Algebraic and relational models for a system based on a poset of two elements

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Abstract: The aim of this paper is to present a very simple set of conditions, necessary for the management of knowledge of a poset $T$ of two agents, which are partially ordered by the capabilities available in the system. We build up a formal system and we elaborate suitable semantic models in order to derive information from the poset. The system is related to three-valued Heyting algebras with Boolean operators.

Key Words: Distributive lattices with Boolean operators, $T$-structures, three-valued Heyting algebras, algebraic and relational models, knowledge representation

1. Introduction

The purpose of this paper is to provide a propositional logical framework for representing and reasoning about knowledge of a poset of two agents (e.g. in robotics). The situation we have in mind may be described as follows.

Assume $T$ is a poset of two agents $t_1$ and $t_2$. We denote $t_1 \leq t_2$ to express the fact that agent $t_2$ has more possibilities than agent $t_1$. In the applications, $T$ may be considered to be a poset of two co-operating intelligent agents partially ordered by the competences about a particular domain, as for example a “knower” and a “learner”.

We suppose that a minimal necessary ingredient of a formal system that is capable of simulating a practical reasoning must include a lattice structure to manage the connectives “and” and “or”.

For agent $t_i$, the intuitive meaning of the connective $S_{t_i}a$ is: “agent $t_i$ perceives the information $a$”. Related to the lattice structure, perception operators are asked to be compositional.

Mathematical simple structures that we explore in modelling our ideas may be presented in the following way.

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On a distributive lattice \((A, 0, 1, \land, \lor)\) with zero and unit we are going to define three unary operators, denoted \(C, S_{t_1}, S_{t_2}\). Perception operators \(S_{t_1}, S_{t_2}\) are asked to be compositional, Boolean, and accepting individual opinions without any change; \(C\) which is considered here only to give a neat definition below, is understood to satisfy the equalities: \(S_{t_1} a \land Ca = 0\) and \(S_{t_1} a \lor Ca = 1\), for all \(a \in A\).

Thus, the required properties for these operators are the following, for all \(a \in A\):

- the operators \(S_t\), for \(t \in \{t_1, t_2\}\), are \((0,1)\)-lattice homomorphisms from \(A\) onto the sublattice \(B(A)\) of all complemented elements of \(A\) such that \(S_t S_w a = S_w a\), for all \(t, w \in \{t_1, t_2\}\),
- \(S_{t_1} a \leq S_{t_2} a\),
- \(S_{t_1}\) is related to the operation \(C\) by the equations: \(S_{t_1} a \land Ca = 0\) and \(S_{t_1} a \lor Ca = 1\).

We remark that, for arbitrary elements \(a, b \in A\), the relation “\(\equiv\)” defined in the following way:

\[ a \equiv b \text{ if and only if } S_t a = S_t b, \quad \text{for } t \in \{t_1, t_2\}. \]

is an equivalence relation on \(A\). With respect to the connectives \(\land, \lor, C, S_{t_1}, S_{t_2}\) it is an easy calculation to check that it is a congruence in \(A\).

In view of this fact, we can identify elements in \(A\) if and only if agents in \(T\) have the same insights on them.

The paper is organised as follows. In Section 2, the definition of the algebraic structure is derived and a fundamental example is exhibited. Other examples are in [4] and [6]. As the definition is not suitable for logic considerations, we give an equational definition in Section 3. In Section 4 a formalized propositional language is introduced as well as two adapted semantics. The equivalence of algebraic and relational semantics is shown in Section 5. Finally, in Section 6, the question of the decidability of the system is answered.

## 2. An algebraic structure and a fundamental example

All the above constraints suggest to consider a three-valued structure that we have studied in [4] and [6]. This structure emerged from a fundamental example presented later and is related to ideas of Moisil [7], [8], [1].
For notational convenience, we sometimes replace $t_1$ and $t_2$ by their indices (i.e. one and two).

