Semi-infinite $q$-wedge construction of the level 2 Fock Space of $U_q(\hat{\mathfrak{sl}}_2)$

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Abstract

In this proceedings a particular example from [KMPY] is presented: the construction of the level 2 Fock space of $U_q(\hat{\mathfrak{sl}}_2)$. The generating ideal of the wedge relations is given and the wedge space defined. Normally ordering of wedges is defined in terms of the energy function. Normally ordered wedges form a base of the wedge space.

The $q$-deformed Fock space is defined as the space of semi-infinite wedges with a finite number of vectors in the wedge product differing from a ground state sequence and endowed with a separated $q$-adic topology. Normally ordered wedges form a base of the Fock space. The action of $U_q(\hat{\mathfrak{sl}}_2)$ on the Fock space converges in the $q$-adic topology. On the Fock space the action of bosons, which commute with the $U_q(\hat{\mathfrak{sl}}_2)$-action, also converges in the $q$-adic topology. Hence follows the decomposition of the Fock space into irreducible $U_q(\hat{\mathfrak{sl}}_2)$-modules.

1 Introduction

The classical semi-infinite wedge construction [DJKM] originates from the study of the representation theory of affine Lie algebras and the soliton theory of integrable hierarchies during the 80’s (for a review see for example [KR]). In the last couple of years the subject has been revived by the consideration of its $q$-deformation in the context of the representation theory of quantum affine algebras.

In [S, KMS] the semi-infinite wedge space construction of the level 1 Fock space of $U_q(\mathfrak{sl}_n)$ and its decomposition were given. In [KMPY] under certain assumptions a general scheme for the wedge construction of $q$-deformed Fock spaces using the theory of perfect crystals was presented. Let $U_q(\mathfrak{g})$ be a

*This proceedings is based on the paper “Perfect crystals and $q$-Fock spaces” [KMPY] by Masaki Kashiwara, Tetsuji Miwa, Chung Ming Yung and the speaker.

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Let $V$ be a finite dimensional $U_q(g')$-module with a perfect crystal base $[KMN]$ of level $k$. Let $V_{\text{aff}}$ denote its affinization. In $[KMPY]$ the wedge space $\bigwedge^r V_{\text{aff}}$ was constructed using the $U_q(g)$-action and the following $U_q(g)$-linear map was given

$$\bigwedge^r V_{\text{aff}} \otimes \bigwedge^s V_{\text{aff}} \rightarrow \bigwedge^{r+s} V_{\text{aff}}.$$ Normal ordering of wedges was defined and it was proven that normally ordered wedges form a base of $\bigwedge^r V_{\text{aff}}$. The level $k$ Fock space $F_m$ ($m \in \mathbb{Z}$) was constructed and the corresponding $U_q(g)$-linear map

$$\bigwedge^r V_{\text{aff}} \otimes F_m \rightarrow F_{m-r}$$

was defined. The decomposition of the Fock space was given.

In $[KMPY]$ examples of the theory for level 1 $A(1)_n$, $B(1)_n$, $D(1)_n$, $A(2)_2$, $A(2)_{2n-1}$, $D(2)_{n+1}$ and level $k$ $A(1)_1$ were also given.

In this talk I summarize some of the results of $[KMPY]$, describing the case of level 2 $U_q(\hat{\mathfrak{sl}}_2)$ in detail. This fairly simple example is sufficient to illustrate the differences between the level 1 $U_q(\hat{\mathfrak{sl}}_n)$ case ($[KMS]$) and other cases ($[KMPY]$): in particular the need to endow the Fock space with a separated $q$-adic topology. For fuller details and proofs, please see $[KMPY]$.

1.1 Thanks

I would like to thank the organisers — Professor Mo-Lin Ge, Professor Yvan Saint-Aubin and Professor Luc Vinet — for the invitation and a most enjoyable meeting at Nankai University. I thank Professor Ge and all the local organisers for the excellent hospitality. I thank R.I.M.S. for travel support. Finally I thank my coauthors of $[KMPY]$ for the collaboration, which made this paper possible.

