LINEARIZATION OF TRANSITION FUNCTIONS ALONG A CERTAIN CLASS OF LEVI-FLAT HYPERSURFACES

SATOSHI OGAWA

ABSTRACT. We pose a normal form of transition functions along some Levi-flat hypersurfaces obtained by suspension. By focusing on methods in circle dynamics and linearization theorems, we give a sufficient condition to obtain a normal form as a geometrical analogue of Arnol’d’s linearization theorem.

1. INTRODUCTION

We study a neighborhood of a Levi-flat hypersurface. Let $X$ be a non-singular complex surface. We say a real hypersurface $M$ of $X$ is Levi-flat if and only if the Levi-form of $M$ vanishes identically. Especially, $M$ is Levi-flat when $M$ admits a system of local defining functions $\{\rho_j\}_{j}$ such that each $\rho_j$ is pluriharmonic (i.e. $\bar{\partial}\partial\rho_j = 0$).

One of our interests is the linearization of transition functions on a neighborhood of a Levi-flat hypersurface. In the study of 1-dimensional complex dynamical systems, it is important to consider whether or not one can find a coordinate by which a function can be regarded as a linear map at a neighborhood of a given fixed point or an invariant curve (linearization, see in §2). In this paper, we will find a normal form of the complex structure of a neighborhood of a Levi-flat hypersurface $M$ by applying a technique for linearization around a circle to the transition functions of a coordinate functions on a neighborhood of $M$. In order to apply such a technique for the linearization around a circle, we will focus on a certain class of Levi-flat hypersurfaces, which are constructed by suspension construction.

Let $Y$ be a non-singular compact complex curve and Diff$^\omega_+(S^1)$ be the group of orientation preserving $C^\omega$-diffeomorphisms of $S^1$, where $S^1$ is the unit circle $\{z \in \mathbb{C} | |z| = 1\}$. For a given action of the fundamental group $\kappa: \pi_1(Y, \ast) \to $ Diff$^\omega_+(S^1)$, we consider the quotient space $M$ defined by $Y_{\text{min}} \times S^1 / \sim$, where $\sim$ is the relation induced from the action $\kappa$ (i.e. $(z, x) \sim (z \cdot \gamma, \kappa(\gamma)(x))$ for $\gamma \in \pi_1(Y, \ast)$). Then $M$ is said to be obtained by suspension construction of $\kappa: \pi_1(Y, \ast) \to $ Diff$^\omega_+(S^1)$.

Assume that $M$ is embedded into a non-singular complex surface $X$. Let $\mathcal{U} = \{U_j\}$ be a finite covering of $Y$ and $\pi: M \to Y$ be the projection. For technical reasons, we assume that there exists a holomorphic submersion $P: V \to Y$ on a neighborhood $V$ of $M$ in $X$ which satisfies $P|_{M} = \pi$.

DEFINITION 1.1. We say that $\{(V_j, (z_j, w_j))\}$ is a good system of local functions of width $\sigma > 0$ if and only if it satisfies the following conditions.

(i) For each $U_j$, a local coordinate $z_j$ of $U_j$ and $M_j := \pi^{-1}(U_j)$, let $V_j$ be a neighborhood of $M_j$ in $X$ which satisfies $\bigcup_j V_j \subset V$, and $(z_j, w_j)$ be a local coordinate on $V_j$, where the coordinate $(z_j, w_j)$ on $V_j$ is given by pullback of the local coordinate of $U_j$ by $P$.

(ii) For each $j$ and $k$, $V_j \cap \bar{M} = M_j$ holds and $V_j \cap V_k = \emptyset$ holds if $M_j \cap M_k = \emptyset$.

(iii) There exists a positive number $\sigma_j \geq \sigma$ which satisfies the following condition for
each \( j \): There exists a biholomorphism from \( V_j \) to \( U_j \times \{ e^{-\sigma_j} < |w_j| < e^{\sigma_j} \} \) which makes the following diagram commutative.

\[
\begin{array}{ccc}
V_j & \xrightarrow{=} & U_j \times \{ e^{-\sigma_j} < |w_j| < e^{\sigma_j} \} \\
\downarrow{P} & & \downarrow{Pr_1} \\
U_j & & \\
\end{array}
\]

(iv) For each \( j \), \( M_j = \{ (z_j, w_j) \in V_j \mid |w_j| = 1 \} \) holds.

In what follows, we always assume that a system \( \{Y, M, U, X, \pi, P\} \) has a good system \( \{V_j, (z_j, w_j)\} \) of local functions of width \( \sigma \). Then, the hypersurface \( M \) has the local defining function determined by \( \log |w_j| \), from which it follows that \( M \) is a Levi-flat hypersurface of \( X \). We recall that \( M \) has a structure of \( S^1 \)-bundle over \( Y \). We will say that a system \( \{Y, U, \kappa, X, P, \{V_j(z_j, w_j)\}, \sigma\} \) is linearizable if there exists a good system of local function system \( \{(V'_j, (z_j, w'_j))\} \) of width \( \sigma' \) for \( \{Y, U, \kappa, X, P\} \) such that the transition function is written as \( w'_k = t_{jk}w_j \) on each \( V'_{jk} := V'_j \cap V'_k \), where \( t_{jk} \in U(1) \).

Let

\[
\begin{aligned}
z_k &= z_k(z_j) \\
w_k &= f_{kj}(w_j)
\end{aligned}
\]

be the transition on \( V_{jk} := V_j \cap V_k \). Note that \( z_k \) does not depend on \( w_j \) by the condition (iii) in Definition 1.1. Note also that the transition function \( f_{kj} \) does not depend on \( z_j \) (see Lemma 3.2). We call \( f_{kj} \) a transversal transition function of \( V_{jk} \) for a good system of local functions \( \{V_j, (z_j, w_j)\} \). The function \( f_{kj}|_{S^1} \) is an element of \( \text{Diff}^\omega(S^1) \) with a variable \( w_j \), where we regard \( S^1 \) as \( \{ |w_j| = 1 \} \). Our aim in this paper is to investigate the linearization of transition functions \( \{f_{kj}\}_{j,k} \).

Let \( \alpha_{kj} \) and \( b_{kj|n} \) be coefficients of the expansion

\[
\log \frac{f_{kj}(w_j)}{w_j} \equiv \alpha_{kj} + \sum_{n \neq 0} b_{kj|n} n^a (\text{mod } 2\pi \sqrt{-1} \mathbb{Z}).
\]

Note that \( \alpha_{kj} \in \sqrt{-1} \mathbb{R} \), since \( f_{kj} \in \text{Diff}^\omega(S^1) \) (see the argument in the proof of Lemma 3.9). If \( \alpha(\{(V_j, (z_j, w_j))\}) := \{(U_{jk}, e^{\alpha_{kj}})\} \in \check{C}^1(U, U(1)) \) satisfies the 1-cocycle condition, \( N = \{[(U_{jk}, e^{\alpha_{kj}})]\} \in H^1(U, U(1)) \) can be regarded as a unitary flat line bundle over \( Y \), where \( U(1) = \{ t \in \mathbb{C} \mid |t| = 1 \} \). We denote by \( b_{kj|n}(\{(V_j, (z_j, w_j))\}) \) the non-zero order coefficients \( b_{kj|n} \) for a good system of local functions \( \{(V_j, (z_j, w_j))\} \).

The main result is the following.

**Theorem 1.2.** Let \( Y \) be a compact complex curve, \( U = \{U_j\} \) a finite open covering of \( Y \), and \( \pi: M \to Y \) an \( S^1 \)-bundle over \( Y \) constructed by suspension associated to an action \( \kappa: \pi_V(Y, *) \to \text{Diff}^\omega(S^1) \). Let \( X \) be a complex surface which has \( M \) as a Levi-flat hypersurface. Assume that there exists a holomorphic submersion \( P: V \to Y \) which satisfies \( P|_M = \pi \), where \( V \) is a neighborhood of \( M \) in \( X \). Assume also that there exists a good system of local functions \( \{(V_{j0}, (z_j, w_{j0}))\} \) of width \( \sigma_0 \) for \( Y, U, \kappa, X \) and \( P \). Then, the system \( \{Y, U, \kappa, X, P, \{(V_{j0}, (z_j, w_{j0}))\}, \sigma_0\} \) is linearizable if the following conditions (i), (ii), (iii) hold.

(i) The 1-cochain \( \alpha(\{(V_{j0}, (z_j, w_{j0}))\}) = \{(U_{jk}, e^{\alpha_{kj}})\} \in \check{C}^1(U, U(1)) \) satisfies the 1-cocycle condition and \( N = \{[(U_{jk}, e^{\alpha_{kj}})]\} \) satisfies \((C_0, \mu, K)\)-Diophantine condition, in
the sense of Definition 2.7 (see §2), where \( C_0 > 0, \mu > 1, \) and \( K \) is constant determined only by \( Y \) and \( U \).

