Multifractality meets entanglement: relation for non-ergodic extended states

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In this work we establish a relation between entanglement entropy and fractal dimension $D$ of generic many-body wavefunctions, by generalizing the result of Don N. Page [Phys. Rev. Lett. 71, 1291] to the case of sparse random pure states (S-RPS). These S-RPS living in a Hilbert space of size $N$ are defined as normalized vectors with only $N^D$ ($0 \leq D \leq 1$) random non-zero elements. For $D = 1$ these states used by Page represent ergodic states at infinite temperature. However, for $0 < D < 1$ the S-RPS are non-ergodic and fractal as they are confined in a vanishing ratio $N^D$ of the full Hilbert space. Both analytically and numerically, we show that the mean entanglement entropy $S_1(A)$ of a sub-system $A$, with Hilbert space dimension $N_A$, scales as $S_1(A) \sim D \ln N$ for small fractal dimensions $D$, $N^D < N_A$. Remarkably, $S_1(A)$ saturates at its thermal (Page) value at infinite temperature, $S_1(A) \sim \ln N_A$ at larger $D$. Consequently, we provide an example when the entanglement entropy takes an ergodic value even though the wavefunction is highly non-ergodic.

Finally, we generalize our results to Renyi entropies $S_q(A)$ with $q > 1$ and also show that their fluctuations have ergodic behavior in narrower vicinity of the ergodic state, $D = 1$.

Introduction— The success of classical statistical physics is based on the concept of ergodicity, which allows the description of complex systems by the knowledge of only few thermodynamic parameters [1, 2]. In quantum realm the paradigm of ergodicity is much less understood and its characterization is now an active research front.

The most accredited theory, which gives an attempt to explain equilibration in closed quantum systems, relies on the eigenstate thermalization hypothesis (ETH) [3–6]. ETH asserts that the system thermalizes locally at the level of single eigenstates and has been tested numerically in a wide variety of generic interacting systems [6, 7].

It is now well established that entanglement plays a fundamental role on the thermalization process [7–11]. Thermal states are locally highly entangled with the rest of the system, which acts as a bath. Consequently, the measurement of entanglement entropy (EE) has been found to be a resounding resource to probe ergodic/thermal phases, both theoretically [12–17] and recently also experimentally [18–21], and to test the validity of ETH in strongly correlated systems [7]. For instance, infinite temperature ergodic states are believed to behave like random vectors [3, 7] and their EE reaches a precise value often referred as Page value [22].

On the other hand, ergodicity is deeply connected to the notion of chaos [7, 23], which implies also an equipartition of the many-body wavefunction over the available many-body Fock states, usually quantified by multifractal analysis, e.g., by scaling of the inverse participation ratio (IPR) [24]. In this case, infinite temperature ergodic states span homogeneously the entire Hilbert space [25]. The latter states should be distinguished from the so-called non-ergodic extended (NEE) states. These NEE states live on a fractal in the Fock space, which is a vanishing portion of the total Hilbert space. Recently, the NEE have been invoked to understand new phases of matter like bad metals [26–33], which are neither insulators nor conventional diffusive metals and also found it in chaotic many-body quantum system like in the Sachdev-Ye-Kitaev model [34–36]. Furthermore, these states, living in a small portion of the Hilbert space, could be seen as a natural prototype for eigenstates of strongly kinematically constrained Hamiltonians, where ergodicity breaks by Hilbert/Fock space fragmentation [37–42].

Very recently, the two aforementioned probes, EE and IPR, have been used to describe thermal phases (specially at infinite temperature), and to detect ergodic-breaking quantum phase transitions (e.g. many-body localization transition) [12, 43–46]. Nevertheless, the relations between these two probes has not been studied extensively so far [47, 48], which leads to the natural question: to what extend do they lead to the same description?

In this work, we build up a bridge between ergodic properties extracted from EE and the ones from multifractal analysis. With this aim, we generalize the seminal work of Page [22], computing EE and its fluctuations for fractal NEE states. We introduce the concept of sparse random pure states (S-RPS), which are fractal NEE states and determined by $N^D$ ($0 < D < 1$) random non-zero elements in the corresponding computational basis of dimension $N$ [49]. The limiting case, $D = 1$, represents an ensemble of infinite temperature ergodic states, for which EE is given by the Page value [50–53].

By studying EE for S-RPS, we derive a precise upper bound for EE for the eigenstates with a fixed fractal dimension $D$. Remarkably, we show, both analytically and numerically, that EE of a sub-system $A$ can still be ergodic (Page value), even though the states are highly non-ergodic $D < 1$. Consequently, the mean value of EE might be not enough to state ergodicity, though EE reaches the Page value.

General definitions— The Renyi entropy, $S_q(A)$, of a sub-system $A$ with Hilbert space dimensions $N_A = N^p$,
\[ p \leq 1/2, \text{ is defined as:} \]
\[ S_q(A) = \frac{\ln \Sigma_q}{1-q}, \text{ with } \Sigma_q = \text{Tr}_B[\rho_A^q] = \sum_{M=1}^{N_A} \lambda_M^q, \quad (1) \]

where \( \rho_A = \text{Tr}_B[\rho] \) is the reduced density matrix of the sub-system \( A \), obtained tracing over the degrees of freedom of the complementary sub-system \( B = A^c \). \( \{\lambda_M\} \) are Schmidt eigenvalues of \( \rho_A \). The von Neumann EE, \( S_1(A) \), corresponds to the limit \( q \to 1 \), \( S_1(A) = -\text{Tr}_B[\rho_A \ln \rho_A] = -\sum_{M=1}^{N_A} \lambda_M \ln \lambda_M \).

