Classification of no-signaling correlations and the guess your neighbor’s input game

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(Dated: September 12, 2014)

We formulate a series of non-trivial equalities which are satisfied by all no-signaling correlations, meaning that no faster-than-light communication is allowed with the resource of these correlations. All quantum and classical correlations satisfy these equalities since they are no-signaling. By applying these equalities, we provide a general framework for solving the multipartite “guess your neighbor’s input” (GYNI) game, which is naturally no-signaling but shows conversely that general no-signaling correlations are actually more non-local than those allowed by quantum mechanics. We confirm the validity of our method for number of players from 3 up to 19, thus providing convincing evidence that it works for the general case. In addition, we solve analytically the tripartite GYNI and obtain a computable measure of supra-quantum correlations. This result simplifies the defined optimization procedure to an analytic formula, thus characterizing explicitly the boundary between quantum and supra-quantum correlations. In addition, we show that the gap between quantum and no-signaling boundaries containing supra-quantum correlations can be closed by local orthogonality conditions in the tripartite case. Our results provide a computable classification of no-signaling correlations.

PACS numbers: 03.67.Mn, 03.65.Ud

I. INTRODUCTION

Quantum mechanics allows non-local correlations such as the Einstein, Podolsky, and Rosen (EPR) pairs [1]. Entanglement like EPR pairs can be used as a valuable resource for quantum information processing [2, 3] such as the well-known quantum teleportation [4, 5]. However, quantum teleportation relies on classical communication for state transmission and thus will not violate the no-signaling condition, meaning that signals cannot be sent faster-than-light. In fact, no-signaling is a general principle of quantum mechanics and it is thus satisfied by all non-local quantum correlations. It is also closely related but different from quantum causality [6–9]. A broad class of theories exist which can characterize the nonlocality of quantum physics, such as the Bell inequalities [10, 11] and the temporal analogue Leggett-Garg inequality [12], see also results in Ref. [13]. In particular, due to the recent advent of quantum information, the extremely intense study of quantum correlations such as entanglement and discord has made non-locality widely appreciated as a fundamental property of various quantum systems, see [14–24] and the references therein for related topics.

On the other hand, it is known that conversely there exist no-signaling correlations more nonlocal than those allowed in quantum mechanics [25], see a recent review paper by Popescu and the references therein [26]. Recently, a nonlocal multipartite scheme GYNI, “guess your neighbor’s input”, has been presented and investigated in Refs. [27–35]. It demonstrates that the no-signaling correlations provide a clear advantage over both classical and quantum correlations, while these two correlations have a common ground in this scheme. This scheme leads to a facet Bell inequality which is true for quantum correlations and is not implied by any other Bell inequalities. Yet its violation is consistent with no-signalling, see a views paper [28] for the implications and importance of the GYNI scheme. Despite the significant role of GYNI in clarifying the concepts of quantum correlations and fundamentals of quantum mechanics, the scheme itself is largely unsolved, even for the simplest tripartite scenario. The optimal advantage of no-signaling in GYNI has been demonstrated analytically for $N = 3$, numerically for $N = 5, 7$ cases, under the assumption of a given probability distribution. For years, with much progress and understandings related to this game, the solution of the GYNI game still seems challenging. By studying the GYNI scheme, we can distinguish quantum correlations from other supra-quantum no-signaling correlations and find the borderline between them. We can also explore the upper boundary of all no-signaling correlations. The parallel situation is the Bell inequality which can distinguish quantum correlations from classical correlations and can also be used to explore the upper bound of quantum correlations. Besides, the GYNI scheme may have important implications in understanding quantum physics and information theory.

