THE AVERAGE MORDELL-WEIL RANK OF ELLIPTIC SURFACES OVER NUMBER FIELDS

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Abstract. Let \( K \) be a finitely generated field over \( \mathbb{Q} \). Let \( \mathcal{X} \rightarrow \mathcal{B} \) be a family of elliptic surfaces over \( K \) such that each elliptic fibration has the same configuration of singular fibers. Let \( r \) be the minimum of the Mordell-Weil rank in this family. Then we show that the locus inside \( |\mathcal{B}| \) where the Mordell-Weil rank is at least \( r + 1 \) is a sparse subset.

In this way we prove Cowan’s conjecture on the average Mordell-Weil rank of elliptic surfaces over \( \mathbb{Q} \) and prove a similar result for elliptic surfaces over arbitrary number fields.

1. Introduction

In a recent preprint Alex Cowan \cite{4} formulated the following conjecture on the average rank of elliptic curves over \( \mathbb{Q}(t) \). Let \( \mu \) be the Mahler measure on \( \mathbb{Z}[t] \) and let \( P_d(M) = \{ p \in \mathbb{Z}[t] \mid \deg(p) \leq d, \mu(p) < M \} \). Define \( S_{m,n}(M) \) to be the set

\[
\left\{ E_{A,B} : y^2 = x^3 + A(t)x + B(t) \right\} \quad \left( A \in P_m(M^2), B \in P_n(M^3), 4A(t)^3 + 27B(t)^2 \neq 0 \right).
\]

Conjecture 1.1 (Cowan). For every pair of positive integers \( m, n \) we have

\[
\lim_{M \to \infty} \frac{1}{\# S_{m,n}(M)} \sum_{E \in S_{m,n}(M)} \text{rank} E(\mathbb{Q}(t)) = 0.
\]

Battistoni, Bettin and Delaunay proved this conjecture for \( 1 \leq m, n \leq 2 \) and for certain unirational subfamilies of \( S_{2,2} \), see \cite{2}.

In this paper we will prove Cowan’s conjecture for all \( (m, n) \) with \( m, n \geq 1 \), and generalize this to arbitrary number fields. The main ingredient holds in a more general context: Fix a field \( K \), finitely generated over \( \mathbb{Q} \). Fix a smooth geometrically irreducible base variety \( \mathcal{B}/K \). Let \( \mathcal{X} \rightarrow \mathcal{B} \) be a family of elliptic surfaces with a section, with \( \mathcal{C} \rightarrow \mathcal{B} \) the base curve of the elliptic fibration (cf. Section 2). Let \( \eta \) be the generic point of \( \mathcal{B} \). Then the generic fiber of \( \mathcal{X}_\eta \rightarrow \mathcal{C}_\eta \) is an elliptic curve \( E_\eta/K(\eta)(\mathcal{C}_\eta) \). Similarly, for a closed point \( b \in |\mathcal{B}| \) the generic fiber of \( \mathcal{X}_b \rightarrow \mathcal{C}_b \) is an elliptic curve \( E_b/K(\mathcal{C}_b) \). There is a specialization map

\[
E_\eta(K(\eta)(\mathcal{C}_\eta)) \rightarrow E_b(K(\mathcal{C}_b))
\]

and a second map if one passes to algebraic closures of \( K(\eta) \) and \( K(b) \). We prove the following result on these specialization maps:

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Theorem 1.2. Let $K$ be a finitely generated field over $\mathbb{Q}$, and let $X \to B$ be a family of elliptic surfaces with a section over $K$. Then there exists a sparse set $Z \subset |B|$ such that for all $b \in |B| \setminus Z$ the specialisation maps

$$E_\eta(K(\eta)(C_\eta)) \to E_b(K(b)(C_b))$$

and

$$E_\eta(K(\eta)(C_\eta)) \to E_b(K(b)(C_b))$$

are bijective.

(This result is a combination of Proposition 2.16 and Theorem 2.18. For a definition of thin subsets and of sparse subsets see Section 2. The existence of the first isomorphism relies heavily on [6] Theorem 8.3] and the main result of [1]. We obtain the second isomorphism from the first by using an appropriate form of Hilbert irreducibility theorem.

In the proof of Cowan’s conjecture we need to take for $B$ some Zariski open subset of $S_{m,n}$. Let $X$ be the universal elliptic curve over $B$. In order to prove Cowan’s conjecture it suffices to show that the associated generic Mordell-Weil group $E_\eta(K(\eta)(C_\eta))$ is finite. If $m = 4k$ and $n = 6k$ for some integer $k > 1$ then using Noether-Lefschetz theory [5] one can show that $E_\eta(K(\eta)(C_\eta))$ is trivial. This yields Cowan’s conjecture almost immediately. However, for certain values of $(m,n)$ we were not able to determine $E_\eta(K(\eta)(C_\eta))$ and, moreover, in the range $1 \leq m \leq 4; 1 \leq n \leq 6$ this larger group has positive rank. We will use quadratic twisting to show that for all choices of $m$ and $n$ there is a $b \in |B| \setminus Z$ such that the arithmetic Mordell-Weil group $E_b(K(b)(C_b))$ is finite. More precisely, given a rational point $b \in B(K)$ consider

$$\text{tw}(b) := \left\{ d \in K^*/(K^*)^2 \left| \begin{array}{c} \text{there exists a point } b' \in B(K) \\ \text{such that } X_b/C_b \neq X_{b'}/C_{b'} \\ \text{and } (X_b/C_b)_{K(\sqrt{d})} \cong (X_{b'}/C_{b'})_{K(\sqrt{d})} \end{array} \right. \right\}$$

the set of $d$ such that the quadratic twist of $X_b$ by $\sqrt{d}$ is also contained in the family $X \to B$.

