Open quantum systems and Random Matrix Theory

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Abstract

A simple model for open quantum systems is analyzed with Random Matrix Theory. The system is coupled to the continuum in a minimal way. In this paper we see the effect of opening the system on the level statistics, in particular the $\Delta_3(L)$ statistic, width distribution and level spacing are examined as a function of the strength of this coupling. A super-radiant transition is observed, and it is seen that as it is formed, the level spacing and $\Delta_3(L)$ statistic exhibit the signatures of missed levels.

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I. INTRODUCTION

Random matrix theory is used to analyse chaotic quantum systems. The statistics of the discrete energies of the system can yield information on the completeness of the data or the presence of intruder states. The standard procedure is straightforward. After preparing the data by rescaling it so that the level density is unity across its whole range (a process known as “unfolding”) the spectral statistics of the system are compared with the RMT results for the appropriate ensemble. We would like to see what the effect of coupling a chaotic system to the continuum might have on the RMT statistics. The canonical example of this process is the analysis of neutron resonance data. A free neutron is incident on a target nucleus, and they combine to make an excited compound nucleus. The incident channel is but one configuration of many, and the initial wave function is simple and consists solely of this component. Through a series of random collisions of the nucleons, the initial wave function “melts” and de-excites, emitting gamma rays finding the ground state. The initial configuration of a free neutron and a ground state target corresponds to one of the discrete excited states of the compound nucleus. We have come to the picture of a discrete state buried in the continuum. The system is an open quantum system, the states of the compound nucleus have a width. It is the effect of the openness of the system on the level statistics that is the main question addressed in this paper.

There is a well developed method for dealing with open quantum systems whose main feature is a Hermtian hamiltonian and coupling to the continuum is modeled by the addition of an imaginary part, making an effective non-hermitian Hamiltonian, see [1], [2] and [3]. The energies of the original Hamiltonian acquire widths. A common feature of these open quantum systems is the appearance of a super-radiant state. The SR state appears as the coupling to the continuum increases. There is a restructuring of the states and one special state acquires all the width.

We will make a very simple model of an open quantum system, namely an $N \times N$ GOE matrix with an imaginary part $\kappa \sqrt{N}$ added to the diagonal. Commonly used RMT statistics are calculated and their behavior as a function of $\kappa$ is explored. The biggest effects happen around $\kappa = 1$ which is when the SR transition happens. A plot of the energies of the opened GOE matrix vs $\kappa$ consistently show the migration of a few levels to the center of the spectrum, and a plot of the level density reveals a deviation from the RMT semicircle in
the middle of the energy range consistent with there being more energies close to zero as $\kappa$ grows. The entropy of the states evolves also, with the SR state clearly emerging at $\kappa = 1$.

Next the $\Delta_3(L)$ statistic or spectral rigidity is examined. The effect of opening the system was to increase the value of the spectral rigidity. The increase was maximum at $\kappa = 1$ and then decreased, but not to zero. The shape of the $\Delta_3(L)$ curves for individual opened spectra looked like those of incomplete spectra. A search for missed levels was performed on the opened spectra using RMT methods. The spectra are complete, but the tests suggested that there was a fraction of the levels missed. This fraction was biggest at $\kappa = 1$ where it reached a value of about 3%. The distribution of widths also undergoes a transformation at $\kappa = 1$, and for large values of $\kappa$ the SR state accounts for all the width and the remaining levels have widths consistent with the Porter-Thomas distribution.

In the next section we describe the system and how it is opened and the effect this has on the energies and entropies. The SR state is seen already at this stage. In Sect. III we look at the density of states and address issues of unfolding the spectra in anticipation of RMT analysis. This is followed in Sect. IV by an analysis of the width distribution and a look at the SR transition. The spectral rigidity is introduced in Sect. V and the effect of opening the system is seen. In Sect. VI we perform an RMT analysis on the ensemble of open spectra using three tests for missed levels and see how open systems give false positives for missed levels. We end with concluding remarks in Sect. VII.

