THE SMALLEST MATROIDS WITH NO LARGE INDEPENDENT FLAT

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Abstract. We show that a simple rank-r matroid with no \((t+1)\)-element independent flat has at least as many elements as the matroid \(M_{r,t}\) defined to be the direct sum of \(t\) binary projective geometries whose ranks pairwise differ by at most 1. We also show for \(r \geq 2t\) that \(M_{r,t}\) is the unique example for which equality holds.

1. Introduction

Call a set \(S\) in a matroid \(M\) a claw of \(M\) if \(S\) is both a flat and an independent set of \(M\). A \(k\)-claw is a claw of size \(k\). These objects were introduced by Bonamy et al. [2] and studied by Nelson and Nomoto [3]: both of these papers consider the structure of 3-claw-free binary matroids.

Here we deal with general matroids, and address the simple extremal question of determining the smallest simple rank-\(r\) matroids omitting a given claw; we solve this problem and characterize the tight examples.

Theorem 1.8 of [3] shows that, for \(r \geq 4\), the unique smallest simple rank-\(r\) binary matroid with no 3-claw is the direct sum of two binary projective geometries of ranks \(\lfloor r/2 \rfloor\) and \(\lceil r/2 \rceil\). We show that, perhaps surprisingly, the exact same construction is also extremal for general matroids, and that its natural generalization is still extremal for excluding larger claws. For integers \(r \geq 1\) and \(t \geq 1\), let \(M_{r,t}\) denote the matroid that is the direct sum of \(t\) (possibly empty) binary projective geometries, whose ranks sum to \(r\) and pairwise differ by at most 1. We prove the following, which was conjectured for the special case of binary matroids in [3].

**Theorem 1.1.** Let \(r, t \geq 1\) be integers. If \(M\) is a simple rank-\(r\) matroid with no \((t+1)\)-claw, then \(|M| \geq |M_{r,t}|\). If equality holds and \(r \geq 2t\), then \(M \cong M_{r,t}\).

Note that for \(r \leq t\), the matroid \(M_{r,t}\) is free and therefore the theorem is trivial. For \(t < r < 2t\), there is a rather tame family of exceptional tight examples, which we describe in Theorem 3.3. One can also ask a similar question with ‘simple’ relaxed to ‘loopless’. (One must still
insist that $M$ is loopless, since any matroid with a loop has no claw.)
In this case the answer is much less interesting; the direct sum of $r - t$
parallel pairs and $t$ coloops has $2r - t$ elements and is the unique smallest
loopless rank-$r$ matroid with no $(t + 1)$-claw, this is a consequence of
Lemma 3.2 below.

**Graph Theory.** The study of the structure of 3-claw-free binary ma-
troids in [28] was motivated by structural results in graph theory. In
the context of these works, the graph-theoretic notions of induced sub-
graphs, cliques, chromatic number and forests have analogies in the
setting of simple binary matroids: cliques are analogous to projective
geometries in the sense of being maximal with a given rank, while
claws correspond to induced forests. A graph-theoretic analogue of
Theorem 1.1 using this correspondence would characterize graphs on
$r$ vertices with minimum number of edges and no induced forests of
given size. From the matroidal point of view, the natural measure of
the size of a forest is the number of edges, but there seem to exist no
direct analogue of Theorem 1.1 using this measure. Defining the size
of a forest as the number of its vertices works much better, as follows.

Let $G_{n,t}$ denote the graph on $n$ vertices which is a disjoint union of $t$
complete subgraphs, whose sizes pairwise differ by at most one. Turán’s
classical theorem [3] is equivalent to the statement that $|E(G)| \geq
|E(G_{n,t})|$ for every graph $G$ on $n$ vertices with no stable set of size
$t + 1$. This observation implies that the following graph-theoretic ana-
logue of Theorem 1.1 generalizes Turán’s theorem for $n \geq 3t$.

**Theorem 1.2.** Let $n,t \geq 1$ be integers such that $n \geq 3t$. If $G$ is a
graph on $n$ vertices having no forest on $2t + 1$ vertices as an induced
subgraph, then $|E(G)| \geq |E(G_{n,t})|$ and the equality holds only if $G$ If
equality holds then $G \cong G_{n,t}$.

We give a short proof of Theorem 1.2 obtained by adapting one of
the standard proofs of Turán’s theorem, in Section 4.

