The partition dimension of cycle books graph

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**Abstract.** Let \(G\) be a nontrivial and connected graph with vertex set \(V(G)\), edge set \(E(G)\) and \(S \subseteq V(G)\) with \(v \in V(G)\), the distance between \(v\) and \(S\) is \(d(v, S) = \min\{d(v, x)|x \in S\}\). For an ordered partition \(\Pi = \{S_1, S_2, S_3, ..., S_k\}\) of \(V(G)\), the representation of \(v\) with respect to \(\Pi\) is defined by \(r(v|\Pi) = (d(v, S_1), d(v, S_2), ..., d(v, S_k))\). The partition \(\Pi\) is called a resolving partition of \(G\) if all representations of vertices are distinct. The partition dimension \(pd(G)\) is the smallest integer \(k\) such that \(G\) has a resolving partition set with \(k\) members. In this research, we will determine the partition dimension of Cycle Books \(B_{C_r, m}\). Cycle books graph \(B_{C_r, m}\) is a graph consisting of \(m\) copies cycle \(C_r\) with the common path \(P_2\). It is shown that the partition dimension of cycle books graph, \(pd(B_{C_r, m})\), is 3 for \(m = 2, 3, \) and \(m \geq 4\). \(pd(B_{C_r, m})\) is 3 + 2\(k\) for \(m = 3k + 2\), 4 + 2\((k - 1)\) for \(m = 3k + 1\), and 3 + 2\((k - 1)\) for \(m = 3k\). \(pd(B_{C_r, m})\) is \(m + 1\).

1. Introduction

One of interesting topic in discrete mathematics is graph theory. One of the topics in graph theory is the dimension metric and partition dimension. Chartrand et.al [1] introduced the concept of partition dimension of \(G\). They introduced the same concept of resolving partition with term partition dimension of graph denoted by \(pd(G)\). For \(S \subseteq V(G)\) with vertex \(v \in V(G)\), the distance between \(v\) and \(S\) is \(d(v, S) = \min\{d(v, x)|x \in S\}\).

Given \(k\) partition \(\Pi = \{S_1, S_2, S_3, ..., S_k\}\) of \(V(G)\) and vertex \(v \in V(G)\). Representation of \(v\) with respect to \(\Pi\) defined by \(r(v|\Pi) = (d(v, S_1), d(v, S_2), ..., d(v, S_k))\) for every \(v \in V(G)\). The minimum of \(k\) such that resolving partition \(\Pi\) is partition dimension of \(G\), denoted by \(pd(G)\). In this research, we will determine the partition dimension of cycle books \(B_{C_r, m}\).

2. Preliminaries

Cycle books graph \(B_{C_r, m}\) is a graph consisting of \(m\) copies cycle \(C_r\) with the common path \(P_2\). In his paper, Ayhan, et al.[2] has discussed the dominating numbers of the book graph \(B_m\). Prasetyo [3] has discussed the metric dimension and partition dimension of stacked book graph \(B_{3, 3}\). Grigorious, et al.[4] has discussed the partition dimension of a class of circulant graphs. Amrullah et al.[5] has shown that partition dimension of a subdivision of a complete graph. Haryeni et al.[6] has shown that partition dimension of some classes of homogenous disconnected graph.

Useful properties in determining \(pd(G)\) are given in the following Lemma 2.1 and Theorema 2.2.

**Lemma 2.1** [7] Let \(\Pi\) be a resolving partition of \(G\) and \(u, v \in V(G)\). If \(d(u, w) = d(v, w)\) for all \(w \in V(G) - \{u, v\}\), then \(u\) and \(v\) belong to distinct elements of \(\Pi\).
3.1. The partition dimension of cycle books graph

In this section, we will show the partition dimension of cycle books graph.

3. Result and discussions

In this section, we will show the partition dimension of cycle books graph.

3.1. The partition dimension of cycle books graph $B_{C_3,m}$

Cycle books graph $B_{C_3,m}$ is a graph consisting of $m$ copies cycle $C_3$ with the common path $P_2$.

