THE RAINBOW TURÁN NUMBER OF $P_5$

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Abstract. An edge-colored graph $F$ is rainbow if each edge of $F$ has a unique color. The rainbow Turán number $\text{ex}^*(n, F)$ of a graph $F$ is the maximum possible number of edges in a properly edge-colored $n$-vertex graph with no rainbow copy of $F$. The study of rainbow Turán numbers was introduced by Keevash, Mubayi, Sudakov, and Verstraëte in 2007.

In this paper we focus on $\text{ex}^*(n, P_5)$. While several recent papers have investigated rainbow Turán numbers for $\ell$-edge paths $P_\ell$, exact results have only been obtained for $\ell < 5$, and $P_5$ represents one of the smallest cases left open in rainbow Turán theory. In this paper, we prove that $\text{ex}^*(n, P_5) \leq \frac{5n}{2}$. Combined with a lower-bound construction due to Johnston and Rombach, this result shows that $\text{ex}^*(n, P_5) = \frac{5n}{2}$ when $n$ is divisible by 16, thereby settling the question asymptotically for all $n$. In addition, this result strengthens the conjecture that $\text{ex}^*(n, P_5) = \frac{5n}{2} + O(1)$ for all $\ell \geq 3$.

1. Introduction

An edge-coloring $c$ of a graph $G$ with edge set $E(G)$ is a function $c : E(G) \to \mathbb{N}$; for $e \in E(G)$ we call $c(e)$ the color of $e$. We say that an edge-colored graph is properly edge-colored if no two incident edges receive the same color, and is rainbow if no two edges receive the same color. Often, we build (or infer) a proper edge-coloring of $G$ in steps, by beginning with a proper edge-coloring of a subgraph $H$ of $G$ and then repeatedly selecting (or deducing) the colors of edges in $E(G) \setminus E(H)$. We say that our color selections obey coloring rules (or are legal) if, for each edge $e \in E(G) \setminus E(H)$, the selected color $c(e)$ is not equal to $c(f)$ for any edge $f$ which is incident to $e$ and already colored. Thus, if we begin with a properly edge-colored subgraph $H$ of $G$ and then select colors for each edge in $E(G) \setminus E(H)$ in a manner which obeys coloring rules, we will return a proper edge-coloring of $G$.

Given graphs $G$ and $F$, an $F$-copy in $G$ is a (not necessarily induced) subgraph of $G$ which is isomorphic to $F$; if $G$ is edge-colored, then a rainbow $F$-copy in $G$ is an $F$-copy in $G$ which is rainbow under the given coloring of $G$. An edge-colored graph is rainbow-$F$-free if it contains no rainbow $F$-copy.

The rainbow Turán number of a fixed graph $F$ is the maximum possible number of edges in a properly edge-colored $n$-vertex rainbow-$F$-free graph $G$. We denote this maximum by $\text{ex}^*(n, F)$, and we say that an $n$-vertex graph $G$ achieves $\text{ex}^*(n, F)$ if $G$ has $\text{ex}^*(n, F)$ edges and there exists a proper edge-coloring of $G$ under which $G$ is rainbow-$F$-free. The study of rainbow Turán numbers was introduced by Keevash, Mubayi, Sudakov, and Verstraëte [9].

Observe that $\text{ex}(n, F) \leq \text{ex}^*(n, F)$, since any properly edge-colored $F$-free graph clearly contains no rainbow $F$-copy. In fact, it was proved in [9] that for any $F$,

\[
\text{ex}(n, F) \leq \text{ex}^*(n, F) \leq \text{ex}(n, F) + o(n^2).
\]

However, for bipartite $F$, $\text{ex}(n, F)$ and $\text{ex}^*(n, F)$ are not asymptotic in general. For example, in [9] it was shown that asymptotically $\text{ex}^*(n, C_6)$ is a constant factor larger than $\text{ex}(n, C_6)$. Thus, as in classical Turán theory, the difficult question is to determine rainbow Turán
numbers for bipartite graphs. Various such problems have received attention; see, for example [2, 6, 7, 8]. Here, we focus on rainbow Turán numbers of paths.

We denote by $P_{\ell}$ the path on $\ell$ edges, i.e., $\ell + 1$ vertices. The behavior of $\text{ex}^*(n, P_{\ell})$ previously has been determined asymptotically only for $\ell \leq 4$. For $\ell = 1$ and $\ell = 2$, the result $\text{ex}^*(n, P_1) = \text{ex}(n, P_1)$ is trivial, since any properly colored $P_1$ or $P_2$-copy is rainbow. For $\ell = 3$ and $\ell = 4$, Johnston, Palmer and Sarkar [7] determined $\text{ex}^*(n, P_{\ell})$ asymptotically for all $n$, with exact values given certain divisibility criteria.

**Theorem 1.1** (Johnston-Palmer-Sarkar [7]). If $n$ is divisible by 4, then

$$\text{ex}^*(n, P_3) = \frac{3}{2}n.$$ 

If $n$ is divisible by 8, then

$$\text{ex}^*(n, P_4) = 2n.$$

These results leave $P_5$ as one of the smallest graphs whose rainbow Turán number has not been determined (the other notable example being $C_4$). Previously, the best known bounds on $\text{ex}^*(n, P_5)$ were due to Johnston and Rombach [8] and Halfpap and Palmer [5], respectively.

**Theorem 1.2** (Johnston-Rombach [8]; Halfpap-Palmer [5]).

$$\frac{5}{2}n + O(1) \leq \text{ex}^*(n, P_5) \leq 4n.$$

The lower bound in Theorem 1.2 is a special case of a lower bound due to Johnston and Rombach [8], which is the best known in general. The upper bound in Theorem 1.2 is obtained by case analysis which does not generalize to longer paths; the best known general upper bound on $\text{ex}^*(n, P_\ell)$ is due to Ergemlidze, Győri and Methuku [3].

**Theorem 1.3** (Johnston-Rombach [8]; Ergemlidze-Győri-Methuku [3]). For $\ell \geq 3$,

$$\frac{\ell}{2}n + O(1) \leq \text{ex}^*(n, P_{\ell}) \leq \left(\frac{9\ell + 5}{7}\right)n.$$

The lower bound on $\text{ex}^*(n, P_\ell)$ from [8] is achieved by taking disjoint copies of the following construction.

**Construction 1.** Let $Q_{\ell-1}$ be the $\ell - 1$ dimensional cube, i.e., the graph whose vertex set is the set of 01-strings of length $\ell - 1$ and two vertices are joined by an edge if and only if their Hamming distance is exactly 1.

We color the edges of $Q_{\ell-1}$ by the position in which their corresponding strings differ. For each vertex $x$ of $Q_{\ell-1}$, let $x$ be the antipode of $x$. That is, $x$ is the unique vertex of Hamming distance $\ell - 1$ from $x$. Now add all edges $xx$ to this graph and color these edges with a new color $\ell$. Call these edges diagonal edges and denote the resulting edge-colored graph $D_{2\ell-1}^*$. The underlying (uncolored) graph of $D_{2\ell-1}^*$ is often referred to as a folded cube graph.

