INITIAL-BOUNDARY VALUE PROBLEMS
FOR LINEAR PDEs:
THE ANALYTICITY APPROACH

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Abstract

It is well-known that the main difficulties associated with the study of initial-boundary value problems for linear PDEs is given by the presence of unknown boundary values in any method of solution. To deal efficiently with this difficulty, we have recently proposed two alternative (but interrelated) methods in Fourier space: the Analyticity approach and the Elimination by Restriction approach. In this work we present the Analyticity approach and we illustrate its power in studying the well-posedness of initial-boundary value problems for second and third order evolutional PDEs, and in constructing their solution. We also show the connection between the Analyticity approach and the Elimination by Restriction approach in the study of the Dirichelet and Neumann problems for the Schrödinger equation in the \(n\)-dimensional quadrant.

1 Introduction

It is well-known that the main difficulties associated with Initial-Boundary Value (IBV) problems for linear PDEs of the type

\[
\mathcal{L}(\nabla, \frac{\partial}{\partial t})u(x, t) = f(x, t), \quad u(x, 0) = u_0(x), \quad x \in V \subset \mathbb{R}^n, \quad t > 0, \quad (1)
\]

where \(\nabla = (\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n})\), \(\mathcal{L}\) is a constant coefficients partial differential operator, \(u(x, t)\) is the unknown field, \(f(x, t)\) is a given forcing and \(u_0(x)\) is the given initial condition, with Dirichelet, or Neumann, or Robin, or mixed, or periodic
boundary data on \( \partial V \), is given by the presence of unknown Boundary Values (BVVs) in any method of solution. To deal efficiently with this difficulty, we have recently proposed two alternative (but interrelated) methods in Fourier space: the Analyticity approach and the Elimination by Restriction approach.

The first step, common to both methods, consists in rewriting the PDE (1), defined in a space-time domain \( D \), in the corresponding Fourier space, using the Green’s formula. The PDE in Fourier space takes the form of a linear relation among the Fourier Transforms (FTs) of the solution, of the initial condition and of a set of BVVs, only a subset of which is given a priori. This relation is always supplemented by strong analyticity requirements on all the FTs involved, consequence of the geometric properties of the space-time domain \( D \).

The second step is where the two methods separate; once the problem is formulated in Fourier space, we propose the following two alternative strategies.

i) The Analyticity approach, which consists in using systematically the analyticity properties of all the FTs involved in the above relation, to derive a system of linear equations which allows one to express the unknown BVVs in terms of the known ones, and therefore to solve the problem.

ii) The Elimination by Restriction (EbR) approach, which consists, instead, in applying to the above linear relation in Fourier space a suitable annihilation operator, which eliminates all the unknown BVVs, generating a new transform, well-suited to the specific IBV problem under scrutiny. The inversion of this new transform (if it exists) leads to the solution.

The Analyticity approach is inspired by Fokas’ recent discovery of the global relation, obtained first within the \( x - t \) transform approach [1] and more recently using differential forms [2]. The use of the global relation to study the well-posedness and solve IBV problems is illustrated, for instance, in [3], [4], [5]. In [5], in particular, general results on the well-posedness of IBV problems for dispersive 1 + 1 dimensional equations of arbitrary order are announced.

Our main contribution to the method consists, after formulating the IBV problem in Fourier space using Green’s formula, in imposing systematically the analyticity properties of all the Fourier transforms involved in the problem, to derive a cascade of analyticity constraints which allow one to express the unknown BVVs in terms of the known ones, and therefore to solve the problem. In particular, Fokas’ global relation appears, in the methodology we propose, as a “zero residue condition” for the FT of the solution.

The Analyticity approach in the form we propose is very elementary and, above all, has the great conceptual advantage to originate from a single guiding principle: the analyticity of all the Fourier transforms involved in the problem. It is the type of approach that can be easily taught in elementary University courses, combining nicely standard PDE theory tools, like the Green’s formula and the Fourier transform, with elementary notions in Complex Functions theory.

The essential aspects of the Analyticity approach were first presented by the authors at the Workshop “Boundary value problems” in Cambridge, De-
November 2001, inside the programme: “Integrable Systems”. The method in its final form is presented for the first time in this paper, illustrated on the study of IBV problems of various type (Dirichelet, Neumann, mixed, periodic) for some second and third order classical PDEs of the Mathematical Physics: the Schrödinger, the heat and the linear Korteweg-de Vries equations. Also its connections with the EbR approach are illustrated in this work, on the particular example of the Schrödinger equation in the n-dimensional quadrant. A good account of the EbR approach is given instead in [6]. A different approach, valid for semicompact domains, has been recently presented in [7]. A general review of the basic spectral methods of solution of IBV problems for linear and soliton PDEs is presented in [8].

§2 is devoted to the presentation of the Analyticity approach, while §3 is dedicated to its application to the solution of some IBV problems for second and third order evolutionary PDEs in 1+1 and in n+1 dimensions. §4 is finally devoted to the study of the connections between the Analyticity approach and the EbR approach.

2 The Analyticity Approach

2.1 The Fourier transform and its analyticity properties

The natural FT associated with the space-time domain \( D = V \otimes (0, \infty) \) (in short: \( FT_D \)) is defined by

\[
\hat{F}(k, q) = \int_D dxe^{-i(k \cdot x + qt)} F(x, t)
\] (2)

for any smooth function \( F(x, t) \), \((x, t) \in D\), assuming that \( F(x, t) \to 0, \ t \to \infty \) fast enough; where \( k = (k_1, \ldots, k_n) \in \mathbb{R}^n \), \( q \in \mathbb{R} \) and \( k \cdot x = \sum_{j=1}^{n} k_j x_j \). Its inverse:

\[
F(x, t)\chi_D(x, t) = \int_{\mathbb{R}^{n+1}} \frac{dkdq}{(2\pi)^{n+1}} e^{i(k \cdot x + qt)} \hat{F}(k, q)
\] (3)

reconstructs \( F(x, t) \) in \( D \) and zero outside, where \( \chi_D(x, t) \) is the characteristic function of the domain \( D \): \( \chi_D(x, t) = 1, \ (x, t) \in D \), \( \chi_D(x, t) = 0, \ (x, t) \notin D \) (therefore: \( \chi_D(x, t) = \chi_V(x)H(t) \), where \( H(t) \) is the usual Heaviside (step) function).

If the space domain is the whole space: \( V = \mathbb{R}^n \), the \( FT_D \) (2) is defined in \( A = \mathbb{R}^n \otimes \overline{I}_q \), where \( \overline{I}_q \) is the closure of the lower half \( q \)-plane \( I_q \), analytic in \( q \in I_q, \ \forall k \in \mathbb{R}^n \) and exhibits a proper asymptotic behaviour for large \( q \) in the analyticity region. If the space domain \( V \) is compact, the \( FT_D \) acquires strong analyticity properties in all the Fourier variables: it is defined in \( A = \mathbb{C}^n \otimes \overline{I}_q \), analytic in \( q \in I_q, \ \forall k \in \mathbb{C}^n \), entire in every complex \( k_j, \ j = 1, \ldots, n \) \( \forall q \in \overline{I}_q \) and exhibits a proper asymptotic behaviour, for large \((k, q)\), in the analyticity regions. If the space domain is semi-compact, then the analyticity in the
Fourier variables $k_j, j = 1, \ldots, n$ is limited to open regions of the complex plane, depending on the geometric properties of the domain $V$.

We are therefore led to the following definition:

**Definition of admissibility.** Given a space-time domain $D$, a function of $(k, q)$ is an *admissible Fourier transform for the domain* $D$ (an admissible $FT_D$) iff it possesses the analyticity properties and the asymptotic behaviour corresponding to that domain.

### 2.2 The IBV problem in Fourier space

We find it convenient to rewrite the IBV problem (1) in Fourier space. This goal is conveniently achieved using the well-known *Green’s formula (identity)*:

$$bL a - a \hat{L} b = \text{div} J(x, t),$$

and its integral consequence, the celebrated *Green’s integral identity*:

$$\int_D (bL a - a \hat{L} b) dxdt = \int_{\partial D} J(x, t) \cdot \nu d\sigma,$$

obtained by integrating (4) over the domain $D$ and by using the divergence theorem. In equation (4), $\hat{L}$ is the formal adjoint of $L$: $\hat{L} = L(-\nabla, -\frac{\partial}{\partial t})$, $J(x, t)$ is an $(n+1)$-dimensional vector field, $\text{div}$ is the $(n+1)$-dimensional divergence operator and $a(x, t)$ and $b(x, t)$ are arbitrary functions. In equation (5), $d\sigma$ is the hypersurface element of the boundary and $\nu$ is its outward unit normal. We remark that, given $L$, its formal adjoint $\hat{L}$ and two arbitrary functions $a$ and $b$, an $(n+1)$-dimensional vector field $J(x, t)$ satisfying the Green’s formula (4) always exists and can be algorithmically found to be a linear expression of $a$, $b$ and their partial derivatives of order up to $N - 1$, if $L$ is of order $N$.

