Theoretical Calculation and Numerical Simulation Based on Radial Fin of Triangular Profile

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Abstract. In most heat exchange equipment, in order to increase heat dissipation between solid surface and external fluid, the heat transfer surface is often made into finned form. This is because the method of increasing the surface area of the fin wall is used to reduce the heat resistance of the convective heat transfer and enhance the heat transfer. There are various types of fin wall, including straight fin, radial fin and needle fin. The radial fin of triangular profile belongs to the radial fin with variable thickness profile. Based on the theoretical calculation and numerical simulation of the excess temperature distribution and fin efficiency of the radial fin with triangular profile, the excess temperature distribution equation and the calculation formula of fin efficiency on the wall of the radial fin with triangular profile are derived, and the curve of excess temperature distribution and fin efficiency is fitted and output by MATLAB, the theoretical basis for the design and application of this kind of radial fin is provided.

1. Introduction

In general, in order to increase the heat transfer on the surface of the object, the surface structure with fins or other protrusions of other shapes is often used, so that the heat transfer surface can be expanded and the heat transfer area can be effectively increased. When the fluid whose temperature is different from the surface temperature of the object flows in the environment, the heat inside the object will not only transfer along the direction of the expanded surface, but also continuously exchange heat with the surface in the direction perpendicular to the direction of the expanded surface in the way of convective heat transfer, which makes the temperature distribution of the expanded surface, that is, the excess temperature distribution of the fin, change accordingly with the direction of its extension. At the same time, the efficiency of the extended surface, that is, the fin efficiency, will change when the extended surface is continuously extended and changed. Therefore, it is the main task to understand the excess temperature distribution and fin efficiency of the fin when analyzing the thermal conductivity of the extended surface.

When analyzing the heat conduction of the fin, the differential equation of heat conduction of the fin should be derived first, and the following assumptions should be made[1]:

(1) The fin is very thin and small, and the temperature of any cross-section along the extension direction of the fin is considered to be uniform, so the heat conduction of the fin can be regarded as one-dimensional.

(2) The thermal conductivity and surface convective heat transfer coefficient of the finned material are constant.

Based on the above assumptions, the thermodynamics analysis of the fin can be carried out to construct the energy balance equation. If the length of the fin is \( l \), the thermal conductivity is \( \lambda \), the...
convection heat transfer coefficient is $h$, the ambient temperature is $T_r$, and the fin temperature is $T$, then the excess temperature of the fin is $\theta = T - T_r$. The cross-sectional area $A$ is a function of the variable $x$ on the extension direction of the fin, and $A_s$ is the side area of the fin. The micro element body with the length of $dx$ is taken as the control body for analysis, and its energy balance equation can be obtained:

$$\phi_x = \phi_{x+dx} + d\phi_c$$

In equation (1), $\phi_x$ and $\phi_{x+dx}$ are respectively the thermal heat flux of the control body entering and leaving, which is determined by Fourier law; $d\phi_c$ is the convective heat transfer flux between the surface of the micro element body and the surrounding ambient fluid, which is determined by Newton's cooling law.

$$\phi_x = -\lambda A \frac{dT}{dx}$$

$$\phi_{x+dx} = \phi_x + \frac{d\phi_x}{dx} dx = -\lambda A \frac{dT}{dx} - \frac{d(\lambda A \frac{dT}{dx})}{dx}$$

$$d\phi_c = h(T - T_r) dA_s$$

With substituted equations (2), (3) and (4) into equation (1), can get:

$$\frac{d}{dx} \left( A \frac{dT}{dx} \right) - \frac{h}{\lambda} \frac{dA_s}{dx} (T - T_r) = 0$$

Equation (5) is a one-dimensional differential equation of heat conduction in the fin. In most of the fin heat conduction problems, the above equation can be used to construct the differential equation of fin heat conduction with different shapes, and the distribution of excess temperature of fin can be obtained by solving the differential equation of heat conduction.

