To snipe or not to snipe, that’s the question!
Transitions in sniping behaviour among competing algorithmic traders

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Abstract In this paper we re-analyse the transition from sure to probabilistic sniping as explored in Menkveld and Zoican [14]. In that paper, the authors introduce a stylized version of a competitive game in which high frequency traders (HFTs) interact with each other and liquidity traders. The authors show that risk aversion plays an important role in the transition from sure to mixed (or probabilistic) sniping. In this paper, we re-interpret and extend these conclusions in the context of repeated games and highlight some differences in results. In particular, we identify situations in which probabilistic sniping is genuinely profitable that are qualitatively different from the ones obtained in [14].

Keywords algorithmic trading · sniping · electronic exchange · high-frequency traders · Nash equilibrium · repeated games · bandits · subgame-perfect equilibrium · transition

1 Introduction

1.1 Motivation

This paper revisits the seminal article by Menkveld and Zoican [14] in which these authors investigate the impact of exchange latency on the sniping be-
haviour of high frequency traders (HFTs). To this end, [14] introduces a stylized model of a stock exchange and describes a race game (hereafter referred to as the MZ game) in which a group of HFTs attempt to attract trade from liquidity traders (LT) by competing on spread. The game strategies are complicated by the fact that news items, published at random times, push the intrinsic value of the financial assets either up or down. The ensuing jump in value causes all the HFTs to race to the exchange, albeit with different intentions. The market maker attempts to cancel his own now stale quotes, while the other HFTs (dubbed bandits hereafter) race to take advantage of the arbitrage opportunity created by the outdated quotes.

Within this framework, the authors in [14] show how the interplay between uncertainty and opportunity give rise to interesting behaviour. They conclude (among other things) that increasing risk aversion in the market induces a qualitative transition in the sniping behaviour of the bandits. More specifically, if risk aversion exceeds a well-defined threshold, bandits change from sure sniping to probabilistic sniping, resulting in a potentially discontinuous jump in the market maker’s utility.

While these results are surprising and highly non-trivial, they hinge on some subtle but important features of the game. In particular, the authors study this problem in the context of a single-shot game, and focus on Nash equilibria as their main solution concept. In this paper we espouse an alternative view by interpreting the market activities of the HFTs as a repeated game with an infinite horizon. In this alternative setting a player can adopt more far-sighted strategies that attempt to maximise long-term gains. This change of viewpoint has a number of important consequences:

- It identifies situations in which probabilistic sniping is genuinely profitable and therefore extends the conclusions in [14] by introducing qualitatively different solutions. More specifically, it builds on the Folk Theorems to find additional subgame perfect equilibria (SPE) that yield a strictly better utility for all HFTs involved.
- It explains why bandits are willing to engage in probabilistic sniping, for which there seems to be no compelling reason in the original MZ model.

**Note on terminology** In this paper we will use the term *equilibrium* in the weak sense, to indicate (roughly speaking) conditions under which “different forces equilibrate”. However, this equilibrium need not be stable in the sense that it will resist change. When stability is indeed guaranteed, we will mention this explicitly by indicating which type of stability holds (e.g. Nash or subgame-perfect).

1.2 Overview of contributions in this paper

The contributions in this paper can be summarized as follows:
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1. We show how the MZ model of the stock exchange can be re-interpreted as a repeated (sequential) game against nature. In addition, we think of the choice of sniping probability (for the bandits) as a choice of a pure action from a continuous action space, rather than a probabilistic mixing of two pure strategies (i.e. sniping and non-sniping). This change of viewpoint allows us to identify a wider set of conditions under which probabilistic sniping makes sense (see section 4). Put succinctly, although these new equilibria are not Nash equilibria for a one-shot game, they are genuine subgame-perfect equilibria in the corresponding repeated game with infinite horizon.

2. We identify two threshold values (viz. $\gamma_K$ and $\gamma_L$) for the risk aversion factor $\gamma$ that govern the transition to probabilistic sniping. More specifically, when $\gamma$ increases beyond $\gamma_K$ probabilistic sniping becomes advantageous for all HFTs involved. Conversely, when $\gamma$ exceeds $\gamma_L$, even probabilistic sniping is no longer profitable.

3. We show that under conditions for which probabilistic sniping is advantageous, there is an optimal sniping probability $p_K^*$ that maximises expected utilities for both the market maker and the bandits. Both the sniping probability and the corresponding optimal utility vary continuously as functions of the game parameters.

The remainder of this paper is organised as follows.

In section 2 we give a brief summary of the MZ paper, paying special attention to some subtle points that are of crucial importance for the exposition of the results. Section 3 takes a detailed look at the concept of probabilistic sniping and derives some explicit results for its impact on the HFTs utilities. In section 4 we show how a simple geometric argument allows us to characterise the conditions under which the transitions between the different types of sniping occur. Section 5 then zooms in on the specific role of risk aversion in the sniping transitions and identifies the relevant thresholds.

Up to this point in the paper, the equilibria obtained under various conditions were optimal, but not necessarily Nash equilibria for the single-shot MZ game. In section 6 we explain what the implications are of shifting our viewpoint from one-shot to repeated games. In particular, we argue that probabilistic sniping will give rise to a new set of playable subgame-perfect Nash equilibria.

Finally, in section 7, we discuss how our work compares to related research and finish by offering some conclusions and suggestions for further research in section 8.

2 The Menkveld-Zoican (MZ) game revisited

2.1 A bird’s eye view of the MZ game

As this paper builds on the MZ paper [14], we briefly highlight some of the main arguments and conclusions from that paper. In it, the authors define
a stylized version of the behaviour and strategies of high frequency traders (HFT) interacting with an (high-frequency) exchange. We defer some details to section 2.2 but, roughly speaking, they envisage a game in which, during an initial pre-game stage, the HFTs have to pick the value of the (half) spread \( s \) in order to post a single bid-ask quote \( (v \pm s) \) for a financial asset of value \( v \). One of the HFTs is then selected as market maker, whereas the others are relegated to the role of bandit. Only the quotes of the market maker will enter the order book, so he is the only one who can benefit from trading with liquidity traders. The bandits will try to gain some payoff by attempting to snipe stale quotes (see section 2.2 for more details).

The game is initiated by a trigger event which can be either the publication of news item, or the interaction of a liquidity trader with the market maker. The latter event does not evoke any reaction from the HFTs, but the former alters the intrinsic value of the asset and causes all the HFTs to race towards the exchange, albeit for different reasons. The market maker attempts to cancel his now stale quotes, whereas the bandits hope to obtain financial gain by sniping the market maker. The outcome of this race if further complicated by the fact that, during the race, another event (news or liquidity trade) might occur. This is less likely when the exchange fast (low exchange latency \( \delta \)).

In [14] the authors investigate in detail how the interplay between various game parameters affect the behaviour and strategies of the HFTs. Although this model is highly stylized and therefore somewhat unrealistic, it is the contention of the MZ authors that even under these simplifying assumptions interesting and non-trivial conclusions can be drawn. This suggests that a more realistic model will give rise to even more intriguing insights.

2.2 Primitives for the MZ game

Because they are important to understand the rest of the paper, we briefly recapitulate the main ingredients and assumptions governing the MZ game. For more details we refer to the original paper [14].

– **Agents or Players** The players in this game is the group of \( H > 2 \) high frequency traders (HFTs) who operate in an environment populated by an infinite number of (informed) liquidity traders (LT). All HFTs have simultaneous and instantaneous access to all public information affecting the market. At the start of the game, one HFT is assigned to the role of market maker. The remaining \( H - 1 \) HFTs take up the role of high frequency bandits (HFB), intent on financial gain by sniping the stale quotes whenever an opportunity presents itself. A final important characteristic of all HFTs is that they are risk-averse (characterized by factor \( \gamma \geq 1 \)), i.e. the utility of negative pay-offs is inflated by a factor \( \gamma \).

– **Exchange latency** \( \delta \) measures the time delay \( \delta \) between arriving at the "front door" of the stock exchange and arriving at the matching engine where each order is actually processed. This delay determines the time span over which the race game is played out and, as such has a direct impact
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on the expected number of additional events that might occur during the race.

- **Exogenous events**
  1. **Common value shock (due to public news):** this shock is observed by everyone, and changes the intrinsic value of an asset. This process is modelled as a Poisson process with rate $\alpha$ (number of news events per time unit). The size of the shock is fixed at $\pm \sigma$.
  2. **Private value shock:** These are only known to the LTs and cause them to join the matcher queue. These events are modelled as an independent Poisson process with rate $\mu$.

- **Detailed chronology of the MZ game** The description below is based on the MZ paper, but attempts to clarify the process in more detail (cf. Fig. 1).

  ![](image)

  **Fig. 1** Chronology of the MZ game

- **At $t = -1$, Strategy selection:** all HFTs are informed about the game’s parameters (i.e. $\alpha, \mu, \delta, \sigma$ and $H$). Based on this information they can then compute the expected utility as a function of the spread $s$ (and the sniping behaviour) for both the market maker and bandits (more details in section 2.3), but are still unsure about which one applies to them. Taking this uncertainty into account, each HFT posts a single bid and ask quote in an empty book (i.e. initially each HFT has a zero position).