In this paper, we propose and formulate a series of non-trivial equalities. These equalities capture the common properties of no-signaling correlations in a precise way. Based on these equalities, a general framework to solve the GYNI problem is provided. We confirm the validity of our solution for a number of players from $N = 3$ up to $N = 19$. This provides convincing evidence that the framework works. We show that the advantage of no-signaling correlations over quantum or classical cases scales to the proven bound $2$ [27, 29] and the correlations achieving the optimal bound are given. Additionally, we solve analytically the tripartite case completely. A concise form of the winning probability ratio between no-
signaling and classical or quantum correlations is obtained, which is computable analytically and thus avoids the optimization procedure. This identifies clearly the boundary between quantum correlations and no-signaling supra-quantum correlations. We also notice that with the local orthogonality condition [26], the gap harboring the existence of supra-quantum correlations can be closed in the case of the tripartite system. This fact confirms the necessity of local orthogonality in the GYNI game for quantum mechanics.

II. GYNI AND THE NO-SIGNALING EQUALITIES

Let us begin with the game of GYNI [27] shown in FIG.1. A number $N$ of players are in a round-table meeting and each receives a poker of ‘heart’ or ‘spade’ representing input bit $x_i \in \{0,1\}$. The aim is that each player provides an output bit $a_i \in \{0,1\}$ representing the guess about his/her right-hand neighbor’s input. No communication is allowed after the inputs are distributed and thus no-signaling is ensured. The input strings $x = x_1,...,x_N$ are chosen according to some prior fixed probability distributions $q(x)$ known to all players, where $P(a_i|x_{i+1}x_{i-1})$ is the probability of obtaining the output $a_i$ when the input $x_{i+1}x_{i-1}$ is given. The probabilities satisfy the identity $P(x,...,0,0,...,0) = 1$, meaning that the probability summation over all possible outputs for a given input is 1, where ‘x’ is assumed to be a summation of all possible outputs at each position. The correct output probability is denoted as $P(a_i = x_{i+1}|x)$. The average winning probability is thus quantified as $\omega = \sum q(x)P(a_i = x_{i+1}|x)$. The winning probabilities by classical strategies and general no-signaling ones are denoted as $\omega_c, \omega_{ns}$, respectively. No quantum advantage over the classical case is available in this game [27], meaning that $\omega_c$ is applicable for the quantum case. This is due to the condition that no communication is allowed in this scheme. We remark that the $N = 2$ case is trivial.

We study the GYNI game first by considering the odd $N$ case and the specified input distribution, $q(x) = 1/2^{N-1}$ when $x_1 \oplus x_2 \oplus ... \oplus x_N = 0$ and $q(x) = 0$ otherwise, which we stick to in this work unless stressed explicitly. We know that $\omega_c = 1/2^{N-1}$ for both classical and quantum resources [27]. Now we show that for no-signaling resources,

$$\max \omega_{ns} = \frac{2}{1 + C_{N-1}^N 2^{N-1} - \omega_c}. \quad (1)$$

We can directly verify that $\max \omega_{ns}/\omega_c$ is larger than 1 and scales to 2 for large $N$, thus saturating the upper bound [27], see FIG.2. Hereafter, we generally explore the upper bound of $\omega_{ns}$ and the notion ‘max’ will be dropped with no confusion.

Our proof of Eq.(1) is based on a series of no-signaling equalities presented below. These equalities belong to facet Bell inequalities, meaning that they are not violable by quantum mechanics and are not implied by other Bell inequalities. On the other hand, they are equalities instead of inequalities and we may name them as Bell equalities, see FIG.3 for explanations about their role in classifying no-signaling correlation. Our proposed no-signaling Bell equalities for the concerned probabilities in the GYNI game take the form:

$$\sum_{x,\omega} (P(x_2,...,x_N|x_1x_2,...,x_N) + P(x'_2,...,x'_N|x_1x_2,...,x_N)) = 1,$$

where the summation is under the condition that the sum of $x_i$ is less than $N/2$. The first half terms $P(x_2,...,x_N|x_1x_1,...,x_N)$ are of GYNI interest, and the second half of the terms $P(x'_2,...,x'_N|x_1x_2,...,x_N)$ are the pairing terms corresponding to the first half. The one-to-one correspondence of $x'_2,...,x'_N$ and $x_1x_2,...,x_N$ is given by

$$x'_i = \begin{cases} 0 & \text{if } x_i = 0 \text{ and } \exists j_i \sum_{k=1}^i (2x_{i+k} - 1) > 0 \\ 1 & \text{otherwise}. \end{cases} \quad (3)$$