![Figure 1. cycle book graph $B_{C_3,m}$](image)

**Theorem 3.1** Let $G$ be cycle books graph $B_{C_3,m}$ for $m = 2, 3, 4, 5, ...$, then

$$pd(G) = \begin{cases} 
3 & \text{if } m = 2, 3 \\
 m & \text{if } m \geq 4
\end{cases}$$

**Proof:** Consider three cases as follows:

(i) For $m = 2$

Let $\Pi = \{S_1, S_2, S_3\}$ be a partition of $V(G)$ with $S_1 = \{v_{c_1}, v_1\}$, $S_2 = \{v_{c_2}\}$, and $S_3 = \{v_2\}$. For $V(G)$ will be shown that all vertices $G$ have distinct representation with respect to $\Pi$. The representation of all vertices of $G$ are as follows:

$r(v_{c_1}|\Pi) = (0, 1, 1) \quad r(v_1|\Pi) = (0, 1, 2)$

$r(v_{c_2}|\Pi) = (1, 0, 1) \quad r(v_2|\Pi) = (1, 1, 0)$

Since all vertices of $G$ have distinct representation. So, $\Pi = \{S_1, S_2, S_3\}$ is resolving partition of $V(G)$ then $pd(G) \leq 3$. By Theorem 2.2.(i), we have lower bound $pd(G) \geq 3$. So we get $pd(G) = 3$.

(ii) For $m = 3$

Let $\Pi = \{S_1, S_2, S_3\}$ be a partition of $V(G)$ with $S_1 = \{v_{c_1}, v_1\}$, $S_2 = \{v_{c_2}, v_2\}$, and $S_3 = \{v_3\}$. For $V(G)$ will be shown that all vertices $G$ have distinct representation with respect to $\Pi$. The representation of all vertices of $G$ are as follows:

$r(v_{c_1}|\Pi) = (0, 1, 1) \quad r(v_1|\Pi) = (0, 1, 2)$

$r(v_{c_2}|\Pi) = (1, 0, 1) \quad r(v_2|\Pi) = (1, 1, 0)$

$r(v_3|\Pi) = (1, 0, 2)$

Since all vertices of $G$ have distinct representation. So, $\Pi = \{S_1, S_2, S_3\}$ is resolving partition of $V(G)$ then $pd(G) \leq 3$. By Theorem 2.2.(i), we have lower bound $pd(G) \geq 3$. So we get $pd(G) = 3$. 

\[\text{Figure 1. cycle book graph } B_{C_3,m}\]
(iii) For $m \geq 4$.
Let $\Pi = \{S_1, S_2, S_3, ..., S_m\}$ be a partition of $V(G)$ with $S_1 = \{v_{c1}, v_1\}$, $S_2 = \{v_{c2}, v_2\}$, $S_j = \{v_j | 3 \leq j \leq m\}$ will be shown that all vertices $G$ have distinct representation with respect to $\Pi$. The representation of all vertices of $G$ are as follows:

- $r(v_{c1}|\Pi) = (0, 1, ..., 1)$
- $r(v_{c2}|\Pi) = (1, 0, 1, ..., 1)$
- $r(v_1|\Pi) = (0, 1, 2, ..., 2)$
- $r(v_2|\Pi) = (1, 0, 2, ..., 2)$
- $r(v_3|\Pi) = (1, 1, 0, 2, ..., 2)$
- $r(v_j|\Pi) = (1, 1, 2, ..., 2, 2, ..., 2); 4 \leq j \leq m$

Since all vertices of $G$ have distinct representation. So, $\Pi = \{S_1, S_2, S_3, ..., S_m\}$ is resolving partition of $V(G)$ then $pd(G) \leq m$.

The lower bound of the partition dimension in cycle books graph $BC_{3,m}$ can be obtained by using Lemma 2.1. According to Lemma 2.1 vertex in common path $v_{c1}$ and $v_{c2}$ must be in a different partition class, and other vertices are $v_j; 1 \leq j \leq m$ must be in a different partition class. Therefore, by placing the vertex $v_{c1}$ and one other vertex (which is not $v_{c2}$) in a partition class (say $S_1$), and placing the vertex $v_{c2}$ and one vertex others are in a partition class (say $S_2$), then the other vertex are each placed in a different partition class (say $S_3, S_4, S_5, ..., S_m$), then obtained the construction of the minimum resolving partition with cardinality $m$ or $pd(BC_{3,m}) \geq m$. So we get $pd(G) = m$.

3.2. The partition dimension of cycle books graph $BC_{4,m}$

Cycle books graph $BC_{4,m}$ is a graph consisting of $m$ copies cycle $C_4$ with the common path $P_2$.

