New Kazhdan groups

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Introduction

A locally compact, second countable group $G$ is called Kazhdan if for any unitary representation of $G$, the first continuous cohomology group is trivial $H^1_{ct}(G, \rho) = 0$. There are several other equivalent definitions, the reader should consult [6], esp. 1.14 and 4.7.

For some time now Kazhdan groups have attracted attention. One of the main challenges is to understand them geometrically.

Recently Pansu [7], Žuk [10], and Ballmann–Świątkowski [1], went back to Garland’s paper [5], improved it in several respects and produced among other things new examples of Kazhdan’s groups. These examples, especially those in [1], are explicit and significantly different from classical ones.

We also go back to Garland’s paper, but instead of euclidean buildings, we study hyperbolic ones.

An interesting class of hyperbolic buildings with cocompact groups of automorphisms were constructed by Tits [9]. He associates with a ring $\Lambda$ and a generalised Cartan matrix $M$ a Kac–Moody group. These groups provide $BN$ pairs for buildings. A special case of particular interest to us is that of $\Lambda$ a finite field and generalised Cartan matrix coming from hyperbolic reflection groups for which the fundamental domain is a simplex; there are 10 of them in dimension 2, two in dimension 3 and one in dimension 4. Buildings associated to these data are locally finite and their automorphism groups are locally compact topological groups. It turns out that they are Kazhdan’ (and more).

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**Theorem 1.** Let $X_q$ be an $n$-dimensional building of thickness $(q + 1)$, associated to a cocompact hyperbolic group with the fundamental domain a simplex. Suppose $G$ is a closed in the compact open topology, unimodular subgroup of the simplicial automorphism group which acts cocompactly on the building.

Then for large $q$ and $1 \leq k \leq n - 1$

$$H^k_{ct}(G, \rho) = 0,$$

that is the continuous cohomology groups of $G$ with coefficients in any unitary representation vanish. In particular $G$, considered as a topological group is a Kazhdan group.

Several comments are in order:

1. The theorem holds for any hyperbolic building. However at present Tits’ Kac–Moody buildings are the only examples where we can verify the assumptions.
2. For Tits’ Kac–Moody building, simplicial automorphism groups which are *uncountable*, are bigger than Kac–Moody groups (given by countably many generators and relations), and Tits’ Kac–Moody groups are not discrete as subgroups of automorphism groups. In [8], B. Rémy using twin buildings exhibits Kac–Moody groups as discrete cofinite volume groups acting on product of buildings, and also constructs discrete cofinite volume groups acting on the building itself. His examples are not cocompact.
3. Unimodularity, brought in by the topology of the group, is an essential assumption. Kazhdan groups are unimodular. On the other hand Świątkowski pointed out to us nonunimodular groups acting cocompactly on classical euclidean buildings: the upper triangular subgroup of $SL_n(Q_p)$ acting on its building. In Tits’ Kac–Moody examples, it is easy to establish that the group of simplicial automorphisms is unimodular.
4. Three and four dimensional dimensional buildings provide first examples of Kazhdan groups of large dimension, not coming from locally symmetric spaces or euclidean buildings (they are also not products of lower dimensional examples, since they are hyperbolic). Here ”dimension” may be understood either as ”continuous cohomological dimension” or as ”large scale dimension”. The argument here requires the computation of $H^n_{ct}(G, \text{St})$, where $\text{St}$ is the Steinberg representation of $G$ on the space of $l_2$-harmonic $n$-cochains on the building, computation which is not essentially not different form the euclidean building case. If we have discrete subgroups of automorphisms of these buildings, they are necessarily Gromov hyperbolic, and the ”dimension” is one more than the dimension of their Gromov boundaries.
5. Bourdon [4] noticed that several two dimensional hyperbolic buildings admit cocompact actions of discrete groups and thus one can use results of [1], [10] to show that some of them are Kazhdan. He does not use Tits’ construction, but builds his buildings as complexes of groups. Most of his buildings are not Kac–Moody.

There are two ingredients we use in the proof.

First is the Garland’s method (which we take from [1], but actually [5] implicitely contains almost all we need) for proving vanishing of cohomology groups of a simplicial complex. Second is the use of continuous cohomology of topological groups, in particular of Borel–Wallach result relating the cohomology of a complex on which the topological group acts with compact stabilizers, to its continuous cohomology.
The progress we obtain is that one does not have to worry about the existence of discrete subgroups. This is very handy, since bare existence of Tits examples is nontrivial, let alone their subtle properties.

In a future paper we construct more examples of Kazhdan groups, all related to buildings.