Intuitively, this construction starts from the terms with the most 1’s and changes one of the $x_i$’s from 1 to 0; it then cascades down until all probability terms have an even number of 1’s to match the terms of GYNI interest, see FIG.3 for explanations about their role in classifying no-signaling correlation. Our proposed no-signaling Bell equalities for the concerned

![FIG. 1: (color online) The game of “guess your neighbor’s input”](image)

The aim is that each player provides an output bit $a_i \in \{0,1\}$ representing the guess about his/her right-hand neighbor’s input. Here $[0,1]$ are represented by ‘spade’ and ‘heart’ of the pokers. No communication is allowed after the inputs are distributed in this game.

With $N = 5$ as a simple example, the 11 terms with zero $x_i$’s from 1 to 0; it then cascade down until all probability terms have an even number of 1’s to match the terms of GYNI interest, see FIG.3 for explanations about their role in classifying no-signaling correlation. Our proposed no-signaling Bell equalities for the concerned
or $4m + 2, 4m, 4m - 2... 2m + 2$ 1s in the case of $N = 4m + 3$, where $m$ is a positive integer. The number of these terms, as a function of odd $N$, can be expressed as $\sum_{i=0}^{m} C_{4m+1}^{2i}$, or $\sum_{i=0}^{m} C_{4m+1}^{2m+2i}$, which can both be reduced to $2^{N-2} + C_{N-1}^{(N-1)/2}/2$ using the combinatorial relation $C_{m+1}^{m+1} = C_{m}^{m} + C_{m}^{m+1}$. This shows the upper bound of the no-signaling winning probability.

$$\omega_{ns} \leq \frac{2}{2^{N-1} + C_{N-1}^{(N-1)/2}}.$$  

### III. THE TIGHT UPPER BOUND AND THE CONSTITUENTS OF NO-SIGNALING CORRELATIONS

Now we demonstrate that no-signaling correlations saturating the inequality (4) can be found. Thus the inequality turns out to be an equality in the optimal case. Generally the number of inequalities is less than the degrees of freedom of the correlations, and such correlations are not unique. However, under the restrictions of basic symmetry (invariance under relabeling inputs, outputs and states), only one solution can be found for $N = 3, 5$ and 7. For $N = 9$, the solutions have two degrees of freedom, and when $N$ becomes larger the degrees of freedom increase exponentially.

The only no-signaling symmetric correlation for $N = 3$ can be expressed as:

$$P(ab|xyz) = \frac{1}{6} (x \otimes y \otimes z \otimes xy \otimes zy \otimes zx \otimes ab \otimes bc \otimes ca$$

$$\otimes cb \otimes ya \otimes xc) + \frac{1}{3} abcxyz + \frac{1}{3} abcxyz + \frac{1}{3} abcxz$$

In these correlations the GYNI probability terms, $P(x_1...x_N | x_1^1...x_1^N)$ with $x_1 \otimes x_2 \otimes ... \otimes x_N = 0$, are all equal to $2/(2^{N-1} + C_{N-1}^{(N-1)/2})$, while the Bell equality (2) can be satisfied. This means that the winning probabilities achieve the upper bound. The advantage of no-signaling over quantum and classical correlations takes the form (3). We remark that this result is confirmed for a number of players up to 19 by a computer workstation (16-core, 384G-memory). The calculations involve the proof of the Bell equality (2) and the saturating of the bound (4) both for $N = 3$ and for $N = 3$ up to 19.

FIG. 2 shows the asymptotical behavior of the winning ratio of no-signaling correlations over quantum or classical correlations (Eq. (1)).

