Work and work-energy theorem in curved spacetime

Shaofan Liu and Liu Zhao∗
School of Physics, Nankai University, Tianjin 300071, China
email: 2120190132@mail.nankai.edu.cn and lzhao@nankai.edu.cn

Abstract

The definitions of gravitational work as well as work done by the total external force on a massive probe particle moving in generic spacetime backgrounds are proposed. These definitions are given in the form of scalar integrals and thus, are independent of coordinate choices. However, the dependence on the choice of observer field is essential and inevitable. The definitions are checked in the case of Minkowski, Schwarzschild, Reisner-Nordström and Kerr-Newman spacetimes and agreements with Newtonian mechanical definitions are verified in the slow motion or the far field limit.

1 Introduction

In an attempt to extend classical thermodynamics and statistical physics into curved spacetime, we encounter the following problem: what is the work and heat in curved spacetime? Thinking a little bit further, we found that even on the microscopic level (i.e. on the level of mechanics rather than statistical physics), the concept of work done by a force is still not clearly defined in the context of relativity. Discussions about the motion of a test particle in curved spacetimes can be easily found in most standard textbooks about general relativity, e.g. Section 4.3 in [1], however no clear and systematic definition of work done on a particle has been mentioned in any textbook on general relativity.

In Newtonian mechanics, the work done by a force exerted on a particle is one of the essential concepts defined as

\[ W = \int_C \mathbf{F} \cdot \, \mathrm{d}\mathbf{x}, \tag{1} \]

where \( \mathbf{x} \) denotes the displacement along the path \( C \) and the force \( \mathbf{F} \) is in general dependent on \( \mathbf{x} \): \( \mathbf{F} = \mathbf{F}(\mathbf{x}) \). This definition can also be recast in a number of different

∗Correspondence author.
forms, e.g.

\[ W = \int_{t_i}^{t_f} F \cdot v \, dt = \int_{t_i}^{t_f} ma \cdot v \, dt, \]  \hspace{1cm} \text{(2)}

due to the second law of Newtonian mechanics, \( F = ma \), with \( m \) the mass of the particle. For conservative forces, the work is path independent, because

\[ W = \int_C F \cdot \, d\mathbf{x} = -\int_{x(t_i)}^{x(t_f)} \nabla U(x) \cdot \, d\mathbf{x} = U[x(t_i)] - U[x(t_f)]. \]  \hspace{1cm} \text{(3)}

In particular, the work of gravity exerted by a mass \( M \) on another mass \( m \) is

\[ W = -\int_{r(t_i)}^{r(t_f)} \frac{GMm}{r^3} r \cdot \, d\mathbf{r} = GMm \left( \frac{1}{r(t_f)} - \frac{1}{r(t_i)} \right). \]  \hspace{1cm} \text{(4)}

The above results only hold in inertial frames. For noninertial frames, however, we have to include a fictitious force \( F_I \) in order that Newton’s second law still holds:

\[ F + F_I = ma, \quad F_I \equiv -ma_I. \]

where \( a_I \) is acceleration of the noninertial frame. The force \( F_I \) is generally called a pseudo force, or inertial force (see e.g. [2]). As the particle moves, nonzero fictitious work can be done on the particle by the inertial force. In the case of a uniform gravitational field with gravitational acceleration \( g \), the inertial force in the frame of a freely falling elevator is \(-mg\), so that the total force exerted on a freely falling particle in this frame is zero.

In the context of general relativity, the concept of work becomes quite obscure. This obscurity comes into reality for a number of reasons. First of all, general relativity requires a covariant formalism. One can introduce a general covariant analogue of the second law of Newtonian mechanics, i.e.

\[ F^\mu = ma^\mu, \quad a^\mu \equiv u^\nu \nabla_\nu u^\mu. \]

However, since the proper acceleration \( a^\mu \) is always spacelike and is normal to the particle’s proper velocity \( u^\mu \),

\[ a^\mu u_\mu = 0, \]

the naive covariantization

\[ W = \int_{t_i}^{t_f} F \cdot v \, dt \rightarrow W = \int_{\tau_i}^{\tau_f} F^\mu u_\mu d\tau \]  \hspace{1cm} \text{(5)}

of eq.(2) does not work, because

\[ \int_{\tau_i}^{\tau_f} F^\mu u_\mu d\tau = \int_{\tau_i}^{\tau_f} ma^\mu u_\mu d\tau = 0. \]
where $\tau$ is the particle’s proper time, which is a natural parameter for the worldline of the particle. Secondly, as a physically measurable quantity, the definition of work must involve observer dependence, but the naive correspondence \( (5) \) clearly does not accommodate this information. The dependence of Newton’s laws on inertial frames apparently violates the principles of relativity as no priority should be given to the inertial observers in a relativistic theory, and the measurement of work for all observers needs to be treated on equal footing. Lastly, when considering the work exerted by gravity, we encounter the most severe problem because gravity in general relativity is not viewed as a force but rather is encoded in the spacetime geometry.

Discussions about observers in curved spacetimes have a long history. For example, in 1909 Erenfest considered the Lorentz contraction of a rigid body observed by an observer at rest \([3]\), Doughty analyzed the proper acceleration of a static observer near the horizon of a black hole \([4]\), Crawford and his coworkers discussed generalized observer sets and measurements of velocity \([5]\). There are also discussions about dependence of quasi-local energy on observers \([6, 7]\). Dahia and da Silva considered the relation between observers in curved spacetimes and those in Minkowski spacetime \([8]\). Berezin and Victor discussed the possibility to build up a set of static observers outside a Schwarzschild black hole \([9]\). Refs.\([6, 7]\) also studied the work and binding energy associated to a massive shell \([6, 7]\). However no previous literature has provided a clear definition of work done on a moving relativistic particle.

In this paper, we propose a relativistic invariant definition of gravitational work as well as work done by the total external forces exerted on a probe particle in a generic curved spacetime. The crucial idea comes from the relativistic work-energy theorem, which asserts that the change in the energy of the particle is the sum of works done by the total external force and by gravity. In differential form, this means that the proper time rate of the energy of the particle equals the sum of the local power of the total external force and that of gravity. It is crucial to notice that, in order to define works along a segment of the particle’s worldline, a single observer is insufficient. Rather, a smooth observer field is a necessary input. Our definitions are checked in the case of several well-known spacetimes and the correctness in the non-relativistic limit is verified satisfactorily.

## 2 The relativistic definition of work

In this section we analyze gravitational work as well as work done by the total external force separately, and then add them up to get the total work. In order to define work in a curved spacetime, we need to analyze the local power measured by an on-the-spot observer as a preparation, and then integrate the power along a segment of the particle’s worldline. This implies the inclusion of an observer field in the definition of work. We shall see that the expression of the local power of gravity contains the directional derivative of the proper velocity of the observer field along the tangent direction of the particle’s worldline, thus the observer field needs to be smoothly distributed at least
along the segment of the particle’s worldline of our interest.

