UNMIXED $r$-PARTITE GRAPHS

REZA JAFARPOUR-GOLZARI AND RASHID ZAARE-NAHANDI

Abstract. Unmixed bipartite graphs have been characterized by Ravada and Villarreal independently. Our aim in this paper is to characterize unmixed $r$-partite graphs under a certain condition, which is a generalization of Villarreal’s theorem on bipartite graphs. Also we give some examples and counterexamples in relevance this subject.

1. Introduction

In the sequel, we use [4] as reference for terminology and notation on graph theory.

Let $G$ be a simple finite graph with vertex set $V(G)$ and edge set $E(G)$. A subset $C$ of $V(G)$ is said to be a vertex cover of $G$ if every edge of $G$, is adjacent with some vertices in $C$. A vertex cover $C$ is called minimal, if there is no proper subset of $C$ which is a vertex cover. A graph is called unmixed, if all minimal vertex covers of $G$ have the same number of elements. A subset $H$ of $V(G)$ is said to be independent, if $G$ has not any edge $\{x, y\}$ such that $\{x, y\} \subseteq H$. A maximal independent set of $G$, is an independent set $I$ of $G$, such that for every $H \supseteq I$, $H$ is not an independent set of $G$. Notice that $C$ is a minimal vertex cover if and only if $V(G) \setminus C$ is a maximal independent set. A graph $G$ is called well-covered if all the maximal independent sets of $G$ have the same cardinality. Therefore a graph is unmixed if and only if it is well-covered. The minimum cardinality of all minimal vertex covers of $G$ is called the covering number of $G$, and the maximum cardinality of all maximal independent sets of $G$ is called the independence number of $G$.
For determining the independence number see [6]. For relation between unmixedness of a graph and other graph properties see [1, 5, 9, 12].

Well-covered graphs were introduced by Plummer. See [7] for a survey on well-covered graphs and properties of them. For an integer \( r \geq 2 \), a graph \( G \) is said to be \( r \)-partite, if \( V(G) \) can be partitioned into \( r \) disjoint parts such that for every \( \{x, y\} \in E(G) \), \( x \) and \( y \) do not lie in the same part. If \( r = 2, 3 \), \( G \) is said to be bipartite and tripartite, respectively.

Let \( G \) be an \( r \)-partite graph. For a vertex \( v \in V(G) \), let \( N(v) \) be the set of all vertices \( u \in V(G) \) where \( \{u, v\} \) be an edge of \( G \). Let \( G \) be a bipartite graph, and let \( e = \{u, v\} \) be an edge of \( G \). Then \( G_e \) is the subgraph induced on \( N(u) \cup N(v) \). If \( G \) is connected, the distance between \( x \) and \( y \) where \( x, y \in V(G) \), denoted by \( d(x, y) \), is the length of the shortest path between \( x \) and \( y \). A set \( M \subseteq E(G) \) is said to be a matching of \( G \), if for any two \( \{x, y\}, \{x', y'\} \in M \), \( \{x, y\} \cap \{x', y'\} = \emptyset \). A matching \( M \) of \( G \) is called perfect if for every \( v \in V(G) \), there exists an edge \( \{x, y\} \in M \) such that \( v \in \{x, y\} \). A clique in \( G \) is a set \( Q \) of vertices such that for every \( x, y \in Q \), if \( x \neq y \), \( x, y \) lie in an edge. An \( r \)-clique is a clique of size \( r \).

Unmixed bipartite graphs have already been characterized by Ravindra and villarreal in a combinatorial way independently [8, 11]. Also these graphs have been characterize in an algebraic method [10].

In 1977, Ravindra gave the following criteria for unmixedness of bipartite graphs.

**Theorem 1.1.** [8] Let \( G \) be a connected bipartite graph. Then \( G \) is unmixed if and only if \( G \) contains a perfect matching \( F \) such that for every edge \( e = \{x, y\} \in F \), the induced subgraph \( G_e \) is a complete bipartite graph.

Villarreal in 2007, gave the following characterization of unmixed bipartite graphs.

**Theorem 1.2.** [11, Theorem 1.1] Let \( G \) be a bipartite graph without isolated vertices. Then \( G \) is unmixed if and only if there is a partition \( V_1 = \{x_1, \ldots, x_g\}, V_2 = \{y_1, \ldots, y_g\} \) of \( G \) such that: (a) \( \{x_i, y_i\} \in E(G) \), for all \( i \), and (b) if \( \{x_i, y_j\} \) and \( \{x_j, y_k\} \) are in \( E(G) \), and \( i, j, k \) are distinct, then \( \{x_i, y_k\} \in E(G) \).