2 Preliminaries

Let $g$ be an affine Lie algebra with associated weight lattice $P := \sum_{i \in I} \mathbb{Z} \Lambda_i \oplus \mathbb{Z} \delta$ and $g'$ its derived Lie subalgebra with associated weight lattice $P_{\text{cl}} := \sum_{i \in I} \mathbb{Z} \Lambda_i^{\text{cl}}$. Define $h_i \in P^*$ by $\langle h_i, \delta \rangle = 0$ and $\langle h_i, \Lambda_j \rangle = 6 \delta_i, j (i, j \in I)$. Similarly define $h_i \in P_{\text{cl}}^*$ by $\langle h_i, \Lambda_j^{\text{cl}} \rangle = 6 \delta_i, j (i, j \in I)$. Let $W$ be the Weyl group of $g$. Let $c$ denote the canonical central element in $g$ and $g'$. Define the projection $\text{cl} : P \rightarrow P_{\text{cl}}$ by

$$\text{cl}(\sum_{i \in I} \omega_i \Lambda_i + \omega_\delta \delta) := \sum_{i \in I} \omega_i \Lambda_i^{\text{cl}} \quad (\omega_i, \omega_\delta \in \mathbb{Z}).$$

Define $P_{\text{cl}}^+ := \{ \lambda \in P_{\text{cl}} \mid \langle h_i, \lambda \rangle \geq 0 \}$. 

2
Let \( U_q(\mathfrak{g}) \) (respectively \( U_q(\mathfrak{g}') \)) be the quantum universal enveloping algebra of \( \mathfrak{g} \) (\( \mathfrak{g}' \)) over \( \mathbb{Q}(q) \) with generators \( e_i, f_i, q^h \) (\( i \in I \) and \( h \in P^* (P_{cl}^*) \)). Write \( t_i \) for \( q^h_i \).

The coproduct of \( U_q(\mathfrak{g}) \) (\( U_q(\mathfrak{g}') \)) is taken to be \( \Delta = \bar{\Delta}_+ + \Delta_+ \):

\[
\Delta : \begin{cases} 
q^h \mapsto q^h \otimes q^h \quad (h \in P^* (P_{cl}^*)) \\
e_i \mapsto e_i \otimes 1 + t_i^{-1} \otimes e_i \quad (i \in I) \\
f_i \mapsto f_i \otimes t_i + 1 \otimes f_i \quad (i \in I)
\end{cases}
\]

In this talk I take \( \mathfrak{g} = \hat{\mathfrak{sl}_2} \), so \( I = \{0, 1\} \) and \( c = h_0 + h_1 \).

### 2.1 Perfect crystal base

I recall briefly some facts from the theory of crystals bases (see [K] for the definitions). Recall that a crystal is a set \( B \) with maps \( \tilde{e}_i, \tilde{f}_i : B \sqcup \{0\} \to B \sqcup \{0\} \) (\( i \in I \)), such that

1. \( \tilde{e}_i 0 = 0 = \tilde{f}_i 0 \),
2. there exists \( n \in \mathbb{Z}_{>0} \) such that \( \tilde{e}_i^n b = 0 = \tilde{f}_i^n b \) (\( \forall b \in B, i \in I \)),
3. \( b' = \tilde{f}_i b \iff b = \tilde{e}_i b' \) (\( b, b' \in B, i \in I \))

Let \( V \) be a finite dimensional \( U_q(\mathfrak{g}') \) module with crystal base \( (L, B) \). We take the \( U_q(\hat{\mathfrak{sl}_2}) \)-module to be \( V := \bigoplus_{j \in J = \{0, 2\}} \mathbb{Q}(q) v_j \) with crystal \( B = \bigsqcup_{j \in J} \{b_j\} \) and crystal graph

\[
\begin{array}{c}
b_0 \\
\uparrow 1 \\
b_1 \\
\downarrow 0 \\
b_2 
\end{array}
\]

In the crystal graph an arrow \( b \xrightarrow{i} b' \iff b' = \tilde{f}_i b \).

Let \( A := \{ f \in \mathbb{Q}(q) \mid f \text{ has no pole at } q = 0 \} \). The crystal lattice \( L = A \otimes_{\mathbb{Q}} \text{span}_{\mathbb{Q}}(B) \).

