Note On The Maximal Prime Gaps
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**Abstract:** This note presents a result for the prime gap of the form $p_{n+1} - p_n \leq c \log (p_n)^{1+\epsilon}$, where $c > 0$ is a constant, for any arbitrarily small real number $\epsilon > 0$, and all sufficiently large integers $n \geq n_0(\epsilon)$. Equivalently, the result shows that short intervals $[x, x + y]$ contain prime numbers for all sufficiently large real numbers $x \geq x_0(\epsilon)$, and $y \geq c \log(x)^{1+\epsilon}$ unconditionally. An application demonstrates that a prime $p \geq x \geq 2$ can be determined in deterministic polynomial time $O \left( \log(x)^8 \right)$.

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**Keywords:** Prime Number, Prime Gap, Short Interval, Prime Complexity, Primality Test, Deterministic Polynomial Time.

1 Introduction

Let $n \geq 2$ be an integer, and let $p_n \geq 2$ denotes the $n$th prime number. The $n$th prime gap is defined as the difference $d_n = p_{n+1} - p_n$ of two consecutive prime numbers. This note investigates the maximal size of the prime gap function. For a large real number $x \geq 1$, the maximal prime gap function is defined by

$$d_{\text{max}} = \max \{ p_{n+1} - p_n \leq x \}.$$  \hfill (1)

The average prime gap for a short sequence of consecutive primes $2, 3, 5, 7, ..., p_n, p_{n+1} \leq x$ is given by the asymptotic formula

$$\frac{x}{\pi(x)} = \frac{1}{x} \sum_{p_n \leq x} (p_{n+1} - p_n) = \log x + O(\log \log x).$$ \hfill (2)

This immediately shows that $d_{\text{max}} \geq \log x + O(\log \log x)$. Some of the leading prime gaps open problems provide upper bounds and lower bounds of the maximal prime gap.

**Cramer Prime Gap Problem.** This problem claims that

$$p_{n+1} - p_n \leq c_0 \log (p_n)^2$$ \hfill (3)

for some constant $c_0 > 0$, see [6]. The precise implied constant in this formula has been debated in the literature, this conjecture is discussed in [15].

**Erdos-Rankin Prime Gap Problem.** This problem claims that

$$p_{n+1} - p_n \geq c_1 \log (p_n) \log_2 (p_n) \log_4 (p_n) \log_3 (p_n)^{-2}$$ \hfill (4)

for some constant $c_1 > 0$, and that the constant cannot be replaced with any increasing function of $n \geq 1$, see [9], [10], [22], [33]. The notation $\log_k(x) = \log(\cdots \log(x) \cdots)$ denotes the $k$-fold iterations of the logarithm.
The Cramer conjecture provides an upper bound of the maximal prime gap function \( d_{\text{max}} \leq c_0 (\log p_n)^2 \). In comparison, the Erdos conjecture provides a lower bound of the maximal size of the prime gap function \( d_{\text{max}} \geq c_1 \log (p_n) \log_2 (p_n) \log_4 (p_n) \log_3 (p_n)^{-2} \). Extensive details on the theory of maximal prime gap and primes in short intervals are given in [34], [3], [18], [35], [36], [13], [15], [14], [30], et alii. Some information on the calculations of the ratio \( \lim_{n \to \infty} d_n / \log p_n \), called the merit factor, appears in [29], p. 340, and [14].

Note: A pair of new results, proved by independent authors, have disproved the Erdos-Rankin prime gap problem, see [11] and [23].

2 The Proof of the Result

The numerical evidence, [28], [31], and the result within suggest that the Erdos conjecture is closer to the correct order of magnitude of the maximal prime gap function:

\[
\frac{c_1 \log (p_n) \log_2 (p_n) \log_4 (p_n)}{\log_3 (p_n)^2} \leq d_{\text{max}} \leq \log (p_n)^{1+\epsilon}
\]

as seemed to be confirmed below.

**Theorem 1.** Let \( \epsilon > 0 \) be an arbitrarily small real number, and let \( n \geq 1 \) be an integer. Then, the \( n \)th prime gap satisfies

\[
p_{n+1} - p_n \leq c \log (p_n)^{1+\epsilon}
\]

where \( c > 0 \) is a constant, for all sufficiently large integer \( n \geq n_0(\epsilon) \) unconditionally.