FIG. 3 shows the above proposed Bell equality and GYNI game in describing various correlations. The largest volume in the figure represents no-signaling correlations. The no-signaling correlations contain quantum correlations as a subset, while the quantum correlations contain classical correlations as a subset. They all share a partly common boundary which is the bottom of the classical volume in this figure. It is described by our proposed Bell equality and thus is marked as ‘Bell equality’. The Bell equality can be checked for general tripartite qubit state which is proven to take a simple form in

### IV. ANALYTIC FORMULA, NO-SIGNALING INEQUALITIES AND LOCAL ORTHOGONALITY

We next consider arbitrary given inputs and assume $N \geq 3$ being both odd and even numbers. The necessary probability inequalities in the tripartite case come from various Bell

![Image](image3.png)

FIG. 3: (color online) Schematic representation of various correlations. Those correlations include classical, quantum and no-signaling correlations. The Bell equality and the GYNI game can identify different boundaries between those correlations. The well-known Bell inequality is also marked.

Ref. [36]. The violation of the Bell equality means that no-signaling correlations cannot accommodate this phenomenon. Quantum correlations and classical correlations may share a common part of boundary distinguishing them from supraquantum correlations which satisfy the no-signaling condition but are beyond quantum mechanics. This boundary can be identified by $\omega_{ns}/\omega_c = 1$ which is marked as ‘GYNI=1’ in this figure. For the tripartite case, we can identify this boundary analytically by using Eq. (6) presented later when it equals to 1. This is the first computable measure of supraquantum correlations. The $\omega_{ns}/\omega_c > 1$ part belongs solely to no-signaling supra-quantum correlation with the boundary identified by Eq. (1). The well-known Bell inequality distinguishes quantum correlations and classical correlations which are also marked in this figure.

![Image](image2.png)

FIG. 2: (color online) Winning probability ratio for no-signaling correlations over classical or quantum correlations. We assume $N$ is odd. The star symbols represent our result of Eq. (1), which will approach 2 asymptotically when $N$ is large.
equalities and have a clear geometric representation.

FIG. 4 shows those inequalities for the tripartite case. A total of 14 restrictions on the probability terms can be found using the hypercube-hyperplane representation of inequalities, which can be divided into two classes. Left panel (a) represents the first class of 6 inequalities that can be represented by $P(000|000) + P(010|001) + P(001|100) + P(011|101) \leq 1$ which corresponds to the normal vector $(0, 1, 0)$. Right panel (b) represents the second class of 8 inequalities that can be represented by $P(000|000) + P(010|001) + P(100|010) + P(001|100) \leq 1$, which corresponds to the normal vector $(1, 1, 1)$. In general, we can construct a hypercube of dimension $N$ in a Cartesian space and all the vertices of the hypercube have a coordinate $(x_1, x_2, ..., x_N)$ corresponding to GYNI interest probability $P(x_2...x_N|x_1)$ for all $x_1$. We propose that, for every hyperplane passing through the center of the $N$-dimensional hypercube while not passing through any vertex, there exists a pair of corresponding inequalities that limit the sums of the probability terms on the two sides of the hyperplane. Explicitly, given a real vector $s_1, s_2, ..., s_N$, if the equations $\sum_i s_i x_i - 1/2 = 0$ and $x_i \in [0, 1]$ have no solutions then we should have $\sum_i s_i x_i \geq P(x_2...x_N|x_1) \geq 1$. This has been proven to be true for $N = 3, 4$ and 5 by direct verification. Instead of the original no-signaling principles, these inequalities can be used to estimate the upper bound of $\omega_{ns}$. However, we remark that they are not complete for general $N$. Using these inequalities, we can find that, for a given input distribution $q(x)$, the maximum ratio of winning probability is

$$\max_{\omega_{ns} \leq \omega_c} = \max \left( \frac{q(000) + q(110) + q(101) + q(011)}{3\omega_c}, \frac{q(100) + q(010) + q(001) + q(111)}{3\omega_c} \right)$$

(6)

where $\omega_c = \max(q(x) + q(\bar{x}))$ is the classical winning probability (see Appendix B for a detailed proof). This is the first analytic measure of supra-quantum correlations without optimization.