- **At $t = -1/2$, Type assignation:** one of the HFTs is chosen to be the market maker, whereas all the others will become "bandits" (eager to snipe). The selection process of the market maker proceeds in two steps:
  1. First, the HFT(s) that posted the smallest spread $s$ are selected;
  2. If there are multiple HFTs that posted the same minimal spread, then one of them is selected at random (uniformly).

  The fact that the outcome of this selection process is uncertain, is an important aspect of the game that the HFTs need to take into account when selecting a strategy at $t = -1$ as it is impossible for them to change their position at a later time.

- **At $t = -1/4$, Nash check of no regret:** at this stage every agent knows its role in the game. This is therefore the point at which the
Nash equilibrium test operates: each individual agent is asked if at this point, he could profitably deviate from his original strategy. If none of the agents regrets the strategy he’s committed to, then the proposed solution is a Nash equilibrium.

- **At** $t = 0$. **Initial trigger starts game:** one of two possible trigger events occurs:
  1. *News event* (public) that changes the value of the asset (i.e. value shock of $\pm \sigma$). This immediately triggers a race among the HFTs: the market maker will try to update his stale quotes, while the bandits will attempt to snipe. The winner of this race is randomly (uniform) chosen among all contestants. The winner is known at time $t = \delta$.
  2. *LT arrives at matcher (queue).* This will result in a transaction, which is invariably profitable for the market maker as he will cash in the spread ($s$). This event does not elicit any reaction from the HFBs.

- Interval $0 < t < \delta$ During this interval one (and only one) of three things can happen:
  1. An additional news event becomes (publicly) known (rate $\alpha$).
  2. Another LT arrives at the matcher, intent on interacting with the market maker (rate $\mu$);
  3. Nothing happens.

- **At** $t = \delta$. **Conclusion and pay-off:** The game concludes and the positions are used to compute the pay-off and corresponding utility for both the bandits and market maker (see section 2.3 for more details). Importantly, the utility of a negative pay-off is further inflated by the risk-aversion factor $\gamma$:

$$
\text{utility} = \begin{cases} 
\text{pay-off} & \text{if pay-off} \geq 0 \\
\gamma \cdot \text{pay-off} & \text{if pay-off} < 0 
\end{cases} \quad (1)
$$

where $\gamma \geq 1$.

In the next sections we will explore some of the concepts in more detail, starting with the detailed computation of the pay-off.

2.3 Pay-off and utility computation

2.3.1 General principles

To compute the pay-off for each HFT we observe that there are three sources of payoff:

- *Changes in position:* obtaining more (or less) of a financial asset results in a corresponding change in wealth;
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– Change of the intrinsic value of an asset: if the intrinsic value of an asset changes (due to good or bad news), the owner benefits (or suffers) accordingly;
– Income from a successful transaction (either buy or sell). The income can be both positive or negative depending on whether the HFT gets paid or needs to pay to complete the transaction.

These observations can be summarized in the following equations:

\[
\text{payoff} = \text{position}(t = \delta) \times \text{value}(t = \delta) + \text{income}(t = \delta) - \text{position}(t = 0) \times \text{value}(t = 0)
\]

but since we are assuming that all HFTs start with a zero position, the last term vanishes and we therefore get the simplified formula:

\[
\text{payoff} = \text{position}(t = \delta) \times \text{value}(t = \delta) + \text{income}(t = \delta).
\] (2)

The corresponding utilities \(U_M\) (for the market maker) and \(U_B\) (for a bandit) can then deduced using eq. 1.

Remarks

– Dependence of utility on spread Since the market makers earn money (spread) when their quotes are hit, their income increases with size of the spread. Conversely, bandits (if successful in sniping) cash the difference between the adjusted value and the spread. So their utility decreases as a function of the spread \(s\) (see also Fig. 2).
– News and LT arrivals are modelled as Poisson point processes with rates \(\alpha\) and \(\mu\), respectively. These rates measure the expected number of events per time unit. Therefore, \(\alpha \delta\) is the average (or expected) number of news arrivals in the time interval \(\delta\) and we are assuming that \(\delta\) (characterizing the exchange latency) is sufficiently small so, that on average, there will be at most one news event (within the interval): \(\alpha \delta \leq 1\). The same assumption holds for LT arrivals: \(\mu \delta \leq 1\).
– Intrinsic value changes of assets are equally likely to go up or down. However, downturns could lead to negative pay-offs which result in worse utilities (when \(\gamma > 1\)). Risk aversion therefore breaks the symmetry and results in downward pressure on the utility.
– The market maker earns a positive pay-off — equal to the (half-)spread \(s\) — each time he successfully interacts with an LT. If he’s fortunate, he will interact with two LTs (one at \(t = 0\) and one in the interval \(0 < t < \delta\)) that are interested in the opposite sides of the order book, yielding a pay-off of \(2s\). The other way for the market maker to earn money is by having a positive (negative) position when news arrives that increases (decreases) the intrinsic value of the asset.
– Bandits earn utility (or pay-off) from sniping. So if they are unsuccessful in sniping, their pay-off is zero. If they are successful, they get a pay-off which is usually positive. However, when the value of an asset is altered during
the sniping-race (due to additional news), their pay-off can be negative. Hence high news-rates \((\alpha \uparrow)\) have a potentially negative impact on HFTs.

2.3.2 Computational details

A detailed analysis of possible events and their corresponding probabilities in [14] shows that the utilities for both types of HFTs are linear functions in the spread \(s\). It therefore suffices to specify two points to pin down each of the functions. Natural choices are \(s = 0\) and \(s = \sigma\) and consequently:

\[
U_B(s) = \left(1 - \frac{s}{\sigma}\right) U_B(0) + \frac{s}{\sigma} U_B(\sigma)
\] (3)

and similarly,

\[
U_M(s) = \left(1 - \frac{s}{\sigma}\right) U_M(0) + \frac{s}{\sigma} U_M(\sigma)
\] (4)

Furthermore, it can be shown (again see [14]) that the utility for the market maker \(U_M\) is a convex combination of the utility due to LT-interaction \(U_M^{LT}\) on the one hand, and weighted sum of the utility due to news \(U_M^{News}\) and the (negative) utility suffered when getting sniped \(U_M^{Snipe}\) by a bandit. More precisely, using the short hand \(\beta := \alpha/\left(\alpha + \mu\right)\) and denoting the probability that the market maker will lose the MZ race by \(h := \left(H - 1\right)/H\), we get the following expression:

\[
U_M(s) = \left(1 - \beta\right) U_M^{LT} + \beta \left(U_M^{News} + h U_M^{Snipe}\right).
\] (5)

The actual values for the end points are summarized in the table below where we used the following abbreviations for notational convenience:

\[
\bar{\alpha} := \alpha \delta / 2, \quad \bar{\mu} := \mu \delta / 2, \quad q := \gamma - 1,
\]

and

\[
m := (1 - \bar{\mu}), \quad F := q \bar{\pi}, \quad G := q \bar{\pi}, \quad \bar{\pi} := m(\alpha + \mu).
\]

allowing us to rewrite the endpoints (for \(s = 0\) and \(s = \sigma\)) more succinctly:

| Table 1 Utility endpoints for market maker and bandit |
|------------------------------------------|
| \(s = 0\) | \(s = \sigma\) |
| \(U_B(s)\) | \(m \alpha \beta / H - F \alpha \beta / H\) |
| \(U_M^{LT}(s)\) | \(-F \alpha (1 + \bar{\pi})\sigma\) |
| \(U_M^{News}(s)\) | \(-G \sigma - 2 \bar{\pi} \sigma\) |
| \(U_M^{Snipe}(s)\) | \(-\gamma m \sigma - F \sigma\) |

Notice that both the endpoints for the bandits as well as the market maker scale linearly with \(\sigma\). As a consequence, we can redefine both \(s\) and \(u\) by using
\( \sigma \) as a natural scale (i.e. \( \tilde{s} = s/\sigma \) and \( \tilde{u} = u/\sigma \)). In terms of these new variables, the equations simplify somewhat:

\[
\tilde{U}_B(\tilde{s}) = (1 - \tilde{s}) \tilde{U}_B(0) + \tilde{s} \tilde{U}_B(1) \quad \& \quad \tilde{U}_M(\tilde{s}) = (1 - \tilde{s}) \tilde{U}_M(0) + \tilde{s} \tilde{U}_M(1)
\]

An equivalent way of thinking about this observation is that \( \sigma \) is a scale factor that has no impact on the qualitative results.