Note that the lower bound from Theorem 1.3 is shown to be tight by Theorem 1.1 when $n = 3, 4$. While these small cases provide limited data, they suggest the following.

**Conjecture 1.4.** For all $\ell \geq 3$, $\text{ex}^*(n, P_{\ell}) = \frac{\ell}{2}n + O(1)$.

The goal of this paper is to prove an upper bound on $\text{ex}^*(n, P_5)$ which asymptotically matches the lower bound from Theorem 1.3 thereby adding further weight to the conjecture that this lower bound is asymptotically correct in all non-trivial cases.
Theorem 1.5. \( \frac{5n}{2} + O(1) \leq \text{ex}^*(n, P_5) \leq \frac{5n}{2} \).

Johnston and Rombach \([8]\) also considered a rainbow version of the generalized Turán problems popularized by Alon and Shikhelman \([1]\). For fixed graphs \( H \) and \( F \), let \( \text{ex}^*(n, H, F) \) denote the maximum possible number of rainbow \( H \)-copies in an \( n \)-vertex properly edge-colored graph which is rainbow-\( F \)-free. (For a different formulation combining rainbow Turán and generalized Turán problems, see \([4]\).) We say that an \( n \) vertex graph \( G \) achieves \( \text{ex}^*(n, H, F) \) if there exists a proper edge-coloring of \( G \) under which \( G \) is rainbow-\( F \)-free and contains \( \text{ex}^*(n, H, F) \) rainbow \( H \)-copies. Following the generalized rainbow framework in \([8]\), Halfpap and Palmer \([5]\) asymptotically determined \( \text{ex}^*(n, C_5, P_4) \) for \( \ell = 3, 4, 5 \):

**Theorem 1.6** (Halfpap-Palmer \([5]\)). Let \( \ell \in \{3, 4, 5\} \). Then, when \( n \) is divisible by \( 2^{\ell-1} \), we have

\[
\text{ex}^*(n, C_\ell, P_\ell) = \frac{(\ell - 1)!}{2} n;
\]

moreover, when \( n \) is divisible by \( 2^{\ell-1} \), the graph consisting of \( \frac{n}{2^{\ell-1}} \) disjoint copies of \( \text{ex}^*(n, C_\ell, P_\ell) \) achieves \( \text{ex}^*(n, C_\ell, P_\ell) \).

The key step in the determination of \( \text{ex}^*(n, P_5, C_5) \) was the proof of the following fact, which we present here as a separate lemma.

**Lemma 1.7.** Let \( G \) be an \( n \)-vertex, properly edge-colored graph which is rainbow-\( P_5 \)-free. Let \( V' \subseteq V(G) \) be the set of vertices of \( G \) which are contained in at least one rainbow \( C_5 \)-copy. Then

\[
\frac{\sum_{v \in V'} d(v)}{|V'|} \leq 5.
\]

Consideration of \( \text{ex}^*(n, C_\ell, P_\ell) \) was motivated by the problem of determining \( \text{ex}^*(n, P_\ell) \), as the extremal constructions avoiding rainbow \( P_3 \) and \( P_4 \)-copies contain many rainbow \( C_3 \) and \( C_4 \)-copies. Intuitively, we expect a rainbow-\( P_\ell \)-free graph with high average degree to contain many short rainbow cycles, since long rainbow walks are unavoidable. So, it is reasonable to conjecture that graphs achieving \( \text{ex}^*(n, P_\ell) \) also achieve \( \text{ex}^*(n, C_\ell, P_\ell) \).

On the other hand, Lemma 1.7 suggests an approach to finding \( \text{ex}^*(n, P_5) \). Denote by \( d(G) \) and \( \delta(G) \) the average degree and minimum degree, respectively, of a graph \( G \). Note that, to prove Theorem 1.5, it would suffice to show that if \( G \) is an \( n \)-vertex, properly edge-colored, rainbow-\( P_5 \)-free graph, then \( d(G) \leq 5 \). We may therefore consider Lemma 1.7 as a partial result towards our desired theorem. The strategy of our proof is thus to focus on vertices of \( G \) which do not lie in any rainbow \( C_5 \)-copy. We will need to show that the average degree across these vertices is at most 5.

Lemma 1.7 is proved through a “vertex pairing” argument. Consider a properly edge-colored graph \( G \) which is rainbow-\( P_5 \)-free. Partition the vertex set \( V \) of \( G \) as \( V = V' \cup \overline{V'} \), where \( V' \) is the set of vertices of \( G \) which lie in some rainbow \( C_5 \)-copy under the given edge-coloring. For \( v \in V \), let \( N(v) \) denote the neighborhood of \( v \). We define the set of **high degree** vertices in \( V' \) as

\[
H = \{ v \in V' : d(v) \geq 6 \}.
\]

For \( v \in H \), it is shown that there exists a **low degree** subset \( L(v) \subseteq N(v) \cap V' \) such that

\[
\frac{d(v) + \sum_{u \in L(v)} d(u)}{|L(v)| + 1} \leq 5.
\]
We call \( \{v\} \cup L(v) \) a local pairing in \( V' \). By construction, the average degree across a local pairing in \( V' \) is at most 5. Moreover, it is shown that we can find a set of pairwise disjoint local pairings whose union contains \( H \). This implies that the average degree over all of \( V' \) must be at most 5.

We shall use a similar approach to estimate the average degree of vertices in \( \overline{V'} \). A local pairing in \( \overline{V'} \) is defined analogously, as a set \( \{v\} \cup L(v) \), where \( d(v) \geq 6 \), \( L(v) \subseteq N(v) \), \( \{v\} \cup L(v) \subseteq \overline{V'} \), and

\[
\frac{d(v) + \sum_{u \in L(v)} d(u)}{|L(u)| + 1} \leq 5.
\]

The crux of the argument is to find a set of pairwise disjoint local pairings in \( \overline{V'} \) which contain all high degree vertices of \( \overline{V'} \).

The paper is organized as follows. In Section 2, we prove a number of structural lemmas relating to vertices which do not lie in any rainbow \( C_5 \)-copy. In Section 3, we employ these lemmas to prove our main result.

2. Structural Lemmas

Fix \( n \geq 1 \) and let \( G_0 \) be a properly edge-colored, \( n \)-vertex graph achieving \( \text{ex}^*(n, P_5) \). We may assume that \( e(G_0) > \frac{5n}{2} \), since otherwise we will have nothing to prove. Thus, \( d(G_0) > 5 \). We will modify \( G_0 \) by repeatedly pruning vertices of degree 1 or 2. By a standard argument, pruning in this way yields a subgraph with minimum degree at least 3, whose average degree is at least that of \( G_0 \). From this subgraph, we will moreover delete any components which have average degree at most 5. Denote by \( G \) the resulting subgraph of \( G_0 \). Observe that

\[
d(G) \geq d(G_0) > 5.
\]

Throughout, we work with this graph \( G \). We will ultimately achieve a contradiction by arguing that in fact, \( d(G) \leq 5 \), thus implying our main result.

