A strong renewal theorem for relatively stable variables

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Abstract

Let \( X \) be a non-negative random variable that is relatively stable, or what amounts to the same, \( \int_0^x P[X > t]dt \sim \ell(x) \) for a slowly varying function \( \ell \). We show that if \( X \) is not arithmetic, the renewal function \( U(x) \) associated with \( X \) satisfies \( \lim_{x \to \infty} [U(x + h) - U(x)]\ell(x) = h \) for \( h > 0 \). An obvious analogue also holds for the arithmetic variable. The extension to not necessarily non-negative \( X \) is obtained.

Keywords: renewal measure; relative stability; infinite mean; slowly varying.

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1 Introduction and results

Let \( X \) be a real valued random variable with distribution function \( F(x), x \in \mathbb{R} \) and \( F(dx) \) the probability measure of \( X \). Let \( U \) be the associated renewal measure (or what is the same, the Green measure of the random walk generated by \( F \)) that is given by

\[
U\{I\} = \sum_{n=0}^{\infty} F^{n*}\{I\},
\]

for finite intervals \( I \subset \mathbb{R} \), where \( F^{n*} \) denotes the \( n \)-fold convolution. The above sum may be infinite. Following [5, Section VI.10] we call \( F \) transient if \( U\{[-y, y]\} < \infty \) for all \( y > 0 \) (otherwise \( U\{[-y, y]\} = \infty \) for all \( y \) large enough). If \( F \) is transient and non-arithmetic, the classical theorem [5, Section XI.9] says that for each \( h > 0 \),

\[
\lim_{x \to \infty} U[x, x+h] = \begin{cases} h/E[X] & \text{if } E|X| < \infty \text{ and } E[X] > 0, \\ 0 & \text{otherwise.} \end{cases}
\]

Here \( U[x, x+h] \) is written for \( U\{[x, x+h]\} \). The random variable \( X \) (or its distribution \( F \)) is called relatively stable (r.s.) if there exists a real norming sequence \( (a_n) \) such that for any \( \varepsilon > 0 \), \( \lim_{n \to \infty} |F^{n*}(a_n(1+\varepsilon)) - F^{n*}(a_n(1-\varepsilon))| = 1 \). In this note we always suppose

\[
E|X| = \infty
\]

and find the exact asymptotic form of \( U[x, x+h] \) when \( F \) is r.s. and transient.

For non-negative \( X \) with \( E[X] = \infty \) the asymptotic behaviour of \( U[x, x+h] \) has been studied when \( U[0, x] \) varies regularly with exponent \( -\alpha \), \( 0 < \alpha \leq 1 \) [8], [4], [3] etc. but the case \( \alpha = 1 \) (necessarily \( X \) is r.s.) were excluded in most of them except in Erickson [4] and [7] to the knowledge of the author: under the assumption that \( x[1 - F(x)] \) is slowly varying (s.v.), entailing the slow variation of \( U[0, x]/x \), it is shown that \( U[x, x+h]/U[0, x] \to h \), or what amounts to the same, \( U[x, x+1] \) is s.v., in [4] and that \( U[x,x+h] \) belongs to the de Haan class \( \Pi \) in [7], a kind of second-order asymptotic result. When \( X \) assumes both positive and negative values but only of the integer lattice \( \mathbb{Z} \) with \( F \) transient and belonging to the Cauchy

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domain of attraction Berger [1, Theorem 3.6] obtained an asymptotic form of the function $U\{n\}$ under some additional assumption on $P[X = n]$ in a case when $X$ is r.s. Our Theorems 1 and 2 improves results of [4] and [1], respectively, by removing the extra conditions on $F$ (see Section 4.3 for the result of [1]) so as to establish that $U[x, x + h]$ is s.v., if $F$ is positively relatively stable ($a_n > 0$ for all large $n$; see Remark [1]) for every transient random walk.

First we consider the case when $X$ assumes only non-negative values because of the importance and simplicity of the result. It is known that in order for $X(\geq 0)$ to be r.s. it is necessary and sufficient that the function

$$\ell(x) := \int_0^x (1 - F(t))dt$$

is s.v. at $+\infty$ (namely, $\ell(x) > 0$ for $x$ large enough and $\lim_{x \to \infty} \ell(\lambda x) / \ell(x) = 1$ for any $\lambda > 1$); and if this is the case, then $U[0, x]/x \sim 1/\ell(x)$ as $x \to \infty$; and conversely if $U[0, x]/x$ is s.v., then so is $\ell[15, 2]$ ($\sim$ means the ratio of its two sides converges to unity). Erickson [4] proved

$$\lim_{x \to \infty} \ell(x)U[x, x + h] = h \text{ for each } h > 0,$$

under the condition that for some s.v. function $L$

$$(*) \quad 1 - F(x) \sim L(x)/x,$$

which is stronger than the slow variation of $\ell$. Our first theorem asserts that the condition $(*)$ is superfluous for (3), if $\ell$ is s.v. We call $X$ (or $F$) arithmetic if $X/h \in \mathbb{Z}$ almost surely (a.s.) for some $h > 0$ and non-arithmetic otherwise.

**Theorem 1.** If $X$ is non-negative a.s. and non-arithmetic and $\ell$ is s.v., then (3) holds.

For arithmetic distributions the analogous result is verified in [19, Appendix(B)] with the proof much simpler [if $X \geq 0$ is arithmetic of span $h > 0$, then (3) holds if $\ell$ is s.v.]. Our proof of Theorem 1 will depend largely on the arguments of [4]. In this respect the contribution of the present note concerning Theorem 1 is Lemma 1 given in the next section.

