Moving-Frame Approach to Nonlinear Internal Waves in Oceans

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Abstract

In this article, we introduce a moving-frame approach to the geophysical equation of two-dimensional uniformly stratified rotational fluid in oceans and find a family of exact solutions containing ten arbitrary parameter functions.

1 Introduction

The system of nonlinear equations

\[
\begin{align*}
\Delta \psi_t - g \rho_x - f v_z &= \psi_x \Delta \psi_z - \psi_z \Delta \psi_x, \\
v_t + f \psi_z &= \psi_x v_z - \psi_z v_x, \\
\rho_t + \frac{N^2}{g} \psi_x &= \psi_x \rho_z - \psi_z \rho_x
\end{align*}
\]

(1.1, 1.2, 1.3)

are used in geophysical fluid dynamics for investigating internal waves in uniformly stratified incompressible fluids (oceans), where \( \psi, v, \rho \) are functions in \( t, x, z \), \( \Delta = \partial_x^2 + \partial_z^2 \) is the two-dimensional Laplacian, \( g \) is the gravitational acceleration, \( f \) is the Coriolis parameter and the quantity \( N \) appears due to the density stratification of a fluid and is constant under the linear stratification hypothesis.

Kistovich and Chashechkin (1991) [KC] used the system (1.1)-(1.3) with \( f = 0 \) to investigate two non-unidirectional wave beams propagating and interacting in stratified fluid. Moreover, Lombard and Riley (1996) found an exact solution of the system (1.1)-(1.3) with \( f = 0 \) that describes stability of a single internal plane wave. Using the system with \( f = 0 \), Tabaei and Akylas (2003) [TA] study certain nonlinear internal gravity wave beams. Furthermore, Tabaei, Akylas and Lamb (2005) [TAL] studied the nonlinear effects in reflecting and colliding internal wave beams. The above system with \( f \neq 0 \) was used by Ibragimov [I] to model weakly nonlinear interactions governing the time behavior of the oceanic energy spectrum.

In [II1], N. Ibragimov and R. Ibragimov proved that the system (1.1)-(1.3) is self-adjoint and obtained various conservation laws. Using separation of variables, they also found some generalized invariant solutions of the system. Invoking the software DYMSM 2.3, N. Ibragimov, R. Ibragimov and Kovalev [IIK] obtained the Lie point symmetries of the system (1.1)-(1.3) in terms of vector fields. Moreover, they found certain rotational

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invariant solutions. Based on the maximal Lie subalgebra of the vector fields, N. Ibragimov and R. Ibragimov [II2] got two additional conservation law.

In this article, we introduce a moving-frame approach to the system (1.1)-(1.3), motivated from our earlier works on fluid equations (cf. [X]) and find a family of exact solutions containing ten arbitrary parameter functions. Our approach captures more rotational features of the fluid. The parameter functions also make our solutions more applicable to practical models. In order to let more people understand the symmetries of the system, we present intuitive derivations of certain easily-using symmetry transformations, some of which are used to obtain more general form of exact solutions.

2 Intuitive Symmetry Analysis

Let $\alpha$ be a function in $t$. The transformation

$$\psi(t, x, z) \mapsto \psi(t, x + \alpha, z), \ v(t, x, z) \mapsto v(t, x + \alpha, z), \ \rho(t, x, z) \mapsto \rho(t, x + \alpha, z) \quad (2.1)$$

changes the equations (1.1)-(1.3) to:

$$\alpha' \Delta \psi_x + \Delta \psi_t - g \rho_x - f v_z = \psi_x \Delta \psi_z - \psi_z \Delta \psi_x, \quad (2.2)$$

$$\alpha' v_x + v_t + f \psi_z = \psi_x v_z - \psi_z v_x, \quad (2.3)$$

$$\alpha' \rho_x + \rho_t + \frac{N^2}{g} \psi_x = \psi_x \rho_z - \psi_z \rho_x, \quad (2.4)$$

where the independent variable $x$ is replaced by $x + \alpha$ and the subindices denote the partial derivatives with respect to the original independent variables. Moreover, the transformation

