Further Considerations on the CP Asymmetry in Heavy Majorana Neutrino Decays

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Abstract

We work out the thermodynamic equations for the decays and scatterings of heavy Majorana neutrinos including the constraints from unitarity. The Boltzmann equations depend on the CP asymmetry parameter which contains both, a self-energy and a vertex correction. At thermal equilibrium there is no net lepton asymmetry due to the CPT theorem and the unitarity constraint. We show explicitly that deviations from thermal equilibrium create the lepton asymmetry.
1 Introduction

Over the past few years it has been shown that a lepton asymmetry can be generated by the mixing of heavy Majorana neutrinos. Majorana neutrinos have the remarkable property of being mixed states of particles and antiparticles. By definition they induce $\Delta L = 2$ and consequently $\Delta (B - L) = -2$ transitions. In addition they could have couplings to scalar particles, which allow them to decay into Higgs particles and leptons, i.e. $N \rightarrow \phi^\dagger \ell, \phi \ell^c$ [1].

In these models $CP$–violation is introduced through complex couplings and appears in the self-energy [2, 3] or vertex corrections [1, 4, 5, 6]. We shall classify the effects using the terminology of $K^0$ mesons [3]. We shall call direct or $\varepsilon'$–type effects, the $CP$–violation which arises from the interference of tree diagrams with vertex corrections. Similarly we call indirect or $\delta$–type effects, those which arise through the self-energies. The self-energies play the role of the box diagram in the $K^0$ system and define the physical states. A consequence of the observation in [2, 3] is that the physical Majorana states $\Psi$ are not $CP$ eigenstates and that the $CP$–asymmetries produce observable effects. Finally, the mass splitting between the states also plays a role and when the mass difference is of the order of the width there is a resonance enhancement [3].

The processes under discussion must satisfy unitarity constraints so that the sum of the probabilities for all transitions to and from a state $i$ should sum to one and yields [7, 8]:

$$\sum_j |M(i \rightarrow j)|^2 = \sum_j |M(j \rightarrow i)|^2 \quad .$$

(1)

For our case this means that when we consider the scatterings $\ell \phi^\dagger \leftrightarrow \ell^c \phi$ equation (1) turns into

$$|M(\ell \phi^\dagger \rightarrow \ell \phi^\dagger)|^2 + |M(\ell \phi^\dagger \rightarrow \ell^c \phi)|^2$$

(2)

$$= |M(\ell \phi^\dagger \rightarrow \ell \phi^\dagger)|^2 + |M(\ell^c \phi \rightarrow \ell \phi^\dagger)|^2 \quad .$$
so that the probabilities for the direct and inverse process should be equal. For the early universe a summation over all possible physical states with Boltzmann factors weighting the initial states is required. Boltzmann factors are introduced according to the thermal properties of the universe. At thermal equilibrium, for example, the weighting factors are all equal, which implies that the lepton asymmetry averages to zero [[7]]. This issue and its implications are explicitly proved in [[9]] and further discussed in [[10]].

The generation of a lepton or baryon asymmetry is a combination of the Sakharov effects [[11]] (i) baryon or lepton violation, (ii) \( C \) and \( CP \)-violation and (iii) the thermodynamic properties of the ensemble of particles in the early universe. For this reason several early papers studied the non-equilibrium equations of the ensemble [[8], [[12], [[13]]. These thermal considerations together with several subtleties of the phenomenon are mentioned in [[4]]. In this article we present an explicit calculation of the modification of the unitarity constraints as implied by the thermodynamics of the system and point out the meaning of the various terms. In fact we shall show that the \( s \)-channel physical poles play a special role and are responsible for the development of asymmetries.

To begin with we consider massive unstable particles which interact and decay. As the particles interact some of them occur in intermediate states. When the lifetimes of the particles are long relative to the interaction times, the intermediate states interact many times with the thermal bath. For this reason they should not be considered as virtual states, but as ensembles of particles with their own thermodynamic distributions. This phenomenon modifies the unitarity sum, i.e. the various terms in equation (2) are weighted by different factors, and a lepton asymmetry is created.

