COMMUTATION UP TO A FACTOR FOR BOUNDED AND UNBOUNDED OPERATORS AND APPLICATIONS

MOHAMMED HICHEM MORTAD* AND CHÉRIFA CHELLALI

Abstract. In this paper, we further investigate the problem of commutation up to a factor (or \(\lambda\)-commutativity) in the setting of bounded and unbounded linear operators in a complex Hilbert space. The results are based on a new approach to the problem. We finish the paper by a conjecture on the commutativity of self-adjoint operators.

1. Introduction

Commutativity up to a factor has been explored by many authors. See e.g. [1], [2], [7], [10] and [17]. The purpose of the present paper is twofold. First, we recover known results by using the question of bounded normal products of self-adjoint operators. Second, we extend this method to unbounded operators.

Let us say a little more about details of this technique. It is well-known that two bounded, normal and commuting operators have a normal product. The proof uses the celebrated Fuglede theorem (we note that this question has been generalized to the case of unbounded operators in e.g. [6], [12], [13] and [14]). In a very similar manner, we also notice that -this time via the Fuglede-Putnam theorem- the product of two anti-commuting normal operators remains normal. So, we conjectured that the product of normal operators which commute up to a factor would be normal. This is in effect the case and the reason why we want to use the normality of the product in question is that we may exploit results on the bounded normal product of self-adjoint operators (as those in [8] and [9], and the references therein). The asset of this approach is that it also extends to unbounded operators so that we may again take advantage of the results in [8] and [9].

To make the paper as self-contained as possible, we recall the following results:

Theorem 1.1. Let \(A\) be a densely defined unbounded operator.

1. \((BA)^* = A^*B^*\) if \(B\) is bounded.
2. \(A^*B^* \subset (BA)^*\) for any densely unbounded \(B\) and if \(BA\) is densely defined.
3. Both \(AA^*\) and \(A^*A\) are self-adjoint whenever \(A\) is closed.

Lemma 1.1 ([13]). If \(A\) and \(B\) are densely defined and \(A\) is invertible with inverse \(A^{-1}\) in \(B(H)\), then \((BA)^* = A^*B^*\).

Proposition 1.1 ([3]). Let \(A\), \(B\) and \(C\) be unbounded self-adjoint operators. Then \(A \subseteq BC \implies A = BC\).

Key words and phrases. Commutativity up to a factor; Normal and self-adjoint operators; Fuglede-Putnam theorem; Bounded and unbounded operators.

* Corresponding author, who is partially supported by "Laboratoire d’Analyse Mathématique et Applications".
Theorem 1.2. [Fuglede-Putnam theorem, for a proof see e.g. [3]] If $A$ is a bounded operator and if $M$ and $N$ are normal operators, then

$$AN \subseteq MA \implies AN^* \subseteq M^*A$$

(if $N$ and $M$ are bounded, then we replace "$\subseteq$" by "=").

Theorem 1.3. [8] Let $A$ and $B$ be two self-adjoint operators such that $AB$ is normal. Then

1. If $A$ and $B$ are both bounded, and if further $\sigma(A) \cap \sigma(-A) \subseteq \{0\}$ or $\sigma(B) \cap \sigma(-B) \subseteq \{0\}$, then $AB$ is self-adjoint.
2. If only $B$ is bounded, and if $\sigma(B) \cap \sigma(-B) \subseteq \{0\}$, then $AB$ is self-adjoint.

Theorem 1.4. [1] Let $A$, $B$ be bounded operators such that $AB \neq 0$ and $AB = \lambda BA$, $\lambda \in \mathbb{C}^*$. Then

1. if $A$ or $B$ is self-adjoint, then $\lambda \in \mathbb{R}$;
2. if both $A$ and $B$ are self-adjoint, then $\lambda \in \{-1, 1\}$; and
3. if $A$ and $B$ are self-adjoint and one of them is positive, then $\lambda = 1$.

Theorem 1.5. [17] Let $A$, $B$ be bounded operators such that $AB = \lambda BA \neq 0$, $\lambda \in \mathbb{C}^*$. Then

1. if $A$ or $B$ is self-adjoint, then $\lambda \in \mathbb{R}$;
2. if either $A$ or $B$ is self-adjoint and the other is normal, then $\lambda \in \{-1, 1\}$; and
3. if $A$ and $B$ are both normal, then $|\lambda| = 1$.