2.1 Decomposition of the particle’s proper velocity

Consider a probe particle of mass \( m \) moving in a generic spacetime \( \mathcal{M} \) endowed with a metric \( g_{\mu \nu} \) with mostly positive signature. A crucial quantity to be used throughout this paper is the energy \( E(O) \) of the particle measured by a generic observer \( O \). The observer \( O \) is characterized by a future-directed timelike curve with tangent vector \( Z^\mu \) normalized as \( Z^\mu Z_\mu = -c^2 \). Since parallel transport is curve dependent in curved spacetimes, there is generally no way for an observer to define the energy of a distant particle \([1]\). Suppose that the worldline of the observer intersects with the worldline of the particle at the event \( P \). Then the energy of the particle at \( P \) is defined as \([1]\)

\[
E(O) \equiv -Z^\mu p_\mu = -mZ^\mu u_\mu. \tag{6}
\]

Introducing a scalar object

\[
\gamma \equiv -\frac{1}{c^2}Z^\mu u_\mu, \tag{7}
\]

the energy can be rewritten as

\[
E(O) = \gamma mc^2. \tag{8}
\]

It is always possible to make an orthogonal decomposition of \( u^\mu \) with respect to \( Z^\mu \), i.e.

\[
\frac{u^\mu}{\gamma} = \gamma(Z^\mu + v^\mu), \quad Z^\mu v_\mu = 0, \tag{9}
\]

where

\[
v^\mu \equiv \frac{\Delta^\mu \nu u_\nu}{\gamma}, \quad \Delta_{\mu \nu} \equiv g_{\mu \nu} + \frac{1}{c^2}Z_\mu Z_\nu. \tag{10}
\]

We may introduce a local orthonormal tetrad \( \{ e_\hat{\mu} \} \) \([1]\) at \( P \) with \( (e_\hat{\mu})^\mu = Z^\mu \), which implies the introduction of Cartesian coordinates in the (co)tangent space of the space-time which is naturally endowed with a Minkowski metric. Under this choice, \( v^\mu \) can be expressed as \( v^\mu = (0, \mathbf{v}) \), and correspondingly, \( u^\mu = \gamma(c, \mathbf{v}) \). It is evident that \( v^\mu \) is the spatial velocity of the particle in coordinate time with respect to the observer at the event \( P \). Using the normalization condition \( \hat{u}^\mu u_\mu = -c^2 \), we have

\[
\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}, \quad v^2 = \mathbf{v} \cdot \mathbf{v}, \tag{11}
\]

where \( \cdot \) means the Cartesian scalar product. It turns out that \( \gamma \) is nothing but the local Lorentz factor in the tangent space of \( \mathcal{M} \) at \( P \). Naturally, \( Z^\mu \) and \( u^\mu \) are connected by a local Lorentz boost.
2.2 Power and work of the total external force

Let us start from the simple case of Minkowski spacetime. We assume that there is a nonvanishing external force $F^\mu$ exerted on the particle. The worldline of the particle is then non-geodesic and is described by the equation

$$F^\mu = \frac{dp^\mu}{d\tau} = mu^\nu \nabla_\nu u^\mu,$$  \hspace{1cm} (12)

where $\tau$ is the proper time of the particle. Let us remind that in Minkowski spacetime, there is no difference between the covariant derivative $\nabla_\mu$ and the ordinary derivative $\partial_\mu$. We use solely the notation $\nabla_\mu$ in order that some of the results can be smoothly shifted to the cases of curved spacetimes.

For an inertial observer $O_i$ whose worldline intersects with the particle’s worldline at the event $P$, we can extract the zeroth component of the force $F^\mu$ in Cartesian coordinates $x^\mu$ via

$$F^0 = -z^\mu F_\mu,$$

where $z^\mu = (c, 0, \cdots, 0)$ is the proper velocity of the observer $O_i$. It is easy to recognize that $F^0$ is just the power of the external force $F^\mu$ defined as the proper time rate of the particle’s energy $E(O_i)$ observed by the observer $O_i$,

$$F^0 = -z^\mu F_\mu = \frac{dE(O_i)}{d\tau}, \hspace{1cm} E(O_i) = -z^\mu p_\mu = -mz^\mu u^\mu.$$

Now if we change the spacetime from Minkowskian to a generic curved manifold $\mathcal{M}$ and replace the inertial observer $O_i$ by a generic observer $O$, the equation of motion of the particle is still described by (12), but the combination $-Z^\mu F_\mu$ is no longer the zeroth component of $F^\mu$. In spite of this difference, it is still reasonable to define the power of the total external force at event $P$ as

$$P_{\text{ex}}(O) \equiv -Z^\mu F_\mu.$$  \hspace{1cm} (13)

We can also make an orthogonal decomposition for $F^\mu$ with respect to $Z^\mu$, just like what we have done for $u^\mu$ in eq.(9),

$$F^\mu = -\frac{1}{c^2} (Z^\nu F_\nu) Z^\mu + \gamma f^\mu,$$  \hspace{1cm} (14)

where

$$f^\mu \equiv \frac{\Delta^\mu_\nu F^\nu}{\gamma}$$  \hspace{1cm} (15)

is the spatial external force. Inserting eqs.(9) and (14) into eq.(13) yields

$$P_{\text{ex}}(O) = \gamma f^\mu v_\mu.$$  \hspace{1cm} (16)

Using the tetrad $\{e_\mu\}$, $f^\mu$ can be written as $(0, f)$, and eq.(16) becomes

$$P_{\text{ex}}(O) = \gamma \cdot f.$$  \hspace{1cm} (17)
This form of the power of the total external force in (13) is more conceivable due to the explicit resemblance to the corresponding Newtonian expression.

We have seen that, in order to define the local power of the total external force, only a single observer $O$ is needed. However, the situation is different while considering the work done by total external force, because the latter ought to be defined via integrating the local power along the particle’s worldline. In order that the integration of the local power is well defined, a densely distributed set of observers needs to be introduced, and the worldline of each observer from this observer set must intersect the particle’s worldline at a single event. Following [7], we define an observer field $Q$ on $\mathcal{M}$ as a smooth timelike future directed vector field, each of whose integral curves is an observer. Form now on, $Z^\mu$ will be used to denote the proper velocity field of $Q$. With the above preparations, we can now write the work done by the total external force during the proper time interval $[\tau_i, \tau_f]$ as

$$W_{\text{ex}}(Q) = \int_{\tau_i}^{\tau_f} P_{\text{ex}}(Q) d\tau = -\int_{\tau_i}^{\tau_f} Z^\mu F_{\mu\nu} d\tau. \quad (18)$$

### 2.3 Power and work of gravity via work-energy theorem

As mentioned in the introduction, gravity is not viewed as a force in general relativity. Therefore, it looks hard to define the power of and the work done by gravity. To find a way out, let us recall that, in Newtonian mechanics, there is a well established law, i.e. the work-energy theorem, which states that the change in the energy of a particle equals the total work done on it. In other words, the only way to change the energy of a particle is to exert mechanical work on it. This statement is among the very few basic principles on top of which classical mechanics is established. There is no reason why such a statement could get changed simply by shifting to relativistic systems. Therefore, we assume the work-energy theorem holds and take it as the tool for defining the power of and the work done by gravity.

