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TOPOLOGICAL AND CARDINALITY PROPERTIES OF CERTAIN SETS OF CLASSES OF BANACH SPACES

EUGENE TOKAREV

Dedicated to the memory of S. Banach.

Abstract. Classes of Banach spaces that are finitely, strongly finitely or elementary equivalent are introduced. On sets of these classes topologies are defined in such a way that sets of defined classes become compact totally disconnected topological spaces. Results are used in the problem of synthesis of Banach spaces, and to describe omittable spaces that are defined below.

A general classification scheme, suggested by author for Banach spaces includes their partition by dimension (which may be regarded as horizontal strips of the proposed coordinate system) and a partition by the relation $\sim_f$ of finite equivalence (defined below), that consists of classes of finitely equivalent Banach spaces (and which may be regarded as vertical strips of the mentioned generalized coordinate system). That class of finite equivalence, which contains a given Banach space $X$ will be denoted by $X_f$. These classes may be further participated in a few ways. In the paper we consider two of such partition: by the relation $\approx$ of strong finite equivalence (that consists of classes $X^\phi$ of strong finite equivalence) and by the

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relation of elementary equivalence \(\equiv\), that generates a partition of the class \(\mathcal{B}\) of all
Banach spaces in classes \(X^\xi\) of pairwise elementary equivalent Banach spaces.

Certainly, the partition by dimension enjoys such nice property as well-ordering. If we fix a cardinal \(\kappa\), then the set \(\{\mathcal{B}_\tau : \tau \leq \kappa\}\) may be equipped with the order topology, under which it became compact.

Although other mentioned partitions are not well-ordered (in the partition into classes of elementary equivalence no order will be defined at all), on these ‘coordinate strips’ suitable topologies that turn them in topological compact spaces may be defined as well.

Dependence of properties of Banach spaces on their position in the classes \(X^f, X^\phi\) and \(X^\xi\) is studied in other papers of author. Here only cardinal and topological properties of the classification will be studied.

1. Classification of Banach spaces by dimension

1.1. Definitions. Ordinals will be denoted by small Greece letters \(\alpha, \beta, \gamma\). Cardinals are identified with the least ordinals of given cardinality and are denoted either by \(\iota, \tau, \kappa, \nu\), or by using Hebrew letter \(\mathfrak{j}\) (may be with indices). As usual, \(\omega\) and \(\omega_1\) denote respectively the first infinite and the first uncountable cardinal (= ordinal).

For a cardinal \(\tau\) its predecessor (i.e. the least cardinal, strongly greater then \(\tau\)) is denoted by \(\tau^+\).

The cofinality of \(\tau\), \(\text{cf}(\tau)\) is the least cardinality of a set \(A \subset \tau\) such that \(\tau = \sup A\).

Let \(A, B\) be sets. The symbol \(B^A\) denotes the set of all functions from \(B\) to \(A\).

In a general case, the cardinality of the set \(B^A\) is denoted either by \(\text{card}(A)^\text{card}(B)\) or by \(\kappa^\tau\), if \(\text{card}(A) = \kappa; \text{card}(B) = \tau\).

For an ordinal \(\gamma\) the cardinal \(\mathfrak{j}(\gamma)\) is given by induction:
\[
\mathfrak{j}(0) = \omega; \quad \mathfrak{j}(\alpha + 1) = 2^{\mathfrak{j}(\alpha)}; \quad \mathfrak{j}(\gamma) = \bigcup\{\mathfrak{j}(\alpha) : \alpha < \gamma\},
\]
if \(\gamma\) is a limit ordinal.

The symbol \(\exp(\tau)\) (or, equivalently, \(2^\tau\)) denotes the cardinality of the set \(\text{Pow}(\tau)\) of all subsets of \(\tau\).

The cardinal \(\exp(\omega)\) (i.e., the cardinality of continuum) will be also denoted by \(c\).

Let \(X\) be a Banach space. Below all Banach spaces will be considered over the field \(\mathbb{R}\) of real scalars.

Any set of elements \(\{w_\beta : \beta < \alpha\}\) (of arbitrary nature), indexed by ordinals \(\alpha\) will be called an \(\alpha\)-sequence.

An \(\alpha\)-sequence \(\{x_\beta : \beta < \alpha\}\) of elements of \(X\) is said to be

- **Spreading**, if for any \(n < \omega\), any \(\varepsilon > 0\), any scalars \(\{a_k : k < n\}\) and any choosing of \(i_0 < i_1 < \ldots < i_{n-1} < \alpha; j_0 < j_1 < \ldots < j_{n-1} < \alpha\) the following equality holds:

\[
\left\| \sum_{k<n} a_k x_{i_k} \right\| = \left\| \sum_{k<n} a_k x_{j_k} \right\|.
\]

- **Symmetric**, if for any \(n < \omega\), any finite subset \(I \subset \alpha\) of cardinality \(n\), any rearrangement \(\varsigma\) of elements of \(I\) and any scalars \(\{a_i : i \in I\}, \)

\[
\left\| \sum_{i \in I} a_i z_i \right\| = \left\| \sum_{i \in I} a_{\varsigma(i)} z_i \right\|.
\]
• **Subsymmetric**, if it is both spreading and 1-unconditional.

Let $C < \infty$ be a constant. Two $\alpha$-sequences $\{x_\beta : \beta < \alpha\}$ and $\{y_\beta : \beta < \alpha\}$ are said to be $C$-equivalent if for any finite subset $I = \{i_0 < i_1 < \ldots < i_{n-1}\}$ of $\alpha$ and for any choosing of scalars $\{a_k : k < n\}$

$$C^{-1} \left\| \sum_{k<n} a_k x_{i_k} \right\| \leq \left\| \sum_{k<n} a_k y_{i_k} \right\| \leq C \left\| \sum_{k<n} a_k x_{i_k} \right\|.$$

Two $\alpha$-sequences $\{x_\beta : \beta < \alpha\}$ and $\{y_\beta : \beta < \alpha\}$ are said to be equivalent if they are $C$-equivalent for some $C < \infty$.

### 1.2. Banach spaces spanned by subsymmetric sequences

It may be shown that if $r$ is not symmetric (resp., is not equivalent to a symmetric sequence) then the cardinality $\text{card}([r] \cap B_\sigma) = 2^\kappa$ (resp., the cardinality card $(([r] \cap B_\sigma)^\omega) = 2^\kappa$).

Let $\sigma$ be a cardinal; $\sigma : \sigma \rightarrow \sigma$ be a transposition (i.e. one-to-one mapping of $\sigma$ onto $\sigma$).

It will be said that $\sigma$ is almost identical if the correlation

$$\gamma_1 < \gamma_2 \Rightarrow \sigma \gamma_1 < \sigma \gamma_2,$$

where $\gamma_1, \gamma_2 < \sigma$, may get broken at most finitely many times.

**Lemma 1.** Let $r = \{x_n : n < \omega\}$ be a spreading sequence that is not symmetric.

Let $\sigma : \omega \rightarrow \omega$ be a transposition, which is not almost identical.

If sequences $\{x_n : n < \omega\}$ and $\{x_{\sigma n} : n < \omega\}$ are equivalent then both of them are equivalent to a symmetric sequences.

**Proof.** Consider a sequence $\{y_\alpha : \alpha < \omega^2\}$, which is given by

$$\{y_{\omega 2k+n} = x_n; \ y_{\omega (2k+1)+n} = x_{\sigma n} : k < \omega; \ n < \omega\}.$$

This sequence is equivalent to a sequence $\{y'_\alpha : \alpha < \omega^2\}$ that belongs to the tower $[r]$, generated by $r = \{x_n : n < \omega\}$.

Certainly, this is equivalent to

$$C^{-1} \left\| \sum_{k<n} a_k x_{i_k} \right\| < \left\| \sum_{k<n} a_k y_{\alpha_k} \right\| < C \left\| \sum_{k<n} a_k x_{i_k} \right\|$$

for every $n < \omega$; every scalars $(a_k)_{k<n}$; every choice $\alpha_0 < \alpha_1 < \ldots < \alpha_{n-1} < \omega^2$,

$$i_0 < i_1 < \ldots < i_{n-1} < \omega$$

and some $C < \infty$ that depends only on equivalence constant between $\{x_n : n < \omega\}$ and $\{x_{\sigma n} : n < \omega\}$. Since $\sigma$ has only a finite number of inversions, our definition of $\{y_\alpha : \alpha < \omega^2\}$ implies that $r$ is equivalent to a symmetric sequence.

**Theorem 1.** Let $\alpha, \beta$ be ordinals; $\omega \leq \beta \leq \alpha$. Let $\{x_\gamma : \gamma < \alpha\}$ be a subsymmetric $\alpha$-sequence which is not equivalent to any $\alpha$-symmetric sequence. Let $X_\alpha = \text{span}(\{x_\gamma : \gamma < \alpha\})$; $X_\beta = \text{span}(\{x_\gamma : \gamma < \beta\})$. If $\beta^\omega < \alpha$ then spaces $X_\alpha$ and $X_\beta$ are not isomorphic.

**Proof.** Assume that $X_\alpha$ is isomorphic to a subspace $Z$ of $X_\beta$.

Let $I : X_\alpha \rightarrow X_\beta$ be the corresponding operator of isomorphic embedding. Without loss of generality it may be assumed that an image of element $x_\gamma$ $(\gamma < \alpha)$ in $X_\beta$ is a finite linear combination with rational coefficients of some $x_\zeta$’s $(\zeta < \beta)$:

$$I x_\gamma = \sum_{k=0}^{n(\gamma)} a_k^\gamma x_{\zeta_k(\gamma)}; \ \ z_0(\gamma) < z_1(\gamma) < \ldots < z_{n(\gamma)}(\gamma) < \beta.$$

Thus, to any $x_\gamma$ corresponds a finite sequence of rational numbers $(a_k^\gamma)_{k=0}^{n(\gamma)}$. 

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Let \( (p_n)_{n<\omega} \) be a numeration of all finite sequences of rationals.

The set \( (p_n)_{n<\omega} \) generates a partition of \( \alpha \) on parts \( (P_n)_{n<\omega} \) in the following way:

an ordinal \( \gamma < \alpha \) belongs to \( P_n \) if \( (a_k^\gamma)_{k=0}^{n(\gamma)} = p_n \).