For a pure state \( \rho = |\psi\rangle \langle \psi| \) [54], \( \rho_A \) takes the form of a Wishart matrix
\[ \rho_{A,M'} = \sum_{m=1}^{N_B} |\psi_{M,m}\rangle \langle \psi_{M',m}|, \quad (2) \]

where \( |\psi_{M,m}\rangle \) are the wavefunction coefficients \( |\psi\rangle = \sum_{M=0}^{N_A-1} \sum_{m=0}^{N_B-1} |\psi_{M,m}\rangle |M\rangle_A |m\rangle_B \) in the computational basis \( |M\rangle_A \), \( 1 \leq M \leq N_A \), and \( |m\rangle_B \), \( 1 \leq m \leq N_B \), of the two sub-systems \( A \) and \( B \), respectively.

For fully random states, \( D = 1 \), the mean von Neumann EE is given by the Page value [22]
\[ \overline{S}_{\text{Page}}(A) = \ln N_A - \frac{N_A}{2N_B}, \quad (3) \]

and its fluctuations decays to zero as [50, 51, 53]
\[ \overline{S}_{\text{Page}}(A) = (\overline{S}^2(A) - \overline{S}^2(A))^{1/2} \sim N_B^{-1}, \quad (4) \]

where the overline indicates the random vector average.

Moreover, the ergodic properties of the wavefunction \( \{\psi_n=|M,m\rangle\} \) can be characterized also in terms of multifractal analysis [24] via an infinite sequence of fractal dimensions \( D_q, q \geq 0 \), defined through the scaling of the inverse participation ratios \( \text{IPR}_q \) with \( N \),
\[ D_q \ln N = \ln \left( \frac{\text{IPR}_q}{1-q} \right), \text{ with } \text{IPR}_q = \sum_n |\psi_n|^2q, \quad (5) \]

giving in the limit \( q \to 1 \), \( D_1 \ln N = -\sum_n |\psi_n|^2 \ln |\psi_n|^2 \).

The exponent \( D_1 \) provides important information on the dimension of the support set of the wavefunction in the Fock space, which scales as \( \sim N^{D_1} \) [55]. Fully ergodic states are characterized by \( D_q = 1 \), meaning that the state is homogeneously spread over the entire Hilbert space [25]. Instead, NEE states are usually multifractal with \( D_q < 1 \) and their support set is a vanishing ratio of the full Hilbert space \( \sim N^{D_1}/N \).

In this work, we consider entanglement properties of NEE states employed by S-RPS. The S-RPS are normalized random vectors \( \{\psi_n\} \) with only \( N^D \) non-zero elements, that form the wave function support set. The S-RPS are described by both any fractal dimension \( D_q = D < 1 \), \( \text{IPR}_q = N^{D(1-q)}, q > 0 \). Thus, the S-RPS are homogeneously spread, but only in a vanishing ratio of the total Hilbert space.

**Results**— We start to outline our results, by computing numerically the mean EE for S-RPS with fractal dimension \( 0 < D < 1 \) in a Hilbert space of dimension \( N = 2^L \).

In this case, the S-RPS could be thought as eigenstates in the middle of the spectrum [56] of some strongly interacting 1/2−spin chain with \( L \) sites.

First, let’s consider two limiting cases: For \( D = 1 \), \( S_1(A) \) is given by the Page value in Eq. (3) \( \sim \ln N_A \) as the system is ergodic. While for \( D = 0 \), the wavefunction is localized in the Fock-space and EE shows area-law \( S_1(A) \sim O(1) \). For generic \( 0 < D < 1 \), one may naively expect the natural interpolation \( S_1(A) \sim D \ln N_A \), as S-RPS are random states in a sub-Hilbert space of dimension \( N^D \). However, as we will show, this intuitive picture is misleading.

Figure 1 presents the mean value of the half-partition EE, \( S_1(L/2) \), \( N_A = 2^{L/2} \). \( S_1(L/2) \) follows a volume law \( S_1(L/2) \sim D_{\text{ent}} \ln 2^{L/2} \) for any \( D > 0 \) and the slope \( D_{\text{ent}} \) grows with increasing \( D \). However, the curves approach the Page value \( S_{1,\text{Page}}(L/2) = L/2 \ln 2 - 1/2 \) (dashed line in Fig. 1 (a)), i.e., \( D_{\text{ent}} = 1 \) for \( D > 1/2 \). Instead, for \( D < 1/2 \), \( S_1(L/2) \) grows slower than \( S_{1,\text{Page}}(L/2) \) and we found \( D_{\text{ent}} = 2D \), Fig. 1 (a),(b). Thus, basing only on the mean EE, one might erroneously conclude that the system is ergodic at \( D > 1/2 \), even though the wavefunction is confined in an exponentially small ratio of the total Hilbert space \( \sim N^{2-(1-D)L} \).

To understand this behavior of \( S_1(L/2) \) as function of \( D \), we consider in details the structure of the reduced density matrix \( \rho_A \) in Eq. (2), determined by the scalar products between the vectors \( \psi_M = \langle \psi_{M,1}, \ldots, \psi_{M,N_A} \rangle \). For \( M \neq M' \), these vectors are independent [57] and the off-diagonal elements of \( \rho_A \) are almost negligible for \( D < 1 \) due to the sparsity properties of \( \psi_M \), which has only \( N^D/N \) fraction of non-zero elements. Instead, the diagonal elements of \( \rho_A \) are given by the norms of the vectors \( \psi_M \) and cannot be neglected [58].

