Dynamic Programming Principle for Stochastic Recursive Optimal Control Problem under $G$-framework

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Abstract. In this paper, we study a stochastic recursive optimal control problem in which the cost functional is described by the solution of a backward stochastic differential equation driven by $G$-Brownian motion. Under standard assumptions, we establish the dynamic programming principle and the related fully nonlinear HJB equation in the framework of $G$-expectation. Finally, we show that the value function is the viscosity solution of the obtained HJB equation.

Key words. $G$-expectation, backward stochastic differential equations, stochastic recursive optimal control, robust control, dynamic programming principle

AMS subject classifications. 93E20, 60H10, 35K15

1 Introduction

It is well known that Duffie and Epstein [7] introduced a stochastic differential recursive utility which corresponds to the solution of a particular backward stochastic differential equation (BSDE). Thus the BSDE point of view gives a simple formulation of recursive utilities (see [5]). Since then, the classical stochastic optimal control problem is generalized to a so called "stochastic recursive optimal control problem" in which the cost functional is defined by the solution of BSDE. The stochastic maximum principle and dynamic programming principle for this problem were first established in Peng [18] and [23] respectively.

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Recently Hu et. al studied a new kind of BSDE which is driven by $G$-Brownian motion in \cite{13} and \cite{12}:

$$
Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds + \int_t^T g(s, Y_s, Z_s)d\langle B \rangle_s (1)
$$

- $\int_t^T Z_sdB_s - (K_T - K_t)$.

They proved that there exists a unique triple of processes $(Y, Z, K)$ which solves (1) under the standard Lipschitz conditions. This new kind of BSDE is based on the $G$-expectation theory which is introduced by Peng (see \cite{19}, \cite{22} and the references therein). This $G$-expectation framework ($G$-framework for short) does not require the probability space and is convenient to study financial problems involving volatility uncertainty. Let us mention that there are other recent advances in this direction. Denis, Martini \cite{5} and Denis, Hu, Peng \cite{6} developed quasi-sure stochastic analysis. Soner et al. \cite{26} have obtained a existence and uniqueness theorem for a new type of fully nonlinear BSDE, called 2BSDE.

An important property of the solution $Y$ of (1) is that it can be represented as the "supremum of expectations" over a set of nondominated probability measures. For example, the solution $Y$ of (1) at time 0 can be written as

$$
Y_0 = \hat{E}[\xi + \int_0^T f(s, Y_s, Z_s)ds + \int_0^T g(s, Y_s, Z_s)d\langle B \rangle_s] (2)
$$

where $P$ is a family of weakly compact nondominant probability measures. Then, (1) can be used to define recursive utility under volatility uncertainty. It is worth to point out that the recursive utility under mean uncertainty was developed in Chen and Epstein \cite{3}. Epstein and Ji \cite{9, 10} introduced a particular recursive utility under both mean and volatility uncertainty.

Motivated by the recursive utility optimization under volatility uncertainty, we explore a stochastic recursive optimal control problem in which the cost functional is defined by the solution of the above new type of BSDE. In more details, the state equation is governed by the following controlled SDE driven by $G$-Brownian motion

$$
dX^{t,x,u} = b(s, X^{t,x,u}, u_s)ds + h_{ij}(s, X^{t,x,u}, u_s)d\langle B^i, B^j \rangle_s + \sigma(s, X^{t,x,u}, u_s)dB_s, \quad X^{t,x,u}_t = x.
$$

The cost functional is introduced by the solution $Y^{t,x,u}$ of the following BSDE driven by $G$-Brownian motion at time $t$:

$$
-dY^{t,x,u}_t = f(s, X^{t,x,u}_s, Y^{t,x,u}_s, Z^{t,x,u}_s, u_s)ds + g_{ij}(s, X^{t,x,u}_s, Y^{t,x,u}_s, Z^{t,x,u}_s, u_s)d\langle B^i, B^j \rangle_s
$$

- $Z^{t,x,u}_sdB_s - dK^{t,x,u}_s$

$$
Y^{t,x,u}_T = \Phi(X^{t,x,u}_T), \quad s \in [t, T].
$$
We define the value function of our stochastic recursive optimal control problem as follows:

\[ V(t, x) = \text{ess inf}_{u(\cdot) \in U[t,T]} Y_{t,x,u}, \]

where the control set is in the G-framework. In view of (2), we essentially have to solve a "inf sup problem". Such problem is known as the robust optimal control problem, i.e., we consider the worst scenario by maximizing over a set of probability measures and then we minimize the cost functional. For recent development of robust utility maximization under volatility uncertainty, we refer the interested readers to [27], [17] and [4]. Tevzadze, Toronjadze, Uzunashvili [27] studied robust exponential and power utilities. Matoussi, Possamai, Zhou [17] related robust utility maximization problem to a particular 2BSDE with quadratic growth. In [4], Denis and Kervarec established a duality theory for this problem in non-dominated models.

The objective of our paper is to establish the dynamic programming principle (DPP) for this stochastic recursive optimal control problem and investigate the value function in G-framework.

It is well known that DPP and related HJB equations is a powerful approach to solving optimal control problems (see [11], [28] and [23]). For the classical stochastic recursive optimal control problem, Peng [23] obtained the Hamilton–Jacobi–Bellman equation and proved that the value function is its viscosity solution. In [24], Peng generalized his results and originally introduced the notion of stochastic backward semigroups which allows him to prove DPP in a very straightforward way. This backward semigroup approach is also introduced in the theory of stochastic differential games by Buckdahn and Li in [1]. Note that Buckdahn et al. [2] obtained an existence result of the stochastic recursive optimal control problem.

In this paper, we adopt the backward semigroup approach to build the DPP in our context. At first, we need to define the essential infimum of a family of random variables in the “quasi-surely” sense (q.s. for short). Compared with classical case in [24], this kind of essential infimum may not exist in our case (the q.s. case). We define the essential infimum and prove its existence in this paper. Under a family of non-dominated probability measures, it is far from being trivial to prove that the value function \( V \) is wellposed and deterministic. Due to a new result in [16], we construct the approximation of an element of the admissible control set which is the key step to prove that \( \text{ess inf}_{u(\cdot) \in U[t,T]} Y_{t,x,u} \) is a deterministic function. At last, we adopt an “implied partition” approach to prove DPP (see Lemma 22) which is completely new in the literature.

We states that \( V \) is deterministic continuous viscosity solution of the following fully nonlinear HJB equation

\[
\begin{aligned}
\partial_t V(t, x) + \inf_{u \in U} H(t, x, V, \partial_x V, \partial_{xx}^2 V, u) &= 0, \\
V(T, x) &= \Phi(x), \quad x \in \mathbb{R}^n,
\end{aligned}
\]
where

\[ H(t, x, v, p, A, u) = G(F(t, x, v, p, A, u)) + \langle p, b(t, x, u) \rangle + f(t, x, v, \sigma^T(t, x, u)p, u), \]

\[ F_{ij}(t, x, v, p, A, u) = (\sigma^T(t, x, u)A\sigma(t, x, u))_{ij} + 2\langle p, h_{ij}(t, x, u) \rangle + 2g_{ij}(t, x, v, \sigma^T(t, x, u)p, u), \]

\((t, x, v, p, A, u) \in [0, T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{S}_n \times U\). The main difficulty to prove this statement lies in the appearance of two decreasing \(G\)-martingale terms. Applying a property of decreasing \(G\)-martingale proved in Lemma 30, we overcome this difficulty (see Lemma 29) and obtain the result.

In conclusion, since there is no reference probability measure under the \(G\)-framework, our results generalize the results in Peng [23] and [24] which was only considered in the Wiener space (corresponding to \(G\) is linear in our paper). Compared with our earlier article [15], the problem in [15] is essentially a "sup sup problem" which is easier to deal with. And the techniques developed in this paper can also used to solve the problem in [15]. Note that \(G\) has the representation (4) which leads to that the above HJB equation can also be understood as a kind of Bellman-Issac equation. Then, it is meaningful to show the difference between our paper and some related references (see [1] and [25]) in game theory. Needless to say, the game problem is more complicated than the robust control problem since it needs to study the value of game. Buckdahn, Li [1] employed strategies and Pham, Zhang [25] formulated their game problem in a weak framework. In contrast, we use controls and our formulation is a "strong" framework under the \(G\)-framework. Different from [25], as revealed in [16], our admissible control set has quasi-continuous property and in particular, it does not change with time. It is worth mentioning that, in our context, the coefficients of the state equation include the state variable \(X\).

The paper is organized as follows. In section 2, we present some fundamental results on \(G\)-expectation theory. We formulate our stochastic recursive optimal control problem in section 3. We prove the properties of the value function in section 4 and establish the dynamic programming principle in section 5. In section 6, we first derive the fully nonlinear HJB equation and prove that the value function is the viscosity solution of the obtained HJB equation.

### 2 Preliminaries

We review some basic notions and results of \(G\)-expectation and the related spaces of random variables. The readers may refer to [19], [20], [21], [22] for more details.

Let \(\Omega_T = C_0([0, T]; \mathbb{R}^d)\), the space of \(R^d\)-valued continuous functions on \([0, T]\) with \(\omega_0 = 0\), and \(B_t(\omega) = \omega_t\) be the canonical process. Set

\[ L_{lip}(\Omega_T) := \{ \varphi(B_{t_1}, ..., B_{t_n}) : n \geq 1, t_1, ..., t_n \in [0, T], \varphi \in C_{b,Lip}(\mathbb{R}^{d \times n}) \}, \]

where \(C_{b,Lip}(\mathbb{R}^{d \times n})\) denotes the set of bounded Lipschitz functions on \(\mathbb{R}^{d \times n}\).
We denote the $G$-expectation space by $(\Omega_T, L_{ip}(\Omega_T), \hat{E})$. The function $G : S_d \to \mathbb{R}$ is defined by

$$G(A) := \frac{1}{2} \hat{E}[\langle AB_1, B_1 \rangle],$$

where $S_d$ denotes the collection of $d \times d$ symmetric matrices. Note that there exists a bounded and closed subset $\Gamma \subset \mathbb{R}^{d \times d}$ such that

$$G(A) = \frac{1}{2} \sup_{Q \in \Gamma} \text{tr}[AQQT].$$

(3)

In this paper, we only consider non-degenerate $G$-normal distribution, i.e., there exists some $\sigma^2 > 0$ such that $G(A) - G(B) \geq \frac{1}{2} \sigma^2 \text{tr}[A - B]$ for any $A \geq B$.

We denote by $L_{G}(\Omega)$ the completion of $L_{ip}(\Omega)$ under the norm $\|X\|_{p,G} = (\hat{E}[|X|^p])^{1/p}$ for $p \geq 1$. For each $t \geq 0$, the conditional $G$-expectation $\hat{E}_t[\cdot]$ can be extended continuously to $L_{1,G}(\Omega)$ under the norm $\| \cdot \|_{1,G}$.

**Definition 1** Let $M_{G}^0(0,T)$ be the collection of processes in the following form: for a given partition $\{t_0, \cdots, t_N\} = \pi_T$ of $[0,T]$,

$$\eta_t(\omega) = \sum_{j=0}^{N-1} \xi_j(\omega) I_{[t_j, t_{j+1})}(t),$$

where $\xi_i \in L_{ip}(\Omega)$, $i = 0, 1, 2, \cdots, N - 1$.

We denote by $M_{G}^p(0,T)$ the completion of $M_{G}^0(0,T)$ under the norm $\|\eta\|_{M_{G}^p} = (\hat{E}[\int_0^T |\eta_s|^p \, ds])^{1/p}$ for $p \geq 1$.

