CRITICAL SLOPE $p$-ADIC $L$-FUNCTIONS OF CM MODULAR FORMS

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Abstract. For ordinary modular forms, there are two constructions of a $p$-adic $L$-function attached to the non-unit root of the Hecke polynomial, which are conjectured but not known to coincide. We prove this conjecture for modular forms of CM type, by calculating the critical-slope $L$-function arising from Kato’s Euler system and comparing this with results of Bellaïche on the critical-slope $L$-function defined using overconvergent modular symbols.

1. Setup

1.1. Introduction. Let $f$ be a cuspidal new modular eigenform of weight $\geq 2$, and $p$ a prime not dividing the level of $f$. It has long been known that if $\alpha$ is any root of the Hecke polynomial of $f$ at $p$ such that $v_p(\alpha) < k - 1$, then there is a $p$-adic $L$-function $L_{p,\alpha}(f)$ interpolating the critical $L$-values of $f$ and its twists by Dirichlet characters of $p$-power conductor; see [MSD74, AV75, Viš76].

If $f$ is non-ordinary (the Hecke eigenvalue of $f$ at $p$ has valuation $> 0$) then both roots of the Hecke polynomial satisfy this condition, but if $f$ is ordinary, then there is one root with valuation $k - 1$ (“critical slope”), to which the classical modular symbol constructions do not apply. Two approaches exist to rectify this injustice to the ordinary forms by constructing a critical-slope $p$-adic $L$-function. Firstly, there is an approach using $p$-adic modular symbols [PS11, PS09, Bell11a]. Secondly, there is an approach using Kato’s Euler system [Kat04] and Perrin-Riou’s $p$-adic regulator map [PR95] (cf. [Col04, Remarque 9.4]). Although it is natural to conjecture that the objects arising from these two constructions coincide (cf. [PS09, Remark 9.7]), and the results of [LZ11b] are strong evidence for this conjecture, prior to the present work this was not known in a single example.

In this paper, we show that the two critical-slope $L$-functions coincide for modular forms of CM type. In this case, Bellaïche has shown [Bell11b] that the “modular symbol” critical-slope $p$-adic $L$-function is related to the Katz $p$-adic $L$-function for the corresponding imaginary quadratic field. We show here that the same relation holds for the Kato critical slope $p$-adic $L$-function, by comparing Kato’s Euler system with another Euler system: that arising from elliptic units. Using the results of [Yag82] and [dS87] relating elliptic units to Katz’s $L$-function, we obtain a formula (Theorem 3.2) for the Kato $L$-function, which coincides with Bellaïche’s formula for its modular symbol counterpart (up to a scalar factor corresponding to the choice of periods). This establishes the equality of the two critical-slope $p$-adic $L$-functions for ordinary eigenforms of CM type (Theorem 3.4).

1.2. Notation. Let $K$ be a finite extension of either $\mathbb{Q}$ or $\mathbb{Q}_p$, where $p$ is an odd prime. We write $K_{\infty} = K(\mu_{p^\infty})$, $\overline{K}$ for an algebraic closure of $K$ and $K^{ab}$ for the maximal abelian extension of $K$ in $\overline{K}$. A $p$-adic representation of the absolute...
Galois group $\text{Gal}(\overline{K}/K)$ is a finite-dimensional $\mathbb{Q}_p$-vector space with a continuous linear action of $\text{Gal}(\overline{K}/K)$.

A Galois extension $L$ of $K$ will be called a $p$-adic Lie extension if $G = \text{Gal}(L/K)$ is a compact $p$-adic Lie group of finite dimension. In this case, we denote by $\Lambda(G)$ its Iwasawa algebra; it is defined to be the completed group ring

$$\Lambda(G) = \varprojlim \mathbb{Z}_p[G/U],$$

where $U$ runs over all open normal subgroups of $G$. We write $Q(G)$ for the total quotient ring of $\Lambda(G)$. If $R$ is a $p$-adically complete $\mathbb{Z}_p$-algebra, we shall write $\Lambda_R(G)$ for $R \otimes \Lambda(G)$, the Iwasawa algebra with coefficients in $R$.

If $L$ is a complete discretely valued subfield of $\mathbb{C}_p$, we write $\mathcal{H}_L(G)$ for the algebra of $L$-valued distributions on $G$ (the continuous dual of the space of locally $L$-analytic functions). This naturally contains $\Lambda_L(G)$ as a subalgebra. When $G$ is the cyclotomic Galois group $\Gamma$ (isomorphic to $\mathbb{Z}_p^\times$), and $i \in \mathbb{Z}$, we shall write $\ell_i$ for the element $\frac{\log(\gamma)}{\log(\gamma_i)} - i$ of $\mathcal{H}_{Q_p}(\Gamma)$ (where $\gamma$ is any element of $\Gamma$ of infinite order).

Assume now that $K$ is a number field, and let $S$ be a finite set of places of $K$ (which we shall always assume to contain the infinite places). Let $K^S$ be the maximal extension of $K$ which is unramified outside $S$, and let $V$ be a $p$-adic representation of $\text{Gal}(K^S/K)$. For an extension $L$ of $K$ contained in $K^S$, write $H^1_{\text{Iw}}(L, V)$ for the Galois cohomology group $H^1(\text{Gal}(K^S/L), V)$. Let $T$ be a $\text{Gal}(\overline{K}/K)$-stable lattice in $V$. If $L \subset K^S$ is a $p$-adic Lie extension of $K$, define

$$H^1_{\text{Iw},S}(L, T) = \varprojlim_n H^1_{\text{Iw}}(L_n, T),$$

where $L_n$ is a sequence of finite Galois extensions of $K$ such that $L = \bigcup_n L_n$ and the inverse limit is taken with respect to the corestriction maps. Note that $H^1_{\text{Iw},S}(L, T)$ is equipped with a continuous action of $G = \text{Gal}(L/K)$, which extends to an action of $\Lambda(G)$. We also define $H^1_{\text{Iw},S}(L, V) = H^1_{\text{Iw},S}(L, T) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$, which is independent of the choice of lattice $T$.

Similarly, let $F$ be a finite extension of $\mathbb{Q}_p$, $V$ a $p$-adic representation of $\text{Gal}(\overline{F}/F)$ and $T$ a $\text{Gal}(\overline{F}/F)$-invariant lattice in $V$. For a $p$-adic Lie extension $L$ of $F$ such that $L = \bigcup_n L_n$ with $L_n/F$ finite Galois, define

$$H^1_{\text{Iw}}(L, T) = \varprojlim_n H^1(L_n, T)$$

and

$$H^1_{\text{Iw}}(L, V) = H^1_{\text{Iw}}(L, T) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.$$

For a finite extension $K$ of $\mathbb{Q}$, denote by $\mathbb{A}_K$ the ring of adèles of $K$. If $\mathfrak{f}$ is an integral ideal of $K$, write $K(\mathfrak{f})$ for the ray class field modulo $\mathfrak{f}$. Let $K(\mathfrak{f}^\infty) = \bigcup_n K(\mathfrak{f}^p^n)$, and define the Galois group $G_{\mathfrak{f}^\infty} = \text{Gal}(K(\mathfrak{f}^\infty)/K)$.

