ESTIMATE OF THE SQUEEZING FUNCTION FOR A CLASS OF BOUNDED DOMAINS

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Abstract. We construct a class of bounded domains, on which the squeezing function is not uniformly bounded from below near a smooth and pseudoconvex boundary point.

1. Introduction

In [14, 15], the authors introduced the notion of holomorphic homogeneous regular. Then in [16], the equivalent notion of uniformly squeezing was introduced. Motivated by these studies, in [3], the authors introduced the squeezing function as follows.

Denote by $B(r)$ the ball of radius $r > 0$ centered at the origin 0. Let $\Omega$ be a bounded domain in $\mathbb{C}^n$, and $p \in \Omega$. For any holomorphic embedding $f : \Omega \to B(1)$, with $f(p) = 0$, set

$$s_{\Omega,f}(p) := \sup \{ r > 0 : B(r) \subset f(\Omega) \}. $$

Then, the squeezing function of $\Omega$ at $p$ is defined as

$$s_\Omega(p) := \sup \limits_f \{ s_{\Omega,f}(p) \}. $$

Many properties and applications of the squeezing function have been explored by various authors, see e.g. [3, 4, 6, 8, 10, 11, 12].

It is clear that squeezing functions are invariant under biholomorphisms, and they are positive and bounded above by 1. It is a natural and interesting problem to study the uniform lower and upper bounds of the squeezing function.

It was shown recently in [12] that the squeezing function is uniformly bounded below for bounded convex domains. On the other hand, in [3], the authors showed that the squeezing function is not uniformly bounded below on certain domains with non-smooth boundaries, such as punctured balls. In [5], the authors constructed a smooth pseudoconvex domain in $\mathbb{C}^3$ on which the quotient of the Bergman metric and the Kobayashi metric is not bounded above near an infinite type point. By [4, Theorem 3.3], the squeezing function is not uniformly bounded below on this domain.

These studies raise the question: Is the squeezing function always uniformly bounded below near a smooth finite type point? In this paper, we answer the question negatively. More precisely, we have the following

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Let \( \Omega \) be a bounded domain in \( \mathbb{C}^3 \), and \( q \in \partial \Omega \). Assume that \( \Omega \) is smooth and pseudoconvex in a neighborhood of \( q \) and the Bloom-Graham type of \( \Omega \) at \( q \) is \( d < \infty \). Moreover, assume that the regular order of contact at \( q \) is greater than \( 2d \) along two smooth complex curves not tangent to each other. Then the squeezing function \( s_{\Omega}(p) \) has no uniform lower bound near \( q \).

Remark 1. The proof gives the estimate \( s_{\Omega}(p) \leq C s^{2 \frac{1}{1+d}} \) for some points approaching the boundary.

In section 2, we recall some preliminary notions and results. In section 3, we prove Theorem 1.

2. Preliminaries

Let \( \Omega \) be a bounded domain in \( \mathbb{C}^n \), \( n \geq 2 \), and \( q \in \partial \Omega \). Assume that \( \Omega \) is smooth and pseudoconvex in a neighborhood of \( q \). The Bloom-Graham type of \( \Omega \) at \( q \) is the maximal order of contact of complex manifolds of dimension \( n-1 \) tangent to \( \partial \Omega \) at \( q \) (see e.g. [1]). Choose local coordinates \((z,t) \in \mathbb{C}^{n-1} \times \mathbb{C} \) such that the complex manifold of dimension \( n-1 \) with the maximal order of contact is given by \( \{ t = 0 \} \). Then \( \Omega \) is locally given by \( \rho(z,t) < 0 \), where \( \rho(z,t) = \text{Re} + P(z) + Q(z,t) \) with \( Q(z,0) \equiv 0 \) and \( \text{deg} \ P(z) = d \). (We say that the degree of \( P \) is \( d \) if the Taylor expansion of \( P \) has no nonzero term of degree less than \( d \).) Since \( \Omega \) is pseudoconvex, we actually have \( d = 2k \) (see e.g. [2]).