Theorem 1.3. Let $K$ be a finitely generated field over $\mathbb{Q}$ and let $X \to B$ be a family of elliptic surfaces with a section.

Let $b \in B(K)$ be a point such that $\text{tw}(b)$ is infinite and such that the configuration of singular fibers of $X_b \to C_b$ is the same as $X_\eta \to C_\eta$. Then $E_\eta(K(\eta)(C_\eta))$ is finite.

These two results combined show that on the complement of a thin subset of $S_{m,n}$ the Mordell-Weil rank is zero. This is sufficient to prove Cowan’s conjecture:

Corollary 1.4. Let $K$ be a number field. Fix positive integers $m, n$. The set of elliptic surfaces in $S_{m,n}$ with Mordell-Weil rank zero has density one.

Corollary 1.5. Let $K = \mathbb{Q}$. Fix positive integers $m, n$. Then the average rank of elliptic surfaces in $S_{m,n}$ equals zero.

We will also prove a generalization of Cowan’s conjecture to arbitrary number fields, see Corollary 4.13. However, in this case we have to adjust the definition of $S_{m,n}$ in order to exclude trivial elliptic fibrations, i.e., elliptic surfaces which are birational to $E \times \mathbb{P}^1$ as fibered surfaces.

The organization of this paper is as follows. In Section 2 we recall some standard results on families of elliptic surfaces. Moreover, we use one of the results from [6]
to determine the locus in the family where the specialization map on the Mordell-Weil group is injective, and then to describe the locus where this map is bijective. In Section 3 we study the universal Weierstrass equation over $S_{m,n}$ and use the results from the previous section to show that for $K = \mathbb{Q}$ the average rank is zero.

2. Families of elliptic surfaces

Let $K$ be a field.

**Definition 2.1.** An elliptic surface $\pi : X \to C$ over $K$ consists of a geometrically irreducible smooth projective surface $X/K$, a geometrically irreducible smooth projective curve $C/K$ and a flat $K$-morphism $\pi : X \to C$ such that the generic fiber of $\pi$ is a smooth projective curve of genus 1, and none of the fibers of $\pi$ contains a $(-1)$-curve.

An elliptic surface with a section is an elliptic surface $\pi : X \to C$ together with a section $\sigma_0 : C \to X$, defined over $K$.

Let $E/K(C)$ be the generic fiber. Then the group $E(K(C))$ of $K(C)$-valued points of $E$ can be naturally identified the set of rational sections of $\pi$. This latter set can be into a group by fiberwise addition, where the zero element of the fiber over a point $p \in C$ is $\sigma_0(p)$.

We call an elliptic surface $\pi : X \to C$ trivial if there is an elliptic curve $E/K$ and a birational map $\psi : X \to E \times C$ such that $\pi = \text{pr}_2 \circ \psi$ as rational maps.

Since $C$ is a smooth curve we can extend every rational section to a section of $\pi$, hence $E(K(C))$ is also the set of sections of $\pi$.

**Definition 2.2.** Let $S$ be a smooth projective surface over an algebraically closed field $K$. Denote with $\text{NS}(S)$ the Néron-Severi group of $S$, the group of divisors on $S$ modulo algebraic equivalence.

**Definition 2.3.** Let $\pi : X \to C$ be an elliptic surface with a section. Let $T \subset \text{NS}(X_K)$ be the trivial subgroup, generated by the irreducible components (over $K$) of the singular fibers not intersecting $\sigma_0(C)$, the class of a smooth fiber and the image of the zero section. ([8, Section 6.1])

We have the following results, which contains both the Mordell-Weil theorem and the Shioda-Tate formula:

**Proposition 2.4.** Let $\pi : X \to C$ be an elliptic surface with a section. If $X$ is not a trivial elliptic surface then there is a natural isomorphism of groups

$$\text{NS}(X_K)/T \cong E(K(C)).$$

In particular, $E(K(C))$ is finitely generated.

**Proof.** See [8] Theorem 6.5] □

**Notation 2.5.** Let $B$ be a $K$-variety. Then denote with $|B|$ the set of closed points of $B$ and with $\eta$ the generic point of $B$, with residue field $K(\eta) = K(B)$

**Definition 2.6.** A family of elliptic surfaces with a section consists of a smooth geometrically irreducible $K$-variety $B$, a smooth projective surface $X \to B$, and a smooth curve $C \to B$ together with morphisms $\pi : X \to C$ and $\sigma_0 : C \to X$ of $B$-schemes, such that for each closed point $b \in |B|$ we have that $\pi_b : X_b \to C_b$ is an elliptic surface over $K(b)$ with zero-section $(\sigma_0)_b : C_b \to X_b$. 
Remark 2.7. Let $X \to C \to B$ be a family of elliptic surfaces. Then $X_\eta$ is an elliptic surface over $K(\eta)$, and for every $b \in \mathcal{B}$ the surface $X_b$ is an elliptic surface over $K(b)$. We denote with $X_\eta$ the base change of $X_\eta$ to $\overline{K(\eta)}$ and with $X_b$ the base change of $X_b$ to $\overline{K(b)}$.

**Proposition 2.8.** Let $X \to C \to B$ be a family of elliptic surfaces. Suppose $X_\eta$ is not a trivial elliptic surface. Then for all $b \in B$ we have that $X_b$ is not a trivial elliptic surface.

