Isospectral Dirac operators

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Abstract. We give the description of self-adjoint regular Dirac operators, on \([0, \pi]\), with the same spectra.

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1 Introduction and statement of result

Let \(p\) and \(q\) are real-valued, summable on \([0, \pi]\) functions, i.e. \(p, q \in L_1^1[0, \pi]\). By \(L(p, q, \alpha) = L(\Omega, \alpha)\) we denote the boundary-value problem for canonical Dirac system (see [5,6,9,13,14]):

\[
\ell y \equiv \left\{ B \frac{d}{dx} + \Omega(x) \right\} y = \lambda y, \quad x \in (0, \pi), \quad y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \quad \lambda \in \mathbb{C},
\]

(1.1)

\[
y_1(0) \cos \alpha + y_2(0) \sin \alpha = 0, \quad \alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right),
\]

(1.2)

\[
y_1(\pi) = 0,
\]

(1.3)

where

\[
B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \Omega(x) = \begin{pmatrix} p(x) & q(x) \\ q(x) & -p(x) \end{pmatrix}.
\]

By the same \(L(p, q, \alpha)\) we also denote a self-adjoint operator generated by differential expression \(\ell\) in Hilbert space of two component vector-function \(L^2([0, \pi]; \mathbb{C}^2)\) on the domain

\[
D = \left\{ y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}; \ y_k \in AC[0, \pi], \ (\ell y)_k \in L^2[0, \pi], \ k = 1, 2; \right\}
\]

\[
y_1(0) \cos \alpha + y_2(0) \sin \alpha = 0, \quad y_1(\pi) = 0
\]

where \(AC[0, \pi]\) is the set of absolutely continuous functions on \([0, \pi]\) (see, e.g. [13,16]). It is well known (see [1,5,9]) that under these conditions the spectra of the operator \(L(p, q, \alpha)\) is purely discrete and consists of simple, real eigenvalues, which we denote by \(\lambda_n = \lambda_n(p, q, \alpha) = \ldots\)

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Let $\lambda_n(\Omega, \alpha), \ n \in \mathbb{Z}$, to emphasize the dependence of $\lambda_n$ on quantities $p, q$ and $\alpha$. It is also well known (see, e.g. [1,5,9]) that the eigenvalues form a sequence, unbounded below as well as above. So we will enumerate it as $\lambda_k < \lambda_{k+1}, k \in \mathbb{Z}, \lambda_k > 0, \text{when} \ k > 0 \text{and} \lambda_k < 0, \text{when} \ k < 0$, and the nearest to zero eigenvalue we will denote by $\lambda_0$. If there are two nearest to zero eigenvalue, then by $\lambda_0$ we will denote the negative one. With this enumeration it is proved (see [1,5,9]), that the eigenvalues have the asymptotics:

$$\lambda_n(\Omega, \alpha) = n - \frac{\alpha}{\pi} + r_n, \quad r_n = o(1), \quad n \to \pm \infty. \quad (1.4)$$

In what follows, writing $\Omega \in A$ will mean $p, q \in A$. If $\Omega \in L^2_{\mathbb{R}}[0, \pi]$, then we know, (see, e.g. [9]), that instead of $r_n = o(1)$ we have:

$$\sum_{n=-\infty}^{\infty} r_n^2 < \infty. \quad (1.5)$$

Let $\varphi(x, \lambda) = \varphi(x, \lambda, \alpha, \Omega)$ be the solution of the Cauchy problem

$$\ell \varphi = \lambda \varphi, \quad \varphi(0, \lambda) = \begin{pmatrix} \sin \alpha \\ -\cos \alpha \end{pmatrix}. \quad (1.6)$$

Since the differential expression $\ell$ self-adjoint, then the components $\varphi_1(x, \lambda)$ and $\varphi_2(x, \lambda)$ of the vector-function $\varphi(x, \lambda)$ we can choose real-valued for real $\lambda$. By $a_n = a_n(\Omega, \alpha)$ we denote the squares of the $L^2$-norm of the eigenfunctions $\varphi_n(x, \Omega) = \varphi(x, \lambda_n(\Omega, \alpha), \alpha, \Omega)$:

$$a_n = \|\varphi_n\|^2 = \int_0^\pi |\varphi_n(x, \Omega)|^2 dx, \quad n \in \mathbb{Z}.$$  