**Definition 2.1** An abstract algebra $(A, 0, 1, \land, \lor, C, S_1, S_2)$ where $0, 1$ are constants, $C, S_1, S_2$ are unary operations and $\land, \lor$ are binary operations is said to be a **Distributive lattice with three unary operators** if

(T1) $(A, 0, 1, \land, \lor)$ is a distributive lattice with zero and unit, and for every $a, b \in A$ and for all $i, j = 1, 2,$ the following equations hold:

(T2) $S_i(a \land b) = S_i a \land S_i b$ ; $S_i(a \lor b) = S_i a \lor S_i b,$

(T3) $S_1 a \land Ca = 0$ ; $S_1 a \lor Ca = 1,$

(T4) $S_i(S_j)a = S_j a,$

(T5) $S_10 = 0$ ; $S_1 1 = 1,$

(T6) If $S_i a = S_i b,$ for all $i = 1, 2,$ then $a = b,$ (Determination Principle)

(T7) $S_1a \leq S_2a.$

We will refer to a **T-structure** $A$, for short (as in [4] and [6]). We remark that this definition is not equational and this fact makes it awkward for us.

**Proposition 2.2** The following properties are true in any $T$-structure:

(T8) $S_2 0 = 0$ ; $S_2 1 = 1,$

(T9) $a \leq b$ if and only if $S_i a \leq S_i b,$ for $i = 1, 2,$

(T10) $S_1 a \leq a \leq S_2 a,$

(T11) $S_i a \land CS_i a = 0$ ; $S_i a \lor CS_i a = 1,$ for $i = 1, 2.$

**Proof.** See [4].

**Remark 2.3** Let $B(A)$ be the Boolean algebra of all complemented elements in $A$ and $S_i(A) = \{x \in A : S_i x = x\}.$

From [4], [5] it is well known that for all $i = 1, 2,$ $S_i(A) = B(A).$ Also, if “$\neg$” denotes the Boolean negation we have $\neg S_i a = CS_i a.$
A fundamental example

For the sake of illustration let us consider a very simple example depicting the introduced notions.

Let $T = \{t_1, t_2\}$ be an ordered set such that $t_1 \leq t_2$. For each $t \in T$ we denote $F(t)$ the increasing subset of $T$, i.e.

$$F(t) = \{w \in T : t \leq w\}.$$ 

Let $A$ be the class of the empty set and all increasing sets, i.e.

$$A = \{\emptyset, F(t_2), F(t_1)\}.$$ 

The class $A$, ordered by inclusion, is an ordered set with three or two elements, and the system $(A, \emptyset, T, \cap, \cup)$, closed under the operations of intersection and union, is a distributive lattice with zero and unit. For each $t \in T$ we define a special operator $S_t$ on $A$ in the following way, for all $X \subseteq A$:

$$S_t(X) = \begin{cases} T & \text{if } t \in X \\ \emptyset & \text{otherwise.} \end{cases}$$

Finally we define $CX = \neg S_{t_1}(X)$. Thus the system $(A, \emptyset, T, \cap, \cup, C, S_{t_1}, S_{t_2})$ is a $T$-structure, called basic $T$-structure and denoted $BT$ or $B$ if it has three or two elements, respectively. Note that $B$ is a subalgebra of $BT$.

For further examples see [4] and [6].

3. An equational definition

In order to develop a logic system of any kind, it is convenient to remember (see for example [10], page 167) that “implication” seems to be the most important connective. This fact suggests what we do here.

In [4] we have introduced an equational definition of a $T$-structure by means of a particular intuitionistic implication.

Definition 3.1 A Heyting algebra with three unary operators (or HT-algebra for short) is an abstract system $A = (A, 0, 1, \wedge, \vee, \Rightarrow, \neg, S_1, S_2)$ such that $0, 1$ are constants, $\neg, S_1, S_2$ are unary operations and $\wedge, \vee, \Rightarrow$ are binary operations satisfying the following conditions, for all $a, b, c \in A$:

$$(HT1) \ (A, 0, 1, \wedge, \vee, \Rightarrow, \neg) \text{ is a Heyting algebra,}$$

and for every $a, b \in A$ and for all $i, j = 1, 2$ the following equations hold:

$$(HT2) \ S_i(a \wedge b) = S_i(a) \wedge S_i(b) \ ; \ S_i(a \vee b) = S_i(a) \vee S_i(b),$$
\( HT3 \) \( S_2(a \Rightarrow b) = (S_2a \Rightarrow S_2b), \)

\( HT4 \) \( S_1(a \Rightarrow b) = (S_1a \Rightarrow S_1b) \land (S_2a \Rightarrow S_2b), \)

\( HT5 \) \( S_iS_ja = S_ja, \)

\( HT6 \) \( S_1a \lor a = a, \)

\( HT7 \) \( S_1a \lor \neg S_1a = 1, \) with \( \neg a = a \Rightarrow 0. \)

The next two theorems state the equivalence between the notion of \( T \)-structure and that of \( HT \)-algebra and are proved in [4].

