Deformed Skyrmions

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The spherically symmetric hedgehog ansatz used in the description of the skyrmion is believed to be inadequate for the rotational states such as the nucleon \( I = J = \frac{1}{2} \) and the \( \Delta \) \( (I = J = \frac{3}{2}) \) due to centrifugal forces. We study here a simple alternative: an oblate spheroidal solution which leads to lower masses for these baryons. As one might expect, the shape of the solution is flatter as one increases \( I = J \) whether the size of the soliton is allowed to change or not.

I. INTRODUCTION

When Skyrme first introduced its model a few decades ago [1] to describe baryons as solitons in a non-linear field theory of mesons, the solution proposed was a spherically symmetric hedgehog ansatz. There are reasons to believe that this solution is not adequate for the rotational states such as the nucleon \( (I = J = \frac{1}{2}) \) and the \( \Delta \) \( (I = J = \frac{3}{2}) \) due to centrifugal forces [2]. Several treatments have been proposed in the past with relative success [3,4]. In this work, we take a naive approach and propose a simple alternative. Instead of the spherically symmetric hedgehog solution, we introduce an oblate spheroidal solution. This leads to lower masses and quadrupole deformations for these baryons. Moreover, the shape of the solution is flatter as one increases \( I = J \) whether one allows the size of the soliton to change or not.

II. THE STATIC OBLATE SOLITON

The oblate spheroidal coordinates \((\eta, \theta, \phi)\) are related to Cartesian coordinates through

\[
(x, y, z) = d(\cosh \eta \sin \theta \cos \phi, \cosh \eta \sin \theta \sin \phi, \sinh \eta \cos \theta).
\]

A surface of constant \( \eta \) corresponds to a sphere of radius \( d \) flattened in the \( z \)-direction by a factor of \( \tanh \eta \). For \( \eta \) small, the shape of the surface is more like that of a pancake of radius \( d \) whereas for large \( \eta \), one recovers a spherical shell of radius \( r = \frac{4c}{\pi} \).

We would like to replace the hedgehog solution for the Skyrme model by an oblate solution. Writing the Lagrangian for the Skyrme model [3] (neglecting the pion mass)

\[
\mathcal{L} = -\frac{F^2}{16} Tr(L_\mu L^\mu) + \frac{1}{32\pi^2} Tr[(L_\mu, L_\nu)^2]
\]

where \( L_\mu = U^\dagger \partial_\mu U \) with \( U \in SU(2) \). Let us now define a static oblate solution by

\[
U = e^{i(\tau \hat{\eta}) f(\eta)}
\]

where \( \hat{\eta} \) is the unit vector \( \hat{\eta} = \frac{\nabla \eta}{|\nabla \eta|} \). The boundary conditions for the winding number \( N = 1 \) solution are \( f(0) = \pi \) and \( f(\infty) = 0 \). Note that this is not a priori a solution of the field equations derived from the Skyrme Lagrangian.

Integrating over the angular variables \( \theta \) and \( \phi \), the static energy reads

\[
E_\eta = \epsilon \int d\eta \left[ \frac{d}{2} (\alpha_{21} f'^2 + \alpha_{22} \sin^2 f) + \frac{1}{4d} (\alpha_{41} f'^2 \sin^2 f + \alpha_{42} \sin^4 f) \right]
\]

with

\[
\alpha_{21}(\eta) = 2 \cosh \eta \quad \alpha_{22}(\eta) = 2 \left(-2 \cosh \eta + (2 \cosh^2 \eta - 1) L(\eta)\right)
\]

\[
\alpha_{41}(\eta) = 2L(\eta) \quad \alpha_{42}(\eta) = \frac{1}{2} \left( \frac{1}{\cosh^4 \eta} (2 \cosh \eta + L(\eta)) + \frac{2 \cosh \eta}{\cosh^2 \eta - 1} \right)
\]

where \( L(\eta) \equiv \ln \left( \frac{\cosh \eta + 1}{\cosh \eta - 1} \right) \) and the constants are defined by \( \epsilon = \frac{2\sqrt{\pi} F}{\epsilon} \) and \( \tilde{d} \equiv \frac{4c}{2\sqrt{2}} d \).
Minimizing the static energy with respect to \( f(\eta) \), we then solve numerically the corresponding non-linear ordinary second-order differential equation. For calculational purposes, we set the value of the parameters of the Skyrme model as \( F_\pi = 129 \text{ MeV}, e = 5.45 \) (and \( m_\pi = 0 \)) which coincide with those of ref. \(^3\) obtained by fitting for the masses of the nucleon and the \( \Delta \) in the hedgehog ansatz.

The solution near \( \eta \to 0 \) has the form \( f(\eta) \sim \pi - a_1\eta \), whereas in the limit \( \eta \to \infty \) one recovers the spherical symmetry with \( f(\eta) \sim k(\delta^{0\eta})^2 \) where \( a_1 \) and \( k \) are constants which depend on \( \tilde{d} \).

The masses of the nucleon and of the \( \Delta \)-isobar get contributions both from the static and rotational energy and will generally depend on the choice of \( \tilde{d} \). We fix the value of \( \tilde{d} \) for each baryon by minimizing its mass with respect to \( \tilde{d} \).

### III. COLLECTIVE VARIABLES

Using the oblate solution, we then compute the masses of the nucleon and of the \( \Delta \)-isobar. However, when one departs from the spherical symmetry of the hedgehog ansatz, it is customary to introduce extra collective variables for isorotation in addition to those characterizing spatial rotation since these are no longer equivalent, in general. The spin and isospin contributions to the rotational energy are however equal in our case since we use solution \(^3\) and we are only interested in ground states with \( K = J + I = 0 \). As a result, we need only consider one set of collective variables.

