SHARP WEIGHTED BOUNDS FOR FRACTIONAL INTEGRAL OPERATORS IN A SPACE OF HOMOGENEOUS TYPE

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Abstract. We consider a version of M. Riesz fractional integral operator on a space of homogeneous type and show an analogue of the well-known Hardy–Littlewood–Sobolev theorem in this context. We investigate the dependence of the operator norm on weighted spaces on the weight constant, and find the sharp relationship between these two quantities. Our result generalizes the recent Euclidean result by Lacey, Moen, Pérez and Torres [9].

1. Introduction

In the Euclidean space $\mathbb{R}^n$, weighted norm inequalities for fractional integral operators of order $\alpha$ or Riesz potential, defined by

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy, \quad 0 < \alpha < n,$$

have been studied in depth by several authors. At a formal level, the limit $\alpha \to 0$ corresponds to the Calderón–Zygmund case, and for $\alpha > 0$ one deals with a positive operator. The qualitative one weight problem was solved in the early 1970’s in the work of B. Muckenhoupt and R. L. Wheeden [10] giving a characterization of weights $w$ for which $I_\alpha : L^p(\mathbb{R}^n, w^p dx) \to L^q(\mathbb{R}^n, w^q dx)$ is bounded: For $1/p - 1/q = \alpha/n$, the inequality

$$\|I_\alpha f\|_{L^q(w^q)} \leq C \|f\|_{L^p(w^p)}$$

holds if and only if

$$[w]_{A_{p,q}} := \sup_Q \left( \frac{1}{|Q|} \int_Q w^p dx \right) \left( \frac{1}{|Q|} \int_Q w^{-p'} dx \right)^{q/p'} < \infty.$$

Nevertheless, the precise dependence of the operator norm on the weight constant $[w]_{A_{p,q}}$ was not considered in detail until the very recent times. The original interest on such sharp estimates was motivated by applications in other areas of analysis, and the interest in understanding such quantitative questions has been a general trend in the last decade in the study of integral operators.

The central $A_2$ conjecture for Calderón–Zygmund operators was only recently solved by Hytönen [6]. The sharp estimates for the fractional integral operators were obtained somewhat earlier by Lacey et al. [9]: For $1/p - 1/q = \alpha/n$,

$$\|I_\alpha\|_{L^p(w^p) \to L^q(w^q)} \lesssim [w]_{A_{p,q}}^{(1-\alpha/n) \max\{1, \frac{2}{q} \}}$$

(1.1)

and the estimate is sharp. Our main result is the generalization of this quantitative result into the context of spaces of homogeneous type.

The paper is organized as follows. First, we introduce our version $T_\gamma, 0 < \gamma < 1$, of the operator $I_\alpha$ in a space of homogeneous type $(X, \mu)$. Second, we show an analogue of the well-known Euclidean Hardy–Littlewood–Sobolev theorem in our context. This preliminary unweighted result motivates us to only consider exponents which satisfy the identity $1/p - 1/q = \gamma$. Our main result shows that for $1/p - 1/q = \gamma$, the estimate (1.1) holds for $T_\gamma$. Finally, we show that the estimate is sharp in any given space $(X, \mu)$ with infinitely many points. We do this by showing

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that any such space supports functions which, at least locally, behave sufficiently similarly to the basic power functions $|x|^{-\alpha}$ on the Euclidean space.

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2. Preliminaries

2.1. Set-up. Let $(X, \rho, \mu)$ be a space of homogeneous type with a quasi-metric $\rho$ and a doubling measure $\mu$. By a quasi-metric we mean a mapping that satisfies the axioms of a metric except for the triangle inequality, which is assumed in the weaker form

$$\rho(x, y) \leq A_0(\rho(x, z) + \rho(z, y))$$

with a quasi-metric constant $A_0 \geq 1$. As usual, for a ball $B = B(x, r) := \{y \in X : \rho(x, y) < r\}$ and $a > 0$, the notation $aB := B(x, ar)$ stands for the concentric dilation of $B$. By a doubling measure we mean a positive Borel-measure $\mu$ which has the doubling property that there exists a constant $C \geq 1$ such that

$$0 < \mu(2B) \leq C\mu(B) < \infty \quad \text{for all balls } B.$$ 

The smallest constant satisfying (2.2) is denoted by $C_\mu$ and referred to as the doubling constant.

We assume that all balls are measurable.

We recall the following properties of a doubling measure.

2.3. Lemma. Suppose $\mu$ is a doubling measure on $X$. Then the following is true.

(i) Let $x \in X$. Then $\mu(\{x\}) > 0$ if and only if there exists $\varepsilon > 0$ such that $\{x\} = B(x, \varepsilon)$.

(ii) If $\mu(\{x\}) \geq \delta > 0$ for all $x \in X$, then $X$ is countable.

(iii) $\mu(X) < \infty$ if and only if $X = B(x, R)$ for some $x \in X$ and $R < \infty$.

We further recall the following well-known result.

2.4. Lemma. Suppose $\mu$ is a doubling measure. Then for every $x \in X$ and $0 < r \leq R$ we have

$$\frac{\mu(B(x, R))}{\mu(B(x, r))} \leq C_\mu \left(\frac{R}{r}\right)^{c_\mu},$$

where $c_\mu = \log_2 C_\mu$.

2.5. A space of homogeneous type with infinitely many points. Some of our results, most importantly the example in Section 8, require that $X$ is sufficiently non-trivial in that it contains infinitely many points. Then we have the following.

2.6. Lemma. Let $(X, \rho, \mu)$ be a space of homogeneous type. The property $\# X = \infty$ is equivalent to the property that

$$\text{(2.7)} \quad \text{for any } N > 0 \text{ there exists balls } B_0 \text{ and } B_1 \text{ such that } \mu(B_1) > N\mu(B_0).$$

Proof. It is clear that (2.7) implies $\# X = \infty$. Indeed, if $X = \{x_1, \ldots, x_N\}$ and thereby, $\mu(\{x_i\}) \geq \delta > 0$ and $\mu(X) = \sum \mu(\{x_i\}) < \infty$, then $X$ can not have the property (2.7).

We are left to show that the property $\# X = \infty$ implies (2.7). Let $N > 0$. First suppose that there exists $x_0 \in X$ with the property that $\mu(\{x_0\}) = 0$. Choose $B_1 = B(x_0, 1)$ and $B_0 = B(x_0, \varepsilon)$ where $\varepsilon > 0$ is small so that $\mu(B(x_0, \varepsilon)) < 1/(N\mu(B_1))$. Then $\mu(B_1)/\mu(B_0) > N$.

Then assume that $\mu(\{x_i\}) > 0$ for all $i \in X$ and thus, $\{x\} = B(x, \varepsilon)$ for some $\varepsilon = \varepsilon(x) > 0$.

Case 1. Suppose that there exists $\delta > 0$ such that $\mu(\{x\}) \geq \delta$ for all $x$. Given $N > 0$, choose $B_0 := \{x_0\}$ and $B_1 := B(x_0, M)$ where $M$ is large so that $x_i \in B_1$ for at least $N\mu(B_0)/\delta + 1$ different $i$ (since $\# X = \infty$, such an $M$ exists). Then

$$\frac{\mu(B_1)}{\mu(B_0)} \geq \frac{(N\mu(B_0)/\delta + 1)\delta}{\mu(B_0)} > N.$$

Case 2. Then assume that $\mu(\{x_i\}) \to 0$ as $i \to \infty$. Given $N > 0$, choose $B_1 := \{x_1\}$ and $B_0 := \{x_i\}$ where $i$ is large so that $\mu(\{x_i\}) < \mu(B_1)/N$. Then $\mu(B_1)/\mu(B_0) > N$. □
2.8. Remarks. (1) If $X$ has the property (2.7), then the balls $B_0$ and $B_1$ may be assumed to have a mutual centre point. Indeed, suppose $B_i = B(x_i, r_i), i = 0, 1,$ and let $R \geq r_1$ be large so that $B_1 \subseteq \tilde{B}_1 := B(x, R)$. Then $\mu(\tilde{B}_1) \geq \mu(B_1) > N\mu(B_0)$.

(2) Lemma 2.10 shows that if $X$ has infinitely many points, then at least one of the following two conditions is satisfied by balls $B$ in $X$: (i) $\mu(B)$ can have arbitrarily small values; (ii) $\mu(B)$ can have arbitrarily large values which is equivalent to $\mu(X) = \infty$. Conversely, both (i) and (ii) imply (2.7) and thereby also that $\# X = \infty$. This observation leads to the three categories of spaces listed in Lemma 2.9

(3) The basic intuition behind (2.7) is that the quantities $\mu(\{x\})$ are, in some sense, vanishing. Indeed, the property (2.7) entails points $x \in X$ with the property that $\mu(\{x\}) \ll \mu(B(x, R))$ for some $R > 0$. Thus, by working on this larger scale (we informally re-scale the measure so that $\mu(B(x, R)) \approx 1$), the measure of the singleton $\{x\}$ becomes negligible.

