Throwing away data improves worst-class error in imbalanced classification

Martin Arjovsky 1  Kamalika Chaudhuri 2  David Lopez-Paz 2

Abstract

Class imbalances pervade classification problems, yet their treatment differs in theory and practice. On the one hand, learning theory instructs us that more data is better, as sample size relates inversely to the average test error over the entire data distribution. On the other hand, practitioners have long developed a plethora of tricks to improve the performance of learning machines over imbalanced data. These include data reweighting and subsampling, synthetic construction of additional samples from minority classes, ensembling expensive one-versus all architectures, and tweaking classification losses and thresholds. All of these are efforts to minimize the worst-class error, which is often associated to the minority group in the training data, and finds additional motivation in the robustness, fairness, and out-of-distribution literatures. Here we take on the challenge of developing learning theory able to describe the worst-class error of classifiers over linearly-separable data when fitted either on (i) the full training set, or (ii) a subset where the majority class is subsampled to match in size the minority class. We borrow tools from extreme value theory to show that, under distributions with certain tail properties, throwing away most data from the majority class leads to better worst-class error.

1. Introduction

Consider the one-dimensional binary classification problem in Figure 1 (a). It concerns the separation of two imbalanced classes: a majority class with many examples in blue, and a minority class with few examples in orange. Since this is a linearly separable problem, we train a linear hard-margin SVM on the available data, as this learner enjoys optimal performance guarantees for this type of problems (Steinwart & Christmann, 2008). As illustrated in every textbook, Figure 1 (b) shows that the average test error of the SVM decays as we increase the amount of training data. However, a closer look at the composition of this error sparks the investigation in this work. In particular, Figure 1 (c) shows the worst-class test error—the maximum of the two per-class test errors—of the SVM as we vary the amount of examples in the majority class. This worst-class error is...
Throwing away data improves worst-class error in imbalanced classification

minimal when we subsample the majority class to match in size the minority class. So, throwing away most data from the majority class improves performance robustness.

This simple but striking observation questions classic theory and practice in machine learning, where we were repeatedly told that one way out of our problems is to simply collect more data (Shalev-Shwartz & Ben-David, 2014; Mohri et al., 2018). The practical implications of this observation are far-reaching. Qualities such as robustness, fairness, and out-of-distribution generalization are becoming key questions in the debate on how to advance towards artificial intelligence. For instance, disparate error rates across different protected groups is a widely-accepted measure of fairness (Barocas et al., 2019). Measuring worst-class error is commonplace to estimate the robustness of machine learning systems against certain distributional shifts (Idrissi et al., 2021). As AI permeates critical applications such as healthcare, vehicle driving, and governance (Engstrom et al., 2020), researchers are increasingly wary of simply measuring average test error. Instead, approaches are being developed to craft machine learning systems that perform well across (known or unknown) subpopulations of data (Sagawa et al., 2019; Arjovsky et al., 2019; Gulrajani & Lopez-Paz, 2020; Idrissi et al., 2021). In short, advancing towards a more responsible evaluation of classification systems does not only count errors, but is also concerned with what types of mistakes these systems make—learning machines may incur catastrophic mistakes hidden in an innocuous 1% average test error.

Our learning theory tools fall short to study this phenomenon. Traditional learning bounds describe decaying average test error—computed over the distribution of training examples—as the number of data increases (Shalev-Shwartz & Ben-David, 2014; Mohri et al., 2018). Coarse descriptions about average test error do not inform us about the performance of our classification rule at each of the classes forming the learning task at hand. This is a point of disconnect between theory and practice, since multiple tricks are commonly deployed in machine learning engineering to address class imbalance. These include data reweighting and subsampling (Idrissi et al., 2021), as well as the synthetic construction of additional samples from the small class (Chawla et al., 2002). The central importance of class-balancing techniques in practice and their blunt omission in theory motivates the present work.

Contribution We develop novel theory to characterize the worst-class test error of Support Vector Machines (Steinwart & Christmann, 2008, SVMs) when trained via the Empirical Risk Minimization (Vapnik, 1992, ERM) principle on linearly-separable one-dimensional data (Section 2). We study learning these machines on either the full available training data (ERM), or on a reduced version where the large class is subsampled to match in size the small class (SUB). We leverage tools from Extreme Value Theory (EVT) to include information about the tails of the data distribution when describing the relationship between subsampling and worst-class error (Section 3). For example, we show that when the class-conditional distributions are Gaussian or Laplace, SUB has strictly better worst-class error than ERM, while for uniform distributions both perform equally well. Therefore, or main (perhaps counter-intuitive) takeaway is:

When dealing with imbalanced classification problems consider subsampling the majority class to match in size the minority class, as this leads to optimal worst-class error.

We also offer valuable insights for multivariate data (Section 5), verify our findings with some numerical experiments (Section 6), review related work (Section 7) and offer some lines of research for future work (Section 8).

2. Learning setup

We consider the binary classification of examples \((x_i, y_i)\), where inputs \(x_i \in \mathbb{R}^d\) and labels \(y_i \in \{-1, +1\}\). We denote the two class-conditional distributions by

\[
D^+(x) = P(X = x \mid Y = +1) \quad \text{and} \quad D^-(x) = P(X = x \mid Y = -1).
\]

Given a classifier \(c : \mathbb{R}^d \to \mathbb{R}\), the main quantity of study in this work is the worst-class error

\[
\text{wce}(c) = \max \left( \Pr_{x \sim D^+}(c(x) < 0), \ Pr_{x \sim D^-}(c(x) > 0) \right).
\]

(1)

We place four assumptions on our data. First, we consider imbalanced classification problems. The remainder assumes \(n\) examples from the positive (majority) class, and \(\beta n\) examples from the negative (minority) class, where \(\beta < 1\) and \(\beta n\) is an integer. Second, we assume that the training data—not necessarily the entire distribution—is linearly separable. Third, we require \(D^+ = D(\mu)\) and \(D^- = D(-\mu)\) to be of the same family and symmetric around their means, assumed to be \(+\mu\) and \(-\mu\), respectively. When necessary, we write \(\mu_n\) to allow these means to depend on the number of majority examples. Fourth, and unless stated otherwise, we consider one-dimensional data \((d = 1)\). We denote our training data by \(\mathcal{D} = \{(x_i, y_i)\}_{i=1}^{n+\beta n}\).

2.1. Training algorithm

Given our assumptions on the data, we choose a linear model for our classifier, namely \(c(x) = w^\top x + b\). We follow the Empirical Risk Minimization (ERM) principle to learn some parameters \((w, b)\) separating the training data \(\mathcal{D}\). Mathematically, the learning process to find \((w, b)\) amounts

\[
\frac{1}{n+\beta n}\sum_{i=1}^{n+\beta n} \mathbb{1}_{y_i = +1} (w^\top x_i + b > 0) + \beta \frac{1}{n+\beta n}\sum_{i=1}^{n+\beta n} \mathbb{1}_{y_i = -1} (w^\top x_i + b < 0). 
\]
We denote by $\theta$ the input value below which we predict negative class, and above which we predict positive class.

The SVM separator is equidistant from the convex hulls of minority/negative class and majority/positive class.

Formally, the weight vector $w$ to $\theta$ is of norm one-dimensional case.

The most popular loss for classification in deep learning is the hinge loss ($\ell(u) = \max(0, 1 - u)$). The SVM solution becomes the mid-point between the two closest points from each class.

Our main goal is to minimize the worst-class error (1); but, the ERM (2) minimizes average error. These two objectives differ for imbalanced classification problems, where the majority examples drive the average error and drown the contribution of minority examples. For instance, a classifier can exhibit 1% average error and 100% worst-class error in a dataset with 99 correctly classified majority examples and 1 misclassified minority example.

Practitioners have noticed this discrepancy over the years and proposed a myriad of alternatives to address this issue. These include data reweighting and subsampling (Idrissi et al., 2021), as well as the synthetic construction of additional samples from the small class (Chawla et al., 2002). It is not hard to see that reweighting examples with nonzero weights does not affect the solution of a hard-margin SVM under linearly-separable training data (Soudry et al.). How-ever, subsampling—removing examples from the small class (Chawla et al., 2002)—may lead to a different SVM solution. This is because we could be removing support vectors, that is, examples closest to the decision boundary (see Figure 2). It has been shown in practice that subsampling leads to improved worst-class error (Idrissi et al., 2021). How could this be theoretically proven?