$$\psi(t, x, z) \mapsto \psi(t, x, z) - \alpha' z, \ v(t, x, z) \mapsto v(t, x, z), \ \rho(t, x, z) \mapsto \rho(t, x, z) \quad (2.5)$$

changes the equations (1.1)-(1.3) to:

$$\Delta \psi_t - g \rho_x - f v_z = \psi_x \Delta \psi_z - \psi_z \Delta \psi_x + \alpha' \Delta \psi_x, \quad (2.6)$$

$$v_t + f \psi_z - f \alpha' = \psi_x v_z - \psi_z v_x + \alpha' v_x, \quad (2.7)$$

$$\rho_t + \frac{N^2}{g} \psi_x = \psi_x \rho_z - \psi_z \rho_x + \alpha' \rho_x. \quad (2.8)$$

Furthermore, the transformation

$$\psi(t, x, z) \mapsto \psi(t, x, z), \ v(t, x, z) \mapsto v(t, x, z) + f \alpha, \ \rho(t, x, z) \mapsto \rho(t, x, z) \quad (2.9)$$

leaves the equations (1.1) and (1.3) invariant, and changes the equation (1.2) to

$$f \alpha' + v_t + f \psi_z = \psi_x v_z - \psi_z v_x. \quad (2.10)$$

Thus the transformation

$$T_{1, \alpha} (\psi(t, x, z)) = \psi(t, x + \alpha, z) - \alpha' z, \ T_{1, \alpha} (v(t, x, z)) = v(t, x + \alpha, z) + f \alpha, \quad (2.11)$$
\[ T_{1,\alpha}(\rho(t, x, z)) = \rho(t, x + \alpha, z) \]  

is a symmetry transformation of the system (1.1)-(1.3). Similarly, we have the following symmetry transformation

\[ T_{2,\alpha}(\psi(t, x, z)) = \psi(t, x, z + \alpha) + \alpha'x, \quad T_{2,\alpha}(v(t, x, z)) = v(t, x, z + \alpha), \]  

\[ T_{2,\alpha}(\rho(t, x, z)) = \rho(t, x, z + \alpha) - \frac{N^2}{g^2} \alpha. \]  

Since \( \psi \) appears in the equations (1.1)-(1.3) with spacial partial derivatives, we have the following symmetry of the equations (1.1)-(1.3):

\[ S_{\alpha,\beta,\gamma}(\psi(t, x, z)) = \psi(t, x + \alpha, z + \beta) + \beta'x - \alpha'z + \gamma, \]  

\[ S_{\alpha,\beta,\gamma}(v(t, x, z)) = v(t, x + \alpha, z + \beta) + f\alpha, \]  

\[ S_{\alpha,\beta,\gamma}(\rho(t, x, z)) = \rho(t, x + \alpha, z + \beta) - \frac{N^2}{g^2} \beta, \]  

where \( \alpha, \beta, \gamma \) are any functions in \( t \).

Let us do degree analysis. Take

\[ \deg \psi = \ell_1, \quad \deg x = \deg z = \ell_2. \]  

To make all the terms in (1.1) having the same degree, we have to take

\[ \deg \Delta \psi_t = \deg \psi_x \Delta \psi_z \implies \ell_1 - 2\ell_2 - \deg t = 2\ell_1 - 4\ell_2 \implies \deg t = 2\ell_2 - \ell_1, \]  

\[ \deg \rho_x = \deg \psi_x \Delta \psi_z \implies -\ell_2 + \deg \rho = 2\ell_1 - 4\ell_2 \implies \deg \rho = 2\ell_1 - 3\ell_2, \]  

\[ \deg v_z = \deg \psi_x \Delta \psi_z \implies -\ell_2 + \deg v = 2\ell_1 - 4\ell_2 \implies \deg v = 2\ell_1 - 3\ell_2. \]  

To make (1.2) homogeneous, we take

\[ \deg v_t = \deg \psi_v = \deg \psi_x v_z \implies \ell_1 - \ell_2 = 3\ell_1 - 5\ell_2 \implies \ell_1 = 2\ell_2. \]  