The following lemma shows that every space of homogeneous type with infinitely many points belongs to one of the three categories listed in the lemma. The most basic examples of such three categories are provided by $([0, 1], dx), (\mathbb{Z}, \mu)$ and $(\mathbb{R}, dx)$, respectively, where $dx$ denotes the one-dimensional Lebesgue measure and $\mu$ is a counting measure.

2.9. Lemma. Suppose that $(X, \mu)$ is a space of homogeneous type and $\# X = \infty$. Then precisely one of the following is satisfied:

1) $\mu(X) < \infty$;
2) $X$ is countably infinite and $\mu(\{x\}) \geq \delta > 0$ for all $x \in X$;
3) $\mu(B)$ can have arbitrarily small and large values.

Proof. First note that if $\mu(X) < \infty$, then 3 cannot be satisfied. The property $\#X = \infty$ implies (2.7), and thereby we must have that $\mu(B)$ can have arbitrarily small values so that also 2 fails. Also note that 2 and 3 are mutually exclusive properties.

Then suppose $\mu(X) = \infty$. Thus, $\mu(B)$ can have arbitrarily large values. If 3 is not satisfied, then $\mu(B) \geq \delta > 0$ for all balls which implies that $\mu(\{x\}) \geq \delta > 0$ for all $x \in X$ by $\sigma$-finiteness. Hence $X$ is countable by the doubling property, and 2 is satisfied.

2.10. Remark. We further record the easy observations that the two properties $\mu(X) < \infty$ and $\mu(\{x\}) \geq \delta > 0$ for all $x \in X$ organize the $L^p(X, \mu)$ spaces in mutually reversed order:

1. Suppose that $X$ belongs to the category 1 so that $\mu(X) < \infty$. Hölder’s inequality then implies that $L^{p_1}(X) \subseteq L^{p_0}(X)$ for all $p_1 \geq p_0$.
2. Suppose that $X$ belongs to the category 2 so that $\mu(\{x\}) \geq \delta > 0$ for all $x \in X$. Then $X$ is a countable set. Now we have that $L^{p_0}(X) \subseteq L^{p_1}(X)$ for all $p_1 \geq p_0$. Indeed,

$$
\|f\|_{L^{p_1}} = \left(\sum_{x \in X} |f(x)|^{p_1} \mu(\{x\})\right)^{1/p_1} \leq \left(\sum_{x \in X} |f(x)|^{p_0} \mu(\{x\})^{p_0/p_1}\right)^{1/p_0} = \delta^{1/p_1-1/p_0} \left(\sum_{x \in X} |f(x)|^{p_0} \mu(\{x\}) \left(\frac{\mu(\{x\})}{\delta}\right)^{p_0/p_1-1}\right)^{1/p_0} \leq \delta^{1/p_1-1/p_0} \left(\sum_{x \in X} |f(x)|^{p_0} \mu(\{x\})\right)^{1/p_0} = \delta^{1/p_1-1/p_0} \|f\|_{L^{p_0}}$$

since $p_0/p_1 - 1 < 0$ and $\mu(\{x\})/\delta \geq 1$.

2.11. Weight classes of interest. We recall definitions and some easy results concerning the classes of weights relevant in our investigations. A non-negative locally integrable function $w$ is a weight. A weight defines a measure (denoted by the same symbol) $w(E) := \int_E w \, d\mu$. We say that
a weight \( w \) belongs to the \( A_p \) class for \( 1 < p < \infty \) if it satisfies the condition

\[
[w]_{A_p} := \sup_B \left( \frac{1}{\mu(B)} \int_B \frac{1}{w} \left( \frac{1}{\mu(B)} \int_B w^{-\frac{1}{p}} \, d\mu \right)^{p-1} \right) < \infty,
\]
where the supremum is over all balls \( B \) in \( X \). The quantity \([w]_{A_p} \geq 1\) is then called the \( A_p \) constant of the weight \( w \). A weight \( w \) is said to belong to the \( A_1 \) class if

\[
Mw \leq Cw \quad \text{a.e.},
\]
where \( M \) is the Hardy–Littlewood maximal operator defined by

\[
Mf(x) = \sup_B \frac{1}{\mu(B)} \int_B |f| \, d\mu \quad \text{for } f \in L^1_{\text{loc}}(X).
\]
The smallest possible constant \( C \) is then called the \( A_1 \) constant of the weight \( w \), i.e.

\[
[w]_{A_1} := \text{ess sup}_{x \in X} \frac{Mw(x)}{w(x)}.
\]

As is well-known, if \( w \in A_p \) for some \( 1 \leq p < \infty \) then \( w = 0 \) a.e. or \( w > 0 \) a.e. so that the interesting examples of such weights enjoy the latter property. Hence, whenever we have an \( A_p \) weight we may assume that it is strictly positive. It is also easy to check from the definitions that \([w]_{A_1} \geq [w]_{A_p} \geq 1\) for every \( 1 \leq p < \infty \).

For \( 1 \leq p \leq \infty \) we denote by \( p' \) the dual exponent of \( p \), i.e. \( 1/p + 1/p' = 1 \). In this definition \( 1/\infty \) means zero. A weight \( w \) is said to belong to \( A_{p,q} \) class for \( 1 < p \leq q < \infty \) if it satisfies the condition

\[
[w]_{A_{p,q}} := \sup_B \left( \frac{1}{\mu(B)} \int_B w^q \, d\mu \right)^{1/q} \left( \frac{1}{\mu(B)} \int_B w^{-p'} \, d\mu \right)^{p'/q} < \infty.
\]
The quantity \([w]_{A_{p,q}} \geq 1\) is then called the \( A_{p,q} \) constant of the weight \( w \). A weight \( w \) is said to belong to the \( A_{1,q} \) class for \( 1 \leq q < \infty \) if

\[
Mw^q \leq Cw^q \quad \text{a.e.},
\]
and \([w]_{A_{1,q}}\) will again be the smallest constant \( C \) that satisfies the above inequality.

The following lemma is easy to check, and we leave the proof to the reader.

2.12. Lemma. Let \( 1 < p \leq q < \infty \), and denote \( r = 1 + q/p' \) and \( s = 1 + p'/q \). Then

(1) \([w]_{A_{p,q}} = [w^q]_{A_r} \). In particular, \( w \in A_{p,q} \) if and only if \( w^q \in A_r \);
(2) \([w]_{A_{p,q}} = [w^{-1}]_{A_{r'}} \). In particular, \( w \in A_{p,q} \) if and only if \( w^{-1} \in A_{r',q'} \);
(3) \([w^{-p'}]_{A_s} = [w^q]_{A_{p,q}} \). In particular, \( w \in A_{p,q} \) if and only if \( w^{-p'} \in A_s \);
(4) \([w]_{A_{1,q}} = [w^q]_{A_1} \). In particular, \( w \in A_{1,q} \) if and only if \( w^q \in A_1 \).

3. Fractional integral operator and the main result

Let \((X, \rho, \mu)\) be a space of homogeneous type. We consider a fractional integral operator of order \( 0 < \gamma < 1 \), defined by

\[
T_\gamma f(x) = \int_X K_\gamma(x,y) f(y) \, d\mu(y)
\]
where the kernel \( K_\gamma \) is the positive function

\[
K_\gamma(x,y) = \begin{cases} 
1 & \text{when } x \neq y \\
\frac{\mu(B(x, \rho(x,y)))}{\mu(x)}^{1-\gamma} & \text{when } x = y.
\end{cases}
\]

3.3. Remarks. (1) Suppose that \( x \in X \) and \( \mu(\{x\}) = 0 \). Then our definition for \( K_\gamma \) formally gives \( K_\gamma(x, x) = +\infty \). However, we may write

\[
T_\gamma f(x) = \int_{X \setminus \{x\}} K_\gamma(x,y) f(y) \, d\mu(y) + K_\gamma(x,x) f(x) \mu(\{x\}),
\]
and the latter term vanishes in case \( \mu(\{x\}) = 0 \) (by the usual interpretation \( 0 \cdot \infty = 0 \)). In fact, we may give an equivalent definition

\[
T_\gamma f(x) := \int_{X \setminus \{x\}} \frac{f(y) \, d\mu(y)}{\mu(B(x, \rho(x,y)))^{1-\gamma}} + f(x) \mu(\{x\})^\gamma.
\]

In case \( \mu \) vanishes on sets which consist of a single point, then the domain of integration \( X \setminus \{x\} \) may be replaced by \( X \), and the extra term \( f(x) \mu(\{x\})^\gamma \) does not appear.

