Global solutions of compressible Navier–Stokes equations with a density–dependent viscosity coefficient

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Abstract

We prove the global existence and uniqueness of the classical (weak) solution for the 2D or 3D compressible Navier–Stokes equations with a density–dependent viscosity coefficient \( \lambda = \lambda(\rho) \). Initial data and solutions are only small in the energy-norm. We also give a description of the large time behavior of the solution. Then, we study the propagation of singularities in solutions. We obtain that if there is a vacuum domain at initially, then the vacuum domain will exists for all time, and vanishes as time goes to infinity.

1 Introduction

In this paper, we consider the following compressible Navier–Stokes equations

\[
\begin{cases}
\rho_t + \text{div}(\rho u) = 0, \\
(\rho u)_t + \text{div}(\rho u \otimes u) + \nabla P = \mu \Delta u + \nabla ((\mu + \lambda(\rho)) \text{div} u) + \rho f,
\end{cases}
\]

(1.1)

for \( x \in \mathbb{R}^N \) and \( t > 0 \), \( N = 2 \) or \( 3 \), with the boundary and initial conditions

\[
\begin{align*}
&\rho(x,t) \to 0, \; \rho(x,t) \to \bar{\rho} > 0, \; \text{as} \; |x| \to \infty, \; t > 0, \\
&(\rho, u)|_{t=0} = (\rho_0, u_0).
\end{align*}
\]

(1.2)

(1.3)

Here \( \rho(x,t), u(x,t) \) and \( P = P(\rho) \) stand for the fluid density, velocity and pressure respectively, \( f \) is a given external force, the dynamic viscosity coefficient \( \mu \) is a positive constant, the second viscosity coefficient \( \lambda = \lambda(\rho) \) is a function of \( \rho \).

In [20], we proved the global existence of weak solutions for the two-dimensional system, and study the propagation of singularities in solutions. In this paper, we want to obtain the global existence, uniqueness and the large time behavior of the classical solution to the system (1.1)–(1.3) in \( \mathbb{R}^2 \) or \( \mathbb{R}^3 \), also obtain the global existence of weak solutions and study the propagation of singularities in solutions in \( \mathbb{R}^3 \).

At first, we obtain the global existence, uniqueness and the large time behavior of the classical solution, when the energy of initial data is small, but the oscillation is arbitrarily large. Specifically, we fix a positive constant \( \bar{\rho} \), assume that \( (\rho_0 - \bar{\rho}, u_0) \) are small in \( L^2 \), and \( \rho_0 - \bar{\rho}, u_0 \in H^3 \) with no restrictions on their norms, (since we use the classical analysis methods in this paper, we restrict the result of the existence of the classical solutions on the framework of Hilbert space \( H^3(\mathbb{R}^N) \rightarrow C^1(\mathbb{R}^N) \)). Our existence result accommodates a wide class of pressures \( P \), including pressures that are not monotone in \( \rho \). It also generalizes and improves upon earlier results of Danchin [4] and Matsumura-Nishida [13] in a significant way: \( (\rho_0 - \bar{\rho}, u_0) \) are only small in \( L^2 \).

Now, we give a precise formulation of our result. Concerning the pressure \( P \), viscosity coefficients \( \mu \) and \( \lambda \), we fix \( 0 < \bar{\rho} < \tilde{\rho} \) and assume that

\[
\begin{cases}
\mu > 0, \; \lambda \in \{0, \infty\}, & N = 2, \\
\mu > 0, \; \lambda \in [0, 3\mu], & N = 3, \text{ for all } \rho \in [0, \bar{\rho}], \\
P(0) = 0, \; P'(\tilde{\rho}) > 0, \\
(\rho - \bar{\rho})[P(\rho) - P(\tilde{\rho})] > 0, & \rho \in [0, \bar{\rho}) \cup (\bar{\rho}, \tilde{\rho}].
\end{cases}
\]

(1.4)

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Let $G$ be the potential energy density, defined by

$$G(\rho) = \rho \int_\rho^\infty \frac{P(s) - P(\rho)}{s^2} \, ds. \quad (1.5)$$

Then, for any $g \in C^2([0, \bar{\rho}])$ with $g(\bar{\rho}) = g'(\bar{\rho}) = 0$, there is a constant $C$ such that $|g(\rho)| \leq CG(\rho)$, $\rho \in [0, \bar{\rho}]$.

Define

$$C_0 = \int \left( \frac{1}{2} \rho_0 |u_0|^2 + G(\rho_0) \right) \, dx, \quad (1.6)$$

$$C_f = \sup_{t \geq 0} \|f(\cdot, t)\|_{L^2}^2 + \int_0^\infty \left( \|f(\cdot, t)\|_{L^2}^2 + \|f(\cdot, t)\|_{L^2}^2 + \|\nabla f\|_{L^4}^2 + \|\nabla \|_{L^4}^2 \right) \, dt \quad (1.7)$$

and

$$M_q = \int_0^\infty \left( \sigma^2 \|f_t\|_{L^2}^2 + \|\nabla f\|_{L^4}^2 + \|\nabla f\|_{L^4}^2 + \|\nabla f\|_{L^4}^2 \right) \, dt + \|\nabla u_0\|_{L^2}^2 + \sup_{t \geq 0} \|f\|_{L^{2+q}}, \quad (1.8)$$

where $\sigma(t) = \min\{1, t\}$,

$$p_1 = \begin{cases} 2 + \frac{2}{q}, & N = 2, \\ 1 + \frac{2}{q}, & N = 3 \end{cases}, \quad p_2 = \begin{cases} 4 + 2q, & N = 2, \\ \frac{6(2+q)}{4-q}, & N = 3 \end{cases}, \quad (1.9)$$

and $q$ is a constant satisfying

$$q \in \begin{cases} (0, 2), & N = 2, \\ (1, \frac{5}{3}), & N = 3, \end{cases} \quad \text{and} \quad q^2 < \frac{4\mu}{\mu + \lambda(\rho)}, \quad \forall \rho \in [0, \bar{\rho}], \quad (1.10)$$

As in [3, 6], we recall the definition of the vorticity matrix $w^{j,k} = \partial_k w^j - \partial_j w^k$, and definition of the function

$$F = (\lambda + 2\mu)\text{div} u - P(\rho) + P(\bar{\rho}). \quad (1.11)$$

Thus, we have

$$\Delta w^j = \partial_j \left( \frac{F + P - P(\bar{\rho})}{\lambda + 2\mu} \right) + \partial_i (w^{j,i}); \quad (1.12)$$

We also define the convective derivative $\frac{D}{Dt}$ by $\frac{D}{Dt} = \dot{w} = w_t + u \cdot \nabla w$, the Hölder norm

$$<v>^\alpha_A = \sup_{x,y \in A} \frac{|v(x) - v(y)|}{|x - y|^\alpha},$$

and

$$<g>^{\alpha, \beta}_{A \times [t_1, t_2]} = \sup_{(x,t),(y,s) \in A \times [t_1, t_2], x \neq y} \frac{|g(x, t) - g(y, s)|}{|x - y|^\alpha + |t - s|^\beta},$$

where $v : A \subseteq \mathbb{R}^N \rightarrow \mathbb{R}^N$, $g : A \times [t_1, t_2] \rightarrow \mathbb{R}^N$ and $\alpha, \beta \in (0, 1]$.

The following is the main result of this paper.

**Theorem 1.1.** Assume that conditions $\{1.4\} - \{1.8\}$ hold. Then, for a given positive number $M$ (not necessarily small) and $\bar{\rho}_1 \in (\bar{\rho}, \bar{\rho})$, there are positive number $\varepsilon$, such that, the Cauchy problem $\{1.4\} - \{1.8\}$ with the initial data $(\rho_0, u_0)$ and external force $f$ satisfying

$$0 < \rho_1 \leq \inf \rho_0 \leq \sup \rho_0 \leq \bar{\rho}_1, \quad C_0 + C_f \leq \varepsilon, \quad M_q \leq M, \quad \rho_0 - \bar{\rho}, u_0 \in H^3, \quad f_t \in C([0, \infty); L^2), \quad f \in C([0, \infty); H^2), \quad (1.13)$$

on some subintervals of $[0, \bar{T}]$.
has a unique global classical solution \((\rho, u)\) satisfying

\[
\frac{1}{2} \rho_1 \leq \rho(x, t) \leq \tilde{\rho}, \quad x \in \mathbb{R}^N, \quad t \geq 0,
\]
\[
(\rho - \tilde{\rho}, u) \in C^1([0, T]) \cap C([0, T], H^3) \cap C^1([0, T], H^2),
\]
\[
\sup_{t \in [0, T]} (\|\rho - \tilde{\rho}\|_{H^3} + \|(\rho_t, u_t)\|_{H^2}) + \int_0^T \|u\|_{H^4}^2 dt \leq K(T), \quad \forall \, T > 0,
\]
\[
\lim_{t \to +\infty} \int (|\rho - \tilde{\rho}|^4 + \rho|u|^4) (x, t) dx = 0,
\]
\[
\lim_{t \to +\infty} \int |\nabla u|^2 (x, t) dx = 0,
\]where \(K(T)\) is a positive constant dependent on \(\rho_1, \tilde{\rho}, (\rho_0 - \tilde{\rho}, u_0)\|_{H^3} \text{ and } T\).

Remark 1.1. For example, we can choose that \(P = A\rho^\gamma\) and \(\lambda(\rho) = c\rho^\beta\) with \(\gamma \geq 1\) and \(\beta \geq 1\), where \(A\) and \(c\) are two positive constants. Also, we can choose that \(\lambda\) is a non-negative constant.

Remark 1.2. Considering the case that the space domain \(\Omega \subset \mathbb{R}^2\) is bounded and \(\lambda = \rho^\beta, \beta > 3\), Vaigant-Kazhikhov ([16], Theorems 1-2) obtained the global existence of strong solutions when the initial data are large and the initial density is bounded from zero. In this paper, since the initial energy is small, we can use the similar argument as that in the case \(\lambda = \text{constant}\) to obtain some good a priori estimates of the solution, and obtained the global existence of classical solutions when the space domain is \(\mathbb{R}^N\), \(N = 2\) or \(3\), the initial density may vanish in an open set and \(\beta \geq 1\).

The proof of Theorem 1.1 bases on the derivation of a priori estimates for the local solution. Specifically, in Section 2, we fix a smooth, local in time solution for which \(0 \leq \rho \leq \bar{\rho}\) and \(A_1 + A_2 \leq 2(C_0 + \bar{C}_f)^\beta\), then obtain the estimate \(A_1 + A_2 \leq (C_0 + \bar{C}_f)^\beta\), and prove that the density remains in a compact subset of \((0, \bar{\rho})\). Using the classical continuation method, we can close these estimates.

Using the initial condition \(u_0 \in H^1\), we can obtain pointwise bounds for \(F\) in Proposition 2.4, which is the key point of the a priori estimates. Because that the mass equation can be transformed to the following form,

\[
\frac{d}{dt} \Lambda(\rho(x(t), t)) + P(\rho(x(t), t)) - P(\bar{\rho}) = -F(x(t), t),
\]
where \(\Lambda\) satisfies that \(\Lambda(\bar{\rho}) = 0\) and \(\Lambda'(\rho) = \frac{2(\rho + \lambda)}{\rho}\), a curve \(x(t)\) satisfies \(\dot{x}(t) = u(x(t), t)\), thus pointwise bounds for the density will therefore follow from pointwise bounds for \(F\).

In theorem 1.2, the constant \(\varepsilon\) is independent of \(\bar{\rho}\). Thus, we can obtain the global existence of weak solutions to (1.1)-(1.3) with the nonnegative initial density \(\rho_0 \geq 0\) (the two-dimensional result can be found in [20]).

