Local models of almost-toric integrable systems: theory and applications

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Abstract

In this article we show how one can use the local models of integrable Hamiltonian systems near critical points to prove a localization theorem for certain singular loci of integrable semi-toric systems in dimension $\geq 6$.

1 Introduction

The classification of integrable Hamiltonian systems is a great and very difficult task. Yet, for the subclass of the so-called toric systems, it has been achieved. A toric system is an integrable Hamiltonian system whose every critical point of $F$ is non-degenerate and for which every component of the moment map provides a global $S^1$ action. For these systems, the image of its moment map is a rational convex polytope (Atiyah - Guillemin & Steinberg theorem - [Ati82], [GS82], [GS84]), and this image characterizes completely the system (Delzant - [Del88], [Del90]). With this strong result, problems involving toric systems are now essentially algebraic and can be treated as it.

In almost-toric and semi-toric systems, we introduce other types of singularities while trying to stay as close as possible from toric systems. Their study clearly fall under the domains of geometry and analysis but, what is lost in rigidity is recovered in generality: an almost-toric system is much more general than a toric one. This is why community studies almost-toric systems with care. In this paper, our point is to get a description of the image of the moment map by describing sets of critical values.

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Since it is question of singular points and values, we shall first agree on the definition of a singularity. Given a symplectic manifold \((M^{2n}, \omega)\), we define an integrable Hamiltonian system by its momentum map \(F \in C^\infty(M \to \mathbb{R}^n)\), which is such that \(\{f_i, f_j\} = 0\) and \(df_1 \wedge \ldots \wedge df_n \neq 0\) almost everywhere. If not specified, we always assume that \(F\) is proper. We also design by \(\mathcal{F}_H := \{\text{connected components of } H^{-1}(c) | c \in \text{Im}(H)\}\) the foliation associated to \(H\). A leaf, hence, is a connected component of a fiber.

**Definition 1.1.** A critical leaf of \(H\) is a leaf in \(\mathcal{F}_H\) that contains a critical point for \(H\).

A singularity of \(H \in C^\infty(M \to \mathbb{R}^k)\) is defined by Zung in [Zun96] as (a germ of) a tubular neighborhood of a critical leaf.

We start with the following definition given by Symington in [Sym01]:

**Definition 1.2.** Let \(F = (f_1, \ldots, f_n) : (M^{2n}, \omega) \to \mathbb{R}^n\) be an integrable Hamiltonian system. It is said to be almost-toric of complexity \(c \leq n\) if every critical point verifies these conditions:

- all critical points are non-degenerate: in the symplectic context, it means that the subalgebra generated by \(\text{Hess}(F)\) is a Cartan subalgebra of maximal rank,
- there are no singularities of hyperbolic type: \(k_h = 0\),
- the function \(\tilde{F}^c := (f_{c+1}, \ldots, f_n)\) generates a global \(\mathbb{T}^{n-c}\)-action.

Toric case is the case \(c = 0\). We keep the terminology “semi-toric” to the case \(c = 1\) and define then \(\tilde{F} := (f_2, \ldots, f_n)\) as the function that generates the \(\mathbb{T}^{n-1}\)-action.

Since in this paper indexes in calculus can be heavy, we rely on these notations, hoping they shall lighten the reading

**Notations 1.3.** We fix the bold notations for row vectors \(x = (x_1, \ldots, x_r)\) (e.g. \(\theta = (\theta_1, \ldots, \theta_k), I = (I_1, \ldots, I_k)\)). We also fix the convention, given row vectors \(x\) and \(y\) of the same size and \(z\) a single coordinate:

\[
\text{dx} \wedge \text{dy} = \sum_{j=1}^{r} dx_j \wedge dy_j \quad \text{and} \quad \text{dx} \wedge dz = \sum_{j=1}^{r} dx_j \wedge dz \quad (1)
\]

Lastly, for \(A, B \in M_{p,q}(\mathbb{R})\), \(A \bullet B = (a_{ij}b_{ij})\).
1.1 The linear model

In the symplectic context, at a critical point the components can be of three kinds: elliptic, hyperbolic and focus-focus. Here, we give the linear model associated to each of them.

**Definition 1.4.** Let $F = (f_1, \ldots, f_n) : (M^{2n}, \omega) \to \mathbb{R}^n$ be an integrable Hamiltonian system and $m$ a critical point. We say that $m$ has a component of

- **elliptic type $E$** if there is a $f_i$ such that in local coordinates around $m$, $f_i = q_i^e + o(2) = x_i^2 + \xi_i^2 + o(2)$,

- **hyperbolic type $H$** if there is a $f_i$ such that in local coordinates around $m$, $f_i = q_i^h + o(2) = x_i \xi_i + o(2)$,

- **focus-focus type $FF$** if there is a couple $(f_i, f_j)$, $i \neq j$, such that in local coordinates around $m$, $f_i = q_i^{1i} + o(2) = x_i \xi_i + x_j \xi_j + o(2)$ and $f_j = q_j^{1j} + o(2) = x_i \xi_j - x_j \xi_i + o(2)$.

We set then some notations

**Notations 1.5.**

- $\mathcal{k}$—Williamson type: quadruple of integers $(k_e, k_f, k_h, k_x)$,

- $k_e$—number of elliptic components,

- $k_f$—number of focus-focus components,

- $k_h$—number of hyperbolic components,

- $k_x$—number of transverse (non-degenerate) components.

- $(\mathcal{W}^n(F), \preceq)$—Poset of possible Williamson type $\mathcal{k}$ for a given integrable Hamiltonian system $F$ of dimension $n$. We have the partial order relation $\mathcal{k} \preceq \mathcal{k}'$ if and only if $k_e \geq k'_e$, $k_f \geq k'_f$, $k_h \geq k'_h$ and $k_x \leq k'_x$.

We can now define in a suitable context the following linear model, with 0 for critical point

**Definition 1.6.** The linear model $L_\mathcal{k}$ associated to a critical point $m$ of type $\mathcal{k}$ is the following triplet $(L_\mathcal{k}, \omega_\mathcal{k}, Q_\mathcal{k})$ with

- $L_\mathcal{k} = \mathbb{R}^{2k_e} \times \mathbb{R}^{2k_h} \times \mathbb{R}^{4k_f} \times (T \times \mathbb{R})^{k_x}$,
• \( \omega_k \) the natural symplectic form on \( L_k \),

• \( Q_k = (Q_e, Q_h, Q_f, I_1, \ldots, I_{k_x}) \) a function of \( L_k \) to \( \mathbb{R}^n \) that sends \( m \) to 0, \( Q_e = (q_e^1, \ldots, q_e^k) \), \( Q_h = (q_h^1, \ldots, q_h^k) \), \( Q_f = (q_f^1, \ldots, q_f^k) \), \( I_1, \ldots, I_{k_x} \), and the \( I \) the action coordinates.

And again, some notations

Notations 1.7.  
• \( \text{CrP}_k(U) \) — Sheaf of critical points of Williamson type \( k \) on the open set \( U \subseteq M \).

• \( \text{CrV}_k(U) := F(\text{CrP}_k(U)) \) — Sheaf of critical values of Williamson type \( k \) on the open set \( U \subseteq M \).

Note here that since the fibers of \( F \) may not be connected, we can have \( \text{CrV}_k(U) = \text{CrV}_k(U') \) for \( U \cap U' = \emptyset \).

1.2 Location of semi-toric critical values

This subsection is devoted to the formulation and the explanation of our main result.

Definition 1.8. Embedded surfaces in \( F(M^6) \) are called nodal surfaces, and we denote them \( \Gamma_i \).

Let \( P \) be a point in \( \mathbb{R}^n \) and \( \vec{e}_1 \) the first vector of the basis induced by \( F \) - \( \vec{e}_1 \) is the direction of \( f_1 \). We prove here the following “location theorem”:

Theorem 1.9. Let \( F \) be a semi-toric integrable system on a compact symplectic manifold \( M^{2n} \). We have the following statements:

1. The image of the locus of critical points of \( F \) of Williamson type \( k \) is a finite union of nodal surfaces of rank \( k_x \) :

\[
F(\text{CrP}_k(M)) = \text{CrV}_k(M) = \bigcup_{i=1}^{m_k} \Gamma_i,
\]

2. If \( k_i = 1 \), for each \( \Gamma_i \), there exists a local symplectomorphism \( \varphi_i \) of \( M \) and a unique affine plane \( P(\Gamma_i) \) of the form \( P(\Gamma_i) = P + \mathbb{R} \cdot \vec{e}_1 + \mathbb{R} \cdot \vec{v}_1 + \ldots + \mathbb{R} \cdot \vec{v}_{k_x} \) with \( \vec{v}_1, \ldots, \vec{v}_{k_x} \) a free family with integer coefficients, such that \( \varphi_i^* \Gamma_i \subseteq P(\Gamma_i) \cap F(M) \).
3. In $\mathcal{P}(\Gamma_i)$, the nodal surface can be represented as the graph of a smooth function $h$ from an open affine domain $D \subseteq \mathbb{R}^k$ to $\mathbb{R}$:

$$\Gamma_i = \{P + h(t) \cdot \vec{e}_1 + t_1 \cdot \vec{v}_1, \ldots, t_k \cdot \vec{v}_k, t \in D\}.$$  

4. If we assume that the fibers are connected, then the nodal surfaces are isolated: there exists a neighborhood $V(\Gamma_i)$ of $\Gamma_i$ such that the only critical values of Williamson index $k$ in $V(\Gamma_i)$ are $\Gamma_i$.