For one dimensional random walks attracted to a stable law of exponent $1 \leq \alpha \leq 2$ there naturally arises the situation that the ascending ladder height variable is r.s. so that the Erickson’s result does not apply to the corresponding renewal measure without the extra assumption that ensures $(*)$ for the height variable but it is not easy to specify such an assumption in terms of the increment distribution of the walk. The author encountered the inconvenience of this situation in studying the tow-sided exit problem of random walks [20], which commanded the interest on the present problem.

The result can be extended in a natural fashion to the case when $X$ assumes both positive and negative values. As noted previously for $U(x, x + h]$ to be finite for all $x$ and $h > 0$ the random walk must be transient.

To state the result we need to introduce notation. Put for $x \geq 0$,

$$H(x) = 1 - F(x) + F(-x), \quad K(x) = 1 - F(x) - F(-x),$$

$$\ell(x) = \int_0^x H(t)dt \quad \text{and} \quad A(x) = \int_0^x K(t)dt.$$ 

By $E|X| = \infty, H(x) > 0$ for all $x$. For our second theorem we require the following conditions:

$$\begin{cases} (Ha) \quad A(x)/xH(x) \to \infty \quad (x \to \infty). \\
(Hb) \quad \int_{x_0}^\infty H(x)dx/A^2(x) < \infty \quad \text{for some } x_0 > 0. \end{cases}$$

(4)
(Ha) is equivalent to positive relative stability, and (Hb) is equivalent to transience of the walk under (Ha) as asserted in the next theorem. A simple example where both (Ha) and (Hb) are satisfied is provided by any distribution such that \( F(-x) \sim (1-p)L(x)/x \) and \( 1-F(x) \sim pL(x)/x \) as \( x \to \infty \) with \( 1/2 < p \leq 1 \) and \( L \) a s.v. function satisfying \( \int_0^\infty L(t)dt/t = \infty \) (see Section 4.3 for more on this example). We shall suppose \( F \) is non-arithmetic, the arithmetic case being similarly dealt with in a simpler way.

**Theorem 2.** Suppose that condition (Ha) is satisfied. Then \( F \) is transient if and only if (Hb) holds; if this is the case and \( F \) is non-arithmetic, then for each \( h > 0 \),

\[
\frac{U(x, x+h)}{h} = \begin{cases} 
\int_x^\infty \frac{1-F(t)}{A^2(t)}dt\{1+o(1)\} & \text{as } x \to \infty, \\
\int_{|x|}^\infty \frac{F(-t)}{A^2(t)}dt + o\left(\frac{1}{A(|x|)}\right) & \text{as } x \to -\infty.
\end{cases}
\]

(5)

The condition (Hb) entails that the derivative \( (1/A)'(x) = -K(x)/A^2(x) \) is integrable about \( x=\infty \) and \( \limsup A(x) = \infty \). Hence

\[
\int_x^\infty \frac{K(t)}{A^2(t)}dt = \frac{1}{A(x)} \quad (x > x_0) \quad \text{under (Hb)};
\]

(6)

Put \( \ell_+(x) := \int_x^\infty [1-F(t)]dt/A^2(t) \). Then \( 1/A(x) \leq \ell_+(x) \) so that if \( F(-x)/[1-F(x)] \) is bounded away from 0, then the additional term \( o(1/A(x)) \) is negligible, but if this ratio tends to zero, the second case of (5) comes down to merely \( U(x, x+h) = o(1/A(x)) \) \( (x \to -\infty) \).

Note that \( \ell_+ \) is s.v. under (Hab), since \( x[1-F(x)]/[A^2(x)\ell_+(x)] \leq xH(x)/A(x) \to 0 \).

Formula (5) is simplified under the following specific condition

\[(Hc) \quad \kappa := \lim_{x \to \infty} A(x)/\ell(x) \quad \text{exists with } \kappa > 0.\]

[The case \( \kappa = 0 \), where certain subtlety is involved, will be briefly discussed in Remark 4.]

Since \( \ell(x) \to \infty \) (because of (2)), we have the (trivial) identity

\[
\int_x^\infty \frac{H(t)}{\ell^2(t)}dt = \frac{1}{\ell(x)},
\]

showing that (Hb) follows from (Hac)—the conjunction of (Ha) and (Hc). On combining it with (6) Theorem 2 accordingly reduces to the following result.

**Corollary** If (Hac) holds, then for each \( h > 0 \),

\[
\frac{\ell(|x|)U(x, x+h)}{h} \to \begin{cases} 
\frac{1}{2}(1+\kappa)/\kappa^2 & \text{as } x \to +\infty, \\
\frac{1}{2}(1-\kappa)/\kappa^2 & \text{as } x \to -\infty.
\end{cases}
\]

(4)

**Remark 1.** If \( X \) is r.s., the norming constants \( a_n \) are eventually positive or eventually negative. In [10] \( X \) is called positively r.s. for the former case and negatively r.s. for the latter. Condition (Ha) is equivalent to the positive relative stability of \( X \) (cf. [16], [11], [10]) and implies

\[(Ha') \quad \exists x_0 > 0, \quad A(x) \text{ is positive for } x \geq x_0 \text{ and s.v. as } x \to +\infty.\]

[In fact \( \log[A(x)/A(x_0)] = \int_x^\infty \varepsilon(t)dt/t \) with \( \varepsilon(t) := tK(t)/A(t) \) so that \( A \) is a (normalized) s.v. function under (Ha).] If (Hc) holds, then the converse is true, i.e., (Hb) follows from (Ha'c) as readily checked. Thus (Hac) is equivalent to (Ha'c).
Remark 2. The conditions (Ha) and (Hb) are not comparable, i.e., there exist an example of \( F \) that satisfies (Ha) but not (Hb) and another one that satisfies (Hb) but not (Ha) (see Section 4.3).