**Theorem 3.2** Let \((A, 0, 1, \land, \lor, C, S_1, S_2)\) be a \( T \)-structure and \( \Rightarrow \) and \( \neg \) be two operations defined by means of the following equations, for all \( a, b \in A \):

\[
\begin{align*}
  a \Rightarrow b &= b \lor \bigwedge_{k=1}^{2}(CS_k a \lor S_k b), \\
  \neg a &= a \Rightarrow 0.
\end{align*}
\]

Then the algebra \( A = (A, 0, 1, \land, \lor, \Rightarrow, \neg, S_1, S_2) \) is a \( HT \)-algebra.

Conversely:

**Theorem 3.3** Let \( A = (A, 0, 1, \land, \lor, \Rightarrow, \neg, S_1, S_2) \) be a \( HT \)-algebra and let us introduce a new operation \( C \) by means of the following equation, for all \( a \in A \):

\[
Ca = \neg S_1a
\]

Then the abstract algebra \( (A, 0, 1, \land, \lor, C, S_1, S_2) \) is a \( T \)-structure.

**Remark 3.4** Every \( HT \)-algebra satisfies the Ivo Thomas axiom [11], for all \( a, b, c \in A \):

\[
((a \Rightarrow c) \Rightarrow b) \Rightarrow (((b \Rightarrow a) \Rightarrow b) \Rightarrow b) = 1.
\]

This equality implies that every \( HT \)-algebra \( A \) is a three-valued Heyting algebra [9].

4. A formalized propositional language

The logic considered in the following sections is intended to provide a framework to manage a poset of two intelligent agents.

A formal system needs a language. In the applications, this language will be used as a tool to represent knowledge. For notational convenience, we use the same symbols for connectives in the language and operations in algebraic structures.
The language of $HT$-logics is a propositional language whose formulas are built from propositional variables taken from a countable set $\text{VarProp}$ with signs of conjunction ($\land$), disjunction ($\lor$), implication ($\Rightarrow$), negation ($\neg$), and the family $\{S_{t_1}, S_{t_2}\}$ of unary connectives. Implication ($\Rightarrow$) and negation ($\neg$) are intuitionistic connectives, and operators $S_{t_1}$ and $S_{t_2}$ are Boolean operators.

The set $\text{For}$ of formulas of the logic is the least set satisfying the conditions:

- $\text{VarProp} \subseteq \text{For}$,
- if $\alpha, \beta \in \text{For}$, then $\alpha \land \beta, \alpha \lor \beta, \alpha \Rightarrow \beta \in \text{For}$,
- if $\alpha \in \text{For}$ then $\neg \alpha, S_{t_1} \alpha, S_{t_2} \alpha \in \text{For}$

Semantics of the language

In order to formally reason about knowledge, we need a suitable semantic model. We define a meaning of formulas of the given language by means of notions of model and satisfiability of formulas in a model, in a standard way.

**a- Algebraic models**

Let $\text{For}$ be the set of formulas and $A$ a $HT$-algebra. In the set of formulas, the connectives ($\land, \lor, \Rightarrow, \neg, S_{t_1}, S_{t_2}$) are regarded as algebraic operations.

A map $h : \text{For} \to A$ is called a homomorphism provided it preserves all the operations on $\text{For}$.

**Definition 4.1** An algebraic model for the set of formulas $\text{For}$, is a system $(A, h)$ such that $A$ is a $HT$-algebra and $h : \text{For} \to A$ is a homomorphism.

A formula $\alpha$ is algebraically true in the algebraic model $(A, h)$ iff $h(\alpha) = 1$, and $\alpha$ is algebraically valid (denoted $\models_{\text{Alg}} \alpha$) iff $\alpha$ is algebraically true in every algebraic model.

A formula $\alpha$ is an algebraic consequence of a set of formulas $\Gamma$ in the algebraic model $(A, h)$ (denoted by $\Gamma \models_{A} \alpha$) iff whenever all the formulas from $\Gamma$ are algebraic true in $(A, h)$, we have $\alpha$ is algebraic true in $(A, h)$; and $\alpha$ is an algebraic consequence of a set of formulas $\Gamma$ (denoted by $\Gamma \models_{\text{Alg}} \alpha$) iff for every algebraic model $(A, h)$, we have $\Gamma \models_{A} \alpha$.