Let \( V \) be the vertex set of \( G \). We will form the partition \( V = V' \cup \overline{V'} \), where \( V' \) is the set of vertices in \( G \) which appear in at least one rainbow \( C_5 \)-copy, and \( \overline{V'} \) is the set of vertices which do not appear in a rainbow \( C_5 \)-copy. As noted above, since Lemma 1.7 tells us that the average degree over \( V' \) is at most 5, it will suffice to show that the average degree over \( \overline{V'} \) is at most 5.

We will follow a similar approach as in the proof of Lemma 1.7. Rather than prove that \( \overline{V'} \) contains no vertex of degree greater than 5 (which, even if true, seems too difficult to directly prove by case analysis), we shall estimate the average degree in \( \overline{V'} \). We do this by finding local pairings in \( \overline{V'} \); once we have found these, we moreover argue that their existence implies that globally, \( \overline{V'} \) in fact has average degree at most 5.

The first step is to show that, if \( v \in \overline{V'} \) has \( d(v) \geq 6 \), then the neighbors of \( v \) are also in \( \overline{V'} \). Since we will eventually pair \( v \) to its neighbors, this will ensure that we are pairing \( v \) only to other vertices in \( \overline{V'} \).

**Lemma 2.1.** If \( v \in \overline{V'} \) has \( d(v) \geq 6 \), then \( N(v) \subseteq \overline{V'} \).

**Proof.** Suppose for a contradiction that \( v \) has a neighbor, \( u \), which is in \( V' \). So \( u \) lies on a rainbow \( C_5 \)-copy, which we shall call \( C \). We may assume that the edges of \( C \) are colored from \( \{1, 2, 3, 4, 5\} \), as pictured in Figure 1

Observe that there is no edge with precisely one endpoint incident to \( C \) that is colored with a color not in \( \{1, 2, 3, 4, 5\} \), as this immediately creates a rainbow \( P_5 \)-copy. Observe
also that since \( d(v) \geq 6 \), \( v \) is incident to an edge which is not colored from \( \{1, 2, 3, 4, 5\} \). We call the color on this edge 6, and conclude that the other endpoint, say \( w \), of this edge is not depicted in Figure 1. So the situation is as in Figure 2. We will also add, in Figure 2, labels to the remaining vertices on \( C \).

Note now that \( c(uv) \) must equal 4, or else either of \( u_3u_2u_1uw \) or \( u_2u_3u_4uw \) is a rainbow \( P_5 \)-copy.

Now, we shall examine \( w \). Recall that \( \delta(G) \geq 3 \). We will aim to arrive at a contradiction by showing that \( d(w) \leq 2 \).

First, we shall observe that \( w \) can be adjacent to no vertex of \( C \). Indeed, if \( wv \) is an edge, then \( c(wv) \in \{1, 2, 3, 4, 5\} \) to avoid an immediate rainbow \( P_5 \)-copy. Since the coloring must be proper, this means that \( c(uv) \in \{3, 5\} \). But either choice produces a rainbow \( P_5 \)-copy (either \( vwu_4u_3u_2 \) or \( vwu_1u_2u_3 \)). So \( wu \) is not an edge.

Next, observe that if \( wu_1 \) is an edge, then \( c(wu_1) \neq 5 \), since then \( u_4uww_1u_2 \) is a rainbow \( P_5 \)-copy. But this implies that either \( vwu_1u_2u_3u_4 \) or \( vwu_1u_2u_3u_4 \) is a rainbow \( P_5 \)-copy. So \( wu_1 \) is not an edge. Analogously, \( wu_3 \) is not an edge.

Finally, if \( wu_2 \) is an edge, then it must be colored 1, since otherwise either \( vwu_2u_3u_4u \) or \( vwu_2u_1uu_4 \) is a rainbow \( P_5 \)-copy. But if \( c(wu_2) = 1 \), then \( wu_2u_1uw \) is a rainbow \( C_5 \)-copy containing \( v \), a contradiction, since we assume \( v \in V' \). By an analogous argument, \( wu_3 \) is not an edge.
Thus, $w$ is not adjacent to any vertex on $C$. Moreover, if $x$ is a vertex not yet considered such that $wx$ is an edge, then observe that $c(wx)$ must equal 4, or else either $xwvu_1u_2$ or $xwvu_4u_3$ is a rainbow $P_5$-copy. Thus, to avoid a rainbow $P_5$-copy, we must have $d(w) \leq 2$, contradicting the minimum degree condition on $G$.

We conclude that $v$ is not adjacent to any vertex in $V'$. □

We will also make use of the following lemma, which further restricts the subgraphs in which high-degree vertices of $V'$ may appear. In a $P_4$-copy $P$, the endpoints of $P$ are the two vertices whose degree in $P$ is 1.

**Lemma 2.2.** Suppose $v \in V'$ has $d(v) \geq 6$. Then $v$ is not the endpoint of a rainbow $P_4$-copy.

**Proof.** Suppose that $v \in V'$ has $d(v) \geq 6$ and is the endpoint of a rainbow $P_4$-copy, say $P$. Our ultimate strategy is to obtain a contradiction by arguing that, to avoid a rainbow $P_5$-copy, $G$ must contain a vertex of degree strictly less than 3. In order to identify this low-degree vertex, we will first need to perform some case analysis to understand the structure of $G$ near $v$. We begin by drawing $P$ in Figure 3, also labeling the other vertices and indicating edge colors.

![Figure 3](image3.png)

**Figure 3.**

Now, $d(v) \geq 6$, so $v$ is incident to at least two edges which are not colored from $\{1, 2, 3, 4\}$. Clearly, both endpoints of these edges must be on $P$ to avoid a rainbow $P_5$-copy. Also, since we assume that $v$ is not in a rainbow $C_5$-copy, $vw$ cannot be such an edge. So, in fact, the two edges which are incident to $v$ and not colored from $\{1, 2, 3, 4\}$ must be $vy$ and $vz$. Without loss of generality, $c(vy) = 5$ and $c(vz) = 6$. Every other edge incident to $v$ must be colored from $\{1, 2, 3, 4\}$; in particular, to achieve $d(v) \geq 6$, $v$ must be incident to an edge colored 4. The other endpoint of this edge cannot be on $P$, since $v$ is already adjacent to $x, y, z$ via edges of different colors, and $w$ is already incident to another edge of color 4. So $v$ is incident to a vertex not yet drawn, say $u$, with $c(vu) = 4$. We update our drawing in Figure 4.