**Proof.** Recall that $X \to B$ and $C \to B$ are both smooth morphisms. By [7, Lemma IV.1.1], a trivial elliptic surface has $b_1(X_b) = 2g(C_b) + 2$, whereas a nontrivial elliptic surface has $b_1(X_b) = 2g(C_b)$. Since both $g(C_b)$ and $b_1(X_b)$ are constant on $B$ we find that either all closed fibers are trivial and so is the generic fiber trivial or none of the closed fibers is trivial. □

We have the following result from Maulik and Poonen [6, Proposition 3.6(a)]:

**Proposition 2.9.** With the same notation as before, we have that the specialization map

$$\text{sp} : \text{NS}(X_\eta) \to \text{NS}(X_b)$$

is injective with torsion-free cokernel.

**Lemma 2.10.** With the same notation as before, we have that $\text{sp}(T_\eta) \subset T_b$.

**Proof.** This specialization map obviously maps the class of the zero section to the class of the section zero and the class of a fiber to a class of a fiber. Moreover, the specialization of a fiber component is contained in a single fiber of the specialized surface, hence $\text{sp}(T_\eta) \subset T_b$. □

The following statements can be found in [8, Chapter 5 and Section 6.1]: Let $\pi : X \to C$ be an elliptic surface over a field $K$. Let $\Delta$ be the discriminant. For $p \in \Delta(K)$ consider the dual graph of $\pi^{-1}(p)$ and eliminate the component intersecting the image of the zero section. Call this graph $\Gamma_p$. Then this graph is of type $A_n, D_m, E_6, E_7, E_8$. Let $\Lambda_p$ be the associated lattice. Then

$$T = \Lambda' \oplus_{p \in \Delta(K)} \Lambda_p$$

where $\Lambda'$ is rank two lattice generated by classes $F$ and $Z$, with $F^2 = 0, Z^2 = -\chi(O_X)$, $F.Z = 1$.

**Definition 2.11.** Let $\pi : X \to C$ be an elliptic surface with section over a field $K$. The configuration of singular fibers is a finite multiset $M$, whose elements are formal symbols $A_n$, with $n \in \mathbb{Z}_{>0}$, $D_m$, with $m \in \mathbb{Z}_{>0}$, $E_6, E_7, E_8$, such that the multiplicity of $\Lambda$ in $M$ equals the number of $p \in C(K)$ such that $\Gamma_p$ is isomorphic to the graph with label $\Lambda$.

**Lemma 2.12.** Let $B'$ be the subscheme of $B$ such that for all $b \in |B'|$ the configuration of singular fibers of $\pi_b : X_b \to C_b$ equals the configuration of singular fibers of $\pi_\eta : X_\eta \to C_\eta$. Then $B'$ is nonempty open and for all closed points $b \in |B'|$ we have

$$\text{sp}(T_\eta) = \text{sp}(T_b)$$
Proposition 2.16. Let \( Z \) be a Zariski-closed proper subset of \( |B|' \). In particular, we have that for all \( b \in |B|' \) the specialization maps

\[
E_\eta(K(\eta)(C_\eta)) \to E_b(K(b)(C_b)) \quad \text{and} \quad E_\eta(K(\eta)(C_\eta)) \to E_b(K(b)(C_b))
\]

are injective.

Proof. Let \( b \in |B|' \) then \( sp(T_\eta) \) is a sublattice of \( T_b \) by the previous lemma. Since the configuration of singular fibers are the same we have that \( \text{rank}(T_b) = \text{rank}(sp(T_\eta)) \), hence image has finite index, and that \( \det(T_b) = \det(sp(T_\eta)) \), hence this index is one. In particular, \( sp(T_\eta) = T_b \) for all \( b \in B' \).

Using Tate’s algorithm [10] one easily sees that the type of singular fibers can be determined by the valuation of three standard invariants of a Weierstrass equation \((v(j), v(c_4), v(c_6))\). In particular, the locus \( B' \) is nonempty and open in \( B \). Moreover, for all \( b \in |B| \setminus (Z' \cup Z'') \) we have the following chain of morphisms

\[
E_\eta(K(\eta)(C_\eta)) \overset{\sim}{\longrightarrow} \text{NS}(\mathcal{X}_\eta)/T_\eta \overset{\sim}{\longrightarrow} \text{NS}(\mathcal{X}_b)/T_b \overset{\sim}{\longrightarrow} E_b(K(b)(C_b)).
\]

This shows that the first (geometric) specialization map is injective. The second (arithmetic) specialization map is the restriction of the first one and is also injective. \( \square \)

Definition 2.13. Let \( K \) be a finitely generated field over \( \mathbb{Q} \). Let \( B \) be a \( K \)-variety. Call a subset \( S \) of \( |B| \) sparse if there exists a dominant and generically finite morphism \( \pi : B_0 \to B \) of irreducible \( K \)-varieties, such that for each \( s \in S \) the fiber \( \pi^{-1}(s) \) is empty or contains at least two closed points.

Definition 2.14. Let \( K \) be a field of characteristic zero. Let \( V/K \) be a \( K \)-variety. A subset \( S \) of \( V(K) \) is called a thin subset of type I if \( S \) is contained in a Zariski closed subset of \( V(K) \). A subset \( S \) of \( V(K) \) is a called a thin subset of type II, if there exists another \( K \)-variety \( V' \) such that \( \dim V = \dim V' \) and a finite morphism \( \varphi : V' \to V \) of degree at least 2, such that \( S \subseteq \varphi(V'(K)) \).

