Odd-frequency pair density waves in underdoped cuprates

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Pair density waves, identified by Cooper pairs with finite center-of-mass momentum, have recently been observed in copper oxide based high Tc superconductors (cuprates). A charge density modulation or wave is also ubiquitously found in underdoped cuprates. Within a general mean-field one-band model we show that the coexistence of charge density waves and superconductivity in d-wave superconductors like cuprates, generates an odd-frequency spin-singlet pair density wave, in addition to the even-frequency counterparts. The strength of the induced odd-frequency pair density wave depends on the modulation wave vector of the charge density wave, with the odd-frequency pair density waves even becoming comparable to the even-frequency ones in parts of the Brillouin zone. The odd-frequency component of the pair density waves also gets enhanced when the charge density wave is uni-axial. Such a coexistence of superconductivity and uni-axial charge density wave has already been experimentally verified at high magnetic fields in underdoped cuprates. We further discuss the possibility of an odd-frequency spin-triplet pair density wave generated in the coexistence regime of superconductivity and spin density waves, applicable to the iron-based superconductors. Our work thus presents a route to bulk odd-frequency superconductivity in high Tc superconductors.

I. INTRODUCTION

Broken symmetry phases characterize different condensed matter systems and define their phase diagrams. One of the most coveted phases of matter is superconductivity. Many non-superconducting phases lure in the proximity to superconductivity in various materials, making their phase diagram immensely complex and rich, with charge and spin density waves being two of the primary candidates. The interplay of density waves and superconductivity has already been found in transition-metal dichalcogenides, twisted bilayer graphene, twisted double-bilayer graphene, and iron-based and copper oxide based (cuprate) superconductors.

In cuprates, charge density waves (CDW) have been ubiquitously observed in underdoped samples using many experimental probes, such as scanning tunneling microscopy, x-ray scattering, NMR and transport measurements. CDW have drawn significant attention due to its potential ability to explain the mysterious pseudo-gap phase, found at temperatures larger than the coherence temperature Tc. It has been argued that CDW compete with superconductivity (SC) in parts of the doping phase diagram. However, the strength of the competition of CDW and superconductivity clearly varies between different cuprates and a coexistence state is also observed at high magnetic fields.

Apart from modulations in the charge density, spatial modulations in the superconducting pair amplitude have also been observed using Josephson scanning tunneling microscopy in the cuprate compound Bi2Sr2CaCu2O8+δ (BSCCO). Modulating superconducting pair amplitude is often referred to as Cooper-pair density waves (PDW). PDW closely resemble the Fulde-Ferrell-Larkin-Ovchinnikov (FFLO) state, but with no explicit time-reversal symmetry breaking or net magnetization. In contrast to the uniform superconducting state, the spatial average of the superconducting order parameter is zero in the PDW state. The experimental observations of PDW, be it direct or indirect, has always associated it to CDW and a uniform SC. Interestingly, the modulation wave vector of PDW, QPDW, is either same or different than the CDW modulation wave vector QCDW of the cuprate parent compound.

Odd-frequency Cooper pairs have also been observed in twisted bilayer graphene, twisted double-bilayer graphene, and iron-based superconductors. Our work thus presents a route to bulk odd-frequency superconductivity in high Tc superconductors.

The PDW order is characterized by finite center-of-mass momentum Cooper pairs and has so far been described by equal time two electron correlation functions, found within conventional BCS-like theory. If instead also unequal time pair correlation functions are considered, the Fermi-Dirac statistics allows for the exotic possibility that the correlation function becomes odd under the exchange of the time coordinates or, equivalently, odd in frequency. Odd-frequency Cooper pairs with zero center-of-mass momentum are present in several superconducting systems, among which most break translational symmetry. The necessary broken translational symmetry has been achieved in junctions, in the presence of impurities, and also in models with a staggered lattice structure. Bulk odd-frequency correlations are also pre-
dicted to be present in multiband superconductors even without broken translational symmetry. But, the thermodynamic stability of odd-frequency superconductivity has been questioned due to a most often found paramagnetic, or negative, Meissner response. However, a diamagnetic Meissner effect can be restored by having a frequency dependent pair potential or if the odd-frequency pairs have a staggered nature, i.e., in other words have finite center-of-mass momentum. Thus odd-frequency superconductivity with a spatially modulated order parameter can be thermodynamically stable. This naturally leads to the tantalizing possibility: is the mysterious PDW state found in the cuprates an odd-frequency state?

In this work, we show that spin-singlet odd-frequency pair density wave (OPDW) correlations are generically found in cuprates due to the simultaneous presence of SC and CDW. The OPDW in our work is an induced correlation with \( Q_{\text{PDW}} = Q \), i.e. not an order parameter and thus does not need an exotic pairing interaction in the OPDW channel. This makes our work much more general in comparison to the earlier works finding modulated odd-frequency superconductivity in quasi-one-dimensional models. We find that the OPDW is also accompanied with an even-frequency pair density wave (EPDW). By exploring various possible CDW modulation vector and different band structures, we show that the OPDW is a generic feature of all cuprates, whereas the relative strength of OPDW and EPDW depends on the specific microscopic properties. Here we do not assume any particular microscopic model and simply consider \( \chi_k \) and \( \Delta_k \) as parameters in order to keep our results as general as possible. Note that the notation for the CDW order parameter should ideally be \( \chi_{k,k+Q} \), but for brevity we use the short-hand notation \( \chi_k \) with the \( Q \) index being absorbed in the definition.

The Hamiltonian in Eq. (1) can be written in a matrix form in the basis \( \Psi_\uparrow = \left( c_{k\uparrow}^\dagger, c_{k+Q\uparrow}^\dagger, c_{-k\downarrow}, c_{-k-Q\downarrow} \right) \) as,

\[
H = \frac{1}{2} \sum_k \Psi_\uparrow \hat{H} \Psi + \text{constant},
\]

with

\[
\hat{H} = \begin{pmatrix}
\xi_k & \chi_k & \Delta_k & 0 \\
\chi_k & \xi_k+Q & 0 & \Delta_k+Q \\
\Delta_k & 0 & -\xi_k & -\chi_k \\
0 & \Delta_k+Q & -\chi_k & -\xi_k+Q
\end{pmatrix},
\]

where we take \( \chi_k \) and \( \Delta_k \) to be real, without loss of generality. A purely imaginary \( \chi_k \) is often considered to describe current density wave orders. Here, we do not discuss such orders and focus on a purely real \( \chi_k \) describing CDW only. The Green’s function \( G \) corresponding to the Hamiltonian in Eq. (2) is given by \( G^{-1}(i\omega) = i\omega - \hat{H} \) where \( \omega \) are fermionic Matsubara frequencies.

