Optimal Quantum States for Image Sensing in Loss

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(Received 15 July 2011; published 31 October 2011)

We consider a general image sensing framework that includes many quantum sensing problems by an appropriate choice of image set, prior probabilities, and cost function. For any such problem, in the presence of loss and a signal energy constraint, we show that a pure input state of light with the signal modes in a mixture of number states minimizes the cost among all ancilla-assisted parallel strategies. Loss binary phase discrimination with a peak photon number constraint and general lossless image sensing are considered as examples.

DOI: 10.1103/PhysRevLett.107.193602 PACS numbers: 42.50.Ex, 03.67.Hk, 06.20.-f, 42.50.Dv

The use of nonclassical and entangled states of light, i.e., states other than the easily generated coherent states and their classical mixtures [1], for applications such as sub-shot-noise imaging [2] and imaging with sub-Rayleigh resolution [3] has received much attention. In the areas of sensing and metrology [4], there have been recent theoretical studies of quantum-enhanced target detection [5], reading of a digital memory [6,7], and of optical phase estimation [8] with nonclassical states. Given the interest in applications of quantum states of light for sensing, it is important to theoretically establish what state(s) accomplish a sensing task using minimum energy, assuming the most general measurements and postprocessing. This would place a limit on the enhancements obtainable from nonclassical states using experimentally realizable measurements. The ubiquitous linear loss is known to be a bottleneck for harnessing quantum advantage in many communication and metrology applications [8,9]. Although the problems of [5–7] naturally include various degrees of loss, few general results including its effects are available. In this Letter, we first set up a general framework for image sensing in the presence of loss that subsumes many of the above problems. We then identify a class of input states that contains an optimal, i.e., cost-minimizing, state for any problem fitting the framework, and under any form of signal energy constraint.

General image sensing framework—Suppose an image is drawn, unknown to the receiver, from a set \( I = \{I_1, \ldots, I_M\} \) of \( M \) images according to the probability distribution \( \{\pi_1, \ldots, \pi_M\} \). We model each image as a pixelated (transmissive or reflective) optical mask with uniform transmissivity or reflectivity and phase shift within each pixel. For \( P \) the number of pixels in each image, the \( p \)th pixel \( (p \in \{1, \ldots, P\}) \) of image \( I_m \) is modeled as a beam splitter effecting the mode transformation

\[
\begin{pmatrix}
\hat{b}_j^{(p)} \\
\hat{c}_j^{(p)}
\end{pmatrix} = \begin{pmatrix}
\sqrt{\eta_m^{(p)}} e^{i\theta_m^{(p)}} & \sqrt{1 - \eta_m^{(p)}} \\
\sqrt{1 - \eta_m^{(p)}} e^{-i\theta_m^{(p)}} & -\sqrt{\eta_m^{(p)}}
\end{pmatrix}
\begin{pmatrix}
\hat{a}_j^{(p)} \\
\hat{e}_j^{(p)}
\end{pmatrix},
\]

where \( \eta_m^{(p)} \) is the transmittance (or reflectance if reflective probing is used) of the \( p \)th pixel in \( I_m \), and \( \theta_m^{(p)} \) is the phase shift imparted to the input (or “signal”) field modes probing the \( p \)th pixel of \( I_m \) alone.

For probing the unknown image, we consider quantum states of \( J = \sum_{j=1}^J f_j^{(p)} \) signal modes that may be entangled to ancilla (or “idler”) modes that are not sent out to interrogate the image but held losslessly (see Fig. 1). Here, \( f_j^{(p)} \) is the number of modes interrogating pixel \( p \), which, for example, could be successive time modes. Equation (1) includes the annihilation operator \( \hat{a}_j^{(p)} \) of the \( j \)th signal field mode probing pixel \( p \) and the annihilation operator \( \hat{e}_j^{(p)} \) of the \( j \)th input environment mode at \( p \). The input environment modes are taken to be in the vacuum state—the assumption of no thermal noise in the environment is realistic at optical frequencies. We further assume that the output mode corresponding to \( \hat{b}_j^{(p)} \) in (1), but not that corresponding to \( \hat{e}_j^{(p)} \), is available for making quantum measurements. This is almost always the case in practice as the environment input and output modes are not easily accessible to the user in a stand-off imaging scenario, or if the light source and receiver are at different spatial