![Figure 2. cycle books graph $BC_{4,m}$](image)

**Theorem 3.2** Let $G$ is cycle books graph $BC_{4,m}$ for $m = 2, 3, 4, 5, ...$, then

$$pd(G) = \begin{cases} 
3 + 2k & \text{if } m = 3k + 2 \\
4 + 2(k - 1) & \text{if } m = 3k + 1 \\
3 + 2(k - 1) & \text{if } m = 3k 
\end{cases}$$

with $k = 0, 1, 2, 3, ...$

**Proof:** Consider three cases as follows:
(i) For $m = 3k$.
Let $\Pi = \{S_1, S_2, S_3, \ldots, S_{4+2(k-1)}\}$ be a partition of $V(G)$ with $S_1 = \{v_1, v_2, v_3\}$, $S_2 = \{v_4, v_5, v_6\}$, $S_3 = \{v_7, v_8, v_9\}$, ..., $S_{4+2(k-1)} = \{v_{2m-2}, v_{2m-1}, v_{2m}\}$. For $V(G)$ will be shown that all vertices $G$ have distinct representation with respect to $\Pi$. The representation of all vertices of $G$ are as follows:
$$r(v_1|\Pi) = (0, 1, 1, 1, 1, \ldots, 1, 1)$$
$$r(v_2|\Pi) = (0, 1, 2, 2, 2, \ldots, 2, 2)$$
$$r(v_3|\Pi) = (1, 0, 1, 2, 2, \ldots, 2, 2)$$
$$r(v_4|\Pi) = (0, 1, 2, 2, 2, \ldots, 2, 2)$$
$$r(v_5|\Pi) = (0, 1, 2, 2, 2, \ldots, 2, 2)$$
$$r(v_6|\Pi) = (0, 1, 2, 2, 2, \ldots, 2, 2)$$
$$r(v_7|\Pi) = (0, 1, 2, 2, 2, \ldots, 2, 2)$$
$$r(v_8|\Pi) = (0, 1, 2, 2, 2, \ldots, 2, 2)$$
$$r(v_9|\Pi) = (0, 1, 2, 2, 2, \ldots, 2, 2)$$
$$r(v_{10}|\Pi) = (2, 1, 2, 1, 0, \ldots, 2, 2)$$
$$r(v_{2m-2}|\Pi) = (2, 1, 2, 2, 2, \ldots, 1, 0)$$
$$r(v_{2m-1}|\Pi) = (2, 1, 2, 2, 2, \ldots, 2, 0)$$
$$r(v_{2m}|\Pi) = (2, 1, 2, 2, 2, \ldots, 2, 0)$$
Since all vertices of $G$ have distinct representation. So, $\Pi = \{S_1, S_2, S_3, \ldots, S_{4+2(k-1)}\}$ is resolving partition of $V(G)$ then $pd(G) \leq 3 + 2(k-1)$.

The lower bound of the partition dimension, will be shown that the resolving partition of graph $B_{C_4,m}$ has cardinality less than $3 + 2(k-1)$, say $2 + 2(k-1)$. Without loss of generality, let $\Pi = \{S_1, S_2, S_3, \ldots, S_{2+2(k-1)}\}$ be a partition of $V(G)$ with $S_1 = \{v_1, v_2, v_3\}$, $S_2 = \{v_4, v_5, v_6\}$, $S_3 = \{v_7, v_8, v_9\}$, ..., $S_{2+2(k-1)} = \{v_{2m-5}, v_{2m-4}, v_{2m-3}, v_{2m-2}, v_{2m-1}, v_{2m}\}$. Then, we obtain $r(v_{2m-5}|\Pi) = r(v_{2m-4}|\Pi) = r(v_{2m-3}|\Pi) = r(v_{2m}|\Pi)$. So $\Pi = \{S_1, S_2, S_3, \ldots, S_{2+2(k-1)}\}$ is not a resolving partition of $V(G)$ then $pd(G) \leq 3 + 2(k-1)$. So we get $pd(G) = 3 + 2(k-1)$.

(ii) For $m = 3k + 1$.
Let $\Pi = \{S_1, S_2, S_3, \ldots, S_{4+2(k-1)}\}$ be a partition of $V(G)$ with $S_1 = \{v_1, v_2, v_3\}$, $S_2 = \{v_4, v_5, v_6\}$, $S_3 = \{v_7, v_8, v_9\}$, ..., $S_{4+2(k-1)} = \{v_{2m-1}, v_{2m}\}$. For $V(G)$ will be shown that all vertices $G$ have distinct representation with respect to $\Pi$. The representation of all vertices of $G$ are as follows:
$$r(v_1|\Pi) = (0, 1, 1, 1, 1, \ldots, 1, 1)$$
$$r(v_2|\Pi) = (0, 1, 2, 2, 2, \ldots, 2, 2)$$
$$r(v_3|\Pi) = (0, 1, 2, 2, 2, \ldots, 2, 2)$$
$$r(v_4|\Pi) = (0, 1, 2, 2, 2, \ldots, 2, 2)$$
$$r(v_5|\Pi) = (0, 1, 2, 2, 2, \ldots, 2, 2)$$
$$r(v_6|\Pi) = (0, 1, 2, 2, 2, \ldots, 2, 2)$$
$$r(v_7|\Pi) = (0, 1, 2, 2, 2, \ldots, 2, 2)$$
$$r(v_8|\Pi) = (0, 1, 2, 2, 2, \ldots, 2, 2)$$
$$r(v_9|\Pi) = (0, 1, 2, 2, 2, \ldots, 2, 2)$$
$$r(v_{10}|\Pi) = (2, 1, 2, 1, 0, \ldots, 2, 2)$$
$$r(v_{2m-1}|\Pi) = (2, 1, 2, 2, 2, \ldots, 2, 0)$$
Since all vertices of $G$ have distinct representation. So, $\Pi = \{S_1, S_2, S_3, \ldots, S_{4+2(k-1)}\}$ is resolving partition of $V(G)$ then $pd(G) \leq 4 + 2(k - 1)$.

