Coverability in 1-VASS with Disequality Tests

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Abstract
We show that the control-state reachability problem for one-dimensional vector addition systems with disequality tests is solvable in polynomial time. For the test-free case we moreover show that control-state reachability is in NC, i.e., solvable in polylogarithmic parallel time.

2012 ACM Subject Classification
Theory of computation → Logic and verification

Keywords and phrases
Counter automata, Vector addition systems with states, Formal verification

1 Introduction
Algorithmic properties of one-counter automata, including reachability, model checking, and equivalence, have been studied by many authors over several decades [2,3,4,5,6,7,8,10]. The above references are a small subset of the extensive literature on one-counter automata, but they well illustrate that there are many variations on the basic model and that these variations can lead to the model having substantially different algorithmic properties. Particular features mentioned in the references above, driven by applications to automated verification and program analysis, include equality tests, disequality tests, inequality tests, parametric tests, binary updates, polynomial updates, and parametric updates.

Analysing the complexity of reachability in the presence of the features listed above leads to a rich complexity landscape. It is shown in [10] that control-state reachability is decidable in NL for a “plain vanilla” model of one-counter machine—namely with a counter taking values in the nonnegative integers with operations increment, decrement, and zero testing. Thinking of one-counter automata as one-dimensional vector addition systems with states (1-VASS), it is natural to allow the counter to be updated by adding integer constants in binary. In this case, still with equality tests, control-state reachability becomes NP-complete [5]. The NP upper bound here is non-trivial since, due to the binary encoding of integers, a computation that reaches the goal state may have length exponential in the size of the machine. If one enriches the model further by introducing inequality tests (comparing the counter with an integer constant) then control-state reachability becomes PSPACE-complete [5]. A model of intermediate complexity is one with equality and disequality tests (introduced in [4], with applications to temporal-logic model checking). In this case the complexity of control-state reachability is open (between NP and PSPACE).

In this paper we consider 1-VASS with disequality tests, but no equality tests. Our main result is that the control-state reachability problem in this setting is solvable in polynomial time. This result confirms the intuition that disequality tests are weaker than equality tests. The main technical challenge to obtaining a polynomial-time bound is that a run witnessing
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Figure 1 A 1-VASS with disequality tests, derived from a 3-CNF formula \( \varphi \) having propositional variables \( X_1, \ldots, X_m \) and clauses \( C_1, \ldots, C_n \). Let \( p_1, \ldots, p_m \) be the first \( m \) primes and write \( P := p_1 \cdots p_m \) for their product. Define \( \text{val} : \mathbb{N} \times \{X_1, \ldots, X_m\} \to \{0, 1\} \) by \( \text{val}(u, X_i) = 1 \) if and only if \( p_i \mid u \). We have states \( s_1, \ldots, s_n \)—one state for each clause—and an initial state \( s_0 \). Suppose that state \( s \) corresponds to a clause \( C \) that mentions variables \( X_{i_1}, X_{i_2}, X_{i_3} \). Then we place a self-loop on \( s \) with increment \( c_i := p_{i_1}^2 p_{i_2} p_{i_3} \) and define the domain \( D_{s} \subseteq \mathbb{N} \) of allowable counter values in state \( s \) to exclude all values \( u \in \{P, P + 1, \ldots, P + p_{i_1} p_{i_2} p_{i_3} - 1\} \) such that \( \text{val}(u, \cdot) \) satisfies the clause \( C \). Given \( u \in \{0, 1, \ldots, P - 1\} \), the configuration \((s_0, u)\) is bounded iff \( \text{val}(u, \cdot) \) satisfies \( \varphi \). Hence \((s_0, u)\) is unbounded for all \( u \in \mathbb{N} \) iff \( \varphi \) is unsatisfiable (see Appendix A for a complete proof).

that a given control state is reachable may have length exponential in the description of the counter automaton. A standard way to overcome this obstacle in related settings is to show that one may restrict attention to computations that fit a regular pattern (usually in terms of iterating a “small” number of cycles). Here the presence of disequality tests proves to be surprisingly disruptive: it destroys the monotonicity of the transition relation and prevents from freely iterating positive-weight cycles. (For example, the failure of monotonicity means that it is \text{coNP} \hard to determine whether all configurations in a given initial location are unbounded—see Figure 1—whereas the same problem for 1-VASS without tests is easily seen to be decidable in polynomial time.) Resolving the complexity of reachability for 1-VASS with both equality and disequality tests remains open. We hope that the techniques developed here can help solve this challenging problem.

To complement our main result we show that for 1-VASS without tests, control-state reachability (and hence also boundedness) is decidable in \text{NC}, i.e., the subclass of \text{P} consisting of problems solvable in polylogarithmic parallel time. Problems in \text{NC} are in particular solvable in polylogarithmic space. Related to this Rosier and Yen [41] have shown that boundedness for VASS is \text{NL}-complete in case there are absolute bounds on the dimension and bit-size of integer vectors.

Due to constraints on space, most proofs appear in the appendix.

2 Definitions

We write \( \mathbb{N} \) to denote the set of all nonnegative integers \( 0, 1, 2, \ldots \). In presenting our results we assume familiarity of the reader with basic graph theory and computational complexity.

One-Dimensional Vector Addition Systems with States and Tests. A 1-VASS with disequality tests is a tuple \( \mathcal{V} = (Q, D, \Delta, w) \), where \( Q \) is a set of states, \( D = \{D_q\}_{q \in Q} \) is a collection of cofinite subsets \( D_q \subseteq \mathbb{N} \), \( \Delta \subseteq Q \times Q \) is a set of transitions, and \( w : \Delta \to \mathbb{Z} \) is a function that assigns an integer weight to each transition. In the special case that each \( D_q \) equals \( \mathbb{N} \), we simply call \( \mathcal{V} \) a 1-VASS (and we omit the collection \( D \)).

A configuration of \( \mathcal{V} \) is a pair \((q, z)\) comprising a state \( q \in Q \) and a nonnegative integer \( z \in \mathbb{N} \) referred to as the counter value. We write \( \text{Conf} \) for the set \( Q \times \mathbb{N} \) of all configurations. We define a partial order on \( \text{Conf} \) by \((q, z) \leq (q', z')\) if and only if \( q = q' \) and \( z \leq z' \). A configuration \((q, z)\) is \text{valid} if \( z \in D_q \).
A path in $\mathcal{V}$ is a sequence of states $\pi = q_1, \ldots, q_n$ such that $(q_i, q_{i+1}) \in \Delta$ for all $i \in \{1, \ldots, n-1\}$. We sometimes refer to such a path as a $q_1$-$q_n$ path. Let $\pi' = p_1, p_2, \ldots, p_m$ be another path such that $q_0 = p_1$, we define $\pi_1 \cdot \pi_2 := q_1, q_0, q_2, p_2, \ldots, p_m$. Given states $p, q, r$, a set $P$ of $p$-$q$ paths, and a set $R$ of $q$-$r$ paths, we define $P \cdot R := \{ \pi' \cdot \pi \mid \pi \in P, \pi' \in R \}$.

The weight of $\pi$ is defined to be $\text{weight}(\pi) := \sum_{i=1}^{n-1} w(q_i, q_{i+1})$. A (possibly empty) prefix of $\pi$ is said to be minimal if it has minimal weight among all prefixes of $\pi$. Define $\text{pmin}(\pi)$ to be the weight of a minimal prefix of $\pi$.

A run is a sequence $(q_1, z_1), \ldots, (q_n, z_n)$ of configurations of $\mathcal{V}$ such that there is a path $\pi = q_1, \ldots, q_n$ with $z_{i+1} = z_i + w(q_i, q_{i+1})$ for $i = 1, \ldots, n-1$. We write $(q_1, z_1) \xrightarrow{\pi} (q_n, z_n)$ to denote such a run. Observe that runs are not allowed to reach negative counter values. A valid run through $\mathcal{V}$ is a run whose configurations are all valid. Intuitively, a valid run through $q$ can proceed if and only if the current counter value is in $D_q$.

In computational problems all numbers in the description of $\mathcal{V}$ are given in binary. Given a state $q$ we represent the cofinite set $D_q$ as the complement of an explicitly given subset of $\mathbb{N}$. Given this convention, we can assume without loss of generality that for all states $q$ the set $D_q$ is either $\mathbb{N}$ or $\mathbb{N} \setminus \{g\}$ for some $g \in \mathbb{N}$; see Appendix B. For states $q$ with $D_q = \mathbb{N} \setminus \{g\}$, we refer to the single missing value $g$ in the domain as the disequality guard on $q$.

The Coverability and Unboundedness Problems. Let $\mathcal{V} = (Q, \Delta, D, w)$ be a 1-VASS with disequality tests, and let $s$ and $t$ be two distinguished states of $\mathcal{V}$. The Coverability Problem asks whether there exists a valid run in $\mathcal{V}$ from $(s, 0)$ to $(t, z)$ for some $z \in \mathbb{N}$ (in which case we say that $(s, 0)$ can cover $t$). The Unboundedness Problem asks whether the set of configurations reachable from $(s, 0)$ is infinite (in which case we say that $(s, 0)$ is unbounded).

The Coverability problem reduces to the Unboundedness problem by, intuitively, forcing $(t, 0)$ to be unbounded using a positive cycle, and removing all states that cannot reach $t$ in the underlying graph of $\mathcal{V}$. In fact, the following holds.