The remainder of this manuscript sets out to develop the necessary learning theory to answer this question. To this end, we will be comparing two SVM solutions for one-dimensional, linearly-separable data:

$$\theta_{\text{erm}} = \frac{1}{2} (M^-_{\text{erm}} + m^+_{\text{erm}}), \quad \theta_{\text{sub}} = \frac{1}{2} (M^-_{\text{sub}} + m^+_{\text{sub}}),$$

where $(M^-_{\text{erm}}, m^+_{\text{erm}})$ are the largest example value from the positive class and the smallest positive value from the positive class over the original dataset $D$. The pair $(M^-_{\text{sub}}, m^+_{\text{sub}})$ is defined analogously over a reduced version of the dataset.

### 2.2. Comparing ERM and subsampling SVM solutions


tive class over the original dataset

**Lemma 1.** (Simplified Version of (Bennett & Bredensteiner, 2000)) Let the conditions stated above hold. Then, the SVM solution is $\theta = \frac{1}{2} (M^- + m^+)$.

Furthermore, under our data assumptions, any two classification thresholds $\theta$ and $\theta'$ satisfy the following property:

**Lemma 2.** For symmetric class-conditional, $|\theta| \geq |\theta'|$ implies $wce(\theta) \geq wce(\theta')$.

**Proof.** Because the distributions are symmetric around the origin, $wce(\theta) = wce(-\theta)$. Therefore, it is sufficient to consider the case where $\theta \geq \theta' \geq 0$. When both $\theta$ and $\theta' > 0$, the class with the higher error is the positive class. Thus, $wce(\theta) = \Pr(X \leq \theta | X \sim D(\mu))$, and $wce(\theta') = \Pr(X \leq \theta' | X \sim D(-\mu))$. Since $\theta \geq \theta'$, $wce(\theta) \geq wce(\theta')$. 

---

**Figure 2.** Illustration of a linear hard-margin SVM discriminating two one-dimensional linearly-separable classes.
where the majority class has been subsampled to match in size the minority class.

While classical learning theory suggests that $\theta_{\text{erm}}$ has a lower average error than $\theta_{\text{sub}}$ (Vapnik, 1999), it does not characterize the worst-class error of these two SVM solutions. We show that $\theta_{\text{sub}}$ can indeed have better worst-class error than $\theta_{\text{erm}}$ for data distributions satisfying certain tail properties, even if this implies throwing away most of our dataset. To characterize and leverage distributional tails properly, we must first introduce some tools from extreme value theory.

3. Some extreme value theory

Our theoretical results depend on the tail properties of the class-conditional distributions generating the data at hand. Intuitively, the heavier the tail of a distribution, the larger the probability of observing extreme values. The branch of mathematics studying extreme deviations is known as Extreme Value Theory (De Haan et al., 2006, EVT). We borrow tools from EVT to the analysis of worst-group error in SVMs, which is a first in the research literature.

A central result from EVT is the Fisher-Tippett-Gnedenko theorem (De Haan et al., 2006), characterizing the maximum value $M_n$ of $n$ identically and independently distributed examples $X_1, \ldots, X_n$ drawn from an fixed distribution with CDF $F$ belonging to one of the following types.

**Definition 1.** Let $x_F = \sup_x \{x \mid F(x) < 1\}$ be the largest value not saturating $F$. Then $F$ is of the family:

- Frechet, if $x_F = \infty$ and $\lim_{t \to \infty} \frac{1 - F(tx)}{1 - F(t)} = x^{-\alpha}$, for $\alpha > 0$ and $x > 0$.
- Weibull, if $x_F < \infty$ and $\lim_{t \to \infty} \frac{1 - F(tx - x)}{1 - F(tx)} = x^\alpha$, for $\alpha > 0$ and $x > 0$.
- Gumbell, if $\lim_{t \to \infty} \frac{1 - F(t + xg(t))}{1 - F(t)} = e^{-x}$, for all real $x$ where $g(t) = \frac{\int_0^t (1 - F(u)) du}{1 - F(t)}$ for $t < x_F$.

Gumbell-type distributions have light tails, and include Gaussians and Exponentials. Weibull-type distributions have finite maximum points, and include Uniforms. Frechet-type distributions have heavy tails, including the Pareto and Frechet distributions. Some distributions, such as the Bernoulli, do not belong to any of these types.

If the distribution $F$ is of either the Frechet, Gumbell or Weibull type, then the Fisher-Tippett-Gnedenko theorem shows that there exist numbers $a_n$ and $b_n$ such that the CDF $F'$ of $a_n M_n + b_n$ converges to a limit distribution $G$.

$$F'(a_n M_n + b_n) \to G,$$

Each of $a_n$, $b_n$ and $G$ take fixed values that depend on $F$.

Finally, let us define the “tail function” of a distribution with CDF $F$, which measures the spread of its tail.

**Definition 2 (Tail Function).** The tail function $U$ of a distribution with CDF $F$ is defined as $U(t) = F^{-1}(1 - 1/t)$.

We now have all the necessary tools to introduce the Fisher-Tippett-Gnedenko theorem formally.

**Theorem 1** (Fisher-Tippett-Gnedenko Theorem). Considering the concepts introduced in this section, we have:

1. If $F$ is of the Frechet type, then $G(x)$ is the Frechet distribution with the following CDF:

   $$G(x) = \exp(-x^{-\alpha}), \quad x \geq 0$$

   $$= 0, \quad x < 0.$$  

   Additionally, $a_n = U(n)$ and $b_n = 0$.

2. If $F$ is of the Weibull type, then $G(x)$ is the reverse Weibull distribution with the following CDF:

   $$G(x) = 1, \quad x \geq 0$$

   $$= \exp(-(x)^\alpha), \quad x < 0.$$  

   Additionally, $a_n = x_F - U(n)$ and $b_n = x_F$.

3. If $F$ is of the Gumbell type, then $G(x)$ is the Gumbell distribution with the following CDF:

   $$G(x) = \exp(-e^{-x}), \quad x \in [-\infty, \infty].$$  

   Additionally, $a_n = g(U(n))$ and $b_n = U(n)$ where $U(n)$ appears in Definition 1.

4. Results for the one-dimensional case

We start by leveraging the Fisher-Tippett-Gnedenko theorem to describe how the one-dimensional closed-form SVM solutions $\theta_{\text{erm}}$ and $\theta_{\text{sub}}$ (Equation 4) look like as the number of majority examples grows. As a first step, if the class-conditional distributions are of the Frechet, Gumbell or Weibull type, then $\theta_{\text{erm}}$ and $\theta_{\text{sub}}$ will satisfy the following conditions in the large $n$ limit.

**Lemma 3.** Let the class-conditional distributions $D^+$ and $D^-$ be Frechet, Gumbell or Weibull type. Then, as $n \to \infty$, we have that:

$$\lim_{n \to \infty} \theta_{\text{erm}} = \frac{1}{2} (b_3 n - b_n + a_3 n Z_3 - a_n Z_4),$$

$$\lim_{n \to \infty} \theta_{\text{sub}} = \frac{1}{2} a_3 n (Z_1 - Z_2)$$

where $(a_n, b_n, G)$ are defined as in the Fisher-Tippett-Gnedenko (Theorem 1), and the $Z_i$ are independent random variables distributed according to $G$.

Next, we illustrate the differences between the $\theta_{\text{erm}}$ and $\theta_{\text{sub}}$ values in the large sample limit for several examples of distributions of the Weibull, Gumbell, and Frechet types.
4.1. Weibull Type

If the class-conditional distributions \((D^+, D^-)\) are of the Weibull type, then Theorem 1 shows that \(a_n = x_F - U(n)\) and \(b_n = x_F\). Thus, Lemma 3 reduces to:

\[
\lim_{n\to\infty} \theta_{erm} = \frac{1}{2}((x_F - U(\beta n))(Z_3 - Z_4) + (U(n) - U(\beta n))(Z_3 - Z_4)),
\]

\[
\lim_{n\to\infty} \theta_{sub} = \frac{1}{2}(x_F - U(\beta n))(Z_1 - Z_2),
\]

where the \(Z_i\) are independent reverse Weibull random variables. Observe that the two distributional limits differ by the term \(\frac{1}{2}(U(n) - U(\beta n))(Z_3 - Z_4)\), where the values of \(U\) depend on the particular class-conditional distribution at hand. To understand them, we provide values for a typical distribution of the Weibull type—the uniform distribution on a closed interval.