**FIG. 1.** Mean von Neumann entanglement entropy scaling as a function of the fractal dimension \( D \) for S-RPS. (a) \( S_1(L/2) \) of half-system, \( N_A = 2^{L/2} \), as a function of \( L \) for different \( D \), dashed line shows the Page value, Eq. (3). (b) Slope \( D_{\text{ent}} \) of \( S_1(L/2) \) \( \sim D_{\text{ent}} L/2 \ln 2 \) as a function of \( D \) (Exact) and of \( -\sum_M \rho_{M,M} \ln \rho_{M,M} \sim D_{\text{ent}} L/2 \ln 2 \) (Diagonal Approx.); black line represents the theoretical prediction in Eq. (8).
The structure of the half-system reduced density matrix \(|\rho_{A,M'}^{M,M'}|/\max_{M,M'}|\rho_{A,M'}^{M,M'}|\) for S-RPS with different fractal dimension (a) \(D = 1\), (b) \(D = 0.7\), and (c) \(D = 0.4\) for \(N_A = 2^{L/2}\) and \(L = 12\). In all panels \(\rho_A\) is mostly represented by the diagonal elements with almost uniform distribution \(\rho_{A,M}^{M,M} \sim N_A^{-1}\) for \(D > 1/2\) (a, b) and bimodal distribution otherwise (c). The latter case is given by \(\sim 2^{DL}\) non-zero nearly uniform elements normalized as \(\rho_{A,M}^{M,M} \sim 2^{-DL}\) with the rest being negligibly small. The corresponding EE saturates at the ergodic Page value \(S_1(A) = S_{\text{Page}}(A)\) for \(D > 1/2\), while being dominated by \(2^{DL}\) non-zero elements for \(D < 1/2\) leading to \(S_1(A) \approx -\sum_M \rho_{A,M}^{M,M} \ln \rho_{A,M}^{M,M} \sim DL\log 2\).

This analysis can be clearly seen in Fig. 2, which shows \(\rho_{A,M} N_A = 2^{L/2}\), for a given random configuration of the S-RPS. As one can notice \(\rho_A\) is always nearly diagonal. Moreover, for \(D > 1/2\), an extensive number of off-diagonal elements become non-zero and the diagonal ones are homogeneously distributed with amplitude \(\rho_{A,M}^{M,M} \sim 2^{-L/2}\), Fig. 2 (a)-(b). As soon as \(D\) is smaller than 1/2, only few off-diagonal elements of \(\rho_A\) are non-zero, while the distribution of the diagonal ones is bimodal with \(\sim 2^{DL}\) non-zero terms, Fig. 2 (c).

Intuitively, as the first approximation, the scaling of EE can be estimated considering only diagonal elements of \(\rho_A\) (diagonal approximation), \(S_1(L/2) \sim -\sum_i \rho_{M,M} \ln \rho_{M,M}\), thus obtaining \(S_1(L/2) \sim S_{\text{Page}}(L/2)\) for \(D > 1/2\) and \(S_1(L/2) \sim 2D\log 2\) for \(D < 1/2\). We further support the validity of the diagonal approximation in Appendices A, B. In Fig. 1 (b) is shown \(D_{\text{ent}}\) extracted by \(-\sum_{M,M'} \rho_{M,M'} \ln \rho_{M,M'} \sim D_{\text{ent}}\log 2\), where a perfect match with the exact \(D_{\text{ent}}\) is found.

The diagonal approximation has been used to describe thermodynamic entropy out-of-equilibrium [59, 60] and it can be analytically verified in terms of leading scaling behavior. Indeed, as only few off-diagonal elements of \(\rho_A\) are non-zero (say, \(\rho_{M,M'}^{M,M} \) for the \(M\)th row) one can estimate the Schmidt eigenvalues \(\lambda_M\) and \(\lambda_{M'}\) by diagonalizing the \(2 \times 2\)-matrix \((\rho_{M,M}^{M,M} \rho_{M',M'}^{M,M} \rho_{M,M'}^{M,M'} \rho_{M',M}^{M,M'})\). Finally, by the Cauchy-Bunyakovski-Schwarz inequality \(|\rho_{M,M}^{M,M}|^2 \leq \rho_{M,M'}^{M,M'}\), one can conclude, that the Schmidt eigenvalues \(\lambda_M\) and \(\lambda_{M'}\) scale with \(N\) as the diagonal elements \(\rho_{A,M}^{M,M} \rho_{A,M'}^{M,M'}\), see Appendix C.

Furthermore, in this leading approximation the mean EE is given by
\[
S_1(L/2) \approx \sum_M \rho_{M,M}^{M,M} \ln \rho_{A,M}^{A,M} \sim \ln N_0,
\]
where \(N_0\) is the number of non-zero diagonal elements \(\rho_{A,M}^{A,M} = \sum_{m=1}^{N_A} |\psi_{M,m}|^2\) [61], which have almost all the same value (see Fig. 2).

The probability distribution \(P(N_0)\) of \(N_0\) can be calculated combinatorially. Let \(g_M\) be the number of non-zero elements giving contributions to \(\rho_{A,M}^{A,M}\). By construction of the S-RPS we have \(\sum_{M} g_M = N^D\). Now, \(P(N_0)\) is proportional to the product of the number of combinations \(\binom{N_A \ldots 1}{N_0 \ldots 1}\) to realize \(N_0\) non-zero \(g_M > 0\) and the number of combinations \(\binom{N_A}{N_0}\) to place them among \(N_A\).
values of $1 \leq M \leq N_A$. The typical $N_0$ is given by the position of the maximum of its probability distribution
\begin{equation}
N_0^{\text{typ}} = \frac{N_A N^D}{N_A + N^D} \approx N_{\min(p,D)},
\end{equation}
confirming the numerical result, Fig. 1,
\begin{equation}
\mathcal{S}_f(A) \approx \begin{cases} D \ln N, & D < p \\ \ln N_A, & D > p \end{cases}.
\end{equation}

Importantly, the S-RPS do not have any intrinsic notion of locality being the position of the non-zero elements randomly chosen. As a consequence, Eq. (8) gives a natural upper-bound for the maximal EE for generic many-body wavefunction with support set $\sim N^D$.

Now, we further numerically test our main result in Eq. (8), by computing $S_f(A)$ for a different sub-system $A$. Figure 3 (a) shows the slope $D_{\text{ent}}$ of $\mathcal{S}_f(N_A) \sim D_{\text{ent}} \ln N_A$ for $N_A = 2^{L/3}$ as a function of the fractal dimension $D$. For $D > 1/3$, we have $D_{\text{ent}} = 1$ and EE shows ergodic behavior. For smaller $D$, $D_{\text{ent}}$ deviates from the infinite temperature thermal value, $D_{\text{ent}} = 3D$, in agreement with Eq. (8). The difference $S_{\text{Page}}(L/3) - S(A)$ is shown in the inset in Fig. 3 (a) supporting the convergence of EE to the Page value $S_{\text{Page}}(L/3)$ up to exponentially small corrections in $L$ (as well as $S_{\text{Page}}(L/2)$ in Fig. 1 (a)).