(ii) For non-zero order coefficients \( b_{kj|n,0} = b_{kj|n}(\{(V_j, (z_j, w_j, 0))\}) \) associated to the transversal transition function \( f_{kj,0} \) of \( \{(V_j, (z_j, w_j, 0))\} \), there exists a constant \( \eta_0 \in (0, \min\{\pi, (1 - \mu^{-\frac{1}{n+1}})\frac{\pi\sigma}{4}\}) \) such that

\[
\max_{\eta, k} \sup_{\sigma_0 < |p| < e^{\sigma_0}} \left| \sum_{n \neq 0} b_{kj|n,0} p^n \right| < \min \left\{ \eta_0, \frac{\eta_0^{\mu+1}}{(1 + e^{\sigma_0})C_1 \mu} \right\}
\]

holds, where \( C_1 \) is a constant which depends only on \( C_0, \mu \) and \( \sigma_0 \).

(iii) For any good system of local functions \( \{(V_j, (z_j, w_j))\} \) which has \([\alpha(\{(V_j, (z_j, w_j))\})| = N \) as a unitary flat line bundle over \( Y \), a 1-cochain \( \{(U_{jk}, b_{jk|n})\} \in \tilde{\mathcal{C}}^1(U, \mathcal{O}_Y(N^n)) \) satisfies the 1-coboundary condition.

Comparing with Arnol’d’s linearization theorem (Theorem 2.4) and Ueda’s linearization theorem (Theorem 2.3), we will explain the conditions (i), (ii), and (iii). The condition (i) is the more detailed version of the Diophantine condition. The condition (ii) corresponds to the assumption for the estimate of a perturbation in Arnol’d’s linearization theorem. The condition (iii) corresponds to vanishing of obstruction classes in Ueda’s proof.

Main result can be applied to finding a criterion of simultaneous linearization of circle diffeomorphisms (see §4). In this sense, Theorem 1.2 can be regarded as a generalization of Arnol’d’s linearization theorem.

In [KU], Koike and Uehara constructed Levi-flat in K3 surfaces and showed there exist a foliated small neighborhood of it. From the viewpoint of Koike and Uehara’s result, this result also can be regarded as one of a result for a foliated neighborhood along Levi-flat hypersurfaces.

Our idea and main result can be explained as a geometrical analogue of Arnold’s linearization theorem. For proving Theorem 1.2, we use Kolmogorov-Arnol’d-Moser (KAM) theory, which is used in the proof of Arnold’s linearization theorem (Theorem 2.4) in [CG] [SM]. In [U], Ueda investigated linearization on a neighborhood of a compact complex curve embedded holomorphically in a complex surface as a geometrical analogue of Siegel’s linearization theorem (Theorem 2.3).

In §2, we introduce preliminaries about linearization theorems. In §2.1 we will explain the expansion of transition functions. In §2.2 and §2.3, we will see two linearization theorems in one-dimensional dynamics and Ueda’s linearization theorem. In §3, we will apply a method in §2.2 to a Levi-flat hypersurface constructed by suspension and show Theorem 1.2. In §4, I will introduce a simple example on the main result and obtain a sufficient condition for simultaneous linearization of circle diffeomorphisms. In §5, I will discuss the relation between the expansion of transition functions and a rotation number.

2. Preliminaries

2.1. Dynamical system of a circle diffeomorphism. In this section, we will review some fundamental facts on 1-dimensional dynamical systems.

For \( f \in \text{Diff}_+(S^1) \), we say that a homeomorphism \( F : \mathbb{R} \to \mathbb{R} \) is a lift of \( f \) if \( F \) satisfies a relation \( e^{2\pi \sqrt{-1} F(x)} = f(e^{2\pi \sqrt{-1} x}) \). The following limit is called the rotation number of
\[ f: \quad \rho(f) := \lim_{m \to \infty} \frac{F^m(x) - x}{m} \pmod{\mathbb{Z}}. \]

It is known that \( \rho(f) \) exists and is independent of a choice of \( F \) and a point \( x \in \mathbb{R} \). We have the Fourier expansion of the lift of \( f \in \text{Diff}_+^\omega(S^1) \) as below.

\[ F(x) = x + F_0 + \sum_{n \neq 0} F_n e^{2\pi \sqrt{-1} nx}. \]

By letting \( w = e^{2\pi \sqrt{-1} x} \) and taking the logarithm, we obtain

\[ \log f(w) = 2\pi \sqrt{-1} F_0 + \sum_{n \neq 0} 2\pi \sqrt{-1} F_n w^n. \]

Note that there exists a branch of \( \log f(w) \) globally on the annulus \( A := \{ e^{-\sigma} < |w| < e^\sigma \} \) for \( \sigma > 0 \).

**Lemma 2.1.** There exists \( h \in H^0(A, \mathcal{O}_A) \) such that \( e^h = g \) holds, where \( g(w) = \frac{f(w)}{w} \).

**Proof.** By considering the exponential sheaf exact sequence

\[ 0 \longrightarrow 2\pi \sqrt{-1} \mathbb{Z} \longrightarrow \mathcal{O}_A \longrightarrow \mathcal{O}_A^* \longrightarrow 0, \]

one obtains the exact sequence of cohomology groups

\[ \cdots \longrightarrow H^0(A, \mathcal{O}_A^*) \overset{\phi}{\longrightarrow} H^1(A, 2\pi \sqrt{-1} \mathbb{Z}) \longrightarrow \cdots. \]

It is sufficient to check \( \phi(g) = 0 \). We can calculate \( \phi(g) \) as

\[ \phi(g) = \int_{\gamma} \frac{g'(w)}{g(w)} dw, \]

where \( \gamma \) is a loop in the annulus \( A \) which generates the fundamental group \( \pi_1(A, \ast) \). We can easily check

\[ \frac{g'(w)}{g(w)} = \frac{f'(w)}{f(w)} - \frac{1}{w}. \]

Since \( f \in \text{Diff}_+^\omega(S^1) \), it is shown that \( \phi(g) = 0 \). \( \square \)

**2.2. Linearization theorems in 1-dimensional dynamics.** Here we survey some studies of the local behavior of a holomorphic function on a neighborhood of a fixed point. Let \( f \) be a holomorphic function which admits the origin as a fixed point. Suppose that \( f \) has the expansion \( f(z) = \Lambda z + b_2 z^2 + b_3 z^3 + \cdots \) on a neighborhood of the origin. A number \( \Lambda \) is called the *multiplier* of \( f \) at the fixed point. It is known that a classification of a fixed point is given according to the multiplier \( \Lambda \).

**Definition 2.2.** An irrational number \( \theta \) is said to be *Diophantine* if and only if there exist \( c > 0 \) and \( B > 1 \) so that

\[ \left| \theta - \frac{p}{q} \right| \geq \frac{c}{q^B} \]

for any rational number \( p/q \ (p, q \in \mathbb{Z}, q > 0) \).
We assume that $\Lambda = e^{2\pi \sqrt{-1}\theta}$ holds for an irrational number $\theta$. In this case, the fixed point 0 is called an *irrationally neutral fixed point*. The following theorem is known as an important linearization theorem at an irrationally neutral fixed point.

**Theorem 2.3** (Siegel’s linearization theorem [S]). Let $f$ be a holomorphic function which has the origin as an irrational fixed point with multiplier $e^{2\pi \sqrt{-1}\theta}$. If $\theta$ satisfies the Diophantine condition, then there exists a holomorphic map $\psi$ on a neighborhood of the origin such that $\psi$ satisfies $\psi(0) = 0$, $\psi'(0) = 1$, and $(\psi^{-1} \circ f \circ \psi)(z) = e^{2\pi \sqrt{-1}\theta}z$.

When $\psi$ as in Theorem 2.3 exists, we say that $f$ is *linearizable* on a neighborhood of 0. To show the linearizability of $f$ is equivalent to solve the following equation called Schröder’s equation:

$$\psi(e^{2\pi \sqrt{-1}\theta}z) = f(\psi(z)).$$

We define $\hat{f}$ and $\hat{\psi}$ by $f(z) = e^{2\pi \sqrt{-1}\theta}z + \hat{f}(z)$ and $\psi(z) = z + \hat{\psi}(z)$. Furthermore, suppose that the function $\hat{\psi}$ can be written as $\hat{\psi}(z) = a_2z^2 + \cdots$. Then, Schröder’s equation can be rewritten as $\hat{\psi}(e^{2\pi \sqrt{-1}\theta}z) - e^{2\pi \sqrt{-1}\theta}\hat{\psi}(z) = \hat{f}(\psi(z))$, which leads that $a_n$ can be determined by $a_2, \cdots, a_{n-1}, b_2, \cdots, b_n$ inductively. In Siegel’s original method, he estimated $a_n$ and proved the convergence of $\hat{\psi}$.

