Testing fundamental physics with distant star clusters: theoretical models for pressure-supported stellar systems

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ABSTRACT

We investigate the mean velocity dispersion and the velocity dispersion profile of stellar systems in MOND, using the N-body code N-MODY, which is a particle-mesh based code with a numerical MOND potential solver developed by Ciotti, Londrillo and Nipoti (2006). We have calculated mean velocity dispersions for stellar systems following Plummer density distributions with masses in the range of $10^4 M_\odot$ to $10^9 M_\odot$ and which are either isolated or immersed in an external field. Our integrations reproduce previous analytic estimates for stellar velocities in systems in the deep MOND regime ($a_i, a_e \ll a_0$), where the motion of stars is either dominated by internal accelerations ($a_i \gg a_e$) or constant external accelerations ($a_e \gg a_i$). In addition, we derive for the first time analytic formulae for the line-of-sight velocity dispersion in the intermediate regime ($a_i \sim a_e \sim a_0$). This allows for a much improved comparison of MOND with observed velocity dispersions of stellar systems. We finally derive the velocity dispersion of the globular cluster Pal 14 as one of the outer Milky Way halo globular clusters that have recently been proposed as a differentiator between Newtonian and MONDian dynamics.

Key words: galaxies: clusters: general - galaxies: dwarf - gravitation - methods: analytical – methods: N-body simulations

1 INTRODUCTION

The flattening of rotation curves of disk galaxies at large radial distances, i.e. the apparently non-Newtonian motion, is usually explained by invoking the otherwise undetected, so called Cold Dark Matter (CDM) (Bosma 1981, Rubin & Burstein 1983). This hypothesis has successfully explained the internal dynamics of galaxy clusters, gravitational lensing and the standard model of cosmology within the framework of general relativity (GR). Despite the fact that the dark matter model has been notably successful on large scales (Spergel 2003), dark matter particles has not been detected after much experimental efforts and the results of high resolution N-body simulations do not seem to be compatible with observations on galactic scales (Klypin et al. 1999, Moore et al. 1999).

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Metz et al. 2008). Another approach to explain galaxy rotation curves would be an alternative theory of gravity. One promising alternative theory is Modified Gravity (MOG) which has recently been successfully applied for dwarf satellite galaxies (Moffat & Toth 2007a) and distant globular clusters (Moffat & Toth 2007b). One of the most famous alternative theories is the so-called modified newtonian dynamics (MOND) theory, which was introduced by Milgrom (1983). According to MOND, the flat rotation curves of spiral galaxies at large distances can be explained by a modification of Newton’s second law of acceleration below a characteristic scale of $a_0 \approx 10^{-10} \text{ms}^{-2}$ without invoking dark matter (Bekenstein & Milgrom 1984).

It has been shown that on galactic scales MOND can explain many phenomena at least as well as CDM (Sanders & McGaugh 2002). For example, Sanchez-Salcedo and Hernandez (2007) studied the tidal radii of distant globular clusters and dwarf spheroidal satellite galaxies in MONDian dynamics. The most serious challenges for MOND come from clusters of galaxies, where MOND cannot completely explain the galaxy velocities...
and the merging of galaxy clusters, where the baryonic mass is clearly separated from the gravitational mass, as indicated by gravitational lensing (Chow et al. 2006). Both phenomena can be explained in MOND if some kind of hot dark matter is assumed, perhaps in the form of a massive (~2eV) neutrino (Angus et al. 2006).

MOND has recently been generalized to a general-relativistic version (Bekenstein 2004), making it possible to test its predictions for gravitational lensing. In the non-relativistic version of MOND, the acceleration of a particle due to a mass distribution \( \rho \) is given by (Bekenstein & Milgrom 1984):

\[
\nabla \cdot (\mu \frac{a}{a_0}) = 4\pi G \rho = \nabla \cdot a_N,
\]

(1)

where \( a_N \) is the Newtonian acceleration vector, \( a = |a| \) is the MONDian acceleration vector, \( \mu = \frac{a}{|a|} \) is the absolute value of MONDian acceleration and \( \mu \) is an interpolating function which runs smoothly from \( \mu(x) = x \) at \( x \ll 1 \) to \( \mu(x) = 1 \) at \( x \gg 1 \). The standard interpolating function is \( \mu_1(x) = \frac{x}{\sqrt{x^2 + 1}} \), but Famaey & Binney (2005) suggested another function \( \mu_2(x) = \frac{x}{\sqrt{x^2 + \epsilon}} \), which provides a better fit to the rotation curve of the Milky Way. Equation (1) can be transformed into \( \nabla \cdot (\mu(\frac{a}{a_0}) a_N - a_N) = 0 \), where the expression in parentheses is thus a curl field, and we may write

\[
\mu(\frac{a}{a_0}) a = a_N + \nabla \times H.
\]

(2)

The value of the curl field \( H \) depends on the boundary conditions and the mass distribution, but vanishes for some special symmetries. In realistic geometries, the curl field is non-zero and leads to difficulties for standard N-body codes. In other words, the non-linearity of the MOND field equation makes the use of the usual Newtonian N-body simulation codes impossible in the MOND regime.