Furthermore, our results can be generalized also for the Renyi EE in Eq. (1). In Fig. 3 (b) we analyze the scaling of $\mathcal{S}_f(A) \sim D_{\text{ent}}(q) \ln N_A$ at half-partitioning $N_A = 2^{L/2}$ for several $q > 1$. As one can notice, $D_{\text{ent}}(q)$ depends only on the fractal dimension $D$, but not on $q$. Similarly to the limit $q \to 1$ and in accordance with Eq. (8), we obtain $D_{\text{ent}}(q) = 1$ for $D > 1/2$ and $D_{\text{ent}}(q) = 2D$. The independence of $D_{\text{ent}}(q)$ from $q \geq 1$ is an artefact of the S-RPS, as they are characterized by the only fractal dimension $D_q = D$ for $q > 0$, Eq. (5). For genuine multifractal states, characterized by non-trivial exponents $D_q$, we expect at half partition, $N_A = N^{1/2}$, $D_{\text{ent}}(q) = 1$ if $D_q > 1/2$ and $D_{\text{ent}}(q) = 2D_q$ otherwise.

**Fluctuations**— Quantum fluctuations represent another important ingredient to understand ergodicity. According to ETH, they can be related to temporal fluctuations around the equilibrium value in a quench protocol. In particular, the study of entropy fluctuations has given important insights on detecting ergodicity-breaking transition in quantum systems. In ergodic systems the scaling of fluctuations is related to the dimension of the larger sub-system Eq. (4) playing the role of a bath [50–53].

The EE fluctuations can be quantified by its standard deviation
\begin{equation}
\overline{\mathcal{S}_f}(A) = \left(\overline{\mathcal{S}_f^2}(A) - \overline{\mathcal{S}_f}(A)^2\right)^{1/2} \sim N^{-D_{\text{fluc}}/2},
\end{equation}
from the collapse with $L$ of the probability distribution $P(x)$ of the rescaled variable $x = (S - \overline{S})/\overline{\mathcal{S}}$, Figure 4 shows the collapse of $P(x)$ with $L$ for several $D$ and $N_A = 2^{L/2}$. Fluctuations displays three different regimes for a generic cut $N_A = N^p$, $p \leq 1/2$, (see inset in Fig. 4 for $p = 1/2$)
\begin{equation}
D_{\text{fluc}} = \begin{cases} D, & D < p \\ 2D - p, & p < D < 1 - p/2 \\ 2(1 - p), & D > 1 - p/2. \end{cases}
\end{equation}

For $D < p$, both mean EE and its fluctuations show the properties of a local observable: their scaling is related to the equilibration within the fractal support set $N^D$ only and does not depend on the sub-system size. For $p < D < 1 - p/2$ the mean EE saturates at the Page value for the considered sub-system size, Eq. (8), and thus, EE cannot be anymore considered as a local observable for such states. Nevertheless, the fluctuations have fingerprints of a non-ergodic behavior, $\overline{\mathcal{S}_f}(A) \sim N^{-(2D-p)}$. Finally for $1 - p/2 < D < 1$, both mean and its fluctuations are undistinguishable from ergodic states at infinite temperature.
Conclusions and discussion—Now, we turn to the main question posed on the Introduction, where we asked what extend ergodicity properties extracted from entanglement measures and from multifractal analysis provide the same description of thermal phases. To answer, we generalized the result of Page [22] on entanglement entropy for random pure states (ergodic) to the case of NEE states characterized by the fractal dimensions $D_q$.

In particular, we presented an upper bound for the entanglement entropy $S_q$ (both von Neumann and Renyi) related to the fixed fractal dimension $D_q$ (see Table I).

This bound shows that $S_q(A)$ can in principle be equal to the Page value so far the wavefunction support set is larger than the sub-system size, $N^{D_q} > N_A$. An example of the saturation of this bound is given for a new introduced class of sparse random pure states. Our results show that for small fractal dimensions $N^{D_q} < N_A$ EE behaves as a local observable both in terms of the mean value and fluctuations.

Thus, ergodicity viewed as the wavefunction equipartition in the full Hilbert space is more strict than the one imposed by the value of the entanglement entropy.

Our results find immediate application in the theory of many-body localization where EE has been used to probe the transition, or in strongly kinematically constrained models where ergodicity may break down due to Fock/Hilbert space fragmentation. For instance, in spin models in Refs. [37, 38], the eigenstates live on an exponentially small fraction of the full Hilbert, due to dipole conservation [38, 62] and strong interactions [37] (Fock-space fragmentation). Nevertheless, the half-chain entanglement entropy equals to the Page value, provided the wavefunction support set have a fractal dimension $D > 1/2$ (see Table I with $p = 1/2$).

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| $S_q(N^p)$ | $D_q > 1 - p/2$ | $p < D_q < 1 - p/2$ | $D_q < p$ |
|------------|----------------|----------------|----------|
| $= p \ln N$, (Page) | $\sim p \ln N$, (Page) | $\sim p \ln N$, (Page) | $\sim p \ln N$, (Page) |

TABLE I. Summary of mean EE of the subsystem A of size $N_A = N^p$ and its fluctuations for S-RPS.
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Appendix A: Numerical tests of diagonal approximation

In this appendix, we provide further numerical evidence of the validity of the diagonal approximation. In the main text we used the diagonal approximation $S^\text{diagonal}_1(A) = -\sum_M \rho_{M,M}^A \ln \rho_{M,M}^A$ to estimate the scaling of the von Neumann EE $S_1(A)$. Figure 5 show both $S_1(A)$ and $S^\text{diagonal}_1(A)$ for half-partition $N_A = 2L/2$, as function of $L$ for several fractal dimension $D$, giving indication that $S_1(A) \simeq S^\text{diagonal}_1(A)$.