Define maps \( \varepsilon, \varphi : B \to \mathbb{N} \) by

\[
\varepsilon_i(b) := \max \{ n \in \mathbb{N} \mid \tilde{e}_i^n b \neq 0 \}, \\
\varphi_i(b) := \max \{ n \in \mathbb{N} \mid \tilde{f}_i^n b \neq 0 \}.
\]

and maps \( \varepsilon, \varphi : B \to P_{cl} \) by

\[
\varepsilon(b) := \sum_{i \in I} \varepsilon_i(b) \Lambda_i^{cl}, \\
\varphi(b) := \sum_{i \in I} \varphi_i(b) \Lambda_i^{cl}.
\]

In our example

\[
\varepsilon(b_j) = (2 - j) \Lambda_0^{cl} + j \Lambda_1^{cl}, \\
\varphi(b_j) = j \Lambda_0^{cl} + (2 - j) \Lambda_1^{cl} \quad (j \in J).
\]
The weight of $b \in B$ is given by $\text{wt}(b) = \varphi(b) - \varepsilon(b)$:

$$\text{wt}(b_j) = 2(1 - j)(\Lambda^1_j - \Lambda^0_0).$$

We take $\{v_j = G(b_j)\}_{j \in J}$ to be a lower global base $\mathbb{K}$ of $V$. For our example, we have

$$e_i G(b) = [\varphi_i(b) + 1]G(\tilde{e}_i b),$$
$$f_i G(b) = [\varepsilon_i(b) + 1]G(\tilde{f}_i b),$$
$$q^h G(b) = q^{(h, \text{wt}(b))}G(b).$$

The use of a lower global base is essential to our construction: see the remarks after Lemma 3.1 and Theorem 4.1.

Let $B = \bigsqcup_{\lambda \in P^+} B_{\lambda}$ be the weight decomposition.

Let $B_1, B_2$ be crystals. The tensor product $B_1 \otimes B_2$ (corresponding to the coproduct $\Delta$) is defined to be the set $B_1 \times B_2$ with the action of the Kashiwara operators $\tilde{e}_i, \tilde{f}_i$ given by

$$\tilde{e}_i(b \otimes b') = \begin{cases} \tilde{e}_i b \otimes b' & \text{if } \varepsilon_i(b) > \varphi_i(b') \\ b \otimes \tilde{e}_i b' & \text{if } \varepsilon_i(b) \leq \varphi_i(b') \end{cases},$$
$$\tilde{f}_i(b \otimes b') = \begin{cases} \tilde{f}_i b \otimes b' & \text{if } \varepsilon_i(b) \geq \varphi_i(b') \\ b \otimes \tilde{f}_i b' & \text{if } \varepsilon_i(b) < \varphi_i(b') \end{cases}.$$

The crystal base $(L, B)$ is perfect of level $k = 2$, which means that it satisfies the following conditions:

(i) There exists a weight $\lambda^0 \in \text{wt}(B)$ such that $\text{wt}(B)$ is contained in the convex hull of $W\lambda^0$ and that $|B_{w\lambda^0}| = 1$ ($w \in W$). The elements of $B_{w\lambda^0}$ and their weights $w\lambda^0$ are called extremal.

(ii) $B \otimes B$ is connected as a crystal graph.

(iii) $k = \max\{n \in \mathbb{Z}_{>0} \mid \langle c, \varepsilon(b) \rangle \geq n \ (\forall b \in B)\}$.

(iv) $\varepsilon, \varphi : \{b \in B \mid \langle c, \varepsilon(b) \rangle = k\} \rightarrow (P^+_{cl})_k := \{\lambda \in P^+_{cl} \mid \langle c, \lambda \rangle = k\}$ are bijective.

Note that $\langle c, \varphi(b) - \varepsilon(b) \rangle = 0$, since $\langle c, \text{wt}(b) \rangle = 0$ ($b \in B$).

