Sampling-Based Methods for Factored Task and Motion Planning

Caelan Reed Garrett¹, Tomás Lozano-Pérez¹, and Leslie Pack Kaelbling¹

Abstract
This paper presents a general-purpose formulation of a large class of discrete-time planning problems, with hybrid state and control-spaces, as factored transition systems. Factoring allows state transitions to be described as the intersection of several constraints each affecting a subset of the state and control variables. Robotic manipulation problems with many movable objects involve constraints that only affect several variables at a time and therefore exhibit large amounts of factoring. We develop a theoretical framework for solving factored transition systems with sampling-based algorithms. The framework characterizes conditions on the submanifold in which solutions lie, leading to a characterization of robust feasibility that incorporates dimensionality-reducing constraints. It then connects those conditions to corresponding conditional samplers that can be composed to produce values on this submanifold. We present two domain-independent, probabilistically complete planning algorithms that take, as input, a set of conditional samplers. We demonstrate the empirical efficiency of these algorithms on a set of challenging task and motion planning problems involving picking, placing, and pushing.

Keywords
task and motion planning, manipulation planning, AI reasoning

1 Introduction
Many important robotic domains require planning in a high-dimensional space that includes not just the robot configuration, but also the “configuration” of the external world, including poses and attributes of objects. There has been a great deal of progress in developing probabilistically complete sampling-based methods that move beyond motion planning to these hybrid problems including various forms of task planning. These new methods each require a new formulation, theoretical framework, sampling method, and search algorithm. We propose to model robot task-and-motion planning problems as factored transition systems, discrete-time planning problems involving continuous and discrete state and control-spaces. This formulation is able to highlight any factoring present within the problem resulting from constraints that only impact a few variables at a time. Directly exposing factoring enables us to design algorithms that are able to more efficiently both sample states and control as well as search the resulting space.

The theoretical contribution of this paper is an analysis of the topology of a problem’s solution space, particularly in the presence of dimensionality-reducing constraints. The key insight is that, in some cases, the intersection of several lower-dimensional constraints lies on a submanifold of the parameter-space that can be identified using only the individual constraints. By understanding the topology of the solution space, we define a property that characterizes a large class of problems for which sampling-based planning methods can be successful.

Figure 1. Experiment 1 (left): the robot must place each block at its corresponding goal pose. Experiment 3 (right): the robot must move the blue block to another table.

¹MIT CSAIL, USA

Corresponding author:
Caelan Reed Garrett, Computer Science and Artificial Intelligence Laboratory, 32 Vassar Street, Cambridge, MA 02139 USA
Email: caelan@csail.mit.edu
The algorithmic contribution is the construction of two sampling-based planning algorithms that exploit the factored, compositional structure of the solution space to draw samples using conditional samplers. These algorithms search in a combined space that includes the discrete structure (which high-level operations, such as “pick” or “place” happen in which order) and parameters (particular continuous parameters of the actions) of a solution. Theoretically, these algorithms are probabilistically complete when given a set of sampling primitives that cover the appropriate spaces. Practically, they can solve complex instances of task-and-motion planning problems as well as problems involving non-prehensile manipulation problems.

2 Related Work

Planning problems in which the goal is not just to move the robot without collision but also to operate on the objects in the world have been addressed from the earliest days of manipulation planning (Lozano-Pérez 1981; Lozano-Pérez et al. 1987; Wilfong 1988). In recent years, there have been a number of approaches to integrating discrete task planning and continuous motion planning (Cambon et al. 2009; Hauser and Latombe 2009; Plaku and Hager 2010) aimed at increasing the capabilities of autonomous robots.

Alami et al. (1990, 1994) decomposed manipulation problems using a manipulation graph that represents connected components of the configuration space given by a particular robot grasp. They observed that solutions are alternating sequences of transit and transfer paths corresponding to the robot moving while its hand is empty and the robot moving while holding an object. Simeon et al. (2004) applied these ideas to manipulation planning using probabilistic roadmaps.

Stilman and Kuffner (2006); Stilman et al. (2007) introduced the problem of robotic navigation among movable obstacles (NAMO). They provide an algorithm for solving monotonic problem instances, problems that require moving each object at most one time. Van Den Berg et al. (2009) developed a probabilistically complete algorithm for robustly feasible NAMO problems. Krontiris and Bekris (2015, 2016) extended the work of Stilman and Kuffner (2006); Stilman et al. (2007) to general rearrangement problems by using their algorithm as a primitive within a larger, complete search.

Hauser and Ng-Thow-Hing (2011) introduced a framework and algorithm for probabilistically complete multimodal motion planning, motion planning in overlapping spaces of non-uniform dimensionality. Barry et al. (2013) considered multi-modal motion planning using bidirectional rapidly-exploring random trees (RRT). Vega-Brown and Roy (2016) extended these ideas to optimal planning with differential constraints.

Kaelbling and Lozano-Pérez (2011) introduced generators to select predecessor states in a goal regression search for task and motion planning. Dornhege et al. (2009, 2013) introduced semantic attachments, predicates evaluated on a geometric representation of the state, to integrate geometric reasoning into artificial intelligence planning. Garrett et al. (2015) gave an algorithm for planning in hybrid spaces by using approximations of the planning problem to guide the sampling of successor actions.

Lagriffoul et al. (2012, 2014) perform a search over plan skeletons, discrete sequences of actions that does not yet assign values for the continuous variables. For each plan skeleton under consideration, they generate a set of approximate linear constraints, e.g., from grasp and placement choices, and use linear programming to compute a valid assignment of continuous values or determine that one does not exist. Lozano-Pérez and Kaelbling (2014) take a similar approach but leverage constraint satisfaction problem (CSP) solvers to identify satisfactory geometric parameters from discretized domains. Pandey et al. (2012); de Silva et al. (2013) use hierarchical task networks (HTNs) to search over plan skeletons and backtrack upon failing to find satisfactory geometric parameters.

Srivastava et al. (2014) integrate task and motion planning by designing an interface allowing an off-the-shelf motion planner to share geometric information with an off-the-shelf task planner. Their approach first plans at the task-level and then attempts to produce motion plans satisfying the discrete actions. If an induced motion planning problem is infeasible, the task-level planning repeats with new action preconditions identifying the source of the infeasibility.

The FFRob algorithm of Garrett et al. (2014, 2016) is related to the INCREMENTAL algorithm discussed in this paper. It also involves sampling a fixed set of object poses and robot configurations and then planning with them using artificial intelligence techniques. An iterative version of FFRob is probabilistically complete and exponentially convergent (Garrett et al. 2016). However, the approach in FFRob is specialized to a particular class of pick-and-place problems and does not use planning to guide sampling.

Toussaint (2015) formulates task and motion planning as a logic-geometric program, a non-linear constrained optimization problem augmented with a logic and knowledge base. He hierarchically optimizes the final state, transfer configurations, and motion trajectories.

Dantam et al. (2016) formulate task and motion planning as a satisfiability modulo theories (SMT) problem. They use an incremental constraint solver to add motion constraints to the task-level logical formula when a candidate task plan is found. Upon failure, they iteratively increase the plan depth and motion planning timeouts resulting in a probabilistically complete algorithm for fixed placements and grasps.
Finally, [Deshpande et al. 2016] proved prehensile task and motion planning is decidable by providing a complete decomposition-based algorithm.

Our work captures many of the insights in these previous approaches in a general framework as well as more thoroughly exploiting the mutual constraints between plan skeletons and geometric parameters.

3 Factored Transition System

We begin by defining a general class of models for controllable discrete-time continuous-space dynamical systems. It is possible to address many continuous-time problems in this framework, as long as they can be solved with a finite sequence of continuous control inputs.

Definition 1. A discrete-time transition system $S = \langle X, U, T \rangle$ is defined by a set of states (state-space) $X$, set of controls (control-space) $U$, and a transition relation $T \subseteq X \times U \times X$.

For many physical systems, $T$ is a transition function from $X \times U$ to $X$.

Definition 2. A problem $P = \langle x^0, X^*, S \rangle$ is an initial state $x^0 \subseteq X$, a set of goal states $X^* \subseteq X$, and a transition system $S$.

Definition 3. A plan for a problem $P$ is finite sequence of $k$ control inputs $(u^1, ..., u^k)$ and $k$ states $(x^1, ..., x^k)$ such that $(x^i−1, u^i, x^i) \in T$ for $i \in \{1, ..., k\}$ and $x^k \in X^*$.

When $T$ is a transition function, a plan can be uniquely identified by its sequence of control inputs.

3.1 Factoring

We are particularly interested in transition systems that are factorizable.

Definition 4. A factored transition system is a transition system with state-space $X = X_1 \times \ldots \times X_m$ and control-space $U = U_1 \times \ldots \times U_n$ that is defined by $m$ state variables $\bar{x} = (x_1, ..., x_m)$ and $n$ control variables $\bar{u} = (u_1, ..., u_n)$.

The transition relation is a subset of the transition parameter-space $T \subseteq X \times U \times X$ Valid transitions are $(x_1, ..., x_m, u_1, ..., u_n, x_1', ..., x_m') \in T$. To simplify notation, we will generically refer to each $x_i, u_i$, or $x'_i$ in a transition as a parameter $z_p$ where $p \in \{1, ..., 2m + n\}$ indexes the entire sequence of variables. For a subset of parameter indices $P = \{p_1, ..., p_k\}$, let $z_P = (z_{p_1}, ..., z_{p_k}) \in Z_P$ be the combined values and $\bar{Z}_P = Z_{p_1} \times \ldots \times Z_{p_k}$ be the combined domain of the parameters.

Many transition relations are hybrid, in that there is a discrete choice between different types of operation, each of which has a different continuous constraint on the relevant parameters. For example, a pick-and-place problem has transitions corresponding to a robot moving its base, picking each object, and placing each object. In order to expose the discrete structure, we decompose the transition relation $T = \bigcup_{a=1}^{\alpha} T_a$ into the union of $\alpha$ smaller transition components $T_a$. A transition relation $T_a$ often is the intersection of several constraints on a subset of the transition parameters.

Definition 5. A constraint is a pair $C = \langle P, R \rangle$ where $P \subseteq \Theta$ is a subset of parameters and $R \subseteq Z_P$ is a relation on these parameters.

A tuple of values that satisfy a constraint is called a constraint element.

Definition 6. A constraint element $C(v_{p_1}, ..., v_{p_k})$ is composed of a constraint $C = \langle P, R \rangle$ and variable values $(v_{p_1}, ..., v_{p_k}) = \bar{v}_P \in R$ for parameters $P = \{p_1, ..., p_k\}$.

For instance, pick transitions involve constraints that the end-effector initially is empty, the target object is placed stably, the robot’s configuration forms a kinematic solution with the placement, and the end-effector ultimately is holding the object. A constraint decomposition is particularly useful when $|P| < 2m + n$: i.e., each individual constraint has low arity. Let $C|_{\Theta} = \{z \in Z_{\Theta} | \bar{z}_P \in R\}$ be the extended form of the constraint over parameters $\Theta = \{1, ..., 2m + n\}$. This alternatively can be seen as a Cartesian product of $R$ with $Z_{\Theta \setminus P}$ followed by a sorting of variable indices.

Definition 7. A transition component $T_a$ is specified as a conjunctive clause of $\beta$ constraints $C_{a} = \{C_1, ..., C_\beta\}$ where $T_a = \bigcap_{C \in C_{a}} C|_\Theta$.

Within a clause, there are implicit domain constraints on each parameter $z_p$ of the form $\langle z_p, Z_p \rangle$. The transition relation $T$ is the union of $\alpha$ conjunctive constraint clauses $\{C_1, ..., C_\alpha\}$.

Factoring the transition relation can expose constraints that have a simple equality form. Equality constraints are important because they transparently reduce the dimensionality of the transition parameter-space.

Definition 8. A constant equality constraint $\langle \{z_p\}, \{\kappa\} \rangle$ (denoted $z_p = \kappa$) indicates that parameter $z_p$ has value $\kappa$.

Definition 9. A pairwise equality constraint $\langle \{z_p, z_p'\}, \{v, v\} | v \in Z_{p} \cap Z_{p'} \rangle$ (denoted $z_p = z_p'$) indicates that parameters $z_p, z_{p'}$ have the same value.

For many high-dimensional systems, the transition relation is sparse, meaning its transitions only alter a small number of the state variables at a time. Sparse transition systems have transition relations where each conjunctive
clause contains pairwise equality constraints \( x_p = x'_p \) for most state variables \( p \). Intuitively, most actions do not change most state variables.

The initial state \( \vec{x}^0 \) and set of goal states \( \vec{X}^* \) can be specified using conjunctive clauses \( C_0 \) and \( C_* \) defined solely on state variables. Because \( \vec{x}^0 \) is a single state, its clause is composed of constant equality constraints.

### 3.2 Constraint Satisfaction

Planning can be thought of as a combined search over discrete clauses and hybrid parameter values. For each action there is a choice of a discrete transition type and of continuous parameters. To select a type is to select a clause from the transition relation; to select its parameters is to select the \( \vec{x}, \vec{u}, \vec{x}' \) values.

**Definition 10.** A finite sequence of clauses \( \vec{a} = (a_1, ..., a_k) \) is a plan skeleton (Lozano-Pérez and Kaelbling 2014).

For example, solutions to pick-and-place problems are sequences of move, pick, move-while-holding, and place clauses involving the same object.

**Definition 11.** The plan parameter-space for a plan skeleton \( \vec{a} = (a_1, ..., a_k) \) is an alternating sequence of states and actions \( \vec{x}^0, \vec{u}^1, \vec{x}^1, ..., \vec{u}^k, \vec{x}^k \) = \( \vec{z} \in \vec{X} \times ([U \times \vec{X}]^k = \vec{Z} \).