The numbers $a_n$ are called norming constants. And by $h_n(x, \Omega)$ we will denote normalized eigenfunctions (i.e. $\|h_n(x)\| = 1$):

$$h_n(x, \Omega) = h_n(x) = \frac{\varphi_n(x, \Omega)}{\sqrt{a_n(\Omega, \alpha)}}. \quad (1.7)$$

It is known (see [5,9]) that in the case of $\Omega \in L^2_{\mathbb{R}}[0, \pi]$ the norming constants have an asymptotic form:

$$a_n(\Omega) = \pi + c_n, \quad \sum_{n=-\infty}^{\infty} c_n^2 < \infty. \quad (1.8)$$

**Definition 1.1.** Two Dirac operators $L(\Omega, \alpha)$ and $L(\tilde{\Omega}, \tilde{\alpha})$ are said to be isospectral, if $\lambda_n(\Omega, \alpha) = \lambda_n(\tilde{\Omega}, \tilde{\alpha})$, for every $n \in \mathbb{Z}$.

**Lemma 1.2.** Let $\Omega, \tilde{\Omega} \in L^2_{\mathbb{R}}[0, \pi]$ and the operators $L(\Omega, \alpha)$ and $L(\tilde{\Omega}, \tilde{\alpha})$ are isospectral. Then $\tilde{\alpha} = \alpha$.

**Proof.** The proof follows from the asymptotics (1.4):

$$\frac{\tilde{\alpha}}{\pi} = \lim_{n \to \infty} \left(n - \lambda_n(\Omega, \alpha)\right) = \lim_{n \to \infty} \left(n - \lambda_n(\tilde{\Omega}, \tilde{\alpha})\right) = \frac{\tilde{\alpha}}{\pi}. \quad \square$$

So, instead of isospectral operators $L(\Omega, \alpha)$ and $L(\tilde{\Omega}, \tilde{\alpha})$, we can talk about “isospectral potentials” $\Omega$ and $\tilde{\Omega}$.
Theorem 1.3 (Uniqueness theorem). The map

$$(\Omega, \alpha) \in L^2_\mathbb{R}[0, \pi] \times \left(\frac{-\pi}{2}, \frac{\pi}{2}\right) \longleftrightarrow \{\lambda_n(\Omega, \alpha), a_n(\Omega, \alpha); n \in \mathbb{Z}\}$$

is one-to-one.

Remark 1.4. It is natural to call this a Marchenko theorem, since it is an analogue of the famous theorem of V. A. Marchenko [15], in the case for Sturm–Liouville problem. The proof of this theorem for the case $p, q \in AC[0, \pi]$ there is in the paper [18]. The detailed proof for the case $p, q \in L^2_\mathbb{R}[0, \pi]$ there is in [7] (see also [4–6, 8, 10, 19]).

Let us fix some $\Omega \in L^2_\mathbb{R}[0, \pi]$ and consider the set of all canonical potentials $\tilde{\Omega} = \left(\frac{p}{q}, -\frac{q}{p}\right)$, with the same spectra as $\Omega$:

$$M^2(\Omega) = \{\tilde{\Omega} \in L^2_\mathbb{R}[0, \pi]: \lambda_n(\tilde{\Omega}, \tilde{\alpha}) = \lambda_n(\Omega, \alpha), n \in \mathbb{Z}\}.$$  

Our main goal is to give the description of the set $M^2(\Omega)$ as explicit as it possible.

From the uniqueness theorem the next corollary easily follows.

Corollary 1.5. The map

$$\tilde{\Omega} \in M^2(\Omega) \leftrightarrow \{a_n(\tilde{\Omega}), n \in \mathbb{Z}\}$$

is one-to-one.

Since $\tilde{\Omega} \in M^2(\Omega)$, then $a_n(\tilde{\Omega})$ have similar to (1.8) asymptotics. Since $a_n(\Omega)$ and $a_n(\tilde{\Omega})$ are positive numbers, there exist real numbers $t_n = t_n(\tilde{\Omega})$, such that $\frac{a_n(\tilde{\Omega})}{a_n(\Omega)} = e^{t_n}$. From the latter equality and from (1.8) follows that

$$e^{t_n} = 1 + d_n, \quad \sum_{n=-\infty}^{\infty} d_n^2 < \infty.$$  \hspace{1cm} (1.9)

It is easy to see, that the sequence $\{t_n; n \in \mathbb{Z}\}$ is also from $l^2$, i.e. $\sum_{n=-\infty}^{\infty} t_n^2 < \infty$. Since all $a_n(\Omega)$ are fixed, then from the corollary 1.5 and the equality $a_n(\tilde{\Omega}) = a_n(\Omega)e^{-t_n}$ we will get the following corollary.