To understand numerically why the diagonal approximation works well, we analyze in more detail the structure of the reduced density matrix $\rho_A$. First, we start to investigate the sparse property of $\rho_A$. For this purpose, we define the sparsity of $\rho_A$ as the number of its non-zero off-diagonal elements, $S = \#\{\rho_{M,M}^A \neq 0\}$. Figure 6(a) and Fig. 7(a) show the sparsity of $\rho_A$ for two different partition $N_A = 2L/2$ and $N_A = 2L/3$, respectively. As one can notice, for large $D$ the number of non-zeros in
\[ S_1^{\text{diagonal}}(L/2) = -\sum_{M} \rho_{M,M} \ln \rho_{M,M} \] (dashed line) as a function of \( M \).

\[ \sum_{M,M} \rho_{M,M} \ln \rho_{M,M} \] (solid-line) and its

\[ \rho_A \text{ grows as the total size of the reduced density matrix } N_A^2 \text{ meaning that the matrix is not sparse for these values of } D. \]

\[ S \sim 2^{D_s L} \] as a function of the fractal dimension \( D \).

\[ D_s \sim 2^{D_s L/2} \] (solid-line) and its

\[ D_s \sim 2^{D_s L/3} \] (solid-line) and its

\[ D_s \sim 2^{D_s L/2} \] (solid-line) and its

\[ D_s \sim 2^{D_s L/3} \] (solid-line) and its

In the next section, we will give an analytical argument showing

\[ D_s \approx \begin{cases} 2^{(D-1+p)/2p}, & D < \frac{1+p}{2} \\ 1, & D \geq \frac{1+p}{2} \end{cases} \] (A1)

and demonstrate that sparsity plays a major role for the validity of the diagonal approximation.
FIG. 8. Mean off-diagonal element of $\rho_A$ for $N_A = 2^{L/2}$. (a) $|\rho_{M,M'}^A|$ as a function of $L$ for several $D$. (b) $D_{off}$ exponent extracted from $|\rho_{M,M'}^A| \sim 2^{-pD_{off}L}$, $p = 1/2$ as a function of $D$.

FIG. 9. Mean off-diagonal element of $\rho_A$ for $N_A = 2^{L/3}$. (a) $|\rho_{M,M'}^A|$ as a function of $L$ for several $D$. (b) $D_{off}$ exponent extracted from $|\rho_{M,M'}^A| \sim 2^{-pD_{off}L}$, $p = 1/3$ as a function of $D$.

Now, we calculate the mean off-diagonal elements of $\rho_A$ (not only non-zero ones). Figure 8 (a) and Fig. 9 (a) show $|\rho_{M,M'}^A|$ as function of $L$ for several $D$ for two different partitions $N_A = 2^{L/2}$ and $N_A = 2^{L/3}$, respectively. In general, we have $|\rho_{M,M'}^A| \sim N^{-D_{off}P}$. Figure 8 (b) and Fig. 9 (b) show $D_{off}$ as a function of $D$. In the next section, we will show that

$$pD_{off} \simeq \begin{cases} 1 + p - D, & D < \frac{1+p}{2}, \\ \frac{1+p}{2}, & D > \frac{1+p}{2}. \end{cases}$$

Appendix B: Structure of reduced density matrix

In this section we consider the structure of diagonal

$$\rho_{M,M}^A = \sum_{m=1}^{N_B} |\psi_{M,m}|^2,$$  \hspace{1cm} (B1)

and off-diagonal

$$\rho_{M,M'}^A = \sum_{m=1}^{N_B} \psi_{M,m}^\ast \psi_{M',m}' , \hspace{1cm} (B2)$$

elements of the reduced density matrix $\rho_A$ assuming the vectors $\psi_M$ and $\psi_{M'}$ to be uncorrelated for $M \neq M'$ with a certain probability distribution of each element

$$P(\psi_{M,m}) = (1 - p_0)\delta(\psi_{M,m}) + p_0 P_1(N^{D/2}/|\psi_{M,m}|^2).$$  \hspace{1cm} (B3)

Here, $p_0 = N^D/N$ is the probability that $\psi_{M,m} \neq 0$. $P_1(y)$ is the probability distribution of non-zero values, which is symmetric $P_1(-y) = P_1(y)$, has a unit variance $\int y^2 P_1(y) dy = 1$ and the fourth cumulant $\sigma^4 = \int (y^2 - 1)^2 P_1(y) dy \sim O(1)$. The latter conditions ensure the scaling $|\psi_{M,m}|^2 \sim N^{-D}$ of non-zero elements and the wavefunction normalization (on average). In the limit of large $N$, we can further neglect the correlations related to the normalization condition.

Next, within the above assumptions one can find the probability distributions of diagonal, Eq. (B1), and off-diagonal, Eq. (B2), elements of the reduced density matrix (similar to [64]). For this purpose we rewrite Eq. (B3) in a short form for $N^{D/2}/|\psi_{M,m}| = y$

$$P(y) = (1 - p_0)\delta(y) + p_0 P_1(y) .$$  \hspace{1cm} (B4)

1. Probability distribution of diagonals $\rho_{M,M}^A$

Here we use the Fourier transform to calculate the $N_B$-fold convolution of the probability distribution $P_1(t') = \frac{N_B}{\sqrt{\pi}}$ of $t' = |\psi_{M,m}|^2$ and obtain

$$P(\rho_{M,M}^A) = \sum_{k=0}^{N_B} \binom{N_B}{k} (1 - p)^{N_B-k} p^k \tilde{P}_k(\rho_{M,M}^A N^D) ,$$  \hspace{1cm} (B5)

with

$$\tilde{P}_k(t) = \frac{1}{2\pi} \int e^{-i\omega t} \left( \int \frac{P_1(\sqrt{t'})}{\sqrt{t'}} e^{i\omega t'} \right)^k d\omega . \hspace{1cm} (B6)$$

The scaling of $p_0 = N^{D-1}$ and $N_B = N^{1-p}$ provide the optimal index

$$k_\ast = N_B p_0 = N^{D-p}$$  \hspace{1cm} (B7)

giving the main contribution to the sum Eq. (B5).