Many stellar systems (e.g. globular clusters) have tidal radii much larger than their sizes, therefore the external field is approximately constant over the cluster area and the motions of stars are not influenced by tidal effects.

In Newtonian dynamics, a stellar system evolving under the influence of a uniform external acceleration, will, in the frame of the system, have the same internal dynamics as an isolated system. In MOND, due to the non-linearity of Poisson’s equation, the strong equivalence principle (SEP) is violated (Bekenstein & Milgrom 1984), and consequently the internal properties and the morphology of a stellar system are affected both by the internal and external field. This so-called external field effect (EFE) significantly affects non-isolated systems and can provide a strict test for MOND. The EFE postulation originated from observations of open clusters in the solar neighborhood, which do not show mass discrepancies even if the internal accelerations are below \( a_0 \) (Milgrom 1983). The EFE has several consequences, for example it allows high velocity stars to escape from the potential of the Milky Way (Famaey et al. 2007; Wu et al. 2007), and it decreases the velocity of satellite galaxies at very large radii, which is in conflict with the asymptotically flattening of rotation curves (Gentile et al. 2007; Wu et al. 2008). The EFE implies that rotation curves of spiral galaxies should fall where the internal acceleration becomes equal to the external acceleration. In addition, if the EFE is taken into account, internal properties of Galaxies such as the Tully-Fisher relation should be changed (Wu et al. 2005).

Milgrom derived the mean velocity dispersion of stellar systems for two special cases of internal or external field dominated systems

analytically, assuming that the systems are everywhere in the deep-MOND regime \( (a_e, a_i \ll a_0) \). If the external acceleration \( a_e \) is much larger than the internal one \( a_i \), the system of mass \( M \) is in the quasi Newtonian regime but with a normalized gravitational constant larger than the standard Newtonian one by a factor \( \frac{a_e}{a_i} \), and therefore the line-of-sight velocity dispersion is (Milgrom 1986)

\[
\sigma_{LOS,M1} = \sigma_{LOS,N} \sqrt{\frac{a_0}{a_e}},
\]

(3)

where \( \sigma_{LOS,N} \) is the Newtonian velocity dispersion. If \( a_e \ll a_i \ll a_0 \), the cluster is isolated and the line-of-sight velocity dispersion is given by

\[
\sigma_{LOS,M2} = 0.471(GM a_0)^\frac{1}{2}.
\]

(4)

Many systems which can be used to test MOND are not completely internally or externally dominated, for example globular clusters or dwarf galaxies of the Milky Way have internal and external accelerations which are of the same order (Baumgardt et al. 2005). Since Milgrom’s relations are valid only for systems that have either \( a_i \gg a_e \) or \( a_i \ll a_e \) and are in the deep-MONDian regime, one has to determine the velocity dispersions numerically for intermediate cases. Milgrom found that for isolated systems (internal acceleration dominated), the mass \( M \) of a system is nearly proportional to the forth power of the line of sight velocity dispersion \( \sigma_{los} \) and the ratio \( \sigma_{los}^4 / GM \) must be somewhere between \( \frac{1}{a_0} \) and \( a_0 \). But how does the velocity dispersion change while the system transits from the Newtonian to the MONDian regime? In an attempt to answer this question, we have performed N-body simulations and present analytical formulae for the velocity dispersion of stellar systems in the intermediate MOND regime. We have calculated the velocity dispersion for a number of isolated systems in which the internal accelerations \( a_i \) are in the range from \( a_i \ll a_0 \) to \( a_i \gg a_0 \). We also give formulae for systems with different strengths of external fields. It should be noted that the isolated systems are in equilibrium only in the Newtonian case, and reach a MONDian equilibrium state after collapse. For non-isolated systems we start from the MONDian equilibrium state which is created as described in section 4.3. These results could be useful for comparison with observational data of several GCs and dSph galaxies that are far away from the host galaxy, so that the external acceleration due to the host galaxy is small \( (a_e < 0.01a_0) \) these objects should therefore provide straightforward possibilities to test MOND. Since the external field affects the velocity dispersion by both tidal effects and EFE, and in order to see the pure MONDian effects, we concentrate on systems in which the tidal radius is much larger than the virial radius and therefore tidal effects are unimportant.