## A. Induced spin-singlet odd-frequency PDW

It has been widely discussed in the literature that a coexistence of SC and CDW can generate PDW with the same wave vector \( Q \) as that of the CDW. Here, we show that the generated PDW in general have both even and odd-frequency components. The induced PDW can be found by looking at the Cooper pair correlator \( F_{k,k+Q}(\tau) = \langle T_\tau e_{k\uparrow}^\dagger(\tau)c_{k+Q\downarrow}(0) \rangle \), where \( \tau \) is the imaginary time and \( T_\tau \) is the \( \tau \)-ordering operator. \( F_{k,k+Q}(\tau) \) has all the symmetries of a PDW field as it describes a Cooper pair with a finite center-of-mass momentum. Using the Hamiltonian in Eq. (2),
the Green’s function is obtained by inverting the $4 \times 4$ matrix $G^{-1}(i\omega)$. The Fourier transformed PDW correlator is then given by,

$$F_{k,k+Q}(i\omega) = G_{14}(i\omega) = F_{k,k+Q}^{even}(i\omega) + F_{k,k+Q}^{odd}(i\omega),$$

where

$$F_{k,k+Q}^{even}(i\omega) = \frac{\chi_k (\xi_k \Delta_k + \xi_k \Delta_k)}{D},$$

$$F_{k,k+Q}^{odd}(i\omega) = \frac{i\omega \chi_k (\Delta_k - \Delta_k)}{D},$$

$$D = (\omega^2 + \xi_k^2 + \Delta_k^2) (\omega^2 + \xi_k^2 + \Delta_k^2 + \xi_k^2 + \omega^2) - 2\chi_k^2 (\xi_k \Delta_k + \Delta_k \xi_k - \Delta_k^2 - \Delta_k^2 - \omega^2) + \chi_k^4.$$ (5)

The induced PDW, per definition given by $G_{14}(i\omega)$, have both even and odd-frequency components. This is seen explicitly in Eqs. (3) and (4), with $F_{k,k+Q}^{even}(i\omega)$ and $F_{k,k+Q}^{odd}(i\omega)$ having an even and odd frequency dependence, respectively, in the nominator, while the denominator $D$ is an even function of frequency.

In order to verify the symmetries of $F_{k,k+Q}(\tau)$, we use the fact that a pair correlation function should always satisfy the Fermi-Dirac statistics. As a result, the correlation function under a joint operation of spin permutation (S), momentum exchange (M), and relative time permutation (T) should satisfy $SMT = -1$. Under $M$, $F_{k,k+Q}(\tau) \rightarrow F_{k,-k,Q}(\tau)$. So, we look at the $G_{23}(i\omega)$ component of the Green’s function, giving,

$$F_{k,k+Q,k}(i\omega) = G_{23}(i\omega) = F_{k,k+Q,k}^{even}(i\omega) + F_{k,k+Q,k}^{odd}(i\omega),$$

where

$$F_{k,k+Q,k}^{even}(i\omega) = \frac{\chi_k (\xi_k \Delta_k + \xi_k \Delta_k)}{D},$$

$$F_{k,k+Q,k}^{odd}(i\omega) = \frac{i\omega \chi_k (\Delta_k - \Delta_k)}{D},$$

$$F_{k,k+Q,k}^{odd}(i\omega) = \frac{i\omega \chi_k (\Delta_k + \Delta_k)}{D}.$$ (6)

The even-frequency component of the induced PDW is thus even under $M$ (seen in Eq. (6)) and also even under $T$. To satisfy $SMT = -1$, it must therefore be odd under $S$, or a spin-singlet state. Similarly, the odd-frequency component is odd under $M$ (from Eq. (7)), odd under $T$ and thus, again odd under $S$. As a consequence, both even and odd-frequency components of the induced PDW are spin-singlet in nature, as we also expect since Eq. (1) is spin-rotation invariant.

From Eq. (4), we can already now gain some insights to the OPDW. If the SC is described by an s-wave, or momentum independent, order parameter, the OPDW is zero, since then $\Delta_{k+Q} = \Delta_k$ for any $Q$. So, the coexistence of s-wave SC and CDW in a single-band system cannot give rise to the OPDW. In contrast, if the superconducting order parameter is momentum dependent, $\Delta_{k+Q} \neq \Delta_k$ in general. For example, a coexistence of d-wave SC (given by $\Delta_k \propto \cos k_x - \cos k_y$) and CDW with $Q = (\pi, \pi)$ gives the highest OPDW as then $\Delta_{k+Q} = -\Delta_k$. In the next section, we investigate the OPDW in the context of cuprates, prototype d-wave superconductors. Although superconductivity in cuprates is achieved by doping a parent antiferromagnetic ($Q = (\pi, \pi)$) insulator, the CDW observed in these materials have a $Q$ different from $(\pi, \pi)$.

Before discussing the case of cuprates, we comment on the similarities of the Hamiltonian in Eq. (2) with that of a multiband superconductor. A simple analogy with the Hamiltonian of multiband superconductors can be drawn if we consider $\xi_{k+Q}$ as an independent second band. In this picture, $\chi_k$ is then the band hybridization between two bands $\xi_k$ and $\xi_{k+Q}$, while $\Delta_k$ and $\Delta_{k+Q}$ become two independent superconducting order parameters in each bands, respectively. As was shown in Ref. [81], odd-frequency pairing arises if the band hybridization is finite and the superconducting order parameters are not equal in two bands, i.e., if $\Delta_{k+Q} \neq \Delta_k$. This is the same criterion as established in Eq. (4) for CDW and thus illustrates the underlying similarity, although the materials and properties are completely different. We also note that in the case of multiband superconductors, the odd-frequency pairing does not have a modulation wave vector.

III. CASE OF CUPRATES

The origin of CDW in cuprates is still a debated question. Two parallel point of views have been proposed: one based on strong real space electron interactions giving CDW with wave vectors commensurate with the lattice and the other based on a momentum space picture where the Fermi surface plays an important role in defining the CDW wave vectors. The former picture discusses the experimentally observed incommensurate CDW wave vectors by disorder-induced discommensuration effects. The latter gives a CDW wave vector that connects points of different branches of the Fermi surface. In this latter picture, it has been postulated that there exists an antiferromagnetic quantum critical point beneath the superconducting dome. Outside the antiferromagnetic phase, short-range antiferromagnetic fluctuations diverge near the ‘hot spots’ ($k$-points where the antiferromagnetic Brillouin zone intersects the Fermi surface) in two spatial dimensions, giving rise to CDW and superconducting correlations. As a result, the CDW wave vectors are connecting different ‘hot spots’. The model in Eq. (1) can also describe the former picture with commensurate CDW wave vectors. Therefore the choice of model is not crucial for our results, although we note that disorder-induced discommensuration effects in the first picture cannot be captured within this model.