As $k$ is integer, one has to distinguish two cases: (i) $D < p$ when $k_\ast = N_B p_0 \ll 1$ and, thus, the probability distribution is nearly bimodal

$$P(\rho_{M,M}^A = x) dx \simeq (1-k_\ast)\delta(x) dx + k_\ast \tilde{P}_1(\rho_{M,M}^A N^D) dxdx ,$$  \hspace{1cm} (B8)
and (ii) \( D > p \) when \( n_* = N_B p_0 \gg 1 \) and the central limit theorem (CLT) works giving

\[
P(\rho^A_{M,M'}) = \frac{e^{-\left(\rho^A_{M,M'} - \bar{N} - N - p\right)^2/(2\sigma^2 N^{-D - p})}}{2\pi^{D/2} N^{-D - p}}.
\]

(B9)

This analysis shows that for \( D > p \) the diagonal \( \rho_A \)-elements are homogeneously distributed with the mean value \( \rho^A_{M,M} = 1/NA \) given by \( \text{Tr}[\rho_A] = 1 \).

2. Probability distribution of off-diagonals \( \rho^A_{M,M'} \)

To obtain \( P(\rho^A_{M,M'}) \) one has to calculate, first, from Eq. (B4)

\[
P(\Psi^D \Psi' = z) = \int P(y)P(y')\delta(z - yy')dydy' = (1 - p^2)\delta(z) + \rho^2_0 \bar{P}(z),
\]

with

\[
\bar{P}_l(z) = \int P_l(y)P_l(y')\delta(z - yy')dydy'.
\]

(B10)

Then, analogously to the previous subsection, one can use the Fourier transform to calculate

\[
P(\rho^A_{M,M'}) = \sum_{l=0}^{N_B} \left( \frac{N_B}{l} \right) (1 - p^2)^{N_B - l} p_0^2 \bar{P}_l(\rho^A_{M,M} N^D),
\]

(B12)

with

\[
\bar{P}_l(t) = \frac{1}{2\pi} \int e^{-i\omega l} \left( \int \bar{P}_l(z') e^{i\omega z'} \right)^l d\omega.
\]

(B13)

The scaling of \( p_0 = N^{D-1} \) and \( N_B = N^{1-p} \) provide the optimal index

\[
l_* = N_B p_0^2 = N^{2D-1-p}
\]

(B14)

giving the main contribution to the sum Eq. (B12).

As \( l \) is integer, one has to distinguish two cases: (i) \( D < \frac{1+p}{2} \) when \( l_* = N_B p_0^2 \ll 1 \) and, thus, the probability distribution is nearly bimodal

\[
P(\rho^A_{M,M'} = x) \, dx \simeq (1 - l_*) \delta(x) \, dx + l_* \bar{P}_l(N^D x) N^D \, dx,
\]

(B15)

and (ii) \( D > \frac{1+p}{2} \) when \( l_* = N_B p_0^2 \gg 1 \) and CLT works giving

\[
P(\rho^A_{M,M'}) = \frac{e^{-\rho^A_{M,M'}^2/(2\sigma^2 N^{-1-p})}}{\sqrt{2\pi} N^{-1-p}}.
\]

(B16)

Here we used the fact that \( \bar{P}_l(z) = \bar{P}_l(-z) \) is symmetric and thus there is no drift in CLT.

The latter analysis confirms the scaling of the off-diagonal elements, Eq. (A2), as well as the number of non-zero off-diagonals, Eq. (A1). Indeed, for \( D > \frac{1+p}{2} \)

\[
\rho^A_{M,M'} \sim N^{-1-p},
\]

(B17)

thus, \( D_* = 1 \) and \( D_{off} = \frac{1+p}{2p} \).

In the opposite limit of \( D < \frac{1+p}{2} \) the distribution is bimodal giving the number of non-zeros

\[
N^{2pD_*} = N^2 l_* = N^{2D - 1+p}
\]

(B18)

as well as the mean value

\[
\langle \rho^A_{M,M'} \rangle = N^{-pD_{off}} = l_* N^{-D} = N^{D - 1-p}.
\]

(B19)

Appendix C: Sparseness of the reduced density matrix for non-ergodic states

Now, we provide an analytical argument to support the validity of the diagonal approximation in the regime in which \( \rho_A \) is sparse. As we are interested in the scaling of the Schmidt values with \( N \) compared to the one of diagonal elements \( \rho^A_{M,M} \), we have to consider two cases: (i) First, when the number of non-zero elements in each row is finite and does not grow with \( N \), the off-diagonal elements can be of the same order as the diagonal ones. (ii) Second, when there are many non-zero off-diagonals which are much smaller than \( \rho^A_{M,M} \).

1. Few non-zero off-diagonal elements \( \rho^A_{M,M'} \), \( D < 1/2 \)

As follows from Eq. (A1) there is at most \( O(1) \) non-zero off-diagonal elements in each row as soon as \( D < 1/2 \) (the total number of off-diagonals \( \sim N_A \)).

