RIEMANN INTEGRABILITY VERSUS WEAK CONTINUITY

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Abstract. In this paper we focus on the relation between Riemann integrability and weak continuity. A Banach space $X$ is said to have the weak Lebesgue property if every Riemann integrable function from $[0,1]$ into $X$ is weakly continuous almost everywhere. We prove that the weak Lebesgue property is stable under $\ell_1$-sums and obtain new examples of Banach spaces with and without this property. Furthermore, we characterize Dunford-Pettis operators in terms of Riemann integrability and provide a quantitative result about the size of the set of $\tau$-continuous non Riemann integrable functions, with $\tau$ a locally convex topology weaker than the norm topology.

1. Introduction

The study of the relation between Riemann integrability and continuity on Banach spaces started on 1927, when Graves showed in [13] the existence of a vector-valued Riemann integrable function not continuous almost everywhere (a.e. for short). Thus, the following problem arises:

Given a Banach space $X$, determine necessary and sufficient conditions for the Riemann integrability of a function $f : [0,1] \to X$.

A Banach space $X$ for which every Riemann integrable function $f : [0,1] \to X$ is continuous a.e. is said to have the Lebesgue property (LP for short). All classical infinite-dimensional Banach spaces except $\ell_1$ do not have the LP. For more details on this topic, we refer the reader to [12], [6], [24], [14] and [19].

Regarding weak continuity, Alexiewicz and Orlicz constructed in 1951 a Riemann integrable function which is not weakly continuous a.e. [2]. A Banach space $X$ is said to have the weak Lebesgue property (WLP for short) if every Riemann integrable function $f : [0,1] \to X$ is weakly continuous a.e. This property was introduced in [27]. Every Banach space with separable dual has the WLP and the example of [2] shows that $C([0,1])$ does not have the WLP. Other spaces with the WLP, such as $L^1([0,1])$, can be found in [5] and [28]. In this paper we focus on the relation between Riemann integrability and weak continuity. In Section 2 we present new results on the WLP. In particular, we prove that the James tree space $JT$ does not have the WLP (Theorem 2.3) and we study when $\ell_p(\Gamma)$ and $c_0(\Gamma)$ have the WLP in the nonseparable case (Theorem 2.8). Moreover, we prove that the WLP is stable

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under \( \ell_1 \)-sums (Theorem 2.13) and we apply this result to obtain that \( C(K)^* \) has the WLP for every compact space \( K \) in the class \( MS \) (Corollary 2.10).

Alexiewicz and Orlicz also provided in their paper an example of a weakly continuous non Riemann integrable function. V. Kadets proved in [13] that a Banach space \( X \) has the Schur property if and only if every weakly continuous function \( f : [0, 1] \to X \) is Riemann integrable. Wang and Yang extended this result in [29] to arbitrary locally convex topologies weaker than the norm topology. In the last section of this paper we give an operator theoretic form of these results that, in particular, provides a positive answer to a question posed by Sofi in [20].

**Terminology and Preliminaries.** All Banach spaces are assumed to be real. In what follows, \( X^* \) denotes the dual of a Banach space \( X \). The weak and weak* topologies of \( X \) and \( X^* \) will be denoted by \( \omega \) and \( \omega^* \) respectively. By an operator we mean a linear continuous mapping between Banach spaces. The Lebesgue measure in \( \mathbb{R} \) is denoted by \( \mu \). The interior of an interval \( I \) will be denoted by \( Int(I) \). The density character \( \text{dens}(T) \) of a topological space \( T \) is the minimal cardinality of a dense subset.

A partition of the interval \([a, b] \subset \mathbb{R} \) is a finite collection of non-overlapping closed subintervals covering \([a, b] \). A tagged partition of the interval \([a, b] \) is a partition \( \{[t_{i-1}, t_i] : 1 \leq i \leq N \} \) of \([a, b] \) together with a set of points \( \{s_i : 1 \leq i \leq N \} \) that satisfy \( s_i \in (t_{i-1}, t_i) \) for each \( i \). Let \( \mathcal{P} = \{(s_i, [t_{i-1}, t_i]) : 1 \leq i \leq N \} \) be a tagged partition of \([a, b] \). For every function \( f : [a, b] \to X \), we denote by \( f(\mathcal{P}) \) the Riemann sum \( \sum_{i=1}^{N} (t_i - t_{i-1}) f(s_i) \). The norm of \( \mathcal{P} \) is \( \|\mathcal{P}\| := \max\{t_i - t_{i-1} : 1 \leq i \leq N \} \). We say that a function \( f : [a, b] \to X \) is Riemann integrable, with integral \( x \in X \), if for every \( \varepsilon > 0 \) there is \( \delta > 0 \) such that \( \|f(\mathcal{P}) - x\| < \varepsilon \) for all tagged partitions \( \mathcal{P} \) of \([a, b] \) with norm less than \( \delta \). In this case, we write \( x = \int_a^b f(t) \, dt \).

The following criterion will be used for proving the existence of the Riemann integral of certain functions:

**Theorem 1.1** ([12]). Let \( f : [0, 1] \to X \). The following statements are equivalent:

1. The function \( f \) is Riemann integrable.
2. For each \( \varepsilon > 0 \) there exists a partition \( \mathcal{P}_\varepsilon \) of \([0, 1] \) with \( \|f(\mathcal{P}_1) - f(\mathcal{P}_2)\| < \varepsilon \) for all tagged partitions \( \mathcal{P}_1 \) and \( \mathcal{P}_2 \) of \([0, 1] \) that have the same intervals as \( \mathcal{P}_\varepsilon \).
3. There is \( x \in X \) such that for every \( \varepsilon > 0 \) there exists a partition \( \mathcal{P}_\varepsilon \) of \([0, 1] \) such that \( \|f(\mathcal{P}) - x\| < \varepsilon \) whenever \( \mathcal{P} \) is a tagged partition of \([0, 1] \) with the same intervals as \( \mathcal{P}_\varepsilon \).

We will also be concerned about cardinality. Throughout this paper, \( \kappa \) denotes the cardinality of the continuum and \( \text{cov}(\mathcal{M}) \) denotes the smallest cardinal such that there exist \( \text{cov}(\mathcal{M}) \) nowhere dense sets in \([0, 1] \) whose union is the interval \([0, 1] \). This cardinal coincides with the smallest cardinal such that there exist \( \text{cov}(\mathcal{M}) \) closed sets in \([0, 1] \) with Lebesgue measure zero whose union does not have Lebesgue measure zero (see [4] Theorem 2.6.14).

A set \( A \subset \mathbb{R} \) is said to be **strongly null** if for every sequence of positive reals \( (\varepsilon_n)_{n=1}^{\infty} \) there exists a sequence of intervals \( (I_n)_{n=1}^{\infty} \) such that \( \mu(I_n) < \varepsilon_n \) for every \( n \in \mathbb{N} \) and \( A \subset \bigcup_{n \in \mathbb{N}} I_n \). We will be interested in the following result:
Theorem 1.2 ([22]). A set $A \subset \mathbb{R}$ is strongly null if and only if for every closed set $F$ with Lebesgue measure zero, the set $A + F = \{a + z : a \in A \text{ and } z \in F\}$ has Lebesgue measure zero.

We will denote by $\text{non}(\mathcal{S}\mathcal{N})$ the smallest cardinal of a non strongly null set. We have that $\aleph_1 \leq \text{cov}(\mathcal{M}) \leq \text{non}(\mathcal{S}\mathcal{N}) \leq \mathfrak{c}$ and, under Martin’s axiom, and therefore under the Continuum Hypothesis, $\text{non}(\mathcal{S}\mathcal{N}) = \text{cov}(\mathcal{M}) = \mathfrak{c}$. Furthermore, if $\mathfrak{b} = \mathfrak{c}$ then $\text{non}(\mathcal{S}\mathcal{N}) = \text{cov}(\mathcal{M})$. However, there exist models of ZFC satisfying $\text{cov}(\mathcal{M}) < \text{non}(\mathcal{S}\mathcal{N})$. For further references and results on this subject we refer the reader to [3].