Since \( \alpha > \beta \), one of \( P_n \)'s must contain a sequence \( \{ \gamma_i : i < \delta \} \) of ordinals, which order type \( \delta \) (in a natural order: \( i < j \) implies that \( \gamma_i < \gamma_j \)) is greater than \( \beta^\omega \). Clearly, for all such \( \gamma_i \),

\[
Ix_{\gamma_i} = \sum_{k=0}^{n} a_k x_{\xi_k(\gamma_i)},
\]

where \( n \) and \( (a_k)_{k=1}^{\infty} \) do not depend on \( i \).

A set of all sequences \( \zeta_0 (\gamma_i) < \zeta_1 (\gamma_i) < ... < \zeta_n (\gamma_i) < \beta \) cannot be ordered to have the order type \( > \beta^\omega \).

Hence, conditions \( \gamma_{i_1} < \gamma_{i_2} \Rightarrow \zeta_0 (\gamma_{i_1}) < \zeta_0 (\gamma_{i_2}) \) must get broken for infinite many pairs \( \gamma_{i_1}, \gamma_{i_2} \). The inequality

\[
C^{-1} \left\| \sum_{k<m} b_k x_{\gamma_k} \right\| \leq \left\| \sum_{k<m} b_k \left( \sum_{k=0}^{n} a_k x_{\xi_k(\gamma_i)} \right) \right\| \leq C \left\| \sum_{k<m} b_k x_{\gamma_k} \right\| ,
\]

where \( C = d(X_\alpha, Z) \), shows that \( \{ x_\gamma : \gamma < \alpha \} \) is equivalent to a symmetric sequence. \( \square \)

**Theorem 2.** Let \( \mathcal{X} = \{ x_n : n < \omega \} \) be a subsymmetric sequence, which is not isomorphic to a symmetric one. Then for any cardinal \( \kappa \geq \omega \) there exists \( 2^\kappa \) pairwise non isomorphic Banach spaces of dimension \( \kappa \), which belongs to the same tower \( [\mathcal{X}] \).

**Proof.** Let \( \{ x_\gamma : \gamma < \mathcal{X} \} \) be a subsymmetric \( \mathcal{X} \)-sequence. Let \( I = \langle I, \ll \rangle \) be a linearly ordered set of cardinality \( \mathcal{X} \); \( J = \langle I, < \rangle \) – another linear ordering on \( I \).

Consider families \( \{ x_i : i \in I \} \) and \( \{ x_j : j \in J \} \) that are indexed (and ordered) by elements of \( I \) and \( J \) respectively.

Let \( X_I = \text{span}\{x_i : i \in I\} \); \( X_J = \text{span}\{x_j : j \in J\} \).

Certainly, \( X_I \) and \( X_J \) are isomorphic if only if there are one-to-one embeddings of embedding \( u : I \to J \) and \( w : J \to I \), which are almost monotone in a following sense:

\[
i_1 < i_2 \Rightarrow u(i_1) < u(i_2) \quad \text{for all but finitely many pairs } i_1, i_2 \in I;
\]

\[
 j_1 < j_2 \Rightarrow w(j_1) < w(j_2) \quad \text{for all but finitely many pairs } j_1, j_2 \in J.
\]

Since there exist \( 2^\kappa \) orderings of \( I \) for any pair of which such almost monotone mappings do not exist, this prove the theorem. \( \square \)

1.3. **Classes** \( \mathcal{B}_\kappa \). Let \( \mathcal{B} \) be a proper class of all Banach spaces (isometric spaces are identified). For any finite or infinite cardinal number \( \kappa \) let \( \mathcal{B}_\kappa \) be a set of all Banach spaces of dimension \( \kappa \):

\[
\mathcal{B}_\kappa = \{ X \in \mathcal{B} : \text{dim } X = \kappa \}
\]

Recall that a *dimension* \( \text{dim}(X) \) of a Banach space \( X \) is the least cardinality of a subset \( A \subset X \), which linear span \( \text{lin}(A) \) is dense in \( X \) (equivalently: such \( A \subset X \) that a closure \( \overline{\text{lin}}(A) \), which will be in the future denoted by \( \text{span}(A) \), is the whole space \( X \)). If \( X \) is of infinite dimension then its dimension \( \text{dim}(X) \) is exactly equal to its density character \( \text{dens}(X) \) – the least cardinality of a subset \( B \subset X \), which is dense in \( X \).

This partition may be regarded as an analogue of horizontal strips of some generalized coordinate system for Banach spaces.
Consider a question on cardinality of classes $B_\kappa$.

Immediately $\text{card } B_0 = \text{card } B_1 = 1$.

Indeed, the inclusion $X \in B_0$ means that $X$ is 0-dimensional, i.e., $X = \{0\}$.

Every 1-dimensional Banach space $X = \text{span } e$ may be identified with the scalar field $\mathbb{R}$ with the norm $\|\lambda e\| = |\lambda|$.

**Theorem 3.** For every finite cardinal $n \geq 2$ the cardinality $\text{card } B_n = \exp (\omega) = \mathfrak{c}$.

**Proof.** Let $n < \omega$. Consider the vector space $\mathbb{R}^n$, equipped with a Hausdorff topology (since all such topologies on finite-dimensional vector space $\mathbb{R}^n$ are equivalent, one may assume, e.g. that $\mathbb{R}^n$ is equipped with the Euclidean metric).

To any $n$-dimensional Banach space $X$ corresponds its unit ball $B(X) = \{x \in X : \|x\| \leq 1\}$, which is the central symmetric closed body in $\mathbb{R}^n$. Because of the set of such bodies in $\mathbb{R}^n$ is at most of cardinality $\mathfrak{c}$, it is clear that $\text{card } B_n \leq \exp (\omega)$.

To show the converse inequality, consider a collections of spaces $l_p^n$ ($1 \leq p \leq \infty$). Recall that the upper index $(n)$ pointed out on dimension. Surely, all these spaces are pairwise non-isometric and, hence

$$\text{card } B_n = \exp (\omega) = \mathfrak{c}. \tag{□}$$

Notice that all Banach spaces of given finite dimension are pairwise isomorphic (= linearly homeomorphic). This is not a case when infinite-dimensional Banach spaces are under consideration.

E.g., the same argument shows that $\text{card } B_{\omega} = \exp (\omega)$. However, because of all spaces $l_p$ ($1 \leq p < \infty$) are of dimension $\omega$ and are pairwise non-isomorphic, the more powerful result is valid. To formulate it we introduce some notations.

Let $X$ be a Banach space. Let $X^{=}$ denotes a set of all Banach spaces $Y$ that are isomorphic to $X$ (shortly: $Y \approx X$ and, thus, $X^{=} = \{Y \in \mathcal{B} : Y \approx X\}$). Surely, all spaces from $X^{=}$ are of the same dimension as $X$. Obviously, $\approx$ is an equivalence relation. which participates $B_\kappa$ into parts modulo $\approx$. Put

$$B^{=}_\kappa = \{X^{=} : X \in B_\kappa\}.$$

Obviously, $\text{card } B^{=}_\kappa = 1$ for all $n < \omega$. Since $l_p$ and $l_q$ are isomorphic if only if $p = q$, it follows that $\text{card } B^{=}_{\omega} = \exp (\omega)$. Indeed,

$$\exp (\omega) = \text{card } \mathbb{R} = \text{card } \{l_p : 1 \leq p < \infty\} \leq \text{card } B^{=}_{\omega} \leq \text{card } B_{\omega} = \exp (\omega).$$

The similar result is valid for every infinite cardinal.

**Theorem 4.** Let $\kappa > \omega$ be a cardinal: $\{x_\alpha : \alpha < \kappa\}$ and $\{y_\alpha : \alpha < \kappa\}$ be subsymmetric sequences; $X = \text{span } \{x_\alpha : \alpha < \kappa\}$; $Y = \text{span } \{y_\alpha : \alpha < \kappa\}$. If spaces $X$ and $Y$ are isomorphic then $\kappa$-sequences $\{x_\alpha : \alpha < \kappa\}$ and $\{y_\alpha : \alpha < \kappa\}$ are equivalent.

**Proof.** Let $u : X \to Y$ be an isomorphism: $\|u\| \|u^{-1}\| = c$. It may be assumed that $ux_\alpha \in Y$ is represented as a block

$$ux_\alpha = \sum_{k=1}^{n(\alpha)} a_k^{n(\alpha)} y_{\beta_k(\alpha)}$$

for some sequence of rational scalars $(a_k^{n(\alpha)})_{k \leq n(\alpha)}$ and finite $n(\alpha)$. Moreover, it may be assumed that this blocks are not intersected, i.e., that any member of a given sequence $(\beta_k(\alpha))_{k \leq n(\alpha)}$ belongs only to this block.
Since $\forall$ is uncountable, among such blocks there is an infinite number of identical ones that differs only in sequences $(\beta_k(\alpha))_{k \leq n(\alpha)}$. Let $A \subset \forall$ be such that all elements $\set{ux_\alpha: \alpha \in A}$ are represented by identical blocks:

$$ux_\alpha = \sum_{k=1}^{\lambda} a_k y_{\beta_k(\alpha)} \text{ for } \alpha \in A.$$ 

Let $(b_i)$ be a sequence of scalars; $a = \max_{1 \leq k \leq n}(\abs{a_k})$.

Then for any finite subset $A' \subset A$, $A' = \set{\alpha_i}_{i=1}^{m}$, because of unconditionality of sequences $(x_\alpha)$ and $(y_\alpha)$,

$$\left\| \sum_{i=1}^{m} b_i x_{\alpha_i} \right\| \geq c \left\| \sum_{i=1}^{m} b_i \left( \sum_{k=1}^{n} a_k y_{\beta_k(i)} \right) \right\| \geq c \left\| \sum_{i=1}^{m} b_i y_{\beta_1(i)} \right\| \geq c n a \left\| \sum_{i=1}^{m} b_i y_{\beta_1(i)} \right\|.$$ 

Analogously, holds the converse inequality that proves the theorem. □

**Corollary 1.** The cardinality of the set of all Banach spaces of dimension $\forall$ is

$$\text{card} \set{B_\forall} = \text{card} \set{B_\omega} = \exp(\omega).$$

*Proof.* The inequality $\text{card} \set{B_\forall} \geq 2^{\forall}$ follows from the previous theorem and results of the previous subsection. The inverse inequality is obvious. □

**Remark 1.** It is of interest that different sets $B_\forall$ and $B_\tau$ may be of the same cardinality. The appearance of a such case depends on the model of the set theory that we chose as the base of all functional analysis.