### 1.3. Grössencharacters

Let $K$ be an imaginary quadratic field. We fix an embedding $K \hookrightarrow \mathbb{C}$. An algebraic Grössencharacter of $K$ of infinity-type $(m, n)$ is a continuous homomorphism $\psi : K^\times \backslash \mathbb{A}_K^\times \rightarrow \mathbb{C}^\times$ whose restriction to $\mathbb{C}^\times$ is given by $z \mapsto z^m \overline{z}^n$.

Let $\theta$ be the Artin map $\widehat{K}^\times/K^\times \rightarrow \text{Gal}(K^{ab}/K)$. We choose the normalizations such that

$$\theta(\pi_q) = [q]^{-1} \mod I_q,$$

where $\pi_q$ is a uniformizer at the prime $q$. $I_q$ is the inertia group and $[q]$ is the arithmetic Frobenius element at $q$. Then we have the following well-known result:

**Theorem 1.1** (Weil, [Wei56]). Let $\psi$ be an algebraic Grössencharacter of $K$, and let $L$ be the finite extension of $\mathbb{Q}$ inside $\mathbb{C}$ generated by $\psi(\widehat{K}^\times)$. Then for any prime $\lambda$ of $L$, there is a (clearly unique) continuous character

$$\psi_{\lambda} : \text{Gal}(\overline{K}/K) \rightarrow L_{\lambda}^\times$$

with the property that

$$\psi_{\lambda} \circ \theta = \psi|_{\widehat{K}^\times}.$$
The character $\psi_\lambda$ is unramified outside the primes dividing $\ell f$, where $\ell$ is the prime of $\mathbb{Q}$ below $\lambda$ and $f$ is the conductor of $\psi$.

The choice of normalization for the Artin map implies that
\[ \psi_\lambda([a]) = \psi(a)^{-1} \]
for each $a$ coprime to $\ell f$. With these conventions, the Hodge–Tate weights of $\psi_\lambda$ are given as follows. Let $\lambda$ be a prime of $L$, and $\mu$ a split prime of $K$, which lie above the same prime of $L \cap K$. Then the decomposition groups of $\mu$ and $\overline{\mu}$ in $\text{Gal}(K^{ab}/K)$ are each isomorphic to $\text{Gal}(\mathbb{Q}_p^{ab}/\mathbb{Q}_p)$, and the Hodge–Tate weight of $\psi_\lambda$ is $m$ at $\mu$ and $n$ at $\overline{\mu}$.

2. Comparison of Euler systems

2.1. Elliptic units. As above, let $K$ be an imaginary quadratic field, with a fixed choice of embedding $K \hookrightarrow \mathbb{C}$. We shall fix, for the remainder of this paper, an embedding $\overline{K} \hookrightarrow \mathbb{C}$ compatible with this choice. In particular, for each integral ideal $\mathfrak{f}$, we regard the ray class field $K(\mathfrak{f})$ as a subfield of $\mathbb{C}$, and we write $K(\mathfrak{f})^+$ for its real subfield.

Definition 2.1. If $L$ is a subfield of $\mathbb{C}$, a CM-pair of modulus $\mathfrak{f}$ over $L$ is a pair $(E,\alpha)$ consisting of an elliptic curve $E/L$ and a point $\alpha \in E(L)_{\text{tors}}$, such that

- there is an isomorphism $\text{End}_{KL}(E) \cong \mathcal{O}_K$, such that the resulting action of $\text{End}_{KL}(E)$ on $\text{colie}(E/KL) \cong KL$ is the natural action of $K$;
- the annihilator of $\alpha$ in $\mathcal{O}_K$ is exactly $\mathfrak{f}$;
- there is an isomorphism $E(\mathbb{C}) \to \mathbb{C}/\mathfrak{f}$ mapping $\alpha$ to 1.

Note that we do not assume that $L \supseteq K$ here, hence the slightly convoluted statement of the first condition.

Theorem 2.2. Let $\mathfrak{f}$ be such that $\mathcal{O}_K^\times \cap (1+\mathfrak{f}) = \{1\}$, $\mathfrak{f} = f$, and the smallest integer in $\mathfrak{f}$ is $\geq 5$. Then there exists a CM-pair of modulus $\mathfrak{f}$ over $K(\mathfrak{f})^+$, and for any field $L$ containing $K(\mathfrak{f})^+$, this CM-pair is the unique CM-pair of modulus $\mathfrak{f}$ over $L$ up to unique isomorphism.

Proof. Consider the canonical CM-pair $(\mathbb{C}/\mathfrak{f},1)$ over $\mathbb{C}$. This corresponds to a point $P_1$ on the modular curve $Y_1(N)(\mathbb{C})$, where $N$ is the smallest integer in $\mathfrak{f}$.

Since $N \geq 5$ by assumption, the curve $Y_1(N)$ has a canonical model over $\mathbb{Q}$ such that $Y_1(N)(\mathbb{Q})$ parametrises elliptic curves over $L$ with a point of order $N$ for each $L \subseteq \mathbb{C}$. Our claim is then precisely that $P_1 \in Y_1(N)(K(\mathfrak{f})^+)$.\[\square\]

It is clear that $P_1 \in Y_1(N)(\mathbb{R})$, since there is a canonical isomorphism from $\mathbb{C}/\mathfrak{f}$ to the elliptic curve $E_\mathbb{R} = \{y^2 = 4x^3 - g_2x - g_3\}$ where $g_2$ and $g_3$ are the usual weight 4 and 6 Eisenstein series, given by $z \mapsto (\psi(z,\mathfrak{f}), \psi'(z,\mathfrak{f}))$. Since $\mathfrak{f} = \mathfrak{f}$, the coefficients $g_2$ and $g_3$ are real, so $E_\mathbb{R}$ is indeed defined over $\mathbb{R}$; and as $\overline{\psi(z,\mathfrak{f})} = \overline{\psi(z,\overline{\mathfrak{f}})}$, this uniformization maps $1 \in \mathbb{C}/\mathfrak{f}$ to a real point of $E_\mathbb{R}$. Hence $P_1 \in Y_1(N)(\mathbb{R})$.

On the other hand, it is well known that there exists a CM-pair of modulus $\mathfrak{f}$ over $K(\mathfrak{f})^+$ (whether or not $\mathfrak{f} = \mathfrak{f}$), so $P_1 \in Y_1(N)(K(\mathfrak{f}))$. Hence $P_1 \in Y_1(N)(K(\mathfrak{f})^+)$.\[\square\]

Remark 2.3. It follows from this construction that the canonical CM pair $(E,\alpha)$ over $K(\mathfrak{f})^+$ becomes isomorphic over $\mathbb{R}$ to $(E_\mathbb{R}, \text{image of } 1 \in \mathbb{C})$. So the complex conjugation automorphism of $E(\mathbb{C})$ arising from this $K(\mathfrak{f})^+$-model corresponds to the natural complex conjugation on $\mathbb{C}/\mathfrak{f}$.

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1 We adopt the convention that the cyclotomic character has Hodge–Tate weight $+1$; this is, of course, the Galois character attached to the norm map $\mathbb{A}_K^{\times} \to \mathbb{R}^\times$, which has infinity-type $(1,1)$.