Theorem 1 follows from Theorem 2 (in fact from Corollary) as a special case so that we have only to prove the latter. However we prove them separately. The proofs of them are both based on certain results obtained by Erickson [4], which however are immediately available only for the proof of Theorem 1. In fact we give two proofs of Theorem 1, one uses a Fourier cosine representation of \( U \) as in [4], while the other uses the corresponding Fourier sine representation unlike [4] and the proof of Theorem 2 is carried out by combining these two approaches by suitably adapting the arguments made in [4].

We shall give a proof of Theorem 1 in Section 2. Another proof of it will be provided in Section 3; to this end we shall describe main steps of the proof under (*) given in [4], which will also prepare for the proof of Theorem 2 that will be given in Section 4.

2 A lemma and Proof of Theorem 1

Suppose \( \ell \) is s.v. and \( X \) is non-arithmetic. It then follows that as \( x \to \infty \),

\[
x[1 - F(x)] = o(\ell(x)), \quad \int_0^x t^2dF(t) \sim \ell(x),
\]

and

\[
\int_x^\infty \frac{t^2dF(t)}{\ell^2(t)} \sim \int_x^\infty \frac{1 - F(t)}{\ell^2(t)} dt = \frac{1}{\ell(x)}. \tag{8}
\]

(The proofs of these relations are standard in view of the fundamental properties of regularly varying functions on which the readers may consult [2] or [5].) Let

\[
\phi(\theta) = \int_0^\infty e^{ix}dF(x),
\]

the characteristic function of \( X \). Then integrating by parts and using (7) lead to

\[
\Re[1 - \phi(\theta)] = \theta \int_0^\infty [1 - F(x)] \sin \theta x \, dx = o(\theta \ell(1/\theta)),
\]

\[
\Im[1 - \phi(\theta)] = -\theta \int_0^\infty [1 - F(x)] \cos \theta x \, dx \sim -\theta \ell(1/\theta))
\]

as \( \theta \to 0 \). Hence

\[
1 - \phi(\theta) = -i\theta [1/|\theta|] \{1 + o(1)\} \quad (\theta \to 0). \tag{9}
\]

In [4] (*) is used to deduce \( \Re[1 - \phi(\theta)] \sim \theta L(1/\theta) \), which cannot be obtained by the present assumption. Fortunately what is actually needed is the result given by Lemma 1 below. Let us write \( C(\theta) \) and \( S(\theta) \) for the real and imaginary parts of \( 1/[1 - \phi(\theta)] \):

\[
C(\theta) = \Re \frac{1}{1 - \phi(\theta)}, \quad S(\theta) = \Im \frac{1}{1 - \phi(\theta)}
\]

so that \( 1/[1 - \phi(\theta)] = C(\theta) + iS(\theta) \).
Lemma 1.

\[ J(\theta) := \int_0^\theta C(t) dt \sim \frac{\pi/2}{\ell(1/\theta)} \quad \text{as } \theta \to +0. \]

Proof. Noting \( J(\theta) = \int_0^\theta dt \int_0^\infty (1 - \cos tx) dF(x) / |1 - \phi(t)|^2 \), split the range of the inner integral on the RHS at \( M/\theta \) with any number \( M > 1 \) and accordingly decompose \( J \) as the sum of two repeated integrals. After interchanging the order of integration for the integral over \( x > M/\theta \), this result in \( J(\theta) = J_1(\theta) + J_2(\theta) \), where

\[ J_1(\theta) = \int_M^\infty dF(x) \int_0^\theta \frac{1 - \cos xt}{|1 - \phi(t)|^2} dt, \]
\[ J_2(\theta) = \int_0^\theta \frac{dt}{|1 - \phi(t)|^2} \int_0^{M/\theta} (1 - \cos tx) dF(x). \]

First we see \( J_2(\theta) \) is negligible. The integrand of the integral defining \( J_2(\theta) \) is less than

\[ \frac{t^2/2}{|1 - \phi(t)|^2} \int_0^{M/\theta} x^2 dF(x) = \frac{1}{\ell^2(1/t)} \times o(\ell^{-1}(1/\theta)), \]

where (9) as well as (12) is used for the equality. This immediately yields \( J_2(\theta) \ell(1/\theta) \rightarrow 0 \).

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In order to estimate \( J_1(\theta) \) we break the inner integral into three parts and using (9) we deduce that uniformly for \( x > M/\theta \), as \( \theta \downarrow 0 \),

\[ \int_0^{1/Mx} \frac{1 - \cos xt}{|1 - \phi(t)|^2} dt \leq x^2 \int_0^{1/Mx} \frac{dt}{\ell^2(1/t)} \sim \frac{x/M}{\ell^2(x)}, \]
\[ \int_{M/x}^\theta \frac{1 - \cos xt}{|1 - \phi(t)|^2} dt \leq \int_{M/x}^\infty \frac{dt}{t^2 \ell^2(1/t)} \sim \frac{x/M}{\ell^2(x)} \]

and

\[ \int_{1/Mx}^{M/x} \frac{1 - \cos xt}{|1 - \phi(t)|^2} dt \sim \frac{1}{\ell^2(x)} \int_{1/Mx}^{M/x} \frac{1 - \cos xt}{t^2} dt = \frac{x}{\ell^2(x)} \int_{1/M}^{M} \frac{1 - \cos t}{t^2} dt. \]