To continue, we consider in Secs. III A-III C a band dispersion mimicking the Fermi surface of the underdoped cuprate, YBa$_2$Cu$_3$O$_{7-x}$ (YBCO) with $\xi_k$ given by,

$$\xi_k = \frac{t_1}{2} (\cos k_x + \cos k_y) + t_2 \cos k_x \cos k_y + t_3 (\cos 2k_x + \cos 2k_y) + \frac{t_4}{2} (\cos 2k_x \cos k_y + \cos k_x \cos 2k_y) + t_5 \cos 2k_x \cos 2k_y + \mu,$$ (8)
with YBCO: \( t_1 = -1.1259 \text{ eV}, t_2 = 0.5540 \text{ eV}, t_3 = -0.1174 \text{ eV}, \)
\( t_4 = -0.0701 \text{ eV}, t_5 = 0.1286 \text{ eV}, \mu = 0.1756 \text{ eV}. \)

This band structure corresponds to an approximate hole doping of 12\%,\textsuperscript{102,103} which is where the intensity of the CDW in x-ray experiments\textsuperscript{17} is close to maximum. We will primarily express all energies corresponding to this band in units of \( t_1 \). Motivated by experiments, superconducting\textsuperscript{104,105} and CDW\textsuperscript{13,106} order parameters are taken to be \( d \)-wave in nature and given by,

\[
\Delta_k = \frac{\Delta_0}{2} (\cos k_x - \cos k_y),
\]

\[
\chi_k = \frac{\chi_0}{2} (\cos k_x - \cos k_y),
\]

where \( \Delta_0 \) and \( \chi_0 \) give the maximum values of the superconducting and CDW order parameters, respectively. We note that recent resonant x-ray scattering measurements suggest an unusual possibility of an \( s \)-wave symmetry of the CDW in YBCO.\textsuperscript{107} Our analysis in Sec. II A holds for any symmetry of the CDW order, however the quantitative results in this section will differ with an \( s \)-wave CDW.

In order to efficiently study the PDW, we use the fact that the EPDW is even and the OPDW is odd under the momentum structure of the CDW in Eqs. (3) and (4). It is evident that the contribution of the OPDW and the EPDW are given by, \( \Delta_k \) and \( \chi_k \) respectively. We note that recent measurements\textsuperscript{111,112} suggest that the energy scales corresponding to the superconducting and CDW order parameters are very close to each other for a large range of doping levels for different cuprate materials. Second, since both \( F^\text{odd}_{k,k+Q}(i\omega) \) and \( F^\text{odd}_{k+Q,k}(i\omega) \) are purely imaginary, see Eq. (7). Looking at the functional form of the EPDW and OPDW correlations in Eqs. (3) and (4), it is evident that the momentum structure of \( F^\text{e}(k,i\omega) \) and \( F^\text{o}(k,i\omega) \) depend on the \( Q \) vector. In addition, we consider both CDW and SC to have \( d \)-wave symmetry. As the induced PDW comes as a product of CDW and SC in Eqs. (3) and (4), we can already now conclude that \( F^\text{e}(k,i\omega) \) and \( F^\text{o}(k,i\omega) \) do not have a simple \( d \)-wave structure. Still, we expect sign changes in \( F^\text{e}(k,i\omega) \) and \( F^\text{o}(k,i\omega) \) as they depend on both \( k \) and \( k+Q \). Keeping in mind that we might encounter sign changes in the induced PDW, we define the following two momentum sums to characterize the total momentum contributions,

\[
F^\text{e/o}(k,i\omega) = \sum_k F^\text{e/o}_{k}(i\omega),
\]

\[
F^\text{e/o}_{k}(i\omega) = \sum_k F^\text{e/o}_{k}(i\omega). \]

If \( F^\text{e/o}_{k}(i\omega) \) has a pure \( d \)-wave structure, \( F^\text{e/o}(k,i\omega) \) will be zero. In next sections, we choose three specific \( Q \) values relevant for cuprates and calculate both the momentum structure and momentum averaged quantities for the induced PDW.

### A. Diagonal CDW

![FIG. 1. Diagonal CDW.](image)

(a) Green lines show the Fermi surface of the underdoped YBCO band in the first Brillouin zone. The antiferromagnetic Brillouin zone is shown by black dashed lines. ‘Hot spots’, defined by \( k \)-points where the antiferromagnetic Brillouin zone intersects the Fermi surface, are marked as blue dots and numbered 1-8. CDW wave vectors \( Q \) are indicated by arrows, connecting diagonally opposite ‘hot spots’. (b) The Brillouin zone is divided into eight regions marked R1-R8 to preserve internal rotational symmetry. All \( k \)-points in a particular region have same diagonal \( Q \) vector, with directions and values indicated in each region.

The ‘hot spots’ theory\textsuperscript{26,27} was originally constructed with wave vectors connecting diagonal parts of the Fermi surface. As shown in Fig. 1(a), the \( Q \) vectors are here given by the ones connecting ‘hot spots’ marked by ‘2’ (‘1’) and ‘6’ (‘5’), lying on diagonally opposite parts of the Fermi surface. In this section, we first start with this diagonal CDW \( Q \) vector and consider the possibility of the OPDW. The CDW order in the ‘hot spots’ theory does not break the \( C_4 \) lattice rotational symmetry. Thus, to ensure the \( C_4 \) symmetry, we divide the Brillouin zone into 8 regions marked as ‘R1’ to ‘R8’ in Fig. 1(b). Each of these octants has a \( Q \) connecting different ‘hot spots’. Directions and values of \( Q \) vectors are shown in Fig. 1(b).

Using the diagonal CDW wave vectors in Fig. 1, we plot in Fig. 2, the PDW amplitudes \( F^\text{e} \) and \( F^\text{o} \) for different realistic values\textsuperscript{102,103,108–110} of \( \Delta_0 = \chi_0 \). We only show results taking \( \Delta_0 = \chi_0 \) for two reasons. First, recent measurements\textsuperscript{111,112} suggest that the energy scales corresponding to the superconducting and CDW order parameters are very close to each other for a large range of doping levels for different cuprate materials. Second, since both \( F^\text{o}(k,i\omega) \) and \( F^\text{e}(k,i\omega) \) are directly proportional to \( \chi_k \), their relative values do not change by only changing \( \chi_0 \). We see in Fig. 2 that \( F^\text{e} \) acquires its maximum value for \( \omega = 0 \) and we call this value \( F^\text{e, max} \). On the other hand, \( F^\text{o} \) is zero at \( \omega = 0 \) by definition of being odd in \( \omega \). \( F^\text{o} \) instead peaks for low but finite \( \omega \) and we call this value \( F^\text{o, max} \). Even for the small value of \( \Delta_0 = 0.05 \), we note that \( F^\text{o} \) is finite, although \( F^\text{o, max} \) is small compared to \( F^\text{e, max} \). However, values of \( F^\text{e} \) and \( F^\text{o} \) become very similar for \( \omega \) greater than a particular value indicated by...
vertical arrows in Fig. 2. Notably, we also find that increasing $\Delta_0$ increases $F^0_{\text{max}}$, while the same decreases $F^e_{\text{max}}$. Therefore, the ratio of $F^0_{\text{max}}$ to $F^e_{\text{max}}$ is strongly increased for stronger SC. For example, for $\Delta_0 = 0.2$, $F^0_{\text{max}}$ is almost half of $F^e_{\text{max}}$. In the inset of Fig. 2, we also show $F^o_{\nu}$ and $F^e_{\nu}$, which are the total momentum sums of the OPDW and EPDW when considering their signs, as defined in Eq. (15). The large differences in the values of $F^o_{\nu}/e$ and $F^e_{\nu}/e$ already suggests that there are notable sign changes in both PDWs.