In relativistic context, the total work done on a particle is consisted of the work done by the total external force and that by gravity (or inertial force, thanks to Einstein’s equivalence principle). Thus we have

$$W_{\text{grav}}(Q) + W_{\text{ex}}(Q) = \Delta E(Q), \quad (19)$$

where $W_{\text{grav}}(Q)$ is the work done by gravity which is yet to be defined. In differential form, the work-energy theorem can be expressed as follows

$$P_{\text{grav}}(Q) + P_{\text{ex}}(Q) = \frac{dE(Q)}{d\tau}, \quad (20)$$

where $P_{\text{grav}}(Q)$ represent the local power of gravity measured by the observer field $Q$. Naturally, $W_{\text{grav}}(Q)$ and $P_{\text{grav}}(Q)$ should be connected via

$$W_{\text{grav}}(Q) = \int_{\tau_i}^{\tau_f} P_{\text{grav}}(Q) d\tau. \quad (21)$$
Recall that \( P_{\text{ex}}(Q) \) is defined in eq. (13). If the proper time rate of \( E(Q) \), i.e. \( \frac{dE(Q)}{d\tau} \) could be evaluated independently, then eq. (20) would give rise to a definition for \( P_{\text{grav}}(Q) \). Fortunately, \( \frac{dE(Q)}{d\tau} \) can be evaluated right away from eq. (6), yielding

\[
\frac{dE(Q)}{d\tau} = -u^\nu \nabla_\nu (Z^\mu p_\mu) = -m u_\mu (u^\nu \nabla_\nu Z^\mu) - m Z^\mu (u^\nu \nabla_\nu u_\mu)
\]

\[= -m u_\mu (u^\nu \nabla_\nu Z^\mu) - Z^\mu F_\mu. \tag{22} \]

The last term on the RHS of eq. (22) is precisely equal to \( P_{\text{ex}}(Q) \). Therefore, the first term on the RHS of eq. (22) needs to be equal to \( P_{\text{grav}}(Q) \) in order for the work-energy theorem to hold,

\[
P_{\text{grav}}(Q) = -m u_\mu (u^\nu \nabla_\nu Z^\mu). \tag{23} \]

Inserting eq. (23) into (21), we get

\[
W_{\text{grav}}(Q) = \int_{\tau_i}^{\tau_f} P_{\text{grav}}(Q) \, d\tau = - \int_{\tau_i}^{\tau_f} m u^\mu u_\nu \nabla_\nu Z_\mu \, d\tau. \tag{24} \]

This completes the definition for the work done by gravity.

Before finishing this subsection, let us stress that the work-energy theorem should be regarded as a law of Nature which does not depend on the choice of observer field. The inclusion of the notation \( Q \) in eq. (19) is simply intended for reminding the fact that the values of the quantities \( W_{\text{grav}}(Q) \), \( W_{\text{ex}}(Q) \) and \( \Delta E(Q) \) are all dependent on the observer field, however the identity (19) holds for any \( Q \).

\section{Examples}

In order to justify the definitions made in the last section, we now consider the motion of a probe particle in several well-known spacetime solutions of general relativity and calculate the corresponding gravitational work as well as work done by external forces (when applicable). It will be clear that in the far field limit, the works calculated using our definitions agree with the well-known non-relativistic result. These example cases may serve as a justification to our definitions.

\subsection{1+1 dimensional Minkowski spacetime}

To begin with, let us study two simple cases in Minkowski spacetime in order to have a quick intuitive understanding about our previous results.

The first case involves a particle of mass \( m \) moving in Minkowski spacetime with a nonzero external force exerted on it. We use the Cartesian coordinates \( x^\mu = (ct, x) \),
and as the first and simplest example case, the observer field is taken to be inertial with $Z^\mu = (c, 0)$. The proper velocity of the particle can be written as $u^\mu = \gamma c (1, \tanh(a\tau/c))$, where $\gamma = \cosh(a\tau/c)$. Clearly, the external force takes the form

$$F^\mu = ma^\mu = mu^\nu \nabla_\nu u^\mu = ma \left[ \sinh \left( \frac{a\tau}{c} \right), \cosh \left( \frac{a\tau}{c} \right) \right].$$

Using the decomposition (14), we get the spatial part of the force

$$f^\mu = (0, ma). \quad (25)$$

The constant $a$ is nothing but the magnitude of the proper acceleration of the particle,

$$\sqrt{a^\mu a_\mu} = a. \quad (26)$$

The inertial observers perceive neither gravity nor inertial forces, hence the only relevant power comes from the external force which reads

$$- Z^\mu F_\mu = mac \sinh \left( \frac{a\tau}{c} \right). \quad (27)$$

Consequently the work done by external force is

$$W_{\text{ex}}(Q) = - \int_{\tau_i}^{\tau_f} Z^\mu F_\mu d\tau = mc^2 \left[ \cosh \left( \frac{a\tau_f}{c} \right) - \cosh \left( \frac{a\tau_i}{c} \right) \right]. \quad (28)$$

In the slow motion limit $a\tau/c \to 0$, $\tau$ is approximately equal to the coordinate time $t$, and we have

$$u^\mu \simeq (c, at),$$

and $W_{\text{ex}}(Q)$ becomes

$$W_{\text{ex}}(Q) \simeq \frac{1}{2} ma^2 (t_f^2 - t_i^2),$$

wherein we recognize that the RHS is the change in the kinematic energy of the particle, which is the correct Newtonian result.

In the second case, the particle is kept fixed at the origin, with its worldline parametrized as $x^\mu = (c\tau, 0)$, so that $u^\mu = (c, 0)$, while the observer field $Q$ is subject to nontrivial acceleration. The proper velocity field of $Q$ is given by $Z^\mu = \gamma c (1, - \tanh(a\tau/c))$, where $\gamma = \cosh(a\tau/c)$. Due to the accelerated motion of $Q$, the observers in $Q$ should observe an effective gravity (which is actually the inertial force) exerted on the particle. The work done by this fictitious gravity can be easily evaluated using eq.(24), yielding

$$W_{\text{grav}}(Q) = - \int_{\tau_i}^{\tau_f} mu^\mu u^\nu \nabla_\nu Z^\mu d\tau = mc^2 \left[ \cosh \left( \frac{a\tau_f}{c} \right) - \cosh \left( \frac{a\tau_i}{c} \right) \right]. \quad (29)$$

Notice that in both cases the amount of works are the same, though they have completely different interpretations. We can of course take the slow motion limit in this case and verify that $W_{\text{grav}}(Q) \simeq \frac{1}{2} ma^2 (t_f^2 - t_i^2)$ in this limit.
3.2 Schwarzschild spacetime

The previous example is trivial in the sense that there is no actual gravity involved in it. The fictitious gravity is solely triggered by the accelerated motion of the observer field. The next example is nontrivial because the system involves real gravity.

Consider a probe particle of mass $m$ moving in Schwarzschild spacetime with line element

$$ds^2 = - \left(1 - \frac{r_g}{r}\right)c^2 dt^2 + \left(1 - \frac{r_g}{r}\right)^{-1} dr^2 + r^2 d\Omega_2,$$

where $r_g = 2GM/c^2$. The Lagrangian of the probe particle reads

$$L = \frac{m}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = \frac{m}{2} \left[- \left(1 - \frac{r_g}{r}\right)c^2 \dot{t}^2 + \left(1 - \frac{r_g}{r}\right)^{-1} \dot{r}^2 + r^2 \left(\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2\right)\right],$$

where the over dots represent derivatives with respect to the proper time $\tau$. This Lagrangian describes a “free” (i.e. in the absence of external forces besides gravity) probe particle, whose worldline follows one of the geodesics. The equations of motion can be obtained by straightforward variations of eq. (31). However, it is better to consider the first integrals and the on-shell condition as alternatives for the Euler-Lagrangian equations.