A subset \( S \) of \( V(K) \) is thin if it is a subset of a finite union of thin subsets of type I and type II.

Remark 2.15. If \( S \) is a sparse subset of \( |A^n| \) then \( S \cap A^n(K) \) is a thin set.

Proposition 2.16. Let \( K \) be a finitely generated field over \( \mathbb{Q} \). Then there is a sparse subset \( Z \subset |B| \) such that for each \( b \in |B| \setminus Z \) the specialization map

\[
E_\eta(K(\eta)(C_\eta)) \to E_b(K(b)(C_b))
\]

is an isomorphism.

Proof. Let \( B' \) be as in the previous lemma. Let \( Z' = |B| \setminus |B'| \). Since \( X \to B \) is smooth and projective, we have by [6] Proposition 3.6 and Theorem 8.3 that there is a sparse subset \( Z'' \subset |B| \), such that

\[
sp : \text{NS}(\mathcal{X}_\eta) \to \text{NS}(\mathcal{X}_b)
\]

is an isomorphism for all \( b \in |B| \setminus Z'' \). Then for all \( b \in |B| \setminus (Z' \cup Z'') \) we have isomorphisms

\[
E_\eta(K(\eta)(C_\eta)) \overset{\sim}{\longrightarrow} \text{NS}(\mathcal{X}_\eta)/T_\eta \overset{\sim}{\longrightarrow} \text{NS}(\mathcal{X}_b)/T_b \overset{\sim}{\longrightarrow} E_b(K(b)(C_b)).
\]

Since \( Z' \) is a Zariski-closed proper subset of \( |B| \) and \( Z'' \) is sparse it follows that \( Z' \cup Z'' \) is sparse in \( |B| \). \( \square \)
We will now prove a similar statement for the specialization on the arithmetic Mordell–Weil groups, \( E_\eta(K(\eta)(C_\eta)) \to E_b(K(b)(C_b)) \). For this we aim to compare the \( \text{Gal}(k(\eta)/k(\eta)) \)-action on \( E_\eta(K(\eta)(C_\eta)) \) and the \( \text{Gal}(k(b)/k(b)) \)-action on \( E_b(K(b)(C_b)) \).

**Remark 2.17.** Fix a finite Galois extension \( L = K(\eta) = K(B) \) with Galois group \( G \). Let \( L_0 \) be the algebraic closure of \( K \) in \( L \). Suppose first that \( L_0 \neq L \).

Then \( L = K(B') \), for some integral \( K \)-variety \( B' \), which may be not geometrically integral. From the fact that \( L_0 \neq L \) it follows that there is a finite rational map \( \tau : B' \to B \) of degree at least 2. By shrinking \( B \) and \( B' \) if necessary we may assume that for \( \tau \) is an unramified finite flat morphism. In particular, for every \( p \in |B| \) we have that \( \tau^{-1}(p) \) consists of finitely many closed points \( p_1, \ldots, p_t \). Each point \( p_i \) yields a unramified field extension \( K(p_i)/K(p) \). Now, the locus where \( t > 1 \) holds, defines a sparse subset \( S \) of \( |B| \setminus Z \).

Hence for each point \( p \in |B| \setminus (Z \cup S) \) we have that \( t = 1 \) and that \( K(p_1)/K(p) \) is Galois with group \( G \). In this case we call \( K(p_1)/K(p) \) the specialization of the Galois extension \( L/K(\eta) \).

If \( L_0 = L \) then for every \( b \in |B| \) we have that the extension \( L(b)/K(b) \) is Galois with group \( G \).

**Theorem 2.18.** Let \( K \) be a finitely generated field over \( \mathbb{Q} \). Let \( \mathcal{X} \to B \) be a family of elliptic surfaces with base curve \( C \). Then there is a sparse subset \( Z \subset |B| \), such that for every \( b \in |B| \setminus Z \) the specialization map

\[
E_\eta(K(\eta)(C_\eta)) \to E_b(K(b)(C_b))
\]

is an isomorphism.

**Proof.** Let \( Z' \) be the sparse subset of \( |B| \) such that on \( |B| \setminus Z' \) the geometric specialization map

\[
E_\eta(K(\eta)(C_\eta)) \to E_b(K(b)(C_b))
\]

is an isomorphism.

Recall that the group \( E_\eta(K(\eta)(C_\eta)) \) is finitely generated. Moreover, for each \( P \in E_\eta(K(\eta)(C_\eta)) \) we have that \( K(\eta)(C_\eta)(P) \subset K(\eta)(C_\eta) \) is algebraic and finitely generated over \( K(\eta)(C_\eta) \). In particular, there exists a minimal finite extension \( L \) of \( K(\eta) \) such that \( L/K(\eta) \) is Galois and

\[
E_\eta(L(C_\eta)) = E_\eta(K(\eta)(C_\eta)).
\]

Let \( G = \text{Gal}(L/K(\eta)) \). Then there is a thin set \( Z' \) such that for all \( b \in |B| \setminus Z' \) the specialization \( M \) of \( L \) to \( K(b) \) exists. Then \( M/K(b) \) is Galois with group \( G \) and \( E_b(M(C_b)) = E_b(K(b)(C_b)) \).

Let \( P_1, \ldots, P_t \) be generators for \( E_\eta(K(\eta)(C_\eta)) \). Let \( H \) be a subgroup of the group \( \text{Gal}(L/K(\eta)) \). If \( E(K(\eta)(C_\eta))^H = E_\eta(K(\eta)(C_\eta)) \) then we call \( H \) irrelevant, otherwise we call \( H \) relevant.