The lower bound of the partition dimension, will be shown that the resolving partition of graph $B_{C_{4,m}}$ has cardinality less than $4 + 2(k - 1)$, say $3 + 2(k - 1)$ . Without loss of generality, let $\Pi = \{S_1, S_2, S_3, \ldots, S_{3+2(k-1)}\}$ be a partition of $V(G)$ with $S_1 = \{v_{c1}, v_1, v_2\}$, $S_2 = \{v_{c2}, v_3\}$, $S_3 = \{v_4, v_5, v_6\}$, $S_4 = \{v_7, v_8, v_9\}$, ..., $S_{3+2(k-1)} = \{v_{2m-4}, v_{2m-2}, v_{2m-1}, v_{2m}\}$. Then, we obtain $r(v_{2m-4}\Pi) = r(v_{2m-3}\Pi) = r(v_{2m-1}\Pi)$. Let $\Pi = \{S_1, S_2, S_3, \ldots, S_{3+2(k-1)}\}$ be a partition of $V(G)$ with $S_1 = \{v_{c1}, v_1, v_2\}$, $S_2 = \{v_{c2}, v_3\}$, $S_3 = \{v_4, v_5, v_6\}$, ..., $S_{3+2k} = \{v_{2m}\}$. For $V(G)$ will be shown that all vertices $G$ have distinct representation with respect to $\Pi$. The representation of all vertices of $G$ are as follows:

$r(v_{c1}\Pi) = (0, 1, 1, 1, 1, \ldots, 1, 1, 1)$
$r(v_1\Pi) = (0, 1, 2, 2, 2, \ldots, 2, 2, 3)$
$r(v_2\Pi) = (0, 2, 1, 2, 2, \ldots, 2, 2, 3)$
$r(v_{c2}\Pi) = (1, 0, 1, 1, 1, \ldots, 1, 1, 1)$
$r(v_3\Pi) = (1, 0, 2, 2, 2, \ldots, 2, 2, 2)$
$r(v_4\Pi) = (1, 1, 0, 2, 2, \ldots, 2, 2, 2)$
$r(v_5\Pi) = (2, 1, 0, 2, 2, \ldots, 2, 2, 2)$
$r(v_6\Pi) = (1, 2, 0, 2, 2, \ldots, 2, 2, 2)$
$r(v_7\Pi) = (2, 1, 2, 0, 2, \ldots, 2, 2, 2)$
$r(v_8\Pi) = (1, 2, 2, 0, 2, \ldots, 2, 2, 3)$
$r(v_9\Pi) = (1, 2, 2, 0, 1, \ldots, 2, 2, 3)$

$\vdots$

$r(v_{2m}\Pi) = (2, 1, 2, 2, 2, \ldots, 1, 0)$

Since all vertices of $G$ have distinct representation. So, $\Pi = \{S_1, S_2, S_3, \ldots, S_{3+2k}\}$ is resolving partition of $V(G)$ then $pd(G) \leq 3 + 2k$.

The lower bound of the partition dimension of cycle books graph $B_{C_{4,m}}$ will be shown that the resolving partition of graph $B_{C_{4,m}}$ has cardinality less than $3 + 2k$, say $2 + 2k$ . Without loss of generality, let $\Pi = \{S_1, S_2, S_3, \ldots, S_{2+2k}\}$ be a partition of $V(G)$ with $S_1 = \{v_{c1}, v_1, v_2\}$, $S_2 = \{v_{c2}, v_3\}$, $S_3 = \{v_4, v_5, v_6\}$, ..., $S_{2+2k} = \{v_{2m-3}, v_{2m-2}, v_{2m-1}, v_{2m}\}$. Then, we obtain $r(v_{2m-3}\Pi) = r(v_{2m}\Pi), r(v_{2m-2}\Pi) = r(v_{2m-1}\Pi)$. So $\Pi = \{S_1, S_2, S_3, \ldots, S_{2+2k}\}$ is not a resolving partition of $V(G)$ then $pd(G) \geq 3 + 2k$. So we get $pd(G) = 3 + 2k$.

