Non-gravitational exceptional supermultiplets

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Abstract: We examine non-gravitational minimal supermultiplets which are based on the tensor gauge fields appearing as matter fields in exceptional generalised geometry. When possible, off-shell multiplets are given. The fields in the multiplets describe non-gravitational parts of the internal dynamics of compactifications of M-theory. In flat backgrounds, they enjoy a global U-duality symmetry, but also provide multiplets with a possibility of coupling to a generalised exceptional geometry.
1. Background

Generalised geometry provides a way of extending the geometric picture of the gravity field to massless tensor fields in string theory or M-theory, thus manifesting and giving a geometric framework to T-duality or U-duality [1-4]. There has recently been progress in the formulation of generalised geometric models, both for the doubled field theories (manifesting T-duality) [5-22] and exceptional theories (U-duality) [23-36]. In particular, the tensor gauge fields for the exceptional setting were described in an accompanying paper, ref. [36], together with a tensor calculus for exceptional generalised geometry. The purpose of the present letter is to construct supermultiplets, not containing generalised gravity, based on the known tensor fields.

Bosonic matter fields, apart from scalars, should come in the modules $R_k$, some of which are listed in Table 1. These modules play a number of roles in exceptional geometry [34,37-39]. In ref. [34], they were shown to describe the reducibility of generalised diffeomorphisms. Their dynamics, as tensor fields, was examined in ref. [36]. Below, we will realise minimal global supersymmetry in $n = 4, 5, 6$ on matter multiplets. As will be clear, “$N = 1$” supersymmetry in $n = 4, 5, 6, 7$ contains 4, 8, 16 and 32 supersymmetries, respectively. The real forms may need to be chosen differently in order to have real fields. We will constrain ourselves to tensor potentials in the modules $R_2, \ldots, R_{8-n}$. There may also be fields in $R_1$, which is the “generalised graviphoton”, but its number of degrees of freedom is too large to allow for minimal set of fermions only.

The sequences $\{R_k\}$ are infinite, but relevant tensor fields come in $R_1, \ldots, R_{8-n}$, which is the part of the sequence where derivatives from $R_k$ to $R_{k-1}$ is connection-free. These modules are analogous to forms, both in the absence of connection, and in the nilpotency of the derivative, which makes the concepts of field strengths and gauge transformations natural. Details are found in ref. [36].

| $n$ | $G$ | $\overline{\mathbb{P}}$ | $R_1$ | $R_2$ | $R_3$ | $R_4$ | $R_5$ |
|-----|------|----------------|-------|-------|-------|-------|-------|
| 3   | $SL(3) \times SL(2)$ | $SU(2) \times U(1)$ | $[3, 2]$ | $\overline{[3, 1]}$ | $[1, 2]$ | $[3, 1]$ | $[\overline{3}, \overline{2}]$ |
| 4   | $SL(5)$ | $Spin(5)$ | 10 | 5 | 5 | 10 | 24 |
| 5   | $SO(5, 5)$ | $Spin(5) \times Spin(5)$ | 16 | 10 | 16 | 45 |
| 6   | $E_6(6)$ | $USp(8)$ | 27 | $2\overline{7}$ | 78 |
| 7   | $E_7(7)$ | $SU(8)$ | 56 | 133 |

*Table 1: A partial list of modules $R_k^{(n)}$.***
2. Construction of the multiplets

As a guide for the construction, it is informative to list the counting of off-shell and on-shell degrees of freedom in the tensor fields. In the cases we consider, the number of off-shell and on-shell degrees of freedom of $R_k$ are given by the dimensions of the corresponding modules when $n$ is lowered by one and two units, respectively [36].

| $n$ | $R_2$ | $R_3$ | $R_4$ |
|-----|-------|-------|-------|
| 4   | $3 \to 2$ | $2 \to 1$ | $3 \to 1$ |
| 5   | $5 \to 3$ | $5 \to 2$ |       |
| 6   | $10 \to 5$ |       |       |

Table 2: Off-shell and on-shell counting for $R_k$.