![Figure 4](image4.png)

**Figure 4.**

We will now investigate potential neighbors of $w$. Firstly, we claim that $w$ can be adjacent only to vertices already depicted in Figure 3. Indeed, suppose $w'$ is a vertex not yet represented, and that $ww'$ is an edge. Obeying coloring rules, we must have $c(ww') \in \{1, 2, 3\}$, or else $vxyzww'$ is a rainbow $P_5$-copy. However, whichever of these colors we chose, either...
\(xyvzwv'\) or \(xvyzww'\) is rainbow. Thus, \(N(w) \subseteq \{x, y, z, v, u\}\). Since none of \(vxyzwv', xvyzww'\), \(xyvzwv'\) use the vertex \(u\), these observations also imply that \(wu\) is not an edge.

We also claim that \(wx\) is not an edge. Indeed, suppose edge \(wx\) is present. Then \(vyxwzv\) is a \(C_5\)-copy containing \(v\), so must not be rainbow. Thus, \(c(wx)\) must be in \(\{5, 6\}\) to obey coloring rules and avoid a rainbow \(C_5\)-copy. But then either \(xvyzwv\) is a rainbow \(C_5\)-copy containing \(v\), or \(uvzyxw\) is a rainbow \(P_5\)-copy. We conclude that \(wx\) is not an edge.

Thus, the only possible neighbors of \(w\) are \(z, y, \) and \(v\). Since \(\delta(G) \geq 3\), \(w\) must be adjacent to all three of these vertices to avoid a contradiction. We can check that to obey coloring rules and avoid rainbow-\(C_5\) copies containing \(v\), we must have \(c(wv) \in \{2, 3\}\) and \(c(wy) \in \{1, 6\}\). We will set \(c(wu) = a\) and \(c(wv) = b\).

Note also that we have now accounted for five neighbors of \(v\); another must exist, and must be a vertex not depicted in Figure 4. We shall call this neighbor \(s\). To ensure that \(svxyzw\) is not a rainbow \(P_5\)-copy, we must have \(c(sv) \in \{1, 2, 3, 4\}\). By coloring rules, \(c(sv) \in \{2, 3\}\). We shall set \(c(sv) = c\). Note also that \(b \neq c\), so if one of \(b, c\) is determined, then the other is also.

We shall reflect our progress in Figure 5.

![Figure 5](https://via.placeholder.com/150)

We shall next examine \(x\). We begin by observing that \(x\) is not a neighbor of \(z\) or \(u\). Recall that we have already shown that \(x\) is not a neighbor or \(w\).

Suppose \(xz\) is an edge. By coloring rules, we have \(c(xz) \in \{5, 7\}\). Therefore, \(uxzyw\) is a rainbow-\(P_5\)-copy unless \(a = 1\). Given \(a = 1\), we have that \(svzxyw\) is a rainbow \(P_5\)-copy unless \(c = 2\). This implies \(b = 3\). But now \(uwxyz\) is a rainbow-\(P_5\)-copy. We conclude that \(xz\) is not an edge.

Next, suppose \(xu\) is an edge. We must have \(c(xu) \in \{2, 4, 5, 6\}\), else \(wzvxyu\) is a rainbow \(P_5\)-copy, and \(c(xu) \in \{1, 3, 4, 5\}\), else \(uxvyzw\) is a rainbow \(P_5\)-copy. Thus, \(c(xu) \in \{4, 5\}\). By coloring rules, \(c(xu) \neq 4\). Moreover, if \(c(xu) = 5\), then \(uxvyzw\) is a rainbow \(C_5\)-copy containing \(v\). Thus, \(xu\) also is not an edge.

Now, if \(d(x) \geq 3\), then either \(xs\) is an edge, or \(x\) has a neighbor, say \(t\), which is not depicted in Figure 5. We examine the cases separately; in each, we show that to avoid a rainbow \(P_5\)-copy, \(G\) must contain a vertex of degree at most 2.
Case 1: $xs$ is an edge.

Observe that $c(xs) = 4$, else one of $sxvyzw$, $sxyvz$, $sxyzv$ is a rainbow $P_5$-copy. Consider potential neighbors of $s$. Observe that $su$ is not an edge, since if so, one of $usxvy$, $usxyv$, $usxyz$ is a rainbow $P_5$-copy. Analogously, $s$ can be adjacent to no vertex which is not depicted in Figure 5. We have also argued previously that $w$ is only adjacent to $z, y$, and $v$, so $sw$ is not an edge.

We next consider $z$. If $sz$ is an edge, observe that $c(sz) \in \{1, 2\}$, since $uvxyzs$ is a rainbow $P_5$-copy under any other legal color assignment. Moreover, if $c(sz) = 2$, then $c = 3$, and $zsvwy$ is a rainbow $C_5$-copy containing $v$. So $c(sz) = 1$. Now, consider the paths $szwvyx$ and $xyvszw$. The colors on these paths are, respectively, $1, 4, b, 5, 2$ and $2, 5, c, 1, 4$. Recall that either $b = 2$ or $c = 2$, so the colors on one of these two paths are (not in order) $1, 2, 3, 4, c(sz)$. By coloring rules, $c(sz)$ is not in $\{2, 3, 4\}$ (since $c(yx) = 2, c(yz) = 3$, and $c(sx) = 4$), so we must have $c(sz) = 1$. Since $c(sz) \neq a$ by coloring rules, and $a \in \{1, 6\}$, this means $a = 6$. Now, $zywxs$ is a $P_5$-copy colored $3, 6, b, 1, 4$, so we must have $b = 3$ to avoid a rainbow $P_5$-copy, forcing $c = 2$.

Thus, if $xs$ is an edge, then $sy$ must also be an edge, and we can fix the colors of all edges indicated in Figure 5. We reflect this state of affairs in Figure 6.

Given the configuration in Figure 6, we shall observe that $u$ cannot satisfy the minimum degree condition on $G$, $sy$ must be an edge. We first claim that $c(sy) = 1$. Observe that both $xsyzw$ and $zyxsw$ are $P_5$-copies, respectively colored $1, c(sy), 3, 4$ and $3, c(sy), 4, 1, b$. Recall that either $b = 2$ or $c = 2$, so the colors on one of these two paths are (not in order) $1, 2, 3, 4, c(sy)$. By coloring rules, $c(sy)$ is not in $\{2, 3, 4\}$ (since $c(yx) = 2, c(yz) = 3$, and $c(sx) = 4$), so we must have $c(sy) = 1$. Since $c(sy) \neq a$ by coloring rules, and $a \in \{1, 6\}$, this means $a = 6$. Now, $zywxs$ is a $P_5$-copy colored $3, 6, b, 1, 4$, so we must have $b = 3$ to avoid a rainbow $P_5$-copy, forcing $c = 2$.