**Triangle-free matroids.** The extremal examples in Theorem 1.1
have many triangles, and our proof techniques analyze triangles closely. It
seems plausible that if $M$ is required to be triangle-free, then the spars-
est examples, instead of projective geometries, come from binary affine
geometries, which are triangle-free and have 2-claws but no 3-claws.
(An affine geometry $AG(r - 1, 2)$ is obtained from a projective geom-
try $PG(r - 1, 2)$ by deleting a hyperplane.) This leads us to conjecture
the following.

**Conjecture 1.3.** Let $t,r$ be integers with $t \geq 1$ and $t|r$. If $M$ is a
simple triangle-free matroid with no $(2t + 1)$-claw, then $|M| \geq t^{2r/t-1}$. 
This conjectured bound holds with equality when \( M \) is the direct sum of \( t \) copies of a rank-\((r/t)\) binary affine geometry; these should be the only cases where equality holds. We prove this in the easy case where \( t = 1 \); see Lemma 3.3.

In what follows, we use the notation of Oxley [4]: flats of a matroid of rank 1 and 2 are points and lines respectively. We additionally write \(|M|\) for \( E(M)\). A simplification of \( M \) is any matroid obtained from \( M \) by deleting all loops and all but one element from each parallel class. All such matroids are clearly isomorphic; we write \( si(M) \) for a generic matroid isomorphic to a simplification of \( M \), and write \( \varepsilon(M) \) for \(|si(M)|\), the number of points of \( M \). A family \( \mathcal{X} \) of sets are skew in a matroid \( M \) if \( r_M(\bigcup \mathcal{X}) = \sum_{X \in \mathcal{X}} r_M(X) \), and we say that a set \( X \) is skew to a set \( Y \) if \( \{X, Y\} \) is a skew family: i.e. \( r_M(X \cup Y) = r_M(X) + r_M(Y) \).

2. The Bound

In this section we give the easy proof of the lower bound in Theorem 1.1. Our first lemma shows that the property of being \((r,t)\) claw-free is essentially closed under contraction; if \( F \) is a \( k\)-claw of some simplification of \( M \), call \( F \) a \( k\)-pseudoclaw of \( M \).

**Lemma 2.1.** Let \( k \geq 1 \). If \( M \) is a simple matroid and \( X \subseteq E(M) \), then every \( k\)-pseudoclaw of \( M/X \) is a \( k\)-claw of \( M \).

**Proof.** Let \( M' = (M/X) \setminus P \) be a simplification of \( M/X \), and suppose that \( M' \) has a \( k\)-claw \( F \). Since \( F \) is independent in \( M/X \), it is independent in \( M \), and is skew to \( X \) in \( M \). Since \( F \) is a flat of \( M/X \), we have \( \emptyset = cl_{M'}(F) - F = cl_M(X \cup F) - (F \cup X \cup P) \supseteq (cl_M(F) - F) - (X \cup P) \), giving \( cl_M(F) - F \subseteq X \cup P \).

The sets \( cl_M(F) \) and \( X \) are skew in \( M \), so \( cl_M(F) - F \subseteq P \). Suppose that \( e \in (cl_M(F) - F) \cap P \). Then there exists \( e' \in E(M') \) parallel to \( e \) in \( M/X \); since \( e' \in cl_M(F) - X \subseteq cl_{M/X}(F) \) we also have \( e \in cl_{M/X}(F) \) and so \( e \in cl_{M'}(F) = F \), contrary to the choice of \( e \). It follows that \( cl_M(F) - F \) intersects neither \( X \) nor \( P \) so is empty; therefore \( F \) is a \( k\)-claw of \( M \). \( \square \)

Let \( f(r,t) = |M_{r,t}| \). Since a rank-\( n \) projective geometry has \( 2^n - 1 \) elements, we clearly have \( f(r,t) = (t-a)2^{\lfloor r/t \rfloor} + a2^{\lceil r/t \rceil} - t \), where \( a \in \{0, \ldots, t-1\} \) is the integer with \( a \equiv r \) (mod \( t \)). More importantly for our purposes, we can define \( f \) recursively; it is easy to check that \( f(r,t) = r \) for all \( 0 \leq r \leq t \) and \( f(r,t) = 2f(r-t,t) + t \) for \( r > t \). We use this recurrence and the previous lemma to prove the lower bound in our main theorem.
**Theorem 2.2.** If \( t \geq 1 \) is an integer and \( M \) is a simple rank-\( r \) matroid with no \((t+1)\)-claw, then \( |M| \geq f(r, t) \).

**Proof.** Let \( M \) be a counterexample for which \( r + |M| \) is minimized. If \( M \) is a free matroid then clearly \( r \leq t \), in which case \( f(r, t) = r \leq |M| \) so \( M \) is not a counterexample. Therefore \( M \) has a non-coloop \( e \).