2. The weak Lebesgue property

It is known that every Banach space with separable dual has the WLP [28]. Next theorem gives a generalization in terms of $\text{cov}(\mathcal{M})$.

Theorem 2.1. Every Banach space $X$ such that $\text{dens}(X^*) < \text{cov}(\mathcal{M})$ has the WLP.

Proof. Let $D = \{x_i^*\}_{i \in I}$ be a dense subset in $X^*$ with $|I| < \text{cov}(\mathcal{M})$ and take $f : [0, 1] \to X$ a Riemann integrable function. Then every function $x_i^*f$ is Riemann integrable. Let $E_i$ be the set of points of discontinuity of $x_i^*f$ for every $i \in I$. Each $E_i$ is a countable union of closed sets with measure zero, so $E = \bigcup_{i \in I} E_i$ has measure zero since $|I| < \text{cov}(\mathcal{M})$. We claim that $f$ is weakly continuous at every point of $E^c$. Let $x^* \in X^*$ and let $M$ be an upper bound for $\{\|f(t)\| : t \in [0, 1]\}$. Fix $\varepsilon > 0$ and $t \in E^c$. Then, there exists $x_i^* \in D$ such that $\|x_i^* - x^*\| < \frac{\varepsilon}{3M}$. Since $t \notin E_i$, there exists a neighbourhood $U$ of $t$ such that $|x_i^*f(t) - x_i^*f(t')| < \frac{\varepsilon}{3}$ for every $t' \in U$. Thus,

$|x^*f(t) - x^*f(t')| \leq |x^*f(t) - x_i^*f(t)| + |x_i^*f(t) - x_i^*f(t')| + |x_i^*f(t') - x^*f(t')| < \varepsilon$

for every $t' \in U$. \hfill \□

Corollary 2.2. Every Banach space with separable dual has the WLP.

The space $\ell_1$ has the WLP because it has the LP. Since every asymptotic $\ell_1$ space has the LP [19], the space $A_T$ defined by Odell in [21] is a separable Banach space with nonseparable dual such that it does not contain an isomorphic copy of $\ell_1$ but it has the WLP (it is asymptotic $\ell_1$). On the other hand, the James tree space $JT$ (see [1] Section 13.4]) is a separable Banach space with nonseparable dual such that it does not contain an isomorphic copy of $\ell_1$ and it does not have the WLP:

Theorem 2.3. The James tree space does not have the WLP.

Proof. We represent the dyadic tree by

$T = \{(n, k) : n = 0, 1, 2, \ldots \text{ and } k = 1, 2, \ldots, 2^n\}$.

A node $(n, k) \in T$ has two immediate successors $(n + 1, 2k - 1)$ and $(n + 1, 2k)$. Then, a segment of $T$ is a finite sequence $\{p_i, \ldots, p_m\}$ such that $p_{j+1}$ is an immediate successor of $p_j$ for every $j = 1, 2, \ldots, m - 1$. The James tree space $JT$ is the completion of $c_{00}(T)$ with the norm

$\|x\| = \sup_{j=1}^t \left( \sum_{(n, k) \in S_j} x(n, k) \right)^2 < \infty,$
where the supremum is taken over all \( l \in \mathbb{N} \) and all sets of pairwise disjoint segments \( S_1, S_2, \ldots, S_t \). Let \( \{e_{(n,k)}\}_{(n,k) \in T} \) be the canonical basis of \( JT \), i.e. \( e_{(n,k)} \) is the characteristic function of \((n,k) \in T\). Define \( f : [0,1] \to JT \) as follows:

\[
f(t) = \begin{cases} 
  e_{(n-1,k)} & \text{if } t = \frac{2k-1}{2^n} \text{ with } n \in \mathbb{N} \text{ and } k = 1,2,\ldots,2^n-1 \\
  0 & \text{in any other case.}
\end{cases}
\]

We claim that \( f \) is Riemann integrable. Fix \( N \in \mathbb{N} \) and let \( \{I_1, I_2, \ldots, I_{2^N-1}\} \) be a family of closed disjoint intervals of \([0,1]\) with

\[
\sum_{1 \leq n \leq 2^N-1} \mu(I_n) \leq \frac{1}{2^n} \quad \text{and} \quad \frac{n}{2^N} \in Int(I_n) \quad \text{for each } 1 \leq n \leq 2^N - 1.
\]

Let \( J_1, J_2, \ldots, J_{2^N} \) be the closed disjoint intervals of \([0,1]\) determined by

\[
[0,1] \setminus \bigcup_{1 \leq n \leq 2^N-1} \text{Int}(I_n).
\]

Then, \( \mu(J_n) \leq \frac{1}{2^n} \) and \( \|\sum_{n=1}^{2^N} a_n f(t_n)\| \leq \sqrt{\sum_{n=1}^{2^N} a_n^2} \) for every \( a_n \in \mathbb{R} \) and every \( t_n \in J_n \) due to the definition of the norm in \( JT \). Thus, any tagged partition \( P_N \) with intervals \( J_1, J_2, \ldots, J_{2^N-1}, J_{2^N} \) and points \( t_1, t'_1, t_2, \ldots, t'_{2^N-1}, t_{2^N} \) satisfies

\[
\|f(P_N)\| \leq \left|\sum_{n=1}^{2^N} \mu(J_n) f(t_{2n-1})\right| + \sum_{n=1}^{2^N-1} \mu(I_n) \leq \sqrt{\sum_{n=1}^{2^N} \mu(J_n)^2} + \frac{1}{2^n} \leq \sum_{n=1}^{2^N} \frac{1}{2^n} + \frac{1}{2^n} \leq \frac{2}{\sqrt{2^N}}.
\]

Hence, \( \|f(P_N)\| \xrightarrow{N \to \infty} 0 \) and \( f \) is Riemann integrable with integral zero.

We show that \( f \) is not weakly continuous at any irrational point \( t \in [0,1] \). Fix an irrational point \( t \in [0,1] \). There exists a sequence of dyadic points \( \left(\frac{2k-1}{2^n}\right)_{j=1}^{\infty} \) converging to \( t \) with \( (n_j - 1, k_j)_{j=1}^{\infty} \) a sequence in \( T \) such that \((n_{j+1} - 1, k_{j+1})\) is an immediate successor of \((n_j - 1, k_j)\) for every \( j \in \mathbb{N} \). Then, \( \sum_{j=1}^{\infty} e^*_{(n_j - 1, k_j)} \) is a functional in \( JT^* \), so the sequence \( f\left(\frac{2k-1}{2^n}\right) = e_{(n_j - 1, k_j)} \) is not weakly null and \( f \) is not weakly continuous at \( t \).

\[ \square \]

**Corollary 2.4 (\textbf{2}).** \( C([0,1]) \) does not have the WLP.

**Proof.** Since every subspace of a Banach space with the WLP has the WLP and every separable Banach space is isometrically isomorphic to a subspace of \( C([0,1]) \), it follows from the previous theorem and the separability of \( JT \) that \( C([0,1]) \) does not have the WLP. \( \square \)

