k+l}}{(q; q)_k(q; q)_l} q^{\binom{k+l}{2} + \binom{1}{2} (k+l)} F_{k+l}(x, y, z; q).
\tag{3.8}
\]
where
\[
\lambda_{u,t} (\xi, \zeta; y, z; q) := e_q \left[ \xi \left( u \oplus_q t \right) \right] E_q \left[ -\xi \left( u \oplus_q t \right) \right] e_q \left[ y \left( u \oplus_q t \right) \right] E_q \left[ -y \left( u \oplus_q t \right) \right]. \tag{3.9}
\]
By using the rules in (1.8), on can check that the quantity (3.9) also reads
\[
\lambda_{u,t} (\xi, \zeta; y, z; q) = e_q \left[ \left( \xi \oplus_q y \right) \left( u \oplus_q t \right) \right] E_q \left[ \left( \xi \oplus_q z \right) \left( u \oplus_q t \right) \right]. \tag{3.10}
\]
On another hand, the r.h.s of (3.10) coincides with the generating function
\[
\sum_{n,r=0}^{\infty} \frac{u^{n+r}}{(q; q)_n(q; q)_r} q^{\binom{n}{2}} P_{n+r}(\xi \oplus_q y, \xi \oplus_q z) \tag{3.11}
\]
involving the Cauchy polynomials. Summarizing the above calculations in (3.8), (3.11), we arrive at the sum
\[
\sum_{n,r=0}^{\infty} \frac{u^{n+r}}{(q; q)_n(q; q)_r} q^{\binom{n}{2}} P_{n+r}(\xi \oplus_q y, \xi \oplus_q z) \sum_{k,l=0}^{\infty} \frac{(-1)^{k+l} u^{k+l}}{(q; q)_k(q; q)_l} q^{\binom{k+l}{2} + \binom{1}{2} (k+l)} F_{k+l}(x, y, z; q)
\]
\[
= \sum_{k,l=0}^{\infty} \frac{(-1)^{k+l} u^{k+l}}{(q; q)_k(q; q)_l} q^{\binom{k+l}{2} + \binom{1}{2} (k+l)} F_{k+l}(x, \xi, \zeta; q). \tag{3.12}
\]
Next, applying the series manipulation [17] p.100]
\[
\sum_{p=0}^{\infty} \sum_{s=0}^{r} A(p, s) = \sum_{p=0}^{\infty} \sum_{s=0}^{r} A(s, p - s) \tag{3.13}
\]
to the l.h.s of (3.12), we obtain that
\[
\sum_{k,l=0}^{\infty} \sum_{n=0}^{r} \sum_{r=0}^{l} \frac{(-1)^{k+l-n-r} u^{k+l-n-r}}{(q; q)_{k-n}(q; q)_{l-r}(q; q)_n(q; q)_r} P_{n+r}(\xi \oplus_q y, \xi \oplus_q z) F_{k+l-n-r}(x, y, z; q)
\]
\[
= \sum_{k,l=0}^{\infty} \frac{(-1)^{k+l} u^{k+l}}{(q; q)_k(q; q)_l} q^{\binom{k+l}{2} + \binom{1}{2} (k+l)} F_{k+l}(x, \xi, \zeta; q). \tag{3.14}
\]
By equating terms with $u^l l^j/(q; q)_n(q; q)_r$ and using the simple combinatorial fact we arrive at the following result

$$F_{k+l}(x, \xi, \zeta; q) = \sum_{n=0}^{k+l} \sum_{r=0}^{l} \left[ \begin{array}{c} n \\ m \end{array} \right] \left( \begin{array}{c} l \\ r \end{array} \right)_q (-1)^{n+r} q^{-(\xi^2)+r(\xi^2+n+1)-(k+l)(n+r)} P_{n+r}(\xi \Theta_q y, \xi \Theta_q z|q) F_{k+l-n-r}(x, y, z; q).$$

As particular cases, by putting $y = \xi = ax$, $z = y$ and $\zeta = \xi$ in Theorem 5 we ave the connection formulae for second Hahn polynomials $\psi_n^{(a)}(x, y|q)$ as

**Corollary 7.**

$$\psi_{k+l}^{(a)}(x, \xi|q) = \sum_{n=0}^{k+l} \sum_{r=0}^{l} \left[ \begin{array}{c} n \\ m \end{array} \right] \left( \begin{array}{c} l \\ r \end{array} \right)_q (-1)^{n+r} q^{-(\xi^2)+r(\xi^2+n+1)-(k+l)(n+r)} (\xi \Theta_q y)^{n+r} \psi_k^{(a)}(x, y|q)$$ (3.15)

and

$$\psi_l^{(a)}(x, \xi|q) = \sum_{r=0}^{l} \left[ \begin{array}{c} l \\ r \end{array} \right] q^{(\xi^2)-r(2l+1)} (\xi \Theta_q y)^r \psi_l^{(a)}(x, y|q).$$ (3.16)

**Theorem 8.** The following summation formula for the product of trivariate $q$-polynomials

$$F_n(x, \xi, \zeta; q)F_r(X, \Omega, U; q) = \sum_{k,m=0}^{n} \left[ \begin{array}{c} n \\ k \end{array} \right] \left[ \begin{array}{c} r \\ m \end{array} \right] (-1)^{k+m} q^{(\xi^2)+(\zeta^2)-mr-nk} P_k(\xi \Theta_q \zeta, y \Theta z) F_{n-k}(x, y, z; q) \times P_m(\Omega \Theta_q U, Y \Theta q Z) F_{r-m}(X, Y, Z; q).$$ (3.17)

holds true.