To understand the sign change and the detailed momentum structure of the induced PDW, we plot color density maps of $F^o_{\nu}(i\omega)$ and $F^e_{\nu}(i\omega)$ in Fig. 3 for $\Delta_0 = \chi_0 = 0.2$ at a specific frequency, $\omega = 0.19$. We choose these parameters because the momentum averaged OPDW and EPDW in Fig. 2 are there of comparable magnitude. We also overlay the Fermi surface of the YBCO band considered in green. We see that the OPDW $F^o_{\nu}$ in Fig. 3(a) changes sign along the Fermi surface, going from the anti-nodal region (near $(\pi, 0)$ and three other $C_4$ symmetric regions) to the nodal region (near $k_x = k_y$ or $k_x = -k_y$ lines). In contrast, $F^e_{\nu}$ in Fig. 3(b) does not change signs along the Fermi surface, instead it changes sign as we go away from the Fermi surface. Plotting the absolute values, $|F^o_{\nu}|$ and $|F^e_{\nu}|$ in Figs. 3(c) and (d), respectively, help to better visualize the zeros. Since $\chi_k$ is assumed to have $d$-wave character, both $F^o_{\nu}$ and $F^e_{\nu}$ have zeros along the $k_x = k_y$ or $k_x = -k_y$ lines. Additionally, $F^o_{\nu}$ is also zero at ‘hot spots’, since $\Delta_{k+Q} = \Delta_k$, giving sign change across these spots. Despite the difference in nodal structure, the regions of the Brillouin zone with high values of PDW correlations are very similar for both the even- and odd-frequency parts.

### B. Bi-axial CDW

Although diagonal CDW is the primary charge instability in models with short-range antiferromagnetic interactions,\textsuperscript{28} recent experiments in cuprates suggest that the CDW are actually bi-axial in nature with $Q = (Q_x, 0)$ and $Q = (0, Q_y)$.\textsuperscript{112,114} The magnitudes of the observed $Q$ vectors are very close to the wave vectors connecting neighboring ‘hot spots’\textsuperscript{114} as shown in Fig. 4(a). CDW with such axial wave vectors have also been found as a competing instability in models with antiferromagnetic interactions\textsuperscript{28} and can furthermore be enhanced by including an off-site Coulomb interaction.\textsuperscript{47,115} Thus, in this section, we consider the CDW wave vector to be bi-axial connecting neighboring ‘hot spots’.
as in Fig. 4(a). As there is no experimental evidence that bi-axial CDW break the $C_4$ lattice rotational symmetry, we again separate the Brillouin zone into eight regions as shown in Fig. 4(b) to ensure the $C_4$ symmetry.

![Momentum-averaged absolute values of EPDW, $F^e$, and OPDW, $F^o$, (dashed lines) and OPDW, $F^o$, (solid lines) plotted as a function of frequency $\omega$ with bi-axial $Q$ CDW given in Fig. 4, and for the same realistic values of $\Delta_0 = \chi_0$ as used in Fig. 2. Arrow indicates the $\omega$ value above which $F^o > F^e$. Inset: EPDW (dashed line) and OPDW (solid line) obtained when the momentum averaging is done considering the actual sign.

We plot the frequency dependence of the EPDW $F^e$ and the OPDW $F^o$ with bi-axial CDW wave vectors in Fig. 5, for the same values of $\Delta_0 = \chi_0$ as used for the diagonal CDW. We again find that for all $\Delta_0$, $F^o$ attains a finite value, with a frequency dependence that is very similar to the case of diagonal CDW. With increasing $\Delta_0$, the $F_{\text{max}}$ increases. The $F^e_{\text{max}}$ on the other hand, initially decreases with an increase in $\Delta_0$, but does not change with further increase in $\Delta_0$. The ratio of the $F^e_{\text{max}}$ to the $F^o_{\text{max}}$ therefore increases with increasing $\Delta_0$, but the ratio is, however, somewhat smaller compared to the case with diagonal CDW case reported in Fig. 2. Still, values of $F^e_{\text{max}}$ and $F^o_{\text{max}}$, shown in the inset of Fig. 5, are comparable and also illustrated the very different frequency dependencies of the EPDW and OPDW states.

We again gain additional insights into the relative strengths of $F^e_k$ and $F^o_k$ by looking at their momentum structure. Color density maps of $F^e_k$, $F^o_k$, $|F^e_k|$ and $|F^o_k|$ for $\omega = 0.19$ and $\Delta_0 = 0.2$ are plotted in Fig. 6 (a), (b), (c), and (d), respectively. Since the choice of $Q$ is different here compared to the diagonal CDW, lines where $\Delta_{k+Q} = \Delta_k$ also change. This gives a very different momentum space structure, especially for $F^o_k$. For example, in the region ‘R2’, $\Delta_{k+Q} = \Delta_k$ along a line $k_y = k_{HS}$ parallel to the $k_y$-axis, where $k_{HS}$ is the x-coordinate of ‘hot spot’ 2, see Fig. 4. In contrast, the line $\Delta_{k+Q} = \Delta_k$ in the diagonal CDW case is not parallel to the $k_y$-axis. Similar features are observed in the other $C_4$ symmetric regions of the Brillouin zone. $F^e_k$ shows a much more similar momentum space structure to the diagonal CDW case. In addition, both $F^e_k$ and $F^o_k$ have usual zeros along nodal directions, i.e. along $k_x = \pm k_y$.

To provide a better understanding of the relative magnitude of the EPDW and OPDW in different parts of the Brillouin zone, we make two line cuts along the red and blue arrows in Fig. 6(a). $F^e_k$ and $F^o_k$ are shown along the red arrow ($k_y = \pi$) in Fig. 6(c) and along the blue arrow ($k_y = 0.4\pi$) in Fig. 6(f). In the anti-nodal region, $k_y = \pi$, the induced PDW is clearly dominated by the even-frequency component with the maximum of OPDW being at most 25% of the EPDW. However, close to the nodal region ($k_y = 0.4\pi$), the OPDW and EPDW have very similar maximum values. Thus, even if momentum integrated values of $F^e_k$ are small compared to their even-frequency counterparts, as indicated in Fig. 5, the magnitudes of OPDW and EPDW become clearly very comparable in some parts of the Brillouin zone.