**Theorem 2** ([6, 14]) There exists a family of weakly compact probability measures $\mathcal{P}$ on $(\Omega, \mathcal{B}(\Omega))$ such that

$$\hat{E}[\xi] = \sup_{P \in \mathcal{P}} E_P[\xi] \quad \text{for all } \xi \in L_{1,G}(\Omega).$$

$\mathcal{P}$ is called a set that represents $\hat{E}$.

For this $\mathcal{P}$, we define capacity

$$c(A) := \sup_{P \in \mathcal{P}} P(A), \ A \in \mathcal{B}(\Omega_T).$$

A set $A \subset \Omega_T$ is polar if $c(A) = 0$. A property holds “quasi-surely” (q.s. for short) if it holds outside a polar set. In the following, we do not distinguish two random variables $X$ and $Y$ if $X = Y$ q.s. We set

$$L_{p}(\Omega_T) := \{X \in \mathcal{B}(\Omega_T) : \sup_{P \in \mathcal{P}} E_P[|X|^p] < \infty\} \text{ for } p \geq 1.$$ 

It is important to note that $L_{G}(\Omega_T) \subset L_{p}(\Omega_T)$. We extend $G$-expectation $\hat{E}$ to $L_{p}(\Omega_T)$ and still denote it by $\hat{E}$, for each $X \in L_{1}(\Omega_T)$, we set

$$\hat{E}[X] = \sup_{P \in \mathcal{P}} E_P[X].$$
For $p \geq 1$, $L^p(\Omega_T)$ is a Banach space under the norm $(\hat{E}[|\cdot|^p])^{1/p}$.

Furthermore, we extend the definition of conditional $G$-expectation. For each fixed $t \in [0,T]$, let $(A_i)_{i=1}^n$ be a partition of $\mathcal{B}(\Omega_t)$, and set

$$\xi = \sum_{i=1}^n \eta_i I_{A_i},$$

where $\eta_i \in L^1_G(\Omega_T)$, $i = 1, \ldots, n$. We define the corresponding generalized conditional $G$-expectation, still denoted by $\hat{E}_s[\cdot]$, by setting

$$\hat{E}_s[\sum_{i=1}^n \eta_i I_{A_i}] := \sum_{i=1}^n \hat{E}_s[\eta_i I_{A_i}] \text{ for } s \in [t,T].$$

Then, many properties of the conditional $G$-expectation still hold (refer to Proposition 2.5 in [12]).

3 Problem

3.1 State equations

We first give the definition of admissible controls.

**Definition 3** For each $t \in [0,T]$, $u$ is said to be an admissible control on $[t,T]$, if it satisfies the following conditions:

(i) $u : [t,T] \times \Omega \to U$ where $U$ is a given compact set of $\mathbb{R}^m$;

(ii) $u \in M^2_G(t,T;\mathbb{R}^m)$.

The set of admissible controls on $[t,T]$ is denoted by $U[t,T]$. In the rest of this paper, we use Einstein summation convention.

Let $t \in [0,T]$, $\xi \in \cup_{\varepsilon > 0} L^{2+\varepsilon}_G(\mathbb{R}^n)$ and $u \in U[t,T]$. Consider the following forward and backward SDEs driven by $G$-Brownian motion:

$$dX^{t,\xi,u}_s = b(s, X^{t,\xi,u}_s, u_s)ds + h_{ij}(s, X^{t,\xi,u}_s, u_s)d\langle B^i, B^j \rangle_s + \sigma(s, X^{t,\xi,u}_s, u_s)dB_s, \quad s \in [t,T],$$

$$X^{t,\xi,u}_t = \xi, \quad (5)$$

and

$$-dY^{t,\xi,u}_s = f(s, X^{t,\xi,u}_s, Y^{t,\xi,u}_s, Z^{t,\xi,u}_s, u_s)ds + g_{ij}(s, X^{t,\xi,u}_s, Y^{t,\xi,u}_s, Z^{t,\xi,u}_s, u_s)d\langle B^i, B^j \rangle_s$$

$$- Z^{t,\xi,u}_s dB_s - dK^{t,\xi,u}_s,$$

$$Y^{t,\xi,u}_t = \Phi(X^{t,\xi,u}_t), \quad s \in [t,T].$$

Set

$$S^0_G(0,T) := \{h(t, B_{t_1, \ldots, t_n}) : t_1, \ldots, t_n \in [0,T], h \in C_b,\text{Lip}(\mathbb{R}^{n+1})\}.$$
For \( p \geq 1 \) and \( \eta \in S^p_G(0,T) \), let \( \| \eta \|_{S^p_G} = \left\{ \mathbb{E} \left[ \sup_{t \in [0,T]} |\eta_t|^p \right] \right\}^{\frac{1}{p}} \). Denote by \( S^p_G(0,T) \) the completion of \( S^p_G(0,T) \) under the norm \( \| \cdot \|_{S^p_G} \).

For given \( t \) and \( \xi \), \((X^{t,\xi,u})\) and \((Y^{t,\xi,u},Z^{t,\xi,u},K^{t,\xi,u})\) are called solutions of the above forward and backward SDEs respectively if

1. \((X^{t,\xi,u}) \in M^2_G(t,T;\mathbb{R}^n); -\) \((X^{t,\xi,u}) \in M^2_G(0,T;\mathbb{R}^d); -\)
2. \(K^{t,\xi,u}\) is a decreasing \(G\)-martingale with \(K^{t,\xi,u}_t = 0\), \(K^{t,\xi,u} \in L^2_G\).

Theorem 4 \((\text{[22]}\)) Let Assumptions (A1) and (A2) hold. Then there exists a unique adapted solution \(X\) for equation (5).

Theorem 5 \((\text{[22]}\)) Let \(\xi, \xi' \in L^p_G(\Omega_t;\mathbb{R}^n)\) with \( p \geq 2 \) and \( u, v \in \mathcal{U}[t,T] \). Then we have, for each \( \delta \in [0,T-t] \),

\[
\mathbb{E}_t[|X^{t,\xi,u}_t - X^{t,\xi',u}_t|^2] \leq C(|\xi - \xi'|^2 + \mathbb{E}_t[\int_t^{t+\delta} |u_s - v_s|^2 ds])
\]

\[
\mathbb{E}_t[|X^{t,\xi,u}_t|^p] \leq \tilde{C}(1 + |\xi|^p),
\]

\[
\mathbb{E}_t\left[ \sup_{s \in [t,t+\delta]} |X^{t,\xi,u}_s - \xi|^p \right] \leq \tilde{C}(1 + |\xi|^p)\delta^{p/2},
\]

where the constant \( \tilde{C} \) depends on \( C, G, p, n, U \) and \( T \).

Theorem 6 \((\text{[12]}\)) Let Assumptions (A1) and (A2) hold. Then there exists a unique adapted solution \((Y,Z,K)\) for equation (6).
Theorem 7 \((12)\) Let \(\xi, \xi' \in \cup_{t>0} L^{2+\epsilon}_{G}(\Omega_t; \mathbb{R}^n)\) and \(u, v \in \mathcal{U}[t, T]\). Then there exist two positive constants \(\bar{C}_1\) and \(\bar{C}_2\) depending on \(C\), \(G\) and \(T\) such that
\[
|Y_t^{t,\xi, u} - Y_t^{t,\xi', v}|^2 \leq \bar{C}_1 \mathbb{E}_t[|\Phi(X_T^{t,\xi, u}) - \Phi(X_T^{t,\xi', v})|^2 + \int_t^T \hat{F}_s|ds|^2]
\]
\[
\leq \bar{C}_2 \mathbb{E}_t[|\Phi(X_T^{t,\xi, u}) - \Phi(X_T^{t,\xi', v})|^2 + \int_t^T |\hat{F}_s|^2|ds|, \]
where
\[
\hat{F}_s = |f(s, X_s^{t,\xi, u}, Y_s^{t,\xi, u}, Z_s^{t,\xi, u}, u_s) - f(s, X_s^{t,\xi', v}, Y_s^{t,\xi', v}, Z_s^{t,\xi', v}, v_s)| + \sum_{i,j=1}^d |g_{ij}(s, X_s^{t,\xi, u}, Y_s^{t,\xi, u}, Z_s^{t,\xi, u}, u_s) - g_{ij}(s, X_s^{t,\xi', v}, Y_s^{t,\xi', v}, Z_s^{t,\xi', v}, v_s)|.
\]

Theorem 8 \((10)\) Let \(b, h_{ij}, \sigma\) be independent of \(u\) and satisfy (A1) and (A2). Assume further that there exist constants \(L > 0, 0 < \lambda < \Lambda\) such that \(|b| \leq L, |h_{ij}| \leq L, \lambda \leq |\sigma_i| \leq \Lambda\) for \(i \leq n\), where \(\sigma_i\) is the \(i\)-th row of \(\sigma\). Then for each \(x, a' \in \mathbb{R}^n\) with \(a \leq a', s \geq t\), we have \(|I_{\{s < t\}}(X_s^{t,x,u} - X_t^{t,x,u})| \in L^2_{G}(\Omega_s)\). In particular, for each \(c, c' \in \mathbb{R}^{d \times k}\), \(c \leq c'\) and \(t \leq s_1 \leq \cdots \leq s_k\), we have \(I_{\{(B_{s_1} - B_t, \ldots, B_{s_k} - B_t) \in [c,c']\}} \in L^2_{G}(\Omega_{s_k})\).

Remark 9 If there exists a \(t_0 < T\) such that \(b, h_{ij}, \sigma\) are continuous in \(s\) just on \([t_0, T]\), then the above theorem still holds by the proof in [10].

### 3.2 Stochastic optimal control problem

The state equation of our stochastic optimal control problem is governed by the above forward SDE \([5]\) and the objective functional is introduced by the solution of the BSDE \([6]\) at time \(t\). Let \(\xi\) equals a constant \(x \in \mathbb{R}^n\). When \(u\) changes, \(Y_t^{t,x,u}\) (the solution \(Y_t^{t,x,u}\) at time \(t\)) also changes. In order to study the value function of our stochastic optimal control problem, we need to define the essential infimum of \(\{Y_t^{t,x,u} \mid u \in \mathcal{U}[t, T]\}\).

**Definition 10** The essential infimum of \(\{Y_t^{t,x,u} \mid u \in \mathcal{U}[t, T]\}\), denoted by
\[
\text{ess inf}_{u(t) \in \mathcal{U}[t, T]} Y_t^{t,x,u},
\]
is a random variable \(\zeta \in L^2_{G}(\Omega_t)\) satisfying:

(i) \(\forall u \in \mathcal{U}[t, T], \zeta \leq Y_t^{t,x,u} \ q.s.;\)

(ii) if \(\eta\) is a random variable satisfying \(\eta \leq Y_t^{t,x,u}\) \(q.s.\) for any \(u \in \mathcal{U}[t, T]\), then \(\zeta \geq \eta\) \(q.s.\).

Similarly, we can define the essential infimum of \(\{Y_t^{t,\xi, u} \mid u \in \mathcal{U}[t, T]\}\), where \(\xi \in \cup_{t>0} L^{2+\epsilon}_{G}(\Omega_t; \mathbb{R}^n)\).

The following example shows that the essential infimum may not exist.

