Dependence Properties of B-Spline Copulas

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**Abstract**

We construct by using B-spline functions a class of copulas that includes the Bernstein copulas arising in Baker’s distributions. The range of correlation of the B-spline copulas is examined, and the Fréchet–Hoeffding upper bound is proved to be attained when the number of B-spline functions goes to infinity. As the B-spline functions are well-known to be an order-complete weak Tchebycheff system from which the property of total positivity of any order follows for the maximum correlation case, the results given here extend classical results for the Bernstein copulas. In addition, we derive in terms of the Stirling numbers of the second kind an explicit formula for the moments of the related B-spline functions on the right half-line.

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1 **Introduction: A Review of the Bernstein Copulas**

A novel method that applied the theory of order statistics to construct multivariate distributions with given marginal distributions was developed by Baker (2008). We refer to Lin et al. (2014) for a recent survey of this topic.

Baker’s idea, applied to univariate cumulative distribution functions \(F\) and \(G\), can be described as follows: Let \(\{X_1, \ldots, X_n\}\) and \(\{Y_1, \ldots, Y_n\}\) be
independent random samples from the distributions $F$ and $G$, respectively. Let $X_{k,n}$ be the $k$th smallest order statistic of $\{X_1, \ldots, X_n\}$, and denote by $F_{k,n}$ the distribution of $X_{k,n}$; we write this as $X_{k,n} \sim F_{k,n}$. Similarly, we denote by $Y_{k,n}$ the $k$th smallest order statistic of $\{Y_1, \ldots, Y_n\}$ and we let $G_{k,n}$ be its corresponding distribution, written $Y_{k,n} \sim G_{k,n}$. (Note that $F$ and $G$ can be discrete distributions.)

Let $R = (r_{k\ell})_{1 \leq k, \ell \leq n}$ be a parameter matrix whose matrix entries $r_{k\ell}$ satisfy the conditions

$$\sum_{k=1}^{n} r_{k\ell} = \sum_{\ell=1}^{n} r_{k\ell} = \frac{1}{n}, \quad r_{k\ell} \geq 0, \quad k, \ell = 1, 2, \ldots, n. \quad (1.1)$$

Now choose the pair $(X_{k,n}, Y_{\ell,n})$ with probability $r_{k\ell}$, $k, \ell = 1, 2, \ldots, n$. Then $(X_{k,n}, Y_{\ell,n})$ follows Baker’s bivariate distribution: For $x, y \in \mathbb{R}$, the joint cumulative distribution function $H(x, y) := \Pr(X_{k,n} \leq x, Y_{\ell,n} \leq y)$ satisfies

$$H(x, y) = \sum_{k=1}^{n} \sum_{\ell=1}^{n} r_{k\ell} F_{k,n}(x) G_{\ell,n}(y)$$

$$= \begin{pmatrix} F_{1,n}(x), \ldots, F_{n,n}(x) \end{pmatrix} R \begin{pmatrix} G_{1,n}(y), \ldots, G_{n,n}(y) \end{pmatrix}^T, \quad (1.2)$$

where “$T$” denotes transpose. It is immediately evident that $H$ has marginal distributions $F$ and $G$.

Let $B_{k,n}$ be the distribution function of the $k$th smallest order statistic of a random sample of size $n$ from the uniform distribution $U$ on $[0, 1]$. It is well-known that

$$B_{k,n}(u) = \int_{0}^{u} b_{k,n}(t) \, dt \quad (1.3)$$

where

$$b_{k,n}(t) = n \binom{n-1}{k-1} t^{k-1} (1-t)^{n-k},$$

$t \in [0, 1]$. Furthermore, $F_{k,n}$ equals the composition $B_{k,n} \circ F$ (see, e.g., Hwang and Lin 1984) and Baker’s bivariate distribution (1.2) can be rewritten as

$$H(x, y) = C(F(x), G(y); R),$$

$x, y \in \mathbb{R}$, where, for $u, v \in [0, 1]$,

$$C(u, v; R) = \sum_{k=1}^{n} \sum_{\ell=1}^{n} r_{k\ell} B_{k,n}(u) B_{\ell,n}(v) \quad (1.4)$$
is a copula function with parameter matrix \( R \) satisfying (1.1). Conversely, if the marginals \( F \) and \( G \) are equal to \( \mathcal{U} \) then Baker’s bivariate distribution (1.2) reduces to the copula (1.4).

The copula \( C(u, v; R) \) in Eq. (1.4) is called the **Bernstein copula with parameter matrix \( R \)** because \( b_{k,n}/n \) is a Bernstein polynomial (see Dou et al. 2016). By differentiating (1.4) with respect to \( u \) and \( v \), we obtain the Bernstein copula density:

\[
c(u, v; R) = \sum_{k=1}^{n} \sum_{\ell=1}^{n} r_{k\ell} b_{k,n}(u)b_{\ell,n}(v),
\]

\( u, v \in [0, 1] \).

Within the parameter space (1.1) of \( R \), the maximum correlation is attained when \( r_{k\ell} = (1/n)\delta_{k\ell} \), i.e., \( R = (1/n)I_n \):

\[
C^*(u, v) := C(u, v; (1/n)I_n) = \frac{1}{n} \sum_{k=1}^{n} B_{k,n}(u)B_{k,n}(v),
\]

with corresponding density

\[
c^*(u, v) := c(u, v; (1/n)I_n) = \frac{1}{n} \sum_{k=1}^{n} b_{k,n}(u)b_{k,n}(v),
\]

\( u, v \in [0, 1] \).

Dou et al. (2013) proved that the maximum correlation copula \( C^*(u, v) \) and its density \( c^*(u, v) \) both are totally positive of order 2 (TP\(_2\)) in \((u, v)\) (Karlin and Studden 1966; Karlin 1968; Pinkus 2010). One of the main purposes of the present paper is to show further that both \( C^* \) and \( c^* \) are TP\(_\infty\), i.e., TP\(_r\) for all \( r \geq 2 \).

In Section 2, we introduce first the general order-complete weak Tchebycheff (OCWT) systems and then a class of copulas, based on B-spline functions, that includes the Bernstein copulas \( C(u, v; R) \) in Eq. (1.4). The maximum correlation copula \( C^*(u, v) \) and its total positivity properties are investigated in Sections 3 and 4, respectively. Finally, in Section 5 we calculate the moments of the related B-spline functions on \([0, \infty)\) and make the connection with the Stirling numbers of the second kind. The proof is given in the Appendix.

## 2 B-Spline Copulas

We consider first a general setting based on OCWT systems, and then we define a class of B-spline copulas that includes the Bernstein copulas as
special cases. In other words, as the Bernstein function is a special case of the B-spline function, using the B-spline function, we can construct the B-spline copula whose form is similar to the Bernstein one \( C(u, v; R) \) in Eq. (1.4).

We also find that the new class of copulas retains the desired properties of the Bernstein copula.

Let \( q_k \geq 0, \, k = 1, \ldots, n, \sum_{k=1}^{n} q_k = 1 \), and let \( \phi_1, \ldots, \phi_n \) be probability densities on \([0, 1]\) such that

\[
\sum_{k=1}^{n} q_k \phi_k(t) = 1, \quad t \in [0, 1].
\]

We assume further that \( \{\phi_1, \ldots, \phi_n\} \) is an order-complete weak Tchebycheff system (OCWT-system), i.e.,

(i) \( \phi_1, \ldots, \phi_n \) are linearly independent, and

(ii) \( \phi_k(t) \) is totally positive of order \( n \) (TP\( n \)) in \((k, t)\), i.e., for each \( r = 1, \ldots, n \),

\[
\det(\phi_{k_i}(t_j))_{r \times r} \geq 0
\]

for all \( k_1 > \cdots > k_r \) and \( t_1 > \cdots > t_r \).

See Karlin and Studden (1966, Chapter 1) or Schumaker (2007, Chapter 2) for examples of OCWT systems.