Thus, if $xs$ is an edge, then $sy$ must also be an edge, and we can fix the colors of all edges indicated in Figure 5. We reflect this state of affairs in Figure 6.

Case 2: $xs$ is not an edge; thus, $x$ has a neighbor $t$ which is not depicted in Figure 5.
Observe that $c(xt) = 4$, else one of $txyzw, txyvzw, txyzvu$ is a rainbow $P_5$-copy. Therefore, $G$ contains the subgraph drawn in Figure 7.

![Figure 7.](image)

We argue that $d(t) \leq 2$.

Observe that $t$ is not adjacent to any vertex which is not yet drawn (say $r$), else one of $rtxyz, rtxyz, rtvxy$ is a rainbow-$P_5$ copy. Analogously, $t$ is not adjacent to $s$ or $u$. We have already seen that $w$ is not adjacent to $t$ (or indeed, to any vertex not in $\{z, y, v\}$). $v$ already has incident edges of every color from $\{1, 2, 3, 4, 5, 6\}$ (since $b, c \in \{2, 3\}$), so $tv$ cannot be an edge, as it would receive a new color, say 7, making $txyzw$ a rainbow-$P_5$ copy. So, the only vertices to which $t$ can be adjacent (aside from $x$) are $y$ and $z$.

Suppose $ty$ is an edge. By coloring rules, $c(ty)$ is not in $\{a, 2, 3, 4\}$. Observe that $svwytx$ is now a $P_5$-copy, with colors $c, b, a, c(ty), 4$. We know that $b, c$ are in $\{2, 3\}$ and are not equal, and that $a \in \{1, 6\}$ is not equal to $c(ty)$. So, $c, b, a, c(ty)$, and 4 must all be distinct colors, and thus $ty$ is not an edge, as its presence yields a rainbow $P_5$-copy. Thus, $d(t) \leq 2$, since its only neighbors are $x$ and possibly $z$.

Thus, to avoid a rainbow $P_5$-copy in $G$, either $d(x) \leq 2$ or $x$ has a neighbor of degree at most 2. In any case, we achieve a contradiction to the minimum degree hypothesis on $G$. We conclude that, if $v \in \overline{V}$ has $d(v) \geq 6$, then $v$ is not the endpoint of a rainbow $P_4$-copy. □

From Lemma 2.2 we can quickly derive the following corollary, which will also be of use in our proof of the main result. The distance between vertices $x, y$ in a graph $G$ is the smallest $\ell$ such that $G$ contains a $P_\ell$-copy with endpoints $x, y$. (If no such $\ell$ exists, we say that $x, y$ are at infinite distance.)

**Corollary 2.3.** Suppose $v \in \overline{V}$ has $d(v) \geq 6$ and $u$ is a vertex at distance 2 from $v$ with $d(u) \geq 6$. Then $u$ is also in $\overline{V}$.

**Proof.** Suppose for a contradiction that $u$ is contained in a rainbow $C_5$-copy, $C$. By assumption, there exists a $P_2$-copy connecting $v$ and $u$, which is necessarily rainbow. We shall label the edges of $P$ with colors 1, 2, and the edges of $C$ with colors $a, b, c, d, e$. Note that, since $d(u) \geq 6$, $u$ must be incident to an edge of a color $f$ which is not contained in $\{a, b, c, d, e\}$. 


The other endpoint of this edge must be on $C$, or a rainbow $P_5$-copy is immediately created. We depict this in Figure 8, adding labels to previously unnamed vertices for convenience.

![Diagram](image)

**Figure 8.**

Now, since $C$ is rainbow and $f$ is distinct from $a, b, c, d, e$, at most two of $a, b, c, d, e, f$ are in $\{1, 2\}$. Thus, one of $vuwu_1u_2$, $vwuu_2u_3$, $vwuu_4u_3$ is a rainbow $P_4$-copy ending at $v$, a contradiction by Lemma 2.2.

Finally, we will need the following result. Although the next lemma holds for any $v \in V$, we will apply it in particular to vertices of high degree in $V'$, in order to more easily build rainbow $P_4$-copies.

**Lemma 2.4.** Let $v \in V$. Then $v$ is the endpoint of a rainbow $P_3$-copy.

**Proof.** Suppose $v \in V$ is not the endpoint of a rainbow $P_3$-copy. We claim that the following holds. If $u$ is a neighbor of $v$, then $d(u) = 3$ and $N(u) \subseteq N(v) \cup \{v\}$. If we can establish this claim, then the lemma is proved, as follows. Given that for any $u \in N(v)$, we have $N(u) \subseteq N(v) \cup \{v\}$, we observe that $\{v\} \cup N(v)$ must induce a component of $G$. Moreover, since every vertex in $N(v)$ has degree 3, the average degree in this component is

$$\frac{d(v) + 3d(u)}{d(v) + 1} < 5,$$

a contradiction, as we suppose that every component of $G$ has average degree greater than 5. Thus, we will have that every $v \in V$ is the endpoint of a rainbow $P_3$-copy.

To establish the claim, suppose $v \in V$ is not the endpoint of a rainbow $P_3$-copy, and let $u$ be a neighbor of $v$. Without loss of generality, $c(uv) = 1$. Since $\delta(G) \geq 3$, $u$ has another neighbor, say $x$, and $c(ux)$ cannot equal $c(uv)$. Say $c(ux) = 2$. Now, $x$ has at least two more neighbors, so must have at least one neighbor not equal to $v$, say $y$. Since we assume $v$ is not the endpoint of a rainbow $P_3$-copy, we must have $c(xy) = 1$. Observe that $x$ can have no other neighbor except $v$ without creating a rainbow $P_3$-copy ending in $v$, so $xv$ must be an edge. We must have $c(xv) = 3$, since $x$ is already incident to edges of colors 1 and 2.

We consider the neighbors of $u$; there must be at least one more. Either $uy$ is an edge, or $u$ is adjacent to some vertex $w$ not already considered. We wish to show that there is no such edge $uw$. If there is, then $c(uw) = 3$ to avoid a rainbow-$P_3$ ending at $v$. Thus, $u$ is adjacent to only one new vertex $w$. Now, $uy$ cannot be an edge, since then $c(uy)$ would be a new color, say 4, which would create a rainbow-$P_3$ copy ending at $v$. So $y$ is adjacent to two
more vertices, neither of which are $u$. If $y$ is adjacent to a vertex $z$ not already considered, then we must have $c(yz) = 3$ to avoid a rainbow $P_3$-copy ending at $v$. $y$ cannot be adjacent to $w$, as $c(yw)$ would be 2 or 4, creating a rainbow $P_3$-copy ending at $v$. So to achieve degree 3, $y$ must be adjacent to $v$ and a new vertex, $z$. We have $c(yz) = 3$, and must have $c(yv) = 2$ to avoid a rainbow $P_3$-copy ending at $v$.