C. Uni-axial CDW

The bi-axial CDW discussed in the previous section is observed in most cuprates at zero or low magnetic fields. A strong magnetic field $B$, particularly in YBCO. For $B > 17$ T, the correlation length of CDW jumps from ~20 to ~100 lattice spacings, indicating a transition to a ‘true’ long-range CDW phase. Long-range CDW are uni-axial in nature with $Q = (Q_x, 0)$ or $Q = (0, Q_y)$, but not both. In other words, a transition occurs from a checkerboard CDW to a stripe CDW with increasing magnetic field. Long-range CDW have also been found to coexist with SC in a window of magnetic field $B_{c2} > B > 17$ T, where $B_{c2}$ is the upper critical field of the superconducting order. Although the magnetic field introduces vortex-induced inhomogeneities in a strongly type-II cuprate superconductor, an effective homogeneous Hamiltonian, as in Eq. (1), still gives a reasonable description of the coexistence phase close to $B_{c2}$. We therefore also study uni-axial CDW, where we consider $Q$ vectors only along one axis as shown in Fig. 7(a). The uni-axial nature of CDW breaks the $C_4$ rotational symmetry of the Brillouin zone. We thus here need to separate the Brillouin zone into four regions, as shown in Fig. 7(b), instead of eight for the previously treated CDW.

Similar to the findings in the two previous sections for diagonal and bi-axial CDW, the OPDW $F^o$ increases with increasing $\Delta_0$ also for a uni-axial CDW, as shown in Fig. 8. Interestingly, $F^e$ with uni-axial CDW wave vectors show a considerable increase compared to $F^o$ with bi-axial CDW (cp. Fig. 5) for all values of $\Delta_0$. For example, for $\Delta_0 = 0.2$, $F^e_{\text{max}} \approx 0.1$ in Fig. 8, whereas $F^o_{\text{max}} \approx 0.04$ for a bi-axial CDW in Fig. 5. Compared to the other investigated CDW, $F^e_{\text{max}}$ for a uni-axial CDW behaves differently and increases with increasing $\Delta_0$. Thus, even though the $F^o_{\text{max}}$ is increased with a uni-axial CDW, the ratio of the $F^e_{\text{max}}$ to the $F^o_{\text{max}}$ remains about same as for the bi-axial CDW. Notably, for high $\omega$ and $\Delta_0$, the OPDW becomes comparable and even larger than the EPDW. However, for $\Delta_0 = 0.2$ gives $F^o > F^e$ for all $\omega > 0.4$ and $F^e > F^o$ for $\omega > 0.25$.

The broken $C_4$ symmetry with a uni-axial CDW is clearly reflected in the momentum structure of the OPDW and the EPDW plotted Fig. 9. In the anti-nodal regions near $k_y = \pm \pi$, both $F^e_k$ and $F^o_k$ behave similar to the case of bi-axial CDW, as
the choice of \( Q \) is the same in these regions for the uni-axial and bi-axial CDW. The distinction between uni-axial CDW and bi-axial CDW instead comes in the anti-nodal regions near \( k_x = \pm \pi \). Values of \( F^o_k \) and \( F^e_k \) near \( k_x = \pm \pi \) are significantly enhanced compared to the values near \( k_y = \pm \pi \), with both displaying very high values near the Fermi surface. We note that \( |F^o_k| \) decays faster than \( |F^e_k| \) as we go away from the Fermi surface to a region close to \((\pm \pi, 0)\), giving \( |F^o_k| < |F^e_k| \) at \((\pm \pi, 0)\). Thus, after momentum averaging, \( F^o < F^e \) as found in Fig. 8, but where the two different PDW show very similar values around the Fermi surface.

### D. Band structure robustness

Our results above for different choices of \( Q \) vectors for the CDW show that the induced OPDW is a common feature when there exists a coexistence of SC and CDW. One might,

![Diagram](image-url)
however, ask how sensitive these results are to the band structure considered. Till now, we have considered only a YBCO band structure, see Eq. (9). In this section we further investigate two different bands with contrasting features. In this endeavor, we take the parameters,

\[
\text{BSCCO: } t_1 = -0.6798 \text{ eV}, t_2 = 0.2368 \text{ eV}, t_3 = -0.0794 \text{ eV}, \quad t_4 = 0.0343 \text{ eV}, t_5 = 0.0011 \text{ eV}, \quad \mu = 0.196 \text{ eV}
\]

and

\[
\text{LSCO: } t_1 = -0.7823 \text{ eV}, t_2 = 0.0740 \text{ eV}, t_3 = -0.0587 \text{ eV}, \quad t_4 = -0.1398 \text{ eV}, t_5 = -0.0174 \text{ eV}, \quad \mu = 0.0801 \text{ eV},
\]

which mimic the band structure of underdoped Bi$_2$Sr$_2$CaCu$_2$O$_{8+x}$ (BSCCO) and La$_{2-x}$Sr$_x$CuO$_4$ (LSCO), respectively. The Fermi surfaces of these BSCCO and LSCO band structures are shown in Fig. 10 (a) and (b), respectively. While the BSCCO band has a Fermi surface with isotropic Fermi velocities and no nesting regions, the LSCO band has ‘approximate’ nesting only in a small region near the anti-nodes. In contrast, the earlier considered YBCO band has long ‘approximately’ nested necks around the ‘hot spots’, as seen in Fig. 1(a).

While the high magnetic field long range uni-axial CDW is observed only in YBCO, both YBCO and BSCCO show the bi-axial CDW at low or zero magnetic fields. In contrast, LSCO features uni-axial stripe CDW even at zero magnetic fields. In order to focus only on the band structure dependencies for the induced PDW, we here consider the case of CDW with bi-axial \( Q \) vectors for all three cuprates and thus divide the Brillouin zone into eight regions as in Fig. 4(b). The magnitude of \( Q \) is given by the distance between neighboring ‘hot spots’ in the corresponding bands, indicated in Fig. 10.