Here, we generically refer to each variable in the plan parameter-space as \( z_p \) where now \( p \in \{1, ..., m + k(m + n)\} = \Theta \). When applying the constraints for clause \( a_i \), plan state \( \vec{x}^i-1 \) is transition state \( \vec{x} \) and likewise \( \vec{x}^i \) is \( \vec{x}' \). Solutions using this skeleton must satisfy a single conjunctive clause of all plan-wide constraints \( C_{\vec{a}} = C_0 \cap C_{a_1} \cap ... \cap C_{a_k} \cap C_* \).

**Definition 12.** A set of constraints \( C \) is satisfiable if there exists parameter values \( \vec{z} \in \vec{Z} \) such that \( \vec{z} \in \cap_{C \in C} C|_\Theta \).

**Definition 13.** A problem \( \mathcal{P} \) is feasible if there exists a plan skeleton \( \vec{a} \) that \( C_{\vec{a}} \) is satisfiable.

Given a plan skeleton, finding satisfying parameter values \( \vec{z} \) is a hybrid constraint satisfaction problem. The joint set of constraints forms a constraint network, a bipartite graph between constraints and parameters (Decter 1992; Lagriffoul et al. 2014). An edge between a constraint node \( C = (P, R) \) and state or control node \( x_p^{i-1}, u_p^i, \) or \( x_p^i \) is defined if and only if \( p \in P \). Figure 2 displays a general constraint network. Many transition systems in practice will have constraint networks with many fewer edges because each \( P \) contains only a small number of parameters.

### 4 Example Domains

We are interested in a general algorithmic framework that can be applied in many domains. A domain \( \mathcal{D} = \{\mathcal{P}, ...\} \) is loosely defined as a set of problems that share similar variable, constraint, and transition forms. Consider the following two domains and their representation as factored transition systems. We begin with a motion planning domain to illustrate the approach, and then describe a pick-and-place domain.

#### 4.1 Motion Planning

Many motion planning problems may be defined by a bounded configuration space \( Q \subseteq \mathbb{R}^d \) and collision-free configuration space \( Q_{\text{free}} \subseteq Q \). We will consider planning motions composed of a finite sequence of straight-line trajectories \( t \) between waypoints \( q, q' \). Problems are given by an initial configuration \( q_0 \in Q \) and a goal configuration \( q_\ast \in Q \). Motion planning can be modeled as a transition system with state-space \( \vec{X} = Q \) and control-space \( \vec{U} = Q^2 \). The transition relation \( \mathcal{T} = \{\text{CMove}\} \) has a single clause

\[
\mathcal{C}_{\text{Move}} = \{\text{Motion}, \text{CFree}\}.
\]

The transition relation does not exhibit any useful factoring. A motion constraint \( \text{Motion} \) enforces \( u_t \) is a straight-line trajectory between \( x_q \) and \( x_q' \).

\[
\text{Motion} = (x_q, u_t, x_q') \in \{(q, t, q') \mid q, q' \in Q^2, t(\lambda) = \lambda q + (1 - \lambda)q'\}
\]

A collision-free constraint \( \text{CFree} \) ensures all configurations on the trajectory are not in collision.

\[
\text{CFree} = \{u_t \mid \forall \lambda \in [0, 1], t(\lambda) \in Q_{\text{free}}\}.
\]

The initial clause is \( \mathcal{C}_0 = \vec{x}^0 = \{x_q = q_0\} \) and the goal clause is \( \mathcal{C}_* = \{x_q = q_\ast\} \).

The system could alternatively be described as \( \vec{x} = (x_{j1}, ..., x_{jd}) \) where \( j \) is a single robot degree-of-freedom. For simplicity, we combine individual degrees of freedom included within the same constraints into a single variable. This is possible because the set of robot joints always occurs together when mentioned as parameters within motion and kinematic constraints.
Figure 3 displays a constraint network for a plan skeleton of length $k$. Because the transition relation has a single clause, all solutions have this form. Dark gray circles are parameters, such as the initial and final configurations, that are constrained by constant equality. Free parameters are yellow circles. Constraints are orange rectangles.

\begin{align*}
\text{CFree}_0 \cup \text{CFree}_{k-2} \cup \text{CFree}_{k-1} \\
\text{Motion}_{k-1} \cup \text{Motion}_k \\
\text{Rob}_{k-1} \cup \text{Rob}_k
\end{align*}

Figure 3. Motion planning plan skeleton of length $k$.

### 4.2 Pick-and-Place Planning

A pick-and-place domain is defined by a single robot with configuration space $Q \subseteq \mathbb{R}^d$, a finite set of moveable objects $O$, a set of stable placement poses $S_o \subseteq SE(3)$ for each object $o \in O$, and a set of grasps relative to the end-effector $G_o \subseteq SE(3)$ for each object $o \in O$. The robot has a single manipulator that is able to rigidly attach to a single object at a time when the end-effector $g$ performs a grasping operation. As before, the robot can execute straight-line trajectories $t$ between waypoints $q, q'$. This domain can be modeled as a transition system with state-space $\mathcal{X} = Q \times SE(3)^{|O|} \times \{(\text{None}) \cup O\}$. States are $\bar{x} = (x_0, x_0, \ldots, x_o, x_h)$. Let $h \in O$ indicate that the robot is holding object $h$ and $h = \text{None}$ indicate that the robot’s gripper is empty. When $h = o$, the pose $x_o$ of object $o$ is relative to the end-effector frame. Otherwise, $x_o$ is relative to the world frame. By representing attachment as a change in frame, the pose of an object remains fixed, relative to the gripper, as the robot moves. Controls are pairs $\bar{u} = (u_t, u_q)$ composed of trajectories $u_t$ and boolean gripper force commands $u_q$. Let $u_q = \text{True}$ correspond to sustaining a grasp force and $u_q = \text{False}$ indicate no force.

The transition relation $\mathcal{T}$ has $1 + 3|O|$ clauses because pick, move-while-holding, and place depend on $o$:

$$
\mathcal{T} = \{\text{C}_{\text{Move}}\} \cup \{\text{C}^0_{\text{MoveH}}, \text{C}^0_{\text{Pick}}, \text{C}^0_{\text{Place}} | o \in O\}.
$$

$\text{C}_{\text{Move}}$ and $\text{C}^0_{\text{MoveH}}$ clauses correspond to the robot executing a trajectory $u_t$ while its gripper is empty or holding object $o$:

$$
\text{C}_{\text{Move}} = \{\text{Motion}, \text{CFree}, x_h = \text{None}, x_h = x'_h, u_q = \text{False}\} \cup \{x_o' = x'_o, \text{CFree}_o' | o' \in O\}
$$

$$
\text{C}^0_{\text{MoveH}} = \{\text{Motion}, \text{CFree}_o, x_h = o, x_h = x'_h, u_q = \text{True}\} \cup \{\text{CFree}_o, x_o' = x'_o, \text{CFree}_o' | o' \in O, o \neq o'\}
$$

$\text{CFree}_o$ is a constraint containing robot trajectories $u_t$ and object $o$ poses $x_h$ that are not in collision with each other. $\text{CFree}_o$ is a constraint composed of robot trajectories $u_t$ and object $o$ grasps $x_h$ relative to the end-effector that are not in collision with the environment. $\text{CFree}_o$ is a constraint containing robot trajectories $u_t$, object $o$ grasps $x_h$ relative to the end-effector, and object $o'$ poses $x'_o$, that are not in collision with each other.

$\text{C}^0_{\text{Pick}}$ and $\text{C}^0_{\text{Place}}$ clauses correspond to instantaneous changes in what the gripper is holding:

$$
\text{C}^0_{\text{Pick}} = \{\text{Stable}_o, \text{Grasp}_o, \text{Kin}_o, x_q = x'_q, x_h = \text{None}, x_h = o' \cup \{x_o' = x'_o | o' \in O, o \neq o'\}
$$

$$
\text{C}^0_{\text{Place}} = \{\text{Grasp}_o, \text{Stable}_o, \text{Kin}_o, x_q = x'_q, x_h = o, x_h = \text{None} \cup \{x_o' = x'_o | o' \in O, o \neq o'\}
$$

As a result, they do not involve any control variables. $\text{Grasp}_o = ((x_o'), G_o)$ is a constraint that $x_o$ is a grasp. Let $\text{Grasp}_o = ((x_o'), G_o)$ be the same constraint but on $x_o'$. Similarly, $\text{Stable}_o = ((x_o), S_o)$ is a constraint that $x_o$ is a stable placement, and $\text{Stable}_o' = ((x_o'), S_o)$. Finally, $\text{Kin}_o$ is a constraint composed of kinematic solutions involving object $o$ for a grasp, pose, and robot configuration:

$$
\text{Kin}_o = ((x_o', x_o, x_q), \{(g, p, q) | \text{KIN}(q) = pg^{-1}\})
$$

$\text{Kin}_o$ is the equivalent constraint but on state variables $(x_o, x_o', x_q)$, where $x_o$ and $x_o'$ are swapped. Because $\text{Kin}_o$ and $\text{Kin}_o'$ refer to the same relation and only involve different parameters, we will primarily just refer to $\text{Kin}_o$.

Pick-and-place domains are substantially factorable. Each constraint involves at most 3 variables. Additionally, in each clause, many variables are entirely constrained by equality. For $\text{C}_{\text{Move}}, \text{C}^0_{\text{MoveH}}$ clauses, only half of the variables are not constrained by equality: $\{x_q, u_t, x'_q\} \cup \{x_o' | o' \in O\}$. For $\text{C}^0_{\text{Pick}}, \text{C}^0_{\text{Place}}$ clauses, only $\{x_o, x_o', x_q\}$ variables are not constrained by equality.

![Figure 4](image-url)
We will use the pick-and-place problem shown in figure 4 with two movable objects A, B as a running example. The initial state $C_0 = x^0 = (q_0, o_0, b_0, \text{None})$ is fully specified using equality constraints and $x^*$ is given as constraints $C_s = \{\text{Region}, x_q = q_s\}$ where the region constraint $\text{Region}_A = ((x_A), R_A)$, for $R_A \subseteq S_A$ is a subset of the stable placements for object A.

A useful consequence of factoring is that the same control values $\bar{u}$ can be considered in many transitions. Consider the two candidate transitions in figure 5, both using the same control trajectory $u$. The application of $u_t$ in the left figure results in a valid transition for clause Move. In fact, for a majority of the combinations of placements of $x_A$, $x_B$, $u_t$ is a valid transition. Thus, for a single value of $u_t$, we are implicitly representing many possible transitions. The left figure, however, shows an instance in which this $u_t$ does not correspond to a legal transition as it would result in a collision with B, thus violating $C_{\text{Free}_B}$.

Figure 5. A valid transition (left) and an invalid transition (right) for the same control trajectory $u_t$. The right transition is invalid because it violates the $C_{\text{Free}_B}$ collision-free constraint.

Figure 6 displays the constraint network for a plan skeleton $\bar{a} = (\text{Move}, \text{Pick}^A, \text{Move}H^A, \text{Place}^A, \text{Move})$ that grabs A, places A in Region A, and moves the robot to $q_s$. Thick edges indicate pairwise equality constraints. Light gray parameters are transitively fixed by pairwise equality. We will omit the constraint subscripts for simplicity. Despite having 29 total parameters, only 7 are free parameters. This highlights the strong impact of equality constraints on the dimensionality of the plan parameter-space.

5 Sampling-Based Planning

Constraints involving continuous variables are generally uncountably infinite sets, which are often difficult to characterize and reason with explicitly. Instead, each constraint can be described using a blackbox, implicit test. A test for constraint $C = (P, R)$ is a boolean-valued function $t_C : \mathcal{Z}_P \to \{0, 1\}$ where $t_C(\mathcal{Z}_P) = [\mathcal{Z}_P \in R]$. Implicit representations are used in sampling-based motion planning, where they replace explicit representations of complicated robot and environment geometries with collision-checking procedures.

In order to use tests, we need to produce potentially satisfying values for $\mathcal{Z}_P = (z_{p1}, \ldots, z_{pk})$ by sampling $\mathcal{Z}_{p1}, \ldots, \mathcal{Z}_{pk}$. Thus, we still require an explicit representation for $x_1, \ldots, x_m$ and $u_1, \ldots, u_n$; however, these are typically less difficult to characterize. We will assume $x_1, \ldots, x_m, u_1, \ldots, u_n$ are each bounded manifolds. This strategy of sampling variable domains and testing constraints is the basis of sampling-based planning (Kavraki et al. 1996). These methods draw values from $x_1, \ldots, x_m$ and $u_1, \ldots, u_n$ using deterministic or random samplers for each space and test which combinations of sampled values satisfy required constraints.

Sampling-based techniques are usually not complete over all problem instances. First, they cannot generally identify and terminate on infeasible instances. Second, they are often unable to find solutions to instances that require identifying values from a set that has very small or even zero measure in the space from which samples are being drawn (this is referred to as the “narrow passage” problem in motion planning). Thus, sampling-based algorithms are typically only complete over robustly feasibly problems. A problem is robustly feasible if there exists a plan skeleton $\bar{a}$ such that $\mu(\bigcap_{C \in \mathcal{C}_a} C(\bar{a})) > 0$ where $\mu$ is a product measure on the plan-parameter-space $\mathcal{Z}$.

5.1 Dimensionality-reducing constraints

Some domains of interest involve constraints that only admit a set of values on a lower-dimensional subset of its parameter-space. A dimensionality-reducing constraint $C$ is one in which $\mu(C) = 0$ for all problems in the domain. Consider the $\text{Stable}_a$ constraint. The set of satisfying values lies on a 3-dimensional manifold. By our current definition, all plans involving this constraint are not robustly feasible. When a problem involves dimensionality-reducing constraints, we have no choice but to sample at their intersection. This, in general, requires an explicit characterization of their intersection, which we may not have. Moreover, the number of dimensionality-reducing constraint combinations can be unbounded as plan skeletons may be arbitrarily long. However, in some cases, we can produce this intersection automatically using explicit characterizations for only a few spaces.