2 We stress that $K(\mathfrak{f})$ is not a CM field in general, so the definition of $K(\mathfrak{f})^+$ depends on the choice of embedding, and in particular $K(\mathfrak{f})^+$ is not a totally real field.
We recall the theory of elliptic units, as described in [Kat04, §15.5-6].

**Theorem 2.4.** For each pair $(f, a)$ of ideals of $K$ such that $O_K^\times \cap (1 + f) = \{1\}$ and $a$ is coprime to $6f$, there is a canonical element

$$ae_f \in K(f)^\times,$$

the elliptic unit of modulus $f$ and twist $a$. If $f$ has at least two prime factors, $ae_f \in O_K^\times (f)$; and for any two ideals $a, b$ coprime to $6f$, we have

$$(N(b) - |b|) \cdot ae_f = (N(a) - |a|) \cdot be_f,$$

where $[a] = \left(\frac{a}{K(f)/K}\right) \in \text{Gal}(K(f)/K)$ is the arithmetic Frobenius element at $a$.

Vital for our purposes is the following complex conjugation symmetry of the elliptic units:

**Proposition 2.5.** If $f$ satisfies the hypotheses of Theorem 2.2, then we have

$$\overline{ae_f} = \bar{a}e_{\bar{f}}.$$

**Proof.** This follows from the construction of the elliptic units. We have

$$ae_f = a \theta_E(\alpha)^{-1}$$

where $(E, \alpha)$ is the canonical CM pair over $K(f)$, and $a \theta_E$ is the element of the function field of $E$ constructed in [Kat04, §15.4].

By Theorem 2.2, $E$ admits a model over $K(f)^+$, and it is clear that if $e$ is the nontrivial element of $\text{Gal}(K(f)/K)^+$ arising from complex conjugation, we have $e(aE) = aE$ and hence (by the uniqueness of $a \theta_E$) we have $(a \theta_E)^e = a \theta_E$. Since $\alpha \in E(K(f)^+)$, we deduce that

$$\overline{ae_f} = (a \theta_E)^e(\alpha)^{-1} = a \theta_E(\alpha)^{-1} = ae_f$$

as required. \qed

**Remark 2.6.** Modulo differing choices of conventions, this is the formula labelled “Transport of Structure” in §2.5 of [Gro80].

2.2. Elliptic units in Iwasawa cohomology. Let $p$ be a rational prime which splits in $K$. For fixed $f$ (which we shall assume prime to $p$), the ideal $g = fp^n$ satisfies the condition $O_K^\times \cap (1 + g) = \{1\}$ for all $n \gg 0$, so if $(a, 6pf) = 1$ we may define the elements $ae_{fp^n}$. These are norm-compatible (c.f. [Kat04, §15.5]), and we may extend their definition to all $n \geq 0$ using the norm maps.

**Note 2.7.** Since $fp^n$ has at least two prime factors for $n \geq 1$, we have $ae_{fp^n} \in O_K^\times (fp^n)$.

Let $S$ be a set of places of $K$ containing the infinite places and the primes above $p$. Then we have the Kummer maps

$$\kappa_L : \mathbb{Z}_p \otimes_{\mathbb{Z}} O_{L,S}^\times \rightarrow H^1_S(L, \mathbb{Z}_p(1)).$$

Since the sequence of elements $ae_{fp^n} = (ae_{fp^n})_{n \geq 0}$ is a norm-compatible sequence of units, their images under the Kummer maps are corestriction-compatible, so we obtain an element

$$ae_{fp^n} \in H^1_{Iw,S}(K(fp^n), \mathbb{Z}_p(1)) = \varprojlim_n H^1_S(K(fp^n), \mathbb{Z}_p(1)).$$

**Theorem 2.8.** If $f$ is Galois-stable, then we have

$$\iota_* (ae_{fp^n}) = ae_{fp^n},$$

where $\iota_*$ is the involution of $H^1_{Iw,S}(K(fp^n), \mathbb{Z}_p(1))$ induced by complex conjugation.
Proof. Immediate from Proposition 2.5, since $f_p^n$ satisfies the conditions of Theorem 2.2 for all $n \gg 0$. □

Definition 2.9. We also define the element
\[
e_{f_p^\infty} = (N(a) - [a])^{-1} \cdot ae_{f_p^\infty} \in Q(G_{f_p^\infty} \otimes_{\Lambda(G_{f_p^\infty})} H^1_{Iw,S}(K(f_p^\infty), \mathbb{Z}_p(1))),
\]
where $\Lambda(G_{f_p^\infty})$ is the Iwasawa algebra of $G_{f_p^\infty} = \text{Gal}(K(f_p^\infty)/K)$ and $Q(G_{f_p^\infty})$ its total ring of quotients.

Note 2.10. The element $e_{f_p^\infty}$ is independent of the choice of $a$.

Corollary 2.11. We have $\iota_*(e_{f_p^\infty}) = e_{f_p^\infty}$.

Proof. The automorphism $\iota_*$ of $H^1_{Iw,S}(K(f_p^\infty), \mathbb{Z}_p(1))$ is $\Lambda(G_{f_p^\infty})$-semilinear, with the action of $\iota$ on $G_{f_p^\infty}$ being given by conjugation in $\text{Gal}(\overline{K}/\mathbb{Q})$; hence $\iota_*$ extends canonically to the tensor product with $Q(G_{f_p^\infty})$; and since $\iota(a) = a$, this finishes the proof by Theorem 2.8. □

Let $W$ be any continuous representation of $G_{f_p^\infty}$ on a one-dimensional vector space over some finite extension $L$ of $\mathbb{Q}_p$. Then we have an isomorphism
\[
H^1_{Iw,S}(K(f_p^\infty), \mathbb{Z}_p(1)) \otimes_{\mathbb{Z}_p} W \overset{\sim}{\longrightarrow} H^1_{Iw,S}(K(f_p^\infty), W(1)).
\]

Definition 2.12. For an element $w \in W$, let $e_{f_p^\infty}(w)$ be the image of $e_{f_p^\infty} \otimes w$ under (1), which is an element of
\[
Q(G_{f_p^\infty}) \otimes_{\Lambda(G_{f_p^\infty})} H^1_{Iw,S}(K(f_p^\infty), W(1)).
\]
Define
\[
e_{\infty}(w) \in Q(\Gamma) \otimes_{\Lambda(\Gamma)} H^1_{Iw,S}(K_{\infty}, W(1))
\]
to be the image of $e_{f_p^\infty}(w)$ under the corestriction map
\[
H^1_{Iw,S}(K(f_p^\infty), W(1)) \longrightarrow H^1_{Iw,S}(K_{\infty}, W(1)).
\]

Lemma 2.13. If $W$ has no fixed points under $\text{Gal}(K(f_p^\infty)/K_{\infty})$, then we have
\[
e_{\infty}(w) \in H^1_{Iw,S}(K_{\infty}, W(1)).
\]

Proof. Let $I$ be the ideal in $\Lambda(f_p^\infty)$ generated by the elements $(Na - [a])$ for integral ideals $a$ prime to $6f$. Suppose $G_{f_p^\infty}$ acts on $W$ via the character $\tau : G_{f_p^\infty} \longrightarrow L$. Then we must show that the ideal in $\Lambda(\Gamma)$ generated by the elements
\[
\{(Na - \tau([a])^{-1}[a]) : a \text{ is an integral ideal coprime to } 6f\}
\]
contains a power of $p$. However, if this is not the case, it must consist of elements of $\Lambda(\Gamma)$ which all vanish at some character $\eta$ of $\Gamma$. Then $\chi([a])\tau([a]) - \eta([a])$ vanishes for every $a$. By the Chebotarev density theorem, we must have $\tau = \chi^{-1}\eta$, which contradicts the assumption that $\tau$ does not factor through $\Gamma$. □

We write $\iota W$ for the representation of $G_{f_p^\infty}$ that acts on $\{\iota w : w \in W\}$ via $g \cdot (\iota w) = \iota(g) \cdot w$.