**Corollary 2.5.** Let \( K \) be a compact Hausdorff space.

1. If \( K \) is metrizable, then \( C(K) \) has the WLP if and only if \( K \) is countable.
2. If \( C(K) \) has the WLP then \( K \) is scattered. The converse is not true since \( c_0(\kappa) \) does not have the WLP (Theorem 2.8) and it is isomorphic to a \( C(K) \) space with \( K \) scattered.
Proof. If $K$ is a countable compact metric space, then $C(K)^*$ is separable [10] Theorem 14.24, so $C(K)$ has the WLP (Theorem 2.1). If $K$ is an uncountable compact metric space, then $C(K)$ is isomorphic to $C([0, 1])$ [11] Theorem 4.4.8, so $C(K)$ does not have the WLP (Corollary 2.3). Finally, if $K$ is not scattered, then $C(K)$ has a subspace isomorphic to $C([0, 1])$ (see the proof of [10] Theorem 14.26), so $C(K)$ does not have the WLP. □

Remark 2.6. Let $\{X_i\}_{i \in I}$ be a family of Banach spaces. Define $X := (\bigoplus_{i \in I} X_i)_{\ell_p}$ with $1 < p < \infty$ or $X := (\bigoplus_{i \in I} X_i)_{\ell_0}$. If $f : [0, 1] \to X$ is a bounded function, then its set of points of weak discontinuity is $E = \bigcup_{i \in I} E_i$, where each $E_i$ is the set of points of weak discontinuity of $f_i$ and $f_i$ is the $i'$th coordinate of $f$. Thus, the countable $\ell_p$-sum or $\ell_0$-sum of Banach spaces with the WLP has the WLP. We cannot extend this result to uncountable $\ell_p$-sums or $\ell_0$-sums even when $X_i = \mathbb{R}$ for every $i \in I$ (Theorem 2.6).

Now, we study the WLP for the spaces of the form $c_0(\kappa)$ and $\ell_p(\kappa)$ with $\kappa$ a cardinal.

Theorem 2.7. For any cardinal $\kappa$ and any $1 < p < \infty$, $c_0(\kappa)$ has the WLP if and only if $\ell_p(\kappa)$ has the WLP.

Proof. If $\ell_p(\kappa)$ does not have the WLP, then there exists a Riemann integrable function $f : [0, 1] \to \ell_p(\kappa)$ which is not weakly continuous a.e. If $I : \ell_p(\kappa) \to c_0(\kappa)$ is the canonical inclusion, then the function $I \circ f$ is weakly continuous at a point $t \in [0, 1]$ if and only if $f$ is weakly continuous at $t$ by Remark 2.6. Therefore, $I \circ f$ is not weakly continuous a.e. Since $f$ is an operator, $I \circ f$ is also Riemann integrable. Thus, $c_0(\kappa)$ does not have the WLP.

To prove the other implication, suppose $c_0(\kappa)$ does not have the WLP. Then, there exists a Riemann integrable function $f : [0, 1] \to c_0(\kappa)$ which is not weakly continuous a.e. Let $f_\alpha$ be the $\alpha'$th coordinate of $f$ for every $\alpha \in \kappa$ and $E_\alpha$ be the set of points where $f_\alpha$ has oscillation strictly bigger than $\frac{1}{n}$ for every $n \in \mathbb{N}$. Note that each $E_\alpha$ has Lebesgue measure zero. Since $f$ is not weakly continuous a.e., $\bigcup_{\alpha \in \kappa} \left(\bigcup_{n \in \mathbb{N}} E^n_\alpha\right)$ has not Lebesgue measure zero, so there exists $n \in \mathbb{N}$ such that $\bigcup_{\alpha \in \kappa} E^n_\alpha$ has not Lebesgue measure zero.

Set $F_0 := E^n_0$ and $F_\alpha := E^n_\alpha \setminus \left(\bigcup_{\beta < \alpha} E^n_\beta\right)$ for every $\alpha \in \kappa \setminus \{0\}$. The sets $F_\alpha$ are pairwise disjoint. Let $\chi_{F_\alpha} : [0, 1] \to \{0, 1\}$ be the characteristic function of $F_\alpha$ for every $\alpha < \kappa$ and $g : [0, 1] \to c_0(\kappa)$ the function defined by the formula $g(t) = \sum_{\alpha < \kappa} \chi_{F_\alpha}(t)e_\alpha$ for every $t \in [0, 1]$, where $\{e_\alpha\}_{\alpha < \kappa}$ is the canonical basis of $X$.

Notice that $g$ is not weakly continuous a.e. since each $\chi_{F_\alpha}$ is not continuous at any point of $F_\alpha$ (because $\mu(F_\alpha) = 0$) and $\bigcup_{\alpha < \kappa} F_\alpha = \bigcup_{\alpha < \kappa} E^n_\alpha$ is not Lebesgue null. We claim that $g$ is Riemann integrable. Let $\varepsilon > 0$. Since $f$ is Riemann integrable, there exists a partition $P_\varepsilon$ of $[0, 1]$ such that $\|f(P_1) - f(P_2)\| < \frac{\varepsilon}{n}$ for all tagged partitions $P_1$ and $P_2$ of $[0, 1]$ that have the same intervals as $P_\varepsilon$. For every $\alpha < \kappa$ and any tagged partitions $P_1$ and $P_2$ of $[0, 1]$ that have the same intervals as $P_\varepsilon$,

$$|\chi_{F_\alpha}(P_1) - \chi_{F_\alpha}(P_2)| \leq \sum_{i=1}^{N} \mu(I_i) \leq n|f_\alpha(P_1) - f_\alpha(P_2)| \leq n\|f(P_1) - f(P_2)\| < \varepsilon$$
for suitable tagged partitions $\mathcal{P}_1^*$ and $\mathcal{P}_2^*$ of $[0, 1]$ with the same intervals as $\mathcal{P}_x$, where $I_1, I_2, \ldots, I_N$ are the intervals of $\mathcal{P}_x$ whose interior has non-empty intersection with $E$.

Therefore, $q$ is Riemann integrable. Let $h: [0, 1] \to \ell_p(\kappa)$ be the function given by the formula $h(t) = \sum_{\alpha<\kappa} \chi_{F_\alpha}(t) e_\alpha$. Since the sets $F_\alpha$ are pairwise disjoint, the function $h$ is well-defined. Moreover, $h$ is not weakly continuous a.e. because $F$ is not continuous at $E$. Without loss of generality, we may assume that $E, F$ be the characteristic function of $F$ null set.

Thus, for any tagged partition $\mathcal{P} = \{(s_i, I_i)\}_{i=1}^{M}$ the following inequalities hold:

\[
\|h(\mathcal{P})\| = \left\| \sum_{s_i \in F} \mu(I_i) e_{\phi(s_i)} \right\| = \left\| \sum_{\alpha<\kappa} \mu \left( \bigcup_{\phi(s_i) = \alpha} I_i \right) e_\alpha \right\| = \left( \sum_{\alpha<\kappa} \mu \left( \bigcup_{\phi(s_i) = \alpha} I_i \right) \right)^{\frac{1}{p}} \leq \left( \sum_{\alpha<\kappa} \mu \left( I_i \right) \right)^{\frac{1}{p}} \leq \varepsilon \left( \sum_{\alpha<\kappa} \mu \left( I_i \right) \right)^{\frac{1}{p}} \leq \varepsilon \mu(I).
\]

Therefore, $h$ is Riemann integrable with Riemann integral zero. \hfill \Box

The LP is separably determined \cite{24}. Nevertheless, it follows from the following theorem that the WLP is not separably determined, since every separable infinite-dimensional subspace of $\ell_2(\kappa)$ is isomorphic to $\ell_2$ (which has separable dual).

**Theorem 2.8.** Let $\kappa$ be a cardinal and $X = c_0(\kappa)$ or $X = \ell_p(\kappa)$ with $1 < p < \infty$.

1. If $\kappa < \text{cov}(\mathcal{M})$ then $X$ has the WLP.
2. If $\kappa \geq \text{non}(\mathcal{S}\mathcal{N})$ then $X$ does not have the WLP.