We motivate these ideas with an example. Consider a plan skeleton $\bar{a}$ with parameters $\Theta = (z_x, z_y)$ where $Z_x = Z_y = (-2, +2)$ and constraints $C_{\bar{a}} = \{C_1, C_2, C_3\}$ where $C_1 = \langle y \rangle, \{-1, 0, 1\}$ $C_2 = \langle (x, y), \{(x, y) \mid x + y = 0\}\rangle$ $C_3 = \langle (x, y), \{(x, y) \mid x - y \geq 0\}\rangle$. 

Prepared using sagej.cls
The set of solutions $C_1\cap C_2\cap C_3 = \{(1, -1), (0, 0)\}$ is 0-dimensional while the parameter-space is 2-dimensional. This is because $C_1$ and $C_2$ are both dimensionality-reducing constraints. A uniform sampling strategy where $X, Y \sim \text{Uniform}(-2, +2)$ has zero probability of producing a solution.

To solve this problem using a sampling-based approach, we must sample from $C_1\cap C_2$. Suppose we are unable to analytically compute $C_1\cap C_2$, but we do have explicit representations of $C_1$ and $C_2$ independently. In particular, suppose we know $C_2$ conditioned on values of $y$, $C_2(y) = \{(x, y)\}$. Now, we can characterize $C_1\cap C_2 = \{(x, y)\}$. With respect to a counting measure on this discrete space, $C_1\cap C_2$ has positive measure. This not only gives a representation for the intersection but also suggests the following way to sample the intersection: $Y \sim \text{Uniform}\{(-1, 0, +1)\}$, $X = -Y$, and reject $(X, Y)$ that does not satisfy $C_3$. This strategy is not effective for all combinations of dimensionality-reducing constraints. Suppose that instead $C_1 = \{(x, y), (x, y) \mid x - y = 1\}$. Because both constraints involve $x$ and $y$, we are unable to condition on the value of one parameter to sample the other.

### 5.2 Intersection of Manifolds

In this section, we develop the topological tools to generalize the previous example. Our objective is to show that by making assumptions on each dimensionality reducing constraint individually, we can understand the space formed by the intersection of many dimensionality constraints. First, we overview several topological ideas that we will use.

#### 5.2.1 Topological Tools

A $d$-dimensional manifold $M$ is a topological space that is locally homeomorphic to $d$-dimensional Euclidean coordinate space. Let $d = \dim M$ be the dimension of the coordinate space of $M$. An atlas for an $d$-dimensional manifold $M \subseteq \mathbb{R}^m$ is a set of charts $\{(U_\alpha, \varphi_\alpha)\}$ such that $\bigcup_\alpha U_\alpha = M$. Each chart $(U_\alpha, \varphi_\alpha)$ is given by an open set $U_\alpha \subseteq M$ and a homeomorphism, a continuous bijection with a continuous inverse, $\varphi : U_\alpha \to \mathbb{R}^d$. Let $N$ be a regular submanifold of ambient manifold $M$. Then, $\dim N = \dim M - \dim N$ is the codimension of $N$.

Define $T_x(M)$ to be the tangent space of manifold $M$ at $x \in M$. A smooth map of manifolds $f : M \to N$ is a submersion at $x \in M$ if its differential $df_x : T_x(M) \to T_{f(x)}(N)$...
$T_f(x)(N)$ is surjective. When $\dim M \geq \dim N$, this is equivalent to the Jacobian matrix of $f$ at $x$ having maximal rank equal to $\dim N$. When $f$ is a submersion, by the preimage theorem (also called the implicit function theorem and the regular level set theorem), the preimage $f^{-1}(y)$ for any $y \subseteq N$ is a submanifold of $M$ of dimension $\dim M - \dim N$. Similarly, by the local normal submersion theorem, there exists coordinate charts $\varphi$ and $\varphi'$ local to $x \in M$ and $f(x) \in N$ such that $f$ is a projection in coordinate space. Thus, the first $\dim N$ coordinates of $\varphi(x)$ and $\varphi(f(x))$ are the same and $\varphi(x)$ contains an extra codim $N$ coordinates.

For manifolds $N_1$ and $N_2$, consider projections $\text{proj}_1 : (N_1 \times N_2) \rightarrow N_1$ and $\text{proj}_2 : (N_1 \times N_2) \rightarrow N_2$. For all $\bar{y} = (y_1, y_2) \in N_1 \times N_2$, the map

$$(d \text{proj}_1, d \text{proj}_2) : T_{\bar{y}}(N_1 \times N_2) \rightarrow T_{y_1}(N_1) \times T_{y_2}(N_2)$$

is an isomorphism. Thus, the tangent space of a product manifold is isomorphic to the Cartesian product of the tangent spaces of its components.

Let $N_1, N_2$ both be submanifolds of the same ambient manifold $M$. An intersection is transverse if $\forall x \in (N_1 \cap N_2), T_x(N_1) + T_x(N_2) = T_x(M)$ where

$T_x(N_1) + T_x(N_2) = \{ v + w \mid v \in T_x(N_1), w \in T_x(N_2) \}$. If the intersection $N_1 \cap N_2$ is transverse, then $N_1 \cap N_2$ is a submanifold of codimension

$$\text{codim}(N_1 \cap N_2) = \text{codim} N_1 + \text{codim} N_2.$$ Intuitively, the intersection is transverse if their combined tangent spaces produce the tangent space of the ambient manifold.

### 5.2.2 Conditional Constraints

We start by defining conditional constraints, a binary partition of $P$ for a constraint $(P, R)$.

**Definition 14.** A conditional constraint $(I, O, R)$ for a constraint $C = (P, R)$ is a partition of $P$ into a set of input parameters $I$ and a set of output parameters $O$.

Define $\text{proj}_I(\bar{z}) = \bar{z}_P$ to be the set-theoretic projection of $\bar{z}$ onto parameters $P$. Let its inverse, the projection preimage $\text{proj}_I^{-1}(\bar{z}_I)$, be the following:

$$\text{proj}_I^{-1}(\bar{z}_I) = \{ \bar{z}_P \in R \mid \bar{z}_P = (\bar{z}_I, \bar{z}_O) \}.$$ Conditional constraints will allow us to implicitly reason about intersections of $\text{proj}_I(R)$ rather than $R$ directly. In order to cleanly describe a lower-dimensional space produced by the intersection of several constraints, we will consider a simplified manifold $M$ that is a subset of $R$. Additionally, we will make an assumption regarding the relationship between $\text{proj}_I(M)$ and $M$ to use the preimage theorem.

**Definition 15.** A conditional constraint manifold $(I, O, M)$ is a nonempty smooth manifold $M \subseteq R$ such that $\text{proj}_I : M \rightarrow \bar{Z}_I$ is a submersion $\forall \bar{z}_P \in M$.

In our context, $M$ is a submanifold of $\bar{Z}_P = \bar{Z}_{P_1} \times ... \times \bar{Z}_{P_p}$ for some $R \subset \bar{Z}_P$. The projection $\text{proj}_I = f$ is trivially smooth map between manifold $M$ and product manifold $\bar{Z}_I$. Thus, $\text{proj}_I$ is a submersion at $\bar{z}_P \in M$ when $df_{\text{proj}_I} : T_{\bar{z}_P}(M) \rightarrow T_{\bar{z}_I}(\bar{Z}_I)$ is surjective. When $\bar{Z}_I = \mathbb{R}^d$, this equivalent to the subspace of tangent space $T_{\bar{z}_P}(M)$ on parameters $I$ having rank $d$. A more intuitive interpretation is locally at $\bar{z}_P$, there exists coordinates to control $M$ in any direction of $I$. This restriction is used to ensure that the intersection of $M$ and other conditional constraint manifolds will not be transdimensional. This implies the preimage $\text{proj}_I^{-1}(\bar{z}_I)$ for particular values of input parameters $\bar{z}_I \in \text{proj}_I(M)$ is a submanifold of $M$ of codimension $\dim \bar{Z}_I$. Importantly, $\text{proj}_I(M)$ is an open set within $\bar{Z}_I$. This is the key consequence of the submersion assumption which is useful when intersecting arbitrary conditional constraint manifolds.

Suppose we are given a set of conditional constraint manifolds $\{ (I_1, O_1, M_1), ..., (I_n, O_n, M_n) \}$. Let $\Theta = \bigcup_{j=1}^{n} P_j$ be the set of parameters they collectively mention. Let $S = \bigcap_{j=1}^{n} M_j|_\Theta$ be their intersection when each constraint is extended on $\Theta$. We now present the main theorem which gives a sufficient condition for when $S$ is a submanifold of $\bar{Z}_\Theta$. This theorem is useful because it identifies when the intersection of several possibly dimensionality-reducing constraints is a space that we can easily characterize.

**Theorem 1.** If $\{ O_1, ..., O_n \}$ is a partition of $\Theta$ and there exists an ordering of $\{ (I_1, O_1, M_1), ..., (I_n, O_n, M_n) \}$ such that $\forall i \in \{1, ..., n\}, I_i \subseteq \bigcup_{j=1}^{i-1} O_j$, then $S$ is a submanifold of $\bar{Z}_\Theta$ of codimension $\sum_{j=1}^{n} \text{codim} M_j$.

**Proof.** Define $\Theta_i = \bigcup_{j=1}^{i} P_j$ to be union of the first $i$ sets of parameters. Notice that $\Theta_i = \bigcup_{j=1}^{i} O_j$ follows from the second assumption. Let $S_i = \bigcap_{j=1}^{i} M_j|_{\Theta_i}$ be the intersection of the first $i$ constraint manifolds over parameters $\Theta_i$. On the final intersection, $\Theta_n = \Theta$ and $S_n = S$. We can also write $S_i$ recursively as $S_i = S_{i-1}|_{\Theta_i} \cap M_i|_{\Theta_i}$, where $S_0 = \emptyset$. Here, $S_{i-1}$ and $M_i$ are extended by Cartesian products with $\bar{Z}_O_i$ and $\bar{Z}_{\Theta_i \setminus P}$, respectively.

We proceed by induction on $i$. For the base case, $I_1 = \bigcup_{j=1}^{1} O_j = \emptyset$. Thus, $S_0|_{\Theta_1} = \bar{Z}_{O_1}$ and $M_1|_{\Theta_1} = M_1$. Since, $M_1 \subseteq \bar{Z}_{O_1}$.

$$S_1 = (S_0|_{\Theta_1} \cap M_1|_{\Theta_1}) = (\bar{Z}_{O_1} \cap M_1) = M_1$$ is a submanifold of codimension $\text{codim} M_1$ within $\bar{Z}_{O_1}$.

For the inductive step, we assume that after the $(i-1)$th intersection, $S_{i-1}$ is a submanifold of codimensionality $\text{codim} S_{i-1}$ on parameters $\Theta_{i-1}$. We will show that
$S_i$ is a manifold of codimension $\text{codim } S_{i-1} + \dim M_i$ on parameters $\Theta_i$. By isomorphism of product manifold tangent spaces, $\forall \bar{z} \in S_{i-1}|_{\Theta_i}$, $T_{\bar{z}}(S_{i-1}|_{\Theta_i})$ is isomorphic to $T_{\bar{z}}(S_{i-1}) \times T_{\bar{z}}(\hat{\bar{z}}_{\Theta_i})$. Similarly, $\forall \bar{z} \in M_i|_{\Theta_i}$, $T_{\bar{z}}(M_i|_{\Theta_i})$ is isomorphic to $T_{\bar{z}}(M_i|_{\Theta_i}) \times T_{\bar{z}}(\hat{\bar{z}}_{\Theta_i\setminus P})$. As a result, the projection for the extended constraint $\text{dproj}_{\Theta_i-1} : T_{\bar{z}}(M_i|_{\Theta_i}) \rightarrow T_{\bar{z}}(S_{i-1})|_{\Theta_i}$ is itself surjective for all $\bar{z}$ and therefore is a submersion.