The following Theorem 2.4 is a counterpart of Theorem 2.3 in circle dynamics.

**Theorem 2.4** (Arnold’s linearization theorem [A], Theorem 7.2 of §2.7 in [CG], cf. Theorem 12.3.1 [KH]). Let $\alpha \in (0, 1)$ be a number which satisfies the Diophantine condition and $\sigma$ be a positive constant. Then there exists a positive constant $\delta$ such that, if $f$ is any element of $\text{Diff}_1^\alpha(S^1)$ with $\rho(f) = \alpha$ which extends to be analytic and univalent on the annulus $\{e^{-\sigma} < |z| < e^\sigma\}$ and satisfies $|f(z) - e^{2\pi \sqrt{-1}\alpha z}| < \delta$ on $\{e^{-\sigma} < |z| < e^\sigma\}$, then $f$ is linearizable on the annulus $\{e^{-\sigma'} < |z| < e^\sigma\}$, where $0 < \sigma' < \sigma$.

Arnol’d’s theorem can be regarded as the linearization along the unit circle. This theorem can be proven by a different strategy from that of Siegel for Theorem 2.3. The proof in [CG] is based on a simple case in KAM theory. In Arnol’d’s proof, we inductively change the coordinates along the unit circle and estimate the non-linear part of $f$. Details of this technique will be explained in §3 (cf. [H]).

### 2.3. Ueda’s linearization theorem

Siegel’s linearization theorem can be generalized in a geometric sense, which is known as Ueda’s linearization theorem ([U]). Let $C$ be a compact complex curve which is holomorphically embedded in a complex surface $S$ with the topologically trivial normal bundle $N_{C/S}$. Note that $\{S_j\}$ is a finite open covering of a neighborhood of $C$ in $S$. For $S_j$, let $s_j$ be a defining function of $S_j \cap C$ in $S_j$. We suppose that, for any $j$ and $k$, there exists $t_{kj} \in U(1)$ such that $s_k = t_{kj}s_j + O(s_j^2)$. Ueda gave a sufficient condition for the existence of an open covering $\{S'_j\}$ and a system of defining functions $\{s'_j\}$ such that $s'_k = t_{kj}s'_j$ holds by using the Diophantine condition of the normal bundle $N_{C/S}$. The Diophantine condition in the sense of Ueda is defined by focusing on an invariant distance $d$ of $\text{Pic}^0(C)$, where the invariant property of the distance $d$ means that the following holds for any $E_1, E_2, G \in \text{Pic}^0(C)$:

$$d(E_1, E_2) = d(E_1^{-1}, E_2^{-1}) = d(E_1 \otimes G, E_2 \otimes G).$$

**Definition 2.5.** For $E \in \text{Pic}^0(C)$ which satisfies $E^\otimes n \neq 1$ for any $n$, $E$ is said to be *Diophantine* if and only if the following holds:

$$- \log d(1, E^\otimes n) = O(\log n) \ (n \to \infty).$$
By using the Diophantine condition of a flat line bundle and the following theorem, he found some estimates of coefficients of transition functions at a neighborhood on $C$ and showed the linearization theorem ([U]).

**Theorem 2.6** (Lemma 4 in [U]). Let $\mathcal{W} = \{W_j\}$ be a finite open covering of $C$. There exists a positive constant $K = K(C, \mathcal{W})$ such that the following holds for any flat line bundle $E$ and any $\mathcal{G} \in C^0(\mathcal{W}, \mathcal{O}(E))$:

$$d(1, E)\|\mathcal{G}\| \leq K\|\delta \mathcal{G}\|,$$

where $\delta$ is the coboundary map from $C^0(\mathcal{W}, \mathcal{O}(E))$ to $C^1(\mathcal{W}, \mathcal{O}(E))$ and the norms are defined by

$$\|\mathcal{G}^0\| := \max_{j} \sup_{p \in W_j} |g_j(p)|$$

and

$$\|\mathcal{G}^1\| := \max_{j,k} \sup_{p \in W_j \cap W_k} |g_{jk}(p)|$$

for a 0-cochain $\mathcal{G}^0 = \{(W_j, g_j)\} \in C^0(\mathcal{W}, \mathcal{O}(E))$ and a 1-cochain $\mathcal{G}^1 = \{(W_j \cap W_k, g_{jk})\} \in C^1(\mathcal{W}, \mathcal{O}(E))$.

In this paper, by using $K$ in Theorem 2.6, we will classify the Diophantine condition in more detail as follows.

**Definition 2.7.** Let $C$ be a compact complex curve, $\mathcal{W}$ be a finite open covering of $C$, and $K$ be the constant as in Theorem 2.6. The unitary flat bundle $E$ on $C$ is said to satisfy $(C_0, \mu, K)$-Diophantine condition for $C_0 > 0$ and $\mu > 1$ if and only if

$$C_0 n^{\mu-1} d(1, E^{\otimes n}) \geq K$$

holds for any $n = 1, 2, \ldots$.

3. **Proof of main theorem**

3.1. **Outline of proof.** In this section, for $Y, \mathcal{U}, \kappa, X$, and $P$ given in §1, we will prove Theorem 1.2. From Theorem 2.6, we obtain a constant $K = K(Y, \mathcal{U})$.

Let $\{(V_{j,0}, (z_j, w_{j,0}))\}$ be an initial good system of local functions of width $\sigma_0 > 0$ over $Y, \mathcal{U}, \kappa, X$, and $P$. Assume that $\{Y, \mathcal{U}, \kappa, X, P, \{(V_{j,0}, (z_j, w_{j,0}))\}, \sigma_0\}$ satisfies the assumption $(i), (ii), (iii)$ in Theorem 1.2. In §3.3, we will explain how to retake of coordinates so that the $L^\infty$-norm of the non-linear part of the transversal transition function becomes smaller. By making width $\sigma_0$ slightly smaller, together with the assumption $(iii)$ in Theorem 1.2, one can obtain a function which renew a coordinate $w_{j,0} \mapsto w_{j,1}$ on a neighborhood of $M_j$ not by changing a unitary flat line bundle $N = [\{(U_{jk}, e^{\alpha_{jk}})\}]$ over $\tilde{Y}$. By an inductive procedure, we obtain $\{(V_{j,m}, (z_j, w_{j,m}))\}$ as a good system of local functions of width $\sigma_m$ retaken $m$-times from the initial system $\{(V_{j,0}, (z_j, w_{j,0}))\}$. Since $N$ is not changed by the procedure, a transversal transition $f_{kj,m} \in \text{Diff}_+^\omega(S^1)$ on $V_{kj,m} = V_{j,m} \cap V_{k,m}$ has the same non-zero order part of $f_{k,j,0}$ as $\alpha_{kj}$. Therefore, by Lemma 2.1, it is checked that $f_{kj,m}$ has the Laurent expansion

$$\log \frac{f_{kj,m}(w_{j,m})}{w_{j,m}} = \alpha_{kj} + \sum_{n \neq 0} b_{kj,m} w_{j,m}^n \pmod{2\pi\sqrt{-1} \mathbb{Z}}.$$
We will denote the sum \( \sum_{n \neq 0} b_{kj|n,m} w_{j,m}^n \) by \( \hat{f}_{kj,m} \). We define the norm by
\[
||\hat{f}_{kj,m}||_{\sigma_m} := \sup_{e^{-\sigma_m} < |w_{j,m}| < e^{\sigma_m}} |\hat{f}_{kj,m}(w_{j,m})|.
\]

Our goal is to prove that \( ||\hat{f}_{kj,m}||_{\sigma_m} \) converges to zero. For proving the main theorem, it is sufficient to show the following statement.

**Theorem 3.1.** Assume that \( \{Y, \mathcal{U}, \kappa, X, P, \{\langle V_{j,0}, (z_j, w_{j,0}) \rangle\}, \sigma_0 \} \) satisfies the assumption in Theorem 1.2. Define \( \delta_m, \sigma_m \) and \( \eta_m \) inductively by
\[
\delta_0 = \min \left\{ \eta_0, \frac{\eta_0^{\mu+1}}{(1 + e^{\sigma_0})C_1\mu} \right\},
\]
\[
\eta_{m+1} = \mu^{-\mu+1} \eta_m,
\]
\[
\delta_{m+1} = (1 + e^{\sigma_0})C_1 \frac{\delta_m^2}{\eta_{m+1}},
\]
and
\[
\sigma_{m+1} = \sigma_m - 4\eta_m,
\]
where \( C_0 > 0 \) and \( \mu > 1 \) are constants such that \( N \) is \((C_0, \mu, K)\)-Diophantine, \( \sigma_0 > 0 \) is the initial width, \( \eta_0 \) satisfies \( 0 < \eta_0 < \min \{ \pi, (1 - \mu^{-\mu+1})^{\frac{\mu}{\mu+1}} \} \) and \( C_1 \) is a constant which depends only on \( C_0, \mu \) and \( \sigma_0 \) (see Lemma 3.6).