Without loss of generality, we assume that the probe particle moves in the equatorial plane $\theta = \pi/2$. Then the Lagrangian is reduced into

$$L = \frac{m}{2} \left[- \left(1 - \frac{r_g}{r}\right)c^2 \dot{t}^2 + \left(1 - \frac{r_g}{r}\right)^{-1} \dot{r}^2 + r^2 \dot{\phi}^2\right].$$

Since both $t$ and $\phi$ are Killing coordinates, we have the corresponding integrals of motion

$$\frac{\partial L}{\partial \dot{t}} = -m \left(1 - \frac{r_g}{r}\right)c^2 \dot{t} = -E,$$

$$\frac{\partial L}{\partial \dot{\phi}} = mr^2 \dot{\phi} = J.$$

The constants $E, J$ respectively can be understood as the energy and the angular momentum “measured” by the Killing vector fields $k = \partial_t$ and $\chi = \partial_\phi$, and their values can only be settled by initial conditions.

The on-shell condition is expressed as follows:

$$u^\mu u_\mu = - \left(1 - \frac{r_g}{r}\right)c^2 \dot{t}^2 + \left(1 - \frac{r_g}{r}\right)^{-1} \dot{r}^2 + r^2 \dot{\phi}^2 = -c^2.$$

Using eqs. (33), (34) and (35), we get

$$\dot{t} = \frac{E}{mc^2} \left(1 - \frac{r_g}{r}\right)^{-1},$$
\[ \dot{r} = -\sqrt{\left( \frac{E}{mc} \right)^2 - \left( 1 - \frac{r_g}{r} \right) \left( c^2 + \frac{J^2}{m^2 r^2} \right)}, \quad (37) \]

where the total minus sign on the RHS of eq. (37) indicates that the particle falls inwardly.

We assume the following initial conditions
\[ x^\mu(\tau_i) = (ct_i, r_i, \frac{\pi}{2}, \phi_i), \]
\[ u^\mu(\tau_i) = \left[ c \left( 1 - \frac{r_g}{r_i} \right)^{-1/2}, 0, 0, 0 \right], \]
where \( t_i, \phi_i \) are arbitrary constants whose values are taken within the allowed ranges for the respective coordinates and meanwhile the constant \( r_i > r_g \). The above choice for \( u^\mu(\tau_i) \) is compatible with the on-shell condition. Using the above initial conditions and recalling the constancy of \( E, J \), we can determine that
\[ E = mc^2 \left( 1 - \frac{r_g}{r_i} \right)^{1/2}, \quad J = 0. \quad (38) \]

In order to exemplify that the power and work can be defined and evaluated for generic observer fields rather than just for the static observer field, let us take the observer field \( Q \) to be moving outward, with proper velocity
\[ Z^\mu = c \left[ \left( 1 - \frac{r_g}{r} \right)^{-1} \left( 1 + C - \frac{r_g}{r} \right)^{1/2}, \sqrt{C}, 0, 0 \right], \quad (r > r_g) \quad (39) \]
where \( C > 0 \) is a dimensionless constant.

The power of the gravity can be evaluated straightforwardly using eq. (23), which results in
\[ P_{\text{grav}}(Q) = \frac{mr_g c^3}{2(r - r_g)^2 \left( 1 + C - \frac{r_g}{r} \right)^{1/2}} \left[ \sqrt{C} \left( 1 + C - \frac{r_g}{r} \right)^{1/2} \left( \frac{2E^2}{m^2 c^4} - 1 + \frac{r_g}{r} \right) \right. \]
\[ + \left. \frac{E}{mc^2} \left( \frac{E^2}{m^2 c^4} - 1 + \frac{r_g}{r} \right)^{1/2} \left( 2C + 1 - \frac{r_g}{r} \right) \right]. \quad (40) \]

The corresponding work is
\[ W_{\text{grav}}(Q) = \int_{\tau_i}^{\tau_f} P_{\text{grav}}(Q) d\tau \]
\[ = mc^2 \left( 1 - \frac{r_g}{r} \right)^{-1} \left[ \frac{E}{mc^2} \left( 1 + C - \frac{r_g}{r} \right)^{1/2} + \sqrt{C} \left( \frac{E^2}{m^2 c^4} - 1 + \frac{r_g}{r} \right)^{1/2} \right] \bigg|_{\tau_i}^{\tau_f}. \quad (41) \]
where \( r_f = r(\tau_f) > r_g \) in order to avoid the horizon.

In the far field limit \( r_i > r_f \gg r_g \), the work \( W_{\text{grav}}(Q) \) can be expand as Taylor series in \( r_g/r_f \) and \( r_g/r_i \). At the order \( O(r_g/r_i,f) \), we have

\[
W_{\text{grav}}(Q) = mc^2 \left[ \frac{2C + 1}{\sqrt{C} + 1} \frac{r_g}{r_f} - \frac{r_g}{r_i} \right] + \sqrt{C} \sqrt{\frac{r_g}{r_f} - \frac{r_g}{r_i}},
\]

(42)

where we have used eq.(38) to express \( E \) in terms of \( r_g/r_i,f \).

Alternatively, if we take the limit \( C \to 0 \), \( Q \) degenerates into the static observer field, whose proper velocity field is proportional to the timelike Killing vector field \( k = \partial_t \), i.e.

\[
\lim_{C \to 0} Z^\mu = [c(-g_{tt})^{-1/2}, 0, 0, 0] = \left[ c \left( 1 - \frac{r_g}{r} \right)^{-1/2}, 0, 0, 0 \right].
\]

(43)

In this case, the gravitational work becomes

\[
\lim_{C \to 0} W_{\text{grav}}(Q) = E \left[ \left( 1 - \frac{r_g}{r_f} \right)^{-1/2} - \left( 1 - \frac{r_g}{r_i} \right)^{-1/2} \right].
\]

(44)

Notice that this is the exact result for gravitational work measured by static observer field. Now if we further take the far field limit \( r_i > r_f \gg r_g \), the work becomes

\[
W_{\text{grav}}(Q) \simeq GMm \left( \frac{1}{r_f} - \frac{1}{r_i} \right).
\]

(45)

This result agrees with eq.(4) which describes the work done by Newtonian gravity. The same result will arise if we directly set \( C \to 0 \) in the far field limit result (42).

Notice that, in eqs.(41) and (44), there is an explicit dependence on \( E \), i.e. on the initial conditions. This dependence seems to be indicating that the gravitational work in Schwarzschild spacetime is path dependent. Since Newtonian gravity is understood as a conservative force, the path dependence of gravitational work in the context of general relativity needs some explanations.