Now, consider $w$. We have already noted that $wy$ is not an edge; nor are $wv$ or $wx$, since these would necessarily receive a new color and thus create a rainbow-$P_3$ copy ending at $v$. Any coloring of $wz$ which is permitted by coloring rules also creates a rainbow $P_3$-copy ending at $v$, so neither is $wz$ an edge. Finally, if $w$ is adjacent to a new vertex, say $s$, then $c(ws) = 1$ to avoid a rainbow $P_3$-copy ending at $v$, so $w$ has at most one neighbor not already considered. But this implies that $d(w) \leq 2$, a contradiction. We conclude that $uw$ is not an edge.

Thus, to achieve degree 3, $u$ is adjacent to $y$, and $c(uy) = 3$ is forced. Given this edge, $y$ must be adjacent to $v$ to achieve degree 3, with $c(yv) = 2$. Now, $v, u, x, y$ form a properly colored $K_4$. It is clear that if any of $u, x, y$ have another neighbor, the incident edge will create a rainbow $P_3$-copy ending at $v$, so in particular, $N(u) = \{ v, x, y \} \subset N(v) \cup \{v\}$, and $d(u) = 3$. Since $u$ was chosen from $N(v)$ arbitrarily, we are done. \[ \square \]

3. Main Result

We are now ready to prove our main result. As in Section 2, we may assume that we work in a properly edge-colored, rainbow-$P_3$-free graph $G$ with $\delta(G) \geq 3$ and such that every component of $G$ has average degree greater than 5. The vertex set of $G$ is again partitioned as $V' \cup \overline{V}$, where $V'$ is the set of vertices which lie in some rainbow $C_5$-copy in $G$.

**Theorem 3.1.** $\text{ex}^*(n, P_5) \leq \frac{5n}{2}$.

**Proof.** Our goal is to show, for a contradiction, that $d(G) \leq 5$. By Lemma 1.7, we know that the average degree in $V'$ is at most 5, so we will be done if we can also show that $\overline{V}$ has average degree at most 5. Define $H(\overline{V}) := \{ v \in \overline{V} : d(v) > 5 \}$. For each $v \in H(\overline{V})$, we aim to find a set $L(v) \subseteq N(v)$ such that the following hold:

1. $\{v\} \cup L(v)$ is a local pairing in $\overline{V}$
2. For any $u \in L(v)$, we have $d(u) \leq 5$ and $N(u) \cap H(\overline{V}) = \{v\}$.

Suppose that for every $v \in H(\overline{V})$, we can find such a set $L(v)$. Then it immediately follows that $\overline{V}$ has average degree at most 5. Indeed, by the definition of a local pairing in $\overline{V}$, condition (1) gives that each $\{v\} \cup L(v)$ is a subset of $\overline{V}$ with average degree at most 5. By condition (2), each vertex in $\overline{V}$ appears in at most one local pairing $\{v\} \cup L(v)$. Thus, setting $S$ to be the set of vertices in $\overline{V}$ which are not contained in any of the selected local pairings in $\overline{V}$, we have

$$\sum_{v \in \overline{V}} d(v) = \sum_{v \in H(\overline{V})} \left( d(v) + \sum_{u \in L(v)} d(u) \right) + \sum_{v \in S} d(v) \leq \sum_{v \in H(\overline{V})} 5(|L(v)| + 1) + 5|S| = 5|\overline{V}|.$$ 

It thus only remains to show that such a set $L(v)$ can be found for every vertex $v \in H(\overline{V})$. By Lemma 2.1, we have that $\{v\} \cup N(v) \subseteq \overline{V}$ for any $v \in H(\overline{V})$, so any potential local pairing will indeed be contained in $\overline{V}$. Fix $v \in H(\overline{V})$ and consider $N(v)$. We distinguish two cases. Throughout both, recall that by Lemma 2.2, we may assume that $v$ is not the endpoint of a rainbow $P_4$-copy.
Case 1: $v$ lies in a rainbow $C_4$-copy, $C$.

We'll label the vertices of $C$, so that $C = vxyzv$, and the edge colors on $C$ are from $\{1, 2, 3, 4\}$. Since $d(v) \geq 6$, $v$ is incident to at least two edges which are not colored from $\{1, 2, 3, 4\}$. One of these edges may be incident to $y$, the vertex on $C$ to which we have not already specified an adjacency, but one must be incident to a new vertex, say $w$. We draw the situation in Figure 9.

\begin{figure}[h]
\centering
\begin{tikzpicture}
    \node (v) at (0,0) [shape=circle, fill=black] {$v$};
    \node (x) at (-1,-1) [shape=circle, fill=black] {$x$};
    \node (y) at (1,-1) [shape=circle, fill=black] {$y$};
    \node (z) at (0,-2) [shape=circle, fill=black] {$z$};
    \node (w) at (0,-3) [shape=circle, fill=black] {$w$};
    \draw (v) -- (x) node [midway, above] {1};
    \draw (v) -- (y) node [midway, above] {2};
    \draw (v) -- (z) node [midway, above] {3};
    \draw (v) -- (w) node [midway, above] {4};
    \draw (w) -- (x) node [midway, above] {5};
    \draw (w) -- (y) node [midway, above] {6};
\end{tikzpicture}
\caption{Figure 9.}
\end{figure}

Consider the possible neighbors of $w$. If $w$ is adjacent to $x$, then $c(wx)$ must be 3, else $vwxyz$ is a rainbow $P_4$-copy ending at $v$. Similarly, if $wz$ is an edge, then $c(wz) = 2$. If $w$ is adjacent to a new vertex, say $u$, then we have $c(wu) \in \{2, 3\}$, or else one of $uvwyz$, $uwvzyx$ is a rainbow $P_5$-copy. Thus, if $w$ is adjacent to $u$, then it is not adjacent to one of $x, z$; if $w$ is adjacent to two new vertices, then it is not adjacent to either $x$ or $z$. Finally, $w$ may be adjacent to $y$. We conclude from this analysis that $d(w) \leq 4$.

The vertex $w$ is one of the neighbors of $v$ which we will ultimately include in $L(v)$, so we must verify that $w$ has no other neighbor of degree greater than 5. Observe first $d(x) \leq 5$, since any edge incident to $x$ whose other endpoint is not on $C$ must be colored from $\{1, 2, 3, 4\}$ to avoid creating a rainbow $P_4$-copy ending at $v$. Analogously, $d(z) \leq 5$.

Suppose next that $w$ is adjacent to a vertex $u$ which is not on $C$. We have seen that $c(uw) \in \{2, 3\}$; since the colors are to this point symmetric, we may assume that $c(uw) = 2$. We now bound $d(u)$.