In this case, we can show that in terms of multifractal scaling with the total Hilbert space dimension \( N \) in the above regime the Schmidt values \( \lambda_M \) scale in the same way as the diagonal elements of \( \rho_A \) and, thus, EE can be approximated by its diagonal counterpart [59, 60]

\[
\Sigma_q(p) = \frac{\ln \Sigma_q}{1 - q}, \quad \Sigma_q = \sum_M \lambda^q_M \simeq \sum_M \langle \rho^A_{M,M} \rangle^q.
\]

(C1)

Indeed, if in each row of \( \rho_A \) there are only few significantly non-zero off-diagonal matrix elements (say, for \( M \text{th} \) and \( M' \text{th} \) diagonals), then Schmidt eigenvalues can be approximated by diagonalizing a 2-by-2 matrix

\[
\begin{pmatrix}
\rho^A_{M,M} & \rho^A_{M,M'} \\
\rho^A_{M',M} & \rho^A_{M',M'}
\end{pmatrix}
\]

(C2)

Assuming the following scaling \( \rho^A_{M,M} \sim N^{-\alpha_M} \), and \( \rho^A_{M,M'} \sim N^{-\beta} \), with \( \alpha_M \leq \alpha_{M'} \) without loss of gener-
ality, we obtain for the corresponding Schmidt values
\[
\lambda_{M/M'} = \frac{N^{-\alpha_M} + N^{-\alpha_{M'}} + \sqrt{(N^{-\alpha_M} + N^{-\alpha_{M'}})^2 + 4N^{-2\beta}}}{2}
\]
The latter approximation is based on the inequality \(\beta \geq (\alpha_M + \alpha_{M'})/2\) leading from the Cauchy-Bunyakovski-Schwarz inequality for the off-diagonal element by the geometric mean of diagonals
\[
|\rho_{M,M'}^A| = \left| \sum_{m=1}^{\mathcal{N}_R} \psi_{M,m} \psi_{M',m}^* \right| \leq \sqrt{\sum_{m=1}^{\mathcal{N}_R} |\psi_{M,m}|^2 \sum_{m'=0}^{\mathcal{N}_R} |\psi_{M',m'}|^2} = \sqrt{\rho_{M,M'}^A \rho_{M',M'}^A}.
\]
As a result, the scaling of Schmidt values \(\lambda_M\) with \(N\) is shown to be the same as for the diagonal elements \(\rho_{M,M}\) in the nearly diagonal sparse regime of \(\rho_A\) \((D < 1/2)\). In next sections we will use this fact to calculate the Renyi and entanglement entropies.

2. Many non-zero off-diagonal elements \(\rho_{M,M'}^A\), \((D > 1/2)\)

In the case of \(D > 1/2\) there is an extensive number of non-zero off-diagonal elements of the reduced density matrix. In order to estimate them we assume their statistical independence from each other and from the diagonal elements following the case \(D = 1\) considered in [53].

In the case of \(D > 1/2\) both diagonal and off-diagonal elements of the reduced density matrix are homogeneously distributed and the latter has the form similar to the Rosenzweig-Porter random matrix ensemble [71]. Then the Schmidt eigenspectrum is not affected by the off-diagonal elements when [78]
\[
|\rho_{M,M'}^A| \sim N^{-\frac{1-\beta}{2}} \ll N^{-p/2},
\]
which is the case as \(p \leq 1/2\).

In general for \(D > 1/2\) one can apply the Mott’s principle of delocalization [79] recently generalized in [65] which reads for \(N_A \times N_A\) matrix, \(N_A = N^p\), as follows: the spectrum is not affected by the off-diagonal elements as soon as
\[
N^p \frac{|\rho_{M,M'}^A|^2}{|\rho_{M,M}^A|^2} \ll 1.
\]
In our case it leads to
\[
N^p \frac{|\rho_{M,M'}^A|^2}{|\rho_{M,M}^A|^2} \sim \begin{cases} N^{2D-2+p}, & D < \frac{1+p}{2} \\ N^{2-p}, & D > \frac{1+p}{2} \end{cases},
\]
and works for any \(0 \leq D \leq 1\).

**Appendix D: Entanglement entropy for fractal states**

In this section we consider mean and fluctuations of Renyi and von Neumann EE within the approximations of two previous sections.

The simplest way to calculate the Renyi entropy, Eq. (C1), in the diagonal approximation
\[
\lambda_M \simeq \rho_{M,M}^A = \sum_{m=1}^{\mathcal{N}_R} |\psi_{M,m}|^2
\]
is to use the probability distributions, Eq. (B8), and, Eq. (B9). Indeed,
\[
\Sigma_q = N_A \left( \rho_{M,M}^A \right)^q = \begin{cases} N^{D(1-q)}, & D < p \\ N^{p(1-q)}, & D > p \end{cases},
\]
leading straightforwardly to Eq. (8) of the main text.

The fluctuations can be also estimates from the moments as soon as the variance
\[
\sigma_{\Sigma}^2 = \Sigma_{\Sigma}^2 - \Sigma_{\Sigma}^{-2} = N_A \left[ \left( \rho_{M,M}^A \right)^q - \left( \rho_{M,M}^A \right)^{q^2} \right]
\]
is small compared to \(\Sigma_{\Sigma}^{-2}\). Indeed, as
\[
(1-q)\Sigma_q(p) = \ln \Sigma_q = \ln \Sigma_{\Sigma} + \ln \left[ 1 + \frac{\sigma_{\Sigma}^2}{\Sigma_{\Sigma}} g_q \right] \sim \ln \Sigma_{\Sigma} + \frac{\sigma_{\Sigma}^2}{2 \Sigma_{\Sigma}} g_q^2
\]
it gives
\[
(1-q)\Sigma_q(p) = \ln \Sigma_q - \frac{\sigma_{\Sigma}^2}{2 \Sigma_{\Sigma}} g_q^2 \simeq \ln \Sigma_q
\]
within the leading approximation, and
\[
(1-q)^2 \left[ \Sigma_q^2(p) - \Sigma_q(p)^2 \right] = \frac{\sigma_{\Sigma}^4}{\Sigma_{\Sigma}^2} .
\]
Here we introduced dimensionless variable \(g_q = \frac{\Sigma_{\Sigma} - \Sigma_q}{\sigma_{\Sigma}}\) with zero mean and unit variance
\[
\bar{g}_q = 0, \quad g_q^2 = 1.
\]
In our case one obtains
\[
\frac{\sigma_{\Sigma}^2}{\Sigma_{\Sigma}} = N^{-D} ,
\]
giving the correct approximation for \(D < p\).
1. **Alternative way to calculate entanglement entropies**