Dropping the tildes for notational convenience, we arrive at the following equations:

\[
\begin{align*}
U_B(s) &= (1 - s) U_B(0) + s U_B(1) \\
U_M(s) &= (1 - s) U_M(0) + s U_M(1)
\end{align*}
\] (6)

where the values at the endpoints can be determined from the following table:

| s   | \( A \) := \( m \beta \) | \( B \) := \( -F \beta \) |
|-----|-----------------|-----------------|
| s = 0 | \( U^L_T \) | \( -F \) |
| s = 1 | \( U^N_{\text{News}} \) | \( (1 + \mu) \) |
|       | \( U^{\text{Snipe}}_M \) | \( -\gamma m \) |
|       | \( -F \) |

Table 2 Value of utilities’ endpoints

Using eq. (6) we see that we can express the endpoints \( C := U_M(0) \) and \( D := U_M(1) \) for the market maker as (using the shorthand \( h := (H - 1)/H \)) as:

\[
C := (1 - \beta)(-F) + \beta(-G + h(-\gamma m)) = -\{(1 - \beta)F + \beta(G + hm\gamma)\}
\] (7)

Similarly:

\[
D := (1 - \beta)(1 + \mu) + \beta(2\mu + h(-F)) = \{(1 - \beta)(1 + \mu) + \beta(2\mu - hF)\}
\] (8)

For completeness’ sake we recall that:

\[
A := (1 - \frac{1}{\mu}) \frac{\beta}{H} = m \frac{\beta}{H}, \quad \text{ and } \quad B := -\frac{q\theta \beta}{H} = -F \frac{\beta}{H}.
\] (9)

As an aside we point out that

\[
(1 - \beta)F + \beta G = q \frac{\alpha \mu}{\alpha + \mu} \delta = \frac{1}{2} q \theta \delta \quad \text{ where } \quad \theta := 2 \frac{\alpha \mu}{\alpha + \mu}.
\]

Notice that \( \theta \) is the harmonic mean of the rates \( \alpha \) and \( \mu \) and therefore closer to the smaller of the two. Since \( \theta \) can be interpreted as a rate, we introduce \( \overline{\gamma} = \theta \gamma / 2 \) by analogy with \( \overline{\alpha} \) and \( \overline{\mu} \). In terms of this parameter, the endpoint \( C \) can be expressed alternatively as:

\[
C = -\left\{ q \overline{\gamma} + hm\beta \gamma \right\}
\] (10)

In the same vein:

\[
D = (1 + \overline{\mu}) - \beta(1 - \overline{\mu}) - \sigma \beta q \overline{\mu}.
\] (11)
2.4 Computing the equilibrium

Recall from the description of the MZ-game in section 2.2 that at the inception of the game (i.e. at \( t = -1 \)), the HFTs are agnostic about their eventual role in the game as this is assigned at \( t = -\frac{1}{2} \). The intersection point between the two utilities will therefore play a pivotal role in the HFT’s decision making as it represents the spread (hereafter denoted by \( s_K^* \)) that makes them indifferent as to which role they will play. In terms of the quantities \( A, B, C \) and \( D \) as defined above, it is straightforward to determine the equilibrium spread \( s_K^* \) and corresponding utility \( u_K^* \) by computing the intersection of the linear utility functions:

\[
\begin{align*}
  s_K^* &= \frac{(A - C)}{(A - C) + (D - B)} \\
  u_K^* &= \frac{(AD - BC)}{(A - C) + (D - B)}
\end{align*}
\]

(12)

It is shown in [14] that, if \( u_K^* > 0 \) (intersection lies above x-axis), the MZ game has a unique pure Nash equilibrium in which all HFTs will pick the optimal spread \( s_K^* \) and expect to earn the utility \( u_K^* := U_B(s_K^*) \equiv U_M(s_K^*) \).

However, if \( u_K^* \leq 0 \) (intersection lies on or below x-axis), things are less straightforward as pursuing the above strategy will now result in losses for both the market maker and the bandits. One of the contributions in [14] is the realisation that probabilistic sniping might suggest an additional equilibrium. This will be explored in detail in the next section.

**Fig. 2** Graph representing the expected utilities \( U_B(s) \) (line AB) and \( U_M(s) \) (line CD) for a bandit and the market maker respectively. Equilibrium spread \( s^* \) and utility \( u^* \) are obtained by determining the intersection of the two utilities (point of indifference). For the time being, we only consider sure sniping which corresponds to \( p = 1 \). For more details on probabilistic sniping (for which \( p < 1 \)), see section 3.
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3 Probabilistic versus sure sniping

3.1 The impact of probabilistic sniping

When the intersection point of the utility lines lies below the x-axis, the agents are no longer interested in posting a spread equal to the intersection value $s^*_K$ because they would expect to lose money. However, the MZ-paper identifies a scenario in which probabilistic sniping offers a viable alternative. In probabilistic sniping the bandit will only react to a sniping opportunity with probability $p < 1$. Although the precise quantitative impact of probabilistic sniping on the utilities $U_B$ and $U_M$ is somewhat intricate (see section 3.4), in qualitative terms the effect is fairly straightforward. Indeed, if a bandit is only going to take advantage of a fraction ($p$) of the sniping opportunities, his utility will decrease by a proportional amount. Geometrically (see Fig 3), this means that his utility function (line AB) will pivot about the zero crossing $s^{**}$, towards the x-axis. In the same vein, it is clear that if bandits refrain from sniping some of the time, the market maker’s utility $U_B(s)$, (line CD) will shift upwards as the negative impact of sniping is reduced. This suggests that there is a level of probabilistic sniping (denoted by $p^*$ in [14]) for which the market maker’s utility line (CD) has its zero-crossing at $s^{**}$ (see Fig. 4). With this in mind, it becomes clear why the MZ paper identifies $s^{**}$ as an additional equilibrium.

![Fig. 3 Schematic representation of the effect of probabilistic sniping on the utility functions of the market maker and bandit.](image-url)
– If $s < s^{**}$, the bandit prefers sure sniping as this results in a strictly better utility. However, the market maker refuses to play as he expects to suffer loses.

– If $s > s^{**}$, the bandit refuses to participate as his expected utility is negative. This situation is mitigated by probabilistic sniping, but at best, his losses will be reduced to zero, so there still is no incentive for the bandit to participate.

– If $s = s^{**}$ an interesting opportunity opens up: since the bandit is now indifferent between sniping and not sniping, he will therefore be equally receptive to mixed (or probabilistic) sniping. This has consequences for the behaviour of the market maker. Indeed, when the bandits agree to snipe with a probability $p \leq p^*$ then the market maker stands to earn zero (if $p = p^*$) or even positive utility (if $p < p^*$). Put differently (see Fig 5), as long as $s < s^{**}$, the MM is restricted to the line CD, but at $s = s^{**}$, his utility could experience a discontinuity as it might jump from F to E (if $p = p^*$), or even higher (if $p < p^*$).

Fig. 4 The probability $p^*$ is defined as the value for which probabilistic version of the market maker utility crosses the x-axis at $s = s^{**}$.
3.2 What compels bandits to snipe probabilistically?

Whereas the optimal spread $s^*_K$ corresponding to positive utility $u^*_K := U_B(s^*_K) \equiv U_B(s^*_K) > 0$ seems straightforward and yields a conventional Nash equilibrium, the significance of the novel equilibrium at $s^{**}$ is more problematic as it relies on the willingness of the bandits to snipe at probabilities $p \leq p^*$. In the MZ paper [14] this is stated explicitly (p.1191):

*To restore equilibrium, the HFBs snipe only probabilistically to reduce adverse-selection cost for the HFM who can then reduce spread to a level at which HFBs earn zero utility on sniping and therefore are OK with sniping only probabilistically.*

From a game theoretical standpoint this is puzzling because it suggests that the bandits would take steps that ultimately don’t bring them any benefit. Why would they do something like that?

To recapitulate: in the initial phase ($t = -1$) each player knows all relevant game parameters and can therefore determine that $\gamma > \pi$. From this, he will deduce (mindful of the MZ-logic) that the only rational spread posting would be $s = s^{**}$, which all HFTs therefore duly commit to. However, once the market maker has been selected, there is no incentive for the remaining HFTs (bandits) to continue with the game, as their expected utility is zero no matter what happens next. As a consequence, it is impossible to predict what the behaviour of these bandits is going to be.

But this observation has consequences for the market maker too! He will only be interested in playing the game if he is sure that the bandits will snipe with a probability $p \leq p^*$. But there is nothing in the game that ensures this, as there is no way of predicting how the bandits will behave. So with
this intrinsic uncertainty in mind, why would the market maker (or any other
HFT for that matter) commit to posting the spread $s = s^{**}$?

The conclusion that follows from this is that in the initial phase of the
game, all HFTs are informed about the game parameters and can therefore
compute $s_K^{*}$ and $u_K^{*}$ using eq. (12). If $u_K^{*} > 0$ they will all post spread $s_K^{*}$, and if they find out (at $t = -\frac{1}{2}$) that they were not selected as market maker, they will subsequently react to every sniping opportunity (i.e. sure sniping). If on the other hand, they conclude that $u_K^{*} \leq 0$ they might simply refrain from playing. So, although probabilistic sniping is a possibility, it’s not a compelling one, and the model is therefore unlikely to reflect actual phenomena.