Theorem 1.6. [10] Let $A$ be an unbounded operator and let $B$ be a bounded one. Assume that $BA \subset \lambda AB \neq 0$ where $\lambda \in \mathbb{C}$. Then

1. $\lambda$ is real if $A$ is self-adjoint.
2. $\lambda = 1$ if $0 \notin W(B)$ (the numerical range of $B$) and if $A$ is normal; hence $\lambda = 1$ if $B$ is strictly positive and $A$ is normal.
3. $\lambda \in \{-1, 1\}$ if $A$ is normal and $B$ is self-adjoint.

In the end, we assume other notions and results on both bounded and unbounded operators (some general textbooks are [3], [15] and [16]). In particular, the reader should be aware that invertible operators are taken to have an everywhere bounded inverse, and that if $A$ and $B$ are two densely defined unbounded operators, then

$$(\lambda A)B = A(\lambda B) = \lambda(AB)$$

whenever $\lambda \in \mathbb{C}^*$.

2. Main Results

2.1. The Bounded Case.

Definition 2.1. Two bounded operators $A$ and $B$ are said to to commute up to a factor $\lambda$ (or $\lambda$-commute) if $AB = \lambda BA$ for some complex $\lambda$.

As alluded to in the introduction, we take on the problem of commutativity up to a factor differently, that is, we first prove that the product of two bounded normal $\lambda$-commuting operators is normal. We have

Theorem 2.1. Let $A$ and $B$ be two bounded normal such that $AB = \lambda BA \neq 0$ where $\lambda \in \mathbb{C}$. Then $AB$ (and also $BA$) is normal for any $\lambda$. 

Proof. Since $A$ and $B$ are both normal, so are $\lambda A$ and $\lambda B$. So by Theorem 1.2 we have

$$AB = \lambda BA \implies AB^* = \overline{\lambda} B^* A$$

and $A^* B = \overline{\lambda} B A^*$. Then we have on the one hand

$$(AB)^* AB = B^* A^* AB = B^* AA^* B = |\lambda|^2 B^* BAA^*.$$ 

On the other hand we obtain

$$AB(AB)^* = ABB^* A^* = AB^* BA^* = |\lambda|^2 B^* BAA^*.$$ 

Thus $AB$ is normal. 

\[ \Box \]

Corollary 2.1. Let $A$ and $B$ be two self-adjoint operators verifying $AB = \lambda BA \neq 0$ where $\lambda \in \mathbb{C}$. If either $\sigma(A) \cap \sigma(-A) \subseteq \{0\}$ or $\sigma(B) \cap \sigma(-B) \subseteq \{0\}$, then $\lambda = 1$.

Proof. By Theorem 2.1 $AB$ (or $BA$) is normal. By Theorem 1.3 $AB$ (or $BA$) is self-adjoint. Hence

$$BA = (AB)^* = AB = \lambda BA,$$

yielding $\lambda = 1$. The proof is thus complete. \[ \Box \]

Corollary 2.2. Let $A$ and $B$ be two self-adjoint operators verifying $AB = \lambda BA \neq 0$ where $\lambda \in \mathbb{C}$. Then $\lambda = 1$ if one of the following occurs:

1. $A$ is positive;
2. $-A$ is positive;
3. $B$ is positive;
4. $-B$ is positive.

Corollary 2.3. If both $A$ and $B$ are self-adjoint, then

$$AB = \lambda BA \implies \lambda \in \{-1, 1\}.$$

Proof. We have

$$A^2 B = \lambda ABA = \lambda^2 BA^2.$$ 

Since $A$ is self-adjoint, $A^2$ is positive so that Corollary 2.2 gives $\lambda^2 = 1$ or $\lambda \in \{-1, 1\}$. \[ \Box \]

2.2. A digression.

Definition 2.2. Two bounded operators $A$ and $B$ are said to commute unitarily if $AB = UBA$ for some unitary operator $U$.

Proposition 2.1. Let $A$ and $B$ be two bounded normal operators such that $AB = UBA$ for some unitary operator $U$. If $U$ commutes with $B$, then $UB$ and $AB$ are both normal.