Since \( |M\setminus e| < |M| \leq f(r, t) \) but \( M\setminus e \) is not a counterexample, there must be a \((t+1)\)-claw \( S' \) in \( M\setminus e \). Now the matroid \( M/\text{cl}_M(S') \) has rank \( t+1 \) and has at most \( t+2 \) elements, so has at most one circuit. Therefore \( M \) has a \((t+1)\)-claw, a contradiction.

If there is some rank-\((t+1)\) flat \( F \) containing \( S \) for which \( |F - S| = 1 \), then \( F \) is a \((t+1)\)-claw. Therefore every such flat satisfies \( |F - S| \geq 2 \); since \( S \) is a flat, it follows that every parallel class of \( M/S \) has size at least 2. Moreover, \( \text{si}(M/S) \) is a rank-\((r-t)\) matroid that by Lemma 2.1 has no \((t+1)\)-claw; inductively we have \( |\text{si}(M/S)| \geq f(r - t, t) \). Now

\[
|M| = |S| + |M/S| \geq t + 2|\text{si}(M/S)| \geq t + 2f(r - t, t) = f(r, t),
\]
as required.

\( \square \)

### 3. Equality

In this section we characterize matroids for which the bound in Theorem 2.2 holds with equality. We require two lemmas: the first (which uses Tutte’s characterization of binary matroids as those with no \( U_{2,4}\)-minor [6]) corresponds to the case \( t = 1 \) of Theorem 2.2.

**Lemma 3.1.** If \( M \) is a simple rank-\( r \) matroid with no \( 2 \)-claw, then \( |M| \geq 2^r - 1 \). If equality holds then \( M \cong \text{PG}(r - 1, 2) \).

**Proof.** Let \( M \) be a minor-minimal counterexample. Clearly \( r(M) \geq 3 \).

Let \( e \in E(M) \) and let \( H \) be a hyperplane of \( M \) not containing \( e \). Since \( M|H \) has no \( 2 \)-claw, we have \( |H| \geq 2^{r-1} - 1 \) by the minimality of \( M \).

For each \( x \in H \), the line spanned by \( x \) and \( e \) contains an element of \( E(M) - \{e, x\} \), and these lines pairwise intersect only in \( e \), so we see that \( |M| \geq 2|H| + 1 \geq 2^r - 1 \) as required.

If \( |M| = 2^r - 1 \), then equality holds above, so \( |H| = 2^{r-1} - 1 \) and thus \( M|H \cong \text{PG}(r - 2, 2) \). Moreover, for each \( x \in H \) we have \( |\text{cl}_M(\{e, x\})| = 3 \) and \( E(M) = \bigcup_{x \in H} \text{cl}_M(\{e, x\}) \), which implies that \( \text{si}(M/e) \cong M|H \cong \text{PG}(r - 2, 2) \) so \( M/e \) is binary. The choice of \( e \) was arbitrary, so \( M/e \) is binary for all \( e \); since \( r \geq 3 \) this gives that \( M \) has no \( U_{2,4}\)-minor so is binary. Since \( M \) is simple with \( 2^r - 1 \) elements, this implies \( M \cong \text{PG}(r - 1, 2) \), a contradiction.  

\( \square \)
Note that if \( t < r < 2t \) then \( |M_{r,t}| = 2r - t \). In this range, the matroids \( M_{r,t} \) are not the only ones satisfying the bound in Theorem 2.2 with equality. The other examples include direct sums of circuits and coloops, and the matroid \( M_{r,t} \) is the special case where all these circuits are triangles. The following lemma shows that these are the only examples. It also implies the characterization of the smallest \((t + 1)\)-claw-free matroids that are not required to be simple that was claimed in the introduction.

**Lemma 3.2.** Let \( r \geq t \geq 1 \) be integers. If \( M \) is a loopless rank-\( r \) matroid with no \((t + 1)\)-claw, then \( |M| \geq 2r - t \). If equality holds, then \( M \) is the direct sum of \( r - t \) circuits and some number of coloops.