(2) Consider \( X = \mathbb{R}^n \) with the usual \( n \)-dimensional Lebesgue measure \( d\mu = dx \). Then \( \mu(B(x, \rho(x,y))) = |B(x, |x-y|)| = C_n |x-y|^n \) with a positive dimensional constant \( C_n \). By the notation \( \alpha := n \gamma \in (0, n) \), our definition for \( T_\gamma \) yields (up to a dimensional constant) the operator \( I_\alpha \),

\[
I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} \, dy
\]

which is the fractional integration (or the Riesz potential) of order \( \alpha \) on \( \mathbb{R}^n \). Thus, \( T_\gamma \) is one of the possible analogues of an integral of fractional order in the context of spaces of homogeneous type.

The following lemma shows that the operator \( T_\gamma \) is an example of more general potential type operators studied in [8]. The proof of the Lemma only involves elementary estimations by the triangle inequality and Lemma 3.4 and we leave the details to the reader.

3.4. Lemma. The operator \( T_\gamma \) is an operator of potential type, i.e. the kernel \( K_\gamma \), defined in (3.2), satisfies the following conditions: For every \( k_2 > 1 \) there exists \( k_1 > 1 \) such that

\[
K_\gamma(x, y) \leq k_1 K_\gamma(x', y) \quad \text{whenever} \quad \rho(x', y) \leq k_2 \rho(x, y),
\]

(3.5)

\[
K_\gamma(x, y) \leq k_1 K_\gamma(x, y') \quad \text{whenever} \quad \rho(x, y') \leq k_2 \rho(x, y).
\]

Moreover, there exists a geometric constant \( C > 0 \) such that for all \( x, y \in X, x \neq y \), there holds

\[
\frac{1}{C} K(x, y) \leq K(y, x) \leq CK(x, y).
\]

We investigate the dependence of the operator norm of \( T_\gamma \) on the \( A_{p,q} \) constant of the weight in weighted spaces. Sharp weighted inequalities for the Riesz potentials \( I_\alpha \) in the Euclidean spaces, acting on weighted Lebesgue spaces were obtained recently in [9] Theorem 2.6. We use the ideas introduced there to extend this result into general spaces of homogeneous type. Our main result is the following.

3.7. Theorem. Suppose \((X, \rho, \mu)\) is a space of homogeneous type. Let \( 0 < \gamma < 1 \) and suppose \( 1 < p \leq q < \infty \) satisfy \( 1/p - 1/q = \gamma \). Then

\[
\|T_\gamma\|_{L^p(w^{\gamma}) \to L^q(w^{\gamma})} \lesssim [w]_{A_{p,q}}^{(1-\gamma) \max \{1, \frac{\rho}{\mu}\}},
\]

and the estimate is sharp in any space \( X \) with infinitely many points in the sense described in Section 3.

4. A PRELIMINARY RESULT

Let us begin our investigations by motivating the restriction \( 1/p - 1/q = \gamma \) imposed on exponents in Theorem 3.7.

Recall the well-known Hardy–Littlewood–Sobolev Theorem in the Euclidean space that if the Riesz potential \( I_\alpha \) maps \( L^p(\mathbb{R}^n) \) to \( L^q(\mathbb{R}^n) \) for some \( p \) and \( q \), then we must have that the exponents are related by \( 1/p - 1/q = \alpha/n \), and this condition is also sufficient to have a bounded operator. In particular, \( q > p \) so that such an operator improves the integrability of a function.

We record an analogous result for \( T_\gamma \) in the context of spaces of homogeneous type. The following non-weighted result describes a necessary and sufficient condition for the exponents \( p \) and \( q \) for which the fractional integration is a bounded operator from \( L^p(\mu) \) to \( L^q(\mu) \) with general doubling measures \( \mu \). This easy observation can probably be found elsewhere but in the lack of a suitable reference, we shall also provide the proof.
4.1. **Proposition.** Let $(X, \rho, \mu)$ be a space of homogeneous type. Let $0 < \gamma < 1$ and $1 \leq p, q < \infty$, and suppose $T_\gamma : L^p(X) \to L^q(X)$ is bounded. Then

$$\mu(B)^{1/q - 1/p + \gamma} \leq C < \infty$$

for all balls $B$. Moreover, the following is true:

i) If $X$ belongs to the category 1 of Lemma 2.9, then $T_\gamma : L^p(X) \to L^q(X)$ is bounded if and only if $1/p - 1/q \leq \gamma$.

ii) If $X$ belongs to the category 2 of Lemma 2.9, then $T_\gamma : L^p(X) \to L^q(X)$ is bounded if and only if $1/p - 1/q \geq \gamma$.

iii) If $X$ belongs to the category 3 of Lemma 2.9, then $T_\gamma : L^p(X) \to L^q(X)$ is bounded if and only if $1/p - 1/q = \gamma$.

4.2. **Remark.** If none of the cases i–iii holds, then $X$ has finitely many points, and the boundedness of $T_\gamma$ is trivial.

**Proof.** First assume that $T_\gamma : L^p(X) \to L^q(X)$ is bounded. Fix a ball $B = B(x_0, r)$ and suppose $x, y \in B$. Then

$$\mu(B(x, \rho(x, y))) \leq \mu(B(x_0, 3A^2(r))) \leq \mu(B(x_0, r))$$

so that for $y \neq x$,

$$K_\gamma(x, y) = \frac{1}{\mu(B(x, \rho(x, y)))^{1-\gamma}} \geq \frac{1}{\mu(B)^{1-\gamma}}.$$

For $y = x$,

$$K_\gamma(x, y) = \frac{1}{\mu(\{x\})^{1-\gamma}} \geq \frac{1}{\mu(B)^{1-\gamma}}.$$

Thus,

$$T_\gamma \chi_B(x) = \int_B K_\gamma(x, y) d\mu \geq \frac{\mu(B)}{\mu(B)^{1-\gamma}} = \mu(B)^\gamma.$$

It follows that

$$\|T_\gamma\|_{L^p(X)} \leq \|T_\gamma\|_{L^q(X)} \geq \|\chi_B T_\gamma \chi_B\|_{L^q(X)} \geq \mu(B)^{1/q}.$$

so that

$$\mu(B)^{1/q - 1/p + \gamma} \leq \|T_\gamma\| < \infty.$$

This shows the necessity of the conditions imposed on the exponents in i–iii. The sufficiency in iii follows from Theorem 5.7 by choosing $w \equiv 1$.

For the sufficiency in i, let $1 \leq p, q < \infty$ be exponents such that $1/p - 1/q \leq \gamma$, and choose $q_0$ such that $1/p - 1/q_0 = \gamma$. Then $q_0 \geq \gamma$. First, iii implies that $T_\gamma : L^p(X) \to L^{q_0}(X)$ is bounded, and the claimed boundedness follows since $L^{q_0} \subseteq L^q$ for $q_0 \geq q$ by Remark 2.10.

For the sufficiency in ii, let $1 \leq p, q < \infty$ be exponents such that $1/p - 1/q \geq \gamma$, and choose $q_0$ such that $1/p - 1/q_0 = \gamma$. Then $q_0 \leq q$. The claimed boundedness follows again by iii and Remark 2.10. □

5. **First steps of the proof: Reduction to the weak-type result**

We start by observing that in order to obtain sharp bounds for the strong-type estimates it is sufficient to show sharp bounds for the weak-type ones. This follows from the investigations of \[8\] (cf. \[11, 12\]), where a large class of potential type operators were studied.

In \[8\] Theorem 1.5, it was shown that if $\sigma$ and $\omega$ are positive Borel-measures in a quasi-metric space $(X, \rho)$ which are finite on balls, and $T$ is a positive operator of the form

$$T(f \, d\sigma)(x) = \int_X K(x, y) f(y) \, d\sigma(y), \quad x \in X,$$

where the kernel $K$ is a non-negative function which satisfies the conditions \[3.5\], and $1 < p \leq q < \infty$, then the boundedness

$$T : L^p_\sigma \to L^q_\omega$$

is characterized by Sawyer-type testing conditions, which we recall below.
In the present note, we investigate the case \( T = T_\gamma \), \( d\sigma = w^p \dv \) and \( d\omega = w^q \dv \) where \( w \) is an \( A_{p,q} \)-weight and \( (X, \rho, \mu) \) is a space of homogeneous type. In this particular case, Theorem 1.5 of \([8]\) says that
\[
T_\gamma : L^p(w^p) \to L^q(w^q),
\]
if and only if \( \omega := w^q \) and the function \( \sigma := w^{-p/(p-1)} = w^{p(1-p')} \) satisfy the (local) testing conditions that
\[
[\sigma, \omega]_{S_{p,q}} := \sup_Q \sigma(Q)^{-1/p} \| \chi_Q T_\gamma(\chi_Q \sigma) \|_{L^q(\omega)} < \infty
\]
and
\[
[\omega, \sigma]_{S_{q',p'}} := \sup_Q \omega(Q)^{-1/q} \| \chi_Q T_\gamma(\chi_Q \omega) \|_{L^{q'}(\sigma)} < \infty,
\]
where the supremum is over all so-called dyadic cubes \( Q \in \bigcup_{k=1}^K \mathcal{D}^t \) (for precise definitions, see \([8, \text{Section 2.2}])\). The proof further shows that
\[
\| T_\gamma \|_{L^p(w^p) \to L^{q'}(w^{q'})} \approx \| \sigma, \omega \| S_{p,q} + [\omega, \sigma]_{S_{q',p'}}.
\]
Moreover, in the characterization of the weak-type two-weight estimate \([8, \text{Theorem 5.4}]\) it was shown that
\[
\| T_\gamma \|_{L^p(w^p) \to L^{q',\infty}(w^{q'})} \approx \| \omega, \sigma \| S_{q',p'}.
\]

Using these characterizations and the fact that \( T_\gamma \) is self-adjoint, we have the following.