E.g., if one assume the Martin axiom $\text{MA}$ with the negation of continuum hypothesis $\neg \text{CH} \ (\text{MA}+\neg \text{CH})$ then for all cardinals $\forall$ such that $\omega < \forall < 2^{\omega}$

$$\text{card} B_\forall = \text{card} B_\omega = \exp(\omega).$$

As it was noted in the introduction, for a given cardinal $\tau$ one may define a set $\mathfrak{B}(\tau)$, which elements are sets $B_\forall$ for all cardinals $\forall \leq \tau$, namely:

$$\mathfrak{B}(\tau) = \set{B_\forall: \forall \leq \tau; \ \forall \text{ is a cardinal}}.$$ 

This set may be identified with the set of all ordinals $\set{\alpha: \alpha \leq \tau}$ and, as it is well known, may be endowed with the order topology (say, $T_{\langle \alpha \rangle}$), under which it becomes a compact topological space. This result we formulate as follows.

**Theorem 5.** Let $\mathfrak{B}(\tau)$ be defined as above. Then $\langle \mathfrak{B}(\tau), T_{\langle \alpha \rangle} \rangle$ is a compact topological space for every cardinal $\tau$.

2. **Classification of Banach spaces by elementary equivalence**

2.1. **Properties of ultrapowers.** The notion of ultraproducts and ultrapowers of Banach spaces was introduced by D. Dacunha-Castelle and J.-L. Krivine [4] as an analogue of the model-theoretical notion of ultraproducts and ultrapowers of models for the first-order language (more details see in the next section).

General properties of ultraproducts and ultrapowers were luxuriously exposed in the S. Heinrich’s paper [5]. Here will be presented results that are not contained in [5].

**Definition 1.** Let $I$ be a set; $\text{Pow}(I)$ be a set of all its subsets. An ultrafilter $D$ over $I$ is a subset of $\text{Pow}(I)$ with following properties:

- $I \in D$;
If $A \subseteq D$ and $A \subseteq B \subseteq I$, then $B \subseteq I$;

• If $A, B \subseteq D$ then $A \cap B \subseteq D$;

• If $A \subseteq D$, then $I \setminus A \notin D$.

**Definition 2.** Let $I$ be a set; $D$ be an ultrafilter over $I$; $\{X_i : i \in I\}$ be a family of Banach spaces. An ultraproduct $(X)_D$ is given by a quotient space

$$(X)_D = l_\infty(X_i, I) / N(X_i, D),$$

where $l_\infty(X_i, I)$ is a Banach space of all families $x = \{x_i \in X_i : i \in I\}$, for which

$$\|x\| = \sup\{\|x_i\|_{X_i} : i \in I\} < \infty;$$

$N(X_i, D)$ is a subspace of $l_\infty(X_i, I)$, which consists of such $x$’s that

$$\lim_{D} \|x_i\|_{X_i} = 0.$$

If all $X_i$’s are all equal to a space $X \in B$ then an ultraproduct is said to be an ultrapower and is denoted by $(X)_D$.

An operator $d_X : X \to (X)_D$ that asserts to any $x \in X$ an element $(x)_D \in (X)_D$, which is generated by a stationary family $\{x_i = x : i \in I\}$ is called the canonical embedding of $X$ into its ultrapower $(X)_D$.

**Definition 3.** Let $D$ be an ultrafilter, $\kappa$ be a cardinal. An ultrafilter $D$ is called

• $\kappa$-regular if there exists a subset $G \subseteq D$ of cardinality $\text{card}(G) = \kappa$ such that any $i \in I$ belongs only to a finite number of sets $e \in G$;

• $\kappa$-complete if an intersection of a nonempty subset $G \subseteq D$ of cardinality $\kappa$ belongs to $D$, i.e., if

$$G \subseteq D \quad \text{and} \quad \text{card}(G) < \kappa \quad \text{implies that} \quad \cap G \in D;$$

• Principal (or non-free), if there exists $i \in I$ such that $D = \{e \subseteq I : i \in e\}$;

• Free, if it is not a principal one;

• Countably incomplete if it is not $\omega_1$-complete.

**Theorem 6.** Let $X$ be a Banach space, $\dim(X) = \kappa$; $D$ be an ultrafilter.

The canonical embedding $d_X : X \to (X)_D$ maps $X$ on $(X)_D$ if and only if $D$ is $\kappa^+$-complete.

**Proof.** Let $D$ be $\kappa^+$-complete. If $\text{card}(I) \leq \kappa$ then $D$ is non-free and, hence, $d_X X = (X)_D$. Assume that $\text{card}(I) > \kappa$ and that $(x_i)_D \in (X)_D \setminus d_X X$. Of course, $(x_i)_D$ is generated by a such family $(x_i)_{i \in I}$ that for any $\varepsilon > 0$

$$I_0(\varepsilon) = \{i \in I : \|x_i - x\| > \varepsilon\} \subseteq D$$

for any $x \in X$. Consider a function $F : I_0 \to X$, which is given by $f(i) = x_i$. Since $\dim(X) = \kappa$, a partition

$$I = \cup \{f^{-1}(x) : x \in I_0(\varepsilon)\} \cup \{I \setminus I_0(\varepsilon)\}$$

participate $I$ in $\tau$ sets where $\tau < \kappa^+$. Since $D$ is $\kappa^+$-complete, one of sets of the partition belongs to $D$. Because of $I_0(\varepsilon) \notin D$, $I \setminus I_0(\varepsilon) \notin D$. Hence there exists such $x \in X$ that $f^{-1}(x) \in D$. Clearly, this contradicts with the assumption $(x_i)_D \in (X)_D \setminus d_X X$.

Conversely, assume that $d_X X = (X)_D$. Since $\dim(X) = \kappa$ there exists $\varepsilon > 0$ and a set $A \subseteq X$ such that $\text{card}(A) = \kappa$ and $\|a - b\| \geq \varepsilon$ for all $a \neq b \in A$. 
Let $I = \cup\{I_a : a \in B\}$ be a partition of $I$ in $\tau = \text{card}(B) < \kappa^+$ parts. Let a function $F : I \to A$ be given by:

$$F(i) = a \quad \text{if and only if} \quad i \in I_a.$$ 

Then $(F(i))_D \in (X)_D = d_X X$, and, hence, $(F(i))_D = d_X (a)$ for some $a \in A$. Because of $\|a - b\| \geq \epsilon$ for all $a \neq b \in A$, $F^{-1}(a) \in D$. However, by our assumption, $F^{-1}(a) = I_a$. Thus, $I_a \in D$ and, since a partition of $I$ by $\tau$ parts was arbitrary, $D$ is $\kappa^+$-complete.

**Corollary 2.** Let $X$ be a Banach space, $D$ be an ultrafilter.

1. If $\dim(X) < \omega$ then $(X)_D = X$;
2. If $\omega \leq \dim(X) < \nu$ where $\nu$ is the first measurable cardinal, and $D$ is a free ultrafilter then $(X)_D \setminus d_X X \neq \emptyset$;
3. If $X$ is of infinite dimension and ultrafilter $D$ is countably incomplete, then $(X)_D \setminus d_X X \neq \emptyset$ too.

**Proof.** Since any ultrafilter is $\omega$-complete (by definitions), all results follows from the preceding theorem.

Nevertheless, by choosing an ultrafilter, a difference $(X)_D \setminus d_X X$ may be maiden arbitrary large.

**Theorem 7.** Let $X \in B_\kappa$; $D$ be a $\tau$-regular ultrafilter over a set $I$ of cardinality $\text{card}(I) = \tau$. Then

$$\dim((X)_D) = (\dim(X))^\tau = \kappa^\tau.$$

**Proof.** Let $A' \subset X$ be a set of cardinality $\kappa$ which is dense in $X$. Immediately,

$$\dim ((X)_D) \leq \dim (l_\infty (X, I)) \leq \text{card}(I_A) = (\text{card}(A))^\text{card}(I) = \kappa^\tau.$$

Assume now that $A \subset A'$ is a set of the same cardinality $\kappa$ such that for some $\varepsilon > 0$ and any $a, b \in A$, $a \neq b$, $\|a - b\| \geq \varepsilon$. Let $\{a_i : i < \kappa\}$ be a numeration of elements of $A$. Any finite subset $s = \{x_0, ..., x_{n-1}\} \subset A$ spans a finite dimensional subspace $X_s$ of $X$.

A set $B$ of all subspaces of kind $X_s$ of $X$ is also of cardinality $\kappa$.

Let $\{X_{\alpha} : \alpha < \kappa\}$ be a numeration of $B$. Consider an ultraproduct $(X_{\alpha})_D$. Since $X_{\alpha} \hookrightarrow X$ for all $\alpha < \kappa$, it may be assumed that $(X_{\alpha})_D \hookrightarrow (X)_D$ and, hence, $\dim((X_{\alpha})_D) \leq \dim((X)_D)$. So, to prove the theorem it is enough to show that $\dim((X_{\alpha})_D) \geq \kappa^\tau$.

From $\tau$-regularity of $D$ it follows that there exists a subset $G \subset D$, $\text{card}(G) = \tau$, such that any $i \in I$ belongs only to a finite number of sets $e \in D$.

Define on $G$ a linear order, say, $\ll$.

Let $g : G \to A$ be a function.

Let a function $f_g : I \to B$ be given by

$$f_g(i) = \text{span}\{a_{g(e_k(i))} : k < n_i\} = B_i,$$

where $n_i$ is a cardinality of a set of those sets $e \in G$, to which $i$ belongs;

$$\{e_0(i) \ll ... \ll e_{n_i}(i)\}$$

be a list of them.