**Proof.** It is enough to prove the result when $X = c_0(\kappa)$ due to Theorem 2.7. Since $c_0(\kappa)^* = \ell_1(\kappa)$ has density character $\kappa$, it follows from Theorem 2.7 that $c_0(\kappa)$ has the WLP if $\kappa < \text{cov}(\mathcal{M})$.

Suppose $\text{non}(\mathcal{S}\mathcal{N}) \leq \kappa \leq \ell$. Due to Theorem 1.2, there exist a closed Lebesgue null set $F$ and a set $E = \{x_\alpha\}_{\alpha<\kappa}$ in $\mathbb{R}$ such that $E + F$ does not have Lebesgue measure zero. Without loss of generality, we may assume that $E, F \subset [0, 1]$ and $(E + F) \cap [0, 1]$ does not have Lebesgue measure zero. Set $F_0 := (x_0 + F) \cap [0, 1]$ and $F_\alpha := (x_\alpha + F) \cap [0, 1] \setminus \left( \bigcup_{\beta<\alpha} F_\beta \right)$ for every $0 < \alpha < \kappa$. Let $\chi_{F_\alpha} : [0, 1] \to \{0, 1\}$ be the characteristic function of $F_\alpha$ for every $\alpha < \kappa$ and $f : [0, 1] \to X$ the function defined by the formula $f(t) = \sum_{\alpha<\kappa} \chi_{F_\alpha}(t) e_\alpha$ for every $t \in [0, 1]$, where $\{e_\alpha\}_{\alpha<\kappa}$ is the canonical basis of $c_0(\kappa)$.

Since the sets $F_\alpha$ are pairwise disjoint, the function $f$ is well-defined. Each $\chi_{F_\alpha}$ is not continuous at $F_\alpha$, since $F_\alpha$ cannot contain an interval of $[0, 1]$. Therefore, $f$ is not weakly continuous a.e. because $\bigcup_{\alpha<\kappa} F_\alpha = (E + F) \cap [0, 1]$ does not have Lebesgue measure zero.
We claim that \( f \) is Riemann integrable. For every \( \alpha < \kappa \) and every tagged partition \( \mathcal{P} = \{(s_i, I_i)\}_{i=1}^N \) we have

\[
\chi_{F_\alpha}(\mathcal{P}) = \sum_{i=1}^N \mu(I_i)\chi_{F_\alpha}(s_i) \leq \sum_{i=1}^N \mu(I_i - x_\alpha)\chi_F(s_i - x_\alpha) = \chi_F(\mathcal{P}')
\]

for a suitable tagged partition \( \mathcal{P}' \) with \( \|\mathcal{P}\| = \|\mathcal{P}'\| \). Since \( F \subset [0,1] \) is a closed Lebesgue measure zero set, the characteristic function \( \chi_F \) is Riemann integrable due to Lebesgue’s Theorem. Then, for every \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that \( \chi_F(\mathcal{P}) < \varepsilon \) for every tagged partition \( \mathcal{P} \) with \( \|\mathcal{P}\| < \delta \). Therefore, for every \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that \( \chi_{F_\alpha}(\mathcal{P}) < \varepsilon \) for all tagged partitions \( \mathcal{P} \) with \( \|\mathcal{P}\| < \delta \) and for every \( \alpha < \kappa \). Thus, \( f \) is Riemann integrable since \( \|f(\mathcal{P})\| = \sup_{\alpha < \kappa} \chi_{F_\alpha}(\mathcal{P}) < \varepsilon \) for every tagged partition \( \mathcal{P} \) of \([0,1] \) with \( \|\mathcal{P}\| < \delta \). \( \square \)

The facts that the countable \( \ell_1 \)-sum of spaces with the WLP has the WLP (Theorem 2.11) and that \( L^1(\lambda) \) has the WLP if \( \text{dens}(L^1(\lambda)) < \text{cov}(\mathcal{M}) \) (Theorem 2.12) will be a consequence of the following lemma.

**Lemma 2.9.** Let \((\Omega, \Sigma, \lambda)\) be a probability space and \( \mathfrak{A} = \{P_A : A \in \Sigma\} \) a family of operators on a Banach space \( X \) such that

1. \( P_A + P_{\Omega \setminus A} = P_\Omega = \text{id}_X \) for every \( A \in \Sigma \);
2. \( \|P_A(x)\| \leq \|x\| \) for every \( x \in X \) and every \( A \in \Sigma \);
3. \( \|P_A(x)\| + \|P_B(x')\| \leq \max\{\|x+x'\|, \|x-x'\|\} \) for every \( x, x' \in X \) whenever \( A \cap B = \emptyset \);
4. \( \lim_{\lambda(A) \to 0} \|P_A(x)\| = 0 \) for every \( x \in X \).

Let \( f : [0,1] \to X \) be a Riemann integrable function. Then there is a measurable set \( E \subseteq [0,1] \) with \( \mu(E) = 1 \) such that, for every sequence \((t_n)_{n=1}^\infty \) in \([0,1] \) converging to some \( t \in E \), the set \( \{f(t_n) : n \in \mathbb{N}\} \) is \( \mathfrak{A} \)-uniformly integrable, in the sense that

\[
\lim_{\lambda(A) \to 0} \sup_{n \in \mathbb{N}} \|P_A(f(t_n))\| = 0.
\]

**Proof.** The proof is similar to that of [3] Lemma 2.3 and [28] Lemma 3. Fix \( \beta > 0 \) and denote by \( E_\beta \) the set of points \( t \in [0,1] \) such that for every \( \delta > 0 \) there exist \( t' \in [0,1] \) with \( |t' - t| < \delta \) and a set \( A \in \Sigma \) with \( \lambda(A) < \delta \) such that

\[
\|P_A(f(t) - f(t'))\| > \beta.
\]

Let \( \mu^* \) be the Lebesgue outer measure in \([0,1] \). We show that \( \mu^*(E_\beta) = 0 \) with a proof by contradiction. Suppose \( \mu^*(E_\beta) > 0 \). Since \( f \) is Riemann integrable, we can choose a partition \( \mathcal{P} = \{J_1, \ldots, J_m\} \) of \([0,1] \) such that

\[
\left\| \sum_{j=1}^m \mu(J_j)(f(\xi_j) - f(\xi'_j)) \right\| < \beta \mu^*(E_\beta)
\]

for all choices \( \xi_j, \xi'_j \in J_j, 1 \leq j \leq m \). Let \( S = \{j \in \{1, \ldots, m\} : I_j \cap E_\beta \neq \emptyset\} \), where \( I_j = \text{Int}(J_j) \) for each \( j = 1, \ldots, m \). Thus,

\[
\sum_{j \in S} \mu^*(I_j \cap E_\beta) = \mu^*(E_\beta).
\]

It is not restrictive to suppose \( S = \{1, \ldots, n\} \) for some \( 1 \leq n \leq m \).
Because of the definition of $E_\beta$ and $I_1$, there exist points $t_1 \in I_1 \cap E_\beta$ and $t_1' \in I_1$ such that $\|f(t_1) - f(t_1')\| \geq \|P_A(f(t_1) - f(t_1'))\| > \beta$ for some $A \in \Sigma$, hence $\|\mu(I_1)(f(t_1) - f(t_1'))\| > \beta \mu(I_1)$.