\textbf{Lemma 1.} There is an $\text{NC}^2$-computable many-one reduction from the Coverability Problem to the Unboundedness Problem for 1-VASS with disequality tests.

Henceforth, we focus on the complexity of deciding the Unboundedness Problem. In Section 3 we prove that the Unboundedness Problem for 1-VASS with disequality tests is decidable in polynomial time. Since $\text{NC}^2 \subseteq \text{P}$, by Lemma 1 we also have that the Coverability Problem in this setting is decidable in polynomial time. In Section 4 we prove that the Unboundedness Problem for 1-VASS (without disequality tests) is in $\text{NC}^2$, and we deduce that the Coverability Problem for 1-VASS is decidable in $\text{NC}^2$.

3 Unboundedness for 1-VASS with Disequality Tests

Fix a 1-VASS $\mathcal{V} = (Q, D, \Delta, w)$ with disequality tests and a distinguished state $s \in Q$. We are interested in determining whether the configuration $(s, 0)$ is unbounded.

For a (possibly infinite) path $\pi = q_1, q_2, \ldots$, denote by $\text{blocked}(\pi)$ the set of $z \in \mathbb{N}$ such that $\pi$ does not lift to a valid run from the configuration $(q_1, z)$, i.e., the unique induced run either contains a negative counter value or violates a disequality guard.

\textbf{Example.} In Figure 2, since 41 is the guard on $s_5$ the run $(s_4, 93), (s_5, 41), (s_6, 93)$ is not valid and 93 $\in \text{blocked}(s_4, s_5, s_6)$. Observe that $\text{blocked}(s_4, s_5, s_6) = [0, 52] \cup \{90, 93, 96\}$ and $\text{blocked}(s_4, s_5, s_6^{\infty}) = [0, 52] \cup \{52 \leq z \leq 96 \mid z \equiv 0, 3, 6 \pmod{9}\}$.

Recall that for a path $\pi$, $\text{pmin}(\pi)$ is the weight of a minimum-weight prefix of $\pi$. Let $Q_+ \subseteq Q$ be the set of states $q \in Q$ such that there is a positive-weight simple cycle
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Example. In Figure 2, all configurations if we do not want to specify the state are connected by a valid run obtained by iterating the cycle subset of the state given q.

Finally, in Section 3.4 we show how to compute inductive construction. This enables us to reason about the structure of the building blocks of our analysis. In Section 3.2 we characterize in as the limit of an inductive construction. This enables us to reason about the structure of in Section 3.3. Finally, in Section 3.4 we show how to compute and decide unboundedness.

3.1 Residue Classes and Chains

Given q ∈ Q+ and 0 ≤ r < Wq, we call the set of configurations \( \{(q, z) \in Conf_+ \mid z \equiv r \pmod{W_q}\} \) a q-residue class. We simply speak of a residue class if we do not want to specify the state q. Given a q-residue class R, a set C ⊆ R is called a q-chain if it is a maximal subset of R with the property that every pair of configurations \((q, z), (q, z') \in C\) with \(z < z'\) are connected by a valid run obtained by iterating the cycle γq. Again, we speak of a chain if we do not want to specify the state q.
We focus on states $s_1$, $s_4$, and $s_{10}$ in the 1-VASS in Figure 2, each of which lies on a simple positive cycle. We also indicate which counter values prevent taking the associated positive cycle. For example, state $s_4$ has the simple cycle $\gamma_{s_4}$ with $W_{s_4} = 9$ and taking $\gamma_{s_4}$ from $\{s_4\} \times \{90, 93, 96\}$ is not allowed due to disequality guards along $\gamma_{s_4}$. The columns underneath each state represent residue classes of that state in $Conf_{+}$. We colour all unbounded chains in blue and all bounded chains in pink; thus all blue configurations form the set $U_0$.

We draw a distinction between bounded chains and unbounded chains, where a chain is bounded if and only if the associated set of counter values is bounded. An unbounded $q$-chain $C$ is contained in $Conf_\infty$ since the cycle $\gamma_q$ can be taken arbitrarily many times from any configuration in $C$ to yield a valid run.

**Remark 3.** For each $q$-residue class $R$, each guard value $z$ induces at most two bounded chains, namely configurations below $(q, z)$, and the singleton $\{(q, z)\}$ (which is vacuously a chain). Since there are at most $|Q|$ guards, each residue class decomposes as a disjoint union of at most $2|Q|$ bounded chains and a single unbounded chain.

Intuitively, within each bounded chain we can iterate the cycle $\gamma_q$ until hitting a guard. We call a residue class $R$ trivial if it consists solely of a single unbounded chain. Note that the union of all bounded $q$-chains is equal to $Conf_+ \cap \{q\} \times \text{blocked}(\gamma_q^\omega)$.

**Example.** As indicated in Figure 3 for the running example, the residue classes $\{s_4\} \times \{(52 + i + 9N)\}$ with $i \in \{0, 1, 3, 4, 7\}$ are indeed trivial, while each residue class $\{s_4\} \times \{(52 + i + 9N)\}$ with $i \in \{2, 5, 8\}$ consists of two bounded chains $\{s_4\} \times \{52 \leq z < 88 + i \mid \Delta \equiv i \pmod{9}\}$ and $\{s_4\} \times \{88 + i\}$, and a single unbounded chain $\{s_4\} \times \{(88 + i) + 9N\}$.

One of the main ideas in this section is to show that a configuration is unbounded if and only if it can reach an unbounded chain via a valid run whose underlying path $\pi$ has the form

$$\pi = \pi_0 \cdot \gamma_{q_1}^{n_1} \cdot \pi_1 \cdot \cdots \cdot \pi_{k-1} \cdot \gamma_{q_k}^{n_k} \cdot \pi_k,$$

where $\pi_0, \ldots, \pi_k$ are primitive paths and $n_1, \ldots, n_k$ are non-negative integers. Moreover, we give a polynomial bound on the length of the $\pi_i$ and the magnitude of $k$ in terms of the size of the underlying 1-VASS (in general, the exponents $n_i$ may be exponential in the size of the 1-VASS). We also show how to detect the existence of such a path in polynomial time.
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We now give an inductive backward-reachability construction of the set of all configurations \( U \), which, as noted above, is equal to \( \bigcup \). Then we essentially consider meta-transitions of the form \( \gamma \). Example. Figure 3 indicates the set \( U \). Remark 4. \( \bigcup \) is \( \{ \text{all trivial residue classes} \} \). Observe that \( U = \{ (s_4, 63), (s_4, 69) \} \); see Figure 4. These two configurations belong to two distinct chains. The downward closure of \( \{ (s_4, 63) \} \) in its chain is \( \{ s_4 \} \times \{ 54, 63 \} \), and the downward closure of \( \{ (s_4, 69) \} \) in its chain is \( \{ s_4 \} \times \{ 60, 69 \} \). We

Recall the structure of \( Conf \) as a partially ordered set. We will use standard order-theoretic terminology and notation to refer to sets of configurations: in particular given sets of configurations \( S, S' \subseteq Conf \), we say that \( S \) is downward closed in \( S' \) if for all \( (q, z) \in S \cap S' \) and \( (q, z') \in S' \) with \( z' \leq z \), we have \( (q, z') \in S \).

### 3.2 Inductive Characterization of \( Conf_\infty \)

We now give an inductive backward-reachability construction of the set of all configurations in \( Conf_+ \) that can reach an unbounded chain. Since unbounded configurations can, in particular, reach unbounded chains, this set is exactly \( Conf_\infty \).

In order for our inductive construction to converge in a polynomial number of steps, we essentially consider meta-transitions of the form \( \gamma \). Formally, we define an increasing sequence \( U_0 \subseteq U_1 \subseteq U_2 \subseteq \cdots \) of subsets of \( Conf_+ \) such that \( \bigcup_{n \in N} U_n = Conf_\infty \). Define \( U_0 \) to be the union of the collection of unbounded chains. Given \( n \in N \) we inductively construct \( U_{n+1} \) as follows. First, define \( U'_n \subseteq Conf_+ \) as the set of configurations \( (q, z) \notin U_n \) whose distance to \( U_n \) is minimal among all configurations in \( Conf_+ \setminus U_n \) (here the distance of a configuration \( (q, z) \) to \( U_n \) is the length of the shortest valid run from \( (q, z) \) to \( U_n \)). Now define \( U_{n+1} \subseteq Conf_+ \) to be the smallest set such that \( U_n \subseteq U_{n+1} \subseteq U_{n+1} \cap C \) and \( U_{n+1} \cap C \) is downward closed in every chain \( C \). Then \( \bigcup_{n \in N} U_n \) is the set of configurations in \( Conf_+ \) that can reach an unbounded chain which, as noted above, is equal to \( Conf_\infty \).

Remark 4. By definition, a shortest run from a configuration \( (q, z) \in U_{n+1} \setminus U_n \) to \( U_n \) has no internal configurations in \( Conf_+ \), and is therefore primitive.