This is the first of a series of papers that deals with the numerical calculations for stellar systems. In the forthcoming papers, the observational constraints on mass and velocity dispersion of Pal 14 will be studied by Hilker et al. (2008) and Jordi et al. (2008).
The paper is organized as follows: In Section 2 we introduce theoretical predictions for the velocity dispersion in different regimes. In Section 3, we give a brief review of the N-MODY code which we use for our modelling. The numerical results for isolated and non-isolated systems and comparison with observational data are discussed in Section 4. We present our conclusion in Section 5.

2 LINE-OF-SIGHT VELOCITY DISPERSION IN DIFFERENT REGIMES

2.1 Newtonian regime

In Newtonian gravity, the mean-square velocity, $\sigma^2$, of a stellar system of mass $M$ is given by the following equation (Equation (4-80a) of Binney and Tremaine (1987)):

$$\sigma^2 = \frac{G M}{r_g},$$

where $r_g$ is the gravitational radius defined as (Equation (2-132) of Binney and Tremaine (1987)):

$$r_g = \frac{GM^2}{W}.$$

Here $W$ is the total potential energy. In the case of a Plummer model (Plummer 1911), and if we assume an isotropic velocity distribution, the line-of-sight velocity dispersion becomes

$$\sigma_{\text{LOS},N} = 0.36\sqrt[4]{\frac{GM}{R_h}},$$

where $R_h$ is the half-mass radius. If we define the half-mass-radius acceleration as $a_h = \frac{GM}{2R_h}$, we can re-write the above relation as

$$\frac{\sigma_{\text{LOS},N}^4}{GM} = 0.035a_h.$$

2.2 MOND regime

In the case of MOND, and in the presence of an external field, the total acceleration, which is the sum of the internal $a_i$ and external $a_e$ acceleration, satisfies the modified Poisson equation (Bekenstein & Milgrom 1984),

$$\nabla \left[ \mu \left( \frac{a_e + a_i}{a_0} \right) (a_i + a_e) \right] \simeq 4\pi G \rho,$$

where $a_e$ is approximately constant, $a_i = \nabla \phi$ is the non-external part of the potential and $\rho$ is the density of the star cluster. The boundary condition is $\nabla \phi = a_e x$ for $r \to \infty$. Equation (9) was postulated by Milgrom (1983) to explain the dynamical properties of nearby open clusters in the Milky Way and is an outcome of the MOND phenomenology. As an approximation for a spherical system one can write equation (9) as:

$$a_i \mu \left( \frac{a_e + a_i}{a_0} \right) = a_N.$$

Note however that Eq. (10) is only an approximate and effective way to take into account the external field effect (EFE), in order to avoid solving the modified Poisson equation with an external source term $\rho_{\text{ext}}$ on the right-hand side. The EFE is indeed a phenomenological requirement of MOND, which has important consequences for non-isolated systems. For example, if $a_e \ll a_i \ll a_0$, then the dynamics is in the MOND regime, and the external field can be neglected. When $a_i \ll a_e \ll a_0$, $\mu$ tends to its asymptotic value $\mu(a_e/a_0) = a_e/a_0$ (saturation of the $\mu$ function), and the gravitational potential is thus Newtonian with a renormalized gravitational constant to $(G_{\text{eff}} = G/\mu(a_e/a_0) \simeq G a_0/a_e$ (Milgrom 1986)).

Recently, several papers were published using this formulation to take into account the EFE (Gentile et al. 2007; Wu et al. 2007; Wu et al. 2008; Angus 2008; Famaey et al. 2007; Klypin & Prada 2008). For example, in order to estimate the order of magnitude of the EFE, Famaey et al. (2007) and Gentile et al. (2007), pointed out $a a_e(a + a_e)/(a_0 + a + a_e) = a_N$ using a simple $\mu$-function. Other authors replaced $a_i + a_e = \sqrt{a_i^2 + a_e^2}$ (Angus 2008; Klypin & Prada 2008). A more rigorous treatment of EFE on galactic rotation curves was made by Wu et al. (2007, 2008). In the work by Wu et al. (2007), for the mass density of the internal system, the MOND Poisson equation was solved as if the system was isolated, but the boundary condition on the last grid point was changed to be nonzero.

As a first approximation, we considered that a cluster is in a non-inertial frame, which free-falls with a uniform systematic acceleration. Since the calculation of $\phi$ is done for an isolated cluster, we did not change the boundary condition and at each step of potential solving, we added the constant external field with $a_e$ inside the $\mu$ function. This method might be only an approximation but as it is clear from the figures 1, 3 and 4, the transition region from the Newtonian to MONDian case is reproduced reasonably well by our method.