Alternatively in the main text we parameterize Schmidt values as follows

\[ \lambda_M \approx \rho_{M,M}^A = \sum_{m=1}^{N_B} |\psi_{M,m}|^2 = g_M/N^D, \]  
(D9)

where \( 0 \leq g_M \leq N^D \) are integer values summed to the support set \( N^D \):

\[ \sum_{M=1}^{N_A} g_M = N^D. \]  
(D10)

The entanglement entropy in this case can be estimated as the logarithm of the number \( N_0 \) of non-zero \( g_M \)

\[ S_1 \sim \ln N_0. \]  
(D11)

As we show below, this approximation is good for mean EE for any \( D \), but fails to capture fluctuations for \( D > p \).

The probability distribution \( P_{N_0} \) of \( N_0 \) can be calculated combinatorially in the assumption of homogeneous distribution of \( g_M \)'s. Indeed, the total number of combinations of \( N_A \) values of \( g_M \), \( 1 \leq M \leq N_A \), taken with repetitions \( (g_M \) can be larger than 1) and with the normalization Eq. (D10) is given by

\[ \mathcal{M} = \left( \frac{N_A + N^D - 1}{N^D} \right). \]  
(D12)

At the same time the combinations with \( N_0 \) non-zero \( g_M \) can be counted as the number of combination to realize \( N_0 \) non-zeros

\[ \mathcal{M}_{N_0} = \left( \frac{N^D - 1}{N_0 - 1} \right) \]  
(D13)

\[ \mathcal{M}_{N_0} = \left( \frac{N_A}{N_0} \right). \] \hspace{1cm} (D14)

As a result

\[ P_{N_0} = \frac{\mathcal{M}_{N_0} \mathcal{M}}{\mathcal{M}} \approx A(N) e^{N_A f(\rho)}, \]  
(D15)

where

\[ f(\rho, \alpha) = -2\rho \ln(\rho) - (\alpha - \rho) \ln(\alpha - \rho) - (1 - \rho) \ln(1 - \rho), \] \hspace{1cm} (D16)

\( N^D = \alpha N_A \), and \( 0 \leq \rho = N_0/N_A \leq 1, \alpha \) and we neglected \(-1\) comparing both to \( N^D \) and \( N_0 \). The expression for \( f(\rho) \) is calculated in the large-\( N \) limit with help of Stirling’s approximation.

The maximum of \( f(\rho) \) is achieved at the typical \( N_0^* = N_A \rho^* \) with

\[ \rho^* = \frac{\alpha}{1 + \alpha} < 1, \]  
(D17)

leading to \( N_0^* = \frac{N_A N^D}{N_A + N^D} \) from the main text.

The relative fluctuations \( \delta N_0/N_0^* = \delta \rho/\rho^* \) can be written in the following form

\[ \frac{\delta N_0}{N_0^*} = \frac{\delta \rho_0}{\rho_0^*} = \frac{1}{\sqrt{-N_A f''(\rho^*)\rho^*_0}} = (N_A + N^D)^{-1/2} \]  
(D18)

in the Gaussian approximation

\[ P_{N_0} = \frac{e^{-(N_A + N^D)(\rho - \rho^*)^2/2}}{\sqrt{2\pi/(N_A + N^D)}}, \]  
(D19)

derived from Eq. (D15) and Eq. (D16) provided \( \rho^* \gg (N_A + N^D)^{-1/2} \).

In the same approximation

\[ \ln N_0 = \ln N_0^* - \frac{1}{2(N_A + N^D)}, \]  
(D20a)

\[ \ln^2 N_0 = \ln^2 N_0^* - \frac{1 - \ln N_0^*}{(N_A + N^D)}. \]  
(D20b)

According to Eq. (D11) and Eq. (D20a) mean EE is given by

\[ \mathcal{S}_1 \approx \ln N_0 = \ln N_0^* - \frac{1}{2(N^D + N_A)} \sim \begin{cases} \ln N = D \ln N, & \text{for } D_1 < p \\ \ln N_A = p \ln N, & \text{for } D_1 > p. \end{cases} \]  
(D21)

In the latter equality we neglected subleading terms.

At the same time according to Eq. (D20a) and Eq. (D20b) EE fluctuations are given mostly by the relative fluctuations of \( N_0 \)

\[ \mathcal{S}_1 - \mathcal{S}_1^2 \approx \ln^2(N_0) - \ln(N_0)^2 \approx \frac{1}{N^D + N_A}. \]  
(D22)

As mentioned above the approximation Eq. (D11) works both for the mean and fluctuations provided \( D < p \). This is the case as well for all Renyi entropies. It is caused by the fact that \( N_0 \approx N^D \) and, thus, all \( g_M \sim O(1) \) leading to

\[ \sum_q^{M} = \sum_{M} \left( \frac{g_M}{N^D} \right)^q \sim N_0 N^{-D q} = N^{D(1 - q)} \]  
(D23)

and

\[ \sum_q^{M} - \sum_q^{2} = \sum_{M} \left( \frac{g_M}{N^D} \right)^2 - \left( \frac{g_M}{N^D} \right)^2 \sim N^{D(1 - 2q)}. \]  
(D24)
However, in the opposite case \( D > p \) when \( N_0 \simeq N_A \ll N^D \) there is a non-trivial distribution of \( g_M \) with \( g_M = \frac{N^D}{N_A} \gg 1 \) and

\[
\ln N_0 \neq \ln \left[ \sum_M \left( \frac{g_M}{N^D} \right)^q \right] \quad \frac{1}{1-q}.
\] (D25)

Nevertheless, as we have shown in Sec. A, on average both sides of the latter equation give the same Page value as Eq. (8) in the main text.