CYCLE INTEGRALS OF THE PARSON POINCARÉ SERIES AND INTERSECTION ANGLES OF GEODESICS ON MODULAR CURVES

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Abstract. We prove a geometric formula for the cycle integrals of Parson’s weight 2k modular integrals in terms of the intersection angles of geodesics on modular curves. Our result is an analog for modular integrals of a classical formula for the cycle integrals of certain hyperbolic Poincaré series, due to Katok. On the other hand, it extends a recent geometric formula of Matsusaka and Duke, Imamoglu, and Tóth for the cycle integrals of weight 2 modular integrals.

1. Introduction and statement of the main results

A classical result of Katok [6] states that for integers $k \geq 2$ the space $S_{2k}(\Gamma)$ of cusp forms of weight 2k for a cofinite discrete subgroup $\Gamma \subseteq \text{SL}_2(\mathbb{R})$ is generated by a family of Poincaré series associated with primitive hyperbolic matrices $\gamma \in \Gamma$. Explicitly, these Poincaré series are defined by

$$f_{k,\gamma}(z) = -\frac{D_{\gamma}^{k-1}}{\pi} \sum_{g \in \Gamma_\gamma \backslash \Gamma} \frac{1}{(Q_{\gamma} \circ g)(z, 1)^k},$$

where $\Gamma_\gamma = \{ \pm \gamma^n : n \in \mathbb{Z} \}$, $Q_{\gamma}(x, y) = cx^2 + (d - a)xy - by^2$ is the binary quadratic form corresponding to $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $D_{\gamma} = \text{tr}(\gamma)^2 - 4$ is the discriminant of $Q_{\gamma}$, and $\Gamma$ acts on binary quadratic forms in the usual way. These hyperbolic Poincaré series have many other interesting applications, the most prominent one being Kohnen’s [7] construction of the holomorphic kernel function for the Shimura correspondence.

Katok [6] also gave a beautiful geometric formula for (the imaginary part of) the geodesic cycle integrals of the cusp forms $f_{k,\gamma}(z)$. If $\gamma, \sigma \in \Gamma$ are primitive hyperbolic elements in $\Gamma$ which have positive trace and which are not conjugacy equivalent, and $S_\gamma$ denotes the geodesic semi-circle in $\mathbb{H}$ connecting the two real fixed points of $\gamma$, then Katok’s formula [6, Theorem 3] states that

$$\text{Im} \left( \int_{z_0}^{z_0} f_{k,\gamma}(z) Q_{\sigma}(z, 1)^{k-1} dz \right) = (D_{\gamma}D_{\sigma})^{\frac{k-1}{2}} \sum_{p \in [S_\gamma] \cap [S_\sigma]} \mu_p^\sigma P_{k-1}(\cos \theta_p),$$

where $z_0 \in \mathbb{H}$ and the path of integration can be chosen arbitrarily, the sum runs over the finitely many intersection points $p$ of the closed geodesics $[S_\gamma] = \Gamma_\gamma \backslash S_\gamma$ and $[S_\sigma] = \Gamma_\sigma \backslash S_\sigma$

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1. We use a slightly different normalization than Katok [6] to simplify our formulas.

2. We will assume throughout that $-1 \in \Gamma$. 

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in $\Gamma \backslash \mathbb{H}$, and $P_r$ denotes the $r$-th Legendre polynomial. Moreover, $\theta_p = \theta_p(\gamma, \sigma) \in [0, \pi]$ denotes the intersection angle at $p$, which is measured counterclockwise from the tangent at $S_\gamma$ to the tangent at $S_\sigma$ at $p$, and $\mu_p = \mu_p(\gamma, \sigma) \in \{ \pm 1 \}$ denotes the sign of the intersection at $p$, which is defined as follows: let $g \in \Gamma$ be chosen such that the intersection point $p$ corresponds to the intersection point of $S_\gamma$ and $S_{g\sigma g^{-1}}$ in $\mathbb{H}$, and suppose that $S_\gamma$ and $S_{g\sigma g^{-1}}$ are oriented clockwise (which means that the lower left entries of $\gamma$ and $g\sigma g^{-1}$ are positive). Then $\mu_p(\gamma, \sigma) = +1$ if the left endpoint of $S_{g\sigma g^{-1}}$ lies between the two endpoints of $S_\gamma$, and $\mu_p(\gamma, \sigma) = -1$ otherwise. The sign of $\mu_p$ changes if the orientation of either $S_\gamma$ or $S_{g\sigma g^{-1}}$ is reversed. Note that $\theta_p(\gamma, \sigma)$ does not depend on the orientation of $S_\gamma$ or $S_\sigma$, but it does depend on the order of $\gamma, \sigma$, that is, we have $\theta_p(\sigma, \gamma) = \pi - \theta_p(\gamma, \sigma)$. Similarly, we have $\mu_p(\gamma, \sigma) = -\mu_p(\sigma, \gamma)$.