Analytical solutions exist only for some special cases that can be subdivided as follows:

1 - If $a_i \gg a_0$ or $a_e \gg a_0$ then the value of the interpolating function is equal to one and the system is in the Newtonian regime and the velocity dispersion is given by equation (8).

2 - If $a_e \ll a_i \ll a_0$, the system is in the deep MOND regime and the external field can be neglected (isolated system).

In this case, the line-of-sight velocity dispersion is given by Equation (9), which can be re-written as

$$\frac{\sigma_{\text{LOS},M1}^4}{GM} = 0.049a_0.$$

3 - If $a_i \ll a_e \ll a_0$, the system is externally field dominated. $\mu(x)$ becomes in this case $\mu(a_e/a_0) = a_e/a_0 = \text{const}$ (saturation of the $\mu$ function) and the system goes to a quasi-Newtonian regime but with an effective gravitational constant $G_e = G\mu^{-1}(a_e/a_0) \simeq G a_0/a_e$ that is larger than the standard Newtonian one. Using equation (3), the line-of-sight velocity dispersion is therefore equal to,

$$\frac{\sigma_{\text{LOS},M2}^4}{GM} = 0.035\left( \frac{a_0}{a_e} \right)^2 a_h.$$

Comparing equations (8), (11) and (12) with each other suggests that the line-of-sight velocity dispersion in the general case should be given by

$$\frac{\sigma_{\text{LOS}}^4}{GM} = f(a_h),$$

where $f(a_h) \sim a_h$ if $a_h \gg a_0$ and $f(a_h) \sim a_0$ if $a_h \ll a_0$. 

Velocity dispersion in MOND
In this paper, we attempt to investigate the universal functional form for $f(a_h)$ for systems with a wide range of internal and external accelerations.

3 N-MODY CODE

In order to numerically solve the non-linear MOND field equations, recently two N-body codes have been developed (Ciotti et al. 2006; Tiret & Combes 2007). In the present work we apply the N-MODY code developed by the first group, which can be used to do numerical experiments in either MONDian or Newtonian dynamics. N-MODY is a parallel, three-dimensional particle-mesh code for the time-integration of collision-less N-body systems (Londrillo & Nipoti 2008). The potential solver of N-MODY is based on a grid in spherical coordinates and is best suited for modeling isolated systems. N-MODY uses the leap-frog method to advance the particles. The code and the potential solver have been presented and tested by Ciotti et al. (2006) and Nipoti et al. (2007).

In the present study we used a spherical grid $(r, \theta, \varphi)$ made of $N_r \times N_\theta \times N_\varphi = 64 \times 64 \times 128$ grid cells for the integration. We use twice as many cells in the $\varphi$ direction since $\varphi$ runs from $0 < \varphi < 2\pi$ while $\theta$ runs only from $0 < \theta < \pi$. The total number of particles was in the range $N_p = 10^5 - 10^6$. The details of the scaling of the numerical MOND models and code units are discussed in Nipoti, Londrillo & Ciotti (2007). In order to include the EFE for non-isolated systems, we changed the N-MODY code and put a constant external field in the MONDian potential solver. We also chose $a = a_e + a_w$ within the interpolating function as the total acceleration of particles.

In the present work, the Plummer model (Plummer 1911) was used as the initial cluster model. It has a density distribution

$$\rho(r) = \frac{3M}{4\pi r_{p1}^3} \left(1 + \frac{r^2}{r_{p1}^2}\right)^{-5/2}$$

(14)

where $M$ is the total mass and $r_{p1}$ is the 'scale radius'. The half-mass radius of a Plummer model is $R_h \approx 1.305 r_{p1}$ and the virial radius is $R_v = \frac{16}{3} r_{p1}$. The total potential energy, $|W| = \frac{3}{2} \frac{GM^2}{2 r_{p1}}$, is used in equation (13) to calculate $r_g$.

4 RESULTS

In this Section, we present N-MODY solutions for stellar systems that are both isolated and non-isolated, allowing for different values of the external field.

4.1 Isolated systems

We have performed a large set of dissipationless N-MODY computations for isolated systems. Since the modeled systems are in equilibrium in the Newtonian case, in the MONDian case, they initially collapse. In order to have a MONDian equilibrium initial system, we rescaled the velocities by an amount given by our fitting formulae (Section 4.2 Equations 15) to prevent collapse. In order to create the initial condition, there is another substantial method developed by Nipoti et al. (2007b, 2008), in which the distribution

Figure 1. Line-of-sight (LOS) velocity dispersion profiles for isolated stellar systems as calculated by N-MODY. In order to have different internal half-mass accelerations $a_h$, several cases with different half-mass radii were calculated. As expected from equation (13), all curves follow the same functional form. The dashed line shows the asymptotic behavior in the Newtonian (equation 5) regime. The solid line shows the asymptotic behavior in deep MONDian (equation 14) regime. For high internal acceleration ($a_h \gg a_0$), the models are consistent with the Newtonian result and for low acceleration ($a_h \ll a_0$), they are consistent with the deep MONDian prediction. ($\log \equiv \log_{10}$).