The arbitrariness of $a$ and $b$ allows one to extract from (4) and (5) several important informations on the IBV problem; with the particular choice

$$a = u(x, t), \quad b = e^{-i(k \cdot x + qt)} / L(ik, iq),$$

where $L(ik, iq)$ is the eigenvalue of the operator $L$, corresponding to the eigenfunction $e^{i(k \cdot x + qt)}$, the vector field $J$ takes the following form:

$$J = e^{-i(k \cdot x + qt)} J'(x, t, k, q) / L(ik, iq)$$

and the Green’s integral identity (5) gives the $FT_D$ of the solution in terms of the $FT_D$’s (or, maybe, of generalized FT’s) of the forcing and of all the IBVs:

$$\hat{u}(k, q) = \hat{f}(k, q) - \int_{\partial D} \frac{e^{-i(k \cdot x + qt)} J'(x, t, k, q) \cdot \nu d\sigma}{L(ik, iq)} \cdot \hat{N}(k, q) / L(ik, iq), \quad (k, q) \in \mathcal{A}.$$  

(7)

If the PDE has the following evolutionary form:

$$L(\nabla, \frac{\partial}{\partial t}) = \frac{\partial}{\partial t} - K(\nabla)$$

(8)
then
\[ \hat{u}(k, q) = \hat{f}(k, q) + \hat{u}_0(k) + \hat{B}(k, q) =: \hat{N}(k, q), \quad (k, q) \in \mathcal{A} \] (9)
and the linear relation (9) makes clear how the different contributions coming from the equation (the denominator \( \mathcal{L} \)), from the forcing \( \hat{f} \), from the initial condition \( \hat{u}_0 \) and from the set of boundary values \( \hat{B} \) separate in Fourier space.

Its inverse transform (3) gives the corresponding **Fourier representation** of the solution:

\[ U(x, t) = u(x, t) \chi_D(x, t) = \int_{\mathbb{R}^{n+1}} \frac{dkdq}{(2\pi)^{n+1}} e^{i(k\cdot x + qt)} \frac{\hat{N}(k, q)}{\mathcal{L}(ik, iq)} \], \quad (x, t) \in \mathbb{R}^{n+1}. \] (10)

Two sources of problems arise at a first glance of equation (7):

i) the RHS of the equation depends on known and unknown BVs;

ii) apparently the RHS of the equation is not an admissible \( FT_D \).

It is very satisfactory that the analyticity constraints which make the RHS of (7) an admissible \( FT_D \) provide also a number of relations among the IBVs which are sufficient to express the unknown BVs in terms of known boundary data.

### 2.3 The analyticity constraints and their resolution

In general, \( \mathcal{L}(ik, iq) \), the denominator of equation (7), is an entire and, most frequently, polynomial function of all its complex variables. Let \( \mathcal{S} \) be the manifold in which this entire function is zero:

\[ \mathcal{S} = \{(k, q) \in \mathbb{C}^{n+1} : \mathcal{L}(ik, iq) = 0\}. \] (11)

Then the RHS of equation (7) provides an admissible \( FT_D \) of the solution of the IBV problem under investigation if the numerator \( \hat{N}(k, q) \) of \( \hat{u} \) in (7) satisfies in \( A \cap S \), hereafter called the **singularity manifold (SM)** of the IBV problem, the following **Zero Residue Condition (ZRC)**:

\[ \hat{N}(k, q) = 0, \quad (k, q) \in A \cap S. \] (12)

If the singularity manifold \( A \cap S \) contains the real axis (which is usually a part of the boundary of \( A \)) and if this singularity is not already taken care of by the ZRC (12), then we must also proceed to the **Denominator Regularization (DR)**:

\[ \mathcal{L}(ik, iq) \to \mathcal{L}_{reg}(ik, iq), \] (13)
which consists in moving a bit the singularity off the real axis, outside the domain \( \mathcal{A} \).

The ZRC plus the DR constitute the main set of **Analyticity Constraints (ACs)** that must be imposed to the RHS of (7) in order to obtain an admissible...
$FT_D$ of the solution of the IBV problem under investigation. The ZRC (12) provides a (linear) relation among the FTs of the forcing, of the initial condition and of all the BVs. The analyticity properties of all these FTs generate, through the admissibility argument, a cascade of further analyticity constraints, until all these conditions are finally met. This procedure defines, in principle, a set of relations (a system of equations) among the IBVs. Therefore:

a) The unique solvability of such a system, together with the admissibility of the obtained solution, are equivalent to the study of the unique solvability of all the IBV problems associated with (1).

b) By solving this system for a set of BVs in terms of the remaining ones, one expresses all quantities in terms of known data and, from equation (10), one obtains the Fourier representation of the solution.

In most of the examples considered in this paper, this system of equations is algebraic, with entire coefficients. Therefore, if $M$ is the squared matrix of the coefficients of the unknown BVs, the admissibility argument imposes that the countable number of zeroes of $\det M$:

$$\{q_m\}_{m \in \mathbb{N}}, \quad \det M(q_m) = 0 \quad (14)$$

lie outside the analyticity domain of an admissible $FT_D$:

$$q_m \notin \mathcal{I}_q, \quad m \in \mathbb{N}. \quad (15)$$

It turns out that the set (14) coincides with the spectrum arising in the eigenfunction expansion approach [9] and coincides also with the restricted domain in which the EbR method works. These deep connections justify for (14) the name of spectrum of the IBV problem.

The admissibility argument imposes also that the constructed solution of the system exhibit the proper asymptotic behaviour in the analyticity domain. It is actually convenient to impose first this asymptotic admissibility, the easiest to be checked, which enables one to disregard without effort all the IBV problems ill-posed because incompatible with asymptotics.

### 2.4 General remarks

**Remark 1. Analyticity vs Causality.** It is well-known that there are definite connections between the analyticity properties of the FT of the solution of evolution equations and the causality principle. In our general setting it is straightforward to show that:

The analyticity properties of the $FT_D$ of the solution of the IBV problem (1) imply the causality principle.

Indeed, using the convolution theorem, the inverse FT (10) of the RHS of equation (7) (in which all the analyticity constraints have been preliminary imposed)
is equivalent to the following Green’s representation of the solution:

\[
u(x, t) = \int_0^t dt' \int_V d\mathbf{x}' G_{RF}(\mathbf{x} - \mathbf{x}', t - t') \mathcal{N}(\mathbf{x}', t'), \quad (x, t) \in \mathcal{D}, \quad (16)\]

where \(\mathcal{N}(\mathbf{x}, t)\chi_D(\mathbf{x}, t)\) is the inverse FT (3) of \(\hat{\mathcal{N}}(\mathbf{k}, q)\) and \(G_{RF}\) is the celebrated retarded fundamental Green’s function of the operator \(\mathcal{L}\):

\[
G_{RF}(x, t) = \int_{\mathbb{R}^{n+1}} \frac{dk dq e^{i(k \cdot x + qt)}}{(2\pi)^n L_{reg}(i\mathbf{k}, iq)}, \quad (17)\]

which satisfies the important property: \(G_{RF}(x, t) = 0, \ t < 0\), due to the regularization of \(\mathcal{L}(i\mathbf{k}, iq)\). Equation (16) is the usual way in which the causality principle becomes transparent.

**Remark 2. Regularization and Fourier representation.** As we have already written, if the zeroes of the denominator on the real axis are all cured by the ZRC, no regularization is needed. On the other hand, some regularization must be introduced also in this case, in the calculation the Fourier representation (10), before splitting \(\hat{\mathcal{N}}\) in the sum of the different contributions (each one singular on the real axis) in (9) coming from the forcing, from the initial condition and from the BVs. The most convenient regularization is obviously that in (13) and leads to the following Fourier representation:

\[
\mathcal{U}(x, t) = \mathcal{U}(x, t)\chi_D(x, t) = \int_{\mathbb{R}^{n+1}} \frac{dk dq e^{i(k \cdot x + qt)}}{(2\pi)^n L_{reg}(i\mathbf{k}, iq)} f(k, q) + \int_{\mathbb{R}^{n+1}} \frac{dk dq e^{i(k \cdot x + qt)}}{(2\pi)^n L_{reg}(i\mathbf{k}, iq)} \overline{a}_0(k) + \int_{\mathbb{R}^{n+1}} \frac{dk dq e^{i(k \cdot x + qt)}}{(2\pi)^n L_{reg}(i\mathbf{k}, iq)} \overline{a}_{\mathcal{G}}(k, q), \quad (x, t) \in \mathbb{R}^{n+1}, \quad (18)\]