We estimate the collision of the particles in relation to their lifetimes. The width of the particles with energy \( E \) and mass \( M_N \) is given by

\[
\Gamma_N = \frac{M_N}{E} \Gamma_0 = \frac{|h\ell|^2 M_N^2}{16\pi T} \quad \text{for} \quad E = T .
\]

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Their frequent interactions are like $e^- + t \rightarrow N + b$ with a Higgs particle exchanged in the t-channel. The cross-section at temperatures $T = E \gg M_N$ is

$$\sigma = \frac{|h_t|^2|h_\ell|^2}{16\pi} \frac{1}{E^2}.$$  \hspace{1cm} (4)

We now calculate the interaction of one particle $N$ with a density of quarks. At a temperature $T \gg M_N$ and assuming Boltzmann statistics the particle density is given by

$$n = \frac{2}{\pi^2} T^3.$$ \hspace{1cm} (5)

Therefore the interaction rate for relativistic velocities is

$$n \cdot \sigma \cdot \nu = \frac{|h_t|^2|h_\ell|^2}{8\pi^3} T.$$ \hspace{1cm} (6)

Thus there are many more interactions per decay width

$$\frac{n \cdot \sigma \cdot \nu}{\Gamma_N} \sim O \left( \left( \frac{T}{M_N} \right)^2 |h_t|^2 \right),$$ \hspace{1cm} (7)

with the last factor being of the order of $10^6$ or larger. Since many scatterings are taking place during the lifetime of the particles, each state has a different history and the particle states are incoherent. At this epoch when we consider many scatterings, decays and recombinations of particles, there is no asymmetry, because the unitarity of the S-matrix guarantees the equality of direct and inverse processes. An asymmetry is generated when specific channels begin to decouple.

As usual we work on an extension of the standard model where we include one heavy right-handed Majorana field $N_i$ per generation of light-lepton ($i = 1, 2, 3$). The new fields are singlets with respect to the standard model $[1]$. Extensions of this model have been constructed in gauge theories $[14, 15]$, supersymmetric theories $[16, 17]$ and theories with electroweak singlet neutrinos $[18]$.

We consider the contribution from the interference of the diagrams in Figure 1, which appears in the asymmetry of the reaction $\ell^c \phi \rightarrow \ell \phi^\dagger$. In
the figure, we have indicated the momenta explicitly and have taken the absorbptive part of the self-energy. In addition we consider many generations indicated by the indices $\alpha$ and $\beta$ for the external particles, the index $\gamma$ for the loop-momentum and $i, j$ and $k$ for the heavy particle propagators. The contribution to the asymmetry is

$$2\text{Re}(\mathcal{M}_L^*\mathcal{M}) = \frac{EE'}{2\pi^2} \left( kp + pp' - kp' \right) \frac{M_i M_k \text{Im}(h_{\beta_i}^* h_{\alpha_i}^* h_{\alpha_k} h_{\gamma_k} h_{\gamma_j}^* h_{\beta_j})}{(s - M_i^2)(s - M_k^2)(s - M_j^2)}$$

with $\mathcal{M}_L$ is the loop diagram and $\mathcal{M}$ is the Born diagram amplitude (see Figure 1). Carrying out the same calculation for the process $\ell\phi^+ \to \ell^c \phi$ we obtain the following expression for the asymmetry:

$$2\text{Re}(\mathcal{M}_L^*\mathcal{M}) = \frac{EE'}{2\pi^2} \left( kp + pp' - kp' \right) \frac{M_i M_k \text{Im}(h_{\gamma_k} h_{\alpha_k} h_{\alpha_i}^* h_{\beta_i}^* h_{\gamma_j}^* h_{\beta_j}^*)}{(s - M_i^2)(s - M_k^2)(s - M_j^2)}$$

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persists when we keep all the masses of the propagators or when we replace the propagators with $\delta$–functions in the narrow-width approximation. The two expressions $2\text{Re}(\mathcal{M}_L^*\mathcal{M})$ and $2\text{Re}(\overline{\mathcal{M}_L^*}\mathcal{M})$ are different when we select specific components. This is in fact what happens in the development of the universe, as the temperature reaches the mass of a heavy neutrino specific components are multiplied by special Boltzmann factors, which single them out as demonstrated below.