For every relevant subgroup \( H \) of \( \text{Gal}(L/K(\eta)) \), let

\[
Q_{H,1}, \ldots, Q_{H,t_H} \in E(K(\eta)(C_\eta))^H
\]

be points such that \( P_1, \ldots, P_t, Q_{H,1}, \ldots, Q_{H,t_H} \) generate \( E(K(\eta)(C_\eta))^H \).

For a relevant subgroup \( H \) of \( G \) let \( Z_H \subset |B| \) be the locus of all \( b \in |B| \) such that \( \text{sp}(Q_{H,i}) \in K(b) \) for all \( i = 1, \ldots, t_H \). Then for every \( b \in |B| \setminus Z_H \) we have that...
\[ E_b(K(b)(C_b)) \neq E_\eta(L(\mathcal{C}_\eta))^H. \] Let
\[ Z = Z' \bigcup_{H \text{relevant}} Z_H. \]

Then for all \( b \in |\mathcal{B}| \setminus Z \) we have \( E_b(K(b)(C_b)) = E_\eta(L(\mathcal{C}_\eta))^H \) for some subgroup \( H \) of \( G \) but since \( b \not\in Z \), this group \( H \) is irrelevant. In particular, \( E_b(K(b)(C_b)) = E_\eta(L(\mathcal{C}_\eta))^H = E_\eta(K(\eta)(\mathcal{C}_\eta)) \)

From the construction of \( Z_H \) it follows that \( Z_H \) is sparse and therefore \( Z \) is sparse, since it is a finite union of sparse subsets. \( \square \)

3. Cowan’s conjecture

In this section we want to use Theorem 2.18 to prove Cowan’s conjecture and generalize this result to number fields. As explained in the Introduction we will use quadratic twists for this.

**Proposition 3.1.** Let \( V \) be a \( \mathbb{Q} \)-vector space. Let \( \rho : \text{Gal}(\bar{K}/K) \to \text{Aut}(V) \) be a Galois representation, such that \( \rho \) factors through a finite group. For any \( d \in K^*/(K^*)^2 \), let \( \chi_d \) be the quadratic character associated with the field extension \( K(\sqrt{d})/K \). Let \( \rho^{(d)} : \text{Gal}(\bar{K}/K) \to \text{Aut}(V) \) be the twisted Galois representation \( \rho^{(d)}(g) = \rho(g) \circ (\chi_d(g) \text{Id}_V) \)

Then there exists at most \( \dim V \) many \( d \in K^*/(K^*)^2 \) such that \( \rho^{(d)} \) has an invariant subspace.

**Proof.** Suppose first that \( V \) is the trivial representation then for all \( d \neq 1 \) we have that \( \rho^{(d)}(\cdot) = \chi_d(\cdot) \text{Id}_V \) has no invariant subspace. Similarly, if \( V \) has a twist which is trivial then all other twists (including the trivial one) have no invariant subspace.

We proceed now by induction on \( \dim V \). The case \( \dim V = 1 \) is covered by the above. Suppose now that \( \dim V > 1 \). Suppose that for some \( d \) there is an invariant subspace then by Maschke’s theorem we can decompose \( V^{(d)} = \mathbb{Q} \oplus V' \). Then there are at most \( d-1 \) twists of \( V' \) which have an invariant subspace and only the trivial twist of \( \mathbb{Q} \) has an invariant subspace. \( \square \)

Consider now a family of elliptic surfaces \( \mathcal{X} \to C \to \mathcal{B} \). Given a rational point \( b \in B(K) \) consider

\[
t \text{tw}(b) := \left\{ d \in K^*/(K^*)^2 \right\} \quad \begin{cases} \text{there exists a point } b' \in B(K) \\ \text{such that } \mathcal{X}_b/C_b \neq \mathcal{X}_{b'}/C_{b'} \\ \text{and } (\mathcal{X}_b/C_b)_{K(\sqrt{d})} \cong (\mathcal{X}_{b'}/C_{b'})_{K(\sqrt{d})} \end{cases}
\]

the set of \( d \) such that the quadratic twist of \( \mathcal{X}_b \) by \( \sqrt{d} \) is also contained in the family \( \mathcal{X} \to \mathcal{B} \).

**Proposition 3.2.** Let \( K \) be a finitely generated field over \( \mathbb{Q} \) and let \( \mathcal{X} \to \mathcal{B} \) be a family of non-trivial elliptic surfaces with a section.

Let \( b \in B(K) \) be a rational point such that \( \text{tw}(b) \) is infinite and the configuration of singular fibers of \( \mathcal{X}_b \to \mathcal{B} \) is the same as \( \mathcal{X}_{\eta} \to \mathcal{C}_{\eta} \). Then \( E_\eta(K(\eta)(\mathcal{C}_\eta)) \) is finite.

**Proof.** Our assumption on the configuration of singular fibers and Lemma 2.12 yield that the specialization map

\[ E_\eta(K(\eta)(\mathcal{C}_\eta)) \to E_b(K(C_b)) \]
is injective. Moreover, the assumption on the configuration of singular fibers is formulated over \( \overline{K} \), hence for all \( d \in \text{tw}(b) \) we have that
\[
E_0(K(\eta))(\mathcal{C}_\eta)) \to E_b^{(d)}(K(\mathcal{C}_b))
\]
is injective.