Let us recall that, in Newtonian mechanics, the mechanical energy is divided into two parts, i.e. the kinematic and the potential energies. However, there is no corresponding description in general relativity. For simplicity, let us temporarily assume \( C = 0 \) and \( \dot{\phi} = 0 \). Then, using eqs.(6) and (36), we get

\[
E(Q) = mc^2 i \left( 1 - \frac{r_g}{r} \right)^{1/2} = E \left( 1 - \frac{r_g}{r} \right)^{-1/2}
= mc^2 \left[ 1 - \frac{r_i^2}{c^2 i^2 (1 - \frac{r_g}{r})^2} \right]^{-1/2} = \frac{mc^2}{\sqrt{1 - |\upsilon|^2/c(r)^2}}.
\]

(46)
where $|v| = \frac{\dot{r}}{t}$ is the spatial coordinate speed of the particle, and $c(r) = c \left| 1 - \frac{r_g}{r} \right|$ is the coordinate speed of light. Obviously eq. (46) has the form of (8), with the local Lorentz factor $\gamma = \frac{1}{\sqrt{1 - \frac{|v|}{c(r)}}}$. According to eq. (46), the initial energy of the particle can be written as

$$E(Q)\bigg|_{\text{init}} = \gamma_{\text{init}} mc^2 = \frac{mc^2}{\sqrt{1 - \frac{|v(\tau_i)|^2}{c^2(1 - \frac{2}{\gamma})}}}.$$  

The initial Lorentz factor $\gamma_{\text{init}}$ is clearly dependent on the initial condition. Since gravity acts on the physical mass rather than on the rest mass, it is reasonable that the work also relies on the initial condition. In fact, the mass of the particle changes as it moves towards the source, the effect of gravitational work also changes. If the energy of the particle becomes large enough, it should no longer be regarded as a probe, as its energy could make significant changes to the spacetime metric. This makes an important difference between general relativity and Newtonian mechanics. To see this more clearly, we assume $|v(\tau_i)|^2 \neq 0$, so the initial energy is larger than $mc^2$ but not too large to break the probe approximation, the gravitational work done on such a particle can be evaluated to be

$$W_{\text{grav}}(Q) = \frac{mc^2}{\sqrt{1 - \frac{|v(\tau_i)|^2}{c^2(1 - \frac{2}{\gamma})}}} \left[ \left( 1 - \frac{r_g}{r_i} \right)^{1/2} \left( 1 - \frac{r_g}{r_f} \right)^{-1/2} - 1 \right].$$  

(47)

Now we assume that there is another particle with rest mass $m'$,

$$m' = \frac{m}{\sqrt{1 - \frac{|v(\tau_i)|^2}{c^2(1 - \frac{2}{\gamma})}}}$$

which carries on a “freely falling” procedure, beginning with zero spatial velocity at $r = r_i$. In this case, the initial energy of the second probe particle is just $m'c^2$. Then the work of gravity is

$$W'_{\text{grav}}(Q) = - \int_{\tau_i}^{\tau_f} m' u'_\mu u'^\nu \nabla_\nu Z^\mu d\tau'$$

$$= \frac{mc^2}{\sqrt{1 - \frac{|v(\tau_i)|^2}{c^2(1 - \frac{2}{\gamma})}}} \left[ \left( 1 - \frac{r_g}{r_i} \right)^{1/2} \left( 1 - \frac{r_g}{r_f} \right)^{-1/2} - 1 \right],$$  

(48)

which is identical to (47). Therefore, the gravitational work exerted on a probe particle with nonzero initial velocity can be equivalently regarded as work exerted on a particle with zero initial velocity with rest mass equal to the physical mass of the former one. If the initial condition changes, the gravitational work will also change.
3.3 Reisner-Nordström (RN) spacetime

Now let us consider the third example, i.e. a particle with mass $m$ and charge $e$ moving in RN spacetime. The metric is given by

$$\mathrm{d}s^2 = - \left(1 - \frac{r_g}{r} + \frac{r_Q^2}{r^2}\right)c^2 \mathrm{d}t^2 + \left(1 - \frac{r_g}{r} + \frac{r_Q^2}{r^2}\right)^{-1} \mathrm{d}r^2 + r^2 \mathrm{d}\Omega^2,$$

which is accompanied by the electric potential

$$A_\mu = \left(\frac{Q}{4\pi \varepsilon_0 cr}, 0, 0, 0\right),$$

where $r_g$ and $r_Q$ are given respectively by

$$r_g = \frac{2GM}{c^2}, \quad r_Q^2 = \frac{GQ^2}{4\pi \varepsilon_0 c^3}.$$

The Lagrangian of a charged probe particle can be written as

$$L(x, \dot{x}) = \frac{m}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu - eA_\mu \dot{x}^\mu;$$

and the corresponding equation of motion is given by

$$m u^\nu \nabla_\nu u^\mu = -e F^\mu\nu u_\nu \equiv F^\mu,$$

where $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$ is the field strength tensor and $F^\mu$ is the total electromagnetic force acting on the charged probe particle. Since the background spacetime is spherically symmetric, we can still assume that the probe particle moves in the equatorial plane $\theta = \pi/2$, $\dot{\theta} = 0$. Then in explicit form the Lagrangian can be re-written as

$$L(x, \dot{x}) = \frac{m}{2} \left[ - \left(1 - \frac{r_g}{r} + \frac{r_Q^2}{r^2}\right)c^2 \dot{t}^2 + \left(1 - \frac{r_g}{r} + \frac{r_Q^2}{r^2}\right)^{-1} \dot{r}^2 + r^2 \dot{\phi}^2 \right] - \frac{eQ}{4\pi \varepsilon_0 r} \dot{t}.\quad (54)$$

The on-shell condition now reads

$$u^\mu u_\mu = - \left(1 - \frac{r_g}{r} + \frac{r_Q^2}{r^2}\right)c^2 \dot{t}^2 + \left(1 - \frac{r_g}{r} + \frac{r_Q^2}{r^2}\right)^{-1} \dot{r}^2 + r^2 \dot{\phi}^2 = -c^2.\quad (55)$$

The integrals of motion associated with the Killing coordinates $t, \phi$ are given by

$$\frac{\partial L}{\partial \dot{t}} = -m \left(1 - \frac{r_g}{r} + \frac{r_Q^2}{r^2}\right)c^2 \dot{t} - \frac{eQ}{4\pi \varepsilon_0 r} \equiv -\mathcal{E}_1, \quad (56)$$

$$\frac{\partial L}{\partial \phi} = mr^2 \dot{\phi} \equiv \mathcal{J}_1, \quad (57)$$
and from the above equations we can get

\[ i = \left( \frac{E_1 - \frac{eQ}{4\pi\varepsilon_0}}{mc^2} \right) \left( 1 - \frac{r_g}{r} + \frac{r_g^2}{r^2} \right)^{-1}, \quad \dot{\phi} = \frac{J_1}{m^2 r^2}. \]  

(58)

Using (58) and (55) we get

\[ \dot{r} = - \left[ \frac{\left( E_1 - \frac{eQ}{4\pi\varepsilon_0} \right)^2}{m^2 c^2} - \left( 1 - \frac{r_g}{r} + \frac{r_g^2}{r^2} \right) \left( c^2 + \frac{J_1^2}{m^2 r^2} \right) \right]^{1/2}. \]  

(59)

The initial conditions are now chosen as

\[ x^\mu(\tau_i) = \left( c t_i, r_i, \frac{\pi}{2}, \phi_i \right), \]  

(60)

\[ u^\mu(\tau_i) = \left[ c \left( 1 - \frac{r_g}{r_i} + \frac{r_g^2}{r_i^2} \right)^{-1/2}, 0, 0, 0 \right], \]  

(61)

where again \( t_i, \phi_i \) take arbitrary allowed constant values within the ranges for the respective coordinates and \( r_i > r_+ \) (\( r_+ \) is the radius of the event horizon of RN black hole). The proper velocity field of \( Q \) is chosen to be

\[ Z^\mu = c \left[ \left( 1 - \frac{r_g}{r} + \frac{r_g^2}{r^2} \right)^{-1} \left( 1 + C_1 - \frac{r_g}{r} + \frac{r_g^2}{r^2} \right)^{1/2}, \sqrt{C_1}, 0, 0 \right], \quad (r > r_+) \]  

