Large $N$ 2D Yang-Mills Theory and Topological String Theory

Stefan Cordes, Gregory Moore, and Sanjay Ramgoolam

stefan@waldzell.physics.yale.edu
moore@castalia.physics.yale.edu
skr@genesis2.physics.yale.edu

Dept. of Physics
Yale University
New Haven, CT 06511

Abstract
We describe a topological string theory which reproduces many aspects of the $1/N$ expansion of $SU(N)$ Yang-Mills theory in two spacetime dimensions in the zero coupling ($A = 0$) limit. The string theory is a modified version of topological gravity coupled to a topological sigma model with spacetime as target. The derivation of the string theory relies on a new interpretation of Gross and Taylor’s “Ω−1 points.” We describe how inclusion of the area, coupling of chiral sectors, and Wilson loop expectation values can be incorporated in the topological string approach.

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1 Currently visiting the Rutgers University Dept. of Physics
1. Introduction

The possibility that the strong interactions might be described by a theory of strings has been an enduring source of fascination and frustration to particle theorists for the past twenty-five years [3–9]. In the early 80’s some interesting progress on this question was made in the case of large N Yang-Mills theory in two dimensions (YM\textsubscript{2}) [10],[11]. Recently this work has been revived, considerably extended, and deepened. Exact results are now available for partition functions \(Z(G, \Sigma_T)\) and Wilson loop averages for a compact gauge group \(G\) on two-dimensional spacetimes \(\Sigma_T\) of arbitrary topology [12–14].

Building on the results [10,12–14] D. Gross and W. Taylor returned to the problem of strings and \(YM\textsubscript{2}\) in a beautiful series of papers [15]. (See also [16]). In particular, [15] derives the \(N \to \infty\) asymptotic expansion for the partition function \(Z(A, G, N)\). Moreover many aspects of the expansion in \(1/N\) have a natural geometrical explanation in terms of weighted sums over maps from a worldsheet \(\Sigma_W\) to the spacetime \(\Sigma_T\). In some cases (e.g. when \(\Sigma_T\) is a torus) Gross and Taylor were able to write the weighted sum explicitly as a sum over covering maps with weights given by symmetry factors for the cover. Given the results of [15] no one could seriously doubt that \(YM\textsubscript{2}\) is equivalent to a string theory. Nevertheless, [15] left untied some loose ends, such as the following two problems:

(1.) The problem of the true meaning of the “\(\Omega^{-1}\) points”.

(2.) The problem of finding the appropriate string action.

With regard to problem (1), the geometrical interpretation of the \(1/N\) expansion necessitated the introduction of \(|2 - 2G|\) “twist points,” (“\(\Omega\) points,” or “\(\Omega^{-1}\) points”). In contrast to the clear and natural geometrical interpretation of all the other aspects of the series, the nature of the \(\Omega^{-1}\) points was fraught with mystery. Related difficulties had already presented themselves ten years earlier in the work of Kazakov and Kostov [11]. Several authors have emphasized the importance of a proper understanding of the \(\Omega^{\pm 1}\) points.

As for problem (2), one of the key motivations for Gross and Taylor’s work was that the action for a string interpretation of \(YM\textsubscript{2}\) might have a natural generalization to four-dimensional targets, or might suggest essential features of a string interpretation of \(YM\textsubscript{4}\). However, difficulties associated with problem (1) presented a serious obstacle to finding

\[\text{footnote 2:} YM\textsubscript{2}\text{ has area-preserving diffeomorphism symmetry so } Z \text{ only depends on the gauge group, topology and total area of } \Sigma_T. \text{ For gauge group } G = SU(N) \text{ and } \Sigma_T \text{ of genus } G \text{ we denote the partition function by } Z(A, G, N).\]
the action for $YM_2$. Indeed, after the appearance of the first papers of [15] it was quickly noted in [17] [18] that, for the case of the partition function of a toroidal target, (where there are no $\Omega^{-1}$ points) the interpretation in terms of covering maps naturally suggests that the string action principle for $YM_2$ will involve a topological sigma model, with $\Sigma_T$ as target, coupled to topological gravity. However, even in the genus one case, the evaluation of Wilson loops necessitates consideration of $\Omega^{-1}$-points. No theory of $YM_2$ can go very far without an understanding of these factors.

In the present paper we will solve problem (1). The solution of this problem allows us to make some definite progress on problem (2). The solution to problem (1) is simple: there is no such thing as an “$\Omega^{-1}$ point!” We have not completely solved problem (2) in the sense that we have not reproduced all known results on $YM_2$ from the string approach. Nevertheless, we have reproduced enough to say that (a.) a description in terms of topological string theory is possible but (b.) the action is more elaborate than the standard coupling of topological gravity to the topological sigma model for $\Sigma_T$, and (c) a careful analysis of contact terms is needed to reproduce the $YM_2$ results.

In more detail, the paper is organized as follows. We review some aspects of [15] and establish notation in sec. 2. We will discuss both the “chiral” partition function $Z^+(A,G,N)$ (eq. (2.4)) as well as the “nonchiral” partition function (eq. (2.2)), both of which we view as asymptotic expansions in $1/N$.

In sections 3,4 we review some necessary background material from mathematics, in particular, we describe the Hurwitz moduli space $H(h,G)$ of holomorphic maps $\Sigma_W \to \Sigma_T$ from a connected Riemann surface $\Sigma_W$ of genus $h$ to a Riemann surface $\Sigma_T$ of genus $G$ with fixed complex structure. In sections 4.3, 4.4 we explain how $H(h,G)$ can be thought of as the base of a principal fibre bundle for $Diff^+ (\Sigma_W) \ltimes Weyl(\Sigma_W)$.

In section 5 we begin with the simplest quantity in $YM_2$: the chiral partition function at zero area: $Z^+(A=0,G,N)$. This expansion can be interpreted as a sum over branched covers [15]. Taking proper account of the $\Omega^{-1}$ factors leads to our first main result, stated as Proposition 5.2 (section 5.2): $Z^+(A=0,G,N)$ is the generating function for the orbifold Euler characteristic $Z$ of the compactified Hurwitz moduli space, $\overline{H(h,G)}$:

$$Z^+(0,N,G) = \exp \left[ \sum_{h=0}^{\infty} \left( \frac{1}{N} \right)^{2h-2} \chi(\overline{H(h,G)}) \right]$$

(1.1)
A branched cover of surfaces $\Sigma_W \to \Sigma_T$ can always be interpreted as a holomorphic map for appropriate complex structures on $\Sigma_W, \Sigma_T$. Thus, the appropriate category of maps with which to formulate the chiral $1/N$ expansion of $YM_2$ is the category of holomorphic maps. This is precisely the situation best suited to an introduction of topological sigma models. Accordingly, in section 6 we introduce a topological string theory which counts holomorphic (and antiholomorphic) maps $\Sigma_W \to \Sigma_T$. Our central claim is that this string theory is the underlying string theory of $YM_2$. The action is schematically of the form:

$$I_{\text{chiral YM}_2 \text{ string}} = I_{tg} + I_{t\sigma} + I_{c\sigma}$$

The first two terms give the (standard) action of 2D topological gravity coupled to a topological $\sigma$-model with $\Sigma_T$ as target. The action $I_{c\sigma}$ turns out to be complicated but can be deduced using a procedure which is in principle straightforward. This procedure is based on the point of view that topological field theory path integrals are related to infinite-dimensional generalisations of the Mathai-Quillen representative of equivariant Thom classes \cite{19,21}. In section 7, we use this point of view to construct $I_{c\sigma}$ explicitly.

In sections 8 and 9 we show how many of the results of chiral $YM_2$ can be derived from the topological string theory \eqref{1.2}.

In section 8 we describe how the area can be restored by a perturbation of the topological action \eqref{1.2} by the area operator:

$$A = \int f^*\omega$$

where $\omega$ is the Kähler class of the target space Riemann surface. This is, of course, just the standard Nambu action, as one might well expect. The novelty in the present context is that the Nambu action is regarded as a perturbation of a previously constructed theory. Calculating this perturbation of the topological string theory is rather subtle because of contact terms. These are responsible for the polynomial dependence on the area in $YM_2$. In section 8.1 we isolate what we believe are “the most important” area polynomials and, after some preliminary analysis of the contact terms between the area operator \eqref{1.3} and curvature insertions arising from $I_{c\sigma}$ in \eqref{1.2}, as well as those between area operators themselves, we show in section 8.7 how these polynomials follow from the string picture.

In section 9 we discuss Wilson loop expectation values in the case of nonintersecting Wilson loops. Following the lead of \cite{15} we show that these may be incorporated in the string approach by computing macroscopic loop amplitudes. The data of the representation
index on the Wilson line $\Gamma$ is translated into covering data of the boundary of the worldsheet over the lines $\Gamma$.

In section 10 we repeat the discussion of section 5 for the partition function of the “full nonchiral $YM_2$.” We follow closely the geometrical picture introduced in [15]. In order to state the analog of (1.1) it is necessary to introduce both holomorphic and antiholomorphic maps, as well as “degenerating coupled covers” (see Definitions 10.3,10.4). We introduce a Hurwitz space for such maps, called coupled Hurwitz space and, in Proposition 10.3, we state the result for the nonchiral theory analogous to (1.1).

In section 11 we explain how the nonchiral partition function can be incorporated in topological string theory. The path integral localizes on both holomorphic and antiholomorphic maps. It also localizes on singular maps (“degenerated coupled covers”) and the contributions from these singular geometries must be defined carefully. The answer for the partition function of the string theory (1.2) depends on how we choose our contact terms for these singular geometries. We discuss two choices which lead to two distinct answers:

$$\exp\left\{ \sum_{h \geq 0} \left( \frac{1}{N} \right)^{2h-2} Z_{\text{string}}(\Sigma_W \to \Sigma_T) \right\} = \begin{cases} Z^+(A = 0, N)Z^-(A = 0, N) \\ Z(A = 0, N) \end{cases}$$ (1.4)

In the first case we choose contact terms so that singular geometries make no contribution (we “set all contact terms to zero”). This reproduces answers of the chiral theory. A more non-trivial choice of contact terms reproduces the full zero-area theory.

Some technical arguments are contained in appendices.

Finally the reader should note that while we were preparing this paper a closely related paper appeared on hep-th [22]. In this paper P. Horava proposes a formulation of $YM_2$ in terms of topological string theory. The theory in [22] is based on counting of harmonic maps, rather than holomorphic maps, (or degenerated coupled covers) and, at least superficially, appears to be different from the proposal of this paper.

2. The Gross-Taylor Asymptotic Series

2.1. Partition Functions

The partition function of two dimensional Yang-Mills theory on an orientable closed manifold $\Sigma_T$ of genus $G$ is [11,12]

$$Z(SU(N), \Sigma_T) = \int [DA^a] \exp \left[ -\frac{1}{4e^2} \int_{\Sigma_T} d^2 x \sqrt{\det G_{ij}} \, TrF_{ij}F^{ij} \right]$$

$$= \sum_R (\dim R)^{2G} e^{-\frac{A}{4N} C_2(R)},$$ (2.1)
where the gauge coupling $\lambda = e^2 N$ is held fixed in the large $N$ limit, the sum runs over all unitary irreducible representations $R$ of the gauge group $G = SU(N)$, $C_2(R)$ is the second casimir, and $A$ is the area of the spacetime in the metric $G_{ij}$. We will henceforth absorb $\lambda$ into $A$.

Using the Frobenius relations between representations of symmetric groups and representations of $SU(N)$ Gross and Taylor derived an expression for the $1/N$ asymptotics of (2.1) in terms of a sum over elements of symmetric groups. The result of [15] is:

\[
Z(A,G,N) \sim \sum_{n^+,i^+=0}^{\infty} \sum_{p^+_1,\ldots,p^+_i} (-1)^{i^+} e^{-\frac{1}{2}(n^++n^-)A} \frac{1}{n^+!n^-!} G_{n^+,n^-} s_{n^+} t_{n^-} \prod_{j=1}^i \prod_{k=1}^G [s_j^+, t_j^+] [s_k^-, t_k^-],
\]

where $[s,t] = st^{-1}$. Here $\delta$ is the delta function on the group algebra of the product of symmetric groups $S_{n^+} \times S_{n^-}$, $T_2$ is the class of elements of $S_{n^\pm}$ consisting of transpositions, and $\Omega_{n^+,n^-}$ are certain elements of the group algebra of the symmetric group $S_{n^+} \times S_{n^-}$ with coefficients in $\mathbb{R}((1/N))$. These will be discussed in detail below.

One of the striking features of (2.2) is that it nearly factorizes, splitting into a sum over $n^+,i^+$, $\cdots$ and $n^-,i^-,\cdots$. Gross and Taylor interpreted the contributions of the $(+)$ and $(-)$ sums as arising from two "sectors" of a hypothetical worldsheets theory. These sectors correspond to orientation reversing and preserving maps, respectively. One views the $n^+ = 0$ and $n^- = 0$ terms as leading order terms in a $1/N$ expansion. At higher orders the two sectors are coupled via the $n^+n^-$ term in the exponential and via terms in $\Omega_{n^+,n^-}$. The latter are described by a simple set of rules in [15], and will be addressed in detail in section 1 below.

The expression (2.2) simplifies considerably if we concentrate on one chiral (or antichiral) sector. In general we define chiral expectation amplitudes in $YM_2$ by translating $SU(N)$ representation theory into representation theory of symmetric groups and making the replacement:

\[
\sum_{R \in \text{Rep}(SU(N))} \to \sum_{n \geq 0} \sum_{R \in Y_n}
\]

(2.3)
where $Y_n$ stands for the set of Young Tableaux with $n$ boxes. For example, in the case of the partition function we may define $\Omega_n \equiv \Omega_{n,0}$ and write the “chiral Gross-Taylor series” (CGTS) as \[15\]:

$$Z^+(A,G,N) = \sum_{n,i,t,h=0}^{\infty} e^{-nA/2} \left( \frac{-1}{i!t!h!} \right) n^{2G-2} + 2h + i + 2t n^h (n^2 - n)^t$$ \hspace{1cm} (2.4)

\[\sum_{p_1, \ldots, p_t \in T} \sum_{s_1, t_1, \ldots, s_G, t_G \in S_n} \left[ \frac{1}{n^t} \delta(p_1 \cdots p_t \Omega_n^{2-2G} \prod_{j=1}^G s_j t_j s_j^{-1} t_j^{-1}) \right].\]

2.2. $\Omega$ factors

Let us now define $\Omega$. We postpone describing $\Omega_{n+,n-}$ to section 10.4 and concentrate on the ”chiral $\Omega$ factors” $\Omega_n$. This is the element of the group algebra of $S_n$ defined by the equation

$$\dim R_Y = \frac{N^n}{n!} \chi_Y(\Omega_n)$$

Here $R_Y$ is an $SU(N)$ representation associated to a Young tableaux $Y$ with $n$ boxes, and $\chi_Y$ is the character in the corresponding representation of $S_n$. Explicitly, $\Omega_n$ is given by

$$\Omega_n = \sum_{\nu \in S_n} \left( \frac{1}{N^t} \right)^{n-K_\nu}.$$ \hspace{1cm} (2.5)

Here, $K_\nu$ is the number of cycles in the permutation $\nu$.

When $G > 1$ one must introduce $\Omega^{-1}$ which is the inverse of $\Omega$ in the group algebra. For example, we have

$$\Omega_2 = 1 + \frac{1}{N}$$

$$\Omega_2^{-1} = \sum_{i=0}^{\infty} (-1/N)^i v^i$$

$$= \sum_{i=0}^{\infty} (1/N)^{2i} - v \sum_{i=0}^{\infty} (1/N)^{(2i+1)}$$

$$= \frac{1}{(1-1/N^2)} \cdot 1 - \frac{1}{(1-1/N^2)} \cdot v$$

for $n = 2$. In the first line we have written it as an element of the free algebra generated by elements of $S_2$. In the second we have reduced it to an element of the group algebra of $S_2$ whose coefficients are expansions in $1/N$.

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\[3\] Note that for some values of $N$, $\Omega_n$ will fail to have an inverse. This does not happen when $N > n$. Hence $\Omega$ may always be considered as invertible in the $1/N$ expansion.
3. Maps and Coverings

We would like to interpret the terms in the $1/N$ expansion as weighted sums of maps $\Sigma_{W} \to \Sigma_{T}$ between compact orientable surfaces without boundary, of genus $h, G$, respectively. In the next two sections we summarize some relevant mathematical results pertaining to such maps.

3.1. Homotopy groups

We will use heavily the properties of homotopy groups of punctured Riemann surfaces. As abstract groups these are “F-groups.” The group $F_{G,L}$ may be described in terms of generators and relations by

$$F_{G,L} \equiv \left\langle \{\alpha_i, \beta_i\}_{i=1,G}, \{\gamma_s\}_{s=1,L} \mid \prod_{i=1}^{G} \prod_{s=1}^{L} \gamma_s = 1 \right\rangle$$

(3.1)

(The product is ordered, say, lexicographically.)

Consider a compact orientable surface $\Sigma_T$ of genus $G$. If we remove $L$ distinct points, and choose a basepoint $y_0$, then there is an isomorphism

$$F_{G,L} \cong \pi_1(\Sigma_T - \{P_1, \ldots, P_L\}, y_0).$$

(3.2)

This isomorphism is not canonical. The choices are parametrized by the infinite group $\text{Aut}(F_{G,L})$.

On several occasions we will make use of a set of generators $\alpha_i, \beta_i, \gamma_i$ of $\pi_1$ so that, if we cut along curves in the homotopy class the surface looks like

![Diagram of generators for the homotopy group of a punctured surface.](image)

**Fig. 1:** A choice of generators for the homotopy group of a punctured surface. The curves $\gamma(P)$ become trivial if we fill in the puncture $P$. 

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The conjugacy class of the curves $\gamma(P)$ can be characterised intrinsically as follows. The process of filling in a point $P_1$ defines an inclusion

$$i : \Sigma_T - \{P_1, P_2, \cdots P_L\} \to \Sigma_T - \{P_2, \cdots P_L\}$$

(3.3)

with an induced map $i_*$ on $\pi_1$. $[\gamma(P_1)]$ is the kernel of $i_*$. 

### 3.2. Branched coverings

Maps of particular importance to us are branched coverings.

**Definition 3.1.**

a.) A continuous map $f : \Sigma_W \to \Sigma_T$ is a branched cover if for any open set $U \subset \Sigma_T$, the inverse $f^{-1}(U)$ is a union of disjoint open sets on each of which $f$ is topologically equivalent to the complex map $z \mapsto z^n$ for some $n$.

b.) Two branched covers $f_1$ and $f_2$ are said to be equivalent if there exists a homeomorphism $\phi : \Sigma_W \to \Sigma_W$ such that $f_1 \circ \phi = f_2$.

For $Q \in \Sigma_W$, the integer $n$ will be called the ramification index of $Q$ and will be denoted $\text{Ram}(f, Q)$. For any $P \in \Sigma_T$ the sum

$$\sum_{Q \in f^{-1}(P)} \text{Ram}(f, Q)$$

is independent of $P$ and will be called the index of $f$ (sometimes the degree). Points $Q$ for which the integer $n$ in condition (a) is bigger than 1 will be called ramification points. Points $P \in \Sigma_T$ which are images of ramification points will be called branch points. The set of branch points is the branch locus $S$. The branching number at $P$ is

$$B_P = \sum_{Q \in f^{-1}(P)} [\text{Ram}(f, Q) - 1]$$

The branching number of the map $f$ is $B(f) = \sum_{P \in S(f)} B_P$. A branch point $P$ for which the branching number is 1 will be called a simple branch point. Above a simple branch point all the inverse images have ramification index $= 1$, with the exception of one point $Q$ with index $= 2$.

We will often use the Riemann-Hurwitz formula. If $f : \Sigma_W \to \Sigma_T$ is a branched cover of index $n$ and branching number $B$, $\Sigma_W$ has genus $h$, $\Sigma_T$ has genus $G$, then:

$$2h - 2 = n(2G - 2) + B.$$  

(3.4)

4 Unfortunately, several authors use these terms in inequivalent ways.
Equivalence classes of branched covers may be related to group homomorphisms through the following construction. Choose a point \( y_0 \) which is not a branch point and label the inverse images \( f^{-1}(y_0) \) by the ordered set \( \{x_1, \ldots, x_n\} \). Following the lift of elements of \( \pi_1(\Sigma_T - S, y_0) \) the map \( f \) induces a homomorphism

\[
f_\#: \pi_1(\Sigma_T - S, y_0) \to S_n.
\]

Suppose \( \gamma(P) \) is a curve surrounding a branch point \( P \) as in fig. 1. There is a close relation between the cycle structure of \( v_P = f_\#(\gamma(P)) \) and the topology of the covering space over a neighbourhood of \( P \). If the cycle decomposition of \( v_P \) has \( r \) distinct cycles then \( f^{-1}(P) \) has \( r \) distinct points. Moreover, a cycle of length \( k \) corresponds to a ramification point \( Q \) of index \( k \).

With an appropriate notion of equivalence the homomorphisms are in 1-1 correspondence with equivalence classes of branched covers.

**Definition 3.2.** Two homomorphisms \( \psi_1, \psi_2 : \pi_1(\Sigma_T - S, y_0) \to S_n \) are said to be equivalent if they differ by an inner automorphism of \( S_n \), i.e., if \( \exists g \) such that \( \forall x, \psi_1(x) = g\psi_2(x)g^{-1} \).

**Theorem 3.1.**\(^{[23]}\)\(^{[24]}\). Let \( S \subset \Sigma_T \) be a finite set and \( n \) a positive integer. There is a one to one correspondence between equivalence classes of homomorphisms

\[
\psi : \pi_1(\Sigma_T - S, y_0) \to S_n
\]

and equivalence classes of \( n \)-fold branched coverings of \( \Sigma_T \) with branching locus \( S \).

**Proof.** We outline the proof which is described in\(^{[24]}\). The first step shows that equivalent homomorphisms determine equivalent branched coverings. Given a branched cover, we can delete the branch points from \( \Sigma_T \) and the inverse images of the branch points from \( \Sigma_W \) giving surfaces \( \Sigma_W \) and \( \Sigma_T \) respectively. The branched cover restricts to a topological (unbranched) cover of \( \Sigma_T \) by \( \Sigma_W \). To this map we can apply the theorem\(^{[25]}\) which establishes a one-to-one correspondence between conjugacy classes of subgroups of \( \pi_1(\Sigma_T) \) and equivalence classes of topological coverings of \( \Sigma_T \).

Similarly the second step proves that equivalent covers determine equivalent homomorphisms. The restriction of \( \phi \) to the inverse images of \( y_0 \) determines the permutation which conjugates one homomorphism into the other.

Finally one proves that the map from equivalence classes of homomorphisms to equivalence classes of branched covers is onto. We cut \( n \) copies of \( \Sigma_T \) along chosen generators
of $\pi_1(\Sigma_T - S)$ (illustrated in Figure 1), and we glue them together according to the data of the homomorphism. ♠

This theorem goes back to Riemann. Since the $YM_2$ partition function sums over covering surfaces which are not necessarily connected we do not restrict to homomorphisms whose images are transitive subgroups of $S_n$.

**Definition 3.3.** An automorphism of a branched covering $f$ is a homeomorphism $\phi$ such that $f \circ \phi = f$.

It follows from the above that the number of such automorphisms of a given equivalence class of branched coverings is equal to the order of the centraliser of the subgroup generated by the image of $\pi_1(\Sigma_T - S, y_0)$ in $S_n$. For a homomorphism $\psi$ we call this $|C(\psi)|$. Then $n!/|C(\psi)|$, the number of cosets of this subgroup, is the number of distinct homomorphisms related to the given homomorphism by conjugation in $S_n$.

### 3.3. Continuous maps

The space of branched coverings is not the only space of maps of surfaces one might wish to consider in rewriting $YM_2$ as a string theory. Another natural choice of category is the category of continuous maps. We mention here two theorems concerning the classification of these maps, and their relation to the category of branched coverings.

Since Riemann surfaces have a contractible universal cover, continuous maps between Riemann surfaces are topologically classified by their action on homotopy groups:

**Theorem 3.2.** [26] Homotopy classes of continuous maps $f : (\Sigma_W, x_0) \to (\Sigma_T, y_0)$ of fixed degree are in 1-1 correspondence with homomorphisms $f_* : \pi_1(\Sigma_W, x_0) \to \pi_1(\Sigma_T, y_0)$.

A map $\Sigma_W \to \Sigma_T$ is said to be a pinch map if there is a compact connected submanifold $H \subset \Sigma_W$, with boundary consisting of a simple closed curve in the interior of $\Sigma_W$, such that the $\Sigma_T$ is $\Sigma_W/H$, the quotient of $\Sigma_W$ with $H$ identified to a point, and such that $f$ is the quotient map. Pinch maps can collapse entire regions of surface to a single point as in

**Theorem 3.3.** [27] In each homotopy class of maps $f : \Sigma_W \to \Sigma_T$ there is a representative $f = p \circ \pi$ where $p$ is a pinch map and $\pi$ is a branched covering.

Notice that pinch maps can only decrease the Euler character of $\Sigma_W$. It therefore follows from theorem 3.3 and [B.3] that the existence of a nonconstant $f : \Sigma_W \to \Sigma_T$ implies that $2h - 2 \geq n(2G - 2)$. This is the now-famous Kneser bound.
4. The Hurwitz space $H$ of branched coverings

4.1. Definition

The Hurwitz space of branched coverings is nicely described in [23] [28]. Let $H(n, B, G; S)$ be the set of equivalence classes of branched coverings of $\Sigma_T$, with degree $n$, branching number $B$, and branch locus $S$, where $S$ is a set of distinct points on $\Sigma_T$. $H(n, B, G; S)$ is a finite set. The union of these spaces over sets $S$ with $L$ elements is the Hurwitz space $H(n, B, G, L)$ of equivalence classes of branched coverings of $\Sigma_T$ with degree $n$, branching number $B$ and $L$ branch points. Finally let $C_L(\Sigma_T)$ be the configuration space of ordered $L$-tuples of distinct points on $\Sigma_T$, that is

$$C_L(\Sigma_T) = \{(z_1, \ldots, z_L) \in \Sigma_T^L | z_i \in \Sigma_T, z_i \neq z_j \text{ for } i \neq j\}.$$ 

The permutation group $S_L$ acts naturally on $C_L$ and we denote the quotient $C_L(\Sigma_T) = C_L(\Sigma_T)/S_L$. There is a map

$$p : H(n, B, G, L) \rightarrow C_L(\Sigma_T) \quad (4.1)$$

which assigns to each covering its branching locus. This map can be made a topological (unbranched) covering map [23] with discrete fiber $H(n, B, G; S)$ over $S \in C_L$.

The lifting of closed curves in $C_L$ will in general permute different elements of the fibers $H(n, B, G, S)$. Note however that $Aut f$ is invariant along any lifted curve so that $Aut f$ is an invariant of the different components of $H(n, B, G, L)$. 

Fig. 2: Example of a pinch map.
4.2. $H$ as an analytic variety

One great advantage of branched covers is that they allow us to introduce the powerful methods of complex analysis, which are crucial to introducing ideas from topological field theory.

Choose a complex structure $J$ on $\Sigma_T$. Then given a branched cover $f : \Sigma_W \to \Sigma_T$ there is a unique complex structure on $\Sigma_W$ making $f$ holomorphic. (use the complex structure $f^*(J)$ on $\Sigma_W$ ) [29]. (This is far from true for pinch maps). Conversely, any nonconstant holomorphic map $f : \Sigma_W \to \Sigma_T$ defines a branched cover. It follows that we can consider the Hurwitz space $H(n,B,G,L)$ as a space of holomorphic maps. The complex structure $J$ on $\Sigma_T$ induces a complex structure on $H(n,B,G,L)$ such that $p$ is a holomorphic fibration. Moreover, the induced complex structure on $\Sigma_W$ defines a holomorphic map $m : H(n,B,G,L) \to M_{h,0}$ where $M_{h,0}$ is the Riemann moduli space of curves of genus $h$, where $h$ is given by (3.4). The image of $H$ is a subvariety of $M_{h,0}$.

4.3. Fiber Bundle approach to Hurwitz space

For comparison with topological field theory we will need another description of Hurwitz space as the base space of an infinite-dimensional fiber bundle.

Let $\Sigma_W$ be an orientable surface, and suppose $\Sigma_T$ is a Riemann surface with a choice of Kähler metric and complex structure $J$. Let us begin with the configuration space

$$\tilde{\mathcal{M}} = \{(f,h) | f \in C^\infty(\Sigma_W,\Sigma_T), h \in \text{Met}(\Sigma_W)\}. \quad (4.2)$$

where $C^\infty(\Sigma_W,\Sigma_T)$ is the space of smooth ($C^\infty$) maps, $f : \Sigma_W \to \Sigma_T$ and $\text{Met}(\Sigma_W)$ is the space of smooth metrics on $\Sigma_W$. A choice of metric $h$ induces a complex structure: $\epsilon(h) \in \Gamma[\text{End}(T\Sigma_W)]$, $\epsilon^2 = -1$. If we choose a basepoint $h_0$ in the space of metrics, and choose oriented isothermal coordinates relative to $h_0$ then we can define a basepoint complex structure to be the standard antisymmetric tensor $\hat{\epsilon}_{\alpha\gamma}$,

$$\hat{\epsilon}_{\alpha\gamma} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad (4.3)$$

in the isothermal coordinates. In these terms we define $\epsilon_{\alpha}^{\beta}(h) = h^{1/2}\hat{\epsilon}_{\alpha\gamma}h^{\gamma\beta}$.