We omit the proof as it is very similar to that of Theorem 2.1 and hence we leave it to the interested reader.
2.3. The Unbounded Case. We may split the main result in this subsection into two parts for the first one of the two is important in its own right. It also generalizes known results for two normal operators (where at least one of them is bounded, see e.g. \[6\] and \[12\]).

**Theorem 2.2.** Let $A$ and $B$ be two normal operators where only $B$ is bounded. Assume that $BA \subset \lambda AB \neq 0$ where $\lambda \in \mathbb{C}$. Then $AB$ is normal iff $|\lambda| = 1$.

**Proof.** First, and since $A$ is closed and $B$ is bounded, $AB$ is automatically closed.

Since $A$ is normal, so is $\lambda A$. Hence the Fuglede-Putnam (see e.g. \[3\]) theorem gives

$$BA \subset \lambda AB \implies BA^* \subset \lambda A^*B \text{ or } \lambda B^*A \subset AB^*.$$  

Using the above "inclusions" we have on the one hand

$$(AB)^*AB \supset B^*A^*AB$$

$$= B^*AA^*B \text{ (since } A \text{ is normal)}$$

$$\supset \frac{1}{\lambda} B^*ABA^*$$

$$\supset \frac{1}{\lambda} B^*BAA^*$$

$$= \frac{1}{|\lambda|^2} B^*BAA^*.$$  

Since $A$ and $AB$ are closed, and $B$ is bounded, all of $(AB)^*AB$, $A^*A$ and $B^*B$ are self-adjoint so that "adjointing" the previous inclusion yields

$$(AB)^*AB \subset \frac{1}{|\lambda|^2} AA^*B^*B.$$  

As $|\lambda|$ is real, the conditions of Proposition \[1\] are met and we finally obtain

$$(1) \quad (AB)^*AB = \frac{1}{|\lambda|^2} AA^*B^*B.$$  

On the other hand, we may write

$$AB(AB)^* \supset ABB^*A^*$$

$$= AB^*B^*A^* \text{ (because } B \text{ is normal)}$$

$$\supset \lambda B^*ABA^*$$

$$= B^*(\lambda AB)A^*$$

$$\supset B^*BAA^*.$$  

As above, we obtain

$$AA^*B^*B \supset AB(AB)^*$$

and by Proposition \[1\] we end up with

$$(2) \quad AB(AB)^* = AA^*B^*B.$$  

Accordingly, we clearly see that $AB$ is normal iff $|\lambda| = 1$, completing the proof. \[\square\]

**Corollary 2.4.** Let $A$ and $B$ be two normal operators where only $B$ is bounded. If $BA \subset \lambda AB \neq 0$ where $|\lambda| = 1$, then

$$BA = \lambda AB,$$

where $BA$ denotes the closure of the operator $BA$. 
Proof. Since $B$ is bounded, $(BA)^* = AB$ so that $(BA)^*$ is normal. Whence $(BA)^* = BA$ too is normal. Now, by Theorem 2.2, $AB$ is normal and so is $\lambda AB$. As normal operators are maximally normal, we get that $BA = \lambda AB$, establishing the result. □

Proposition 2.2. Let $A$ and $B$ be two self-adjoint operators where only $B$ is bounded. Assume that $BA \subset \lambda AB \neq 0$ where $\lambda \in \mathbb{C}$. Then $AB$ is normal for any $\lambda$.

Proof. Since $A$ and $B$ are self-adjoint, $BA \subset \lambda AB$ implies the following three "inclusions"

$$BA \subset \lambda AB, \lambda BA \subset AB \text{ and } \lambda BA \subset AB.$$ 

Proceeding as in the proof of Theorem 2.2, we obtain

$$(AB)^*AB \supset |\lambda|^2B^2A^2 \text{ and } AB(AB)^* \supset |\lambda|^2B^2A^2.$$ 

Again as in the proof of Theorem 2.2, $AB$ is normal. □

Now we apply the foregoing results to give spectral properties of $\lambda$-commuting self-adjoint operators (in the unbounded case).

Corollary 2.5. Let $A$ and $B$ be self-adjoint operators where only $B$ is bounded. Assume that $BA \subset \lambda AB \neq 0$ where $\lambda \in \mathbb{C}$. If further $\sigma(B) \cap \sigma(-B) \subseteq \{0\}$, then $\lambda = 1$.