Let $h : G \to A$ be any other function: $h \neq g$. Then there exists $e \in G$ such that $h(e) \neq g(e)$. Corresponding functions $f_g$ and $f_h$ are differ also. Indeed, for any
$i \in e$ the set $e$ is contained in a finite sequence $e_0 (i) \ll ... \ll e_n (i)$ of all sets that contain $i$. If its order number in a such sequence is equal to $k$, then

$$f_g (i) = \text{span} \{ ..., g (e_k), ... \} \neq \text{span} \{ ..., h (e_k), ... \} = f_h (i).$$

Note that $e \in D$ and that $f_g (i) \neq f_h (i)$ for all $i \in e$.

Hence $f_g (i)$ and $f_h (i)$ generate different elements $\xi (f)$ and $\xi (h)$ of $\langle X_\alpha \rangle_D$; moreover $\| \xi (f) - \xi (h) \| \geq \epsilon$.

Since $\text{card}(^G A) = \kappa^\tau$, different elements $f, h$ of the set $^G A$ of all functions from $G$ to $A$ generate different elements $\xi (f)$ and $\xi (h)$ of the ultraproduct $\langle X_\alpha \rangle_D$ and, in addition, $\| \xi (f) - \xi (h) \| \geq \epsilon$, it is clear that $\dim (\langle X_\alpha \rangle_D) \geq \kappa^\tau$. \hfill $\Box$

**Remark 2.** It may be proved that for any infinite-dimensional Banach space $X$ and any countably incomplete ultrafilter $D$,

$$\dim (\langle X \rangle_D) = \left( \dim (\langle X \rangle_D) \right)^\omega.$$

Hence, by using ultrapowers, cannot be obtained any space $\langle X \rangle_D$, which dimension $\kappa$ is of countable confinality (i.e. such that $\text{cf}(\kappa) = \omega$; e.g., $\omega_\omega$, $\omega_{\omega_\omega}$, $\prod \omega$ and so on).

### 2.2. Elementary equivalent Banach spaces.**

Let $X, Y \in \mathcal{B}$. These spaces are said to be **elementary equivalent** (in symbol: $X \equiv Y$) if there exists such ultrafilter $D$ that ultrapowers $\langle X \rangle_D$ and $\langle Y \rangle_D$ are isometric.

It easy to prove that the relation $\equiv$ is reflexive and symmetric:

$$X \equiv X \quad \text{and} \quad X \equiv Y \iff Y \equiv X.$$  

To show its transitivity it needs much more work.

Namely, it will be used the result that was presented without proof in [3], which was called there the Keisler-Shelah theorem. In [3] this result was formulated as follows:

- For every Banach space $X$ and a pair of ultrafilters $E$ and $D$ there exists a such ultrafilter $G$ that ultrapowers $\langle X \rangle_D$ and $\langle X \rangle_E$ are isometric.

We shall need the more powerful result, which is a consequence of the S. Shelah’s one (cf. [3]), and which will be called the full Keisler-Shelah theorem.

**Theorem 8. (Keisler-Shelah)** Let $\kappa$ be a cardinal. Let $X$ be a Banach space and a pair of ultrafilters $E$ and $D$ be such that $\max \dim \{ \langle X \rangle_D, \langle X \rangle_E \} < \tau$, where $\tau$ is the least cardinal such that $\kappa^\tau > \kappa$. Then there exists an ultrafilter $G$ over the set of cardinality $\kappa$, which does not depend on $X$, $E$ and $D$, such that ultrapowers $\langle X \rangle_D$ and $\langle X \rangle_E$ are isometric. In other words, $\langle X \rangle_D \equiv \langle X \rangle_E$ (and, in particular, $X \equiv \langle X \rangle_D$ for every ultrafilter $D$).

As a corollary we obtain the following result.

**Theorem 9.** The relation $X \equiv Y$ is an equivalence relation on the class $\mathcal{B}$ of all Banach spaces.

**Proof.** As it was noted before, the relation $\equiv$ is reflexive ($X \equiv X$) and symmetric ($X \equiv Y \iff Y \equiv X$).

Assume that $X \equiv Y$ and $Y \equiv Z$. This means that there exists such ultrafilters $D$ and $E$ that $\langle X \rangle_D = \langle Y \rangle_D$ and $\langle Y \rangle_E = \langle Z \rangle_E$ (the symbol $\equiv$ means the isometry). By the preceding theorem there exists such ultrafilter $G$ over the set $\kappa$, where $\kappa^\tau > \tau$. 

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and \( \max \dim \{ (Y)_D, (Y)_E \} < \tau \), that \( ((Y)_D)_G = ((Y)_E)_G \). Hence, \( ((X)_D)_G = ((Z)_E)_G \).

Since \( X \equiv ((X)_D)_G \), there exists such ultrafilter \( F \) that \( (X)_F = ((X)_D)_G_F \). Notice that \( F \) depends only on dimension.

So, one may chose \( F \) such that \( (Z)_F = ((Z)_E)_G \).

Surely, this implies that \( (X)_F = (Z)_F \), i.e., that \( X \equiv Z \). \( \square \)

Thus, every Banach space \( X \) generates a class (of elementary equivalence)

\[ X^\xi = \{ Y \in B : X \equiv Y \}. \]

The second result, presented in \[5\] (also, without proof) is the result that was called in \[5\] the Löwenheim-Skolem theorem. This result for its formulation needs the following

**Definition 4.** Let \( X \) be a Banach space; \( Y \) be its subspace \( (Y \hookrightarrow X) \).

\( Y \) is said to be an elementary subspace of \( X \) (in symbols: \( Y \prec X \)) if there exists an ultrafilter \( D \) and an isometry \( i : (Y)_D \to (X)_D \) such that \( i \circ d_Y = d_X \circ i \circ d_Y \), where \( i \circ d_Y : Y \hookrightarrow X \) is the identical embedding; \( d_Y \) and \( d_X \) are canonical embeddings of \( Y \) and \( X \) respectively into their ultrapowers.

In other words, \( Y \prec X \) if for an ultrafilter \( D \) and an isometry \( i : (Y)_D \to (X)_D \) the diagram

\[ \begin{array}{c}
(Y)_D \downarrow \quad (X)_D \\
\uparrow \hspace{1cm} \uparrow \\
Y \quad \rightarrow \quad X
\end{array} \]

where vertical arrows denote respective canonical embeddings commutes.

After this definition one can formulate the Löwenheim-Skolem theorem:

**Theorem 10.** (Löwenheim-Skolem) For every Banach space \( X \) of infinite dimension \( \kappa \) and each subset \( A \subset X \) of cardinality \( \text{card} A < \kappa \) there exists an elementary subspace \( Y_A \prec X \) of dimension \( \dim Y_A = \max \{ \omega, \text{card} A \} \) such that \( A \subset Y_A \).

Certainly, if \( Y \prec X \) then \( Y \equiv X \). So, the following result is true.

**Theorem 11.** For every infinite-dimensional Banach space \( X \) the corresponding class \( X^\xi \) contains spaces of arbitrary given dimension.

If \( X \) is of finite dimension, \( X^\xi \) contains exactly one member – the space \( X \) itself.

*Proof.* From previous results it follows the second part of the theorem. If \( X \) is of infinite dimension (say, \( \kappa \)), then, for every infinite cardinal \( \tau \) and regular ultrafilter \( D \) over \( \tau \), the ultrapower \( (X)_D \) belongs to \( X^\xi \). Its dimension \( \dim (X)_D = \kappa^\tau \) may be maiden arbitrary large by choosing of \( D \).

By the Löwenheim-Skolem theorem, for every infinite cardinal \( \kappa < \kappa^\tau \) there exists an elementary subspace \( X_0 \prec (X)_D \) of dimension \( \kappa \).

Since \( \kappa \) is arbitrary and \( X_0 \equiv X \), the theorem is done. \( \square \)

So, any such class \( X^\xi \) contains at least one separable space and it may be considered the set

\[ \xi(B) = \{ X^\xi \cap B_\omega : X \in B \}. \]

Since there is one-to-one correspondence between classes \( X^\xi \) and sets \( X^\xi \cap B_\omega \), it will be assumed (for convenience) that elements (members) of \( \xi(B) \) are just classes \( X^\xi \).
As it was already noted, both theorems – the Keisler-Shelah and the L"owenheim-Skolem were appeared in literature in the Banach space setting without proofs. For the completeness we present in the following subsection their proofs, which use some model theory.

2.3. Some model theory. To express the notion of a Banach space in the logic consider a language \( \mathcal{L} \) - a set of symbols, which includes besides of logical symbols (connectives \& \lor, \neg, quantifiers \forall, \exists and variables \( u, v, x_1, x_2, \) etc) non-logical primary symbols as well:

- A binary functional symbol +;
- A countable number of unary functional symbols \( \{ f_q : q \in \mathbb{Q} \} \);
- An unary predicate symbol \( B \).

Any Banach space \( X = (X, \| \cdot \|) \) may be regarded as a model for all logical propositions (or formulae) that are satisfied in \( X \). A set \( |X| = X \) is called a support (or an absolute) of a model \( X \), in which all non logical symbols of the language \( \mathcal{L} \) are interpreted as follows:

- \( +^X \) is interpreted as the addition of vectors from \( X \);
- \( (f_q)^X \) for a given \( q \in \mathbb{Q} \) is interpreted as a multiplication of vectors from \( X \) by a rational scalar \( q \);
- \( B^X \) is interpreted as the unit ball \( B(X) = \{ x \in X : \|x\| \leq 1 \} \).

The language \( \mathcal{L} \) was introduced by J. Stern \[9\], who used it for examine the Banach space theory in the first order logic. In this logic any quantifier acts only on elements of a support of corresponding structure (e.g., a quantification over either all countable subsets of \( X \) or over all natural \( n \)'s is forbidden); only formulae that contains only finite strings of symbols are admissible.

For a Banach space \( X \) (which is identified with the model \( X \)) define a set \( \text{Th}(X) \) of all formulae of the first order logic, expressed in the language \( \mathcal{L} \), which are satisfied in \( X \) (in this case we shall write \( X \models \text{Th}(X) \)). Certainly, in a general case \( X \) is not the unique model of \( \text{Th}(X) \): there may be exist other structures, different from \( X \) (say, \( A \)) such that \( A \) satisfies the same formulae as \( X \) (in symbol: \( A \models \text{Th}(X) \)). Of course, \( A \) and \( X \) cannot be distinguished by tools of the first order logic.