**Example 1: Uniform.** To ensure that the training data is linearly separable, let \(\mu = \frac{1}{2}\), so \(D^+\) is the uniform distribution on \([\mu - 1/2, \mu + 1/2]\). Thus the positive class is uniform on \([0, 1]\) and the negative class is uniform on \([-1, 0]\). Since the uniform distribution is of the Weiball type (De Haan et al., 2006), each \(Z_i\) is distributed according to the reverse Weibull distribution with \(\alpha = 1\) and \(\beta_n = 2/\beta\) and \(\beta_n = 1/2\). Then, we can prove:

**Theorem 2 (Uniform).** If \(\mu = \frac{1}{2}\), \(D^+ = \text{Uniform}[\mu - 1/2, \mu + 1/2]\), \(D^- = \text{Uniform}[\mu - 1/2, -\mu + 1/2]\), and \(\beta_n \to \infty\) as \(n \to \infty\), then, the ERM SVM solution converges in distribution to the subsampling SVM solution.

Theorem 2 shows that for a certain family of uniform distributions, the SVM solution under ERM and the SVM solution under Subsampling converge to the same value as the number of training data tends to infinity. Thus, we can expect the same balanced error for both learning strategies.

4.2. Gumbell Type

Next, we turn our attention to class-conditional distributions of the Gumbell type, a scenario where Subsampling leads to better balancing error than ERM. Here, \(a_n = f(U(n))\) and \(b_n = U(n)\), so the limits in Lemma 3 reduce to:

\[
\lim_{n\to\infty} \theta_{erm} = \frac{1}{2}(f(U(\beta n)) - U(n)) + f(U(\beta n))Z_3 - f(U(n))Z_4),
\]

\[
\lim_{n\to\infty} \theta_{sub} = \frac{1}{2}f(U(\beta n))(Z_1 - Z_2),
\]

Observe that \(\theta_{erm}\) has an extra offset term \(\frac{1}{2}(U(\beta n) - U(n))\) – this is the term that pulls \(\theta_{erm}\) away from the majority class center, leading to higher minority class error. Once again, the precise value of this term relative to the remainder depends on the exact class-conditional distribution. Let us exemplify with two well-known Gumbell distributions—the Laplace and Gaussian distributions.

**Example 2: Laplace.** Let \(D^+\) be the Laplace distribution with mean \(\mu_n = \log(n/\epsilon)\) and scale \(1\). Similarly, let \(D^-\) be the Laplace distribution with mean \(-\mu_n\) and scale \(1\). Then, \(a_n = 1\) and \(b_n = \log(n)\) (De Haan et al., 2006), leading to the following result.

**Theorem 3.** Let \(\theta_{erm}\) moves further and further away. Thus, unlike the uniform distribution, the gap between \(\theta_{sub}\) and \(\theta_{erm}\) widens with increasing \(n\), with \(\theta_{sub}\) being closer to the origin due to symmetry. This, in turn, implies that \(\theta_{erm}\) has considerably higher worst-class error than \(\theta_{sub}\). In particular, if \(\beta \to 0\) with \(\beta_n \to \infty\), the worst-class errors differ by a factor of \(\approx 1/\sqrt{3}\), which could be arbitrarily large. Finally, we note that while it might seem slightly unusual to have worst-class error of \(\theta_{sub}\) be \(O(1/n)\) since \(\theta_{sub}\) is based on \(O(\beta n)\) samples. This is an artifact of the separation \(\mu_n = \log(n/\epsilon)\) between the classes and the symmetry of the problem. In general, in problems like this, max-margin classification will achieve average (as well as worst-group) error of \(\approx 1 - F(\mu_n - \theta)\).

4.3. Gaussian

A more complicated example concerns Gaussian class-conditional. Let \(D^+ = \mathcal{N}(\mu_n, 1)\) and \(D^- = \mathcal{N}(-\mu_n, 1)\). To ensure linearly-separable training data, let \(\mu_n\) to be a solution to \(\frac{\epsilon - n^{2/3}}{2\pi \mu_n} = \frac{\epsilon}{n}\). The Gaussian distribution is of the Gumbell type (De Haan et al., 2006), with \(a_n = \sqrt{\frac{1}{\log n}}\), \(b_n = \sqrt{\log n - \log \log n + \log(4\pi)}\), and \(Z_i\) following a Gumbell distribution. These constants lead to the following result.

**Theorem 4.** Let \(0 < \epsilon, \delta, \gamma < 1\) be constants. There exists an \(n_0\) such that the following holds. If \(n \geq n_0\) and \(\beta \geq 1/n^{3/4}\), then with probability greater or equal than \(1 - 2\epsilon -

Throwing away data improves worst-class error in imbalanced classification
When \( n\beta \geq \epsilon \), this implies:
\[
\text{wce}(\theta_{\text{sub}}) \leq \frac{2\epsilon}{\gamma n}, \quad \text{wce}(\theta_{\text{erm}}) \geq \frac{\epsilon\gamma^{1/4}}{2n\beta^{1/12}}
\]

Whilst slightly more complex, the result is similar to the one for Laplace class-conditionals. For Gaussians, if \( \beta \to 0 \) with \( \beta n \to \infty \), then the relative gap between \( \theta_{\text{sub}} \) and \( \theta_{\text{erm}} \) widens. In particular, \( \theta_{\text{sub}} \) lies in an interval of length \( \approx \frac{1}{2\sqrt{2\log(\beta n)}} \) around the origin, while \( \theta_{\text{erm}} \) lies \( \approx \frac{\log(1/\beta)}{2\sqrt{2\log(\beta n)}} \). This leads to a widening of the balanced error between the two SVM solutions, with \( \theta_{\text{sub}} \) having considerably lower error than \( \theta_{\text{erm}} \). Specifically, the two balanced errors differ by a factor of \((1/\beta)^{1/12}\), which can grow arbitrarily large.

### 4.3. Frechet Type

For Frechet-type class-conditionals, we have \( a_n = U(n) \) and \( b_n = 0 \). Thus, Lemma 3 reduces to:
\[
\lim_{n \to \infty} \theta_{\text{erm}} = \frac{1}{2} U(\beta n) Z_3 - U(n) Z_4
\]
\[
\lim_{n \to \infty} \theta_{\text{sub}} = \frac{1}{2} U(\beta n) (Z_1 - Z_2),
\]
where \( Z_i \) are random variables following a Frechet distribution. Observe that unlike the Gumbell Type, here there is no offset term. However, \( U(\beta n) \) is typically less than \( U(n) \), which might pull \( \theta_{\text{erm}} \) away from the majority class with some probability when \( Z_3 \) is not too much smaller than \( Z_4 \). We study the values of the constants for one example of Frechet-type class-conditionals, the Frechet distribution.