In particular, this theorem answers negatively to a question asked to me by Colin de Verdière in 2010: “Can we have a ‘loop’ of focus-focus-transverse singular values in dimension $2n = 6$?”. We must thank him deeply for this simple question that acted both as a compass and as an incentive in my research during the years 2010-2011. We first developed the techniques of local models to answer this question, and then figured out that we could generalize the result using more “conceptual” theorems. This is the object of two articles to be published soon: [Wac14a] and [ZW14].

In the theorem above, we speak of critical values of a given Williamson type, but the Williamson type of a fiber is not well defined. As a result, a value can belong to different $CrV_k(M)$’s. Yet, we chose to give a result describing the image of the moment map rather than the base space of the foliation, because our “local model” results describe the former. Description of the latter requires the introduction of new structures and a study on its own. This is what we actually work on in a paper with N.T. Zung that is to be published in the following months, hopefully. Another reason is that the image of the moment map is the space that physicists directly have access to by experimentation.
Organization of this paper is the following: first we set up background to discuss local models, and then remind existing results concerning the existence of these models, or "normal forms". Then we give precisions concerning the local model of the image of a semi-toric critical point, and we finish with the proof of the global result Theorem 1.9 using the results about local models proved in section 3.

2 Normal forms for points, orbits and leaves

In this section, we remind different results to explain the context and how our result must be understood. We start with Eliasson normal form, with the Focus-Focus-Elliptic case as an example. We pass by the orbital Miranda-Zung’s theorem of linearization by orbits and finish with Zung’s leaf-wise theorem “Arnold-Liouville with singularities”.

2.1 Eliasson normal form

In 1984, Eliasson proved the following theorems although he only published the first one. We state the theorem and give the necessary references for the reader for a full proof:

**Theorem 2.1** (Eliasson Normal Form - Semi-toric case). Let \((M^{2n}, \omega, F)\) a semi-toric integrable system with \(n \geq 3\), and \(m\) a critical point of Williamson type \(k = FF - E^k - X^k\). That is, on an open set near \(m\), there exists a symplectomorphism \(\varphi_0 : (U_0 \subseteq M^{2n}, \omega) \rightarrow (\mathbb{R}^{2n}, \omega_0)\) such that \(F \circ \varphi_0^{-1} = Q_k + o(2)\) with

- \(Q_k = (q_1^f, q_2^f, q_1^e, q_2^e, \theta_1^x, \ldots, \theta_k^x, I_1^x, \ldots, I_k^x)\),
- \(q_1^f = x_1^{f}\xi_1 + x_2^{f}\xi_2, q_2^f = x_1^{f}\xi_1 - x_2^{f}\xi_2\),
- \(q_i^e = (x_i)\xi_i^2 + (\xi_i)\xi_i^2\),
- \(\theta_i^x, I_i^x\) action-angle coordinates.

Then there exists a triplet \((U_m, \varphi_k, G_k)\) with \(U_m\) an open set of \(M\), \(\varphi_k\) a symplectomorphism of \(U_m\) to a neighborhood of \((0 \in \mathbb{R}^{2n}, \omega_0)\) and \(G_k\) a local diffeomorphism of \(0 \in \mathbb{R}^n\) such that:

\[ \varphi_k^*F = G_k \circ Q_k \]
It was the contribution of many people that allowed eventually the statement of the theorems above. The first works to be cited here are those of Birkhoff, Vey \cite{Vey78}, Colin de Verdière and Vey \cite{dVV79}, and of course Eliasson in \cite{Eli84} and \cite{Eli90}. More recently, Chaperon in \cite{Cha12} and \cite{Cha96}, Zung in \cite{Zun97} and \cite{Zun02}, and San Vũ Ngọc & Wacheux in \cite{VW13} provided new proofs and filled the technical gaps that remained in the original proof.

2.2 Semi-local normal form

We go on with the semi-toric assumption: \( k_l = 0 \) or \( 1 \) and \( k_h = 0 \). Eliasson normal form is the first of many results generalizing the symplectic linearization of integrable systems.

Let \( F : (M^{2n}, \omega) \to \mathbb{R}^n \) be a proper integrable semi-toric system. The orbit \( O_m \) of a critical point \( m \in M \) by the local Poisson \( \mathbb{R}^n \)-action is a submanifold of dimension equal to the rank \( k_x \) of the action at the point \( m \). For this section, we can assume without loss of generality that \( df_1 \wedge \ldots \wedge df_{k_x} \neq 0 \).

**Definition 2.2.** The orbit \( O_m \) is called non-degenerate if, when we take the symplectic quotient of a neighborhood of \( O_m \) by the Poisson action of \( \mathbb{R}^{k_x} \) generated by \( F_X := (f_1, \ldots, f_{k_x}) \), the image of \( m \) is a non-degenerate fixed point.

A non-degenerate orbit has only non-degenerate critical points of the same Williamson index. Thus it makes sense to talk of an orbit of a given Williamson index. The linear model of a non-degenerate orbit is the same as the linear model of a point. Of course, a non-degenerate Hamiltonian system has only non-degenerate orbits and non-degenerate leaves. Non-degeneracy is an open property.

**Theorem 2.3** (Miranda & Zung, \cite{MZ04}). Let \( m \) be a non-degenerate critical point of Williamson type \( k \) of an integrable system \( (M, \omega, F) \).

Then there exists a neighborhood \( \mathcal{U}_m \) saturated with respect to the action of \( F_X \), the transverse components of \( F \), a symplectic group action of a finite group \( \Gamma \) and a symplectomorphism:

\[
\varphi : (\mathcal{U}_m, \omega) \to \varphi(\mathcal{U}_m) \subset L_k/\Gamma
\]

such that:

- \( \varphi^* F = Q_k \),
• The transverse orbit $O_{FX}(m)$ is sent to the zero-torus

$$T = \{ x^{e,f} = \xi^{e,f} = 0, \ I = 0 \}$$

of dimension $k_x$ (remember that $k_h = 0$).

Moreover, if there exists a symplectic action of a compact group $G \subset M$ that preserves the moment map $F$, the action can be linearized equivariantly with respect to that group action.

**Remark 2.4.** The finite group action $\Gamma$ is trivial in particular if $k_h = 0$ and $k_l = 1$ or $0$, that is, if we have a semi-toric integrable system like in this article for instance.

This theorem is an extension of Eliasson normal form: it means that we can linearize the singular Lagrangian foliation of an integrable semi-toric system symplectically on an orbital neighborhood of a non-degenerate critical point. The normal form is valid in a neighborhood that is larger than a ‘point-wise’ $\varepsilon$-ball of a critical point: it is saturated with respect to the transverse action of the system.

### 2.3 Arnold-Liouville with singularities

#### 2.3.1 Stratification by the orbits

One of the consequence of the existence of focus-focus and hyperbolic critical points is that there is a distinction between the leaf containing a point and the orbit through that point. The following proposition describes precisely how non-degenerate critical leaves are stratified by orbits of different Williamson types:

**Proposition 2.5.** Let $m \in M$ be a point of Williamson type $k$ of a proper, non-degenerate integrable system $F$. Then:

1. $O_F(m)$ is diffeomorphic to a direct product $T^c \times \mathbb{R}^o$ (and $c + o = k_x$).
2. For any point $m'$ in the closure of $O_F(m)$, $k_e(m') = k_e$, $k_h(m') \geq k_h$, and $k_l(m') \geq k_l$.
3. The quantities $k_e$, $k_e + c$ and $k_l + o$ are invariants of the leaf.
4. For a non-degenerate proper semi-toric system, a leaf $\Lambda$ contains a finite number of $F$-orbits with a minimal $k$ for $\prec$, and the Williamson type for these $F$-orbits is the same.
All these assertions are proven by Zung in [Zun96] and essentially rely on the stratification of the leaf by orbits of varying partial order. The last statement asserts that, in a non-degenerate critical leaf, a point with minimal Williamson type is not unique in general, but the minimal Williamson type is.

This allows us to give the following definition:

**Definition 2.6.** For a non-degenerate semi-toric system $F$, and $b$ an element of the base space $B$ of the foliation $\mathcal{F}$, the Williamson type of a leaf $\Lambda_b$ is defined as

$$k(\Lambda_b) := \min_{\succeq} \{k(m) \mid m \in \Lambda_b\}.$$ 

That quantity is well defined because the only critical leaves that differ from those of toric systems, are the ones that contain singularities with focus-focus components. Since leaves are closed, points with minimal Williamson type must contain one $\text{FF}$ component and the maximal number of $\text{E}$ components the leaf has. That ensures us the unicity of the Williamson type of the leaf. Let us mention here that once we have defined the Williamson type of non-degenerate orbits and leaves, there is no relevant notion of a Williamson type of a fiber.