Furthermore, the ratio derived here for arbitrary input distributions can always be reached. Consider two no-signaling resources: the first one is given by $P(a\oplus c|x\oplus x') = (a \oplus y \oplus x \oplus x')(b \oplus z \oplus y \oplus y')(c \oplus x \oplus y \oplus y')$, the classical strategy described in [27], where $x = x' y' z'$ maximizes $(q(x) + q(\bar{x}))$, and the second one is the extreme no-signaling correlation [5]. It can be easily checked that after suitable relabeling, the first one gives $\omega_{ns} = \omega_c$, and the second one gives $3\omega_{ns} = \max(q(000) + q(110) + q(101) + q(011), q(100) + q(010) + q(001) + q(111))$.

We can also look at the no-signaling GYNI game with four parties for arbitrary input distributions. We first show that no input distributions can achieve a higher probability ratio than the distributions satisfying $q(x) \in [0, 1/2^{N-1}]$ and $q(x) + q(\bar{x}) = 1/2^{N-1}$. From [27], we know that $\omega_c = \max_{\omega} q(x) + q(\bar{x})$. If for some $x$, $q(x) + q(\bar{x}) < \omega_c$, then we can increase $q(x)$ so that $q(x) + q(\bar{x}) = \omega_c$ and renormalize $q(x)$. In this process $\omega_{ns}$ does not decrease and $\omega_c$ does not change, so the ratio does not decrease. When $q(x) + q(\bar{x}) = \omega_c$ for all $x$, we must have $\omega_{ns} = 1/2^{N-1}$. Since $\omega_{ns}$ is linear in $q(x)$ for a fixed no-signaling correlation, the maximum can only be achieved when either $q(x) = 0$, $q(\bar{x}) = 1/2^{N-1}$ or $q(\bar{x}) = 0$, $q(x) = 1/2^{N-1}$.

By the above reasoning, we only need to check a small number of input distributions. This then becomes some linear programming problems with several target functions. There are three inequivalent normal vectors, $(1, 0, 0, 0), (5, 2, 2, 2)$, and $(1, 1, 1, 0)$ in the hypercube-hyperplane (geometric) representation, corresponding to a total of 104 inequalities. Under these restrictions, we tested all 256 input distributions in the form $q(x) \in [0, 1/8]$ and $q(x) + q(\bar{x}) = 1/8$. Calculations show that $\max_{\omega_{ns}, \omega_c, N = 4} = 4/3$. The result shows that no matter how the input distribution is given, adding a party into the game cannot help no-signaling resources doing better.

Remarkably, it has recently been proposed that local orthogonality may play a critical role for the characterization of quantum mechanics, in particular for distinguishing supra-quantum correlations [30]. Explicitly, local orthogonality offers new inequalities satisfied by the correlations. By some calculations (presented in Appendix B), we find that the gap between boundaries of no-signaling and quantum in GYNI which harbors supra-quantum correlation will be completely closed by local orthogonality conditions. This fact confirms that local orthogonality is necessary in distinguishing supra-quantum correlation.

V. CONCLUSIONS

In summary, we provide a valid framework for solving the general GYNI game by introducing a series of non-trivial equalities which might be named as Bell equalities. All no-signaling correlations satisfy these equalities and the violation of them means the violation of no-signaling. We also obtain a concise form for measure of supra-quantum correlations without relying on an optimization procedure which is generally a hard task. We remark that the supra-quantum correlations will be removed by local orthogonality conditions. Our results offer a classification of no-signaling correlations. A lot of new questions arise related to the results in this article. For example, the proof of general Bell equalities and their applications, measures of supra-quantum correlation for more general cases, the explicit relationship between Bell inequality and Bell inequality. These are still open problems which are...
worth studying further.