We work in the body-fixed system and assume that the time dependence can be introduced using the usual substitution \( U \to A(t)U A^*(t) \) where \( A(t) \) is a time-dependent \( SU(2) \) matrix. We can then go on and treat \( A(t) \) approximately as quantum mechanical variables.

The quantization procedure is fairly standard and leads to principal moments of inertia \( I_{11} \) and \( I_{33} \) in the body-fixed system. We get a representation analog to a symmetrical top with the rotational kinetic energy in space and isospace

\[
E_{\text{rot}}^{J,J_3} = \frac{1}{2I_{11}} \left( |J|^2 + |I|^2 \right) + \frac{1}{2} \left( \frac{1}{I_{33}} - \frac{1}{I_{11}} \right) J_3^2.
\]

where \( |J|^2 \) and \( |I|^2 \) are the spin and the isospin respectively and, \( J_3 \), the \( z \)-component of the spin. We have already used the relation \( J_3 = -I_3 \) here which follows from axial symmetry of the ansatz. Added to the static energy \( E_s \), it leads to the total energy \( M^{J,J_3} = E_s + E_{\text{rot}}^{J,J_3} \) identified with the mass of the baryon.

Observables states, however, must be eigenstates of \( |J|^2, J_3, |I|^2, I_3 \) with eigenvalues \( J(J + 1), m_J, I(I + 1), m_I \) where the operators now refer to the laboratory system (as opposed to body-fixed operators in (??) and above). These eigenstates are taken into account by direct products of rotation matrices

\[
\langle \Omega|J, m_J, m_I\rangle \langle \omega|I, m_I, -m_I\rangle = D^J_{m_J m_I} \langle \Omega \rangle D^I_{m_I -m_I} \langle \omega \rangle
\]

where \( \Omega \) and \( \omega \) are, respectively, the Euler angles for the rotation and isorotation from the body-fixed frame to the laboratory system. The explicit calculation of the energy of rotation requires in general the diagonalization \( E_{\text{rot}}^{J,J_3} \).

(see ref. \(^3\) for more details)

The minimization of the static energy for the spherical symmetric ansatz gives \( E_s = 8.20675\epsilon, M_N = 8.906\epsilon \) and \( M_{\Delta} = 11.703\epsilon \). For the oblate spheroidal ansatz, the parameter \( \tilde{d} \) is chosen in order to minimize the mass of the corresponding baryon. In general, as \( \tilde{d} \) increases, the static energy \( E_s \) deviates from its lowest energy value given by the spherical hedgehog configuration. On the other hand, oblate configurations have larger moments of inertia which tend to decrease the rotational kinetic energy. The existence of a non-trivial oblate spheroidal ground state for the nucleon and the \( \Delta \)-isobar, as it turns out, depends mostly on the relative importance of static and rotational energy.

We find that the ground state for the nucleon is almost spherical but nonetheless oblate with \( \tilde{d} = 0.0013 \) thus exhibiting a small quadrupole deformation and a slightly lower mass with respect to a spherical configuration. For the \( \Delta \)-isobar, the oblateness or quadrupole deformation is even more important and accounts for a \( 4\% \) decrease in mass. We obtain a minimum mass for a value of \( \tilde{d} = 0.32 \) with \( M_{\Delta} = 11.293\epsilon \).

Since the minimum of the ground state is affected by the oblate shape of the solution, the parameters \( F_\pi \) and \( e \) as given in ref. \(^2\) no longer reproduce the quantities they were designed to fit. We must readjust \( F_\pi \) and \( e \) which determine the value of \( \tilde{d} \) for the nucleon and \( \Delta \)-isobar respectively. After several iterations, we find \( F_\pi = 118.4 \text{ MeV} \) and \( e = 5.10 \) with \( \tilde{d} = 0.0014 \) (\( \tilde{d} = 0.40 \)) for the nucleon (\( \Delta \)-isobar).
IV. DISCUSSION

Quadrupole deformations were found previously in the context of rotationally improved skyrmions. Contrary to our approach, these solutions involve the minimization of an Hamiltonian which also includes the (iso) rotational kinetic energy. Yet, we found that the oblate spheroidal ansatz gives lower energy than the spherical one for baryon ground states. Of course, ansatz (3) is not necessarily the lowest energy solution.

It may also be interesting to consider deformations of the oblate skyrmions under scaling of the unitary transformations $U(r)$ such that $U(r) = U_0(\rho r)$ to minimize the total energy of the nucleon and $\Delta$-isobar. The total energies $M_N(\rho)$ and $M_\Delta(\rho)$ can be minimized with respect to the scaling parameter $\rho$, i.e. to the energetically favored size of the oblate skyrmion. The energies are found to be $M_N(\rho_{\text{min}} = 0.868) = 8.797\epsilon$ and $M_\Delta(\rho_{\text{min}} = 0.670) = 10.064\epsilon$ for both the oblate case compared with $M_N(\rho_{\text{min}} = 0.867) = 8.799\epsilon$ and $M_\Delta(\rho_{\text{min}} = 0.668) = 10.238\epsilon$ for spherical ansatz. The baryon ground states are now swollen oblate solutions. Again, one should in principle readjust the $F_\pi$ and $e$ parameters to fit the masses of the nucleon and $\Delta$-isobar.

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