II. OPENING THE SYSTEM, ENERGIES AND ENTROPY

Open quantum systems have been treated very successfully with an effective Hamiltonian approach. The main idea is the Hamiltonian of a loosely bound system is connected to continuum channels via a factorizable non-Hermitian term. The details are worked out in [1], [2] and [4]. This method provides a general framework applicable to a broad range of systems from loosely bound nuclei [4] to transport of electrons in nanosystems [5]. The approach taken here is to make the most minimal adjustment to the GOE that would mimic openness and see how the RMT results are affected. We take a GOE matrix, $H^0$, and add an imaginary part to the diagonal elements, $H_{ii} \rightarrow H_{ii}^0 - \frac{i\kappa}{\sqrt{N}}$, where $\kappa$ is the strength of the coupling. Because the matrix is random, it is sufficient to just make the replacement $H_{11} \rightarrow H_{11}^0 - \kappa\sqrt{N}$ and leave other matrix elements unchanged. The resulting spectrum
will be a set of complex energies $\varepsilon_n = E_n + i\Gamma_n$

The evolution of the energies with $\kappa$ show robust and interesting features. There are a small number of energies that migrate, then settle down for $\kappa$ in the range $0.5 \rightarrow 1.5$. In Fig. 1 we see a specific example of this generic behavior. If we look at the entropy of the corresponding wave functions one state in particular emerges. Starting with a wave function $\psi = \sum_i c_i |n\rangle$ we define the entropy as $S = \sum_i |c_i|^2 \ln(|c_i|^2)$. We can calculate $S$ in the original basis in which we wrote out $H$, or in the energy basis, where $H$ is diagonal and $H_{ii}^0 = E_i$. In Fig. 2 we see the results for an $N = 50$ system. The SR state emerges with a very simple structure in the original basis (blue lines), having a very low entropy. The other states stay at the GOE predicted average value $S = \ln(0.48N)$, which in this case is 3.2. We see that in the energy basis (black lines), where the hamiltonian is diagonal for $\kappa = 0$, we have the complementary situation with entropy. Now the SR state is a complicated mixture of energy eigenstates with an entropy of around 3, and the other states have lower entropy, indeed for many they entropy is close to zero.

III. THE DENSITY OF STATES

The migration of a few energies to the middle of the spectrum as $\kappa$ increases is reflected in the excess of small magnitude energies in the density of states (note we use level density and density of states interchangeably here as there is no degeneracy in the eigenvalues of random matrices). The semicircle DOS of the GOE is no longer valid. In Fig. 3 we see in the resulting empirical densities a clear deviation of $\rho(E)$ from the semicircle for small $E$.

This change in $\rho(E)$ raises an important practical question for how to do an RMT analysis, mainly how do we unfold the spectrum. First a comment on unfolding spectra. The RMT results for various spectral statistics depend on global symmetries of the systems under study, specifically invariance under time reversal for the GOE. The statistics themselves involve operations on the level density, which depends on the specifics of the system. The semicircle level density of the GOE, for example, bares no relation to the exponential level density of realistic nuclear systems. To remove the system specific (secular) features of the level density we must unfold the spectrum [6, 7], which means rescaling the energies in units of the local average density. The unfolded spectrum has a level density of unity, but all the fluctuations are preserved. To go from a set of energies $\{E\}$, with density $\rho(E)$ to an
FIG. 1: The evolution of the energy levels with $\kappa$. We just take the 300 unfolded energies of one particular matrix, and plot $E_n$ vs $\kappa$ for levels 130 to 195 of the $N = 300$ unfolded spectrum. The curves, or “trajectories” around $\kappa = 1$ vary from matrix to matrix.

The unfolded spectrum $\{\xi\}$ with density $\rho(\xi) = 1$, we need to integrate $\rho(E)$ to get a smooth cumulative level number $N(E)$:

$$N(E) = \int_{-\infty}^{E} \rho(E')dE'.$$

The $i^{th}$ unfolded energy is simply $\xi_i = N(E_i)$. In this analysis we unfolded the spectra using the semicircular level density and also a fit to the empirical level density and the results were the same for all statistics we evaluated. So for simplicity we used the semicircle to unfold in our analysis.
FIG. 2: The evolution of the energy and entropy as $\kappa$ increase for an $N = 50$ system. The blue lines correspond to the basis which the original hamiltonian $H^0$ is written in. The black points correspond to the basis in which $H^0$ is diagonal. In both cases, the superradiant state is obvious. One can think of Fig. 1 as a birds eye view of this plot, where we just see the energy and $\kappa$ values. Note that here the GOE average value of the entropy is $S = 3.2$.