Figure 2. Fit of our best fitting curve (equation 15) to the numerical solution for an isolated stellar system. The difference of each point from the fit function is presented in the inset. The average residual of this function from our numerical solution, which is defined as $\Delta = |f_{\text{theory}} - f_{\text{fit}}|$, is less than $10^{-3}$, ($\log \equiv \log_{10}$).
function is obtained numerically with an Eddington inversion with the far field logarithmic behavior of the MOND potential. Here we have used our method to set up MONDian initial condition. This method generalizes to non isolated systems easily (see section 4.2).

As discussed in Nipoti et al. (2007a), all simulations with different masses but with the same value of \(a_0\) are identical, in the sense that they can be simply rescaled to different masses, provided that \(M/r_h^2 = 2a_h/G\) remains constant. As a consequence, systems of any mass with \(a_0\) in the explored range follow the same functional form. Therefore, we consider only one simulation for given \(a_h\). In order to produce different internal acceleration regimes, we changed the half-mass radii of the system from 1 pc to 1 kpc. The models are evolved for several crossing times to reach the equilibrium state, which is identified by stationary Lagrange radii (e.g. Fig 6).

The resulting global velocity dispersions as a function of internal acceleration of the stellar systems are plotted in Fig. 1. As expected from equation (12), all of them follow the same functional form. The dashed line shows the Newtonian prediction for the velocity dispersion (equation 5). The asymptotic behavior of the models in the Newtonian regime are compatible with this analytical prediction. The solid line shows the analytical velocity dispersion in the deep MONDian regime (equation 11). In the low acceleration region, the numerical solutions are compatible with the analytical formula. At \(a_h = a_0\), the difference between the numerical model and the MONDian prediction is about 0.2 in \(\log \sigma\), which means that \(\sigma(a_h) \approx 1.3 \times \sigma_{\text{LOS},M1}\).

We now try to find an expression for a function \(f_0(x)\) where \(x = \frac{a_h}{a_0}\) which fits the numerical results. In the Newtonian regime \((x \gg 1)\), the function \(f_0\) has to approach \(f_0(x) = x + \text{const}\), while in the deep MONDian regime \((x \ll 1)\), \(f_0\) has to be constant. We therefore make an ansatz,

\[
f_0(x) = a \ln\left(\frac{a}{a_0}\right) + b + c, \tag{15}
\]

for the function \(f_0\). The best-fitting coefficients are then determined by a least-squares fit to the data and are found to be \(a = 0.3314\), \(b = 1.78\), and \(c = -1.48\). This function is shown in Fig. 2 as a solid line. The average residual of this function from our numerical results, which is defined as \(\Delta = |f_{\text{theory}} - f_{\text{fit}}|\), is less then \(10^{-3}\). Therefore, for any isolated system, if the internal half-mass-radius acceleration, \(a_h\), can be measured, it is possible to find out the MONDian prediction by this function. This is especially useful
Log (a
\textit{tion from the MONDian to quasi Newtonian regime occurs when
\textit{a} \textit{transition. The functional form of each fit curve is given in Table (1).

The dispersion is 
\textit{in MOND. The corresponding formula for the velocity dispersion
\textit{for the intermediate case which has no analytical prediction
\textit{in MOND. The corresponding formula for the velocity dispersion
\textit{as calculated by N-MODY. The x-axis gives the Newtonian internal
\textit{acceleration of the system. In order to see the transition regime we assume several values of \textit{a}\textsubscript{c}. When the internal acceleration of the system decreases, there are two transitions in the velocity treatment. The first transition occurs near \(a_{h} \approx \text{a}_0\) from Newtonian into MONDian regime and the second transition from the MONDian to quasi Newtonian regime occurs when the internal acceleration becomes equal to the external acceleration. The functional form of each fit curve is given in Table (1). (\textit{Log } \equiv \textit{ log}_{10}).

for the intermediate case which has no analytical prediction
\textit{in MOND. The corresponding formula for the velocity dispersion
\textit{is

\[
\text{log}_{10}(\sigma_{LOS}) = 0.25(0.331\ln(\exp\left(\frac{1.51GM}{a_0R_h^2}\right)) + 1.78 -1.48 + \log_{10}(GMa_0))
\]

A simple relation exists between the three-dimensional half-
\textit{mass radius and easier to observe two-dimensional, projected half-mass radius \(R_{hp} : R_{hp} = \gamma R_h\) with \(\gamma \approx 0.74\), which can be used in this formulae.