Definition 1.1. We say that \((\rho, u)\) is a weak solution of (1.1)-(1.3), if \(\rho\) and \(u\) are suitably integrable and satisfy that

\[
\int_{t_1}^{t_2} \rho \phi dx = \int_{t_1}^{t_2} \int (\rho \phi_t + \rho u \cdot \nabla \phi) dx dt
\]
for all times \(t_2 \geq t_1 \geq 0\) and all \(\phi \in C^1_0(\mathbb{R}^N \times [t_1, t_2])\),

\[
\int_{t_1}^{t_2} \rho u \psi dx = \int_{t_1}^{t_2} \int \{\rho u \phi_t + \rho (u \cdot \nabla \psi) \cdot u + P \text{div} \psi\} dx dt
\]
\[
= - \int_{t_1}^{t_2} \int \{\rho \partial_k u \partial_k \psi + (\mu + \lambda) \text{div} u \text{div} \psi - \rho f \psi\} dx dt
\]
for all times \(t_2 \geq t_1 \geq 0\) and all \(\psi \in (C^1_0(\mathbb{R}^N \times [t_1, t_2]))^N\).
Concerning the pressure \( P \), viscosity coefficients \( \mu \) and \( \lambda \), we fix \( 0 < \tilde{\rho} < \bar{\rho} \) and assume that
\[
\begin{align*}
P \in C^1([0, \bar{\rho}]), \quad &\lambda \in C^2([0, \bar{\rho}]), \\
\mu > 0, \quad &\lambda \in \begin{cases} [0, \infty), & N = 2, \\
[0, 3\mu], & N = 3, \end{cases} \quad \text{for all } \rho \in [0, \bar{\rho}], \\
P(0) = 0, \quad &P'(\tilde{\rho}) > 0, \\
(\rho - \tilde{\rho})[P(\rho) - P(\tilde{\rho})] > 0, \quad &\rho \in [0, \bar{\rho}] \cup (\tilde{\rho}, \bar{\rho}], \\
P \in C^2([0, \bar{\rho}]) \quad \text{or} \quad &\frac{P_c(\cdot)}{2\mu + \lambda(\cdot)} \quad \text{is a monotone function on } [0, \bar{\rho}].
\end{align*}
\]

Theorem 1.2. Assume that conditions (1.1)–(1.3) and (1.22) hold. Then, for a given positive number \( M \) (not necessarily small) and \( \tilde{\rho}_1 \in (\tilde{\rho}, \bar{\rho}) \), there are positive numbers \( \varepsilon \) and \( \theta \), such that, the Cauchy problem (1.1)–(1.3) with the initial data \((\rho_0, u_0)\) and external force \( f \) satisfying
\[
0 \leq \inf \rho_0 \leq \sup \rho_0 \leq \tilde{\rho}_1, \\
0 \leq C_0 + C_f \leq \varepsilon, \\
M_q \leq M,
\]
has a global weak solution \((\rho, u)\) in the sense of (1.20)–(1.21) satisfying
\[
C^{-1} \inf \rho_0 \leq \rho \leq \bar{\rho}, \quad \text{a.e.}
\]
\[
\begin{align*}
\rho - \tilde{\rho}, \quad &\rho u \in C([0, \infty); H^{-1}), \\
\nabla u \in L^2(\mathbb{R}^N \times [0, \infty)), \\
\end{align*}
\]
\[
< u >_{\mathbb{R}^N \times [r, \infty)} + \sup_{t \geq r} (||\nabla F(\cdot, t)||_{L^2} + ||\nabla w(\cdot, t)||_{L^2}) \leq C(\tau) (C_0 + C_f)^\theta,
\]
where \( \alpha \in (0, 1) \) when \( N = 2 \), \( \alpha \in (0, \frac{1}{2}) \) when \( N = 3 \), \( \tau > 0 \) and \( \alpha' \in (0, \frac{2 + \frac{N}{2}}{2 + q}) \),
\[
\sup_{t \geq 0} \int \left( \frac{1}{2} |\rho|^2 + |\rho - \tilde{\rho}|^2 + \sigma |\nabla u|^2 \right) dx
\]
\[
\quad + \int_0^\infty \int \left( |\nabla u|^2 + \sigma |(\rho u)_t + \text{div}(\rho u \otimes u)|^2 + \sigma^N |\nabla u|^2 \right) dx dt
\]
\[
\quad \leq (C_0 + C_f)^\theta,
\]
where \( \lim_{t \to +\infty} \int (|\rho - \tilde{\rho}|^4 + \rho |u|^4) (x, t) dx = 0. \]

In addition, in the case that \( \inf \rho_0 > 0 \), the term \( \int_0^\infty \int \sigma |\dot{u}|^2 dx dt \) may be included on the left hand side of (1.30).

Remark 1.3. Considering the case that \( \lambda = \text{constant} \), Hoff-Santos [5] and Hoff [6, 7, 8] obtained the existence of global weak solutions. In this paper, since the viscosity coefficient \( \lambda \) is a function of the density \( \rho \), we need a higher regularity condition \( \nabla u_0 \in L^2 \), and use some new methods to obtain a priori estimates of the solution. Using the initial condition \( \nabla u_0 \in L^2 \), we can obtain pointwise bounds for \( F \) in Proposition 2.6 which is the key point of the a priori estimates. Using the compensated compactness method [10] and the estimate \( \int_0^T |F(\cdot, t)|_{L^\infty} dt \leq C(T) \), we can obtain the strong limit of approximate densities \( \{\rho^i\} \), see Section 4.
Then, we study the propagation of singularities in solutions obtained in Theorem 1.2. Under the regularity estimates of the solution in Theorem 1.2, and similar arguments as that in [5, 20], we can obtain Theorems 1.3–1.7 and omit the details.

In Theorem 1.3 we obtain that each point of \( \mathbb{R}^N \) determines a unique integral curve of the velocity field at the initial time \( t = 0 \), and that this system of integral curves defines a locally bi-Hölder homeomorphism of any open subset \( \Omega \) onto its image \( \Omega' \) at each time \( t > 0 \). From this Lagrangean structure, we can obtain that if there is a vacuum domain at the initial time, then the vacuum domain will exist for all time, and vanishes as time goes to infinity, see Theorem 1.5. Also, in Theorem 1.6 we obtain that, if the initial density has a limit at a point from a given side of a continuous hypersurface, then at each later time both the density and the divergence of the velocity have limits at the transported point from the corresponding side of the transported hypersurface, which is also a continuous manifold. If the limits from both sides exist, then the Rankine-Hugoniot conditions hold in a strict pointwise sense, showing that the jump in the \( (\lambda + 2\mu)\text{div} \) is proportional to the jump in the pressure (Theorem 1.7). This leads to a derivation of an explicit representation for the strength of the jump in \( \Lambda(\rho) \) in non-vacuum domain.

**Theorem 1.3.** Assume that the conditions of Theorem 1.2 hold.

1. For each \( t_0 \geq 0 \) and \( x_0 \in \mathbb{R}^N \), there is a unique curve \( X(\cdot, t_0, t_0) \in C^1((0, \infty); \mathbb{R}^N) \cap C^{\frac{\alpha}{\alpha - \beta}}((0, \infty); \mathbb{R}^N) \), satisfying
   \[
   X(t; x_0, t_0) = x_0 + \int_{t_0}^{t} u(X(s; x_0, t_0), s)ds.
   \] (1.32)

2. Denote \( X(t, x_0) = X(t; x_0, 0) \). For each \( t > 0 \) and any open set \( \Omega \subset \mathbb{R}^N \), \( \Omega' = X(t, \cdot)\Omega \) is open and the map \( x_0 \mapsto X(t, x_0) \) is a homeomorphism of \( \Omega \) onto \( \Omega' \).

3. For any \( 0 \leq t_1, t_2 \leq T \), the map \( X(t_1, y) \to X(t_2, y) \) is Hölder continuous from \( \mathbb{R}^N \) onto \( \mathbb{R}^N \). Specifically, for any \( y_1, y_2 \in \mathbb{R}^N \),
   \[
   |X(t_2, y_2) - X(t_2, y_1)| \leq \exp(1 - e^{-C(1 + T)})|X(t_1, y_2) - X(t_1, y_1)|e^{-C(1 + T)}. \] (1.33)

4. Let \( \mathcal{M} \subset \mathbb{R}^N \) be a \( C^\beta \) \( (N - 1) \)-manifold, where \( \beta \in [0, 1) \). Then, for any \( t > 0 \), \( \mathcal{M} = X(t, \cdot)\mathcal{M} \) is a \( C^{\beta'} \) \( (N - 1) \)-manifold, where \( \beta' = \beta e^{C(1 + t)} \).

**Theorem 1.4.** Assume that the conditions of Theorem 1.2 hold. Let \( V \) be a nonempty open set in \( \mathbb{R}^N \). If \( \text{essinf}_{\partial \Omega} \rho \geq \rho > 0 \), then there is a positive number \( \rho^* \) such that,
   \[
   \rho(x, t)|_{V^t} \leq \rho^*.
   \]

for all \( t > 0 \), where \( V^t = X(t, \cdot)V \).

**Theorem 1.5.** Assume that the conditions of Theorem 1.2 hold. Let \( U \) be a nonempty open set in \( \mathbb{R}^N \). Assume that \( \rho_0|_{\partial U} = 0 \). Then,
   \[
   \rho(x, t)|_{U^t} = 0,
   \]

for all \( t > 0 \), where \( U^t = X(t, \cdot)U \). Furthermore, we have
   \[
   \lim_{t \to \infty} |\left\{ x \in \mathbb{R}^N | \rho(x, t) = 0 \right\}| = 0. \] (1.34)

Recall that the oscillation of \( g \) at \( x \) with respect to \( E \) is defined by (as in [2])
   \[
   \text{osc}(g; x, E) = \lim_{R \to 0} \left( \text{esssup}_{E \cap B_R(x)} g - \text{essinf}_{E \cap B_R(x)} g \right),
   \]

where \( x \in \overline{E} \) and \( g \) maps an open set \( E \subset \mathbb{R}^N \) into \( \mathbb{R} \). We shall say that \( g \) is continuous at an interior point \( x \) of \( E \), if \( \text{osc}(g; x, E) = 0 \).

**Theorem 1.6.** Assume that the conditions of Theorem 1.2 hold. Let \( E \subset \mathbb{R}^N \) be open and \( x_0 \in \overline{E} \). If \( \text{osc}(\rho_0; x_0, E) = 0 \), then \( \text{osc}(\rho(x, t); X(t, x_0), X(t, \cdot)E) = 0 \). In particular, if \( x_0 \in E \) and \( \rho_0 \) is continuous at \( x_0 \), then \( \rho(x, t) \) is continuous at \( X(t, x_0) \).
Now, let $\mathcal{M}$ be a $C^0$ $(N - 1)$-manifold in $\mathbb{R}^N$ and $x_0 \in \mathcal{M}$. Then there is a neighborhood $G$ of $x_0$ which is the disjoint union $G = (G \cap \mathcal{M}) \cup E_+ \cup E_-$, where $E_\pm$ are open and $x_0$ is a limit point of each. If $\text{osc}(g; x_0, E_+) = 0$, then the common value $g(x_0^+, t)$ is the one-sided limit of $g$ at $x_0$ from the plus-side of $\mathcal{M}$, and similar for the one-sided limit $g(x_0^-, t)$ from the minus-side of $\mathcal{M}$. If both of these limits exist, then the difference $[g(x_0)] := g(x_0^+) - g(x_0^-)$ is the jump in $g$ at $x_0$ with respect to $\mathcal{M}$ (see [5]). Then, we can obtain the following result about the propagation of singularities in solutions.

**Theorem 1.7.** Let $(\rho, u)$ as in Theorem 1.2, $\mathcal{M}$ be a $C^0$ $(N - 1)$-manifold and $x_0 \in \mathcal{M}$.

(a) If $\rho_0$ has a one-sided limit at $x_0$ from the plus-side of $\mathcal{M}$, then for each $t > 0$, $\rho(\cdot, t)$ and $\text{div} u(\cdot, t)$ have one-sided limits at $X(t, x_0)$ from the plus-side of the $C^0$ $(N - 1)$-manifold $X(t, \cdot)\mathcal{M}$ corresponding to the choice $E^+_t = X(t, \cdot)E_+$. The map $t \mapsto \rho(X(t, x_0) +, t)$ is in $C^1([0, \infty)) \cap C^1([0, \infty))$ and the map $t \mapsto \text{div}(X(t, x_0) +, t)$ is locally Hölder continuous on $(0, \infty)$.

(b) If both one-sided limits $\rho_0(x_0 \pm)$ of $\rho_0$ at $x_0$ with respect to $\mathcal{M}$ exist, then for each $t > 0$, the jumps in $P(\rho(\cdot, t))$ and $\text{div} u(\cdot, t)$ at $X(t, x_0)$ satisfy the Rankine-Hugoniot condition

$$[(2\mu + \lambda(\rho(X(t, x_0), t)))\text{div} u(X(t, x_0), t)] = [P(\rho(X(t, x_0), t))].$$

(1.35)

(c) Furthermore, if $\rho_0(x_0 \pm) \geq Z > 0$, then the jump in $\Lambda(\rho)$ satisfies the representation

$$[\Lambda(\rho(X(t, x_0), t))] = \exp \left( - \int_0^t a(\tau, x_0) d\tau \right) [\Lambda(\rho_0(x_0))]$$

(1.36)

where $a(t, x_0) = \frac{|P(\rho(X(t, x_0), t))|}{|\Lambda(\rho(X(t, x_0), t))|}$.

