Statistics of temperature fluctuations in an electron system out of equilibrium

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We study the statistics of the fluctuating electron temperature in a metallic island coupled to reservoirs via resistive contacts and driven out of equilibrium by either a temperature or voltage difference between the reservoirs. The fluctuations of temperature are well-defined provided that the energy relaxation rate inside the island exceeds the rate of energy exchange with the reservoirs. We quantify these fluctuations in the regime beyond the Gaussian approximation and elucidate their dependence on the nature of the electronic contacts.

The temperature of a given system is well-defined in the case when the system is coupled to and in equilibrium with a reservoir at that temperature. Out of equilibrium, the temperature is determined by a balance of the different heat currents from/to the system [1]. However, this applies only to the average temperature: the heat currents fluctuate giving rise to temperature fluctuations. Although the equilibrium fluctuations have been discussed in textbooks [2], their existence was still debated around the turn of 1990’s [3].

In this Letter we generalize the concept of temperature fluctuations to the nonequilibrium case by quantifying their statistics in an exemplary system: a metal island coupled to two reservoirs (see Fig. 1). The island can be biased either by a voltage or temperature difference between the reservoirs. In this case, the temperature of the electrons is not necessarily well defined. The electron-electron scattering inside the island may however provide an efficient relaxation mechanism to drive the energy distribution of the electrons towards a Fermi distribution with a well-defined, but fluctuating temperature [1-4]. Here we assume this quasiequilibrium limit where the time scale $\tau_{e-e}$ of internal relaxation is much smaller than the scale $\tau_E$ related to the energy exchange with the reservoirs.

In equilibrium, the only relevant parameters characterizing the temperature fluctuation statistics (TFS) are the average temperature $T_a$, fixed by the reservoirs, and the heat capacity $C = \pi^2 k_B^2 T_a/(3\delta_I)$ of the system. The latter is inversely proportional to the effective level spacing $\delta_I$ on the island. In terms of these quantities, the probability of the electrons being at temperature $T_e$ reads [2-5]

$$P_{eq}(T_e) \propto \exp \left[ -\frac{C(T_e - T_a)^2}{k_B T_a^2} \right] = \exp \left[ -\frac{\pi^2 k_B (T_e - T_a)^2}{3T_a \delta_I} \right]$$

(1)

corresponding to Boltzmann distribution of the total energy of the island. The probability has a Gaussian form even for large deviations from $T_a$, apart from that the probability naturally vanishes for $T_e < 0$. From this distribution we can for example infer the variance, $\langle(\Delta T_e)^2\rangle = k_B T^2/C$. As we show below, the scale of the probability log, $\ln P \sim T_a/\delta_I$ is the same for the nonequilibrium case while its dependence on $(T_e/T_a)$ is essentially different.

To generalize the concept of temperature fluctuations to the nonequilibrium case we examine the probability that the temperature of the island measured within a time interval $\tau_0 \ldots \tau_0 + \Delta\tau$ and averaged over the interval, equals $T_e$:

$$P(T_e) = \left\langle \delta_I \left( \frac{1}{\Delta\tau} \int_{\tau_0}^{\tau_0 + \Delta\tau} T_I(t) dt - T_e \right) \right\rangle$$

$$= \left\langle \int \frac{dk}{2\pi} \exp \left[ ik \left( \int_{\tau_0}^{\tau_0 + \Delta\tau} (T_I(t) - T_e) dt \right) \right] \right\rangle$$

(2)

The average $\langle \cdot \rangle$ is over the nonequilibrium state of the
system. The latter is evaluated using an extension of the Keldysh technique [6] where the fluctuations of charge and heat are associated with two counting fields, $\chi$ and $\xi$, respectively [7, 8, 9]. The technique allows one to evaluate the full statistics of current fluctuations both for charge [7] and heat current [8] in an arbitrary multiterminal system. In terms of the fluctuating temperature and chemical potential of the island, $T_I(t)$ and $\mu_I(t)$, and the associated counting fields $\xi_I(t)$ and $\chi_I(t)$ the average in Eq. (2) is presented in the form

$$P(T_c) \propto \int D\xi_I(t)D\chi_I(t) D\mu_I(t) dk \times \exp \left\{ -A + ik \left[ \int_{\tau_0}^{\tau_0 + \Delta \tau} dt(T_I(t) - T_c) \right] \right\}. \quad (3)$$

Here $A = A[\xi_I(t), T_I(t), \chi_I(t), \mu_I(t)]$ is the Keldysh action of the system. The counting fields $\xi_I(t)$ and $\chi_I(t)$ enter as Lagrange multipliers that ensure the conservation of charge and energy [9].