There is a method to produce such models \( A \). It is called ” the model-theoretical ultrapower" and is denoted by \( \Pi_D X \), where \( D \) is an ultrafilter over a set \( I \) and \( \Pi_D X \) is a model (or, equivalently, structure) for \( \mathcal{L} \), whose absolute (or support) is the set-theoretical ultrapower of set \( X \) (i.e. a quotient \( \Pi_I X \) of the Cartesian product \( \Pi_I X \) of card \( I \) copies of \( X \) by the equivalence relation: \( \langle x_i \rangle_D =_D \langle y_i \rangle \) that means that \( \{ i \in I : x_i = y_i \} \in D \) ), and non-logical primary symbols of \( \mathcal{L} \) are interpreted in \( \Pi_D X \) in the following way:

- Addition of vectors and multiplying by a rational scalar are defined coordinate-wise;
- \( B^{\Pi_D X} \) is interpreted by

\[
B^{\Pi_D X} = \{ \langle x_i \rangle_D \in \Pi_D X : \{ i \in I : \|x_i\| \leq 1 \} \in D \}.
\]

It is well-known that \( \Pi_D X \models \text{Th}(X) \). However, \( \Pi_D X \) is not necessary a Banach space because of it may contain nonstandard elements (non-zero elements with zero norm or elements for which their norms, calculated by the Minkowski’s functional, are infinite).
J. Stern\cite{9} suggested a procedure of elimination of nonstandard elements, which looks as follows. Let $A \models Th(X)$ be a (may be - nonstandard) model. Put

\begin{align*}
\text{Fin}(A) &= \{a \in A : B^\mathfrak{A}(n^{-1}a) \text{ for some } n \in \mathbb{N}\}; \\
\text{Null}(A) &= \{a \in A : B^\mathfrak{A}(n^{-1}a) \text{ for all } n \in \mathbb{N}\}; \\
\|a\|^\mathfrak{A} &= \inf\{q \in \mathbb{Q} : B^\mathfrak{A}(q^{-1}a)\}
\end{align*}

and define the procedure $\llbracket \cdot \rrbracket : A \to \mathfrak{A}$ by

$$
\mathfrak{A} = \left( \text{Fin}(A) / \text{Null}(A) \right)^\wedge,
$$

where $\wedge$ means the completion by the norm $\| \cdot \|^\mathfrak{A}$.

By \cite{9}, this procedure sends the model-theoretical ultrapower $\Pi_DX$ of a Banach space $X$, which is regarded as an $\mathcal{L}$-model $\mathfrak{X}$, to its Banach-space ultrapower $(X)_D$.

After the preparation we are ready to present proofs of both results mentioned before.

Proof. The Keisler-Shelah theorem.

Let $X$ be a Banach space regarding as an $\mathcal{L}$-model $\mathfrak{X}$. Let $D, E$ be ultrafilters. Consider model-theoretical ultrapowers $\Pi_D\mathfrak{X}$ and $\Pi_E\mathfrak{X}$. Since, as it was noted before, $\mathfrak{X} \models Th(X)$; $\Pi_D\mathfrak{X} \models Th(X)$ and $\Pi_E\mathfrak{X} \models Th(X)$ as well, by the result of J.Keisler \cite{7} (in assumption the GCH) and of S. Shelah \cite{2} (in a general case) there exists a such ultrafilter $F$ that $\Pi_F(\Pi_D\mathfrak{X}) = \Pi_F(\Pi_E\mathfrak{X})$. Moreover, $F$ may be chosen in a such way, that it depends only of maximal cardinality $\tau = \max \dim\{\Pi_E\mathfrak{X}, \Pi_D\mathfrak{X}\}$ (to be an ultrafilter over a set $\kappa$ where $\kappa$ is the minimal cardinal such that $\tau^\kappa > \kappa$; see \cite{2}).

Hence

$$
\llbracket \cdot \rrbracket : \Pi_F(\Pi_D\mathfrak{X}) \to \llbracket \Pi_F(\Pi_D\mathfrak{X}) \rrbracket = \llbracket (\Pi_D\mathfrak{X}) \rrbracket_F = \llbracket (X)_D \rrbracket_F.
$$

From the other hand,

$$
\llbracket \cdot \rrbracket : \Pi_F(\Pi_E\mathfrak{X}) \to \llbracket \Pi_F(\Pi_E\mathfrak{X}) \rrbracket = \llbracket (\Pi_E\mathfrak{X}) \rrbracket_F = \llbracket (X)_E \rrbracket_F.
$$

So, $\llbracket (X)_D \rrbracket_F = \llbracket (X)_E \rrbracket_F$ and the theorem is proved. \hfill $\Box$

Proof. The Löwenheim-Skolem theorem.

For $\kappa = \omega$ the result is obvious: as $Y_A$ it may be chosen the space $X$ itself.

Let $\kappa > \omega$. Let $X$ be a Banach space of dimension $\kappa > \omega$; $A \subset X$ be a subset of cardinality $\tau$; $\omega \leq \tau < \kappa$. Consider $X$ as an $\mathcal{L}$-model $\mathfrak{X}$. By the (model theoretical) Löwenheim-Skolem theorem there exists a submodel (say $\mathfrak{Y}$) of $\mathfrak{X}$ such that its absolute $\llbracket \mathfrak{Y} \rrbracket$ contains $A$, $\mathfrak{Y} \models Th(X)$ and $\text{card} \llbracket \mathfrak{Y} \rrbracket = \text{card} A$. Moreover, $\mathfrak{Y}$ is an elementary submodel of $\mathfrak{X}$, i.e. there are an ultrafilter $D$ and a model-theoretical isomorphism $m : \Pi_D\mathfrak{Y} \to \Pi_D\mathfrak{X}$ such that the following diagram commutes:

$$
\begin{array}{ccc}
\Pi_D\mathfrak{Y} & \xrightarrow{m} & \Pi_D\mathfrak{X} \\
\uparrow & & \uparrow \\
\mathfrak{Y} & \hookrightarrow & \mathfrak{X}
\end{array}
$$

Using the Stern’s procedure $\llbracket \cdot \rrbracket$ it immediately follows that $\llbracket \mathfrak{Y} \rrbracket \prec \llbracket \mathfrak{X} \rrbracket$ and that $\dim \llbracket \mathfrak{Y} \rrbracket = \dim A$.

Put $Y_A = \llbracket \mathfrak{Y} \rrbracket$. Clearly, $A \subset Y_A \prec X$. \hfill $\Box$
2.4. Topological properties of $\xi(\mathcal{B})$. In this subsection it will be used another
definition of the notion of elementary equivalence, that belongs to S. Heinrich and
C.W.Henson, see [3]. Namely,

- Banach spaces $X$ and $Y$ are elementary equivalent, $X \equiv Y$, if and only
  if for any natural $n$, any $\varepsilon > 0$ and any chain of finite dimensional
  subspaces $A_1 \hookrightarrow A_2 \hookrightarrow \ldots \hookrightarrow A_n$ of $X$
  there exists a chain of finite dimensional subspaces $B_1 \hookrightarrow B_2 \hookrightarrow \ldots \hookrightarrow B_n$ of $Y$
such that there exists an isomorphism $u : B_n \rightarrow A_n$ with $u(B_i) = A_i$ for all $i < n$, which
satisfies $\|u\| \|u^{-1}\| < 1 + \varepsilon$, corresponding restrictions $u_{|B_i}$ have the same
estimates of norms and for any finite dimensional subspace $A_{n+1}$, $A_n \hookrightarrow A_{n+1} \hookrightarrow X$
there exists a subspace $B_{n+1}$, $B_n \hookrightarrow B_{n+1} \hookrightarrow Y$ with an isomorphism $\pi$
between $B_{n+1}$ and $A_{n+1}$, which restriction to $B_n$ is equal to $u$ and which
satisfies the estimate $\|u\| \|u^{-1}\| < 1 + \varepsilon$ and, conversely, for any $B_{n+1}$,
$B_n \hookrightarrow B_{n+1} \hookrightarrow Y$ there exists $A_{n+1}$, $A_n \hookrightarrow A_{n+1} \hookrightarrow X$ and an isomor-
phism $\pi$ between $B_{n+1}$ and $A_{n+1}$ which restriction to $B_n$ is equal to $u$ and which
norm satisfies the same estimate.

Let $\mathfrak{A} \subset \xi(\mathcal{B})$. Recall that we assume elements of $\mathfrak{A}$ are classes $X^\xi$.
The set $\mathfrak{A}$ is said to be ultraclosed if for any family $\{X_i : i \in I\}$, where $(X_i)^\xi$
belongs to $\mathfrak{A}$, their ultraproduct $Z = (X_i)^D$ generates a class $Z^\xi \in \mathfrak{A}$.

It will be said that $\mathfrak{A}$ is double ultraclosed if both sets $\mathfrak{A}$ and $\xi(\mathcal{B}) \setminus \mathfrak{A}$ are ultraclosed.

Define on $\xi(\mathcal{B})$ a topology $\mathcal{T}_\xi$ by choosing for its base of open sets a set $\mathfrak{M}$ of all
double ultraclosed sets $A \subset \xi(\mathcal{B})$.

Theorem 12. The topological space $\langle \xi(\mathcal{B}), \mathcal{T}_\xi \rangle$ is totally disconnected, Hausdorff
and compact.

Proof. The first assertion about $\xi(\mathcal{B})$ follows from the definition of the topology.

Compactness. Let $\mathfrak{B} \in \mathfrak{A}$ has the property: for any finite subset $\mathfrak{B}_0$ of $\mathfrak{B}$ its
intersection $\cap \mathfrak{B}_0$ is nonempty. It may be defined such ultrafilter $D$ that for any family $(X_i)_{i \in \mathfrak{B}}$, where each $X_i$
generates a member of $\mathfrak{B}$, their ultraproduct $(X_i)^D$ generates a member of the intersection $\cap \mathfrak{B}$. Hence, $\xi(\mathcal{B})$ has the finite intersection
property and, consequently, is compact.

Let us show that $\langle \xi(\mathcal{B}), \mathcal{T}_\xi \rangle$ is Hausdorff.