2 The Boltzmann Equations

In this section we explicitly discuss the thermodynamic development of the system. In this way we explain how deviations from thermal equilibrium modify the sum of unitary contributions and render them different from zero. The lepton asymmetry is a sum of several terms:

First, there are the decays of the heavy Majorana neutrinos:

$$\Psi_i \rightarrow \ell\phi^\dagger$$ which creates an excess of leptons, and

$$\Psi_i \rightarrow \ell^c\phi$$ which reduces the amount of leptons

and the corresponding recombination terms. In addition there are two scattering processes:

$$\ell^c\phi \rightarrow \ell\phi^\dagger$$ which creates an excess of leptons, and

$$\ell\phi^\dagger \rightarrow \ell^c\phi$$ which reduces the amount of leptons.

The result of all these terms is the following differential equation for the lepton asymmetry density $n_L = n_\ell - n_{\ell^c}$:

$$\dot{n}_L + 3Hn_L = f_{\Psi_1}\left[ |\mathcal{M}(\Psi_1 \rightarrow \ell\phi^\dagger)|^2 - |\mathcal{M}(\Psi_1 \rightarrow \ell^c\phi)|^2 \right] \Lambda_{12}^3$$

$$+ \left[ -f_\ell f_{\phi^\dagger} |\mathcal{M}(\ell\phi^\dagger \rightarrow \Psi_1)|^2 + f_{\ell^c} f_{\phi} |\mathcal{M}(\ell^c\phi \rightarrow \Psi_1)|^2 \right] \Lambda_{12}^3$$

(10)
\[
+2 \Lambda_{12}^{34} \left\{ f_\ell f_\phi |M(\ell^c \phi \to \ell^c \phi^\dagger)|^2 - |M_{RIS}(\ell^c \phi \to \ell^c \phi^\dagger)|^2 \right\} \\
- f_\ell f_\phi^\dagger \left[ |M(\ell^c \phi \to \ell^c \phi^\dagger)|^2 - |M_{RIS}(\ell^c \phi \to \ell^c \phi)|^2 \right]
\]

where \( \Lambda_{12}^{34} \) is the four-particle phase space factor depending on the momenta \( p_1 \) to \( p_4 \), \( f_\ell \) and \( f_\phi^\dagger \) are the Boltzmann distributions for leptons and scalar particles, respectively. \( f_\Psi \) is the distribution for the heavy Majorana states. Since the \( \Psi_1 \)'s are produced copiously through the multiple scatterings as discussed in the previous section they have their own distribution. For the decays and recombinations we introduce the three-particle phase space \( \Lambda_{12}^3 \). The first two lines already include the decays of the real intermediate states (RIS) and their recombinations. For this reason they are subtracted from the scattering amplitudes.

The interference of the tree with the one loop graph gives the \( CP \)-violating factor:

\[
\left[ |M(\Psi_1 \to \ell^c \phi^\dagger)|^2 - |M(\Psi_1 \to \ell^c \phi)|^2 \right] = (\varepsilon' + \delta) |M_0|^2
\]

with \( |M_0|^2 \) being the tree level amplitude. Both \( \varepsilon' \) and \( \delta \) are \( CP \) violating parameters with \( \varepsilon' \) is produced by the vertices (direct \( CP \)-violation) and \( \delta \) from the self energies. Combining the terms and neglecting those ones of \( O(|\varepsilon' + \delta|^2, |\mu(\varepsilon' + \delta)|/T) \), the first two lines of equation 10 give:

\[
n_{\Psi_1} (\varepsilon' + \delta) \langle \Gamma_{\Psi_1} \rangle + n_{\Psi_1}^{EQ} (\varepsilon' + \delta) \langle \Gamma_{\Psi_1} \rangle - \left( \frac{\mu}{T} \right) n_{\Psi_1}^{EQ} \langle \Gamma_{\Psi_1} \rangle \ ,
\]

where the phase space was combined with the amplitude \( |M_0|^2 \) to produce the thermally-averaged decay width \( \langle \Gamma_{\Psi_1} \rangle \).