### 3 Illustrative Examples

In this section we apply the Analyticity approach to the following classical equations of the Mathematical Physics, the Schrödinger, the heat and the linear Korteweg-de Vries (KdV) equations:

\[
\frac{\partial u}{\partial t} - \alpha \frac{\partial^2 u}{\partial x^2} = f, \quad \alpha = i, 1 \quad x \in V, \ t > 0, \quad (19)\]

\[
\frac{\partial u}{\partial t} - \eta \frac{\partial^3 u}{\partial x^3} = f, \quad \eta = \pm 1, \quad x \in V, \ t > 0, \quad (20)\]

prototype examples respectively of second and third order evolutionary PDEs and basic universal models for the description of dispersive and diffusive phenomena, where the space domain \(V\) is either the segment \((0, L)\) or the semiline \((0, \infty)\). Hereafter the BVs will be indicated by

\[
v^{(j)}_0(t) := \frac{\partial^j u}{\partial x^j}(x, t)|_{x=0}, \quad v^{(j)}_L(t) := \frac{\partial^j u}{\partial x^j}(x, t)|_{x=L}, \quad j \in \mathcal{N} \quad (21)\]
and their Fourier transforms by $\hat{v}_0^{(j)}(q), \hat{v}_L^{(j)}(q)$:

$$
\hat{v}_0^{(j)}(q) := \int_0^\infty dt e^{-iqt} v_0^{(j)}(t), \quad \hat{v}_L^{(j)}(q) := \int_0^\infty dt e^{-iqt} v_L^{(j)}(t).
$$

We also apply the method to the study of IBV problems for the multimeimensional analogue of equation (19), for $\alpha = i$:

$$
\frac{\partial u}{\partial t} - i \Delta u = f, \quad x \in V, \; t > 0, \quad \Delta := \nabla \cdot \nabla = \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2}
$$

in the $n$-dimensional Quadrant

$$
V = \{x : x_j \geq 0, \; j = 1, \ldots, n\}. \tag{24}
$$

The corresponding BVs will be indicated by:

$$
v_0^{(0)}(x_j, t) = u(x, t)|_{x_j=0}, \quad v_0^{(1)}(x_j, t) = \frac{\partial u}{\partial x_j}(x, t)|_{x_j=0} \tag{25}
$$

and their FTs by:

$$
\hat{v}_0^{(m)}(k_j, q) = \int_0^\infty dt \int \mathbb{V} \mathbb{J} e^{-i(k_j \cdot x_j + qt)} v_0^{(m)}(x_j, t), \quad m = 0, 1. \tag{26}
$$

In equations (25)-(26) $x_j = (x_1, \ldots, \hat{x}_j, \ldots, x_n) \in \mathbb{R}^{n-1}$, $k_j = (k_1, \ldots, \hat{k}_j, \ldots, k_n) \in \mathbb{R}^{n-1}$, $\int_\mathbb{V} \mathbb{J} \mathbb{X} = \int_0^{L_1} dx'_1 \cdots (\int_0^{L_j} dx'_j) \cdots \int_0^{L_n} dx'_n$ and the superscript $-$ indicates that the quantity underneath is removed.

The application of the Analyticity approach to higher order problems and to other relevant examples will be presented in [8].

### 3.1 The second order PDEs (19)

In this case, equations (4) and (6) imply:

$$
\mathcal{L} = -\frac{\partial}{\partial t} - a \frac{\partial^2}{\partial x^2}, \quad J = (ab, a[\alpha \frac{\partial b}{\partial x} - b \frac{\partial a}{\partial x}]), \tag{27}
$$

In addition, if $V$ is the segment $(0, L)$, equation (9) yields

$$
\hat{\mathcal{L}}(k, q) = \hat{f}(k, q) + \hat{u}_0(k) - \alpha \left( [\hat{v}_0^{(1)}(q) + i\alpha \hat{v}_0^{(0)}(q)] - e^{-ikL} [\hat{v}_0^{(1)}(q) + i\alpha \hat{v}_0^{(0)}(q)] \right) \tag{28}
$$
and the Fourier representation (18) of the solution takes the following form:

\[
\begin{align*}
    u(x,t) &= \int_{\mathbb{R}^2} \frac{dqdk}{(2\pi)^2} e^{i(kx+qt)} \frac{f(k,q)}{q-i\alpha k^2-i0} + \int_{\mathbb{R}} \frac{dk}{2\pi} e^{ikx-\alpha k^2t} \hat{u}_0(k) + \\
    &\int_{\partial K^{(-1)}_1} \frac{dk}{2\pi} e^{ikx-\alpha k^2t} \hat{v}_L^{(1)}(i\alpha k^2) + i k\hat{v}_0^{(0)}(i\alpha k^2) + \\
    &\int_{\partial K^{(-1)}_0} \frac{dk}{2\pi} e^{ik(x-L)-\alpha k^2t} \hat{v}_L^{(1)}(i\alpha k^2) + i k\hat{v}_0^{(0)}(i\alpha k^2), \ x \in (0,L), \ t > 0,
\end{align*}
\]

(29)

where \(K^{(-1)}_1\) and \(K^{(-1)}_0\) are respectively the first and third quadrant of the complex \(k\)-plane, \(K_m = \hat{\rho}_{q\pi/4} K_m\), \(m = 0,1\), where \(\hat{\rho}_{\pi/4}\) is the \(\pi/4\) rotation operator:

\[\hat{\rho}_{\pi/4}: k \rightarrow e^{i\pi/4} k\]

and \(\partial K\) is the counterclockwise oriented boundary of \(K\) (see Figs 1a,b).

The corresponding expressions for the semiline or for the infinite line cases, with rapidly decreasing conditions at \(\infty\), follow immediately from the ones above, setting \(\hat{v}_L^{(0)} = \hat{v}_L^{(1)} = 0\) in the semiline case, or setting \(\hat{v}_0^{(0)} = \hat{v}_0^{(1)} = \hat{v}_L^{(0)} = \hat{v}_L^{(1)} = 0\) in the infinite line case.

It is instructive to first apply the Analyticity approach to the simplest case in which the space domain is the whole space, with rapidly decreasing BVs at \(x = \pm \infty\).

### 3.1.1 The whole line \(V = (-\infty, \infty)\)

Equation (28b) reduces to

\[
\hat{N}(k,q) = \hat{f}(k,q) + \hat{u}_0(k)
\]

(30)

and the admissibility argument imposes that \(\hat{N}(k,q)/\mathcal{L}\) be defined in \((k,q) \in \mathcal{A} = \mathbb{R} \otimes \mathbb{I}_q\) and be analytic in \(q \in \mathbb{I}_q, \forall k \in \mathbb{R}\). If \(\alpha = 1\), the denominator is singular for \(q = -k^2, k \in \mathbb{R}\), outside the definition domain, and no regularization is needed. If, instead, \(\alpha = i\), the denominator is singular for \(q = -k^2 < 0\), on the real negative axis, at the boundary of the analyticity domain, and the only analyticity constraint to be fulfilled is the Denominator Regularization (13):

\[
\mathcal{L}(ik,iq) = i(q - i\alpha k^2) \rightarrow \mathcal{L}_{\text{reg}}(ik,iq) = i(q - i\alpha k^2 - i0).
\]

(31)

The regularization (31) is sufficient to make the RHS of (28a) an admissible \(FT_D\), from which we recover the well-known Fourier representation of the solution of equations (19):

\[
\begin{align*}
    u(x,t) &= \int_{\mathbb{R}^2} \frac{dqdk e^{i(kx+qt)}}{q-i\alpha k^2-i0} \hat{f}(k,q) + \int_{\mathbb{R}} \frac{dk}{2\pi} e^{ikx-\alpha k^2t} \hat{u}_0(k), \ x \in \mathbb{R}, \ t > 0.
\end{align*}
\]

(32)

We now proceed considering semi-compact and compact domains.
3.1.2 The semiline $V = (0, \infty)$

In this case:

$$\hat{N}(k, q) = \hat{f}(k, q) + \hat{u}_0(k) - \alpha [\hat{v}_0^{(1)}(q) + i k \hat{v}_0^{(0)}(q)]$$  \hspace{1cm} (33)

and admissibility imposes that $\hat{N}/\mathcal{L}$ be defined in $\mathcal{A} = \mathcal{I}_k \otimes \mathcal{I}_q$, be analytic in $q \in \mathcal{I}_q$, $\forall k \in \mathcal{I}_k$ and be analytic in $k \in \mathcal{I}_k$, $\forall q \in \mathcal{I}_q$. Therefore the singularity manifolds $\mathcal{A} \cap \mathcal{S}^{(\alpha)}$, corresponding to $\alpha = i, 1$, are parametrizable either in terms of $k$ or in terms of $q$ in the following way:

$$\mathcal{A} \cap \mathcal{S}^{(\alpha)} = \left\{ q = i \alpha k^2, \ k \in \bar{k}_0^{(\alpha)} \right\} = \{ k = k_0^{(\alpha)}(q), \ \pi \leq \arg q \leq 2\pi \}, \hspace{1cm} (34)$$

where

$$k_0^{(\alpha)}(q) = \left\{ \begin{array}{ll} iq^{\frac{1}{2}}, & \alpha = i \\ e^{\pi \alpha i q^{\frac{1}{2}}}, & \alpha = 1. \end{array} \right. \hspace{1cm} (35)$$

If $\alpha = 1$, there is no singularity on the real axis and no regularization is needed. If $\alpha = i$, there are two singularities for $k \in \mathcal{R}$; that corresponding to $k < 0$ is cured by the ZRC (12), while that corresponding to $k > 0$ is cured instead by the regularization (31).