Let $X^\xi$, $Y^\xi$ be distinct classes. There exist natural $n$, real $\varepsilon > 0$ and a chain
$A_1 \hookrightarrow A_2 \hookrightarrow \ldots \hookrightarrow A_n$ of finite dimensional subspaces of $X$ such that for any
$Y \in Y^\xi$ and for any chain $B_1 \hookrightarrow B_2 \hookrightarrow \ldots \hookrightarrow B_n$ of finite dimensional subspaces of
$Y$, which is $(1 + \varepsilon)$-isomorphic to a chain $A \hookrightarrow A \hookrightarrow \ldots \hookrightarrow A_n$ there exists $A_{n+1}$,
$A_n \hookrightarrow A_{n+1} \hookrightarrow X$ such that $Y_0$ does not contain any $B_{n+1}$, $B_n \hookrightarrow B_{n+1} \hookrightarrow Y$,
that extended $B_1 \hookrightarrow B_2 \hookrightarrow \ldots \hookrightarrow B_n$ to the chain which is $(1 + \varepsilon)$-isomorphic to
$A_1 \hookrightarrow A_2 \hookrightarrow \ldots \hookrightarrow A_{n+1}$ (or respectively such $B_{n+1}$, $B_n \hookrightarrow B_{n+1} \hookrightarrow Y$ that no one
of $A_{n+1}$, $A_n \hookrightarrow A_{n+1} \hookrightarrow X$ forms a desired extension).

Let $\mathfrak{G}_{k,n}$ be a class of all $Z \in \mathcal{B}$ which have the property:

Any $(1 + 1/k)$-isomorphism between given chains $A_1 \hookrightarrow A_2 \hookrightarrow \ldots \hookrightarrow A_n$ of $X$
and $B_1 \hookrightarrow B_2 \hookrightarrow \ldots \hookrightarrow B_n$ of $Z$ cannot be extended to a $(1 + 1/k)$-isomorphism
between $A_{n+1}$, $A_n \hookrightarrow A_{n+1} \hookrightarrow X$ and $B_{n+1}$, $B_n \hookrightarrow B_{n+1} \hookrightarrow Z$ and conversely.

It is clear that the class $\mathfrak{G} = \cup \{\mathfrak{G}_{k,n} : n, k < \infty\}$ is doubly ultraclosed. Certainly,
this class contains $Y^\xi$. The class $X^\xi$ belongs to $\xi(\mathcal{B}) \setminus \mathfrak{G}$.
3. Classification of Banach spaces by finite equivalence

**Definition 5.** Let $X, Y$ be Banach spaces. $X$ is said to be finitely representable in $Y$, shortly: $X <_f Y$, if for every $\varepsilon > 0$ and for every finite dimensional subspace $A$ of $X$ there is a subspace $B$ of $Y$ and an isomorphism $u : A \rightarrow B$ such that $\|u\| \|u^{-1}\| < 1 + \varepsilon$. $X$ and $Y$ are said to be finitely equivalent, $X \sim_f Y$, if $X <_f Y$ and $Y <_f X$. So, every Banach space $X$ generates a class of finite equivalence

$$X^f = \{ Y \in B : X \sim_f Y \}.$$

For any infinite dimensional Banach space $X$ the corresponding class $X^f$ is proper: it contains spaces of arbitrary large dimension.

**Remark 3.** It is well known that $X <_f Y$ if and only if $X$ is isometric to a subspace of some ultrapower $(Y)_D$.

**Theorem 13.** For any infinite dimensional Banach space $X$ the class $X^f$ contains at least one space $Y_\kappa$ of dimension $\kappa$ for each infinite cardinal number $\kappa$ (particularly, at least one separable space $Y_\omega$). In other words, for any cardinal $\kappa \geq \omega$, $X^f \cap B_\kappa \neq \emptyset$.

**Proof.** Let $\dim(X) = \tau$. If $\kappa = \omega$ then we choose a countable sequence $\{A_i : i < \omega\}$ of finite dimensional subspaces of $X$ which is dense in a set $H(X)$ of all different finite dimensional subspaces of $X$ (isometric subspaces in $H(X)$ are identified), equipped with a metric topology, which is induced by the Banach-Mazur distance. Clearly, $X_0 = \text{span}\{A_i : i < \omega\} \hookrightarrow X$ is separable and is finitely equivalent to $X$.

If $\kappa < \tau$ then as a representative of $X^f \cap B_\kappa$ may be chosen any subspace of $X$ of dimension $\tau$ that contains a subspace $X_0$.

If $\tau < \kappa$ then, by the theorem 8 it may be found an ultrapower $(X)_D$ of dimension $\geq \kappa$. Certainly, $(X)_D \sim_f X$. Choose any subspace of $(X)_D$ of dimension $\kappa$, which contains a subspace $d_X X$. $\square$

**Remark 4.** This result is an obvious consequence of the Löwenheim-Skolem theorem.

For any two Banach spaces $X, Y$ their Banach-Mazur distance is given by

$$d(X, Y) = \inf \{ \|u\| \|u^{-1}\| : u : X \rightarrow Y \},$$

where $u$ runs all isomorphisms between $X$ and $Y$ and is assumed, as usual, that $\inf \emptyset = \infty$.

It is well known that $\log d(X, Y)$ forms a metric on each class of isomorphic Banach spaces, where almost isometric Banach spaces are identified.

Recall that Banach spaces $X$ and $Y$ are almost isometric if $d(X, Y) = 1$. Surely, any almost isometric finite dimensional Banach spaces are isometric.

The set $\mathfrak{M}_n$ of all $n$-dimensional Banach spaces, equipped with this metric, is the compact metric space, called the Minkowski compact $\mathfrak{M}_n$.

The disjoint union $\bigcup \{ \mathfrak{M}_n : n < \infty \} = \mathfrak{M}$ is a separable metric space, which is called the Minkowski space.

Consider a Banach space $X$. Let $H(X)$ be a set of all its different finite dimensional subspaces (isometric finite dimensional subspaces of $X$ in $H(X)$ are identified). Thus, $H(X)$ may be regarded as a subset of $\mathfrak{M}$, equipped with a restriction of the metric topology of $\mathfrak{M}$. 
Of course, $H(X)$ need not to be a closed subset of $\mathcal{M}$. Its closure in $\mathcal{M}$ will be denoted by $\overline{H(X)}$. From definitions it follows that $X <_f Y$ if and only if $\overline{H(X)} \subseteq \overline{H(Y)}$ and therefore, $X \sim_f Y$ if and only if $\overline{H(X)} = \overline{H(Y)}$. Thus, there is a one to one correspondence between classes of finite equivalence
\[ X^f = \{ Y \in \mathcal{B} : X \sim_f Y \} \]
and closed subspaces of $\mathcal{M}$ of kind $\overline{H(X)}$. Indeed, all spaces $Y$ from $X^f$ have the same set $\overline{H(X)}$. This set, uniquely determined by $X$ (or, equivalently, by $X^f$), will be denoted $\mathcal{M}(X^f)$ and will be referred to as the Minkowski's base of the class $X^f$. Let $f(\mathcal{B})$ be a set of all different classes of kind $X^f$.

To avoid set-theoretical difficulties note that each class $X^f$ is in one-to-one correspondence with the set $\mathcal{M}(X^f)$ and $f(\mathcal{B})$ may be regarded as the set of sets of kind $\mathcal{M}(X^f)$.

The following result is obvious.

**Theorem 14.** $\text{card } f(\mathcal{B}) = \mathfrak{c} \ (= \exp(\omega))$.

*Proof.* There is just the continuum number of closed subsets of the Minkowski's space $\mathcal{M}$.

At the same time, there is the continuum number of pairwise distinct classes of finite equivalence, e.g., classes $(l_p)^f \ (1 \leq p \leq \infty)$.

3.1. **Topological properties of $\mathcal{M}(X^f)$**. Let $X$ be a finite dimensional Banach space; $\dim X = n$. In this subsection it will be shown that for $n \geq 2$ either $\text{card } \mathcal{M}(X^f) = \mathfrak{c}$ or $X$ is isomorphic to the Euclidean space $l_2^{(n)}$ (in this case $\text{card } \mathcal{M}(X^f) = n + 1$).

Of course, $\text{card } \mathcal{M}(X^f) = 3$ for any 2-dimensional space $X$ since each of its subspaces is either $X$ itself or is 0- or 1-dimensional.

Along with $H(X)$ it will be considered a set $G(X)$ of all distinct finite-dimensional subspaces of $X$ ($A, B \in G(X)$ are considered as distinct if either they are of different dimension or $A \cap B \neq A$). A subset
\[ G_m(X) = \{ A \in G(X) : \dim A = m \}, \]
endowed with the metric
\[ \Theta(A, B) = \max \{ r(A, B), r(B, A) \}, \]
where
\[ r(A, B) = \sup \{ \inf \{|a - b| : \|b\| = 1\} : \|a\| = 1 \} \]
is called the Grassman manifold. It is known that $(G_m(X), \Theta)$ is a compact connected metric space for any finite dimensional $X \in \mathcal{B}$ and every $m < \dim(X)$.

**Theorem 15.** $(H_m(X), \theta)$ is a connected (compact, metric) space.

*Proof.* Let $(A_i)_{i < \infty} \subset G_m(X)$ be a fundamental sequence with respect to the metric $\Theta$, i.e. such that $\lim_{i, k \to \infty} \Theta(A_i, A_k) = 0$.

Because of compactness of $G_m(X)$ there is such $A \in G_m(X)$ and a subsequence $(A_{i_k})_{k < \infty} \subset (A_i)_{i < \infty}$ that $\Theta(A_{i_k}, A) \leq m^{-(k+1)} = (\dim A)^{-k-1}$.

Let $(e_j)_{j < m}$ be the Auerbach basis of $A$, i.e., such that for any element $a \in A$, $a = \sum_{j=1}^m a_j e_j$, the following inequality holds:
\[ \max_{1 \leq j \leq m} |a_j| \leq \|a\| \leq \sum_{1 \leq j \leq m} |a_j|. \]
Chose in $A_{i_k}$ vectors $(e^k_j)$ of norm one such that for all $j = 1, 2, ..., m$
$$\|e_j - e^k_j\| \leq \Theta(A_{i_k}, A) \leq m^{-(k+1)}$$

Clearly,
$$\sum_{j=1}^m \|e_j - e^k_j\| \leq m m^{-(k+1)} = m^{-k} < 1.$$  
Hence, by the Krein-Milman-Rutman theorem, $(e^k_j)_{j=1}^m$ is a basis of $A_{i_k}$.