Theorem 2.14. If $W$ has no fixed points under $\text{Gal}(K(f_p^\infty)/K_{\infty})$, the element
\[
e_{\infty}(w) \in H^1_{Iw,S}(K_{\infty}/K, W(1))
\]
satisfies
\[
\iota_*(e_{\infty}(w)) = e_{\infty}(\iota w)
\]
where $\iota_*$ is induced from the maps
\[
H^1_{f_p^\infty}(K(f_p^n), W(1)) \longrightarrow H^1_{f_p^\infty}(K(f_p^n), (\iota W)(1))
\]
sending a cocycle $\tau$ to the cocycle $g \mapsto \iota\tau(\iota g)$, for each $n \geq 0$.

We split the proof of the theorem into a number of steps.
Definition 2.15. Let \( \Lambda^4(G_{p^n}) \) denote \( \Lambda(G_{p^n}) \) endowed with the action of \( \text{Gal}(K^S/K) \) via the product of the cyclotomic character with the inverse of the canonical character \( \text{Gal}(K^S/K) \to G_{p^n} \to \Lambda(G_{p^n})^\times \), i.e. \( g \omega = \chi(g) \bar{g}^{-1} \omega \) for any \( g \in \text{Gal}(K^S/K) \) and \( \omega \in \Lambda^2(G) \). Here, \( \bar{g} \) denotes the image of \( g \) in \( G_{p^n} \).

Lemma 2.16. We have a commutative diagram

\[
\begin{array}{ccc}
H^1_{Iw,S}(K(f_p\infty), \mathbb{Z}_p(1)) \otimes_{\mathbb{Z}_p} W & \xrightarrow{\psi} & H^1_{Iw,S}(K(f_p\infty), W(1)) \\
\downarrow \circlearrowleft & & \downarrow \circlearrowleft \\
H^1_{Iw,S}(K(f_p\infty), \mathbb{Z}_p(1)) \otimes_{\mathbb{Z}_p} iW & \xrightarrow{\sim} & H^1_{Iw,S}(K(f_p\infty), (iW)(1))
\end{array}
\]

where the left-hand vertical map is the tensor product of the automorphism \( \iota_* \) of \( H^1_{Iw,S}(K(\mathbb{Z}_p(1)), iW) \) and the canonical map \( (\cdot)(1) \) and the canonical map \( : W \to iW \), and the right-hand vertical map is as defined in the statement of Theorem 2.14.

Proof. We will deduce this isomorphism by using an alternative definition of the Iwasawa cohomology which renders the horizontal maps in the diagram easier to handle. By Shapiro’s lemma, we have a canonical isomorphism of \( \Lambda(G_{p^n}) \)-modules

\[
H^1_{Iw,S}(K(f_p\infty), M(1)) \cong H^1_S(K, M \otimes_{\mathbb{Z}_p} \Lambda^2(G_{p^n})(1))
\]

for any \( \text{Gal}(K^S/K) \)-module \( M \) which is finite-rank over \( \mathbb{Z}_p \) or \( \mathbb{Q}_p \).

Let \( \tau \) be the character by which \( G_{p^n} \) acts on \( W \), and define \( \tau_* : \Lambda^2(G) \to \Lambda^2(G) \) to be the map induced by \( g \to \tau(g)^{-1} g \). Then the natural twisting map

\[
j : H^1_S(K, \Lambda^2(G)(1)) \otimes W \xrightarrow{\sim} H^1_S(K, \Lambda^2(G)(1) \otimes W),
\]

is explicitly given as follows: if \( c : \text{Gal}(K^S/K) \to \Lambda^2(G)(1) \) is a cocycle and \( w \in W \), define

\[
j(c \otimes w)(g) = \tau_*(c(g)) \otimes w.
\]

We check that \( j(c \otimes w) \) is a cocycle. Let \( h, g \in \text{Gal}(K^S/K) \). Then

\[
j(c \otimes w)(gh) = \tau_*(c(gh)) \otimes w
\]

\[
= \tau_*(g.c(h)) \otimes w + \tau_*(c(g)) \otimes w
\]

\[
= \chi(g) \tau_*(g^{-1}c(h)) \otimes w + \tau_*(c(g)) \otimes w
\]

\[
= \chi(g) \tau_*(g^{-1}[\tau_*(c(h))]) \otimes w + \tau_*(c(g)) \otimes w
\]

\[
= g.[j(c \otimes w)(h)] + j(c \otimes w)(g)
\]

Rewrite the diagram (2) as

\[
\begin{array}{ccc}
H^1_S(K, \Lambda^2(G)(1)) \otimes_{\mathbb{Z}_p} W & \xrightarrow{jW} & H^1_S(K, \Lambda^2(G)(1) \otimes iW) \\
\downarrow \circlearrowleft & & \downarrow \circlearrowleft \\
H^1_S(K, \Lambda^2(G)(1)) \otimes_{\mathbb{Z}_p} iW & \xrightarrow{jW} & H^1_S(K, \Lambda^2(G)(1) \otimes iW)
\end{array}
\]

It is then immediate from the description of \( j \) that the diagram commutes, which finishes the proof.

\( \square \)

Proof of Theorem 2.14. By Corollary 2.11 and Lemma 2.16, we have

\[
\iota_*(e_{fp^n}(w)) = e_{fp^n}(yw).
\]
The action of $\iota_\ast$ is clearly compatible with corestriction, so we have a commutative diagram
\[
\begin{array}{ccc}
H^1_{Iw,S}(K(\mathfrak{p}^\infty), W(1)) & \longrightarrow & H^1_{Iw,S}(K_{\infty}, W(1)) \\
\iota^* & & \iota_* \\
H^1_{Iw,S}(K(\mathfrak{p}^\infty), (\iota W)(1)) & \longrightarrow & H^1_{Iw,S}(K_{\infty}, (\iota W)(1))
\end{array}
\]
which implies that $\iota_\ast(e_\infty(w)) = e_\infty(\iota w)$, completing the proof.