H. Haghighi in [3] gives the following characterization of unmixed tripartite graphs under certain conditions.
Theorem 1.3. [3, Theorem 3.2] Let $G$ be a tripartite graph which satisfies the condition (*). Then the graph $G$ is unmixed if and only if the following conditions hold:

1. If $\{u_i, x_q\}, \{v_j, y_q\}, \{w_k, z_q\} \in E(G)$, where no two vertices of $\{x_q, y_q, z_q\}$ lie in one of the tree parts of $V(G)$ and $i, j, k, q$ are distinct, then the set $\{u_i, v_j, w_k\}$ contains an edge of $G$.

2. If $\{r, x_q\}, \{s, y_q\}, \{t, z_q\}$ are edges of $G$, where $r$ and $S$ belong to one of the three parts of $V(G)$ and $t$ belongs to another part, then the set $\{r, s, t\}$ contains an edge of $G$ (here $r$ and $s$ may be equal).

In the above theorem, he has considered the condition (*) as:

being a tripartite graph with partitions $U = \{u_1, \ldots, u_n\}, V = \{v_1, \ldots, v_n\}, W = \{w_1, \ldots, w_n\}$, in which $\{u_i, v_i\}, \{u_i, w_i\}, \{v_i, w_i\} \in E(G)$, for all $i = 1, \ldots, n$.

Also to simplify the notations, he has used $\{x_i, y_i, z_i\}$ and $\{r_i, s_i, t_i\}$ as two permutations of $\{u_i, v_i, w_i\}$.

We give a characterization of unmixed $r$-partite graphs under certain condition which we name it (*) (see Theorem 2.3).

In both theorems 2.1 and 2.2 in an unmixed connected bipartite graph, there is a perfect matching, with cardinality equal to the cardinality of a minimal vertex cover, i.e. $\frac{|V(G)|}{2}$. An unmixed graph with $n$ vertices such that its independence number is $\frac{n}{2}$, is said to be very well-covered. The unmixed connected bipartite graphs are contained in the class of very well-covered graphs. A characterization of very well-covered graphs is given in [2].

2. A generalization

By the following proposition, bipartition in connected bipartite graphs is unique.

Proposition 2.1. Let $G$ be a connected bipartite graph with bipartition $\{A, B\}$, and let $\{X, Y\}$ be any bipartition of $G$. Then $\{A, B\} = \{X, Y\}$.

Proof. Let $x \in A$ be an arbitrary vertex of $G$. Then $x \in X$ or $x \in Y$, without loss of generality let $x$ be in $X$. Let $a \in A$. then $d(x, a)$ is even. Then $a$ and $x$ are in the same part (of partition $\{X, Y\}$). Then $A \subseteq X$, ...
and by the same argument we have $X \subseteq A$. Therefore $A = X$, and then $\{A, B\} = \{X, Y\}$. □

The above fact for bipartite graphs, is not true in case of tripartite graphs, as shown in the following example.

In the above graph there are two different tripartitions:

$$\{\{a_1, a_2, a_3\}, \{a_4, a_5\}, \{a_6\}\}$$

and

$$\{\{a_1, a_2\}, \{a_4, a_5\}, \{a_3, a_6\}\}.$$

A natural question refers to find criteria which characterize a special class of unmixed $r$-partite ($r \geq 2$) graphs.

In the above two characterizations of bipartite graphs, having a perfect matching is essential in both proofs. This motivates us to impose the following condition.

*We say a graph $G$ satisfies the condition $(\ast)$ for an integer $r \geq 2$, if $G$ can be partitioned to $r$ parts $V_i = \{x_{1i}, \ldots, x_{ni}\}, (1 \leq i \leq r)$, such that for all $1 \leq j \leq n$, $\{x_{j1}, \ldots, x_{jr}\}$ is a clique.*

**Lemma 2.2.** Let $G$ be a graph which satisfies $(\ast)$ for $r \geq 2$. If $G$ is unmixed, then every minimal vertex cover of $G$, contains $(r - 1)n$ vertices. Moreover the independence number of $G$ is $n = \frac{|V(G)|}{r}$