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§1 Generalities about automorphism groups.

We recall basic facts about simplicial automorphism groups of simplicial complexes. They are all fairly standard.

Let $X$ be a countable locally finite simplicial complex, let $\text{Aut}(X)$ be the group of its simplicial automorphisms. The compact–open topology on $\text{Aut}(X)$ is defined using the basis of open neighbourhoods of the identity $U(K) = \{g : g|_K = \text{id}_K\}$, where $K$ runs over compact subsets in $X$.

Let $G$ be a closed subgroup of $\text{Aut}(X)$, with induced topology. Since $X$ is a countable complex, $\text{Aut}(X)$, thus $G$, has countable basis and hence it is metrizable by a left invariant metric.

**Proposition 1.1.** $G$ is locally compact. In fact stabilizers of compact subcomplexes in $X$ are compact and open.

**Proposition 1.2.** $G$ is separable.
Thus, being metrizable and separable, $G$ is second countable (has a countable basis).

**Proposition 1.3.** $G$ is countable at infinity, or $\sigma$-compact: the sum of countably many compact subsets.

**Proposition 1.4.** Stabilizers of compact subcomplexes are either all finite or all uncountable.

**Proposition 1.5.** $G$ is totally disconnected.

Unimodularity of $G$ will play important role. Observe first that if the group $G$ is generated by compact subgroups then it is unimodular, since all generators go to identity under the modular homomorphism.

Suppose that a subgroup $G \subset \text{Aut}(X)$ generated by compact subgroups of $\text{Aut}(X)$ acts transitively on $n$-simplices. Then $\text{Aut}(X)$ is unimodular, since it is generated by $G$ and a stabilizer of a simplex. A situation of interest to us where this happens is this:

**Lemma 1.6.** Suppose $X$ is a connected locally finite simplicial complex, and suppose that links of simplices of codimension $\geq 2$ in $X$ are connected. Suppose that stabilizers of
\((n - 1)\)-simplices act transitively on their respective links. Then \(\text{Aut}(X)\) acts transitively on \(X\) and is unimodular.

The proof is clear from the above discussion. Observe that locally finite buildings coming from \(BN\) pairs satisfy the assumptions of the Lemma.

\section{2 Borel–Wallach Lemma}

Assume that \(G\), a closed subgroup of the group of simplicial automorphisms of \(X\), acts cocompactly. Sometimes one can identify \(H^*_\text{ct}(G, \rho)\) with the cohomology of \(X\) with coefficients in \(\rho\). Specifically:

Consider all alternating maps \(\phi\) from ordered \(k\)-simplices in \(X\) to \(\mathcal{H}\), satisfying for all \(g \in G\) and \(\sigma \in X\)

\[
\phi(g\sigma) = \rho(g)\phi(\sigma)
\]

Call the space of such maps \(C^k(X, \rho)\). There is a natural differential making \(C^*(X, \rho)\) into a complex

\[
d\phi(\sigma) = \sum_{i=0}^{k} (-1)^i \sigma_i
\]

where \(\sigma = (v_0, \ldots, v_k)\) and \(\sigma_i = (v_0, \ldots, v_{i-1}, v_{i+1}, \ldots, v_k)\)

\begin{lemma}
([2], lemma X.1.12 page 297) Let \((X, G)\) be an acyclic locally finite complex with a cocompact action of a group of its simplicial automorphisms. Suppose \(\rho\) is a representation of \(G\) on a quasi-complete (for example Hilbert) space. Then

\[
H^i\text{ct}(G, \rho) = H^i(C^*(X, \rho))
\]

The assumptions of this theorem are satisfied for locally finite buildings coming from \(BN\) pairs.

\section{3 Vanishing theorem}

Here we adapt Ballmann–Świątkowski presentation of Garland’s method to greater generality of not necessarily discrete group actions. To keep the exposition short we refer to their paper for the notation.

\begin{theorem}
Let \(X\) be a locally finite simplicial complex, and \(G\) a cocompact unimodular group of its simplicial automorphisms. Assume that for any simplex \(\tau\) of \(X\) the link \(X_\tau\) is connected and

\[
\kappa_\tau > \frac{k(n - k)}{k + 1}
\]

where \(\kappa_\tau\) is the smallest positive eigenvalue of the Laplacian \(\Delta_\tau\) on \(C^0(X_\tau, R)\).

\end{theorem}
Then for $1 \leq k \leq n - 1$, $H^k(C^*(X, \rho)) = 0$ for any unitary representation $\rho$ of $G$.

Then from Lemma 2.1, we immediately get.