**Proof.** Suppose first that every circuit of \( M^* \) has at most two elements. Then, since \( M^* \) is coloopless, it is the direct sum of loops and parallel classes of size at least two; let \( \mathcal{P} \) be its set of parallel classes, so \( r(M^*) = |\mathcal{P}| \) and \( r = |M| - |\mathcal{P}| \). Let \( U \) be a set comprising exactly two elements from each \( P \in \mathcal{P} \). Since \( r(M^*|U) = r(M^*) \), the set \( (E - U) \) is independent in \( M \), and since \( M^*|U \) is coloopless, the set \( (E - U) \) is also a flat of \( M \), so is a claw of \( M \). By hypothesis, it follows that \( t \geq |E - U| = |M| - 2|\mathcal{P}| \). Therefore \( 2r - t \leq 2(|M| - |\mathcal{P}|) - (|M| - 2|\mathcal{P}|) = |M| \), as required. If equality holds, then \( t = |M| - 2|\mathcal{P}| = r - |\mathcal{P}| \), so \(|\mathcal{P}| = r - t \). By the definition of \( \mathcal{P} \), each \( P \in \mathcal{P} \) is a circuit of \( M \), and each other element of \( M \) is a coloop; this gives the required structure.

We may therefore assume \( M^* \) has a circuit \( C \) of size at least 3. Let \( B \) be a basis of \( M^* \) containing all but one element of \( C \). Since \( M^* \) has no coloops, for each \( x \in B \), there is a circuit \( C_x \) of \( M^* \) for which \( x \in C_x \) and \( |C_x \cap B| = |C_x| - 1 \); choose the \( C_x \) so that \( C_x = C \) for each \( x \in C \). Let \( X = \cup_{x \in C} C_x \). Since each \( C_x \) contains only one element outside \( B \) and the element of \( C_0 - B \) is chosen at least twice, we have \( |X| < 2r(M^*) \).

By construction, the set \( X \) contains a basis and, since \( X \) is a union of circuits of \( M^* \), the matroid \( M^*|X \) has no coloops. Let \( Y = E(M) - X \); by construction the set \( Y \) is independent in \( M \), and \( M/Y \) has no loops, so \( Y \) is a flat, and thus a claw, of \( M \). Hence \( |Y| \leq t \) and so \( |X| \geq |M| - t \). By our upper bound on \( |X| \), this gives \( |M| - t < 2r(M^*) = 2(|M| - r) \) and so \( |M| > 2r - t \), as required. \( \square \)

We are now ready to strengthen Theorem 2.2 with an equality characterisation. Note that both outcomes in the equality case imply that \( M \) has a \( t \)-claw.
Theorem 3.3. Let \( t \geq 1 \). If \( M \) is a simple rank-\( r \) matroid with no \((t+1)\)-claw, then \( |M| \geq f(r,t) \). If equality holds, then either

- \( M \cong M_{r,t} \), or
- \( t < r < 2t \) and \( M \) is the direct sum of coloops and exactly \( r-t \) circuits, not all of which are triangles.

Proof. Consider a counterexample \( M \) for which \( |M| + r \) is minimized. Clearly \( r > t \), as otherwise \( |M| \geq r = f(r,t) \) and there is nothing to prove. Therefore \( M \) is not a free matroid, since otherwise any \((t+1)\)-element subset of \( E(M) \) is a claw.

3.3.1. \( M \) has a \( t \)-claw.

Subproof: Let \( e \) be a non-coloop of \( M \); by the minimality of \( M \), the matroid \( M\setminus e \) has a \((t+1)\)-claw \( F \); now \( M_{\text{cl}}(F) \) has rank at least \( t+1 \) and has at most \( t+2 \) elements, so has at most one circuit. There is thus a \( t \)-element subset of \( \text{cl}_M(F) \) that contains at most \( |C| - 2 \) elements of each circuit \( C \) of \( M_{\text{cl}}(F) \); this set is a \( t \)-claw of \( M \).

Call a \( t \)-claw \( S \) of \( M \) generic if no four-point line of \( M \) intersects \( S \), and exactly \( f(r-t, t) \) triangles of \( M \) intersect \( S \). Let \( S \) be a \( t \)-claw of \( M \), chosen not to be generic if such a choice is possible.

3.3.2. \( |M| = f(r,t) \), each parallel class of \( M/S \) has size 2, and \( \varepsilon(M/S) = f(r-t, t) \).

Subproof: The matroid \( M/S \) has rank \( r-t \) and, by Lemma \([21]\), has no \((t+1)\)-pseudoclaw. Therefore \( \text{si}(M/S) \) has no \((t+1)\)-claw, so \( \varepsilon(M/S) \geq f(r-t, t) \). Moreover, if some parallel class \( Y \) of \( M/S \) has size 1, then \( S \cup Y \) is a \((t+1)\)-claw of \( M \), so every parallel class of \( M/S \) has size at least 2, giving \( |M/S| \geq 2\varepsilon(M/S) \). Therefore

\[
f(r,t) \geq |M| \geq 2\varepsilon(M/S) + |S| \geq 2f(r-t, t) + t = f(r,t).
\]

Equality holds throughout, which gives the claim.