**Example 11** Let \(d = 1\) and \((B_t)_{t \geq 0}\) be a 1-dimensional \(G\)-Brownian motion with \(G(a) = \frac{1}{2}(a^{+} - \frac{1}{3}a^{-})\). We first show that \(I_{\{(B_{t})_{t} = \frac{1}{2}\}}, I_{\{(B_{t})_{t} \geq \frac{1}{2}\}} \not\in L^2_{G}(\Omega_t)\).
It is easy to verify that $h_k((B)_1) \downarrow I_{\{(B)_1=\frac{1}{2}\}}$, where

$$h_k(x) = k \left( x - \frac{1}{2} \right) + \frac{1}{k} I_{\left( \frac{1}{2} - \frac{1}{k}, \frac{1}{2} \right)}(x) + \left( 1 - k \left( x - \frac{1}{2} \right) \right) I_{\left( \frac{1}{2}, \frac{1}{2} + \frac{1}{k} \right)}(x).$$

If $I_{\{(B)_1=\frac{1}{2}\}} \in L^2_G(\Omega_1)$, then

$$h_k((B)_1) - I_{\{(B)_1=\frac{1}{2}\}} \in L^2_G(\Omega_1) \text{ and } h_k((B)_1) - I_{\{(B)_1=\frac{1}{2}\}} \downarrow 0.$$

By Corollary 33 in [6], we have $\hat{\mathbb{E}}[h_k((B)_1) - I_{\{(B)_1=\frac{1}{2}\}}] \downarrow 0$. On the other hand,

$$\hat{\mathbb{E}}[h_k((B)_1) - I_{\{(B)_1=\frac{1}{2}\}}] \geq \lim_{j \to \infty} \hat{\mathbb{E}}[h_k((B)_1) - h_j((B)_1)] = \lim_{j \to \infty} \sup \{h_k(x) - h_j(x) : x \in [0, 1] \} = 1.$$

Thus $I_{\{(B)_1=\frac{1}{2}\}} \notin L^2_G(\Omega_1)$. Similarly, we can prove that $I_{\{(B)_1 \geq \frac{1}{2}\}}$ and $I_{\{(B)_1 \leq \frac{1}{2}\}} \notin L^2_G(\Omega_1)$, which implies that

$$I_{\{(B)_1 > \frac{1}{2}\}} = 1 - I_{\{(B)_1 \leq \frac{1}{2}\}} \notin L^2_G(\Omega_1).$$

Set $\mathcal{H}_1 = \{h_k((B)_1) : k \geq 1\}$ and $\mathcal{H}_2 = \{g_k((B)_1) : k \geq 1\}$, where

$$g_k(x) = k \left( x - \frac{1}{2} \right) + \frac{1}{k} I_{\left( \frac{1}{2} - \frac{1}{k}, \frac{1}{2} \right)}(x) + I_{\left( \frac{1}{2}, \infty \right)}(x).$$

We assert that either $\text{ess inf}_{\xi \in \mathcal{H}_1} \xi$ or $\text{ess inf}_{\xi \in \mathcal{H}_2} \xi$ does not exist. Otherwise, $\text{ess inf}_{\xi \in \mathcal{H}_1} \xi$ and $\text{ess inf}_{\xi \in \mathcal{H}_2} \xi$ belong to $L^1_G(\Omega_1)$.

By the definition we get

$$\text{ess inf}_{\xi \in \mathcal{H}_1} \xi \leq I_{\{(B)_1=\frac{1}{2}\}} \text{ q.s.;}$$

$$\text{ess inf}_{\xi \in \mathcal{H}_2} \xi \leq I_{\{(B)_1 \geq \frac{1}{2}\}} \text{ q.s.;}$$

$$(\text{ess inf}_{\xi \in \mathcal{H}_1} \xi - \text{ess inf}_{\xi \in \mathcal{H}_2} \xi)^+ \in L^2_G(\Omega_1);$$

$$(\text{ess inf}_{\xi \in \mathcal{H}_1} \xi - \text{ess inf}_{\xi \in \mathcal{H}_2} \xi)^- \in L^2_G(\Omega_1),$$

which implies that

$$I_{\{(B)_1=\frac{1}{2}\}} \text{ ess inf}_{\xi \in \mathcal{H}_2} \xi = \text{ ess inf}_{\xi \in \mathcal{H}_1} \xi.$$

Note that $\tilde{h}_k((B)_1) \leq \text{ess inf}_{\xi \in \mathcal{H}_2} \xi$ for $k \geq 1$, where

$$\tilde{h}_k(x) = k \left( x - \frac{1}{2} \right) I_{\left( \frac{1}{2} - \frac{1}{k}, \frac{1}{2} \right)}(x) + I_{\left( \frac{1}{2}, \infty \right)}(x).$$

It yields that $I_{\{(B)_1 > \frac{1}{2}\}} \leq \text{ess inf}_{\xi \in \mathcal{H}_2} \xi$ q.s. Then $\text{ess inf}_{\xi \in \mathcal{H}_2} \xi = \text{ess inf}_{\xi \in \mathcal{H}_1} \xi + I_{\{(B)_1 > \frac{1}{2}\}}$ which implies $I_{\{(B)_1 > \frac{1}{2}\}} \in L^1_G(\Omega_1)$.

But this contradicts to $I_{\{(B)_1 > \frac{1}{2}\}} \notin L^2_G(\Omega_1)$. □
Our stochastic optimal control problem is: for given $x \in \mathbb{R}^n$, to find $u(\cdot) \in \mathcal{U}[t, T]$ so as to minimize the objective function $Y^{t,x,u}_t$.

For $x \in \mathbb{R}^n$, we define the value function

$$V(t, x) := \text{ess inf}_{u \in \mathcal{U}[t, T]} Y^{t,x,u}_t \text{ for } x \in \mathbb{R}^n.$$ (7)

In the following we will prove that $V(\cdot, \cdot)$ exists and is deterministic and for each $\xi \in \bigcup_{\varepsilon > 0} L^{2+\varepsilon}_G(\Omega_t; \mathbb{R}^n)$, $V(t, \xi) = \text{ess inf}_{u \in \mathcal{U}[t, T]} Y^{t,\xi,u}_t$. Furthermore, we will obtain the dynamic programming principle and the related fully nonlinear HJB equation.

4 Properties of the value function

We first give some notations:

$L_{ip}(\Omega^t) := \{\varphi(B_{t_1} - B_t, \ldots, B_{t_n} - B_t) : n \geq 1, t_1, \ldots, t_n \in [t, s], \varphi \in C_b Lip(\mathbb{R}^{d \times n})\}$;

$L^2_G(\Omega^t) := \text{the completion of } L_{ip}(\Omega^t)$ under the norm $\| \cdot \|_{2,G}$;

$M^0_G(t, T) := \{\eta = \sum_{i=0}^{N-1} \xi_i I_{[t_i, t_{i+1})} : t = t_0 < \cdots < t_N = T, \xi_i \in L_{ip}(\Omega^t_i)\}$;

$M^2_G(t, T) := \text{the completion of } M^0_G(t, T)$ under the norm $\| \cdot \|_{M^2_G}$;

$\mathcal{U}[t, T] := \{u : u \in M^2_G(t, T; \mathbb{R}^m) \text{ with values in } U\}$;

$\mathcal{U}[t, T] := \{u : u \in M^2_G(t, T; \mathbb{R}^m) \text{ with values in } U\}$;

$\mathcal{U}[t, T] := \{u = \sum_{i=0}^{N-1} I_{A_i} u^i : n \in \mathbb{N}, u^i \in \mathcal{U}[t, T], I_{A_i} \in L^2_G(\Omega_t), \Omega = \bigcup_{i=1}^n A_i\}$;

$\mathcal{U}[t, T] := \{u = \sum_{i=0}^{N-1} \sum_{j=1}^{l_i} a^i_j I_{A^j} I_{[t_i, t_{i+1})} : l_i \in \mathbb{N}, a^i_j \in U, I_{A^j} \in L^2_G(\Omega^t_i), \Omega = \bigcup_{j=1}^{l_i} A^j\}$.

Remark 12 For $t = t_0 < \cdots < t_N = T$, $\xi_i \in L^2_G(\Omega^t_i)$, it is easy to check that $\sum_{i=0}^{N-1} \xi_i I_{[t_i, t_{i+1})}(s) \in M^2_G(t, T)$. From this we can deduce that $\mathcal{U}[t, T] \subset \mathcal{U}[t, T] \subset \mathcal{U}[t, T] \subset \mathcal{U}[t, T]$.

In order to prove

$$V(t, x) = \inf_{u \in \mathcal{U}[t, T]} Y^{t,x,u}_t = \inf_{u \in \mathcal{U}[t, T]} Y^{t,x,u}_t,$$

we need the following lemmas.

Lemma 13 Let $u \in \mathcal{U}[t, T]$ be given. Then there exists a sequence $(u^k)_{k \geq 1}$ in $\mathcal{U}[t, T]$ such that

$$\lim_{k \to \infty} \mathbb{E}\left[\int_t^T |u_s - u^k_s|^2 ds\right] = 0.$$
Proof. For each $\varepsilon > 0$, we only need to prove that there exists a process $v \in \mathbb{U}[t, T]$ such that $\hat{E}\left[\int_t^T |u_s - v_s|^2 ds\right] \leq \varepsilon$. Since $u \in M^2(t, T; \mathbb{R}^m)$, there exists a sequence processes $v^k \in M^0(t, T; \mathbb{R}^m)$ such that $\hat{E}\left[\int_t^T |u_s - v^k_s|^2 ds\right] \to 0$. Set $U_\varepsilon := \{a \in \mathbb{R}^m : d(a, U) \leq \frac{\varepsilon}{16} \}$, then

$$
\hat{E}\left[\int_t^T |u_s - v^k_s|^2 ds\right] \geq \hat{E}\left[\int_t^T |u_s - v^k_s|^2 I_{\{\hat{v} \notin U_\varepsilon\}} ds\right] \geq \frac{\varepsilon}{16} \hat{E}\left[\int_t^T I_{\{\hat{v} \notin U_\varepsilon\}} ds\right],
$$

which implies that $\hat{E}\left[\int_t^T I_{\{\hat{v} \notin U_\varepsilon\}} ds\right] \to 0$. Thus there exists a $k_0 \geq 1$ such that

$$
\hat{E}\left[\int_t^T |u_s - v^k_0|^2 ds\right] \leq \frac{\varepsilon}{4} \hat{E}\left[\int_t^T I_{\{\hat{v} \notin U_\varepsilon\}} ds\right] \leq \frac{\varepsilon}{16M^2},
$$

where $M = \sup\{|a| : a \in U\}$. Set $\tilde{v} = v^{k_0}$, we can write $\tilde{v}$ as

$$
\tilde{v}_s = \varphi_0(\xi_0)I_{\{t_0, t_1\}}(s) + \sum_{i=1}^{N-1} \varphi_i(\xi_0, \xi_i)I_{\{t_i, t_{i+1}\}}(s),
$$

where $t = t_0 < t_1 < \cdots < t_N = T$, $\xi_0 = (B_s^{0,1}, \ldots, B_s^{0, k_0})$ for $s_0^0 \in [0, t]$, $\xi_i = (B_s^{i,1} - B_{t_i}, \ldots, B_s^{i, k_i} - B_t)$ for $s^i_j \in [t, t_i]$, $i \geq 1$, $\varphi_i \in C_b(Lip(\mathbb{R}^n; \mathbb{R}^m))$ with $n_0 = dk_0$, $n_i = d(k_0 + k_i)$ for $i \geq 1$. Obviously, we can find two constants $M > 0$ and $L > 0$ such that for $i \leq N - 1$,

$$|\varphi_i| \leq M, \ |\varphi_i(x^i) - \varphi_i(\bar{x}^i)| \leq L|x^i - \bar{x}^i| \text{ for } x^i, \bar{x}^i \in \mathbb{R}^n_i.$$