For \( 1 \leq k \leq n-1 \), let \( \varphi : \mathbb{C}^k \to \mathbb{C}^n \) be analytic with \( \varphi(0) = q \) and \( \text{rank} k \varphi = k \). Then the regular order of contact at \( q \) along the \( k \)-dimensional complex manifold defined by \( \varphi \) is defined as \( \text{deg} \rho \circ \varphi \) (see e.g. [2]).

Denote by \( \Delta \) the unit disc in \( \mathbb{C} \). Let \( p \in \Omega \) and \( \zeta \in \mathbb{C}^n \). The Kobayashi metric is defined as

\[
K_{\Omega}(p,\zeta) := \inf \{ \alpha : \alpha > 0, \exists \phi : \Delta \to \Omega, \phi(0) = p, \alpha \phi'(0) = \zeta \}.
\]

Then the Kobayashi indicatrix is defined as (see e.g. [13])

\[
D_{\Omega}(p) := \{ \zeta \in \mathbb{C}^n : K_{\Omega}(p,\zeta) < 1 \}.
\]

For each unit vector \( e \in \mathbb{C}^n \), set \( D_{\Omega}(p,e) := \max \{ \| \eta \| : \eta \in \mathbb{C}, \eta e \in D_{\Omega}(p) \} \). By the definition of Kobayashi indicatrix, the following three lemmas are clear.

Lemma 1. \( D_{\mathbb{B}(r)}(0) = \mathbb{B}(r) \).

Lemma 2. Let \( \Omega_1 \) and \( \Omega_2 \) be two domains in \( \mathbb{C}^n \) with \( \Omega_1 \subset \Omega_2 \). Then for each \( p \in \Omega_1 \), \( D_{\Omega_1}(p) \subset D_{\Omega_2}(p) \).

Lemma 3. Let \( \Omega \) be a domain in \( \mathbb{C}^n \) and \( f : \Omega \to \mathbb{C}^n \) a biholomorphic map. Then for each \( p \in \Omega \), \( D_{f(\Omega)}(f(p)) = f'(p)D_{\Omega}(p) \).

We also need the following localization lemma (see e.g. [9, Lemma 3]).

Lemma 4. Let \( \Omega \) be a domain in \( \mathbb{C}^n \), \( q \in \partial \Omega \) and \( U \) a neighborhood of \( q \). If \( V \subset \subset U \) and \( q \in V \), then

\[
K_{\Omega}(p,\zeta) \simeq K_{\Omega \cap V}(p,\zeta), \quad \forall \ p \in V, \ \zeta \in \mathbb{C}^n.
\]

By the above lemma, when we consider the size of the Kobayashi indicatrix in the next section, we will work in \( \Omega \cap U \).
3. Estimate of the squeezing function

We first choose local coordinates adapted to our purpose. We will use \( \gtrsim \) (resp. \( \lesssim \)) to mean \( \geq \) (resp. \( \leq \)) up to a positive constant.

**Lemma 5.** Let \( \Omega \) be a bounded domain in \( \mathbb{C}^{n+1} \), \( n \geq 1 \), and \( q \in \partial \Omega \). Assume that \( \Omega \) is smooth and pseudoconvex in a neighborhood of \( q \) and the Bloom-Graham type of \( \Omega \) at \( q = 2k, k \geq 1 \). Then there exist local coordinates \( (z, t) = (z_1, \cdots, z_n, u + iv) \) such that \( q = (0, 0) \) and \( \Omega \) is locally given by \( \rho(z, t) < 0 \) with

\[
\rho(z, t) = u + P(z) + Q(z) + vR(z) + t^2 + o(|t|^2),
\]

where \( P(z) \) is plurisubharmonic, homogeneous of degree \( 2k \), but not pluriharmonic, \( \deg Q(z) \geq 2k + 1 \) and \( \deg R(z) \geq k + 1 \).