Example. Figure 4 indicates the set \( U_0 \) for the running example. Note that \( U_0 \) contains all trivial residue classes. Observe that \( U'_1 = \{ (s_4, 63), (s_4, 69) \} \); see Figure 4. These two configurations belong to two distinct chains. The downward closure of \( \{ (s_4, 63) \} \) in its chain is \( \{ s_4 \} \times \{ 54, 63 \} \), and the downward closure of \( \{ (s_4, 69) \} \) in its chain is \( \{ s_4 \} \times \{ 60, 69 \} \). We
have that $U_1 = U_0 \cup \{(s_4) \times \{54, 60, 63, 69\}\}$. The second iteration to compute $U_2$ only adds the configuration $(s_1, 12)$ to $U_1$; see Figure 4B. The sequence stabilizes in this iteration.

3.3 The Structure of Conf$_\infty$

In this section we analyze the structure of Conf$_\infty$, based on its inductive characterization. This analysis will be key in obtaining a polynomial-time algorithm to compute Conf$_\infty$.

The guiding intuition is that for all $n$ the set $U_n$ is almost upward closed in each residue class $R$. By this we mean that if $(q, z)$ is the least configuration in $R \cap U_n$, then all but polynomially many configurations of $R$ above $(q, z)$ are also in $U_n$. More specifically, we show that for any bounded chain $C$ in $R$ that lies above $(q, z)$, although the number of configurations in $C$ may be exponential in $|Q|$, the size of $C \setminus U_n$ is bounded by a polynomial in $|Q|$. (Note here that the unique unbounded chain in $R$ is contained in $U_0$ and hence is contained in $U_n$ for all $n \in \mathbb{N}$.) Using this observation, we provide a polynomial bound on the number of iterations until the inductive construction converges. Indeed, in every iteration, unless a fixed point has been reached, there must exist some bounded chain $C$ such that the size of $C \setminus U_n$ strictly decreases. After showing that $C \setminus U_n$ is of polynomial size, we obtain a polynomial bound on the number of iterations until $U_n$ convergence by Remark 3.

We start by characterizing the paths between chains.

$\textbf{Proposition 5.}$ Let $(q, z), (q', z') \in \text{Conf}_+$ and let $(q, z) \overset{\pi}{\rightarrow} (q', z')$ be a (not necessarily valid) run such that $\pi$ is a primitive path. Then there exists a run $(q, z) \overset{\pi'}{\rightarrow} (q', z'')$ of length at most $|Q|^2 + 2$ such that (1) $\text{pmin}(\pi') \geq \text{pmin}(\pi)$, (2) $z'' \geq z'$, and (3) the $q'$-residue class of $(q', z'')$ is either trivial or identical to that of $(q', z')$.

Given a $q$-residue class $R$, in general $U_n$ is not an upward closed subset of $R$. The following definitions are intended to measure the defect of $U_n$ in this regard.

We say that a bounded chain $C$ that is contained in a residue class $R$ is $n$-active if there exists a configuration in $U_n \cap R$ that lies below some configuration in $C$. Let $C$ be an $n$-active chain. Recall that $U_n$ is downward closed in $C$ and hence $C \setminus U_n$ is upward closed in $C$. Suppose that $C \setminus U_n$ is non-empty, write $m_1 := \min\{x : (q, x) \in C \setminus U_n\}$ and $m_2 := \max\{x : (q, x) \in C \setminus U_n\}$, and define $\delta_n(C) := \{(q, x) \in \text{Conf}_+ : m_1 \leq x \leq m_2 \text{ and } (q, x) \notin U_n\}$. Thus $\delta_n(C)$ contains all configurations in $C \setminus U_n$, as well as all configurations “between” elements of $C \setminus U_n$, apart from those that are themselves in $U_n$. If $C \setminus U_n = \emptyset$ then we define $\delta_n(C) := \emptyset$. Finally for a residue class $R$ we write

$$\delta_n(R) := \bigcup \{\delta_n(C) : C \subseteq R \text{ an } n\text{-active chain}\}.$$  \hspace{1cm} (1)

For $(q, x_{\min})$, the least element in $R \cap U_n$ we have that $|\{(q, x) \in R \setminus U_n : x_{\min} \leq x\}| \leq |\delta_n(R)|$.

$\textbf{Example.}$ In Figure 4A consider the 1-active chain $C := \{54, 63, 72, 81\}$. Since $C \setminus U_1 = \{72, 81\}$ we have that $\delta_1(C) = \{72, 75, 78, 81\}$.

$\textbf{Lemma 6.}$ For all $n \in \mathbb{N}$ and every chain $C$ we have that $|\delta_n(C)| \leq |Q| \cdot |C \setminus U_n|$.

We now come to the central technical part of the paper, controlling the growth of $\delta_n(R)$ as a function of $n$:

$\textbf{Lemma 7.}$ There exists a polynomial $\text{poly}_2$ such that for each residue class $R$ and all $n \in \mathbb{N}$ we have $|\delta_{n+1}(R)| \leq \max\{|\delta_n(R')| : R' \text{ a residue class} \} + \text{poly}_2(|Q|)$ if $R$ contains a chain that is $(n+1)$-active but not $n$-active.
Before proceeding to prove Lemma 7, we demonstrate the underlying intuition. Consider a configuration \((q, z) \in R \cap U_n^+\) that has a primitive path \(\pi\) to a configuration \((q', z') \in U_{n-1}\). To prove Lemma 7, we argue that \(\pi\) lifts to a valid run from a “dense” subset of configurations in \(\{(q, z'') \in R : z'' \geq z\}\). There are two main cases in this argument based on whether one of the larger configurations in the chain induces a valid run ending in a trivial residue class.

**Example.** The first case occurs in obtaining \(U_1\) from \(U_0\) in the running example; see Figure 4a. Consider the \(s_4\)-chain \(C := \{54, 63, 72, 81\}\). The primitive path \(s_4, s_7, s_8, s_9, s_{10}\) from the largest configuration \((s_4, 81)\) in \(C\) leads to a non-trivial \(s_{10}\)-residue class (out of \(U_0\)). However, one among the \(n\)-next largest configurations in \(C\), for \(n = |\text{blocked}(s_4, s_7, s_8, s_9, s_{10})| \cdot |Q|\), lifts to a valid run in a trivial \(s_{10}\)-residue class. In the example, this is the case for \((s_4, 63)\).

The second case occurs in obtaining \(U_2\) from \(U_1\) in the running example; see Figure 4b. Consider the \(s_1\)-chain \(C' := \{12, 18, 24, \ldots, 54\}\). The primitive path \(s_1, s_3, s_4\), from none of the configurations in this chain, ends in a trivial \(s_4\)-residue class. However, we provide a subtle argument to bound \(|C' \setminus U_2|\) with \(|\delta_1(C)| + \text{poly}_2(|Q|)\).

**Proof of Lemma 7.** Pick the minimal element \((q, z_0) \in R \cap U_{n+1}^+\). Moreover, let \((q', z') \in U_n\) and \((q, z_0) \xrightarrow{\delta_n} (q', z')\) be such that \(\pi\) is a shortest run from \((q, z_0)\) to \(U_n\). By Remark 4, \(\pi\) is a primitive path. By Proposition 5, there is a run \((q, z_0) \xrightarrow{\delta_n} (q', z'')\), for some \(z'' \geq z',\) such that \(\pi'\) has length at most \(|Q|^2 + 2,\) and the residue class \(R'\) of \((q', z'')\) is either trivial or the same as the residue class of \((q', z')\). (Note that we do not claim that \((q', z'') \in U_n,\) nor that \(\pi'\) lifts to a valid run.) We now identify two cases according to the order of \(W_q\) in the group \(\mathbb{Z}/\mathbb{Z}W_{q'}\) of integers modulo \(W_{q'}\), which is \(\frac{W_{q'}}{\gcd(W_{q}, W_{q'})}\). Recall that this quantity is the smallest \(c \in \mathbb{N}\) such that \(W_q \cdot c \equiv 0 \pmod{W_{q'}}\).

**Case (i):** \(\frac{W_{q'}}{\gcd(W_{q}, W_{q'})} > |Q|\). We first show that \(|C \setminus U_{n+1}| \leq (|Q|^2 + 2)(|Q| + 1)\) for every \((n+1)\)-active chain \(C\) in \(R\).

Let \(C\) be an \((n+1)\)-active chain of \(R\) and suppose for a contradiction that \(|C \setminus U_{n+1}| > (|Q|^2 + 2)(|Q| + 1)\). Since \(C\) is \((n+1)\)-active, for every configuration \((q, z) \in C \setminus U_{n+1}\) we have \(z \geq z_0\). Further, since \(\text{min}(\pi') + z_0 \geq 0, \pi'\) can only be blocked on a configuration due to a violation of a disequality guard. Since the length of \(\pi'\) is at most \(|Q|^2 + 2,\) it follows that at most \(|Q|^2 + 2\) elements of \(C \setminus U_{n+1}\) lie in \(\text{blocked}(\pi')\).