When Schmidt solved the problem of the best shape and size of the fin, he found that the fin had better be bounded by two parabolas. In order to approach this type and be easy to manufacture, the fin was often made into trapezoid or triangular section shape.[1]

In the calculation of fin heat conduction differential equation, it is found that most of the differential equations belong to the form of Bessel equation, and the solution of fin heat conduction differential equation can be obtained according to the form of Bessel equation solution. In previous study for the radial fin, because the heat transfer area is along the fin height direction changes, the analysis result is much more complex than straight fin. The radial fin can be replaced by the corresponding straight fin without changing the heat transfer effect, and a simplified formula for calculating the efficiency of the radial fin is derived, which is similar to the formula for calculating the efficiency of the straight fin.[2] However, the radial fin and the straight fin are fundamentally different, in which the differential equation of heat conduction of the radial fin of triangular profile does not belong to Bessel equation, and must be calculated in other ways.

2. Calculation model

2.1. Physical model

There are many forms of fin, usually can be divided into needle fin, straight fin and radial fin. The needle fin is divided into cylindrical fin with equal profile, conical fin and parabolic fin. The straight fin is divided into straight fin with equal profile, triangular fin and parabolic fin. The radial fin are also divided into radial fin with equal thickness, radial fin of triangular profile and radial fin of hyperbolic profile.[3] This paper mainly analyzes the heat transfer characteristics of the radial fin of triangular profile. The radial fin of triangular profile heat exchanger (Figure 1) was used as its physical model for research and analysis. The structural diagram of the radial fin of triangular profile is shown in Figure 2,
where the fin thickness is \( \delta_b \) (mm), fin height is \( b \) (mm) and pipe radius is \( r_b \) (mm). In order to facilitate the heat transfer analysis of the radial fin of triangular profile, the following assumptions need to be made:[4-5]

1. The thermal conductivity of the finned material is uniform, the thermal conductivity coefficient is constant along the direction of the fin height, and the temperature in the finned material changes little along the direction of thickness, so it can be considered as a one-dimensional temperature field.
2. The end of the fin is adiabatic.
3. The radiative heat transfer from and to the finned surface is ignored.
4. There is no condensate on the fin, that is, the system is in full dry condition.

2.2. Mathematical model

Take the radial fin profile to establish a rectangular coordinate system, as shown in Figure 3. Take a micro element segment with length \( dr \) from the center line \( r \) of the tube.

The function expression of the triangle section line is:

\[
f(r) = \frac{\delta_b}{2b} (r - r_a)
\]  

(6)

The energy balance of radial fin is constructed by thermodynamic analysis. The heat going into the radial fin:
The heat loss of the radial fin:

\[ q = k \cdot 2\pi r \cdot 2f(r) \frac{d\theta}{dr} \]  

The heat coming out of the radial fin:

\[ q_{r+dr} = q_r + \frac{dq}{dr} \, dr \]  

The energy balance equation of the radial fin:

\[ q_r = q_{r+dr} + q \]  

By substituting equations (8) and (9) into equation (10), that

\[ \frac{dq}{dr} \, dr + 2h(2\pi r)dr \theta = 0 \]  

By substituting equation (7) into Equation (11), the energy balance equation (12) of the radial fin can be obtained:

\[ f(r) \frac{d^2\theta}{dr^2} + \frac{f(r)}{r} \frac{d\theta}{dr} + \frac{df(r)}{dr} \frac{d\theta}{dr} - \frac{h}{k} \theta = 0 \]  