The matroid \( \text{si}(M/S) \) has no \((t+1)\)-claw and has \( f(r-t, t) \) elements, so inductively satisfies one of the conclusions of the theorem. For each component \( N \) of \( M/S \), the matroid \( \text{si}(N) \) is either a circuit or a binary projective geometry.

3.3.3. Let \( e_1, e_2 \in E(M/S) \). If \( e_1 \) and \( e_2 \) are in different components, then \( M/S \) has a \( t \)-pseudoclaw containing \( e_1 \) and \( e_2 \). If \( e_1 \) and \( e_2 \) are in the same component, then there is a \((t-1)\)-element set \( U \) such that \( U \cup \{e_1\} \) and \( U \cup \{e_2\} \) are both \( t \)-pseudoclaws of \( M/S \).
Subproof: We first argue that $M/S$ has a $t$-pseudoclaw. Since $\text{si}(M/S)$ satisfies one of the outcomes of the theorem, this only fails if $\text{si}(M/S) \cong M_{r-t,t}$ and $r - t < t$. If this holds then $|M| = f(r,t) = 2r - t$ and $M$ satisfies the hypothesis of Lemma 3.2 so is the direct sum of coloops and $r - t$ circuits. If these circuits are all triangles then $M \cong M_{r,t}$, and otherwise $M$ satisfies the second outcome of the theorem; both are contrary to the choice of $M$ as a counterexample. Therefore $M/S$ has a $t$-pseudoclaw.

Since every component of $\text{si}(M/S)$ is a circuit or projective geometry, given any pseudoclaw $K$ of $M$ and any $e, e'\in$ the same component of $M/S$ for which $e \in K$ and $e' \notin K$, the set $(K - e) \cup \{e'\}$ is also a pseudoclaw. Since $M/S$ has at least one $t$-pseudoclaw, both conclusions of the claim easily follow. ■

The above claim implies in particular that every element of $M/S$ is in a $t$-pseudoclaw.

3.3.4. For each $t$-pseudoclaw $U$ of $M/S$, there is a bijection $\psi_U$ from $U$ to $S$ so that for each $e \in U$, the flat $T_e = \text{cl}_M(e, \psi_U(e))$ is a triangle of $M$, and so that $M|\text{cl}_M(S \cup U) = \oplus_{e\in U}(M|T_e)$.

Subproof: Since the closure of $U$ in $M/S$ is obtained from $U$ by extending each element of $U$ once in parallel, we have $|\text{cl}_M(S \cup U)| = |S| + 2|U| = 3t$. By Lemma 3.2 it follows that the simple rank-2t matroid $M' = M|\text{cl}_M(S \cup U)$ is the direct sum of $t$ circuits and some set of coloops, and therefore that is precisely the direct sum of $t$ triangles. Since $S$ is a $t$-claw of $M'$ and $U$ is a $t$-pseudoclaw of $M'/S$, both $S$ and $U$ must be transversals of this set of triangles. The claim follows. ■

Every element $e$ of $M/S$ is contained in a $t$-pseudoclaw, so the above claim implies that each such $e$ is in exactly one triangle that intersects $S$. Write $\psi(e)$ for the unique element of $S$ for which $e$ and $\psi(e)$ are contained in a triangle; we have $\psi_U(e) = \psi(e)$ for each $t$-pseudoclaw $U$ of $M/S$ containing $e$.

Since $S$ is a claw and no rank-1 flat of $M/S$ has more than two elements, no line of $M$ that intersects $S$ has more than three elements. Moreover, each $e \in E(M/S)$ is in exactly one triangle of $M$ that intersects $S$, so the number of triangles of $M$ that intersect $S$ is exactly $\frac{1}{2}|M/S| = \frac{1}{2}(2\varepsilon(M/S)) = f(r - t,t)$. Therefore $S$ is generic. It follows from the choice of $S$ that every $t$-claw of $M$ is generic.

3.3.5. For all $e_1, e_2 \in E(M/S)$, we have $\psi(e_1) = \psi(e_2)$ if and only if $e_1$ and $e_2$ are in the same component of $M/S$. Moreover, $M/S$ has exactly $t$ components.
Subproof: Suppose that \( e_1 \) and \( e_2 \) are in the same component of \( M/S \). By 3.3.3 there is a set \( U \subseteq E(M/S) \) such that \( U \cup \{ e_1 \} \) and \( U \cup \{ e_2 \} \) are both \( t \)-pseudoclaws of \( M/S \). For each \( i \in \{1, 2\} \), there is a bijection \( \psi_i = \psi_{U \cup \{ e_i \}} \) from \( U \cup \{ e_i \} \) to \( S \), and moreover for each \( e \in U \) we have \( \psi_1(e) = \psi(e) = \psi_2(e) \). Therefore \( \psi_1 \) and \( \psi_2 \) agree on all \( t - 1 \) elements of \( U \); thus \( \psi_1(e_1) = \psi_1(e_2) \) and so \( \psi(e_1) = \psi(e_2) \).