Let an isomorphism $u_k : A \to A_k (k < \infty)$ be given by $u_k(e_j) = e^k_j (j = 1, 2, ..., m)$.

Since $(e_j)_{j<m}$ is the Auerbach basis of $A$,
$$\|u_k\| = \sup \left\{ \frac{\|\sum_{j=1}^m a_j e_j\|}{\|\sum_{j=1}^m a_j e^k_j\|} \right\} = \sup \left\{ \frac{\|\sum_{j=1}^m a_j e_j\|}{\|\sum_{j=1}^m a_j e_j + \sum_{j=1}^m a_j (e^k_j - e_j)\|} \right\} \leq \sup \left\{ \frac{\|\sum_{j=1}^m a_j e_j\|}{\left( \|\sum_{j=1}^m a_j e^k_j\| - \|\sum_{j=1}^m a_j (e^k_j - e_j)\| \right)^{-1}} \right\} \leq \sup \left\{ \frac{1}{1 - m^{-k-1} \sum_{j=1}^m |a_j| / \max_{1 \leq j \leq m} |a_j|} \right\} \leq (1 - m^{-k})^{-1}.$$  
Similarly, the norm of the inverse operator $u_k^{-1}$ is estimated:
$$\|u_k^{-1}\| = \sup \left\{ \frac{\|\sum_{j=1}^m a_j e^k_j\|}{\|\sum_{j=1}^m a_j e_j\|} \right\} \leq \sup \left\{ \left( \|\sum_{j=1}^m a_j e_j\| + \|\sum_{j=1}^m a_j (e^k_j - e_j)\| \right)/\|\sum_{j=1}^m a_j e_j\| \right\} \leq \frac{1 + m^{-k}}{1}.$$  
Hence, $\|u_k\| \|u_k^{-1}\| \leq (1 + m^{-k}) / (1 - m^{-k}) \to 1$ as $k \to \infty$.

Hence, the convergence of sequences with respect to the metric $\Theta$ implies the convergence with respect to the metric $\varrho$. Moreover, this implies that $H_m(X)$ is the connected space because it is a continuous image of the connected space $G_m(X)$ (under the mapping that 'gluing' isometric subspaces).  

\[\textbf{Corollary 3.} \text{ If } X \text{ is not equal to } l_2^{(n)} \text{ and } \dim (X) \geq 3 \text{ then card } H(X) = \infty.\]

\textit{Proof.} Let $m < n$. Then either card $H_m(X) = 1$ or card $H_m(X) = \infty$. In the first case the condition $\dim(X) \geq 3$ yields that there exists an even $m < n$. According to the M. Gromov’s theorem \[\textbf{3}], conditions card $H_m(X) = 1, m < n$ and $m$ is even imply that $X$ is isometric to the Euclidean space.  

\[\textbf{Corollary 4.} \text{ For every infinite dimensional Banach space } X \text{ and each } n < \omega \text{ the set } H(X) \text{ is a connected subset of } \mathfrak{M}_n.\]

\textit{Proof.} Fix $n < \omega$ and assume that $A$ and $B$ belong to different components of $H_n(X)$. These spaces may be regarded as subspaces of $X$. Let $C = \text{span}\{A, B\}$. Certainly, $C \in H(X); A \in H_n(C)$ and $B \in H_n(C)$. However, $H_n(C)$ is a connected set according to the theorem. Surely, this contradicts to our assumption.  

Corollary 5. For every infinite dimensional Banach space $X$ and each $n < \omega$ the set $\mathcal{M}_n(X^f)$ is a connected compact perfect space.

Proof. It is enough to notice that for every (non trivial) ultrafilter $D$ and every $X \in \mathcal{B}$ its ultrapower $(X)_D$ satisfies $H((X)_D) = \mathcal{M}(X^f)$.

Hence, $H_n((X)_D) = \mathcal{M}_n(X^f)$ for all $n < \omega$. So, by the previous corollary, $\mathcal{M}_n(X^f)$ is compact it is perfect. □

3.2. A synthesis of Banach spaces. A problem of synthesis of Banach spaces is to describe sets $\mathcal{R} \in \mathcal{M}$ with the property: there is a Banach space $X_{\mathcal{R}}$ with $\mathcal{M}(X^f) = \mathcal{R}$ and to give a procedure for constructing such $X_{\mathcal{R}}$. Obviously, necessary conditions on $\mathcal{R}$ that yields the existence of $X_{\mathcal{R}}$ with $\mathcal{M}(X^f) = \mathcal{R}$ are:

(H) If $A \in \mathcal{R}$ and $B \in H(A)$ then $B \in \mathcal{R}$.

(A0) If $A, B \in \mathcal{R}$ then there exists $C \in \mathcal{R}$ such that $A \in H(C)$ and $B \in H(C)$.

(C) $\mathcal{R}$ is closed in $\mathcal{M}$.

Theorem 16. Let $\mathcal{R} \in \mathcal{M}$. There exists a Banach space $X_{\mathcal{R}}$ with $\mathcal{M}(X^f) = \mathcal{R}$ if and only if $\mathcal{R}$ satisfies conditions (H), (A0), (C).

Proof. Assume that $\mathcal{R} \in \mathcal{M}$ is closed and satisfies conditions (H) and (A0). Define on $\mathcal{R}$ a partial order assuming that $A < B$ if $A \in H(B)$ $(A, B \in \mathcal{R})$.

To every $A \in \mathcal{R}$ corresponds a set

$$A^> = \{B \in \mathcal{R} : A < B\}.$$  

From the property (A0) it follows that $\mathcal{F} = \{A^> : A \in \mathcal{R}\}$ has the finite intersection property:

- The intersection of any finite subset of $\mathcal{F}$ is nonempty.

Indeed, if $A_0, A_1, ..., A_{m-1} \in \mathcal{R}$ then there exists a such $B_m \in \mathcal{R}$ that $A_i < B_m$ for all $i < m$. Hence, $\cap_{i < m} A^>_i \supseteq B^>_m \neq \emptyset$. So, $\mathcal{F}$ may be extended to some ultrafilter $D$ over $\mathcal{R}$.

Let $\mathcal{R}$ be considered as a set $\{A_A : A \in \mathcal{R}\}$, indexed by itself.

By a standard technique of ultrapowers it follows that the ultraproduct $W = (A_A)_D$ of all spaces from $\mathcal{R}$ has desired properties.

Indeed, for every $B \in H(W)$ and every $\varepsilon > 0$ there exists a space $A \in \mathcal{R}$, which is $(1 + \varepsilon)$-isomorphic to $B$. Surely the converse is also true: $\mathcal{R} \subset H(W)$. Since $\mathcal{R}$ is closed, $W = H(W)$.

Conversely, every $\mathcal{R} \in \mathcal{M}$ that satisfy conditions (H) and (A0) corresponds to some unique maximal centered system of closed subsets of $\langle F(B), T \rangle$ and, hence, to a class $(X_{\mathcal{R}})^f$. Any space of this class has desired properties. □

3.3. Topological properties of $f(B)$. Let $A \in \mathcal{M}(X^f)$. Let $T_f$ be a topology on $f(B)$, which base of open sets is formed by finite unions and intersections of sets of kind

$$U_+(A) = \{Y^f : A \in \mathcal{M}(X^f)\}; \quad U_-(A) = \{Y^f : A \notin \mathcal{M}(X^f)\}.$$  

Theorem 17. A space $\langle f(B), T_f \rangle$ is a totally disconnected Hausdorff compact topological space.

Proof. 1. Since $U_+(A) = f(B) \setminus U_-(A)$, every open set $U \in T_f$ is also closed, i.e., $\langle f(B), T_f \rangle$ is totally disconnected.
and $T$ has the nonempty intersection. Then $f(\langle f(B), T_f \rangle$ is Hausdorff.

2. If $X^f \neq Y^f$ then $\mathcal{M}(X^f) \neq \mathcal{M}(Y^f)$, i.e., there exists such $A \in \mathcal{M}$ that $A \in \mathcal{M}(X^f)$ and $A \notin \mathcal{M}(Y^f)$. From definitions it follows that $X^f \in U_+(A)$; $Y^f \in U_-(A)$ and, hence, $\langle f(B), T_f \rangle$ is Hausdorff.

3. The compactness of $\langle f(B), T_f \rangle$ is a consequence of the theorem on synthesis. Indeed, let $T_0 \subset T_f$ be a centered system, i.e., every finite collection of sets of $T_0$ has the nonempty intersection. Then $T_0$ may be extended to an ultrafilter $D$ and the ultrapower $(T_0)_D = W$ generates a class $W^f$.

Immediately, the intersection $\cap T_0$ contains $W^f$ and, hence, is nonempty as well. Since $T_0$ is arbitrary, $\langle f(B), T_f \rangle$ is compact. $\square$

Remark 5. The same result may be obtained in other way: sets of kind $U_+(A)$ and $U_-(A)$ generate a Boolean algebra (say, $\mathcal{B}$), which Stonian space $\mathcal{S}(\mathcal{B})$ is a totally disconnected Hausdorff compact space according to the well-known M. Stone’s theorem $\mathbb{3}$. Obviously, $\mathcal{S}(\mathcal{B})$ and $\langle f(B), T_f \rangle$ are homeomorphic.

4. Classification of Banach spaces by strong finite equivalence

Definition 6. Let $X, Y \in \mathcal{B}$. $X$ is said to be strongly finitely representable in $Y$; shortly: $X \preceq Y$, if $H(X) \subseteq H(Y)$. $X, Y \in \mathcal{B}$ are said to be strongly finitely equivalent; in symbol: $X \approx Y$, if $H(X) = H(Y)$.

We shall write

$$X^\phi = \{Y \in B : X \approx Y\}.$$ 

It is clear that $X^\phi \subseteq X^f$. In the general case these classes are not coincide. The set of all different classes of kind $X^\phi$ may be regarded as the set of suitable subsets of $\mathcal{M}$ and will be denoted by $\Phi(\mathcal{B})$.