**Lemma 2.17.** Let $V$ be any $p$-adic representation of $\text{Gal}(K^S/\mathbb{Q})$. Then the restriction map induces an isomorphism
\[
H^1_{Iw,S}(\mathbb{Q}_\infty, V) \longrightarrow H^1_{Iw,S}(K_{\infty}, V)^{\text{Gal}(K^S/\mathbb{Q}_\infty)}.
\]

**Proof.** The restriction map is induced from the restriction maps on finite level, which fit into the exact sequence
\[
0 \longrightarrow H^1(\text{Gal}(K_n/\mathbb{Q}_n), V^{\text{Gal}(K^S/K_n)}) \longrightarrow H^1_S(\mathbb{Q}_n, V) \longrightarrow H^1_S(K_n, V)^{\text{Gal}(K_n/\mathbb{Q}_n)} \longrightarrow H^2(\text{Gal}(K_n/\mathbb{Q}_n), V^{\text{Gal}(K^S/K_n)}).
\]
Since $\mathbb{Q}_p$ has characteristic 0, the higher cohomology groups of any $\mathbb{Q}_p$-linear representation of the cyclic group of order 2 are zero. This gives the claim at each finite level, and hence in the inverse limit. $$\square$$

Let $\alpha$ be the unique nontrivial element of $\text{Gal}(K_{\infty}/\mathbb{Q}_\infty)$.

**Lemma 2.18.** We have $\alpha = \delta_\ast$, where $\delta$ is the unique element of $\text{Gal}(K_{\infty}/K)$ which acts on $\mathbb{Q}_\infty$ as complex conjugation. In particular, $\delta$ is of order 2.

**Corollary 2.19.** If $\alpha$ is the unique nontrivial element of $\text{Gal}(K_{\infty}/\mathbb{Q}_\infty)$, then for any $w \in W$, $\alpha_\ast(e_\infty(w)) = \delta \cdot e_\infty(\iota w)$. $\square$

2.3. The two-variable $L$-function of $K$. We recall the construction (originally due to Yager [Yag82]) of a two-variable $p$-adic $L$-function from the elliptic units. Let $\mathfrak{p}$ be one of the two primes of $K$ above $p$. We choose an embedding $\overline{K} \hookrightarrow \overline{\mathbb{Q}}_p$ inducing the $p$-adic valuation on $K$. Then for any finite extension $L/K$, and any $\text{Gal}(\overline{K}/K)$-module $M$, we may define
\[
Z^1_{\mathfrak{p}}(L, M) = \bigoplus_{q|\mathfrak{p}} H^1(L_q, M) = H^1(K_{\mathfrak{p}}, \text{Ind}_{\mathfrak{p}}^K M),
\]
which is a $\text{Gal}(L/K)$-module. We also define
\[
Z^1_{Iw,\mathfrak{p}}(K(\mathfrak{p}^\infty), M) = \lim_{\substack{\longrightarrow \\mathfrak{p}}} Z^1_{\mathfrak{p}}(L, M)
\]
where the limit is taken over finite extensions $L/K$ contained in $K(\mathfrak{p}^\infty)$.

We now recall the theory of two-variable Coleman series, as introduced, under certain additional hypotheses, by Yager [Yag82], and generalized to the semi-local situation here by de Shalit [dS87, §II.4.6]. Let $\zeta = (\zeta_p^n)_{n \geq 0}$ be a compatible system of $p$-power roots of unity in $\overline{K}$; and let $\hat{\mathbb{F}}_{\infty}$ be the completion of $K(\mathfrak{p}^\infty)$ with respect to the prime $\mathfrak{p}$ of $\overline{K}$ above $\mathfrak{p}$ induced by our choice of embedding $\overline{K} \hookrightarrow \overline{\mathbb{Q}}_p$, and $\hat{\mathbb{O}}_{\infty}$ the ring of integers of $\hat{\mathbb{F}}_{\infty}$. (Thus $\hat{\mathbb{O}}_{\infty}$ is a complete discrete valuation ring
with maximal ideal generated by \( p \), and its residue field is a finite extension of the unique \( \mathbb{Z}_p \)-extension of \( \mathbb{F}_p \).

**Proposition 2.20.** There is a unique morphism of \( \Lambda(G_{fp}) \)-modules

\[
\text{Col}^\Sigma : \hat{Z}_{\text{ur},p}(K(\mathcal{f}p^{\infty}), \mathbb{Z}_p(1)) \rightarrow \Lambda_{\hat{\mathcal{O}}}(G_{fp^{\infty}})
\]

with the following property:

For each finite-order character \( \eta \) of \( G_{fp^{\infty}} \) which is not unramified at \( p \), we have

\[
\text{Col}^\Sigma(u)(\eta) = \tau(\eta, \zeta)^{-1}\eta(\tilde{\varphi})^n \left( \sum_{\sigma \in G_{fp^n}} \eta(\sigma)^{-1} \log_p(u_m^\sigma) \right).
\]

Here \( \tilde{\varphi} \) is the unique lifting of the arithmetic Frobenius of \( \text{Gal}(K(\mathcal{f}p^{\infty})/K) \) to \( \text{Gal}(K(\mathcal{f}p^{\infty})/K_{\infty}) \), \( n \) is any integer such that \( \eta \) factors through the quotient \( G_{fp^n} = \text{Gal}(K(\mathcal{f}p^{n})/K) \), \( \log_p \) is the logarithm map

\[
\mathcal{O}_{K(\mathcal{f}p^n)}^\times \rightarrow K(\mathcal{f}p^n) \mathcal{p},
\]

and

\[
\tau(\eta, \zeta) = \sum_{\sigma \in \text{Gal}(K(\mathcal{f}p^{n})/\mathcal{O}_{K(\mathcal{f}p^n))} } \omega(\sigma)^{-1}\zeta^\sigma,
\]

where \( n \) is the exact power of \( p \) dividing the conductor of \( \eta \).

**Definition 2.21.** We let

\[
\mathbb{L}_{fp^{\infty}} = \text{Col}^\Sigma(e_{fp^{\infty}}) \in \hat{\mathcal{O}}_{\infty} \hat{\otimes}_{\mathbb{Z}_p} Q(G_{fp^{\infty}}).
\]

**Proposition 2.22.** The element \( \mathbb{L}_{fp^{\infty}} \) lies in \( \Lambda_{\hat{\mathcal{O}}_{\infty}}(G_{fp^{\infty}}) \), and it coincides with the measure \( \mu(\mathcal{f}p^{\infty}) \) in [dS87, Theorem II.4.14].

**Proof.** We have \( (N\mathfrak{a} - [\mathfrak{a}]) \cdot \mathbb{L}_{fp^{\infty}} \in \Lambda_{\hat{\mathcal{O}}_{\infty}}(G_{fp^{\infty}}) \) for all \( \mathfrak{a} \). Since the ideal generated by \( N\mathfrak{a} - [\mathfrak{a}] \) for all integral ideals \( \mathfrak{a} \) coprime to 6\( \mathcal{f} \) has height 2, this implies that \( \mathbb{L}_{fp^{\infty}} \in \Lambda_{\hat{\mathcal{O}}_{\infty}}(G_{fp^{\infty}}) \) (cf. [dS87, II.5.5]).