Let \( q_{1k} \geq 0, \, k = 1, \ldots, n_1, \) such that \( \sum_{k=1}^{n_1} q_{1k} = 1 \). Also, let \( q_{2\ell} \geq 0, \, \ell = 1, \ldots, n_2, \) such that \( \sum_{\ell=1}^{n_2} q_{2\ell} = 1 \). Letting

\[
\Phi_k(u) = \int_0^u \phi_k(t) \, dt,
\]

\( u \in [0, 1] \), we define the B-spline copula, a generalization of the Bernstein copula (1.4), by

\[
C(u, v; R) = \sum_{k=1}^{n_1} \sum_{\ell=1}^{n_2} r_{k\ell} \Phi_k(u) \Phi_\ell(v),
\]

\( u, v \in [0, 1] \), with parameter matrix

\[
R = (r_{k\ell})_{1 \leq k \leq n_1; 1 \leq \ell \leq n_2}, \quad r_{k\ell} \geq 0,
\]

\[
\sum_{k=1}^{n_1} r_{k\ell} = q_{2\ell}, \quad \sum_{\ell=1}^{n_2} r_{k\ell} = q_{1k}, \quad k = 1, 2, \ldots, n_1, \quad \ell = 1, 2, \ldots, n_2.
\]
The copula (2.3) is a bona fide copula since, for any \( u \in [0, 1] \),
\[
    C(u, 1; R) = \sum_{k=1}^{n_1} \sum_{\ell=1}^{n_2} r_{k\ell} \Phi_k(u) = \sum_{k=1}^{n_1} q_{1k} \Phi_k(u)
\]
\[
    = \int_0^u \sum_{k=1}^{n_1} q_{1k} \phi_k(t) \, dt = \int_0^u 1 \, dt = u;
\]
and similarly, \( C(1, v; R) = v \), \( v \in [0, 1] \).

Throughout the paper, we restrict our attention to the case in which \( n_1 = n_2 = n \) and \( q_{1k} = q_{2k} = q_k \); further, we use the notation \( Q = \text{diag}(q_k)_{1 \leq k \leq n} \) for the diagonal matrix with diagonal entries \( q_1, \ldots, q_n \).

**Theorem 1.** For the copula (2.3) with the parameter space (2.4), the maximum correlation is attained when \( r_{k\ell} = q_k \delta_{k\ell} \), equivalently, \( R = Q \).

In the maximum correlation case, \( C(u, v; R) \) becomes
\[
    C^*(u, v) := C(u, v; Q) = \sum_{k=1}^{n} q_k \Phi_k(u) \Phi_k(v), \tag{2.5}
\]
\( u, v \in [0, 1] \).

To prove Theorem 1, we need the following crucial lemma. This result is a generalization of the weak majorization inequality on the closed simplicial cone
\[
    \mathcal{D}_+ = \{ (x_1, \ldots, x_n) : x_1 \geq \cdots \geq x_n \geq 0 \} \subset \mathbb{R}^n_+
\]
for doubly stochastic matrices (Marshall et al. 2011, p. 639).

**Lemma 1.** Let \( a_1 \geq \cdots \geq a_n \geq 0 \) and \( b_1 \geq \cdots \geq b_n \geq 0 \) be given. Let \( q_1, \ldots, q_n \geq 0 \) satisfy \( \sum_{k=1}^{n} q_k = 1 \). Then,
\[
    \max_{\sum_k r_{k\ell} = q_k} \sum_{k=1}^{n} \sum_{\ell=1}^{n} r_{k\ell} a_k b_\ell = \sum_{k=1}^{n} q_k a_k b_k. \tag{2.6}
\]

**Proof.** Let \( a = (a_1, \ldots, a_n) \) and \( b = (b_1, \ldots, b_n) \). Define
\[
    p_{k\ell} = \begin{cases} 
        r_{k\ell}, & k \neq \ell, \\
        r_{kk} + 1 - q_k, & k = \ell.
    \end{cases}
\]
For given \( Q = \text{diag}(q_k)_{1 \leq k \leq n} \), it is straightforward to see that \( P = R + I_n - Q \) is a \( n \times n \) doubly stochastic matrix. Hence, by the famous characterization
of majorization due to Hardy et al. (1929), the vector $c = aP$ is majorized by $a$, denoted $c \prec a$.

We now rearrange the components of $c = (c_1, \ldots, c_n)$ in descending order, listing them as $c_{[1]} \geq \cdots \geq c_{[n]}$, and let $c^* = (c_{[1]}, \ldots, c_{[n]})$. Then we have also $c^* \prec a$ and hence $c^*b^T \leq ab^T$ because $a, b, c^* \in D_+$ (see Marshall et al. 2011, p. 133). On the other hand, note that $cb^T \leq c^*b^T$ because $b, c^* \in D_+$. These results together imply that $aPb^T = cb^T \leq c^*b^T \leq ab^T$.

Consequently, $\max_P aPb^T = ab^T$, which we can write alternatively as

$$\max_{\sum_k p_{k\ell} = 1} \sum_{k=1}^n \sum_{\ell=1}^n p_{k\ell} a_k b_\ell = \sum_{k=1}^n a_k b_k.$$ 

Equivalently, $\max_R aRb^T = aQb^T$ by canceling the common term $ab^T$ out on both sides above. This is exactly (2.6), so the proof now is complete.

**Proof of Theorem 1.** Since $\{\phi_1, \ldots, \phi_n\}$ is an OCWT-system, for all $i < j$ and $s < t$,

$$\phi_i(s)\phi_j(t) - \phi_j(s)\phi_i(t) = \det \begin{pmatrix} \phi_i(s) & \phi_i(t) \\ \phi_j(s) & \phi_j(t) \end{pmatrix} \geq 0.$$ 

Integrating this inequality with respect to $(s, t)$ over $s \in (0,u)$ and $t \in (u,1)$, we obtain

$$\Phi_i(u)(1 - \Phi_j(u)) - \Phi_j(u)(1 - \Phi_i(u)) = \Phi_i(u) - \Phi_j(u) \geq 0,$$

$u \in [0,1]$. Therefore, we obtain the stochastic order,

$$\Phi_1(u) \geq \Phi_2(u) \geq \cdots \geq \Phi_k(u)$$

for all $u \in [0,1]$. Combining this result with Lemma 1, we obtain the inequality

$$C^*(u, v) := C(u, v; Q) \geq C(u, v; R)$$

for all $u, v \in [0,1]$ and $R$ satisfying (2.4). The theorem now follows from Hoeffding’s covariance formula,

$$\text{Cov}(X, Y) = \int_{-\infty}^\infty \int_{-\infty}^\infty \left[ \Pr(X \leq x, Y \leq y) - \Pr(X \leq x) \Pr(Y \leq y) \right] \, dx \, dy$$
Dependence Properties of B-Spline Copulas (see, e.g., Lin et al. 2014). The proof is complete.

Functions $\phi_k$ satisfying (2.1) and (2.2) can be constructed by B-spline functions as we now show. Let $N_i^d$ be a B-spline function on $[0, 1]$ of degree $d \geq 0$ defined as a non-zero B-spline basis with $m + 2d + 1$ knots:

$$t_{-d} = \cdots = t_{-1} = t_0 = 0 < t_1 < \cdots < t_{m-1} < 1 = t_m = t_{m+1} = \cdots = t_{m+d}.$$  

(2.7)

Then, $N_i^d(t)$ is generated by the recursion formula,

$$N_i^d(t) = \frac{t - t_i}{t_{i+d} - t_i} N_i^{d-1}(t) + \frac{t_{i+d+1} - t}{t_{i+d+1} - t_{i+1}} N_{i+1}^{d-1}(t),$$

$$= \frac{t - t_i}{t_{i+d} - t_i} N_i^{d-1}(t) + \left(1 - \frac{t - t_{i+1}}{t_{i+d+1} - t_{i+1}}\right) N_{i+1}^{d-1}(t),$$

$t \in [0, 1]$, for $i = -d, \ldots, -1, 0, 1, \ldots, m - 1$, with initial conditions

$$N_i^0(t) = \begin{cases} 1, & i < m \text{ and } t \in [t_i, t_{i+1}), \\ 0, & \text{otherwise} \end{cases}$$

(see de Boor 1972; de Boor 2001; Nürnberg 1989). The number of non-zero bases is

$$n = m + d.$$

The B-spline is known to satisfy

(i) $N_i^d(t) \geq 0$, $t \in [0, 1]$,

(ii) The support is given by

$$\text{supp}N_i^d = \{t \mid N_i^d(t) > 0\} = [t_i, t_{i+d+1}],$$

$i = -d, \ldots, -1, 0, 1, \ldots, m - 1$, and

(iii) The “partition of unity” property:

$$\sum_{i=-d}^{m-1} N_i^d(t) = 1 \text{ for all } t \in [0, 1].$$

For given $d$ and $m$, let

$$q_k = q_{k,d} = \int_0^1 N_{k-d-1}(t) \, dt, \quad \phi_k(t) = \phi_{k,d}(t) = \frac{1}{q_k} N_{k-d-1}(t),$$

(2.8)
where \( t \in [0,1] \) and \( k = 1, 2, \ldots, n (= m + d) \). Then, Eq. (2.1) holds, and we have the following result (see de Boor 1976, or Schumaker 2007, Theorems 4.18 and 4.65).