Hence

\[ J_1(\theta) = \int_{M/\theta}^\infty \frac{xdF(x)}{\ell^2(x)} \left\{ \frac{\pi}{2} + O(1/M) \right\}. \]

and in view of (8) \( J_1(\theta) \ell(1/\theta) \rightarrow \frac{1}{2} \pi \). This concludes the proof of the lemma, for \( M \) may be chosen arbitrarily large. \( \square \)

In [4] the auxiliary condition (\( \ast \)) is used to obtain the estimate of Lemma [1] (as well as of [2]) and once it is established the proof of [3] proceeds without directly using (\( \ast \)) (see Section 3.1 of this paper). With Lemma [1] at hand we can accordingly follow the arguments given in Section 5 of [4] to complete the proof of Theorem 1.

3 Proof of Theorem 1 based on the estimate of \( S(\theta) = \mathfrak{R}[1 - \phi(\theta)]^{-1} \)

In this section, consisting of three subsections, we point out that the proof of Theorem 1 can be based on [3] rather than Lemma 1; this way is rather simpler, being performed without
resorting any estimate of \( C(\theta) \) except for its summability in a neighbourhood of zero that is valid quite generally (see \( 23 \)). The exposition given in this section also prepares for the proof of Theorem \( 2 \). In the first two subsections the entire mass of \( F \) is supposed to concentrate on \([0, \infty)\). In the last subsection \( F \) is allowed to have mass on the negative half line and we obtain a general analytic result involving the characteristic function of \( F \) and fundamental in our proof of Theorem \( 2 \).

3.1. In this subsection we describe the main steps of the proof of Theorem \( 1 \) given by \([4]\) under \((*)\). For a real function \( g \in L_1(\mathbb{R}) \) let \( \gamma \) be its Fourier transform:

\[
\gamma(x) = \int_{\mathbb{R}} e^{-ix\theta} g(\theta) d\theta.
\]

Then an elementary manipulation leads to

\[
\int_{\mathbb{R}} e^{i\theta \xi} \gamma(\xi) U\{ x + d\xi \} = \int_{\mathbb{R}} U\{ dy \} \int_{\mathbb{R}} e^{i(y-x)u} g(\theta - u) du,
\]

provided that the integral on the LHS is absolutely convergent for which it is sufficient that

\[
\gamma(x) = O(x^{-2}) \quad (|x| \to \infty)
\]

in view of \( 11 \). Below \( g \) is always assumed to be continuous and piece-wise smooth \((C^2\text{-class is enough})\) and vanish outside a compact set, which in particular ensures \( 11 \).

Consider the particular function \( g = g_a, \ a > 0 \) defined by

\[
g_a(\theta) = \begin{cases} a^{-1}(1 - |\theta|/a) & |\theta| \leq a, \\ 0 & |\theta| > a, \end{cases}
\]

whose Fourier transform is equal to \( \gamma_a(x) = 2(1 - \cos(ax))/a^2x^2 \). It is shown \([4, \text{Lemma 8}]\) that given a constant \( \nu \geq 0 \) and a family of measures \((\mu_x)_{x>0}\) on \( \mathbb{R} \) such that \( \mu_x\{[-y, y]\} < \infty \) for every \( y > 0 \),

\[
\text{if} \quad \forall a > 0, \forall \theta \in \mathbb{R}, \quad \lim_{x \to \infty} \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\theta \xi} \gamma_a(\xi) \mu_x\{ d\xi \} = \nu g_a(\theta),
\]

\[
\text{then} \quad \forall h > 0, \forall y \in \mathbb{R}, \quad \lim_{x \to \infty} \mu_x\{ [y, y + h] \} = \nu h.
\]

(Here the case \( \nu = 0 \)—trivial from the special case \( \theta = 0 \)—is included, although it is not explicitly stated in \([4]\).) On applying this result to \( \mu_x\{ d\xi \} = \ell(x) U\{ x + d\xi \} \) it suffices for the proof of Theorem \( 1 \) to show that for each \( a > 0 \) and \( \theta \in \mathbb{R} \), as \( x \to \infty \)

\[
\frac{1}{2\pi} \int_{\mathbb{R}} e^{i\theta \xi} \gamma_a(\xi) U\{ x + d\xi \} \sim \frac{g_a(\theta)}{\ell(x)}.
\]

In order to verify \( 13 \) the relation \( 10 \) or its variant will be used. On formal level one might interchange the order of the repeated integral on its RHS and derive \( \int_{\mathbb{R}} e^{iyu} U\{ dy \} = \sum_0^\infty \phi^n(t) = 1/[1 - \phi(u)] \) to find

\[
\int_{\mathbb{R}} e^{i\theta \xi} \gamma(\xi) U\{ x + d\xi \} = \int_{\mathbb{R}} \frac{g(\theta - u)}{1 - \phi(u)} e^{-ixu} du.
\]

We shall see this is valid but the difficulty of direct approach would be obvious: actually \( U\{ \mathbb{R} \} = \infty \), so that the Fourier-Stieltjes transform of \( U(d\xi) \) does not exist in a usual sense.
Following the standard approach we make the decomposition $U\{I\} = V\{I\} + V^*\{I\}$, where

$$V\{I\} = \frac{1}{2}(U\{I\} + U\{-I\}), \quad V^* = \frac{1}{2}(U\{I\} - U\{-I\}).$$