Fix $1 \leq k < n$ and assume that we have already chosen points $t_j, t_j' \in I_j$ for all $1 \leq j \leq k$ with the property that

$$\left\| \sum_{j=1}^{k} \mu(I_j)(f(t_j) - f(t_j')) \right\| > \beta \left( \sum_{j=1}^{k} \mu(I_j) \right).$$

Define $x := \sum_{j=1}^{k} \mu(I_j)(f(t_j) - f(t_j')) \in \mathcal{X}$ and

$$\alpha := \|x\| - \beta \left( \sum_{j=1}^{k} \mu(I_j) \right) > 0.$$

Due to (4), we can choose $\delta > 0$ such that $\|P_A(x)\| < \alpha$ whenever $A \in \Sigma$ satisfies $\lambda(A) < \delta$. Take $t_{k+1}, t_{k+1}' \in I_{k+1}$ and a set $A \in \Sigma$ with $\lambda(A) < \delta$ such that $\|P_A(f(t_{k+1}) - f(t_{k+1}'))\| > \beta$, so $y := \mu(I_{k+1})(f(t_{k+1}) - f(t_{k+1}'))$ satisfies $\|P_A(y)\| > \beta \mu(I_{k+1})$.

By the choice of $A$, (1) and (3), we also have (interchanging the role of $t_{k+1}$ and $t_{k+1}'$ if necessary)

$$\left\| \sum_{j=1}^{k+1} \mu(I_j)(f(t_j) - f(t_j')) \right\| \geq \|P_A(y)\| + \|P_A(X)\| \geq \|P_A(y)\| + \|x\| - \|P_A(x)\| >$$

$$> \beta \mu(I_{k+1}) + \alpha + \beta \sum_{j=1}^{k} \mu(I_j) - \|P_A(x)\| > \beta \sum_{j=1}^{k+1} \mu(I_j).$$

Thus, there exist $t_j, t_j' \in I_j$ for all $1 \leq j \leq n$ such that

$$\left\| \sum_{j=1}^{n} \mu(I_j)(f(t_j) - f(t_j')) \right\| > \beta \left( \sum_{j=1}^{n} \mu(I_j) \right) \geq \beta \mu^*(E_\beta),$$

which contradicts the inequality (2). So we can conclude that $\mu^*(E_\beta) = 0$.

Therefore, $E := [0,1] \setminus \bigcup_{m \in \mathbb{N}} E_{\frac{1}{m}}$ is measurable with $\mu(E) = 1$. Fix $t \in E$ and $m \in \mathbb{N}$. Since $t \not\in E_{\frac{1}{m}}$, there exists $\delta_m > 0$ such that for every $t' \in [0,1]$ with $|t' - t| < \delta_m$ and every set $A \in \Sigma$ with $\lambda(A) < \delta_m$,

$$\|P_A(f(t) - f(t'))\| \leq \frac{1}{m}.$$ 

Thus, for every $m \in \mathbb{N}$, every sequence $(t_n)_{n=1}^\infty$ converging to $t$ and every $A \in \Sigma$ with $\lambda(A) < \delta_m$,

$$\|P_A(f(t_n))\| \leq \|P_A(f(t))\| + \frac{1}{m}$$

for $n$ big enough depending only on $m$.

Now the conclusion follows from (4). $\square$

Let $\{X_j\}_{j \in \Gamma}$ be a family of Banach spaces. We denote by $\pi_j : (\bigoplus_{i \in \Gamma} X_i) \to X_j$ the canonical projection onto $X_j$ for each $j \in \Gamma$.

We will need the following property of $\ell_1$-sums and the space $L_1(\lambda)$ for Theorems 2.11 and 2.12.
Lemma 2.10. Let \((\Omega, \Sigma, \lambda)\) be a probability space and \(\{X_i\}_{i \in \Gamma}\) a family of Banach spaces. Then:

1. \(\max\{\|x + y\|, \|x - y\|\} \geq \sum_{i \in A} \|\pi_i(x)\| + \sum_{i \in B} \|\pi_i(y)\|\) for every vectors \(x, y \in (\bigoplus_{i \in \Gamma} X_i)_{\ell_1}\) and any disjoint sets \(A, B \subset \Gamma\).
2. \(\max\{\|f + g\|, \|f - g\|\} \geq \int_A |f|d\lambda + \int_B |g|d\lambda\) for every \(f, g \in L_1(\lambda)\) and any disjoint sets \(A, B \subset \Sigma\).

Proof. The second part is essentially Lemma 2 of [28]. The proof of the first part is analogous and we include it for the sake of completeness. Let \(A = \{a_n : n \in \mathbb{N}\}\) and \(B = \{b_n : n \in \mathbb{N}\}\) are countable subsets. Consider the functionals \(x^*, y^* \in (\bigoplus_{i \in \Gamma} X_i)_{\ell_1}\) defined by \(x^*(u) = \sum_{i \in A} x_i^*(\pi_i(u))\) and \(y^*(u) = \sum_{i \in B} y_i^*(\pi_i(u))\) for every \(u \in (\bigoplus_{i \in \Gamma} X_i)_{\ell_1}\), where each \(x_i^*, y_i^* \in X_i^*\) satisfies \(\pi_i = \pi_i\) and \(x_i^*(\pi_i(x)) = \|\pi_i(x)\|\) if \(i = a_n\) and \(y_i^*(\pi_i(y)) = \|\pi_i(y)\|\) if \(i = b_n\). Then, since \(A, B\) are disjoint, \(\|x^* + y^*\| \leq \|x^* - y^*\| = 1\). Therefore,

\[
\|x + y\| + \|x - y\| \geq (\|x + x^* + y^*\| + \|x - y^*\|) = 2(\|x^*\| + \|y^*\|) = 2\left(\sum_{i \in A} \pi_i(x)\right) + 2\left(\sum_{i \in B} \pi_i(y)\right),
\]

so \(\max\{\|x + y\|, \|x - y\|\} \geq \sum_{i \in A} \|\pi_i(x)\| + \sum_{i \in B} \|\pi_i(y)\|\).

\(\square\)

Theorem 2.11. Let \(\{X_i\}_{i \in \mathbb{N}}\) be Banach spaces with the WLP. Then the space \(X := (\bigoplus_{i \in \mathbb{N}} X_i)_{\ell_1}\) has the WLP.

Proof. We are going to apply Lemma 2.9. Take \(\Omega := \mathbb{N}, \Sigma := \mathcal{P}(\mathbb{N})\) the power set of \(\mathbb{N}\), \(\lambda(A) := \sum_{n \in A} 2^{-n}\) and \(\Psi = \{P_A : A \in \Sigma\}\) with

\[
\pi_i(P_A(x)) = \begin{cases} 
\pi_i(x) & \text{if } i \in A \\
0 & \text{if } i \notin A
\end{cases}
\]

for every \(A \in \Sigma\) and every \(x \in X\). Property (3) of Lemma 2.9 is Lemma 2.10 and property (4) holds because if \(\lambda(A) < \frac{1}{k}\), then \(A \subset \{n, n + 1, \ldots\}\), so

\[
\|P_A(x)\| = \sum_{i \in A} \|\pi_i(x)\| \leq \sum_{i \geq n} \|\pi_i(x)\|
\]

for every \(x \in X\). Therefore, we can apply Lemma 2.9 so there exists a measurable set \(E \subset [0, 1]\) with \(\mu(E) = 1\) such that for every sequence \(\{t_n\}_{n=1}^{\infty}\) in \([0, 1]\) converging to some \(t \in E\) the set \(\{f(t_n) : n \in \mathbb{N}\}\) is \(\Psi\)-uniformly integrable. We can assume that, for each \(i \in \mathbb{N}\), the map \(t \mapsto \pi_i(f(t))\) is weakly continuous at each point of \(E\) because each \(X_i\) has the WLP.

It is a well known fact that a sequence \(\{x_n\}_{n=1}^{\infty}\) in \(X\) converges weakly to \(x \in X\) if and only if it satisfies the following two conditions:

1. \(\pi_i(x_n) \to \pi_i(x)\) weakly in \(X_i\) for every \(i \in \mathbb{N}\);
2. for every \(\varepsilon > 0\) there is a finite set \(J \subseteq \mathbb{N}\) such that \(\sup_{n \in \mathbb{N}} \|P_{\mathbb{N}\setminus J}(x_n)\| \leq \varepsilon\).