(62)

where \( C_1 > 0 \) is a dimensionless constant. Therefore, the powers of the electromagnetic force and that of the gravity are given respectively as

\[ P_{em}(Q) = -\frac{eQc}{4\pi\varepsilon_0 \left( r^2 - r_g r + \frac{r_g^2}{4} \right)} \left[ \frac{\sqrt{C_1}}{mc^2} \left( E_1 - \frac{eQ}{4\pi\varepsilon_0 r} \right)^2 \right. \]

\[ + \left. \left( 1 + C_1 - \frac{r_g}{r} + \frac{r_g^2}{r^2} \right)^{1/2} \left( \frac{E_1 - \frac{eQ}{4\pi\varepsilon_0}}{mc^2} \right)^2 - 1 + \frac{r_g}{r} - \frac{r_g^2}{r^2} \right)^{1/2} \]

\[ P_{grav}(Q) = \frac{mc^2 r \left( r_g r - 2 r_g^2 \right)}{2 \left( r^2 - r_g r + \frac{r_g^2}{4} \right)^2 \left( 1 + C_1 - \frac{r_g}{r} + \frac{r_g^2}{r^2} \right)^{1/2}} \left[ \frac{E_1 - \frac{eQ}{4\pi\varepsilon_0}}{mc^2} \right. \]

\[ \times \left( \frac{E_1 - \frac{eQ}{4\pi\varepsilon_0}}{mc^2} \right)^2 - 1 + \frac{r_g}{r} - \frac{r_g^2}{r^2} \right)^{1/2} \]

\[ + \sqrt{C_1} \left( 1 + C_1 - \frac{r_g}{r} + \frac{r_g^2}{r^2} \right)^{1/2} \left( \frac{2 \left( E_1 - \frac{eQ}{4\pi\varepsilon_0} \right)^2}{mc^2} - 1 + \frac{r_g}{r} - \frac{r_g^2}{r^2} \right) \]  

The corresponding works are given by the integrals of the above powers over the proper time interval \([\tau_i, \tau_f]\), however these integrals cannot be worked out explicitly due to the overwhelming complexities of the integrands.
The situation will get simplified drastically in the following two special cases.

The first special case is when the probe particle is neutral, i.e. $e = 0$. In this case, $P_{em}(Q) = 0$ and hence $W_{em}(Q) = 0$, and $W_{grav}(Q)$ can be evaluated explicitly,

$$W_{grav}(Q) = \left( 1 - \frac{r_g}{r} + \frac{r_Q^2}{r_i^2} \right)^{-1} \left[ \mathcal{E}_1 \left( 1 + C_1 - \frac{r_g}{r} + \frac{r_Q^2}{r^2} \right)^{1/2} + \sqrt{C_1 mc^2 \left( \frac{\mathcal{E}_1^2}{m^2 c^4} - 1 + \frac{r_g}{r} - \frac{r_Q^2}{r^2} \right)^{1/2}} \right] \biggr|_{r_i}^{r_f}. \quad (63)$$

The gravitational work still contains contribution from the charge $Q$ because $Q$ not only produces electromagnetic field but also contributes to the gravity of the background spacetime. It remains to determine the integral of motion $\mathcal{E}_1$ using the initial data (61). Using eqs. (61) and (58), we get

$$\mathcal{E}_1 = mc^2 \left( 1 - \frac{r_g}{r} + \frac{r_Q^2}{r_i^2} \right)^{1/2} + \frac{eQ}{4\pi \varepsilon_0 r_i}. \quad (64)$$

Inserting this result into eq. (63) would finish the evaluation of the work done by gravity in RN spacetime. If $Q = 0$, then eq. (63) will fall back to the corresponding result (41) in the Schwarzschild case if we identify $C_1$ with $C$.

The second special case is to take the static observer field from the very beginning, i.e. choosing $C_1 = 0$ and

$$Z^\mu = \left[ e \left( 1 - \frac{r_g}{r} + \frac{r_Q^2}{r^2} \right)^{-1/2}, 0, 0, 0 \right]. \quad (65)$$

Then we will have

$$W_{em}(Q) = \int_{r_i}^{r_f} \frac{eQ}{4\pi \varepsilon_0 r^2} \left( 1 - \frac{r_g}{r} + \frac{r_Q^2}{r_i^2} \right)^{-1/2} dr$$

$$= \frac{eQ}{8\pi \varepsilon_0 r_Q} \ln \left( \frac{2r_Q^2 - r_g r - 2r_Q \sqrt{r^2 - r_g r + r_Q^2}}{2r_Q^2 - r_g r + 2r_Q \sqrt{r^2 - r_g r + r_Q^2}} \right) \biggr|_{r_i}^{r_f}, \quad (66)$$

and

$$W_{grav}(Q) = \int_{r_i}^{r_f} - \left( \frac{4\pi \mathcal{E}_1 r_0}{-eQ} - \frac{eQ}{8\pi \varepsilon_0 r_i^4} \right) \left( r_g r - 2r_Q^2 \right) \left( 1 - \frac{r_g}{r} + \frac{r_Q^2}{r_i^2} \right)^{-3/2} dr$$

$$= \left[ \mathcal{E}_1 - \frac{eQ}{4\pi \varepsilon_0} \left( \frac{1}{r} - \frac{1}{r_i} \right) \right] \left[ \left( 1 - \frac{r_g}{r} + \frac{r_Q^2}{r_i^2} \right)^{-1/2} \biggr|_{r_i}^{r_f} \right. - \frac{eQ}{8\pi \varepsilon_0 r_Q} \ln \left( \frac{2r_Q^2 - r_g r - 2r_Q \sqrt{r^2 - r_g r + r_Q^2}}{2r_Q^2 - r_g r + 2r_Q \sqrt{r^2 - r_g r + r_Q^2}} \right) \biggr|_{r_i}^{r_f}. \quad (67)$$
Notice that the last term of eq. (67) is equal to \(-W_{\text{em}}(Q)\). Therefore, the total work becomes

\[
\Delta E(Q) = \left[ \mathcal{E}_1 - \frac{eQ}{4\pi \varepsilon_0} \left( \frac{1}{r} - \frac{1}{r_i} \right) \right] \left( 1 - \frac{r_g}{r} + \frac{r^2 Q}{r^2} \right)^{-1/2} r_f^{r_f} r_i^{r_i}.
\]

If the probe particle is neutral, \(e = 0\), then

\[
W_{\text{grav}}(Q) = \mathcal{E}_1 \left[ \left( 1 - \frac{r_g}{r_f} \right)^{-1/2} - \left( 1 - \frac{r_g}{r_i} \right)^{-1/2} \right].
\]

If, in addition, \(Q = 0\), then

\[
W_{\text{grav}}(Q) = \mathcal{E}_1 \left[ \left( 1 - \frac{r_g}{r_f} \right)^{-1/2} - \left( 1 - \frac{r_g}{r_i} \right)^{-1/2} \right],
\]

which reproduces the result (44) in the Schwarzschild case if we set \(\mathcal{E}_1 = \mathcal{E}\). Taking expansion of eq. (66) and keeping the leading order yields

\[
W_{\text{em}}(Q) \simeq -\frac{eQ}{4\pi \varepsilon_0} \left( \frac{1}{r_f} - \frac{1}{r_i} \right), \quad (68)
\]

This approximate result recovers the work done by the static electric field in Newtonian case.