Since for $\xi < x$, $U\{-x - d\xi\} = 0$—entailing

$$U\{x + d\xi\} = 2V\{x + d\xi\} \quad (\xi < x),$$

$U$ can be replaced by $2V$ in (13) because of (11). Introducing

$$V_s\{dx\} = \sum_{n=0}^{\infty} s^n[F^{n*}\{dx\} + F^{n*}\{-dx\}] \quad (0 < s < 1)$$

one sees that the identity (10) holds for $V_s$ replacing $U$ on both sides of it. By letting $s \uparrow 1$ in the identity that accordingly appears it is shown that

$$\int e^{i\theta x + \gamma(x)}V\{x + d\xi\} = \int_{\mathbb{R}} g_a(\theta - u)C(u)e^{-i\theta u}du$$

(Eq(4.3) of [1]; see also [6] (where roles of $g$ and $\gamma$ interchanged)). With the help of the bound

$$|\phi(\theta) - \phi(\theta')| \leq 2|\theta - \theta'|(1/|\theta - \theta'|) \quad (\theta \neq \theta')$$

(Lemma 5 of [1]) the integral on the RHS restricted to $|\theta| \geq M/x$ is shown to be bounded in absolute value by a constant multiple of $1/M\ell(x)$ for any large number $M$. If one applies Lemma 1 of the preceding section, it is easy to see that

$$\int_{|\theta| < M/x} g_a(\theta - u)C(u)e^{-i\theta u}du \sim \frac{g_a(\theta)}{\ell(x)}$$

for each $M$ fixed. Because of (15) and (17) this verifies (13).

3.2. According to (12) it suffices for the proof of Theorem 1 to show that as $x \to \infty$

$$\frac{1}{\pi} \int_{\mathbb{R}} e^{i\theta x + \gamma(x)}V^*\{x + d\xi\} \sim \frac{g(\theta)}{\ell(x)}$$

since $U(x + d\xi) = 2V^*(x + d\xi)$ ($x > 1, \xi < -x$). In the next subsection we show the identity

$$\int_{\mathbb{R}} e^{i\theta x + \gamma(x)}V^*\{x + d\xi\} = i \int_{\mathbb{R}} g(\theta - u)S(u)e^{-i\theta u}du$$

(for every non-arithmetic $F$ on $\mathbb{R}$). By (9) we have

$$S(u) \sim 1/|u|(1/u) \quad (u \to 0).$$

Since $S$ is an odd function, the RHS of (20) restricted to $|u| < M/x, M > 1$ is written as

$$g(\theta) \int_{|u| < M/x} S(u)\sin xu du + i \int_{|u| < M/x} [g(\theta - u) - g(\theta)] S(u)e^{-i\theta u}du,$$

and easily evaluated to be $\pi [g(\theta)/\ell(x)]\{1 + o(1) + O(1/M)\}$. On using (18) the other integral can be shown to be bounded in absolute value by a constant multiple of $M^{-1}g(\theta)/\ell(x)$ as in [1]. Thus (19) follows, for $M$ can be made arbitrarily large, and the proof of Theorem 1 will be complete if we can show (20).
Remark 3. The identities (17) and (20) are valid for every non-arithmetic $X \geq 0$. Adding them gives (14). In fact these are all true whenever $F$ is transient (cf. [6], (38)).

3.3. Proof of (20). Taking $\theta = x = 0$ in (17) yields

$$\int_0^1 C(u)du < \infty$$

(23)

(at least for every non-arithmetic $X \geq 0$). Identity (20) will be derived solely from (23), the derivation being valid irrelevantly to the support of $F$. For application done in the next section we formulate the result as the following

**Lemma 2.** For any non-arithmetic distribution $F$ on $\mathbb{R}$ (23) implies (20) as well as the summability $\int_0^1 |uS(u)|du < \infty$.

**Proof.** Put $V_s^*\{dx\} = \frac{1}{2} \sum_{n=0}^{\infty} s^n \left[ F^n\{dx\} - F^n\{-dx\} \right] (0 < s < 1)$. In the identity (10) $U$ may be replaced by $V_s^*$ simultaneously on both sides. The order of the repeated integral on the RHS of the resulting identity may be interchanged (since the total variation of $V_s^*$ is finite), yielding

$$\int_{\mathbb{R}} e^{i\theta \xi} \gamma(\xi) V_s^*\{x + d\xi\} = i \int_{\mathbb{R}} e^{-ixu} g(\theta - u) \Im \frac{1}{1 - s\phi(u)} du$$

(24)

because of the identity

$$\int_{\mathbb{R}} e^{iuy} V_s^*\{dy\} = i \int_{\mathbb{R}} \sin uy V_s^*\{dy\} = i \Im \frac{1}{1 - s\phi(u)}.$$

Thus formal manipulation gives (20) which may be written as

$$\int_{\mathbb{R}} e^{i\theta \xi} \gamma(\xi) V_s^*\{x + d\xi\} = i \int_{\mathbb{R}} e^{-ixu} g(\theta - u) \Im \frac{1}{1 - \phi(u)} du.$$  

(25)

We let $s \uparrow 1$ in (24). The convergence of its LHS to that of (25) is easily verified by (11), i.e., $\gamma(x) = O(x^{-2})$. Thus our task is reduced to verifying that if (23) holds, then

$$i \int_{\mathbb{R}} e^{-ixu} g(\theta - u) \Im \frac{1}{1 - s\phi(u)} du \longrightarrow \int_{\mathbb{R}} g(\theta - u) S(u)e^{-ixu} du.$$  