For each $k \geq 1$, we can find finite nonempty cubes $A_{j}^{i,k} \subset \mathbb{R}^{dk_i}$, $i \geq 0$, $j = 1, \ldots, l_i - 1$, such that $[-ke^i, ke^i) = \bigcup_{j \leq l_i - 1} A_{j}^{i,k}$ with $e^i = [1, \ldots, 1]^T \in \mathbb{R}^{dk_i}$ and $\rho(A_{j}^{i,k}) := \sup\{|x^i - \bar{x}^i| : x^i, \bar{x}^i \in A_{j}^{i,k}\} \leq \frac{1}{k}$. Set $A_{j}^{i,k} = \mathbb{R}^{dk_i} \setminus [-ke^i, ke^i)$ and

$$
v^k_s = \left(\sum_{j_0 \leq l_0^0} \varphi_0(x_{j_0}^0)I_{\{\xi_0 \in A_{j_0}^{0,k}\}}I_{\{t_0, t_1\}}(s)\right)
+ \sum_{i=1}^{N-1} \left(\sum_{j_0 \leq l_0^i, j_i \leq l_i^i} \varphi_i(x_{j_0}^0, x_{j_i}^i)I_{\{\xi_0 \in A_{j_0}^{0,k}\}}I_{\{\xi_i \in A_{j_i}^{i,k}\}}I_{\{t_i, t_{i+1}\}}(s)\right)

= \sum_{j_0 \leq l_0^0} I_{\{\xi_0 \in A_{j_0}^{0,k}\}} \left(\varphi_0(x_{j_0}^0)I_{\{t_0, t_1\}}(s) + \sum_{i=1}^{N-1} \left(\sum_{j_i \leq l_i^i} \varphi_i(x_{j_0}^0, x_{j_i}^i)I_{\{\xi_i \in A_{j_i}^{i,k}\}}I_{\{t_i, t_{i+1}\}}(s)\right)\right),
$$
where \( x_{i,k} \) is one point belonging to \( A_{i,k} \) for \( i \geq 0 \) and \( j \leq t_i^k \). By Theorem we can get \( I_{(0,\xi)} A_{i,k} \in L^2(\Omega_t) \) and \( I_{(i, \xi)} A_{j,k} \in L^2(\Omega_t) \) for \( i \geq 1 \). Then we have

\[
\mathbb{E}[\int_t^T |\tilde{v}_s - \tilde{v}_s^{k}|^2 ds] \leq \sum_{i=0}^{N-1} \mathbb{E}[|\tilde{v}_{t_i} - \tilde{v}_{t_i}^{k}|^2](t_{i+1} - t_i) \\
\leq \sum_{i=0}^{N-1} \mathbb{E} \left[ \frac{2L^2}{k^2} + \frac{4M^2}{k^2} (|\xi_0|^2 + |\xi_1|^2) \right] (t_{i+1} - t_i) \\
\rightarrow 0 \text{ as } k \rightarrow \infty.
\]

We set

\[
\tilde{v}_s^k = \sum_{j_0 \leq t_i^k} I_{(0,\xi)} A_{i,j_0} \left( \tilde{\varphi}_0(x_{j_0}^0, I_{(t_0,t_1)}(s)) + \sum_{i=1}^{N-1} \left( \sum_{j, \leq t_i^k} \tilde{\varphi}_i(x_{j_0}^i, x_{j_i}^i) I_{(i, \xi)} A_{j_i, k} \right) I_{(t_i, t_{i+1})}(s) \right),
\]

where \( \tilde{\varphi}_i(x_{j_0}^i, x_{j_i}^i) \) is one point in \( U \) such that \( |\tilde{\varphi}_i(x_{j_0}^i, x_{j_i}^i) - \tilde{\varphi}_i(x_{j_0}^i, x_{j_i}^i)| = d(\varphi_i(x_{j_0}^i, x_{j_i}^i), U) \). By Remark it is easy to verify that \( \tilde{v}^k \in U[t, T] \) and

\[
\tilde{\varphi}_0(x_{j_0}^0, I_{(t_0,t_1)}(s)) + \sum_{i=1}^{N-1} \left( \sum_{j, \leq t_i^k} \tilde{\varphi}_i(x_{j_0}^i, x_{j_i}^i) I_{(i, \xi)} A_{j_i, k} \right) I_{(t_i, t_{i+1})}(s) \in U'[t, T].
\]

Note that

\[
|u_s - \tilde{v}_s^k|^2 = |u_s - \tilde{v}_s^k|^2 I_{\{\tilde{v}_s \in U'\}} + |u_s - \tilde{v}_s^k|^2 I_{\{\tilde{v}_s \not\in U'\}} \\
\leq 4M^2 I_{\{\tilde{v}_s \in U'\}} + 2|u_s - \tilde{v}_s|^2 + 2|\tilde{v}_s - \tilde{v}_s^k|^2 I_{\{\tilde{v}_s \in U'\}} \\
\leq 4M^2 I_{\{\tilde{v}_s \in U'\}} + 2|u_s - \tilde{v}_s|^2 + 2 \left( \frac{\sqrt{\varepsilon}}{4} + \frac{2\sqrt{2L}}{k} \right)^2,
\]

then we get

\[
\limsup_{k \to \infty} \mathbb{E} \left[ \int_t^T |u_s - \tilde{v}_s^k|^2 ds \right] \leq \frac{7\varepsilon}{8}.
\]

Thus there exists a \( k_1 \geq 1 \) such that \( \mathbb{E}[\int_t^T |u_s - \tilde{v}_s^k|^2 ds] \leq \varepsilon \). The proof is complete by taking \( v = \tilde{v}^{k_1} \in U[t, T] \).

**Lemma 14** Let \( u \in U'[t, T] \) be given. Then there exists a sequence \((u^k)_{k \geq 1}\) in \( U'[t, T] \) such that

\[
\lim_{k \to \infty} \mathbb{E}[\int_t^T |u_s - u^k_s|^2 ds] = 0.
\]

**Proof.** The proof is the same as Lemma we omit it.

**Lemma 15** Let \( \xi \in \cup_{\varepsilon > 0} L^\infty_t(\Omega_t; \mathbb{R}^n) \), \( u \in U[t, T] \) and \( v_s = \sum_{i=1}^{N} I_{A_i} u^i_s \in U[t, T] \). Then there exists a constant \( L_1 \) depending on \( T, G \) and \( C \) such that

\[
\mathbb{E}[|Y^\xi_t u - \sum_{i=1}^{N} I_{A_i} Y^\xi_t u^i|^2] \leq L_1 \mathbb{E}[\int_t^T |u_s - v_s|^2 ds].
\]
Proof. Consider the following equations:

\[ dX_{t,i}^{\xi,v} = b(s, X_{t,i}^{\xi,v}, v_i) ds + h_{ij}(s, X_{t,i}^{\xi,v}, v_i) d(B^j_s, B^i_s) + \sigma(s, X_{t,i}^{\xi,v}, v_i) dB_s, \]

\[ X_{t,i}^{\xi,v} = \xi, \quad s \in [t, T], \quad i = 1, \ldots, N. \]

\[ -dY_{t,i}^{\xi,v} = f(s, X_{t,i}^{\xi,v}, Y_{t,i}^{\xi,v}, Z_{t,i}^{\xi,v}, v_i) ds + g_{ij}(s, X_{t,i}^{\xi,v}, Y_{t,i}^{\xi,v}, Z_{t,i}^{\xi,v}, v_i) d(B^j_s, B^i_s) \]

\[ -Z_{t,i}^{\xi,v} dB_s - dK_{t,i}^{\xi,v}, \]

\[ Y_{T,i}^{\xi,v} = \Phi(X_{T,i}^{\xi,v}), \quad s \in [t, T], \quad i = 1, \ldots, N. \]

For \( s \in [t, T], \) we set

\[ \bar{X}_{t,i}^{\xi,v} = \sum_{i=1}^{N} I_{A_i} X_{t,i}^{\xi,v}, \quad \bar{Y}_{t,i}^{\xi,v} = \sum_{i=1}^{N} I_{A_i} Y_{t,i}^{\xi,v}, \quad \bar{Z}_{t,i}^{\xi,v} = \sum_{i=1}^{N} I_{A_i} Z_{t,i}^{\xi,v}, \quad \bar{K}_{t,i}^{\xi,v} = \sum_{i=1}^{N} I_{A_i} K_{t,i}^{\xi,v}. \]

Multiplying \( I_{A_i} \) on both sides of the above equations and summing up, we have

\[ d\bar{X}_{t,i}^{\xi,v} = b(s, \bar{X}_{t,i}^{\xi,v}, v_i) ds + h_{ij}(s, \bar{X}_{t,i}^{\xi,v}, v_i) d(B^j_s, B^i_s) + \sigma(s, \bar{X}_{t,i}^{\xi,v}, v_s) dB_s, \]

\[ \bar{X}_{t,i}^{\xi,v} = \xi, \quad s \in [t, T], \]

\[ -d\bar{Y}_{t,i}^{\xi,v} = f(s, \bar{X}_{t,i}^{\xi,v}, \bar{Y}_{t,i}^{\xi,v}, \bar{Z}_{t,i}^{\xi,v}, v_i) ds + g_{ij}(s, \bar{X}_{t,i}^{\xi,v}, \bar{Y}_{t,i}^{\xi,v}, \bar{Z}_{t,i}^{\xi,v}, v_i) d(B^j_s, B^i_s) \]

\[ -\bar{Z}_{t,i}^{\xi,v} dB_s - d\bar{K}_{t,i}^{\xi,v}, \]

\[ \bar{Y}_{T,i}^{\xi,v} = \Phi(\bar{X}_{T,i}^{\xi,v}), \quad s \in [t, T]. \]

By Theorem 7, we can obtain that there exists a constant \( C_1 > 0 \) depending on \( T, G \) and \( C \) such that

\[ |Y_{t,i}^{\xi,v} - \sum_{i=1}^{N} I_{A_i} Y_{t,i}^{\xi,v}|^2 \leq C_1 \mathbb{E}_s |\Phi(X_{t,i}^{\xi,v}) - \Phi(X_{T,i}^{\xi,v})|^2 + \int_t^T (|X_{t,i}^{\xi,u} - X_{t,i}^{\xi,v}|^2 + |u_s - v_s|^2) ds \]

\[ \leq C_1 \mathbb{E}_s (|X_{t,i}^{\xi,u} - X_{T,i}^{\xi,v}|^2 + \int_t^T (|X_{t,i}^{\xi,u} - X_{t,i}^{\xi,v}|^2 + |u_s - v_s|^2) ds), \quad (8) \]

where \( C \) is the Lipschitz constant of \( \Phi. \) By Theorem 8, there exists a constant \( C_2 > 0 \) depending on \( T, n, G \) and \( C \) such that

\[ \mathbb{E}_s |X_{t,i}^{\xi,u} - \bar{X}_{t,i}^{\xi,v}|^2 \leq C_2 \mathbb{E}_s \int_t^T |u_s - v_s|^2 ds. \]

Taking \( G \)-expectation on both sides of (8), we obtain the result. \( \blacksquare \)

Remark 16 By the definition of generalized conditional \( G \)-expectation and Proposition 2.5 in [12], the above analysis still holds for the case that \( \{A_i\}_{i=1}^N \) is a \( \mathcal{B}(\Omega_i) \)-partition of \( \Omega. \)
Theorem 17 The value function \( V(t, x) \) exists and
\[
V(t, x) = \inf_{u \in \mathcal{U}[t, T]} Y^t,x,u = \inf_{v \in \mathcal{U}[t, T]} Y^t,x,v.
\]