**Theorem 2.** Under the hypotheses (2.7) and (2.8), the set \( \{N_i^d\}_{i=-d}^{m-1} \) of B-spline functions, and hence also the B-spline system \( \{\phi_1, \ldots, \phi_n\} \), forms an OCWT-system satisfying (2.2).

To illustrate the use of B-spline systems, we now provide some examples.

**Theorem 3.** Let \( m = 1 \) and the degree \( d = n - 1(= n - m) \). Then the B-splines (2.8) reduce to the Bernstein system (1.3). Specifically, for \( k = 1, \ldots, n \) and \( t \in [0,1] \),

\[
q_k = q_{k,d} = \frac{1}{n}, \quad \phi_k(t) = \phi_{k,d}(t) = b_{k,n}(t). \tag{2.9}
\]

**Proof.** We prove the result by induction on \( d \). Note that for \( d = 0 \) (i.e., \( n = 1 \), \( N_0^0(t) = 1, t \in [0,1] \), and hence

\[
q_1 = q_{1,0} = \int_0^1 N_0^0(t) \, dt = 1, \quad \phi_1(t) = \phi_{1,0}(t) = N_0^0(t) = b_{1,1}(t),
\]

\( t \in [0,1] \). For \( d = 1 \) (i.e., \( n = 2 \)), we have the required \( N_1^1 \), \( q_{1,1} \), and \( \phi_{1,1} \) as follows:

\[
N_{-1}^1(t) = (1-t)N_0^0(t) = 1-t, \quad N_0^1(t) = tN_0^0(t) = t, \quad t \in [0,1],
\]

\[
q_1 = q_{1,1} = \int_0^1 N_{-1}^1(t) \, dt = 1/2, \quad q_2 = q_{2,1} = \int_0^1 N_0^1(t) \, dt = 1/2,
\]

\[
\phi_1(t) = \phi_{1,1}(t) = 2N_{-1}^1(t) = b_{1,2}(t), \quad \phi_2(t) = \phi_{2,1}(t) = 2N_0^1(t) = b_{2,2}(t), \quad t \in [0,1].
\]

Assume that the theorem holds true for \( d = n - 2 \), then we want to prove (2.9) for \( d = n - 1 \). In this case, the B-spline functions are of the form

\[
N_{k-n}^{n-1}(t) = \begin{cases} 
(1-t)N_{2-n}^{n-2}(t) = (n-1)^{-1}(1-t)b_{1,n-1}(t), & k = 1, \\
tN_{k-n}^{n-2}(t) + (1-t)N_{k-n+1}^{n-2}(t) = (n-1)^{-1}[tb_{k-1,n-1}(t) + (1-t)b_{k,n-1}(t)], & 1 < k < n, \\
tN_0^{n-2}(t) = (n-1)^{-1}tb_{n-1,n-1}(t), & k = n.
\end{cases}
\]

It can be shown that for \( k = 1, 2, \ldots, n \),

\[
q_k = q_{k,n-1} = \int_0^1 N_{k-n}^{n-1}(t) \, dt = 1/n
\]
and
\[ N_{k-n}^{n-1}(t) = q_k b_{k,n}(t), \]
\[ t \in [0,1]. \] This completes the proof.

From now on, for simplicity, we consider only the B-spline with equally-spaced knots, i.e., the B-spline functions on \([0,1]\) of order \(d\) having knots given in Eq. (2.7) with \(t_i = i/m, i = 1,2,\ldots,m - 1.\)

**Example 1.** Suppose \(d = 0\), i.e., \(n = m\); then the B-spline system becomes a “histogram”. Namely, for \(k = 1,2,\ldots,n,\)
\[ q_k = q_{k,0} = \frac{1}{n}, \quad \phi_k(t) = \phi_{k,0}(t) = n\mathbb{1}_{[\frac{k-1}{n}, \frac{k}{n})}(t), \tag{2.10} \]
\[ t \in [0,1], \] where \(\mathbb{1}_A\) denotes the indicator function of the set \(A\).

**Example 2.** For \(d = 1\), i.e., \(n = m + 1\), we have
\[ q_1 = q_{1,1} = \frac{1}{2m}, \quad q_2 = q_3 = \cdots = q_{m-1} = q_{m} = \frac{1}{m}, \quad q_{n} = q_{n,1} = \frac{1}{2m}, \]
and the B-spline system is
\[ \phi_1(t) = \phi_{1,1}(t) = q_1^{-1}N_1^1(t) = 2m(1 - mt)\mathbb{1}_{[0, \frac{1}{m})}(t), \]
\[ \phi_n(t) = \phi_{n,1}(t) = q_n^{-1}N_{m-1}^1(t) = 2m(mt - m + 1)\mathbb{1}_{[1 - \frac{1}{m}, 1)}(t), \]
and, for \(k = 2,3,\ldots,n - 1,\)
\[ \phi_k(t) = \phi_{k,1}(t) = q_k^{-1}N_{k-2}^1(t) \]
\[ = q_k^{-1}[(mt - k + 2)N_{k-2}^0(t) + (k - mt)N_{k-1}^0(t)] \]
\[ = m\left[(mt - k + 2)\mathbb{1}_{[\frac{k-2}{m}, \frac{k-1}{m})}(t) + (k - mt)\mathbb{1}_{[\frac{k-1}{m}, \frac{k}{m})}(t)\right], \]
\[ t \in [0,1]. \]

We remark that Shen et al. (2008) earlier proposed the “linear B-spline copula”, which corresponds to the case \(d = 1.\)

**Example 3.** For \(d = 3\) and \(m = 2\), i.e., \(n = 5\), we have
\[ q_1 = 1/8, \quad q_2 = q_3 = q_4 = 1/4, \quad q_5 = 1/8, \]
and the B-spline system is

\[
\begin{align*}
\phi_1(t) &= q_1^{-1}N_{-3}^3(t) = 8(1 - 2t)^31_{[0,1/2)}(t), \\
\phi_2(t) &= q_2^{-1}N_{-2}^3(t) = 8t(7t^2 - 9t + 3)1_{[0,1/2)}(t) + 8(1 - t)^31_{[1/2,1)}(t), \\
\phi_3(t) &= q_3^{-1}N_{-1}^3(t) = 8t^2(3 - 4t)1_{[0,1/2)}(t) + 8(1 - t)^2(4t - 1)1_{[1/2,1)}(t), \\
\phi_4(t) &= \phi_2(1 - t), \\
\phi_5(t) &= \phi_1(1 - t),
\end{align*}
\]

\(t \in [0,1]\). The means of the densities \(\phi_1, \ldots, \phi_5\) are \(1/10, 3/10, 1/2, 7/10,\) and \(9/10\), respectively. We use these values in the computation of Table 1 of maximum correlations for \((n, d) = (5, 3)\).

3 The Maximum Correlation Copula: Range of Correlation

For copula functions, the range of the correlation is of particular importance. In particular, great attention is paid to the maximum achievable correlation (see, e.g., Lin and Huang 2010). By Theorem 1, the maximum is attained when the copula density is

\[
c^*(u, v) = \sum_{k=1}^{n} q_k \phi_k(u)\phi_k(v), \quad u, v \in [0, 1]. \tag{3.1}
\]

Suppose that \((U, V)\) is from the copula density (3.1). Then,

\[
E[UV] = \sum_{k=1}^{n} q_k \left( \int_0^1 u\phi_k(u) \, du \right)^2. \tag{3.2}
\]

Noting that \(E[U] = E[V] = 1/2\) and \(\text{Var}(U) = \text{Var}(V) = 1/12\), it follows that

\[
corr(U, V) = 12 \left( E[UV] - \frac{1}{4} \right). \tag{3.3}
\]

In the Bernstein case \((m = 1)\), it follows from Theorem 3 that \(E[UV] = (2n + 1)/[6(n + 1)]\) and hence

\[
corr(U, V) = 1 - \frac{2}{n + 1}.
\]

In the case of the B-spline of order zero, given in Eq. (2.10),

\[
corr(U, V) = 12 \left( \frac{1}{n} \sum_{k=1}^{n} \left( n \int_{(k-1)/n}^{k/n} t \, dt \right)^2 - \frac{1}{4} \right) = 1 - \frac{1}{n^2}.
\]
In particular, when \( d = 0 \) and \( n = m = 1 \), \( \phi_1(t) = 1 \) on [0, 1], and hence \( C^*(u,v) = uv \), \( u,v \in [0, 1] \). This is the independent case, so \( \text{corr}(U,V) = 0 \).