**Definition 2.7.** A non-degenerate critical leaf $\Lambda$ is called topologically stable if there exists a saturated neighborhood $\mathcal{V}(\Lambda)$ and a $\mathcal{U} \subset \mathcal{V}(\Lambda)$ a small neighborhood of a point $m$ of minimal rank in $\lambda$ such that

$$\forall k \in \mathcal{W}_0^\rho, \ F(\text{CrP}_k(\mathcal{V}(\Lambda))) = F(\text{CrP}_k(\mathcal{U})).$$

An integrable system will be called topologically stable if all its critical points are non-degenerate and topologically stable.

The assumption of topological stability rules out some pathological behaviours that can occur for general foliations. Note however that for all known examples, the non-degenerate critical leaves are all topologically stable, and it is conjectured that it is also the case for all analytic systems.

Since the papers [Zun96] and [Zun03] of Zung, the terminology concerning the assumption of topological stability has evolved. One speaks now of the transversality assumption, or the non-splitting condition. This terminology was proposed first by Bolsinov and Fomenko in [BF04].

Yet, the expression “topological stability” was originally justified by the following theorem

**Theorem 2.8 (Zung, [Zun96]).** Let $F$ be an integrable Hamiltonian system, $\Lambda$ a non-degenerate topologically stable critical leaf of rank $k_\times$, and $\mathcal{U}(\Lambda)$ a
saturated neighborhood of it. Then all critical leaves of rank $k_x$ are topologically equivalent (i.e. homeomorphic), all closed orbits of the Poisson action of $\mathbb{R}^n$ given by the moment map have the same dimension, and all critical sub-leaves are topologically stable.

A normal form theorem can be seen, loosely, as a decomposition of a function into “simpler” blocks with respect to the general goal. This is what we set with the next definition:

**Definition 2.9.** We say that two singularities are isomorphic if they are leaf-wise homeomorphic. We name the following singularities isomorphism classes “simple”:

- A singularity is called of (simple) elliptic type if it is isomorphic to $L^e: \text{a plane } \mathbb{R}^2 \text{ foliated by } q_e$.
- A singularity is called of (simple) focus-focus type if it is isomorphic to $L^f$, where $L^f$ is given by $\mathbb{R}^4$ locally foliated by $q_1$ and $q_2$. One can show (see Proposition 6.2 in [VN00]) that the focus-focus critical leaf must be homeomorphic to a pinched torus $\mathbb{T}^2$ : a 2-sphere with two points identified. The regular leaves around are regular tori.

Properties of elliptic and focus-focus singularities are discussed in details in [Zun96]. In particular, the fact that we can extend the Hamiltonian $S^1$-action that exists near a focus-focus point to a tubular neighborhood of the focus-focus singularity guarantees that the focus-focus critical fiber is indeed a pinched torus.

**Assumption 2.10.** From now on, we will assume that all the systems we consider are simple and topologically stable. In particular, simplicity implies that for the semi-toric systems we consider, focus-focus leaves will only have one vanishing cycle.

### 2.3.2 Statement of the theorem

Now we can formulate an extension of Liouville-Arnold-Mineur theorem to singular leaves. We call the next theorem a “leaf-wise” result, as we obtain a normal form for a leaf of the system. However we won’t say that assertion 2. of the theorem extends Eliasson normal form, since here the normal form of the leaf doesn’t preserve the symplectic structure.

**Theorem 2.11** (Arnold-Liouville with semi-toric singularities, [Zun96]). Let $F$ be a proper semi-toric system, $\Lambda$ be a non-degenerate critical leaf of Williamson type $k$ and $V(\Lambda)$ a saturated neighborhood of $\Lambda$ with respect to $F$.

Then the following statements are true:
1. There exists an effective Hamiltonian action of $\mathbb{T}^{k_e + k_f + k_x}$ on $V(\Lambda)$. There is a locally free $\mathbb{T}^{k_e}$-subaction. The number $k_e + k_f + k_x$ is the maximal possible for an effective Hamiltonian action.

2. If $\Lambda$ is topologically stable, $(V(\Lambda), F)$ is leaf-wise homeomorphic (and even diffeomorphic) to an almost-direct product of elliptic, hyperbolic and focus-focus elementary singularities: 

$$(V(\Lambda), F) \simeq (\mathcal{U}(\mathbb{T}^{k_e}), F_r) \times \mathcal{L}_1^e \times \ldots \times \mathcal{L}_{k_e}^e \times \mathcal{L}_1^f \times \ldots \times \mathcal{L}_{k_f}^f$$

where $(\mathcal{U}(\mathbb{T}^{k_e}), F_r)$ is a regular foliation by tori of a saturated neighborhood of $\mathbb{T}^{k_e}$.

3. There exists partial action-angle coordinates on $V(\Lambda)$: there exists a diffeomorphism $\varphi$ such that

$$\varphi^* \omega = \sum_{i=1}^{k_x} d\theta_i \wedge dI_i + P^* \omega_1$$

where $(\theta, I)$ are the action-angle coordinates on $T^* \mathcal{T}$, where $\mathcal{T}$ is the zero torus in Miranda-Zung equivariant Normal form theorem stated in [MZ04] and $\omega_1$ is a symplectic form on $\mathbb{R}^{2(n-k_e)} \simeq \mathbb{R}^{2(k_e+2k_f)}$.

Theorem 2.11 says that in particular for semi-toric systems, under this mild assumption that is topological stability on the leaves, a critical leaf $\Lambda$ is diffeomorphic to a product of the “simplest” regular, elliptic and focus-focus leaves one can find.

**Remark 2.12.** Assertion 3. of Theorem 2.11 explains that we do not have a symplectomorphism in the assertion 2. This is the “raison d’être” of all the work done in the topic of integrable systems with singularities, including this article.

Also, one should notice that in Theorem 2.11, it is only because we made the Assumption 2.10 that the singularity is leaf-wise diffeomorphic to an almost-direct product of simple singularities, and not only homeomorphic to it. For instance, were there more than one pinch on the singularity, one could only guarantee the existence of an homeomorphism between the two.

### 2.4 Stratification by Williamson index

We give now the following results, consequences of Zung’s article [Zun96]. It is an extension of stratification of $M$ by the rank of $F$ to the semi-toric case. To this end, we give a definition of a stratification
Definition 2.13. A stratification of a topological manifold $M^n$ is a finite partition $\mathcal{S}$ of it indexed by a poset $(\mathcal{I}, \preceq)$ such that

- **Decomposition**: for $i \in \mathcal{I}$ the $S_i$’s are smooth manifolds and for $i, j \in \mathcal{I}$, $i \preceq j$ if and only if $S_i \subseteq S_j$.
- **Splitting condition**: by induction over the dimension of the stratified manifold $n$.

If $x \in S_i$, for a neighborhood $U_x$ of $x$ in $\mathbb{R}^n$, there exists a disk $D_i \subset \mathbb{R}^{\dim(S_i)}$ and a cone $C(J) = \text{over a } (n - \dim S_i - 1)$-dimensional stratified smooth manifold $J$ (and $(n - \dim S_i - 1) \leq n - 1$) such that: $U_x$ and $D_i \times C(J)$ are isomorphic as stratifolds.

The triplet $(M, S, \mathcal{I})$ is called a stratified manifold.

The definition above is consistent, since stratafications of manifolds of dimension 0 and 1 are obvious. Usually, one defines stratification over larger classes of objects that are not regular enough for the given topology, while the strata are. Here, the example to remember is the cone, or the manifolds with corners: they won’t be smooth manifolds but their strata will be.

**Theorem 2.14.** Let $(M^{2n}, \omega, F)$ be a semi-toric integrable Hamiltonian system.

Then we have the following results:

- For any $k \in \mathcal{W}(F)$, $\text{CrP}_k(M)$ is a smooth open manifold locally diffeomorphic to $(S^{k_1} \times \mathbb{R}^{k_2})$. On $\text{CrP}_k(M)$, the 2-form $\omega_k := \sum_{i=1}^{k_2} d\theta_i \wedge dI_i$ is a symplectic manifold.
- The triplet $(\text{CrP}_k(M), \omega_k, F_k)$, with $F_k := F|_{\text{CrP}_k(M)}$ is an integrable system with no critical points.
- The triplet $(M, \text{CrP}^F(M), \mathcal{W}(F))$ is a stratified manifold.