Proof. For each \( v \in \mathcal{U}'[t, T] \), it is easy to check that \( Y^t,x,v \) is a constant. In the following, we prove that \( \varepsilon \inf_{u \in \mathcal{U}[t, T]} Y^t,x,u = \inf_{v \in \mathcal{U}'[t, T]} Y^t,x,v \), q.s.. Since \( \mathcal{U}'[t, T] \subset \mathcal{U}[t, T] \), we only need to show that \( Y^t,x,u \geq \inf_{v \in \mathcal{U}'[t, T]} Y^t,x,v \) q.s. for each \( u \in \mathcal{U}[t, T] \). For each fixed \( u \in \mathcal{U}[t, T] \), by Lemma 13 there exists a sequence \( u^k = \sum_{i=1}^{N_k} I_{A_i}^t v^i,k \in \mathcal{U}[t, T] \), \( k = 1, 2, \ldots \), such that
\[
\hat{\mathbb{E}} \left[ \int_t^T \left| u_s - \sum_{i=1}^{N_k} I_{A_i}^t v^i,k \right|^2 ds \right] \to 0.
\]
By Lemma 15,
\[
\hat{\mathbb{E}} \left[ \left| Y^t,x,u - \sum_{i=1}^{N_k} I_{A_i}^t Y_t^t,x,v^i,k \right|^2 \right] \leq L_t \hat{\mathbb{E}} \left[ \int_t^T \left| u_s - \sum_{i=1}^{N_k} I_{A_i}^t v^i,k \right|^2 ds \right].
\]
It yields that \( \sum_{i=1}^{N_k} I_{A_i}^t Y_t^t,x,v^i,k \) converges to \( Y_t^t,x,u \) in \( L^2_G \). Then there exists a subsequence (for simplicity, we still denote it by \( \{\sum_{i=1}^{N_k} I_{A_i}^t Y_t^t,x,v^i,k\} \)) which converges to \( Y_t^t,x,u \) q.s.. Note that
\[
\sum_{i=1}^{N_k} I_{A_i}^t Y_t^t,x,v^i,k \geq \inf_{v \in \mathcal{U}'[t, T]} Y_t^t,x,v \text{ q.s.,}
\]
then we have \( Y_t^t,x,u \geq \inf_{v \in \mathcal{U}'[t, T]} Y_t^t,x,v \) q.s.. Thus
\[
V(t, x) = \varepsilon \inf_{u \in \mathcal{U}[t, T]} Y_t^t,x,u = \inf_{v \in \mathcal{U}'[t, T]} Y_t^t,x,v.
\]
Similarly, by Lemmas 14 and 15 we can get \( \inf_{u \in \mathcal{U}'[t, T]} Y_t^t,x,u = \inf_{v \in \mathcal{U}'[t, T]} Y_t^t,x,v \). The proof is complete. ■

Lemma 18 There exists a constant \( L_2 > 0 \) depending on \( T, G \) and \( C \) such that
\[
|V(t, x) - V(t, y)| \leq L_2 |x - y| \text{ for any } x, y \in \mathbb{R}^n.
\]

Proof. By Theorems 6 and 7 for any \( x, y \in \mathbb{R}^n \) and \( u \in \mathcal{U}[t, T] \),
\[
|Y_t^t,x,u - Y_t^t,y,u| \leq C_1 \hat{\mathbb{E}} \left[ |\Phi(X_t^t,x,u) - \Phi(X_t^t,y,u)|^2 + \int_t^T |X_s^t,x,u - X_s^t,y,u|^2 ds \right] \leq C_2 |x - y|^2.
\]
It is easy to verify that \( \inf_{v \in \mathcal{U}'[t, T]} Y_t^t,x,v - \inf_{v \in \mathcal{U}'[t, T]} Y_t^t,y,v \leq \sup_{v \in \mathcal{U}'[t, T]} |Y_t^t,x,v - Y_t^t,y,v| \). Thus by the above estimate and Theorem 17 we obtain the result. ■
Lemma 19 There exists a constant \( L_3 > 0 \) depending on \( T, G \) and \( C \) such that

\[ |V(t, x)| \leq L_3(1 + |x|) \text{ for any } x \in \mathbb{R}^n. \]

Proof. The proof is similar to Lemma 18, we omit it. ■

Theorem 20 For any \( \xi \in \cup_{\varepsilon > 0} L_G^2(\Omega_t; \mathbb{R}^n) \), we have

\[ V(t, \xi) = \text{ess inf}_{u \in \mathcal{U}[t, T]} Y_{t, \xi, u}^t. \]

Proof. First, we prove that \( \forall u \in \mathcal{U}[t, T], V(t, \xi) \leq Y_{t, \xi, u}^t \text{ q.s.} \)

For a fixed \( \xi \in \cup_{\varepsilon > 0} L_G^2(\Omega_t; \mathbb{R}^n) \), we can find a sequence \( \xi^k = \sum_{i=1}^{N_k} x_i^k I_{A_i^k} \),

\( k = 1, 2, ..., \) where \( x_i^k \in \mathbb{R}^n \) and \( \{A_i^k\}_{i=1}^{N_k} \) is a \( \mathcal{B}(\Omega_t) \)-partition of \( \Omega \), such that

\[ \lim_{k \to \infty} \mathbb{E}[|\xi - \xi^k|^2] = 0. \]

Here \( I_{A_i^k} \) may not not in \( L_G^2(\Omega_t) \). By Lemma 18, we have

\[ \mathbb{E}[|V(t, \xi) - V(t, \xi^k)|^2] \leq L_2^2 \mathbb{E}[|\xi - \xi^k|^2] \to 0. \]

By similar analysis as in Lemma 15 and the definition of generalized conditional \( G \)-expectation,

\[ |Y_{t, \xi, u}^t - \sum_{i=1}^{N_k} I_{A_i^k} Y_{t, x_i^k, u}^t|^2 \]

\[ \leq C_1 \mathbb{E}[|\Phi(X_{t, \xi, u}^t) - \Phi(\sum_{i=1}^{N_k} I_{A_i^k} X_{t, x_i^k, u}^t)|^2 + \int_t^T |X_{s, \xi, u}^t - \sum_{i=1}^{N_k} I_{A_i^k} X_{s, x_i^k, u}^t|^2 ds] \]

\[ \leq C_2 \mathbb{E}[|\xi - \xi^k|^2]. \]

Then

\[ \lim_{k \to \infty} \mathbb{E}[|Y_{t, \xi, u}^t - \sum_{i=1}^{N_k} I_{A_i^k} Y_{t, x_i^k, u}^t|^2] = 0. \]

Note that

\[ V(t, \xi^k) = \sum_{i=1}^{N_k} I_{A_i^k} V(t, x_i^k) \leq \sum_{i=1}^{N_k} I_{A_i^k} Y_{t, x_i^k, u}^t \text{ q.s.} \]

Thus

\[ V(t, \xi) \leq Y_{t, \xi, u}^t \text{ q.s.}. \]

Note that

Second, we prove that for a given \( \eta \in L_G^2(\Omega_t) \), if \( \forall u \in \mathcal{U}[t, T], \eta \leq Y_{t, \xi, u}^t \text{ q.s.} \),

then \( \eta \leq V(t, \xi) \text{ q.s.} \).

By the above analysis, we know that

\[ |Y_{t, \xi, u}^t - \sum_{i=1}^{N_k} I_{A_i^k} Y_{t, x_i^k, u}^t|^2 \leq C_2 \mathbb{E}[|\xi - \xi^k|^2] \text{ q.s.} \]
Then, for any \( u \in \mathcal{U}[t,T] \),

\[
\eta \leq \sum_{i=1}^{N_k} I_{A^k_i} Y^{t,x^k,u}_{t} + \sqrt{C_2 \tilde{E}_t[|\xi - \xi^k|^2]} \text{ q.s.}
\]

For each fixed \( N_k \), it yields that

\[
\eta \leq \sum_{i=1}^{N_k} I_{A^k_i} V(t, x^k_i) + \sqrt{C_2 \tilde{E}_t[|\xi - \xi^k|^2]} \text{ q.s.}
\]

Note that

\[
\lim_{k \to \infty} \tilde{E}_t[|V(t, \xi) - V(t, \xi^k)|^2] = 0
\]

and

\[
\lim_{k \to \infty} \tilde{E}_t[|\xi - \xi^k|^2] = 0.
\]

Then there exists a subsequence \((\xi^{k_i})\) of \((\xi^k)\) such that as \( k_i \to \infty \),

\[
V(t, \xi^{k_i}) \to V(t, \xi), \quad \tilde{E}_t[|\xi - \xi^{k_i}|^2] \to 0, \text{ q.s.}
\]

Thus \( \eta \leq V(t, \xi) \text{ q.s.} \). This completes the proof. \( \blacksquare \)

5 Dynamic programming principle

For given initial data \((t, x)\), a positive real number \( \delta \leq T-t \) and \( \eta \in \cup_{\varepsilon>0} L_{G}^{2+\varepsilon}(\Omega_{t+\delta}) \), we define

\[
G_{t,t+\delta}[\eta] := \tilde{Y}^{t,x,u}_t,
\]

where \((X^{t,x,u}_s, \tilde{Y}^{t,x,u}_s, \tilde{Z}^{t,x,u}_s)_{t \leq s \leq t+\delta}\) is the solution of the following forward and backward equations:

\[
\begin{align*}
\frac{dX^{t,x,u}_s}{ds} &= b(s, X^{t,x,u}_s, u_s)ds + h_{ij}(s, X^{t,x,u}_s, u_s)d(B^i_s, B^j_s) + \sigma(s, X^{t,x,u}_s, u_s)dB_s, \\
X^{t,x,u}_t &= x,
\end{align*}
\]

and

\[
\begin{align*}
\frac{d\tilde{Y}^{t,x,u}_s}{ds} &= f(s, X^{t,x,u}_s, \tilde{Y}^{t,x,u}_s, \tilde{Z}^{t,x,u}_s, u_s)ds + g_{ij}(s, X^{t,x,u}_s, \tilde{Y}^{t,x,u}_s, \tilde{Z}^{t,x,u}_s, u_s)d(B^i_s, B^j_s) \\
-\tilde{Z}^{t,x,u}_s dB_s - dK^{t,x,u}_s, \\
\tilde{Y}^{t,x,u}_{t+\delta} &= \eta, \quad s \in [t, t+\delta].
\end{align*}
\]

Note that \( G_{t,t+\delta}[\cdot] \) is a (backward) semigroup which was first introduced by Peng in [21].

Our main result in this section is the following dynamic programming principle.
Theorem 21 Let Assumptions (A1) and (A2) hold. Then for any \( t \leq s \leq T, x \in \mathbb{R}^n \), we have
\[
V(t, x) = \inf_{u(\cdot) \in \mathcal{U}_c\{t, s\}} G_{t, s}^{t, x, u}[V(s, X_s^{t, x, u})] 
\]
\[
= \inf_{u(\cdot) \in \mathcal{U}_c\{t, s\}} G_{t, s}^{t, x, u}[V(s, X_s^{t, x, u})].
\]

In order to prove Theorem 21, we need the following lemmas.