3. Theoretical calculation

3.1. The temperature excess of the radial fin of triangular profile

By solving the energy balance equation of the radial fin of triangular profile and substituted the function equation (6) of the triangle section line of the radial fin into the energy balance equation (12), the governing differential equation of the temperature excess of the radial fin of triangular profile can be obtained:

\[ r \left( r_a - r \right) \frac{d^2\theta}{dr^2} + \left( r_a - 2r \right) \frac{d\theta}{dr} - bm^2r \theta = 0 \]  

where, here too,

If a transformation is made,

\[ v = r_a - r \]

so that

\[ dv = -dr \]

the differential equation for temperature excess can be transformed to

\[ v \left( r_a - v \right) \frac{d^2\theta}{dv^2} + \left( 2v - r_a \right) \frac{d\theta}{dv} + bm^2 \left( v - r_a \right) \theta = 0 \]  

According to the Frobenius & Fuchs theorem, the differential equation must have a solution of the following form:[6]

\[ \theta = v^p \sum_{k=0}^{\infty} a_k v^k \]  

where, \( a_0 \neq 0 \), (If it is a constant point, then the corresponding \( p = 0 \) )

so that

\[ \frac{d\theta}{dr} = \sum_{k=0}^{\infty} \left( p + k \right) a_k v^{k+p-1} \]
\[
\frac{d^2 \theta}{dr^2} = \sum_{k=0}^{\infty} (k + p)(k + p - 1)a_k r^{k+p-2} \tag{17}
\]

When the assumed values of \( \theta \) and its derivatives are substituted into equation (14), a series involving \( v \) to various powers of \( p \) results:
\[
C_1 v^{p+1} + C_2 v^p + C_3 v^{p+1} + \cdots + C_k v^{k+p-2} = 0
\]

where,
\[
C_1 = p^2 r_a a_0
\]
\[
C_2 = \left[ (p^2 + 2p + 1) r_a \right] a_1 - \left( p^2 + p + bm^2 r_a \right) a_0
\]
\[
C_3 = \left[ (p^2 + 4p + 4) r_a \right] a_2 - \left( p^2 + 3p + 2 + bm^2 r_a \right) a_1 + bm^2 a_0
\]

Because it is the expansion of the analytic function, by the uniqueness theorem, the coefficients \( C_k \) be identically equal to zero, it is observed from \( C_1 \) that,
\[
p^2 r_a a_0 = 0
\]

and because \( r_a \) is a physical dimension that cannot equal zero, and \( a \neq 0 \). Thus the only alternative is that \( p = 0 \).

In theory that the temperature excess control solution of the radial fin of triangular profile for differential equations will be:[7]
\[
\theta = A_1 \sum_{k=0}^{\infty} a_k v^k + A_2 \ln \left( \sum_{k=0}^{\infty} a_k v^k + \sum_{k=0}^{\infty} b_k v^k \right) \tag{18}
\]

But to keep \( \theta \) finite at \( r = r_a \), where \( v = r_a - r_v = 0 \), \( A_2 \) must be zero, the particular solution will be:
\[
\theta = A_1 \sum_{k=0}^{\infty} a_k v^k \tag{19}
\]

Because,
\[
C_2 = \left[ (p^2 + 2p + 1) r_a \right] a_1 - \left( p^2 + p + bm^2 r_a \right) a_0 = 0 \quad \text{and} \quad p = 0, \quad \text{so that}
\]
\[
r_a a_1 = bm^2 r_a a_0
\]
or
\[
a_i = bm^2 a_0
\]

It can be shown after analyzing and transforming that for \( k \geq 2 \),
\[
a_k = \frac{k(k-1)+br_m m^2}{k^2 r_a} \left[ a_{k-1} - bm^2 a_{k-2} \right]
\]

so that
\[
\frac{a_k}{a_0} = \frac{k(k-1)+br_m m^2}{k^2 r_a} \left[ \frac{a_{k-1}}{a_0} - bm^2 \left( \frac{a_{k-2}}{a_0} \right) \right]
\]