#### Example 4: Frechet

Assume that \( D^+ \) is the two-sided Frechet distribution with parameter \( \alpha \) and CDF:
\[
F(x) = \begin{cases} 
\frac{1}{2} + \frac{1}{2} e^{-(x-\mu_n)^{-\alpha}}, & x \geq \mu_n, \alpha > 0 \\
\frac{1}{2} - \frac{1}{2} e^{-(\mu_n-x)^{-\alpha}}, & x \leq \mu_n,
\end{cases}
\]
where \( \alpha_n = n^{1/\alpha} \), \( b_n = 0 \), and \( Z_i \) follow a Frechet distribution with parameter \( \alpha \). We let \( D^- \) be similarly defined, using \(-\mu_n\). Then, it follows:

#### Theorem 5

Let \( \epsilon, \delta > 0 \). There exists an \( n_0 \) such that the following holds. If \( n \geq n_0 \) and \( \mu_n \approx (n/2\epsilon)^{1/\alpha} \) then, with probability greater or equal than \( 1 - 2(\epsilon + \delta) - 5\gamma \), we have that:
\[
|\theta_{\text{erm}}| \geq \frac{1}{2} \left( \frac{n}{\log \frac{1}{\theta}} \right)^{1/\alpha} - \frac{1}{2} \left( \frac{n\beta}{\gamma} \right)^{1/\alpha},
\]
\[
|\theta_{\text{sub}}| \leq \frac{1}{2} \left( \frac{n\beta}{\gamma} \right)^{1/\alpha},
\]

and
\[
\text{wce}(\theta_{\text{sub}}) \leq \frac{\epsilon}{n \left( 1 - \frac{1}{2} (2\epsilon/\log(1/\gamma))^{1/\alpha} \right)}
\]
\[
\text{wce}(\theta_{\text{erm}}) \geq \frac{\epsilon}{n \left( 1 - \frac{1}{2} (2\epsilon/\log(1/\gamma))^{1/\alpha} + \frac{1}{2} (2\epsilon/\log(1/\gamma))^{1/\alpha} \right)}.
\]

If \( \beta \to 0 \) and \( \beta n \to \infty \) then, for fixed \( \gamma \), the gap between \( \theta_{\text{erm}} \) and \( \theta_{\text{sub}} \) widens. More specifically, \( \theta_{\text{sub}} \) lies in an interval of length \( \approx (n\beta/\gamma)^{1/\alpha} \), while \( \theta_{\text{erm}} \) is \( \approx (n/\log(1/\gamma))^{1/\alpha} \) away from the origin. With growing \( n \), the gap between the two SVM solutions grows bigger.

This also leads to a difference in the error of \( \theta_{\text{erm}} \) and \( \theta_{\text{sub}} \) when \( \beta \to 0 \), while \( (\epsilon/\gamma) \) and \( (\epsilon/\log(1/\gamma)) \) is a constant not too close to zero. In this case, the lower bound on \( \text{wce}(\theta_{\text{erm}}) \) is approximately \( (1 - 1/\epsilon)e/\epsilon^\alpha n \), while the upper bound on \( \text{wce}(\theta_{\text{sub}}) \) is \( \approx 1/\epsilon n \). Adjusting the constants gives \( \theta_{\text{sub}} \) a constant-factor edge over \( \theta_{\text{erm}} \) in error. The difference from Gumbell-type class-conditionals (Theorems 3 and 4)—for the Frechet-type, the difference between the two SVM solutions is visible only when \( \gamma \) is moderate and \( \beta \) is small. Error difference is also smaller for heavy-tailed Frechet class-conditionals, where changing the offset by a little does not make a big difference to the tail probabilities.

### 5. Results for high-dimensions

We perform a similar analysis for multivariate data, where \( d > 1 \). A fine analysis is challenging, but we offer valuable insights by reducing the multivariate case to one dimension.

Again, there are two classes, \( Y = \{-1, +1\} \). We assume that \( X = Y \cdot (\mu + \xi) \) where \( \mu \in \mathbb{R}^d \) is a fixed vector and \( \xi \sim \Xi \) follows a spherically symmetric distribution. Let \( \bar{Y} \) be the distribution of \( \xi^\top u \) for any unit vector \( u \), and let \( F \) be the corresponding CDF. As in Section 4, we assume that there are \( n \) examples \( S_+ = \{(x_1, 1), \ldots, (x_n, 1)\} \) with positive labels, and \( n \) examples \( S_- = \{(x_1', -1), \ldots, (x_n', -1)\} \) with negative labels. We also assume that the training data is linearly separable; as in Section 4, we will set \( \mu = \mu_n \) so that this happens with high probability.

Let \((w^*, b^*) \in \mathbb{R}^{d+1} \) correspond to the maximum margin classifier, meaning the solution to the optimization problem (3). Let us denote \( \bar{w}^* = \frac{w^*}{\|w^*\|} \) and \( \bar{\mu} = \frac{\mu}{\mu} \). Letting \( n \to \infty \), by consistency we will get \( \bar{w}^* \to \bar{\mu} \), under certain conditions on the distributions. Since analyzing the precise rate of convergence of \( \bar{w}^* \) to \( \bar{\mu} \) is difficult...
in general (without distributional assumptions), we split the analysis assuming our theory that \( \bar{w}^* \) is close to \( \bar{\mu} \), which we verify experimentally. To start, observe that for any \( w \), its optimal \( b \) in (3) (that we denote \( b(w) \)) satisfies
\[
\min_{t \in S_+} \xi_t^w w + b = \min_{t \in S_-} \xi_t^w w - b.
\]
This implies:
\[
b(w) = \frac{1}{2} \min_{t \in S_-} \xi_t^T w - \frac{1}{2} \min_{t \in S_+} \xi_t^T w
\]  
(5)

A close look at Equation 5 reveals that \( b(w) \) is essentially the solution to (one-dimensional) SVM on the data points when they are projected on to \( w \). Thus if \( \bar{w}^* \) is close to \( \bar{\mu} \), then \( b^* = b(\bar{w}^*) \) is also close to \( b(\bar{\mu}) \).

We make this concrete by assuming \( \varphi_w^* \) is the angle between \( \bar{\mu} \) and \( \bar{w}^* \), and that \( |b(\bar{w}^*) - b(\bar{\mu})| < r_b \). Observe that as \( \bar{w}^* \) approaches \( \bar{\mu} \), \( \varphi_w^* \) approaches 0 (and its cosine approaches 1) and \( b(\bar{w}^*) \) approaches \( b(\bar{\mu}) \) since \( w \mapsto b(w) \) is a continuous function. This allows us to transform the high dimensional case to a 1-dimensional case with marginals \( \bar{\mu} \). Precisely, what this means is that the Theorems 2, 3, 4 and 5 from Section 4 will apply, up to a margin of error that depends on \( \varphi_w^* \) and \( r_b \).

To be more formal, suppose that \( (w_{\text{erm}}, b_{\text{erm}}) \in \mathbb{R}^{d+1} \) is solution to optimization problem 3 on ERM, and \( \theta_{\text{erm}} \) is the solution to the corresponding SVM when the data points are projected along the unit vector along \( \mu \). Similarly, we define the notations \( (w_{\text{sub}}, b_{\text{sub}}) \) and \( \theta_{\text{sub}} \). Then, \( b_{\text{sub}} - b_{\text{erm}} \geq \theta_{\text{sub}} - \theta_{\text{erm}} + 2r_b \). In other words, the gap between the offsets in the actual solutions narrows by at most \( 2r_b \). This implies
\[
\text{wce}(w_{\text{erm}}, b_{\text{erm}}) - \text{wce}(\theta_{\text{erm}}) 
= F(-\mu + \theta_{\text{erm}}) - F(-\mu \cos \varphi_{w_{\text{erm}}} + b_{\text{erm}})
\leq F(-\mu + \theta_{\text{erm}}) - F(-\mu \cos \varphi_{w_{\text{erm}}} + \theta_{\text{erm}} + r_b).
\]

A similar relationship applies to the subsampling case. This implies that the gap between the error estimates narrow by an amount that depends on how well \( w^* \) converges to \( \mu \) and \( b^* \) converges to \( b(\mu) \).

In figure 3 we see that \( \cos \varphi_{w_{\text{erm}}} \) is indeed close to 1 experimentally, which, as mentioned, also means \( b(\bar{w}^*) \) approaches \( b(\bar{\mu}) \). This qualitatively closes the gap in our analysis, showing that we can reduce the multivariate case to the one-dimensional one at a small expense by projecting the classification problem into \( \bar{\mu} \).

6. Numerical simulations

We conduct some numerical simulations to verify empirically our theoretical results. Our theory only considers hard margin SVMs, but given their popularity in practice, we here consider two additional classifiers—soft SVM and logistic regression. These are all implemented in terms of liblinear. Hard

margin SVM sets loss="hinge", C=1000 and max_iter=1e8; soft SVM sets loss="hinge"; logistic sets loss="logistic". We sample data from the four class-conditioned distributions studied in the previous sections: uniform, Gaussian, Laplace, and Frechet. Finally, we explore one-dimensional and ten-dimensional setups \( (d = 1, 10) \), as well as linearly-separable and non-linearly-separable training data \( (\mu = 3, 1, \text{respectively}) \).