Corollary 1.6. The map

$$\tilde{\Omega} \in M^2(\Omega) \leftrightarrow \{t_n(\tilde{\Omega}), n \in \mathbb{Z}\} \in l^2$$

is one-to-one.

Thus, each isospectral potential is uniquely determined by a sequence $\{t_n; n \in \mathbb{Z}\}$. Note, that the problem of description of isospectral Sturm–Liouville operators was solved in [3, 11, 12, 17].

For Dirac operators the description of $M^2(\Omega)$ is given in [8]. This description has a “recurrent” form, i.e. at the first in [8] is given the description of a family of isospectral potentials $\Omega(x, t), t \in \mathbb{R}$, for which only one norming constant $a_m(\Omega(\cdot, t))$ different from $a_m(\Omega)$ (namely, $a_m(\Omega(\cdot, t)) = a_m(\Omega)e^{-t}$), while the others are equal, i.e. $a_m(\Omega(\cdot, t)) = a_m(\Omega)$, when $n \neq m$. 

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Theorem 1.7 ([8]). Let $t \in \mathbb{R}$, $a \in \left( -\frac{T}{2}, \frac{T}{2} \right]$ and
\[
\Omega(x, t) = \Omega(x) + \frac{e^t - 1}{\theta_m(x, t, \Omega)} \left\{ Bh_m(x, \Omega)h_m^*(x, \Omega) - h_m(x, \Omega)h_m^*(x, \Omega)B \right\},
\]
where $\theta_n(x, t, \Omega) = 1 + (e^t - 1) \int_0^1 |h_n(s, \Omega)|^2 ds$, and $*$ is a sign of transponation, e.g. $h_n^* = \left( h_{n+1}, h_{n-2} \right)^*$. Then, for arbitrary $t \in \mathbb{R}$, $\lambda_n(\Omega, t) = \lambda_n(\Omega)$ for all $n \in \mathbb{Z}$, $a_n(\Omega, t) = a_n(\Omega)$ for all $n \in \mathbb{Z} \setminus \{m\}$ and $a_m(\Omega, t) = a_m(\Omega)e^{-t}$. The normalized eigenfunctions of the problem $L(\Omega(t), \alpha)$ are given by the formulae:
\[
h_n(x, \Omega(t), t) = \begin{cases} e^{-t/2} \frac{\theta_m(x, t, \Omega)}{\theta_m(x, t, \Omega)} h_m(x, \Omega), & \text{if } n = m, \\ h_n(x, \Omega) - \frac{(e^t - 1) \int_0^1 h_m^*(s, \Omega)h_n(s, \Omega) ds}{\theta_m(x, t, \Omega)} h_m(x, \Omega), & \text{if } n \neq m. \end{cases}
\]

Theorem 1.7 shows that it is possible to change exactly one norming constant, keeping the others. As examples of isospectral potentials $\Omega$ and $\tilde{\Omega}$ we can present $\Omega(x) \equiv 0 = \left( \begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix} \right)$ and
\[
\tilde{\Omega}(x) = \Omega_m(x) = \frac{\pi(e^t - 1)}{\pi + (e^t - 1)x} \begin{pmatrix} -\sin 2mx & \cos 2mx \\ \cos 2mx & \sin 2mx \end{pmatrix},
\]
where $t \in \mathbb{R}$ is an arbitrary real number and $m \in \mathbb{Z}$ is an arbitrary integer.

Changing successively each $a_m(\Omega)$ by $a_m(\Omega)e^{-t_m}$, we can obtain any isospectral potential, corresponding to the sequence $\{t_m; m \in \mathbb{Z}\} \subset \mathbb{R}$. It follows from the uniqueness Theorem 1.3 that the sequence, in which we change the norming constants, is not important.