**b- Relational models**

Motivated by some results in [5], p.135, we introduce the following notion.
Definition 4.2 A **HT-frame** is a system

\[ K = (W, R, s_1, s_2) \]

where, for all \( w \in W \)

(K0) \( W \) is a nonempty set (of states \( w \)), \( R \) is a binary relation on \( W \) and \( s_1, s_2 \) are functions on \( W \),

(K1) \( R \) is a preorder, that is \( R \) is reflexive and transitive,

(K2) \( s_j(s_i(w)) = s_j(w) \), for all \( i, j = 1, 2 \),

(K3) \( R(s_1(w), w) \),

(K4) \( R(w, s_2(w)) \),

(K5) \( R(w, w') \) implies \( R(s_1(w), s_1(w')) \) and \( R(s_i(w'), s_i(w)) \), for \( i = 1, 2 \),

(K6) If \( w \in W \) then there are \( i \in \{1, 2\} \) and \( w' \in W \) such that \( w = s_i(w') \).

Definition 4.3 A **HT-model** based on a HT-frame \( K \) is a system \( M = (K, m) \)
such that \( m : \text{VarProp} \to \mathcal{P}(W) \) is a meaning function that assigns subsets of states to propositional variables, and satisfies the atomic heredity condition:

\[
(\text{her at}) \quad R(w, w') \text{ and } w \in m(p) \text{ imply } w' \in m(p).
\]

We say that in a HT-model \( M \) a state \( w \) satisfies a formula \( \alpha \) (denoted \( M, w \text{ sat } \alpha \)) whenever the following conditions are satisfied:

\[
M, w \text{ sat } p \text{ iff } w \in m(p), \text{ for } p \in \text{VarProp},
\]

\[
M, w \text{ sat } \alpha \land \beta \text{ iff } M, w \text{ sat } \alpha \text{ and } M, w \text{ sat } \beta,
\]

\[
M, w \text{ sat } \alpha \lor \beta \text{ iff } M, w \text{ sat } \alpha \text{ or } M, w \text{ sat } \beta,
\]

\[
M, w \text{ sat } \alpha \Rightarrow \beta \text{ iff } \text{for all } w', \text{ if } R(w, w') \text{ and } M, w' \text{ sat } \alpha \text{ then, } M, w' \text{ sat } \beta,
\]

\[
M, w \text{ sat } \neg \alpha \text{ iff } \text{for all } w', \text{ if } R(w, w') \text{ then, not } M, w' \text{ sat } \alpha,
\]

\[
M, w \text{ sat } S_i \alpha \text{ iff } M, s_i(w) \text{ sat } \alpha.
\]

Given a HT-model \( M \), we extend the meaning function \( m \) to all formulas:

\[
m(\alpha) = \{ w \in W : M, w \text{ sat } \alpha \}
\]

A formula \( \alpha \) is **true in a HT-model** \( M = (K, m) \) (denoted \( M \text{ sat } \alpha \)) iff \( M, w \text{ sat } \alpha \), for every \( w \in W \) (i.e. \( m(\alpha) = W \)), \( \alpha \) is **true in a HT-frame** \( K \) iff it is true in every HT-model based on \( K \), and \( \alpha \) is **HT-valid** (denoted \( \models_{\text{Rel}} \alpha \)) iff it is true in every HT-frame.
A formula $\alpha$ is a relational $HT$-consequence of a set of formulas $\Gamma$ in a $HT$-model $M = (K, m)$ (denoted by $\Gamma \models_M \alpha$) iff whenever all the formulas from $\Gamma$ are true in $M$, we have $\alpha$ is true in $M$; and $\alpha$ is a relational $HT$-consequence of a set of formulas $\Gamma$ (denoted by $\Gamma \models_{Rel} \alpha$) iff for every $HT$-model $M$, $\Gamma \models_M \alpha$).

**Proposition 4.4** For every $HT$-model $M = (K, m)$ and for every formula $\alpha$ the following heredity condition holds:

(her) \quad if $R(w, w')$ and $M, w$ sat $\alpha$, then $M, w'$ sat $\alpha$.