![Schematic of procedure for sensing of an unknown image](image)

FIG. 1. Schematic of procedure for sensing of an unknown image from a set \( I \) with pixels described by \( (\eta_m^{(p)}, \theta_m^{(p)}), p = 1, \ldots, P \), via (1). The source generates signal modes \( \{\hat{a}_j^{(p)}\} \) for probing the image and idler modes which are retained losslessly. An optimal measurement for \( x \in X \) is made on the \( J \) return modes \( \{\hat{b}_j^{(p)}\} \) and \( J' \) idler modes jointly.
locations. Additional loss during state propagation may be
included as a multiplicative factor in the \( \{ \eta_m^{(p)} \} \).

We first consider pure input states; we will return to the
mixed input state case later. An arbitrary pure quantum
state \( |\psi\rangle_{IS} \) of the signal and idler modes may be written in the form

\[
|\psi\rangle_{IS} = \sum_n c_n |\phi_n\rangle |n\rangle_S.
\]

(2)

Here \( n = (n^{(1)}, \ldots, n^{(p)}, \ldots, n^{(p)}) \) is a
\( J \)-dimensional vector whose component \( n^{(p)} \) indexes the
photon number in the \( j \)th mode interrogating the \( p \)th pixel,
and \(|n\rangle_S\) are Fock states of the signal modes. We do not
restrict the number \( J' \) of the idler modes, nor the form of
the idler states \(|\phi_n\rangle\), only requiring that they be normalized.
Allowing an arbitrary input state (2) corresponds to the
most general ancilla-assisted parallel strategy (see [4]
for discussion on parallel strategies). Little is known about
nonparallel strategies (but see [10]) that may include adap-
tive selection of inputs, which is known to assist some
channel discrimination problems [11]. Irrespective of the
form of the \( \{ |\phi_n\rangle \} \), the probability mass function (PMF) of
the photon number in the signal modes is \( p_n = |c_n|^2 \),
which determines quantities of interest such as the mean
total signal energy

\[
\left\langle \sum_{p-1}^P \sum_{j_1}^{j(p)} N_{j}^{p(p)} \right\rangle = \sum_{n} n p_n,
\]

where

\[
n = \sum_{p-1}^P \sum_{j_1}^{j(p)} n^{(p)} = \sum_{p-1}^P n^{(p)}.
\]

In practice, the mean total signal energy may be upper
bounded by a given number \( N_S \).

Once an input state \(|\psi\rangle_{IS}\) is chosen and the signal modes
are sent to probe the image, the return + idler states con-
stitute an ensemble \( \mathcal{E} = \{(\pi_m, \rho_m)\}_{m=1}^M \), where \( \rho_m = \text{id}_{J} \otimes \mathcal{K}_m(|\psi\rangle_{IS} |\psi\rangle) \) is the density operator on the return + idler
Hilbert space at the output of the quantum channel
\( \text{id}_{J} \otimes \mathcal{K}_m \) resulting from the interaction of the signal
modes with \( I_m \) via (1) and the identity map on the idler
modes. Depending on the imaging task, we attempt to
extract a parameter lying in an observation space \( \mathcal{X} \) by
making a quantum measurement that is represented by a
positive operator-valued measure (POVM) [12] with out-
comes \( x \in \mathcal{X} \) and corresponding operators \( \{ E_x \}_{x \in \mathcal{X}} \).
The task also specifies a cost function \( C(m, x) \), and we are
interested in the minimum average cost \( \bar{C} \)

\[
\bar{C}[\mathcal{E}] = \min_{\{ E_x \}_{x \in \mathcal{X}}} \sum_{x \in \mathcal{X}} \sum_{m=1}^M \pi_m \text{tr}(\rho_m E_x) C(m, x),
\]