**Remark 1.4.** Using similar arguments as that in [20], we also can show that the condition of $\mu =$ constant will induce a singularity of the system at vacuum in the following two aspects: 1) considering the special case where two fluid regions initially separated by a vacuum region, the solution we obtained is a non-physical weak solution in which separate kinetic energies of the two fluids need not to be conserved; 2) smooth solutions for the spherically symmetric system will blowup when the initial density is compactly supported. Therefore, the viscosity coefficient $\mu$ plays a key role in the Navier-Stokes equations.

We now briefly review some previous works about the Navier-Stokes equations with density-dependent viscosity coefficients. For the free boundary problem of one-dimensional or spherically symmetric isentropic fluids, there are many works, please see [9] [10] [12] [17] [18] [19] and the references cited therein. Under a special condition between $\mu$ and $\lambda$, $\lambda = 2\mu\mu' - 2\mu$, there are some existence results of global weak solutions for the system with the Korteweg stress tensor or the additional quadratic friction term, see [1] [2]. Also see Lions [11] for multidimensional isentropic fluids.

We should mention that the methods introduced by Hoff in [17] and Vaigant-Kazhikhov in [10] will play a crucial role in our proof here.

### 2 Global existence

Standard local existence results now apply to show that there is a smooth local solution $(\rho, u)$ to $(1.1)$–(1.3), defined up to a positive time $T_0$, such that

$$\rho, \ u \in C^1(\mathbb{R}^N \times [0, T_0]) \quad \text{with} \quad \rho > 0 \quad \text{for all} \ t \in [0, T_0]$$

and

$$(\rho - \bar{\rho}, u) \in C([0, T_0], H^3) \cap C^1([0, T_0], H^2).$$

(See for example Matsumura-Nishida [14] and Nash [15].) Let $[0, T^*)$ be the maximal existence interval of the above solution to $(1.1)$–(1.3).

**Claim 1:** For any $T > 0$, if $(\rho, u)$ satisfies

$$0 \leq \rho \leq \bar{\rho}$$

(2.1)
\[ A_1 + A_2 \leq 2(C_0 + C_f)^\theta, \quad \forall \ t \in [0, T] \cap [0, T^*), \]  
(2.2)

where \( \theta \in (0, 1) \),

\[ A_1(T) = \sup_{t \in (0, T] \cap (0, T^*)} \sigma \int_0^{T \wedge T^*} |\nabla u|^2 dx + \int_0^{T \wedge T^*} \sigma \rho |\dot{u}|^2 dx dt \]

and

\[ A_2(T) = \sup_{t \in (0, T] \cap (0, T^*)} \sigma^N \int_0^{T \wedge T^*} |\nabla u|^2 dx + \int_0^{T \wedge T^*} \sigma^N |\nabla \dot{u}|^2 dx dt, \quad T \wedge T^* = \min\{T, T^*\}, \]

then we have

\[ \frac{1}{2} \rho_1 < \rho < \bar{\rho}, \quad A_1 + A_2 \leq (C_0 + C_f)^\theta, \quad \forall \ t \in [0, T] \cap [0, T^*). \]  
(2.3)

In this paper, we assume that \( \varepsilon \leq 1 \).

We can rewrite the momentum equation in the form,

\[ \rho \ddot{u} = \partial_j F + \mu \partial_k u^{j,k} + \rho f^j. \]  
(2.4)

Stated differently, the decomposition (2.4) implies that

\[ \Delta F = \text{div}(\rho \ddot{u} - \rho f). \]  
(2.5)

Similarly, we have

\[ \mu \Delta u^{j,k} = \partial_k (\rho \ddot{u}^j) - \partial_j (\rho \dot{u}^k) + \partial_j (\rho f^k) - \partial_k (\rho f^j). \]  
(2.6)

Thus \( L^2 \) estimates for \( \rho \ddot{u} \), immediately imply \( L^2 \) bounds for \( \nabla F \) and \( \nabla u \). These three relations (2.4)–(2.6) will play the important role in this section.

From now on, the constant \( C \) (or \( C(T) \)) will be independent of \( \rho_1 \).

**Proposition 2.1.** There is a positive constant \( C = C(\bar{\rho}) \) independent of \( \rho_1 \), such that if \( (\rho, u) \) is a smooth solution of (1.1)–(1.3) satisfying (2.1)–(2.2), then

\[ \sup_{t \in [0, T] \cap [0, T^*)} \int \left[ \frac{1}{2} \rho |u|^2 + G(\rho) \right] dx + \int_0^{T \wedge T^*} \left\| \nabla u \right\|^2 dx dt \leq C(C_0 + C_f). \]  
(2.7)

**Proof.** Using the energy estimate, we can easily obtain (2.7), and omit the details. \( \square \)

The following lemma contains preliminary versions of \( L^2 \) bounds for \( \nabla u \) and \( \dot{u} \).

**Lemma 2.1.** If \( (\rho, u) \) is a smooth solution of (1.1)–(1.3) as in Proposition 2.1, then there is a constant \( C = C(\bar{\rho}) \) independent of \( \rho_1 \), such that

\[ \sup_{t \in (0, T] \cap (0, T^*)} \sigma \int_0^{T \wedge T^*} |\nabla u|^2 dx + \int_0^{T \wedge T^*} \sigma \rho |\dot{u}|^2 dx dt \leq C(C_0 + C_f + O_1), \]  
(2.8)

where \( O_1 = \int_0^{T \wedge T^*} \sigma |\nabla u|^3 dx dt \), and

\[ \sup_{t \in (0, T] \cap (0, T^*)} \sigma^N \int_0^{T \wedge T^*} |\nabla u|^2 dx + \int_0^{T \wedge T^*} \|D^2 u\| dx dt \]

\[ \leq C(C_0 + C_f + O_1(T)) + C \int_0^{T \wedge T^*} \sigma^N (|u|^4 + |\nabla u|^4) dx dt. \]  
(2.9)

**Proof.** In [20] (Lemma 2.1), we obtain this lemma in \( \mathbb{R}^2 \). Using the similar argument as that in [20] (Lemma 2.1) and [7] (Lemma 2.1), we can easily obtain this lemma in \( \mathbb{R}^3 \) and omit the details. \( \square \)

The following lemmas will be applied to bound the higher order terms occurring on the right hand sides of (2.1)–(2.2).
Lemma 2.2. If \((\rho, u)\) is a smooth solution of (1.1)–(1.3) as in Proposition 2.1, then there is a constant \(C = C(\tilde{\rho})\) independent of \(\tilde{\rho}\), such that,

\[
\|u\|_{L^p} \leq C_p \|u\|^\frac{2N-N_p+2p}{2} \|\nabla u\|^\frac{N_p-2N}{p} \ , \quad p \in \left\{ \begin{array}{ll}
[2, \infty), & N = 2, \\
[2, 6], & N = 3.
\end{array} \right.
\] (2.10)

\[
\|u\|^p_{L^p} \leq C_p(C_0 + C_f) \|\nabla u\|^\frac{N_p-2N}{p} + C_p(C_0 + C_f) \frac{2N-N_p+2p}{4} \|\nabla u\|^p_{L^p}, \quad p \in \left\{ \begin{array}{ll}
[2, \infty), & N = 2, \\
[2, 6], & N = 3.
\end{array} \right.
\] (2.11)

\[
\||\nabla u||_{L^p} \leq C_p(||F||_{L^p} + ||w||_{L^p} + ||P - P(\tilde{\rho})||_{L^p}), \quad p \in (1, \infty),
\] (2.12)

\[
\|\nabla F\|_{L^p} + \|\nabla w\|_{L^p} \leq C_p(||\rho u||_{L^p} + ||f||_{L^p}), \quad p \in \left\{ \begin{array}{ll}
[2, \infty), & N = 2, \\
[2, 6], & N = 3.
\end{array} \right.
\] (2.13)

Also, for \(0 \leq t_1 \leq t_2 \leq T\), \(p \geq 2\) and \(s \geq 0\),

\[
\int_{t_1}^{t_2} \sigma^s |\rho - \tilde{\rho}|^p \, dt \leq C \left( \int_{t_1}^{(t_2)} \sigma^{s-1} |\rho - \tilde{\rho}|^p \, dt \right)^{-1} \int_{t_1}^{t_2} \sigma^s |\nabla u|^p \, dx ds \leq C \left( \int_{t_1}^{t_2} |\nabla u|^p \, dx ds \right),
\] (2.15)

\[
\int_{t_1}^{t_2} \sigma^s |\rho - \tilde{\rho}|^p \, dt \leq C \left( \int_{t_1}^{t_2} \sigma^s |\nabla F|^p \, dx ds + C_0 + C_f \right).
\] (2.16)

Proof. Using the similar argument as that in [20] (Lemma 2.2) and [7] (Lemma 2.3), we can easily obtain this lemma and omit the details.

To bound the higher order term \(\sigma^s |\nabla u|^3\) occurring on the right hand sides of (2.8) in \(\mathbb{R}^3\), we need to obtain the estimate of \(\|u\|_{H^2}\) near \(t = 0\).

Lemma 2.3. If \(N = 3\), \(u_0 \in H^1\), \((\rho, u)\) is a smooth solution of (1.1)–(1.3) as in Proposition 2.1, then there is a positive constant \(T_1\) independent of \(\rho\), such that

\[
\sup_{t \in [0, T_1]} \int_{\mathbb{R}^3} |\nabla u|^2 \, dx + \int_0^{T_1} \int_{\mathbb{R}^3} \rho |\ddot{u}|^2 \, dx dt \leq C(1 + M_q).
\] (2.17)

Proof. Using a similar argument as that in the proof of [8], we have

\[
\int_{\mathbb{R}^3} |\nabla u|^2 \, dx + \int_0^t \int_{\mathbb{R}^3} \rho |\ddot{u}|^2 \, dx dt \leq C(C_0 + C_f + M_q) + C \int_0^t \int_{\mathbb{R}^3} |\nabla u|^3 \, dx ds.
\]

From (2.12) and (2.16), we have

\[
\int_0^t \int_{\mathbb{R}^3} |\nabla u|^3 \, dx ds \leq C + C \int_0^t \int_{\mathbb{R}^3} (|F|^3 + |w|^3) \, dx ds.
\]

From (2.13)–(2.14) and (2.10), we obtain

\[
\int_{\mathbb{R}^3} (|F|^3 + |w|^3) \, dx
\leq C \left( \int_{\mathbb{R}^3} |F|^2 \, dx \right) \frac{2}{3} \left( \int_{\mathbb{R}^3} |\nabla F|^2 \, dx \right) + C \left( \int_{\mathbb{R}^3} |w|^2 \, dx \right) \frac{2}{3} \left( \int_{\mathbb{R}^3} |\nabla w|^2 \, dx \right).
\[
C \left( \int_{\mathbb{R}^3} (|\nabla u|^2 + |\rho - \bar{\rho}|^2) dx \right) ^{\frac{2}{3}} \left( \int_{\mathbb{R}^3} (\rho |\dot{u}|^2 + |f|^2) dx \right) ^{\frac{1}{3}}.
\]
(2.18)

Thus, from Proposition 2.1, we have
\[
\int_{\mathbb{R}^3} |\nabla u|^2 dx + \int_0^T \int_{\mathbb{R}^3} \rho |\dot{u}|^2 dx dt \leq C(1 + M_g) + C t \sup_{s \in [0,t]} ||\nabla u(\cdot, s)||_{L^2}^0, \ t \in [0,1].
\]

Thus, when \( T_1 = \min\{ \frac{1}{C(1 + M_g)^2}, 1 \} \), we can easily obtain (2.17).

Now, we apply the estimates of Lemma 2.2 to close the bounds in Lemma 2.1.

**Proposition 2.2.** If \((\rho, u)\) is a smooth solution of (1.1)–(1.3) as in Proposition 2.1 and \(\varepsilon\) is small enough, then we have

\[
\sup_{t \in [0,T]} \int_{\mathbb{R}^3} (\sigma |\nabla u|^2 + \sigma^N |\rho| |\dot{u}|^2) dx + \int_0^{T\wedge T^*} \int_{\mathbb{R}^3} (\sigma |\dot{u}|^2 + \sigma^N |\nabla \dot{u}|^2) dx dt \leq (C_0 + C_f)^9. \tag{2.19}
\]

**Proof.** Since we obtain this proposition in \(\mathbb{R}^2\) in [20] (Proposition 2.2), then we only prove this proposition in \(\mathbb{R}^3\) in this paper.