The ZRC (12) is conveniently parametrized in terms of $q$ in the following way:

$$\hat{N} (k_0^{(\alpha)}(q), q) = \hat{f}(k_0^{(\alpha)}(q), q) + \hat{u}_0(k_0^{(\alpha)}(q)) - \alpha [\hat{v}_0^{(1)}(q) + i k_0^{(\alpha)}(q) \hat{v}_0^{(0)}(q)] = 0,$$ \hspace{1cm} (36)

for $\pi \leq \arg q \leq 2\pi$. It is one equation involving 4 FT’s which are therefore dependent. If we are interested in solving the Dirichelet and Neumann problems, we use this ZRC to express the unknown BVs in terms of the known ones:

**Dirichelet problem:** \hspace{1cm} $\alpha^*_0^{(1)}(q) = \hat{f}(k_0^{(\alpha)}(q), q) + \hat{u}_0(k_0^{(\alpha)}(q)) - \alpha i k_0^{(\alpha)}(q) \hat{v}_0^{(0)}(q),$  \hspace{1cm} (37)

**Neumann problem:** \hspace{1cm} $i k_0^{(\alpha)}(q) \hat{v}_0^{(0)}(q) = \hat{f}(k_0^{(\alpha)}(q), q) + \hat{u}_0(k_0^{(\alpha)}(q)) - \alpha \hat{v}_0^{(1)}(q),$  \hspace{1cm} (37)

for $\pi \leq \arg q \leq 2\pi$. It is easy to see from (37) that the unknown BVs define admissible FTs which, inserted in (29), give the wanted solution of the Dirichelet and Neumann problems.

We remark that the ZRC (36) could also be solved for $\hat{u}_0$ (using now, for convenience, the variable $k$):

$$\hat{u}_0(k) = -\hat{f}(k, i k^2) + \alpha [\hat{v}_0^{(1)}(i k^2) + i k \hat{v}_0^{(0)}(i k^2)], \ k \in \bar{k}_0^{(\alpha)} \hspace{1cm} (38)$$

but, in this case, the solution would not be, in general, an admissible FT, since the RHS of (38) cannot be extended to the rest of the lower half $k$-plane. Even in the special case in which the forcing and the assigned BVs were on a compact support in $t$, corresponding to entire FTs, the solution $\hat{u}_0(k)$ would not
be admissible, because it would not possess, in general, the proper asymptotics. This means that the (unphysical) problem in which we assign arbitrarily \( u \) and its space derivative at \( x = 0 \) cannot be treated by this method, unless the above BVs are suitably constrained.

### 3.1.3 The segment \( V = (0, L) \)

Now admissibility implies that \( \hat{N} / \mathcal{L} \) be defined in \( \mathcal{A} = \mathcal{C} \otimes \mathcal{I}_q \), be analytic in \( q \in \mathcal{I}_q \), \( \forall k \in \mathcal{C} \) and be analytic in \( k \in \mathcal{C} \), \( \forall q \in \mathcal{I}_q \), with proper asymptotics for large \( |k| \) and/or \( |q| \) in the analyticity regions. Therefore the singularity manifolds on which the ZRC (12) is defined are now the unions of two sectors:

\[
\mathcal{A} \cap \mathcal{S}^{(\alpha)} = \bigcup_{m=0}^{1} \{ q = i\alpha k^2, \ k \in K_m^{(\alpha)} \} = \bigcup_{m=0}^{1} \{ k = k_m^{(\alpha)}(q) = (-)^m k_0^{(\alpha)}(q), \ \pi \leq \arg q \leq 2\pi \}. \tag{39}
\]

![Fig.1a The SM \( \mathcal{A} \cap \mathcal{S}^{(\alpha)} (\alpha = i) \)

![Fig.1b The SM \( \mathcal{A} \cap \mathcal{S}^{(\alpha)} (\alpha = 1) \)

Both singularities on the real axis are cured by the ZRC and no regularization is needed. The regularization (31), however, is still introduced, according to the Remark 2 of \( \S 2.4 \), in computing the Fourier representation (65) of the solution. The ZRC (12), conveniently parametrized using \( q \), consists of the following system of two linear algebraic equations:

\[
\hat{N}(k_m^{(\alpha)}(q), q) = 0, \quad m = 0, 1 \quad \pi \leq \arg q \leq 2\pi \tag{40}
\]

containing four BVs. Therefore we expect to be allowed to assign arbitrarily two out of four BVs. To establish which pairs of BVs can be assigned arbitrarily, one should impose that the corresponding solutions of the algebraic system define admissible FTs; i.e., the following two conditions must be satisfied.

1) The system must be uniquely solvable for the unknown pair of BVs in its definition domain. More precisely, indicating by \( M \) the \( 2 \times 2 \) matrix of the coefficients of the unknown BVs, the admissibility condition imposes that the
countable set \( \{ q_j \}_{j \in \mathcal{N}} \) of zeroes of \( \det M \), the spectrum of the IBV problem, lie outside the analyticity domain:

\[
q_j \notin \mathcal{I}_q, \quad j \in \mathcal{N}.
\]  

ii) The solution of the system must define admissible Fourier Transforms; in particular, it must exhibit the proper asymptotics in the analyticity domain.

Studying first the asymptotics of (40), one infers without any effort which pairs of BVs cannot be assigned arbitrarily. The asymptotics of (40) imply immediately that the following expressions:

\[
\hat{f}(k^{(\alpha)}_0(q), q) + \hat{u}_0(k^{(\alpha)}_0(q)) - \alpha [\hat{v}^{(1)}_0(q) + ik^{(\alpha)}_0(q)\hat{v}^{(0)}_0(q)],
\]

\[
e^{ik^{(\alpha)}_0(q)L}[f(-k^{(\alpha)}_0(q), q) + \hat{u}_0(-k^{(\alpha)}_0(q))] + \alpha [\hat{v}^{(1)}_L(q) - ik^{(\alpha)}_L(q)\hat{v}^{(0)}_L(q)]
\]  

are exponentially small for \( q \sim \infty \quad \pi \leq \arg q \leq 2\pi \). Since the asymptotic series of the admissible FTs appearing in the LHS of equations (42) impose severe constraints on the involved functions, implying that:

asymptotic admissibility is compatible with assigning at \( x = 0 \) any BV between \( (v^{(0)}_0, v^{(1)}_0) \) and, at \( x = L \), any BV between \( (v^{(0)}_L, v^{(1)}_L) \). It is not compatible instead with assigning arbitrarily the pairs \( (v^{(1)}_0, v^{(0)}_0) \) or \( (v^{(1)}_L, v^{(0)}_L) \).

To complete our analysis, we must check if the spectrum associated with the IBV problems compatible with the asymptotics lie outside the definition domain. The analysis is straightforward and produces the following results.

**Proposition (the spectrum).** Assigning arbitrarily \((v^{(0)}_0, v^{(0)}_L)\) (the Dirichlet problem) or \((v^{(1)}_0, v^{(1)}_L)\) (the Neumann problem), the spectrum is characterized by the equation \( \sin(kL) = 0 \iff k_m = \frac{2m\pi}{L}, \quad m \in \mathbb{Z} \) and is given by the negative eigenvalues \( \{q_m\}_{m \in \mathcal{N}} \), \( q_m = -k_m^2 = -(\frac{m\pi}{L})^2 \), \( m \in \mathcal{N}, \) if \( \alpha = i \), and by the purely imaginary eigenvalues \( q_m = i k_m^2 = i(\frac{m\pi}{L})^2 \), \( m \in \mathcal{N}, \) if \( \alpha = 1 \).

Assigning instead \((v^{(0)}_0, v^{(1)}_L)\) or \((v^{(1)}_0, v^{(0)}_L)\) (the mixed problems), the spectrum is characterized by the equation \( \cos(kL) = 0 \iff k_m = \frac{\pi}{L}(2m + 1), \quad m \in \mathbb{Z} \) and is given by the negative eigenvalues \( \{q_m\}_{m \in \mathcal{N}} \), \( q_m = -k_m^2 = -(\frac{n\pi}{L})^2 \), \( m \in \mathcal{N}, \) if \( \alpha = i \), and by the purely imaginary eigenvalues \( q_m = i k_m^2 = i(\frac{n\pi}{L})^2 \), \( m \in \mathcal{N}, \) if \( \alpha = 1 \).

For \( \alpha = 1 \) the spectrum lies outside the analyticity region and the solutions of the algebraic system (40) define directly admissible FTs; if \( \alpha = i \) the solutions of the algebraic system (40) define admissible FTs after moving these singularities a bit off the real \( q \)-axis, outside the definition domain (again a regularization!).