We plot the results in Fig. 11, where \( F^o \) and \( F^e \) are shown for all three cuprates at \( \Delta_0 = 0.2 \). In all three cases, \( F^o \) is finite and behave very similarly with frequency. \( F^e \) increases notably from YBCO to BSCCO to LSCO. This is primarily due to the fact that \( \Delta_{k+Q} - \Delta_k \) is larger in most parts of the Fermi surface in BSCCO and LSCO compared to YBCO. In order to illustrate this statement, we focus on a representative point \( k_1 \) on the Fermi surface, indicated in Fig. 10. For LSCO, \( k_1 + Q \) is found at \( k_x = 0 \). As a result, \( \Delta_{k_1+Q} \) is maximum due to the \( d \)-wave nature (note that \( Q = (-Q_x,0) \) does not change \( k_{1y} \)). On the other hand, \( k_1 + Q \) for BSCCO lies away from \( k_x = 0 \). Although not shown in Fig. 10, \( k_1 + Q \) for YBCO lies even further away from \( k_x = 0 \). So, we clearly find \( \Delta_{k_1+Q}(\text{LSCO}) > \Delta_{k_1+Q}(\text{BSCCO}) > \Delta_{k_1+Q}(\text{YBCO}) \). Furthermore, \( \Delta_{k_1}(\text{LSCO}) \approx \Delta_{k_1}(\text{BSCCO}) \approx \Delta_{k_1}(\text{YBCO}) \) and \( F^e_{k_1} \propto \Delta_{k_1+Q} - \Delta_{k_1} \). As a direct consequence, \( F^e_{\text{max}} \) is the largest in LSCO and smallest in YBCO. However, for YBCO, \( F^o \) decays a bit more slowly with \( \omega \), such that its value actually become the highest among the three bands for large frequencies. We also note that these band structure results show that the OPDW presented in the earlier three sections, III A-III C, is likely underestimated in terms of its magnitude. In spite of these band effects, the ratio of \( F^o_{\text{max}} / F^e_{\text{max}} \) does not change with the change in the band structure as \( F^e_{\text{max}} \) also increases in BSCCO and LSCO. Based on these results for three different band structures, representing three different underdoped cuprates, we conclude that the OPDW is a robust and ubiquitous feature in cuprate superconductors, although there exist some quantitative differences.
IV. SPIN-TRIPLET ODD-FREQUENCY PDW: COEXISTENCE OF SC AND SDW

Having established that CDW generically give rise to OPDW correlations, and especially in the cuprates, we also explore whether a spin density wave (SDW) can induce such correlations. In order to investigate the coexistence of SC and SDW, we write a mean field Hamiltonian in momentum space,

\[ H_{\text{SDW}} = \sum_{k,\sigma} \xi_k c_{k\sigma}^\dagger c_{k\sigma} + \sum_{k,\sigma} \left( m_k c_{k\sigma}^\dagger c_{k+Q\sigma} + \text{H.c.} \right) + \text{constant}, \] 

where \( m_k \) is the SDW order parameter with a modulation wave vector \( Q \) and all other notation is the same as in Eq. (1). This Hamiltonian can be written in a matrix form, in the basis as for the CDW case: \( \Psi = \left( c_{k\uparrow}^\dagger, c_{k+Q\uparrow}^\dagger, c_{-k\downarrow}, c_{-k-Q\downarrow} \right) \) as,

\[ H_{\text{SDW}} = \frac{1}{2} \sum_k \Psi^\dagger \hat{H}_{\text{SDW}} \Psi + \text{constant}, \]

with

\[ \hat{H}_{\text{SDW}} = \begin{pmatrix} \xi_k & m_k & \Delta_k & 0 \\ m_k & \xi_{k+Q} & 0 & \Delta_{k+Q} \\ \Delta_k & 0 & -\xi_k & m_k \\ 0 & \Delta_{k+Q} & m_k & -\xi_{k+Q} \end{pmatrix}. \] 

We can obtain the Green’s function \( G^s \) for Eq. (19) in the same way as in Sec. II A and the PDW correlator is given by,

\[ F^s_{k+Q,k}(i\omega) = G^s_{14}(i\omega) = F^s_{k+Q,k}(i\omega) + F^{s,\text{odd}}_{k+Q,k}(i\omega), \]

where

\[ F^s_{k+Q,k}(i\omega) = \frac{m_k (\xi_k\Delta_{k+Q} - \xi_{k+Q}\Delta_k)}{D}, \] 

\[ F^{s,\text{odd}}_{k+Q,k}(i\omega) = \frac{i\omega m_k (\Delta_k + \Delta_{k+Q})}{D}, \]

\[ D = (\omega^2 + \xi_k^2 + \Delta_k^2) (\omega^2 + \xi_{k+Q}^2 + \Delta_{k+Q}^2) - 2m_k^2 (\xi_k\xi_{k+Q} - \Delta_k\Delta_{k+Q} - \omega^2) + m_k^4. \]

The functional form of \( D \) here is same as that in Eq. (5), only with \( \eta_k \) replaced by \( m_k \). Here again, we see that the induced PDW, very generically has both even and odd-frequency components. However, we note directly that there is a change of signs in the definitions of the PDW generated by SDW compared to the earlier CDW case; the ‘+’ sign in Eq. (3) is now given by a ‘-’ sign in Eq. (20) and the ‘-’ sign in Eq. (4) is now given by a ‘+’ sign in Eq. (21).

To investigate the spin symmetries for the SDW-generated PDW, we look at the \( G^s_{23}(i\omega) \) component of the Green’s function, similarly as in Sec. II A. We obtain

\[ F^s_{k+Q,k}(i\omega) = G^s_{23}(i\omega) = F^{s,\text{even}}_{k+Q,k}(i\omega) + F^{s,\text{odd}}_{k+Q,k}(i\omega), \]

where

\[ F^{s,\text{even}}_{k+Q,k}(i\omega) = \frac{m_k (-\xi_k\Delta_{k+Q} + \xi_{k+Q}\Delta_k)}{D}, \]

\[ F^{s,\text{odd}}_{k+Q,k}(i\omega) = \frac{i\omega m_k (\Delta_k + \Delta_{k+Q})}{D}. \] 

Thus, in contrast to Sec. II A, the even-frequency component of the PDW is now odd under \( M \) (from Eq. (23)). As the EDPW is even under \( T \), to satisfy \( SMT = -1 \), \( S \) should be even or in a spin-triplet configuration. Similarly, the odd-frequency component is even under \( M \) (from Eq. (24)), odd under \( T \), and thus \( S \) should also be even or spin-triplet. So, both the even and odd-frequency components of the induced PDW are spin-triplet in nature when the SC coexists with the SDW.

It is actually not surprising that the PDW correlations in the coexistence state of the SC and the SDW are spin-triplet in nature. The coexistence or competition of the SC and the AFM was discussed within an SO(5) model in Ref. [122]. Within this field theoretic picture, a PDW operator rotates a superconducting field to an AFM field. Since we consider the superconducting state to be spin-singlet, the spin transfer from the superconducting state to the AFM state is unity. Thus, the PDW operator has to be triplet with net spin unity, in order to conserve the spin. With \( Q = (\pi, \pi) \), this PDW operator is famously known as the ‘II’ operator. PDW correlations, obtained in Eqs. (20) and (21), are equivalent to the ‘II’ operator if we take \( Q = (\pi, \pi) \). While the even frequency is discussed in the literature in the context of SO(5) model, the odd-frequency component has, to our knowledge, not been previously explored.

Another striking difference between the OPDW generated in the two coexistence states is that, while the OPDW from CDW in Eq. (4) is zero when \( \Delta_k = \Delta_{k+Q} \), the OPDW from SDW in Eq. (21) is maximum when \( \Delta_k = -\Delta_{k+Q} \) and zero when \( \Delta_k = \Delta_{k+Q} \). So, a coexistence of \( d \)-wave SC and SDW with \( Q = (\pi, \pi) \) always lead to a zero OPDW. Thus, a coexistence of \( d \)-wave SC and antiferromagnetism in cuprates cannot induce any OPDW. On the other hand, spin-triplet OPDW correlations can exist in an \( s \)-wave superconductor, where the spin-singlet CDW-generated OPDW instead cannot be found.