**Corollary 3.2.** Under the assumptions of Theorem 3.1, $G$ is a Kazhdan group, provided $X$ is acyclic.

Proof: Theorem 3.1 corresponds to Theorem 2.5 of [1]. Their calculation goes through as it stands, except for two changes.

1. $|G_\sigma|$, the cardinality of the stabilizer of $\sigma$, should now be understood as the Haar measure of that stabilizer inside $G$.

2. Their Lemma 1.3 should be shifted from the discrete to locally compact setting. Here is how this can be done.

Let $\Sigma(k)$ denote the set of ordered $k$-simplices in $X$, let $\Sigma(k, G)$ be a set of representatives of $G$ orbits on $\Sigma(k)$. Modified Lemma 1.3 of [1] reads now as follows:

**Lemma 3.3.** Let $X, G$ be as in Theorem 3.1. For $0 \leq l < k \leq n$, let $f = f(\tau, \sigma)$ be a $G$-invariant function on the set of pairs $(\tau, \sigma)$, where $\tau$ is an ordered $l$-simplex and $\sigma$ is an ordered $k$-simplex with $\tau \subset \sigma$, that is vertices of $\tau$ are vertices of $\sigma$. Then

$$
\sum_{\sigma \in \Sigma(k, G)} \sum_{\tau \in \Sigma(l, G)} \frac{f(\tau, \sigma)}{|G_\sigma|} = \sum_{\tau \in \Sigma(l, G)} \sum_{\sigma \in \Sigma(k, G)} \frac{f(\tau, \sigma)}{|G_\tau|}.
$$

Proof:

$$
\sum_{\sigma \in \Sigma(k, G)} \sum_{\tau \in \Sigma(l, G)} \frac{f(\tau, \sigma)}{|G_\sigma|} = \sum_{\sigma \in \Sigma(k, G)} \sum_{\tau \in \Sigma(l, G)} \frac{f(\gamma_i \tau, \sigma)}{|G_\sigma|}
$$

$$
= \sum_{\sigma \in \Sigma(k, G)} \int_{\gamma_i G_\tau} \frac{f(\gamma \tau, \sigma)}{|G_\tau|} d\gamma
$$

$$
= \sum_{\sigma \in \Sigma(k, G)} \int_{\gamma \gamma \tau \subset \sigma} \frac{f(\tau, \gamma^{-1} \sigma)}{|G_\tau|} d\gamma
$$

$$
= \sum_{\sigma \in \Sigma(k, G)} \int_{\gamma \tau \subset \gamma \sigma} f(\tau, \gamma \sigma) d\gamma
$$

$$
= \sum_{\tau \in \Sigma(l, G)} \sum_{\sigma \in \Sigma(k, G)} \frac{f(\tau, \sigma)}{|G_\tau|}.
$$

\section*{§4 Hyperbolic buildings}

Here we rely on results of Tits [9]. We need the existence of buildings with cocompact proper group action, with unimodular automorphism group, and arbitrarily large thickness. Tits provides us with what we need as follows. Take a Coxeter group, whose Dynkin diagram is a triangle, a square or a pentagon. For triangles we allow labels $m, n, k$ on edges,
such that $m, n, k = 2, 3, 4, 6$, and $\frac{1}{m} + \frac{1}{n} + \frac{1}{k} < 1$. For a square one of the edges is labelled 4, and remaining ones are labelled 3, or two opposite edges are labelled 4 and remaining ones are labelled 3. For pentagon one of the edges is labelled 4 and remaining ones are labelled 3. These are all cocompact hyperbolic reflection group with the fundamental domain a simplex, and edges labelled $2, 3, 4, 6$ [3].

For each such diagram and a finite field $F_q$, Tits constructs a Kac-Moody group, acting cocompactly (in fact transitively on simplices of maximal dimension) on a hyperbolic building, with links of vertices being spherical buildings of thickness $q + 1$ corresponding to parabolic subgroups of the Coxeter system. Moreover the group is generated by elements stabilizing codimension 1 simplices. Thus taking the closure of the Kac–Moody group in the full automorphism group we obtain a cocompact unimodular group acting on a hyperbolic building.

As far as we know, the existence of discrete cocompact subgroups in these groups has not been established except for some two dimensional examples.

Now all we have to do to finish the proof of the theorem 1 is to check that the spectral condition holds for the links. But this has been done (for large thickness) already by Garland [5, Sections 6–8] (see also remark at the end of the Section 3.1 of [1]).

It seems to us that Garland could have included these hyperbolic examples in his original paper.

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