Next we consider the last two lines of equation 10, where we expand the particle distributions in powers of \( \mu/T \) :

\[
2 \Lambda_{12}^{34} f_\ell f_\phi \left[ |M(\ell^c \phi \to \ell^c \phi^\dagger)|^2 - f_\ell f_\phi^\dagger |M(\ell^c \phi \to \ell^c \phi^\dagger)|^2 \right] \\
+ f_\ell f_\phi^\dagger \left[ |M_{RIS}(\ell^c \phi \to \ell^c \phi)|^2 - f_\ell f_\phi^\dagger |M_{RIS}(\ell^c \phi \to \ell^c \phi)|^2 \right]
\]

\[
= 2 \Lambda_{12}^{34} f_{\Psi_1}^{EQ} \left[ (1 - \frac{\mu}{T}) |M(\ell^c \phi \to \ell^c \phi^\dagger)|^2 - (1 + \frac{\mu}{T}) |M(\ell^c \phi \to \ell^c \phi)|^2 \right]
\]
\[ + (1 + \frac{\mu}{T}) |M_{RIS}(\ell \phi^\dagger \rightarrow \ell \phi)|^2 - (1 - \frac{\mu}{T}) |M_{RIS}(\ell \phi \rightarrow \ell \phi^\dagger)|^2 \]

\[ + O \left( \frac{\mu^2}{T^2} \right) \]

The leading terms proportional to unity involve the difference of the complete amplitudes \(|M|^2\) and vanish by virtue of unitarity. This is the only place where unitarity is effective in bringing a complete cancellation. The remaining terms arise from interactions which change the number of particles (chemical potentials) and through deviations from equilibrium. The former is the origin of terms proportional to the chemical potentials which are multiplied with \(CP\)-conserving amplitudes. The leading term from the real intermediate states is of special interest, because it is twice as big as the second term in equation (12), but has the opposite sign. Its net effect is to change the sign of the second term in equation (12). After some algebra, described in the appendix, equation (13) leads to:

\[ -2 n_{\psi_1}^E (\varepsilon' + \delta) \langle \Gamma_{\psi_1} \rangle \]

\[ -2 \left( \frac{\mu}{T} \right) n_\gamma^2 \left( \langle \sigma' (\ell \phi^\dagger \rightarrow \ell \phi) \cdot v \rangle + \langle \sigma' (\ell \phi \rightarrow \ell \phi^\dagger) \cdot v \rangle \right) . \]

We can substitute the chemical potential by the density for the lepton asymmetry using the equation

\[ \frac{n_L}{n_\gamma} = 2 \frac{\mu}{T} + O \left( \frac{\mu^3}{T^3} \right) . \]

Combining the various terms we arrive at the final Boltzmann equation:

\[ \dot{n}_L + 3H n_L = (\varepsilon' + \delta) \left[ n_{\psi_1} - n_{\psi_1}^E \right] \langle \Gamma_{\psi_1} \rangle - \frac{1}{2} \left( \frac{n_L}{n_\gamma} \right) n_{\psi_1}^E \langle \Gamma_{\psi_1} \rangle (16) \]

\[ - n_L n_\gamma \left[ \langle \sigma' (\ell \phi^\dagger \rightarrow \ell \phi) \cdot v \rangle + \langle \sigma' (\ell \phi \rightarrow \ell \phi^\dagger) \cdot v \rangle \right] \]

This is coupled with the differential equation for the density of the Majorana neutrinos, which reads:

\[ \dot{n}_{\psi_1} + 3H n_{\psi_1} = -\langle \Gamma_{\psi_1} \rangle (n_{\psi_1} - n_{\psi_1}^E) - \frac{1}{2} \left( \frac{n_L}{n_\gamma} \right) n_{\psi_1}^E (\varepsilon' + \delta) \langle \Gamma_{\psi_1} \rangle . \]
The integration is now straight forward. We can integrate equation (17), then introduce the answer in equation (16) whose numerical integration gives the lepton asymmetry. The terms proportional to $n_L$ in equation (16) are multiplied by negative coefficients and after integration, if they were alone, they would wash out any lepton asymmetry exponentially.