One way out of this conundrum might be the introduction of a small trans-
action cost that would dissuade bandits from sniping [18]. However, in this
paper we argue that probabilistic sniping will play a major role once we
realise that an interpretation in terms of repeated games provides a more ac-
curate model for an (high-frequency) exchange. More specifically, we propose
to model the activities at the high-frequency exchange as an infinite repetition
of the MZ game. This means that we are looking at a repeated game (with an
infinite horizon) for which the MZ game is the one-shot (stage) game.

In this alternative setting a player can adopt more far-sighted strategies
that attempt to maximise long-term gains. This change of viewpoint has a
number of important consequences:

- It identifies situations in which probabilistic sniping is genuinely profitable
  and therefore extends the original MZ conclusions by introducing qualita-
tively different solutions. More specifically, it builds on the Folk Theorems
to find additional subgame perfect equilibria (SPE) that yield a strictly
better utility for all HFTs involved.
- It explains why bandits are willing to engage in probabilistic sniping, for
  which there seems to be no compelling reason in the single-shot MZ game.

3.3 The MZ stage game as a sequential game and the corresponding
strategies

Before we explore the strategies that are available in the (indefinitely) repeated
MZ game (a.k.a. $MZ^\infty$), we need to pause for a minute and clarify the nature
of the strategies that are available to the players in the single shot stage game
(a.k.a. $MZ^1$). To this end, it is helpful to realise that the stage involves different
interdependent steps and can therefore be seen as a sequential game involving a
(possibly) random move by “nature”. Below, we give an explicit representation
of the decision tree for the game and clarify a subtle but important point in
the interpretation of probabilistic sniping.

3.3.1 Game tree

In section 2.2, we already outlined the detailed chronology of the various defining
events in the game. For the interpretation of the MZ game as a sequential
game it is also helpful to highlight the various steps for an individual HFT in the corresponding decision tree (also see Fig. 6):

1. HFT (at time $t = -1$) has to choose a spread $s$ from the continuous action space $0 \leq s \leq \sigma$. Alternatively, he can refrain from playing altogether (NULL action resulting in zero utility).

2. Nature (at time $t = -1/2$) selects one of the HFTs as MM (using the rule specified earlier in section 2.2), relegating the others to the role of bandit.

3. At $t = 0$, each HFT knows its type (market maker or bandit) and can therefore make his reaction to the trigger event (news or LT arrival) contingent on his type. This results in a new bifurcation of the decision tree:
   - if he’s bandit: he has to choose between quitting the game (NULL-action with utility 0), or some form of sniping: sure $p = 1$, no ($p = 0$) or mixed sniping ($0 < p < 1$) resulting in a utility $u_B(s, p)$.
   - if he’s MM: he either proceeds and the game ends with utility $u_M(s, p)$ (which actually also depends on $p$), or he withdraws from the game (NULL with utility 0):

![Fig. 6](image_url) The MZ game as a sequential game against nature. Notice that in the last decision point, the MM is uncertain about his pay-off as his utility $u_M(s, p)$ as it depends on the bandit’s sniping probability which at the moment of decision is unknown to him!

### 3.3.2 Actions and Strategies

Disregarding the trivial NULL action, what actions and strategies are available to the players? At the first decision point, it is clear that each HFT can choose a spread $s$, which corresponds to a **pure but continuous action**.

At the second HFT decision point, the HFTs know their type. The strategy for the market maker is straightforward: if the trigger event is the publication
Fig. 7 Interpretation of the sniping probability. **Left:** In the MZ paper, the pure actions available to the bandit are *sniping* and *not sniping*. Mixed or (probabilistic) sniping is therefore only possible when the bandit is indifferent between the pure actions. **Right:** In this paper, we interpret the sniping probability as a *pure but continuous* action that the bandit can take. In this interpretation the bandit need not be indifferent between the extremes in order to adopt a probabilistic strategy.

of a *news item*, the intrinsic value of the asset jumps and he will race to cancel his stale quotes. The choice of strategies for the other HFTs (bandits) is more subtle. Indeed, in the MZ paper [14] the authors make some tacit assumptions regarding the pure strategies that are available to the HFTs once they know their type. Since we will take an alternative point of view in this paper, we will spell out this difference in detail:

- In [14], the authors describe mixed sniping as randomly mixing between the pure and discrete strategies of *sniping* and *not sniping*. As a consequence, mixed sniping will only feature as a Nash equilibrium when the bandits are indifferent between sniping and not sniping!
- In our interpretation, we think of each bandit selecting a sniping probability $p$ as him choosing a pure strategy from a continuous action space. This allows us to introduce probabilistic sniping in situations where the expected utilities at the extremes (i.e. $p = 0$ or 1) are not necessarily the same.

3.4 Utility functions for probabilistic sniping

Up to this point we have assumed that HFBs (“bandits”) will always attempt to snipe whenever a sniping opportunity presents itself (i.e. when the triggering event at $t = 0$ is *news*). We will say that the bandits are engaged in probabilistic sniping (with sniping probability $p$) when they only react to a sniping opportunity with probability $p$. In order to appreciate the effect of probabilistic sniping we need to re-evaluate the utilities for both market makers and bandits.
Impact of sniping on market maker and bandits  The market maker (MM) does not act probabilistically (i.e. he races for sure to cancel his stale quotes) but his utility depends on the number of bandits that do attempt to out-race him. Indeed, the third term in the expression for the market maker’s utility (eq. 5) basically multiplies the (negative) utility due to sniping by the probability that the market maker will get sniped, which — in the case of sure sniping — equals \( h := (H-1)/H \). However, when individual bandits snipe (independently!) with a probability \( p < 1 \), the probability that the market maker will get sniped is more complicated and depends on \( p \). We will therefore denote it as \( h(p) \), defined formally as:

\[
 h(p) := P \{ \text{MM will lose MZ race} \mid \text{individual bandits snipe with prob } p \} \tag{13}
\]

This factor plays a crucial role in the utility computation for both the market maker and the bandits:

- **Market maker:** the probability \( h(p) \) extends (and supplants) the multiplication factor \( h \equiv h(1) \) in eq. (5), yielding the more general expression:

\[
 U_M(s, p) = (1 - \beta) U_{\text{LT}}^M(s) + \beta \left( U_{\text{News}}^M(s) + h(p) U_{\text{Snipe}}^M(s) \right). \tag{14}
\]

- **Bandit:** To determine the utility of the individual bandit \( (U_B) \) under probabilistic sniping, we need his probability of winning the sniping race, since under all other circumstances, his pay-off will be zero. Notice that this probability is identical for all bandits (symmetric game), and therefore equals:

\[
 \frac{1}{H-1} h(p). \tag{15}
\]

Consequently, the utility for an individual bandit equals a linear function (in \( s \)) defined by:

\[
 U_B(s, p) = (1 - s) A g(p) + s B g(p) \tag{16}
\]

where \( A \) and \( B \) are defined in eq. (9), while

(The additional factor \( H \) in the numerator is necessary to cancel that same factor which has been included in the definition of the points \( A \) and \( B \).)

Notice that \( g(1) = 1 \) and hence we recover the solution for sure sniping. Furthermore, eq. (16) shows that probabilistic sniping induces the utility function to pivot (towards smaller values) about its intersection with the \( x \)-axis (recall Fig. 3).

**Computation of \( h(p) \)**  It is conceptually easier to focus on the complementary probability \( h(p) := 1 - h(p) \) that the market maker will win the MZ-race and therefore thwart the sniping attempts of the bandits. Specifically, since the \( H - 1 \) bandits decide independently and with equal probability \( p \), whether or
not they will snipe, the number of bandits that will enter the race is a random variable \((N_B\text{ say})\) that is distributed according to a binomial distribution:

\[
N_B \sim \text{Bin}(H - 1, p).
\]

This is equivalent to saying that the probability of having \(b = 0, 1, \ldots, H - 1\) bandits participating in a race, equals:

\[
P(N_B = b) = \binom{H - 1}{b} p^b (1 - p)^{H - b-1} \text{ for } b = 0, 1, \ldots, H - 1 \tag{17}
\]

When there are \(b\) bandits racing, the probability that the market maker will win the race equals \(1/(1 + b)\) since each participant has the same probability of winning. As a consequence, the probability \(h(p)\) that market maker will win the race (and therefore escape sniping) equals

\[
\bar{h}(p) = \sum_{b=0}^{H-1} \frac{1}{b+1} \binom{H - 1}{b} p^b (1 - p)^{H - b-1} \tag{18}
\]

which after some straightforward manipulation can be reduced to

\[
\bar{h}(p) = \frac{1 - (1 - p)^H}{pH} \tag{19}
\]