In our example $\lambda^0 = \pm 2(\Lambda^1_1 - \Lambda^0_0)$,

$\begin{array}{ccc}
    b_0 \otimes b_0 & \xrightarrow{1} & b_0 \otimes b_1 & \xrightarrow{1} & b_0 \otimes b_2 \\
    \uparrow{0} & & \uparrow{0} & & \downarrow{1} \\
    B \otimes B = & b_1 \otimes b_0 & \xrightarrow{1} & b_1 \otimes b_1 & \xleftarrow{0} & b_1 \otimes b_2 \\
    \uparrow{0} & & \downarrow{1} & & \downarrow{1} \\
    & b_2 \otimes b_0 & \xleftarrow{0} & b_2 \otimes b_1 & \xleftarrow{0} & b_2 \otimes b_2
\end{array}$

and (2.1.1) implies that $\varepsilon, \varphi$ map bijectively. $b_0, b_2$ are extremal.
2.2 Affinization

Let $V_{\text{aff}} := V \otimes \mathbb{Q}[z, z^{-1}]$ be the $U_q(\mathfrak{g})$-module, which is the affinization of $V$ such that

$\begin{align*}
e_i(v \otimes \xi) := (e_i v) \otimes z^{\delta_{i,0}} \xi, \\
f_i(v \otimes \xi) := (f_i v) \otimes z^{-\delta_{i,0}} \xi, \\
\text{wt}(v_j \otimes z^a) := \text{wt}(v_j) + a\delta.
\end{align*}$

Let $(L_{\text{aff}}, B_{\text{aff}})$ be the crystal base of $V_{\text{aff}}$. Let $z^a : V_{\text{aff}} \to V_{\text{aff}}$ denote the $U_q(\mathfrak{g}')$-linear endomorphism $v \otimes \xi \mapsto v \otimes z^a \xi$. Then $z^a v_j := z^a (v_j \otimes 1) \equiv v_j \otimes z^a$ ($j \in J, a \in \mathbb{Z}$).

Let $(L_{\text{aff}}, B_{\text{aff}})$ be the crystal base of $V_{\text{aff}}$. Let $z^a : V_{\text{aff}} \to V_{\text{aff}}$ denote the $U_q(\mathfrak{g}')$-linear endomorphism $v \otimes \xi \mapsto v \otimes z^a \xi$ ($v \otimes \xi \in V_{\text{aff}}$). Then $z^a v_j := z^a (v_j \otimes 1) \equiv v_j \otimes z^a$ ($j \in J, a \in \mathbb{Z}$).

The crystal $B_{\text{aff}} = \bigsqcup_{j \in J, a \in \mathbb{Z}} \{z^a b_j\}$. The crystal graph of $B_{\text{aff}}$ has the following structure.

```
      0 1  zb2 0 1  b2 0 1  \
   zb1  b1  zb1  b0  zb0  \\
      1 0 1  0 1  
```

The arrows of the crystal graph induce a partial ordering of $B_{\text{aff}}$.

Define the map $l : B_{\text{aff}} \to \mathbb{Z}$ by

\[ l(z^a b_j) := 2a - j. \]

Define the energy function $(\text{[KMN]}) H : B_{\text{aff}} \otimes B_{\text{aff}} \to \mathbb{Z}$ by

(i) $H(zb \otimes b') = H(b \otimes b') + 1$ and $H(b \otimes zb') = H(b \otimes b') - 1$, $b, b' \in B_{\text{aff}}$.

(ii) $H$ is constant on every connected component of the crystal graph $B_{\text{aff}} \otimes B_{\text{aff}}$.

(iii) $H(z^a b^o, z^a b^o) = 0$ for any extremal $b^o \in B$ and $a \in \mathbb{Z}$.

For our example we have

\[ H(b_i \otimes b_j) = \min\{i, 2 - j\} \quad (i,j \in J). \]

If $H(b \otimes b') \leq 0$, then $l(b) \geq l(b')$.

3 Wedge space

For an extremal $b^o \in B$, let $v^o := G(b^o)$. Define

\[ N := U_q(\mathfrak{g})[z \otimes z, z^{-1} \otimes z^{-1}, z \otimes 1 + 1 \otimes z](v^o \otimes v^o). \]
It is not too difficult to prove that $N$ is independent of the choice of extremal element $b'$: indeed for any extremal $b' \in B$, $G(z^a b') \otimes G(z^a b') \in N$ ($a \in \mathbb{Z}$).