We briefly also discuss the influence of a different \(\mu\) function on the results. In Fig. 3 we plot the velocity dispersion for an isolated system using the simple interpolation function \(\mu_2\), and compare it with the results obtained for \(\mu_1\). Since the simple function has a stronger MONDian effect, the velocity dispersion is higher than that of the standard function. The difference of the velocity dispersion between both functions at \(a = a_0\) is about 20\%, so in order to determine the \(\mu\) function from observations, one needs to measure the mass and overall structure of a stellar system very accurately. As expected, in the extreme limit of \(a_h \ll a_0\) or \(a_h \gg a_0\), both functions predict the same value for the velocity dispersion.

4.2 Non isolated systems

Systems relevant for testing MOND (e.g. globular clusters or
dwarf galaxies) usually move through the gravitational field of a host galaxy. Therefore, the internal dynamics is often influenced by the host galaxy due to the EFE of MOND. We assume that coriolis forces that arise in the rotating reference frame of the cluster and that tidal forces arising from a gradient of the external field can be neglected. We believe this to be a a good first approximation that allows us to focus on the effects of the (constant) external field, therewith allowing us to for the first time venture into the intermediate MOND regime in order to study the external field effect numerically.

As an example which shows the EFE on the predicted \textit{LOS velocity dispersion for non-isolated stellar systems with different internal accelerations, we choose an external acceleration of \(a_e = 0.01a_0\). This corresponds to a cluster or dwarf galaxy being at a distance of about 1\textit{Mpc} from the Galactic center for an enclosed Milky Way mass of \(10^{12}\text{M}_\odot\). The resulting velocity dispersion as calculated by N-MODY is shown in Fig. 4. The first transition occurs near \(a_h \approx a_0\), when the systems enter the MONDian regime and the velocity dispersion deviates from the Newtonian prediction. The velocity dispersion remains close to the MOND prediction for the isolated case until the internal acceleration reaches \(a_i \approx a_e \approx 0.01a_0\) at which point a second transition occurs. Due to the saturation of the \(\mu\) function in the external field dominated regime, the velocity dispersion curve falls in a quasi Newtonian way if \(a_h < a_e\) and therefore deviates from the prediction of MOND for isolated systems.

Note that the internal acceleration shown in Fig. 4 is the Newtonian internal acceleration, \(a_h = \frac{GM}{R_h}\), of the system. In the deep MOND regime, since the Newtonian acceleration is the square of the MONDian acceleration \(a_{\text{M}} = \sqrt{a_Na_0}\), the point \(a_{h,N} = 0.0001a_0\) corresponds to \(a_{h,M} = 0.01a_0\).

In order to see the effect of different external accelerations, the MONDian velocity dispersion as a function of internal acceleration is plotted in Fig. 3 from a weak to a strong external field. Note that the transition point is determined by the strength of the external field. For a smaller external field, the transition point occurs at a smaller acceleration. As predicted by theory, for a strong external field \((a_e \gg a_0)\) the system is completely in the Newtonian regime, even for a low internal acceleration.

In order to find out the best functional form for the velocity dispersion as a function of the strength of the external field, we use the same procedure as in the isolated case, but take into account the different asymptotic behavior in Fig. 5. In the Newtonian regime, \((x \gg 1), f(x)\) still changes as \(f(x) = x + \text{const}\). In the deep MONDian regime \((x \ll 1),\) the systems are in the quasi-Newtonian regime and \(f(x)\) is again linear with the same slope as in the Newtonian regime. In the intermediate regime, in which the system is internal-acceleration dominated, \(\sigma_{LOS}/GM\) has to be constant. A general function satisfying all these constraints is therefore given by

\[
f(x) = f_0(x) - a \ln(\exp(-\frac{x}{a}) + b) + c.
\]

The coefficients \(a, b,\) and \(c\) depend on the external acceleration. Table 1 gives their values for several values of the external acceleration \(a_e\). Since the asymptotic value of dif-
different interpolating functions is the same, the choice of the $\mu$-function does not affect systems which are in the low acceleration regime. However for the higher acceleration systems ($a_h \sim a_0$), the $\mu$-function plays a more important role. Equation (17) will for example allow it to test MOND against the observed global velocity dispersions of dwarf galaxies. The relevant external acceleration can be determined by interpolating between the points computed with N-MODY (Table 1).