### 3.4 Kerr-Newman spacetime

As the last example case, let us consider a probe particle with mass \(m\) and charge \(e\) moving in Kerr-Newman spacetime. The Lagrangian of the particle can be still written in the form (52), but with different underlying spacetime geometry and different electromagnetic potential. In Boyer-Linquist coordinates, the metric reads:

\[
ds^2 = -\frac{\Delta}{\rho^2} \left( c dt - a \sin^2 \theta d\phi \right)^2 + \frac{\sin^2 \theta}{\rho^2} \left( \rho^2 + a^2 \right) d\phi - ac dt \right)^2 + \rho^2 \left( \frac{dr^2}{\Delta} + d\theta^2 \right),
\]

where

\[
\rho^2 = \rho^2 + a^2 \cos^2 \theta, \quad (70)
\]

\[
\Delta = \rho^2 - r_g \rho + r^2 Q + a^2, \quad (71)
\]

and the parameters \(r_g, r_Q\) are respectively connected with the mass \(M\) and the charge \(Q\) of the gravitational source and \(a\) is the ratio of the angular momentum \(J\) over the mass \(M\) of the source,

\[
a = \frac{J}{Mc}, \quad (72)
\]
In the above coordinate system, the electromagnetic potential of the source is given by

$$A_\mu = \left( \frac{Qr}{4\pi \varepsilon_0 c \rho^2}, 0, 0, -\frac{aQr \sin^2 \theta}{4\pi \varepsilon_0 c \rho^2} \right). \quad (73)$$

Substituting eqs.\((69), (73)\) into eq.\((52)\), we get the explicit form for the Lagrangian,

$$L = -\frac{m}{2} \left( 1 - \frac{r g r - \tilde{r}_Q^2}{\rho^2} \right) c^2 \dot{t}^2 + \frac{ma}{\rho^2} \sin^2 \theta \left( -r g r + \tilde{r}_Q^2 \right) \dot{c} \dot{\phi}$$
$$+ \frac{m}{2} \sin^2 \theta \left( r^2 + a^2 + \frac{a^2}{\rho^2} \sin^2 \theta (r g r - \tilde{r}_Q^2) \right) \dot{\phi}^2 + \frac{m}{2} \rho^2 \left( \dot{\theta}^2 + \dot{\rho}^2 \left( \frac{\dot{\rho}}{\Delta} + \dot{\theta}^2 \right) \right)$$
$$- \frac{eQr}{4\pi \varepsilon_0 c \rho^2} \left( c \dot{t} - a \sin^2 \theta \dot{\phi} \right). \quad (74)$$

This Lagrangian depends on neither \(t\) nor \(\phi\), therefore we have two integrals of motion:

$$\frac{\partial L}{\partial \dot{t}} = -m \left( 1 - \frac{r g r - \tilde{r}_Q^2}{\rho^2} \right) c^2 \dot{t} + \frac{ma}{\rho^2} \sin^2 \theta \left( -r g r + \tilde{r}_Q^2 \right) \dot{c} - \frac{eQr}{4\pi \varepsilon_0 \rho^2} = -E_2, \quad (75)$$
$$\frac{\partial L}{\partial \dot{\phi}} = m \sin^2 \theta \left( r^2 + a^2 + \frac{a^2}{\rho^2} \sin^2 \theta (r g r - \tilde{r}_Q^2) \right) \dot{\phi}$$
$$+ \frac{ma}{\rho^2} \sin^2 \theta \left( -r g r + \tilde{r}_Q^2 \right) \dot{c} + \frac{eQr a \sin^2 \theta}{4\pi \varepsilon_0 c \rho^2} = J_2. \quad (76)$$

It would be convenient if we restrict \(\theta\) to \(\pi/2\) so that \(\rho = r\). Then we can solve \(\dot{t}\) and \(\dot{\phi}\) form eqs.\((75)\) and \((76)\), yielding

$$\dot{t} = \frac{E_2}{mc^2} \left[ r^2 + a^2 \left( 1 + \frac{r g}{r} + \frac{\tilde{r}_Q^2}{r^2} \right) \right] - \frac{eQ}{4\pi \varepsilon_0 c mc} \left( r^2 + a^2 + \frac{2a^2 \tilde{r}_Q^2}{r^2} \right) \left( -\frac{r g}{r} + \frac{\tilde{r}_Q^2}{r^2} \right)$$
$$\Delta + \frac{2a^2 \tilde{r}_Q^2}{r^2} \left( 1 - \frac{r g}{r} + \frac{\tilde{r}_Q^2}{r^2} \right), \quad (77)$$
$$\dot{\phi} = \frac{\sqrt{\frac{2E_2}{m c}} \left( 1 - \frac{r g}{r} + \frac{\tilde{r}_Q^2}{r^2} \right) - \frac{aE_2}{mc} \left( -\frac{r g}{r} + \frac{\tilde{r}_Q^2}{r^2} \right) - \frac{aeQ}{4\pi \varepsilon_0 c mc}}{\Delta + \frac{2a^2 \tilde{r}_Q^2}{r^2} \left( 1 - \frac{r g}{r} + \frac{\tilde{r}_Q^2}{r^2} \right)}. \quad (78)$$

By applying the on-shell condition

$$u_\mu u^\mu = -\left( 1 - \frac{r g}{r} + \frac{\tilde{r}_Q^2}{r^2} \right) c^2 \dot{t}^2 + \frac{r^2}{\Delta} \dot{\phi}^2 + \left[ r^2 + a^2 \left( 1 + \frac{r g}{r} - \frac{\tilde{r}_Q^2}{r^2} \right) \right] \dot{\phi}^2$$
$$+ 2ac \left( -\frac{r g}{r} + \frac{\tilde{r}_Q^2}{r^2} \right) \dot{c} \dot{\phi} = -c^2, \quad (79)$$

we can work out \(\dot{r}\), but the expression is too tedious to be shown here. However, the asymptotic value looks quite simple,

$$\lim_{r \to +\infty} \dot{r}^2 = c^2 \left( 1 + \frac{E_2^2}{m^2} \right). \quad (80)$$
We assume that the probe particle starts from $r(\tau_i) = +\infty$, $\theta(\tau_i) = \pi/2$ with initial 4-velocity $u^\mu(\tau_i) = (c,0,0,0)$. These initial conditions imply that the integrals of motion are

$$J_2 = 0, \quad E_2 = mc^2. \quad (81)$$

Even under such initial conditions we find that the explicit evaluation for the works done by electromagnetic force and by gravity on a charged particle is extremely hard if we take a generic observer field. However, since $k = \partial_t$ and $\chi = \partial_\phi$ are both Killing vector fields, it is always possible to choose the observer field $Q$ to be static in the region beyond the ergosphere. Thus, we shall restrict ourselves solely to the static observer field $Q$ with $Z^\mu = \left[c \left(1 - \frac{r_g r - r_Q^2}{r^2}\right)^{-1/2}, 0, 0, 0\right]$.