Acknowledgments

This work was supported by the ’973’ Program (2010CB922904), NSFC (11175248), NFFTBS (J1030310, J1103205) and grants from the Chinese Academy of Sciences.

Appendix A: Proof and the correspondence of the no-signaling equality

We first present the proof for $N = 5$. We write the equality explicitly and get:

$$P(00000|00000) + \sum_{c} (P(10000|01000) + P(11000|01100) + P(10100|01010) + P(11100|01100)) = 1,$$  \quad (A1)

where the subscript ‘c’ means cyclic summation of the input and output.

To prove this equality, note that

$$P(10000|01000) + P(11000|01100) + P(10100|01010) + P(11100|01100) = P(10000|01000) + P(11000|01100) + P(10100|01110) + P(11100|00110)$$

$$= P(10000|00100) + P(11000|00100) + P(10100|00110) + P(11100|00110), \quad (A2)$$

where an ‘x’ in the output stands for summation of all possible states at that position, and the no-signalling principle has been used several times. Substitute this into the left hand side and the equation can be transformed into:

$$P(00000|00000) + \sum_{c} (P(1x00|00000) + P(11110|00000)) = P(xxxx|00000) = 1,$$  \quad (A3)

which completes the proof of this equality for $N = 5$. Proofs for larger $N$s can be constructed similarly with the aid of a computer.

The correspondence between the pair $x'_1, x'_2, ..., x'_N$ and $x_1, x_2, ..., x_N$ was described as an existence criterion. Now we give a different way of constructing the correspondence. Consider the following procedures:

1. Copy the input string onto a piece of cyclic paper (which ensures that $x_{N+1}$ is equivalent to $x_1$).

2. For each ‘1’ on the paper, cross it out, then find the nearest ‘0’ not crossed out on the left of this ‘1’ and cross this ‘0’ out.

3. For the numbers not crossed-out (which must be ‘0’), change them to ‘1’.

4. Now the paper contains a new string, which is the desired $x'_1, x'_2, ..., x'_N$.

We have to clarify some points of this procedure. First, it should be obvious that step 2 of this procedure is well-defined; that is, the result is the same regardless of the sequence we cross out 1s. Second, it is always possible to finish step 2, since for $x_i$ we have the restriction $\sum x_i < \frac{N}{2}$, and there are more zeros than ones in the input string. Each one ‘cancels’ a zero, and there should be an equal number of crossed-out zeros and ones.

We now show that the procedure is equivalent to the existence criteria we have given. The statement ‘for some $j$ there are more ones than zeros in $x_{i+1}...x_{i+j}$’ by step 2 of the procedure we described, the zero at this position will be ‘canceled’ by a one from its right and remain as a zero in its pair string. If this is not the case, then either $x_i = 1$, or $x_i = 0$ which is not crossed out and $x'_i = 1$.

Using this procedural description, we can also show that this correspondence is one-to-one. Given $x'_1, x'_2, ..., x'_N$, we can go through a similar procedure in the reverse way to recover the original string $x_1, x_2, ..., x_N$. For all ‘0’s in the string, cross it out, and find the nearest ‘1’s not crossed out on the right of this ‘0’ and cross them out. Then the remaining ‘1’s should be changed to ‘0’s.

We notice that the pairing process always introduce new terms satisfying $\sum'_{i=1}(x_i + x'_i) = N$. We believe that this pairing process should have physical implications, but by now we have not found a clear explanation of this.

Appendix B: Derivation of the maximum ratio

Here we prove the maximum ratio of winning probability for $N = 3$, expressed as a maximizing function of the input distribution.