A coexistence regime of an \( s \)-wave SC and SDW with \( Q = (\pi, 0) \) or \( Q = (0, \pi) \) is often observed in the iron-based superconductors (FeSC), especially in the ferropnictides. Although FeSC are multi-orbital in nature, we can gain a simplistic understanding of the phenomenology by looking only...
at a single orbital described by the Hamiltonian in Eq. (18). FeSC are considered to be sign changing $s$-wave superconductors ($s^{++}$ between the electron and the hole bands),\textsuperscript{123} with the intra-orbital pair symmetry taking forms such as $s_{xy} \sim \cos(k_x) \cos(k_y)$.\textsuperscript{124} From Eqs. (21) we then directly see that $s_{xy}$ intra-orbital pairing cannot give rise to OPDW in this simplified one-band picture. However, the SDW order has been shown to be able to promote $s^{++}$ intra-orbital pairing in the coexistence phase.\textsuperscript{125}

In the $s^{++}$ state, induced spin-triplet OPDW are enhanced as $\Delta_k = -\Delta_{k+Q}$. Furthermore, a transition from the $s^{++}$ to the $s^{+}$ state is expected with the increase of impurity scatterings.\textsuperscript{126,128} This might then lead to the emergence of a spin-triplet OPDW with increasing disorder in FeSC. A complete understanding of the OPDW in the coexisting state of the SC and the SDW in the FeSC will require the consideration of multiple orbitals and is left to future work.

V. CONCLUSION AND DISCUSSION

In summary, we showed that the $d$-wave nature of the superconducting state in the cuprate high-temperature superconductors leads to induced odd-frequency pair density wave (OPDW) correlations in the region of the phase diagram where the SC coexists with a CDW. We considered several different CDW wave vectors relevant to the cuprates and showed that the existence of the OPDW is extremely robust to the choice of the wave vector and also to the variations in the band structure found between different families of cuprates. We find that the OPDW is often significant in magnitude and becomes even equal or larger than the even-frequency PDW (EPDW) near the nodal regions in momentum space. Moreover, we find that breaking the $C_4$ lattice symmetry in the CDW wave vector with a uni-axial wave vector further enhances the OPDW amplitude. We also do not restrict ourselves to the cuprates, but also show that the OPDW can also be found in the coexistent state of SC and SDW in the iron-based superconductors.

Direct experimental detection of odd-frequency superconductivity has been challenging in the past. In our results, the induced PDW correlations do not directly influence the one-electron spectral function because the quasi-particle energy spectrum is not affected. So, one-electron experimental probes, such as angle-resolved photoemission spectroscopy (ARPES) or scanning tunneling spectroscopy (STS), will not detect the induced PDW. However, two-electron response functions, will have signatures of these PDW correlations,\textsuperscript{129-133} and even be able to distinguish odd-frequency components. For example, the imaginary part of the density response function $\chi''(q, \Omega)$ can characterize various bosonic excitations or correlations at different momentum ($q$) and energies or frequencies ($\Omega$) depending on the experimental probe. Note that there are two sets of momentum and frequencies, one internal ($k$ and $\omega$) and the other external ($q$ and $\Omega$). While calculating the response function, we integrate over internal frequencies only and arrive at $\chi''(q, \Omega)$ as a function of momentum and frequency of the external probe. The contribution of the OPDW to this response function comes with terms proportional to $F_{\Delta k \rightarrow k-eq}^\dagger (i\omega)$ and $\chi''(q, \Omega)$.

Now, the question remains how to distinguish between the EPDW and the OPDW contributions? To approach an answer to this question, we use the combined facts that the OPDW and the EPDW have different momentum space structure and different frequency dependence (EPDW peaks at $\omega = 0$, while OPDW peaks at finite $\omega$). So, we should look for experimental probes which resolve the response functions both in momentum and frequency spaces. Probes like resonant inelastic x-ray scattering (RIXS), momentum-resolved electron-loss spectroscopy (M-EELS), and Raman spectroscopy can fulfill our needs. Each of these has their own advantages. RIXS is a well-established method in the literature, with RIXS experiments in underdoped BSCCO showing intensity peaks near $q = Q$ (where $Q$ is the CDW wave vector) both at $\Omega = 0$ and finite $\Omega$.\textsuperscript{134} In fact, the momentum-dispersion of the finite $\Omega$ peak has been predicted to be signatures of the role of phonons\textsuperscript{134} or collective modes.\textsuperscript{133} Due to the difference in the frequency dependence of the OPDW and the EPDW, we expect the frequency-dispersion to play a key role in detecting the OPDW. At the same time, RIXS lacks good energy resolution, where instead M-EELS derives a clear advantage in having a very high energy resolution of 1 meV compared to a resolution of 40 meV in RIXS.\textsuperscript{135} But, to the best of our knowledge, M-EELS have so far only been used to investigate optimal or overdoped cuprates.\textsuperscript{136} Future M-EELS experiments in the underdoped regime are hence necessary. On the other hand, Raman spectroscopy has a unique advantage of preferentially probing parts of the Brillouin zone.\textsuperscript{312} For example, one way to distinguish the OPDW from the EPDW is Raman measurements in the $B_{2g}$ channel (which preferentially probes the nodal region), as both OPDW and EPDW are equally dominant in the nodal region. It is possible that the so-called peak-dip-hump structure\textsuperscript{38} in Raman intensity as a function of frequency can find its explanation in terms of the OPDW. Finally, the OPDW might also leave distinct signatures in Josephson scanning tunneling measurements,\textsuperscript{94} Meissner response\textsuperscript{132} and other transport measurements.\textsuperscript{95}

Let us also comment on disorder-induced inhomogeneities that are intrinsic to all cuprate materials. Due to the Imry-Ma criterion,\textsuperscript{136} any strength of disorder disrupts the long-range phase coherence of CDW in dimensions $d \leq 4$. Thus, at low or zero magnetic fields, CDW in two-dimensional cuprates, are only short-ranged correlations with no long-range phase coherence. We have ignored these fluctuations due to disorder in our mean-field calculations. However, the correlation lengths of CDW in cuprates are still long enough\textsuperscript{17} to allow for a finite CDW mean-field amplitude, validating the bulk of our analysis in this work. Even if disorder fluctuations were to be included on top of the mean-field analysis, we expect the OPDW to be present locally in regions of local coexistence of
SC and CDW. In addition, the high-field CDW is experimentally shown to be a ‘true’ long-range order.\textsuperscript{18,19,31} The high-field coexistence of SC and uni-axial CDW will thus lead to a long-range OPDW.