Recall that \(C \setminus U_{n+1}\) is upward closed in \(C\), so by the assumption that \(|C \setminus U_{n+1}| > (|Q|^2 + 2)(|Q| + 1)\), there exists a set \(S := \{(q, z_1) + iW_q : 0 \leq i \leq |Q|\}\) of \(|Q| + 1\) “consecutive” elements of \(C \setminus U_{n+1}\), for some \(z_1\), such that no element of \(S\) lies in \(\text{blocked}(\pi')\). Then \(\pi'\) lifts to a valid run from each element of \(S\). Moreover, since the order of \(W_q\) in \(\mathbb{Z}/\mathbb{Z}W_{q'}\) is assumed to be greater than \(|Q|\), the images of the elements of \(S\), after following \(\pi'\), lie in pairwise distinct \(q'\)-residue classes. But the number of non-trivial \(q'\)-residue classes is at most \(|Q|\) and hence some configuration in \(S\) has a run over \(\pi'\) to a trivial \(q'\)-residue class and hence to \(U_n\). But then such a configuration lies in \(U_{n+1}\), which is a contradiction.

We conclude that \(|C \setminus U_{n+1}| \leq (|Q|^2 + 2)(|Q| + 1)\) for every \((n+1)\)-active chain \(C\) in \(R\). But then \(|\delta_{n+1}(C)| \leq |Q|(2|Q|^2 + 2(|Q| + 1)\) by Lemma 6. Finally, since \(R\) comprises at most \(2|Q|\) bounded chains by Remark 3, we have that \(|\delta_{n+1}(R)| \leq 2|Q|^2(|Q|^2 + 2(|Q| + 1)\).

**Case (ii):** \(\frac{W_{q'}}{\gcd(W_{q}, W_{q'})} \leq |Q|\). For the residue classes \(R\) and \(R'\) as above, define an injective partial mapping \(\Phi : \delta_{n+1}(R) \to \delta_n(R')\) by \(\Phi(q, x) = (q', x')\) if and only if \(x' = x + \text{weight}(\pi')\) and \((q', x') \in \delta_n(R')\). We will prove that \(\Phi\) is defined on all but \(\text{poly}_3(|Q|)\) many configurations in \(\delta_{n+1}(R)\), for some polynomial \(\text{poly}_3\), thereby showing that \(|\delta_{n+1}(R)| \leq |\delta_n(R')| + \text{poly}_3(|Q|)\).

To this end, it suffices to show that \(\Phi\) is defined on all but \(\text{poly}_3(|Q|)\) many configurations in \(\delta_{n+1}(C)\) for every \((n+1)\)-active chain \(C\) in \(R\), for some polynomial \(\text{poly}_4\).
Let $C$ be an $(n+1)$-active chain in $R$ and let $C_1, \ldots, C_s$ be a list, given in increasing order, of the chains in $R'$ that are mapped into by $\Phi$ from some configuration in $\delta_{n+1}(C)$. Then $C_1, \ldots, C_s$ are all $n$-active (as they are above $(q', z') \in U_n$). For $i \in \{1, \ldots, s\}$, write $(q_i, x_i)$ for the minimum configuration in $\delta_{n+1}(C)$ that is mapped by $\Phi$ to $C_i$ and write $(q_i, x_i)$ for the maximum configuration in $\delta_{n+1}(C)$ that is mapped to $C_i$. Then for each $i = 1, \ldots, s$, every configuration $(q, x) \in \delta_{n+1}(C)$ such that $x_{\min} \leq x \leq x_{\max}$ and $(q, x) \not\in \text{blocked}(\pi')$ is mapped by $\Phi$ to $\delta_n(R')$. Thus, writing $(q_i, x_{\max})$ and $(q_i, x_{\min})$ respectively for maximum and minimum configurations in $\delta_{n+1}(C)$, we have that $\Phi$ is defined on all elements of $\delta_{n+1}(C)$ lying outside the set

$$\left\{(q, x) \in \delta_{n+1}(C) : x \in \left(x_{\max}^{(i)}, x_{\max}\right) \cup \left[x_{\min}, x_{\min}^{(i)}\right] \cup \bigcup_{i=1}^{s-1} \left(x_{\max}^{(i)}, x_{\max}^{(i+1)}\right)\right\} \quad (2)$$

We claim that the set (2) has cardinality at most $(2|Q| + 1) \cdot \text{poly}_5(|Q|)$, for some polynomial $\text{poly}_5$. For this it will suffice to show that any sub-interval $I$ of $\delta_{n+1}(C)$ of the form $\{(q, x) \in \delta_{n+1}(C) : a \leq x \leq b\}$, where $a, b \geq x_{\min}$ and such that it does not meet the domain of $\Phi$, has cardinality at most $\text{poly}_5(|Q|)$. Indeed, note that (2) is a union of at most $2|Q| + 1$ such intervals since there are at most $2|Q|$ chains in $R$ by Remark 3.

Let $\text{poly}_6(x) := (x^2 + 2)(x + 1) + 1$. Since $\text{blocked}(\pi')$ has cardinality at most $|Q|^2 + 2$, if we take $\text{poly}_6(|Q|)$ consecutive elements of $C \setminus U_{n+1}$ then there are at least $|Q| + 1$ consecutive elements that lie outside $\text{blocked}(\pi')$ and at least one of these elements — say $(q, x)$ — has a valid run over $\pi'$ to the residue class $R'$ by the assumption that $\frac{W_{q', z'}}{\text{gcd}(W_{q', z'}, W_{q, x})} \leq |Q|$. Since $(q, x) \not\in U_{n+1}$ we have that $(q', x + \text{weight}(\pi')) \not\in U_n$ and hence $(q, x)$ is in the domain of $\Phi$. We conclude that any sequence of at least $\text{poly}_6(|Q|)$ consecutive elements of $C \setminus U_{n+1}$ meets the domain of $\Phi$. Hence any sub-interval $I$, as defined above, contains at most $\text{poly}_6(|Q|)$ elements of $C \setminus U_{n+1}$ and, by Lemma 3, contains at most $|Q| \cdot \text{poly}_6(|Q|)$ elements in total. ▶

Proposition 8 follows from Lemma 7 by induction, as follows.

▶ Proposition 8. There exists a polynomial $\text{poly}_1$ such that for each residue class $R$ and all $n \in \mathbb{N}$ we have $|\delta_n(R)| \leq \text{poly}_1(|Q|)$.

Proof. Let $\alpha_n$ be the number of chains in $\text{Conf}_+$ that are $n$-active. Since $n$-active chains are by definition bounded, we have that $\alpha_n \leq 2|Q|^2$ for all $n \in \mathbb{N}$ (see Remark 3). We argue by induction on $n$ that $|\delta_n(R)| \leq \alpha_n \cdot \text{poly}_2(|Q|)$ for all $n \in \mathbb{N}$ and all residue classes $R$. We conclude that $|\delta_n(R)| \leq 2|Q|^2 \cdot \text{poly}_2(|Q|)$.

The base case is trivial as there are no 0-active chains and $\delta_0(R)$ is empty for all residue classes. The induction step has two cases. First, suppose that $\alpha_{n+1} = \alpha_n$, i.e., all chains in $\text{Conf}_+$ that are $(n + 1)$-active were already $n$-active. Since $U_n \subseteq U_{n+1}$, we have that $\delta_{n+1}(C) \subseteq \delta_n(C)$ for all chains $C$ in $R$. We conclude that $\delta_{n+1}(R) \subseteq \delta_n(R)$ and so $|\delta_{n+1}(R)| \leq |\delta_n(R)|$. Since $|\delta_n(R)| \leq \alpha_n \cdot \text{poly}_2(|Q|)$ by induction hypothesis, and $\alpha_n = \alpha_{n+1}$ we get that $|\delta_{n+1}(R)| \leq \alpha_{n+1} \cdot \text{poly}_2(|Q|)$.

The second case is that $\alpha_{n+1} > \alpha_n$. Then by Lemma 7 we have $|\delta_{n+1}(R)| \leq \max\{|\delta_n(R')| : R' \text{ a residue class}\} + \text{poly}_2(|Q|)$. Since the right-hand side of the latter is at most $\leq \alpha_n \cdot \text{poly}_2(|Q|) + \text{poly}_2(|Q|)$, by induction hypothesis, and $\alpha_{n+1} > \alpha_n$ we get that $|\delta_{n+1}(R)| \leq \alpha_{n+1} \cdot \text{poly}_2(|Q|)$.

As a consequence of Proposition 8 we have:

▶ Corollary 9. The sequence $(U_n)_{n \in \mathbb{N}}$ stabilizes in at most $\text{poly}_1(|Q|)$ steps.
3.4 Computing \( \text{Conf}_\infty \) and Deciding Unboundedness

In this section we show how to compute \( \text{Conf}_\infty \) in polynomial time and how to decide in polynomial time whether the initial configuration \((s, 0)\) can reach \( \text{Conf}_\infty \).

We start by showing that if a configuration can reach \( U_n \) via a primitive run, then it can also reach \( U_n \) via a polynomial-length run (see Appendix B for the proof).

▶ **Proposition 10.** There exists a polynomial \( \text{poly}_\gamma \) such that the following holds. Let \((q, z), (q', z') \in \text{Conf}_+\) and let \((q, z) \xrightarrow{\pi} (q', z')\) a valid run such that \((q', z') \in U_n\) and \(\pi\) is primitive. Then there is a valid run \((q, z) \xrightarrow{\pi'} (q', z'')\) such that \((q', z'') \in U_n\) and \(\pi'\) has length at most \(\text{poly}_\gamma(|Q|)\).