Then, the following holds for any \( m \):
\[
\max_{j,k} ||\hat{f}_{kj,m}||_{\sigma_m} < \delta_m \leq \min \left\{ \eta_m, \frac{\eta_m^{\mu+1}}{(1 + e^{\sigma_0})C_1\mu} \right\}.
\]

Recall that \( K = K(Y, \mathcal{U}) \) is invariant under the inductive procedure. Note that \( \delta_m \to 0 \), since \( \eta_m \to 0 \) and \( \delta_m < \eta_m \). Vanishing of the limit of \( \delta_m \) allows to deduce that \( ||\hat{f}_{kj,m}||_{\sigma_m} \to 0 \) as \( m \to \infty \). From the definition of \( \{\sigma_m\} \), one can check directly that the limit of \( \sigma_m \) is a positive constant.

In what follows, \( \{(V_{j,m}, (z_j, w_{j,m}))\} \) is a good system of local functions retaken \( m \)-times and satisfies the inductive assertion as above. In §3.2, we will give some estimates of transition functions \( f_{kj,m} \). In §3.3, we will explain a function of retaking coordinates \( w_{j,m} \to w_{j,m+1} \). In §3.4, we will define renewed transition functions \( f_{kj,m+1} \) from \( f_{kj,m} \) and give an estimate of \( f_{kj,m+1} \).

### 3.2. Review of transition function

The transition on \( V_{jk,m} := V_{j,m} \cap V_{k,m} \) is given by
\[
\begin{cases}
z_k = z_k(z_j) \\
w_{k,m} = f_{kj,m}(w_{j,m})
\end{cases}
\]

We denote an expansion of \( f_{kj,m} \in \text{Diff}^\omega_+(S^1) \) by
\[
\log \frac{w_{k,m}}{w_{j,m}} = \log \frac{f_{kj,m}(w_{j,m})}{w_{j,m}} \equiv \alpha_{kj} + \sum_{n \neq 0} b_{kj|n,m} w_{j,m}^n \quad (\mod 2\pi \sqrt{-1} \mathbb{Z}).
\]

Firstly, we check that the transition function \( f_{kj,m} \) does not depend on \( z_j \). In the following lemma, we denote an expansion of \( f_{kj,m} \in \text{Diff}^\omega_+(S^1) \) by
\[
\log \frac{f_{kj,m}(w_{j,m}, z_j)}{w_{j,m}} \equiv \alpha_{kj} + \sum_{n \neq 0} b_{kj|n,m}(z_j) w_{j,m}^n \quad (\mod 2\pi \sqrt{-1} \mathbb{Z}).
\]
**Lemma 3.2.** For any \( n \neq 0 \), \( b_{kj,n,m}(z_j) = -b_{kj,-n,m}(z_j) \) holds. Especially, the transition function \( f_{kj,m} \) does not depend on \( z_j \).

**Proof.** By considering a contour integral over the unit circle, one has
\[
\int_0^1 \log \left( \frac{f_{kj,m}(e^{2\pi \sqrt{-1}\theta}, z_j)}{e^{2\pi \sqrt{-1}\theta}} \right) e^{-2\pi n\sqrt{-1}\theta} d\theta.
\]
Taking conjugation of \( b_{kj,n,m} \),
\[
\overline{b_{kj,n,m}(z_j)} = \int_0^1 \log \left( \frac{\overline{f_{kj,m}(e^{2\pi \sqrt{-1}\theta}, z_j)}}{e^{2\pi \sqrt{-1}\theta}} \right) e^{2\pi n\sqrt{-1}\theta} d\theta.
\]
Since \( f_{kj,m} \in \text{Diff}_+^\omega(S^1) \), \( b_{kj,n,m}(z_j) = -b_{kj,-n,m}(z_j) \) holds. Therefore, \( b_{kj,n,m} \) is holomorphic and antiholomorphic function of \( z_j \). This implies that \( b_{kj,m} \) is a constant with respect to \( z_j \).

**Lemma 3.3.** For any \( n \neq 0 \), the following holds:
\[
|b_{kj,n,m}| \leq ||f_{kj}||_{\sigma_m} e^{-|n|\sigma_m}.
\]

**Proof.** From the definition,
\[
|b_{kj,n,m}| \leq \frac{1}{2\pi} \int_\gamma \left| \log \frac{f_{kj,m}(w_{j,m})}{w_{j,m}} - \alpha_{kj} \right| \frac{d|w_{j,m}|}{|w_{j,m}|^{n+1}},
\]
where \( \gamma \) is a generating loop. Considering the loop \( \gamma = \{|w_{j,m}| = e^{\pm\sigma_m}\} \), we obtain the inequality.

### 3.3. A function of retaking coordinates.
In this section, we will define the renewed coordinate \( \{(z_j, w_{j,m+1})\} \) from the coordinate \( \{(z_j, w_{j,m})\} \) by suitably constructing the function \( \psi_{j,m} \):
\[
w_{j,m} = \psi_{j,m}(w_{j,m+1}).
\]
The function \( \psi_{j,m} \) will be constructed by using the function
\[
\hat{\psi}_{j,m}(w') = \sum_{n \neq 0} a_{j,n,m} w'^n,
\]
where \( a_{j,n,m} \) are suitably chosen constants to satisfy
\[
\hat{\psi}_{j,m}(w') = \log \frac{\psi_{j,m}(w')}{w'}.
\]
Let us explain how to construct of \( a_{j,n,m} \). We obtain \( \psi_{j,m} \) from the simplified Schröder’s equation
\[
(1) \quad \hat{f}_{kj,m}(w_{j,m+1}) + \hat{\psi}_{j,m}(w_{j,m+1}) - \hat{\psi}_{k,m}(e^{\alpha_{kj}} w_{j,m+1}) = 0
\]

**Observation 3.4.** Let us explain the simplified Schröder’s equation (1). For simplicity assume that transitions on \( V_{j,k,m+1} \) is linear: \( w_{k,m+1} = e^{\alpha_{kj}} w_{j,m+1} \). Then, functions of retaking coordinates satisfy
\[
\alpha_{kj} = \log \frac{w_{k,m+1}}{w_{j,m+1}} = \log \left( \frac{w_{k,m+1}}{w_{k,m}} \cdot \frac{w_{k,m}}{w_{j,m}} \cdot \frac{w_{j,m}}{w_{j,m+1}} \right) = -\hat{\psi}_{k,m}(w_{k,m+1}) + \log \frac{w_{k,m}}{w_{j,m}} + \hat{\psi}_{j,m}(w_{j,m+1}).
\]
Thus, we obtain $\tilde{f}_{kj,m}(w_{j,m}) + \tilde{\psi}_{j,m}(w_{j,m+1}) - \tilde{\psi}_{k,m}(w_{k,m+1}) = 0$ as Schröder’s equation. All we have to do is to find the solution of this, which is not easy. Instead of solving Schröder’s equation, we consider (1) as a simplified Schröder’s equation (replace $w_{j,m}$ with $w_{j,m} + 1$ and $w_{k,m+1}$ with $e^{\alpha_{kj}}w_{j,m+1}$). This idea came from the KAM theoretical proof of Theorem 2.4.

By using power series, the simplified Schröder’s equation (1) turns out that $a_{j|n,m}$ should satisfy

$$b_{kj,n,m} + a_{j|n,m} - (e^{\alpha_{kj}})^na_{k|n,m} = 0.$$  

It is easily checked that this condition is equivalent to the existence of $\{a_{j|n,m}\}$ which satisfies $\delta^0_{n,m}(\{(U_j, a_{j|n,m})\}) = \{(U_{jk}, b_{kj|n,m})\}$, where

$$\delta^0_{n,m} : \tilde{C}^0(\mathcal{U}, \mathcal{O}_Y(N^n)) \to \tilde{C}^1(\mathcal{U}, \mathcal{O}_Y(N^n))$$

is the coboundary map. From the assumption (iii) in Theorem 1.2, one can find $\{a_{j|n,m}\}$ which satisfies the condition above. In this situation, since the compactness of $Y$ and the unitary-flatness of $N$, we can apply Theorem 2.6 to $\{a_{j|n,m}\}$ and $\{b_{kj|n,m}\}$ to conclude that

$$d(1, N^{\otimes n}) \max_j |a_{j|n,m}| \leq K \max_{j,k} |b_{kj|n,m}|.$$  

Recall that $K$ does not depend on $n$ and $m$. From an invariant property $d(1, N^{\otimes n}) = d(1, N^{\otimes (-n)})$,

$$C_0|n|^\mu d(1, N^{\otimes n}) \geq K$$

holds for any $n \neq 0$ if $N$ satisfies $(C_0, \mu, K)$-Diophantine condition. By combining these estimates, we obtain the following:

**Lemma 3.5.** If $N$ satisfies $(C_0, \mu, K)$-Diophantine condition,

$$\max_j |a_{j|n,m}| \leq C_0|n|^\mu \max_{j,k} |b_{kj|n,m}|$$

holds for any $n \neq 0$.