Consider now the Gal(\( \overline{K}/K \))-representation \( V = E_0(\overline{K}(t)) \otimes \mathbb{Z} \mathbb{Q} \). The Galois representation on \( E_b^{(d)}(K(\mathcal{C}_b)) \) is precisely \( V^{(d)} \). Hence if the rank of \( E_b^{(d)}(K(\mathcal{C}_b)) \) is positive then \( V^{(d)} \) has an invariant subspace. Since there are only finitely many \( d \in K^*/(K^*)^2 \) with this property it follows that there is a \( d \in \text{tw}(b) \) with \( E_b^{(d)}(K(\mathcal{C}_b)) \) finite and therefore \( E_0(K(\eta))(\mathcal{C}_\eta)) \) is finite. \( \square \)

**Notation 3.3.** For a commutative ring \( R \) let \( R[t]_d \) be the \( R \) module of polynomials in \( t \) with coefficients from \( R \) and of degree at most \( d \). Let \( m, n \) be integers and let \( S_{m,n}(R) = \langle (A, B) \in R[t]_m \times R[t]_n | 4A^3 + 27B^2 \neq 0 \rangle \).

Let \( k = \lceil \max(m/4, n/6) \rceil \). Suppose now that \( R \) is an integral domain not of characteristic 2 or 3. Let \( K \) be its quotient field. To a pair \( (A, B) \in S_{m,n} \) we can associate an elliptic curve \( E_{(A,B)}/K(t) \) with Weierstrass equation
\[
y^2 = x^3 + A(t)x + B(t).
\]
Moreover we can associate a hypersurface \( W_{A,B} \) in in the \( \mathbb{P}^2 \)-bundle \( \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(-2k) \oplus \mathcal{O}(-3k)) \) over \( \mathbb{P}^1 \), given by
\[
-Y^2Z + X^3 + s^{4k}A \left( \frac{t}{s} \right) XZ^2 + s^{6k}B \left( \frac{t}{s} \right) Z^3 = 0
\]

**Remark 3.4.** Consider the projection morphism \( \psi : W_{A,B} \to \mathbb{P}^1 \). Then for all \( p \in \mathbb{P}^1 \) the fiber over \( p \) is irreducible and the general fiber is an elliptic curve. If we resolve the singularities of \( W_{A,B} \) and then collapse all \(-1\) curve then we obtain an elliptic surface \( X_{A,B} \to \mathbb{P}^1 \) \( \square \) Lecture III.3. The collapsing of \(-1\) curves is only necessary if the original Weierstrass equation is not minimal, or if both \( \text{deg}(A) \leq 4k - 4 \) and \( \text{deg}(B) \leq 6k - 6 \) hold.

**Notation 3.5.** We define now the following subset
\[
U_{m,n}(R) = \left\{ (A, B) \in S_{m,n}(R) \mid 4A^3 + 27B^2 \text{ is smooth in } K[t] \text{ and } \text{deg}(4A^3 + 27B^2) = \text{max}(3m, 2n) \right\}.
\]
In the following we will identify \( R[t]_{\leq n} \times R[t]_{\leq m} \) with \( A^{m+n+2}_R \).

**Lemma 3.6.** There exists subvarieties \( Z' \) and \( Z'' \) of \( A^{m+n+2} \) such that \( S_{m,n}(R) = \mathcal{A}^{m+n+2}(R) \setminus Z'(R) \) and \( U_{m,n}(R) = \mathcal{A}^{m+n+2}(R) \setminus Z''(R) \). Moreover, \( U_{m,n}(R) \neq \emptyset \).

**Proof.** The subvariety \( Z' \) is defined by the property that all coefficients in \( 4A^3 + 27B^2 \) vanish. Recall that for every \( (A, B) \in S_{m,n}(R) \) we have that \( \text{deg}(4A^3 + 27B^2) \leq \text{max}(2n, 3m) \). Hence \( Z'' \) is defined by the vanishing of the coefficient of \( t^d \) in \( 4A^3 + 27B^2 \), where \( d = \text{max}(2n, 3m) \). It remains to show that \( U_{m,n} \) is nonempty: if \( 2n \geq 3m \) then \((1, t^n) \in U_{m,n}(R)\), if \( 2n < 3m \) then \((t^m, 1) \in U_{m,n}(R)\). \( \square \)

**Notation 3.7.** For every field \( K \) of characteristic zero there is a subscheme of \( U_{m,n} \subset A^{m+n+2}_K \) such that \( U_{m,n}(L) = U_{m,n}(L) \), for every field extension \( L/K \). One easily checks that \( U_{m,n} \) admits a model over \( \mathbb{Z} \) such that \( U_{m,n}(R) = U_{m,n}(R) \) for every integral domain \( R \) of characteristic zero.
Consider the universal Weierstrass model. Proposition 3.9. Let \((A, B) \in U_{m,n}(K)\) the Weierstrass model \(W_{A,B}\) has at most one singular point, if for one pair \((A, B)\) this model is smooth then all \(W_{A,B}\) are smooth and if \(W_{A,B}\) is singular then the type of singularity depends only on \((m, n)\). We will use this to show that \((X_{A,B})(A, B) \in U_{m,n}(K)\) yields a family of elliptic surfaces with section and constant trivial lattice.

Recall that \(k = \lceil \max(m/4, n/6) \rceil \). Let \(\alpha = 4k - m\) and let \(\beta = 6k - n\). Note that \(\alpha, \beta \geq 0\) and that \(\alpha < 4\) or \(\beta < 6\).