For our example, define the following elements of $N$:

\[
\begin{align*}
C_{b_0, b_0} &:= v_0 \otimes v_0, \\
C_{b_0, b_1} &:= v_0 \otimes v_1 + q^2 v_1 \otimes v_0, \\
C_{b_0, b_2} &:= v_0 \otimes v_2 + q v_1 \otimes v_1 + q^4 v_2 \otimes v_0, \\
C_{b_1, b_2} &:= v_1 \otimes v_2 + q^2 v_2 \otimes v_1, \\
C_{b_2, b_2} &:= v_2 \otimes v_2, \\
C_{z b_2, b_1} &:= z v_2 \otimes v_1 + q^2 v_1 \otimes v_2, \\
C_{z^2 b_2, b_0} &:= z^2 v_2 \otimes v_0 + q z v_1 \otimes v_1 + q^4 v_0 \otimes z^2 v_2, \\
C_{z b_1, b_0} &:= z v_1 \otimes v_0 + q^2 v_0 \otimes z v_1, \\
C_{z b_1, b_1} &:= z v_1 \otimes v_1 + q^2 v_1 \otimes z v_1 + q^2 [2](v_0 \otimes z v_2 + z v_2 \otimes v_0).
\end{align*}
\]

(These elements are constructed by the action of $U_q(sI_2)$ on $v_1 \otimes v_1$.)

**Lemma 3.1.** Let $B_{i,j} := \{(b, b') \in B_{\text{aff}} \otimes B_{\text{aff}} \mid H(b \otimes b') > 0, l(b_j) \leq l(b) < l(z^{H(b_i \otimes b_j)} b_j), l(b_j) < l(b') \leq l(z^{H(b_i \otimes b_j)} b_j)\}$. Each element $C_{z^{H(b_i \otimes b_j)} b_i, b_j}$ has the form

\[
G(z^{H(b_i \otimes b_j)} b_i) \otimes G(b_j) - \sum_{(b, b') \in B_{i,j}} a_{b, b'} G(b) \otimes G(b')
\]

The coefficients $a_{b, b'}$ lie in $q \mathbb{Z}[q]$. $H(z^{H(b_i \otimes b_j)} b_i \otimes b_j) = 0$.

The conditions in this Lemma are required for the construction of the Fock space. Note that this Lemma does not hold for the corresponding elements of $N$ in an upper global base.

The vectors $\{C_{z^{H(b_i \otimes b_j)} b_i, b_j}\}$ are linearly independent. We have

\[
\sum_{i,j \in J} Q(q)[z \otimes z, z^{-1} \otimes z^{-1}, z \otimes 1 + 1 \otimes z]C_{z^{H(b_i \otimes b_j)} b_i, b_j} = N.
\]

Define the wedge product by

\[
\wedge^2 V_{\text{aff}} := V_{\text{aff}} \otimes V_{\text{aff}} / N.
\]

For $v, v' \in V_{\text{aff}}$, denote the image of $v \otimes v' \in V_{\text{aff}}^\otimes 2$ in $\wedge^2 V_{\text{aff}}$ by $v \wedge v'$.

We call a pair $(b, b') \in B_{\text{aff}}^\otimes 2$ normally ordered, if $H(b \otimes b') > 0$. For such a pair, $G(b) \wedge G(b')$ is called a normally ordered wedge.

Note that $v_1 \wedge v_1$ is normally ordered and is not equal to 0 even at $q = 1$.

The elements $C_{b, b'} \in N (b, b' \in B_{\text{aff}}$ such that $H(b \otimes b') = 0$) should be thought of as a rule for writing $G(b) \wedge G(b')$ as a linear combination of normally ordered wedges.
Proposition 3.2. \( \{z^{a_1}v_{j_1} \wedge z^{a_2}v_{j_2}\}_{H(z^{a_1}b_{j_1} \otimes z^{a_2}b_{j_2})>0} \) is a base of \( \bigwedge V_{\text{aff}} \).

