Ground state energy and mass gap of a generalised quantum spin ladder

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Abstract

We show that a 2-leg ladder hamiltonian introduced recently by Albeverio and Fei can be made to satisfy the Hecke algebra. As a result we have found an equivalent representation of the eigenspectrum in terms of the spin-$\frac{1}{2}$ antiferromagnetic XXZ chain at $\Delta = -\frac{5}{3}$. The values of thermodynamic quantities such as the ground state energy and mass gap follow from the known XXZ results.
1 Introduction

A number of solvable models of quantum spin ladders have recently been found. In some models certain combinations of couplings exist such that ground states can be constructed in simple form \([1]\). Other models have an underlying \(R\)-matrix and are thus amenable to the machinery of exactly solvable lattice models in statistical mechanics. Of particular relevance here are the 2-leg Heisenberg ladders of Albeverio and Fei \([2]\) and Wang \([3]\). The two models discussed by Wang are related to known \(su(4)\) and \(su(1\vert 3)\) \(R\)-matrices, and thus to one-dimensional chains. These symmetries are broken by the inclusion of rung interactions of arbitrary strength which preserve the integrability. The rung interactions appear as chemical potentials in the equivalent one-dimensional chains, with only a slight modification to their Bethe Ansatz solution. On the other hand, the model introduced by Albeverio and Fei has rung interactions already included in the given interactions round an elementary face of the ladder. The model has not been solved in general. Those authors found the eigenvalues in the sector of the hamiltonian in which one spin is flipped from the ferromagnetic ground state. Here we address how this model relates to other known models and deduce some exact results from this equivalence.

The 2-leg ladder model is based on the symmetric 16 \(\times\) 16 \(R\)-matrix \([2]\)

\[
\tilde{R}(x) = \begin{pmatrix}
  a_1 & a_3 & 9a_2 & 3a_2 & 3b_2 & 3a_2 \\
 9a_2 & a_3 & 3b_2 & 3a_2 & 3a_2 & a_5 \\
 3a_2 & 3b_2 & a_5 & a_2 & 16x & 4a_2 \\
 & & & & & 4a_2 & a_4 & 8a_2 & 4b_2 \\
 & & & & & & & & a_5 & a_2 & 3a_2 & 3b_2 \\
 3b_2 & 3a_2 & a_5 & & & 4b_2 & 8a_2 & a_4 & 4a_2 \\
 & & & & & & & & & & & 2a_2 & 4b_2 & 4a_2 & 16x \\
 & & & & & & & & & & & & a_2 & a_5 & 3b_2 & 3a_2 \\
 & & & & & & & & & & & & & & a_1 \\
 3a_2 & 3b_2 & a_3 & 9a_2 & & & & & & & & & & & a_1 \\
 3b_2 & 3a_2 & 9a_2 & a_3 & & & & & & & & & & & & & & a_1
\end{pmatrix}
\]

where \(a_1 = 2(-1 + 9x)\), \(a_2 = -b_2 = (-1 + x)\), \(a_3 = 7 + 9x\), \(a_4 = 2(3 + 5x)\), \(a_5 = -1 + 17x\) with \(x\) arbitrary.
The matrix $\hat{R}(x)$ takes values in $V \otimes V$, where $V$ denotes a 4-dimensional vector space. It satisfies the Yang-Baxter equation

$$\hat{R}_i(x)\hat{R}_{i+1}(xy)\hat{R}_i(y) = \hat{R}_{i+1}(y)\hat{R}_i(xy)\hat{R}_i(x),$$

(2)

where $\hat{R}_i$ denotes the matrix on the vector space $V \otimes V \otimes V$, where $\hat{R}_i = \hat{R} \otimes 1$ and $\hat{R}_{i+1} = 1 \otimes \hat{R}$ and 1 is the identity operator on $V$.