We conclude that all the IBV problems compatible with admissible asymptotics turn out to be well-posed:

**IBV problems for the Schrödinger and heat equations (19) are well-posed assigning at \( x = 0 \) any BV among \((v^{(0)}_0, v^{(1)}_0)\) and at \( x = L \) any BV among \((v^{(0)}_L, v^{(1)}_L)\).**
It is interesting to remark that, if one insisted, instead, in solving an IBV problem in which the BVs ($\tilde{v}_{0}^{(1)}(q)$, $\tilde{v}_{0}^{(0)}(q)$) are assigned, the corresponding algebraic system would be always uniquely solvable (no point spectrum would arise), but the solution would exhibit an exponential blow up at $q \sim \infty$ in the analyticity region, that cannot be accepted. This undesired blow up could be cured if the assigned BVs were related by the (additional) analyticity constraint:

$$\hat{f}(k_{0}^{(a)}(q), q) + \hat{u}_{0}(k_{0}^{(a)}(q)) - \alpha [\tilde{v}_{1}^{(1)}(q) + ik_{0}^{(a)}(q)\tilde{v}_{0}^{(0)}(q)] = 0,$$

implying the following admissible solutions of the algebraic system (40):

$$\hat{v}_{L}^{(1)}(q) = -ik_{0}^{(a)}(q)\hat{v}_{0}^{(0)}(q) = -e^{-ik_{0}^{(a)}(q)L}\left[\hat{f}(k_{0}^{(a)}(q), q) + \hat{f}(-k_{0}^{(a)}(q), q) + \hat{u}_{0}(k_{0}^{(a)}(q)) + \hat{u}_{0}(-k_{0}^{(a)}(q)) - 2\alpha \tilde{v}_{0}^{(1)}(q)\right].$$

(44)

The additional analyticity constraint (43) is not surprising at all, since it is nothing but the ZRC of the semiline problem. Similarly, assigning the right boundary conditions ($\hat{v}_{L}^{(0)}$, $\hat{v}_{L}^{(1)}$), the unknowns $\hat{v}_{0}^{(0)}$ and $\hat{v}_{0}^{(1)}$ would exhibit again an exponential blow up which cannot be accepted; an admissible asymptotics would be guaranteed now by the (additional) analyticity constraint:

$$e^{-ik_{0}^{(a)}(q)L}\left[\hat{f}(-k_{0}^{(a)}(q), q) + \hat{u}_{0}(-k_{0}^{(a)}(q))\right] + \alpha [\hat{v}_{1}^{(1)}(q) - ik_{0}^{(a)}(q)\hat{v}_{1}^{(0)}(q)] = 0,$$

implying the following solution of the algebraic system:

$$\hat{v}_{0}^{(1)}(q) = ik_{0}^{(a)}(q)\hat{v}_{0}^{(0)}(q) = \frac{1}{2\alpha}\hat{f}(k_{0}^{(a)}(q), q) + \hat{f}(-k_{0}^{(a)}(q), q) + \hat{u}_{0}(k_{0}^{(a)}(q)) + \hat{u}_{0}(-k_{0}^{(a)}(q)) + 2\alpha e^{-ik_{0}^{(a)}(q)L}\hat{v}_{1}^{(1)}(q).$$

(45)

3.1.4 The periodic problem

If we assume $L$-periodicity of $u$ and $u_{x}$, then $\tilde{v}_{0}^{(1)} = \hat{v}_{L}^{(1)} = : \check{v}_{1}^{(1)}$, $\tilde{v}_{0}^{(0)} = \hat{v}_{L}^{(0)} = \check{v}_{0}^{(0)}$ and the algebraic system (40) consists now of two equations for two BVs, which have to be treated therefore as unknowns. The solutions of this system read:

$$\check{v}^{(1)}(q) = \frac{1}{2\alpha} \left( \frac{\hat{f}(k_{0}^{(a)}(q), q) + \hat{u}_{0}(k_{0}^{(a)}(q))}{1 - e^{-ik_{0}^{(a)}(q)L}} + \frac{\hat{f}(-k_{0}^{(a)}(q), q) + \hat{u}_{0}(-k_{0}^{(a)}(q))}{1 - e^{ik_{0}^{(a)}(q)L}} \right),$$

$$\check{v}^{(0)}(q) = \frac{1}{2ik_{0}^{(a)}(q)} \left( \frac{\hat{f}(k_{0}^{(a)}(q), q) + \hat{u}_{0}(k_{0}^{(a)}(q))}{1 - e^{-ik_{0}^{(a)}(q)L}} - \frac{\hat{f}(-k_{0}^{(a)}(q), q) + \hat{u}_{0}(-k_{0}^{(a)}(q))}{1 - e^{ik_{0}^{(a)}(q)L}} \right).$$

(47)

They satisfy asymptotic admissibility and the spectrum, characterized by the equation $1 - e^{\pm ik_{0}L} = 0$, (⇒ $k_{n} = \frac{2\pi}{L}n$, $n \in \mathbb{Z}$), is given by $q_{n} = -k_{n}^{2} = -(\frac{2\pi}{L})^{2}n^{2}$, $n \in \mathbb{N}$, for $\alpha = 1$, and by $q_{n} = ik_{n}^{2} = i(\frac{2\pi}{L})^{2}n^{2}$, $n \in \mathbb{N}$, for $\alpha = 1$; therefore the usual regularization is needed again in the Schrödinger case. We conclude that
the periodic problem for equations (19), in which one imposes the
$L$-periodicity of $u$ and $u_x$, is well-posed and no BV can be assigned
arbitrarily.

**Remark** We remark that the Fourier transforms of the unknown boundary
functions exhibit generically a branch point at $q = 0$, due to the well-known
slow decay as $t \to \infty$ of the solutions of the dispersive evolution equation under
investigation.

The above procedure generalizes with no difficulties to higher order prob-
lems. In the following we concentrate on a third order problem only.

### 3.2 The linear KdV equation

In this section we investigate IBV problems for $3^\cdot d$ order operators, illustrating
the method on the simplest possible example (20).

Since the group velocity $v_g = 3\eta k^2$ of the associated wave packet is positive
(positive) for $\eta$ positive (negative), we have the following expectations. In the
semiline case, one should be able to assign at $x = 0$ more BVs for positive $\eta$
than for negative $\eta$. In the segment case, for $\eta$ positive one can assign arbi-
trarily more BVs at $x = 0$ than at $x = L$ (and vice versa for $\eta$ negative). The
precise indication of “how many” and “which” BVs can be assigned in order to
have a well-posed IBV problem follows again in a straightforward way from the
Analyticity approach.

Equations (4) and (6) imply:

$$\hat{J} = -\mathcal{L}, \quad J = (ab, -\eta[b\frac{\partial^2 a}{\partial x^2} - \frac{\partial b}{\partial x} + \frac{\partial^2 b}{\partial x^2} a]),$$

$$\mathcal{L}(ik, iq) = i(q + \eta k^3).$$

(48)

In addition, if $V$ is the segment $(0, L)$, equation (9) yields

$$\hat{u}(k, q) = -i\gamma \hat{N}(k, q),$$

$$\hat{N}(k, q) = \mathcal{F}(k) + \hat{u}_0(k) - \eta \left( [\phi_0^{(2)}(q) + i\gamma \hat{v}_0^{(1)}(q) - k^2 \hat{v}_0^{(0)}(q)] - \right.$$  

$e^{-ikL}[\phi_L^{(2)}(q) + i\gamma \hat{v}_L^{(1)}(q) - k^2 \hat{v}_L^{(0)}(q)] \bigg)$

(49)

and the Fourier representation (18) of the solution takes the following form:

$$u(x, t) = \int_{\mathbb{R}^2} \frac{\delta dk}{2\pi} e^{i(kx + qt)} \frac{f(k, q)}{q + \eta k^3 - \alpha} + \int_{\mathbb{R}} \frac{dk}{2\pi} e^{i(kx - \eta k^3 t)} \hat{u}_0(k) -$$

$$\eta \left( \int_{\mathbb{R}^2} \frac{dk}{2\pi} e^{i(kx - \eta k^3 t)} [\phi_0^{(2)}(-\eta k^3) + i\gamma \hat{v}_0^{(1)}(-\eta k^3) - k^2 \hat{v}_0^{(0)}(-\eta k^3)] + \right.$$  

$\int_{\mathbb{R}^2} \frac{dk}{2\pi} e^{i(kx - \eta k^3 t)} [\phi_L^{(2)}(-\eta k^3) + i\gamma \hat{v}_L^{(1)}(-\eta k^3) - k^2 \hat{v}_L^{(0)}(-\eta k^3)], \ x \in (0, L), \ t > 0,$

(50)

where $\gamma_0 = \partial \mathcal{K}_0^{(-)}, \ \gamma_0 = \partial \mathcal{K}_1^{(-)} \cup \partial \mathcal{K}_2^{(-)}, \ \gamma_0 = \partial \mathcal{K}_1^{(+)}, \ \partial \mathcal{K}_2^{(+)}, \ \gamma_L = \partial \mathcal{K}_0^{(+)},$

$\mathcal{K}_m^{(-)} = \{ k : \frac{\pi}{3}(2m + 1) \leq \text{arg} \ k \leq \frac{\pi}{3}(2m + 2) \}, \ \mathcal{K}_m^{(+)} = \rho_\eta \mathcal{K}_m^{(-)}, \ m = 0, 1, 2,$

(51)
and \( \hat{\rho}_\pi \) is the involution \( \hat{\rho}_\pi : k \rightarrow -k \).