5.4. Lemma.
\[
\| T_\gamma \|_{L^p(w^p) \to L^{q}(w^{q})} \approx \| T_\gamma \|_{L^p(w^p) \to L^{q',\infty}(w^{q'})} + \| T_\gamma \|_{L^{q'}(w^{q'}) \to L^{p'}(w^{p'})}
\]

Proof. Denote \( u := w^p \) so that \( \sigma = u^{1-p'} \) which is equivalent to \( u = \sigma^{1-p} \). With this notation, \((5.3)\) becomes
\[
\| T_\gamma \|_{L^p(u) \to L^{q}(\sigma)} \approx \| \omega, u^{1-p'} \| S_{q',p'}.
\]
Thus,
\[
[\sigma, \omega]_{S_{p,q}} = [u^{1-p'}, \omega^{(1-q')(1-q)}]_{S_{p,q}} \approx \| T_\gamma \|_{L^p(u^{1-p'}) \to L^{q',\infty}(u^{1-p'})}.
\]
Then combine this and \((5.3)\) with \((5.2)\) to make the final conclusion. \( \square \)

By Lemma 5.4, the proof of Theorem 5.1 is completed by the following proposition.

5.5. Proposition. Let \( 0 < \gamma < 1 \) and suppose \( 1 \leq p \leq q < \infty \) satisfy \( 1/p - 1/q = \gamma \). Then
\[
\| T_\gamma \|_{L^p(w^p) \to L^{q',\infty}(w^{q'})} \lesssim [w]^{-1-\gamma}_{A_{p,q}},
\]
and the estimate is sharp in any space \( X \) in the sense described in Section 8.

Proof of Theorem 5.1 assuming Proposition 5.5. Note that if \( 1/p - 1/q = \gamma \), then also \( 1/q' - 1/p' = \gamma \). By Lemma 5.1, we have that
\[
\| T_\gamma \|_{L^p(w^p) \to L^{q}(w^{q})} \lesssim \| T_\gamma \|_{L^p(w^p) \to L^{q',\infty}(w^{q'})} + \| T_\gamma \|_{L^{q'}(w^{q'}) \to L^{p'}(w^{p'})}
\]
\[
\lesssim [w]^{-1-\gamma}_{A_{p,q}} + [w^{-1}]^{-1-\gamma}_{A_{q',p'}} = [w]^{-1-\gamma}_{A_{p,q}} + [w]^{1-\gamma}_{A_{p,q}}
\]
\[
\lesssim 2[w]^{(1-\gamma)}_{A_{p,q}} \max\{1, \frac{1}{\gamma} \},
\]
where we used Lemma 4.12(2). \( \square \)
6. Proof of the Weak-type Result Via Extrapolation

We will use the following sharp weak-type version of an extrapolation theorem for \( A_{p,q} \) weights.

6.1. Theorem. Let \( T \) be an operator defined on an appropriate class of functions (e.g. bounded functions with bounded support). Suppose that for some pair \((p_0, q_0)\) of exponents \(1 \leq p_0 \leq q_0 < \infty\), \( T \) satisfies the weak-type inequality

\[
\|T f\|_{L^{p_0}((w^{q_0})^{1/q_0})} \leq C[w]_{A_{p_0,q_0}}^{\alpha} \|f\|_{L^{p_0}(w^{p_0})}
\]

for all weights \( w \in A_{p_0,q_0} \) and some \( \alpha > 0 \). Then,

\[
\|T f\|_{L^{p}((w^{q})^{1/q})} \leq C[w]_{A_{p,q}}^{\alpha} \max\{1, \frac{p}{p_0}\}^{\frac{\gamma}{\alpha}} \|f\|_{L^{p}(w^{p})}
\]

for all weights \( w \in A_{p,q} \) and all pairs \((p, q)\) of exponents that satisfy

\[
\frac{1}{p} - \frac{1}{q} = \frac{1}{p_0} - \frac{1}{q_0}.
\]

The Euclidean version of this extrapolation theorem can be found in [9, Corollary 2.2] where it is shown to follow from the corresponding sharp strong-type extrapolation result. The Euclidean proofs can be adapted into spaces of homogeneous type. We will comment on this in Section 7.

As for now, assume Theorem 6.1. We note that in order to successfully apply the Theorem and obtain the desired exponent \( 1 - \gamma \) for all pairs \((p, q)\) of exponents in the norm estimate of Proposition 6.3 it entails that we show the weak-type inequality (6.2) for \( T_\gamma \) with exponents \( p_0 = 1 \) and \( q_0 = 1/(1 - \gamma) \); for any other pair, the extrapolation theorem would only give the positive result for a limited range of exponents, i.e. for ones with \( q_0 p'/p_0 q \leq 1 \), and for large \( p \), the exponent obtained by the extrapolation theorem would be strictly larger. Thus, we need the following lemma.

6.3. Lemma. Let \( 0 \leq u \in L^1_{loc}(X, \mu) \) be a weight. The fractional integral operator \( T_\gamma \) satisfies the weak-type estimate

\[
\|T_\gamma f\|_{L^{q_0}((u^{q_0})^{1/q_0})} \leq C\|f\|_{L^1((Mu)^{1/q_0})}
\]

with \( q_0 = 1/(1 - \gamma) \). As a consequence,

\[
\|T_\gamma f\|_{L^{q_0}((u^{q_0})^{1/q_0})} \leq C[w]_{A_{1,q_0}}^{1-\gamma} \|f\|_{L^1(u)}
\]

for all weights \( w \in A_{1,q_0} \).

The proof of Lemma 6.3 follows the Euclidean argument given in [9] except that we need to put out some extra work with the technical details when working with general doubling measures. Note that the Lemma together with the extrapolation result gives Proposition 5.5 which in turn leads to strong-type estimates and complete the proof of our main result, Theorem 3.7, as already described.

Proof. We recall that \( \| \cdot \|_{L^{p}((u^{q})^{1/q})} \) is equivalent to a norm when \( p > 1 \). Hence, we may use the triangle inequality as follows

\[
\|T_\gamma f\|_{L^{q_0}((u^{q_0})^{1/q_0})} \leq C_{q_0} \int_X |f(y)| \|K_\gamma(\cdot, y)\|_{L^{q_0}((u^{q_0})^{1/q_0})} d\mu(y).
\]

Fix \( y \in X \). First note that for all \( x \in X \)

\[
K_\gamma(x, y) \leq \frac{1}{\mu(\{y\})^{1-\gamma}}.
\]
We then calculate
\[ \|K_x(\cdot, y)\|_{L^{q_0, \infty}(u)} = \sup_{\lambda > 0} \lambda \left[ \sup_{0 < \lambda < \mu(y)} \lambda [u \{ \{ x \in X : K_x(x, y) > \lambda \} \}]^{1/q_0} \right] \]
\[ = \sup_{0 < \lambda < \mu(y)} \lambda [u \{ \{ x \in X : K_x(x, y) > \lambda \} \}]^{1/q_0} \]
\[ = \left[ \sup_{0 < \lambda < \mu(y)} \lambda^{q_0} u \left( \left\{ x \in X : K_x(x, y) = \frac{1}{\lambda} \right\} \right) \right]^{1/q_0} \]
\[ = \sup_{t > \mu(y)} \frac{1}{t} u \left( \{ x \in X, x \neq y : \mu(B(x, \rho(x, y))) < t \} \right) \]
\[ \leq C^{1/q_0} \left[ \sup_{t > \mu(y)} \frac{1}{t} u \left( \{ x \in X : \mu(B(y, \rho(x, y))) < Ct \} \right) \right]^{1/q_0} \]
\[ \leq C \sup_{t > \mu(y)} \frac{1}{t} u \left( \{ x \in X : \mu(B(y, \rho(x, y))) < Ct \} \right) \]

The second to last estimate is true since \( \mu(B(x, \rho(x, y))) < t \) implies that \( \mu(B(y, \rho(x, y))) \leq \mu(B(x, 2\rho(x, y))) < Ct, C = C(A_0, \mu) \), by the doubling property.