IV. THE WIDTH OF THE ENERGIES.

The addition of an imaginary part to $H_{11}$ gives a width to all the levels. These widths can be treated as random variables, and their distribution examined. An ensemble of 200 matrices was prepared and opened as in Sect. II. Each random matrix was the start of a sequence of 301 opened matrices with $0 \leq \kappa \leq 3.0$ in steps of 0.01. The complex energies $\varepsilon_n(\kappa)$ were calculated. The widths $\Gamma(\kappa)$ of the levels were sorted and their size as a function of $\kappa$ was examined, see in Fig. 4. Immediately we see the emergence of the SR state that absorbs all the width. If this state is excluded from the plot we get a completely different behaviour, the remaining widths have an exponential dependance on $\kappa$. If we plot $\bar{\Gamma}$, the average of all but the biggest widths vs $\kappa$ on a log-log plot we get a very simple picture,
FIG. 3: Here we have the density of an ensemble of 2000 with $N = 100$ and $\kappa = 0, 0.4, 1.2$ and 2.8. The level density is close to the semicircle of the GOE even for $\kappa = 0.4$. The deviations are consistent with Fig. 1 where levels migrate to the center of the energy range.

shown in Fig. 5. The straight line sections are roughly $\ln \bar{\Gamma} = \pm \ln \kappa - 2.5$. Here the range of $\kappa$ was from $10^{-3}$ to $10^6$. A qualitatively identical SR transition is seen in a different context in [8] where the interplay of disorder and SR was examined in the context of the Anderson model. 

This picture of the emergence of the SR state is further reinforced by an analysis of the reduced widths. When the special state is excluded from the analysis we recover the Porter-Thomas distribution (PTD). In Fig. 6 we see a log-log plot for the distribution of $\gamma/\bar{\gamma}$ with and without the 2 largest widths for an ensemble with $\kappa = 1.5$. We stress that this is as deep as we went in analyzing the distribution of widths. There could well be deviations from the PTD and for a derivation of alternative width distribution see [9].

V. THE SPECTRAL RIGIDITY, $\Delta_3(L)$. 

The spectral rigidity or $\Delta_3(L)$ statistic is a common diagnostic for statistical analysis of data based on RMT. It is a robust statistic and can be used to gauge the completeness of data, detecting the fraction of missed or spurious levels, or the degree that the system is chaotic [7, 10, 11]. $\Delta_3(L)$ is defined in terms of fluctuations in the cumulative level number, $\mathcal{N}(E)$, the number of levels with energy $\leq E$. $\mathcal{N}(E)$ is a staircase with each step being one unit high, and its slope is the level density $\rho(E)$. A harmonic oscillator will have a
FIG. 4: When $\kappa$ is turned on, the levels acquire a width, $\Gamma$. Here we sorted the set of 300 $\Gamma$ for each spectrum in an ensemble of 200. For example, $\Gamma_3$ is the 3rd largest width. The 9 lines in this plot are the ensemble average of $\Gamma_i$ vs $\kappa$, with $i = 2 \ldots 10$. The insert includes the largest width, $\Gamma_1$ which eventually becomes linear in $\kappa$.

A regular staircase, with each step being one unit wide, whereas the quantum equivalent of a classically regular system has a random but uncorrelated spectrum will have steps whose width have a poissonian distribution. $\Delta_3(L)$ is a measure of the spectral average deviation of $N(E)$ from a regular (constant slope) staircase, within an energy interval of length $L$. The spectral average means that the deviation is averaged over the location in the spectrum of the window. The definition is

$$\Delta_3(L) = \left\langle \min_{A,B} \frac{1}{L} \int_{E_i}^{E_i+L} dE' [N(E') - AE' - B]^2 \right\rangle = \langle \delta_3^i(L) \rangle.$$  

(2)

$A$ and $B$ are calculated for each $i$ to minimize $\delta_3^i(L)$. The details of the exact calculation of $A$ and $B$ in terms of the energies $\{E_i\}$ are in [11]. The harmonic oscillator has $\Delta_3(L) = 1/12$. At the other extreme, a classically regular system will lead to a quantum mechanical spectrum with no level repulsion, the fluctuations will be far greater because there is no long range correlation, and $\Delta_3(L) = L/15$. The angle brackets mean the average is to be taken
FIG. 5: This is a log-log plot of the average of the lines in Fig. 4.