We have also not explored the prospects of the OPDW being responsible for the anomalous properties of the pseudogap phase in cuprates. Our results should motivate future research in this direction as its dynamical character naturally make it a hard to detect, or even hidden, order. Moreover, in the pseudogap phase, there are additional broken symmetry orders, such as nematic or time-reversal symmetry broken orders. As we found in Sec. III C, the magnitude of the OPDW is significantly increased when the $C_4$ symmetry is broken by the CDW wave vectors. We believe an additional nematic distortion of the Fermi surface will likely further enhance the OPDW correlations.

We finally note two possible extensions. First, the translational symmetry breaking due to the CDW order reconstructs a single band superconductor into effectively two bands, one shifted from the other by the CDW wave vector. This shift makes the system conceptually analogous to a multiband superconductor with different superconducting order parameters in different bands. As a result, odd-frequency correlations are induced in the same spirit as in multiband superconductors, but with a major difference being the modulations in the induced order. As a consequence, our work on coexisting CDW and SC can easily be generalized to other translational symmetry breaking orders, which might thus also host significant odd-frequency components. Second, in the one-band model considered in this work, the coexistence of SC and CDW cannot give rise to an odd-frequency component of the induced PDW when the superconducting order parameter is momentum-independent or $s$-wave. But the analysis does not hold true for multi-band superconductors, such as transition-metal dichalcogenides (TMDs) with coexistence of SC and CDW. A PDW state in TMDs has already been proposed theoretically\textsuperscript{141} and also very recently observed experimentally.\textsuperscript{142} Even though TMDs host $s$-wave SC, the multiband nature might still induce OPDW. The search for OPDW in TMDs is a part of ongoing research.

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1. Y. I. Joe, X. M. Chen, P. Ghaemi, K. D. Finkelstein, G. A. de la Peña, Y. Gan, J. C. T. Lee, S. Yuan, J. Heck, G. J. MacDougall, T. C. Chiang, S. L. Cooper, E. Fradkin, and P. Abbamonte, Nat. Phys. 10, 421 (2014).

2. G. Gye, E. Oh, and H. W. Yeom, Phys. Rev. Lett. 122, 016403 (2019).

3. K. Cho, M. Kończykowski, S. Teknowijoyo, M. A. Tanatar, J. Guss, P. B. Gartin, J. M. Wilde, A. Kreysig, R. J. McQueeney, A. I. Goldman, V. Mishra, P. J. Hirschfeld, and R. Prozorov, Nat. Commun. 9, 2796 (2018).

4. H. Isobe, N. F. Q. Yuan, and L. Fu, Phys. Rev. X 8, 041041 (2018).

5. P. Rickhaus, F. de Vries, J. Zhu, E. Portolés, G. Zheng, M. Masseroni, A. Kurzmann, T. Taniguchi, K. Wantanabe, A. H. MacDonald, T. Ihn, and K. Ensslin, arXiv:2005.05373 (2020).

6. P. Dai, Rev. Mod. Phys. 87, 855 (2015).

7. B. Keimer, S. A. Kivelson, M. R. Norman, S. Uchida, and J. Zaanen, Nature 518, 179 (2015).

8. E. Fradkin, S. A. Kivelson, and J. M. Tranquada, Rev. Mod. Phys. 87, 457 (2015).

9. J. E. Hoffman, E. W. Hudson, K. M. Lang, V. Madhavan, H. Eisaki, S. Uchida, and J. C. Davis, Science 295, 466 (2002).

10. K. Matsuura, S. Yoshizawa, Y. Mochizuki, T. Mochiku, K. Hirata, and N. Nishida, J. Phys. Soc. Jpn. 76, 063704 (2007).

11. S. Yoshizawa, T. Koseki, K. Matsuura, T. Mochiku, K. Hirata, and N. Nishida, J. Phys. Soc. Jpn. 82, 083706 (2013).

12. T. Machida, Y. Kohsaka, K. Matsuoaka, K. Iwaya, T. Hanaguri, and T. Tamegai, Nat. Commun. 7, 11747 (2016).

13. M. H. Hamidian, S. D. Edkins, C. K. Kim, J. C. Davis, A. P. Mackenzie, H. Eisaki, S. Uchida, M. J. Lawler, E.-A. Kim, S. Sachdev, and K. Fujita, Nat. Phys. 12, 150 (2015).

14. J. Chang, E. Blackburn, A. T. Holmes, N. B. Christensen, J. Larsen, J. Mesot, R. Liang, D. A. Bonn, W. N. Hardy, A. Watenphul, M. v. Zimmermann, E. M. Forgan, and S. M. Hayden, Nat. Phys. 8, 871 (2012).

15. S. Blanco-Canosa, A. Frano, T. Loew, Y. Lu, J. Porras, G. Ghiringhelli, M. Minola, C. Mazzoli, L. Braicovich, E. Schierle, E. Weschke, M. Le Tacon, and B. Keimer, Phys. Rev. Lett. 110, 187001 (2013).

16. E. Blackburn, J. Chang, M. Hücke, A. T. Holmes, N. B. Christensen, R. Liang, D. A. Bonn, W. N. Hardy, U. Rütt, O. Gutowski, M. v. Zimmermann, E. M. Forgan, and S. M. Hayden, Phys. Rev. Lett. 110, 137004 (2013).

17. G. Ghiringhelli, M. Le Tacon, M. Minola, S. Blanco-Canosa, C. Mazzoli, N. B. Brookes, G. M. De Luca, A. Frano, D. G. Hawthorn, F. He, T. Loew, M. M. Sala, D. C. Peets, M. Salluzzo, E. Schierle, R. Sutarto, G. A. Sawatzky, E. Weschke, B. Keimer, and L. Braicovich, Science 337, 821 (2012).

18. S. Gerber, H. Jang, H. Nojiri, S. Matsuzawa, Y. Yamashita, D. A. Bonn, R. Liang, W. N. Hardy, Z. Islam, A. Mehta, S. Song, M. Sikorski, D. Stefancic, Y. Feng, S. A. Kivelson, T. P. Devereaux, Z.-X. Shen, C.-C. Kao, W. S. Lee, D. Zhu, and J. S. Lee, Science 350, 949 (2015).

19. J. Chang, E. Blackburn, O. Iwashko, A. T. Holmes, N. B. Christensen, M. Hücke, R. Liang, D. A. Bonn, W. N. Hardy, U. Rütt, M. v. Zimmermann, E. M. Forgan, and H. S. M., Nat. Commun. 7, 11494 (2016).

20. T. Wu, H. Mayaffre, S. Krämer, M. Horvatic, C. Berthier, W. N. Hardy, R. Liang, D. A. Bonn, and M.-H. Julien, Nature 477, 1911 (2011).

21. T. Wu, H. Mayaffre, S. Krämer, M. Horvatic, C. Berthier, P. L. Kuhns, A. P. Reyes, R. Liang, W. N. Hardy, D. A. Bonn, and...