(26)

Observe that the odd part of

$$ie^{-ixu} g(\theta - u) - g(\theta) \sin xu$$

is bounded in absolute value by a constant multiple of $|u|$ (because of the assumption on $g$) while the integral on the LHS of the even part vanishes (since $\Im[1 - s\phi(u)]^{-1}$ is odd). We then infer that (26) follows if we can show that

$$\lim_{s \uparrow 1} \int_{\mathbb{R}} \left| \Im \frac{1}{1 - s\phi(u)} - S(u) \right| g(u)|du = 0 \quad \text{if} \quad g(u) = O(|u|), |u| < 1.$$  

(27)

We claim that

$$\left| u \Im \frac{1}{1 - s\phi(u)} \right| \leq \frac{|u|E|\sin uX|}{s|1 - \phi(u)|^2} \leq MC(u) \quad (|u| < 1, 1/2 < s < 1).$$  

(28)
The first inequality is obvious. For the proof of the second one we see that 
\[ 1 - \Re \phi(u) = 2E \sin^2(uX/2) \text{ and } |u| E \sin uX \leq 2(E \sin^2(uX/2))^{1/2}, \]
so that
\[ |u| E \sin uX \leq \left| u \right| \sqrt{E \sin^2(uX/2)} \left(1 - \Re \phi(u)\right) < \delta^{-1}(1 - \Re \phi(u)) \]
for some \( \delta > 0 \). For non-arithmetic \( F \) we have \( |1 - \phi(u)| > 0 \). Hence (28) is verified. Thus condition (23) implies (27) as well as the summability \( \int_0^1 |uS(u)|du < \infty \). The proof of Lemma 2 is complete.

4 Proof of Theorem 2.

In this section \( X \) assumes both positive and negative values and \( F \) is supposed to be non-arithmetic. Let \( C \) and \( S \) be the even and odd parts of \( 1/(1 - \phi) \) as before so that
\[ 1/(1 - \phi(\theta)) = C(\theta) + iS(\theta) \text{ for } \theta \in \mathbb{R} \text{ such that } \phi(\theta) \neq 1. \] (29)

4.1. Preliminary lemmas.

Lemma 3. If (Ha) holds, then as \( \theta \to 0 \)
\[ 1 - \phi(\theta) = -i\theta A(1/|\theta|)\{1 + o(1)\}. \] (30)

Proof. This is an easy part of the criteria for relative stability obtained by Maller [12, Theorem 1]. (Note that \( A(x) \sim \int_{-x}^x t dF(t) \text{ (} x \to \infty \text{) under (Ha).} \)

Let (Ha) hold. It then follows that \( A \) is s.v. under (Ha) as mentioned in Remark 1, and instead of (7)
\[ \int_0^x t^2 (-H(t)) dt \leq 2 \int_0^x tH(t)dt = o(xA(x)). \] (31)

Note that \( 1/A(x) \leq \int_{-\infty}^x H(t)dt/A^2(t) \) because of (6) and observe that
\[ \int_{x}^{\infty} \frac{td(-H(t))}{A^2(t)} = \int_{x}^{\infty} \frac{H(t)}{A^2(t)}dt\{1 + o(1)\}, \] (32)
where the two integrals are simultaneously finite or infinite. Recall that (Hb) is the integrability condition \( \int_{-\infty}^0 H(t)dt/A^2(t) < \infty \).

Lemma 4. Suppose (Ha) holds. Then (Hb) is equivalent to \( \int_0^1 C(\theta)d\theta < \infty \) and implies
\[ \int_0^\theta C(u)du \sim \frac{\pi}{2} \int_{1/\theta}^\infty \frac{H(x)}{A^2(x)}dx \quad (\theta \downarrow 0). \] (33)

Proof. We compute \( \int_0^\theta C(u)du \) by following the proof of Lemma 1. Define \( J_1 \) and \( J_2 \) as in it but with \( dF(x) \) replaced by \( d(-H(x)) \) so that \( J(\theta) = J_1(\theta) + J_2(\theta) \). Given \( M > 1 \) choose \( \theta > 0 \) so small that \( \theta < M/x_0 \). Then on using (30) and (31)
\[ J_2(\theta) \leq \frac{1}{2} \int_0^\theta \frac{dt}{A^2(1/t)\{1 + o(1)\}} \int_0^{M/\theta} x^2 d(-H(x)) = o\left( \frac{1}{A(1/\theta)} \right), \] (34)
while by (30) again uniformly for \( x \geq M/\theta \)
\[
\int_0^\theta \frac{1 - \cos xt}{|1 - \phi(t)|^2} dt = x \int_0^{x\theta} \frac{(1 - \cos t)dt}{t^2 A^2(x/t) \{1 + o(1)\}} = \frac{x}{A^2(x)} \left\{ \frac{\pi}{2} + o(1) + O\left(\frac{1}{M}\right) \right\},
\]
and
\[
J_1(\theta) = \int_{M/\theta}^{\infty} x d(-H(x)) \left\{ \frac{\pi}{2} + o(1) + O\left(\frac{1}{M}\right) \right\}.
\] (35)

Finally we combine (34), (35) and (32) to conclude that \( J(\theta) < \infty \) if and only if (Hb) holds and either one implies (33).