**Proof.** By assumption, we have a local defining function of the form

\[
\rho(z, t) = u + P(z) + Q(z) + au^2 + buv + cv^2 + uA(z) + vB(z) + o(|t|^2),
\]

where \( P(z) \) is plurisubharmonic, homogeneous of degree \( 2k \), but not pluriharmonic, and \( \deg Q(z) \geq 2k + 1 \). By changing \( t \) to \( t + dt \) and multiplying with \( 1 + ev \) or \( 1 + eu \), we can freely change the quadratic terms in \( u, v \). Thus, we can assume that

\[
\rho(z, t) = u + P(z) + Q(z) + u^2 + v^2 + uA(z) + vB(z) + o(|t|^2).
\]

Multiplying with \( 1 - uA(z) \), we can further assume that

\[
\rho(z, t) = u + P(z) + Q(z) + u^2 + v^2 + vB(z) + o(|t|^2).
\]

Write \( B(z) = B_s(z) + B'(z) \), where \( B_s(z) \) is the lowest order homogeneous term of degree \( s \geq 1 \). Assume that \( B_s(z) \) is pluriharmonic. Then there exists a holomorphic function \( F(z) = A(z) - iB_s(z) \). Change again \( \rho \) to add the term \( uA(z) \) with this new \( A(z) \). Then \( \rho \) takes the form

\[
\rho(z, t) = u + P(z) + Q(z) + u^2 + v^2 + uA(z) + vB(z) + o(|t|^2)
= u + P(z) + Q(z) + u^2 + v^2 + \text{Re}(tF(z)) + vB'(z) + o(|t|^2).
\]

By absorbing \( \text{Re}(tF(z)) \) into \( u \), we get

\[
\rho(z, t) = u + P(z) + Q(z) + u^2 + v^2 + vB'(z) + o(|t|^2).
\]

Continuing this process, we can assume that \( \rho \) takes the form

\[
\rho(z, t) = u + P(z) + Q(z) + u^2 + v^2 + vB_l(z) + vB'(z) + o(|t|^2),
\]

where \( B_l(z) \) is not pluriharmonic. Suppose that \( l \leq k \). We will arrive at a contradiction to pseudoconvexity.

Note that \( P(z) \) is plurisubharmonic but not pluriharmonic. This implies that there exists a complex line through the origin on which the restriction of \( P \) is subharmonic, but not harmonic. Pick a tangent vector \( \xi = (\xi_1, \cdots, \xi_n) \) so that the Levi form of \( P \) calculated at a point \( \eta \xi, \eta = |\eta|e^{i\theta} \), in the direction of \( \xi \) is \( |\eta|^{2k-2}G(\theta)||\xi||^2 \). Here \( G \) is a smooth nonnegative function which at most vanishes at finitely many angles. Choose \( \lambda \) such that \( \sigma = (\xi, \lambda) \) is a complex tangent vector to \( \partial \Omega \), i.e.

\[
\sum_{j=1}^n \frac{\partial \rho}{\partial z_j} \xi_j + \frac{\partial \rho}{\partial \bar{t}} \lambda = 0.
\]

Then we have \( |\lambda| = O(|\eta|^{2k-1} + v|\eta|^{l-1} + |t|^2)||\xi|| \).
scaling in each variable, we can assume that a variable.

Proof. Lemma 7. Let \( \phi: \Delta \rightarrow \mathbb{C} \) be a point so that \( \phi(\tau) = (\beta \tau, 0, -\delta) \) for \( \tau \in \Delta \), and \( \|\beta\| = \epsilon \delta^{-\frac{1}{2}} \) for \( 0 < \epsilon \ll 1 \). Then

\[
\rho \circ \phi(\tau) \leq -\delta + C|\beta \tau|^{4k+1} + o(\delta) < -\delta + \epsilon \delta + o(\delta) < 0.
\]

Therefore, \( K_\Omega(p, u) \lesssim \delta^{-\frac{1}{2}} \). The argument in the direction \( v \) is similar. \( \square \)

Let \((a, b, 0)\) be a point so that \( P(a \tau, b \tau) \) is a subharmonic homogeneous polynomial of degree \( 2k \) which is not harmonic. Then both \( a \) and \( b \) must be nonzero. By scaling in each variable, we can assume that \( a = b = 1/\sqrt{2} \).

Lemma 7. Let \( \zeta = \frac{1}{\sqrt{2}}(1, 1, 0) \). Then \( K_\Omega(p, \zeta) \gtrsim \delta^{-\frac{1}{2}} \).