Figure 7. Upper panel: Density profile of Pal 14 for different cluster masses obtained by N-MODY computations. The shapes of the profiles are the same for all masses. Lower panel: Surface density profile of Pal 14 for masses as in the upper panel scaled to the level of data. The surface density profile shapes compare well with the observed density profile of Pal 14 (blue dots) as traced by giant stars (Hilker 2006). The meaning of the different lines is as in the upper panel. ($\log \equiv \log_{10}$).

4.3 The velocity dispersion of Galactic globular clusters

In order to decide whether MOND or dark matter is the right theory to explain the dynamics of the universe, it is desirable to study MOND for objects in which no dark matter is supposed to exist and where the characteristic acceleration of the stars is less than the MOND critical acceleration parameter $a_0$. GCs are a perfect candidate since they are the largest virialized structure that does not contain dark matter (Moore 1996), and their internal accelerations can be lower than $a_0$. Hence, GCs may provide a good laboratory to test the law of gravity (Baumgardt et al. 2005).

We choose the globular cluster Pal 14, for which there is a current observational effort to determine its velocity dispersion (Jordi et al. 2008). We initially choose a Newtonian equilibrium Plummer model initially. While the half-mass radius of Pal 14 is about 33 pc (Hilker 2006), the mass is not actually known, but an observing campaign is underway to constrain it (Hilker et al. 2008). We change the mass in the wide range from $10^2$ to $10^7 M_\odot$ and consider the half-mass radius to be constant. We perform numerical modeling to obtain the mean velocity dispersion as well as the density profile and velocity dispersion profile.

Since the modeled systems are in equilibrium in the Newtonian case, in the MONDian case, they initially collapse and $R_h$ is decreased before the systems virialize again. In order to have a MONDian equilibrium initial system, we increased the velocity of the Newtonian system by an
Figure 6. Upper left: Evolution of Lagrangian radii for a low mass cluster with stellar velocities corresponding to Newtonian virial equilibrium but in the deep-MOND limit. After rapid collapse, the system reaches an equilibrium state. Upper right: By increasing the initial velocities of particles by a factor of 2.8, calculated from equation (15), the system reaches equilibrium in MOND without collapsing. Lower panel: Evolution of the Lagrangian radius for a high mass cluster in the intermediate MOND regime. In the left panel, the velocities are not adjusted and the system still collapses. After adjusting the velocities by a factor of 1.38, calculated from equation (15), there is no collapse (right panel).

amount given by our fitting formulae (Equations (15) and (17)) to avoid a collapse. For low mass systems that are in the deep-MOND regime \(a_i < a_0\) the increase is larger than for massive systems that are in the intermediate regime \(a_i \sim a_0\). In Fig. we plot the evolution of the Lagrangian radii for two clusters with the same half-mass radius and different mass in the deep-MOND and intermediate MOND regime. In deep-MOND (low mass cluster), after a rapid collapse, the system reaches an equilibrium state. By increasing the initial velocity of the particles, the collapse can be prevented.

In Figs. and we show the numerical solution for Pal 14. We plot the density profile of Pal 14 for different masses and compare it with the observed profile in Fig. (observational data from Hilker (2006)). The shape of the density profiles is the same for all masses, but the central density differs significantly. All calculated surface density profiles compare well with the observed density profile. It should be mentioned that the full observed density profile is not known for Pal 14 and that the surface density profile shown in Fig. is based only on giant stars.

Fig. shows the numerical solutions for the line-of-sight velocity dispersion and compares it with Milgrom’s analytical predictions for the extreme limits (to see how analytical predictions differ from the numerical solution). \(M1\) refers to equation (11) which is for isolated systems and \(M2\) refers
to the quasi-Newtonian case which is for the external field dominated case (equation (12)). As expected, the analytical estimates are consistent with the numerical solution either in the external field dominated (small mass) or internal field dominated case, but have significant deviations in the intermediate regime. For Pal 14 the external acceleration of the Galaxy is about \(a_e \sim 0.1a_0\). For low cluster masses which means low internal accelerations, the prediction of \(M2\) is compatible with the numerical solution. As the mass increases, the internal acceleration grows and the system enters the internally dominated regime and the numerical solution gets close to the \(M1\) prediction.

The velocity dispersion profiles of Pal 14 for different cluster masses are also plotted in Fig. 3. The velocity dispersion changes slightly (about 10%) from the center to the half-mass-radius of the cluster.

We would finally like to mention that the line-of-sight velocity dispersion in the direction of the external field is almost 5% lower than that perpendicular to the external field. Such a small difference would be unobservable. This can be understood as arising from the external field negating MONDian gravity in the direction of the field. Also, the cluster is elongated by a few percent along the vector \(\mathbf{R}_g\). However for larger systems such as galaxies, the anisotropy could be observable for a case in which \(a_i \ll a_e \ll a_0\).