More recently, Matsusaka [9] investigated the (homogenized) cycle integrals of certain modular integrals of weight 2 for $\text{SL}_2(\mathbb{Z})$ with rational period functions. These modular integrals were constructed by Duke, Imamoğlu, and Tóth in [1, 2], and are defined for primitive hyperbolic $\gamma \in \text{SL}_2(\mathbb{Z})$ by

$$F_\gamma(z) = \frac{2\sqrt{D_\gamma}}{\pi} \sum_{n=0}^{\infty} \left( \int_{z_0}^{z_{2n}} j_n(\tau) \frac{d\tau}{Q_\gamma(\tau, 1)} \right) e^{2\pi i nz},$$

where $j_n(\tau)$ denotes the unique weakly holomorphic modular function for $\text{SL}_2(\mathbb{Z})$ whose Fourier expansion has the shape $j_n(\tau) = q^{-n} + O(q)$ with $q = e^{2\pi i \tau}$. It was shown in [1] that the series defining $F_\gamma(z)$ converges to a holomorphic function on $\mathbb{H}$, and satisfies for any $\sigma \in \text{SL}_2(\mathbb{Z})$ the transformation formula

$$r_\gamma(\sigma, z) = (F_\gamma|2\sigma)(z) - F_\gamma(z) = \frac{2\sqrt{D_\gamma}}{\pi} \sum_{g \in \Gamma \backslash \Gamma} \frac{\text{sign}(Q_\gamma \circ g)}{(Q_\gamma \circ g)(z, 1)},$$

where $w'_\gamma < w_\gamma$ denote the two real fixed points of $\gamma$, and we put $\text{sign}(Q) = \text{sign}(A)$ for a binary quadratic form $Q(x, y) = Ax^2 + Bxy + Cy^2$. Note that the sum on the right-hand side is finite. The function $\sigma \mapsto r_\gamma(\sigma, z)$ defines a holomorphic weight 2 cocycle for $\text{SL}_2(\mathbb{Z})$ with values in the rational functions on $\mathbb{C}$, and the function $F_\gamma(z)$ is called a modular integral of weight 2 for $r_\gamma(\sigma, z)$. Matsusaka proved the remarkable formula

$$\text{Im} \left( \lim_{n \to \infty} \int_{\sigma^n \cdot z_0}^{\sigma^{n+1} \cdot z_0} F_\gamma(z) dz \right) = \sum_{p \in [S_\gamma] \cap [S_\sigma]} 1,$$

where $z_0 \in \mathbb{H}$ and the path of integration can be chosen arbitrarily, and $\gamma, \sigma \in \text{SL}_2(\mathbb{Z})$ are primitive hyperbolic matrices with positive trace which are not conjugacy equivalent (compare [9, Corollary 3.8, Theorem 3.3]). Note that the homogenization of the cycle integral on the left-hand side is necessary to make it independent of $z_0$, and a conjugacy class invariant in $\sigma$. The right-hand side of the formula (1.5) counts the number of

\[\text{Again, our normalization differs from [9] [1] [2].}\]
intersections of $[S_r]$ and $[S_s]$ in $\text{SL}_2(\mathbb{Z}) \backslash \mathbb{H}$, which by \cite[Theorem 3]{3} can also be interpreted as the linking number of certain modular knots associated with $\gamma$ and $\sigma$.

Notice that there is a striking similarity between Matsusaka’s formula (1.5) and (the formal specialization to $k = 1$ of) Katok’s formula (1.2). Motivated by this observation, in the present work we extend Matsusaka’s formula (1.5) to certain modular integrals of higher weight $2k$ (with $k \geq 2$) for cofinite discrete subgroups $\Gamma \subset \text{SL}_2(\mathbb{R})$, by evaluating their homogenized cycle integrals in terms of intersection angles of geodesics in $\Gamma \backslash \mathbb{H}$, much in the spirit of Katok’s formula (1.2).

The modular integrals we consider here were introduced by Parson in \cite{10}, and are defined for integers $k \geq 2$ and primitive hyperbolic $\gamma \in \Gamma$ by

\begin{equation}
F_{k,\gamma}(z) := -\frac{D_k^{k-\frac{1}{2}}}{\pi} \sum_{g \in \Gamma \backslash \Gamma} \text{sign}(Q_\gamma \circ g) (Q_\gamma \circ g)(z, 1)^k.
\end{equation}

A direct computation shows that the holomorphic function $F_{k,\gamma}(z)$ satisfies for any $\sigma \in \Gamma$ the transformation law

\begin{equation}
r_{k,\gamma}(\sigma, z) := (F_{k,\gamma}|_{2k}\sigma)(z) - F_{k,\gamma}(z) = \frac{2D_k^{k-\frac{1}{2}}}{\pi} \sum_{g \in \Gamma \backslash \Gamma} \text{sign}(Q_\gamma \circ g) (Q_\gamma \circ g)(z, 1)^k u_{\gamma,\sigma}^{-1} \sigma^{-1} \in \sigma^{-1} \langle \gamma \rangle \subset u_{\gamma,\sigma}^{-1} \langle \gamma \rangle.
\end{equation}

The sum on the right hand side is finite. In particular, the map $\sigma \mapsto r_{k,\gamma}(\sigma, z)$ is a holomorphic weight $2k$ cocycle for $\Gamma$ with values in the rational functions on $\mathbb{C}$, and $F_{k,\gamma}(z)$ is a modular integral for $r_{k,\gamma}(\sigma, z)$.