(3)

where the minimization is over all POVMs \( \{ E_x \}_{x \in \mathcal{X}} \).
Therefore, adaptive measurements are included in our
model. Note that the input state and image parameters
enter into the cost via the ensemble \( \mathcal{E} \) while the imaging
task determines \( \mathcal{X} \) and the cost function. Thus, choosing
\( \mathcal{X} = \{1, \ldots, M\} \) and \( C(m, x) = 1 - \delta_{m, x} \) makes
(\( \bar{C}[\mathcal{E}] \) equal the minimum probability of error (MPE) in discrimi-
nating the \( M \) images. For the same cost function, \( M = 2 \),
\( P = 1 \), and \( \theta_m^{(1)} = 0 \) correspond to the quantum reading
and target detection problems of [5–7]. For \( P = 1 \) and
\( \eta_m^{(1)} = \eta \), choosing \( \mathcal{X} = [0, 2\pi] \) and \( C(m, x) = |x - \theta_m^{(1)}|^2 \)
corresponds to minimum-mean-square-error (MMSE)
phase discrimination in the presence of loss. As \( M \to \infty \),
we approach MMSE phase estimation in loss. The usual
interferometric setup for phase estimation [4,8] is recov-
ered using \( P = 2, m = (\delta, \phi) \in [0, 2\pi]^2 \), \( \theta_m^{(1)} = \phi + \delta \),
\( \theta_m^{(2)} = \phi \), and the cost function \( C(m, x) = |x - \delta|^2 \). Here
\( \delta \) is the relative phase shift of interest and \( \phi \) is the
undesired common phase shift in both arms of the interfer-
ometer (see Fig. 2).

In the state (2), if \( |\phi_n\rangle \) = \( |\phi\rangle \) for all \( n \) and some idler
state \(|\phi\rangle\), the state factorizes so that the idler is effectively
absent. The other extreme is the case of \( \langle \phi_n | \phi_n \rangle = \delta_{n,n'} \)
so that the density operator on the signal modes is diagonal
in the multimode Fock basis. Such states are called
number-diagonal signal (NDS) states in [7], well-known
elements being the two-mode squeezed vacuum of [5,6]
and the NOON state [13]. In [7], the error probability and
other quantities of interest were computed for NDS inputs
in the \( M = 2 \) case. Our main result is that the NDS states
are optimally “matched” to the general imaging problem
described above.

Theorem 1.—(NDS state lower bound.) Let \( I \) be a set of \( M \)
P-pixel images described via transformations of the form
(1) with prior probabilities \( \{ \pi_m \}_{m=1}^M \). For any imaging
task with cost function \( C(m, x) \), the minimum cost \( \bar{C} \)
achieved by the input state \(|\psi\rangle_{IS} = \sum_n c_n |\phi_n\rangle |n\rangle_S \) is
lower bounded by that achieved by a corresponding NDS
state \(|\Psi\rangle_{IS} = \sum_n c_n |\Phi_n\rangle |n\rangle_S \), where \( \{ |\Phi_n\rangle \} \) is any orthon-
ormal set, with the same signal photon PMF.

General results in quantum decision theory.—The proof
of Theorem 1 requires two simple but general results in
quantum decision theory. To state them, we define the
notion of mixture of ensembles. For each value of an

![FIG. 2. Image sensing problems in linear loss. Left: Quantum
reading—digital reader composed of a transmitter T and receiver
R. Right: Interferometer for discrimination or estimation of the
relative phase shift δ with two-mode signal-only source S and
detector D.](image-url)
For cables.) Consider a sensing task with cost function

\[ C(x) = \sum_{l} \lambda_l C(E_l) := \left( \sum_{l} \lambda_l \pi_m(l) \rho_m \right)_{m=1}^{M} \]  

is also an ensemble, called the mixture of the ensembles \{E_l\}—the \{E_l\} are subensembles of \( C \). A mixture of ensembles can arise from a two-step procedure in which \( l \) is chosen with probability \( \lambda_l \), following which the state \( \rho_m(l) \) is prepared with probability \( \pi_m(l) \). In our proof of Theorem 1, subensembles arise as the conditional output states of a measurement with outcomes \( \{l\} \) on a given ensemble. We have the following basic result.