**Proof.** The proof is by induction with respect to complexity of $\alpha$. By the way of example we show (her) for formulas of the form $S_i \alpha$. Let $R(w, w')$ and $M, w$ sat $S_i \alpha$, hence by (K5) we have $R(s_i(w), s_i(w'))$ and by Definition 4.3 we deduce $M, s_i(w)$ sat $\alpha$. From the inductive hypothesis we obtain $M, s_i(w')$ sat $\alpha$, i.e. $M, w'$ sat $S_i \alpha$.

**Proposition 4.5** In every $HT$-frame $K = (W, R, s_1, s_2)$, for every $w \in W$, there is $i \in \{1, 2\}$ such that $w = s_i(w)$, i.e. each $w$ is a fixed point of a function $s_i$.

**Proof.** Let $w \in W$. By (K6) there are $i \in \{1, 2\}$ and $w' \in W$ such that $w = s_i(w')$. Hence $s_i(w) = s_i(s_i(w')) = s_i(w') = w$.

5. Equivalence of algebraic and relational model validity

First let us suppose that we have a $HT$-model $M = (W, R, s_1, s_2, m)$. We will define an algebraic model $(A, 0, 1, \wedge, \vee, \Rightarrow, \neg, S_1, S_2, h)$ such that for any formula $\alpha$:

\[ h(\alpha) = 1 \quad \text{if and only if} \quad M \text{ sat } \alpha \]

A subset $X \subseteq W$ will be called $R$-closed if whenever $w \in X$ and $R(w, w')$, then $w' \in X$.

Let $RC$ be the collection, ordered by inclusion ($\subseteq$), of all $R$-closed subsets of $W$:

\[ RC = \{ X \subseteq W : X \text{ is } R\text{-closed} \} \]

We can consider on $RC$ the operations of intersection $\cap$ and union $\cup$. The system $(RC, \emptyset, W, \cap, \cup)$ is a distributive lattice with zero and unit.
Also, if $X, Y \in RC$, let us consider the sets:

$$S_iX = \{ w \in W : s_i(w) \in X \} = s_i^{-1}(X), \text{ for } i = 1, 2,$$

$$CX = C_WS_1X,$$

where $C_W$ is the ordinary set complementation.

If $X, Y$ are $R$-closed then $S_1X$, $S_2X$ and $CX$ are $R$-closed. In fact, assume $w' \in S_iX$ and $R(w', w'')$. By (K5) in Definition 4.2 we obtain $R(s_i(w'), s_i(w''))$. Since $X$ is $R$-closed and $s_i(w') \in X$ we deduce $s_i(w'') \in X$, i.e. $w'' \in S_iX$.

To prove $C_WS_1X \in RC$ assume $w' \in C_WS_1X$ and $R(w', w'')$. From (K3) we have $R(s_1(w'), w')$. By transitivity of $R$ we get $R(s_1(w'), w'')$. By (K5), we have $R(s_1(w''), s_1s_1(w'))$ and by (K2) we get $R(s_1(w''), s_1(w'))$. Since $X$ is $R$-closed and $s_1(w') \notin X$ it follows that $s_1(w'') \notin X$, hence $w'' \notin S_1X$, i.e. $w'' \in C_WS_1X$.

Moreover $S_1\emptyset = \emptyset$ and $S_1W = W$.

**Proposition 5.1** The system $(RC, \emptyset, W, \cap, \cup, C, S_1, S_2)$ is a $T$-structure.

**Proof.** We show that the operations defined above fulfill the properties (T1)–(T7) indicated in Definition 2.1.

In fact, (T1) and (T5) have been indicated above; (T2) follows at once from properties of the inverse image; (T3) is a consequence of definitions; (T4) is a consequence of (K2) and a property of the inverse image.

To prove (T6) suppose $S_iX = S_iY$, for all $i = 1, 2$. Let $w \in X$; by (K6) there is $i \in \{1, 2\}$ and $w' \in W$ such that $w = s_i(w') \in X$. It follows that $w' \in S_iX = S_iY$, that is $s_i(w') = w \in Y$ and thus $X \subseteq Y$. The proof of the other half is similar.

Finally, to prove (T7) let $w' \in S_iX$, that is $s_i(w') \in X$. By (K3) we have $R(s_1(w'), w')$ and since $X$ is $R$-closed we obtain $w' \in X$. Suppose now $w \in X$. By (K4) we have $R(w, s_2(w))$ so $s_2(w) \in X$ and $w \in S_2X$.

The proof of the proposition is now complete.