The subspace of pairs defining a holomorphic map $\Sigma_W \to \Sigma_T$ is then given by

$$\tilde{\mathcal{F}} = \{(f,h) : df \epsilon = Jdf \} \subset \tilde{\mathcal{M}} \quad (4.4)$$
The defining equation \( df \epsilon = Jdf \) is an equation in \( \Gamma[\text{End}(T_{x} \Sigma W, T_{f(x)} \Sigma T)] \).

Let \( Diff^{+}(\Sigma W) \ltimes Weyl(\Sigma W) \) be the semidirect product of the group of orientation preserving diffeomorphisms of \( \Sigma W \) and the group of Weyl transformations on \( \Sigma W \). There is a natural action of this group on \( \tilde{\mathcal{M}} \). There is an action of \( Diff^{+}(\Sigma W) \ltimes Weyl(\Sigma W) \) on \( \tilde{F} \). The quotient space
\[
\mathcal{F} \equiv \tilde{F}/(Diff^{+}(\Sigma W) \ltimes Weyl(\Sigma W))
\]
parametrizes holomorphic maps \( \Sigma W \to \Sigma T \).

We have now provided two descriptions of the space of holomorphic maps: Hurwitz space \( H \) and (4.5). Let \( H(h, G) = \Pi n(2-2G-B=2-2h)H(n, B, G, L) \) where the disjoint union runs over \( n, B, L \geq 0 \) compatible with (3.4). There is a map \( \sigma : H(h, G, L) \to \mathcal{F} \) which is generically smooth and one-one. The space \( \mathcal{F} \) has orbifold singularities where the group \( Diff^{+}(\Sigma W) \) fails to act freely. Because we divide by \( Diff^{+}(\Sigma W) \), the local orbifold group at \((f, h) \in \mathcal{F}\) is \( \text{Autf} \). This will be important when we introduce the orbifold Euler character of \( H \). The map \( \tilde{\mathcal{M}} \to \text{Met}(\Sigma W) \) obtained from \((f, h) \to h\) induces the map \( m : H(h, G) \to \mathcal{M}_{h,0} \) of sec. 4.2 and relates the bundle description of Hurwitz space to the bundle description of \( \mathcal{M}_{h,0} \). Since \( \text{Autf} \subset \text{Aut}(\Sigma W) \), orbifold points of \( \mathcal{F} \) map to orbifold points of \( \mathcal{M}_{h,0} \).

4.4. Geometry of \( \mathcal{F} \)

Our discussion of the topological string theory approach to \( YM_{2} \) requires a brief discussion of the geometry of \( \mathcal{F} \). In particular, we need to define a connection on \( TF \) and compute its curvature.

Let us first make the tangent space to \( \tilde{\mathcal{F}} \) more explicit. The tangent space to \( \tilde{\mathcal{M}} \) is
\[
T_{f, h} \tilde{\mathcal{M}} = \Gamma[f^{*}(T_{f} \Sigma T)] \oplus \Gamma[S^{\otimes 2}(T^{*} \Sigma W)],
\]
where \( \Gamma \) is the space of \( C^{\infty} \) sections and \( S^{\otimes n} \) is the \( n \)th symmetric power. The tangent space to \( \tilde{\mathcal{F}} \) is the subspace of pairs \((\delta f, \delta h)\) which preserve the equation \( df \epsilon = Jdf \). In order to characterize this subspace by a differential equation we identify the differential \( df \) with a section of \( T^{*} \Sigma W \otimes f^{*}(T_{f} \Sigma T) \) \((= \Gamma[\text{End}(T_{x} \Sigma W, T_{f(x)} \Sigma T)])\). Then, in order to vary with respect to \( f \) we must compare \( T\Sigma_{T} \) at different points. We do this using the Kähler metric on \( \Sigma_{T} \) to define a pullback connection \( \tilde{\nabla} \) on \( f^{*}(T_{f} \Sigma T) \):
\[
\nabla : \Gamma[f^{*}(T_{f} \Sigma T)] \to \Gamma[T^{*} \Sigma W \otimes f^{*}(T^{*} \Sigma T)].
\]

\(^{5}\) See appendix B for a careful derivation of this connection.
Finally, let \( k(\delta h) = \delta \epsilon \) be the variation of complex structure on \( \Sigma_W \) induced from a variation of metric \( \delta h \in S^{\otimes 2}(T^* \Sigma_W) \). The tangent space \( \tilde{F} \) at \((f, h)\) is the subspace of \( T\tilde{M} \) defined by

\[
T_{f,h} \tilde{F} = \{ (\delta f, \delta h) : \nabla(\delta f) + J\nabla(\delta f)\epsilon + Jdfk(\delta h) = 0 \}. \tag{4.7}
\]

See appendix B for the proof.

We now separate out “pure gauge” deformations. The action of the gauge group on \( \tilde{F} \) defines a subbundle \( T^{vert} \tilde{F} \subset T\tilde{F} \) with fibers isomorphic to \( \text{im} C \) where

\[
C : T[Diff^+(\Sigma_W) \ltimes Weyl(\Sigma_W)] \to T^{vert} \tilde{F}
\]

\[
C \left( \frac{\xi^\alpha}{\delta \sigma} \right) \mapsto \left( (P\xi)_{\alpha\beta} + (\delta \sigma + \nabla \cdot \xi) h_{\alpha\beta} \right) \tag{4.8}
\]

and

\[
(P\xi)_{\alpha\beta} = \nabla(\alpha \xi_\beta) - h_{\alpha\beta} \nabla \cdot \xi, \tag{4.9}
\]

as is familiar from string theory. In general \( \ker C = 0 \) and we have an isomorphism \( T_{[f,h]} F \cong T_{f,h} \tilde{F}/\text{im} C \).

To go further we use the natural \( Diff^+(\Sigma_W) \)-invariant metric on \( \tilde{M} \) (hence on \( \tilde{F} \)), given by

\[
\langle (\delta f_1, \delta h_1), (\delta f_2, \delta h_2) \rangle_{\tilde{M}}
= \int d^2z \ h^{1/2} \left\{ G_{ij} \delta f_1^i \delta f_2^j + (h^{\alpha\gamma} h^{\beta\delta} + c h^{\alpha\beta} h^{\gamma\delta} ) \delta h_1_{\alpha\beta} \delta h_2_{\gamma\delta} \right\}. \tag{4.10}
\]

with \( c \in \mathbb{R}^+ \) arbitrary. This allows us to define adjoints and orthogonal projections. We now introduce the operator:

\[
\Phi = \left( \frac{d}{df} J dfk \right) : T_{f,h} \tilde{M} \to \Gamma[T\Sigma_W \otimes f^*(T\Sigma_T)] \oplus \Gamma[T\Sigma_W]. \tag{4.11}
\]

where the components are given by:

\[
D \chi^i = \nabla \chi^i + J(\nabla \chi^i)\epsilon
\]

\[
\partial f \chi^i = h^{\alpha\beta}(\partial f^i)^\alpha \chi_j^\beta
\]

Consider deformations \((\delta f, \delta h)\) in the kernel of \( \Phi \). The first line of \((E.11)\) ensures that \((\delta f, \delta h) \in T\tilde{F} \) and the second ensures that \((\delta f, \delta h) \notin T^{vert} \tilde{F} \). An index theorem shows that \( \dim_{\mathbb{Q}} \ker \Phi - \dim_{\mathbb{Q}} \text{coker} \Phi = B = 2h - 2 - (2G - 2) \) is the total branching number.
On the other hand, a generalisation of Kodaira-Spencer theory described in appendix A shows that $\dim \mathbf{\mathcal{F}} = B$. In the generic case ($G, h > 2$), $\dim \mathbf{coker} \Phi = 0$. Moreover we have an orthogonal decomposition

$$T_{f,h} \tilde{\mathcal{F}} = T(\text{Diff}^+(\Sigma_W) \ltimes \text{Weyl}(\Sigma_W)) \oplus \ker \Phi.$$  \hfill (4.13)

Even though the metrics are not $\text{Weyl}(\Sigma_W)$ invariant, the orthogonal decomposition (4.13) is invariant. Therefore $\ker \Phi$ is isomorphic to the tangent space $T_{[f,h]} \mathcal{F}$.

The natural metric (4.10) on $\tilde{\mathcal{F}}$ also defines connections on the principal $\text{Diff}^+(\Sigma_W) \ltimes \text{Weyl}(\Sigma_W)$ bundle $\pi : \tilde{\mathcal{F}} \to \mathcal{F}$ as well as on the tangent $T \tilde{\mathcal{F}} \to \tilde{\mathcal{F}}$. In the first case we define a lift of a curve $\gamma(t) \subset \mathcal{F}$ to be $\tilde{\gamma}(t) \subset \tilde{\mathcal{F}}$ defined by the conditions:

$$\frac{d}{dt} \tilde{\gamma} \in \ker \Phi,$$

$$d\pi \left( \frac{d}{dt} \tilde{\gamma} \right) = \frac{d}{dt} \gamma.$$

(4.14)

In the second case we define $\tilde{\nabla}$ on $T \tilde{\mathcal{F}}$ by declaring $\ker \Phi$ to be the horizontal subspace of the fiber. Finally we use these connections to define a connection $\nabla$ on $T \mathcal{F} \to \mathcal{F}$. It suffices to define the parallel transport $X(t)$ of $X(0) \in T_{\gamma(0)} \mathcal{F}$ along a path $\gamma(t) \subset \mathcal{F}$. Choose lifts $\tilde{\gamma}(t) \subset \tilde{\mathcal{F}}$ and $\tilde{X}(0) \in T_{\tilde{\gamma}(0)} \tilde{\mathcal{F}}$. Use $\tilde{\nabla}$ to define the parallel-transported $\tilde{X}(t) \in T_{\tilde{\gamma}(t)} \tilde{\mathcal{F}}$ and define $X(t) = d\pi(\tilde{X}(t))$. Since $\tilde{\nabla}$ preserves the orthogonal decomposition (4.13) our definition is independent of the choices made in lifting. See Figure 3.

**Fig. 3:** Construction of a connection on $T \mathcal{F} \to \mathcal{F} : \delta(t)$ is the lift of $\gamma(t)$, determined by choosing as initial point a lift $Y(0)$ of $X(0)$. 

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Finally, let us describe the curvature of the connection $\tilde{\nabla}$ on $T\tilde{F}$ more explicitly. In local coordinates we may describe the connection on $T_{f,h}\tilde{F}$ as follows. Introduce local tangent vector fields
\[ X = \left( \chi^i \psi_{\alpha\beta} \right), \quad \tilde{X} = \left( \tilde{\chi}^i \psi_{\alpha\beta} \right), \]
of as elements of ker $\Phi$. Then
\[ \tilde{\nabla}_X \tilde{X} = \delta_X \tilde{X} + \Phi^\dagger \frac{1}{\Phi \Phi^\dagger} \delta_X \Phi \tilde{X}. \quad (4.15) \]
where $\delta_X \tilde{X} = X \circ \tilde{X}$ and $\delta \Phi = X \cdot \Phi$. Equation (4.13) makes sense since $\Phi \Phi^\dagger$ is invertible.
A simple calculation, using repeatedly the fact that $\Phi \tilde{X} = 0$ shows that the curvature on horizontal vectors is given by
\[ (\tilde{X}_1, R_{\left[ X_1, X_2 \right]} \tilde{X}_2) \]
\[ = \left( \tilde{X}_1, (\delta_1 \Phi^\dagger)(\Phi \Phi^\dagger)^{-1}(\delta_2 \Phi) \tilde{X}_2 \right) - \left( \tilde{X}_1, (\delta_2 \Phi^\dagger)(\Phi \Phi^\dagger)^{-1}(\delta_1 \Phi) \tilde{X}_2 \right) \quad (4.16) \]
When we descend to $\mathcal{F}$ we are working on an analytic space, and, from (4.10) we see that $T\mathcal{F}$ is a holomorphic Hermitian vector bundle. Thus if we choose a local holomorphic framing $G_I$ we may use (4.10) to form the positive definite matrix of inner products $h_{IJ} = \langle G_I, G_J \rangle$. In these terms the connection and curvature are given by [30]:
\[ \nabla = \partial \log h \quad R = \partial \bar{\partial} \log h \quad (4.17) \]
We will return to this formula in our discussion of contact terms.

4.5. Compactification of Hurwitz Space

Consider $H(n, B, G)$ the space of branched coverings with a given degree and branching number of a surface of genus $G$. A $B$-dimensional open subset of this space is $H(n, B, G, L = B)$, which consists of maps where all the branch points are simple. We will refer to this space as simple Hurwitz space.

Simple Hurwitz space can be (partially) compactified to form the Hurwitz space $H(n, B, G)$ by adding $L$-dimensional compactification varieties of the form $H(n, B, G, L < B)$:
\[ H(n, B, G) = \bigcup_L H(n, B, G, L) \quad (4.18) \]
We thus have in mind the following schematic description of Hurwitz space:
Compactified Hurwitz space is a bundle over the compactified configuration space where the branch points on the target are allowed to collide. The compactification subvarieties may be described in terms of basic degenerations of branched coverings. We will use the facts in sections 3.1 and 3.2 to describe some properties of degenerations that happen when two branch points collide. The following observation is basic. Let \( f \in H(n, B, G; S) \). Choosing a set of generators for \( \pi_1 \) we may then associate a cut surface as in the proof of Theorem 3.1, and as in the LHS of Fig. 5.

Choose two generators of \( \pi_1(\Sigma_G - S, y_0), \gamma_j \) and \( \gamma_{j+1} \) (see (3.1)). Consider a closed path \( \gamma^* \) homotopic to the product \( \gamma_j \gamma_{j+1} \). As we let the points enclosed by \( \gamma_j \) and \( \gamma_{j+1} \) approach each other, the image of \( \gamma^* \) in \( S_n \) does not change and remains the product \( u_j u_{j+1} \) where \( u_j \) is the image of \( \gamma_j \) and \( u_{j+1} \) is the image of \( \gamma_{j+1} \) under \( f_\# \).
Using the above rule we see that there are three types of collisions of simple branch points. They are classified by the behaviour of the inverse ramification points, and illustrated in Figure 6.

**Type 1.** A collision of type 1 produces a single ramification point of index 3. For example if \( u_j = (12), \) \( u_{j+1} = (23) \) then collision of \( P_j, P_{j+1} \) produces a ramification point with \( u = (123). \) Starting from the space of generic branched covers successive collisions can lead to multiple branch points; a collision of \( \ell \) simple branch points can lead to a ramification point of index \( \ell + 1. \)

**Type 2.** A collision of type 2 produces two ramification points. This occurs when \( u_j \) and \( u_{j+1} \) are disjoint transpositions.

**Type 3.** A collision of type 3 produces no ramification point but instead produces a double point. This occurs when \( u_j \) and \( u_{j+1} \) are the same transposition. Collisions of any two branch points in a hyperelliptic curve produce degenerations of the third type.

Collisions of types 1, 2 and their generalisations explain why \( H(n, B, G, L < B) \) are used as some of the compactification varieties in the compactification of Hurwitz space.

We now describe a class of collisions of branch points generalising collisions of type 3, which will be useful in the discussion of the nonchiral theory. Let

\[
u_j = (1, 2 \cdots k, k+1)(k+2) \cdots (n)\]

and

\[
u_{j+1} = (1) \cdots (k-1)(k, k+1, \cdots, 2k)(2k+1) \cdots (n)\]

The monodromy \( u^* \) around \( \gamma^* \) is

\[
(1, 2, \cdots, k-1, k+1)(k+2, \cdots, 2k)(2k+1) \cdots (n).
\]

The product permutation has two cycles of length \( k \) and remaining cycles of length 1. Before collision the total branching number at the ramification points is \( (n - K_{u_j}) + (n - K_{u_{j+1}}) = 2k. \) After collision the branching occurs at a single point so the branching number is \( K_{u^*} - n = 2k - 2. \) But the genus of the worldsheet does not change during the collision. So there is a collapsed tube connecting the two cycles of length \( k, \) the other sheets labelled \( 2k + 1, \cdots, n \) do not participate in the collision. The fact that collision of branch points can produce tubes connecting ramification points of equal index will be used in section 11.3 (More complicated collisions can occur but do not seem to contribute to the \( YM_2 \) partition function). Note that the deficiency in total branching number can only
be even when branch points collide, which is clear geometrically. This also follows from the fact that $(-1)^{K_u-n}$ is equal to the parity of the permutation $u$.

**Remark.** Beware. Compactifications of moduli spaces are not unique. Moreover, the construction of compactifications is a tricky business. Compactifications of the base of Hurwitz space are described in [31]. The complete mathematical description of families of maps associated with collisions of arbitrary branch points is rather complicated. A compactification of Hurwitz spaces and its relation to the Deligne-Mumford compactification of moduli spaces of complex structures [32] and to the compactification of configuration spaces of [33] is discussed in detail in [28].

5. The CGTS and the space of branched coverings

In this section we make our first connection between the topology of Hurwitz space
and YM$_2$ amplitudes. Consider the CGTS (2,4). As in 2D gravity, relations to topological field theory become most transparent in the limit $A \to 0$. Accordingly in this section we will study the series

$$Z^+(0,G,N) = \sum_{n=0}^{\infty} N^n (2-2G) \sum_{s_1, t_1, \ldots, s_G, t_G \in S_n} \left[ \frac{1}{n!} \delta(\Omega_n^{2-2G} \prod_{j=1}^{G} s_j t_s^{-1} t_j^{-1}) \right].$$  \hfill (5.1)

### 5.1. Recasting the CGTS as a sum over branched coverings

The first step in rewriting (5.1) is to count the weight of a given power of $1/N$. To this end we expand the $\Omega^{-1}$ point as an element of the free algebra generated by elements of the symmetric group,

$$\Omega_n^{-1} = 1 + \sum_{k=1}^{\infty} \sum'_{v_1 \cdots v_k \in S_n} \left( \frac{1}{N} \right)^{\sum_{j=1}^{k} n - K_{v_j}} (v_1 v_2 \cdots v_k) (-1)^k.$$  \hfill (5.2)

where the primed sum means no $v_i = 1$. We could rewrite (5.1) by imposing relations of the symmetric group of $S_n$ as in (2,6). However, we decline to do this and rather substitute the expansion (5.2) into (5.1) to obtain

$$Z^+(0,G,N) = \sum_{n=0}^{\infty} \sum_{L=0}^{\infty} N^n (2-2G) \sum_{s_1, t_1, \ldots, s_G, t_G \in S_n} \sum'_{v_1 \cdots v_L \in S_n} N \sum_{j=1}^{L} (K_{v_j} - n) \left[ \frac{d(2-2G,L)}{n!} \delta(v_1 v_2 \cdots v_L \prod_{j=1}^{G} s_j t_s^{-1} t_j^{-1}) \right].$$  \hfill (5.3)

where $d(m, L)$ is defined by

$$(1 + x)^m = \sum_{L=0}^{\infty} d(m, L) x^L.$$  \hfill (5.4)

Explicitly we have

$$d(2 - 2G, L) = (-1)^L \frac{(2G + L - 3)!}{(2G - 3)!L!}, \quad \text{for} \quad G > 1$$

$$d(0, L) = 0, \text{unless} \quad L = 0$$

$$d(2, L) = 0, \text{unless} \quad L = 0, 1, 2.$$  \hfill (5.5)

For $G > 1$, $d(2 - 2G, L)$ is the number of ways of collecting $L$ objects into $2G - 2$ distinct sets. The equation (5.3) correctly gives the the partition function for any $G$ including
zero and one. For example the vanishing of $d(0, L)$ for $L > 0$ means that in the zero area limit only maps with no branch points contribute to the torus partition function. And for genus zero the vanishing of $d(2, L)$ for $L > 2$ means that only maps with no more than two branch points contribute to the CGTS for the sphere.

To each nonvanishing term in the sum (5.3) we may associate a homomorphism $\psi : F_{G,L} \to S_n$, where $F_{G,L}$ is an $F$ group (3.1), since, if the permutations $v_1, \ldots, v_L, s_1, t_1, \ldots, s_G, t_G$ in $S_n$ satisfy $v_1 \cdots v_L \prod_{i=1}^{G} s_i t_i s_i^{-1} t_i^{-1} = 1$ we may define

$$\psi : \alpha_i \to s_i \quad \psi : \beta_i \to t_i \quad \psi : \gamma_i \to v_i \quad (5.6)$$

Moreover, if there exists a $g \in S_n$ such that $g\{v_1, \ldots, v_L; s_1, t_1 \cdots s_G, t_G\}g^{-1} = \{v'_1, \cdots v'_L; s'_1, t'_1 \cdots s'_G, t'_G\}$ as ordered sets then by definition 3.2 the induced homomorphisms are equivalent. Evidently the class of $\psi$ will appear in the sum $n!/[C(\psi)]$ times in (5.3). Therefore, we may write (5.3) as

$$Z^+(0, G, N) = \sum_{n=0}^{\infty} \sum_{B=0}^{\infty} N^{n(2-2G)-B} \sum_{L=0}^{B} d(2-2G, L) \sum_{\psi \in \Psi(n, B, G, L)} \frac{1}{|C(\psi)|} \quad (5.7)$$

where $\Psi(n, B, G, L)$ is the set of equivalence classes of homomorphisms $F_{G,L} \to S_n$, with the condition that the $\gamma_i$ all map to elements of $S_n$ not equal to the identity. We have collected terms with fixed value of:

$$B \equiv \sum_{i=1}^{L} (n - K_{vi}) \quad (5.8)$$

Now we use Theorem 3.1 to rewrite the sum (5.7) as a sum over branched coverings. To do this we must make several choices. We choose a point $y_0 \in \Sigma_T$ and for each $n, B, L, \psi$ we also make a choice of:

1. some set $S$ of $L$ distinct points on $\Sigma_T$.
2. an isomorphism (3.2).

To each $\psi, S$ we may then associate a homomorphism $\pi_1(\Sigma_T - S, y_0) \to S_n$. By theorem 3.1 we see that, given a choice of $S$, to each class $[\psi]$ we associate the equivalence class of a branched covering $f \in H(n, B, G; S)$, where $f : \Sigma_W \to \Sigma_T$. The genus of the covering surface $h = h(G, n, B)$ is given by the Riemann-Hurwitz formula (3.4). Note that the power of $\frac{1}{N}$ in (5.7) is simply $2h - 2$. Finally, the centralizer $C(\psi) \subset S_n$ is isomorphic to the automorphism group of the associated branched covering map $f$. The
order of this group, $|\text{Aut}(f)|$, does not depend on the choice of points $S$ used to construct $f$. Accordingly, we can write $Z^+$ as a sum over equivalence classes of branched coverings:

$$Z^+(0, G, N) = \sum_{n=0}^{\infty} \sum_{B=0}^{\infty} \sum_{L=0}^{B} \left( \frac{1}{N} \right)^{2h-2} d(2 - 2G, L) \sum_{f \in H(n, B, G, S)} \frac{1}{|\text{Aut}(f)|}$$  \hspace{1cm} (5.9)

5.2. Euler characters

We have now expressed the CGTS as a sum over equivalence classes of branched coverings. We now interpret the weights in terms of the Euler characters of the Hurwitz space $H$.

To begin we write

$$d(2 - 2G, L) = \frac{(\chi_G)(\chi_G - 1) \cdots (\chi_G - L + 1)}{L!},$$  \hspace{1cm} (5.10)

where $\chi_G = 2 - 2G$. The RHS of (5.10) is the Euler character of the space $C_L(\Sigma_T) = C_L(\Sigma_T)/S_L$. This may be easily proved in two ways. Recall that it is general property of fibre bundles with connected base that their Euler character is the product of Euler characters of base and fibre [26][34]. Let $\mathcal{M}_{G,L}$ be the uncompactified moduli space of complex structures of a surface of genus $G$ with $L$ punctures. The fibration:

$$C_L(\Sigma_T) \rightarrow \mathcal{M}_{G,L} \rightarrow \mathcal{M}_{G,0}$$  \hspace{1cm} (5.11)

together with the celebrated Harer-Zagier-Penner formula: [35]

$$\chi(\mathcal{M}_{G,L}) = (-1)^L \frac{(2G - 3 + L)!(2G - 1)}{L!(2G)!} B_{2G}$$  \hspace{1cm} (5.12)

(where $G \geq 0, 2G - 2 + L > 0$, and $B_{2G}$ is a Bernoulli number) gives the result:

$$\chi(C_L(\Sigma_T)) = \frac{\chi(\mathcal{M}_{G,L})}{\chi(\mathcal{M}_{G,0})} = (-1)^L \frac{(2G + L - 3)!}{(2G - 3)!L!},$$  \hspace{1cm} (5.13)

An alternative proof, (which also covers the cases of interest at genus zero) uses the fibration of configuration spaces described in [36]. Let $C_{m,n}(\Sigma_T)$ be the configuration space of $n$ labelled points on a surface $\Sigma_T$ of genus $G$ with $m$ fixed punctures. There is a fibration

$$C_{L-1,1}(\Sigma_T) \rightarrow C_{0,L}(\Sigma_T) \rightarrow C_{0,L-1}(\Sigma_T).$$  \hspace{1cm} (5.14)
Using the product formula for Euler characters of a fibration we get

$$\chi(C_{0,L}(\Sigma_T)) = (2 - 2G - (L - 1))\chi(C_{0,L-1}(\Sigma_T))$$  \hspace{1cm} (5.15)

This recursion relation together with \( \chi(C_{0,1}(\Sigma_T)) = \chi(\Sigma_T) \) gives \( \chi(C_{0,L}(\Sigma_T)) = (\chi_G)(\chi_G - 1) \cdots (\chi_G - L + 1) \). But \( C_{0,L}(\Sigma_T) \) is a topological covering space of \( C_L(\Sigma_T) \) of degree \( L! \) so this leads to

$$\chi(C_L(\Sigma_T)) = d(2 - 2G, L).$$ (5.16)

Using (5.16), we can further rewrite the CGTS as

$$Z^+(0, G, N) = \sum_{n=0}^{\infty} \sum_{B=0}^{\infty} N^{n(2-2G)-B} \sum_{L=0}^{B} \chi(C_L(\Sigma_T)) \sum_{f \in H(n,B,G,S)} \frac{1}{|Autf|}$$ \hspace{1cm} (5.17)

Let us now return to the fibration (4.1). Consider first the case where the covering surface \( \Sigma_W \) has no automorphisms. From the results of [37], it follows that this will happen for primitive branched coverings of surfaces with \( G > 2 \) with \( B > n/2 \) simple branch points. In such cases we can identify

$$\chi(C_L(\Sigma_T)) \sum_{f \in H(n,B,G,S)} \frac{1}{|Autf|} = \chi(C_L(\Sigma_T))|H(n,B,G,S)|$$ \hspace{1cm} (5.18)

$$= \chi(H(n, B, G, L))$$

where we have again used the fact that the Euler character of a bundle is the product of that of the base and that of the fibre [20] (the Euler character of the fiber is \( \chi(H(n, B, G, S)) = |H(n, B, G; S)| \)).

When \( H(n, B, G, L) \) contains coverings with automorphisms the corresponding space \( \mathcal{F} \) has orbifold singularities. We introduce the orbifold Euler character \( \chi_{orb}(H) \) as the Euler character of \( \chi(\mathcal{F}) \) calculated by resolving its orbifold singularities. The division by the factor \( |Aut(f)| \) is the correct factor for calculating the orbifold Euler characteristic of the subvariety \( H(n, B, G, L) \) since \( Aut(f) \) is the local orbifold group of the corresponding point in \( \mathcal{F} \). With this understood we naturally define:

$$\chi_{orb}((H(n, B, G, L))) \equiv \chi(C_G(L)) \sum_{f \in H(n,B,G,S)} \frac{1}{|Autf|}$$ \hspace{1cm} (5.19)

in the general case. Thus we finally arrive at our first main result:
Proposition 5.1. The CGTS is the generating functional for the orbifold Euler characters of the Hurwitz spaces:

$$Z^+(0, G, N) = \sum_{n=0}^{\infty} \sum_{B=0}^{\infty} \left( \frac{1}{N} \right)^{2h-2} \sum_{L=0}^{B} \chi_{\text{orb}}(H(n, B, G, L))$$  \hspace{1cm} (5.20)

where $h$ is determined from $n, G$ and $B$ via the Riemann-Hurwitz theorem.

The $L = B$ contribution in the sum is the Euler character of the space of generic branched coverings. As described in section 4.4 compactification of this space involves addition of boundaries corresponding to the space of maps with higher branch points, i.e., where $L < B$.

Quite generally, suppose $X$ is a closed manifold with boundary $\partial X$. The inclusion $\partial X \hookrightarrow X$ gives rise to a long exact sequence in homology

$$\cdots \rightarrow H_i(\partial X) \rightarrow H_i(X) \rightarrow H_i(X, \partial X) \rightarrow H_{i-1}(\partial X) \rightarrow \cdots$$  \hspace{1cm} (5.21)

By Lefschetz duality we may write the relative cohomology groups in terms of the homology of the interior $X^0$:

$$H^i(X, \partial X) = H_{n-i}(X^0),$$  \hspace{1cm} (5.22)

where $n$ is the dimension of $X$. If $n$ is even then $\chi(X, \partial X) = \chi(X^0)$. Applying the above discussion to Hurwitz space we see that we can interpret

$$\sum_{L=0}^{B} \chi_{\text{orb}}(H(n, B, G, L))$$

as the Euler character of a partially compactified Hurwitz space $(H(h, B, G))$ obtaining by adding degenerations of type 1 and 2 and their generalizations.