### 3.3. The partition dimension of cycle books graph $B_{C_{5,m}}$

Cycle books graph $B_{C_{5,m}}$ is a graph consisting of $m$ copies Cycle $C_5$ with the common path $P_2$.

Cycle books graph $B_{C_{5,m}}$ has $3m + 2$ vertices (say $v_{c1}, v_{c2}, v_1, v_2, \ldots, v_{3m}$).
Theorem 3.3 Let $G$ is cycle books graph $B_{C_5,m}$ for $m = 2, 3, 4, 5, \ldots$, then $pd(G) = m + 1$

Proof: The upper bound of the partition dimension of cycle books graph $B_{C_5,m}$ can be obtained by constructing a resolving partition $\Pi$. Let $\Pi = \{S_1, S_2, S_3, \ldots, S_m, S_{m+1}\}$ be a partition of $V(G)$ with $S_1 = \{v_{c1}, v_1, v_2\}$, $S_2 = \{v_{c2}, v_3\}$, $S_3 = \{v_4, v_5, v_6\}$, ..., $S_m = \{v_{3m-5}, v_{3m-4}, v_{3m-3}, v_{3m-2}, v_{3m-1}, v_{3m}\}$. For $V(G)$ will be shown that all vertices $G$ have distinct representation with respect to $\Pi$. The representation of all vertices of $G$ are as follows:

$$r(v_{c1})|\Pi) = (0, 1, 1, 1, 1, \ldots, 1, 1)$$
$$r(v_1)\Pi) = (0, 2, 2, 2, 2, \ldots, 2, 2)$$
$$r(v_2)\Pi) = (0, 1, 3, 3, 3, \ldots, 3, 3)$$
$$r(v_3)\Pi) = (1, 0, 1, 1, 1, \ldots, 1, 1)$$
$$r(v_4)\Pi) = (2, 1, 0, 2, 2, \ldots, 2, 2)$$
$$r(v_5)\Pi) = (2, 2, 0, 3, 3, \ldots, 3, 3)$$
$$r(v_6)\Pi) = (2, 1, 0, 3, 3, \ldots, 3, 3)$$
$$r(v_7)\Pi) = (2, 1, 2, 0, 2, \ldots, 2, 2)$$
$$r(v_8)\Pi) = (2, 2, 3, 0, 3, \ldots, 3, 3)$$
$$r(v_9)\Pi) = (1, 2, 2, 0, 2, \ldots, 2, 2)$$
$$r(v_{10})\Pi) = (2, 1, 2, 2, 0, \ldots, 2, 2)$$

Since all vertices of $G$ have distinct representation. So, $\Pi = \{S_1, S_2, S_3, \ldots, S_m, S_{m+1}\}$ is resolving partition of $V(G)$ then $pd(G) \leq m + 1$.

The lower bound of the partition dimension of cycle books graph $B_{C_5,m}$ will be shown that the resolving partition of graph $B_{C_5,m}$ has cardinality less than $m + 1$, say $m$. Without loss of generality, let $\Pi = \{S_1, S_2, S_3, \ldots, S_m\}$ be a partition of $V(G)$ with $S_1 = \{v_{c1}, v_1, v_2\}$, $S_2 = \{v_{c2}, v_3\}$, $S_3 = \{v_4, v_5, v_6\}$, ..., $S_m = \{v_{3m-5}, v_{3m-4}, v_{3m-3}, v_{3m-2}, v_{3m-1}, v_{3m}\}$. Then, we obtain $r(v_{3m-5})|\Pi) = r(v_{3m-4})|\Pi)$ and $r(v_{3m-3})|\Pi) = r(v_{3m})|\Pi)$. So $\Pi = \{S_1, S_2, S_3, \ldots, S_m\}$ is not a resolving partition of $V(G)$ then $pd(G) \geq m + 1$. So we get $pd(G) = m + 1$.

4. Conclusion
In this paper, we have the partition dimension of cycle books graf as follows: The partition dimension of cycle books graf, $pd(B_{C_4,m})$ is 3 for $m = 2$, and $m$ for $m \geq 3$. $pd(B_{C_4,m})$ is $3 + 2k$ for $m = 3k + 2$, $4 + 2(k - 1)$ for $m = 3k + 1$, and $3 + 2(k - 1)$ for $m = 3k$. $pd(B_{C_5,m})$ is $m + 1$.

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