*Proof.* Let $C$ be a minimal vertex cover of $G$. Since for every $1 \leq j \leq n$, the vertices $x_{j1}, \ldots, x_{jr}$ are in a clique, $C$ must contain at least $r - 1$ vertices in $\{x_{j1}, \ldots, x_{jr}\}$. Therefore $C$ contains at least $(r - 1)n$ vertices. By hypothesis $\bigcup_{i=1}^{r-1} V_i$ is minimal vertex cover with $(r - 1)n$ vertices, and $G$ is unmixed. Then every minimal vertex cover of $G$ contains exactly $(r - 1)n$ elements. The last claim can be concluded from this fact that the complement of a minimal vertex cover, is an independent set. □

Now we are ready for the main theorem.
Theorem 2.3. Let $G$ be an $r$-partite graph which satisfies the condition $(\ast)$ for $r$. Then $G$ is unmixed if and only if the following condition holds: For every $1 \leq q \leq n$, if there is a set $\{x_{k_1s_1}, \ldots, x_{k_rs_r}\}$ such that

$$x_{k_1s_1} \sim x_{q1}, \ldots, x_{k_rs_r} \sim x_{qr},$$

then the set $\{x_{k_1s_1}, \ldots, x_{k_rs_r}\}$ is not independent.

Proof. Let $G$ be an arbitrary $r$-partite graph which satisfies the condition $(\ast)$ for $r$.

Let $G$ be unmixed. We prove that mentioned condition holds. Assume the contrary. Let

$$x_{k_1s_1} \sim x_{q1}, \ldots, x_{k_rs_r} \sim x_{qr},$$

but the set $\{x_{k_1s_1}, \ldots, x_{k_rs_r}\}$ is independent. Then there is a maximal independent set $M$, such that $M$ contains this set. Since $M$ is maximal, $C = V(G) \setminus M$ is a minimal vertex cover of $G$. Since the set $\{x_{k_1s_1}, \ldots, x_{k_rs_r}\}$ is contained in $M$, then its elements are not in $C$, and since $C$ is a cover of $G$, then all vertices $x_{qi}$, $(1 \leq i \leq r)$ are in $C$. But by Lemma 3.2, every minimal vertex cover, contains $n - 1$ vertices of clique $q$ th, a contradiction.

Conversely let the condition hold. We have to prove that $G$ is unmixed. We show that all minimal vertex covers of $G$, intersect the set $\{x_{q1}, \ldots, x_{qr}\}$ in exactly $r - 1$ elements (for every $1 \leq q \leq n$). Let $C$ be a minimal vertex cover and $q$ be arbitrary. Since $C$ is a vertex cover and $\{x_{q1}, \ldots, x_{qr}\}$ is a clique, then $C$ intersects this set at least in $r - 1$ elements. Let the contrary. Let the cardinality of $C \cap \{x_{q1}, \ldots, x_{qr}\}$ be $r$. Attending to minimality of $C$, for every $1 \leq i \leq r, N(x_{qi})$ contains at least one element, distinct from the elements of $\{x_{q1}, \ldots, x_{qr}\} \setminus \{x_{qi}\}$, which is not in $C$, because we can not remove $x_{qi}$ of cover. Let this element be $x_{k_is_i}$ where $s_i \neq i$ and $k_i \neq q$. Then $x_{k_is_i} \notin C$ and $\{x_{k_is_i}, x_{qi}\}$ is in $E(G)$. There is at least two elements $i$ and $j$ such that $1 \leq i < j \leq r$ and $s_i \neq s_j$, because $x_{qi}$ can not choose its adjacent vertex from the part $i$. Therefore the set $\{x_{k_is_1}, \ldots, x_{k_is_r}\}$ contain at least two elements. Then by hypothesis, at least two elements, say $a, b$ of $\{x_{k_is_1}, \ldots, x_{k_is_r}\}$ are adjacent by an edge. Now $C$ is a cover but $a, b$ are not in $C$, a contradiction.

Remark 2.4. Villareal’s theorem (Theorem 1.2) for bipartite graphs, and Haghighi’s theorem (Theorem 1.3) for tripartite graphs, are special cases of Theorem 2.3 (where $r = 2$, and $r = 3$).
3. Examples and counterexamples

In this section, we give examples of two classes of unmixed graphs, and an example which shows that it is not necessary that an unmixed \( r \)-partite graph satisfies condition (*)..

**Example 3.1.** By Theorem 2.3, the following 4-partite graphs are unmixed.

\[
\begin{align*}
&\begin{array}{c}
\text{Graph 1} \\
\text{Graph 2}
\end{array} \\
&\begin{array}{c}
(t_2, x_1, y_1, z_1) \\
(x_2, y_2, z_2, t_2)
\end{array}
\end{align*}
\]

In each of the above graphs, there are two complete graphs of order 4 and some edges between them.