Figure 4 summarizes our findings, where we observe that the subsampling strategy \( \theta_{\text{sub}} \) out-performs the ERM baseline \( \theta_{\text{erm}} \), while never performing worse. The advantage of subsampling seems most pronounced in low-dimensional, non-linearly separable cases. This motivates future research to develop theory for non-separable data, which admittedly is often the case in real-world applications. It is noteworthy to mention that the recent work of (Idrissi et al., 2021) exploring the worst-class error across a multitude of real-world datasets, ratifying the superiority of the subsampling strategy with respect to the ERM baseline.

7. Related work

Imbalanced classification is ubiquitous across machine learning applications (Japkowicz, 2000). Researchers have also long been aware that class imbalance poses special challenges. The two standard approaches to addressing these challenges are reweighting and subsampling (Idrissi et al., 2021), which have been empirically observed to work more or less equally well when regularization parameters are properly tuned. Without regularization, subsampling tends to dominate since otherwise reweighting doesn’t differ much from ERM (see (Idrissi et al., 2021) for more details). Other approaches include (Chawla et al., 2002), which constructs synthetic inputs for augmenting the minority class, and (Ertekin et al., 2007) that uses active learning.
Throwing away data improves worst-class error in imbalanced classification

Figure 4. We train different linear models (hard SVM, soft SVM, logistic regression) on data with four different types of class-conditional distributions (uniform, Gaussian, Laplace, Fréchet), and different configurations for the dimensionality of the problem ($d \in \{1, 10\}$) and the mean vector separating the classes ($\mu \in \{1, 3\}$). Each point illustrates the average and standard deviation of $\delta = \text{wce}(\theta_{\text{erm}}) - \text{wce}(\theta_{\text{sub}})$, the worst-class error of ERM minus the one of subsampling. Subsampling provides the same or better worst-class error than ERM.

8. Conclusions

In conclusion, we develop novel theory to characterize the worst-class error of Support Vector Machines trained via the Empirical Risk Minimization under class imbalance. We study learning these machines on either the full available training data, or on a reduced version where the large class is subsampled to match in size the minority class. Our results show that under certain distributional assumptions, the subsampling strategy (throwing away most of the training data) outperforms vanilla empirical risk minimization.

Our key contribution is to leverage tools from Extreme Value Theory (EVT) to include information about the tails of the data distribution when describing the relationship between subsampling and balanced test error. We show that when the class-conditional distributions are Gaussians or Laplaces, subsampling has strictly better worst-group error than reweighting, while for Uniforms, both perform equally well. While our main analysis concerns the one-dimensional case, we also offer valuable insights for multivariate data.
Throwing away data improves worst-class error in imbalanced classification

References

Arjovsky, M., Bottou, L., Gulrajani, I., and Lopez-Paz, D. Invariant risk minimization. arXiv, 2019.

Barocas, S., Hardt, M., and Narayanan, A. Fairness and Machine Learning. fairmlbook.org, 2019. http://www.fairmlbook.org.

Bennett, K. P. and Bredensteiner, E. J. Duality and geometry in svm classifiers. In ICML, volume 2000, pp. 57–64. Citeseer, 2000.

Chasalow, K. and Levy, K. Representativeness in statistics, politics, and machine learning. In FAccT, 2021.

Chawla, N. V., Bowyer, K. W., Hall, L. O., and Kegelmeyer, W. P. Smote: synthetic minority over-sampling technique. Journal of artificial intelligence research, 2002.

De Haan, L., Ferreira, A., and Ferreira, A. Extreme value theory: an introduction, volume 21. Springer, 2006.

Drosou, M., Jagadish, H., Pitoura, E., and Stoyanovich, J. Diversity in big data: A review. Big data, 2017.

Engstrom, D. F., Ho, D. E., Sharkey, C. M., and Cuéllar, M.-F. Government by algorithm: Artificial intelligence in federal administrative agencies. NYU School of Law, Public Law Research Paper, 2020.

Ertekin, S., Huang, J., Bottou, L., and Giles, L. Learning on the border: active learning in imbalanced data classification. In Proceedings of the sixteenth ACM conference on Conference on information and knowledge management, pp. 127–136, 2007.

Feller, W. An Introduction to Probability Theory and Its Applications. Wiley, 1968.

Gulrajani, I. and Lopez-Paz, D. In search of lost domain generalization. ICLR, 2020.

Hsu, D. and Sabato, S. Loss minimization and parameter estimation with heavy tails. The Journal of Machine Learning Research, 17(1):543–582, 2016.

Idrissi, B. Y., Arjovsky, M., Pezeshki, M., and Lopez-Paz, D. Simple data balancing achieves competitive worst-group-accuracy. CLeaR, 2021.

Japkowicz, N. The class imbalance problem: Significance and strategies. In IJCAI, 2000.

Mohri, M., Rostamizadeh, A., and Talwalkar, A. Foundations of machine learning. MIT press, 2018.

Sagawa, S., Koh, P. W., Hashimoto, T. B., and Liang, P. Distributionally robust neural networks for group shifts: On the importance of regularization for worst-case generalization. arXiv preprint arXiv:1911.08731, 2019.

Sagawa, S., Raghunathan, A., Koh, P. W., and Liang, P. An investigation of why overparameterization exacerbates spurious correlations. In ICML, 2020.

Shalev-Shwartz, S. and Ben-David, S. Understanding machine learning: From theory to algorithms. Cambridge university press, 2014.

Soudry, D., Hoffer, E., Nacson, M. S., Gunasekar, S., and Srebro, N. The Implicit Bias of Gradient Descent on Separable Data. URL http://arxiv.org/abs/1710.10345.

Srinivasan, V., Prasad, A., Balakrishnan, S., and Ravikumar, P. K. Efficient estimators for heavy-tailed machine learning. 2020.

Steinwart, I. and Christmann, A. Support vector machines. Springer Science & Business Media, 2008.

Vapnik, V. Principles of risk minimization for learning theory. 1992.

Vapnik, V. The nature of statistical learning theory. Springer science & business media, 1999.
A. Proofs

Proof. (Of Lemma 3) Let for any \( n \), \( M_n \) denote the maximum of \( n \) iid random variables drawn from the distribution \( D(0) \). Since the two classes are symmetric around their mean and drawn independently of each other, \( \theta_{\text{sub}} \) can be written as:

\[
\theta_{\text{sub}} = \frac{1}{2}(-\mu + M_{\beta n} + \mu - M'_{\beta n}),
\]

where \( M'_{\beta n} \) is an independent copy of \( M_{\beta n} \). Since \( D(0) \) is of the Frechet or Weibull or Gumbell type, we can now apply the Fisher-Tippett-Gnedenko theorem to conclude that as \( n \to \infty \),

\[
\theta_{\text{sub}} \to \frac{1}{2} a_{\beta n} (Z_3 - Z_4)
\]

where each \( Z_i \) is drawn from the appropriate limit distribution. We can also use a similar argument on \( \theta_{\text{ERM}} \) to conclude that

\[
\theta_{\text{erm}} = \frac{1}{2}(-\mu + M_{\beta n} + \mu - M'_{n}),
\]

(6)

where \( M_{\beta n} \) and \( M'_n \) are independent. A similar argument gives that as \( n \to \infty \), \( \theta_{\text{erm}} \to \frac{1}{2}(b_{\beta n} - b_n + a_{\beta n} Z_1 - a_n Z_2) \).

Proof. (Of Lemma 2) Observe that because of the symmetry of the distributions, \( wce(\theta) \) is symmetric – that is, \( wce(\theta) = wce(-\theta) \). Therefore, it is sufficient to consider the case where \( \theta \geq \theta' \geq 0 \).

When both \( \theta \) and \( \theta' > 0 \), the worst group is the positive class. Thus,

\[
wce(\theta) = \Pr(X \leq \theta | X \sim D(\mu))
\]

and

\[
wce(\theta') = \Pr(X \leq \theta' | X \sim D(-\mu))
\]

Since \( \theta \geq \theta' \), \( wce(\theta) \geq wce(\theta') \) as well. The lemma follows.