To show that the resulting measure coincides with de Shalit’s \( \mu(\mathcal{f}p^{\infty}) \), we compare the defining property of the map \( \text{Col} \) above with [dS87, Theorem II.5.2]. For a finite-order character \( \eta \) of \( G_{fp^{\infty}} \), whose conductor \( \mathfrak{c} \) is divisible by \( p \) and satisfies \( \mathcal{O}_K \cap (1 + \mathfrak{c}) = \{1\} \), de Shalit shows that

\[
\eta(\mu(\mathcal{f}p^{\infty})) = \frac{-1}{12g} G(\eta) \sum_{\mathfrak{c} \in \text{Cl}(\mathfrak{c})} \eta^{-1}(\mathfrak{c}) \log \phi_\mathfrak{c}(\mathfrak{c}),
\]

where \( g \) is the smallest rational integer in \( \mathfrak{c} \), \( \phi_\mathfrak{c}(\mathfrak{c}) \) is Robert’s invariant and the quantity \( G(\eta) \) coincides with what we have called \( \tau(\eta, \zeta)^{-1} \eta(\tilde{\varphi})^n \). Since

\[
(N\mathfrak{a} - [\mathfrak{a}])\phi_\mathfrak{c}(\mathfrak{c}) = [\mathfrak{c}] \cdot (\mathfrak{a}\mathfrak{c})^{-12g},
\]

this shows that the two measures coincide at every finite-order character, and hence they are equal in \( \Lambda_{\hat{\mathcal{O}}_{\infty}}(G_{fp^{\infty}}) \). \( \square \)

**Note 2.23.** If one identifies \( G(\mathcal{f}p^{\infty}) \) with the ray class group modulo \( \mathcal{f}p^{\infty} \) via the Artin map, normalized as in §1.3 above, then this measure coincides with the pull-back of the Katz two-variable \( L \)-function of \( K \) (cf. [HT93, §4]) up to a difference of signs. This remark will be important in the proof of Theorem 3.4 below.
2.4. Kato’s zeta element. Let $f = \sum a_nq^n$ be a modular form of CM type, corresponding to a Grössencharacter $\psi$ of $K$ with infinity-type $(1-k,0)$ where $k$ is the weight of $f$. It is clear that the coefficient field $F = \mathbb{Q}(a_n : n \geq 1)$ of $f$ is contained in the finite extension $L/K$ contained in $\mathbb{C}$ generated by $\psi(\hat{K}^\times)$.

Following [Kat04, §6.3], we write $S(f)$ and $V(f)$ for the subspaces of the de Rham and Betti cohomology of the Kuga–Sato variety attached to $f$. Note that both of these are $F$-vector spaces, and $S(f)$ is 1-dimensional over $F$ while $V(f)$ is 2-dimensional. For a commutative ring $A$ over $F$, define $S_A(f) = S(f) \otimes_F A$ and $V_A(f) = V(f) \otimes_F A$. If $\lambda$ is a place of $F$ above $p$, we may identify $V_{F_\lambda}(f)$ with the $p$-adic representation associated to $f$ of Deligne [Del69] and $S_{F_\lambda}(f)$ may be identified with $\mathrm{Fil}^1 D_{\mathrm{cris}}(V_{F_\lambda}(f))$.

**Definition 2.24.** Let $\chi$ be a Dirichlet character of conductor $p^n$. We define the maps $\theta^\pm_{\chi,f}$ by

$$\theta^\pm_{\chi,f} : S(f) \otimes_{\mathbb{Q}} \mathbb{Q}(\mu_{p^n}) \to V_{\mathbb{C}}(f)^\pm$$

$$x \otimes y \to \sum_{\sigma \in G_\chi} \chi(\sigma)\sigma(y) \per_f(x)^\pm$$

where $G_\chi = \text{Gal}(\mathbb{Q}(\mu_{p^n})/\mathbb{Q})$, $\per_f : S(f) \to V_{\mathbb{C}}(f)$ is the period map as defined in [Kat04, §6.3] and $\gamma \mapsto \gamma^\pm$ is the projection from $V_{\mathbb{C}}(f)$ to its (1-dimensional) $\pm1$-eigenspace for the complex conjugation.

**Theorem 2.25 ([Kat04, Theorem 12.5(1)]).** We have a $L_\lambda$-linear map

$$V_{L_\lambda}(f) \to H^1_{\text{dR},S}(\mathbb{Q}_\infty, V_\lambda(f))$$

which satisfies the following. Let $\chi$ be a Dirichlet character of conductor $p^n$, $\gamma \in V_{L_\lambda}(f)$ and $1 \leq r \leq k - 1$, then

$$\theta^\pm_{\chi,f} \circ \exp^* \left( \mathbb{Z}_\gamma^\text{Kato} \otimes (\mu_{p^n})^{\otimes (k-r)} \right) = (2\pi i)^{k-r-1} L_{(\rho)}(f^*, \chi, r) \cdot \gamma^\pm$$

where $\pm = (-1)^{k-r-1} \chi(-1)$.

Let $\mathfrak{f}$ be an ideal of $\mathcal{O}_K$ satisfying the conditions in Theorem 2.2 which is contained in the conductor of $\psi$. Let $(E, \alpha)$ be the canonical CM-pair over $K(\mathfrak{f})$. Following [Kat04, §15.8], we define $V_\mathfrak{f}(\psi) = H^1(E(\mathbb{C}), \mathbb{Q})^{\otimes (k-1)} \otimes_K L$ and $S(\psi) = H^0(\text{coLie}(E)^{\otimes (k-1)} \otimes_K L)$, where the action of $\text{Gal}(K(\mathfrak{f})/K)$ on the space $\text{coLie}(E)^{\otimes (k-1)} \otimes_K L$ is as described in op.cit.. Both of these are 1-dimensional $L$-vector spaces. For any commutative ring $A$ over $L$, we write $V_A(\psi) = V_\mathfrak{f}(\psi) \otimes_L A$ and $S_A(\psi) = S(\psi) \otimes_L A$. The Galois group $\text{Gal}(\overline{K}/K)$ acts on $V_{L_\lambda}(\psi) \otimes_L L_\lambda$ via $\psi_\lambda$, and there exists a period map

$$\per_{\psi} : S(\psi) \to V_{\mathbb{C}}(\psi)$$

induced by passing to the $(k-1)$-st tensor power from the comparison isomorphism $\per_{\psi}$ described above.

We now recall Kato’s results on the relation between this zeta element and the elliptic units.

**Lemma 2.26 ([Kat04, Lemma 15.11]).** Fix a choice of isomorphism of $L$-vector spaces

$$s : S(\psi) \to S_L(\psi).$$

(a) There exists a unique isomorphism of representations of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ over $L_\lambda$

$$V_{L_\lambda}(\psi) \to V_{L_\lambda}(f)$$

such that the isomorphism $S_{L_\lambda}(\psi) \to S_L(\psi)$ induced by the functoriality of $\mathcal{D}_{\text{MR}}$ is compatible with $s$. 


(b) There exists a unique isomorphism of representations of $\text{Gal}(\mathbb{C}/\mathbb{R})$ over $L$

$$\widetilde{V}_L(\psi) \sim V_L(f)$$

for which the diagram

$$\begin{array}{ccc} S(\psi) & \xrightarrow{\text{per}_\psi} & \widetilde{V}_C(\psi) \\ \downarrow & & \downarrow \\ S_L(f) & \xrightarrow{\text{per}_f} & V_C(f) \end{array}$$

commutes.

Note that the isomorphism of part (b) implies an isomorphism $V_L(\lambda)(\psi) \sim V_L(\lambda)(f)$ on extending scalars to $L$, but one does not know that this coincides with the isomorphism of part (a), as remarked in [Kat04, §15.11].