Proof. For \( z, w \) small, we have

\[
v^2 + vR(z, w) \geq v^2 - 2CV \|z, w\|^2 + C^2 \|z, w\|^2 - C^2 \|z, w\|^2 \geq -C \|z, w\|^2 + 2k + 2.
\]

Therefore,

\[
\rho \geq u + P(z, w) + Q(z, w) - C^2 \|z, w\|^2 + o(u, v, w, u, v, z, w) = u + P(z, w) + \tilde{Q}(z, w) + o(u, v, w, u, v, z, w) =: \tilde{\rho}.
\]

Consider an analytic map \( \phi: \Delta \rightarrow \Omega \) with

\[
\phi(\tau) = (\beta \tau + f(\tau), \beta \tau + g(\tau), -\delta + h(\tau)), \quad f(\tau), g(\tau), h(\tau) = O(\tau^2).
\]

Then \( \tilde{\rho} \circ \phi(\tau) \leq \rho \circ \phi(\tau) < 0 \). For terms containing \( u \), the dominant term of \( \tilde{\rho} \circ \phi(\tau) \) is \( -\delta \). Thus, we have

\[
\varphi(\tau) := \Re h(\tau) + P(\beta \tau + f(\tau), \beta \tau + g(\tau)) + \tilde{Q}(\beta \tau + f(\tau), \beta \tau + g(\tau)) + o(v^2) \lesssim \delta,
\]

and

\[
\frac{1}{2\pi} \int_0^{2\pi} \varphi(\tau e^{i\theta})d\theta \lesssim \delta.
\]
Therefore, we have $c > \varepsilon > 0$ for some $\varepsilon$. Let $p; u; v; w$ be a linear map with $F(\varepsilon) = \sum_{i=0}^{\delta} |\beta|^i |\tau|^{4k-i} \lesssim \delta$.

Choose $|\tau| = \frac{1}{2}|\beta|$. Then (4) gives

$$\left| \frac{\beta}{2} \right|^{4k} \lesssim \delta.$$ 

Hence, $K_{\Omega}(p, u) \gtrsim \delta^{-\frac{1}{4k}}$. 

\begin{proof}
Lemma 8. Let $D$ be a bounded domain in $\mathbb{C}^n$, $n \geq 2$, containing the origin. Assume that there exist two linearly independent nonzero vectors $\zeta_1, \zeta_2 \in D$ and $\varepsilon > 0$ such that $\varepsilon(\zeta_1 + \zeta_2) \notin D$. Then there does not exist a linear map $L : D \to \mathbb{C}^n$, with $L(0) = 0$, such that $B(3\varepsilon) \subset L(D) \subset B(1)$.

Proof. Let $L : D \to \mathbb{C}^n$ be a linear map with $L(0) = 0$ and suppose $B(3\varepsilon) \subset L(D)$. Since $\varepsilon(\zeta_1 + \zeta_2) \notin D$ and $L$ is linear, we have $\varepsilon(L(\zeta_1) + L(\zeta_2)) \notin B(3\varepsilon)$ and thus $\|L(\zeta_1) + L(\zeta_2)\| \geq 3$. However, $\|L(\zeta_1) + L(\zeta_2)\| \leq \|L(\zeta_1)\| + \|L(\zeta_2)\| \leq 1 + 1 = 2$. This completes the proof. 

\end{proof}

Proof of Theorem 1. Choose local coordinates $(z, w, t)$ such that $q = (0, 0, 0)$ and let $p = (-\delta, 0, 0)$ for $\delta > 0$ small. Let $\zeta_1 = (1, 0, 0)$ and $\zeta_2 = (0, 1, 0)$. By Lemma 6, $K_{\Omega}(p, \zeta_1), K_{\Omega}(p, \zeta_2) \lesssim \delta^{-\frac{1}{4k}}$. By Lemma 7, $K_{\Omega}(p, \frac{1}{\sqrt{2}}(\zeta_1 + \zeta_2)) \gtrsim \delta^{-\frac{1}{4k}}$.