Proposition 5.2: The CGTS is the generating functional for the orbifold Euler characters of the analytically compactified Hurwitz spaces:

$$Z^+(0, G, N) = \sum_{n=0}^{\infty} \sum_{B=0}^{\infty} \left( \frac{1}{N} \right)^{2h-2} \chi_{\text{orb}}((H(n, B, G)))$$

$$= \exp \left[ \sum_{n=0}^{\infty} \left( \frac{1}{N} \right)^{2h-2} \chi_{\text{orb}}((H(h, G))) \right]$$  \hspace{1cm} (5.23)

\[6\] We thank E. Getzler for these clarifying remarks.
Remarks:

1. Recall that we allow for the possibility of disconnected worldsheets. Expressing the result in terms of connected coverings leads to the final equation.

2. The importance of the high-codimension compactification varieties in the $YM_2$ partition sum (Hurwitz spaces with $L < B - 1$) is extremely intriguing from the point of view of the topological field theory discussed in the remainder of this paper. $YM_2$ appears to be an example where the “higher contact terms” of the topological field theory are extremely important in getting the correct answer. This will be a recurring theme throughout this paper.

3. There is more than one way to interpret the expansion $Z^+$ of (5.1) as a sum over maps. The ambiguity arises from the treatment of the $\Omega^{-1}$ terms. Had we used the relations of the symmetric group in writing a formula for $\Omega^{-1}$ we would have found that the coefficient of any permutation $v$ multiplies an infinite series $(1/N)^{n-K_v} + \ldots$. All powers in the series, but the leading one, would be too large to be accounted for by branching alone. We could still describe the $YM_2$ partition function in terms of maps but we would need to invoke the pinch maps of section 3.3. As an example, consider the expansion of $\Omega^{-1}$ in (2.6). In the first line we have written the inverse omega point as a sum where $i$ transpositions come with a factor $N^{-i}$. Interpreting each factor $v$ as the data of some branch point leads to a description in terms of branched covers. Using the relations of the symmetric group we obtain the last line of (2.6). The higher powers of $1/N$ must be accounted for by collapsed handles, for example, by pinch maps. The advantage of excluding pinch maps and associating each term in the CGTS with branched coverings is that, as in sec. 4.2, when the target is equipped with a complex structure such maps can always be interpreted as holomorphic or antiholomorphic. This is an encouraging sign because topological sigma models count (anti)-holomorphic maps [39]. This remark is the first step on the road to the construction of the equivalent string theory in section 6.

4. The paper of Gross and Taylor already showed that by expanding $\Omega^{-1}$ one could interpret all contributions to $Z^+$ in terms of branched covers. However, in the picture of [15] there are $|2 - 2G|$ special points: “twist points,” on the target space, and one imagines that all the branch points $v_1 \cdots v_L$ are somehow “anchored” to these special twist points, for all values of $L$. In this paper we allow the branch points to “sail” over the entire target space $\Sigma_T$. From the latter point of view the combinatorial factors $d(2 - 2G, L)$ are more natural.
6. Synopsis of Topological Field Theory

In this section we briefly review topological field theory. A number of reviews of this subject already exist\[40,41,42,43,44,45\] and it is to these that we refer the interested reader for more detail and further references.

Topological Field Theories (TFT) study the topology of moduli spaces. There are usually a number of different descriptions of a moduli space, \(\mathcal{M}\). For the purpose of formulating a TFT, it is convenient to characterize \(\mathcal{M}\) as a subspace within a \(\mathcal{G}\)-manifold\[7\], \(\tilde{\mathcal{C}}\):

\[
\mathcal{M} = \tilde{\mathcal{M}}/\mathcal{G} \\
\tilde{\mathcal{M}} = \{\varphi \in \tilde{\mathcal{C}} \mid s(\varphi) = 0\}
\]

Here \(s\) is a \(\mathcal{G}\)-equivariant map between \(\tilde{\mathcal{C}}\) and an auxiliary \(\mathcal{G}\)-manifold, \(\tilde{\mathcal{V}}\):

\[
s: \tilde{\mathcal{C}} \rightarrow \tilde{\mathcal{V}}.
\]

The precise natures of the map, \(s\), and the auxiliary \(\mathcal{G}\)-manifold, \(\tilde{\mathcal{V}}\), depend on the moduli space under consideration.

Remarks:

6.1.1. We may regard 

\[
\begin{array}{ccc}
\tilde{\mathcal{E}} & \leftarrow & \tilde{\mathcal{V}} \\
\downarrow^{\pi} & & \\
\tilde{\mathcal{C}} & & 
\end{array}
\]

as an equivariant \(\mathcal{G}\)-bundle which may be either trivial (as in case of topological Yang-Mills theory\[46\]) or non-trivial (as in case of topological sigma models\[39\]). \(s\) induces a section, \(\tilde{s}\), of \(\tilde{\mathcal{E}}\).

6.1.2. The action of \(\mathcal{G}\) on \(\tilde{\mathcal{E}}\) may fail to be free, in which case the quotient \(\tilde{\mathcal{E}}/\mathcal{G}\) is not a manifold. For example, let \(\tilde{\mathcal{E}} = \mathcal{A}\) be the space of gauge connections on a principal \(G\)-bundle, \(P\). The group of gauge transformations acts on \(\mathcal{A}\). For a reducible connection, \(A \in \mathcal{A}\), \(\mathcal{G}\) possesses a non-trivial isotropy group.

6.1.3 \(\mathcal{G}\) may, of course, be trivial, in which case \(\mathcal{M} = \tilde{\mathcal{M}}\). This is the case for the topological sigma model\[39\].

Since the quotient space, \(\tilde{\mathcal{E}}/\mathcal{G}\), is potentially a singular space, we cannot in general study its cohomology directly. Instead one must examine the \(\mathcal{G}\)-equivariant cohomology of \(\tilde{\mathcal{E}}\).

---

\[7\] Recall that for any group \(\mathcal{G}\), a \(\mathcal{G}\)-manifold is a manifold on which the action of \(\mathcal{G}\) is defined at every point. In the context of TFTs, \(\tilde{\mathcal{C}}\) and \(\mathcal{G}\) are typically infinite dimensional. We will remain formal and largely ignore this fact in our discussion.
6.1. Topological Description of Equivariant Cohomology

A sketch of equivariant cohomology necessitates a brief discussion of universal bundles. Imagine that we are given a contractible $G$-manifold, $X$, on which $G$ acts freely, then

1. Since $EG$ is contractible, $\tilde{E}$ and $\tilde{E} \times X$ have the same homotopy type and thus have identical de Rham cohomology.

2. Since $G$ acts freely on $X$, $\tilde{E} \times X$ inherits a free $G$ action:

$$g \cdot (e, x) = (ge, xg^{-1}). \quad (6.1)$$

Then the quotient

$$\tilde{E} \times_G X \overset{\text{def}}{=} \frac{\tilde{E} \times X}{G} \quad (6.2)$$

defines a manifold.

The case when $X$ is a principal $G$-bundle is of particular importance. Such spaces are to a large degree unique:

**Definition 6.1.4:** To a compact, finite dimensional group, $G$ we can associate a universal $G$-bundle, $EG$,

$$EG \overset{\pi_G}{\longrightarrow} G \quad (6.3)$$

$EG$ is a contractible principal $G$-bundle over the classifying space, $BG = EG/G$. $EG$ is unique up to equivariant homotopy type, while $BG$ is unique up to homotopy type.

Note that if $G$ acts freely on $\tilde{E}$, then $\tilde{E} \times G$ and $\tilde{E}/G$ have the same homotopy type and therefore have identical de Rham cohomologies. More generally,

$$\tilde{E} \times EG \overset{\text{def}}{=} \frac{\tilde{E} \times EG}{\pi_G} \quad (6.4)$$

may be viewed as a $G$-bundle over $BG$. These properties motivate the following

**Definition 6.1.5:** The $G$-equivariant cohomology of $\tilde{E}$ is defined to be

$$H_{G, \text{top}}^\bullet(\tilde{E}) \overset{\text{def}}{=} H^\bullet \left( \Omega(\tilde{E} \times G EG), d \right). \quad (6.5)$$

Though this definition offers a perfectly sensible way in which to define the topology of $\tilde{E}/G$, the manifold $\tilde{E} \times G EG$ is generally quite difficult to study directly. In certain cases,
finite dimensional and compact, the cohomology of this space may be studied very effectively via indirect means.

The pullback by the projection, \( \tilde{\pi}_G \), induces an injective homomorphism:

\[
\tilde{\pi}_G^*: \Omega^\bullet(\tilde{\mathcal{E}} \times_G E\mathcal{G}) \longrightarrow \Omega^\bullet(\tilde{\mathcal{E}} \times E\mathcal{G})
\]  

(6.6)

Since \( \tilde{\mathcal{E}} \times E\mathcal{G} \) is topologically a simpler space, it is very useful to characterize \( \tilde{\pi}_G^* H^\bullet(\tilde{\mathcal{E}} \times G E\mathcal{G}) \) as a subset of \( H^\bullet(\tilde{\mathcal{E}} \times E\mathcal{G}) \). Since \( \mathcal{G} \) acts freely on \( \tilde{\mathcal{E}} \times E\mathcal{G} \), there is at each point \( x \in \tilde{\mathcal{E}} \times E\mathcal{G} \) a map of \( \mathcal{G} \) into the fiber over \( y = \tilde{\pi}_G(x) \),

\[
R_x: \mathcal{G} \longrightarrow \tilde{\mathcal{E}} \times E\mathcal{G}|_y.
\]  

(6.7)

where \( \tilde{\mathcal{E}} \times E\mathcal{G}|_y \overset{\text{def}}{=} \pi^{-1}_G(y) \). The differential of \( R_x \) defines a map from \( \mathfrak{g} = \text{Lie } \mathcal{G} \) into the vertical tangent space

\[
C_x \overset{\text{def}}{=} dR_x: \mathfrak{g} \longrightarrow T_x(\tilde{\mathcal{E}} \times E\mathcal{G})^{\text{vert}}
\]

\[
C_x: \gamma \overset{\text{def}}{=} X_\gamma|_x
\]

(6.8)

This defines two actions of \( \mathfrak{g} \) on \( \Omega^\bullet(\tilde{\mathcal{E}} \times E\mathcal{G}) \):

1. **Contraction**: For all \( g \in \mathfrak{g} \),

\[
i(g): \Omega^k(\tilde{\mathcal{E}} \times E\mathcal{G}) \longrightarrow \Omega^{k-1}(\tilde{\mathcal{E}} \times E\mathcal{G})
\]

\[
i(g): \omega \longmapsto i_{X_g}\omega
\]

where \( i_{X_g} \) is the usual interior product with a vector field \( X_g \).

2. **Lie Derivative**: For all \( g \in \mathfrak{g} \),

\[
\mathcal{L}(g): \Omega^k(\tilde{\mathcal{E}} \times E\mathcal{G}) \longrightarrow \Omega^k(\tilde{\mathcal{E}} \times E\mathcal{G})
\]

\[
\mathcal{L}(g): \omega \longmapsto \mathcal{L}_{X_g}\omega
\]

where \( \mathcal{L}_{X_g} \equiv [i_{X_g}, d]_+ \) is the usual Lie derivative with the vector field \( X_g \).

These derivations characterize a special subcomplex of \( \Omega^\bullet(\tilde{\mathcal{E}} \times E\mathcal{G}) \).

**Definition 6.1.6**: Let \( \{ T_i \} \) be a basis for \( \mathfrak{g} \). A form \( \eta \in \Omega^\bullet(\tilde{\mathcal{E}} \times E\mathcal{G}) \) is called \( \mathfrak{g} \)-basic, if it is both

1. **Horizontal**, i.e. \( \eta \in \bigcap_{i=1}^{\dim \mathcal{G}} \ker i(T_i) \), and
2. **Invariant**, i.e. \( \eta \in \bigcap_{i=1}^{\dim \mathcal{G}} \ker \mathcal{L}(T_i) \).

The \( \mathfrak{g} \)-basic subcomplex will be denoted by \( \Omega(\tilde{\mathcal{E}} \times E\mathcal{G})^\mathfrak{g}_{\text{basic}} \).
**Theorem 6.1.7:** In the case that $\mathcal{G}$ is finite dimensional and compact, the de Rham cohomology of $\tilde{\mathcal{E}} \times_{\mathcal{G}} E\mathcal{G}$ is precisely the basic cohomology of $\tilde{\mathcal{E}} \times E\mathcal{G}$.

**Proof:** See Mathai and Quillen [17]

It follows from Definition 6.1.5 and Theorem 6.1.7, that the $\mathcal{G}$-equivariant cohomology of $\tilde{\mathcal{E}}$ can be computed as follows:

\[
H_{\mathcal{G}\text{-top}}^*(\tilde{\mathcal{E}}) \cong H^*\left(\Omega(\tilde{\mathcal{E}} \times_{\mathcal{G}} E\mathcal{G}), d\right) \\
\cong H^*\left(\Omega(\tilde{\mathcal{E}} \times E\mathcal{G})_{\mathcal{G}\text{-basic}}, d\right)
\]

(6.9)

**Remarks:**

6.1.8 The local gauge groups, $\mathcal{G}$, encountered in TFT are neither finite dimensional nor compact. Nevertheless the cohomologies found are remarkably similar to those of related compact groups.

6.1.9 If $\tilde{\mathcal{E}} \rightarrow_{\rho} \tilde{\mathcal{C}}$ is a non-trivial bundle with standard fiber $\tilde{\mathcal{V}}$, and fiber metric, $(\cdot, \cdot)_{\tilde{\mathcal{V}}}$, then $\tilde{\mathcal{E}}$ is associated to a principal $SO(\tilde{\mathcal{V}})$ bundle, $\tilde{\mathcal{F}}$, the bundle of all orthonormal frames on $\tilde{\mathcal{E}}$:

\[
\tilde{\mathcal{E}} = \tilde{\mathcal{F}} \times_{SO(\tilde{\mathcal{V}})} \tilde{\mathcal{V}}
\]

(6.10)

It is then convenient to express the cohomology of $\tilde{\mathcal{E}}$ in terms of the basic cohomology of $\tilde{\mathcal{F}} \times \tilde{\mathcal{V}}$, i.e. from Theorem 6.1.7

\[
H^*\left(\Omega(\tilde{\mathcal{E}}), d\right) \cong H^*\left(\Omega(\tilde{\mathcal{F}} \times_{SO(\tilde{\mathcal{V}})} \tilde{\mathcal{V}}), d\right) \\
\cong H^*\left(\Omega(\tilde{\mathcal{F}} \times \tilde{\mathcal{V}})_{so(\tilde{\mathcal{V}})\text{-basic}}, d\right)
\]

(6.11)

Note also that in this case the $\mathcal{G}$-equivariant cohomology of $\tilde{\mathcal{E}}$ is given by

\[
H_{\mathcal{G}\text{-top}}^*(\tilde{\mathcal{E}}) = H^*\left(\Omega(\tilde{\mathcal{F}} \times \tilde{\mathcal{V}} \times E\mathcal{G})_{\mathbf{g} \oplus so(\tilde{\mathcal{V}})\text{-basic}}, d\right)
\]

(6.12)

so that effectively there are two gauge groups $SO(\tilde{\mathcal{V}})$ and $\mathcal{G}$.

6.2. **Algebraic Description of Equivariant Cohomology**

There are also algebraic models for the $\mathcal{G}$-equivariant cohomology of $\tilde{\mathcal{E}}$. These are far from unique, a fact which partially accounts for the abundance of TFT for a given moduli space. For reasons of brevity we shall direct most of our attention to the *Cartan model*. As
was shown by Kalkman \cite{21}, this model is closely related to the BRST model which is often used in physics. We shall discuss other models only briefly; we refer the interested reader to \cite{21,45,47} for a fuller description of the various algebraic models and their equivalence to one another.

The Cartan model proceeds from the complex

\[ S^\bullet(\mathfrak{g}^*) \otimes \Omega^\bullet(\tilde{E}) \]  

(6.13)

where \( \mathfrak{g}^* \) is the dual to \( \mathfrak{g} \) and \( S^\bullet(\mathfrak{g}^*) \) is the symmetric algebra on \( \mathfrak{g}^* \), which is freely generated by \( \{ \phi^i \}_{i=1,...,\dim \mathfrak{g}} \). A differential, \( d_C \), may be defined via its action on the generators of the complex

\[
\begin{align*}
    d_C \phi^i &= 0 & \forall \phi^i & \in S^2(\mathfrak{g}^*) \\
    d_C \omega &= (1 \otimes d - \phi^i \otimes i(T_i))\omega & \forall \omega & \in \Omega^\bullet(\tilde{C})
\end{align*}
\]

(6.14)

In analogy to the geometric Lie derivative, one may define an algebraic Lie derivative on \( S(\mathfrak{g}^*) \otimes \Omega^\bullet(\tilde{E}) \) to be \( L_i \overset{\text{def}}{=} [1 \otimes i(T_i), d_C]_+ \). Note that \( d_C^2 = -\phi^i \otimes \mathcal{L}(T_i) = \phi^i L_i \otimes 1 \); so that \( d_C \) is nilpotent only on the subcomplex of \textit{equivariant differential forms}, defined as

\[
\Omega^g(\tilde{E}) \overset{\text{def}}{=} \left( S^\bullet(\mathfrak{g}^*) \otimes \Omega^\bullet(\tilde{E}) \right)^G
\]

(6.15)

where the superscript \((\cdot)^G\) denotes the \( G \)-invariant subcomplex. That is \( \eta \in S(\mathfrak{g}^*) \otimes \Omega^\bullet(\tilde{E}) \) is an equivariant differential form iff \( \eta \in \bigcap_{i=1}^{\dim G} \ker L_i \). This subcomplex corresponds to the basic subcomplex \cite{21,45}. This motivates the following

**Definition 6.2.1:** The algebraic definition of the \( G \)-equivariant cohomology of \( \tilde{E} \) is

\[
H^\bullet_{\mathcal{G},\text{alg}}(\tilde{E}) \overset{\text{def}}{=} H^\bullet \left( \Omega^g(\tilde{E}), d_C \right)
\]

(6.16)

**Theorem 6.2.2:** For \( G \) finite dimensional and compact:

\[
H^\bullet_{\mathcal{G},\text{top}}(\tilde{E}) \cong H^\bullet_{\mathcal{G},\text{alg}}(\tilde{E})
\]

(6.17)

**Proof:** Please see \cite{47,21,45}.

The relationship between the de Rham and Cartan models can be made very concrete. Given a connection, \( \nabla \), and its curvature, \( F_{\nabla} \), we may define the Chern-Weil homomorphism\(^8\), \( w_{\nabla,\nabla} \):

\[
\begin{align*}
    w_{\nabla,\nabla} : \Omega^g(\tilde{E}) & \longrightarrow \Omega^\bullet(\tilde{E})_{\mathfrak{g}-\text{basic}} \\
    w_{\nabla,\nabla} : \mathcal{P}(\phi) & \longrightarrow (\mathcal{P}(F_{\nabla}))^{\text{hor}}
\end{align*}
\]

(6.18)

The superscript \((\cdots)^{\text{hor}}\) indicates projection onto the horizontal subcomplex via \( \nabla \).

\(^8\) The subscript \( \mathcal{C} \) indicates that this homomorphism acts on the Cartan complex. We will in section 6.3.2 define an analogous homomorphism for the Weil model.
6.3. The Mathai-Quillen Representative of the Thom Class

The Thom class is central to the construction of TFT actions\[20\]. In this subsection we shall outline constructions of representatives of this class in the context of various models of equivariant cohomology.

Let \( \tilde{E} \) be an orientable vector bundle

\[
\begin{array}{c}
\tilde{E} \\
\tilde{\pi} \\
\tilde{C}
\end{array}
\begin{array}{c}
\downarrow \\
\tilde{\nabla}
\end{array}
\begin{array}{c}
\tilde{V}
\end{array}
\]

We consider \( H^\bullet_{\text{vrd}}(\tilde{E}) \), the cohomology of forms that are rapidly decreasing in the vertical direction\[9\]. On such forms integration along the fiber is well-defined:

\[
\tilde{\pi}_*: \Omega^{\bullet + \text{rank} \tilde{E}}_{\text{vrd}}(\tilde{E}) \to \Omega^\bullet(\tilde{C}) \quad (6.19)
\]

In fact, the cohomologies of these two complexes are isomorphic.

**Theorem 6.3.1: [Thom Isomorphism]** Integration along the fiber defines an isomorphism

\[
\tilde{\pi}_*: H^\bullet_{\text{vrd}}^{\bullet + \text{rank} \tilde{E}}(\tilde{E}) \cong H^\bullet(\tilde{C}) \quad (6.20)
\]

**Proof:** Please see Bott and Tu\[34\].

The **Thom class**, of \( \tilde{E} \) is defined as

\[
\left[ \Phi(\tilde{E}) \right] \overset{\text{def}}{=} (\tilde{\pi}_*)^{-1}(1) \in H^\text{rank}_{\text{vrd}}(\tilde{E}) \quad (6.21)
\]

A **Thom form**, \( \Phi_\nabla(\tilde{E}) \), i.e. a particular representative of the Thom class, will in general depend on a connection, \( \nabla \), on \( \tilde{E} \). In terms of such a \( \Phi_\nabla(\tilde{E}) \), the Thom isomorphism is explicitly given by:

\[
\mathcal{T}: H^\bullet(\tilde{C}) \to H^\bullet_{\text{vrd}}^{\bullet + \text{rank} \tilde{E}}(\tilde{E})
\]

\[
\mathcal{T}(\omega) \mapsto \tilde{\pi}^*(\omega) \wedge \Phi_\nabla(\tilde{E})
\]

From remark 6.1.9 we know that \( \tilde{E} \) is associated to a principal \( SO(\tilde{V}) \) bundle, \( \tilde{F} \). A Thom form may therefore be constructed in the context of an algebraic model of \( SO(\tilde{V}) \)-equivariant cohomology. An element\[10\] \( U_{\text{Cartan}} \) is called a **universal Thom form**, if it is

---

\[9\] This is equivalent to \( H^\bullet_{\text{cv}}(\tilde{E}) \), the cohomology of forms that are compactly supported along the (vertical) fiber direction\[47\].

\[10\] The subscript \( \text{Cartan} \) indicates that we are working within the Cartan model for \( SO(\tilde{V}) \).
related via the Chern-Weil homomorphism to a Thom form. The relevant complexes fit together as indicated in the diagram below:

\[
\left( S(so(\tilde{V})^*) \otimes \Omega^*(\tilde{V}) \right)^{SO(\tilde{V})} \xrightarrow{w_{c,\nabla}} \Omega^* \left( \tilde{V} \times E \ SO(\tilde{V}) \right)_{SO(\tilde{V})-\text{basic}} \]

and the various forms are related as follows

\[
U_{c-SO(\tilde{V})} \xrightarrow{w_{c,\nabla}} w_{c,\nabla}(U_{c-SO(\tilde{V})}) = \tilde{\pi}_{SO(\tilde{V})}^* (\Phi_{\nabla}(\tilde{E})) \]

\[
\Phi_{\nabla}(\tilde{E}) = \tilde{\omega}_{c,\nabla} \left( U_{c-SO(\tilde{V})} \right) \]

Diagram 6.3.2

Mathai and Quillen\[47\] constructed an explicit representative of the universal Thom class. Let \(x^i\) be orthonormal coordinates for \(\tilde{V}\) and introduce anticommuting orthonormal coordinates, \(\rho_i\), for \(\Pi\tilde{V}^*\). Here \(\Pi\) is the parity change functor\[48\] and indicates that we are to regard the coordinates of \(\Pi\tilde{V}^*\) as being anti-commuting. Then a representative of the universal Thom form may be written as

\[
U_{c-SO(\tilde{V})} = \left( \frac{1}{4\pi t} \right)^{\frac{1}{2} \text{rank } \tilde{E}} \int_{\Pi\tilde{V}^*} d\rho \ \exp \left\{ -\frac{1}{4t} (x, x)_{\tilde{V}} + i\langle \rho, dx \rangle + t\langle \rho, \phi \rho \rangle_{\tilde{V}^*} \right\} \quad (6.22) \]

where \((\cdot, \cdot)_X\) denotes the inner product on \(X\), while \(\langle \cdot, \cdot \rangle\) denotes the dual pairing. Note that we actually have a one parameter family of representatives that depend on \(t \in \mathbb{R}\).

Remarks:

6.3.4. We are often interested in constructing the Thom class of \(\tilde{E} \times_G E\mathcal{G} \to \tilde{C} \times_G E\mathcal{G}\). From Remark 6.1.9 we know that in this case we need to consider \(SO(\tilde{V}) \times \mathcal{G}\)-equivariant cohomology. In the following we shall construct a number of other universal Thom forms; we shall leave the total local symmetry group unspecified, to allow for the possibility of non-trivial \(\mathcal{G}\).

6.3.5. Mathai-Quillen representatives of Thom classes play a central role in the construction of topological field theories\[20\]. The argument of the exponential in the

32
Mathai-Quillen representative is interpreted as the action of a TFT. For example, the term $\langle \rho, dx \rangle$ is the kinetic term for the ghost/anti-ghost system.

6.3.6. $\ker \nabla s$ are the ghost zero modes, while $\coker \nabla s$ are the anti-ghost zero modes. When $\coker \nabla s \neq 0$, the Grassmann integral over the anti-ghosts in the Mathai-Quillen representative of the Thom class brings down powers of the curvature from the argument of the exponential. This fact will be made precise in the localization theorems.

The main use of Thom classes stems from the following

**Proposition 6.3.7:** The pullback of the Thom class, $\Phi_{\nabla}(\tilde{E})$, by any section, $\tilde{s}: \tilde{C} \to \tilde{E}$, is the Euler class of $\tilde{E}$.

*Proof:* Please see [34,45].

We conclude this subsection by briefly indicating a few other representatives of the Thom class. Please see [45] for more details.

6.3.1. Another Representative of the Universal Thom Class in the Cartan Model

If we introduce further commuting coordinates $\pi$ for $\tilde{V}^*$, and extend the Cartan differential to $S^\bullet(g^*) \otimes \Omega^\bullet(\tilde{V}) \otimes \Omega^\bullet(\Pi\tilde{V}^*)$, via

$$
Q_C = 1 \otimes d \otimes 1 + 1 \otimes 1 \otimes d - \phi^i \otimes i(T_i) \otimes 1 - \phi^i \otimes 1 \otimes i(T_i)
$$

then we may compactly write

$$
U_{C-\mathcal{G}} = \left( \frac{1}{2\pi} \right)^{\text{rank } \mathcal{E}} \int_{\tilde{V}^* \times \Pi\tilde{V}^*} d\pi \, d\rho \, \exp -Q_C \left( -i \langle \rho, x \rangle - t(\rho, \pi) \tilde{\gamma}^* \right) \quad (6.24)
$$

The significant feature of this formulation is that the argument of the exponential (the TFT action) is $Q_C$-exact.

11 Notation: Conventions and the paucity of alphabets force us to use $\pi$ for both the Lagrange multiplier fields and the projections. Projections between (in general) infinite dimensional spaces will have a tilde; projections between (in general) finite dimensional spaces have a bar. Lagrange multiplier fields have neither accent.
6.3.2. A Representative of the Universal Thom Class in the Weil Model

The Weil model of the $\mathcal{G}$-equivariant cohomology of $\tilde{E}$ starts from the complex:

$$ W(\mathfrak{g}) = \Lambda(\mathfrak{g}^*) \otimes S(\mathfrak{g}^*) $$

where $\Lambda(\mathfrak{g}^*)$ is the exterior algebra of $\mathfrak{g}^*$ which is freely generated by $\{\theta^i\}_{i=1,\ldots,\dim \mathfrak{g}}$. The differential of the Weil complex need not concern us here. A fuller discussion may be found in [45,21,47].

The universal Thom form within the Weil model is by:

$$ U_{\mathcal{W}-\mathcal{G}} = \frac{1}{\pi^{\frac{\dim \mathcal{E}}{2}}} e^{-(x,x)_{\mathcal{V}}} \int_{\Pi_{\mathcal{V}*}} d\rho \exp \left\{ \frac{1}{4}(\rho,\phi\rho)_{\mathcal{V}}, + i\langle \nabla x, \rho \rangle \right\} $$

(6.25)

where $\nabla x = dx + \theta \cdot x$.

In the context of the Weil model, the Chern-Weil homomorphism, $w_{\mathcal{W},\mathcal{V}}$, simply makes the replacement $\begin{pmatrix} \theta \\ \phi \end{pmatrix} \rightarrow \begin{pmatrix} A \\ F_A \end{pmatrix}$, where $A$ is a connection on $\tilde{E}$ and $F_A$ is its curvature. The absence of horizontal projection is often convenient. On the other hand the Chern-Weil homomorphism explicitly introduces a connection which in many situations is non-local. In these cases the Weil model is unsuitable for the construction of TFT actions.