Recall that \(U_{n+1}\) consists of all configurations in \(\text{Conf}_+\) with minimal distance to \(U_n\). Combining Remark 4 and Proposition 10, a configuration in \(U_n\) has minimal distance to \(U_n\) and is at most \(\text{poly}_\gamma(|Q|)\). It follows that we can restrict the search for configurations that can reach \(U_n\), to those within a polynomially-bounded distance to \(U_n\). By itself this is not sufficient to obtain a polynomial-time algorithm to decide whether \(U_n\) is reachable. Indeed, it is easy to see that for many classes or target sets of configurations of a 1-VASS, bounded reachability is NP-hard, e.g., reaching singleton sets is NP-hard by reduction from SUBSET-SUM. However, using our analysis of the structure of \(U_n\) in Section 3.3, we are able to formulate the bounded reachability problem above in a form that admits a polynomial-time algorithm.

Specifically, we consider the Bounded Coverability problem with a Disequality Objective: Given as input a 1-VASS \(\mathcal{V} = (Q, D, \Delta, w)\) with a distinguished state \(q_f\), a positive integer \(L\) (written in unary), an initial configuration \((q_0, x_0)\), and a coverability objective of the form

\[
O = \{(q_f, x) \mid x \geq \ell \land \left( \bigwedge_{i=1}^{m} (x \not\equiv a_i \mod W) \land \bigwedge_{i=1}^{n} (x \not\equiv b_i) \right) \},
\]

where \(\ell, W\) and the \(a_i\) and \(b_i\) are non-negative integers given in binary, decide whether \(O\) is reachable from \((q_0, x_0)\) via a valid run of length at most \(L\).

▶ **Proposition 11.** The Bounded Coverability problem with a Disequality Objective is decidable in polynomial time.

We now show how to compute \(\text{Conf}_\infty\) in polynomial time. By Corollary 9, the sequence \(\{U_n\}_{n \in \mathbb{N}}\) converges in at most \(\text{poly}_\gamma(|Q|)\) steps. It remains to show how to compute \(U_{n+1}\) from \(U_n\) in polynomial time for each \(n\).

Recall that all unbounded chains are contained in \(U_0\) and hence are contained in \(U_n\) for all \(n\). Recall also that the total number of bounded chains is at most \(2|Q|\) and that \(U_n\) is downward closed in each bounded chain. Thus \(U_n\) is determined by giving, for every bounded chain \(C\) such that \(U_n \cap C \neq \emptyset\), the maximum configuration in \(U_n \cap C\). In particular, \(U_n\) can be described in space polynomial in the description of the given 1-VASS.

Recall that \(U_{n+1}\) is obtained from \(U_n\) by adding the configurations in \(\text{Conf}_+ \setminus U_n\) that have minimum distance to \(U_n\) and then closing downward in each bounded chain. By Remark 4 and Proposition 10, a configuration in \(\text{Conf}_+ \setminus U_n\) that has minimum distance to \(U_n\) has distance at most \(\text{poly}_\gamma(|Q|)\). The idea to compute \(U_{n+1}\) from \(U_n\) is as follows:

For each bounded chain \(C\), and each configuration \((q, x) \in C \setminus U_n\) that is among the top \(\text{poly}_\gamma(|Q|)\) configurations in \(C\), we determine the distance of \((q, x)\) to \(U_n\) up to a bound of \(\text{poly}_\gamma(|Q|)\). To do this we use the procedure described in Proposition 11 having first written \(U_n\) as a polynomial-size union of sets of the form \(\mathcal{C}_a\)—see below for details. The reason that it suffices to look only among the top \(\text{poly}_\gamma(|Q|)\) configurations in each bounded chain is...
because we know from Proposition 8 that $|C \setminus U_{n+1}| \leq \text{poly}_1(|Q|)$ for every $(n+1)$-active chain $C$.

We next show how to decompose $U_n$ into a polynomial union of sets of the form (5) in order to apply Proposition 11. Fixing $q \in Q_+$, let $R_1, \ldots, R_m$ be a list of the non-trivial $q$-residue classes and for each $i \in \{1, \ldots, m\}$, write $a_i$ for the corresponding residue modulo $W_q$ and define $\ell_i := \min(R_i \cap U_n)$. Moreover, let $b_1, \ldots, b_q$ be a list of the counter values $x$ such that $x \geq \ell_i$ and $(q, x) \in R_i \setminus U_n$ for some $i$. Note that $m \leq |Q|$ and $k \leq m \cdot \text{poly}_1(|Q|)$. We decompose the set of configurations $\{(q, z) \in U_n\}$ into the following two components:

1. $\{(q, z) : z \geq \min(|q_i|) \land \bigwedge_{i=1}^m z \not\equiv a_i \pmod{W_q}\}$, i.e., all configurations in trivial $q$-residue classes,
2. for all $j \in \{1, \ldots, m\}$, the set $\{(q, z) : z \geq \ell_j \land \bigwedge_{i \neq j} z \not\equiv a_i \pmod{W_q} \land \bigwedge_{i=1}^k z \not\equiv b_i\}$, which includes $R_j \cap U_n$ for the non-trivial residue class $R_j$.

Finally, it remains to decide whether the configuration $(s, 0)$ is unbounded. By Proposition 2, $(s, 0)$ is unbounded if and only if it can reach $\text{Conf}_\infty$. Now a shortest run from $(s, 0)$ to $\text{Conf}_\infty$ is necessarily primitive: if an internal configuration in such a run lies in $\text{Conf}_\infty$ then it is also in $\text{Conf}_\infty$—a contradiction. By Proposition 11, a shortest run from $(s, 0)$ to $\text{Conf}_\infty$ has length at most $\text{poly}_2(|Q|)$. Thus we can decide whether such a run exists in polynomial time using Proposition 11. In conclusion we have

\begin{theorem}
The Unboundedness Problem and the Coverability Problem for 1-VASS with disequality tests are decidable in polynomial time.
\end{theorem}

4. Unboundedness for 1-VASS

In this section we show that the Unboundedness Problem for 1-VASS (i.e., with no disequality tests) is in NC$^2$. Recall that NC$^i$ is the class of decision problems solvable in time $O(\log^i n)$, with $n$ the size of the input, on a parallel computer with a polynomial number of processors.

Let $V = (Q, \Delta, w)$ be a 1-VASS with a distinguished state $s \in Q$. We want to decide whether the configuration $(s, 0)$ is unbounded. Since $V$ has no disequality tests, deleting a negative-weight or zero-weight cycle that appears as an infix of a valid run yields another valid run. It follows that $(s, 0)$ is unbounded if and only if there is a valid run from $(s, 0)$ consisting of a simple path (of length at most $|Q|$) followed by a positive-weight simple cycle (again, of length at most $|Q|$). We call such a run a lasso.

Let $V = (Q, \Delta, w)$ be a 1-VASS and let $\pi = q_1, \ldots, q_n$ be a path in $V$. Recall that a (possibly empty) prefix of $\pi$ is said to be minimal if it has minimal weight among all prefixes of $\pi$. Likewise a (possibly empty) suffix of $\pi$ is said to be maximal if it has maximal weight among all suffixes. It is clear that $q_1, \ldots, q_m$ is a minimal prefix of $\pi$ if and only if $q_m, \ldots, q_n$ is a maximal suffix. In such a case let us call $q_m$ a nadir of $\pi$ (the nadir is the lowest point reached in any run over $\pi$). Recall that $\text{pmin}(\pi)$ is the weight of a minimal prefix of $\pi$; correspondingly we define $\text{smax}(\pi)$ to be the weight of a maximal suffix.

Given paths $\pi$ and $\pi'$, say that $\pi$ is dominated by $\pi'$ if $\text{pmin}(\pi') \leq \text{pmin}(\pi)$ and $\text{smax}(\pi') \leq \text{smax}(\pi)$. Observe that if $\pi$ is dominated by $\pi'$ then weight($\pi') \leq \text{weight}(\pi')$. Fix two states $p, q \in Q$ and let $P$ be a set of $p$-$q$ paths. We say that a set $P'$ of $p$-$q$ paths is a Pareto set for $P$ if for every $\pi \in P$ there exists $\pi' \in P'$ such that $\pi$ is dominated by $\pi'$.

We observe some simple properties of Pareto sets:

\begin{lemma}
Let $p, q, r \in Q$. Then all of the following statements hold:

1. If $P_1, P_2, P_3$ are sets of $p$-$q$ paths such that $P_1$ is a Pareto set of $P_2$ and $P_2$ is a Pareto set of $P_3$, then $P_1$ is a Pareto set of $P_3$.
\end{lemma}
2. If \( P, R \) are sets of \( p-q \) paths with respective Pareto sets \( P', R' \), then \( P' \cup R' \) is a Pareto set for \( P \cup R \).
3. If \( P \) is a set of \( p-q \) paths and \( R \) is a set of \( q-r \) paths with respective Pareto sets \( P', R' \), then \( P' \cdot R' \) is a Pareto set of \( P \cdot R \).

\[ \text{Proposition 14.} \text{ Let } p, q \in Q. \text{ Then every set } P \text{ of } p-q \text{ paths of length at most } k \text{ has a Pareto set } P' \text{ of cardinality at most } |Q| \text{ such that each path in } P' \text{ has length at most } 2k. \text{ Moreover such a set } P' \text{ can be computed from } P \text{ in } NC^1. \]

\[ \text{An } NC^2 \text{ Upper Bound} \]

\[ \text{Theorem 15.} \text{ The Unboundedness Problem and the Coverability Problem for 1-VASS are decidable in } NC^2. \]

\[ \text{Proof.} \text{ By Lemma 1, it will suffice to show that Unboundedness is in } NC^2. \]

Let \( V = (Q, \Delta, w) \) be a 1-VASS. Given \( p, q \in Q \) and \( m \in \mathbb{N} \), denote by \( \text{Paths}_{p,q,m} \) the set of all \( p-q \) paths in \( V \) of length at most \( m \).