In order to calculate the maximum correlation for general \( d \), we present first a lemma in which it is understood that the vectors \((q_k)\) and \((r_k)\) reduce to the central parts when \( d = 0 \).

**Lemma 2.** Suppose that \( m \geq d \geq 0 \), i.e., \( n = m + d \geq 2d \geq 0 \). Let \( N^d_i \), \( i = -d,-d+1,\ldots,m-1 \), be the B-spline functions on [0, 1] of order \( d \) having knots (2.7) with \( t_i = i/m \), \( i = 0,1,\ldots,m \). In addition, denote the integral and the first moment of \( N^d_{k-d-1} \) by

\[
q_k = \int_0^1 N^d_{k-d-1}(t) \, dt \quad \text{and} \quad r_k = \int_0^1 tN^d_{k-d-1}(t) \, dt,
\]

\( k = 1,\ldots,n \). Then,

\[
(q_k)_{1 \leq k \leq n} = \frac{1}{m} \begin{pmatrix} 1 & 2 & d & \cdots & d \choose d+1 & d+1 & d+1 & \cdots & d+1 m-d \end{pmatrix} \cdot \begin{pmatrix} \frac{d}{d+1} & \frac{d}{d+1} & \frac{d}{d+1} & \cdots & \frac{d}{d+1} \end{pmatrix},
\]

\[
(r_k)_{1 \leq k \leq n} = \frac{1}{m^2} \begin{pmatrix} 1^2(1+1) & 2^2(2+1) & d^2(d+1) & \cdots & d^2(d+1) \choose 2(d+1)(d+2) & 2(d+1)(d+2) & 2(d+1)(d+2) & \cdots & 2(d+1)(d+2) \end{pmatrix}\cdot \begin{pmatrix} \frac{d+1}{2} & \frac{d+3}{2} & \cdots & \frac{2m-1-d}{2} m-d \end{pmatrix} \cdot \begin{pmatrix} m^2(q_d-r_d) & \cdots & m^2(q_1-r_1) \end{pmatrix}.
\]

**Proof.** For \( 1 \leq k \leq m \), we have \( q_k = \gamma^d_{k-d-1}(0)/m \) and \( r_k = \gamma^d_{k-d-1}(1)/m^2 \), where \( \gamma^d_i(0) \) and \( \gamma^d_i(1) \) are given below in Eqs. (5.8) and (5.9), respectively. Also, for \( k = m+1,\ldots,m+d = n \), we have the relations

\[
q_k = q_{n+1-k}, \quad r_k = q_{n+1-k} - r_{n+1-k},
\]

because \( N^d_{k-d-1}(t) = N^d_{m-k}(1-t) = N^d_{(n+1-k)-d-1}(1-t), t \in [0,1] \). Solving the equations (3.4) in a successive manner, we obtain the stated results.

**Theorem 4.** Under the assumptions of Lemma 2, suppose that \((U,V)\) have the copula density \( c^* \) in Eq. (3.1) with \( \phi_k \) defined through the B-spline functions (2.8) having knots given in Lemma 2. Then the correlation of \((U,V)\) is

\[
\text{corr}(U,V) = 1 - \frac{d+1}{(n-d)^2} + \frac{d(d+3)(2d+3)}{5(d+2)(n-d)^3}.
\]
Proof. Using (3.2), (3.4), and the notations in Lemma 2, write first

$$E[UV] = \sum_{k=1}^{n} \frac{r_k^2}{q_k} = \sum_{k=1}^{m} \frac{r_k^2}{q_k} + \sum_{k=1}^{d} \frac{(q_k - r_k)^2}{q_k}$$

$$= \sum_{k=1}^{m} \frac{r_k^2}{q_k} + \sum_{k=1}^{d} \left( q_k - 2r_k + \frac{r_k^2}{q_k} \right)$$

$$= 2\sum_{k=1}^{d} \frac{r_k^2}{q_k} + \sum_{k=1}^{d} q_k - 2\sum_{k=1}^{d} r_k + \sum_{k=d+1}^{m} \frac{r_k^2}{q_k}. \quad (3.6)$$

The final result is obtained by substituting (3.6) in (3.3) and carrying out the calculations to obtain (3.5) with the help of Lemma 2.

The maximum correlation in Eq. (3.5) remains valid for all cases $m \geq d \geq 0$. Further, the maximum correlation converges to 1 as $n \to \infty$, so we obtain $\text{Var}(U-V) = (1 - \text{corr}(U,V))/6 \to 0$, or $V-U \to 0$ in probability. Therefore, as $n \to \infty$, the random variable $(U,V) = (U,U + (V-U))$ converges in law to $(U,U)$, a bivariate random variable whose joint distribution, remarkably, happens to provide the Fréchet–Hoeffding upper bound, $\min\{u,v\}$. Thus, we have the following result.

Theorem 5. Let $C^*$ be the maximum correlation copula function (2.5) that is constructed by the B-spline

$$\{ N_k^{d-d-1} \}_{k=1}^{n} = \{ N_i^d : i = -d, -d+1, \ldots, m-2, m-1 \}$$

on $[0,1]$ of degree $d \geq 0$, having equally-spaced knots (2.7) with $t_i = i/m$, $i = 0, 1, \ldots, m$, where $m \geq d$. As $m \to \infty$, $C^*(u,v) \to \min\{u,v\}$ for all $u,v$, the Fréchet–Hoeffding upper bound.

Table 1 shows the maximum correlations when the number of basis functions is $n$. In view of Table 1, the range of correlation for the B-spline copulas of small order $d$ is wider than that of the Bernstein copula. Indeed,

$$\text{corr}(U,V) \approx 1 - \frac{d+1}{n^2}.$$ 

On the other hand, $d$ determines the smoothness of the copula density. This makes B-spline copulas more flexible than the Bernstein copula and allow B-spline copulas model distributions more accurately. Consequently, some criterion is needed to evaluate data fitness so as to balance the accuracy of the approximation with the smoothness of the density; this problem will be studied in future work.

We conjecture that Theorem 5 holds in more general settings.
Dependence Properties of B-Spline Copulas

Table 1: Maximum correlations

| n  | d = 0          | d = 1          | d = 2          | d = 3          |
|----|---------------|---------------|---------------|---------------|
| 2  | 0.333         | 0.75          | 0.333         | NA            |
| 3  | 0.5           | 0.889         | 0.667         | 0.5*          |
| 4  | 0.6           | 0.938         | 0.827         | 0.688         |
| 5  | 0.667         | 0.96          | 0.896         | 0.796         |
| 6  | 0.714         | 0.972         | 0.931         | 0.796         |
| 7  | 0.75          | 0.98          | 0.951         | 0.908         |
| 8  | 0.778         | 0.984         | 0.963         | 0.892         |
| 9  | 0.8           | 0.988         | 0.971         | 0.919         |
| 10 | 0.818         | 0.99          | 0.977         | 0.937         |

\(n\) is the number of knots, and \(d\) is the degree of the B-spline.

*: Bernstein case \((m = n - d + 1)\), **: Example 3

Conjecture 1. Let \((U, V)\) be distributed as the maximum correlation distribution \((2.5)\) constructed by the B-spline

\[
\{N^d_{k-d-1}\}_{k=1}^n = \{N^d_i : i = -d, -d+1, \ldots, m-2, m-1\}
\]

on \([0, 1]\), of degree \(d \geq 0\), with the knots \((2.7)\). As \(m \to \infty\) with \(\max_{1 \leq i \leq m} |t_i - 1 - t_i| \to 0\), \(\text{corr}(U, V)\) converges to 1; hence, for all \(u, v\), \(C^*(u, v)\) converges to \(\min\{u, v\}\), the Fréchet–Hoeffding upper bound.