From Proposition 2.1 and Lemmas 2.1–2.2, we have

\[
\text{LHS of (2.19)} \leq C(C_0 + C_f) + C \int_0^{T\wedge T^*} \int_{\mathbb{R}^3} (\sigma |\nabla u|^3 + \sigma^3 |u|^4 + \sigma^3 |\nabla u|^4) dx ds. \tag{2.20}
\]

From (2.12), we get

\[
\int_0^{T\wedge T^*} \int_{\mathbb{R}^3} \sigma^3 |\nabla u|^4 dx ds \leq \int_0^{T\wedge T^*} \int_{\mathbb{R}^3} \sigma^3 (|F|^4 + |u|^4 + |P - P(\bar{\rho})|^4) dx ds. \tag{2.21}
\]

From (2.2), (2.7), (2.11), (2.10) and (2.13)–(2.16) we obtain

\[
\int_0^{T\wedge T^*} \int_{\mathbb{R}^3} \sigma^3 (|F|^4 + |u|^4) dx ds \leq C \left( \int_0^{T\wedge T^*} \int_{\mathbb{R}^3} \sigma^3 |F|^4 dx ds + C_0 + C_f \right) \leq C(C_0 + C_f)^{2\theta} + C(C_0 + C_f)^2, \tag{2.22}
\]

\[
\int_0^{T\wedge T^*} \int_{\mathbb{R}^3} \sigma^3 |\rho - \bar{\rho}|^4 dx ds \leq C \left( \int_0^{T\wedge T^*} \int_{\mathbb{R}^3} \sigma^3 |F|^4 dx ds + C_0 + C_f \right) \leq C(C_0 + C_f)^{2\theta} + C(C_0 + C_f), \tag{2.23}
\]

\[
\int_0^{T\wedge T^*} \int_{\mathbb{R}^3} \sigma^3 |u|^4 dx ds \leq C(C_0 + C_f)^{\frac{2}{3}} \int_0^{T\wedge T^*} \sigma^3 (||\nabla u||_{L^2}^3 + ||\nabla u||_{L^2}^s) ds \leq CA \left( C_0 + C_f \right)^{\frac{2}{3}} + CA \left( C_0 + C_f \right)^{\frac{1}{3}}. \tag{2.24}
\]

From (2.21)–(2.24), we have

\[
\int_0^{T\wedge T^*} \int_{\mathbb{R}^3} \sigma^3 (|u|^4 + |\nabla u|^4) dx ds \leq C(C_0 + C_f)^{2\theta} + C(C_0 + C_f). \tag{2.25}
\]
Similarly, we get
\[
\int_{T_1 \cap T^*} \int_{\mathbb{R}^3} \sigma \lvert \nabla u \rvert^3 dx ds \\
\leq \int_{T_1 \cap T^*} \int_{\mathbb{R}^3} (\sigma^2 \lvert \nabla u \rvert^4 + \lvert \nabla u \rvert^2) dx ds \\
\leq C(T_1) \int_{T_1 \cap T^*} \int_{\mathbb{R}^3} (\sigma^3 \lvert \nabla u \rvert^4 + \lvert \nabla u \rvert^2) dx ds \\
\leq C(M_q)(C_0 + C_f)^{2\theta} + C(M_q)(C_0 + C_f),
\] (2.26)

\[
\int_{T_1 \cap T^*} \int_{\mathbb{R}^3} \sigma \lvert \nabla u \rvert^3 dx ds \\
\leq C(C_0 + C_f) + \int_{T_1 \cap T^*} \int_{\mathbb{R}^3} \sigma (|F|^3 + |w|^3) dx ds \\
\leq C(C_0 + C_f) + \int_{T_1 \cap T^*} \sigma \left( \int_{\mathbb{R}^3} (|\nabla u|^2 + |\rho - \tilde{\rho}|^2) dx \right) \left( \int_{\mathbb{R}^3} (|\rho|^2 + |f|^2) dx \right) ds \\
\leq C(M_q)(C_0 + C_f) + C(M_q) (C_0 + C_f)^{\frac{3}{4}} + \left( \int_{T^*} \sigma \left\lVert \sqrt{\rho} \nu \right\rVert_{L^2}^2 ds \right)^{\frac{1}{4}} \\
\times \left( \sup_{t \in [0, T]} \left\lVert \nabla u \right\rVert_{L^2}^2 \int_{T_1 \cap T^*} \left\lVert \nabla u \right\rVert_{L^2}^2 ds + C \int_{T_1 \cap T^*} \left\lVert \rho - \tilde{\rho} \right\rVert_{L^2}^2 dt \right)^{\frac{1}{4}} \\
\leq C(M_q)(C_0 + C_f) + C(M_q)(C_0 + C_f)^{\frac{3}{4} + \theta} + C(M_q)(C_0 + C_f)^{\frac{3}{4} + \theta}. \tag{2.27}
\]

Then, from (2.2), (2.20) and (2.23)–(2.27), we obtain
\[
\text{LHS of (2.19)} \leq C(M_q)(C_0 + C_f)^{1 + \theta} + C(M_q)(C_0 + C_f)^{\frac{3}{4} + \theta}, \tag{2.28}
\]

when
\[
C(M_q) \epsilon^{1+\theta} \leq 1. \tag{2.29}
\]

Then, we consider the Hölder continuity of $u$ in the following lemma.

**Lemma 2.4.** Let $\alpha \in (0, 1)$ when $N = 2$, $\alpha \in (0, \frac{1}{2}]$ when $N = 3$. When $t \in (0, T] \cap (0, T^*)$, we have
\[
\langle u(\cdot, t) \rangle^\alpha \leq C \left( \left\lVert \rho \nu \right\rVert_{L^2}^{\frac{N-2+2\theta}{2}} \left( \left\lVert \nabla u \right\rVert_{L^2}^{\frac{4-N-2\theta}{2}} + (C_0 + C_f)^{\frac{4-N-2\theta}{4}} \right) + \left\lVert \nabla u \right\rVert_{L^2} + (C_0 + C_f)^{\frac{1+\theta}{2}} \right). \tag{2.30}
\]

**Proof.** Let $p = \frac{N}{1-\alpha}$. From (2.12), (2.14) and Sobolev’s embedding theorem, we have
\[
\langle u(\cdot, t) \rangle^\alpha \leq C \left( \left\lVert \rho \nu \right\rVert_{L^2}^{\frac{N-2+2\theta}{2}} \left( \left\lVert \nabla u \right\rVert_{L^2}^{\frac{N-2+2\theta}{2}} + (C_0 + C_f)^{\frac{N-2+2\theta}{2p}} \right) + \left\lVert \nabla u \right\rVert_{L^2} + (C_0 + C_f)^{\frac{1+\theta}{2}} \right). \tag{2.30}
\]

\[\blacksquare\]
**Proposition 2.3.** If $u_0 \in H^1$, $(\rho, u)$ is a smooth solution of (1.1)–(1.3) as in Proposition 2.1, then we have

$$\sup_{t \in [0, T]} \int_0^T \int |\nabla u|^2 \, dx + \int_0^T \int \rho |\dot{u}|^2 \, dx \, dt \leq C(M_q). \tag{2.31}$$

**Proof.** Using a similar argument as that in the proof of (2.3), we have

$$\sup_{t \in [0, T]} \int_0^T \int |\nabla u|^2 \, dx + \int_0^T \int \rho |\dot{u}|^2 \, dx \leq C(C_0 + C_f + M_q) + C \int_0^T \int |\nabla u|^3 \, dx ds.$$

From Lemma 2.3 and (2.20), we can easily obtain (2.31). □

**Proposition 2.4.** If $u_0 \in H^1$, $(\rho, u)$ is a smooth solution of (1.1)–(1.3) as in Proposition 2.1, then we have

$$\sup_{t \in [0, T]} \int_0^T \sigma \int |\dot{u}|^2 \, dx + \int_0^T \int |\nabla \dot{u}|^2 \, dx \leq C(M_q). \tag{2.32}$$

**Proof.** Since we obtain this proposition in $\mathbb{R}^2$ in [20] (Proposition 2.4), then we only prove this proposition in $\mathbb{R}^3$ in this paper.

Using a similar argument as that in the proof of (2.10), from (2.21), we have

$$\sup_{t \in [0, T]} \int_0^T \int |\nabla \dot{u}|^2 \, dx \leq C(C_0 + C_f + M_q) + C \int_0^T \int |\nabla u|^3 \, dx ds.$$

Without loss of generality, assume that $T > 1$. From (2.25), we get

$$\sup_{t \in [0, T]} \int_0^T \sigma \int |\dot{u}|^2 \, dx + \int_0^T \int |\nabla \dot{u}|^2 \, dx \leq C(M_q) + C \int_0^T \sigma (|u|^4 + |\nabla u|^4) \, dx ds.$$

From (2.7), (2.11)–(2.12) and (2.31), we have

$$\sup_{t \in [0, T]} \sigma \int_0^T \int |\nabla \dot{u}|^2 \, dx \leq C(M_q) + C \int_0^T \sigma (|F|^4 + |w|^4) \, dx ds.$$

From (2.7), (2.13), (2.10) and (2.31), we obtain

$$\int_0^{1 \wedge T} \sigma (|F|^4 + |w|^4) \, dx ds \leq C \int_0^{1 \wedge T} \sigma \left( \int |F|^2 \, dx \right)^{1/2} \left( \int |\nabla F|^2 \, dx \right)^{1/2} \, dx ds + C \int_0^{1 \wedge T} \sigma \left( \int |w|^2 \, dx \right)^{1/2} \left( \int |\nabla w|^2 \, dx \right)^{1/2} \, dx ds \leq C(M_q) + C(M_q) \sup_{t \in [0, T]} \sigma^{1/2} \|\sqrt{\rho} \dot{u}\|_{L^2}. \tag{2.33}$$

Using Young’s inequality, we can finish the proof of this proposition. □

**Lemma 2.5.** For any $p \in [2, \infty)$ when $N = 2$, $p \in [2, 6]$ when $N = 3$, we have

$$\|\dot{u}\|_{L^p} \leq C_p \|\sqrt{\rho} \dot{u}\|_{L^2}^{2N/4p} \|\nabla \dot{u}\|_{L^2}^{Np/2} + C_p \|\nabla \dot{u}\|_{L^2}. \tag{2.34}$$

**Proof.** Since

$$\rho \int |\dot{u}|^2 \, dx \leq \int \rho |\dot{u}|^2 \, dx + \left( \int |\rho - \overline{\rho}|^2 \, dx \right)^{1/2} \left( \int |\dot{u}|^4 \, dx \right)^{1/4},$$

applying (2.10), we get

$$\|\dot{u}\|_{L^2}^2 \leq C \|\sqrt{\rho} \dot{u}\|_{L^2}^2 + C \|\nabla \dot{u}\|_{L^2}^2.$$

From (2.10), we can immediately obtain (2.34). □
Lemma 2.6. For any $q \in (0, 2)$ when $N = 2$, $q \in (1, \frac{4}{2})$ when $N = 3$, we have
\[
\int_0^{T \wedge T^*} \int_\mathbb{R}^3 \sigma^{p_1-1} \rho |\dot{u}|^{2+q} dx ds \leq C(M_q).
\]

Proof. Since we obtain this lemma in $\mathbb{R}^2$ in [20] (lemma 2.5), then we only prove this lemma in $\mathbb{R}^3$ in this paper.