Using the above data, we are now in a position to calculate the works $W_{\text{em}}(Q)$ and $W_{\text{grav}}(Q)$ in Kerr-Newman spacetime. In integral form, these works can be expressed as

$$W_{\text{em}}(Q) = \int_{+\infty}^{r_f} \frac{eQ}{4\pi \varepsilon_0 r^2} \left(1 - \frac{r_g}{r} + \frac{r_Q^2}{r^2}\right)^{-1/2} dr, \quad (82)$$

$$W_{\text{grav}}(Q) = \int_{+\infty}^{r_f} \frac{(eQ - 4\pi \varepsilon_0 mc^2 r)(r_g r - 2r_Q^2)}{8\pi \varepsilon_0 r^4} \left(1 - \frac{r_g}{r} + \frac{r_Q^2}{r^2}\right)^{-3/2} dr, \quad (83)$$

provided

$$r_f > \frac{1}{2} \left[r_g + \sqrt{r_g^2 - 4r_Q^2}\right],$$

i.e. the final position of the particle is located outside the ergosphere. Clearly, the results (82) and (83) coincide with eqs.(66) and (67) if $r_i$ were taken to be equal to $+\infty$. So, we conclude that, under the above choice of initial condition and observer field, the results for Kerr-Newman spacetime are the same as those for RN spacetime. The rotation of the source affects neither the work done by gravity, nor the work done by electromagnetic field.

### 4 Concluding remarks

To summarize, we have proposed a coordinate independent, however observer dependent definition for gravitational work and work done by external forces on a massive particle in curved spacetime. The definitions are then checked in Minkowski, Schwarzschild, RN and Kerr-Newman spacetimes with appropriate choice of observer fields, and their validities are justified in the far field limit.

The result of the present paper constitutes the first step towards a macroscopic definition of work and heat in curved spacetime backgrounds. In order to achieve the macroscopic definitions in mind, the next step may include considerations involving
relativistic kinetic theory \cite{10, 11, 12, 13}. In this regard, the Jüttner distribution \cite{14, 15} may serve as a starting point.

Alternatively, the gravitational work defined in this paper may serve as a tool for determining the gravitational binding energy for a self-gravitating system in the context of general relativity. The gravitational binding energy has been considered earlier in \cite{16, 17, 18} using different methods. We shall come back later on this subject elsewhere.

Moreover, the relativistic rocket has long been an interesting subject where the gravitational work done on the rocket should be taken into consideration. However, the previous discussions are always restricted to special relativity in Minkowski spacetime. Recently, Henriques and his coworkers discussed about the rocket problem in curved spacetime, but they did not discuss about the concept of work \cite{19}. Our results can be applied to the rocket problem in curved spacetime.

Another possible field of application of our results is in the study of Brownian motion and its connection to fluctuation theorems \cite{20}. In this field, the work and free energy, as well as the non-equilibrium work relations are central concepts. There are already some discussions about relativistic Brownian motion \cite{21} and relativistic fluctuation theorems \cite{22}. Therefore, our results may also find applications in the study of these topics.

As a final remark, let us mention that, in an interview with www.guokr.com, A. Zee, a renowned theoretical physicist, stated \cite{23}, “Relativity is perhaps the worst name ever in the history of physics.” What he meant is that the central subject of concern in the theory of relativity is what remains invariant irrespective of the choice of observers, i.e. what we call the natural laws. This is certainly correct, but reflects only one side of relativistic physics. There is another – often neglected – side of relativistic physics, i.e. the observed quantities of physical observables do depend on the choice of observers, albeit independent of the coordinate choices\footnote{An equivalent statement for this observation that the values of physical observables are foliation dependent \cite{24}. We thank B.P. Kosyakov for letting us know about ref.\cite{24}.}. Without this latter side, our understanding about relativistic physics may be incomplete. This two-sidedness of relativistic physics is best exemplified by the work-energy theorem \cite{19}, in which all quantities $\Delta E(Q), W_{\text{ex}}(Q)$ and $W_{\text{grav}}(Q)$ are observer dependent, whereas the identity \cite{19} holds for every choice of observer field $Q$.

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References

[1] R. Wald, General Relativity, Chicago Univ. Pr., 1984.

[2] C. Kittel, D. Knight, A. Ruderman, A. Carl Helmholz, J. Moyer, Mechanics (Berkeley Physics Course vol. 1) (2nd), McGraw-Hill Book Company, 1973.

[3] P. Ehrenfest, Gleichförmige Rotation starrer Körper und Relativitätstheorie (Uniform rotation of rigid bodies and theory of relativity.), Physik. Z. 10 (1909) 918.

[4] N. A. Doughty, Acceleration of a static observer near the event horizon of a static isolated black hole, American Journal of Physics 49(5) (1981) 412-416.

[5] P. Crawford, I. Tereno, Generalized Observers and Velocity Measurements in General Relativity, General Relativity and Gravitation 34 (2002) 2075–2088.

[6] I. S. Booth, and R. B. Mann, Moving Observers, Non-orthogonal Boundaries, and Quasilocal Energies, Phys. Rev. D 59 (1999) 064021.

[7] P. P. Yu, R. R. Caldwell, Observer dependence of the quasilocal energy and momentum in Schwarzschild space-time, General Relativity and Gravitation 41 (2009) 559–570.

[8] F. Dahia, P. J. Felix da Silva Static observers in curved spaces and non-inertial frames in Minkowski spacetime, General Relativity and Gravitation 43 (2011) 269–292.

[9] V. Berezin, Could a real (not virtual) static observer exist outside a Schwarzschild black hole?, General Relativity and Gravitation 44 (2012) 1555–1561.

[10] O. Sarbach and T. Zannias, Relativistic kinetic theory: An introduction, AIP Conference Proceedings 1548 (2013) 134–155.

[11] O. Sarbach and T. Zannias, The geometry of the tangent bundle and the relativistic kinetic theory of gases, Classical and Quantum Gravity 31 (2014) 085013.

[12] C. Cercignani and G. Kremer, The relativistic Boltzmann equation: theory and application, Birkhäuser; Basel, 2002.

[13] R. Hakim, Introduction to relativistic statistical mechanics, World Scientific Publishing, 2011.

[14] F. Jüttner, Das maxwellsche gesetz der geschwindigkeitsverteilung in der relativtheorie, Annalen der Physik 339 No.5 (1911) 856–882.

[15] F. Jüttner, Die relativistische quantentheorie des idealen gases, Zeitschrift für Physik 47 (1928) 542–566.

[16] R. Arnowitt, S. Deser, C. W. Misner, Minimum size of dense source distributions in general relativity, Annals of Physics 33(1) (1965) 88-107.
[17] W. Israel, *Singular hypersurfaces and thin shells in general relativity*, *Il Nuovo Cimento* B (1965-1970) 44, 1–14 (1966).

[18] P. Bizon, E. Malec, N. O. Murchadha, *Binding energy for spherical stars*, *Class. Quant. Gravity* 7(11), (1990) 1953.

[19] P. G. Henriques, J. Natário, *The Rocket Problem in General Relativity*, *J. Optim. Theory. Appl.* 154 (2012) 500–524.

[20] U. Seifert, *Stochastic thermodynamics, fluctuation theorems, and molecular machines*, *Reports on Progress in Physics* 75 (2012) 126001.

[21] J. Dunkel, P. Hänggi, *Relativistic Brownian motion*, *Physics Reports* 471 (2009) 1-73.

[22] A. Fingerle, *Relativistic fluctuation theorems*, *Comptes Rendus Physique* 8 (5-6) (2007) 696-713.

[23] See https://www.guokr.com/article/441062/.

[24] B.P. Kosyakov, *Self-interaction in classical gauge theories and gravitation*, *Physics Reports* 812 (2019) 1–55.