In [8] were used the following designations:
\[
T_{-1} = \{ \ldots, 0, \ldots \}, \\
T_0 = \{ \ldots, 0, 0, t_0, 0, \ldots, 0, \ldots \}, \\
T_1 = \{ \ldots, 0, 0, 0, t_0, t_1, 0, \ldots, 0, \ldots \}, \\
T_2 = \{ \ldots, 0, 0, 0, t_{-1}, t_0, t_1, 0, \ldots, 0, \ldots \}, \\
\vdots \\
T_{2n} = \{ \ldots, 0, 0, t_{-n}, \ldots, t_{-1}, t_0, t_1, \ldots, t_{n-1}, t_n, 0, \ldots \}, \\
T_{2n+1} = \{ \ldots, 0, t_{-n}, t_{-n+1}, \ldots, t_{-1}, t_0, t_1, \ldots, t_n, t_{n+1}, 0, \ldots \}, \\
\vdots 
\]

Let $\Omega(x, T_{-1}) \equiv \Omega(x)$ and
\[
\Omega(x, T_m) = \Omega(x, T_{m-1}) + \Delta \Omega(x, T_m), \quad m = 0, 1, 2, \ldots,
\]
where
\[
\Delta \Omega(x, T_m) = \frac{e^t - 1}{\theta_m(x, t, \Omega; T_{m-1})} \begin{pmatrix} Bh_m(x, \Omega(\cdot, T_{m-1}))h_m^*(\cdot) - h_m(\cdot)h_m^*(\cdot)B \end{pmatrix},
\]
where $\tilde{m} = \frac{m+1}{2}$, if $m$ is odd and $\tilde{m} = \frac{m}{2}$, if $m$ is even. The arguments in others $h_m(\cdot)$ and $h_m^*(\cdot)$ are the same as in the first. And after that in [8] the following theorem was proved.
\textbf{Theorem 1.8 ([8])}. Let \( T = \{ t_n, n \in \mathbb{Z} \} \in L^2 \) and \( \Omega \in L^2_\mathbb{R}[0, \pi] \). Then

\[
\Omega(x, T) \equiv \Omega(x) + \sum_{m=0}^{\infty} \Delta \Omega(x, T_m) \in M^2(\Omega).
\] (1.10)

We see, that each potential matrix \( \Delta \Omega(x, T_m) \) defined by normalized eigenfunctions \( h_n(x, \Omega(x, T_{m-1})) \) of the previous operator \( L(\Omega(\cdot, T_{m-1}), \alpha) \). This approach we call “recursive” description.

In this paper, we want to give a description of the set \( M^2(\Omega) \) only in terms of eigenfunctions \( h_n(x, \Omega) \) of the initial operator \( L(\Omega, \alpha) \) and sequence \( T \in L^2 \). With this aim, let us denote by \( N(T_m) \) the set of the positions of the numbers in \( T_m \), which are not necessary zero, i.e.

\[
N(T_0) = \{ 0 \},
N(T_1) = \{ 0, 1 \},
N(T_2) = \{ -1, 0, 1 \},
\]

\[
\vdots
N(T_{2n}) = \{ -n, -(n-1), \ldots, 0, \ldots, n-1, n \},
N(T_{2n+1}) = \{ -n, -(n-1), \ldots, 0, \ldots, n, n+1 \},
\]

in particular \( N(T) \equiv \mathbb{Z} \). By \( S(x, T_m) \) we denote the \((m+1) \times (m+1)\) square matrix

\[
S(x, T_m) = \left( \delta_{ij} + (e^l - 1) \int_0^x h^*_i(s) h_j(s) ds \right)_{i,j \in N(T_m)}
\] (1.11)

where \( \delta_{ij} \) is a Kronecker symbol. By \( S_p^{(k)}(x, T_m) \) we denote a matrix which is obtained from the matrix \( S(x, T_m) \) by replacing the \( k \)th column of \( S(x, T_m) \) by \( H_p(x, T_m) = \{ -(e^l - 1)h_k(x) \} \) column, \( p = 1, 2 \). Now we can formulate our result as follows.

\textbf{Theorem 1.9}. Let \( T = \{ t_k \}_{k \in \mathbb{Z}} \in L^2 \) and \( \Omega \in L^2_\mathbb{R}[0, \pi] \). Then the isospectral potential from \( M^2(\Omega) \), corresponding to \( T \), is given by the formula

\[
\Omega(x, T) = \Omega(x) + G(x, x, T)B - BG(x, x, T) = \begin{pmatrix} p(x, T) & q(x, T) \\ q(x, T) & -p(x, T) \end{pmatrix},
\] (1.12)

where

\[
G(x, x, T) = \frac{1}{\det S(x, T)} \sum_{k \in \mathbb{Z}} \left( \frac{\det S_p^{(k)}(x, T)}{\det S_p^{(k)}(x, T)} \right) h^*_k(x),
\]

and \( \det S(x, T) = \lim_{m \to \infty} \det S(x, T_m) \) (the same for \( \det S_p^{(k)}(x, T), \ p = 1, 2 \).