**Proof.** Concerning points 1. and 2., if we take a critical point $p \in M$ of Williamson index $k$, with items 2. and 3. of Theorem 2.11 we have a tubular neighborhood $\mathcal{V}$ of the leaf containing $p$ such that $\mathcal{V}$ is leaf-wise diffeomorphic to

$$(\mathcal{V}(\Lambda), \mathcal{F}) \simeq (\mathcal{U}(T^{k_1}), \mathcal{F}_r) \times \mathcal{L}_{k_1}^e \times \ldots \times \mathcal{L}_{k_2}^e \times \mathcal{L}_{k_1}^l \times \ldots \times \mathcal{L}_{k_2}^l$$

and a diffeomorphism $\varphi$ such that $\varphi^* \omega = \sum_{i=1}^{k_2} d\theta_i \wedge dI_i + P^* \omega_1$. The subset $\text{CrP}_k(\mathcal{V})$ is diffeomorphic in these local coordinates to $\{q_1 = q_2 = \ldots = q_{k_2} = \ldots = q_{k_1} = 0\}$. The
\(0, x_1^e = y_1^e = \ldots = x_k^e = y_k^e = 0\}, that is, to an open subset of \(T^*T^k\) of the form \(T^k \times \hat{D}^k\). On it, \(\omega_1\) vanishes so \(\text{CrP}_k(V)\) is described by the partial action-angle coordinates given by Miranda and Zung in Theorem 2.3. This proves item 1.

Next, we have that \(F_k \in C^\infty(\text{CrP}_k(M) \to \mathbb{R}^k)\), it is clearly integrable as an Hamiltonian system of \(\text{CrP}_k(M)\). Moreover, a critical point for \(F_k\) is a critical point for \(F\) with a smaller Williamson index, which is impossible on \(\text{CrP}_k(M)\) by definition. Hence \(F_k\) has no critical point on \(\text{CrP}_k(M)\). This proves point 2.

To conclude with point 3., we first have to prove the decomposition condition. It is clear that \(\{\text{CrP}_k(M)\}_{k \in \mathcal{W}(F)}\) is a partition of \(M\) by smooth manifolds. The indexing set is the poset \(\mathcal{W}(F)\). Remembering Proposition 2.5 for \(k, k'\) in \(\mathcal{W}(F)\), if \(\text{CrP}_k(M) \subseteq \text{CrP}_{k'}(M)\), then \(k \preceq k'\). To prove the converse statement, one can notice with the local models that for a critical point \(m\) of Williamson index \(k\), \(m\) can always be attained by a sequence of points of the same Williamson index, provided that this index is bigger than \(k\). In particular, \(m\) is in \(\text{CrP}_{k'}(M)\).

For the splitting condition, with Item 2. of Theorem 2.11 we see that we only need to treat the simple elliptic and focus-focus cases with local models. In the elliptic case \(\text{CrP}_E\mathbb{R}^2\) is just a point : a neighborhood of the critical point is a disk, it is homeomorphic to the critical point times a cone over a small circle. For the focus-focus case, it is not more complicated : \(\text{CrP}_{FF}(\mathbb{R}^4)\) is again a point, and we need to show there exists a 3-dimensional stradispace \(L\) such that a neighborhood of the focus-focus point is homeomorphic to this point times the cone over \(L\). We can just take the 3-sphere \(S^3\) and take the cone over it : it is homeomorphic to the 4-ball, and hence is a neighborhood of a focus-focus point.
3 Local models for semi-toric values

Eliasson normal form theorem 2.1 and Miranda-Zung theorem 2.3 give us a general understanding of the commutant of an almost-toric system: we have a local model of the image of the moment map near a critical value, that can be precised in the semi-toric case. We remind first these two lemmas.

Lemma 3.1. Let $\mathcal{F}$ be a singular Liouville foliation given by a momentum map $F : M \to \mathbb{R}^k$. Let $\mathcal{F}'$ be a singular Liouville foliation given by a momentum map $F' : N \to \mathbb{R}^k$. If the level sets are locally connected, then for every smooth symplectomorphism
\[ \varphi : U \subset M \to V \subset N \]
where $U$ is an open neighborhood of $p \in M$, $V$ a neighborhood of $p' = \varphi(p) \in N$, and such that
\[ \varphi^* \mathcal{F} = \mathcal{F}' , \]
there exists a unique local diffeomorphism
\[ G : (\mathbb{R}^k, F(p)) \to (\mathbb{R}^k, F'(p')) \]
such that
\[ F \circ \varphi = G \circ F'. \]

Lemma 3.2. Let $(M, \omega, F)$ be a proper, non-degenerate almost-toric integrable system. Then its fibers has a finite number of connected components.

Proof. of Lemma 3.2 Let $c$ be a value of $F$, and $L$ be a connected component of $F^{-1}(c)$. On each point of $L$ we can apply Eliasson normal form. Since $F$ is compact and observing at the local models that can occur on $L$, this gives us the existence of an open neighborhood $\mathcal{V}(L)$ of $L$ in which there is no other connected component of $F^{-1}(c)$.

Now, we have that $\bigcup_{L \subseteq F^{-1}(c)} \mathcal{V}(L)$ is an open covering of $F^{-1}(c)$, which is compact by the properness of $F$. We can thus extract a finite sub-covering of it. It implies that there is only a finite number of connected components. □
3.1 Symplectomorphisms preserving a semi-toric foliation

Eliasson-Miranda-Zung normal form gave us the existence of $G$ a local diffeomorphism of the linear model associated to any integrable system. The theorem presented here gives precisions about the form of $G$ in the semi-toric case.

**Theorem 3.3.** Let $F$ be a semi-toric integrable system, and $m$ be a critical point of Williamson type $k$, with $k_i = 1$. Let $\varphi : (U_m, \omega) \to (\varphi(U) \subseteq L_k, \omega_k)$ be a symplectomorphism sending the transverse orbit of $m$ on the zero-torus and such that the foliations $\varphi^*F$ and $Q_k$ are equivalent on $U_m$.

Then the diffeomorphism $G : U \to G(U) \subseteq F(M)$ given by Eliasson-Miranda-Zung normal form and lemma 3.1 are such that:

1. $F \circ \varphi^{-1} = G \circ Q_k$,
2. $\tilde{F} = A \cdot \tilde{Q}_k + \tilde{F}(c)$, with $A \in GL_{n-1}(\mathbb{Z})$.

That is, the Jacobian of $G$ is of the form:

$$
\begin{pmatrix}
\partial q_1 G_1 & \partial q_2 G_1 & \cdots & \partial q_{k_e} G_1 & \partial I_1 G_1 & \cdots & \partial I_{k_e} G_1 \\
0 & F^d & \cdots & F_{k_e}^d & F^e & \cdots & F_{k_e}^e \\
0 & E^d_1 & \cdots & E^d_{k_e} & E^e & \cdots & E^e_{k_e} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & X^d_1 & \cdots & X^d_{k_e} & X^e & \cdots & X^e_{k_e}
\end{pmatrix}
\begin{pmatrix}
F^d & F^e & F^x \\
E^d & E^e & E^x \\
X^d & X^e & X^x
\end{pmatrix}
$$

Or, put in another way,

$$A = \begin{pmatrix}
F^d & F^e & F^x \\
E^d & E^e & E^x \\
X^d & X^e & X^x
\end{pmatrix},$$

with:

- $F^d \in \mathbb{Z}$, $F^e \in M_{1,k_e}(C^\infty(\mathbb{R} \to \mathbb{R}))$, $F^x \in M_{1,k_e}(C^\infty(\mathbb{R} \to \mathbb{R}))$,
- $E^d \in M_{k_e,1}(\mathbb{Z})$, $E^e \in M_{k_e}(\mathbb{Z})$, $E^x \in M_{k_e}(\mathbb{Z})$. 

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• \( X^f \in M_{k_1}(\mathbb{Z}) \), \( X^e \in M_{k_2}(\mathbb{Z}) \), \( X^x \in M_{k_3}(\mathbb{Z}) \).

This theorem includes the particular case where \( F = Q^k \), that is, when we consider symplectomorphisms that start and end with \( Q^k \). One more convention we shall use from here : we multiply periodic vector fields by \( 2\pi \) so that in calculus, we do not have to bring along a \( 2\pi \) constant.