\textbf{Lemma 1.}—(Concavity of \( C \) under mixing of ensembles.) Consider a sensing task with cost function \( C(m, x) \). For \( M \)-ary ensembles \( \{E_l\} \) indexed by \( l \), and probability distribution \( \{\lambda_l\} \),

\[ C \left( \sum_{l} \lambda_l E_l \right) \geq \sum_{l} \lambda_l C[E_l] \]  

The notion of orthogonal ensembles provides a sufficient condition for equality in (5). The support of an ensemble \( E = \{(\pi_m, \rho_m)^{M}_{m=1}\} \) is defined to be supp \( E := \sum_{m=1}^{M} \text{ran} \rho_m \), where ran \( \rho_m \) is the range of \( \rho_m \). Then ensembles \( E \) and \( F \) are said to be orthogonal if the support spaces supp \( E \) and supp \( F \) are orthogonal.

\textbf{Lemma 2.}—If \( \{E_l\} \) are pairwise orthogonal ensembles on \( \mathcal{H} \), we have

\[ C \left( \sum_{l} \lambda_l E_l \right) = \sum_{l} \lambda_l C[E_l] \]  

The proofs of Lemma 1 and 2 are given in [14] along with their physical interpretation.

\textbf{Proof of Theorem 1.}—Suppose the general state (2) is used as input. To calculate the output states \( \{\rho_m\}_{m=1}^{M} \), we may use the Schrödinger-picture form of (1) to get the purification

\[ |\psi_m\rangle = \sum_{n} \sum_{l} c_{m}^{(n)} A_{m}^{(n,l)} |\phi_{n}\rangle_{I} |n - I\rangle_{S} |I_{E}\rangle \]  

where \( |I_{E}\rangle \) is a Fock state of the environment modes and

\[ A_{m}^{(n,l)} = \prod_{p=1}^{L} e^{i\eta_{p}^{(m)}} \prod_{j=1}^{n_{p}^{(m)}} \left( \eta_{j}^{(p)} \right)_{I_{E}} \eta_{j}^{(p)} \left( 1 - \eta_{j}^{(p)} \right)_{I_{E}} \]  

The output state \( \rho_m \) is then given by (note that \( |\psi_m\rangle_{IS} \) are non-normalized states)

\[ \rho_m = \sum_{l} |\psi_m(l)\rangle_{IS} \langle \psi_m(l) | \].

In (7)–(9), \( I \) is the (random and unknown) pattern of the number of photons leaked from the signal modes into the environment modes during interrogation of the image. The probability that the leaked photon pattern is \( I \) is

\[ \lambda_I = \sum_{m=1}^{M} \pi_m(l) |\psi_m(l)\rangle_{IS} \langle \psi_m(l) | \]  

so that the conditional probability of hypothesis \( m \) given \( I \) is

\[ \pi_m(l) = \frac{\pi_m(l) |\psi_m(l)\rangle_{IS} \langle \psi_m(l) |}{\lambda_I} \]  

Thus, the ensemble

\[ E = \{ (\pi_m, \rho_m) \}_{m=1}^{M} = \sum_{l} \lambda_l E_l \]  

for the subensembles \( \{E_l\} \) given by

\[ E_l = \left( \left( \pi_m, \rho_m \right) \right)_{m=1}^{M} \]  

According to Lemma 1, the mixture (12) satisfies

\[ C[E_l] = \sum_{l} \lambda_l C[E_l] \]  