Notice that the cocycle $r_\gamma(\sigma, z)$ in (1.4) is the specialization to $k = 1$ of the cocycle $r_{k,\gamma}(\sigma, z)$ in (1.7). Hence, we may view the weight $2$ modular integral $F_{1,\gamma}(z)$ defined in (1.3) as the $k = 1$ analog of Parson’s modular integral $F_{k,\gamma}(z)$ defined in (1.6). However, $F_{k,\gamma}(z)$ does not converge for $k = 1$, although it is probably possible to extend the definition (1.6) to $k = 1$ using Hecke’s trick, and to show that $F_{1,\gamma}(z) = F_\gamma(z)$.

Our main result is the following geometric formula for (the imaginary part of) the cycle integrals of Parson’s modular integrals $F_{k,\gamma}(z)$. It is an analog of Katok’s formula (1.2) for modular integrals, and a higher weight analog of Matsusaka’s formula (1.5).

**Theorem 1.1.** Let $k \geq 2$ be an integer and let $\gamma$, $\sigma$ be primitive hyperbolic elements in $\Gamma$ with positive trace which are not conjugacy equivalent. We have

\begin{equation}
\text{Im} \left( \lim_{n \to \infty} \int_{\sigma^n \cdot z_0} (F_{k,\gamma}(z)Q_\sigma(z, 1)^{k-1}dz) \right) = (D_\gamma D_{\sigma})^{k-\frac{1}{2}} \sum_{p \in [S_r] \backslash [S_s]} \mu_p^{k-1} F_{k-1}(\cos \theta_p),
\end{equation}

where the notation is as in (1.2).

The proof of Theorem 1.1 will be given in Section 2 below. We would like to give a quick proof of the fact that the limit on the left-hand side exists, and is independent of

\footnote{In fact, our definition is not precisely Parson’s, but differs from her original Poincaré series by a cusp form. Moreover, we use a different normalization than Parson to match our normalization of (1.1).}
z_0. First, note that the limit of \( \sigma^n z_0 \) as \( n \to \infty \) is independent of the choice of \( z_0 \in \mathbb{H} \) and converges to \( w_\sigma \) (if we assume for the moment that \( \sigma \) has positive trace and positive lower left entry; see [9, Lemma 2.7]). Now the function \( x \mapsto \int_{x_0}^{x_1} F_{k,\gamma}(z)Q_\sigma(z,1)^{k-1}dz \) is Lipschitz-continuous when \( x \) is approaching \( w_\sigma \). Indeed, note that

\[
\left| \int_{x_0}^{x_1} F_{k,\gamma}(z)Q_\sigma(z,1)^{k-1}dz - \int_{x_1}^{x_2} F_{k,\gamma}(z)Q_\sigma(z,1)^{k-1}dz \right| \leq L_{k,\gamma,\sigma} |x_0 - x_1|,
\]

for some constant \( L_{k,\gamma,\sigma} > 0 \) and \( x_0, x_1 \) close to \( w_\sigma \), as \( r_{k,\gamma}(\sigma, z) \) is holomorphic at \( w_\sigma \) if \( \gamma \) and \( \sigma \) are not conjugacy equivalent, so \( |r_{k,\gamma}(\sigma, z)Q_\sigma(z,1)^{k-1}| \) is bounded in a neighbourhood of \( w_\sigma \). This implies that \( \int_{\sigma^n z_0}^{\sigma^{n+1} z_0} F_{k,\gamma}(z)Q_\sigma(z,1)^{k-1}dz \) is a Cauchy sequence.

Moreover, if we put \( x_0 = \sigma^n z_0 \) and \( x_1 = \sigma^n z_1 \) in (1.9) and take the limit as \( n \to \infty \), we see that the left-hand side in Theorem 1.1 is independent of \( z_0 \). Note that this also implies that the homogenized cycle integral of \( F_{k,\gamma}(z) \) is a conjugacy class invariant in \( \sigma \). We would also like to remark that the right-hand side in Theorem 1.1 is a finite sum, which can be explicitly computed (numerically) as explained in [11].

Previous to Matsusaka’s formula (1.5), Duke, Imamoğlu, and Tóth proved its ”parabolic version” which expresses the number of intersections of the net of the geodesics which can be explicitly computed (numerically) as explained in [11]. We generalize (1.10) to the higher weight case.