**Proof.** From the generating function (1.10), we have

$$e_q(yt)E_q(-xt)E_q(-zt)e_q(YT)E_q(-XT)E_q(-ZT) = \sum_{n,r=0}^{\infty} \frac{(-1)^{n+r} q^{(\xi^2)+(\zeta^2)} T^r}{(q; q)_n(q; q)_r} F_n(x, y, z; q) F_r(x, Y, Z; q).$$ (3.18)

Replacing in (3.18) $y$ by $\xi$, $z$ by $\zeta$, $Y$ by $\Omega$ and $Z$ by $U$ we get

$$e_q(\xi t)E_q(-xt)E_q(-\zeta t)e_q(\Omega T)E_q(-XT)E_q(-UT) = \sum_{n,r=0}^{\infty} \frac{(-1)^{n+r} q^{(\xi^2)+(\zeta^2)} T^r}{(q; q)_n(q; q)_r} F_n(x, \xi, \zeta; q) F_r(x, \Omega, U; q).$$ (3.19)

By replacing

$$E_q(-xt)E_q(-XT) = \frac{1}{e_q(yt)E_q(-zt)e_q(YT)E_q(-ZT)} \sum_{n,r=0}^{\infty} \frac{(-1)^{n+r} q^{(\xi^2)+(\zeta^2)} T^r}{(q; q)_n(q; q)_r} F_n(x, y, z; q) F_r(x, Y, Z; q).$$

in the l.h.s. of (3.19) and using (1.8) one gets, after expanding the exponentials in series, the following

$$\sum_{n,r=0}^{\infty} \frac{(-1)^{n+r} q^{(\xi^2)+(\zeta^2)} T^r}{(q; q)_n(q; q)_r} F_n(x, \xi, \zeta; q) F_r(x, \Omega, U; q)$$

$$= \frac{e_q(\xi t)E_q(-\zeta t)e_q(\Omega T)E_q(-UT)}{e_q(yt)E_q(-zt)e_q(YT)E_q(-ZT)} \sum_{n,r=0}^{\infty} \frac{(-1)^{n+r} q^{(\xi^2)+(\zeta^2)} T^r}{(q; q)_n(q; q)_r} F_n(x, y, z; q) F_r(x, Y, Z; q)$$

$$= \frac{e_q(\xi \Theta_q \zeta t)}{e_q(y \Theta q z t)} e_q(\Omega \Theta_q U T) \sum_{n,r=0}^{\infty} \frac{(-1)^{n+r} q^{(\xi^2)+(\zeta^2)} T^r}{(q; q)_n(q; q)_r} F_n(x, y, z; q) F_r(x, Y, Z; q).$$
Finally, by replacing $n$ by $n - k$ and $r$ by $r - m$ in the r.h.s. of the last relation, the proof is completed. \qed

4. Particular cases

(1) Putting $y = \xi = ax$, $z = y$ and $\zeta = \xi$ in Theorem 5 leads to a connection formulae for second Hahn polynomials $\psi_n^{(a)}(x, y|q)\)\)
\[
\psi_{k+1}^{(a)}(x, \xi|q) = \sum_{n=0}^{\infty} \sum_{r=0}^{l} \left[ \begin{array}{c} k \\ n \\ q \\ r \\ q \\ \end{array} \right] P_{k}(\xi \otimes q , \zeta ) F_{m}(x, y, z; q) \\
\times \sum_{r,m=0}^{\infty} (-1)^{r} q^{r(m+1)} P_{m}(\Omega \otimes q U, Y \otimes q Z) F_{r}(X, Y, Z; q).
\] (3.20)

(2) Putting $y = \xi = ax$, $Y = \Omega = aX$, $z = y$, $\zeta = \xi$ and $U = \Omega$ in Theorem 8 we get
\[
\psi_{n}^{(a)}(x, \xi|q)\psi_{l}^{(a)}(X, \Omega|q) = \sum_{k,m=0}^{n,r} \left[ \begin{array}{c} n \\ k \\ q \\ m \\ q \\ \end{array} \right] (-1)^{k+m} q^{(k+1)+l+1} \psi_{n-k}^{(a)}(x, y|q) \\
\times P_{m}(\Omega \otimes q \leq \rho, \rho \otimes q \leq \rho) \psi_{l-m}(X, Y|q).
\] (4.3)

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