In this manner, we obtain a function of retaking coordinates

$$\tilde{\psi}_{j,m}(w_{j,m+1}) = \sum_{n \neq 0} a_{j|n,m}w^n_{j,m+1}$$

(the convergence of $\tilde{\psi}_{j,m}$ will be proven later in this section). For $\sigma' > 0$, we define the norm $\| \cdot \|_{\sigma'}$ by

$$\|\tilde{\psi}_{j,m}\|_{\sigma'} := \sup_{e^{-\sigma'} < |w_{j,m+1}| < e^{\sigma'}} |\tilde{\psi}_{j,m}(w_{j,m+1})|$$

Next, we estimate the norm of the function of retaking coordinates.

**Lemma 3.6.** There exists a constant $C_1$ which depends only on $C_0, \mu$, and $\sigma_0$ such that

$$\|\tilde{\psi}_{j,m}\|_{\sigma_m - \lambda} \leq C_1 \max_{j,k} \|\tilde{f}_{kj,m}\|_{\sigma_m} \cdot \lambda^{-\mu}.$$  

holds for any $\lambda \in (0, \sigma_m)$. 

PROOF. From Lemma 3.3 and Lemma 3.5, we have

\[ ||\psi_{j,m}||_{\sigma_m - \lambda} = \sup_{e^{-\sigma_m - \lambda} < |w_{j,m}| < e^\sigma_m - \lambda} \left| \sum_{n \neq 0} a_{j|m,n} w_{j,m+1}^n \right| \]

\[ \leq \sum_{n \neq 0} |a_{j|m,n}| e^{(\sigma_m - \lambda)|n|} \]

\[ \leq \sum_{n \neq 0} C_0 |n|^{\mu - 1} \cdot \max_{j,k} |b_{k|m,n}| \cdot e^{(\sigma_m - \lambda)|n|} \]

\[ \leq 2C_0 \cdot \max_{j,k} ||\hat{f}_{kj,m}||_{\sigma_m} \cdot \sum_{n \geq 1} n^{\mu - 1} e^{-n\lambda}. \]

The sum in the right hand side can be calculated as follows:

\[ \sum_{n \geq 1} n^{\mu - 1} e^{-n\lambda} \leq e^{-\lambda} \sum_{n \geq 1} n(n + 1) \cdots (n + \mu - 2)(e^{-\lambda})^{n-1} = \frac{e^{-\lambda} \cdot (\mu - 1)!}{(1 - e^{-\lambda})^\mu} \leq \frac{(\mu - 1)!}{(1 - e^{-\lambda})^\mu}. \]

From the convexity of the function \( x \mapsto 1 - e^{-x} \), for any \( \lambda \) which satisfies \( 0 < \lambda < \sigma_m < \sigma_0 \), the following holds:

\[ \frac{1 - e^{-\lambda}}{\lambda} > \frac{1 - e^{-\sigma_0}}{\sigma_0}. \]

Hence, we obtain

\[ ||\psi_{j,m}||_{\sigma_m - \lambda} \leq 2C_0 \cdot \max_{j,k} ||\hat{f}_{kj,m}||_{\sigma_m} \cdot \frac{\sigma_0 \mu^{\mu - 1}!}{(1 - e^{-\sigma_0})^\mu} \cdot \lambda^{-\mu}. \]

Letting \( C_1 = \frac{2C_0 \sigma_0^{\mu - 1}!}{(1 - e^{-\sigma_0})^\mu} \), the statement is proven. \( \square \)

We shall check the well-definedness of retaking coordinates. Let \( V_{j,m}(\sigma') := \{ e^{-\sigma'} < |w_{j,m}| < e^{\sigma'} \} \) and \( V_{j,m+1}(\sigma') := \{ e^{-\sigma'} < |w_{j,m+1}| < e^{\sigma'} \} \) for a positive real number \( \sigma' \). For coordinates \( w_{j,m} \) and \( w_{j,m+1} \), note that the renewed transversal transition function \( f_{kj,m+1} \) is defined by

\[ w_{j,m+1} \mapsto f_{kj,m+1}(w_{j,m+1}) = (\psi_{k,m}^{-1} \circ f_{kj,m} \circ \psi_{j,m})(w_{j,m+1}). \]

Recall that \( \sigma_m - \nu \eta_m \) is positive for any \( \nu \in \{1, 2, 3, 4\} \) by definitions of \( \{\sigma_m\} \) and \( \{\eta_m\} \) in Theorem 3.1.

**Proposition 3.7.** The function \( f_{kj,m+1} \) is well-defined as a map from \( V_{j,m+1}(\sigma_m - 4\eta_m) \) to \( V_{k,m+1}(\sigma_m - \eta_m) \).

**Proof.** We prove this theorem by checking the following properties.

1. \( \psi_{j,m}(V_{j,m+1}(\sigma_m - 4\eta_m)) \subset V_{j,m}(\sigma_m - 3\eta_m) \).
2. \( f_{kj,m}(V_{j,m}(\sigma_m - 3\eta_m)) \subset V_{k,m}(\sigma_m - 2\eta_m) \).
3. \( \psi_{k,m}^{-1} \) is well-defined on \( V_{k,m}(\sigma_m - 2\eta_m) \).
4. \( \psi_{k,m}(V_{k,m}(\sigma_m - 2\eta_m)) \subset V_{k,m+1}(\sigma_m - \eta_m) \).
PROOF OF (1) AND (2). Note that \( \psi_{j,m} \) is well-defined on \( V_{j,m+1}(\sigma_m - \lambda) \) for \( 0 < \lambda < \sigma_m \) by Lemma 3.6. From Lemma 3.6, for \( \nu = 1, 2, 3, 4 \), we obtain

\[
||\hat{\psi}_{j,m}||_{\sigma_m - \nu \eta_m} \leq C_1 \cdot (\nu \eta_m)^{-\mu} \cdot \left( \max_{j,k} ||\hat{f}_{k,j}||_{\sigma_m} \right)
\]

\[
< C_1 \cdot (\nu \eta_m)^{-\mu} \cdot \frac{\eta_m^{\mu+1}}{(1 + e^{\sigma_0})C_1 \mu} < \eta_m
\]

by the inductive assumption \( \max_{j,k} ||\hat{f}_{k,j}||_{\sigma_m} < \frac{\eta_m^{\mu+1}}{(1 + e^{\sigma_0})C_1 \mu} \). Therefore one has

\[
| \log |\hat{\psi}_{j,m}(w_{j,m+1})| - \log |w_{j,m+1}| | = | \Re (\hat{\psi}_{j,m}(w_{j,m+1})) |
\]

\[
< |\hat{\psi}_{j,m}(w_{j,m+1})|
\]

\[
< \eta_m
\]

on \( V_{j,m+1}(\sigma_m - 4 \eta_m) \), which proves the assertion (1).

The assertion (2) is proven from the following inequality for \( w_{j,m} \in V_{j,m}(\sigma_m - 3 \eta_m) \) and the inductive assumption:

\[
| \log |f_{k,j,m}(w_{j})| - \log |w_{j,m}| | = | \Re (\hat{f}_{k,j,m}(w_{j,m})) |
\]

\[
< |\hat{f}_{k,j,m}(w_{j,m})|
\]

\[
< \max_{j,k} ||\hat{f}_{k,j,m}||_{\sigma_m}
\]

\[
< \delta_m < \eta_m. \quad \square
\]

To prove (3) and (4), we shall use the following lemma.