3 Numerical Integrations

Following the usual method to evaluate the coupled Boltzmann equations (13) we introduce the dimensionless quantity

$$z = \frac{M_1}{T},$$

(18)

to change the integration variable from the time $t$ to $z$, which has a temperature dependence. Furthermore it is convenient to look at the particle density per comoving volume, which is:

$$Y = \frac{n}{s} \approx \frac{n}{g_* n_\gamma},$$

(19)

with $s$ being the entropy density and $g_*$ the number of effectively massless degrees of freedom (particles with mass $m \ll T$). Now the differential equations (16) and (17) read:

$$\frac{dY_L}{dz} = \frac{\langle \Gamma_{\Psi_1}(z = 1) \rangle}{H(z = 1)} z \left\{ \left( \varepsilon' + \delta \right) \left( Y_{\Psi_1} - Y_{\Psi_1}^{EQ} \right) \gamma - \frac{1}{2} Y_L^2 \gamma L \right\}$$

$$\frac{dY_{\Psi_1}}{dz} = -\frac{\langle \Gamma_{\Psi_1}(z = 1) \rangle}{H(z = 1)} z \gamma \left\{ Y_{\Psi_1} - Y_{\Psi_1}^{EQ} (1 + O(\varepsilon', \delta)) \right\}$$

(20)

with

$$Y_{\Psi_1}^{EQ} = \frac{1}{2g_*} \int_z^\infty dx \sqrt{x^2 - z^2} \frac{x}{e^x + 1} = \left\{ \begin{array}{ll} \frac{1}{g_*} & z \ll 1 \\ \frac{1}{g_*} \sqrt{\pi / 2} z^{3/2} e^{-z} & z \gg 1 \end{array} \right.$$
where \( K_n \) are the modified Bessel functions \( [8] \), and

\[
\gamma_L(z) = \frac{\left\{ \frac{1}{7} g_* Y_{\psi_1}^{EO} (\Gamma_{\psi_1}(z)) + 2 \langle \sigma' v \rangle n_\gamma \right\}}{\langle \Gamma_{\psi_1}(z = 1) \rangle} 
\cong \begin{cases} 
  z + 0.1 z^{-1} & z \ll 1 \\
  z^{3/2} e^{-z} + 0.1 z^{-5} & z \gg 1
\end{cases}
\]

Here we approximate the cross-section terms as in \([13]\). For a detailed calculation of the cross-section terms see \([6]\).

The overall factor

\[
K = \frac{\langle \Gamma_{\psi_1}(z = 1) \rangle}{H(z = 1)} \tag{21}
\]

is an important measure for the efficiency to create a lepton asymmetry. In the case \( K \ll 1 \), which corresponds to \( \langle \Gamma_{\psi_1}(z = 1) \rangle \ll H(z = 1) \), the decay rate is much smaller than the expansion rate of the universe and the particles come out of equilibrium and create a lepton asymmetry. In the other case, where \( K \gg 1 \), the particle decay fast and recombine and are in thermal equilibrium in comparison to the expansion rate of the universe. In this case the lepton asymmetry will go to zero.

Figure 2 shows this dependence of the lepton asymmetry, which we obtain by integrating the Boltzmann equations \((20)\) numerically. The shape of the curves is in direct correspondence to \([13]\), but the asymptotic value of the curves is now bigger, because of the \( CP \)-parameter \( \delta \) coming out of the interference of the tree with the self energy diagram:

\[
Y_L(z \to \infty) = \frac{\varepsilon' + \delta}{g_*}, \quad \text{for} \quad K \ll 1 \tag{22}
\]

If \( K \) increases the asymptotic value of the lepton asymmetry decreases and finally for \( K \gg 1 \) goes to zero. Similar numerical results were obtained in \([19]\) where the energy scale for the generation of the lepton asymmetry was studied.
Figure 2: The development of the lepton asymmetry $Y_L/[(\varepsilon' + \delta)/g_*]$ with the expansion of the universe given by $x = M_1/T$.