The Parson Poincaré series \( F_{k,\gamma}(z) \) defined in (1.6) has a Fourier expansion of the shape

\[
F_{k,\gamma}(z) = \sum_{n \in \frac{1}{c} \mathbb{Z}, n > 0} a_{k,\gamma}(n)e^{2\pi inz},
\]

where \( N \) denotes the width of the cusp \( \infty \) with respect to the group \( \Gamma \). The Fourier coefficients \( a_{k,\gamma}(n) \) can be explicitly computed in terms of Kloosterman sums and Bessel functions as in [10, Theorem 3.1], or in terms of cycle integrals of weakly holomorphic
modular forms as in [2, Theorem 3]. We define the twisted \( L \)-function of \( F_{k,\gamma}(z) \) by

\[
L_{k,\gamma}\left(s, -\frac{d}{c}\right) = \sum_{n \in \mathbb{Z}, n > 0} \frac{a_{k,\gamma}(n)e^{-2\pi i \frac{dn}{c}}}{n^s}, \quad (c, d) = 1, \ c > 0, \ \text{Re}(s) \gg 1,
\]

At its central value, the twisted \( L \)-function satisfies the following analog of (1.10).

**Theorem 1.2.** Let \( k \geq 2 \) be an odd integer, \( \gamma \) be a hyperbolic element with positive trace and \( d, c \) be coprime integers with \( c > 0 \). We have

\[
(1.11) \quad (-1)^{\frac{k-1}{2}} \frac{(k-1)!}{(2\pi)^k} \text{Re}L_{k,\gamma}\left(k, -\frac{d}{c}\right) = D_{\gamma}^{\frac{k-1}{2}} \sum_{p \in [S_\gamma] \cap S_{-d/c}} P_{k-1}(\cos \theta_p),
\]

where \( \theta_p \) denotes the angle of intersection at the point \( p \) between the geodesic in the equivalence class of \( S_\gamma \) and the geodesic \( S_{-d/c} \).

The left-hand side of (1.11) can be written as the imaginary part of the cycle integral of \( F_{k,\gamma}(z) \) along the non-compact geodesic from \(-d/c\) to \( i\infty\), so Theorem 1.2 can be viewed as a "parabolic" analog of Theorem 1.1.

2. The proofs of Theorem 1.1 and Theorem 1.2

2.1. Proof of Theorem 1.1. The proof is similar to the proof of Katok’s formula (1.2), compare [3, Theorem 3]. The main difference is that we can’t unfold the Parson Poincaré series as such.

Up to interchanging \( \gamma \) with \( \gamma^{-1} \) and \( \sigma \) with \( \sigma^{-1} \) we may assume that the lower left entries of \( \gamma, \sigma \) are positive. First, we rewrite

\[
\int_{\sigma^n_z}^{\sigma^{n+1}_z} F_{k,\gamma}(z)Q_\sigma(z, 1)^{k-1}dz = \int_{z_0}^{\sigma^n_z} (c_nz + d_n)^{-2k}F_{k,\gamma}(\sigma^n_z)Q_\sigma(z, 1)^{k-1}dz,
\]

where we put \( \sigma^n = (c_n^*, d_n^*) \). A direct computation shows that

\[
(2.1) \quad \text{sign}(Q_\gamma \circ \sigma^{-n}(1, 0)) \to \text{sign}(Q_\gamma(u'_\sigma, 1)) \quad \text{as} \quad n \to +\infty;
\]

see [3, Lemma 2.7]. If the two geodesics \( S_\gamma \) and \( S_\sigma \) intersect, we have \( \text{sign}(Q_\gamma(u'_\sigma, 1)) = -\mu_p(\gamma, \sigma) \). Now the Parson Poincaré series becomes

\[
(c_nz + d_n)^{-2k}F_{k,\gamma}(\sigma^n_z) = -\frac{D_{\gamma}^{k-1/2}}{\pi} \sum_{g \in \Gamma_n \setminus \Gamma} \frac{\text{sign}(Q_\gamma \circ g)}{(c_nz + d_n)^{2k}(Q_\gamma \circ g)(\sigma^n_z, 1)^k}
\]

\[
= -\frac{D_{\gamma}^{k-1/2}}{\pi} \sum_{g \in \Gamma_n \setminus \Gamma} \frac{\text{sign}(Q_\gamma \circ g \sigma^{-n})}{(Q_\gamma \circ g)(z, 1)^k}
\]

\[
\to -\frac{D_{\gamma}^{k-1/2}}{\pi} \sum_{g \in \Gamma_n \setminus \Gamma} \frac{\text{sign}(Q_\gamma \circ g)(u'_\sigma, 1))}{(Q_\gamma \circ g)(z, 1)^k}, \quad \text{as} \quad n \to +\infty,
\]