4 The Maximum Correlation Copula: Total Positivity

The next two results improve significantly the previous ones about the Bernstein copulas.

Theorem 6. The copula \(C^*\) in Eq. \((2.5)\) is TP\(_\infty\), i.e., for any \(r \geq 1\),

\[
\det(C^*(u_i, v_j))_{r \times r} \geq 0
\]

for all \(u_1 > \cdots > u_r\) and \(v_1 > \cdots > v_r\).

Proof. All determinants arising in the proof are of order \(r\), unless otherwise specified. Further, we consider two cases: (I) \(r > n\), and (II) \(r \leq n\).

Case I. \(r > n\). In this case, the \(r \times r\) matrix \((C^*(u_i, v_j))_{1 \leq i, j \leq r}\) satisfies

\[
(C^*(u_i, v_j))_{1 \leq i, j \leq r} = \left(\sum_{k=1}^n q_k \Phi_k(u_i) \Phi_k(v_j)\right)_{1 \leq i, j \leq r} = (q_j \Phi_j(u_i))_{1 \leq i \leq r; 1 \leq j \leq n} (\Phi_i(v_j))_{1 \leq i \leq n; 1 \leq j \leq r}.
\]
Consequently, the rank of this matrix is at most $n$, and hence is degenerate. Therefore, it follows obviously that $\det (C^*(u_i, v_j)) = 0$.

Case II. $r \leq n$. We will show that $\det (C^*(u_i, v_j)) \geq 0$. By the Binet–Cauchy formula,

$$\det (C^*(u_i, v_j)) = \det \left( \sum_{k=1}^{n} q_k \Phi_k(u_i) \Phi_k(v_j) \right)$$

$$= \sum_{n \geq k_1 > \cdots > k_r \geq 1} \left( \prod_{i=1}^{r} q_{k_i} \right) \det (\Phi_{k_i}(u_j)) \det (\Phi_{k_i}(v_j)). \quad (4.1)$$

Writing

$$\det (\Phi_{k_i}(u_j)) = \det \left( \int_{0}^{u_j} \phi_{k_i}(t) \, dt \right) = \det \left( \int_{0}^{1} \phi_{k_i}(t) 1_{(0, u_i)}(t) \, dt \right), \quad (4.2)$$

it follows by the continuous version of the Binet–Cauchy formula (Gross and Richards 1998; Karlin 1968) that

$$\det \left( \int_{0}^{1} \phi_{k_i}(t) 1_{(0, u_i)}(t) \, dt \right)$$

$$= \int \cdots \int_{1 > t_1 > \cdots > t_r > 0} \det (\phi_{k_j}(t_i)) \det (1_{(0, u_i)}(t_j)) \prod_{i=1}^{r} dt_i. \quad (4.3)$$

By Theorem 2, \{\phi_{k_1}, \ldots, \phi_{k_r}\} is an OCWT-system, hence

$$\det (\phi_{k_j}(t_i)) \geq 0 \quad (4.4)$$

for all $k_1 > \cdots > k_r$ and $t_1 > \cdots > t_r$. Also, it is well-known from Karlin (1968) & Karlin and Studden (1966) that

$$\det (1_{(0, u_i)}(t_j)) \geq 0$$

for all $u_1 > \cdots > u_r$ and $t_1 > \cdots > t_r$.

Therefore, we deduce from Eqs. (4.2) and (4.3) that $\det (\Phi_{k_i}(u_j)) \geq 0$ for all $k_1 > \cdots > k_r$ and $u_1 > \cdots > u_r$. Similarly, we obtain $\det (\Phi_{k_i}(v_j)) \geq 0$ for $k_1 > \cdots > k_r$ and $v_1 > \cdots > v_r$. Hence, it follows from Eq. (4.1) that $\det (C^*(u_i, v_j)) \geq 0$ for $u_1 > \cdots > u_r$ and $v_1 > \cdots > v_r$. The proof is complete.
Remark 1. For the case of the Bernstein copula, we note that (4.4) is proved as follows. Consider
\[ \phi_k(t) = b_{k,n}(t) = k\binom{n}{k} t^{k-1}(1-t)^{n-k}, \]
\[ t \in [0,1]. \] Then,
\[ \det(\phi_k(t)) = \left( \prod_{i=1}^{r} k_i \binom{n}{k_i} \right) \det \left( t_i^{k_j-1}(1-t_i)^{n-k_j} \right) \]
and
\[ \det \left( t_i^{k_j-1}(1-t_i)^{n-k_j} \right) = \det \left( \left( \frac{t_i}{1-t_i} \right)^{k_j-1} (1-t_i)^{n-1} \right) \]
\[ = \prod_{i=1}^{r} (1-t_i)^{n-1} \cdot \det \left( \left( \frac{t_i}{1-t_i} \right)^{k_j-1} \right). \]

For \( k_1 > \cdots > k_r \geq 1 \), set \( k_j - 1 = \kappa_j + r - j \), \( j = 1, \ldots, r \), and define the partition \( \kappa = (\kappa_1, \ldots, \kappa_r) \), i.e., \( \kappa_1, \ldots, \kappa_r \) are nonnegative integers and \( \kappa_1 \geq \cdots \geq \kappa_r \). Also, let \( z_i = t_i/(1-t_i) \), \( i = 1, \ldots, r \), and let \( z = (z_1, \ldots, z_r) \). Recall from Macdonald (2015) that the Schur function corresponding to the partition \( \kappa \) is defined as
\[ \chi_{\kappa}(z) = \frac{\det(z_i^{k_j-1})}{\prod_{i<j} (z_i - z_j)}. \]
Then we obtain
\[ \det(z_i^{k_j-1}) = \chi_{\kappa}(z) \cdot \prod_{i<j} (z_i - z_j) \]
\[ = \chi_{\kappa}(z) \cdot \prod_{i<j} \left( \frac{t_i}{1-t_i} - \frac{t_j}{1-t_j} \right) \]
\[ = \chi_{\kappa}(z) \cdot \prod_{i<j} \frac{t_i - t_j}{(1-t_i)(1-t_j)}. \]
It is well-known that \( \chi_{\kappa}(z) \geq 0 \) for all \( z_1, \ldots, z_r \geq 0 \) (Gross and Richards 1998; Macdonald 2015), and hence
\[ \det \left( t_i^{k_j-1}(1-t_i)^{n-k_j} \right) \geq 0 \]
for all \( k_1 > \cdots > k_r \) and \( t_1 > \cdots > t_r \). This completes the proof of Eq. (4.4).
As a consequence of Theorem 6, we obtain a new proof of the total positivity of the function \( \min\{u, v\} \); see (Karlin 1968, Chapter 2).

**Corollary 1.** The Fréchet-Hoeffding upper bound, \( \min\{u, v\} \), is TP\(_\infty\).

**Proof.** Recall that for the Bernstein copula, \( C^*(u, v) \) increases to \( \min\{u, v\} \) as the number \( n \) of basis functions goes to infinity (see Huang et al. 2013). Moreover, it follows from Theorem 5 that for the equally-spaced knot B-spline copula, \( C^*(u, v) \) converges to \( \min\{u, v\} \). In either case, by taking the limit, as \( n \to \infty \), of the \( r \times r \) nonnegative determinant, \( \det(C^*(u_i, v_j)) \), we obtain

\[
\min\{u, v\} = \lim_{n \to \infty} C^*(u, v) \geq 0,
\]

which proves that the function \( \min\{u, v\} \) is TP\(_r\). Finally, since \( r \) is arbitrary it follows that the function \( \min\{u, v\} \) is TP\(_\infty\).

By mimicking the proof of Theorem 6, we actually have the following stronger result. The proof is omitted.

**Theorem 7.** The copula density \( c^* \) in Eq. (3.1) is TP\(_\infty\).

Theorem 6 is in fact a consequence of Theorem 7 by using Lemma 3 below, but we provide a direct proof there.

Let \((X, Y) \sim H\) with marginal distributions \( F \) and \( G \), and copula function \( C \). Using the language of reliability theory, define the survival functions

\[
\overline{F}(x) = 1 - F(x), \quad \overline{G}(y) = 1 - G(y),
\]

and

\[
\overline{H}(x, y) = \Pr(X > x, Y > y) = 1 - F(x) - G(y) + H(x, y),
\]

\(x, y \in \mathbb{R}\). It follows from the definition of the copula function that \( H(x, y) = C(F(x), G(y)) \) and \( \overline{H}(x, y) = \overline{C}(\overline{F}(x), \overline{G}(y)) \) for all \( x, y \in \mathbb{R}\). Recently, Lin et al. (2018) proved the following result.