Let \( n \in \mathbb{Z}_{>0} \). Define the \( n \)-wedge space by
\[
N_n := \sum_{r=0}^{n-2} V_{\text{aff}}^{\otimes r} \otimes N \otimes V_{\text{aff}}^{\otimes (n-r-2)},
\]
\[
\bigwedge^n V_{\text{aff}} := V_{\text{aff}}^{\otimes n}/N_n.
\]
By construction \( \bigwedge^n V_{\text{aff}} \) is a \( U_q(g) \)-module.

A sequence \( (b_{j_1}, b_{j_2}, \ldots, b_{j_n}) \) is called normally ordered if
\[
H(b_{j_m} \otimes b_{j_{m+1}}) > 0 \quad \text{for every} \quad m \in [0, n-1].
\]
In this case the corresponding wedge \( v_{j_1} \wedge v_{j_2} \wedge \cdots \wedge v_{j_n} \) is called a normally ordered wedge.

The homomorphism
\[
\bigwedge^r V_{\text{aff}} \otimes \bigwedge^s V_{\text{aff}} \rightarrow \bigwedge^{r+s} V_{\text{aff}},
\]
\[
\begin{array}{cc}
\otimes & \mapsto \wedge
\end{array}
\]
is \( U_q(g) \)-linear.

Theorem 3.3. Normally ordered \( n \)-wedges form a base of \( \bigwedge^n V_{\text{aff}} \).

Define \( L(\bigwedge^n V_{\text{aff}}) \) to be the image of \( L(V_{\text{aff}}^{\otimes n}) \) in \( \bigwedge^n V_{\text{aff}} \) and \( B(\bigwedge^n V_{\text{aff}}) := \{(b_{j_1}, b_{j_2}, \ldots, b_{j_n}) \mid \text{normally ordered}\} \).

4 Level 2 Fock space

4.1 Ground state sequence

Recall that \( B_{\text{aff}} \) is the affinization of the perfect level \( k = 2 \) crystal \( B \).

Extend the maps \( \varepsilon, \varphi : B \rightarrow P_{\text{cl}} \) to \( B_{\text{aff}} \rightarrow P_{\text{cl}} \) by \( \varepsilon(z^a b_j) := \varepsilon(b_j) \) and \( \varphi(z^a b_j) = \varphi(b_j) \).

Fix a sequence \( b_m^o \in B_{\text{aff}} \) (\( m \in \mathbb{Z} \)) such that
\[
\langle c, \varepsilon(b_m^o) \rangle = k, \quad \varepsilon(b_m^o) = \varphi(b_{m+1}), \quad H(b_m^o \otimes b_{m+1}^o) = 1.
\]
We call this sequence a ground state sequence. Take also a sequence of weights \( \lambda_m \) (\( m \in \mathbb{Z} \)) such that
\[
\langle c, \lambda_m \rangle = k, \quad \text{cl}(\lambda_m) = \varphi(b_m^o), \quad \lambda_m = \text{wt}(b_m^o) + \lambda_{m+1}.
\]
Define \( v_m^o := G(b_m^o) \).
For our example of level 2 $U_q(\mathfrak{sl}_2)$ there are, up to equivalence, only two possible ground state sequences

\[ b_m^o = \begin{cases} \varepsilon b_2 & \text{for } m \in 2\mathbb{Z}, \\ b_0 & \text{for } m \in 2\mathbb{Z} + 1, \end{cases} \]

\[ \lambda_m = \begin{cases} 2\Lambda_0 + \delta & \text{for } m \in 2\mathbb{Z}, \\ 2\Lambda_1 & \text{for } m \in 2\mathbb{Z} + 1, \end{cases} \] (A)

and

\[ b_m^o = b_1 \quad (m \in \mathbb{Z}), \]

\[ \lambda_m = \Lambda_0 + \Lambda_1 \quad (m \in \mathbb{Z}). \] (B)

### 4.2 Fock space

In the formal semi-infinite wedge space $\bigwedge^\infty V_{\text{aff}}$, define the vacuum vector

\[ |m\rangle := v_m^o \wedge v_{m+1}^o \wedge v_{m+2}^o \wedge \cdots \quad (m \in \mathbb{Z}). \]