For a fixed \( y \in X \), denote \( E_t := \{ x \in X : \mu(B(y, \rho(y, x))) < t \} \). Note that \( y \in E_t \) for all \( t > 0 \).

We make the following technical observation.

6.5. Lemma. Given \( y \in X \) and \( t > 0 \), consider the set \( E_t := \{ x \in X : \mu(B(y, \rho(y, x))) < t \} \) and the quantity \( r_y(t) := \sup\{ r > 0 : \mu(B(y, r)) < t \} \in [0, \infty] \).

Here it is understood that \( B(y, 0) = \emptyset \) so that the supremum always exists. Then the following is true.

1) If \( x_1 \in E_t \) for some \( x_1 \neq y \), then \( x \in E_t \) for all \( x \) with \( \rho(y, x) \leq \rho(y, x_1) \), and \( r_y(t) > 0 \).
2) If \( x_2 \notin E_t \) for some \( x_2 \), then \( x \notin E_t \) for all \( x \) with \( \rho(y, x) \geq \rho(y, x_2) \), and \( r_y(t) < \infty \).
3) If \( r_y(t) = 0 \), then \( E_t = \{ y \} \).
4) If \( r_y(t) = \infty \), then \( E_t = X \) and \( \mu(X) \leq t \).
5) If \( 0 < r_y(t) < \infty \), then the set \( E_t \) is one of the two choices \( B(y, r_y(t)) \) and \( B(y, r_y(t)) \).

Moreover, \( \mu(E_t) \leq t \).

Case 1: Consider such \( t > \mu(\{ y \}) \) for which there exists \( x \in X, x \neq y \), with \( x \in E_t \). Then \( r_y(t) > 0 \), and \( E_t \) is one of the three choices, \( B(y, r_y(t)) \) or \( B(y, r_y(t)) \) or \( X \), and \( \mu(E_t) \leq t \). Recall that in case \( E_t = X \), the condition \( \mu(E_t) \leq t \) is infinite implies that \( X = B(y, R) \) for some \( 0 < R < \infty \).

Hence, for such \( t \) we have
\[ \frac{1}{t} u(E_t) \leq \frac{u(E_t)}{\mu(E_t)} \leq M u(y) \]

Case 2: Then consider \( t > \mu(\{ y \}) \) with \( E_t = \{ y \} \). If \( \mu(\{ y \}) = 0 \), then \( u(E_t)/t = 0 \). Recall that \( \mu(\{ y \}) > 0 \) implies that \( \{ y \} = B(y, \varepsilon) \) for some \( \varepsilon > 0 \). For such \( t \) we have
\[ \frac{1}{t} u(E_t) \leq \frac{u(B(y, \varepsilon))}{\mu(B(y, \varepsilon))} \leq M u(y) \]

Altogether we have obtained that the inner norm in \([5.4]\) satisfies
\[ \|K_x(\cdot, y)\|_{L^{q_0, \infty}(u)} \leq C_{q_0} \left( \sup_{t > \mu(y)} \frac{1}{t} u(E_t) \right)^{1/q_0} \]
\[ \leq C_{q_0} (M u(y))^{1/q_0} \]
and consequently,
\[ \|T_x f\|_{L^{q_0, \infty}(u)} \leq C_{q_0} \int_X |f(y)| (M u(y))^{1/q_0} \, d\mu(y) = C_{q_0} \|f\|_{L^1(M u(y))} \]

This is the first assertion.
Then suppose that \( w \in A_{1,q_0} \) and denote \( u := w^{q_0} \). Recall the \( A_{1,q_0} \) condition for \( w \),
\[
Mu \leq |w|_{A_{1,q_0}} u \quad \text{a.e.,}
\]
and that \( q_0 = 1/(1 - \gamma) \). From this and the first assertion we may deduce
\[
\|T_\gamma f\|_{L^{q_0,\infty}(w^{q_0})} \leq C q_0 \int_X |f|(Mu)^{1/q_0} d\mu \leq C q_0 |w|_{A_{1,q_0}}^{1/q_0} \int_X |f|wd\mu
\]
which completes the proof.

\[\square\]

Proof of Proposition 5.5. We apply Theorem 6.1 with exponents \( p_0 = 1 \) and \( q_0 = 1/(1 - \gamma) \), and \( \alpha = 1 - \gamma \). First note that then
\[
\alpha \max \left\{ 1, \frac{q_0 p'}{p_0 q} \right\} = 1 - \gamma.
\]

Theorem 6.1 together with Lemma 6.3 show that
\[
\|T_\gamma f\|_{L^{q,\infty}(w^q)} \leq C[w]_{A_{p,q}} \|f\|_{L^p(w^p)}
\]
for all weights \( w \in A_{p,q} \) and all exponents \( 1 < p \leq q < \infty \) that satisfy
\[
\frac{1}{p} - \frac{1}{q} = \frac{1}{p_0} - \frac{1}{q_0} = 1 - (1 - \gamma) = \gamma.
\]

\[\square\]

7. Extrapolation

In [9] Corollary 2.2] it was shown that the sharp weak-type extrapolation theorem 6.1, which we used in Section 6, follows from the following sharp strong-type extrapolation result. To prove this implication, the authors, working in the Euclidean space, used an idea from Grafakos and Harboure, Macías and Segovia [5]. The sharp version in the Euclidean space can be found in [9, Theorem 6.1], which is very general and applies to our situation. Thus, we may complete the proof of Theorem 6.1 by the following theorem.

7.1. Theorem. Let \( T \) be an operator defined on an appropriate class of functions (e.g. bounded functions with bounded support). Suppose that for some exponents \( 1 \leq p_0 \leq q_0 < \infty \), \( T \) satisfies
\[
\|Tf\|_{L^{p_0}(w^{p_0})} \leq C[w]_{A_{p_0,q_0}} \|f\|_{L^{p_0}(w^{p_0})}
\]
for all weights \( w \in A_{p_0,q_0} \) and some \( \alpha > 0 \). Then,
\[
\|Tf\|_{L^q(w^q)} \leq C[w]_{A_{p,q}} \alpha \max \left\{ \frac{q_0 p'}{p_0 q}, \frac{q_0 p}{p_0 q} \right\} \|f\|_{L^p(w^p)}
\]
holds for all weights \( w \in A_{p,q} \) and all exponents \( 1 < p \leq q < \infty \) that satisfy
\[
\frac{1}{p} - \frac{1}{q} = \frac{1}{p_0} - \frac{1}{q_0} = 1 - (1 - \gamma) = \gamma.
\]

The original qualitative version of this extrapolation result in the Euclidean space is due to Harboure, Macías and Segovia [5]. The sharp version in the Euclidean space can be found in [9] Theorem 2.1]. The proof for the metric space version follows, from line to line, the Euclidean proof in [9] except that we need the following result for \( A_p \) weights which is the main tool in the proof.

7.2. Lemma. Suppose \( \mu \) is a doubling measure on \( X \). Let \( 1 \leq r_0 < r < \infty \) and \( w \in A_r \). Then for any \( g \geq 0 \), \( g \in L^{(r/r_0)'}(w) \), there exists a function \( G \in L^{(r/r_0)'}(w) \) with the properties that

1. \( G \geq g \);
2. \( \|G\|_{L^{(r/r_0)'}(w)} \leq 2\|g\|_{L^{(r/r_0)'}(w)} \);
3. \( Gw \in A_{r_0} \); moreover, \( [Gw]_{A_{r_0}} \leq C[w]_A \), where \( C > 0 \) depends only on \( X, \mu, r_0 \) and \( r \).
A qualitative version of Lemma 7.2 in the Euclidean space first appeared in [3]. A quantitative version (see the results in [2]) uses a suitable sharp version of the celebrated Rubio de Francia algorithm, a very general technique, and Buckley’s theorem [11] on the sharp dependence of $\|M\|_{L^p(w)}$ on $[w]_{A_p}$ in Muckenhoupt’s theorem for Hardy–Littlewood maximal operator. Buckley’s result remains true in a space of homogeneous type; see [7, Proposition 7.13]. After this, the proof of Lemma 7.2 in the metric setting follows, from line to line, the Euclidean proof, and we may refer the reader to the original proof given in [2]. Finally, we may refer to [9, Theorem 2.1] for the proof of Theorem 7.1.

8. Sharpness of the result

In this final section, we show that the exponent $1 - \gamma$ in the estimate

$$\|T_\gamma f\|_{L^{p,q}(w)} \approx [w]_{A_{p,q}}^{1-\gamma} \|f\|_{L^p(w)}$$

from Proposition 5.5 is best possible in the sense described as follows. This also implies that the exponent $(1 - \gamma) \max\{1,p'/q\}$ in the norm estimate in Theorem 3.7 is sharp. In fact, we will show the following.