6.3.3. A Representative of the Universal Thom Class in Hybrid Cartan and Weil Models

To construct a universal Thom form on $\tilde{E} \times E\mathcal{G} \rightarrow \tilde{E} \times \mathcal{G} E\mathcal{G}$, we know from Remark 6.1.9 that it is useful to work within $SO(\tilde{V}) \times \mathcal{G}$-equivariant cohomology. It is useful to use different algebraic models for the equivariant cohomologies of these groups. Since the $SO(\tilde{V})$ connection is local, it is convenient to work in the context of the Weil model, thereby explicitly introducing the connection, but obviating the horizontal projection. On the other hand, the $\mathcal{G}$-connection is generally non-local, so that for purposes of constructing a TFT action, it is essential that we work in the context of the Cartan model for $\mathcal{G}$. The diagram depicting the interrelation of the various complexes is the natural generalization of Diagram 6.3.2.
Diagram 6.3.8

The universal Thom form, $U_{W-SO(\tilde{V}),c-G}$, takes its values in the complex $(S^{*}(g^{*}) \otimes (W(so(\tilde{V}))) \otimes \Omega(\tilde{V}))_{so(\tilde{V})-basic}$\(^G\). Applying the Chern-Weil homomorphism we obtain

$$w_{W,\nabla_{SO(\tilde{V})}}(U_{W-SO(\tilde{V}),c-G}) = \pi^{*}_{SO(\tilde{V})}(\Upsilon_{c-G})$$

where $\Upsilon_{c-G} \in \Omega_{G}(E)$ will play an important role in the general localization theorem (Proposition 6.5.4). Altogether, we have

$$U_{W-SO(\tilde{V}),c-G} \longrightarrow w_{W,\nabla_{SO(\tilde{V})}}(U_{W-SO(\tilde{V}),c-G}) = \pi^{*}_{SO(\tilde{V})}(\Upsilon_{c-G})$$

Explicitly

$$\Upsilon_{c-G} = \int_{\tilde{V}^* \times \Pi \tilde{V}^*} d\pi \, dp \, \exp \, -Q_{c} \Psi_{Loc} \quad (6.26)$$

where $Q_{c}$ is the Cartan differential for $G$-equivariant cohomology of $\tilde{E}$, analogous to $(6.23)$; and where

$$\Psi_{Loc} = -i \langle \rho, x \rangle + t(\rho, \Gamma_{SO(\tilde{V})} \cdot \rho)_{\tilde{V}^*} - t(\rho, \pi)_{\tilde{V}^*} \quad (6.27)$$

6.4. The Localization Formula for trivial $G$

We shall now apply the construction of Thom classes to sketch a localization formula for the simpler case when $G$ is trivial. Consider an orientable vector bundle

$$\tilde{E} \xrightarrow{\pi} \tilde{V}$$

Let $\tilde{s}: \tilde{C} \to \tilde{E}$ be a section. In terms of a local trivialization, we may write this as

$$\tilde{s} = (id, s) \quad \text{where} \quad s: \tilde{C} \to \tilde{V}$$

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The subspace of interest is characterized by
\[ \tilde{M} = \{ \varphi \in \tilde{C} \mid s(\varphi) = 0 \}. \] (6.28)

For every \( \varphi \in \tilde{C} \), the differential of \( s \) is a map
\[ ds|_{\varphi}: T_{\varphi}\tilde{C} \to T_{s(\varphi)}\tilde{V}. \] (6.29)

Actually, since \( \tilde{V} \) is a linear space, \( T\tilde{V} \cong \tilde{V} \) and we may view \( ds|_{\varphi} \) as a linear operator:
\[ ds|_{\varphi}: T_{\varphi}\tilde{C} \to \tilde{V} \]

Clearly
\[ \ker ds|_{\varphi} = \ker \nabla s|_{\varphi} \subset T_{\varphi}\tilde{M} \quad \forall \varphi \in \tilde{M} \]
Moreover, if \( ds|_{\varphi} \) is injective, then
\[ \ker \nabla s|_{\varphi} \cong T_{\varphi}\tilde{M} \quad \forall \varphi \in \tilde{M}. \] (6.30)

It is also clear that
\[ \text{Im} \ ds|_{\varphi} = \text{Im} \ \nabla s|_{\varphi} \quad \forall \varphi \in \tilde{M} \]

Now consider the exact sequence of bundles over \( \tilde{M} \):
\[ 0 \to \text{Im} \ \nabla s \to \tilde{E} \to \text{coker} \ \nabla s \to 0 \] (6.31)

**Proposition 6.4.1**: Let \( \tilde{s} = (\text{id}, s) \) be a section of the orientable vector bundle \( \tilde{E} \) as above. Let \( i \) denote the inclusion, \( i: \tilde{M} \to \tilde{C} \). If \( P \subset \tilde{M} \) is the Poincaré dual to \( e(\text{coker} \ \nabla s \to \tilde{M}) \), then \( i(P) \subset \tilde{C} \) is Poincaré dual to \( e(\tilde{E} \to \tilde{C}) = \tilde{s}^* \Phi_{\nabla}(\tilde{E}) \) in \( \tilde{C} \).

**Proof**: For a physical proof, please see [15, 50].

It follows from Propositions 6.3.7 and 6.4.1 that

**Proposition 6.4.2**: For \( \tilde{O} \in H^{\text{index}} \nabla s(\tilde{C}) \), we have
\[ \int_{\tilde{C}} \tilde{s}^* \Phi_{\nabla}(\tilde{E}) \wedge \tilde{O} = \int_{\tilde{M}} e(\text{coker} \ \nabla s \to \tilde{M}) \wedge i^* \tilde{O} \] (6.32)

where we have used the fact that for trivial \( \tilde{G}, \tilde{M} = \tilde{M} \).

**Proof**: For a discussion, please see section 11.10.3 of [15].

Proposition 6.4.2 is the basis for the construction of TFTs without local symmetries. There are a number of applications of this localization theorem to super quantum mechanics and topological sigma models. For a survey of such applications as well as a more extensive list of references, please see [15].

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6.5. The Localization Formula for Non-trivial $G$

In order to give a description of $YM_2$ as a topological string theory, we need to consider a more general localization theorem with $G$ non-trivial. Let $\tilde{E}$ be an orientable $G$-equivariant vector bundle

$$\tilde{E} \leftarrow \tilde{V}
\tilde{\pi}$$

and $\tilde{s}: \tilde{C} \rightarrow \tilde{E}$ a $G$-equivariant section. Then $\tilde{s}$ induces a section, $\bar{s}$:

$$\tilde{E} \times EG \leftarrow \tilde{C} \times EG
\tilde{\pi}_G
\bar{s}$$

$$\tilde{E} \times \bar{G}EG \leftarrow \tilde{C} \times \bar{G}EG
\bar{\pi}_G$$

These maps induce pullback maps between the corresponding de Rham complexes:

$$\Omega^\bullet(\tilde{E} \times EG) \xrightarrow{\tilde{s}^*} \Omega^\bullet(\tilde{C} \times EG)
\tilde{\pi}_G^*$$

$$\Omega^\bullet(\tilde{E} \times \bar{G}EG) \xrightarrow{\bar{s}^*} \Omega(\tilde{C} \times \bar{G}EG)
\bar{\pi}_G^*$$

Though $\tilde{E} \times EG \rightarrow \tilde{E} \times \bar{G}EG$ is a principal $G$-bundle, we may define an analogue of the Thom isomorphism for vector bundles

$$T_G: H^\bullet(\tilde{E}) \rightarrow H^{\bullet+\dim \bar{G}}(\tilde{E})
(6.33)$$

$$T_G(\omega) \mapsto \tilde{\pi}_G^*(\omega) \wedge \Phi_G(\tilde{E})$$

where $\Phi_G(\tilde{E}) \in \Omega_G(\tilde{E})$ is partially characterized by the fact that for all $x \in \tilde{E} \times EG$, $R_x^* \Phi_G(\tilde{E})$ is the normalized Haar measure of $G$. Please see [45] for a fuller discussion.

Now $\tilde{\pi}_G^*(\omega) \in \Omega(\tilde{E} \times EG)_{g\text{-basic}}$, so that by Theorem 6.1.7 it is related to $\varpi \in \Omega_G(\tilde{E})$ via the Chern-Weil homomorphism:

$$w_{C,\nabla_g}(\varpi) = \tilde{\pi}_G^*(\omega).
(6.34)$$

We may then define a map

$$S_G: \Omega_G^\bullet(\tilde{E}) \rightarrow \Omega^\bullet(\tilde{E} \times EG)
S_G(\varpi) \mapsto w_{C,\nabla_g}(\varpi) \wedge \Phi_G$$

(6.35)

The virtue of this map is that it may be readily interpreted in the context of a TFT.
Introduce $\lambda_a$ and $\eta_a$ as generators of $S(\mathfrak{g})$ and $\Lambda(\mathfrak{g})$, respectively and extend the action of the Cartan differential on these generators as follows:

$$Q_C \left( \begin{array}{c} \lambda \\ \eta \end{array} \right) = \left( \begin{array}{cc} 0 & 1 \\ -\phi_i \otimes \mathcal{L}(T_i) & 0 \end{array} \right) \left( \begin{array}{c} \lambda \\ \eta \end{array} \right)$$

Then we have the following

**Proposition 6.5.1:** For all $\varpi \in \Omega^* G(\tilde{E})$,

$$S_G(\varpi) = \int d\phi \left\{ \varpi \wedge \left( \frac{1}{2\pi i} \right)^{\dim G} \int_{g \times \Pi G} d\lambda \, d\eta \, \exp -Q_C \Psi_{\text{Proj}} \right\}$$

where

$$\Psi_{\text{Proj}} = -i(\lambda, C^\dagger)_{\mathfrak{g}}$$

**Proof:** Please see [45].

**Remarks:**

6.5.2 We have used the metric on $\tilde{E}$ to define the adjoint, $C^\dagger_x$ for all $x \in \tilde{E} \times E G$:

$$C_x: \mathfrak{g} \rightarrow T_x \tilde{E}$$

$$C^\dagger_x: T_x \tilde{E} \rightarrow \mathfrak{g}$$

We may view $C^\dagger$ as a $\mathfrak{g}$-valued 1-form and so $(\lambda, C^\dagger)_{\mathfrak{g}} \in T^*_\mathfrak{g} \tilde{E}$.

6.5.3 This procedure is distinct from Faddeev-Popov gauge fixing. Note that no section of $\tilde{E} \times E G \rightarrow \tilde{E} \times_G E G$ enters into our discussion. For a more careful comparison, please see [45].

Again it is useful to depict the interrelation of the various complexes and maps diagramatically:

$$\left( S(g^*) \otimes \Omega(\tilde{E}) \right) \xrightarrow{\mathcal{G}} \Omega(\tilde{E} \times E G) \xrightarrow{s^*} \Omega(\tilde{E} \times E G)$$

$$\downarrow \bar{w}_G \quad \uparrow \tau_G \quad \downarrow (\pi_G)_* \quad \Omega(\tilde{E} \times_G E G) \xrightarrow{s^*} \Omega(\tilde{E} \times_G E G)$$

**Diagram 6.5.4**

The object $\Upsilon_{\nabla SO(\tilde{V}), G} \in \Omega_G(\tilde{E})$ which we constructed in (6.26) is related to the equivariant Thom class, $\Phi_{\nabla}(\tilde{E} \times_G E G)$, via the following diagram
\[ \Upsilon_{\nabla_{SO(\bar{V}),c-g}} \xrightarrow{S_\Upsilon} S_\Upsilon \left( \Upsilon_{\nabla_{SO(\bar{V}),c-g}} \right) \xrightarrow{\tilde{s}^*} \tilde{s}^* S_\Upsilon \left( \Upsilon_{\nabla_{SO(\bar{V}),c-g}} \right) \]

\[ = \Upsilon_{\nabla_{\nabla_{SO(\bar{V}),c-g}}} \left( \Phi(\tilde{\Upsilon} \times_\Upsilon E\mathcal{G}) \right) \]

\[ \Upsilon_{\nabla_{\nabla_{SO(\bar{V}),c-g}}} \left( \Phi(\tilde{\Upsilon} \times_\Upsilon E\mathcal{G}) \right) \xrightarrow{\tilde{s}^*} \tilde{s}^* \Phi(\tilde{\Upsilon} \times_\Upsilon E\mathcal{G}) \]

\[ = \tilde{w}_{C,\nabla_\Upsilon} \left( \Upsilon_{\nabla_{SO(\bar{V}),c-g}} \right) \]

Diagram 6.5.5

From this it is apparent that

\[ \tilde{s}^* \Phi(\tilde{\Upsilon} \times_\Upsilon E\mathcal{G}) = \tilde{s}^* S_\Upsilon \left( \Upsilon_{\nabla_{SO(\bar{V}),c-g}} \right) \] (6.39)

where \( \Upsilon_{\nabla_{SO(\bar{V}),c-g}} \) is given by (6.26).

We shall assume that \( \mathcal{G} \) acts freely on \( \tilde{\Upsilon} \) and \( \tilde{\Upsilon} \):

\[ \mathcal{E} = \tilde{\Upsilon}/\mathcal{G} \]
\[ \mathcal{C} = \tilde{\Upsilon}/\mathcal{G} \]

in order that \( \mathcal{M} \) be a manifold\(^{12}\). Then we know from Proposition 6.4.2 that for \( \mathcal{O} \in H^{\text{index} \nabla_{\tilde{s}}}(\mathcal{C}), \)

\[ \int_{\mathcal{C}} \tilde{s}^* \Phi(\tilde{\Upsilon} \times_\Upsilon E\mathcal{G}) \land \mathcal{O} = \int_{\mathcal{M}} e(\text{coker } \nabla \tilde{s} \rightarrow \mathcal{M}) \land i^* \mathcal{O} \] (6.40)

and from (6.39)

\[ \int_{\mathcal{C}} \tilde{s}^* \Phi(\tilde{\Upsilon} \times_\Upsilon E\mathcal{G}) \land \mathcal{O} = \int_{\tilde{\mathcal{C}}} \tilde{s}^* S_\Upsilon \left( \Upsilon_{\mathcal{C}-g} \right) \land \tilde{\pi}_G^* \mathcal{O} \] (6.41)

Combining (6.40) and (6.41) we arrive at the following

**Proposition 6.5.6:** For \( \mathcal{O} \in H^{\text{index} \nabla_{\tilde{s}}}(\mathcal{C}), \)

\[ \int_{\tilde{\mathcal{C}}} S_\Upsilon \left( \Upsilon_{\nabla_{SO(\bar{V}),c-g}} \right) \land \tilde{\pi}_G^* \mathcal{O} = \int_{\mathcal{M}} e(\text{coker } \nabla \tilde{s} \rightarrow \mathcal{M}) \land i^* \mathcal{O} \] (6.42)

**Remarks:**

\(^{12}\) Hurwitz space is, in fact, not smooth but possesses orbifold singularities. In this case, we actually compute orbifold Euler characters. For more singular spaces, we do not know a general prescription.
6.5.7. If we introduce a supermanifold \( \tilde{C} \), whose odd coordinates are generated from the fibers of \( T^*\tilde{C} \), then
\[
\mathcal{C}^\infty(\tilde{C}) \cong \Omega^*\tilde{C}.
\]
On the other hand, \( \tilde{C} \) has a natural measure
\[
\mu = d\varphi^1 \wedge \cdots \wedge d\varphi^n d\psi^1 \wedge \cdots \wedge d\psi^n
\]
where \((\varphi_i, \psi_i)\) are local coordinates on \( \tilde{C} \). If \( \tilde{\omega} \in \mathcal{C}^\infty(\tilde{C}) \) corresponds to the differential form \( \omega \in \Omega^*\tilde{C} \), then
\[
\int_{\tilde{C}} \omega = \int_{\tilde{C}} \mu \tilde{\omega}
\]
so that we can rewrite the integral over \( \tilde{C} \) in superspace form.

6.5.8. The vector bundle coker \( \nabla s \to M \), though crucial in the general localization formula, is difficult to work with directly. \( \nabla s: T\tilde{C} \to (T(s)\tilde{E})^{\text{vert}} \) is a simpler operator. Since \( s \) is \( G \)-equivariant, ker \( \nabla s \) and coker \( \nabla s \) are in general infinite dimensional. However the operator
\[
\Phi = \left( \nabla s |_{C^1} \right): T\tilde{C} \to (T(s)\tilde{E})^{\text{vert}} \oplus g
\]
defines equivariant vector bundles of finite rank, ker \( \Phi \) and coker \( \Phi \), over \( \tilde{M} \). These descend to vector bundles over \( M \). The operator \( \Phi \) is of direct importance to TFT as it appears as the fermionic kinetic term of the complete lagrangian.

Finally using remarks 6.5.7 and 6.5.7 we may rewrite (6.42) in a way that makes the TFT action more apparent:
\[
\int_M e(\text{coker } \nabla s \to M) \land i^*O
\]
\[
= \int_M e((\text{coker } \Phi \to \tilde{M}/G) \land i^*O
\]
\[
\sim \int_{g^*}[d\phi] \int_{\tilde{C} \times \Pi\tilde{C}} [d\varphi] [d\psi] \int_{\tilde{V}^* \times \Pi\tilde{V}^*} [d\lambda] [d\eta] \int_{g \times \Pi g} [d\lambda] [d\eta] \exp -Q_C (\Psi_{\text{Loc}} + \Psi_{\text{Proj}})
\]
where we have absorbed the normalizations into the measures \([d\cdots]\). The TFT action may be identified with
\[
I_{\text{Top}} = Q_C (\Psi_{\text{Loc}} + \Psi_{\text{Proj}})
\]
where \( \Psi_{\text{Loc}} \) is given by (6.27) and \( \Psi_{\text{Proj}} \) is given by (6.38).
7. Topological String Theory and the Chiral Theory

7.1. Standard Topological String Theory

The basic configuration space is given by

\[ \tilde{C} = \{ (f, h) \mid f \in \mathcal{C}^\infty(\Sigma_W, \Sigma_T) \text{ and } h \in \text{Met}_{-1}(\Sigma_W) \} \]  

(7.5)

where Met\(_{-1}(\Sigma_W)\) is the space of metrics on \(\Sigma_W\) with constant Ricci scalar curvature\(^{13}\). \(-1\) Hurwitz space may be described by the Gromov equation for (pseudo-) holomorphic maps:

\[ \mathcal{M} = \tilde{\mathcal{M}} / \text{Diff}^+(\Sigma_W) \]

\[ \tilde{\mathcal{M}} = \{ (f, h) \in \tilde{C} \mid df + J df \epsilon[h] = 0 \} \]

(7.6)

At \((f, h) \in \tilde{\mathcal{M}}\), the tangent space is described by (See Appendix B for a derivation.)

\[ T_{(f,h)}\tilde{\mathcal{M}} = \{ (\delta f, \delta h) \in T_{(f,h)}\tilde{C} \mid D(\delta f) + J D(\delta f) \epsilon[h] + J df k[\delta h] = 0 \} \]

(7.7)

\(^{13}\) The localization to Met\(_{-1}(\Sigma_W) \subset\) Met\((\Sigma_W)\) is standard (For a review and more extensive references, please see [45].) For the localization Lagrangian we introduce a scalar antighost \(\rho\) and its Lagrange multiplier \(\pi\). Then the gauge fermion for localizing to Met\((\Sigma)k\) is:

\[ \Psi_{\text{Weyl Loc}} = \int d^2 z \sqrt{h} \rho \ (R + 1) \]

(7.1)

Using the following two relations:

\[ Q_c \Gamma^\alpha_{\beta\gamma} = \frac{1}{2} h^{\alpha\delta} (D_\beta Q h_{\gamma\delta} + D_\gamma Q h_{\beta\delta} - D_\delta Q h_{\beta\gamma}) \]

\[ QCR = - \frac{1}{2} D_\alpha D^\alpha (h^{\gamma\beta} Q h_{\beta\gamma}) + D^\alpha D^\beta Q h_{\alpha\beta} - \frac{1}{2} R h^{\alpha\beta} Q h_{\alpha\beta} \]

(7.2)

we may write this action as

\[ I_{\text{Weyl Loc}} = \int d^2 z \sqrt{h} \ \{ \pi \ (R + 1) - \rho \ L^{\alpha\beta} \psi_{\alpha\beta} \} \]

(7.3)

where

\[ L^{\alpha\beta} = D^\alpha D^\beta - \frac{1}{2} h^{\alpha\beta} D^2 + \frac{1}{2} h^{\alpha\beta} \]

(7.4)

\(^{14}\) We assume for simplicity that the genus of the world sheet is greater than one.
where $D$ is the pulled-back connection $(D_\alpha \delta f)^\mu = \partial_\alpha \delta f^\mu + \Gamma^\mu_{\kappa\lambda} \partial_\alpha f^\kappa \delta f^\lambda$; and $k[\delta h]$ is the variation of the complex structure. (7.7) suggests that we introduce an operator

\[
\mathbb{I}_D(f,h) : T(f,h) \tilde{\mathcal{C}} \longrightarrow \tilde{\mathcal{V}}(f,h)
\]

\[
\mathbb{I}_D(f,h)(\delta f, \delta h) \longrightarrow D(\delta f) + JD(\delta f) \epsilon[h] + J df k[\delta h]
\]

(7.8)

where $\tilde{\mathcal{V}}(f,h)$ will be defined shortly.

To construct a topological string theory action, we regard the Gromov equation, (7.6), as a $\text{Diff}^+(\Sigma_W)$-equivariant section

\[
s : \tilde{\mathcal{C}} \longrightarrow \tilde{\mathcal{E}}
\]

\[
s(f,h) \longmapsto df + J df \epsilon[h]
\]

(7.9)

where $\tilde{\mathcal{E}}$ is a $\text{Diff}(\Sigma_W)$-equivariant vector bundle whose fiber above $(f,h) \in \tilde{\mathcal{C}}$ is given by

\[
\tilde{\mathcal{V}}(f,h) := \Gamma[T^* \Sigma_W \otimes f^*(T\Sigma_T)]^+
\]

(7.10)

The superscript $(\cdots)^+$ indicates that the sections must satisfy the self-duality constraint:

\[
\rho \in \Gamma[T^* \Sigma_W \otimes f^*(T\Sigma_T)]^+ \iff \rho - J \rho \epsilon[h] = 0
\]

(7.11)

$\tilde{\mathcal{E}}$ admits an $\text{SO}(\tilde{\mathcal{V}})$-connection, $\nabla_{\text{SO}(\tilde{\mathcal{V}})}$, characterized by (6.30)

\[
\ker \nabla_{\text{SO}(\tilde{\mathcal{V}})} s|_{(f,h)} = T(f,h)\tilde{\mathcal{M}} \quad \forall (f,h) \in \tilde{\mathcal{M}}
\]

Let $s_\alpha^\mu[f,h]$ be a local section of $\tilde{\mathcal{E}} \rightarrow \tilde{\mathcal{C}}$, and define

\[
(\nabla_{\text{SO}(\tilde{\mathcal{V}})} s)_\alpha^\mu = \int d^2 \sigma \sqrt{h} \left( \frac{\delta s_\alpha^\mu}{\delta f^\kappa(\sigma)} \delta f^\kappa(\sigma) + \frac{\delta s_\alpha^\mu}{\delta h_{\beta\gamma}(\sigma)} \delta h_{\beta\gamma}(\sigma) \right.
\]

\[
- \Gamma^\mu_{\kappa\lambda}[f(\sigma), h(\sigma)] s_\alpha^\kappa \delta f^\lambda(\sigma))
\]

(7.12)

Then for $s[f,h] = df + J df \epsilon$, one may readily check that

\[
\nabla_{\text{SO}(\tilde{\mathcal{V}})} s = \mathbb{I}_D(\delta f, \delta h)
\]

(7.13)

Having determined $\nabla_{\text{SO}(\tilde{\mathcal{V}})}$, we may construct $\Upsilon_{\nabla_{\text{SO}(\tilde{\mathcal{V}})} \cdot C - G}$:

\[
\Upsilon_{C - G} = \int_{\tilde{\mathcal{V}}^* \times \Pi\tilde{\mathcal{V}}^*} d\pi d\rho \exp -I_{\text{Top}} \sigma
\]

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where from (6.26) and (6.27):

$$I_{\text{Top}\sigma} = Q_C \left[ i \langle \rho, s(f, h) \rangle - t(\rho, \Gamma_{SO(\tilde{V})} \cdot \rho)_{\tilde{V}^*} + t(\rho, \pi)_{\tilde{V}^*} \right]$$

$\tilde{C}$ is a Diff($\Sigma_W$)-manifold, so that for all $(f, h) \in \tilde{C}$, there is a canonical isomorphism between $\text{diff}(\Sigma_W)$ and $(T\tilde{C})^{\text{vert}}$ given by

$$C_{(f, h)}: \text{diff}(\Sigma_W) \rightarrow (T\tilde{C})^{\text{vert}}$$

$$C_{(f, h)}(\gamma) \rightarrow \left( L_\gamma f, L_\gamma h \right)$$

(7.14)

Hence from (6.38) the projection fermion is given by

$$\Psi_{\text{Proj}} = -i(\lambda, C^\dagger)_{\tilde{g}}$$

$$\Psi_{\text{Weyl Loc}} + \Psi_{\text{Top}\sigma} + \Psi_{\text{Proj}}$$

and therefore the action we have produced thus far, is Diff($\Sigma_W$) invariant. Therefore, in order to compute anything using the standard methods of local quantum field theory, we still need to fix this symmetry. This may be done by adding the (gauge-fixing) action

$$\Psi_{\text{GF}} = \int d^2z \sqrt{h} b^{\alpha\beta} (h_{\alpha\beta} - h_{\alpha\beta}^{(0)})$$

(7.15)

where the action of $Q_C$ extends to the symmetric tensor fields, $b^{\alpha\beta}$ and $d^{\alpha\beta}$ as $Q_C b^{\alpha\beta} = d^{\alpha\beta}$. Altogether the action of standard topological string theory is given by

$$I_{\text{TS}} = I_{\text{Weyl Loc}} + I_{\text{Top}\sigma} + I_{\text{GF}} + I_{\text{Proj}}$$

(7.16)

Our objective is to find a string theory whose connected partition function is

$$Z_{\text{string}} \sim \chi_{\text{orb}}(\mathcal{M})$$

$$= \int_{\mathcal{M}} e(T\mathcal{M} \rightarrow \mathcal{M})$$

(7.17)

If we pursue the construction outlined in section 6, it is apparent that we indeed obtain a theory that localizes to Hurwitz space. From (6.44) the measure is given by $e(\text{coker } \Phi / \mathcal{G})$, where (7.12) and (7.14) together define $\Phi_{(f, h)}$. Unfortunately, however, (7.14),

$$\ker \Phi_{(f, h)} / \mathcal{G} \cong T_{(f, h)} \mathcal{M}$$

$$\text{coker } \Phi_{(f, h)} \cong \{0\}$$

(7.18)
so that the standard topological string theory clearly does not produce the desired measure, 
e(TM \to M). It is clear that we have to modify this theory somewhat. The following
gives a clue about what this modification ought to entail.

Since \( \tilde{V} \) is endowed with a metric, we may define the adjoint of \( \Phi_{(f,h)} \), which we may
view as an operator
\[
\Phi^\dagger_{(f,h)} : T_{s(f,h)} \tilde{V} \oplus g \to T_{(f,h)} \tilde{C}
\]
From (7.18) it follows that
\[
\ker \Phi^\dagger \cong \{0\}
\]
\[
coker \Phi^\dagger / G \cong T_{(f,h)} M \tag{7.19}
\]
Clearly we want to produce a TFT wherein the fermion kinetic term is
\[
\Phi^{\text{Total}} = \Phi \oplus \Phi^\dagger \tag{7.20}
\]
In this case, \( \ker \Phi^{\text{Total}} = \coker \Phi^{\text{Total}} = TM \), so that such a theory would produce the
 correct measure.

7.2. “Cofields”

In order to obtain (7.20) as the fermion kinetic operator, we must extend the field space
relative to that of the standard topological string theory. The new fields are completely
determined by two requirements
1. \( \Phi^\dagger \) maps ghosts to antighosts,
2. \( Q_C \) extends to act on the new fields as the Cartan differential of \( \text{Diff}(\Sigma_W) \)-equivariant
cohomology.
To describe the additional fields, it is easiest to begin with the new set of ghosts, \( \hat{G} \); we
shall sometimes refer to these as the “Co-Ghosts”. These take values in the domain of \( \Phi^\dagger \),
so that
\[
\hat{G} \in \Gamma(T\Sigma W \otimes f^*(T^*\Sigma_T))^+ \oplus \Gamma(T\Sigma W) \tag{7.21}
\]
and their index structure is given by
\[
\hat{G} = \begin{pmatrix} \hat{\chi}_\mu^\alpha \\ \hat{\psi}^\alpha \end{pmatrix}
\]
As usual, the superscript \((\cdots)^+\) indicates that the \( \hat{\chi}_\mu^\alpha \) satisfy a self-duality constraint:
\[
\hat{\chi}_w^z = 0
\]
The Co-Ghosts represent differential forms on an enlarged field space, \( \widetilde{D} \). This enlarged field space may itself be viewed as the total space of a vector bundle, \( \widetilde{D} \rightarrow \widetilde{C} \), where the fibre at \((f, h) \in \widetilde{C}\) is given by

\[
\widetilde{D}_{(f, h)} = \Gamma(T \Sigma W \otimes f^*(T^* \Sigma T))^{\perp} \oplus \Gamma(T \Sigma W)
\]

We refer to the additional fields as “Co-Fields”; like the Co-Ghosts, their index structure is given by

\[
\hat{I} F = (\hat{f}_\mu^\alpha \hat{h}_\alpha)
\]  \( (7.22) \)

where \( \hat{f}_{w^z} = 0 \).