**Proof of Theorem.** 3.3

Since \( \varphi \) is a symplectomorphism, it preserves the dynamics induced by the Hamiltonian vector fields of \( \hat{F} \). We have assumed that the components of \( \hat{F} \) have \( 2\pi \)-periodic flows. So, once pushed forward by \( \varphi \), the vector fields must remain \( 2\pi \)-periodic. We have the expression

\[
\chi_{f \circ \varphi^{-1}} = \partial_{q_1}G_i \cdot \chi_{q_1} + \partial_{q_2}G_i \cdot \chi_{q_2} + \sum_{j=1}^{k_x} \partial_{q_{(j)}}G_i \cdot \chi_{q_{(j)}} + \sum_{j=1}^{k_2} \partial_{I_j}G_i \cdot \chi_{I_j}
\]

The partial derivatives of \( G \) are constant under the action by the Hamiltonian flow of \( Q^k \). The classical complex variables of singular flows are the following

| Type | Elliptic | Focus-Focus |
|------|----------|-------------|
| Coordinates | \( z^e := x^e + i\xi^e \) | \[
\begin{align*}
    z_{f_{1}}^f &:= x_{f_{1}}^f + ix_{f_{2}}^f, \\
    z_{f_{2}}^f &:= \xi_{f_{1}}^f + i\xi_{f_{2}}^f
\end{align*}
\] |
| Formula | \( \phi_{q}^t(z^e) = e^{iqt}z^e \) | \[
\begin{align*}
    \phi_{q_{1}}^t(z_{1}, z_{2}) &:= (e^{-q_{1}t}z_{1}, e^{q_{1}t}z_{2}), \\
    \phi_{q_{2}}^t(z_{1}, z_{2}) &:= (e^{iq_{2}t}z_{1}, e^{iq_{2}t}z_{2})
\end{align*}
\]

gives us the following expression for the flow on a neighborhood \( \mathcal{U} \):

\[
\phi_{f \circ \varphi^{-1}}^t(z_1, z_2, x_1^e, x_2^e, x_3^e, \cdots, x_{k_1}^e, x_{k_2}^e, x_{k_3}^e, \theta_1, I_1, \ldots, \theta_{k_x}, I_{k_x}) =
\left( e^{(\partial q_{1}G_i + i\partial q_{2}G_i)t}z_{f_{1}}^f, e^{(-\partial q_{1}G_i + i\partial q_{2}G_i)t}z_{f_{2}}^f, e^{i\partial q_{1}G_i t}z_{q_{1}}^e, \cdots, e^{i\partial q_{(k_2)}G_i t}z_{q_{(k_2)}}^e, \theta_1 + \partial_{I_1}G_i t, \ldots, \theta_{k_x} + \partial_{I_{k_x}}G_i t, I_1, \ldots, I_{k_x} \right).
\]

So necessarily, for \( i = 2, \ldots, n \) and \( j = 1, \ldots, n \), we have \( \partial q_{i}G_i = 0 \) and

\[
\partial q_{j}G_i, \partial q_{j}G_i \in \mathbb{Z} \quad \text{and} \quad \partial q_{k_x}G_i \in \mathbb{Z}.
\]

If a coefficient of the Jacobian is integer on \( \varphi(\mathcal{U}) \), it must be constant on it. This shows that \( \hat{F} = A \circ \hat{Q} \) with \( A \in M_{n-1}(\mathbb{Z}) \). Now, \( A \) is invertible.
because \( G \) is a local diffeomorphism, and since the components of \( \hat{F} \) are \( 2\pi \)-periodic, we have that necessarily \( A^{-1} \in M_{n-1}(\mathbb{Z}) \).

Note that this result here only uses the \( 2\pi \)-periodicity of the flow, and no other assumption about the dynamics of \( F \). In the next theorem, we have the same foliation before and after composing with \( \varphi \). This stronger statement will get us precisions about the form of \( \text{Jac}(G) \), in particular the unicity of its infinite jet on the set of critical values.

### 3.2 Transition functions between the semi-toric local models

In this section, we need to precise our notion of flat function. To this end, let’s introduce the following set :

**Definition 3.4.** Let \( S \subseteq \mathbb{R}^k \). We define the set \( \mathcal{F}_{\ell}^Q(S)(U) \) as the set of real-valued smooth functions on \( U \) an open subset of \( \mathbb{R}^k \) which are flat in all directions given by the family of vectors \( v \), for all the points \( x \in S \).

#### 3.2.1 Symplectomorphisms preserving a linear semi-toric foliation

We prove here a kind of *uniqueness* theorem of the \( G \) introduced in Theorem 3.3. For this reason, we shall call the diffeomorphism \( B \) here, because the constraints of \( G \) in that case is an information about the possible changes of the Basis \( Q_k \) that can occur.

**Theorem 3.5.** Let \((L_k, \omega_k, Q_k)\) be a linear model with \( k_l = 1 \). Let \( \psi \) be a symplectomorphism of \( U \subset L_k \) an open neighborhood of 0 saturated with respect to the orbit of \( Q_x \), and which preserves the foliation \( Q_k \).

Then the diffeomorphism \( B : U \to B(U) \) introduced in 3.3 is such that there exists \( \epsilon_1, \epsilon_2 \in \{-1, +1\} \), a matrix \( e^e \in \text{Diag}_{k_e}(\{-1, +1\}) \) and a function \( u \in \mathcal{F}_{\ell}^Q(S)(U \to \mathbb{R}) \), constant in \( Q_e, Q_x \) on \( S \), where \( S := \text{Cr}_k^Q(U) \subseteq U := Q_k(U) \), so that we have :

1. \( \text{Jac}(B)_1(Q_k) = (\epsilon_1 q_1 + \partial_{q_1} u, \partial_{q_2} u, 0, \ldots, 0) \),
2. \( E^l = 0, E^e = e^e \) and \( E^x = 0 \),
3. \( F^l = \epsilon_2, F^e = 0 \) and \( F^x = 0 \),
4. \( X^l \in M_{k_s,1}(\mathbb{Z}), X^e \in M_{k_e}(\mathbb{Z}), X^x \in GL_{k_x}(\mathbb{Z}) \),
That is, we have, for $\tilde{x} = x \circ \psi^{-1}$:

- $\tilde{q}_1 = \epsilon^f_1 q_1 + u$, $\tilde{q}_2 = \epsilon^f_2 q_2$
- $\tilde{q}_e^{(i)} = \epsilon^e_i q^{(i)}_e$
- $\tilde{I}_1, \ldots, \tilde{I}_{k_s} = (0|0|id_{k_s}) \circ \tilde{Q}_k$

For the Jacobian, it means that

$$Jac(B) = \begin{pmatrix}
\epsilon^f_1 + u_1 & u_2 & \ldots & \ldots & \ldots & \ldots & \ldots & u_n \\
0 & \epsilon^f_2 & 0 & \ldots & 0 & 0 & \ldots & 0 \\
0 & 0 & \epsilon^e_1 & \ddots & \vdots & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \ddots & 0 & \vdots & \ldots & \vdots \\
0 & 0 & \ldots & 0 & \epsilon^e_{k} & 0 & \ldots & 0 \\
\vdots & \vdots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & X^f_1 & \vdots & \ldots & X^e & \ldots & X^x \\
0 & X^f_{k_s} & \vdots & \ldots & \vdots & \ldots & \vdots & \vdots
\end{pmatrix}$$

(★)

with the $u_i$’s being in $\mathcal{F}^{Q_s}(U)$ (they are the derivatives of $u$ in all the variables).

For practical uses, we are not interested in the precise form of $u$, but just by the fact that it is flat on $S$.

The theorem was proved in the focus-focus case for $2n = 4$ by San Vũ Ngọc in [VN03]. We follow the same ideas to give here a proof in the general case.

**Proof of Theorem.**

As a particular case of theorem 3.3, we already know that $Jac(B)$ is of the form (★). The point here is to exploit the fact that the linear model $Q_k$ has specific dynamical features conserved by a canonical transformation. Note also that the fibers of $Q_k$ are connected.

**Points (2), (3) and (4):**

A point fixed by the flow of a Hamiltonian $H$ is preserved by a symplectomorphism: its image will be a fixed point for the precomposition of $H$ by a symplectomorphism. Theorem 2.3 tells us how critical loci of a given Williamson type come as “intersections” of other critical loci. In particular, we know that for $i = 1, \ldots, k_s$, there exists a $c_i \in U$ such that for all $p \in (Q_k)^{-1}(c_i)$, $\chi_{q_1}(p) = 0$, $\chi_{q_e^{(i)}}(P) \neq 0$ for $j \neq i$ and $\chi_{q_2}(p) \neq 0$ (the $\chi_I$’s never cancel). Now, we have the following formula

$$\chi_{q_e^{(i)} \circ \psi^{-1}}(p) = E^f_{ij} \chi_{q_2}(p) + \sum_{j=1, j \neq i}^{k_s} E^e_{ij} \chi_{q_e^{(i)}}(p) + \sum_{j=1}^{k_t} E^x_{ij} \chi_{I_j}(p) = 0.$$
The family

\[ (\chi_{q_2}(p), \chi_{q_1(p)}, \ldots, \chi_{q_{i-1}(p)}, \chi_{q_{i+1}(p)}, \ldots, \chi_{q_{k_e}(p)}, \chi_{I_1}, \ldots, \chi_{I_{k_e}}) \]

is a free family \((Q_k\) is an integrable system), so we have for \(j \neq i\) that \(E_{ij}^c = 0\), and \(E_{ij}^x = 0\) for all \(j = 1, \ldots, k_x\). This is true for all \(i = 1, \ldots, k_e\).

If we take a \(c\) in \(\{q_1 = q_2 = 0\}\), we get by a similar reasoning that \(F^c = 0\) and \(F^x = 0\).
Point (1), part 1:

Now, we need to deal with the flow of $q_1$, that is, the “pseudo-hyperbolic” component of the focus-focus singularity.