**Proof:** Given a small real number \( \epsilon > 0 \), let \( n \in \mathbb{N} \) be an integer, and let \( p_{n+1} = p_n + d_n \). By Lemma 4, in Section 3, the inequality

\[
(p_n + d_n)^n = p_{n+1}^n < p^n_{n+1} < p_{n+1}^{n + (\log n)^{-1+\epsilon}}
\]

is satisfied for all sufficiently large integer \( n \geq n_0(\epsilon) \). Moreover, taking logarithm of both sides, and simplifying yield

\[
c_0 \frac{nd_n}{p_n} \leq n \log \left( 1 + \frac{d_n}{p_n} \right) < ((\log n)^{-1+\epsilon}) \log p_n.
\]

Equivalently, it can be rewritten in the simpler form

\[
d_n \leq c_1 \frac{p_n \log p_n (\log n)^{\epsilon}}{n \log n},
\]

where \( c_0, c_1 > 0 \) are constants. Apply the asymptotic formula for the \( n \)th prime \( p_n = (1+o(1))n \log n \), see [17], and the references [8], [29], p. 254, and simplify to obtain

\[
p_{n+1} - p_n \leq c \log (p_n)^{1+\epsilon}
\]

where \( c > 0 \) is a constant.
The result in Theorem 1 is equivalent to the existence of prime numbers in very short intervals \([x, x+y]\) for all sufficiently large real numbers \(x \geq x_0(\epsilon)\) and \(y \geq c \log(x)^{1+\epsilon}\), unconditionally. This assertion follows from the inequality
\[
p_{n+1} - p_n \leq c \log (p_n)^{1+\epsilon} \leq c_2 \log(x)^{1+\epsilon},
\]
with \(c_2 > 0\) constant. Here \(p_n \leq x\), and \(p_{n+1} \leq x + c_2 \log(x)^{1+\epsilon}\). The prime pairs with large prime gaps near \(\log (p_n) \log_2 (p_n)\) are studied in [26], and large prime gaps near \(q \log (p_n) \log_2 (p_n)\) in arithmetic progressions \(\{qn + a : \gcd(a, q) = 1, n \in \mathbb{N}\}\) are studied in [16].

### 2.1 Density of Primes in Short Intervals

The density of primes on large intervals \([x, x+y]\), where \(x \geq x_0(\epsilon)\) and \(y \geq cx^{1/2+\epsilon}\), has the expected order of magnitude
\[
\pi(x+y) - \pi(x) = y/\log x + o(x/\log x),
\]
see [17, p. 70], and [17]. In contrast, the precise form of the counting function for primes in short intervals with \(y < cx^{1/2+\epsilon}\) remains unknown. In fact, the density of prime numbers in very short intervals does not satisfy (12) since it has large fluctuations. More precisely, for any \(m > 1\), the inequalities
\[
\log^{m-1} x < \pi(x + \log^m x) - \pi(x) \quad \text{and} \quad \log^{m-1} x > \pi(x + \log^m x) - \pi(x)
\]
occur infinitely often as \(x \to \infty\), tends to infinity, see [26].

**Corollary 2.** Let \(\epsilon > 0\) be an arbitrarily small real number, let \(m > 1\) be an integer, and let \(x \geq x_0(\epsilon)\) be a sufficiently large real number. Then, there are at least
\[
\log^{m-1-\epsilon} x < \pi(x + \log^{m+\epsilon} x) - \pi(x)
\]
prime numbers in the very short interval \([x, x + \log^{m+\epsilon} x]\).

A related problem is the resolution of the correct inequality \(\log^{m-1-\epsilon} x < \pi(x + \log^{m+\epsilon} x) - \pi(x)\) for \(x > y > 1\) as \(x \to \infty\). This is an important problem in the theory of primes numbers, known as the Hardy-Littlewood conjecture. Equivalently, it asks whether or not the intervals \([y, y+x]\) of lengths \(x \geq 1\) contain more or less primes than the intervals \([1, x]\) as \(x, y \to \infty\) tend to infinity, see [9, p. 6], [12, p. 24], [20].

The numerical calculations for a potential counterexample is beyond the reach of current machine computations, [17], p. 217. For small multiple \(kx = y\), there is an equality \(\pi(x) = \pi(y + x) - \pi(y)\).