Given a state \( s \in Q \), recall that \( (s,0) \) is unbounded if and only if there exists a lasso run that starts at \( (s,0) \). To determine the existence of such a run we compute a Pareto set \( P_q \) for \( \text{Paths}_{s,q,|Q|} \) and a Pareto set \( P'_q \) for \( \text{Paths}_{s,q,|Q|} \) for every state \( q \in Q \). Having done this we look for \( q \in Q \) and paths \( \pi \in P_q \) and \( \pi' \in P'_q \) such that \( \pi \cdot \pi' \) induces a valid run from \( (s,0) \) and \( \pi' \) has positive weight.

It remains to show how to compute a Pareto set of \( \text{Paths}_{p,q,|Q|} \) for all pairs of states \( p, q \in Q \) (together with the values \( \text{weight}(\pi) \) and \( \text{pmin}(\pi) \) for every path \( \pi \) in the Pareto set) in \( NC^2 \).

For \( k = 1, \ldots, \lfloor \log |Q| \rfloor \), we show how to compute a family \( \mathcal{P}_k = \{ P_{p,q,k} \}_{p,q \in Q} \) such that for all \( p, q \in Q \):

1. \( P_{p,q,k} \) is a Pareto set for \( \text{Paths}_{p,q,2^k} \);
2. \( P_{p,q,k} \subseteq \text{Paths}_{p,q,4^k} \);
3. \( |P_{p,q,k}| \leq |Q| \).

By Item 1, if \( k = \lfloor \log |Q| \rfloor \) then \( P_{p,q,k} \) is a Pareto set for \( \text{Paths}_{p,q,|Q|} \). \( \text{Note that for } k = \lfloor \log |Q| \rfloor, \mathcal{P}_k \) consists of paths of length at most \( |Q|^2 \).

The construction of \( \mathcal{P}_k \) is by induction on \( k \). Suppose we have computed \( \mathcal{P}_k \) with Properties 1-3 above. Fix \( p, q \in Q \). In order to compute \( P_{p,q,k+1} \) we observe that

\[ P := \{ \pi_1 \cdot \pi_2 : \exists r \in Q (\pi_1 \in P_{p,r,k} \land \pi_2 \in P_{r,q,k}) \} \tag{4} \]

is a Pareto set for \( \text{Paths}_{p,q,2^{k+1}} \) by Items 2 and 3 of Lemma 13. Moreover \( |P| \) is at most \( \sum_{r \in Q} |P_{p,r,k}| |P_{r,q,k}| \leq |Q|^3 \). Applying Proposition 14 we obtain a Pareto set \( P' \) for \( P \) of cardinality at most \( |Q| \). By Item 1 of Lemma 13, \( P' \) is a Pareto set for \( \text{Paths}_{p,q,2^{k+1}} \). Finally, it is clear from the length bound in Proposition 14 that all paths in \( P' \) have length at most \( 4^{k+1} \). Thus we define \( P_{p,q,k+1} := P' \).

It remains to establish the \( NC^2 \) complexity bound for computing \( \mathcal{P}_{\lfloor \log |Q| \rfloor} \). For this it suffices to show that for all \( k \) the computation of \( \mathcal{P}_{k+1} \) from \( \mathcal{P}_k \) can be carried out in \( NC^1 \). But we may compute each set \( P_{p,q,k+1} \) in parallel (over \( p, q \in Q \)), and the computation of each such set can be done in \( NC^1 \) by Proposition 14.

\[ \square \]
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A Proof of the reduction in Figure 1

We associate with every value \( u \in \mathbb{N} \), an assignment \( \pi^u : \{X_1, \ldots, X_m\} \to \{0, 1\} \) defined by \( \pi(X_i) = \text{val}(u, X_i) \).

The key observation is the following: let \( u \in \{0, \ldots, P - 1\} \), and consider a clause \( C_i = \ell_{i_1} \lor \ell_{i_2} \lor \ell_{i_3} \), where \( \ell_{i_j} \) is a literal of variable \( X_{i_j} \), then \( \pi^u \) satisfies \( C_i \) iff there exists some \( k \in \mathbb{N} \) such that \( u + kp_1p_2 = 0 \mod D_i \).

Indeed, note that for every \( j \in \{1, 2, 3\} \) and every \( k \in \mathbb{N} \) we have that \( p_j | u \) iff \( p_j | u + kp_1p_2p_3 \). Recall that \( \text{val}(u, X_{i_j}) = 1 \) iff \( p_j | u \), and observe that since \( u < P \), there exists \( k \in \mathbb{N} \) such that \( u + kp_1p_2p_3 \in \{P, P + 1, \ldots, P + p_1p_2p_3 - 1\} \). We have that \( \pi^u \) satisfies \( C_i \) iff \( \pi^u + kp_1p_2p_3 \) satisfies \( C_i \), if \( u + kp_1p_2p_3 \notin D_i \).

Now, assume \( \varphi \) is satisfiable, and let \( \pi \) be a satisfying assignment. We associate with \( \pi \) the number \( u = \prod_{j: \pi(X_j) = 1} p_j \mod P \) (note that taking modulo \( P \) simply means that if the product is exactly \( P \), we take \( u = 0 \)). Clearly \( \pi = \pi^u \). We claim that \( (s_0, u) \) is bounded. Indeed, the only paths possible from \( (s_0, u) \) start by choosing a state \( s_1 \), and then repeatedly applying the cycle of cost \( c_i \). However, since \( \pi^u \) satisfies all clauses, then by the above, all such paths are blocked by a disequality guard after taking the \( c_i \) for \( k \) times, for some \( k \in \mathbb{N} \) (which depends on \( i \)). Thus, \( (s_0, u) \) is bounded.

Conversely, assume \( (s_0, u) \) is bounded for some value \( u \), we claim that \( \pi^u \) satisfies \( \varphi \). Indeed, by the same reasoning above, it follows that for every cycle of cost \( c_i \), we have \( u + kc_i \notin D_i \) for some \( k \in \mathbb{N} \), so \( \pi^u \) satisfies \( C_i \). Since this is true for all clauses, we have that \( \pi^u \) satisfies \( \varphi \).

We conclude that \( \varphi \) is satisfiable iff some configuration \( (s_0, u) \) is bounded, which completes the reduction.

Finally, we note that the reduction indeed takes polynomial time — indeed, the construction clearly has polynomially many states. Also, the first \( m \) primes \( p_1, \ldots, p_m \) can be listed in time polynomial in \( m \), and are representable in polynomially many bits. Therefore, the binary representation of the transition values and the amount of missing elements in the domain of each state are both polynomial.

B Single disequality guards suffice

Given a 1-VASS \( V = (Q, \Delta, D, w) \) with disequality tests, we can assume that for all states \( q \) the set \( D_q \) is either \( \mathbb{N} \) or \( \mathbb{N} \setminus \{q\} \) for some \( q \in \mathbb{N} \). This assumption is without loss of generality, as a state \( q \) with \( D_q = \mathbb{N} \setminus \{a_1, \ldots, a_n\} \) can be replaced with a sequence of new states \( q_1, \cdots, q_n \), connected with 0-weight transitions, such that \( D_{q_i} = \mathbb{N} \setminus \{a_i\} \) for \( i \in [1, \ldots, n] \). The transformation yields only a polynomial blow-up in the size of the 1-VASS, and there is a natural correspondence between runs in the original 1-VASS and the modified one.

C Proof of Lemma 1

Consider a 1-VASS \( V = (Q, \Delta, D, w) \) with disequality tests, and let \( s, t \in Q \). We reduce the Coverability problem to the Unboundedness problem as follows.

We obtain from \( V \) a new 1-VASS \( V' \) as follows. First, we remove from \( V \) all the states that cannot reach \( t \) in the underlying graph. Second, we introduce a new state \( t' \) with a self-loop of weight +1, that is reachable from \( t \) with a transition of weight 0. The output of the reduction is \( V' \) with the distinguished state \( s \).

Recall that reachability in directed graphs can be decided in \( \text{NL} \subseteq \text{NC}^2 \), and hence this reduction is \( \text{NC}^2 \)-computable.
Henceforth assume that $s$ can reach $t$ in the underlying graph of $V$ (otherwise $s$ cannot cover $t$, and the reduction can output a trivial negative instance). We proceed to prove the correctness of the reduction.

First, if $(s, 0)$ can cover $t$ in $V$, then in particular it can only cover $t$ using states in $V'$. We now have that $(s, 0)$ is unbounded in $V'$, by covering $t$, and then taking the transition to $t'$ and repeating the self loop unboundedly. Note that crucially, there are no disequality guards on $t'$, and therefore once $t$ is reached, we can take the transition to $t$ and repeat the self loop unboundedly.