On an open set $\mathcal{U}$, we have the explicit expression for the field of $q_1 \circ \psi^{-1}$

$$x_{q_1 \circ \psi^{-1}} = \partial_{q_1} B_1 \cdot x_{q_1} + \partial_{q_2} B_1 \cdot x_{q_2} + \sum_{j=1}^{k_e} \partial_{q_1^{(j)}} B_1 \cdot x_{q_1^{(j)}} + \sum_{j=1}^{k_x} \partial_{I_j} B_1 \cdot x_{I_j}.$$  

When evaluated on $p \in \text{CrP}_{k_e}^{Q_k} (\mathcal{U})$, comes

$$0 = \partial_{q_2} B_1 \cdot x_{q_2} (p) + \sum_{j=1}^{k_e} \partial_{q_1^{(j)}} B_1 \cdot x_{q_1^{(j)}} (p) + \sum_{j=1}^{k_x} \partial_{I_j} B_1 \cdot x_{I_j} (p).$$

First, since the $x_{I_i}$'s have no fixed point, we necessarily have that

$$\forall c \in \text{CrV}_{k_e}^{Q_k} (\mathcal{U}) \text{ and } \forall j = 1, \ldots, k_e, \partial_{I_j} B_1 (c) = 0 \quad (2)$$

The result is true for all $k$ with $k_t = 1$, so if $k_e \geq 1$, we can apply the same reasoning and get equation 2 for $\text{CrV}_{k}^{Q_k} (\mathcal{U})$, where $\bar{k} = (0, k_t, 0, k_x + k_e) \geq k$.

Since $\text{CrV}_{k}^{Q_k} (U) \subseteq \text{CrV}_{k_e}^{Q_k} (U)$, we have

$$\forall c \in \text{CrV}_{k}^{Q_k} (U) \text{ and } \forall j = 1, \ldots, k_e, \partial_{q_1^{(j)}} B_1 (c) = 0. \quad (3)$$

Now that we know there is no transverse nor elliptic component in the flow of $q_1 \circ \psi^{-1}$ for critical leaves with focus-focus component, let’s focus on the $q_2$-component.

A leaf $\Lambda'$ of $Q_k$ of Wiliamson type $k' \ll k$ is stable by the flow of $q_1$. On it, the flow is radial: for a point $m' \in \Lambda$ of Williamson type $k' = (0, 1, k_x)$, there exists a unique point $m$ on the zero-torus of the leaf $\Lambda$ such that the segment $[m', m]$ is a trajectory for $q_1$. Depending on whether $m'$ is on the stable (+) or the unstable (-) manifold, we have that $[m', m] = \{\phi_{q_1}^t (m') | t \in [0, \infty]\}$. Remembering that $m$ is a fixed point for $q_1$, we have that $\psi([m', m])$ is an infinite trajectory of $q_1 \circ \psi^{-1} = B_1 \circ Q_k$.

The image trajectory $\psi([m', m])$ is contained in a 2-dimensional plane (the stable or unstable manifold). Since $\psi$ is smooth at $m'$, $\psi([m', m])$ is even contained in a sector of this plane that also contains $m'$. Remembering that $\psi([m', m])$ is a trajectory for an infinite time, the only linear combinations of $x_{q_1}, x_{q_2}$ which yields trajectories confined in a fixed sector are multiples of $x_{q_1}$ only. So we have that

$$\partial_{q_2} B_1 (c) = 0. \quad (4)$$
This shows that $B_1$ is constant in the variables $(q^c, I)$ on $\text{CrV}^Q_k(U)$, but does not tell us more information.

**Point (1), part 2:**

To show that $B_1 - q_1$ is flat on $\text{CrV}^Q_k(U)$ in the variables $q'_1$ and $q'_2$, we can now treat the variables $(q^c, I)$ as parameters. We can always suppose that $\psi$ preserves the stable and unstable manifold of $q_1$: this assumption is equivalent to fix the sign of $\partial_1 B_1$ to be positive on $U$. As a result we’ll have $\epsilon'_1 = 1$. And again, we can assume that $k_e = 0$, as flatness is a closed property: here, it is stable when taking the limit $q'_1 \rightarrow 0$.

With the explicit expression of the flow of $q_1$ and $q_2$, if we set $z_1 z_2 = c$ and $z_2 = \delta$, we have for the joint flow of $q_1$ and $q_2$ at respective times $s = \ln |\frac{\delta}{c}|$ and $t = \arg(\delta) - \arg(c)$

$$
\phi_{\psi_1} \circ \phi_{\psi_2} (c, \delta, \theta, I) = (\delta, c, \theta, I)
$$

(5)

One can then state the fact that $\Upsilon$ is a smooth and single-valued function in a neighborhood $W$ containing $\{(0, \bar{\delta}, \theta, I), \theta \in \mathbb{T}^{k_e}, I \in \mathcal{B}^{k_s}(0, \eta)\}$. Now, we know that $\psi^{-1}(0, \delta, \theta, I)$ is of the form $(0, a, \theta', I)$ and $\psi^{-1}(\delta, 0, \theta, I)$ is of the form $(b, 0, \theta''', I)$, since $\psi$ preserve the level sets and the stable and unstable manifolds. Hence, for $\psi^{-1} \circ \Upsilon \circ \psi$

$$(0, a, \theta', I) \xrightarrow{\psi} (0, \bar{\delta}, \theta, I) \xrightarrow{\Upsilon} (\delta, 0, \theta, I) \xrightarrow{\psi^{-1}} (b, 0, \theta'''', I).$$

With the expression of $\Upsilon$ in (5), we know that in the complementary set of $\{z_1 = 0\}$, $\psi^{-1} \circ \Upsilon \circ \psi$ is equal to the joint flow of $B(q_1, q_2, I)$ at the multi-time $(\ln |\frac{\delta}{c}|, \arg(\delta) - \arg(c), 0, \ldots, 0)$. With what we already know about the flow of $q_2 \circ \psi^{-1}$, when we write the joint flow in terms of the flows of the components of $Q_k$ we get:

$$
\psi^{-1} \circ \Upsilon \circ \psi = (\ln |\frac{\delta}{c}|, \arg(\delta) - \arg(c)), \\
\phi_{B(q_1, q_2, I)} = \phi_{\psi_1} \circ \phi_{\psi_2} \circ \phi_{q_1} \circ \phi_{q_2} \circ \phi_{q_1} \circ \phi_{q_2} \circ \ldots \circ \phi_{q_1} = \text{id} \quad \text{(c.f. Point 4.(1))}
$$

Since $\psi^{-1} \circ \Upsilon \circ \psi$ is smooth at the origin, it’s also smooth in a neighborhood of the origin; for $c$ small enough, we can look at the first component of the flow on $(c, a, \theta, I)$: here $c$ shall be the variable while $a, \theta$ and $I$ are parameters. We have the application
\[ c \mapsto e^{\partial_1 B_1 \ln |c| + i(\partial_2 B_1 \ln |c| + \arg(\delta) - \arg(c))} \]

The terms in brackets are obviously a smooth function of \( c \), and so the last exponential term is also smooth as a function of \( c \) on 0. Hence, the real part and the imaginary part are both smooth functions of \((c_1, c_2)\) in \((0, 0, I)\).

We then have the following lemma:

**Lemma 3.6.** Let \( f \in C^\infty(\mathbb{R}^k \to \mathbb{R}) \) be a smooth function such that: \( x \mapsto f(x) \ln \| x \| \) is also a smooth function.

Then \( f \) is necessarily flat in 0 in the \( k \) variables \((x_1, \ldots, x_k)\).

With this elementary lemma, we have that \((1 - \partial_1 B_1) \circ \alpha \) and \( \partial_2 B_1 \circ \alpha \) are flat for all \((0, 0, I)\), where \( \alpha(c_1, c_2, I) = (c_1 \delta_1 + c_2 \delta_2, c_1 \delta_2 - c_2 \delta_1, I) \). The function \( \alpha \) is a linear function, and it is invertible since \( \delta \neq 0 \). This gives us the flatness of \( 1 - \partial_1 B_1 \) and \( \partial_2 B_1 \), as functions of \( c_1 \) and \( c_2 \) for all the \((0, 0, I)\), and thus, as functions of all the \( n \) variables for all the \((0, 0, I)\). We have thus \( u \in \mathcal{F}_S(U \to \mathbb{R}) \) with \( S = \{(0, 0, I) | I \in \mathbb{R}^k\} \).

Supposing that \( \psi \) exchanges stable and unstable manifolds yields the same demonstration \textit{mutatis mutandis}, that is, in the last part of the proof, if we look at the first component of the flow on \((\bar{c}, a, \theta, \xi)\) with \( a, \theta \) and \( \xi \) understood as parameters. \( \square \)

In Theorem 3.3, we associate to a symplectomorphism that preserves a semi-toric foliation a unique \( G \) of the form \((\star)\). It would be interesting to have more knowledge about the restrictions on such symplectomorphism, but here we want to describe the moment map. We’d like to know for instance to what extent \( G \) is unique in Theorem 3.3.