Suppose now that \( e_1 \) and \( e_2 \) are in different components of \( M/S \). By 3.3.3 there is a \( t \)-pseudoclaw \( U \) containing \( e_1 \) and \( e_2 \). Since \( \psi_U \) is a bijection we have \( \psi(e_1) = \psi_U(e_1) \neq \psi_U(e_2) = \psi(e_2) \), as required.

It follows from the first part that the image of \( \psi \) has size equal to the number of components of \( M/S \). But clearly the image of \( \psi \) contains the image of \( \psi_U \), which is equal to \( S \), for each \( t \)-pseudoclaw \( U \). Therefore \( \psi \) has image \( S \), so \( M/S \) has exactly \( |S| = t \) components.

Let \( \mathcal{N} \) be the set of components of \( M/S \). By 3.3.5 for each \( N \in \mathcal{N} \) there is some \( \psi(N) \) for which \( \psi(e) = \psi(N) \) for each \( e \in E(N) \). Since \( |S| = |\mathcal{N}| = t \), the \( t \)-tuple \( \psi(N) : N \in \mathcal{N} \) is a permutation of \( S \). For each \( N \in \mathcal{N} \), let \( \widehat{N} = M|(E(N) \cup \psi(N)) \).

3.3.6. If \( N \in \mathcal{N} \) and \( L \) is a line of \( M \) intersecting \( E(N) \), then either \( |L| = 2 \), or \( |L| = 3 \) and \( L \subseteq E(\widehat{N}) \).

Subproof: Let \( U \) be a \( t \)-pseudoclaw of \( M/S \) containing an element \( e \in E(N) \cap L \). Note that \( U \) is a generic \( t \)-claw in \( M \), which gives \( |L| \leq 3 \).

Suppose that \( |L| = 3 \). Note that \( 1 = r_{M/S}(e) \leq r_{M/S}(L - S) \leq r_M(L) = 2 \). If \( r_{M/S}(L - S) = 2 \) then \( L \) is a triangle of \( M/S \) that intersects the component \( N \) of \( M/S \), so obviously \( L \subseteq E(N) \). If \( r_{M/S}(L - S) = 1 \) then \( L \subseteq cl_M(S \cup \{ e \}) \subseteq cl_M(S \cup U) \), so 3.3.4 gives \( L = cl_M(\{ e, \psi(e) \}) \). Since \( L - \psi(e) \) is a two-element rank-1 set in \( M/S \), the third element of \( L \) is the element of \( N \) parallel to \( e \), so \( L \subseteq E(\widehat{N}) \) as required.

For each \( N \in \mathcal{N} \) and \( e \in E(N) \), let \( \tau(e) \) be the number of \( 3 \)-element lines of \( M \) containing \( e \), and let \( \sigma(e) \) be the number of elements of \( E(\widehat{N}\setminus e) \) that are not in a \( 3 \)-element line of \( M \) with \( e \). By the previous claim we have \( 2\tau(e) + \sigma(e) = |\widehat{N}\setminus e| = |N| \).

3.3.7. For each \( N \in \mathcal{N} \), every line of \( E(\widehat{N}) \) has size 3.

Subproof: Suppose not, so there is some \( e \in E(N) \) for which \( \sigma(e) > 0 \). Let \( U \) be a \( t \)-pseudoclaw of \( M/S \) containing \( e \). Since \( U \) is a generic
Let $N \in \mathcal{N}$. It is clear, since $N$ is obtained from $\widehat{N} / \psi(N)$ by $t - 1$ successive extension-contraction operations, that $r(N) \leq r(\widehat{N}) - 1$. The matroid $\text{si}(N)$ is a circuit or a binary projective geometry, so

$$|\widehat{N}| = |N| + 1 \leq 2(2^{r(N)} - 1) + 1 = 2^{r(N)} + 1 \leq 2^{r(\widehat{N})} - 1.$$ 