Consider the right-hand side (rhs) of (14). For each \( l \), \( E_l \) is a pure-state ensemble, so \( C[E_l] \) is a function of just the pairwise inner products \( \langle \psi_m | \psi_m(l) \rangle_{IS} = G_{m,m'}^{(l)} \), the \( M \times M \) Gram matrix [15]

\[ G_{m,m'}^{(l)} = \sum_{n} c_{m}^{(n)} c_{m'}^{(n)} A_{m}^{(n,l)} A_{m'}^{(n,l)} \]  

The crucial point is that, owing to the form (7) of the beam splitter transformation, \( G_{m,m'}^{(l)} \) is independent of the choice of the \( \{\phi_n\} \). From (10) and (11), so are \( \lambda_l \) and \( \pi_m(l) \). Thus, the hypothetical measurement scenario in which one has knowledge of \( I \) (or alternatively, one is allowed to make a photon number measurement on all the output environment modes) and whose \( C \) is given by the rhs of (14), has the same \( C \) for any choice of the \( \{\phi_n\} \).

Finally, we consider the NDS input state \( |\Psi\rangle_{IS} = \sum_{n} c_{n} |\phi_n\rangle_{I} |n\rangle_{S} \) corresponding to (2) satisfying

\[ \langle \Phi_n | \Phi_n \rangle_{I} = \delta_{n,n'}. \]  

It is readily verified using (7) and (16) that

\[ \langle \psi_m | \psi_m(l) \rangle_{IS} = \delta_{I,I'} \langle \psi_m | \psi_m(l) \rangle_{IS} \]  

so that the \( \{E_l\} \) are pairwise orthogonal ensembles. Therefore, by Lemma 2, the NDS input \( |\Psi\rangle_{IS} \) attains
Σλ_j |ψ⟩⟨ψ|_IS.

Discussion and implications.—The Ĉ of the NDS state |Ψ⟩_IS of Theorem 1 is a function of only the J-mode photon PMF {p_0}. Thus, for a given J, the search for an optimal input state for a given imaging task may be confined to the set of {p_n} satisfying given constraints, e.g., an average or peak signal energy constraint or a mode-by-mode signal energy constraint. As illustrated below, the problem of finding the optimal quantum state reduces to the classical problem of finding an optimal probability distribution. To see that mixed input states ρ_IS do not help, we first purify ρ_IS using added idler modes. As (2) contains no restriction on the idlers, the NDS state corresponding to the purification—which has the same {p_n} as ρ_IS—has Ĉ not larger than that of the purification (which in turn beats ρ_IS). Note also that the performance achieved by an arbitrary state of signal energy N_S can be achieved by an NDS state of total (signal + idler) energy not larger than 2N_S by choosing |ϕ_n⟩_I = |n⟩_I, the Fock state of the idler modes.

Theorem 1 strongly suggests that ancilla-assisted parallel strategies for image sensing are superior to signal-only parallel strategies. This is known for discrimination between some pairs of channels [16] and is also true for our phase discrimination example below. We conjecture that they are strictly better whenever nonzero loss is present (η_m < 1) because for a signal-only input state, the {E_l} are not orthogonal and are unlikely to appear the lower bound of (14). Such ancilla-assisted schemes appear to be unexplored for some problems of interest—e.g., studies of the optimal state for phase estimation in loss have hitherto been confined to two-mode signal-only states [8]. At the same time, Theorem 1 implies that the best possible performance can be obtained without ancillary modes if the value of I is known. This result should place interesting limitations on the quantum advantage obtainable in any sensing problem.