In turn \( \widetilde{D} \) forms the base space of a Diff(\( \Sigma W \))-equivariant vector bundle,

\[
\tilde{E} \oplus \tilde{E} \leftarrow \tilde{V} \oplus \tilde{V}
\]

Now consider the following section

\[
S: \widetilde{D} \longrightarrow \tilde{V} \oplus \tilde{V}_{cf}
\]

\[
S: (F, \tilde{F}) \longmapsto (df + J df \epsilon[h], \Phi^\dagger \tilde{F})
\]  \( (7.23) \)

The zero set of this section is still Hurwitz space

\[
\{(F, \tilde{F}) \in \widetilde{D} \mid s(F, \tilde{F}) = 0 \} = \mathcal{M} \times \{0\}
\]  \( (7.24) \)

since ker \( \Phi^\dagger = \{0\} \). Moreover, when restricted to \( \mathcal{M} \times \{0\} \), the operator appearing in the total fermion kinetic term is given by \( \Phi^{\text{Total}} = \Phi \oplus \Phi^\dagger \). Our choice of section dictates that the antighost bundle be dual to \( \tilde{V} \oplus \tilde{V}_{cf} \) where \( \tilde{V} \) is defined in \( (7.10) \), while the range of \( \Phi^\dagger \) defines

\[
\tilde{V}_{cf} = \Gamma(f^*(T \Sigma T)) \oplus \Gamma(\text{Sym}(T \Sigma W^2)).
\]  \( (7.25) \)

or

\[
\tilde{A} = \begin{pmatrix} \hat{\rho}^\mu \\ \hat{\eta}_{\alpha\beta} \end{pmatrix}.
\]  \( (7.26) \)

\( Q_C \) extends to the Co-Fields as the Cartan differential for Diff(\( \Sigma W \))-equivariant cohomology.

\[
Q_C \begin{pmatrix} \tilde{F} \\ \tilde{G} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\mathcal{L}_\gamma & 0 \end{pmatrix} \begin{pmatrix} \tilde{F} \\ \tilde{G} \end{pmatrix}
\]

\[
Q_C \begin{pmatrix} \tilde{A} \\ \tilde{\Pi} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\mathcal{L}_\gamma & 0 \end{pmatrix} \begin{pmatrix} \tilde{A} \\ \tilde{\Pi} \end{pmatrix}
\]
The addition of the cofields does not change the $Q_c$-cohomology, so we expect to have the same observables as in topological string theory. (Please see section 7.3).

The Lagrangian for the YM$_2$ string will be a sum of a Lagrangian for the topological string theory $\Sigma_W \to \Sigma_T$ plus a Lagrangian for localizing to $\mathbf{F} = 0$:

$$I_{YM_2} = I_{TS} + I_{cf}$$

(7.27)

Following section 6, we write down the gauge fermion for the co-fields:

$$\Psi_{cf} = \left( \tilde{A}, \Phi^\dagger \mathbf{F} \right) - t \left( \tilde{A}, \mathbf{H} \right)$$

(7.28)

where $t \in \mathbb{R}$ and

$$\Phi^\dagger = \begin{pmatrix} -G_{\mu\nu} \partial_{\gamma} f^{\mu \nu} - \frac{1}{2} (\delta_{\gamma}^\alpha D^{\beta} + \delta_{\gamma}^\beta D^\alpha) \\ -\delta^\alpha_{\gamma} J^{\nu}_{\mu} \partial f^\mu_{\nu} \epsilon_{\beta \gamma} \end{pmatrix}$$

(7.29)

To derive the localization theorem for this theory, we must analyze the exact bundle sequence analogous to (6.31). For the choice of section, $S$, and connection, $\nabla_{SO(\tilde{V})}$, we have We find

$$0 \longrightarrow \text{Im} (\Phi \oplus \Phi^\dagger) \longrightarrow \tilde{V} \oplus \tilde{V}_{cf} \longrightarrow \text{coker} (\Phi \oplus \Phi^\dagger) \longrightarrow 0$$

as a sequence of bundles over $M \times \{0\}$. Then by the general principles we have explained in the previous section, we see, by combining (7.18) and (7.19), with (6.44), that the path integral computes the Euler character of the cokernel bundle, $TM$, which is the problem we set out to solve.

7.3. Observables in the theory

The most obvious observables in topological string theories are made from gravitational descendents of the primaries of the corresponding topological sigma model. The observables in topological string theories are of two types: (a) homology observables and (b) homotopy observables[46,52,53,54].

Homology Observables:

These observables are built from cohomology classes of the target space. For a target space a Riemann surface of genus $G$, the cohomology classes are described by:

$$\{1\} \in H^0(\Sigma_T)$$

$$\{\xi^A, \xi^{\dot{A}}\} \in H^1(\Sigma_T) \quad A = 1, \ldots, G$$

$$\{\omega\} \in H^2(\Sigma_T)$$

(7.30)
where \( \omega \) is the Kähler class. So the homology observables of the topological string theory are given by:

\[
\begin{align*}
\sigma_n(1) \\
\sigma_n(\xi^A_i \chi^i), \sigma_n(\bar{\xi}^\bar{A}_i \bar{\chi}^i) & A = 1, \ldots, G \\
\sigma_n(\omega_{ij} \chi^i \chi^j)
\end{align*}
\]

(7.31)

where \( \sigma_n(\cdots) \) represents the gravitational dressing of the operator. In essence

\[
\sigma_n(O) = (\epsilon_{\alpha \beta} \partial^\alpha \gamma^\beta)^n O.
\]

Homotopy Observables:

There is a ring of homotopy observables with \( 2G \) generators:

\[
\mathcal{O}_{\vec{k}} = \exp \left\{ 2\pi i \left( \sum_{A=1}^{G} k_A \cdot \int_{w_0} f(z) \xi^A + \bar{k}_{\bar{A}} \cdot \int_{\bar{w}_0} \bar{f}(z) \bar{\xi}^{\bar{A}} \right) \right\}
\]

(7.32)

where \( \vec{k} = (k_{[1]}, \ldots, k_{[G]}) \) are vectors in the dual to the period lattice, \( \Lambda \).

We expect that these operators will form a ring related to the group ring of the fundamental group of the target manifold. (Since the matter is not a topological conformal field theory we expect it will involve a nontrivial deformation of that ring.)

Via the descent equations we may obtain 1-form versions of the operators (7.32), which generate an algebra of symmetries which we naturally expect to be related to \( w_{\infty} \). The operators relevant to the topological string theory are, of course, the gravitational dressings of the above. Indeed, 2D string theories are famous for having \( w_{\infty} \)-type symmetries in the target space theory. This should be true in our case and should explain the area-preserving diffeomorphism invariance of \( YM_2 \) from the string perspective. We have not carried this out in detail.

8. Turning on the area

8.1. Area Polynomials in \( YM_2 \)

The same basic reasoning we have used in the \( A = 0 \) case can be applied to the \( A > 0 \) case. We begin with the \( 1/N \) expansion of the chiral \( YM_2 \) partition function.
Manipulations identical to those leading to (5.7) give

\[ Z^+(A, G, N) = \sum_{n, \ell > 0} N^{n(2-2G)-\ell} e^{-\frac{n^2}{2N^2}} \frac{(-A)^\ell}{\ell!} \sum_{s_i, t_i \in S_n} \frac{1}{n!} \delta(\Omega_{n}^{2-2G}T_{2,n}^\ell \prod_{1}^{G}[s_i, t_i]) \]

\[ = \sum_{n=0}^{\infty} \sum_{\ell=0}^{\infty} e^{-\frac{nA}{2}} \frac{(-A)^\ell}{\ell!} \sum_{p_1, \ldots, p_k \in T_2 \subset S_n} \sum_{L'=0}^{\infty} \sum_{v_1, \ldots, v_{L'} \in S_n} \left( \frac{1}{N} \right)^{n(2G-2)+\ell+\sum_{j=1}^{L'} (K_{v_j} - n)} \sum_{s_1, t_1, \ldots, s_{G} \in S_n} \frac{d(2-2G, L')}{n!} \delta(p_1 \cdots p_k v_1 \cdots v_{L'} \prod_{i=1}^{G}[s_i, t_i]). \]

where \( T_{2,n} \in \mathcal{C}[S_n] \) is the sum of transpositions. Recall that to establish a correspondence between homomorphisms from \( F_{G,L} \) to \( S_n \) and branched coverings over a set \( S \) we make a choice of generators of \( \pi_1(\Sigma_T - S, y_0) \). This choice leads to an association of a set of \( k \) points with the permutations \( p_1 \cdots p_k \). After collecting powers of \( N \) we get

\[ Z^+(A, G, N) = \sum_{n=0}^{\infty} \sum_{B=0}^{\infty} e^{-nA/2} \frac{n^2}{2N^2} A \left( \frac{1}{N} \right)^{2h-2} \sum_{L=0}^{B} P_{n,B,L}(A) \]  

(8.2)

where \( P_{n,B,L}(A) \) is a polynomial defined by:

\[ P_{n,B,L}(A) = \sum_{k=0}^{B} \frac{(-A)^k}{k!} \sum_{L=k}^{B} \chi(C_{L-k}(\Sigma_T)) \sum_{\Psi(n,B,G,L,k)} \frac{1}{|\Psi|^C(\psi)|}. \]

(8.3)

where \( \Psi(n, B, G, L, k) \) is the set of homomorphisms \( F_{L,G} \rightarrow S_n \) which, via Theorem 3.1, correspond to branched coverings over some set \( S \) of branch points, with the property that over a fixed subset of \( k \) points the branching is simple.

For example, if \( \psi \) has only simple branch points, i.e., corresponds to a map in the simple Hurwitz space then the formula becomes:

\[ \psi \in \Psi(n, B, G, L = B) \implies P_{\psi}(A) = \sum_{k=0}^{B} \frac{(-A)^k}{k!} \chi(C_{B-k}(\Sigma_T)). \]

(8.4)

In order to obtain the full \( 1/N \) expansion we must also expand the factor

\[ e^{\frac{n^2}{2N^2}} A \]

(8.5)

in (8.2). This has been interpreted in [16] and in [15] in terms of contributions of “collapsed tubes and handles.”
8.2. Area Polynomials from Perturbations

In the topological field theory there is a natural mechanism by which the area can be included: perturbation by BRST closed but nonexact operators. This is how, e.g., one explores the more physical phases of 2d gravity, studied in the double scaling limit of matrix models, in the framework of topological 2D gravity. In the present case we will study the perturbation of the action by a BRST invariant operator $\frac{1}{2} \int A^{(2)}$. Here $A^{(2)}$ fits into the area operator descent multiplet:

\[
\begin{align*}
A^{(0)} &= \sigma_0(\omega_{ij}(f(x)) \chi^i \chi^j) \\
A^{(1)} &= \sigma_0(dx^\alpha \omega_{ij}(f(x)) \partial_\alpha f^i \chi^j) \\
A^{(2)} &= \sigma_0(dx^\alpha \wedge dx^\beta \omega_{ij}(f(x)) \partial_\alpha f^i \partial_\beta f^j)
\end{align*}
\]  

Here $\omega$ is the Kähler two-form from the target space. The form degree 0, ghost number 2 member of this multiplet has a geometric interpretation as a 2-form on $F$. Thus insertions of $A^{(0)}$ compute intersection numbers on $F$. The deformed action is

\[
I_0 \rightarrow I_0 + \frac{1}{2} \int A^{(2)} \quad (8.7)
\]

Naively, the contribution of $\frac{1}{2} \int A^{(2)}$ in a path integral over maps $f$ of index $n$ is $e^{-\frac{1}{2} n A}$. This accounts nicely for the genus-independent exponential factors in (8.2), but fails to explain the polynomial of $A$ in (8.2).

We can understand some features of the polynomial in $A$ in (8.2) by considering more carefully the “conformal perturbation series” in question:

\[
Z^+(A, G, N) = \left\langle e^{-\frac{1}{2} \int A^{(2)}} \right\rangle_{A=0} = \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{\ell!} \left\langle \left( \frac{1}{2} \int_{\Sigma_W} f^* \omega \right)^\ell \right\rangle_{A=0} \quad (8.8)
\]

The measure $\langle \cdots \rangle$ implicitly contains further operators, such as the four fermion terms in the curvature, which arise from the co-sigma model. It is important to understand that the expression (8.8) is ill-defined. Evaluation of the terms in the series involves integration over operators inserted at coincident points. As in all theories of gravity, merely identifying the operators as in (8.6) does not fully specify their correlators, because we must choose contact terms, i.e., we must carefully specify the terms in the correlators of $A^{(2)}$ and $\langle \tilde{A}, R[G, G] \tilde{A} \rangle$ which have delta-function support on two or more points.

In the following subsections we will show how a consideration of contact terms can account for the area polynomials (8.4) which arise from the contributions of simple Hurwitz
space. We will not try to account for the other types of coverings in the sum over $\Psi$ in (8.2). Similarly we do not try to account for the the terms arising from expanding (8.3). We firmly believe that these more complicated polynomials can also be explained by looking at more complicated contact terms from the higher codimension boundaries of Hurwitz space and the space of Maps $\times$ Metrics.

8.3. Measure on the space of simple covers

Let $\mathcal{F}^{(1)}$ be the simple Hurwitz space of maps with $B$ simple branch points. Denote these simple branch points by $P_I$ with corresponding ramification points of index 2 at $R_I$: these are the unique ramification points above $P_I$. We can choose a basis $\{G_I\}_{I=1,\ldots,2B}$ for $TF$, such that $G_{I-1}$ and $G_I$ have support only at the $I$-th ramification point. The analogue in ordinary string theory is a choice of Beltrami differentials which have support only at punctures. This is a well-defined choice away from the boundary of moduli space.

Now consider the curvature insertions in these local coordinates:

$$\int \mathcal{D}[\tilde{A}] \exp \left[ -\frac{1}{4} \tilde{A}^I R_{IJ} \tilde{A}^J \right] = \frac{(-1)^B}{2^B} \text{Pfaff}(R_{IJ})$$

where $B$ is even and the matrix, $R_{IJ}$, takes the following form in an oriented orthonormal basis

$$R_{IJ} = \begin{pmatrix} 0 & R_{12} & \cdots & 0 \\ -R_{12} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ -R_{2B-1,2B} & 0 & \cdots & 0 \end{pmatrix}$$

so that

$$\text{Pfaff}(R_{IJ}) = \prod_{I=1}^{2B} R_{2I-1,2I}[G^{2I-1},G^{2I}](R_I)$$

and the full measure for the topological string theory is

$$\frac{(-1)^B}{(2\pi)^B} \int_{\mathcal{F}(1)} \mathcal{D}[F,G] \prod_{I=1}^{B} R_{2I-1,2I}[G^{2I-1},G^{2I}](R_I) \exp \left[ -\frac{1}{2} \int_{\Sigma_W} f^* \omega \right]$$

$$= \frac{1}{(2\pi)^B} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \int_{\mathcal{F}(1)} \mathcal{D}[F,G] \prod_{I=1}^{B} R_{2I-1,2I}[G^{2I-1},G^{2I}] \left( \frac{1}{2} \int_{\Sigma_W} f^* \omega \right)^k$$

$$= \frac{1}{(2\pi)^B} \sum_{k=0}^{\infty} \frac{(-1)^k}{2^k k!} \langle A^{(2)} \cdots A^{(2)} \rangle_{\mathcal{F}(B,k)}$$
In the last line we have introduced a space \( \mathcal{F}(B, k) \), which is the product space

\[
\mathcal{F}(B, k) = \mathcal{F}^{(1)} \times (\Sigma_W)^k
\]

(8.13)

The integral over this space, \( \langle \cdots \rangle_{\mathcal{F}(B, k)} \) is formally defined by (8.12). In order to define the integrated correlators we have to describe possible delta-function supported contributions at places where area operators collide with curvature insertions, and where area operators collide with themselves. Collisions of curvature operators \( R_i = R_j \) belong to higher codimension boundaries outside the space of simple covers and contribute to other terms in (8.2). In the next three sections we analyze the other collisions.

8.4. Plumbing Fixtures

Suppose a simple ramification point \( R \) and a marked unramified point \( S \) collide on the worldsheet. The corresponding images of these points \( P \) and \( Q \) collide on the target space. Let \( U_1 \subset \Sigma_T \) be a disk containing \( P \) and \( Q \). Let the disk \( V_1 \subset \Sigma_W \) be the preimage of \( U_1 \). \( V_1 \) contains \( R \) and \( S \). It is a double covering of the disk \( U_1 \subset \Sigma_T \). The disk \( U_1 \) glues into the annulus \( U_2 \) on \( \Sigma_T \). \( V_2 \) is the double-covering preimage of \( U_2 \). The figure below describes this collision.

Fig. 7: Collision of Area and Curvature Operators
In the plumbing fixture description of the degeneration, the points are kept fixed in their local coordinates. The transition functions depend on the modulus which describe the relative position of $R$ and $S$.

To describe the plumbing fixture more explicitly, we shall need to specify transition maps between coordinate patches both on the worldsheet as well as on the target.

\[
\begin{align*}
V_1 & \xrightarrow{h_{21}} V_2 \\
U_1 & \xrightarrow{g_{21}} U_2
\end{align*}
\]

(8.14)

We introduce local coordinates $w_{1,2}$ on $U_{1,2}$ and $z_1, z_2$ on $V_1, V_2$, respectively. Then the transition functions of the holomorphic family of degenerations are given by

\[
\begin{align*}
g_{21}(w_1) &= q \frac{w_1}{h_{21}(z_1)} = \eta q^{1/2} z_1,
\end{align*}
\]

(8.15)

where $\eta = \pm$. Note that the transition function $g_{21}$ does not change the complex structure of the target Riemann surface without marked points. Similarly the function $f : \Sigma_W \rightarrow \Sigma_T$ is locally defined by

\[
\begin{align*}
f_1(z_1) &= z_1^2
\end{align*}
\]

(8.16)

The modular deformation which leads to this degeneration corresponds to a diffeomorphism generated by a quasi conformal vector field with support on $V_2$ and a discontinuity along $C$. This quasi conformal vector field is given by

\[
V_2 = z_2^2 \frac{\delta q}{q} \frac{\partial}{\partial z_2}
\]

(8.17)

Integrating against the stress tensor, we obtain

\[
\delta q \frac{L_0}{q}
\]

(8.18)

We exponentiate this to obtain the plumbing fixture:

\[
\begin{align*}
&\int d^2 \tau d^2 \bar{\tau} \exp 2\pi i (\tau L_0 + \delta l_0) \exp -2\pi i (\bar{\tau} \bar{L}_0 + \bar{\delta} \bar{l}_0) \\
&= \int \frac{d^2 q}{|q|^2} q^{L_0} q^{\bar{L}_0} l_0 \bar{l}_0
\end{align*}
\]

(8.19)
where $l_0$ and $\bar{l}_0$ are modes of the topological superstress tensor, $t$, which is the BRST partner to the ordinary stress energy tensor, $T$.

$$\{Q, t\} = T \quad (8.20)$$

### 8.5. Area-Curvature Contact Terms

To begin, let us note that simple considerations of quantum numbers and scaling constrain the contact term to be of the form:

$$\int \mathbf{R}_{2I-1,2I}[G,G]A^{(2)} = cA^{(0)} \quad (8.21)$$

for some constant $c$. The integral is over an infinitesimal disc surrounding the curvature insertion, which is nonzero because of a delta function contribution from the collision of two operators. Since $R$ is a two form on $F$, it has ghost number two. $A^{(2)}$ is a two form on the target pulled back to the worldsheet. After doing the integral, it is clear that the result should have ghost number two and should be a zero form on the worldsheet. Moreover, the LHS scales like the area of the target space. The unique operator in section 6.5 which satisfies all these criteria is $A^{(0)}$.

We now describe how the above contact term can be directly derived using ideas along the lines of those used in the case of pure topological gravity [53]. The derivation is only heuristic. The contact term we wish to compute here is the collision of an integrated area operator and a curvature insertion

$$\frac{1}{2\pi} \int_{|q| \leq \epsilon} \frac{d^2q}{|q|^2} q^{L_0} L_0 |l_0, \bar{l}_0\rangle \{A^{(0)}(1)R_{IJ} [G,G](0)\}$$

$$= \frac{1}{2\pi} \int_{|q| \leq \epsilon} d^2q \partial_q \partial_{\bar{q}} \left\{ q^{L_0} L_0 [A^{(0)}(1)\Phi_{IJ}(0)] \right\} \quad (8.22)$$

where $A^{(0)}$ is BRST invariant, 0-form descendant of $A^{(2)}$. $\epsilon << 1$ is some small positive number. We leave the zero mode projections $b_0$ implicit. In writing the second line we have used a contour-deformation argument for the BRST current, the equation (8.20), and the fact that

$$R_{IJ}[G_K, G_L] = \delta_K \delta_L \Phi_{IJ}. \quad (8.23)$$

However, as noted at the end of section 4.4 in [4.17], since $T\mathcal{F}$ is a holomorphic Hermitian vector bundle we may identify the matrix $\Phi_{IJ} = \log h_{IJ}$ where $h_{IJ}$ is the matrix of inner products of the fields projected onto the zero-mode sector.
We evaluate the total derivative by first expanding:

\[
\frac{1}{2\pi} \int_{|q| \leq \epsilon} d^2q \frac{\partial}{\partial q} \frac{\partial}{\partial \bar{q}} \left\{ q^{L_0} \bar{q}^{\bar{L}_0} [A^{(0)} \Phi_{IJ}] \right\} = \frac{1}{2\pi} \int_{|q| \leq \epsilon} d^2q \frac{\partial}{\partial q} \frac{\partial}{\partial \bar{q}} \left\{ (1 + \log q L_0 + \cdots)(1 + \log \bar{q} \bar{L}_0 + \cdots) [A^{(0)} \Phi_{IJ}] \right\}
\]

(8.24)

At this point we do not integrate by parts (this is part of our choice of contact term). Instead, we argue that if \( L_0 + \bar{L}_0 \) does not annihilate the term in brackets we will pick up a \( \delta \)-function when \( q \to 0 \). Therefore, we must evaluate

\[
(L_0 + \bar{L}_0) [A^{(0)} \Phi_{IJ}] = -2A^{(0)}
\]

(8.25)

This may be done - at least heuristically - by remarking that \((L_0 + \bar{L}_0)\) is the generator of scaling transformations. From (4.10) we see that under a Weyl transformation \( h_{\alpha \beta} \to \lambda^2 h_{\alpha \beta} \), \( A^{(0)} \) scales as \( h_{IJ} \to \lambda^2 h_{IJ} \) so that \( \Phi_{IJ} \to \Phi_{IJ} + 2 \log \lambda \) leading to

\[
(L_0 + \bar{L}_0) [A^{(0)} \Phi_{IJ}] = -2A^{(0)}
\]

(8.26)

Note also that \((L_0 + \bar{L}_0)^n [A^{(0)} \Phi_{IJ}] = 0\) for all \( n > 1 \), so that (8.24) becomes

\[
\to \int_{|q| \leq \epsilon} \delta^{(2)}(q) A^{(0)}(1)
\]

(8.27)

which corresponds to an area operator inserted at a ramification point. In conclusion, we have derived (8.21) with \( c = -2 \).

8.6. Area-Area Contact Terms

The argument (8.22) to (8.27) can be repeated for the collision of two integrated area operators. The plumbing fixture in this case is the one familiar from bosonic string theory

\[
\frac{1}{2\pi} \int_{|q| \leq \epsilon} d^2q \frac{\partial}{\partial q} \frac{\partial}{\partial \bar{q}} \left\{ q^{L_0} \bar{q}^{\bar{L}_0} [A^{(0)}(1) A^{(0)}(0)] \right\}
\]

\[
= \frac{1}{2\pi} \int_{|q| \leq \epsilon} d^2q \frac{\partial}{\partial q} \frac{\partial}{\partial \bar{q}} \left\{ q^{L_0} \bar{q}^{\bar{L}_0} [A^{(0)}(1) k(0)] \right\}
\]

(8.28)

The extra minus sign arises because we have computed the change in \( \Phi_{IJ} \) under an active transformation on \( h_{\alpha \beta} \) whereas the eigenvalue of \( L_0 + \bar{L}_0 \) measures the response of an operator under a passive transformation \( z \to \lambda z \).
where $k$ is the Kähler potential,

$$\omega = \partial \bar{\partial} k$$  \hfill (8.29)

The calculation now proceeds as in the previous subsection. The analog of (8.26) is

$$(L_0 + \bar{L}_0) [A^{(0)}k] = 0$$  \hfill (8.30)

since both $A^{(0)}$ and $k$ are invariant under worldsheet scale transformations.

**Remark:**

The absence of $A \cdot A$ contact terms may at first seem a bit counterintuitive. Indeed, the collisions of analogous operators in conformal field theory are well known to play an important role [56]. For example, when changing the “compactification data” of a product of Gaussian models by conformal perturbation theory it is exactly these contact terms which account for the dependence of the conformal weights and operator product coefficients on the compactification data (see e.g. [57].) In the present case we have made implicit choices in our evaluation of contact terms. Our choices are related to the preservation of BRST invariance of the theory.

**Conjecture 8.1.** Within the family of contact terms preserving the BRST Ward identities the area polynomials will remain unchanged and will be given by (8.3) above.

### 8.7. Recursion Relations and Calculation of an Area Polynomial

We now combine the above results on contact terms to derive recursion relations for the integrated area correlators. We attempt to remove the area operators $A^{(2)}$ successively. Each operator contributes a bulk term and a contact term. If $r$ such operators have collided with curvature operators producing a contact term of type (8.21) then the remaining correlator is integrated over a space $\mathcal{F}(B, \ell; r) = \mathcal{F}^{(1)} \times (\Sigma_W)^{\ell}$ where $B - r$ copies of $A^{(0)}$ are inserted at simple ramification points, $r$ copies of the curvature operator are inserted at the remaining simple ramification points, and $\ell$ area operators $A^{(2)}$ are integrated over the worldsheet $\Sigma_W$. If we try to remove an area operator $A^{(2)}$ we obtain the recursion relation:

$$\langle\langle \overbrace{A^{(0)} \cdots A^{(0)}}^{B-r} \overbrace{A^{(2)} \cdots A^{(2)}}^{k} \rangle\rangle_{\mathcal{F}(B,k;r)}$$

$$= nA \langle\langle \overbrace{A^{(0)} \cdots A^{(0)}}^{B-r} \overbrace{A^{(2)} \cdots A^{(2)}}^{k-1} \rangle\rangle_{\mathcal{F}(B,k-1;r)}$$

$$- 2r \langle\langle \overbrace{A^{(0)} \cdots A^{(0)}}^{B-r+1} \overbrace{A^{(2)} \cdots A^{(2)}}^{k-1} \rangle\rangle_{\mathcal{F}(B,k-1;r-1)}$$  \hfill (8.31)
The first term represents the bulk contribution. In the second term there is one extra insertion of \( A^{(0)} \) which has replaced a curvature operator at a ramification point, and there is one fewer \( A^{(2)} \) operator. The coefficient \( r \) in the second term comes from the fact that for each area integral there are \( r \) collisions with curvature insertions at ramification points. The factor of \(-2\) comes from the normalization of the contact term. Iterating this recursion relation, we are led to the following

\[
\langle \langle A^{(2)} \cdots A^{(2)} \rangle \rangle \mathcal{F}(B,k) = \sum_{l=0}^{k} \binom{k}{l} \frac{2^l B! (-1)^l}{(B-l)!} (nA)^{k-l} \langle \langle A^{(0)}(R_1) \cdots A^{(0)}(R_l) \rangle \rangle \mathcal{F}(B,0;B-l).
\]  

(8.32)

When \( l > B \) it is clear that the correlation function on the right vanishes, by ghost number counting. So that altogether

\[
\frac{1}{(2\pi)^B} \int_{\mathcal{F}^{(1)}} D[\mathcal{I},G] \prod_{l=1}^{B} \mathbf{R}_{2l-1} \mathbf{2l}[G^{2l-1},G^{2l}](Q_{l}) \exp -\frac{1}{2} \int_{\Sigma} f^* \omega \bigg|_{\mathcal{N}} = \sum_{k=0}^{\infty} \frac{B!}{k!(B-k)!} \sum_{l=0}^{\min[k,B]} \binom{k}{l} (-\frac{1}{2} nA)^{k-l} \langle \langle A^{(0)}(R_1) \cdots A^{(0)}(R_l) \rangle \rangle \mathcal{F}(B,0;B-l).
\]

(8.33)

Substituting in the RHS of (8.33) we obtain

\[
e^{-\frac{1}{2} nA} \sum_{k=0}^{B} \frac{B!}{k!(B-k)!} \langle \langle A^{(0)}(R_1) \cdots A^{(0)}(R_k) \rangle \rangle \mathcal{F}(B,0;r=B-k).
\]

(8.34)

So we are left with the integral

\[
\int_{\mathcal{F}^{(1)}} D[\mathcal{I},G] \prod_{l=1}^{B-k} \mathbf{R}_{2l-1} \mathbf{2l}[G^{2l-1},G^{2l}](R_{l}) A^{(0)}(R_{B-k+1}) \cdots A^{(0)}(R_{B})
\]

(8.35)

Now we use again the fact that we are only interested in the contribution of simple Hurwitz space. This space is a bundle over \( C_{0,B}/S_B \) with discrete fiber the set \( \Psi(n,B,G,L = B) \). Further the measure on Hurwitz space inherited from the path integral divides out by diffeomorphisms. Therefore the correlator in (8.34) is:

\[
\sum_{\psi \in \Psi(n,B,G,L = B)} \frac{1}{|C(\psi)|} \times \frac{1}{B!} \int_{C_{0,B}} \prod_{l=1}^{B-k} \mathbf{R}_{2l-1} \mathbf{2l}[G^{2l-1},G^{2l}](R_{l}) A^{(0)}(R_{B-k+1}) \cdots A^{(0)}(R_{B})
\]

(8.36)
The correlation function has singularities when any two ramification points \( R_I \) collide. In isolating the contributions of simple Hurwitz space we must ignore the singularities from the collisions of \( R_I, I \leq B - k \) with \( R_J, J \geq B - k + 1 \). Thus we replace (8.36) by the expression:

\[
\sum_{\psi \in \Psi(n,B,G,L=B)} \frac{1}{|C(\psi)|} \times \frac{1}{B!} \times \\
\int_{C_0,B-k \times (\Sigma_T)^k} \langle \prod_{I=1}^{B-k} R_{2I-1} \ 2I[G_{2I-1},G_{2I}](R_I) \rangle \wedge \omega(P_{B-k+1}) \wedge \cdots \wedge \omega(P_B) \tag{8.37}
\]

where \( P_J \in \Sigma_T \) are the images of the simple ramification points \( R_J \). We can do the integrals over the the wedge product of Kahler classes separately to get \( A^k \) (the area of \( \Sigma_T \times k \)). The remaining integral over the \( B - k \) curvature insertions is the same correlator appearing in the partition function. Thus we have:

\[
\langle \langle A^{(0)}(R_1) \cdots A^{(0)}(R_k) \rangle \rangle_{\mathcal{F}(B,0;r=B-k)} = \frac{(-A)^k}{B!} \frac{1}{|C(\psi)|} \sum_{\psi \in \Psi(n,B,G,L=B)} \chi(C_{B-k}(\Sigma_T)) \tag{8.38}
\]

Substituting in (8.34), and comparing with (8.4), we see that the contribution of simple Hurwitz space to the Path integral perturbed as in (8.7) agrees with the conjecture that the perturbation in (8.7) is equivalent to 2D Yang Mills at finite area.