Under the above assumption, (1.3) is homogeneous. In summary, we have

\[ \deg \psi = 2\ell_2, \quad \deg \rho = \deg v = \ell_2, \quad \deg t = 0. \]  

Since the equation (1.1)-(1.3) do not contain variable coefficients, they are translation invariant. Therefore we have the symmetry transformation:

\[ T_{a,b}(\psi(t, x, z)) = b^{-2} \psi(t + a, bx, bz), \quad T_{a,b}(v(t, x, z)) = b^{-1} v(t + a, bx, bz), \]  

\[ T_{a,b}(\rho(t, x, z)) = b^{-1} \rho(t + a, bx, bz), \]  

where \( a, b \in \mathbb{R} \) such that \( b \neq 0 \).
3 Moving-Frame Approach

Let $\gamma$ be a function in $t$. Denote the moving frame

$$X = x \cos \gamma + z \sin \gamma, \quad Z = z \cos \gamma - x \sin \gamma. \quad (3.1)$$

Then

$$\partial_t(X) = \gamma' Z, \quad \partial_t(Z) = -\gamma' X. \quad (3.2)$$

By the chain rule of taking partial derivatives,

$$\partial_x = \cos \gamma \partial_X - \sin \gamma \partial_Z, \quad \partial_z = \sin \gamma \partial_X + \cos \gamma \partial_Z. \quad (3.3)$$

Solving the above system, we get

$$\partial_X = \cos \gamma \partial_x + \sin \gamma \partial_z, \quad \partial_Z = -\sin \gamma \partial_x + \cos \gamma \partial_z. \quad (3.4)$$

Moreover, (3.1) and (3.4) imply

$$\partial_X(Z) = 0, \quad \partial_Z(X) = 0. \quad (3.5)$$

In particular,

$$\Delta = \partial_x^2 + \partial_z^2 = \partial_X^2 + \partial_Z^2, \quad x^2 + z^2 = X^2 + Z^2. \quad (3.6)$$

Assume

$$\psi = \vartheta(t, x, z) + \xi(t, X), \quad v = \kappa(t, x, z) + \eta(t, X), \quad \rho = \varphi(t, x, z) + \zeta(t, X), \quad (3.7)$$

where $\xi, \eta, \zeta$ are function in $t, X$, and $\vartheta, \kappa, \varphi$ are functions in $t, x, z$ that $\vartheta$ is quadratic and $\kappa, \varphi$ are linear in $x, z$. Now

$$\Delta(\psi) = \Delta \vartheta + \xi_{XX} \quad (3.8)$$

by (3.6), and $\Delta \vartheta$ is a function in $t$. Moreover,

$$\Delta(\psi_t) = \Delta \vartheta_t + \xi_{XXt} + \partial_t(X)\xi_{XXX} = \Delta \vartheta_t + \xi_{XXt} + \gamma' Z \xi_{XXX} \quad (3.9)$$

by (3.2), and

$$\Delta(\psi_x) = [\Delta(\psi)]_x = \xi_{XXX} \cos \gamma, \quad \Delta(\psi_z) = [\Delta(\psi)]_z = \xi_{XXX} \sin \gamma. \quad (3.10)$$

Thus

$$\psi_x \Delta \psi_z - \psi_z \Delta \psi_x = (\partial_x + \xi_X \cos \gamma)\xi_{XX} \sin \gamma - (\partial_z + \xi_X \sin \gamma)\xi_{XXX} \cos \gamma$$

$$= (\partial_x \sin \gamma - \partial_z \cos \gamma)\xi_{XXX} = -\partial_Z(\vartheta)\xi_{XXX} \quad (3.11)$$

by (3.4).