### 3.2.1 The segment \( V = (0, L) \)

Now \( \hat{\mathcal{N}} / \mathcal{L} \) must be defined in \( \mathcal{A} = \mathcal{C} \otimes \hat{\mathcal{I}}_q \), analytic in \( q \in \mathcal{I}_q \), \( \forall k \in \mathcal{C} \); and analytic in \( k \in \mathcal{C}, \forall q \in \hat{\mathcal{I}}_q \), with proper asymptotics for large \( |k| \) and/or \( |q| \) in the analyticity regions. Therefore the singularity manifolds \( \mathcal{A} \cap \mathcal{S}^{(\eta)} \), corresponding to \( \eta = \pm 1 \), are given by (see Figs 2a,b):

\[
\mathcal{A} \cap \mathcal{S}^{(\eta)} = \bigcup_{m=0}^{2} \{ q = -\eta k^3, k \in \mathcal{K}_m^{(\eta)} \} = \bigcup_{m=0}^{2} \{ k = \kappa_m^{(\eta)}(q), \pi \leq \arg q \leq 2\pi \}
\]

where \( \kappa_m^{(\eta)}(q) = -\eta \rho_m q^2 \) and \( \rho_m \), \( m = 0, 1, 2 \) are the 3 roots of unity:

\[
\rho_m = e^{\frac{2\pi i m}{3}}, \ m = 0, 1, 2.
\]

![Fig.2a](image1.png) The SM \( \mathcal{A} \cap \mathcal{S}^{(\eta)} (\eta = -1) 

![Fig.2b](image2.png) The SM \( \mathcal{A} \cap \mathcal{S}^{(\eta)} (\eta = 1) 

The ZRC (12) consists of the following three equations:

\[
\hat{\mathcal{N}}(\kappa_m^{(\eta)}(q), q) = 0, \ m = 0, 1, 2, \ \pi \leq \arg q \leq 2\pi.
\]

For \( q \in \mathcal{R} \) there is one singularity on the real \( k \)-axis, which is cured by one of the three equations (54) and no denominator regularization is then needed. The regularization (31), however, is still introduced, according to the Remark 2 of §2.2.2, in writing the Fourier representation (50) of the solution.

The 3 algebraic equations (54) contain 6 BVs; therefore we expect to be allowed to assign independently only 3 BVs. As before, a quick asymptotic estimate selects the sets of 3 BVs which can be assigned independently, compatibly with asymptotic admissibility. The asymptotics of equations (54) imply that the following expressions, respectively, for \( \eta = -1 \):

\[
\begin{align*}
\ e^{i q \frac{\pi}{L}} [f(q, q, u_0(q), \tilde{u}_0(q))] - [v_0^{(1)}(q) + i q \frac{\pi}{L} v_0^{(2)}(q) - \rho_1 \frac{\pi}{L} v_0^{(0)}(q)], \\
\ f(p_1 q, q, u_0(p_1 q, \tilde{u}_0(q))] + [v_0^{(2)}(q) + i p_1 q \frac{\pi}{L} v_0^{(1)}(q) - \rho_1 \frac{\pi}{L} v_0^{(0)}(q)], \\
\ f(p_2 q, q, u_0(p_2 q, \tilde{u}_0(q))] + [v_0^{(2)}(q) + i p_2 q \frac{\pi}{L} v_0^{(1)}(q) - \rho_1 \frac{\pi}{L} v_0^{(0)}(q)],
\end{align*}
\]

\[
(55)
\]
and for $\eta = 1$:

$$
\hat{f}(-q^+, q) + \hat{u}_0(-q^+) \left[-\hat{v}_0^{(2)}(q) - iq^+\hat{v}_0^{(1)}(q) - q^+\hat{v}_0^{(0)}(q)\right],
$$

$$
\exp(-i\rho_1 q^+ L)\left[\hat{f}(-\rho_1 q^+, q) + \hat{u}_0(-\rho_1 q^+)\right] + \left[i\hat{v}_L^{(2)}(q) - i\rho_1 q^+\hat{v}_L^{(1)}(q) - q^+\hat{v}_L^{(0)}(q)\right],
$$

(56)

$$
\exp(-i\rho_2 q^+ L)\left[\hat{f}(-\rho_2 q^+, q) + \hat{u}_0(-\rho_2 q^+)\right] + \left[i\hat{v}_L^{(2)}(q) - i\rho_2 q^+\hat{v}_L^{(1)}(q) - q^+\hat{v}_L^{(0)}(q)\right],
$$

are exponentially small for $q \sim \infty$ in $\pi \leq \arg q \leq 2\pi$. Therefore, reasoning as before, we see that:

i) for $\eta = -1$, a necessary and sufficient condition to obtain FTs with admissible asymptotics is to assign at $x = 0$ any BV among $v_0^{(0)}$, $v_0^{(1)}$, $v_0^{(2)}$ (consequence of equations (55b,c)) and, at $x = L$, any two BVs among $v_L^{(0)}$, $v_L^{(1)}$, $v_L^{(2)}$ (consequence of equation (55a));

ii) for $\eta = 1$, a necessary and sufficient condition to obtain FTs with admissible asymptotics is to assign at $x = 0$ any two BVs among $v_0^{(0)}$, $v_0^{(1)}$, $v_0^{(2)}$ (consequence of equation (56a)) and, at $x = L$, any BV among $v_L^{(0)}$, $v_L^{(1)}$, $v_L^{(2)}$ (consequence of equations (56a,b)).

Again, to complete our investigation, we must check if the spectrum associated with the above IBV problems selected by the asymptotic admissibility, lie entirely outside the analyticity domain of an admissible FT. It is easy to prove that it is indeed the case.

**Proposition (the spectrum of the IBV problem)** Consider any IBV problem on the segment for equation (20) compatible with the asymptotic admissibility established above; i.e., in which, for $\eta = -1$, one assigns arbitrarily at $x = 0$ any BV among $v_0^{(0)}$, $v_0^{(1)}$, $v_0^{(2)}$ and any two BVs at $x = L$ among $v_L^{(0)}$, $v_L^{(1)}$, $v_L^{(2)}$, and in which, for $\eta = 1$, one assigns arbitrarily at $x = 0$ any two BVs among $v_0^{(0)}$, $v_0^{(1)}$, $v_0^{(2)}$ and any BV at $x = L$ among $v_L^{(0)}$, $v_L^{(1)}$, $v_L^{(2)}$. For $\eta = -1$, let $v_0^{(n)}$ be the given BV at $x = 0$ and $v_L^{(m)}$ be the unknown BV at $x = L$ while, for $\eta = 1$, let $v_0^{(n)}$ be the unknown BV at $x = 0$ and $v_L^{(m)}$ be the given BV at $x = L$. Then the corresponding spectrum is characterized by the following equation:

$$
\Delta^{(\eta(m-n))}(k) = 0,
$$

(57)

where:

$$
\Delta^{(j)}(k) := \exp(-ikL) + \rho_1^j \exp(-\rho_1 ikL) + \rho_2^j \exp(-\rho_2 ikL).
$$

(58)

The proof is tedious but straightforward and makes essential use of the well-known algebra of the roots of unity, which implies also that all the above IBV problems lead only to three (similar) purely imaginary discrete spectra $\{k_n^{(j)}\}_{n \in \mathcal{N}}$, characterized by the three equations $\Delta^{(j)}(k) = 0$, $j = 0, 1, 2$. More precisely: 1) the spectrum characterized by equation $\Delta^{(0)}(k) = 0$ is given by:

$$
k_n^{(0)} = -i(\zeta_n^{(0)}/L), \quad n \in \mathcal{N}^+ : \{\zeta_n^{(0)}\}_{n \in \mathcal{N}^+} : \frac{\zeta_n^{(0)}}{2} = -\cos\left(\frac{\sqrt{2}\pi}{2} \zeta_n^{(0)}\right),
$$

\[\zeta_n^{(0)} \sim \frac{\pi}{\sqrt{3}}(2n - 1), \quad n \geq 1.\] 

(59)
2) The spectrum characterized by equation $\Delta^{(1)}(\zeta) = 0$ is:

$$k_n^{(1)} = -i(\zeta_n^{(1)}/L), \ n \in N : \ \{\zeta_n^{(1)}\}_{n \in N} : \ \zeta_0^{(1)} = 0, \ \zeta_n^{(1)} \sim \frac{2\pi}{\sqrt{3}}(n - \frac{5}{6}), \ n \geq 2.$$  (60)