Theorem 3.5 can be applied to answer this question. If we think of the \( G \)’s as “local models” or “maps” of the image of the moment map, then Theorem 3.5 describes the “transition functions” \( B \). Indeed, if we have two local models given by Theorem 3.3 on \( U \subseteq M \), a neighborhood of a point \( m \) of Williamson type \( k \), one has:

\[
\begin{align*}
F \circ \varphi &= G \circ Q_k, \\
F \circ \varphi' &= G' \circ Q_k.
\end{align*}
\]

Then we get:

\[
Q_k \circ (\varphi^{-1} \circ \varphi') = (G^{-1} \circ G') \circ Q_k.
\]
We can apply Theorem 3.5 to the pair \((\psi = \varphi^{-1} \circ \varphi', B = G^{-1} \circ G')\) and then get the relation:

\[
G' = (\epsilon_1^f G_1 + u, \epsilon_2^e G_2, \epsilon_1^e G_3, \ldots, \epsilon_k^e G_{k+2}, (X^f | X^e | X^x) \circ \tilde{G}), \quad u \in \mathcal{F}_S(U).
\]

(6)

### 3.2.2 Symplectic invariants for the transition functions

We have given some restriction on the transition function \(B\) between two local models of a semi-toric system. If we now authorize ourselves to change the symplectomorphism in the local models, we can have a “nicer” \(G\). This amounts to determine what in \(B\) is a semi-local symplectic invariant of the system. Let us set

\[
E_1 = \frac{1}{2} \begin{pmatrix}
1 + \epsilon_1^f & 1 - \epsilon_1^f & 0 & 0 \\
-1 + \epsilon_1^f & 1 + \epsilon_1^f & 0 & 0 \\
0 & 0 & 1 + \epsilon_1^e & 1 - \epsilon_1^e \\
0 & 0 & -1 + \epsilon_1^e & 1 + \epsilon_1^e
\end{pmatrix},
\]

\[
E_2 = \frac{1}{2} \begin{pmatrix}
1 + \epsilon_2^f & 0 & 1 - \epsilon_2^f & 0 \\
0 & 1 + \epsilon_2^f & 0 & 1 - \epsilon_2^f \\
1 - \epsilon_2^f & 0 & 1 + \epsilon_2^f & 0 \\
0 & 1 - \epsilon_2^f & 0 & 1 + \epsilon_2^f
\end{pmatrix}.
\]

**Theorem 3.7.** Let \(\psi\) be a symplectomorphism of \(L_k\) preserving \(Q_k\) and \(B\) one of the possible associated diffeomorphisms of \(\mathbb{R}^n\) introduced in Theorem 3.5 of the form \((\star)\). Consider the diffeomorphisms

\[
\zeta_B(z_1, z_2, x^e, \xi^e, \theta, I) = (e^{-i \theta \cdot (X^f)} z_1, e^{-i \theta \cdot (X^e)} z_2, e^{-i \theta \cdot (X^x)^t} \bullet z^e, \theta, I + Q_k^e \cdot (X^e)^t + q_2 \cdot (X^f)^t)
\]

and

\[
\eta_B(x_1, x_2, \xi^e, x^e, \xi^e, \theta, I) = ((x_1, \xi_1, x_2, \xi_2) E_B^t, x^e, \theta \cdot (X^x)^{-1}, I \cdot (X^x)^t)
\]

where \(E_B = E_1 E_2\).

Then we have that \(\zeta_B\) and \(\eta_B\) are symplectomorphisms of \(L_k\) which preserve the foliation \(Q_k\) and

\[
B \circ Q_k = (\psi \circ \zeta_B \circ \eta_B)^* ((1, 1, \epsilon_1^e, \ldots, \epsilon_k^e, 1, \ldots, 1) \bullet Q_k + (u \circ Q_k, 0, \ldots, 0))
\]

with \(u\) as introduced in Theorem 3.5.

(7)
The symplectomorphisms $\zeta_B$ and $\eta_B$ are admissible modifications of semi-toric local models. If we have two local models $(\varphi, G)$ and $(\varphi', G')$, Theorem \[3.7\] tells us we can always modify the symplectomorphism of one of them, for instance $\varphi'$, to get another local model $(\tilde{\varphi}', \tilde{G}')$ such that:

\[(\tilde{G}'^{-1} \circ G) \circ Q_k = ((1, 1, \epsilon_1^k, \ldots, \epsilon_k^k, 1, \ldots, 1) \bullet Q_k + (u \circ Q_k, 0, \ldots, 0))\].

**Proof.** of Theorem \[3.7\]

First let’s prove equation 7.

For $\zeta_B(q_1 + iq_2) = \zeta_B(\tilde{z}_1z_2) = e^{i\theta X^i} \tilde{z}_1e^{-i\theta X^i} z_2 = \tilde{z}_1z_2 = q_1 + iq_2$ so $q_1, q_2$ are preserved.

$$
\zeta_B^*Q_k^e = (|e^{-i\theta(X_1^e)}|^2 \cdot z_1^{e^k_1} \ldots, |e^{-i\theta(X_{k-1}^e)}|^2 \cdot z_{k-1}^{e^k_{k-1}}) = (|z_1^{e^k_1}|^2 \ldots, |z_{k-1}^{e^k_{k-1}}|^2) = Q_k^e
$$

$$
\zeta_B^*I = q_2 \cdot X^f + Q_k^e \cdot (X^e)^f + I.
$$

For $\eta$, we have: $\eta^*Q_k = (E_1 E_2)^*Q_k^e + (X^e)^*Q_k^e$, so we can treat each action separately. We can also treat $E_1$ and $E_2$ separately, as the two matrices commute, and treat only the case when $\epsilon_1^f$ (respectively $\epsilon_2^f$) is equal to $-1$, for when $\epsilon_i^f = +1$, $E_i = id$.

- $\epsilon_1^f = -1$:

$$E_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \text{ so } E_1 \cdot \begin{pmatrix} x_1 \\ \xi_1 \\ x_2 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} \xi_1 \\ -x_1 \\ \xi_2 \\ -x_2 \end{pmatrix} = \begin{pmatrix} \hat{x}_1 \\ \hat{\xi}_1 \\ \hat{x}_2 \\ \hat{\xi}_2 \end{pmatrix}
$$

$E_1^*q_1 = \hat{x}_1\hat{\xi}_1 + \hat{x}_2\hat{\xi}_2 = -\xi_1x_1 - \xi_2x_2 = -q_1 = \epsilon_1^f q_1,$

$E_1^*q_2 = \hat{x}_1\hat{\xi}_2 - \hat{x}_2\hat{\xi}_1 = \xi_1(-x_2) - \xi_2(-x_1) = q_2.$

- $\epsilon_2^f = -1$:

$$E_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}
$$

$E_2^*q_1 = \hat{x}_1\hat{\xi}_1 + \hat{x}_2\hat{\xi}_2 = x_2\xi_2 + x_1\xi_1 = q_1$

$E_2^*q_2 = \hat{x}_1\hat{\xi}_2 - \hat{x}_2\hat{\xi}_1 = x_2\xi_1 - x_1\xi_2 = -q_2 = \epsilon_2^f q_2.$

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And \((X^x)^* Q_k^x = I \cdot (X^x)^t\).
What is left is to prove the preservation of \(\omega\). For \(\zeta\) we have:

\[
\zeta^* \omega^k_k = \zeta^* \omega^I_k + \zeta^* \omega^e_k + \zeta^* \omega^x_k.
\]

\[
\zeta^* \omega^I_k = \mathfrak{Re} \left[ (e^{-i \theta \cdot X^t} dz_1 - i z_1 d\theta \cdot X^t e^{-i \theta \cdot X^t}) \wedge (e^{i \theta \cdot X^t} d\bar{z}_2 + i \bar{z}_2 d\theta \cdot X^t e^{i \theta \cdot X^t}) \right]
\]

\[
= \mathfrak{Re} \left[ dz_1 \wedge d\bar{z}_2 - d\theta \cdot X^t \wedge i(z_1 d\bar{z}_2 + \bar{z}_2 d\bar{z}_1) \right]
\]

\[
= \omega^I_k - d[\theta \cdot X^t] \wedge dq_2
\]

\[
\zeta^* \omega^e_k = \sum_{j=1}^{k_e} \mathfrak{Im} \left[ (e^{-i \theta \cdot (X^x)^t_j} dz^e_j - i e^{-i \theta \cdot (X^x)^t_j} z^e_j d\theta \cdot (X^x)^t_j) \right]
\]

\[
\wedge (e^{i \theta \cdot (X^x)^t_j} d\bar{z}^e_j + i e^{i \theta \cdot (X^x)^t_j} \bar{z}^e_j d\theta \cdot (X^x)^t_j) \right]
\]

\[
= \sum_{j=1}^{k_e} \mathfrak{Im} \left[ dz^e_j \wedge d\bar{z}^e_j - d\theta \cdot (X^x)^t_j \wedge i(z^e_j d\bar{z}^e_j + \bar{z}^e_j d\bar{z}^e_j) \right]
\]

\[
= \omega^e_k - \sum_{j=1}^{k_e} d\theta \cdot (X^x)^t_j \wedge dq^e_j
\]

\[
\zeta^* \omega^x_k = d\theta \wedge d(I + Q^e_k \cdot (X^x)^t + q_2(X^y)^t)
\]

( with formula [1] \( = \omega^x_k + [d\theta \cdot X^t] \wedge dq_2 + \sum_{j=1}^{k_e} d\theta \cdot (X^x)^t_j \wedge dq^e_j \)

So when we sum \(\zeta^* \omega^I_k, \zeta^* \omega^e_k\) and \(\zeta^* \omega^x_k\), we get that \(\zeta^* \omega = \omega\).