We see that $\hat{R}$ obeys the properties

$$\hat{R}(1) = 16 1 \otimes 1,$$

$$\hat{R}(x)\hat{R}(\frac{1}{x}) = -\frac{1}{4}(x-9)(9x-1) 1 \otimes 1.$$  
(3)

The hamiltonian can be defined as

$$H_{ladder} = J \sum_{i=1}^{L} h_i,$$

(4)

where the local interactions follow from $h_i = \frac{1}{16}\hat{R}'(x)$. We find that $h$ has eigenvalues $-\frac{1}{8}$ (degeneracy 3) and $\frac{3}{8}$ (deg. 13), in agreement with Albeverio and Fei. However, we disagree with the precise form of the ladder hamiltonian, which here reads

$$H_{ladder} = J \frac{1}{16} \sum_{i=1}^{L-1} [-3 S_i \cdot T_i + 13 S_{i+1} \cdot T_{i+1} + 3 (S_i \cdot S_{i+1} + T_i \cdot T_{i+1})$$

$$-3 (S_i \cdot T_{i+1} + T_i \cdot S_{i+1}) - 12 (T_i \cdot S_{i+1})(S_i \cdot T_{i+1})$$

$$+ 20 (S_i \cdot T_i)(S_{i+1} \cdot T_{i+1}) + 12 (S_i \cdot S_{i+1})(T_i \cdot T_{i+1})] + \frac{57}{4} 1 \otimes 1].$$

(5)

For simplicity, we have considered open boundary conditions. Here $S = \frac{1}{2}(\sigma^x, \sigma^y, \sigma^z)$ and $T = \frac{1}{2}(\sigma^x, \sigma^y, \sigma^z)$ are the usual spin-$\frac{1}{2}$ operators, with $S_i$ and $T_i$ spin operators on the $i$-th rung of each leg of the ladder. The ladder has $L$ rungs.

## 2 Hecke algebra

Our starting point is to note that the operator $h_j$ obeys an algebra. Specifically, if we define

$$U_j = \frac{8}{3} \left(h_j + \frac{1}{8}\right)$$

(6)

then $U_j$ obeys the well-known Hecke algebra, which we write here as

$$U_j^2 = (q + q^{-1}) U_j,$$

$$U_j U_{j+1} + U_{j+1} U_j - U_j = U_{j+1} U_j U_{j+1} - U_{j+1},$$

(8)

$$[U_i, U_j] = 0, \quad \text{for} \quad |i - j| > 1.$$ 

(9)

with $q + q^{-1} = \frac{10}{3}$. Thus $q = \frac{1}{3}$ or 3.

\(^1\)The first two terms of the ladder hamiltonian in Ref. \[2\] have co-efficient 5.
A number of models satisfy the Hecke algebra \([4]\). The \(4 \times 4\) representation of interest here is \([3]\)

\[
U_j = \begin{pmatrix}
q + q^{-1} & 0 & 1 \\
q & 1 & q^{-1} \\
1 & q^{-1} & q + q^{-1}
\end{pmatrix}.
\] (10)

This is the co-product of the Casimir element belonging to the centre of \(U_q(su(2))\). The representation \([10]\) is to be compared with the more well-known representation

\[
U_j = \begin{pmatrix}
0 & 1 & 0 \\
q & 1 & q^{-1} \\
1 & q^{-1} & 0
\end{pmatrix}
\] (11)

of the Temperley-Lieb algebra. The latter satisfies the relations (7)-(9), but with \(U_j U_{j \pm 1} U_j - U_j = 0\), and is thus a quotient of Hecke. The representation \([10]\) may be written in terms of spin-\(\frac{1}{2}\) operators as

\[
U_j = \frac{1}{2} (\sigma^x_j \sigma^x_{j+1} + \sigma^y_j \sigma^y_{j+1}) + \frac{1}{4} (q + q^{-1}) (\sigma^x_j \sigma^x_{j+1} - 1)
+ \frac{1}{4} (q^{-1} - q) (\sigma^z_{j+1} - \sigma^z_j) + (q + q^{-1}) 1.
\] (12)

It follows that the Hecke hamiltonian made up of spin-\(\frac{1}{2}\) operators can be written

\[
H_{\text{Hecke}} = \sum_{j=1}^{L-1} U_j
= \frac{3}{4} (q + q^{-1}) (L - 1) + \frac{1}{2} \sum_{j=1}^{L-1} (\sigma^x_j \sigma^x_{j+1} + \sigma^y_j \sigma^y_{j+1} - \Delta \sigma^z_j \sigma^z_{j+1})
+ \frac{1}{4} (q^{-1} - q) (\sigma^z_{L} - \sigma^z_1),
\] (13)

where

\[
\Delta = -\frac{1}{2} (q + q^{-1}).
\] (14)