**Remark 5.2** Taking into account the equivalence between Definitions 2.1 and 3.1 we find in particular the well known result that the system $(RC, \emptyset, W, \cap, \cup, \Rightarrow, \sim)$ is a Heyting algebra (2, page 24).

For sets $X, Y \in RC$, the set $X \Rightarrow Y$ is given by the equation (1) in Theorem 3.3.

That is:

$$X \Rightarrow Y = Y \cup \bigcup_{k=1}^{2} (C_WS_1S_kX \cup S_kY) = Y \cup \bigcup_{k=1}^{2} (s_k^{-1}(C_WX) \cup s_k^{-1}(Y))$$

$$= Y \cup s_1^{-1}(C_WX) \cup s_1^{-1}(Y) = Y \cup s_1^{-1}(C_WX)$$

Thus $Z = S_1(C_WX) \cup Y$ is the largest $R$-closed subset such that $X \cap Z \subseteq Y$. 
We define \( h : For \rightarrow \mathcal{P}(W) \) by

\[
h(\alpha) = \{ w \in W : M, w \text{ sat } \alpha \}\]

Let \( w \in h(\alpha) \) and \( R(w,w') \). By (her) (Proposition 4.4), \( M, w' \text{ sat } \alpha \), i.e. \( w' \in h(\alpha) \). Thus \( h(\alpha) \) is \( R \)-closed.

From a result in ([2], page 24), we know that \( h \) is a Heyting homomorphism. Moreover we have the equality \( h(S_1 \alpha) = S_1 h(\alpha) \). This fact is a consequence of the following equivalent conditions:

\[
\begin{align*}
w \in h(S_i a) & \iff M, w \text{ sat } S_i \alpha \iff M, s_i(w) \text{ sat } \alpha \\
& \iff s_i(w) \in h(\alpha) \iff w \in S_i h(\alpha)
\end{align*}
\]

Thus \( (RC, \emptyset, W, \cap, \cup, \Rightarrow, \neg, S_1, S_2, h) \) is an algebraic model.

Concerning the validity of a formula, we have the desired equivalence:

\[
h(\alpha) = W(\in RC) \text{ if and only if } m(\alpha) = W
\]

Conversely, suppose we have an algebraic model \((A, h)\). We will define a HT-model \( M = (W, R, s_1, s_2, m) \) such that for any formula \( \alpha \):

\[
M \text{ sat } \alpha \text{ if and only if } h(\alpha) = 1
\]

Let \( W \) be the class of all prime filters in \( A \). Let \( R \) be the inclusion relation \( \subseteq \) and \( s_i : W \rightarrow W \) be the maps defined as follows, for \( i = 1, 2 \) and \( P \in W \):

\[
s_i(P) = \{ x \in A : S_i x \in P \}.
\]

This set is a prime filter.

If \( p \in \text{VarProp} \) and \( P \) is a prime filter, we define

\[
M, P \text{ sat } p \text{ if and only if } h(p) \in P.
\]

**Proposition 5.3** If \( A \) is a HT-algebra, the system \( K = (W, R, s_1, s_2) \) defined above is a HT-frame.

**Proof.** (K0) follows from the definition of \( K \). The relation \( \subseteq \) satisfies (K1); (K2) is a consequence of (HT5) and a property of the inverse image; (K3) and (K4) are consequence of (HT6), (T10) and a property of prime filters.

To prove (K5) suppose \( P, Q \in W \) and \( P \subseteq Q \). Let \( x \in s_i(P) \) then \( S_i x \in P \subseteq Q \), hence \( x \in s_i(Q) \). In addition, let \( x \in s_i(Q) \), i.e. \( S_i x \in Q \). If \( S_i x \notin P \) then \( \neg S_i x \in P \subseteq Q \) and \( S_i x \land \neg S_i x = 0 \in Q \), which is impossible; hence \( S_i x \in P \), i.e. \( x \in s_i(P) \).