\[
\text{Re}L_{k,\gamma}\left(k, -\frac{d}{c}\right) = D_{\gamma}^{k-1/2} \sum_{p \in [S_\gamma] \cap S_{-d/c}} P_{k-1}(\cos \theta_p),
\]

as \( n \to +\infty \).
where in the last line we applied \((2.1)\). The resulting series is modular of weight \(2k\) for the group \(\Gamma_\sigma = \{ \pm \sigma^n : n \in \mathbb{Z} \}\). Hence, the cycle integral
\[
\int_{z_0}^{z_0} \sum_{g \in \Gamma_\gamma} \frac{\text{sign}((Q_\gamma \circ g)(w'_\sigma, 1))}{(Q_\gamma \circ g)(z, 1)^k} Q_\sigma(z, 1)^{k-1} dz
\]
is independent of the choice of \(z_0\).

Now a typical unfolding argument yields
\[
\int_{z_0}^{z_0} \sum_{g \in \Gamma_\gamma \setminus \Gamma_\sigma} \frac{\text{sign}((Q_\gamma \circ g)(w'_\sigma, 1))}{(Q_\gamma \circ g)(z, 1)^k} Q_\sigma(z, 1)^{k-1} dz
= \sum_{g \in \Gamma_\gamma \setminus \Gamma_\sigma} \sum_{m \in \mathbb{Z}} \int_{\sigma \cdot z_0}^{\sigma \cdot z_0} \frac{\text{sign}((Q_\gamma \circ g)(w'_\sigma, 1))}{(Q_\gamma \circ g)(z, 1)^k} Q_\sigma(z, 1)^{k-1} dz
= \sum_{g \in \Gamma_\gamma \setminus \Gamma_\sigma} \int_{S_\sigma} \frac{\text{sign}((Q_\gamma \circ g)(w'_\sigma, 1))}{(Q_\gamma \circ g)(z, 1)^k} Q_\sigma(z, 1)^{k-1} dz.
\]

Subtracting the complex conjugate from the last equation, gives the sum of integrals
\[
\sum_{g \in \Gamma_\gamma \setminus \Gamma_\sigma} \int_{C(\sigma)} \frac{\text{sign}((Q_\gamma \circ g)(w'_\sigma, 1))}{(Q_\gamma \circ g)(z, 1)^k} Q_\sigma(z, 1)^{k-1} dz
\]
over the circle \(C(\sigma)\) through the roots of \(Q_\sigma(z, 1) = 0\). The integrands are all meromorphic, with poles only at the real roots of \((Q_\gamma \circ g)(z, 1)\). Hence, by Cauchy’s theorem, they vanish if the geodesic connecting the roots of \((Q_\gamma \circ g)(z, 1)\) does not intersect \(S_\sigma\). Thus, we are left with
\[
\frac{D^k}{\pi} \sum_{p \in [S_\gamma] \cap [S_\sigma]} \mu_\sigma(\gamma, \sigma) \int_{C(\sigma)} \frac{1}{(Q_\gamma \circ g)(z, 1)^k} Q_\sigma(z, 1)^{k-1} dz,
\]
as \(\mu_\sigma(\gamma, \sigma) = -\text{sign}((Q \circ g)(w'_\sigma, 1))\) if the geodesics \(S_\sigma\) and \(S_{g^{-1}g}\) intersect.

The integrals evaluate to
\[
\int_{C(\sigma)} \frac{1}{(Q_\gamma \circ g)(z, 1)^k} Q_\sigma(z, 1)^{k-1} dz = 2\pi i D^{-k/2} D_{\sigma^\gamma}^{-1} \mu_\sigma^k P_{k-1}(\cos \theta_p);
\]
see [6] p. 478]. This finishes the proof of Theorem 1.1.

2.2. Proof of Theorem 1.2. The proof is a careful application of [6] Lemma 2], which asserts that for odd \(k \in \mathbb{Z}, k \geq 3\), and any \(A, B, C \in \mathbb{R}\) with \(D = B^2 - 4AC > 0\), we have
\[
(2.2) \quad \int_{-\infty}^{\infty} \frac{t^{k-1}}{(-At^2 + Bt + C)^k} dt = \begin{cases} 0, & AC > 0, \\ (-1)^{k+1} \frac{1}{2} \text{sign}(A) 2\pi D^{-k/2} P_{k-1} \left( \frac{B}{\sqrt{A}} \right), & AC < 0. \end{cases}
\]