The Keldysh action consists of two types of terms, $A = \int dt(S_I(t) + S_c(t))$, with $S_I(t) = Q_I \Delta \chi_I + E_I \xi_I$ describing the storage of charge on the island and $S_c$ describing the contacts to the reservoirs. Here $Q_I = C_c \mu_I$ is the charge on the island, $E_I = C(T_I)T_I/2 + C_c \mu_I^2/2$ gives the total electron energy of the island and $C_c$ is the electrical capacitance of the island. For the electrical contacts, the action can be expressed in terms of the Keldysh Green’s functions as [10] (we set $\hbar = e = k_B = 1$ for intermediate results)

$$S_{c,cl} = \frac{1}{2} \sum_{n} \sum_{\alpha} \text{Tr} \ln \left[ 1 + T_n \left\{ \tilde{G}_\alpha, \tilde{G}_I \right\} - 2 \right]. \quad (4)$$

The sums run over the lead and channel indices $\alpha$ and $n$. All products are convolutions over the inner time variables. The trace is taken over the Keldysh indices and the action is evaluated with equal outer times. This action is a functional of the Keldysh Green’s functions $\tilde{G}_\alpha$ and $\tilde{G}_I$ of the reservoirs and the island, respectively. It also depends on the transmission eigenvalues $\{T_n \}$, characterizing each contact. The counting fields enter the action by the gauge transformation of Green’s function [8]

$$\tilde{G}(t, t') = e^{-\frac{i}{2}(\chi_I(t) - \chi_I(t'))\Delta \tau} \tilde{G}_0(t, t') e^{\frac{i}{2}(\chi_I(t) - \chi_I(t'))\Delta \tau}, \quad (5)$$

where the Keldysh Green’s function reads

$$\tilde{G}_0(t, t') = \int \frac{d\epsilon}{2\pi} e^{-i\epsilon(t-t')} \left\{ \frac{1 - 2f(\epsilon)}{2 - 2f(\epsilon)} - 1 + 2f(\epsilon) \right\}. \quad (6)$$

For quasiequilibrium $f(\epsilon) = \{ \exp[(\epsilon - \mu)/T] + 1 \}^{-1}$ is a Fermi distribution. In what follows, we assume the fields $\chi_I(t), T_I(t)$ to vary slowly at the time scale $T^{-1}$, in which case we can approximate $-i\chi_I(t)\partial_t \mapsto \chi_I(t)\epsilon$.

The saddle point of the total action at $\chi = \xi = 0$ yields the balance equations for charge and energy. Assuming that the electrical contacts dominate the energy transport, we get

$$\frac{\partial Q_I}{\partial t} = C_c \partial_t \mu_I = \sum_{\alpha} \text{Tr} \tau_3 \sum_n T_n^\alpha \left[ \tilde{G}_\alpha, \tilde{G}_I \right] + 4 + T_n^\alpha (\left\{ \tilde{G}_I, \tilde{G}_\alpha \right\} - 2) \quad (7a)$$

$$\frac{\partial E_I}{\partial t} = C \partial_t T_I = \sum_{\alpha} \text{Tr} (\epsilon - \mu_I) \tau_3 \sum_n T_n^\alpha \left[ \tilde{G}_\alpha, \tilde{G}_I \right] + 4 + T_n^\alpha (\left\{ \tilde{G}_I, \tilde{G}_\alpha \right\} - 2). \quad (7b)$$

The right-hand sides are sums of the charge and heat currents, respectively, flowing through the contacts $\alpha$ [11].

The time scale for the charge transport is given by $\tau_c = C_c / G$, with $G = \sum_n \sum_n T_n^\alpha / (2\pi)$. This is typically much smaller than the corresponding time scale for heat transport, $\tau_h = C_h / G_{th}$, where $G_{th} = \pi^2 G T / 3$. We assume that the measurement takes place between these time scales, $\tau_c \ll \Delta \tau \ll \tau_h$. In this limit the potential and its counting field $\mu_I$ and $\chi_I$ follow adiabatically the $T_I(t)$ and $\xi_I(t)$ and there is no charge accumulation on the island. As a result, we can neglect the charge capacitance $C_c$ concentrating on the zero-frequency limit of charge transport.