B. Proofs for Uniform Distribution

Proof. (Of Lemma 2) From the conditions of the lemma, with probability 1, we are in the realizable case – that is, the positive and negative training samples are separable by a threshold. Therefore, Lemma 1 implies that the solution to SVM under ERM is \( \theta_{\text{ERM}} \) and the corresponding solution under subsampling is \( \theta_{\text{SS}} \).

Again, from the conditions of the problem and Lemma 3, we have that here \( b_n = 0 \), and \( a_n = x_F - U(n) = \frac{1}{2} - \frac{1}{n} \). This gives that \( \theta_{\text{SS}} \) converges in distribution to \( \left( \frac{1}{2} - \frac{1}{\beta n} \right)(Z_3 - Z_4) \), and \( \theta_{\text{ERM}} \) converges in distribution to \( \left( \frac{1}{2} - \frac{1}{\beta n} \right)Z_1 - \left( \frac{1}{2} - \frac{1}{n} \right)Z_2 \) = \( \left( \frac{1}{2} - \frac{1}{\beta n} \right)(Z - Z') + \left( \frac{1}{\beta n} - \frac{1}{n} \right)Z' \). Observe that here,

\[
\frac{1}{\beta n} - \frac{1}{n} = \frac{1 - \beta}{\beta n} \to 0,
\]

as \( \beta n \to \infty \) as \( n \to \infty \). The lemma follows by combining this with lemma 4 and the fact that convergence in probability implies convergence in distribution.

Lemma 4. Let \( A \) and \( B \) be two random variables, and let \( \lambda_n \to 0 \) be a positive scalar. Then, \( A + \lambda_n B \to A \) in probability.

Proof. Pick a particular \( \epsilon \). Then, by definition of convergence in probability, \( \Pr(|A + \lambda_n B - A| \geq \epsilon) = \Pr(\lambda_n |B| \geq \epsilon) = \Pr(|B| \geq \epsilon/\lambda_n) \). As \( n \to \infty \), \( \epsilon/\lambda_n \to \infty \) and hence this probability tends to zero. The lemma follows.

C. Proofs for Laplace Distribution

Lemma 5. Let \( Z_3 \) and \( Z_4 \) be two independent standard Gumbell variables. Then with probability \( 1 - \frac{2}{1 + \tau^2} \), \( |Z_3 - Z_4| \leq \tau \).
Proof. Since $Z_3$ and $Z_4$ are independent standard Gumbell variables, $Z_3 - Z_4$ is distributed according to a logistic distribution with location parameter $0$ and scale parameter $1$. This means that $Z_3 - Z_4$ is symmetric about $0$, and also that for any $\tau$,

$$\Pr(Z_3 - Z_4 \in [-\tau, \tau]) = 1 - \frac{2}{1 + e^{-\tau}} \quad (7)$$

The lemma follows. $\square$

Proof. (Of Theorem 3)

Fix $\epsilon$, $\delta$ and $\gamma$. We first show that with the given value of $\mu_n$, the probability that the training data is linearly separable is at least $1 - 2\epsilon$. To see this, observe that the probability a sample $z$ from the positive class has value $< 0$ is at most $\frac{2}{e^{\mu_n}} = \frac{2}{e^n}$. An union bound over all $n$ samples establishes that the probability that any of the $n$ positive samples lie below zero is at most $\epsilon$. A similar argument can also be applied to the negative class to show that with probability $\geq 1 - 2\epsilon$, the training data is linearly separable.

To show the first part of the theorem, observe from Lemma 3 that for $n$ large enough, $\theta_{\text{sub}} \rightarrow \frac{1}{2}(Z_3 - Z_4)$ where each $Z_i$ is a standard Gumbell distribution. This implies that

$$\Pr(|\theta_{\text{sub}}| \geq \frac{1}{2} \log(1/\gamma)) \leq \Pr(|Z_3 - Z_4| \geq \log(1/\gamma)) + \delta$$

From Lemma 5, this probability is at most $\delta + \frac{2}{1 + \sqrt{\gamma}} \geq \delta + \frac{2\sqrt{\pi}}{1 + \gamma} \leq \delta + 2\gamma$. Similarly, from Lemma 3, for $n$ large enough, we have that $\theta_{\text{erm}} \rightarrow \frac{1}{2}(- \log(1/\beta) + Z_1 - Z_2)$. This implies that

$$\Pr(\theta_{\text{erm}} \leq \frac{1}{2}(- \log(1/\beta) + \log(1/\gamma))) \leq \Pr(Z_1 - Z_2 \geq \log(1/\gamma)) + \delta$$

which is at most $\frac{1}{1 + \sqrt{\gamma}} + \delta \leq \delta + \gamma$. The first part of the theorem follows from an union bound, and the fact that for any threshold $\tau$, $\Pr(|\theta_{\text{erm}}| \geq \tau) = \Pr(\theta_{\text{erm}} \leq -\tau) + \Pr(\theta_{\text{erm}} \geq \tau) \geq \Pr(\theta_{\text{erm}} \leq -\tau)$.

To show the second part, we observe that $\text{wce}(\theta_{\text{erm}}) \geq \frac{1}{2}e^{-\mu_n} - \frac{1}{2}(\log(1/\beta) - \log(1/\gamma)) \geq \frac{\epsilon \sqrt{n}}{2n\sqrt{\beta}} \cdot \frac{\sqrt{n}}{\sqrt{\gamma}}$. Similarly, $\text{wce}(\theta_{\text{sub}}) \leq e^{-\mu_n} + \frac{1}{2} \log(1/\gamma) \leq \frac{\epsilon}{n \sqrt{\gamma}}$. The theorem follows. $\square$

D. Proofs for Gaussian Distribution

Lemma 6. (Feller, 1968) Let $X \sim N(0, 1)$. Then, for any $t$,

$$\frac{1}{\sqrt{2\pi}} e^{-t^2/2} \left(1 - \frac{1}{t^2} \right) \leq \Pr(X \geq t) \leq \frac{1}{\sqrt{2\pi}} e^{-t^2/2} t$$

Proof. (Of Theorem 4)

Fix $\epsilon$, $\gamma$ and $\delta$. We first show that with the given value of $\mu_n$, the probability that the training samples are realizable is at least $1 - 2\epsilon$. To see this, observe that from Lemma 6, the probability a sample $z$ from the positive class has value $< 0$ is at most:

$$\Pr(X \geq \sqrt{2 \log(n/\epsilon)} | X \sim N(0, 1)) \leq \frac{e^{-\mu_n^2/2}}{\sqrt{2\pi \mu_n}} \leq \epsilon/n$$

An union bound over all $n$ samples establishes that the probability that any of the $n$ positive samples lie below zero is at most $\epsilon$. A similar argument can also be applied to the negative class to show that with probability $\geq 1 - 2\epsilon$, the training data is linearly separable.

To establish the first part of the theorem, we will separately look at $\theta_{\text{sub}}$ and $\theta_{\text{erm}}$. From Lemma 3 that for $n$ large enough,

$$\Pr \left( |\theta_{\text{sub}}| \geq \frac{1}{2} \frac{\log(1/\gamma)}{\sqrt{2 \log(\beta n)}} \right) \leq \Pr \left( |Z_3 - Z_4| \geq \log(1/\gamma) \right) + \delta$$
From Lemma 5, this probability is at most \( \delta + \frac{2}{1+\gamma} \leq \delta + 2\gamma \).