However, writing the XXZ term as \(H(\Delta)\), the eigenspectrum of \([13]\) is invariant under the transformation \(H(\Delta) = -H(-\Delta)\) and the interchange \(q \leftrightarrow q^{-1}\). This gives the eigenvalue equivalence

\[
E_{\text{Hecke}} \Leftrightarrow E_{\text{XXZ}} + \frac{3}{4} (q + q^{-1}) (L - 1),
\] (15)

in which the XXZ hamiltonian is defined as

\[
H_{\text{XXZ}} = -\frac{1}{2} \sum_{j=1}^{L-1} (\sigma^x_j \sigma^x_{j+1} + \sigma^y_j \sigma^y_{j+1} + \Delta \sigma^z_j \sigma^z_{j+1})
+ \frac{1}{2} p (\sigma^z_L - \sigma^z_1),
\] (16)

where

\[
p = \frac{1}{2} (q^{-1} - q)
\] (17)
This latter hamiltonian is precisely that of the open antiferromagnetic spin-$\frac{1}{2}$ XXZ chain with fields $\pm p$ at the ends of the chain. Recalling that $q = \frac{1}{3}$, the eigenspectrum of the open spin ladder hamiltonian (3) is thus equivalent to that of the open XXZ chain with $\Delta = -\frac{5}{3}$, after appropriate rescaling through eq. (6). In particular, for $J > 0$ the hamiltonian (3) is equivalent to the antiferromagnetic XXZ chain, whilst for $J < 0$ the equivalence is with the ferromagnetic XXZ chain. For simplicity we take $J = \pm 1$.

The eigenvalue equivalence (15) assumes that the two representations of the Hecke algebra are faithful, i.e. although the representations differ in size, they share all eigenvalues in common. Only the multiplicity of eigenvalues differ. For given number of rungs $L$, the ladder hamiltonian (3) is of size $16^{L-1} \times 16^{L-1}$, whilst the equivalent XXZ hamiltonian is of size $4^{L-1} \times 4^{L-1}$. We have compared the eigenspectrum of each hamiltonian with increasing $L$ and believe that the representations are indeed faithful. In fact, the entire situation is somewhat analogous to the history of the spin-1 biquadratic chain. That model [6] was mapped to the XXZ chain via the Temperley-Lieb algebra, from which such quantities as the mass gap and the groundstate energy etc were obtained [7]. These were seen to be in agreement with exact inversion relation calculations on the model itself [8]. The spin-1 biquadratic chain was later solved via the Bethe Ansatz [9].

From (6) we have
\[ H_{\text{ladder}} = \frac{3}{8} H_{\text{Hecke}} - \frac{1}{8} (L - 1). \]  
(18) 
On the other hand, from (15) we have the eigenvalue equivalence $E_{\text{Hecke}} \Leftrightarrow E_{\text{XXZ}} + \frac{5}{2} (L - 1)$ and thus
\[ E_{\text{ladder}} \Leftrightarrow \frac{3}{8} E_{\text{XXZ}} + \frac{13}{16} (L - 1). \]  
(19) 
This is our key result.

3 Ground state energy and mass gap

The open XXZ chain with arbitrary boundary fields has been solved by means of the Bethe Ansatz [10]. In particular, the solution for the case $\Delta^2 - p^2 = 1$, as applies here, simplifies considerably. We shall not reproduce the equations here, but content ourselves with recalling the relevant results. Consider the antiferromagnetic case first. In the massive region $\Delta < -1$ it is convenient to define $q = e^{-\theta}$. Here the ground state energy per site, the surface free energy and the mass gap have all been derived [11]. For the given XXZ normalisation, the mass gap is

\[ \Lambda_{\text{XXZ}} = 2 \sinh \theta \prod_{n=1}^{\infty} \left( \frac{1 - q^n}{1 + q^n} \right)^2 = \frac{8}{3} \prod_{n=1}^{\infty} \left( \frac{3^n - 1}{3^n + 1} \right)^2 = \frac{8}{3} 0.128108 \ldots \]  
(20) 

\(^2\) Of course, the expressions for the ground state energy and the mass gap are in agreement with those obtained originally [12, 13] for periodic boundary conditions.
It thus follows from (19) that the ladder Hamiltonian (5) has gap \( \Lambda_{\text{ladder}} = 0.128108 \ldots \) More generally we expect all massive excitations in the ladder eigenspectrum to be multiples of this elementary gap.