4 Conclusions

We have shown explicitly how a lepton asymmetry is created through the decay of Majorana neutrinos. The asymmetry is proportional to the sum of the self energy and the vertex contribution. We emphasize that at thermal equilibrium, the unitarity of the amplitudes implies the vanishing of the lepton asymmetry, as it is demonstrated in section 1. On the other hand, the thermal development of the Majorana densities is the reason for the unitarity cancellation to become ineffective and allow the generation of the lepton asymmetry (see section 2). Finally, we solve the Boltzmann equations numerically and create a final asymmetry, which is proportional to the sum of the $CP$—parameters coming from the vertex contribution $\varepsilon'$ and from the self energy $\delta$. The latter one has a resonant behaviour depending on the mass.
difference of the Majorana particles [3].

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A The Mixed Majorana States

The problem of mixing of Majorana neutrinos has been treated at several places. One formalism is to discuss the development of the various components in the wavefunction, where the absorptive part has a definite effect [2, 3]. Another method is to construct the inverse propagator and their renormalization [14, 15]. In both cases the absorptive part is a physical observable, which can not be removed by renormalization [15]. We follow here the approach of references [2, 3], where to $O(h^2)$ the $CP$–effects were computed exactly for all relative values of the masses [3].

We assume a mass hierarchy and consider an epoch in the development of the universe, where the Majorana neutrinos are decoupled from each other. When the temperature of the universe becomes of the order of the lightest mass $M_1$, there is only this particle left, which can only decay bringing the universe out-of-equilibrium. The physical heavy Majorana Neutrino $\Psi_1$ is a mixed state of the interaction states:

$$|\Psi_1\rangle = \frac{1}{\sqrt{2}} \left\{ |N_{R1}\rangle + |(N_{R1})^\text{c}\rangle + \alpha_2 |N_{R2}\rangle + \alpha_1 |(N_{R2})^\text{c}\rangle \right\}$$

(23)

It has a definite mass and equation (23) gives the couplings of the various components. The transition amplitudes of the mixed state receive contributions from the vertex corrections through the $\varepsilon'$ and from the self energies through the mixing of the states:

$$|\mathcal{M}(\Psi_1 \to \ell_\alpha \phi^\dagger)|^2 = |\langle \Psi_1 | N_{R1} \rangle \langle N_{R1} | \ell_\alpha \phi^\dagger \rangle + \langle \Psi_1 | N_{R2} \rangle \langle N_{R2} | \ell_\alpha \phi^\dagger \rangle|^2$$
\[ |M(0)|^2 = \left\{ \frac{1}{2} (1 + \varepsilon') + 2 \text{Re} \left( \alpha_1^* h_2 h_1 \alpha_2 \right) \right\} \frac{1}{|h_1|^2} + O \left( |\alpha_2|^2 \right) \]

\[ |M(\ell^c \phi \rightarrow \Psi_1)|^2 = |M(\Psi_1 | N_{R1}^c \rangle \langle N_{R1}^c | \ell^c \phi) + \langle \Psi_1 | N_{R2}^c \rangle \langle N_{R2}^c | \ell^c \phi)|^2 \]

\[ = |M(0)|^2 \left\{ \frac{1}{2} (1 - \varepsilon') + 2 \text{Re} \left( \alpha_1^* h_1^* h_2 \right) \right\} \frac{1}{|h_1|^2} + O \left( |\alpha_1|^2 \right) \]

\[ = |M(\ell \phi \dagger \rightarrow \Psi_1)|^2 \]  

with \( |M_0|^2 = \sum_\alpha |h_\alpha|^2 M_1^2 \). These are the relevant amplitudes for the Boltzmann equation (10), which are given now by:

\[ \dot{n}_{L} + 3H n_{L} = \Lambda_{12}^3 |M_0|^2 \left[ -f_{\ell^c f_{\phi^*}} \left\{ \frac{1}{2} (1 - \varepsilon') + \text{Re} \left( \alpha_1^* h_1^* h_2 \right) \right\} \frac{1}{|h_1|^2} + O \left( |\alpha_1|^2 \right) \right] 
+ f_{\ell^c \phi} \left\{ \frac{1}{2} (1 + \varepsilon') + \frac{\text{Re} \left( \alpha_1^* h_1^* h_2 \right)}{|h_1|^2} \right\} 
+ f_{\Psi_1} \left\{ \varepsilon' + \frac{\text{Re} \left( h_1^* h_2 (\alpha_2 - \alpha_1^*) \right)}{|h_1|^2} \right\} 
+ 2 \Lambda_{12}^3 \left\{ |M(\ell \phi \dagger \rightarrow \ell^c \phi)|^2 - |M_{RIS}(\ell \phi \dagger \rightarrow \ell^c \phi)|^2 \right\} \]  

\[ = \Lambda_{12}^3 |M_0|^2 \left[ -f_{\ell^c f_{\phi^*}} \left\{ \frac{1}{2} (1 - \varepsilon') + \text{Re} \left( \alpha_1^* h_1^* h_2 \right) \right\} \frac{1}{|h_1|^2} + O \left( |\alpha_1|^2 \right) \right] 
+ f_{\ell \phi \dagger} \left\{ \frac{1}{2} (1 + \varepsilon') + \frac{\text{Re} \left( \alpha_1^* h_1^* h_2 \right)}{|h_1|^2} \right\} 
+ f_{\Psi_1} \left\{ \varepsilon' + \frac{\text{Re} \left( h_1^* h_2 (\alpha_2 - \alpha_1^*) \right)}{|h_1|^2} \right\} 
+ 2 \Lambda_{12}^3 \left\{ |M(\ell \phi \dagger \rightarrow \ell^c \phi)|^2 - |M_{RIS}(\ell \phi \dagger \rightarrow \ell^c \phi)|^2 \right\} \]  

Next we substitute the phase space densities and develop them in the parameter \( \mu/T \):

\[ f_{\ell^c f_{\phi^*}} = f_{\Psi_1}^E \mathrm{e}^{\mu/T} = f_{\Psi_1}^{\mathrm{EQ}} \left[ 1 + \frac{\mu}{T} + O \left( \frac{\mu^2}{T^2} \right) \right] \]

\[ f_{\ell \phi \dagger} = f_{\Psi_1}^E \mathrm{e}^{-\mu/T} = f_{\Psi_1}^{\mathrm{EQ}} \left[ 1 - \frac{\mu}{T} + O \left( \frac{\mu^2}{T^2} \right) \right] \]  

As usual we assume that the particles are in kinetic equilibrium. Finally, we use the definition of the thermally-averaged decay width \( \langle \Gamma \rangle \) which is:

\[ \langle \Gamma \rangle = \frac{g/(2\pi)^3 \int d^3 p f(p) \Gamma(p)}{g/(2\pi)^3 \int d^3 p f(p)} = \frac{g/(2\pi)^3 \int d^3 p f(p) \Gamma(p)}{n} \]

where \( g \) are the degrees of freedom and \( n \) is the particle density. With all this information we arrive at equation (12) and equation (13). In the next section we study the scattering terms in the Boltzmann equation.
B The Scattering Amplitudes

In this section we study the scattering terms in the Boltzmann equation:

$$-2\Lambda_{12} f_{\Psi_1}^E \{ \left| \mathcal{M}_{RIS}(\ell\phi \rightarrow \ell\phi^\dagger) \right|^2 - \left| \mathcal{M}_{RIS}(\ell\phi^\dagger \rightarrow \ell\phi) \right|^2 \}
+ \frac{\mu}{T} \left[ \left| \mathcal{M}'(\ell\phi^\dagger \rightarrow (\ell\phi) \right|^2 + \left| \mathcal{M}'(\ell\phi \rightarrow \ell\phi^\dagger) \right|^2 \right]$$

(29)

with

$$\left| \mathcal{M}' \right|^2 = \left| \mathcal{M} \right|^2 - \left| \mathcal{M}_{RIS} \right|^2$$

(30)