Using Hölder’s inequality, (2.31), (2.32) and (2.34) with $p = 6$, we have
\[
\int_0^{T \wedge T^*} \int_\mathbb{R}^3 \sigma^{\frac{q}{2}} \rho |\dot{u}|^{2+q} dx ds \\
\leq C \int_0^{T \wedge T^*} \sigma^{\frac{q}{2}} \|\sqrt{\rho u}\|_{L^2_x}^{\frac{4-q}{2}} \|\dot{u}\|_{L^2_x}^{\frac{2q}{2(q+3)}} ds \\
\leq C \int_0^{T \wedge T^*} \sigma^{\frac{q}{2}} \|\sqrt{\rho u}\|_{L^2_x}^{\frac{4-q}{2}} \|\nabla \dot{u}\|_{L^2_x}^{\frac{3q}{2}} ds \\
\leq C \left( \int_0^{T \wedge T^*} \sigma \|\nabla \dot{u}\|_{L^2_x}^2 dt \right)^{\frac{q}{2}} \left( \int_0^{T \wedge T^*} \|\sqrt{\rho u}\|_{L^2_x}^2 dt \right)^{\frac{4-q}{2}} \left( \sup_{t \in [0, T]} \sigma \|\sqrt{\rho u}\|_{L^2_x}^2 \right)^{\frac{q}{2}} \\
\leq C(M_q).
\]

Proposition 2.5. If $u_0 \in H^1$, $(\rho, u)$ is a smooth solution of \((E.1)-(E.3)\) as in Proposition 2.4 and
\[
q^2 \leq \frac{4\mu}{\lambda(\rho) + \mu}, \quad \forall \rho \in [0, \bar{\rho}],
\]
then we have
\[
\sup_{t \in (0, T] \cap (0, T^*)} \sigma^{p_1} \int_0^{T \wedge T^*} \rho |\dot{u}|^{2+q} dx ds + \int_0^{T \wedge T^*} \int_\mathbb{R}^3 \sigma^{p_1} |\dot{u}|^q |\nabla \dot{u}|^2 dx ds \leq C(M_q).
\]

Proposition 2.6. If $f \in L^\infty \cap L_x^{2+q}$, $(\rho, u)$ is a smooth solution of \((E.1)-(E.3)\) as in Proposition 2.5, then we have
\[
\|F\|_{L^\infty} + \|w\|_{L^\infty} \leq C(\|\nabla u\|_{L^2} + \|\rho - \bar{\rho}\|_{L^2})^{\frac{2(2+q-N)}{4+2q-N}} (\|\rho \dot{u}\|_{L^2} + \|f\|_{L^2})^{\frac{2N+4N_q}{2(2q-N)N}}
\]
and
\[
\int_0^{T \wedge T^*} (\|F\|_{L^\infty} + \|w\|_{L^\infty}) ds \leq C(M_q)(\bar{C}_0 + C_f) \frac{4(2+q-N)}{4+2q-N} (1 + T).
\]

Proof. From (2.7), (2.13), (2.19), (2.37) and the Galiardo-Nirenberg inequality, we have
\[
\|F\|_{L^\infty}^{\frac{2(2+q-N)}{2(2+q-N)}} \leq C \|F\|_{L^2_x}^{\frac{2N+4N_q}{2(2+q-N)}} \\
\leq C(\|\nabla u\|_{L^2} + \|\rho - \bar{\rho}\|_{L^2})^{\frac{2(2+q-N)}{2(2+q-N)}} (\|\rho \dot{u}\|_{L^2} + \|f\|_{L^2})^{\frac{2N+4N_q}{2(2q-N)N}}.
\]

and
\[
\int_0^{T \wedge T^*} \|F\|_{L^\infty} ds \\
\leq C(M_q) \int_0^{T \wedge T^*} (\sigma^{-\frac{1}{2}}(C_0 + C_f)^{\frac{2(2+q-N)}{4+2q-N}} (\sigma^{-\frac{1}{2}}) \frac{2N+4N_q}{2(2q-N)N} ds \\
\leq C(M_q)(\bar{C}_0 + C_f) \frac{4(2+q-N)}{4+2q-N} (1 + T).
\]

Similarly, we can obtain the same estimates for $w$. \(\square\)
Proposition 2.7. Given numbers \(0 < \rho_1 < \rho < \rho_1 < \rho_2 < \rho\), there is an \(\varepsilon > 0\) such that, if \((\rho, u)\) is a smooth solution of (1.1)--(1.3) with \(C_0 + C_f \leq \varepsilon\) and \(\rho_1 \leq \rho_0 \leq \rho_1\), then

\[
\frac{1}{2} \rho_1 \leq \rho \leq \rho_2, \quad (x, t) \in \mathbb{R}^N \times [0, T] \cap [0, T^*],
\]

(2.40)

for any \(T > 0\). Furthermore, Claim 1 and the estimates in Propositions 2.1--2.6 hold for any \(T > 0\).

Proof. At first, we prove that (2.1) and (2.2) hold, then estimate (2.40) holds.

We fix a curve \(x(t)\) satisfying \(\dot{x} = u(x(t), t)\) and \(x(0) = x\). From (2.1), we have

\[
\frac{d}{dt} \Lambda(\rho(x(t), t)) + P(\rho(x(t), t)) - P(\rho) = -F(x(t), t),
\]

(2.41)

where \(\Lambda\) satisfies that \(\Lambda(\rho) = 0\) and \(\Lambda'(\rho) = \frac{2\mu + \Lambda(\rho)}{\rho}\).

(I) For the small time, we estimate the pointwise bounds of the density as follows. From (2.1) and (2.30), we have, for all \(t \in [0, 1]\),

\[
|\Lambda(\rho(x(t), t)) - \Lambda(\rho_0(x))| \leq C(M_\theta)(C_0 + C_f) \frac{e^{\frac{4}{3+2\theta-N}+\frac{4}{3}}}{e^{\frac{4}{3+2\theta-N}+\frac{4}{3}} + Ct}.
\]

When

\[
2C(M_\theta)e^{\frac{4}{3+2\theta-N}} \leq \Lambda(\bar{\rho}_1 + \frac{1}{3}(\bar{\rho}_2 - \bar{\rho}_1) - \Lambda(\bar{\rho}_1),
\]

we get

\[
\Lambda(\rho(x(t), t)) \leq \Lambda(\bar{\rho}_1 + \frac{1}{3}(\bar{\rho}_2 - \bar{\rho}_1)), \quad t \in [0, \tau],
\]

and

\[
\rho \leq \bar{\rho}_1 + \frac{1}{3}(\bar{\rho}_2 - \bar{\rho}_1), \quad (x, t) \in \mathbb{R}^N \times [0, \tau],
\]

(2.43)

where \(\tau = \min\{1, \frac{1}{\mu_0}[\Lambda(\bar{\rho}_1 + \frac{1}{3}(\bar{\rho}_2 - \bar{\rho}_1)) - \Lambda(\bar{\rho}_1)]\}\). Similarly, since

\[
\Lambda(\bar{\rho}_1) - \Lambda(\bar{\rho}_1) \geq \int_{\rho_1^{\theta}}^{\rho_1^{\theta}} \frac{2}{s} ds = \mu \ln \frac{6}{5},
\]

then, if

\[
2C(M_\theta)e^{\frac{4}{3+2\theta-N}} \leq 2\mu \ln \frac{6}{5} \leq \Lambda(\bar{\rho}_1) - \Lambda(\bar{\rho}_1),
\]

(2.44)

we get

\[
\rho \geq \frac{5}{6}\bar{\rho}_1, \quad (x, t) \in \mathbb{R}^N \times [0, \tau_1],
\]

(2.45)

where \(\tau_1 = \min\{\tau, \frac{1}{\mu_0} \ln \frac{6}{5}\}\).

(II) For the large time \(t \geq \tau_1\), we estimate the pointwise bounds of density as follows. From (2.7), (2.19), (2.37), (2.38) and (2.41), we have

\[
\frac{d\Lambda(\rho(x(t), t))}{dt} + P(\rho(x(t), t)) - P(\rho) = O_\theta(t),
\]

(2.46)

where

\[
|O_\theta(t)| \leq C(\tau_1, M_\theta)(C_0 + C_f) \frac{e^{\frac{4}{3+2\theta-N}}}{e^{\frac{4}{3+2\theta-N}} + t} \geq \tau_1.
\]

Now, we apply a standard maximum principle argument to estimate the upper bounds of density. Let

\[
t_0 = \sup\{t \in (\tau, T) \cap (\tau, T^*)|\Lambda(\rho(x(s), s)) \leq \Lambda(\bar{\rho}_2), \quad \text{for all } s \in [0, t]\}.
\]

If \(t_0 < T\) and \(t_0 < T^*\), we have

\[
\Lambda(\rho(x(t_0), t_0)) = \Lambda(\bar{\rho}_2),
\]

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\[
\frac{d\Lambda(\rho(x(t), t))}{dt} \bigg|_{t=t_0} \geq 0,
\]
and
\[
\rho(x(t_0), t_0) = \bar{\rho}.
\]
From (2.46), we have
\[
O_5(t_0) \geq P(\bar{\rho}) - P(\tilde{\rho}).
\]
On the other hand, when
\[
C(\tau_1, M_q) \varepsilon^{\frac{q(2+q-N)}{2+q-N+q}} < P(\bar{\rho}) - P(\tilde{\rho}),
\]
we have
\[
O_5(t_0) < P(\bar{\rho}) - P(\tilde{\rho}).
\]
It is a contradiction. Thus, we have
\[
\rho \leq \bar{\rho}, \quad (x, t) \in \mathbb{R}^N \times \{[0, T] \cap [0, T^*]\}. \tag{2.48}
\]
Similarly, let
\[
t_1 = \sup \{t \in (\tau, T) \cap (\tau, T^*) | \Lambda(\rho(x(s), s)) \geq \Lambda(\frac{1}{2\rho_1}), \text{ for all } s \in [0, t]\}.
\]
If \(t_1 < T\) and \(t_1 < T^*\), we have
\[
\Lambda(\rho(x(t_1), t_1)) = \Lambda(\frac{1}{2\rho_1}),
\]
\[
\frac{d\Lambda(\rho(x(t), t))}{dt} \bigg|_{t=t_1} \leq 0,
\]
and
\[
\rho(x(t_1), t_1) = \frac{1}{2\rho_1}.
\]
From (2.46), we have
\[
O_5(t_1) \leq P(\frac{1}{2\rho_1}) - P(\tilde{\rho}) \leq \max_{s \in [0, \frac{1}{2}\tilde{\rho}]} P(s) - P(\tilde{\rho}).
\]
On the other hand, when
\[
C(\tau_1, M_q) \varepsilon^{\frac{q(2+q-N)}{2+q-N+q}} < P(\tilde{\rho}) - \max_{s \in [0, \frac{1}{2}\tilde{\rho}]} P(s),
\]
we have
\[
O_5(t_1) > \max_{s \in [0, \frac{1}{2}\tilde{\rho}]} P(s) - P(\tilde{\rho}).
\]
It is a contradiction. Thus, we have
\[
\rho \geq \frac{1}{2\rho_1}, \quad (x, t) \in \mathbb{R}^N \times \{[0, T] \cap [0, T^*]\}.
\]
Using the classical continuation method, (2.19) and (2.40), we can finish the proof of this proposition. \(\square\)

Lemma 2.7. For any \(T > 0\), we have
\[
\sup_{t \in [0, T]} \|\nabla \rho(\cdot, t)\|_{L^2} + \int_0^{T \wedge T^*} (\|\nabla u\|_{L^\infty} + \|\Delta u\|_{L^2}) dt \leq K(T), \tag{2.49}
\]
\[
\sup_{t \in [0, T]} \|\nabla \rho(\cdot, t)\|_{L^2} \leq K(T). \tag{2.50}
\]
Proof. From (1.19), we have
\[ \partial_t \Lambda(\rho(x,t)) + u \cdot \nabla \Lambda + P(\rho(x,t)) - P(\tilde{\rho}) = -F(x,t), \]
where \( \Lambda \) satisfies that \( \Lambda(\tilde{\rho}) = 0 \) and \( \Lambda'(\rho) = \frac{2 \mu + \rho \Lambda(\rho)}{\rho} \). By the simple computation, we have
\[ \| \nabla \Lambda(t) \|_{L_{2+q}^q}^2 \leq \| \nabla \Lambda(0) \|_{L_{2+q}^q}^2 + K \int_0^t \left( \| \nabla \Lambda \|_{L_{2+q}^q}^2 + \| \nabla F \|_{L_{2+q}^q} \| \nabla \Lambda \|_{L_{2+q}^q} \right) \| \nabla \Lambda \|_{L_{2+q}^q}^2 \, ds. \]
Using the Fourier analysis methods, one can obtain the following estimate.
\[ \| \nabla u \|_{L_{\infty}^\infty} \leq C\| u \|_{L_2^2} + C\left( \| \nabla u \|_{B_2^0} + 1 \right) \log \left( 1 + \| \Delta u \|_{L_{2+q}^q} \right). \]
(For the convenience of reader’s reading, we also give the proof in Appendix (5.1).) From (1.19), we have
\[ \| \Delta u \|_{L_{2+q}^q} \leq K \left( \| \nabla F \|_{L_{2+q}^q} + \| F \|_{L_\infty^\infty} \| \nabla \Lambda \|_{L_{2+q}^q} + \| \nabla \Lambda \|_{L_{2+q}^q} + \| \nabla w \|_{L_{2+q}^q} \right), \]
and
\[ \| \nabla u \|_{B_2^0} \leq C \left( \| F \|_{L_\infty^\infty} + \| \rho - \tilde{\rho} \|_{L_\infty^\infty} + \| u \|_{L_\infty^\infty} \right). \]
Thus, we have
\[ \| \nabla \Lambda(t) \|_{L_{2+q}^q}^2 \leq \| \nabla \Lambda(0) \|_{L_{2+q}^q}^2 + K \int_0^t \| \nabla F \|_{L_{2+q}^q} \| \nabla \Lambda \|_{L_{2+q}^q}^1 \| \nabla \Lambda \|_{L_{2+q}^q} \| \nabla \Lambda \|_{L_{2+q}^q} \, ds + K \int_0^t A \log \left( \| \nabla \Lambda \|_{L_{2+q}^q} + 1 \right) \| \nabla \Lambda \|_{L_{2+q}^q}^2 \, ds, \]
and
\[ \sup_{t \in [0,T] \cap [0,T^*]} \| \nabla \Lambda(t) \|_{L_{2+q}^q} \leq \left( K + \| \nabla \Lambda(0) \|_{L_{2+q}^q} \int_0^T \| \nabla F \|_{L_{2+q}^q} \, ds \right) \exp \left( K \int_0^T A(s) \, ds \right), \]
where \( A = \left( \| F \|_{L_\infty^\infty} + \| u \|_{L_\infty^\infty} + 1 \right) \log \left( \| F \|_{L_\infty^\infty} + \| u \|_{L_\infty^\infty} + \| \nabla F \|_{L_{2+q}^q} + \| \nabla w \|_{L_{2+q}^q} + 1 \right) \). From (2.7), (2.13), (2.31), (2.32), (2.37), (2.38) and (2.40), we can immediately obtain (2.49). Similarly, we can obtain (2.50).