**Maier-Pomerance Integer Gap Problem.** Let \(J(x) = \max_{n \leq x} j(n)\) be the maximal of the Jacobsthal function \(j(n)\). This problem claims that
\[
J(x) = O \left( (\log x)(\log \log x)^{2+o(1)} \right)
\]
confer [27, p. 202] and [19]. The relevance of this problem arises from the inequality \(J(x) \leq d_{\max}(x)\). Since \((\log x)(\log \log x)^{2+o(1)} \leq c(\log x)^{1+\epsilon}\) for any constants \(c > 0\), and \(\epsilon > 0\), the result in Theorem 1
is in line with this conjecture.

**Least Quadratic Nonresidue Problem.** Let \( q \geq 2 \), and let \( \chi \neq 1 \) be a character modulo \( q \), the least quadratic nonresidue is defined by \( n_\chi = \min\{n \in \mathbb{N} : \chi(n) \neq 0, 1\} \). Some partial new results for the least quadratic nonresidue for some characters are demonstrated in [4], Corollary 2, for example, \( n_\chi \ll (\log q)^{1.37+o(1)} \). This result is strikingly similar to the upper bound stated in Theorem 1.

### 3 Supporting Materials

The basic background information and a few Lemmas are provided in this section. The proofs of these Lemmas use elementary methods. At most a weak form of the Prime Number Theorem

\[
\pi(x) = \frac{x}{\log x} + o\left(\frac{x}{\log x}\right)
\]  

is required, see [34]. The innovation here is splicing together these elementary concepts into a coherent and effective result in the theory of primes numbers in very short intervals.

#### 3.1 Formula for the \( n \)th Prime, And Associated Series

The Prime Number Theorem (16) implies that the \( n \)th prime has an asymptotic expression of the form \( p_n = n \log n + O(n \log \log n) \). The Cipolla formula specifies the exact formula for the \( n \)th prime:

\[
p_n = (n \log n)^f(n),
\]  

where

\[
f_s(x) = a_s x^s + a_{s+1} x^{s+1} + \cdots + a_1 x + a_0 \in \mathbb{Q}[x]
\]  

is a polynomial, \( \log^s n = (\log n)^s \) is an abbreviated notation for the \( s \)th power of the logarithm or its iterated form respectively. For example,

\[
f_0(x) = x - 1, f_1(x) = x - 2, f_2(x) = -(x^2 - 6x + 11)/2.
\]  

The leading coefficient \( a_s = (-1)^{s+1}/s \) of the \( s \)th polynomial \( f_s(x) \) has alternating sign for \( s \geq 1 \). Other details on the asymptotic representation of the \( n \)th prime \( p_n \) are discussed in [2], p. 27, [5], and [8].

The function \( f(n) \) is quite similar to the simpler function

\[
g(x) = 1 + \log \log x \log x + \frac{1}{\log x} \sum_{s \geq 1} (-1)^{s+1} \frac{(\log \log x)^s}{s \log^s s} = 1 + \frac{\log \log x}{\log x} + \frac{1}{\log x} \log \left(1 + \frac{\log \log x}{\log x}\right)
\]  

and the asymptotic analysis of these two functions are quite similar.

An important component in the proof of Lemma 3 is the behavior of the logarithmic ratio

\[
\log \left(\frac{f(x)}{f(x+1)}\right) \text{ as } x \to \infty.
\]  

(21)
The result below works out a lower and an upper estimates of the logarithm ratio of the function \( f(x) \) associated with the \( n \)th prime \( p_n \).

**Lemma 3.** Let \( n \geq 1 \) be a sufficiently large integer. Then

(i) \( 0 < \log f(n) \leq c_1 \frac{\log \log n}{\log n} \),

(ii) \( 0 < f(n) - f(n + 1) \leq c_1 \frac{\log \log n}{n \log^2 n} \),

(iii) \( 0 < c_2 \frac{\log \log n}{n \log^2 n} \leq \log f(n) - f(n + 1) \leq c_3 \frac{\log \log n}{n \log^2 n} \)

where \( c_1, c_2, \text{and} c_3 > 0 \) are constants.

**Proof:** (i) This follows from the power series \([17]\), and the estimate \( \log(1 + z) \leq z \), where \(|z| < 1\).

(ii) The derivative of the function \( f(x) \) satisfies \( f'(x) < 0 \) on the interval \([x_0, \infty)\) for \( x_0 \geq 1619 \). Hence, the difference \( f(x) - f(x + 1) > 0 \) is nonnegative on the interval \([x_0, \infty)\).