The fermions (“spinor” fields and supersymmetry generators) transform in a module $S$ of the double cover $\overline{H}$ of the compact subgroup of the U-duality group. For $n = 4$ the spinor module is $S = 4 = (01)$ of $Spin(5)$, for $n = 5$ $S = (4,1) \oplus (1,4) = (01)(00) \oplus (00)(01)$ of $Spin(5) \times Spin(5)$, for $n = 6$ $S = 8 = (1000)$ of $USp(8)$, and for $n = 7$ $S = 8 \oplus \overline{8} = (100000) \oplus (000001)$ of $SU(8)$. These modules may also come together with some $R$-symmetry. In fact, some non-trivial $R$-symmetry must be present in $n = 6, 7$. This is because the momentum module $27 = (0100)$ or $28 \oplus 2\overline{8} = (010000) \oplus (000010)$ only comes in the antisymmetric product $\wedge^2 S$. So in those cases, there should be at least an $SU(2)$ or $SL(2)$ R-symmetry providing a two-index $\varepsilon$ tensor. What we here call $R$-symmetry is of course a Lorentz symmetry, from the usual perspective of compactification. It is not possible to have “chiral” spinors in $n = 5$, since the momenta are in $16 \to (4,4)$. This means that minimal supersymmetry implies 4, 8, 16 and 32 supersymmetries for $n = 4, 5, 6$ and 7, respectively. There may also be restrictions connected to the real form of the U-duality group and the local subgroup. For $n = 4$ and compact $Spin(5)$, $S = 4$ is not real. We will nevertheless construct the minimal supermultiplets based on $R_2$, $R_3$ and $R_4$ for $n = 4$, on $R_2$ and $R_3$ for $n = 5$ and on $R_2$ for $n = 6$. For $n = 7$, minimal supersymmetry will imply 32 supercharges, unless one may build a model with “chiral” spinors.

We always assume flat backgrounds. The coupling to some non-trivial generalised geometry should be straightforward, along the lines of ref. [36]. The derivatives used to construct field strengths from gauge potentials and equations of motion from dual field strengths are free of connection.
2.1. \( n = 4, R_2 \)

We take the bosonic field to be a potential \( A_m \) in \( R_2 = \mathbb{R} \) of \( SL(5) \) and the fermion to be a spinor \( \chi^\alpha \) in \( 4 \) of \( Spin(5) \). In view of Table 2, this should be enough to match the 2 on-shell degrees of freedom of a spinor. One auxiliary scalar should help supersymmetry close off-shell. The potential has a field strength in \( R_1 = 10 \) of \( G \):

\[
F_{mnp} = \frac{3}{4} \partial [ A^m A^n A^p ] .
\]

The gauge transformations with \( \Lambda \) in \( R_3 = \mathbb{R} \) are

\[
\delta_A = \partial [ A^m A^n A^p ] .
\]

The supersymmetry transformations are

\[
\begin{align*}
\delta_\varepsilon A_m &= (\varepsilon \gamma_m \chi) , \\
\delta_\varepsilon \chi^\alpha &= \frac{1}{4} F_{mnp} (\gamma^{mnp} \varepsilon)^\alpha .
\end{align*}
\]

We will check the supersymmetry algebra. The general symmetric Fierz identity is

\[
A_(\alpha B^\beta) = -\frac{1}{8} (\gamma_{ab})^{\alpha \beta} (A_{\gamma^{ab}} B),
\]

which simplifies Fierz rearrangements. Commuting two supersymmetry transformations on \( A_m \) gives

\[
[\delta_\varepsilon, \delta_\varepsilon^\prime] A_m = \frac{1}{4} (\varepsilon \gamma_m \gamma^{npq} \varepsilon') F_{npq} - (\varepsilon \leftrightarrow \varepsilon')
\]

\[
= (\varepsilon \gamma^{npq} \varepsilon') \partial_{np} A_m + 2 \partial_{mn} [ (\varepsilon \gamma^{np} \varepsilon') A_p ] ,
\]

\[
\text{(2.1)}
\]

\[
\text{i.e., a translation and a gauge transformation.}
\]

Acting on the fermion, one gets

\[
[\delta_\varepsilon, \delta_\varepsilon^\prime] \chi^\alpha = \frac{1}{4} (\gamma^{mnp} \varepsilon') (\gamma^m \partial_{np} \chi) - (\varepsilon \leftrightarrow \varepsilon')
\]

\[
= \frac{1}{8} (\varepsilon \gamma^{rs} \varepsilon') (\gamma^{mnp} \gamma^{rs} \gamma^m \partial_{np} \chi)^\alpha
\]

\[
= (\varepsilon \gamma^{np} \varepsilon') \partial_{np} \chi^\alpha + \frac{1}{8} (\varepsilon \gamma^{rs} \varepsilon') (\gamma^{mnp} \gamma^{rs} \partial_{np} \chi)^\alpha .
\]

\[
\text{(2.2)}
\]

In the last step, we have used the identity \( \gamma^{mnp} \gamma^{rs} \gamma^m + \gamma^{rs} \gamma^{np} = -8 \delta^{np}_{rs} \), which happens to be valid in five dimensions. The first term on the last line is the translation, with the same coefficient as on \( A_m \), and the second one an equation of motion.