**Lemma 2.8.** If \( \rho_0 - \tilde{\rho} \in H^1 \) and \( u_0 \in H^2 \), then for any \( T > 0 \), we have
\[ \sup_{t \in [0,T] \cap [0,T^*]} \int_0^T \int \rho |\dot{u}|^2 \, dx + \int_0^T \left( |\nabla \dot{u}|^2 + \left| \frac{D}{Dt} \text{div} u \right|^2 \right) \, dx \, dt \leq K, \]
\[ \sup_{t \in [0,T] \cap [0,T^*]} (\| u \|_{L_\infty^\infty} + \| u \|_{H^2}) \leq K(T). \]

**Proof.** Using the similar argument as that in the proof of Proposition 2.4, we can obtain (2.53). From (2.13), we get
\[ \sup_{t \in [0,T] \cap [0,T^*]} (\| \nabla F \|_{L_2^2} + \| \nabla w \|_{L_2^2}) \leq K. \]
From (2.7), (1.19), (2.31), (2.32), (2.33), (2.40), (2.41), (2.44), (2.50), we have
\[ \| u(\cdot, t) \|_{L_2^2} \leq K(\| u \|_{L_2^2} + \| \nabla F \|_{L_2^2} + \| F \|_{L_2^2} \| \nabla \rho \|_{L_2^2} + \| \nabla w \|_{L_2^2}) \]
\[ \leq K(\| u \|_{L_2^2} + \| \nabla F \|_{L_2^2} + \| F \|_{L_2^2} \| \nabla \rho \|_{L_{2+q}^q} + \| \nabla \rho \|_{L_{2+q}^q} + \| \nabla w \|_{L_2^2}) \]
\[ \leq K(T), \quad t \in [0, T] \cap [0, T^*]. \]
Then, using Sobolev’s embedding theorem, we can finish this proof.

**Lemma 2.9.** For any \( T > 0 \), we have
\[ \sup_{t \in [0,T] \cap [0,T^*]} \| \rho(\cdot, t) - \tilde{\rho} \|_{H^2} \leq K(T). \]
Proof. From (2.51) and the simple computation, we have
\[
\|\Lambda(t)\|_{H^2}^2 \leq \|\Lambda(0)\|_{H^2}^2 + K\int_0^t \left( \|F\|_{H^2}^2 \|\Lambda\|_{H^2} + (1 + \|\nabla u\|_{L^\infty} + \|u\|_{H^2}^2)\|\Lambda\|_{H^2}^2 \right) ds. \tag{2.57}
\]
From (2.5), (2.31), (2.40), (2.50), (2.53), the Gagliardo-Nirenberg inequality and Sobolev’s embedding theorem, we get
\[
\|F\|_{H^2} \leq K \left( \|F\|_{L^2} + \|\nabla \rho \|_{L^2}^2 + \|\rho \nabla \hat{u}\|_{L^2} + \|\rho \nabla \hat{u}\|_{L^2} + \|\nabla f\|_{L^2} + \|f \cdot \nabla \rho\|_{L^2} \right)
\leq K(T) \left( 1 + \|\nabla \rho\|_{L^\infty}^2 \|\rho\|_{L^2}^2 + \|\nabla \hat{u}\|_{L^2}^2 + \|\rho \nabla \hat{u}\|_{L^2} + \|\nabla \hat{u}\|_{L^2} \right)
\leq K(T) \left( 1 + \|\Lambda\|_{H^2} + (1 + \|\nabla \hat{u}\|_{L^2}) + \|\nabla \hat{u}\|_{L^2} \right). \tag{2.58}
\]
Thus, from (2.51), (2.53), (2.56) and (2.58), we have
\[
\|\Lambda(t)\|_{H^2}^2 \leq K(T) + K(T) \int_0^t (1 + \|\nabla u\|_{L^\infty} + \|\nabla \hat{u}\|_{L^2})\|\Lambda\|_{H^2}^2 ds
\]
and
\[
\|\Lambda(t)\|_{H^2}^2 \leq K(T).
\]
Using (2.40) and (2.50), we can immediately obtain (2.56). \hfill \square

Lemma 2.10. For any \( T > 0 \), we have
\[
\int_0^{T \wedge T^*} \|u\|_{H^3}^2 dt \leq K(T). \tag{2.59}
\]
Proof. From (2.6), (2.58), (2.55) and (2.58), we have
\[
\int_0^{T \wedge T^*} (\|F\|_{H^2} + \|u\|_{H^2})^2 dt \leq K(T). \tag{2.60}
\]
From (2.51), (2.52), (2.50) and (2.58), we have
\[
\int_0^{T \wedge T^*} \|u\|_{H^3}^2 dt \leq K \int_0^{T \wedge T^*} (\|u\|_{L^2}^2 + (1 + \|F\|_{H^2}^2)(1 + \|\rho - \hat{\rho}\|_{H^2}^2) + \|u\|_{H^2}^2)^2 dt \leq K(T). \tag{2.61}
\]

Proposition 2.8. For any \( T > 0 \), we have
\[
\sup_{t \in [0,T] \cap [0,T^*]} \left( \int \|\nabla \hat{u}\|_{H^3}^2 dx + \int_0^{T \wedge T^*} \|\nabla^2 \hat{u}\|_{L^2}^2 dt \right) \leq K(T), \tag{2.61}
\]
\[
\sup_{t \in [0,T] \cap [0,T^*]} \left( \|\rho - \hat{\rho}, u\|_{H^3} + \|\rho_t, u_t\|_{H^2} + \int_0^{T \wedge T^*} \|u\|_{H^4}^2 ds \right) \leq K(T). \tag{2.62}
\]
Proof. Taking the operator \( \nabla \partial_t + \nabla \text{div}(u) \) in (2.12), multiplying by \( \nabla \hat{u} \) and integrating, we obtain
\[
\frac{1}{2} \int \rho |\nabla \hat{u}|^2 dx
= \frac{1}{2} \int \rho_0 |\nabla \hat{u}_0|^2 dx + \int_0^t \left\{ -\nabla \rho \partial_t \hat{u} \nabla \hat{u} - \nabla (\rho \partial_j \hat{u}) \partial_j \hat{u} \nabla \hat{u}
- \nabla \hat{u} \nabla (\partial_j P_i + \text{div}(\partial_j P u) - \mu \Delta \hat{u}^j [\Delta u_i^j + \text{div}(u \Delta u_i^j)]
- \Delta \hat{u}^j [\partial_j (\lambda + \mu) \text{div} u] + \text{div}(u \partial_j ((\lambda + \mu) \text{div} u)) - \Delta \hat{u}^j [\rho f^j_t + \text{div}(u \rho f^j)] \right\} dx ds
\]
\[ J_1 = \frac{1}{2} \int \rho_0 |\nabla \mathbf{u}_0|^2 dx \leq K. \]  

(2.64)

Using (2.41), (2.53)–(2.54), (2.56), (2.59), the integration by parts and Hölder’s inequality, we have

\[
J_2 = -\int_0^t \int \nabla \rho \partial_t \mathbf{u} \cdot \nabla \mathbf{u} dx ds
\]

\[
= -\int_0^t \int \nabla \rho \partial_t \left( \frac{\mu \Delta \mathbf{u} + \nabla ((\mu + \lambda) \nabla \mathbf{u}) - \nabla P - \rho f}{\rho} \right) \cdot \nabla \mathbf{u} dx ds
\]

\[
\leq K(T) \int_0^t \left\{ \| \nabla \mathbf{u} \|_{L^6} \| \nabla \mathbf{u} \|_{L^3} \left( \| \nabla \rho \|_{L^2} + \| \nabla P \|_{L^2} + \| f_i \|_{L^2} \right) \right\} ds
\]

\[
\leq K(T) + \frac{\mu}{10} \int_0^t \left( \| \nabla \mathbf{u} \|_{L^2}^2 + \| \nabla P \|_{L^2}^2 + \| f_i \|_{L^2} \right) ds,
\]

(2.65)

\[
J_3 = -\int_0^t \int \nabla (\rho \mathbf{u}_j) \partial_j \mathbf{u} \cdot \nabla \mathbf{u} dx ds
\]

\[
\leq C \int_0^t \| \nabla \mathbf{u} \|_{L^4}^2 \left( \| \nabla \rho \|_{L^2} \| \mathbf{u} \|_{L^8} + \| \nabla \mathbf{u} \|_{L^2} \right) ds,
\]

(2.66)

\[
J_4 = \int_0^t \int \Delta \mathbf{u}^i \left[ \partial_j P_{ij} + \text{div}(\partial_j P_{ij}) \right] dx ds
\]

\[
= -\int_0^t \int \left[ \partial_j \Delta \mathbf{u}^i P^j \rho + \partial_k \Delta \mathbf{u}^i \partial_j P_{kli} \right] dx ds
\]

\[
= \int_0^t \int \left[ P^j \text{div} \partial_j \Delta \mathbf{u}^i - \partial_k (\partial_j \Delta \mathbf{u}^i u^k) P + P \partial_j (\partial_k \Delta \mathbf{u}^i u^k) \right] dx ds
\]

\[
\leq K \left( \int_0^t \int (|\nabla \mathbf{u}|^2 + |\nabla \rho|^2 |\nabla \mathbf{u}|^2) \right) \frac{1}{2} \left( \int_0^t \int |\nabla \mathbf{u}|^2 dx ds \right)^{\frac{1}{2}}
\]

\[
\leq K(T) + \frac{\mu}{10} \int_0^t \| \nabla \mathbf{u} \|_{L^2}^2 ds,
\]

(2.67)

\[
J_5 = -\int_0^t \int \mu \Delta \mathbf{u}^j \left[ \partial_j \mathbf{u}^i + \text{div}(\mathbf{u} \Delta \mathbf{u}^i) \right] dx ds
\]

\[
= \int_0^t \int \mu [\partial_j \Delta \mathbf{u}^i \partial_j \mathbf{u}^i + \Delta \mathbf{u}^i \cdot \nabla \Delta \mathbf{u}^i] dx ds
\]

\[
= \int_0^t \int \mu [-|\nabla \mathbf{u}|^2 - \partial_i \Delta \mathbf{u}^i u^k \partial_i \partial_k \mathbf{u}^i - \partial_i \Delta \mathbf{u}^i \partial_i \partial_k \mathbf{u}^k + \Delta \mathbf{u}^i \cdot \nabla \Delta \mathbf{u}^i] dx ds
\]

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From (2.15), we have

From (2.16), (2.19) and (2.22), we have

any term dominated by proof of Lemmas 2.9–2.10, we can easily get (2.62).