Lemma 22 Let Assumptions (A1) and (A2) hold. Assume further that there exist constants \( L > 0, \Lambda > 0 \) such that \( |b| \leq L \), \( |h_{ij}| \leq L \) and \( |\sigma| \leq \Lambda \) for \( i \leq n \) and \( (t, x, u) \in [0, T] \times \mathbb{R}^n \times U \), where \( \sigma_i \) is the \( i \)-th row of \( \sigma \). Then for any \( t < s \leq T, x \in \mathbb{R}^n \), we have
\[
V(t, x) \leq \inf_{u(\cdot) \in \mathcal{U}_c\{t, s\}} G_{t, s}^{t, x, u}[V(s, X_s^{t, x, u})].
\]

Proof. For each \( \varepsilon > 0 \), there exists a \( u(\cdot) \in \mathcal{U}_c\{t, s\} \) such that
\[
G_{t, s}^{t, x, u}[V(s, X_s^{t, x, u})] - \varepsilon \leq \inf_{v(\cdot) \in \mathcal{U}_c\{t, s\}} G_{t, s}^{t, x, v}[V(s, X_s^{t, x, v})].
\]

We can write \( u(\cdot) \) as
\[
u_r = \sum_{i=1}^{N-1} \sum_{j=1}^{l_i} a_j^i I_{[t_j, t_{j+1}]}(r) + a_0^i I_{[t_0, t_1]}(r),
\]
where \( t = t_0 < t_1 < \cdots < t_N = s \), \( l_i \in \mathbb{N} \), \( a_j^i \in U \), \( I_{A_j^i} \in L_0^2(\Omega_{t_i}) \) and \( \{A_j^i\}_{j=1}^{l_i} \) is a partition of \( \Omega \). Consider the following SDE: for any \( v(\cdot) \in \mathcal{U}_c\{t, s\} \),
\[
\begin{align*}
\frac{d\tilde{X}_r^{t, x, v}}{dr} &= b(r, \tilde{X}_r^{t, x, v} - \tilde{X}_r^e, e)dr + h_{ij}(r, \tilde{X}_r^{t, x, v} - \tilde{X}_r^e, e)dB^i_r, \\
\frac{d\tilde{X}_r^{t, x, v}}{dr} &= \frac{1}{2} \sigma(r, \tilde{X}_r^{t, x, v} - \tilde{X}_r^e, e, v_r)dB^j_r, \\
\tilde{X}_t^{t, x, v} &= x, \quad \tilde{X}_t = 0, \quad r \in [t, s],
\end{align*}
\]
where \( e = [1, \ldots, 1]_T \in \mathbb{R}^n \). It is easy to verify that
\[
\tilde{X}_r^{t, x, v} = X_r^{t, x, v} + (\Lambda + 1)(B_t^1 - B_r^1) + \tilde{X}_r, \quad \tilde{X}_r = (\Lambda + 1)(B_t^1 - B_r^1), \quad r \in [t, s]
\]
is the solution of the above SDE. We set
\[
I_{i+1} = \{J_{i+1} = (j_0, j_1, \ldots, j_i) : 1 \leq j_k \leq l_k, 0 \leq k \leq i\},
\]
where \( l_0 = 1, i \leq N - 1 \). For each given \( J_{i+1} = (j_0, j_1, \ldots, j_i) \in I_{i+1} \), we denote
\[
\tilde{X}_r^{t, x, J_{i+1}} := \tilde{X}_r^{t, x, u^{J_{i+1}}}, \quad r \in [t, t_{i+1}],
\]
where \( u^{J_{i+1}} = \sum_{k=0}^{i} a_{j_k}^k I_{[t_k, t_{k+1}]}(r) \) is deterministic. We claim that
\[
\tilde{X}_r^{t, x, u} = \sum_{J_{i+1} \in I_{i+1}} \left( \prod_{k=1}^{i} \tilde{X}_r^{t, x, J_{i+1}} \right)^2 \quad \text{for} \quad 0 \leq i \leq N - 1,
\]
(11)
where $\Pi_{k=1}^{N} I_{A_{kj}} = 1$. It is easy to check that the equality (11) holds for $i = 0$. Suppose that the equality (11) holds for $i_0 \geq 0$, then by the similar analysis as in the proof of Lemma [15], we can get

$$\tilde{X}_{i_0 + 1} = \sum_{j=1}^{i_0 + 1} I_{A_{kj}} \tilde{X}_{i_0 + 1, i_{j}}^{i_0 + 1, i_{j}}, \quad \tilde{X}_{i_0 + 1, i_{j}}^{i_0 + 1, i_{j}} = \sum_{J_{i_0 + 1} \subseteq I_{i_0 + 1}} \prod_{k=1}^{i_0} I_{A_{kj}} \tilde{X}_{i_0 + 1, i_{j}}^{i_0 + 1, i_{j}},$$

where $\xi = \tilde{X}_{i_0 + 1}^{i_0 + 1} \xi_{i_0 + 1}^{i_0 + 1}$. It is easy to verify that $\tilde{X}_{i_0 + 1, i_{j}}^{i_0 + 1, i_{j}} = \tilde{X}_{i_0 + 2}^{i_0 + 1, i_{j}}$. Thus the equality (11) holds for $i_0 + 1$. From this we can deduce that

$$\tilde{X}_{s}^{i, x, u} = \sum_{J_{N} \subseteq I_{N}} \prod_{k=1}^{N-1} I_{A_{kj}} \tilde{X}_{s}^{i, x, J_{N}}.$$

It is easy to check that

$$\sqrt{|\sigma_{i_1} + (\Lambda + 1)|^2 + |\sigma_{i_2}|^2 + \cdots + |\sigma_{i_d}|^2} \geq 1,$$

where $\sigma_{ij}$ is the $i$-th row and $j$-th column of $\sigma$. Then by Theorem [8] we have $I_{\{\tilde{X}_{s}^{i, x, J_{N} \in [a, a']}\}} \in L_{2}^{\Omega_{k}}$ for $a, a' \in \mathbb{R}^{n}$ with $a \leq a'$. Thus

$$I_{\{\tilde{X}_{s}^{i, x, u} \in [a, a']\}} = \sum_{J_{N} \subseteq I_{N}} \prod_{k=1}^{N-1} I_{A_{kj}} I_{\{\tilde{X}_{s}^{i, x, J_{N} \in [a, a']}\}} \in L_{2}^{\Omega_{k}}.$$

For each integer $k \geq 1$, we can choose finite nonempty cubes $D_{j}^{k} \subseteq \mathbb{R}^{n}$, $E_{j}^{k} \subseteq \mathbb{R}$ and $x_{j}^{k} \in D_{j}^{k}$, $q_{j}^{k} \in E_{j}^{k}$ for $j = 1, \ldots, k$, such that $[-ke, ke] = \cup_{j \leq k} D_{j}^{k}$, $[-k, k] = \cup_{j \leq k} E_{j}^{k}$, $\rho(D_{j}^{k}) = \sup \{|x - \bar{x}| : x, \bar{x} \in D_{j}^{k}\} \leq \frac{1}{k}$ and $\rho(E_{j}^{k}) = \sup \{|q - \bar{q}| : q, \bar{q} \in E_{j}^{k}\} \leq \frac{1}{\sqrt{n}(\Lambda + 1)k}$. Set

$$\xi_{k, u} = \sum_{j_{1}=1}^{k} \sum_{j_{2}=1}^{k} (x_{j_{1}}^{k} - (\Lambda + 1)q_{j_{2}}^{k}) I_{\{\tilde{X}_{s}^{i, x, u} \in D_{j_{1}}^{k} \cap \{B_{1}^{k} - B_{1}^{k} \in E_{j_{2}}^{k}\}\}}.$$

Note that $\tilde{X}_{s}^{i, x, u} = X_{i, x, u}^{s} + (\Lambda + 1)(B_{1}^{k} - B_{1}^{k})e$, then we get

$$|X_{i, x, u}^{s} - \xi_{k, u}| \leq \frac{2}{k} + |X_{i, x, u}^{s}||X_{i, x, u}^{s}| + \sqrt{n}(\Lambda + 1)|B_{1}^{k} - B_{1}^{k}|.$$

By Theorem [5] there exists a constant $C_{1} > 0$ depending on $T, n, U, G$ and $C$ such that

$$\hat{E}[|X_{i, x, u}^{s}|] \leq C_{1}(1 + |x|^{4}).$$

Thus we obtain

$$\hat{E}[|X_{i, x, u}^{s} - \xi_{k, u}|^{2}] \leq \frac{8(1 + |X_{i, x, u}^{s}|^{4}) + n^{2}(\Lambda + 1)|B_{1}^{k} - B_{1}^{k}|^{4}}{k^{2}} \leq \frac{C_{2}(1 + |x|^{4})}{k^{2}}.$$
Lemma 24

Let Assumptions (A1) and (A2) hold. Then for any $t < s \leq T$, by Theorem 17, $V(s, \hat{x}_{j_1 j_2}^k) = \inf_{u \in \mathbb{U}[s, T]} Y_s \hat{x}_{j_1 j_2}^k u$ and $V(s, 0) = \inf_{u \in \mathbb{U}[s, T]} Y_s u$, then we can choose $\tilde{u}_{j_1 j_2}^k$, $\tilde{u}_{0, k}^k \in \mathbb{U}[s, T]$ such that

$$V(s, \hat{x}_{j_1 j_2}^k) \leq V(s, \tilde{u}_{j_1 j_2}^k, \tilde{u}_{0, k}^k) \leq V(s, \hat{x}_{j_1 j_2}^k) + \varepsilon, \ V(s, 0) \leq Y_s 0, \tilde{u}_{0, k}^k \leq V(s, 0) + \varepsilon.$$

Set

$$\bar{u}_{j_1 j_2} = \sum_{j_1 = 1}^k \sum_{j_2 = 1} I(\hat{x}_{j_1 j_2}^k \in D_{j_1 j_2}) \hat{I}(B_{j_1 j_2}^k \in E_{j_1 j_2}^k) \tilde{u}_{j_1 j_2}^k + \tilde{I}(\hat{x}_{j_1 j_2}^k \in [-k, k]^r_{j_1 j_2} \cap (B_{j_1 j_2}^k \in [-k, k]^r_{j_1 j_2} \cap E_{j_1 j_2}^k)) \tilde{u}_{0, k}^k,$$

by the similar analysis as in the proof of Lemma 15, we can get

$$V(s, \xi^k u) \leq \bar{Y}_{t, x, u} \bar{u}_{j_1 j_2} \leq V(s, \xi^k u) + \varepsilon.$$

By Theorem 7 there exists a constant $C_3 > 0$ depending on $T$, $G$ and $C$ such that

$$\hat{E}[[Y_s X_t^k u - Y_{s, t, x, u}]^{2}] \leq C_3 \hat{E}[[X_s X_t^k u - \xi^k u]^{2}].$$

Set $\tilde{u}(r) = u(r) I_{[t, s]}(r) + \bar{u}(r) I_{s(T)}(r)$, it is easy to check that $\tilde{u} \in \mathbb{U}[t, T]$ and

$$V(t, x) \leq Y_t^k X_t^k u = \mathbb{G}_{t, x, u}^k [Y_s X_t^k u, \bar{u}_{j_1 j_2}].$$

By Theorem 7, we get

\[
\begin{align*}
|\mathbb{G}_{t, x, u}^k [Y_s X_t^k u, \bar{u}_{j_1 j_2}] - \mathbb{G}_{t, x, u}^k [V(s, X_t^k u)]| & \leq C_3 \hat{E}[[Y_s X_t^k u, \bar{u}_{j_1 j_2} - V(s, X_t^k u)]^{2}] \\
& \leq 2C_3 (\hat{E}[[Y_s X_t^k u, \bar{u}_{j_1 j_2} - Y_s X_t^k u, \bar{u}_{j_1 j_2}]^{2}] + \hat{E}[[Y_s X_t^k u, \bar{u}_{j_1 j_2} - V(s, X_t^k u)]^{2}]) \\
& \leq 2C_3 (C_3 \hat{E}[[X_s X_t^k u - \xi^k u]^{2}] + \hat{E}[[V(s, \xi^k u) - V(s, X_t^k u)] + \varepsilon]^{2}) \\
& \leq \frac{C_4 (1 + |x|^4)}{k^2} + 4C_3 \varepsilon^2.
\end{align*}
\]

Thus

$$V(t, x) - \frac{\sqrt{C_4 (1 + |x|^4)}}{k} - (2 \sqrt{C_3} + 1) \varepsilon \leq \inf_{v(\cdot) \in \mathbb{U}[t, s]} \mathbb{G}_{t, x, u}^k [V(s, X_t^k u)].$$

Letting $k \to \infty$ first and then $\varepsilon \downarrow 0$, we obtain $V(t, x) \leq \inf_{v(\cdot) \in \mathbb{U}[t, s]} \mathbb{G}_{t, x, u}^k [V(s, X_t^k u)].$

The proof is complete. \[\blacksquare\]