Define the pre-Fock space

\[ \bar{F}_m := \sum_{r \in \mathbb{N}} \bigwedge^r V_{\text{aff}} \wedge |m+r\rangle. \]

Let $L(\bar{F}_m)$ be the crystal lattice of $\bar{F}_m$. Define the Fock space $F_m$ to be

\[ F_m := \bar{F}_m / \bigcap_{n>0} q^n L(\bar{F}_m). \]

Let $|m\rangle$ be the image of $|m\rangle$ in $F_m$. Taking $\{q^n L(\bar{F}_m)\}_{n \in \mathbb{N}}$ as a neighborhood system of 0, $F_m$ is endowed with a $q$-adic topology. By construction the $q$-adic topology is separated, since $\bigcap_{n>0} q^n L(\bar{F}_m) = 0$.

We have an algebra homomorphism $\bigwedge^* V_{\text{aff}} \otimes F_m \to F_{m-r}$.

Denote the vacuum vector $|m\rangle$ associated to ground sequences (A) and (B) by $|m\rangle^A$ and $|m\rangle^B$ respectively. Similarly denote the Fock space $F_m$ associated to ground sequences (A) and (B) by $F_m^A$ and $F_m^B$ respectively.

Note that $G(b) \wedge |m\rangle = 0$, if and only if there exists $r \in \mathbb{N}$ such that $G(b) \wedge v_m^o \wedge v_{m+1}^o \wedge \cdots \wedge v_{m+r}^o = 0$.

**Theorem 4.1.** Let $b \in B_{\text{aff}}$ be such that $H(b \otimes b_m^o) \leq 0$. In $F_m$ the equality

\[ G(b) \wedge |m\rangle = 0 \]

holds in the $q$-adic topology.
Note that this theorem only holds for a lower global base.
For example in $F^B_m$ the vacuum vector is $|m\rangle^B = v_1 \wedge v_1 \wedge v_1 \wedge \ldots$ and
\[
v_0 \wedge |m\rangle^B = \lim_{r \to \infty} (-q^2)^r v_1^{\wedge r} \wedge v_0 \wedge |m + r\rangle^B = 0 \quad \text{(in the } q\text{-adic topology)}. \tag{4.2.1}
\]

For a normally ordered sequence $(z^{a_m}b_{j_m}, z^{a_{m+1}}b_{j_{m+1}}, \ldots)$ in $B_{\text{aff}}$ such that $z^{a_r}b_{jr} = b^c_r$ for all $r \gg m$, the wedge $z^{a_m}v_{j_m} \wedge z^{a_{m+1}}v_{j_{m+1}} \wedge \ldots$ is called a normally ordered wedge.

**Theorem 4.2.** The normally ordered wedges in $F^B_m$ form a base of $F^B_m$.

### 4.3 $U_q(g)$-module structure

First we assign weights to the Fock space by setting
\[
\text{wt}(|m\rangle) := \lambda_m.
\]
This fixes the action of the $q^h$ ($h \in P^*$). For example $\text{wt}(|m\rangle^B) = \Lambda_0 + \Lambda_1$ ($m \in \mathbb{Z}$).

**Proposition 4.3.** Let $V(\lambda_m)$ denote the irreducible integrable $U_q(g)$-module of highest weight $\lambda_m$. The character of $F^B_m$ is
\[
\text{ch}(F^B_m) = \text{ch}(V(\lambda_m)) \prod_{r > 0} (1 - q^{-r})^{-1}. \tag{4.3.1}
\]

**Theorem 4.4.** $F^B_m$ has the structure of an integrable $U_q(g)$-module.

There are two steps to proving this: (i) proving that the action of $e_i, f_i$ is well-defined (converges in the $q$-adic topology) and is integrable, and (ii) checking that the commutation relations are satisfied. In fact it is sufficient to prove it on the vacuum vector $|m\rangle$.