**Definition 2.27.** We write $\Phi_{\psi,f}$ for the canonical map

$$H^1_{Iw,S}(K(\mathfrak{p}^\infty), V_L(\psi)) \rightarrow H^1_{Iw,S}(Q_\infty, V_L(f))$$

as defined in [Kat04, (15.12.1)].

Concretely, this map can be defined as follows:

$$H^1_{Iw,S}(K(\mathfrak{p}^\infty), V_L(\psi)) \rightarrow H^1_S(K, \Lambda^2(\Gamma) \otimes V_L(\psi)) \rightarrow H^1_S(Q, \Lambda^2(\Gamma) \otimes V_L(f)).$$

**Theorem 2.28.** Let $\gamma \in V_L(\psi)$ and write $\gamma'$ for its image in $V_L(f)$ under the map given by Lemma 2.26(b). Then we have

$$\Phi_{\psi,f} \left( e_\infty(\gamma) \otimes (\zeta_{\mathfrak{p}^\infty})^\otimes(-1) \right) = z^K_{\psi' \gamma}.$$

**Proof.** This is [Kat04, (15.16.1)]; it is immediate from a comparison the interpolating properties of the two zeta elements, since an element of $H^1_{Iw}(Q_\infty/Q, V_L(f))$ is uniquely determined by its images under the dual exponential maps at each finite level in the tower $Q_\infty/Q$. \qed

**Proposition 2.29.** We have a commutative diagram

$$\begin{array}{ccc} H^1_{Iw,S}(K_\infty, V_L(\psi)) & \xrightarrow{\Phi_{\psi,f}} & H^1_{Iw,S}(Q_\infty, V_L(f)) \\ \downarrow & & \downarrow \\ H^1_{Iw,S}(K_\infty, V_L(\psi) \oplus \iota V_L(\psi))^{\alpha=1} & \cong & \end{array}$$

where the left-hand vertical map sends $x$ to $x \oplus \delta \cdot \iota_*(x)$, and the diagonal isomorphism is given by restriction.

**Proof.** Clear. \qed
3. Critical-slope $L$-functions

Let $f$ be a modular form of CM type, as above, and $\psi$ the corresponding Grössencharacter. We choose a basis $\gamma$ of $V_L(\psi)$, and let $\gamma'$ be its image in $V_L(f)$ under the isomorphism of Lemma 2.26(b).

We fix an embedding $\overline{K} \hookrightarrow \overline{\mathbb{Q}}_p$, which induces the $\lambda$-adic valuation on $L$. This gives an embedding $\text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \hookrightarrow \text{Gal}(\overline{K}/\mathbb{Q})$, whose image is contained in the subgroup $\text{Gal}(\overline{K}/K)$. This gives a localization map

$$\text{loc}_p : H^1_{Iw,S}(\mathbb{Q}_\infty, M) \longrightarrow H^1_{Iw}(\mathbb{Q}_p, M)$$

for each $\text{Gal}(K^S/\mathbb{Q})$-module $M$. Moreover, we have a map

$$\text{loc}_p : H^1_{Iw,S}(K, M) \longrightarrow H^1_{Iw}(\mathbb{Q}_p, M)$$

for each $\text{Gal}(K^S/K)$-module $M$, and we clearly have $\text{loc}_p = \text{loc}_p \circ \text{res}_{K/\mathbb{Q}}$.

Via the isomorphism of Lemma 2.26(a), the space $V_{L,\gamma}(f)$ is isomorphic as a representation of $\text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ to $V_{L,\gamma}(\psi) \oplus \iota(V_{L,\gamma}(\psi))$. Note that $\iota$ does not normalize the image of $\text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$, so the two factors are non-isomorphic; indeed $V_{L,\gamma}(\psi)$ has Hodge–Tate weight $1 - k$, while $\iota(V_{L,\gamma}(\psi))$ has Hodge–Tate weight $0$. Hence we have

$$\text{loc}_p(z^\text{Kato}_{\gamma}) \in H^1_{Iw}(\mathbb{Q}_p, V_{L,\gamma}(\psi)) \oplus H^1_{Iw}(\mathbb{Q}_p, \iota(V_{L,\gamma}(\psi))).$$

Let us write $\text{pr}_1$ and $\text{pr}_2$ for the projections to the two direct summands above. By Corollary 2.28, the projection $\text{pr}_1 \circ \text{loc}_p(z^\text{Kato}_{\gamma})$ to $H^1_{Iw}(\mathbb{Q}_p, V_{L,\gamma}(\psi))$ is

$$\text{loc}_p \left( e_\infty(\gamma) \otimes (\zeta_p^n)^{\otimes 1} \right).$$

By Proposition 2.29, we see that the projection of $\text{loc}_p(z^\text{Kato}_{\gamma})$ to the other direct summand is

$$\delta \cdot \text{loc}_p \left[ \iota_* \left( e_\infty(\gamma) \otimes (\zeta_p^n)^{\otimes 1} \right) \right] = [\delta \cdot \text{loc}_p (\iota_* (e_\infty(\gamma)))] \otimes (\zeta_p^n)^{\otimes 1}.$$

We have

$$\iota_* (e_\infty(\gamma)) = e_\infty(\iota \gamma),$$

so this simplifies to

$$\text{pr}_2 \left( \text{loc}_p(z^\text{Kato}_{\gamma}) \right) = \delta \cdot [\text{loc}_p (e_\infty(\iota \gamma))] \otimes (\zeta_p^n)^{\otimes 1}.$$
Hodge-Tate weights, then for any \( x \in H^1_{\text{cr}}(K(\fp^{\infty}), V) \) and any choice of basis \( e_\xi \) of \( \Q_p(\xi) \) we have
\[
L_{\fp}^{G_{\fp}}(x \otimes e_\xi)(\eta) = (t_0 \ldots t_{j-1})(\eta) \cdot L_{\fp}^{G_{\fp}}(x)(\eta^{\xi-1}) \otimes t^{-j} e_\xi.
\]
Note that if \( \xi \) takes values in the finite extension \( L/\Q_p \), this is an equality of two elements of \( L \otimes \hat{F}_\infty \otimes \hat{\Q} \): the element \( t^{-j} e_\xi \in \hat{B}_{\text{cris}} \otimes \Q_p \) transforms via \( \tau \) under \( G_{\Q_p} \), and hence lies in \( \hat{F}_\infty \otimes \hat{\Q} \), since the periods of unramified characters lie in \( \hat{F}_\infty \subseteq \hat{B}_{\text{cris}} \).