**Lemma 3.** If the bivariate distribution \( H \) has a TP\(_r\) density with \( r \geq 2 \), then both \( H \) and \( \overline{H} \) are TP\(_r\). Consequently, if \( H \) has a TP\(_\infty\) density, both \( H \) and \( \overline{H} \) are TP\(_\infty\).

An immediate consequence of the last two theorems is the following result. In part (ii) of this result, we apply the fact that both \( F \) and \( G \) are non-decreasing, while both \( \overline{F}, \overline{G} \) are non-increasing (see Marshall et al. 2011, p. 758).

**Corollary 2.** Let \( C^* \) be the copula defined in Eq. (2.5).

(i) The survival function \( \overline{C^*} \) is TP\(_\infty\).

(ii) If \((X, Y) \sim H\) with copula \( C^* \), then both \( H \) and \( \overline{H} \) are TP\(_\infty\).
We next discuss some implications of the total positivity. By the results of Gross and Richards (1998, Section 3, Example 3.7), we can obtain some inequalities for moments and probabilities, which might be useful in statistical inference and reliability theory (see, e.g., Block and Sampson 2006 and the references therein). In general, these inequalities are hard to derive directly, even for specific bivariate distributions.

**Corollary 3.** Let \((X, Y) \sim H\) with marginals \(F, G\) and copula \(C^*\) in Eq. (2.5) and \(r \geq 2\). Then the matrix

\[
(E[X^{i-1}Y^{j-1}])_{1 \leq i, j \leq r} = \begin{pmatrix}
1 & E[Y] & \cdots & E[Y^{r-1}]
E[X] & E[XY] & \cdots & E[XY^{r-1}]
\vdots & \vdots & \ddots & \vdots
E[X^{r-1}] & E[X^{r-1}Y] & \cdots & E[X^{r-1}Y^{r-1}]
\end{pmatrix}
\]  

(4.5)
is TP\(_r\), provided the expectations exist.

Let \(x_1 < \cdots < x_r\) and \(y_1 < \cdots < y_r\). The matrix

\[
(\overline{H}(x_i, y_j))_{1 \leq i, j \leq r} = \begin{pmatrix}
\overline{H}(x_1, y_1) & \cdots & \overline{H}(x_1, y_r)
\vdots & \ddots & \vdots
\overline{H}(x_r, y_1) & \cdots & \overline{H}(x_r, y_r)
\end{pmatrix}
\]  

(4.6)
is TP\(_r\). Equivalently, \(\overline{H}\) is TP\(_r\).

In particular, when \(r = 3\), it follows from Eq. (4.5) that

\[
\det \begin{pmatrix}
1 & E[Y] & E[Y^2]
E[X] & E[XY] & E[XY^2]
E[X^2] & E[X^2Y] & E[X^2Y^2]
\end{pmatrix} \geq 0,
\]

an inequality that is equivalent to

\[
E[X^2Y^2]E[XY] - E[X^2Y]E[XY^2] - E[X]E[X^2Y^2]E[Y] + E[X^2]E[XY^2]E[Y] + E[X]E[X^2Y]E[Y^2] - E[X^2]E[XY]E[Y^2] \geq 0.
\]

When \(E[X] = E[Y] = 0\), it further reduces to

\[
\text{Cov}(X^2, Y^2) E[XY] \geq E[X^2Y]E[XY^2],
\]

from which we can estimate the lower (or upper) bound of covariance of \(X^2\) and \(Y^2\) in terms of the lower order product moments.
Let \( x_1 = -\infty < x_2 = x < x_3 = x' \) and \( y_1 = -\infty < y_2 = y < y_3 = y' \). Note that \( F(x) = \overline{H}(x, -\infty) \) and \( G(y) = \overline{H}(-\infty, y) \). By Eq. (4.6), the matrix
\[
\begin{pmatrix}
1 & \overline{G}(y) & \overline{G}(y') \\
F(x) & \overline{H}(x, y) & \overline{H}(x, y') \\
F(x') & \overline{H}(x', y) & \overline{H}(x', y')
\end{pmatrix}
\]
is totally positive of order 3. By calculating the 2 \times 2 principal minor of this matrix, we find that
\[
\overline{H}(x, y) \geq F(x)G(y); \text{ equivalently, } \overline{H}(x, y) \geq F(x)G(y),
\]
\( x, y \in \mathbb{R} \), i.e., the distribution function \( \overline{H} \) is positively quadrant dependent. Further, by calculating the determinant of this matrix, we obtain
\[
\overline{H}(x', y') \overline{H}(x, y) - \overline{H}(x', y) \overline{H}(x, y') - F(x) \overline{H}(x', y') \overline{G}(y) + F(x') \overline{H}(x', y) \overline{G}(y') - F(x') \overline{H}(x, y) \overline{G}(y') \geq 0
\]
for \( x < x' \), \( y < y' \). This example shows that TP\(_3\) includes TP\(_2\); furthermore, it provides inequalities on moments and probabilities of its own accord. We remark that more general inequalities can be deduced from Gross and Richards (1998, Example 3.11).

5 Moments of the B-Spline Functions with Initial Boundary

In this section, we provide the moment formula for the B-spline functions with initial boundary at \( t = 0 \) defined on \( \mathbb{R}_+ = [0, \infty) \). The expressions for \( q_k \) and \( r_k \) in Lemma 2 are obtained, in Corollary 4 below, as a consequence of the moment formula.

Let \( N^d_i \) be a B-spline function of degree \( d \geq 0 \) on \( \mathbb{R}_+ \) with knots:
\[
\underbrace{t_{-d} = \cdots = t_{-1}}_d = t_0 = 0 < t_1 = 1 < t_2 = 2 < \cdots \tag{5.1}
\]
(compare with the previously studied B-spline function defined in Eq. (2.7)). Here, we have \( t_i = (i)_+ = \max\{i, 0\} \) and, as before, \( N^d_i(t) \) is generated by the following recursion formula:
\[
N^d_i(t) = \frac{t - (i)_+}{(i + d)_+ - (i)_+} N^{d-1}_i(t) + \frac{(i + d + 1)_+ - t}{(i + d + 1)_+ - (i + 1)_+} N^{d-1}_{i+1}(t), \tag{5.2}
\]
d \( \geq 1 \), with initial conditions
\[
N^0_i(t) = \begin{cases} 
1, & i \geq 0 \text{ and } t \in [i, i + 1), \\
0, & \text{otherwise}.
\end{cases}
\]
For each $i \geq -d$, $N^d_i$ is a non-zero function with support $[\max\{i, 0\}, i+d+1]$. The recurrence (5.2) can be written more concretely as

$$N^d_i(t) = \begin{cases} 
\frac{t-i}{d} N^d_{i-1}(t) + \frac{i+d+1-t}{d} N^d_{i+1}(t), & i \geq 0, \\
\frac{i}{i+d} N^d_{i-1}(t) + \frac{i+d+1-t}{i+d+1} N^d_{i+1}(t), & -d < i \leq -1, \\
(1-t) N^d_{i-d+1}(t), & i = -d, \\
0, & i < -d.
\end{cases}$$

For $h \geq 0$, denote the $h$-moment of $N^d_i$,

$$\gamma^d_i(h) := \int_{-\infty}^{\infty} t^h N^d_i(t) \, dt = \int_{\max\{i, 0\}}^{i+d+1} t^h N^d_i(t) \, dt;$$

this quantity was used in the proof of Lemma 2 above. Then, we have the following recurrence relation for these moments.

$$\gamma^d_i(h) = \begin{cases} 
\gamma^d_{i-1}(h+1) - \frac{i^{d-1}}{d} \gamma^{d-1}_{i+1}(h), & i \geq 0, \\
\gamma^d_{i-1}(h+1) - (i+d+1) \gamma^{d-1}_{i+1}(h) - \gamma^d_{i+1}(h+1), & -d < i < 0, \\
\gamma^d_{i-d+1}(h) - \gamma^d_{i-d+1}(h+1), & i = -d, \\
0, & i < -d,
\end{cases}$$

with boundary condition

$$\gamma^0_i(h) = \begin{cases} 
\frac{(i+1)^{h+1} - i^{h+1}}{h+1}, & i \geq 0, \\
0, & i < 0.
\end{cases}$$