When the boundary condition is \( r = r_b \), \( \theta = \theta_b \). Substituted \( \theta = \theta_b \) into equation (19), where \( r_a - r_b = b \), that
substituted $A_i$ into equation (19), that

$$
A_i = \frac{\theta_b}{\sum_{k=0}^{\infty} a_k b^k} = \frac{\theta_b}{a_0 + a_1 b + \sum_{k=2}^{\infty} a_k b^k}
$$

substituted $A_i$ into equation (19), that

$$
\theta = \frac{\theta_b \left( a_0 + a_1 v + \sum_{k=2}^{\infty} a_k v^k \right)}{a_0 + a_1 b + \sum_{k=2}^{\infty} a_k b^k}
$$

(20)

substituted $v = r_a - r$ back into equation (20) and simplify, that

$$
\theta = \frac{\theta_b \left[ 1 + b m^2 (r_a - r) + \sum_{k=2}^{\infty} \left( \frac{a_k}{a_0} \right) (r_a - r)^k \right]}{1 + (b m)^2 + \sum_{k=2}^{\infty} \left( \frac{a_k}{a_0} \right) b^k}
$$

(21)

Equation (21) is the temperature excess of the radial fin of triangular profile.

3.2. The fin efficiency of the radial fin of triangular profile

The fin efficiency is the ratio of the actual heat dissipation of the fin to the ideal heat dissipation of the fin at the surface temperature of the fin. The ideal heat loss is the maximum heat loss that you can assume that the entire surface of the fin is at the temperature of the fin.[8]

$$
\eta = \frac{\text{the actual heat dissipation}}{\text{the ideal heat dissipation}}
$$

(22)

The actual heat dissipation is

$$
q_b = -2 \pi r_b \lambda \delta_b \left. \frac{d \theta}{dr} \right|_{r=r_b}
$$

(23)

The ideal heat dissipation is

$$
q_{id} = 2 \pi \left( r_a^2 - r_b^2 \right) h \theta_b
$$

(24)

Take the derivative of equation (21), that

$$
\frac{d \theta}{dr} = \frac{\theta_b \left[ -b m^2 - 2 \frac{a_2}{a_0} (r_a - r) - \cdots - k \frac{a_k}{a_0} (r_a - r)^{k-1} \right]}{1 + (b m)^2 + \sum_{k=2}^{\infty} \left( \frac{a_k}{a_0} \right) b^k}
$$

(25)

Where $r = r_b$, $r_a - r_b = b$, that

$$
\left. \frac{d \theta}{dr} \right|_{r=r_b} = \frac{\theta_b \left[ -b m^2 - 2 \left( \frac{a_2}{a_0} \right) b - \cdots - k \left( \frac{a_k}{a_0} \right) b^{k-1} \right]}{1 + (b m)^2 + \sum_{k=2}^{\infty} \left( \frac{a_k}{a_0} \right) b^k} = -\theta_b \left[ b m^2 + \sum_{k=2}^{\infty} k \left( \frac{a_k}{a_0} \right) b^{k-1} \right]
$$

(26)

substituted equations (23), (24) and (26) into equation (22), that
\eta = \frac{r_x \lambda \delta_b}{h \left( r_a^2 - r_b^2 \right)} \left[ b m^2 + \sum_{k=2}^{\infty} k \left( \frac{a_k}{a_0} \right) b^{k-1} \right] \left[ 1 + (b) \left( \frac{a_k}{a_0} \right) b^k \right] \quad (27)

Equation (27) is the fin efficiency of the radial fin of triangular profile.

4. Numerical simulation

Both $\theta$ and $\eta$ are series solutions of second-order differential equations, which are not only complicated in form, but also cannot be guaranteed in accuracy. MATLAB can be used to curve fit the calculated series solution, and output the matching curve, and then analyze the solution[9].