Choose $\lambda > 0$ with $\lambda \gtrsim \delta^{-\frac{1}{4k}}$ such that $\lambda \zeta_1, \lambda \zeta_2 \in D_{\Omega}(p)$. Then for $\varepsilon \approx \delta^{-\frac{1}{4k+1}}$, we have $\varepsilon(\lambda \zeta_1 + \lambda \zeta_2) \notin D_{\Omega}(p)$. Thus, by Lemma 8, there does not exist a linear map $L : D_{\Omega}(p) \to \mathbb{C}^n$ such that $B(3\varepsilon) \subset L(D_{\Omega}(p)) \subset B(1)$.

Let $f$ be a biholomorphism of $\Omega$ into $B(1)$ such that $f(p) = 0$ and $B(c) \subset f(\Omega)$ for some $c > 0$. Set $L = f'(p)$. Then, by Lemmas 1, 2 and 3, $B(c) \subset L(D_{\Omega}(p)) \subset B(1)$. Therefore, we have $c \lesssim \delta^{-\frac{1}{4k+1}}$. Since $f$ is arbitrary, we get $s_{\Omega}(p) \lesssim \delta^{-\frac{1}{4k+1}}$. Since $\delta$ can be arbitrarily small, this completes the proof.

\begin{remark}
Remark 2. Theorem 1 does not hold if only assuming that the regular order of contact at $q$ is greater than $2d$ along one smooth complex curve. For instance, consider $\Omega$ given by

$$\{(z, w, t) \in \mathbb{C}^3 : |t|^2 + |z|^2 + |w|^6 < 1\}.$$ 

Then at $q = (0, 0, 1)$, the Bloom-Graham type is 2 and the regular order of contact along $(0, 1, 0)$ is $6 > 4$. But $\Omega$ is a bounded convex domain and thus the squeezing function has a uniform lower bound by [12].

\end{remark}

\begin{remark}
Remark 3. Using similar arguments, one can extend Theorem 1 to higher dimensions as follows.

\end{remark}
Theorem 2. Let $\Omega$ be a bounded domain in $\mathbb{C}^n$, $n \geq 4$, and $q \in \partial \Omega$. Assume that $\Omega$ is smooth and pseudoconvex in a neighborhood of $q$ and the Bloom-Graham type of $\Omega$ at $q$ is $d$. Moreover, assume that the regular order of contact at $q$ is $d$ along a two-dimensional complex surface $\Sigma$ and the regular order of contact at $q$ is greater than $2d$ along two smooth complex curves not tangent to each other contained in $\Sigma$. Then the squeezing function $s_\Omega(p)$ has no uniform lower bound near $q$.

Remark 4. After the completion of this work, it was brought to our attention by Gregor Herbort that a similar comparison result to [5] was obtained for the following domain in [7]:

$$
\Omega := \{(z, w, t) \in \mathbb{C}^3 : \Re t + |z|^{12} + |w|^{12} + |z|^2|w|^4 + |z|^6|w|^2 < 0\}.
$$

Therefore, by our remark in the introduction, the squeezing function does not have a uniform lower bound on this domain. More generally, we have the following

Theorem 3. Let $\Omega$ be a bounded domain in $\mathbb{C}^3$, and $q \in \partial \Omega$. Assume that $\Omega$ is smooth and pseudoconvex in a neighborhood of $q$ and the Bloom-Graham type of $\Omega$ at $q$ is $d < \infty$. Let $p$ be a defining function of $\Omega$ near $q$ in the normal form (1) and assume that the leading homogeneous term $P(z)$ only contains positive terms. Moreover, assume that the regular order of contact at $q$ is greater than $d$ along two smooth complex curves not tangent to each other. Then the squeezing function $s_\Omega(p)$ has no uniform lower bound near $q$.

Sketch of proof. In Lemma 6, we get $K_{\Omega}(p, u), K_{\Omega}(p, v) \lesssim \delta^{-\frac{1}{2d+1}}$, by the same argument. In Lemma 7, we get $K_{\Omega}(p, u) \gtrsim \delta^{-\frac{1}{d+1}}$, by noticing that instead of (4) we have $|\xi|^2 \approx \delta$ since all terms of $P(z)$ are positive. Then arguing exactly as in the proof of Theorem 1, we get $s_\Omega(p) \lesssim \delta^{-\frac{1}{2d+1}}$. □

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