Taking the supersymmetry variation on the fermion equation of motion should give the one for the bosons.

\[
\delta_\varepsilon (\gamma^{mn} \partial_{mn} \chi)^\alpha = \frac{1}{8} (\gamma^{mnpq} \varepsilon') \partial_{mn} F_{npq}
\]

\[
= - (\gamma^m) \partial^{np} F_{mnp} - (\gamma^{npq} \varepsilon') \partial_{np} F_{mpq} + \frac{1}{16} (\gamma^{mnpq} \varepsilon') \partial_{mn} F_{npq}
\]

\[
= - (\gamma^m) \partial^{np} F_{mnp} - (\gamma^{npq} \varepsilon') \partial_{np} F_{mpq} + \frac{1}{16} (\gamma^{mnpq} \varepsilon') \partial_{mn} \partial_{pq} A_r .
\]

\[
\text{(2.4)}
\]
So, no “extra” conditions are produced, except for the equation of motion $\partial^{np} F_{mnp} = 0$, if the section condition (the last two terms) is fulfilled.

One can indeed introduce a single scalar auxiliary field $H$, with the supersymmetry transformations

$$
\delta_\varepsilon A_m = (\varepsilon \gamma_m \chi), \\
\delta_\varepsilon \chi^\alpha = \frac{1}{6} F_{mnp} (\gamma^{mnp} \varepsilon)^\alpha + H \varepsilon^\alpha, \\
\delta_\varepsilon H = \frac{1}{2} (\varepsilon \gamma^{mn} \partial_{mn} \chi).
$$

This cancels the equation of motion term in eq. (2.4), does not affect the commutator acting on $A_m$ and gives the correct algebra on $H$. The closure on $H$ is provided by

$$
[\delta_\varepsilon', \delta_\varepsilon] H = \frac{1}{2} (\varepsilon \gamma^{mn} \varepsilon') \partial_{mn} H + \frac{1}{12} (\varepsilon \gamma^{mn} \gamma^{pqr} \varepsilon') \partial_{mn} F_{pqr} - (\varepsilon \leftrightarrow \varepsilon')
$$

$$
= (\varepsilon \gamma^{mn} \varepsilon') \partial_{mn} H + (\varepsilon \gamma^{mnp} \varepsilon') \partial_m F_{pqn}.
$$

(2.6)

The expression $\partial_{[m} F_{p]n}$ vanishes due to the section condition. The equations of motion follow by supersymmetry from $H = 0$.

2.2. $n = 4$, $R_3$

Consider a potential $B_m$ in $R_3 = 5$, with field strength $G_m = \partial_{mn} B^n$ in $R_2 = \mathbf{5}$ and gauge transformation $\delta_\Lambda B_m = \partial_{np} \Lambda^{[mnp]}$ with $\Lambda$ in $R_4 = \mathbf{10}$. In addition to a fermion $\chi^\alpha$, one needs a physical scalar field $\phi$, and for off-shell supersymmetry an auxiliary field $H$. The transformations are

$$
\delta_\varepsilon B^m = (\varepsilon \gamma^m \chi), \\
\delta_\varepsilon \phi = (\varepsilon \chi), \\
\delta_\varepsilon \chi^\alpha = -G_m (\gamma^m \varepsilon)^\alpha + \frac{1}{2} \partial_{mn} \phi (\gamma^{mnp} \varepsilon)^\alpha + H \varepsilon^\alpha, \\
\delta_\varepsilon H = \frac{1}{2} (\varepsilon \gamma^{mn} \partial_{mn} \chi).
$$

(2.7)

The commutator on $\chi^\alpha$ gives

$$
[\delta_\varepsilon', \delta_\varepsilon] \chi^\alpha = (\varepsilon \gamma^{pq} \varepsilon') \left(\left(\frac{1}{2} \gamma^m \gamma_{pq} \gamma^n - \frac{1}{8} \gamma^{mnp} \gamma_{pq} - \frac{1}{8} \gamma_{pq} \gamma^{mn}\right) \partial_{mn} \chi\right)^\alpha \\
= (\varepsilon \gamma^{mn} \varepsilon') \partial_{mn} \chi^\alpha.
$$