For example in $F^B_m$ we have
\[
t_i |m\rangle^B = q |m\rangle^B \quad (i \in I),
\]
\[
e_1 |m\rangle^B = [2] \sum_{r \in \mathbb{N}} v_1^{\wedge r} \wedge v_0 \wedge |m + 1 + r\rangle^B = 0 \quad \text{(by (4.2.1))},
\]
\[
f_1 |m\rangle = [2] \sum_{r \in \mathbb{N}} v_1^{\wedge r} \wedge v_2 \wedge q |m + 1 + r\rangle^B
\]
\[
= q [2] \left( \sum_{r \in \mathbb{N}} (-q^2)^r \right) v_2 \wedge |m + 1\rangle^B = v_2 \wedge |m + 1\rangle^B.
\]

Then one can check that
\[
[e_1, f_1] |m\rangle^B = e_1 \cdot f_1 |m\rangle^B = e_1 v_2 \wedge |m + 1\rangle^B
\]
\[
= |m\rangle^B = \frac{t_1 - t_1^{-1}}{q - q^{-1}} |m\rangle^B.
\]
4.4 Bosons

Define boson operators $B_a$

\[ B_a := \sum_{r \in \mathbb{N}} 1^\otimes r \otimes z^a \otimes 1^\otimes \infty \quad (a \in \mathbb{Z} \setminus \{0\}) \]
\[ \equiv z^a \otimes 1^\otimes \infty + 1 \otimes z^a \otimes 1^\otimes \infty + 1 \otimes 1 \otimes z^a \otimes 1^\otimes \infty + \ldots . \]

**Proposition 4.5.** The action of the operators $B_a$ on $F_m$ converges in the $q$-adic topology.

**Proposition 4.6.** $B_a |m\rangle = 0$ for all $a \in \mathbb{Z}_{>0}$.

**Proposition 4.7.** There exists $\gamma_a \in \mathbb{Q}(q)$ (independent of $m$) such that

\[ [B_a, B_{a'}] |m\rangle = \gamma_a \delta_{a+a',0} |m\rangle. \]

At $q = 0$, $\gamma_a = a$.

Let $H$ be the Heisenberg algebra generated by $\{B_a\}_{a \in \mathbb{Z} \setminus \{0\}}$ with the defining relations $[B_a, B_{a'}] = \delta_{a+a',0} \gamma_a$. Then $H$ acts on the Fock space $F_m$ commuting with the action of $U_q(\mathfrak{g}')$. Let $\mathbb{Q}[H_-] := \mathbb{Q}[B_{-a}]_{a \in \mathbb{Z}_{>0}} \cdot 1$ be the Fock space for $H$ with vacuum vector $1$ and the defining relation $B_a \cdot 1 = 0$ ($a \in \mathbb{Z}_{>0}$). Let $u_{\lambda_m}$ denote the highest weight vector in $V(\lambda_m)$. Since $|m\rangle$ is annihilated by $e_i$ ($i \in I$) and $B_a$ ($a \in \mathbb{Z}_{>0}$), we have an injective $U_q(\mathfrak{g}') \otimes H$-linear homomorphism

\[ \iota_m : V(\lambda_m) \otimes \mathbb{Q}[H_-] \to F_m \]
\[ u_{\lambda_m} \otimes 1 \mapsto |m\rangle. \]

**Theorem 4.8.** $\iota_m : V(\lambda_m) \otimes \mathbb{Q}[H_-] \to F_m$ is an isomorphism.

The proof is by comparing the characters (4.3.1). This gives the decomposition of $F_m$ into irreducible $U_q(\mathfrak{g})$-modules.

$\gamma_a$ can be calculated by using the decomposition via $\iota_m$ of the wedge vertex operator $(\text{Vaff} \otimes F_m \to F_{m-1})$ to a product of the usual vertex operator $(\text{Vaff} \otimes V(\lambda_m) \to V(\lambda_{m-1}))$ and the boson vertex operator, and then calculating the equality of two-point functions corresponding to this decomposition (see [KMPY]).

**Proposition 4.9.** In our example

\[ [B_a, B_{a'}] |m\rangle^B = \frac{a}{1 - q^{2a}} \delta_{a+a',0} |m\rangle^B. \]
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