Binary phase discrimination.—As an application of Theorem 1, we obtain the single-pass (J = 1) state that discriminates between a 0 and π phase shift with minimum error probability among states with a peak signal photon constraint of N_peak = 2 in the presence of loss. In the terminology of our framework, we have M = 2, P = 1, η_1 = η_2 = η < 1, and θ_1 = 0, θ_2 = π. We also assume σ_1 = σ_2 = 1/2. An arbitrary state |ψ⟩_IS satisfying these constraints is

|ψ⟩_IS = \sqrt{p_0} |ϕ_0⟩_I |0⟩_S + \sqrt{p_1} |ϕ_1⟩_I |1⟩_S + \sqrt{p_2} |ϕ_2⟩_I |2⟩_S,

where phase factors have been absorbed into the normalized kets \{|ϕ_n⟩_I\}. In terms of the density operators ρ_1 and ρ_2 defined earlier, the minimum error probability is given by the Helstrom formula [12]

\[ P_e = \frac{1}{2} - \frac{1}{4} \| ρ_1 - ρ_2 \|_1 , \]

where \| · \|_1 is the trace norm. According to Theorem 1, we may confine our search for optimal states to the NDS class for which \{|ϕ_n⟩_I\} are orthonormal. For such states, the minimum error probability is given in closed form by Eq. (39) of [7]:

\[ P_e^{NDS} = \frac{1}{2} - \sqrt{p_1^0(p_0 + p_2^3(1 - η^2)) + (2p_2(1 - η^2))^{1/2}} \]

Since p_0 + p_1 + p_2 = 1, we may consider p_0 and p_1 as independent variables taking values in the triangle T whose vertices have the (p_0, p_1) values (0,0), (1,0), and (0,1). It is easy to show that P_e^{NDS} is identically 1/2 on the p_0 axis and that it has local minima on the p_1 axis at p_1 = p_2 = 1/2 and on the remaining boundary of the triangle at p_0 = p_1 = 1/2. There also exists a local extremum of P_e^{NDS} in the interior of T at the point:

\[ (p_0^*, p_1^*, p_2^*) = \left( \frac{1 + 2η - η^2}{2(1 + η)(3 - η)}, \frac{1}{2(1 + η)(3 - η)} \right) \]

Figure 3 (left panel) shows P_e^{NDS} plotted over T for η = 0.6. The interior extremum point (p_0^*, p_1^*, p_2^*) achieves the minimum error probability. For comparison, we consider also a signal-only input state of the form

\[ |ψ⟩_S = \sqrt{p_0} |0⟩_S + \sqrt{p_1} |1⟩_S + \sqrt{p_2} |2⟩_S. \]

For each choice of (p_0, p_1) in T, the error probability P_e^{signal-only} is computed numerically using (19), and in Fig. 3 (right panel), the difference P_e^{signal-only} − P_e^{NDS} is plotted on T. The difference is everywhere non-negative, being zero on the two boundaries of T other than the p_1 axis.

Lossless image sensing.—In the lossless case, I = 0 with probability one, so that the performance of the hypothetical measurement described in Theorem 1 is attainable with any choice of idler states. That performance is determined by the Gram matrix elements from (15)

FIG. 3. Left: The error probability P_e^{NDS} of the NDS state of the form of Eq. (18) for η = 0.6. Right: The difference P_e^{signal-only} − P_e^{NDS} between the error probabilities of corresponding signal-only and NDS states as a function of (p_0, p_1).
\[ G_{m,m'}^{(0)} = \sum_{n} \left( p_n \prod_{p=1}^{P} e^{i(\theta_{m,n'}^{(p)} - \theta_{m,n}^{(p)})n^{(p)}} \right) \]
\[ = \sum_{\nu} p_{\nu} e^{i \sum_{p=1}^{P} (\theta_{\nu,n'}^{(p)} - \theta_{\nu,n}^{(p)})n^{(p)}}, \]

where \( \nu = (n^{(1)}, \ldots, n^{(P)}) \) has PMF \( p_{\nu} \). Choosing \( |\phi_{\nu}\rangle \equiv |\phi\rangle \), the signal-only state

\[ |\psi\rangle_S = \sum_{\nu} \sqrt{p_{\nu}} |n^{(1)}, \ldots, n^{(P)}\rangle_S \]