Using the standard arguments based on the local existence results together with the estimates (2.40) 

\[ \int_0^t \int \mu[|\nabla^2 \bar{u}|^2 + \partial_t \Delta \bar{u} \partial_t u^j - \partial_j \Delta \bar{u} \partial_k u^j - \partial_j u^j \partial_k \partial_k \Delta \bar{u}^j]dxds \]

\[ \leq -\frac{1}{2} \int_0^t \int \mu |\nabla^2 \bar{u}|^2dxds + K \int_0^t \int |\nabla u|^2 |\nabla^2 u|^2 dxds \]

\[ \leq -\frac{1}{2} \int_0^t \int \mu |\nabla^2 \bar{u}|^2 dxds + K(T), \tag{2.68} \]

\[ J_6 = -\int_0^t \int \Delta \bar{u}^j [\partial_j \partial_t ((\lambda + \mu) \text{div} u) + \text{div}(u \partial_j ((\lambda + \mu) \text{div} u))]dxds \]

\[ = \int_0^t \int \{ \partial_j \Delta \bar{u}^j [\partial_t ((\lambda + \mu) \text{div} u) + \text{div}(u(\lambda + \mu) \text{div} u)] \]

\[ + \Delta \bar{u}^j \text{div}(\partial_t u (\lambda + \mu) \text{div} u) \} dxds \]

\[ = \int_0^t \int \partial_j \Delta \bar{u}^j [\partial_t ((\lambda + \mu) \text{div} u) + u^k \lambda' \partial_k \text{div} u + (\lambda + \mu) u^k \partial_k \text{div} u] dxds + O_6 \]

\[ = \int_0^t \int \partial_j \Delta \bar{u}^j ((\lambda + \mu) \frac{D}{Dt} \text{div} u + \lambda' \rho \text{div} u + u^k \lambda' \partial_k \text{div} u] dxds + O_6 \]

\[ = \int_0^t \int \partial_j \Delta \bar{u}^j (\lambda + \mu) \frac{D}{Dt} \text{div} u dxds + O_6 + O_7 \]

\[ = - \int_0^t \int (\lambda + \mu) |\nabla \frac{D}{Dt} \text{div} u|^2 dxds + K(T) + \frac{\mu}{10} \int_0^t \|\nabla^2 \bar{u}\|^2_{L^2} ds, \tag{2.69} \]

\[ J_7 = -\int_0^t \int \Delta \bar{u}^j [\rho f^j_t + \text{div}(uf^j)] dxds \]

\[ = -\int_0^t \int \Delta \bar{u}^j [\rho f^j + \rho u \cdot \nabla f^j] dxds \]

\[ \leq \frac{1}{10} \int_0^t \int \mu |\nabla^2 \bar{u}|^2 dxds + K(T), \tag{2.70} \]

where \( O_6 \) denotes any term dominated by \( C \int_0^t \int (|\nabla \rho| |\nabla u|^2 + |\nabla u| |\nabla^2 u|)|\nabla^2 \bar{u}| dxds \) and \( O_7 \) denotes any term dominated by \( C \int_0^t \int (|\nabla \rho| |\nabla u|^2 + |\nabla u| |\nabla^2 u|)|\nabla \frac{D}{Dt} \text{div} u| dxds \), \( t \in [0, T] \cap [0, T^*). \) From (2.63) and (2.70), we can immediately obtain (2.61). Using similar arguments as that in the proof of Lemmas 2.9–2.10, we can easily get (2.62).

Using the standard arguments based on the local existence results together with the estimates (2.40) and (2.62), we can obtain that \( T^* = \infty \). Since the uniqueness of the solution \( (\rho - \bar{\rho}, u) \in C([0, \infty); H^2) \cap C^1([0, \infty); H^2) \) is classical, we omit the detail. Thus, we finish the proof of the existence and uniqueness parts of Theorem 1.1.

3 Large time Behavior

From (2.16), (2.19) and (2.22), we have

\[ \int_1^\infty \int (|\rho - \bar{\rho}|^4 + |F|^4) dxdt \leq C. \tag{3.1} \]

From (2.15), we have

\[ \int |\rho - \bar{\rho}|^4(x, t) dx \leq \int |\rho - \bar{\rho}|^4(x, s) dx + C \int_N^{N+2} \int |F|^4 dxdr, \]
where \( t \in [N + 1, N + 2] \) and \( s \in [N, N + 1], N > 1. \) Integrating it with \( s \) in \([N, N + 1], \) we obtain

\[
\sup_{t \in [N+1,N+2]} \int \rho - \tilde{\rho} |^4(x,t)dx \leq C \int_N^{N+2} \int (|\rho - \tilde{\rho}|^4 + |F|^4)dxdr.
\]

Letting \( N \to \infty, \) using (3.1), we can easily obtain

\[
\lim_{t \to +\infty} \int |\rho - \tilde{\rho}|^4(x,t)dx = 0.
\]

From (2.25), we can obtain

\[
\int_{1}^{\infty} \int (|u|^4 + |\nabla u|^4 )dxds \leq C.
\]

From (2.7), we have

\[
\int_{0}^{\infty} \int |\nabla u|^2 dxds \leq C.
\]

Thus, for all \( \epsilon \in (0,1), \) there is a positive constant \( T_\epsilon, \) such that for all \( \tau > T_\epsilon, \) we have

\[
\int_{\tau}^{\infty} \int (|u|^4 + |\nabla u|^4 + |\nabla u|^2 + |f|^4) dxds < \epsilon.
\]

For all \( t > T_\epsilon + 2 \) and \( \tau \in [t - 1, t - 2], \) from (1.1), (3.4) and Hölder’s inequality, we get

\[
\int \frac{1}{4} |\rho u|^4 (x,t)dx + \int_{\tau}^{t} \int |u|^2 [\mu |\nabla u|^2 + (\lambda + \mu)(\text{div}u)^2] dxds
\]

\[
= \int \frac{1}{4} |\rho u|^4 (x,t)dx + \int_{\tau}^{t} \int [P\text{div}(|u|^2 u) - \frac{1}{2} \mu |\nabla u|^2|^2 - (\lambda + \mu)\text{div}u \cdot \nabla |u|^2 + \rho f \cdot |u|^2] dxdt
\]

\[
\leq \int \frac{1}{4} |\rho u|^4 (x,t)dx + C \left( \int_{\tau}^{t} \int |\nabla u|^4 dxds \right)^{\frac{1}{4}} \left( \int_{\tau}^{t} \int |u|^4 dxds \right)^{\frac{3}{4}}
\]

\[
+ C \left( \int_{\tau}^{t} \int |\nabla u|^4 dxds \right)^{\frac{1}{4}} \left( \int_{\tau}^{t} \int |u|^4 dxds \right)^{\frac{1}{4}} + C \left( \int_{\tau}^{t} \int |f|^4 dxds \right)^{\frac{1}{4}} \left( \int_{\tau}^{t} \int |u|^4 dxds \right)^{\frac{1}{4}}
\]

\[
\leq \int \frac{1}{4} |\rho u|^4 (x,t)dx + C \epsilon.
\]

Integrating it with \( \tau \) in \([t - 1, t - 2], \) we obtain

\[
\int \frac{1}{4} |\rho u|^4 (x,t)dx \leq \int_{t-2}^{t-1} \int \frac{1}{4} |\rho u|^4 (x,\tau)d\tau + C \epsilon \leq C \epsilon.
\]

Thus, we immediately obtain (1.17).

From (2.34), we have

\[
||\dot{u}||_{L^2} \leq C ||\sqrt{\rho} \dot{u}||_{L^2} + C ||\nabla \dot{u}||_{L^2}.
\]

From (2.19), (3.2) – (3.3) and (3.5), we have that for all \( \epsilon \in (0,1), \) there is a positive constant \( T_{2\epsilon}, \) such that for all \( \tau > T_{2\epsilon}, \) we have

\[
\int_{\tau}^{\infty} \int (|u|^4 + |\nabla u|^4 + |\nabla u|^2 + |\dot{u}|^2 + |\nabla \dot{u}|^2) dxds < \epsilon.
\]

For all \( t > T_{2\epsilon} + 2 \) and \( \tau \in [t - 1, t - 2], \) multiplying (1.1) by \( \dot{u}, \) integrating it over \( \mathbb{R}^N \times [\tau,t], \) we obtain

\[
\int_{\tau}^{t} \int \rho |\dot{u}|^2 dxds
\]

\[
= \int_{\tau}^{t} \int (-\dot{u} \cdot \nabla P + \mu \Delta u \cdot \dot{u} + \nabla (\lambda + \mu)\text{div}u \cdot \dot{u} + \rho f \cdot \dot{u}) dxds
\]

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\[ J_i := \sum_{i=1}^{4} J_i. \]  

(3.7)

Using the integration by parts and Hölder’s inequality, we have

\[ J_1 = -\int_{\tau}^{t} \int \dot{u} \cdot \nabla P dxds \]
\[ = \int_{\tau}^{t} \int \text{div}(\dot{u}(P - P(\bar{\rho}))) dxds \]
\[ \leq C \left[ \int_{\tau}^{t} \int |\nabla \dot{u}|^2 dxds \right]^\frac{1}{2} \]
\[ \leq C \epsilon^{\frac{1}{2}}, \quad (3.8) \]

\[ J_2 = \int_{\tau}^{t} \int \mu \Delta u \cdot \dot{u} dxds \]
\[ = -\frac{\mu}{2} \int \nabla u|^{2}(x,t) dx + \frac{\mu}{2} \int \nabla u|^{2}(x,\tau) dx - \int_{\tau}^{t} \int \mu \partial_i u^j \partial_i (u^k \partial_k u^j) dxds \]
\[ \leq -\frac{\mu}{2} \int \nabla u|^{2}(x,t) dx + \frac{\mu}{2} \int \nabla u|^{2}(x,\tau) dx + C \int_{\tau}^{t} \int |\nabla u|^3 dxds \]
\[ \leq -\frac{\mu}{2} \int \nabla u|^{2}(x,t) dx + \frac{\mu}{2} \int \nabla u|^{2}(x,\tau) dx + C \epsilon, \quad (3.9) \]

\[ J_3 = \int_{\tau}^{t} \int \nabla((\lambda + \mu)\text{div}u) \cdot \dot{u} dxds \]
\[ = -\frac{1}{2} \int \left[ (\lambda + \mu)\text{div}u \right]^{2}(x,t) dx + \frac{1}{2} \int \left[ (\lambda + \mu)\text{div}u \right]^{2}(x,\tau) dx + \int_{\tau}^{t} \int \frac{1}{2} \lambda' \rho_i |\text{div}u|^2 dxds \]
\[ - \int_{\tau}^{t} \int (\lambda + \mu)\text{div}u \text{div}(u \cdot \nabla u) dxds \]
\[ \leq -\frac{1}{2} \int \left[ (\lambda + \mu)\text{div}u \right]^{2}(x,t) dx + \frac{1}{2} \int \left[ (\lambda + \mu)\text{div}u \right]^{2}(x,\tau) dx + C \int_{\tau}^{t} \int |\nabla u|^3 dxds \]
\[ \leq -\frac{1}{2} \int \left[ (\lambda + \mu)\text{div}u \right]^{2}(x,t) dx + \frac{1}{2} \int \left[ (\lambda + \mu)\text{div}u \right]^{2}(x,\tau) dx + C \epsilon, \quad (3.10) \]

\[ J_4 = \int_{\tau}^{t} \int \rho f \cdot \dot{u} dxds \leq C \left( \int_{\tau}^{t} \int |\dot{u}|^2 dxds \right)^\frac{1}{2} \leq C \epsilon^{\frac{1}{2}}. \]  

(3.11)

From (2.7), (3.7)–(3.11), we obtain

\[ \frac{\mu}{2} \int |\nabla u|^2(t) dx + \int_{\tau}^{t} \int \rho |\dot{u}|^2 dxds \leq C \epsilon^{\frac{1}{2}} + C \int |\nabla u|^2(x,\tau) dx \]

Integrating it with \( \tau \) in \( [t - 1, t - 2] \), we obtain

\[ \frac{\mu}{2} \int |\nabla u|^2(t) dx \leq C \epsilon^{\frac{1}{2}} + C \int_{t-2}^{t-1} \int |\nabla u|^2(x,\tau) dx \leq C \epsilon^{\frac{1}{2}}. \]

Thus, we immediately obtain (1.18).

Thus, we finish the proof of Theorem 1.1.
4 Proof of Theorem 1.2

Let \( j_\delta(x) \) be a standard mollifying kernel of width \( \delta \). Define the approximate initial data \( (\rho^\delta_0, u^\delta_0) \) by

\[
\rho^\delta_0 = j_\delta * \rho_0 + \delta, \quad u^\delta_0 = j_\delta * u_0.
\]

Assuming that similar smooth approximations have been constructed for functions \( P, f \) and \( \lambda \), we may then apply Theorem 1.1 to obtain a global smooth solution \( (\rho^\delta, u^\delta) \) of (1.1)–(1.3) with the initial data \( (\rho^\delta_0, u^\delta_0) \), satisfying the bound estimates of Propositions 2.1–2.7 with constants independent of \( \delta \).