For the real intermediate states one can use the narrow-width approximation:

$$\left| \mathcal{M}_{RIS}(\ell\phi^\dagger \rightarrow (\ell\phi) \right|^2 = \frac{\pi}{M_1 \Gamma_{\Psi_1}} \delta(s - M_1^2) \left| \mathcal{M}(\ell\phi^\dagger \rightarrow \Psi_1) \right|^2 \left| \mathcal{M}(\Psi_1 \rightarrow \ell\phi) \right|^2$$

(31)

$$\left| \mathcal{M}_{RIS}(\ell\phi \rightarrow \ell\phi^\dagger) \right|^2 = \frac{\pi}{M_1 \Gamma_{\Psi_1}} \delta(s - M_1^2) \left| \mathcal{M}(\ell\phi \rightarrow \Psi_1) \right|^2 \left| \mathcal{M}(\Psi_1 \rightarrow \ell\phi^\dagger) \right|^2$$

Using the following integral [8]:

$$\int d\Pi_1 d\Pi_2 f_{\Psi_1}^E \delta(s - M_1^2) = \frac{1}{(2\pi)^6} \frac{2\pi^4 n_{\Psi_1}^E}{M_1 \Gamma_{\Psi_1}} \langle \Gamma_{\Psi_1} \rangle$$

(32)

and the definition of the thermally-averaged cross section

$$\langle \sigma \cdot v \rangle = \frac{g_A/2(2\pi)^3 g_B/2(2\pi)^3 \int d^3p_A \int d^3p_B f(\vec{p}_A) f(\vec{p}_B) \sigma(AB \rightarrow CD) v}{g_A/2(2\pi)^3 g_B/2(2\pi)^3 \int d^3p_A \int d^3p_B f(\vec{p}_A) f(\vec{p}_B) \sigma(AB \rightarrow CD) v}$$

$$= \frac{g_A/2(2\pi)^3 g_B/2(2\pi)^3 \int d^3p_A \int d^3p_B f(\vec{p}_A) f(\vec{p}_B) \sigma(AB \rightarrow CD) v}{n_A n_B}$$

(33)

as well as the approximations \( n_{\ell} \approx n_{\ell^c} \approx n_{\phi} \approx n_{\phi^c} \approx n_{\gamma} \), expression (14) can be derived from equation (29).
References

[1] M. Fukugita and T. Yanagida, Phys. Lett. B 174 45 (1986)
[2] M. Flanz, E.A. Paschos and U. Sarkar, Phys. Lett. B 345 248 (1995)
[3] M. Flanz, E.A. Paschos, U. Sarkar and J. Weiss, Phys. Lett. B 389 693 (1996)
[4] M.A. Luty, Phys. Rev. D 45 455 (1992)
[5] W. Buchmüller and M. Plümmacher, Phys. Lett. B 389 73 (1996)
[6] M. Plümmacher, Z. Phys. C 74 549 (1997)
[7] A.D. Dolgov, JETP Lett., Vol. 29, No. 4, 254 (1979)
[8] E.W. Kolb and S. Wolfram, Nucl. Phys. B 172 224 (1980)
[9] E. Roulet, L. Covi and F. Vissani, Phys. Lett. 424 101 (1998)
[10] W. Buchmüller and M. Plümmacher, hep-ph/9710460 (revised version)
[11] A.D. Sakharov, ZhETF Pis’ma 5 32 (1967)
[12] A.D. Dolgov, Sov. J. Nucl. Phys. 32 851 (1980)
[13] E.W. Kolb and M.S. Turner, The Early Universe (Addison-Wesley, 1990)
[14] A. Pilaftsis, Phys. Rev. D 56 5431 (1997)
[15] A. Pilaftsis, Nucl. Phys. B 504 61 (1997)
[16] L. Covi and E. Roulet, Phys. Lett. B 399 113 (1997)
[17] M. Plümmacher, hep-ph/9704231
[18] E.Kh. Akhmedov, V.A. Rubakov and A.Yu. Smirnov, \texttt{hep-ph/9803253}

[19] J. Faridani, S. Lola, P.J. O’Donnell and U. Sarkar, \texttt{hep-ph/9804261}