Notice that for sure sniping \(\bar{h}(p = 1) = 1/H\) which is consistent with our earlier observations. Similarly, \(\lim_{p\downarrow 0} \bar{h}(p) = 1\) as expected. As a consequence, the probability \(h(p)\) that the market maker will get sniped depends on the sniping probability \(p\) for the individual bandit and equals:

\[
h(p) = 1 - \bar{h}(p) = \frac{pH - 1 + (1 - p)^H}{pH} \tag{20}
\]

which for sure sniping (i.e. \(p = 1\)) reduces to the familiar factor \(h := h(1) = (H - 1)/H\). For completeness we point out that tedious but straightforward calculations show that:

\[
\lim_{p\rightarrow 0} h'(p) = \frac{H - 1}{2}. \tag{21}
\]

Endpoints as function of sniping probability Since we want to establish a common notational framework for the sure and mixed (probabilistic) sniping approach, we re-express the endpoints of the utility lines (i.e. A, B, C and D) as functions of \(p\). To this end, we draw on eqs. \([16]\) and \([14]\) and notice that the dependence on the sniping probability for the bandit (parameters \(A\) and \(B\)) is captured by the common proportionality factor \(g(p)\):

\[
A(p) := Ag(p) \quad \text{and} \quad B(p) := Bg(p), \tag{22}
\]

This dependence of both \(A\) and \(B\) on the common proportionality factor \(g(p) \leq 1\) implies that probabilistic sniping causes the bandit’s utility to pivot about its zero-crossing towards the \(x\)-axis (see Fig 9).
To snipe or not to snipe, that’s the question!

Fig. 8 Probability \( h(p) \) the market maker will get sniped as a function of the sniping probability \( p \) of the individual bandits (cf. eq. 20). We also show the derivative \( h'(p) \) (red curve).

Fig. 9 Basic geometry of probabilistic sniping: The solid lines represent the case of sure sniping resulting in an equilibrium \( (s^*, u^*) \) (which is non-playable since \( u^* < 0 \)). Reducing the probability of sniping to \( p < 1 \) pushes the MM utility function CD upwards (as the MM benefits from less sniping), while the bandit’s utility pivots about its zero crossing. As a consequence, the intersection moves to a new intersection point \( (s^*(p), u^*(p)) \) which is now playable as \( u^*(p) > 0 \).
For the market maker (parameters C and D) the dependence on sniping probability $p$ enters via the factor $h = h(p)$:

$$C(p) = -\{q\theta + h(p)\beta\gamma\}$$

(23)

and similarly

$$D(p) = (1 - \beta)(1 + \theta) + \beta[2\pi - h(p)\beta]$$

(24)

Slightly abusing notation, we will denote $A(1) = A$, $B(1) = B$, $C(1) = C$ and $D(1) = D$. Finally, we point out that

$$C(p) = C + (h(1) - h(p))\beta\gamma = C + m\beta\gamma\psi(p).$$

(25)

where $\psi(p) := h(1) - h(p)$ is a positive but decreasing function. In the same vein,

$$D(p) = D + (h(1) - h(p))\beta\gamma - 1 = D + \beta\gamma - 1\psi(p).$$

(26)

From eqs. (25) and (26) we learn that reducing the sniping probability pushes the market maker’s utility upwards (again, see Fig 9). As we will explore in the next session, this is essentially the reason why probabilistic sniping can result in a better utility.

### 3.5 When is probabilistic sniping advantageous?

Is there an advantage in switching from sure sniping to probabilistic sniping? Obviously, less sniping is advantageous for the market maker as his pay-off is reduced by sniping. However the answer is less clear for the HF bandits. Some basic intuition is gleaned from Fig. 10, which illustrates the difference between sure and probabilistic sniping. More specifically, let us assume that sure sniping results in an equilibrium $E$ such that $u^* := u(s^*) = 0$. Sure sniping ($p = 1$) is represented by the two blue lines (AB and CD, expected utility for bandit and market maker, respectively). Changing to probabilistic sniping ($p < 1$) results in two different lines ($A'B'$ and $C'D'$, respectively). Notice that $A'B'$ pivots about the equilibrium point $E$, while $C'D'$ shifts upwards (as less sniping means less risk for the market maker). As a consequence, the new intersection point $E'$ will yield a (strictly) positive utility.

Another way to put this is that the derivative $du^*/dp$ (evaluated at $p = 1$) is negative: smaller values for $p$ result in higher utility. For a concrete, numerical example, see Fig 12 below. In the next section, we will recast this geometric insight into an algebraic expression, but first we will show that this new equilibrium is not a Nash equilibrium for the one-shot MZ.
Fig. 10 Geometric representation that illustrates why probabilistic sniping is advantageous when $u^* = 0$. See main text for more information.

Fig. 11 An example where probabilistic sniping is advantageous even though sure sniping results in a strictly positive utility. In this example, the probability of sniping $p$ starts at $p = 1$ and is decremented with steps of 0.05 to $p = 0$. Notice how the expected utility rises from 0.02 for sure sniping ($p = 1$) to a maximum of 0.05 for $p = 0.15$.

*Does $(s^*(p), u^*(p))$ constitute a Nash equilibrium of the MZ game?* One of the reasons why for $p < 1$ the new intersection point $(s^*(p), u^*(p))$ is not considered in the MZ paper (even if $u^*(p) > 0$), is that it does not constitute a Nash equilibrium for the (single-shot) MZ game. To see this, it suffices to realise that when $p < 1$, a bandit will always benefit from unilaterally deviating to sure sniping (assuming $u^*(p) \equiv U_B(s^*(p)) > 0$). As a consequence, these intersection points are uninteresting in terms of the single-shot version of the MZ game.
Fig. 12 Example of transition from sure to probabilistic sniping (for a definition of $\gamma_K$ and $\gamma_L$ see section 4). In the above example we fix the parameters $\alpha = .45$, $\mu = .5$, $\delta = .5$, $H = 5$. Next we set $\gamma$ to four different values: $\gamma = 1.7575$ (slightly risk-averse, sure sniping), $\gamma = \gamma_K = 2.5150$ (risk averse, knife edge) and $\gamma = \gamma_L = 7.8313$ (strongly risk-averse) and $\gamma_K < \gamma < \gamma_L = 3.5$ (probabilistic sniping). Notice that in the bottom left picture probabilistic sniping yields a better utility, even though the utility for sure sniping is positive!

3.6 Equilibrium spread and utility for probabilistic sniping

From the discussion above we know that for probabilistic sniping the endpoints of the linear utility functions are replaced by their probabilistic counterparts:

$$A \rightarrow Ag(p), \quad B \rightarrow Bg(p), \quad C \rightarrow C(p), \quad \text{and} \quad D \rightarrow D(p).$$

Using the same logic as before we conclude that the equilibrium for probabilistic sniping is given by (cf. eqs. [12]):

$$s^*(p) = \frac{Ag(p) - C(p)}{(Ag(p) - C(p)) + (D(p) - Bg(p))} \quad (27)$$

$$u^*(p) = \frac{Ag(p) D(p) - Bg(p) C(p)}{(Ag(p) - C(p)) + (D(p) - Bg(p))} \quad (28)$$

For ease of reference, we will denote the common denominator by

$$Q(p) := Ag(p) - C(p) + D(p) - Bg(p),$$

and

$$N(p) := Ag(p) D(p) - Bg(p) C(p).$$
Recall that – in contradistinction to the MZ paper – we do not restrict the equilibrium solution to the case of positive utilities. Of course, it is obvious that traders will not participate when the expected utilities are negative.

4 Characterising the transitions in sniping behaviour

In this section we introduce a geometric argument to deduce the general conditions under which

– the transition from sure to probabilistic sniping occurs, and
– probabilistic sniping is no longer advantageous.

In section 5 below we will then show how these general conditions can be related to risk aversion and how they give rise to two thresholds ($\gamma_K$ and $\gamma_L$) that govern these transitions.

4.1 Transition from pure to probabilistic sniping

As argued above, the equilibrium $(s^*, u^*)$ can be seen as a function of the sniping probability $p$, i.e. $(s^*(p), u^*(p))$. To decide whether probabilistic sniping is advantageous, we need to determine when the slope of the tangent to the curve $u^*(p)$ at $p = 1$ changes sign (see Fig. 12):

$$\left. \frac{du^*(p)}{dp} \right|_{p=1} > 0 \quad \text{(sure sniping)} \quad \rightarrow \quad \left. \frac{du^*(p)}{dp} \right|_{p=1} < 0 \quad \text{(probabilistic sniping)}$$

Indeed, if the slope of this tangent is positive, it means that reducing $p$ from 1 (sure sniping) to a lower value $p < 1$ decreases the value of $u^*(p)$. Hence, sure sniping is better than probabilistic sniping. Conversely, if the slope of the tangent is negative, moving from sure to probabilistic sniping (i.e. reducing $p$) does improve $u^*(p)$ and therefore probabilistic sniping is to be preferred.