For simplicity, we introduce some short-form notations: $P(000|000)$ will be shortened to $P_0$, $P(010|010)$ to $P_1$, and $q(010)$ to $q_2$, and so on. Then the maximum ratio becomes:

$$\max \frac{\omega_{ns}}{\omega_c} = \max (1, \frac{q_0 + q_3 + q_5 + q_6}{3\omega_c}, \frac{q_1 + q_2 + q_4 + q_7}{3\omega_c}),$$  \quad (B1)

where $\omega_c = \max(q_0 + q_7, q_1 + q_6, q_2 + q_5, q_3 + q_4)$ is the classical winning probability.

The proof is divided into two parts. The first part assumes $q_0 + q_3 + q_5 + q_6 \geq 3\omega_c$. We start by choosing 4 inequalities out of 14, namely

$$P_0 + P_1 + P_3 + P_5 \leq 1,$$
$$P_0 + P_3 + P_5 + P_7 \leq 1,$$
$$P_0 + P_4 + P_5 + P_6 \leq 1,$$
$$P_3 + P_5 + P_6 + P_7 \leq 1.$$  \quad (B2)
We multiply these 4 inequalities by \( q_0 + q_3 + q_5 - 2q_6, q_0 + q_3 + q_5 - 2q_4, q_0 + q_5 + q_6 - 2q_1 \) and \( q_3 + q_4 - 2q_2 \) respectively. Since \( q_0 + q_3 + q_5 + q_6 \geq 3\omega_c \geq 3q_1 + 3q_6, q_0 + q_3 + q_5 - 2q_6 \geq 3q_1 \geq 0 \) and by symmetry all the coefficients here are nonnegative. Adding these together and we get

\[
3(q_0 P_0 + q_3 P_3 + q_5 P_5 + q_6 P_6) + (q_0 + q_3 + q_5 - 2q_4)P_1 + (q_0 + q_3 + q_6 - 2q_3)P_2 \\
+ (q_0 + q_5 + q_6 - 2q_2)P_4 + (q_3 + q_5 + q_6 - 2q_0)P_7 \\
\leq q_0 + q_3 + q_5 + q_6. \tag{B3}
\]

Since \( q_0 + q_3 + q_5 - 2q_6 \geq 3q_1 \), we have \( (q_0 + q_3 + q_5 - 2q_6) P_1 \geq 3q_1 P_1 \), thus changing the left hand side gives

\[
3(q_0 P_0 + q_3 P_3 + q_5 P_5 + q_6 P_6 + q_1 P_1 + q_2 P_2 + q_4 P_4 + q_7 P_7) \leq q_0 + q_3 + q_5 + q_6. \tag{B4}
\]

Noticing that \( \omega_{\text{ns}} = \sum_{i=0}^{7} q_i P_i \), we have

\[
\omega_{\text{ns}} \leq \frac{q_0 + q_3 + q_5 + q_6}{3} \tag{B5}
\]

The proof for \( q_1 + q_2 + q_4 + q_7 \geq 3\omega_c \) is similar and we have \( \omega_{\text{ns}} \leq (q_1 + q_2 + q_4 + q_7)/3 \).

The second part assumes \( q_0 + q_3 + q_5 + q_6 < 3\omega_c \) and \( q_1 + q_2 + q_4 + q_7 < 3\omega_c \). We prove that \( \omega_{\text{ns}}/\omega_c \leq 1 \) by constructing a new input distributions so that \( \omega_{\text{ns}}/\omega_c \geq \omega_{\text{ns}}/\omega_c \) and then show that \( \omega_{\text{ns}} \leq \omega_c \).

\( q' \) is constructed from \( q \) as follows:

\[
Q'_i = q_i + (\omega_c - q_1 - q_7 - \cdots) s_i (s_1 + s_7 - \cdots),
\]

\[
\omega'_{\text{ns}} = \sum_{i=0}^{7} Q'_i P_i / \omega_c = \omega_{\text{ns}} / \omega_c. \tag{B8}
\]

Now without loss of generality, we assume that \( \min q'_i = q'_1 \). Because \( Q'_1 + Q'_2 + Q'_5 + Q'_6 \leq 3\omega_c \), \( q'_1 + q'_5 + q'_6 \leq 3\omega_c \) and \( q'_0 \leq \omega'_{\text{ns}} \).