Since \(\Psi\)-uniform integrability is equivalent to (ii), it follows that \(f\) is weakly continuous at each point of \(E\).

A similar idea to that of Theorem 2.11 let us prove the following theorem, which improves [28] Theorem 5 and [5] Proposition 2.10.

Theorem 2.12. Let \((\Omega, \Sigma, \lambda)\) be a probability space.
Let $\mu \in \mathcal{M}$. If $\mu = \frac{1}{2} \mathbb{N}$ then $\ell^1(\mathbb{N})$ has the WLP.

If $\mu$ is any other measure then $\mu$ is isomorphic to a countable $\ell^1$.

Proof. Fix a Riemann integrable function $f : [0, 1] \to L^1(\mu)$. Take $P_A(x) := x\chi_A$ for every $A \in \Sigma$ and every $x \in L^1(\mu)$. The family of operators $\{P_A : A \in \Sigma\}$ fulfills the requirements of Lemma 2.11 (bear in mind Lemma 2.10). Then $\mathfrak{P}$-uniform integrability is the usual uniform integrability and therefore a set is bounded and $\mathfrak{P}$-uniformly integrable if and only if it is relatively weakly compact due to Dunford’s Theorem (see [1] Theorem 5.2.9). Lemma 2.9 ensures that there exist measurable $E \subset [0, 1]$ with $\mu(E) = 1$ such that for every sequence $(t_n)_{n=1}^{\infty}$ in $[0, 1]$ converging to some $t \in E$, the set $\{f(t_n) : n \in \mathbb{N}\}$ is relatively weakly compact.

Let $C \subset \Sigma$ be a dense family of $\lambda$-measurable sets, i.e. such that

$$\inf_{C \in \mathcal{C}} \lambda(A \triangle C) = 0$$

for every $A \in \Sigma$.

Let $(h_n)_{n=1}^{\infty}$ be a relatively weakly compact sequence in $L^1(\lambda)$ and $h \in L^1(\lambda)$. Since $C$ is a dense family of $\lambda$-measurable sets, if $\int_{C_n} h_n \, d\mu \to \int_{C_n} h \, d\mu$ for every $C_n \in C$, then $h = \omega-lim n h_n$.

Suppose $\mu(\{l_n\}) = 0$. Then $\mathcal{C}$ can be taken such that $|\mathcal{C}| < \mu(\mathcal{C})$. Therefore, we can assume that, for each $C \in \mathcal{C}$, the Riemann integrable map $t \mapsto \int_{C_n} f(t) \, d\lambda$ is continuous at each point of $E$. Then, for every sequence $(t_n)_{n=1}^{\infty}$ in $[0, 1]$ converging to a point $t \in E$, we have $f(t) = \omega-lim f(t_n)$.

Now suppose $\upsilon = \mu(\{l_n\}) > 0$. Due to Maharam’s Theorem (see [16] p. 127, Theorem 9), $L^1(\lambda)$ contains an isometric copy of $L^1(\mu)$, where $\mu$ is the usual product probability measure on $[0, 1]^\nu$. Since $L^1(\mu)$ contains an isomorphic copy of $\ell_2(\nu)$ (see [16] p. 128, Theorem 12) and $\ell_2(\nu)$ does not have the WLP (Theorem 2.8), we conclude that $L^1(\lambda)$ does not have the WLP.

Theorem 2.11 can be extended to arbitrary $\ell_1$-sums:

**Theorem 2.13.** The arbitrary $\ell_1$-sum of a family of Banach spaces with the WLP has the WLP.

Proof. The proof uses some ideas of [18]. Let $f : [0, 1] \to X := (\bigoplus_{i \in \Gamma} X_i)_{\ell_1}$ be a Riemann integrable function, where $X_i$ is a family of Banach spaces with the WLP. For each $J \subset \Gamma$, we denote by $P_J : X \to X$ the function defined by $P_J(x) = x_i$ for every $x \in X$. Then, for every sequence $(r_n)_{n=1}^{\infty}$ in $[0, 1]$ converging to a point $t \in E$, we have $f(t) = \omega-lim f(r_n)$.

Therefore, we can assume that $\int_0^1 f(t) \, dt = 0$ and that $f$ is null over a dense set. Let

$$A_n^J := \{t \in [0, 1] : \|P_J(f(t))\| < \frac{1}{n}\}$$

for each $n \in \mathbb{N}$ and each subset $J \subset \Gamma$. If $J_1 \subset J_2 \subset \Gamma$, then $A_n^{J_2} \subset A_n^{J_1}$.

**Claim:** For every $n \in \mathbb{N}$ there exists a countable set $J \subset \Gamma$ with $\mu(A_n^J) = 0$.

Suppose this is not the case. Then, there exist $n \in \mathbb{N}$ and $\delta > 0$ with $\mu(A_n^J) > \delta$
for every countable subset \( J \subset \Gamma \) (if for every \( m \in \mathbb{N} \) we can take a countable set \( J_m \subset \Gamma \) with \( \mu \left( \overline{A^m_n} \right) < \frac{1}{m} \), then \( J = \bigcup_{m \in \mathbb{N}} J_m \) verifies \( \mu \left( \overline{A^m_n} \right) = 0 \). Let \( \mathcal{P} = \{ I_1, I_2, \ldots, I_N \} \) be a partition of \([0, 1]\) such that

\[
\left\| \sum_{j=1}^{N} \mu(I_j)(f(\xi_j) - f(\xi'_j)) \right\| < \frac{\delta}{n} \quad \text{for all choices } \xi_j, \xi'_j \in I_j, 1 \leq j \leq N.
\]

Let \( J \subset \Gamma \) be a countable subset. Since \( \sum_{j=1}^{N} \mu \left( I_j \cap \overline{A^m_n} \right) = \mu \left( \overline{A^m_n} \right) > \delta \) and \( f \) is null over a dense set, we can suppose that there exist \( \xi_1 \in \text{Int}(I_1) \cap A^m_n \) and \( \xi'_1 \in I_1 \) such that \( \| \mu(I_1)(f(\xi_1) - f(\xi'_1)) \| \geq \frac{1}{n} \mu(I_1) \). Let \( J_1 = \text{supp}(f(\xi_1) \cup \text{supp}(f(\xi'_1))) \). By (4) we have \( \mu(I_1) < \delta < \sum_{j=1}^{N} \mu \left( I_j \cap \overline{A^m_n} \right) \) and so it is not restrictive to suppose \( \text{Int}(I_2) \cap \overline{A^m_n} \neq \emptyset \). Thus, due to Lemma 2.10 we can choose \( \xi_2, \xi'_2 \in I_2 \) such that

\[
\| \mu(I_1)(f(\xi_1) - f(\xi'_1)) + \mu(I_2)(f(\xi_2) - f(\xi'_2)) \| \geq \frac{1}{n} \mu(I_1) + \mu(I_2).
\]

Fix \( 1 \leq k < N \) and assume that we have already chosen points \( \xi_j, \xi'_j \in I_j \) for all \( 1 \leq j \leq k \) with the property that

\[
\left\| \sum_{j=1}^{k} \mu(I_j)(f(\xi_j) - f(\xi'_j)) \right\| \geq \frac{1}{n} \left( \sum_{j=1}^{k} \mu(I_j) \right).
\]

Set \( J_k := \bigcup_{j=1}^{k} \text{supp}(f(\xi_j) \cup \text{supp}(f(\xi'_j))) \), which is countable. By (4) we have \( \sum_{j=1}^{k} \mu(I_j) < \delta < \sum_{j=1}^{N} \mu \left( I_j \cap \overline{A^m_n} \right) \), hence it is not restrictive to suppose that \( \text{Int}(I_{k+1}) \cap \overline{A^m_n} \neq \emptyset \) and therefore that there exist points \( \xi_{k+1}, \xi'_{k+1} \in I_{k+1} \) such that

\[
\left\| \sum_{j=1}^{k+1} \mu(I_j)(f(\xi_j) - f(\xi'_j)) \right\| \geq \frac{1}{n} \left( \sum_{j=1}^{k+1} \mu(I_j) \right).
\]

Since \( \sum_{j=1}^{N} \mu(I_j) = 1 > \delta \), it follows that there exist \( \xi_j, \xi'_j \in I_j \) for every \( 1 \leq j \leq N \) such that

\[
\left\| \sum_{j=1}^{N} \mu(I_j)(f(\xi_j) - f(\xi'_j)) \right\| \geq \frac{\delta}{n}.
\]

But this is a contradiction with (4). Therefore, the Claim is proved.