**Lemma 3.8.**

\[
\sup_{|\Re \zeta| < \sigma_m - \eta_m} \left| \frac{d}{d\zeta} \hat{\psi}_{k,m}(e^{\zeta}) \right| \leq \frac{1}{1 + e^{\sigma_0}}.
\]

**Proof.** By using power series expression of \( \hat{\psi}_{k,m} \) and the same argument as in the proof of Lemma 3.6, one has

\[
\sup_{|\Re \zeta| < \sigma_m - \eta_m} \left| \frac{d}{d\zeta} \hat{\psi}_{k,m}(e^{\zeta}) \right| \leq 2C_0 \cdot \max_{j,k} ||\hat{f}_{k,j,m}||_{\sigma_m} \cdot \sum_{n \geq 1} n^\mu e^{-n \eta_m}
\]

\[
\leq 2C_0 \cdot \max_{j,k} ||\hat{f}_{k,j,m}||_{\sigma_m} \cdot \frac{\mu! e^{-\eta_m}}{(1 - e^{-\eta_m})^{\mu+1}}
\]

\[
= C_1 \mu \cdot \max_{j,k} ||\hat{f}_{k,j,m}||_{\sigma_m} \cdot \left( \frac{1 - e^{-\sigma_0}}{\sigma_0(1 - e^{-\eta_m})} \right)^\mu \cdot \frac{1}{e^{\eta_m} - 1}.
\]

Thus we have

\[
\sup_{|\Re \zeta| < \sigma_m - \eta_m} \left| \frac{d}{d\zeta} \hat{\psi}_{k,m}(e^{\zeta}) \right| \leq C_1 \mu \cdot \max_{j,k} ||\hat{f}_{k,j,m}||_{\sigma_m} \cdot \frac{1}{\eta_m^{\mu} (e^{\eta_m} - 1)}.
\]

Therefore the assertion follows from \( \max_{j,k} ||\hat{f}_{k,j,m}||_{\sigma_m} < \frac{\eta_m^{\mu+1}}{(1 + e^{\sigma_0})C_1 \mu} \) and the inequality \( \frac{\eta_m}{e^{\eta_m} - 1} < 1. \) \quad \square

Furthermore we shall prove the following.
It follows that $\psi_{k,m}(S^1) = S^1$, where $S^1$ is identified with $\{|w_{k,m+1}| = 1\}$.

**Proof.** From lemma 3.2, the relation $b_{kj}[n,m] = a_{kj}[j,n,m]$ holds. This relation leads
\[
\delta_{n,m}^0 \left( \{(U_j, a_{j[n,m]}(z))\} \right) = \delta_{n,m}^0 \left( \{(U_j, -a_{j[n,m]}(z))\} \right) = \{(U_j, b_{kj}[n,m])\}.
\]
One also has $H^0(U, \mathcal{O}_Y(N^\otimes n)) = 0$ since $N^\otimes n \neq \mathbb{1}$ holds for any $n \neq 0$ and $Y$ is compact. Hence it follows that $a_{j[n,m]} = -a_{j[n,m]}$ holds from $\{(U_j, a_{j[n,m]} - (-a_{j[n,m]})\} \in H^0(U, \mathcal{O}_Y(N^\otimes n))$. One can easily check
\[
\frac{d}{dw^k_{m+1}} \psi_{k,m}(w_{k,m+1}) = \left(1 + w_{k,m+1} \frac{d}{dw^k_{m+1}} \hat{\psi}_{k,m}(w_{k,m+1})\right) e^\hat{\psi}_{k,m}(w_{k,m+1}).
\]
This calculation and Lemma 3.8 lead that $\psi_{k,m}$ has no critical point in $V_{k,m+1}(\sigma_m - 2\eta_m)$. Thus $\psi_{k,m}$ is a local homeomorphism on a neighborhood of the unit circle, from which it follows that $\psi_{k,m}(S^1) = S^1$ holds.

**Lemma 3.10.** The function $\psi_{k,m}$ is one-to-one on the unit circle $S^1 = \{|w_{k,m+1}| = 1\}$.

**Proof.** Let $\Psi_{k,m}$ be a lift of $\psi_{k,m}$ which is described as below:
\[
\Psi_{k,m}(x) = \frac{1}{2\pi \sqrt{-1}} \log \psi_{k,m}(e^{2\pi \sqrt{-1}x}) (x \in \mathbb{R}).
\]
Calculating directly, one has
\[
\Psi_{k,m}(x) = x + \frac{1}{2\pi \sqrt{-1}} \hat{\psi}_{k,m}(e^{2\pi \sqrt{-1}x}).
\]
From Lemma 3.8,
\[
\left. \frac{d\Psi_{k,m}(x)}{dx} \right|_{\zeta = 2\pi \sqrt{-1}x} < 1 + \frac{1}{1 + e^{\sigma_0}} < 2
\]
and
\[
\frac{d\Psi_{k,m}(x)}{dx} > 0
\]
hold. Supposing the degree of $\psi_{k,m}|_{S^1}$ is larger than 2, it contradicts this estimate from the mean value theorem.

To combine Lemma 3.9 and Lemma 3.10, we can see $\psi_{k,m} \in \text{Diff}^{\omega}_{\omega}(S^1)$. By considering Rouché’s theorem, we can check the well-definedness of $\psi^{-1}_{k,m}$ to prove (3) and (4) of Proposition 3.7. We will show the consequence of checking the well-definedness of the map $\psi^{-1}_{k,m}$ on $V_{k,m}(\sigma_m - 2\eta_m)$. In what follows, we use the notation in proof of Lemma 3.10. Let $\Psi_{k,m}$ be a lift of the map $\psi_{k,m}|_{S^1}$. For a given point $w_0 \in V_{k,m}(\sigma_m - 2\eta_m)$, let $\tilde{w}_0$ be a point in $B(\sigma_m - 2\eta_m) := \{|\text{Im } z| < (\sigma_m - 2\eta_m)/2\pi\}$ which satisfies $e^{2\pi \sqrt{-1}w_0} = w_0$. We define a domain $D \subset \mathbb{C}$ by
\[
D = \left\{ z \in \mathbb{C} : |\text{Im } z| < \frac{\sigma_m - \eta_m}{2\pi}, |\text{Re } z - \text{Re } \tilde{w}_0| < \frac{1}{2} \right\}
\]
(see Figure 1). We define holomorphic functions $g_1 , g_2$ on a neighborhood of $D$ by
\[
g_1(z) = z - \tilde{w}_0
\]
\[
g_2(z) = \Psi_{k,m}(z) - z
\]

**Lemma 3.11.** For any $z \in \partial D$, $|g_1(z)| > |g_2(z)|$ holds.
Proof. One has
\[ |g_1(z)| \geq \min \left\{ \frac{1}{2}, \frac{1}{2\pi \eta_m} \right\}. \]
From the definition, \( \{\eta_m\} \) is monotonically decreasing. Thus, one obtains
\[ \frac{1}{2\pi \eta_m} < \frac{1}{2\pi \eta_0} < \frac{1}{2} \]
under the assumption \( \eta_0 < \pi \). Considering the relation
\[ \Psi_{k,m}(z) = z + \frac{1}{2\pi \sqrt{-1}} \hat{\psi}_{k,m}(e^{2\pi \sqrt{-1}z}), \]
one has
\[ |g_2(z)| \leq \frac{1}{2\pi} \sup_{z \in \partial D} |\hat{\psi}_{k,m}(e^{2\pi \sqrt{-1}z})| < \frac{1}{2\pi \eta_m}. \]
Therefore, \( |g_1(z)| \geq \frac{1}{2\pi \eta_m} > |g_2(z)| \) holds. \( \square \)

Figure 1. The domain \( D \) is independent of the imaginary part of \( \tilde{w}_0 \).

Proof of (3) and (4). From Lemma 3.11, applying Rouché’s theorem, one finds that \( \Psi_{k,m}(z) = \tilde{w}_0 \) and \( z = \tilde{w}_0 \) have the same number of zeros in \( D \). Thus the relation
\[ \#\{z \in D \mid \Psi_{k,m}(z) = z\} = \#\{z \in D \mid z = \tilde{w}_0\} = 1 \]
holds. Consequently one can define \( \psi_{k,m}^{-1} \) on \( V_{k,m}(\sigma_m - 2\eta_m) \).