Consider the integral \(\int_{-\frac{d}{c}}^{\infty} F_{k, \gamma}(z)(cz + d)^{k-1} dz\). A standard argument gives that
\[
\int_{-\frac{d}{c}}^{\infty} F_{k, \gamma}(z)(cz + d)^{k-1} dz = \left( \frac{c}{2\pi} \right)^k \Gamma(k) \frac{1}{c} L_{\gamma,k} \left( k, -\frac{d}{c} \right).
\]
The imaginary part of the integral is equal to

\[ \text{Im} \int_{-\frac{d}{c}}^{\infty} F_{k,\gamma}(z)(cz + d)^{k-1} dz = e^{k-1} \text{Im} \int_{0}^{\infty} F_{k,\gamma} \left( z - \frac{d}{c} \right) z^{k-1} dz \]

\[ = -e^{k-1} \frac{D_{\gamma}^{k-\frac{c}{2}}}{\pi} \sum_{g \in \Gamma_{\gamma}} \text{Im} \left( \int_{0}^{\infty} \frac{\text{sign}(Q_{\gamma} \circ g)(z - d/c, 1)^{k-1}}{(Q_{\gamma} \circ g)(z - d/c, 1)^{k}} dz \right), \]

where we can exchange sum and integral by Fubini’s theorem, as the integrand is absolutely convergent.

Fix \( g \in \Gamma_{\gamma} \setminus \Gamma \) for the moment and write \( Az^2 + Bz + C \) for \( (Q_{\gamma} \circ g)(z, 1) \). The quadratic form

\[ (Q_{\gamma} \circ g)(z - d/c, 1) = Az^2 + (B - 2Ad/c) z + \left( A \left( \frac{d}{c} \right)^2 - Bd/c + C \right) = A'z^2 + B'z + C' \]

intersects with the non-compact geodesic \( S_{-d/c} \) if and only if

\[ A'C' = A \left( \left( \frac{d}{c} \right)^2 - Bd/c + C \right) < 0. \]

With \([2.2]\), we get

\[ \text{Im} \left( \int_{0}^{\infty} \frac{\text{sign}(A')z^{k-1}}{(A'z^2 + B'z + C')^k} dz \right) = \frac{(-1)^{\frac{k-1}{2}}}{2} \int_{-\infty}^{\infty} \frac{\text{sign}(A')t^{k-1}}{(-A't^2 + B't + C')^k} dt \]

\[ = -\pi D_{\gamma}^{-k/2} P_{k-1} \left( \frac{B'}{\sqrt{D_{\gamma}}} \right) \]

if and only if the geodesic \( S_{g'g^{-1}} \) intersects the non-compact geodesic \( S_{-d/c} \) associated to the quadratic form \( cz + d \). Otherwise, the integral evaluates to zero by \( [2.2] \). By \([11] \) Proposition 2.2], the intersection angle \( \theta_p \) between these two geodesics is given by

\[ \cos \theta_p = \frac{Bc - 2Ad}{c\sqrt{D_{\gamma}}}. \]

This finishes the proof.

3. Additional Remarks

In this section, we present some further properties and possible applications of the homogenized cycle integrals and periods of Parson’s modular integrals \( F_{k,\gamma}(z) \).

3.1. Explicit representation of the cycle integral. As the Parson Poincaré series is no longer modular, the cycle integral \( \int_{z_0}^{z_0} F_{k,\gamma}(z)Q_{\sigma}(z, 1)^{k-1} dz \) depends on the choice of the point \( z_0 \) and is a complicated function in terms of \( z_0 \). Only when taking the homogenization we get a conjugacy class invariant object independent of the choice of \( z_0 \), which has the nice representation on the right hand side of \([1.8] \).
Explicitly, we have
\[
\int_{z_0}^{\sigma_{20}} F_{k,\gamma}(z)Q_\sigma(z,1)^{k-1}dz = \int_{i\infty}^{\sigma_{i\infty}} F_{k,\gamma}(z)Q_\sigma(z,1)^{k-1}dz
+ \sum_{n=0}^{2k-2} \sum_{\substack{g \in \Gamma \gamma \Gamma, \\ w_{g'y'\gamma} < 1 \in \sigma^{-1}, i\infty < w_{g'y'\gamma}} \rho_{n,g,\gamma,\sigma}(z_0) \ _2F_1\left(k,2k-1-n;2k;1-\frac{z_0-wQ}{z_0-wQ}\right),
\]
where \( \rho_{n,g,\gamma,\sigma}(z_0) \) is a rational function given by
\[
\rho_{n,g,\gamma,\sigma}(z_0) = \frac{\Gamma(2k-n-1)\Gamma(n+1)}{\Gamma(2k)\Gamma(z_0-w_{g'y'\gamma}^{\sigma^{-1}})^{n-2k+1}}.
\]
One can also see from this representation that the integral converges as \( z_0 \to w_\sigma \).