FIG. 6: The ensemble result for the distribution of reduced widths $p(\gamma/\bar{\gamma})$. In red we see the full set of 2.5 million widths which are a superposition of 250000 matrices with $N = 100$ and $\kappa = 1.5$. The mean of this set is $\bar{\gamma} = 0.15$. In black we see the results for the smallest 100 widths of 88646 matrices with $N = 102$. The mean of this subset is $\bar{\gamma} = 0.0549$. In blue we have a plot of the function $P(x) = \frac{1}{\sqrt{2\pi x}} \exp(-\frac{x}{2})$. 
over all positions \( E_i \) of the window of length \( L \), this is a spectral average. It is an amazing fact of RMT that the spectra of the GOE have huge long range correlations, indeed the GOE result is:

\[
\Delta_3(L) = \frac{1}{\pi^2} \left[ \log(2\pi L) + \gamma - \frac{5}{4} - \frac{\pi^2}{8} \right] \\
= (\log L - 0.0678)/\pi^2,
\]

with \( \gamma \) being Euler’s constant. We stress that this is the RMT value for the ensemble average of \( \Delta_3(L) \). A given GOE matrix will give a graph of \( \Delta_3(L) \) that is different, but the average of many such lines, the ensemble average, will rapidly converge onto Eq. 4. Putting in values we get \( \Delta_3(L) = (\log L - 0.0678)/\pi^2 \). In our opened ensemble, \( \Delta_3(L) \) deviates from the GOE result. In Fig. 7 we take various fixed values of \( L \) and we see how the value of \( \Delta_3(L) \) evolves with \( \kappa \). Deviations from the GOE result \((\kappa = 0)\) increases slowly as \( \kappa \) changes from 0 to 1, then the deviations start to decrease. The effect is similar for all \( L \) in the range we looked at, so in Fig. 8 we plot all the lines of Fig. 7 but divide them by their value at \( \kappa = 0 \). Now we see that \( \Delta_3(L) \) increases by a similar factor with \( \kappa \) for a broad range of \( L \), and furthermore this factor has a maximum value of around 1.15 which happens when \( \kappa = 1 \).

### VI. RMT TESTS FOR MISSED LEVELS

The increase in the value of \( \Delta_3(L) \) due to opening of the system could be misconstrued as evidence of spurious or missed levels. There are RMT tests for missed levels, we will concentrate on 3 RMT tests, two of them based on \( \Delta_3(L) \) and another based the the nearest neighbor distribution (nnd). We apply these tests to the open spectra and see if there is any consistent picture that emerges. In [12] these methods were described in detail and used on a variety of experimental data.

In [12] a maximum likelihood method based on \( \Delta_3(L) \) was developed. The \( \Delta_3(L) \) statistic is the spectral average of the set of random numbers \( \delta_i^3(L) \), and the distribution of these numbers \( p(\delta) \) was used as the basis of a likelihood function. The basic idea is that \( \mathcal{N}(\delta) \), the cdf for \( p(\delta) \), is a simple function of \( \log \delta \). This led to the following parameterization:

\[
\mathcal{N}(\delta) = \frac{1}{2}(1 - \text{Erf}[a + b \log \delta + c(\log \delta)^2])
\]
FIG. 7: Here we see ensemble average of $\Delta_3(L)$ vs $\kappa$. The 46 lines are for the 46 values of $L$, with $5 \leq L \leq 50$ in steps of 2. There are 200 spectra in the ensemble, with $N = 300$.