Note that

$$\psi_x v_z - \psi_z v_x = (\partial_x + \xi_X \cos \gamma)(\kappa_x + \eta_x \sin \gamma) - (\partial_z + \xi_X \sin \gamma)(\kappa_x + \eta_x \cos \gamma)$$

$$= -\partial_Z(\vartheta)\eta_x + \partial_Z(\kappa)\xi_X + \vartheta_x \kappa_x - \vartheta_z \kappa_x \quad (3.12)$$
by (3.4). Similarly,
\[ \psi_x \psi_z - \psi_z \psi_x = -\partial_z (\vartheta) \zeta_x + \partial_z (\varphi) \xi_x + \vartheta_x \varphi_z - \vartheta_z \varphi_x. \] (3.13)

Moreover,
\[ v_t = \kappa_t + \eta_t + \gamma' \zeta \eta_x, \quad \rho_t = \varphi_t + \zeta_t + \gamma' \zeta \eta_x. \] (3.14)

Now the equations (1.1)-(1.3) become
\[ \Delta \vartheta_t + \xi_{xx} + \gamma' \zeta \xi_{xx} - g (\varphi_x + \eta_x \cos \gamma) - f (\kappa_x + \zeta_x \sin \gamma) = -\partial_z (\vartheta) \zeta_{xx}, \] (3.15)
\[ \kappa_t + \eta_t + \gamma' \zeta \eta_x + f (\vartheta_x + \xi_x \sin \gamma) = -\partial_z (\varphi) \eta_x + \partial_z (\varphi) \xi_x + \vartheta_x \kappa_x - \vartheta_z \kappa_x, \] (3.16)
\[ \varphi_t + \zeta_t + \gamma' \zeta \zeta_x + \frac{N^2}{\alpha} (\vartheta_x + \xi_x \cos \gamma) = -\partial_z (\varphi) \xi_x + \partial_z (\varphi) \zeta_x + \vartheta_x \varphi_x - \vartheta_z \varphi_x. \] (3.17)

We rewrite them as
\[ \Delta \vartheta_t - g \varphi_x - f \kappa_x + \xi_{xx} - g \eta_x \cos \gamma - f \xi_x \sin \gamma + (\gamma' \zeta + \partial_z (\vartheta)) \zeta_{xx} = 0, \] (3.18)
\[ \kappa_t + \eta_t + f \vartheta_x - \vartheta_x \kappa_x + \vartheta_z \kappa_x + (\gamma' \zeta + \partial_z (\vartheta)) \eta_x + (f \sin \gamma - \partial_z (\kappa)) \xi_x = 0, \] (3.19)
\[ \varphi_t + \zeta_t + \frac{N^2}{\alpha} \vartheta_x - \vartheta_x \varphi_x + \vartheta_z \varphi_x + (\gamma' \zeta + \partial_z (\vartheta)) \zeta_x + \left( \frac{N^2}{\alpha} \cos \gamma - \partial_z (\varphi) \right) \xi_x = 0. \] (3.20)

In order to solve the above system, we assume
\[ \vartheta = -\frac{\gamma'}{2} \zeta^2 - \frac{(\alpha' \zeta + \beta') \zeta}{\alpha} + \frac{\alpha_1}{2} \zeta^2 + \beta_1 \zeta \] (3.21)
for some functions \( \alpha, \alpha_1, \beta, \beta_1 \) in \( t \), and
\[ \Delta \vartheta_t - g \varphi_x - f \kappa_x = 0, \] (3.22)
\[ \kappa_t + f \vartheta_x - \vartheta_x \kappa_x + \vartheta_z \kappa_x = 0, \] (3.23)
\[ \varphi_t + \frac{N^2}{\alpha} \vartheta_x - \vartheta_x \varphi_x + \vartheta_z \varphi_x = 0. \] (3.24)

According to (3.1), (3.5) and (3.21),
\[ \gamma' \zeta + \partial_z (\vartheta) = -\frac{\alpha' \zeta + \beta'}{\alpha}, \] (3.25)
\[ \vartheta_x = \left( \gamma' \zeta + \frac{\alpha' \zeta + \beta'}{\alpha} \right) \sin \gamma + \left( -\frac{\alpha'}{\alpha} \zeta + \alpha_1 \zeta + \beta_1 \right) \cos \gamma, \] (3.25)
\[ \vartheta_z = -\left( \gamma' \zeta + \frac{\alpha' \zeta + \beta'}{\alpha} \right) \cos \gamma + \left( -\frac{\alpha'}{\alpha} \zeta + \alpha_1 \zeta + \beta_1 \right) \sin \gamma. \] (3.26)