To prove this, we use
\[
\int_{z_0}^{\sigma_{20}} F_{k,\gamma}(z)Q_\sigma(z,1)^{k-1}dz = \int_{i\infty}^{\sigma_{i\infty}} F_{k,\gamma}(z)Q_\sigma(z,1)^{k-1}dz + \int_{i\infty}^{z_0} r_{k,\gamma}(\sigma,z)Q_\sigma(z,1)^{k-1}dz.
\]
This follows from differentiating both sides in \( z_0 \) and observing that both sides are equal at \( z_0 \to i\infty \). Rewriting the polynomial \( Q_\sigma(z,1)^{k-1} = \sum_{n=1}^{2k-2} a_{n,\sigma}(z_0)(z-z_0)^n \) in its Taylor expansion about \( z_0 \), we obtain a finite sum of integrals
\[
\int_{i\infty}^{z_0} r_{k,\gamma}(\sigma,z)Q_\sigma(z,1)^{k-1}dz = \sum_{n=1}^{2k-2} a_{n,\sigma}(z_0) \sum_{\sigma Q \gamma \gamma} \int_{z_0}^{i\infty} \frac{(z-z_0)^n}{(z-wQ)^k(z-w'Q)^k}dz,
\]
which can be solved. Standard integral transformations give
\[
\int_{z_0}^{z_{i\infty}} \frac{(z-z_0)^n}{(z-wQ)^k(z-w'Q)^k}dz = \frac{\Gamma(2k-n-1)\Gamma(n+1)}{\Gamma(2k)}(z-w')^{n-2k+1} \ _2F_1\left(k,2k-n-1,2k;1-\frac{z-wQ}{z-w'}\right).
\]
That the right hand side of (3.1) is symmetric in \( w'_Q \) and \( w_Q \) also follows from the identity
\[
\ _2F_1\left(c-a,b,c;\frac{z}{z-1}\right) = (1-z)^b \ _2F_1\left(a,b,c;\frac{z}{z-1}\right).
\]

The integral evaluation (3.1) can also be used to prove that the weight \( 2-2k \) cocycle
\[
R_\gamma(\sigma,z) = \frac{(-2\pi i)^{2k-1}D^{k-1/2}}{\pi(2k-1)!} \sum_{w_{Q'Q} < \frac{1}{2} < w_Q} \frac{1}{(Q(1,0)|(z-w'_Q)^{n-1}} \ _2F_1\left(k,1,2k;1-\frac{z-wQ}{z-w'}\right)
+ \frac{(-2\pi i)^{2k-1}i}{(2k-2)!} \sum_{n=0}^{2k-2} \left(\frac{2k-2}{n}\right)^{n+1} \frac{c}{2\pi} \Gamma(n+1) L_{k,\gamma}(n+1,a/c)(cz+d)^n
\]
for \( \sigma = (a,b,c,d) \in \Gamma \) and \( Q \) running over the equivalence class \([Q_\gamma]\) is a \((2k-1)\)-th primitive of \( r_\gamma(g,z) \), i.e. \( D^{2k-1}R_\gamma(g,z) = r_\gamma(g,z) \) with \( D = \frac{1}{2\pi i cz} \). The Fourier coefficients \( a_{k,\gamma}(n) \) can be explicitly calculated as in [10, Theorem 3].

3.2. Periods of modular integrals. In this subsection we let \( \Gamma = \text{SL}_2(\mathbb{Z}) \). Kohnen and Zagier [8] studied the periods of the cusp forms \( f_{k,\gamma}(z) \) defined in [11], and showed that certain linear combinations of these periods are rational numbers.
A natural follow-up question to Theorem 1.1 would be to study the periods of the Parson Poincaré series, i.e. $p_n(F_{k,\gamma}) = \int_0^\infty F_{k,\gamma}(it)t^n \, dt$ for $0 \leq n \leq 2k - 2$. As in [8] we define the symmetrizations

$$F_{k,\gamma}^+ = F_{k,\gamma} + F_{k,\gamma}'(z), \quad F_{k,\gamma}^- = i(F_{k,\gamma}(z) - F_{k,\gamma}'(z)),$$

where $\gamma' = (\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}) \gamma (\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix})$. We split the period polynomial

$$p(F_{k,\gamma})(z) = \int_0^{\infty} F_{k,\gamma}(z)(x - z)^{2k-2} \, dz = \sum_{n=0}^{2k-2} i^{-n+1} \binom{2k - 2}{n} p_n(F_{k,\gamma}) x^{2k-2-n}$$

of $F_{k,\gamma}$ into its even and odd part as $p(F_{k,\gamma}) = ip^+(F_{k,\gamma}) + p^-(F_{k,\gamma})$ with

$$p^+(F_{k,\gamma})(x) = \sum_{0 \leq n < 2k-2 \atop n \text{ even}} (-1)^n (2k - 2) \binom{2k - 2}{n} p_n(F_{k,\gamma}) x^{2k-2-n},$$

$$p^-(F_{k,\gamma})(x) = \sum_{0 \leq n < 2k-2 \atop n \text{ odd}} (-1)^{n-1/2} (2k - 2) \binom{2k - 2}{n} p_n(F_{k,\gamma}) x^{2k-2-n}.$$