Similarly, from Lemma 3, for \( n \) large enough, we have that:

\[
\theta_{\text{erm}} \to \frac{1}{2} (b_\beta n - b_n) + \frac{1}{2} a_\beta n(Z_1 - Z_2) + \frac{1}{2} (a_n - a_\beta n)Z_2
\]

where \( Z_1 \) and \( Z_2 \) are standard Gumbell random variables. This means that for \( n \) large enough, and for any threshold \( \tau \), we have that:

\[
\Pr(\theta_{\text{erm}} \leq \tau) \leq \Pr(\frac{1}{2} (b_\beta n - b_n) + \frac{1}{2} a_\beta n(Z_1 - Z_2) + \frac{1}{2} (a_n - a_\beta n)Z_2 \leq \tau) + \delta
\]  

(8)

Now, observe that for Gaussians:

\[
b_\beta n - b_n = \sqrt{2 \log \beta n} - \frac{\log(\beta n) + \log 4\pi}{\sqrt{2 \log \beta n}} - \sqrt{2 \log n} + \frac{\log n + \log 4\pi}{\sqrt{2 \log n}}
\]

\[
\leq -\sqrt{2 \log \beta n} - \sqrt{2 \log n}
\]

\[
= -\sqrt{2 \log \beta n} (1 - \frac{\log n}{\log \beta n})
\]

\[
= -\sqrt{2 \log \beta n} (1 - (1 + \frac{\log(1/\beta)}{\log \beta n})^{1/2})
\]

\[
\leq -\frac{2 \log(1/\beta)}{3 \sqrt{2 \log \beta n}}
\]

(9)

Here the first step follows as \( \frac{\log n + \log(4\pi)}{\sqrt{\log n}} \) is a decreasing function of \( n \), and the second and third steps follow from algebra. The fourth step follows from the fact that \( (1 + x)^{1/2} \geq 1 + x/3 \) when \( x \leq 3 \); as \( n^3 \geq \beta^4 \), \( \frac{\log(1/\beta)}{\log \beta n} \leq 3 \). The final step then follows from algebra. Additionally, from Lemma 5,

\[
\Pr(a_\beta n(Z_1 - Z_2) \geq \frac{\log(1/\gamma)}{\sqrt{2 \log \beta n}}) \leq \Pr(Z_1 - Z_2 \geq \log(1/\gamma)) \leq \frac{1}{1 + 1/\gamma} \leq \gamma
\]

Finally, observe that

\[
a_\beta n - a_n = \frac{1}{\sqrt{2 \log \beta n}} - \frac{1}{\sqrt{2 \log n}}
\]

\[
= \frac{1}{\sqrt{2 \log \beta n}} (1 - \sqrt{\frac{\log \beta n}{\log n}})
\]

\[
= \frac{1}{\sqrt{2 \log \beta n}} (1 - (1 - \frac{\log(1/\beta)}{\log n})^{1/2})
\]

\[
\leq \frac{1}{\sqrt{2 \log \beta n}} \frac{\log(1/\beta)}{\log n}
\]

(10)

where the last step follows because for \( 0 < x < 1 \), \( \sqrt{1-x} \geq 1 - x \). Therefore,

\[
\Pr((a_\beta n - a_n)Z_2 \geq \frac{\log(1/\gamma)}{\sqrt{2 \log \beta n}}) \leq \Pr(Z_2 \geq \log(1/\gamma) \cdot \frac{\log n}{\log(1/\beta)})
\]

\[
\leq \Pr(Z_2 \geq \log(1/\gamma))
\]

\[
\leq 1 - e^{-7} \leq \gamma
\]

(11)

where the first step follows from Equation 10, the second step from the fact that \( \log(1/\beta) \leq \log n \), and the final step from the standard Gumbell cdf and the fact that \( 1 - e^{-x} \leq x \). The first part of the theorem follows from combining Equations 9, 10 and 11.
To prove the third part of the theorem, we use Lemma 6. From the second part of the theorem and Lemma 6, observe that

\[ \text{wce}(\theta_{\text{sub}}) \leq \Phi\left( \mu_n - \frac{\log(1/\gamma)}{2\sqrt{2 \log \beta n}} \right) \]

From Lemma 6, the right hand side is, in turn, at most:

\[ \leq \frac{1}{\sqrt{2\pi}(\mu_n - \frac{\log(1/\gamma)}{2\sqrt{2 \log \beta n}})} \cdot \exp\left( -\frac{1}{2} \left( \mu_n - \frac{\log(1/\gamma)}{2\sqrt{2 \log \beta n}} \right)^2 \right) \]

\[ \leq \frac{2}{\sqrt{2\pi} \mu_n} \cdot \exp\left( -\frac{1}{2} \mu_n^2 + \frac{\mu_n \log(1/\gamma)}{2\sqrt{2 \log \beta n}} \right) \]

\[ \leq \frac{2\epsilon}{n} \cdot \exp\left( \frac{\mu_n \log(1/\gamma)}{2\sqrt{2 \log \beta n}} \right) \]

Here the first step follows because for \( n \) large enough, \( \mu_n - \frac{\log(1/\gamma)}{2\sqrt{2 \log \beta n}} \geq \frac{1}{\sqrt{2\pi}} \); this is because \( \mu_n \) is an increasing function of \( n \). The second step follows because by design \( \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\mu_n^2/2} = \epsilon/n \) and the third step follows from simple algebra.

We observe that \( \mu_n \leq \sqrt{2 \log(n/\epsilon)} \) – this is because \( e^{-\frac{1}{2}\log(n/\epsilon)^2/\sqrt{1/2 \log(n/\epsilon)}} < \epsilon/n \). This implies that:

\[ \frac{\mu_n}{\sqrt{2 \log \beta n}} \leq \left( 1 + \frac{\log(1/\beta)}{\log(1/\gamma)} \right)^{1/2} \leq 2, \]

provided \( \frac{1}{\epsilon} \leq \beta^2 n \). Therefore,

\[ \text{wce}(\theta_{\text{sub}}) \leq \frac{2\epsilon}{\gamma n} \]

In contrast, from Lemma 6,

\[ \text{wce}(\theta_{\text{erm}}) \geq \Phi\left( \mu_n - \frac{3}{2} \frac{\log(1/\beta) - 2 \log(1/\gamma)}{2\sqrt{2 \log \beta n}} \right) \]

\[ = \Phi\left( \mu_n - \frac{\log(\gamma^2/\beta^{2/3})}{2\sqrt{2 \log \beta n}} \right) \]

From Lemma 6, this is at least:

\[ \frac{1}{\sqrt{2\pi}} \left( \frac{1}{\mu_n} - \frac{1}{2\sqrt{2 \log \beta n}} \right) \cdot \exp\left( -\frac{1}{2} \left( \mu_n - \frac{\log(\gamma^2/\beta^{2/3})}{2\sqrt{2 \log \beta n}} \right)^2 \right) \]

For \( n \) large enough, \( \frac{1}{\mu_n} \leq \frac{1}{\sqrt{2 \log \beta n}} \); also \( \frac{1}{\mu_n} \geq \frac{1}{\sqrt{2 \log \beta n}} \). This implies that the right hand side is at least:

\[ \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{2\mu_n} \cdot \exp\left( -\frac{1}{2} \left( \mu_n - \frac{\log(\gamma^2/\beta^{2/3})}{2\sqrt{2 \log \beta n}} \right)^2 \right) \]

Observe that:

\[ \geq \frac{1}{\sqrt{2\pi} \mu_n} \cdot \exp\left( -\frac{1}{2} \left( \mu_n - \frac{\log(\gamma^2/\beta^{2/3})}{2\sqrt{2 \log \beta n}} \right)^2 \right) \]

\[ \geq \frac{1}{2} \frac{\epsilon}{n} \cdot \exp\left( \frac{\mu_n \log(\gamma^2/\beta^{2/3})}{2\sqrt{2 \log \beta n}} \right) \]
where the first step follows since \( \frac{1}{\sqrt{2\pi}\mu_n} \cdot e^{-n^2/2} = \epsilon/n \) and the second step follows since for large enough \( n, \mu_n/2 \geq \log(\gamma^2/\beta^2/3) \). 

Also: observe that for \( n \) large enough \( \frac{\mu_n}{\sqrt{2\log 3n}} \geq \frac{1}{2} \) since \( \mu_n \geq \frac{1}{2} \sqrt{2\log(n/\epsilon)} \). Putting these all together, the entire error is at least:

\[
\frac{\epsilon}{2n} \exp(\frac{1}{8} \log(\gamma^2/\beta^2/3)) \geq \frac{\epsilon^2}{2n^{3/4}}.
\]

The theorem now follows.

\[\square\]

**Lemma 7.** Suppose we are given \( \mu > 0 \), and \( 0 \leq t \leq \mu \). Then,

\[
wce(t) = \Phi(\mu - t)
\]

where \( \Phi(x) \) is the Gaussian tail function.