Thus, for every \( n \in \mathbb{N} \) there exists a countable set \( J_n \) such that \( \mu \left( \overline{A^m_n} \right) = 0 \). Fix \( J := \bigcup_{n \in \mathbb{N}} J_n \). Theorem 2.11 guarantees the existence of a set \( F \subset [0, 1] \) of measure one such that \( P_f(f) \) is weakly continuous at every point of \( F \). Let \( E = F \setminus \bigcup_{n \in \mathbb{N}} \overline{A^m_n} \).

Then, \( \mu(E) = 1 \), \( f = P_{f}(f) + P_f(f) \), \( P_f(f) \) is weakly continuous at each point of \( E \) and \( P_{f_r}(f) \) is norm continuous at each point of \( E \) (if \( t_n \to t \in E \), then, for every \( m \in \mathbb{N} \), \( t_n \notin A^m_n \) for \( n \) big enough so \( \| P_{f_r}(f)(t_n) \| < \frac{1}{m} \)).

\[\square\]

**Corollary 2.14** (24 [20]). \( \ell_1(\kappa) \) has the LP for any cardinal \( \kappa \).

**Proof.** Since \( \ell_1(\kappa) \) has the Schur property, \( \ell_1(\kappa) \) has the LP if and only it has the WLP. Therefore, the conclusion follows from Theorem 2.13. \[\square\]
As an application of Corollary 2.13, we also obtain the following result:

**Corollary 2.15.** Let $K$ be a compact Hausdorff space. Then, $C(K)^*$ has the WLP if $\text{dens}(L^1(\lambda)) < \text{cov}(\mathcal{M})$ for every regular Borel probability $\lambda$ on $K$.

**Proof.** For every compact Hausdorff space $K$, the Banach space $C(K)^*$ is isometric to a $\ell_1$-sum of $L^1(\lambda)$ spaces, where each $\lambda$ is a regular Borel probability measure on $K$ (see the proof of [11, Proposition 4.3.8]). Thus, $C(K)^*$ has the WLP if each space $L^1(\lambda)$ has the WLP, due to Theorem 2.13. Hence, the result follows from Theorem 2.12. $\square$

**Corollary 2.16.** If $K$ is a compact Hausdorff space in the class $MS$ (i.e., $L^1(\lambda)$ is separable for every regular Borel probability on $K$), then $C(K)^*$ has the WLP.

Some classes of compact spaces in the class $MS$ are metric compacta, Eberlein compacta, Radon-Nikodym compacta, Rosenthal compacta and scattered compacta. For more details on this class, we refer the reader to [8], [17], and [25].

The LP is a three-space property, i.e., if $X$ is a Banach space and $Y$ is a subspace of $X$ such that $Y$ and $X/Y$ have the LP, then $X$ has the LP [24, Proposition 1.19]. This result follows from Michael’s Selection Theorem. However, as far as we are concerned, it is not known whether the WLP is a three-space property. We have a positive result in the following case:

**Theorem 2.17.** Let $X$ be a Banach space and $Y$ a subspace of $X$. If $Y$ is reflexive, $\text{dens}(Y) < \text{cov}(\mathcal{M})$ and $X/Y$ has the WLP, then $X$ has the WLP.

**Proof.** Let $Q : X \to X/Y$ be the quotient operator and $\phi : X/Y \to X$ be a norm-norm continuous map such that $Q\phi = 1_{X/Y}$ given by Michael’s Selection Theorem (see [10, Section 7.6]). Let $f : [0, 1] \to X$ be a Riemann integrable function. Then, since $Qf$ is Riemann integrable and $X/Y$ has the WLP, there exists a set $E \subset [0, 1]$ with $\mu(E) = 1$ such that $Qf$ is weakly continuous at every point of $E$. Set

\[(5) \quad C = \{x \in X : \exists (t_n)_{n=1}^{\infty} \text{ converging to some } t \in E \text{ with } x = \omega\text{-lim } f(t_n)\}.
\]

First we are going to see that $\text{dens}(C) < \text{cov}(\mathcal{M})$. Let $x \in C$ and $(t_n)_{n=1}^{\infty}$ as in (5). Then $Qx = \omega\text{-lim } Qf(t_n) = Qf(t)$. Therefore, $x = \phi(Qx) + (x - \phi(Qx))$ with $\phi(Qx) \in \phi(Qf(E))$ and $x - \phi(Qx) \in Y$. Notice that $\phi(Qf(E))$ is separable because of the $\omega$-separability of $Qf(E)$ and Mazur’s Lemma. Thus, $C \subset \phi(Qf(E)) + Y$ satisfies $\text{dens}(C) < \text{cov}(\mathcal{M})$.

Let $\{x^*_\alpha\}_{\alpha \in \Gamma} \subset X^*$ be a set separating points of $C$ with $|\Gamma| < \text{cov}(\mathcal{M})$. Set $E_0 \subset E$ with $\mu(E_0) = 1$ such that $x^*_\alpha \circ f$ is continuous at every point of $E_0$ for every $\alpha \in \Gamma$. Notice that this can be done because the set of discontinuity points of each $x^*_\alpha \circ f$ is an $F_\sigma$ Lebesgue null set and $|\Gamma| < \text{cov}(\mathcal{M})$. We claim that $f$ is weakly continuous at each point of $E_0$. Let $t \in E_0$ and $(t_n)_{n=1}^{\infty}$ be a sequence converging to $t$. Since $Qf(t) = \omega\text{-lim } Qf(t_n)$, the set $\{Qf(t_n) : n \in \mathbb{N}\}$ is relatively weakly compact in $X/Y$. From the reflexivity of $Y$, it follows that $Q$ is a Tauberian operator, so $\{f(t_n) : n \in \mathbb{N}\}$ is relatively weakly compact in $X$ (see [11, Theorem 2.1.5 and Corollary 2.2.5]). Therefore, it is enough to prove the uniqueness of the limit of the subsequences of $(f(t_n))_{n=1}^{\infty}$. Let $x = \omega\text{-lim } \lim_{k} f(t_{n_k})$. Then, $x, f(t) \in C$ and $x^*_\alpha(x) = \lim_{k} x^*_\alpha(f(t_{n_k})) = x^*_\alpha(f(t))$ for every $\alpha \in \Gamma$, so $x = f(t)$. $\square$
3. Weak continuity does not imply integrability

It is not true that every weakly continuous function is Riemann integrable [2]. In fact, V. Kadets proved the following theorem:

**Theorem 3.1** ([14]). If $X$ is a Banach space without the Schur property, then there is a weakly continuous function $f : [0, 1] \to X$ which is not Riemann integrable.

The proof of the previous theorem together with Josefson-Nissenzweig Theorem (see [2] Chapter XII) gives the following corollary:

**Corollary 3.2.** Given an infinite-dimensional Banach space $X$, there always exists a weak* continuous function $f : [0, 1] \to X^*$ which is not Riemann integrable.