First, we obtain the strong limit of \( \{u^\delta\} \). From (2.19) and (2.30), we have

\[
< u^\delta(\cdot,t) >^\alpha \leq C(\tau), \quad t \geq \tau > 0,
\]

where \( \alpha \in (0,1) \) when \( N = 2 \), \( \alpha \in (0,\frac{1}{2}) \) when \( N = 3 \). From (4.1), we have

\[
|u^\delta(x,t) - \frac{1}{|B_R(x)|} \int_{B_R(x)} u^\delta(y,t)dy| \leq C(\tau)R^\alpha, \quad t \geq \tau > 0.
\]

Taking \( R = 1 \), from (2.11) and (2.19), we have

\[
\|u^\delta\|_{L^\infty(\mathbb{R}^N \times [\tau, \infty))} \leq C(\tau).
\]

Then, we need only to derive a modulus of Hölder continuity in time. For all \( t_2 \geq t_1 \geq \tau \), from (2.17), (2.19), (2.34) and (4.2), we have

\[
\begin{align*}
|u^\delta(x,t_2) - u^\delta(x,t_1)| & \leq \frac{1}{|B_R(x)|} \int_{t_1}^{t_2} \int_{B_R(x)} |u^\delta(y,s)|dyds + C(\tau)R^\alpha \\
& \leq CR^{-\frac{\alpha}{2}}|t_2 - t_1|^\frac{\gamma}{2} \left( \int_{t_1}^{t_2} |u^\delta_t|^2dyds \right)^\frac{1}{2} + C(\tau)R^\alpha \\
& \leq CR^{-\frac{\alpha}{2}}|t_2 - t_1|^\frac{\gamma}{2} \left( \int_{t_1}^{t_2} |\Delta u^\delta|^2 + |\nabla u^\delta|^2dyds \right)^\frac{1}{2} + C(\tau)R^\alpha \\
& \leq C(\tau) (R^{-\frac{\alpha}{2}}|t_2 - t_1|^\frac{\gamma}{2} + R^\alpha).
\end{align*}
\]

Choosing \( R = |t_2 - t_1|^\frac{1}{\alpha + \gamma} \), we have

\[
< u^\delta >^\alpha \mathbb{R}^N \times [\tau, \infty)) \leq C(\tau), \quad \tau > 0.
\]

From the Ascoli-Arzela theorem, we have (extract a subsequence)

\[
u^\delta \to u \quad \text{uniformly on compact sets in } \mathbb{R}^N \times (0, \infty).
\]

Second, we obtain the strong limits of \( \{F^\delta\} \) and \( \{w^\delta\} \). From (2.13), (2.14), (2.19) and (2.37), using similar arguments as that in the proof of (4.1)–(4.2), we have

\[
< F^\delta(\cdot,t) >^\alpha + \|F^\delta\|_{L^\infty(\mathbb{R}^N \times [\tau,T])} < u^\delta(\cdot,t) >^\alpha + \|w^\delta\|_{L^\infty(\mathbb{R}^N \times [\tau,T])} \leq C(\tau,T),
\]

where \( 0 < \tau \leq t \leq T \) and \( \alpha' \in (0,\frac{2+2q-N}{2+q}) \). The simple computation implies that

\[
F^\delta_t = \rho^\delta(2\mu + \lambda(\rho^\delta)) \left( \frac{F^\delta d}{ds} \left( \frac{1}{2\mu + \lambda(s)} \right) \bigg|_{s=\rho^\delta} + \frac{d}{ds} \left( \frac{P(s) - P(\rho^\delta)}{2\mu + \lambda(s)} \right) \bigg|_{s=\rho^\delta} \right) \text{div} u^\delta \\
- u^\delta \cdot \nabla F^\delta + (2\mu + \lambda(\rho^\delta)) \text{div} \dot{u}^\delta - (2\mu + \lambda(\rho^\delta))\partial_i u^\delta_j \partial_j u^\delta_i \tag{4.6}
\]

and

\[
w^\delta_t = -u^\delta \cdot \nabla (w^\delta)^k \cdot \partial_i \dot{u}^\delta_j - \partial_j u^\delta_k - \partial_i u^\delta_j \partial_j u^\delta_i + \partial_k u^\delta_i \partial_i u^\delta_j. \tag{4.7}
\]
Then, from \((2.13)\), \((2.19)\), \((2.25)\), \((4.2)\) and \((4.5)\), we have
\[
\|F^\delta_t\|_{L^2(\mathbb{R}^N \times [\tau,T])} + \|w^\delta_t\|_{L^2(\mathbb{R}^N \times [\tau,T])} \leq C(\tau,T), \ T > \tau > 0.
\]
Using a similar argument as that in the proof of \((2.3)\), we obtain
\[
< F^\delta >_{\mathbb{R}^N \times [\tau,T]} + < w^\delta >_{\mathbb{R}^N \times [\tau,T]} \leq C(\tau,T), \ T > \tau > 0.
\] (4.8)
and (extract a subsequence)
\[
F^\delta \rightharpoonup F, \ w^\delta \rightharpoonup w, \text{ uniformly on compact sets in } \mathbb{R}^N \times (0,\infty).
\] (4.9)

Third, we obtain the strong limit of \(\{\rho^\delta\}\). From \((2.40)\), we get (extract a subsequence)
\[
\rho^\delta \rightharpoonup \rho, \text{ weak-* in } L^\infty(\mathbb{R}^N).
\]
Let \(\Phi(s)\) be an arbitrary continuous function on \([0,\overline{\rho}]\). Then, we have that (extract a subsequence) \(\Phi(\rho^\delta)\) converges weak-* in \(L^\infty(\mathbb{R}^N)\). Denote the weak-* limit by \(\bar{\Phi}\):
\[
\Phi(\rho^\delta) \rightharpoonup \bar{\Phi}, \text{ weak-* in } L^\infty(\mathbb{R}^N).
\]
From the definition of \(F\), we have
\[
\text{div} u = \nu F + \overline{T_0},
\] (4.10)
where
\[
\nu(\rho) = \frac{1}{2\mu + \lambda(\rho)}, \ P_0(\rho) = \nu(\rho)(P(\rho) - P(\overline{\rho})).
\]
From \((1.1)\), we have
\[
\partial_t \rho \ln \rho + \text{div}(\rho \ln \rho u) + F\rho \ln \rho + \rho P_0 = 0
\]
and
\[
\partial_t (\rho \ln \rho) + \text{div}(\rho \ln \rho u) + F \rho + \rho P_0 = 0.
\]
Letting \(\Psi = \rho \ln \rho - \rho \ln \rho \geq 0\), we obtain
\[
\partial_t \Psi + \text{div}(\Psi u) + F(\rho \ln \rho - \rho \nu) + F\rho(\nu - \nu) + \rho P_0 - \rho P_0 = 0.
\] (4.11)
with the initial condition \(\Psi|_{t=0} = 0\) almost everywhere in \(\mathbb{R}^N\). Let \(\phi(s) = s \ln r\). Since
\[
\phi''(s) = \frac{1}{s} \geq \frac{1}{\overline{\rho}}, \ s \in [0,\overline{\rho}],
\]
we get
\[
\phi(\rho^\delta) - \phi(\rho) = \phi'(\rho)(\rho^\delta - \rho) + \frac{1}{2}\phi''(\rho + \xi(\rho^\delta - \rho))(\rho^\delta - \rho)^2, \ \xi \in [0,1],
\]
and
\[
\lim_{\delta \to 0} \|\rho^\delta - \rho\|_{L^2} \leq C\|\Psi\|_{L^1}.
\] (4.12)
Similarly, every function \(f \in C^2([0,\overline{\rho}])\) satisfies
\[
\left| \int g(\tilde{f} - f(\rho))dx \right| \leq C \int |g|\Psi dx,
\] (4.13)
where \(g\) is any function such that the integrations exist. Then, when \(\nu \in C^2([0,\overline{\rho}])\), we have
\[
\left| \int F(\rho \ln \rho - \rho \nu)dx \right| \leq C \int |F|\Psi dx
\] (4.14)
and
\[
\left| \int F\rho(\nu - \nu)dx \right| \leq C \int |F|\Psi dx.
\] (4.15)
When $P_0 \in C^2([0, \bar{\rho}])$, we have
\[
\left| \int (\rho P_0 - \rho \overline{P_0}) dx \right| \leq \int (\rho P_0 - \rho P_0) dx + \int \rho(\overline{P_0} - P_0) dx \leq C \int \Psi dx. \tag{4.16}
\]
When $P_0$ is monotone function on $[0, \bar{\rho}]$, using the Lemma 5 in [16], we have
\[
\overline{\rho P_0} \geq \rho \overline{P_0}. \tag{4.17}
\]
From (4.11)–(4.17), we obtain
\[
\int \Psi dx \leq \int_0^t (1 + |F|) \Psi dx ds.
\]
Using (2.39) and Gronwall’s inequality, we get
\[
\Psi = 0, \ (t, x) \in [0, T] \times \mathbb{R}^N,
\]
and (extract a subsequence)
\[
\rho^\delta - \hat{\rho} \to \rho - \hat{\rho}, \ \text{strongly in} \ L^k(\mathbb{R}^N \times [0, \infty)),
\]
for all $k \in [2, \infty)$.

Thus, it is easy to show that the limit function $(\rho, u)$ are indeed a weak solution of the system (1.1)–(1.3). Using a similar argument as that in the proof of (1.17), we get (1.31). This finishes the proof of Theorem 1.2. \hfill \Box

## 5 Appendix

It requires a dyadic decomposition of the Fourier space, so let us start by recalling the definition of the following operators of localization in Fourier space (see [23]):
\[
\Delta_q a \triangleq \mathcal{F}^{-1}(\varphi(2^{-q}|\xi|) \hat{a}), \quad \text{for } q \in \mathbb{Z},
\]
where $\mathcal{F}a$ and $\hat{a}$ denote the Fourier transform of any function $a$. The function $\varphi$ is smooth, and satisfies
\[
\text{supp} \varphi \subset \left\{ \xi \in \mathbb{R}^2 \left| \frac{3}{4} \leq |\xi| \leq \frac{8}{3} \right. \right\}, \quad \sum_{j \in \mathbb{Z}} \varphi(2^{-j} t) = 1, \ \forall \ t \in \mathbb{R}\setminus\{0\}.
\]
Let us note that if $|j - j'| \geq 5$, then $\text{supp} \varphi(2^{-j} t) \cap \text{supp} \varphi(2^{-j'} t) = \emptyset$.

**Definition 5.1.** We denote by $\hat{B}^s_{p, q}$ the space of distributions, which is the completion of $\mathcal{S}(\mathbb{R}^N)$, $N \geq 2$, by the following norm:
\[
\|a\|_{\hat{B}^s_{p, q}} \triangleq \left\| 2^{sk}\|\Delta_k a\|_{L^p(\mathbb{R}^N)} \right\|_{l^q}.
\]

**Lemma 5.1** ([3]). Denote $\mathcal{B}$ a ball of $\mathbb{R}^N$, and $\mathcal{C}$ a ring of $\mathbb{R}^N$. Assume that $1 \leq p_2 \leq p_1 \leq \infty$ and $1 \leq q_2 \leq q_1 \leq \infty$. If the support of $\hat{a}$ is included in $2^k \mathcal{B}$, then
\[
\|\partial^\alpha a\|_{L^{p_1}} \lesssim 2^{k(|\alpha| + N(\frac{1}{p_2} - \frac{1}{p_1}))} \|a\|_{L^{p_2}}.
\]
If the support of $\hat{a}$ is included in $2^k \mathcal{C}$, then
\[
\|a\|_{L^{p_1}} \lesssim 2^{-kM} \sup_{|\alpha| = M} \|\partial^\alpha a\|_{L^{p_1}}.
\]

**Lemma 5.2.** For any $p > N$ and $N \geq 2$, there exists a positive constant $C_p$ such that
\[
\|\nabla u\|_{L^\infty} \leq C\|u\|_{L^2} + C_p(\|\nabla u\|_{\hat{B}^0_{1, \infty}} + 1) \log \left( e + \frac{\|\Delta u\|_{L^p}}{\|\nabla u\|_{\hat{B}^0_{1, \infty}} + 1} \right). \tag{5.1}
\]
Proof.

\[ \|\nabla u\|_{L^\infty} \leq C\|u\|_{L^2} + \sum_{k \geq 0} \|\nabla \Delta_k u\|_{L^\infty} = C\|u\|_{L^2} + \sum_{k \geq 0} \|\nabla \Delta_k u\|_{L^\infty} \]

\[ \leq C\|u\|_{L^2} + M\|\nabla u\|_{\dot{B}^{0}_{\infty,\infty}} + \sum_{k \geq M} 2^k (\frac{N}{p} - 1)\|\nabla \Delta_k u\|_{L^p} \]

\[ \leq C\|u\|_{L^2} + M\|\nabla u\|_{\dot{B}^{0}_{\infty,\infty}} + \frac{C2^M(\frac{N}{p} - 1)}{1 - 2^{\frac{N}{p} - 1}}\|\nabla u\|_{L^p}. \]

Choosing \( M \sim \frac{p}{p-N} \log \left(e + \frac{\|\nabla u\|_{\dot{B}^{0}_{\infty,\infty}}}{\|\Delta u\|_{\dot{B}^{0}_{\infty,\infty}} + 1}\right) \), we can finish the proof.

\[ \square \]

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