By 3.3.7 and Lemma 3.1 we have $|\widehat{N}| \geq 2^{r(\widehat{N})} - 1$, so equality holds and therefore each matroid $\widehat{N}$ is a binary projective geometry of rank $r(N) + 1$. The sets $E(\widehat{N}) : N \in \mathcal{N}$ partition $E(M)$, so

$$r \leq \sum_{N \in \mathcal{N}} r(\widehat{N}) = \sum_{N \in \mathcal{N}} (r(N) + 1) = r(M/S) + |\mathcal{N}| = t + r(M/S) = r,$$

so equality holds throughout, and the sets $E(\widehat{N})$ are mutually skew in $M$. Thus $M$ is the direct sum of $t$ nonempty binary projective geometries. If $M$ has components of ranks $r_1, r_2$ with $r_2 \geq r_1 + 2$, then deleting both and replacing them with projective geometries of rank $r_2 - 1$ and $r_1 + 1$ respectively gives a matroid $M'$ with no $(t + 1)$-claw satisfying

$$|M| - |M'| = 2^{r_2} + 2^{r_1} - 2^{r_2 - 1} - 2^{r_1 + 1} = 2^{r_2 - 1} - 2^{r_1 + 1} > 0,$$

which contradicts the minimality of $|M|$. It follows that no two components of $M$ have ranks differing by more than 1, so $M \cong M_{r,t}$, contrary to the choice of $M$ as a counterexample. □

Finally, we prove the $t = 1$ case of Conjecture 1.3 as promised.

**Lemma 3.4.** If $M$ is a simple rank-$r$ triangle-free matroid with no 3-claw, then $|M| \geq 2^{r-1}$. If equality holds, then $M \cong AG(r - 1, 2)$.

**Proof.** We may assume that $r \geq 3$. We first show that every triple of distinct elements of $M$ is contained in a four-element circuit; indeed, given such a triple $I$, since $I$ is not a triangle or a 3-claw, we have $r_M(I) = 3$ and $\text{cl}_M(I) \neq I$. Thus there is some $x \in \text{cl}_M(I) - I$. Since
Let $e \in E(M)$. Since $M$ is triangle-free, the matroid $M/e$ is simple. If $M/e$ has a 2-claw $I$, then $I \cup \{e\}$ is clearly a 3-claw of $M$; therefore $M/e$ is 2-claw-free and so $|M/e| \geq 2^{r-1} - 1$ by Lemma 3.1. It follows that $|M| \geq 2^{r-1}$ as required.

If equality holds, then $M/e \cong PG(r - 2, 2)$ so $M/e$ is binary. This holds for arbitrary $e \in E(M)$; it follows (since $r \geq 3$) that $M$ has no $U_{2,4}$-minor so is also binary. A simple rank-$r$ triangle-free binary matroid has at most $2^{r-1}$ elements and equality holds only for binary affine geometries (see [1], for example); therefore $M \cong AG(r-1, 2)$. 

4. Graphs

Let $g(n,t): \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ be defined recursively by

- $g(n,t) = 0$ for $n < 2t$,
- $g(n,t) = 3(n - 2t)$ for $2t \leq n \leq 4t$,
- $g(n,t) = g(n - 1,t) + \left\lceil \frac{n}{3} \right\rceil - 1$, for $n > 4t$.

It is easy to check that $|E(G_{n,t})| = g(n,t)$ for $n \geq 3t$ (although not for smaller $n$). The recursion for $n > 4t$ in fact also holds when $3t < n \leq 4t$. Thus the next theorem implies Theorem 1.2. Like in the matroid setting, there is a range of cases ($2t \leq n < 4t$) with non-unique tight examples, while the tight examples are just the expected ones for larger $n$.

**Theorem 4.1.** Let $n,t \geq 1$ be integers. Let $G$ be a simple graph on $n$ vertices such that no forest on $2t + 1$ vertices is an induced subgraph of $G$. Then $|E(G)| \geq g(n,t)$.

If equality holds and $n < 4t$, then every component of $G$ is a complete graph on 1, 3 or 4 vertices. If equality holds and $n \geq 4t$, then $G \cong G_{n,t}$.

**Proof.** We prove the theorem by induction on $|V(G)|$. We may clearly assume that $n \geq 2t + 1$, as otherwise the result is easy. Let $v$ be a vertex of $G$ of maximum degree.