Lemma 3.8. Let \((A, B) \in U_{m,n}(K)\) then \(p_g(X_{A,B}) = k - 1\). Moreover, all singular fibers are of type \(I_1\), except possibly for the fiber at \(t = \infty\). At \(t = \infty\) we have the following fiber depending only on \((\alpha, \beta)\):

| \(\alpha\) | \(\beta\) | 0 | \(\geq 0\) | \(\geq 1\) | \(\geq 2\) | \(\geq 3\) | \(\geq 4\) | \(\geq 5\) |
| --- | --- | --- | --- | --- | --- | --- | --- | --- |
| fiber-type | singularity | \(I_0\) | \(A_0\) | \(I_0\) | \(A_0\) | \(I_1\) | \(A_1\) | \(I_2\) | \(A_2\) | \(I_3\) | \(A_3\) | \(I_4\) | \(A_4\) | \(I_5\) | \(A_5\) | \(I_6\) | \(A_6\) | \(I_7\) | \(A_7\) | \(I_8\) | \(A_8\) |

Proof. The statement about \(p_g(X)\) is [1, Lemma IV.1.1]. For the fiber over infinity we have \(v(c_3) = \alpha\), \(v(c_5) = \beta\) and \(v(\Delta) = 12k - \max(3m, 2n) = \min(3\alpha, 2\beta)\). A straight-forward application of Tate’s algorithm yields the type of fiber and the type of singularity, see [10].

For a given \((A, B) \in U_{m,n}\) we constructed a Weierstrass model \(W_{A,B}\). This yields a universal family over \(U_{m,n}\), i.e., a morphism of schemes \(W_{m,n} \rightarrow U_{m,n}\).

Proposition 3.9. Consider the universal Weierstrass model \(\psi : W_{m,n} \rightarrow U_{m,n}\). Then the resolution of the singularity of the generic fiber yields a family of elliptic surfaces with a section \(X_{m,n} \rightarrow U_{m,n}\) such that for all \(b \in U_{m,n}\) we have \(T_n = T_b\).

Proof. From the previous lemma it follows that over \(U_{m,n}\) all Weierstrass models are smooth (if \(\alpha = 0\) or if \(\beta \in \{0, 1\}\)) or all Weierstrass models have the same type of singularity (if \(\alpha > 0\) and \(\beta > 1\)). In particular, if we manage to resolve the singularities of the morphism \(W_{m,n} \rightarrow U_{m,n}\) uniformly then we have that the trivial lattice is constant, i.e., \(\text{sp}(T_n) = T_b\) for all \(b \in U_{m,n}\).

Suppose that the singularity is of type \(A_1\) or type \(A_2\) then a single blow-up of the point \(((X : Y : Z), (s : t)) = ((0 : 0 : 1), (0 : 1))\) suffices to resolve the singularity, hence this can be done simultaneously for the whole family. In case the singularity is of type \(D_4\) then we have a local equation of type

\[-y^2 + x^3 + s^2g(s)x + s^3h(s) = 0\]

where \(4g(s)^3 + 27h(s)^2\) does not vanish at \(s = 0\). If substitute \(y = sy, x = sx\) then the strict transform has equation

\[-y^2 + s(x^3 + g(s)x + h(s)) = 0.\]

Since \(4g(0) + 27h(0) \neq 0\) holds, this surface has \(A_1\) singularities at the three points in \(y = s = x^3 + g(0)x + h(0) = 0\). Hence this singularity can be also resolved simultaneously.

One can proceed similarly for the remaining singularities \(E_6, E_7, E_8\). In particular, if we resolve the singularity at \(((0 : 0 : 1), (0 : 1))\) then we obtain a family \(X_{m,n} \rightarrow U_{m,n}\) of elliptic surface over \(P^1\). \(\square\)
Let $K$ be a finitely generated field over $\mathbb{Q}$. Let $m,n$ be positive integers. Let $X_{m,n}$ be the universal elliptic curve over $B=U_{m,n}$. Then $E_\eta(K(\eta)(t))$ is finite.

**Proof.** Fix a pair $(A,B) \in U_{m,n}(K)$. Take now a $d \in K^*$. Then the Weierstrass equation of $E_{A,B}^{(d)}$ equals

$$-YZ^2 + X^3 + d^2AXZ^2 + d^3Z^3$$

In particular, we have that $E_{A,B}^{(d)} \cong E_{dA,dB}$ as elliptic curves over $K(t)$.

Recall that from $(A,B) \in U_{m,n}(K)$ it follows that $\Delta(A,B) := 4A^3 + 27B^2$ has degree $\text{max}(3m,2n)$ and is smooth. Since $\Delta(d^2A,d^3B) = d^6\Delta(A,B)$ holds, also $\Delta(d^2A,d^3B)$ is smooth and of degree $\text{max}(3m,2n)$. Hence $(d^2A,d^3B) \in U_{m,n}(K)$ and therefore $\text{tw}((A,B)) = K^*/(K^*)^2$. From Proposition 3.2 it follows now that $E_\eta(K(\eta)(t))$ is finite. \qed

**Remark 3.11.** In the following we will use results from [9] concerning the density of thin subsets. In these statements the measure on $K[t]_n \times K[t]_n$ is the naive height. Cowan [4] uses the Mahler measure. Since these measures are equivalent we are free to use these result from [9].

**Corollary 3.12.** Suppose $K$ is a number field. Then the set $(A,B) \in S_{m,n}(K)$ such that $E_{A,B}(K(t))$ has rank zero is the complement of a thin set. In particular, this set has density one in $S_{m,n}(K)$.