The next result, which is interesting in its own right, presents the solution of the recurrence system in terms of the Stirling numbers of the second kind:

$$S(n, k) = \frac{1}{k!} \sum_{j=0}^{k} (-1)^j \binom{k}{j} (k-j)^n.$$
Here, $S(n, 0) = \delta_{n0}$, $S(n, k) = 0$ for $n < k$, and $0^0 \equiv 1$ whenever it arises. Note also that $S(n, 1) = S(n, n) = 1$ and $S(n, n - 1) = n(n - 1)/2$. The Stirling numbers of the second kind satisfy the recurrence formula

$$S(n + 1, k) = kS(n, k) + S(n, k - 1),$$

(5.5)

and the identity

$$S(n + 1, k + 1) = \sum_{j=k}^{n} \left( \begin{array}{c} n \\ j \end{array} \right) S(j, k) = \sum_{j=0}^{n-k} \left( \begin{array}{c} n \\ n-j \end{array} \right) S(n - j, k),$$

(5.6)

which will be used later. For the identity (5.6), see Wagner (1996) and the end of Remark 2 below.

In terms of the Stirling number of the second kind, the recurrence Eq. (5.3) with (5.4) can be represented explicitly as follows. The proof is given in the Appendix.

**Theorem 8.** For $d \geq 0$, the $h$-moment of the B-spline function $N_i^d$ in Eq. (5.8) is of the form

$$\gamma_i^d(h) = \begin{cases} 
\sum_{\ell=0}^{h} \ell^i \left( \begin{array}{c} h \\ \ell \end{array} \right) S(h + d + 1 - \ell, d + 1), & i \geq 0, \\
\frac{i + d + 1}{d + 1} S(h + i + d + 1, i + d + 1) \left( \begin{array}{c} h + d + 1 \\ d + 1 \end{array} \right), & -d \leq i \leq 0, \\
0, & i < -d.
\end{cases}$$

(5.7)

**Corollary 4.** For $h = 0, 1$, we have

$$\gamma_i^d(0) = \begin{cases} 
1, & i \geq 0, \\
\frac{i + d + 1}{d + 1}, & -d \leq i \leq 0, \\
0, & i < -d
\end{cases}$$

(5.8)

and

$$\gamma_i^d(1) = \begin{cases} 
\frac{d + 2i + 1}{2}, & i \geq 0, \\
\frac{(i + d + 1)^2(i + d + 2)}{2(d + 1)(d + 2)}, & -d \leq i \leq 0, \\
0, & i < -d.
\end{cases}$$

(5.9)
The formulas (5.8) and (5.9) can be applied to obtain the formula for
the maximum correlation (3.5).

**Corollary 5.** For $i = 0, 1$,

\[
\gamma_i^d(h) = \frac{S(h+i+d+1, i+d+1)}{\binom{h+d+1}{d+1}}. \tag{5.10}
\]

**Proof.** For the case $i = 1$,

\[
\gamma_1^d(h) = \sum_{\ell=0}^{h} \binom{h}{\ell} \frac{S(h+d+1-\ell, d+1)}{\binom{h+d+1-\ell}{d+1}}
\]

\[
= \frac{1}{\binom{h+d+1}{d+1}} \sum_{\ell=0}^{h} \binom{h+d+1}{h+d+1-\ell} S(h+d+1-\ell, d+1)
\]

\[
= \frac{S(h+d+2, d+2)}{\binom{h+d+1}{d+1}},
\]

by the identity (5.6).

**Remark 2.** Recall the generalized (higher-order) Bernoulli polynomial $B^{(x)}_\ell$ defined by the generating function

\[
\left(\frac{t}{e^t - 1}\right)^x = \sum_{\ell=0}^{\infty} B^{(x)}_\ell \frac{t^\ell}{\ell!}, \tag{5.11}
\]

$|t| < 2\pi$, $x \in \mathbb{R}$, where $B^{(x)}_\ell$ is a polynomial of degree $\ell$ in $x$ with rational
coefficients. Neuman (1981, Proposition 3.5) showed that $\gamma_0^d(h) = B_h^{(-d+1)}$, from which (5.10) for $i = 0$ in Corollary 5 also follows by the relationship between the Stirling number and the generalized Bernoulli polynomial:

\[
S(n+k, k) = \binom{n+k}{k} B_n^{(-k)}. \tag{5.12}
\]

The latter can be verified by Eq. (5.11) and the exponential generating function,

\[
\frac{1}{k!} (e^t - 1)^k = \sum_{n=k}^{\infty} S(n, k) \frac{t^n}{n!}, \tag{5.13}
\]
\[ t \in \mathbb{R}, \ k \geq 0. \text{ See Carlitz (1960) and Branson (2000) for Eqs. (5.12) and (5.13), respectively. The generating function (5.13) can also be established by induction on } k \text{ and this is equivalent to verifying the above useful identity (5.6).} \]

**Remark 3.** By iteration, we have

\[ N^d_{-d}(t) = (1 - t)N^d_{-d+1}(t) = \cdots = (1 - t)^d N^0_0(t) = (1 - t)^d \mathbb{1}_{[0,1)}(t), \]

and hence its \( h \)-moment is equal to

\[ \gamma^d_{-d}(h) = \int_0^1 t^h (1 - t)^d \, dt = \frac{\Gamma(h + 1) \Gamma(d + 1)}{\Gamma(h + d + 2)} = \frac{h! d!}{(h + d + 1)!}, \]

as shown in Eq. (6.5).

**Remark 4.** It can be shown that for \( i \geq 0, \)

\[ N^d_{i+1}(t) = N^d_i(t - 1), \quad i + 1 \leq t < i + d + 2. \]

Therefore, the \( h \)-moment of \( N^d_{i+1} \) can be calculated as

\[
\gamma^d_{i+1}(h) = \int_{i+1}^{i+d+2} t^h N^d_{i+1}(t) \, dt = \int_{i+1}^{i+d+2} t^h N^d_i(t - 1) \, dt \\
= \int_{i}^{i+d+1} (t + 1)^h N^d_i(t) \, dt = \sum_{j=0}^{h} \binom{h}{j} \gamma^d_j(i), \quad i \geq 0.
\]

This is equivalent to the first formula of Eq. (5.7) (with \( i \geq 0 \)). Indeed, for \( i \geq 1, \) it follows from Eq. (5.7) that

\[
\sum_{j=0}^{h} \binom{h}{j} \gamma^d_{i-1}(j) = \sum_{j=0}^{h} \binom{h}{j} \sum_{k=0}^{j} (i - 1)^k \binom{j}{k} \frac{S(j + d + 1 - k, d + 1)}{(j + d + 1 - k) (d + 1)},
\]

and if we now change variables from \( 0 \leq k \leq j \leq h \) to \( 0 \leq k \leq \ell \leq h, \) where \( \ell = h - j + k, \) then we obtain

\[
\sum_{j=0}^{h} \binom{h}{j} \gamma^d_{i-1}(j) = \sum_{\ell=0}^{h} \sum_{k=0}^{\ell} (i - 1)^k \binom{\ell}{k} \binom{h}{\ell} \frac{S(h + d + 1 - \ell, d + 1)}{(h + d + 1 - \ell) (d + 1)} \\
= \sum_{\ell=0}^{h} i^\ell \binom{h}{\ell} \frac{S(h + d + 1 - \ell, d + 1)}{(h + d + 1 - \ell) (d + 1)} \\
= \gamma^d_i(h).
\]
6 Discussion

The copula was originally proposed by Sklar (1959), and is now widely used to describe dependence structures of multivariate data in various fields, including quantitative finance, civil engineering, reliability engineering, and medicine. To model varying dependence structures of data, flexible copulas are required. In this paper, as a generalization of the Bernstein copula, we defined the B-spline copula and examined its correlation range. The B-spline copula is shown to represent a wider range of correlation than the Bernstein copula, which suggests that the B-spline copula is sufficiently flexible for practical use. Although we only provided the explicit form of the bivariate B-spline copula, this actually can be extended to multivariate cases by replacing the parameter matrix \( R \) in Eq. (2.3) with a multidimensional array of parameters as demonstrated in Dou et al. (2013, 2016) for the Bernstein copula.