Suppose the radius of the pipe is 50mm, the height of the fin is 75mm, the thickness of the fin is 8mm, the ambient temperature is 35℃, the wall temperature is 80℃, the convection heat transfer coefficient is 40W/(m².℃), the thermal conductivity coefficient is 40W/(m.℃), and the thermal conductivity does not change with the temperature change. By using MATLAB curve fitting method to perform curve fitting for $\theta$ and $\eta$ solutions, the curve of the temperature excess of the radial fin of triangular profile (Figure 4) and the curve of the fin efficiency of the radial fin of triangular profile (Figure 5) can be obtained.

Figure 4 The curve of the temperature excess of the radial fin of triangular profile

Figure 5 The curve of the fin efficiency of the radial fin of triangular profile
According to the curve of the temperature excess of the radial fin of triangular profile (Figure 4), the relationship between the temperature distribution of the fin and the radius of the fin can be explained. With the increasing of the radius of the fin, the temperature of the whole fin decreases continuously. It can also be explained from the curve of the fin efficiency of the radial fin of triangular profile (Figure 5) that the relationship between the efficiency of fin and the height of fin is shown. With the height of fin increases, the efficiency of fin shows a trend of gradual decline.

Thus it can be seen that the fin setting also needs certain conditions. The fin height is the main factor affecting the heat transfer enhancement effect, the fin height increases, the heat transfer area also increases, the heat resistance decreases, and the heat transfer increases. However, the continuous increase in the height of the fin will increase the heat conduction and thermal resistance of the fin, resulting in the deterioration of heat transfer effect and the reduction of the service life of the fin. Therefore, comprehensive consideration should be given.[10-11]

5. Conclusion
Firstly, the one-dimensional differential equation of heat conduction of the fin is obtained by thermodynamic analysis of the heat transfer process of the fin. Secondly, the heat transfer of the radial fin of triangular profile is analyzed thermodynamically, and the one-dimensional heat conduction differential equation of the radial fin is established. By numerical solution and theoretical derivation, the temperature excess equation and the fin efficiency formula are obtained. Finally, using MATLAB software through the curve fitting method to get the radial fin of triangular profile with the change of the height of the fin caused by the curve of the change of the temperature excess and the change of the fin efficiency. As can be clearly seen from the figure, when the height of the fin continuously increases, the temperature excess and efficiency of the fin decrease. Therefore, it can be seen that the height of the fin is the main factor affecting the heat transfer of the fin, and the adjustment should be taken into comprehensive consideration.

References
[1] Zhang, J.Z. (2015) Advanced Heat Transfer. Science Press, Beijing.
[2] Liu, J.F., Yu, H.W. (2011) A Simplified Calculation Method for Efficiencies of Circumferential Fins. Energy Conservation Technology. 29(03): 245-247.
[3] Li, Y.R. (2013) Advanced Heat Transfer. Science Press, Beijing.
[4] Dong, S.F., Li, Y.T. (2009) Numerical Analyze About Natural Convection of Fins with Heat Source. Energy Technology. 30(06): 331-334.
[5] Yang, H., Wen, G.Y., Zhao, B.F. (2012) Using Curve Fitting to Optimize of Fin and Tube Heat Exchanger Based on MATLAB. Heating & Refrigeration. (01): 64-66.
[6] Guo, Y.C. (2006) Method of Mathematical Physics. Tsinghua University Press, Beijing.
[7] Allan D, K., Abdul A., James W. (2000) Transient Heat Transfer in Extended Surfaces. John Wiley & Sons Inc, New York.
[8] Cheng, J.G. (1991) Advanced Heat Transfer. Chongqing University Press, Chongqing.
[9] Li, J.X., Lu, T. (2014) Optimization Analysis of Finned Tube Heat Exchanger Based on Numerical Simulation. District Heating. (06):68-71+94.
[10] Li, J.K., Xia, Y.F., Zhu, Q.Y. (2018) The Theoretical Research and Design of The Annular Heat Transfer Fin. Journal of Shenyang Institute of Engineering(Natural Science). 14(03):216-222+246.
[11] Yang, S.M., Tao, W.Q. (2006) Heat Transfer. Higher Education Press., Beijing.