3.4. The estimate of renewed transition functions and proof of Theorem 3.1. The renewed transition function \( f_{k,j,m+1} \) satisfies
\[ \hat{f}_{k,j,m+1}(w_{j,m+1}) = \hat{f}_{k,j,m}(w_{j,m}) + \hat{\psi}_{j,m}(w_{j,m+1}) - \hat{\psi}_{k,m}(w_{k,m+1}). \]
Combining (1) and this equation, we obtain
\[ |\hat{f}_{k,j,m+1}(w_{j,m+1})| \leq |\hat{f}_{k,j,m}(w_{j,m}) - \hat{f}_{k,j,m}(w_{j,m+1})| \]
\[ + |\hat{\psi}_{k,m}(e^{\alpha_k w_{j,m+1}}) - \hat{\psi}_{k,m}(w_{k,m+1})|. \]

Claim 3.12. If the inductive assumption \( \max_{j,k} ||\hat{f}_{k,j,m}||_{\sigma_m} < \delta_m \) holds, then the renewed function \( f_{k,j,m+1} \) satisfies \( \max_{j,k} ||\hat{f}_{k,j,m+1}||_{\sigma_{m+1}} < \delta_{m+1} \).
Hence one has
\[ |\hat{f}_{kj,m}(w_{j,m}) - \hat{f}_{kj,m}(w_{j,m+1})| = \left| \int_{\log w_{j,m+1}}^{\log w_{j,m}} \frac{d}{d\zeta} \hat{f}_{kj,m}(e^\zeta) d\zeta \right| \]
\[ \leq \log \frac{w_{j,m}}{w_{j,m+1}} \cdot \sup_{|\Re \zeta| < \sigma_m - \eta_m} \left| \frac{d}{d\zeta} \hat{f}_{kj,m}(e^\zeta) \right| \]
\[ \leq \|\hat{\psi}_{j,m}\|_{\sigma_m - 4\eta_m} \cdot \sup_{|\Re \zeta| < \sigma_m - \eta_m} \left| \frac{d}{d\zeta} \hat{f}_{kj,m}(e^\zeta) \right|. \]

From Lemma 3.6, we can easily show \( \|\hat{\psi}_{j,m}\|_{\sigma_m - 4\eta_m} < \frac{C_1 \delta_m}{\eta_m} \). By using Cauchy’s integral expression, the following holds:
\[ \sup_{|\Re \zeta| < \sigma_m - \eta_m} \left| \frac{d}{d\zeta} \hat{f}_{kj,m}(e^\zeta) \right| \leq \sup_{|\Re \zeta| < \sigma_m - \eta_m} \left| e^\zeta \right| \cdot \frac{1}{2\pi \sqrt{-1}} \int_{|\zeta - e^\zeta| = \eta_m} \hat{f}_{kj,m}(\xi) \frac{d\xi}{(\zeta - e^\zeta)^2} \]
\[ \leq e^{\sigma_m \delta_m} \int_0^{2\pi} \frac{1}{\eta_m^2} \eta_m d\theta \]
\[ = e^{\sigma_m \delta_m} \frac{\eta_m^2}{\eta_m} < e^{\sigma_0 \delta_m}. \]

Hence one has
\[ |\hat{f}_{kj,m}(w_{j,m}) - \hat{f}_{kj,m}(w_{j,m+1})| < \frac{C_1 \delta_m}{\eta_m} \cdot e^{\sigma_0 \delta_m} = \frac{e^{\sigma_0} C_1 \delta_m^2}{\eta_m^{\mu+1}}. \]

Next, for the second term of (2), we have the estimate as below by using Lemma 3.8:
\[ |\hat{\psi}_{k,m}(e^{2\pi \sqrt{-1}\alpha_{kj} w_{j,m+1}}) - \hat{\psi}_{k,m}(w_{k,m+1})| = \left| \int_{\alpha_{kj} + \log w_{j,m+1}}^{\log w_{k,m+1}} \frac{d}{d\zeta} \hat{\psi}_{k,m}(e^\zeta) d\zeta \right| \]
\[ \leq \log w_{k,m+1} - (\alpha_{kj} + \log w_{j,m+1}) \cdot \sup_{|\Re \zeta| < \sigma_m - \eta_m} \left| \frac{d}{d\zeta} \hat{\psi}_{k,m}(e^\zeta) \right| \]
\[ < \|\hat{f}_{kj,m+1}\|_{\sigma_m - 4\eta_m} \cdot \frac{1}{1 + e^{\sigma_0}}. \]

Therefore, it follows that
\[ \|\hat{f}_{kj,m+1}\|_{\sigma_m - 4\eta_m} \leq \frac{e^{\sigma_0} C_1 \delta_m^2}{\eta_m^{\mu+1}} + \frac{\|\hat{f}_{kj,m+1}\|_{\sigma_m - 4\eta_m}}{1 + e^{\sigma_0}}. \]

Solving for \( \|\hat{f}_{kj,m+1}\|_{\sigma_m - 4\eta_m} \) gives
\[ \max_{j,k} \|\hat{f}_{kj,m+1}\|_{\sigma_{m+1}} < (1 + e^{\sigma_0}) C_1 \frac{\delta_m^2}{\eta_m^{\mu+1}} = \delta_{m+1}. \]

\[ \square \]
Finally we need to prove $\delta_{m+1} < \eta_{m+1}$ and $\delta_{m+1} < \frac{\eta_{m+1}^{\mu+1}}{(1 + e^{\sigma_0})C_1 \mu}$ under the inductive assumption. The latter inequality can be proven as below:

\[
\delta_{m+1} = (1 + e^{\sigma_0})C_1 \frac{1}{\eta_{m+1}} \delta_m^2 < (1 + e^{\sigma_0})C_1 \frac{1}{\eta_{m+1}^{\mu+1}} \left( \frac{\eta_{m+1}^{\mu+1}}{(1 + e^{\sigma_0})C_1 \mu} \right)^2
\]

\[
= \frac{1}{(1 + e^{\sigma_0})C_1 \mu} \cdot \frac{1}{\eta_{m+1}^{\mu+1}}
\]

\[
= \frac{\eta_{m+1}^{\mu+1}}{(1 + e^{\sigma_0})C_1 \mu}.
\]

The former inequality is shown as

\[
\delta_{m+1} < (1 + e^{\sigma_0})C_1 \frac{1}{\eta_{m+1}^{\mu+1}} \cdot \frac{\eta_{m+1}^{\mu+1}}{(1 + e^{\sigma_0})C_1 \mu} \cdot \delta_m
\]

\[
= \mu^{-1} \delta_m
\]

\[
< \mu^{-\mu+1} \eta_m
\]

\[
= \eta_{m+1}.
\]

From the assumption of Theorem 1.2, the initial transition function $f_{kj,0}$ satisfies

\[
\max_{j,k} ||\widehat{f}_{kj,0}||_{\sigma_0} = \max_{j,k} \sup_{e^{-\sigma_0} < |p| < e^{\sigma_0}} \left| \sum_{n \neq 0} b_{kj|n,0} p^n \right| < \min \left\{ \eta_0, \frac{\eta_0^{\mu+1}}{(1 + e^{\sigma_0})C_1 \mu} \right\} = \delta_0.
\]

Hence, we can obtain the inequality $\max_{j,k} ||\widehat{f}_{kj,m}||_{\sigma_m} < \delta_m$ inductively.

It is easily checked that $\delta_m$ converges to zero by $\delta_m < \eta_m$ and the limit of $\sigma_m$ is non-zero as follows.

Since $\mu^{-\mu+1} < 1$ for any $\mu > 1$, $\lim_{m \to \infty} \eta_m = 0$ holds from the definition of $\{\eta_m\}_{m=0}^{\infty}$. Therefore we can see $\lim_{m \to \infty} \delta_m = 0$ from the inequality $\delta_m < \eta_m$. The limit of $\sigma_m$ is computed as

\[
\lim_{m \to \infty} \sigma_m = \sigma_0 - 4(\eta_0 + \eta_1 + \cdots) = \sigma_0 - \frac{4\eta_0}{1 - \mu^{-\mu+1}}.
\]

From the condition which $\eta_0$ satisfies in Theorem 3.1, this limit is a non-zero constant. Therefore the main theorem follows from Theorem 3.1

\[
\square
\]

4. Example

In [FK, Cor 1. ], B. Fayad and K. Khanin showed that the family of commuting circle diffeomorphisms is simultaneous linearizable when rotation numbers of them satisfy the simultaneously Diophantine condition. In this section, we give a simple example and see that we get a sufficient condition for simultaneous linearization of the pair of circle diffeomorphisms not necessarily commutative as a consequence of the main theorem. Let $f_1$ and $f_2$ be elements of Diff$^\omega_+(S^1)$. For the simultaneous linearization of $f_1$ and $f_2$, we construct a Levi-flat hypersurface which has a structure of $S^1$-bundle as below. Let $Y$ be a compact Riemann surface with genus 2. We give a finite covering $U = \{U_j\}_{j=0,1,2}$ of $Y$ as below (see Figure 2). The intersections are denoted by $\{U^+_0, U^-_0\}_{j=1,2}$ as Figure 3.
Define a fundamental group action $\kappa: \pi_1(Y, \ast) \to \text{Diff}^+(S^1)$ by letting $\kappa(\alpha_1) = f_1$, $\kappa(\alpha_2) = f_2$, and $\kappa(\beta_1) = \kappa(\beta_2) = \text{id}_{S^1}$ for generating loops $\alpha_1, \alpha_2, \beta_1$, and $\beta_2$ (see Figure 4).