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Appendix

Proof of Theorem 8. We prove the statement by mathematical induction on $d$. Note first that (5.7) with $d = 0$ coincides with the boundary conditions (5.4) for all $i$ and $h$.

Suppose that (5.7) is true for the case $d - 1$ and for all $i$ and $h$, then we wish to prove that it also holds true for the case $d$ and for all $i$ and $h$.

(i) For $i \geq 0$, by the assumption of induction,

$$
\gamma_i^{d-1}(h) = \sum_{\ell=0}^{h} i^\ell \binom{h}{\ell} \frac{S(h + d - \ell, d)}{\binom{h + d - \ell}{d}}.
$$

Then, we have

$$
\gamma_{i+1}^{d-1}(h) = \sum_{k=0}^{h} (i + 1)^k \binom{h}{k} \frac{S(h + d - k, d)}{\binom{h + d - k}{d}}.
$$
and by expanding \((i + 1)^k\) using the binomial theorem, we obtain

\[
\gamma_{i+1}^{d-1}(h) = \sum_{k=0}^{h} \sum_{\ell=0}^{k} i^\ell \binom{k}{\ell} \left( \frac{h}{k} \right) S(h + d - k, d) \left( \frac{h + d - k}{d} \right).
\]

Interchanging the order of summation and using the identity,

\[
\binom{k}{\ell} \left( \frac{h}{k} \right) = \binom{h}{\ell} \left( \frac{h - \ell}{k - \ell} \right),
\]

we find that

\[
\gamma_{i+1}^{d-1}(h) = \sum_{\ell=0}^{h} i^\ell \binom{h}{\ell} \sum_{k=\ell}^{h} \left( \frac{h - \ell}{k - \ell} \right) S(h + d - k, d) \left( \frac{h + d - k}{d} \right).
\]

Replacing \(k\) by \(k - \ell\), we have

\[
\gamma_{i+1}^{d-1}(h) = \sum_{\ell=0}^{h} i^\ell \binom{h}{\ell} \sum_{k=0}^{h-\ell} \left( \frac{h - \ell}{k} \right) S(h + d - \ell - k, d) \left( \frac{h + d - \ell - k}{d} \right),
\]

and using the identity,

\[
\frac{\left( \frac{h - \ell}{k} \right)}{\left( \frac{h + d - \ell - k}{d} \right)} = \frac{\left( \frac{h + d - \ell}{h + d - \ell - k} \right)}{\left( \frac{h + d - \ell}{d} \right)},
\]

we deduce that

\[
\gamma_{i+1}^{d-1}(h) = \sum_{\ell=0}^{h} i^\ell \binom{h}{\ell} \frac{1}{\left( \frac{h + d - \ell}{d} \right)} \sum_{k=0}^{h-\ell} \left( \frac{h + d - \ell}{h + d - \ell - k} \right) \\
\times S(h + d - \ell - k, d)
\]

\[
= \sum_{\ell=0}^{h} i^\ell \binom{h}{\ell} S(h + d - \ell + 1, d + 1) \left( \frac{h + d - \ell}{d} \right), \tag{6.1}
\]

where the last equality follows from the identity (5.6). Moreover, using the identity,

\[
\binom{h + 1}{\ell} = \binom{h}{\ell - 1} + \binom{h}{\ell},
\]
we obtain

\[ \gamma_i^{d-1}(h+1) - i \gamma_i^{d-1}(h) = i^{h+1} + \sum_{\ell=0}^{h} \left[ i^\ell \binom{h+1}{\ell} \frac{S(h + d - \ell + 1, d)}{d} \right] - i^{\ell+1} \binom{h}{\ell} \frac{S(h + d - \ell, d)}{d} \]

\[ = i^{h+1} + \sum_{\ell=0}^{h} \left[ i^\ell \left\{ \binom{h}{\ell} - \binom{h}{\ell-1} \right\} \frac{S(h + d - \ell + 1, d)}{d} \right] - i^{\ell+1} \binom{h}{\ell} \frac{S(h + d - \ell, d)}{d} \]. \tag{6.2} \]

Since \( S(d, d) = 1 \) then

\[ i^{h+1} + \sum_{\ell=0}^{h} i^\ell \binom{h}{\ell-1} \frac{S(h + d - \ell + 1, d)}{d} = \sum_{\ell=1}^{h+1} i^\ell \binom{h}{\ell-1} \frac{S(h + d - \ell + 1, d)}{d} \]

\[ = \sum_{\ell=0}^{h} i^\ell \binom{h}{\ell} \frac{S(h + d - \ell, d)}{d}, \]

and substituting this result into (6.2), we obtain

\[ \gamma_i^{d-1}(h+1) - i \gamma_i^{d-1}(h) = \sum_{\ell=0}^{h} i^\ell \binom{h}{\ell} \frac{S(h + d - \ell + 1, d)}{d}. \tag{6.3} \]
Similarly, it follows from Eq. (6.1) that
\[
\gamma_{i+1}^{d}(h+1) - i\gamma_{i+1}^{d-1}(h) \\
= i^{h+1} + \sum_{\ell=0}^{h} i^{\ell} \frac{S(h+d-\ell+2,d+1)}{d} \left( \frac{h+d-\ell+1}{d} \right) \\
- i^{\ell+1} \frac{S(h+d-\ell+1,d+1)}{d} \left( \frac{h+d-\ell}{d} \right) \\
= \sum_{\ell=0}^{h} i^{\ell} \frac{S(h+d-\ell+2,d+1)}{d} \left( \frac{h+d-\ell+1}{d} \right).
\]

Hence, by substituting (6.3) and (6.4) into (5.3), we find that
\[
\gamma_{i}^{d}(h) = \frac{\gamma_{i}^{d-1}(h+1) - i\gamma_{i}^{d-1}(h)}{d} - \frac{\gamma_{i+1}^{d-1}(h+1) - i\gamma_{i+1}^{d-1}(h)}{d} + \frac{d+1}{d} \gamma_{i+1}^{d-1}(h) \\
= \sum_{\ell=0}^{h} i^{\ell} \frac{S(h+d-\ell+1,d+1)}{d} \left( \frac{h+d-\ell+1}{d+1} \right) \\
+ \frac{(d+1)S(h+d-\ell+1,d+1)}{d} \left( \frac{h+d-\ell}{d} \right)
\]

where the last equality follows from the recurrence formula (5.5).

(ii) When \(-d < i < 0\), by the assumption of induction,
\[
\gamma_{i}^{d-1}(h) = \frac{i + d}{d} \cdot \frac{S(h+i+d,i+d)}{(h+d)}.
\]
It then follows from Eq. (5.3) that

\[
\gamma^d_i(h) = \frac{\gamma^{d-1}_{i+1}(h+1)}{i+d+1} + \frac{(i + d + 1)\gamma^{d-1}_{i+1}(h) - \gamma^{d-1}_i(h+1)}{i+d+1} \\
= \frac{1}{i+d+1} \frac{S(h+i+d+1,i+d)}{\binom{h+d+1}{d}} \\
+ \frac{i+d+1}{d} \frac{S(h+i+d+1,i+d+1)}{\binom{h+d}{d}} \\
- \frac{1}{i+d+1} \frac{S(h+i+d+2,i+d+1)}{\binom{h+d+1}{d}} \\
= \frac{i+d+1}{d+1} \frac{S(h+i+d+1,i+d+1)}{\binom{h+d+1}{d+1}}.
\]

Here we used the recurrence formula (5.5) with \( n = h + i + d + 1 \) and \( k = i + d + 1 \), viz.,

\[
S(h+i+d+2,i+d+1) = (i+d+1)S(h+i+d+1,i+d+1) + S(h+i+d+1,i+d).
\]

(iii) When \( i = -d \), by the inductive hypothesis,

\[
\gamma^{d-1}_{-d+1}(h) = \frac{h!(d-1)!}{(h+d)!}.
\]

Then, by Eq. (5.3), we have

\[
\gamma^d_{-d}(h) = \gamma^{d-1}_{-d+1}(h) - \gamma^{d-1}_{-d+1}(h+1) = \frac{h!d!}{(h+d+1)!}, \quad (6.5)
\]

which coincides with Eq. (5.7) with \( i = -d \).

The proof is completed by induction on \( d \).

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