Using the notation introduced in section 3.6 we know that $u^*(p) = N(p)/Q(p)$ and hence

$$\left. \frac{du^*(p)}{dp} \right|_{p=1} = \frac{N'(1)Q(1) - N(1)Q'(1)}{Q^2(1)}$$

(30)

Since $N(p) = Ag(p)D(p) -Bg(p)C(p)$ we get:

$$N'(p) = A \left\{ g'(p)D(p) + g(p)D'(p) \right\} - B \left\{ g'(p)C(p) + g(p)C'(p) \right\},$$

(31)

and likewise:

$$Q'(p) = Ag'(p) - C'(p) + D'(p) - Bg'(p).$$

(32)

From eq. 20 we compute

$$h'(p) = \frac{1 - (1-p)^{H-1}}{p^H},$$

(33)
whence
\[ h'(1) = \frac{1}{H}, \quad \text{and} \quad g'(1) = \frac{1}{H - 1}. \] (34)

Consequently,
\[ C'(p) = -\beta m \gamma h'(p) = -AH\gamma h'(p) \] (35)
whence
\[ C'(1) = -AH\gamma h'(1) = -A\gamma \] (36)
Likewise,
\[ D'(p) = -\beta F h'(p) \] (37)
whence:
\[ D'(1) = -\beta F h'(1) = -F \frac{\beta}{H} = B. \] (38)

Plugging all these values into the equations above we obtain:

- \( N_1 := N(1) = AD - BC \)
- \( N'(1) = \left( \frac{AD - BC}{H - 1} \right) + AB(1 + \gamma) \) Notice that \( AB(1 + \gamma)h < 0 \) (as \( B < 0 \))
- \( Q_1 := Q(1) = A - C + D - B > 0 \)
- \( Q'(1) = \frac{A - B}{H - 1} + D'(1) - C'(1) = \frac{A - B}{H - 1} + (A\gamma + B) \)

To find out what conditions govern the transition from sure to probabilistic sniping, we have to determine for which parameter-combination the derivative \( du^*/dp \) (evaluated at \( p = 1 \)) changes sign. To this end we hark back to eq. (30) and conclude that we have to find the conditions under which:

\[ N'(1) Q(1) - N(1) Q'(1) = 0 \] (39)

or again (after multiplication by \( H - 1 \) for notational convenience):

\[ N_1(D - C) + (H - 1) \{ ABQ_1(\gamma + 1) - N_1(A\gamma + B) \} = 0 \] (40)

In section 5 we will revisit this condition and interpret it as a (cubic) equation in the risk aversion factor \( \gamma \). This will allow us to identify the specific risk aversion threshold \( \gamma_K \) that governs this transition.

4.2 Transition from probabilistic to non-sniping

If the expected utility associated with the sure sniping equilibrium is sufficiently negative (i.e. \( u^*(s^*, p = 1) \ll 0 \)), even probabilistic sniping will not be able to turn this into profits. Geometrically, this corresponds to a situation in which the path traced by \((s^*(p), u^*(p))\) will never enter (strictly) positive territory before it hits the x-axis at \( p = 0 \) (see Fig. 12 bottom-right panel).
To snipe or not to snipe, that’s the question!

This transition can therefore be characterised by a condition which is analogous to the characterisation of the transition from pure to probabilistic sniping in eq. (29), this time however focusing on the tangent at \( p = 0 \):

\[
\left. \frac{du^*(p)}{dp} \right|_{p=0} > 0 \quad \text{(probabilistic sniping)} \quad \longrightarrow \quad \left. \frac{du^*(p)}{dp} \right|_{p=0} < 0 \quad \text{(no sniping)}
\]

This transition therefore occurs when

\[
\left. \frac{du^*(p)}{dp} \right|_{p=0} = 0
\]

(41)

Expanding

\[
\left. \frac{du^*(p)}{dp} \right|_{p=0} = \frac{N'(0)Q(0) - N(0)Q'(0)}{Q^2(0)}
\]

and using that \( N(0) = 0 \) while \( Q(0) = D - C > 0 \), we see that eq. (42) simplifies to:

\[
N'(0) = 0
\]

or, after expansion:

\[
N'(0) = Ag'(0)D(0) + Ag(0)D'(0) -Bg'(0)C(0) -Bg(0)C'(0)
\]

\[
= g'(0)AD(0) - BC(0) + g(0)AD'(0) - BC'(0)
\]

Since we know that \( g(0) = 0 \) and \( g'(0) = H/2 \), it follows that transition threshold is governed by the equation

\[
N'(0) = \frac{H}{2} \{ AD(0) - BC(0) \}
\]

\[
= \frac{H}{2} \{ A(D + n\beta(\gamma - 1)h) - B(C + m\beta\gamma h) \} = 0
\]

or again:

\[
A \{ D + n\beta(\gamma - 1)h \} - B \{ C + m\beta\gamma \} = 0.
\]

(43)

In the next section we will interpret both conditions (43) and (40) as functions of \( \gamma \) and deduce appropriate threshold values for the risk aversion factor.
5 Risk aversion induces a transition to probabilistic sniping

5.1 Expanding game parameters as functions of risk aversion $\gamma$

One of the key insights expounded in [14] is the realisation that increasing risk aversion induces a transition from sure to probabilistic sniping. Indeed, increasing risk aversion $\gamma$ puts downward pressure on the utilities, and will induce a transition from sure to probabilistic sniping. This can be seen by expanding the endpoints $B, C$ and $D$ as functions of $\gamma$ from which it becomes clear that $\gamma$ appears with negative coefficients:

- $A$ is independent of $\gamma$
- $B = -E(\gamma - 1)$ where $E = \overline{\pi} \beta / H$
- $C = -M \gamma + \bar{\theta}$ where $M = \bar{\theta} + h m \beta$, and $\bar{\theta} := \theta \delta / 2$
- $D = -E(H - 1)(\gamma - 1) + Z$ where $Z = (1 + \bar{\theta}) - \beta(1 - \bar{\theta})$

Downward migration of the endpoints $B, C$ and $D$, results in a corresponding downward movement of the equilibrium at the intersection of the lines, and consequently, for $\gamma$ sufficiently large, the point of intersection will hit, and then cross, the zero-axis. Including the $\gamma$-dependencies outlined above, we conclude:

$$N_1 = -EM \gamma^2 + E \{ (\theta \delta + h m \beta) - A(H - 1) \} \gamma + \{ AZ + AE(H - 1) - E \overline{\theta} \} \quad (44)$$

and

$$Q_1 = \{ M - E(H - 2) \} \gamma + \{ A + Z + E(H - 2) - \overline{\theta} \} \quad (45)$$

5.2 Transition from pure to probabilistic sniping as a function of $\gamma$

5.2.1 Transition threshold $\overline{\gamma}_K$

Plugging expressions (44) and (45) into eq. (40) yields a cubic equation

$$K(\gamma) := K_3 \gamma^3 + K_2 \gamma^2 + K_1 \gamma + K_0 = 0, \quad (46)$$

where

- Coef for $\gamma^3$
  $$K_3 = EM (A(H - 1) - E(H - 2))$$
  $$= (EM + AE(H - 1)) (E + M - E(H - 1))$$
  $$= EAE(H - 1) (H - 2) - M^2$$

- Coef for $\gamma^2$
  $$K_2 = E(H - 1)(H - 2) (A - E) A$$
  $$+ M^2 - M (Z - 2 \overline{\theta} + 2 A(H - 1)) - AZ (H - 1)$$
To snipe or not to snipe, that’s the question!

\[ K_1 = E \{ (\bar{\theta} + M - A(H - 1)) (A + Z - \bar{\theta} + E(H - 2)) \} \\
- E (A + E (H - 2)) (\bar{\theta} + M - A(H - 1)) \\
+ (M - E(H - 2)) \{ AZ + 2AE(H - 1) - E\bar{\theta} \} + \\
( E\bar{\theta} - AZ - AE(H - 1)) (A(H - 1) - E(H - 2)) \\
= E \{ \bar{\theta} + M - A(H - 1) \} (Z - \bar{\theta}) \\
+ (M - E(H - 2)) \{ AZ + 2AE(H - 1) - E\bar{\theta} \} + \\
( E\bar{\theta} - AZ - AE(H - 1)) (A(H - 1) - E(H - 2)) \\
\]

Coef for \( \gamma^0 \)

\[ K_0 = (E\bar{\theta} - AZ - AE(H - 1)) (A + E(H - 2)) \\
+ \{ A(Z + E(H - 1)) - E\bar{\theta} + AE(H - 1) \} (A + Z - \bar{\theta} + E(H - 2)) \\
\]

To determine the critical value \( \gamma^K \) at which the transition between pure and probabilistic (mixed) sniping occurs, we need to find the zero-crossing of eq. (46) that exceeds 1.

Recall that a cubic equation has at least one solution (zero-crossing). However, we need to make sure that this solution satisfies \( \gamma \geq 1 \) can therefore be interpreted as a risk aversion factor. This brings us to the following definition:

**Definition of transition threshold \( \gamma^K \)**

We will denote by \( \gamma^K \) the unique solution of the cubic equation (46) for which \( \gamma \geq 1 \). Put differently, \( \gamma^K \) is defined by the conditions:

\[ K(\gamma^K) = 0 \quad \text{and} \quad \gamma^K > 1. \quad (47) \]

This parameter determines the threshold that governs the transition from pure to probabilistic sniping.