**Remark 23** In the above proof, $\xi^k u$ is called an “implied partition” of $X_t^k u$.

**Lemma 24** Let Assumptions (A1) and (A2) hold. Then for any $t < s \leq T$, $x \in \mathbb{R}^n$, we have

$$V(t, x) \leq \inf_{v(\cdot) \in \mathbb{U}[t, s]} \mathbb{G}_{t, x, u}^k [V(s, X_t^k u)].$$
We define \( G \) and \( dX^{t,x,u,N} = b^{N}(s, X^{t,x,u,N}, u_s)ds + h^{N}(s, X^{t,x,u,N}, u_s)d(B^{i}, B^{j})_s + \sigma^{N}(s, X^{t,x,u,N}, u_s)dB_s, \) 
\( X^{t,x,u,N} = x, \)

and

\[
-dY^{t,x,u,N} = f(s, X^{t,x,u,N}, Y^{t,x,u,N}, Z^{t,x,u,N}, u_s)ds \\
+ g_{ij}(s, X^{t,x,u,N}, Y^{t,x,u,N}, Z^{t,x,u,N}, u_s)d(B^{i}, B^{j})_s - Z^{t,x,u,N}dB_s - dK^{t,x,u,N}, \\
Y^{t,x,u,N}_T = \Phi(X^{t,x,u,N}_s), \quad s \in [t,T].
\]

We define

\[
V^{N}(t,x) = \inf_{u(\cdot) \in \mathbb{U}[t,T]} Y^{t,x,u,N}_t.
\]

By Lemma \[22\] we get for any \( t < s \leq T, \) \( x \in \mathbb{R}^n, \)

\[
V^{N}(t,x) \leq \inf_{u(\cdot) \in \mathbb{U}[t,s]} G^{t,x,u,N}_{t,s}[V^{N}(s, X^{t,x,u,N}_s)], 
\]

(12)

where \( G^{t,x,u,N}_{t,s}[\cdot] \) is defined as in \( G^{t,x,u,N}_{t,s} \). By Theorem \[5\] there exists a constant \( C_1 > 0 \) depending on \( T, n, U, G \) and \( C \) such that for any \( u(\cdot) \in \mathbb{U}[t,T], \)

\[
\mathbb{E}[|X^{t,x,u,N}_t|^4] \leq C_1(1 + |x|^4),
\]

\[
\mathbb{E}[|X^{t,x,u,N}_t - X^{t,x,u,N}_s|^2] \leq C_2 \mathbb{E}\left[ \int_t^S (|b^{N}|^2 + |h^{N}_{ij}|^2 + |\sigma - \sigma^{N}|^2)(s, X^{t,x,u,N}_s, u_s)ds \right] \\
\leq C_2 \mathbb{E}\left[ \int_t^S \frac{1}{N^2}(|b|^4 + |h_{ij}|^4 + |\sigma|^4)(s, X^{t,x,u,N}_s, u_s)ds \right] \\
\leq C_2(1 + |x|^4) \frac{1}{N^2},
\]

where \( C_2 \) depending on \( T, n, U, G \) and \( C \). By Theorem \[7\] there exists a constant \( C_3 > 0 \) depending on \( T, G \) and \( C \) such that for any \( u(\cdot) \in \mathbb{U}[t,T], \)

\[
|Y^{t,x,u,N}_t - Y^{t,x,u,N}_s|^2 \leq C_3 \mathbb{E}[|\Phi(X^{t,x,u,N}_s) - \Phi(X^{t,x,u,N}_s)|^2 + \int_t^S |X^{t,x,u,N}_s - X^{t,x,u,N}_r|^2 dr] \\
\leq C_4 \frac{(1 + |x|^4)}{N^2},
\]

where \( C_4 \) depending on \( T, n, U, G \) and \( C \). Thus we get

\[
|V^{N}(t,x) - V(t,x)| \leq \sup_{u(\cdot) \in \mathbb{U}[t,T]} |Y^{t,x,u,N}_t - Y^{t,x,u}_t| \leq \frac{\sqrt{C_4(1 + |x|^4)}}{N}.
\]

20
It is easy to verify that Lemma \[18\] still holds for \(V^N\). Then we can get
\[
|V^N(s, X^t,x,u) - V(s, X^t_s)|^2 \\
\leq 2(|V^N(s, X^t,x,u) - V^N(s, X^t_s)|^2 + |V^N(s, X^t_s) - V(s, X^t_s)|^2) \\
\leq 2(L^2_2|X^t,x,u - X^t_s|^2 + C_4(1 + |X^t_s|^4)) \cdot \frac{N^2}{N^2}.
\]

Similar to the proof of Lemma \[22\], we can obtain for any \(u(\cdot) \in U[t, s]\),
\[
|G^{t,x,u,N}(V^N(s, X^t,x,u,N)) - G^{t,x,u}(V(s, X^t_s)|^2 \\
\leq C_3\mathbb{E}[(V^N(s, X^t,x,u,N) - V(s, X^t_s))^2 + \int_t^s |X^t,x,u,N - X^t_s|^2 dr].
\]

Thus
\[
\inf_{u(\cdot) \in U[t, s]} G^{t,x,u,N}(V^N(s, X^t,x,u,N)) - G^{t,x,u}(V(s, X^t_s)| \leq \frac{\sqrt{C_5}(1 + |x|^4)}{N},
\]
where \(C_5\) depending on \(T, n, U, G\) and \(C\). Taking \(N \to \infty\) in inequality \[12\], we obtain the result. The proof is complete. ■

Now we give the proof of Theorem \[21\]

**Proof.** (1) By Lemma \[18\], we have for any \(u(\cdot) \in U[t, T]\),
\[
Y^s_s, X^t,x,u = V(s, X^t_s) \text{ q.s.,}
\]
where \(Y^s_s, X^t,x,u = Y^t_s\) is the solution of equation \[9\] at time \(s\). Then, by the comparison theorem of G-BSDE, we obtain
\[
Y^t,x,u \geq G^{t,x,u}(V(s, X^t_s)) \text{ q.s.,}
\]
which leads to
\[
V(t, x) \geq \text{ess inf}_{u(\cdot) \in U[t, s]} G^{t,x,u}(V(s, X^t_s)).
\]

(2) Now we prove the converse inequality.

By Theorem \[17\], we get
\[
\text{ess inf}_{u(\cdot) \in U[t, s]} G^{t,x,u}(V(s, X^t_s)) = \text{inf}_{u(\cdot) \in U[t, s]} G^{t,x,u}(V(s, X^t_s)) = \text{inf}_{u(\cdot) \in U[t, s]} G^{t,x,u}(V(s, X^t_s)).
\]

By Lemma \[24\], we obtain
\[
V(t, x) \leq \text{inf}_{u(\cdot) \in U[t, s]} G^{t,x,u}(V(s, X^t_s)).
\]

This completes the proof. ■

The following lemma shows the continuity of \(V\) in \(t\).
Lemma 25 The value function $V$ is $\frac{1}{2}$ Hölder continuous in $t$.

Proof. Set $(t, x) \in \mathbb{R}^n \times [0, T]$ and $\delta > 0$. By dynamic programming principle, we have

$$V(t, x) = \inf_{u(\cdot) \in \mathcal{U}[t, t+\delta]} \mathbb{E}^t_{t,x,u}[V(t+\delta, X_{t+\delta}^{t,x,u})].$$

Set $\tilde{f} = \tilde{g}_{ij} = 0$, it is easy to verify that $(V(t+\delta, x), 0, 0)_{x \in [t, t+\delta]}$ is the solution of $G$-BSDE (9) with terminal condition $\tilde{Y}_{t+\delta} = V(t+\delta, x)$. Thus by Proposition 5.1 in [12], there exists a constant $C_1 > 0$ depending on $T$, $G$ and such that for any $u(\cdot) \in \mathcal{U}_{t}^{t+\delta}$,

$$|\mathbb{E}^t_{t,x,u}[V(t+\delta, X_{t+\delta}^{t,x,u})] - V(t+\delta, x)|^2 \leq C_1 \mathbb{E}^t_{t,x,u}[|X_{t+\delta}^{t,x,u} - x|^2].$$

By Lemmas 15 and 18, we can get

$$|\mathbb{E}^t_{t,x,u}[V(t+\delta, X_{t+\delta}^{t,x,u})] - V(t+\delta, x)|^2 \leq C_2 \mathbb{E}^t_{t,x,u}[|X_{t+\delta}^{t,x,u} - x|^2].$$

where $C_2$ depending on $T$, $G$ and $C$. By Theorem 5 there exists a constant $C_3 > 0$ depending on $T$, $n$, $U$, $G$ and $C$ such that for any $u(\cdot) \in \mathcal{U}_{t}^{t+\delta}$,

$$\mathbb{E}^t_{t,x,u}[|X_{t+\delta}^{t,x,u} - x|^2] \leq C_3(1 + |x|^2).$$

Then we obtain

$$|\mathbb{E}^t_{t,x,u}[V(t+\delta, X_{t+\delta}^{t,x,u})] - V(t+\delta, x)|^2 \leq C_4(1 + |x|^2)\delta,$$

where $C_4$ depending on $T$, $n$, $U$, $G$ and $C$. Thus

$$|V(t, x) - V(t+\delta, x)| \leq \sup_{u(\cdot) \in \mathcal{U}_{t}^{t+\delta}} |\mathbb{E}^t_{t,x,u}[V(t+\delta, X_{t+\delta}^{t,x,u})] - V(t+\delta, x)| \leq \sqrt{C_4(1 + |x|^2)\delta}.$$

The proof is complete.

6 The viscosity solution of HJB equation

The following theorem gives the relationship between the value function $V$ and the second-order partial differential equation (13).
Theorem 26 Let Assumptions (A1) and (A2) hold. \( V \) is the value function defined by \([7]\). Then \( V \) is the unique viscosity solution of the following second-order partial differential equation:

\[
\begin{align*}
\partial_t V(t, x) + & \inf_{u \in U} H(t, x, V, \partial_x V, \partial^2_{xx} V, u) = 0, \\
V(T, x) = & \Phi(x), \quad x \in \mathbb{R}^n,
\end{align*}
\]

where

\[
\begin{align*}
H(t, x, v, A, u) = & \ G(F(t, x, v, A, u)) + \langle p, b(t, x, u) \rangle + f(t, x, v, \sigma^T(t, x, u)p, u), \\
F_{ij}(t, x, v, A, u) = & \ (\sigma^T(t, x, u)A\sigma(t, x, u))_{ij} + 2\langle p, h_{ij}(t, x, u) \rangle \\
& + 2g_{ij}(t, x, v, \sigma^T(t, x, u)p, u),
\end{align*}
\]

\((t, v, A, u) \in [0, T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{S}_n \times U, \ G \) is defined by equation \([3]\).

For simplicity, we only consider the case \( h_{ij} = g_{ij} = 0 \).