**Remark.** Our discussion has only focused on the contact terms needed to reproduce the area polynomial of simple Hurwitz space. Higher contact terms will be affected by the presence of gravitational descendents of the area operator in the action. Thus we should consider perturbations generalizing (8.7) like:

\[
I_0 \rightarrow I_0 + \sum_{n \geq 0} \tau_n \int \sigma_n(A^{(2)}) \tag{8.39}
\]

In the original \( YM_2 \) theory there is a similar class of deformations of the theory obtained by adding higher Casimirs to the heat kernel Boltzman weight:

\[
C_2(R) \rightarrow \sum_{n \geq 2} t_n C_n(R) \tag{8.40}
\]

It is natural to conjecture that these classes of deformed theories are in fact equivalent. Experience from 2D gravity [58] leads us to expect that the change of variables \( \{t_n\} \rightarrow \{\tau_n\} \) can involve complicated nonlinear terms. Indeed nothing in the present discussion precludes the possibility that the pure \( C_2(R) \) \( YM_2 \) theory is equivalent to a perturbation of type (8.39) with \( \tau_n \neq 0 \) for \( n > 0 \).
9. Wilson Loops

The techniques of [15] extend to Wilson loop expectation values. In general the answer is expressed in terms of rather intricate gluing rules [15]. In this section we will restrict attention to the simplified case of the chiral theory. The string interpretation of these quantities is given by macroscopic loop amplitudes (familiar from gravity) with certain Dirichlet boundary data on the boundary of the worldsheet.

9.1. Observables

The natural observables in gauge theory are the Wilson loops. Let \( R \) be a finite-dimensional representation of \( SU(N) \) and let \( \Gamma \) be a piecewise-differentiable oriented curve \( \Gamma: S^1 \to \Sigma_T \). Such curves generically have at most double points as self-intersections and we will assume this to be the case. We define:

\[
W(R, \Gamma) \equiv \text{tr}_R(U_{\Gamma})
\]

\[
U_{\Gamma} = P \exp \oint_{\Gamma} A
\]  

(9.1)

we will often denote the image \( \Gamma \subset \Sigma_T \) by the same symbol.

As pointed out in [15] a more natural basis of observables for the \( 1/N \) expansion are the loop functions:

\[
\Upsilon(\vec{k}_\Gamma, \Gamma) \equiv \prod_{j=1}^{\infty} (\text{tr} U_{\Gamma}^j)^{k_j}
\]  

(9.2)

The vector \( \vec{k}_\Gamma = (k_{\Gamma}^1, k_{\Gamma}^2, \ldots) \) determines a conjugacy class (via cycle decomposition) in \( S_{m_{\Gamma}} \) where \( m_{\Gamma} = \sum jk_{\Gamma}^j \). By Frobenius reciprocity we have

\[
\Upsilon(\vec{k}_\Gamma, \Gamma) = \sum_{R \in Y_{m_{\Gamma}}} \chi_R(p_{\Gamma}) W(R, \Gamma)
\]  

(9.3)

where \( p_{\Gamma} \) is any element in the conjugacy class \( \vec{k}_\Gamma \).

9.2. Exact Answer: Nonintersecting Loops

Suppose that we have a collection \( \{ \Gamma \} \) of nonintersecting curves in \( \Sigma_T \). Let

\[
\Sigma_T - \Pi \Gamma = \Pi_c \Sigma_T^c
\]  

(9.4)

be the decomposition into disjoint connected components. Each component has \( G_c \) handles and \( b_c \) boundaries. Since \( \Sigma_T \) and \( \Gamma \) are each oriented, each curve \( \Gamma \) can be deformed into two curves \( \Gamma^\pm \) as in
Fig. 8: Using the orientation of the surface and of the Wilson line we can define two infinitesimal deformations of the Wilson line $\Gamma^\pm$.

We let $c^\pm_{\Gamma}$ denote the label of the component $\Sigma^c_{\Gamma}$ which contains $\Gamma^\pm$. The exact answer for correlation functions of Wilson loops is easily obtained from standard cutting and gluing techniques. One finds:

$$\left\langle \prod_{\Gamma} W(R_{\Gamma}, \Gamma) \right\rangle = \sum_{R(c)} \prod_{c} (\dim R(c))^{\chi(\Sigma^c_{\Gamma})} e^{-\frac{1}{2}A_c c_2(R(c))/N} \prod_{\Gamma} N^{R_{\Gamma}(c^+)}_{R(c^+_\Gamma), R_{\Gamma}}$$

(9.5)

where we sum over unitary irreps $R(c)$ for each component $c$, $N^{R_1}_{R_1, R_2}$ are the “fusion numbers” defined by the decomposition of a tensor product into irreducible representations

$$R_1 \otimes R_2 = \oplus_{R_3} N^{R_3}_{R_1, R_2} R_3$$

(9.6)

and $A_c$ denote the areas of the components $\Sigma^c_{\Gamma}$. Note that $\Sigma^c_{\Gamma}$ are open manifolds. When we speak of the Euler character we glue back in the $b_c$ boundary circles.

9.3. Chiral Expansion: Nonintersecting Loops

The chiral expansion of Wilson loop averages may be obtained directly from (9.5) without recourse to the gluing rules of [13]. To begin one derives a formula for the fusion numbers in terms of a sum over the symmetric group. This may be done by expressing them as integrals of characters, passing to the loop function basis, and then expressing in answer in terms of the symmetric group. The result, in a form useful for us, is:

$$\sum_{R \in Y_{m_{\Gamma}}} \chi_R(p_{\Gamma}) N^{R(c^+_\Gamma)}_{R(c^-_{\Gamma}), R} = \delta_{n(c^+_\Gamma), n(c^-_{\Gamma})+m_{\Gamma}} \frac{d^{R(c^+_\Gamma)}}{(n(c^+_\Gamma))!} \frac{d^{R(c^-_{\Gamma})}}{(n(c^-_{\Gamma}))!}$$

$$\sum_{u^-_{\Gamma} \in S_{n(c^-_{\Gamma})}} \sum_{u^+_{\Gamma}} \frac{\chi_{R(c^+_\Gamma)}(u^+_{\Gamma}) \chi_{R(c^-_{\Gamma})}(u^-_{\Gamma})}{d^{R(c^+_\Gamma)}} \frac{d^{R(c^-_{\Gamma})}}{d^{R(c^-_{\Gamma})}} \delta_{n(c^+_\Gamma)} [u^+_\Gamma, x_{\Gamma}(u^-_{\Gamma} \cdot p_{\Gamma}) x^{-1}_{\Gamma}]$$

(9.7)
where $d_R$ is the dimension of the representation of the symmetric group, and, in the last factor $(u^-_\Gamma \cdot p\Gamma)$ is the image under the natural embedding of symmetric groups

$$
i^\pm_\Gamma : S^R_{n(c^-_\Gamma)} \times S_{m\Gamma} \rightarrow S^R_{n(c^+_\Gamma)} \quad (9.8)$$

which takes the first permutation to a permutation of the first $n(c^-_\Gamma)$ entries and the second permutation to a permutation of the last $m\Gamma$ entries.

As usual we obtain the chiral sum by making the replacement in (2.3). Using the standard set of identities from [15] together with (9.7), the chiral expansion of (9.5) becomes:

$$\left\langle \prod_{\Gamma} \left[ k_{\Gamma}\right]^m Y(k_{\Gamma}, \Gamma) \right\rangle = \sum_{n(c) \geq 0} \sum_{\ell(c) \geq 0} \sum_{L(c) \geq 0} \sum_{v_1(c), \ldots, v_{L(c)}(c) \in S_{n(c)} s_1(c), \ldots, t_{G_E}(c) \in S_{n(c)} u^+_\Gamma \in S_{n(c^+_\Gamma)} x_\Gamma \in S_{n(c^+_\Gamma)} p\Gamma \in S_{m\Gamma}}$$

$$\prod_c e^{-\frac{1}{2} A_c(n(c) - \frac{n(c)^2}{3})} \left( -A_c \right)^{\ell(c)} (1) \frac{n(c) \chi(\Sigma_\Gamma^c) - \sum_i (n(c) - k_{\nu_i(c)}) \chi(\Sigma_{L(c)}(\Sigma_\Gamma^c))}{\ell(c)!} \prod_{\Gamma : c^-_\Gamma = c} u^-_\Gamma \prod_{\Gamma : c^+_\Gamma = c} u^+_\Gamma \prod_{i} v_i(c) T_{2, n(c)} G_c \prod_{i} [s_i(c), t_i(c)] \right] \prod_{\Gamma} \frac{1}{m_{\Gamma}!} \left[ \delta_{n(c^+_\Gamma), n(c^-_\Gamma) + m_{\Gamma}} \delta_{n(c^+_\Gamma)}(u^+_\Gamma, x_\Gamma(u^-_\Gamma \cdot p\Gamma) x^{-1}_\Gamma) \right] \right.$$ 

The normalization factors in front of $Y$ are chosen for later convenience; $|k_{\Gamma}|$ is the order of the conjugacy class determined by $p\Gamma$. Despite its extremely cumbersome appearance, this expression has an elegant geometrical content as we shall see. The fourth line defines the coverings $\Sigma_W^c$ of components $\Sigma_\Gamma^c$. The last line describes how these covering spaces $\Sigma_W^c$ are glued together.

### 9.4. Chiral Expansion: Intersecting Loops

If the loops $\Pi\Gamma$ have intersections (including self-intersections) then the exact answer for $YM_2$ is much more complicated than (9.3) and involves summing over 6$j$ symbols at the intersection vertices of the loops [13]. Nevertheless the chiral $1/N$ expansion for intersecting Wilson loops has a relatively simple set of rules which have been worked out in [15]. The only modification of (9.9) is the replacement:

$$\Sigma_\Gamma^c \rightarrow \tilde{\Sigma}_\Gamma^c \quad (9.10)$$
where $\Sigma^c_T$ is constructed from the open manifold $\Sigma^c_T$ by gluing in open intervals and vertices along $\partial \Sigma^c_T$. The rules for constructing $\tilde{\Sigma}^c_T$ are as follows. Consider $\Pi \Gamma$ as a graph. It has open edges $E_j$ and vertices $v_j$. Using the orientation we can define deformations $E_j^+$ and $v_j^{++}$. The edge $E_j^+$ is the deformation of the edge in the direction of $\Gamma^+$, the vertex is obtained by deforming into the $+$ region for each of the two intersecting curves. We glue the edges $E_j$ to the boundary of the component containing $E_j^+$ and we glue the vertices $v_j$ to the boundary of the component containing $v_j^{++}$. We may define the Euler character of $\tilde{\Sigma}^c_T$ to be

$$\chi(\tilde{\Sigma}^c_T) = \chi(\Sigma^c_T) + \sum_{E_j \in \tilde{\Sigma}^c_T} (-1) + \sum_{v_j \in \tilde{\Sigma}^c_T} (+1) \quad (9.11)$$

This is not a homotopy invariant, but it is a homeomorphism invariant. In the previous case of nonintersecting Wilson loops the modification $\Sigma^c_T \rightarrow \tilde{\Sigma}^c_T$ makes no difference since $\chi(S^1) = 0$. Gross and Taylor’s rule says that the only change we must make in (9.9) is the change $\Sigma^c_T \rightarrow \tilde{\Sigma}^c_T$ of (9.10)!

### 9.5. String interpretation

The string interpretation of the chiral nonintersecting Wilson loop averages in the $\Upsilon$ basis is stated very simply. The vectors $\vec{k}_\Gamma$ may be thought of as specifying the homotopy class of a map from a disjoint union of circles to $\Gamma$: We have $k^j_\Gamma$ $j$-fold coverings of the circle by the circle. The only change that is needed in the path integral of sections 6,7 is that we have a macroscopic loop amplitude: The worldsheet $\Sigma_W$ has a boundary. Data specifying the Wilson loops is encoded in the boundary conditions on $f : \Sigma_W \rightarrow \Sigma_T$. These boundary conditions state that $f : \partial \Sigma_W \rightarrow \Pi \Gamma$ is in the homotopy class $\{k_\Gamma\}$. Boundary conditions on the metric are standard [59], and follow from the requirements that 1) The loop $\Gamma$ is unparametrized, and 2) $P^\dagger$ is the adjoint of $P$. Let $n$ denote a normal vector and $t$ a tangent vector to $\partial \Sigma_W$. We take $g(n, t) = 0$ on $\partial \Sigma_W$. Correspondingly, vector fields $\xi$ generating diffeomorphisms satisfy $n.\xi = 0$ and $n^a t^b \nabla_{(a} \xi_{b)} = 0$. Boundary conditions for other fields follows from BRST invariance and invariance of the action. This string interpretation will be justified in the next section.

**Conjecture 9.1.** The string interpretation for the case of chiral intersecting Wilson loop amplitudes is obtained by the boundary condition that $f : \partial \Sigma_W \rightarrow \Pi \Gamma$ is in the homotopy class

$$\partial \Sigma_W \xrightarrow{\{k_\Gamma\}} \Pi \Gamma \xrightarrow{S^1 \Pi \Gamma} \Pi \Gamma \quad (9.12)$$

\[16\] Since $j \geq 1$ the homotopy class has an orientation preserving representative.
The first arrow describes a covering of circles by circles. The second is the homotopy class of the curves defining the Wilson loops.

9.6. Hurwitz spaces for surfaces with boundary

We now give an argument for the claim of the previous subsection.

**Definition 9.1.** A *boundary-preserving branched covering* is a map

\[ f : (\Sigma_W, \partial \Sigma_W) \to (\Sigma_T, \partial \Sigma_T) \]  

such that

1. \( f : \partial \Sigma_W \to \partial \Sigma_T \) is a covering map.
2. \( f : \Sigma_W - \partial \Sigma_W \to \Sigma_T - \partial \Sigma_T \) is a branched covering.

Equivalence and automorphism of such maps are defined in the obvious way. Note that the boundary components \( \partial \Sigma_W \) are unlabelled so \( \phi : (\Sigma_W, \partial \Sigma_W) \to (\Sigma_W, \partial \Sigma_W) \) can permute the boundaries.

By (1) \( f \) determines a class \( \vec{k}_T \) for each component \( \Gamma \) of \( \partial \Sigma_T \). Let us assume that \( \Sigma_T - \partial \Sigma_T \) is connected. Then, by (2) \( f \) determines a branch locus \( S(f) \subset \Sigma_T - \partial \Sigma_T \), an index \( n \), and an equivalence class of a homomorphism \( \psi_f : \pi_1(\Sigma_T - S(f), y_0) \to S_n \). We have the direct analog of the Riemann existence theorem Theorem 3.1:

**Proposition 9.1.** Let \( \Sigma_T \) be a connected, closed surface with boundary. Let \( S \subset \Sigma_T - \partial \Sigma_T \) be a finite set, and let \( n \) be a positive integer. There is a one-one correspondence between equivalence classes of boundary-preserving branched covers \( (9.13) \) with branch locus \( S \) and equivalence classes of homomorphisms \( \psi : \pi_1(\Sigma_T - S(f), y_0) \to S_n \).

**Proof.** The proof proceeds as before. We choose a representative for \( \psi \) and a basis of generators for \( \pi_1 \). Then we glue together \( n \) copies \( \Sigma_T \) according to the data given by the homomorphism. \( \blacklozenge \)

The maps that we will need for nonintersecting Wilson loops are considerably more complicated than boundary-preserving covers: we must allow for the possibility that the inverse images of the loops \( \Gamma \) contain loops which lie in the interior of \( \Sigma_W \). This leads us to introduce:

**Definition 9.2.** Suppose \( \Sigma_W \) is a closed oriented surface with boundary and \( \{\Gamma\} \) is a collection of nonintersecting oriented closed curves in \( \Sigma_T \). By a *covering map* \( f : \Sigma_W \to \Sigma_T \) with boundaries over \( \{\Gamma\} \) we mean a continuous orientation-preserving map \( f \) such that

1. \( f : \partial \Sigma_W \to \bigcup \Gamma \) is a covering map.

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2. $f^{-1}(\Pi \Gamma)$ is a disjoint union of circles.
3. $f : \Sigma_W - f^{-1}[\Pi \Gamma] \to \Sigma_T - \Pi \Gamma$ is a branched covering.
   
   Equivalence and automorphism are defined as before.

   We now describe these maps in some detail. As before, by (1) $f$ determines a homotopy class $\{\vec{k}_\Gamma\}$ of $\partial \Sigma_W \to \Pi \Gamma$. By (3), we have a covering

   $$f^c : \Sigma^c_W \to \Sigma^c_T$$

   (9.14)

   where $\Pi_c \Sigma^c_W = \Sigma_W - f^{-1}(\Pi \Gamma)$. The number of sheets of a covering will be different for different components of $\Sigma^c_T$. An elementary example is:

   ![Diagram showing different components of the target space can be covered by different number of sheets.](image)

   **Fig. 9**: Different components of the target space can be covered by different number of sheets.

   In general, the coverings $f^c$ are boundary-preserving branched coverings, the boundaries of $\Sigma^c_W$ covering the boundaries of $\Sigma^c_T$. From each covering $f^c$ we obtain a branch locus $S_c \subset \Sigma^c_T$, index $n(c)$, and equivalence classes of homomorphisms: $\psi_c : \pi_1(\Sigma^c_T - S_c, y_0,c) \to S_{n(c)}$. By (2) the inverse image under $f$ of any loop $\Gamma$ may be divided into interior loops and boundary loops, that latter living in $\partial \Sigma_W$. The different surfaces $\Sigma^c_W$ must be smoothly glued together along the interior loops of $f^{-1}(\Gamma)$. This requirement results in the gluing conditions (9.5) to (9.20) below.
First, above a loop $\Gamma$ there are $\sum_j j^j$ components with $m_\Gamma$ sheets belonging to $\partial \Sigma_W$. (Recall that $m_\Gamma = \sum j^j$.) The remaining components lie in the interior of $\Sigma_W$. Using the orientation we see that if we perturb the curves $f^{-1}(\Gamma)$ in the plus direction we get an $n(c^+_\Gamma)$-sheeted covering of $\Gamma^+$. On the other hand, since a perturbation of the boundary curves of $\Sigma_W$ in the minus direction takes us off the surface $\Sigma_W$ we can only perturb the interior curves of $f^{-1}(\Gamma)$ in the minus direction. Thus we get an $n(c^-_\Gamma)$-sheeted covering of $\Gamma^-$. The situation may be summarized in the following figure:

![Diagram showing the covering map](image)

**Fig. 10:** Locally the covering map looks like this. Above $\Gamma^-$ there are only interior curves. Above $\Gamma^+$ there are interior and boundary curves.

From the figure the first gluing condition

$$n(c^+_\Gamma) = n(c^-_\Gamma) + m_\Gamma$$

becomes evident.

We obtain the second gluing condition by starting from Proposition 9.1. As in Fig. 1 of sec. 3.1 we can choose generators

$$\alpha_i(c), \beta_i(c), \sigma_i(c), \gamma^+_i, \gamma^-_i$$

(9.16)
of \( \pi_1(\Sigma^c_T - S_c, y_{0,c}) \), such that \( \alpha_i(c), \beta_i(c) \), run around handles, \( \sigma_i(c) \) become trivial if we fill in branch points, \( \gamma_i^\pm \) become trivial if we fill in \( \Gamma^\pm \), we have the relation:

\[
\prod \gamma_i^+ \prod \gamma_i^- \prod \sigma_i(c) \prod [\alpha_i(c), \beta_i(c)] = 1
\]  

(9.17)

and the covering \( \Sigma^c_W \rightarrow \Sigma^c_T \) can be constructed by gluing together copies of \( \Sigma^c_T \) using the homomorphism \( \psi \) as in the LHS or RHS of Figure 12.

Consider now two components \( \Sigma^c_T \) and \( \Sigma^{c'}_T \) which must be glued along a loop \( \Gamma \), as in Figure 12. Suppose that \( c^+ = c, c^- = c' \), and that \( \gamma^\pm \) surrounds \( \Gamma^\pm \). The homotopy type of the covering of the interior circles above \( \Gamma^- \) is given by the conjugacy class of

\[
\psi^- = \psi^{-\Gamma}(\gamma^-)
\]  

(9.18)
while the homotopy type of the covering of the interior circles above $\Gamma^+$ is given by the conjugacy class of
\[
\psi^+_\Gamma = \psi^+_c(\gamma^+)
\] (9.19)
Because the interior curves must be smoothly glued together there must exist a re-labelling $x_\Gamma \in S_{n(c^+_\Gamma)}$ of the sheets above $\Sigma^+_T$ such that
\[
\psi^+_\Gamma = x_\Gamma [\iota^+_T(\psi^-_T p_\Gamma)] x_\Gamma^{-1}
\] (9.20)
where $\iota^+_T$ is the embedding (9.8) and $p_\Gamma$ is any element in the conjugacy class $k_\Gamma$. Finally two equivalent coverings $\tilde{f} = f \circ \phi$ will lead to the same data but with $\psi_c, \tilde{\psi}_c$ differing by an inner automorphism of $S_{n(c)}$.

In summary, to any map satisfying Definition 9.2 we can unambiguously associate an equivalence class of covering data $D$. This data is composed of

D1. Branch loci $S_c \subset \Sigma^+_T$, basepoints $y_{0,c}$, indices $n(c)$, boundary data $k_\Gamma$.
D2. Generators (9.16) of $\pi_1(\Sigma^+_T - S_c, y_{0,c})$
D3. Homomorphisms $\psi_c : \pi_1(\Sigma^+_T - S_c, y_{0,c}) \to S_{n(c)}$. These are obtained after choosing a labelling of the inverse images of $y_{0,c}$.
D4. Elements $p_\Gamma \in S_{m_\Gamma}$. These are obtained after choosing basepoints $y_\Gamma \in \Gamma$ and choosing a labelling of the points lying on the boundaries of $\Sigma^+_W$ in $f^{-1}(y_\Gamma)$.
D5. Elements $x_\Gamma \in S_{n(c^+_\Gamma)}$

These data are required to satisfy the conditions:

C1. $n(c^+_\Gamma) = n(c^-_\Gamma) + m_\Gamma$
C2. $\forall \Gamma \exists p_\Gamma \in S_{m_\Gamma}$ such that $[p_\Gamma] = k_\Gamma$ and $\psi^+_\Gamma = x_\Gamma [\iota^+_T(\psi^-_T p_\Gamma)] x_\Gamma^{-1}$

Two sets of data $D$ and $\tilde{D}$ satisfying these conditions will be considered to be equivalent if the following relations holds:

E1. The bases $\alpha_i(c)$, ..., $\tilde{\alpha}_i(c)$, ..., differ by $Aut(\pi_1(\Sigma^+_T - S_c, y_{0,c}))$.
E2. $\tilde{\alpha}_i = \alpha_i$, ..., and $\exists w(c) \in S_{n(c)}$, $w(\Gamma) \in S_{m_\Gamma}$ such that
\[
\tilde{\psi}_c(\cdot) = w(c)\psi_c(\cdot) w(c)^{-1}
\]
\[
\tilde{x}_\Gamma = w(c^+_\Gamma) x_\Gamma w(c^-_\Gamma)^{-1} w(\Gamma)^{-1}
\]
\[
\tilde{p}_\Gamma = w(\Gamma) p_\Gamma w(\Gamma)^{-1}
\] (9.21)

We define $C(\psi_c, x_\Gamma, p_\Gamma)$ to be the subgroup of $\prod S_{n(c)} \prod S_{m_\Gamma}$ which leaves the set $\{\psi_c, x_\Gamma, p_\Gamma\}$ fixed under the action defined above.

The above discussion has proved half of:
**Proposition 9.2.** There is a one-one correspondence between equivalence classes of covering maps with boundaries over $\Pi \Gamma$ with prescribed data:

1. $f$ defines an $n(c)$-fold cover of $\Sigma_T^c$, with branch locus $S_c$.
2. $\partial \Sigma_W \to \Pi \Gamma$ is of homotopy type $\tilde{\kappa}_\Gamma$.

and equivalence classes of covering data $D$ as defined above.

**Proof.** Given the data $D$ we first construct boundary-preserving branched coverings $f^c : \Sigma_W^c \to \Sigma_T^c$ in the standard way. Then, with the labelling of the sheets specified by the data we glue the loops covering $\Gamma^-$ to the interior loops covering $\Gamma^+$ using the relabelling $x_\Gamma$. This gluing is smooth by the conditions defining $D$. Equivalent data induce equivalent coverings. Note in particular that $\prod_c S_{n(c)} \prod \Gamma S_{m_\Gamma}$ acts transitively on equivalence classes of relation $E^2$ and that the number of distinct elements in a class is given by

$$\frac{\prod n(c)! \prod m_\Gamma!}{|C(\psi_c, x_\Gamma, p_\Gamma)|}$$

It is also clear from this construction that the automorphism group of the covering is $\text{Aut}(f) = C(\psi_c, x_\Gamma, p_\Gamma)$. ♠

**Definition 9.3.** The Hurwitz space $H(h, b, n(c), S_c, \tilde{\kappa}_\Gamma)$ is the space of equivalence classes of covers $f$ with boundaries above $\Pi \Gamma$, where $\Sigma_W$ has $h$ handles, $b$ boundaries, such that $f$ is an $n(c)$-fold covering of $\Sigma_T^c$ with branch locus $S_c$, and restricts to $\partial \Sigma_W \to \Pi \Gamma$ of homotopy type $\tilde{\kappa}_\Gamma$. The union of these spaces over sets of branch points $S_c$ with $L(c)$ elements defines the Hurwitz space $H(h, b, n(c), L(c), \tilde{\kappa}_\Gamma)$. The compactification of this space with different $L(c)$ defines the Hurwitz space $H(h, b, n(c), \tilde{\kappa}_\Gamma)$.

A corollary of Proposition 9.2 is that $H(h, b, n(c), L(c), \tilde{\kappa}_\Gamma)$ is a discrete fibration over $\prod_c C_{L(c)}(\Sigma_T^c)$. Therefore, the orbifold Euler characteristic is defined as usual:

$$\chi_{\text{orb}} \left( H(h, b, n(c), L(c), \tilde{\kappa}_\Gamma) \right) = \prod_c \chi(C_{L(c)}(\Sigma_T^c)) \sum_{[f] \in H(h, b, n(c), S_c, \tilde{\kappa}_\Gamma)} \frac{1}{|\text{Aut}(f)|}$$

where, on the RHS we may choose any set $S_c$ of $L(c)$ distinct points in $\Sigma_T^c$.