(2.8)

and acting on $B^m$ one gets

$$
[\delta_\varepsilon', \delta_\varepsilon] B^m = -(\varepsilon \gamma^m \gamma^n \varepsilon') \partial_{np} B^p + \frac{1}{2} (\varepsilon \gamma^m \gamma^{np} \varepsilon') \partial_{np} \phi + (\varepsilon \gamma^m \varepsilon') H - (\varepsilon \leftrightarrow \varepsilon')
$$

$$
= (\varepsilon \gamma^{np} \varepsilon') \partial_{np} B^m + \partial_{np} \left[ -3(\varepsilon \gamma^{[mn]} \varepsilon') B^p + (\varepsilon \gamma^{mnp} \varepsilon') \phi \right].
$$

(2.9)
The commutators on $\phi$ and $H$ also work, the latter thanks to the section condition, giving
the Bianchi identity $\partial_{[mn}G_{p]} = 0$.

All equations of motion for the physical fields are generated by supersymmetry from $H = 0$, and are, as expected:

$$\partial_{mn}G^n = 0,$$
$$\partial^{mn}\partial_{mn}\phi = 0,$$
$$\left(\gamma^{mn}\partial_{mn}\chi\right)^\alpha = 0.$$  \hfill (2.10)

2.3. $n = 4$, $R_4$

The last example for $n = 4$ is a potential $C_{mn}$ in $R_4 = \mathbf{10}$ with a field strength $H_{mnpq} = 6\partial_{[mn}C_{pq]}$ in $R_3 = \mathbf{5}$. Gauge transformations leaving $H$ invariant are $\delta_{\Lambda}C_{mn} = \partial_{(m|[p|}(\Lambda_{n])^{p]}$, with $\Lambda$ in $1 \oplus 24$. We also know that this action of a derivative will contain connection, which is no problem in flat space, but still somewhat confusing (see ref. [36] for comments). Until this is cleared out, a covariant formulation of gauge transformations in a non-trivial
generalised gravity background may be problematic. A scalar $\phi$ is also needed.

Let the fields transform according to

$$\delta_{\varepsilon}C_{mn} = (\varepsilon\gamma_{mn}\chi),$$
$$\delta_{\varepsilon}\phi = (\varepsilon\chi),$$
$$\delta_{\varepsilon}\chi^\alpha = -\frac{1}{24}H_{mnpq}(\gamma^{mnpq}\varepsilon)^\alpha + \frac{1}{2}\partial_{mn}\phi(\gamma^{mn}\varepsilon)^\alpha.$$  \hfill (2.11)

This gives a commutator of supersymmetries on $\chi$:

$$[\delta_{\varepsilon'}, \delta_{\varepsilon}]\chi^\alpha = (\varepsilon\gamma^{pq}\varepsilon')(\left(\frac{1}{10}\gamma^{mnrn}\gamma_{pq}\gamma_{rs} - \frac{1}{8}\gamma^{mnpq}\partial_{mn}\chi\right)^\alpha
= (\varepsilon\gamma^{mn}\varepsilon')\partial_{mn}\chi^\alpha.$$  \hfill (2.12)

This is an off-shell multiplet, without the inclusion of auxiliary fields. It is straightforwardly
checked that the commutators on the bosonic fields work out, e.g.:

$$[\delta_{\varepsilon'}, \delta_{\varepsilon}]C_{mn} = (\varepsilon\gamma^{pq}\varepsilon')(\partial_{pq}C_{mn}
+ \partial_{[m[p|}(\delta_{\varepsilon})C_{qr} - 4(\varepsilon\gamma^{pq}\varepsilon')C_{n]q} - 4(\varepsilon\gamma_{n]}p\varepsilon')\phi].$$  \hfill (2.13)

Assuming the equation of motion for $\chi$ to be the same as above, we can apply a supersymmetry transformation, and get the equation of motion for $C$, $\partial^{pq}H_{mnpq} = 0$, as expected.
2.4. \( n = 5, R_2 \)

In \( n = 5 \), consider a potential \( A_m \) in \( R_2 = 10 \) with a field strength \( F^A \) in \( R_1 = 16 \) and gauge parameter in \( R_3 = \mathbf{16} \). The potential splits into \( (5, 1) \oplus (1, 5) \) under \( H = SO(5) \times SO(5) \), while \( 16, \mathbf{16} \to (4, 4) \). The supersymmetry parameters \( \varepsilon, \bar{\varepsilon} \) are in \( S = (4, 1) \oplus (1, 4) \). In addition to the gauge potential, there will be spinors \( \chi, \bar{\chi} \) in \( S \) and a scalar \( \phi \). We know that \( R_2 \) has 3 degrees of freedom on-shell, which together with the scalar matches the \( \frac{1}{2} \times 8 = 4 \) fermions. Off-shell supersymmetry demands 2 auxiliary fields \( H, \bar{H} \).