### 5.2.2 Existence of transition threshold \( \gamma^K \)

To prove the existence of the zero-crossing \( \gamma^K \) we show that (also see Fig [13])

1. \( K_3 < 0 \) indicating that \( \lim_{\gamma \to \infty} K(\gamma) = -\infty \), i.e. \( K(\gamma) \) will be negative for large enough \( \gamma \);
2. \( K(1) = K_3 + K_2 + K_1 + K_0 > 0 \).

These two observations imply that the cubic polynomial \( K \) has to have at least one zero crossing \( \gamma^K > 1 \). To prove these assertions above, we proceed as follows:

1. The sign of \( K_3 \) is determined by the sign of \( AE(H - 1)(H - 2) - M^2 \). We observe that
– Using the definitions we get:

\[ M = \bar{\theta} + m\beta h = \beta [\pi + (1 - \pi)h] = \frac{\beta}{H} [\pi H + (1 - \pi)(H - 1)], \]

and hence:

\[ M^2 = \left(\frac{\beta}{H}\right)^2 [\pi H + (1 - \pi)(H - 1)]^2. \]

– Similarly,

\[ AE(H - 1)(H - 2) = m\beta \frac{H}{H} \left(\frac{\beta}{H}\right)^2 [\pi m(H - 1)(H - 2)]. \]

Cancelling the common factor \((\beta/H)^2\) we see that the second expression is less than \((H - 1)(H - 2)\) since \(\pi m < 1\). On the other hand, the first expression is the square of a convex combination of \(H\) and \(H - 1\) and therefore exceeds \((H - 1)^2\). From this it is straightforward to conclude that \(K_3 < 0\).

2. Using Matlab's symbolic toolbox allows us to simplify the expression for \(K(1)\) considerably:

\[ K(1) = K_3 + K_2 + K_1 + K_0 = AZ(A + M + Z - \bar{\theta} - AH) \quad (48) \]

– Using the definition we get:

\[ (A + M + Z - \bar{\theta} - AH) = \frac{m\beta}{H} + \bar{\theta} + hm\beta + (1 + \pi) - \beta(1 - \pi) - \bar{\theta} - \frac{m\beta}{H} H \]

– by simplifying and cancelling \(\bar{\theta}\), we get:

\[ K(1) = -m\beta + (1 + \pi) = (1 - \beta) + (1 + \beta)\pi \]

As \(AZ > 0\) and \(0 < \beta < 1\), the sign of the expression is positive, we conclude that \(K(1) > 0\).

5.3 Transition from probabilistic to no sniping as function of \(\gamma\)

**Definition of threshold \(\gamma_L\)** The threshold \(\gamma_L\) is defined as the lowest threshold for risk aversion among the HFTs beyond which both sure and probabilistic sniping results in negative utility \(u^*(s^*, p) < 0, \quad (\forall 0 \leq p \leq 1)\).

The threshold satisfies eq. (43) which can be expanded in a quadratic equation in \(\gamma\) (after factoring out a common factor \(H\)):

\[ L(\gamma) := L_2 \gamma^2 + L_1 \gamma + L_0 \quad (49) \]

where:

– Coef for \(\gamma^2\)

\[ L_2 = E(hm\beta - M) = -\pi \beta \bar{\theta}/H \]
To snipe or not to snipe, that’s the question!

– Coef for $\gamma^1$

\[
L_1 = h\beta A\pi - h\beta mE + E \left\{ \bar{\theta} - A(H - 1) + M \right\} \\
= 2\pi \beta \bar{\theta}/H
\]

– Coef for $\gamma^0$

\[
L_0 = - \left( E\bar{\theta} - AZ + AE \left( H - 1 \right) + \frac{A\pi \beta \left( H - 1 \right)}{H} \right) \\
= AZ - \pi \beta \bar{\theta}/H
\]

Introducing the notation $L = -\pi \beta \bar{\theta}$ the quadratic equation simplifies to:

\[L\gamma^2 - 2L\gamma + (AZ + L) = 0\]

which can be solved explicitly:

\[
\gamma = 1 \pm \sqrt{-AZ/L} = 1 \pm \sqrt{1 - \frac{\mu}{\pi \gamma} Z}.
\] (50)

The largest root is guaranteed to be positive and corresponds to the threshold $\tau_L$ which is therefore defined as:

\[
\tau_L := 1 + \sqrt{1 - \frac{\mu}{\pi \gamma} Z}.
\] (51)

A numerical example is shown in Fig.13 where $\alpha = .45, \mu = .5, \delta = .5, H = 5$ resulting in a quadratic equation for which the roots are $-5.8313$ and $7.8313$, whence we conclude that $\tau_L = 7.8313$.

5.4 Optimal sniping probability $p^*_K$

In the preceding sections we have shown that increasing risk aversion ($\gamma > 1$) creates multiple sniping regimes for the bandits:

– $1 \leq \gamma < \tau_K$: Sure sniping is most profitable; This situation corresponds to the top left panel in Fig.12

– $\tau_K \leq \gamma < \tau_L$: Probabilistic sniping results in higher utilities; in fact, there is an optimal sniping probability $p^*_K$ (see below) which results in a optimal (positive) utility. This situation corresponds to the bottom left panel in Fig.12

– $\gamma \geq \tau_L$: No sniping (resulting in zero utility) is the best option. This situation corresponds to the bottom right panel in Fig.12

The situation in the second case $\tau_K < \gamma < \tau_L$ is schematically illustrated in Fig.14 where $\tau_K < \gamma < \tau_L$ is an optimal sniping probability that yields the largest utility:

\[
p^*_K := \arg\max_p u^*(p)
\]

Since this is a unique optimal point it provides a natural focal point for the game.
Fig. 13 Probabilistic sniping is profitable for values of the risk aversion parameter $\gamma$ between the thresholds $\gamma_K$ that is the positive zero-crossing (exceeding 1) of cubic equation (47) and $\gamma_L$ that is the positive zero-crossing of the quadratic equation (51) for MZ game parameters $H = 5$, $\alpha = 0.45$, $\mu = 0.5$, $\delta = 0.5$.

Fig. 14 Definition of $p^*_K$: When probabilistic sniping is advantageous (i.e. $\gamma_K < \gamma < \gamma_L$), reducing the sniping probability $p$ shifts the equilibrium position $(s^*(p), u^*(p))$ along a concave arc, the top of which corresponds to the optimal sniping probability $p^*_K$. 

5.5 Summary: Transition from Sure to Probabilistic Sniping

- The transition from sure to probabilistic sniping is governed by the condition in eq. (40).
- There are several ways to satisfy this condition, but one (sure) way is to increase the risk aversion $\gamma$.
- Expanding eq. (40) as a function of $\gamma$ we end up with a cubic polynomial $K(\gamma)$ that has a unique root $\tau_K > 1$. This threshold $\tau_K$ governs the transition from sure sniping ($\gamma < \tau_K$) to probabilistic sniping ($\gamma > \tau_K$).
- The second relevant threshold is $\tau_L$ beyond which even probabilistic sniping no longer yields positive utility. Hence if risk aversion $\gamma > \tau_L$ then all HFTs no longer expect to benefit from the game.
- For values of the risk aversion factor $\tau_K < \gamma < \tau_L$, the optimal sniping probability is given by $p^*_K$.

Fig. 15 The figure shows the equilibrium utility spread as a function of risk aversion in both sure sniping ($u^*_K(p = 1)$ blue line) and optimal probabilistic sniping ($u^*_K(p = p^*_K)$ red line).

6 Optimal probabilistic sniping as SPE in repeated MZ$^\infty$ game

6.1 Subgame-Perfect Equilibria (SPE) in repeated games

The emergence of collaboration in repeated games In the sections above we identified the risk aversion threshold $\tau_K$ above which probabilistic sniping has the potential of yielding better utilities for both the market maker and the bandits. When $\gamma > \tau_K$ there is a corresponding optimal sniping probability $p^*_K$ that yields the most favourable outcome for all parties. The problem with
this equilibrium is that it does not constitute a Nash equilibrium for the single-shot MZ\(^1\) game (as argued in section 3.5). However, things are different when we consider the infinite horizon repeated game version (MZ\(^\infty\)). Indeed in this case, we can invoke the so-called folk theorems that show that any equilibrium that is strictly better than a Nash equilibrium in the single-shot (stage) game gives rise to a new Nash equilibrium (NE) in the repeated game (at least, under the mild assumption that all the players are sufficiently patient).