First, if $u$ is adjacent to a vertex not yet described, say $s$, then $c(us) \in \{3, 4, 5\}$, since otherwise $suwzy$ is a rainbow $P_5$-copy. Note that $u$ is not adjacent to $z$, since if so, either $uxzu$ or $uux$ is a rainbow $P_4$-copy ending at $u$. If $ux$ is an edge, then $c(ux) \in \{3, 4\}$, since otherwise $uvzyx$ is a rainbow $P_4$-copy ending at $v$. Thus, $d(u) \leq 5$, since $u$ is adjacent to $w$, may be adjacent to $y$, and any other incident edge to $u$ must be colored from $\{3, 4, 5\}$.

Finally, $w$ may be adjacent to $y$. We wish to show that, if $wy$ is an edge, then $d(y) \leq 5$. Suppose for a contradiction that $wy$ is an edge and $d(y) \geq 6$. By Corollary 2.3, $y \in V'$, and so by Lemma 2.2, $y$ is not the endpoint of a rainbow $P_4$-copy.

We re-draw the situation in Figure 10, setting $c(yw) = c$.

Now, consider $w$. If $w$ is adjacent to a vertex not depicted in Figure 10, say $s$, then one of $yxwv$, $yzwv$ is a rainbow $P_4$-copy ending at $y$. So $w$ is adjacent to no vertex which is not drawn in Figure 10. If $wx$ is an edge, then either $vwxyz$ is a rainbow $P_4$-copy ending at $v$, or $yxwvz$ is a rainbow $P_4$-copy ending at $y$. Finally, if $wz$ is an edge, then either $yzwv$
is a rainbow $P_4$-copy ending at $y$ or $vwzyx$ is a rainbow $P_4$-copy ending at $v$. We conclude that $d(w) = 2$, a contradiction. So, if $wy$ is an edge, then $d(y) \leq 5$.

We have thus shown that $d(w) \leq 4$ and $v$ is the only vertex in $N(w)$ of degree greater than 5. We are now ready to build $L(v)$.

Observe that $v$ has at least $d(v) - 5$ neighbors of the same type as $w$, namely, neighbors which do not lie on $C$ and are incident to $v$ by an edge whose color is not from $\{1, 2, 3, 4\}$. Let $L(v)$ be the set of such vertices. The above argument then shows that the maximum degree in $L(v)$ is at most 4, and that if $w$ is any vertex in $L(v)$, then $v$ is the only neighbor of $w$ with degree greater than 5.

We now observe that the average degree in $L(v) \cup \{v\}$ is at most 5. Say $|L(v)| = d(v) - k$; we have noted that $k \leq 5$. The average degree in $L(v) \cup \{v\}$ is then

$$d(v) + \sum_{u \in L(v)} d(u) \leq \frac{d(v) + 4(d(v) - k)}{d(v) - k + 1} = \frac{5(d(v) - 4k/5)}{d(v) - (k - 1)}.$$ 

Observe that

$$\frac{5(d(v) - 4k/5)}{d(v) - (k - 1)} \leq 5$$

as long as $d(v) - (k - 1) \geq d(v) - 4k/5$, i.e., $k - 1 \leq 4k/5$, which holds precisely when $k \leq 5$.

So, $\{v\} \cup L(v)$ is a local pairing as desired.

**Case 2:** $v$ does not lie in a rainbow $C_4$-copy.

Using Lemma $2.4$ we know that $v$ is the endpoint of a rainbow $P_3$-copy, say $P = vxyz$, with edge colors $1, 2, 3$. Moreover, since $d(v) \geq 6$, $v$ is incident to at least three edges which are not colored from $\{1, 2, 3\}$. Clearly, one of these does not have its other endpoint on $P$. So there exists a new vertex $w$ such that $vw$ is an edge which receives a new color, say $4$.

Consider the possible neighbors of $w$. If $w$ is adjacent to a vertex not on $P$, say $u$, then $c(wu) \in \{1, 2, 3\}$ in order to obey coloring rules and ensure that $uwvwyz$ is not a rainbow $P_5$-copy. If $wx$ is an edge, then $vwxyz$ is a $P_4$-copy ending at $v$, so must not be rainbow, which means $c(wx) = 3$. If $wy$ is an edge, then $wyxwv$ is a $C_4$-copy containing $v$, so is not rainbow, meaning $c(wy) = 1$. If $wz$ is an edge, then $vwzyx$ is a $P_4$-copy ending at $v$, so cannot be rainbow, meaning $c(wz) = 2$. Thus, $w$ can only be incident to edges colored from $\{1, 2, 3, 4\}$, so we conclude that $d(w) \leq 4$.

Our goal is to use $w$ as one of the low-degree vertices in $L(v)$, so we must verify that $w$ has no neighbor other than $v$ of degree greater than 6.

![Figure 10](image-url)
Firstly, we examine $z$. If $z$ is adjacent to a vertex not in $\{w, v, x, y\}$, say $u$, then $c(zu) \in \{1, 2\}$, else $vxyzu$ is a rainbow $P_4$-copy ending at $v$. We have seen that if $zw$ is an edge, then $c(zw) = 2$. If $vz$ is an edge, then $c(vz) = 2$, or else $vxyvz$ is a rainbow $C_4$-copy containing $v$. So, there is at most one vertex, namely $x$, which is incident to $z$ via an edge which is not colored from $\{1, 2, 3\}$. Thus, $d(z) \leq 4$.

Next, consider $x$. Suppose $wx$ is an edge and (for a contradiction) that $d(x) \geq 6$. Since $x$ is adjacent to $v$, we know by Lemma 2.1 that $x \in V'$. Thus, by Lemma 2.2 $x$ is not the endpoint of a rainbow $P_4$-copy.

We have seen that $c(wx)$ must be 3. Observe that, since $d(v) \geq 6$, $v$ must be incident to an edge of a new color, say 5, whose other endpoint is a vertex not yet considered, say $u$. We draw this in Figure 11.

Figure 11.

We shall achieve a contradiction by showing that $d(u) \leq 2$. First, we argue that $u$ is not adjacent to any vertex from $\{w, x, y, z\}$. Observe that $uw$ is not an edge, since if so, either $vuwxy$ is a rainbow $P_4$-copy ending at $v$, or $vuxwv$ is a rainbow $C_4$-copy containing $v$. Next, $ux$ is not an edge, since if so, any legal choice for $c(ux)$ makes $vuxyz$ a rainbow $P_4$-copy ending at $v$. Observe that $uy$ is not an edge, since if so, either $vuyxw$ is a rainbow $C_5$-copy containing $v$, or $vuxv$ is a rainbow $C_4$-copy containing $v$. Finally, $uz$ is not an edge, since if so, either $vuzx$ is a rainbow $P_4$-copy ending at $v$ or $xvuz$ is a rainbow $P_4$-copy ending at $z$.