Remark. The previous result generalizes 2) of Theorem 1.6. Besides the proof of Theorem 1.6 contained a small error which, by the present result, has now been fixed.

Proof. By Proposition 2.2, $AB$ is normal. Thanks to the condition on the spectrum of $B$ and Theorem 1.6, we get that $AB$ is self-adjoint. Hence $(AB)^* = AB$ so that $AB = (AB)^* \subset \frac{1}{\lambda}AB$.

But $D(AB) = D(\alpha AB)$ for any $\alpha \neq 0$. Therefore, $AB = \frac{1}{\lambda}AB$ or merely $\lambda = 1$. □

Corollary 2.6. Let $A$ and $B$ be self-adjoint operators where only $B$ is bounded. Assume that $BA \subset \lambda AB \neq 0$ where $\lambda \in \mathbb{C}$. Then $\lambda = 1$ if $B$ (or $-B$) is positive.

Corollary 2.7. Let $A$ and $B$ be self-adjoint operators where only $B$ is bounded. Assume that $BA \subset \lambda AB \neq 0$ where $\lambda \in \mathbb{C}$. Then $\lambda \in \{-1, 1\}$.

Proof. We may write $B^2A \subset \lambda BAB \subset \lambda^2AB^2$.

Since $B$ is self-adjoint, $B^2$ is positive so that Corollary 2.6 applies and gives $\lambda^2 = 1$ or $\lambda \in \{-1, 1\}$. □

We finish this paper with the case of two unbounded operators. As should have been expected, this case is quite delicate to handle unless strong assumptions are made. But first, we start by a version of the Fuglede-Putnam theorem.
Theorem 2.3. Let $A$, $N$ and $M$ three unbounded invertible operators on a Hilbert space such that $N$ and $M$ are normal. If $AN = MA$, then

$$A^*N = MA^*$$

and

$$AN^* = M^*A.$$ 

Proof. We have

$$AN = MA \Rightarrow A^{-1}M \subset NA^{-1}.$$ 

Since $A^{-1}$ is bounded, by Theorem 1.2 we have $A^{-1}M^* \subset N^*A^{-1}$ and hence

$$M^*A \subset AN^*.$$ 

By taking adjoints (and applying Lemma 1.1) we obtain

$$NA^* \subset A^*M.$$ 

Now since $A^{-1}M \subset NA^{-1}$, we can get $N^{-1}A^{-1} \subset A^{-1}M^{-1}$. But all operators (in the previous inclusion) are bounded. Therefore, we get

$$N^{-1}A^{-1} = A^{-1}M^{-1}.$$ 

Applying the Fuglede-Putnam theorem (the all-bounded-operators version) once more yields

$$(N^{-1})^*A^{-1} = A^{-1}(M^{-1})^*.$$ 

Whence

$$(M^{-1})^*A \subset A(N^{-1})^* \text{ or } AN^* \subset M^*A.$$ 

By Lemma 1.1 again, $A^*M \subset NA^*$. Thus $AN^* = M^*A$ and $A^*N = MA^*$, establishing the result. \qed

Corollary 2.8. If $A$ and $B$ are two unbounded normal and invertible operators such that $AB = \lambda BA$, then

$$A^*B = \lambda BA^*$$

or

$$AB^* = \lambda B^*A.$$ 

With Lemma 1.1 and Theorem 2.3 in hand, we may just mimic the proof of Yang-Du (in [17]) to prove the following result:

Theorem 2.4. Let $A$, $B$ two unbounded invertible operators such that $AB = \lambda BA \neq 0$, $\lambda \in \mathbb{C}$. Then

(1) If $A$ or $B$ is self-adjoint, then $\lambda \in \mathbb{R}$.

(2) If either $A$ or $B$ is self-adjoint and the other is normal, then $\lambda \in \{-1, 1\}$.

(3) If both $A$ and $B$ are normal, then $|\lambda| = 1$.

3. A Conjecture

In Corollary 2.5 (for example), we said nothing about the spectrum of $A$. This is in fact due to an (a natural) open question from [8] which the first author of this paper has been working on lately. Let us state it as a conjecture:

Conjecture. Let $A$ and $B$ be two self-adjoint operators such that only $B$ is bounded and $A$ is positive. Then $AB$ is self-adjoint whenever $AB$ is normal.