Now finally let us compare with the chiral expansion (9.9). If we first put $A(c) = 0$ then the effects of homotopically trivial loops $\Gamma$ can be shown to be trivial, simply contributing overall powers of $1/N$. Nevertheless, homotopically nontrivial loops have nontrivial correlators. Combining (9.9) with Proposition 9.2 we see that the chiral expansion becomes:

$$\sum_{n(c), h, b \geq 0} \left( \frac{1}{N} \right)^{2h+b-2} \chi_{\text{orb}} \left( H(h, b, n(c), \tilde{\kappa}_\Gamma) \right)$$

(9.24)
as in the previous sections. Thus, with an appropriate choice of contact terms decoupling the $\pm$ sectors \[17\] the macroscopic loop path integral described in section 9.4 will produce the product of chiral Wilson loop averages, in complete analogy to the partition function.

Finally, we include the effects of the area in the Wilson loop averages, as computed in (9.3). The structure is exactly the same as that found in sec. 8 and the same discussion shows that - at least on the simple Hurwitz sub-space $H(h, b, n(c), L(c) = B(c), \vec{k}_\Gamma)$ the contact terms account for the area.

9.7. Divers Remarks on Wilson Loop Averages

We gather several miscellaneous remarks in this subsection.

1. First, even in the chiral theory the other $A_c$-dependent polynomials in (9.9) associated with non-simple Hurwitz spaces remain to be analyzed.

2. The discussion should be generalized to the chiral case allowing intersections of the graph $\Pi\Gamma$. The conjectural string formulation (Conjecture 9.1) will follow from Conjecture 9.2. The chiral intersecting loop amplitudes may be discussed as in section 9.6 with by simply modifying condition 2 in definition 9.2 by allowing $f^{-1}(\Pi\Gamma)$ to be a graph and modifying Proposition 9.2 and Definition 9.3 by allowing the branch locus $S_c$ to intersect $\Gamma$. In order to avoid double-counting we must let $S_c$ approach from the + side only. The only effect on Proposition 9.2 and (9.23) is the replacement: $\Sigma_c \rightarrow \tilde{\Sigma_c}$.

3. The extension of our discussion to the coupled theory is very nontrivial. The rules of [15] become considerably more elaborate. Some of the issues arising in this extension appear to be quite relevant to establishing a string picture of four-dimensional QCD along the lines of [60].

4. An interesting open problem is the derivation of the Migdal-Makeenko loop equations from the topological string theory point of view. Given our experience with 2D gravity, we may guess that there is an analog of $W_\infty$ constraints on the partition function which is equivalent to the loop equations, and that these $W_\infty$ constraints follow from a contact term analysis.

5. Further investigation of Wilson loop amplitudes also promises to yield some extremely interesting insights in mathematics. Recently, V.I. Arnold has discovered new invariants of plane curves (immersions) [61]. We remark that if $S_k(x_1, x_2, \ldots)$ are elementary symmetric polynomials then $S_l(\frac{\partial}{\partial A_c})|A_c=0 \langle \prod \Upsilon \rangle$ are also invariants of immersions (by

\[17\] See section 11 below.
the area-preserving diffeomorphism invariance of $YM_2$.) The relations of these invariants to the mathematics of covering spaces may well be very rich.

6. Finally we mention that, following [92], [93], [15], one should be able to incorporate dynamical quarks into the present framework. One must modify the string theory by turning it into an open-closed (Dirichlet) string.

We hope to return to these issues in future work.

10. The coupled theory

We have seen that some aspects of the chiral $YM_2$ theory are related to the topological string theory of sec. 6. In the next two sections we will show how the full ("coupled") theory also fits into the framework of the theory of a topological string theory. We will restrict attention to reproducing the partition function (2.2). The key observation is that $Z$ in (2.2) differs from a product of chiral partition functions $Z^+ Z^-$ through the contribution of boundaries of the space $\text{Maps} \times \text{Met}(\Sigma_W)$.

In sections 10.1-10.5 we will use the geometrical picture of [15] to show that the $A=0$ partition function (2.2) computes Euler characters of spaces of maps from singular surfaces $\Sigma_W$ to $\Sigma_T$. In 10.1, 10.2 we describe in detail the maps and worldsheets in question. In 10.3 we develop a combinatoric approach to the space of maps. In 10.4 we describe the relation of the space of maps to configuration spaces. In 10.5 we write the zero area QCD sum in terms of Euler characters of the spaces of maps.

10.1. Degenerated coupled covers

Let $\Sigma_W$ be a smooth surface, perhaps with double points. Topologically this means that there is a set of points $\{P_1, \ldots, P_d\} \subset \Sigma_W$ such that a local neighborhood $D_i$ of $P_i$ is the one-point union of disks $D_i^{(1)}, D_i^{(2)}$:

$$D_i = D_i^{(1)} \sqcup D_i^{(2)}/(P_i^{(1)} \sim P_i^{(2)})$$

(10.1)

The normalization of $\Sigma_W$, $N(\Sigma_W)$ is the smooth surface obtained by replacing $D_i \rightarrow D_i^{(1)} \sqcup D_i^{(2)}$. $N(\Sigma_W)$ may be connected or disconnected. A map $f : \Sigma_W \rightarrow \Sigma_T$ defines a normalized map $N(f)$ in a natural way.

**Definition 10.1** Let $\Sigma_W$ be a surface with double points. A *degenerate branched cover* $f : \Sigma_W \rightarrow \Sigma_T$ is a continuous map such that

1. $N(f) : N(\Sigma_W) \rightarrow \Sigma_T$ is a branched cover, and
2. If \( P_i^{(1)}, P_i^{(2)} \) are the normalizations of the double points \( P_i \) then
\[
\text{Ram}(N(f), P_i^{(1)}) = \text{Ram}(N(f), P_i^{(2)}) = e_i
\]
where \( \text{Ram} \) is the ramification index.

The cover in the neighborhood of the double point may be thought of as a degeneration of a family of 2\(e_i\)-sheeted covers of annuli by annuli degenerating to the cover of one disk by two disks.

One of the very strange aspects of the \( YM_2 \) partition function is that it appears to involve maps which are branched covers which are neither holomorphic nor anti-holomorphic.

**Definition 10.2** A coupled map \( f : \Sigma_W \to \Sigma_T \) of Riemann surfaces is a continuous map such that there are circles \( S_i \) which separate \( \Sigma_W \) into two disjoint surfaces
\[
\Sigma_W = \Sigma_W^+ \sqcup \Sigma_W^- / (S_i^+ \sim S_i^-)
\]
and such that \( f : \Sigma_W^+ \to \Sigma_T \) is holomorphic while \( f : \Sigma_W^- \to \Sigma_T \) is antiholomorphic.

An example of such a map is a mapping of the complex plane to a closed disk given by \( f(z) = z^n \) for \(|z| \leq 1\) and \( = 1/\bar{z}^n \) for \(|z| \geq 1\). Note that \( df \) is discontinuous along the unit circle.

Our main object of interest combines the above two notions and will be called a degenerated coupled cover. It is a coupled map, where the circles \( S_i \) have been shrunk to points. More formally, we state

**Definition 10.3** A degenerated coupled cover (dcc) \( f : \Sigma_W \to \Sigma_T \) of Riemann surfaces is a map such that if we take the normalization of the double points \( \{Q_1, \ldots Q_d\} \) then we have a disjoint decomposition into smooth surfaces \( N(\Sigma_W) = N^+(\Sigma_W) \sqcup N^-(\Sigma_W) \) such that \( f^+ : N^+(\Sigma_W) \to \Sigma_T \) is holomorphic, \( f^- : N^-(\Sigma_W) \to \Sigma_T \) is antiholomorphic and such that
\[
\text{Ram}(f^+, Q_i^+) = \text{Ram}(f^-, Q_i^-)
\]
Two dcc’s \( f_1 \) and \( f_2 \) are said to be equivalent if there is a homeomorphism \( \phi : \Sigma_W \to \Sigma_W \) such that \( f_1 \circ \phi = f_2 \). A homeomorphism \( \phi \) of \( \Sigma_W \) is an automorphism of the dcc if \( f \circ \phi = f \). If \( \Sigma_W \) has no double points then the map is just a branched cover, either holomorphic or antiholomorphic.

As with branched covers we may associate several natural quantities to a dcc. \( f^\pm \) define indices \( n^\pm(f) \), branch loci:
\[
S^\pm(f) = \{ f^\pm(Q) | \text{Ram}(f^\pm, Q) > 1 \}
\]
double point locus:

\[ S^T(f) = \{ f(Q) | Q \text{ is a double point} \} \tag{10.6} \]

tube number

\[ d(f, P) = Card[\{ Q | P = f(Q), Q \text{ is a double point} \}] \tag{10.7} \]

ramification vectors:

\[ \bar{r}^\pm(f, P) = (r_1^\pm, r_2^\pm, \ldots) \]

\[ r_j^\pm(f, P) = Card\left[ \{ Q | f^\pm(Q) = P, Ram(f^\pm, Q) = j \} \right] \tag{10.8} \]

and, finally, homomorphisms:

\[ \psi^\pm: \pi_1(\Sigma_T - S^\pm(f), y_0) \to S_n^\pm \tag{10.9} \]

In the ordinary case the specification of the branch locus and the homomorphism \( \psi \) essentially specified the equivalence class of the map \( f \) as in Theorem 3.1. This is no longer the case for dcc’s because there are many ways in which the “double points” connecting the different ramification points can be introduced. This leads us to the combinatoric discussion of subsection 10.3.

10.2. Degenerating coupled covers

The answer provided by \( YM_2 \) demands a further refinement of the above ideas. We must take into account the way in which a family of covering maps has degenerated to a dcc.

We will define a local degenerating family of coupled covers to be specified by the following data.

1. We have a plumbing fixture degenerating to the double point of \( \Sigma_W \):

\[ U_q = \{ (z_1, z_2) | z_1z_2 = \eta q, q \leq |z_1|, |z_2| < 1 \} \tag{10.10} \]

where \( 0 \leq q < 1 \) and \( \eta \) is an \( n^{th} \) root of unity for a positive integer \( n \).

2. On the plumbing fixture we have a family of covering maps

\[ f^{q,n}(z) = \begin{cases} z_1^n & \text{for } q^{1/2} \leq |z_1| < 1 \\ z_2^n & \text{for } q^{1/2} \leq |z_2| < 1 \end{cases} \tag{10.11} \]

Notice that \( n \) different degenerating complex structures on \( \Sigma_W \) determine maps \( f^{q,n} \) projecting to the same target space. Fig. 13 illustrates the model in the case where \( n = 2 \).

\[ \]
Fig. 12: Local model for a degenerating coupled cover with $n=2$. The region between stripes single-covers the annulus.

We will want to distinguish different $f$’s corresponding to these $n$ different degenerations, so we introduce:

**Definition 10.4** A *degenerating* coupled cover (Dcc) is a dcc equipped with a choice of a locally degenerating family of coupled covers ($f^{q,n}$) for each double point.

**Remarks.**
1. In a degenerating family $f^{q,n}$ is not differentiable along $|z_i| = q^{1/2}$ because the normal derivative is discontinuous. It lives in the space of piecewise differentiable maps rather than in the space of $C^\infty$ maps. The discontinuity is proportional to $q^{n-1/2}$ and goes to zero when $q$ goes to zero, for $n > 1$.

2. Our definition is admittedly somewhat *ad hoc* and could be considerably improved.

The partition function of $YM_2$ computes the Euler characters of spaces of ‘degenerating coupled covers’, as opposed to those of dcc’s (see Proposition 10.1). The introduction of the discrete choice of degenerating family accounts for an important combinatoric factor proportional to the product over all the double points, of the index of the ramification points being joined at each double point.

3. Much has been made on the suppression of “folds” and “fold degrees of freedom” in $YM_2$. It is perhaps worth noting that at $q \neq 0$ the map $f^{11}$ has a fold, but this
fold disappears for $q \to 0$. In general, in the formulation of this paper folds are suppressed because they are incompatible with holomorphy of the map $f$.

10.3. Combinatoric description of degenerated coupled covers

We give now a combinatoric description of dcc’s, establishing a 1-1 correspondence between data defined in terms of symmetric groups and maps defined geometrically in the previous subsection. We will discuss here dcc’s with parameters $n^\pm, B^\pm, S = S^+ \cup S^- \cup S^T$ fixed. We denote $L = |S|$. When we wish to emphasize the cardinality of $S$ we write $S(L)$.

Consider a dcc. Pick a base point $y_0$ on the target space and label the inverse images $x^+_1$ to $x^+_n$ on the holomorphic side and $x^-_1$ to $x^-_n$ on the antiholomorphic side. Inclusion gives natural maps:

$$\pi_1(\Sigma_T - S(L), y_0) \xrightarrow{i_*} \pi_1(\Sigma_T - S^\pm(f), y_0) \xrightarrow{j_*} \pi_1(\Sigma_T, y_0)$$

(10.12)

The map $i_*$ naturally defines homomorphisms $\psi^\pm_L : \pi_1(\Sigma_T - S(L), y_0) \to S_n^\pm$ which factor through the homomorphisms $\psi^\pm$ of the previous section. Now choose a set of generators $\alpha_i, \beta_i, \gamma(P), P \in S(L)$ of $\pi_1(\Sigma_T - S(L), y_0)$. Once we have chosen a set of generators, each loop $\alpha_i, \beta_i, \gamma(P)$ defines a pair of permutations $(s_i, \tilde{s}_i), (t_i, \tilde{t}_i)$, and $(v(P), w(P))$ in $S_{n^+} \times S_{n^-}$.

The behavior of a dcc at a double point determines some further data from the following construction. Let us choose a representation of $\Sigma_T - S$ as in Figure 1 of section 3.1. If $\gamma(y_0, P)$ is a curve from $y_0$ to $P$ then we may lift this curve with $f^\pm$. We denote the endpoint of the lifted curve by $x^\pm_a \cdot \gamma(y_0, P)$, where we choose $x^\pm_a$ as the lift of the initial point. If $v(P)$ has a cycle $(a_1, \cdots, a_k)$ of length $k$ then $x^\pm_a \cdot \gamma(y_0, P)$ will be ramification point $Q^+$ over $P$ of index $k$. Thus, a choice of curves $\gamma(y_0, P)$ allows us to define a pairing of the cycles in $v(P)$ with those in $w(P)$. To be precise, we introduce the following definition.

**Definition 10.5** Let $v \in S_{n^+}$ and $w \in S_{n^-}$. Let $Cyc(v)$ be the set of cycles in the cycle decomposition of $v$, and $Cyc(w)$ be the set of cycles of $w$. A pairing of $(v, w)$ is a subset $K \subseteq Cyc(v) \times Cyc(w)$ such that

1. $(\alpha, \beta) \in K$ only for cycles $\alpha, \beta$ of equal length.
2. Projections $K \to Cyc(v)$ and $K \to Cyc(w)$ are injective.

The second condition expresses the fact that a ramification point in the holomorphic sector can be connected to at most one ramification point in the antiholomorphic sector.
i.e there are only double points and no higher singularities. The cardinality $|K|$ is the number of pairings. Let $J_{vw}$ denote the set of all pairings of $(v, w)$.

**Example** Suppose $v = (12)^+ (3)^+ (4)^+$ and $w = (12)^- (34)^-$. If $K = \{(12)^+, (12)^-\}$, then one pairing has been made and the other cycles have been left unpaired. This pairing is illustrated by the figure below. This means that in the degenerate branched cover inducing this map the point $x_1^+ \cdot \gamma(y_0, P)$ is a double point coinciding with $x_1^- \cdot \gamma(y_0, P)$.

![Fig. 13: Glueing together ramification points according to the data of a pairing.](image)

Thus, in summary, a dcc, together with a choice of basis of $\pi_1$ and curves $\gamma(y_0, P), P \in S(L)$ describes elements

1. $(s_i, t_i) \in S_{n+}$
2. $(\tilde{s}_i, \tilde{t}_i) \in S_{n-}$
3. $(v(P), w(P)) \in S_{n+} \times S_{n-} \forall P \in S(L)$.
4. $K_P \in J_{v(P), w(P)} \forall P \in ST$

This data we call a *configuration*. Moreover, conjugation by $S_{n+} \times S_{n-}$ acts on the data 1–4, and defines an equivalence relation. We let CFG stand for the set of equivalence classes. The subgroup of $S_{n+} \times S_{n-}$ which leaves an element $e$ of CFG invariant is called $C(e)$.

**Example**: Suppose $L$ is 1, $n^+ = n^- = 4$, and there is one tube over the branch point $P$. Let $v(P) = v$ and $w(P) = w$, where $v$ and $w$ are the same as in the previous example. A possible CFG, $e$, has as representative the *configuration* defined by $K = \{(12)^+, (12)^-\}$ together with the $s$’s, $t$’s and $\tilde{s}$’s, $\tilde{t}$’s. Conjugating by the permutation $(12)^-(34)^-$ leaves this pairing invariant. Suppose it also leaves the $\tilde{s}$’s and $\tilde{t}$’s invariant. Then $(1, (12)^-(34)^-)$ is an element of $C(e)$. The permutation $(1, (14)^-(23)^-)$, on the other hand, does not leave
the pairing $K$ invariant although it does leave $w$ invariant. So it cannot be an element of $C(e)$.

**Proposition 10.1** There is a one-one correspondence between elements of CFG and equivalence classes of degenerated coupled covers (dcc’s).

**Proof**
We describe the proof in three steps.

1. Equivalent *configurations* come from equivalent dcc’s. Suppose two maps $f_1 : \Sigma_1 \to \Sigma_T$ and $f_2 : \Sigma_2 \to \Sigma_T$ determine equivalent *configurations* related by a permutation $g$ in $S_{n^+} \times S_{n^-}$. Delete the set $S(L)$ from $\Sigma_T$ and its inverse images from $\Sigma_1$ and $\Sigma_2$ to give $\tilde{\Sigma}_1$ and $\tilde{\Sigma}_2$. Then $f_1, f_2$ restrict to unbranched covers of $\Sigma_T - S(L)$ by $\tilde{\Sigma}_1$ and $\tilde{\Sigma}_2$ respectively. The proof that $f_1^\pm$ and $f_2^\pm$ give equivalent branched covers follows from Theorem 3.1. In the inverse image of $y_0$, $\phi$ restricts to the permutation $g$ which conjugates the *configuration* associated to $f_1$ into that associated with $f_2$. In the case of ordinary branched covers we just use continuity to complete the homeomorphism $\phi$ over the deleted points. In the case of dcc’s we have to prove that when two points of $\tilde{\Sigma}_1$ over $P$ get identified their images under the homeomorphism $\phi$ also get identified as $\Sigma_2$ is reconstructed from $\tilde{\Sigma}_2$. This follows from the uniqueness of lifted paths \[25\] with given starting point on the cover, which implies that for any $x_i$

$$ (\phi(x_i)) \cdot \gamma(y_0, P^i) = \phi(x_i \cdot \gamma(y_0, P^i)) \quad (10.13) $$

where $P^i$ is a point infinitesimally close to $P$.

2. Equivalent dcc’s determine equivalent *configurations*. This part again uses the proof in the case of ordinary branched covers together with uniqueness of lifted paths.

3. The map from CFG to equivalence classes of dcc’s is onto. This third part follows from the usual proof in the case of branched covers together with our choice of allowed pairings in CFG.

A corollary of Proposition 10.1 is that the group $\text{Aut} f$ is isomorphic to $C(e)$, where $e \in \text{CFG}$ represents the equivalence class of $f$.

**Remarks**

1. Each element $e$ of CFG corresponds to one *degenerated* coupled cover. We denote by $m(e)$, the number of *degenerating* coupled covers associated to it. $m(e)$ is the product of the common cycle lengths over all the pairings.
2. \textit{Aut} is a subgroup of \textit{Aut}^+ \times \textit{Aut}^−, and therefore \textit{C}(e) is a subgroup of \textit{C}(ψ^+) \times \textit{C}(ψ^-) where \( e \) is an equivalence class of dcc’s corresponding to a pair of equivalence classes \( ψ^+, ψ^- \) of branched covers. In general \textit{C}(e) is a proper subgroup. As in section 4.2 we see that \textit{Aut} is a subgroup of \textit{Aut}(Σ_W), where \( Σ_W \) can now have double points.

10.4. Coupled Hurwitz Space

Now let \( CHS(h, G) \) be the space of degenerating coupled covers from a surface of genus \( h \) to a surface of genus \( G \). Let \( S \) be the union of the branch and tube loci. Define \( CHS(h, G, L) \) to be the space of dcc’s for which the set \( S \) has \( L \) points. We isolate the subspace of coupled branched covers where \( n^+ \) is the total degree of the map from holomorphic-sector, \( B^+ \) is the holomorphic branching number, \( n^−, B^- \) are corresponding quantities for the anti-holomorphic sector; and \( D \) is the total number of double points. We call this \( CHS(n^±, B^±, L, D) \). If we specify the locus \( S(L) \) then we define the finite set \( CHS(n^±, B^±, S(L), D) \). The space \( CHS(n^±, B^±, L, D) \) is a bundle over the configuration space of \( L \) points on \( Σ_T \), with discrete fibre \( CHS(n^±, B^±, S(L), T) \).

Following the reasoning of section 5 we may write a formula for the orbifold Euler characteristic of \( CHS(n^±, B^±, L, D) \). By definition this may be taken to be:

\[
\chi\left(CHS(n^±, B^±, L, D)\right)_{\text{orb}} \equiv \chi(\mathcal{C}_L(Σ_T)) \sum_{[f] ∈ CHS(n^±, B^±, S(L), D)} \frac{1}{|Aut(f)|} \tag{10.14}
\]

We first show how to count equivalence classes of degenerating coupled covers \([f]\) inducing the data (10.7)–(10.9) compatible with \( n^±, B^±, L, D \).

It will be convenient to introduce the following quantities. Given two vectors \( \vec{r}^± \), of nonnegative integers with almost all entries zero, we define the polynomial

\[
φ(\vec{r}^+, \vec{r}^-, x) = \prod_{j=1}^{∞} \left[ 1 + \sum_{ℓ=1}^{∞} x^ℓ ℓ! \binom{r^+_j}{ℓ} \binom{r^-_j}{ℓ} \right] \tag{10.15}
\]

\[
= \sum_{t=0}^{∞} x^t φ(t, \vec{r}^+, \vec{r}^-).
\]

The binomial coefficient \( \binom{r^+_j}{ℓ} \) is defined to be zero for \( r^+_j < ℓ \). If \( v \) and \( w \) are in symmetric groups we also define \( p^\text{d}_{v ⊗ w} \) to be equal to \( φ(d, \vec{r}^+, \vec{r}^-) − δ(v, w) \) where \( r^± \) encode the cycle decompositions of \( v, w \) and \( δ(v, w) = 1 \) if \( (v = 1, w = 1) \) and zero otherwise.
Proposition 10.2. Suppose we are given \( n^\pm > 0, B^\pm, D \) and the set \( S(L) \). Then, we have:

\[
\sum_{e \in \text{CFG}} \frac{m(e)}{|C(e)|} = \sum_{[f] \in CHS(n^\pm, B^\pm, S(L), D)} \frac{1}{|\text{Aut} f|} = \sum_{d_1, d_2, \ldots, d_L} \sum_{s^\pm_i, t^\pm_i, v_i, w_i} \frac{1}{n! n^{-!}}
\]

(10.16)

where the first sum is over elements of CFG compatible with the data, the second sum is over equivalence classes of degenerating coupled covers inducing the specified data, and the \( v_i, w_i \) in the third sum are compatible with total branching numbers \( B^\pm \).

Proof. The first equality follows from Proposition 10.1. To derive the second equality we note that for a given pair of homomorphisms which determine (anti-)holomorphic maps \( f^\pm : N^\pm(\Sigma_W) \to \Sigma_T \), there are many ways to introduce double points to define a dcc. We sum over all possible numbers of double points \( d_1 \cdots d_L \) compatible with \( d_1 + d_2 \cdots d_L = D \). Now there are a number of ways of introducing double points \( d_1, \cdots, d_L \) to define dccs compatible with the maps \( f^\pm \). Let \( d(f, P) \) denote the tube number above \( P \). The case \( d(P) = 2 \) is illustrated in Figure 15.

Fig. 14: \( d(P) \) is the number of tubes above the point \( P \)

Because we count distinct degenerations separately, a double point joining ramification points of index \( j \) is counted \( j \)-times. Thus there are exactly \( \prod_P p^{d(f, P)}(v(P), w(P)) \) ways of introducing double points above points in \( S(L) \) compatible with the specified homomorphisms. Therefore each pair of homomorphisms in the third line is weighted by the
total number configurations compatible with it, multiplied by the multiplicity appropriate for counting degenerating coupled covers. $S_{n^+} \times S_{n^-}$ acts on the set of configurations and the number of times a given equivalence class occurs is $\frac{n^+!n^-!}{\text{[C(e)]}}$. This establishes the second equality.

Now, combining Proposition 10.2 and (10.14), we may write the Euler character as

$$\chi(CHS(n^\pm, B^\pm, L, D))_{\text{orb}} = \chi(C_L(\Sigma_T)) \sum_{d_1, d_2, \ldots, d_L} \sum_{s^\pm_i, t^\pm_i, v_i, w_i} \frac{1}{n^+!n^-!} \delta \left( \prod_{i=1}^{L} v_i \otimes w_i \prod_{i=1}^{G} [s^+_i \otimes s^-_i, t^+_i \otimes t^-_i] \right) \prod_{i=1}^{L} \rho^{(d_i)}_{v_i \otimes w_i}. \tag{10.17}$$

10.5. The nonchiral YM$_2$ sum and Euler characters of CHS

Having completed our geometrical preliminaries we return to the $1/N$ expansion of YM$_2$. The full partition function (2.2) can be written, in the zero area limit, as

$$Z(0, G, N) = \sum_{n^\pm=0}^{\infty} N^{(n^+ + n^-)(2-2G)} \sum_{s^\pm_1, t^\pm_1, \ldots, s^\pm_G, t^\pm_G \in S_n} \left[ \frac{1}{n^+!n^-!} \delta \left( \Omega_{n^+,-n^-}^{2-2G} \prod_{j=1}^{G} [s^+_j \otimes t^+_j \otimes s^-_j \otimes t^-_j] \right) \right], \tag{10.18}$$

The delta function is over the group $S_{n^+} \times S_{n^-}$. The element $\Omega_{n^+,-n^-}$, introduced in (13), is related to the dimension of $SU(N)$ representations by

$$\dim(R \overline{S}) = \frac{N^{n^+ + n^-}}{n^+!n^-!} \chi_{RS}(\Omega_{n^+,-n^-}) \tag{10.19}$$

where $R$ has $n^+$ boxes and $S$ has $n^-$ boxes, and $(R \overline{S})$ is the irreducible representation of largest dimension in the tensor product of $R$ with the complex conjugate of $S$. Explicitly, $\Omega_{n^+,-n^-}$ is an element of the group algebra of $S_{n^+} \times S_{n^-}$ given by

$$\Omega_{n^+,-n^-} = \sum_{v \in S_{n^+}, w \in S_{n^-}} (v \otimes w) P_{v, w} \left( \frac{1}{N^2} \right) \left( \frac{1}{N} \right)^{(n^+ - K_v) + (n^- - K_w)} \tag{10.20}$$

The polynomials $P_{v, w}(\frac{1}{N^2})$ are given by $P_{v, w}(\frac{1}{N^2}) = \varphi(\tilde{r}(v), \tilde{r}(w), -1/N^2)$ where $\tilde{r}$ is the vector of non-negative integers describing the cycle decomposition of the permutations $v, w$ and $\varphi$ was defined in (10.15).
We write
\[
\Omega_{n^+,n^-} = 1 \otimes 1 + \sum_{v \in S_{n^+}, w \in S_{n^-}} v \otimes w \left( \frac{1}{N} \right)^{(n^+ - K_v) + (n^- - K_w)} p_{v,w} \left( \frac{1}{N^2} \right)
\]
\[
= 1 \otimes 1 + \sum_d \left( \frac{-1}{N^2} \right)^d \sum_{v \in S_{n^+}, w \in S_{n^-}} v \otimes w \left( \frac{1}{N} \right)^{(n^+ - K_v) + (n^- - K_w)} p_{v,w}^{(d)}
\]
where we have pulled out the leading term of $1 \otimes 1$ so that $p_{v,w} = P_{v,w} - \delta(v, w)$. In the second line we have collected terms with a given power of $1/N$.