For \( r > 4 \), also \( r = 3 \), using two complete graphs of order \( r \), we can construct \( r \)-partite unmixed graphs which are natural generalization of the above graphs.

**Example 3.2.** For every \( n \), \( n \geq 3 \), the complete graph \( K_n \), is an \( n \)-partite graph which satisfies the condition (*). By Theorem 2.3, \( K_n \) is unmixed.

Theorem 2.3 does not characterize all unmixed \( r \)-partite graphs. More precisely, the condition (*) is not valid for all unmixed graphs. In the following, we give an example of an unmixed \( r \)-partite graph which does not satisfy the condition (*).

**Example 3.3.** The following graph is a 4-partite graph with partition \( \{y_1\}, \{y_2, y_4\}, \{y_3\}, \) and \( \{y_5, y_6\} \). This graph does not satisfy the condition (*) because 6 is not a multiple of 4.
We show that this graph is unmixed. Let $C$ be an arbitrary minimal vertex cover of $G$. We show that $C$ is of size 4.

Since $C$ is a cover, it selects at least one element of $\{y_4, y_6\}$. Now we consider the following cases:

**case 1:** $y_6 \in C$ and $y_4 \notin C$. In this case, since $C$ is a vertex cover, $y_1, y_3, y_5 \in C$. Now $\{y_1, y_3, y_5, y_6\}$ is a vertex cover of $G$, and since $C$ is minimal, $C = \{y_1, y_3, y_5, y_6\}$.

**case 2:** $y_4 \in C$ and $y_6 \notin C$. In this case, $y_2, y_3 \in C$, and at least one vertex of $y_1, y_5$ and by minimality, only one is in $C$. Now since $\{y_2, y_3, y_4, y_i\}$ where $i \in \{1, 5\}$ is one of two vertices $y_1$ and $y_5$, is a cover of $G$, by minimality of $C$, $C = \{y_2, y_3, y_4, y_i\}$.

**case 3:** $y_4, y_6 \in C$. In this case, at least one of two vertices $y_1, y_5$ and by minimality of $C$, only one is in $C$. Now if $y_5 \in C$, $y_3$ should be in $C$ (because the edge $\{y_1, y_3\}$ should be covered). Also $y_2 \in C$ (because the edge $\{y_1, y_2\}$ should be covered). Now $\{y_2, y_3, y_4, y_6\}$ is a cover, and since $C$ is minimal, $C = \{y_2, y_3, y_4, y_6\}$, that is a contradiction because $y_6$ can be removed. If $y_1 \in C$, at least one of $y_2$ and $y_3$, and by minimality only one, is in $C$. Now since $\{y_1, y_4, y_6, y_j\}$, where $j \in \{2, 3\}$ is one of two vertices $y_2$ and $y_3$, is a vertex cover, by minimality of $C$, $C = \{y_1, y_4, y_6, y_j\}$.

**References**

[1] M. Estrada and R. H. Villarreal, Cohen-Macaulay bipartite graphs, *Arc. Math.*, 68 (1997), 124-128.

[2] O. Fanaron, Very well covered graphs, *Discrete. Math.*, 42 (1982), no. 2-3, 177-187.

[3] H. Haghighi, A generalization of Villarreal’s result for unmixed tripartite graphs, *Bull. Iranian Math. Soc.*, 40 (2014), no. 6, 1505-1514.

[4] F. Harary, *Graph Theory*, Addison-Wesley, Reading, MA, 1972.
[5] J. Herzog and T. Hibi, Distributive lattices, bipartite graphs, and Alexander duality, *J. Algebraic Combin.*, 22 (2005), no. 3, 289-302.

[6] R. M. Karp, Complexity of computer computation, *Plenum Press*, New York, (1972), 85-103.

[7] M. D. Plummer, Well-covered graphs: A survey, *Questions Math.*, 16 (1993), no. 3, 253-287.

[8] G. Ravindra, Well-covered graphs, *J. Combin. Inform. , System Sci.* 2 (1977), no. 1, 20-21.

[9] R. H. Villarreal, Cohen-Macaulay graphs, *Manuscripta Math.*, 66 (1990), 277-293.

[10] R. H. Villarreal, *Monomial Algebras*, Marcel Dekker, Inc. New York, 2001.

[11] R. H. Villarreal, Unmixed bipartite graphs, *Rev. Colombiana Mat.*, 41 (2007), no. 2, 393-395.

[12] R. Zaree-Nahandi, Pure simplicial complexes and well-covered graphs, *Rocky Mountain Journal of Mathematics*, 45 (2015), no. 2, 695-702.