In [29], Wang and Yang extend the previous result to a general locally convex topology weaker than the norm topology. In this section, we generalize these results in Theorem 3.4.

Following the terminology used in [9], we say that a subset $M$ of a Banach space is spaceable if $M \cup \{0\}$ contains a closed infinite-dimensional subspace.

We start with the definitions of $\tau$-Dunford-Pettis operator and the $\tau$-Schur property, that coincide with the classical definitions of Dunford-Pettis or completely continuous operator and the Schur property when $\tau$ is the weak topology.

**Definition 3.3.** Let $X$ and $Y$ be Banach spaces and $\tau$ a locally convex topology on $X$ weaker than the norm topology. An operator $T : X \to Y$ is said to be $\tau$-Dunford-Pettis ($\tau$-DP for short) if it carries bounded $\tau$-null sequences to norm null sequences. A Banach space $X$ is said to have the $\tau$-Schur property if the identity operator $I : X \to X$ is $\tau$-DP.

**Theorem 3.4.** Let $X$ and $Y$ be Banach spaces and $\tau$ be a locally convex topology on $X$ weaker than the norm topology. If $T : X \to Y$ is an operator which is not $\tau$-DP, then the family of all bounded $\tau$-continuous functions $f : [0, 1] \to X$ such that $Tf$ is not Riemann integrable is spaceable in $\ell_\infty([0, 1], X)$, the space of all bounded functions from $[0, 1]$ to $X$ with the supremum norm.

**Proof.** The proof uses ideas from [14]. Since $T$ is not $\tau$-DP, we can take a bounded sequence $(x_n)_{n=1}^\infty$ that is $\tau$-convergent to zero such that $\|Tx_n\| = 1$ for all $n \in \mathbb{N}$.

Let $K \subset [0, 1]$ be a copy of the Cantor set constructed by removing from $[0, 1]$ an open interval $I_1$ in the middle of $[0, 1]$ and removing open intervals $I_1^n, I_2^n, \ldots I_{2^n}$ from the middles of the remaining intervals in each step. Suppose that the removed intervals are so small that $\mu(K) > \frac{1}{2}$. Let $C_a([0, 1])$ be the closed subspace of $C([0, 1])$ consisting of all continuous functions $g : [0, 1] \to \mathbb{R}$ antisymmetric with respect to the axe $x = \frac{1}{2}$ and with $g(0) = g(1) = 0$. For every $g \in C_a([0, 1])$ and every open interval $I = (a, b)$ in $[0, 1]$, we define the functions $g_I : [0, 1] \to \mathbb{R}$ and $f_g : [0, 1] \to X$ as follows

$$g_I(t) = \begin{cases} 0 & \text{if } t \notin (a, b), \\ g(\frac{t-a}{b-a}) & \text{if } t \in [a, b]. \end{cases}$$

$$f_g(t) = \begin{cases} 0 & \text{if } t \in K, \\ g_I(t)x_n & \text{if } t \in I_{2^n}. \end{cases}$$

The function $\phi : C_a([0, 1]) \to \ell_\infty([0, 1], X)$ given by the formula $\phi(g) := f_g$ for every $g \in C_a([0, 1])$ is a linear map and satisfies $\|\phi(g)\| = (\sup_n \|x_n\|)\|g\|$ for every
g ∈ C_a([0, 1]). Therefore, φ is a multiple of an isometry. Thus, V := φ(C_a([0, 1]) is an infinite-dimensional closed subspace of L_∞([0, 1], X).

We are going to check that each function f_g ≠ 0 is τ-continuous but T f_g is not Riemann integrable. Since g is continuous, g(0) = g(1) = 0 and x_n \to 0, f_g is τ-continuous. Suppose T f_g is Riemann integrable. Then,

\[ y^* \left( \int_0^1 T f_g(t) dt \right) = \int_0^1 y^* T f_g(t) dt = \sum_{k,m} y^*(T x_n) \int_{I_k^m} g f_g(t) dt = 0 \]

for each y^* ∈ Y^*. The only possible value for the Riemann integral of T f_g is 0 due to the above equality. Choose a partition P = \{A_j \} of [0, 1]. Let A = \{j : 1 \leq j \leq N, \text{ Int } J_j \cap K \neq \emptyset \}. We can take m ∈ N such that if j ∈ A then J_j contains some interval I_k^m. Hence, if j ∈ A, there is t_j ∈ J_j such that f_g(t_j) = ||g||x_m. If j /∈ A, then we pick any t_j ∈ Int J_j. From the inequality \( \sum_{j \in A} \mu(J_j) \geq \mu(K) > \frac{3}{4} \), we deduce

\[ \left\| \sum_{j=1}^N \mu(J_j) T f_g(t_j) \right\| = \left\| \sum_{j \in A} \mu(J_j) T f_g(t_j) + \sum_{j \notin A} \mu(J_j) T f_g(t_j) \right\| \geq \]

\[ \left\| \sum_{j \in A} \mu(J_j) T f_g(t_j) \right\| - \left\| \sum_{j \notin A} \mu(J_j) T f_g(t_j) \right\| > \frac{2}{3} ||g|| - \frac{1}{3} \sup_{t \in [0,1]} \|T f_g(t)\| = \frac{1}{3} ||g||. \]

Then, T f_g is Riemann integrable if and only if g = 0 if and only if f_g = 0. □

The next corollary gives an affirmative answer to a question posed by Sofi in [26].

**Corollary 3.5.** Given an infinite-dimensional Banach space X, the set of all weak* continuous functions f : [0, 1] → X* which are not Riemann integrable is spaceable in L_∞([0, 1], X*).

**Proof.** X* is not ω*-Schur for any infinite-dimensional Banach space X due to the Josefson-Nissenzweig Theorem. Thus, the conclusion follows from Theorem 3.4. □

Given a Banach space X, a function f : [0, 1] → X is said to be scalarly Riemann integrable if every composition x^* f with x^* ∈ X* is Riemann integrable.

We can also characterize Dunford-Pettis operators thanks to Theorem 3.4. The equivalence (1) ⇔ (3) in the following corollary was mentioned without proof in [23].

**Corollary 3.6.** Let X and Y be Banach spaces and T : X → Y be an operator. The following statements are equivalent:

1. T is Dunford-Pettis.
2. T f is Riemann integrable for every ω-continuous function f : [0, 1] → X.
3. T f is Riemann integrable for every scalarly Riemann integrable function f : [0, 1] → X.

**Proof.** (2) ⇒ (1) is a consequence of Theorem 3.4. Since every ω-continuous function f : [0, 1] → X is scalarly Riemann integrable, (3) implies (2). Therefore, it remains to prove (1) ⇒ (3). Suppose T is Dunford-Pettis and fix (P_n)_{n=1}^∞ a sequence of tagged partitions of [0, 1] with \( \|P_n\| \to 0 \). Let f : [0, 1] → X be a scalarly Riemann integrable function. Then, x^* f(P_n) \to \int_0^1 x^* f(t) dt for every
Thus, \( f(\mathcal{P}_n) \) is a \( \omega \)-Cauchy sequence in \( X \), so \( T f(\mathcal{P}_n) \) is norm convergent to some \( y \in Y \). The limit \( y \) does not depend on the sequence of tagged partitions, since if \( (\mathcal{P}'_n)_{n=1}^\infty \) is any other sequence of tagged partitions with \( \|\mathcal{P}'_n\| \xrightarrow{n \to \infty} 0 \), then \( f(\mathcal{P}_n) - f(\mathcal{P}'_n) \) is weakly null and this in turn implies that \( \|T f(\mathcal{P}_n) - T f(\mathcal{P}'_n)\| \xrightarrow{n \to \infty} 0 \). Thus, \( T f \) is Riemann integrable.

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