4.1. Topological properties of $\Phi(\mathcal{B})$. Define on $\Phi(\mathcal{B})$ a topology $T$, which base of open sets consists of finite unions and intersections of sets of kind

$$V_+ \langle A \rangle = \{X^\phi : A \in H(X)\}; \quad V_- \langle A \rangle = \{X^\phi : A \notin H(X)\},$$

where $A$ runs $\mathcal{M}$.

In the sequel there will be needed following notions.

Let $A_1, A_2, ..., A_p$ be finite dimensional Banach spaces. This collection is said to be independent if there is no pair $i, j$ for which $A_i \in H(A_j)$.

Definition 7. Let $X \in \mathcal{B}$. The set $\mathcal{M}(X^f)$ has the compactness property (shortly: c. p.) if for any independent collection $A_1, A_2, ..., A_n, B_1, B_2, ..., B_m$ there exists $C \in \mathcal{M}(X^f)$ such that $A_1, A_2, ..., A_n \in H(C)$ and $B_1, B_2, ..., B_m \notin H(C)$.

Clearly, $\mathcal{M}(l_\infty^f)$ has the c. p. For any finite dimensional $X \in \mathcal{B}$ the set $\mathcal{M}(X^f)$ fails to have this property.

Theorem 18. The space $\langle \Phi(\mathcal{B}), T \rangle$ is totally disconnected Hausdorff compact topological space.

Proof. $V_+ \langle A \rangle = \Phi(\mathcal{B}) \setminus V_- \langle A \rangle$ for every $A \in \mathcal{M}$. Hence, every open set $U \in T$ is also closed, i.e. $\langle \Phi(\mathcal{B}), T \rangle$ is totally disconnected.

If $X^\phi \neq Y^\phi$ then $H(X) \neq H(Y)$, i.e. there exists $A \in \mathcal{M}$ such that $A \in H(X)$ and $A \notin H(Y)$. It is clear that $X^\phi \in V_+ \langle A \rangle$, $Y^\phi \in V_- \langle A \rangle$ and, consequently, $\langle \Phi(\mathcal{B}), T \rangle$ is Hausdorff.
To show that each centered system of closed sets of $\Phi(B)$ has a nonempty intersection, using the c. p. and the Zorn lemma, extend a given centered system $F$ of closed sets of $\Phi(B)$ to a maximal centered system $F_{\text{max}}$. It is clear that the intersection of $F_{\text{max}}$ consists of the unique point - some class $X^\phi$, i.e. is not empty. \hfill \Box

**Remark 6.** The compactness property means that the sets $V_+(A)$ and $V_-(A)$ generate the Boolean algebra, say $\mathfrak{B}'$ with operations

\[V_\pm(A_1) \wedge V_\pm(A_2) \overset{\text{def}}{=} V_\pm(A_1) \cap V_\pm(A_2),\]
\[V_\pm(A_1) \vee V_\pm(A_2) \overset{\text{def}}{=} V_\pm(A_1) \cup V_\pm(A_2),\]
\[\mathbf{C}V_+(A_1) \overset{\text{def}}{=} V_-(A_1);\]
\[\mathbf{C}V_-(A_1) \overset{\text{def}}{=} V_+(A_1).\]

Its Stonian space $\mathfrak{S}(\mathfrak{B}')$ is just the topological space $\langle \Phi(B), T \rangle$, as it follows from the definition.

Any Hausdorff compact space is normal and the same is true for $\mathfrak{M}$ when the last set is regarded as the topological subspace of $\Phi(B)$ endowed with the induced topology (but not the metric $d$). Hence the Wallman compact extension of $\mathfrak{M}$ (which may be identified with $\Phi(B)$ according to definitions) is homeomorphic to its Stone-Čech compactification. From this observation the theorem follows

**Theorem 19.** $\text{card } \Phi(B) = 2^c$.

*Proof.* From the c. p. and definitions it follows the existence of an infinite subset of $\Phi(B)$ which is discrete in the induced topology. Thus, the Stone-Čech compactification $\beta \Phi(B)$ contains the Stone-Čech compactification of naturals $\beta \mathbb{N}$ and is of cardinality $2^c$ because of $\text{card } \beta \mathbb{N} = 2^c$. \hfill \Box

**Remark 7.** It is unknown: whether every class $X^\phi$, which is generated by an infinitely dimensional Banach space $X$, is a proper class (i.e. contains a space of arbitrary large dimension).

Using some model theory it may be shown that there exists such cardinal number $\tau$ that any Banach space of dimension $\geq \tau$ generates the proper class $X^\phi$. As $\tau$ may be chosen so called the Hunf number $h(\omega_1, \omega)$ of the infinitary logic $\mathfrak{L}_{\omega_1, \omega}$. It is known that $h(\omega_1, \omega) = \mathfrak{I}(\omega_1)$.

So, in the case $\dim(X) \geq \mathfrak{I}(\omega_1)$, the class $X^\phi$ contains spaces of arbitrary dimension larger then $\dim(X)$.

**Remark 8.** Let $\mathfrak{M}$ be regarded as the topological subspace of $\langle \Phi(B), T \rangle$ (resp., of $\langle f(B), T_f \rangle$ or $\langle \xi(B), T_\xi \rangle$, equipped with the corresponding restriction of topology $(T |_{\mathfrak{M}}, T_f |_{\mathfrak{M}}$ or $T_\xi |_{\mathfrak{M}}$ respectively). Then all these restrictions $-T |_{\mathfrak{M}}, T_f |_{\mathfrak{M}}$ or $T_\xi |_{\mathfrak{M}}$ are coincide. The topological spaces $\langle \Phi(B), T \rangle$, $\langle f(B), T_f \rangle$ and $\langle \xi(B), T_\xi \rangle$ may be regarded as corresponding compactifications of $\mathfrak{M}$.

E. g., $\langle \Phi(B), T \rangle$ is just the Stone-Čech compactification of $\langle \mathfrak{M}, T |_{\mathfrak{M}} \rangle$.

Notice that there exists a continuous surjection $h_f : \langle \Phi(B), T \rangle \to \langle f(B), T_f \rangle$ that holds $\mathfrak{M}$ and glues finitely equivalent elements of $\Phi(B)$.

Similarly, there exists a continuous surjection $h_\xi : \langle \Phi(B), T \rangle \to \langle \xi(B), T_\xi \rangle$ that holds $\mathfrak{M}$ and glues finitely equivalent elements of $\xi(B)$.
Surely, from cardinality arguments it follows the existence of such different classes $X^\phi$ and $Y^\phi$ that are finitely equivalent and, as well, the existence of such different classes $X_1^\phi$ and $Y_1^\phi$ that are elementary equivalent. It is easy to give examples of such classes.

**Example 1.** Classes $(c_0)^\phi$ and $(l_\infty)^\phi$ are finitely equivalent. At the same time, for every finite-dimensional subspace of $c_0$ its unit ball is a polyhedron. Certainly, $(c_0)^\phi <_\phi (l_\infty)^\phi$ and $(c_0)^\phi$ is not strongly finitely equivalent to $(l_\infty)^\phi$.

**Example 2.** Another example may be obtained if instead $(l_\infty)^\phi$ it will be regarded a class $\langle c_0 \rangle$, where $c_0$ is the space $c_0$, regarded under the equivalent strictly convex norm. Surely, $(c_0)^\phi$ and $\langle c_0 \rangle$ are finitely equivalent and are strongly finitely incomparable.

**Example 3.** Classes $(l_p)^\phi$ and $(l_p \oplus l_p)^\phi$ are distinct (since $l_p$ does not contain $l_2^{(n)}$ for large $n$). Nevertheless, these classes are elementary equivalent (see e.g. [3])

4.2. **Omittable spaces in the class $X^f$.** Put \[ \Phi(X^f) = \{ Y^\phi : X \sim_f Y \}. \]

**Definition 8.** A space $A \in \mathfrak{M}(X^f)$ is said to be omittable in the class $X^f$ if there exists such $Y^\phi \in \Phi(X^f)$ that $A \notin H(Y)$. A space $A \in \mathfrak{M}(X^f)$ is said to be non omittable in the contrary case.

It is obvious that if $X$ is of finite dimension then every space $A \in H(X)$ is non omittable. Another example gives the class $(l_\infty)^f$, in which every nontrivial (i.e. more then one-dimensional) space from $\mathfrak{M}((l_\infty)^f)$ is omittable.

**Theorem 20.** A space $A \in \mathfrak{M}(X^f)$ is non-omittable if and only if $A$ is an isolated point in $\Phi(B, T)$.

**Proof.** If $X$ is finite dimensional then every $A \in \mathfrak{M}(X^f)$ is an isolated point. Indeed, $V_+(X) = X$ and $X$ is isolated. If $A \in \mathfrak{M}(X^f)$ and $\dim A = \dim X - 1$ then $\{A\} = V_+(A) \cap V_-(X)$ and, hence, $A$ is an isolated point. Then we are proceeding by induction. Let $X$ be infinite dimensional. Let $\mathfrak{A}$ be a set of all non omittable spaces in $\mathfrak{M}(X^f)$. It is clear that $A$ satisfies properties (H) and (A0). From the theorem 3 follows that there exists a space $Y \in \mathfrak{B}$ with $H(Y) = \mathfrak{A}$. If $Y$ is infinite dimensional then there exists such $B \in H(Y)$ that $A \in H(B)$ and $B$ is non omittable. Then $\{A\} = V_+(A) \cap V_-(B)$ is an isolated point. If $Y$ is of finite dimension and $A \in \mathfrak{A}$ then either $A \neq Y$ and is isolated as below or $A = Y$. Consider a set of all $B \in \mathfrak{M}(X^f)$ such that $A \notin H(B)$. This set does not have the property (A0). Indeed, in the contrary case the set $\{B \in \mathfrak{M}(X^f) : A \notin H(B)\}$ may be regarded as $H(Z)$ for some $Z \in \mathfrak{B}$. However, this contradicts with the non-omittability of $A$. Hence, there exists a space $B \in \mathfrak{M}(X^f)$ such that $B$ contains a subspace, isometric to $A$. In this case $A = V_+(A) \cap V_-(B)$ and, hence, is an isolated point too. 

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B.E. Ukrecolan, 33-81 Iskrinskaya str., 61005, Kharkiv-5, Ukraine
E-mail address: tokarev@univer.kharkov.ua