3) The spectrum characterized by equation $\Delta^{(2)}(\zeta) = 0$ is:

$$k_n^{(2)} = -i(\zeta_n^{(2)}/L), \ n \in N : \ \{\zeta_n^{(2)}\}_{n \in N} : \ \zeta_0^{(2)} = 0, \ \zeta_n^{(2)} \sim \frac{2\pi}{\sqrt{3}}(n - \frac{1}{6}), \ n \geq 1.$$  (61)

We conclude that all the three discrete spectra

$$\{q_n^{(j)}\}_{n \in N}, \ q_n^{(j)} = k_n^{(j)}3 = i \left(\frac{\zeta_n^{(j)}}{L}\right)^3, \ j = 0, 1, 2,$$  (62)

associated with the above IBV problems lie on the positive imaginary axis of the complex $q$ plane, outside the analyticity domain of an admissible FT. Therefore:

**IBV problems for equation (20) on the segment $(0, L)$ are well-posed iff:**

i) for $\eta = -1$, one assigns at $x = 0$ any BV among $v_0^{(0)}$, $v_0^{(1)}$, $v_0^{(2)}$ and at $x = L$ any two BVs among $v_L^{(0)}$, $v_L^{(1)}$, $v_L^{(2)}$;

ii) for $\eta = 1$, one assigns at $x = 0$ any two BVs among $v_0^{(0)}$, $v_0^{(1)}$, $v_0^{(2)}$ and any BV at $x = L$ among $v_L^{(0)}$, $v_L^{(1)}$, $v_L^{(2)}$.

### 3.2.2 The periodic problem

If we assume $L$-periodicity of $u$, $u_x$ and $u_{xx}$, then $v^{(j)}_0 = v^{(j)}_L$, $j = 0, 1, 2$, the algebraic system (54) consists now of three equations for three BVs, which have to be treated then as unknowns. The solution of this system satisfy asymptotic admissibility and the spectrum, characterized by the equations $1 - e^{-i\rho k} = 0$, $j = 0, 1, 2$ ($\Rightarrow k_n = \frac{2\pi}{\sqrt{3}}j^{-1}n$, $n \in Z$, $j = 0, 1, 2$), is given by the real numbers $q_n = -\eta k_n^3 = -\eta(\frac{2\pi}{\sqrt{3}})3n^3$, $n \in Z$ and must be regularized in the usual way. We conclude that:

**the periodic problem for the linear KdV equation (20), in which one imposes $L$-periodicity to $u$, $u_x$ and $u_{xx}$, is well-posed and no BV can be assigned.**

### 3.2.3 The semiline $V = (0, \infty)$

Taking the limit $L \to \infty$ of the results of §3.2.1 we immediately obtain the results on the semiline. In this case, the singularity manifolds are the restrictions of the
above ones to the lower half \( k \) plane. No spectrum arises and the asymptotic admissibility implies that:

**IBV problems for equation (20) on the semiline \((0, \infty)\) are well-posed iff, for \( \eta = -1 \), one assigns at \( x = 0 \) any BV among \((v_0^{(0)}, v_0^{(1)}, v_0^{(2)})\) and, for \( \eta = 1 \), one assigns at \( x = 0 \) any two BVs among \((v_0^{(0)}, v_0^{(1)}, v_0^{(2)})\).**

We remark that, in the cases treated so far, the spectra of all the IBV problems compatible with asymptotic admissibility lie always outside the analyticity domain. We do not have, however, a general argument excluding the situation in which part of the spectrum lie inside. Therefore the complete characterization of the spectrum, the only part of the method in which some technicality is involved, seems to be unavoidable and makes it difficult to prove general results for operators of arbitrary order.

The Analyticity approach applies nicely also to an arbitrary number of dimensions and next section is devoted to an illustration of it. The application of the method to higher order problems and to other relevant examples will be presented in [8].

### 3.3 Multidimensional Schrödinger equation

In this section we study the Dirichelet and Neumann problems for the Schrödinger equation (23) in the \( n \)-dimensional quadrant (24). Then:

\[
\hat{L} = -\frac{\partial}{\partial t} - i\triangle, \quad J = (ab, ia \nabla b - b \nabla a), \quad \mathcal{L}(i\mathbf{k}, iq) = i(q + k^2),
\]

where \( k^2 = \mathbf{k} \cdot \mathbf{k} \). Equations (7) and (63) give the following expression of the Fourier transform of the solution in terms of the Fourier transforms of the forcing and of all the IBVs:

\[
\hat{u}(\mathbf{k}, q) = \frac{\hat{N}(\mathbf{k}, q)}{p(q^2 + k^2)},
\]

\[
\hat{N}(\mathbf{k}, q) := \hat{f}(\mathbf{k}, q) + \hat{u}_0(\mathbf{k}) - i \sum_{j=1}^n \left[ \hat{v}_0^{(1)}(\mathbf{k}_j, q) + ik_j \hat{v}_0^{(0)}(\mathbf{k}_j, q) \right].
\]

The Fourier representation (18) of the solution reads:

\[
u(x, t) = \int_{R^{n+1}} \frac{d\mathbf{k}}{(2\pi)^{n+1}} e^{i(\mathbf{k} \cdot \mathbf{x} + qt)} \frac{\hat{f}(\mathbf{k}, q)}{q + k^2} + \int_{R^n} \frac{d\mathbf{k}}{(2\pi)^n} e^{i(\mathbf{k} \cdot \mathbf{x} - k^2 t)} \hat{u}_0(\mathbf{k}) + \sum_{j=1}^n \int_{R^{n-1}} \frac{d\mathbf{k}_j}{(2\pi)^{n-1}} \int_{\partial K_j^{(i)}} \frac{d\mathbf{k}}{2\pi} e^{i(\mathbf{k} \cdot \mathbf{x} - k^2 t)} [\hat{v}_0^{(1)}(\mathbf{k}_j, -k^2) + ik_j \hat{v}_0^{(0)}(\mathbf{k}_j, -k^2)],
\]

where \( d\mathbf{k}_j = dk_1 \cdots dk_{j-1} \cdots dk_{n-1} \cdots dk_n \).
In view of the distinguished parity properties of the Fourier transforms in (64), we shall make an extensive use of the parity operators:

\[ \Delta_{\pm} = \prod_{i=1}^{n} (1 \pm \hat{\sigma}_{i}), \quad \Delta_{\pm}^{(j)} = \prod_{i=1 \atop i \neq j}^{n} (1 \pm \hat{\sigma}_{i}), \]  

(66)

where \( \hat{\sigma}_{j} \) is the involution \( \hat{\sigma}_{j} : k_{j} \rightarrow -k_{j} \).

In this multidimensional case, the FT of the solution is defined in \( \mathcal{A} = \mathcal{I}_{k_{1}} \otimes \cdots \otimes \mathcal{I}_{k_{n}} \), analytic for \( q \in \mathcal{I}_{q} \), \( \forall k \in \mathcal{I}_{k_{i}} \otimes \cdots \otimes \mathcal{I}_{k_{n}} \) and in \( k_{j} \in \mathcal{I}_{k_{j}} \), \( \forall k_{j} \in \mathcal{I}_{k_{j}} \otimes \cdots \otimes \mathcal{I}_{k_{j}} \otimes \cdots \otimes \mathcal{I}_{k_{n}} \) and \( \forall q \in \mathcal{I}_{q} \). We found it convenient to study the ZRC in the \( n \) different regions \( \mathcal{Q}_{j}^{-} \subset \mathcal{A} \cap \mathcal{S}, j = 1, \ldots, n \) defined by:

\[ \mathcal{Q}_{j}^{-} := \{(k, q) \in \mathbb{C}^{n+1} : k_{j} \in \mathbb{R}^{n-1}, \pi \leq \arg q \leq 2\pi, k_{j} = \chi_{j}(k_{j}, q)\}, \quad \chi_{j}(k_{j}, q) := i(q + k_{j} \cdot \hat{k}_{j})^{-1} \in \mathcal{I}_{k}, \quad j = 1, \ldots, n. \]  

(67)

Therefore the starting point of the analysis is the set of \( n \) equations

\[ \hat{N}(k, q)|_{k_{j}=\chi_{j}} = 0, \quad k_{j} \in \mathbb{R}^{n-1}, \quad \pi \leq \arg q \leq 2\pi, \quad j = 1, \ldots, n. \]  

(68)

**Dirichlet problem.** The parity properties in \( k \) of the BV terms imply that the application of the parity operator \( \Delta_{\pm}^{(j)} \) defined in (66) to the \( j^{th} \) equation (68) eliminates all the \( \hat{v}_{0}^{(1)} \)'s except \( \hat{v}_{0j}^{(1)} \):

\[ \Delta_{\pm}^{(j)} \hat{v}_{0j}^{(1)}(k, q) = -\Delta_{\pm}^{(j)} (W(k, q)|_{k_{j}=\chi_{j}}), \quad j = 1, \ldots, n, \]

\[ W(k, q) := f(k, q) + i\hat{u}_{0}(k) + i \sum_{j=1}^{n} k_{j} \hat{v}_{0j}^{(0)}(k_{j}, q) \]  

(69)

and the analyticity properties of the \( \hat{v}_{0}^{(1)} \)'s allow one to express them in terms of known quantities:

\[ \hat{v}_{0j}^{(1)}(k, q) = \mathcal{P}^{(j)} \Delta_{\pm}^{(j)} \hat{v}_{0j}^{(1)}(k_{j}, q) = -\mathcal{P}^{(j)} \Delta_{\pm}^{(j)} (W(k, q)|_{k_{j}=\chi_{j}}), \quad j = 1, \ldots, n \]  

(70)

applying the lower half plane analyticity projectors in all the \( k \)-variables (except \( k_{j} \)):

\[ \mathcal{P}^{(j)} = \prod_{m \neq j}^{n} \mathcal{P}_{m}, \quad \mathcal{P}_{m} = -\frac{1}{2\pi i} \int_{\mathcal{R}} \frac{dk_{m}'}{k_{m}' - (k_{m} - i\Omega)}. \]  

(71)

Equations (70) summarize all the analyticity informations contained in the ZRC, allow one to express the unknown BVs in terms of given data and, via (65), to solve the Dirichelet problem.