If \( q'_1 + q'_2 + q'_4 - q_0 \leq \omega_c \), we use the following inequalities:

\[
q'_1 + q'_2 + q'_4 - q_0 \leq \max(1, \frac{q_0 + q_2 + q_5}{3\omega_c}, \frac{q_1 + q_2 + q_4}{3\omega_c}), \tag{B10}
\]

which completes the proof.

We can be easily verified that all coefficients are nonnegative. Adding all of these together and we get

\[
q_0 P_0 + q_1 P_1 + q_2 P_2 + q_4 P_4 + (q'_0 + q'_1 - q'_4) P_5 \leq q'_0 + q'_4.
\]

Similarly, if \( q'_1 + q'_2 + q'_4 - q_0 \geq \frac{\omega_c}{2} \), then we use the following inequalities:

\[
q'_1 + q'_2 + q'_4 - q_0 \leq 
\]

Adding all of these together and we get

\[
q'_0 P_0 + q'_1 P_1 + q'_2 P_2 + q'_4 P_4 + (q'_0 + q'_1 - q'_4) P_5 \leq q'_0 + q'_4.
\]

Putting everything together, we have

\[
\max \frac{\omega_{\text{ns}}}{\omega_c} = \max(1, \frac{q_0 + q_2 + q_5}{3\omega_c}, \frac{q_1 + q_2 + q_4}{3\omega_c}), \tag{B14}
\]

By studying the no-signaling equalities, we will find multipartite no-signaling correlations that violate the quantum bound. On the one hand, we may wonder whether no-signaling theories other than quantum mechanics are necessary and this will motivate us to explore how much quantum mechanics can be violated by no-signaling correlations, as we have already done. On the other hand, it is also an interesting question what additional principles we need to constrain no-signaling theories down to quantum mechanics.

Here, we show explicitly that if we use the Local Orthogonality (LO) restrictions [30], we will recover the common boundary of classical and quantum mechanics:
which means that LO inequalities are complete for this input-undetermined GYNI problem. The proof is straightforward: LO adds two new inequalities, \( P_0 + P_3 + P_5 + P_6 \leq 1 \) and \( P_1 + P_2 + P_3 + P_7 \leq 1 \) to the list of inequalities. By the previous proof we only need to consider the case \( q_0 + q_3 + q_5 + q_6 \geq 3\omega_c \). Now we set \( \min(q_0, q_3, q_5, q_6) = q_0 \) without the loss of generality. Then we have

\[
\omega_{LO} = \sum_{i=0}^{7} q_i P_i \leq q_0 + (q_1 P_1 + q_2 P_2 + (q_3 - q_0) P_3 + q_4 P_4 + (q_5 - q_0) P_5 + (q_6 - q_0) P_6 + q_7 P_7. \tag{B16}
\]

Take \( (0, q_1, q_2, q_3 - q_0, q_4, q_5 - q_0, q_6 - q_0, q_7) \) as a new input distribution, with classical winning probability \( \omega_c - q_0 \) and the relationship \( q_1 + q_5 + q_6 - 3q_0 \leq 3\omega_c - 3q_0 = 3(\omega_c - q_0) \). Using the same reasoning of the second part of the previous proof, we have

\[
q_1 P_1 + q_2 P_2 + (q_3 - q_0) P_3 + q_4 P_4 + (q_5 - q_0) P_5 + (q_6 - q_0) P_6 + q_7 P_7 \leq \omega_c - q_0. \tag{B17}
\]

which means that \( \omega_{LO} \leq \omega_c \).

As we already know that \( \omega_c \) is reachable, we conclude that LO will close completely the gap between no-signaling and quantum. This fact is proven for tripartite state with arbitrary probability distributions, extending the results of fixed input distributions found in Ref [20].