By closely following the proof of [8, Theorem 5], one obtains

$$p^+(F_{k,\gamma})(x) + p^-(F_{k,\gamma})(x) \overset{3.2}{=} -2 \sum_{\substack{a,b,c \in [Q_\gamma] \\ \gcd(a,b,c) = 1 \atop a \neq b \neq c}} (ax^2 - bx + c)x^{k-1} - \frac{2D^{k-1/2} \zeta(Q_\gamma)(k)}{(2k - 2)(2k - 1)(2k)} x^{2k-2 - 1},$$

where $\zeta(Q_\gamma)(s)$ is the $\zeta$-function associated with $Q_\gamma$ as in [8, p. 222], $\zeta(s)$ is the Riemann $\zeta$-function, and $\overset{3.2}{=} \overset{3.2}{=}$ means equality up to a non-zero multiplicative constant. In particular, the periods $p_n(F_{k,\gamma}^+)$ for even $0 < n < 2k-2$ and the periods $p_n(F_{k,\gamma}^-)$ for odd $0 < n < 2k-2$ are rational.

We let $F_{k,D}(z) = \sum_{D_\gamma \equiv D} F_{k,\gamma}(z)$ where the sum ranges over a system of representatives $\gamma$ of the conjugacy classes of primitive hyperbolic elements in $\text{SL}_2(\mathbb{Z})$ with discriminant $D_\gamma = D$. From (3.2) it follows that for odd $k \in \mathbb{Z}$ we have

$$p^+(F_{k,D})(x) \overset{3.3}{=} -\sum_{\substack{a,b,c \in [Q_D] \\ \gcd(a,b,c) = 1 \atop a \neq b \neq c}} (ax^2 + bx + c)x^{k-1} - \frac{D^{k-1/2} \zeta(k) L_D(k)}{(2k - 2)(2k - 1)(2k)} x^{2k-2 - 1},$$

where $L_D(s)$ is the Dirichlet $L$-function associated to the Kronecker symbol $(\frac{D}{\cdot})$. Formula (3.3) is the analog of [8, Theorem 4]. In particular, the periods $p_n(F_{k,D})$ for even $0 < n < 2k-2$ are rational and satisfy the symmetry $p_{2k-2-n}(F_{k,D}) = p_n(F_{k,D})$.

\footnote{The formulas (3.2) and (3.3) are correct if we normalize $F_{k,\gamma}$ as in [8]. Since our normalization of $F_{k,\gamma}$ is different, we get some simple but unpleasant extra factors.}
3.3. The homogenized cycle integral as an "inner product". By unfolding and [7, Proposition 7], one can also show that
\[
\lim_{n \to \infty} \int_{\gamma_n \to z_0} F_{k,\gamma}(z)Q_\sigma(z,1)^{k-1}dz
= \int_{\Gamma \backslash \mathbb{H}} \sum_{\gamma \in \Gamma} \sum_{h \in \Gamma \cap \Gamma_0} \frac{\text{sign}(Q_\gamma \circ g)(w_{\kappa h}^{-1},1))}{(Q_\gamma \circ g)(z,1)^k(Q_\sigma \circ h)(z,1)^k} 2k dxdy y^2.
\]
Since $w_{\kappa h}^{-1} = h^{-1}w_{\kappa}^{-1}$, the integrand is $\Gamma$-invariant.

One should compare this with the fact that the cycle integral $\int_{\gamma_0} f_{k,\gamma}(z)Q_\sigma(z,1)^{k-1}dz$ is up to constants equal to the Petersson inner product $\langle f_{k,\sigma}, f_{k,\gamma} \rangle$ of the hyperbolic Poincaré series [11].

3.4. Equidistribution of intersection angles. A possible application of Theorem [11] could be to give another proof the fact that the intersection angles $h_\theta$ of a fixed geodesic $[S_\gamma]$ with the geodesics of discriminant $D$ (for $\text{SL}_2(\mathbb{Z})$) equidistribute to the measure $\frac{1}{2} \sin \theta \ d\theta$ as $D \to +\infty$. This was conjectured by Rickards [11, Conjecture 4.2] and recently proved by Jung and Sardari [5]. To show this, it suffices to prove that $\cos h_\theta$ equidistribute to the Lebesgue measure on $[-1,1]$. The Legendre polynomials $P_{k-1}$ with $k \geq 1$ form a complete orthonormal system of the space of continuous functions on $[-1,1]$. For $k > 1$ even, the Weyl sum on the right hand side of (1.2) can be estimated using the Shimura-theory of cusp forms [7] and non-trivial bounds on the Fourier coefficients of half-integral weight cusp forms [4]. A Siegel-type bound for the number of intersections can be obtained using [13, Theorems 3.3 and 4.7, Remark 4.8], and some elementary considerations on continued fractions. For $k > 1$ odd, this approach does not work due to the appearance of the $\mu_\sigma$-factor, so the Weyl sum here is given by the right hand side in (1.3). However, it appears to be difficult to estimate the traces of the homogenized cycle integrals. We plan to come back to this in the future.

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