(iii) Start with the ratio

\[
\frac{f(n)}{f(n + 1)} = 1 + \frac{f(n) - f(n + 1)}{f(n + 1)}. \tag{22}
\]

Taking logarithm and applying the estimate \( \log(1 + z) \leq z \), where \(|z| < 1\), lead to

\[
c_0 \frac{f(n) - f(n + 1)}{f(n + 1)} \leq \log \left( 1 + \frac{f(n) - f(n + 1)}{f(n + 1)} \right) \leq \frac{f(n) - f(n + 1)}{f(n + 1)}. \tag{23}
\]

where \( c_0 > 0 \) is a constant. In addition, \( 1/2 \leq f(x) \leq 2 \). Consequently,

\[
c_1 (f(n) - f(n + 1)) \leq \log \left( 1 + \frac{f(n) - f(n + 1)}{f(n + 1)} \right) \leq c_2 (f(n) - f(n + 1)). \tag{24}
\]

where \( c_1, c_2 > 0 \) are constants. The lower and upper bounds are computed in a term by term basis. For example, the difference of the first term of the power series \([17]\) satisfies

\[
c_3 \frac{\log \log n}{n \log^2 n} \leq \frac{\log \log n - 1}{\log n} - \frac{\log \log(n + 1) - 1}{\log(n + 1)} \leq c_4 \frac{\log \log n}{n \log^2 n}, \tag{25}
\]

where \( c_3, c_4 > 0 \) are constants. Lastly, as the function \( f(n) - f(n + 1) \) is an alternating power series, the difference of the first term is sufficient. ■

The information on the lower and upper bound of the logarithm ratio \( \log f(x)/f(x + 1) \) provides an effective way of estimating the order of magnitude of the error term in the difference \( p_{n+1} - p_n \). It sidesteps the difficulty encountered while using the asymptotic formula \( p_n = n \log n + O(n \log \log n) \).

### 3.2 The Prime Numbers Equations And Inequalities

The exponential prime numbers equations and inequalities

\[
p_{n+1}^x \pm p_n^y = r(n), \quad p_{n+1}^x \pm p_n^y < r(n) \tag{26}
\]

and other variations of these equations and inequalities have been studied by many workers in the field, confer \([12], \text{p.} 19\), and \([34], \text{p.} 256\) for discussions and references. The specific case \( p_{n+1}^x \pm p_n^y < 0 \), which implies the Cramer conjecture, can be established using the elementary method employed in
this work. A stronger result considered here seems to be plausible.

**Lemma 4.** Let $\epsilon > 0$ be an arbitrarily small real number, and let $n \geq 1$ be a sufficiently large integer. Then

\[ p_n^{n+1} < p_n^{n+(\log n)^{-1+\epsilon}}. \quad (27) \]

**Proof:** Write a pair of consecutive primes as

\[ p_n = (n \log n) f(n) \quad \text{and} \quad p_{n+1} = ((n + 1) \log(n + 1)) f(n + 1). \quad (28) \]

where $f(n)$ is the power series \([17]\), and consider the inequality

\[ \log p_{n+1}^n < \log p_n^{n+k}, \quad (29) \]

where $k = k(n, \epsilon)$ is a function of $n$ and $\epsilon$. Replacing (28) into (29) transforms it into

\[ \log \left(1 + \frac{1}{n}\right) + \log \left(1 + \frac{(1 + 1/n) \log n}{\log n}\right) + \log f(n) + 1 < \frac{k}{n} (\log n + \log \log n + \log f(n)) + \log f(n). \quad (30) \]

Now use $\log(1 + z) \leq z$, where $|z| < 1$, to reduce it to

\[ \frac{c_0}{n} + \frac{c_1}{n \log n} < \frac{k}{n} (\log n + \log \log n + \log f(n)) + \log \frac{f(n)}{f(n + 1)}. \quad (31) \]

where $c_0, c_1 > 0$ are constants. In Lemma 3 it is shown that

\[ 0 < \log f(n) \leq c_2 \frac{\log \log n}{\log n} \quad \text{and} \quad 0 < \log \frac{f(n)}{f(n + 1)} \leq c_3 \frac{\log \log n}{n \log^2 n}, \quad (32) \]

where $c_2, c_3 > 0$ are constants. Accordingly, the penultimate inequality can be written as

\[ \frac{c_0}{n} + \frac{c_1}{n \log n} < \frac{k}{n} \left(\log n + \log \log n + c_2 \frac{\log \log n}{\log n}\right) + c_3 \frac{\log \log n}{n \log^2 n}. \quad (33) \]