8.2. Proposition. Let $(X,\rho,\mu)$ be a space of homogeneous type with the property that $\#X = \infty$. Then, there exists a family $\{w_t: 0 < t < 1\}$ of weights such that

$$[w_t]_{A_{p,q}} \approx \frac{1}{t},$$

and

$$\|T_\gamma f\|_{L^{p,q}(w_t)} \approx [w_t]_{A_{p,q}}^{1-\gamma} \|f\|_{L^p(w_t)}.$$  \hspace{1cm} (8.3)

Consequently, if $\|T_\gamma f\|_{L^{p,q}(w_t)} \leq \phi([w]_{A_{p,q}})$ for some increasing $\phi: [1,\infty) \to (0,\infty)$, then $\phi(s) \geq s^{1-\gamma}$. In particular, for any $\varepsilon > 0$, we have that

$$\sup_{w\in A_{p,q}} \frac{\|T_\gamma f\|_{L^{p,q}(w_t)}}{[w]_{A_{p,q}}^{1-\gamma-\varepsilon}} = \infty.$$

We will first observe that Proposition 8.2 follows from the following lemma.

8.4. Lemma (Reduction). Let $(X,\rho,\mu)$ be a space of homogeneous type with the property that $\#X = \infty$. Then, for every $0 < t < 1$, there exists a weight $u_t$ and a function $f_t \neq 0$ such that $[u_t]_{A_1} \approx 1/t$ and

$$\|T_\gamma (f_t u_t^{1/\gamma})\|_{L^{p,q}(u_t)} \gtrsim [u_t]_{A_1}^{1-\gamma} \|f_t\|_{L^p(u_t)}.$$ \hspace{1cm} (8.5)

Proof of Proposition 8.2 assuming Lemma 8.4. First, note that $\|f_t\|_{L^p(\mu)} = \|f_t u_t^{1/\gamma}\|_{L^p(u_t^{p/\gamma})}$ since $1/p - 1/q = \gamma$. By replacing $f_t$ with $f_t u_t^{-\gamma}$, (8.5) becomes

$$\|T_\gamma f_t\|_{L^{p,q}(u_t)} \gtrsim [u_t]_{A_1}^{1-\gamma} \|f_t\|_{L^p(u_t^{p/\gamma})}. \hspace{1cm} (8.6)$$

We denote $u_t := u_t^{1/q}$ and observe that, by Lemma 2.12, $[u_t]_{A_1} \geq [u_t]_{A_1+\eta/p'} = [u_t]_{A_1+\eta/p'} = [u_t]_{A_{p,q}}$. Thus, (8.6) yields

$$\|T_\gamma f_t\|_{L^{p,q}(u_t^{p/\gamma})} \gtrsim [u_t^{p/\gamma}]_{A_1}^{1-\gamma} \|f_t\|_{L^p(u_t^{p/\gamma})} \gtrsim [u_t]_{A_{p,q}}^{1-\gamma} \|f_t\|_{L^p(u_t^{p/\gamma})}$$

This shows the estimate (8.3). Moreover, we have that

$$[u_t]_{A_{p,q}}^{1-\gamma} \|f_t\|_{L^p(w_t)} \approx \|T_\gamma f_t\|_{L^{p,q}(w_t)} \approx [u_t]_{A_{p,q}}^{1-\gamma} \|f_t\|_{L^p(w_t)} \gtrsim [u_t]_{A_{p,q}}^{1-\gamma} \|f_t\|_{L^p(w_t)}$$

so that

$$[u_t]_{A_{p,q}} \approx [u_t]_{A_1} = [u_t]_{A_1} \approx \frac{1}{t}.$$  \hspace{1cm} (8.8)

Finally, let $\phi: [1,\infty) \to (0,\infty)$ be an increasing function such that $\|T_\gamma f\|_{L^{p,q}(w_t)} \leq \phi([w]_{A_{p,q}})$. Then, in particular, for every $0 < t < 1$, $\phi(C/t) \geq \phi([w]_{A_{p,q}}) \geq \|T_\gamma f\|_{L^{p,q}(w_t)} \gtrsim [w_t]_{A_{p,q}} \gtrsim t^{\gamma-1}$.
so that for every $s := C/t \in (C, \infty)$,
\[\phi(s) \geq c(C/s)^{\gamma-1} \geq s^{1-\gamma}.
\]

The proof of Lemma 8.4 consists of several steps. We start with the following definitions.

8.7. Definition ($\varepsilon$-point). We say that a point $x \in X$ is an $\varepsilon$-point for $\varepsilon > 0$, if there exists a ball $B(x, R)$ such that
\[\mu(B(x, R)) > \varepsilon^{-1} \mu(\{x\}).
\]
Note that the property $\# X = \infty$ implies, by Lemma 2.9, that there exists an $\varepsilon$-point for every $\varepsilon > 0$.

8.9. Definition (Power weights). For $0 < t < 1$, let $x_t \in X$ be an $\varepsilon = \varepsilon(t)$-point (for a small $\varepsilon(t) > 0$ to be fixed). We define
\[u_t(x) := \frac{1}{\mu(B(x_t, \rho(x, x_t)))^{1-t}}.
\]
We agree that $B(x, 0) = \{x\}$ for all $x \in X$.

The small positive number $\varepsilon(t)$ will vary in the different lemmata below, until we fix it at the end of the proof.

8.11. Lemma. Let $0 < t < 1$ and suppose that $x_t$ is an $\varepsilon$-point with $\varepsilon = (2C_\mu)^{-3/t}$. Then, for any ball $B = B(x_t, R)$,
\[u_t(B) = \int_B u_t \, d\mu \lesssim \frac{\mu(B)^t}{t}.
\]
Moreover, if $\mu(B(x_t, R)) > \varepsilon^{-1} \mu(\{x_t\})$, then
\[u_t(B) = \int_B u_t \, d\mu \gtrsim \frac{\mu(B)^t}{t}.
\]

Proof. Fix $B = B(x_t, R)$.

Case 1: First assume that $\mu(B) \leq 2\mu(\{x_t\})$, so that only the first assertion requires a proof. Note that now $\mu(\{x_t\}) > 0$, and thus $\{x_t\} = B(x_t, \varepsilon)$ for some $\varepsilon > 0$. We have
\[\int_B u_t \, d\mu = \int_{\{x_t\}} u_t \, d\mu + \int_{B \setminus B(x_t, \varepsilon)} u_t \, d\mu \leq \mu(\{x_t\})^t + \frac{\mu(B) - \mu(\{x_t\})}{\mu(\{x_t\})^{1-t}} \lesssim \frac{\mu(B)^t}{t}.
\]

Case 2: We may then assume that $\mu(B) > 2\mu(\{x_t\})$. We choose a decreasing sequence $(r_k)$ of radii as follows: Let $r_0 = R$. Then let $k_1 \geq 1$ be the smallest integer such that $\mu(B(x_t, 2^{-k_1} R)) < 2^{-1} \mu(B(x_t, r_0))$, and set $r_1 := 2^{-k_1} R$. Note that since $\mu(B(x_t, r_0)) > 2\mu(\{x_t\})$, such an $r_1$ exists. Having chosen $k_m$ and $r_m$ in this fashion, let $k_{m+1} > k_m$ be the smallest integer such that $\mu(B(x_t, 2^{-k_{m+1}} R)) < 2^{-1} \mu(B(x_t, r_m))$, and set $r_{m+1} := 2^{-k_{m+1}} R$. Note that if $\mu(\{x_t\}) = 0$, we may keep sub-dividing infinitely many times. Otherwise, we stop iterating at the step $K \geq 1$ for which $\mu(B(x_t, r_{K-1})) > 2\mu(\{x_t\})$ and $\mu(B(x_t, r_K)) \leq 2\mu(\{x_t\})$.

Note that we have
\[\mu(B(x_t, r_{K-1})) < 2^{-1} \mu(B(x_t, r_1)).
\]

On the other hand, since $r_{i+1}$ is, by choice, the largest number of the form $2^{-k_i} R$ with the property
\[2r_{i+1} \text{ satisfies the inverse estimate, and thus}\]
\[\mu(B(x_t, r_i)) \leq 2\mu(B(x_t, 2r_{i+1})) \leq 2C\mu(B(x_t, r_{i+1})),
\]
where the second estimate follows by the doubling property.
For the first assertion, we consider two cases: First assume that \( \mu(\{x_t\}) = 0 \). We may then keep sub-divining infinitely many times, and
\[
\int_B u_t \, d\mu = \sum_{i=0}^{K-1} \int_{B(x_t, r_i) \setminus B(x_t, r_{i+1})} \frac{d\mu(y)}{\mu(B(x_t, \rho(y, x_t)))^{1-t}} \lesssim \sum_{i=0}^{K-1} \frac{\mu(B(x_t, r_i))}{\mu(B(x_t, r_{i+1}))^{1-t}} \lesssim \sum_{i=0}^{\infty} \mu(B(x_t, r_{i+1}))
\]
\[
\leq \mu(B(x_t, r_0)) (1 + 2^{-t} + 2^{-2t} + \ldots) = \frac{\mu(B(x_t, R))^t}{1 - 2^{-t}} \lesssim \frac{\mu(B)}{t}
\]
where we used (8.13) in the second estimate and (8.12) in the second-to-last estimate.