Suppose \( \varphi \in C^{2,3}_{b,Lip,1}([0, T] \times \mathbb{R}^n) \). Define

\[
\begin{align*}
F_1(s, x, y, z, u) = & \ \partial_x \varphi(s, x) + f(s, x, y + \varphi(s, x), z + \sigma^T(s, x, u)\partial_x \varphi(s, x), u) \\
& + \langle b(s, x, u), \partial_x \varphi(s, x) \rangle, \\
F_{ij}^2(s, x, u) = & \ \frac{1}{2}(\partial^2_{xx} \varphi(s, x)\sigma_i(s, x, u), \sigma_j(s, x, u)).
\end{align*}
\]

Consider the following G-BSDEs: \( \forall s \in [t, t + \delta] \),

\[
\begin{align*}
Y^{1, u}_s = & \ \int_s^{t + \delta} F_1(r, X^{t,x,u}_r, Y^{1, u}_r, Z^{1, u}_r, u_r)dr + \int_s^{t + \delta} F_{ij}^2(r, X^{t,x,u}_r, u_r)d\langle B^i, B^j \rangle_r \\
& - \int_s^{t + \delta} Z^{1, u}_r dB_r - (K^{1, u}_t - K^{1, u}_s),
\end{align*}
\]

and

\[
\begin{align*}
Y^u_s = & \ \varphi(t + \delta, X^{t,x,u}_{t+\delta}) + \int_s^{t + \delta} f(r, X^{t,x,u}_r, Y^u_r, Z^u_r, u_r)dr - \int_s^{t + \delta} Z^u_r dB_r - (K^u_t - K^u_s).
\end{align*}
\]

Lemma 27 For each \( s \in [t, t + \delta] \), we have

\[
Y^{1, u}_s = Y^u_s - \varphi(s, X^{t,x,u}_s).
\]

Proof. Applying Itô’s formula to \( \varphi(s, X^{t,x,u}_s) \), we have

\[
d(Y^u_s - \varphi(s, X^{t,x,u}_s)) = dY^{1, u}_s.
\]

Since \( Y^{u}_{t + \delta} - \varphi(t + \delta, X^{t,x,u}_{t+\delta}) = Y^{1, u}_{t+\delta} = 0 \), we obtain

\[
Y^{1, u}_s = Y^u_s - \varphi(s, X^{t,x,u}_s), \quad \forall s \in [t, t + \delta].
\]

The proof is completed. \( \blacksquare \)

Consider the G-BSDE: \( \forall s \in [t, t + \delta] \),
\[
Y_s^{2,u} = \int_s^{t+\delta} F_1(r, x, Y_r^{2,u}, Z_r^{2,u}, u_r) dr + \int_s^{t+\delta} F_2^i(r, x, u_r) d(B^i)_r \\
- \int_s^{t+\delta} Z_r^{2,u} dB_r - (K_r^{2,u} - K_r^{2,u}).
\]  

(18)

We have the following estimates.

**Lemma 28** For each \( u(\cdot) \in U^t[t, t + \delta] \), we have

\[
|Y_1^{1,u} - Y_2^{2,u}| \leq L_4 \delta^{3/2},
\]

(19)

where \( L_4 \) is a positive constant dependent on \( x \) and independent of \( u(\cdot) \).

**Proof.** By Proposition 5.1 in [12], there exists a constant \( C_1 > 0 \) depending on \( T, G \) and \( C \) such that for any \( u(\cdot) \in U^t[t, t + \delta] \),

\[
|Y_1^{1,u} - Y_2^{2,u}|^2 \leq C_1 \hat{E} [ (\int_t^{t+\delta} \hat{F}_r dr)^2 ],
\]

where

\[
\hat{F}_r = |F_1(r, X_r^{t,x,u}, Y_r^{2,u}, Z_r^{2,u}, u_r) - F_1(r, x, Y_r^{2,u}, Z_r^{2,u}, u_r)|
\]

\[
+ \sum_{i,j=1}^d |F_2^{ij}(r, X_r^{t,x,u}, u_r) - F_2^{ij}(r, x, u_r)|.
\]

Note that \( \varphi \in C^{2,3}_{b,Lip}([0, T] \times \mathbb{R}^n) \), it is easy to verify that

\[
\hat{F}_r \leq C_2(|X_r^{t,x,u} - x| + |X_r^{t,x,u} - x|^2),
\]

where \( C_2 \) is dependent on \( x \) and independent of \( u(\cdot) \). By Theorem 5, we can obtain that for any \( p \geq 2 \),

\[
\hat{E} \left[ \sup_{r \in [t, t+\delta]} |X_r^{t,x,u} - x|^p \right] \leq C_3 (1 + |x|^p) \delta^{p/2},
\]

where \( C_3 \) is independent of \( u(\cdot) \). Then by Hölder’s inequality we can deduce that \( |Y_1^{1,u} - Y_2^{2,u}| \leq L_4 \delta^{3/2} \), where \( L_4 \) is dependent on \( x \) and independent of \( u(\cdot) \). This completes the proof. \( \blacksquare \)

Now we compute \( \inf_{u(\cdot) \in U^t[t, t+\delta]} Y_1^{2,u} \).

**Lemma 29** We have

\[
\inf_{u(\cdot) \in U^t[t, t+\delta]} Y_1^{2,u} = Y_1^{0},
\]

where \( Y_1^{0} \) is the solution of the following ordinary differential equation

\[
-dY_s^0 = F_0(s, x, Y_s^0, 0) ds, \quad Y_{t+\delta}^{0} = 0, \quad s \in [t, t + \delta]
\]

(20)

and

\[
F_0(s, x, y, z) := \inf_{u \in U^s} [F_1(s, x, y, z, u) + 2G(F_2(s, x, u))].
\]
In order to prove Lemma 29, we need the following property of the decreasing $G$-martingale.

**Lemma 30** Suppose that $(M_s)_{t \leq s \leq t+\delta}$ is a decreasing $G$-martingale. Then there exists a $Q \in \mathcal{P}$ such that

\[ M_{t+\delta} = M_t, \quad Q-a.s. \]

**Proof.** By the representation of $G$-expectation, we know that

\[ \hat{\mathbb{E}}[M_{t+\delta} - M_t] = \sup_{P \in \mathcal{P}} E_P[M_{t+\delta} - M_t]. \]

Thus there exist $Q_k \in \mathcal{P}$, $k = 1, 2, \ldots$, such that

\[ E_{Q_k}[M_{t+\delta} - M_t] \uparrow \hat{\mathbb{E}}[M_{t+\delta} - M_t] = \hat{\mathbb{E}}]\mathbb{E}_t[M_{t+\delta} - M_t] = 0. \]

Since $\mathcal{P}$ is weakly compact, there exist $Q \in \mathcal{P}$ and a subsequence $(Q_{k_i})$ of $(Q_k)$ such that $Q_{k_i}$ converges weakly to $Q$. By Lemma 29 in [6], we get

\[ E_Q[M_{t+\delta} - M_t] = \lim_{i \to \infty} E_{Q_{k_i}}[M_{t+\delta} - M_t] = 0. \]

Note that $M_{t+\delta} - M_t \leq 0$, q.s.. Thus, we obtain that

\[ M_{t+\delta} = M_t, \quad Q-a.s. \]

This completes the proof. □

Now we give the proof of Lemma 29.

**Proof.** (1) We first prove that for any $u(\cdot) \in \mathcal{U}[t, t + \delta]$, $Y_{t+\delta}^{2,u} \geq Y_t^0$.

Note that

\[ Y_s^{2,u} = \int_t^{s+\delta} [F_1(r, x, Y_r^{2,u}, Z_r^{2,u}, u_r) + 2G(F_2(r, x, u_r))]dr \\
+ [\int_t^{s+\delta} F_2^{ij}(r, x, u_r)d(B^i_r, B^j_r) - \int_t^{s+\delta} 2G(F_2(r, x, u_r))dr] \\
- \int_t^{s+\delta} Z_r^{2,u}dB_r - (K_{t+\delta}^{2,u} - K_s^{2,u}), \quad Q-a.s.. \]

It is easy to verify that

\[ M_s = \int_t^s F_2^{ij}(r, x, u_r)d(B^i_r, B^j_r) - \int_t^s 2G(F_2(r, x, u_r))dr \]

is a decreasing $G$-martingale.

By Lemma 30 there exists a $Q \in \mathcal{P}$ such that for $s \in [t, t + \delta]$, $M_s = 0, \quad Q-a.s.$

Then

\[ Y_s^{2,u} = \int_t^{s+\delta} [F_1(r, x, Y_r^{2,u}, Z_r^{2,u}, u_r) + 2G(F_2(r, x, u_r))]dr \\
- \int_t^{s+\delta} Z_r^{2,u}dB_r - (K_{t+\delta}^{2,u} - K_s^{2,u}), \quad Q-a.s.. \]
Consider the following BSDE: for \( s \in \left[t, t + \delta\right] \),
\[
Y^0_s = \int_t^{t+\delta} F_0(r, x, Y^0_r, Z^0_r)dr - \int_t^{t+\delta} Z^0_r dB_r - (K^0_{t+\delta} - K^0_t), \quad Q - \text{a.s.}
\]
Since \( F_0 \) is a deterministic function, we obtain that \( Z^0_s = 0 \), \( K^0_s = 0 \) and \( Y^0_s \) is just the solution of equation (20). Note that \(-K^2_s u_s\) is an increasing process and \( F_1(r, x, y, z, u_r) + 2G(F_2(r, x, u_r)) \geq F_0(r, x, y, z)\), then by the comparison theorem of classical BSDE (under the reference probability measure \( Q \)), we deduce that
\[
Y^{2, u}_{t+\delta} \geq Y^0_t.
\]

(2) We denote the class of all deterministic controls in \( \mathcal{U}^t[t, t + \delta] \) by \( \mathcal{U}_1 \). Then, for every \( u(\cdot) \in \mathcal{U}_1 \), \( Y^{2, u} \) is the solution of the following ordinary differential equation:
\[
-dY^2_{s, u} = [F_1(s, x, Y^{2, u}_s, 0, u_s) + 2G(F_2(s, x, u_s))]ds, \quad s \in [t, t + \delta], \quad Y^{2, u}_{t+\delta} = 0.
\]
It is easy to check that
\[
Y^0_t = \inf_{u(\cdot) \in \mathcal{U}_1} Y^{2, u}_t \geq \inf_{u(\cdot) \in \mathcal{U}^t[t, t + \delta]} Y^{2, u}_t.
\]
This completes the proof. ■

Finally we give the proof of Theorem 26.

**Proof.** The uniqueness of viscosity solution of equation (13) can be proved similarly as in Theorem 6.1 in [1], we only prove that \( V \) is a viscosity solution of equation (13). By Lemmas 18 and 25, \( V \) is a continuous function on \([0, T] \times \mathbb{R}^n\).

We first prove that \( V \) is the subsolution of (13). Given \( t \leq T \) and \( x \in \mathbb{R}^n \), suppose \( \varphi \in C_{b, \text{Lip}}([0, T] \times \mathbb{R}^n) \) such that \( \varphi(t, x) = V(t, x) \) and \( \varphi \geq V \) on \([0, T] \times \mathbb{R}^n\). By Theorem 21, we have
\[
V(t, x) = \inf_{u(\cdot) \in \mathcal{U}^t[t, t + \delta]} \mathcal{G}^{t, x, u}_{t, t + \delta}[V(t + \delta, X^{t, x, u}_{t+\delta})].
\]
Note that \( \varphi \geq V \) on \([0, T] \times \mathbb{R}^n\). Then by comparison theorem, we get
\[
\inf_{u(\cdot) \in \mathcal{U}^t[t, t + \delta]} \{\mathcal{G}^{t, x, u}_{t, t + \delta}[\varphi(t + \delta, X^{t, x, u}_{t+\delta})] - \varphi(t, x)\} \geq 0.
\]
By equality (17), we have
\[
\inf_{u(\cdot) \in \mathcal{U}^t[t, t + \delta]} Y^{1, u}_t \geq 0.
\]
By inequality (19) and Lemma 29, we get
\[
\inf_{u(\cdot) \in \mathcal{U}^t[t, t + \delta]} Y^{2, u}_t \geq -L_4 \delta^{3/2}
\]
and
\[
Y^0_t \geq -L_4 \delta^{3/2}.
\]
Thus
\[-L_{\delta}^{\delta^{1/2}} \leq \delta^{-1}Y_t^0 = \delta^{-1} \int_t^{t+\delta} F_0(r, x, Y_r^0, 0)dr.\]

Letting \(\delta \to 0\), we get
\[F_0(t, x, 0, 0) = \inf_{u \in U} (F_1(t, x, 0, 0, u) + G(F_2(t, x, u))) \geq 0,\]
which implies that \(V\) is a subsolution of (13). Using the same method, we can prove \(V\) is the supersolution of (13). This completes the proof.

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