Because $S_i \subseteq \bar{Z}_{\Theta_i}$, for any $\bar{z} \in S_i$, $T_{\bar{z}}(M_i|_{\Theta_i}) \cup T_{\bar{z}}(S_{i-1}|_{\Theta_i}) \subseteq T_{\bar{z}}(\bar{Z}_{\Theta_i})$. For each $\bar{z} \in S_i$, consider any tangent vector $v \in T_{\bar{z}}(\bar{Z}_{\Theta_i})$. Let $v = x + y$ be an orthogonal decomposition of $v$ into a component $x$ defined on parameters $\Theta_i - 1$ and component $y$ defined on $O_i$. The first component $x$ is isomorphic to an element of $T_{\bar{z}}(M_i|_{\Theta_i})$ because its tangent space is surjective to $\bar{Z}_{\Theta_i-1}$. The second component $y$ is isomorphic to an element of $T_{\bar{z}}(S_{i-1}|_{\Theta_i})$ by the isomorphism to the product space involving $\bar{Z}_{\Theta_i}$. Thus, $T_{\bar{z}}(\bar{Z}_{\Theta_i}) \subseteq T_{\bar{z}}(M_i|_{\Theta_i}) \cup T_{\bar{z}}(S_{i-1}|_{\Theta_i})$. As a result, $M_i|_{\Theta_i}$ and $S_{i-1}|_{\Theta_i}$ intersect transversally implying that $S_i$ is a smooth submanifold of $\bar{Z}_{\Theta_i}$ with codimension

$$\text{codim } M_i|_{\Theta_i} + \text{codim } S_{i-1}|_{\Theta_i} = \text{codim } M_i + \text{codim } S_{i-1}.$$  

After $n$ iterations, the entire intersection is a submanifold of codimension $\dim S = \dim M_1 + \ldots + \dim M_n$ in $\Theta$. Let $d_i = \dim M_i - \dim \bar{Z}_{I_i}$ be the dimension of the submanifold defined by the projection preimage $\text{proj}_{\bar{z}}^{-1}(\bar{z})$. Finally, $\dim S$ can be seen alternatively as the sum of $\sum_{i=1}^{n} d_i$, the number of coordinates introduced on each iteration.

$$\dim S \geq \dim \bar{Z}_{\Theta_i} - \sum_{i=1}^{n} \text{codim } M_i$$

$$= \dim \bar{Z}_{\Theta_i} - \sum_{i=1}^{n} (\dim \bar{Z}_P_i - \dim M_i)$$

$$= \dim \bar{Z}_{\Theta_i} - \sum_{i=1}^{n} (\dim \bar{Z}_{O_i} + \dim \bar{Z}_{I_i} - \dim M_i)$$

$$= \sum_{i=1}^{n} (\dim M_i - \dim \bar{Z}_{I_i}) = \sum_{i=1}^{n} d_i$$

We will call $S$ a sample space when theorem[1] holds. From the partition condition, each parameter must be the output of exactly one conditional constraint manifold. Intuitively, a parameter can only be “chosen” once. From subset condition, each input parameter must be an output parameter for some conditional constraint manifold earlier in the sequence. Intuitively, a parameter must be “chosen” before it can be used to produce values for other parameters. Theorem[2] can be understood graphically using sampling networks. A sampling network is a subgraph of a constraint network using constraints corresponding to conditional constraint manifolds. Each parameter node has exactly one incoming edge. Directed edges go from input parameters to constraints or constraints to output parameters. Each parameter is the output of exactly one constraint. Additionally, the graph is acyclic.

For a given domain, we will assume we have a set of conditional constraint manifolds $\mathcal{M}$ defined on $(\bar{z}, \bar{u}, \bar{x})$. These are typically identified for dimensionality-reducing constraints that have a simple analytic form across the domain. We may have multiple conditional constraint manifolds per constraint $C$, resulting from the different ways of conditioning $C$. Implicit domain constraints $(I, (x_i), (x'_i), (u_j), (u'_j)) \in \mathcal{M}$ for each variable are always present within $\mathcal{M}$. Given a set of constraints $C$ defined on parameters $\Theta$, we can produce the corresponding set of conditional constraint manifolds on $\Theta$ by substituting constraints for the conditional constraint manifolds. For a constraint $C = (P, R)$, let $\mathcal{M}_C$ be the set of conditional constraint manifolds $\{I, O, M\} \subseteq \mathcal{M}$ associated with $C$ by substituting the input and output parameters $I, O$ for each conditional constraint manifold for with the parameters for $P$. For a set of constraints, let $\mathcal{M}_C = \bigcup_{C \in \mathcal{C}} \mathcal{M}_C$ be the union for each constraint $C \in \mathcal{C}$.

Now we can provide a more general definition of robust feasibility. We will define robustness properties with respect to a set of conditional constraint manifolds $\mathcal{M}$. This allows us to analyze the set of solutions in a lower-dimensional space $S$ given by the constraint manifolds. Intuitively, a set of constraints is robustly satisfiable if for some parameter values $\bar{z} \in S$, all parameter values $\bar{z}'$ in a neighborhood of $\bar{z}$ satisfy $C$.

**Definition 16.** A set of constraints $C$ is robustly satisfiable with respect to $\mathcal{M}$ if there exists a sample space $S = \bigcap_{i=1}^{n} M_i$ formed from conditional constraint manifolds $\{C_1, C_2, \ldots, C_m\} \subseteq \mathcal{M}$ where there exists a satisfying $\bar{z}$ with a neighborhood of parameter values $\mathcal{N}(\bar{z})$ in $S$ such that $\mathcal{N}(\bar{z}) \subseteq C_{\mathcal{E}|\mathcal{C}} C_{\mathcal{E}|\mathcal{C}}$.

**Definition 17.** A factored transition problem $\mathcal{P}$ is robustly feasible with respect to $\mathcal{M}$ if there exists a plan skeleton $\bar{a}$ such that $C_{\mathcal{E}|\mathcal{C}}$ is robustly satisfiable with respect to $\mathcal{M}$.

### 5.3 Robust Motion Planning

Our motion planning domain involves a single dimensionality-reducing constraint $\text{Motion}$. The implicit variable domain constraint $\text{Var}_q = (q_x, q^2)$ has full dimensionality by default. Each straight-line trajectory $t$ is uniquely described by its start configuration $q = t(0)$ and end configuration $q' = t(1)$. Thus, we can notate a straight-line trajectory as $t = (q, q') \in Q^{2d}$. As a result, $\text{Motion}$ is a 2$d$-dimensional submanifold of $Q^{2d}$. We will consider...
sample spaces resulting from constraint manifolds \( \mathcal{M} = \{ \text{Var}_q, \text{Motion} \} \). The collision-free \( \text{CFree} \) constraint is not included within \( \mathcal{M} \) because its composition varies from problem to problem within a domain.

Figure 8 shows a sampling network for the generic motion planning constraint network in figure 3. This sampling network uses conditional constraint manifolds \( \{ (x, q), \text{Var}_q, ((x, q), (u, t), \text{Motion}) \} \). The projection \( \text{proj}_{x^i} \text{Motion} = Q^{2d} \) is and is thus trivially a submanifold. The projection preimage \( \text{proj}_{x^i} \text{Motion} (q, q') = \{ t \} \) is a single point corresponding to the straight-line trajectory. Thus, the sample space \( S \subseteq Q^{(k-1)+2dk} \) is a submanifold of dimensionality \( d(k-1) \). Intuitively, the space of satisfying values is parametrizable by each configuration \( x^i_q \) for \( i = 1, \ldots, (k-1) \). As a result, a possible coordinate space of \( S \) is \( Q^{(k-1)} \) corresponding to Cartesian product of \( \mathcal{Q} \), the domain for each \( x^i_q \).

Subject to this sample space, we can analyze robustly feasible motion planning problems. Figure 9 fixes the plan skeleton \( \tilde{a} = (\text{Move}, \text{Move}) \) and investigates robustness properties of four problems varying the environment geometry. The plan skeleton has the following free parameters: \( u^1_q, x^1_q, u^2_q \). However, \( u^1_q, u^2_q \) can be uniquely determined given \( x^1_q \). The choices of these parameters must satisfy the following constraints:

\[
\{ \text{Motion}(q_0, u^1_q, x^1_q), \text{CFree}(u^1_q), \text{Motion}(x^1_q, u^2_q, q_0), \text{CFree}(u^2_q) \}.
\]

Varying the environment only affects the \( \text{CFree} \) constraint. The top row displays a top-down image of the scene and the bottom image shows the robot’s collision-free configuration space \( Q_{\text{free}} \) in light grey. Linear trajectories contained within the light grey regions satisfy their \( \text{CFree} \) constraint. The yellow region indicates values of \( x^1_q \) that will result in a plan. Problem 1 is unsatisfiable because no values of \( x^1_q \) result in a plan. Problem 2 has only a 1 dimensional interval of plan, thus it is satisfiable but not robustly satisfiable. Problem 3 has a 2 dimensional region of solutions, so it is robustly satisfiable. Problem 4 is unsatisfiable for the current plan skeleton. However, it is robustly satisfiable for a plan skeleton \( \tilde{a} = (\text{Move}, \text{Move}, \text{Move}) \).

5.4 Robust Pick-and-Place

In pick-and-place problems, \( \text{Stable}, \text{Region}, \text{Grasp}, \text{Kin} \), and \( \text{Motion} \) are all individually dimensionality-reducing constraints. Fortunately, we generally understand explicit representations of these sets. We will consider the following constraint manifolds:

\( \mathcal{M} = \{ \text{Var}_q, \text{Motion} \} \cup \{ \text{Stable}_o, \text{Grasp}_o, \text{Kin}_o \mid o \in \mathcal{O} \} \).

We will only consider problems in which \( \text{dim} \text{Stable}_o = \text{dim} \text{Region}_o \), in which case \( \text{Stable}_o \) captures the reduction of dimensionality from \( \text{Region}_o \). Again, \( \text{CFree}, \text{CFree}_o \), \( \text{CFreeH}_o \), and \( \text{CFreeH}_{o, \text{o}} \) are not assumed to be dimensionality-reducing constraints.

Figure 10 shows a sampling network for the pick-and-place constraint network in figure 5. It uses the following conditional constraint manifolds:

\[
\{ (x, q, (u, t), \text{Motion}) \} \cup \{ (x, q, \text{Stable}_o), (x, q, \text{Grasp}_o), (x, q, \text{Kin}_o) \mid o \in \mathcal{O} \}.
\]

The sampling network satisfies the graph theoretic conditions in theorem 1. Additionally, each conditional constraint manifold has full dimensionality in its input parameter-space. \( \text{Var}_q, \text{Stable}_o \) and \( \text{Grasp}_o \) have no input parameters and therefore trivially have full input dimensionality. The projection \( \text{proj}_{x, q, (u, t)} \text{Motion} \) has full dimensionality under the assumption that the robot gripper’s workspace has positive measure in \( \text{SE}(3) \). We will consider a manifold subset of \( \text{Kin}_o \) that satisfies the submersion conditions by omitting kinematic singularities. As before, \( \text{proj}_{x, q, (u, t)} \text{Motion} \) has full input dimensionality.

This sampling network structure generalizes to all pick-and-place problems with goal constraints on object poses and robot configurations. Solutions are alternating sequences of \( \text{Move}, \text{Pick}, \text{MoveH}, \) and \( \text{Place} \) transitions where \( \text{Move} \) and \( \text{MoveH} \) may be repeated zero to arbitrarily many times. Each new cycle introduces a new grasp parameter, pose parameter, two configuration parameters, and two trajectory parameters. However, the only interaction with the next cycle is through the beginning and ending configurations which serve as the input parameters for the next move action. Thus, this small set of conditional constraint manifolds is enough to define sample spaces for a large set of pick-and-place problems involving many objects.

To better visualize a pick-and-place sample space, we investigate a 1-dimensional example where \( \mathcal{Q}, \mathcal{S}_A, \mathcal{S}_B \subseteq \mathbb{R} \). Figure 11 fixes the plan skeleton \( \tilde{a} = (\text{MoveH}, \text{Place}) \) and investigates robustness properties of three problems varying the initial pose \( b_0 \) of block \( B \). The robot starts off holding block \( A \) with grasp \( a_0 \), so in the initial state, \( x^A_0 = A \) and \( x^B_0 = a_0 \). The robot...
may only grasp block A when A touches its left or right side. Therefore, the set of grasps \( G_A = \{-a_0, a_0\} \) is finite. Additionally, the kinematic constraint on \( (x_o, x'_o, x_q) \) results in a plane within \( \mathbb{R}^3 \).

\[
\text{Kin}_{o} = \langle (x'_o, x_o, x_q), \{q + g - p = 0 \ (g, p, q) \in \mathbb{R}^3 \} \rangle
\]

The goal constraint \( C_s = \{\text{Region}_A\} \) requires that A be placed within a goal region. The plan skeleton has the following free parameters: \( \{x^1_q, u^1_t, x^2_A\} \). Once again, \( u^1_t \) can be uniquely determined given \( x^1_q \). The choices of these parameters must satisfy the following constraints:

\[
\{\text{Motion}(q_0, u^1_t, x^1_q), \text{CFree}_H(u^1_t, a_0, a_0), \text{CFree}_H(u^1_t, a_0, b_0), \text{Grasp}(a_0), \text{Stable}(x^1_A), \text{Kin}(a_0, x^1_A, x^1_q), \text{Region}(x^1_A)\}
\]

Varying \( b_0 \) only affects the \( \text{CFree}_H(u^1_t, a_0, b_0) \) constraint. The sample space \( S \) is the 1-dimensional manifold given by \( \text{Kin}(a_0, x^2_A, x^1_q) \).

Each plot visualizes values of \( (x^2_A, x^1_q) \) that satisfy the \( \text{Kin}_A \) (blue line), \( \text{Region}_A \) (red rectangle) and \( \text{CFree}_H \) (green rectangle) constraints. For this example, we use \( x^2_A \) as a surrogate for \( u^1_t \) with respect to \( \text{CFree}_H(u^1_t, a_0, b_0) \). The yellow region indicates values of \( (x^2_A, x^1_q) \) that satisfy all the constraints and therefore result in a plan. Problem 1 is unsatisfiable because the constraints have no intersection. Problem 2 has only a single plan and therefore has zero measure with respect to \( S \). Because of this, Problem 2 satisfiable but not robustly satisfiable. Problem 3 has a 1-dimensional interval of plans on \( S \), so it is robustly satisfiable. However, this set of solutions has zero measure with respect to \( \mathbb{R}^2 \). Without the identification of the sample space \( S \), this problem would not be deemed robustly satisfiable.
6 Conditional Samplers

Now that we have identified sample spaces that arise from dimensionality-reducing constraints, we can design samplers to draw values from these spaces. Traditional samplers either deterministically or nondeterministically draw a sequence of values \( s = (v^1_p, v^2_p, \ldots) \) from the domain \( Z_p \) of a single parameter \( p \). In order to solve problems involving dimensionality-reducing constraints using sampling, we must extend this paradigm in two ways. First, we must intentionally design samplers that draw values of several variables involved in one or more dimensionality-reducing constraints. Second, we need to construct samplers conditioned on particular values of other variables in order to sample values at the intersection of several dimensionality-reducing constraints. Thus, our conditional treatment of samplers will closely mirror the treatment of constraints.