In addition, for \( p(x, T) \) and \( q(x, T) \) we get explicit representations:

\[
p(x, T) = p(x) - \frac{1}{\det S(x, T)} \sum_{k \in \mathbb{Z}} \sum_{p=1}^{2} \det S_p^{(k)}(x, T) h_{k(p-1)}(x),
\]

\[
q(x, T) = q(x) + \frac{1}{\det S(x, T)} \sum_{k \in \mathbb{Z}} \sum_{p=1}^{2} (-1)^{p-1} S_p^{(k)}(x, T) h_{k(p)}(x).
\]
2 Proof of Theorem 1.9

The spectral function of an operator $L(\Omega, \alpha)$ defined as

$$
\rho(\lambda) = \begin{cases} 
\sum_{0 < \lambda_n \leq \lambda} \frac{1}{a_n(\Omega)} & \lambda > 0, \\
- \sum_{\lambda < \lambda_n \leq 0} \frac{1}{a_n(\Omega)} & \lambda < 0,
\end{cases}
$$

i.e. $\rho(\lambda)$ is left-continuous, step function with jumps in points $\lambda = \lambda_n$ equals $\frac{1}{a_n}$ and $\rho(0) = 0$.

Let $\Omega, \tilde{\Omega} \in L^2_\mathbb{R}[0, \pi]$ and they are isospectral. It is known (see [1, 2, 6, 13]), that there exists a function $G(x, y)$ such that:

$$
\varphi(x, \lambda, \alpha, \tilde{\Omega}) = \varphi(x, \lambda, \alpha, \Omega) + \int_0^x G(x, s) \varphi(s, \lambda, \alpha, \Omega) dt.
$$

(2.1)

It is also known (see, e.g. [1, 6, 13]), that the function $G(x, y)$ satisfies to the Gelfand–Levitan integral equation:

$$
G(x, y) + F(x, y) + \int_0^x G(x, s) F(s, y) ds = 0, \quad 0 \leq y \leq x,
$$

(2.2)

where

$$
F(x, y) = \int_{-\infty}^{\infty} \varphi(x, \lambda, \alpha, \Omega) \varphi^*(y, \lambda, \alpha, \Omega) d[\tilde{\rho}(\lambda) - \rho(\lambda)].
$$

(2.3)

If the potential $\tilde{\Omega}$ from $M^2(\Omega)$ is such that only finite norming constants of the operator $L(\tilde{\Omega}, \alpha)$ are different from the norming constants of the operator $L(\Omega, \alpha)$, i.e. $a_n(\tilde{\Omega}) = a_n(\Omega)e^{-\lambda_n}$, $n \in N(T_m)$ and the others are equal, then it means, that

$$
d\tilde{\rho}(\lambda) - d\rho(\lambda) = \sum_{k \in N(T_m)} \left( \frac{1}{a_k} - \frac{1}{\tilde{a}_k} \right) \delta(\lambda - \lambda_k) d\lambda = \sum_{k \in N(T_m)} \left( \frac{\lambda_k}{a_k} - 1 \right) \delta(\lambda - \lambda_k) d\lambda,
$$

(2.4)

where $\delta$ is Dirac $\delta$-function. In this case the kernel $F(x, y)$ can be written in a form of a finite sum (using notation (1.7)):

$$
F(x, y) = F(x, y, T_m) = \sum_{k \in N(T_m)} (e^{\lambda_k} - 1) h_k(x, \Omega) h_k^*(y, \Omega),
$$

(2.5)

and consequently, the integral equation (2.2) becomes to an integral equation with degenerated kernel, i.e. it becomes to a system of linear equations and we will look for the solution in the following form:

$$
G(x, y, T_m) = \sum_{k \in N(T_m)} g_k(x) h_k^*(y),
$$

(2.6)

where $g_k(x) = \left( \frac{g_{k_1}(x)}{g_{k_2}(x)} \right)$ is an unknown vector-function. Putting the expressions (2.5) and (2.6) into the integral equation (2.2) we will obtain a system of algebraic equations for determining the functions $g_k(x)$:
The systems (2.8) might be written in matrix form. It would be better if we consider the equations (2.7) for the vectors \( g \) where the column vectors \( g \) found in the form (Cramer’s rule):

\[
\vec{g}(x) + \sum_{i \in N(T_m)} s_{ik}(x)g_i(x) = -(e^{ik} - 1)h_k(x), \quad k \in N(T_m),
\]

where

\[
s_{ik}(x) = (e^{ik} - 1) \int_{0}^{x} h^*_k(s)h_k(s)ds.
\]