On the other hand, the ground state energy of the open XXZ chain scales for large \( N \) as

\[
E_{\text{XXZ}} \sim N e_{\text{XXZ}} + f_{\text{XXZ}}. \tag{21}
\]

The surface free energy contribution is given by \( f_{\text{XXZ}} = g - \frac{1}{4} \Lambda_{\text{XXZ}} \), where \( g \) is a known, though complicated, expression [11]. The ground state energy per site is given by

\[
e_{\text{XXZ}} = \frac{1}{2} \cosh \theta - \sinh \theta \left( 1 + 4 \sum_{n=1}^{\infty} \frac{1}{1 + e^{2n\theta}} \right). \tag{22}
\]

It follows from (19) that the ground state energy per site of the ladder is given by

\[
e_{\text{ladder}} = \frac{5}{8} - 2 \sum_{n=1}^{\infty} \frac{1}{1 + 9^n} = 0.397527 \ldots \tag{23}
\]

The surface free energy relation is \( f_{\text{ladder}} = \frac{3}{8} f_{\text{XXZ}} - \frac{13}{16} \).

In the ferromagnetic regime \( J < 0 \) the groundstate energy of the ladder corresponds to the trivial ferromagnetic groundstate \( \frac{1}{4} (q + q^{-1})(L - 1) \) of the open XXZ chain. It follows from (19) that the groundstate energy of the ferromagnetic ladder is given by \( E_{\text{ladder}} = -\frac{9}{8} (L - 1) \). This value is in agreement with the observation of Albeverio and Fei [2].

4 Discussion

We have shown that a 2-leg ladder Hamiltonian introduced recently by Albeverio and Fei [2] can be made to satisfy the Hecke algebra for \( q = \frac{1}{3} \). As a result we have found an equivalent representation of the eigenspectrum in terms of the spin-\( \frac{1}{2} \) XXZ chain at \( \Delta = -\frac{5}{3} \). We considered open boundary conditions for which the equivalent chain has surface fields \( \pm \frac{2}{3} \) at the ends of the chain. The values of thermodynamic quantities such as the mass gap (20) and ground state energy per site (23) followed from the known XXZ results. Periodic boundary conditions can also be considered by imposing a twist on the periodic XXZ chain, as was done, e.g., for the XXZ chain equivalent to the spin-1 biquadratic model [7].

Albeverio and Fei have noted that like the well-known spin-1 Affleck-Kennedy-Lieb-Tasaki chain [14] the 2-leg ladder Hamiltonian has no free parameter. However, a free parameter has been introduced into the AKLT model via \( q \)-deformation [15, 16]. A form of \( q \)-deformation should also exist for the 2-leg ladder, corresponding to variable \( q \) in the XXZ chain, again equivalent via the Hecke algebra. However, the precise form of the 2-leg Hamiltonian would be very complicated. Nevertheless a phase transition should exist at which the model becomes massless at the critical value \( q = 1 \).

Another related point is that just as Wang’s 2-leg ladders can be generalised to \( n \)-leg ladders [17], we might ask whether there is another faithful representation of Hecke, this
time of size $64^{L-1} \times 64^{L-1}$, corresponding to a 3-leg ladder. Again the hamiltonian would most likely include all possible interactions.

Although we have seen that the 2-leg ladder hamiltonian provides a representation of the Hecke algebra, the equivalence ultimately lies with the XXZ chain, and thus with the Temperley-Lieb algebra. We thus expect that the Hecke representation we have found here is also a quotient of Hecke. On another tack, considering instead the Temperley-Lieb representation (11) with $q = \frac{1}{3}$, we observe that as to be expected only part of the eigenspectrum of the 2-leg ladder is recovered. Similarly if we use Saleur’s Hecke representation [18]

$$U_j = \begin{pmatrix} 0 & q & 1 \\ 1 & q^{-1} & \end{pmatrix} \begin{pmatrix} q^{-1} \\ q + q^{-1} \end{pmatrix}$$  (24)

with $q = \frac{1}{3}$ we recover part of the ladder eigenspectrum. The latter is known to be a quotient of Hecke [19], but more importantly it is free-fermionic, being equivalent to an XX chain [18]. This gives the free-fermionic part of the ladder eigenspectrum. As a point of further interest Saleur’s representation can also be used to give the free-fermionic part of the eigenspectrum of the XXZ chain.

Finally we note that although the 2-leg ladder (3) includes complicated interactions it is nevertheless a model for which exact results can be obtained. It thus provides a useful testbed for numerical calculations on more realistic models.

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