**Definition 18.** A conditional sampler \( \psi = (I, O, C, f) \) is given by a function \( f(\vec{v}_I) = (\vec{v}_{I_1}, \vec{v}_{I_2}, \ldots) \) from input values \( \vec{v}_I \) for parameters \( I \) to a sequence of output values \( \vec{v}_O \) for parameters \( O \). The graph of \( f \) satisfies a set of constraints \( C \) on \( I \cup O \).

We will call any \( \psi \) with no inputs \( I = () \) an unconditional sampler. The graph implicitly contains output parameter constraints \( Var_p \) for \( p \in O \). The function \( f \) may produce sequences that are enumeratedly infinite or finite. Let \( \text{NEXT}(f(\vec{v}_I)) = \vec{v}_O \) produce the next output values in the sequence. The function \( f \) may be implemented to nondeterministically produce a sequence using random sampling. It is helpful algorithmically to reason with conditional samplers for particular input values.

**Definition 19.** A conditional sampler instance \( s = \psi(\vec{v}_I) \) is a conditional sampler paired with input values \( \vec{v}_I \).

We frequently design conditional samplers to draw values from the conditional constraints present on sample networks for the domain. For example, consider the conditional sampler \( \psi^0_{IK} \) for the kinematic constraint \( \text{Kin}_o, \)

\[
\psi^0_{IK} = ((x_o, x'_o), (x_q), \{\text{Kin}_o\}, \text{INVERSE-KIN})
\]

\( \psi^0_{IK} \) has input parameters \( I = (x_o, x'_o) \) and output parameters \( O = (x_q) \). Its function \( f = \text{INVERSE-KIN} \) performs inverse kinematics, producing configurations \( q \) that have end-effector transform \( pg^{-1} \) for world pose \( p \) and grasp pose \( g \). For a 7 degree-of-freedom manipulator in \( \text{SE}(3) \), this would sample from a 1-dimensional manifold.

Like conditional constraints, conditional samplers can be composed in a sampler sequence \( \vec{\psi} = (\psi_1, \ldots, \psi_k) \) to produce a vector of values for several parameters jointly. A well-formed sampler sequence for a set of parameters \( \Theta \) satisfies \( \Theta = \bigcup_{j=1}^k O_j \) as well the conditions from theorem[1]. Each parameter must be an output of exactly one conditional sampler and later conditional samplers must only depend on earlier samplers. The set of values generated by the sampler sequence \( \vec{\psi} \) is given by

\[
F(\vec{\psi}) = \{ \vec{v} \mid \vec{v}_{O_1} \in f_1(), \vec{v}_{O_2} \in f_2(\vec{v}_{I_2}), \ldots, \vec{v}_{O_k} \in f_k(\vec{v}_{I_k}) \}.
\]

We are interested in identifying combinations of conditional samplers that will provably produce a solution.
for robustly feasible problems. In particular, we are interested in properties of the samplers that are almost surely true, i.e., properties that hold with probability one. Similar to $\mathcal{M}_C$, for a set of constraints $\mathcal{C}$ and set of conditional samplers $\Psi$, let $\Psi_C$ be the set of conditional samplers appropriate for each constraint.

**Definition 20.** A set of conditional samplers $\Psi$ is sufficient for a robustly satisfiable plan skeleton $\vec{a}$ with respect to $\mathcal{M}$ if there exists a sampler sequence $\vec{\psi} \subseteq \Psi_C$ such that almost surely $F(\vec{\psi}) \cap \mathcal{C}_\vec{a} \neq \emptyset$.

**Definition 21.** $\Psi$ is sufficient for a domain $\mathcal{D}$ with respect to $\mathcal{M}$ if for all robustly feasible $\mathcal{P} \in \mathcal{D}$, there exists a robustly satisfiable plan skeleton $\vec{a}$ for which $\Psi$ is sufficient.

A conditional sampler must generally produce values covering its constraint manifold to guarantee completeness across a domain. This ensures that the sampler can produce values within every neighborhood on the constraint manifold. This property is advantageous because it is robust to other adversarial, worst-case constraints that only admit solutions for a neighborhood of values on the constraint manifold that the sampler is unable to reach. In motion planning, a traditional sampler $s = (v^i)^{i=1,\ldots}$... is dense with respect to a conditional topology space $Z$ if the topological closure of its output sequence is $Z$ (LaValle 2006). The topological closure of $s$ is the union of $s$ and its limit points, points $z \in Z$ for which every neighborhood of $z$ contains a point in $s$. We extend this idea to conditional samplers for conditional constraint manifolds.

**Definition 22.** A conditional sampler $\psi = (I,O,C,f)$ is dense with respect to a conditional constraint manifold $(I,O,M)$ if $\forall \bar{v}_t \in \text{proj}_I^{-1}(M), f(\bar{v}_t)$ is dense with high probability in $\text{proj}_O(\text{proj}_I^{-1}(\bar{v}_t))$.

The following theorem indicates that dense conditional samplers for the appropriate conditional constraints will result in a sufficient collection of samplers. Thus, a set of dense conditional samplers for individual conditional constraint manifolds can be leveraged to be sufficient for any robustly satisfiable plan skeleton.

**Theorem 2.** A set of conditional samplers $\Psi$ is sufficient for a domain $\mathcal{D}$ with respect to $\mathcal{M}$ if for each conditional constraint manifold $(I,O,M)$ in $\mathcal{M}$, there exists a conditional sampler $\psi \in \Psi$ that is dense for $(I,O,M)$.

**Proof.** Consider any robustly feasible $\mathcal{P} \in \mathcal{D}$ and by definition there exists a sampler sequence $\vec{\psi} \subseteq \Psi_C$ such that almost surely $F(\vec{\psi}) \cap \mathcal{C}_\vec{a} \neq \emptyset$. Let $\bar{y}_i \in \mathbb{R}^d$, be the coordinates introduced on the $i$th projection preimage where $d_i = \text{dim } M_i - \text{dim } \hat{Z}_I$. Consider the atlas for $\mathcal{S}$ constructed by concatenating combinations of the charts for each projection preimage in the sequence. There exists an open set in the coordinate space of $\mathcal{S}$ centered around $\bar{z}$ that satisfies $\mathcal{C}_\bar{a}$. Consider a subset of the coordinate space of $\mathcal{N}(\bar{z})$ given as $\bar{Y}_1 \times \ldots \times \bar{Y}_n \subset \mathbb{R}^{\dim \mathcal{S}}$ where each $\bar{Y}_i \subset \mathbb{R}^{d_i}$ is an open set. Any combination of values $(\bar{y}_1, \ldots, \bar{y}_n)$ where $\bar{y}_i \in \bar{Y}_i$ is contained within this set.

Consider a procedure for sampling $(\bar{y}_1, \ldots, \bar{y}_n)$ that chooses values for $\bar{y}_i$ in a progression of $i$ iterations. On iteration $i$, the value of $\bar{z}_{\bar{q}_{i-1}}$ is fixed from the choices of $\bar{y}_1, \ldots, \bar{y}_{i-1}$ on previous iterations. Consider the submanifold defined by the projection preimage $\text{proj}_I^{-1}(\bar{z}_I)$ for the $i$th conditional constraint manifold $(I_i, O_i, M_i)$. Recall that $I_i \subseteq \Theta_{i-1}$. By assumption, there exists a conditional sampler $\psi \in \Psi$ that is dense for $(I,O,M)$. Thus, $\psi(\bar{z}_I)$ densely samples $\text{proj}_I^{-1}(\bar{z}_I)$ producing output values $\bar{z}_O$. Correspondingly, $\psi(\bar{z}_I)$ densely samples the coordinate space of $\bar{y}_i$. Upon producing $\bar{y}_i \in \bar{Y}_i$, the procedure moves to the next iteration. Because $\bar{Y}_i$ is open within $\mathbb{R}^{d_i}$ and the sampling is dense, the sampler will produce a satisfying coordinate values $\bar{y}_i$ within a finite number of steps with probability one. After the $n$th iteration, $\bar{z}$ given by coordinates $(\bar{y}_1, \ldots, \bar{y}_n)$ in $\bar{Y}_1 \times \ldots \times \bar{Y}_n$ will satisfy constraints $\mathcal{C}_\bar{a}$.

### 6.1 Motion Planning Samplers

In order to apply theorem 2 to the sampling network (figure 3), we require conditional sampler for the $\text{Var}_q$ and Motion conditional constraint manifolds. For $\text{Var}_q$, we provide an unconditional sampler $\psi_{\text{conf}}$ that is equivalent to a traditional configuration sampler in motion planning.

$$\psi_Q = ((),(x_a),\text{Var}_q),\text{SAMPLE-CONF}$$

Its function $\text{SAMPLE-CONF}$ densely samples $Q$ using traditional methods (LaValle 2006). For $\text{Motion}$, we provide a conditional sampler $\psi_T$ that simply computes the straight-line trajectory between two configurations.

$$\psi_T = ((x_0, x_0'), (u_0), \text{Motion}), \text{STRAIGHT-LINE}$$

STRAIGHT-LINE generates a sequence with just a single value corresponding to the trajectory $t$ between $q,q'$.

Because our motion planning transition system exhibits little factoring, $\psi_Q$ samples entire states $(x_q, \bar{q})$. Additionally, each trajectory $(u_t) = \bar{u}$ computed by $\psi_T$ corresponds to either a single transition when $u_t \in \text{CFree}$ or otherwise no transitions. Thus, $\psi_Q$ and $\psi_T$ can be seen as directly sampling the entire state and control-spaces.

### 6.2 Pick-and-Place Samplers

In addition to $\psi_{\text{PK}}, \psi_{\text{Q}}, \psi_T$, the pick-and-place sampling network (figure 10) requires the following unconditional
Definition 23. An algorithm is probabilistically complete with respect to \( \mathcal{D} \) and \( \mathcal{M} \) if for all robustly feasible \( \mathcal{P} \in \mathcal{D} \), it will almost surely return a plan in finite time.

We present algorithms that take as an input a set of conditional samplers \( \Psi \) for the domain. The algorithms are therefore domain-independent because the problem-specific knowledge is restricted to the constraints and samplers. We will show that these algorithms are probabilistically complete, given a set of sufficient conditional samplers \( \Psi \) for conditional constraint manifolds \( \mathcal{M} \). Thus, the completeness of the resulting algorithm is entirely dependent on the conditionals samplers. We give two algorithms, INCREMENTAL and FOCUSED.

7.1 Incremental Algorithm

The incremental algorithm alternates between generating samples and checking whether the current set of samples admits a solution. It can be seen as a generalization of the probabilistic roadmap (PRM) (Kavraki et al. 1996) for motion planning and the FFRob algorithm for task and motion planning (Garrett et al. 2016) which both alternate between exhaustive sampling and search phases for their respective domains.

The pseudocode for the incremental algorithm is displayed in figure 12. INCREMENTAL maintains a set of constraint elements, tuples of values annotated with the constraint they satisfy, and a queue of sampler instances. Define an iteration of INCREMENTAL to be the set of commands in body of the while loop. On each iteration, INCREMENTAL first calls DISCRETE-SEARCH to attempt to find a plan \( \langle \vec{a}, \vec{x}, \vec{u} \rangle \) using elements. The procedure DISCRETE-SEARCH searches a discretized transition system for problem \( \mathcal{P} = \{ C_0, C_\ast, \{ C_1, \ldots, C_n \} \} \), using the samples from each \( X_i \) and \( U_i \) given by elements. We outline several possible implementations of DISCRETE-SEARCH in section 8. If DISCRETE-SEARCH is successful, the sequence of control inputs \( \vec{u} \) is returned. Otherwise, DISCRETE-SEARCH produces \( \vec{a} = \text{None} \). In which case, INCREMENTAL calls the PROCESS-SAMPLERS subroutine to sample values from at most \( \text{len}(\text{queue}) \) sampler instances using the function SAMPLE. The procedure SAMPLE in figure 14 has a single sampler-instance argument \( s \) and queries the next set of output values \( \bar{\nu}_i \) in the sequence \( f(\vec{a}_i) \). If the sequence has not been enumerated, i.e. \( \text{NEXT}(f(\bar{\nu}_i)) \neq \text{None} \), it returns the constraint elements that \( \bar{\nu}_j + \bar{\nu}_o \) together satisfy using the helper function ELEMENTS.

The procedure PROCESS-SAMPLERS iteratively instantiates and processes sampler instances \( s \). Its inputs are a queue of sampler instances, a set of already processed sampler instances, the set of constraint elements, and two additional parameters that are used differently by INCREMENTAL and FOCUSED. PROCESS is a procedure that takes as input a sampler instance and returns a set of elements and \( k \) is the maximum number of sampler instances to process. On each iteration, PROCESS-SAMPLERS pops a sampler instance \( s \) off of queue, adds the result of PROCESS
Theorem 3. INCREMENTAL is probabilistically complete for a domain given a sufficient set of conditional samplers.