We apply this result with \( V = \Q_p \) (the trivial representation), \( x = e_{p^{\infty}} \otimes e_{-1} \), and various values of \( \xi \). Firstly, taking \( \xi \) to be the cyclotomic character, we have
\[
\I_{\fp} = t_0^{-1} L_{\Q_p}^{G_{\fp}}(e_{p^{\infty}}),
\]
and thus
\[
L_{\fp}^{G_{\fp}}(\eta) = L_{\Q_p}^{G_{\fp}}(e_{p^{\infty}} \otimes e_{-1})(\chi(1)^{\eta}) \otimes t^{-1} e_1.
\]
On the other hand we have
\[
L_{p,1}^\gamma(\eta) = L_{V_{L_1}(\psi)(k-1)}^{G_{p^{\infty}}}(\text{pr}_1(\Z_{p^{k_{\text{Kato}}}}(\chi_{p^{k_{-1}}})) (\eta) = L_{V_{L_1}(\psi)(k-1)}^{G_{p^{\infty}}}(e_{\infty}(\chi_{p^{k_{-1}}}) \otimes e_{k-2}) (\eta).
\]
The group \( G_{\Q_p} \) acts on \( V_{L_1}(\psi)(k-1) \) via the unramified character \( \chi^{-k-1} \psi_{L_1} \), so this is
\[
L_{p,1}^\gamma(\eta) = L_{V_{L_1}(\psi)(k-1)^{\infty}}^{G_{p^{\infty}}}(e_{\infty} \otimes e_{-1})(\chi^{-k-1} \psi_{L_1}(1)^{\eta}) \otimes (t^{-k-1} \psi_{L_1}) \otimes (t^{-k} e_{k-2}).
\]
Comparing this with (4), we deduce that
\[
L_{p,1}^\gamma(\eta) = L_{\fp}^{G_{\fp}}(\chi^{-k-1} \psi_{L_1}(1)^{\eta}) \cdot \Omega_{p}^{\omega}.
\]
We now turn to \( L_{p,2}^\gamma \). We have
\[
L_{p,2}^\gamma(\eta) = L_{V_{L_1}(\psi)(k-1)^{\infty}}^{G_{p^{\infty}}}(\text{pr}_2(\Z_{p^{k_{\text{Kato}}}}(\chi_{p^{k_{-1}}}) (\eta) = L_{V_{L_1}(\psi)(k-1)^{\infty}}^{G_{p^{\infty}}}(\delta \cdot e_{\infty}(\gamma) \otimes e_{k-2}) (\eta) = (-1)^{k-2} \eta(\delta) L_{V_{L_1}(\psi)(k-1)^{\infty}}^{G_{p^{\infty}}}(e_{\infty}(\gamma) \otimes e_{k-2}) (\eta).
\]
The group \( G_{\Q_p} \) acts on \( V_{L_1}(\psi)(1) \) by the character \( \psi_{L_1} \), which is unramified; so this is
\[
L_{p,2}^\gamma(\eta) = (-1)^{k-2} \eta(\delta)(\xi_0 \ldots \xi_{k-2})(\eta) \cdot L_{\Q_p}^{G_{\fp}}(e_{\infty} \otimes e_{-1})(\chi^{-k-1} \psi_{L_1}(1)^{\eta}) \otimes t^{1-k} e_{k-1} \otimes \gamma.
\]
As above, we identify \( t^{2-k} e_{k-2} \in \hat{\D}_{\text{cris}}(\Q_p(k-2)) \) with \( 1 \in \Q_p \); and if \( \omega \) is a basis of \( S_L(\psi) \), the image of \( \omega \) under the comparison isomorphism is a basis of \( \hat{\D}_{\text{cris}}(\ell(V_{L_1}(\psi))) \), so if we define \( \Omega_{p}^\prime = (\gamma)(\omega) \) this becomes
\[
L_{p,2}^\gamma(\eta) = (-1)^{k-2} \eta(\delta)(\xi_0 \ldots \xi_{k-2})(\eta) \cdot L_{\fp}^{G_{\fp}}(\chi^{-k-1} \psi_{L_1}(1)^{\eta}) \cdot \Omega_{p}^\prime \psi_{L_1}(1)^{\eta}.
\]
\( \square \)
Definition 3.3. Let $\omega$ be a basis of $S_L(\psi)$ as above, and let $L_{p,\alpha}(g)$ and $L_{p,\beta}(g)$ be the elements of $\mathcal{H}_{L,G}(\Gamma)$ defined by

$$L_{p,1}^\gamma = L_{p,\alpha}(g) \cdot \omega$$

and

$$L_{p,2}^\gamma = L_{p,\beta}(g) \cdot \omega.$$ 

Then $L_{p,\alpha}$ and $L_{p,\beta}$ are the $p$-adic $L$-functions attached to $g$, where $\alpha$ and $\beta$ are respectively the unit and non-unit roots of the Hecke polynomial of $g$.

As shown in [Kat04, §16], this is consistent with the classical Amice–Velu–Vishik construction of the ordinary $p$-adic $L$-function $L_{p,\alpha}(g)$, and thus it is natural to regard $L_{p,\beta}(g)$ as a candidate for a critical-slope $p$-adic $L$-function. This is the definition of the Kato critical-slope $L$-function used in [LZ11a].

Theorem 3.4. Up to multiplication by two nonzero scalars, one for each sign, $L_{p,\beta}(g)$ coincides with the modular symbol critical-slope $L$-function $L_{p,\beta}^{MS}(g)$ attached to the non-ordinary $p$-stabilization of $f$ in [Bel11b].

Proof. This follows by comparing the formulae of Theorem 3.2 with Theorem 2 of [Bel11b]. Note that Bellaïche shows that if $p_1$ and $p_2$ are the two characters by which $\text{Gal}(\overline{K}/K)$ acts on $V_\varphi^*$, then

$$L_{p,\alpha}(g)(\eta) = L_{p,\alpha}(\rho_2\eta^{-1}) \cdot (\text{constant}^\pm),$$

$$L_{p,\beta}^{MS}(g)(\eta) = (\ell_0 \cdots \ell_{k-2})(\eta) \cdot L_{p,\alpha}(\rho_2\eta^{-1}) \cdot (\text{constant}^\pm).$$

Here constant$^\pm$ indicates an equality of distributions on $\Gamma$ up to multiplication by two nonzero constants (one for each sign). On the other hand, we have proved that

$$L_{p,\alpha}(g)(\eta) = L_{p,\alpha}(\chi\rho_1^{-1}\eta) \cdot (\text{constant}),$$

$$L_{p,\beta}(g)(\eta) = (\ell_0 \cdots \ell_{k-2})(\eta) \cdot L_{p,\alpha}(\chi\rho_1^{-1}\eta) \cdot \text{(constant)}.$$ 

To reconcile these formulae, we note that the $p$-adic $L$-function $L_{p,\alpha}$ satisfies a functional equation [dS87, §II.6]

$$L_{p,\alpha}(\iota(\eta)) = C(\eta) \cdot L_{p,\alpha}(\chi\eta^{-1}),$$

for a function $C(\eta)$ (involving a $p$-adic root number and various other correction terms) which depends only on the coset of $\eta$ modulo characters factoring through $\text{Gal}(\mathbb{Q}_p/\mathbb{Q})$. Since $\iota(p_1) = p_2$ and vice versa, we deduce that

$$L_{p,\beta}(g) = L_{p,\beta}^{MS}(g) \cdot (\text{constant}^\pm).$$

Since the modular symbol $L$-function is only defined up to scalars, this completes the proof. \qed

Note 3.5. Both Kato’s and Bellaïche’s critical-slope $p$-adic $L$-functions are only defined up to multiplication by a nonzero constant for characters of each sign; in Kato’s construction these constants correspond to the choice of $\gamma$, whose projection to each of the $\pm$ eigenspaces of complex conjugation must be non-zero. It seems natural to ask whether one can choose normalizations for both in a compatible fashion so Theorem 3.4 holds exactly, but the present authors do not feel sufficiently familiar with the modular symbol construction to comment further.
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