By considering extending a domain of $f_1$ and $f_2$ along the unit circle, one obtains a non-singular complex surface $X$ which has a Levi-flat hypersurface $M$ constructed by suspension of $\kappa$. Let $\pi: M \to Y$ be the projection and $P$ be a holomorphic submersion as in §1. For the good system of local functions $\{(V_j, (z_j, w_j))\}$ of $\sigma > 0$ over $\{Y, U, \kappa, X, P\}$, we define the transversal transition on each $V^+_0, V^-_0, V^+_{01}, V^-_{01}, V^+_{02}, \text{and } V^-_{02}$, where $V^0_j = V_0 \cap V_j = P^{-1}(U_0) \cap P^{-1}(U_j) (j = 1, 2)$. Denote by $w_j = f^+_j(w_0) = f^+_j(w_0)$ the transition on $V^+_0$ and by $w_j = f^-_j(w_0) = w_0$ the transition $V^-_0$ for each $j = 1, 2$. Then, the Laurent expansions for transition functions are written as below:

\[
\log \frac{f_j^+(w_0)}{w_0} = \log \frac{f_j(w_0)}{w_0} \equiv \alpha_{j0} + \sum_{n \neq 0} b^+_j w_0^n,
\]

\[
\log \frac{f_j^-(w_0)}{w_0} = \log \frac{\text{id}(w_0)}{w_0} \equiv \alpha_{j0} \equiv 0.
\]

In this situation, since $U_0 \cap U_1 \cap U_2 = \emptyset$, any Čech 1-cochain over $Y$ satisfies 1-cocycle condition. Therefore one obtains

\[
N = [\alpha(\{(V_j, (z_j, w_j))\})] = [(U^+_0, e^{\alpha^+_0}), (U^-_0, 1)]_{j=1,2} \in \hat{H}^1(U, U(1))
\]

and

\[
[b_n(\{(V_j, (z_j, w_j))\})] = [(U^+_0, b^+_j n), (U^-_0, 0)]_{j=1,2}.
\]

**Proposition 4.1.** Let $f_1, f_2, Y, U, \kappa, X, \text{and } P$ be as above and $\{(V_j, (z_j, w_j))\}_{j=0,1,2}$ be the good system of local functions of width $\sigma$. If the following conditions (i), (ii)...

---

**Figure 2.**

**Figure 3.**

**Figure 4.**
and (iii) hold, then $f_1$ and $f_2$ are simultaneous linearizable.

(i) The unitary flat line bundle $N$ over $Y$ satisfies $(C_0, \mu, K)$-Diophantine condition with the constant $K = K(Y, U)$ obtained by Theorem 2.6, where $C_0 > 0$ and $\mu > 1$ is a constant determined only by $Y$ and $U$.

(ii) For non-zero order coefficients $b^+_{j0n}$ associated to the transversal transition function $f_j$ of $\{(V^\varepsilon_j(z_j, w_j))\}_{\varepsilon = +, -, j = 1, 2}$, there exists a constant $\eta_0 \in (0, \min\{\pi, (1 - \mu^{-1})2\pi\})$ such that

$$\max_{j=1,2} \sup_{e^{-\sigma} < \eta < e^\sigma} \left| \sum_{n \neq 0} b^+_{j0n} \eta^n \right| < \min \left\{ \eta_0, \frac{\eta_0^{\mu+1}}{(1 + e^\sigma)C_1 \mu} \right\}$$

holds, where the constant $C_1$ is obtained by Lemma 3.6.

(iii) The 1-cohomology group $[b_n(\{(V^\varepsilon_j(z_j, w_j))\})]$ is cohomologous to 0 for any good system of local functions $\{(V^\varepsilon_j(z_j, w_j))\}$ which satisfies $[\alpha(\{(V^\varepsilon_j(z_j, w_j))\})] = N$.

**Proof.** From Theorem 1.2, the system $\{Y, U, \kappa, X, P, \{(V_j, (z_j, w_j))\}, \sigma\}$ is linearizable. Then, there exist the functions of retaking coordinates $\psi_j(u_j) = w_j$ ($j = 0, 1, 2$) and the good system of local function $\{(V^\varepsilon_j, (z_j, u_j))\}_{j=0,1,2}$ whose transition is the linear map $u_j = e^{\alpha_j^\varepsilon}u_0$ for each $\varepsilon = +, -$ and $j = 1, 2$. One obtains the following relations:

$$(\psi^{-1}_{j0} \circ f_{j0}^\varepsilon \circ \psi_0)(u_0) = e^{\alpha_j^\varepsilon}u_0.$$ 

Therefore we obtain the following for each $j = 1, 2$:

$$(\psi^{-1}_{j0} \circ f_j \circ \psi_0)(u_0) = e^{\alpha_j^0}u_0$$

$$(\psi^{-1}_{j0} \circ \psi_0)(u_0) = u_0.$$ 

Hence, one can check $(\psi^{-1}_{j0} \circ f_j \circ \psi_0)(u_0) = e^{\alpha_j^+}u_0$ for each $j = 1, 2$. That is simultaneous linearization of $f_1$ and $f_2$. Since $f_j$ is conjugated to the rotation $w \mapsto e^{\alpha_j^+}w$, $\alpha_j^+ \equiv 2\pi \sqrt{-1} \rho(f_j)$ (mod $2\pi \sqrt{-1}\mathbb{Z}$) holds, where $\rho(f_j)$ is the rotation number of $f_j$. \qed

5. Discussion

In this paper, we considered a function $f \in \text{Diff}_+^\omega(S^1)$ which has the Laurent expansion

$$\log \frac{f(w)}{w} = \alpha + \sum_{n \neq 0} b_n w^n.$$ 

Comparing the condition of main theorem with the assumption in Arnol’d’s linearization theorem, we should check a relation of the constant term of Laurent expansion as above and the rotation number of $f$. As is seen the example in §4, sometimes $\alpha$ turns out to be equal to $2\pi \sqrt{-1} \rho(f)$ modulo $2\pi \sqrt{-1}\mathbb{Z}$.

**Question 5.1.** Assume that $\alpha$ is a constant term Laurent expansion of $\log \frac{f(w)}{w}$. Then, will $e^{\alpha} = e^{2\pi \sqrt{-1} \rho(f)}$ always hold?

**Acknowledgment.** The author would like to thank Takayuki Koike for fruitful comments. This work was partly supported by Osaka Central Advanced Mathematical Institute: MEXT Joint Usage/Research Center on Mathematics and Theoretical Physics JPMXP0619217849.
REFERENCES

[A] V. I. Arnold, Small denominators. I: On the mappings of the circumference onto itself, Isv. Akad. Nauk, Math series, 25, 1, (1961), p. 21–96. Transl. A. M. S, 2nd series, 46, p. 213–284.

[CG] L. Carleson and T. W. Gamelin, Complex Dynamics, Universitext: Tracts in Mathematics Springer–Verlag (1993).

[FK] B. Fayad and K. Khanin, Smooth linearization of commuting circle diffeomorphisms, Ann. Math. 170 961–980 (2009).

[H] M. R-Herman, Sur la conjugaison différentiable des difféomorphismes du cercle à des rotations, Inst. Hautes Études Sci. Publ. Math. 49 50–233. (1979).

[KH] A. Katok and B. Hasselblatt, Introduction to the Modern Theory of Dynamical Systems, Cambridge University Press Encyclopedia Math. Appl., 54, Cambridge Univ. Press, (1995).

[KU] T. Koike and T. Uehara, A gluing construction of K3 surfaces, arXiv:1903.01444.

[S] C. L. Siegel, Iteration of analytic functions, Ann. Math. 43, 607–612 (1942).

[SM] C.L. Siegel and J.K. Moser, Lectures on Celestial Mechanics, Springer-Verlag (1971).

[U] T. Ueda, On the neighborhood of a compact complex curve with topologically trivial normal bundle, Math. Kyoto Univ., 22 (1983), 583–607.

[Y] Jean-Christophe Yoccoz, Analytic linearization of circle diffeomorphisms Dynamical systems and small divisors (Cetraro, 1998), 125-173, Lecture Notes in Math., 1784, Fond.CIME/CIME Found. Subser., Springer, Berlin, (2002).

DEPARTMENT OF MATHEMATICS, GRADUATE SCHOOL OF SCIENCE, OSAKA METROPOLITAN UNIVERSITY
3-3-138, SUGIMOTO, SUMIYOSHI-KU OSAKA, 558-8585
JAPAN
Email address: sn22894n@st.omu.ac.jp