Conversely, suppose $(s, 0)$ is unbounded in $V'$, then either there is a valid run in $V$ from $(s, 0)$ to $(t', z)$ for some $z$, in which case $(s, 0)$ can cover $t$ in $V$, or $(s, 0)$ is unbounded already in $V$ and, moreover, it is unbounded in $V$ using only states that can reach $t$ in the underlying graph. We claim that in the latter case, $(s, 0)$ can cover $t$ in $V$. Indeed, from $(s, 0)$ there is a valid run to a configuration $(q, z)$ with $z$ that is large enough, such that a simple path from $q$ to $t$ in the underlying graph lifts to a valid run from $(q, z)$ to $(t, z')$ for some $z'$. Specifically, taking $z > |Q| \cdot W \cdot G$ where $W$ is the maximal absolute value of the weight of a transition in $V$, and $G$ is the maximal disequality guard, suffices for such a run.

**D Proof of Proposition 2**

Clearly if $(s, 0)$ can reach an unbounded configuration in $Conf_{+}$, then it is unbounded.

Conversely, if $(s, 0)$ is unbounded, then there is a state $q$ such that for all $z_0 \in \mathbb{N}$, there exist $z, z' \geq z_0$ and a valid run starting in $(s, 0)$ that visits $(q, z)$ and ends in $(q, z')$. Thus, there is a positive cycle $\gamma$ on $q$. The positive cycle $\gamma$ on $q$ may not be simple, but it certainly visits a state $p$ with a simple positive cycle $\gamma_p$ on it. Pick $z_0$ such that $z_0 > p\min(\gamma) + x$. for all $x \in \text{blocked}(\gamma_p^w)$ (Note that \text{blocked}(\gamma_p^w) is finite since $\gamma_p$ is a positive cycle. The maximum is thus well-defined.) Hence, there is a valid run from $(s, 0)$ to $(p, y)$ where $y > \max(\text{blocked}(\gamma_p^w))$. Observe that $(p, y) \in Conf_{+}$ and it is unbounded.

**E Proof of Proposition 5**

Suppose that $\pi$ has length strictly greater than $|Q|^2 + 2$. Then we can find $|Q| + 1$ distinct proper prefixes (i.e. prefixes that are not just the initial state, or the entire path) of $\pi$ that end in the same state. That is, $|Q|$ proper cycles on the same state. Let $\pi_1, \ldots, \pi_{|Q|+1}$ be a list of these prefixes, given in order of increasing length, and let the corresponding suffixes be $\pi_1', \ldots, \pi_{|Q|+1}'$. We now consider two cases.

First, suppose that there exist $i < j$ such that $\text{weight}(\pi_i)$ and $\text{weight}(\pi_j)$ have the same residue modulo $W_q'$. Then define $\pi' := \pi_i \cdot \pi_j^s$. In this case path $\pi'$ lifts to a run from $(q, z)$ to $(q', z')$ such that $(q', z')$ lies in the same $q'$-residue class as $(q', z'')$. The second case is that the respective residue classes of $\text{weight}(\pi_1), \ldots, \text{weight}(\pi_{|Q|+1})$ modulo $W_q'$ are all distinct. Then there exists $i > 1$ such that, defining $\pi' := \pi_1 \cdot \pi_i^s$, the path $\pi'$ lifts to a run from $(q, z)$ to $(q', z'')$ such that $(q', z'')$ lies in a trivial $q'$-residue class (as there are at most $|Q|$ non-trivial residue classes).

Continuing in this fashion we can recursively remove cycles from the original path $\pi$ to eventually obtain a path $\pi'$ that has length at most $|Q|^2 + 2$ and such that Item 3 is satisfied. Consider all maximal infixes that were removed from $\pi$ to obtain $\pi'$. Note that each such infix must necessarily be a cycle as they arise from iteratively removing cycles. Since $\pi$ was primitive, all of them must have non-positive weight. Hence, Items 1 and 2 also hold.
**F  Proof of Lemma 6**

Consider two “consecutive” configurations \((q, z), (q, z + W_q)\) ∈ \(C \setminus U_n\), then all configurations \((q, z')\) for \(z \leq z' < z + W_q\) lie in pairwise-distinct \(q\)-residue classes. In particular, since there are at most \(|Q|\) non-trivial residue classes, and since trivial residue classes are contained in \(U_0\), we have that at most \(|Q|\) such elements are in \(\delta_n(C)\).

**G  Proof of Proposition 10**

By Proposition 8 we can find a polynomial \(\text{poly}^\prime_\gamma\) such that

\[
\text{poly}^\prime_\gamma(|Q|) \geq |Q|^2 + |Q| + 3 + \sum_{R \text{ non-trivial}} |\delta_n(R)|
\]

for all \(n \in \mathbb{N}\).

Set \(\text{poly}_\gamma(|Q|) := |Q| \cdot (\text{poly}^\prime_\gamma(|Q|))^2 + |Q|^2 + 4\), and consider a valid, primitive path \(\pi\) such that \(\text{length}(\pi) > \text{poly}_\gamma(|Q|)\) and \((q, z) \xrightarrow{\pi} (q', z')\).

Since \(\pi\) has length greater than \(|Q| \cdot (\text{poly}^\prime_\gamma(|Q|))^2 + 2\), there exists a state \(q'' \in Q\) that occurs at least \((\text{poly}^\prime_\gamma(|Q|))^2 + 2\) times in internal configurations within the first \(|Q| \cdot (\text{poly}^\prime_\gamma(|Q|))^2 + 2\) configurations of \(\pi\). Thus, there exists a sequence of proper prefixes \(\pi_1 < \ldots < \pi_{\text{poly}^\prime_\gamma(|Q|)}\) of \(\pi\) that all end in \(q''\) and such that one of the following two cases holds.

(i) The numbers \(\text{weight}(\pi_i)\) all have the same residue modulo \(W_q\).

(ii) The numbers \(\text{weight}(\pi_i)\) have pairwise distinct residues modulo \(W_q\).

Indeed, since there are \((\text{poly}^\prime_\gamma(|Q|))^2\) prefixes to choose from, either Case (i) holds, or there are strictly less than \(\text{poly}^\prime_\gamma(|Q|)\) prefixes per residue class. If the latter holds then there must be least \(\text{poly}^\prime_\gamma(|Q|)\) such distinct residue classes, so Case (ii) holds.

In either case, we decompose the computation \(\pi\) as \(\pi = \pi_{\text{poly}^\prime_\gamma(|Q|)} \cdot \pi'\). Observe that since \(\pi\) is primitive, then so is \(\pi'\). Applying Proposition 8 to \(\pi'\) we obtain a path \(\pi''\) of length at most \(|Q|^2 + 1\) such that \(\pi_{\text{poly}^\prime_\gamma(|Q|)} \cdot \pi''\) leads from \((q, x)\) to either the same residue class as \((q', z')\) or to a trivial \(q\)-residue class.

It is important to note that we cannot assume \(\pi''\) is not blocked after the prefix \(\pi_{\text{poly}^\prime_\gamma(|Q|)}\). However, since \(|\text{blocked}(\pi'')| \leq |Q|^2\), we can remove from the list of prefixes at most \(|Q|^2\) prefixes such that the remaining prefixes do not cause \(\pi''\) to block. (Indeed, we will not modify the path by literally removing prefixes but rather cycles which correspond to the path from a prefix to a longer prefix. For now, we are only speaking about removing elements from the collection of prefixes we can choose from.) W.l.o.g., let \(\pi_1, \ldots, \pi_d\) be the remaining prefixes.

Consider the family of paths \(\theta_i := \pi_i \cdot \pi''\) for \(i \in \{1, \ldots, d\}\). Note that every \(\theta_i\) is of length at most \(\text{poly}^\prime_\gamma(|Q|)\), and since the \(\theta_i\) are obtained by removing \(q''\)-cycles, and since \(\pi\) is primitive, the configurations reached by \(\theta_i\) are above \((q', z')\). We claim that one of the \(\theta_i\) is a valid run from \((q, z)\) to \(U_n\).

We separate the analysis according to the cases above.

- In Case (i), if \(\pi''\) leads to a trivial residue class, then all the \(\theta_i\) reach \(U_n\), and we are done. Otherwise, \(\pi''\) leads to the same residue class as \((q', z')\). By our choice of \(\text{poly}^\prime_\gamma(|Q|)\) in (5), we have that \(d > \sum_{R \text{ non-trivial}} |\delta_n(R)|\). That is, there are more prefixes that do not cause \(\pi''\) to block than there are missing elements above \((q', z')\) in \(U_n\). We conclude that some \(\theta_i\) reaches \(U_n\).

- In Case (ii), the paths \(\theta_i\) all reach distinct residue classes. In particular, since there are more than \(|Q|\) such prefixes — i.e. \(d > |Q|\) by our choice of \(\text{poly}^\prime_\gamma(|Q|)\) — then some \(\theta_i\) reach trivial residue classes, and thus reach \(U_n\).
We carry out a forward reachability analysis starting from the initial configuration \((q_0, x_0)\). The algorithm runs for \(L + 1\) rounds. In the \(k\)-th round, we maintain for each state \(q\) a set \(S_{q,k}\) of configurations \((q, x)\) that are reachable from \((q_0, x_0)\) by valid runs of length \(k\). Let \(R_{q,k}\) denote the set of all configurations \((q, x)\) that are reachable from \((q_0, x_0)\) by valid runs of length \(k\). We maintain the invariant that if some configuration \((q, x) \in R_{q,k}\) can reach the objective \(O\) in \(L - k\) steps via a path \(\pi\) then some configuration \((q, x') \in S_{q,k}\) can also reach \(O\) via the same path \(\pi\). We output that the objective is reachable if and only if one of the sets \(S_{q,k}\) for some \(k \in \{0, \ldots, L\}\) intersects \(O\). This last step is clearly sound, given the invariant.