To determine the probability, we evaluate the path integral in Eq. (3) in the saddle-point approximation. There are four saddle-point equations,

$$\partial_{\chi_I} S_c = 0, \quad \partial_{\mu_I} S_c = 0 \quad (8a)$$

$$\frac{\pi^2}{6} \xi_I = -\partial_{T_I} \frac{i}{2} M_0(t; \tau_0, \Delta \tau) \quad (8b)$$

$$\frac{\pi^2}{6} \frac{\Delta \tau}{T_I} = \partial_{\xi_I} S_c. \quad (8c)$$

Here $M_0(t) = 1$ inside the measurement interval $(\tau_0, \tau_0 + \Delta \tau)$ and zero otherwise. Equations [8a] express the chemical potential and charge counting field in terms of instant values of temperature $T_I$ and energy counting field $\xi_I$, $\mu_I = \mu_I(\xi_I, T_I)$, $\chi_I = \chi_I(\xi_I, T_I)$. The third and fourth equations give the evolution of these variables. It is crucial for our analysis that these equations are of Hamilton form, $\xi_I$ and $T_I^2$ being conjugate variables, the total connector action $S_c$ being an integral of motion. Boundary conditions at $t \pm \infty$ correspond to most probable configuration $T_c = T_a$. This implies $S_c = 0$ at trajectories of interest.

The zeros of $S_c$ in $\xi_I - T_I$ plane are concentrated in two branches that cross at the equilibrium point $\xi_I = 0, T_c = T_a$ (Fig. 2a). Branch B ($\xi = 0$) corresponds to the usual “classical” relaxation to the equilibrium point from either higher or lower temperatures. Branch A corresponds to “anti-relaxation”: the trajectories following
The curve quickly depart from equilibrium to either higher or lower temperatures. The solution of the saddle-point equations follows A before the measurement and B after the measurement (Figs. 2 b-e).

Since $S_c = 0$, the only contribution to path integral \[ \exp \{ 2 \mathcal{I} \} \] comes from the island term $C \xi^2 T^2$ and is evaluated as

\[ P(T_c) \exp \left[ \frac{\pi^2}{3 \delta I} \int T^2 dt \right] = \exp \left[ -\frac{2\pi^2}{3 \delta I} \int T_a \xi^2(T)a(T)dt \right] \]  

Thus, in order to find $P(T_c)$, we only need a function $\xi^2(T)$ satisfying $S_c(\xi^2(T), T) = 0$ at branch A.

The connector action can generally be written in the form

\[ S_c = \sum_{\alpha} \sum_{n \in \alpha} \int \frac{de}{2\pi} \ln \{ 1 + T_n^a \} \{ f_I(1 - f_o) \} \times (e^{\chi_I - \xi^2(I)} - 1) + f_o(1 - f_I)(e^{\chi_I + \xi^2(I)} - 1) \],

with $f_o/1 = \{ \exp[(\mu - \chi/I)/T_o] + 1 \}^{-1}$.

To prove the validity of the method for the equilibrium case, let us set all the chemical potentials to 0 and all the reservoir temperatures to $T_o$. This implies $\mu = \chi/I = 0$. Using the fact that for a Fermi function $f = -e^{\chi/I}(1 - f)$, we observe that $S_c = 0$ regardless of contact properties provided $\xi_I = \xi^2(T)/T = 0$. Substituting this to Eq. \[ \] reproduces the equilibrium distribution, Eq. \[ (1) \].

Out of equilibrium, the further analytical progress can be made in the case when the connectors are ballistic, $T_o = 1$. Such a situation can be realized in a chaotic cavity connected to terminals via open quantum point contacts. The connector action reads \[ (1) \],

\[ S_c = \sum_{\alpha} G_{\alpha} \left[ \frac{2\mu \chi I + T_o \chi^2 + \frac{\pi^2 T^2/3 + \mu^2}{1 - T_o \chi I} \chi I}{1 + T_o \chi I} \right] - \frac{2\mu \chi I - T_o \chi^2 + \frac{\pi^2 T^2/3 + \mu^2}{1 + T_o \chi I}}{1 + T_o \chi I}. \]

Let us first assume two reservoirs with $T_1 = T_2 \equiv T_L$. In this case the general saddle-point solution for the potential follows from Kirchhoff law: $\mu_I = \mu_1 + \mu_2)/(1 + g)$ with $g = g_{L}/g_{R}$. For the charge counting field we get $\chi = -\mu \xi$. The most probable temperature $T_o$ is given by $T_o^2 = T_L^2 + 3g(\mu_1 - \mu_2)/[\pi^2(1 + g)^2]$, and function $\xi^2(T)$ is expressed as

\[ \xi_I = \frac{T_o^2 - T_a^2}{T^2 + T_o^2}. \]