Then assume that \( \mu(\{x_t\}) > 0 \) and let \( K \) denote the step when the iteration ends. Then
\[
\int_B u_t \, d\mu = \sum_{i=0}^{K-1} \int_{B(x_t, r_i) \setminus B(x_t, r_{i+1})} \frac{d\mu(y)}{\mu(B(x_t, \rho(y, x_t)))^{1-t}} + \int_{B(x_t, r_K)} \frac{d\mu(y)}{\mu(B(x_t, \rho(y, x_t)))^{1-t}}
\]
\[
=: I_1 + I_2.
\]
The term \( I_1 \) is estimated as in the first case only that we now have a finite sum instead of an infinite one. Recall that the iteration stops when \( \mu(B(x_t, r_K)) \leq 2\mu(\{x_t\}) \), so that the ball \( B(x_t, r_K) \) in the term \( I_2 \) is in the regime of the Case 1. This completes the proof for the first assertion.

For the second assertion, suppose \( \mu(B(x_t, R)) > (2C_\mu)^{3/t} \mu(\{x_t\}) \). Let \( K \) be an integer such that
\[
K - 1 < \frac{1}{t} \leq K.
\]

By iterating (8.13), we see that
\[
\mu(B(x_t, r_K)) \geq (2C_\mu)^{-K} \mu(B(x_t, r_0)) \geq (2C_\mu)^{-1}(2C_\mu)^{-1/t} \mu(B(x_t, R)) > 2\mu(\{x_t\}),
\]
which shows that the iteration proceeds at least \( K \) times. Also note that by \( K \geq 1/t \), we have
\[
(2C_\mu)^{-Kt} \leq 2^{-1}.
\]
Thus,
\[
\int_B u_t \, d\mu = \sum_{i=0}^{K-1} \int_{B(x_t, r_i) \setminus B(x_t, r_{i+1})} \frac{d\mu(y)}{\mu(B(x_t, \rho(y, x_t)))^{1-t}} + \int_{B(x_t, r_K)} \frac{d\mu(y)}{\mu(B(x_t, \rho(y, x_t)))^{1-t}} \geq \sum_{i=0}^{K-1} \frac{\mu(B(x_t, r_i))}{\mu(B(x_t, r_{i+1}))^{1-t}}.
\]

Note that here \( \mu(B(x_t, r_i)) - \mu(B(x_t, r_{i+1})) \geq 2^{-1} \mu(B(x_t, r_i)) \) by (8.12). Thus, by (8.13) and (8.14),
\[
\int_B u_t \, d\mu \geq 2^{-1} \sum_{i=0}^{K-1} \mu(B(x_t, r_i)) \geq 2^{-1} \mu(B(x_t, r_0))(1 + (2C_\mu)^{-t} + \ldots + (2C_\mu)^{-(K-1)t})
\]
\[
= 2^{-1} \mu(B(x_t, R)) \frac{1 - (2C_\mu)^{-Kt}}{1 - (2C_\mu)^{-t}} \gg \frac{\mu(B)}{t}.
\]

\[
\square
\]

8.15. Lemma. Let \( 0 < t < 1 \) and suppose that \( x_t \) is an \( \varepsilon \)-point with \( \varepsilon = (2C_\mu)^{-3/t} \). Then
\[
[u_t]_{A_1} \approx \frac{1}{t}.
\]

Proof. To show the estimate \( \lesssim \), it suffices to show that for a.e. \( x \in X \) and all balls \( B \ni x \),
\[
\frac{1}{\mu(B)} \int_B u_t \, d\mu \lesssim \frac{u_t(x)}{t}.
\]
To this end, fix \( x \in X \) and a ball \( B = B(y, r) \ni x \).

Case 1: First assume that \( r \leq (4A_2)^{-1} \rho(x, x_t) \). Note that if \( x_t \) is not a point mass, it suffices to consider points \( x \neq x_t \); otherwise, for \( x = x_t \), the restriction on \( r \) (formally) reduces to considering
only the ball $B(x_t, 0)$ which is interpreted as the singleton $\{x_t\}$. Then, for $z \in B$ we have that $\rho(z, x_t) \geq (2A_0)^{-1}\rho(x_t, x_t)$, and thus

$$\frac{1}{\mu(B)} \int_B u_t \, d\mu = \frac{1}{\mu(B)} \int_B \frac{d\mu(z)}{(\mu(B(x_t, \rho(z, x_t))))^{1-t} \leq \frac{1}{\mu(B(x_t, (2A_0)^{-1}\rho(x_t, x_t)))^{1-t}$$

$$= \left( \frac{\mu(B(x_t, \rho(x_t, x_t)))}{\mu(B(x_t, (2A_0)^{-1}\rho(x_t, x_t)))} \right)^{1-t} u_t(x) \leq u_t(x) \leq \frac{u_t(x)}{t}.$$ 

Case 2: Then assume that $r > (4A_0^2)^{-1}\rho(x_t, x_t)$. (This also includes the case $x = x_t$ if $x_t$ is a point mass, and in this case we consider any $r > 0$.) Consider the balls $\hat{B} := B(x_t, R), R := 6A_0^2r$, and $\hat{B} := B(y, 3A_0^2R)$. It is easy to see that $B \subseteq \hat{B} \subseteq \hat{B}$, and the doubling property implies that

$$\mu(B) \leq \mu(\hat{B}) \leq \mu(B) \leq C\mu(B), \quad C = C(A_0, \mu).$$

Thus, by Lemma 8.11 we conclude with

$$\frac{1}{\mu(B)} \int_B u_t \, d\mu \leq \frac{C}{\mu(B)} \int_B u_t \, d\mu \leq \frac{1}{\mu(B)} \mu(\hat{B})^t \int_B u_t \, d\mu = \frac{1}{t\mu(B(x_t, R))^{1-t}} \leq \frac{u_t(x)}{t}$$

since $B(x_t, R) \supseteq B(x_t, \rho(x_t, x_t))$ by the choice of $R$.

We are left to show the estimate $\gtrsim$, and it suffices to show that there exists a set $E$ with $\mu(E) > 0$ such that for every $x \in E$ and some ball $B \ni x$ we have that

$$\frac{1}{\mu(B)} \int_B u_t \, d\mu \geq \frac{u_t(x)}{t}.$$ 

To see this, recall that $x_t$ is a $(2C_\mu)^{-3/1}$-point so that there exists a ball $B = B(x_t, R)$ so that $\mu(B(x_t, R)) > (2C_\mu)^{3/1}\mu(\{x_t\})$. Let $k \geq 1$ be the first integer such that $\mu(B(x_t, 2^{-k}R)) < 2^{-1}\mu(B)$, and set $r := 2^{-k}R$. Then the ball $B(x_t, 2r)$ satisfies the inverse estimate, i.e.

$$\mu(B) \leq 2\mu(B(x_t, 2r)) \leq 2C_\mu\mu(B(x_t, r)).$$

Set $E := B \setminus B(x_t, r)$. Then $\mu(E) = \mu(B) - \mu(B(x_t, r)) > 2^{-1}\mu(B) > 0$, and for every $x \in E$, the ball $B \ni x$ satisfies, by Lemma 8.11 and (8.17), the estimate

$$\frac{1}{\mu(B)} \int_B u_t \, d\mu \geq \frac{1}{t\mu(B)^{1-t}} \geq \frac{1}{t\mu(B(x_t, r))^{1-t}} \geq \frac{1}{t\mu(B(x_t, \rho(x, x_t)))^{1-t}} = \frac{u_t(x)}{t}.$$

8.18. Lemma. Let $0 < \eta < 1$ and suppose that $x_t$ is an $\epsilon$-point with $\epsilon = (2C_\mu)^{-3/\eta}$, and let $B = B(x_t, R)$ be a ball so that $\mu(B) > \epsilon^{-1}\mu(\{x_t\})$. Then, for the function $f_t = \chi_B$,

$$\|f_t\|_{L^p(\mu_t)} = u_t(B)^{1/p} \approx \left( \frac{\mu(B)^t}{t} \right)^{1/p}.$$ 

Proof. This is a special case of Lemma 8.11. \qed

8.19. Lemma. Given $0 < \eta < 1$, suppose $x_t$ is an $\epsilon$-point with $\epsilon = (2C_\mu)^{-2(2C_\mu)^{-1/\gamma}}$, and let $B = B(x_t, R)$ be a ball so that $\mu(B) > \epsilon^{-1}\mu(\{x_t\})$. Let $u_t$ be a power weight defined in 8.9 and set $f_t = \chi_B$. Then

$$\|T_{f_t}(u_t^\gamma)\|_{L^\infty(u_t)} \gtrsim \|u_t\|_{A_\gamma}^{-\gamma} \|f_t\|_{L^p(u_t)}.$$ 

Note that Lemma 8.19 completes the proof of Lemma 8.3.