5 CONCLUSIONS

In this work we have calculated global line-of-sight velocity dispersions of stellar systems in MOND for both isolated and non-isolated stellar systems. The velocity dispersion of stellar systems in MOND was so far only known in the case of the deep MONDian limit where all accelerations are much smaller than the critical acceleration, \(a_0\), and even in this case only if either the internal acceleration is much larger than the external acceleration or the internal acceleration is much lower than the external acceleration. We used the N-MODY code to calculate for the first time the line-of-sight velocity dispersions of stellar systems also for the intermediate regime.

We have obtained a large set of dissipationless N-MODY numerical solutions for isolated systems with masses in the range \(10^4 M_\odot\) to \(10^9 M_\odot\) and with the Plummer model as the initial condition. In order to produce different internal acceleration regimes, for each mass, we changed the half-mass radius of the system. We deduce the analytical formulae for the velocity dispersion of a stellar system as a function of its half-mass-radius-internal-acceleration, \(a_h\) (equation (13)), and investigate the universal functional form for the velocity dispersion of isolated systems (equation (15)).

We have also studied the effect of a different choice of the interpolation function on the line-of-sight velocity dispersion for systems with different internal accelerations. We found that the simple function suggested by Famaey and Binney (2005) produces a larger velocity dispersion than the prediction with the standard interpolation function suggested by Milgrom (1984), with the maximum difference occurring at \(a_h \sim a_0\) and being of order 20%.

Since most stellar systems (e.g. globular clusters or dwarf galaxies) are not isolated and usually move through the gravitational field of a host galaxy, the internal dynamics is often influenced by the host galaxy due to the external field effect (EFE) of MOND. Therefore, we have investigated non-isolated systems, adding the external field to N-MODY. Our simulations reproduce previous analytic estimates for stellar velocities in systems in the deep MOND regime \((a_i, a_e \ll a_0)\), where the motion of the stars is either dominated by internal accelerations \((a_i \gg a_e)\) or external accelerations \((a_e \gg a_i)\). In addition, we calculate the line-of-sight velocity dispersion for intermediate cases and derive for the first time analytic formulae for the line-of-sight velocity dispersion in the intermediate regime \((a_i \sim a_e \sim a_0)\) and found a smooth functional form for the velocity dispersion of stellar systems under the EFE. These formulae will allow to test MOND more thoroughly than was hitherto possible.

We finally calculated the velocity dispersion of the globular cluster Pal 14, and will compare it with observational data in a forthcoming paper (Jordi et al. 2008). An additional observational study in order to constrain the mass of Pal 14 is also underway by our team (Hilker et al. 2008). In a future contribution we will also discuss the fascinating possibility of ”freezing” a cluster on a highly eccentric orbit: as a cluster moves from the Newtonian regime (small \(\sigma_N\)) to the MONDian regime on a time scale comparable to or faster than the internal crossing time it will retain a Newtonian velocity dispersion (Haghi et al. 2008).

Additional observational efforts to determine the velocity dispersion of stellar systems such as GCs or dSph satellites would be highly important as such data also provide a strict test of MOND. On the other hand, if we believe in MOND, these observations could be used to constrain the external field and consequently to put constrains on the potential in which the systems are embedded. Moreover it would be worthwhile to observe the stellar system in the intermediate regime to constrain the \(\mu\)-function and \(a_0\).

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| External acceleration | a | b | c |
|-----------------------|---|---|---|
| \(a_e = 0.01a_0\)     | 0.489 | 2242 | -7.694 |
| \(a_e = 0.03a_0\)     | 0.496 | 292 | -5.680 |
| \(a_e = 0.1a_0\)      | 0.495 | 35.32 | -3.585 |
| \(a_e = 0.3a_0\)      | 0.342 | 8.38 | -2.119 |
| \(a_e = 1.0a_0\)      | 0.281 | 1.00 | -0.006 |
| \(a_e = 10.0a_0\)     | 0.378 | 0.56 | 0.477 |

Table 1. Best fitting coefficients for the velocity dispersion predicted by N-MODY simulation for various values of the external acceleration. The general form is given by equation (17), where \(f_0(x)\) is due to an isolated cluster (equation (13)). In case \(a_e = 10a_0\), the function is nearly linear and the best fit function can also be obtained by \(f(x) = x - 1.467\), i.e. the Newtonian prediction.
Figure 9. Line-of-sight velocity dispersion profiles of Pal 14 for different cluster masses obtained by N-MODY computation. The horizontal blue (dashed) line indicates the mean global velocity dispersion and the vertical red (dashed) line indicates the half-mass-radius of Pal 14. The velocity dispersion changes only slightly (about 10%) from the center to the half-mass-radius.

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