Proof. Consider any robustly feasible problem $P \in \mathcal{D}$. By definitions 6 and 7, there exists a sampler sequence $\psi = (\psi_1, \ldots, \psi_k)$ that with probability one, a finite number of calls to SAMPLE produces values that are parameters in $C_{\tilde{a}}$ for some robustly satisfiable plan skeleton $\tilde{a}$. In its initialization, INCREMENTAL adds all sampler instances $s$ available from the $P$'s constants. On each iteration, INCREMENTAL performs SAMPLE($s$) for each sampler instance $s$ in queue at the start of the iteration. There are a finite number of calls to SAMPLE each iteration. The resulting constraints elements SAMPLE($s$) are added to elements and all new sampler instances $s'$ are added to queue to be later sampled. The output values from each sampler instance will be later become input values for all other appropriate conditional samplers. This process will indirectly sample all appropriate sampler sequences including $\tilde{\psi}$. Moreover, because $s$ is re-added to queue, it will be revisited on each iteration. Thus, SAMPLE($s$) will be computed until a solution is found. Therefore, each sampler sequence will also be sampled not only once but arbitrarily many times. INCREMENTAL will produce satisfying values from $\psi$ within a finite number of iterations.
We will assume discrete-search is any sound and complete discrete search algorithm. Thus, discrete-search will run in finite time and return a correct plan if one exists. On the first iteration in which a solution exists within samples, discrete-search will produce a plan \( \bar{\vec{a}} \neq \text{None} \). And incremental will itself return the corresponding sequence of control inputs \( \bar{\vec{u}} \) as a solution.

Because incremental creates sampler instances exhaustively, it will produce many unnecessary samples. This results in a combinatorial growth in the number of sampled states as well as the size of the discretized state-space. This motivates our second algorithm, which is able to guide the selection of samplers by integrating the search over structure and search over samples.

7.2 Focused Algorithm

The focused algorithm uses lazy samples as placeholders for actual concrete sample values. Lazy samples are similar in spirit to symbolic concrete references [Srivastava et al. 2014]. The lazy samples are optimistically assumed to satisfy constraints with concrete samples and other lazy samples via lazy constraint elements. This allows discrete-search to reason about plan skeletons without some concrete parameters. After finding a plan, focused calls samplers that can produce values for the lazy samples used. This algorithm is related to a lazy PRM [Bohin and Kavraki 2000] [Dellin and Srinivasa 2016], which defers collision checks until a path is found. However instead of just collision checks, focused defers generation of sample values until an optimistic plan is found. In a pick-and-place domain, this means lazily sampling poses, inverse kinematic solutions, and trajectories. Because focused plans using both lazy samples and concrete samples, it is able to construct plans that respect constraints on the actual sample values. And by using lazy samples, it can indicate the need to produce new samples when the existing samples are insufficient.

The pseudocode for the focused algorithm is shown in figure 15. The focused algorithm uses the same subroutines in figure 14 as the incremental algorithm. Once again, let an iteration of focused be the set of commands in body of the while loop. Define an episode of focused to be the set of iterations between the last sampled reset and the next sampled reset. Let the initialization of sampled in line 1 also be a reset. On each iteration, the focused algorithm creates a new queue and calls process-samplers to produce mixed-elements. It passes the procedure sample-lazy rather than sample in to process-samplers. For each output \( o \) of \( \psi \), sample-lazy creates a unique lazy sample \( \bar{\vec{l}}^o_o \) for the combination of \( \psi \) and \( o \). Then, for each

```latex
\begin{verbatim}
SAMPLE-LAZY(s = \langle I, O, C, f \rangle): 1 lazy_elements = ELEMENTS(C, \bar{\vec{v}}_1 + (\bar{\vec{l}}^o_o \mid o \in O)) 2 for e in lazy_elements: 3 e.instance = s 4 return lazy_elements
\end{verbatim}
```

Figure 15. The pseudocode for the focused algorithm.

lazy constraint element \( e \) formed using \( \bar{\vec{l}}^o_o \), \( s \) is recorded as the sampler instance that produces values satisfying the element using \( e.instance \). For a pose sampler instance \( \psi_P^o() \), sample-lazy creates a single lazy sample \( \bar{\vec{l}}^o_o \) and returns a single lazy constraint element \( \text{Stable}(\bar{\vec{l}}_o^o) \):

```latex
\text{LAZY-SAMPLE}(\psi_P^o()) = \{ \text{Stable}(\bar{\vec{l}}_o^o) \}.
```

Discrete-search performs its search using mixed_elements, a mixed set of elements and lazy_elements. If discrete-search returns a plan, focused first checks whether it does not require any lazy_elements, in which case it returns the sequence of control inputs \( \bar{\vec{u}} \). Otherwise, it calls retrace-instances to recursively extract the set of sampler instances used to produce the lazy_elements. Retrace-instances returns just the set of ancestor sampler instances that do not contain lazy samples in their inputs. For each ancestor sampler instance \( s \), focused samples new output values
and adds any new constraint elements to new\_elements. To ensure all relevant sampler instances are fairly sampled, each s is then added to sampled. This prevents these sampler instances from constructing lazy samples within process-samplers on subsequent iterations. Additionally, elements are added to new\_elements before they are moved to elements to limit the growth in sampler instances. When DISCRETE-SEARCH fails to find a plan, new\_samples are added to elements, sampled is reset, and this process repeats on the next episode.

While not displayed in the pseudocode, the focused algorithm has the capacity to identify infeasibility for some problems. When DISCRETE-SEARCH fails to find a plan and sampled is empty, the problem is infeasible because the discretized problem with optimistic lazy samples is infeasible. If no graph on conditional samplers \( \Psi \) contains cycles, then a lazy sample can be created for each sampler instance rather than each sampler. Then, retrace-instances can sample values for lazy elements that depend on the values of other lazy elements. Finally, new\_elements can be safely added directly to elements. These modifications can speed up planning time by requiring fewer calls to solve-discrete. For satisficing planning, to bias DISCRETE-SEARCH to use few lazy samples, we add a non-negative cost to each transition instance corresponding to the number of lazy samples used and use a cost sensitive version of solve-discrete. In this context solve-discrete can be thought of optimizing for a plan that requires the least amount of additional sampler effort. Thus, plans without lazy samples have low cost while plans with many lazy samples have high cost.

**Theorem 4.** FOCUSED is probabilistically complete for a domain given a sufficient set of conditional samplers.

**Proof.** As in theorem 5 consider any robustly feasible problem \( \mathcal{P} \in \mathcal{D} \). By definitions 20 and 21 there exists a sampler sequence \( \bar{\psi} = (\psi_1, \ldots, \psi_k) \) that with probability one, in a finite number of calls to sample produces values that are parameters in \( C_\bar{a} \) for some robustly satisfiable plan skeleton \( \bar{a} \). At the start of an episode, elements implicitly represents a set of partially computed sampler sequences. We will show that between each episode, for each partially computed sampler sequence that corresponds to some plan skeleton, a next sampler in the sampler sequence will be called. And both the new partial sampler sequence as well as the old one will be present within elements during the next episode.

On each iteration, FOCUSED calls DISCRETE-SEARCH to find a plan that uses both real samples and lazy samples. FOCUSED calls sample for each sampler instance s corresponding to a lazy sample along the plan. Additionally, by adding s to sampled, FOCUSED prevents the lazy samples resulting from s from being used for any future iteration within this episode. This also prevents DISCRETE-SEARCH from returning the same plan for any future iteration in this episode. The set elements is fixed for each episode because new samples are added to new\_elements rather than elements. Thus, there are a finite number of plans possible within each episode. And the number of iterations within the episode is upper bounded by the initial number of plans. Each plan will either be returned on some iteration within the episode and then be blocked or it will be incidentally blocked when search returns another plan. Either way, sample will be called for a sampler instance s on its remaining sampler sequence. When no plans remain, search will fail to find a plan. Then, focused resets, allowing each s \( \in \) sampled to be used again, and the next episode begins.

Because each episode calls sample for at least one \( \psi \) along each possible partial sampler sequence, \( \bar{\psi} \) will be fully sampled once after at most k episodes. Moreover, each subsequent episode will sample \( \psi \) again as new partial sampler sequences are fully computed. Thus, \( \bar{\psi} \) will be fully sampled arbitrarily many times. Consider the first episode in which a solution exists within elements. DISCRETE-SEARCH is guaranteed to return a plan only using only elements within this episode. This will happen, at latest, when all plans using lazy elements are blocked by sampled. Then, focused will itself return the corresponding sequence of control inputs as a solution.

It is possible to merge the behaviors of the incremental and focused algorithms and toggle whether to eagerly or lazily sample per conditional sampler. This allows inexpensive conditional samplers to be immediately evaluated while deferring sampling of expensive conditional samplers. This fusion leads to a variant of the focused algorithm where sampler instances switch from being lazily evaluated to eagerly evaluated when they are added to sampled. In this case, sampled need not be reset upon DISCRETE-SEARCH failing to identify a plan.

**8 Discrete Search**

A straightforward implementation of DISCRETE-SEARCH is a breadth-first search (BFS) from \( \bar{x}_0 \). The directed, outgoing edges from a state \( \bar{x} \) can be dynamically computed by considering each clause \( C_\alpha \), substituting the values for \( \bar{x} \), determining all combinations of \( \bar{u} \) and \( \bar{x'} \) that satisfy

\[
\text{ELEMENTS}(C_\alpha, \bar{x} + \bar{u} + \bar{x'}) \subseteq \text{elements}. \tag{1}
\]

The resulting state is \( \bar{x'} \). The control variables \( \bar{u} \) are only used in the resulting plan. This can be further optimized by fixing the values of any \( \bar{u}, \bar{x'} \) constrained by equality. Thus, the number of outgoing edges is bounded by the number of combinations of sampled values for each free control or subsequent state variable. While a BFS avoids...
explicitly constructing the full discretized state-space, it will still search the entire state-space reachable from $\bar{x}_0$ in fewer transitions than the length of the shortest plan. This can be prohibitively expensive for problems with significant factoring such as pick-and-place problems where many choices of objects to manipulate result in a large branching factor.

8.1 Factored Planning

The artificial intelligence community has developed many algorithms that are much more efficient than classical graph search algorithms for high-dimensional, factored problems. These algorithms exploit both the factored state representation and transitions with many equality constraints to guide search using domain-independent heuristics. Many heuristics are derived by solving an easier approximation of the original search problem. This leads to both admissible (Bonet and Geffner 2001) and empirically effective heuristics (Hoffmann and Nebel 2001; Helmert 2006). These heuristics can frequently avoid exploring most of the discrete state-space and even efficiently identify many infeasible problem instances.

In our experiments, we use the efficient FastDownward planning toolkit (Helmert 2006) which contains implementations of many of these algorithms. FastDownward, as well as many other planners, operate on states described as a finite set of discrete variables. Discrete transitions are often described using a precondition and effect action model such as in SAS+ (Backström and Nebel 1995). Definitions of many of these algorithms. FastDownward, as well as many other planners, operate on states described as a finite set of discrete variables. Discrete transitions are often described using a precondition and effect action model such as in SAS+ (Backström and Nebel 1995).

Definition 24. An action $\langle$pre, eff$\rangle$ is given by sets of constant equality conditions pre on $\bar{x}$ and eff on $\bar{x}'$. State variables $i$ omitted from eff are assumed to be constrained by pairwise equality constraints $x_i = x_i'$. A single action will represent many different transitions if some state variables are not mentioned within pre. Thus, action models can be advantageous because they compactly describe many transitions using a small set of actions.

In order to use these algorithms, we automatically compile discretized factored transition systems into SAS+. We could instead automatically compile to Planning Domain Definition Language (PDDL) (McDermott et al. 1998; Edelkamp 2004), a standardized artificial intelligence planning language used in competitions. However, many planners compile problem instances into a form similar to SAS+, so we jump directly to this representation.

8.2 Action Compilation

A factored transition system with a discretized set of constraint elements can be compiled into SAS+ as follows. First, the goal constraints $C_g$ are converted into a ‘goal transition’

$$C_g \cup \{x'_\text{goal} = \text{True}\} \cup \{x_1 = x'_1, \ldots, x_m = x'_m\}$$

by adding a state variable $x_{\text{goal}}$ that is true when the goal constraints have been satisfied. This allows the new goal $C'_g = \{x'_\text{goal} = \text{True}\}$ to be represented with a single equality constraint. Because of this additional state variable, all existing transitions are augmented with an equality constraint $\{x_{\text{goal}} = x'_\text{goal}\}$.

Each $C_a$ is compiled into a set of actions by first identifying its full set of parameters $P_a$ comprised of $x_i, u_j$ mentioned within $C_a$ as well as all of $\bar{x}'$. The free parameters $F_a \subseteq P_a$ include a single variable for each connected component of variables transitively constrained by pairwise equality. $F_a$ excludes variables transitively constrained by constant equality. For every combination of the free parameters that satisfy equation [1] create an action $\langle$pre, eff$\rangle$. Let pre contain an equality constraint from each $x_i \in F_a$ to its corresponding constant or free parameter. And let eff contain an equality constraint from each $x'_i$ to its corresponding constant or free parameter if $[x_i = x'_i] \notin C_a$.

Consider the following actions generated for $C_{\text{pick}}$:  

$$\langle\{x_o = p, x_h = \text{None}, x_q = q\}, \{x_o = g, x_h = o\}\rangle \quad | \exists g, p, q. \quad \text{Kin}_o(g, p, q) \in \text{elements} \rangle$$

$C_{\text{pick}}$ and $C_{\text{place}}$ only have 3 free parameters, so $F_{\text{pick}} = \{p, g, q\}$. This results in a compact description of their transitions. Now consider the actions generated for $C_{\text{move}}$:

$$\langle\{x_q = q, x_h = \text{None}\} \cup \{x_o = p, o \in O\}, \{x'_q = q'\}\rangle \quad | \exists q, t, q', p_1, \ldots, p_{|O|}, \{\text{Motion}(q, t, q')\}, C_{\text{Free}}(t) \cup \{C_{\text{Free}}(t, p) \mid o \in O \subseteq \text{elements}\}$$

$C_{\text{move}}$ and $C_{\text{moveH}}$ have $3 + |O|$ free parameters because $F_{\text{move}} = \{q, t, q', p_1, \ldots, p_{|O|}\}$. For non-unary discretization of each object variable, the number of transitions grows exponentially in $|O|$. Despite this, each eff only involves one variable and there is only one control variable. The rest of the state variables are solely used to determine action feasibility. Additionally, each constraint has low arity: Motion involves 3 parameters and each $C_{\text{Free}}$ constraint only involves 2 parameters.