Proof. We pick a decreasing sequence $(r_k)$ of radii as in the proof of Lemma 8.11. Let $r_0 := R$. Then let $k_1 \geq 1$ be the smallest integer such that $\mu(B(x_t, 2^{-k_1}r_0)) < 2^{-1}\mu(B(x_t, r_0))$, and set $r_1 := 2^{-k_1}r_0$. Having chosen $k_{m+1} > k_m$ in this fashion, let $k_{m+1} := k_m$ be the smallest integer such that $\mu(B(x_t, 2^{-k_{m+1}}r_0)) < 2^{-1}\mu(B(x_t, r_{m+1}))$, and set $r_{m+1} := 2^{-k_{m+1}}r_0$. Again, if $\mu(\{x_t\}) = 0$, we may keep sub-dividing infinitely many times, and otherwise, we stop at the step $K \geq 1$ for which $\mu(B(x_t, r_{K-1})) > 2\mu(\{x_t\})$ and $\mu(B(x_t, r_K)) \leq 2\mu(\{x_t\})$. 
We have the estimate
\[(8.20) \quad \mu(B(x_t, r_t)) \leq 2C_\mu \mu(B(x_t, r_{t+1})); \]
cf. \[\text{[8.13]} \] (with \(r_0 = R\) in stead of \(r_0 = r\)). Let \(K \geq 1\) be an integer such that
\[K - 1 < \frac{1}{t_\gamma} \leq K.\]
Let us show that the sub-division into smaller balls by reducing to half the mass proceeds at least \(K\) times. Indeed, by iterating \[(8.21)\] and by the choice of \(K\), we see that
\[(8.21) \quad \mu(B(x_t, r_{K})) \geq (2C_\mu)^{-K} \mu(B(x_t, r_0)) \geq (2C_\mu)^{-1/(t_\gamma)} \mu(B).\]
First, \[(8.21)\] implies that
\[(8.22) \quad \mu(B(x_t, r_{K})) > 2C_\mu (2C_\mu)^{3/(t_\gamma)} \mu(\{x_t\}).\]
In particular, \(\mu(B(x_t, r_{K})) > 2\mu(\{x_t\})\) and thus, we may sub-divide \(K\) times, as claimed. Set \(r_t := 2^{-1} r_{K}\). Then \(\mu(B(x_t, r_t)) = \mu(B(x_t, 2^{-1} r_{K})) \geq (C_\mu)^{-1} \mu(B(x_t, r_{K}))\), and by \[(8.22)\] we have that
\[(8.23) \quad \mu(B(x_t, r_t)) \geq (2C_\mu)^{3/(t_\gamma)} \mu(\{x_t\}) > (2C_\mu)^{3/4} \mu(\{x_t\}).\]
Second, \[(8.21)\] implies that
\[(8.24) \quad \mu(B(x_t, r_t)) > C \mu(B), \quad C = (\mu, \gamma).\]
We then estimate
\[
\|T_\gamma(f_1 u_t^\gamma)\|_{L^{q_\infty}(u_\theta)} = \sup_{\lambda > 0} \lambda u_t \left( \{ x \in X : T_\gamma(f_1 u_t^\gamma)(x) > \lambda \} \right)^{1/q} \\
\geq \sup_{\lambda > 0} \lambda u_t \left( \{ x \in B(x_t, r_t) : T_\gamma(f_1 u_t^\gamma)(x) > \lambda \} \right)^{1/q}.
\]
Let \(x \in B(x_t, r_t)\). Then
\[
T_\gamma(f_1 u_t^\gamma)(x) \geq \int_{X \setminus \{x\}} \frac{\chi_B(y) u_t^\gamma(y) d\mu(y)}{\mu(B(x, \rho(x, y)))^{1-\gamma}} \geq \int_{B(x_t, r_t) \setminus B(x_t, 2\rho(x_t, x))} \frac{u_t^\gamma(y) d\mu(y)}{\mu(B(x, \rho(x, y)))^{1-\gamma}}.
\]
Observe that for \(y \notin B(x_t, 2\rho(x_t, x))\), we have \(B(x, \rho(x, y)) \subseteq B(x_t, 2A_\mu^\gamma \rho(x_t, y))\), and the doubling property implies that \(\mu(B(x, \rho(x, y))) \leq \mu(B(x_t, 2A_\mu^\gamma \rho(x_t, y))) \leq \mu(B(x_t, \rho(x_t, y)))\). Thus,
\[
T_\gamma(f_1 u_t^\gamma)(x) \geq \int_{B(x_t, R) \setminus B(x_t, 2\rho(x_t, x))} \frac{u_t^\gamma(y) d\mu(y)}{\mu(B(x, \rho(x, y)))^{1-\gamma}} \geq \int_{B(x_t, R) \setminus B(x_t, 2\rho(x_t, x))} \frac{d\mu(y)}{\mu(B(x_t, \rho(x_t, y)))^{1-\gamma}}
\]
\[\geq \sum_{i=0}^{K-1} \int_{B(x_t, r_i) \setminus B(x_t, r_{i+1})} \frac{d\mu(y)}{\mu(B(x_t, \rho(x_t, y)))^{1-\gamma}}.
\]
since \(x \in B(x_t, r_t)\) and thereby, \(2\rho(x_t, x) < r_t = r_K\) by the choice of \(r_t\) so that \(B(x_t, 2\rho(x_t, x)) \subseteq B(x_t, r_K)\). From now on the estimates are very similar to the ones performed when proving the estimate \(\geq\) of Lemma \[\text{[8.11]}\] with the only deviation that here the exponent of the quantities \(\mu(B(x_t, r_t))\) is \(t_\gamma\) in place of \(t\). We may conclude with
\[
T_\gamma(f_1 u_t^\gamma)(x) \geq \mu(B(x_t, R))^{t_\gamma} \frac{1 - (2C_\mu)^{-K t_\gamma}}{1 - (2C_\mu)^{-t_\gamma}},
\]
Recall from the beginning of the proof that \(K\) is chosen to satisfy \(K \geq 1/(t_\gamma)\). Thus, \((2C_\mu)^{-K t_\gamma} < (2C_\mu)^{-1} < 1/2\), so that
\[
T_\gamma(f_1 u_t^\gamma)(x) \geq \mu(B(x_t, R))^{t_\gamma} \frac{1 - (2C_\mu)^{-K t_\gamma}}{1 - (2C_\mu)^{-t_\gamma}} \geq \frac{\mu(B)^{t_\gamma}}{t_\gamma} \quad \text{for } 0 < t_\gamma < 1 \text{ and } x \in B(x_t, r_t).
We have shown that
\[
\|T_\gamma(f_\lambda u^\gamma_t)\|_{L^{q,\infty}(u_t)} \gtrsim \sup_{\lambda > 0} \lambda u_t \left( \{ x \in B(x_t, r_t) : T_\gamma(f_t u^\gamma_t)(x) > \lambda \} \right)^{1/q} \\
\gtrsim \frac{\mu(B)}{2 t \gamma} u_t \left( \{ x \in B(x_t, r_t) : T_\gamma(f_t u^\gamma_t)(x) > \frac{\mu(B)}{2 t \gamma} \} \right)^{1/q} \\
= \frac{\mu(B)}{2 t \gamma} u_t(B(x_t, r_t))^{1/q}.
\]
By \[8.23\], we may use Lemma \[8.11\] to estimate \( u_t(B(x_t, r_t)) \) from below. Recalling \[8.24\], we see that
\[
u_t(B(x_t, r_t)) \gtrsim \frac{\mu(B(x_t, r_t))}{t} \gtrsim \frac{\mu(B)}{t}.
\]
Thus,
\[
\|T_\gamma(f_\lambda u^\gamma_t)\|_{L^{3,\infty}(u_t)} \gtrsim C \left( \frac{1}{l} \right)^{1/q} \left( \frac{1}{l} \right)^{1/p} \mu(B) \left( \frac{1}{l} \right)^{1-\gamma} \left( \frac{\mu(B)}{l} \right)^{1/q} \left( \frac{\mu(B)}{l} \right)^{1/p}, \quad C = C(A_0, C_\mu, \gamma),
\]
where we used the identity \( 1/p - 1/q = \gamma \). Lemma \[8.15\] and Lemma \[8.18\] now complete the proof. \[\square\]

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