An ensemble of depleted spectra was made and the parameters for $\mathcal{N}(\delta)$ were empirically found for a range of $L$, and $x$ the fraction of levels missed, and fitted to smooth functions $a_L(x), b_L(x)$ and $c_L(x)$. Differentiation of $\mathcal{N}(\delta)$ gives probability density for $\delta^3_3(L)$ with $x$ as a continuous parameter:

$$p(\delta, x) = -\frac{1}{\sqrt{\pi}} \exp \left[-(a_L(x) + b_L(x) \log \delta + c_L(x) \log \delta^3) \right] \left(\frac{b_L(x)}{\delta} + \frac{2 \ c_L(x) \log \delta}{\delta}\right).$$

This is then used as the basis for a maximum likelihood method for determining $x$.

In [13], Bohigas and Pato gave an expression is given for $\Delta_3(L)$ for incomplete spectra. The fraction of missed levels $x$ is both a scaling factor and a weighting factor and $\Delta_3(L, x)$ is the sum of the GOE and poissonian result:

$$\Delta_3(L, x) = x^2 \frac{L}{15} + (1 - x)^2 \Delta_{3}^{\text{GOE}}(L/(1 - x)).$$

The $\Delta_3(L)$ statistic of an open spectrum can be compared with this expression and the best $x$ found.

The nearest neighbor distribution (nnd) is another commonly used statistic. The nnd for
FIG. 8: Here we see ensemble average of $\Delta_3(L, \kappa)$ vs $\kappa$ for $20 \leq L \leq 50$, but now each line is divided by $\Delta_3(L, 0)$. The lower $L$ values are not as sensitive to $\kappa$ as the window of $L$ levels is too narrow to probe long range correlations. So it is best to not include the range $L < 20$. The legend refers to the 3 bold lines in the plot. The lines for various $L$ become closer as $L$ increases.

A pure spectrum follows the Wigner distribution,

$$P(s) = \frac{\pi}{2} s e^{-\pi s^2/4},$$

where $s = S/D$, $S$ being the spacing between adjacent levels, and $D$ is the average spacing ($D = 1$ for an unfolded spectrum). The nd of a spectrum incomplete by a fraction $x$ is given by

$$P(s) = \sum_{k=0}^{\infty} (1 - x)x^k P(k; s);$$

where $P(k; s)$ is the $k^{th}$ nearest neighbor spacing, $E_{k+i} - E_i$. This was first introduced as an ansatz in [14], and rederived in [15] and [13]. Eq. 9 was used by Agvaanluvsan et al as the basis for a maximum likelihood method (MLM) to determine $x$ for incomplete spectra [15].

These three tests for missed levels were applied to complete opened GOE spectra of dimension $N = 300$. The value of $\kappa$ went from 0 to 3 in steps of 0.01. The values $x$ for the fraction depleted vs $\kappa$ are shown in Fig. 9. It appears that the spectra look incomplete when the system is opened, and the effect is strongest to the tune of about 3% when $\kappa = 1$. 
FIG. 9: Here we see the results of tests to determine the fraction of levels missed in incomplete spectra vs $\kappa$ for open but complete GOE spectra. $N$ is 300 in all cases.

VII. CONCLUSION

A simple model for an open quantum system was realized by adding an imaginary number $i\sqrt{N}\kappa$ to the trace of an $N \times N$ GOE matrix. The level density deviated from the semicircle of Wigner and the deviation grew with $\kappa$. There was a drifting of some levels to the center of the spectrum at around $\kappa = 1$ and the emergence of a very low entropy state which we identify with a super-radiant state. The widths $\Gamma(\kappa)$ of the levels were sorted and a graph of the biggest 10 showed the emergence of the SR state that absorbs all the width. If the largest width is excluded the remaining widths were consistent with a Porter-Thomas distribution, and their average value had a simple exponential dependence on $\kappa$. A plot of $\bar{\Gamma}$, the average of all but the biggest widths vs $\kappa$ on a log-log plot reveals a SR transition, Fig. $\text{VI}$. The $\Delta_3(L)$ statistic deviated from the GOE value and looked like that of an incomplete spectra. Three separate tests for missed levels based on RMT consistently showed that at $\kappa \approx 1$ the spectra appeared incomplete to the tune of about 3%.

It is interesting that such a simple system can capture so much of the systematics of super-radiance in open quantum systems.
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