8.3 Axiom Compilation

Low constraint arity allows us to further factor transitions by introducing derived variables, variables evaluated from the core state variables $\bar{x}$ using rules known as axioms (Helmert 2006). Axioms are known to be useful compactly expressing planning problems (Thiébaux et al. 2005). Axioms have the same form $\langle$pre, eff$\rangle$ as actions. However, they are automatically applied upon reaching a new state unlike actions, which are chosen by a planner.

For each non-equality constraint $C = (P, R) \in C_a$, we compute a parameterized derived variable $d_C(\bar{z}_D)$. The parameterization $D = P \setminus \bar{x}$ includes parameters for the
control $\bar{u}$ and subsequent state $\bar{x}'$ but excludes the current state variables $\bar{x}$. An axiom $\langle \text{pre}, \text{eff} \rangle$ is computed for each constraint element $C(\bar{v}) \in \text{elements}$ where $\text{pre} = \{ p : v_p \mid p \in (P \cap \bar{x}) \}$ and $\text{eff} = \{ d_C(\bar{v}_D) : \text{True} \}$. This allows $d_C(\bar{z}_D) = \text{True}$ to be substituted for $C$ within the action representation of $C_o$. We include $\bar{u}$ and $\bar{x}'$ within $D$ to ensure that same control and subsequent state values are used in each derived variable precondition $d_C(\bar{z}_D) = \text{True}$. Axioms allow us to remove $\bar{x}$ from the parameters $P_o$ of $C_o$ by individually testing if each constraint is satisfied. Intuitively, an action is performable if, for each constraint $C$, the values of state variables $(P \cap \bar{x})$ complete a constraint element within $\text{elements}$. Because each action and axiom now only involve a few parameters, the set of their instances is generally much smaller than before.

The axioms computed for $\text{Motion}$ and each $C_{Free/o}$ within $C_{Move}$ are as follows:

$$\left\{ \langle \{ x_q = q \}, \langle \text{Motion}(\cdot, t, q') = \text{True} \rangle \right\}$$

$$\mid \exists q, t, q'. \text{Motion}(q, t, q') \in \text{elements}$$

$$\left\{ \langle \{ x_p = p \}, \langle C_{Free/o}(t, \cdot) = \text{True} \rangle \right\}$$

$$\mid \exists t, p. C_{Free/o}(t, p) \in \text{elements}.$$ 

And $C_{Move}$ can be modified to be the following:

$$\left\{ \langle \{ \text{Motion}(\cdot, t, q') = \text{True}, x_h = \text{None} \rangle \right\}$$

$$\cup \langle \{ C_{Free/o}(t, \cdot) : \text{True} \mid o \in \mathcal{O} \} \right\}$$

$$\mid \exists t, q'. \{ \text{Var}(t), \text{Var}(q') \} \subseteq \text{elements}.$$ 

The resulting $\text{Motion}$ and $C_{Free/o}$ axioms as well as $C_{Move}$ actions all have 3 or fewer parameters. And the number of actions and axioms need to describe a pick-and-place transition is linear in $|\mathcal{O}|$ rather than exponential in $|\mathcal{O}|$.

9 Tabletop Manipulation

We seek to model tabletop manipulation problems involving a manipulator attached to a movable base as a factored transition system. The previously presented pick-and-place factored transition system encompasses this domain, and the specified samplers lead to probabilistically complete algorithms. However, the previous formulation leads to poor performance in practice for high-dimensional robot configuration spaces as it attempts to construct control trajectories between all pairs of robot configurations. The resulting control-space is similar to a simplified Probabilistic Roadmap (sPRM) [Kavraki and Latombe 1998], which is known to be inefficient for high dimensional robot configuration spaces. Instead, we model tabletop manipulation problems as a transition systems in which multi-waypoint robot trajectories $u_m$ are control parameters. This allows us to design samplers that call efficient motion planners to produce trajectories between pairs of configurations. In our experiments, we use RRT-Connect (Bidirectional Rapidly-exploring Randomized Trees) [Kuffner and LaValle 2000] to sample these trajectories.

Rather than specify $C^o_{\text{Pick}}$ and $C^o_{\text{Place}}$ transitions as instantaneous contacts with each object, we represent robot trajectories moving to, manipulating, and returning from an object as a single transition $C^o_{\text{MPick}}$ or $C^o_{\text{MPlace}}$.

$$C^o_{\text{MPick}} = \{ \text{Stable}_o, \text{Grasp}'_o, \text{Manip},$$

$$x_h = \text{None}, x_o = \text{None},$$

$$\{ x_0' = x_0', C_{Free/o} \} \mid o \in \mathcal{O}, o \neq o' \}$$

$$C^o_{\text{MPlace}} = \{ \text{Grasp}_o, \text{Stable'}_o, \text{Manip}',$$

$$x_h = o, x_h = \text{None},$$

$$\{ x_0' = x_0', C_{Free/o} \} \mid o \in \mathcal{O}, o \neq o' \}$$

This behavior is enforced be a manipulation constraint $\text{Manip}$ on parameters $x_o, x_o', u_m$ representing poses $x_o, x_o'$ for object $o$ and trajectory $u_m$. Let $q_0$ be the initial robot configuration and $q_{Kin}$ be a kinematic solution for end-effector transform $x_o'x_o^{-1}$ grasping object $o$ with grasp $x_o^{-1}$ at placement $x_o'$. The trajectory $u_m$ is the concatenation of a motion plan from $q_0 \rightarrow q_{Kin}$, a grasp plan, and a motion plan $q_{Kin} \rightarrow q_0$. Both motion plans are computed to be free of collisions with fixed obstacles. Additionally, one motion plan avoids collisions with $o$ placed at $x_o'$, and the other avoids collisions between $o$ held at grasp $x_o$ and fixed obstacles. Because the robot always returns to $q_0$, the robot configuration need not be included as a state variable.

We structure the transition system this way based on the insight that the bulk of the robot’s configuration space is unaffected by the placements of movable obstacles. The moveable obstacles mostly only prevent the safe execution of manipulator trajectories above tabletops. Rather than directly plan paths between pairs of configurations manipulating objects, we instead plan paths to a home configuration chosen arbitrarily as the initial configuration $q_0$. This approach guarantees the feasibility of the resulting plan while not inducing significant overhead. Shorter, direct trajectories between pairs of base configurations can later be produced when post-processing a solution. Figure 17 visualizes the set of $u_m$ trajectories as edges in a star roadmap [Garrett et al. 2016] with $q_0$ as the root.

We specify the following samplers to produce values satisfying the constraints in this transition system. The grasp sampler $\psi^g_o$ and placement $\psi^p_o$ sampler are the same as before. The manipulation sampler $\psi^M_o$ is similar to $\psi^F_K$:

$$\psi^M_o = \langle (x_o, x_o'), (u_m), \{ \text{Manip}_o \}, \text{SAMPLE-MANIP} \rangle.$$
The procedure \texttt{SAMPLE-MANIP} samples a base pose and performs manipulator inverse kinematics to identify a grasping configuration to perform the pick or place. Then, it calls \texttt{RRT-CONNECT} twice to find motion plans to this grasping configuration and back. These motion plans are computed to not be in collision with fixed obstacles or both when it is on the table and when it is held. Additionally, each call has a timeout to ensure termination. This timeout is increased on subsequent calls to each sampler instance of $\psi^o_M$.

### 10 Example Mobile Manipulation Problem

To illustrate both the incremental and focused algorithms, we work through their steps on an example mobile manipulation problem. Consider the problem in figure 16 with two movable objects $A, B$ and two tables $T_1, T_2$. States are $\bar{x} = (x_A, x_B, x_h)$ and controls are $\bar{u} = u_m$. The initial state is $\bar{x}_0 = (a_0, b_0, \text{None})$ and the goal constraints are $C_s = \{\text{Region}_A\}$ indicating that object $A$ is placed on $T_1$. For simplicity, we assume that each moveable object has a single grasp $a_g$ or $b_g$. The values $a_0, b_0, a_g, b_g \in \text{SE}(3)$ represent continuous transformations. Similarly, manipulations $m$ are full body motion plans from $q_0$ to a grasping configuration and back. For the example, we assume that the conditional samplers never fail to produce an appropriate value.

#### 10.1 Incremental Algorithm

Table 1 traces the sampler instances $S_i$ for which \texttt{SAMPLE} is called paired with the resulting element for each iteration $i$ of the incremental algorithm. The set of elements available on each iteration is the union of the previously sampled elements $S_j, j < i$. The incremental algorithm fails to find a plan for 2 iterations and finally finds the following plan $\pi_3$ on the 3rd iteration.

$$\bar{a}_3 = [C^A_{\text{M_pick}}, C^A_{\text{M_place}}]$$
$$\bar{x}_3 = [(a_0, b_0, \text{None}), (a_g, b_0, A), (a_1, b_0, \text{None})]$$
$$\bar{u}_3 = [m_1, m_3]$$

Notice that the number of sampler instances sampled per iteration grows quickly. Additionally, samples are generated for both objects $A$ and $B$ despite the task only requiring manipulating $A$.

#### 10.2 Focused Example

Table 2 traces each iteration $i$ of the incremental algorithm. We will assume that new elements are directly added to elements. As before, $S_i$ contains the sampler instances \texttt{SAMPLE} paired with the resulting elements. Elements certified by $\text{CFree}$ and $\text{Region}$ are individually added to $S_i$. The set of sampled sampler instances are contained in $S_j, i < j$. Let $\mathcal{E}_i$ be the set of elements using lazy samples generated by \texttt{PROCESS-SAMPLERS} on each iteration. The union of $\mathcal{E}_i$ and elements is $\text{mixed elements}$. The plan returned on each iteration is denoted by $\pi_i = (\bar{a}_i, \bar{x}_i, \bar{u}_i)$. We denote the lazy sampler for each sampler as follows:

$$\text{LAZY-SAMPLE}(\psi^o_M()) = \{\text{Grasp}(l^G_{i,j})\}$$
LAZY-SAMPLE($\psi_P^0()$) = \{Stable($t^0_P$)\}

LAZY-SAMPLE($\psi^A_M(a_0, a_g)$) = \{Manip($a_0, a_g, m_1$)\}

On the first iteration, the sampler instances $\psi^A_M(x_o, a_o', x_o'^0)$ are sampled to produce values for $t^0_A, t^0_B$ respectively. On the second iteration, $\psi^A_M(a_0, a_g)$ and $\psi^B_M(a_1, a_g)$ generate the manipulations required for the $C_{\text{MPick}}^A$ and $C_{\text{MPlace}}^A$ transitions given the new grasp $a_g$ and placement $a_1$. On the final iteration, a plan $\pi_3$ not requiring any lazy samples is generated, resulting in a solution.

The focused algorithm samples fewer values than incremental by only sampling values determined to be useful for completing lazy samples and satisfying constraints along a plan. More specifically, it avoids sampling values for object $B$ altogether, saving time by not computing expensive motion plans. This behavior becomes even more prevalent in problems with many moveable objects, such as ones arising from human environments.

### 10.3 Additional Example Scenarios

We sketch out several additional problems and outline how, in particular, the focused strategy will proceed in each of these scenarios.

#### 10.3.1 Sampling Failure: We previously assumed that each sampler successfully generated output values satisfying its constraints. In general, samplers may fail to do so because of timeouts or even because no sample exists. For example, suppose $\psi^A_M(a_0, a_g)$ fails to produce a collision-free inverse kinematic solution, resulting in a failure. After the failure, $\psi^A_M(a_0, a_g)$ will be added to sampled, preventing it from being sampled on the next iteration. Without $\text{Manip}(a_0, a_g, m_1)$ or $\text{Manip}(a_0, a_g, t^0_A)$, the focused algorithm will fail to find a plan. In that case, sampled is reset allowing $\psi^A_M(a_0, a_g)$ to be sampled again on the next episode. This cycle will automatically repeat with increased time limits for as long as $\psi^A_M(a_0, a_g)$ fails to produce a manipulation, as picking object $A$ is required for any solution to this problem.

#### 10.3.2 Obstructions: Suppose that object $A$ is initially obstructed by object $B$ as in figure 18. While $\psi^A_M(a_0, a_g)$ can produce a manipulation $m_1$, it cannot be performed because it violates the collision constraint $C_{\text{Free}}(m_1, b_3)$. However, the lazy pose $t^0_B$ is optimistically assumed to not be in collision with $m_1$. Thus, a valid plan involves first moving $B$ to $t^0_B$ before picking $A$. In the event where the sampled value for $t^0_B$ is still in collision with $m_1$, an additional value can be generated on the next episode. Lazy samples allow the focused algorithm to reason about trajectories $u_m$ that do not yet correspond to a feasible transition because they violate one or more collision constraints. By sampling concrete values for the lazy samples corresponding to these violated constraints, it can attempt to find transitions for which the control is feasible.

![Figure 18. Mobile manipulation problem where object $B$ is obstructing manipulations that pick object $A$.](image)

#### 10.3.3 Regrasp: Consider the regrasp experiment in figure 23 where the robot is unable to pick and place the goal object using the same grasp. Because of this, it is forced to place the goal object at an intermediate location to change grasps. The focused algorithm will only create one lazy grasp sample $t^G_A$ for object $A$. On the subsequent iteration, at least one of $\psi^A_M(a_0, a_g)$ and $\psi^A_M(a_1, a_g)$ will fail to sample a manipulation. Both will be added to...
Grasp \( \vec{a} \)

We performed three scaling experiments on pick-and-place problems. All experiments considered five problem sizes, varying the number of objects. We performed five trials using randomly (with the exception of Experiment 2) generated problem instances for each problem size. Each scatter plot in figures [19], [21] and [22] display the total runtime of each configuration per trial. Timeouts are indicated by the omission of a trial.