Now for \(\eta_B\), we can again treat separately the action on the different types \(E_i\)'s, and just treat the case when the \(\epsilon\)’s are \(-1\). We have that \(E^*_{1,2} \omega_k = E^*_{1,2} \omega^I_k + \omega^e_k + \omega^x_k\) and:

\[
E^*_1 \omega^I_k = d\hat{x}_1 \wedge d\hat{\xi}_1 + d\hat{x}_2 \wedge d\hat{\xi}_2 + d\hat{\theta}_3 \wedge d\hat{\xi}_3
\]

\[
= d\xi_1 \wedge d(-x_1) + d\xi_2 \wedge d(-x_2) + d\theta_3 \wedge d\xi_3 = \omega^I_k
\]

\[
E^*_2 \omega^I_k = d\hat{x}_1 \wedge d\hat{\xi}_1 + d\hat{x}_2 \wedge d\hat{\xi}_2 + d\hat{\theta}_3 \wedge d\hat{\xi}_3
\]

\[
= dx_2 \wedge d\xi_2 + dx_1 \wedge d\xi_1 + d\theta_3 \wedge d\xi_3 = \omega^I_k.
\]

Lastly, the transformation \((\theta, I) \mapsto (\theta \cdot (X^x)^{-1}, I \cdot (X^x)^t)\) is a linear symplectomorphism with respect to the symplectic form \(\omega^x_k = \sum_{j=1}^{k_e} d\theta_j \wedge dI_j = d\theta \wedge dI\). \(\square\)
This theorem means that the only symplectic invariants of the local model of a semi-toric critical value are the orientations of the half-spaces given by its elliptic components and its jet in $q_1, q_2$. Conservation of plans orientations is only natural: symplectic structure is exactly the algebraic area on specific plans. Conservation of the jet of the focus-focus value, is the specificity here for semi-toric systems. It is related to the Taylor expansion of action coordinates near a semi-toric critical value (see [Wac14b]).
4 Image of moment map for a semi-toric integrable system

Now that we have gathered enough results concerning local models, we can prove the principal result of this paper relying a local-to-global principle. One can notice that although the main result is given here in dimension 3, we have proved all the local results for any dimension. This opens perspectives for a more general result.

**Proof. of Theorem 1.9**

**Local proof of 1., 2. and 3.:**

Let \( p \) be a critical point of Williamson type \( k \) with \( k_1 = 1 \) of a semi-toric integrable system \((M, \omega, F)\). Applying Theorem 3.3, with the correct system of local coordinates \( \varphi \) in a neighborhood \( \mathcal{U} \) of \( p \), we have a smooth function \( G \) and a matrix \( A \in \text{GL}_{n-1}(\mathbb{Z}) \) such that \( F \circ \varphi = (G_1(Q_k), A \circ \tilde{Q}_k) \). So, the surface \( \Gamma := \text{CrV}_k(\mathcal{U}) \) is parametrized as follows: let \( t \) be here the values of \( I \). With Theorem 2.14, we know that \( t \in \tilde{D} \subseteq \mathbb{R}^{k_x} \). We define now \( h_{C^\infty}(\tilde{D} \to \mathbb{R}) \) as \( \tilde{h}(t) := G_1(0,0,0,\ldots,0, t) \). With notations of Theorem 3.3, we set

\[
\Gamma := \text{CrV}_k(\mathcal{U}) = \left\{ \tilde{h}(t) = (\tilde{h}(t), F^x \circ t, E^x \circ t, X^x \circ t) | t \in D \right\}.
\]

On \( \text{CrP}_k(\mathcal{U}) \), \( F \) is of rank \( k_x \) by definition, and in Theorem 3.3, first column of \( \text{Jac}(G) \) is \((\partial G_1, 0, \ldots, 0)^t \), so there exists a linear map \( S \) of \( \mathbb{R}^n \) such that \( S \circ (0,\ldots,0, t) \neq 0 \), \( Im(h) \) is not in \( Im(T) \).

**Global proof of 1., 2. and 3.:**

If we call \( \mathcal{U}_p \) the open set given for a point \( p \in M \) by Theorem 2.11, the family \( \{\mathcal{U}_p\}_{p \in M} \) is an open covering of \( M \), so we can extract a finite one of it. If we now fix \( k \), each open set \( \mathcal{U}_k \) gives a surface \( \Gamma_i \) in \( \text{CrV}_k(M) \) there is at most a finite number \( m_k \) of surfaces: \( \text{CrV}_k(M) = \bigcup_{i=1}^{m_k} \text{CrV}_k(U^i) = \bigcup_{i=1}^{m_k} \Gamma_i \).

We want to show that the covering \( \{\mathcal{U}_k\}_{i=1}^{m_k} \) of \( \text{CrP}_k(M) \) can be optimized in the sense that we can take a \( \mathcal{U}_k \) containing a connected component of \( \text{CrP}_k(M) \). A consequence is that \( m_k \) is then minimal.
Two open sets $\mathcal{U}_i$ and $\mathcal{U}_j$ may intersect. In this case, with Theorems 3.5 and 3.7 we can always modify one of the two local models so that we have a local model on the union $\mathcal{U}_i \cup \mathcal{U}_j$ that is a natural extension of each local model. We will see that it extends also results of Theorem 1.9.

On each open set we can apply the results proved before, and get a $h_i$ and $\mathcal{P}(\Gamma_i)$ for each $\Gamma_i$. Since the change of local model between $\mathcal{U}_i$ and $\mathcal{U}_j$ is the identity on $\text{CrP}_k(\mathcal{U}_i \cap \mathcal{U}_j)$, we have first that $\mathcal{P}(\Gamma_i) = \mathcal{P}(\Gamma_j)$. We can also set a function $h$ on the union as following

$$h_{\mathcal{U}_i \cup \mathcal{U}_j} := \begin{cases} h_i & \text{on } \mathcal{U}_i \\ h_j & \text{on } \mathcal{U}_j \end{cases}.$$ 

It is consistent because of Theorems 3.5 and 3.7. From this, we can extend step by step Theorem 1.9 from a $\mathcal{U}_i^k$ containing a $p$ to the reunion of all $\mathcal{U}_j^k$ path connected to $p$. It is an open set, the finite union of $\mathcal{U}_j^k$ that contains the connected component of $\text{CrP}_k(M)$ containing $p$. This family of open sets is finite and disjoint. Thus, $\text{CrP}_k(M)$ has a finite number of connected components which are strongly separated. This proves 1., 2. and 3. globally.

**Proof of Item 4.**

If we suppose that the fibers are connected, then for a critical value $v$, it now makes sense to talk of its Williamson index $k$. Taking $p \in F^{-1}(v)$, there is a unique connected component of $\text{CrP}_k(M)$ that contains $p$, and hence a unique $\Gamma_i$ in $\text{CrV}_k(M)$ that contains $v$. The connected components of $\text{CrP}_k(M)$ being strongly isolated, we have that $\Gamma_i$ is strongly isolated as well.

5 Conclusion

In the case $2n = 6$, Theorem 1.9 is very visual: $FF - X$ critical values are a union of nodal paths; each nodal path is contained in a plane. All planes share a common direction, and a nodal path is the embedding of the graph of a smooth function from $\mathbb{R}$ to $\mathbb{R}$.

In this article, we have presented the local tools that exists for the investigation of semi-toric and almost-toric systems. In the description of the image of the moment maps, one shall think about the $Q_k$ as “singular local coordinates” for the image of the moment map. We have showed how these singular local coordinates can be used to describe “singular manifolds”, by analogy with local coordinates and manifolds.
As calculus in local coordinates for regular manifolds, the calculus in local singular coordinates becomes heavy quickly. Nevertheless, it remains one of the most efficient techniques to provide results, even in unfriendly settings. For instance, although we don’t know the form it may take, there must be an extension of Theorem 1.9 to almost-toric systems.

Local models are one technique, it is not the only one. In upcoming articles, we give another description of the $CrV_k$, using more general arguments like Atiyah - Guillemin & Steinberg theorem. We shall rely on it to prove that the fibers of semi-toric systems are connected.

Annexe

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