with \( J^{(p)} = 1, J' = 0 \) suffices to attain \( \tilde{C} \). In the absence of loss, the \( \{ \mathcal{K}_m \}_{m=1}^{M} \) are unitary channels. This result shows that, among parallel strategies, ancillae do not improve sensing of \( M \) unitary phase images under a signal energy constraint. This is unlike the case of minimum error probability discrimination of finite-dimensional unitaries in [17], although ancillae are not required for discriminating two unitaries [17,18]. The fact that single-pass imaging \( (J^{(p)} = 1) \) suffices is also remarkable as there are examples of pairs of unitaries that are better (even perfectly) discriminated if multiple shots are allowed [17,19].

We acknowledge useful discussions with Masoud Mohseni and Jeffrey H. Shapiro. This work was supported by DARPA’s Quantum Sensor Program under AFRL Contract No. FA8750-09-C-0194.

[1] L. Mandel and E. Wolf, *Optical Coherence and Quantum Optics* (Cambridge University Press, Cambridge, U.K., 1995).
[2] G. Brida, M. Genovese, and I. Ruo Berchera, Nat. Photon. 4, 227 (2010).
[3] V. Giovannetti, S. Lloyd, L. Maccone, and J. H. Shapiro, Phys. Rev. A 79, 013827 (2009); M. Tsang, Phys. Rev. Lett. 102, 253601 (2009); C. Thiel, T. Bastin, J. von Zanthier, and G. S. Agarwal, Phys. Rev. A 80, 013820 (2009).
[4] V. Giovannetti, S. Lloyd, and L. Maccone, Nat. Photon. 5, 222 (2011).
[5] S.-H. Tan, B. I. Erkmen, V. Giovannetti, S. Guha, S. Lloyd, L. Maccone, S. Pirandola, and J. H. Shapiro, Phys. Rev. Lett. 101, 253601 (2008).
[6] S. Pirandola, Phys. Rev. Lett. 106, 090504 (2011).
[7] R. Nair, Phys. Rev. A 84, 032312 (2011).
[8] R. Demkowicz-Dobrzański, Phys. Rev. A 83, 061802 (2011); J. Kołodyński and R. Demkowicz-Dobrzański, *ibid.* 82, 053804 (2010).
[9] H. P. Yuen, in *Quantum Squeezing*, edited by P. D. Drummond and Z. Ficek (Springer-Verlag, Berlin, 2004), Chap. 7.
[10] W. van Dam, G. M. D’Ariano, A. Ekert, C. Macchiavello, and M. Mosca, Phys. Rev. Lett. 98, 090501 (2007); G. Chiribella, G. M. D’Ariano, and P. Perinotti, *ibid.* 101, 060401 (2008).
[11] A. W. Harrow, A. Hassidim, D. W. Leung, and J. Watrous, Phys. Rev. A 81, 032339 (2010).
[12] C. W. Helstrom, *Quantum Detection and Estimation Theory* (Academic Press, New York, 1976).
[13] J. P. Dowling, Contemp. Phys. 49, 125 (2008).
[14] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.107.193602 for proofs of Lemmas 1 and 2.
[15] Two pure-state ensembles with the same prior probabilities and the same Gram matrix have the same \( C \) because there exists a unitary \( \tilde{U} \) connecting the two ensembles (and also the POVMs on the two ensembles). This is seen by performing Gram-Schmidt orthogonalization on the two state sets separately—\( \tilde{U} \) is chosen so as to map one Gram-Schmidt basis to the other.
[16] M. F. Sacchi, Phys. Rev. A 72, 014305 (2005).
[17] G. M. D’Ariano, P. Lo Presti, and M. G. A. Paris, Phys. Rev. Lett. 87, 270404 (2001).
[18] A. M. Childs, J. Preskill, and J. Renes, J. Mod. Opt. 47, 155 (2000).
[19] A. Acín, Phys. Rev. Lett. 87, 177901 (2001).