**Proof.** The first part of the lemma follows from our definition of error and from geometry. \[\square\]

**E. Proofs for Frechet Distribution**

For this example, we assume that \( D(\mu) \) is the two-sided Frechet distribution with parameter \( \alpha \); in other words, the CDF \( F \) is:

\[
F(x) = \begin{cases} 
\frac{1}{2} + \frac{1}{2} e^{-(x-\mu)^{-\alpha}}, & x \geq \mu, \alpha > 0 \\
\frac{1}{2} - \frac{1}{2} e^{-(\mu-x)^{-\alpha}}, & x \leq \mu 
\end{cases}
\]

Here, each \( Z_i \) is distributed according to a Frechet distribution with parameter \( \alpha \); additionally, \( b_n = 0 \) and \( a_n = n^{1/\alpha} \). Accordingly, we can get the following theorem.

**Lemma 8.** If \( 0 \leq x \leq 1 \), then,

\[
x(1-e^{-1}) \leq 1 - e^{-x} \leq x
\]

**Proof.** First we show the upper bound. Let \( f(x) = 1 - x - e^{-x} \); \( f(0) = 0 \), and \( f'(x) = -1 + e^{-x} \leq 0 \) for \( x \geq 0 \). Therefore \( f \) is a decreasing function for \( x \geq 0 \); this means that for \( x \geq 0 \), \( f(x) = 1 - x - e^{-x} \leq f(0) = 0 \). The bound follows.

For the lower bound, let \( g(x) = 1 - e^{-x} \). Observe that \( g(0) = 0 \). Additionally, \( g''(x) = -e^{-x} < 0 \) which makes \( g \) a concave function – which means that for \( x \in [0,1] \), \( g(x) \) lies above the line segment connecting \( g(0) = 0 \) and \( g(1) = 1 - e^{-1} \). Formally,

\[
g(x) \geq (1-x)g(0) + xg(1) \geq (1-e^{-1})x
\]

The lower bound follows. \[\square\]

**Proof.** (Of Theorem 5) We first show that with the given value of \( \mu_n \), the probability that the training data is linearly separable is at least \( 1 - 2\epsilon \). To see this, observe that the probability a sample \( z \) from the positive class has value \( < 0 \) is at most \( \frac{1}{2}(1 - e^{-\mu_n^2/2}) \leq \frac{1}{2} \mu_n^{-\alpha} \leq (\epsilon/n) \). An union bound over all \( n \) samples establishes that the probability that any of the \( n \) positive samples lie below zero is at most \( \epsilon \). A similar argument can also be applied to the negative class to show that with probability \( \geq 1 - 2\epsilon \), the training data is linearly separable.

To prove the first part of the theorem, observe that Lemma 3 suggests that \( \theta_{\text{sub}} \rightarrow \frac{1}{2}(\beta n)^{1/\alpha}(Z_3 - Z_4) \) and \( \theta_{\text{erm}} \rightarrow \frac{1}{2}(\beta n)^{1/\alpha}Z_1 - n^{1/\alpha}Z_2 \) where each \( Z_i \) is a Frechet random variable with parameter \( \alpha \).

Let us first address \( \theta_{\text{sub}} \). Since \( Z_3 \) and \( Z_4 \) are standard Frechet variables and are always positive, we have that:

\[
\Pr(|Z_3 - Z_4| \geq (1/\gamma)^{1/\alpha}) \leq \Pr(Z_3 \geq (1/\gamma)^{1/\alpha}) + \Pr(Z_4 \geq (1/\gamma)^{1/\alpha}) \leq 2\gamma
\]
Throwing away data improves worst-class error in imbalanced classification

Since $\theta_{\text{sub}}$ converges in distribution to $\frac{1}{2}(n/\beta)^{1/\alpha}(Z_3 - Z_4)$, for $n$ large enough, with probability $\geq 1 - 2\gamma - \delta$, $|\theta_{\text{sub}}| \leq \frac{1}{2}(\beta n/\gamma)^{1/\alpha}$.

Next we look at $\theta_{\text{erm}}$, which in this case converges in distribution to:

$$
\lim_{n \to \infty} \theta_{\text{erm}} = \frac{1}{2}(\beta n)^{1/\alpha}Z_1 - \frac{1}{2}n^{1/\alpha}Z_2
$$

We will now bound the two terms on the right hand side of the equation separately. To bound the first term, note that:

$$
\Pr(Z_1 \geq \frac{1}{\alpha}((1/\gamma)^{1/\alpha} - \frac{1}{2}(\beta n/\gamma)^{1/\alpha})) = 1 - \exp(-\frac{1}{\alpha}((1/\gamma)^{1/\alpha} - \frac{1}{2}(\beta n/\gamma)^{1/\alpha})) \leq \gamma
$$

To bound the second term, note that:

$$
\Pr(Z_2 \geq \frac{1}{\log(1/\gamma)^{1/\alpha}}) = 1 - \exp(-\frac{1}{\log(1/\gamma)^{1/\alpha}}) = 1 - \exp(-\log(1/\gamma)) = 1 - \gamma
$$

Therefore, for $n$ large enough, with probability at least $1 - 2\gamma - \delta$, we have that:

$$
\theta_{\text{erm}} \leq -\frac{1}{2}\left(\frac{n}{\log^{1/\alpha}}\right)^{1/\alpha} + \frac{1}{2}(\beta n/\gamma)^{1/\alpha}
$$

The first part now follows by an union bound, and the fact that for any $\tau$, $\Pr(|\theta_{\text{erm}}| \geq \tau) \leq \Pr(\theta_{\text{erm}} \leq -\tau)$.

For the second part of the theorem, for $\theta_{\text{sub}}$, we have:

$$
\text{wce}(\theta_{\text{sub}}) \leq \frac{1}{2}\left(1 - \exp(-\frac{1}{\mu_n - \frac{1}{2}(\beta n/\gamma)^{1/\alpha}})\right);
$$

where the factor of $1/2$ comes from the relevant tail of the majority class. Observe that:

$$
\frac{1}{\mu_n - \frac{1}{2}(\beta n/\gamma)^{1/\alpha}} = \frac{1}{(n/2\epsilon)^{1/\alpha} - \frac{1}{2}(\beta n/\gamma)^{1/\alpha}} = \frac{2\epsilon}{n} \cdot \frac{1}{(1 - \frac{1}{2}(2\epsilon/\log(1/\gamma))^{1/\alpha})}
$$

For large enough $n$, this is $\leq 1$; therefore, from Lemma 8, we can write:

$$
\text{wce}(\theta_{\text{sub}}) \leq \frac{\epsilon}{n} \cdot \frac{1}{(1 - \frac{1}{2}(2\epsilon/\log(1/\gamma))^{1/\alpha})}
$$

Similarly,

$$
\text{wce}(\theta_{\text{erm}}) \geq \frac{1}{2}\left(1 - \exp(-\frac{1}{\mu_n - \frac{1}{2}(n/\log(1/\gamma))^{1/\alpha} + \frac{1}{2}(\beta n/\gamma)^{1/\alpha}})\right)
$$

Observe that

$$
\frac{1}{\mu_n - \frac{1}{2}(n/\log(1/\gamma))^{1/\alpha} + \frac{1}{2}(\beta n/\gamma)^{1/\alpha}} = \frac{1}{((n/2\epsilon)^{1/\alpha} - \frac{1}{2}(n/\log(1/\gamma))^{1/\alpha} + \frac{1}{2}(\beta n/\gamma)^{1/\alpha}} = \frac{2\epsilon}{n} \cdot \frac{1}{(1 - \frac{1}{2}(2\epsilon/\log(1/\gamma))^{1/\alpha} + \frac{1}{2}(2\epsilon/\gamma)^{1/\alpha})}
$$

For large enough $n$, this is $\leq 1$; therefore, from Lemma 8, we can write that $\text{wce}(\theta_{\text{erm}})$ is at least:

$$
\frac{\epsilon}{n} \cdot \frac{1 - e^{-1}}{(1 - \frac{1}{2}(2\epsilon/\log(1/\gamma))^{1/\alpha} + \frac{1}{2}(2\epsilon/\gamma)^{1/\alpha})}
$$

The theorem follows.