The supersymmetry transformations are

\[
\begin{align*}
\delta_{\varepsilon, \bar{\varepsilon}} A_a &= (\varepsilon \sigma_a \chi), \quad \delta_{\varepsilon, \bar{\varepsilon}} \bar{A_a} = (\bar{\varepsilon} \sigma_a \bar{\chi}), \\
\delta_{\varepsilon, \bar{\varepsilon}} \phi &= (\varepsilon \chi) + (\bar{\varepsilon} \bar{\chi}), \\
\delta_{\varepsilon, \bar{\varepsilon}} \alpha^a &= ((F + \partial \phi) \varepsilon)^a + \varepsilon^a H, \quad \delta_{\varepsilon, \bar{\varepsilon}} \bar{\alpha}^{\bar{a}} = ((-F^t + \partial^t \phi) \bar{\varepsilon})^{\bar{a}} + \bar{\varepsilon}^{\bar{a}} \bar{H}
\end{align*}
\]

\( \delta_{\varepsilon, \bar{\varepsilon}} H = 2(\bar{\varepsilon} \partial^t \chi), \quad \delta_{\varepsilon, \bar{\varepsilon}} \bar{H} = 2(\varepsilon \partial \bar{\chi}) \),

where \( F \) and \( \partial \) are matrices in \( (4, 4) \) and \( F^A = (\gamma^m \partial)^A A_m \) gives \( F_{a\bar{a}} = (\sigma^a \partial)_{a\bar{a}} A_a - (\partial \sigma^a)_{a\bar{a}} A_{\bar{a}} \). It is straightforward to derive the commutation relations

\([\delta_{\varepsilon, \bar{\varepsilon}}, \delta_{\varepsilon', \bar{\varepsilon}'}] = 2\xi^A \partial A + \ldots\]

where the translation parameter is \( \xi = \varepsilon \otimes \bar{\varepsilon}' - \varepsilon' \otimes \bar{\varepsilon} \) and the ellipsis denotes a gauge transformation \( \delta A_m = (\partial \gamma^m \Lambda) \) with parameter\(^1 \) \( \Lambda_A = - (\gamma^m \xi)_{A m} - \xi_A \phi \).

2.5. \( n = 5, R_3 \)

There should also be a multiplet with potential in \( R_3 = \mathbf{16} \) and field strength in \( R_2 = 10 \). The number of off-shell degrees of freedom in \( R_3 \) is 5, and reduces on-shell to 2. The multiplet turns out to contain two scalars, and can be taken off-shell by the introduction of a single auxiliary field. The transformations are

\[
\begin{align*}
\delta_{\varepsilon, \bar{\varepsilon}} A &= \varepsilon \otimes \tilde{\chi} + \chi \otimes \tilde{\varepsilon}, \\
\delta_{\varepsilon, \bar{\varepsilon}} \phi &= 2(\varepsilon \chi), \quad \delta_{\varepsilon, \bar{\varepsilon}} \tilde{\phi} = 2(\bar{\varepsilon} \tilde{\chi}), \\
\delta_{\varepsilon, \bar{\varepsilon}} \alpha^a &= F_a(\sigma^a \varepsilon)^a + (\partial \phi \varepsilon)^a + H \varepsilon^a, \\
\delta_{\varepsilon, \bar{\varepsilon}} \bar{\alpha}^{\bar{a}} &= \bar{F}_{\bar{a}}(\sigma^{\bar{a}} \bar{\varepsilon})^{\bar{a}} + (\partial^t \phi \bar{\varepsilon})^{\bar{a}} + \bar{H} \bar{\varepsilon}^{\bar{a}}, \\
\delta_{\varepsilon, \bar{\varepsilon}} H &= (\varepsilon \partial \tilde{\chi}) + (\bar{\varepsilon} \partial \chi).
\end{align*}
\]

\(^1 \) This expression is not \( SO(10) \) covariant, but there will be a metric converting \( 16 \) to \( \mathbf{16} \).
2.6. $n = 6$, $R_2$