Finally, to prove (K6), assume \( P \in W \). By theorem 5.10 in ([3], p.149) there exists a unique ultrafilter \( P' = (P \cap B(A)) \) in \( B(A) \) and an integer \( i \in \{1, 2\} \) such that \( P = P'_i = \{ x \in A : S_i x \in P \} = s_i(P) \).
Proposition 5.4 For formulas $\alpha, \beta$, and prime filters $P, Q$ we have:

1. If $M, P \text{ sat } \alpha$ and $P \subseteq Q$ then $M, Q \text{ sat } \alpha$

2. $M, P \text{ sat } (\alpha \land \beta)$ iff $M, P \text{ sat } \alpha$ and $M, P \text{ sat } \beta$

3. $M, P \text{ sat } (\alpha \lor \beta)$ iff $M, P \text{ sat } \alpha$ or $M, P \text{ sat } \beta$

4. $M, P \text{ sat } (\alpha \Rightarrow \beta)$ iff for every $Q \in W$, if $P \subseteq Q$, and $M, P \text{ sat } \alpha$ then $M, Q \text{ sat } \beta$

5. $M, P \text{ sat } \neg \alpha$ iff for every $Q \in W$ such that $P \subseteq Q$ then not $M, Q \text{ sat } \alpha$

6. $M, P \text{ sat } S_i \alpha$ iff $M, s_i(P) \text{ sat } \alpha$

Proof. We show, for example, the reverse implication of (4) and the statement (6).

Suppose not $M, P \text{ sat } (\alpha \Rightarrow \beta)$, i.e. $h(\alpha \Rightarrow \beta) = h(\alpha) \Rightarrow h(\beta) \notin P$. Since $h(\beta) \subseteq h(\alpha) \Rightarrow h(\beta)$ we deduce that $h(\beta) \notin P$. Let $F(P, h(\alpha))$ be the filter generated by $P$ and $h(\alpha)$. This filter is proper because for example $h(\beta) \notin F(P, h(\alpha))$. In fact, if $h(\beta) \in F(P, h(\alpha))$, then there would be some $p \in P$ such that $p \land h(\alpha) \leq h(\beta)$ which is equivalent to $p \leq h(\alpha) \Rightarrow h(\beta) \in P$, a contradiction. Since $A$ is a distributive lattice then there is a prime filter $Q$ such that $F(P, h(\alpha)) \subseteq Q$ and $h(\beta) \notin Q$. By construction, $P \subseteq Q$ and $h(\alpha) \in Q$. That is $M, Q \text{ sat } \alpha$. Hence, by hypothesis, $M, Q \text{ sat } \beta$, i.e. $h(\beta) \in Q$, a contradiction.

Statement (6) is a consequence of the following equivalent conditions:

$$M, P \text{ sat } S_i \alpha \iff h(S_i(\alpha)) = S_i h(\alpha) \in P \iff h(\alpha) \in s_i(P) \iff M, s_i(P) \text{ sat } \alpha$$

We define $m : For \rightarrow P(W)$ such that $m(\alpha) = \{ P \in W : M, P \text{ sat } \alpha \}$. Thus the obtained system $M = (K, m)$ is a HT-model.

Concerning the validity of a formula $\alpha$, we have:

$$m(\alpha) = \{ P \in W : M, P \text{ sat } \alpha \} = W \iff h(\alpha) \in P, \text{ for every } P \in W \iff h(\alpha) \in \bigcap_{P \in W} P \iff h(\alpha) = 1$$

Summing up the above results we will provide the expected result, which is useful in applications:

Theorem 5.5 A formula $\alpha$ is a relational consequence of a set of formulas $\Gamma$ if and only if $\alpha$ is an algebraic consequence of $\Gamma$. 

Proof. The statement can be formally written in the following way:

\[ \Gamma \models_{Rel} \alpha \quad \text{if and only if} \quad \Gamma \models_{Alg} \alpha \]

(\rightarrow) Assume \( \Gamma \models_{Rel} \alpha \). If \( \Gamma \not\models_{Alg} \alpha \), there would be an algebraic model \((A, h)\) such that \( h(\gamma) = 1 \), for all \( \gamma \in \Gamma \) but \( h(\alpha) \neq 1 \). Let \( M = (K, m) \) be the \( HT \)-model for \( \Gamma \) defined in Propositions 5.3 and 5.4. Since \( h(\alpha) \neq 1 \) there is a prime filter \( P \) in \( A \) such that \( h(\alpha) \not\in P \), i.e. \( m(\alpha) = \{ P \in W : h(\alpha) \in P \} \neq W \), a contradiction.

(\leftarrow) Conversely, suppose \( \Gamma \models_{Alg} \alpha \). If \( \Gamma \not\models_{Rel} \alpha \), there would be a \( HT \)-model \( M = (K, m) \) for \( \Gamma \) such that \( M \text{ sat } \gamma \), for all \( \gamma \in \Gamma \) but \( not M \text{ sat } \alpha \).