11 Experiments

We implemented both algorithms and tested them on a suite of tabletop manipulation problems. All experiments used the factored transition system described in section 9. The conditional samplers for poses, grasps, inverse kinematics, and motion plans were implemented using OpenRAVE [Diankov and Kuffner 2008]. We used Open Dynamics Engine [Smith 2005] for collision checking.

We considered two FastDownward [Hellert 2006] configurations for the INCREMENTAL and FOCUSED algorithms: H uses the FastForward heuristic [Hoffmann and Nebel 2001] in a lazy greedy search and No-H is a breadth-first search.

All trials were run on 2.8 GHz Intel Core i7 processor with a 120 second time limit. Our Python implementation of the algorithms can be found here: [https://github.com/caelan/factored-transition-systems](https://github.com/caelan/factored-transition-systems)

We also have developed a similar suite of algorithms for an extension of the PDDL [McDermott et al. 1998] called STRIPStream [Garrett et al. 2017] available at [https://github.com/caelan/stripstream](https://github.com/caelan/stripstream). Videos of the experiments are available at [https://youtu.be/xJ3OemAfmgc](https://youtu.be/xJ3OemAfmgc). Base trajectories are post-processed by computing a direct trajectory between pairs of base configurations.

11.1 Scaling Experiments

We performed three scaling experiments on pick-and-place problems. All experiments considered five problem sizes, varying the number of objects. We performed five sampled causing the DISCRETE-SEARCH to fail to find a plan on the next iteration. After sampled is reset, the focused algorithm is able to use \( l^A_{m} \) to produce the second grasp and arrive at a solution to the problem.

| \( s_0 \) | initial-elements(\( P \)) : \{Stable(\( a_0 \)), Stable(\( b_0 \))\) |
|---|---|
| \( \epsilon_1 \) | Grasp(\( l^A_{m} \)), Grasp(\( l^B_{b} \)), \text{Stable}(l^B_{b}), \text{Region}(l^A_{m}), \text{Stable}(l^B_{b}), \text{Manip}(\( a_0, l^A_{m}, t^A_{m} \)), \text{Manip}(\( b_0, l^B_{b}, t^B_{b} \)), \text{Manip}(\( a_1, l^A_{m}, t^A_{m} \)), \text{Manip}(\( b_1, l^B_{b}, t^B_{b} \)), \text{CFree}(l^A_{m}, b_0), \text{CFree}(l^B_{b}, b_0), \text{CFree}(t^{B}_{m}, a_0), \text{CFree}(t^{B}_{m}, b_0) |
| \( \pi_1 \) | \( \psi_1^A(\vec{L}) : \text{Grasp}(\vec{a}), \psi_1^A(\vec{L}) : \text{Stable}(\vec{a}), \text{Region}(\vec{a}) \) |
| \( \epsilon_2 \) | Grasp(\( l^A_{m} \)), \text{Stable}(l^B_{b}), \text{Manip}(\( a_0, a_1, t^A_{m} \)), \text{Manip}(\( a_1, a_2, t^A_{m} \)), \text{CFree}(l^A_{m}, b_0), \text{CFree}(l^A_{m}, b_1), \text{CFree}(l^B_{b}, a_0), \text{CFree}(l^B_{b}, a_1), \text{CFree}(l^B_{b}, t^B_{b}) |
| \( \pi_2 \) | \( \psi_2^A(\vec{L}) : \text{Manip}(\vec{a}, a_0, a_1, m_1), \psi_2^A(\vec{L}) : \text{Manip}(\vec{a}, a_0, a_1, m_2), \text{CFree}(m_1, p_0), \text{CFree}(m_2, p_0) \) |
| \( \epsilon_3 \) | Grasp(\( l^A_{m} \)), \text{CFree}(l^B_{b}, t^B_{b}), \text{Manip}(\( b_0, l^B_{b}, t^B_{b} \)), \text{Manip}(\( b_1, l^B_{b}, t^B_{b} \)), \text{CFree}(l^{B}_{m}, a_0), \text{CFree}(l^{B}_{m}, a_1) |
| \( \pi_3 \) | \( \psi_3^A(\vec{L}) : \text{Manip}(\vec{b}, b_0, b_1, m_1), \text{Manip}(\vec{b}, b_0, b_1, m_2) \) |

Table 2: Example walkthrough of the focused algorithm. Each \( \epsilon_i \) displays the set of lazy elements at the start of the iteration. Each \( S_i \) displays the set of sampler instances for which \( \text{SAMPLE} \) is called along with the new elements produced.

![Figure 19. Experiment 1: total runtime of the algorithms over 5 trials per problem size.](image-url)
Figure 20. Experiment 2: the robot must place the green object in the green region.

Figure 21. Experiment 2: total runtime of the algorithms over 5 trials per problem size.

Each object has four side-grasps. This experiment reflects many real-world environments where the state-space is enormous but many objects do not substantially affect a task. As can be seen in figure 21 both Focused-No-H and Focused-H solved all problem instances showing that the Focused algorithm is able to avoid producing samples for objects until they are relevant to the task.

Experiment 3 in figure 1 has the goal that a single blue object be moved to a different table. The blue object starts at the center of the visible table, and the red objects are randomly placed on the table. The table size scales with the number of objects. Each object has four side-grasps. As shown in figure 22 Focused-H solved all instances and Focused-No-H solved all but one (size=21).

11.2 Additional Experiments

We experimented on several additional problems to show that the factored transition system framework and algorithms can be successfully applied to problems involving pushing, stacking, and discrete state variables. We also experimented on two tricky pick-and-place problems that require regrasping and require violating several goal constraints along a plan to achieve the goal. We conducted 40 trials per problem and algorithm, and each trial once again had a 120 second time limit. The success percentage of each algorithm (%) and mean runtime for successful trials are displayed in table 3. Each algorithm performed comparably on the first four problems (Regrasp, Push, Wall, Stacking). Only the heuristically informed algorithms where able to consistently solve the larger last two problems (Nonmon., Dinner).

The regrasp problem (Regrasp) in figure 20 is Problem 3 of Garrett et al. (2015). The goal constraints are that the green object be at the green pose and the blue object remain at its current pose. The robot must place the green object at an intermediate pose to obtain a new grasp in order to insert it in the thin, right cupboard. This indicates that the algorithms can solve pick-and-place problems where even non-collision constraints affect the plan skeleton of solutions.

The first pushing problem (Push) in figure 23 has the goal constraint that the short blue cylinder on the left table be placed at the blue point on the right table. Because the blue cylinder is short and wide, the robot is unable to grasp it except by side grasps at the edges of each table. Thus, the robot must first push the cylinder to the edge of the left table, pick the cylinder, place the cylinder on the edge of the right table, and push the cylinder to the goal point. This problem introduces an additional transition relating to trajectories corresponding movements between two poses. Additionally, it requires a conditional sampler to generate push trajectories and motion plans per pairs of poses on the same table.

The second pushing problem (Wall) in figure 24 is Problem 2 of Garrett et al. (2015) where the goal constraint is that the short green cylinder be placed at the green point.
Figure 23. **Regrasp** (left): a forced regrasp problem. **Push** (right): a problem requiring pushing, picking, and placing.

A wall of moveable objects initially blocks the robot from pushing the green cylinder to the goal point. However, if several of these blocks are moved, the robot can execute a sequence of pushes to push the green cylinder directly to its goal.

Figure 24. **Wall**: a pushing problem involving a wall of blocks.

The stacking problem (**Stacking**) in figure 24 is Problem 4 of Garrett et al. (2015). The goal constraints are that the blue block be contained within the blue region, the green block be contained within the green region, and the black block be on top of the blue block. The robot must unstack the red block to safely move the green block. This problem requires a modification of the transition system to account for stability constraints. Additionally, it requires a conditional sampler that produces poses of the black block on the blue block given poses of the blue block.

The pick-and-place problem (**Nonmon.**) in figure 26 is Problem 3-2 of Garrett et al. (2016). The goal constraints are that the green blocks be moved from their initial pose on the left table to their corresponding pose on the right table. Additionally, there are goal constraints that each blue and cyan block end at its initial pose. This is a highly nonmonotonic problem as solutions require violating several goal constraints satisfied by the initial state. Incremental-\(H\) slightly outperformed Focused-\(H\) because solving this problem required manipulating all but objects. Thus, Incremental-\(H\) and Focused-\(H\) produce comparable sets of samples, but Incremental-\(H\) has less overhead. Both algorithms performed significantly better than best algorithm of Garrett et al. (2016) which had a 72% success rate and took on average 135 seconds.

The task and motion planning problem (**Dinner**) in figure 27 is Problem 5 of Garrett et al. (2016). The state contains an additional discrete state variable for each block indicating whether it is dirty, clean, or cooked. The transition relation contains additional clauses to clean a dirty block when it is placed on the dishwasher and to cook a clean block when it is placed on the microwave. The goal constraints are that the green blocks (“cabbage”) be cooked and placed on the plates, the blue blocks (“cups”) be cleaned and placed at the blue points, the cyan block (an unnecessary “cup”) be cleaned, and the pink blocks (“turnips”) remain placed on the shelf. To reach the cabbage, the robot must first move several turnips and later replace them to keep the kitchen tidy. The heuristic guided Incremental-\(H\) and Focused-\(H\) planners were able to quickly solve the problem reinforcing the point that search guidance is necessary for problems over long horizons. These algorithms compare favorably to the best algorithm of Garrett et al. (2016) which had a 76% success rate and took on average 44 seconds.
Figure 27. *Dinner*: a task and motion planning problem.

| Problem | Incr. | Incr. - H | Focus | Focus - H |
|---------|-------|-----------|-------|-----------|
|         | %     | t         | %     | t         | %     | t         |
| Regrasp | 98    | 1         | 100   | 2         | 98    | 1         | 95    | 1         |
| Push    | 100   | 11        | 100   | 13        | 100   | 13        | 100   | 9         |
| Wall    | 95    | 10        | 98    | 13        | 100   | 6         | 100   | 8         |
| Stacking| 100   | 9         | 100   | 9         | 100   | 2         | 100   | 3         |
| Nonmon. | 25    | 21        | 98    | 15        | 0     | -         | 88    | 43        |
| Dinner  | 0     | -         | 100   | 27        | 0     | -         | 98    | 22        |

Table 3. The success percentage (%) and mean runtime (t) for the additional experiments over 40 trials.

12 Conclusion

We introduced factored transition systems for modeling discrete-time planning problems for hybrid systems. Factored transition systems can model motion planning, pick-and-place planning, and task and motion planning domains. Manipulation domains are significantly factorable. Legal transitions can be expressed as the conjunction of constraints each involving only several state or control variables.

Conditional constraint manifolds enabled us to give a general definition of robust feasibility for factored transition systems. Under certain conditions, they allow us to describe a submanifold of plan parameter-space resulting from the intersection of dimensionality-reducing constraints. Thus, robustness properties can be examined relative to this submanifold rather than the for plan parameter-space. We introduced the idea of conditional samplers: samplers that given input values, produce a sequence of output values satisfying a constraint with the input values. When appropriate conditional samplers are specified for each conditional constraint manifold, the resulting collection of samplers is sufficient for solving any robustly feasible problem. Sampling benefits from factoring because a small collection of samples for each variable can correspond to a large number of combined states and transitions.

We gave two general-purpose algorithms that are probabilistically complete given sufficient samplers. The incremental algorithm iteratively calls each conditional sampler and tests whether the set of samples is sufficient using a blackbox, discrete search. The focused algorithm first creates lazy samples representing hypothetical outputs of conditional samplers and then uses a blackbox, discrete search to identify which lazy samples could be useful. We empirically demonstrated that these algorithms are effective at solving challenging pick-and-place, pushing, and task and motion planning problems. The focused algorithm is more effective than the incremental algorithm in domains with many movable objects as it can selectively produce samples for only objects affecting the feasibility of a solution. Additionally, both algorithms were more efficient when using a discrete search subroutine that exploited factoring in the search through domain-independent heuristics.

12.1 Future Work

Future work involves developing additional algorithms for solving factored transition systems. In particular, both the incremental and focused algorithms treat DISCRETE-SEARCH as a blackbox. By directly integrating the search and sampling, an algorithm may be able to more directly target sampling based on the search state and possible transitions. For example, Backward-Forward Search (Garrett et al. 2015) performs its search directly in the hybrid state-space instead of a discretized state-space.

Our formulation gives rise to several new opportunities for learning to improve sampling and search runtimes. For instance, one could learn a policy to decide how frequently to sample each conditional sampler. A high performing policy would balance the likelihood of a conditional sampler to produce useful samples, the overhead of computing samples, and the impact additional samples have on subsequent discrete searches. Similarly, in the focused algorithm, one could learn costs associated with using lazy samples reflective of the expected future planning time resulting from sampling from a particular conditional stream. These costs could cause DISCRETE-SEARCH to produce plans that are likely realizable without too much overhead.

Finally, this work can be extended to optimal planning settings where there are nonnegative costs $c(\overline{u})$ on control inputs. This will require adapting properties such as asymptotic optimality (Karaman and Frazzoli 2011) to the factored transition system setting and modifying the incremental and focused algorithms to achieve these properties.
13 Acknowledgements

We thank Beomjoon Kim and Ferran Alet for their feedback on this manuscript. We gratefully acknowledge support from NSF grants 1420316, 1523767 and 1723381, from AFOSR FA9550-17-1-0165, from ONR grant N00014-14-1-0486. Caelan Garrett is supported by an NSF GRFP fellowship with primary award number 1122374. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of our sponsors.

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