For $n = 6$, we only consider the $R_2$ module. One has a potential $A$ in $R_2 = \mathbf{27}$ with a field strength in $R_1 = \mathbf{27}$. When $E_6 \rightarrow USp(8)$, $\mathbf{27} \rightarrow \mathbf{27}$, $\mathbf{27} \rightarrow \mathbf{27}$, which is an $\varepsilon$-traceless antisymmetric tensor. As mentioned in section 1, at least an $SU(2)$ $R$-symmetry is needed, and the fermions come in $(8, 2)$ under $USp(8) \times SU(2)$. The number of off-shell degrees of freedom in $R_2$ is 10, and on-shell 5 (it contains effectively a six-dimensional 1-form and a 4-form, dual to a scalar). It seems reasonable to expect that $R_2$ is accompanied by an $SU(2)$ triplet of scalars, so that the 8 on-shell bosonic degrees of freedom match the fermionic ones. This would correspond to the degrees of freedom of $N = (1, 1)$ SYM in six dimensions, where the $SO(4)$ $R$-symmetry is broken to $SU(2)$ by the dualisation of one scalar.

The transformations are

\[ \delta \varepsilon A_{ab} = 4 \varepsilon_{ij} \varepsilon_{ab}^i \chi^j, \]
\[ \delta \varepsilon \chi^i_a = \varepsilon^{bc} (F_{ac} \varepsilon^i_e + \varepsilon_{jk} \partial_{ab} \phi^{ij} \varepsilon^k_e), \]
\[ \delta \varepsilon \phi^{ij} = 2 \varepsilon^{ab} \varepsilon^{(i} \chi_j^{j)}, \] (2.16)

where $F_{ab} = 2 \varepsilon^{cd} \partial_{[ab]} A_{cd}$ and where $X_{[ab]} = X_{[ab]} - \frac{1}{8} \varepsilon_{abc} \varepsilon^{cd} X_{cd}$ denotes $\varepsilon$-traceless antisymmetrisation. The supersymmetry transformations commute to

\[ [\delta \varepsilon^1, \delta \varepsilon^2] = -\varepsilon_{ij} \varepsilon^{ab} \varepsilon^{cd} \varepsilon^i_d \varepsilon^j_e \partial_{bd}. \] (2.17)

modulo gauge transformations and equations of motion.

3. Summary and Outlook

We have constructed a number of non-gravitational supermultiplets, which in the present formulation enjoy global U-duality. Coupling to a generalised geometric background will be straightforward, along the lines of ref. [36].

Let us analyse the effective physical content of the multiplets. The bosonic fields are listed in Table 3. The scalars appearing in addition to the fields from $R_k$ are given in brackets. Fields without local degrees of freedom ($(n-1)$-forms and $n$-forms) are not listed. Notice how dualisation of some fields makes manifest U-duality possible. Take for example the $n = 6$ multiplet. The physical content agrees with $N = (1, 1)$ super-Yang–Mills theory in six dimensions, which has an $SU(2) \times SU(2)$ $R$-symmetry, with the scalars transforming as $(2, 2)$. By only considering the diagonal subgroup, $(2, 2) \rightarrow 1 \oplus 3$, the singlet can be
dualised into a 4-form, which together with the vector (and an unphysical vector) build up the $R_2$ module.

$$
\begin{array}{ccc}
 n & R_2 & R_3 \\
 4 & \text{vector} & \text{scalar + [ scalar ]} & \text{2-form + [ scalar ]} \\
 5 & \text{vector + [ scalar ]} & \text{scalar + 3-form + [ 2 scalars ]} \\
 6 & \text{vector + 4-form + [ 3 scalars ]} & \\
\end{array}
$$

Table 3: The effective content of physical bosonic fields in the multiplets.

As mentioned in Section 2, we have not put emphasis on the real forms of the U-duality groups and their locally realised subgroups. The reality properties of the fields in the multiplets will depend on the choice of real form.

All the multiples we have described are minimal. Extended non-gravitational supermultiplets may be obtained by dimensional reduction from higher to lower $n$. The general branching pattern for the modules under consideration here is $R_{k}^{(n)} \to R_{k}^{(n-1)} \oplus R_{k+1}^{(n-1)}$. We have not touched on the issue of multiplets containing generalised gravity. The minimal exceptional gravity multiplets were described in ref. [33], and the content of the maximal supermultiplets was sketched in ref. [36]. A detailed investigation, including a superspace formulation would be interesting, and might lead towards a formalism, generalising that of refs. [40,41], where U-duality and supersymmetry are simultaneously manifested.

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