If $\deg(v) \leq 2$, every component of $G$ is a path or a cycle. Let $S$ be the set of vertices of cycles of $G$, and $b$ be the number of cycles of $G$; note that $b \leq \frac{1}{3}|S| \leq \frac{1}{3}n$. Clearly $G$ contains an induced forest on $n - b$ vertices, so $n - b \leq 2t$. This gives $n \leq 2t + \frac{n}{3}$, so $n \leq 3t$, which in turn implies that $g(n,t) = 3(n - 2t) \leq 3b$. On the other hand, we have $|E(G[S])| = |S|$, so

$$|E(G)| \geq |E(G[S])| = |S| \geq 3b \geq g(n,t),$$

giving the bound. If equality holds, then $E(G) = E(G[S])$ and $b = \frac{1}{3}|S|$, so every component of $G$ is an isolated vertex or triangle. We
have argued that $n \leq 3t$; thus $G$ has the claimed structure. We may therefore assume that $\deg(v) \geq 3$.

Let $X \subseteq V(G)$ be maximal so that $G[X]$ is a forest, so $|X| \leq 2t$. Let $Z$ be the set of non-isolated vertices of $G[X]$. As $G[X \cup \{w\}]$ contains a cycle for every $w \in V(G) \setminus X$, every such $w$ has at least two neighbors in $Z$. Thus

$$\sum_{z \in Z} \deg(z) \geq |Z| + 2|V(G) - X| \geq |Z| + 2(n - 2t).$$

Hence there exists $z_0 \in Z$ such that $\deg(z_0) \geq 2(n - 2t)/|Z| + 1 \geq (n - 2t)/t + 1 = n/t - 1$; thus $\deg(v) \geq \left\lceil \frac{n}{t} \right\rceil - 1$ by the choice of $v$.

By the above, we can assume that $\deg(v) \geq \max(3, \left\lceil \frac{n}{t} \right\rceil - 1)$. Let $H = G - v$. It follows that

$$|E(G)| = |E(H)| + \deg(v) \geq g(n - 1,t) + \max(3, \left\lceil \frac{n}{t} \right\rceil - 1) = g(n,t);$$

the last equality is easy to check. This gives the desired bound.

Suppose now that equality holds. We have $\deg(v) = \max(3, \left\lceil \frac{n}{t} \right\rceil - 1)$, and $|E(H)| = g(n - 1,t)$, so equality also holds for $H$. For each $d \geq 1$, let $n_d$ be the number of components of $H$ having exactly $d$ vertices. Call a component of $H$ with at least two vertices big.

Let $U$ be a set containing all isolated vertices of $H$, and exactly two vertices from each big component of $H$, chosen so that $U$ contains as few neighbors of $v$ as possible. We argue that $|U| = 2t$. If $2t + 1 \leq n \leq 4t$, by induction, each component of $H$ is a complete graph on $1$, $3$ or $4$ vertices. It follows that $n - 1 = n_1 + 3n_3 + 4n_4$ and $3(n - 2t) = g(n,t) = 3n_1 + 6n_4$. This gives $|U| = n_1 + 2(n_3 + n_4) = (n - 1) - \frac{1}{3}(3(n - 2t)) = 2t$.

If $n > 4t$, then $H \cong G_{n-1,t}$, so $H$ has $t$ components, all big; thus $|U| = 2t$.

If each big component of $H$ contains a non-neighbour of $v$, then $U \cup \{v\}$ induces a forest on $2t + 1$ vertices, a contradiction. Therefore $H$ has a big component $C$ with $V(C) \subseteq N(v)$. Induction implies that each big component of $H$ has at least $\max(3, \left\lceil \frac{n}{t} \right\rceil - 1) = \deg(v)$ vertices; it follows that $|V(C)| = \deg(v) = \max(3, \left\lceil \frac{n}{t} \right\rceil - 1)$, and that $G$ is obtained from $H$ by adding a new vertex with neighbourhood $V(C)$.

If $2t + 1 \leq n \leq 4t$, then $|V(C)| = \deg(v) = \max(3, \left\lceil \frac{n}{t} \right\rceil - 1) = 3$, so $G$ is obtained from $H$ by turning a triangle into a $K_4$; thus, every component of $G$ is complete with $1$, $3$ or $4$ vertices. If $n < 4t$ then this implies that $M$ has the claimed structure, and if $n = 4t$ then the fact that $|E(G)| = g(4t,t) = 6t = \frac{3}{2}|V(G)|$ and $G$ has maximum degree $3$ implies that $G$ is $3$-regular, and so $G \cong G_{4t,t}$, as required.

If $n > 4t$, then $H \cong G_{n-1,t}$, then $|V(C)| = \deg(v) = \left\lceil \frac{n}{t} \right\rceil - 1 = \left\lceil \frac{n-1}{t} \right\rceil$, so $C$ is a smallest component of $H$. Thus $G \cong G_{n,t}$, as required.

\[\Box\]
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