**Proof.** Combining Proposition 3.9 with Theorem 2.18 yields that there is a thin subset $Z$ of $U_{m,n}(K)$ such that for all $(A,B) \in U_{m,n}(K) \setminus Z$ we have that the rank of $E_{A,B}(K(t))$ vanishes. Now the complement of $U_{m,n}(K)$ in $K[t]_m \times K[t]_n \cong \mathbb{A}^{m+n+2}$ is a Zariski closed proper subset. From [9], Theorem 13.1.3 it follows that $U_{m,n}(K) \setminus Z$ has density one in $K[t]_m \times K[t]_n$, and therefore also density one in the smaller set $S_{m,n}(K)$. \qed

To prove Cowan’s conjecture we have to deal with some subtle points. First of all, Cowan’s conjecture is formulated over $\mathbb{Z}$ rather than over $\mathbb{Q}$. Moreover, there is a density zero subset of Weierstrass equations whose minimal model is trivial. Let $s = \lfloor \min(m/4,n/6) \rfloor$. For an integral domain $R$, let

$$Z_{m,n}^{\text{tr}}(R) = \{ (\lambda u^4,\mu u^6) \mid \lambda,\mu \in R, 4\lambda^3 + 27\mu^2 \neq 0, u \in R[t]_s \setminus \{0\} \} .$$

**Proposition 3.13.** Let $K$ be a field. Let $m,n$ be positive integers and $k = \lfloor \max(m/4,n/6) \rfloor$. Let $(A,B) \in S_{m,n}(K) \setminus Z_{m,n}^{\text{tr}}(K)$. Then

$$\text{rank } E_{A,B}(K(t)) \leq 10k - 2.$$

**Proof.** For $(A',B') \in U_{m,n}$ we have $p_g(X) = k - 1$. Using semi-continuity we find that $p_g(X_{A,B}) \leq k - 1$.

If $X_{A,B}$ is trivial then a minimal Weierstrass equation of the generic fiber $E_{A,B}/K(t)$ is

$$y^2 = x^3 + \lambda x + \mu,$$

with $\lambda,\mu \in K$. Every other short Weierstrass equation for $E$, in particular the equation for $W_{A,B}$, is of the form

$$y^2 = x^3 + \lambda u^4 x + \mu u^6,$$

with $u \in K(t)$. In particular $(A,B) \in Z_{m,n}^{\text{tr}}$, which we excluded.
Hence we may apply the Shioda-Tate formula (Proposition 2.4) and obtain rank \( E(K(t)) \leq h^{1,1} - 2 \). From [7, Lemma IV.1.1] it follows that \( h^{1,1} = 10(p_g + 1) \leq 10k \).

**Corollary 3.14.** Let \( m,n \) be positive integers. Then the average rank of \( S_{m,n}(\mathbb{Z}) \) is zero.

**Proof.** From Corollary 3.12 it follows that there a thin subset \( Z \subset S_{m,n}(\mathbb{Z}) \) such that on \( S_{m,n}(\mathbb{Z}) \setminus Z \) the rank is zero. Let \( Z' = Z \cap Z_{m,n}^{tr}(\mathbb{Z}) \). Let \( Z'' = Z \setminus Z' \). By [9, Theorem 13.1.1] both sets \( Z' \) and \( Z'' \) have density zero in \( S_{m,n}(\mathbb{Z}) \).

On \( Z''(\mathbb{Z}) \) the Mordell-Weil rank can be bounded by \( 10k - 2 \) by Proposition 3.13. Since this set has density zero it does not contribute to the average rank.

It is not known whether the rank on \( Z'(\mathbb{Z}) \) is bounded or not. However, from [3] it follows that on \( Z_{m,n}^{tr}(\mathbb{Z}) \) the average rank exists and is finite (cf. [2, Section 3]). Since this set has density zero, it also does not contribute to the average rank.

We will now generalize Cowan’s conjecture to an arbitrary number field \( K \). However we cannot control the average rank on \( Z_{m,n}^{tr}(\mathcal{O}_K) \).

**Corollary 3.15.** Let \( m,n \) be positive integers. Let \( K \) be a number field with ring of integers \( \mathcal{O}_K \). Then the average rank of \( S_{m,n}(\mathcal{O}_K) \setminus Z_{m,n}^{tr}(\mathcal{O}_K) \) is zero.

**Proof.** As in the proof of the previous corollary there a thin subset \( Z \subset S_{m,n}(\mathcal{O}_K) \setminus Z_{m,n}^{tr}(\mathcal{O}_K) \) such that on \( S_{m,n}(\mathcal{O}_K) \setminus Z \) the rank is zero.

By Proposition 3.13 we can bound the Mordell-Weil rank on \( Z \) by \( 10k - 2 \). From [9, Theorem 13.1.1] it follows that the density of \( Z \) equals zero. Hence the average rank is zero.

This approach to Cowan’s conjecture shows that if for a given member in the family there are many twists contained in the family then the average rank is zero. We will now give an example of a family where the average rank is positive.

**Example 3.16.** For \( k \) a positive integer. Consider the following subfamily \( \mathcal{X} \) of \( S_{2k,3k} \) given by

\[
\{ y^2 - x^3 + g^3 - h^2 \mid g \in \mathbb{Q}[t]_{2k}, h \in \mathbb{Q}[t]_{3k}, g^3 - h^2 \text{ has } 6k \text{ distinct factors} \}.
\]

In this case we have that for any \( b \in |\mathcal{B}| \) the set \( \text{tw}(b) = \{1\} \).

In this case one easily checks that every member of \( \mathcal{X} \) has 6 fibers of type II, and therefore a torsion-free Mordell-Weil group. Moreover, the Mordell-Weil group contains a non-trivial point \((x,y) = (g,h)\). In particular, the average rank is at least one.

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