The reason for this is fairly straightforward: the promise of higher payoffs means even purely selfish agents are incentivised to collaborate if they expect this cooperation to be profitable. This new equilibrium is stabilized by the implicit threat that, if anyone breaks the tacit plan, opponents will do the same and opportunities will instantly evaporate. Working out whether cooperation is the rational choice involves estimating what is most favourable: the exceptional but one-off payoff one stands to gain from defecting, or the smaller but accumulating payoffs that result from continuing collaboration and, that over time, will eclipse the former. The precise nature of these results is captured in the so-called Folk Theorems. Basically, these theorems (of which there are different versions) confirm the common sense notion that it is possible to engender mutually beneficial collaboration among players by the threat of retaliation when someone deviates from an implicitly agreed action profile. Such strategies are appropriately called trigger strategies, or even grim trigger strategies if after a single deviation all players irrevocably renege on their intention to collaborate [13].

For a formal statement of the folk theorems, we start from the observation that any Nash equilibrium payoff in a repeated game must satisfy the following two properties [9]:

- **Individual rationality (IR)** The Nash payoff must weakly dominate the minimax payoff profile of the constituent stage game. That is, the equilibrium payoff of each player must be at least as large as the minimax payoff of that player. (Recall that the minimax payoff is the minimum payoff a player can guarantee himself even if all opponents are trying to inflict as much damage on him as possible.) This is because a player achieving less than his minimax payoff always has incentive to deviate by simply playing his minimax strategy at every history.

- **Feasibility:** Since the payoff in a repeated game is just a weighted average of payoffs in the stage games, it must be a convex combination of possible payoff profiles of the stage game.

Folk theorems essentially state the reverse: they contend that (under certain conditions, which are different in each folk theorem), every payoff that is both IR and feasible can be realized as a Nash equilibrium payoff profile in the repeated game. One caveat in this respect is that grim trigger strategy might be very costly and therefore result in non-credible threats. The following result guarantees that for sufficiently patient players (i.e. with discount factor \(\delta \approx 1\)) it is possible to devise a strategy that yields a subgame perfect equilibrium (SPE) with appealing payoffs.
To snipe or not to snipe, that’s the question!

In what follows we will denote by $G$ a $n$-player one-shot (stage) game that is repeated an infinite number of times. At repetition $t = 0, 1, 2, \ldots$ the action profile $a^t = (a^t_1, a^t_2, \ldots, a^t_n)$ is played, resulting in corresponding payoff $u^t = (u^t_1, u^t_2, \ldots, u^t_n)$. We are assuming that for each player $i$, future rewards are discounted by a factor $0 < \delta_i < 1$ which implies that the total expected payoff for player $i$ when playing the (infinite) action sequence $a = (a^0, a^1, \ldots, a^t, \ldots)$ is given by

$$u_i(a) = (1 - \delta_i) \sum_{t=0}^{\infty} \delta_i^t u_i(a^t_1, a^t_{-i}).$$

Finally, denoting by $G^\infty(\delta)$ the infinitely repeated game with individual discount factors $\delta = (\delta_1, \delta_2, \ldots, \delta_n)$ we get the following important result:

**Theorem 1** (Friedman) Let $a^{NE}$ be a static Nash equilibrium of the stage game $G$ with payoffs $u^{NE}$. For any feasible payoff $v$ with $v_i > u_i^{NE}$ for all $i = 1, 2, \ldots, n$, there exists some discount threshold $\delta_0 < 1$ such that if the discount factor $\delta_i$ for every agent $i$ exceeds this threshold (i.e. $\delta_i > \delta_0$ for all $i = 1, 2, \ldots, n$), then there exists a subgame perfect equilibrium of the infinitely repeated game $G^\infty(\delta)$ with payoffs $v$.

Probabilistic sniping gives rise to subgame-perfect equilibrium for $MZ^\infty$. The relevance of this result for the current paper is that we have shown that for $\gamma_L < \gamma < \gamma_K$ applying a sniping probability $p^*_K$ results in a strictly better utility $u^*(p^*)$ for all HFTs. The above folk theorem now guarantees that this can be turned into a stable equilibrium which is unique and therefore results in predictable behaviour.

7 Related work

*High-frequency trading* Over the last decades, financial markets have changed significantly and become fragmented. Hence, traders can search across many markets, but this requires costly infrastructure and technology to make trading profitable [8]. In addition, high speed trading is now considered to be a crucial part of the trading technology. In 2010, the speed of round-trip between NASDAQ and the Chicago Mercantile Exchange decreased from over 14.5 milliseconds to under 8.1 milliseconds in 2014 [5]. Also NASDAQ supports high frequency traders by offering faster access to their infrastructure and trade data transmission [11]. In response to these drastic changes in financial markets, *algorithmic trading* harnesses advanced technologies to connect to different markets directly and trade ever faster at lower costs. Therefore, high-frequency traders (HFTs) can trade continuously and benefit from serial order processing. This continuous time trading has given rise to a speed race and sniping has emerged as an opportunity for HFTs to benefit from a stale quote [4]. In [1] it is shown that HFTs will tend to post thin but strictly positive bid-ask spreads to compensate the costs of getting sniped even without asymmetric information about fundamentals.
The emergence of high-speed trading has spurred on a lot of research regarding the effects on important market parameters such as liquidity. The seminal paper by Menkveld and Zoican [14] has been the starting point for this research and has been discussed extensively in this paper. The focus in both this and the MZ paper has been on a stylized version of the interactions occurring at high frequency electronic exchanges. However, real market are competitive and continuous, and algorithmic traders need to choose their strategies from scratch and for a longer period in order to benefit from the dynamic of the market and speed (arm race). Moreover, traders should have a set of predefined strategies [6] and traders’ behaviour may be influenced by observing their competitors [16]. Therefore, we explain agent’s strategies and the reason why repeated game instead of one shot Nash equilibrium is a better choice to explain behaviours in the electric exchange. The case for repeated games is further strengthened by research by Bruce (2007) that shows that a one-shot stage game brings significant surplus loss to the traders in a dynamic setting [3]. More recently, some studies showed that repetition in the game allows decision makers (agents, players in the market) to construct their strategies explicitly and benefit in the long-run [7,16].

Breitmoser (2015) analyzed individual strategies that are cooperate after mutual cooperation, defect after mutual defection, otherwise randomize. He finds a semi-grim equilibrium as a threshold that agents start cooperating in the first round by knowing treatment parameters and switch to semi-Grim with equal probability in the repeated game. At the end, sustain mutual cooperation led to the long-run welfare [2] and [10]. Furthermore, agents gain experience from the history of the game and trust is built [12] and an indefinitely repeated games with high continuation probability helps agents to mix strategies with minimal restrictions on the type and lengths of pure strategies [16].

8 Conclusions and further research

In this paper we re-analysed the MZ game as described in [14] and argued that a recasting of the problem in terms of repeated games offers a natural interpretation in which the solution space of equilibria accommodates genuine probabilistic sniping. Probabilistic sniping was introduced in the original paper to account for changes in the behavior of HFTs when risk aversion increases, but its practical significance in the single-shot game is unconvincing. However, in the context of repeated games probabilistic sniping provides genuinely new and interesting insights. In particular:

– We give a simple geometric argument that allows one to deduce the general conditions that govern the transitions in sniping behaviour. By translating these conditions in terms of the risk aversion parameter $\gamma$ we obtain two new thresholds ($\gamma_K$ and $\gamma_L$).
– Contrary to the situation in the single-shot game, we predict that probabilistic sniping will start playing a role even before the equilibrium utility for sure sniping is reduced to zero (by increasing risk aversion).
To snipe or not to snipe, that’s the question!

This also eliminates the discontinuous jump in the market maker’s utility that was reported in the single-shot game.

Some notes on extensions for further research In both this paper and the original MZ paper it was shown how simple and highly stylized models can already provide unexpected insights and interesting conclusions. We therefore expect that more realistic models will yield even more intriguing insights. Some of the challenges we intend to tackle in an upcoming paper are:

- We showed how important quantities that determine the qualitative transitions in the results (such as $s^*_K$ or $\pi_K$) can be computed based on the knowledge of the game parameters, i.e. $\alpha, \mu, \delta$ and $H$. However, in most realistic situations, these parameters are unknown and need to estimated from observations. As a result there is an amount of uncertainty and noise associated with these results. How stable are the conclusions with respect to noisy game parameters?

- The collaboration that emerges in repeated games is predicated on the assumption that players can monitor each other’s actions and retaliate if necessary. But in the case of probabilistic sniping this monitoring is complicated by the fact that it is intrinsically error-prone to judge the probability of an uncertain event. In order to estimate a probability $p$ we need to observe a number of rounds that is a (sizeable) multiple of $1/p$. How does this uncertainty affect the game and corresponding results?

- In the repeated game scenario expounded in this paper, each stage game starts from zero position. This is unrealistic, and should be modified, as the traders want to regress back to zero-positions.

- Risk aversion varies depending on position. One reason for this is that traders dread reporting bad news. Hence, when they find themselves in a negative position, they are willing to take more risk since, if they are lucky, they can redeem themselves, and if they are unlucky, the "size" of their bad news does not make all that much difference.

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Conflict of interest

The authors declare that they have no conflict of interest.

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