The key to obtaining a polynomial-time runtime bound is to suitably prune the sets \(S_{q,k}\) to keep them of polynomial size. In order to compute \(\{S_{q,k+1}\}_{q \in Q}\) from \(\{S_{q,k}\}_{q \in Q}\) we proceed as follows. First define \(\{S'_{q,k}\}_{q \in Q}\) to be the indexed set of all valid configurations reachable in one step from \(\{S_{q,k}\}_{q \in Q}\). Now we obtain \(S_{q,k+1}\) from \(S'_{q,k}\) by the following two steps:

1. First, we delete from \(S'_{q,k}\) all configurations \((q, x)\) such that there are \((n + L)\) configurations \((q, x')\) in \(S'_{q,k}\) with \(x' > x\) and \(x' \equiv x\pmod{W}\).

2. Secondly, we delete from \(S'_{q,k}\) all configurations \((q, x)\) such that there are \((n + L)(m + 1)\) configurations \((q, x')\) in \(S'_{q,k}\) with \(x' > x\).

Clearly each set \(S_{q,k}\) has cardinality at most \((n + L)(m + 1)\), and moreover, it can be computed from the collection of sets \(\{S_{q',k-1} \mid q' \in Q\}\) in polynomial time.

It remains to argue that the invariant is maintained between rounds. To this end, suppose some state \((q, x) \in R_{q,k+1}\) can reach the objective in \(L - k - 1\) steps via a path \(\pi\). Then there exists a state \((q', x') \in R_{q',k}\) that can reach the objective in \(L - k\) steps via the path \(q' \pi\). By the loop invariant there exists a state \((q'', x'') \in S_{q',k}\) that can also reach the objective via the path \(q' \pi\). Hence there is a state \((q, y) \in S_{q',k}\) that can reach the objective via the path \(\pi\). Now if \((q, y)\) is deleted in the first stage of pruning then there is some configuration \((q, y')\) such that \(y' > y\), \(y' \equiv y\pmod{W}\), and \(\pi\) yields a valid computation from \((q, y')\) to the objective \(O\). After the first stage of pruning, each residue class in \(S'_{q,k}\) contains at most \(n + L\) elements. Hence if \((q, y)\) is deleted in the second stage of pruning, there are at least \(n + L\) configurations \((q, y'')\) in \(S_{q,k+1}\) that are above \((q, y')\) and are such that the run over \(\pi\) from \((q, y')\) leads to a configuration \((qf, z)\) with \(\bigwedge_{i=1}^n z \not\equiv a_i \pmod{W}\). Now from one of these configurations \(\pi\) yields a valid run that reaches \(O\) since one of \(n + L\) choices of \((q, y')\) will avoid blocked(\(\pi\)) and lead to a configuration \((qf, z)\) such that \(\bigwedge_{i=1}^n z \not\equiv b_i\).

## Proof of Lemma 13

Items 1 and 2 are obvious. Item 3 follows from the fact that if \(\pi_1 \in P\) is dominated by \(\pi'_1 \in P'\) and \(\pi_2 \in R\) is dominated by \(\pi'_2 \in R'\) then \(\pi_1 \cdot \pi_2\) is dominated by \(\pi'_1 \cdot \pi'_2\). Indeed,

\[
\begin{align*}
\text{pmin}(\pi_1 \cdot \pi_2) &= \min(\text{pmin}(\pi_1), \text{weight}(\pi_1) + \text{pmin}(\pi_2)) \\
&\leq \min(\text{pmin}(\pi'_1), \text{weight}(\pi'_1) + \text{pmin}(\pi'_2)) \\
&= \text{pmin}(\pi'_1 \cdot \pi'_2).
\end{align*}
\]

We can similarly argue that \(\text{smax}(\pi_1 \cdot \pi_2) \leq \text{smax}(\pi'_1 \cdot \pi'_2)\).
Proof of Proposition 14

Fix a state \( r \in Q \). Consider all \( p \to r \) paths that appear as a minimal prefix of some path in \( P \). Pick a single such prefix \( \pi_1 \) of maximum weight. Likewise consider all \( r \to q \) paths that appear as a maximal suffix of some path in \( P \) and pick a single such a prefix \( \pi_2 \) of maximum weight. Now form the path \( \pi := \pi_1 \cdot \pi_2 \). This path dominates any path in \( P \) with nadir \( r \).

We define \( P' \) to be the set of paths \( \pi \) formed in this way as \( r \) runs through \( Q \). Without loss of generality, we will henceforth suppose the absolute weight of all paths in \( P' \) is at most \( 2^k \). That is, it can be encoded in binary using \( k + 1 \) bits.

The \( \mathsf{NC}^1 \) bound on computing \( P' \) relies on the well-known fact that the sum of a list of binary integers can be computed in \( \mathsf{NC}^1 \) [12, Chapter 1]. To obtain \( P' \) we compute the weight of each prefix and suffix of every path in \( P \) in parallel. According to [12], this can be done in time \( O(\log k) \) on a parallel computer with \( |P|k \) processors: one for each element of \( P \) and each midpoint \( 0 \leq m \leq k \). Finally, for each state \( r \in Q \) in parallel, we find a maximum-weight prefix of a path in \( P \) that connects \( p \) and \( r \) and a maximum-weight suffix of a path in \( P \) that connects \( r \) and \( q \). It is straightforward to prove the latter is also in \( \mathsf{NC}^1 \) (see Appendix K) thus completing the proof.

Computing the maximum of a list of numbers is in \( \mathsf{NC}^1 \)

We will actually prove that the problem is in \( \mathsf{AC}^0 \). Since \( \mathsf{AC}^0 \) is known to be strictly contained in \( \mathsf{NC}^1 \) [1], this is slightly stronger than what we require.

Let us formalize the problem we focus on and define the complexity class \( \mathsf{AC}^0 \).

Given \( n \) numbers \( x_0, \ldots, x_{n-1} \) encoded in binary as \( m \)-bit strings, the ITMAX problem asks to compute the maximum of the given list of numbers in binary.

The complexity class \( \mathsf{AC}^0 \) consists of all decision problems decidable by a logspace-uniform family of Boolean circuits with unbounded fan-in \( \{ \land, \lor \} \)-gates, polynomial size, and constant depth [1]. We will make use of the following equivalent descriptive-complexity definition: The class \( \mathsf{AC}^0 \) consists of the set of all languages describable in first-order logic with the addition of the \( \mathsf{BIT} \) predicate [9]. The latter predicate is defined as follows

\[
\mathsf{BIT}(x, i) \overset{\text{def}}{=} \text{the } (i + 1)\text{-th bit of } x \text{ is set to } 1.
\]

In the sequel, let 0 be the index of the most significant bit in the binary representation of the given integers. That is, the first bit of the binary string is the most significant one.

Proposition 16. ITMAX can be computed in \( \mathsf{AC}^0 \).

Proof. We proceed by first defining \( \mathsf{AC}^0 \) circuits \( \mathsf{GEQ}_{i,j} \) (or rather first-order logic predicates using BIT) that output 1 for the input if and only if \( x_i \geq x_j \) holds. That is, for all \( i \neq j \) we define the following.

\[
\mathsf{GT}_{i,j} \overset{\text{def}}{=} \exists \ell \in [0, m] : \mathsf{BIT}(x_i, \ell) \land \neg \mathsf{BIT}(x_j, \ell) \land \\
(\forall k \in [0, \ell] : \mathsf{BIT}(x_j, k) \leftrightarrow \mathsf{BIT}(x_i, k))
\]

\[
\mathsf{GEQ}_{i,j} \overset{\text{def}}{=} (\forall k \in [0, m] : \mathsf{BIT}(x_i, k) \leftrightarrow \mathsf{BIT}(x_j, k)) \lor \mathsf{GT}_{i,j}
\]

Then, we make use of those circuits to define new \( \mathsf{AC}^0 \) circuits \( M_i \) which recognize whether the \( i \)-th element \( x_i \) from the input list of integers is a maximal one

\[
M_i \overset{\text{def}}{=} \forall j \in [0, m] : \mathsf{GEQ}_{i,j}.
\]
Finally, in order to output a single maximal element, we build an $\text{AC}^0$ circuit $M'_i$, per input integer, which outputs 1 if $i$ is the first index such that $M_i$ outputs 1. Concretely, for all $0 \leq i \leq n$ and all $0 \leq k \leq m$, we let
\[
M'_i \overset{\text{def}}{=} M_i \land (\forall j \in [0, i) : \neg M_j)
\]
\[
b_k \overset{\text{def}}{=} \exists i \in [0, m] : (M'_i \land \text{BIT}(x_i, k))
\]
where $b_k$ is the $k$-th output bit of the circuit. Since $M'_i$ clearly holds only if $M'_j$ does not hold, for all $i \neq j$, the circuit correctly outputs the binary representation of the first maximal element of the given list.
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