In particular,
\[ -\vartheta_x \vartheta_z + \vartheta_z \vartheta_x = -\left( \gamma' \zeta + \frac{\alpha' \zeta + \beta'}{\alpha} \right) \partial_x + \left( \frac{\alpha'}{\alpha} \zeta - \alpha_1 \zeta - \beta_1 \right) \partial_z \] (3.27)

This motivates us to assume
\[ \kappa = \alpha_2 \zeta + \beta_2 \zeta + \nu_2, \quad \varphi = \alpha_3 \zeta + \beta_3 \zeta + \nu_3, \] (3.28)
where $\alpha_2, \alpha_3, \beta_3, \beta_3, \nu_2, \nu_3$ are functions in $t$ to be determined. Then (3.22)-(3.24) become

$$f(\alpha_3 \cos \gamma - \beta_3 \sin \gamma) + g(\alpha_2 \sin \gamma + \beta_2 \cos \gamma) = \alpha_1' - \gamma', \quad (3.29)$$

$$(\alpha'_2 - \beta_2' \gamma')\mathcal{X} + (\beta'_2 + \alpha_2' \gamma')\mathcal{Z} - \left(\gamma' \mathcal{Z} + \frac{\alpha' \mathcal{X} + \beta'}{\alpha}\right) \alpha_2 + \left(\frac{\alpha'}{\alpha} \mathcal{Z} - \alpha_1 \mathcal{X} - \beta_1\right) \beta_2$$

$$+ f \left[- \left(\gamma' \mathcal{Z} + \frac{\alpha' \mathcal{X} + \beta'}{\alpha}\right) \cos \gamma + \left(-\frac{\alpha'}{\alpha} \mathcal{Z} + \alpha_1 \mathcal{X} + \beta_1\right) \sin \gamma\right] + \nu_2' = 0, \quad (3.30)$$

$$(\alpha'_3 - \beta_3' \gamma')\mathcal{X} + (\beta'_3 + \alpha_3' \gamma')\mathcal{Z} - \left(\gamma' \mathcal{Z} + \frac{\alpha' \mathcal{X} + \beta'}{\alpha}\right) \alpha_3 + \left(\frac{\alpha'}{\alpha} \mathcal{Z} - \alpha_1 \mathcal{X} - \beta_1\right) \beta_3$$

$$+ \frac{N^2}{g} \left[\left(\gamma' \mathcal{Z} + \frac{\alpha' \mathcal{X} + \beta'}{\alpha}\right) \sin \gamma + \left(-\frac{\alpha'}{\alpha} \mathcal{Z} + \alpha_1 \mathcal{X} + \beta_1\right) \cos \gamma\right] + \nu_3' = 0. \quad (3.31)$$

Observe that (3.30) is equivalent to

$$\alpha'_2 - \frac{\alpha'}{\alpha} \alpha_2 - (\alpha_1 + \gamma') \beta_2 + f \left(-\frac{\alpha'}{\alpha} \cos \gamma + \alpha_1 \sin \gamma\right) = 0, \quad (3.32)$$

$$\beta'_2 + \frac{\alpha'}{\alpha} \beta_2 - f \left(\frac{\alpha'}{\alpha} \sin \gamma + \gamma' \cos \gamma\right) = 0, \quad (3.33)$$

$$\nu_2' - \frac{(\alpha_2 + f \cos \gamma) \beta'}{\alpha} + (f \sin \gamma - \beta_2) \beta_1 = 0. \quad (3.34)$$

By (3.33), we take

$$\beta_2 = f \sin \gamma. \quad (3.35)$$

Substituting it to (3.32), we get

$$\alpha'_2 - \frac{\alpha'}{\alpha} \alpha_2 - f \left(\frac{\alpha'}{\alpha} \cos \gamma + \gamma' \sin \gamma\right) = 0. \quad (3.36)$$

So we take

$$\alpha_2 = - f \cos \gamma. \quad (3.37)$$

According to (3.31), (3.35) and (3.37), we take $\nu_2 = 0$.