The summability \( \int_0^1 C(t) dt < \infty \) is equivalent to transience of \( F \) (cf. [14], [17]). By Lemma 2 it also implies \( \int_0^1 |\theta S(\theta)| d\theta < \infty \). For clarity and convenience of later citation we state these consequences of Lemma 4 as a lemma.

**Lemma 5.** Suppose (Ha). Then (Hb) is a necessary and sufficient condition for the transience of \( F \), and implies
\[
\int_0^1 [C(\theta) + |\theta S(\theta)|] d\theta < \infty.
\] (36)

Of the equivalence assertion in this lemma the proof of the sufficiency part, due to Ornstein, is considerably involved. In the present case we have \(|3\phi(\theta)| \geq \delta|\theta|\) in a neighbourhood of zero (by (6) as well as (30)), and on using \( \lim_{\theta \to 0} \phi(\theta) = 1 \) and the inequality
\[
\Re \left\{ \frac{1}{1 - s\phi} \right\} \leq \frac{1}{s^2} \Re \left\{ \frac{1}{1 - \phi} \right\} + \Re \left\{ \frac{1}{(1 - s)\xi - is\eta} \right\} \quad (|\xi| \leq 1, 0 < s < 1)
\]
valid for any complex number \( \phi = \xi + i\eta \), from the same summability of \( C(\theta) \) as above it is easy to deduce that
\[
\sup_{0 < s < 1} \int_0^1 \Re \left\{ \frac{1}{1 - s\phi(\theta)} \right\} d\theta < \infty,
\] (37)
the necessary and sufficient condition for the transience of \( F \) that is found in standard text books of probability theory.

**4.2. Proof of Theorem 3**. Theorem 3 will be proved along the same lines as in the proof of Theorem 1 with the help of Lemma 3. The proof will also rest on the general statement (12) valid independently of the present situation.

Lemma 5 ensures that the repeated integral on the RHS of (10) is interchangeable so as to yield (14), or what is the same thing
\[
\int_R e^{i\theta \gamma(\xi)} U \{ x + d\xi \} = \int_R g(\theta - u) e^{-ixu} [C(u) + iS(u)] du.
\] (38)

Indeed, by (37) valid under (Hab) the family of measures \( \Re [1 - s\phi(\xi)]^{-1} d\xi, |\xi| \leq 1 \) converges in total variation, as \( s \uparrow 1 \), to a finite measure of the form \( C(\xi) d\xi + \nu \delta_0 \) with some constant \( \nu \geq 0 \) (\( \delta_0 \) denotes the Dirac delta measure) and by taking \( V_s \) in place of \( U \) on the LHS of (14) and letting \( s \uparrow 1 \) in the corresponding equality we find that
\[
\int_R e^{i\theta \gamma(\xi)} V \{ x + d\xi \} = \int_R g(\theta - u) e^{-ixu} C(u) du + \nu g(\theta).
\] (39)
Then (40) holds at least under some side condition: if \( K \) and \( x \) in the present setting. Since (20) follows from (36) in view of Lemma 2 we obtain (38).

(See also [6, Eq(15)] where roles of \( \gamma \) and \( g \) is interchanged.) By \( E|X| = \infty \) we have \( U(x, x + h) \to 0 \) as \( x \to \pm \infty \) according to (3), by which we deduce that \( \int_\mathbb{R} |\gamma(\xi)|V\{x + d\xi\} \to 0 \). The first term on the RHS of (39) tends to zero by Riemann’s lemma. Thus \( \nu = 0 \) and we see (17) in the present setting. Since (20) follows from (36) in view of Lemma 2 we obtain (38).

It is also noted that the inequality (18) can be generalized to any probability distribution on \( \mathbb{R} \) (essentially the same proof as in [4] applies).

Now we can easily complete the proof of Theorem 2. By (33) (entailing the slow variation of \( \int_0^{1/x} C(t)dt \)) it follows that as \( x \to \infty \)

\[
\int_{-M/x}^{M/x} C(u) \cos xu \, du \sim \pi \int_x^\infty \frac{H(t)}{A^2(t)} \, dt;
\]

also by (30) \( S(\theta) \sim [\theta A(1/\theta)]^{-1} \), and on using this

\[
\int_{-M/x}^{M/x} S(u) \sin xu \, du = \frac{1}{A(x)} \{\pi + o(1) + O(1/M)\}.
\]

Making the same argument as given at (22) and employing (18) as well as the above estimates we compute the RHS of (38) to see that if \( g(\theta) \neq 0 \),

\[
\int_\mathbb{R} g(\theta - u)e^{-ixu} C(u) \, du \sim g(\theta) \int_\mathbb{R} \cos xu C(u) \, du \sim \pi g(\theta) \int_{|x|}^\infty \frac{H(t)}{A^2(t)} \, dt \quad (x \to \pm \infty)
\]

and

\[
i \int_\mathbb{R} g(\theta - u)e^{-ixu} S(u) \, du \sim g(\theta) \int_\mathbb{R} \sin xu S(u) \, du \sim \frac{\pi g(\theta)}{A(|x|)} \quad (x \to \pm \infty).
\]

Hence, on recalling \( 1/A(|x|) = \int_{|x|}^\infty K(t)\, dt/A^2(t) \), as \( x \to \pm \infty \)

\[
\int_\mathbb{R} e^{-i\theta u - i\theta} \gamma(\xi) U\{x + d\xi\} = \pi \left( \int_{|x|}^\infty \frac{H(t)}{A^2(t)} \, dt\{1 + o(1)\} \pm \int_{|x|}^\infty \frac{K(t)}{A^2(t)} \, dt\{1 + o(1)\} \right) g(\theta),
\]

which shows the formula of Theorem 2 in view of (12). \( \square \)