Using \((10.21)\) and expanding the inverse $\Omega$ point as in section five leads to the following expression for the partition function
\[
Z(0, G, N) = \sum_h N^{2-2h} \sum_{n^+ \pm, B^\pm, D} (-1)^D \sum_L \chi(C_L(\Sigma_T)) \sum_{d_1, d_2, \ldots, d_L \atop d_1 + \cdots + d_L = D} \delta\left( \prod_{i=1}^L v_i \otimes w_i \prod_{i=1}^G [s^+_i \otimes s^-_i, t^+_i \otimes t^-_i] \right) \prod_{i=1}^L p_{v_i, w_i}^{(d_i)}.
\]

Since we have collected together the contributions with fixed branching number $B^+$ and $B^-$, the sum over the permutations in the last line is required to obey the condition $\sum_i (n^\pm - K_{v_i}) = B^\pm$. Note that the sum on $L$ is actually finite, and can be bounded above by $B^+ + B^- + D$. In the above, the sum appearing after the Euler character of the configuration space of $L$ points in $\Sigma_T$ is a sum over a discrete set which we have described in the previous subsection as a sum over equivalence classes of dcc’s. Indeed, using \((10.17)\) we finally arrive at

**Proposition 10.3** The full $A = 0$ partition function of $YM_2$ is a generating functional for the orbifold Euler characters of coupled Hurwitz spaces:
\[
Z(0, G, N) = \sum_h N^{2-2h} \sum_{n^+ \pm, B^\pm, D} (-1)^D \sum_{B^+ + B^- + D \atop L=0} \chi_{\text{orb}}(CHS(n^\pm, B^\pm, L, D)),
\]

Note that the Euler character of configuration spaces which appears involves the configurations of both branch points and images of double points on the target.
11. The Nonchiral Topological String Theory

The nonchiral analog of the theory of section 7 must localize on both the space of holomorphic and antiholomorphic maps. When we regard the topological string path integral as an infinite dimensional version of an equivariant Thom class, it becomes clear that we need a section $T$ of some bundle which localizes on the submanifolds $\tilde{M}^\pm$ of $\tilde{C}$ defined by $df \pm J df \epsilon[h] = 0$. It is therefore natural to choose a section of the form:

$$\mathbf{T}: \tilde{C} \longrightarrow \tilde{V}_{nc} \oplus \tilde{V}_{nc}^{ef}$$

$$\mathbf{T}(f, h) \longmapsto (df + J df \epsilon[h]) \otimes (df - J df \epsilon[h])$$ (11.1)

Following the considerations for the construction of a general TFT, we have the following fields, ghosts, antighosts, and Lagrange-multipliers:

$$\mathbf{F} = \left( \begin{array}{c} f^\mu \\ h_{\alpha \beta} \end{array} \right) \quad \mathbf{G} = \left( \begin{array}{c} \chi^\mu \\ \psi_{\alpha \beta} \end{array} \right) \quad A = \rho_{\alpha \beta}^{\mu \nu} \quad \Pi = \pi_{\alpha \beta}^{\mu \nu}.$$

Only the anti-ghosts and Lagrange multipliers of the sigma model have changed relative to the chiral theory. In particular, the appropriate bundle for the antighosts $\rho$ has fiber:

$$\tilde{V}_{nc}^{\rho(f, h)} = \Gamma \left[ (T^*\Sigma_W)^{\otimes 2} \otimes (f^*(T^*T))^\otimes \right]_\pm$$ (11.2)

where the subscript $\pm$ indicates that the sections must satisfy “self-duality” constraints:

$$\rho \in \tilde{V}_{nc}^{\rho(f, h)} \Longleftrightarrow \begin{cases} \rho - (J \otimes 1) \rho (\epsilon \otimes 1) = 0 \\
\text{or} \\
\rho + (1 \otimes J) \rho (1 \otimes \epsilon) = 0 \end{cases}$$ (11.3)

The BRST transformations are the same as above. The nonchiral theory has an action

$$I_{YM_{2\text{string}}} = I_{tg} + I_{t^\sigma}^{nc} + I_{\text{cofield}}^{nc}$$ (11.4)

The gravity part of the action is the same as before. The topological sigma model part is

$$I_{t^\sigma}^{nc} = Q \int d^2 z \sqrt{h} \left\{ \rho^{\alpha \beta}_{\mu \nu} \left[ i h_{\alpha \beta}^{\mu \nu} - \Gamma^{\mu}_{\lambda \rho} \chi^\lambda \rho^{\rho \nu}_{\alpha \beta} - \Gamma^{\nu}_{\lambda \rho} \chi^\lambda \rho^{\mu \rho}_{\alpha \beta} + \frac{1}{2} \pi^{\mu \nu}_{\alpha \beta} \right] \right\}$$ (11.5)

where the indices on $\rho$ and $\pi$ are raised and lowered with the metrics on the worldsheet ($h$), and target space ($G$).
If we expand (11.5) and integrate out the Lagrange multiplier then the bosonic term becomes (in local conformal coordinates)

\[ I_{nc}^{inc} = \int h^{z\bar{z}} G_{w\bar{w}}^2 |\partial_z f^w|^2 |\partial_{\bar{z}} f^w|^2 + \cdots \]  

(11.6)

thus clearly localizing on both holomorphic and antiholomorphic maps. Moreover, when we work out the quadratic terms in the fermions we find that many components of \( \rho \) do not enter the Lagrangian. These components are eliminated by the constraints (11.3). In locally conformal coordinates the only non-trivial components of \( \rho \in \tilde{\mathbb{V}}_{nc}(f,h) \) are \( \rho_{ww\bar{w}z\bar{z}}, \rho_{\bar{w}wz\bar{z}}, \rho_{ww\bar{z}\bar{z}}, \) and \( \rho_{\bar{w}wz\bar{z}} \). (Note that \( \rho_{\alpha\beta}^{\mu\nu} \) is not symmetric in interchanging \( \{(\alpha\beta)(\mu\nu)\} \leftrightarrow \{(\beta\alpha)(\nu\mu)\} \).)

The kinetic term for the fermions is given by

\[ I_{t\sigma}^{nc} = i \int d^2 z \sqrt{h} (\rho \cdot \eta) O_{nc}^{inc} \left( \chi \psi \right) + \cdots \]  

(11.7)

where \( O_{nc} \) is a 2 \( \times \) 2 matrix operator with entries:

\[
O_{11}^{nc} = D^+ \otimes [df - J df \epsilon] + [df + J df \epsilon] \otimes D^- \\
O_{12}^{nc} = J df k \otimes [df - J df \epsilon] - [df + J df \epsilon] \otimes J df k \\
O_{21}^{nc} = \partial f \\
O_{22}^{nc} = P^\dagger
\]

(11.8)

where \( (P^\dagger \delta h)_\beta = D^\alpha \delta h_{\alpha\beta} \), \( D^\pm \chi^\mu = D \chi^\mu \pm J(D \chi^\mu) \epsilon \) and, as usual, \( k[\delta h] \) is the variation of the complex structure on \( \Sigma_W \) induced from a variation of the metric \( \delta h \). The co-model is introduced using the same principles as before.

11.1. Singular Geometries

The full path integral of the topological string theory involves field configurations \((f, h) \in \text{Map}(\Sigma_W, \Sigma_T) \times \text{Met}(\Sigma_W)\), which are not necessarily \( \mathcal{C}^\infty \). An appropriate completion of this space will introduce, among other things, piecewise continuous or even singular maps and geometries. Usually in field theory such considerations of analysis are of interest only in constructive quantum field theory \[ 64 \] but in the present case the contribution of singular field configurations \((f, h)\) in the path integration domain becomes an issue of great importance because the path integral can localize on subspaces of singular geometries. By allowing piecewise differentiable maps and metrics we can incorporate the coupled Hurwitz space in the fiber bundle picture we described for ordinary Hurwitz space in section 4.4.
In the theory of analytic functions one can show\textsuperscript{19} that the weak assumption of
\textit{differentiability} of a solution to the Cauchy Riemann equations implies the function is $C^\infty$
and even analytic\textsuperscript{35}. Thus it might seem that one gains nothing by replacing the $C^\infty$
assumption by the assumption of differentiability when searching for zeroes of $df \pm Jdf\epsilon$.
This reasoning breaks down if the place where derivatives of $f$ are discontinuous is also a
singular point in the geometry of the worldsheet. This is precisely what happens in a dcc.
Indeed, the following simple reasoning shows the presence of solutions of $\tilde{w} = 0$ in (11.1)
which are not in $\tilde{F}^\pm$.

Consider the space $\Xi = \text{Map}(\Sigma_W, \Sigma_T) \times \text{Met}(\Sigma_W)$ where we have added singular geometries to form a “boundary.”\textsuperscript{20} The situation is illustrated schematically in Fig. 16.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure16.png}
\caption{$\tilde{F}^+$ and $\tilde{F}^-$ in $\text{Map} \times \text{Met}$}
\end{figure}

\textsuperscript{19} This is sometimes called Goursat’s theorem.
\textsuperscript{20} We have sketched how dcc’s arise in the bundle approach to Hurwitz space when we consider the full space of maps and metrics. It would be very interesting to construct a careful compactification of the space of Maps $\times$ Metrics in such a way that it is manifest why degenerated coupled covers should be counted with the degeneracy $m(\epsilon)$.
The Hurwitz spaces $\tilde{F}^{\pm}$ lie in the interior of $\Xi$, but extend out to the boundary because, as we have already seen in section 4.5, type 3 collisions between branch points can give rise to a singular worldsheets. We saw there that we can obtain ramification points of equal index on each of the components which are joined by a tube. Such degenerations of holomorphic covers are labeled by $(++)$ on the LHS of Figure 16.

Now, the theory of section 11.1 is invariant only under the group of orientations preserving diffeomorphisms $Diff^+(\Sigma_W)$, so configurations related by diffeomorphisms of type $(\pm, \mp)$, which are orientation preserving on one component but orientation reversing on the other are considered gauge-inequivalent. The 4 points we have indicated on the boundary of $\Xi$ which lie on the outer circle are related to one another by such diffeomorphisms. The configurations indicated at the top and bottom of Fig. 16 are additional, singular configurations which can contribute to the localisation of the path integral. The way in which we handle these surface contributions corresponds to a choice of contact terms, similar to the contact terms that accounted for area polynomials in section 8.

One choice of contact terms simply declares that the dcc’s do not contribute: we cut out all singular field configurations and define the integral by a limiting procedure. From the results of section 7 we can immediately conclude that with this choice of contact terms the partition function of the topological string theory becomes:

$$exp\left\{ \sum_{h \geq 0} \left( \frac{1}{N} \right)^{2h-2} Z_{\text{string}}(\Sigma_W \to \Sigma_T) \right\} = Z^+(A = 0, N)Z^-(A = 0, N).$$

That is, we produce the “chiral” $YM_2$ theory obtained by replacing:

$$\Omega_{n^+,n^-} \longrightarrow \Omega_{n^+,0} \Omega_{0,n^-}$$

in the full $YM_2$ sum. (Note this is a product of chiral theories in the sense that “chiral” is usually employed.) It is not at all clear that this is a sensible (e.g., BRST invariant) choice of contact terms from the point of view of the topological string theory. A contact term analysis similar to that used in section 8 is needed to explain why dcc’s contribute whereas degenerations of $(++)$ or $(--)$ type do not contribute.
12. Conclusions

12.1. General Remarks

In this paper we have used the results of [15] to make some progress towards a formulation of $YM_2$ as a topological string theory. We have seen that the $1/N$ expansion of $YM_2$ may always be formulated in terms of quantities associated to branched covers, provided we admit sufficiently singular geometries. In sections 6-9 we formulated a string theory which reproduces chiral $YM_2$ when proper account is taken of singular geometries. In sections 10, 11 we have partially extended the results to nonchiral $YM_2$.

Reproducing all $YM_2$ in complete detail, involves highly complicated considerations of the contributions of multiple contact terms. In some cases, for example, for the area polynomials associated with simple Hurwitz space a heuristic analysis of contact terms allows us to derive the singular contributions to the string path integral, as in section 8. Amplitudes we have not explicitly calculated from the string theory picture are:

1.) Area polynomials associated with non-simple coverings, and the contributions of handles and tubes (from expanding (8.5)).

2.) Modifications of area polynomials from including dcc’s, and expanding $\exp(-n^+n^- A/N^2)$ in (2.2).

3.) Nonchiral Wilson loop amplitudes.

There is an interesting analogy between the topological string formulation of $YM_2$ and the topological approach to 2D gravity. The exact $YM_2$ answers like (2.1)(9.3) play a role analogous to the results of the double-scaled matrix models. As in 2D gravity, these nontrivial and exact answers show the importance of singular geometries to any topological formulation. We are lucky to have these answers, since they guide us through the dense thicket of singular boundary contributions which are, presumably, ubiquitous in all theories of gravity. Contact terms are at once the Achilles heel and the rock of salvation of topological gravity. They make the truly interesting theories nearly impossible to analyze, yet provide a mechanism whereby the theory can be nontrivial enough to be worthy of attention.

21 An interesting related example where the exact answers have not been previously available is curve-counting in Calabi-Yau three-folds [18].
12.2. Open problems, future directions

The present work suggests several possible generalizations and further directions for research.

One important generalization is the string-theoretic version of $YM_2$ based on the other series of classical compact gauge groups $Sp(N)$, $O(N)$. The $1/N$ expansion of these theories has been worked out in \cite{66,67}. For $O(N)$ and $Sp(N)$ Yang Mills one has to deal with new subtleties associated with string theories on non-orientable surfaces. A given branched cover (possibly with double points) in these theories appears only once. There is no analog of the two sectors in the $SU(N)$ theory. In a bundle description of the corresponding Hurwitz space analogous to section 4.3, one would replace $Diff^+(\Sigma_W)$ with $Diff(\Sigma_W)$.

Douglas \cite{8} has observed that for $\Sigma_T$ of genus one there is a “near” $A \to 1/A$ duality of $YM_2$ since the amplitudes are expressed in terms of Eisenstein series in $q = e^{-A}$. As noted by many physicists, target space duality is quite natural for a topological string theory based on a topological conformal field theory (TCFT). In order to make the answer truly modular covariant it is necessary \cite{8} to modify $YM_2$ in seemingly unmotivated ways. One may search for an explanation of this in the string formulation. Further, when $\Sigma_T$ does not have genus one we are coupling topological gravity to a topological $\sigma$-model which is not a TCFT.

As is well known, an important challenge to the string approach is the derivation of quantities such as the meson spectrum. It is nontrivial to rederive the standard results of the 't Hooft model.

Finally, the construction of the anomaly-cancelling co-model $S_c$, can be applied to a wide class of topological field theories. One may wonder if the resulting theories are of any interest. For example, the analogous construction with Donaldson theory would compute the Euler character of the moduli space of instantons. What is the physical interpretation of this theory?

12.3. What About $QCD_4$?

Aside from the intrinsic beauty of the subject, one of the main reasons we are interested in a string action for $YM_2$ is the hope that this action might contain essential features of a hypothetical string action for $YM_4$. In this respect our construction is disappointing since it relies on topological field theory. Our construction does have a natural generalization
to 4 dimensions: Choose an almost complex structure on the four-dimensional target and form the appropriate topological sigma model. Then follow the construction of the co-model to cancel the anomalous $R$-symmetry and guarantee that we get the Euler density for the moduli space of curves in the target. Following the general ideas outlined above one will find that the topological string theory localizes onto the family of holomorphic maps, $\mathcal{F}(\Sigma_h, X)$. A formula for the dimension of this moduli space is given in eq. (A.16) below.

Of course, any given topological theory admits several different formulations, and it might be that these different formulations admit different generalizations to four dimensions. Perhaps one of these will teach us something about $YM_4$. Some indication that this might be possible is given by the construction of [60] which also has a natural generalization to four dimensions.

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Appendix A. The Deformation Theory Approach to Hurwitz Space

In this appendix we derive a formula for the dimension of the space of families of holomorphic maps. Our treatment here is valid also for the case of maps into higher dimensional target spaces.

The deformation theory of holomorphic maps is a subject developed by Horikawa[68], Mijayima[69] and Namba[70]. Let $U$ be a fixed compact complex manifold. A family of holomorphic maps into $U$ is, by definition a family $(X, \pi, S) = \{V_s\}_{s \in S}$ of compact complex manifolds, together with a holomorphic map $\mathcal{F} : X \to U$. We denote it by $(X, \pi, S, \mathcal{F})$. Set

$$f_s = \mathcal{F}|_{V_s} : V_s \to U.$$
Sometimes this family of maps is denoted by \( \{ V_s, f_s \}_{s \in S} \).

Next we define an infinitesimal deformation of a family \((X, \pi, S, \mathcal{F}) = \{ V_s, f_s \}_{s \in S} \) at \( o \in S \). Let \( f = f_o \). The data which characterize \( \mathcal{F} \) locally are:

(a) An open covering of \( V, \mathcal{V} = \{ V_m \} \) with local coordinates \( z^\alpha_{(m)} \) and transition functions \( g_{mn} : V_n \times S \to V_m \) which vary with \( s \in S \),
(b) An open covering of \( U, \mathcal{U} = \{ U_m \} \) with local coordinates \( u^i_{(m)} \) and fixed transition functions \( h_{mn} : U_n \to U_m \),
(c) A family of holomorphic maps \( f_{(m)} : V_m \times S \to U_m \) which vary with \( s \in S \).

The \( \{ f_m \}, \{ g_{mn} \}, \) and \( \{ h_{mn} \} \) must satisfy the compatibility condition

\[
  h_{mn} \circ f_n = f_m \circ g_{mn},
\]

which ensures commutativity of the following:

\[
\begin{array}{ccc}
  U_n \cap U_m & \xrightarrow{h_{mn}} & U_m \cap U_n \\
  V_n \cap V_m & \xrightarrow{g_{mn}} & V_m \cap V_n \\
  \downarrow f_n & & \downarrow f_m \\
  \end{array}
\]

Differentiating (A.1) with respect to \( s \) and contracting with \( \frac{\partial}{\partial w^i_{(m)}} \), we find

\[
  \sum \alpha \partial_s f^i_{(m)} \frac{\partial}{\partial w^i_{(m)}} - \sum \alpha, \beta \partial_s f^i_{(n)} \frac{\partial h^i_{mn}}{\partial w^i_{(n)}} \circ f_n \frac{\partial}{\partial w^i_{(m)}} = - \sum \alpha, \beta \partial_s g_{mn} \frac{\partial f^\alpha_{(m)}}{\partial z^\beta_{(m)}} \circ g_{mn} \frac{\partial}{\partial w^i_{(m)}}. \tag{A.2}
\]

Now define the Čech 1-cocycle, \( \theta \), valued in the sheaf of germs of holomorphic sections of \( TV \) to be:

\[
  \theta = \{ \theta_{mn} \} \in Z^1(\mathcal{V}, \Theta_V), \quad \theta_{mn} = \partial_s g_{mn}. \tag{A.3}
\]

Further define a Čech 0-cochain, \( \eta \), valued in the inverse image sheaf \( f^* \Theta_U \), by

\[
  \eta = \{ \eta_m \} \in C^0(\mathcal{V}, f^* \Theta_U), \quad \eta_m = \partial_s f_m. \tag{A.5}
\]

22 There is a natural extension of the deformation theory of holomorphic maps developed by Namba [7], wherein the complex structures of both \( V \) and \( U \) are varied. This, however, would only become relevant if we studied YM coupled to 2d (spacetime) gravity.

23 Recall that if \( f : V \to U \) is a continuous map of topological spaces, then the inverse image sheaf, \( f^* \Theta_U \), of \( \Theta_U \) by the map \( f \) is defined as

\[
  f^* \Theta_U = \mathcal{O}(f^*TU), \tag{A.4}
\]

the sheaf of germs of holomorphic sections of the pullback \( f^*TU \) of the holomorphic tangent bundle over \( f \).

\( f_* : \Theta_V \to \Theta_U \) is the push-forward map.
Then (A.2) expresses the fact that

$$(\delta \eta)_{mn} = -f_* \theta_{mn}. \quad (A.6)$$

The deformation theory of holomorphic maps has a very concise formulation in terms of a *characteristic map* from a tangent vector $\frac{\partial}{\partial s} \in T_0 S$ to a certain cohomology class. This is the analogue of the Kodaira-Spencer map which arises in the study of deformations of the complex structures of complex manifolds. In order to present this description of the tangent space to $\mathcal{F}$, we need to introduce a few more notions: First we introduce the following complex of sheaves:

$\mathcal{L}^* : 0 \rightarrow f_* \mathcal{L}^0 = \Theta_V \rightarrow f_* \mathcal{L}^1 = f^* \Theta_U \rightarrow 0, \quad (A.7)$

where $f_*$ is the sheaf map which tautologically satisfies $(f_*)^2 = 0$. Associated to this complex of sheaves are the cohomology sheaves $\mathcal{H}^q = H^q(\mathcal{L})$. Setting $L^q(U) = H^0(V, \mathcal{L}^q)$, the presheaf

$$V \mapsto \ker \frac{\{f_* : \mathcal{L}^q(V) \rightarrow \mathcal{L}^{q+1}(V)\}}{f_* \mathcal{L}^{q-1}(V),}$$

gives rise to a sheaf $\mathcal{H}^q$ whose stalk is

$$\mathcal{H}^q_x = \lim_{V \ni x} \frac{\ker \{f_* : \mathcal{L}^q(V) \rightarrow \mathcal{L}^{q+1}(V)\}}{f_* \mathcal{L}^{q-1}(V).}$$

A section $\eta$ of $\mathcal{H}^q$ over $V$ is given by a covering $\{V_m\}$ of $V$ and $\eta_m \in \mathcal{L}^q(V_m)$ such that

$$f_* \eta_m = 0 \quad (A.8)$$

A section is zero in the case when

$$\eta_m = f_* \theta_m, \quad \theta_m \in \mathcal{L}^{q-1}(V_m), \quad (A.9)$$

after perhaps refining the cover.

Let $C^p(V, \mathcal{L}^q)$ be Čech cochains valued in $\mathcal{L}^q$. Then we have the two operators:

$$\delta : C^p(V, \mathcal{L}^q) \rightarrow C^{p+1}(V, \mathcal{L}^q),$$

$$f_* : C^p(V, \mathcal{L}^q) \rightarrow C^p(V, \mathcal{L}^{q+1}), \quad (A.10)$$

which satisfy $(f_*)^2 = \delta^2 = \{f_*, \delta\} = 0$, so we have a double complex $\{C^{p,q} = C^p(V, \mathcal{L}^q), f_*, \delta\}$. The associated single complex $(C^*, D)$ is defined by

$$C^n = \bigoplus_{p+q=n} C^{p,q}, \quad D = f_* + \delta. \quad (A.11)$$

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We define the hypercohomology as follows:

$$\mathbb{H}^*(V, L^*) = \lim_V H^*(C^*(V), D). \quad (A.12)$$

The pair \((\eta, \theta) \in \mathbb{H}^1(V, L^*)\) is called an infinitesimal deformation of the family \(\{V_s, f_s\}_{s \in S}\) at \(s \in S\) in the direction \(\frac{\partial}{\partial s} \in T_o S\).

One denotes \(\alpha_o(\frac{\partial}{\partial s}) \equiv (\eta, \theta)\). \(\alpha_o\) is a linear map

$$\alpha_o : T_o S \longrightarrow \mathbb{H}^1(U, L^*). \quad (A.13)$$

called the characteristic map.

**Definition:** The family \(\{V_s, f_s\}_{s \in S}\) is said to be effectively parametrized at \(o \in S\) if \(\alpha_o\) is injective.

**Definition:** A morphism of \((X', \pi', S', \mathcal{F}')\) to \((X, \pi, S, \mathcal{F})\) is by definition a morphism \((h, \tilde{h})\) which makes the following diagram commutative

![Diagram](https://example.com/diagram.png)

**Definition:** A family \(\{V_s, f_s\}_{s \in S}\) is said to be complete at \(o \in S\) if for every family \(\{V_{s'}, f_{s'}\}_{s' \in S'}\) with a point \(o' \in S'\) and a biholomorphic map \(i : V_{o'} \rightarrow V_o\) which makes the following diagram commutative

![Diagram](https://example.com/diagram.png)

there is an open neighborhood \(U'\) of \(o' \in S'\) and a morphism \((h, \tilde{h})\) of \(\{V_{s'}, f_{s'}\}_{s' \in S'}\) such that

(i) \(h(o') = o\),
(ii) \(\tilde{h}_{o'} = i : V_{o'} \rightarrow V_o\).

If a family is complete at \(o \in S\), then it contains all small deformations of \(f\). \(\{V_s, f_s\}_{s \in S}\) is complete, if it is complete at every point of \(S\).

If a family \(\{V_s, f_s\}_{s \in S}\) is complete and effectively parametrized at \(o \in S\), then it is said to be versal. In this case, it is the smallest among complete families. One important
property of $F$ which is of obvious interest is its dimension. For this purpose the following two theorems due to Horikawa\[68] and Namba\[70] are useful:

**Theorem:** [Horikawa] For a holomorphic map $f : V \to U$, if

(A) $H^1(V, \Theta_V) \to H^1(V, f^*\Theta_U)$ is surjective,

(B) $H^2(V, \Theta_V) \to H^2(V, f^*\Theta_U)$ is injective,

there exists a complete family $\{V_s, f_s\}_{s \in S}$ of holomorphic maps into $U$ with a point $o \in S$, such that

1. $V_o = V$,
2. $f_o = f$,
3. it is effectively parametrized at $o$,
4. $o$ is a non-singular point of $S$ and $\dim_o S = \dim H^1(V, \mathcal{L}^*)$.

**Theorem:** [Namba] The following sequence is exact:

$$0 \to H^0(V, \mathcal{L}) \to H^0(V, \Theta_V) \to H^0(V, f^*\Theta_U) \to$$

$$\to H^1(V, \mathcal{L}) \to H^1(V, \Theta_V) \to H^1(V, f^*\Theta_U) \to$$

$$\to H^2(V, \mathcal{L}) \to H^2(V, \Theta_V) \to H^2(V, f^*\Theta_U) \to$$

(A.14)

For $h \geq 2$, $H^0(V, \mathcal{L}^*) = 0$, we find for versal families $F(\Sigma_h, \Sigma_G, J)$ that

$$\dim T F(\Sigma_h, \Sigma_G, J) = \dim H^1(V, \mathcal{L}^*)$$

$$= - \dim H^0(V, \Theta_V) + \dim H^0(V, f^*\Theta_U)$$

$$+ \dim H^1(V, \Theta_V) - \dim H^1(V, f^*\Theta_U) = B$$

(A.15)

where $B$ is precisely the branching number!

As another application, one can use (A.15) to determine the dimension of the moduli space of holomorphic maps, $f : \Sigma_W \to X$, from a Riemann surface, $\Sigma_W$, into a higher dimensional target space, $X$. If $\Sigma_W$ has genus $h$, then it is easy to establish that

$$\dim F(\Sigma_h, X) = 3(h - 1) + \dim H^0(\Sigma_h, f^*\Theta_X) - \dim H^0(\Sigma_h, K_h \otimes f^*K_X).$$

(A.16)

**Appendix B. Derivation of the Variation of Gromov’s Equation**

We shall remain general and consider the target space to be an arbitrary complex manifold, $X$. It is important to note that the Gromov equation is non-linear in $f$. We can make this clear by explicitly indicating that $J$ is evaluated at $f(\sigma)$:

$$df(\sigma) + J[f(\sigma)]df(\sigma)e(\sigma) = 0$$

(B.1)
Now consider a one parameter family of holomorphic maps

\[ F : \Sigma \times I \to X \]
\[ F(\sigma; t) \mapsto f_t(\sigma) \]

with \( f_0(\sigma) = f(\sigma) \). This family must also satisfy the Gromov equation

\[ df_t(\sigma) + J[f_t(\sigma)] df_t(\sigma) \epsilon(\sigma) = 0 \quad \forall \sigma \in \Sigma \text{ and } \forall t \in I \]

Now take the derivative with respect to \( t \) and evaluate at \( t = 0 \). We suppress worldsheet indices where they are obvious.

\[
\left[ df_0^\mu(\sigma) + \partial_\nu J^\mu in[fi(\sigma)]j^\nu_i(\sigma) df_t^\nu(\sigma) \epsilon(\sigma) + J^\mu in[fi(\sigma)]df_t^\nu(\sigma) \epsilon(\sigma) \right]_{t=0} = 0
\]

Now consider the covariant derivative of \( J \)

\[
\nabla_\kappa J^\mu_\nu = \partial_\kappa J^\mu_\nu + \Gamma^\mu_{\kappa\lambda} J^\lambda_\nu - \Gamma^\lambda_{\kappa\nu} J^\mu_\lambda
\]

Then we may write (setting \( \delta f^\mu = j^\mu_i \mid_{t=0} \))

\[
\partial_\kappa J^\mu_\nu \delta f^\kappa df^\nu \epsilon = -\Gamma^\mu_{\kappa\lambda} J^\lambda_\nu \delta f^\kappa df^\nu \epsilon + \Gamma^\lambda_{\kappa\nu} J^\mu_\lambda \delta f^\kappa df^\nu \epsilon + \nabla_\kappa J^\mu_\nu \delta f^\kappa df^\nu \epsilon
\]

Now since \( f_t(\sigma) \) is, by fiat, a family of holomorphic maps, \( df_t^\mu \epsilon = J^\mu_\nu df_t^\nu \), for all \( t \), so that

\[
\partial_\kappa J^\mu_\nu \delta f^\kappa df^\nu \epsilon = \Gamma^\lambda_{\kappa\nu} J^\mu_\lambda \delta f^\kappa df^\nu \epsilon - \Gamma^\mu_{\kappa\nu} \delta f^\kappa df^\lambda + \nabla_\kappa J^\mu_\nu \delta f^\kappa df^\nu \epsilon
\]

In the case that \( X \) is a complex manifold, \( \nabla_\kappa J^\mu_\nu = 0 \) and we deduce the following equation for the tangent space:

\[
D(\delta f) + J D(\delta f) \epsilon[h] + J df \ k[\delta h] = 0 \quad \text{(B.2)}
\]

where \( D \) is the pulled-back connection \((D_\alpha \delta f)^\mu = \partial_\alpha \delta f^\mu + \Gamma^\mu_{\kappa\lambda} \partial_\alpha f^\kappa f^\lambda\).
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