**Neumann problem.** Similar considerations can be made in solving the Neumann BV problem. In this case:

\[ i\chi_{j}(k_{j}, q) \Delta_{\pm}^{(j)} \hat{v}_{0j}^{(0)}(k_{j}, q) = -\Delta_{\pm}^{(j)} (V(k, q)|_{k_{j}=\chi_{j}}), \quad j = 1, \ldots, n, \]

\[ V(k, q) := f(k, q) + i\hat{u}_{0}(k) + \sum_{j=1}^{n} \hat{v}_{0j}^{(1)}(k_{j}, q) \]  

(72)
and
\[ \hat{v}_{0j}^{(0)}(k_j, q) = i P(j) \left( \frac{1}{\chi_j(k_j, q)} \Delta^{(j)}(V(k, q)|_{k_j=\chi_j}) \right), \quad j = 1, \ldots, n. \tag{73} \]

In this multidimensional context, for the presence of the analyticity projectors, the unknown BVs in Fourier space turn out to be nonlocal expressions of the given data. It is however possible to show that, due to the analyticity properties of the involved FTs, it is not really necessary to apply the above analyticity projectors to construct the unknown BVs and the solution \( u(x, t) \) in configuration space. The strategy to avoid unpleasant nonlocalities is outlined in the next section and leads to a Fourier representation of the solution already obtained in [6] using the EbR approach. Therefore this strategy is also the way to establish the connection between the Analyticity and the EbR approaches.

4 Connections between the Analyticity and the EbR approaches

Dirichelet problem

We first remark that the unknown BVs can be constructed directly in terms of known data from the RHS of (69b):
\[ v_{0j}^{(1)}(x_j, t) = - \int_{\mathbb{R}^n} \frac{dk_j dq}{(2\pi)^n} e^{i(k_j \cdot x_j + qt)} \Delta^{(j)}(W(k, q)|_{k_j=\chi_j}), \quad t > 0, \quad x_k \geq 0, \quad k \neq j. \tag{74} \]

Indeed, from the analyticity properties of \( \hat{v}_{0j}^{(1)} \) we know that its inverse FTs (3) is zero outside the domain of definition in configuration space (i.e., for \( x_k < 0, \quad k \neq j \)); this implies the formula
\[ \int_{\mathbb{R}^n} d\mathbf{k}_j dq e^{i(k_j \cdot x_j + qt)} [\hat{v}_{0j}^{(1)}(k_j, q) - \Delta^{(j)} \hat{v}_{0j}^{(1)}(k_j, q)] = 0, \quad t > 0, \quad x_j > 0, \quad j = 1, \ldots, n \tag{75} \]

and, through (69a), equation (74).

Also the solution \( u(x, t) \) can be reconstructed without going through the nonlocalities associated with the analyticity projectors. Indeed it is possible to show that the following relation holds true:
\[ \sum_{j=1}^{n} \hat{v}_{0j}^{(1)}(k_j, q) \equiv (\Delta_- - 1)W(k, q), \tag{76} \]

where the equivalence \( \hat{A}(k, q) \equiv \hat{B}(k, q) \) means that the FTs \( \hat{A}(k, q) \) and \( \hat{B}(k, q) \) are equal under the following Fourier integral projector:
\[ \int_{\mathbb{R}^{n+1}} d\mathbf{k} dq e^{i(k \cdot x + qt)} \frac{1}{q + k^2 - i0} [\hat{A}(k, q) - \hat{B}(k, q)] = 0, \quad (x, t) \in D. \tag{77} \]
The equivalence (76) and equation (69) imply
\[ \hat{N}(k, q) \equiv \Delta_- W(k, q) \]  
and the following spectral representation of the solution in terms of known data:
\[ u(x, t) = \int_{\mathbb{R}^{n+1}} \frac{dq}{(2\pi)^{n+1}} e^{i(k \cdot x + qt)} \Delta_+ f(k, q) + \int_{\mathbb{R}^n} \frac{dk}{(2\pi)^n} e^{i(k \cdot x - k^2 t)} \Delta_- \hat{u}_0(k) + \sum_{j=1}^n \int_{\mathbb{R}^{n-1}} \frac{dk}{(2\pi)^{n-1}} \left( \int_{K_j} dk_j \sum_{j=1}^n \frac{d\chi_j}{\pi} e^{i(k \cdot x - k^2 t)} k_j \Delta_-^{(j)} \hat{v}_{0j}^{(0)}(k_j, -k^2), \quad (x, t) \in \mathcal{D}, \right) \]  
already obtained in [6] using the EbR approach.

The proof of (76) is based on the important fact that all the admissible Fourier transforms \( \hat{A}(k, q) \) under consideration satisfy the equivalence
\[ \hat{\sigma}_j \hat{A}(k, q) \equiv \hat{A}(k, q)|_{k_j = \chi_j}, \quad j = 1, \ldots, n \]
and goes as follows. For \( n = 2 \), the 2 ZRCs (68) and their consequences (69) yield the 4 equivalence relations:
\[ \hat{v}_{0j}^{(1)}(k_j, q) + \hat{\sigma}_j \hat{v}_{0j}^{(1)}(k_j, q) \equiv - \hat{\sigma}_j W(k, q), \quad \hat{\sigma}_j W(k, q) = \hat{\sigma}_j W(k, q), \]
\[ (1 - \hat{\sigma}_j) \hat{v}_{0j}^{(1)}(k_j, q) + \hat{\sigma}_j (1 - \hat{\sigma}_j) \sum_{i \neq j} \hat{v}_{0j}^{(1)}(k_i, q) \equiv - \hat{\sigma}_j (1 - \hat{\sigma}_j) W(k, q), \quad i \neq j, \quad j = 1, \ldots, n, \]
\[ \Delta_-^{(j)} \hat{v}_{0j}^{(1)}(k_j, q) \equiv - \hat{\sigma}_j \Delta_-^{(j)} W(k, q), \quad j = 1, \ldots, n. \]
The sum of all these equations with weights \( 1/(\frac{n-1}{m}) \) (\( m \) is the number of parity operators appearing in the equation) yields the result (76).

**Neumann problem**

Similar considerations can be made in the case of the Neumann IBV problem. Now the unknown BVs are recovered via:
\[ \hat{v}_{0j}^{(0)}(x_j, t) = - \int_{\mathbb{R}^n} \frac{dk}{(2\pi)^n} e^{i(k \cdot x_j + qt)} (\Delta_+^j V(k, q)|_{k_j = \chi_j}), \quad t > 0, \quad x_k \geq 0, \quad k \neq j \]
and the spectral representation of the solution reads:
\[ u(x, t) = - \int_{\mathbb{R}^{n+1}} \frac{dq}{(2\pi)^{n+1}} e^{i(k \cdot x + qt)} \Delta_+ f(k, q) + \int_{\mathbb{R}^n} \frac{dk}{(2\pi)^n} e^{i(k \cdot x - k^2 t)} \Delta_- \hat{u}_0(k) - \sum_{j=1}^n \int_{\mathbb{R}^{n-1}} \frac{dk}{(2\pi)^{n-1}} \left( \int_{K_j} \frac{d\chi_j}{\pi} e^{i(k \cdot x - k^2 t)} \Delta_-^{(j)} \hat{v}_{0j}^{(1)}(k_j, -k^2), \quad (x, t) \in \mathcal{D}, \right) \]
a formula already derived in [6] using the EbR approach.

Acknowledgments

The present work has been carried out during several visits and meetings. We gratefully acknowledge the financial contributions provided by the RFBR Grant 01-01-00929, the INTAS Grant 99-1782 and by the following Institutions: the University of Rome “La Sapienza” (Italy), the Istituto Nazionale di Fisica Nucleare (Sezione di Roma), the Landau Institute for Theoretical Physics, Moscow (Russia) and the Isaac Newton Institute, Cambridge (UK), within the programme “Integrable Systems”.

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