**Remark 4.** If

\[
A(x) \int_x^\infty \frac{H(t)}{A^2(t)} \, dt \to \infty,
\]

then the two integrals on the RHS of the formula of Theorem 2 are asymptotically the same so that

\[
U(x, x + h) \sim \frac{1}{2} \int_{|x|}^\infty \frac{H(t)}{A^2(t)} \, dt \quad \text{as} \quad x \to \pm \infty.
\]

(In fact (40) is equivalent to (11) from the last step of the proof given above.) Suppose that \( \kappa = 0 \) in the formula involved in (Hc) and that (Hab) holds, namely

\[
\lim_{x \to \infty} A(x) \times H(x) = \infty, \quad \lim_{x \to \infty} A(x) / \ell(x) = 0 \quad \text{and} \quad \int_{x_0}^\infty \frac{H(t)}{A^2(t)} \, dt < \infty. \quad (42)
\]

Then (40) holds at least under some side condition: if \( K(\xi)/H(x) \to 0 \) (40) plainly holds; (40) is also true if \( \lim_{x \to \infty} \int_{|x|}^\infty [A(x)/\ell(x)] \, dt \to 1/2 \) as is deduced by decomposing the integral \( \int_x^\infty H(t) \, dt/A^2(t) \) into the sum \( \ell(x)/A(x) + \int_x^\infty \ell(t) K(t) \, dt/A^2(t) \). [The author do not know whether (40) hold generally under (42) but it would do in practically almost all cases.]
4.3. Example. Suppose that $X$ belongs to the domain of attraction of a stable law of exponent 1, or equivalently that as $x \to \infty$,

$$1 - F(x) \sim pL(x)/x \quad \text{and} \quad F(-x) \sim (1-p)L(x)/x$$

for some constant $0 \leq p \leq 1$ and s.v. function $L$. Then (Ha) holds if and only if

$$\rho_n =: P[S_n > 0] \to 1$$

(cf. [18], [9]). Here $S_n$ is a random walk with $S_0 = 0$ and the increment distribution given by $F$. (Ha) holds whenever $p > 1/2$ and in a (small) sub-case of $p = 1/2$. Suppose $E|X| = \infty$. Then for $p > 1/2$ condition (Hac) is satisfied with $\kappa = (2p - 1)$ so that Corollary is applied to yield that

$$(2p - 1)^2 \ell(x)U(x, x + h)/h \text{ converges to } p \text{ or } 1 - p \text{ according as } x \to \infty \text{ or } - \infty.$$ 

This refines Theorem 3.6 of [1] where $F$ is assumed to be arithmetic of span 1 and some auxiliary restriction is imposed on $P[X = y]$. When $p = 1/2$ and (Ha) holds, $F$ is transient if and only if $\int_{x_0}^{\infty} [L(t)/tA^2(t)]dt < \infty$ for some $x_0$, and if this is the case, (A/H/A) holds since $K(x)/H(x) \to 0$ and (43) accordingly follows.

Here we give a criterion for the transience: under (43) $F$ is transient if and only if

$$\int_1^{\infty} \frac{H(t)}{(L(t) \vee |A(t)|)^2} dt < \infty,$$  

(44)

If $p \neq 1/2$, the random walk is transient as we know and this conforms to the above criterion. Let $p = 1/2$. Then, similarly to [9] as $\theta \downarrow 0$

$$1 - E[\cos \theta X] \sim \frac{1}{2} \pi \theta L(1/\theta) \quad \text{and} \quad E[\sin \theta X] = \theta [A(1/\theta) + o(L(1/\theta))]$$

(cf. [13]) so that

$$\Re \frac{1}{1 - \phi(\theta)} \propto \frac{\frac{1}{2} \pi L(1/\theta)}{[\frac{1}{4} \pi^2 L^2(1/\theta) + A^2(1/\theta)] \theta}.$$

and hence we have the asserted criterion in view of Ornstein’s theorem. [In the particular case when $\rho \to 0$ or 1 the criterion (44) is obtained from Lemma 5 if $\rho_n$ remains in a closed interval of $(0, 1)$, then (44) is reduced to $\int_1^{\infty} [tL(t)]^{-1} dt < \infty$, and the result is obtained in [21] by using the Gnedenko local limit theorem.]

For an arithmetic walk satisfying (43) Berger [1, Theorem 3.5] shows that if $\rho_n$ tends to $\rho \in (0, 1)$, then $U\{n\}/\sum_{k=n}^{\infty} 1/kL(k) \to c(\rho)$ where $c(\rho)$ is a positive function of $\rho$ (expressed explicitly), provided $F$ is transient. The result can be extended to non-arithmetic walks. Thus under (43) the problem of finding an analytic expression of the asymptotic form of $U(x, x + h)$ is settled unless $\rho_n$ oscillates.

Finally we present two particular cases of (43) that verify incomparability of (Ha) and (Hb). Let $p = 1/2$ in (43). If $L(x) \sim (\log x)^2$, we may have $K(x) \sim (\log x)^{\delta}/x$ with $0 < \delta < 2$ so that $A(x) \sim (\log x)^{\delta+1}/(\delta + 1)$, showing that if $1/2 < \delta \leq 1$, (Hb) holds whereas (Ha) fails. Similarly if $L(x) \sim 1$ and $K(x) \sim (\log x)^{-\delta}/x$ with $1/2 < \delta < 1$, then (Ha) holds but (Hb) does not.
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