Let \((RC, \emptyset, W, \cap, \cup, C, S_1, S_2, h)\) be the algebraic model of \( R \)-closed subsets of \( W \), where \( h : \text{For} \to RC \) is the homomorphism: \( h(\alpha) = \{ w \in W : M, w \text{ sat } \alpha \} \), for \( \alpha \in \text{For} \). We have \( h(\gamma) = W \) for every \( \gamma \in \Gamma \) but \( h(\alpha) \neq W \), a contradiction.

In particular if \( \Gamma \) is empty we can conclude the following fact:

**Theorem 5.6** A formula \( \alpha \) is valid in every relational model if and only if \( \alpha \) is algebraically valid.

6. A finite algebraic model

In this section we show that there is an effective method whereby, for any given formula \( \alpha \), it can be determined in a finite number of steps whether or not \( \alpha \) is an algebraic consequence of a finite set of formulas \( \Gamma \). Thus, the formalised propositional system introduced in Section 4 is decidable.

This result is a consequence of the following theorem.

**Theorem 6.1** A formula \( \alpha \) is an algebraic consequence of a finite set of formulas \( \Gamma \) if and only if we have \( \Gamma \models_{BT} \alpha \), for every algebraic model \((\text{BT}, h)\) based on the finite \( HT \)-algebra \( \text{BT} \).

Proof. The statement can be formally written in the following way, for a finite set \( \Gamma \) of formulas:

\[ \Gamma \models_{Alg} \alpha \quad \text{if and only if} \quad \Gamma \models_{BT} \alpha \]

(\rightarrow) Assume \( \Gamma \models_{Alg} \alpha \). Thus, for every algebraic model \((A, h)\), if \( h(\gamma) = 1 \) for all \( \gamma \in \Gamma \), then \( h(\alpha) = 1 \). In particular in the case \( A = \text{BT} \).

(\leftarrow) Conversely, suppose \( \Gamma \models_{BT} \alpha \) and let \((A, h)\) be any algebraic model such that \( h(\gamma) = 1 \), for all \( \gamma \in \Gamma \).

If \( \alpha \) is not true in \((A, h)\), there would be a minimal prime filter \( P \) in \( A \) (see [3]) such that \( h(\gamma) \in P \), for all \( \gamma \in \Gamma \) but \( h(\alpha) \not\in P \). Let \( f : A \to \text{BT} \) be the canonical homomorphism defined -via the quotient algebra \( A/P \), isomorphic to a subalgebra de \( \text{BT} \)- as in the proof of Proposition 6.2 in ([4], p.152).
The composition $g = f \circ h : For \rightarrow BT$ is a homomorphism which satisfies $g(\gamma) = 1$, for all $\gamma \in \Gamma$ and $g(\alpha) \neq 1$. This means that $\Gamma \not\models_{BT} \alpha$, a contradiction.

Finally we point out another link between finite models. The system $K^0 = (W^0, R^0, s_1^0, s_2^0)$ related to the finite HT-algebra $BT$ can be defined in the following way:

- $W^0 = T$,
- $R^0 = \{(w, w') : w, w' \in W^0$ and $w \leq w'\}$, i.e. $R^0$ is the order on $T$,
- if $t \in T$ and $w \in W^0$ then $s_i^0(w) = t_i$, for $i = 1, 2$.

**Proposition 6.2** The system $K^0 = (W^0, R^0, s_1^0, s_2^0)$ satisfies the properties $(K0)-(K6)$, that is $K^0$ is a HT-frame.

**Proof.** The proof is straightforward.

We note that, given the HT-frame $K^0$, the collection of $R$-closed sets of $W^0$ is $\{\emptyset, F(t_2), F(t_1)\}$.

As in Section 5, we can construct the $T$-structure $(RC, \emptyset, W^0, \cap, \cup, C, S_1, S_2)$, which is isomorphic to the basic $T$-structure $BT$.

In view of the results above, we conclude the paper with the following statement.

**Proposition 6.3** Let $K^0$ be the HT-frame defined above, $\Gamma$ a finite set of formulas, and $\alpha$ a formula. It follows that:

$$\Gamma \models_{HT} \alpha \quad \text{iff} \quad \text{for every HT-model } M^0 = (K^0, m), \text{ we have: } \Gamma \models_{M^0} \alpha.$$ 

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