Thus, $u$ is not adjacent to $w, x, y, z$. Suppose now that $u$ has a neighbor $s$ which is not yet considered. Then $swux$ is a $P_4$-copy ending in $x$, so cannot be rainbow, which means $c(us) \notin \{3, 4\}$. We also have that $suwxyz$ is a $P_5$-copy, so cannot be rainbow, which means $c(us) \neq 4$. Thus, if $u$ is adjacent to a new vertex, the edge used must be colored 3, meaning that $u$ has at most one neighbor not yet drawn. So $d(u) \leq 2$, a contradiction. We conclude that if $wx$ is an edge, then $d(x) \leq 5$.

Next, suppose $wy$ is an edge. We wish to show that $d(y) \leq 5$. Suppose for a contradiction that $d(y) \geq 6$. We have seen that $c(wy) = 1$. Note that by Corollary 2.3 $y$ is in $V'$, so is not the endpoint of a rainbow $P_3$-copy.

We now examine $z$. Note first that if $z$ has a neighbor not in $\{w, v, x\}$, say $u$, then $c(zu) = 1$, or else one of $vxyzu, vuxzu$ is a rainbow $P_4$-copy ending at $v$. Thus, to achieve degree 3, $z$ must have at least one neighbor among $\{w, v, x\}$. We shall show that, to the contrary, $z$ cannot be adjacent to any of these vertices.
Suppose $wz$ is an edge. We have seen that $c(wz)$ must be 2. But now $yzwvx$ is a rainbow $P_4$-copy ending in $y$, a contradiction. So $wz$ is not an edge. Observe that $zv$ is not an edge, since for any legal choice of $c(zv)$, $vzyuv$ is a rainbow $C_4$-copy containing $v$. Finally, suppose $xz$ is an edge. We must have $c(xz) = 4$, else $vwyxz$ is a rainbow $P_4$-copy ending at $v$. Now, since $d(v) = 6$, $v$ must be incident to an edge of a new color, say 5, whose other endpoint is not in $\{w, x, y, z\}$. Say this edge is $vs$. Observe that $yzxvu$ is now a rainbow $P_4$-copy ending at $y$. We conclude that $z$ is adjacent to none of $v, y, w$, so $d(z) \leq 2$, a contradiction. Thus, if $wy$ is an edge, then $d(y) \leq 5$.

Finally, we must show that if $w$ is adjacent to a vertex not in $\{v, x, y, z\}$, say $u$, then $d(u) \leq 5$. Suppose to the contrary that $d(u) \geq 6$. We can again apply Corollary 2.3 to give that $u$ is in $\overline{V'}$, so $u$ is not the endpoint of a rainbow $P_4$-copy. We must have $c(uw) \in \{1, 2\}$, else $uwvx$ is a rainbow $P_4$-copy ending at $u$; however, unlike the other cases, we cannot at the moment fix $c(uw)$. We shall set $c(uw) = a$.

Now, since $d(u) \geq 6$, $u$ is incident to an edge of a new color, 5. It is simple to check that the other endpoint of this edge must be a vertex not in $\{v, x, y, z\}$, say $s$. Similarly, $v$ must be incident to an edge of another new color, 6, and the endpoint of this edge must be a vertex not in $\{s, u, w, x, y, z\}$, say $t$. We illustrate the situation in Figure 12.

![Figure 12.](image)

We will argue that $d(t) \leq 2$. We claim first that $t$ is not adjacent to $s, u, w,$ or $x$. Indeed, if $ts$ is an edge, then one of the paths $ustw$, $ustx$ is a rainbow $P_4$-copy ending at $u$. If $tu$ is an edge, then either $v$ is in a rainbow $C_4$-copy or $utvx$ is a rainbow $P_4$-copy ending at $u$. If $tw$ is an edge, then either $vtwus$ is a rainbow $P_4$-copy ending at $v$, or $vtwxyz$ is a rainbow $P_5$-copy. And if $tx$ is an edge, then either $vtxy$ is a rainbow $P_4$-copy ending at $v$, or $uwtx$ is a rainbow $P_4$-copy ending at $u$. Thus, the only possible neighbors of $t$ are $y, z$, and vertices not depicted in Figure 12.

Observe, if $ty$ is an edge, then $c(ty) \in \{a, 4\}$, else $uwty$ is a rainbow $P_4$-copy ending at $u$. But $c(ty) \neq 4$, since then $vtyxv$ is a rainbow $C_4$-copy containing $v$. Thus, if $ty$ is an edge, then $c(ty) = a$. Similarly, if $tz$ is an edge, then $c(tz) = a$. And if $t$ is adjacent to a vertex not yet drawn, then the incident edge also must be colored $a$ to avoid forming either a rainbow $P_5$-copy with $txyz$ or a rainbow $P_4$-copy ending at $u$.

Thus, if $t$ is adjacent to any vertex other than $v$, the incident edge must be colored $a$, so $d(t) \leq 2$, a contradiction.

We have thus argued that $d(w) \leq 4$ and that $v$ is the only vertex in $N(w)$ of degree greater than 5. We are now ready to build $L(v)$.
Note that $v$ has at least $d(v) - 4$ neighbors of the same type as $w$, that is, neighbors which are not on the path $vxyz$ and which are incident to $v$ by an edge whose color is not from $\{1, 2, 3\}$. (Note that if $vz$ is an edge, then $c(vz) = 2$, else $v$ is in a rainbow-$C_4$ copy, and thus we can find $d(v) - 4$ vertices of the same type as $w$, instead of the expected $d(v) - 5$.) Let $L(v)$ be the set of such vertices. As in the previous case, it follows that the average degree in $L(v) \cup \{v\}$ is at most 5, (and actually will be strictly less than 5, as $|L(v)| > d(v) - 5$).

4. Concluding remarks

While the details of the proofs are largely routine case analysis, it is worth reiterating the central idea which makes this approach to our problem tractable. The choice to partition $V$ as $V' \cup \overline{V'}$ is not arbitrary. Since rainbow $C_\ell$-copies arise naturally when we attempt to avoid rainbow $P_\ell$-copies, it is much easier to show that rainbow $P_\ell$-copies are unavoidable if we start with a vertex in $\overline{V'}$ than with a vertex whose situation we do not know. On the other hand, every vertex in $V'$ lies on a rainbow $C_5$-copy, meaning that we can begin any case analysis concerning vertices in $V'$ with a rainbow $C_5$-copy, rather than with a single vertex. This additional guaranteed structure likewise makes case analysis arguments tractable where they seem not to be if we pick a vertex from $V$ without assuming that it lies (or does not lie) in a rainbow $C_5$-copy.

Given the increase of required analysis from previous results (the proof that $\text{ex}^*(n, P_4) = 2n + O(1)$ is about two pages long), it seems clear that these naive methods will not be sufficient to obtain exact results for longer paths. However, the result for $P_3$ alone adds weight to the conjecture that $\text{ex}^*(n, P_\ell) = \frac{\ell}{2}n + O(1)$ in general.

Acknowledgements

The author would like to thank Cory Palmer for his advice and support in the preparation of this paper.

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