Substituting this to Eq. \[ \] yields for the full probability

\[ -\ln P_{ball} = \frac{\pi^2 k_B}{3 \delta I T_o^2} \left[ T_L(T_c - T_o)((T_c + T_o)T_L - 2T_o^2) \right. \]

\[ + \left. 2T_o^2(2T_o^2 - T_L^2) \ln \left( \frac{T_o^2 + T_L^2}{T_o + T_oT_L} \right) \right] \]

In the strong nonequilibrium limit $V \equiv (\mu_1 - \mu_2) \gg T_L$, i.e., $T_o \gg T_L$, this reduces to

\[ P_{ball} \propto \exp \left\{ -\frac{2\pi^2 k_B}{3 \delta I} \left[ \frac{T_o^2 + 2T_o(T_c - T_o)^2}{3T_o^2} \right] \right\} \]

The logarithm of this probability is plotted as the lowermost line in Fig. \[ (1) \].

If the island is biased by temperature difference, $T_1 \equiv T_L \gg T_2$, $V = 0$, the probability obeys the same Eq. \[ (14) \] with $T_o^2 = gT_o^2/(1 + g)$.
For rare fluctuations of temperature, $|T_e - T_a| \approx T_a$, the probability distribution is essentially non-Gaussian in contrast to the equilibrium case. The skewness of the distribution is negative in the case of voltage driving: low-temperature fluctuations ($T_e < T_a$) are preferred to the high-temperature ones ($T_e > T_a$). In contrast, biasing with a temperature difference (uppermost curve in Fig. 3) favours high-temperature fluctuations.

The non-Gaussian features of the temperature fluctuations can be accessed at best in islands with a large level spacing, that is smaller than the average temperature say, by an order of magnitude. Many-electron quantum dots with spacing up to 0.1 K$/k_B$ seem natural candidates for the measurement of the phenomenon. The most natural way to detect the rare fluctuations is through a threshold detector [12], which produces a response only for temperatures exceeding or going under a certain threshold value. Besides the direct measurement of temperature, one can use the correlation of fluctuations. For example, Fig. 2(e) shows that the fluctuation of the temperature also causes a fluctuation in the charge current. Observing the latter may thus yield information about the former.

To conclude, we have evaluated non-equilibrium temperature fluctuations of an example system beyond the Gaussian regime. The method makes use of saddle-point trajectories and allows to describe electric contacts of arbitrary transparency.

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FIG. 3: (Color online.) Logarithm of TFS probability $P(T_e)$ in a few example cases. Solid lines from top to bottom: temperature bias with symmetric tunneling contacts, $T_a = T_1/\sqrt{2}$, $T_2 = 0$ (magenta); Gaussian equilibrium fluctuations (black), nonequilibrium fluctuations with $T_a = \sqrt{3}eV/(2\pi k_BT)$, $T_1 = T_2 = 0$ for symmetric tunneling and ballistic contacts (blue and red lines, respectively). The dashed lines are Gaussian fits to small fluctuations $(T_e - T_a) \ll T_a$, described by the heat current noise $S_Q$ at $T_e \approx T_a$.

For general contacts, the connector action and its saddle-point trajectories have to be calculated numerically. For tunnel contacts, the full probability distribution is plotted in two regimes in Fig. 3. The distribution takes values between the ballistic and equilibrium cases. Let us understand this concentrating on Gaussian regime and inspecting the variance of the temperature fluctuations for various contacts. This variance is related to the zero-frequency heat current noise $S_Q$ via

$$2G_{th}C(\delta T^2) = S_Q = \partial^2 S|_{\xi=0}. \quad (15)$$

In equilibrium, $S_Q^{(eq)} = 2G_{th}T^2$ by virtue of the fluctuation-dissipation theorem. For an island with equal ballistic contacts driven far from equilibrium, $V \gg T_L$, $S_Q^{bal} = \sqrt{3}GV^3/(8\pi) = G_{th}(T_a)/T_a^2$, i.e., only half of $S_Q^{(eq)}$. The reduction manifests vanishing temperature of the reservoirs. Most generally, for contacts of any nature, the heat current noise reads

$$S_Q/S_Q^{(eq)} = \frac{1}{2} + a_Q \sum_{\alpha} F_{\alpha}, \quad (16)$$

where $F_{\alpha} = \sum_n T_n^\alpha (1 - T_n^\alpha)/\sum_n T_n^\alpha$ is the Fano factor for a contact $\alpha$, $a_Q \approx 0.112$ being a numerical factor. For two tunnel contacts we hence obtain $S_Q^{num} \approx 0.723 S_Q^{(eq)}$, a value between the ballistic and equilibrium values. For contacts of any type, the variation of temperature fluctuations is between the ballistic and tunneling values.

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