The first term $k \log(n)/n$ on the right side of the previous inequality dominates if the parameter is set to $k = (\log n)^{-1+\epsilon}$ for any arbitrarily small real number $\epsilon > 0$. Therefore, the inequality (33) is satisfied by any sufficiently large integer $n \geq n_0(\epsilon)$. ■

4 Applications

There are many practical and theoretical applications of Theorem 1 in the Mathematics, Cryptography, and Physics. Two obvious applications are considered here.

4.1 Generating Prime Numbers

The result sketched in this subsection improves a recent work on the complexity of generating prime numbers in \([38]\).

**Corollary 5.** Given any real number $x \in \mathbb{R}$, a prime $p \geq x \geq 2$ can be constructed in deterministic polynomial time complexity of $O \left(\left((\log x)^8\right)\right)$ arithmetic operations.
\textbf{Proof:} Apply the AKS primality test algorithm to the sequence of $k \geq 1$ consecutive odd numbers

$$n + 1, n + 3, n + 5, \ldots, n + k,$$

where $k = O \left( \log^2 x \right)$, and $n = 2[x/2] + 2$. By Theorem 1, at least one of these integers is a prime number. Thus, repeating the primality test $k$ times, result in the running time complexity of $O \left( \left( \log x \right)^8 \right)$ arithmetic operations.

The analysis of the AKS primality test algorithm appears in [1], it has deterministic polynomial time complexity of $O \left( \left( \log x \right)^8 \right)$ arithmetic operations. An improved version of faster running time appears in [21].

**4.2 Scherk Expansions of Primes**

The binary expansion of prime number is computable in polynomial time complexity, but the Scherk expansion of a prime has exponential time complexity, and it seems to be impossible to reduce it to a lower time complexity. This is an expansion of the form, [29, p. 353],

$$p_{2n} = 2p_{2n-1} + \sum_{0 \leq k \leq 2n-2} \epsilon_k p_k \text{ or } p_{2n+1} = p_{2n} + \sum_{0 \leq k \leq 2n-1} \epsilon_k p_k$$

(35)

where $\epsilon_k \in \{-1, 0, 1\}$ and $p_0 = 1$. A proof of this expansion is given in the American Mathematics Monthly. For example, the Scherk expansion of the prime number

$$10000000000000000151 = (1 \text{ or } 2) \cdot 10000000000000000129 + \sum_{0 \leq k \leq 2n-2} \epsilon_k p_k \text{ or } + \sum_{0 \leq k \leq 2n-1} \epsilon_k p_k$$

(36)

requires about $2 \times 10^{18}$ primes to complete. This amounts to a very large scale computation. However, the short Scherk expansion

$$10000000000000000151 = 10000000000000000129 + 19 + 3$$

has polynomial time complexity, this follows from Corollary 5.

**5 Numerical Data**

Significant amount of data have been compiled by several authors over the past decades, see [27], [28], [31], [39], et alii, for full details. A few examples of prime gaps, and the estimated upper bound as stated in Theorem 1, using the inequality

$$d(n) = 15 \log (p_n) \left( \log \log (p_n) \right)^2 - (p_{n+1} - p_n)$$

(37)

with $c = 15$, see the Maier-Pomerance problem in Section 1 for explanation.

1. The largest prime gap for the prime

$$p_n = 46242083809774032061673394226721$$
listed on the tables in [25], and [26] satisfies the inequality:

\[ 1998 = p_{n+1} - p_n \leq 15 \log (p_n) (\log \log (p_n))^2 = 4691.026292. \] (38)

2. The recently reported very large prime gap of 337446 for the 7976 digits prime \( p_n \), see [40], satisfies the inequality:

\[ 337446 = p_{n+1} - p_n \leq 15 \log (p_n) (\log \log (p_n))^2 = 2704737.1290. \] (39)

3. A few others reported examples for the prime gaps of very large probable primes also satisfy the inequality, in particular,

\[ 2254930 = p_{n+1} - p_n \leq 15 \log (p_n) (\log \log (p_n))^2 = 36615528.48. \] (40)

for a 86853 decimal digits probable prime \( p_n \), see [29], [40].

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