Neither a proof nor a counterexample have been reached yet. However, we can state the following:
(1) The "counterexample": \( A = -f'' \) on \( H^2(\mathbb{R}) \) (the Sobolev space) and \( B \) a multiplication operator by an essentially bounded real-valued function \( \varphi \) on \( \mathbb{R} \) does not work for the conditions of the conjecture will force \( \varphi \) to vanish (and so \( AB = 0 \)).

(2) The conjecture is true with \( BA \) in lieu of \( AB \). But, the normality of \( BA \) is stronger than that of \( AB \) because \( BA \) normal will then imply that \( AB \) is normal too, and \( AB = BA \)!

(3) The conjecture is true if one assumes further that \( BA \) is closed (in such case this will follow from the previous point).

(4) The conjecture seems to be a hard one. Indeed, the Fuglede (-Putnam) theorem is the tool par excellence when dealing with products involving normal (bounded or unbounded) operators. However, none of the known versions of the Fuglede theorem (such as [5], [8] and [11]) helps us in the proof to get \( BA \subset AB \), a sufficient condition to make \( AB \) is self-adjoint.

References

1. J. A. Brooke, P. Busch, D. B. Pearson, Commutativity up to a factor of bounded operators in complex Hilbert space, R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci., 458/2017 (2002) 109-118.
2. M. Cho, J. I. Lee, T. Yamazaki, On the operator equation \( AB = zBA \), Sci. Math. Jpn., 69/2 (2009), 257-263.
3. J. B. Conway, A Course in Functional Analysis, Springer, 1990 (2nd edition).
4. A. Devinatz, A. E. Nussbaum, J. von Neumann, On the Permutability of Self-adjoint Operators, Ann. of Math. (2), 62 (1955) 199-203.
5. B. Fuglede, A Commutativity Theorem for Normal Operators, Proc. Nati. Acad. Sci., 36 (1950) 35-40.
6. K. Gustafson, M. H. Mortad, Unbounded Products of Operators and Connections to Dirac-Type Operators, Bull. Sci. Math., (to appear). DOI: 10.1016/j.bulsci.2013.10.007
7. J. M. Khalagai, M. Kavila, On spectral properties of \( \lambda \)-commuting operators in Hilbert spaces, Int. Electron. J. Pure Appl. Math., 6/4 (2013), 283-291.
8. M. H. Mortad, An application of the Putnam-Fuglede Theorem to normal products of self-adjoint operators, Proc. Amer. Math. Soc. 131 (2003), 3135-3141.
9. M. H. Mortad, On some product of two unbounded self-adjoint operators, Integral Equations Operator Theory 64 (2009), 399-408.
10. M. H. Mortad, Commutativity up to a Factor: More Results and the Unbounded Case, Z. Anal. Anwendungen: Journal for Analysis and its Applications, 29/3 (2010), 303-307.
11. M. H. Mortad, An All-Unbounded-Operator Version of the Fuglede-Putnam Theorem, Complex Anal. Oper. Theory, 6/6 (2012), 1269-1273. DOI: 10.1007/s11785-011-0133-6.
12. M. H. Mortad, On the closedness, the self-adjointness and the normality of the product of two unbounded operators, Demonstratio Math., 45/1 (2012), 161-167.
13. M. H. Mortad, On the Normality of the Unbounded Product of Two Normal Operators, Concrete Operators, 1 (2012), 11-18. DOI: 10.2478/conop-2012-0002.
14. M. H. Mortad, Commutativity of Unbounded Normal and Self-adjoint Operators and Applications, Operators and Matrices, (to appear).
15. W. Rudin, Functional Analysis, McGraw-Hill, 1991 (2nd edition).
16. J. Weidmann, Linear operators in Hilbert spaces (translated from the German by J. Szücs), Springer-Verlag, GTM 68 (1980).
17. J. Yang, Hong-Ke Du, A Note on Commutativity up to a Factor of Bounded Operators, Proc. Amer. Math. Soc., 132/6 (2004) 1713-1720.
Department of Mathematics, University of Oran, B.P. 1524, El Menouar, Oran 31000, Algeria.

Mailing address (for the corresponding author):
Prof. Dr. Mohammed Hichem Mortad
BP 7085 Es-Seddikia
Oran
31013
Algeria

E-mail address: mhmortad@gmail.com.
E-mail address: chchellali@gmail.com.