Next (3.31) is equivalent to

$$\alpha'_3 = \frac{\alpha'}{\alpha} \alpha_3 - (\alpha_1 + \gamma') \beta_3 + \frac{N^2}{g} \left(\alpha_1 \cos \gamma + \frac{\alpha'}{\alpha} \sin \gamma\right) = 0, \quad (3.38)$$

$$\beta'_3 + \frac{\alpha'}{\alpha} \beta_3 + \frac{N^2}{g} \left(\gamma' \sin \gamma - \frac{\alpha'}{\alpha} \cos \gamma\right) = 0, \quad (3.39)$$

$$\nu'_3 - \left(\alpha_3 - \frac{N^2}{g} \sin \gamma\right) \frac{\beta'}{\alpha} + \left(\frac{N^2}{g} \cos \gamma - \beta_3\right) \beta_1 = 0. \quad (3.40)$$

Similarly, we have the solutions:

$$\beta_3 = \frac{N^2}{g} \cos \gamma, \quad \alpha_3 = \frac{N^2}{g} \sin \gamma, \quad \nu_3 = 0. \quad (3.41)$$
Now (3.29) becomes $\alpha'_1 - \gamma'' = 0$. So we take

$$\alpha_1 = \gamma'. \quad (3.42)$$

Observe that (3.18)-(3.20) become

$$\xi_{xx} + gn_x \cos \gamma - f\zeta \sin \gamma - \frac{\alpha' X + \beta'}{\alpha} \xi_{xx} = 0, \quad (3.43)$$

$$\eta_t - \frac{\alpha' X + \beta'}{\alpha} \eta_x = 0, \quad (3.44)$$

$$\zeta_t - \frac{\alpha' X + \beta'}{\alpha} \zeta_x = 0. \quad (3.45)$$

By (3.44) and (3.45), we have

$$\eta = \phi'(\alpha X + \beta), \quad \zeta = \mu'(\alpha X + \beta) \quad (3.46)$$

for some one-variable functions $\phi$ and $\mu$. Moreover, (3.43) yields

$$\xi = \frac{h(\alpha X + \beta)}{\alpha^2} + \int \frac{f\phi(\alpha X + \beta) \sin \gamma + g\mu(\alpha X + \beta) \cos \gamma}{\alpha} dt \quad (3.47)$$

for some one-variable function $h$, where $X$ and $t$ should be treated as independent variables in the integral.

By (3.1),

$$Z \sin \gamma - X \cos \alpha = -x, \quad Z \cos \gamma + X \sin \alpha = z. \quad (3.48)$$

In summary, we have:

**Theorem 3.1.** Let $\alpha, \beta, \beta_1, \gamma, \theta_1, \theta_2, \theta_3$ be any differentiable functions in $t$ such that $\alpha \neq 0$, and let $\phi, \mu, h$ be any differentiable one-variable functions. Denote $X = x \cos \gamma + z \sin \gamma$ and $Z = -x \sin \gamma + z \cos \gamma$. We have the following solution of the equation (1.1)-(1.3):

$$\psi = \frac{\gamma'}{2}(x^2 + z^2) - \frac{(\alpha' X + \beta')Z}{\alpha} + \beta_1 X + \frac{h(\alpha X + \beta)}{\alpha^2} + \int \frac{f\phi(\alpha X + \beta) \sin \gamma + g\mu(\alpha X + \beta) \cos \gamma}{\alpha} dt, \quad (3.49)$$

$$v = -fx + \phi'(\alpha X + \beta), \quad \rho = \frac{N^2 z}{g} + \mu'(\alpha X + \beta). \quad (3.50)$$

Applying the symmetry transformation $S_{\theta_1, \theta_2, \theta_3}$ to the above solution, we get a solution of the equation (1.1)-(1.3) with ten arbitrary parameter functions.

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