It would be better if we consider the equations (2.7) for the vectors \( g_k = \left( \vec{g}_{k1}, \vec{g}_{k2} \right) \) by coordinates \( g_{k1} \) and \( g_{k2} \) to be a system of scalar linear equations:

\[
g_{kp}(x) + \sum_{i \in N(T_m)} s_{ik}(x)g_{ip}(x) = -(e^{ik} - 1)h_{kp}(x), \quad k \in N(T_m), \quad p = 1, 2.
\]

The systems (2.8) might be written in matrix form

\[
S(x, T_m)g_{p}(x, T_m) = H_{p}(x, T_m), \quad p = 1, 2,
\]

where the column vectors \( g_{p}(x, T_m) = \{g_{kp}(x, T_m)\}_{k \in N(T_m)}, \quad p = 1, 2 \), and the solution can be found in the form (Cramer’s rule):

\[
g_{kp}(x, T_m) = \frac{\det S_{p}^{(k)}(x, T_m)}{\det S(x, T_m)}, \quad k \in N(T_m), \quad p = 1, 2.
\]

Thus we have obtained for \( g_{k}(x) \) the following representation:

\[
g_{k}(x, T_m) = \frac{1}{\det S(x, T_m)} \left( \frac{\det S_{1}^{(k)}(x, T_m)}{\det S_{2}^{(k)}(x, T_m)} \right)
\]

and then by putting (2.10) into (2.6) we find the \( G(x, y, T_m) \) function. If the potential \( \Omega \) is from \( L_{2}^{1} \), then such is also the kernel \( G(x, x, T_m) \) (see [8]), and the relation between them gives as follows:

\[
\Omega(x, T_m) = \Omega(x) + G(x, x, T_m)BG - BG(x, x, T_m).
\]

On the other hand we have

\[
\Omega(x, T_m) = \Omega(x) + \sum_{k=0}^{m} \Delta\Omega(x, T_k).
\]

So, using the Theorem 1.8 and the equality (2.12) we can pass to the limit in (2.11), when \( m \to \infty \):

\[
\Omega(x, T) = \Omega(x) + G(x, x, T)BG - BG(x, x, T).
\]

The potentials \( \Omega(x, T) \) in (1.10) and (2.13) have the same spectral data \( \{\lambda_{n}(T), a_{n}(T)\}_{n \in Z} \), and therefore they are the same and \( \Omega(\cdot, T) \) defined by (2.13) is also from \( M^{2}(\Omega) \).

Using (2.6) and (2.10) we calculate the expression \( G(x, x, T_m)BG - BG(x, x, T_m) \) and pass to the limit, obtaining for the \( p(x, T) \) and \( q(x, T) \) the representations:
\[ p(x, T) = p(x) - \frac{1}{\det S(x, T)} \sum_{k \in \mathbb{N}(T)} \sum_{p=1}^{2} \det S_{p}^{(k)}(x, T) h_{k(3-p)}(x), \]
\[ q(x, T) = q(x) + \frac{1}{\det S(x, T)} \sum_{k \in \mathbb{N}(T)} \sum_{p=1}^{2} (-1)^{p-1} S_{p}^{(k)}(x, T) h_{k(p)}(x). \]

Theorem 1.9 is proved.

For example, when we change just one norming constant (e.g. for \( T_0 \)) we get two independent linear equations:
\[ (1 + s_{00}(x)) g_0(x) = -(e^{t_0} - 1) h_0(x), \]
\[ (1 + s_{00}(x)) g_2(x) = -(e^{t_0} - 1) h_2(x). \]

For the solutions we get:
\[ g_0(x) = -\frac{(e^{t_0} - 1) h_0(x)}{1 + s_{00}(x)}, \]
\[ g_2(x) = -\frac{(e^{t_0} - 1) h_2(x)}{1 + s_{00}(x)}, \]
and for the potentials \( p(x, T_0) \) and \( q(x, T_0) \):
\[ p(x, T_0) = p(x) + \frac{e^{t_0} - 1}{1 + s_{00}(x)} (2h_0(x)h_2(x)), \]
\[ q(x, T_0) = q(x) + \frac{e^{t_0} - 1}{1 + s_{00}(x)} (h_0^2(x) - h_0^2(x)). \]

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